A SHELLING OF THE ODD SECOND HYPERSIMPLEX

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Abstract. Hypersimplices are well-studied objects in combinatorics, optimization, and representation theory. For each hypersimplex, we define a new family of subpolytopes, called $r$-stable hypersimplices, and show that a standard unimodular triangulation of the hypersimplex restricts to a unimodular triangulation of each $r$-stable hypersimplex. For the case of the second hypersimplex defined by the two-element subsets of an $n$-set with $n$ odd, we provide a shelling of this triangulation that sequentially shells each $r$-stable sub-hypersimplex. In this case, we also investigate connections between the Ehrhart $h^*$-vector of the second hypersimplex and the $r$-stable sub-hypersimplices.

1. Introduction

Fix integers $0 < k < n$. We let $[n] := \{1, 2, \ldots, n\}$ and let $\binom{[n]}{k}$ denote the collection of $k$-element subsets of $[n]$. The characteristic vector of a $k$-subset $I$ of $[n]$ is the $(0, 1)$-vector $\epsilon_I := (\epsilon_1, \ldots, \epsilon_n)$ such that $\epsilon_i = 1$ for $i \in I$ and $\epsilon_i = 0$ for $i \notin I$. The $n,k$-hypersimplex, denoted $\Delta_{n,k}$, is the $(n-1)$-dimensional polytope that is the convex hull of the characteristic vectors of all $k$-subsets of $[n]$. That is, $\Delta_{n,k}$ is the convex hull of all $(0,1)$-vectors in $\mathbb{R}^n$ containing precisely $k$ nonzero terms. Hypersimplices appear naturally in algebraic and geometric contexts, as well as in pure and applied combinatorial contexts. In [10], Stanley proved that the volume of the $n,k$-hypersimplex is the Eulerian number $A_{k,n} - 1$. De Loera, Sturmfels, and Thomas studied the connection between triangulations of the hypersimplex and Gröbner bases via toric algebra [2]. In [7], Lam and Postnikov showed four useful triangulations of the hypersimplex are all identical. In [5], Katzman gave an algebraic description of the Ehrhart $h^*$-vector of the hypersimplex, and in [8], Li gave a second interpretation in terms of exceedences and descents. These many investigations have proven fruitful for our understanding of these fundamental polytopes, but many interesting questions about $\Delta_{n,k}$ still remain unanswered.

Many unanswered questions pertaining to the hypersimplex lie in the field of Ehrhart theory. A lattice polytope $P$ of dimension $d$ is the convex hull in $\mathbb{R}^n$ of finitely many points in $\mathbb{Z}^n$ that together affinely span a $d$-dimensional hyperplane in $\mathbb{R}^n$. For $t \in \mathbb{Z}_{>0}$, set $tP := \{tp : p \in P\}$, and let $L_{P}(t) = |\mathbb{Z}^n \cap t\mathbb{P}|$. In [3], Ehrhart proved that with the polynomial basis $\{\binom{t+d-i}{d} : i \in [0,d] \cap \mathbb{Z}\}$,

$$L_{P}(t) = \sum_{i=0}^{d} h_i^* \binom{t+d-i}{d}.$$ 

Stanley then proved that the coefficients $h_i^*$ are all nonnegative integers [11]. The polynomial $L_{P}(t)$ is called the Ehrhart polynomial of $P$ and has connections to commutative algebra, algebraic geometry, combinatorics, and discrete and convex geometry. However, these polynomials are not well understood in many cases. Given the above polynomial representation of $L_{P}(t)$ it is common to study the coefficient vector $h^*(P) := (h_1^*, \ldots, h_d^*)$, which is often referred to as the $h$-star vector or $\delta$-vector of $P$. This vector encodes useful information about the polytope $P$. For example,
vol(\mathcal{P}) = \sum_i h_i^*, where vol(\mathcal{P}) denotes the Euclidean Volume (Lebesgue measure) of \mathcal{P} with respect to the integer lattice contained in the hyperplane spanned by \mathcal{P}. We call the sum \sum_i h_i^* the normalized volume of \mathcal{P}. Another useful (but not always achieved) property of these vectors is unimodality. A vector \((x_0, x_1, \ldots, x_d)\) is called unimodal if there exists an index \(j, 0 \leq j \leq d\), such that \(x_{i-1} \leq x_i\) for \(i \leq j\), and \(x_i \geq x_{i+1}\) for \(i \geq j\). Unimodality of the \(h^*\)-vector has interesting algebraic implications and consequently is a widely sought property of these vectors [4, 5, 9]. In the special case of the hypersimplex, Haws, De Loera, and Köppe computationally verified that the \(h^*\)-vector of \(\Delta_{n,k}\) is unimodal for \(n\) as large as 75 [1]. Since the \(h^*\)-vector of \(\Delta_{n,k}\) is not symmetric, these results are intriguing from both the perspective of investigation of the \(h^*\)-vectors of hypersimplices as well as the perspective of investigation of unimodal sequences.

In the following, we identify a subpolytope of the \(n, k\)-hypersimplex for each integer \(r\) satisfying \(0 < r \leq \left\lfloor \frac{n}{k} \right\rfloor\) that we call the \(r\)-stable hypersimplex. In section 2, we show that a well-studied regular unimodular triangulation of the hypersimplex induces a regular unimodular triangulation of the \(r\)-stable hypersimplex. In section 3, we provide a shelling of this triangulation for \(\Delta_{n,2}\) with \(n\) odd that proceeds with respect to these subpolytopes. Then in section 4, we investigate some of the Ehrhart theoretical consequences of this shelling. In particular, this work suggests that the Ehrhart theory of the odd second hypersimplex can be studied via the enumeration of lattice paths lying in various ladder-shaped regions of the plane.

2. The \(r\)-stable \(n, k\)-hypersimplex

Label the vertices of a regular \(n\)-gon embedded in \(\mathbb{R}^2\) in a clockwise fashion from 1 to \(n\). We define the circular distance between two elements \(i\) and \(j\) of \([n]\), denoted \(\text{cd}(i, j)\), to be the number of edges in the shortest path between the vertices \(i\) and \(j\) of the \(n\)-gon. We also denote the path of shortest length from \(i\) to \(j\) by \(\text{arc}(i, j)\). A subset \(S \subset [n]\) is called \(r\)-stable if each pair \(i, j \in S\) satisfies \(\text{cd}(i, j) \geq r\). The \(r\)-stable \(n, k\)-hypersimplex, denoted \(\Delta_{n,k}^{\text{stab}(r)}\), is the convex hull of the characteristic vectors of all \(r\)-stable \(k\)-subsets of \([n]\). For fixed \(n\) and \(k\), these polytopes form the nested chain

\[
\Delta_{n,k} \supset \Delta_{n,k}^{\text{stab}(2)} \supset \Delta_{n,k}^{\text{stab}(3)} \supset \cdots \supset \Delta_{n,k}^{\text{stab}\left(\left\lfloor \frac{n}{k} \right\rfloor\right)}.
\]

2.1. A well-studied triangulation of the hypersimplex. In [7], Lam and Postnikov compare four different triangulations of the hypersimplex, and show that they are identical. While these triangulations possess the same geometric structure the constructions are all quite different, and consequently each one reveals different information about the triangulation’s geometry. Here, we utilize properties of two of these four constructions. The first is a construction given by Sturmfels in [12] using techniques from toric algebra. The second construction, known as the circuit triangulation, is introduced in [7] by Lam and Postnikov. We will show that this triangulation restricts to a triangulation of the \(r\)-stable hypersimplex.

2.1.1. Sturmfels’ Triangulation. We recall the description of this triangulation presented in [7]. Let \(I\) and \(J\) be two \(k\)-subsets of \([n]\) and consider their multi-union \(I \cup J\). Let \(\text{sort}(I \cup J) = (a_1, a_2, \ldots, a_{2k})\) be the unique nondecreasing sequence obtained by ordering the elements of the multiset \(I \cup J\) from least-to-greatest. Now let \(U(I, J) := \{a_1, a_3, \ldots, a_{2k-1}\}\) and \(V(I, J) := \{a_2, a_4, \ldots, a_{2k}\}\). As an example consider the 4-subsets of \([8]\), \(I = \{1, 3, 4, 6\}\) and \(J = \{3, 5, 7, 8\}\). For this pair of subsets we have that \(\text{sort}(I \cup J) = (1, 3, 3, 4, 5, 6, 7, 8)\), \(U(I, J) = \{1, 3, 5, 7\}\), and \(V(I, J) = \{3, 4, 6, 8\}\). The ordered pair of \(k\)-subsets \((I, J)\) is said to be sorted if \(I = U(I, J)\) and \(J = V(I, J)\). Moreover, an ordered \(d\)-collection \(\mathcal{I} = (I_1, I_2, \ldots, I_d)\) of \(k\)-subsets is called sorted if each pair \((I_i, I_j)\) is sorted for all \(1 \leq i < j \leq d\). For a sorted \(d\)-collection \(\mathcal{I}\) we let \(\sigma_{\mathcal{I}}\) denote the \((d - 1)\)-dimensional simplex with vertices \(\epsilon_{I_1}, \epsilon_{I_2}, \ldots, \epsilon_{I_d}\).
Theorem 2.1. [12] Sturmfels] The collection of simplices $\sigma_{\mathcal{I}}$, where $\mathcal{I}$ varies over the sorted collections of $k$-element subsets of $[n]$, forms a triangulation of $\Delta_{n,k}$.

Notice that the maximal simplices in this triangulation correspond to the maximal-by-inclusion sorted collections, which all have $d = n$.

This triangulation of $\Delta_{n,k}$ was identified by Sturmfels’ via the correspondence between Gröbner bases for the toric ideal associated to $\Delta_{n,k}$ and regular triangulations of $\Delta_{n,k}$. To construct the toric ideal for $\Delta_{n,k}$ let $k[x_I]$ denote the polynomial ring in the $\binom{n}{k}$ variables $x_I$ labeled by the $k$-subsets of $[n]$, and define the semigroup algebra homomorphism

$$\varphi : k[x_I] \longrightarrow k[z_1, z_2, \ldots, z_n]; \quad \varphi : x_I \longmapsto z_{i_1}z_{i_2}\cdots z_{i_k}, \quad \text{for } I = \{i_1, i_2, \ldots, i_k\}.$$ 

The kernel of this homomorphism, ker $\varphi$, is the toric ideal of $\Delta_{n,k}$. The correspondence between Gröbner bases for ker $\varphi$ and regular triangulations of $\Delta_{n,k}$ is given as follows. Any sufficiently generic height vector induces a regular triangulation of $\Delta_{n,k}$. On the other hand, such a height vector induces a term order $<$ on the monomials in the polynomial ring $k[x_I]$. Thus, we may identify a Gröbner basis for ker $\varphi$ with respect to this term order, say $G_\prec$. Moreover, the initial ideal associated to a Gröbner basis is square-free if and only if the corresponding regular triangulation is unimodular. The details of this correspondence are outlined nicely in [12].

Theorem 2.2. [12] Sturmfels] The set of quadratic binomials

$$G_\prec := \left\{ x_I x_J - x_{U(I,J)} x_{V(I,J)} : I, J \in \binom{[n]}{k} \right\}$$

is a Gröbner basis for ker $\varphi$ under some term order $<$ on $k[x_I]$ such that the underlined term is the initial monomial. In particular, the initial ideal of $G_\prec$ is square-free, and the simplices of the corresponding unimodular triangulation are $\sigma_{\mathcal{I}}$, where $\mathcal{I}$ varies over the sorted collections of $k$-element subsets of $[n]$.

We denote this triangulation of $\Delta_{n,k}$ by $\nabla_{n,k}$, and we let $\max \nabla_{n,k}$ denote the set of maximal simplices in $\nabla_{n,k}$. In [7], Lam and Postnikov prove a more general version of Theorem 2.2 which we will utilize to show that this triangulation restricts to a triangulation of the $r$-stable hypersimplex $\Delta_{n,k}^{\text{stab}(r)}$.

2.1.2. The Circuit Triangulation. The second construction of this triangulation that we will utilize first appeared in [7], and it arises from examining minimal length circuits in a particular direct graph with labeled edges. We construct this directed graph as follows. Let $G_{n,k}$ be the directed graph with vertices $\epsilon_I$, where $I$ varies over all $k$-subsets of $[n]$. For a vertex $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ of $G_{n,k}$ we think of the coordinate indices $i$ as elements of the cyclic group $\mathbb{Z}/n\mathbb{Z}$. Hence, $\epsilon_{i+1} = \epsilon_1$. We construct the directed, labeled edges of $G_{n,k}$ as follows. Suppose $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ and $\epsilon'$ are vertices of $G_{n,k}$ for which $(\epsilon_i, \epsilon_{i+1}) = (1, 0)$ and the vector $\epsilon'$ is obtained from $\epsilon$ by switching $\epsilon_i$ and $\epsilon_{i+1}$. Then we include the directed labeled edge $\epsilon \xrightarrow{1} \epsilon'$ in $G_{n,k}$. Hence, each edge of $G_{n,k}$ is given by shifting a 1 in a vertex $\epsilon$ exactly one entry to the right (modulo $n$), and this can happen if and only if the next place is occupied by a 0.

We are interested in the circuits of minimal possible length in the graph $G_{n,k}$. We will call such a circuit minimal. A minimal circuit in $G_{n,k}$ containing the vertex $\epsilon$ is given by a sequence of edges moving each 1 in $\epsilon$ into the position of the 1 directly to its right. Hence, the length of such a circuit is precisely $n$. An example of a minimal circuit is given in Figure 1.

For a fixed initial vertex, the sequence of labels of edges in a minimal circuit forms a permutation $\omega = \omega_1 \omega_2 \cdots \omega_n \in S_n$, the symmetric group on $n$ elements. There is one such permutation for each choice of initial vertex in the minimal circuit. Hence, a minimal circuit in $G_{n,k}$ corresponds to an equivalence class of permutations in $S_n$ where permutations are equivalent modulo cyclic shifts $\omega_1 \cdots \omega_n \sim \omega_n \omega_1 \cdots \omega_{n-1}$. In the following, we choose the representative $\omega$ of the class of
permutations associated to the minimal circuit for which \( \omega_n = n \). We remark that this corresponds to picking the initial vertex of the minimal circuit to be the lexicographically maximal \((0,1)\)-vector in the circuit. For example, the lexicographic ordering on the \((0,1)\)-vectors in the circuit depicted in Figure 1 is

\[
(1,0,1,0,0) > (1,0,0,1,0) > (0,1,0,1,0) > (0,1,0,0,1) > (0,0,1,0,1),
\]

and the permutation given by reading the edge labels of this circuit beginning at vertex \((1,0,1,0,0)\) is \( \omega = 31425 \). Thus, we see that \( \omega_n = n \) as desired.

**Theorem 2.3.** [7 Lam and Postnikov] A minimal circuit in the graph \( G_{n,k} \) corresponds uniquely to a permutation \( \omega \in S_n \) modulo cyclic shifts. Moreover, a permutation \( \omega \in S_n \) with \( \omega_n = n \) corresponds to a minimal circuit in \( G_{n,k} \) if and only if the inverse permutation \( \omega^{-1} \) has exactly \( k-1 \) descents.

We label the minimal circuit in the graph \( G_{n,k} \) corresponding to the permutation \( \omega \in S_n \) with \( \omega_n = n \) by \( (\omega) \). Let \( v(\omega) \) denote the set of all vertices \( \epsilon_I \) of \( \Delta_{n,k} \) used by the circuit \( (\omega) \), and let \( \sigma(\omega) \) denote the convex hull of \( v(\omega) \).

**Theorem 2.4.** [7 Lam and Postnikov] The collection of simplices \( \sigma(\omega) \) corresponding to all minimal circuits in \( G_{n,k} \) forms the collection of maximal simplices of a triangulation of the hypersimplex \( \Delta_{n,k} \). This triangulation is identical to the triangulation \( \nabla_{n,k} \).

We call this construction of \( \nabla_{n,k} \) the circuit triangulation. To simplify notation we will often write \( \omega \) for the simplex \( \sigma(\omega) \in \nabla_{n,k} \).

### 2.2. The induced triangulation of the \( r \)-stable hypersimplex

Let \( \mathcal{M} \) be a collection of \( k \)-subsets of \([n] \), and let \( \mathcal{P}_{\mathcal{M}} \) denote the convex hull in \( \mathbb{R}^n \) of the \((0,1)\)-vectors \( \{\epsilon_I : I \in \mathcal{M} \} \). Notice that \( \mathcal{P}_{\mathcal{M}} \) is a subpolytope of \( \Delta_{n,k} \). The collection \( \mathcal{M} \) is said to be sort-closed if for every pair of subsets \( I \) and \( J \) in \( \mathcal{M} \) the subsets \( U(I,J) \) and \( V(I,J) \) are also in \( \mathcal{M} \). In [7], Lam and Postnikov proved the following theorem.

**Theorem 2.5.** [7 Lam and Postnikov] The triangulation \( \nabla_{n,k} \) of the hypersimplex \( \Delta_{n,k} \) induces a triangulation of the polytope \( \mathcal{P}_{\mathcal{M}} \) if and only if \( \mathcal{M} \) is sort-closed.

We then have the following corollary to this theorem.

**Corollary 2.6.** Fix an integer \( 0 < r \leq \begin{pmatrix} n \end{pmatrix} \). Let \( \mathcal{M} \) be the collection of \( r \)-stable \( k \)-subsets of \([n]\). The triangulation \( \nabla_{n,k} \) induces a triangulation of the \( r \)-stable hypersimplex \( \mathcal{P}_{\mathcal{M}} = \Delta_{n,k}^{\text{stabil}(r)} \).

**Proof.** By Theorem 2.5 it suffices to show that the collection \( \mathcal{M} \) is sort-closed. Let \( I \) and \( J \) be two elements of \( \mathcal{M} \), and consider \( \text{sort}(I \cup J) = (a_1, a_2, \ldots, a_{2k}) \). Suppose for the sake of contradiction that for some \( i \), \( a_{i+2} = a_i + t \) for some \( t \in [r-1] \). Here we think of our indices and addition modulo \( n \). We remark that \( t \) must be nonzero since the multiplicity of each element of \([n]\) in \( \text{sort}(I \cup J) \) is at most two. Without loss of generality, we assume that \( a_i \in I \). Hence, \( a_{i+2} \in J \). Since \( I \) is \( r \)-stable and \( \text{cd}(a_i, a_{i+2}) < r \). Since \( \text{sort}(I \cup J) \) is nondecreasing it follows that \( a_{i+1} = a_i + j \) for some \( j \in \{0, 1, \ldots, t\} \). First consider the cases where \( j = 0 \) and \( j = t \). In the former case we have that \( a_{i+1} = a_i \), and in the latter case \( a_{i+1} = a_{i+2} \). Hence, in the former case, the multiplicity of
a_i in $\text{sort}(I \cup J)$ is two. Thus, $a_i$ appeared in both $I$ and $J$. Since $a_{i+2} \in J$, this contradicts the assumption that $J$ is $r$-stable. Similarly, in the latter case the multiplicity of $a_{i+2}$ in $\text{sort}(I \cup J)$ is two, so $a_{i+2}$ must also appear in $I$, and this contradicts the assumption that $I$ is $r$-stable. Now suppose that $0 < j < t$. Then since $I$ is $r$-stable and $a_i \in I$, it must be that $a_{i+1} \in J$. But since $J$ is $r$-stable and $a_{i+2} \in J$, then $a_{i+1} \in I$, a contradiction. □

We let $\nabla_{n,k}$ denote the triangulation of $\Delta^\text{stab}_{n,k}$ induced by $\nabla_{n,k}$. This gives the following nesting of triangulations

$$\nabla_{n,k} \supset \nabla^2_{n,k} \supset \nabla^3_{n,k} \supset \cdots \supset \nabla^{\lfloor \frac{n}{k}\rfloor}_{n,k}.$$  

**Lemma 2.7.** If $n \equiv 1 \mod k$, then $\Delta^\text{stab}_{n,k}$ is $(n-1)$-dimensional for all $r \in \lfloor \frac{n}{k}\rfloor$. In particular, $\Delta^{\lfloor \frac{n}{k}\rfloor}_{n,k}$ is a unimodular $(n-1)$-simplex.

Proof. Notice first that for $r = \lfloor \frac{n}{k}\rfloor$ there are precisely $n$ $r$-stable $k$-subsets of $[n]$. Hence, $\Delta^\text{stab}_{n,k}$ is an $(n-1)$-dimensional simplex. Now suppose $\epsilon$ is a vertex of this simplex. Then precisely $k$ entries in $\epsilon$ are occupied by $1$'s, $k-1$ pairs of these $1$'s are separated by $r-1$ $0$'s, and the remaining pair is separated by $r$ $0$'s. Hence, the only $1$ that can be moved to the right and result in another $r$-stable vertex is the left-most $1$ in the pair of $1$'s separated by $r$ $0$'s. Making this move $n$ times results in returning to the vertex $\epsilon$, and produces a minimal circuit $(\omega)$ in $G_{n,k}$ using only $r$-stable vertices.

Since there are only $n$ such vertices it must be that $\sigma(\omega) = \Delta^\text{stab}_{n,k}$. We may also prove this result using Sturmfels' construction of this triangulation. Simply notice that there are precisely $n$ $\lfloor \frac{n}{k}\rfloor$-stable $k$-subsets of $[n]$, namely

$$\{1, 1 + \lfloor \frac{n}{k}\rfloor, 1 + 2 \lfloor \frac{n}{k}\rfloor, \ldots, 1 + (k - 1) \lfloor \frac{n}{k}\rfloor\},$$

$$\{2, 2 + \lfloor \frac{n}{k}\rfloor, 2 + 2 \lfloor \frac{n}{k}\rfloor, \ldots, 2 + (k - 1) \lfloor \frac{n}{k}\rfloor\},$$

$$\vdots$$

$$\{n, n + \lfloor \frac{n}{k}\rfloor, n + 2 \lfloor \frac{n}{k}\rfloor, \ldots, n + (k - 1) \lfloor \frac{n}{k}\rfloor\}.$$  

It is easy to see that these subsets form a sorted collection of $k$-subsets of $[n]$. Hence, they correspond to a unimodular $(n-1)$-simplex in the triangulation $\nabla_{n,k}$. □

In the coming sections we utilize the triangulation $\nabla^r_{n,k}$ to investigate geometric properties of the subpolytope $\Delta^\text{stab}_{n,k}$ and its relationship with $\Delta_{n,k}$.

3. A Stable Shelling of the $r$-stable Odd Second Hypersimplex

Triangulations have many useful properties and well-studied applications in Ehrhart Theory. Given a triangulation $\nabla$ of a $d$-dimensional polytope $P$ let $\max \nabla$ denote the set of $d$-dimensional simplicies in $\nabla$. We call an ordering of the simplicies in $\max \nabla$, $(\alpha_1, \ldots, \alpha_s)$, a *shelling* of $\nabla$ if for each $2 \leq i \leq s$, $\alpha_i \cap (\alpha_1 \cup \cdots \cup \alpha_{i-1})$ is a union of facets ($(d-1)$-dimensional faces) of $\alpha_i$. An equivalent condition for a shelling is that every $\alpha_i$ has a unique minimal (with respect to dimension) face that is not a face of the previous simplicies [9]. A triangulation with a shelling is called *shellable*. For a shelling and a maximal simplex $\alpha$ in the triangulation define the *shelling number* of $\alpha$, denoted $\#(\alpha)$, to be the number of facets shared by $\alpha$ and some previous simplex. The following theorem is due to Stanley.

**Theorem 3.1.** [11 Stanley] Let $\nabla$ be a unimodular shellable triangulation of a $d$-dimensional polytope $P$. Then

$$\sum_{j=0}^{d} h_j^* z^j = \sum_{\alpha \in \max \nabla} z^{\#(\alpha)}.$$
Proof. Lemma 3.4 indicates that \( n \) of \( \Delta_{n,2} \). Consider a simplex \( \text{Corollary 3.5.} \) Since \( \epsilon \) Let \( \epsilon \Leftarrow \). \( \omega \) this choice, we then associate to \( \omega \) \( \omega \) \( r \)-adjacent vertices used, and then ordering the elements within these sets via the colexicographic ordering applied to their associated compositions. We then utilize their associated lattice paths to identify the unique minimal new face for each simplex. In particular, we shell the simplices in terms of least \( r \)-adjacent vertices used to most \( r \)-adjacent vertices used.

3.1. \( r \)-adjacent vertices. For \( \ell \in [n] \), let \( \text{adj}_r(\ell) \) denote the vertex \( \epsilon \) where \( I = \{ \ell, r \} \in \binom{[n]}{2} \). We call a vertex \( \text{adj}_r(\ell) \) an \( r \)-adjacent vertex. Let \( \text{Adj}_r[n] := \{ \text{adj}_r(\ell) : \ell \in [n] \} \). So \( \text{Adj}_r[n] \) is precisely the set of vertices that are \( r \)-stable but not \((r+1)\)-stable.

Lemma 3.4. Let \( \epsilon \) and \( \epsilon' \) be two vertices in \( (\omega) \) for some simplex \( \omega \in \max \nabla_{n,2} \). Suppose that \( \epsilon \) has entries \( \epsilon_i = \epsilon_j = 1 \) with \( i < j \), and \( \epsilon_t = 0 \) for all \( t \neq i,j \). Suppose also that \( \epsilon' \) has entries \( \epsilon'_k = \epsilon'_l = 1 \) with \( k < l \), and \( \epsilon'_t = 0 \) for all \( t \neq k,l \). Then \( (\omega) \) we have that \( i \leq k \leq j \leq l \).

Proof. Since \( \epsilon \) and \( \epsilon' \) are vertices of a simplex in \( \nabla_{n,2} \) they correspond to a sorted pair of 2-subsets of \([n]\). \( \square \)

Corollary 3.5. Let \( \omega \in \max \nabla_{n,2} \), and suppose that \( \text{adj}_r(\ell) \) and \( \text{adj}_r(\ell') \) are vertices in \( v(\omega) \cap \text{Adj}_r[n] \). Then \( \text{cd}(\ell, \ell') \leq r \).

Proof. Lemma 3.4 indicates that
\[
\ell \leq \ell' \leq \ell + r \leq \ell' + r, \quad \text{or}
\ell' \leq \ell \leq \ell' + r \leq \ell + r.
\]

Remark 3.6. Consider a simplex \( \omega \in \max \nabla_{n,2} \). Notice that for a fixed \( 0 < r < \left\lfloor \frac{n}{2} \right\rfloor \) we may order the elements of the set \( v(\omega) \cap \text{Adj}_r[n] \) as \( \text{adj}_r(\ell) <_{\text{adj}} \text{adj}_r(\ell') \) if and only if \( \ell < \ell' \). In this way, there exists a unique maximal element of the set \( v(\omega) \cap \text{Adj}_r[n] \).
The minimal circuit corresponding to the simplex
\( \omega = 5671892(10)(11)(12)(13)3(14)4(15) \).

3.2. **Associating a composition to** \( \omega \in \max \, \nabla_{n,2}^r \setminus \nabla_{n,2}^{r+1} \). Fix \( \omega \in \max \, \nabla_{n,2}^r \setminus \nabla_{n,2}^{r+1} \). Then \( \omega \) uses at least one element of \( \text{Adj}_i[n] \). We may fix one such \( \text{Adj}_i(\ell) \), and consider the circuit \( (\omega) \) as having initial vertex \( \text{Adj}_i(\ell) \). We refer to the 1 in entry \( \ell \) of the vertex \( \text{Adj}_i(\ell) \) as the left 1 and the 1 in entry \( \ell + r \) as the right 1. Then, in \( (\omega) \) each edge corresponds to a move of the left 1 or of the right 1. In particular,

(\#) the left 1 makes \( r \) moves,

(\#) the right 1 makes \( n - r \) moves, and

(\#) the left 1 cannot move first or last.

Note that the first two conditions are immediate from the definition of \( (\omega) \) and the fact that \( k = 2 \). The third condition holds since \( \omega \) uses only vertices that are \( r \)-stable. It follows that for a fixed \( \text{Adj}_i(\ell) \in v(\omega) \) we can think of the circuit \( (\omega) \) as a sequence of moves of the left 1 and moves of the right 1 satisfying these conditions. Moreover, we may encode this as a composition

\[ \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n-r-1}) \]

do to \( r \) into \( n - r - 1 \) parts, where part \( \lambda_i \) denotes the number of moves of the left 1 after the \( i \)th move of the right 1 and before the \( (i + 1) \)th move of the right 1.

**Example 3.7.** Consider the minimal circuit in \( G_{15,2} \) depicted in Figure 2. This circuit corresponds to a simplex \( \omega \in \max \, \nabla_{15,2}^4 \setminus \nabla_{15,2}^5 \). If we choose the initial vertex of this circuit to be the unique maximal element of the set \( v(\omega) \cap \text{Adj}_4[15] \), namely \( \text{Adj}_4(15) \), then this circuit has associated composition \( \lambda = (1, 0, 0, 1, 0) \).

**Example 3.8.** Next consider the minimal circuit in \( G_{9,2} \) depicted in Figure 3. This circuit corresponds to a simplex \( \omega \in \max \, \nabla_{9,2}^3 \setminus \nabla_{9,2}^4 \). If we choose the initial vertex of this circuit to be the unique maximal element of the set \( v(\omega) \cap \text{Adj}_3[9] \), namely \( \text{Adj}_3(7) \), then this circuit has associated composition \( \lambda = (0, 0, 3, 0) \).
Figure 3. The minimal circuit corresponding to the simplex $\omega = 456123789$.

**Proposition 3.9.** Fix $\mathrm{adj}_r(\ell) \in \mathrm{Adj}_r[n]$. Each simplex $\omega \in \max \mathcal{V}_{n,2} \setminus \max \mathcal{V}_{n,2}^{r+1}$ that uses the vertex $\mathrm{adj}_r(\ell)$ corresponds uniquely to a composition

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n-r-1})$$

of $r$ into $n - r - 1$ parts that satisfies

$$i + 1 + 2r - n \leq \sum_{j=1}^{i} \lambda_j \leq i$$

for all $i = 1, 2, \ldots, n - r - 1$.

**Proof.** Let $\omega \in \max \mathcal{V}_{n,2} \setminus \max \mathcal{V}_{n,2}^{r+1}$ that uses vertex $\mathrm{adj}_r(\ell)$. Then (\omega) is a minimal circuit in the directed graph $G_{n,2}$, one of whose vertices is $\mathrm{adj}_r(\ell)$. Thinking of $\mathrm{adj}_r(\ell)$ as the initial vertex we consider the 1 in place $\ell$ as the left 1 and the 1 in place $\ell + r$ as the right 1. By the above conditions it is clear that we may construct the partition $\lambda$ of $r$ into $n - r - 1$ parts, where part $\lambda_i$ denotes the number of moves of the left 1 after the $i^{th}$ move of the right 1 and before the $(i + 1)^{st}$ move of the right 1. Since $\omega \in \max \mathcal{V}_{n,2}$ the left 1 can never have made more moves that the right 1. This gives the upper bound on $\sum_{j=1}^{i} \lambda_j$ for each $i = 1, 2, \ldots, n - r - 1$.

Similarly, since $\omega \in \max \mathcal{V}_{n,2}$ after the $(r+1)^{st}$-to-last move of the right 1 and before its $r^{th}$-to-last move we must have that the left 1 moved at least once. More generally, after the $n - r - t + 1^{st}$ move of the right 1 we must have that the left 1 moved at least $r - t + 2$ times for $t = r + 1, r, r - 1, \ldots, 2$. Hence, the number of left moves that occur after the $i^{th}$ right move and before the $(i + 1)^{st}$ right move is at least $i + 1 + 2r - n$. This gives the lower bound on $\sum_{j=1}^{i} \lambda_j$.

Conversely, suppose that we have a composition $\lambda$ satisfying the given conditions. We can construct a simplex $\omega(\lambda) \in \max \mathcal{V}_{n,2} \setminus \max \mathcal{V}_{n,2}^{r+1}$ that uses the vertex $\mathrm{adj}_r(\ell)$ by constructing a minimal circuit in $G_{n,2}$ as follows. Starting with $\mathrm{adj}_r(\ell)$, and labeling the left 1 and right 1 as we have been, after the $i^{th}$ move of the right 1 move the left 1 $\lambda_i$ times. Once this has been done for all $i = 1, 2, \ldots, n - r - 1$, move the right 1 once more. The upper bound ensures that the right distance between the 1s is always at least $r$. Similarly, the lower bound ensures that the left distance is always at least $r$. Since $\mathrm{adj}_r(\ell)$ is in the circuit $(\omega(\lambda))$ then this corresponds to a simplex $\omega(\lambda) \in \max \mathcal{V}_{n,2} \setminus \max \mathcal{V}_{n,2}^{r+1}$.

**Remark 3.10.** By Remark 3.6 we can identify each simplex $\omega \in \max \mathcal{V}_{n,2} \setminus \max \mathcal{V}_{n,2}^{r+1}$ with the unique maximal element of $\mathcal{V}(\omega) \cap \mathrm{Adj}_r[n]$, say $\mathrm{adj}_r(\ell)$. Let $\lambda$ be the composition associated to $\omega$ via $\mathrm{adj}_r(\ell)$ by Proposition 3.9. Then we may uniquely label the simplex $\omega$ as $\omega_{\ell,\lambda}$.

**Definition 3.11.** For a simplex $\omega_{\ell,\lambda}$ recall that we think of the 1 in entry $\ell$ of $\mathrm{adj}_r(\ell)$ as the left 1, and the 1 in entry $\ell + r$ as the right 1.
A SHELLING OF THE ODD SECOND HYPERSIMPLEX

\begin{align*}
y &= x \quad y &= x + n - 2r \\
\end{align*}

adj (1) 4
adj (2) 4
adj (3) 4
adj (4) 4
adj (5) 4
adj (6) 4
adj (7) 4
adj (8) 4
adj (9) 4
adj (10) 4
adj (11) 4
adj (12) 4
adj (13) 4
adj (14) 4
adj (15) 4

Figure 4. The lattice path \( p(\omega_{15, \lambda}) \), where \( \lambda = (1, 0, 0, 1, 0, 0, 0, 0, 0, 1) \).

• A left move in \((\omega_{\ell, \lambda})\) is an edge in \((\omega_{\ell, \lambda})\) corresponding to a move of the left 1, and
• A right move in \((\omega_{\ell, \lambda})\) is an edge in \((\omega_{\ell, \lambda})\) corresponding to a move of the right 1.
• The parity of an edge in \((\omega_{\ell, \lambda})\) is left if the edge is a left move, and right if the edge is a right move.

Remark 3.12 (Lattice Path Correspondence). Notice that each simplex \( \omega_{\ell, \lambda} \) in the set \( \max \nabla^{r+1}_{n, 2} \setminus \max \nabla^{r+2}_{n, 2} \) corresponds to a lattice path, \( p(\omega_{\ell, \lambda}) \), from \((0, 0)\) to \((n - r, r)\) that uses only North \((0,1)\) and East \((1,0)\) moves. Here, right moves in the circuit \((\omega_{\ell, \lambda})\) correspond to East moves in \( p(\omega_{\ell, \lambda}) \), and left moves in \((\omega_{\ell, \lambda})\) correspond to North moves in \( p(\omega_{\ell, \lambda}) \). By Proposition 3.9 the lattice path \( p(\omega_{\ell, \lambda}) \) is bounded between the lines \( y = x \) and \( y = x - n + 2r \). Each vertex in \((\omega_{\ell, \lambda})\) corresponds uniquely to a lattice point on \( p(\omega_{\ell, \lambda}) \). In particular, for \( 0 \leq t \leq r \) the vertex \( \text{adj}_{\ell}(t, r + t) \) corresponds to the lattice point \((t, n - 2r + t)\).

Here are some examples of simplices and their corresponding lattice paths.

Example 3.13. Let \( n = 15 \) and \( r = 4 \). Recall the simplex from Example 3.7

\[ \omega = 5671892(10)(11)(12)(13)3(14)4(15) \in \max \nabla^{4}_{15, 2} \setminus \max \nabla^{5}_{15, 2}. \]

This simplex corresponds to the minimal circuit in the graph \( G_{15, 2} \) depicted in Figure 2.

From this, we can see that \( \omega \) uses the vertices \( \text{adj}_{4}(15) \), \( \text{adj}_{4}(1) \), and \( \text{adj}_{4}(14) \). Hence, we label \( \omega \) as \( \omega_{15, \lambda} \), where

\[ \lambda = (1, 0, 0, 1, 0, 1, 0, 0, 0, 1). \]

The lattice path corresponding to \( \omega \) via this labeling is depicted in Figure 4.

Example 3.14. Let \( n = 9 \) and \( r = 3 \). Recall the simplex from Example 3.8

\[ \omega = 456123789 \in \max \nabla^{3}_{9, 2} \setminus \max \nabla^{4}_{9, 2}. \]

This simplex corresponds to the minimal circuit in the graph \( G_{9, 2} \) depicted in Figure 3.

From this, we can see that \( \omega \) uses the vertices \( \text{adj}_{3}(7) \), \( \text{adj}_{3}(4) \), and \( \text{adj}_{3}(1) \). Hence, we label \( \omega \) as \( \omega_{7, \lambda} \), where

\[ \lambda = (0, 0, 3, 0, 0). \]

The lattice path corresponding to \( \omega \) via this labeling is depicted in Figure 5.
3.3. The shelling order. Recall that the *colexicographic* order on a pair of ordered \( m \)-tuples \( a = (a_1, \ldots, a_m) \) and \( b = (b_1, \ldots, b_m) \) is defined by \( b <_{\text{colex}} a \) if and only if the right-most nonzero entry in \( a - b \) is positive. Let \( W_{\ell,s} \) denote the set of all simplices with label \( \omega_{\ell,\lambda} \) that use precisely \( s \) elements of \( \text{Adj}_r[n] \). Order the elements in each set \( W_{\ell,s} \) with respect to the colexicographic ordering on their associated compositions (from least to greatest). We write \( \omega_{\ell,\lambda} <_{\text{colex}} \omega_{\ell,\lambda'} \) if and only if \( \lambda <_{\text{colex}} \lambda' \). Next order the sets \( W_{\ell,s} \) (from least to greatest) with respect to the colexicographic ordering on the labels \( (\ell, s) \). We then write \( \omega_{\ell,\lambda} < \omega_{\ell',s'} \) if and only \( \omega_{\ell,\lambda} \in W_{\ell,s} \) and \( \omega_{\ell',s'} \in W_{\ell',s'} \) with \( (\ell, s) <_{\text{colex}} (\ell', s') \) or if \( (\ell, s) = (\ell', s') \) and \( \omega_{\ell,\lambda} <_{\text{colex}} \omega_{\ell,\lambda'} \).

**Theorem 3.15.** The order \( < \) on the simplices \( \omega \in \max_{n,2}^{\nabla} \max_{n,2}^{\nabla +1} \) (from least to greatest) extends the shelling of \( \nabla_{n,2}^{\nabla +1} \) to a shelling of \( \nabla_{n,2} \).

Theorem 3.2 will follow immediately from Theorem 3.15. To prove Theorem 3.15 it suffices to identify the unique minimal new face associated to each simplex in the shelling order. To do so, we first prove a sequence of lemmas.

**Lemma 3.16.** Suppose the \( \omega_{\ell,\lambda} \) uses \( \text{adj}_r(\ell') \) for \( \ell' \neq \ell \). Then \( \text{adj}_r(\ell') \) is a vertex in \( \omega_{\ell,\lambda} \) that is either

(i) produced by a right move for which the preceding number of left moves is minimal and not maximal with respect to equation (1), or

(ii) produced by a left move and followed by a right move for which the number of left moves preceding the right move is maximal and not minimal with respect to equation (1).

**Proof.** Since \( \text{adj}_r(\ell) \) is selected to be the greatest element of \( v(\omega) \cap \text{Adj}_r[n] \) then each other \( \text{adj}_r(\ell') \) used by \( \omega_{\ell,\lambda} \) is produced in \( \omega_{\ell,\lambda} \) by doing \( n - 2r + t \) right moves for some number \( t \) of left moves, or \( \text{adj}_r(\ell') \) is produced by doing \( 0 < t < r \) right moves and the same number of left moves. In the former case, such a vertex corresponds to an entry \( \lambda_m \) in the composition \( \lambda \) with \( m = n - 2r + t \) for which

\[
t = m + 2r - n = (m - 1) + 1 + 2r - n \leq \sum_{j=1}^{m-1} \lambda_j = t.
\]

Hence, the number of left moves preceding the \( m^{th} \) right move is minimal. Moreover, the number of left moves preceding the \( m^{th} \) right move is maximal only if

\[
m + 2r - n = k = \sum_{j=1}^{m-1} \lambda_j = m - 1.
\]
Thus,
\[ r = \frac{n - 1}{2} = \left\lfloor \frac{n}{2} \right\rfloor. \]

But recall that since \( \Delta_{n,2}^{stab}(\left\lfloor \frac{n}{2} \right\rfloor) \) is a unimodular \((n-1)\)-simplex we are only completing the shelling of \( \nabla^{r+1} \) to a shelling of \( \nabla^r \) for \( r < \left\lfloor \frac{n}{2} \right\rfloor \). So we conclude that the sum is minimal and not maximal.

In the latter case, the vertex \( \adj_r(\ell_j) \) is produced by doing \( 0 < t < r \) right moves and the same number of left moves. Thus, following this vertex with another left move results in a vertex that is no longer \( r \)-stable. So the move following \( \adj_r(\ell_j) \) must be a right move. Such a vertex corresponds to an entry \( \lambda_m \) in the composition \( \lambda \) for which \( m = t \), and the right move following the vertex is the \((m+1)st\) right move in the circuit. Thus,
\[ m + 1 + 2r - n \leq \sum_{j=1}^{m} \lambda_j = t = m. \]

Hence, the number of left moves preceding the right move following the vertex is maximal. If this number is also minimal then it must be that
\[ m + 1 + 2r - n = m, \]
\[ r = \frac{n - 1}{2} = \left\lfloor \frac{n}{2} \right\rfloor, \]
and so we conclude that the sum is not minimal just as in the previous case. It remains to show that the move preceding the vertex \( \adj_r(\ell_j) \) is a left move. Suppose for the sake of contradiction that \( \adj_r(\ell_j) \) is preceded by a right move. Then \( \lambda_m = 0 \). Thus, since the number of left moves preceding the \((m+1)st\) right move is maximal we have that
\[ m = \sum_{j=1}^{m} \lambda_j = \sum_{j=1}^{m-1} \lambda_j \leq m - 1, \]
which is a contradiction. Thus, we conclude that \( \adj_r(\ell_j) \) is produced by a left move and followed by a right move for which the number of left moves preceding the right move is not minimal. □

**Lemma 3.17.** Suppose that the simplex \( \omega_{\ell,\lambda} \) uses the elements \( \adj_r(\ell_1) \prec_{\adj} \adj_r(\ell_2) \prec_{\adj} \cdots \prec_{\adj} \adj_r(\ell_s) = \adj_r(\ell) \) of \( \text{Adj}_r[n] \). For \( j \neq s \), the parities of the edges preceding \( \adj_r(\ell_j) \) in \( (\omega_{\ell,\lambda}) \) and following \( \adj_r(\ell_j) \) are opposite. Also, the parity of the edges about \( \adj_r(\ell) \) is right.

**Proof.** First recall that we have already noted that the first and last moves of \( (\omega_{\ell,\lambda}) \) must be right moves. Hence, the parity of the edges about \( \adj_r(\ell) \) is right.

Now consider \( \adj_r(\ell_j) \) for \( j \neq s \). By Lemma 3.16 we have two cases. In case (ii), \( \adj_r(\ell_j) \) is produced by a left move and followed by a right move for which the number of left moves preceding the right move is not minimal. Hence, the result is immediate.

In case (i), \( \adj_r(\ell_j) \) is produced by a right move for which the preceding number of left moves is minimal and not maximal. Suppose for the sake of contradiction that the parities of the moves about \( \adj_r(\ell_j) \) are the same. So if \( \adj_r(\ell_j) \) is produced by the \( m^{th} \) right move we have that
\[ m + 2r - n = (m - 1) + 1 + 2r - n = \sum_{j=1}^{m-1} \lambda_j. \]
Since the parities of the edges about $\text{adj}_r(\ell_j)$ are the same it is followed by a right move, and so it must be that $\lambda_m = 0$. Hence, by equation (11)
\[
m + 1 + 2r - n \leq \sum_{j=1}^{m} \lambda_j = \sum_{j=1}^{m-1} \lambda_j = m + 2r - n,
\]
which is a contradiction. \hfill \Box

**Lemma 3.18.** Suppose that the simplex $\omega_{\ell, \lambda}$ uses the vertex $\text{adj}_r(\ell')$. Then switching the parities of the moves about $\text{adj}_r(\ell')$ does not replace $\text{adj}_r(\ell')$ with another vertex in $\text{Adj}_r[n]$.

**Proof.** First consider the case where $\ell' \neq \ell$. By Remark 3.12 the simplex $\omega_{\ell, \lambda}$ corresponds to a lattice path $p(\omega_{\ell, \lambda})$ that is bounded between the lines $y = x$ and $y = x - n + 2r$, and the elements of $\text{Adj}_r[n]$ reachable from $\text{adj}_r(\ell')$ all lie on these two lines. Suppose for the sake of contradiction that switching the parities of the moves about $\text{adj}_r(\ell')$ replaced this vertex with another element of $\text{Adj}_r[n]$, say $\text{adj}_r(\ell'')$. Then the resulting change in the lattice path $p(\omega_{\ell, \lambda})$ implies that $\text{adj}_r(\ell'')$ lies on the opposite of these two lines from that on which $\text{adj}_r(\ell')$ lies. It then follows that $n - 2r = 2$, or equivalently, $n = 2r + 2$. Since we have chosen $n$ to be odd this is a contradiction.

Now consider the case where $\ell' = \ell$. Suppose for the sake of contradiction that switching the parities of the moves about $\text{adj}_r(\ell)$ replaces $\text{adj}_r(\ell)$ with another vertex $\text{adj}_r(\ell'')$. Consider the vertex before the right move producing $\text{adj}_r(\ell)$. Since this right move produces $\text{adj}_r(\ell)$ then in this preceding vertex there must be precisely $r$ 0’s to the right of the right 1 and before the left 1. Similarly, since the left move produces the vertex $\text{adj}_r(\ell'')$ it must be that there are $r$ 0’s to the right of the left 1 and before the right 1. Hence, $n = 2r + 2$, a contradiction. \hfill \Box

**Lemma 3.19.** Suppose that the simplex $\omega_{\ell, \lambda}$ uses the elements
\[
\text{adj}_r(\ell_1) <_{\text{adj}} \text{adj}_r(\ell_2) <_{\text{adj}} \cdots <_{\text{adj}} \text{adj}_r(\ell_s) = \text{adj}_r(\ell)
\]
of $\text{Adj}_r[n]$. Switching the parity of the edges about $\text{adj}_r(\ell_j)$ replaces the vertex $\text{adj}_r(\ell_j)$ with an $(r + 1)$-stable vertex not in $\text{Adj}_r[n]$, and leaves all other vertices in $(\omega_{\ell, \lambda})$ fixed.

**Proof.** First fix $\text{adj}_r(\ell_j)$ for $j \neq s$, and switch the parity of the moves directly before and after $\text{adj}_r(\ell_j)$ in $(\omega_{\ell, \lambda})$. By Lemma 3.16 there are two cases. In case (i), Lemma 3.17 implies that the parity switch changes the move before from a right to a left, and the move after from a left to a right. Since each vertex in the circuit is determined by the number of left moves and right moves by which it differs from $\text{adj}_r(\ell)$ this switching does not change any of the vertices preceding $\text{adj}_r(\ell_j)$ in $(\omega_{\ell, \lambda})$. Similarly, it does not change any of the vertices following $\text{adj}_r(\ell_j)$. The reader should also note that this switch changes the composition $\lambda$. However, Lemma 3.16 ensures that the resulting composition, say $\lambda'$, still satisfies the bounds of equation (11). Hence, by Proposition 3.9 this switch produces a circuit $(\omega_{\ell', \lambda'})$ for which $\omega_{\ell', \lambda'} \in \max \nabla^{r-1}_{n, 2} \setminus \max \nabla^{r+1}_{n, 2}$. Moreover, the vertex which replaces $\text{adj}_r(\ell_j)$ is not an element of $\text{Adj}_r[n]$ by Lemma 3.18. As an example, consider the following scenario for $r = 3$:

\[
\begin{array}{c}
(0, 1^R, 0, 0, 0, 1^L, 0, \ldots, 0) \\
\downarrow R \\
(0, 0, 1^R, 0, 0, 0, 1^L, 0, \ldots, 0) \in \text{Adj}_3[n] \\
\downarrow L \\
(0, 0, 0, 1^R, 0, 0, 0, 1^L, 0, \ldots, 0) \\
\downarrow R \\
(0, 0, 1^R, 0, 0, 0, 0, 1^L, 0, \ldots, 0) \notin \text{Adj}_3[n]
\end{array}
\]

We remark that $\omega_{\ell, \lambda} \in W_{\ell, s}$, so $\omega_{\ell', \lambda'} \in W_{\ell, s-1}$. 

In case (ii), Lemma 3.17 implies that the parity switch changes the move before $\text{adj}_r(\ell_j)$ from a left to a right, and the move after $\text{adj}_r(\ell_j)$ from a right to a left. Now apply the same argument as for case (i), and the result follows. We again remark that since $\omega_{\ell,\lambda} \in W_{\ell,s}$ then the parity switch results in a simplex $\omega_{\ell,\lambda'} \in W_{\ell,s-1}$.

Now consider $\text{adj}_r(\ell)$. By the same argument as before, switching the parities of the moves about $\text{adj}_r(\ell)$ replaces $\text{adj}_r(\ell)$ with an $(r + 1)$-stable vertex that is not in $\text{Adj}_r[n]$. This scenario is depicted in the following diagram for $r = 3$.

$$
\begin{array}{c}
(0, 1, 0, 0, 0, 1, 0, \ldots, 0) \\
(0, 1, 0, 0, 1, 0, \ldots, 0) = \text{adj}_3(\ell) \\
(1, 0, 0, 0, 1, 0, \ldots, 0) \notin \text{Adj}_3[n] \\
(1, 0, 0, 0, 1, 0, \ldots, 0)
\end{array}
$$

Notice that we omit the labels $R$ and $L$. This is because removing $\text{adj}_r(\ell)$, the vertex which defines the labels, demands a relabeling of the resulting circuit, and in general the new labels will not agree with the old. However, this is acceptable since to switch the parities of the moves about $\text{adj}_r(\ell)$ we simply note that the vertices directly before and after $\text{adj}_r(\ell)$ in the circuit are completely determined by $\text{adj}_r(\ell)$. Moreover, the edges before and after $\text{adj}_r(\ell)$ each correspond to a move of a different 1 in $\text{adj}_r(\ell)$. Hence, to switch the parities, starting at the vertex preceding $\text{adj}_r(\ell)$ simply switch the order in which the 1’s move.

\[ \square \]

**Corollary 3.20.** For the simplex $\omega_{\ell,\lambda}$, switching the parities of the edges about $\text{adj}_r(\ell_j)$, for $j \neq s$, reduces the simplex $\omega_{\ell,\lambda} \in W_{\ell,s}$ to a simplex $\omega_{\ell,\lambda'} \in W_{\ell,s-1}$. For $s \neq 1$, switching the parities of the edges about $\text{adj}_r(\ell)$ reduces the simplex $\omega_{\ell,\lambda} \in W_{\ell,s}$ to a simplex $\omega_{\ell,s-1,\lambda'} \in W_{\ell,s-1,s-1}$. For $s = 1$, switching the parities of the edges about $\text{adj}_r(\ell)$ reduces the simplex $\omega_{\ell,\lambda} \in W_{\ell,s}$ to a simplex in $\max \nabla_{n,2}^{r+1}$.

3.3.1. **Proof of Theorem 3.15.** We are now ready to prove Theorem 3.15. Recall, to prove Theorem 3.15 it suffices to identify the unique minimal new face associated to each simplex in the shelling order. Given $\omega_{\ell,\lambda} \in W_{\ell,s}$ recall that we can associate to $\omega_{\ell,\lambda}$ a lattice path $p(\omega_{\ell,\lambda})$. Notice also that for each $\text{adj}_r(\ell) \in \text{Adj}_r[n]$ the first simplex in our order that uses $\text{adj}_r(\ell)$ is $\omega_{\ell,\lambda'}$ where

$$
\lambda^* = (0, 1, 1, \ldots, 1, 0, 0, \ldots, 0).
$$

A picture of the lattice path $p(\omega_{15,\lambda^*})$ corresponding to $\lambda^*$ for $n = 15$, $r = 4$, and $\ell = 15$ is given in Figure 6.

We claim that the unique minimal new face of $\omega_{\ell,\lambda}$ is the collection of vertices

- $\text{adj}_r(\ell)$,
- those vertices corresponding to lattice points on $p(\omega_{\ell,\lambda})$ that lie on $y = x$, and
- those vertices corresponding to lattice points on $p(\omega_{\ell,\lambda})$ that are
  - right-most in their row of the lattice,
  - corners of $p(\omega_{\ell,\lambda})$, and
  - do not lie on $p(\omega_{\ell,\lambda'})$.

That is to say, the corners of $p(\omega_{\ell,\lambda})$ that are “furthest away” or “point away” from the path $p(\omega_{\ell,\lambda'})$. 

\[ \square \]
Figure 6. The lattice path $p(\omega_{15, \lambda^*})$. Here, $\lambda^* = (0, 1, 1, 1, 0, 0, 0, 0, 0)$.

Figure 7. The lattice path $p(\omega_{15, \lambda})$ compared to $p(\omega_{15, \lambda^*})$. Here, $\lambda = (1, 0, 0, 1, 0, 1, 0, 0, 1)$.

The unique minimal new face of $\omega_{15, \lambda}$ is given by the open lattice points.

**Example 3.21.** Let $n = 15$ and $r = 4$. Consider the simplex $\omega_{15, \lambda}$ from Example 3.13. The unique minimal new face for $\omega_{15, \lambda}$ is given by the vertices

$$(0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)$$

$$(1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$$

$$(1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0)$$

$$(0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0)$$

$$(0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0)$$

These vertices correspond to the open lattice points on the path $p(\omega_{15, \lambda})$ depicted in Figure 7. The reader should note the position of these points relative to the lattice path $p(\omega_{15, \lambda^*})$.

To see these vertices form the unique minimal new face fix a simplex $\omega_{\ell, \lambda} \in W_{\ell, s}$, and suppose that this set of vertices is

$$G_\omega = \{ v_0 := \text{adj}_r(\ell), v_1, v_2, \ldots, v_q \}.$$
We will show that any face of \( \omega_{\ell, \lambda} \) not using \( G_\omega \) has previously appeared, and that \( G_\omega \) is indeed a new face.

First consider a face \( F \) of \( \omega_{\ell, \lambda} \) that does not use vertex \( v_t \in G_\omega \) for \( t \neq 0 \). There are then two cases:

1. \( v_t \in \text{Adj}_r[n] \), or
2. \( v_t \notin \text{Adj}_r[n] \).

In case (1), we saw in Corollary 3.20 that \( \omega_{\ell, \lambda} \) reduces to a previously shelled simplex that only differs for \( \omega_{\ell, \lambda} \) by the vertex \( v_t \). That is, to construct a simplex \( \omega_{\ell, \lambda'} \) that uses the face \( F \) and was shelled before \( \omega_{\ell, \lambda} \) switch the parities of the moves about \( v_t \). This results in a simplex \( \omega_{\ell, \lambda'} \in W_{\ell, \lambda - 1} \), which was therefore shelled before \( \omega_{\ell, \lambda} \).

In case (2), we can identify a previously shelled simplex \( \omega_{\ell, \lambda'} \in W_{\ell, s} \) that uses the face \( F \) and for which \( \omega_{\ell, \lambda'} < \text{colex} \omega_{\ell, \lambda} \) as follows. Since \( v_t \notin \text{Adj}_r[n] \) then it must be that \( v_t \) corresponds to a lattice point on \( p(\omega_{\ell, \lambda}) \) that is a right-most vertex on a row of the lattice, that is a corner of the path, and is also not a point on the path \( p(\omega_{\ell, \lambda'}) \) (since all points on the line \( y = x \) correspond to elements of \( \text{Adj}_r[n] \)). Hence, the vertex \( v_t \) is produced by a right move and followed by a left move in \( (\omega_{\ell, \lambda}) \). Switching the parities of these moves results in replacing \( v_t \) with a vertex \( v_t' \) which is produced by a left move and followed by a right move in the resulting cycle, say \( (\omega_{\ell, \lambda'}) \). Notice that the vertex \( v_t' \) is not an element of \( \text{Adj}_r[n] \). To see this, assume otherwise. Then by Lemma 3.16 \( v_t' \) is a vertex produced by a left move and followed by a right move for which the number of left moves preceding the right move is maximal and not minimal with respect to equation (1). Hence, the lattice point corresponding to \( v_t' \) lies on the line \( y = x \). But this implies that \( v_t \) is a vertex on \( p(\omega_{\ell, \lambda'}) \), which is a contradiction. Thus, \( v_t' \) is not an element of \( \text{Adj}_r[n] \). Notice also that it is immediate from the parity switch of the moves about \( v_t \) that \( \omega_{\ell, \lambda'} < \text{colex} \omega_{\ell, \lambda} \). Hence, \( \omega_{\ell, \lambda'} \in W_{\ell, s} \) with \( \omega_{\ell, \lambda'} < \text{colex} \omega_{\ell, \lambda} \). Moreover, since \( \omega_{\ell, \lambda'} \) uses the face \( F \) since this simplex only differs from \( \omega_{\ell, \lambda} \) by the vertex \( v_t \), which is not used in \( F \).

Now suppose \( v_t = v_0 = \text{adj}_r(\ell) \). By Corollary 3.20 we know that switching the parities about \( v_0 \) reduces to a previously shelled simplex, which only differs from the simplex \( \omega_{\ell, \lambda} \) by the vertex \( v_0 \). Hence, the face \( F \) also appears in the previously shelled simplex.

We next show that \( G_\omega \) is indeed a new face. Notice that by Remark 3.12 \( G_\omega \) contains all the vertices in \( v(\omega) \cap \text{Adj}_r[n] \). For the sake of contradiction, suppose that \( G_\omega \) appeared in a previously shelled simplex, say \( \omega_{\ell', \lambda'} \). That is, \( \omega_{\ell', \lambda'} < \omega_{\ell, \lambda} \). Since \( \omega_{\ell, \lambda} \in W_{\ell, s} \) and we are assuming \( \omega_{\ell', \lambda'} < \omega_{\ell, \lambda} \) then \( \omega_{\ell', \lambda'} \) uses at most \( s \) elements of \( \text{Adj}_r[n] \). But since \( G_\omega \) contains \( s \) elements of \( \text{Adj}_r[n] \) we have that \( \omega_{\ell', \lambda'} \in W_{\ell', s} \). In particular, \( \omega_{\ell', \lambda'} \) uses precisely the same elements of \( \text{Adj}_r[n] \) as \( \omega_{\ell, \lambda} \), and so \( \ell' = \ell \). Hence, \( \omega_{\ell', \lambda'} = \omega_{\ell, \lambda} \in W_{\ell, s} \). So it must be that \( \lambda' < \text{colex} \lambda \). That is, the right-most nonzero entry in \( \lambda' - \lambda \) is positive, say \( \lambda_m - \lambda'_m > 0 \).

Consider the vertex in \( \omega_{\ell, \lambda} \), say \( v_t \), produced by the \( m \)th right move in \( (\omega_{\ell, \lambda}) \). In \( p(\omega_{\ell, \lambda}) \) \( v_t \) corresponds to the right-most corner vertex in a row of the lattice since \( \lambda_m > 0 \). We then have two cases:

1. The vertex \( v_t \) does not correspond to a point on \( p(\omega_{\ell, \lambda'}) \).
2. The vertex \( v_t \) does correspond to a point on \( p(\omega_{\ell, \lambda'}) \).

In case (1), it follows that \( v_t \in G_\omega \). Since \( \omega_{\ell, \lambda} \) and \( \omega_{\ell, \lambda'} \) both have the same largest element in the set \( \text{Adj}_r[n] \), namely \( \text{adj}_r(\ell) \), every element of \( G_\omega \) is uniquely determined by the number of left and right moves needed to produce it from \( \text{adj}_r(\ell) \). In particular, we have that \( v_t \) is produced by \( m \) right moves and \( \sum_{j=1}^{m-1} \lambda_j \) left moves in \( \omega_{\ell, \lambda} \), and \( v_t' \) is produced by \( m \) right moves and \( \sum_{j=1}^{m-1} \lambda'_j \) left moves in \( \omega_{\ell, \lambda'} \). But since \( \lambda_m - \lambda'_m > 0 \) and \( \lambda_j - \lambda'_j = 0 \) for all \( j > m \) we have that

\[
\sum_{j=1}^{m-1} \lambda_j < \sum_{j=1}^{m-1} \lambda'_j,
\]
a contradiction.

In case (2), it follows that \( \omega_{\ell, \lambda} \) does not contain the vertex \( \text{adj}_{\ell'}(l') \) corresponding to the lattice point diagonally across from \( v_t \) on the line \( y = x \). This scenario is depicted in Figure 8. However, since \( \lambda_m - \lambda_m' \) is the right-most nonzero entry in \( \lambda - \lambda' \) then \( \omega_{\ell, \lambda'} \) does contain \( \text{adj}_{\ell'}(l') \), since one of the left moves accounted for by \( \lambda_m \) must now be accounted for in \( \lambda_m' \) for \( t < m \). But this contradicts the fact that

\[
\nu_{(\omega_{\ell, \lambda'})} \cap \text{Adj}_r[n] = \nu_{(\omega_{\ell, \lambda})} \cap \text{Adj}_r[n].
\]

Thus, \( G_\omega \) is indeed a new face. This completes the proof of Theorem 3.15.

4. SOME CONSEQUENCES OF THE STABLE SHELLING

We first note that inductively Theorem 3.15 results in a shelling of the odd second hypersimplex \( \Delta_{n,2} \), thereby proving Theorem 3.2. This shelling is interesting in the sense that it begins with simplices that use only the “most stable” vertices of the polytope and at each stage adds a simplex that uses more and more of the “less stable” vertices. We now investigate some of the nice properties of \( \Delta_{n,2}^{\text{stab}}(r) \) and \( \Delta_{n,2} \) we earn via this shelling.

**Corollary 4.1.** Let \( \omega_{\ell, \lambda} \in \max \nabla_{n,2}^r \setminus \max \nabla_{n,2}^{r+1} \). The maximum dimension of the minimum new face \( G_\omega \) is \( r + 1 \).

**Proof.** Consider the lattice path \( p(\omega_{\ell, \lambda}) \) and recall that the vertices of the minimal new face \( G_\omega \) correspond to lattice points on \( p(\omega_{\ell, \lambda}) \) that lie on the line \( y = x \) together with those that are the right-most such points in their row of the lattice, that are corners of \( p(\omega_{\ell, \lambda}) \), and are not on the path \( p(\omega_{\ell, \lambda'}) \). In particular, these are the lattice points \((x, x)\) for \( x \in [r] \) (which we will call *type 1*), and \((x, y)\) where \( y \geq x - 3 \), \( y \in \{0, 1, \ldots, r - 1\} \), and \((x, y)\) is a corner of \( p(\omega_{\ell, \lambda}) \) that is not also on the path \( p(\omega_{\ell, \lambda'}) \) (which we will call *type 2*). The lattice path \( p(\omega_{\ell, \lambda}) \) can have at most one such point on each line \( y = \alpha \) for \( \alpha \in [r] \), and at most two such points on the line \( y = 0 \) (one of which is always \( \text{adj}_{\ell}(\ell) \)). This gives a maximum possible dimension of \( r + 2 \) for \( G_\omega \).

Assume now that there exists \( \omega_{\ell, \lambda} \) such that \( G_\omega \) has dimension \( r + 2 \). Consider the vertices of \( G_\omega \) corresponding to lattice points \((x, x)\) for \( x \in [r] \). Label this set of vertices by \( V \), and for \( v \in V \) label its corresponding lattice point by \( p_v := (x_v, y_v) \). Notice that \( V \) is nonempty since by our assumption \( G_\omega \) contains a vertex corresponding to a lattice point on the line \( y = r \), and (since we do not wish to over-count the vertex \( \text{adj}_{\ell}(\ell) \)) the only option is \( \text{adj}_{\ell}(\ell + r) \in V \). So let \( v \in V \). Then \( p(\omega_{\ell, \lambda}) \) cannot contain any type 2 points on the lines \( y = y_v - 1 \) or \( y = y_v - 2 \) (this is because \( p(\omega_{\ell, \lambda}) \) may only use North and East moves). But by our assumption the lines \( y = y_v - 1 \) and \( y = y_v - 2 \) must each contain a point corresponding to an element of \( G_\omega \). Hence, they are type 1 points, and therefore elements of \( V \).

**Figure 8.** The case when vertex \( v_t \) is a point on \( p(\omega_{\ell, \lambda'}) \).
Beginning with \( v = \text{adj}_i(\ell + r) \), which has corresponding lattice point \( p_v = (r, r) \), iterating this argument shows that for each line \( y = \alpha, \alpha \in \{0, 1, 2, \ldots, r - 1\} \), the path \( p(\omega_{\ell, \lambda}) \) uses the point \((\alpha, \alpha)\), and no type 2 points on \( y = \alpha \). Hence, \( #(G_\omega) = r + 1 \), a contradiction.

For the case when \( r = 1 \), the next corollary is also a corollary to the algebraic formula given for the \( h^* \)-polynomial of \( \Delta_{n, 2} \) by Katzman in [5]. However, we are now able to give an entirely combinatorial proof of this result.

**Corollary 4.2.** Let \( n \) be odd and \( r < \left\lfloor \frac{n}{2} \right\rfloor \). The degree of the \( h^* \)-polynomial of \( \Delta_{n, 2}^{\text{stab}(r)} \) is \( \left\lfloor \frac{n}{2} \right\rfloor \), and it has leading coefficient \( n \).

**Proof.** Since \( r < \left\lfloor \frac{n}{2} \right\rfloor \) we have shelled the simplices in \( \max \nabla_{n, 2}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} \setminus \max \nabla_{n, 2}^{\frac{n}{2}} \) in order to build the hypersimplex \( \Delta_{n, 2}^{\text{stab}(r)} \). By Corollary 4.1 for a simplex \( \omega \in \max \nabla_{n, 2}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} \setminus \max \nabla_{n, 2}^{\frac{n}{2}} \) the maximum dimension of \( G_\omega \) is \( \left\lfloor \frac{n}{2} \right\rfloor \). It remains to show that this maximum dimension is achieved precisely \( n \) times.

Notice first that for \( r = \left\lfloor \frac{n}{2} \right\rfloor - 1 \) we have that the lattice paths labeling simplices in \( \max \nabla_{n, 2}^{\frac{n}{2}} \setminus \max \nabla_{n, 2}^{n - 1} \) are bounded between the lines \( y = x \) and \( y = x - 3 \). Also, for \( \omega \) to satisfy \( #(G_\omega) = r + 1 \) then there must be a total of \( r \) points of \( p(\omega_{\ell, \lambda}) \) on the lines \( y = x \) and \( y = x - 3 \) other than \((0, 0)\) and \((n - r, r)\).

For \( \max(v_\omega \cap \text{Adj}_r[n]) = \text{adj}_l(\ell) \) with \( \ell < r \) this is impossible since there are less than \( r \) points on these lines that we may use without violating the choice of \( \max(v_\omega \cap \text{Adj}_r[n]) = \text{adj}_l(\ell) \).

For \( r \leq \ell < n \) consider the following. Suppose \( p(\omega_{\ell, \lambda}) \) uses a point \((\alpha, \alpha)\) for \( 0 < \alpha \leq r \). Then by the same argument as in Corollary 4.1 this implies that \( p(\omega_{\ell, \lambda}) \) uses \((\alpha - 1, \alpha - 1)\). Iterating this just as before we get that \( p(\omega_{\ell, \lambda}) \) uses the points

\[ \{(0, 0), (1, 1), (2, 2), \ldots, (\alpha, \alpha)\}. \]

However, this contradicts the fact that \( r \leq \ell < n \) and \( \max(v_\omega \cap \text{Adj}_r[n]) = \text{adj}_l(\ell) \). Hence, the only point used by \( p(\omega_{\ell, \lambda}) \) on the line \( y = x \) is \((0, 0)\). Therefore, \( p(\omega_{\ell, \lambda}) \) must be the path using all the points on the line \( y = x - 3 \).

For \( \ell = n \) there are exactly \( r + 1 \) paths such that \( #(G_\omega) = r + 1 \). This is also seen from the iterative argument we used in Corollary 4.1. Suppose the path \( p(\omega_{\ell, \lambda}) \) uses \( s \leq r \) points of the set \( \{(1, 1), (2, 2), \ldots, (r, r)\} \), and let \((\alpha, \alpha)\) be the point in this collection for which the value of \( \alpha \) is maximal. It then follows that \( p(\omega_{\ell, \lambda}) \) uses all the points

\[ \{(0, 0), (1, 1), (2, 2), \ldots, (\alpha, \alpha)\}. \]

Hence, it must be that \( \alpha = s \). Since there is exactly one path that uses \( r + 1 \) points on the lines \( y = x \) and \( y = x - 3 \) and uses the points \( \{(0, 0), (1, 1), (2, 2), \ldots, (\alpha, \alpha)\} \), then we conclude that there are exactly \( r + 1 \) simplices \( \omega_{n, \lambda} \) with \( #(G_\omega) = r + 1 \).

Considering all of these cases together we conclude that there are exactly \( n \) simplices with \( #(G_\omega) = r + 1 \).

□

**Corollary 4.3.** The \( h^* \)-vector of \( \Delta_{n, 2}^{\text{stab}(2)} \) is unimodal for \( n \) odd.

**Proof.** Consider the completion of the shelling of \( \nabla_{n, 2}^2 \) to a shelling of \( \nabla_{n, 2} \). Here, \( r = 1 \), and so the simplices \( \omega_{\ell, \lambda} \in \max \nabla_{n, 2} \setminus \max \nabla_{n, 2}^2 \) are labeled with compositions of length \( n - 2 \). The composition \( \lambda \) is a composition of 1 that must satisfy equation (1). We now determine which compositions are admissible for a fixed \( \text{adj}_1(\ell) = \max v_\omega \cap \text{Adj}_1[n] \).

For \( \ell \in \{2, 3, 4, \ldots, n - 1\} \) the composition \((1, 0, 0, \ldots, 0)\) does not label a simplex with \( \text{adj}_1(\ell) = \max v_\omega \cap \text{Adj}_1[n] \), since such a composition would necessarily have \( \max v_\omega \cap \text{Adj}_1[n] = \text{adj}_1(\ell + 1) \). This is depicted in Figure 9. On the other hand, each other composition of 1 into \( n - 2 \) parts does
And corresponding to a simplex with $\text{adj}_1(1) = \max v(\omega) \cap \text{Adj}_1[n]$. By considering the associated lattice paths in Figure 10 it is easy to see that the simplex $\omega_{\ell, \lambda}$ has $\#(G_\omega) = 1$, and the other $n - 4$ simplices $\omega_{\ell, \lambda}$ with $\lambda = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ have $\#(G_\omega) = 2$.

For $\ell = 1$, the compositions $(1, 0, 0, \ldots, 0)$ and $(0, 0, \ldots, 0, 1)$ do not label a simplex with $\text{adj}_1(1) = \max v(\omega) \cap \text{Adj}_1[n]$, since such a simplex necessarily has $\max v(\omega) \cap \text{Adj}_1[n] \in \{2, n\}$. This is depicted in Figure 10. Again, the simplex $\omega_{1, \lambda}$ has $\#(G_\omega) = 1$, and the remaining $n - 3$ simplices $\omega_{1, \lambda}$, for $\lambda = (0, 0, \ldots, 0, 0, 0, \ldots, 0)$ have $\#(G_\omega) = 2$.

Finally, for $\ell = n$ both the compositions $(1, 0, 0, \ldots, 0)$ and $(0, 0, \ldots, 0, 1)$ label a simplex with $\text{adj}_1(1) = \max v(\omega) \cap \text{Adj}_1[n]$, since $\text{adj}_1(n) = \max \text{Adj}_1[n]$. Hence there is one simplex, namely $\omega_{n, \lambda}$, with $\#(G_\omega) = 1$, and the remaining $n - 3$ simplices have $\#(G_\omega) = 2$. Summarizing this analysis we have $n$ simplices with $\#(G_\omega) = 1$, and $n(n - 4)$ simplices with $\#(G_\omega) = 2$.

In [5], Katzman computed that for $\Delta_{n,2}$

$$h^*_i = \binom{n}{2i}$$

for $i \neq 1$, and

$$h^*_1 = \binom{n}{2} - n.$$

Thus, since there are $n$ elements in $\text{Adj}_1[n]$ we have that

$$(h^*_1)^{\text{stab}(2)} = h^*_1 - n = \binom{n}{2} - 2n,$$

$$(h^*_2)^{\text{stab}(2)} = h^*_2 - n(n - 4) = \binom{n}{4} - n(n - 4),$$

and for $i \neq 1, 2$

$$(h^*_i)^{\text{stab}(2)} = h^*_i.$$

It is then easy to verify that $(h^*_1)^{\text{stab}(2)} - (h^*_0)^{\text{stab}(2)} \geq 0$ and $(h^*_2)^{\text{stab}(2)} - (h^*_1)^{\text{stab}(2)} \geq 0$ for all $n \geq 0$. As well, $(h^*_3)^{\text{stab}(2)} - (h^*_2)^{\text{stab}(2)} \geq 0$ for all $n \neq 6, 7, 8$. But this is fine since the $h^*$-vector for $\Delta_{7,2}^{\text{stab}(2)}$...
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\[ h^* \left( \Delta_{7,2}^{\text{stab}(2)} \right) = (1, 7, 14, 7, 0, 0). \]

We also remark that

\[ h^* \left( \Delta_{6,2}^{\text{stab}(2)} \right) = (1, 3, 3, 1, 0), \]
\[ h^* \left( \Delta_{8,2}^{\text{stab}(2)} \right) = (1, 12, 38, 28, 1, 0, 0). \]

\[ \square \]

Remark 4.4. We remark that the proofs of the previous corollaries are intriguing since they point out that this shelling allows us to study the Ehrhart Theory of the odd second hypersimplices, as well as the r-stable odd second hypersimplices, by enumerating lattice paths in various ladder-shaped regions of the plane. However, this enumeration problem, in general, is not trivial as suggested by the work of Krattenthaler in [6].

It is possible to apply the same strategy used in the proof of Corollary 4.3 to show that the \( h^* \)-vector of \( \Delta_{n,2}^{\text{stab}(3)} \) is unimodal. In short, we count the lattice paths corresponding to simplices in the set \( \max \nabla_{n,2}^2 \setminus \max \nabla_{n,2}^3 \) with unique minimal new face of cardinality \( i = 1, 2, 3 \) for each choice of maximal \( \text{adj}_i(\ell), \ell \in [n] \). We then subtract these values from the corresponding coefficients in the \( h^* \)-vector of \( \Delta_{n,2}^{\text{stab}(2)} \), and check that the unimodality condition is satisfied for the resulting \( h^* \)-vector. However, the details of this computation are quite unpleasant, so we omit them.

Corollary 4.5. The \( h^* \)-vector of \( \Delta_{n,2}^{\text{stab}(3)} \) is given by

\[ (h^*_1)^{\text{stab}(3)} = \binom{n}{2} - 3n, \]
\[ (h^*_2)^{\text{stab}(3)} = \binom{n}{4} - \frac{1}{2}(n(7n - 55) + 94), \]
\[ (h^*_3)^{\text{stab}(3)} = \binom{n}{6} - \frac{1}{2}(n^3 - 13n^2 + 40n + 16), \text{and for } i \neq 1, 2, 3 \]
\[ (h^*_i)^{\text{stab}(3)} = \binom{n}{2i}. \]

Moreover, \( h^* \left( \Delta_{n,2}^{\text{stab}(3)} \right) \) is unimodal for \( n \) odd.

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