Lepton Flavor Violation in Models with \( A_4 \) and \( S_4 \) Flavor Symmetries

Gui-Jun Ding\(^\ast\), Jia-Feng Liu

Department of Modern Physics, University of Science and Technology of China, Hefei, Anhui 230026, China

ABSTRACT: The lepton flavor violation \( \mu \rightarrow e\gamma, \tau \rightarrow e\gamma, \tau \rightarrow \mu\gamma, \mu \rightarrow 3e, \tau \rightarrow 3e, \tau \rightarrow 3\mu \) and \( \mu - e \) conversion in Al and Ti are studied in both the Altarelli-Feruglio \( A_4 \) model and the \( S_4 \) model of Ding. The rates of these lepton flavor violation process for the inverted hierarchy neutrino mass spectrum are enhanced considerably by the next to leading order contributions. For both models, we find that the \( \mu - e \) conversion in Ti is within the precision of next generation experiments in all the parameter space considered, the \( \mu - e \) conversion in Al should be observable at least in a very significant part of the parameter space, whereas \( \tau \rightarrow e\gamma, \tau \rightarrow \mu\gamma, \tau \rightarrow 3e \) and \( \tau \rightarrow 3\mu \) are below the expected future sensitivity. The detectability of \( \mu \rightarrow e\gamma \) and \( \mu \rightarrow 3e \) in near future depends on the models and the neutrino mass spectrum. We suggest that a comprehensive consideration of the lepton flavor violation processes is important to test and distinguish both discrete flavor symmetry models.

KEYWORDS: Lepton Flavor Violation, Flavor symmetry, Seesaw Mechanism.

*dinggj@ustc.edu.cn
1. Introduction

In the past decades, one of the most striking developments in particle physics beyond the standard model (SM) is the experimental establishment of neutrino masses and the large mixing property, which is quite different from the small mixing in the quark sector. A global fit to the current neutrino oscillation data from the solar, atmospheric, reactor and long baseline neutrino experiments gives the following 3σ interval for the mixing parameters \(1.1\)

\[
\sin^2 \theta_{12} = 0.304^{+0.066}_{-0.054}, \quad \sin^2 \theta_{23} = 0.50^{+0.17}_{-0.13}, \quad \sin^2 \theta_{13} < 0.056 \\
\Delta m_{21}^2 = 7.65^{+0.69}_{-0.50} \times 10^{-5} \text{eV}^2, \quad \Delta m_{31}^2 = \pm 2.40^{+0.35}_{-0.33} \times 10^{-3} \text{eV}^2 \quad (1.1)
\]

For the overall scale of the neutrino and the mixing angle \(\theta_{13}\) currently only upper limit exists. But considerable progress is expected from future double beta decay [3] and reactor neutrino oscillation experiments [4, 5]. It is remarkable that such a mixing pattern is very close to the so-called tri-bimaximal (TB) mixing [6], which leads to \(\sin^2 \theta_{12} = 1/3\), \(\sin^2 \theta_{23} = 1/2\) and \(\sin^2 \theta_{13} = 0\). Thus it is an important subject to realize the TB mixing from the theoretical viewpoint.

Recently it is found that the TB mixing appears naturally in models with discrete flavor symmetry such as \(A_4\), \(T'\) and \(S_4\), and many models based on these flavor symmetries have been proposed [7, 8, 9, 10, 11, 12, 13, 14]. We note that the \(T'\) model usually has the same structure as that of the \(A_4\) model in the lepton sector. In these models, the flavor

| Contents                                                                 | Page |
|------------------------------------------------------------------------|------|
| 1. Introduction                                                        | 1    |
| 2. LFV in SUSY Seesaw                                                  | 3    |
| 3. Flavor models with \(A_4\) and \(S_4\) symmetries                   | 6    |
| \hspace{1cm} 3.1 AF \(A_4\) model                                      | 6    |
| \hspace{1cm} 3.2 \(S_4\) model of Ding                                 | 11   |
| 4. Predictions for LFV in AF model                                      | 15   |
| 5. Predictions for LFV in \(S_4\) model                                | 21   |
| 6. Conclusions and discussions                                         | 24   |
| 7. Acknowledgments                                                     | 26   |
symmetry is realized at a higher energy scale, the lepton fields transform nontrivially under the corresponding flavor group, and the flavor symmetry is spontaneously broken by a set of flavons along appropriate directions to provide a realistic description of the lepton masses and mixing angles. To be concrete, in this work we will concentrate on the Altarelli-Feruglio $A_4$ model \cite{9} and the $S_4$ model of Ding \cite{12}, which represent a typical class of flavor models. The structure of the model is tightly constrained by the flavor symmetry, as a result only few parameters are involved at leading order (LO), and the models are rather predictive. The common features of both models are as follows: they are supersymmetry (SUSY) models and the neutrino masses are generated via the type-I Seesaw mechanism. At LO the charged lepton mass matrix is diagonal, the light neutrino mass matrix is controlled only by two complex parameters, and it is diagonalized by the TB mixing matrix exactly. Despite of the tight constraints on the model parameters, and in particular the fact that there is only one real parameter left after taking into account $\Delta m^2_{21}$ and $\Delta m^2_{31}$ measured from the neutrino oscillation experiments, it has been shown that the $A_4$ models can reproduce the observed matter-anti matter asymmetry for natural values of the parameters in a quite satisfactory way \cite{15, 16, 17, 18, 19}.

Since this kind of $A_4$ and $S_4$ models are so attractive and predictive, it is highly desirable to verify or exclude these model experimentally. There is no doubt that the precise measurement of $\theta_{13}$ is a crucial test of the models. One of the implications of the observation of neutrino oscillation is the possibility of measurable branching ratio for charged lepton flavor violating (LFV) decays. While the LFV processes are still highly suppressed in the SM, predictions of the SUSY Seesaw for these rare decays are much more enhanced, as these processes are suppressed by the SUSY scale rather than the Planck scale. Furthermore, as different models obtain large neutrino mixing angles through different mechanisms, their predictions for the LFV decays can be very distinct. In the past years, LFV processes have been explored extensively from experiments, the upper bound of the branch ratio would be improved considerably in future, which could strongly constrain the new physics beyond the SM. Consequently, LFV rare decays may provide a way to test the $A_4$ and $S_4$ flavor symmetries.

In this work, we shall investigate the predictions for the various LFV processes in the Altarelli-Feruglio model and the Ding’s $S_4$ model, assuming the so-called minimal supergravity (mSUGRA) boundary conditions. Although our analysis is performed for the two specific models, it has generic features which are universal to the models based on the discrete flavor symmetry. We find that these models are so predictive that the branching ratios of the LFV processes are closely related to light neutrino mass, and there are some characteristic relations between various LFV branching ratios. Present work is different from the previous studies of the LFV decay branching ratio within the SUSY Seesaw framework. Since the low energy data allow to reconstruct only partially the high energy couplings of the theory, they generally inversed the Seesaw formula following the procedure of Casas and Ibarra \cite{20}, then they carried out the Monte Carlo studies by scanning the unknown right handed neutrino mass spectrum and the angles and phases of the inversion matrix in order to present scatter plots of the rare branching ratios \cite{21, 22, 23}. Another method is the top-down approach, where the neutrino Yukawa coupling matrix and the
right handed neutrino mass matrix is determined by a specific SUSY grand unified theory [24].

The paper is organized as follows: in the second section we briefly review the LFV within the framework of the SUSY Seesaw. Then we give a concise introduction to the Altarelli-Feruglio model and the Ding’s $S_4$ model in section 3. Our analysis of LFV decay branching ratios for these two interesting models is presented in section 4 and section 5 respectively. Finally we summarize our results and draw the conclusions.

2. LFV in SUSY Seesaw

SUSY is a well motivated possibility for new physics [25, 26], and the supersymmetric Seesaw mechanism can accommodate simultaneously tiny neutrino masses and large hierarchy between the electroweak scale and the high Seesaw scale without serious fine-tunings. In this framework the particle content of the minimal supersymmetric standard model (MSSM) is extended by including three gauge singlet right handed neutrino superfields $\nu^c_i$ ($i=1,2,3$). Imposing $R-$ parity conservation, the leptonic part of the superpotential is thus given by

$$W_{\text{lep}} = e^c_i Y_{eij} L_j H_d + \nu^c_{ij} Y_{\nu Lj} L_j H_u + \frac{1}{2} \nu^c_i M_{\nu ij} \nu_j^c$$  \hspace{1cm} (2.1)

where $L_i$ denotes the left handed lepton doublet, $e^c_i$ is the right handed charged lepton, $H_u$ and $H_d$ are the hypercharge $+1/2$ and $-1/2$ Higgs doublets respectively. $Y_e$ and $Y_\nu$ are the charged lepton and neutrino Yukawa coupling matrices respectively, and $M_\nu$ represents the $3 \times 3$ majorana mass matrix for the heavy right handed neutrinos. In general, the overall scale of $M_\nu$, which we shall denote by $M_{maj}$, is much larger than the electroweak scale. At energies below $M_{maj}$, the theory will be well described by the following effective superpotential

$$W_{\text{eff}} = e^c_i Y_{eij} L_j H_d + \frac{1}{2} M_{\nu ij} (L_i H_u)(L_j H_u)$$  \hspace{1cm} (2.2)

where the light neutrino effective mass matrix $M_\nu$ is given by the well-known Seesaw formula

$$M_\nu = -Y_\nu^T M_{\nu}^{-1} Y_\nu v_u^2 \hspace{1cm} (2.3)$$

where $v_u$ is the vacuum expectation value of the neutral component of the Higgs doublet $H_u$. In nature SUSY must be a broken symmetry, we thus introduce the soft SUSY breaking (SSB) Lagrangian $\mathcal{L}_{\text{soft}}$, which could be new sources of flavor violation. The leptonic part of $\mathcal{L}_{\text{soft}}$ has the following form,

$$-\mathcal{L}_{\text{soft}} = \begin{array}{c}
(m_{\tilde{L}}^2)_{ij} \tilde{L}_i \tilde{L}_j + (m_{\tilde{e}}^2)_{ij} \tilde{e}_i \tilde{e}_j + (m_{\tilde{\nu}}^2)_{ij} \tilde{\nu}_i \tilde{\nu}_j + \left( A_e \right)_{ij} \tilde{e}_i \tilde{L}_j H_d \\
+ \left( A_\nu \right)_{ij} \tilde{\nu}_i \tilde{L}_j H_u + h.c.
\end{array}$$  \hspace{1cm} (2.4)

where $\tilde{L}_i, \tilde{e}_i$ and $\tilde{\nu}_i$ are the supersymmetric partners of the left handed lepton doublet, right handed charged lepton and right handed neutrinos respectively. $m_{\tilde{L}}^2, m_{\tilde{e}}^2$ and $m_{\tilde{\nu}}^2$ are the corresponding soft mass matrices squared, $A_e$ and $A_\nu$ are the charged lepton and neutrino soft trilinear couplings respectively. In general, above SSB terms can have arbitrary flavor
structures which would induce unacceptably large flavor violating effects. The simplest solution to this SUSY flavor problem is to assume a flavor-blind SUSY breaking mediation mechanism, which will generate flavor universal SSB terms at some high scale. In the present work, we will restrict ourselves to the so-called minimal supergravity scenario (mSUGRA). It assumes that, at the GUT scale, the slepton mass matrices are diagonal and universal in flavor, and that the trilinear couplings are proportional to the Yukawa couplings

\[
(m_S^2)_{ij} = (m_L^2)_{ij} = (m_R^2)_{ij} = m_0^2 \delta_{ij}, \quad m_{H_u}^2 = m_{H_d}^2 = m_0^2
\]

\[
(A_e)_{ij} = A_0 Y_{ei j}, \quad (A_\nu)_{ij} = A_0 Y_{\nu ij}
\]

(2.5)

where \(m_0\) is the common scalar mass, and \(A_0\) is the common trilinear parameter. In addition, there are still three parameters characterizing the mSUGRA: the common gaugino mass \(M_{1/2}\), \(\tan \beta = v_u/v_d\) and the sign of the Higgs mixing parameters \(\mu\), where \(v_d\) is the vacuum expectation value of the neutral component of the Higgs doublet \(H_d\). However, these SSB terms will not be universal at the weak scale. Since the Yukawa coupling matrices \(Y_e\) and \(Y_\nu\) can not be simultaneously diagonalized, non-vanishing off-diagonal elements would be generated in the SSB mass-squared matrices and the trilinear couplings due to renormalization effect. The one-loop renormalization group equations for \(m_L^2, m_e^2\) and \(A_e\) are as follows [22, 27],

\[
\mu \frac{d m_L^2}{d \mu} = \frac{1}{16 \pi^2} \left[ (m_L^2 + 2 m_{H_d}^2) Y_e^\dagger Y_e + (m_L^2 + 2 m_{H_u}^2) Y_\nu^\dagger Y_\nu + 2 Y_e^\dagger m_L^2 Y_e + 2 Y_\nu^\dagger m_\nu^2 Y_\nu + 2 Y_e^\dagger m_e^2 Y_e \right. \\
+ (Y_e^\dagger Y_\nu + Y_\nu^\dagger Y_e) m_L^2 + 2 A_e^\dagger A_e + 2 A_\nu^\dagger A_\nu - 6 g_2^2 |M_2|^2 - \frac{6}{5} g_1^2 |M_1|^2 - \frac{3}{5} g_1^2 S \\
\left. - \frac{24}{5} g_1^2 |M_1|^2 + \frac{6}{5} g_1^2 S \right]
\]

\[
\mu \frac{d m_e^2}{d \mu} = \frac{1}{16 \pi^2} \left[ (2 m_L^2 + 4 m_{H_d}^2) Y_e^\dagger Y_e + 4 Y_e^\dagger m_L^2 Y_e + 2 Y_e^\dagger m_e^2 Y_e + 4 A_e A_e^\dagger \right. \\
- \frac{24}{5} g_1^2 |M_1|^2 + \frac{6}{5} g_1^2 S \\
\left. + Y_e^\dagger \left\{ \text{Tr}(3 Y_d Y_d^\dagger + Y_e Y_e^\dagger) + 5 Y_e^\dagger Y_e + Y_\nu^\dagger Y_\nu - 3 g_2^2 - \frac{9}{5} g_1^2 S \right\} \\
+ Y_e^\dagger \left\{ \text{Tr}(6 A_d A_d^\dagger + 2 A_e A_e^\dagger) + 4 Y_e^\dagger A_e + 2 Y_\nu^\dagger A_\nu + 6 g_2^2 M_2 + \frac{18}{5} g_1^2 M_1 \right\} \right] \\
\]

(2.6)

where \(\mu\) is the renormalization point, and we have defined

\[
S = m_{H_u}^2 - m_{H_d}^2 + \text{Tr}[m_Q^2 - m_L^2 - 2 m_u^2 + m_d^2 + m_e^2]
\]

(2.7)

Here \(m_Q^2, m_L^2\) and \(m_d^2\) are respectively the SSB mass-squared matrices for the supersymmetric partner of the left handed quark doublet, right handed up type quark and right handed down type quark. In the phenomenological studies it is convenient to work in the leptonic basis where both the charged lepton mass matrix and the mass matrix of the right handed neutrino are real and diagonal. In the leading-logarithmic approximation with universal boundary conditions Eq.(2.5), the off-diagonal elements of the above SSB slepton
mass matrices and trilinear coupling are given by [28, 29]

\[(m_L^2)_{ij} \simeq -\frac{1}{8\pi^2} (3m_0^2 + A_0^2) (\hat{Y}_e^T L \hat{Y}_\nu)_{ij}\]
\[(A_e)_{ij} \simeq -\frac{3A_0}{16\pi^2} (\hat{Y}_e \hat{Y}_\nu^T L \hat{Y}_\nu)_{ij}\]
\[(m_h^2)_{ij} \simeq 0\]  

where the hat denotes the basis in which the right handed neutrino and the charged lepton mass matrix are real and diagonal, and the factor \(L\) is defined as

\[L_{ij} = \log(\frac{M_G}{M_i}) \delta_{ij}\]  

Here \(M_i\) is the \(i\)th heavy right handed neutrino mass, and \(M_G\) is the GUT scale typically equal to \(2 \times 10^{16}\) GeV. Eq.(2.8) shows that, within the type-I Seesaw mechanism the off-diagonal elements of the right handed charged slepton matrix approximately don’t run in the leading-log approximation, and the running of the trilinear parameters \(A_e\) is suppressed by the charged lepton masses. At low energy, the flavor off-diagonal correction \((m_L^2)_{ij}\) induces the LFV processes such as \(l_i \rightarrow l_j \gamma\), \(l_i \rightarrow 3l_j\) and \(l_i - l_j\) conversion in nuclei, with \(i > j = 1, 2, 3\). A very useful tool to treat analytically the complicated expression for the branching ratio is the mass insertion approximation, where the small off-diagonal elements of the soft mass matrix are treated as insertions in the sfermion propagators in the loop. Then we obtain the compact expression for the branching ratio of the charged lepton LFV radiative decay

\[Br(l_i \rightarrow l_j \gamma) \simeq \frac{G_F}{\sqrt{2}m_s} \frac{\alpha}{\Gamma_{l_i}} [(m_L^2)_{ij}]^2 \tan^2 \beta\]  

where \(G_F\) is the Fermi coupling constant and \(m_s\) is the characteristic mass scale of the SUSY particle in the loop. It has been show that the expression (2.10) with \((m_L^2)_{ij}\) given by Eq.(2.8) represents an excellent approximation to the exact renormalization group result if one sets [27]

\[m_s^8 \simeq 0.5m_0^2m_{1/2}^2(m_0^2 + 0.6M_{1/2}^2)^2\]  

The LFV processes \(l_i \rightarrow l_j \gamma\), \(l_i \rightarrow 3l_j\) and \(l_i - l_j\) conversion in nuclei occur via \(\gamma\), \(Z\) and Higgs-penguins as well as squark/slepton box diagrams. It has been shown that the contribution from the \(\gamma\) - penguin diagrams are almost undistinguishable from the total rates in the universal mSUGRA scenario, although the contribution of Higgs-penguins becomes large in the region of large \(\tan \beta \sim 60\) and light Higgs boson mass\(\sim 100\) GeV, which is not allowed by the current experimental lower bounds on the MSSM particle masses [34]. In this approximation, the branching ratio for the trilepton decays is approximately given by [21, 30]

\[Br(l_i \rightarrow 3l_j) \simeq \frac{\alpha}{3\pi} (\log \frac{m_{l_i}}{m_{l_j}} - \frac{11}{4})Br(l_i \rightarrow l_j \gamma)\]  

For \(\mu - e\) conversion, the \(\gamma\)-penguins dominance gives

\[CR(\mu N \rightarrow eN) = \frac{\Gamma(\mu N \rightarrow eN)}{\Gamma_{cap}} \simeq \frac{\alpha^4m_\mu^2G_F^2}{12\pi^4\Gamma_{cap}} \frac{ZZ_{eff}(q^2)^2}{F(q^2)^2} Br(\mu \rightarrow e\gamma)\]  

- 5 -
Table 1: Present and upcoming experimental limits on various lepton flavor violation processes [34].

| Process          | Present      | Future       |
|------------------|--------------|--------------|
| $\text{BR}(\mu \to e\gamma)$ | $1.2 \times 10^{-11}$ | $10^{-13}$ |
| $\text{BR}(\tau \to \mu\gamma)$ | $4.5 \times 10^{-8}$ | $10^{-9}$ |
| $\text{BR}(\tau \to e\gamma)$ | $3.3 \times 10^{-8}$ | $10^{-9}$ |
| $\text{BR}(\mu \to eee)$ | $1.0 \times 10^{-12}$ | $10^{-14}$ |
| $\text{BR}(\tau \to \mu\mu\mu)$ | $3.6 \times 10^{-8}$ | $10^{-9}$ |
| $\text{CR}(\mu Ti \to e Ti)$ | $3.6 \times 10^{-12}$ | $10^{-18}$ |
| $\text{CR}(\mu Al \to e Al)$ | – | $10^{-16}$ |

where $\Gamma_{\text{cap}}$ is the measured total muon capture rate, $Z$ is the proton number of the nucleus, $Z_{\text{eff}}$ is the effective atomic charge obtained by averaging the muon wavefunction over the nuclear density, and $F(q^2)$ denotes the nuclear form factor at momentum transfer $q$ [31].

In this work, we will consider $\mu - e$ conversion in two nuclei $^{27}$Al and $^{48}$Ti. $^{27}$Al would be used by the proposed Mu2e experiment at Fermilab [32], $Ze_{\text{ff}} = 11.5$, the values of the relevant parameters are $F(q^2 \approx -m_{\mu}^2) = 0.64$ and $\Gamma_{\text{cap}} = 4.64079 \times 10^{-19}$ GeV. For the nucleus $^{48}$Ti used by the proposed PRIME experiment at J-PARC [33], $Ze_{\text{ff}} = 17.6$, $F(q^2 \approx -m_{\mu}^2) \approx 0.54$ and $\Gamma_{\text{cap}} = 1.70422 \times 10^{-18}$ GeV.

Experimental discovery of LFV is one of smoking gun signatures of new physics beyond the SM, thus many experiments have been developed to detect the rare LFV processes. The present and projected bounds on these processes are summarized in Table 1. In the following, we shall investigate in details the predictions for the rare processes $l_i \to l_j \gamma$, $l_i \to 3l_j$ and $l_i - l_j$ conversion in $^{27}$Al and $^{48}$Ti in the AF model and the $S_4$ model of Ding, and discuss the possibility of testing these discrete flavor symmetry models by LFV.

3. Flavor models with $A_4$ and $S_4$ symmetries

In this section, we will briefly present the Altarelli-Feruglio $A_4$ model [9] and the $S_4$ model of Ding [12], the next to leading order (NLO) corrections to the lepton mixing parameters will be discussed in detail, especially the allowed region of the reactor neutrino angle $\theta_{13}$ and the Jarlskog invariant $J$ in the lepton sector [35].

3.1 AF $A_4$ model

The AF model is a typical supersymmetric flavor model with $A_4$ symmetry, and the auxiliary symmetries are $Z_3$ and the Froggatt-Nielsen symmetry $U(1)_{FN}$. The $U(1)_{FN}$ is to reproduce the observed charged lepton mass hierarchies, and the $Z_3$ symmetry is to guarantee that the neutrino and charged lepton couple with different flavons at LO, so that the $A_4$ symmetry is broken down to $Z_3$ and $Z_2$ subgroups in the charged lepton and neutrino sectors respectively at LO, this misalignment of symmetry breaking is exactly the origin of
TB mixing. $A_4$ is the group of the even permutation of four objects, geometrically it is the invariant group of the regular tetrahedron in 3-dimensional space. It has three independent one-dimensional representations 1, 1' and 1'' and one three-dimensional representations 3. The multiplication rule for two triplet representations is the following

$$3 \otimes 3 = 1 \oplus 1' \oplus 1'' \oplus 3_S \oplus 3_A$$  \hspace{1cm} (3.1)$$

If we denote the two triplets as $(a_1, a_2, a_3)$ and $(b_1, b_2, b_3)$, the singlets and triplets contained in the product are given by

$$1 \sim a_1 b_1 + a_2 b_3 + a_3 b_2$$
$$1' \sim a_3 b_3 + a_1 b_2 + a_2 b_1$$
$$1'' \sim a_2 b_2 + a_1 b_1 + a_3 b_3$$
$$3_S \sim (2a_1 b_1 - a_2 b_3 - a_3 b_2, 2a_3 b_3 - a_1 b_2 - a_2 b_1, 2a_2 b_2 - a_1 b_3 - a_3 b_1)$$
$$3_A \sim (a_2 b_3 - a_3 b_2, a_1 b_2 - a_2 b_1, a_3 b_1 - a_1 b_3)$$  \hspace{1cm} (3.2)$$

The fields in the model and their transformation properties under the flavor group are listed in Table 2. At LO the flavon fields develop the following vacuum expectation values (VEVs),

$$\langle \varphi_T \rangle = (V_T, 0, 0), \quad \langle \theta \rangle = V_\theta,$$
$$\langle \xi \rangle = V_\xi, \quad \langle \bar{\xi} \rangle = 0,$$
$$\langle \varphi_S \rangle = (V_S, V_S, V_S)$$  \hspace{1cm} (3.3)$$

|   | $\ell$ | $e^c$ | $\mu^c$ | $\tau^c$ | $\nu^c$ | $h_{a,d}$ | $\varphi_T$ | $\varphi_S$ | $\xi$ | $\bar{\xi}$ | $\varphi_0^T$ | $\varphi_0^S$ | $\xi_0$ |
|---|------|------|--------|--------|--------|--------|-----------|-----------|------|---------|-------------|-------------|------|
| $A_4$ | 3    | 1    | $1''$  | 1'     | 3      | 1      | 3         | 3         | 1    | 1       | 3           | 3           | 1    |
| $Z_3$ | $\omega$ | $\omega^2$ | $\omega^2$ | $\omega^2$ | $1$    | 1      | $\omega^2$ | $\omega^2$ | 1    | 1       | $\omega^2$ | $\omega^2$ |      |
| $U(1)_{FN}$ | 0    | 2    | 1      | 0      | 0      | 0      | 0         | 0         | 0    | -1      | 0           | 0           | 0    |
| $U(1)_R$ | 1    | 1    | 1      | 1      | 0      | 0      | 0         | 0         | 0    | 2       | 2           | 2           |      |

Table 2: The fields of the $A_4 \times Z_3 \times U(1)_{FN}$ model and their representations, where $\omega$ is the third root of unity $\omega = e^{i2\pi/3}$.

At LO the neutrino masses are generated by the following superpotential

$$w_\nu = y_\nu (\nu^c \ell) b_u + (x_A \xi + \bar{x}_A \bar{\xi}) (\nu^c \nu^c) + x_B (\nu^c \nu^c \varphi_S)$$  \hspace{1cm} (3.4)$$

Here and in the following, we denote an invariant under $A_4$ by a parenthesis (...). After the electroweak and flavor symmetry breaking, we have

$$w_\nu = y_\nu (\nu^c_1 \nu_1 + \nu^c_2 \nu_2 + \nu^c_3 \nu_3) v_u + x_A V_\xi (\nu^c_1 \nu^c_2 + 2 \nu^c_1 \nu^c_3 + 2 \nu^c_2 \nu^c_3 + \nu^c_2 \nu^c_3 - \nu^c_1 \nu^c_3 - \nu^c_1 \nu^c_3)$$  \hspace{1cm} (3.5)$$
where \( v_u = \langle h_u \rangle \) is the VEV of the up type Higgs, and one can always set \( y_\nu \) to be real by a global phase transformation of the lepton doublet field. The first term in Eq.(3.5) contributes to the neutrino Dirac mass matrix

\[
M_D = y_\nu v_u \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}
\] (3.6)

The remaining terms lead to neutrino Majorana mass matrix

\[
M_M = \begin{pmatrix} A + \frac{2}{3}B & -\frac{1}{3}B & -\frac{1}{3}B \\ -\frac{1}{3}B & \frac{2}{3}B & A - \frac{1}{3}B \\ -\frac{1}{3}B & A - \frac{1}{3}B & \frac{2}{3}B \end{pmatrix}
\] (3.7)

where \( A = 2x_A V_\xi \) and \( B = 6x_B V_S \). The right handed neutrino mass matrix \( M_M \) can be diagonalized by a unitary matrix \( U \) as usual

\[
U^T M_M U = \text{diag}(|A + B|, |A|, |-A + B|)
\] (3.8)

The unitary matrix \( U \) is given by

\[
U = U_{TB} U_\phi
\] (3.9)

where \( U_{TB} \) exactly is the well-known TB mixing matrix

\[
U_{TB} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}
\] (3.10)

and \( U_\phi = \text{diag}(e^{-i\phi_1/2}, e^{-i\phi_2/2}, e^{-i\phi_3/2}) \) is a matrix of phase with \( \phi_1 = \text{arg}(A + B) \), \( \phi_2 = \text{arg}(A) \) and \( \phi_3 = \text{arg}(-A + B) \). The light neutrino mass matrix is given by the type-I Seesaw formula

\[
m_\nu = -M_D^T M_M^{-1} M_D
\] (3.11)

It is diagonalized by the unitary transformation \( U_\nu \), i.e.,

\[
U_\nu^T m_\nu U_\nu = \text{diag}(m_1, m_2, m_3)
\] (3.12)

where \( U_\nu = iU_{TB} U^*_\phi \), and \( m_{1,2,3} \) are the light neutrino masses

\[
m_1 = \frac{y_\nu^2 v_u^2}{|A + B|}, \quad m_2 = \frac{y_\nu^2 v_u^2}{|A|}, \quad m_3 = \frac{y_\nu^2 v_u^2}{|-A + B|}
\] (3.13)

It is obvious that the AF model is strongly constrained, the neutrino part of the Lagrangian depends on only three parameters: \( y_\nu \), \( A \) and \( B \), the latter two parameters are in general complex numbers. Concerning the light neutrino at low energy, only two parameters \( A/y_\nu^2 \) and \( B/y_\nu^2 \) are involved. Imposing the constraints of \( \Delta m_{21}^2 \) and \( \Delta m_{31}^2 \), measured from the neutrino oscillation experiments, only one real parameter is left, and it is usually chose to be the lightest neutrino mass. Detailed studies showed that the flavon VEVs \( V_T \), \( V_\xi \) and
$V_S$ should be approximately of the same order $\mathcal{O}(\lambda^2 \Lambda)$ [9, 16], where $\lambda$ is the Cabibbo angle, and $\Lambda$ is the cutoff scale of the theory. The neutrino mass spectrum can be either normal hierarchy (NH) or inverted hierarchy (IH), and the lightest neutrino mass is tightly constrained as follows [16, 17]

$$
\begin{align*}
0.0044 \text{ eV} & \leq m_1 \leq 0.0060 \text{ eV}, & \text{NH} \\
0.017 \text{ eV} & \leq m_3, & \text{IH}
\end{align*}
$$

The above LO predictions for the lepton masses and mixing parameters could be corrected by both the higher dimensional operators in the superpotential $w$, compatible with the symmetry of the model and the shift of the vacuum alignment. Including the NLO operators, the charged lepton masses are described by the following superpotential,

$$
\begin{align*}
w_{\ell} &= y_\ell \frac{1}{\Lambda} \bar{e}(\ell \varphi_T) \theta^2 h_d + y_{\mu} \frac{1}{\Lambda^2} \bar{\mu}(\ell \varphi_T) \theta h_d + y_{\tau} \frac{1}{\Lambda^2} \bar{\tau}(\ell \varphi_T) \theta h_d \nonumber \\
&+ y_{\ell 1} \frac{1}{\Lambda^2} \bar{e}(\ell \delta \varphi_T) \theta h_d + y_{\mu 1} \frac{1}{\Lambda^2} \bar{\mu}(\ell \delta \varphi_T) \theta h_d + y_{\tau 1} \frac{1}{\Lambda^2} \bar{\tau}(\ell \delta \varphi_T) \theta h_d
\end{align*}
$$

where $\delta \varphi_T$ denotes the shifted vacuum of the flavon $\varphi_T$. The first line represents the LO contributions, which leads to a diagonal charged lepton mass matrix, and the latter two lines are the NLO corrections. Since $\langle (\varphi_T \varphi_T)_3 S \rangle = (V_T^2, 0, 0)$ which has the same alignment directions as $\langle \varphi_T \rangle$, consequently the contributions of $y_{\ell 1}, y_{\mu 1}$ and $y_{\tau 1}$ terms can be absorbed into the LO results. Taking into account $\langle \delta \varphi_T \rangle \simeq (1, 1, 1) \delta V_T$ [9], the charged lepton mass matrix at NLO is given by

$$
m_{\ell} = \left( \begin{array}{ccc} y_e \frac{V_T^2}{\Lambda} & y_e \frac{V_T}{\Lambda} \frac{\delta V_T}{V_T} & y_e \frac{V_T^2}{\Lambda} \frac{\delta V_T}{V_T} \\
 y_{\mu} \frac{V_T^2}{\Lambda} & y_{\mu} \frac{V_T}{\Lambda} \frac{\delta V_T}{V_T} & y_{\mu} \frac{V_T^2}{\Lambda} \frac{\delta V_T}{V_T} \\
 y_{\tau} \frac{V_T^2}{\Lambda} & y_{\tau} \frac{V_T}{\Lambda} \frac{\delta V_T}{V_T} & y_{\tau} \frac{V_T^2}{\Lambda} \frac{\delta V_T}{V_T} \end{array} \right) v_d \tag{3.16}
$$

where we have redefined $V_T + \delta V_T \rightarrow V_T$. The charged lepton mass matrix $m_{\ell}$ can be diagonalized by the bi-unitary transformation, i.e., $U_{\ell}^\dagger m_{\ell} U_{\ell} = \text{diag}(m_e, m_\mu, m_\tau)$, and the unitary matrix $U_{\ell}$ approximately is

$$
U_{\ell} \simeq \left( \begin{array}{ccc} 1 & \frac{\delta V_T}{V_T} & \frac{\delta V_T}{V_T} \\
-\frac{\delta V_T}{V_T} & 1 & \frac{\delta V_T}{V_T} \\
\frac{\delta V_T}{V_T} & -\frac{\delta V_T}{V_T} & 1 \end{array} \right) \tag{3.17}
$$

The NLO corrections to the neutrino Dirac mass terms are as follows,

$$
\frac{y_A}{\Lambda} \langle (\nu^c \ell)_3 S \varphi_T \rangle h_u + \frac{y_B}{\Lambda} \langle (\nu^c \ell)_3 S \varphi_T \rangle h_u \tag{3.18}
$$

where $\langle ... \rangle_{3S(A)}$ denotes the $3_{S(A)}$ product of the two triplet representations, which can be read directly from the multiplication rule Eq.(3.2). Substituting the VEVs in Eq.(3.3), we obtain the NLO correction $\delta M_D$

$$
\delta M_D = \left( \begin{array}{ccc} 2y_A & 0 & 0 \\
 0 & 0 & -y_A + y_B \\
 0 & -y_A - y_B & 0 \end{array} \right) \frac{V_T}{\Lambda} v_u \tag{3.19}
$$
The Majorana mass matrix of the right handed neutrino are corrected by the following terms at NLO

\[
(x_A \delta \xi + \tilde{x}_A \delta \tilde{\xi})(\nu \nu^c) + x_B (\nu \nu^c \delta \phi_S) + \frac{x_A^T(\nu \nu^c \phi_T)}{\Lambda} + \frac{x_B^T(\nu \nu^c \phi_T)}{\Lambda} + \frac{x_C^T(\nu \nu^c)(\phi_T \phi_S)}{\Lambda} + \frac{x_D^T(\nu \nu^c)'(\phi_T \phi_S)'}{\Lambda} + \frac{x_E^T(\nu \nu^c)''(\phi_T \phi_S)''}{\Lambda} + \frac{x_F^T((\nu \nu^c)_3S(\phi_T \phi_S)_3s)}{\Lambda} \]  

(3.20)

where \(\delta \xi\), \(\delta \tilde{\xi}\) and \(\delta \phi_S\) denote the shifted vacuum of \(\xi\), \(\tilde{\xi}\) and \(\phi_S\) respectively. Absorbing the above corrections partly into the LO results, then \(\delta M_M\) can be parameterized by four independent parameters as follows

\[
\delta M_M = \left( \begin{array}{ccc}
2\tilde{x}_D & \tilde{x}_B - \tilde{x}_E & \tilde{x}_C \\
\tilde{x}_B - \tilde{x}_E & \tilde{x}_C & -\tilde{x}_D \\
\tilde{x}_C & -\tilde{x}_D & \tilde{x}_B + 2\tilde{x}_E \\
\end{array} \right) \frac{V_T}{\Lambda} B
\]  

(3.21)

Therefore at NLO the light neutrino mass matrix is given by \(m_\nu = -(M_D^2 + \delta M_D^2)(M_M + \delta M_M)^{-1}(M_D + \delta M_D)\). Performing the standard perturbation diagonalization, \(m_\nu\) can be diagonalized by the unitary transformation \(U'_\nu\) as \(U'_\nu m_\nu U'_\nu = \text{diag}(m_{1'}, m_{2'}, m_{3'})\). To first order in the expansion parameter \(V_T/\Lambda \equiv \varepsilon\), the light neutrino masses \(m_{1',2,3}\) are

\[
m_{1'} = -\frac{y_{\nu}^2 v_u^2}{A + B} - \frac{y_{\nu} v_u^2}{2(A + B)} \left[ 4y_A A + 4y_A B + (\tilde{x}_B + \tilde{x}_C - 2\tilde{x}_D - 2\tilde{x}_E)y_u B \right] \varepsilon
\]

\[
m_{2'} = -\frac{y_{\nu}^2 v_u^2}{A} + \frac{y_{\nu}^2 v_u^2}{A^2} (\tilde{x}_B + \tilde{x}_C) \varepsilon
\]

\[
m_{3'} = -\frac{y_{\nu}^2 v_u^2}{A - B} + \frac{y_{\nu} v_u^2}{2(A - B)} \left[ -4y_A A + 4y_A B + (\tilde{x}_B + \tilde{x}_C + 2\tilde{x}_D + 2\tilde{x}_E)y_u B \right] \varepsilon
\]

(3.22)

The mixing matrix \(U'_\nu\) is given by

\[
U'_\nu = \left( \begin{array}{ccc}
\sqrt{\frac{2}{3}} - \frac{1}{\sqrt{3}} s_{12} & \frac{1}{\sqrt{3}} + \sqrt{\frac{2}{3}} s_{12} & \frac{1}{\sqrt{3}} + \sqrt{\frac{2}{3}} s_{13} \\
-\frac{1}{\sqrt{3}} s_{12} - \frac{1}{\sqrt{3}} s_{13} & \frac{1}{\sqrt{3}} + \sqrt{\frac{2}{3}} s_{12} - \frac{1}{\sqrt{3}} s_{23} & \frac{1}{\sqrt{3}} + \sqrt{\frac{2}{3}} s_{13} - \frac{1}{\sqrt{3}} s_{23} \\
-\frac{1}{\sqrt{3}} s_{12} + \frac{1}{\sqrt{3}} s_{13} & \frac{1}{\sqrt{3}} + \sqrt{\frac{2}{3}} s_{13} - \frac{1}{\sqrt{3}} s_{23} & \frac{1}{\sqrt{3}} + \sqrt{\frac{2}{3}} s_{13} + \frac{1}{\sqrt{3}} s_{23} \\
\end{array} \right)
\]  

(3.23)

The small parameters \(s_{12}, s_{13}\) and \(s_{23}\) are

\[
s_{12} = \frac{1}{\sqrt{2}(|B|^2 + AB^* + A^*B)y_\nu} \left\{ A^*[4y_A A + 2y_B B + (\tilde{x}_D + \tilde{x}_E)y_u B] \varepsilon + (A + B)[4y_A^* A^* + 2y_B^* B^* + (-2\tilde{x}_D + \tilde{x}_E)y_u B^*] \varepsilon \right\} + 2y_B^* B^* + (-2\tilde{x}_D + \tilde{x}_E)y_u B^* \varepsilon
\]

\[
s_{13} = \frac{1}{4\sqrt{3}(AB^* + A^*B)y_\nu} \left\{ -B(A^* - B^*)[4y_B + 3(\tilde{x}_B - \tilde{x}_C)y_u] \varepsilon + B^*(A + B)[4y_B^* + 3(\tilde{x}_B^* - \tilde{x}_C^*)y_u] \varepsilon \right\}
\]

\[
s_{23} = \frac{1}{\sqrt{6}(AB^* + A^*B - |B|^2)y_\nu} \left\{ B(A^* - B^*)[2y_B + 3\tilde{x}_E y_u] \varepsilon - B^* A(2y_B^* + 3\tilde{x}_E^* y_u) \varepsilon \right\}
\]  

(3.24)
Taking into account the NLO corrections to the charged lepton mass matrix, the leptonic PMNS matrix becomes \( U_{PMNS} = U^\dagger U_{\nu} \), consequently the parameters of the lepton mixing matrix are modified as

\[
|U_{e3}| = \left| \frac{1}{6\sqrt{2}(AB^* + A*B)y_{\nu}} \left\{ -B(A^* - B^*)[4y_B + 3(\bar{x}_B - \bar{x}_C)y_{\nu}]\varepsilon + B^*(A + B)[4y_B^* + 3(\bar{x}_B - \bar{x}_C)\varepsilon^*] \\
-\bar{x}_C^* y_{\nu}\varepsilon^* \right\} \right| + \frac{1}{3\sqrt{2}(AB^* + A*B - |B|^2)y_{\nu}} \left\{ B(A^* - B^*)(2y_B + 3\bar{x}_E y_{\nu})\varepsilon - B^* A(2y_B^* + 3\bar{x}_E y_{\nu})\varepsilon^* \right\} \\
\sin^2 \theta_{21} = \frac{1}{2} + \frac{1}{2} \left[ \frac{\delta V_T}{V_T} \right]^* - \frac{|B|^2}{12(AB^* + A*B)y_{\nu}} \left\{ 4y_B + 3(\bar{x}_B - \bar{x}_C)y_{\nu}\varepsilon + [4y_B^* + 3(\bar{x}_B - \bar{x}_C)y_{\nu}]\varepsilon^* \right\} \\
\sin^2 \theta_{23} = \frac{1}{2} + \frac{1}{2} \left[ \frac{\delta V_T}{V_T} \right]^* - \frac{|B|^2}{6(AB^* + A*B - |B|^2)y_{\nu}} \left\{ 2y_B + 3\bar{x}_E y_{\nu}\varepsilon + (2y_B^* + 3\bar{x}_E y_{\nu})\varepsilon^* \right\}
\]

It is obvious that all the three mixing angles receive corrections of the same order of magnitude about \( \lambda^2 \), which is allowed by the current neutrino oscillation data in Eq.(2.1). In particular, the reactor mixing angle \( \theta_{13} \) becomes of order \( \lambda^2 \) after including the NLO corrections. In order to see clearly the behavior of the mixing parameters after including the subleading contributions, we perform a numerical analysis. The LO and NLO parameters \( y_{\nu}, y_A, y_B, \bar{x}_B, \bar{x}_C, \bar{x}_D \) and \( \bar{x}_E \) are treated as random complex number with absolute value between 0 and 2, and \( A \) and \( B \) in the right handed neutrino mass matrix Eq.(3.7) are random complex number with absolute value in the range of \( 10^{12} - 10^{16} \) GeV, and the parameters \( \delta V_T/V_T \) and \( V_T/A \) have been fixed at the indicative value of 0.04. Furthermore, we require the oscillation parameters to lie in their allowed 3\( \sigma \) range given in Eq.(2.1). The results are presented in Fig. 1, where the Jarlskog invariant \( J \) of CP violation in neutrino oscillation is defined as

\[
J = Im[(U_{PMNS})_{e1}(U_{PMNS})_{\mu2}(U_{PMNS})_{\mu3}^*(U_{PMNS})_{e3}^*]
\]

As we can see from this figure, in AF model \( \theta_{13} \) should be much smaller than the present upper bound, and it prefer to lie in the 1\( \sigma \) range. Our analytical estimates are confirmed.

### 3.2 \( S_4 \) model of Ding

The total flavor symmetry of this model is \( S_4 \times Z_3 \times Z_4 \) [12], where the auxiliary symmetry \( Z_3 \times Z_4 \) is crucial to eliminate unwanted couplings and to insure the needed vacuum alignment. It is remarkable that the realistic pattern of fermion masses and flavor mixing in both the lepton and quark sector have been reproduced in this model, and the mass hierarchies are controlled by the spontaneous breaking of the flavor symmetry instead of the FN mechanism. The matter fields and the flavons of the model and their classification under the flavor symmetry are shown in Table 3, where the quark fields have been omitted. In this section, the convention for the group theory of \( S_4 \) is the same as that in Ref.[12].
Taking into account the vacuum alignment in Eq. (3.29), explicit calculation demonstrated that the LO vacuum alignment is as follows \[ \omega = e^{i \frac{2\pi}{3}} = (-1 + i\sqrt{3})/2. \]

The transformation rules of the matter fields and the flavons under the symmetry groups \( S_4, Z_3 \) and \( Z_4 \) in the \( S_4 \) model of Ref. [12]. \( \omega \) is the third root of unity, i.e. \( \omega = e^{i \frac{2\pi}{3}} = (-1 + i\sqrt{3})/2. \)

Explicit calculation demonstrated that the LO vacuum alignment is as follows \[ \begin{align*}
\langle \varphi \rangle &= (0, V_\varphi, 0), \quad \langle \chi \rangle = (0, V_\chi, 0) \quad \langle \zeta \rangle = V_\zeta, \\
\langle \eta \rangle &= (V_\eta, V_\eta), \quad \langle \phi \rangle = (V_\phi, V_\phi, V_\phi), \quad \langle \Delta \rangle = V_\Delta.
\end{align*} \] (3.27)

In this model, the charged lepton masses are described by the following superpotential

\[
\begin{align*}
w_\ell & = \frac{ye_1}{\Lambda^3} e^{c_1(\varphi)}_{11}(\varphi)_{11} h_\ell + \frac{ye_2}{\Lambda^3} e^{c_2(\varphi)}_{21}(\varphi)_{21} h_d + \frac{ye_3}{\Lambda^3} e^{c_3(\varphi)}_{31}(\varphi)_{31} h_d \\
& + \frac{ye_7}{\Lambda^3} e^{c_7(\varphi)}_{71}(\varphi)_{71} h_d + \frac{ye_8}{\Lambda^3} e^{c_8(\varphi)}_{81}(\varphi)_{81} h_d + \frac{ye_9}{\Lambda^3} e^{c_9(\varphi)}_{91}(\varphi)_{91} h_d \\
& + \frac{ye_{10}}{\Lambda^3} e^{c_{10}(\varphi)}_{101}(\varphi)_{101} h_d + \frac{\mu_\ell}{2\Lambda^3} e^{c_{11}(\varphi)}_{11} h_\ell + \frac{\mu_\Delta}{\Lambda^3} e^{c_{12}(\varphi)}_{12} h_d + \ldots
\end{align*}
\] (3.28)

Taking into account the vacuum alignment in Eq.(3.27), we find that the charged lepton mass matrix is diagonal at LO, and the charged lepton masses are given by

\[
\begin{align*}
m_\ell &= \frac{V_\ell}{\Lambda^3} v_d, \quad m_\mu &= \frac{V_\mu}{\Lambda^2} v_d, \quad m_\tau &= \frac{V_\tau}{\Lambda} v_d
\end{align*}
\] (3.29)

| \( \ell \) | \( e^c \) | \( \mu^c \) | \( \tau^c \) | \( \nu^c \) | \( h_{u,d} \) | \( \varphi \) | \( \chi \) | \( \zeta \) | \( \eta \) | \( \phi \) | \( \Delta \) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| \( S_4 \) | 3_1 | 1_1 | 1_2 | 1_1 | 3_1 | 1_1 | 3_1 | 3_2 | 1_2 | 2_1 | 3_1 | 1_2 |
| \( Z_3 \) | \( \omega \) | \( \omega^2 \) | \( \omega^3 \) | \( \omega^2 \) | 1_1 | 1_1 | 1_1 | \( \omega^2 \) | \( \omega^2 \) | \( \omega^2 \) |
| \( Z_4 \) | 1_1 | i | -i | -i | 1_1 | 1_1 | i | i | i | 1_1 | 1_1 | -1 |

Table 3: The transformation rules of the matter fields and the flavons under the symmetry groups \( S_4, Z_3 \) and \( Z_4 \) in the \( S_4 \) model of Ref. [12]. \( \omega \) is the third root of unity, i.e. \( \omega = e^{i \frac{2\pi}{3}} = (-1 + i\sqrt{3})/2. \)
where \( y_e \) is the linear combination of \( y_{e1} \) (\( i = 1\)–10). At LO the superpotential contributing to the neutrino mass is as follows
\[
w_\nu = \frac{y_{e1}}{\Lambda}((\nu^c \ell) e \eta)_1, h_u + \frac{y_{e2}}{\Lambda}((\nu^c \ell)_3, \phi)_1 h_u + \frac{1}{2} M (\nu^c \nu^c)_1
\]
(3.30)

The neutrino Dirac and Majorana mass matrices can be straightforwardly read out as
\[
M_D = \begin{pmatrix} 2b & a - b & a - b \\ a - b & a + 2b & -b \\ a - b & -b & a + 2b \end{pmatrix} v_u, \quad M_M = \begin{pmatrix} M & 0 & 0 \\ 0 & 0 & M \\ 0 & M & 0 \end{pmatrix}
\]
(3.31)

where we denote \( a = y_{e1} V_{\eta}/\Lambda \) and \( b = y_{e2} V_{\phi}/\Lambda \), both of them should be of order \( \lambda^2 c \). The heavy right handed neutrino mass matrix \( M_M \) can be diagonalized as follows
\[
U^T M_M U = \text{diag}(|M|, |M|, |M|)
\]
(3.32)

It is obvious that the right handed neutrinos are exactly degenerate, and the unitary matrix \( U \) is
\[
U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} U_\alpha
\]
(3.33)

Here \( U_\alpha = \text{diag}(e^{-i\alpha/2}, e^{-i\alpha/2}, e^{-i\alpha/2}) \), and \( \alpha = \text{arg}(M) \) is the complex phase of \( M \). The light neutrino mass matrix is given by the Seesaw relation \( m_\nu = -M_D^T M_M^{-1} M_D \), which is exactly diagonalized by the TB mixing matrix
\[
U^T_\nu m_\nu U_\nu = \text{diag}(m_1, m_2, m_3)
\]
(3.34)

The unitary matrix \( U_\nu \) is written as
\[
U_\nu = U_{TB} \text{diag}(e^{-i\alpha_1/2}, e^{-i\alpha_2/2}, e^{-i\alpha_3/2})
\]
(3.35)

where the phase \( \alpha_{1,2,3} \) and the light neutrino masses \( m_{1,2,3} \) are given by
\[
\alpha_1 = \text{arg}(-(a - 3b)^2/M), \quad \alpha_2 = \text{arg}(-4a^2/M), \quad \alpha_3 = \text{arg}((a + 3b)^2/M)
\]
\[
m_1 = |(a - 3b)^2| v_u^2 \left| \frac{v^2}{M} \right|, \quad m_2 = 4|a^2| v_u^2 \left| \frac{v^2}{M} \right|, \quad m_3 = |(a + 3b)^2| v_u^2 \left| \frac{v^2}{M} \right|
\]
(3.36)

Similar to the AF model, this model is very predictive, there are only three independent parameter \( a, b \) and \( M \) in the neutrino sector. The neutrino mass spectrum can be normal hierarchy or inverted hierarchy as well. Taking into account the mass square difference \( \Delta m^2_{21} \) and \( \Delta m^2_{31} \) measured in the neutrino oscillation experiments, we obtain the following limit for the lightest neutrino mass
\[
m_1 \geq 0.011 \text{ eV}, \quad \text{NH}
\]
\[
m_3 > 0.0 \text{ eV}, \quad \text{IH}
\]
(3.37)

In the following, we briefly discuss the NLO corrections, please read Ref.[12] for details. After including the higher dimensional operators allowed by the symmetry at NLO, the
off-diagonal entries of the charged lepton mass matrix become non-zero and of the order of the diagonal term in each row multiplied by $\epsilon$, which parameterizes the ratio $VEV/\Lambda$ with order $O(\lambda_2^2)$. Therefore the charged lepton mass matrix can generally be written as

$$m_\ell = \begin{pmatrix} m_{11}^\ell & m_{12}^\ell & m_{13}^\ell \\ m_{21}^\ell & m_{22}^\ell & m_{23}^\ell \\ m_{31}^\ell & m_{32}^\ell & m_{33}^\ell \end{pmatrix} \epsilon v_d$$ (3.38)

where the coefficients $m_{ij}^\ell (i, j = 1, 2, 3)$ are order one unspecified constants. The matrix $m_\ell^d m_\ell$ is diagonalized by setting $\ell$ to $U_\ell \ell$, where $U_\ell$ approximately is given by

$$U_\ell \simeq \begin{pmatrix} 1 & \left( \frac{m_{12}^\ell}{m_{22}^\ell} \right)^* \left( \frac{m_{13}^\ell}{m_{33}^\ell} \right)^* \\ -\frac{m_{12}^\ell}{m_{22}^\ell} & 1 \\ -\frac{m_{13}^\ell}{m_{33}^\ell} & -\frac{m_{13}^\ell}{m_{33}^\ell} \end{pmatrix}$$ (3.39)

The NLO correction to the Majorana masses of the right handed neutrino arises at order $1/\Lambda$, the corresponding higher dimensional operator is $(\nu^c \nu^c)_{11} \zeta^2/\Lambda$, whose contribution can be completely absorbed into the redefinition of the mass parameter $M$. Consequently the right handed neutrinos are highly degenerate, and we only need to consider the NLO corrections to the neutrino Dirac couplings as follows

$$\frac{y_{11}}{\Lambda} (\nu^c \ell \delta \eta)_{11} h_u + \frac{y_{22}}{\Lambda} (\nu^c \ell \delta \phi)_{11} h_u + \frac{x_{11}}{\Lambda^2} (\nu^c \ell \eta)_{12} \zeta h_u + \frac{x_{22}}{\Lambda^2} (\nu^c \ell \phi)_{12} \zeta h_u$$ (3.40)

A part of the above corrections can be absorbed into the LO results, then the NLO corrections to $M_D$ can be parameterized as

$$\delta M_D = \begin{pmatrix} 0 & b \tilde{e} & \tilde{a} \epsilon - \tilde{b} \epsilon \\ -\tilde{b} \epsilon & \tilde{a} \epsilon & \tilde{b} \epsilon \\ \tilde{a} \epsilon + \tilde{b} \epsilon & -\tilde{b} \epsilon & 0 \end{pmatrix} v_u$$ (3.41)

where the magnitudes of $\tilde{a}$ and $\tilde{b}$ are expected to be of the same order as those of $a$ and $b$. In first order of $\epsilon$, the parameters of the lepton mixing matrix are modified as

$$|U_{\ell3}| = \frac{1}{\sqrt{2}} \frac{1}{6|a|^2 + 9|b|^2} \left[ (ab^* + a^* b) (ab^* + a^* b) + (a + 3b)^2 (a^* \tilde{a}^* + 6b^* \tilde{b}^*) \epsilon - (a^* - 3b^*)^2 (a \tilde{a} + 6b \tilde{b}) \epsilon \right]$$

$$+ \left( \frac{m_{21}^\ell}{m_{22}^\ell} \right)^* - \left( \frac{m_{21}^\ell}{m_{22}^\ell} \right)^*$$

$$\sin^2 \theta_{12} = \frac{1}{3} \left[ 1 - \frac{m_{21}^\ell}{m_{22}^\ell} \epsilon - \frac{m_{21}^\ell}{m_{23}^\ell} \epsilon - \left( \frac{m_{21}^\ell}{m_{22}^\ell} \right)^* - \left( \frac{m_{21}^\ell}{m_{23}^\ell} \right)^* \right]$$

$$\sin^2 \theta_{23} = \frac{1}{2} + \frac{1}{2 (|a|^2 + 9|b|^2) (ab^* + a^* b)} \left[ ab (a^* \tilde{a}^* + 6b^* \tilde{b}^*) + a^* b (a \tilde{a} + 6b \tilde{b}) \epsilon \right]$$

$$+ \frac{1}{2} \left( \frac{m_{31}^\ell}{m_{33}^\ell} \epsilon + \left( \frac{m_{31}^\ell}{m_{33}^\ell} \epsilon \right)^* \right)$$ (3.42)

As we can see, NLO contributions introduce corrections of order $\lambda_2^2$ to all mixing angles, and the reactor angle $\theta_{13}$ becomes non-zero. Then we turn to a numerical analysis. Since
\(a, b, \tilde{a}\) and \(\tilde{b}\) are expected to be of order \(\lambda^2\), they are treated as complex numbers with absolute value between 0.01 and 0.1, the parameters \(m^2_{11}/m^2_{22}\), \(m^2_{31}/m^2_{33}\) and \(m^2_{32}/m^2_{33}\) in the charged lepton mixing matrix \(U_e\) are taken to be complex numbers with absolute value in the range of 0-2, the heavy neutrino mass \(|M|\) is allowed to vary from \(10^{11}\) GeV to \(10^{14}\) GeV, and the expansion parameter \(\epsilon\) is set to the indicative value 0.04. The correlation for \(\sin^2 \theta_{23} - \sin^2 \theta_{13}\) and \(J - \sin^2 \theta_{13}\) is showed in Fig. 2. It is obvious that rather small \(\theta_{13}\) is favored, which is consistent with our theoretical analysis.

![Figure 2: Scatter plot of sin^2 \theta_{23} and the Jarlskog invariant J against sin^2 \theta_{13} for both normal hierarchy and inverted hierarchy neutrino mass spectrum in the S_4 model of Ref.[12].](image)

4. Predictions for LFV in AF model

We first study the branch ratios of the rare LFV processes at LO of the model. Since the number of independent parameters is rather small in this case, these branching ratios are closely related to the neutrino oscillation parameters. As has been shown in section 3.1, The charged lepton mass matrix is diagonal at LO, and the right handed neutrino mass matrix \(M_R\) is diagonalized by the unitary matrix \(U = U_{TB}U_{\phi}\), consequently in the base where both the right handed neutrino and charged lepton mass matrices are real and diagonal, the neutrino Yukawa coupling matrix is given by

\[
\begin{align*}
\hat{Y}_\nu &= U^T \hat{Y}_\nu = y_\nu \\
&= \left( \begin{array}{ccc}
\frac{\sqrt{2}}{3} e^{-i\phi_1/2} & -\frac{1}{\sqrt{6}} e^{-i\phi_1/2} & -\frac{1}{\sqrt{6}} e^{-i\phi_2/2} \\
\frac{1}{\sqrt{3}} e^{-i\phi_2/2} & \frac{1}{\sqrt{2}} e^{-i\phi_2/2} & \frac{1}{\sqrt{2}} e^{-i\phi_2/2} \\
0 & -\frac{1}{\sqrt{2}} e^{-i\phi_3/2} & \frac{1}{\sqrt{2}} e^{-i\phi_3/2}
\end{array} \right)
\end{align*}
\]

(4.1)

As a result we can straightforwardly obtain the off-diagonal elements of \(\hat{Y}_\nu^T \hat{L} \hat{Y}_\nu\), which is
directly related to the LFV branching ratios via \eqref{eq:2.8}, \eqref{eq:2.10}, \eqref{eq:2.12} and \eqref{eq:2.13}

\begin{align}
(\hat{Y}_L^\dagger \hat{L} \hat{Y}_H)_{12} &= (\hat{Y}_L^\dagger \hat{L} \hat{Y}_H)_{21} = \frac{1}{3} y_{\nu}^2 \ln \frac{m_2}{m_1} \\
(\hat{Y}_L^\dagger \hat{L} \hat{Y}_H)_{13} &= (\hat{Y}_L^\dagger \hat{L} \hat{Y}_H)_{31} = \frac{1}{3} y_{\nu}^2 \ln \frac{m_2}{m_1} \\
(\hat{Y}_L^\dagger \hat{L} \hat{Y}_H)_{23} &= (\hat{Y}_L^\dagger \hat{L} \hat{Y}_H)_{32} = \frac{1}{6} y_{\nu}^2 \ln \frac{m_1 m_2^2}{m_3^2} \tag{4.2}
\end{align}

Obviously the LFV processes are tightly related to the light neutrino mass, and the branching ratio is proportional to \( y_{\nu}^2 \). We notice that the branching ratio is independent of the grand unification scale \( M_G \), and it tends to zero if the neutrino mass spectrum is degenerate. Whereas the branching ratio could become considerable large if the neutrino mass spectrum is strongly normal hierarchy or strongly inverted hierarchy. It is remarkable that we have \( (\hat{Y}_L^\dagger \hat{L} \hat{Y}_H)_{12} = (\hat{Y}_L^\dagger \hat{L} \hat{Y}_H)_{13} \), which is related to the \( \mu - \tau \) symmetry of the light neutrino mass matrix. As a result, the ratio of the branching ratios for \( \tau \rightarrow e\gamma \) and \( \mu \rightarrow e\gamma \) approximately is

\[ \frac{\text{Br}(\tau \rightarrow e\gamma)}{\text{Br}(\mu \rightarrow e\gamma)} \approx \frac{\text{Br}(\tau \rightarrow e\bar{\nu}_e \nu_\tau)}{\text{Br}(\mu \rightarrow e\bar{\nu}_e \nu_\mu)} \approx 17.84\% \tag{4.3} \]

This ratio is almost independent of the SUSY breaking parameters. Given the present experimental bound \( \text{Br}(\mu \rightarrow e\gamma) < 1.2 \times 10^{-11} \) \cite{34}, this result implies that \( \tau \rightarrow e\gamma \) has rates much below the present and expected future sensitivity \cite{34}. We note that this ratio would be corrected drastically by the NLO contributions. For the IH neutrino mass spectrum, we have

\[ m_1 = \sqrt{m_3^2 + |\Delta m_{31}^2|}, \quad m_2 = \sqrt{m_3^2 + |\Delta m_{31}^2| + \Delta m_{21}^2}, \]

\[ \ln \frac{m_2}{m_1} = \frac{1}{2} \ln \left( \frac{m_3^2 + |\Delta m_{31}^2| + \Delta m_{21}^2}{m_3^2 + |\Delta m_{31}^2|} \right) \approx \frac{\Delta m_{21}^2}{2(m_3^2 + |\Delta m_{31}^2|)} \tag{4.4} \]

As a result, in the case of IH spectrum, the LFV processes \( \mu \rightarrow e\gamma, \tau \rightarrow e\gamma, \mu \rightarrow 3e, \tau \rightarrow 3e \) and \( \mu - e \) conversion in nuclein generally have rather small branching ratios at LO, which should be smaller than those of the NH case. While for \( \tau \rightarrow \mu\gamma \) and \( \tau \rightarrow 3\mu \), the branching ratios for NH and IH spectrum should be approximately of the same order.

The analytical results in \eqref{eq:4.2} allow us to estimate the branching ratios of the LFV processes via the formula \eqref{eq:2.10}-\eqref{eq:2.13}. For definiteness, we will present our results only for the well-known mSUGRA point SPS3 \cite{36}, which is taken as the reference example. The SPS3 point is in the co-annihilation region for the SUSY dark matter, and the values of the universal SSB parameters are as follows

\[ m_0 = 90 \text{ GeV}, \quad M_{1/2} = 400 \text{ GeV}, \quad A_0 = 0 \text{ GeV}, \quad \tan \beta = 10 \tag{4.5} \]

Since no suppression is expected for parameter that is unrelated to the breaking of the flavor symmetry, the coupling \( y_{\nu} \) should be of order 1. We set \( y_{\nu} \) to be equal to 0.5 in the following numerical analysis, the corresponding predictions for \( \text{Br}(\ell_i \rightarrow \ell_j \gamma), \text{Br}(\ell_i \rightarrow 3\ell_j), CR(\mu - e, Al) \) and \( CR(\mu - e, Ti) \) are shown in Fig.3 and Fig.4 for NH and IH spectrum.
respectively. The uncertainties of both $\Delta m_{21}^2$ and $\Delta m_{31}^2$ are taken into account, they are allowed to vary within their $3\sigma$ allowed range in Eq.(2.1). We can clearly see that the LFV processes $\tau \rightarrow e\gamma$, $\tau \rightarrow \mu\gamma$, $\tau \rightarrow 3e$ and $\tau \rightarrow 3\mu$ are below the present and future experimental precision at $B$ factory for both the NH and IH spectrum. As has been stressed above, the predictions of $Br(\mu \rightarrow e\gamma)$, $Br(\tau \rightarrow e\gamma)$, $Br(\mu \rightarrow 3e)$, $Br(\tau \rightarrow 3e)$, $CR(\mu - e, Al)$ and $CR(\mu - e, Ti)$ for IH are really much smaller than those of NH case, the rates of $\tau \rightarrow \mu\gamma$ and $\tau \rightarrow 3\mu$ for NH and IH roughly have the same order.

**Figure 3:** Dependence of $Br(\ell_i \rightarrow \ell_j\gamma)$, $Br(\ell_i \rightarrow 3\ell_j)$, $CR(\mu - e, Al)$ and $CR(\mu - e, Ti)$ on the lightest neutrino mass $m_1$ in AF model for the normal hierarchy spectrum. The bands have been obtained by varying $\Delta m_{21}^2$ and $\Delta m_{31}^2$ in their $3\sigma$ experimental range. The dashed and dotted lines represent the present and future experimental sensitivity respectively. There is still no upper bound for $CR(\mu - e, Al)$ so far.

**Figure 4:** Dependence of $Br(\ell_i \rightarrow \ell_j\gamma)$, $Br(\ell_i \rightarrow 3\ell_j)$, $CR(\mu - e, Al)$ and $CR(\mu - e, Ti)$ on the lightest neutrino mass $m_1$ in AF model for the inverted hierarchy spectrum. The bands have been obtained by varying $\Delta m_{21}^2$ and $\Delta m_{31}^2$ in their $3\sigma$ experimental range. The dashed and dotted lines represent the present and future experimental sensitivity respectively.
For the NH mass spectrum, numerical calculations demonstrate that the ratio between $Br(\tau \to e\gamma)$ and $Br(\mu \to e\gamma)$ indeed is 17.84%, and $Br(\tau \to \mu\gamma)$ is about 2-4 times as large as $Br(\mu \to e\gamma)$. Therefore we conclude that the branching ratios of $\mu \to e\gamma$, $\tau \to e\gamma$ and $\tau \to \mu\gamma$ roughly are of the same order, the same result has been obtained in Ref. [37]. It is notable that the current experiment limit $Br(\mu \to e\gamma) < 1.2 \times 10^{-11}$ already constrains the model strongly, the lightest neutrino mass $m_1$ very close to the lower bound $0.0044$ eV is disfavored. The rates of $\mu \to 3e$, $\mu-e$ conversion in Al and $\mu-e$ conversion in Ti are below the present upper bound, while they are above the expected future experimental sensitivity, consequently it is very promising to detect these three rare processes in near future. If this turns out to be true, the parameter space of the model would be strongly constrained. We plot the constraint on the parameters $m_1$ and $y_\nu$, imposed by the observation of $\mu \to e\gamma$, $\mu \to 3e$ and $\mu-e$ conversion in Al and Ti in Fig.5. It is obvious that the branching ratios of these LFV processes are more sensitive to the parameter $y_\nu$ than $m_1$. For $y_\nu$ of order 1, we should be able to observe $\mu \to e\gamma$, $\mu \to 3e$ and $\mu-e$ conversion in Al and Ti at next generation experiments in the case of NH spectrum, in particularly the $\mu-e$ conversion processes, which is an even more robust test to the AF model.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{contour.png}
\caption{The contour plots for $Br(\mu \to e\gamma)$, $Br(\mu \to 3e)$, $CR(\mu-e, Al)$ and $CR(\mu-e, Ti)$ in the parameter space of $y_\nu$ and $m_1$ for the NH spectrum in AF model, where $\Delta m^2_{21}$ and $\Delta m^2_{31}$ are chose to be the best fit values $7.65 \times 10^{-5}$ eV$^2$ and $2.40 \times 10^{-3}$ eV$^2$ respectively.}
\end{figure}

For the IH spectrum, the branching ratios for the $\mu e$ and $\tau e$ involved LFV processes are
indeed smaller than those of NH case, as has been stressed below Eq. (4.4). We notice that all the LFV processes are below both the present and future experiment sensitivity except the $\mu - e$ conversion in Ti. However, the rates for $\mu \rightarrow e\gamma$, $\tau \rightarrow e\gamma$, $\mu \rightarrow 3e$, $\tau \rightarrow 3e$ and $\mu - e$ conversion in Al and Ti at LO are so small that they may be corrected considerably by the NLO terms. Therefore it is very necessary to include the NLO contributions and exploit the physical effects on the LFV observables.

We perform a numerical analysis by treating all the NLO effective couplings $y_A$, $y_B$, $\tilde{x}_B$, $\tilde{x}_C$, $\tilde{x}_D$ and $\tilde{x}_E$ in Eq. (3.21) and Eq. (3.23) as random complex numbers with absolute value between 0 and 2. The LO parameters $A$ and $B$ in the heavy right handed Majorana mass matrix Eq. (3.7) are taken to be random complex numbers with absolute value between $10^{12}$ GeV and $10^{16}$ GeV, while $\frac{\delta V_T}{V_T}$ and $\frac{V_T}{\Lambda}$ has been fixed at the indicative values of 0.04. In order to compare with the leading order predictions, the coupling $y_e$ is set equal to 0.5 as well in the numerical analysis. The scatter plot of the LFV branching ratios vs the lightest neutrino mass for NH and IH spectrum are showed in Fig. 6 and Fig. 7 respectively. These plots display only the points corresponding to choices of the parameters reproducing $\Delta m_{21}^2$, $\Delta m_{31}^2$ and the mixing angles within their allowed $3\sigma$ interval given in Eq. (2.1).

![Scatter plots](image)

**Figure 6:** Scatter plot of $Br(\ell_i \rightarrow \ell_j\gamma)$, $Br(\ell_i \rightarrow 3\ell_j)$, $CR(\mu - e, Al)$ and $CR(\mu - e, Ti)$ against the lightest neutrino mass $m_1$ in AF model for the normal hierarchy spectrum. The dashed and dotted lines denote the present and future experimental sensitivity respectively. There is no upper bound for $CR(\mu - e, Al)$ so far.

We see that the LFV branching ratios for NH are modified slightly by the NLO contribution, the above discussions for NH case at LO still apply here, and the next generation experiments of $\mu \rightarrow e\gamma$, $\mu \rightarrow 3e$ and $\mu - e$ conversion in Al and Ti are crucial tests to the AF model. However, for IH spectrum the rates of the processes $\mu \rightarrow e\gamma$, $\tau \rightarrow e\gamma$, $\mu \rightarrow 3e$, $\tau \rightarrow 3e$ and $\mu - e$ conversion in Al and Ti are enhanced considerably. We note that $Br(\tau \rightarrow \mu\gamma)$ and $Br(\tau \rightarrow 3\mu)$ are qualitatively the same as the LO results, this is because that $\tau \rightarrow \mu\gamma$ and $\tau \rightarrow 3\mu$ are related with the elements $(\tilde{Y}_L^\dagger L\tilde{Y}_\nu)_{23}$, which are not suppressed at LO and are much larger than NLO corrections. It is notable that $\mu - e$ conversion in Ti is within the sensitivity of the next generation experiment for the whole
parameter space considered, the signals of $\mu \rightarrow e\gamma$ and $\mu - e$ conversion in Al should be detected in near future in a very large part of the parameter space, and $\mu \rightarrow 3e$ could be observed only in a marginal part of the parameter space. However, $\tau \rightarrow e\gamma$, $\tau \rightarrow \mu\gamma$, $\tau \rightarrow 3e$ and $\tau \rightarrow 3\mu$ are still below the expected future sensitivity. We note that the LO result $\frac{Br(\tau \rightarrow e\gamma)}{Br(\mu \rightarrow e\gamma)} \approx 17.84\%$ is destroyed completely after including the NLO corrections, the variation of $\frac{Br(\tau \rightarrow e\gamma)}{Br(\mu \rightarrow e\gamma)}$ vs the lightest neutrino mass is presented in Fig. 8.

In short summary, for both NH and IH spectrum of AF model, $\mu - e$ conversion in Ti can be observed in all the parameter space considered, $\mu \rightarrow e\gamma$ and $\mu - e$ conversion in Al should be observed at least in a very significant part of the parameter space, and $\mu \rightarrow 3e$ may be observed on for NH spectrum. Of all the LFV processes, the $\mu - e$ conversion in Ti should be an even more robust one, since its sensitivity would be improved drastically in near future.

Figure 8: Scatter plots of $Br(\tau \rightarrow e\gamma)/Br(\mu \rightarrow e\gamma)$ against the lightest neutrino mass for both NH and IH in AF model.
5. Predictions for LFV in $S_4$ model

Similar to section 4, we first study the predictions for the LFV branching ratios at LO of the model [12], then present the NLO corrections. From the discussion in section 3.2, we learn that the charged lepton mass matrix is diagonal, and the heavy right handed neutrinos are degenerate at LO. In the base where both the right handed neutrino and charged lepton mass matrices are diagonal and real, the neutrino Yukawa coupling matrix is $\hat{Y}_\nu = U^T Y_\nu$, where the unitary matrix $U$ is given by Eq. (3.33). Straightforwardly we can calculate the hermitian matrix $\hat{Y}_\nu^\dagger \hat{Y}_\nu$ as follows

$$
\hat{Y}_\nu^\dagger \hat{Y}_\nu = \begin{pmatrix}
2|a|^2 + 6|b|^2 - 4|a||b|\cos \Phi & |a|^2 - 3|b|^2 + 2|a||b|\cos \Phi & |a|^2 - 3|b|^2 + 2|a||b|\cos \Phi \\
|a|^2 - 3|b|^2 + 2|a||b|\cos \Phi & 2|a|^2 + 6|b|^2 - 2|a||b|\cos \Phi & |a|^2 - 3|b|^2 - 4|a||b|\cos \Phi \\
|a|^2 - 3|b|^2 + 2|a||b|\cos \Phi & |a|^2 - 3|b|^2 - 4|a||b|\cos \Phi & 2|a|^2 + 6|b|^2 + 2|a||b|\cos \Phi
\end{pmatrix} \ln \frac{M_G}{|M|}
$$

where $\Phi$ is the relative phase between $a$ and $b$, and $m_{1,2,3}$ are the light neutrino masses given in Eq.(3.36). The off-diagonal elements of $\hat{Y}_\nu^\dagger \hat{Y}_\nu$ can be read out directly

$$
(\hat{Y}_\nu^\dagger \hat{Y}_\nu)_{12} = (\hat{Y}_\nu^\dagger \hat{Y}_\nu)_{21} = (m_2 - m_1) \frac{|M|}{3v_u^2} \ln \frac{M_G}{|M|} = \frac{\Delta m_{21}^2}{3(m_1 + m_2)v_u^2} |M| \ln \frac{M_G}{|M|}
$$

$$
(\hat{Y}_\nu^\dagger \hat{Y}_\nu)_{13} = (\hat{Y}_\nu^\dagger \hat{Y}_\nu)_{31} = (m_2 - m_1) \frac{|M|}{3v_u^2} \ln \frac{M_G}{|M|} = \frac{\Delta m_{21}^2}{3(m_1 + m_2)v_u^2} |M| \ln \frac{M_G}{|M|}
$$

$$
(\hat{Y}_\nu^\dagger \hat{Y}_\nu)_{23} = (\hat{Y}_\nu^\dagger \hat{Y}_\nu)_{32} = (m_1 + m_2 - 3m_3) \frac{|M|}{6v_u^2} \ln \frac{M_G}{|M|} = \frac{\Delta m_{31}^2}{3(m_1 + m_2)v_u^2} |M| \ln \frac{M_G}{|M|}
$$

(5.2)

It is obvious that the off-diagonal elements of $\hat{Y}_\nu^\dagger \hat{Y}_\nu$ are closely related to the light neutrino mass difference and the right handed neutrino mass $|M|$, and they increase with $|M|$. Interestingly we notice $(\hat{Y}_\nu^\dagger \hat{Y}_\nu)_{12} = (\hat{Y}_\nu^\dagger \hat{Y}_\nu)_{13}$, which turns out to be true in AF model at LO as well. The relation $(\hat{Y}_\nu^\dagger \hat{Y}_\nu)_{12} = (\hat{Y}_\nu^\dagger \hat{Y}_\nu)_{13}$ seems universal in models which reproduce TB mixing at LO. Consequently we also have exactly $Br(\tau \to e\gamma)/Br(\mu \to e\gamma) \approx 17.84\%$ at LO, which is not satisfied anymore after including the subleading corrections. Since the factor $\frac{\Delta m_{31}^2}{3(m_1 + m_2)}$ is rather small for IH spectrum, the branching ratios of $\mu \to e\gamma$, $\tau \to e\gamma$, $\mu \to 3e$, $\tau \to 3e$ and $\mu - e$ conversion in Al and Ti should be small substantially at leading order. In the limit of degenerate light neutrino mass spectrum, the off-diagonal elements tend to be zero, the LFV processes would be highly suppressed.

It is interesting to estimate the order of magnitude for the right handed neutrino mass $|M|$. Since the parameters $a$ and $b$ are expected to be of order $\lambda^2$, with this and using $\sqrt{\Delta m_{31}^2} \approx 0.05$ eV as the light neutrino mass scale in the Seesaw formula, we obtain

$$
|M| \sim 10^{12} - 10^{13}\text{GeV}
$$

(5.3)

We set $|M|$ to be equal to $1.0 \times 10^{13}$ GeV, the LO predictions for the branching ratios of LFV processes are plotted in Fig.9 and Fig.10 for NH and IH spectrum respectively,
where the SSB parameters are chosen to be the SPS3 point as well. We see that the LFV branching ratio is smaller the corresponding one of AF model, all the LFV processes are below the planned experiment sensitivity except $\mu - e$ conversion in Ti. Consequently the next generation experiment of $\mu - e$ conversion in Ti is very important to test this $S_4$ model, and the parameter space of the model could be tightly constrained by $CR(\mu - e, Ti)$. The contour plot of $CR(\mu - e, Ti)$ in the $m_1 - |M|$ ($m_3 - |M|$) plane is showed in Fig.11, where both NH and IH mass spectrum are considered. It is obvious that the contour curves move toward the left with the increase of the sensitivity. We notice that in this $S_4$ model the signal of $\mu - e$ conversion in Ti could be observed in a large part of the allowed parameter space. As we will demonstrate that in case of IH spectrum, $CR(\mu - e, Ti)$ receives relative large correction from the NLO contributions, as a result, the contour plot for IH should be taken with a grain of salt.

**Figure 9:** Dependence of $Br(\ell_i \rightarrow \ell_j \gamma)$, $Br(\ell_i \rightarrow 3\ell_j)$, $CR(\mu - e, Al)$ and $CR(\mu - e, Ti)$ on the lightest neutrino mass $m_1$ in the Ding’s $S_4$ model for normal hierarchy spectrum. The bands have been obtained by varying $\Delta m^2_{21}$ and $\Delta m^2_{31}$ in their $3\sigma$ allowed range. The dashed and dotted lines represent the present and future experimental sensitivity respectively. There is still no upper bound for $CR(\mu - e, Al)$ at present.

Since the $\mu e$ and $\tau e$ involved LFV processes are predicted to be suppressed at LO in the case of IH spectrum, the NLO contributions may be comparable to the leading ones. Therefore we should take into account the NLO contributions in order to reach a much more solid conclusion. In the following numerical analysis, the NLO effective parameters $m^\ell_{21}/m^\ell_{22}$, $m^\ell_{31}/m^\ell_{33}$ and $m^\ell_{32}/m^\ell_{33}$ in Eq.(3.38) are treated as random complex numbers with absolute value between 0 and 2. $a$, $b$, $\tilde{a}$ and $\tilde{b}$ are taken to be complex random number with absolute value in the range of 0.01-0.1. The heavy neutrino mass $|M|$ is fixed at $1.0 \times 10^{13}$ GeV in order to compare with LO results, and the expansion parameters $\epsilon$ is set equal to 0.04. The scatter plots of LFV branching ratios for the NH and IH spectrum are displayed in Fig.12 and Fig.13 respectively.

We see that the LFV branching ratios can vary within a larger region than the LO results in Fig.9 and Fig.10, and the rates for $\mu e$ and $\tau e$ involved processes could be enhanced.
Figure 10: Dependence of $Br(\ell_i \rightarrow \ell_j \gamma)$, $Br(\ell_i \rightarrow 3\ell_j)$, $CR(\mu - e, Al)$ and $CR(\mu - e, Ti)$ on the lightest neutrino mass $m_3$ in the Ding’s $S_4$ model for inverted hierarchy spectrum. The bands have been obtained by varying $\Delta m^2_{21}$ and $\Delta m^2_{31}$ in their 3$\sigma$ allowed range. The dashed and dotted lines represent the present and future experimental sensitivity respectively.

Figure 11: The contour plot for $CR(\mu - e, Ti)$ in the parameter space of the lightest neutrino mass and right handed neutrino mass $|M|$ in the Ding’s $S_4$ model, where $\Delta m^2_{31}$ and $\Delta m^2_{31}$ are chose to be the best fit values $7.65 \times 10^{-5}$ eV$^2$ and $\pm 2.40 \times 10^{-3}$ eV$^2$ respectively.

by a factor a 10-100 for the IH case. However, $Br(\tau \rightarrow \mu \gamma)$ and $Br(\tau \rightarrow 3\mu)$ are only modified slightly by NLO contributions. It is remarkable that in this model the $\mu - e$ conversion in Ti is completely within the precision of the next generation experiment, the $\mu - e$ conversion in Al should be observable in a large part of the parameter space, the radiative decay $\mu \rightarrow e\gamma$ can only be observed in a marginal part of the parameter space at near future experiment, and $\tau \rightarrow e\gamma$, $\tau \rightarrow e\mu\gamma$, $\mu \rightarrow 3e$, $\tau \rightarrow 3e$ and $\tau \rightarrow 3\mu$ are below the expected future sensitivities. This fact can be used to distinguish this $S_4$ model from the AF model.
Figure 12: Scatter plot of $Br(\ell_i \to \ell_j \gamma)$, $Br(\ell_i \to 3\ell_j)$, $CR(\mu - e, Al)$ and $CR(\mu - e, Ti)$ against the lightest neutrino mass $m_1$ in Ding’s $S_4$ model for normal hierarchy spectrum. The dashed and dotted lines represent the present and future experimental sensitivity respectively. There is no upper bound for $CR(\mu - e, Al)$ so far.

Figure 13: Scatter plot of $Br(\ell_i \to \ell_j \gamma)$, $Br(\ell_i \to 3\ell_j)$, $CR(\mu - e, Al)$ and $CR(\mu - e, Ti)$ against the lightest neutrino mass $m_3$ in Ding’s $S_4$ model for inverted hierarchy spectrum. The dashed and dotted lines represent the present and future experimental sensitivity respectively.

6. Conclusions and discussions

Recently some models with discrete flavor symmetry such as $A_4$ and $S_4$ have been showed to be able to naturally produce TB mixing at leading order. Since only few parameters are involved at LO in these models, they are rather predictive. In this work, we have studied the LFV $\mu \to e\gamma$, $\tau \to e\gamma$, $\tau \to \mu\gamma$, $\mu \to 3e$, $\tau \to 3e$, $\tau \to 3\mu$ and $\mu - e$ conversion in Al and Ti in the AF $A_4$ model [9] and the $S_4$ model of Ding [12] within the framework of mSUGRA.

At LO the branching ratio for LFV process is closely related to the light neutrino mass. For inverted hierarchy neutrino mass spectrum, $\mu \to e\gamma$, $\tau \to e\gamma$, $\mu \to 3e$, $\tau \to 3e$ and $\mu - e$
Table 4: Summary of predictions for LFV processes in AF model [9] and Ding’s $S_4$ model [12].
The symbol √ denotes the rare process is above the sensitivity of next generation experiments for
the whole parameter space considered, ✓ represents the LFV process should be observed in a very
large part of parameter space in near future, ? for the process only observable in a marginal part of
parameter space, and × denotes the process is below the sensitivity of next generation experiments.

|                 | AF NH | AF IH | $S_4$ NH | $S_4$ IH |
|-----------------|-------|-------|----------|----------|
| $\mu \to e\gamma$ | √     | √     | ?        | ?        |
| $\tau \to e\gamma$ | ×     | ×     | ×        | ×        |
| $\tau \to \mu\gamma$ | ×     | ×     | ×        | ×        |
| $\mu \to 3e$ | √     | ?     | ×        | ×        |
| $\tau \to 3e$ | ×     | ×     | ×        | ×        |
| $\tau \to 3\mu$ | ×     | ×     | ×        | ×        |
| $\mu - e$ in Al | √     | √     | ✓        | ✓        |
| $\mu - e$ in Ti | √     | ✓     | ✓        | ✓        |

Our predictions for the rare LFV processes in the two models are summarized in Table 4. From detailed numerical analysis, we find that for NH spectrum of AF model, $\mu \to e\gamma$, $\mu \to 3e$ and $\mu - e$ conversion in Al and Ti are within the reach of next generation experiments (Fig. 6). While for IH spectrum $\mu - e$ conversion in Ti is above the expected future sensitivity in all the parameter space considered, $\mu \to e\gamma$ and $\mu - e$ conversion in Al could be observed by near future experiments in a very significant proportion of the parameter space, nevertheless the signal of $\mu \to 3e$ could be only detected in a marginal part of the parameter space (Fig. 7). Then we turn to the Ding’s $S_4$ model, for both NH and IH spectrum, we find that only $\mu - e$ conversion in Ti is within the precision of the future experiment, $\mu - e$ conversion in Al should be observed in a large part of the parameter space, while $\mu \to e\gamma$ observable in a marginal part of the parameter space (Fig. 12 and Fig. 13). In both AF model and the $S_4$ model of Ding, $\tau \to e\gamma$, $\tau \to \mu\gamma$, $\tau \to 3e$ and $\tau \to 3\mu$ are below the sensitivity of next generation experiments. We conclude that $\mu - e$ conversion in Ti is a particularly robust test to the AF model and the Ding’s $S_4$ model. If it is really observed in future, it would be a great support to these discrete flavor models.

We note that our consideration are not at all restricted to the AF model and Ding’s $S_4$ model, but could apply to a much wider class of theories. Models with discrete flavor symmetry can be strongly constrained or be excluded by existing or future LFV bounds, a combined analysis of LFV processes provide us a way to distinguish different models.
Note Added: near the completion of this work, papers that address similar issues appear [38, 39].

7. Acknowledgments

We are grateful to Prof. Mu-Lin Yan for stimulating discussion, and we acknowledge Dr. Jia-Wei Zhao and Wei Li for their help on numerical calculations. This work is supported by the National Natural Science Foundation of China under Grant No.10905053, Chinese Academy KJCX2-YW-N29 and the 973 project with Grant No. 2009CB825200. Jia-Feng Liu is supported in part by the National Natural Science Foundation of China under Grant No.10775124.
References

[1] T. Schwetz, M. A. Tortola and J. W. F. Valle, New J. Phys. 10, 113011 (2008) [arXiv:0808.2016 [hep-ph]]; M. Maltoni and T. Schwetz, arXiv:0812.3161 [hep-ph].

[2] G. L. Fogli, E. Lisi, A. Marrone, A. Palazzo and A. M. Rotunno, arXiv:0905.3549 [hep-ph]; M. C. Gonzalez-Garcia and M. Maltoni, Phys. Rept. 460, 1 (2008) [arXiv:0704.1800 [hep-ph]].

[3] F. T. . Avignone, S. R. Elliott and J. Engel, Rev. Mod. Phys. 80, 481 (2008) [arXiv:0708.1033 [nucl-ex]].

[4] F. Ardellier et al. [Double Chooz Collaboration], arXiv:hep-ex/0606025.

[5] Y. f. Wang, arXiv:hep-ex/0610024.

[6] P. F. Harrison, D. H. Perkins and W. G. Scott, Phys. Lett. B 530, 167 (2002), hep-ph/0202074; P. F. Harrison and W. G. Scott, Phys. Lett. B 535, 163 (2002), hep-ph/0203209; Z. Z. Xing, Phys. Lett. B 533, 85 (2002), hep-ph/0204049; X. G. He and A. Zee, Phys. Lett. B 560, 87 (2003), hep-ph/0301092.

[7] E. Ma and G. Rajasekaran, Phys. Rev. D 64 (2001) 113012, arXiv:hep-ph/0106291; E. Ma, Mod. Phys. Lett. A 17 (2002) 627 [arXiv:hep-ph/0203238]; K. S. Babu, E. Ma and J. W. F. Valle, Phys. Lett. B 552 (2003) 207, arXiv:hep-ph/0206292; M. Hirsch, J. C. Romao, S. Skadhauge, J. W. F. Valle and A. Villanova del Moral, arXiv:hep-ph/0312244; Phys. Rev. D 69 (2004) 093006 [arXiv:hep-ph/0312265]; E. Ma, Phys. Rev. D 70 (2004) 031901; Phys. Rev. D 70 (2004) 031901 [arXiv:hep-ph/0404199]; New J. Phys. 6 (2004) 104 [arXiv:hep-ph/0405152]; arXiv:hep-ph/0409075; S. L. Chen, M. Frigerio and E. Ma, Nucl. Phys. B 724 (2005) 423 [arXiv:hep-ph/0504181]; E. Ma, Phys. Rev. D 72 (2005) 037301 [arXiv:hep-ph/0505209]; M. Hirsch, A. Villanova del Moral, J. W. F. Valle and E. Ma, Phys. Rev. D 72 (2005) 091301 [Erratum-ibid. D 72 (2005) 119904] [arXiv:hep-ph/0507148]. K. S. Babu and X. G. He, arXiv:hep-ph/0507217; E. Ma, Mod. Phys. Lett. A 20 (2005) 2601 [arXiv:hep-ph/0508099]; A. Zee, Phys. Lett. B 630 (2005) 58 [arXiv:hep-ph/0508278]; E. Ma, Phys. Rev. D 73 (2006) 057304 [arXiv:hep-ph/0511133]; X. G. He, Y. Y. Keum and R. R. Volkas, JHEP 0604 (2006) 039 [arXiv:hep-ph/0601001]; B. Adhikary, B. Brahmachari, A. Ghosal, E. Ma and M. K. Parida, Phys. Lett. B 638 (2006) 345 [arXiv:hep-ph/0603059]; E. Ma, Mod. Phys. Lett. A 21 (2006) 2931 [arXiv:hep-ph/0607190]; Mod. Phys. Lett. A 22 (2007) 101 [arXiv:hep-ph/0610342]; L.avoura and H. Kuboock, Mod. Phys. Lett. A 22 (2007) 181 [arXiv:hep-ph/0610050]; E. Ma, H. Sawanaka and M. Tanimoto, Phys. Lett. B 641, 301 (2006), hep-ph/0606103; S. F. King and M. Malinsky, Phys. Lett. B 645 (2007) 351 [arXiv:hep-ph/0610250]; S. Morisi, M. Picariello and E. Torrente-Lujan, Phys. Rev. D 75 (2007) 075015 [arXiv:hep-ph/0702034]; M. Hirsch, A. S. Joshipura, S. Kaneko and J. W. F. Valle, Phys. Rev. Lett. 99, 151802 (2007) [arXiv:hep-ph/0703346]; F. Yin, Phys. Rev. D 75 (2007) 073010 [arXiv:0704.3827 [hep-ph]]; F. Bazzocchi, S. Kaneko and S. Morisi, JHEP 0803 (2008) 063 [arXiv:0707.3032 [hep-ph]]; F. Bazzocchi, S. Morisi, S. Morisi and M. Picariello, Phys. Lett. B 659 (2008) 628 [arXiv:0710.2928 [hep-ph]]; M. Honda and M. Tanimoto, Prog. Theor. Phys. 119 (2008) 583 [arXiv:0801.0181 [hep-ph]]; B. Brahmachari, S. Choubey and M. Mitra, Phys. Rev. D 77 (2008) 073008 [Erratum-ibid. D 77 (2008) 119901] [arXiv:0801.3554 [hep-ph]]; G. Altarelli, F. Feruglio and C. Hagedorn, JHEP 0803, 052 (2008) [arXiv:0802.0090 [hep-ph]]; F. Bazzocchi, S. Morisi, M. Picariello and E. Torrente-Lujan, J. Phys. G 36 (2009) 015002 [arXiv:0802.1693 [hep-ph]]; B. Adhikary...
and A. Ghosal, Phys. Rev. D 78 (2008) 073007 [arXiv:0803.3582 [hep-ph]]; M. Hirsch, S. Morisi and J. W. F. Valle, Phys. Rev. D 78 (2008) 093007 [arXiv:0804.1521 [hep-ph]]; Y. Lin, Nucl. Phys. B 813, 91 (2009) [arXiv:0804.2867 [hep-ph]]; P. H. Frampton and S. Matsuzuki, arXiv:0806.4592 [hep-ph]; F. Feruglio, C. Hagedorn, Y. Lin and L. Merlo, Nucl. Phys. B 809, 218 (2009) [arXiv:0807.3160 [hep-ph]]; H. Ishimori, T. Kobayashi, Y. Omura and M. Tanimoto, JHEP 0812, 082 (2008) [arXiv:0807.4625 [hep-ph]]; F. Bazzocchi, M. Frigerio and S. Morisi, Phys. Rev. D 78, 116018 (2008) [arXiv:0809.3573 [hep-ph]]; S. Morisi, Phys. Rev. D 79, 033008 (2009) [arXiv:0901.1080 [hep-ph]]; P. Ciafaloni, M. Picariello, E. Torrente-Lujan and A. Urbano, Phys. Rev. D 79, 116010 (2009) [arXiv:0901.2236 [hep-ph]]; M. C. Chen and S. F. King, JHEP 0906, 072 (2009) [arXiv:0903.0125 [hep-ph]]; T. J. Burrows and S. F. King, arXiv:0909.1433 [hep-ph]; F. Feruglio, C. Hagedorn and L. Merlo, arXiv:0910.4058 [hep-ph]; S. Morisi and E. Peinado, arXiv:0910.4389 [hep-ph]; Y. Lin, L. Merlo and A. Paris, arXiv:0911.3037 [hep-ph].

[8] G. Altarelli and F. Feruglio, Nucl. Phys. B 720, 64 (2005), hep-ph/0504165.
[9] G. Altarelli and F. Feruglio, Nucl. Phys. B 741, 215 (2006), hep-ph/0512103.
[10] G. Altarelli and D. Meloni, J. Phys. G 36, 085005 (2009) [arXiv:0905.0620 [hep-ph]].
[11] G. J. Ding, Phys. Rev. D 78, 036011 (2008) [arXiv:0803.2278 [hep-ph]]; P. D. Carr and P. H. Frampton, arXiv:hep-ph/0701034; F. Feruglio, C. Hagedorn, Y. Lin and L. Merlo, Nucl. Phys. B 775, 120 (2007), hep-ph/0702194; M. C. Chen and K. T. Mahanthappa, Phys. Lett. B 652, 34 (2007), arXiv:0705.0714 [hep-ph]; P. H. Frampton and T. W. Kephart, JHEP 0709, 110 (2007), arXiv:0706.1186 [hep-ph]; A. Aranda, Phys. Rev. D 76, 111301 (2007), arXiv:0707.3661 [hep-ph]; P. H. Frampton, T. W. Kephart and S. Matsuzuki, Phys. Rev. D 78, 073004 (2008) [arXiv:0807.4713 [hep-ph]]; D. A. Eby, P. H. Frampton and S. Matsuzuki, Phys. Lett. B 671, 386 (2009) [arXiv:0810.4899 [hep-ph]].
[12] G. J. Ding, arXiv:0909.2210 [hep-ph], to appear on Nucl. Phys. B.
[13] E. Ma, Phys. Lett. B 632, 352 (2006) [arXiv:hep-ph/0508231]; C. S. Lam, Phys. Rev. Lett. 101, 121602 (2008) [arXiv:0804.2622 [hep-ph]]; C. S. Lam, Phys. Rev. D 78, 073015 (2008) [arXiv:0809.1185 [hep-ph]]; C. S. Lam, arXiv:0907.2206 [hep-ph]; F. Bazzocchi and S. Morisi, arXiv:0811.0345 [hep-ph]; F. Bazzocchi, L. Merlo and S. Morisi, arXiv:0901.2086 [hep-ph]; F. Bazzocchi, L. Merlo and S. Morisi, arXiv:0902.2849 [hep-ph]; H. Ishimori, Y. Shimizu and M. Tanimoto, Prog. Theor. Phys. 121, 769 (2009) [arXiv:0812.5031 [hep-ph]]; G. Altarelli, F. Feruglio and L. Merlo, JHEP 0905, 020 (2009) [arXiv:0903.1940 [hep-ph]]; W. Grimus, L. Lavoura and P. O. Ludl, arXiv:0906.2689 [hep-ph]; B. Dutta, Y. Mimura and R. N. Mohapatra, arXiv:0911.2242 [hep-ph]; D. Meloni, arXiv:0911.3591 [hep-ph].
[14] D. G. Lee and R. N. Mohapatra, Phys. Lett. B 329, 463 (1994) [arXiv:hep-ph/9403201]; C. Hagedorn, M. Lindner and R. N. Mohapatra, JHEP 0606, 042 (2006) [arXiv:hep-ph/0602244]; Y. Cai and H. B. Yu, Phys. Rev. D 74, 115005 (2006) [arXiv:hep-ph/0608022]; H. Zhang, Phys. Lett. B 655, 132 (2007) [arXiv:hep-ph/0612214]; Y. Koide, JHEP 0708, 086 (2007) [arXiv:0705.2275 [hep-ph]]; M. K. Parida, Phys. Rev. D 78, 053004 (2008) [arXiv:0804.4571 [hep-ph]].
[15] E. E. Jenkins and A. V. Manohar, Phys. Lett. B 668, 210 (2008) [arXiv:0807.4176 [hep-ph]].
[18] D. Aristizabal Sierra, F. Bazzocchi, I. de Medeiros Varzielas, L. Merlo and S. Morisi, arXiv:0908.0907 [hep-ph].
[19] R. G. Felipe and H. Serodio, arXiv:0908.2947 [hep-ph].
[20] J. A. Casas and A. Ibarra, Nucl. Phys. B 618, 171 (2001) [arXiv:hep-ph/0103065].
[21] E. Arganda and M. J. Herrero, Phys. Rev. D 73, 055003 (2006) [arXiv:hep-ph/0510405].
[22] A. Ibarra and C. Simonetto, JHEP 0804, 102 (2008) [arXiv:0802.3858 [hep-ph]].
[23] M. Hirsch, J. W. F. Valle, W. Porod, J. C. Romao and A. Villanova del Moral, Phys. Rev. D 78, 013006 (2008) [arXiv:0804.4072 [hep-ph]].
[24] A. Masiero, S. K. Vempati and O. Vives, Nucl. Phys. B 649, 189 (2003) [arXiv:hep-ph/0209303]; A. Masiero, S. K. Vempati and O. Vives, New J. Phys. 6, 202 (2004) [arXiv:hep-ph/0407325]; L. Calibbi, A. Faccia, A. Masiero and S. K. Vempati, JHEP 0707, 012 (2007) [arXiv:hep-ph/0610241]; C. H. Albright and M. C. Chen, Phys. Rev. D 77, 113010 (2008) [arXiv:0802.4228 [hep-ph]].
[25] M. Drees, R. Godbole and P. Roy, “Theory and phenomenology of sparticles: An account of four-dimensional N=1 supersymmetry in high energy physics,” Hackensack, USA: World Scientific (2004) 555 p
[26] S. P. Martin, arXiv:hep-ph/9709356.
[27] S. T. Petcov, S. Profumo, Y. Takanishi and C. E. Yaguna, Nucl. Phys. B 676, 453 (2004) [arXiv:hep-ph/0306195].
[28] J. Hisano, T. Moroi, K. Tobe and M. Yamaguchi, Phys. Rev. D 53, 2442 (1996) [arXiv:hep-ph/9510309].
[29] J. Hisano and D. Nomura, Phys. Rev. D 59, 116005 (1999) [arXiv:hep-ph/9810479].
[30] E. Arganda, M. J. Herrero and A. M. Teixeira, JHEP 0710, 104 (2007) [arXiv:0707.2955 [hep-ph]].
[31] A. Czarnecki, W. J. Marciano and K. Melnikov, AIP Conf. Proc. 435, 409 (1998) [arXiv:hep-ph/9801218]; R. Kitano, M. Koike and Y. Okada, Phys. Rev. D 66, 096002 (2002) [Erratum-ibid. D 76, 059902 (2007)] [arXiv:hep-ph/0203110]; Y. Kuno and Y. Okada, Rev. Mod. Phys. 73, 151 (2001) [arXiv:hep-ph/9909265].
[32] E.C. Dukes et al. [Mu2e Collaboration], “Proposal to Search for $\mu^-N \to e^-N$ with a Single Event Sensitivity Below 10$^{-16}$”, http://mu2e.fnal.gov/public/hep/index.shtml.
[33] Y. Mori et al. [The PRIME Working Group], “An Experimental Search for the $\mu^- \to e^-$ Conversion Process at an Ultimate Sensitivity of the Order of 10$^{-18}$ with PRISM”, LOI-25, http://www-ps.kek.jp/jhf-np/LOIlist/LOIlist.html.
[34] C. Amsler et al. (Particle Data Group), Phys. Lett. B667, 1 (2008).
[35] C. Jarlskog, Z. Phys. C 29, 491 (1985).
[36] B. C. Allanach et al., in Proc. of the APS/DPF/DPB Summer Study on the Future of Particle Physics (Snowmass 2001) ed. N. Graf, Eur. Phys. J. C 25, 113 (2002) [arXiv:hep-ph/0202233].
[37] F. Feruglio, C. Hagedorn, Y. Lin and L. Merlo, Nucl. Phys. B 809, 218 (2009) [arXiv:0807.3160 [hep-ph]].
[38] C. Hagedorn, E. Molinaro and S. T. Petcov, arXiv:0911.3605 [hep-ph].

[39] F. Feruglio, C. Hagedorn, Y. Lin and L. Merlo, arXiv:0911.3874 [hep-ph].