Bounds on the conditional and average treatment effect in the presence of unobserved confounders

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Abstract

The causal effect of an intervention can not be consistently estimated when the treatment assignment is influenced by unknown confounding factors. However, we can still study the causal effect when the dependence of treatment assignment on unobserved confounding factors is bounded by performing a sensitivity analysis. In such a case, the treatment effect is partially identifiable in that the bound of the treatment effect is still estimable based on the observed data. Here, we propose a sensitivity analysis approach to bound the conditional average treatment effect over observed covariates under bounded selection on unobservables. Additionally, we propose a semi-parametric method to estimate bounds on the average treatment effect and derive confidence intervals for these bounds. Combining the confidence intervals of the lower and upper bound gives a confidence region that includes the average treatment effect when the bounded selection on unobservables holds. This method scales to settings where the dimension of observed covariates is too high to apply a traditional sensitivity analysis based on covariate matching. Finally, we provide evidence from simulations and real data to illustrate the accuracy of the confidence intervals and value of our approach in practical finite sample regimes.

1 Introduction

In conventional causal inference, we have a binary treatment indicator $Z = 1$ and $0$, representing the intervention and control, respectively, potential outcomes $\{Y(1), Y(0)\}$, where $Y(1) \in \mathbb{R}$ is the outcome under intervention and $Y(0) \in \mathbb{R}$ is the outcome under control and a set of observed covariates $X \in \mathcal{X} \subseteq \mathbb{R}^d$ [32]. The objective of inference is the average treatment effect (ATE)

$$\tau = \mathbb{E}[Y(1)] - \mathbb{E}[Y(0)],$$

(1.1)

and the conditional average treatment effect (CATE)

$$\tau(x) = \mathbb{E}[Y(1)|X = x] - \mathbb{E}[Y(0)|X = x]$$

(1.2)

using $n$ independent identically distributed observed data of $(Y(Z), Z, X)$. The key assumption for constructing consistent estimators for the ATE or CATE is

$$\{Y(1), Y(0)\} \perp \! \! \! \perp Z \mid X.$$

This assumption excludes the presence of unobserved confounding factors and facilitates the development of methods such as propensity score matching and double robustness estimation. However, the assumption (1) is too restrictive: there are often situations where a confounding factor $U \in \mathcal{U}$ might that effects treatment selection is be unobserved by researchers. In such a case, the “correct” assumption with respect to confounding is

$$\{Y(1), Y(0)\} \perp \! \! \! \perp Z \mid X, U,$$

(1.3)

where, again, $U$ is unobserved. Note that this model is general enough to allow $U = (Y(1), Y(0))$, for which (1.3) will hold trivially. This allows for the case where treatment selection partially depends on the unobserved potential outcome. In the rest of paper, we assume that condition (1.3) is satisfied. Under this general assumption, neither the ATE nor the CATE is identifiable based on observed data [28, 18, 26]. However, restrictions on the effect of $U$ on the treatment assignment $Z$, may allow us to estimate bounds of the aforementioned treatment effect even though $U$ is unavailable.

Such a bound can be useful in interpreting the estimated treatment effect from an observational study. For instance, the first known sensitivity analysis to unobserved confounding was presented by
Cornfield et al. [9] to demonstrate that if the observed association between lung cancer and smoking can be explained by a hormone, then the hormone needs to increase the chance of smoking by a unrealistic ninefold. While the effect size of smoking on lung cancer is large and the conclusion is clear, contemporary epidemiological studies focusing on smaller effect sizes often require more nuanced analyses and sensitive approach for estimating the causal effect in the presence of potential unobserved confounding.

The objective of the present work is to refine the approach to estimating bounds on the average treatment effect as proposed by Rosenbaum [28] and extended in [29, 30, 31]. Using the same model for unobserved confounding, we extend the idea of bounding the ATE to bounding the CATE. Then, we present a semi-parametric approach to solve the problem of bounding the average treatment effect, as well. We show that for certain distributions over the potential outcomes, our proposed bounds on the treatment effect are tight: if the bounds shrank any further, they would no longer be guaranteed to include the treatment effect.

1.1 Bounded selection on unobservables condition

The condition first presented in [9] and later formalized in [28, 29] provides a way of quantifying how much a latent variable can affect the treatment assignment. Specifically, the distribution \( P \) for \((Y(1), Y(0), X, U, Z)\) has \( \Gamma \)-selection bias, if

\[
1 \leq \frac{P(Z = 1 | X = x, U = u) \cdot P(Z = 0 | X = x, U = \tilde{u})}{P(Z = 0 | X = x, U = u) \cdot P(Z = 1 | X = x, U = \tilde{u})} \leq \Gamma
\]

for \( P \) almost every \( u, \tilde{u} \in U, \ x \in X \). It is motivated by its equivalence to the following simple regression model for the treatment selection probability [28, Proposition 12],

\[
\log \left( \frac{P(Z = 1 | X = x, U = u)}{P(Z = 0 | X = x, U = u)} \right) = \kappa(x) + \log(\Gamma)b(u),
\]

where \( \kappa(\cdot) : \mathcal{X} \to \mathbb{R} \) is an appropriate measurable transformation of covariate \( X \), and \( b(\cdot) : U \to [0, 1] \) is a bounded function of the unobserved variable \( U \).

1.2 Goal: Bounding treatment effects

Our goal is to provide a bound on ATE under the condition (1.4). Denote \( L[P] \) as a lower bound the ATE if for any distribution \( P \) over \( Y(1), Y(0), X, Z \) with at most \( \Gamma \)-selection bias,

\[
L[P] \leq \mathbb{E}_P [Y(1) - Y(0)].
\]

Note that the lower bound \( L[P] \) is not unique and some are more conservative than others. In order for such a lower bound to be useful in practice, it must be estimable based on the observed data. Therefore, \( L[P] \) can only depend on \( P \) through the joint distribution of the observable quantities \( (Y(Z), Z, X) \). The one-sided confidence interval for \( L[P] \) can also serve as a conservative yet valid one-sided confidence interval for ATE in the presence of unobserved confounding; Such a bound accounts for statistical uncertainty in estimating the lower bound \( L[P] \) as well as uncertainty about the relationship between \( Y(1), Y(0) \) and \( Z \) due to the missing \( U \). Of course, the same idea can be applied to the upper bound of ATE, defined as

\[
U[P] \geq \mathbb{E}_P [Y(1) - Y(0)].
\]

Combining two one-sided confidence intervals for lower and upper bounds of ATE, we may form a conservative two-sided confidence interval for ATE. Unlike the conventional confidence interval, this interval would not shrink to a single point, even if the sample size increases to infinity, since the uncertainty caused by the latent variable would remain regardless of the sample size.

Suppose that our data consist of \( n \) observations \( (Y_i, Z_i, X_i)_{i=1}^n \), where \( Y_i = Y_i(Z_i) \) is the observed potential outcome. An estimate of the lower bound on the ATE is a function of the random data \( \tau_n \left( (Y_i, Z_i, X_i)_{i=1}^n \right) \). The consistency of the estimator means that

\[
\hat{L} \left( (Y_i, Z_i, X_i)_{i=1}^n \right) \to L[P]
\]

2
in probability, where $L[P]$ is a valid lower bound satisfying (1.5). Note that this consistency implies that

$$P \left\{ \hat{L} (Y_i, Z_i, X_i)_{i=1}^n \leq E_P \left[ Y(1) - Y(0) \right] + \epsilon \right\} \to 0$$

for any $\epsilon > 0$ as $n \to \infty$. Similarly, an estimate of the upper bound can be defined.

This model can be extended to a bound on the CATE: a functional $L[P](x)$ is a lower bound on the CATE if

$$L[P](x) \leq E_P [Y(1) - Y(0) | X = x],$$

(1.7)

for $P$-almost every $x$. A functional $U[P](x)$ is an upper bound on the CATE if

$$U[P](x) \geq E_P [Y(1) - Y(0) | X = x],$$

(1.8)

for $P$-almost every $x$. Throughout, we will only focus on lower bounds, as all concepts can be easily adapted to upper bounds.

We propose an estimation procedure for lower and upper bounds on the ATE under the $\Gamma$-selection bias condition. This procedure has $\sqrt{n}$ rates of convergence, and does not depend on having exact matched pairs of observations with the same level of covariates $x$. Furthermore, we propose a related procedure that provides lower and upper bounds on the CATE in the sense of (1.7).

**Notation** We will use the $P_n$ to denote the empirical distribution on $n$ data and $E_n$ to denote an empirical expectation over $P_n$. Often, we will use the notation $E_n[|Z = 1] = n_1^{-1} \sum_{i=1}^n Z_i$, where $n_1 = \sum_{i=1}^n Z_i$. On occasion, we will use the notation $P_1(\cdot)$ to denote the conditional distribution $P(\cdot | Z = 1)$. Of course, the methods provided here are meaningless when $n = 1$, so there is no risk of conflict between $P_1$ and $P_n$. However, for clarity we will try to use $P(\cdot | Z = 1)$ whenever it can reasonably be written out.

### 1.3 Related Work

The sensitivity analysis in the presence of unobserved confounding has been extensively studied in the literature. We focus on the model described in Rosenbaum [28], because of its transparent interpretation. Robins et al. [26] suggests that such a model should only be used when a known unobserved variable is believed to exist. The potential outcomes themselves or their functions can be used as the unobserved variable to induce the strongest association between the latent variable and the outcome. In this case, the sensitivity model allows for direct dependence of the treatment assignment on the unobserved potential outcome, or partial information about the unobserved potential outcome, a situation that may occur in medical and economic settings.

Various authors have considered nonparametric models for sensitivity analysis under selection on unobservables [25, 26, 40, 35]. Zhao et al. [40] and Shen et al. [35] consider the sensitivity of inverse probability weighted estimates to unobserved confounder, as well as mis-specification of the propensity score model. The method in Zhao et al. [40] is closely related to this paper, but they considered a marginal sensitivity model that is more conservative than (1.4). Furthermore, the related statistical inference relies on the computational intensive bootstrap method due to the complexity of the asymptotic distribution of their estimator of the lower bound. Many technical tools developed in the present work may be used for deriving the asymptotic properties of related estimators in their sensitivity analysis. [5] develops a sensitivity analysis that varies both the effect of selection bias on the treatment assignment and on the outcome in a marginal structural model.

Many matching methods have been developed for the model (1.4), [28, 29, 30, 31, 12]. These results assume that exact matched pairs on all observed covariates can be formed. Unfortunately, it is oftentimes not feasible in practice even with covariates vector of a moderate dimension. When considering continuous covariates, it is often impossible to find pairs with exact matched covariates. Abadie and Imbens [1] shows that common matching estimators using matches based on unit with the nearest covariates have a $O(n^{-1/d})$ bias, where $d$ is the dimension of the observed continuous covariates. However, the matching approaches do share some similarities to the presented approach. Among them, the M-estimate model considered in [31] is most closely related. Fogarty and Small [12] also convert the sensitivity analysis into an optimization problem over weights on each pair.

Rosenbaum [29] defines the power of a sensitivity analysis for an alternative with no unobserved confounding and a positive treatment effect. They also define a notion of *design sensitivity*: the threshold
of \( \Gamma \) such that the power of a test for the null of no treatment effect goes to 0 when \( \Gamma > \hat{\Gamma} \), but goes to 1 when \( \Gamma < \hat{\Gamma} \). [30] improves the design sensitivity of [29] by developing a test with a tighter bound on the treatment effect. [31] shows that using multiple controls can improve the design sensitivity further in some settings. In Section 4.4, we derive the design sensitivity of our proposed method, and show that it is optimal in certain cases: in particular, including the case when the potential outcomes are continuous and follow Gaussian distribution.

[25] provides an in-depth literature review of sensitivity analyses that make relatively few assumptions about the structural model. Ding and VanderWeele [11] recently extended this idea to define a sensitivity analysis method with few assumptions beyond that the outcome is binary.

The CATE has been studied to some degree as a nuisance parameter for the estimation of the ATE in semi-parametric models [18, 15, 33, 8]. Recently, interest in identifying treatment effect heterogeneity has lead to a number of estimators targeted at the CATE [16, 2, 20, 39, 24]. See Künnel et al. [20] for a more complete literature review and discussion of these methods. To our knowledge, no sensitivity analysis models for bounding the CATE in the presence of unobserved confounding exist in the literature. Kallus and Zhou [19] recently presented a model for personalized decision policy learning in the presence of unobserved confounding that is closely related to the CATE. Their model is based on the marginal sensitivity analysis model of Zhao et al. [40], which was shown to lead to conservative bounds under the \( \Gamma \)-selection bias condition.

1.4 Outline

Before considering covariates \( X \), Section 2 begins with the simple setting where \( \mathcal{X} = \emptyset \), i.e., there are no observed covariates. It is similar to the case, where \( \mathcal{X} \neq \emptyset \), but the sensitivity analysis is conditional on a subgroup of patients with \( X = x \) for a specific \( x \). However, statistical inference for the later is more challenging when the covariates are continuous, because a smoothing technique is needed to borrow information from observations with covariates similar to \( x \). The formal estimation of the lower bound of the CATE conditional on \( X = x \) is presented in Section 3. Finally, the method for bounding the the average treatment effect in the presence of covariate \( \mathcal{X} \) is presented in Section 4.

2 With no observed covariates

In this section, we assume that \( \mathcal{X} = \emptyset \), so that (1.3) simplifies to the condition

\[
Y(1), Y(0) \perp \perp Z|U,
\]

and (1.4) simplifies to

\[
\frac{1}{\hat{\Gamma}} \leq \frac{P(Z = 1|U = u)}{P(Z = 0|U = u)} \frac{P(Z = 0)}{P(Z = 1)} \leq \Gamma. \tag{2.2}
\]

2.1 A lower bound on the ATE

Bound \( \tau = \mathbb{E}[Y(1) - Y(0)] \) by separately bounding \( \mu_1 = \mathbb{E}[Y(1)] \) and \( \mu_1 = \mathbb{E}[Y(1)] \), and combining. Decompose \( \mu_1 \) as

\[
\mu_1 = \mathbb{E}[Y(1)|Z = 1]P(Z = 1) + \mathbb{E}[Y(1)|Z = 0]P(Z = 0), \tag{2.3}
\]

and note that except for \( \mathbb{E}[Y(1)|Z = 0] \), all other quantities in the decomposition are estimable based on observed data. Therefore, only \( \mathbb{E}[Y(1)|Z = 0] \) requires a bound under (2.2). Assume, for a moment, that \( P_Y(Y(1)|Z = 1) \) is absolutely continuous with respect to \( P_Y(Y(1)|Z = 0) \) (this is formally verified in Lemma 3.1). Then, re-write \( \mathbb{E}[Y(1)|Z = 0] \) as an expectation over the distribution \( P(\cdot|Z = 1) \),

\[
\mathbb{E}[Y(1)|Z = 0] = \mathbb{E} \left[ Y(1)L(Y(1))|Z = 1 \right], \tag{2.4}
\]

where

\[
L(y) = \frac{dP(Y(1) = y|Z = 0)}{dP(Y(1) = y|Z = 1)}.
\]

\( L(y) \) is unknown, but can be bounded under the \( \Gamma \)-selection bias condition.
Lemma 2.1. If $P$ has $\Gamma$-selection bias, $P_{Y(1)}(\cdot | Z = 0)$ is absolutely continuous with respect to $P_{Y(1)}(\cdot | Z = 1)$, and the likelihood ratio $L := \frac{dP_{Y(1)}(\cdot | Z = 0)}{dP_{Y(1)}(\cdot | Z = 1)}$ satisfies $L(y) \leq \Gamma L(\tilde{y})$ for $P_{Y(1)}(Z = 1)$-almost every $y, \tilde{y}$.

The proof can be found in Appendix A.

Lemma 2.1 and the decomposition (2.1) imply that $\mu_1$ can be bounded by $\mu_1^-$, defined by the linear optimization problem

$$\mu_1^- = \inf_{L(y)} \mathbb{E}[Y(1)|Z = 1]P(Z = 1) + \mathbb{E}[Y(1)L(Y(1))|Z = 1]P(Z = 0).$$

(2.5)

s.t. $L(y) \geq 0$, for $P_{Y(1)}$ a.e. $y$

$\mathbb{E}[L(Y(1))|Z = 1] = 1$

$L(y) \leq \Gamma L(\tilde{y})$, for $P_{Y(1)}$ a.e. $(y, \tilde{y})$.

Note the optimizer $\mu_1^- = \mathbb{E}[Y(1)|Z = 1]P(Z = 1) + \theta_1 P(Z = 0)$, where

$$\theta_1 = \inf_{L(y)} \mathbb{E}[Y(1)L(Y(1))|Z = 1].$$

(2.6)

s.t. $L(y) \geq 0$, for $P_{Y(1)}$ a.e. $y$

$\mathbb{E}[L(Y(1))|Z = 1] = 1$

$L(y) \leq \Gamma L(\tilde{y})$, for $P_{Y(1)}$ a.e. $(y, \tilde{y})$.

First consider the following special case, where $Y(1)$ is uniformly distributed over a discrete set. The solution to this simple problem will provide intuition about the general case. Let $\mathcal{Y} = \{y_j\}_{j=1}^m$, and let $P(Y(1) = y_j|Z = 1) = m^{-1}$. (2.6) degenerates into

$$\theta_1 = \inf_{L \in \mathbb{R}^m} m^{-1} \sum_{j=1}^m y_j L_j.$$

s.t. $L_j \geq 0$

$m^{-1} \sum_{j=1}^m L_j = 1$

$L_j \leq \Gamma L_k \forall j, k = 1, \ldots, m$

It is clearly a linear programming problem. In this case, the optimal solution $L^*_j$s will be on the boundary of the constraint set, i.e., $L^*_j = \Gamma L_k^*$ or $L^*_j = \Gamma L_k^*$ for any $1 \leq j < k \leq m$. The optimal weights should also be ordered in the opposite order of the observed $y_j$s to minimize the objective function. These observations motivate the following simple algorithm: bisection the support $\mathcal{Y}$ using a threshold $y^*$ and assign weight $\Gamma$ for each $y_j < y^*$ and 1 otherwise; normalize all weights so that the total weight is 1. This intuition carries over to the general case. Specifically, the following lemma presents the dual problem to characterizing $\theta_1$. All subsequent infima over $L$ are implicitly taken in the space of $Y(1)$-measurable functions.

Lemma 2.2. If $\theta_1 < \infty$, then

$$\theta_1 = \inf_{L(y)} \mathbb{E}[Y(1)L(Y(1))|Z = 1]$$

s.t. $\mathbb{E}[L(Y(1))|Z = 1] = 1$

$L(y) \geq 0, L(y) \leq \Gamma L(\tilde{y})$ for all $y, \tilde{y} \in \mathbb{R}$

$$= \sup_{\mu} \mu$$

(2.8)

s.t. $\mathbb{E}[(Y(1) - \mu)_+ - \Gamma(Y(1) - \mu)_-|Z = 1] \geq 0,$

where $a_+ = \max\{a, 0\}$ and $a_- = \min\{a, 0\}$. Therefore $\mu_1^+$ can be obtained by solving the optimization problem given in (2.8).

By symmetry, a similar conclusion holds for $\mu_0^+$, the upper bound of $\mu_0$ under the $\Gamma$-selection bias condition. Specifically,

$$\mu_0^+ = \sup_{L(y)} \mathbb{E}[Y(0)|Z = 0]P(Z = 0) + \mathbb{E}[Y(0)L(Y(0))|Z = 0]P(Z = 1).$$

(2.9)

s.t. $L(y) \geq 0$, for $P_{Y(1)}$ a.e. $y$

$\mathbb{E}[L(Y(1))|Z = 1] = 1$

$L(y) \leq \Gamma L(\tilde{y})$, for $P_{Y(1)}$ a.e. $y$

Then, the lower bound on $\tau$ is simply

$$\tau^- = \mu_1^- - \mu_0^+.$$ 

(2.10)

The following proposition shows that this is a valid lower bound for any hidden variable $U$ satisfying (1.3) and (1.4). We emphasize the dependence of $\tau^-$ on $P$ through the notation $\tau^-[P]$. 


\textbf{Theorem 2.1.} Let $\Gamma \geq 1$ be fixed, and let $P$ be a distribution over $Y(1), Y(0), Z, U$ satisfying (1.4). Let $\tau^-$ be as in (2.10) for the optimization problems solved with the same choice of $\Gamma$. When $E[Y(1)]$ and $E[Y(0)]$ are finite, and $P(Z = 1) \in (\epsilon, 1 - \epsilon)$ for some $\epsilon > 0$,

$$\tau^-[P] \leq E_P[Y(1) - Y(0)].$$

\section{Proposed estimation methodology}

To estimate $\mu^-_1$ and $\mu^+_0$ based on observed data, simply replace $P(\cdot | Z = 1)$ and $P(\cdot | Z = 0)$ with their empirical counterparts in (2.5) and (2.9). That is,

$$\hat{\mu}^-_1 = \inf_{L(y)} E_n[Y(1) | Z = 1] P_n(Z = 1) + E_n[Y(1) L(Y(1)) | Z = 1] P_n(Z = 0),$$

s.t. $L(y) \geq 0$, for $P_n$ a.e. $Y(1) = y$

$P_n[L(Y(1)) | Z = 1] = 1$

$L(y) \leq \Gamma L(\hat{y})$, for $P_n$ a.e. $Y(1) = y$

and

$$\hat{\mu}^+_0 = \sup_{L(y)} E_n[Y(0) | Z = 0] P_n(Z = 0) + E_n[Y(0) L(Y(0)) | Z = 0] P_n(Z = 1).$$

s.t. $L(y) \geq 0$, for $P_n$ a.e. $Y(0) = y$

$E_n[L(Y(0)) | Z = 0] = 1$

$L(y) \leq \Gamma L(\hat{y})$, for $P_n$ a.e. $Y(0) = y$

The following sections detail the convergence guarantees of these estimators.

\section{Asymptotic properties}

The following propositions establish the consistency and asymptotic normality of (2.11) and (2.12). We only present results for the case of $\mu^-_1$, as the results for $\mu^+_0$ are parallel. Since

$$\hat{\mu}^-_1 = E_n[Y(1) | Z = 1] P_n(Z = 1) + \hat{\theta}_1 P_n(Z = 0),$$

we only need to derive the asymptotic properties of $\hat{\theta}_1$, where

$$\hat{\theta}_1 = \inf_{L(y)} E_n[Y(1) L(Y(1)) | Z = 1],$$

s.t. $L(y) \geq 0$, for $P_{Y(1)}$ a.e. $y$

$P_n[L(Y(1)) | Z = 1] = 1$

$L(y) \leq \Gamma L(\hat{y})$, for $P_{Y(1)}$ a.e. $y$

which estimates $\theta_1$ given in (2.6). To this end, we have the following two propositions.

\textbf{Proposition 2.1.} Assume that $Y(1)$ has finite expectation $\mu_1$. Then $\hat{\theta}_1 \xrightarrow{P} \theta_1$ as $n \to \infty$.

\textbf{Proposition 2.2.} Assume that conditional on $Z = 1$, $Y(1)$ has finite expectation $\mu_1$ and finite variance $\sigma_1^2$. Then, $\sqrt{n} \left( \hat{\theta} - \theta_1 \right) \xrightarrow{D} V$ as $n \to \infty$, with

$$V = \frac{1}{P(Y(1) > \theta_1) + \Gamma P(Y(1) \leq \theta_1)} G_+ - \frac{1}{P(Y(1) > \theta_1) + \Gamma P(Y(1) < \theta_1)} G_-, \quad \text{for } G \sim N \left( 0, \text{Var} \left( \psi_{\theta_1} \left( Z, Y(1) \right) \right) \right),$$

where $\psi_{\theta}(z, y) = z \left[ (y - \theta)_+ + \Gamma (y - \theta)_- \right]$.

A direct consequence of the Proposition (2.2) is that if $P(Y(1) = \theta_1) = 0$, then $\sqrt{n} \left( \hat{\theta} - \theta_1 \right)$ converges weakly to a mean zero normal distribution with a finite variance. To approximate the distribution of $\sqrt{n} (\hat{\theta}_1 - \theta_1)$, one may easily estimate $P(Y(1) > \theta_1) + \Gamma P(Y(1) \leq \theta_1)$, $P(Y(1) > \theta_1) + \Gamma P(Y(1) < \theta_1)$ and $\text{Var} \left( \psi_{\theta_1} \left( Z, Y(1) \right) \right)$, in proposition (2.2) by their empirical counterparts.

Ultimately, a consistent estimator of $\tau^-$ can be constructed as

$$\hat{\tau} := \hat{\mu}_1 E_n[Y(1) | Z = 1] + (1 - \hat{\mu}_1) \hat{\theta}_1 - (1 - \hat{\mu}_1) E_n[Y(0) | Z = 0] - \hat{\mu}_1 \hat{\theta}_0,$$
where \( \hat{\nu} := P_n(Z = 1) \), \( \hat{\theta}_0 \) is defined analogously to \( \hat{\theta}_1 \). \( \hat{\tau} \) is consistent to the lower bound \( \tau^- (P) \) and \( \hat{\tau} - \tau^- (P) \) can be approximated by a limiting distribution. Since Proposition (2.2) has established the asymptotic properties of \( \sqrt{n} (\hat{\theta}_1 - \theta_1) \) (and similarly for \( \sqrt{n} (\hat{\theta}_0 - \theta_0) \)). Especially, if \( P(Y_1 = \theta_1) = 0 \), then this limiting distribution is a mean zero Gaussian distribution, whose variance can be easily estimated by coupling the delta method and the fact that

\[
\hat{\tau} - \tau^- (P) = (\mu_1 + \mu_0 - \theta_1 - \theta_0) (\hat{\nu} - p) + \mathbb{E}_n [Y(1) - \mu_1 | Z = 1] p_1 + (1 - p_1) (\hat{\theta}_1 - \theta_1) - (1 - p_1) \mathbb{E}_n [Y(0) - \mu_0 | Z = 0] - p_1 (\hat{\theta}_0 - \theta_0) + o_p (n^{-1/2}).
\]

In addition to the large sample property of \( \hat{\tau}^- \), the following proposition characterizes the distance between \( \hat{\tau}^- \) and \( \tau^- \) in finite sample using a standard concentration bound argument.

**Proposition 2.3.** Let \( n_1 \) and \( n_0 \) denote the number of samples such that \( Z = 1 \) and \( Z = 0 \) respectively, and \( \text{Var}_1 Y(1) \leq \sigma_1^2 \), \( \text{Var}_0 Y(0) \leq \sigma_0^2 \). Then, with probability at least \( 1 - 5 \delta \)

\[
| \tau^- (P) - \hat{\tau} | \leq \frac{\Gamma \sigma_1}{\sqrt{n_1} \delta} + \frac{1}{2} \sqrt{\frac{\log 1}{n_1}} \delta + \frac{\Gamma \sigma_0}{\sqrt{n_0} \delta} + \frac{1}{2} \sqrt{\frac{\log 1}{n_0}} \delta.
\]  

(2.14)

3 Sensitivity of the conditional average treatment effect (CATE)

In this section, we study the lower bound of CATE conditional on \( X = x \), for \( x \in \mathcal{X} \), where we assume that \( \mathcal{X} \) is bounded. To begin, Lemma 2.1 can be generalized when \( P_{Y(1)|Z=1,X} \) and \( P_{Y(0)|Z=0,X} \) are regular conditional probabilities and the independence (1.3) and the bound (1.4) hold.

**Lemma 3.1.** If under \( P \), (1.3) and (1.4) hold, then \( P_{Y(1)|Z=0} \) is absolutely continuous with respect to \( P_{Y(1)|Z=1} \), and the likelihood ratio

\[
L_x(y) = \frac{dP_{Y(1)|Z=0,X=x}}{dP_{Y(1)|Z=1,X=x}}
\]

satisfies \( L_x(y) \leq \Gamma L_y(\bar{y}) \) for \( P_{Y(1)|Z=1,X=x}^- \) almost every \( y, \bar{y}, \) and \( P \)-almost every \( x \).

The proof follows the same argument as for Lemma 2.1, found in Section A, but under the measure defined by the measure for every \( x \), \( P(\cdot | X = x) \).

Following the same arguments used in Section 2, a valid lower bound on the CATE is

\[
\tau^- [P](x) = \mu_1^-(x) - \mu_0^+(x),
\]  

(3.1)

where

\[
\mu_1^-(x) = \inf_{L \geq 0} \mathbb{E} [Y(1)| X = x, Z = 1] P(Z = 1| X = x) + \mathbb{E} [Y(1)L| X = x, Z = 1] P(Z = 0| X = x),
\]

s.t. \( \mathbb{E}[L| X = x, Z = 1] = 1 \)

\[
L(\bar{y}) \leq \Gamma \mathbb{L}(y) \text{ for } P(\cdot | X = x, Z = 1)^- \text{-a.e. } y, \bar{y}
\]

\[= e_1(x) \mu_{1,1}(x) + (1 - e_1(x)) \theta_1(x),
\]  

(3.2)

\[
\mu_0^+(x) = \sup_{L \geq 0} \mathbb{E} [Y(0)| X = x, Z = 0] P(Z = 0| X = x) + \mathbb{E} [Y(1)L| X = x, Z = 0] P(Z = 1| X = x)
\]

s.t. \( \mathbb{E}[L| X = x, Z = 0] = 1 \)

\[
L(\bar{y}) \leq \Gamma \mathbb{L}(y) \text{ for } P(\cdot | X = x, Z = 1)^- \text{-a.e. } y, \bar{y}
\]

\[= (1 - e_1(x)) \mu_{0,0}(x) + e_1(x) \theta_0(x)
\]  

(3.3)

for

\[
\theta_1(x) = \inf_{L \geq 0} \mathbb{E} [Y(1)L| X = x, Z = 1],
\]

s.t. \( \mathbb{E}[L| X = x, Z = 1] = 1 \)

\[L(\bar{y}) \leq \Gamma \mathbb{L}(y) \text{ for } P(\cdot | X = x, Z = 1)^- \text{-a.e. } y, \bar{y}
\]  

(3.4)
\[\theta_0(x) = \sup_{L \geq 0} \mathbb{E}[Y(0)L|X = x, Z = 0], \tag{3.5}\]
\[\text{s.t. } \mathbb{E}[L|X = x, Z = 0] = 1 \]
\[L(\hat{y}) \leq \Gamma L(y) \text{ for } P(|X = x, Z = 0)-\text{a.e. } y, \hat{y} \]
\[e_1(x) = P(Z = 1|X = x), \]
\[\mu_{1,1}(x) = \mathbb{E}[Y(1)|Z = 1, X = x], \text{ and } \mu_{0,0}(x) = \mathbb{E}[Y(0)|Z = 0, X = x].\]

Note that in this section and Section 4, \(\theta_1(\cdot)\) is a function, an infinite dimensional parameter, in contrast to Section 2 where it was a one-dimensional scalar.

As before, we will proceed by studying only \(\theta_1(x)\), as \(\theta_0(x)\) is a symmetric problem, and the inferences for other terms are standard. For instance \(\mu_{1,1}(x)\) is the mean function among the treated and \(e_1(x) = P(Z = 1|X = x)\) is the pseudo-propensity score (pseudo- denoting the fact that this is not a true propensity score since \(U\) is not included). Non-parametric estimation of these functions, and associated inference procedures, can be found in many references, such as [18, 24, 33].

Section 3.1 characterizes a lower bound on the CATE as the solution to a loss minimization problem. Section 3.2 provides an approach to solving the empirical optimization problem via the use of sieve estimation method [14]. Section 3.3 then derives the asymptotic properties of the sieve estimator presented in Section 3.2. Throughout these sections, we will use the notation \(\mathbb{E}_1[\cdot] = \mathbb{E}[\cdot|Z = 1]\) and drop the explicit dependence of \(\tau^{-}[P](x)\) on \(P\).

### 3.1 Proposed estimation methodology

Define the loss function \(\ell : \mathbb{R}^k \times (\mathcal{X}, \mathbb{R}) \to \mathbb{R}\)
\[\ell(\theta; (x, y)) := \frac{1}{2} \left( [y - \theta(x)]^2_+ + \Gamma [y - \theta(x)]^2_- \right),\]

and consider the following loss minimization problem
\[\inf_{\theta(\cdot)} \{ \mathbb{E}[\ell(\theta; (X, Y(1)))]|Z = 1] | \theta : \mathcal{X} \to \mathbb{R} \text{ measurable} \} \tag{3.6}\]

where the infimum is over measurable functions. The loss \(\theta \mapsto \ell(\theta; (x, y))\) is convex in \(\theta\) for all \((x, y)\), and so first order conditions for the optimization problem (3.6) implies that the unique optimizer of the loss function is also the solution to the estimating equation (see Lemma B.1 in Appendix for formal proof):
\[\mathbb{E}[(Y(1) - \theta(x))^+ - \Gamma (Y(1) - \theta(x))^- | X = x, Z = 1] = 0, \tag{3.7}\]

for \(P\)-almost every \(x \in \mathcal{X}\), if \(\mathbb{E}_1[\ell(\theta_1; (X, Y(1)))] < \infty\). Following the same arguments in the previous section, the solution of the estimating equation is \(\theta_1(x)\) given in (3.4). Therefore, \(\theta_1(x)\) is the minimizer of the loss function (3.6). The new perspective provides a number of methods for estimating \(\theta(x)\) under different conditions. For example, many popular machine learning method such as random forest can be used to minimizing the loss function. In the present work, we will focus on sieve / series estimation for their intuitive interpretation and formal convergence guarantees[23, 7, 17, 6].

### 3.2 Sieve Estimators

It is appealing to estimate \(\theta_1(x)\) via minimizing the empirical loss function corresponding to (3.6). However, operationally, \(\hat{\theta}_1(\cdot)\) must be restricted to an appropriate functional space, i.e., sieve space. Otherwise, there are pathological choices of \(\hat{\theta}_1(x)\) that minimize the empirical loss but are 0 everywhere except a measure 0 subset of \(\mathcal{X}\). Specifically, we propose to estimate \(\theta_1(x)\) by solving the population version of the optimization problem
\[\min_{\theta \in \Theta_n} \mathbb{E}_n[\ell(\theta; (X, Y(1)))|Z = 1] = \frac{1}{n_1} \sum_{i=1}^{n_1} \ell(\theta; (X_i, Y(1)_i)), \tag{3.8}\]

where
\[\Theta_1 \subseteq \cdots \subseteq \Theta_n \subseteq \cdots \subseteq \Theta,\]
is a sequence of approximating parameter spaces. The follow provide some examples of approximating parameter spaces for which good approximation guarantees exist. In these examples, let $\mathcal{X} = [0, 1]^d$ and use $J$ to modulate the size of the approximation spaces.

**Example 1** (Polynomials): Let $\text{Pol}(J)$ denote the space of $J$-th order polynomials on $[0, 1]$

$$\text{Pol}(J) := \left\{ x \mapsto \sum_{k=0}^{J} a_k x^k, x \in [0, 1] : a_k \in \mathbb{R} \right\}.$$ 

Then for a sequence $(J_n)n=1^{\infty}$, define the sieve, $\Theta_n := \text{Pol}(J_n) \times \cdots \times \text{Pol}(J_n)$. 

**Example 2** (Splines): Let $0 = t_0 < \cdots < t_{J+1} = 1$ be knots that satisfy bounded mesh ratio

$$\frac{\max_{0 \leq j \leq J} (t_{j+1} - t_j)}{\min_{0 \leq j \leq J} (t_{j+1} - t_j)} \leq c$$

for some $c > 0$. Then, the space of $r$-th order splines is given by

$$\text{Spl}(r, J) := \left\{ x \mapsto \sum_{k=0}^{r-1} a_k x^k + \sum_{j=1}^{J} b_j [x - t_j]_+^{r-1}, x \in [0, 1] : a_k, b_k \in \mathbb{R} \right\}.$$ 

Then, for some integer $r \geq |p| + 1$ and sequence $(J_n)n=1^{\infty}$, let $\Theta_n := \text{Spl}(r, J_n) \times \cdots \times \text{Spl}(r, J_n)$. 

Note that choosing $\Theta_n$ as a finite dimensional linear sieves such Examples 1, or 2 results in a convex empirical optimization problem (3.8), which facilitate the relevant numeric computation [4, Chapter 2.3].

### 3.3 Convergence rate of $\hat{\theta}_1(x)$

Let $\hat{\theta}_1(\cdot)$ be the minimizer of (3.8). This section establishes the asymptotic convergence rate of $\hat{\theta}_1(x)$ to $\theta_1$ under appropriate regularity conditions. To this end, assume the following.

**Assumption 1.** $\theta_1 \in \Lambda_{p_1}^p(\mathcal{X}) := \Theta$ for some $c > 0$, where $\Lambda_{p_1}^p(\mathcal{X})$ denotes the holder class of $p$-smooth functions

$$\Lambda_{p_1}^p(\mathcal{X}) := \left\{ h \in C^{p_1}(\mathcal{X}) : \sup_{\sum_{i=1}^{d} \alpha_i \leq p_1} \sup_{x \in \mathcal{X}} |D^{\alpha} h(x)| \leq c, \sup_{\sum_{i=1}^{d} \alpha_i = p_1} \sup_{x \in \mathcal{X}, x \neq x'} \frac{|D^{\alpha} h(x) - D^{\alpha} h(x')|}{\|x - x'\|^{p_1}} \leq c \right\},$$

$C^{p_1}(\mathcal{X})$ denotes the space of $p_1$-times continuously differentiable functions on $\mathcal{X}$, and

$$D^{\alpha} = \frac{\partial^{\alpha}}{\partial^{\alpha_1} \cdots \partial^{\alpha_d}},$$

for any $d$-tuple of nonnegative integers $\alpha = (\alpha_1, \ldots, \alpha_d)$.

**Assumption 2.** $\mathbb{E}_1[(Y(1) - \theta_1(\mathcal{X}))^2|\mathcal{X}]$ is bounded.

**Assumption 3.** $X|Z = 1$ has density $p_1 = \frac{d \lambda}{d \mathcal{X}}$ with respect to the Lebesgue measure $\lambda$, with $0 < \inf_{x \in \mathcal{X}} p_1(x) \leq \sup_{x \in \mathcal{X}} p_1(x) < \infty$.

Assumption 1 assumes that $\theta_1$ is in a $p$-smooth Holder space. The class of $p$-smooth Holder functions forms a natural parameter space since it is rich yet can be well-approximated by a finite dimensional linear span of simple basis functions, such as those presented in Section 3.2. Assumption 2 ensures the existence of a finite second moment of the loss function at its optimum, which is a standard condition to ensure the convergence rate of the empirical loss function. Lastly, sieve estimation typically requires control over the modulus of continuity at $\theta_1$, with respect to the supremum norm $\sup_{\|\theta(\cdot) - \theta_1(\cdot)\|_{L^2} \leq \delta} |\ell(\theta; (X, Y)) - \ell(\theta_1; (X, Y))|$ [6]. Assumption 3 requires that $X|Z = 1$ has a density function bounded above and below, i.e., equivalent to the Lebesgue measure. Since supremum norms can be controlled by the Lebesgue $L^2$-norm $\|\cdot\|_{L^2}$ in Holder spaces, this assumption allows control over the above modulus of continuity by controlling its $\|\cdot\|_{L^2}$-counterpart.
The main theorem in this section states that the accuracy of \( \hat{\theta}_1(\cdot) \) is dictated by the tradeoff between the random estimation error and approximation strength of the sieve space \( \Theta_n \). Before stating the result, we need to state the formal definition of covering numbers used to measure complexity of parameter spaces.

Let \( \mathcal{V} \) be a vector space with (semi)norm \( \|\cdot\| \) on \( \mathcal{V} \), and let \( V \subset \mathcal{V} \). A collection \( v_1, \ldots, v_N \subset V \) is an \( \epsilon \)-cover of \( V \) if for each \( v \in V \), there exists \( v_i \) such that \( \|v - v_i\| \leq \epsilon \). The covering number of \( V \) with respect to \( \|\cdot\| \) is then \( N(V, \epsilon, \|\cdot\|) := \inf \{N \in \mathbb{N}: \text{there is an } \epsilon \text{-cover of } V \text{ with respect to } \|\cdot\|\} \).

Furthermore, for some fixed \( b > 0 \), define the sequence

\[
\delta_n := \inf \left\{ \delta \in (0, 1): \frac{1}{\sqrt{n \delta^2}} \int_0^\delta \sqrt{\log N \left( e^{1+b/2\delta^2 \cdot 2}, \Theta_n, \|\cdot\|_2, P_n \right) d\delta \leq 1 \right\},
\]

where \( P_n(\cdot) = P(|V| = 1) \). The following convergence result is a consequence of general results on sieve estimators \([7, 17, 6]\) adapted for the optimization problem in (3.6). The proof is given in Appendix B.2.

**Theorem 3.1.** Let Assumptions 1, 2, 3 hold, and let \( \hat{\theta}_1 \) be an approximate empirical minimizer to the problem (3.8) satisfying

\[
\mathbb{E}_n [\ell(\hat{\theta}_1; (X, Y(1))) | Z = 1] \leq \inf_{\theta \in \Theta_n} \mathbb{E}_n [\ell(\theta; (X, Y(1))) | Z = 1] + O_p \left( \epsilon_n^2 \right)
\]

where \( \epsilon_n := \max \{\delta_n, \inf_{\theta \in \Theta_n} \|\theta_1 - \theta\|_{2, P} \} \). Then, \( \|\hat{\theta}_1 - \theta\|_{2, P} = O_p(\epsilon_n) \).

Concretely, using finite dimensional linear sieves considered in Section 3.2 yields standard non-parametric rates for estimating \( \theta_1 \).

**Corollary 3.1.** For \( \mathcal{X} = [0, 1]^d \), let \( \Theta_n \) be given by finite dimensional linear sieves considered in Examples 1, and 2 (assume \( \theta_1 \) can be extended periodically for trigonometric polynomial bases) with \( J_n \approx n^{\frac{p}{p+1}} \). Let Assumptions 1, 2, 3 hold, and let \( \hat{\theta}_1 \) be an approximate empirical minimizer to the problem (3.8)

\[
\mathbb{E}_n [\ell(\hat{\theta}_1; (X, Y(1))) | Z = 1] \leq \inf_{\theta \in \Theta_n} \mathbb{E}_n [\ell(\theta; (X, Y(1))) | Z = 1] + O_p \left( n^{-\frac{p}{p+1}} \right).
\]

Then, \( \|\hat{\theta}_1 - \theta\|_{2, P} = O_p \left( n^{-\frac{p}{p+1}} \right) \).

See Appendix B.3 for the proof.

As we discussed earlier, to estimate \( \tau^{-}(x) \), we also need to estimate \( e(x), \mu_{1,1}(x), \) and \( \mu_{0,0}(x) \). Let \( \tilde{e}(\cdot): \mathcal{X} \rightarrow [0, 1], \tilde{\mu}_{1,1}(\cdot): \mathcal{X} \rightarrow \mathbb{R}, \tilde{\mu}_{0,0}(\cdot): \mathcal{X} \rightarrow \mathbb{R} \) be suitable estimators for \( e(\cdot), \mu_{1,1}(\cdot) \) and \( \mu_{0,0}(\cdot) \), respectively, such that

\[
\|\tilde{e}(\cdot) - e(\cdot)\|_{2, P} = O_p(r_{n,1}), \|\tilde{\mu}_{1,1}(\cdot) - \mu_{1,1}(\cdot)\|_{2, P} = O_p(r_{n,2}), \|\tilde{\mu}_{0,0}(\cdot) - \mu_{0,0}(\cdot)\|_{2, P} = O_p(r_{n,3}),
\]

where \( r_{n,j} = o(1), j = 1, 2, 3 \) measure the convergence rates of relevant estimators and depend on the model assumptions and estimation method. For example, under assumptions parallel to Assumptions 1-3, we may employ the sieve method for regression to estimate those functions and the convergence rates can be \( O(r_{n,j}) = O \left( n^{-\frac{p}{p+1}} \right) \).

Lastly, a natural estimator for \( \tau^{-}(x) \) is

\[
\hat{\tau}^{-}(x) = \tilde{\mu}_{1,1}(x) \tilde{e}(x) + \hat{\theta}_1(x) (1 - \tilde{e}(x)) - \tilde{\mu}_{0,0}(x) (1 - \tilde{e}(x)) + \hat{\theta}_0(x) \tilde{e}(x).
\]

It is clear that under Assumptions 1-3, \( \hat{\tau}^{-}(\cdot) \) is consistent as an estimator of \( \tau^{-}(\cdot) \) and

\[
\left\| \hat{\tau}^{-}(\cdot) - \tau^{-}(\cdot) \right\|_{2, P} = O_p \left( n^{-\frac{p}{p+1}} \right) + O_p \left( r_{n,1} + r_{n,2} + r_{n,3} \right),
\]

i.e., the convergence rate of \( \hat{\tau}^{-}(\cdot) \) is dominated by those of \( \hat{\theta}_j(\cdot), \tilde{e}(\cdot) \) and \( \tilde{\mu}_{j, j}(\cdot), j = 0, 1. \)
4 Sensitivity of the ATE with observed covariates

In this section, we will study the lower bound of the ATE in the presence of covariates $X$. A lower bound on the ATE follows easily from marginalizing the lower bound on the CATE given $X$ developed in the previous section over. Specifically, let

$$\tau^-[P] = E_P[\tau^-(X)]$$
$$= E_P[\mu_1^-(X) - \mu_0^+(X)] \tag{4.1}$$

for $\mu_1^-(x)$ and $\mu_0^+(x)$ defined as in (3.2) and (3.3), respectively. For the remainder of the section we will drop the explicit notation $\tau^-[P]$, except in cases where it requires clarity that $\tau^-$ depends on the distribution $P$. It is clear that such a $\tau^-$ is a lower bound of ATE, since

$$\tau^- = E_P[\tau^-(X)] \leq E_P[E_P[Y(1) - Y(0)|X]]$$
$$= E[Y(1) - Y(0)]. \tag{4.2}$$

$\tau^-$ in (4.1) can be rewritten as

$$\tau^- = E \left[ E[Y(1)|X, Z = 1]P(Z = 1|X) + \theta_1(X)P(Z = 0|X) \right]$$
$$- E \left[ E[Y(0)|X, Z = 0]P(Z = 0|X) - \theta_0(X)P(Z = 1|X) \right]$$
$$= E \left[ ZY(1) + (1 - Z)\theta_1(X) - (1 - Z)Y(0) - Z\theta_0(X) \right]$$
$$= \mu_1^- - \mu_0^+, \tag{4.3}$$

where

$$\mu_1^- = E \left[ ZY(1) + (1 - Z)\theta_1(X) \right]$$
$$\mu_0^+ = E \left[ (1 - Z)Y(0) + Z\theta_0(X) \right].$$

Therefore, to estimate $\tau^-$, one only needs to estimate $\mu_1^-$ and $\mu_0^+$ separately. Also note that although the function $\theta_1(\cdot)$ cannot be estimated at the root $n$ in general, it is still possible to construct a regular $\sqrt{n}$-consistent estimator for $\mu_1^-$, which is a functional of $\theta_1(\cdot)$.

As in the previous sections, we divide this section up into subsections. Section 4.1 provides a method for semi-parametric estimation of $\tau^-$. This depends on estimating a number of functional nuisance parameters, including $\theta_1(x)$ used to estimate the lower bound on the CATE from Section 3 and two other nuisance parameters. Section 4.1 then provides a method for estimating the unknown nuisance parameter, $\nu_1(x)$, which can not be estimated by standard nonparametric methods as other parameters such as $e_1(\cdot)$. Section 4.2 provides $\sqrt{n}$-consistent and asymptotically normal convergence result for the semi-parametric estimator derived in Section 4.1, and Section 4.3 provides convergence guarantees for the remaining nuisance parameter. Finally, Section 4.4 establishes the optimality of these results in the framework of testing the null hypothesis of no treatment effect against an positive alternative with unobserved confounding under the $\Gamma$-selection bias condition.

4.1 Proposed semi-parametric estimation methodology

First, consider an ideal estimator for $\mu_1^-:

$$\hat{\mu}_1^- = \frac{1}{n} \sum_{i=1}^n Z_i Y_i + (1 - Z_i)\theta_1(X_i) + E \left[ \frac{Y_i - \theta_1(X_i)}{\nu_1(X_i)} - 1 - e_1(X_i) \right]/e_1(X_i) \tag{4.3}$$

where $e_1(x)$ is the (pseudo-)propensity score parameter

$$e_1(x) = P(Z = 1|X = x), \tag{4.4}$$
and \( \nu_1(x) \) is the weight normalization

\[
\nu_1(x) = P(Y(1) \geq \hat{\theta}_1(x)|X = x, Z = 1) + \Gamma P(Y < \hat{\theta}_1(x)|X = x, Z = 1).
\]  

(4.5)

If \( \hat{\theta}_1(x), \epsilon_1(x), \) and \( \nu_1(x) \) were all known, this estimator would be asymptotically unbiased, which can be checked by noting that \( \mathbb{E}[(Y_i - \theta(X_i))] + \Gamma(Y_i - \theta(X_i))|X_i = 0 \). Furthermore, it is a root \( n \) regular estimator in that \( \sqrt{n}(\hat{\mu}_i - \mu_1) \) converges weakly to a mean zero Gaussian distribution with a finite variance as the sample size goes to infinity.

Similarly, we may define

\[
\nu_0(x) = P(Y(0) \leq \hat{\theta}_0(x)|X = x, Z = 1) + \Gamma P(Y > \hat{\theta}_0(x)|X = x, Z = 1),
\]  

(4.6)

and construct a similar estimator of \( \mu_{0i}^\nu \). It is clear that all subsequent estimates with respect to \( \mu_{1i}^\nu \) in terms of \( Y(1) \) can be symmetrically derived for \( \mu_{0i}^\nu \) in terms of \( Y(0) \).

In practice, the nuisance parameters \( \hat{\theta}_1(x), \epsilon_1(x) \) and \( \nu_1(x) \) are not known a-priori in general, so that \( \hat{\mu}_i^\nu \) is unavailable. In order to construct a usable estimator, we need to replace aforementioned nuisance parameters by their consistent estimators based on the observed data. Section 3 already presented an estimator for \( \hat{\theta}_1(x), \epsilon_1(x) \) is the conditional mean of \( Z \) given \( X = x \), so it can be estimated by standard nonparametric or machine learning methods. \( \nu_1(x) \), on the other hand, depends on \( \hat{\theta}_1(x) \), whose estimation is more complicated.

Furthermore, if we replace all the nuisance parameters in (4.3) by their respective estimators, then the convergence rate of the resulting estimator may be altered and become slower than that of the oracle estimator based on the true nuisance parameters. In the current case, since the relevant nuisance parameters can not be estimated at the regular root \( n \) rate, it is not even clear if the convergence rate of the new estimator is still root \( n \). However, the special construction of the estimator \( \hat{\mu}_i^\nu \) suggests that its influence function satisfies the Neyman orthogonality condition \([8]\) (see the proof of Theorem 4.1). Therefore, the corresponding new estimator based on estimated nuisance parameters in the same form of \( \hat{\mu}_i^\nu \) can still be root \( n \) consistent, provided that nuisance parameters are estimated via special techniques such as out of sample cross fitting. The formal discussion of this phenomena can be found in Chernozhukov et al. [8].

In the current case, let \( J \in \mathbb{N} \) be a number of folds for cross-fitting. For simplicity of notation, assume that \( n/J \) is an integer. Randomly split the data into \( J \) folds, and let \( I_j \) be the indices corresponding to the samples in the \( j \)-th fold. Let \( I_{-j} \) be the indices of the samples not in the \( j \)-th fold. For each fold \( j \), compute \( \hat{\theta}_{1j}(x) \) using the procedure developed in Section 3 on the data in \( I_{-j} \). Similarly, estimate \( \bar{e}_j(x) \) and \( \bar{\nu}_j(x) \) on \( I_{-j} \) using an nonparametric estimator that satisfies \( \|\bar{e}(\cdot) - e_1(\cdot)\|_{2, P} = \text{op}(n^{-1/4}) \) and \( \|\bar{\nu}(\cdot) - \nu(\cdot)\|_{2, P} = \text{op}(n^{-1/4}) \).

As we discussed earlier, \( e_1(\cdot) \) can be estimated via standard nonparametric methods. The construction of \( \bar{\nu}_{1j}(\cdot) \) (and \( \bar{\nu}(\cdot) \)) is more involved. Specifically, we propose to employ sieve estimation method again. Let \( \nu_1(x) = 1 + (\Gamma - 1)\bar{P}(Y(1) \geq \hat{\theta}_1(x)|X = x) \). Define loss function \( \tilde{\ell} : [1, \Gamma]^Y \times \mathbb{R}^Y \times (\mathcal{X}, \mathbb{R}) \to \mathbb{R} \)

\[
\tilde{\ell}(\nu; \theta_1, (x, y)) := \frac{1}{2} \left( 1 + (\Gamma - 1)1 \{ y \geq \theta(x) \} - \nu(x) \right)^2,
\]

and consider the minimization problem

\[
\inf_{\nu(\cdot)} \left\{ \mathbb{E}_1[\tilde{\ell}(\nu; \theta_1, (X, Y(1)))] \mid \nu : \mathcal{X} \to \mathbb{R} \text{ measurable} \right\}.
\]  

(4.7)

Since \( \nu_1 \) is the unique minimizer of the problem (4.7), an sieve estimator for \( \nu_1 \) may be obtained by minimizing the empirical version of (4.7). However, this requires knowledge of \( \theta_1 \), which itself needs to be estimated. Therefore, we consider the following (nested) cross-fitting approach. Partition the samples in \( I_{-j} \) into two independent sets, and let \( \hat{\theta}_{1j} \) be an estimator of \( \theta_1 \) trained on the first set. For some sequence of sieve parameter spaces

\[
\Pi_1 \subseteq \cdots \subseteq \Pi_n \subseteq \cdots \subseteq \Pi,
\]

estimate \( \hat{\nu}_{1j}(x) \) by finding an (approximate) minimizer of the following sieve approximation for the plug-in version of the population problem (4.7)

\[
\min_{\nu(\cdot) \in \Pi_{1j}, 1 + (\Gamma - 1)\Pi_n} \mathbb{E}_{n_{1j}}[\tilde{\ell}(\nu; \hat{\theta}_{1j}^\nu, (X, Y(1)))],
\]  

(4.8)
where the expectation is with respect to the empirical distribution of the observations in the second set of $Z_{.i}$. Section 3.2 provides examples of reasonable choices of sieves that may be used.

In the end, our proposed estimator of $\mu^{-}_1$ is

$$\hat{\mu}^{-}_1 = \frac{1}{n} \sum_{j=1}^{J} \sum_{i \in I_{j}} Z_i Y_i + (1 - Z_i)\hat{\theta}_{1,j}(X_i)_{+} + Z_i \frac{Y_i - \hat{\theta}_{1,j}(X_i)}{\hat{c}_{j}(X_i)} - \frac{1}{n} \hat{\nu}(X_i). \quad (4.9)$$

A estimator for estimate $\mu^{-}_1$ can be constructed similarly.

### 4.2 Asymptotic convergence of the semi-parametric estimator

In this section, we will present the asymptotic property of the the proposed estimator $\hat{\mu}_1^{-}$. To this end, we need the following assumptions.

**Assumption 4.** For some $q > 2$, let $P \in \mathcal{P}$ satisfy

(a) $e_1(x) \in (\epsilon, 1 - \epsilon)$, for fixed $\epsilon > 0$,

(b) $Y(1)$ has finite expectation $\mu_1$, variance $\sigma_1^2$, and $E[|Y(1)|^q] \leq C_q$, and

(c) $P(Y(1) < t|X = x, Z = 1)$ has a density $p_{Y(1)}(y|X = x, Z = 1)$ with respect to the Lebesgue measure, such that $\sup_{y,x} p_{Y(1)}(y|X = x, Z = 1) = R < \infty$.

**Assumption 5.** For some $q > 2$, let $P \in \mathcal{P}$ and $\eta(x) = (\hat{\theta}(x), \hat{\nu}(x), \hat{c}(x))^\top$ satisfy

(a) $\|\hat{\theta}() - \theta_1()\|_{2,P} = O_P(n^{-1/4})$,

(b) $\|\hat{c}() - c_1()\|_{2,P} = O_P(n^{-1/4})$,

(c) $\|\hat{\nu}() - \nu_1()\|_{2,P} = O_P(n^{-1/4})$,

(d) $\|\hat{\theta}() - \theta_1()\|_{q,P} = O_P(1)$,

(e) $\|\hat{c}() - c_1()\|_{q,P} = O_P(1)$, and

(f) $\|\hat{\nu}() - \nu_1()\|_{q,P} = O_P(1)$.

Assumption 4(a,b) are standard regularity conditions needed for estimation of the ATE, even without unobserved confounding [8]. Assumption 4(c) is needed to ensure that the term

$$\theta() \rightarrow E[Z((Y(1) - \theta(X))_{+} - \Gamma(Y(1) - \theta(X))_{-})|X = x]$$

is smooth enough to control the effect of the variation in estimating the nuisance parameter on the score function. Given the intuition from Section 2.3, specifically the discussion of Proposition 2.2, it is possible that without this assumption, $\sqrt{n}$ consistency will still hold for estimating $\mu_1$, but with a different limiting distribution. Future work generalizing the Neyman orthogonality condition and functional differentiability to a functional analog of sub-differentials may allow such a result, but is outside the scope of the current work. Note that if $\theta_1(x)$ and $\theta_1$ have a uniformly bounded range $\mathcal{I}$ for every $x$, then Assumption 4(c) can be relaxed to the condition

$$\sup_{y \in \mathcal{I}_x} p_{Y(1)}(y|X = x, Z = 1) < \infty.$$

The rate conditions on estimating the nuisance parameters in Assumption 5 are relatively standard rates in semi-parametric estimation [22, 8]. Because $c_1(x)$ is the conditional mean of observed random variables, a variety of machine learning and nonparametric methods can be used to guarantee $n^{-1/4}$ consistency [39, 8]. The methods provided in Section 3 and 4.1 can achieve the rates of convergence required of Assumption 5 under appropriate smoothness conditions on $\theta_1(x)$ and $\nu_1(x)$. For instance, if Assumptions 1, 2, and 3 hold with $p > d/2$, then Corollary 3.1 shows that estimating $\theta_1(x)$ as described in Section 3 with a finite dimensional linear sieve (see Examples 1 and 2) will satisfy Assumption 5(a,d). As described below in Section 4.3, estimating $\nu_1(x)$ according to Section 4.1 with a finite dimensional linear sieve will satisfy Assumption 5(c,f) when $p > d/2$, as well.

Under these assumptions, the following theorem gives the asymptotic distribution of the proposed estimator $\hat{\mu}_1^{-}$ in (4.9).
Theorem 4.1. Let \( P \), \( \eta_1(x) = (\theta_1(x), \nu_1(x), e_1(x))^\top \), and \( \hat{\eta}(x) \) satisfy Assumptions 4 and 5 for some choice of \( q > 2 \) that satisfies both assumptions. Then, for any \( P \in P \), \( \hat{\mu}_1 \) estimated as in (4.9) is asymptotically normal with
\[
\sqrt{n}(\hat{\mu}_1 - \mu_1) \xrightarrow{d} N(0, \sigma^2),
\]
where
\[
\sigma^2 = \text{Var}\left[Z_i Y_i + (1 - Z_i) \theta_1(X_i) + Z_i \frac{(Y_i - \theta_1(X_i))^+ - \Gamma(Y_i - \theta_1(X_i))^- - e(X_i)}{\nu_1(X_i)} \right].
\]
Let \( z_{1-\alpha} = \mathbb{P}(\left| N(0, 1) \right| \leq 1 - \alpha) \). If we estimate
\[
\hat{\sigma}^2 = \text{Var}_{P_n}\left[Z_i Y_i + (1 - Z_i) \theta_1(X_i) + Z_i \frac{(Y_i - \theta_1(X_i))^+ - \Gamma(Y_i - \theta_1(X_i))^- - e(X_i)}{\nu_1(X_i)} \right],
\]
where \( j \) has is implicitly a function \( j = j(i) \) that is the sample such that \( i \in I_j \), then the confidence interval
\[
\hat{\mu}_1 \in [\hat{\mu}_1 \pm z_{1-\alpha} \hat{\sigma} / \sqrt{n}]
\]
satisfies
\[
P(\mu_1 \in \hat{\mu}_1) \rightarrow 1 - \alpha,
\]
as \( n \rightarrow \infty \).

See Appendix C for proof.

Let
\[
\hat{\tau}^-(Y_i, Z_i, X_i)_{i=1}^n = \hat{\mu}_1 - \hat{\mu}_0^+.
\]
A simple extension of Theorem 4.1 gives the following results.

Corollary 4.1.
\[
\sqrt{n}(\hat{\tau}^- - \tau^-) \rightarrow N(0, \sigma^2_{\tau^-}), \tag{4.11}
\]
where
\[
\sigma^2_{\tau^-} = \text{Var}\left[Z_i Y_i + (1 - Z_i) \theta_1(X_i) + Z_i \frac{(Y_i - \theta_1(X_i))^+ - \Gamma(Y_i - \theta_1(X_i))^- - e(X_i)}{\nu_1(X_i)} \right],
\]
Let \( z_{1-\alpha} = \mathbb{P}(\left| N(0, 1) \right| \leq 1 - \alpha) \). If we estimate
\[
\hat{\sigma}^2_{\tau^-} = \text{Var}_{P_n}\left[Z_i Y_i + (1 - Z_i) \theta_1(X_i) + Z_i \frac{(Y_i - \theta_1(X_i))^+ - \Gamma(Y_i - \theta_1(X_i))^- - e(X_i)}{\nu_1(X_i)} \right],
\]
where \( j \) has is implicitly a function \( j = j(i) \) that is the sample such that \( i \in I_j \), then the confidence interval
\[
\hat{\tau}^- \in [\hat{\tau}^- \pm z_{1-\alpha} \hat{\sigma}_{\tau^-} / \sqrt{n}]
\]
satisfies
\[
P(\tau^- \in \hat{\tau}^-) \rightarrow 1 - \alpha,
\]
as \( n \rightarrow \infty \).

The proof is almost identical to that of Theorem 4.1, because no cross terms exist between the nuisance parameters of \( \mu_1 \) and \( \mu_0^+ \). Therefore, we omit the proof.

Here, we break from our focus on lower bounds to provide an important result about bounding the average treatment effect from both sides. By applying the same approach as in Section 3 and this section to upper bound the average treatment effect as \( \tau^+ [P] \), we can create a lower CI (as in Corollary 4.1) and an upper CI that include the lower bound \( \tau^- \) and upper bound \( \tau^+ \), respectively. Because \( \tau^- \leq E[Y(1) - Y(0)] \leq \tau^+ \), the following result shows that we can produce a confidence interval that includes the true ATE when the \( \Gamma \)-selection bias condition holds. Specifically, let \( \hat{\tau}^- \) be defined as
in Corollary 4.1 as well as \( \hat{\varphi}_{r-}^2 \), and let \( \hat{\varphi}_{r}^+ \) and \( \hat{\varphi}_{r+}^+ \) be their counterparts for an upper bound on the ATE. Then the confidence interval
\[
\hat{\mathcal{C}}_r = [\hat{\varphi}^- - z_{1-\alpha} \frac{\hat{\varphi}_{r-}^2}{\sqrt{n}}, \hat{\varphi}^+ + z_{1-\alpha} \frac{\hat{\varphi}_{r+}^2}{\sqrt{n}}]
\]
would cover the true ATE with a probability greater than \( 1 - \alpha \) as the sample size goes to infinity, i.e.,
\[
\liminf_{n \to \infty} P(\mathbb{E}_P[Y(1) - Y(0)] \in \hat{\mathcal{C}}_r) \geq 1 - \alpha.
\]

### 4.3 Convergence guarantees for estimation of \( \nu_1(x) \)

In this section, we provide convergence guarantees for the estimator in Section 4.1 for completeness. In order for the empirical plug-in for the problem (4.8) to converge, the map \( x \mapsto P(Y(1) \geq \theta(x)|X = x, Z = 1) \) must be smooth for smooth functions \( \theta \) around \( \theta_1 \).

**Assumption 6.** There exists \( q, r > 0 \), and a set \( S \subset \Lambda^l_\delta(X) \) with \( \theta_1 \in S \) such that

1. \( P(\hat{\theta}_1 \in S|Z = 1) \to 1 \) as \( n \to \infty \)
2. for all \( \theta \in S \), \( x \mapsto P(Y(1) \geq \theta(x)|X = x, Z = 1) \) belongs in \( \Pi := \Lambda^l_\delta(X) \setminus \{\nu : X \to [0, 1]\} \).

For example, above assumption holds if \( Y \) would cover the true ATE with a probability greater than 1 - \( \alpha \) as the sample size goes to infinity, i.e.,
\[
\liminf_{n \to \infty} P(\mathbb{E}_P[Y(1) - Y(0)] \in \hat{\mathcal{C}}_r) \geq 1 - \alpha.
\]

**Assumption 7.** There exists \( L_\nu > 0 \) such that \( t \mapsto P_1(Y(1) \geq t|X) \) is \( L_\nu \)-Lipschitz almost surely.

For example, above assumption holds if \( Y(1)|X, Z = 1 \) has a Lipschitz cumulative distribution guarantees this.

To define an appropriate notion of model complexity and estimation error, for some \( b > 0 \), let
\[
\delta_n := \inf \left\{ \delta \in (0, 1) : \frac{1}{\sqrt{n} \delta^2} \int_{B_2} \log N \left( \epsilon^{1+d/2p}, \Pi_n, \|\cdot\|_{2, P_1} \right) d\epsilon \leq 1 \right\}.
\]

The following proposition quantifies the trade-off between the estimation/approximation error, and \( \|\hat{\theta}_1 - \theta_1\|_{2, P_1} \) when approximating \( \nu_1 \). We defer its proof to Appendix C.1.

**Proposition 4.1.** Let Assumptions 3, 6, 7 hold, and let \( \hat{\nu}_1 \) be an approximate empirical minimizer to the problem (4.8) satisfying
\[
\mathbb{E}_n[\bar{f}(\hat{\nu}_1, \hat{\theta}_1^1, (X, Y(1)))|Z = 1] \leq \inf_{\nu \in \Lambda_{1+(r-1)} \Pi_n} \mathbb{E}_n[\bar{f}(\nu, \hat{\theta}_1^1, (X, Y(1)))|Z = 1] + O_p \left( \epsilon_n^2 \right)
\]

where \( \epsilon_n := \max \left\{ \delta_n, \inf_{\nu \in \Lambda_{1+(r-1)} \Pi_n} \|\nu_1 - \nu\|_{2, P_1} \right\} \). If \( n \epsilon_n \to \infty \), then \( \|\hat{\nu}_1 - \nu_1\|_{2, P_1} = O_p \left( \epsilon_n + \|\hat{\theta}_1^1 - \theta_1\|_{2, P_1} \right) \).

The following standard non-parametric rates for estimating \( \nu_1 \) are a consequence of using the finite dimensional linear sieves given in Section 3.2.

**Corollary 4.2.** For \( X = [0, 1]^d \), let \( \Pi_n \) be given by finite dimensional linear sieves considered in Examples 1 or 2 with \( J_\rho \approx n^{1/(d+2)} \). Let Assumptions 3, 6, 7 hold, and let \( \hat{\nu}_1 \) be an approximate empirical minimizer to the problem (4.8) satisfying
\[
\mathbb{E}_n[\bar{f}(\hat{\nu}_1, \hat{\theta}_1^1, (X, Y(1)))|Z = 1] \leq \inf_{\nu \in \Lambda_{1+(r-1)} \Pi_n} \mathbb{E}_n[\bar{f}(\nu, \hat{\theta}_1^1, (X, Y(1)))|Z = 1] + O_p \left( n^{-(d+2)/(d+2)} \right).
\]

Then, \( \|\hat{\nu}_1 - \nu_1\|_{2, P_1} = O_p \left( n^{-\frac{2d}{d+2}} + \|\hat{\theta}_1^1 - \theta_1\|_{2, P_1} \right) \).
For example, consider a procedure that partitions the sample into two sets, and uses the first subset to estimate \( \hat{\theta}_1 \) as in Corollary 3.1. Using the other subset to compute \( \hat{\nu}_1 \), Corollary 4.2 yields \( \|\hat{\nu}_1 - \nu_1\|_{2,P_\nu} = O_p\left(\frac{1}{\sqrt{n}}\right) \).

### 4.4 Hypothesis testing and design sensitivity

In this section, we provide an asymptotic level \( \alpha \) hypothesis test for the null

\[
H_0 : \begin{cases} 
\mathbb{E}[Y(1)] \leq \mathbb{E}[Y(0)], \\
(Z,U) : \Gamma \text{ selection bias,} \\
\text{Assumptions 4 and 5 hold,}
\end{cases}
\]

and analyze its design sensitivity [29]. A level \( \alpha \) test is easy to construct based on the confidence intervals in Corollary 4.1. Specifically, Let

\[
\psi_n((Y_i, Z_i, X_i)_{i=1}^n) = 1_{\left\{ \hat{\sigma}_- > 0 \right\}},
\]

where \( \hat{\sigma}_- \) are defined in Corollary 4.1. Then the following Corollary guarantees that the test \( \psi_n \) is a valid level \( \alpha \) test.

**Corollary 4.3.** Let \( H_0 \) be defined as in (4.14). For \( \psi_n((Y_i, Z_i, X_i)_{i=1}^n) \) given in (4.15) and any \( P \in H_0 \),

\[
\lim_{n \to \infty} \inf \mathbb{P}(\psi_n \leq 0) \geq 1 - \alpha.
\]

In the next subsection, we are going to study its design sensitivity and demonstrate that its design sensitivity is optimal under additional conditions.

#### 4.4.1 Design sensitivity

A test \( t \) is said to be level \( \alpha \) for a composite null hypothesis \( H_0 \) if for any \( P \in H_0 \), \( P(t((Y_i, Z_i, X_i)_{i=1}^n = 0) \geq 1 - \alpha \). The power of a test \( t \) against an alternative \( H_1 = \{ Q \} \) is \( \mathbb{P}(t((Y_i, Z_i, X_i)_{i=1}^n) = 1) \), where we use \( Q \) to denote the probability measure under alternative in general.

In the current setting, let \( H_1 = \{ Q \} \) be an alternative hypothesis, such that the causal treatment effect is positive \( \mathbb{E}_Q[Y(1)] - \mathbb{E}_Q[Y(0)] = \tau > 0 \), and there is no selection bias (ie. \( \Gamma = 1 \)). Without loss of generality, assume \( \mathbb{Q}(Z = 1) = \frac{1}{2} \). The design sensitivity of a sequence of level \( \alpha \) tests \( t_n((Y_i, Z_i, X_i)_{i=1}^n) \in \{0,1\} \) of the null \( H_0(\Gamma) \) defined in (4.14) with respect to the alternative \( H_1 \) is the threshold \( \hat{\Gamma} \) such that the power \( \mathbb{P}(t_n((Y_i, Z_i, X_i)_{i=1}^n) = 1) \to 0 \) for \( \Gamma > \hat{\Gamma} \) and \( \mathbb{P}(t_n((Y_i, Z_i, X_i)_{i=1}^n) = 1) \to 1 \) for \( \Gamma < \hat{\Gamma} \) [29, 30].

Loosely speaking, if the selection bias is above the design sensitivity, the test can never differentiate the alternative from the null; if the selection bias is below that design sensitivity, the test can always differentiate the alternative from the null given adequate sample size. Note that in order for \( t_n \) to be a level \( \alpha \) test for \( H_0(\Gamma) \), it will generally depend on \( \Gamma \). As a result, it may be convenient to write \( t_n^\Gamma \) to denote this dependence.

**Proposition 4.2.** Let \( \psi_n^\Gamma \) be defined as in (4.15), so that \( \psi_n^\Gamma \) is asymptotically level \( \alpha \) for \( H_0(\Gamma) \) in (4.14) (see Corollary 4.3). Then, for an alternative \( H_1 = \{ Q \} \), \( \psi_n^\Gamma \) has design sensitivity \( \hat{\Gamma} \) for \( \hat{\Gamma} \) satisfying

\[
0 = -\frac{1}{2} \mathbb{E}_Q[Y(0)] + \sup_{\hat{\theta}_0(\cdot)} \frac{1}{2} \mathbb{E}_Q \left[ \frac{1\{Y(1) > \hat{\theta}_0(X)\} + \hat{\Gamma}1\{Y(1) < \hat{\theta}_0(X)\}}{\mathbb{Q}(Y(1) > \hat{\theta}_0(X)|X) + \hat{\Gamma} \mathbb{Q}(Y(1) < \hat{\theta}_0(X)|X)} Y(1) \right] \]
\[
- \frac{1}{2} \mathbb{E}_Q[Y(1)] + \inf_{\hat{\theta}_1(\cdot)} \frac{1}{2} \mathbb{E}_Q \left[ \frac{1\{Y(0) > \hat{\theta}_1(X)\} + \hat{\Gamma}1\{Y(0) < \hat{\theta}_1(X)\}}{\mathbb{Q}(Y(0) > \hat{\theta}_0(X)|X) + \hat{\Gamma} \mathbb{Q}(Y(0) < \hat{\theta}_0(X)|X)} Y(0) \right].
\]

While there is no simplified expression for \( \hat{\Gamma} \), in general, it can be evaluated for a given distribution \( Q \). For instance the following corollary provides the simple form for a Gaussian alternative.
Corollary 4.4. Let \( \psi^R_n \) be defined as in (4.15), so that \( \psi^R_n \) is asymptotically level \( \alpha \) for \( H_0(\Gamma) \) in (4.14) (see Corollary 4.3). Then, for the alternative \( H_1 = \{Y(1) \sim N(1/2, \sigma^2), Y(0) \sim N(-1/2, \sigma^2), Z \sim \text{Bernoulli}(1/2)\} \), \( \psi^R_n \) has design sensitivity

\[
\tilde{\Gamma} \leq \frac{\int_{-\infty}^{0} y \exp \left( -\frac{(y-\tau)^2}{2\sigma^2} \right) dy}{\int_{-\infty}^{0} y \exp \left( -\frac{(y+\tau)^2}{2\sigma^2} \right) dy}.
\]

(4.16)

For the case of a Gaussian alternative, the following proposition provides a lower bound for the design sensitivity, which is the same as that of the proposed test. It suggests the optimality of the proposed test in this case.

Proposition 4.3. Let \( H_0(\Gamma) \) be defined as in (4.14), and let \( H_1 = \{Y(1) \sim N(1/2, \sigma^2), Y(0) \sim N(-1/2, \sigma^2), Z \sim \text{Bernoulli}(1/2)\} \). If

\[
\Gamma \geq \frac{\int_{-\infty}^{0} y \exp \left( -\frac{(y-\tau)^2}{2\sigma^2} \right) dy}{\int_{-\infty}^{0} y \exp \left( -\frac{(y+\tau)^2}{2\sigma^2} \right) dy}.
\]

(4.17)

then no level \( \alpha \) test \( t^*_n \) for \( H_0(\Gamma) \) will have power \( Q(t^*_n) = 1 > \alpha \). That is, a design sensitivity \( \tilde{\Gamma} \) of any level \( \alpha \) test for \( H_1 \) must satisfy the inequality

\[
\tilde{\Gamma} \geq \frac{\int_{-\infty}^{0} y \exp \left( -\frac{(y-\tau)^2}{2\sigma^2} \right) dy}{\int_{-\infty}^{0} y \exp \left( -\frac{(y+\tau)^2}{2\sigma^2} \right) dy}.
\]

Note that these results can be extended to any alternative such that \( Y(0) \overset{d}{=} C(1 - Y(1)) \), for some constant \( C > 0 \).

5 Numerical experiments

While the theoretical properties of the DML procedure ensure that the estimator is asymptotically normal and has valid asymptotic confidence intervals, we have examined their finite sample performance based on Monte-Carlo simulation as well as real data from observational study examining the effect of fish consumption on blood mercury level. The details of numerical study are given in the next two subsections. But in summary, the Monte-Carlo simulation study supports the validity of the inference procedure in realistic settings. The analysis of the real observational study also shows the practical improvements over existing methods. In particular, the method gives a tighter control over the bounds on the treatment effect.

5.1 Simulations

The purpose of the simulation study is to demonstrate that the proposed confidence intervals have a good coverage level for reasonable choices of sample size \( n \) and covariate dimension \( d \) for supporting the theoretical analysis in previous section. In all simulations, \( X \sim \text{Uniform}(0, 1)^d \). Conditional on \( X \),

\[ U \sim N(0, (1 + \frac{1}{2} \sin(2.5X_1))^2). \]

Conditional on \( X \) and \( U \),

\[ Y(0) = X^\top \beta + U, \]
\[ Y(1) = \tau + X^\top \beta + U, \]

and finally the treatment assignment

\[ Z \sim \text{Bernoulli} \left( \frac{1}{1 + \exp \left( -X^\top \mu - \log(\Gamma)I_{U>0} - C \right)} \right). \]
Table 1. Coverage statistics from \( B = 2000 \) simulations generated from a randomly selected linear model with \( d = 4 \) observed covariates. The true ATE in the simulation is \( \tau = 1 \), and unobserved confounding is chosen such that simulation satisfies (1.4) for \( \Gamma = \exp(1) \). All nuisance parameters are estimated non-parametrically, using kernel smoothing for \( \hat{\theta}(x) \) and random forests for the propensity score, \( \hat{\ell}(x) \). Std. dev. of \( \hat{\tau}^- \) refers to standard deviation between simulation runs (likewise for \( \hat{\tau}^+ \)). Coverage is for estimated 95% confidence intervals \((\hat{\tau}^- - 1.96\hat{\sigma}^-, \hat{\tau}^+ + 1.96\hat{\sigma}^+)\). These are conservative because the true model cannot simultaneously be upward biased and downward biased. Therefore, we expect a model under worst-case confounding to converge to 97.5% coverage.

| n  | \( \hat{\tau}^- \) | \( \hat{\sigma}^- \) | Std. dev. of \( \hat{\tau}^- \) | \( \hat{\tau}^+ \) | \( \hat{\sigma}^+ \) | Std. dev. of \( \hat{\tau}^+ \) | Coverage |
|----|-----------------|----------------|-----------------|----------------|----------------|----------------|---------|
| 100 | 1.001           | 0.241          | 0.332           | 1.837          | 0.208          | 0.249          | 0.929    |
| 200 | 0.980           | 0.168          | 0.185           | 1.801          | 0.159          | 0.172          | 0.967    |
| 400 | 0.985           | 0.122          | 0.132           | 1.784          | 0.117          | 0.126          | 0.965    |
| 800 | 0.985           | 0.087          | 0.094           | 1.766          | 0.083          | 0.089          | 0.971    |
| 1600| 0.991           | 0.063          | 0.065           | 1.766          | 0.061          | 0.060          | 0.975    |

Table 2. Coverage statistics from \( B = 2000 \) simulations generated from a randomly selected linear model with \( d = 8 \) observed covariates. The true ATE in the simulation was \( \tau = 1 \), and unobserved confounding was chosen such that simulation satisfies (1.4) for \( \Gamma = \exp(1) \). To match a realistic observational analysis, \( n = 1100 \) was chosen, 21% of the observations were assigned to treatment. Std. dev. of \( \hat{\tau}^- \) refers to standard deviation between simulation runs (likewise for \( \hat{\tau}^+ \)). See Table 1 for notes on coverage. The estimator was compared with a parametric propensity score model and a non-parametric propensity score model, and compared to the matching estimator \cite{31} in package \texttt{sensitivitymv}.

| Approach | \( \hat{\tau}^- \) | \( \hat{\sigma}^- \) | Std. dev. of \( \hat{\tau}^- \) | \( \hat{\tau}^+ \) | \( \hat{\sigma}^+ \) | Std. dev. of \( \hat{\tau}^+ \) | Coverage |
|----------|-----------------|----------------|-----------------|----------------|----------------|----------------|---------|
| Parametric | 0.988           | 0.071          | 0.081           | 1.775          | 0.068          | 0.076          | 0.970    |
| Nonparametric | 0.995          | 0.073          | 0.081           | 1.775          | 0.069          | 0.076          | 0.960    |
| Matching   | 0.869           | 0.067          | 0.097           | 2.125          | 0.068          | 0.097          | 0.996    |

The parameters \( \beta \) and \( \mu \) are fixed for each of the settings below. \( C \) is a constant chosen to adjust the marginal distribution \( P(Z = 1) \). Note that because of the heteroskedasticity of \( U \), a linear model will not be correctly specified for the CATE sensitivity analysis, requiring the use of a nonparametric model to consistently estimate the lower bound.

In the first setting, we simulate data with a small number of observed covariates \( (d = 4) \) to demonstrate that the confidence intervals are well calibrated as \( n \) grows large. All nuisance parameter estimators are non-parametric sieve estimators using the polynomial sieve, and tuning parameters (sieve size and regularization) are selected via 10-fold cross-validation. Because the true parameter is known from the simulation design, coverage statistics are estimable, and are presented for a number of choices of \( n \) varying from 100 to 1600 in Table 1.

In the second setting, we simulate data to match the size of data often found in practical observational studies. Specifically, the parameters of the simulation mimic those from the analysis of the real observational study examining the effect of fish consumption on blood mercury level considered in the next subsection \((d = 8, n = 1100, P(Z = 1) = 0.21) \). The results are presented in Table 2. In this setting, parametric and non-parametric estimators of the nuisance parameters both work reasonably well, although the non-parametric estimator of the propensity score increases the variance of the estimator and requires mild weight clipping to bound the influence on the estimator from potential influential points. The statistical model used to derive the parametric estimator is not correctly specified, and therefore the resulting estimator has a non-vanishing asymptotic bias. Standard errors are slightly underestimated (off by a factor of 10%), due to finite sample errors from higher order effects of estimating the nuisance parameters on the score equation. Finally, the matching method based on M-estimates \cite{31} implemented in \texttt{sensitivitymv} are compared to the proposed approaches. While the confidence intervals for ATE had a good coverage, this was partially due to an overly conservative lower bound estimate; the standard errors were severely underestimated by a factor of 31%.
Table 3. Comparison to sensitivity results of [40] using the same data set. Because the same sensitivity model as the matched analysis was used, results can be compared directly. We demonstrate that the method can achieve tighter bounds on the average treatment effect both in point estimates and confidence intervals.

| \( \Gamma \) | Proposed method | Matching |
|---|---|---|
| \( \text{Lower} \) | \( \text{Upper} \) | \( \text{Lower} \) | \( \text{Upper} \) | \( \text{Lower} \) | \( \text{Upper} \) | \( \text{Lower} \) | \( \text{Upper} \) |
| 1 | 1.74 | 1.74 | 1.51 | 1.97 | 2.08 | 2.08 | 1.90 | 2.25 |
| \( \exp(0.5) \) | 1.53 | 2.03 | 1.31 | 2.26 | 1.75 | 2.41 | 1.57 | 2.59 |
| \( \exp(1) \) | 1.27 | 2.27 | 1.07 | 2.47 | 1.45 | 2.74 | 1.25 | 2.94 |
| \( \exp(2) \) | 0.91 | 2.77 | 0.74 | 2.89 | 0.87 | 3.36 | 0.58 | 3.65 |
| \( \exp(3) \) | 0.60 | 3.19 | 0.47 | 3.29 | 0.28 | 3.97 | -0.23 | 4.48 |
| \( \exp(4) \) | 0.29 | 3.55 | 0.18 | 3.63 | - | - | - | - |

5.2 Real observational data

To illustrate our method on real observed data, we compare the results of our proposed method to a prior analysis using matching to infer the effect of fish consumption on blood mercury levels [40]. The real data are from \( N = 2,512 \) adults in the United States who participated in a single cross-sectional wave of the National Health and Nutrition Examination Survey (2013-2014), in which participants answered a questionnaire regarding their demographics and food consumption, and had blood tests conducted including measured mercury concentration (data available in the R package CrossScreening). To match the prior analysis [40], no missing data was imputed for the 1 individual with missing education data and 7 individuals with missing smoking data, while the median income was used to impute missing income data for the 175 individuals missing this variable, and a binary variable was assigned to them to indicate if the income data were missing.

High fish consumption is defined as individuals who reported > 12 servings of fish or shellfish in the previous month per their questionnaire, low fish consumption as 0 or 1 servings of fish, and the outcome as \( \log_2 \) of total blood mercury concentration (ug/L). The data provides a total of 234 treated individuals (those with high fish consumption), 873 control individuals (low fish consumption), and 8 covariates on which estimation or matching is performed (gender, age, income, whether income is missing, race, education, ever smoked, and number of cigarettes smoked last month). Our approach uses the same sensitivity model as the previous matched-pair analysis so it is reasonable to directly compare results between our proposed method and sensitivity analysis based on the 234 matched pairs. As shown in Table 3, when \( \Gamma > \exp(1) \), our proposed method achieves tighter confidence intervals around the effect size estimate of fish consumption on blood mercury level: our confidence intervals are nested within the confidence intervals based on matching method. When \( \Gamma \leq \exp(1) \), the confidence intervals of the proposed method are not nested within those from the matching methodology. However, the length of the confidence interval based the proposed method is still substantially smaller. The only exception is when \( \Gamma = 1 \), i.e, no unobserved confounding, the matching method generates a slightly narrower confidence interval.

6 Discussion

The model (1.3) and (1.4) relax the unconfoundedness assumption required for identification of treatment effects. We have refined the approach taken by Rosenbaum [28] and others based on matched pairs. Section 3 extends the idea of bounding the average treatment effect to bounding the CATE, conditional on the observed covariates, \( X \). We propose estimators for lower and upper bounds for the ATE and associated inference procedure without matching the observed covariates, which is often infeasible in the presence of multiple continuous covariates. On the other hand, approximate matching in with more than 2 observed covariates may induce higher order bias. In contrast, the methodology developed here uses an orthogonal moments score function [8], which allows estimating the bound on the ATE at the regular root \( n \) rate, even when the infinite dimensional nuisance parameters are estimated at a much slower rate \( o_p(n^{-1/4}) \). Furthermore, we have also demonstrated that the proposed sensitivity analysis achieves the
optimal design sensitivity in special cases including normally distributed potential outcomes. We are investigating whether this approach can be extended to provide optimality guarantees over more general distributions such as binary or survival outcomes.

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A Proofs for results without covariates

Lemma 2.1. If \( P \) has \( \Gamma \)-selection bias, \( P_{Y(1)}(\cdot | Z = 0) \) is absolutely continuous with respect to \( P_{Y(1)}(\cdot | Z = 1) \), and the likelihood ratio \( L := \frac{dP_{Y(1)}(\cdot | Z = 0)}{dP_{Y(1)}(\cdot | Z = 1)} \) satisfies \( L(y) \leq \Gamma L(\tilde{y}) \) for \( P_{Y(1)} | Z = 1 \)-almost every \( y, \tilde{y} \).

Proof. First, we show that \( P_U(\cdot | Z = 1) \) is absolutely continuous with respect to \( P_U(\cdot | Z = 0) \), and that the two measures are equivalent, meaning that

\[
C_1 P_U(A | Z = 1) \leq P_U(A | Z = 0) \leq C_2 P_U(A | Z = 0),
\]

for \( 0 < C_1, C_2 < \infty \). Let \( A \in \sigma(U) \) with \( P(A | Z = 1) > 0 \). Then,

\[
P(A | Z = 1) = \frac{P(A, Z = 1)}{P(Z = 1)} = \frac{P(Z = 1 | A) P(A)}{P(Z = 1)}. \tag{A.1}
\]

Similarly,

\[
P(A | Z = 0) = \frac{P(A, Z = 0)}{P(Z = 0)} = \frac{P(Z = 0 | A) P(A)}{P(Z = 0)}.
\]

By plugging in \( P(A) \) from (A.1) and using (1.4),

\[
P(A | Z = 0) = \frac{P(Z = 0 | A) P(Z = 1)}{P(Z = 0) P(Z = 1 | A)} P(A | Z = 1) \\
\geq \frac{1}{\Gamma} P(A | Z = 1) > 0.
\]

Choosing \( A \) such that \( P(A | Z = 0) > 0 \) and applying the argument in the other direction shows

\[
P(A | Z = 1) \geq \frac{1}{\Gamma} P(A | Z = 0) > 0,
\]

which together satisfy the definition of mutual absolute continuity [3]. Therefore, \( P_U(\cdot | Z = 1) \) and \( P_U(\cdot | Z = 0) \) are equivalent.

Let \( R := \frac{dP_{Y(1)}(\cdot | Z = 0)}{dP_{Y(1)}(\cdot | Z = 1)} \) denote the likelihood ratio. Applying Bayes rule to (2.2),

\[
\frac{1}{\Gamma} \leq \frac{R(u)}{R(\tilde{u})} \leq \Gamma \tag{A.2}
\]

for almost any \( u, \tilde{u} \). Now, we show that \( P_{Y(1)}(\cdot | Z = 1) \) and \( P_{Y(1)}(\cdot | Z = 0) \) are absolutely continuous, and derive a bound on their likelihood ratio. For \( B \in \sigma(Y(1)) \), because of the independence assumption (2.1)

\[
\mathbb{E} \left[ 1_{(B)} | Z = 0 \right] = \mathbb{E} \left[ \mathbb{E} \left[ 1_{(B)} | U \right] | Z = 0 \right] \\
= \mathbb{E} \left[ R(U) \mathbb{E} \left[ 1_{(B)} | U \right] | Z = 1 \right] \\
= \mathbb{E} \left[ \mathbb{E} \left[ R(U) 1_{(B)} | U \right] | Z = 1 \right] \\
= \mathbb{E} \left[ \mathbb{E} \left[ R(U) Y(1), Z = 1 \right] 1_{(B)} | Z = 1 \right] = \mathbb{E} \left[ \mathbb{E} \left[ R(U)|Y(1), Z = 1 \right] 1_{(B)} | Z = 1 \right],
\]

because \( 1_{(B)} \) is \( Y(1) \)-measurable. Since the bound (A.2) and \( \mathbb{E} \left[ R(U) | Z = 1 \right] = 1 \) imply that \( \frac{1}{\Gamma} \leq R(u) \leq \Gamma \) almost everywhere, we have

\[
\mathbb{E} \left[ \mathbb{E} \left[ R(U)|Y(1), Z = 1 \right] 1_{(B)} | Z = 1 \right] > 0
\]

whenever \( \mathbb{E}[1_{(B)} | Z = 1] > 0 \). This implies the absolute continuity result.

Now, noting that \( P_{Y(1)}(\cdot | Z = 1) \) is absolutely continuous with respect to \( P_{Y(1)}(\cdot | Z = 0) \), and

\[
\mathbb{E} \left[ 1_{(B)} | Z = 0 \right] = \mathbb{E} \left[ \mathbb{E} \left[ R(U)|Y(1), Z = 1 \right] 1_{(B)} | Z = 1 \right],
\]

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the likelihood ratio \( L(y) \) satisfies
\[
L = \mathbb{E} \left[ R(U) | Y(1), Z = 1 \right] \quad \mathbb{P}_{Y(1)}|Z=1 \text{ almost surely}
\]
by the Radon-Nikodym Theorem.

Finally, show \( L(y) \leq \Gamma L(\tilde{y}) \). Let \( \delta > 0 \) and let \( u^\delta \) be such that
\[
R(u^\delta) < \inf_u R(u) + \delta.
\]
Then, for almost every \( y \),
\[
L(y) = \mathbb{E}[R(U)|Y(1) = y, Z = 1] = R(u^\delta)\mathbb{E}[R(U)/R(u^\delta)|Y(1) = y, Z = 1] \leq \Gamma R(u^\delta),
\]
and
\[
L(y) \geq \inf_u R(u) > R(u^\delta) - \delta.
\]
Therefore,
\[
L(y) \leq \Gamma R(u^\delta) < \Gamma (L(\tilde{y}) + \delta).
\]
\( \delta \) is arbitrary, so taking \( \delta \rightarrow 0 \) gives \( L(y) \leq \Gamma L(\tilde{y}) \).

**Lemma 2.2.** If \( \theta_1 < \infty \), then
\[
\theta_1 = \inf_{L(y)} \sup_{\mu} \mathbb{E}[Y(1) L(Y(1)) | Z = 1] \tag{2.7}
\]
\[
\text{s.t.} \quad \begin{align*}
\mathbb{E}[L(Y(1)) | Z = 1] &= 1 \\
L(y) &\geq 0, L(y) \leq \Gamma L(\tilde{y}) \text{ for almost every } y, \tilde{y} \in \mathbb{R}
\end{align*}
\]
\[
= \sup_{\mu} \inf_{L(y)} \mathbb{E}[(Y(1) - \mu)_+ - \Gamma(Y(1) - \mu)_- | Z = 1] \geq 0, \tag{2.8}
\]
\[
\text{s.t.} \quad L(y) \geq 0, L(y) \leq \Gamma L(\tilde{y}) \text{ for all } y, \tilde{y} \in \mathbb{R}
\]
**Proof of Lemma** First, we show that
\[
\inf_{L(y)} \mathbb{E}[Y(1) L(Y(1)) | Z = 1] \tag{A.3}
\]
\[
\text{s.t.} \quad \begin{align*}
\mathbb{E}[L(Y(1)) | Z = 1] &= 1 \\
L(y) &\geq 0, L(y) \leq \Gamma L(\tilde{y}) \text{ for almost every } y, \tilde{y} \in \mathbb{R}
\end{align*}
\]
\[
= \sup_{\mu} \inf_{L(y)} \mathbb{E}[(Y(1) - \mu)_+ - \Gamma(Y(1) - \mu)_- | Z = 1] \geq 0, \tag{A.4}
\]
\[
\text{s.t.} \quad L(y) \geq 0, L(y) \leq \Gamma L(\tilde{y}) \text{ for all } y, \tilde{y} \in \mathbb{R}
\]
which is a consequence of standard functional duality. Since \( \{ L \text{ measurable} : L(y) \leq \Gamma L(\tilde{y}) \text{ for all } y, \tilde{y} \} \) is convex, and \( L \equiv 1 \) satisfies \( \mathbb{E}[L(Y(1)) | Z = 1] = 1 \) and \( L > 0 \), extended Slater’s condition holds. The equality \( (A.4) \) then follows from strong duality (see, e.g., Luenberger [21, Theorem 8.6.1 and Problem 8.7]).

Now, note that for each \( \mu \), the optimal likelihood ratio that attains the inner infimum
\[
\inf_{L(y)} \left[ \mathbb{E}[(Y(1) - \mu)_+ | Z = 1] : L(y) \geq 0, L(y) \leq \Gamma L(\tilde{y}) \text{ for all } y, \tilde{y} \in \mathbb{R} \right]
\]
satisfies
\[
L^*(y) \propto \mathbb{1}_{(y-\mu \leq 0)} + \mathbb{1}_{(y-\mu > 0)} \tag{A.5}
\]
for some choice of \( t \) that depends on \( \mu \). Indeed, due to the constraint \( L(y) \leq \Gamma L(\tilde{y}) \), \( L \) can necessarily take on two values \( c \{ 1, \Gamma \} \) for some \( c \geq 0 \), and it should be monotone in \( y - \mu \). By inspection, \( L^* \) then yields the minimal value possible. Plugging this parameterization of \( L^* \) in the inner problem \( (A.4) \), we obtain
\[
\theta_1 = \sup_{\mu} \inf_{c \geq 0} \mathbb{E}[c(Y(1) - \mu)_+ - c\Gamma(Y(1) - \mu)_- | Z = 1] + \mu.
\]
Finally, applying standard duality to \( c \) (Slater’s condition holds with \( c = 1 \)), obtains the result.
**Theorem 2.1.** Let $\Gamma \geq 1$ be fixed, and let $P$ be a distribution over $Y(1), Y(0), Z, U$ satisfying (1.4). Let $\tau^-$ be as in (2.10) for the optimization problems solved with the same choice of $\Gamma$. When $E[Y(1)]$ and $E[Y(0)]$ are finite, and $P(Z = 1) \in (\epsilon, 1 - \epsilon)$ for some $\epsilon > 0$, 

$$\tau^- [P] \leq E[P | Y(1) - Y(0)].$$

**Proof** By Lemma 2.1,

$$L(y) = \frac{dP_{Y(1)}(|Z = 0)}{dP_{Y(1)}(|Z = 1)}(y)$$

satisfies the constraints of (2.5). This implies that 

$$\theta_1 \leq E \left[ L(Y(1)) | Y(1) = 1 \right] = E[Y(1) | Z = 0].$$

A similar analysis shows that 

$$\theta_0 \geq E[Y(0) | Z = 1].$$

Plugging these inequalities into the definition of $\tau^- (P)$ shows that

$$\tau^- (P) \leq E \left[ Y(1) | Z = 1 \right] P(Z = 1) + \theta_1 P(Z = 0) - E \left[ Y(0) | Z = 0 \right] P(Z = 1) - \theta_0 P(Z = 1)$$

$$\leq E[Y(1)] P(Z = 1) + E[Y(1)] P(Z = 0) - E[Y(0)] P(Z = 1) - E[Y(0)] P(Z = 0)$$

$$\leq E[Y(1) - Y(0)].$$

☐

**Lemma A.1.** Estimating $\theta_1$ is equivalent to solving the estimating equation

$$\Psi_\theta (\theta_1) = P_n \psi_\theta (Z, Y(1)) = 0,$$

where

$$\psi_\theta (z, y) = z \left[ (z - \theta)_+ - \Gamma (z - \theta)_- \right].$$

Furthermore, $\Psi (\theta_1) = E[\psi_\theta (Z, Y(1))] = 0$.

**Proof** The first order optimality conditions of (2.8) imply the claim—because the objective is linear, the constraint on $\mu$ must be tight. Therefore, finding $\mu$ that satisfies the constraint with equality is an equivalent definition of the estimator. ☐

**Proposition 2.1.** Assume $Y(1)$ has finite expectation $\mu_1$. Then $\hat{\theta}_1 \overset{P}{\rightarrow} \theta_1$ as $n \rightarrow \infty$.

**Proof** The proof is relatively straightforward once Lemma A.1 is established. Because $\theta \mapsto \Psi (\theta)$ is the average of monotone functions whose derivatives are bounded between $-1$ and $-\Gamma$, and $E[Y(1)]$ is finite, dominated convergence holds and $\Psi (\theta)$ is a monotone function with derivatives bounded between $-1$ and $-\Gamma$. This implies that $\Psi (\theta_1 + \epsilon) < -\epsilon$ and $\Psi (\theta_1 - \epsilon) > \epsilon$. Similarly, $\Psi_n (\theta)$ is a finite average of strictly monotone decreasing continuous functions. Likewise, its derivatives are bounded between $-1$ and $-\Gamma$ everywhere. Therefore, it has a unique zero. By [38, Lemma 5.10], $\hat{\theta}$ is consistent for $\theta_1$. ☐

**Lemma A.2.** Let $K$ be a compact subset of $\mathbb{R}$. Then, $\{\psi_\theta (Z, Y) : \theta \in K\}$ is a Donsker class.

**Proof** Note that $|\psi_\theta (Z, Y) - \psi_{\theta_2}| \leq \Gamma |	heta_1 - \theta_2|$. Then, the class meets the conditions of Example 19.7 in Van der Vaart [38], and therefore is Donsker. ☐

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Proposition 2.2. Assume that conditional on \( Z = 1 \), \( Y(1) \) has finite expectation \( \mu_1 \) and finite variance \( \sigma_1^2 \). Then, \( \sqrt{n} \left( \hat{\theta} - \theta_1 \right) \xrightarrow{d} V \) as \( n \to \infty \), with

\[
V = \frac{1}{P(Y(1) > \theta_1) + \Gamma P(Y(1) \leq \theta_1)} G_+ - \frac{1}{P(Y(1) \geq \theta_1) + \Gamma P(Y(1) < \theta_1)} G_-,
\]

for \( G \sim N \left( 0, \text{Var} \left( \psi_{\theta_1}(Z,Y(1)) \right) \right) \), where \( \psi_{\theta}(z,y) = z \left[ (y - \theta)_+ - \Gamma(y - \theta)_- \right] \).

Proof First, establish that \( \hat{\theta} \) converges to \( \theta_1 \) in the variance semi-metric. Indeed, by Proposition 2.1, \( \hat{\theta} \xrightarrow{p} \theta_1 \), and so when \( \hat{\theta} < \theta_1 \),

\[
\mathbb{E} \left[ (\psi_{\hat{\theta}}(Y) - \psi_{\theta_1}(Y))^2 \right] = \mathbb{E}_1 \left( (1 - \Gamma)Y - \Gamma \hat{\theta} + \theta_1 \right)^2 ; Y \in [\hat{\theta}, \theta_1] + (\hat{\theta} - \theta_1)^2
\]

\[
\leq [(1 - \Gamma)\theta_1]^2 + [(\Gamma \hat{\theta} - \theta_1)^2 - 2(1 - \Gamma)\Gamma \hat{\theta} (\Gamma \hat{\theta} - \theta_1) + (\hat{\theta} - \theta_1)^2
\]

\[
= o_P(1).
\]

When \( \hat{\theta} > \theta_1 \),

\[
\mathbb{E} \left[ (\psi_{\hat{\theta}}(Y) - \psi_{\theta_1}(Y))^2 \right] = \mathbb{E}_1 \left( (1 - \Gamma)Y - \Gamma \hat{\theta} + \theta_1 \right)^2 ; Y \in [\theta_1, \hat{\theta}] + (\hat{\theta} - \theta_1)^2
\]

\[
\leq [(1 - \Gamma)\theta_1]^2 + [\Gamma \theta_1 - \theta_1)^2 - 2(1 - \Gamma)\theta_1 (\Gamma \theta_1 - \hat{\theta}) + (\theta_1 - \hat{\theta})^2
\]

\[
= o_P(1).
\]

Proposition 2.1 implies that there is some \( K \) such that \( \hat{\theta} \in K \) eventually. From Lemma A.2, \( \{\psi_{\theta}(z,y) : \theta \in K \} \) is a Donsker class, so [38, Lemma 19.24] shows that

\[
\sqrt{n} \left( \Psi_{\theta_1}(\hat{\theta}) - \Psi(\theta_1) + \Psi(\hat{\theta}) \right) \xrightarrow{p} 0.
\]

Because \( \Psi_{\theta}(\hat{\theta}) = 0 \) and \( \Psi(\theta_1) = 0 \), (A.12) can be simplified,

\[
\sqrt{n} \Psi(\hat{\theta}) = -\sqrt{n} \left( \Psi_{\theta_1}(\hat{\theta}) - \Psi(\theta_1) \right) + o_P(1),
\]

where \( \sqrt{n}(\Psi_{\theta_1}(\hat{\theta}) - \Psi(\theta_1)) \rightarrow N(0, \text{Var}(\psi_{\theta_1}(Y))) \) as it is a sum of iid random variables with mean 0. Expanding \( \Psi(\hat{\theta}) \),

\[
\sqrt{n} \Psi(\hat{\theta}) = \sqrt{n} \left( \Psi(\hat{\theta}) - \Psi(\theta_1) \right)
\]

\[
= \sqrt{n} \left( \left( P_1(Y(1) > \theta_1) + \Gamma P_1(Y(1) \leq \theta_1) \right) \frac{\hat{\theta} - \theta_1}{\varepsilon_\theta} \right)
\]

where

\[
|\varepsilon_\theta| \leq \Gamma \hat{\theta} - \theta_1 |P_1(\theta_1 \land \hat{\theta} < Y(1) < \theta_1 \lor \hat{\theta}).
\]

Applying Proposition 2.1, consistency of \( \hat{\theta} \) implies \( \lim_{n\to\infty} P_1(\theta_1 \land \hat{\theta} < Y(1) \leq \theta_1 \lor \hat{\theta}) = P_1(Y(1) = \theta_1) \) and \( |\hat{\theta} - \theta_1| = o_P(1) \), which together imply \( \varepsilon = o_P(1) \).

Considering the other terms, we consider two cases. When \( \hat{\theta} > \theta_1 \), \( P_1(Y(1) < \theta_1 \lor \hat{\theta} \rightarrow P_1(Y(1) > \theta_1) \) and \( P_1(Y(1) < \theta_1 \land \hat{\theta}) \rightarrow P_1(Y(1) < \theta_1) \). However, when \( \hat{\theta} < \theta_1 \), \( P_1(Y(1) < \theta_1 \lor \hat{\theta} \rightarrow P_1(Y(1) \geq \theta_1) \) and \( P_1(Y(1) < \theta_1 \land \hat{\theta}) \rightarrow P_1(Y(1) < \theta_1) \).

Altogether,

\[
\sqrt{n} \Psi(\hat{\theta}) = \sqrt{n} \left( \left( P_1(Y(1) < \theta_1) + \Gamma P_1(Y(1) \leq \theta_1) \right) \frac{\hat{\theta} - \theta_1}{\varepsilon_\theta} \right)
\]

\[
- \left( P_1(Y(1) \geq \theta_1) + \Gamma P_1(Y(1) < \theta_1) \right) \frac{\hat{\theta} - \theta_1}{\varepsilon_\theta} \right)
\]

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Combining (A.13) and (A.17) and applying Slutsky's Theorem, we have
\[
\sqrt{n} (\hat{\theta} - \theta_1) \xrightarrow{d} \frac{1}{P(Y(1) > \theta_1) + \Gamma P(Y(1) \leq \theta_1)} G_+ - \frac{1}{P(Y(1) \geq \theta_1) + \Gamma P(Y(1) < \theta_1)} G_-, \tag{A.18}
\]
where \(G \sim N\left(0, \sigma^2 \left(\psi_0(Z, Y(1))\right)\right)\).

**Proposition 2.3.** Let \(n_1\) and \(n_0\) denote the number of samples such that \(Z = 1\) and \(Z = 0\) respectively, and \(\text{Var}_1 Y(1) \leq \sigma_1^2\), \(\text{Var}_0 Y(0) \leq \sigma_0^2\). Then, with probability at least \(1 - 5\delta\)
\[
\left|\tau^- (P) - \hat{\tau}\right| \leq \frac{\Gamma \sigma_1}{\sqrt{n_1} \delta} + \frac{1}{\theta_1} \sqrt{\frac{\Gamma}{n_1} \log \frac{1}{\delta}} + \frac{\Gamma \sigma_0}{\sqrt{n_0} \delta} + \frac{1}{\theta_0} \sqrt{\frac{1}{n_0} \log \frac{1}{\delta}}. \tag{2.14}
\]

**Proof** Recall \(\hat{\theta}_1\), the Z-estimator of \(\theta_1\) in Lemma A.1, and define \(\hat{\theta}_0\) analogously. Then,
\[
\hat{\tau}(U) := \hat{\rho}_1 \psi_{1,n}[Y(1)] + (1 - \hat{\rho}_1) \hat{\theta}_1 - (1 - \hat{\rho}_1) \psi_{0,n}[Y(0)] - \hat{\rho}_1 \hat{\theta}_0
\]
for \(\hat{\rho}_1 := \hat{\rho}_n(Z = 1)\).

The following lemma bounds variation in \(\hat{\theta}_1 - \hat{\theta}_0\) by variation in the estimating equation \(\psi(\theta_1) - \psi_n(\theta_1)\).

**Lemma A.3.**
\[
|\hat{\theta}_1 - \theta_1| \leq |\psi_n(\theta_1)| = |\psi_n(\theta_1) - \psi(\theta_1)|. \tag{A.19}
\]

**Proof of Lemma** First, note that \(\psi_n\) is convex since \(\theta \mapsto (y - \theta)_+\) and \(\theta \mapsto -\Gamma(y - \theta)_-\) are convex for every \(y \in \mathbb{R}\). By convexity of \(\psi_n\),
\[
\psi_n(\hat{\theta}_1) \geq \psi_n(\theta_1) + \partial \psi_n(\theta_1)(\hat{\theta}_1 - \theta_1)
\]
\[
\psi_n(\theta_1) \geq \psi_n(\hat{\theta}_1) + \partial \psi_n(\hat{\theta}_1)(\theta_1 - \hat{\theta}_1).
\]

Since \(\psi_n\) is non-increasing with \(\partial \psi_n \in [-1, -\Gamma]\), the preceding display implies that
\[
|\hat{\theta}_1 - \theta_1| = \begin{cases} 
\hat{\theta}_1 - \theta_1 & \text{if } \hat{\theta}_1 \geq \theta_1 \\
\theta_1 - \hat{\theta}_1 & \text{if } \hat{\theta}_1 < \theta_1 
\end{cases} \leq \begin{cases} 
\psi_n(\theta_1) & \text{if } \hat{\theta}_1 \geq \theta_1 \\
-\psi_n(\theta_1) & \text{if } \hat{\theta}_1 < \theta_1 
\end{cases}
\]
\[
= |\psi_n(\theta_1)| = |\psi_n(\theta_1) - \psi(\theta_1)|, \tag{A.22}
\]
using \(\psi(\theta_1) = 0\) in the last equality.

Apply any concentration bound to \(|\psi_n(\theta^*) - \psi(\theta^*)|\), to obtain a finite sample concentration result for \(|\hat{\theta}_1 - \theta_1|\) (e.g. Chebyshev, sub-Gaussian or sub-exponential). The exact statement of the proposition follows from applying Chebyshev's inequality to control terms containing \(Y(1)\) and a Chernoff bound to control terms containing \(Z\).

**B Proofs for sieve estimation**

**B.1 Proof of Lemma B.1**

**Lemma B.1.** If \(E_i [f(\theta_1; X, Y(1))] < \infty\), then the unique solution \(\theta_1\) to the estimating equation (3.7) is also a unique argmin of the minimization problem (3.6).
Proof

We use the following lemma based on normal integrand theory [27, Section 14.D] which allows swapping integrals and infimum over measurable mappings. Recall that a map \( f: \mathbb{R} \times \mathcal{X} \to \mathbb{R} \) is a normal integrand if its epigraphical mapping—viewed as a set-valued mapping—\( S_f: \mathcal{X} \to \mathbb{R} \times \mathbb{R}, x \mapsto \text{epi} \{ f(\cdot; x) \} = \{(t, \alpha) \in \mathbb{R} \times \mathbb{R} : f(t; x) \leq \alpha \} \) is closed-valued (i.e. \( S_f \) is closed for all \( x \in \mathcal{X} \)) and measurable (i.e. for any open set \( O \subseteq \mathbb{R}^2, S_f^{-1}(O) := \bigcup_{o \in O} S_f^{-1}(o) \subseteq \mathcal{A} \)).

Lemma 1 (Rockafellar and Wets [27, Theorem 14.60]). If \( f: \mathbb{R} \times \mathcal{X} \to \mathbb{R} \) is a normal integrand, and \( \int_X f(\theta(x); x) \, dP(x) < \infty \) for some measurable \( \theta_1 \), then

\[
\inf_{\theta} \left\{ \int_{\mathcal{X}} f(\theta(x); x) \, dP(x) \mid \theta : \mathcal{X} \to \mathbb{R} \text{ measurable} \right\} = \int_{\mathcal{X}} \inf_{\theta} f(t; x) \, dP(x).
\]

If this common value is not \(-\infty\), \( \theta^* : \mathcal{X} \to \mathbb{R} \) attains the minimum of the left-hand side iff \( \theta^*(x) \in \arg\min_{\theta \in \mathcal{A}} f(t; x) \) for \( P\)-almost every \( x \in \mathcal{X} \).

Let \( f(t, x) := \frac{1}{2} \mathbb{E}_{1}[y - t]^2 + \Gamma [y - t]^2 | X = x \). Since \( f \) is jointly continuous in \((y, t)\), \( f \) is a normal integrand [27, Examples 14.31]. Rewrite the minimization problem (3.6) using the tower property

\[
\inf_{\theta} \left\{ \mathbb{E}_1[\mathbb{E}_1[f(\theta(X); X)] | X] \right\} = \mathbb{E}_1[f(\theta(X); X)] \mid \theta : \mathcal{X} \to \mathbb{R} \text{ measurable} \right\}.
\]

Applying Lemma 1, \( \theta^*(x) := \arg\min_{\theta \in \mathcal{A}} f(t; x) \) is the solution to the preceding display.

Since \( t \mapsto f(t; x) \) is convex, the first order condition \( \frac{1}{2} \mathbb{E}_1[f(t; x)] \) shows that \( \theta^*(x) = \theta_1(x) \). By strict monotonicity of the derivative, we conclude that \( \theta_1 \) is an unique optimum to the optimization problem (3.6).

\( \square \)

B.2 Proof of Theorem 3.1

Theorem 3.1. Let Assumptions 1, 2, 3 hold, and let \( \hat{\theta}_1 \) be an approximate empirical minimizer to the problem (3.8) satisfying

\[
\mathbb{E}_n[\ell(\hat{\theta}_1; (X, Y(1))) | Z = 1] \leq \inf_{\theta \in \Theta_n} \mathbb{E}_n[\ell(\theta; (X, Y(1))) | Z = 1] + O_p \left( \epsilon_n^2 \right)
\]

where \( \epsilon_n := \max \left\{ \delta_n, \inf_{\theta \in \Theta_n} \| \theta - 1 \|_{2, p_1} \right\} \). Then, \( \| \hat{\theta}_1 - \theta_1 \|_{2, p_1} = O_p (\epsilon_n) \).

Proof

Throughout this proof, let \( \mathbb{E}_1[\cdot] = \mathbb{E}[\cdot | Z = 1] \) and \( \mathbb{E}_{1,n}[\cdot] = \mathbb{E}_n[\cdot | Z = 1] \).

We use a general result for sieve estimation due to Chen and Shen [7] (see, also [6, 17]). For any two functions \( f_1(\theta) \) and \( f_2(\theta) \), we say that \( f_1 \approx f_2 \) if there exists universal constants \( C, C' \) such that \( C f_1(\theta) \leq f_2(\theta) \leq C' f_1(\theta) \) for all \( \theta \).

Lemma 2 (Chen [6, Theorem 3.2]). Let \( \theta_1 \in \mathcal{X}_c^p(X) \) for some \( p, c > 0 \), and for \( \theta \) in some neighborhood of \( \theta_1 \),

\[
\mathbb{E}_1[\ell(\theta; (X, Y(1)))] - \mathbb{E}_1[\ell(\theta_1; (X, Y(1)))] \approx \| \theta - \theta_1 \|_{2, p_1}^2.
\]

For \( \delta \) small enough, let

\[
\sup_{\theta \in \Theta_n, \| \theta - \theta_1 \|_{2, p_1} \leq \delta} \text{Var}_P \left( \ell(\theta; (X, Y(1))) - \ell(\theta_1; (X, Y(1))) \right) \leq \delta^2 \quad (B.1)
\]

\[
\sup_{\theta \in \Theta_n, \| \theta - \theta_1 \|_{2, p_1} \leq \delta} \left| \ell(\theta; (X, Y(1))) - \ell(\theta_1; (X, Y(1))) \right| \leq \delta^* U(X, Y(1)) \quad (B.2)
\]

for some \( s \in (0, 2) \) and \( \mathbb{E}_1[U(X, Y(1))^2] < \infty \). Then, we have \( \| \hat{\theta}_1 - \theta_1 \|_{2, p_1} = O_p (\epsilon_n) \).
We show our desired result by verifying hypothesis of the above lemma. First, we check that
\[E_1[\ell(\theta; (X, Y(1)))] - E_1[\ell(\theta_1; (X, Y(1)))] \approx \|\theta - \theta_1\|_{2, P_1}^2.\]
For convenience, let \(g(t; y) := \frac{1}{2} \left(\|y - t\|_\theta^2 + \Gamma \|y - t\|_\theta^2\right).\)
Rewriting \(g(t; y)\) as
\[g(t; y) = \frac{1}{2}(y - t)^2 + \frac{1}{2}(\Gamma - 1)[y - t]_1^2,\]
t \mapsto g(t; y) is 1-strongly convex since \(t \mapsto \frac{1}{2}(y - t)^2\) is 1-strongly convex, and \(t \mapsto \frac{1}{2}(\Gamma - 1)[y - t]_1^2\) is convex. Hence, we have for any \(t, t' \in \mathbb{R}\)
\[g(t; y) - g(t'; y) \geq g_1(t'; y)(t - t') + \frac{1}{2}|t - t'|^2\]
where we denoted by \(g_1(t; y) := [y - t]_1 + \Gamma [y - t]_\theta\) the partial derivative of \(g\) with respect to \(t\). Noting that
\[E_1[g_1(\theta_1(X); Y(1))|X] = 0\] almost surely, let \(t = \theta(X)\) and \(t' = \theta(X)\), and take expectations to obtain
\[E_1[\ell(\theta; (X, Y(1)))] - E_1[\ell(\theta_1; (X, Y(1)))] \geq \frac{1}{2}\|\theta - \theta_1\|_{2, P_1}^2.\]
To show the other direction, note that \(t \mapsto g_1(\theta; y)\) is \(\Gamma\)-Lipschitz. From strong smoothness, we have for any \(t, t' \in \mathbb{R}\)
\[g(t; y) - g(t'; y) \leq g_1(t'; y)(t - t') + \Gamma |t - t'|^2.\]
Again, letting \(t = \theta(X)\) and \(t' = \theta(X)\), and taking expectations yield
\[E_1[\ell(\theta; (X, Y(1)))] - E_1[\ell(\theta_1; (X, Y(1)))] \leq \frac{1}{2}\|\theta - \theta_1\|_{2, P_1}^2.\]
We now show the bounds (B.1), (B.2). Since \(\theta \mapsto g_1(\theta; y)\) is \(\Gamma\)-Lipschitz, for all \(t, t'\)
\[|g(t; y) - g(t'; y)| \leq \left(g_1(t'; y) + \Gamma |t - t'|\right)|t - t'|.\]
Letting \(t = \theta(x)\) and \(t' = \theta_1(x)\) again, we have
\[|\ell(\theta; (x, y)) - \ell(\theta_1; (x, y))| \leq \Gamma |\theta - \theta_1(x)||\theta(x) - \theta_1(x)| + \Gamma |\theta(x) - \theta_1(x)|^2.\]  \hfill (B.3)
Next, we use the following lemma [7, Lemma 2]. For \(\theta \in \Lambda^p(X)\), we have \(\|\theta\|_\infty \leq 2^{1+\frac{2p}{\delta+1}}\|\theta\|_{2, \lambda}^\alpha\).
Noting that \(\|\cdot\|_{2, \lambda} \approx \|\cdot\|_{2, P_1}\) by Assumption 3, we conclude \(\|\theta\|_\infty \lesssim \|\theta\|_{2, P_1}^\alpha\).
Taking squares on both sides in the inequality (B.3) and using convexity of \(t \mapsto t^2\), we get
\[|\ell(\theta; (x, y)) - \ell(\theta_1; (x, y))|^2 \leq 2\Gamma^2|\theta - \theta_1(x)|^2|\theta(x) - \theta_1(x)|^2 + 2\Gamma^2|\theta(x) - \theta_1(x)|^4.\]
Take expectations and recalling that \(\mathbb{E}_1[(Y(1) - \theta_1(X))^2|X] \leq M\) for some \(M > 0\), Lemma 3 yields
\[\sup_{\theta \in \Theta_n, \|\theta - \theta_1\|_{2, P_1} \leq \delta} \var_{P_1}(\ell(\theta; (X, Y(1)))/\ell(\theta_1; (X, Y(1)))) \lesssim \Gamma^2 M \delta^2 + \Gamma^2 \delta^2 + \frac{2p}{2p + d} \lesssim \delta^{2p}\]
whenever \(\delta \in (0, 1)\). This verifies the condition (B.1). Similarly, we have
\[\sup_{\theta \in \Theta_n, \|\theta - \theta_1\|_{2, P_1} \leq \delta} \left|\ell(\theta; (X, Y(1)))/\ell(\theta_1; (X, Y(1))) - \ell(\theta; (X, Y(1)))/\ell(\theta_1; (X, Y(1)))\right| \lesssim \Gamma \delta^{\frac{2p}{2p+d}} (M + \delta^{\frac{2p}{2p+d}}) \lesssim \delta^{\frac{2p}{2p+d}}\]
which verifies the bound (B.2) with \(s = 2p/(2p + d)\).
B.3 Proof of Corollary 3.1

Corollary 3.1. For $\mathcal{X} = [0, 1]^d$, let $\Theta_n$ be given by finite dimensional linear sieves considered in Examples 1 and 2 (assume $\theta_1$ can be extended periodically for trigonometric polynomial bases) with $J_n \approx n^{\frac{1}{p} + \varepsilon}$. Let Assumptions 1, 2, 3 hold, and let $\theta_1$ be an approximate empirical minimizer to the problem (3.8)

$$
\mathbb{E}_n[\ell(\hat{\theta}_1; (X, Y(1))) | Z = 1] \leq \inf_{\theta} \mathbb{E}_n[\ell(\theta; (X, Y(1))) | Z = 1] + O_p \left( n^{\frac{2p}{d + 2}} \right).
$$

Then, $\|\hat{\theta}_1 - \theta_1\|_{p,1} = O_p \left( n^{\frac{1}{p} + \varepsilon} \right)$.

Proof

It suffices to bound $\delta_n$ and the approximation error $\inf_{\theta \in \Theta_n} \|\theta_1 - \theta\|_{p,1}$. Again, define the following shorthand notation,

$$
E_1[\cdot] = \mathbb{E}[\cdot | Z = 1]
$$

$$
E_{1,n}[\cdot] = \mathbb{E}_n[\cdot | Z = 1].
$$

First, we note from Chen and Shen [7], van de Geer [37] that

$$
\log N \left( c, \Theta_n, \|\cdot\|_{2, p} \right) \lesssim \dim(\Theta_n) \log \frac{1}{\epsilon},
$$

where $\dim(\Theta_n) = J_n^d$. Then, we have

$$
\frac{1}{\sqrt{n\delta^2}} \int_{\delta^2}^{\frac{1}{\delta^2}} \frac{\log N \left( c^{1 + d/2p}, \Theta_n, \|\cdot\|_{2, p} \right) d\epsilon}{\sqrt{\dim(\Theta_n)}} \lesssim \frac{\sqrt{\dim(\Theta_n)}}{\delta\sqrt{n}}
$$

so that $\delta_n \approx \sqrt{\frac{\dim(\Theta_n)}{n}} = \sqrt{\frac{J_n}{n}}$.

When $\Theta_n$ is defined as in Example 1 with $J = J_n$, standard function approximation results yield $\inf_{\theta \in \Theta_n} \|\theta - \theta_1\|_{\infty} = O(J_n^{-p})$. See, for example, Timan [36, Section 5.3.1].

When $\Theta_n$ is defined as in Example 2 with $J = J_n$, it is well-known (see, for example, Schumaker [34, Theorem 12.8]) that $\inf_{\theta \in \Theta_n} \|\theta - \theta_1\|_{\infty} = O(J_n^{-p})$. Similar approximation guarantees hold for wavelet bases (see [10]), which we omit for brevity. We refer the reader to Chen [6] and references therein for a more comprehensive overview of finite-dimensional linear sieves.

Therefore, for any of these choices of approximating functions,

$$
\inf_{\theta \in \Theta_n} \|\theta_1 - \theta\|_{2, p} = O(J_n^{-p}).
$$

Setting $J_n \approx n^{\frac{1}{p} + \varepsilon}$, we obtain the result from Theorem 3.1.

C Proofs for semiparametric results

Theorem 4.1. Let $\mathcal{P}$, $\eta_i(x) = (\theta_1(x), \nu_1(x), e_1(x))^\top$, and $\tilde{\eta}(x)$ satisfy Assumptions 4 and 5 for some choice of $q > 2$ that satisfies both assumptions. Then, for any $P \in \mathcal{P}$, $\mu_1$ estimated as in (4.9) is asymptotically normal with

$$
\sqrt{n}(\hat{\mu}_1 - \mu_1) \overset{d}{\rightarrow} \mathcal{N}(0, \sigma^2),
$$

where

$$
\sigma^2 = \text{Var} \left( Z_i Y_i + (1 - Z_i) \theta(X_i) + Z_i \frac{\left( Y_i - \hat{\theta}(X_i) \right)_+ - \Gamma (Y_i - \hat{\theta}(X_i)) + 1 - e(X_i) \right) \nu(X_i), \frac{\left( Y_i - \hat{\theta}(X_i) \right)_+ - \Gamma (Y_i - \hat{\theta}(X_i)) + 1 - e(X_i) \right) e(X_i) \right),
$$

Let $z_{1-\alpha} = \mathbb{P}(\|N(0, 1)\| \leq 1 - \alpha)$. If we estimate

$$
\tilde{\sigma}^2 = \text{Var}_{P_n} \left( Z_i Y_i + (1 - Z_i) \theta_{1,j}(X_i) + Z_i \frac{\left( Y_i - \theta_{1,j}(X_i) \right)_+ - \Gamma (Y_i - \theta_{1,j}(X_i)) + 1 - e(X_i) \right) \nu_{1,j}(X_i), \frac{\left( Y_i - \theta_{1,j}(X_i) \right)_+ - \Gamma (Y_i - \hat{\theta}_{1,j}(X_i)) + 1 - e(X_i) \right) e_j(X_i) \right),
$$

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where $j$ has is implicitly a function $j = j(i)$ that is the sample such that $i \in \mathcal{I}_j$, then the confidence interval

$$\hat{CI} = [\hat{\mu}_1 \pm z_{1-\alpha} \hat{\sigma}/\sqrt{n}]$$

satisfies

$$P(\mu_1^2 \in \hat{CI}) \to 1 - \alpha,$$

as $n \to \infty$.

**Proof** Let $Y = Y(Z)$ be the observed potential outcome. Define $W_i = (Y_i, X_i, Z_i)'$ as the $d + 2$ dimensional random vector containing all the observed random variables. Similarly $w = (y, x, z)'$ for a fixed or temporary variable. Define the score

$$m(w, \mu, \theta, \nu) = -\mu + zy + (1 - z)\theta(x) + z\left(\frac{(y - \theta(x)) + \nu}{\nu(x)} - 1 - e(x)\right). \quad (C.1)$$

The proof depends heavily on Theorem 3.1, 3.2 and Corollary 3.1 of Chernozhukov et al. [Chernozhukov et al. Assumption 9], and depends primarily on checking that their Assumptions 3.1 and 3.2 hold for the proposed estimator. To simplify proof verification, the assumptions are repeated here, with slight notational adaptation to match the other results presented here and avoid notational conflict. The score $m$ satisfies the Neyman orthogonality condition with $\lambda_n = 0$, so the most general conditions (for example in Assumption 3.1(d)) are omitted. Finally, the following constraints on constants and variables appearing in these assumptions are previously established: $c_0 > 0, c_1 > 0, q > 2, c_0 \leq c_1$, and $\delta_n$ and $\Delta_n$ are sequences with $\delta_n \geq n^{-1/2}$ and $\lim_{n \to \infty} \delta_n = \lim_{n \to \infty} \Delta_n = 0$. Let $\mathcal{T} = \{\eta(\cdot) : \eta \text{ measurable, } 1 \leq \nu(x) \leq \Gamma, \epsilon \leq e(x) \leq 1 - \epsilon\}$, so that $\eta_1 \in \mathcal{T}$, and $\hat{\eta} \in \mathcal{T}$ almost surely.

**Assumption 8** (Chernozhukov et al. [Chernozhukov et al. Assumption 3.1]). For all $n \geq 3$ and $P \in \mathcal{P}_n$, the following conditions hold.

(a) The true parameter $\mu_1^- \in \mathcal{T}$ obeys

$$E_{P} \left[ m(W, \mu_1^-, \eta) \right] = 0.$$

(b) The score $m$ is linear in $\mu_1^-$, so that

$$m(w, \mu, \eta) = m^u(w; \eta)\mu + m^v(w; \eta),$$

for all $w \in W, \mu \in \Theta, \eta \in T$.

(c) The map $\eta \mapsto E[m(W, \mu, \eta)]$ is twice continuously Gateaux-differentiable on $T$.

(d) The score $m$ obeys the Neyman orthogonality condition (definition 2.1 of Chernozhukov et al. [Chernozhukov et al. Assumption 3.2]),

$$\partial_rE \left[ m(W, \mu_1^-, \eta_1 + r(\eta - \eta_1)) \right]$$

exists for all $\eta \in \mathcal{T}$ and $r \in (0, 1)$, and

$$\partial_rE \left[ m(W, \mu_1^-, \eta_1 + r(\eta - \eta_1)) \right]_{r=0} = 0.$$

(e) The singular values of the matrix

$$J_0 = E[m^u(W; \eta_1)]$$

are bounded between $c_0$ and $c_1$.

**Assumption 9** (Chernozhukov et al. [Chernozhukov et al. Assumption 3.2]). For all $n \geq 3$ and $P \in \mathcal{P}_n$, the following conditions hold.

(a) Given a random subset $I$ of $[n]$, of size $n/K$, the nuisance parameter $\bar{\eta}((W_i)_{i \in I})$ belongs to a set $\mathcal{T}_n$ with probability at least $1 - \Delta_n$, where $\mathcal{T}_n$ contains $\eta_1$ and satisfies the next conditions.

(b) The moment conditions hold:

$$m_n = \sup_{\eta \in \mathcal{T}_n} \left( E[\|m(W, \mu_1^-, \eta)\|^q] \right)^{1/q} \leq c_1$$

and

$$m'_n = \sup_{\eta \in \mathcal{T}_n} \left( E[\|m^u(W, \mu_1^-, \eta)\|^q] \right)^{1/q} \leq c_1.$$
(c) The following conditions on the statistical rates $r_n, \, r'_n, \text{ and } \lambda'_n$ hold:

$$r_n = \sup_{n \in \mathbb{N}} E[|m_n(W, \mu_1, \eta) - m_n(W, \mu_1, \eta_1)|] \leq \delta_n,$$

$$r'_{n} = \sup_{n \in \mathbb{N}} \left( E[|m(W, \mu_1, \eta) - m(W, \mu_1, \eta_1)|^2] \right)^{1/2} \leq \delta_n,$$

$$\lambda'_n = \sup_{n \in \mathbb{N}} E[\partial^2 \|m(W, \mu_1, \eta_1 + r(\eta - \eta_1))\|] \leq \delta_n/\sqrt{n}.$$

(d) All eigenvalues of

$$E \left[ m(W, \mu_1, \eta_1) m(W, \mu_1, \eta_1)^T \right]$$

are bounded from below by $c_0$.

Step 1 checks that Assumptions 8(a-e) are met, and step 2 checks that Assumptions 9(a-d) are met.

Step 1

The score satisfies

$$E[m(W, \mu_1, \theta_1, \nu_1, e_1)] = -\mu_1 + E[ZY + (1 - Z)\theta_1(X)]$$

$$+ E \left[ \frac{1}{r_1(X)} E[Z \left( Y - \theta_1(X) \right)_+ - \Gamma(Y - \theta_1(X))_- \right] |X| \left( \frac{1 - e_1(X)}{e_1(X)} \right) = 0$$

(C.2)

because $E[Z \left( Y - \theta_1(X) \right)_+ - \Gamma(Y - \theta_1(X))_- |X] = 0$ almost everywhere, $\nu_1(x) \geq 1$, and $e(x) > \epsilon$. This satisfies Assumption 8(a). The score is linear in $\mu_1$ with non-random slope $-1$. Hence, $m = -1$, satisfying 8(b).

The main difficulty when checking Assumption 8(c) is verifying the twice Gateaux differentiability with respect to $\theta(x)$. To simplify notation for the algebra, it is useful to note that $\nu_1(x)$ and $e(x)$ are bounded below and above. So therefore, it suffices to check that for $g(x)$ an arbitrary measurable function bounded $g(x) \in [-\frac{1}{2}, \frac{1}{2}]$ almost everywhere,

$$\theta(x) \rightarrow E[g(X)E[Z \left( Y - \theta(X) \right) - \Gamma(Y - \theta(X))_- |X]$$

must be Gateaux differentiable. When $P(Y|X = x, Z = 1)$ has a density for almost every $x$, this is satisfied by observing that for any $x$,

$$\frac{d}{dt} E[Z \left( (Y - \theta(X) - \Gamma(Y - \theta(X))_- |X = x, Z = 1 = -P(Y > t|X = x, Z = 1) - \Gamma P(Y < t|X = x, Z = 1),$$

as long as $P(Y = t|X = x, Z = 1) = 0$. Then,

$$\frac{d^2}{dt^2} E[Z \left( (Y - \theta(X) - \Gamma(Y - \theta(X))_- |X = x, Z = 1 = p_Y(t|X = x, Z = 1) - \Gamma p_Y(t|X = x, Z = 1)$$

$$= (1 - \Gamma)p_Y(t|X = x, Z = 1).$$

Differentiability for each $x$ and the boundedness of $g$ imply Gateaux differentiability.

For $\theta(x)$,

$$\frac{d}{dt} E[m(W, \mu_1, \theta_1 + r(\theta - \theta_1), \nu_1, e_1)] |_{r=0} \quad (C.4)$$

$$= \frac{d}{dr} E[\left( 1 - Z \right) (\theta_1(X) + r(\theta(X) - \theta_1(X))] |_{r=0} \quad (C.5)$$

$$+ \frac{d}{dr} E[Z \left( Y - \theta_1(X) - r(\theta(X) - \theta_1(X)) \right)_+ - \Gamma(Y - \theta_1(X) - r(\theta(X) - \theta_1(X))_- \right] |X| \left( \frac{1 - e_1(X)}{e_1(X)} \right)$$

$$= \frac{d}{dr} E[(1 - e_1(X))(\theta_1(X) + r(\theta(X) - \theta_1(X))] |_{r=0} \quad (C.6)$$

$$+ \left[ \frac{d}{dr} E[Z \left( (Y - \theta_1(X) - r(\theta(X) - \theta_1(X)) \right)_+ - \Gamma(Y - \theta_1(X) - r(\theta(X) - \theta_1(X))_- |X] \left( \frac{1 - e_1(X)}{e_1(X)} \right) \right] |_{r=0} \quad (C.7)$$

$$+ \left[ \frac{d}{dr} E[Z \left( (Y - \theta_1(X) - r(\theta(X) - \theta_1(X)) \right)_+ - \Gamma(Y - \theta_1(X) - r(\theta(X) - \theta_1(X))_- |X] \left( \frac{1 - e_1(X)}{e_1(X)} \right) \right] |_{r=0} \quad (C.8)$$

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\[
\begin{align*}
E[(1 - e_1(X))(\theta(X) - \theta_1(X))] &= E\left[\left(\frac{e_1(X)\nu_1(X) (\theta(X) - \theta_1(X))}{\nu_1(X)}\right) (1 - e_1(X))\right] \\
E[(1 - e_1(X))(\theta(X) - \theta_1(X))] - E[(\theta(X) - \theta_1(X)) (1 - e_1(X))] &= 0.
\end{align*}
\]

For \(\nu(x),\)
\[
\partial \nu E[m(W, \mu_1^2, \theta_1, \nu_1 + r(\nu - \nu_1), e_1)]\bigg|_{r = 0} = \frac{d}{dr} E[Z (Y - \theta_1(X))^4 - \Gamma (Y - \theta_1(X))^4 - \frac{1}{e_1(X)} - \frac{\nu_1(X)}{e_1(X)}] = 0,
\]

because \(E[Z ((Y - \theta(X))^4 - \Gamma(Y - \theta(X))^4)] = 0\) almost everywhere and \(\nu(X)\) is \(X\)-measurable.

Similarly, for \(e(X),\)
\[
\partial e E[m(W, \mu_1^2, \theta_1, \nu_1 + r(\nu - \nu_1), e_1)] = \frac{d}{dr} E[Z (Y - \theta_1(X))^4 - \Gamma (Y - \theta_1(X))^4 - \frac{1}{e_1(X)} - \frac{\nu_1(X)}{e_1(X)}] = 0,
\]

because \(E[Z ((Y - \theta_1(X))^4 - \Gamma(Y - \theta_1(X))^4)] = 0\) almost everywhere and \(e(X), e_1(X)\) are \(X\)-measurable.

This verifies Assumption 8(d) with \(\lambda_n = 0\). That is, the score is Neyman orthogonal. Finally, \(J_0 = -1\) as \(m^* = -1\) is non-random, which satisfies Assumption 8(e), as long as \(c_0 \leq 1 \leq c_1.\)

**Step 2** Estimators \(\hat{\theta}, \hat{\nu}\) and \(\hat{e}\) are assumed to satisfy, \(\|\hat{\theta}(\cdot) - \theta(\cdot)\|_2, P = \text{op}(n^{-1/4}), \|\hat{\nu}(\cdot) - \nu_1(\cdot)\|_2, P = \text{op}(n^{-1/4}), \) and \(\|\hat{e}(\cdot) - e_1(\cdot)\|_2, P = \text{op}(n^{-1/4}).\) Therefore, there exists sequences \(a_n \to 0\) and \(\Delta_n \to 0\) such that
\[
\begin{align*}
\|\hat{\theta}(\cdot) - \theta(\cdot)\|_2, P &\leq a_n n^{-1/4}, \\
\|\hat{\nu}(\cdot) - \nu_1(\cdot)\|_2, P &\leq a_n n^{-1/4}, \text{ and} \\
\|\hat{e}(\cdot) - e_1(\cdot)\|_2, P &\leq a_n n^{-1/4}
\end{align*}
\]
with probability \(1 - \Delta_n/2.\) Note that \(a_n\) can be chosen so that these are satisfied with \(\hat{\eta}\) only fit using \((1 - \frac{1}{n}) n\) examples. Similarly, Assumption 5 implies there exists \(C_1\) such that
\[
\begin{align*}
\|\hat{\theta}(\cdot) - \theta(\cdot)\|_2, P &\leq C_1, \\
\|\hat{\nu}(\cdot) - \nu_1(\cdot)\|_2, P &\leq C_1, \text{ and} \\
\|\hat{e}(\cdot) - e_1(\cdot)\|_2, P &\leq C_1
\end{align*}
\]
with probability \(1 - \Delta_n/2.\) Let \(\mathcal{T}_n = \{\eta: \|\hat{\theta}(\cdot) - \theta(\cdot)\|_2, P \leq a_n n^{-1/4}, \|\nu(x) - \nu_1(x)\|_2, P \leq a_n n^{-1/4}, \|e(\cdot) - e_1(\cdot)\|_2, P \leq a_n n^{-1/4}, \|\nu_1(\cdot) - \nu_1(\cdot)\|_2, P \leq C_1, \|e(\cdot) - e_1(\cdot)\|_2, P \leq C_1\}.\) Union bounding the probability of events (C.16) and (C.17) failing shows that \(\hat{\eta} \in \mathcal{T}_n\) with probability at least \(1 - \Delta_n.\) This verifies Assumption 9(a).

To check Assumption 9(b), bound the score with the true nuisance parameter \(\eta_1\) plugged in, and apply the triangle inequality to bound the difference.

Using the construction of \(\theta_1(x)\) and its relationship to (2.6) for any \(x,\) note that for any \(\delta > 0,\) there exists \(L_\delta^x(y)\) such that
\[
\theta_1(x) = \mathbb{E}_x [L_\delta^x(Y(1)|Y(1)|X = x) + \delta,
\]
with \(\mathbb{E}[L_\delta^x(Y(1))|Y(1)|] = 1, L_\delta^x \geq 0, \) and \(L_\delta^x(y) \leq \Gamma L_\delta^x(y)\) for almost every \(y, \tilde{y}.\) Together, these imply \(\frac{\delta}{\Gamma} \leq L_\delta^x(y) \leq \Gamma.\)

Therefore, Assumption 4(b) and Holder’s inequality imply
\[
\mathbb{E} \left[\|\theta_1(X)\|_2\right] \leq \mathbb{E} \left[\left(\mathbb{E}_x \left[|L_\delta^{x}(Y(1))|Y(1)|X\right]\right)^{\frac{1}{2}} + \delta^{\frac{1}{2}}\right]
\]
\[
\leq \mathbb{E} \left[\left(\mathbb{E}_x \left[|L_\delta^{x}(Y(1))|Y(1)|X\right]\right)^{\frac{1}{2}} + \delta^{\frac{1}{2}}\right]
\]

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\[
\delta \text{ was arbitrary, so taking } \delta \to 0 \text{ gives } \mathbb{E}[|\theta_1(X)|^q] \leq C_q' = \Gamma C_q.
\]

Let \( a(y,t) = (y-t)_+ - \Gamma (y-t)_- \).

\[
\mathbb{E}\left[|m(W, \mu_1^- , \eta_1)|^q\right] \leq \mathbb{E}\left[|\mu_1^-|^q + |ZY|^q + (1-Z)|\theta_1(X)|^q + Z\frac{\mathbb{E}[a(Y, \theta_1(X))(1-e(X))]}{\nu(X)e(X)}\right]
\]
\[
\leq |\theta_1|^q + (1-\epsilon)\mathbb{E}[|Y|^q | Z = 1] + (1-\epsilon)\mathbb{E}\left[|\theta_1(X)|^q + (1-\epsilon)\mathbb{E}\left[|\theta_1(X)|^q + (1-\epsilon)2^{q-1} \Gamma \left( \mathbb{E}[|Y(1)|^q] + \mathbb{E}[|\theta(X)|^q] \right) \right] \right]
\]
\[
\leq C_q' + (1-\epsilon)C_q + (1-\epsilon)C_q' + (1-\epsilon)2^{q-1} \Gamma \left( C_q + C_q' \right) = \tilde{C}.
\]

Then, bound the difference,

\[
\mathbb{E}\left[|m(W, \mu_1^- , \eta_1) - m(W, \mu_1^-, \eta_1)|^q\right] \leq \mathbb{E}\left[(1-Z)|\theta_1(X) - \theta(X)|^q + Z\frac{\mathbb{E}[a(Y, \theta_1(X))(1-e(X))]}{\nu(X)e(X)}\right]
\]
\[
\leq (1-\epsilon)C_q' + \mathbb{E}\left[Z\frac{\mathbb{E}[a(Y, \theta_1(X))(1-e(X))\nu(X)e(X)]}{\nu(X)e(X)} - \frac{\mathbb{E}[a(Y, \theta(X))(1-e(X))\nu(X)e(X)]}{\nu(X)e(X)}\right]
\]
\[
\leq (1-\epsilon)C_q' + \frac{1}{\epsilon} \mathbb{E}\left[\frac{\mathbb{E}[a(Y, \theta_1(X))(1-e(X))\nu(X)e(X)]}{\nu(X)e(X)} - \frac{\mathbb{E}[a(Y, \theta(X))(1-e(X))\nu(X)e(X)]}{\nu(X)e(X)}\right]
\]
\[
\leq (1-\epsilon)C_q' + \frac{2^{q-1}}{\epsilon} \mathbb{E}\left[\frac{\mathbb{E}[a(Y, \theta_1(X))(1-e(X)) - a(Y, \theta(X))(1-e(X))\nu(X)e(X)]}{\nu(X)e(X)}\right]
\]
\[
\leq (1-\epsilon)C_q' + \frac{2^{q-1}}{\epsilon} \mathbb{E}\left[\frac{\mathbb{E}[a(Y, \theta_1(X))(1-e(X))\nu(X)e(X)]}{\nu(X)e(X)}\right]
\]

uniformly over \( \mathcal{T}_n \) by using that all the nuisance parameters in \( \mathcal{T}_n \) have a bounded distance from the true nuisance parameter in \( L_q, p \)-norm. Finally, combine to verify Assumption 9(b) by

\[
\mathbb{E}\left[|m(W, \mu_1^- , \eta_1)|^q\right] \leq \tilde{C} + \tilde{C} \leq c_1.
\]

If \( \tilde{C} + \tilde{C} < 1 \), choose \( c_1 = 1 \) so that Assumption 8 remains satisfied.

For Assumption 9(c), \( r_n = 0 \) follows easily, as \( m^* = -1 \). Use the construction of \( \mathcal{T}_n \) to show

\[
\sup_{\eta \in \mathcal{T}_n} \mathbb{E}\left[(m(W, \mu_1^-, \eta) - m(W, \mu_1^-, \eta_1))^2\right]^{1/2} \leq \sup_{\eta \in \mathcal{T}_n} \|(\cdot) - \theta_1(\cdot)\|_{2,p}^2 + \frac{\Gamma(1-\epsilon)}{\epsilon} \mathbb{E}\left[2\|e(\cdot) - e_1(\cdot)\|_{2,p} + \|\nu(\cdot) - \nu_1(\cdot)\|_{2,p}^2\right]
\]
remains satisfied.

Bounding $\lambda_n$ is more involved, and is done in Lemma 5. This gives the constant $C_3 > 0$ so that

$$\lambda_n \leq C_3 a_n / \sqrt{n}.$$ 

Now, let $\delta_n = \max\{C_2 a_n, C_3 a_n^2, n^{-1/2}\}$, where max refers to the pointwise maximum for each $n$. Note that each of the terms $a_n, a_n^2,$ and $n^{-1/2} \to 0$, so that this satisfies the conditions on $\delta_n$ in Chernozhukov et al. [8].

To show that Assumption 9(d) is satisfied, note that

$$\text{Var} \left( m(W, \mu_1, \eta_1)^2 \right) |X = x, Z = 1 = \text{Var} \left[ Y(1) + \frac{(Y(1) - \theta_1(X)) + \Gamma(Y(1) - \theta_1(X)) - 1 - e(X)}{\nu_1(X)} - \frac{1 - e(X)}{\nu_1(X)} \right] |X = x \geq \text{Var}(Y | X = x, Z = 1)$$

Use the conditional variance law to conclude

$$\text{Var} \left( m(W, \mu_1, \eta_1)^2 \right) \geq \mathbb{E} \left[ \text{Var}(Y | X = x, Z = 1) \right] \geq c_0 > 0,$$

whenever $\text{Var}(Y | X = x, Z = 1) > 0$. If $\mathbb{E} \left[ \text{Var}(Y | X = x, Z = 1) \right] > 1$, choose $c_0 = 1$ so that Assumption 8 remains satisfied.

The following lemma will be useful for reducing the amount of tedious algebra needed to bound second derivatives of the score function.

**Lemma 4.** Let $f(x, r)$ be a function uniformly bounded in $L_2, r$ for all $r \in (0, 1)$, and let $g(x)$ be a function uniformly bounded in $L_\infty, r$ for all $r \in (0, 1)$, both with well-defined second derivatives. Additionally, assume the Dominated Convergence holds. Then,

$$\frac{\partial^2}{\partial r^2} \mathbb{E} \left[ \frac{f(x, r)}{g(x, r)} \right] \leq \mathbb{E} \left[ C_1 \left| \frac{\partial^2}{\partial r^2} f(x, r) \right| + C_2 \left| \frac{\partial^2}{\partial r^2} g(x, r) \right| + C_3 \left| \frac{\partial}{\partial r} f(x, r) \frac{\partial}{\partial r} g(x, r) \right| + C_4 \left( \frac{\partial}{\partial r} g(x, r) \right)^2 \right].$$

(C.18)

**Proof** Begin by using dominated convergence to exchange differentiation and integration,

$$\frac{\partial^2}{\partial r^2} \mathbb{E} \left[ \frac{f(x, r)}{g(x, r)} \right] = \frac{\partial}{\partial r} \mathbb{E} \left[ \frac{\partial}{\partial r} f(x, r) \right] \frac{g(x, r)^2}{g^3(x, r)}$$

(C.19)

$$= \mathbb{E} \left[ - \frac{\partial}{\partial r} g(x, r) \frac{\partial}{\partial r} f(x, r) - f(x, r) \frac{\partial}{\partial r} \frac{g(x, r)^2}{g^3(x, r)} \right]$$

(C.20)

$$= \mathbb{E} \left[ - \frac{1}{g^3(X, r)} \left( f(x, r) \frac{\partial}{\partial r} g(x, r) + g(x, r) \frac{\partial}{\partial r} f(x, r) \right) \frac{g(X, r)^2}{g^3(X, r)} \left( \frac{\partial}{\partial r} g(x, r) \right)^2 \right]$$

(C.21)

$$+ \mathbb{E} \left[ \frac{g(X, r)}{g^3(X, r)} \frac{\partial}{\partial r} \frac{g(x, r)^2}{g^3(x, r)} f(x, r) - f(x, r) \frac{\partial}{\partial r} g(x, r) \right]$$

(C.22)
Using the fact that $g$ is bounded from above and below, there exists $\tilde{C}$ such that

$$\left| \frac{\partial^2}{\partial r^2} \mathbb{E} \left[ \frac{f(X,r)}{g(X,r)} \right] \right| \leq \tilde{C} \mathbb{E} \left[ \left| \frac{\partial^2}{\partial r^2} g(X,r) \right| f(X,r)^2 + \left| \frac{\partial}{\partial r} f(X,r) \right| \left| \frac{\partial^2}{\partial r^2} g(X,r) \right| \right] \quad (C.23)$$

$$+ \mathbb{E} \left[ \left| \frac{\partial}{\partial r} g(X,r) \right| f(X,r)^2 + \left| \frac{\partial}{\partial r} f(X,r) \right| \left| \frac{\partial^2}{\partial r^2} g(X,r) \right| \right] \quad (C.24)$$

Using the Cauchy-Schwarz inequality and the fact that for any $r \in [0,1)$, $f(X,r)$ is bounded in $L_{2,p},$

$$\left| \frac{\partial^2}{\partial r^2} \mathbb{E} \left[ \frac{f(X,r)}{g(X,r)} \right] \right| \leq \mathbb{E} \left[ C_3 \left| \frac{\partial}{\partial r} g(X,r) \right| f(X,r)^2 + C_4 \left( \frac{\partial}{\partial r} g(X,r) \right)^2 \right]$$

$$+ C_1 \left| \frac{\partial}{\partial r} f(X,r) \right| + C_2 \left| \frac{\partial^2}{\partial r^2} g(X,r) \right| \quad (C.25)$$

**Lemma 5.** Assume the conditions of Theorem 4.1. If $T_n \subseteq \{ \eta : \| \theta(\cdot) - \theta_1(\cdot) \|_{2,p} \leq a_n n^{-1/4}, \| \nu(\cdot) - \nu_1(\cdot) \|_{2,p} \leq a_n n^{-1/4}, \| e(\cdot) - e_1(\cdot) \|_{2,p} \leq a_n n^{-1/4} \}$, then the following rate condition holds

$$\sup_{r \in (0,1), \eta \in T_n} \partial^2 \mathbb{E} \left[ m(W, \mu_1, \eta; + r(\eta - \eta_1)) \right] \leq C a_n^2 n^{-1/2}. \quad (C.26)$$

**Proof** For notational convenience, let $a(y,t) = (y-t)_+ - \Gamma (y-t)_-$. Then, let

$$h(x,r) = \mathbb{E} \left[ Z \left( a \left( Y, \theta_1(X) + r(\theta(X) - \theta_1(X)) \right) \right) \bigg| X = x \right].$$

Differentiate once to get

$$\frac{d}{dr} h(x,r) = \frac{d}{dr} \mathbb{E} \left[ Z \left( a \left( Y, \theta_1(X) + r(\theta(X) - \theta_1(X)) \right) \right) \bigg| X = x \right]$$

$$= (\theta(X) - \theta_1(X)) \frac{d}{dr} \mathbb{E} \left[ Z a(Y,t) \bigg| X = x \right]_{t=\theta_1(X)+r(\theta(X)-\theta_1(X))}$$

$$= (\theta(X) - \theta_1(X)) \left[ -1 - (\Gamma - 1) P(Y < \theta_1(X) + r(\theta(X) - \theta_1(X)) \big| Z = 1, X = x) \right] e_1(X)$$

and again to get

$$\frac{d^2}{dr^2} h(x,r) = \frac{d}{dr} \left( \theta(X) - \theta_1(X) \right) \left[ -1 - (\Gamma - 1) P(Y < \theta_1(X) + r(\theta(X) - \theta_1(X)) \big| Z = 1, X = x) \right] e_1(X)$$

$$= - (\Gamma - 1) e_1(X) \left( \theta(X) - \theta_1(X) \right)^2 \frac{d}{dt} \left[ P(Y < t \big| Z = 1, X = x) \right]_{t=\theta_1(X)+r(\theta(X)-\theta_1(X))}$$

$$= - (\Gamma - 1) e_1(X) \left( \theta(X) - \theta_1(X) \right)^2 \nu_Y (\theta_1(X) + r(\theta(X) - \theta_1(X)) \big| Z = 1, X = x).$$
Let $f(x, r) = h(x, r) \left( 1 - e_1(x) - r(e(x) - e_1(x)) \right)$. Then, the above implies

\[
\frac{d}{dr} f(x, r) = \left( 1 - e_1(x) - r(e(x) - e_1(x)) \right) \frac{d}{dr} h(x, r) + h(x, r) \frac{d}{dr} \left( 1 - e_1(x) - r(e(x) - e_1(x)) \right)
\]

\[
= \left( 1 - e_1(x) - r(e(x) - e_1(x)) \right) \frac{d}{dr} h(x, r) - (e(x) - e_1(x)) h(x, r)
\]

and

\[
\frac{d^2}{dr^2} f(x, r) = \frac{d}{dr} \left( 1 - e_1(x) - r(e(x) - e_1(x)) \right) \frac{d}{dr} h(x, r) - \frac{d}{dr} (e(x) - e_1(x)) h(x, r)
\]

\[
= -(e(x) - e_1(x)) \frac{d}{dr} h(x, r) + \left( 1 - e_1(x) - r(e(x) - e_1(x)) \right) \frac{d^2}{dr^2} h(x, r) - (e(x) - e_1(x)) \frac{d}{dr} h(x, r)
\]

\[
= -2(e(x) - e_1(x)) \frac{d}{dr} h(x, r) + \left( 1 - e_1(x) - r(e(x) - e_1(x)) \right) \frac{d^2}{dr^2} h(x, r)
\]

Bounding these with the assumption $e(x) \in (e, 1 - e)$ gives

\[
\left| \frac{d}{dr} f(x, r) \right| \leq \left| (1 - e) \Gamma |\theta(X) - \theta_1(X)| + |e(x) - e_1(x)||h(x, r)| \right|
\]

and

\[
\left| \frac{d^2}{dr^2} f(x, r) \right| \leq 2\Gamma |e(x) - e_1(x)||\theta(x) - \theta_1(x)|
\]

\[
+ (1 - e)(\Gamma - 1) \left( \theta(X) - \theta_1(X) \right)^2 p_Y (\theta_1(X) + r(\theta(X) - \theta_1(X))) Z = 1, X = x).
\]

Because $p_Y(t|Z = 1, X = x)$ is uniformly bounded, this simplifies further,

\[
\left| \frac{d^2}{dr^2} f(x, r) \right| \leq 2\Gamma |e(x) - e_1(x)||\theta(x) - \theta_1(x)| + (1 - e)(\Gamma - 1) R (\theta(X) - \theta_1(X))^2.
\]

Next, let

\[
g(x, r) = (\nu_1(x) + r(\nu(x) - \nu_1(x))) \left( e_1(x) + r(e(x) - e_1(x)) \right)
\]

Differentiate once to get

\[
\frac{d}{dr} g(x, r) = \frac{d}{dr} (\nu_1(x) + r(\nu(x) - \nu_1(x))) \left( e_1(x) + r(e(x) - e_1(x)) \right)
\]

\[
= (\nu(x) - \nu_1(x)) \left( e_1(x) + r(e(x) - e_1(x)) \right) + (\nu_1(x) + r(\nu(x) - \nu_1(x))) (e(x) - e_1(x))
\]

and again to get

\[
\frac{d^2}{dr^2} g(x, r) = \frac{d}{dr} (\nu(x) - \nu_2(x)) \left( e_1(x) + r(e(x) - e_1(x)) \right) + \frac{d}{dr} (\nu_1(x) + r(\nu(x) - \nu_2(x))) (e(x) - e_1(x))
\]

\[
= (\nu(x) - \nu_1(x))(e(x) - e_1(x)) + (\nu(x) - \nu_1(x))(e(x) - e_1(x))
\]

\[
= 2(\nu(x) - \nu_1(x))(e(x) - e_1(x))
\]

Bound

\[
\left| \frac{d}{dr} g(x, r) \right| \leq (1 - e)|\nu(x) - \nu_1(x)| + \Gamma |e(x) - e_1(x)|.
\]

$\nu(x) \in [1, \Gamma]$ for almost every $x$, and, by assumption, $e(x) \in (e, 1 - e)$. Therefore, $f$ and $g$ meet the conditions of Lemma 4, because $f(x, r)$ is bounded in $L_2,p$ by assumption. Applying this lemma,

\[
\frac{\partial^2}{\partial r^2} \mathbb{E} \left[ \frac{f(X, r)}{g(X, r)} \right] \leq \mathbb{E} \left[ C_1 \frac{\partial^2}{\partial r^2} f(X, r) + C_2 \frac{\partial^2}{\partial r^2} g(X, r) + C_3 \left( \frac{\partial}{\partial r} f(X, r) \frac{\partial}{\partial r} g(X, r) \right) + C_4 \left( \frac{\partial}{\partial r} g(X, r) \right)^2 \right]
\]

\[
\leq 2C_1 \Gamma ||e(\cdot) - e_1(\cdot)||_{2,p} ||\theta(\cdot) - \theta_1(\cdot)||_{2,p} + C_1 R(1 - e)(\Gamma - 1)||\theta(\cdot) - \theta_1(\cdot)||_{2,p}^2
\]

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\[+ 2C_2 \|\nu(\cdot) - \nu_1(\cdot)\|_{2,p} \|e(\cdot) - e_1(\cdot)\|_{2,p}
+ C_3 \left(1 - e\|\nu(\cdot) - \nu_1(\cdot)\|_{2,p} + \Gamma\|e(\cdot) - e_1(\cdot)\|_{2,p}\right)
\cdot \left((1 - e)\Gamma\|\theta(\cdot) - \theta_1(\cdot)\|_{2,p} + \|e(\cdot) - e_1(\cdot)\|_{2,p}\|h(\cdot, r)\|_{2,p}\right)
+ 2C_4 \left((1 - e)^2\|\nu(\cdot) - \nu_1(\cdot)\|_{2,p}^2 + \Gamma^2\|e(\cdot) - e_1(\cdot)\|_{2,p}^2\right).\]

\[\|h(\cdot, r)\|_{2,p}\] is bounded, because \[\|\theta_1(\cdot) + r(\theta(\cdot) - \theta_1(\cdot))\|_{2,p}\] is bounded and \(\text{Var}(Y|X = x)\) is bounded in \(L_{2,p}\). This, along with the construction of \(T_n\) so that for each of the nuisance parameters, \(\|\cdot - \cdot\|_{2,p} = a_{n^\alpha}n^{-\beta}\) implies

\[\sup_{\eta \in T_n, r \in (0,1)} \frac{\partial^2}{\partial^2 r} \left[ \frac{f(X, r)}{g(X, r)} \right] \leq C a_{n^\alpha} n^{-1/2}.\]

Bounding \(\partial^2_{\eta^2} \left[m(W, \mu_1^\alpha, \eta)\right]\) follows easily from the above, because for any \(r \in (0,1)\) and \(\eta \in T_n\),

\[\partial^2_{\eta^2} \left[m(W, \mu_1^\alpha, \eta_1 + r(\eta - \eta_1))\right] = \frac{\partial^2}{\partial^2 \eta^2} \left[-\mu_1^\alpha + ZY + (1 - Z)\theta_1(X) + (1 - Z)r(\theta(X) - \theta_1(X)) + \frac{f(X, r)}{g(X, r)}\right]\]

\[= \frac{\partial^2}{\partial^2 \eta^2} \left[-\mu_1^\alpha + ZY + (1 - Z)\theta_1(X) + (1 - Z)r(\theta(X) - \theta_1(X))\right] + \frac{\partial^2}{\partial^2 r} \left[\frac{f(X, r)}{g(X, r)}\right]\]

\[= \frac{\partial^2}{\partial^2 \eta^2} \left[f(X, r)\right]_{g(X, r)} \leq C a_{n^\alpha} n^{-1/2}.\]

\[\square\]

**C.1 Proof of Proposition 4.1**

**Proposition 4.1.** Let Assumptions 3, 6, 7 hold, and let \(\hat{\nu}_1\) be an approximate empirical minimizer to the problem (4.8) satisfying

\[\mathbb{E}_n[\hat{\nu}(\hat{\theta}_1^\alpha, (X, Y(1)))|Z = 1] \leq \inf_{\nu \in 1 + (\Gamma - 1)P_1} \mathbb{E}_n[\hat{\nu}(\nu; \hat{\theta}_1^\alpha, (X, Y(1)))|Z = 1] + O_p\left(\epsilon_n^2\right)\]

where \(\epsilon_n := \max\{\delta_n, \inf_{\nu \in 1 + (\Gamma - 1)P_1} \|\nu_1 - \nu\|_{2,p}\}\). If \(n\epsilon_n^2 \rightarrow \infty\), then \(\|\hat{\nu}_1 - \nu_1\|_{2,p} = O_p\left(\epsilon_n + \|\hat{\theta}_1^\alpha - \theta_1\|_{2,p}\right)\).

**Proof** As before, in this proof we shall use the shorthand notation \(E_1[\cdot] = \mathbb{E}[\cdot|Z = 1]\) and \(E_1[\cdot] = \mathbb{E}_n[\cdot|Z = 1]\).

Define \(\tilde{\nu}'\) as the solution to the optimization problem (4.7) with \(\theta_1\) replaced with its estimate \(\hat{\theta}_1^\alpha\)

\[\tilde{\nu}'(x) := 1 + (\Gamma - 1)P_1(Y(1) \geq \hat{\theta}_1^\alpha(X), X)\]

where the probability is taken only over \(Y\), and not the randomness in \(\hat{\theta}_1^\alpha\). From triangle inequality,

\[\|\hat{\nu}_1 - \nu_1\|_{2,p} \leq \|\hat{\nu}_1 - \tilde{\nu}'\|_{2,p} + \|\tilde{\nu}' - \nu_1\|_{2,p},\]

and we now bound the two terms separately.

To bound the second term in expression (C.27), we use Assumption 7 to get

\[\|\tilde{\nu}' - \nu_1\|_{2,p} = (\Gamma - 1) \left(E_{X \sim P_1} \left[P_1(Y(1) \geq \hat{\theta}_1^\alpha(X)|X) + P_1(Y(1) \geq \theta_1(X)|X)\right]^2\right)^{1/2}\]

\[\leq (\Gamma - 1)L \|\hat{\theta}_1 - \theta_1\|_{2,p}\]

where the expectations are taken only with respect to \(X \sim P_1\).
It now remains to show that \( \| \hat{\nu}_1 - \hat{\nu}' \|_{2, P_1} = O_p(\epsilon_n) \). To this end, we use the fact that \( \hat{\nu}' \) is the unique minimizer of
\[
\min_{\nu \in \mathbb{E}_{1 + (\Gamma-1)\Pi_n}} \mathbb{E}_1[\ell_2(\nu; \hat{\theta}^{\varsigma}_{i_1^1}, (X, Y(1)))],
\]
and apply a general result for sieve estimators again. Although the constant terms in the asymptotics now depend on \( \hat{\theta}^{\varsigma}_{i_1^1} \), we use a variant of Lemma 2 with explicit constants [7, Corollary 1 and Remark 1] that allows us to establish the usual rate \( O_p(\epsilon_n) \). In the below lemma, all expectations are only over the samples used to estimate \( \hat{\nu}_1 \), and not over the randomness in \( \hat{\theta}^{\varsigma}_{i_1^1} \).

**Lemma 6** (Chen and Shen [7, Corollary 1]). Let \( \hat{\nu}' \in 1 + (\Gamma - 1)\Pi_n^q(X) \) for some \( q, r > 0 \), and let
\[
\mathbb{E}_1[\ell(\hat{\nu}; \hat{\theta}^{\varsigma}_{i_1^1}, (X, Y(1)))] \approx \mathbb{E}_1[\ell(\hat{\nu}'; \hat{\theta}^{\varsigma}_{i_1^1}, (X, Y(1)))] \approx \| \nu - \nu_1 \|^2_{2, P_1}.
\]
For \( \delta \) small enough, let
\[
\sup_{\nu \in \mathbb{E}_{1 + (\Gamma-1)\Pi_n}} \mathbb{E}_1[\| \ell(\nu; \hat{\theta}^{\varsigma}_{i_1^1}, (X, Y(1))) - \ell(\hat{\nu}'; \hat{\theta}^{\varsigma}_{i_1^1}, (X, Y(1))) \|^2] \lesssim \delta^2 \quad (C.28)
\]
\[
\sup_{\nu \in \mathbb{E}_{1 + (\Gamma-1)\Pi_n}} \mathbb{E}_1[|\ell(\nu; \hat{\theta}^{\varsigma}_{i_1^1}, (X, Y(1))) - \ell(\hat{\nu}'; \hat{\theta}^{\varsigma}_{i_1^1}, (X, Y(1)))|] \lesssim \delta^s \quad (C.29)
\]
for some \( s \in (0, 2) \). If the loss \( \ell \) is uniformly bounded, then there exists a universal constant \( C > 0 \) (that does not depend on \( \hat{\theta}^{\varsigma}_{i_1^1} \)) such that for any \( t > 0 \)
\[
P_t\left( \| \hat{\nu}_1 - \hat{\nu}' \|_{2, P_1} \geq \epsilon\right) \leq C \exp\left( -n_2 t^2 \right).
\]

On the event \( \mathcal{E}_n := \{ \hat{\theta}_1 \in S \} \), we have \( \hat{\nu}' \in 1 + (\Gamma - 1)\Pi \) by Assumption 6. Since the event \( \mathcal{E}_n \) is independent of the samples used in the sieve procedure (4.8) for computing \( \hat{\nu}_1 \), we can apply Lemma 6 conditioned on this event; we verify remaining hypotheses of Lemma 6 below.

From \( \mathbb{E}_1[\| \hat{\nu}'(X) - 1 - (\Gamma - 1)1 \{ Y(1) \geq \hat{\theta}_1(X) \} \| |X] = 0 \) almost surely, we have
\[
\mathbb{E}_1[\ell(\hat{\nu}; \hat{\theta}^{\varsigma}_{i_1^1}, (X, Y(1))) - \mathbb{E}_1[\ell(\hat{\nu}'; \hat{\theta}^{\varsigma}_{i_1^1}, (X, Y(1)))] = \frac{1}{2} \mathbb{E}_1[(\hat{\nu}(X) - \hat{\nu}'(X))^2],
\]
which verifies the second condition in Lemma 6. To verify the bounds (C.28), (C.29), we begin by noting that
\[
\ell(\nu; \hat{\theta}^{\varsigma}_{i_1^1}, (X, Y(1))) - \ell(\hat{\nu}; \hat{\theta}^{\varsigma}_{i_1^1}, (X, Y(1))) = \frac{1}{2} \left( \nu(x)^2 - \hat{\nu}'(x)^2 \right) + \left( 1 + (\Gamma - 1)1 \{ y \geq \hat{\theta}^{\varsigma}_{i_1^1}(x) \} \right) \left( \nu(x) - \hat{\nu}'(x) \right).
\]
Squaring both sides and taking expectations, we have
\[
\mathbb{E}_1[\ell(\nu; \hat{\theta}^{\varsigma}_{i_1^1}, (X, Y(1))) - \ell(\hat{\nu}; \hat{\theta}^{\varsigma}_{i_1^1}, (X, Y(1)))] \lesssim \| \nu - \hat{\nu}' \|^2_{2, P_1} \lesssim \delta^2
\]
for any \( \nu \in \Pi_n \) such that \( \| \nu - \hat{\nu}' \|_{2, P_1} \leq \delta \). Taking supremum over this set, we have condition (C.28).

For the bound (C.29), similarly note that
\[
\ell(\nu; \hat{\theta}^{\varsigma}_{i_1^1}, (X, Y(1))) - \ell(\hat{\nu}; \hat{\theta}^{\varsigma}_{i_1^1}, (X, Y(1))) \lesssim \| \nu - \hat{\nu}' \|_{\infty}.
\]

Applying Lemma 3 and Assumption 3, we have that condition (C.29) holds with \( s = 2q/(2q + d) \).

**Corollary 4.3.** Let \( H_0 \) be defined as in (4.14). For \( \psi_n(Y, Z_i, X_i)_{i=1}^n \) given in (4.15) and any \( P \in H_0, \)
\[
\lim inf_{n \to \infty} P(\psi_n \leq 0) \geq 1 - \alpha.
\]
Proof Corollary 4.1 implies
\[ \lim_{n \to \infty} P(\hat{\tau}^+ - \frac{\hat{\tau}^-}{\sqrt{n}} \leq \tau^-) = (1 - \alpha). \]

The lower bound property of \( \tau^-[P] \) (see Eq. (4.2)) along with the definition of the hypothesis \( H_0 \) implies \( \tau^-[P] \leq \text{E}_P[Y(1) - Y(0)] \leq 0 \). Therefore, the monotonicity of probability and the above imply the claim. □

**Proposition 4.2.** Let \( \psi_n^\circ \) be defined as in (4.15), so that \( \psi_n^\circ \) is asymptotically level \( \alpha \) for \( H_0(\Gamma) \) in (4.14) (see Corollary 4.3). Then, for an alternative \( H_1 = \{ Q \} \), \( \psi_n^\circ \) has design sensitivity \( \hat{\Gamma} \) for \( \hat{\Gamma} \) satisfying
\[
0 = \frac{1}{n} \text{E}_Q[Y(1)] + \inf_{\theta_1} \frac{1}{n} \text{E}_Q \left[ \frac{1 \{ Y(1) \geq \theta_1(X) \} + \hat{\Gamma}_1 \{ Y(1) < \theta_1(X) \}}{Q(Y(1) \geq \theta_1(X)|X) + \hat{\Gamma} Q(Y(1) < \theta_1(X)|X) Y(1)} \right] \\
- \frac{1}{n} \text{E}_Q[Y(0)] - \sup_{\theta_0} \frac{1}{n} \text{E}_Q \left[ \frac{\hat{\Gamma}_1 \{ Y(0) > \theta_0(X) \} + 1 \{ Y(0) \leq \theta_0(X) \}}{\hat{\Gamma} Q(Y(0) > \theta_0(X)|X) + \hat{\Gamma} Q(Y(0) < \theta_0(X)|X) Y(0)} \right].
\]

Proof By Corollary 4.3, for data \((Y_i, Z_i, X_i)_{i=1}^n\) drawn from \( Q \), \( \psi_n^\circ((Y_i, Z_i, X_i)_{i=1}^n) \overset{d}{\to} \tau^-[Q] \). Therefore, the power of \( \psi_n \) is defined by the condition \( \tau^-[Q] > 0 \). When \( \tau^-[Q] > 0 \), \( \lim_{n \to \infty} Q(\psi_n^\circ \leq 0) = 0 \), and when \( \tau^-[Q] \leq 0 \), \( \lim_{n \to \infty} Q(\psi_n^\circ \leq 0) = 1 \). Expanding the definition of \( \tau^-[Q] \) gives the result. □

**Corollary 4.4.** Let \( \psi_n^\circ \) be defined as in (4.15), so that \( \psi_n^\circ \) is asymptotically level \( \alpha \) for \( H_0(\Gamma) \) in (4.14) (see Corollary 4.3). Then, for the alternative \( H_1 = \{ Y(1) \sim N(\frac{1}{2}, \sigma^2), Y(0) \sim N(-\frac{1}{2}, \sigma^2), Z \sim \text{Bernoulli}(1/2) \} \), \( \psi_n^\circ \) has design sensitivity
\[
\hat{\Gamma} \leq \frac{\int_{\infty}^{\infty} y \exp \left( \frac{-\text{e}^{-(y)}{2\sigma^2}}{2\sigma^2} \right) dy}{\int_{-\infty}^{0} y \exp \left( \frac{-\text{e}^{-(y)}{2\sigma^2}}{2\sigma^2} \right) dy}.
\]

Proof Note that in Proposition 4.2, the design sensitivity is at least \( \hat{\Gamma} \) satisfying
\[
0 \geq \frac{1}{n} \text{E}_Q[Y(1)] + \inf_{\theta_1} \frac{1}{n} \text{E}_Q \left[ \frac{1 \{ Y(1) \geq \theta_1(X) \} + \hat{\Gamma}_1 \{ Y(1) < \theta_1(X) \}}{Q(Y(1) \geq \theta_1(X)|X) + \hat{\Gamma} Q(Y(1) < \theta_1(X)|X) Y(1)} \right] \\
- \frac{1}{n} \text{E}_Q[Y(0)] - \sup_{\theta_0} \frac{1}{n} \text{E}_Q \left[ \frac{\hat{\Gamma}_1 \{ Y(0) > \theta_0(X) \} + 1 \{ Y(0) \leq \theta_0(X) \}}{\hat{\Gamma} Q(Y(0) > \theta_0(X)|X) + \hat{\Gamma} Q(Y(0) < \theta_0(X)|X) Y(0)} \right].
\]

Indeed, if for such a \( \hat{\Gamma} \), this term is strictly negative, then \( \tau^-[P] < 0 \) in the proof of Proposition 4.2, so \( \lim_{n \to \infty} Q(\psi_n^\circ \leq 0) = 1 \). Therefore, the true design sensitivity is at least \( \hat{\Gamma} \).

Remove the dependence on \( X \), as \( Y(1) \) and \( Y(0) \) do not depend on \( X \) under \( Q \), and plug in the form of the density of \( Y(1) \) under \( Q \). Use that \( Y(0) \overset{d}{=} -Y(1) \) to show
\[
0 \geq \frac{1}{2} \left( \frac{1}{2} \int_{-\infty}^{\infty} \left( 1 \{ Y(1) \geq \theta_1 \} + \hat{\Gamma}_1 \{ Y(1) < \theta_1 \} \right) y \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left( \frac{-1}{2\sigma^2} \left( y - \frac{1}{2} \right)^2 \right) dy \right),
\]
for \( C \) the appropriate normalizing constant \( C = Q(Y(1) \geq \theta_1) + \hat{\Gamma} Q(Y(1) < \theta_1) \). Choosing \( \theta_1 = \frac{1}{2} \) gives the bound
\[
\frac{1}{2} \left( \frac{1}{2} \int_{-\infty}^{\infty} \left( 1 \{ Y(1) \geq \frac{1}{2} \} + \hat{\Gamma}_1 \{ Y(1) < \frac{1}{2} \} \right) y \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left( \frac{-1}{2\sigma^2} \left( y - \frac{1}{2} \right)^2 \right) dy \right).
\]
\[ \frac{1}{2} \left( \frac{\tau}{2} + \inf_{\theta_1 \in \Theta} \int_{-\infty}^{\infty} \left( 1_{\{Y(1) \geq \theta_1\}} + \Gamma 1_{\{Y(1) < \theta_1\}} \right) y \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left( -\frac{1}{2\sigma^2} (y - \frac{x}{\sigma})^2 \right) \, dy \right). \]

Using the assumption (4.16) gives
\[ \frac{1}{2} \left( \frac{\tau}{2} + \frac{1}{C} \int_{-\infty}^{\infty} \left( 1_{\{Y(1) \geq -\frac{x}{\sigma}\}} + \Gamma 1_{\{Y(1) < -\frac{x}{\sigma}\}} \right) y \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left( -\frac{1}{2\sigma^2} (y - \frac{x}{\sigma})^2 \right) \, dy \right) = 0. \]

\[ \text{Proposition 4.3. Let } H_0(\Gamma) \text{ be defined as in (4.14), and let } H_1 = \{ Y(1) \sim N(\frac{x}{2}, \sigma^2), Y(0) \sim N(-\frac{x}{2}, \sigma^2), Z \sim \text{Bernoulli}(1/2) \}. \text{ If}
\]
\[ \Gamma \geq \frac{\int_0^\infty y \exp \left( -\frac{(y-x)^2}{2\sigma^2} \right) \, dy}{\int_{-\infty}^\infty y \exp \left( -\frac{(y-x)^2}{2\sigma^2} \right) \, dy} \quad (4.17) \]

then no level \( \alpha \) test \( t_n^* \) for \( H_0(\Gamma) \) will have power \( Q(t_n^* = 1) > \alpha \). That is, a design sensitivity \( \Gamma \) of any level \( \alpha \) test for \( H_1 \) must satisfy the inequality
\[ \tilde{\Gamma} \geq \frac{\int_0^\infty y \exp \left( -\frac{(y-x)^2}{2\sigma^2} \right) \, dy}{\int_{-\infty}^\infty y \exp \left( -\frac{(y-x)^2}{2\sigma^2} \right) \, dy}. \]

\[ \text{Proof} \quad \text{The proof proceeds by verifying the following claim, and then verifying the result based on this observation.}
\]
\[ \text{Claim 7. For } H_0 \text{ and } H_1 \text{ as in Proposition 4.3, and (4.17), there exists } P \in H_1 \text{ such that } \|P(Y(Z), Z, X) - Q(Y(Z), Z, X)\|_{TV} = 0.
\]

We defer proof of this claim to below. Let \( t_n^{\Gamma} \) be any level \( \alpha \) test under \( H_0(\Gamma) \). Then,
\[ P(t_n^{\Gamma} (Y_i, Z_i, X_i)_{i=1}^n = 0) \geq 1 - \alpha. \]

The claim implies
\[ Q(t_n^{\Gamma} (Y_i, Z_i, X_i)_{i=1}^n = 0) \geq 1 - \alpha, \]
and so
\[ Q(t_n^{\Gamma} (Y_i, Z_i, X_i)_{i=1}^n = 1) \leq \alpha. \]

\[ \text{Proof of Claim 7} \quad \text{Draw } Z \sim \text{Bernoulli}(a). \text{ Choose } t^* \text{ to attain}
\]
\[ \inf_1 \mathbb{E}_Q \left[ \frac{1 + \Gamma 1_{\{Y(1) \leq t^*\}}}{1 + \Gamma 1_{\{Y(1) < t^*\}}} Y(1) \right]. \]

Let \( q_1(t) \) be the density of \( Y(1) \) under \( Q \), and \( q_0(t) \) be the density of \( Y(0) \) under \( Q \). Define \( q_a = Q(Y(1) > t^*) \), and with this define
\[ p_a = \frac{\sqrt{a}}{1 + \sqrt{1 - a}}. \]

Define \( P_1(t) = P(Y(1) < t) \) via the density
\[ p_1(t) = \frac{1}{C_1} \left( \frac{\sqrt{1} + 1}{\sqrt{1} p_a 1_{(t > t^*)}} + \frac{\sqrt{1} + 1}{1 - p_a 1_{(t \leq t^*)}} \right) q_1(t). \quad (C.30) \]

where \( C_1 \) is chosen so that \( p_1(t) \) integrates to 1.

Then, draw \( Y(1) \) according to this distribution and set \( Y(0) = -Y(1) \). Finally, let \( U = 1_{\{Y(1) < t^*\}} \).

Then,
\[ U = 1_{\{Y(1) > t^*\}}. \]

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\[ = 1_{\{Y(0) < -t^*\}}. \]

Note that
\[
P(U = 1) = P(Y(1) > t^*)
\]
\[
= \frac{\sqrt{T+1}}{\sqrt{T+1}} Q(Y(1) > t^*) + \frac{1-\sqrt{T}}{\sqrt{T+1}} Q(Y(1) \leq t^*)
\]
\[
= \frac{\sqrt{T+1}}{\sqrt{T+1}} q_u + \frac{1-\sqrt{T}}{\sqrt{T+1}} (1-q_u)
\]
\[
= \frac{1}{\sqrt{T+1}} q_u + \frac{1}{\sqrt{T+1}} (1-q_u)
\]
\[
= \frac{1}{\sqrt{T+1} q_u + \frac{1}{\sqrt{T+1} (1-q_u)} = p_u.
\]

because
\[
\frac{1-q_p}{q} = \frac{1}{\sqrt{T}} \left( \frac{1-p}{p} \right)^2.
\]

Define the distribution of \(Z\) under \(P\) by the following conditional probabilities
\[
P(Z = 1|U = 1) = \frac{1}{P(U = 1)} \sqrt{T} + 1
\]
\[
P(Z = 0|U = 1) = \frac{1}{P(U = 1)} \sqrt{T} + 1
\]
\[
P(Z = 1|U = 0) = \frac{1}{P(U = 0)} \sqrt{T} + 1
\]
\[
P(Z = 0|U = 0) = \frac{1}{P(U = 0)} \sqrt{T} + 1
\]

Note that the marginal probability of \(Z\) satisfy \(P(Z = 1) = a\), and \(P\) satisfies the \(\Gamma\)-selection bias condition.

\[
P(Z = 1|U = 1) P(Z = 0|U = 0)\]
\[
P(Z = 0|U = 1) P(Z = 1|U = 0) = \frac{1}{P(U = 1)} \sqrt{T} + 1 \frac{1}{P(U = 0)} \sqrt{T} + 1 = \Gamma.
\]

Therefore, \(P \in H_0\) if \(E_P[Y(1) - Y(0)] \leq 0\). To check this, first calculate the conditional likelihood ratios of under \(P\) to \(Q\). First, note
\[
\frac{dP_{Y(1)|Z=1}(t)}{dP_{Y(1)}} = \frac{P(Z = 1|Y(1) = t)}{P(Z = 1)}
\]
\[
= \frac{P(Z = 1|U = 1) P(U = 1|Y(1) = t) + P(Z = 1|U = 0) P(U = 0|Y(1) = t)}{P(Z = 1)}
\]
\[
= \frac{1}{P(U = 1)} \sqrt{T} + 1 1_{\{t > t^*\}} + \frac{1}{P(U = 0)} \sqrt{T} + 1 1_{\{t \leq t^*\}};
\]

and
\[
\frac{dP_{Y(0)|Z=a}(t)}{dP_{Y(0)}} = \frac{P(Z = 1|Y(1) = t)}{P(Z = 1)}
\]
\[
= \frac{1}{P(U = 1)} \sqrt{T} + 1 1_{\{t < -t^*\}} + \frac{1}{P(U = 0)} \sqrt{T} + 1 1_{\{t \geq -t^*\}};
\]

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Therefore,

\[
\frac{dP_{Y(1)|Z=1}}{dQ_1}(t) = \frac{dP_{Y(1)|Z=1}}{dP_{Y(1)}}(t) \frac{dP_0}{dQ_0}(t) = 1, \tag{C.31}
\]

\[
\frac{dP_{Y(0)|Z=0}}{dQ_0}(t) = 1. \tag{C.32}
\]

Finally,

\[
\frac{dP_{Y(1)|Z=0}}{dQ_1}(t) \propto 1_{\{t \geq t^\ast\}} + \Gamma_1 \{Y(1) < t^\ast\},
\]

\[
\frac{dP_{Y(0)|Z=1}}{dQ_0}(t) \propto 1_{\{t \leq -t^\ast\}} + \Gamma_1 \{Y(0) > -t^\ast\}.
\]

Therefore,

\[
E_P [Y(1) - Y(0)] = E_P [Y(1) - Y(0)|Z = 1] \frac{1}{2} + E_P [Y(1) - Y(0)|Z = 1] \frac{1}{2} \leq 0,
\]

because \(Y(0) = -Y(1)\). Let \(C = Q(Y(1) \geq t^\ast) + \Gamma Q(Y(1) < t^\ast)\) and write the expectation in terms of the density of \(Y(1)\) under \(Q\),

\[
= \frac{\tau}{2} + \frac{1}{C} \int_{-\infty}^{\infty} y q_{Y(1)}(y) dy + \frac{\Gamma}{C} \int_{-\infty}^{-\frac{\tau}{2}} y q_{Y(1)}(y) dy = \frac{1}{C} \int_{-\infty}^{\infty} (y + \frac{\tau}{2}) q_{Y(1)}(y) dy + \frac{\Gamma}{C} \int_{-\infty}^{-\frac{\tau}{2}} q_{Y(1)}(y) dy
\]

\[
= \frac{1}{C} \int_{0}^{\infty} y q_{Y(1)}(y - \frac{\tau}{2}) dy + \frac{\Gamma}{C} \int_{-\infty}^{-\frac{\tau}{2}} y q_{Y(1)}(y - \frac{\tau}{2}) dy
\]

\[
\leq 0,
\]

when (4.17) holds.

Verifying that \((Y(Z), Z, X)\) has the same distribution under \(Q\) and \(P\) verifies the claim. \(Q\) does not specify a distribution on \(X\), and the potential outcomes do not depend on this distribution, so let \(X = \emptyset\). Alternatively, one could choose any distribution that matches the regularity conditions in \(H_0\).

The marginals over \((Y(Z), Z)\) are equal under \(Q\) and \(P\) because of (C.31) and (C.32).