Optimal Allocation of Randomly Selected Redundancies to $k$–Out–of–$n$ System With Independent but Nonidentical Components

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This work was supported in part by the National Natural Science Foundation of China under Grant 11501162, and in part by the Young Talent Support Plan of Hebei Province.

ABSTRACT The quality of manufactured components in different batches may be different because of the influence of various stochastic factors. This paper studies the optimal allocation of redundancies drawn from these randomly selected batches to $k$–out–of–$n$ system composed of $n$ independent but nonidentical components. When the total number of redundancies is fixed, we show how the theory of majorization can be used to solve the problem of optimal redundancy allocation problem in $k$–out–of–$n$ system. We give an investigation on how the batches heterogeneity affects the reliability of $k$–out–of–$n$ system.

INDEX TERMS Reliability, redundancy allocation, heterogeneous population, majorization order.

I. INTRODUCTION

In the area of reliability engineering, some redundancies are usually allocated to a system to increase the reliability of the system [1]. Several authors used various of algorithms to solve the redundancy allocation problem (RAP) [2]–[6]. Some drawbacks of these algorithms are that they are time consuming and difficult to find an exact solution.

The $k$–out–of–$n$ system is a very important fault-tolerant system, and it functions if and only if at least $k$ of the $n$ components operate. The $k$–out–of–$n$ system has many applications in both industrial and military systems [7]. Redundancy allocation policies usually include two types which are cold standby redundancy and parallel (or hot) redundancy. Several papers have studied the effect of adding a cold standby redundancy to a system, one can refer to [8]–[11] for recent research. However, due to the difficulty of component fault detection, it is not easy to carry out the cold standby redundancy in reality. Therefore, parallel redundancy has generally been adopted to solve the RAP. Several authors have studied optimal allocation of parallel redundancies to the $k$–out–of–$n$ system by employing a powerful tool of majorization theory in recent years. Hu and Wang [12] considered the problems of optimal allocation of $r$ parallel redundancies to the $k$–out–of–$n$ system with independent and identical components. Li and Ding [13] studied the allocation of independent and identical parallel redundancies to the $k$–out–of–$n$ system with heterogeneous components by means of the usual stochastic order. Ding and Li [14] dealt with the allocation of parallel redundancies to the $k$–out–of–$n$ system with independent and identical components by means of the hazard rate order. Zhuang and Li [15] studied the allocation of independent and identical parallel redundancies to the $k$–out–of–$n$ system with independent but nonidentical components. Ding et al. [16] studied the problem of optimal allocation of $r$ parallel redundancies to a multi-state $k$–out–of–$n$ system. Zhang [17] investigated the optimal allocation of parallel redundancies to a weighted $k$–out–of–$n$ system. You and Li [18] studied the optimal allocation of one parallel redundancy to the $k$–out–of–$n$ system with statistically dependent components. Assume that components and redundancies have independent lifetimes, Fang and Li [19] studied the allocation policy of parallel redundancies to a coherent system.

In practice, the quality of manufactured components in different batches may be different because of various factors (e.g., human errors, defective resources, instability of production processes, etc.). Then, a heterogeneous population can be
formed by large lots of batches of manufactured components. Assume that each group of components, belonging to a subpopulation, is drawn from the heterogeneous population with a certain probability. Hazra et al. [20] firstly gave an investigation on the optimal grouping of components in series and parallel systems; Ding et al. [21] further gave an investigation on the optimal grouping of components in \( k \)-out-of-\( n \) system. Assume that all components of each subsystem are drawn from a randomly selected batch of manufactured components, Ling et al. [22] studied the optimal grouping of components in series-parallel and parallel-series systems; Ling et al. [23] further studied the optimal grouping of components among these subsystems with components selected randomly from a heterogeneous population.

In this paper, we study the problem of allocating redundancies to \( k \)-out-of-\( n \) system with independent but non-identical components. We assume that these redundancies allocated to each component are selected from a randomly selected batch of manufactured components. To the best knowledge of the authors, most of the existing papers, concentrating on allocating redundancies to the \( k \)-out-of-\( n \) system, assume that all redundancies are selected from a homogeneous population. Assume that there exists the maximum number of redundancies allowed for each component and the entire system, the RAP in this paper is to determine the number of parallel redundancies to each component in order to maximize the reliability of the \( k \)-out-of-\( n \) system.

This paper also investigates the influence of the batches heterogeneity on the reliability of the \( k \)-out-of-\( n \) system. It is well known that the parallel redundancy can increase the reliability of a system (chances of a ‘good event’). Sometimes, we may be interested in reducing the reliability of a ‘bad event’ which can be achieved by allocating redundancies to the component in series. Boland et al. [24], [25] presented many examples in practical situations where series subsystems in the \( k \)-out-of-\( n \) system should be considered. Therefore, in this paper, we also consider the case for series subsystems in the \( k \)-out-of-\( n \) system. The model of this paper can be used to describe a system composed of \( n \) independent but nonidentical components separated geographically. For example, electronic monitoring devices, purchased from different manufacturers, are installed in different places of an area. These devices send the observed data to the management center. An accurate data of the area information can be obtained if at least \( k \) places send information accurately. To increase the reliability of accurate data acquisition, each place needs to purchase some electronic monitoring devices to form parallel redundancies, and these purchased redundancies are selected from a randomly selected batch. Assume that the total number of purchased redundancies is fixed and the maximum number of redundancies allowed for each place is fixed. Now, we are interested in how to allocate these redundancies in each place in order to maximize the reliability of accurate data acquisition.

This paper is organized as follows. Section II formulates the RAP. Section III compares different parallel redundancy allocation policies and investigates the influence of batches heterogeneity on the reliability of the \( k \)-out-of-\( n \) system. Section IV presents the corresponding results when redundancies and component are arranged in series of each subsystem. Finally, Section V concludes this paper.

II. PRELIMINARIES

The necessary notations are given in TABLE I.

| Symbol | Meaning |
|--------|---------|
| \( Y \) | \( Y = (Y_1, Y_2, \ldots, Y_n) \). \( Y_i \) denotes the random lifetime of the \( i \)th component of the \( k \)-out-of-\( n \) system, \( i = 1, 2, \ldots, n \). |
| \( H_i(t) \); \( \overline{H}_i(t) \) | distribution function of \( Y_i \); reliability function of \( Y_i \). |
| \( a \) | \( a = (a_1, a_2, \ldots, a_n) \), allocation policy in which \( a_i \) denotes the number of redundancies allocated to the \( i \)th component, \( i = 1, 2, \ldots, n \). |
| \( T_i(a, p, Y, X) \) | the random lifetime of the \( i \)th parallel redundancies [series subsystem] composed of the \( i \)th component and \( a_i \) redundancies. |
| \( S_i(a, p, Y, X) \) | reliability function of \( T_i(a, p, Y, X) \) |
| \( F_{T_i(a, p, Y, X)}(t) \) | distribution function of \( T_i(a, p, Y, X) \). |
| \( F_{S_i(a, p, Y, X)}(t) \) | \( S_i(a, p, Y, X) \). |
| \( F_T(a, p, Y, X)(t) \) | random lifetime of the \( k \)-out-of-\( n \) system having parallel redundancies [series subsystems]. |
| \( F_S(a, p, Y, X)(t) \) | reliability function of \( T(a, p, Y, X) \) |
| \( A = (A_1, A_2, \ldots, A_n) \), \( A_i \) denotes the maximum number of redundancies for the \( i \)th component. |
| \( s \) | total number of redundancies. |
| \( \mathcal{A} \) | \( \mathcal{A} = \{ (a_1, a_2, \ldots, a_n) \mid \sum_{i=1}^{n} a_i = s, a_i \leq A_i, i = 1, 2, \ldots, n \} \) denotes the set of all redundancy allocation policies. |
| \( \overline{\mathcal{A}} \) | \( \overline{\mathcal{A}} = \{ (a_1, a_2, \ldots, a_n) \mid \sum_{i=1}^{n} a_i = s, a_i \leq A_i, a_1 \leq a_2 \leq \cdots \leq a_n, i = 1, 2, \ldots, n \} \) denotes the set of all admissible redundancy allocation policies. |

Let the heterogeneous population be composed of \( m \) different subpopulations, and the proportion of the \( i \)th subpopulation be denoted by \( p_i \), \( i = 1, 2, \ldots, m \). Let \( X_i(i = 1, 2, \ldots, m) \) denote the random lifetime of the \( i \)th
subpopulation with distribution function \( G_i(t) \) and reliability function \( \overline{G}_i(t) \), respectively. Then, a randomly selected batch of components belongs to the \( i \)th subpopulation with probability \( p_i \), \( i = 1, 2, \ldots, m \). Then, \( p = (p_1, p_2, \ldots, p_m) \) and \( X = (X_1, X_2, \ldots, X_m) \) can be used to describe the batches heterogeneity. Assume that we choose \( a_i \) redundancies in a randomly selected batch of components from the heterogeneous population. Then, the \( i \)th parallel [series] subsystem can be formed by putting these \( a_i \) redundancies in parallel [series] with the \( i \)th component, \( i = 1, 2, \ldots, n \). This paper is to obtain an optimal allocation policy \( \alpha^* \in \mathcal{A} \) in order to maximize [minimize] the reliability of \( k-out-of-n \) system with parallel redundancies [series subsystems]. In addition, we also study the influence of the batches heterogeneity, i.e. \( p \) and \( X \) on the reliability of \( k-out-of-n \) system.

### III. Parallel Redundancy

In this section, we put \( a_i \) redundancies in parallel with the \( i \)th component to form the \( i \)th subsystem, \( i = 1, 2, \ldots, n \). Redundancies in each subsystem belong to the \( i \)th subpopulation with probability \( p_i \), \( i = 1, 2, \ldots, m \). Figure 1 presents the structural diagram of \( n \) subsystems having parallel redundancies.

![Parallel subsystems](image)

**FIGURE 1.** Parallel subsystems.

It is easy to see that the reliability function of the \( i \)th subsystem is given by

\[
F_{T_i(a, p, X, Y)}(t) = 1 - \sum_{l=1}^{m} p_l H_l(t) \overline{G}_i^{a_l}(t), \quad i = 1, 2, \ldots, n.
\]

Then, the reliability function of \( k-out-of-n \) system is given by

\[
F_{T(a, p, X, Y)}(t) = \sum_{i=k}^{n} \prod_{l=1}^{i} F_{T_i(a, p, X, Y)}(t) \times \prod_{j=i+1}^{n} F_{T_j(a, p, X, Y)}(t),
\]

where \( \sum_{i} \) is taken over all permutations \( (j_1, j_2, \ldots, j_n) \) of \( (1, 2, \ldots, n) \), and \( (j_1, j_2, \ldots, j_n) \) satisfies \( j_1 < j_2 < \cdots < j_i \) and \( j_{i+1} < j_{i+2} \cdots < j_n \). In this section, we give several counterexamples to illustrate the theoretical results, and the system considered in each counterexample is a \( 3-out-of-4 \) system.

Before proceeding to the next result, we recall the majorization order which is used to compare the degrees of dispersion among the elements between two vectors [26]. For any two \( n \)-dimensional vectors \( x = (x_1, x_2, \ldots, x_n) \) and \( y = (y_1, y_2, \ldots, y_n) \), \( x_1 \leq x_2 \leq \cdots \leq x_n \) and \( y_1 \leq y_2 \leq \cdots \leq y_n \) represent the increasing order of the elements of \( x \) and \( y \), respectively. Then \( y \) is said to be majorized by \( x \) (denoted by \( y \preceq x \)) if

\[
\sum_{j=1}^{n} x(j) \leq \sum_{j=1}^{n} y(j) \text{ and } \sum_{i=1}^{n} x(i) = \sum_{i=1}^{n} y(i)
\]

for all \( j = 1, 2, \ldots, n-1 \). In practice, one may wonder how to draw these redundancies from the heterogeneous population in order to maximize the reliability of \( k-out-of-n \) system. The next result shows that more redundancies should be allocated to the weaker component in order to maximize the reliability of \( k-out-of-n \) system (see appendix for the proof). That is, in order to increase the system reliability, the allocation policy \( (a_1, a_2, \ldots, a_n) \) with \( a_1 \geq a_2 \geq \cdots \geq a_n \) is better than the allocation policy with \( a_i \leq a_j \) for some \( i < j \) under the condition that \( \overline{H}_i(t) \leq \overline{H}_j(t) \leq \cdots \leq \overline{H}_n(t) \).

**Theorem 1:** For \( a \in \mathcal{A} \) and \( a' \in \mathcal{A} \), suppose that there exist some \( i \neq j \) such that \( a_i = a'_j \), \( a_j = a'_i \) and \( a_d = a'_d \) for \( d \neq i, j \). If \( a_i \leq a_j \) and \( \overline{H}_i(t) \geq \overline{H}_j(t) \) for all \( t \in \mathbb{R}_+ \), then

\[
F_{T(a, p, X, Y)}(t) \geq F_{T(a', p, X, Y)}(t)
\]

for all \( t \in \mathbb{R}_+ \).

Let the random lifetime of the \( 1-out-of-n \) (parallel) system be denoted by \( T(a, p, X, Y) \). The next result compares two redundancy allocation policies of parallel system. It shows that the reliability of the parallel system can be improved by balancing the allocation of these redundancies. Its proof is omitted since it can be obtained immediately from the Theorem 9 of [20].

**Theorem 2:** For any \( a \in \mathcal{A} \) and \( a' \in \mathcal{A} \) such that \( a \preceq a' \), we have

\[
F_{T(a, p, X, Y)}(t) \geq F_{T(a', p, X, Y)}(t)
\]

for all \( t \in \mathbb{R}_+ \).

The following counterexample indicates that the result of Theorem 2 does not hold for \( 3-out-of-4 \) system. It tells us that the result of Theorem 2 may not be true for \( k-out-of-n \) system with \( k \geq 2 \).

**Counterexample 1:** Let \( a = (5, 5, 5, 5) \), \( a' = (1, 3, 7, 9) \), \( p = (0.25, 0.15, 0.1, 0.4, 0.1) \), \( G_i(t) = e^{-\lambda_i t} \), \( i = 1, 2, \ldots, 5 \), \( \overline{H}_i(t) = e^{-\lambda_i t} \), \( i = 1, 2, \ldots, 4 \), \( (\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (2.6, 2.2, 2.8, 2.4, 2.0) \), and \( (\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0.2, 0.5, 1.2, 1.6) \). Obviously, \( a \preceq a' \). From Figure 2, we can see that

\[
F_{T(a, p, X, Y)}(t) \geq F_{T(a', p, X, Y)}(t)
\]

does not hold.

The next result shows that when redundancies are more reliable than the components of system, the reliability of \( k-out-of-n \) system can be improved by balancing the allocation of these redundancies (see appendix for the proof). In addition, Theorem 3 of [13] and Theorem 3.1 of [23] are two special cases of the next result.

**Theorem 3:** For \( a \in \mathcal{A} \) and \( a' \in \mathcal{A} \), suppose that \( \overline{H}_i(t) \leq \overline{G}_i(t) \) for all \( t \in \mathbb{R}_+ \), \( i = 1, 2, \ldots, n \) and \( j = 1, 2, \ldots, m \). If \( a \preceq a' \), then

\[
F_{T(a, p, X, Y)}(t) \geq F_{T(a', p, X, Y)}(t)
\]

for all \( t \in \mathbb{R}_+ \).
Let $A(1) \leq A(2) \leq \cdots \leq A(n)$ be the increasing order of the elements of $A$. According to Theorem 1, Theorem 3 and Corollary 3.3 of [23], it is easy to get the following result which gives the optimal allocation policy that maximizes the reliability of $k$–out–of–$n$ system.

**Corollary 1:** Let $\overline{H}_i(t) \leq \overline{H}_2(t) \leq \cdots \leq \overline{H}_n(t)$ for all $t \in \mathbb{R}_+$, and $\overline{H}_i(t) \leq \overline{G}_j(t)$ for all $t \in \mathbb{R}_+$, $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

(i) Let $a^* = (a_1^*, a_2^*, \ldots, a_n^*)$ satisfy that $0 \leq a_i^* - a_j^* \leq 1$ for any $1 \leq i < j \leq n$. If $A(1) = \frac{a_1}{a_2}$, then $a^*$ is the optimal allocation policy in order to maximize the reliability of $k$–out–of–$n$ system.

(ii) Let $A(l) \leq \frac{s}{n}$, $A(2) \leq \frac{s-A(l)}{n-l+1}$, ... , $A(i) \leq \frac{s-A(j-1)}{n-j+1}$, $\alpha = s - \sum_{l=j}^{i} A(l) - (n-i)\Gamma_1$, where $\Gamma_1 = \text{INT}\left(\frac{s-A(l)}{n-l+1}\right)$ and $s = \sum_{l=1}^{m} A(l)$. It is clear that $a^* = (a_1^*, a_2^*, \ldots, a_n^*)$ is the optimal allocation policy in order to maximize the system reliability.

Assume that $\overline{H}_1(t) \leq \overline{H}_2(t) \leq \cdots \leq \overline{H}_n(t)$, Theorem 1 shows that more redundancies should be allocated to the weaker component. However, there may be situations in which the allocation policy can only belong to $\overline{A}$. For example, when $s = 9$ and the constraint vector $A = (1, 4, 6)$, the allocation policy $(a_1, a_2, a_3)$ must satisfy $a_1 \leq a_2 \leq a_3$. Under this situation, the next result shows that balancing the allocation of redundancies can improve the system reliability.

**Theorem 4:** Assume that $\overline{H}_1(t) \leq \overline{H}_2(t) \leq \cdots \leq \overline{H}_n(t)$ for all $t \in \mathbb{R}_+$. For any $a \in \overline{A}$ and $a' \in \overline{A}$ such that $a \preceq a'$, we have $\overline{F}_{T(a,p,Y,X)}(t) \geq \overline{F}_{T(a',p,Y,X)}(t)$ for all $t \in \mathbb{R}_+$.

**Counterexample 2:** $p = (0.25, 0.15, 0.1, 0.4, 0.1)$, $\overline{G}_i(t) = e^{-\gamma_i t}$, $i = 1, 2, \ldots, 5$, $\overline{H}_i(t) = e^{-\lambda_i t}$, $i = 1, 2, \ldots, 5, (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5) = (0.6, 0.2, 0.8, 0.6, 0.5)$, $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (1.5, 1.4, 1.3, 1.2)$, $a = (8, 12, 20, 60)$ and $a' = (1, 25, 27, 47)$. It is easy to see that $\overline{H}_1(t) \leq \overline{G}_i(t)$, $i = 1, 2, \ldots, 4$, $l = 1, 2, \ldots, 5$, and $\overline{H}_1(t) \leq \overline{H}_2(t) \leq \overline{H}_3(t) \leq \overline{H}_4(t)$, but $a \preceq a'$ does not hold. From Figure 3, we have that $\overline{F}_{T(a,p,Y,X)}(t) \geq \overline{F}_{T(a',p,Y,X)}(t)$ does not hold. Hence, $a \preceq a'$ of Theorem 3 and Theorem 4 cannot be dropped.

**Counterexample 3:** $a = (20, 20, 20, 20)$, $a' = (5, 15, 35, 45)$, $p = (0.25, 0.15, 0.1, 0.4, 0.1)$, $\overline{G}_i(t) = e^{-\gamma_i t}$, $l = 1, 2, \ldots, 5$, $\overline{H}_i(t) = e^{-\lambda_i t}$, $i = 1, 2, \ldots, 4$, $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0.5, 0.2, 1.2, 1.6)$ and $(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5) = (2.6, 2.2, 2.8, 2.4, 2.0)$. It is clear that $a \preceq a'$, $\overline{H}_2(t) \leq \overline{H}_3(t) \leq \overline{H}_1(t) \leq \overline{H}_4(t)$, $\overline{H}_1(t) \geq \overline{G}_i(t)$, $i = 1, 2, \ldots, 4$, $l = 1, 2, \ldots, 5$. From Figure 4, we can see that $\overline{F}_{T(a,p,Y,X)}(t) \geq \overline{F}_{T(a',p,Y,X)}(t)$ does not hold. Hence, $\overline{H}_1(t) \leq \overline{G}_i(t)$ for all $i = 1, 2, \ldots, 4$ and $l = 1, 2, \ldots, 5$ of Theorem 3, and $\overline{H}_1(t) \leq \overline{H}_2(t) \leq \overline{H}_3(t) \leq \overline{H}_4(t)$ of Theorem 4 cannot be dropped.
We define \( A' = (A'_1, A'_2, \ldots, A'_n) \), where \( A'_i = \min (A_i, A_{i+1}, \ldots, A_n) \), \( i = 1, 2, \ldots, n \). It is clear that \( A'_1 \leq A'_2 \leq \cdots \leq A'_n \) and \( A'_1 \leq A_i \). On one hand, if \( a \in \tilde{A} \), then for any \( 1 \leq i < j \leq n \), \( a_i \leq a_j \leq A_j \), that is \( a_i \leq A'_i \). On the other hand, if \( a \in \tilde{A}' \), then for any \( 1 \leq i < j \leq n \), \( a_i \leq A'_j \leq A_i \), hence \( a \in \tilde{A}' \). In summary, \( a \in \tilde{A} \) and \( a \in \tilde{A}' \) are equivalent. According to Theorem 4 and Corollary 3.3 of [23], the following result gives the optimal allocation policy that maximizes the reliability of \( k \)-out-of-\( n \) system.

**Corollary 2:** Suppose that \( a \in \tilde{A} \) and \( \overline{H}_1(t) \leq \overline{H}_2(t) \leq \cdots \leq \overline{H}_n(t) \) for all \( t \in \mathbb{R}^+ \).

(i) Let \( a^*(t) = (a^*_1, a^*_2, \ldots, a^*_n) = \tilde{A}' \), where \( a^*_j - a^*_i \leq 1 \) for any \( 1 \leq i < j \leq n \). If \( A'_i - \frac{s}{n} \), then \( a^* \) is the optimal allocation policy in order to improve the system reliability.

(ii) If \( A'_i + 1 \leq \frac{s}{n} \), then \( a^*(t) = (A_1, A_2, \ldots, A_i, A_i, \ldots, A_l, \Gamma_l, \Gamma_{l+1}, \Gamma_l+1, \ldots, \Gamma_l+1) \) is the optimal allocation policy in order to improve the system reliability, where \( \alpha = s - \sum_{i=1}^{l} A'_i - (n-l)\Gamma_l \) and \( \Gamma_l = \operatorname{INT} \left( \frac{\omega - A'_i - A'_i - \cdots - A'_i}{n-l} \right) \).

Under the condition of the ordered subpopulations, the next result shows how the selection probabilities affect the \( k \)-out-of-\( n \) system reliability. It states the fact that the increase of selection probabilities of more reliable subpopulations can improve the \( k \)-out-of-\( n \) system reliability. Its proof is very similar to the proof of Theorem 3.4 of [23] and is therefore omitted. Let \( D_{0 \{1 \}} = \{ x \in [0, 1]^n \mid x_1 \geq x_2 \geq \cdots \geq x_m \} \), and it will appear in the following results.

**Theorem 5:** Let \( \mathbf{p} \in D_{0 \{1 \}}, \mathbf{p}' \in D_{0 \{1 \}} \) and \( \overline{G}_1(t) \geq \cdots \geq \overline{G}_m(t) \) for all \( t \in \mathbb{R}^+ \). Then \( \mathbf{p}' \preceq \mathbf{p} \) implies \( \overline{F}_{T \mathbf{(a,p',Y,X)}}(t) \geq \overline{F}_{T \mathbf{(a,p,Y,X)}}(t) \) for all \( t \in \mathbb{R}^+ \).

**Counterexample 4:** Let \( \mathbf{a} = (5, 5, 5, 5), \overline{G}_1(t) = e^{-\lambda t}, l = 1, 2, \ldots, 5, \overline{H}_1(t) = e^{-\lambda t}, i = 1, 2, \ldots, 4, \mathbf{p} = (0.50, 0.20, 0.15, 0.10, 0.05), \mathbf{p}' = (0.36, 0.30, 0.27, 0.05, 0.02), (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5) = (0.1, 0.2, 0.3, 0.4, 0.5) \) and \( (\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (6, 8, 11, 4) \). It is clear that \( \overline{G}_1(t) \geq \cdots \geq \overline{G}_m(t), t \geq 0 \), but \( \mathbf{p}' \preceq \mathbf{p} \) is not established. From Figure 5, we can see that \( \overline{F}_{T \mathbf{(a,p,Y,X)}}(t) \geq \overline{F}_{T \mathbf{(a,p',Y,X)}}(t) \) does not hold. Hence, \( \mathbf{p}' \preceq \mathbf{p} \) of Theorem 5 cannot be dropped.

**Counterexample 5:** Let \( \mathbf{p} = (0.43, 0.28, 0.17, 0.10, 0.02), \mathbf{p}' = (0.40, 0.25, 0.20, 0.10, 0.05), \mathbf{a} = (5, 5, 5, 5) \). It is easy to see that \( \mathbf{p}' \preceq \mathbf{p} \). Suppose that \( \overline{G}_1(t) = e^{-\lambda t}, l = 1, 2, \ldots, 5, \overline{H}_1(t) = e^{-\lambda t}, i = 1, 2, \ldots, 4, (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5) = (1.5, 0.1, 0.3, 0.6, 0.8) \) and \( (\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0.6, 0.8, 1.1, 0.4) \). It is clear that \( \overline{G}_2(t) \geq \cdots \geq \overline{G}_m(t), t \geq 0 \). From Figure 6, we have that \( \overline{F}_{T \mathbf{(a,p,Y,X)}}(t) \geq \overline{F}_{T \mathbf{(a,p',Y,X)}}(t) \) does not hold.

Hence, \( \overline{G}_1(t) \geq \cdots \geq \overline{G}_m(t) \) cannot be dropped.

By Theorems 4 and 5, we have the following result.

**Theorem 6:** Let \( \mathbf{a} \in \tilde{A}, \mathbf{a}' \in \tilde{A}, \mathbf{p} \in D_{0 \{1 \}}, \mathbf{p}' \in D_{0 \{1 \}}, \mathbf{G}_1(t) \geq \cdots \geq \mathbf{G}_m(t) \) for all \( t \in \mathbb{R}^+ \) and \( \overline{H}_1(t) \leq \overline{H}_2(t) \leq \cdots \leq \overline{H}_n(t) \) for all \( t \in \mathbb{R}^+ \). Then \( \mathbf{a} \preceq \mathbf{a}' \) and \( \mathbf{p}' \preceq \mathbf{p} \) imply \( \overline{F}_{T \mathbf{(a,p,Y,X)}}(t) \geq \overline{F}_{T \mathbf{(a',p',Y,X)}}(t) \) for all \( t \in \mathbb{R}^+ \).

The next result shows how the change of subpopulation distributions affects the system reliability. Its proof is trivial and hence it is omitted.

**Theorem 7:** Let \( \mathbf{X} = (X_1, X_2, \ldots, X_m) \) and \( \mathbf{Z} = (Z_1, Z_2, \ldots, Z_m) \). If \( \overline{G}_X(t) \geq \overline{G}_Z(t) \) for all \( t \in \mathbb{R}^+ \) and \( i = 1, 2, \ldots, m \), then \( \overline{F}_{T \mathbf{(a,p,Y,X)}}(t) \geq \overline{F}_{T \mathbf{(a,p,Y,Z)}}(t) \) for all \( t \in \mathbb{R}^+ \).

**IV. SERIES SUBSYSTEMS**

There are situations in which we are interested in reducing the reliability of a ‘bad event’ by allocating redundancies to the component in series. One may refer to [23]–[25] for several
examples in which series subsystems of the \(k\)-out-of-\(n\) system should be taken into account. In this section, we consider that \(a_i\) redundancies, selected from one of subpopulations, are arranged in series with the \(i\)th component of the \(k\)-out-of-\(n\) system, \(i = 1, 2, \ldots, n\). The structural diagram of \(n\) series subsystems is given by Figure 7.

\[
\begin{array}{c}
\text{Component 1} \quad \text{Redundancy 1} \\
\vdots \\
\text{Component \(n\)} \quad \text{Redundancy \(n\)} \\
\end{array}
\]

\[
\text{Subsystem 1} \quad \text{Subsystem 2} \quad \cdots \quad \text{Subsystem \(n\)}
\]

**FIGURE 7.** Series subsystems.

Then, we have the reliability function of the \(i\)th subsystem given by

\[
F_{S(a,p,Y,X)}(t) = \sum_{i=1}^{m} p_i \hat{H}_1(t)^{G_i}(t), \quad i = 1, 2, \ldots, n.
\]

Therefore, the reliability function of \(k\)-out-of-\(n\) system having series subsystems is presented by

\[
F_{S(a,p,Y,X)}(t) = \sum_{i=1}^{n} \prod_{i=1}^{n} F_{S_i(a,p,Y,X)}(t)
\]

where \(\sum_{p} \) is taken over all permutations \((j_1, j_2, \ldots, j_n)\) of \((1, 2, \ldots, n)\) and \((j_1, j_2, \ldots, j_n)\) satisfies \(j_1 < j_2 < \cdots < j_i\). Since proofs of the following results are similar to those of Section 3, for brevity, these results are given without proof.

The next result shows that more redundancies should be allocated to the stronger components in the system in order to minimize the system reliability. That is, in order to reduce the system reliability, the allocation policy \((a_1, a_2, \ldots, a_n)\) with \(a_1 \geq a_2 \geq \cdots \geq a_n\) is better than the allocation policy with \(a_i \leq a_j\) for some \(i < j\) under the condition that \(H_1(t) \geq H_2(t) \geq \cdots \geq H_n(t)\).

**Theorem 8:** Let \(a \in \mathbb{A}\) and \(a' \in \mathbb{A}\). Suppose that there exist some \(i \neq j\) such that \(a_i = a'_j, a_j = a'_i, a_d = a'_d\) for all \(d \neq i, j\). If \(a_i \leq a_j\) and \(H(t) \geq H(t)\) for all \(t \in R_+\), then \(F_{S(a,p,Y,X)}(t) \geq F_{S(a',p,Y,X)}(t)\) for all \(t \in R_+\).

The next result compares two redundancy allocation policies of \(n\)-out-of-\(n\) (series) system. Let the random lifetime of the series system be denoted by \(S(a, p, Y, X) = \{k | k = n\}\). It shows that the reliability of series system may be reduced by balancing the allocation of these redundancies.

**Theorem 9:** For any \(a \in \mathbb{A}\) and \(a' \in \mathbb{A}\) such that \(a_i \leq a'_i\), we have \(F_{S(a,p,Y,X)}(t) \leq F_{S(a',p,Y,X)}(t)\) for all \(t \in R_+\).

The next result shows that when components of the system are more reliable than redundancies, the reliability of \(k\)-out-of-\(n\) system can be reduced by balancing the allocation of these redundancies. Theorem 4.1 of [23] is a special case of the next result.

**Theorem 10:** For \(a \in \mathbb{A}\) and \(a' \in \mathbb{A}\), suppose that \(H_1(t) \geq H_2(t) \geq \cdots \geq H_n(t)\) for all \(t \in R_+\). Let \(a_{min}^m = a_{min}^m, \ldots, a_{min}^m\) be \(m\) randomly selected redundancies. Then \(a^*\) is taken over all permutations \((j_1, j_2, \ldots, j_m)\) satisfies \(j_1 < j_2 < \cdots < j_m\). Since proofs of the following results are similar to those of Section 3, for brevity, these results are given without proof.

Theorem 8: Let \(a \in \mathbb{A}\) and \(a' \in \mathbb{A}\). Suppose that there exist some \(i \neq j\) such that \(a_i = a'_j, a_j = a'_i, a_d = a'_d\) for all \(d \neq i, j\). If \(a_i \leq a_j\) and \(H(t) \geq H(t)\) for all \(t \in R_+\), then \(F_{S(a,p,Y,X)}(t) \geq F_{S(a',p,Y,X)}(t)\) for all \(t \in R_+\).

Theorem 9: For any \(a \in \mathbb{A}\) and \(a' \in \mathbb{A}\) such that \(a_i \leq a'_i\), we have \(F_{S(a,p,Y,X)}(k = n) \leq F_{S(a',p,Y,X)}(k = n)\) for all \(t \in R_+\).
Assume that the lifetime distributions of subpopulations are ordered, the following result tells us that the reliability of $k$–out–of–$n$ system will be reduced if the selection probabilities of reliable subpopulations are decreased.

**Theorem 12:** Let $p \in D_{[0,1]}$, $p' \in D_{[0,1]}$ and $\overline{G}_1(t) \geq \overline{G}_2(t) \geq \cdots \geq \overline{G}_m(t)$ for all $t \in \mathbb{R}^+$. Then $p \leq p'$ implies $F_{S(a,p,Y,X)}(t) \geq F_{S(a',p',Y,X)}(t)$ for all $t \in \mathbb{R}^+$.

By Theorems 11 and 12, we have the following result.

**Theorem 13:** Let $a, a' \in \mathbb{R}_+, p \in D_{[0,1]}$, $p' \in D_{[0,1]}$, $\overline{G}_1(t) \geq \overline{G}_2(t) \geq \cdots \geq \overline{G}_m(t)$ for all $t \in \mathbb{R}^+$ and $a' \leq a$ and $p' \leq p$ imply $F_{S(a,p,Y,X)}(t) \geq F_{S(a',p',Y,X)}(t)$ for all $t \in \mathbb{R}^+$.

In order to minimize the reliability of $k$–out–of–$n$, the next result shows that we should draw redundancies from the heterogeneous population consisting of unreliable subpopulations.

**Theorem 14:** Let $X = (X_1, X_2, \ldots, X_m)$ and $Z = (Z_1, Z_2, \ldots, Z_m)$. If $\overline{G}_X(t) \geq \overline{G}_Z(t)$ for all $t \in \mathbb{R}^+$ and $i = 1, 2, \ldots, m$, then $F_{S(a,p,Y,X)}(t) \geq F_{S(a',p',Y,X)}(t)$ for all $t \in \mathbb{R}^+$.

**V. CONCLUSIONS**

In the area of reliability engineering, it is of great interest to allocate some redundancies to a system in order to optimize the reliability of the system. In practical production, life distributions of different batches of components are different due to various risk factors. In this paper, we deal with the allocation of redundancies to the $k$–out–of–$n$ system with independent but nonidentical components. These redundancies are selected randomly from the heterogeneous population. Firstly, for $k$–out–of–$n$ system with parallel redundancies [series subsystems], the allocation policy that assigning more redundancies to the weaker [stronger] component can improve [reduce] the system reliability. Secondly, under different conditions, we show that the reliability of $k$–out–of–$n$ system with parallel redundancies [series subsystems] can be improved [reduced] by balancing the allocation of these redundancies. Finally, we also consider the influence of batches heterogeneity on the reliability of the $k$–out–of–$n$ system. This paper considers the reliability comparisons between two systems, hazard rate functions comparisons between two systems may be studied in the future.

**APPENDIX**

Consider the $k$–out–of–$n$ system with random lifetime $T(a,p,Y,X)$, $N_{ii}^{(i)}(a,p,Y,X)(t)$ represents the number of working subsystems at time $t$ except for the $i$th subsystem and the $j$th subsystem.

**Proof of Theorem 1:** For any fixed $i \neq j$, by the proof of Theorem 3.1 in [23], we have

\[
\overline{F}_{T(a,p,Y,X)}(t) = \mathbb{P}(N_{ij}^{(i)}(a,p,Y,X)(t) \geq k) + \mathbb{P}(N_{ij}^{(i)}(a,p,Y,X)(t) = k - 1) + \cdots + \mathbb{P}(N_{ij}^{(i)}(a,p,Y,X)(t) = 1) + \mathbb{P}(N_{ij}^{(i)}(a,p,Y,X)(t) = 0) \times (1 - \overline{F}_{T(a,p,Y,X)}(t))^m.
\]

Since $H_i(t) \leq H_j(t)$ and $a_i \leq a_j$, we have $\overline{F}_{T(a,p,Y,X)}(t) \geq \overline{F}_{T(a',p',Y,X)}(t)$.

**Proof of Theorem 3:** Without loss of generality, for convenience, let $a \in \mathbb{A}$ and $a' \in \mathbb{A}$. By Lemma D.1 of [26] (p. 195),
there exist $\ell - 2$ vectors which satisfy
\[ a = a^{(1)} \leq a^{(2)} \leq \ldots \leq a^{(t)} = a'. \]
Here, for any $b = 1, 2, \ldots, \ell - 1$,
\[ a^{(b)} = (a_1^{(b)}, a_2^{(b)}, \ldots, a_n^{(b)}) \]
and
\[ a^{(b+1)} = (a_1^{(b+1)}, a_2^{(b+1)}, \ldots, a_n^{(b+1)}) \]
satisfy that there exist some $1 \leq i < j \leq n$ such that
\[ a_i^{(b)} = a_i^{(b+1)} + 1, \quad a_j^{(b)} = a_j^{(b+1)} - 1 \]
and
\[ a_d^{(b)} = a_d^{(b+1)}, \quad d \neq i,j. \]
Therefore, without loss of the generality, we only need to consider $a \in \mathbb{A}$ and $a' \in \mathbb{A}$ which satisfy that there exist $1 \leq i < j \leq n$ such that
\[ a_i = a_i' + 1, \quad a_j = a_j' - 1 \quad \text{and} \quad a_d = a_d', \quad d \neq i,j. \]
It is evident that $a \preceq a'$ and $(a_i, a_j) \preceq (a_i', a_j')$. For all $t \geq 0$, we have
\[
\bar{F}_{T(a,p,Y,X)(t)} - \bar{F}_{T(a',p,Y,X)(t)}(t)
= P\left( N^{(i,j)}_{T(a,p,Y,X)(t)} = k - 1 \right)
	imes \left(1 - F_{T(a,p,Y,X)(t)} \right) F_{T(a,p,Y,X)(t)}(t)
+ P\left( N^{(i,j)}_{T(a,p,X)(t)} = k - 2 \right)
	imes \left(1 - F_{T(a,p,Y,X)(t)} \right) (1 - F_{T(a,p,Y,X)(t)})
- P\left( N^{(i,j)}_{T(a,p,X)(t)} = k - 1 \right)
	imes \left(1 - F_{T(a',p,Y,X)(t)} \right) F_{T(a',p,Y,X)(t)}(t)
- P\left( N^{(i,j)}_{T(a,p,X)(t)} = k - 2 \right)
	imes \left(1 - F_{T(a',p,Y,X)(t)} \right) (1 - F_{T(a',p,Y,X)(t)}). \]
By Theorem 2, we have
\[
(1 - F_{T(a,p,Y,X)(t)}(t) F_{T(a,p,Y,X)(t)}) \leq (1 - F_{T(a',p,Y,X)(t)}(t) F_{T(a',p,Y,X)(t)}(t)). \]
Let
\[
\phi(a, b)
= \left(1 - F_{T(a,p,Y,X)(t)}(t) \right) \left(1 - F_{T(a',p,Y,X)(t)}(t) \right)
\quad - \left(1 - F_{T(a',p,Y,X)(t)}(t) \right) \left(1 - F_{T(a',p,Y,X)(t)}(t) \right)

\[
= \left(1 - H(t) \sum_{l=1}^{m} p_l G_{l}^{a_l}(t) \right)
\times \left(1 - H(t) \sum_{l=1}^{m} p_l G_{l}^{a_l}(t) \right)
- \left(1 - H(t) \sum_{l=1}^{m} p_l G_{l}^{a_l}(t) \right)
\times \left(1 - H(t) \sum_{l=1}^{m} p_l G_{l}^{a_l}(t) \right)
\leq \left(1 - H(t) \sum_{l=1}^{m} p_l G_{l}^{a_l}(t) \right)
\times \left(1 - H(t) \sum_{l=1}^{m} p_l G_{l}^{a_l}(t) \right)
- \left(1 - H(t) \sum_{l=1}^{m} p_l G_{l}^{a_l}(t) \right)
\times \left(1 - H(t) \sum_{l=1}^{m} p_l G_{l}^{a_l}(t) \right).
\]
By $H(t) \leq \mathcal{C}_l(t)$, $i = 1, 2, \ldots, n$, $l = 1, 2, \ldots, m$, we have
\[
H_l(t) G_l^{a_l}(t) - H_l(t) G_l^{a_l}(t) \leq H_l(t) G_l^{a_l}(t) - H_l(t) \leq 0.
\]
Then we have
\[
\sum_{l=1}^{m} p_l H_l(t) G_l^{a_l}(t) \sec_{\sech} \leq \sum_{l=1}^{m} p_l H_l(t) G_l^{a_l}(t) \sec_{\sech} \leq 0.
\]
and

\[ 1 - \sum_{l=1}^{m} p_l H_j(t)G_j^{a_l,j_l}(t) \geq 1 - \sum_{l=1}^{m} p_l H_j(t)G_j^{a_l,j_l}(t) \geq 0. \]

Therefore, \( \psi(a_t) \geq 0 \), which means \( \phi(a, b) \geq 0 \). Hence, \( F_{T(a,p,Y,X)}(t) \geq F_{T(a',p,Y,X)}(t) \).

**Proof of Theorem 4:** Note that \( a \in \tilde{A} \) and \( a' \in \tilde{A} \), by Lemma D.1 of [26] (p. 195), there exist \( \ell-2 \) vectors which satisfy

\[ a = a^{(1)} \leq a^{(2)} \leq \ldots \leq a^{(\ell)} = a'. \]

Here, for any \( b = 1, 2, \ldots, \ell - 1, \)

\[ a^{(b)} = (a_1^{(b)}, a_2^{(b)}, \ldots, a_n^{(b)}) \]

and

\[ a^{(b+1)} = (a_1^{(b+1)}, a_2^{(b+1)}, \ldots, a_n^{(b+1)}) \]

satisfy that there exist some \( 1 \leq i < j \leq n \) such that

\[ a_i^{(b)} = a_i^{(b+1)} + 1, \quad a_j^{(b)} = a_j^{(b+1)} - 1 \]

and

\[ a_d^{(b)} = a_d^{(b+1)}, \quad d \neq i, j. \]

Therefore, without loss of the generality, we only need to consider \( a \in \tilde{A} \) and \( a' \in \tilde{A} \) which satisfy that there exist \( 1 \leq i < j \leq n \) such that

\[ a_i = a_i^t + 1, \quad a_j = a_j^t - 1 \quad \text{and} \quad a_d = a_d, \quad d \neq i, j. \]

It is evident that \( a \preceq a' \) and \( (a_i, a_j) \preceq (a_i^t, a_j^t) \). For all \( t \geq 0 \), we have

\[ F_{T(a,p,Y,X)}(t) - F_{T(a',p,Y,X)}(t) = P(N_{T(a,p,Y,X)}(t) = k - 1) \]

\[ \times \left(1 - F_{T(a,p,Y,X)}(t)F_{T(a,p,Y,X)}(t)\right) \]

\[ + P(N_{T(a,p,Y,X)}(t) = k - 2) \]

\[ \times \left(1 - F_{T(a,p,Y,X)}(t)\right)\left(1 - F_{T(a,p,Y,X)}(t)\right) \]

\[ - P(N_{T(a,p,Y,X)}(t) = k - 1) \]

\[ \times \left(1 - F_{T(a',p,Y,X)}(t)\right)F_{T(a',p,Y,X)}(t) \]

\[ - P(N_{T(a,p,Y,X)}(t) = k - 2) \]

\[ \times \left(1 - F_{T(a',p,Y,X)}(t)\right)\left(1 - F_{T(a',p,Y,X)}(t)\right). \]

By Theorem 2, we have

\[ 1 - F_{T(a,p,Y,X)}(t)F_{T(a,p,Y,X)}(t) \geq 1 - F_{T(a',p,Y,X)}(t)F_{T(a',p,Y,X)}(t). \]

Let

\[ \psi(a_i, a_j) = (1 - F_{T(a,p,Y,X)}(t))(1 - F_{T(a,p,Y,X)}(t)). \]

Then

\[ \frac{\partial \psi(a_i, a_j)}{\partial a_i} - \frac{\partial \psi(a_i, a_j)}{\partial a_j} = \left(1 - H_j(t)\right) \sum_{l=1}^{m} p_l G_j^{a_l,j_l}(t) \]

\[ \times \left(-H_j(t)\right) \sum_{l=1}^{m} p_l G_j^{a_l,j_l}(t) \ln G_j(t) \]

\[ - \left(1 - H_j(t)\right) \sum_{l=1}^{m} p_l G_j^{a_l,j_l}(t) \]

\[ \times \left(-H_j(t)\right) \sum_{l=1}^{m} p_l G_j^{a_l,j_l}(t) \ln G_j(t). \]

Because \( a_j - a_i \geq 0 \) and \( H_j(t) \leq H(t) \), so we have, for all \( l = 1, 2, \ldots, m, \)

\[ H_j(t)G_j^{a_l,j_l}(t) - H_j(t)G_j^{a_l,j_l}(t) \geq \psi(a_i, a_j) \]

\[ - H_j(t)G_j^{a_l,j_l}(t) - H_j(t) \leq 0. \]

Then, we have

\[ \sum_{l=1}^{m} p_l H_j(t)G_j^{a_l,j_l}(t) \ln G_j(t) \leq \sum_{l=1}^{m} p_l H_j(t)G_j^{a_l,j_l}(t) \ln G_j(t) \leq 0, \]

and

\[ 1 - \sum_{l=1}^{m} p_l H_j(t)G_j^{a_l,j_l}(t) \geq 1 - \sum_{l=1}^{m} p_l H_j(t)G_j^{a_l,j_l}(t) \geq 0. \]

Therefore,

\[ \frac{\partial \psi(a_i, a_j)}{\partial a_i} - \frac{\partial \psi(a_i, a_j)}{\partial a_j} \geq 0, \]

which means \( \frac{\partial \psi(a_i, a_j)}{\partial a_k} \) is decreasing in \( k, k = i, j \). By Lemma 3 of [20], we have

\[ \psi(a_i, a_j) \geq \psi(a_i^t, a_j^t). \]

Hence, \( F_{T(a,p,Y,X)}(t) \geq F_{T(a',p,Y,X)}(t) \).

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