The Mass Ratio Between Neutrinos and Charged Leptons

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Abstract

In the framework of the recently proposed electroweak theory on a Planck lattice, we are able to solve approximately the lattice Dyson equation for the fermion self-energy functions and show that the large difference of charged lepton and neutrino masses is caused by their very different gauge couplings. The predicted mass ratio ($10^{-5} \sim 10^{-6}$) between neutrinos and charged lepton is fully compatible with present experiments.

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Since their appearance neutrinos have always been extremely peculiar. In the sixty years of their life, their charge neutrality, their apparent masslessness, their left-handedness have been at the centre of a conceptual elaboration and an intensive experimental analysis that have played a major role in donating to mankind the beauty of the electroweak theory. V-A theory and Fermi universality would possibly have eluded us for a long time had the eccentric properties of neutrinos, all tied to their apparent masslessness, not captured the imagination of generations of experimentalists and theorists alike.

However, with the consolidation of the Standard Model (SM) and in particular with the general views on (spontaneous?) mass generation in the SM, the observed (almost) masslessness of the three neutrinos ($\nu_e, \nu_\mu, \nu_\tau$) has recently come to be viewed as a very problematic and bizarre feature of the mechanism(s) that must be at work to produce the very rich mass spectrum of the fundamental fields of the SM. Indeed, in the (somewhat worrying) proliferation of the Yukawa couplings of fermions to the Higgs fields that characterizes the generally accepted SM, no natural reason can be found why the charge-neutral neutrinos are the fundamental particles of lowest mass; for in the generally accepted minimal Higgs mechanism, the actual values of the fermion masses are in direct relation with the strengths of their couplings to the Higgs doublet, and it appears rather bizarre that nature has chosen to create a very sophisticated mass pattern by the mere fine-tuning of a large number of parameters.

On the other hand we believe that it cannot be a simple dynamical accident that the only charge-neutral fermions that populate the SM are also the lightest. In this letter we wish to show that this most remarkable property of Nature can naturally be understood in the framework of the Planck Lattice Standard Model (PLSM) that we have recently introduced [1, 2, 3], based on the general hypothesis that, as a result of the violent quantum fluctuations of the metrical (gravitational) field, at the Planck length $a_p \simeq 10^{-33}$ cm, space-time is no more a 4-dimensional continuum, but can be well represented by a (random) lattice with (average) lattice constant equal to $a = a_p[4]$. Here we would only like to stress that in order to avoid a well known theorem by Nielsen and Ninomiya [5], which prevents a simple transcription of the usual continuum SM lagrangian to the lattice, we have been forced to add new 4-fermi terms of the Nambu-Jona Lasinio type (NJL)[6]. Recently, we analyzed the solutions of the Dyson equations involving the NJL interactions only, with the following results[2]:

1. the fundamental chiral symmetry of the full Lagrangian is spontaneously broken, and as a consequence only one family in the quark sector, which is identified with the top quark and bottom quark doublet, acquires a mass. The
lepton sector remains massless owing to its different colour structure.

2. the consistent solution of the gap equations also produces non-zero Wilson-type parameters \( r_q \) and \( r_l \) and mass counterterms for each quark and lepton.

The effective action that is left after such massive rearrangement of the vacuum is \((a = a_p)\)

\[
S = S_G + S_D + \sum_{xF} \bar{\psi}(x)m_F\psi(x)
- \frac{1}{2a} \sum_{Fx\mu} [\bar{\psi}^F(x) \left( L^F_\mu(x) + R^F_\mu(x) \right) r_F U^F_\mu(x) \psi^F(x + a_\mu) + \text{h.c.}] + \cdots,
\]

where \( S_G \) is the usual Wilson gauge action, \( S_D \) the usual Dirac action, \( F = l(q) \) denotes the lepton (quark) sector, \( m_F, r_F \) are matrices in flavour and weak isospin space, and the dots denote the NJL interaction and the necessary counterterms. Gauge links are given by

\[
L^F_\mu(x) = U^L_\mu(x)V^F_\mu(x), \quad R^F_\mu(x) = \begin{pmatrix} V^F_{\mu_1}(x) & 0 \\ 0 & V^F_{\mu_2}(x) \end{pmatrix},
\]

where \( U^q_\mu(x) \in SU_c(3), \quad U^L_\mu(x) \in SU_L(2) \) and \( V_\mu(x) \in U_Y(1) \). Turning on gauge interactions we approximately analyzed the Dyson equation for the top and bottom doublet and obtained a mass ratio between the top and bottom quarks \([7]\) in agreement with recent experimental indications. In this note, we apply an analogous analysis to the lepton sector. Our aim is, of course, to see whether gauge interactions are capable of yielding the large observed differences between the masses of the charged leptons and the neutrinos.

When we take into consideration in addition to the NJL interaction the gauge interactions of the action \([\ref{1}]\), the Dyson equations for the lepton sector have the structure depicted diagrammatically in Fig.1, which can be written as:

\[
\Sigma^Q_c(p) = m_Q\sigma + \sum_g C^Q_g(p);
\]

\[
\sigma = \frac{2g_1}{N_c} \int_{-\pi}^\pi \frac{d^4q}{(2\pi)^4} \frac{1}{\sin^2 q_\mu + [m_Qa + r_l w(q)]^2} + \cdots,
\]

where \( \Sigma^Q_c(p) (\Sigma^Q_c(0) = m_Q) \) are the leptons’ self-energy functions; the second term in the rhs of \((\ref{3})\) denotes the contributions of the relevant gauge interactions and \( \sigma \) stands for the contributions (dots for high-order contributions) of the NJL interactions, which are supposed to be the same for neutrinos \((Q = 0)\) and charged leptons \((Q = -1)\) because \( m_Qa \simeq 0, \quad r_Q = r_l \ (Q = 0, -1) \). We remark that the terms that diverge like \( \frac{1}{a} \) have all been consistently cancelled by mass counterterms \([2]\).
For external momenta $p_\mu a \ll 1$, we divide the integration domain over the variable $q_\mu$ into two regions: the “continuum” region: $0 \leq |q_\mu a| \leq \epsilon$ and the “lattice” region: $\epsilon \leq |q_\mu a| \leq \pi$, where $|ap_\mu| \ll \epsilon \ll \pi$. With this separation we can write the contribution of the gauge interactions as:

$$C_g^Q(p) = \tilde{C}_g^Q(p, \Lambda) + \theta_g^Q(\epsilon, r_l) + \delta_g^Q(\epsilon, r_l),$$  \hspace{1cm} (5)

$$\tilde{C}_g^Q(p, \Lambda) = \frac{1}{4\pi^2} \int d^4q \frac{\lambda_g(q^2)}{[\Lambda^2 + m_g^2]} \Sigma_c^Q(q),$$  \hspace{1cm} (6)

where the continuum-region integral is up to $\Lambda = \Lambda p$, ($\Lambda p = \frac{\pi}{a} \simeq 10^{10}$GeV). As for the $\ell n\epsilon$-divergent contribution in the lattice-region, one has

$$\theta_g^Q(\epsilon, \Lambda) = \int_{[\epsilon, \pi]} \frac{d^4l}{16\pi^2} \frac{\lambda_g^Q(l) \Sigma_c^Q(l)}{\sin^2 \frac{l}{2} \cos^2 \frac{l}{2} (\sin^2(l) + (m_Qa + r(l))^2)} \simeq \lambda_g^Q(\Lambda_p) m_Q (\tilde{c}_2(r_l) - \frac{1}{2} (\ell n \epsilon),$$  \hspace{1cm} (7)

where we take the asymptotic mean-field value $\Sigma_c^Q(|l| \geq \epsilon) \simeq m_Q$ and numerically calculate $\tilde{c}_2(r_l)$, which is plotted in Fig.2. Note that the $q^2$-dependent renormalized couplings are given by

$$\lambda_g^Q(q^2) = \lambda_g^Q(m_Q^2) Z_3(q^2, m_Q^2); \hspace{1cm} \lambda_g^Q(m_Q^2) = \frac{3 \beta_L^Q \beta_R^Q(m_Q^2)}{4\pi}$$  \hspace{1cm} (8)

where $(\beta_L^Q, R^Q(m))$ are left- and right-handed gauge couplings to different fermions ($Q$) on the mass-shell $m = m_Q$, and $Z_3(q^2, m_Q^2)$ are normal gauge field renormalization functions in the abelian and non-abelian cases ($SU_L(2)$). In eq. (5) the regular contributions from the lattice-region are

$$\delta_{A,Z}^Q(\epsilon, r_l) = -\left(\frac{3}{\pi} \left[ \beta_L^Q + \beta_R^Q \right]^2(\Lambda_p) \right) r_l^2 \int_{[\epsilon, \pi]} \frac{d^4l}{16\pi^2} \frac{\Sigma_c^Q(l)}{(\sin^2(l) + (m_Qa + r(l))^2)}$$  \hspace{1cm} (9)

$$\delta_{W \pm}^{(0, -1)}(\epsilon, r_l) = \left(\frac{3}{\pi} \left[ \beta_L^Q + \beta_R^Q \right]^2(\Lambda_p) \right) r_l^2 \int_{[\epsilon, \pi]} \frac{d^4l}{16\pi^2} \frac{\Sigma_3^Q(l) + \Sigma_{i=1}^3 |V_{KM}^j| \Sigma_{c}^{(-1, 0)}(l_i)}{(\sin^2(l) + (m_{(i,0)}a + r(l))^2)},$$  \hspace{1cm} (10)

where $V_{KM}$ is a KM-type matrix for lepton sector. Since (9) and (10) have no $\ell n\epsilon$ divergence, we may safely take the limit $\epsilon \rightarrow 0$.

Due to the fact that there are extra terms arising from $\tilde{c}_2(r_l)$ [eq.(5)], $\delta_g^Q(r_l)$ [eqs. (9-10)] and NJL terms in (3), the gap-equation (3) can have consistent massive solution ($m_Q \neq 0$) even for small gauge couplings(8). In this preliminary discussion, we take a trivial (unity) KM matrix in (10) and the Landau mean-field approximation $\Sigma_c^Q(q) \simeq m_Q$ in (5-10). Thus, for each lepton family, we have from (5)

$$m_\nu = m_\nu(NJL) + m_\nu \delta_Z^{\nu}(r_l) + m_\nu \delta_W^{\nu}(r_l)$$  \hspace{1cm} (11)

$$m_l = m_l(NJL) + m_l(\tilde{C}_A^{\nu} + \tilde{C}_Z) + m_l(\delta_A^{l}(r_l) + \delta_Z^{l}(r_l)) + m_\nu \delta_W^{\nu}(r_l),$$  \hspace{1cm} (12)
where we can see that the gauge field contributions $C^Q_g(p)$ in eq. (1) play the essential role in distinguishing between neutrinos and charged leptons. We notice that for neutrinos $\tilde{C}^0_g = \theta^0_g = 0 (g = A_{em}, W^\pm$ and $Z^0)$, while for charged leptons $\tilde{C}^{(-1)}_{W^\pm} = \theta^{(-1)}_{W^\pm} = 0$. As for the gap equation for neutrinos (11), the consistent solution must be $C_g = \theta_g = 0$. Where we can see that the gauge field contributions $(\theta_g)$, using the renormalization group relations (8) to estimate the values of $\beta_L(A_p), \beta_R(A_p) (\alpha(A_p) \approx 1.56\alpha(m_e), \sin^2 \theta_w(A_p) \approx 2.32 \sin^2 \theta_w(m^2_Z))$ in (11,10,13) $\alpha \equiv \alpha(m_e) = \frac{1}{137}, \sin^2 \theta_w \equiv \sin^2 \theta_w(m^2_Z) = 0.23$, one gets $(\tilde{C}^{Q}_g = C^{(-1)}_g)$

\begin{align*}
C_A^l & \approx 3 \cdot 10^{-3} (\ell n \frac{\Lambda_p}{m_l})^2 + 0.75 \ell n \frac{\Lambda_p}{m_l} + 1.5 \bar{c}_2(r_i) \alpha; \\
C_Z^l & \approx 3.6 \cdot 10^{-3} (\ell n \frac{\Lambda_p}{m_Z})^2 + 0.37 \ell n \frac{\Lambda_p}{m_Z} + 0.18 \bar{c}_2(r_i) \left( \frac{\sin^2 \theta_w}{4\pi} \right) + 1.2 \cdot 10^{-3} (\ell n \frac{\Lambda_p}{m_Z})^2 + 0.37 \ell n \frac{\Lambda_p}{m_Z} + 0.18 \bar{c}_2(r_i) \alpha, \quad (14)
\end{align*}

where $m_Z = 91.2$GeV. It will be seen soon that $\delta^Q_g(r_i), [eqs.(11,13)],$ are small numbers ($\approx 10^{-5}$) due to the smallness of gauge couplings and of $r_i$. Neglecting $\delta^Q_g(r_i)$ in (12) and adopting the four-fermi coupling $g_1$ determined by the top quark mass, one can show that the gap equation for charged leptons (12) has a consistent solution $m_l \sim O(\text{MeV}) \ll \Lambda_p$ for small gauge couplings [3]. However, for the time being, we are not in a position to obtain the correct spectrum of charged leptons [3]. As for the gap equation for neutrinos (11), the consistent solution must be $m_\nu \ll m_l$ because $(1 - \sigma) \sim O(\alpha \ell n \frac{\Lambda_p}{m_l}) \gg \delta^{(0)}_g(r_i)$. In fact, in eqs. (11) and (12), we have seen the emergence of the hierarchy spectrum of neutrinos and charged leptons due to their very different gauge couplings.

Although we are still not able to calculate the masses of neutrinos and charged leptons based on eqs. (11) and (12), their mass ratio can be estimated. It is easy to see that the solution to (12) is simply

\begin{align*}
\delta^l_W(r_i) m_l^2 - 2\Delta m_\nu m_\nu - \delta^l_W(r_i) m_\nu^2 = 0 \\
2\Delta_l = - \left( C_A^l + C_Z^l \right) - \delta^l_A(r_i) + \left( \delta^l_Z(r_i) - \delta^l_Z(r_i) \right).
\end{align*}

From the definitions (3), one obtains

\begin{align*}
\delta^l_Z(r_i) - \delta^l_Z(r_i) & \approx 82.2 \alpha r_i^2 G(r_i); \\
\delta^l_A(r_i) & \approx 58.9 \alpha r_i^2 G(r_i), \\
G(r_i) & = \int \frac{d^4p}{(2\pi)^4} \frac{1}{\sin^2(l_\mu) + (m^2_{Qa} + r_i w(l))^2}.
\end{align*}
where $G(r_l)$ as a function of $r_l$ is plotted in Fig. 3, while for $\delta_{W}^{\nu}(r_l) \text{[eq.(10)]}$, where $\beta_R = 0, \beta_L = \frac{g_2}{\sqrt{2}}$ and $g_2$ is the $SU_L(2)$ gauge coupling, we have

$$\delta_{W}^{\nu}(r_l) = \delta_{W}^{\nu}(r_l) = -\frac{3\pi}{2} \frac{\alpha(\Lambda_p)}{\sin^2 \theta_w(\Lambda_p)} r_l^2 G(r_l) \approx -13.72 \alpha r_l^2 G(r_l). \tag{19}$$

Thus, solving (15), we obtain

$$m_{\nu} = \frac{\Delta_l + \sqrt{\Delta_l^2 + (\delta_{W}^{\nu}(r_l))^2}}{\delta_{W}^{\nu}(r_l)} m_l. \tag{20}$$

We see that $\Delta_l < 0$ and $\delta_{W} \ll 1$ are crucial for obtaining $m_{\nu} \ll m_l$. For $r_l \simeq 0.01 \sim 0.04$, one gets $\bar{c}_2(r_l) = 0.65 \sim 0.62$ and $G(r_l) = 0.62 \sim 0.6$, which leads approximately to

$$m_{\nu_1} \simeq (1.0 \cdot 10^{-5} \sim 1.6 \cdot 10^{-4}) m_l. \tag{21}$$

Putting the experimental charged lepton masses ($m_e = 0.5\text{MeV}, m_\mu = 105\text{MeV}, m_\tau = 1774\text{MeV}$) into (21), we predict the neutrino masses:

$$m_{\nu_e} \simeq (5 \sim 128)\text{eV},$$
$$m_{\nu_\mu} \simeq (1 \sim 26)\text{keV},$$
$$m_{\nu_\tau} \simeq (17 \sim 284)\text{keV}, \tag{22}$$

which are compatible with present experiments, even though our analysis is still at a rather rudimentary level. Leaving aside the uncertainties of our analysis, which may well exceed the theoretical uncertainty appearing in our predictions, we believe that we can say with a certain degree of confidence that:

1. eqs. (23) represent the first (successful?) attempt to derive within a complete closed dynamical scheme the mass ratios between the members of the lepton doublets;

2. these mass ratios are, to the approximation we work in, universal;

3. the large difference in (22) between the masses of the neutral and charged members of the lepton weak doublets finds its natural and convincing origin in the their gauge-couplings, and in particular in the neutrino lack of electromagnetic interactions, which most likely is at the root of the mechanism by which charged leptons acquire their masses\footnote{In\textsuperscript{[4]} we show that the Wilson parameter for quarks ($r_q = 0.28 \sim 0.3$) is determined by minimizing vacuum energy, yielding for the Wilson parameter for leptons $r_l = 0.01 \sim 0.04$.};

\footnote{Note that the main uncertainty comes from the uncertainty of the Wilson parameter $r_l$.}
4. the crucial role played by the Wilson parameter $r_l$ in yielding a connection between the masses of the charged leptons and their neutrinos, through the high energy ($\sim \Lambda_p$) coupling of the right-handed neutrinos to their left-handed counterparts via the charged leptons and the $W^\pm$-bosons, should be viewed as a clearly unique dynamical feature of our proposal.

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Figure Captions

**Figure 1:** The diagrammatic form of the Dyson equation for the lepton sector.

**Figure 2:** The function $\bar{c}_2(r_l)$ in terms of $r_l$.

**Figure 3:** The function $G(r)$ in terms of $r_l$. 
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