Depolarization of multiple scattered light in atmospheres due to anisotropy of small grains and molecules

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Abstract

Freely oriented small anisotropic grains and molecules depolarize radiation both in single scattering and in the process of multiple scattering. Especially large depolarization occurs for resonant scattering corresponding to the electron transitions between the energy levels with very different quantum numbers. The existence of light absorption also changes essentially the angular distribution and polarization of radiation, outgoing from an atmosphere. In the present paper we consider these effects in detail both for continuum radiation and for resonant lines. Because the term describing the depolarization deals with isotropic radiation, we consider the axially symmetric part of radiation. We derived the formulas for observed intensity and polarization using the invariance-principles both for continuum and resonant scattering. We confine ourselves to two problems - the diffuse reflection of the light beam from semi-infinite atmosphere, and the Milne problem.

Keywords: Radiative transfer, scattering, resonant scattering, polarization

1 Introduction

The observation of polarization of radiation emanating from atmospheres of planets, stars, stellar envelopes and accretion discs gives additional information on these objects. First of all, the polarization demonstrates the existence of various types of anisotropy in observing objects. The observed polarization helps us to construct various models of the object, namely the models of non-spherical atmospheres. If we observe the eclipsing binary, the value of observed polarization is a variable. In all these cases we have to know the local distribution of polarization of emanating radiation in an atmosphere. The calculation of polarization emanating from semi-infinite plane-parallel atmosphere is one of the basic problem in the theory of radiative transfer.

In most papers the scattering particles (molecules, dust grains) are assumed to be small compared to the wavelength of radiation. In this case the dipole scattering is most important. The incident radiation induces the time dependent dipole moment, which is the source of scattered light. The induced moment depends on the structure of grain or molecule. If the scattering particle is anisotropic, the induced dipole moment depends on the orientation of particle. Only for isotropic particle (say, electron) the induced moment is always the same. Scattering on isotropic particle gives rise to the maximal polarization of radiation. The ensemble of chaotically oriented anisotropic grains or molecules gives rise to smaller polarization as compared to the case of isotropic particles. Thus, anisotropic structure of grains or molecules depolarizes scattered radiation. In the present paper, we restrict ourselves to dipole approximation.

Clearly, the radiative transfer in an atmosphere having the anisotropic grains or molecules is more difficult for consideration than the usually assumed scattering on isotropic particles. On the other hand, the real atmospheres consist of anisotropic particles. For this reason, the estimation of depolarizing effect in such atmospheres is interesting and important. Below we present the solution of standard problems of radiative transfer (see Chandrasekhar 1960) taking into account the effect of depolarization. We consider the diffuse reflection of light beam from semi-infinite atmosphere without sources of radiation, and the

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Milne problem, where the sources of radiation are located far from the surface (τ ≫ 1). The solutions of these problems, according to Chandrasekhar (1960), follow from the invariance-principle and the radiative transfer equation without source term.

The axially symmetric part of radiation is described by two intensities - \( I_1(τ, μ, ν) \) and \( I_r(τ, μ, ν) \). Here τ is the optical depth below the surface of semi-infinite plane-parallel atmosphere, \( μ = \cos θ \) with \( θ \) being the angle between the outer normal \( \mathbf{n} \) to the surface and the direction of light propagation \( \mathbf{n} \), \( ν \) is the frequency of light. The intensity \( I_1 \) describes the light linearly polarized in the plane (\( \mathbf{nN} \)), and \( I_r \) has polarization perpendicular to this plane. The total intensity \( I = I_1 + I_r \), and the Stokes parameter \( Q = |I_1 - I_r|/(I_1 + I_r) \). The degree of linear polarization is equal to \( p = |I_1 - I_r|/(I_1 + I_r) \).

Circularly polarized light is described by a separate equation. We do not consider this equation.

Introducing the (column) vector \( \mathbf{I} \) with the components \( (I_1, I_r) \) we obtain the following matrix transfer equation for multiple scattering of continuum radiation on small anisotropic particles (see Chandrasekhar 1960; Dolginov et al. 1995):

\[
\begin{align*}
\mu \frac{d\mathbf{I}(τ, μ)}{dτ} &= \mathbf{I}(τ, μ) - \frac{1 - q}{2} \int_{-1}^{1} dμ' \hat{P}(μ, μ') \mathbf{I}(τ, μ'), \\
\hat{P}(μ, μ') &= \begin{pmatrix}
P_1, P_2 \\
P_3, P_4
\end{pmatrix} = \mathbf{b}_1 \frac{3}{4} \left( 2(1 - μ^2)(1 - μ'^2) + μ^2μ'^2, \frac{μ^2}{μ'^2}, 1 \right) + \mathbf{b}_2 \frac{3}{2} \left( 1, 1, 1 \right).
\end{align*}
\]

Here \( q = N_0σ_a/(N_0σ_s + N_0σ_a) \) is the probability of light absorption, \( N_0 \) and \( N_a \) are the number densities of absorbing (the grains) and scattering particles, respectively; \( dτ = (N_0σ_s + N_0σ_a)dz \) determines the dimensionless optical depth, \( σ_s \) and \( σ_a \) are the cross sections of scattering and absorption.

The scattering cross section of small particles (dust grains, molecules) is \( σ_a = (8π/3)(ω/c)^4(b_1 + 3b_2) \), where \( ω = 2πν \) is cyclic frequency of light, \( c \) is the speed of light. For freely (chaotically) oriented particles \( σ_a \) is independent of the polarization of incident electromagnetic wave \( \mathbf{E}(ω) \). The values \( b_1 \) and \( b_2 \) are related to polarizability tensor \( β_1,2(ω) \) of a particle as a whole. Induced dipole moment of a particle, as whole, is equal to \( p_1(ω) = β_1,2(ω)E_1(ω) \).

Anisotropic particle with axial symmetry is characterized by two polarizabilities - along the symmetry axis \( β_∥(ω) \), and in transverse direction \( β_⊥(ω) \). For such particles

\[
\begin{align*}
b_1 &= \frac{1}{9} |2β_⊥ + β_∥|^2 + \frac{4}{45} |β_∥ - β_⊥|^2, \\
b_2 &= \frac{1}{15} |β_∥ - β_⊥|^2.
\end{align*}
\]

In transfer equation we use the dimensionless parameters

\[
\begin{align*}
\mathbf{b}_1 &= \frac{b_1}{b_1 + 3b_2}, \quad \mathbf{b}_2 = \frac{b_2}{b_1 + 3b_2}, \quad \mathbf{b}_1 + 3\mathbf{b}_2 = 1.
\end{align*}
\]

For needle like particles (\( |β_∥| ≫ |β_⊥| \)) parameters \( \mathbf{b}_1 = 0.4, \mathbf{b}_2 = 0.2 \), and for plate like particles (\( |β_⊥| ≫ |β_∥| \)) we have \( \mathbf{b}_1 = 0.7, \mathbf{b}_2 = 0.1 \).

Parameter \( \mathbf{b}_2 \) describes the depolarization of radiation, scattered by freely oriented anisotropic particles. The integral term in Eq.(1) (the source function \( \mathbf{B}(τ, μ) = (B_1(τ, μ), B_r(τ, μ)) \) for single scattering of non-polarized radiation \( (I_{1,r} = (1/2)I_0δ(μ - μ_0)) \) acquires the form:

\[
\begin{align*}
B_1(μ) &= \frac{3}{16}(1 - q)I_0\left\{2(1 - μ^2)(1 - μ_0^2) + μ^2(1 + μ_0^2)\right\}\mathbf{b}_1 + 4\mathbf{b}_2, \\
B_r(μ) &= \frac{3}{16}(1 - q)I_0\left\{(1 + μ^2)\mathbf{b}_1 + 4\mathbf{b}_2\right\}.
\end{align*}
\]

The degree of polarization of single scattered radiation is then given by

\[
p(μ, μ_0) = \frac{B(μ) - B_r(μ)}{B(μ) + B_r(μ)} = \frac{\mathbf{b}_1(1 - μ^2)(1 - 3μ_0^2)}{2(1 - μ^2)(1 - μ_0^2) + (1 + μ^2)(1 + μ_0^2)\mathbf{b}_1 + 8\mathbf{b}_2}. \tag{5}
\]

The peak polarization is reached for \( μ = 0 \), i.e., for scattering of radiation parallel to the surface of an atmosphere:
\[ p_{\text{max}} = \frac{\bar{b}_1(1 - 3\mu_0^2)}{(3 - \mu_0^2)ar{b}_1 + 8\bar{b}_2}. \]  

When \( \mu_0 = 1 \) the scattering angle is equal to 90° and the polarization degree \( (6) \) takes the value \( p_{\text{max}} = -\bar{b}_1/(\bar{b}_1 + 4\bar{b}_2) \). For needle like particles \( p_{\text{max}} = -33.3\% \), and for plate like particles \( p_{\text{max}} = -63.63\% \). For isotropic particles \( (\bar{b}_2 = 0) \) one has \( p_{\text{max}} = -100\% \). The minus sign denotes that preferable oscillations of wave electric vector are perpendicular to meridional plane \((\text{NN})\).

Note that in Chandrasekhar (1960) the parameter \( \gamma \) is used. Our parameters \( \bar{b}_1 \) and \( \bar{b}_2 \) are related to \( \gamma \) according to formulas: \( \bar{b}_1 = (1 - \gamma)/(1 + 2\gamma) \) and \( \bar{b}_2 = \gamma/(1 + 2\gamma) \).

Usually the light polarization weakly influences the angular distribution of radiation, emanating from an atmosphere. Thus, in conservative Milne’s problem the ratio \( J(\mu) = I(0, \mu)/I(0, 0) \) takes the value 3.06, if the polarization terms in transfer equation are taken into account. If these terms are omitted the value \( J(\mu) = 3.02 \). For this reason, if one is interested in the intensity of radiation, then one uses the scalar transfer equation only for intensity \( I(\tau, \mu) \). This equation in our axially symmetric case can be derived from matrix equation \((1)\) through the summation \((I_l + I_r)\) and substituting \( I_l(\tau, \mu') = I_r(\tau, \mu' = I(\tau, \mu')/2 \) in the integrand of Eq.(1). As a result, we obtain the standard equation:

\[
\mu \frac{dI(\tau \mu)}{d\tau} = I(\tau, \mu) - \frac{1 - q}{2} \int_{-1}^{1} d\mu' P(\mu, \mu') I(\tau, \mu'),
\]

\[
P(\mu, \mu') = \left[ \frac{1}{8} (3 - \mu^2)(3 - \mu'^2) + \mu^2 \mu'^2 \right] \bar{b}_1 + 3\bar{b}_2.
\]

The integrand \( \hat{P}(\mu, \mu') \) in Eq.(7) is the sum of products of type \( \varphi(\mu)\varphi(\mu') \). According to Chandrasekhar (1960), it is possible to derive a system of non-linear integral equations for three \( H \)-functions. These functions can be used in the derivation of the formulas for outgoing intensity of reflected radiation, and in the Milne problem.

The phase matrix \( \hat{P}(\mu, \mu') \) can also be presented analogous to scalar phase function (see Lenoble 1970; Abhyankar & Fymat 1971). In our case \( (\bar{b}_2 \neq 0) \) the phase matrix \( \hat{P}(\mu, \mu') \) can be rewritten in the form:

\[
\hat{P}(\mu, \mu') = \hat{P}(\mu^2, \mu'^2) = \bar{b}_1 \hat{M}(\mu^2)\hat{M}(\mu'^2)^T + \frac{3}{4}\bar{b}_2 \hat{L}\hat{L}^T.
\]

Here the superscript \( T \) stands for matrix transpose.

In the present paper, we consider in detail two basic problems of the radiative transfer theory - the reflection of polarized light from semi-infinite plane parallel atmosphere, and the Milne problem corresponding to the thermal sources in very deep layers of an absorbing atmosphere.

The main features of our investigation are consideration of problems with depolarization parameter \( \bar{b}_2 \), and taking into account the true absorption (parameter \( q \)) (i.e., the existence of absorbing grains in an atmosphere.) Firstly we consider the solution of transfer equation only for intensity \( I_l \), and then the more complex case of matrix equation for intensities \( I_l \) and \( I_r \). Recall that the depolarization parameter \( \bar{b}_2 \) gives the contribution to the axially symmetric part of radiation.

A detailed consideration of problems without the parameter \( \bar{b}_2 \) is presented in many papers (see, for example, Chandrasekhar 1960; Horak & Chandrasahar 1961; Lenoble 1970; Abhyankar & Fymat 1971).

Below we present briefly the standard general description of the problems under consideration and then turn to particular solutions.

### 1.1 The case of resonant scattering

For investigation of multiple scattering of resonant radiation one uses the matrix transfer equation of the general form (see, for example, Hummer 1962; Ivanov et al. 1997a, 1997b; Dementiev 2008):

\[
\mu \frac{dI(\tau, \mu, \nu)}{d\tau} = \alpha(\nu)I(\tau, \mu, \nu) - \frac{1 - q}{2} \int_{-1}^{1} d\mu' \int_{-\infty}^{\infty} dv' \hat{P}(\mu, \nu; \mu', \nu') I(\tau, \mu', \nu'),
\]  

(10)
where $\nu$ is the frequency of light, the absorption factor in a resonant line is $\alpha_{\text{resonant}}(\nu) = \alpha_0 \varphi(\nu)$, the optical depth $d\tau = \alpha_0 dz$ takes into account the mean absorption factor in a line, the dimensionless factor

$$\alpha(\nu) = \varphi(\nu) + \alpha_{\text{cont}}/\alpha_0,$$

with $\alpha_{\text{cont}}$ being the extinction factor in nearby continuum.

The normalized function $\varphi(\nu)$ describes the form of the line. Often one uses the limiting forms - Gaussian or Doppler

$$\varphi(\nu) = \frac{1}{\sqrt{\pi} \Delta \nu_D} \exp \left[ - \left( \frac{\nu - \nu_0}{\Delta \nu_D} \right)^2 \right],$$

and the Lorentz

$$\varphi(\nu) = \frac{\delta}{\pi} \frac{1}{(\nu - \nu_0)^2 + \delta^2}.$$  

The function $\varphi(\nu)$ is normalized to unity, namely:

$$\int_{-\infty}^{\infty} d\nu \varphi(\nu) = 1.$$

Here $\nu_0$ is central frequency of a resonant line, $\Delta \nu_D$ is Doppler width of a line:

$$\Delta \nu_D^2 = \Delta \nu_{\text{th}}^2 + \Delta \nu_{\text{turb}}^2 = \frac{\nu_0^2}{C^2} (u_{\text{th}}^2 + u_{\text{turb}}^2),$$

where the thermal velocity is determined by the temperature $u_{\text{th}}^2 = 2k_B T/m$, and the turbulent velocity is determined as a mean value of chaotic macroscopic motions $u_{\text{turb}}^2 = \langle u^2(\mathbf{r}, t) \rangle$. The value $\delta$ depends on widths of the atomic energy levels. The Gaussian form of a line arises as a result of Doppler frequency shifts due to thermal and turbulent motions of atoms and molecules. This form usually corresponds to the line core. The Lorentz form characterize the wings of a line which are often blanketed by line environment.

The matrix $\hat{P}(\mu, \nu; \mu', \nu')$ in general has very complex form (see McKenna 1985; Landi Degl’Ippocenti & Landolfi 2004). This is the reason why one uses model of fully redistributed frequencies:

$$\hat{P}(\mu, \nu; \mu', \nu') = \varphi(\nu) \varphi(\nu') \hat{P}(\mu, \mu'),$$

where the matrix $\hat{P}(\mu, \mu')$ has the form (8). This means that the scattering law in a spectral line formally coincides with that by scattering on anisotropic freely oriented small particles. The parameters $\bar{b}_1$ and $\bar{b}_2$ in this case are related to the parameters $E_1$ and $E_2$ used in Chandrasekhar (1960) as $\bar{b}_1 = E_1$ and $\bar{b}_2 = E_2/3$ with $E_1 + E_2 = 1$. It should be noted that in this case the depolarization parameter $\bar{b}_2$ can be very large and plays a very important role in the calculation of resonant emission polarization. Large depolarization of spectral lines is both the consequence of chaotic orientations and the mixture of the dipole electron transitions with very different quantum numbers. Thus, the estimates demonstrate that for $H\alpha$ line, consisting of 5 close components, the value $E_2 \simeq 0.3$. Note, that the Doppler broadening of close components overlap the frequency differences between them and the line, as a whole, can be considered as a single line having the Doppler form (Varshalovich et al. 2006; Lekht et al. 2008).

From the matrix equation (10) one can also derive the separate scalar transfer equation for intensity $I(\tau, \mu, \nu)$:

$$\mu \frac{dI(\tau, \mu, \nu)}{d\tau} = \alpha(\nu) I(\tau, \mu, \nu) - \frac{1 - q}{2} \int_{-\infty}^{1} d\nu' \int_{-\infty}^{\infty} d\nu' P(\mu, \nu; \mu' \nu') I(\tau, \mu', \nu'),$$

where the scalar function $P(\mu, \nu; \mu' \nu') = (P_1 + P_2 + P_3 + P_4)/2$ is equal to the sum of all four components of matrix $\hat{P}(\mu, \nu; \mu', \nu')$.

### 2 Basic formulas

In this section we briefly recall the basic theoretical formulas first derived by Chandrasekhar (1960). This is done as a matter of convenience in consideration of particular problems depending on depolarization parameter $\bar{b}_2$ and the degree of true absorption $q$.
2.1 The equation for scattering matrix

Let a parallel beam of light with fluxes \( \pi F_1 \) and \( \pi F_r \) along the direction characterized by \( \cos \theta_0 = -\mu_0 \) and azimuth angle \( \varphi_0 \) be incident on the surface of a semi-infinite plane-parallel atmosphere. The intensity of the light diffusely reflected from the atmosphere can be expressed in terms of scattering matrix:

\[
I(0, \mu, \varphi) = \frac{1}{4\mu} \hat{S}(\mu, \varphi; \mu_0, \varphi_0) F.
\] (15)

The scattering matrix \( \hat{S}(\mu, \varphi; \mu_0, \varphi_0) \) obeys the matrix equation (the invariance principle):

\[
\left( \frac{1}{\mu} + \frac{1}{\mu_0} \right) \hat{S}(\mu, \varphi; \mu_0, \varphi_0) \equiv (1 - q) \left[ \hat{P}(\mu, \varphi; -\mu_0, \varphi_0) + \frac{1}{4\pi} \int_0^{2\pi} d\varphi'' \hat{S}(\mu, \varphi; \mu', \varphi') \hat{P}(-\mu', \varphi'; -\mu_0, \varphi_0) + \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} d\varphi'' d\varphi''' \hat{S}(\mu, \varphi; \mu', \varphi') \hat{P}(-\mu', \varphi'; \mu'', \varphi'') \hat{S}(\mu'', \varphi''; \mu_0, \varphi_0) \right].
\] (16)

In our axially symmetric case the dependence on azimuthal angles will be absent. The integrations over \( \varphi' \) and \( \varphi'' \) give the values \( 2\pi \), and formula (16) takes a simpler form. In formula (16) we take the common factor \( (1 - q) \) out of brackets, i.e. \( \hat{S} \sim (1 - q) \). Note, that in our case \( \hat{P}(\mu, \mu') \equiv \hat{P}(\mu^2, \mu'^2) \).

If the matrix \( \hat{P}(\mu, \mu') \) has the form

\[
\gamma_1 \hat{M}_1(\mu^2) \hat{M}_1(\mu'^2)^T + \gamma_2 \hat{M}_2(\mu^2) \hat{M}_2(\mu'^2)^T,
\] (17)

then the expression for \( \hat{S}(\mu, \mu') \) takes the following form:

\[
\hat{S}(\mu, \mu') = \frac{(1 - q)\mu\mu'}{\mu + \mu'} \left[ \gamma_1 \hat{N}_1(\mu) \hat{N}_1(\mu')^T + \gamma_2 \hat{N}_2(\mu) \hat{N}_2(\mu')^T \right],
\] (18)

where \( \hat{N}_n(\mu) \) obeys the relation:

\[
\hat{N}_n(\mu) = \hat{M}_n(\mu^2) + \frac{1}{2} \int_0^{2\pi} d\mu' \hat{S}(\mu, \mu') \hat{M}_n(\mu'^2).
\] (19)

Substitution of Eq.(18) into Eq.(19) gives rise to the system of non-linear matrix equations for matrices \( \hat{N}_1 \) and \( \hat{N}_2 \). It should be noted that the expression (18) is valid only when \( \hat{M}_n(\mu) = \hat{M}_n(\mu^2) \). In general case the expression for \( \hat{S}(\mu, \mu') \) takes the form:

\[
\hat{S}(\mu, \mu') = \frac{(1 - q)\mu\mu'}{\mu + \mu'} \left[ \gamma_1 \hat{R}_1(\mu) \hat{K}_1(\mu') + \gamma_2 \hat{R}_2(\mu) \hat{K}_2(\mu') \right].
\] (20)

The matrices \( \hat{R}_n(\mu) \) and \( \hat{K}_n(\mu) \) obey the equations:

\[
\hat{R}_n(\mu) = \hat{M}_n(\mu) + \frac{1}{2} \int_0^{2\pi} d\mu' \hat{S}(\mu, \mu') \hat{M}_n(-\mu'),
\]

\[
\hat{K}_n(\mu) = \hat{M}_n^T(-\mu) + \frac{1}{2} \int_0^{2\pi} d\mu' \hat{M}_n^T(\mu') \hat{S}(\mu', \mu).
\] (21)

The system of non-linear matrix equations for \( \hat{R}_n \) and \( \hat{K}_n \) are more complex than those for \( \hat{N}_n \).
2.2 Formulas for Milne’s problem

The Milne problem deals with the solution of Eq.(1) when the sources of thermal radiation are placed in deep layers of an atmosphere. The important part of this problem is the solution of the transfer equation in infinite atmosphere. This solution has the form \( I(\tau, \mu) = g(\mu) \exp(k\tau)/(1-k\mu) \). The (column) vector \( g(\mu) \) obeys the homogeneous matrix equation:

\[
g(\mu) = \frac{1-q}{2} \int_{-1}^{1} d\mu' \hat{P}(\mu, \mu') \frac{g(\mu')}{1-k\mu'}.
\]  

For phase matrix \( \hat{P}(\mu^2, \mu'^2) \) and phase function \( P(\mu^2, \mu'^2) \) the (column) vector \( g(\mu) \) and the analogous scalar \( g(\mu) \) depend on \( \mu^2 \). In this case Eq.(22) acquires more simple form:

\[
g(\mu^2) = (1-q) \int_{0}^{1} d\mu' \hat{P}(\mu^2, \mu'^2) \frac{g(\mu'^2)}{1-k^2\mu'^2}.
\]

Below we consider only such cases.

The homogeneous equation (22) has the solution, if the constant parameter \( k \) obeys the characteristic equation (zero’s determinant of Eq.(22)). The most simple form of the characteristic equation is obtained for the scalar transfer equation for intensity \( I(\tau, \mu) = g(\mu) \exp(k\tau)/(1-k\mu) \) with the isotropic phase function \( (\bar{b}_1 = 0, \bar{b}_2 = 1/3) \):

\[
(1-q)f_0(k) = (1-q)\frac{1}{2k} \ln \frac{1+k}{1-k} = 1. 
\]  

(23)

For the case of Rayleigh phase function \( (\bar{b}_1 = 1, \bar{b}_2 = 0) \) the characteristic equation is:

\[
1 - \frac{3}{8}(1-q)(3f_0 + 3f_4 - 2f_2) + \frac{9}{8}(1-q)^2(f_0 f_4 - f_2^2) = 0. 
\]  

(24)

Here the functions \( f_n(k) \) are determined as:

\[
f_0(k) = \int_{0}^{1} \frac{d\mu}{1-k^2\mu^2} = \frac{1}{2k} \ln \frac{1+k}{1-k},
\]

\[
f_2(k) = \int_{0}^{1} \frac{\mu^2}{1-k^2\mu^2} = \frac{f_0 - 1}{k^2},
\]

\[
f_4(k) = \int_{0}^{1} \frac{\mu^4}{1-k^2\mu^2} = \frac{3f_2 - 1}{3k^2}. 
\]  

(25)

For Rayleigh scattering \( (\bar{b}_1 = 1, \bar{b}_2 = 0) \) with taking into account the polarization terms the characteristic equation is:

\[
1 - \frac{3}{4}(1-q)(3f_0 + 3f_4 - 4f_2) + \frac{9}{8}(1-q)^2(f_0^2 - f_2^2 + 2f_0 f_4 - 2f_0 f_2) = 0. 
\]  

(26)

In Tables 1 and 2 we present the values \( k(q) \) for these three cases. From these tables, we see that the \( k \)-values for isotropic scattering are larger than those corresponding to Rayleigh phase function. The inclusion of polarization terms gives rise to smaller values of \( k \) compared to case where they are neglected. The maximum relative difference of \( \approx 1\% \) between the \( k \)-values obtained from Eq.(23) and Eq.(24) occurs at \( q \approx 0.4 \), while that between Eq.(24) and Eq.(25) occurs at \( q \approx 0.2 \). For small absorption \( (q \ll 1) \) the value of \( k \approx \sqrt{3q} \). This approximation for \( k \) is valid up to \( q \approx 0.05 \), where the relative difference with the exact value is \( \approx 2\% \).

The invariance-principles give rise to the formula (Chandrasekhar, 1960):

\[
I(0, \mu) = \text{Const} \left[ \frac{g(\mu)}{1-k\mu} - \frac{1}{2\mu} \int_{0}^{1} d\mu' \frac{1}{1+k\mu'} S(\mu, \mu') g(\mu') \right]. 
\]  

(27)

The value \( \text{Const} \) is related with the total flux of outgoing radiation.
3 The intensity $I(\tau, \mu)$ for an atmosphere with depolarization parameter $T_2$

The transfer equation for $I(\tau, \mu)$ is presented in Eq.(7). The phase function has the form:

$$P(\mu, \mu') = \frac{1}{8} \mathcal{T}_1(3 - \mu^2)(3 - \mu'^2) + \mathcal{T}_1 \mu^2 \mu'^2 + 3 \mathcal{T}_2.$$  

(28)

For the case of unpolarized radiation, the general formulas (18) and (19), give the following expression for $S(\mu, \mu')$:

$$S(\mu, \mu') = \frac{(1 - q)\mu\mu'}{\mu + \mu'} \left[ \frac{1}{8} \mathcal{B}_1 \psi(\mu)\psi(\mu') + \mathcal{B}_1 \phi(\mu)\phi(\mu') + 3 \mathcal{B}_2 \xi(\mu)\xi(\mu') \right],$$  

(29)

where the functions $\psi(\mu), \phi(\mu)$ and $\xi(\mu)$ are expressed in term $S(\mu, \mu')$:

$$\psi(\mu) = (3 - \mu^2) + \frac{1}{2} \int_0^1 \frac{d\mu'}{\mu'} (3 - \mu'^2) S(\mu, \mu'),$$

$$\phi(\mu) = \mu^2 + \frac{1}{2} \int_0^1 \frac{d\mu'}{\mu'} \mu'^2 S(\mu, \mu'),$$

$$\xi(\mu) = 1 + \frac{1}{2} \int_0^1 \frac{d\mu'}{\mu'} S(\mu, \mu').$$  

(30)

From the above equations, we see that $\xi(\mu) = [\psi(\mu) + \phi(\mu)]/3$. For convenience in the following operations, we introduce new notations:

$$\psi(\mu) = 4[A(\mu) + B(\mu)], \phi(\mu) = 2A(\mu) - B(\mu), \xi(\mu) = 2A(\mu) + B(\mu).$$  

(31)

The system of equations for $A(\mu)$ and $B(\mu)$ is the following:

$$A(\mu) = \frac{1 + \mu^2}{4} + \frac{3}{8} (1 - q) \mu \int_0^1 d\mu' \frac{1 + \mu'^2}{\mu + \mu'} T(\mu, \mu'),$$

$$B(\mu) = \frac{1 - \mu^2}{2} + \frac{3}{4} (1 - q) \mu \int_0^1 d\mu' \frac{1 - \mu'^2}{\mu + \mu'} T(\mu, \mu'),$$  

(32)

where

$$T(\mu, \mu') = \mathcal{B}_1 (2AA' + BB') + \mathcal{B}_2 (2A + B)(2A' + B') \equiv T(\mu', \mu).$$  

(33)

Here and in what follows, for brevity, we use the notations $A(\mu) = A, A(\mu') = A'$ etc. In new notations the scattering function $S(\mu, \mu')$ acquires the form:

$$S(\mu, \mu') = \frac{3(1 - q)\mu\mu'}{\mu + \mu'} T(\mu, \mu').$$  

(34)

The functions $A(\mu)$ and $B(\mu)$ terms of one $H$-function. We now briefly describe the way to achieve this. From the system of equations (32) we obtain:

$$2A(\mu) + B(\mu) = 1 + \frac{3}{2} (1 - q) \mu \int_0^1 d\mu' \frac{T(\mu, \mu')}{\mu + \mu'}. $$  

(35)

Similarly we find the expression for $2A(\mu) - B(\mu)$ and using formula (35), one can obtain the relation:

$$2A(\mu)\phi_A(\mu) = B(\mu)\phi_B(\mu),$$  

(36)
where the functions $\phi_A(\mu)$ and $\phi_B(\mu)$ are:

$$\phi_A(\mu) = 1 - \mu^2 + \frac{3}{2} (1 - q) \mu \{ b_1 (\mu A_0 - A_1) + b_2 [\mu (2 A_0 + B_0) - (2A_1 + B_1)]\},$$

$$\phi_B(\mu) = 1 + \mu^2 - \frac{3}{2} (1 - q) \mu \{ b_1 (\mu B_0 - B_1) + b_2 [\mu (2 A_0 + B_0) - (2A_1 + B_1)]\}. \quad (37)$$

Here we use the notations

$$A_n = \int_0^1 d\mu \mu^n A(\mu), \quad B_n = \int_0^1 d\mu \mu^n B(\mu). \quad (38)$$

The expression (36) is valid if there exist the relations:

$$A(\mu) = \frac{1}{4} \phi_B(\mu) H(\mu), \quad B(\mu) = \frac{1}{2} \phi_A(\mu) H(\mu). \quad (39)$$

Introducing these relations in Eq.(35), one can obtain the following equation for H - function:

$$\frac{1}{2} (\phi_A + \phi_B) H = 1 + \frac{3}{16} (1 - q) \mu \int_0^1 \frac{d\mu'}{\mu + \mu'} \left[ b_1 (2 \phi_A \phi'_A + \phi_B \phi'_B) + 2 b_2 (\phi_A + \phi_B) (\phi'_A + \phi'_B) \right] H'H'. \quad (40)$$

The functions $\phi_A(\mu)$ and $\phi_B(\mu)$ are polynomials of the type $\phi = a + b\mu + c\mu^2$. Hence one can show that

$$\frac{\phi(\mu)}{\mu + \mu'} = \frac{\phi(-\mu')}{\mu + \mu'} + b(c + \mu - \mu'). \quad (41)$$

Substituting Eq.(41) for $\phi_A(\mu)$ and $\phi_B(\mu)$ in Eq.(40), and keeping in mind that terms with moments $A_0, A_1, B_0, B_1$ and $\mu^2$ in the left side of Eq.(40) are canceled by the same terms in the right side of this equation, one can obtain the standard formula for H - function:

$$H(\mu) = 1 + \mu H(\mu) \int_0^1 \frac{d\mu'}{\mu + \mu'} H(\mu'), \quad (42)$$

where

$$\Psi(\mu) = \frac{3}{16} (1 - q) b_1 [2 \phi_A(\mu) \phi_A(-\mu) + \phi_B(\mu) \phi_B(-\mu)] + \frac{3}{8} (1-q) b_2 [\phi_A(\mu) + \phi_B(\mu)] [\phi_A(-\mu) + \phi_B(-\mu)]. \quad (43)$$

For deriving Eq.(42) we have used the relations:

$$2 A_0 + B_0 = 1 + \frac{3}{4} (1 - q) [b_1 (2 A_0^2 + B_0^2) + b_2 (2 A_0 + B_0)^2], \quad (44)$$

$$2 A_0 - B_0 = \frac{1}{3} + \frac{3}{4} (1 - q) [b_1 (2 A_1^2 + B_1^2) + b_2 (2 A_1 + B_1)^2], \quad (44)$$

which follows from Eq.(32), if one uses the symmetry relation $T(\mu, \mu') = T(\mu', \mu)$ and the substitution $\mu \rightarrow \mu'$ and the opposite $\mu' \rightarrow \mu$. The initial double integral over $\mu$ and $\mu'$ coincides with that after substitutions. As a result, in the numerator of the sum of these integrals arises the term $(\mu + \mu')$ which cancels the same term in the denominator.

The explicit form of the function $\Psi(\mu)$ is the following:

$$\Psi(\mu) = \frac{3}{16} (1 - q) \{ 3 b_1 + 8 b_2 - b_1 [2 - (1 - q) b_1 - 3(1 - q) b_2] \mu^2 + \ldots \}$$
This formula follows from Eq.(43) after the substitution of expressions (37) for $\phi_A(\mu)$ and $\phi_B(\mu)$ and using the relations (44). It is of interest that we do not use the explicit values of $A_n$ and $B_n$. It seems such situation reflects some symmetry properties of the transfer equation (7).

For $\overline{\beta}_1 = 0$ and $\overline{\beta}_2 = 1/3$ the expression (45) gives $\Psi(\mu) = (1-q)/2$. In this case Eq.(42) coincides with the standard $H$-function equation for isotropic scattering. For $\overline{\beta}_2 = 0$ and $\overline{\beta}_1 = 1$ the $\Psi(\mu)$ - function is given by

\begin{equation}
\Psi(\mu) = \frac{3}{16}(1-q)[3 -(1+q) \mu^2 + 3q \mu^4].
\end{equation}

For conservative atmosphere ($q = 0$) the expression (46) transforms into the known form, namely $\Psi(\mu) = (3/16)(3 - \mu^2)$.

Taking into account the relations (39), one can write the scattering function $S(\mu, \mu')$ in the form:

\begin{equation}
S(\mu, \mu') = \frac{3(1-q)\mu\mu'}{8(\mu + \mu')}H(\mu)H(\mu') \times
\end{equation}

\begin{equation}
[\overline{\beta}_1(2\phi_A\phi'_A + \phi_B\phi'_B) + 2\overline{\beta}_2(\phi_A + \phi_B)(\phi'_A + \phi'_B)].
\end{equation}

The expression in brackets is polynomial of $\mu$ and $\mu'$.

### 3.1 The Milne problem

The function $g(\mu)$ for Milne problem with depolarization parameter $\overline{\beta}_2$ takes the form

\begin{equation}
g(\mu) = \frac{3}{16}(1-q) \int_{-1}^{1} \frac{d\mu'}{1 - k\mu'} \left[ \overline{\beta}_1(3 - \mu^2 - \mu'^2 + 3\mu^2 \mu'^2) + 8\overline{\beta}_2 \right] g(\mu').
\end{equation}

According to this equation

\begin{equation}
g(\mu) = g_0 + g_2 \mu^2.
\end{equation}

The homogeneous system of equations for parameters $g_0$ and $g_2$ has the form:

\begin{align}
\left\{ 1 - \frac{3}{8}(1-q)[(3f_0 - f_2)\overline{\beta}_1 + 8\overline{\beta}_2 f_0] \right\} g_0 - \frac{3}{8}(1-q)[(3f_2 - f_4)\overline{\beta}_1 + 8\overline{\beta}_2 f_2] g_2 &= 0,
\end{align}

\begin{align}
- \frac{3}{8}(1-q)\overline{\beta}_1(3f_2 - f_0) g_0 + [1 + 3\\2\overline{\beta}_1(3f_4 - f_2)] g_2 &= 0.
\end{align}

The $f_n$ - functions are defined in Eq.(25). The characteristic equation (the zero value for determinant of system (50)) is:

\begin{align}
1 - \frac{3}{8}(1-q)[\overline{\beta}_1(3f_0 + 3f_4 - 2f_2) + 8\overline{\beta}_2 f_0] + \frac{9}{8}(1-q)^2\overline{\beta}_1(\overline{\beta}_1 + 3\overline{\beta}_2)(f_0f_4 - f_2^2) &= 0.
\end{align}

The solution of this algebraic equation determines the parameter $k$.

The formula (27) for the intensity of outgoing radiation takes the form:

\begin{equation}
I(0, \mu) = \frac{\text{Const}}{1 - k\mu} \left[ g(\mu) - \frac{3}{2}(1-q) \int_{0}^{1} \frac{d\mu'}{(\mu + \mu')(1 + k\mu')} T(\mu, \mu') g(\mu') \right].
\end{equation}

Taking into account the equality

\begin{equation}
\frac{\mu'(1-k\mu)}{(\mu + \mu')(1 + k\mu')} = \frac{1}{1 + k\mu'} - \frac{\mu}{\mu + \mu'}
\end{equation}

and Eq.(35) for $(2A + B)$, the expression (52) can be presented as follows:

\begin{equation}
I(0, \mu) = \frac{\text{Const}}{1 - k\mu} \left\{ [2A(\mu) + B(\mu)] g(\mu) + \frac{3}{2}(1-q) \mu [T_1(\mu) - \mu T_0(\mu)] g_2 - 
\right.
\end{equation}
\[
\frac{3}{2} (1 - q) \int_0^1 d\mu' \frac{T(\mu, \mu')}{1 + k\mu' g(\mu')} \left\{ \right.
\]
\[
where we have introduced the notations:
\]
\[
T_n(\mu) = \int_0^1 d\mu' \mu'' T(\mu, \mu').
\]

If one uses the relations of \(A(\mu)\) and \(B(\mu)\) with \(H\) - function (see Eq.(39)), then the main term \(I(0, \mu) \sim H(\mu)/(1 - k\mu)\) appears.

4 The intensities \(I_l\) and \(I_r\) for an atmosphere with depolarization parameter \(b_2\)

The system of equations for \(I_l(\tau, \mu)\) and \(I_r(\tau, \mu)\) is described by the matrix transfer equation (1). According to Eqs.(8) and (9), the matrix \(\hat{P}(\mu, \mu')\) can be presented as the sum of product of matrices \(\hat{M}(\mu^2)\) and \(M^T(\mu'^2)\), and the product of \(\hat{L}\) and \(\hat{L}^T = \hat{L}\). The general theory (see, Eqs.(17) - (21)) gives the following form of scattering matrix \(\hat{S}(\mu, \mu')\):

\[
\hat{S}(\mu, \mu') = \begin{pmatrix} S_1, S_2 \\ S_3, S_4 \end{pmatrix} = \left(1 - q\right) \mu' \mu + \frac{3}{4} \hat{b}_2 \hat{\phi}(\mu) \hat{\phi}^T(\mu'),
\]

where the matrices \(\hat{N}(\mu)\) and \(\hat{\phi}(\mu)\) are related to \(\hat{S}(\mu, \mu')\) as:

\[
\hat{N}(\mu) = \begin{pmatrix} N_1, N_2 \\ N_3, N_4 \end{pmatrix} = \hat{M}(\mu) + \frac{1}{2} \int_0^1 d\mu' \hat{S}(\mu, \mu') \hat{M}(\mu^2),
\]

\[
\hat{\phi}(\mu) \equiv \begin{pmatrix} a, a \\ b, b \end{pmatrix} = \hat{L} + \frac{1}{2} \int_0^1 d\mu' \hat{S}(\mu, \mu') \hat{L}.
\]

The properties \(\phi_1 = \phi_2 = a\) and \(\phi_3 = \phi_4 = b\) are consequences of explicit form of matrix \(\hat{L}\) (\(L_{ik} = 1\), see Eq.(9)). Eq.(58) can be written as

\[
a(\mu) = 1 + \frac{1}{2} \int_0^1 \frac{d\mu'}{\mu'} \left[ S_1(\mu, \mu') + S_2(\mu, \mu') \right],
\]

\[
b(\mu) = 1 + \frac{1}{2} \int_0^1 \frac{d\mu'}{\mu'} \left[ S_3(\mu, \mu') + S_4(\mu, \mu') \right].
\]

It is easy to check that

\[
a = 2 \left( \frac{N_1}{\sqrt{3}} + \frac{N_2}{\sqrt{6}} \right), \quad b = 2 \left( \frac{N_3}{\sqrt{3}} + \frac{N_4}{\sqrt{6}} \right).
\]

So, the matrix \(\hat{\phi}(\mu)\) is expressed in terms of matrix \(\hat{N}(\mu)\).

Let us introduce more convenient notations:

\[
A = \frac{N_1 + N_3}{2\sqrt{3}}, \quad B = \frac{N_2 + N_4}{\sqrt{6}},
\]

\[
C = \frac{N_3 - N_1}{2\sqrt{3}}, \quad D = \frac{N_4 - N_2}{\sqrt{6}}.
\]

In these notations the scattering matrix \(\hat{S}(\mu, \mu')\) acquires the form:
\[ \hat{S}(\mu, \mu') = \frac{3}{2} \left( 1 - q \right) \mu \mu' \times \]
\[
\left\{ [\overline{b}_1 (2AA' + BB') + \overline{b}_2 (2A + B)(2A' + B')] \left( \begin{array}{c} 1, 1 \\ 1, 1 \end{array} \right) + \\
[\overline{b}_1 (2CC' + DD') + \overline{b}_2 (2C + D)(2C' + D')] \left( \begin{array}{c} 1, -1 \\ -1, 1 \end{array} \right) + \\
[\overline{b}_1 (2AC' + BD') + \overline{b}_2 (2A + B)(2C' + D')] \left( \begin{array}{c} -1, 1 \\ -1, 1 \end{array} \right) + \\
[\overline{b}_1 (2CA' + DB') + \overline{b}_2 (2C + D)(2A' + B')] \left( \begin{array}{c} -1, -1 \\ 1, 1 \end{array} \right) \right\}. \quad (62)
\]

The transition to the case of scalar equation, considered in Section 3, can be made according to formula \( S(\mu, \mu') = (S_1 + S_2 + S_3 + S_4)/2 \). As a result, we recover to Eqs.(34) and (33), where the functions \( A(\mu) \) and \( B(\mu) \) obey the system of equations (32).

The substitution of expression (56) into Eqs.(57) and (58) gives rise to explicit form of system of non-linear equations for functions \( A(\mu), B(\mu), C(\mu) \) and \( D(\mu) \):

\[
A(\mu) = \frac{1 + \mu^2}{4} + \frac{3}{8} (1 - q) \mu \int_0^1 \frac{d\mu'}{\mu + \mu'} [(1 + \mu'^2)T(\mu, \mu') + (1 - \mu'^2)R(\mu, \mu')],
\]

\[
B(\mu) = \frac{1 - \mu^2}{2} + \frac{3}{4} (1 - q) \mu \int_0^1 \frac{d\mu'}{\mu + \mu'} [(1 - \mu'^2)T(\mu, \mu') - (1 - \mu'^2)R(\mu, \mu')],
\]

\[
C(\mu) = \frac{1 - \mu^2}{4} + \frac{3}{8} (1 - q) \mu \int_0^1 \frac{d\mu'}{\mu + \mu'} [(1 + \mu'^2)R(\mu', \mu) + (1 - \mu'^2)U(\mu, \mu')],
\]

\[
D(\mu) = -\frac{1 - \mu^2}{2} + \frac{3}{4} (1 - q) \mu \int_0^1 \frac{d\mu'}{\mu + \mu'} [(1 - \mu'^2)R(\mu', \mu) - (1 - \mu'^2)U(\mu, \mu')]. \quad (63)
\]

Here we used the notations:

\[
T(\mu, \mu') = \overline{b}_1 (2AA' + BB') + \overline{b}_2 (2A + B)(2A' + B') = T(\mu, \mu),
\]

\[
R(\mu, \mu') = \overline{b}_1 (2AC' + BD') + \overline{b}_2 (2A + B)(2C' + D'),
\]

\[
U(\mu, \mu') = \overline{b}_1 (2CC' + DD') + \overline{b}_2 (2C + D)(2C' + D') = U(\mu', \mu). \quad (64)
\]

If we neglect the polarization terms \( C(\mu) = 0, D(\mu) = 0 \), then the system (63) transforms to the system (32) for functions \( A \) and \( B \).

For unpolarized incident radiation \( (I_1(-\mu') = I_r(-\mu') = F_0/2) \) the degree of polarization at \( \tau = 0 \) is (see Eq.(15)):

\[
p(\mu, \mu') = \frac{I_1 - I_r}{I_1 + I_r} = -\frac{\overline{b}_1 (2CA' + DB') + \overline{b}_2 (2C + D)(2A' + B')}{T(\mu, \mu')} \quad (65)
\]

Recall that, for brevity, we use the notations \( A = \overline{A}(\mu), A' = \overline{A}(\mu') \) etc. For the same reason in formula (65) we take \( \mu' = \mu_0 \), where \( \mu_0 \) characterizes the incident radiation at the surface (see section 2). Substitution in Eq.(65) instead of functions \( A, B, C \) and \( D \) the corresponding free terms in equations (63), gives rise to the expression (5).

It is of interest that the functions \( A, B, C \) and \( D \) depend on one function, which we denote as \( \overline{H}_0(\mu) \). Let us briefly prove this statement. From the system (63) one can obtain the expression:
\[ 2A + B = 1 + \frac{3}{2}(1-q)\mu \int_0^1 \frac{d\mu'}{\mu + \mu'} T(\mu, \mu'). \]  

(66)

Deriving the difference \( 2A - B \) and using Eq.(66), we obtain the relation between \( A \) and \( B \):

\[ 2A(\mu)\phi_A(\mu) = B(\mu)\phi_B(\mu), \]  

(67)

where

\[
\begin{align*}
\phi_A(\mu) &= 1 - \mu^2 - \frac{3}{2}(1-q)\mu [\bar{b}_1 (A_1 - \mu A_0 + c_0 - c_2) + \bar{b}_2 \beta(\mu)], \\
\phi_B(\mu) &= 1 + \mu^2 + \frac{3}{2}(1-q)\mu [\bar{b}_1 (B_1 - \mu B_0 + d_0 - d_2) + \bar{b}_2 \beta(\mu)], \\
\beta(\mu) &= 2(A_1 - \mu A_0) + (B_1 - \mu B_0) + 2(c_0 - c_2) + d_0 - d_2.
\end{align*}
\]  

(68)

Here we introduce the notations:

\[ c_n(\mu) = \int_0^1 \frac{d\mu'}{\mu + \mu'} C(\mu')\mu^n. \]  

(69)

Analogous formula is defined for \( a_n(\mu), b_n(\mu) \) and \( d_n(\mu) \). The relation (67) is valid if

\[ A(\mu) = \frac{1}{4}\phi_B(\mu)H_0(\mu), \quad B(\mu) = \frac{1}{2}\phi_A(\mu)H_0(\mu). \]  

(70)

Deriving the sum \( 2C + D \), we obtain the relation analogous to Eq.(67):

\[ 2C(\mu)\phi_C(\mu) = D(\mu)\phi_D(\mu), \]  

(71)

where

\[
\begin{align*}
\phi_C(\mu) &= 1 - \frac{3}{2}(1-q)\mu [\bar{b}_1 a_0 + \bar{b}_2 (2a_0 + b_0)], \\
\phi_D(\mu) &= -1 + \frac{3}{2}(1-q)\mu [\bar{b}_1 b_0 + \bar{b}_2 (2a_0 + b_0)].
\end{align*}
\]  

(72)

The relation (71) implies that

\[ C(\mu) = \frac{1}{4}\phi_D(\mu)H_1(\mu), \quad D(\mu) = \frac{1}{2}\phi_C(\mu)H_1(\mu). \]  

(73)

Eq.(66) can also be written as follows:

\[ 2A(\mu)\phi_C(\mu) = 1 + B(\mu)\phi_D(\mu). \]  

(74)

Solving the system of equations (71) and (74), one can obtain \( \phi_C(\mu) \) and \( \phi_D(\mu) \) as functions of \( A, B, C \) and \( D \):

\[
\begin{align*}
\phi_C &= \frac{D}{2(AD - BC)} \equiv \frac{2D}{H_1}, \\
\phi_D &= \frac{2C}{2(AD - BC)} \equiv \frac{4C}{H_1}.
\end{align*}
\]  

(75)

It follows from Eqs.(75) that

\[ H_1(\mu) = 4[A(\mu)D(\mu) - B(\mu)C(\mu)]. \]  

(76)

Derivation of \( 2C - D \) shows that

\[ 2C(\phi_A + \mu^2\phi_C) = 1 - \mu^2 + D(\phi_B + \mu^2\phi_D). \]  

(77)
Joint with Eq.(71) this formula gives rise to relation:

\[ 2C\phi_A = 1 - \mu^2 + D\phi_B. \]  

(78)

Substitution of the equalities \( \phi_A = 2B/H_0 \) and \( \phi_B = 4A/H_0 \) in the above relation, and taking into account the expression (76), gives the relation between \( H_1(\mu) \) and \( H_0(\mu) \):

\[ H_1(\mu) = -(1 - \mu^2)H_0(\mu). \]  

(79)

Thus, all the functions - \( A, B, C \) and \( D \) are expressed in terms of one function \( H_0(\mu) \). Recall, that in the case of scalar transfer equation (7) the substitution of the relations \( A = \phi_B H/4 \) and \( B = \phi_A H/2 \) into Eq.(35) (this is analog of Eq.(66)) gives rise to standard nonlinear equation (42) for \( H \) - function. In the present case of polarized transfer, the functions \( \phi_A \) and \( \phi_B \) depend on functions \( c_0, c_2, d_0 \) and \( d_2 \). In other words the functions \( \phi_A \) and \( \phi_B \) for polarized case are not the polynomials as they were in scalar case (see the expressions (37)). For this reason we could not obtain the closed equation for \( H_0(\mu) \). Equation (66) can serve as the basis for iteration method to calculate the \( H_0 \) - function. The detail consideration of this problem will be given in a future publication.

4.1 The Milne problem

The matrix integral equation for the (column) vector \( \mathbf{g}(\mu) = (g_l, g_r) \) is presented in Eq.(22), where the phase matrix \( \hat{P}(\mu, \mu') \) is given in Eqs.(8) and (9). It follows from this equation that

\[ \mathbf{g}(\mu) = \mathbf{g}_0 + \mu^2 \mathbf{g}_2 = \begin{pmatrix} a \\ b \\ c \\ 0 \end{pmatrix} + \mu^2 \begin{pmatrix} 0 \\ a \\ b \\ c \end{pmatrix}. \]  

(80)

Homogeneous system of algebraic equations for values \( a, b \) and \( c \) have the form:

\[ [1 - \alpha_1 (2f_0 - 2f_2) - 2\alpha_2 f_0] a - 2\alpha_2 f_0 b - 2[\alpha_1 (f_2 - f_4) + \alpha_2 f_2] c = 0, \]

\[ - (\alpha_1 f_2 + 2\alpha_2 f_0) a + (1 - \alpha_1 f_0 - 2\alpha_2 f_0) b - (\alpha_1 f_4 + 2\alpha_2 f_2) c = 0, \]

\[ - \alpha_1 (3f_2 - 2f_0) a - \alpha_1 f_0 b + [1 - \alpha_1 (3f_4 - 2f_2)] c = 0. \]  

(81)

Here, for brevity, we use the notations:

\[ \alpha_1 = \frac{3}{4}(1 - q) \overline{b}_1, \quad \alpha_2 = \frac{3}{4}(1 - q) \overline{b}_2. \]  

(82)

The values of \( f_n \) are given in Eq.(25). The characteristic equation (the zero value for determinant \( \Delta(k) \) of system (81)) allows us to calculate the value of parameter \( k \).

\[ \Delta(k) = \Delta_1(k) - 4\alpha_2 f_0 + 6\alpha_1 \alpha_2 (f_0^2 - f_2^2 + 2f_0f_4 - 2f_0f_2) = 0, \]

\[ \Delta_1(k) = 1 - \alpha_1 (3f_0 + 3f_4 - 4f_2) + 2\alpha_1^2 (f_0^2 - f_2^2 + 2f_0f_4 - 2f_0f_2). \]  

(83)

For dipole scattering (\( \overline{b}_2 = 0, \overline{b}_1 = 1 \)) the expression (83) transforms to Eq.(26). For isotropic scattering (\( \overline{b}_1 = 0, \overline{b}_2 = 1/3 \)) Eq.(83) reduces to the equation \( 4\alpha_2 f_0 = 1 \), which determines the parameter \( k(q) \) for scalar transfer equation with isotropic phase function.

The angular distribution and the polarization of outgoing radiation \( I(0, \mu) \) is described by Eq.(27). Taking into account the equality (53), this equation can be written as

\[ I(0, \mu) = \frac{\text{Const}}{1 - k\mu} \left[ \mathbf{g}(\mu) + \frac{1}{2} \int_0^1 \frac{d\mu'}{\mu'} \hat{S}(\mu, \mu') \mathbf{g}(\mu) - \frac{1}{2\mu} \int_0^1 \frac{d\mu'}{\mu'} \cdot \frac{\mu + \mu'}{1 + k\mu'} \hat{S}(\mu, \mu') \mathbf{g}(\mu') \right]. \]  

(84)
5 Polarization of resonant radiation in a model of full frequency redistribution

In this model the transfer equation for $I = (I_1, I_r)$ has the form:

$$\frac{dI(\tau, \mu, \nu)}{d\tau} = \alpha(\nu)I(\tau, \mu, \nu) - \frac{1}{2} \int_{-\infty}^{1} \int \varphi(\nu')P(\mu^2, \mu'^2)I(\tau, \mu', \nu').$$  \hspace{1cm} (85)

Recall that the absorption factor of resonant radiation is $\alpha_{\text{resonant}}(\nu) = \alpha_0 \varphi(\nu)$, the quantity $\alpha(\nu) = \varphi(\nu) + \frac{\alpha_{\text{cont}}}{\alpha_0}$, and $d\tau = \alpha_0 \, d\tilde{z}$. Eq. (85) has been investigated in detail in several earlier papers (see e.g. Ivanov et al. 1997a, 1997b; Dementyev 2008). For a more complete list of references we refer the reader to Faurobert-Scholl & Frisch (1989) and reviews by Nagendra (2003) and Nagendra & Sampoorana (2009). As opposed to these authors, our consideration is based on the invariance-principles (see Eq. (16)) which are valid both for the continuum radiation and for resonant one. For resonant radiation the phase matrix in Eq. (16) is replaced by $\varphi(\nu) \varphi(\nu')P(\mu^2, \mu'^2)$ and in the integral terms there now appears the integration over frequencies.

In our axially symmetric case Eq. (15) transforms to

$$I(0, \mu, \nu) = \frac{1}{4\mu} S(\mu, \nu; \mu_0, \nu_i) F_0(\mu_0, \nu_i).$$  \hspace{1cm} (86)

Here $\nu_i$ is the arbitrary frequency in an incident flux $F_0(\mu_0, \nu_i)$ of resonant radiation. According to the invariance principle, the matrix $S(\mu, \nu; \mu_0, \nu_i)$ obeys the equation:

$$\left( \frac{\alpha(\nu)}{\mu} + \frac{\alpha(\nu_i)}{\mu_0} \right) S(\mu, \nu; \mu_0, \nu_i) = (1 - q) \left[ \varphi(\nu) \varphi(\nu_i) P(\mu^2, \mu_0^2) + \right.$$

$$\frac{1}{2} \int_{-\infty}^{1} \int \varphi(\nu) \varphi(\nu') \varphi(\nu'') P(\mu^2, \mu'^2) S(\mu', \nu''; \mu_0, \nu_i) + \frac{1}{2} \int_{-\infty}^{1} \int \varphi(\nu') P(\mu'^2, \mu_0^2) \varphi(\nu'') S(\mu, \nu''; \mu_0, \nu_i) \varphi(\nu'') \bigg].$$  \hspace{1cm} (87)

It is convenient to take out from the matrix $S(\mu, \nu; \mu_0, \nu_i)$ the product $\varphi(\nu) \varphi(\nu_i)$:

$$S(\mu, \nu; \mu_0, \nu_i) = \varphi(\nu) \varphi(\nu_i) S_1(\mu, \nu; \mu_0, \nu_i).$$  \hspace{1cm} (88)

The matrix $S_1(\mu, \nu; \mu_0, \nu_i)$ obeys the equation:

$$\left( \frac{\alpha(\nu)}{\mu} + \frac{\alpha(\nu_i)}{\mu_0} \right) S_1(\mu, \nu; \mu_0, \nu_i) = (1 - q) \left[ P(\mu^2, \mu_0^2) + \right.$$

$$\frac{1}{2} \int_{-\infty}^{1} \int \varphi(\nu) \varphi(\nu') P(\mu^2, \mu'^2) S_1(\mu', \nu''; \mu_0, \nu_i) + \frac{1}{2} \int_{-\infty}^{1} \int \varphi(\nu') P(\mu'^2, \mu_0^2) \varphi(\nu'') S_1(\mu, \nu''; \mu_0, \nu_i) \varphi(\nu'') \bigg].$$  \hspace{1cm} (89)

According to general theory (see Eqs. (17) - (21)) one can derive the following formula (here and in what follows, for brevity, we take $\mu_0 = \mu'$ and $\nu_i = \nu'$):

$$\left( \frac{\alpha(\nu)}{\mu} + \frac{\alpha(\nu')}{\mu'} \right) S_1(\mu, \nu; \mu' \nu') = \text{...}$$
Nevertheless, the relation (71) takes place, where in terms of the $H$-functions.

The analog of Eq.(22) for resonant radiation is:

$$
\begin{align*}
\tilde{b}_1 \tilde{N}(\mu, \nu)\tilde{N}^T(\mu', \nu') + \frac{2}{3} \eta(\mu, \nu)\eta^T(\mu', \nu')
\end{align*}
$$

where the matrices $\tilde{N}(\mu, \nu)$ and $\eta(\mu, \nu)$ are related to $\tilde{S}_1(\mu, \nu; \mu', \nu')$ as:

$$
\tilde{N}(\mu, \nu) = \tilde{M}(\mu^2) + \frac{1}{2} \int_0^1 \frac{d\mu'}{\mu'} \int_{-\infty}^{\infty} d\nu' \tilde{S}_1(\mu, \nu; \mu', \nu')\varphi^2(\nu')\tilde{M}(\mu^2),
$$

$$
\eta(\mu, \nu) = \tilde{L} + \frac{1}{2} \int_0^1 \frac{d\mu'}{\mu'} \int_{-\infty}^{\infty} d\nu' \tilde{S}_1(\mu, \nu; \mu', \nu')\varphi^2(\nu')\tilde{L}.
$$

The formulas (91) are analogous to Eqs.(57) and (58) with additional factor $\varphi^2(\nu')$ and the integration over $\nu'$. The matrix $\eta$ has two independent components $\eta_1 = \eta_2 = a$ and $\eta_3 = \eta_4 = b$. The functions $a$ and $b$ are related with components of $\tilde{N}(\mu, \nu)$ according to Eq.(60). Using our convenient notations (61), we obtain the scattering matrix $\tilde{S}$ in the form:

$$
\tilde{S}(\mu, \nu; \mu', \nu') = \frac{3}{2} (1 - q) \mu \mu' \varphi(\nu)\varphi(\nu') \{ g \}.
$$

The brackets $\{ \}$ are analogous to brackets in Eq.(62), where the functions $A, B, C$ and $D$ depend on frequency: $A(\mu) \to A(\mu, \nu)$ etc. According to Eqs.(90) and (91), the functions $A(\mu, \nu), B(\mu, \nu), C(\mu, \nu)$ and $D(\mu, \nu)$ obey the equations (63), where in denominators instead of $(\mu + \mu')$ is to be taken as $[\alpha(\nu')\mu + \alpha(\nu)\mu']$. Besides, the additional integration over $\nu'$ with the weight function $\varphi^2(\nu')$ is to be taken. For example, instead of Eq.(66) the frequency dependent analog has the form:

$$
2A(\mu, \nu) + B(\mu, \nu) = 1 + \frac{3}{2} (1 - q) \mu \mu' \int_0^1 d\mu' \int_{-\infty}^{\infty} d\nu' \frac{\varphi^2(\nu')}{\alpha(\nu')\mu + \alpha(\nu)\mu'} T(\mu, \nu; \mu', \nu'),
$$

where the function $T(\mu, \nu; \mu', \nu')$ generalize that in Eq.(64). It is of interest to note that the existence of denominator $[\alpha(\nu')\mu + \alpha(\nu)\mu']$ does not allow to obtain the expression (67) between $A(\mu, \nu)$ and $B(\mu, \nu)$. Nevertheless, the relation (71) takes place, where

$$
a_\alpha(\mu, \nu) = \int_0^1 d\mu' \int_{-\infty}^{\infty} d\nu' \frac{\varphi^2(\nu')}{\alpha(\nu')\mu + \alpha(\nu)\mu'} A(\mu', \nu'),
$$

Analogous expression is valid for function $b_\alpha(\mu, \nu)$. Thus, the relation (72) continues to hold good for the present case, but the relation (70) does not exist. Contrary to the case of scattering of continuum radiation, the functions $A, B, C$ and $D$ for the case of scattering of resonant radiation cannot be expressed in terms of the $H$-functions.

5.1 The Milne problem for resonant radiation

The analog of Eq.(22) for resonant radiation is:

$$
(\alpha(\nu) - k\mu)g(\mu, \nu) = \frac{1 - q}{2} \int_{-1}^1 d\mu' \int_{-\infty}^{\infty} d\nu' \tilde{P}(\mu^2, \mu') \varphi(\nu')\varphi(\nu)g(\mu', \nu').
$$

It follows from this equation that

$$
g(\mu, \nu) = \frac{\varphi(\nu)}{\alpha(\nu) - k\mu} (g_0 + \mu^2 g_2),
$$

where $g_0 = (a, b)$ and $g_2 = (c, 0)$ are independent of $\mu$ and $\nu$. The equation for $g_0 + \mu^2 g_2$ acquires the form:
\[ g_0 + \mu^2 g_2 = (1 - q) \int_0^1 d\mu' \Phi(\mu', k) \hat{P}(\mu^2, \mu'^2)(g_0 + \mu'^2 g_2). \] (97)

Here
\[ \Phi(\mu', k) = \int_{-\infty}^{\infty} d\nu' \frac{\alpha(\nu')}{\alpha^2(\nu') - k^2 \mu'^2}. \] (98)

Introducing the quantities
\[ \Phi_n(k) = \int_0^1 d\mu \mu^n \Phi(\mu, k), \] (99)
we can derive the characteristic equation in the form (83), where instead of \( f_n(k) \) one has to substitute the functions \( \Phi_n(k) \). The system of algebraic equations for functions \( a, b \) and \( c \) coincides with the system (81) but with the same substitution \( f_n(k) \rightarrow \Phi_n(k) \).

General formula (27) in the case of resonant line acquires the form:
\[ I(0, \mu, \nu) = \frac{\text{Const} \varphi(\nu)}{\alpha(\nu) - k\mu} [g_0 + \mu^2 g_2 - \frac{1}{2\mu} \int_0^1 d\mu' \int_{-\infty}^{\infty} d\nu' \frac{[\alpha(\nu') - k\mu']}{[\alpha(\nu') + k\mu']^2} \hat{S}_1(\mu, \nu; \mu', \nu')(g_0 + \mu^2 g_2)]. \] (100)

Using the equality
\[ \frac{\mu'[\alpha(\nu) - k\mu]}{[\alpha(\nu') + k\mu'][\alpha(\nu')\mu + \alpha(\nu)\mu']} = \frac{1}{\alpha(\nu') + k\mu'} - \frac{\mu}{\alpha(\nu')\mu + \alpha(\nu)\mu'}. \] (101)

Eq.(100) can be written in another form:
\[ I(0, \mu, \nu) = \frac{\text{Const} \varphi(\nu)}{\alpha(\nu) - k\mu} \left[ g_0 + \mu^2 g_2 + \frac{1}{2} \int_0^1 d\mu' \int_{-\infty}^{\infty} d\nu' \frac{\varphi^2(\nu')}{\alpha(\nu') + k\mu'} \hat{S}_1(\mu, \nu; \mu', \nu')(g_0 + \mu^2 g_2) - \frac{1}{2\mu} \int_0^1 d\mu' \int_{-\infty}^{\infty} d\nu' \frac{\varphi^2(\nu')}{\alpha(\nu') + k\mu'} \hat{S}_1(\mu, \nu; \mu', \nu')(g_0 + \mu^2 g_2) \right]. \] (102)

6 Conclusion

Anisotropy of small grains and molecules gives rise to depolarization of light upon both single and multiple scattering. The existence of true absorption of light also changes essentially the angular distribution and polarization of radiation emerging from an atmosphere. In this paper we consider the multiple scattering of radiation on freely (chaotic) oriented small particles.

We derived the explicit formulas for intensity and linear polarization of light, reflected from semi-infinite plane-parallel atmosphere. The standard Milne’s problem is also considered. We considered radiative transfer in both continuum and resonant lines. For both types of radiation we used the technique of invariance principles, which lead to the system of non-linear equation for four \( \Phi \)-functions. We investigated the axially symmetric part of radiation. Only this part depends on depolarization parameter \( b_2 \).

It is shown that depolarization parameter does not increase the degree of polarization as compared with the case of pure dipole scattering. In the case of continuum all four \( H \)-functions are expressed in terms of one function. In the case of resonant radiation only two \( H \)-functions, describing the polarization of light, can be expressed in terms of one function. For Milne’s problems we derived the characteristic equations to calculate unknown parameter \( k \) (\( I \sim \exp(k\tau) \)). These equations depend on usual parameter...
Table 1: The roots of characteristic equations (23), (24) and (26).

| $q$ | 0 | 0.001 | 0.002 | 0.003 | 0.004 | 0.005 | 0.010 | 0.015 | 0.020 |
|-----|---|-------|-------|-------|-------|-------|-------|-------|-------|
| Eq. (23) | 0 | 0.054757 | 0.077398 | 0.094754 | 0.109369 | 0.122229 | 0.172511 | 0.210856 | 0.242983 |
| Eq. (24) | 0 | 0.054748 | 0.077391 | 0.094742 | 0.109350 | 0.122202 | 0.172435 | 0.210716 | 0.242768 |
| Eq. (26) | 0 | 0.054743 | 0.077377 | 0.094717 | 0.109311 | 0.122148 | 0.172284 | 0.210445 | 0.242356 |

Table 2: The roots of characteristic equations (23), (24) and (26) (continue).

| $q$ | 0.03 | 0.04 | 0.05 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 |
|-----|-------|-------|-------|------|------|------|------|------|------|
| Eq. (23) | 0.296381 | 0.340829 | 0.379485 | 0.525429 | 0.710412 | 0.828635 | 0.907332 | 0.957504 | 0.985624 |
| Eq. (24) | 0.295991 | 0.340233 | 0.378659 | 0.523200 | 0.704828 | 0.819984 | 0.896901 | 0.947380 | 0.978166 |
| Eq. (26) | 0.295255 | 0.339133 | 0.377166 | 0.519583 | 0.697604 | 0.811199 | 0.888707 | 0.941298 | 0.974750 |

$b_1$, describing the dipole scattering, and on the depolarization parameter. These equations contain terms proportional to $b_1$, $b_1^2$, $b_2$ and $b_1 b_2$.

Resonant radiation has the additional effective absorption due to transitions of frequencies from the initial value to other frequencies. It means that the parameter $k$ is not zero even for conservative atmosphere, and the outgoing resonant radiation is more elongated than that for continuum. As a result, the polarisation of resonant radiation is greater than that in the case of continuum radiation.

The paper is devoted to theoretical investigation of light depolarization due to anisotropy of grains, and also due to dipole transitions between molecular levels at resonant scattering. Recall that most important effects occur for radiative transfer of resonant radiation. The existence of absorption is also taken into consideration.

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