Unusual corrections to scaling in entanglement entropy

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Abstract. We present a general theory of the corrections to the asymptotic behaviour of the Rényi entropies $S_A^{(n)} = (1 - n)^{-1} \log \text{Tr} \rho_A^n$ which measure the entanglement of an interval $A$ of length $\ell$ with the rest of an infinite one-dimensional system, in the case when this is described by a conformal field theory of central charge $c$. These can be due to bulk irrelevant operators of scaling dimension $x > 2$, in which case the leading corrections are of the expected form $\ell^{-2(x-2)}$ for values of $n$ close to 1. However, for $n > x/(x-2)$ corrections of the form $\ell^{2-x-x/n}$ and $\ell^{-2x/n}$ arise and dominate the conventional terms. We also point out that the last type of corrections can also occur with $x$ less than 2. They arise from relevant operators induced by the conical spacetime singularities necessary to describe the reduced density matrix. These agree with recent analytic and numerical results for quantum spin chains. We also compute the effect of marginally irrelevant bulk operators, which give a correction $O((\log \ell)^{-2})$, with a universal amplitude. We present analogous results for the case when the interval lies at the end of a semi-infinite system.

Keywords: conformal field theory (theory), spin chains, ladders and planes (theory), entanglement in extended quantum systems (theory)

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1. Introduction

Recently there has been much interest in characterizing bipartite quantum entanglement of pure states in extended systems near a quantum critical point in terms of the Rényi entropies [1]. For a given division of the Hilbert space into a part $A$ and its complement $B$, these are defined as

$$S_A^{(n)} = (1 - n)^{-1} \log \text{Tr} \rho_A^n,$$

where $\rho_A = \text{Tr}_B \rho$ is the reduced density matrix of subsystem $A$ and $\rho = |\Psi\rangle\langle\Psi|$ is the density matrix of the whole system in a pure state $|\Psi\rangle$. Knowledge of the $S_A^{(n)}$ for different $n$ characterizes the full spectrum of non-zero eigenvalues of $\rho_A$ (see, e.g., [2]), and gives more information about the entanglement than the widely studied von Neumann entropy $S_A^{(1)}$. It also gives a fundamental insight into understanding the convergence and scaling of algorithms based on matrix product states [3].

In [4]–[6] it was shown that for a one-dimensional critical system whose scaling limit is described by a conformal field theory (CFT) of conformal anomaly number (central charge) $c$, in the case where $A$ is an interval of length $\ell$ embedded in an infinite system, the asymptotic behaviour of the Rényi entropies is given by

$$S_A^{(n)} \approx \frac{c}{6}(1 + n^{-1}) \log \ell + O(1). \tag{1}$$

This result has by now been verified analytically and numerically (see, e.g., [7]–[17], but this list is far from being exhaustive) for a large number of examples of quantum spin chains whose scaling limit is believed to be described by CFT. It gives, in fact, one of the most accurate ways of measuring the conformal anomaly number. However, this asymptotic result is often obscured by large (sometimes oscillating) corrections to scaling [18]–[20] whose origin has, so far, not been clarified in the context of quantum field theory.

In any real system with an ultraviolet cutoff such as a lattice, even when it is tuned to the critical point there will, in general, be operators in the Hamiltonian or action which ensure that the continuum field theory results are only asymptotic on distance scales much
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larger than the cutoff. It is important to understand the form of the corrections to the asymptotic results in order to make accurate fits to analytic and numerical finite-lattice data. Renormalization group (RG) theory shows that these are ordinarily due to irrelevant operators, with scaling dimension \( x > d \) (\( d \) being the dimension of spacetime), which are allowed in the effective action even at the critical point. These are generally responsible for finite-size corrections in integer powers of \( L^{-(x-d)} \) [21], where \( L \) is a characteristic length scale of the system. Since field theory classifies these irrelevant operators and allows the computation of their scaling dimensions, it therefore is capable of quantifying the form of corrections to scaling, if not the values of the non-universal prefactors. Moreover, as \( x \to 2^+ \) these corrections become more important, and, at \( x = 2 \), the so-called marginal case, take the form of logarithms rather than powers. The theoretical advantage is that the amplitudes are then often universal and calculable.

In this paper we analyse the corrections to the Rényi entropies due to irrelevant and marginally irrelevant bulk operators in the Hamiltonian. For the case of an interval of length \( \ell \) in an infinite system, we find the expected terms \( O(\ell^{-2(x-2)}) \). For the case of an interval at the end of a semi-infinite system there can also be terms \( O(\ell^{-(x-2)}) \). However, we also find unusual terms which are \( O(\ell^{-2x/n}) \) and \( O(\ell^{-x/n}) \), in which the exponent depends on \( n \). For \( n > n_c(x) = x/(x-2) \) these dominate the conventional corrections.

While the appearance of \( n \)-dependent exponents for an irrelevant bulk operator is perhaps surprising, in recent analytic and numerical studies of several spin chains [18] even more dramatic corrections of the form \( \ell^{-2x/n} \), with values of \( x < 2 \), have been reported. We argue that these may also be understood from the field theory under the assumption that the conical singularities of the Riemann surfaces on which the path integral must be evaluated in fact locally break the criticality of the system, thus allowing the appearance of operators at these points, which, in the bulk, would be relevant with scaling dimension \( x < 2 \). Such operators do not drive the system away from bulk criticality because they are localized near points in spacetime. It is an old result of CFT [22]\(^4\) that such a bulk operator near a conical singularity of degree \( n \) in fact has its scaling dimension modified to \( x/n \). Thus we predict that such operators should give rise to corrections of the form \( \ell^{-2x/n} \) in the case of an interval in an infinite system and \( \ell^{-x/n} \) in the semi-infinite case.

The marginal case \( x = 2 \) turns out to be even more subtle and difficult. However, we are able to show that the corrections take a universal form in which \( c \) in (1) is replaced by

\[
c - \frac{1}{b^2(\log \ell)^3} + O((\log \ell)^{-4}),
\]

where \( b \) is a universal (and, for many CFTs, known) operator product expansion (OPE) coefficient. Note that, for this leading correction, the \( n \) dependence is the same as that of the leading term. As we discuss below, this is a consequence of Zamolodchikov’s \( c \) theorem [23].

2. Field theory for corrections to scaling

As was argued in [4, 5, 24], for a one-dimensional quantum system on a lattice \( L \) the expression \( \text{Tr} \rho_n^A \) is given by the ratio \( Z_n/Z_1 \), where \( Z_n \) is the partition function defined by the path integral on \( n \) copies of \( L \otimes \) imaginary time \( \tau \in \{-\infty, +\infty\} \), sewn together

\(^4\) In this reference, only corners on the boundary were, in fact, considered. However, the analysis extends trivially to the case of conical singularities in the bulk.
along $\tau = 0$ so that in the interval $B \subset \mathcal{L}$ the $j$th sheet with $\tau > 0$ is connected to the $j$th sheet with $\tau < 0$, while in the interval $A$ the $j$th sheet is connected to the $(j + 1)$th sheet, cyclically. In the continuum limit, where the lattice is replaced by the real line, this defines an $n$-sheeted Riemann surface $\mathcal{R}_n$ with conical singularities at the ends of the interval $A$, but in what follows later it is important to realize that the identification also holds on a spatial lattice. It is also convenient to consider the logarithm $-(F_n - nF_1)$ of the above ratio of partition functions, which is then proportional to the Rényi entropy $S_A^{(n)}$.

In analysing such a lattice system using continuum field theory methods, the leading universal effects of the lattice are usually assumed to be taken into account by imposing a short-distance cutoff $\epsilon$ on the continuum theory: for example, by restricting the asymptotic expressions for correlation functions to be valid only for separations greater than $\epsilon$. While this might appear to be a rather crude approximation to the effect of a lattice, since we are looking only for the universal form of corrections to scaling and not their precise amplitudes, this is, in fact, adequate. The response of the free energy $F$ to a change in $\epsilon$ is, in general, given in the field theory by the integrated trace of the stress tensor [25,26]:

$$\frac{-\epsilon}{\partial \epsilon} = \frac{1}{2\pi} \int (\Theta(z)) \, d^2z. \quad (3)$$

In a CFT in flat spacetime, $\langle \Theta(z) \rangle = 0$, implying that $F$ is scale-invariant. (This is, in fact, strictly correct only after subtracting off the non-universal bulk free energy, which in our case automatically cancels in the combination $F_n - nF_1$.) However, in [25,26] it was shown that this is no longer true at conical singularities: in fact, each one contributes a term $(c/12)(n - n^{-1})$ to the right-hand side of (3). Integrating up and using the fact that $F_n - nF_1$ can depend only on the ratio $\ell/\epsilon$ then gives the result (1), which was derived in slightly different ways in [4,5].

Let us now consider the effect of a bulk irrelevant operator in the Hamiltonian of action. In the field theory this is equivalent to perturbing the CFT by an operator $\Phi(z)$ of scaling dimension $x > 2$, so the action is

$$S = S^* + \lambda \int \Phi(z) \, d^2z, \quad (4)$$

where $S^*$ is the CFT action and $\lambda$ is a coupling constant. It has the dimensional form $g/\epsilon^{2-x}$, where $g$ is dimensionless. The change in the dimensionless free energy is then formally given by the perturbative series

$$-\delta F_n = \sum_{N=1}^{\infty} \frac{(-\lambda)^N}{N!} \int_{\mathcal{R}_n} \cdots \int_{\mathcal{R}_n} \langle \Phi(z_1) \cdots \Phi(z_N) \rangle_{\mathcal{R}_n} \, d^2z_1 \cdots d^2z_N, \quad (5)$$

of integrals of connected correlation functions of the CFT over $\mathcal{R}_n$.

In the case when $A$ is the interval $(0, \ell)$ in an infinite system, $\mathcal{R}_n$ may be conformally mapped to the punctured complex plane $\mathbb{C}' = \mathbb{C} \setminus \{0\}$ by [5]

$$\zeta = \left( \frac{z}{z - \ell} \right)^{1/n} : \quad z = \ell f(\zeta) \equiv \ell - \frac{\zeta^n}{\zeta^n - 1}. \quad \text{ió:10.1088/1742-5468/2010/04/P04023}$$
This maps the ends of the interval to $\zeta = 0$ and $\infty$. The correlation functions in the two geometries are related by

$$\langle \Phi(z_1) \cdots \Phi(z_N) \rangle_{\mathcal{R}_n} = \prod_{j=1}^{N} |\ell f'(\zeta_j)|^{-x} \langle \Phi(\zeta_1) \cdots \Phi(\zeta_N) \rangle_{\mathcal{C}'}.$$  

Since $\langle \Phi \rangle_{\mathcal{C}'} = 0$, the $N = 1$ term in (5) is absent in this case. The second-order term may be transformed to an integral over the $\zeta$ plane:

$$\delta F_n^{(2)} = - \frac{1}{2} g^2 (\ell/\epsilon)^{4-2x} \int_{\mathcal{C}'} \int_{\mathcal{C}'} \frac{|f'(\zeta_1)|^{2-x} |f'(\zeta_2)|^{2-x}}{|\zeta_1 - \zeta_2|^{2x}} \, d^2\zeta_1 \, d^2\zeta_2 \tag{6}$$

$$= - \frac{1}{2} g^2 (n\ell/\epsilon)^{4-2x} \int_{\mathcal{C}'} \int_{\mathcal{C}'} \frac{|\zeta_1 \zeta_2|^{(2-x)(n-1)}}{|\zeta_1^{n-1} - 1|^{4-2x} |\zeta_2^{n-1} - 1|^{4-2x}} \, d^2\zeta_1 \, d^2\zeta_2. \tag{7}$$

Note that this integral makes sense also for non-integer $n$, although we derived it only for integer $n$. We may then consider $n$ as an arbitrary real parameter. The integral now has several potential sources of UV divergence which should be regulated by a cutoff that in the $z$ plane is $O(\epsilon)$. These potential divergences give rise to a further power of $\epsilon$ in $\delta F_n^{(2)}$ and, by scaling, to further powers in $\ell$ in the corrections to scaling.

To elucidate the mechanism let us start from the case when $n-1$ is small and positive. The integral in (7) then converges for $x < 1$. For larger values of $x$, a divergence comes from the region $\zeta_1 \rightarrow \zeta_2$ and it should be regularized with a cutoff $|z_1 - z_2| < \epsilon$. The leading divergence in the integral is $O(\epsilon^{2-2x})$, leading to a dependence in $\delta F_n^{(2)}$ proportional to $\epsilon^{-2} \text{Area}(\mathcal{R}_n)$. This is a contribution to the non-universal bulk free energy, which cancels in the combination $F_n - nF_1$. This subtraction also cancels the apparent singularities at $\zeta^n = 1$, which correspond to $|z| \rightarrow \infty$ on $\mathcal{R}_n$. If the leading divergence is subtracted off from (7), the remainder is analytic at $x = 1$. In fact, the finite part is then given by the analytic continuation of (7) around its pole at $x = 1$. This is then finite all the way up to $x = 3$ and the $\ell$ dependence comes from the explicit prefactor, that is $\ell^{-2(x-2)}$, the standard power law of FSS in the RG framework. Note that, although the amplitudes of these terms are non-universal, depending on $g$, their ratios are universal, and in principle calculable by evaluating the analytic continuation of the integral. We also note that, for an interval near the end of a semi-infinite system, depending on the boundary conditions $\langle \Phi \rangle$ may be non-vanishing, in which case the leading correction will be $O(\ell^{2-x})$.

However, for larger values of $n$ and $x > 2$, (7) may also exhibit divergences as $\zeta_j \rightarrow 0$ or $\infty$. In the original coordinates, these occur as $z_1$ or $z_2$ approach one of the branch points. These are genuine divergences due to the conical singularities and are not present in the bulk. Close to (say) $z = 0$ each integral behaves like $\int |z|^{(1/n)-1} x d^2z$, which diverges if $n > n_c(x) = x/(x-2)$. In this case the integral must be further regulated with a cutoff $|z| < \epsilon$. This leads to a further multiplicative factor $\propto \epsilon^{2-x+(x/n)}$ which, once again, by scaling leads to corrections to scaling in the Rényi entropy of the form $\ell^{4-2x-(2-x+x/n)} = \ell^{2-x-x/n}$. For $n > n_c$ these divergences are further enhanced when $z_1$ and $z_2$ are close to different branch points, leading to a further factor of $\epsilon^{2-x+x/n}$ and a consequent $\ell$ dependence of the form $\ell^{4-2x-2(2-x+x/n)} = \ell^{-2x/n}$. (The singularities when $z_1$ and $z_2$ approach the same branch point are removed by the bulk free energy subtraction.)

To summarize, the presence of bulk irrelevant operators with $x > 2$ leads to corrections to scaling of the form $\ell^{4-2x}$, $\ell^{2-x-x/n}$ and $\ell^{-2x/n}$. In general, all these will be present, but

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for $n < n_c = x/(x - 2)$ the first dominates, while for $n > n_c$ it is the last. For $n \approx n_c$ we expect them all to play a role, with multiplicative logarithmic factors when $n = n_c$.

The appearance of these terms is similar to what happens in the case of a boundary. In that case the bulk operator $\Phi(y)$ at a distance $y$ from the boundary behaves like $y^{-x+z_b}\Phi_b$, where $\Phi_b$ is a boundary operator with scaling dimension $x_b$. This gives rise, on integration, to a term $\lambda^2 e^{1-x+z_b}$ in the free energy per unit length of the boundary which is singular if $x_b < x - 1$, and hence to a term $L^{x-x_b} L^{4-2x}$ in the total free energy of a finite system of size $L$. This boundary contribution can actually overwhelm the normal bulk term. Analogously, close to a conical singularity, $\Phi(z) \sim |z|^{-x+z/n}\Phi^{(n)}(0)$, where $\Phi^{(n)}$ is localized at the tip of the cone, and has scaling dimension $(x/n)$. The difference is that in the boundary case $x_b$ is usually larger than the bulk scaling dimension $x$, while at a conical singularity the scaling dimension $(x/n)$ becomes arbitrarily small for large $n$.

3. Relevant operators at conical singularities

We now argue that corrections to scaling of the form $\ell^{-(2x/n)}$ in the Rényi entropy can arise not only by the mechanism discussed above due to irrelevant bulk operators with $x > 2$, but in a different way which also gives corrections of this form but with $x < 2$. This is because a lattice model which is critical can nevertheless generate operators localized at the conical singularities with scaling dimension $(x/n)$ but with $x < 2$, that is, operators which, if they appeared in the bulk Hamiltonian, would be relevant and therefore drive the system away from criticality. This is most easily seen if we discretize time as well as space. Consider, for example, a 2d classical model on a square lattice. The anisotropic limit of this in general gives rise to a 1d quantum Hamiltonian. Computing $\text{Tr} \rho_\ell^n$ for this quantum model corresponds, as before, to considering the 2d lattice model on an $n$-sheeted surface with branch points of degree $n$. On the lattice, the details of this depend on exactly how the degrees of freedom are divided between $A$ and $B$. In most models like quantum spin chains, the degrees of freedom are on the lattice sites, so the division is along a spatial bond. In the time-discretized picture, the branch point can be considered to lie halfway along a spatial bond, or in the middle of a plaquette. In either case, it is clear that degrees of freedom close to the branch points have an enhanced (if $n > 1$) number of neighbours. If we maintain the same bond interactions as in the bulk, there will therefore be an effective local coupling to the energy density, or to other operators which do not break the internal symmetries of the system. For example, in the case of the nearest-neighbour Ising model, if we put the branch point in the centre of a plaquette, each order variable (Ising spin) still has exactly four nearest neighbours, but the dual spin, which sits on top of the branch point, will have $4n$ nearest neighbours, thus locally breaking the self-duality of the model and, locally, driving it away from criticality. In this case we would therefore expect a coupling to the energy density, with $x = 1$. In the field theory, such an operator close to a conical singularity of degree $n$ has its scaling dimension modified to $x/n$ [22], leading, in this case, to corrections of the form $\ell^{-2/n}$.

The conclusion is that the correct form of the field theory action on the $n$-sheeted surface should be

$$S = S_{\text{CFT}} + \sum_j \lambda_j \int_{\mathcal{R}_n} \Phi_j d^2z + \sum_P \sum_k \lambda_k \Phi^{(n)}_k(P),$$

where

$$S_{\text{CFT}} = -\frac{1}{2} \int_{\mathcal{R}_n} d^2z \left( \frac{\partial \Phi}{\partial \Phi'} \right)^2 + \frac{1}{4} \int_{\mathcal{R}_n} d^2z \left( \frac{\partial \Phi}{\partial \Phi'} \right)^4,$$

and

$$\lambda_j \sim \frac{1}{n^2} \frac{1}{\ln n}$$

for $j > 2$. This action gives rise to corrections of the form $\ell^{-(2x/n)}$ in the Rényi entropy, as desired.
where the second term takes account of bulk irrelevant operators \( \Phi_j \) with \( x_j > 2 \), already discussed, and the third term is a sum over the branch points \( P \) of localized operators \( \Phi^{(n)}_k(P) \) with scaling dimension \( x_k/n \), with all possible values of \( x_k \) allowed by symmetry, including those with \( x_k < 2 \). However, in the perturbative expansion in powers of the \( \lambda_k \), each \( \Phi^{(n)}_k(P) \) at a given branch point should appear at most once (otherwise we can use the OPE to write higher powers in terms of other localized operators). In the case of an infinite system, since \( \langle \Phi^{(n)}(P) \rangle = 0 \), we therefore expect the leading correction to be \( O(\ell^{-2x/n}) \). This should be the case no matter how many branch points \( \geq 2 \) there are. For an interval at the end of a semi-infinite system, depending on the boundary conditions it may be that \( \langle \Phi^{(n)}(P) \rangle \neq 0 \), in which case the leading correction will be \( O(\ell^{-x/n}) \).

### 4. The marginal case and the \( c \) theorem

Next we turn to the marginal case \( x \to 2+ \), which is technically more challenging. The integral in (6) is, in general, very difficult to manipulate into a form in which the necessary analytic continuation around the pole at \( x = 1 \) can be performed. It is easier to extract the limit \( x \to 2 \) by making the subtraction explicitly, and this we now describe. It is, however, quite subtle, as the cutoff and subtraction must be performed in the \( z \) plane. Imposing a cutoff \( |z_1 - z_2| > \epsilon \), in the \( \zeta \) plane we have

\[
\delta F_n = -\frac{1}{2} g^2 (\ell/\epsilon)^{4-2x} \int \frac{f'(\zeta_1)^2 |f'(\zeta_2)|^{2-x} |f'(\zeta_2)|^{2-x}}{|\zeta_1 - \zeta_2|^{2x}} d^2 \zeta_1 d^2 \zeta_2. \tag{8}
\]

The subtraction is

\[
-\frac{1}{2} g^2 (\ell/\epsilon)^{4-2x} \int_{R_n} \frac{2\pi \epsilon^{2-2x}}{2 - 2x} d^2 z_1 = -\frac{1}{2} g^2 (\ell/\epsilon)^{4-2x} \int \frac{|f'(\zeta_1)|^{2-2x}}{2 - 2x} d^2 \zeta_1 = -\frac{1}{2} g^2 (\ell/\epsilon)^{4-2x} \int \frac{|f'(\zeta_1)|^{4-2x}}{|\zeta_2 - \zeta_1|^{2x}} d^2 \zeta_1 d^2 \zeta_2.
\]

It can be shown that, up to terms which vanish as \( x \to 2 \), the cutoff in the last integral can be replaced by \( |f(\zeta_1) - f(\zeta_2)| > (\epsilon/\ell) \). Thus the second-order contribution to the subtracted free energy is

\[
\delta F_n - n \delta F_1 = -\frac{1}{2} g^2 (\ell/\epsilon)^{4-2x} \int \frac{|f'(\zeta_1)|^{2-x} |f'(\zeta_2)|^{2-x} |f'(\zeta_2)|^{4-2x}}{|\zeta_1 - \zeta_2|^{2x}} d^2 \zeta_1 d^2 \zeta_2.
\]

Symmetrizing between \( \zeta_1 \) and \( \zeta_2 \) we then find

\[
\delta F^{(2)}_n - n \delta F^{(2)}_1 = \frac{1}{4} g^2 (\ell/\epsilon)^{4-2x} \int \frac{|f'(\zeta_1)|^{2-x} |f'(\zeta_2)|^{2-x} - |f'(\zeta_1)|^{4-2x}}{|\zeta_1 - \zeta_2|^{2x}} d^2 \zeta_1 d^2 \zeta_2. \tag{9}
\]

The integral is now finite as \( \epsilon \to 0 \) and the cutoff can be removed, at least for \( x \) close to 2. Also, notice that everywhere \( f'(\zeta_1) \) and \( f'(\zeta_2) \) are regular, the integrand vanishes as \( x \to 2 \). However, this is not the case close to the conical singularities. For example, near \( \zeta = 0 \), \( |f'(\zeta)|^{2-x} \sim |\zeta|^{(n-1)(2-x)} \), and the limits \( \zeta \to 0 \) and \( x \to 2 \) do not commute. However,
we would obtain the same limit as \( x \to 2 \) in the integral if we restricted the integration region to, say, \((|\zeta_1| < \rho), \quad (|\zeta_2| < \rho) \cup (|\zeta_1| > \rho^{-1}), \quad (|\zeta_2| > \rho^{-1})\) for any \( 0 < \rho < 1 \). In particular, we can take \( \rho \) arbitrarily small, in which case we can accurately approximate \( f' (\zeta) \) by its asymptotic forms \( n \zeta^{n-1} \) and \( n \zeta^{-n-1} \), up to terms which vanish as \( x \to 2 \). The consequence\(^5\) is that, as \( x \to 2 \), we can replace the integral in (8) by the analytic continuation to \( x \approx 2 \) of

\[
2 \int \frac{|n \zeta_1^{n-1}|^{2-x} |n \zeta_2^{n-1}|^{2-x}}{|\zeta_1 - \zeta_2|^{2x}} \, d^2 \zeta_1 \, d^2 \zeta_2.
\]

This, apart from the \( \ell^{4-2x} \) prefactor, is precisely twice what we would obtain for a single branch point. This integral can be evaluated explicitly. Rescaling \( \zeta_2 = w \zeta_1 \) we have

\[
2 n^{4-2x} I (n, x) \int |\zeta_1|^{-2-2n(x-2)} \, d^2 \zeta_1,
\]

where\(^6\)

\[
I (n, x) = \int |w|^{(n-1)(2-x)} |w - 1|^{-2x} \, d^2 w
= \frac{\pi}{\Gamma (1 + (n + 1) (x - 2)/2) \Gamma (1 - (n - 1) (x - 2)/2) \Gamma (1 - x)}.
\]

Although the equivalence of this result to (9) is valid only as \( x \to 2 \), we note that it does exhibit the required poles at \( x = 1 \) and \( 2n/(n - 1) \), corresponding to the short-distance divergences already discussed above. However, as \( x \to 2 \) we have

\[
I (n, x) \sim - (\pi/4) (n^2 - 1) (x - 2) + O ((x - 2)^2).
\]

The integral over \( \zeta_1 \) in (10), gives, after imposing a short-distance cutoff \( |\zeta| > \epsilon/\sqrt{n} \), a factor

\[
\frac{2 \pi (\epsilon/\sqrt{n})^{2n(x-2)}}{2n(x-2)} \sim \frac{\pi}{n(x-2)} + O (1).
\]

Putting all these factors together we then find that, for \( x \approx 2 \)

\[
F_n - n F_1 = -\frac{c}{6} (n - n^{-1}) \log (\ell/\epsilon) + g^2 (n - n^{-1}) (\ell/\epsilon)^{3-2x} \left( \frac{\pi^2}{4} + O (x - 2) \right) + O (g^2).
\]

For \( x = 2 \) this gives an (apparently) uninteresting constant contribution to the Rényi entropy, and it is therefore necessary to go to the next order, involving an integral over the three-point function of the form

\[
\frac{1}{6} b g^3 (\ell/\epsilon)^{6-3x} \int \frac{|f' (\zeta_1)|^{2-x} |f' (\zeta_2)|^{2-x} |f' (\zeta_3)|^{2-x}}{|\zeta_2 - \zeta_3|^{x} |\zeta_3 - \zeta_1|^{x} |\zeta_1 - \zeta_2|^{x}} \, d^2 \zeta_1 \, d^2 \zeta_2 \, d^2 \zeta_3,
\]

where \( b \) is the universal coefficient in the OPE:

\[
\Phi (\zeta) \cdot \Phi (0) = |\zeta|^{-2x} (1 + b |\zeta|^{2x} \Phi (0) + \cdots).
\]

\(^5\) This also shows that the argument also holds in the presence of a boundary when the form \( |\zeta_1 - \zeta_2|^{-2x} \) of the two-point function holds only at short distances.

\(^6\) This follows using the representation \( |w|^{2n} = \Gamma (-a)^{-1} \int_0^\infty u^{1-a} e^{-u w} w \, du \) for each factor and first performing the Gaussian integral over \( w \). The result can then be reduced to a beta-function integral.
Once again, it can be shown that, after subtracting the short-distance singularities as \( \zeta_j \to \zeta_k \), the measure is concentrated on the conical singularities as \( x \to 2 \), so the result for the interval is twice that found by replacing \( f(\zeta) \) by \( \zeta^n \). The integral can then be performed explicitly. (A similar computation was carried in \([28,29]\) for the corrections to the free energy of a cylinder, which corresponds to the limit \( n \to 0 \) of the present calculation.) However, it turns out that the coefficient of the \( O(g^3) \) term in (11) is determined from the \( O(g^2) \) term by Zamolodchikov's \( c \) theorem \([23]\). To use this, however, it is more correct to consider the logarithmic derivative of the free energy with respect to the cutoff which, from (3), is proportional to the integral of the trace \( \langle \Theta \rangle \): we see from (11) that this takes the form

\[
c_{\text{eff}}(g) = c - 3\pi^2(2 - x)g^2 + O(g^3). \tag{12}
\]

The \( c \) theorem \([23]\) states that there exists a function \( C(g) \) which decreases along RG flows and is stationary at fixed points where it equals the conformal anomaly number of the corresponding CFT. Zamolodchikov’s analysis also implies that \( C'(g) \propto (1 + O(g^2))\beta(g) \), where \( \beta(g) = -\epsilon(\partial g/\partial \epsilon)_{\Lambda_R} \), keeping the renormalized coupling \( \lambda_R \) fixed. For a perturbed CFT, the first two terms in \( \beta(g) \) are universal \([23,30]\):

\[
-\beta(g) = (2 - x)g - \pi bg^2 + O(g^3). \tag{13}
\]

Thus, up to and including terms \( O(g^3) \), all candidates for an interpolating function \( C(g) \) must agree, and in particular \( c_{\text{eff}}'(g) = \beta(g) \) to this order. This fixes the coefficient of the \( O(g^3) \) term in (12) to be \( 2\pi^3b \).\(^7\)

However, the result in (12) disguises the fact that in the neglected higher-order terms logarithmic dependences on \( \ell \) appear, corresponding to poles at \( x = 2 \). These can, however, be absorbed by ‘RG-improving’ the expansion, that is, replacing \( g \) by \( g(\ell) \) where

\[
\ell \frac{dg}{d\ell} = -\beta(g(\ell)).
\]

If \( x > 2 \), that is, the perturbation is slightly irrelevant, then asymptotically \( g(\ell) \propto \ell^{-(x-2)} \) and \( c_{\text{eff}}(\ell) \to c \), with a power law correction, consistent with our earlier analysis. If \( x < 2 \), that is the perturbation is relevant, then, depending on the sign of \( g/b \), either \( g(\ell) \to g^* \) and \( c_{\text{eff}}(\ell) \to c_{\text{new}} = c - (2 - x)^3/b^2 + O((2 - x)^4) \) \([29]\) or \( g(\ell) \) grows beyond the range of this perturbative treatment.

The interesting case is when \( x = 2 \), that is, the perturbation is marginal. Then if \( g/b < 0 \), the perturbation is marginally relevant and \( g(\ell) \) again grows, but if \( g/b > 0 \) it is marginally irrelevant and \( g(\ell) \) flows to zero, albeit logarithmically slowly. In fact

\[
g(\ell) = \frac{g}{1 + \pi bg \log(\ell/\epsilon)} \sim \frac{1}{\pi b \log(\ell/\epsilon)}.
\]

Substituting this into (12) we then see that

\[
c_{\text{eff}}(\ell) = c + \frac{2}{b^2(\log(\ell/\epsilon))^3} + O((\log(\ell/\epsilon))^{-4}). \tag{13}
\]

If we now integrate with respect to \( \epsilon \) to find the free energy \( F_n - nF_1 \), we find the result in (2). Note that this asymptotic value is reached from below, in apparent contradiction to Zamolodchikov's \( c \) theorem. However, the correct definition of \( C(g) \) in this case is through

\[
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\]

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the logarithmic derivative of the entanglement entropy [32], which is ultraviolet-finite at
the fixed points, and in this quantity $c$ is approached from above. We also remark that, for
fitting any finite $\ell$ data, it is better to replace the asymptotic $(\log(\ell/\epsilon))^{-3}$ in (13) by $g(\ell)^3$.

5. Comparison with other studies and discussions

We have shown that the block entanglement in an infinite 1d system (measured by the
Rényi entropies $S_A^{(n)}$) generically displays standard corrections to scaling of the form
$\ell^{-2(x-2)}$ with $x > 2$ and unusual ones of the form $\ell^{-2x/n}$ (and also the ‘combination’
$\ell^{2-x-x/n}$, which, however, is never the leading one). We call these corrections unusual
because of the explicit $n$ dependence of the exponent, a property that seems to contrast
with RG finite-size scaling theory [21]. Clearly there is no contradiction: we showed
that these terms arise from the conical singularities needed to describe $S_A^{(n)}$ in a path
integral formulation. The existence of such geometry-dependent exponents was, in fact,
first noticed a long time ago in [33]. The most surprising effect here is that the unusual
correction $\ell^{-2x/n}$ can be present also for relevant operators with $0 < x < 2$, occasioned by
a local breaking of scale invariance at the conical singularity. Such effects have probably
not be seen in earlier studies of corner critical behaviour [34] because they focused on
effective values of $n < 1$, where the exponent of these corrections is larger.

These unusual effects clearly need direct confirmation from lattice computations.
Large corrections to scaling have been observed in several numerical studies quoted before,
but a quantitative analysis has become available only recently [18]. It has been shown
analytically for the Ising and XX universality class that corrections to the scaling are of the
form $\ell^{-2/n}$ [18], which agrees with our general formula when $x = 1$. For the Ising model,
$x = 1$ corresponds to the energy density operator that indeed we argued to be generated
by the conical singularity. For anisotropic Heisenberg chains, the corrections to the scaling
have been found numerically to be of the form $\ell^{-2K/n}$, where $K$ is the Luttinger liquid
exponent, i.e. the most relevant present in the continuum theory. Again this perfectly
agrees with our result. Similar unusual (i.e. $n$-dependent) corrections have also been
found in other entanglement measures [35,36], but their quantitative understanding in
the framework of quantum field theory requires further investigation.

In the case of systems with boundaries, we showed that these unusual corrections
are of the form $\ell^{-x/n}$. Evidence of such power laws has been reported for $n = 1$ [19,20]
both for XX and Heisenberg chains, but a quantitative study for general $n$ is still lacking.
Preliminary results show that our results are correct [18,37].

However, our theory of the origin of the $\ell^{-2x/n}$ corrections with $x < 2$ (and $\ell^{-x/n}$ in
the semi-infinite case) has a definite prediction, so far untested in analytic and numerical
studies of particular systems: if the origin of these terms is indeed in the local deviation
from criticality near the conical singularities, then a modification of the lattice action close
to the singularities should have the effect of changing the amplitudes of these corrections.
In fact, by tuning the local couplings to a particular value (which may, however, depend
on $n$) it should be possible to eliminate these correction terms altogether. It should
be stressed that this modification needs to be done locally in spacetime, not simply by
modifying the interaction strengths in the Hamiltonian near the ends of the interval. This
could be carried out by starting with the ground state $|0\rangle$ of the full Hamiltonian $\hat{H}$,
and evolving this with the operator $\exp(-\tau\hat{H}')$, where $\hat{H}'$ is $\hat{H}$ with modified interactions
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in a region $O(\epsilon \ll \ell)$ around the ends of the interval. Here $\tau \sim \epsilon/v$, with $v$ being the coefficient in the quasiparticle dispersion relation $\omega \sim v|\mathbf{k}|$. The prediction is that the leading behaviour of the Rényi entropies measured in the modified state is the same as that of the ground state, but that the amplitudes of the $O(\ell^{-2x/n})$ correction terms should be different.

It is also worth commenting on the other correction $\ell^{2-x-x/n}$ that is never the most relevant one and could be obscured by the others in numerical analysis. For $n \sim n_c$ its effect should be more important, but an accurate quantitative analysis is difficult because the various corrections get too close to each other. However, in [18] it has been noticed that, for $n$ close to 1 and for $\Delta < -0.5$ (the anisotropy parameter of the Heisenberg chain), the single correction $\ell^{-2K/n}$ does not describe numerical data accurately. This is a confirmation of the presence of other corrections to the scaling that could be of the form $\ell^{2-x-x/n}$ (but could also be of very different origin).

We stress that these corrections in some lattice systems (those described by Luttinger liquid theory) show a strong oscillating behaviour [13, 18, 19, 38]. While the physical origin of this effect is due to a strong tendency to antiferromagnetic order [19], a general proof of this universal form from the continuum Luttinger liquid field theory is still lacking.

Marginal perturbations deserve separate discussion, because of the logarithmic corrections to the scaling. It is a well-known (and obvious) fact that log-corrections are hard to detect in numerical studies, and in fact, up to now no direct evidence for them has still been provided. However, for the isotropic Heisenberg antiferromagnet (that has a marginal operator) the relations found in [19] between entanglement entropy ($n=1$) and energy of the ground state (known to have a similar kind of log-corrections) is indirect evidence of the correctness of our prediction (cf equation (13)).

Finally we want to comment on the case of more than one interval, corresponding to more than two branch points in the $n$-sheeted Riemann surface. In this case, already the leading term (the analogue of equation (1)) is rather involved [39]–[42], but it is calculable for integer $n$ for the simplest CFTs [40, 41, 43]. Even in this case, the large corrections to the scaling prevent a direct simple analysis [40, 41] and the check of the complicated asymptotic forms. The exact knowledge of the correction to the scaling exponents would greatly simplify the analysis. Up to now, it has been shown numerically for the Ising model, with periodic boundary conditions, and for $n = 2$ that the exponent is $1/2$ [43], compatible with a form such as $2x/n$ ($x = 1/2$ and $n = 2$). Exploiting the exact solvability of the model, an accurate analysis [44] shows also for other low values of $n = 3, 4$ corrections compatible with $1/n$. In the Ising model, the non-local fermion operator has dimension $1/2$ and could be the origin of such scaling. In fact, to build the reduced density of the spin degrees of freedom in the Ising model, one should introduce the non-local Jordan–Wigner string between the two intervals [42]–[44], which plays the rule of the Ising fermion. This does not enter in the single interval reduced density matrix that is the same for spin and fermion degrees of freedom. As a confirmation of this behaviour, we mention that, for fermions in the Ising model, we have the same correction as for the single interval $2/n$ [44], because the Jordan–Wigner string is not present. For the XX model, the same analysis [44] shows unambiguously that the exponent is exactly $2/n$. Thus all evidence confirms that also for two intervals the correction to the scaling exponent is of the form $2x/n$, even if the example of the Ising model shows that some care is needed to fix the appropriate value of $x$. 

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References

[1] Amico L, Fazio R, Osterloh A and Vedral V, Entanglement in many-body systems, 2008 Rev. Mod. Phys. 80 517
Eisert J, Cramer M and Plenio M B, Area laws for the entanglement entropy—a review, 2010 Rev. Mod. Phys. 82 277
Calabrese P, Cardy J and Doyon B (ed), Entanglement entropy in extended systems, 2009 J. Phys. A: Math. Theor. 42 500301
[2] Calabrese P and Lefevre A, Entanglement spectrum in one-dimensional systems, 2008 Phys. Rev. A 78 032329
[3] Schuch N, Wolf M M, Verstraete F and Cirac J I, Entropy scaling and simulability by matrix product states, 2008 Phys. Rev. Lett. 100 030504
Perez-Garcia D, Verstraete F, Wolf M M and Cirac J I, Matrix product state representations, 2007 Quantum Inf. Comput. 7 401
Tagliazucchi L, de Oliveira T R, Iblisdir S and Latorre J I, Scaling of entanglement support for matrix product states, 2008 Phys. Rev. B 78 024410
Pollmann F, Mukerjee S, Turner A M and Moore J E, Theory of finite-entanglement scaling at one-dimensional quantum critical points, 2009 Phys. Rev. Lett. 102 255701
Verstraete F and Cirac J I, Renormalization and tensor product states in spin chains and lattices, 2009 J. Phys. A: Math. Theor. 42 504004
[4] Holzhey C, Larsen F and Wilczek F, Geometric and renormalized entropy in conformal field theory, 1994 Nucl. Phys. B 424 443
[5] Calabrese P and Cardy J, Entanglement entropy and quantum field theory, 2004 J. Stat. Mech. P06002
Calabrese P and Cardy J, Entanglement entropy and conformal field theory, 2009 J. Phys. A: Math. Theor. 42 504005
[6] Vidal G, Latorre J I, Rico E and Kitaev A, Entanglement in quantum critical phenomena, 2003 Phys. Rev. Lett. 90 227902
Latorre J I, Rico E and Vidal G, Ground state entanglement in quantum spin chains, 2004 Quantum Inf. Comput. 4 048
[7] Jin B-Q and Korepin V E, Quantum spin chain, Toeplitz determinants and Fisher-Hartwig conjecture, 2004 J. Stat. Phys. 116 79
Its A R, Jin B-Q and Korepin V E, Entanglement in XY spin chain, 2005 J. Phys. A: Math. Gen. 38 2975
Francolini F, Its A R and Korepin V E, Renyi entropy of the XY spin chain, 2008 J. Phys. A: Math. Theor. 41 025302
[8] Keating J P and Mezzadri F, Entanglement in quantum spin chains, symmetry classes of random matrices, and conformal field theory, 2005 Phys. Rev. Lett. 94 050501
[9] Zhou H-Q, Barthel T, Fjaerestad J O and Schollwoeck U, Entanglement and boundary critical phenomena, 2006 Phys. Rev. A 74 050305
[10] De Chiara G, Montangero S, Calabrese P and Fazio R, Entanglement entropy dynamics in Heisenberg chains, 2006 J. Stat. Mech. P03001
Igloi F and Juhasz R, Exact relationship between the entanglement entropies of XY and quantum Ising chains, 2008 Europhys. Lett. 81 57003
[11] Latorre J I and Riera A, A short review on entanglement in quantum spin systems, 2008 J. Stat. Mech. P05018
[12] Nienhuis B, Campostrini M and Calabrese P, Entanglement, combinatorics and finite-size effects in spin-chains, 2009 J. Stat. Mech. P02063
[13] Alba V, Fagotti M and Calabrese P, Entanglement entropy of excited states, 2009 J. Stat. Mech. P10020
[14] Latorre J I and Riera A, A short review on entanglement in quantum spin systems, 2009 J. Phys. A: Math. Theor. 42 504002
Peschel I and Eisler V, Reduced density matrices and entanglement entropy in free lattice models, 2009 J. Phys. A: Math. Theor. 42 504003
[15] Gliozzi F and Tagliazucchi L, Entanglement entropy and the complex plane of replicas, 2010 J. Stat. Mech. P01002
[16] Calabrese P, Campostrini M, Essler F and Nienhuis B, Parity effects in the scaling of block entanglement in gapless spin chains, 2010 Phys. Rev. Lett. 104 095701

doi:10.1088/1742-5468/2010/04/P04023
Unusual corrections to scaling in entanglement entropy

[19] Laflorencie N, Sorensen E S, Chang M-S and Affleck I, *Boundary effects in the critical scaling of entanglement entropy in 1D systems*, 2006 Phys. Rev. Lett. 96 100603
Sorensen E S, Laflorencie N and Affleck I, *Entanglement entropy in quantum impurity systems and systems with boundaries*, 2009 J. Phys. A: Math. Theor. 42 504009

[20] Song H F, Rachel S and Le Hur K, *General relation between entanglement and fluctuations in one dimension*, 2010 arXiv:1002.0825

[21] Fisher M E and Barber M N, *Scaling theory for finite-size effects in the critical region*, 1972 Phys. Rev. Lett. 28 1516

[22] Cardy J L, *Conformal invariance and surface critical behaviour*, 1984 Nucl. Phys. B 240 514

[23] Zamolodchikov A B, *Irreversibility of the flux of the renormalization group in a 2D field theory*, 1986 JETP Lett. 43 731

Zamolodchikov A B, 1986 Pisma Zh. Eksp. Teor. Fiz. 43 565 (Engl. Transl.)

[24] Cardy J L, Castro-Alvaredo O A and Doyon B, *Form factors of branch-point twist fields in quantum integrable models and entanglement entropy*, 2007 J. Stat. Phys. 130 129

[25] Cardy J and Peschel I, *Finite-size dependence of the free energy in two-dimensional critical systems*, 1988 Nucl. Phys. B 300 377

[26] Cardy J L, *Conformal invariance and statistical mechanics*, 1988 Les Houches Lectures

[27] Diehl H W, *The theory of boundary critical phenomena*, 1986 Phase Transitions and Critical Phenomena vol 10, ed C Domb and J L Lebowitz (London: Academic)

Diehl H W, *The theory of boundary critical phenomena*, 1997 Int. J. Mod. Phys. B 11 3503

[28] Cardy J L, *Logarithmic corrections to finite-size scaling in strips*, 1986 J. Phys. A: Math. Gen. 19 L109

Cardy J L, 1987 J. Phys. A: Math. Gen. 20 5039 (erratum)

[29] Cardy J L and Ludwig A, *Perturbative evaluation of the conformal anomaly at new critical points with applications to random systems*, 1987 Nucl. Phys. B 285 687

[30] Cardy J, 1996 *Scaling and Renormalization in Statistical Physics* (Cambridge Lecture Notes in Physics) (Cambridge: Cambridge University Press)

[31] Cardy J, *Perturbative corrections to the conformal anomaly in two-dimensional field theories*, in preparation

[32] Casini H and Huerta M, *A finite entanglement entropy and the c-theorem*, 2004 Phys. Lett. B 600 142

[33] Cardy J L, *Critical behaviour at an edge*, 1983 J. Phys. A: Math. Gen. 16 3617

[34] Barber M N, Peschel I and Pearce P A, *Magnetization at corners in 2-dimensional Ising models*, 1984 J. Stat. Phys. 37 497

Peschel I, *Some more results for the Ising square lattice with a corner*, 1985 Phys. Lett. A 110 313

[35] Aleraç F C and Rittenberg V, *Shared information in stationary states at criticality*, 2010 J. Stat. Mech. P03024

[36] Kallin A B, González I, Hastings M B and Melko R, *Valence bond and von Neumann entanglement entropy in Heisenberg ladders*, 2009 Phys. Rev. Lett. 103 117203

Hastings M B, Gonzalez I, Kallin A B and Melko R G, *Measuring Renyi entanglement entropy with infinite matrix product states, conformal field theory and the Haldane–Shastry model*, 2010 Phys. Rev. B 81 104431

Xavier J C, *Entanglement entropy, conformal invariance and the critical behavior of the anisotropic spin-S Heisenberg chains: a DMRG study*, 2010 arXiv:1002.2335

[37] Calabrese P and Essler F H L, in preparation

[38] Legeza O, Solyom J, Tincani L and Noack R M, *Entanglement entropy in quantum impurity systems and systems with boundaries*, 2009 J. Phys. A: Math. Theor. 42 504009

[39] Alcaraz F C and Rittenberg V, *Large corrections to finite-size scaling in strips*, 1986 J. Phys. A: Math. Gen. 19 L109

Cardy J L, 1987 J. Phys. A: Math. Gen. 20 5039 (erratum)

[40] Cardy J and Peschel I, *Finite-size dependence of the free energy in two-dimensional critical systems*, 1988 Nucl. Phys. B 300 377

[41] Cardy J L, *Conformal invariance and surface critical behaviour*, 1984 Nucl. Phys. B 240 514

[42] Cardy J L, *Critical behaviour at an edge*, 1983 J. Phys. A: Math. Gen. 16 3617

[43] Barber M N, Peschel I and Pearce P A, *Magnetization at corners in 2-dimensional Ising models*, 1984 J. Stat. Phys. 37 497

Peschel I, *Some more results for the Ising square lattice with a corner*, 1985 Phys. Lett. A 110 313

[35] Aleraç F C and Rittenberg V, *Shared information in stationary states at criticality*, 2010 J. Stat. Mech. P03024

[36] Kallin A B, González I, Hastings M B and Melko R, *Valence bond and von Neumann entanglement entropy in Heisenberg ladders*, 2009 Phys. Rev. Lett. 103 117203

Hastings M B, Gonzalez I, Kallin A B and Melko R G, *Measuring Renyi entanglement entropy with infinite matrix product states, conformal field theory and the Haldane–Shastry model*, 2010 Phys. Rev. B 81 104431

Xavier J C, *Entanglement entropy, conformal invariance and the critical behavior of the anisotropic spin-S Heisenberg chains: a DMRG study*, 2010 arXiv:1002.2335

[37] Calabrese P and Essler F H L, in preparation

[38] Legeza O, Solyom J, Tincani L and Noack R M, *Entanglement entropy in quantum impurity systems and systems with boundaries*, 2009 J. Phys. A: Math. Theor. 42 504009

[39] Alcaraz F C and Rittenberg V, *Large corrections to finite-size scaling in strips*, 1986 J. Phys. A: Math. Gen. 19 L109

Cardy J L, 1987 J. Phys. A: Math. Gen. 20 5039 (erratum)

[40] Cardy J and Peschel I, *Finite-size dependence of the free energy in two-dimensional critical systems*, 1988 Nucl. Phys. B 300 377

[41] Cardy J L, *Conformal invariance and surface critical behaviour*, 1984 Nucl. Phys. B 240 514

[42] Cardy J L, *Critical behaviour at an edge*, 1983 J. Phys. A: Math. Gen. 16 3617

[43] Barber M N, Peschel I and Pearce P A, *Magnetization at corners in 2-dimensional Ising models*, 1984 J. Stat. Phys. 37 497

Peschel I, *Some more results for the Ising square lattice with a corner*, 1985 Phys. Lett. A 110 313

[35] Aleraç F C and Rittenberg V, *Shared information in stationary states at criticality*, 2010 J. Stat. Mech. P03024

[36] Kallin A B, González I, Hastings M B and Melko R, *Valence bond and von Neumann entanglement entropy in Heisenberg ladders*, 2009 Phys. Rev. Lett. 103 117203

Hastings M B, Gonzalez I, Kallin A B and Melko R G, *Measuring Renyi entanglement entropy with infinite matrix product states, conformal field theory and the Haldane–Shastry model*, 2010 Phys. Rev. B 81 104431

Xavier J C, *Entanglement entropy, conformal invariance and the critical behavior of the anisotropic spin-S Heisenberg chains: a DMRG study*, 2010 arXiv:1002.2335

[37] Calabrese P and Essler F H L, in preparation

[38] Legeza O, Solyom J, Tincani L and Noack R M, *Entanglement entropy in quantum impurity systems and systems with boundaries*, 2009 J. Phys. A: Math. Theor. 42 504009