DOUBLING PROPERTY FOR BILIPSCHITZ HOMOGENEOUS GEODESIC SURFACES

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Abstract. In this paper we discuss general properties of geodesic surfaces that are (locally) biLipschitz homogeneous. In particular, we prove that they are locally doubling and there exists a special doubling measure analogous to the Haar measure for locally compact groups.

1. Introduction

According to a consequence of a general theorem by V. N. Berestovskii [Ber88, Ber89a, Ber89b], if a geodesic distance $d$ on a surface $S$ induces the same topology and has the property that the isometries of $(S, d)$ act transitively on $S$, then $(S, d)$ is isometric to a Finsler surface. In particular, such spaces are locally biLipschitz equivalent to a planar Euclidean domain.

On the other hand, some distances on the plane are not locally biLipschitz equivalent to the Euclidean distance. Laakso constructed in [Laa02] geodesic metrics on the plane that are not biLipschitz embeddable into any $\mathbb{R}^n$ but still share many properties with the Euclidean metric, such as Ahlfors 2-regularity, local linear contractibility, and the fact that a Poincaré inequality holds; see [Hei01] for an introduction to these last definitions.

In this paper we begin the study of a property that holds in the case of the Euclidean plane but has never been singled out: the fact that biLipschitz maps act transitively. Since every Riemannian/Finsler surface is locally biLipschitz equivalent to an Euclidean planar domain, every two points on the surface have neighborhoods that are biLipschitz equivalent. Briefly, we say that every Finsler surface is locally biLipschitz homogeneous, see the next section for the general definitions. Thus the natural question that is currently leading our research is whether every geodesic metric on the plane (or on a surface) where the biLipschitz maps act (locally) transitively is biLipschitz equivalent to a Riemannian metric and so, locally, to the Euclidean metric.

General homogeneity appears frequently in different mathematical areas and is as natural to assume as it is hard to handle in proofs. We refer, for example, to the challenging open conjecture of Bing and Borsuk that states that an $n$-dimensional,
homogeneous Euclidean Neighborhood Retract, should be a homology $n$-manifold. See [Bry06] for definitions, progress and references.

Homogeneity by isometries in the case of geodesic metric spaces has been successfully studied and characterized by Berestovskii [Ber88, Ber89a, Ber89b]. The interest in biLipschitz homogeneity is relatively recent. It has been studied in dimension one, by several authors [Bis01, GH99, FH08], for planar curves with metrics induced by the ambient geometry. BiLipschitz homogeneity for geodesic spaces has appeared naturally in Geometric Group Theory for some actions on quasi-planes, i.e., geometric objects that are coarsely 2 dimensional, e.g., in [KK06].

Our purpose is to study the 2-dimensional case together with the hypothesis, as is common in Geometric Group Theory, that the metric is geodesic. Such an assumption in dimension one would give trivial results.

The main result of this paper is that any geodesic metric surface that is locally biLipschitz homogeneous is a locally doubling metric space. This fact leads to plenty of consequences, e.g., the Hausdorff dimension is finite and there exists a doubling measure that, like the Haar measure on Lie groups is preserved by (left) translations, it is “biLipschitz -preserved” by biLipschitz maps.

1.1. Definitions, results and strategies. A metric space $(X, d)$ is **doubling** if there is a constant $N \in \mathbb{N}$ such that each ball $B(x, 2R) \subset X$ is contained in the union of $\leq N$ balls of radius $R$. We say that $(X, d)$ is locally doubling if any point has a neighborhood that is doubling.

A metric space $(X, d)$ is locally **biLipschitz homogeneous**, if for every two points $x_1, x_2 \in X$, there are neighborhoods $U_1$ and $U_2$ of $x_1$ and $x_2$ respectively and a biLipschitz homeomorphism $f: U_1 \to U_2$, such that $f(x_1) = x_2$.

A metric space is said to be **locally linearly contractible** if there is a constant $C \geq 1$ so that each metric ball of radius $0 < R < C^{-1}$ in the space can be contracted to a point inside a ball of same center but radius $CR$. See [Sem96] for more discussion about this condition.

We prove the following.

**Theorem 1.1.** Let $(X, d)$ be a geodesic metric space topologically equivalent to a surface. Assume that $X$ is locally biLipschitz homogeneous. Then

1. $(X, d)$ is locally doubling,
2. the Hausdorff dimension of $(X, d)$ is finite,
3. the Hausdorff 2-measure $\mathcal{H}^2$ of small $r$-ball $B_r$ has a quadratic lower bound: for each point $p \in X$ there are constants $c, \bar{r} > 0$, so that
   \[ \mathcal{H}^2(B(p, r)) \geq cr^2, \quad r < \bar{r}, \]
   and
4. every point of $(X, d)$ has a neighborhood that is locally linearly contractible.
Subsequently, we investigate the properties of general doubling biLipschitz homogeneous spaces. We show that they admit an analog of the Haar measure: there exists a doubling measure that is quasi-preserved by biLipschitz maps and is quasi-unique. We consider two measures $\nu$ and $\mu$ to be $\alpha$-quasi-equivalent, writing $\nu \approx^\alpha \mu$, when, for all measurable sets $A$,

$$\frac{1}{\alpha} \mu(A) \leq \nu(A) \leq \alpha \mu(A).$$

With this notation we can precisely formulate the result.

**Proposition 1.3.** [Existence] Let $X$ be a doubling metric space. Then there exists a (non-zero) Radon measure $\mu$ with the property that, for any $L > 1$, there exists a constant $\alpha$ such that

$$\mu \approx^\alpha f_* \mu, \quad \forall f : X \to X \text{ } L\text{-biLipschitz}.$$  

[Uniqueness] If moreover $(X, d)$ is $L$-biLipschitz homogeneous, then whenever $\mu$ and $\nu$ satisfy (1.4) we have $\mu \approx^\beta \nu$, for some $\beta > 1$.

Measures satisfying (1.4) are called Haar-like. In section 4, we discuss some connections between the existence of a Poincaré inequality and upper bounds on the Hausdorff dimension, cf. Proposition 4.16. We also show that every Haar-like measure satisfies a lower and an upper polynomial bound for the measure of balls in terms of the radius of the ball, cf. Corollary 4.12.

Before summarizing the strategy for proving Theorem 1.1, let us recall some terminology. A geodesic triangle is said to be $\delta$-thin if each edge is in the $\delta$-neighborhood of the other two edges. If every geodesic triangle is $\delta$-thin, the space is said to be $\delta$-hyperbolic. A triangle that is not $\delta$-thin is $\delta$-fat.

Here is the intuition behind the proof of Theorem 1.1: a preliminary argument (cf. Lemma 2.1) asserts that we can suppose that our space is a neighborhood $U$ of the origin $O$ in the plane that is uniformly biLipschitz homogeneous, say with constant $L$. Then we consider two complementary situations, one is going to imply the theorem, the other will result in a contradiction.

**Either:** there exists a $\rho$ such that for any $r$ smaller than $\rho$ there exists an $r/M$-fat triangle in $B(O, r)$; $M$ will be a fixed number depending on the biLipschitz constant $L$.

In this case, (cf. Corollary 2.6) there exists an $r/10M$-ball surrounded by the triangle. The basic idea of the argument is to consider the surrounding function $\text{Sur}(p, r)$ that is the minimum length of loops that surround the metric ball $B(p, r)$. Therefore, the surrounding function for the above ball is less then the length of triangle’s edges, that is less than $6r$. Using “quasi invariance” (cf. Lemma 3.4) of the function, we get (cf. Corollary 3.6) some
constant $k$ s.t., for some $\rho' > 0$,
\[
\text{Sur}(p, r) < kr, \quad \forall p \in U, \forall r < \rho'.
\]

From this, a nice argument implies the local doubling and locally linearly contractible properties, cf. Proposition 3.9 and Proposition 3.8 respectively.

Or: for any natural number $n$ there exists $r_n < 1/n$ such that any triangle in $B(O, r_n)$ is $r_n/M$-thin.

In other words, $B(O, r_n)$ is $r_n/M$-hyperbolic. BiLipschitz homogeneity implies that any $r_n/L$ ball is $Lr_n/M$-hyperbolic. This however implies, via a corollary of Gromov’s coarse version of the Cartan-Hadamard Theorem (Corollary 2.3) that we have global hyperbolicity, if we chose $M$ carefully. Set $M = CL^2$ (the constant $C$ is the universal constant in Gromov’s Theorem): then Gromov’s theorem holds and so our initial neighborhood $U$ is $C''r_n$-hyperbolic for any $n \in \mathbb{N}$ ($C''$ is depending only on $L$, see Remark 2.4). Since $r_n$ goes to 0, this says that $U$ is 0-hyperbolic. Every 0-hyperbolic space is a tree or an $\mathbb{R}$-tree. This is a contradiction since $U$ is an open set of the plane. This second situation could not in fact occur.

The idea of the construction of Haar-like measures is as follows. For each $r > 0$, we consider a maximal $r$-separated net $N_r$. Let $\mu_r$ be a sum of Dirac masses at the elements of $N_r$, and re-scale the result so that the mass of some unit ball $B(x, 1)$ is 1. Then we claim that the measures $\mu_r$ sub-converge weakly to a nice measure on $X$. Now, the existence of a good measure is assured by the doubling property and does not require biLipschitz homogeneity, cf. Proposition 4.3. The equivalence class of such measures is unique when the space is biLipschitz homogeneous, cf. Proposition 4.5.

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2. Preliminaries

Let \((X, d)\) be a geodesic metric space topologically equivalent to a surface.

Assume that \(X\) is locally biLipschitz homogeneous, i.e., suppose that for every two points \(x_1, x_2 \in X\), there is a pointed biLipschitz homeomorphism \((U_1, x_1) \to (U_2, x_2)\), where \(U_i\) is a neighborhood of \(x_i\).

2.1. Uniform biLipschitz homogeneity. A collection \(\Phi\) of homeomorphisms of subsets of \(X\) is said to act transitively on (the points of) \(X\) if, for each pair of points \(p, q \in X\), there exists a map in \(\Phi\) that sends \(p\) to \(q\).

Lemma 2.1. Let \((X, d)\) be any metric space of second category. Suppose \((X, d)\) is locally biLipschitz homogeneous. Then, for any point of \(X\), there exist a compact neighborhood \(U\) of the point and a constant \(L\) such that the family of maps \(L\text{-BiLip}(U, X)\) defined on \(U\) with values on \(X\) that are \(L\)-biLipschitz act transitively on \(U\).

Proof. Fix a base point \(O \in X\) that we will call origin, and let \(W\) be a compact neighborhood of the origin. Consider the sets

\[ S_{n,m} := \left\{ p \in W \mid f(O) = p, \text{ for some } f : B(O, \frac{1}{m}) \to X, \text{n-biLipschitz} \right\}. \]

By transitivity, we have \(W = \bigcup_{m,n \in \mathbb{N}} S_{n,m}\). We claim that each \(S_{n,m}\) is closed. Take a sequence \(p_j \in S_{n,m}\) converging to \(p \in W\). Each \(p_j\) gives a function \(f_j : B(O, \frac{1}{m}) \to X\). The \(f_j\)'s are n-biLipschitz, and \(f_j(O) = p_j\) converges. The Ascoli-Arzelà argument implies that \(f_j\) converges to some \(f\) uniformly on the closed ball \(B(O, \frac{1}{m})\), and the limit function is n-biLipschitz. Therefore, \(f(O) = p\) for a n-biLipschitz map \(f\) on \(B(O, \frac{1}{m})\). Thus \(p \in S_{n,m}\) and so \(S_{n,m}\) is closed.

The Baire Category Theorem implies that there exists an \(S_{N,M}\) that has non-empty interior. Therefore there exists a compact neighborhood \(V \subset U\) of some point \(q\), such that \(V \subset S_{N,M}\). Let \(f_q : B(O, \frac{1}{M}) \to X\) be an \(N\)-biLipschitz map such that \(f_q(O) = q\).

We claim that \(f_q^{-1}(V) \cap B(O, \frac{1}{2MN}) =: U\) is a neighborhood satisfying the conclusion of the lemma if we choose \(L := N^2\). Indeed, for any two points \(p_1, p_2 \in f_q^{-1}(V)\), for \(i = 1, 2\), \(f_q(p_i) \in V \subset S_{N,M}\); so there exists an \(N\)-biLipschitz map \(f_i : B(O, \frac{1}{M}) \to X\), such that \(f_i(0) = f_q(p_i)\). Thus

\[
p_2 = (f_q^{-1} \circ f_2 \circ f_1^{-1} \circ f_q)(p_1)\]
and \( f_q^{-1} \circ f_2 \circ f_1^{-1} \circ f_q \) is \( L \)-biLipschitz. If moreover \( p_1 \in B(O, \frac{1}{2MN^2}) \), the function is defined in all \( B(O, \frac{1}{2MN^2}) \).

\[
\begin{align*}
B(O, \frac{1}{2MN^2}) & \subset B(p_1, \frac{1}{M^2}) \xrightarrow{f_q} B(f_q(p_1), \frac{1}{MN^2}) \xrightarrow{f_1^{-1}} B(O, \frac{1}{MN^2}) \\
B(p_2, \frac{1}{M}) & \leftarrow B(f_q(p_2), \frac{1}{MN}) \xrightarrow{f_2} B(f_q(p_2), \frac{1}{MN})
\end{align*}
\]

\( \square \)

2.2. Gromov’s coarse version of the Cartan-Hadamard theorem. The following generalization of Cartan-Hadamard theorem will be our local-to-global argument: if small balls are \( \delta \)-hyperbolic then the space is \( \delta' \)-hyperbolic.

**Theorem 2.2** (Cf. [Gro87], [Bow91, Theorem 8.1.2]). There are constants \( d_0, C_1, C_2, \) and \( C_3 \) with the following property. Let \( X \) be a metric space of bounded geometry. Assume that for some \( \delta, \) and \( d \geq \max(C_1 \delta, d_0) \), every ball of radius \( C_2d \) in \( X \) is \( \delta \)-hyperbolic, and \( \text{Rips}_d(X) \) is 1-connected. Then \( X \) is \( C_3 d \)-hyperbolic.

What we need is the following:

**Corollary 2.3.** There are constants \( C, C' \) such that if \( Z \) is a simply-connected geodesic metric space, and for some \( R \), every ball of radius \( CR \) is \( R \)-hyperbolic, then \( Z \) is \( C'R \)-hyperbolic.

**Remark 2.4.** We want to conclude now that, as we briefly said in the introduction, if any \( r_n/L \) ball is \( Lr_n/M \)-hyperbolic and \( M = CL^2 \) then the space is \( C''r_n \)-hyperbolic, for \( C'' = C''(L) \). Indeed, if \( C \) and \( C' \) are the constants in the statement of the above Corollary, considering \( R = R(n) = Lr_n/M \), we have that any \( CR \) ball is \( R \)-hyperbolic. Then \( U \) is \( C'R \)-hyperbolic. Setting \( C'' := C'M/L = C'CL \), then \( C'' \) depends only on \( L \) and since \( C'R = C''r_n \), the space \( U \) is \( C''r_n \)-hyperbolic.

2.3. Existence of surrounded balls. The ideas in this subsection have been partially inspired by the paper [Pap05].

**Proposition 2.5.** Let \( p \) and \( q \) two points in a geodesic plane. Let \( \gamma \) be a geodesic from \( p \) to \( q \) and let \( \eta \) be another curve from \( p \) to \( q \). Suppose \( \gamma \) is not contained in the \( R \)-neighborhood of \( \eta \). Then there exists an \( R/10 \)-ball surrounded by \( \gamma \cup \eta \), i.e., it is in one of the bounded components of \( X \setminus (\gamma \cup \eta) \).
Proof. Let \( r := R/10 \). Call \( U \) the ‘inside’ \( r \)-neighborhood of \( \gamma \) and \( V \) the ‘inside’ \( r \)-neighborhood of \( \eta \). The word ‘inside’ means that we consider the intersections of the neighborhoods of the curves with the bounded components of the complements of the curves. If the complement of \( U \cup V \) has a bounded component then the \( r \)-ball centered at any point of that component is surrounded by \( \gamma \cup \eta \). Assume for by contradiction that this union is simply connected. Since both \( U \) and \( V \) are connected, Mayer-Vietoris tells us that the intersection \( U \cap V \) is connected too. (Note that \( p \) and \( q \) are in \( U \cap V \)). Let \( \sigma \) be a curve from \( p \) to \( q \) inside \( U \cap V \). From the hypothesis we know that there exists a ball of radius \( R \) and center at some point \( x \in \gamma \) that do not intersect \( \eta \).

We claim that \( \sigma \) cannot avoid the ball of center \( x \) and radius \( R - r \). Otherwise, take an \( r \)-net along the curve \( \sigma \). To each point in the net we can associate a point on \( \gamma \), different from \( x \), at distance less than \( r \); it is possible since \( \sigma \) is in the \( r \)-neighborhood of \( \gamma \). But, this association has to ‘change sides’ of \( x \) at some point, in the sense that there are two consecutive points \( y \) and \( z \) of the net that have associated points \( y' \) and \( z' \) in disjoin component of \( \gamma \setminus \{x\} \). Now, since both \( y \) and \( z \) are outside the \((R - r)\)-ball,

\[
d(x, y') \geq d(x, y) - d(y, y') \geq (R - r) - r = R,
\]
and similarly \( d(x, z') \geq R \). This tells as that on one hand, since \( y', x, z' \) are in this order on a geodesic, we have \( d(y', z') = d(y', x) + d(x, z') \geq 2R \). On the other hand,

\[
d(y', z') \leq d(y', y) + d(y, z) + d(z, z') \leq 3r.
\]
But we chose \( r \) so that \( 2R \leq 3r \) is false.

Thus \( \sigma \) intersects the ball of radius \( R - r \) center at \( x \). However, each point in \( \sigma \) is no farther than \( r \) from \( \eta \). This would imply that \( x \) is at distance strictly less than \( R \) from \( \eta \). This is a contradiction. \( \square \)

**Corollary 2.6.** In a geodesic plane, each geodesic triangle that is not \( R \)-thin surrounds an \( R/10 \) ball.

**Proof.** Let \( \gamma \) be the geodesic edge that is not in the \( R \)-neighborhood of the other two edges and let \( \eta \) be the concatenation of the other two edges. Now use the previous proposition. \( \square \)

### 2.4. Existence of cutting-through biLipschitz segments.

We may assume \( U \) is the closure of the metric ball \( B(O, \lambda R_0) \subset \mathbb{R}^2 \) for some \( \lambda > 1 \) set in the sequel, and \( R_0 \in \mathbb{R}_+ \), and set \( U_0 := B(O, R_0) \subset \mathbb{R}^2 \).

Since \( U := B(O, \lambda R_0) \) is uniformly biLipschitz homogeneous we have the following property:

**Proposition 2.7.** There exists a constant \( K \) such that for each point \( p \in U_0 := B(O, R_0) \) there is a \( K \)-biLipschitz image into \( X \) of an interval passing through \( p \) and starting and ending outside \( U_0 \).
Proof. Take a geodesic $\eta$ connecting $O$ to a point in $X$ that is $100L^2R_0$ far from $O$. Let $q$ be a point in $\eta \cap (U \setminus B(O, L^2R_0))$. Move $q$ to $O$ with a $L$-biLipschitz map $f$. Since $d(O, q) > L^2R_0$, we have $d(f(O), O) > LR_0$, i.e., both $f(O)$ and the end of the geodesic are quantitatively outside $U_0$. For any other point $p \in U_0$ we can move the curve obtaining an $L^2$-biLipschitz curve passing through $p$ and now the end points have distance greater than $R_0$ from $O$, so they are outside $U_0$. \[ \square \]

3. The surrounding function

**Definition 3.1.** A loop $\gamma \subset X$ surrounds a subset $\Sigma \subset X$ if $\gamma \cap \Sigma = \emptyset$, and $\gamma$ separates $\Sigma$ from infinity, i.e. any proper path $\mathbb{R}_+ \to X$ starting at $\Sigma$ intersects $\gamma$.

If $\gamma$ is a loop in $X$, we let $|\gamma|$ denote the length of $\gamma$ with respect to the metric $d$.

**Definition 3.2** (Surrounding function). Given $p \in X$, $r \in \mathbb{R}_+$, let $\text{Sur}(p, r)$ be the minimum of lengths of loops $\gamma \subset X$ which surround the metric ball $B(p, r) \subset X$.

We actually need a local substitute to control the diameter of the surrounding loops.

**Definition 3.3.** Given $p \in U$, $r < R \in \mathbb{R}_+$, let $\text{Sur}_R(p, r)$ be the minimum of lengths of loops $\gamma \subset B(p, R)$ which surround the metric ball $B(p, r) \subset X$.

Note that if the set of such loops is non empty, then there exists a minimum by Ascoli-Arzelà theorem. We will refer to a loop $\gamma$ which realizes the minimum as a smallest or shortest loop which surrounds $B(p, r)$.

**Lemma 3.4.** The function $\text{Sur}_{r}(\cdot, \cdot)$ is “quasi-invariant”:

\[
\frac{1}{L} \text{Sur}_{LR}(p', \frac{r}{L}) \leq \text{Sur}_R(p, r) \leq L \text{Sur}_{R/L}(p', Lr),
\]

for all $p, p' \in U_0$, $r < R_0$.

**Proof.** Let $p, p' \in U_0$. Let $f : (U, p) \to (X, p')$ be an $L$-biLipschitz map. Choose a smallest loop $\gamma \subset B(p, R)$ which surrounds $B(p, r)$. Then $f(\gamma)$ is a loop that surrounds $B(p', r/L)$, it is in $B(p, LR)$ and its length is no more than $L|\gamma|$ so $\text{Sur}_{LR}(p', r/L) \leq L|\gamma| = L\text{Sur}_R(p, r)$. \[ \square \]

**Remark 3.5.** If a geodesic triangle in $B(O, R)$ surrounds a ball of radius $\frac{r}{10M}$ and center $\tilde{p}$, then we know $\text{Sur}_r(\tilde{p}, \frac{r}{10M}) \leq 6r$. Thus, from the previous lemma, for any other $p' \in U_0$,

\[
\text{Sur} \left( p', \frac{r}{10ML} \right) \leq \text{Sur}_{Lr} \left( p', \frac{r}{10ML} \right) \leq L \text{Sur}_r \left( \tilde{p}, \frac{r}{10M} \right) \leq 6Lr.
\]

Hence we have proved the (upper) bound for the surrounding function.
**Corollary 3.6.** Let \((X,d)\) a biLipschitz homogenous geodesic surface, and \(U\) be a compact neighborhood that is \(L\)-biLipschitz homogeneous. There exist constants \(k\) and \(\rho'\) such that

\[
\text{Sur}(p,r) < kr
\]

for any \(p \in U\) and \(r < \rho'\).

**Lemma 3.7.**

1. For each \(r < R_0\), \(p \in U_0\), and each loop \(\gamma \subset U\) which surrounds \(B(p,r)\) we have \(\text{diam}(\gamma) \geq \frac{2}{K}r\).
2. Let \(C_0 = 2/K^2\). Then for all \(r < R_0\) we have \(\text{Sur}(p,r) \geq C_0r\).
3. Suppose that \(\gamma\) is a graph which surrounds a ball \(B(p,r)\), \(r < R_0\). Then for \(r' < C_0r\) and each \(p' \in \gamma\) the length of \(\gamma \cap B(p',r')\) is at least \(r\).
4. Let \(C_1 = 2K^2\), any loop \(\gamma\) which surrounds an \(r\)-ball \(B(p,r)\) must lie in \(B(p,C_1|\gamma|)\).
5. Let \(C_2 = 4K^2\), if \(\gamma\) surrounds \(B(p,r) \subset X\), then the connected component of \(p\) in \(X \setminus \gamma\) is contained in \(B(p,C_2|\gamma|)\).

**Proof.**

1. Let \(\gamma \subset U\) be a loop which surrounds the ball \(B(p,r)\). Consider (cf. Proposition 2.7) a \(K\) biLipschitz segment \(\sigma\) which divides \(U_0\) and has \(\sigma(0) = p\). Since \(\gamma\) surrounds \(B(p,r)\) there are two points \(p_\pm \in \gamma\) such that \(\sigma(T_\pm) = p_\pm\), with \(T_- < 0 < T_+\). Thus

\[
\text{diam}(\gamma) \geq d(p_-, p_+) \geq \frac{1}{K}(T_+ - T_-) = \frac{1}{K}[d(T_-, 0) + d(0, T_+)]
\]

\[
\geq \frac{1}{K^2}d(p_-, p) + d(p, p_+) \geq \frac{1}{K^2}(r + r) = \frac{2}{K^2}r.
\]

2. Let \(\gamma\) be a smallest loop surrounding \(B(p,r)\), by part (1)

\[
\text{Sur}(p,r) = |\gamma| \geq \text{diam}(\gamma) \geq \frac{2}{K^2}r.
\]

3. According to (1), \(\text{diam}(\gamma) \geq C_0r\). Hence for each \(r' \leq C_0r\) and \(p' \in \gamma\), the metric sphere \(S(p',r')\) has nonempty intersection with \(\gamma\). Thus, the length of \(\gamma \cap B(p',r')\) is at least \(r'\).

4. Let \(p_\pm\) be the points considered in (1). Then

\[
d(p_\pm, p) \leq Kd(T_\pm, 0) = K|T_\pm| \leq K|T_+ - T_-| \leq K^2d(p_+, p_-) \leq K^2|\gamma|.
\]

Thus for any \(z \in \gamma\)

\[
d(z, p) \leq d(z, p_+) + d(p_+, p) \leq |\gamma| + K^2|\gamma| \leq 2K^2|\gamma|.
\]

Thus \(\gamma \subset B(p,C_1|\gamma|)\).
5. Consider a point \( q \in X \setminus \gamma \) which lies in the same component of \( X \setminus \gamma \) as \( p \). Then either \( q \notin N_r(\gamma) \) or \( d(q, \gamma) \leq r \). In the first case \( \gamma \) surrounds both \( B(p, r) \), \( B(q, r) \) and hence, by (4),
\[
\gamma \subset B(p, C_1|\gamma|), \quad \gamma \subset B(q, C_1|\gamma|),
\]
i.e., any point of \( \gamma \) is at distance less then \( C_1|\gamma| \) from both \( p \) and \( q \). By the triangle inequality we conclude that \( d(p, q) \leq 2C_1|\gamma| = 4K^2|\gamma| \). In the second case if \( d(q, \gamma) \leq r \) then \( d(q, \gamma(t)) \leq r \) for some \( t \). Then (by (4) and (2))
\[
d(p, q) \leq d(p, \gamma(t)) + d(\gamma(t), q) \leq C_1|\gamma| + r \leq C_1|\gamma| + |\gamma|/C_0 = 5/2 K^2|\gamma| < 4K^2|\gamma|.
\]
Therefore \( q \in B(p, C_2|\gamma|) \). \( \square \)

**Proposition 3.8.** Suppose a metric surface \( U \) has the property that there are constants \( C, R > 0 \), and a compact neighborhood \( V \) such that \( \text{Sur}(p, r) \subset C r \) for all \( p \in V \), and all \( r < R \). Then any point of \( U \) has a locally linearly contractible neighborhood.

**Proof.** Consider the ball \( B(p, r) \) and a length minimizing surrounding loop \( \gamma \). Note that each bounded component of \( X \setminus \gamma \) is simply connected and homotopic to a point. The ball \( B(p, r) \) is connected so it contained in the connected component of \( X \setminus \gamma \) containing \( p \), and \( B(p, r) \) is homotopic to a point in that component. By point (5) in the previous lemma, this component is contained in \( B(p, C_2|\gamma|) \). The bound on the surrounding function gives \( |\gamma| = \text{Sur}(p, r) < C r \) and so \( B(p, C_2|\gamma|) \subset B(p, C_2Cr) \). In conclusion, \( B(p, r) \) is homotopic to a point in \( B(p, C_2Cr) \). \( \square \)

**Proposition 3.9.** Suppose a metric surface \( U \) has the property that there are constants \( C, R > 0 \), and a compact neighborhood \( V \) such that \( \text{Sur}(p, r) \subset C r \) for all \( p \in V \), and all \( r < R \). Then any point of \( U \) has a doubling neighborhood.

**Proof.** If \( \gamma \) surrounds \( B(p, r) \) and is a minimizer for \( \text{Sur}(p, r) \), then the hypothesis tells us that \( |\gamma| \leq C r \). In this case, Part (5) of Lemma 3.7 says that the connected component of \( p \) in \( X \setminus \gamma \) is contained in \( B(p, C_2Cr) \), since \( C_2Cr \geq C_2|\gamma| \).

Pick \( p \in X \). Choose a loop \( \gamma_1 \subset X \) with length at most \( Cr \) which surrounds \( B(p, r) \), and set \( L_1 = \{ \gamma_1 \} \). Let \( N_1 \) be an \( \frac{r}{2K^2} \)-separated \( \frac{r}{2} \)-net in \( \gamma_1 \). Then, by Lemma 3.7 (3), the cardinality of \( N_1 \) is at most
\[
\frac{|\gamma_1|}{r/(4K^2)} \leq \frac{Cr}{r/(4K^2)} = 4K^2C =: c.
\]
Let \( L_2 \) be a collection of loops (each having size at most \( Cr \)) surrounding the \( r \)-balls centered at points in \( N_1 \). Proceed inductively in this fashion, building up \( k \) layers of surrounding loops in \( X \). The union \( V_k := N_0 \cup \ldots \cup N_k \) has cardinality at most
\[
\ell^{k+1} = (4K^2C)^{k+1}.
\]
We claim that the collection of \( C_2Cr \)-balls centered at points in \( V_k \) covers \( B(p, \frac{kC}{2}) \). To see this, consider a path \( \sigma \) of length at most \( \frac{kC}{2} \) starting at \( p \). We inductively break
σ into a concatenation of at most k sub-paths of length at least $\frac{r}{2}$ as follows. Let $\sigma_1$ be the initial segment of $\sigma$ until it intersects $\gamma_1$. The path $\sigma_1$ has length at least $r$ and terminates within distance $\frac{r}{2}$ of a point $p_1 \in \mathcal{N}_1$. Let $\sigma_2$ be the initial segment of $\sigma \setminus \sigma_1$ until it intersects the surrounding loop for $B(p_1, r)$, et cetera. At each step the segment $\sigma_i$ has length at least $\frac{r}{2}$, and from what we said at the beginning of the proof, each $\sigma_i$ is contained in $\bigcup_{q \in \mathcal{V}_k} B(q, C_2 Cr)$.

Thus

$$B(p, \frac{kr}{2}) \subset \bigcup_{i=1}^{c^k+1} B(p_i, C_2 Cr),$$

for each $p \in U$. Choosing $k$ such that $\frac{k}{2} = 2C_2 C$, (you can suppose $C_2, C \in \mathbb{N}$) and define the constant $N = c^{k+1}$. Writing $\rho$ in the form $\rho = C_2 C r$ we have proved that, for any $p \in U$,

$$B(p, 2\rho) \subset \bigcup_{i=1}^{N} B(p_i, \rho).$$

In other words $V$ is doubling. □

**Corollary 3.10.** Every biLipschitz homogenous geodesic surface is locally doubling.

4. Consequences of the Doubling Property

4.1. Dimension Consequences. Recall that doubling spaces are precisely those spaces with finite Assouad dimension (also known as metric covering dimension or uniform metric dimension in the literature). See Heinonen’s book [Hei01] for the definition. However, the Assouad dimension of a metric space can be defined equivalently as the infimum of all numbers $D > 0$ with the property that every ball of radius $r > 0$ has at most $Ce^{-D}$ disjoint points of mutual distance at least $\epsilon r$, for some $C \geq 1$ independent of the ball.

Let us recall that a set $N \subset X$ is said to be $\epsilon$-separated if $d(x, y) \geq \epsilon$ for each distinct $x, y \in N$. Also, a set $N \subset X$ is said to be an $\epsilon$-net if, for each $x \in X$, $d(x, N) \leq \epsilon$. Clearly an $\epsilon$-separated set that is maximal with respect to inclusions of sets, will be an $\epsilon$-net; we call it a maximal $\epsilon$-separated net.

Thus, a metric space $X$ of Assouad dimension less than $D$ has the property that there exists a constant $C$ such that, for any $p \in X$ and any $r > 0$,

$$N_\delta \text{ is } \delta\text{-separated } \Rightarrow \#((N_\delta \cap B(p, r)) \leq C \left(\frac{\delta}{r}\right)^{-D}$$

(4.1)

Since the Hausdorff dimension of a metric space does not exceed its Assouad dimension, the next corollary is immediate.

**Corollary 4.2.** A locally biLipschitz homogeneous geodesic surfaces has finite Hausdorff dimension.
Proof. By Corollary [3.10], any point has a neighborhood that is doubling. Thus the Hausdorff dimension of such neighborhood is finite, say $\alpha$. Now, since the space is biLipschitz homogeneous and biLipschitz maps preserve Hausdorff dimension, all points have neighborhoods with Hausdorff dimension equal to $\alpha$. Since the Hausdorff dimension depends on local data, the dimension of the space is $\alpha$. □

4.2. Good measure class: the Haar-like measures. We will give now the details about the Haar-like measures. Throughout this section, let $O$ be a fixed point and let $B_r = B_r(O)$. Let $\delta_p$ be the Dirac measure defined by $\delta_p(A) = 1$ if $p \in A$ and $\delta_p(A) = 0$ if $p \notin A$.

Notation $\mu \overset{\alpha}{\approx} \nu$. For $\mu$ and $\nu$ Borel measures and a number $\alpha > 0$, we say that $\mu \overset{\alpha}{\approx} \nu$ if

$$\frac{1}{\alpha} \nu(A) \leq \mu(A) \leq \alpha \nu(A),$$

for each Borel set $A$. Equivalently, if they are absolute continuous with respect to each other and the derivatives are bounded between $\frac{1}{\alpha}$ and $\alpha$, i.e., there exists a function $h : X \to [\frac{1}{\alpha}, \alpha]$ so that $d\nu = hd\mu$.

For a set $N \subset X$ such that $\#(N \cap B_1) < \infty$, define the Radon measure

$$\mu_N := \frac{1}{\#(N \cap B_1)} \sum_{p \in N} \delta_p,$$

i.e.,

$$\mu_N(A) = \frac{\#(N \cap A)}{\#(N \cap B_1)}.$$

The normalization has the purpose of having $\mu_N(B_1) = 1$ for any set $N$.

Now, the existence of a good measure is assured by the doubling property, and does not require homogeneity.

Proposition 4.3 (Existence). Let $(X, d)$ be a $C$-doubling metric space. Then there exist a non-zero Radon measure $\mu$, with the property that for any $L > 1$ there is a constant $\alpha = \alpha(C, L) > 0$ such that $\mu \overset{\alpha}{\approx} f_* \mu$ for each $f \in L$-BiLip$(X, d)$.

Proof. For each $\epsilon > 0$ choose a maximal $\epsilon$-separated net $N_\epsilon$ and consider the associated measure $\mu_\epsilon := \mu_{N_\epsilon}$ defined as above, i.e.,

$$\mu_\epsilon(A) := \frac{\#(N_\epsilon \cap A)}{\#(N \cap B_1)}.$$

By Theorem 1.59 in [AFP00], since the $\mu_\epsilon$ are (finite) Radon measures and $\mu_\epsilon(B_1) = 1$ there is a subsequence $\mu_{\epsilon_n}$ that is weak* convergent to a measure $\mu$. Recall that, if $cl(B)$ is the closure of a set, then

$$\limsup_{\epsilon_n} \mu_{\epsilon_n}(cl(B)) \leq \mu(cl(B)).$$
Let us prove that $\mu$ satisfies the conclusion of the theorem. Take any $f \in L\text{-Bilip}(X, d)$. Then note that $f(N_\epsilon)$ is an $\frac{\epsilon}{L}$-separated $L\epsilon$-net.

Fix any ball $B$. Take two other balls $B'' \subset B' \subset B$ with same center and different radii $r'' < r' < r$. If $\epsilon \leq r' - r''$ and $B(p, \epsilon) \cap B'' \neq \emptyset$, then we have $p \in B'$. Thus

$$B'' \subset \bigcup_{p \in B' \cap N_\epsilon} B(p, \epsilon), \quad \forall \epsilon \leq r' - r'',$$

since $N_\epsilon$ is an $\epsilon$-net. Moreover, since $f(N_\epsilon)$ is $\frac{\epsilon}{L}$-separated, from (4.1), we have

$$
\#(B(p, \epsilon) \cap f(N_\epsilon)) \leq C \left( \frac{\epsilon/L}{\epsilon} \right)^{-D} = CL^D.
$$

Then

$$
\#(B'' \cap f(N_\epsilon)) \leq \sum_{p \in B' \cap N_\epsilon} \#(B(p, \epsilon) \cap f(N_\epsilon)) \leq CL^D \#(B' \cap N_\epsilon).
$$

So,

$$
\begin{align*}
\star & \left( B'' \cap N_\epsilon \right) \\ & \leq \frac{\#(f^{-1}(B'') \cap N_\epsilon)}{\#(B_1 \cap N_\epsilon)} \\ & = \frac{\#(B'' \cap f(N_\epsilon))}{\#(B_1 \cap N_\epsilon)} \\ & \leq CL^D \frac{\#(B' \cap N_\epsilon)}{\#(B_1 \cap N_\epsilon)} \\ & = CL^D \mu_c(B') \\ & \leq CL^D \mu_c(cl(B')).
\end{align*}
$$

Taking the limit for $\epsilon_n \to 0$, we have, from the last estimate and from (4.4)

$$
\begin{align*}
\star & \left( B'' \cap N_\epsilon \right) \\ & \leq \liminf_{\epsilon_n \to 0} f_{*}\mu_{\epsilon_n}(B'') \\ & \leq \limsup_{\epsilon_n \to 0} CL^D \mu_{\epsilon_n}(cl(B')) \\ & \leq CL^D \mu(cl(B')) \\ & \leq CL^D \mu(B).
\end{align*}
$$

Since $B'' \subset B$ was arbitrary, we get

$$f_{*}\mu(B') \leq CL^D \mu(B).$$

In conclusion, $f_{*}\mu \leq \alpha \mu$, for $\alpha = CL^D$, on every (small) ball, so the same inequality holds on every open set and so on every Borel set. Since $f^{-1} \in L\text{-Bilip}(X, d)$, we also get

$$\frac{1}{\alpha} \mu(A) \leq f_{*}\mu(A),$$

for each Borel set $A$. So both the required inequalities are proven. □
The equivalence class of the Haar-like measures is unique when the space is biLipschitz homogeneous.

**Proposition 4.5** (Uniqueness). Let \((X, d)\) be a doubling metric space with a transitive set \(\mathcal{F}\) of \(L\)-bilip maps and suppose that two non-zero Radon measures \(\mu_1\) and \(\mu_2\) on \(X\) are such that \(\mu_i \preccurlyeq f_\ast \mu_i\), for \(i = 1, 2\) and for each \(f \in \mathcal{F}\). Then \(\mu_1 \preccurlyeq \mu_2\), for a constructive \(\beta > 1\).

Let us prepare for the proof of the uniqueness of the class of good measures with a lemma that we will be again useful later in the proof of polynomial growth of measures of balls. The following lemma says that when \(\mu\) is a Haar-like measure, then the \(\mu\) measure of the \(\epsilon\)-balls is approximatively the inverse of the cardinality of a maximal \(\epsilon\)-separated net in the unit ball.

**Lemma 4.6.** Let \((X, d)\) be a doubling metric space with a transitive set \(\mathcal{F}\) of \(L\)-biLipschitz maps. Suppose that a non-zero Radon measures \(\mu\) on \(X\) is such that \(\mu \preccurlyeq f_\ast \mu\), for each \(f \in \mathcal{F}\). Then there are positive constants \(\epsilon_0\), \(k\), and \(h\) such that for any \(\epsilon < \epsilon_0\) and for any maximal \(\epsilon\)-separated net, defining \(c_\epsilon := \# (N_\epsilon \cap B_1)\), we have

\[
\mu \left( B(p, L\epsilon) \right) \geq kc_\epsilon^{-1},
\]

and

\[
\mu \left( B \left( p, \frac{\epsilon}{2L} \right) \right) \leq hc_\epsilon^{-1}.
\]

**Proof.** Set \(\epsilon_0 = 1/2\). Let \(\epsilon < \epsilon_0\) and let \(N_\epsilon\) be a maximal \(\epsilon\)-separated net. Fix \(p \in X\). For any \(p_j \in N_\epsilon\) choose \(f_j \in \mathcal{F}\) so that \(f_j(p) = p_j\). Thus \(B(p_j, \epsilon) \subset f_j \left( B(p, L\epsilon) \right)\).

To show \(\epsilon_0\), consider that, since \(N_\epsilon\) is an \(\epsilon\)-net, the family \(\{B(p_j, \epsilon)\}_{p_j \in N_\epsilon}\) is a cover of \(X\). Therefore

\[
B_{\frac{1}{2}} \subset \bigcup \{B(p_j, \epsilon) : p_j \in N_\epsilon \cap B_1\},
\]
because $\epsilon < \frac{1}{2}$ (we had to reduce to the ball $B_2$ because removing those $\epsilon$-balls with center outside $B_1$, we might fail to cover $B_1 \setminus \hat{B}_{1-\epsilon}$). So

$$0 < \mu(B_2) \leq \mu\left( \bigcup \{B(p_j, \epsilon) : p_j \in N_\epsilon \cap B_1\} \right)$$

$$\leq \sum_{p_j \in N_\epsilon \cap B_1} \mu(B(p_j, \epsilon))$$

$$\leq \sum \mu(f_j(B(p, L\epsilon)))$$

$$= \sum ((f_j^{-1})_* \mu)(B(p, L\epsilon))$$

$$\leq \sum_{p_j \in N_\epsilon \cap B_1} \alpha \mu(B(p, L\epsilon))$$

$$= #(N_\epsilon \cap B_1) \cdot \alpha \mu(B(p, L\epsilon))$$

$$= c_{\epsilon} \alpha \mu(B(p, L\epsilon)).$$

Putting $k = \alpha^{-1} \mu(B_2)$ we obtain (4.7).

Now we show (4.8). Since $N_\epsilon$ is $\epsilon$-separated and $\epsilon < 1/2$, we have that $\{B(p_j, \frac{\epsilon}{2})\}_{p_j \in N_\epsilon \cap B_1}$ is a disjoint family of subsets of $B_2$. Therefore,

$$\mu(B_2) \geq \mu\left( \bigcup \{B\left(p_j, \frac{\epsilon}{2}\right) : p_j \in N_\epsilon \cap B_1\} \right)$$

$$= \sum_{p_j \in N_\epsilon \cap B_1} \mu\left(B\left(p_j, \frac{\epsilon}{2}\right)\right)$$

$$\geq \sum \mu(f_j\left(B\left(p, \frac{\epsilon}{2L}\right)\right))$$

$$= \sum ((f_j^{-1})_* \mu)(B\left(p, \frac{\epsilon}{2L}\right))$$

$$\geq \sum_{p_j \in N_\epsilon \cap B_1} \alpha^{-1} \mu\left(B\left(p, \frac{\epsilon}{2L}\right)\right)$$

$$= #(N_\epsilon \cap B_1) \cdot \alpha^{-1} \mu\left(B\left(p, \frac{\epsilon}{2L}\right)\right)$$

$$= c_{\epsilon} \alpha^{-1} \mu\left(B\left(p, \frac{\epsilon}{2L}\right)\right).$$

Setting $h = \alpha \mu(B_2)$, we obtain (4.8).

Proof of Proposition 4.5. Let $s = h/k$. Then (4.7) and (4.8) imply that for each $\epsilon < \frac{1}{2}$ we have

$$\mu_1\left(B\left(p, \frac{\epsilon}{2L}\right)\right) \leq s \mu_2\left(B\left(p, L\epsilon\right)\right).$$
Now we plan to estimate the measure $\mu_2(B(p, L\epsilon))$ with a constant times $\mu_2 \left( B \left( p, \frac{\epsilon}{2L} \right) \right)$ using the fact that $(X, d)$ is doubling. Indeed, there is a number $m$, not depending on $\epsilon$, so that $m$ balls of radius $\epsilon/L$ cover $B(p, L\epsilon)$. Let $q_1, q_2, \ldots, q_m$ be such that

$$B(p, L\epsilon) \subset \bigcup_{i=1}^{m} B(q_i, \epsilon/L).$$

For each $i = 1, \ldots, m$, choose $g_i \in F$ with $g_i(q_i) = p$.

$$\mu_2(B(p, L\epsilon)) \leq \sum_{i=1}^{m} \mu_2(B(q_i, \epsilon/L)) \leq \sum_{i=1}^{m} \alpha \mu_2(B(q_i, \epsilon/L)) \leq \sum_{i=1}^{m} \alpha \mu_2(B(p, \epsilon/2L)) = m \alpha \mu_2(B(p, \epsilon/2L)).$$

Hence, from (4.9), we have that exists $\gamma > 0$, such that, for all $\epsilon > 0$,

$$\mu_1 \left( B \left( p, \frac{\epsilon}{2L} \right) \right) \leq \gamma \mu_2 \left( B \left( p, \frac{\epsilon}{2L} \right) \right).$$

In conclusion, $\mu_1$ is smaller than $\gamma \mu_2$ on every small ball so the same is true on every open set and on every Borel set. The symmetric hypothesis on $\mu_1$ and $\mu_2$ gives us the other inequality too. \hfill $\square$

**Lemma 4.10.** Let $(X, d)$ be a metric space of Hausdorff dimension $\alpha$. Then, for any $t > 0$ and $c > 0$, there exists an $\epsilon_0 > 0$ such that any $\epsilon$-net $N_\epsilon$, with $\epsilon < \epsilon_0$ has the property that

$$\# (N_\epsilon \cap B_1) \geq \frac{c}{\epsilon^{\alpha-t}}.$$

**Proof.** Since the Hausdorff dimension is $\alpha$, all the Hausdorff measures of dimension less than $\alpha$ are infinite:

$$\mathcal{H}^{\alpha-s}(B_{1/2}) = \infty, \quad \forall s > 0.$$

On the other hand, let assume that the conclusion of the lemma is not true, i.e., there exist $t, c > 0$ so that, for all $\epsilon_0 > 0$, there is an $\epsilon$-net $N_\epsilon$, with $\epsilon < \epsilon_0$ with

$$\# (N_\epsilon \cap B_1) \leq \frac{c}{\epsilon^{\alpha-t}}.$$
for $\epsilon < 1/2$, is a covering of $B_{1/2}$ by sets of diameter less than $2\epsilon$. We can estimate the Hausdorff measure

$$\mathcal{H}_{2\epsilon}^{\alpha-s}(B_{1/2}) := \inf \left\{ \sum (\text{diam} V_i)^{\alpha-s} : \text{diam} V_i \leq 2\epsilon, B_{1/2} \subset \bigcup V_i \right\}$$

$$\leq \sum_{p \in N_{\epsilon} \cap B_1} (2\epsilon)^{\alpha-s}$$

$$\leq \frac{c}{\epsilon^{\alpha-t}} (2\epsilon)^{\alpha-s} = 2^{\alpha-s} \epsilon^{t-s}.$$ 

So, take $0 < s < t$, we have that there exists a sequence of $\epsilon$ going to zero and $\mathcal{H}_{2\epsilon}^{\alpha-s}(B_{1/2}) < 2^{\alpha-s} \epsilon^{t-s}$ goes to zero too. Thus

$$\mathcal{H}^{\alpha-s}(B_{1/2}) := \lim_{\delta \to 0} \mathcal{H}_{\delta}^{\alpha-s}(B_{1/2}) = 0,$$

contradicting (4.11).

Let us remark that since $(X, d)$ is doubling, the cardinality of $N_{\epsilon} \cap B_1$ is finite, in fact, using (4.1), it is bounded by $C\epsilon^{-D}$, for some constants $C > 0$ and any $D$ greater than the Assouad dimension. Using Lemma 4.10 and Lemma 4.6 we conclude the following.

**Corollary 4.12.** For any $t > 0$, there exists $r_0 > 0$ and $K > 1$ such that for all $p \in X$ and any $r < r_0$,

$$\frac{1}{K} r^{\dim_{A}(X,d)+t} < \mu(B(p,r)) < K r^{\dim_{H}(X,d)-t}$$

Recall that $\dim_{\text{top}} \leq \dim_{H} \leq \dim_{A}$, so for $r < 1$, we have $r^{\dim_{A}} \leq r^{\dim_{H}} \leq r^{\dim_{\text{top}}}$.

**Corollary 4.13.** Let $\gamma$ be a rectifiable curve. For any Haar-like measure $\mu$, we have $\mu(\gamma) = 0$

Since any doubling measure is non-atomic and strictly positive on non-empty open sets, we are allowed to use the following Theorem by Oxtoby and Ulam.

**Theorem 4.14 (OUP41 Theorem 2).** Let $\mu$ be a Radon measure on the square $Q = [0,1]^n$, $n > 2$, with the properties that

(i) $\mu$ is zero on points,
(ii) $\mu$ is strictly positive on non-void open sets,
(iii) $\mu(Q) = 1$,
(iv) $\mu(\partial Q) = 0$. 

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Then \( \mu \) is automorphic to the Lebesgue measure, i.e., there exists an homeomorphism \( h : Q \to Q \) such that \( \mu = h_* \mathcal{L} \).

As an immediate consequence we have the following:

**Corollary 4.15.** Any doubling measure on the plane is locally a multiple of the Lebesgue measure up to a continuous change of variables.

### 4.3. Upper bounds for the Hausdorff dimension

It is an open question whether a biLipschitz homogeneous geodesic surface satisfies a Poincaré inequality. However, we now show that the existence of a Poincaré inequality implies bounds on the Hausdorff dimension.

Let \( 1 \leq p < \infty \). We say that a measure metric space \( (X, d, \mu) \) admits a weak \((1, p)\)-Poincaré inequality if there are constants \( \lambda \geq 1 \) and \( C \leq 1 \) so that

\[
\int_B |u - u_B| \, d\mu \leq C(\text{diam } B) \left( \int_{\lambda B} \rho^p \, d\mu \right)^{1/p}
\]

for all balls \( B \subset X \), all bounded continuous functions \( u \) on \( B \), and all upper gradients \( \rho \) of \( u \). That is

\[
|u(x) - u(y)| \leq \int_{\gamma_{xy}} \rho \, ds,
\]

for each rectifiable curve \( \gamma_{xy} \) joining \( x \) and \( y \) in \( X \).

**Proposition 4.16.** Let \( (X, d) \) be a biLipschitz homogeneous geodesic surface. If a weak \((1, p)\)-Poincaré inequality hold for a Haar-like measure \( \mu \), then

\[
\dim_H(X, d) \leq 1 + p.
\]

**Proof.** Fix any geodesic \( \sigma \) in \( X \). Since \( X \) is a plane we can consider a simply connected set \( B \subset X \) that is divided into two parts by \( \sigma \), i.e., \( B \setminus \sigma = A_0 \sqcup A_1 \) with \( A_0 \) and \( A_1 \) simply connected. Define the following functions:

\[
\delta(p) := \begin{cases} 
  d(p, \sigma) & \text{for } p \in A_1 \\
  -d(p, \sigma) & \text{for } p \in A_0
\end{cases}
\]

and \( u_\epsilon(p) := \begin{cases} 
  \frac{\epsilon - \delta(p)}{2\epsilon} & \text{for } -\epsilon \leq \delta(p) \leq \epsilon \\
  0 & \text{for } \delta(p) \leq -\epsilon \\
  1 & \text{for } \delta(p) \geq \epsilon
\end{cases} \).

The function \( u_\epsilon \) is 0 on those points of \( A_0 \) at distance more than \( \epsilon \) from \( \sigma \). In the \( \epsilon \)-neighborhood of \( \sigma \) it increases linearly in the distance from \( \sigma \) to the value 1 at those points of \( A_1 \) at distance more than \( \epsilon \) from \( \sigma \). Therefore the function \( \rho_\epsilon \) defined to be \( \frac{1}{\epsilon} \) on the \( \epsilon \)-neighborhood of \( \sigma \) and 0 otherwise is an upper-gradient for \( u_\epsilon \).

Since \( u_\epsilon \to \chi_{A_1} \) as \( \epsilon \to 0 \), an easy computation gives that

\[
\int_B |u_\epsilon - (u_\epsilon)_B| \, d\mu \to \frac{2\mu(A_0)\mu(A_1)}{(\mu(B))^2} \neq 0.
\]

So the limit is non zero.
On the other hand, one can cover the $\epsilon$-neighborhood of $\sigma$ with $\frac{\text{length}(\sigma)}{2\epsilon}$ balls of radius $2\epsilon$. Thus, if $\alpha$ is any number smaller than the Hausdorff dimension, using Corollary 4.12 and the Poincaré inequality, we get

$$
\left(\int_{\lambda B} \rho^p \, d\mu\right)^{1/p} \leq \left(\sum_{j} (\mu(B(p_j, 2\epsilon)))(\frac{1}{2\epsilon})^p\right)^{1/p}
\leq \left(\frac{\text{length}(\sigma) K(2\epsilon)^\alpha}{2\epsilon^p}\right)^{1/p} = K'(\epsilon^{\alpha-1-p})^{1/p}.
$$

If $\alpha > 1 + p$, then this last term would go to zero, as $\epsilon$ goes to zero, and it would give a contradiction. So $\alpha$ and hence $\dim_H(X, d)$ must be smaller than $1 + p$. □

An immediate consequence of the above proposition is that the existence of a $(1,1)$-Poincaré inequality implies that the Hausdorff dimension is 2.

4.4. Lower bound for the Hausdorff 2-measure. Another consequence of the bound on the surrounding function is a lower density bound on the 2-dimensional Hausdorff measure.

**Proposition 4.17.** Suppose a metric surface $U$ has the property that there are constants $C, R > 0$, and a compact neighborhood $V$ such that $\text{Sur}(p, r) < Cr$ for all $p \in V$, and all $r < R$. Then, for $r < R$, any $r$-ball in $V$ has 2-dimensional Hausdorff measure greater than $Cr^2$.

If the space is countably 2-rectifiable, the Hausdorff 2-measure of an $R$-ball can be calculate integrating from 0 to $R$ the 1-Hausdorff measure of the boundary of the $r$-ball in $dr$; If the space is not countably 2-rectifiable, $\geq$ is always true (up to some factor). See [Fed69]. Let $\mathcal{H}^k(X)$ be the $k$-dimensional Hausdorff measure of a metric space $X$. We will make use of the following theorem:

**Theorem 4.18** (Federer, [Fed69, 2.10.25]). Let $X$ be a metric space and let $f : X \to \mathbb{R}$ be a Lipschitz map. If $A \subset X$ and $k, m \geq 0$, then

$$
(Lip f)^m \frac{\omega(k) \omega(m)}{\omega(k + m)} \mathcal{H}^{k+m}(A) \geq \int_{\mathbb{R}} \mathcal{H}^k(A \cap f^{-1}\{r\}) d\mathcal{H}^m(r),
$$

where $f^*$ is the upper integral and $\omega(k)$ is the measure of the $k$-dimensional unit ball.

**Proof of Proposition 4.17** Using the theorem for $f(\cdot) = d(p, \cdot)$ (that is 1-Lipschitz), $A = B(p, R)$, and $k = m = 1$, we have

$$
\frac{\omega(1)^2}{\omega(2)} \mathcal{H}^2(B(p, R)) \geq \int_{\mathbb{R}} \mathcal{H}^1(B(p, R) \cap f^{-1}\{r\}) d\mathcal{H}^1(r) = \int_{[0,R]} \mathcal{H}^1(\partial B(p, t)) dt.
$$
For the last equality, note that $f^{-1}\{r\} = \partial B(p, r)$. Thus

$$
(4.19) \quad \mathcal{H}^2(B(p, R)) \geq C_1 \int_{[0, R]} \mathcal{H}^1(\partial B(p, r)) \, dr,
$$

where $C_1$ is a suitable constant.

We claim that $\mathcal{H}^1(\partial B(p, r)) \geq Cr$. The rest of the subsection will be devoted to the demonstration of the claim. However, modulo this claim, the theorem is proved. Indeed, using it in (4.19) and integrating, we get $\mathcal{H}^2(B(p, R)) \geq \frac{C}{2} R^2$. \hfill \square

The reason behind the claim is that either $\partial B(p, r)$ has infinite length or it is a curve surrounding the ball $B(p, r)$. If the measure is infinite there is nothing to prove. In the case when the measure is finite, call $\Sigma$ the exterior boundary of $B(p, r)$, i.e., the boundary of the unbounded component of the complement of $B(p, r)$. Note that $\Sigma$ surrounds $B(p, r)$, then if $\Sigma$ were a curve, its 1 dimensional Hausdorff measure would be its length. Thus the assertion of the claim follows from the bound on the surrounding function.

To prove that $\Sigma := \partial_{\text{ext}} B(p, r)$ is a curve, we want to use a general theorem [Maz20]:

**Theorem 4.20** (The Hahn-Mazurkiewicz theorem). A Hausdorff topological space is a continuous image of the unit interval if and only if it is a Peano space, i.e., it is a compact, connected, locally connected metric space.

To apply the theorem we only need to prove that $\Sigma$ is locally connected. By a corollary of the Phragmén-Brouwer theorem, see [Why42, page 106], since $\Sigma$ is a common boundary of two domains, it is a continuum. In order to complete the proof of Proposition 4.17 we just need to recall the following:

**Proposition 4.21.** Each continuum $\Sigma$ with $\mathcal{H}^1(\Sigma) < \infty$ is locally connected.

A proof of the proposition can be argued using Theorem 12.1 in [Why42, page 18]. In what follows we give an alternative and easier proof.

**Proof of Proposition 4.21.** Suppose that $\Sigma$ is not locally connected. Hence there exist a point $p$ and a closed normal neighborhood $V$ of it such that any other neighborhood of $p$ contained in $V$ is not connected.

**Lemma 4.22.** The closed set $Z := \cap\{S \mid p \in S, S \subset V, S \text{ clopen}\}$ is not a neighborhood of $p$.

**Proof.** Suppose $Z$ is a neighborhood of $p$. Since $Z$ has to be disconnected, there are $Z_1$ and $Z_2$ two closed (and so compact), disjoint subsets of $Z$ such that $Z = Z_1 \cup Z_2$ and $p \in Z_1$ but $p \notin Z_2$.

Since $V$ is normal there are $H_1$ and $H_2$ disjoint open neighborhood of $Z_1$ and $Z_2$ in $V$. Let $H = H_1 \cup H_2$. 

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Since $V \setminus H$ is a compact subset of $V \setminus Z$ there is a finite number of clopen subsets $K_1, \ldots, K_n$ of $V$ not containing $p$ which cover $V \setminus H$. Their union $K$ is also a clopen subset of $V$, not containing $p$ which covers $V \setminus H$. Clearly $K \cup H_2$ is a clopen subset of $V$ containing $Z_2$ but not $p$. □

Now fix a closed neighborhood $U \subset V$ of $p$ such that $c := \text{dist}(U, \partial V) > 0$. By Lemma 4.22 there is a clopen set $Y$ of $V$ which intersects $U$ but does not contain $p$. Since $\Sigma$ is connected and $U$ and $V$ are closed (and clearly different from $\Sigma$), $Y$ intersects also $\partial U$ and $\partial V$ non-trivially.

**Lemma 4.23.** $\mathcal{H}^1(Y) \geq c$.

**Proof.** The function $\rho : Y \to \mathbb{R}$ defined by $\rho(y) = \text{dist}(y, \partial V)$ is non-expanding. Suppose there is a point $\xi \in \mathbb{R}$ disconnecting $\rho(Y) \subset \mathbb{R}$. Then the set of all points of $Y$ with distance from $\partial V$ bigger than $\xi$ is a clopen set of $Y$ not intersecting $\partial V$ and so it is a proper clopen set of $\Sigma$. This contradicts the fact that $\Sigma$ is connected. Hence $\rho(Y)$ is a connected subset of the positive real line, and moreover it contains $0$ and $c$. Therefore the image of $\rho$ contains the interval $[0, c]$. Since 1-Lipschitz maps do not increase Hausdorff measures and $\mathcal{H}([0, c]) = c$, we get $\mathcal{H}^1(Y) \geq c$. □

We can now conclude the proof of Proposition 4.21 by contradicting the fact that $\mathcal{H}^1(\Sigma) < +\infty$. We will construct a sequence $Y_i$ of disjoint clopen subsets of $V$ with $\mathcal{H}^1(Y_i) \geq c$ for each $i$ and arrive at a contradiction since $\mathcal{H}^1(\Sigma) \geq \mathcal{H}^1(V) \geq \sum_i \mathcal{H}^1(Y_i) = +\infty$.

Put $U_1 = U$, $V_1 = V$ and $Y_1 = Y$. Inductively, consider $U_{j+1} := U_j \setminus Y_j$ and $V_{j+1} := V_j \setminus Y_j$, they are still closed. Using Lemma 4.22 choose a clopen set $Y_{j+1}$ of $V_{j+1}$ which does not contain $p$ but meets $U_{j+1}$, hence it meets also $\partial U_{j+1}$ and $\partial V_{j+1}$.

Note that since $V_j$ is a clopen subset of $V$ then $V_j \setminus \partial V$ is open and so $\partial V_j \subset \partial V$. Similarly $\partial U_j \subset \partial U$ and hence $\text{dist}(\partial U_j, \partial V_j) \geq \text{dist}(\partial U, \partial V) \geq c$.

As for $Y = Y_1$ we have that $\mathcal{H}^1(Y_j) \geq c$. □

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