Local gauge symmetries are key features of theories describing fundamental interactions \cite{1} and emerging phenomena in condensed matter physics \cite{2}. The large-scale properties of three-dimensional (3D) gauge models and the nature of their thermal or quantum transitions are of interest in several physical contexts. For instance, they are relevant for superconductivity \cite{3}, for topological order and quantum transitions \cite{4,8}, and also in high-energy physics, as they describe the finite-temperature electroweak and strong-interaction transition which occurred in the early universe \cite{4} and which is presently being investigated in heavy-ion collisions \cite{10}.

We discuss a 3D model of interacting scalar fields with a nonabelian gauge symmetry, which we may name scalar chromodynamics or nonabelian Higgs model. It provides a paradigmatic example for the nonabelian Higgs mechanism, which is at the basis of the Standard Model of the fundamental interactions \cite{11}. In condensed matter physics, it may be relevant for systems with emerging nonabelian gauge symmetries, see, e.g., Ref. \cite{6}. It represents the natural extension of abelian Higgs models, which have been extensively studied in various contexts, see, e.g., Refs. \cite{3,5,12,13}. We will focus on the multiflavor case $N_f \geq 2$. For $N_f = 1$ scalar nonabelian models have been carefully investigated, as they are relevant for the finite-temperature behavior of the electroweak theory \cite{13,20}. Much less is known about the phase diagram and the nature of the transitions (symmetry-breaking pattern, universality class, etc.) in the multiflavor case, and about the effective field theory that describes the critical behavior.

In this paper we consider 3D lattice models of complex matricial scalar fields, with $N_c \times N_f$ components, minimally coupled to an SU($N_c$) gauge field \cite{21}. We investigate their phase diagram, for various values of $N_f \geq 2$ and $N_c$, and the nature of their phase transitions. Our numerical results allow us to understand which effective field theory provides the correct description of the phase transition. This study provides therefore information on the field-theoretical approach to be used to analyze thermal and quantum transitions in the presence of emergent nonabelian gauge symmetries. Moreover, it may provide information on the finite-temperature phase diagram of nonabelian gauge models involving scalar fields, that are meant to describe the new physics beyond the Standard Model of fundamental interactions.

Classical and quantum phase transitions have traditionally been studied using statistical field theories \cite{22}. Their properties depend on the global symmetry of the model, the symmetry-breaking pattern, and some other global properties, such as the space dimensionality. In the presence of gauge symmetries, one may think that also the gauge-symmetry group is a distinctive element that should be specified to characterize the transition. As we shall discuss, however, this is not necessarily true, as gauge modes are not necessarily critical at the transition.

For the same reason, at variance with systems that only have global symmetries, there is not a unique natural effective theory for the transition. One can, of course, consider the continuum gauge theory that corresponds to the lattice model. However, one may also consider a Landau-Ginzburg-Wilson (LGW) $\Phi^4$ theory based on a gauge-invariant order-parameter field with the global symmetry of the model \cite{13,22}. Note that, while in the first approach the gauge symmetry is still present in the effective model, in the second one, gauge invariance does not play a particular role beside fixing the order parameter. The LGW approach is expected to be the correct one when the gauge interactions are short-ranged at the transition. In the opposite case, instead, the continuum gauge model should allow for the correct picture. We recall that the LGW approach was used to predict the nature of the finite-temperature phase transition of hadronic matter in the massless limit of quarks, implicitly assuming that the SU(3) gauge modes are not critical \cite{23,24}. We compare the renormalization-group (RG) predictions of the above-
mentioned field-theory approaches with numerical lattice results. This study allows us to deepen our understanding of their effectiveness and limitations, in particular for the widely used LGW approach.

To investigate the above issues, we consider lattice scalar gauge theories obtained by partially gauging a maximally symmetric model of matrix variables $Z_x^{af}$. We start from the lattice action \[ S_s = -J \sum_{x,\mu} \text{Re} \text{Tr} Z_x^\dagger U_{x,\mu} Z_{x+\mu}, \quad \text{Tr} Z_x^\dagger Z_x = 1, \] (1)

where $Z_x^{af}$ are $N_c \times N_f$ complex matrices and the sum is over all links of a cubic lattice ($\mu$ are unit vectors along the three lattice directions). The model has a global O($N$) symmetry with $N = 2N_cN_f$. In particular, it is invariant under the global SU($N_c$) transformations $Z_x \to V Z_x$, $V \in \text{SU}(N_c)$. To make this symmetry a local one, we use the Wilson approach \[21\]. We associate a SU($N_c$) matrix $U_{x,\mu}$ with each link, and consider the action \[ S_g = -\beta N_f \sum_{x,\mu} \text{Re} \text{Tr} \left[ Z_x^\dagger U_{x,\mu} Z_{x+\mu} \right] \]

\[ -\frac{\beta_c}{N_c} \sum_{x,\mu>\nu} \text{Re} \text{Tr} \left[ U_{x,\mu} U_{x+\mu,\nu} U_{x+\nu,\mu}^\dagger U_{x,\mu}^\dagger \right], \] (2)

where the second sum is over all lattice plaquettes. Beside the SU($N_c$) gauge invariance, the model has also a global U($N_f$) symmetry, $Z_x \to Z_x U$ with $U \in \text{U}(N_f)$. For $N_c = 2$ the global symmetry group is larger than U($N_f$). Indeed the action turns out to be invariant under the unitary symplectic group Sp($N_f$) $\supset$ U($N_f$), see also Refs. \[26, 27\]. If one defines $2 \times 2N_f$ matrix $\Gamma^{af}$

\[ \Gamma^{af} = Z^{af}, \quad \Gamma^{(N_f + f)} = \sum_b e^{ab} Z^{bf} \] (3)

($e^{ab} = -e^{ba}, e^{12} = 1$), one can show that the action is invariant under $\Gamma^{af} \to \sum W^{lm} \Gamma^{am}$ where $W \in \text{Sp}(N_f)$, and $l, m = 1, \ldots, 2N_f$. For $\beta_c = \infty$, the gauge fields are equal to the identity (modulo gauge transformations), thus we recover the ungauged model in Eq. (1), i.e. the standard $N$-vector model with $N = 2N_cN_f$.

For $N_f = 1$ no transition \[28\] is expected for finite $\beta_g$ \[29, 31\] (we have verified it numerically up to $\beta_g = 6$), and long-range correlations should only develop for $\beta_g \to \infty$ close to the O($2N_c$) critical point. For $N_f \geq 2$ we find a transition line, that separates two different phases—see Fig. 1—which are characterized by the behavior of gauge-invariant order parameter \[ Q_z^{fg} = \sum_a Z_a^{af} Z_a^{ag} - \frac{1}{N_f} \delta^{fg}, \] (4)

which is a hermitian and traceless $N_f \times N_f$ matrix. The nature of the transition depends on $N_f$ and $N_c$, while it does not depend on the gauge coupling $\beta_g$.

Before presenting the numerical results, we discuss the predictions of the statistical field theories which may describe the behavior along the transition line. We begin considering the continuum scalar chromodynamics defined by the Lagrangian

\[ \mathcal{L} = \frac{1}{4g^2} \text{Tr} F_{\mu\nu}^2 + \text{Tr}[ \{ D_{\mu} Z \} \{ D_\mu Z \} ] + V(\text{Tr} Z Z), \] (5)

where $V(X) = rX + \frac{1}{4} u X^2$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ and $D_{\mu} = \partial_\mu \delta_{ab} + t_{ab}^c A_\mu^c$. The RG flow in the space of the renormalized couplings $u$ and $f \equiv g^2$ can be studied perturbatively within the $\varepsilon = 4 - D$ expansion \[32\]. At one loop, the $\beta$ functions read \[33, 37\]

\[ \beta_f(u, f) \equiv \mu \frac{\partial f}{\partial \mu} = -\varepsilon f - (22N_c - N_f) f^2, \] (6)

\[ \beta_u(u, f) \equiv \mu \frac{\partial u}{\partial \mu} = -\varepsilon u + (N_f N_c + 4) u^2 \]

\[ -\frac{18}{N_c} (N_f - 1) u f + \frac{27(N_f - 1)(N_c^2 + 2N_c - 2)}{N_c^4} f^2. \]

A stable fixed point (FP) is only found for $N_f > N_f^*(N_c)$ with $N_f^*(2) = 359 + O(\varepsilon)$ and $N_f^*(3) = 972.95 + O(\varepsilon)$. Therefore, for any $N_f < N_f^*(N_c)$, and, in particular, for small values of $N_f$, the transition is predicted to be first order. We also note that for large $\beta_g$ the lattice model \[22\] is expected to show significant crossover effects due to the nearby O($N$) transition point ($N = 2N_cN_f$). In the theory \[30\] such a crossover is controlled by the RG flow in the vicinity of the O($N$) fixed point [$f^* = 0$ and $u^*_N = \varepsilon/(N_f N_c + 4)$], which is always unstable with respect to the gauge perturbation \[38\].

An alternative field-theoretical approach is provided by the LGW framework \[15, 32, 39, 42\], in which one assumes that the critical modes are associated with a gauge-invariant composite operator. In the present case, the natural order parameter is the operator $Q_z^{fg}$ defined in Eq. (4). This is a nontrivial assumption, as it postulates that gauge fields do not play a relevant role in the effective theory of the critical modes. The LGW fundamental field is correspondingly a traceless hermitian
matrix $\Psi(x)$, which can be formally considered as the average of $Q_{x}^{fg}$ over a large but finite domain. The LGW theory is obtained by considering the most general 4th-order polynomial consistent with the global symmetry:

$$\text{Tr}(\partial_{\mu} \Psi)^{2} + r \text{Tr} \Psi^{2} + w \text{tr} \Psi^{3} + u (\text{tr} \Psi^{2})^{2} + v \text{Tr} \Psi^{4}$$ (8)

Continuous transitions may only occur if its RG flow has a stable FP. For $N_{f} = 2$, the cubic term vanishes and the two quartic terms are equivalent, leading to the O(3)-symmetric LGW theory. This implies that the phase transition can be continuous, in the O(3) universality class because of the mapping SO(3) = SU(2)/$\mathbb{Z}_{2}$. An explicit O(3) order parameter is obtained by considering the real vector variable $\varphi_{x}^{k} = \sum_{f,g} a_{fg}^{k} Q_{x}^{fg}$, where $a_{fg}^{k}$ are the Pauli matrices. For $N_{f} \geq 3$, the cubic $\Psi^{3}$ term is generically expected to be present. For 3D systems this is usually taken as an indication that phase transitions are generically first order. [43]

The above arguments apply to generic $N_{c} > 2$. Since for $N_{c} = 2$ the global symmetry is Sp($N_{f}$), the order parameter is now a $2 N_{f} \times 2 N_{f}$ matrix given by

$$\mathcal{T}_{x}^{lm} = \sum_{a} \mathcal{T}_{x}^{alf} \mathcal{T}_{x}^{mfa} - \frac{\delta_{lm}}{2 N_{f}} \sum_{an} \mathcal{T}_{x}^{anf} \mathcal{T}_{x}^{man}$$ ,

where is can be expressed in terms of $Q_{x}^{fg}$ defined in Eq. (4) ($\mathcal{T}_{x}^{fg} = Q_{x}^{fg}$ for $f,g,1,...,N_{f}$) and of $D_{x}^{fg} = \sum_{abc} e^{abc} Z_{x}^{af} Z_{x}^{bg}$. For $N_{f} = 2$ the Sp(2) group is isomorphic to the O(5) group. The Sp(2) LGW theory is that of O(5)-symmetric vector model. Indeed, the order parameter can be rewritten in terms of the three-component vector $\varphi_{x} = \sum_{f,g} a_{fg}^{k} Q_{x}^{fg}$ and the real and imaginary parts of the complex variable $\phi_{x} = \frac{1}{2} \sum_{f,g} e^{ifg} D_{x}^{fg}$. They form a five-component order parameter. Thus we predict an O(5) critical behavior. For larger $N_{f}$, the Sp($N_{f}$) LGW theory contains cubic interactions [44], and therefore first-order transitions are predicted.

Summarizing, the LGW approach based on a gauge-invariant order parameter predicts that continuous transitions only occur for $N_{f} = 2$. They belong to the O(3) universality class for any $N_{c} \geq 3$ and to the O(5) universality class for $N_{c} = 2$. Instead, first-order transitions are generically expected for $N_{f} \geq 3$ and any $N_{c}$. Note that for $N_{f} = 2$ the LGW predictions differ from those of the continuum gauge theory [43], as the latter predicts a first-order transition (no FP for $N_{f} = 2$). The two theories apparently also disagree for large values of $N_{f}$. The continuum gauge theory admits the possibility of continuous transitions, since a stable FP exists, while the LGW theory indicates first-order transitions for any $N_{f} > 2$.

In our numerical study [43] we consider the model [22] on a cubic lattice of size $L$ and periodic boundary conditions. We compute the correlation $G(x-y) = \langle \text{Tr} Q_{x} Q_{y} \rangle$ of the composite operator $Q_{x}$ defined in Eq. (4), its susceptibility $\chi = \sum_{x} G(x)$ and correlation length $\xi$,

$$\xi^{2} \equiv \frac{1}{4 \sin^{2}(\pi / L)} \frac{\tilde{G}(0) - G(p_{m})}{G(p_{m})}$$ ,

where $\tilde{G}(p) = \sum_{x} e^{ip \cdot x} G(x)$ and $p_{m} = (2\pi / L, 0, 0)$. We also consider the Binder parameter

$$U = \frac{\langle \mu_{2}^{2} \rangle}{\langle \mu_{2} \rangle^{2}} , \quad \mu_{2} = \frac{1}{L \delta} \sum_{x,y} \text{Tr} Q_{x} Q_{y}$$ .

At continuous transitions, RG invariant quantities, such as $R_{\xi} \equiv \xi / L$ and $U$, behave as [22] [42]

$$R(\beta, L) = f_{R}(X) + O(L^{-\omega}) , \quad X = (\beta - \beta_{c}) L^{1/\nu}$$ ,

where $\nu$ is the correlation-length exponent, $f_{R}(X)$ is a universal function (apart from a normalization of the argument), and $\omega$ is the exponent associated with the leading scaling corrections. Moreover, since $R_{\xi}$ is a monotonic function, Eq. (12) implies $U(\beta, L) \approx F_{U}(R_{\xi})$, where $F_{U}$ depends on the universality class only, without free normalizations (once fixed the boundary conditions and the shape of the lattice).

The results for $N_{f} = 2, N_{c} = 3$, and $\beta_{g} = 0$ up to $L = 64$, see Fig. 2 confirm that the transition at $\beta_{c} = 3.7518(2)$ is continuous, and belongs to the 3D O(3) universality class, characterized by the universal exponents [42], [46] [48] $\nu = 0.7117(5), \eta = 0.0378(3), \omega = 0.782(13)$. We obtained analogous results for $\beta_{g} = 3 \beta_{c} = 3.203(1)$, and for $N_{c} = 4$ at $\beta_{g} = 0 \beta_{c} = 4.896(1)$, see Fig. 3, supporting the O(3) nature of the transition in both cases. These results confirm the LGW predictions. They lead us to conjecture that the phase diagram for

FIG. 2: $R_{\xi}$ versus $(\beta - \beta_{c}) L^{1/\nu}$ for $N_{f} = 2, N_{c} = 3$, and $\beta_{g} = 0, \text{up to } L = 64$. We use the O(3) value $\nu = 0.7117(5)$ and $\beta_{c} = 3.7518(2)$. The data collapse on a unique curve confirms the O(3) critical behavior. The inset reports $R_{c}$ versus $\beta$. The vertical dashed line corresponds to $\beta_{c}$, while the horizontal one corresponds to the critical value $R_{c}^{*} = 0.5639(2)$. 

FIG. 3: $R_{\xi}$ versus $(\beta - \beta_{c}) L^{1/\nu}$ for $N_{f} = 2, N_{c} = 3$, and $\beta_{g} = 0, \text{up to } L = 64$. We use the O(3) value $\nu = 0.7117(5)$ and $\beta_{c} = 3.7518(2)$. The data collapse on a unique curve confirms the O(3) critical behavior. The inset reports $R_{c}$ versus $\beta$. The vertical dashed line corresponds to $\beta_{c}$, while the horizontal one corresponds to the critical value $R_{c}^{*} = 0.5639(2)$. 

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$N_f = 2$ and $N_c \geq 3$ is characterized by a continuous transition line, related to the condensation of the order parameter $Q_{\sigma}$, which belongs to the O(3) universality class for any finite $\beta_g$. For $\beta_g \rightarrow \infty$ the critical behavior turns into that of O($N$) vector model with $N = 4N_c$.

We have also performed a FSS analysis for $N_f = N_c = 2$ at $\beta_g = 0$ (up to $L = 96$) and $\beta_g = 2$ (up to $L = 64$). In both cases we observe continuous transitions, at $\beta_c = 2.68885(5)$ and $\beta_c = 1.767(1)$, respectively [note that $\beta_c = 0.96339(1)$ in the O(8) vector model [49] obtained for $\beta_g \rightarrow \infty$]. Data are consistent with the O(5) universality class, whose critical exponents are $\nu = 0.779(3), \eta = 0.034(1)$, and $\omega = 0.79(2)$. Numerical results [44] for $\beta_g = 0$ are shown in Fig. 4. They confirm the predictions of the LGW theory based on the enlarged global symmetry group Sp(2)$\cong$O(5). We note that the enlarged O(5) symmetry may be seen as emerging from the combination of an O(3) magnetic-like and of a U(1) superfluid-like order parameter [44]. This provides a possible mechanism for emergent O(5) symmetries, as also claimed in various contexts, see, e.g., Refs. [5, 54, 60].

We finally mention that we also studied models for $N_f = 3$, $N_c = 2$ and 3, at $\beta_g = 0$. The numerical results provide evidence of a first-order transition in both cases [44]. This is again consistent with the predictions of the effective LGW theory [8].

In conclusion, we have investigated the phase diagram of the lattice multiflavor scalar chromodynamics [2], for positive couplings $\beta$ and $\beta_g$. This is a paradigmatic 3D model with a nonabelian gauge symmetry. For $N_f \geq 2$ the phase diagram is characterized by two phases: a low-temperature phase in which the order parameter $Q_{\sigma}^g$ condenses, and a high-temperature disordered phase where it vanishes. Gauge and vector observables do not show long-range correlations for any finite $\beta$ and $\beta_g$. The two phases are separated by a transition line driven by the condensation of $Q_{\sigma}^g$, as sketched in Fig. 1, that ends at the unstable O($N$) transition point with $N = 2N_cN_f$ for $\beta_g \rightarrow \infty$. The gauge coupling $\beta_g$ does not play any particular role: the nature of the transition is conjectured to be the same for any $\beta_g$, as numerically checked for some values of $\beta_g$. Along the transition line only correlations of the gauge-invariant operator $Q_{\sigma}^g$ display long-range order. Gauge modes are not critical and only represent a background that gives rise to crossover effects.

The numerical results are compared with the predictions of the continuum scalar gauge theory [2] and of the gauge-invariant LGW theory [8]. They agree with those of the LGW theory, showing that the LGW framework provides the correct description of the large-scale behavior of these systems, predicting first-order transitions for $N_f = 3$, and continuous transitions for $N_f = 2$, which belong to the O(3) universality class for $N_c \geq 3$ and O(5) universality class for $N_c = 2$. On the other hand, the results for $N_f = 2$ are in contradiction with the predictions of the continuum gauge model [2]: as no stable FP exists for $N_f = 2$, one would expect a first-order transition. There are at least two possible explanations for this apparent failure. A first possibility is that it does not encode the relevant modes at the transition. A second possibility is that the perturbative treatment around 4D does not provide the correct description of the 3D behavior. The 3D FP may not be related to a 4D FP, and therefore it escapes any perturbative analysis in powers of $\epsilon$. The analysis of the behavior in the large-$N_f$ limit, where again the two approaches give different results, may help
to shed light on these issues. Similar issues for the multi-component lattice scalar electrodynamics are addressed in Ref. [15, 61].

We have considered a paradigmatic lattice model obtained by gauging a maximally symmetric scalar system. It would be interesting to consider scalar theories with different global and local symmetries, and different symmetry-breaking patterns. Their classification deserves further investigation.

We mention that the LGW approach has been applied to the finite-temperature transition of quantum chromodynamics (QCD) with fermionic matter [23, 24]. The lattice-QCD numerical results have only partially confirmed the LGW predictions, due to the complexity of the simulations with fermions [62]. Our results support the effectiveness of the LGW approach, since the derivation of the LGW theory is essentially independent of the bosonic or fermionic nature of the matter fields. We finally stress that the results for lattice scalar chromodynamics may be particularly useful in condensed matter physics, to understand the effects of emerging nonabelian gauge fields [2, 6] at phase transitions.

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