STABLE STRATA OF GEODESICS IN OUTER SPACE

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Abstract. In this paper we propose an Outer space analogue for the principal stratum of the unit tangent bundle to the Teichmüller space \( T(S) \) of a closed hyperbolic surface \( S \). More specifically, we focus on properties of the geodesics in Teichmüller space determined by the principal stratum. We show that the analogous Outer space “principal” periodic geodesics share certain stability properties with the principal stratum geodesics of Teichmüller space. We also show that the stratification of periodic geodesics in Outer space exhibits some new pathological phenomena not present in the Teichmüller space context.

1. Introduction

Let \( S \) be a closed oriented surface of genus \( \geq 2 \) and let \( T(S) \) be the Teichmüller space of \( S \). Recall that the unit (co)tangent bundle to \( T(S) \) is canonically identified with the space \( Q^1(S) \) of unit area holomorphic quadratic differentials on \( S \). The space \( Q^1(S) \) has a natural stratification, invariant under the Teichmüller geodesic flow, according to the orders of zeros of a quadratic differential, with the principal stratum \( Q_{\text{princ}}^1(S) \) consisting of quadratic differentials where all zeros are simple. This stratification of \( Q^1(S) \) defines the corresponding stratification of the space \( \mathcal{G}T(S) \) of all bi-infinite directed Teichmüller geodesics in \( T(S) \), by looking at the unit tangent vector to \( L \in \mathcal{G}T(S) \) at some (equivalently, any) point \( X \in L \). Thus we also get a subset \( \mathcal{G}_{\text{princ}}T(S) \subseteq \mathcal{G}T(S) \) consisting of all Teichmüller geodesics \( L \in \mathcal{G}T(S) \) with defining tangent vectors in \( Q_{\text{princ}}^1(S) \).

By the classic work of Thurston, the space \( PMF(S) \) of projective measured foliations on \( S \) can be viewed as the Thurston boundary \( \partial T(S) \) of \( T(S) \). Note that there is no canonical stratification of \( \partial T(S) \) corresponding to the stratification of \( Q^1(S) \) discussed above. Indeed, one can show that for every point \( [\mu] \in \partial T(S) \) there exists some Teichmüller geodesic ray \( \rho \subseteq T(S) \) such that the initial tangent vector of \( \rho \) belongs to \( Q_{\text{princ}}^1(S) \). However, for a pseudo-Anosov \( g \in MCG(S) \) there is a well-defined notion of \( g \) being principal, which corresponds to the bi-infinite \( g \)-periodic geodesic in \( T(S) \) being defined by a tangent vector from \( Q_{\text{princ}}^1(S) \), or equivalently, to the stable foliation \( [\mu_+(g)] \) coming from a quadratic differential in the principal stratum \( Q_{\text{princ}}^1(S) \).

An important result of Kaimanovich and Masur [KM96] concerns the boundary behavior of a random walk on the mapping class group \( MCG(S) \), satisfying some mild restrictions. They proved that for every basepoint \( X \in T(S) \), for almost every trajectory \( \omega \) of this random walk, projecting \( \omega \) to \( T(S) \) from the basepoint \( X \) via the orbit map gives a sequence in \( T(S) \) that converges to a uniquely ergodic point of \( \partial T(S) = PMF(S) \). Thus we get an
exit measure $\nu_X$ on $\partial T(S)$ for the projected random walk in $T(S)$ starting at $X$. This exit measure is supported on the set $UE \subseteq PMF(S)$ of uniquely ergodic projective measured foliations. Maher [Mah11] later showed that, again under some mild assumptions on the random walk, the element $g_n \in MCG(S)$, obtained after $n$ steps of the walk, is pseudo-Anosov with probability tending to 1 and $n \to \infty$. (Rivin [Riv08] had earlier proved the same conclusion about $g_n$ for the simple random walk on $MCG(S)$ corresponding to a finite generating set of $MCG(S)$.) However, until recently, little else has been known about the properties of a “random” point of $\partial T(S)$ corresponding to the exit measure $\nu_X$, or about the properties of the stable foliation $\mu_+(g_n)$ of the pseudo-Anosov $g_n$ as above.

In [GM16], Gadre and Maher shed light on these questions. They proved that if the support of a random walk on $MCG(S)$ is “sufficiently large” and contains a principal pseudo-Anosov $g$, then for every $X \in T(S)$ and for $\nu_X$-a.e. point $x \in UE \subseteq T(S)$, the Teichmüller geodesic from $X$ to $x$ has its initial tangent vector in $Q^1_{princ}(S)$. They also proved that in this setting, with probability tending to 1 as $n \to \infty$, after $n$ steps the random walk produces a principal pseudo-Anosov $g_n \in MCG(S)$. Gadre and Maher also obtained the following stability result for principal axes. For $X, Y$ in the axis $L_g \subseteq T(S)$ of $g$ and for $R \geq 0$, denote by $\Gamma_R(X, Y)$ the collection of all oriented bi-infinite Teichmüller geodesics $L \subseteq T(S)$ with uniquely ergodic vertical and horizontal foliations such that $B(X, R) \cap L \neq \emptyset$, $B(Y, R) \cap L \neq \emptyset$ and such that the first point in $L \cap (B(X, R) \cup B(Y, R))$ belongs to $B(X, R)$. Here the balls $B(X, R)$ and $B(Y, R)$ are taken with respect to the Teichmüller metric on $T(S)$. A crucial ingredient in the proof of the main results of [GM16] is the following “stability” property of $L_g$ for a principal pseudo-Anosov $g$; see [GM16] Proposition 2.7:

**Theorem 1.1** (Gadre-Maher). Let $S$ be a closed oriented surface of genus $\geq 2$ and let $g \in MCG(S)$ be a principal pseudo-Anosov. Then for any $R \geq 0$ there exists $D > 0$ such that if $X, Y \in L_g$ have $d(X, Y) \geq D$, then every $L \in \Gamma_R(X, Y)$ is principal.

We are interested in investigating the corresponding questions in the $Out(F_r)$ setting, where $r \geq 2$. Similar to the mapping class group setting, it is by now well-known [MT14] that, under some mild assumption on the support, for a random walk on $Out(F_r)$, an element $\varphi_n \in Out(F_r)$ obtained after $n$ steps of the walk is atoroidal fully irreducible with probability tending to 1 as $n \to \infty$. It is also known that projecting a random orbit of this walk to the Culler-Vogtmann Outer space $CV_r$ (starting at some basepoint $X \in CV_r$) gives a sequence in $CV_r$ that with probability 1 converges to some point in $[T] \in \partial CV_r$ and, moreover, that the $\mathbb{R}$-tree $T$ is uniquely ergodic [NPR14]. The proofs that $\varphi_n$ is generically fully irreducible and atoroidal involve projecting a random walk on $Out(F_r)$ to the free factor complex in the first case and to the co-surface graph in the second case. These are both Gromov hyperbolic and one argues that $\varphi_n$ acts loxodromically on the hyperbolic graph in question. Addressing the index properties, Kapovich and Pfaff [KP15] proved that for a “train-track directed” random walk on $Out(F_r)$, the element $\varphi_n$ is, with an asymptotically positive probability, an ageometric fully irreducible outer automorphism with a 1-element index list $\{\frac{r}{2} - r\}$ and that the corresponding ideal Whitehead graph is complete (the relevant definitions are discussed next below). We wish to understand how this statement generalizes to the case of a more general random walk on $Out(F_r)$.

One of the difficulties in the $Out(F_r)$ setting is finding a suitable notion of a “principal stratum.” In the original context of a closed hyperbolic surface $X$, if $[\mu] \in PMF(S)$ is uniquely ergodic and with the dual $\mathbb{R}$-tree having all branch-points being trivalent, then
[μ] ∈ Q_{princ}. This fact motivates us to use the index properties of a geodesic in Outer space when defining strata in the space of such geodesics.

Given a nongeometric fully irreducible φ ∈ Out(F_r) (where “nongeometric” means that φ is not induced by a homeomorphism of a surface S with π_1(S) ∼= F_r) one can define a conjugacy class invariant called the ideal Whitehead graph IW(φ) of φ (see §2.6 for details). The graph IW(φ) captures essential information about the structure of the attracting lamination of φ and therefore of branch-points of the stable R-tree T_φ of φ (as well as about the interaction of “directions” in T_φ at those branch-points). The graph IW(φ) can be read-off, via an explicit procedure, from any train track representative of φ. In addition, one can also define the index list for φ (recording the sizes of components of IW(φ)), and the index sum i(φ), obtained by summing up the numbers in the index list of φ. Unlike in the surface case, there may be many types of Ideal Whitehead graphs with the same index list and we shall see that stability properties are related to the graph types rather than the index lists (or sums). We say that a finite graph G is r-dominant if G is a union of complete graphs, each with ≥ 3 vertices, and if the index sum of G is \( \frac{3}{2} - r \). Of special interest is the r-dominant graph all of whose components are triangles, we denote it ∆_r.

Let r ≥ 3 and let CV_r denote the (projectivized) Culler-Vogtmann Outer space for the free group F_r. We denote by F_r the set of all bi-infinite fold lines in CV_r, where folds are performed one at a time (see Definition 2.16 below for the precise formulation). Note that all elements of F_r are bi-infinite geodesics for the asymmetric Lipschitz metric on CV_r. We denote by A_r the set of all “axes,” i.e. the set of all L ∈ F_r such that L is a periodic fold line defined by an expanding irreducible train track representative of some φ ∈ Out(F_r). We endow F_r with a natural topology, where for L, L’ ∈ F_r, the line L’ is “close” to L if there exist a “large” R ≥ 1 and a “small” ε > 0 such that some subsegment J of L’ of length R is contained in the ε-neighborhood of L (with respect to the symmetrized Lipschitz metric on CV_r). See Definition 3.2 below for details. All of the various subsets of F_r discussed below are then given the subspace topology.

**Definition 1.2** (Dominant and principal strata, and their basins). Let r ≥ 3 and let G be a graph. We define the G-basin BS_r(G) ⊆ A_r as the set of all L ∈ A_r such that L is φ-periodic for φ ∈ Out(F_r) agemetric fully irreducible and satisfying that IW(φ) is a union of components of G. We define the G-stratum S_r(G) ⊆ BS_r(G) as those lines for which the corresponding φ satisfies IW(φ) ∼= G. Thus S_r(G) ⊆ BPr_r(G) ⊆ A_r.

If G is an r-dominant (resp. r-principal) graph, we say S_r(G) is a dominant stratum (resp. P_r, the r-principal stratum) and BS_r(G) is dominant (resp. BPr_r is the principal basin).

The results of Mosher-Pfaff [MP16] imply that if φ ∈ Out(F_r) is G-dominant for some r-dominant graph G, then φ is a “lone axis” fully irreducible outer automorphism, i.e. φ has a unique axis in CV_r. In particular, this fact applies to all principal φ ∈ Out(F_r).

Recall that CV_r is a simplicial complex of dimension 3r − 4 (with some faces missing). For an integer k ≥ 0, we will denote by CV_r^{(k)} the k-skeleton of CV_r. Our main result is the following attracting/stability property for dominant strata:

**Theorem A.** Let r ≥ 3 and let G be an r-dominant graph. Let L ∈ S_r(G). Then there exist 0 ≤ k ≤ 3r − 4 and a neighborhood U ⊆ A_r of L in A_r with the following properties:

(a) For each L’ ∈ U with L’ ⊆ CV_r^{(k)}, we have L’ ∈ BS_r(G).
(b) For each \( L' \in U \) with \( L' \subseteq CV_r^{(k)} \) and with \( L' \) containing no full folds, we have \( L' \in S_r(\mathcal{G}) \).

See Definition 2.18 for terminology regarding full folds. Note, the conclusion of Theorem A implies that each \( L' \in U \) is an axis of an ageometric fully irreducible element of \( \text{Out}(F_r) \).

Moreover, in the case of (b), \( L' \) is the unique axis of that fully irreducible in \( CV_r^{(k)} \) ([MP16]).

Our results suggest that for a reasonable random walk on \( \text{Out}(F_r) \), for a random fully irreducible \( \varphi_n \in \text{Out}(F_r) \) obtained after \( n \) steps of the walk, there are several possibilities for \( IW(\varphi_n) \) that each occur with an asymptotically positive probability as \( n \to \infty \).

**Question 1.3. Is the conclusion of Theorem A true only for an \( r \)-dominant \( \mathcal{G} \)?**

It turns out that it is, in general, not possible to replace \( BS_r(\mathcal{G}) \) by \( S_r(\mathcal{G}) \) in the conclusion of Theorem A(a) above. We show that certain kinds of pathologies exist that can force \( L' \in U \) to fall out of the dominant \( \mathcal{G} \)-stratum and that the best one can conclude is that \( L' \in BS_r(\mathcal{G}) \):

**Theorem B.** There exists a principal fully irreducible outer automorphism \( \varphi \in \text{Out}(F_3) \) with a train track representative \( f : \Gamma \to \Gamma \) with a Stallings fold decomposition \( \mathfrak{f} \), such that for every \( n \geq 1 \) there exists a nonprincipal fully irreducible outer automorphism \( \psi_n \in \text{Out}(F_3) \) with a train track representative \( g_n : \Gamma \to \Gamma \) with a Stallings fold decomposition \( \mathfrak{g}_n \) such that \( g_n \) starts with \( f^n \).

Theorem B immediately implies:

**Corollary C.** For \( r = 3 \), there exist a principal periodic geodesic \( L \in \mathcal{P}_r \) in \( CV_r \) and a sequence of nonprincipal periodic geodesics \( \{ L_n \}_{n=1}^\infty \subseteq B\mathcal{P}_r - \mathcal{P}_r \) such that \( \lim_{n \to \infty} L_n = L \).

The cause of the pathologies exhibited in Theorem B is that the folding process may identify vertices. Hence, some \( f \)-periodic vertices of \( \Gamma \) may become nonperiodic for \( g_n \). Such vertices contribute to \( IW(\varphi) \) but not to \( IW(\psi_n) \).

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2. **Background & Definitions**

Given a free group \( F_r \) of rank \( r \geq 2 \), we choose once and for all a free basis \( A = \{ X_1, \ldots, X_r \} \). Let \( R_r = \vee_{i=1}^r S^1 \) denote the graph with one vertex and \( r \) edges. We choose also once and for all an orientation on \( R_r \) and identify each positive edge of \( R_r \) with an element of the chosen free basis. Thus, a cyclically reduced word in the basis corresponds to an immersed loop in \( R_r \).

2.1. **Outer space \( CV_r \).**

**Definition 2.1** (Marked metric \( F_r \)-graph). Let \( r \geq 2 \) be an integer. A marked metric graph for \( F_r \) is a triple \( (\Gamma, m, \ell) \) which satisfies:

- \( \Gamma \) is a finite 1-dimensional CW complex, with the 0-cells *vertices* and the 1-cells *edges*.
- For each vertex \( v \), \( \text{deg}(v) \geq 3 \).
Each open 1-cell \( e \) of \( \Gamma \) is given a positive length \( L(e) > 0 \), and \( \Gamma \) is endowed with a metric \( \ell \) such that for each open 1-cell \( e \) of \( \Gamma \) there is a locally isometric bijection between \( e \) and the interval \((0, L(e)) \subseteq \mathbb{R}\).

- \( m \) is a homotopy equivalence \( m: R_r \to \Gamma \), which we call a marking.

**Notation 2.2.** We use the following notation.

1. We sometimes write \((\Gamma, m)\) for \((\Gamma, m, \ell)\) if the metric is otherwise clear or irrelevant.
2. Given a graph \( \Gamma \) (metric or topological), we let \( E(\Gamma) \) denote the set of oriented edges of \( \Gamma \) and let \( V(\Gamma) \) denote the vertex set of \( \Gamma \).

**Definition 2.3** (Change of marking & marked graph equivalence). Let \((\Gamma, m)\) and \((\Gamma', m')\) be marked graphs. Then a change of marking is a continuous map \( f: \Gamma \to \Gamma' \) so that \( m' \) is homotopic to \( f \circ m \). Two \( F_r \)-marked (metric) graphs \((\Gamma, m)\) and \((\Gamma', m')\) are equivalent if there exists an isometric change of marking \( \varphi: \Gamma \to \Gamma' \).

**Definition 2.4** (Unprojectivized Outer space). The \( (\text{rank-}r) \) unprojectivized Outer space \( \hat{CV}_r \) is the space of equivalence classes of \( F_r \)-marked metric graphs. By abuse of notation, we usually still denote the equivalence class of \((\Gamma, m)\) by \((\Gamma, m)\), or of \((\Gamma, m, \ell)\) by \((\Gamma, m, \ell)\).

For a marked metric graph \((\Gamma, m)\) denote by \( vol(\Gamma, m, \ell) \), or just \( vol(\Gamma) \), the sum of the \( \ell \)-lengths of the 1-cells in \( \Gamma \). Note that \( vol(\Gamma, m, \ell) \) is preserved by the above equivalence relation, so that \( vol(\Gamma, m) \) is well-defined for points of \( \hat{CV}_r \).

**Definition 2.5** (Projectivized) Outer space). Let \( r \geq 2 \) be an integer. For each \( r \geq 2 \) the \( (\text{rank-}r) \) Outer space \( CV_r \) is the set of \((\Gamma, m, \ell) \in \hat{CV}_r \) with \( vol(\Gamma, m, \ell) = 1 \). There is a map from \( q: \hat{CV}_r \to CV_r \) normalizing the graph volume, i.e. if \((\Gamma, m, \ell)\) is a marked metric graph, then \( q(\Gamma, m, \ell) = (\Gamma, m, \frac{1}{vol(\Gamma, m, \ell)} \ell) \).

Note that \( \mathbb{R}_{>0} \) has a natural action on \( \hat{CV}_r \) by multiplying the metric on \( \Gamma \) by a positive real number. There is a canonical identification between \( CV_r \) and the quotient set \( \hat{CV}_r / \mathbb{R}_{>0} \) and we will usually not distinguish between these two sets.

**Definition 2.6** (Simplicial structure on \( CV_r \)). Let \( \Gamma \) be a topological graph and \( m: R_r \to \Gamma \) a homotopy equivalence, so that \((\Gamma, m)\) is a marked graph. We denote the simplex \( \sigma \) in \( CV_r \) corresponding to \((\Gamma, m)\) by

\[
\sigma_{(\Gamma, m)} := \{(\Gamma, m, \ell) \in CV_r\}.
\]

By enumerating \( E(\Gamma) \), we can identify \( \sigma_{(\Gamma, m)} \) with the open simplex

\[
\Sigma_{|E|} = \left\{ \overrightarrow{v} \in \mathbb{R}_+^{|E|} \left| \sum_{i=1}^{|E|} v_i = 1 \right. \right\}.
\]

**Definition 2.7** \( CV_r^{(k)} \). We let \( CV_r^{(k)} \) denote the \( k \)-skeleton of \( CV_r \).

**Definition 2.8** (Simplicial metric). Given an open simplex \( \sigma_{(\Gamma, m)} \) in \( CV_r \), the simplicial metric on \( \sigma_{(\Gamma, m)} \) is the Euclidean metric on \( \Sigma_{|E|} \). We also denote by \( d_{simp} \) the extension of this metric to a path metric on \( CV_r \). (There is another (asymmetric) metric on Outer space see Definition 2.21).

**Definition 2.9** (Topology on \( \hat{CV}_r \)). We call the full preimage under \( q \) (see Definition 2.4) of a simplex in \( CV_r \) an unprojectivized simplex in \( \hat{CV}_r \). The unprojectivized Outer space
\( \widehat{CV}_r \) is topologized by giving it the the structure of an ideal simplicial complex built from (unprojectivized) open simplices (see [Vog02] for details). Faces of \( \sigma_{(\Gamma,m)} \) arise by letting the edges of a tree in \( \Gamma \) have length 0. The projectivized outer space \( CV_r \subset \widehat{CV}_r \) is given the subspace topology, and can also be thought of as an ideal simplicial complex built from open simplices. The subspace topology on \( CV_r \) coincides with the quotient topology on \( \widehat{CV}_r/\mathbb{R}_{>0} \).

### 2.2. Train track maps & gate structures.

**Definition 2.10** (Graph maps & train track maps). We call a continuous map of graphs \( g : \Gamma \to \Gamma' \) a **graph map** if it takes vertices to vertices and is locally injective on the interior of each edge. A self graph map \( g : \Gamma \to \Gamma \) is a **train track map** if \( g \) is a homotopy equivalence and if for each \( k \geq 1 \) the map \( g^k \) is locally injective on edge interiors.

We call the train track map \( g \) **expanding** if for each edge \( e \in E(\Gamma) \) we have that \( |g^n(e)| \to \infty \) as \( n \to \infty \), where for a path \( \gamma \) we use \( |\gamma| \) to denote the number of edges \( \gamma \) traverses (with multiplicity).

**Definition 2.11** (Directions). For each \( x \in \Gamma \) we let \( D(x) \) denote the set of **directions** at \( x \), i.e. germs of initial segments of edges emanating from \( x \). For each edge \( e \in E(\Gamma) \), we let \( D(e) \) denote the initial direction of \( e \). For an edge-path \( \gamma = e_1 \ldots e_k \), we let \( D\gamma = D(e_1) \).

Let \( g : \Gamma \to \Gamma' \) be a graph map. We denote by \( Dg \) the map of directions induced by \( g \), i.e. \( Dg(d) = D(g(e)) \) for \( d = D(e) \). For a self-map, i.e. one where \( \Gamma = \Gamma' \), a direction \( d \) is **periodic** if \( Dg^k(d) = d \) for some \( k > 0 \) and **fixed** when \( k = 1 \).

**Definition 2.12** (Turns & gates). Let \( g : \Gamma \to \Gamma' \) be a graph map. We call an unordered pair of directions \( \{d_i, d_j\} \) a **turn**, and a **degenerate turn** if \( d_i = d_j \). We denote by \( Tg \) the map induced by \( Dg \) on the turns of \( \Gamma \). A turn \( \tau \) is called \( g \)-**prenull** if \( Tg(\tau) \) is degenerate. When \( g : \Gamma \to \Gamma \) is a self-map, the turn \( \tau \) is called an **illegal turn** for \( g \) if \( Tg^k(\tau) \) is degenerate for some \( k \) and a **legal turn** otherwise. We call a \( g \) **transparent** if each illegal turn is prenull. Notice that every graph self-map has a transparent power.

Considering the directions of an illegal turn equivalent, one can define an equivalence relation on the set of directions at a vertex. We call the equivalence classes **gates** and call the partitioning of the directions at each vertex into gates the **induced gate structure**.

For a path \( \gamma = e_1e_2 \ldots e_{k-1}e_k \) in \( \Gamma \) where \( e_1 \) and \( e_k \) may be partial edges, we say \( \gamma \) takes \( \{\overline{e}_i, e_{i+1}\} \) for each \( 1 \leq i < k \). For both edges and paths we more generally use an “overline” to denote a reversal of orientation. Given a graph map \( g : \Gamma \to \Gamma' \), we say that a turn \( T \) in \( \Gamma' \) is \( g \)-**taken** if there exists an edge \( e \) so that \( g(e) \) takes \( T \). A path \( \gamma \) is **legal** with respect to a train track structure on \( \Gamma \) if \( \gamma \) only takes turns that are legal in this train track structure.

**Definition 2.13** (Irreducible & fully irreducible). We call a train track map **irreducible** if it has no proper invariant subgraph with a noncontractible component.

An outer automorphism \( \varphi \in \text{Out}(F_r) \) is **fully irreducible** if no positive power preserves the conjugacy class of a proper free factor of \( F_r \). Bestvina and Handel [BH92] proved that every (fully) irreducible outer automorphism admits an irreducible train track representatives.

**Definition 2.14** (Transition matrix, Perron-Frobenius matrix, Perron-Frobenius eigenvalue). The **transition matrix** of a train track map \( g : \Gamma \to \Gamma \) is the square \( |E(\Gamma)| \times |E(\Gamma)| \) matrix \( (a_{ij}) \) such that \( a_{ij} \), for each \( i \) and \( j \), is the number of times \( g(e_i) \) passes over \( e_j \) in either direction. A transition matrix \( A = [a_{ij}] \) is **Perron-Frobenius (PF)** if there exists an \( N \) such
that, for all \( k \geq N \), \( A^k \) is strictly positive. By Perron-Frobenius theory, we know that each such matrix has a unique eigenvalue of maximal modulus and that this eigenvalue is real. This eigenvalue is called the \textit{Perron-Frobenius (PF) eigenvalue} of \( A \).

2.3. Fold lines.

**Definition 2.15** (Fold lines). A \textit{fold line} in \( \hat{CV} \) is a continuous, injective, proper function \( \mathbb{R} \to \hat{CV} \), defined by a continuous 1-parameter family of marked graphs \( t \to \Gamma_t \) and a family of differences of markings \( h_{st} : \Gamma_t \to \Gamma_s \) defined for \( s \leq t \in \mathbb{R} \), satisfying:

1. \( h_{ts} \) is a local isometry on each edge for all \( s \leq t \in \mathbb{R} \).
2. \( h_{us} \circ h_{st} = h_{st} \) for all \( t \leq s \leq u \in \mathbb{R} \) and \( h_{ss} : \Gamma_s \to \Gamma_s \) is the identity for all \( s \in \mathbb{R} \).

A fold line in \( CV \) is the \( q \)-image (where \( q \) is the normalizing map, see Definition 2.5) of a fold line. We shall denote \( q(\hat{\Gamma}_t) \) and \( q \circ h_{s,t} \) by \( \Gamma_t \) and \( h_{s,t} \) respectively.

**Definition 2.16** (Simple fold lines). A fold line in Outer space \( \mathbb{R} \to CV \) is said to be \textit{simple} if there exists a subdivision of \( \mathbb{R} \) by points \( (t_i)_{i \in \mathbb{Z}} \)

\[ \ldots t_{i-1} < t_i < t_{i+1} \ldots \]

such that \( \lim_{t \to \infty} t_i = \infty, \lim_{t \to -\infty} t_i = -\infty \) and such that the following holds:

For each \( i \in \mathbb{Z} \) there exist distinct edges \( e, e' \) in \( \Gamma_{t_i} \), with a common initial vertex, such that: For each \( s \in (t_i, t_{i+1}] \), the map \( h_{s,t} : \Gamma_{t_i} \to \Gamma_s \) identifies an initial segment of \( e \) with an initial segment of \( e' \), with no other identifications (that is, \( h_{s,t} \) is injective on the complement of those two initial segments in \( \Gamma_t \)).

**Remark 2.17** (Simple fold lines). All fold lines that we consider in this paper will be simple.

**Definition 2.18** (Stallings folds). Stallings introduced “folds” in \[Sta83\]. Let \( g : \Gamma \to \Gamma' \) be a homotopy equivalence of marked graphs. Let \( e_1' \subset e_1 \) and \( e_2' \subset e_2 \) be maximal, initial, nontrivial subsegments of edges \( e_1 \) and \( e_2 \) emanating from a common vertex and satisfying that \( g(e_1') = g(e_2') \) as edge paths and that the terminal endpoints of \( e_1' \) and \( e_2' \) are distinct points in \( g^{-1}(\mathcal{V}(\Gamma)) \). Redefine \( \Gamma \) to have vertices at the endpoints of \( e_1' \) and \( e_2' \) if necessary.

One can obtain a graph \( \Gamma_1 \) by identifying the points of \( e_1' \) and \( e_2' \) that have the same image under \( g \), a process we will call \textit{folding}.

Let \( \mathcal{F} \) be a fold of \( e_1 \) and \( e_2 \). We call \( \mathcal{F} \) a \textit{full fold} if the entirety of \( e_1 \) and \( e_2 \) are identified. We call \( \mathcal{F} \) a \textit{proper full fold} if only an initial subsegment of one of \( e_1 \) or \( e_2 \) is folded with the entirety of the other. We call \( \mathcal{F} \) a \textit{partial fold} if neither \( e_1 \) nor \( e_2 \) is entirely folded.

**Definition 2.19** (Stallings fold decomposition). Stallings \[Sta83\] also showed that if \( g : \Gamma \to \Gamma' \) is a homotopy equivalence graph map, then \( g \) factors as a composition of folds and a final homeomorphism. We call such a decomposition a \textit{Stallings fold decomposition}. It can be obtained as follows: At an illegal turn for \( g : \Gamma \to \Gamma' \), one can fold two maximal initial segments having the same image in \( \Gamma' \) to obtain a map \( g_1 : \Gamma_1 \to \Gamma' \) of the quotient graph \( \Gamma_1 \). The process can be repeated for \( g_1 \) to obtain a map \( g_k : \Gamma_{k-1} \to \Gamma' \) has no illegal turn, then \( g_k \) will be a homeomorphism and the fold sequence is complete.

Notice that choices of illegal turns are made in this process and that different choices lead to different Stallings fold decompositions of the same homotopy equivalence.

When \( \Gamma \) is a marked metric graph (of volume 1), we obtain an induced metric on each \( \Gamma_k \), which we may renormalize to be again of volume 1.
Definition 2.20 (Periodic fold lines). Let \( g: \Gamma \to \Gamma \) be an expanding irreducible train track map representing an outer automorphism \( \varphi \in \text{Out}(F_r) \) and let \( \lambda \) be its Perron-Frobenius eigenvalue. If \( g_1, \ldots, g_k \) is a Stallings fold sequence for \( g \), the process of Skora defines a path \( L_0: [0, \log \lambda] \to CV_r \) so that the union of \( \varphi^k \)-translates of \( L_0 \) for all \( k \) gives the entire fold line \( L \) determined by \( g \), see Definition 2.19. That is, \( L: \mathbb{R} \to CV_r \) is defined by \( L(t) = L_0(t - \lfloor t / \log \lambda \rfloor) \varphi^{\lfloor t / \log \lambda \rfloor} \). \( L \) is called a periodic fold line for \( \varphi \) or, if \( \varphi \) is fully irreducible, an axis for \( \varphi \).

2.4. Geodesics in Outer space.

Definition 2.21 (Lipschitz metric). Given an ordered pair of points \((X, Y)\) in the Outer space \(CV_r\), the Lipschitz distance \( d(X, Y) \) from \( X = (\Gamma_X, m_X, \ell_X) \) to \( Y = (\Gamma_Y, m_Y, \ell_Y) \) is defined as the logarithm of the minimal Lipschitz constant of a Lipschitz difference of \( CV_r \)-space markedings. (It is known \[FM11\] this minimum is in fact realized and that \( d(X, Y) \geq 0 \), with \( d(X, Y) = 0 \) if and only if \( X = Y \) in \( CV_r \).) We sometimes also denote \( d(X, Y) \) by \( d_L(X, Y) \).

Let \( \alpha \) be an element of \( F_r \). We also denote by \( \alpha \) the corresponding loop in the base rose \( R_r \). Let \( X = (\Gamma, m) \) be a point in Outer space. Denote by \( \alpha_X \) the immersed loop (unique up to cyclic reparametrization) in \( X \) that is freely homotopic to \( m(\alpha) \).

Definition 2.22 (Witness). It is proved in \[FM11\] that for each ordered pair of points \((X, Y)\) in Outer space there exists an element \( \alpha \) of \( F_r \) so that \( \log \frac{\text{len}(\alpha_Y)}{\text{len}(\alpha_X)} = d(X, Y) \). We call each such \( \alpha \) a witness of \( d(X, Y) \) or of the change of marking from \( X \) to \( Y \).

Definition 2.23 (Candidate). Let \( X \in CV_r \). A loop in \( X \) whose image is an embedded circle, an embedded figure-8, or an embedded barbell is called a candidate of \( X \).

Lemma 2.24. \[FM11\] For each ordered pair of points \((X, Y)\) in Outer space there exists a candidate loop in \( X \) that is a witness of \( d(X, Y) \).

Definition 2.25 (Geodesic). A map \( L: [0, \ell] \to CV_r \) is a Lipschitz geodesic if
(1) for all \( s, t, r \in [0, \ell] \) so that \( s \leq t \leq r \) we have
\[
d(\mathcal{L}(s), \mathcal{L}(r)) = d(\mathcal{L}(s), \mathcal{L}(t)) + d(\mathcal{L}(t), \mathcal{L}(r))\]
and
\[
(2) \text{ there exists no } X_0 \in CV_r \text{ and no nontrivial subinterval } [a, b] \text{ of } [0, \ell] \text{ so that } \mathcal{L}(t) = X_0 \text{ for all } t \in [a, b].
\]

Lemma 2.26. Let \( \mathcal{L} = \{\Gamma_t\}^\infty_{t=0} \) be a Lipschitz geodesic ray. Then there exists an element \( \alpha \in F_r \) so that, for each \( t \geq 0 \), if \( \alpha_t \) denotes the immersed loop representing the conjugacy class of \( \alpha \) in \( \Gamma_t \), then \( \alpha_t \) is a witness to \( d(\Gamma_t, \Gamma_s) \) for each \( t \leq s \).

Proof. For each \( 0 \leq t \leq s \) we have \( d(\Gamma_0, \Gamma_s) = d(\Gamma_0, \Gamma_t) + d(\Gamma_t, \Gamma_s) \). Let \( f_{s,0}, f_{t,0}, f_{s,t} \) be optimal maps, i.e. maps that realize the equality in Definition 2.21 and let \( \alpha \) be a witness loop for \( d(\Gamma_0, \Gamma_s) \). Then \( \frac{\text{len}(\alpha_s)}{\text{len}(\alpha_0)} = \text{Lipschitz}(\alpha_s) \leq \text{Lipschitz}(\alpha_t) \). Hence, \( \alpha_0 \) is a witness for \( f_{t,0} \) for all \( t \leq s \) and \( \alpha_t \) is a witness for \( f_{s,t} \). Moreover, by Lemma 2.24, \( \alpha_0 \) can be chosen to be a candidate in \( \Gamma_0 \). Notice that there are only finitely many candidate loops in \( \Gamma_0 \). Let \( \Theta_s \) be the set of candidate loops that are \( f_{s,0} \)-witnesses. Then the sequence \( \{\Theta_s\}_s^\infty \) consists of a non-empty decreasing finite sets of loops, hence stabilizes as \( s \to \infty \). We let
\[
\Theta_\infty := \bigcap_{s=0}^\infty \Theta_s.
\]
Any \( \alpha_0 \in \Theta_\infty \) is a witness for each \( f_{s,0} \) with \( 0 \leq s \) and, by the discussion above, \( \alpha_t \) is a witness of \( f_{s,t} \) for each \( 0 \leq t \leq s \).

Lemma 2.27. Let \( g \) be an expanding train track representative of \( \varphi \in \text{Out}(F_r) \) and \( \mathcal{L} \) the periodic fold line determined by \( g \) as in Definition 2.20. Then \( \mathcal{L} \) is a Lipschitz geodesic.

Proof. For each interval \( [t, s] \subset \mathbb{R} \), let \( \mathcal{L}_{t,s} \) denote the restriction of \( \mathcal{L} \) to \( [t, s] \). It suffices to show \( \mathcal{L}_{t,s} \) is a geodesic segment for each pair \( t, s \) of positive integer multiples of \( \log \lambda \). For each such \( t, s \) the change of marking map from \( \mathcal{L}(t) \) to \( \mathcal{L}(s) \) is \( g^k \), where \( k = \frac{s-t}{\log \lambda} \).

Given \( \alpha \in F_r \), we denote by \( \alpha_t \) the immersed loop in \( \mathcal{L}(t) \) representing \( \alpha \). Let \( \beta \in F_r \) satisfy that \( \beta_t \) is \( g \)-legal. Then \( h_{u,t}(\beta_t) = \beta_u \) is immersed for each \( t \leq u \leq s \), thus it is a witness for \( d(\mathcal{L}(t), \mathcal{L}(u)) \). Moreover, \( \beta_u \) is still not \( h_{s,u} \)-prenull for each \( t \leq u \leq s \). Taking the logarithm of both sides of \( \frac{\ell_t(\beta_t)}{\ell(t)} = \frac{\ell_u(\beta_u)}{\ell(t)} \), we see that \( \mathcal{L} \) is a geodesic.

2.5. Nielsen paths & principal points.

Definition 2.28 (Nielsen paths). Let \( g : \Gamma \to \Gamma \) be an expanding irreducible train track map. Bestvina and Handel [BH92] define a nontrivial immersed path \( \rho \) in \( \Gamma \) to be a periodic Nielsen path (PNP) if, for some power \( R \geq 1 \), we have \( g^R(\rho) \cong \rho \) rel endpoints (and just a Nielsen path (NP) if \( R = 1 \)). An NP \( \rho \) is called indivisible (hence is an “iNP”) if it cannot be written as \( \rho = \gamma_1 \gamma_2 \), where \( \gamma_1 \) and \( \gamma_2 \) are themselves NPs.

Definition 2.29 (Ageometric). A fully irreducible outer automorphism is called ageometric if it has a train track representative with no NPs.

Bestvina and Handel describe in [BH92, Lemma 3.4] the structure of iNPs:
Lemma 2.30 ([BH92]). Let \( g : \Gamma \to \Gamma \) be an expanding irreducible train track map and \( \rho \) an iNP for \( g \). Then \( \rho = \bar{\rho}_1\rho_2 \), where \( \rho_1 \) and \( \rho_2 \) are nontrivial legal paths originating at a common vertex \( v \) and such that the turn at \( v \) between \( \rho_1 \) and \( \rho_2 \) is a nondegenerate illegal turn for \( g \).

Definition 2.31 (Principal points). Given a train track map \( g : \Gamma \to \Gamma \), following [HM11], we call a point principal that is either the endpoint of a PNP or is a periodic vertex with \( \geq 3 \) periodic directions. Thus, in the absence of PNPs, a point is principal if and only if it is a periodic vertex with \( \geq 3 \) periodic directions.

Definition 2.32 (Rotationless). An expanding irreducible train track map is called rotationless if each principal point and periodic direction is fixed and each PNP is of period one. By [FH11, Proposition 3.24], one then defines a fully irreducible \( \varphi \in \text{Out}(F_r) \) to be rotationless if some (equivalently, all) of its train track representatives is rotationless.

2.6. Whitehead graphs. The following definitions are in [HM11] and [MP16].

Definition 2.33 (Whitehead graphs & indices). Let \( g : \Gamma \to \Gamma \) be a train track map. The local Whitehead graph \( LW(v; \Gamma) \) at a point \( v \in \Gamma \) has a vertex for each direction at \( v \) and an edge connecting the vertices corresponding to a pair of directions \( \{d_1, d_2\} \) if the turn \( \{d_1, d_2\} \) is \( g^k \)-taken for some \( k \geq 0 \). The stable Whitehead graph \( SW(v; \Gamma) \) at a principal point \( v \) is the subgraph of \( LW(v; \Gamma) \) obtained by restricting to the periodic direction vertices.

Let \( g : \Gamma \to \Gamma \) be a PNP-free train track representative of a fully irreducible \( \varphi \in \text{Out}(F_r) \). Then the ideal Whitehead graph \( IW(\varphi) \) of \( \varphi \) is isomorphic to the disjoint union \( \bigsqcup_j SW(g; v_j) \) taken over all principal vertices. Justification of this being an outer automorphism invariant can be found in [HM11] [Pfa12].

Let \( \varphi \in \text{Out}(F_r) \) be fully irreducible. For each component \( C_i \) of \( IW(\varphi) \), let \( k_i \) denote the number of vertices of \( C_i \). Then the index sum is defined as \( i(\varphi) \) := \( \sum 1 - \frac{k_i}{2} \). Since the index sum can be computed as such from the ideal Whitehead graph, we can define an index sum for an ideal Whitehead graph, or in fact any graph. For a graph \( G \), we write the index sum as \( i(G) \). Writing the terms \( 1 - \frac{k_i}{2} \) as a list, we obtain the index list for \( \varphi \).

Remark 2.34. By [GJLL98], we know that all fully irreducible \( \varphi \in \text{Out}(F_r) \) satisfy \( 0 > i(\varphi) \geq 1 - r \). An ageometric fully irreducible \( \varphi \in \text{Out}(F_r) \) can be characterized by satisfying \( 0 > i(\varphi) > 1 - r \). The definition we have given for an ideal Whitehead graph only works for ageometric fully irreducibles. However the index sum is always defined from the ideal Whitehead graph as in Definition 2.33 and general definitions of the ideal Whitehead graph can be found in [Pfa12] or [HM11].

A train track map \( g \) induces a simplicial (hence continuous) map \( Dg : LW(g, v) \to LW(g, g(v)) \) extending the map of vertices defined by the direction map \( Dg \). When \( g \) is rotationless and \( v \) a principal vertex, the map \( Dg : LW(g, v) \to LW(g, v) \) has image in \( SW(g, v) \). Since \( Dg \) acts as the identity on \( SW(g, v) \), when viewed as a subgraph of \( LW(g, v) \), this map is in fact a surjection \( Dg : LW(g, v) \to SW(g, v) \).

2.7. Full irreducibility criterion. The following lemma is essentially [Pfa13 Proposition 4.1], with the added observation that a fully irreducible outer automorphism with a PNP-free train track representative is in fact ageometric (by definition). [Kap14] has a related result.
Proposition 2.35 ([Pfa13]). (The Ageometric Full Irreducibility Criterion (FIC)) Let \( g: \Gamma \to \Gamma \) be a PNP-free, irreducible train track representative of \( \varphi \in \text{Out}(F_r) \). Suppose that the transition matrix for \( g \) is Perron-Frobenius and that all the local Whitehead graphs are connected. Then \( \varphi \) is an ageometric fully irreducible outer automorphism.

2.8. Lone Axis Fully Irreducible Outer Automorphisms. In [MP16] Mosher and Pfaff defined the property of being a lone axis fully irreducible outer automorphism. In lay terms this means that there is only one fold line in \( CV_r \) that is invariant under \( \varphi \).

Theorem 2.36 ([MP16]). Let \( \varphi \in \text{Out}(F_r) \) be an ageometric fully irreducible outer automorphism. Then \( \varphi \) is a lone axis fully irreducible if and only if

1. the rotationless index satisfies \( i(\varphi) = \frac{3}{2} - r \) and
2. no component of the ideal Whitehead graph \( IW(\varphi) \) has a cut vertex.

Remark 2.37. It will be important for our purposes that each train track representative of an ageometric lone axis fully irreducible \( \varphi \) is PNP-free ([MP16] Lemma 4.4).

The unique axis is a periodic fold line and one may choose a particularly nice train track representative to generate it (Definition 2.19).

Proposition 2.38 ([MP16]). Let \( \varphi \) be an ageometric lone axis fully irreducible outer automorphism. Then there exists a train track representative \( g: \Gamma \to \Gamma \) of some power \( \varphi^R \) of \( \varphi \) so that all vertices of \( \Gamma \) are principal, and fixed, and all but one direction is fixed.

Definition 2.39 (\( A_\varphi \)). Given a lone axis fully irreducible outer automorphism \( \varphi \), we denote its axis by \( A_\varphi \). In particular, \( A_\varphi \) will be the periodic fold line determined by any (and every) train track representative of any positive power of \( \varphi \).

3. Stratification of the space of fold lines

3.1. The space of fold lines. We fix a rank \( r \geq 3 \) throughout this section. Notice that the Outer space \( CV_r \) has dimension \( 0 \leq k \leq 3r - 4 \).

Definition 3.1 (\( F_r \& A_r \)). \( F_r \) will denote the set of all simple fold lines in \( CV_r \) (see Definition 2.16). \( A_r \subset F_r \) will denote the set of all periodic fold lines in \( CV_r \) (see Definition 2.20).

Definition 3.2 (Topology on \( F_r \)). Let \( \mathcal{L} \) be a geodesic in \( CV_r \). We let \( N(\mathcal{L}, \varepsilon) \) denote the \( \varepsilon \)-neighborhood of \( \mathcal{L} \) in \( CV_r \) with respect to the symmetrized Lipschitz metric on \( CV_r \). We let \( B(\mathcal{L}, R, \varepsilon) \subset F_r \) denote the set of all \( \mathcal{L}' \in F_r \) such that \( \mathcal{L}' \) has a length-\( R \) subsegment \( \beta \) with \( \beta \subset N(\mathcal{L}, \varepsilon) \). For each integer \( k \) with \( 0 \leq k \leq 3r - 4 \), we denote by \( B^k(\mathcal{L}, R, \varepsilon) \) the set of all \( \mathcal{L}' \in B(\mathcal{L}, R, \varepsilon) \) such that the line \( \mathcal{L}' \) is contained in the \( k \)-skeleton \( CV_r^{(k)} \).

We topologize \( F_r \) by using, for each \( \mathcal{L} \in F_r \), the family of sets \( \{ B(\mathcal{L}, R, \varepsilon) \}_{\varepsilon > 0, R \geq 1} \) as the basis of neighborhoods of \( \mathcal{L} \) in \( F_r \).

Remark 3.3. It is a subtle but rather minor point to decide which metric to use in Definition 3.2. The Lipschitz metric \( d_L \) is not symmetric. Define the symmetrized Lipschitz metric as \( d_s(X,Y) = d_L(X,Y) + d_L(Y,X) \). Consider the 4 possible topologies arising from the following generating sets: balls in the symmetrized metric, balls in the simplicial metric, “incoming balls” in the Lipschitz metric \( B_{in}(X,r) = \{ Y \in CV_r \mid d_L(Y,X) < r \} \), and “outgoing balls” in the Lipschitz metric \( B_{out}(X,r) = \{ Y \in CV_r \mid d_L(X,Y) < r \} \). The four topologies on \( CV_r \) coincide [AK12]. However the same is not true for neighborhoods of geodesics. Let \( \mathcal{L} \)
be a geodesic and consider \( N_{simp}(\mathcal{L}, \varepsilon) = \{ Y \mid d_{simp}(Y, \mathcal{L}) < \varepsilon \} \), similarly define \( N_{sym}(\mathcal{L}, \varepsilon) \).

Define \( N_n(\mathcal{L}, \varepsilon) = \{ Y \mid d(Y, \mathcal{L}) < \varepsilon \} \) and \( N_{out}(\mathcal{L}, \varepsilon) = \{ Y \mid d(\mathcal{L}, Y) < \varepsilon \} \). Then sets of the first three types are equivalent, in that for each of the two types and for all \( \varepsilon \), one may find an \( \varepsilon' \) so that \( N(\mathcal{L}, \varepsilon) \) of one of the types contains \( N(\mathcal{L}, \varepsilon') \) of the other type. The same is not true for \( N_{out}(\mathcal{L}, \varepsilon) \). There exists a geodesic \( \mathcal{L} \) and \( \varepsilon > 0 \) so that for all \( \varepsilon' \), \( N_{sym}(\mathcal{L}, \varepsilon) \) does not contain \( N_{out}(\mathcal{L}, \varepsilon') \). The outgoing neighborhoods are “too big,” hence we use the others (these geodesics \( \mathcal{L} \) necessarily don’t stay in any “thick part” of \( CV_r \)).

**Remark 3.4.** Note, with the topology defined above, the space \( \mathcal{F}_r \) is nonHausdorff: if two distinct fold lines \( \mathcal{L}, \mathcal{L}' \in \mathcal{F}_r \) overlap along a common subray, then each neighborhood of \( \mathcal{L} \) in \( \mathcal{F}_r \) contains \( \mathcal{L}' \) and each neighborhood of \( \mathcal{L}' \) contains \( \mathcal{L} \). Nevertheless, the topology on \( \mathcal{F}_r \) given in Definition 3.2 is natural for our purposes. Moreover, the topology is better behaved when restricted to the subspace \( \mathcal{A}_r \subseteq \mathcal{F}_r \), the main object of interest in this paper.

### 3.2. Strata of geodesics.

**Definition 3.5** (r-Dominant graph). Let \( r \geq 3 \) be fixed. A finite graph \( \mathcal{G} \) is \( r \)-dominant if it is a disjoint union of complete graphs and has index sum \( \frac{3}{2} - r \), see Definition 2.33.

**Definition 3.6** (r-Dominant outer automorphism). Let \( \varphi \in \text{Out}(F_r) \) be fully irreducible and suppose \( IW(\varphi) \) is \( r \)-dominant. Then we say that \( \varphi \) is an \( (r-) \)dominant outer automorphism.

**Remark 3.7.** Notice that if \( \varphi \) is dominant then it is ageometric and by Theorem 2.36 \( \varphi \) is also a lone axis outer automorphism.

**Definition 3.8** (Stratum \( \mathcal{S}_r(\mathcal{G}) \)). For a finite graph \( \mathcal{G} \), we define the stratum \( \mathcal{S}_r(\mathcal{G}) \subset \mathcal{A}_r \) for \( \mathcal{G} \):

\[
\mathcal{S}_r(\mathcal{G}) := \{ \mathcal{L} \mid \mathcal{L} \text{ is an axis for a fully irreducible } \varphi \in \text{Out}(F_r) \text{ with } IW(\varphi) \cong \mathcal{G} \}.
\]

If \( \mathcal{G} \) is an \( r \)-dominant graph, we call \( \mathcal{S}_r(\mathcal{G}) \) a dominant stratum.

**Definition 3.9** (\( \mathcal{G} \)-basin \( BS_r(\mathcal{G}) \)). For an \( r \)-dominant graph \( \mathcal{G} \), we define the \( \mathcal{G} \)-basin:

\[
BS_r(\mathcal{G}) := \{ \mathcal{L} \mid \mathcal{L} \text{ is an axis for an ageometric fully irreducible } \varphi \in \text{Out}(F_r) \text{ with } IW(\varphi) \cong \mathcal{G} \text{ is the union of some subset of the components of } \mathcal{G} \}.
\]

Thus \( BS_r(\mathcal{G}) \subset \mathcal{S}_r(\mathcal{G}) \subset \mathcal{A}_r \). Notice that each element in \( BS_r - \mathcal{S}_r \) is not a lone axis automorphism since its index sum is strictly larger than \( \frac{3}{2} - r \).

We give special names (and attention) to the following dominant strata.

**Definition 3.10** (Principal strata \( \mathcal{P}_r \)). Let \( \Delta_r \) denote the graph that is a disjoint union of \( 2r - 3 \) triangles. Notice that, in particular, \( \Delta_r \) has index sum \( \frac{3}{2} - r \). We define the \( (\text{rank-}r) \) principal stratum of \( \mathcal{A}_r \) as \( \mathcal{P}_r = \mathcal{S}_r(\Delta_r) \).

In light of the above, we call a fully irreducible outer automorphism \( \varphi \in \text{Out}(F_r) \) with \( IW(\varphi) \cong \Delta_r \) a principal outer automorphism.

We define the \( (\text{rank-}r) \) principal stratum basin in \( \mathcal{A}_r \) as \( \mathcal{B}\mathcal{P}_r := BS_r(\Delta_r) \). Outer automorphisms with axes in \( \mathcal{B}\mathcal{P}_r \) will be called principal basin outer automorphisms.

Note that
\[
\mathcal{B}\mathcal{P}_r := \{ \mathcal{L} \mid \mathcal{L} \text{ is an axis for a fully irreducible } \varphi \in \text{Out}(F_r) \text{ having } IW(\varphi) \cong \Delta_{r'} \text{ with } r' \leq r \}.
\]
Remark 3.11. We have noted that every dominant outer automorphism is a lone axis outer automorphism. If an outer automorphism is principal then its axis intersects the interior of a maximum dimensional simplex in \( CV_r \). This can be seen to follow from Proposition 2.38. Moreover, if \( \varphi \) is dominant and not principal, then it will not pass through the interior of a maximum dimensional simplex, as one of its Whitehead graphs comes from a stable whitehead graph of a vertex with more than three stable directions.

Definition 3.12 (Principal index list). Since the index list for a principal outer automorphism is comprised of terms \(- \frac{1}{2}\) summing to \( \frac{3}{2} - r \), we call this the principal index list.

4. Nielsen path prevention

Definition 4.1 (Long turns). Suppose that we have a train track structure on \( \Gamma \) induced by a train track map \( g \) on \( \Gamma \) (see Definition 2.12). By a long turn at a vertex \( v \) we will mean a pair of legal paths \( \{ \alpha, \beta \} \) emanating from \( v \). If \( \{ D(\alpha), D(\beta) \} \) is legal, then we call \( \{ \alpha, \beta \} \) legal. If \( \{ D(\alpha), D(\beta) \} \) is illegal, then we call \( \{ \alpha, \beta \} \) illegal.

If either \( g(\alpha) \) is an initial subpath of \( g(\beta) \) or vice versa, then we call \( \{ \alpha, \beta \} \) extendable. Those long turns that are not extendable can be characterized as either safe or dangerous depending on whether, respectively, \( g#(\alpha \beta) \) is a legal path or not (whether the cancellation of \( g#(\alpha) \) and \( g#(\beta) \) ends with a legal turn or an illegal turn).

The following is a relatively direct consequence of Lemma 2.30.

Lemma 4.2. Let \( g : \Gamma \rightarrow \Gamma \) be an expanding irreducible train track map and \( \rho \) an iNP for \( g \). Then \( \rho = \bar{\rho}_1 \rho_2 \), where \( \{ \rho_1, \rho_2 \} \) is a dangerous long turn for each positive power \( g^k \) of \( g \). More generally, if \( g : \Gamma \rightarrow \Gamma \) has a PNP, then \( \Gamma \) contains dangerous long turns for each positive power \( g^k \) of \( g \). Thus, an expanding irreducible train track map with no dangerous long turns has no PNP.

Definition 4.3 (k-Protected path). Let \( g : \Gamma \rightarrow \Gamma \) be an expanding irreducible train track map. Let \( \gamma \) be a path in \( \Gamma \) and \( \alpha \) a subpath of \( \gamma \) whose endpoints are at vertices. We say that \( \alpha \) is k-protected if

- \( \gamma \) contains \( \geq k \) edges to the right of \( \alpha \) and \( \geq k \) edges to the left of \( \alpha \) and
- the length-\( k \) subpath of \( \gamma \) directly to the right of \( \alpha \) and the length-\( k \) subpath of \( \gamma \) directly to the left of \( \alpha \) are each legal.

Definition 4.4 (Splitting). Let \( g : \Gamma \rightarrow \Gamma \) be an expanding irreducible train track map. Let \( \gamma \) be a path in \( \Gamma \). We say that \( \gamma \) is a splitting if it is a \( k \)-splitting for all \( k > 0 \).

The following is a special case of the definition of \( P_r \) on pg. 558 of [BFH00].

Definition 4.5 (Pth\(_g\)). Let \( g : \Gamma \rightarrow \Gamma \) be an expanding irreducible train track map. We let \( Pth_g \) denote the paths in \( \gamma \) so that:

1. Each \( g^k_#(\gamma) \) contains exactly one illegal turn.
2. The number of edges in \( g^k_#(\gamma) \) is bounded independently of \( k \).
The following is [BFH00] Lemma 4.2.5.

**Lemma 4.6** ([BFH00]). Let \( g: \Gamma \to \Gamma \) be an expanding irreducible train track map. Then \( \text{Pth}_g \) is finite.

The next lemma states that there is a uniform \( k \) so that for each long turn \( \{\alpha, \beta\} \) the iterate \( g^k(\bar{\alpha}\beta) \) splits into at most three well understood parts.

**Lemma 4.7.** Suppose that \( g \) is an expanding irreducible train track map representing a fully irreducible outer automorphism \( \varphi \in \text{Out}(F_r) \). Then there exists some power \( g^k \) of \( g \) so that for each long turn \( \{\alpha, \beta\} \) the iterate \( g^k(\bar{\alpha}\beta) \) splits with respect to \( g \) into paths that are legal paths except for at most a single \( \text{iNP} \).

**Proof.** First notice that, since \( \varphi \) is fully irreducible, \( g: \Gamma \to \Gamma \) has only a single \( \text{EG} \) stratum. Thus, by [BFH00] Lemma 4.2.2, there exists a constant \( K \) so that, if \( \tau \) is a path in \( \Gamma \) and \( \sigma \) is a \( K \)-protected subpath of \( \tau \), then \( \tau \) can be split at the endpoints of \( \sigma \).

There are only finitely many paths of length \( \leq 4K \). Let \( S \) denote the set of all paths of length \( \leq 4K \) with only a single illegal turn. Let \( k \in \mathbb{Z} \) be the power so that, if \( \rho \in S \), then either \( g^k(\rho) \) is legal or for no \( n \in \mathbb{Z} \) is \( g^{nk}(\rho) \) legal. By [BFH00] Lemma 4.2.6 we can then replace \( k \) with a higher power, if necessary, so that for each \( \rho \in S \), we have that \( g^k(\rho) \) splits into subpaths that are either legal or an element of \( \text{Pth}_g \) (a uniform power is possible since \( S \) is finite). The subpaths that are elements of \( \text{Pth}_g \) are permuted. Thus, by replacing \( k \) with a higher power yet, we can assume that they are \( \text{iNP} \)s (hence have only one illegal turn).

Let \( \{\alpha, \beta\} \) be a long turn. Then using trivial paths as \( K \)-protected subpaths, \( \bar{\alpha}\beta \) can be split into legal paths and a path \( \rho \) of length \( \leq 4K \) containing the single illegal turn. Since \( \rho \in S \), we can use the power \( k \) of the previous paragraph and obtain that \( g^k(\bar{\alpha}\beta) \) splits into subpaths that are either legal or \( \text{iNP} \)s. But \( \bar{\alpha}\beta \) had only one illegal turn, and the number of illegal turns cannot increase under \( g^k \). So there can only be one \( \text{iNP} \) in the splitting. \( \square \)

**Lemma 4.8.** Suppose that \( g \) is an expanding irreducible train track map with no \( \text{PNP} \)s. Then there exists some power \( g^k \) of \( g \) with no dangerous long turns.

**Proof.** Let \( k \) be as in Lemma 4.7 and suppose, for the sake of contradiction, that \( g^k \) had a dangerous long turn \( \tau = \{\alpha, \beta\} \). By Lemma 4.7 \( g^k(\bar{\alpha}\beta) \) splits into legal paths and \( \text{iNP} \)s. Since \( g^k \) admits no \( \text{iNP} \)s, \( g^k(\bar{\alpha}\beta) \) is legal, contradicting that \( \tau \) is dangerous. \( \square \)

**Definition 4.9** (Legalizing train track maps). We call a train track map \( g: \Gamma \to \Gamma \) legalizing if it has no \( \text{PNP} \)s.

**Proposition 4.10.** Suppose that \( g \) is a \( \text{PNP} \)-free expanding irreducible train track map. Then there exists some \( p > 0 \) so that \( g^p \) is legalizing.

**Proof.** This follows from Lemmas 4.2 and 4.8. \( \square \)

**Proposition 4.11.** Suppose \( \varphi \) is a lone axis fully irreducible outer automorphism. Then there is a fully stable transparent legalizing train track representative \( g: \Gamma \to \Gamma \) of a power \( \varphi^R \) of \( \varphi \) so that all vertices of \( \Gamma \) are principal and fixed, and all but one direction is fixed.

**Proof.** This follows from Remark 2.37, Proposition 2.38 and Proposition 4.10. \( \square \)

**Definition 4.12** (Convenient train track maps). For a lone axis fully irreducible outer automorphism \( \varphi \) we call a train track representative of a power \( \varphi^R \) of \( \varphi \) satisfying the properties of Proposition 4.11 convenient.
5. Proof of the main result

Lemma 5.1. Let $r \geq 3$ be an integer, so that $n = 3r - 4$ is the dimension of $CV_r$. For each lone axis fully irreducible $\varphi \in \text{Out}(F_r)$ there exists an integer $k \leq n$ so that $A_\varphi \subset CV_r^{(k)} \setminus CV_r^{(k-2)}$. In particular, all folds in $A_\varphi$ are proper full folds.

Proof. Let $g: \Gamma \to \Gamma$ be a convenient train track map representing $\varphi$, guaranteed by Proposition 4.11. Let $k = |E(\Gamma)|$, i.e. $1+$ the dimension of the open simplex containing $\Gamma$. The fold line $A_\varphi$ is a periodic fold line for a Stallings fold decomposition of $g$. It suffices to show that each fold of $A_\varphi$ is a proper full fold. But, if one of the folds were full, then some vertex of $\Gamma$ would not be $g$-periodic, hence not principal. This contradicts that $g$ is convenient. □

If $g$ is a Stallings fold decomposition of $g$ we denote by $g^p$ the Stallings fold sequence obtained by juxtaposing $p$ copies of $g$. Note that $g^p$ is a decomposition of $g^p$. In the next lemma we need not assume the outer automorphism represented by $h$ is fully irreducible.

Lemma 5.2. Let $r \geq 3$ be an integer, so that $n = 3r - 4$ is the dimension of $CV_r$, and let $2 \leq k \leq n$. Suppose that $L \subset CV_r^{(k)} \setminus CV_r^{(k-2)}$ is the periodic fold line for a Stallings fold decomposition $g$ of a train track map $g$. Then there exist constants $R, \varepsilon > 0$ so that: For each fold line $L' \subset B_k(L, R, \varepsilon)$ that is the periodic fold line corresponding to a Stallings fold decomposition $h$ of some train track map $h$, there exists a power $p$ so that $h^p$ contains $g$.

In particular, $h$ and $g$ are self-maps of the same topological graph $\Gamma$.

Proof. Let $R$ be three times the length of a $g$-segment of $L$. Since $L$ is periodic and contained in $CV_r^{(k)} \setminus CV_r^{(k-2)}$, there exists some $\varepsilon_0 > 0$ such that $N(L, \varepsilon_0) \cap CV_r^{(k)} \subset CV_r^{(k)} \setminus CV_r^{(k-2)}$. Therefore, for any $0 < \varepsilon \leq \varepsilon_0$, any geodesic segment $\gamma$ of length $R$ contained in $N(L, \varepsilon)$ passes through the same sequence of simplices as a subsegment of $L$ of length $\geq R - 2\varepsilon$, and hence shares a fold sequence with this subsegment of $L$. Choose $\varepsilon = \min\{\varepsilon_0, \frac{R}{4}\}$. Then any subsegment of $L$ of length $\geq R - 2\varepsilon \geq \frac{3}{2}R$ contains twice the length of a $g$-segment of $L$ so must contain a $g$-segment of $L$. Hence, any periodic fold line $L' \subset B_k(L, R, \varepsilon)$ will in fact contain the full fold sequences $g$. We can now take the power $p$ of $h$ high enough so that $h^p$ contains any length-$R$ subsegment of $L'$ and the conclusion of the theorem will hold. □

Lemma 5.3. Let $r \geq 3$ be an integer, so that $n = 3r - 4$ is the dimension of $CV_r$, and let $2 \leq k \leq n$. Suppose $\varphi$ is a dominant lone axis fully irreducible with axis $A_\varphi \subset CV_r^{(k)} \setminus CV_r^{(k-2)}$. Then there exist constants $R, \varepsilon > 0$ and a convenient train track representative $g: \Gamma \to \Gamma$ of a power $\varphi^p$ of $\varphi$ so that for each periodic fold line $L \subset B_k(A_\varphi, R, \varepsilon)$ there exist and a self-map $h$ on $\Gamma$ with a Stallings fold decomposition yielding $L$ and such that:

(a) $h: \Gamma \to \Gamma$ is a train track map.
(b) $h$ does not admit a PNP.
(c) The transition matrix for $h$ is Perron-Frobenius.
(d) $\bigcup_{v \in V(\Gamma)} LW(g, v) = \bigcup_{v \in V(\Gamma)} LW(h, v)$.
(e) If the vertex $w$ of $\Gamma$ is $h$-periodic then $SW(h, w) = SW(g, w)$.
(f) If $L$ contains no proper full fold, then all vertices of $\Gamma$ are principal with respect to both $g$ and $h$ and $\bigcup_{v \in V(\Gamma)} SW(g, v) = \bigcup_{v \in V(\Gamma)} SW(h, v)$.
**Proof.** Since \( \varphi \) is a lone axis fully irreducible, by Proposition \( \ref{2.38} \), there exists a rotationless power \( \varphi^p \) of \( \varphi \) with a convenient train track representative \( g: \Gamma \to \Gamma \). We call the nonfixed direction \( d \). Since \( \varphi \), hence \( \varphi^p \), is a lone axis fully irreducible, \( A_\varphi \) is the unique periodic fold line for \( g \) and is formed by iterating the fold sequence \( g \) for \( g \). Replace \( g \) with a power so that each turn in \( LW(g) \) is taken by \( g(e) \) for each edge \( e \).

Notice that \( A_\varphi \) is also the periodic fold line for \( g^3 \) and that \( g^3 \) is the fold sequence for \( g^3 \). Applying Lemma \( \ref{5.2} \), there exist \( R, \varepsilon > 0 \) so that for any periodic fold line \( \mathcal{L} \in B^k(A_\varphi, R, \varepsilon) \) corresponding to a train track map \( h' \) and fold sequence \( h' \) of \( h' \), there exists a power \( p \) such that \( (h')^p \) contains \( g^3 \). Thus, replacing \( h \) with this power and possibly applying a cyclic permutation, \( h' \) factors as \( h' = f \circ g^2 \), see \( \cite{2} \).

The trickiest aspect of this proof, and the reason to use \( g^3 \) instead of \( g \), is to prove item (b). We will show all items for the cyclic permutation \( h = g \circ f \circ g^2 \) of \( h' \).

\begin{equation}
\begin{array}{c}
\Gamma \xrightarrow{g} \Gamma \xrightarrow{g} \Gamma \xrightarrow{f} \Gamma \xrightarrow{g} \Gamma \xrightarrow{g} \Gamma \xrightarrow{g} \Gamma \xrightarrow{f} \Gamma \\
\end{array}
\end{equation}  

We first show (a). Suppose \( h \) is not a train track map, i.e. \( h^p(e) \) contains a backtracking segment for some \( e \in E(\Gamma) \) and power \( p \). We parametrize \( \mathcal{L}: \mathbb{R} \to CV_\varepsilon \) so that the graphs appearing in \( \ell \) are \( \mathcal{L}(0), \mathcal{L}(1), \mathcal{L}(2), \ldots \), respectively. Let \( \gamma \in F_\varepsilon \) be the witness guaranteed by Lemma \( \ref{2.26} \), i.e. \( \gamma_t \in \mathcal{L}(t) \) is legal for all \( t \geq 0 \). Note that \( \gamma_4 = g(\gamma_3) \), and since \( g \) maps each edge onto the entire graph, \( \gamma_4 \) contains \( e \). Thus, \( e \) cannot be \( h \)-legal, a contradiction.

We now show (b). Recall that \( g \) is convenient, hence if \( \{\alpha, \beta\} \) is a long turn then either \( g(\alpha) \) is an initial subsegment of \( g(\beta) \) or \( g_#(\bar{\alpha}\beta) = \bar{\alpha}'\beta' \) is legal, where \( \alpha', \beta' \) are nontrivial terminal subsegments of \( \alpha, \beta \). In the second case all turns of \( g(\bar{\alpha}^\prime\beta^\prime) \) are \( g \)-taken (since \( \{Dg\alpha', Dg\beta'\} \) cannot contain \( d \) so it, too, is \( g \)-taken). Notice that each \( g \)-taken turn is \( h^p \circ g \circ f \)-legal since for any witness loop \( \gamma \), \( g^2(\gamma) \) maps over all \( g \)-taken turns and \( h^p \circ g \circ f(g^2(\gamma)) \) is legal. Concluding, we get that the path \( g(\bar{\alpha}^\prime\beta^\prime) \) is legal with respect to \( h^p \circ g \circ f \) for each \( p \). Now if \( \rho = \bar{\alpha}^\prime \) is an iPNP, then for some \( p \), \( h^p_#(\rho) = \rho \), which is illegal. But \( h^p(\bar{\alpha}^\prime) = h^{p-1} \circ g \circ f(g(#(\bar{\alpha}^\prime))) = h^{p-1} \circ g \circ f(g(\alpha'\beta')) \), which is legal. We get a contradiction to the fact that \( \rho \) is an iPNP.

To prove (c) recall that each edge of \( \Gamma \) maps onto \( \Gamma \) under the map \( g \). Since \( h = g \circ (f \circ g^2) \) is a train track map, the same is true for \( h \). Thus the transition matrix of \( h \) is PF.

To prove (d), recall that \( g(e) \) contains all turns in each local Whitehead graph. Note also that for any witness loop \( \gamma \) for \( h(\gamma) \) contains all \( g \)-taken turns. Thus \( LW(h, w) \supset LW(g, w) \). Let \( d \) be the unique \( g \)-nonperiodic direction, and let \( v \) be its initial vertex. Note that \( \bigcup_{w \in \mathbb{E}} LW(g, w) \) contains all turns not involving \( d \). Thus, if \( \bigcup LW(h, w) \setminus \bigcup LW(g, w) = \emptyset \) then \( \{d, d'\} \in LW(h, v) \setminus LW(g, v) \) for some \( d' \neq d \). Therefore, there exists an edge \( e \) and a natural \( p \) so that \( h^p(e) \) crosses \( \tau = \{d, d'\} \). Denoting \( \alpha = f \circ g^2 \circ h^{p-1}(e) \) (which is an immersed \( g \)-legal path) we have \( h^p(e) = g(\alpha) \) contains \( \tau \). But \( \tau \) is not \( g \)-taken and not in \( Im(Tg) \), since \( d \notin Im(Dg) \), a contradiction. So \( LW(h, w) = LW(g, w) \) for each \( w \in \Gamma \).

To prove (e) we again denote by \( v \) the initial vertex of the direction \( d \) that is \( g \)-nonperiodic. First let \( w \neq v \) be \( h \)-periodic. Since all turns at \( w \) are \( h \)-taken, \( Dh \) is injective on the directions at \( w \), hence all directions at \( w \) are periodic and \( SW(h, w) = LW(h, w) = LW(g, w) = \)}
For $v$, since $LW(v, g) = LW(v, h)$, we have that $LW(v, h)$ contains a complete graph $C$ on $\text{deg}(v) - 1$ vertices (all directions except $d$). $Dh$ is injective on $C$, since otherwise a taken turn would be illegal. If $v$ is not $h$-periodic then there is nothing to prove, so we assume that $v$ is $h$-periodic. Let $h^p$ be a rotationless power of $h$. Then $Dh^p : LW(h, v) \to SW(h, v)$ sends $C$ to an isomorphic graph. Moreover, we know that $\text{deg}(v) = V(C) + 1$ and that $d$ is in an $h$-illegal turn, so $SW(h, v)$ cannot contain more than $V(C)$ vertices. Hence, $SW(h, v) \cong C \cong SW(g, v)$.

To prove (f) note that the containment in (e) is proper if and only if not all vertices are principal. Since the local Whitehead graphs contain all edges without $d$, this happens only when for some $w \neq u \in V(\Gamma)$ we have $h^p(w) = h^p(u)$. When this happens, some fold in the fold sequence does not restrict to an injective map on the vertices. This implies that the fold is full. Therefore, if no fold in the fold sequence of $L$ is full, then all vertices of $\Gamma$ are $h$-principal, and the stable Whitehead graphs of $h$ and $g$ are identical.

The basin of any dominant stratum has the following “rigidity” properties:

**Theorem A.** Let $r \geq 3$ and let $\mathcal{G}$ be an $r$-dominant graph. Let $L \in S^r(\mathcal{G})$. Then there exist $0 \leq k \leq 3r - 4$ and a neighborhood $U \subseteq \mathcal{A}_r$ of $L$ with the following properties:

(a) For each $L' \in U$ with $L' \subseteq CV_r^{(k)}$, we have $L' \in BS^r(\mathcal{G})$.

(b) For each $L' \in U$ with $L' \subseteq CV_r^{(k)}$ and with $L'$ containing no full folds, we have $L' \in S^r(\mathcal{G})$.

**Proof.** Using Lemma [5.3], we have that $L$ is a periodic fold line for a Stallings decomposition of the train track map $h$ with no PNPs and with connected local Whitehead graphs. By the FIC (Proposition 2.35) we get that the outer automorphisms $\varphi'$ represented by $h$ is ageometric fully irreducible. Let $W \subset V(\Gamma)$ be the set of $h$-principal vertices of $\Gamma$. Then by Lemma [5.3(e)],

$$IW(\varphi') = \bigcup_{v \in W} SW(h, v) \subset \bigcup_{v \in V(\Gamma)} SW(g, v) = IW(\varphi)$$

and the unions are disjoint. Thus, $IW(\varphi')$ is a union of components of $IW(\varphi)$.

We prove (b). By Proposition [5.3(f)], all vertices are $h$-principal, hence $IW(\varphi') = \bigcup_{v \in V(\Gamma)} SW(h, v) = \bigcup_{v \in V(\Gamma)} SW(g, v) = IW(\varphi)$. $\square$

**Corollary 5.4.** Suppose $L \in \mathcal{P}_r$. Then there exists an open neighborhood $U$ of $L$ so that:

(a) $U \cap \mathcal{A}_r \subset BP_r$ and

(b) each periodic fold line in $U$ containing no proper folds is contained in $\mathcal{P}_r$.

### 6. Examples

**Example 6.1** (Principal outer automorphisms exist). We claim that, for each rank $r \geq 3$, the examples constructed in [CL15] to have the principal index list are in fact principal outer automorphisms. For each $r \geq 3$, we denote this outer automorphism in $Out(F_r)$ by $\varphi_r$. By [CL15 Theorem 6.2] we know that each $\varphi_r$ is an ageometric fully irreducible outer automorphism. To show that $\varphi_r$ is principal we must prove that $IW(\varphi_r) = \Delta_r$.

The proof of [CL15 Proposition 4.3] indicates that the stable Whitehead graph at each vertex is a complete graph. Since there are no periodic Nielsen paths (again by [CL15 Theorem 6.2]), this indicates that each component of the ideal Whitehead graph is a complete
The maps $g_1, g_2, g_3, h$ are described in Figure 6.2. Composing the maps in the diagram yields:

$$k(e_1) = \bar{e}_1 \bar{e}_4, \quad k(e_2) = e_1, \quad k(e_3) = e_2 \bar{e}_5, \quad k(e_4) = e_2 \bar{e}_5, \quad k(e_5) = \bar{e}_3 \bar{e}_5$$

Notice the following facts:

1. The only $k$-prenull turn is $\{\bar{e}_3, \bar{e}_4\}$.
2. $\{\bar{e}_3, \bar{e}_4\}$ is the only $f$-illegal turn, and it is not in $\text{Im}(Tf)$.
3. $\text{Im}(Dk)$ does not contain $\bar{e}_4$.
4. $\{\bar{e}_3, \bar{e}_4\}$ is not a $k$-taken turn. The $k$-taken turns are: $\{\bar{e}_2, \bar{e}_5\}, \{e_3, \bar{e}_5\}, \{e_1, \bar{e}_4\}$.
5. $k(v_1) = k(v_3) = v_2$ and $k(v_2) = v_1$ (see the bottom of the label of Figure 6.2).
6. The $Tk$ images of all of the $f$-legal turns at $v_1$ and $v_2$ are: $\{e_2, \bar{e}_5\}, \{e_1, \bar{e}_5\}, \{e_1, e_2\}$ at the vertex $v_2$ and $\{e_1, e_3\}, \{e_1, e_4\}, \{e_4, e_3\}$ at the vertex $v_1$.
7. $f$ takes all turns not involving $\bar{e}_4$.

**Lemma 6.3.** $f' = k \circ f$ is an irreducible train track map.

**Proof.** We first show that $f'$ is a train track map. Suppose not - then for some $p \in \mathbb{N}$ and edge $e$, $(f')^p(e)$ would contain backtracking. Note that then $f \circ (f')^p(e)$ would contain backtracking. We show by induction on $p$ that $f \circ (f')^p(e)$ does not contain backtracking. Since $f$ is a train track map, $f(e)$ can contain no backtracking. Inductively assume $\beta = f \circ (f')^{p-1}(e)$ has no backtracking. All of the turns in $\beta$ are either $f$-taken or in $\text{Im}(Tf)$, so
also $f$-taken. No $f$-taken turn is $k$-prenull, so $k(\beta)$ has no backtracking. All turns in $k(\beta)$ are either $k$-taken or in $Im(Tk)$ so by properties (3) and (4) above, $k(\beta)$ does not contain $\{\bar{e}_3, \bar{e}_4\}$. So by (2) it is $f$-legal. This completes the induction step. The fact that $f'$ is irreducible follows from the fact that $f(e)$ contains all edges and $k$ is onto. \hfill \Box

Lemma 6.4. $f' = k \circ f$ has no PNPs.

Proof. Recall that $f$ is legalizing and transparent, which implies that if $\alpha$ and $\beta$ are legal paths initiating at the same vertex then, without loss of generality, either $f(\alpha)$ is an initial subpath of $f(\beta)$ (then $\{\alpha, \beta\}$ is an $f$-extendable long turn) or $f(\alpha \beta) = \bar{\alpha}' \bar{\beta}'$ is $f$-legal, where $\alpha', \beta'$ are terminal subsegments of $\alpha, \beta$ respectively.

Now suppose $\rho = \bar{\alpha} \bar{\beta}$ is an iPNP for $f'$. Since there exists a $p > 0$ so that $(f'^p)_{\#}(\rho) = \rho$, we would then have that $\rho$ is not $f'$-extendable, which would imply that it is not $f$-extendable. By the first paragraph we have $f(\rho) = \bar{\alpha}' \bar{\beta}'$, where $\alpha', \beta'$ are $f$-taken paths and the turn $\{D\alpha', D\beta'\}$ is $f$-legal, so it is not equal to $\{\bar{e}_3, \bar{e}_4\}$, the only $k$-prenull turn. Thus $k(\bar{\alpha}' \bar{\beta}')$ contains no backtracking. Therefore, the turns in $f'(\rho) = k(\bar{\alpha}' \bar{\beta}')$ are in the image of $Tk$ or are $k$-taken turns. By properties (3) and (4) above, $f'(\rho)$ does not contain $\{\bar{e}_3, \bar{e}_4\}$, so is

Figure 2. From left to right the graphs are $\Gamma_1$, $\Gamma_2$, $\Gamma_3$, $\Gamma_4$, $\Gamma_1$. Colors indicate the (partial) edges folded by the subsequent maps. $g_1$ is the full fold of $\overline{e}_3$ and $\overline{e}_4$ (and $e'_4$ is the edge formed by the identification of $\overline{e}_3$ and $\overline{e}_4$). $g_2$ is a partial fold of $\overline{e}_4'$ and $\overline{e}_5'$ (and $e''_4$ is the edge formed from the identification of the initial portions of $\overline{e}_4'$ and $\overline{e}_5'$, and $e''_5$ is the portion of $e_5$ not folded, and $e''_4$ is the portion of $e_4$ not folded). $g_3$ is a proper full fold of $e_1$ over $\overline{e}_2$ (and $e''_1$ is the portion of $e_1$ remaining after the fold), i.e. $g_3(e_1) = e'_1 e''_1$. $h$ is a homeomorphism sending $\overline{e}'_1$ to $\overline{e}_3$, and $e'_1$ to $\overline{e}_1$, and $e''_4$ to $e_2$, and $e''_5$ to $\overline{e}_5$, and $e'_5$ to $\overline{e}_3$. The bottom vertex in $\Gamma_1$ is $v_1$, the upper left vertex is $v_2$, and the upper right vertex is $v_3$.  

19
Proof. By the FIC, it suffices to show that each local Whitehead graph is connected.

By item (6), we have that all turns at \( v_2 \) are \( f' \)-taken and all turns at \( v_1 \) not involving \( d = e_4 \) are \( f' \)-taken. It follows that \( LW(f', v_2) \) is a triangle and \( LW(f', v_1) \) contains a triangle. Since \( f' \) is an irreducible train track map and for no such map does \( LW(v_1, f') \) have an isolated vertex, we have that \( e_4 \) in \( LW(v_1, f') \) is also connected via an edge to another vertex, hence \( LW(v_1, f') \) is connected. By (4), \( LW(f', v_3) \) is also connected. \( \square \)

Lemma 6.6. Let \( f' = k \circ f \) and suppose that \( f' \) represents the automorphism \( \varphi' \in \text{Out}(F_3) \). Then the ideal Whitehead graph \( IW(\varphi') \) is a union of two triangles.

Proof. By the proof of Lemma 6.5, \( LW(f', v_2) \) is a triangle and \( LW(f', v_1) \) contains a triangle. Now \( v_1 \) and \( v_2 \) are permuted by \( f \) and \( k \), hence are permuted by \( f' \). Thus they are fixed by a rotationless power \( f^n \) of \( f' \). Moreover, \( Df^n \) cannot identify any of the directions at \( v_2 \), since that would collapse a taken turn. Therefore, the triangle in \( LW(f', v_2) \) is taken by \( f^n \) to a triangle in \( SW(f^n, v_2) \). Similarly, no two of the directions \( \{d_1, d_2, d_3\} \) at \( v_1 \), distinct from \( d \), can be identified by \( Df^n \). Thus their images span a triangle in \( SW(f^n, v_1) \). Thus \( SW(f', v_1) \) contains a triangle. But since \( d \) forms an illegal turn with some other direction, there are only 3 gates, and the graph is in fact a triangle. Lastly note that \( k(v_3) = v_2 \), so \( v_3 \) is not a principal vertex (this is in fact the key for dropping from a union of 3 triangles in \( IW(f) \) to two triangles in \( IW(f') \)). The ideal Whitehead graph is the union of the stable Whitehead graphs of principal vertices glued along PNPs but, since \( f' \) admits none, \( IW(\varphi') \) is a union of two disjoint triangles. \( \square \)

We can now prove one of the main results stated in the introduction:

**Theorem B.** There exists a principal fully irreducible outer automorphism \( \varphi \in \text{Out}(F_3) \) with a train track representative \( f: \Gamma \to \Gamma \) with a Stallings fold decomposition \( f \) such that, for each \( n \geq 1 \), there exists a nonprincipal fully irreducible outer automorphism \( \psi_n \in \text{Out}(F_3) \) with a train track representative \( g_n: \Gamma \to \Gamma \) with a Stallings fold decomposition \( g_n \) such that \( g_n \) starts with \( f^n \).

Proof. The lemmas and proofs above apply verbatim when \( f \) is replaced with \( f^n \). By the FIC (Proposition 2.35), both \( f \) and \( g_n = k \circ f^n \) represent fully irreducible outer automorphisms \( \varphi, \psi_n \) of \( F_3 \). The ideal Whitehead graph of \( \varphi \) is a union of three triangles, while \( IW(\psi_n) \) is a union of two triangles. Hence \( \varphi \) is principal, while \( \psi_n \) is not. \( \square \)

Theorem B immediately implies:

**Corollary C.** For \( r = 3 \), there exist a principal periodic geodesic \( \mathcal{L} \in \mathcal{P}_r \) in \( CV_r \) and a sequence of nonprincipal periodic geodesics \( \{\mathcal{L}_n\}_{n=1}^\infty \subseteq \mathcal{BP}_r - \mathcal{P}_r \) such that \( \lim_{n \to \infty} \mathcal{L}_n = \mathcal{L} \).

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