Simple solutions of relativistic hydrodynamics for longitudinally expanding systems

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Abstract

Simple, self-similar, analytic solutions of 1 + 1 dimensional relativistic hydrodynamics are presented, generalizing Bjorken’s solution to inhomogeneous rapidity distribution.

Key words: Relativistic hydrodynamics, equation of state, Bjorken flow, analytic solutions

1 Introduction

Relativistic hydrodynamics has various applications, including the calculations of single-particle spectra and two-particle correlations in high energy heavy ion collisions. The applications of relativistic hydrodynamics in the field of heavy ion physics have been reviewed in [1]. More recently, there has been an increasing interest in studying RHIC and coming LHC experiments in the framework of relativistic hydrodynamics [2–4]. The hydrodynamical analysis can also be extended to the study of these processes on event-by-event basis [5,6]. However, most works in hydrodynamics are numerical so not always transparent. In this sense, exact solutions would be useful, but are rarely found due to the highly non-linear nature of relativistic hydrodynamics. Actually, Landau’s one-dimensional analytical solution of relativistic hydrodynamics [7] gave rise

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to a new approach in high energy physics. The boost-invariant Bjorken solution [8], found more than 20 years later, is frequently utilized as the basis for estimations of initial energy densities in ultra-relativistic nucleus-nucleus collisions. Recently, Biró has found self-similar exact solutions of relativistic hydrodynamics for cylindrically expanding systems [9,10], however, his solutions are valid only when the pressure is independent of space and time, as e.g. in the case of a rehadronization phase transition in the middle of a relativistic heavy ion collision.

Here we present an analytic approach, which goes back to the data-motivated exact analytic solution of non-relativistic hydrodynamics found by Zimányi, Bondorf and Garpman (ZBG) in 1978 for low energy heavy ion collisions with spherical symmetry [11]. This solution has been extended to the case of elliptic symmetry by Zimányi and collaborators in [12]. In [13,14] a Gaussian parameterization has been introduced to describe the mass dependence of the effective temperature and the radius parameters of the two-particle Bose-Einstein correlation functions in high energy heavy ion collisions. Later it has been realized that this phenomenological parameterization of data corresponds to an exact, Gaussian solution of non-relativistic hydrodynamics with spherical symmetry [15]. The spherically symmetric self-similar solutions of non-relativistic hydrodynamics were obtained in a general manner in [16], that included an arbitrary scaling function for the temperature profile, and expressed the density distribution in terms of the temperature profile function. The ZBG solution and the Gaussian solution of [15] are recovered from the general solution of [16] as special cases, corresponding to different scaling functions of the temperature profile. The Gaussian solution has been generalized to ellipsoidal expansions in [17], that provides analytic insight into the physics of non-central heavy ion collisions [18].

Our approach corresponds to a relativistic generalization of these recently obtained analytic solutions [15,16,18] of non-relativistic fireball hydrodynamics to the case of relativistic flows, based on the success of the analytic approach to parameterize the single particle spectra and the two-particle Bose-Einstein correlations in high energy heavy ion physics in terms of hydrodynamically expanding sources [19].

In particular, we attempt here to solve the 1+1 dimensional relativistic hydrodynamical problem, in trying to overcome two shortcomings of Bjorken’s well-known solution. These two shortcomings of Bjorken’s solutions are that i) it only describes ultra-relativistic limit so that the rapidity distribution is flat; ii) it contains no transverse flow.

Here we present a new family of exact analytic solutions of relativistic hydrodynamics in 1 + 1 dimension (time + longitudinal coordinate), that is able to describe arbitrary inhomogeneous rapidity distributions. The inclusion of the
transverse flow will be treated in a subsequent paper.

An interesting aspect of this solution is that the shapes of the rapidity distribution \(dN/dy\) and temperature distribution are coupled in the way that the larger the rapidity density, the smaller the effective temperature. Choosing the effective temperature distribution \(T_{\text{eff}}(y)\) to be flat, we recover Bjorken’s 1+1 dimensional solution.

2 The equations of relativistic hydrodynamics

In order to get solutions that have simple non-relativistic limiting behavior, we consider a gas of one type of particles whose total number is conserved. We then solve the relativistic continuity and energy-momentum conservation equation:

\[
\partial_\mu (nu^\mu) = 0, \quad \partial_\mu T^{\mu\nu} = 0. \tag{1} \tag{2}
\]

Here \(n \equiv n(x)\) is the number density, the four-velocity is denoted by \(u^\mu \equiv u^\mu(x) = \gamma(1, v)\), normalized to \(u^\mu u_\mu = 1\), and the energy-momentum tensor is denoted by \(T^{\mu\nu}\). We assume perfect fluid,

\[
T^{\mu\nu} = (\varepsilon + p) u^\mu u^\nu - pg^{\mu\nu}, \tag{3}
\]

where \(\varepsilon\) stands for the relativistic energy density and \(p\) denotes the pressure.

We close this set of relativistic hydrodynamical equations with the use of an equation of state. We assume an ideal gas that contains massive conserved quanta,

\[
\varepsilon = mn + \kappa p, \quad p = nT. \tag{4} \tag{5}
\]

This equation of state has two free parameters, \(m\) and \(\kappa\). The case of ultrarelativistic gas is recovered by choosing \(m = 0\) and \(\kappa = 3\). If the thermal motion of the gas is non-relativistic, one has \(m \gg T\). Using this condition, the equations of non-relativistic hydrodynamics are re-obtained from the above set of equations in the \(v^2 \ll 1\) limiting case. The non-relativistic ideal gas equation of state corresponds to the case of \(\kappa = 3/2\).

The energy-momentum conservation equation contains 4 independent equations. These can be projected into a component parallel to \(u^\mu\) and a component
orthogonal to $u^\mu$, that yield the relativistic energy and Euler equations:

\begin{align}
  u^\mu \partial_\mu \epsilon + (\epsilon + p) \partial_\mu u^\mu &= 0, \\
  u_\nu u^\mu \partial_\mu p + (\epsilon + p) u^\mu \partial_\mu u_\nu - \partial_\nu p &= 0.
\end{align}

The relativistic Euler equation contains only three independent components, by construction.

Using the equation of state and the continuity equation, the energy equation can be rewritten as an equation for the temperature,

\begin{equation}
  u^\mu \partial_\mu T + \frac{1}{\kappa} T \partial_\mu u^\mu = 0.
\end{equation}

From now on we assume that the flow is homogeneous in the transverse direction corresponding to infinitely broad target and projectile in high energy heavy ion collisions. We reduce our equations to the $1 + 1$ dimensional case, so the coordinates are $x^\mu = (t, r_z)$, $x^\mu = (t, -r_z)$ and the metric tensor is $g^{\mu\nu} = g_{\mu\nu} = \text{diag}(1, -1)$.

We solve 3 independent equations, the continuity, the temperature equation and the $z$ component of the Euler equation (1, 7, 8). The equations, (4, 5) close this system of equations in terms of 3 variables, $n$, $T$ and $v_z$.

### 3 Self-similar solution

We search for solutions that scale in the $z$ direction. We introduce the time dependent scaling variable

\begin{equation}
  x = \frac{r_z^2}{Z(t)^2},
\end{equation}

and assume that the longitudinal motion corresponds to a Hubble type of self-similar longitudinal expansion,

\begin{equation}
  v_z(t, r_z) = \frac{\dot{Z}(t)}{Z(t)} r_z,
\end{equation}

where $\dot{Z} = dZ(t)/dt$ stands for the time derivative of the scale parameter $Z$. In a relativistic notation, this form is equivalent to
\[ u^\mu = (\cosh \zeta, \sinh \zeta), \quad (11) \]
\[ \tanh \zeta = \frac{\dot{Z}(t)}{Z(t)} r_z, \quad (12) \]
\[ \cosh \zeta = \frac{1}{\sqrt{1 - \dot{Z}^2 x}} \equiv \gamma. \quad (13) \]

Using this ansatz, we find that the continuity equation is solved by the form
\[ n(t, r_z) = n_0 \frac{Z_0}{Z \cosh \zeta} G(x), \quad (14) \]

where \( G(x) \) is an arbitrary non-negative function of the scaling variable \( x \) and \( n_0 \) and \( Z_0 \) are normalization constants. We use the convention \( Z_0 = Z(t_0) \) and \( n_0 = n(t_0, 0) \) which implies that \( G(x = 0) = 1 \). The temperature equation, \( (8) \) is solved by the following form:
\[ T(t, r_z) = T_0 \left( \frac{Z_0}{Z \cosh \zeta} \right)^{1/\kappa} F(x). \quad (15) \]

The constants of normalization are chosen such that \( T_0 = T(t_0, 0) \) and \( F(0) = 1 \). Here again, we find that the solution is independent of the form of the function \( F(x) \), provided that \( F(x) \geq 0 \).

Using the ansatz for the flow profile and the solution for the density and the temperature, the relativistic Euler equation reduces to a complicated non-linear equation that contains \( Z, \dot{Z} \) and \( \ddot{Z} \) and \( x \). Taking this equation at \( x = 0 \) we express \( \dot{Z} \) as a function of \( Z \) and \( \dot{Z} \). Substituting this back to the Euler equation we obtain an equation for \( \dot{Z}, Z \) and \( x \). In particular, for the \( m = 0 \) case, \( Z \) cancels out and this reduces to a second order polynomial equation for \( \dot{Z}^2 \), which has only one positive root. The form of the solution in this case \( (m = 0) \) is
\[ \dot{Z}^2(t) = F(x). \quad (16) \]

Observing that the function \( F \) depends only on the scaling variable \( x \), while \( \dot{Z} \) depends only on the time variable \( t \), we conclude that the only solution of this equation should be a constant
\[ \dot{Z} = \dot{Z}_0. \quad (17) \]

Now we choose the origin of the time axis such that \( Z(t = 0) = 0 \) without loss of generality. The solutions can be cast in a relatively simple form by intro-
ducing the longitudinal proper time $\tau$ and the space-time rapidity $\eta$ defined as

$$\tau = \sqrt{t^2 - r_z^2}, \quad (18)$$

$$\eta = \frac{1}{2} \log \left( \frac{t + r_z}{t - r_z} \right). \quad (19)$$

This implies that

$$Z(t) = \dot{Z}_0 t, \quad (20)$$

$$v_z = \frac{r_z}{t} = \tanh \eta, \quad (21)$$

$$\zeta = \eta. \quad (22)$$

Thus the solution for the flow velocity field corresponds to the flow field of the Bjorken solution. However, in the Bjorken solution the temperature distribution was independent of the $\eta$ variable, while in our case the density and the temperature distributions can be both $\eta$ dependent, or in other words, our solutions are scale dependent. The scale is defined by the parameter $\dot{Z}_0$, in the longitudinal direction.

This special form of the solution for the flow velocity field implies that $\ddot{Z} = 0$. This equation implies that there is no pressure gradient and there is no acceleration in this class of self-similar solutions. The Euler equation is reduced to the following requirement:

$$\left( \partial_z + \frac{r_z}{t} \partial_t \right) \left[ \left( \frac{t_0}{\tau} \right)^{(1+1/\kappa)} (1 - \dot{Z}_0^2 x)^{(1+1/\kappa)} \mathcal{G}(x) \mathcal{F}(x) \right] = 0 \quad (23)$$

This equation is solved by the trivial $\mathcal{G}(x)\mathcal{F}(x) = 0$ as well as by the non-trivial solution of

$$\mathcal{G}(x)\mathcal{F}(x) = (1 - \dot{Z}_0^2 x)^{-(1+1/\kappa)}, \quad (24)$$

which is indeed only a function of $x$ as $\dot{Z}_0$ is a constant of time. With this form, the Euler equation is satisfied. This solution implies that the scaling profile functions for the temperature and the density distribution are not independent. As the constraint is given only for their product, one of them can be still chosen in an arbitrary manner.

At this point it is worthwhile to introduce new forms of the scaling functions. Let us define
\[ T(x) = \mathcal{F}(x)(1 - \dot{Z}_0^2 x)^{1/\kappa}, \]  
(25)
\[ V(x) = \mathcal{G}(x)(1 - \dot{Z}_0^2 x). \]  
(26)

Then the constraint Eq.(24) can be cast to the simplest form of

\[ V(x) T(x) = 1. \]  
(27)

Let us summarize our new family of solutions of the 1+1 dimensional relativistic hydrodynamics by substituting the results in the density, temperature and pressure profiles.

We obtain

\[ v_z = \frac{r_z}{\dot{t}} = \tanh \eta, \]  
(28)
\[ x = \frac{r_z^2}{Z_0^2 t^2} = \frac{\tanh^2 \eta}{Z_0^2}, \]  
(29)
\[ n(t, r_z) = n_0 \frac{t_0}{\tau} V(x), \]  
(30)
\[ p(t, r_z) = p_0 \left( \frac{t_0}{\tau} \right)^{1+1/\kappa} \]  
(31)
\[ T(t, r_z) = T_0 \left( \frac{t_0}{\tau} \right)^{1/\kappa} \frac{1}{V(x)}, \]  
(32)

where \( p_0 = n_0 T_0 \).

This implies that we have generated a new family of exact solutions of relativistic hydrodynamics: a new hydrodynamical solution is assigned to each non-negative function \( T(x) \). It can be checked that the above solutions are valid also for massive particles, the form of the solution is independent of the value of the mass \( m \). The form of solutions depends parametrically on \( \kappa \), that characterizes the equation of state.

4 Analysis of the solutions

We have obtained new solutions of the (1+1) dimensional relativistic hydrodynamical equations which describe a self-similar, streaming flow. The pressure and the flow profiles are the same as in the 1+1 dimensional Bjorken solution. In the case of \( V(x) = 1 \), we recover Bjorken’s solution. In this limiting case, the pressure, the density and the temperature profiles depend only on the longitudinal proper time \( \tau \).
In the general case, our solution contains a characteristic scale defining parameter in the longitudinal direction, $Z_0$, and an arbitrary scaling function $\mathcal{V}(x)$. Thus we have an infinitely rich new family of solutions. Let us try to determine the physical meaning of the scaling function $\mathcal{V}(x)$.

In order to do this we evaluate the single particle spectra corresponding to the new solutions. Here we neglect any possible dynamics in the transverse directions, as usual in case of applications of Bjorken’s solution. The four-velocity field of our solutions thus becomes $u^\mu = (\cosh \eta, 0, 0, \sinh \eta)$. The four-momentum of the observed particles with mass $m$ is denoted by $k^\mu = (m, \cosh y, k_x, k_y, m \sinh y)$. Let us assume that particles freeze out at a constant longitudinal proper-time $\tau_f$, for the sake of simplicity. This implies freeze-out at a constant pressure, but at a space-time rapidity dependent temperature and density, and makes it possible to continue the calculation analytically.

The source function of locally thermalized relativistically flowing particles in a Boltzmann approximation can be written as

$$S(x, k) = C(\eta) m_t \cosh(\eta - y) n(x) \exp \left(-\frac{k^\mu u_\mu}{T}\right) \delta(\tau - \tau_f), \quad (33)$$

where $C(\eta)$ is an $\eta$ dependent normalization factor, given by the condition that $\int d{k}/E S(x, k) = n(x) \delta(\tau - \tau_f)$, which implies that

$$C(\eta) = \left\{ 4\pi m^2 T(\tau_f, \eta) K_2[m/T(\tau_f, \eta)] \right\}^{-1}, \quad (34)$$

where $K_\nu(z) = \int_0^\infty dz \exp(-z \cosh t) \cosh(\nu t)$ is the modified Bessel function of the second kind.

The single particle spectrum can be calculated from the emission function as

$$E \frac{d^3N}{d^3k} = \int \tau d\tau d\eta S(x, k). \quad (35)$$

Substituting our family of new solutions, and using $\mathcal{T}(x) = 1/\mathcal{V}(x)$, we obtain the following form

$$S(x, k) = C(\eta) m_t \cosh(\eta - y) n(x) f_B(x, k) \quad (36)$$

$$f_B(x, k) = \exp \left[ -\frac{m_t \cosh(\eta - y)}{T_0} \left( \frac{\tau}{t_0} \right)^{1/\kappa} \mathcal{V}\left( \frac{\tanh^2 \eta}{Z_0^2} \right) \right] \delta(\tau - \tau_f). \quad (37)$$

We are interested in the coupling between the measurable rapidity distribution and the rapidity dependence of the effective temperature in the transverse directions as obtained from our new family of solutions. We assume that $\mathcal{V}(x)$
is a slowly varying function, i.e. \( d \log V(x)/dx \ll 1 \) in the region of interest. This assumption implies that the point of maximal emissivity is located at \( \eta = y \) with correction terms of \( \mathcal{O}(d \log V(x)/dx) \). The measurable single-particle spectra can be written as

\[
E \frac{d^3 N}{d\mathbf{k}} = 2C(y) n_0 t_0 V \left( \frac{\tanh^2 y}{Z_0^2} \right) K_1[m_t/T_{\text{eff}}(y)], \tag{38}
\]

\[
\frac{dN}{dy} = n_0 t_0 V \left( \frac{\tanh^2 y}{Z_0^2} \right). \tag{39}
\]

where

\[
T_{\text{eff}}(y) = \frac{1}{V \left( \frac{\tanh^2 y}{Z_0^2} \right)} T_0 \left( \frac{t_0}{\tau_f} \right)^{1/\kappa}. \tag{40}
\]

Note that the \( V \) function is a free fit function that describes the measurable rapidity distribution, including characteristic scales of the size of \( \dot{Z}_0 \).

We see that the slope parameter for transverse mass distribution \( T_{\text{eff}} \) is related to the rapidity distribution as

\[
T_{\text{eff}}(y) = T_0 \left( \frac{t_0}{\tau_f} \right)^{1/\kappa} \frac{dN/dy (y = 0)}{dN/dy}. \tag{41}
\]

Figures 1 and 2 illustrate the calculated behavior of the effective temperature distribution as a function of rapidity for a single Gaussian-like and a double Gaussian-like ansatz for the measurable rapidity distribution. In case of a homogeneous rapidity distribution, \( dN/dy = C \) we recover Bjorken’s result that the effective temperature distribution is rapidity independent. This behavior is expected to appear in high energy heavy ion collisions in the infinite bombarding energy limit.

5 Summary

We have found a new family of solutions of \((1+1)\) dimensional relativistic hydrodynamics. This family solves the continuity equation and the conservation of the energy - momentum tensor of a perfect fluid, assuming an ideal gas equation of state. The flow field coincides with that of Bjorken’s solution. However, the shape of the measurable rapidity distribution, \( dN/dy \) plays the
role of an arbitrary scaling function in our solution, and we obtain that the
effective temperature of the transverse momentum distribution becomes ra-
pidity dependent. Assuming that $dN/dy$ is a slowly varying function of the
rapidity $y$, we find that the effective temperature is proportional to the inverse
of the rapidity distribution, $T_{\text{eff}}(y) \propto (dN/dy)^{-1}$.

As compared to the well-known Bjorken’s solution, we have solved one more
equation, the continuity equation. We have considered equations of state that
have two free parameters, the mass $m$ and $\kappa = \partial \epsilon / \partial p$. Interestingly, these
generalizations resulted in additional freedom in the solution. Our solution,
similarly to Bjorken’s case, describes scaling longitudinal flow and a pressure
distribution that depends only on the longitudinal proper time. However, in
our case, the pressure is a product of the local number density and the local
temperature, hence one of these can be chosen in an arbitrary manner. In
principle, we obtained that the measured single particle rapidity distribution
can be arbitrary, and this measurable distribution plays the role of a scaling
function of both the density and the inverse temperature in this family of
new solutions of relativistic hydrodynamics. A generalization of the present
one dimensional solution to 3 dimensional case can also be done in a similar
manner to the nonrelativistic case [17].

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Fig. 1. Rapidity distribution $dN/dy$ and effective temperature distribution $T_{eff}(y)$ as a function of rapidity $y$, as obtained from a new family of solutions of (1+1) dimensional relativistic hydrodynamics. Here we use the scaling function $V(x) = (1 - x)^{1/4}$, using a scale parameter $Z_0 = \tanh(4)$, $n_0t_0 = 900$ and $T_0(t_0/\tau_f)^{1/\kappa} = 200$ MeV, corresponding to a single maximum in the rapidity distribution $dN/dy$. The analytic expressions are given by eqs. (30, 32, 39, 40).

Fig. 2. Same as Fig. 1 but utilizing a different form of the scaling function, $V(x) = \sqrt{1 + 1.6x^4 - 2.6x^8}$, using a scale parameter $Z_0^2 = 1$, $n_0t_0 = 800$ and $T_0(t_0/\tau_f)^{1/\kappa} = 200$ MeV, corresponding to a two-peaked rapidity distribution.