ON THE MOSER–TRUDINGER INEQUALITY IN COMPLEX SPACE

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Abstract. In this paper we prove the pluricomplex counterpart of the Moser-Trudinger and Sobolev inequalities in complex space. We consider these inequalities for plurisubharmonic functions with finite pluricomplex energy, and estimate the concerned constants.

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1. Introduction

Many researchers in partial differential equations and calculus of variation are interested in Sobolev type inequalities, or Sobolev embedding theorems as some wish to call them. The borderline case when the dimension is two is sometimes known as the Moser-Trudinger inequality or Trudinger-Moser inequality after the work of Trudinger [41] in 1967 and Moser [35] in 1971. To this day these ideas are still used in ongoing research (see e.g. [6, 21, 27, 33, 38, 42]). In this paper we shall prove the pluricomplex counterpart to the Moser-Trudinger and Sobolev inequalities. We shall now continue with a brief discussion about the setting, and we refer the reader to Section 2 for a more detailed background.

Let \( \Omega \) be an open set in \( \mathbb{C}^n \). An upper semicontinuous function \( u : \Omega \to \mathbb{R} \cup \{-\infty\} \) is called plurisubharmonic if the Laplacian of \( u \) is, in the sense of distributions, non-negative along each complex line that intersects \( \Omega \). We shall always assume that a plurisubharmonic function is defined on a so-called hyperconvex domain \( \Omega \subset \mathbb{C}^n \). This assumption is made to ensure a satisfying amount of plurisubharmonic functions with certain properties. As the abstract reveal we are interested in plurisubharmonic functions with finite pluricomplex energy. To be able to define these functions we start by defining what we recognize as the pluricomplex counterpart of test functions in the theory of distributions. We say that a plurisubharmonic function \( \varphi \) defined on \( \Omega \) belongs to \( \mathcal{E}_0 (= \mathcal{E}_0(\Omega)) \) if \( \varphi \) is a bounded function, \( \lim_{z \to \xi} \varphi(z) = 0 \), for every \( \xi \in \partial \Omega \), and \( \int_{\Omega} (dd^c \varphi)^n < \infty \), where \( (dd^c \cdot)^n \) is the complex Monge-Ampère operator. Finally, we say that \( u \in \mathcal{E}_p (= \mathcal{E}_p(\Omega)) \) if \( u \) is a plurisubharmonic function defined on \( \Omega \) and there exists a decreasing sequence, \( \{ \varphi_j \}, \varphi_j \in \mathcal{E}_0 \), that converges pointwise to \( u \) on \( \Omega \), as \( j \) tends to \( \infty \), and

\[
\sup_j e_p(\varphi_j) = \sup_j \int_{\Omega} (-\varphi_j)^p (dd^c \varphi_j)^n < \infty.
\]

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This definition implies that if \( u \in \mathcal{E}_p \), then \( e_p(u) < \infty \). This justify that we say that a function \( u \in \mathcal{E}_p \) have finite pluricomplex \( p \)-energy, or simply finite pluricomplex energy. The energy cones, \( \mathcal{E}_p \), were introduced and studied in [15], and the growing use of complex Monge-Ampère techniques in applications makes this framework of significant importance (see e.g. [4, 11, 24, 25, 26]).

The first inequality we prove is the following.

**The pluricomplex Moser-Trudinger inequality.** Let \( \Omega \) be a bounded hyperconvex domain in \( \mathbb{C}^n \), \( n \geq 2 \). Then there exist constants \( A(p,n,\Omega) \) and \( B(p,n,\Omega) \) depending only on \( p, n, \Omega \) such that for any \( u \in \mathcal{E}_p \) we have that

\[
\log \int_{\Omega} e^{-u} d\lambda_{2n} \leq A(p,n,\Omega) + B(p,n,\Omega)e_p(u)^{\frac{p}{p-n}},
\]

(1.1)

where \( d\lambda_{2n} \) is the Lebesgue measure in \( \mathbb{C}^n \). For any given \( 0 < \epsilon < 1 \) we can take the constants \( A(p,n,\Omega) \) and \( B(p,n,\Omega) \) to be

\[
A(p,n,\Omega) = \log \left( \left( \pi^n + \beta(n) \frac{en}{(n-\epsilon n)^n} \right) \text{diam}(\Omega)^{2n} \right)
\]

and

\[
B(p,n,\Omega) = (2\epsilon n)^{-\frac{n}{p}},
\]

where \( \beta(n) \) is a constant depending only on \( n \). Furthermore, we have the following estimates on \( B(p,n,\Omega) \)

\[
\frac{p}{(4\pi)^{\frac{p}{p-n+p}^n}} \leq B(p,n,\Omega),
\]

(1.2)

and if \( \Omega = \mathbb{B} \) is the unit ball, then we have that

\[
\frac{p}{(4\pi)^{\frac{p}{p-n+p}^n}} \leq B(p,n,\mathbb{B}) \leq \left( \frac{n^{-1}(n+1)(n+p)^{p-1}}{4\pi^n(n+1)(n+p)^{p-1}} \right)^{\frac{1}{p}}.
\]

(1.3)

The proof of this theorem is divided into parts. Inequality (1.1) is proved in Theorem [3, 1]. This inequality was first proved for \( p = 1 \) in [17], and two months later another proof appeared [9] that generated slightly better estimates. In Theorem [4, 4] we present the proof of our estimates. In the proof of the upper bound we are using a slightly modified version of Moser’s original inequality.

In fact one can take

\[
C(p,q,n,\Omega) = e^{\frac{1}{n}A(p,n,\Omega)(n+p)B(p,n,\Omega)} \left( \frac{nq}{n+p+1} \right)^{\frac{p}{p-n}} \Gamma \left( \frac{nq}{n+p+1} \right)^{\frac{1}{p}},
\]

(1.5)
where the constants $A(p,n,\Omega)$ and $B(p,n,\Omega)$ are given in the above Moser-Trudinger inequality. Furthermore, $\Gamma$ denotes the gamma function. In addition, inequality (1.4) may be written in the form

$$\|u\|_{L^q} \leq D(p,n,\Omega)q^{\frac{n}{n+p}}e_p(u)^{\frac{1}{n+p}},$$

where the constant $D(p,n,\Omega)$ does not depend on $q$.

The pluricomplex Sobolev inequality is proved in Theorem 5.1. It should be noted that Theorem 5.1 is more general than the above statement, this due to presentational reasons. Our work in Section 5 was inspired by [9, 13, 19, 31]. Next let $C(p,q,n,\mathbb{B})$ denote the optimal constant in (1.4), i.e. the infimum of all admissible constants. This optimal constant is classically of great importance. For example it is connected to the isoperimetric inequality and therefore classically to symmetrization of functions (see e.g. [39]). In pluripotential theory there have been many attempts to symmetrize pluriharmonic functions, but few progress have been made in that direction since plurisubharmonicity might be lost during a symmetrization procedure ([5]). For a positive result see [10]. A strong trend today is to try to prove a pluricomplex counterpart of Talenti’s theorem for the Laplacean ([40], see also [14, Theorem 10.2]). A successful attempt would not only imply simplified proofs, but also many of the biggest unsolved problems would be conquered. For further information and details we refer to Section 10 in the excellent survey [14] written by Błocki. With this in mind we shall in Section 6 prove that

$$C(1,1,n,\mathbb{B}) = \frac{\pi^{\frac{n^2}{n+1}}}{4^{\frac{n}{n+1}}n!}.$$

2. Background

In this section we shall give some necessary background on pluripotential theory. For further information we refer to [16, 22, 23, 30, 32, 37].

A set $\Omega \subseteq \mathbb{C}^n$, $n \geq 1$, is called a bounded hyperconvex domain if it is a bounded, connected, and open set such that there exists a bounded plurisubharmonic function $\varphi : \Omega \to (-\infty, 0)$ such that the closure of the set $\{z \in \Omega : \varphi(z) < c\}$ is compact in $\Omega$, for every $c \in (-\infty, 0)$.

We say that a plurisubharmonic function $\varphi$ defined on $\Omega$ belongs to $E_0 (= E_0(\Omega))$ if it is a bounded function,

$$\lim_{z \to \xi} \varphi(z) = 0 \quad \text{for every } \xi \in \partial \Omega,$$

and

$$\int_\Omega (dd^c\varphi)^n < \infty,$$

where $(dd^c)^n$ is the complex Monge-Ampère operator. Furthermore, we say that $u \in E_p (= E_p(\Omega))$, $p > 0$, if $u$ is a plurisubharmonic function defined on $\Omega$ and there exists a decreasing sequence, $\{\varphi_j\}$, $\varphi_j \in E_0$, that converges pointwise to $u$ on $\Omega$, as $j$ tends to $\infty$, and

$$\sup_j e_p(\varphi_j) = \sup_j \int_\Omega (-\varphi_j)^p(dd^c\varphi_j)^n < \infty.$$
We shall need on several occasions the following two inequalities. The inequality in Lemma 2.1 follows by standard approximation techniques from the work of Błocki in [12].

**Lemma 2.1.** Let \( u \in E_0, v \in E_p, p > 0, 1 \leq k \leq n \). Then

\[
\int_{\Omega} (-v)^{p+k} (dd^c u)^n \leq (p+k) \cdots (p+1) \|u\|_k^k \int_{\Omega} (-v)^p (dd^c) u)^k \wedge (dd^c u)^{n-k}.
\]

Next theorem was proved in [36] for \( p \geq 1 \), and for \( 0 < p < 1 \) in [13] (see also [15, 20]).

**Theorem 2.2.** Let \( p > 0 \) and \( u_0, u_1, \ldots, u_n \in E_p \). If \( n \geq 2 \), then

\[
\int_{\Omega} (-u_0)^p dd^c u_1 \wedge \cdots \wedge dd^c u_n
\leq d(p, n, \Omega) e_p(u_0)^{p/(p+n)} e_p(u_1)^{1/(p+n)} \cdots e_p(u_n)^{1/(p+n)},
\]

where

\[
d(p, n, \Omega) = \begin{cases} 
\left( \frac{k}{p} \right)^{\frac{n}{n-p}} & \text{if } 0 < p < 1, \\
1 & \text{if } p = 1, \\
\frac{\alpha(n, p)}{p^p} & \text{if } p > 1,
\end{cases}
\]

and \( \alpha(n, p) = (p + 2) \left( \frac{n+1}{p} \right)^{n-1} - (p + 1) \).

### 3. The pluricomplex Moser-Trudinger inequality

The aim of this section is to prove the following Moser-Trudinger inequality for \( E_p, p > 0 \).

**Theorem 3.1.** Let \( \Omega \) be a bounded hyperconvex domain in \( \mathbb{C}^n \). Then there exist constants \( A(p, n, \Omega) \) and \( B(p, n, \Omega) \) depending only on \( p, n, \Omega \) such that for any \( u \in E_p \) we have that

\[
\log \int_{\Omega} e^{-u} d\lambda_{2n} \leq A(p, n, \Omega) + B(p, n, \Omega) e_p(u)^\frac{1}{p}.
\]

Furthermore, for any given \( 0 < \epsilon < 1 \) we can take the constants \( A(p, n, \Omega) \) and \( B(p, n, \Omega) \) to be

\[
A(p, n, \Omega) = \log \left( \pi^n + \beta(n) \frac{en}{(n - \epsilon n)^n} \right) \text{diam}(\Omega)^{2n}) \quad \text{and} \quad B(p, n, \Omega) = (2\epsilon n)^{\frac{1}{p}},
\]

where \( \beta(n) \) is a constant depending only on \( n \).

**Proof.** First assume that \( u \in E_0(\Omega) \cap C(\Omega) \). Next, thanks to [15] we can find a uniquely determined function \( w \in E_0 \) that satisfies

\[
(dd^c w)^n = (-u)^p(dd^c u)^n.
\]

We shall now prove that

\[
u \geq t^{-\frac{n}{p}} w - t \quad \text{for all } t > 0.
\]

First notice that on the set \( \{ z \in \Omega : u(z) \geq -t \} \) we have that

\[
u(z) \geq -t \geq t^{-\frac{n}{p}} w - t.
\]
Next, since \( u \in \mathcal{E}_0 \) and \( \lim_{z \to \partial \Omega} u(z) = 0 \) we have that the open set \( \omega = \{ z \in \Omega : u(z) < -t \} \) is relatively compact in \( \Omega \) and therefore
\[
(\partial^c u)^n \leq (-u)^n (\partial^c u)^n = (\partial^c u)^n.
\]
Hence,
\[
(\partial^c u)^n \leq t^{-p}(\partial^c u)^n = \left( \partial^c \left( t^{-\frac{p}{n}} w - t \right) \right)^n,
\]
and furthermore
\[
\liminf_{\omega \ni z \to \partial \omega} \left( u(z) - t^{-\frac{p}{n}} w(z) + t \right) \geq 0.
\]
Therefore, by the comparison principle (see [15]) we get that \( u \geq t^{-\frac{p}{n}} w - t \) on \( \omega \) and (3.1) is valid.

Fix \( 0 < \epsilon < 1 \) and choose \( t \) such that
\[
t = \left( \frac{e_p(u)}{e^n(2n)^n} \right)^\frac{1}{\epsilon}.
\]
With this choice of \( t \) we have
\[
\int_\Omega (\partial^c t^{-\frac{p}{n}} w)^n = t^{-p} \int_\Omega (-u)^n (\partial^c u)^n = t^{-p} e_p(u) = e^n(2n)^n.
\]
By using Corollary 5.2 in [2] for the function \( t^{-\frac{p}{n}} w \) we get
\[
\log \int_\Omega e^{-u} d\lambda_{2n} \leq \log \int_\Omega e^{-t^{-\frac{p}{n}} w} e^t d\lambda_{2n} \leq \log \left( \left( \pi^n + \beta(n) \frac{en}{(n - en)^n} \right) \text{diam}(\Omega)^{2n} \right) + e_p(u)^\frac{\epsilon}{\epsilon - 1}.
\]
By a standard procedure we can now remove the assumption that \( u \in \mathcal{E}_0(\Omega) \cap C(\Omega) \), since for arbitrary \( u \in \mathcal{E}_p \) there exists a sequence \( u_j \in \mathcal{E}_0(\Omega) \cap C(\Omega) \) such that \( u_j \searrow u \) and \( e_p(u_j) \to e_p(u) \), \( j \to \infty \) (see e.g. [18]).

Next in Corollary 3.3 we obtain Theorem 3.1 for functions from the class \( \mathcal{E}_\chi \).

**Definition 3.2.** Let \( \Omega \) be a bounded hyperconvex domain in \( \mathbb{C}^n \), \( n \geq 2 \). Let \( \chi : (-\infty, 0] \to (-\infty, 0] \) be a continuous and nondecreasing function. Furthermore, let \( \mathcal{E}_\chi \) contain those plurisubharmonic functions \( u \) for which there exists a decreasing sequence \( u_j \in \mathcal{E}_0 \) that converges pointwise to \( u \) on \( \Omega \), as \( j \) tends to \( \infty \), and
\[
e^\chi(u) = \sup_j \int_\Omega -\chi(u_j) (\partial^c u_j)^n < \infty.
\]
For example, with this notation if \( \chi = -(t)^p \), then \( \mathcal{E}_\chi = \mathcal{E}_p \). It was proved in [17] and in [28] that if \( \chi : (-\infty, 0] \to (-\infty, 0] \) is continuous, and strictly increasing, then the complex Monge-Ampère operator is well defined on \( \mathcal{E}_\chi \). We are now in position to prove Corollary 3.3.

**Corollary 3.3.** Let \( \chi : (-\infty, 0] \to (-\infty, 0] \) be a continuous and nondecreasing function, and let \( \Omega \) be a bounded hyperconvex domain in \( \mathbb{C}^n \), \( n \geq 2 \). Then for any fixed \( 0 < \epsilon < 1 \) we have for all \( u \in \mathcal{E}_\chi \) that
\[
\log \int_\Omega e^{-u} d\lambda_{2n} \leq \log \left( \left( \pi^n + \beta(n) \frac{en}{(n - en)^n} \right) \text{diam}(\Omega)^{2n} \right) - \chi^{-1} \left( \frac{-e^\chi(u)}{(2en)^n} \right).
\]
Proof. The proof of Theorem 3.1 works here as well with some changes of references. The reference [16] should be replaced with [28], and [18] should be replaced with [7]. □

We shall end this section with a remark about the case when the underlying space is a compact Kähler manifold. Let us first recall some facts. Let \((X, \omega)\) be a Kähler manifold of dimension \(n\) with a Kähler form \(\omega\) such that \(\int_X \omega^n = 1\). We say that \(u \in \mathcal{E}_p(X, \omega) = \mathcal{E}_p\) if there exists a sequence \(u_j \in \mathcal{PSH}(X, \omega) \cap L^\infty(X)\) such that \(u_j \leq 0\), \(u_j \rightharpoonup u\), \(j \to \infty\) and
\[
\sup_j \int_X (-u_j)^p (dd^c u_j + \omega)^n < \infty.
\]
Here \(\mathcal{PSH}(X, \omega)\) denote the set of \(\omega\)-plurisubharmonic functions. For \(u \in \mathcal{E}_p\), set
\[
e_p(u) = \int_X (-u)^p (dd^c u + \omega)^n.
\]
In the case \(p = 1\), we have the following classical functional defined on \(\mathcal{E}_1\) by
\[
E_\omega(u) = \frac{1}{(n+1)!} \sum_{k=0}^n \int_X (-u)(dd^c u + \omega)^k \wedge \omega^{n-k},
\]
and we have the following estimation
\[
E_\omega(u) \leq \frac{1}{n!} \int_X (-u)(dd^c u + \omega)^n = \frac{1}{n!} e_1(u).
\]

**Remark.** Let \((X, \omega)\) be a compact Kähler manifold of dimension \(n\) with a Kähler form \(\omega\) such that \(\int_X \omega^n = 1\). It was proved in [9] that there exist constants \(a, b > 0\) such that for all \(u \in \mathcal{E}_1\) and \(k > 0\) it holds that
\[
\log \left( \int_X e^{-ku} \omega^n \right) \leq ak^{n+1} e_\omega(u) + b.
\]
Now let \(u \in \mathcal{E}_p\), \(p > 1\). By using Hölder inequality we get that
\[
e_1(u) = \int_X (-u)(dd^c u + \omega)^n \leq \left( \int_X (-u)^p (dd^c u + \omega)^n \right)^{\frac{1}{p}} = e_p(u)^{\frac{1}{p}}.
\]
Thus, \(u \in \mathcal{E}_p\), and by (3.2) we arrive at
\[
\log \left( \int_X e^{-ku} \omega^n \right) \leq ak^{n+1} E_\omega(u) + b \leq \frac{ak^{n+1}}{n!} e_1(u) + b
\]
\[
\leq \frac{ak^{n+1}}{n!} e_p(u)^{\frac{1}{p}} + b.
\]
This inequality shall be used on page 14. It should be noted that the case when \(0 < p < 1\) is at this point unknown to the authors.

4. Estimates of the constant \(B(p, n, \Omega)\)

Let us introduce the following notation. For \(r > 0\) let \(B(z_0, r) = \{z \in \mathbb{C}^n : |z - z_0| < r\}\) be the open ball with center \(z_0\) and radius \(r\), and to simplify the notations set \(B = B(0, 1)\).

Now let \(B(p, n, \Omega)\) denotes the optimal constant in the Moser-Trudinger inequality (1.1), i.e. the infimum of all admissible constants. The aim of this section is to estimate the constant \(B(p, n, \Omega)\) for arbitrary hyperconvex domains, inequality...
and in the special case when $\Omega = B$, inequality (1.3). We shall arrive to the following estimates.

**Theorem 4.1.** Let $\Omega$ be a bounded hyperconvex domain in $\mathbb{C}^n$, and $B(p, n, \Omega)$ the constant in Theorem 3.1. Then we have that

$$
\frac{p}{(4\pi)^{n+p+1}} \leq B(p, n, \Omega).
$$

**Proof.** Without loss of generality we can assume that $0 \in \Omega$. Let $g_\Omega(z, 0)$ be the pluricomplex Green function with pole at 0, and for a parameter $\beta \leq 0$ let us define

$$
u(z) = (2n + 2p) \max (g_\Omega(z, 0), \beta).$$

This construction yields that

$$
e_p(u) = \int_\Omega (-u)^p(d^ru)^n = (2\pi)^n (2n + 2p)^{p+n} (-\beta)^p,$$

and then we shall proceed by estimating the integral

$$
\int_\Omega e^{-\nu} d\lambda_{2n}.
$$

(4.1)

From the definition of the pluricomplex Green function it follows that there exist a radius $r > 0$, and a constant $C > 0$, such that $\mathbb{B}(0, r) \subseteq \Omega$ and such that the following inequalities hold for all $z \in \mathbb{B}(0, r)$:

$$
\log |z| - C \leq g_\Omega(z, 0) \leq \log |z| + C.
$$

(4.2)

Choose then $0 \leq \beta_1 \leq 0$ such that it holds $\{z \in \Omega : g_\Omega(z, 0) < \beta_1 \} \subset \mathbb{B}(0, r)$. From now on we shall only consider those $\beta$ with $0 \leq \beta_1$. From (4.2) we now have that

$$
\mathbb{B}(0, e^{-\beta - C}) \subset \{z \in \Omega : g_\Omega(z, 0) < \beta \} \subset \mathbb{B}(0, e^{\beta + C}).
$$

We start by dividing (4.1) as

$$
\int_\Omega e^{-\nu} d\lambda_{2n} = \int_{\{z \in \Omega : g_\Omega(z, 0) < \beta\}} e^{-(2n+2p)\beta} d\lambda_{2n}
$$

$$
+ \int_{\{z \in \Omega : g_\Omega(z, 0) \geq \beta\}} e^{-(2n+2p)g_\Omega(z, 0)} d\lambda_{2n} = I_1 + I_2,
$$

and notice that by (1.2) we have

$$
\frac{\pi^n}{n!} e^{-2nC} e^{-2p\beta} = e^{-(2n+2p)\beta} \lambda_{2n}(\mathbb{B}(0, e^{\beta - C})) \leq I_1
$$

$$
\leq e^{-(2n+2p)\beta} \lambda_{2n}(\mathbb{B}(0, e^{\beta + C}) = \frac{\pi^n}{n!} e^{2nC} e^{-2p\beta}.
$$

(4.3)

Furthermore,

$$
I_2 = \int_{\Omega \setminus \mathbb{B}(0, r)} e^{-(2n+2p)g_\Omega(z, 0)} d\lambda_{2n} + \int_{\mathbb{B}(0, r) \cap \{z \in \Omega : g_\Omega(z, 0) \geq \beta\}} e^{-(2n+2p)g_\Omega(z, 0)} d\lambda_{2n}
$$

$$
= I_3 + I_4.
$$

For $z \in \Omega \setminus \mathbb{B}(0, r)$ we have that

$$
1 \leq e^{-(2n+2p)g_\Omega(z, 0)} \leq e^{-(2n+2p)\beta_1},
$$

and therefore

$$
\lambda_{2n}(\Omega \setminus \mathbb{B}(0, r)) \leq I_3 \leq e^{-(2n+2p)\beta_1} \lambda_{2n}(\Omega \setminus \mathbb{B}(0, r)).
$$

(4.4)
We also get the estimate of \( I_4 \) as
\[
0 \leq I_4 \leq \int_{B(0,r)\setminus B(0,e^{\beta-C})} e^{-(2n+2p)(\log |z|-C)} d\lambda_{2n}
\]
\[
= e^{(2n+2p)C} \frac{2\pi^n}{(n-1)!} \int_{e^{\beta-C}}^{r} t^{-1-2p} dt = \frac{2\pi^n e^{(2n+2p)C}}{(n-1)!(-2p)} \left( r^{-2p} - e^{(\beta-C)(-2p)} \right). 
\]
(4.5)

From (4.1), (4.3), (4.4) and (4.5) it follows that there exist constant \( c_1, c_2, c_3, c_4 \) not depending on \( \beta \) such that
\[
c_1 e^{-2p\beta} + c_2 \leq \int_{\Omega} e^{-u} d\lambda_{2n} \leq c_3 e^{-2p\beta} + c_4,
\]
and therefore
\[
\lim_{\beta \to -\infty} \frac{\log \left( \int_{\Omega} e^{-u} d\lambda_{2n} \right)}{e_p(u)^{\frac{1}{p}}} = \frac{p}{(4\pi)^{\frac{p}{2}} (n+p)^{1+\frac{1}{p}}}. 
\]
Thus,
\[
B(p, n, \Omega) \geq \frac{p}{(4\pi)^{\frac{p}{2}} (n+p)^{1+\frac{1}{p}}}. 
\]
\[\Box\]

To prove the inequality (1.3) we shall make use of radially symmetric plurisubharmonic functions. Let us recall some basic facts here, and we refer the reader to [2, 34] and the references therein for further information. Recall that a function \( u : \mathbb{B} \to [-\infty, \infty) \) is said to be radially symmetric if we have that \( u(z) = u(|z|) \) for all \( z \in \mathbb{B} \).

For each radially symmetric function \( u : \mathbb{B} \to [-\infty, \infty) \) we define the function \( \hat{u} : [0, 1) \to [-\infty, \infty) \) by
\[
\hat{u}(t) = u(|z|), \text{ where } t = |z|. 
\]
(4.6)

On the other hand, to every function \( \hat{v} : [0, 1) \to [-\infty, \infty) \) we can construct a radially symmetric function \( v \) through (4.0). Furthermore, \( u \) is a radially symmetric plurisubharmonic function if and only if \( u(t) \) is an increasing function, and it is convex with respect to \( \log t \).

Let us first show a few elementary lemmas.

**Lemma 4.2.** For any \( \alpha > 1 \), and any \( A > 0 \), there exists a constant \( B \) such that for all \( t \geq 0 \) it holds
\[
At^\alpha + B \geq t.
\]
In fact one can take
\[
B = \frac{\alpha - 1}{\alpha} (\alpha A)^{\frac{1}{\alpha}}.
\]

**Proof.** It is is enough to observe that the function
\[
f(t) = At^\alpha - t
\]
attains its minimum at
\[
t_0 = (\alpha A)^{\frac{1}{\alpha}}. 
\]
and that

$$
\min_{[0, \infty)} f = \frac{1 - \alpha}{\alpha}(\alpha A)^{\frac{1}{1 - \alpha}} = -B.
$$

\[\square\]

In Lemma 4.3 we shall make use of the following equality. For \( f \in L^p(X, \mu) \) we have

$$
\int_X |f|^p \, d\mu = p \int_0^\infty t^{p-1} \mu(\{x \in X : |f(x)| \geq t\}) \, dt.
$$

\textbf{Lemma 4.3.} Let \( p > 0 \), and let \( u(z) = u(|z|) = \tilde{u}(t) \) be a radially symmetric plurisubharmonic function such that \( \lim_{z \to \partial B} u(z) = 0 \) and \( u \in \mathcal{E}_p \), then we have

$$
e_p(u) = (2\pi)^n p \int_0^1 (-\tilde{u}(t))^{p-1} \tilde{u}'(t)^n t^n \, dt.
$$

\textbf{Proof.} If \( u(z) = u(|z|) = \tilde{u}(t) \) is a radially symmetric plurisubharmonic function such that \( \lim_{z \to \partial B} u(z) = 0 \), then for \( t = |z| \) it holds that

$$
F(t) := \frac{1}{(2\pi)^n} (dd^c u)^n(B(0, t)) = t^n \tilde{u}'(t)^n,
$$

where \( \tilde{u}' \) is the left derivative of a convex function \( \tilde{u} \) (see [2]). For \( t \geq 0 \) we have that

$$
\{ z \in \mathbb{B} : u(z) \leq -t \} = B(0, s), \quad \text{where} \ s = \tilde{u}^{-1}(-t),
$$

where \( \tilde{u}^{-1}(\inf u) = \sup\{ x : \tilde{u}(x) = \inf \tilde{u} \} \). Therefore, by using (4.7) we arrive at

$$
e_p(u) = \int_B (-u)^p (dd^c u)^n = p \int_{-\tilde{u}}^{\inf u} t^{p-1} (dd^c u)^n(\{ z \in \mathbb{B} : u(z) \leq -t \}) \, dt
$$

$$
= p(2\pi)^n \int_{-\tilde{u}}^{\inf \tilde{u}} t^{p-1} F(\tilde{u}^{-1}(-t)) \, dt = (2\pi)^n p \int_0^1 (-\tilde{u}(s))^{p-1} \tilde{u}'(s)^n s^n \, ds,
$$

where \( \tilde{u}(s) = t \), and this completes this proof. \[\square\]

We are now in position to prove the inequality (1.3).

\textbf{Theorem 4.4.} Let \( p > 0 \), and let \( u \) be a radially symmetric plurisubharmonic function such that \( \lim_{z \to \partial B} u(z) = 0 \) and \( u \in \mathcal{E}_p \), then we have that

$$
\log \int_B e^{-u(z)} d\lambda_2 \leq d + \left( \frac{e_p(u)p^{p-1}}{(4\pi)^n(n+1)^{n+1}(n+p)^{p-1}} \right)^{\frac{1}{p}},
$$

where the constant \( d \) does not depend on \( u \). Therefore,

$$
B(p, n, \mathbb{B}) \leq \left( \frac{p^{p-1}}{(4\pi)^n(n+1)^{n+1}(n+p)^{p-1}} \right)^{\frac{1}{p}}.
$$

\textbf{Proof.} By Lemma 1.3 we have that the pluricomplex \( p \)-energy of \( u \) is equal to

$$
e_p(u) = (2\pi)^n p \int_0^1 (-\tilde{u}(t))^{p-1} \tilde{u}'(t)^n t^n \, dt
$$

$$
= \frac{(2\pi)^n p}{(n+p)^{n+1}} \int_0^1 \left( \left( (-\tilde{u}(t))^{\frac{n}{n+1}} \right)^{n+1} \right) t^n \, dt.
$$
Therefore, if \( v(t) = -(-x\bar{u}(t))^\frac{n+p}{n} \), where
\[
x = \left( \frac{(2\pi)^n p(n+1)^{n+1}}{e_p(u)(n+p)^{n+1}} \right)^\frac{1}{n+p},
\]
then \( v \) be an increasing function \( v : [0,1) \to (-\infty,0) \) such that \( \lim_{t\to1^-} v(t) = 0 \) and
\[
\int_0^1 (v'(s))^n s^nds \leq 1.
\]

Thanks to a slightly modified version of the classical Moser-Trudinger inequality (cf. [35]), we arrive at
\[
\int_0^1 e^{2n(-v(s))}\frac{e_p^{p-1}}{(4\pi)^n(n+1)^{n+1}(n+p)^{p-1}} s^{2n-1}ds \leq \frac{c}{2n}, \quad (4.9)
\]
where the constant \( c \) does not depend on \( u \). Lemma [1.2] yields that
\[
-\bar{u}(s) \leq 2n(-\bar{u}(s)x)\frac{e_p^{p-1}}{(4\pi)^n(n+1)^{n+1}(n+p)^{p-1}} + y.
\]

Hence by (4.9),
\[
\int_{\mathbb{B}} e^{-u(z)}d\lambda_{2n} = \frac{2\pi^n}{(n-1)!} \int_0^1 e^{-\bar{u}(s)} s^{2n-1}ds \leq \frac{2\pi^n}{(n-1)!} \frac{2\pi^n}{c} e^y,
\]
and finally
\[
\log \int_{\mathbb{B}} e^{-u(z)}d\lambda_{2n} \leq \log \left( \frac{\pi^n c}{n!} \right) + \left( \frac{e_p^{p-1}}{(4\pi)^n(n+1)^{n+1}(n+p)^{p-1}} \right)^\frac{1}{p}.
\]

A direct consequence of (4.9) is the following corollary which was first proved in [10] in the case \( p = 1 \).

**Corollary 4.5.** Let \( p > 0 \), and let \( u \) be a radially symmetric plurisubharmonic function such that \( \lim_{z \to \partial \mathbb{B}} u(z) = 0 \) and \( u \in E_p(\mathbb{B}) \), then we have that
\[
\int_{\mathbb{B}} e^{\alpha(p,n)(-u(z))\frac{e_p^{p-1}}{n+1} c_p(u)^{\frac{1}{n+p}}} d\lambda_{2n} < \infty,
\]
where \( \alpha(p,n) = 4\pi np^n\left( \frac{n+1}{n+p} \right)^{\frac{n+1}{n}} \).

**Proof.** By (4.9) we have
\[
\int_{\mathbb{B}} e^{\alpha(p,n)(-u(z))\frac{e_p^{p-1}}{n+1} c_p(u)^{\frac{1}{n+p}}} d\lambda_{2n} = \frac{2\pi^n}{(n-1)!} \int_0^1 e^{\alpha(p,n)(-\bar{u}(s))\frac{e_p^{p-1}}{n+1} c_p(u)^{\frac{1}{n+p}}} s^{2n-1}ds < \frac{c\pi^n}{n!} < \infty.
\]

\( \square \)
5. THE PLURICOMPLEX SOBOLEV INEQUALITY

In this section we shall prove the pluricomplex Sobolev inequality. We shall prove it for differences of plurisubharmonic functions with finite energy, i.e. for functions in $\delta \mathcal{E}_p = \mathcal{E}_p - \mathcal{E}_p$. If we for $u = u_1 - u_2 \in \delta \mathcal{E}_p$ define $\|u\|_p$ by

$$\|u\|_p = \inf_{a_1, a_2 \in \mathcal{E}_p} e_p(u_1 + u_2)^{\frac{1}{n+p}},$$

then $(\delta \mathcal{E}_p, \|\cdot\|_p)$ becomes a quasi-Banach space, and for $p = 1$ a Banach space (see [1]). Note that in the case $u \in \mathcal{E}_p$ we have that $\|u\|_p = e_p(u)^{\frac{1}{n+p}}$.

**Theorem 5.1.** Let $\Omega$ be a bounded hyperconvex domain in $\mathbb{C}^n$, and let $u \in \delta \mathcal{E}_p$, $p > 0$. Then for all $q > 0$ there exists a constant $C(p, q, n, \Omega) > 0$ depending only on $p, q, n, \Omega$ such that

$$\|u\|_{L^q} \leq C(p, q, n, \Omega) \|u\|_p.$$  

(5.1)

In fact one can take

$$C(p, q, n, \Omega) = e^{\frac{A(p, n, \Omega)}{n+p}} \left( \frac{nq}{n+p} \right)^{\frac{q}{n+p}} \Gamma \left( \frac{nq}{n+p} + 1 \right)^{\frac{1}{q}},$$

(5.2)

where the constants $A(p, n, \Omega)$ and $B(p, n, \Omega)$ are given in Theorem 3.1. In addition, inequality (5.1) may be written, for $q \geq 1$, in the form

$$\|u\|_{L^q} \leq D(p, n, \Omega) q^{\frac{n}{p+n}} \|u\|_p,$$

(5.3)

where the constant $D(p, n, \Omega)$ does not depend on $q$. Furthermore, the identity operator $\iota: \delta \mathcal{E}_p \to L^q$ is compact.

Before we start the proof let us recall the definition of compactness in quasi-Banach spaces.

**Definition 5.2.** Let $X, Y$ be two quasi-Banach spaces. The operator $K : X \to Y$ is called compact if for any sequence $\{x_n\} \subset X$ with $\|x_n\| \leq 1$, then there exists a convergent subsequence $\{y_{n_k}\}$ of $\{K(x_n)\}$.

**Proof of Theorem 5.1.** First assume that $u \in \mathcal{E}_p$, $p > 0$. For $t, s > 0$ define

$$f(t) = \int_{\Omega} e^{-tu} d\lambda_{2n} \text{ and } \lambda(s) = \lambda_{2n}(\{z \in \Omega : u(z) < -s\}).$$

Note that by Theorem 3.1 there exist constants $A = A(p, n, \Omega)$ and $B = B(p, n, \Omega)$ such that

$$f(t) \leq e^A e^{B t^{\frac{n+p}{1+p}}} e_p(u)^{\frac{1}{n+p}},$$

where $g(t) = B t^{\frac{n+p}{n}} e_p(u)^{\frac{1}{n+p}}$. For $s, t > 0$ we have that

$$\lambda(s) \leq \int_{\{z \in \Omega : u(z) < -s\}} e^{-st} e^{-tu} d\lambda_{2n} \leq e^{-st} \int_{\Omega} e^{-tu} d\lambda_{2n} \leq C e^{-st + g(t)}.$$  

(5.4)

By Lemma 4.2 we now have that

$$g(t) - st = B t^{\frac{n+p}{n}} e_p(u)^{\frac{1}{n+p}} - st \geq -s^{\frac{n+p}{1+p}} \frac{np}{(n+p)^{1+p}} B^{-\frac{n}{n+p}} e_p(u)^{-\frac{1}{n+p}}.$$

Therefore, it follows from (5.4) that

$$\lambda(s) \leq C e^{-xs^{\frac{n+p}{1+p}}} \text{, where } x = \frac{np}{(n+p)^{1+p}} B^{-\frac{n}{n+p}} e_p(u)^{-\frac{1}{n+p}}.$$  

(5.5)
By letting \( r = x^{\frac{n+p}{p}} \) in (5.5) we get that
\[
\|u\|_{L^q} = \int_\Omega (-u)^q d\lambda_2 = q \int_0^\infty s^{q-1} \lambda(s) ds \leq qC \int_0^\infty s^{q-1} e^{-C s^{\frac{n+p}{r}}} ds
\]
\[
= \frac{qnC}{(n+p)x^{\frac{np}{n+p}}} \int_0^\infty r^{-1 + \frac{np}{n+p}} e^{-r} dr = \frac{qnC}{(n+p)x^{\frac{np}{n+p}}} \Gamma \left( \frac{np}{n+p} \right)
\]
\[
= Cx^{-\frac{np}{n+p}} \Gamma \left( \frac{np}{n+p} + 1 \right) = C(n+p)^q B^{\frac{np}{n+p}} \Gamma \left( \frac{np}{n+p} + 1 \right) e_p(u)^{\frac{np}{n+p}} .
\] (5.6)
Thus, (5.1) holds for \( u \in E_p \). In the general case, let \( u = u_1 - u_2 \in \delta E_p \) it is enough to note that
\[
\|u\|_{L^q} \leq \|u_1 + u_2\|_{L^q} \leq C(p, q, \Omega) e_p(u_1 + u_2),
\]
and then taking the infimum over all possible decomposition of \( u \).

Next we shall prove (5.3). First assume that \( u \in E_p \). Note that for \( y \geq 1 \) it is a fact that
\[
\Gamma(y + 1) \leq 2y^y,
\]
so by (5.6) it holds that for \( q \geq 1 \)
\[
\|u\|_{L^q} \leq \left( C(n+p)^q B^{\frac{np}{n+p}} \Gamma \left( \frac{np}{n+p} + 1 \right) e_p(u)^{\frac{np}{n+p}} \right)^\frac{1}{q}
\]
\[
\leq 2^{\frac{1}{q}} c^{\frac{1}{q}} \left( \frac{(n+p)B^{\frac{np}{n+p}}}{n^{\frac{np}{n+p}}} \right)^{\frac{1}{q}} e_p(u)^{\frac{1}{n^{\frac{np}{n+p}}}}
\]
\[
\leq 2c A^{\frac{n+p}{n^{\frac{np}{n+p}}}} B^{\frac{np}{n+p}} q^{\frac{np}{n+p}} e_p(u)^{\frac{1}{n^{\frac{np}{n+p}}}} .
\]
To proceed to the general case \( u = u_1 - u_2 \in \delta E_p \), we follow the above procedure and arrive at
\[
\|u\|_{L^q} \leq D(p, n, \Omega) q^{\frac{np}{n+p}} \|u\|_p,
\]
with
\[
D(p, n, \Omega) = 2c A(p, n, \Omega) \frac{(n+p)^{\frac{np}{n+p}} B(p, n, \Omega)^{\frac{np}{n+p}}}{p^{\frac{np}{n+p}}} .
\]
To complete this proof we shall prove that the identity operator \( \iota : \delta E_p \rightarrow L^q \) is compact. Take a sequence \( \{u_n\} = \{u_{n_k}^1 - u_{n_k}^2\} \subset \delta E_p \) with \( \|u\|_p \leq 1 \). Then by the same reasoning as above we get that
\[
\|u_n^j\|_{L^q} \leq \|u_{n_k}^1 + u_{n_k}^2\|_{L^q} \leq C(p, n, \Omega) \quad \text{for} \quad j = 1, 2.
\]
Hence, there exists a subsequence \( \{u_{n_k}^j\} \) converging almost everywhere to some plurisubharmonic function \( \{\upsilon^j\} \). This means that \( \{u_{n_k}^1 - u_{n_k}^2\} \) is a Cauchy sequence in \( L^q \). Thus, \( \iota \) is compact.

The proof of Theorem (5.1) relies on the Moser-Trudinger inequality (Theorem 3.1). In the case when \( q \leq n + p \), we can present an elementary proof only using the inequalities in Lemma 2.1 and Theorem 2.2.

**Proof of Theorem 5.1 for \( q \leq n + p \).** There exists \( \varphi_0 \in E_0 \) such that
\[
(dd^c \varphi_0)^n = d\lambda_{2n} ,
\]
Let \( u = u_1 - u_2 \in \delta E_p \), and let \( 0 < q \leq p + n \). Thanks to Lemma 2.1 and Theorem 2.2 we get that
\[
\| u \|_{L^q} \leq \lambda_{2n}(\Omega)^{\frac{q}{n} - \frac{p}{n}} \| u_1 + u_2 \|_{L^{p+n}} = \\
= \lambda_{2n}(\Omega)^{\frac{q}{n} - \frac{p}{n}} \left( \int_{\Omega} (-u_1 - u_2)^{p+n} (dd^c \varphi_0)^n \right)^{\frac{1}{n+p}} \leq \\
\leq \lambda_{2n}(\Omega)^{\frac{q}{n} - \frac{p}{n}} (p+n) \cdots (p+1) \| \varphi_0 \|_{L^\infty}^n \int_{\Omega} (-u_1 - u_2)^p (dd^c (u_1 + u_2))^n \right)^{\frac{1}{n+p}} \leq \\
\leq \left( \lambda_{2n}(\Omega)^{\frac{q}{n} - \frac{p}{n}} (p+n) \cdots (p+1) \| \varphi_0 \|_{L^\infty}^n \right)^{\frac{1}{n+p}} e_p(u_1 + u_2)^{\frac{1}{n+p}} .
\]

Finally by taking the infimum over all possible decompositions \( u = u_1 - u_2 \) we obtain that
\[
\| u \|_{L^q} \leq C(p, q, n, \Omega) \| u \|_p .
\]

Next we present an example that shows that it is impossible to have an estimate of the type
\[
e_p(u)^{\frac{1}{n+p}} \leq C \| u \|_{L^q} .
\]

**Example 5.3.** Consider the following functions defined on the unit ball \( B \) in \( \mathbb{C}^n \)
\[
u_j(z) = \frac{1}{j} \max \left( \log |z|, -j^{1+\frac{n}{p}} \right) .
\]

Then we have that
\[
u_j(z) = \begin{cases} 
\frac{1}{j} \log |z| & \text{if } \exp \left(-j^{1+\frac{n}{p}} \right) \leq |z| \leq 1 \\
-j^{\frac{n}{p}} & \text{if } 0 \leq |z| \leq \exp \left(-j^{1+\frac{n}{p}} \right) .
\end{cases}
\]
Hence, \( \| u_j \|_{L^\infty} \to 0 \), as \( j \to \infty \), but at the same time we have that
\[
e_p(u) = \frac{1}{j^{n+p}} (2\pi)^n \left( j^{1+\frac{n}{p}} \right)^p = (2\pi)^n,
\]
which is a contradiction. \( \square \)

Example 5.4 shows that it is also impossible to have an estimate of the type
\[
\| u \|_{L^\infty} \leq C e_p(u)^{\frac{1}{n+p}} .
\]

**Example 5.4.** Similarly as in Example 5.3 consider the following functions defined on the unit ball \( B \) in \( \mathbb{C}^n \)
\[
u_j(z) = \frac{1}{j} \max \left( \log |z|, -j \right) .
\]
Then we have that \( \| u_j \|_{L^\infty} = -u_j(0) = j^{\frac{n}{n+p}} \to \infty \), as \( j \to \infty \), and at the same time
\[
e_p(u_j) = (2\pi)^n j^p \left( \frac{1}{j^{n+p}} \right)^{n+p} = (2\pi)^n
\]
and a contradiction is obtained. \( \square \)
Finally we present an example that shows that it is impossible to have an estimate of the type\[ e_p(u) \leq C \|u\|_{L^\infty} \].

**Example 5.5.** Similarly as before we consider the following functions defined on the unit ball $B$ in $\mathbb{C}^n$

\[ u_j(z) = j \max \left( \log |z|, -\frac{1}{j} \right) . \]

Then we have that $\|u_j\|_{L^\infty} = -u_j(0) = 1$ and at the same time

\[ e_p(u_j) = (2\pi)^n j^{n+p} \left( \frac{1}{j} \right)^p = (2\pi)^n j^n \to \infty \]

and a contradiction is obtained.

\[ \square \]

Next in Corollary 5.6 we prove the corresponding Sobolev estimate (5.1) for functions in $E_\chi$. For the definition of $E_\chi$ see Definition 3.2 on page 5.

**Corollary 5.6.** Let $\chi : (-\infty, 0] \to (-\infty, 0]$ be a continuous and nondecreasing function, let $\Omega$ be a bounded hyperconvex domain in $\mathbb{C}^n$, $n \geq 2$, and let $u \in E_\chi$. Then for all $q > 0$ there exists a constant $G(n, \Omega) \geq 0$ depending only on $n$ and $\Omega$ such that

\[ \|u\|_{L^q} \leq G(n, \Omega) \frac{\Gamma(q+1) \Gamma(q+\frac{1}{p})}{(2\pi)^{n+1}} \left( \inf_{s>0} s^{-q} \exp \left( - \left( \frac{e_\chi(u)}{2\pi n} \right)^{-\frac{1}{q}} s^{\frac{1}{q}} + (-s^n) \right) \right)^{\frac{1}{q}} . \]

**Proof.** This is a straight forward modification of the proof of Theorem 5.1. \[ \square \]

We shall end this section with a remark about compact Kähler manifolds. This was first proved in [9] for the case $p = 1$. The notation and background about the Kähler case are stated before the remark on page 6.

**Remark.** Let $(X, \omega)$ be a compact Kähler manifold of dimension $n$ with a Kähler form $\omega$ such that $\int_X \omega^n = 1$. Let $u \in E_p$, $p > 0$, and $k > 0$. From (3.3) we know that

\[ \log \left( \int_X e^{-ku^n} \right) \leq \frac{ak^{n+1}}{n!} e_p(u)^{\frac{1}{p}} + b , \]

for some constants $a$ and $b$. By repeating the argument from the proof of Theorem 5.1 one can prove that there exists a constant $c$ depending only on $p$, $X$, and not on $q$, such that

\[ \|u\|_{L^q} \leq c q^{\frac{1}{p}} e_p(u)^{\frac{1}{p}} . \]

6. **ON THE SOBOLEV CONSTANT FOR THE UNIT BALL $B$**

In this section let $C(p, q, n, \mathbb{B})$ be the infimum of all admissible constants in the Sobolev type inequality given in (5.1). Our aim here is to show that

\[ C(1, 1, n, \mathbb{B}) = \frac{\pi^{\frac{n^2}{2}}}{4^{\frac{n+1}{2}} n! (n+1)^{\frac{1}{2}}} . \]

We shall do it in two part as follows.
In Example [6.1] we derive that for \( q \leq n + 1 \)
\[
C(1, q, n, \mathbb{B}) \leq \pi^{\frac{n(n+1)}{4(n+1)}} \frac{1}{n!} \left[ (1 + [q - 1]) \right]^{\frac{1}{n+q-1}} \frac{1}{4(n+q-1)},
\]
where \([ \cdot ]\) is the ceiling function.

(2) In Example [6.2] we prove that
\[
C(p, 1, n, \mathbb{B}) = \frac{\pi^{\frac{n(n+p-1)}{4(n+p)}}}{4^\frac{n}{n+p} n! (n+p) (n \text{B}(p+1, n))^{\frac{1}{n+p}}},
\]
where \( \text{B} \) is the beta function. We shall actually obtain a bit more general result in this example.

**Example 6.1.** Let \( \Omega = \mathbb{B} \) be the unit ball in \( \mathbb{C}^n \), and on \( \mathbb{B} \) define
\[
\varphi_0 = \frac{1}{4^{\frac{n}{n+1}}}(|z|^2 - 1).
\]
Then \( \varphi_0 \in \mathcal{E}_0 \), \((dd^c \varphi_0)^n = d\lambda_{2n}\), and
\[
e_p(\varphi_0) = \frac{n\pi^n}{4^n(n!)^\frac{1}{n+1}} \text{B}(p+1, n),
\]
where \( \text{B} \) is the classical beta Euler function. Recall that if \( q \leq n + p \), then the ceiling function evaluated at \( q - p \) is defined by
\[
[q - p] = \min \{ k \in \mathbb{N} : k \geq q - p \}.
\]
Once again thanks to Lemma [2.1] and Theorem [2.2] we get that
\[
\|u\|_{L^q} \leq \lambda_{2n}(\mathbb{B})^{\frac{1}{q}} \frac{1}{(q-p)+\frac{1}{p}} \|u\|_{L^{p+[q - p]}} \leq \lambda_{2n}(\mathbb{B})^{\frac{1}{q}} \frac{1}{(q-p)+\frac{1}{p}} \int_\Omega (-u)^p (dd^c u)^{[q - p]} \wedge (dd^c \varphi_0)^{n-[q - p]} d(p, n, \mathbb{B})
\]
\[
\cdot e_p(\varphi_0)^{\frac{n-[q - p]}{n+p}} e_p(\varphi_0)^{\frac{p+[q - p]}{n+p}} =
\]
\[
= \frac{e_p(u)\pi^{\frac{1}{p}}}{4^\frac{n}{n+p} (n!)^\frac{1}{n+p}} \frac{n^{p+n-q}}{n(p+n-q)} d(p, n, \mathbb{B})^{\frac{1}{p+q-p}} (n \text{B}(p + 1, n))^{\frac{n-[q - p]}{n+p} + \frac{p+[q - p]}{n+p}}
\]
\[
\cdot (p+1) \cdots (p + [q - p]). \quad (6.1)
\]
If \( p = 1 \) we know that \( d(1, n, \mathbb{B}) = 1 \), and \( n \text{B}(2, n) = \frac{1}{n+1} \). Hence,
\[
C(1, q, n, \mathbb{B}) \leq \pi^{\frac{n(n+1)}{4(n+1)}} \frac{1}{n!} \left[ (1 + [q - 1]) \right]^{\frac{1}{n+q-1}} \frac{1}{4(n+q-1)},
\]
where [\cdot ] is the ceiling function.
Example 6.2. For $\alpha > 0$, $k > 0$, define on the unit ball $B$ in $\mathbb{C}^n$ the following family of functions $$ u_{\alpha,k}(z) = k(|z|^{2\alpha} - 1) . $$

Then we have that $$ e_p(u_{\alpha}) = \int_{B(0,1)} (-u_{\alpha})^p (dd^c u_{\alpha})^n = k^{n+p} n(4\pi)^n \alpha^n B(p+1,n) , $$
and
$$
\int_B (-u_{\alpha,k}(z))^{n} d\lambda_{2n} = \frac{2\pi^n k^q}{(n-1)!} \int_0^1 (1-t^{2\alpha})^q t^{2n-1} dt 
= \frac{\pi^n k^q}{(n-1)!} \int_0^1 (1-s)^q s^{\frac{n}{2n-1}} dr = \frac{\pi^n k^q}{(n-1)!} B\left(q + 1, \frac{n}{\alpha}\right) .
$$

Hence,
$$
C(p,q,n,B) \geq \frac{||u_{\alpha,k}||_{L^p}}{e_p(u_{\alpha,k})^{\frac{n}{n+p}}} \geq \frac{n \frac{\pi^{n+p} \Gamma(n+p)}{\Gamma(n+1)}}{4 \pi^n (n!)^{\frac{n}{2}} B(p+1,n) \frac{n^{n+p}}{\alpha^{n+p}}} \frac{B\left(q + 1, \frac{n}{\alpha}\right)^{\frac{1}{q}}}{\alpha^{\frac{1}{q} + \frac{n}{n+p}}} .
$$

Now set $\beta = \frac{n}{\alpha}$, and $s = \frac{n}{n+p}$. With these notation we get that
$$
\frac{B(q + 1, \frac{n}{\alpha})^{\frac{1}{q}}}{\alpha^{\frac{1}{q} + \frac{n}{n+p}}} = n^{-\frac{1}{q} - \frac{n}{n+p}} (B(q + 1, \beta) \beta^{1+q \frac{n}{n+p}})^{\frac{1}{q}} = n^{-\frac{1}{q} - \frac{n}{n+p}} f(\beta)^{\frac{1}{q}} ,
$$
where
$$
f(\beta) = B(q + 1, \beta)^{1+qs} .
$$

Next we want to find $\sup_{\beta \in (0,\infty)} f(\beta)$. First we see that
$$
\lim_{\beta \to 0} f(\beta) = \lim_{\beta \to \infty} f(\beta) = 0 ,
$$
and that
$$
f'(\beta) = B(q + 1, \beta)^{1+qs} (\beta(\psi(\beta) - \psi(\beta + q + 1)) + 1 + sq) ,
$$
where $\psi(x)$ is the classical digamma function.

In the case when $q \in \mathbb{N}$, it holds that
$$
\psi(\beta) - \psi(\beta + q + 1) = -\sum_{j=0}^{q} \frac{1}{\beta + q - j} .
$$

Therefore for $q \in \mathbb{N}$ we have that $f'(\beta) = 0$ if, and only if,
$$
\sum_{j=0}^{q} \frac{\beta}{\beta + q - j} = 1 + qs . \tag{6.3}
$$

This implies that in the case $q = 1$ the equation (6.3) have a solution given by
$$
\beta_0 = \frac{s}{1-s} = \frac{n}{p} .
$$

Using the standard equalities
$$
B(x + 1, y) = \frac{x}{x+y} B(x, y) , \quad \text{and} \quad B(1,y) = \frac{1}{y} ,
$$
we derive that
$$
\sup f(\beta) = B(q + 1, \beta_0)^{1+qs} = \frac{\beta_0^s}{1 + \beta_0} = \frac{p \frac{n}{n+p} \frac{\pi^{n+p}}{\pi^n}}{n+p} .
$$
Hence,
\[
C(p, 1, n, B) \geq \frac{n(n+p-1)}{p} \frac{1}{4^{\frac{n}{2p}} n!(n+p)(nB(p+1, n))^{\frac{1}{2p}}}. \tag{6.4}
\]
In the case when \( q \in \mathbb{N}, q \geq 2 \), then we have that
\[
\beta_0 = \frac{q+1}{2} = \frac{(q+1)n}{2p}
\]
is a good approximation of a solution to the equation \( 6.3 \), and therefore
\[
\sup f(\beta) \geq B \left( q+1, \frac{(q+1)n}{2p} \right)^{1+\frac{1}{2p}}. \tag{6.5}
\]
Thus,
\[
C(p, q, n, B) \geq \frac{\pi^{\frac{n(n+p-q)}{2(n+p)}} n^{\frac{2n+q}{q(n+p)}} \frac{n^q}{(q+1)2n+q}(n!)^{\frac{1}{q}} (nB(p+1, n))^{\frac{1}{2p}}}{4^{\frac{n}{2p}} (2p)^{\frac{1}{n+p}} B \left( q+1, \frac{(q+1)n}{2p} \right)^{\frac{1}{q}}}. \tag{6.5}
\]
For \( q \in \mathbb{R}, q \geq 2 \), one can insert the floor function evaluated at \( q, \lfloor q \rfloor \), in \( 6.5 \). □

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