Hitting Time Quasi-metric
and Its Forest Representation

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Abstract

Let \( \hat{m}_{ij} \) be the hitting (mean first passage) time from state \( i \) to state \( j \) in an \( n \)-state ergodic homogeneous Markov chain with transition matrix \( T \). Let \( \Gamma \) be the weighted digraph whose vertex set coincides with the set of states of the Markov chain and arc weights are equal to the corresponding transition probabilities. It holds that

\[
\hat{m}_{ij} = q_j^{-1} \cdot \begin{cases} 
    f_{ij}, & \text{if } i \neq j, \\
    q, & \text{if } i = j,
\end{cases}
\]

where \( f_{ij} \) is the total weight of 2-tree spanning converging forests in \( \Gamma \) that have one tree containing \( i \) and the other tree converging to \( j \), \( q_j \) is the total weight of spanning trees converging to \( j \) in \( \Gamma \), and \( q = \sum_{j=1}^{n} q_j \) is the total weight of all spanning trees in \( \Gamma \). Moreover, \( f_{ij} \) and \( q_j \) can be calculated by an algebraic recurrent procedure. A forest expression for Kemeny’s constant is an immediate consequence of this result. Further, we discuss the properties of the hitting time quasi-metric \( m \) on the set of vertices of \( \Gamma \): \( m(i,j) = \hat{m}_{ij}, i \neq j \), and \( m(i,i) = 0 \). We also consider a number of other metric structures on the set of graph vertices related to the hitting time quasi-metric \( m \)—along with various connections between them. The notions and relationships under study are illustrated by two examples.

Keywords: Mean first passage time; Spanning rooted forest; Hitting time quasi-metric; Resistance metric; Commute time metric; Markov Chain Tree Theorem, Partial metric

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1 Introduction

Let $T = [t_{ij}] \in \mathbb{R}^{n \times n}$ be the transition matrix of an $n$-state ergodic homogeneous Markov chain with states $1, 2, \ldots, n$. Then $T$ is an irreducible stochastic matrix.

The mean first passage time (also called the hitting time) from state $i$ to state $j$ is

$$\hat{m}_{ij} = E(F_{ij}) = \sum_{k=1}^{\infty} k \Pr(F_{ij} = k),$$

where

$$F_{ij} = \min\{p > 0 : X_p = j \mid X_0 = i\}$$

and $X_p$ is the state of the chain at time $p \in \mathbb{N}$. By [29, Theorem 3.3], the matrix $\hat{M} = [\hat{m}_{ij}] \in \mathbb{R}^{n \times n}$ has the following representation:

$$\hat{M} = (I - L^\# + 1\ell_{dg}^\#)\Pi^{-1},$$

where $1 = (1, \ldots, 1)^T \in \mathbb{R}^n$, $I = \text{diag}(1)$, $L^\# = [l^\#_{ij}]_{n \times n}$ is the group inverse [29] of $L$,

$$L = I - T,$$

$$\ell_{dg}^\# = (l^\#_{11}, \ldots, l^\#_{nn}), \quad \Pi = \text{diag}(\pi_1, \ldots, \pi_n), \quad \text{and} \quad (\pi_1, \ldots, \pi_n) = \pi \text{ is the normalized left Perron vector of } T,$$

satisfying

$$\pi T = \pi \quad \text{and} \quad \|\pi\|_1 = \sum_{i=1}^{n} \pi_i = 1.$$

In an entrywise form, (3) reads as follows (see, e.g., [5]):

$$\hat{m}_{ij} = \pi_j^{-1} \left\{ \begin{array}{ll}
(l^\#_{jj} - l^\#_{ij}), & \text{if } i \neq j, \\
1, & \text{if } i = j.
\end{array} \right. \quad (5)$$

In the next section, we present a graph-theoretic interpretation of hitting times related to this formula.

Remark 1. If one replaces $p > 0$ with $p \geq 0$ in the definition (1)–(2) of hitting time, i.e.,

$$m_{ij} = E(\min\{p \geq 0 : X_p = j \mid X_0 = i\}),$$

then $m_{ii} = 0$, $i = 1, \ldots, n$, and (5) and (3) simplify to

$$m_{ij} = \frac{l^\#_{jj} - l^\#_{ij}}{\pi_j}, \quad i, j = 1, \ldots, n \quad (7)$$

and

$$M = [m_{ij}]_{n \times n} = (1\ell_{dg}^\# - L^\#)\Pi^{-1}.$$
2 A forest expression for the hitting times

Let us say that a weighted digraph $\Gamma$ with vertex set $V = \{1, \ldots, n\}$ corresponds to the Markov chain with transition matrix $T$ if $\Gamma$ has an arc $(i, j)$ with $i \neq j$ whenever $t_{ij} \neq 0$, and the weight $w_{ij}$ of this arc is $t_{ij}$. Obviously, in this case the Laplacian (Kirchhoff) matrix $\tilde{L}$ of weighted digraph $\Gamma$,

$$\tilde{L} = \text{diag}(W) - W,$$

(8)

where $W = [w_{ij}]_{n \times n}$, coincides with $L$ of (4).

Recall some graph-theoretic notation. A digraph is weakly connected if the corresponding undirected graph is connected. A weak component of a digraph $\Gamma$ is any maximal weakly connected subdigraph of $\Gamma$. A converging tree is a weakly connected digraph in which one vertex, called the root, has outdegree zero and the remaining vertices have outdegree one. An in-forest of $\Gamma$ is a spanning subdigraph of $\Gamma$ all of whose weak components are converging trees (also called in-arborescences). An in-forest is said to converge to the roots of its converging trees. An in-forest $F$ of a digraph $\Gamma$ is called a maximum in-forest if $\Gamma$ has no in-forest with a greater number of arcs than $F$. The in-forest connectivity of a digraph $\Gamma$ is the number of weak components in any maximum in-forest. Obviously, every maximum in-forest of $\Gamma$ has $n - d$ arcs, where $d$ is the in-forest connectivity of $\Gamma$. A submaximum in-forest of $\Gamma$ is an in-forest of $\Gamma$ that has $d + 1$ weak components; as a consequence, it has $n - d - 1$ arcs. The weight of a weighted digraph is the product of its arc weights; the weight of any digraph that has no arcs is 1. The weight of a set of digraphs is the sum of the weights of its members. In this paper, our main tool is Lemma 1.

Lemma 1 ([9], (iii) of Proposition 15). For any weighted digraph $\Gamma$, it holds that

$$\tilde{L}^\# = \frac{\sigma_{n-d-1}}{\sigma_{n-d}} (P_{n-d+1} - P_{n-d}),$$

(9)

where $\sigma_k$ is the total weight of in-forests with $k$ arcs, $P_k = Q_k/\sigma_k$, and $Q_k$ is the matrix whose $ij$-entry $q_{ij}^{(k)} (i, j = 1, \ldots, n)$ is the total weight of in-forests that have $k$ arcs and vertex $i$ belonging to the tree that converges to vertex $j$.

The following theorem presented in [7] is a forest representation of the hitting times.

Theorem 1. Let $T \in \mathbb{R}^{n \times n}$ be the transition matrix of an $n$-state ergodic homogeneous Markov chain with states $1, \ldots, n$. Let $\Gamma$ be the weighted digraph without loops whose vertices are $1, \ldots, n$ and arc weights are equal to the corresponding transition probabilities in $T$. Then the hitting time from state $i$ to state $j$ in this chain is given by

$$\hat{m}_{ij} = q_j^{-1} \left\{ \begin{array}{ll} f_{ij}, & \text{if } i \neq j, \\ q, & \text{if } i = j, \end{array} \right.$$  

(10)

where $f_{ij}$ is the total weight of 2-tree in-forests of $\Gamma$ that have one tree containing $i$ and the other tree converging to $j$, $q_j$ is the total weight of spanning trees converging to $j$, and $q = \sum_{k=1}^{n} q_k = \sigma_{n-1}$ is the total weight of all converging trees in $\Gamma$. 

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**Proof.** Observe that since the Markov chain under consideration is ergodic, the corresponding digraph $\Gamma$ has a spanning converging tree. Thus, its in-forest connectivity $d$ is 1. Hence, for every $i, j = 1, \ldots, n$, each maximum in-forest converging to $j$ is a spanning converging tree, which contains $i$. Therefore, the $jj$- and $ij$-entries of the matrix $Q_{n-d} = Q_{n-1}$ coincide: $q_{jj}^{(n-1)} = q_{ij}^{(n-1)} = q_j$, $i, j = 1, \ldots, n$, where $q_j$ is the total weight of spanning trees converging to $j$. Thus, the differences $p_{jj}^{(n-1)} - p_{ij}^{(n-1)}$, where $[p_{ij}^{(n-1)}]_{n \times n} = P_{n-1} = P_{n-d}$, are 0, as well as the differences $q_{jj}^{(n-1)} - q_{ij}^{(n-1)}$. Consequently, for any $i \neq j$, substituting (9) into (5) yields

$$
\hat{m}_{ij} = \frac{l_{jj} - l_{ij}}{\pi_j} = \frac{\sigma_{n-2}}{\sigma_{n-1} \pi_j} (p_{jj}^{(n-2)} - p_{ij}^{(n-2)}) = \frac{q_{jj}^{(n-2)} - q_{ij}^{(n-2)}}{\sigma_{n-1} \pi_j} = \frac{f_{ij}}{\sigma_{n-1} \pi_j},
$$

where

$$
f_{ij} \eqdef q_{jj}^{(n-2)} - q_{ij}^{(n-2)}.
$$

It follows from the definition of $Q_{n-2}$ that $f_{ij}$ is the weight of the set of 2-tree in-forests of $\Gamma$ that converge to $j$ and have $i$ and $j$ in different trees.

Furthermore, we know from the Markov Chain Tree Theorem \[27\, 28\] obtained earlier in [39, Lemma 7.1] (see also [40, 17, Lemma 3.1] and the references in [31]) that

$$
\pi_j = q_j / q, \quad i = 1, \ldots, n
$$

where $q = \sum_{k=1}^n q_k = \sigma_{n-1}$. Now (11), (13), and (5) provide $\hat{m}_{ij} = \frac{f_{ij}}{q_j}$ for $i \neq j$ and $\hat{m}_{jj} = \frac{q_j}{q}$. \hfill $\square$

**Corollary 1.** For the version of hitting times introduced by (6), in the notation of Theorem 1

$$
m_{ij} = f_{ij} / q_j, \quad i, j = 1, \ldots, n
$$

and $M = (1q_{dg}^{(n-2)} - Q_{n-2})(\text{diag}(q_1, \ldots, q_n))^{-1}$, where $q_{dg}^{(n-2)} = (q_{11}^{(n-2)}, \ldots, q_{nn}^{(n-2)})$.

**Remark 2.** The values $q_j = q_{jj}^{(n-1)}$ and $f_{ij}$ that satisfy (12) can be calculated by means of elementary matrix algebra, namely, by the following recurrent procedure [9, Proposition 4], which has a polynomial complexity. For $k = 0, 1, \ldots, n - 2$ one has

$$
\sigma_{k+1} = \frac{\text{tr}(LQ_k)}{k+1},
$$

$$
Q_{k+1} = -LQ_k + \sigma_{k+1}I,
$$

where $\sigma_0 = 1$, and $Q_0 = I$. Relations with the GTH algorithm can be found in [37].

**Remark 3.** Theorem 1 can be alternatively derived from [30, Lemma 3.3] or [41, Lemma 3.4], both based on Lemma 3.4 in [10, 17]. The authors are grateful to Raphael Cerf for pointing out Ref. [41]. Some special cases of Theorem 1 were obtained in [21]; see also [31].

The following corollary (recently appeared as [31, Corollary 1.4] and [23, Theorem 2.3]) relates to Kemeny’s constant (see [20]) and is an immediate consequence of Theorem 1.
Corollary 2. Under the conditions of Theorem 1, Kemeny's constant is equal to $1 + \frac{\sigma_{n-2}}{\sigma_{n-1}}$.

Proof. By Theorem 1 and (13), for any $i \in V$,

$$\sum_{j=1}^{n} \pi_j \hat{m}_{ij} = \frac{q_i}{q} \cdot q + \sum_{j \neq i} \frac{q_j}{q} \cdot f_{ij} = 1 + \sum_{j \neq i} f_{ij} = 1 + \frac{\sigma_{n-2}}{\sigma_{n-1}}.$$  \hspace{1cm} (17)

The last transition holds because the weight of any 2-tree in-forest of $\Gamma$ is included in exactly one sum $f_{ij}$: it is the one where $j$ is the root of the tree that does not contain $i$. \hfill \Box

Remark 4. By (17) for any $i \in V$, $\sum_{j=1}^{n} \pi_j m_{ij} = \frac{\sigma_{n-2}}{\sigma_{n-1}}$. Along with (17) and Corollary 1 this convinces that the $m_{ij}$'s lead to simpler expressions than the $\hat{m}_{ij}$'s do.

In Sections 3 and 4 we study metric properties of the hitting times.

3 Hitting time quasi-metric and related metrics

3.1 Hitting time quasi-metric

A function $d : X \times X \to \mathbb{R}$ is a quasi-metric on $X$ \cite{19, 41, 14} if for all $x, y, z \in X$,

1. $d(x, y) \geq 0$;
2. $d(x, y) = 0$ if and only if $x = y$;
3. $d(x, y) \leq d(x, z) + d(z, y)$ (oriented triangle inequality).

As distinct from metrics, quasi-metrics are not generally symmetric.

It was observed in Remark 4 that the $m_{ij}$ version \cite{10} of hitting times leads to more elegant expressions than the $\hat{m}_{ij}$ version \cite{11}. In addition, by \cite{22, Proposition 9-58}, $m(i, j) = m_{ij}$ is a quasi-metric on the set of states of our Markov chain (and on the set of vertices of any corresponding weighted digraph $\Gamma$). It is called the hitting time (or mean first-passage time) quasi-metric.

Moreover, by \cite{22, Proposition 9-58} or \cite{24, Theorem 6.2.1}, this quasi-metric satisfies the cutpoint additivity \cite{8} (also called the graph-geodetic property \cite{25}):

$$m(i, j) = m(i, k) + m(k, j)$$

holds true if and only if all paths in $\Gamma$ from $i$ to $j$ pass through $k$.

3.2 Commute time metric

The commute time metric (or random roundtrip time distance) $c$ on the set of states of our Markov chain (or on $V(\Gamma)$, where $\Gamma$ is any corresponding weighted digraph) is defined by

$$c(i, j) = m(i, j) + m(j, i), \quad i, j = 1, \ldots, n.$$  \hspace{1cm} (18)

The commute time $c(i, j)$ is the average number of steps that takes a random walk to reach $j$ from $i$ and return to $i$. $C = [c_{ij}]$ is the corresponding matrix. Using (14) we have
Corollary 3. Under the conditions of Theorem 1 for all \(i, j \in V\), \(c(i, j) = \frac{f_{ij}}{q_i} + \frac{f_{ji}}{q_j}\).

Since \(m(i, j)\) is a cutpoint additive quasi-metric and \(c(i, j)\) is symmetric, \(c(i, j)\) is a cutpoint additive metric.

### 3.3 Resistance distance

There is a strong connection between random walks in graphs and electric networks [15]. Given a connected weighted undirected graph \(G\), the underlying electrical network is the network obtained by replacing vertices and edges by nodes and electrical resistors, respectively. Edge weights are interpreted as conductances, so the resistances are the reciprocal weights. The effective resistance \(\Omega(i, j) = \Omega_{ij}\) between any two nodes \(i\) and \(j\) is defined as the voltage that develops between \(i\) and \(j\) when a unit current is maintained through them (i.e., enters one and leaves the other node).

Obviously, for all nodes \(i, j, k\), \(\Omega(i, j) \geq 0\), \(\Omega(i, j) = 0\) iff \(i = j\), \(\Omega(i, j) = \Omega(j, i)\), and it can be shown that

\[
\Omega(i, j) + \Omega(j, k) \geq \Omega(i, k),
\]

i.e., \(\Omega(\cdot, \cdot)\) is a metric [35, 18] called the electric metric (or the resistance distance [26]).

Let \(\tilde{L}\) be the symmetric Laplacian matrix of \(G\) defined by (8), where \(W\) is the matrix of edge weights of \(G\). The tilde distinguishes this matrix from \(L\) of (4). The resistance distance in \(G\) can be represented as follows [36, 32]:

\[
\Omega(i, j) = \tilde{\ell}^#_{ii} + \tilde{\ell}^#_{jj} - \tilde{\ell}^#_{ij} - \tilde{\ell}^#_{ji}, \tag{19}
\]

where \(\tilde{L}# = [\tilde{\ell}^#_{ij}]_{n \times n}\) is the group inverse (coinciding in this case with the Moore-Penrose generalized inverse) of \(\tilde{L}\).

Consider a forest representation of the resistance distance ([33, Theorem 7-4]; [34]).

**Corollary 4.** \(\Omega(i, j) = f'_{ij}/q',\) where \(q'\) is the total weight of spanning trees in \(G\) and \(f'_{ij}\) is the total weight of 2-tree spanning forests of \(G\) having \(i\) and \(j\) in different trees.

**Proof.** Let \(\Gamma\) be the directed version of \(G\): for every edge of \(G\), \(\Gamma\) has a pair of opposite arcs carrying the weight of that edge. \(G\) and \(\Gamma\) share the same \(\tilde{L}\). In the same way as in (11), for \(\Gamma\) we get \(\tilde{\ell}^#_{ii} + \tilde{\ell}^#_{jj} - \tilde{\ell}^#_{ij} = (f_{ij} + f_{ji})/q\). Returning to \(G\) observe that \(q = nq'\) (as any spanning tree in \(G\) corresponds to \(n\) trees of the same weight converging to different vertices in \(\Gamma\)) and \(f_{ij} + f_{ji} = nf'_{ij}\) (as any 2-tree spanning forest in \(G\) with \(n_i\) vertices in the tree containing \(i\) and \(n_j = n - n_i\) vertices in the tree containing \(j\) corresponds to \(n_i\) converging forests whose weights are counted in \(f_{ij}\) and \(n_j\) forests whose weights are counted in \(f_{ji}\)). Therefore by (19), \(\Omega(i, j) = (f_{ij} + f_{ji})/q = f'_{ij}/q'.\)

There are two popular ways of attaching a Markov chain to a weighted graph \(G\). The first one is to define the transition matrix (cf. the first paragraph of Section 2) by

\[
T_\tau = I - \tau \tilde{L}, \tag{20}
\]

where \(0 < \tau \leq (\max_i \sum_j w_{ij})^{-1}\), which guarantees the stochasticity of \(T = T_\tau\). Then \(T_\tau\)

\footnote{Sometimes \(\tau = (\max_i \sum_{j \neq i} w_{ij})^{-1}\) or \(\tau = ((n - 1) \max_{i,j} w_{ij})^{-1}\) or \(\tau = (n \max_{i,j} w_{ij})^{-1}\) is chosen.}
is symmetric for any undirected $G$. Moreover, all transition probabilities between distinct vertices are proportional to the edge weights in $G$, as the matrix (4) of $T$ is proportional to $L$. On the other hand, $T$ normally has a nonzero diagonal even when $G$ has no loops, which allows the corresponding Markov chain to preserve its state on adjacent steps.

The second way is to normalize each row of $W$ separately:

$$T_W = (\text{diag}(W1))^{-1}W.$$  \hfill (21)

Here, the symmetry of $W$ does not guarantee the symmetry of $T = T_W$, while the chain alters its state on each step whenever $G$ has no loops.

It is noteworthy that with either way of defining $T$, the resistance distance for $G$ is proportional to the commute time metric for the Markov chain determined by $T$.

**Proposition 1.** For transition matrices (20), $C = n\tau^{-1} \Omega$, where $C = [c_{ij}], \Omega = [\Omega_{ij}]$.

**Proof.** Observe that by (4), $L = \tau L$. Since $T$ is symmetric, $\pi = n^{-1}1^T$ holds. Now comparing (7) and (18) with (19) we have $c(i,j) = n(l^#_{jj} - l^#_{ij} + l^#_{ii} - l^#_{ji}) = n\tau^{-1} \Omega(i,j)$. \hfill $\Box$

**Corollary 5 (16).** For the transition matrix (21), $C = (\sum_{k,t=1}^n w_{kt}) \Omega$.

**Proof.** By Theorem $\Pi$ $m(i,j) = \frac{f_{ij}}{q_j}$ ($i \neq j$). Every spanning tree converging to $j$ in $\Gamma$ has one arc weight in each row of $T$, except for row $j$. Hence by (21), $q_j = q's_jR$, where $q'$ is the total weight of spanning trees in $G$, $s = (s_1, \ldots, s_n)^T = W1$, and $R = (\prod_{k=1}^n s_k)^{-1}$. Every 2-tree in-forest whose weight is a term of the sum $f_{ij}$ has one arc weight in each row of $T$, except for row $j$ and some other row $k$. Hence by (21), $f_{ij} = \sum_{k\neq j} f'_{ik,j} s_j s_k R$, where $f'_{ik,j}$ is the total weight of 2-tree spanning forests in $G$ having $i$ and $k$ in one tree and $j$ in the other tree. Therefore, $m_{ij} = \frac{f_{ij}}{q_j} = \frac{\sum_{k\neq j} f'_{ik,j} s_k}{q_j}$ and $c_{ij} = m_{ij} + m_{ji} = (q')^{-1} (\sum_{k\neq j} f'_{ik,j} s_k + \sum_{k\neq i} f'_{jk,i} s_k)$. The weight of each 2-tree spanning forest of $G$ having $i$ and $j$ in different trees is a term of the first sum on the r.h.s. with multipliers $s_k$ for all vertices $k$ of its tree containing $j$ and enters the second sum with multipliers $s_k$ for all vertices $k$ of its tree containing $j$. Thus, using Corollary 4 we have $c(i,j) = (q')^{-1} f_{ij} \sum_{k=1}^n s_k = (\sum_{k,t=1}^n w_{kt}) \Omega(i,j)$. \hfill $\Box$

For additional relations between the electric metric and Markov chains, we refer to [16] and for relevant identities to [2]. In [12], effective resistance is generalized to directed graphs.

4 A weighted form of hitting times for random walks

Consider the hitting time quasi-metric in the case of random walks on connected positively weighted undirected graphs $G$, when the transition matrix is defined by (21). The class of such walks coincides with that of irreducible reversible Markov chains [1, Section 3.2].

In this case, $\pi = (\pi_1, \ldots, \pi_n)$ is obviously proportional to $1^T W$. Furthermore, the hitting time quasi-metric $m$ is [12, p. 32] a weightable quasi-metric (see also [13] and [14, Chapter 16]), i.e., there exists a weight function $u : V \to \mathbb{R}_{\geq 0}$ such that for all $i, j \in V$ it holds that

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\footnotesize

3On hitting times for random walks on directed graphs, we refer to [8].
\[ m(i, j) + u_i = m(j, i) + u_j, \quad (22) \]

where \( u_i = u(i) \). Consequently, the hitting time quasi-metric \( m \) has the cyclic tour property (also called the relaxed symmetry property): for any \( i, j, k \in V \), it holds that

\[ m(i, j) + m(j, k) + m(k, i) = m(i, k) + m(k, j) + m(j, i). \quad (23) \]

For unweighted graphs \( G \), this property appeared in \([10, \text{Lemma 2}]\). In turn, the cyclic tour property implies the weightability of \( m \). Indeed, for an arbitrary \( k \in V \), set

\[ u_i = m(k, i) - m(i, k), \quad i = 1, \ldots, n. \quad (24) \]

Now for any \( i, j \in V \), \((23)\) gives

\[ m(i, j) - m(j, i) = -m(j, k) - m(k, i) + m(k, j) = -u_i + u_j, \]

yielding \((22)\). It remains to apply (if necessary) a shift that provides the defined function \( u \) with non-negativity. Thus, \((22)\) and \((23)\) are equivalent, and the weightability of hitting times for random walks on undirected weighted graphs follows from the cyclic tour property of any reversible Markov chain \([1]\). Conversely, the cyclic tour property implies reversibility \([38]\) and thus, representability of the chain as a random walk on a weighted undirected graph. Hence, the weightability of hitting times indicates that the chain has the above representation.

A probabilistic interpretation of the weights \( u_i \) is clear from \((24)\): \( u_1, \ldots, u_n \) are, up to a shift, hitting time differences from an arbitrary vertex \( k \) to all vertices and back. They relatively measure hitting asymmetry of the vertices. Corollary \([1]\) supplies a structural description of this relative asymmetry: \( u_i = f_{ki}/q_i - f_{ik}/q_k \). It is worth recalling that the forest representation involves the weighted digraph \( \Gamma \) of Theorem \([1]\) rather than the initial graph \( G \).

The commute time metric \( c \) on \( V \) has now the representation

\[ c(i, j) = m(i, j) + m(j, i) = 2m(i, j) + u_i - u_j, \]

while

\[ m(i, j) = \frac{1}{2}(c(i, j) - u_i + u_j). \]

In this case, the pair \((c, u)\) is a weighted metric on \( V \), i.e., a metric with a weight function \( u : V \to \mathbb{R}_{\geq 0} \) such that the down-weighted condition \( c(i, j) \geq u_i - u_j \) is satisfied \([11, \text{Chapter 6}]\). Furthermore, the function \( p \),

\[ p(i, j) = m(i, j) + u_i = \frac{1}{2}(c(i, j) + u_i + u_j), \]

is a partial metric on \( V \) (cf. \([11]\)), which means that for all \( i, j, k \in V \), it holds that:

1. \( p(i, j) \geq 0; \)
2. \( p(i, j) \geq p(i, i) \) (small self-distances);
3. \( p(i, i) = p(j, j) = p(i, j) \Rightarrow i = j \) (separation axiom);
4. \( p(i, j) = p(j, i) \) (symmetry);
5. \( p(i, j) \leq p(i, k) + p(k, j) - p(k, k) \) (sharp triangle inequality).
It is straightforward to check that
\[
\frac{1}{2}(c(i,k) + c(k,j) - c(i,j)) = p(i,k) + p(k,j) - p(i,j) - p(k,k) = m(i,k) + m(k,j) - m(i,j),
\]
\[c\]
i.e., the respective triangle inequalities are equivalent on all three levels: of the weighted metric \(c\), of the partial metric \(p\), and of the weightable quasi-metric \(m\).

Moreover,
\[m(i,j) \geq 0 \Leftrightarrow c(i,j) \geq u_i - u_j \Leftrightarrow p(i,j) \geq p(i,i).
\]
So, the non-negativity condition \(m(i,j) \geq 0\) for the (weightable) quasi-metric \(m\) is equivalent to the down-weighted condition \(c(i,j) \geq u_i - u_j\) for the weighted metric \(c\), and to the small self-distances condition \(p(i,j) \geq p(i,i)\) for the partial metric \(p\).

Now let us call a weightable quasi-metric \(v\) along with weight function \(u\) a strong weighted quasi-metric if for all \(i,j \in V\), \(v(i,j) \leq u_j\) holds. Similarly, call a weighted metric \((d,u)\) a strong weighted metric if for all \(i,j \in V\), \(d(i,j) \leq u_i + u_j\) holds, i.e., if it is not only down-weighted, but also up-weighted. Finally, call a partial metric \(p\) a strong partial metric if the large self-distance condition holds: \(p(i,j) \leq p(i,i) + p(j,j)\) for all \(i,j \in V\).

It can be observed that
\[m(i,j) \leq u_j \Leftrightarrow c(i,j) \leq u_i + u_j \Leftrightarrow p(i,j) \leq p(i,i) + p(j,j), \quad i,j \in V.
\]

Thus, the weightable quasi-metric \(m\) with weight function \(u\) is a strong weighted quasi-metric if and only if the weighted metric \((c,u)\) is a strong weighted metric, and if and only if the partial metric \(p\) is a strong partial metric.

In this case, the strong weighted metric \((c,u)\) has an additional nice property. Consider the \((n+1) \times (n+1)\) matrix \([c'_{ij}]\), \(0 \leq i,j \leq n\), with \(c'_{00} = 0\), \(c'_{0i} = c'_{i0} = u_i\) for \(i \in V\), and \(c'_{ij} = c(i,j)\) for \(i,j \in V\). In other words, the weight \(u_i\) is considered as a distance from the point \(i \in V\) to an additional point 0: \(u_i = c'(i,0) = c'(0,i)\). In the case of strong weighted metric \(c\), the function \(c'\) turns out to be a metric, since the addition of vertex 0 does not violate the triangle inequality: \(c'(i,j) \leq c'(i,0) + c'(0,j)\) and \(c'(i,0) \leq c'(i,j) + c'(j,0)\).

The results presented in this paper demonstrate fruitful connections between the forest representation of hitting times and their metric properties.

5 Examples

In this section, we illustrate the above concepts and results by two examples.

5.1 Example 1: hitting times and their forest expression

Consider the Markov chain with transition matrix \(T\) and the Laplacian matrix defined by \((4)\):

\[
T = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 4 & 1 & 0 \\
\frac{2}{5} & 0 & \frac{2}{5} & 0 \\
0 & 0 & \frac{1}{4} & \frac{3}{4}
\end{bmatrix}; \quad L = \begin{bmatrix}
1 & -1 & 0 & 0 \\
0 & \frac{1}{5} & -\frac{1}{5} & 0 \\
-\frac{2}{5} & 0 & \frac{1}{5} & -\frac{2}{5} \\
0 & 0 & -\frac{1}{4} & \frac{3}{4}
\end{bmatrix}.
\]
First, let us obtain the matrix $\hat{M}$ of hitting times by the direct use of (3). Finding
\[ \pi = \frac{1}{25}(2, 10, 5, 8) \] and
\[ L^\# = \frac{1}{625} \begin{bmatrix} 463 & 1065 & -280 & -1248 \\ -112 & 1315 & -155 & -1048 \\ 138 & -560 & 470 & -48 \\ -62 & -1560 & -30 & 1652 \end{bmatrix} \]
and substituting these in (3) yields
\[ \hat{M} = \frac{1}{2} \begin{bmatrix} 25 & 2 & 12 & 29 \\ 23 & 5 & 10 & 27 \\ 13 & 15 & 10 & 17 \\ 21 & 23 & 8 & 6\frac{1}{4} \end{bmatrix} \]. \hspace{1cm} (25)

Mention that $L^\#$ can be calculated (see, e.g., [9, (i) of Proposition 15]) by applying
\[ L^\# = (L + 1\pi)^{-1} - 1\pi. \]

Now let us obtain $\hat{M}$ by means of Theorem I. The weighted digraph $\Gamma$ without loops corresponding to the Markov chain under consideration is shown in Fig. 1. The converging trees of $\Gamma$ are shown in Fig. 2, where the roots are given in a boldface font.

Figure 1: A weighted digraph corresponding to the Markov chain.

Figure 2: The converging trees $T_1, T_2, T_3,$ and $T_4$ of $\Gamma.$
Having the weights of these trees, by the definition of \( q_i \) given in Section 2 we obtain:

\[
(q_1, q_2, q_3, q_4) = (w(\{T_1\}), w(\{T_2\}), w(\{T_3\}), w(\{T_4\})) = \frac{1}{100}(2, 10, 5, 8).
\]

Since \( q = \sum_{k=1}^{4} q_i = w(\{T_1, T_2, T_3, T_4\}) = \sigma_3 = \frac{1}{2}, \) \( \sigma_3 \) implies \( \frac{(q_1, q_2, q_3, q_4)}{q} = \frac{1}{20}(2, 10, 5, 8). \)

In concordance with the Markov Chain Tree Theorem, this vector coincides with \( \pi \), the normalized left Perron vector of \( T \).

The 2-tree in-forests of \( \Gamma \) are shown in Fig. 3; the roots are given in a boldface font.

\[
\begin{align*}
\text{Figure 3: The 2-tree in-forests } F_1, \ldots, F_8 \text{ of } \Gamma.
\end{align*}
\]

In Theorem 1, \( f_{ij} \) is defined as the total weight of 2-tree in-forests of \( \Gamma \) that have one tree containing \( i \) and the other tree converging to \( j \). Therefore,

\[
[f_{ij}] = \begin{bmatrix}
0 & w(\{F_3\}) & w(\{F_2, F_5\}) & w(\{F_1, F_4, F_6, F_7, F_9\}) \\
w(\{F_2, F_3, F_7\}) & 0 & w(\{F_3, F_5, F_8\}) & w(\{F_1, F_4, F_6, F_8\}) \\
w(\{F_1, F_2, F_7\}) & w(\{F_3, F_4, F_5, F_8\}) & 0 & w(\{F_1, F_4, F_6\}) \\
w(\{F_1, F_2, F_7\}) & w(\{F_3, F_4, F_5, F_8\}) & w(\{F_6\}) & 0 \\
\end{bmatrix}
\]

\[
= \frac{1}{100} \begin{bmatrix}
0 & 10 & 30 & 116 \\
23 & 0 & 25 & 108 \\
13 & 75 & 0 & 68 \\
21 & 115 & 20 & 0 \\
\end{bmatrix},
\]

where \( w(A) \) is the weight of a set \( A \) of digraphs. Moreover, \( \sigma_2 = w(\{F_1, \ldots, F_8\}) = \frac{39}{25} \).

Substituting (26)–(27) in (19) yields the matrix \( \tilde{M} \) of hitting times coinciding with (25).

Remark 2 enables one to avoid generating the converging trees and 2-tree in-forests of \( \Gamma \). Instead, \( f_{ij} \) and \( q_j \) can be computed by means of the recurrent procedure (15)–(16). Starting with \( Q_0 = I, \sigma_0 = 1 \), for this example we have:
\begin{align*}
Q_1 &= -LQ_0 + \frac{\text{tr}(LQ_0)}{1}I = \frac{1}{20} \begin{bmatrix}
25 & 20 & 0 & 0 \\
0 & 41 & 4 & 0 \\
8 & 0 & 29 & 8 \\
0 & 0 & 5 & 40
\end{bmatrix}, \quad \sigma_1 = \frac{9}{10}; \\
Q_2 &= -LQ_1 + \frac{\text{tr}(LQ_1)}{2}I = \frac{1}{100} \begin{bmatrix}
31 & 105 & 20 & 0 \\
8 & 115 & 25 & 8 \\
18 & 40 & 50 & 48 \\
10 & 0 & 30 & 116
\end{bmatrix}, \quad \sigma_2 = \frac{39}{25}; \\
Q_3 &= -LQ_2 + \frac{\text{tr}(LQ_2)}{3}I = \frac{1}{100} \begin{bmatrix}
2 & 10 & 5 & 8
\end{bmatrix}, \quad \sigma_3 = \frac{1}{4}.
\end{align*}

By (12) we have \( f_{ij} = q_{ij}^{(2)} - q_{ij}^{(1)}, \ i, j = 1, \ldots, 4. \) Thereby (28) provides the matrix \([f_{ij}]_{4 \times 4}, \) which coincides with (27). Eq. (29) yields \((q_1, q_2, q_3, q_4) = \frac{1}{100} (2, 10, 5, 8), \) which coincides with (26). Now using Theorem 1 we obtain the matrix (25) of hitting times again.

By (28), (29), and Corollary 2, Kemeny’s constant of this chain is \( K = 1 + \frac{\sigma_2}{\sigma_3} = \frac{7}{10} \) and by Remark 4, \( \sum_{j=1}^{4} \pi_j m_{ij} = \frac{\sigma_2}{\sigma_3} = \frac{6}{25} \) for any \( i = 1, \ldots, 4. \)

It is easy to observe that the hitting time quasi-metric \( m \) defined by the matrix

\[
M = \frac{1}{2} \begin{bmatrix}
0 & 2 & 12 & 29 \\
23 & 0 & 10 & 27 \\
13 & 15 & 0 & 17 \\
21 & 23 & 8 & 0
\end{bmatrix}
\]

is cutpoint additive. For example, \( m(4, 3) + m(3, 2) = m(4, 2), m(1, 3) + m(3, 4) = m(1, 4), \)
\( m(2, 3) + m(3, 4) = m(2, 4), \) however, \( m(3, 2) + m(2, 4) > m(3, 4). \)

On the other hand, it is not weightable, as the cyclic tour property is violated: \( m(1, 2) + m(2, 3) + m(3, 1) \neq m(1, 3) + m(3, 2) + m(2, 1). \)

The corresponding commute time metric \( c, c(i, j) = m(i, j) + m(j, i) \) is defined by

\[
C = \frac{1}{2} \begin{bmatrix}
0 & 25 & 25 & 49 \\
25 & 0 & 25 & 49 \\
25 & 25 & 0 & 25 \\
49 & 49 & 25 & 0
\end{bmatrix}.
\]

### 5.2 Example 2: hitting metric functions for an undirected graph

To illustrate the concept of weighted metric on a nontrivial example, consider a random walk on the undirected unweighted graph \( G = (V, E) \) with \( V = \{1, \ldots, 6\} \) and \( E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{3, 5\}, \{4, 5\}, \{5, 6\}\}, \) whose automorphism group is trivial (Fig. 4).

Define the transition matrix of the corresponding Markov chain by (21). As the vertex degrees are \( d(1) = d(6) = 1, d(2) = d(4) = 2, \) and \( d(3) = d(5) = 3, \) it holds that
Figure 4: A graph $G$ whose automorphism group is trivial.

$$T = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1/2 & 0 & 1/2 & 0 & 0 & 0 \\
0 & 1/3 & 0 & 1/3 & 1/3 & 0 \\
0 & 0 & 1/2 & 0 & 1/2 & 0 \\
0 & 0 & 1/3 & 1/3 & 0 & 1/3 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}.$$ 

$T$ is the matrix of arc weights of the corresponding digraph $\Gamma$ without loops. As well as for a general Markov chain, the matrix $\hat{M}$ of hitting times can be computed using Theorem 1. However, in the present case, there is no need to enumerate trees for obtaining

$$(q_1, q_2, q_3, q_4, q_5, q_6) = \frac{1}{12} (1, 2, 3, 2, 3, 1),$$

(30)

$q = \sum_{i=1}^{6} q_i = \sigma_5 = 1$ and finally $\pi = q^{-1}(q_1, \ldots, q_6) = \frac{1}{12}(1, 2, 3, 2, 3, 1)$, as we know that for such random walks, $\pi$ is proportional to $1^T W$, where $W$ is the edge weight matrix of $G$.

Using all 76 2-tree in-forests of $\Gamma$ one may obtain the matrix $[f_{ij}]$, where $f_{ij}$ is the total weight of 2-tree in-forests of $\Gamma$, where one tree contains $i$ and the other converges to $j$:

$$[f_{ij}] = \frac{1}{36} \begin{bmatrix}
0 & 6 & 36 & 56 & 78 & 59 \\
33 & 0 & 27 & 50 & 69 & 56 \\
60 & 54 & 0 & 32 & 42 & 47 \\
68 & 70 & 24 & 0 & 30 & 43 \\
70 & 74 & 30 & 28 & 0 & 33 \\
73 & 80 & 39 & 34 & 9 & 0
\end{bmatrix} ; \quad \sigma_4 = \sum_{i=1}^{6} f_{ij} = \frac{235}{36}, \ i = 1, \ldots, 6. \quad (31)$$

Kemeny’s constant is $1 + \frac{\sigma_4}{\sigma_5} = \frac{17}{36}$. Substituting (30)–(31) into (10) and (14) yields the matrix $\hat{M}$ of hitting times and the cutpoint additive quasi-metric $m$ represented by matrix $M$:

$$\hat{M} = \frac{1}{3} \begin{bmatrix}
36 & 3 & 12 & 28 & 26 & 59 \\
33 & 18 & 9 & 25 & 23 & 56 \\
60 & 27 & 12 & 16 & 14 & 47 \\
68 & 35 & 8 & 18 & 10 & 43 \\
70 & 37 & 10 & 14 & 12 & 33 \\
73 & 40 & 13 & 17 & 3 & 36
\end{bmatrix} ; \quad M = \frac{1}{3} \begin{bmatrix}
0 & 3 & 12 & 28 & 26 & 59 \\
33 & 0 & 9 & 25 & 23 & 56 \\
60 & 27 & 0 & 16 & 14 & 47 \\
68 & 35 & 8 & 0 & 10 & 43 \\
70 & 37 & 10 & 14 & 0 & 33 \\
73 & 40 & 13 & 17 & 3 & 0
\end{bmatrix}.$$ 

Furthermore, $m$ is a weightable quasi-metric, whose (non-negative and defined up to a positive shift) weight function is defined by the row vector $\mathbf{u} = \frac{1}{3}(48, 18, 0, 8, 4, 34)$. 

13
The corresponding commute time metric $c$, $c_{ij} = m(i, j) + m(j, i)$, the resistance distance $\Omega(i, j) = (1^T W 1)^{-1} c(i, j)$, and the partial metric $p(i, j) = \frac{c(i, j) + u_i + u_j}{2}$, are given by

$$
C = \begin{bmatrix}
0 & 12 & 24 & 32 & 32 & 44 \\
12 & 0 & 12 & 20 & 20 & 32 \\
24 & 12 & 0 & 8 & 8 & 20 \\
32 & 20 & 8 & 0 & 8 & 20 \\
32 & 20 & 8 & 8 & 0 & 12 \\
44 & 32 & 20 & 20 & 12 & 0
\end{bmatrix}; \quad \Omega = \frac{1}{12} C; \quad P = \frac{1}{3} \begin{bmatrix}
48 & 51 & 60 & 76 & 74 & 107 \\
51 & 18 & 27 & 43 & 41 & 74 \\
60 & 27 & 0 & 16 & 14 & 47 \\
76 & 43 & 16 & 0 & 8 & 20 \\
74 & 41 & 14 & 18 & 0 & 37 \\
107 & 74 & 47 & 51 & 37 & 34
\end{bmatrix}.
$$

For any weight function $u$ such that $m(i, j) \leq u_j$, $i, j = 1, \ldots, 6$, i.e., starting from $u = \frac{1}{3}(73, 43, 25, 33, 29, 59)$, functions $m$, $c$, and $p$ are strong on the corresponding level: $m$ is a strong weighted quasi-metric, $c$ a strong weighted metric, and $p$ a strong partial metric.

Moreover, in this case, the function $c': V' \times V' \to \mathbb{R}$, where $V' = \{0, 1, \ldots, 6\}$, $c'(0, 0) = 0$, $c'(0, i) = c'(i, 0) = u_i$, and $c'(i, j) = c(i, j)$ for $i, j = 1, \ldots, 6$, is a metric on $V'$. For $u = \frac{1}{3}(73, 43, 25, 33, 29, 59)$ its matrix is:

$$
C' = \frac{1}{3} \begin{bmatrix}
0 & 73 & 43 & 25 & 33 & 29 & 59 \\
73 & 0 & 36 & 72 & 96 & 96 & 132 \\
43 & 36 & 0 & 36 & 60 & 60 & 96 \\
25 & 72 & 36 & 0 & 24 & 24 & 60 \\
33 & 96 & 60 & 24 & 0 & 24 & 60 \\
29 & 96 & 60 & 24 & 24 & 0 & 36 \\
59 & 132 & 96 & 60 & 60 & 36 & 0
\end{bmatrix}.
$$

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