On Achromatic coloring of certain classes of transformation graphs $G^{+−}$

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Abstract. Let $G$ be an $n$-vertex graph having $n \geq 3$. The transformation graph $G^{+−}$ of this $G$ is the graph with the union of vertex set and edge set in which the adjacency of two vertices $a$ and $b$ is defined as follows: (i) $a$ and $b$ in $V(G)$ are adjacent if and only if they are non-adjacent in $G$ (ii) $a$ and $b$ in $E(G)$ are adjacent if and only if they are adjacent in $G$ (iii) one of $a$ and $b$ is in $V(G)$ while the other is in $E(G)$, and they are not incident in $G$. In this paper, we obtain the achromatic coloring of transformation graph $G^{+−}$ of path graph, sunlet graph, friendship graph, ladder graph, tadpole graph, bistar graph and cycle graph.

1. Introduction

Throughout the paper, we consider all the graphs are finite, without any loops or multiple edges. Let $G = (V(G), E(G))$ with the vertex set $V(G)$ of order $n(G)$ and the edge set $E(G)$ of size $m(G)$. The minimum and maximum degree of $G$ are denoted by $\delta(G)$ and $\Delta(G)$ respectively. If $\delta(G)$ and $\Delta(G)$ equals $k$, then $G$ is $k$-regular. A complete $k$-coloring of $G$ is a proper vertex coloring of $G$ such that adjacent vertices can be assigned $k$-distinct colors. For a graph $G$ is $k$-colorable when a proper $k$-coloring exists. The chromatic number $\chi(G)$ of $G$ is the minimum of $k$ colors needed in a proper coloring of $G$ [3,11]. Graph $k$-coloring is $NP$ complete, while the problem is solvable in polynomial time for more classes. An achromatic coloring of $G$ is a proper vertex coloring of $k$-colors such that every pair of colors in a color class is adjacent by atleast one edge [9]. For maximum number of colors take a positive integer $k$ which is assigned as a $k$-coloring of $G$ and is defined as a function $\phi$ that assigns to each vertex a color as follows: $\phi : V \rightarrow \{1, 2, \ldots, k\}$. In 1967, Harary, Hedetniemi and Prins [8] was first introduced the achromatic number and has attracted a lot of attention since then. Yannakakis and Gavril [15] was proved the computation of achromatic number of a general graph to be $NP$-complete and Faber et al. [5] proved that the problem is $NP$-hard on bipartite graphs. But, further it was proved that the achromatic number problem remains $NP$-complete even for connected graphs which are both cographs and interval graphs respectively[2]. Geller and Kronk [6] proved that there is almost optimal coloring for families of paths and cycles [4,10]. The approximation of achromatic number on every bipartite graph was studied by Guy Kortsarz and Sunil Shende in 2007 [7]. In this case the generalized transformation graph is denoted as $G^{ppp}$ having the vertex set as $V(G) \cup E(G)$. Let $a$, $b$ be any two elements of $V(G) \cup E(G)$. When these two elements are adjacent or incident in $G$ where the associativity of $a$ and $b$ is assigned as $+$ otherwise it is taken as $−$. Let the three permutation of the set $\{+, −\}$ be $p$, $q$, $r$. The pair $a$ and $b$ is said to
correspond to p or q or r of pqr if a and b are both in \( V(G) \) or in \( E(G) \) or one of the element belongs to \( V(G) \) and the other element belongs to \( E(G) \) respectively [13]. This shows that the transformation graph \( G^{pqr} \) of \( G \) is the transformation graph whose vertex set is \( V(G) \cup E(G) \) and two of its vertices a and b are adjacent if and only if whose associativity in \( G \) is constant with the equivalent element of pqr [1,16]. In 2005, a particular case when \( pqr = -++ \) was studied by Baoyindureng Wu, Li Zhang, and Zhao Zhang [12] and in the year 2008, the transformation graph \( G^{-++} \) was studied by Lan Xu and Baoyindureng Wu [14]. Path graph is a sequence of distinct vertices \( v_1, v_2, \ldots, v_n \) with edges between \( v_i \) and \( v_{i+1}, 1 \leq i \leq n - 1 \). By connecting \( n \) pendent edges to the vertices of a cycle graph \( C_n \) will result a \( n \)-sunlet graph having \( 2n \) vertices. The friendship graph \( F_n \) can be constructed by connecting \( n \) copies of the cycle graph \( C_3 \) with a common vertex. The ladder graph \( L_n \) can be constructed by the Cartesian product of \( n \) copies of a path graph \( P_n \) and a complete graph with two-vertices \( K_2 \). The tadpole graph \( T_{m,n} \) is the graph by constructing a series of \( C_m \) and \( P_n \) with an edge from any vertex of \( C_m \) to a pendent vertex of \( P_n \) for some \( m \geq 3 \) and \( n \geq 0 \). The bistar graph \( B_{m,n} \) is the graph obtained from \( K_2 \) by attaching \( m \) pendent edges to one end and \( n \) pendent edges to the other end of \( K_2 \). Some number of vertices connected in a closed chain is known as cycle graph.

2. Achromatic coloring of transformation graph \( G^{-+} \) of Path graph

2.1. Theorem

The achromatic number for \( G^{-+}(P_n) = n \).

Proof

Let \( G = P_n \) be the path graph having length \( n - 1 \) whose vertex set and edge set is defined as, \( V(G) = \{a_1, a_2, \ldots, a_n\} \) and \( E(G) = \{e_1, e_2, \ldots, e_{n-1}\} \). Every vertex in \( G \), i.e., every \( a_i \) is adjacent with the vertices \( a_{i-1} \) and \( a_{i+1} \) for \( 2 \leq i \leq n - 1 \), \( a_1 \) is adjacent with \( a_2 \) and \( a_n \) is adjacent with \( a_{n-1} \) in the same way the edges \( e_i (2 \leq i \leq n - 1) \) are adjacent with \( e_{i-1} \) and \( e_{i+1} \) and \( e_1 \) is adjacent with \( e_2 \) and \( e_{n-1} \) is adjacent with \( e_{n-2} \).

By taking transformation graph \( G^{-+} \) on \( G \), we have the vertex set of this transformation graph corresponds to both vertex set and edge set of the graph \( G \). The vertex set of \( G^{-+}(P_n) \) is defined as follows:

\[
V(G^{-+}(P_n)) = \{a_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n - 1\}.
\]

Define the vertex labeling \( \sigma : V(G^{-+}(P_n)) \rightarrow N \) in a proper coloring way to get maximum number of pair of colors and to prove such coloring is achromatic as,

\[
\sigma(a_i) = 1, n - 1 \leq i \leq n ; \quad \sigma(a_i) = \sigma(a_2) = n ; \quad \sigma(a_i) = i + 1, 3 \leq i \leq n - 2 ; \quad \sigma(e_i) = i + 1, 1 \leq i \leq n - 2 ; \quad \sigma(e_{n-1}) = 2.
\]

By the above coloring process we have \( \psi(G^{-+}(G)) \leq n \). Now consider for any pair \( (\sigma_i, \sigma_j) \) which satisfy the coloring as achromatic and maximum.

- For \( i = 1 \) and \( 1 \leq k \leq n - 2 \), \( j = i + k \), the edges connecting the vertices \( a_{ni} \) and \( e_{j-1} \) will result for the pair \( (\sigma_i, \sigma_j) \).
- For \( i = n \) and \( j = 1 \), the edges connecting the vertices \( a_i \) and \( e_{j-1} \) will result for the pair \( (\sigma_i, \sigma_j) \).
- For \( 2 \leq i \leq n - 2 \) and \( j = i + k \), the edges connecting the vertices \( e_{i-1} \) and \( e_{j-1} \) will result for the pair \( (\sigma_i, \sigma_j) \).
- For \( 2 \leq i \leq n - 3 \) and \( 2 \leq k \leq n - 3 \), \( j = i + k \), assigning the value for every \( i \) not exceeding \( j \leq n - 1 \), then the edges joining the vertices \( e_{i-1} \) and \( a_{j-1} \) will result for the pair \( (\sigma_i, \sigma_j) \).
- For \( i = 2 \) and \( j = n \), the edges connecting the vertices \( e_{n-i+1} \) and \( v_{j-n+1} \) will result for the pair \( (\sigma_i, \sigma_j) \).
- For \( 3 \leq i \leq n - 1 \), \( j = n \), the edges connecting the vertices \( e_{i-1} \) and \( v_{j-n+1} \) will result for the pair \( (\sigma_i, \sigma_j) \).

3. Achromatic coloring of transformation graph $G^{--}$ of Sunlet graph

3.1. Theorem

The achromatic number for $G^{--}(S_n) = 2n + 1$.

Proof

Consider $G$ is a sunlet graph having $2n$ vertices and $2n$ edges.

Let $V(G) = \{a_1, a_2, \ldots, a_n\} \cup \{b_1, b_2, \ldots, b_n\}$ and

$E(G) = \{e_i : 1 \leq i \leq n - 1\} \cup \{e_n\} \cup \{e_{n+i} : 1 \leq i \leq n - 1\}$.

where $e_i \rightarrow a_ia_{i+1}(1 \leq i \leq n - 1)$

$e_n \rightarrow a_ia_n$

$e_{n+i} \rightarrow a_ib_i(1 \leq i \leq n - 1)$

By taking transformation graph $G^{--}$ on $G$, we have the vertex set, $V(G^{--}(G))$ corresponds to both $V(G)$ and $E(G)$ of $G$, and the vertex set of $G^{--}(G)$ is defined as follows:

$V(G^{--}(G)) = \{a_i : 1 \leq i \leq n\} \cup \{b_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n-1\} \cup \{e_n\} \cup \{e_{n+i} : 1 \leq i \leq n\}$.

Let us define the vertex labeling $\sigma : V(G^{--}(S_n)) \rightarrow N$ in a proper coloring way and which will result the assigned colors that makes the coloring as achromatic.

$\sigma(a_i) = i, 1 \leq i \leq n$, $\sigma(b_i) = n + i, 1 \leq i \leq n$, $\sigma(e_i) = \begin{cases} i, & 1 \leq i \leq n \\ 2n + 1, & 2n + 1 \leq i \leq 2n \end{cases}$

From the above coloring procedure, we conclude that $\psi[G^{--}(S_n)] \leq 2n + 1$. By considering any pair $(\sigma_i, \sigma_j)$ which will lead to the fact of achromatic coloring and also it is maximum.

- For every $1 \leq i \leq n - 1$ and $j = k + i$ where $1 \leq k \leq n - 1$, the edges connecting the vertices $a_i$ and $e_j$ will result for the pair $(\sigma_i, \sigma_j)$.

- For every $1 \leq i \leq n$ and $j = n + k$ where $1 \leq i \leq n$, the edges connecting the vertices $e_i$ and $b_{j-n}$ will result for the pair $(\sigma_i, \sigma_j)$.

- For $1 \leq i \leq n$ and $j = 2n + 1$, the edges connecting the vertices $e_i$ and $e_{\lceil \frac{i}{2} \rceil + k}$ where $1 \leq k \leq n$ will result for the pair $(\sigma_i, \sigma_j)$.

- For every $n + 1 \leq i \leq 2n - 1$ and $j = i + k$ but not exceeding $j \leq 2n$ where $1 \leq k \leq n - 1$, the edges connecting the vertices $b_{i-n}$ and $b_{j-n}$ will result for the pair $(\sigma_i, \sigma_j)$.

- For every $n + 1 \leq i \leq 2n - 1$ and $j = 2n + 1$, the edges connecting the vertices $b_{i-n}$ and $e_{\lceil \frac{i}{2} \rceil + k}(1 \leq k \leq n - 1)$ will result for the pair $(\sigma_i, \sigma_j)$.

- For $i = 2n$ and $j = 2n + 1$, the edge connecting the $b_{i-n}$ and $e_{\lceil \frac{i}{2} \rceil}$ will result for the pair $(\sigma_i, \sigma_j)$.

Thus the above coloring pairs shows that the each pair is adjacent by at least one edge. This proves the assigned color is achromatic which receives maximum number of colors.

Therefore $\psi[G^{--}(S_n)] = 2n + 1$.

4. Achromatic coloring of transformation graph $G^{--}$ of Friendship graph

4.1. Theorem

The achromatic number for $G^{--}(F_n) = 3n$.

Proof

If $G$ is a friendship graph with $|V(G)| = 2n + 1$ and $|E(G)| = 3n$ consisting of triangles of the disjoint union of $n$ complete graphs $k_2 : k_1\nabla nk_2$ and its vertex set and edge set are, $V(G) = \{a, b_i, c_i : 1 \leq i \leq n\}$ and $E(G) = \{ab_i, ac_i, c_i : 1 \leq i \leq n\} \cup \{b_ic_i : 1 \leq i \leq n\}$. 


By taking transformation graph $G^{++}$ on $G$ we have $V[G^{++}(G)]$ corresponds to both $|V(G)|$ and $|E(G)|$ of $G$. Thus the vertex set of this transformation graph is defined as follows: 

$$V[G^{++}(F_n)] = \{a\} \cup \{b_i : 1 \leq i \leq n\} \cup \{c_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq 3n\}.$$

Now define the vertex labeling $\sigma : V[G^{++}(F_n)] \rightarrow N$ according to a proper coloring manner will lead to the coloring is achromatic and it has maximal color class.

$$\sigma(a) = 1; \sigma(b_1) = 2; \sigma(b_i) = 3i - 1, 2 \leq i \leq n; \sigma(c_1) = 2; \sigma(c_i) = 3i - 1, 2 \leq i \leq n; \sigma(e_i) = i, 1 \leq i \leq 3n.$$

From the above proper coloring, we suspect that $\psi[G^{++}(F_n)] \leq 3n$. Now it is needed to consider any pair $(\sigma_i, \sigma_j)$, its implication will lead to the fact that the coloring is achromatic and it is maximum.

- For $1 \leq i \leq 3n - 1, j = i + 1$, the edges joining the vertices $e_i$ and $e_j$ will result for the pair $(\sigma_i, \sigma_j)$.
- For $i = 1$ and $i \equiv 1 \mod 3, j = i + k$ where $2 \leq k \leq 3n - i$ but $k \neq 4, 7, 10, ... 3n - (i + 1)$, the edges joining the vertices $e_i$ and $e_j$ will result for the pair $(\sigma_i, \sigma_j)$.
- For $i \equiv 0 \mod 3, j = i + k$ where $3 \leq k \leq 3n - i$ but $k \neq 5, 8, 11, ... 3n - (i + 1)$, the edges joining the vertices $e_i$ and $e_j$ will result for the pair $(\sigma_i, \sigma_j)$.
- For $i \equiv 1 \mod 3$ with $j = i + 4$; $i \equiv 2 \mod 3$ with $j = i + 3$; $i \equiv 0 \mod 3$ with $j = i + 2$.
- Then the edges connecting the vertices $e_i$ and $b_{j-3}$ will result for the pair $(\sigma_i, \sigma_j)$.
- For each $i$, where $i = 2, 5, 18, ... 3n - 1$ with $j = i + k(1 \leq k \leq 3n - i)$, then the edges connecting the vertices $b_{i-oddk}$ for odd $k < i$ and $e_j$ will result for the pair $(\sigma_i, \sigma_j)$.

Clearly from the above coloring process we conclude that, every pair in the color class is neighboring by at least one edge.

Thus the above said coloring is achromatic and it is maximal.

Therefore $\psi[G^{++}(F_n)] = 3n$.

5. Achromatic coloring of transformation graph $G^{++}$ of Ladder graph

5.1. Theorem

The achromatic number for $G^{++}(L_n) = \begin{cases} 2n & \text{if } n = 2 \\ 2n + 1 & \text{if } n = 3 \\ 2n + 2 & \text{if } n \geq 3 \end{cases}$

Proof

If $G$ is a ladder graph and is defined by $L_n = P_n \times K_2$ i.e., a ladder graph is the Cartesian product of a path graph $P_n$ having $n$ vertices and a complete graph having two vertices. It is a planar undirected graph having $2n$ vertices and $3n - 2$ edges with $\Delta(L_n) = 3$ and $\delta(L_n) = 2$.

Let $V(G) = \{a_i : 1 \leq i \leq n\} \cup \{b_i : 1 \leq i \leq n\}$ and $E(G) = \{e_i : 1 \leq i \leq 3n - 2\}$

By taking transformation graph $G^{++}$ on $G$, we notice that this graph interrelated to both vertex set and edge set of the graph $G$. Thus the vertex set of $G^{++}(L_n)$ is defined as, $V[G^{++}(L_n)] = \{a_i : 1 \leq i \leq n\} \cup \{b_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq 3n - 2\}$

Case 1 For $n = 2$

Define the vertex coloring $\sigma : V[G^{++}(L_n)] \rightarrow N$ in a proper coloring way as follows: $\sigma(a_1) = 1; \sigma(a_2) = 2n; \sigma(b_i) = i + 1, 1 \leq i \leq 2; \sigma(e_i) = i, 1 \leq i \leq 2n$.

By considering the above coloring pattern, we have $\psi[G^{++}(L_n)] \leq 2n$. Now consider for any pair $(\sigma_i, \sigma_j)$ which will satisfy the assigned coloring as achromatic and also it is maximum.

- For $i = 1, j = i + 1$ , the edges adjoining the vertices $a_i$ and $e_j$ will result for the pair $(\sigma_i, \sigma_j)$.
For every $i = 1, j = k + 1$ where $2 \leq k \leq 2n - 1$ the edges joining the vertices $e_i$ and $e_j$ will result for the pair $(\sigma_i, \sigma_j)$.

For every $i = 2, j = k + 1$ where $1 \leq k \leq 2n - 1$ the edges joining the vertices $e_i$ and $e_j$ will result for the pair $(\sigma_i, \sigma_j)$.

For $i = 3, j = i + 1$ , the edges joining the vertices $e_i$ and $a_{j-2}$ will result for the pair $(\sigma_i, \sigma_j)$.

Thus from the above process, it is clear that the resulting pairs in the color class is adjacent by atleast one edge and which satisfies the condition of achromatic coloring.

Hence $\psi[G^{++}(L_n)] = 2n$ for $n = 2$.

**Case 2** For $n = 3$

Define the color class $\sigma : V[G^{++}(L_n)] \to N$ in a proper coloring manner as follows:

$\sigma(a_1) = i, 1 \leq i \leq 2; \sigma(a_2) = 2n + 1; \sigma(b_1) = n + 2; \sigma(b_i) = i + 2, 2 \leq i \leq n; \sigma(e_i) = i, 1 \leq i \leq 2n + 1$.

By considering any pair $(\sigma_i, \sigma_j)$ which will prove the above said coloring is achromatic and maximum.

• For $i = 1, j = i + k$ where $1 \leq k \leq 2n$ but $k \neq 4$, the lines joining the vertices $a_i$ and $e_j$ will result for the pair $(\sigma_i, \sigma_j)$.

• For $i = 2, j = i + k$ where $1 \leq k \leq 2n - 1$ but $k \neq 4$, the lines joining the vertices $a_i$ and $e_j$ will result for the pair $(\sigma_i, \sigma_j)$.

• For $i = 3, j = i + k$ where $1 \leq k \leq n$, the lines joining the vertices $e_i$ and $e_j$ will result for the pair $(\sigma_i, \sigma_j)$.

• For $i = 4, j = i + k$ where $2 \leq k \leq n$, the lines joining the vertices $e_i$ and $e_j$ will result for the pair $(\sigma_i, \sigma_j)$.

• For $i = 5, j = i + k$ where $1 \leq k \leq n - 1$, the lines joining the vertices $b_{i-n-1}$ and $e_j$ will result for the pair $(\sigma_i, \sigma_j)$.

• For $i = 1, j = 2i + n$ where $1 \leq k \leq n - 1$, the lines joining the vertices $e_i$ and $b_{j-n-1}$ will result for the pair $(\sigma_i, \sigma_j)$.

• For $i = 2, j = 3i$, the lines joining the vertices $e_i$ and $e_j$ will result for the pair $(\sigma_i, \sigma_j)$.

• For $i = 3, j = 2i + 1$, the lines joining the vertices $b_{i-1}$ and $e_j$ will result for the pair $(\sigma_i, \sigma_j)$.

• For $i = 4, j = i + 1$, the lines joining the vertices $e_i$ and $b_{j-n-1}$ will result for the pair $(\sigma_i, \sigma_j)$.

• For $i = 6, j = i + 1$, the lines joining the vertices $e_i$ and $a_{j-n-1}$ will result for the pair $(\sigma_i, \sigma_j)$.

From the above cases, it is clear that the resulting pairs satisfies the adjacency condition of the coloring definition. This proves the above coloring is achromatic and it has maximum number of colors.

$\psi[G^{++}(L_n)] = 2n + 1$ for $n = 3$.

**Case 3** For $n \geq 4$

Define the color class $\sigma : V[G^{++}(L_n)] \to N$ in a proper coloring way to prove such coloring is achromatic.

$\sigma(a_1) = 1; \sigma(a_2) = 2n; \sigma(a_i) = i, 3 \leq i \leq n - 1; \sigma(a_n) = 2n + 2; \sigma(b_1) = 2n - 1; \sigma(b_i) + 2i, 2 \leq i \leq n; \sigma(e_i) = i, 1 \leq i \leq 3n - 2$.

• For some $i = 1$ and $i = 3, j = i + k$ where $1 \leq k \leq 3n - 2$ but $k \neq 6$, the edges connecting the vertices $a_i$ and $e_j$ will result for the pair $(\sigma_i, \sigma_j)$.

• For $i = 2, j = i + k$ where $2 \leq k \leq n$ but not exceeding $j \leq n + 2$, the edges connecting the vertices $e_i$ and $b_{j-2}$ will result for the pair $(\sigma_i, \sigma_j)$. 
• For every \( i = 4 \) and \( i = 5, j = i + k \) where \( 2 \leq k \leq 3n - 2 \) but \( k \neq 4 \), the edges connecting the vertices \( b_{i-2} \) and \( e_j \) will result for the pair \((\sigma_i, \sigma_j)\).

• For \( i = 6, j = i + k, 1 \leq k \leq 2n + 1 \), the edges connecting the vertices \( b_{i-2} \) and \( e_j \) will result for the pair \((\sigma_i, \sigma_j)\).

Clearly from the above coloring process we conclude that, every pair in the color class is adjacent by atleast one edge.

Thus the above said coloring is achromatic and it is maximal. Hence \( \psi[G^{++}(L_n)] = 2n + 2 \) for \( n \geq 4 \).

Thus by considering all the cases, the resulting pairs in the color class is adjacent by atleast one edge and this shows that it satisfies the definition of achromatic coloring.

Therefore \( \psi[G^{++}(L_n)] = \begin{cases} 2n & \text{if } n = 2 \\ 2n + 1 & \text{if } n = 3 \\ 2n + 2 & \text{if } n \geq 3 \end{cases} \)

6. Achromatic coloring of transformation graph \( G^{++} \) of Tadpole graph

6.1. Theorem

The achromatic number for \( G^{++}(T_{m,n}) = \begin{cases} m + n + 1 & \text{for } m = 3, n > 0 \\ m + n & \text{for } m \geq 4, n > 0 \end{cases} \)

Proof

Let \( G = T_{m,n} \). Consider this graph \( G \) is obtained by attaching any one vertex of a cycle graph \( C_n \) (having atleast 3 vertices) to a path graph \( P_n \) with a bridge, whose vertex set and edge set is defined as,

\[
V(G) = \{a_i : 1 \leq i \leq m\} \cup \{b_i : 1 \leq i \leq n\} \\
E(G) = \{e_i : 1 \leq i \leq m+n\}
\]

For the transformation graph \( G^{++} \) on \( G \), the vertex set of this transformation graph has a similar to both vertex set and edge set of \( G \).

ie., \( V[G^{++}(T_{m,n})] = \{a_i : 1 \leq i \leq m\} \cup \{b_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq m+n\} \)

Now define the vertex labeling \( \sigma : V[G^{++}(T_{m,n})] \rightarrow N \) according to a proper coloring way to show that this transformation graph receives maximum number of colors for an achromatic coloring.

Case 1 For \( m = 3, n > 0 \)

\( \sigma(e_i) = i, 1 \leq i \leq m+n \); \( \sigma(a_i) = m + n + 1, 1 \leq i \leq 3 \); \( \sigma(b_j) = m + j, 1 \leq j \leq n \).

An easy check shows that the above coloring process receives maximum of \( m+n+1 \) colors unless for \( m = 3 \) and \( n > 0 \). Otherwise some of pairs does not exist, which contradicts the definition of achromatic coloring.

Thus \( \psi[G^{++}(T_{m,n})] = m + n + 1, m = 3 \) and \( n > 0 \).

Case 2 For \( m \geq 4, n > 0 \)

\( \sigma(e_i) = i, 1 \leq i \leq m+n \); \( \sigma(a_i) = i, 1 \leq i \leq m \); \( \sigma(b_j) = m + j, 1 \leq j \leq n \).

From the above coloring process, we conclude that \( \psi[G^{++}(T_{m,n})] \leq m + n \), which will be proved by considering any pair \((\sigma_i, \sigma_j)\) whose adjacency provides the fact of maximal color class.

• For every \( i(1 \leq i \leq m-1) \) with \( j = i + k \) where \( 1 \leq k \leq m \) but not exceeding \( j \leq m + 1 \) and avoiding the pair \((a_1, e_m)\), then the edges connecting the vertices \( a_i \) and \( e_j \) will result for the pair \((\sigma_i, \sigma_j)\).

• For \( i = 1 \) and \( j = m \), the edges connecting the vertices \( e_i \) and \( b_j \) will result for the pair \((\sigma_i, \sigma_j)\).

• For \( i = m + n - 1 \) and \( j = n \), the edges connecting the vertices \( e_i \) and \( b_j \) will result for the pair \((\sigma_i, \sigma_j)\).
Thus from the above process, the set of pairs in the color class is adjacent by at least one edge. Hence this process leads to the fact of achromatic coloring of maximum number of colors.

Therefore \( \psi(G^{++}(T_{m,n})) = \begin{cases} m + n + 1 & \text{for } m = 3, n > 0 \\
 + n & \text{for } m \geq 4, n > 0 \end{cases} \)

7. Achromatic coloring of transformation graph \( G^{++} \) of Bistar graph

7.1. Theorem
The achromatic number for \( G^{++}(B_{m,n}) = m + n + 2 \) for \( m = n, m > n, m < n \).

8. Achromatic coloring of transformation graph \( G^{++} \) of Cycle graph

8.1. Theorem
The achromatic number for \( G^{++}(C_n) = \begin{cases} n & \text{for } 3 \leq n \leq 5 \\
 + 1 & \text{for } n \geq 6 \end{cases} \)

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