Non-integrability of a three-dimensional generalized Hénon-Heiles system

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Abstract In recent paper Fakkousy et al. show that the 3D Hénon-Heiles system with Hamiltonian

\[ H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \frac{1}{2}(Aq_1^2 + Cq_2^2 + Bq_3^2) + (\alpha q_1^2 + \gamma q_2^2)q_3 + \frac{\beta}{3}q_3^3 \]

is integrable in sense of Liouville when \( \alpha = \gamma, \frac{\alpha}{\beta} = 1, A = B = C \); or \( \alpha = \gamma, \frac{\alpha}{\beta} = \frac{1}{6}, A = C, B \)-arbitrary; or \( \alpha = \gamma, \frac{\alpha}{\beta} = \frac{1}{16}, A = C, \frac{A}{B} = \frac{1}{16} \) (and of course, when \( \alpha = \gamma = 0 \), in which case the Hamiltonian is separable). It is known that the second case remains integrable for \( A, C, B \) arbitrary. Using Morales-Ramis theory, we prove that there are no other cases of integrability for this system.

1 Introduction

The generalized two-degrees-of-freedom Hénon-Heiles system (H–H2) is defined by the Hamiltonian

\[ H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(Aq_1^2 + Bq_2^2) + \alpha q_1^2 q_2 + \frac{\beta}{3}q_3^3 \]  

(1)

The original Hénon-Heiles Hamiltonian [1] agrees with (1) when \( A = B = \alpha = 1, \beta = -1 \). This famous system appears as an important model in a large set of problems in celestial, statistical and quantum mechanics, see for example the references in [2–8].

It is well known that the integrable cases of the above Hamiltonian are:

(0) \( \alpha = 0 \) – the Hamiltonian is separable;

(1) \( \beta = \alpha \) and \( A = B \);

(2) \( \beta = 6\alpha \) and \( A, B \)-arbitrary;

(3) \( \beta = 16\alpha \) and \( B = 16A \).

It seems that Ito [9] was the first who rigorously studied the non-integrability of (1) using the Ziglin’s monodromy approach (see Sect. 2). Assuming that \( A = B \), he established that there exists an additional independent integral only in the above cases. Later, Morales-Ruiz [10] studied the non-integrability using differential Galois approach. Again assuming \( A = B \), he showed that (1) is integrable in Liouvillian sense only in the above cases. Finally, Wenlei Li et al [2] proved using differential Galois approach without assuming priory \( A = B \) that (1) is Liouville integrable only in the above cases.
Let us recall some generalizations of (H–H2). In fact, Hénon and Heiles have studied in [1] an axisymmetric three-dimensional natural Hamiltonian system describing galactic motion. By means of the angular momentum integral they have reduced the considered system to a two-degrees-of-freedom one and have carried out numerical experiments in searching for additional integral on the zero level of the angular momentum integral. Later, Ferrer et al [3,4] again for three-dimensional axisymmetric Hénon-Heiles systems have studied normal forms, relative equilibria, bifurcations and existence of periodic orbits.

Several multi-dimensional integrable generalizations of case (2) above are known. In Eilbeck et al [5] a Lax pair for such a generalization is given, and hence, the commuting integrals are obtained. Kostov et al [6] have shown that the complete integrability is preserved in [5] even considering the additional terms of the kind \( C_j q_j^2 \) with arbitrary constants \( C_j \). Zeng [7] has constructed a hierarchy of multi-dimensional Hénon-Heiles systems with the help of the \( x \)- and \( t_\theta \)-constrained flows of the KdV hierarchy.

In recent paper Fakkousy et al. [8] have studied for integrability the following generalization of the Hénon-Heiles system (H–H3)

\[
H = \frac{1}{2} (p_1^2 + p_2^2 + p_3^2) + \frac{1}{2} (A q_1^2 + C q_2^2 + B q_3^2) + (\alpha q_1^2 + \gamma q_2^2) q_3 + \frac{\beta}{3} q_3^3,
\]

(2)

assuming further that \( A = C \) and \( \alpha = \gamma \). Due to this symmetry there exists an additional integral, and this is clear after introducing spherical coordinates. On the zero level of that integral Hamiltonian (2) coincides with (H–H2) Hamiltonian (1). Then, starting from the integrable cases listed above, the authors have succeeded in finding a complete set of commuting first integrals for (H–H3) in the following cases:

(i) \( \alpha = \gamma, \frac{\alpha}{\beta} = 1, A = B = C; \)

(ii) \( \alpha = \gamma, \frac{\alpha}{\beta} = \frac{1}{6}, A = C, B\)-arbitrary;

(iii) \( \alpha = \gamma, \frac{\alpha}{\beta} = \frac{1}{16}, A = C, \frac{A}{B} = \frac{1}{16}. \)

Again we have to add to the above list the case \( \alpha = \gamma = 0 \) in which Hamiltonian (2) is separable. As we pointed out earlier, the integrability in the second case (ii) can be extended for \( A, B, C \) arbitrary.

Numerical study of (H–H3), carried out in [8] near the cases of integrability, reveals chaotic behavior which prevents integrability. But there is still a possibility for existence of other integrable cases far from those already found. The motivation for our study is to establish whether this is true. Without assuming a priori that \( A = C \) and \( \alpha = \gamma \) we establish the following result.

**Theorem 1** *The three-dimensional Hénon-Heiles system corresponding to (2) is non-integrable by means of meromorphic first integrals except for the cases given above.*

The paper is organized as follows. In Section 2 we recall some notions and results from what is now called Ziglin-Morales-Ramis theory, including its generalization with higher-order variational equations. For the reader’s convenience, we also add two appendices in which we summarize known results concerning necessary conditions for integrability of Hamiltonian systems with homogeneous potentials and of Hamiltonian systems with variational equations of Lamé type. In Section 3, the proof of Theorem 1 is carried out in several steps.

First, we study the necessary conditions for integrability of the homogeneous Hamiltonian related to (2). Making use of the results in Appendix A1, we narrow the range of the
values of the parameters, for which an additional integral would exist. In Step 2, already for initial Hamiltonian (2), we study the necessary conditions for integrability of the variational equations along a particular solution, which are of Lamé type. In this way, we recover the integrability case (iii). Still, there are values of the parameters for which we cannot conclude non-integrability. To resolve these cases, in Step 3 we derive the higher variational equations up to order 3. Then, we use these equations in Steps 4 and 5, and along the way we recover the integrable case (i). Finally, in Step 6 we present the additional to \( H \) commuting integrals in the integrable case (ii) for \( A, B, C \) arbitrary.

Thus, we get a complete answer to the question about integrability and non-integrability of the considered system.

2 Theory

In this section, we summarize some notions and results related to Ziglin-Morales-Ramis theory. We refer the reader to [10–12] for a more complete exposition on differential Galois approach to integrability of Hamiltonian systems. A very detailed presentation about differential Galois theory may be found in the book of van der Put and Singer [13].

Given an analytic Hamiltonian \( H \), defined on a complex \( 2n \)-dimensional manifold \( \mathcal{M} \) determining the system

\[
\dot{x} = X_H(x), \quad t \in \mathbb{C}, \quad x \in \mathcal{M}
\]  

(3)

Recall that such Hamiltonian system is completely integrable (or integrable in Liouville sense) if it admits \( n \) independent first integrals \( F_1 = H, F_2, \ldots, F_n \) in involution.

Assume that system (3) has a non-equilibrium solution \( \varphi(t), t \in \tilde{I} \subset \mathbb{C} \). The image of \( \tilde{I} \) by \( \varphi \) is a Riemann surface \( \Gamma \). We can write the equation in variation (VE) along this solution

\[
\dot{\xi} = DX_H(\varphi(t))\xi, \quad \xi \in T_\Gamma M.
\]  

(4)

Further, using the integral \( dH \) we can reduce the variational equation. Consider the normal bundle of \( \Gamma, \mathcal{F} := T_\Gamma M / TM \) and let \( \pi : T_\Gamma M \to \mathcal{F} \) be the natural projection. Equation (4) induces an equation on \( \mathcal{F} \)

\[
\dot{\eta} = \pi_\ast(DX_H(\varphi(t))(\pi^{-1}\eta), \quad \eta \in \mathcal{F}.
\]  

(5)

which is called the normal variational equation (NVE). Each meromorphic first integral of Hamiltonian system (3) in a vicinity of \( \Gamma \) gives rise to a meromorphic first integral of (NVE) [14,15]. Therefore, the problem of complete integrability of (3) reduces to the study of integrability of linear systems (4) or (5).

Consider such a linear system

\[
\dot{x} = A(t)x, \quad x \in \mathbb{C}^n, \quad t \in \Gamma.
\]  

(6)

The continuation of the solutions along non-trivial loops on \( \Gamma \) defines a linear automorphisms of the space of solutions, called monodromy transformations. More precisely, let \( Y(t) \) be a fundamental solution of (6), analytic in some neighborhood of any non-singular point \( t_0 \). The linear automorphism \( \Delta_\gamma \), associated with the loop \( \gamma \in \pi_1(\Gamma, t_0) \), corresponds to a multiplication of \( Y(t) \) from the right by a constant matrix \( M_\gamma \)—monodromy matrix.

\[
\Delta_\gamma Y(t) = Y(t)M_\gamma.
\]

The set of these matrices constitute the monodromy group.
In 1982, Ziglin [14, 15] used the relation between the monodromy group of (VE) or (NVE) and branching of solutions to obtain necessary conditions for integrability of complex Hamiltonian systems. Let us mention one more application of Ziglin’s theory which is related to our study: in 1987, Yoshida [16] found a criterion for non-existence of an additional integral in two-degrees-of-freedom Hamiltonian systems with a homogeneous potential.

Morales-Ruiz and Ramis extended the Ziglin’s approach in studying the integrability of Hamiltonian systems by investigating the structure of the differential Galois group of (NVE) (or (VE)) along certain non-equilibrium solution. We briefly recall some notions and facts.

Denote the coefficient field in (6) by \( K \). A differential field \( K \) is a field with derivation \( \partial = ' \), i.e., an additive mapping satisfying Leibnitz rule. A differential automorphism of \( K \) is an automorphism commuting with the derivation. Let \( y_{ij} \) be the elements of the fundamental matrix \( Y(t) \). Let \( L(y_{ij}) \) be the extension of \( K \) generated by \( K \) and \( y_{ij} \) – a differential field. This extension is called Picard-Vessiot extension. Similarly to classical Galois theory we define the Galois group \( G = \text{Gal}(L/K) \) to be the group of all differential automorphisms of \( L \) leaving the elements of \( K \) fixed. The Galois group is, in fact, an algebraic group. It has a unique connected component \( G_0 \) which contains the identity and which is a normal subgroup of finite index. The Galois group \( G \) can be represented as an algebraic linear subgroup of \( GL(n, \mathbb{C}) \) by

\[
\sigma(Y(t)) = Y(t)R_\sigma,
\]

where \( \sigma \in G \) and \( R_\sigma \in GL(n, \mathbb{C}) \) (see, e.g., [13]).

The solutions of (4) define an extension \( L_1 \) of the coefficient field \( K \) of (VE). This naturally defines a differential Galois group \( G = \text{Gal}(L_1/K) \). Then, the following fundamental result has been established

**Theorem 2** (Morales-Ruiz-Ramis [10]) Suppose that a Hamiltonian system has \( n \) meromorphic first integrals in involution. Then the identity component \( G_0 \) of the Galois group \( G = \text{Gal}(L_1/K) \) is abelian.

If it turns out that \( G_0 \) is not abelian, the studied Hamiltonian system is non-integrable in the Liouville sense, but since the above result is only a necessary condition if \( G_0 \) is abelian that does not imply integrability.

To obtain other obstacles to the integrability, a method based on the higher variational equations has been introduced in [10] and the previous theorem has been extended in [11]. Before formulating this result let us give an idea of higher variational equations. For system (4) with a particular solution \( \varphi(t) \) we put

\[
x = \varphi(t) + \varepsilon\xi^{(1)} + \varepsilon^2\xi^{(2)} + \ldots + \varepsilon^k\xi^{(k)} + \ldots,
\]

where \( \varepsilon \) is a formal small parameter. Substituting the above expression into (3) and comparing terms with the same order in \( \varepsilon \) we obtain the following chain of linear non-homogeneous equations

\[
\dot{\xi}^{(k)} = A(t)\xi^{(k)} + r_k(\xi^{(1)}, \ldots, \xi^{(k-1)}), \quad k = 1, 2, \ldots,
\]

where \( A(t) = DX_H(\varphi(t)) \) and \( r_1 \equiv 0 \). Equation (8) is called \( k \)-th variational equation (VE\(_k\)). Let \( Y(t) \) be the fundamental matrix of (VE\(_1\))

\[
\dot{Y} = A(t)Y.
\]

Then the solutions of (VE\(_k\)), \( k > 1 \) can be found by

\[
\xi^{(k)} = Y(t)c(t),
\]
where \( c(t) \) is a solution of
\[
\dot{c} = Y^{-1}(t)r_k. \tag{10}
\]

As we mention above, the higher variational equations (VE\(_k\)) are not homogeneous equations, but they can be made such, and therefore, one can define successive extensions \( K \subset L_1 \subset L_2 \subset \ldots \subset L_k \), where \( L_k \) is the extension obtained by adjoining the solutions of (VE\(_k\)). Then, naturally one can define the Galois groups \( \text{Gal}(L_1/K), \ldots, \text{Gal}(L_k/K) \). The following theorem is proven in [11].

**Theorem 3** If Hamiltonian system (3) is integrable in Liouville sense, then the identity component of every Galois group \( \text{Gal}(L_k/K) \) is abelian.

Notice that we apply Theorem 3 in the situation when the identity component of the Galois group \( \text{Gal}(L_1/K) \) is abelian. This means that the first variational equation is solvable. Once we have the solution of (VE\(_1\)), then the solutions of (VE\(_k\)) can be found successively by the method of variations of constants in the way explained above. In turn, this implies that the Galois groups \( \text{Gal}(L_k/K) \) are solvable. One possible way to show that some of them is not abelian is to find a logarithmic term in the corresponding local solution (see detailed descriptions and explanations in [10–12]).

### 3 Proof of Theorem 1

For the proof we use the strategy from [2] applied in studying the non-integrability of the two-dimensional Hénon-Heiles system.

Any natural Hamiltonian with a polynomial potential can be represented in the form
\[
H = \frac{1}{2} \sum p_j^2 + V(q) = \frac{1}{2} \sum p_j^2 + V_{\text{min}}(q) + \ldots + V_{\text{max}}(q), \tag{11}
\]
where \( V_{\text{min}}(q)(V_{\text{max}}(q)) \) is the lowest (highest) order term of \( V(q) \). It is established by Hietarinta [17] and Maciejevski, Przybylska [18] that if the Hamiltonian system defined by \( H \) is integrable, then integrable are its subsystems
\[
H_{\text{min}} = \frac{1}{2} \sum p_j^2 + V_{\text{min}}(q), \quad H_{\text{max}} = \frac{1}{2} \sum p_j^2 + V_{\text{max}}(q).
\]

On the other hand, if it turns out that some of the above subsystems are non-integrable, then the original Hamiltonian system is non-integrable.

In our case \( H_{\text{min}} \) is trivially integrable, so we consider
\[
H = T + V_{\text{max}} = \frac{1}{2} (p_1^2 + p_2^2 + p_3^2) + (\alpha q_1^2 + \gamma q_2^2)q_3 + \frac{\beta}{3} q_3^3, \tag{12}
\]
where \( \alpha, \beta, \gamma \) are assumed nonzero. After rescaling \( t \) we get
\[
H = \frac{1}{2} (p_x^2 + p_y^2 + p_z^2) + (\alpha_1 x^2 + \gamma_1 y^2)z + \frac{1}{3} z^3 \tag{13}
\]
with \( \alpha_1 = \alpha/\beta \) and \( \gamma_1 = \gamma/\beta \).
First, we study the integrability of the Hamiltonian system corresponding to Hamiltonian (13).

**Step 1. Necessary conditions for the integrability of (13)**

The equations of motion for Hamiltonian (13) are

\[\ddot{q}_1 = -2\alpha q_1 q_3, \quad \ddot{q}_2 = -2\gamma q_2 q_3, \quad \ddot{q}_3 = -\alpha q_1^2 - \gamma q_2^2 - q_3^2.\]  

(14)

We are looking for a particular solution of the form

\[q_1 = c_1 u(t), \quad q_2 = c_2 u(t), \quad q_3 = c_3 u(t),\]  

(15)

where

\[\ddot{u} = -u^2, \quad \dot{u}^2 = \frac{2}{3}(1 - u^3).\]  

(16)

Then, \(c = (c_1, c_2, c_3)^T\) satisfies the following system

\[c_1 = 2\alpha_1 c_3, \quad c_2 = 2\gamma_1 c_2, \quad c_3 = \alpha_1 c_1^2 + \gamma_1 c_2^2 + c_3^2.\]  

(17)

There are several possibilities for solutions of (17) which we are going to use:

1) \(c_1 = 0, \ c_2 = \frac{1}{2\gamma_1} \sqrt{2 - \frac{1}{\gamma_1}}, \ c_3 = \frac{1}{2\gamma_1}.\)

The variational equations (VE) along the found particular solution read

\[\ddot{\xi} = -u(t)V''(c)\xi,\]  

(18)

where \(\xi = (\xi_1, \xi_2, \xi_3)^T\) and

\[V''(c) = \begin{pmatrix} 2\alpha_1 c_3 & 0 & 2\alpha_1 c_1 \\ 0 & 2\gamma_1 c_3 & 2\gamma_1 c_2 \\ 2\alpha_1 c_1 & 2\gamma_1 c_2 & 2c_3 \end{pmatrix} = \begin{pmatrix} \frac{\alpha_1}{\gamma_1} & 0 & 0 \\ 0 & 1 & \sqrt{2 - \frac{1}{\gamma_1}} \\ 0 & \sqrt{2 - \frac{1}{\gamma_1}} & \frac{1}{\gamma_1} \end{pmatrix}.\]  

(19)

The eigenvalues of \(V''(c)\) are

\[\lambda_1 = \frac{\alpha_1}{\gamma_1}, \quad \lambda_2 = \frac{1}{\gamma_1} - 1, \quad \lambda_3 = 2.\]  

(20)

2) \(c_2 = 0, \ c_1 = \frac{1}{2\alpha_1} \sqrt{2 - \frac{1}{\alpha_1}}, \ c_3 = \frac{1}{2\alpha_1}.\)

The eigenvalues of \(V''(c)\) in this case are

\[\lambda_1 = \frac{\gamma_1}{\alpha_1}, \quad \lambda_2 = \frac{1}{\alpha_1} - 1, \quad \lambda_3 = 2.\]  

(21)

3) \(c_1 = 0, \ c_2 = 0, \ c_3 = 1.\)

The eigenvalues of \(V''(c)\) in this case are

\[\lambda_1 = 2\alpha_1, \quad \lambda_2 = 2\gamma_1, \quad \lambda_3 = 2.\]  

(22)

Now, if the Hamiltonian system is integrable, the pairs \((k = 3, \lambda_j)\) have to belong to one of the following cases (1), (11), (12), (13), (14), (18) of Theorem 4 (see Appendix A1).

Denote

\[g_1(s) = s + s(s - 1)3/2, \quad g_2(s) = -\frac{1}{24} + \frac{1}{24}(2 + 6s)^2, \quad g_3(s) = -\frac{1}{24} + \frac{1}{24} \left(\frac{3}{2} + 6s\right)^2.\]
\[ g_4(s) = -\frac{1}{24} + \frac{1}{24} \left( \frac{6}{5} + 6\xi \right)^2, \quad g_5(s) = -\frac{1}{24} + \frac{1}{24} \left( \frac{12}{5} + 6\xi \right)^2, \quad g_6(s) = \frac{1}{5} + \frac{3}{2}s(s+1) \]

and observe that \( g_j(s) > 0 \) for \( s \in \mathbb{Z} \). Then, if \( \lambda_2 = g_j(s) \) for some integer \( s \) in (20), (21) we get that

\[ \alpha_1, \gamma_1 \in (0, 1], \quad \alpha_1, \gamma_1 \in \mathbb{Q}. \] (23)

Now, to obtain necessary conditions for existence of additional first integral we incorporate condition (89) from Theorem 5 (see Appendix A1) for \( \lambda_2 \) in any of above possibilities. We have

\[ \frac{1}{6} \sqrt{1 + 24 \frac{\alpha_1}{\gamma_1}} = \frac{1}{6} \sqrt{1 + 24 \left( \frac{1}{\gamma_1} - 1 \right) + l_1}, \] (24)

\[ \frac{1}{6} \sqrt{1 + 24 \frac{\gamma_1}{\alpha_1}} = \frac{1}{6} \sqrt{1 + 24 \left( \frac{1}{\alpha_1} - 1 \right) + l_2}, \] (25)

\[ \frac{1}{6} \sqrt{1 + 48 \alpha_1} = \frac{1}{6} \sqrt{1 + 48 \gamma_1 + l_3}, \] (26)

for some \( l_1, l_2, l_3 \in \mathbb{Z} \). It is known from [19] that all radicals in these expressions are in fact rational numbers.

Consider relation (26). Due to (23)

\[ 1 < \sqrt{1 + 48 \alpha_1}, \quad \frac{1}{6} \sqrt{1 + 48 \gamma_1} \leq 7. \]

Hence, (26) is valid only for \( l_3 = 0 \), but this gives

\[ \gamma_1 = \alpha_1. \]

Alternatively, the equality \( \gamma_1 = \alpha_1 \) can be obtained also by exploring the necessary conditions from Theorem 4. Suppose \( \lambda_1 = \frac{\alpha_1}{\gamma_1} := w \) from (20) takes values in some \( g_i(s) \), \( i = 1, \ldots, 6, s \in \mathbb{Z} \), i.e., \( w = g_i(s) \). Similarly, for integrability \( \lambda_2 = \frac{\gamma_1}{\alpha_1} := \frac{1}{w} \) from (21) also has to take values in some \( g_j(q) \), \( j = 1, \ldots, 6, q \in \mathbb{Z} \), that is, \( \frac{1}{w} = g_j(q) \). Then we must have

\[ g_j(q) = \frac{1}{g_i(s)} \]

for some \( i, j = 1, \ldots, 6 \) and some \( q, s \in \mathbb{Z} \). But this can be achieved only for \( i = j = 1 \) and \( q = s = 1 \), which implies that \( w = 1 \), or \( \gamma_1 = \alpha_1 \).

This result has two major consequences: firstly, Hamiltonian system (14) admits an additional integral \( F = q_1p_2 - q_2p_1 \), and secondly, we get a representation from (25) for \( \alpha_1 \) for which we may have complete integrability, namely

\[ \frac{5}{6} = \frac{1}{6} \sqrt{1 + 24 \left( \frac{1}{\alpha_1} - 1 \right) + l}, \quad l \in \mathbb{Z}, \] (27)

which is equivalent to

\[ \frac{1}{\alpha_1} - 1 = -\frac{1}{24} + \frac{1}{24} (5 - 6l)^2 \quad \text{or} \quad \alpha_1 = \frac{24}{23 + (5 - 6l)^2}. \] (28)

We will use the above representation to obtain those \( \alpha_1 \) for which Hamiltonian system (14) is necessarily integrable by investigating whether there are some \( s, l \in \mathbb{Z} \), such that \( \lambda_2 \)
from (21) and (22) takes values in the six families \( g_j(s), \ j = 1, 2, \ldots, 6 \). Starting with \( \lambda_2 \) from (21) we have
\[
\frac{1}{\alpha_1} - 1 = -\frac{1}{24} + \frac{1}{24} (5 - 6l)^2 = g_j(s), \quad j = 1, \ldots, 6.
\] (29)

Trivial computations show that there are no such integers \( s, l \) for \( j = 2, \ldots, 6 \). However,
\[
-\frac{1}{24} + \frac{1}{24} (5 - 6l)^2 = g_1(s) = \frac{3}{2} s^2 - \frac{1}{2} s
\]
is reduced to \( l + s = 1 \), which in turn gives
- \( s = 0, l = 1 \), and therefore, \( \alpha_1 = 1 \);
- \( s = 1, l = 0 \), and therefore, \( \alpha_1 = \frac{1}{2} \).

Further, we turn to \( \lambda_2 \) from (22). Using that \( \alpha_1 = \gamma_1 \) and (28), we have to find \( s, l \in \mathbb{Z} \) for which the following relation
\[
2\alpha_1 = \frac{48}{23 + (5 - 6l)^2} = g_j(s), \quad \text{for some } s, l \in \mathbb{Z}.
\] (30)
is fulfilled. Since \( 0 < \frac{48}{23 + (5 - 6l)^2} < 1 \), we should have \( 0 < g_j(s) \leq 1 \) and this gives a very few possibilities for \( s \). Indeed,
\[
\frac{48}{23 + (5 - 6l)^2} = g_1(s) = \frac{3}{2} s^2 - \frac{1}{2} s
\]
is fulfilled for \( s = 0 \) and \( l = 1 \), which amounts to \( \alpha_1 = 1/2 \), but we know that.

Next,
\[
\frac{48}{23 + (5 - 6l)^2} = g_2(s) = -\frac{1}{24} + \frac{1}{24} (6s + 2)^2
\]
is fulfilled for \( s = 0 \) and \( l = 4 \), which gives \( \alpha_1 = \frac{1}{16} \). There are no \( s, l \in \mathbb{Z} \) which satisfy
\[
\frac{48}{23 + (5 - 6l)^2} = g_j(s), \quad j = 3, 4, 5.
\]
Finally,
\[
\frac{48}{23 + (5 - 6l)^2} = g_6(s) = \frac{1}{2} \left[ \frac{2}{3} + 3s(s + 1) \right]
\]
is fulfilled for \( s = 0 \) and \( l = -1 \), and hence, \( \alpha_1 = \frac{1}{6} \).

Summarizing, the necessary conditions for system (14) to be integrable are \( \alpha_1 = \gamma_1 \) and \( \alpha_1 = 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{16} \). Therefore, we have the following

**Proposition 1** Hamiltonian system (14) corresponding to the Hamiltonian with homogeneous potential (13) is non-integrable if

(i) \( \gamma_1 \neq \alpha_1 \), or

(ii) \( \gamma_1 = \alpha_1, \quad \alpha_1 \in \mathbb{C} \setminus \{1, \frac{1}{2}, \frac{1}{6}, \frac{1}{16}\} \)

Now, consider original Hamiltonian (2) assuming that \( A, B \) and \( C \) are nonzero. Again after rescaling \( t \) we get
\[
H = \frac{1}{2} (p_1^2 + p_2^2 + p_3^2) + \frac{1}{2} (A_1 q_1^2 + C_1 q_2^2 + B_1 q_3^2) + (\alpha_1 q_1^2 + \gamma_1 q_2^2) q_3 + \frac{1}{3} q_3^3, \quad (31)
\]
where \( A_1 = A/\beta, B_1 = B/\beta, C_1 = C/\beta \) and \( \alpha_1 = \gamma_1 \) are as before. The Hamiltonian system corresponding to (31) reads
\[
\ddot{q}_1 = -A_1 q_1 - 2\alpha_1 q_1 q_3, \quad \ddot{q}_2 = -C_1 q_2 - 2\alpha_1 q_2 q_3, \quad \ddot{q}_3 = -B_1 q_3 - \alpha_1 (q_1^2 + q_2^2) - q_3^2.
\]
\[
(32)
\]

The above system admits the following particular solution
\[
\tau \frac{d}{\sqrt{6}} t, \quad i^2 = -1, \quad g_2 = 3B_1^2, \quad g_3 = 12h - B_1^3
\]
The variational equation (VE) along this solution, written with respect to the independent variable \( \tau, (\dot{\tau} = d/d\tau) \), is
\[
\xi_1'' - [12\alpha_1 \phi'(\tau) + 6(A_1 - B_1\alpha_1)]\xi_1 = 0, \quad (34)
\]
\[
\xi_2'' - [12\alpha_1 \phi'(\tau) + 6(C_1 - B_1\alpha_1)]\xi_2 = 0, \quad (35)
\]
\[
\xi_3'' - 12\phi'(\tau)\xi_3 = 0. \quad (36)
\]
Clearly, equation (36) forms the tangential part of (VE), while equations (34) and (35) form (NVE). Notice that (NVE) is given by two independent second-order equations of Lamé type.

**Step 2. Analysis of (NVE)**

In what follows, we apply the results described in Appendix A.2. Consider first (34) written as
\[
\xi_1'' - f_1(\tau, h)\xi_1 = 0
\]
with \( f_1(\tau, h) = 12\alpha_1 \phi'(\tau) + 6(A_1 - B_1\alpha_1) \). After some calculations we get
\[
f_1'' = P(f_1, h) = \frac{1}{3\alpha_1} f_1^3 + \frac{6}{\alpha_1} (A_1 - B_1\alpha_1) f_1^2 + \left[-36\alpha_1 B_1^2 + \frac{36}{\alpha_1} (A_1 - B_1\alpha_1)^2\right] f_1
\]
\[
- 216\alpha_1 B_1^2 (A_1 - B_1\alpha_1) + \frac{72}{\alpha_1} (A_1 - B_1\alpha_1)^3 + (12)^2 \alpha_1^2 B_1^3 - h(12)^3 \alpha_1^2.
\]
\[
(37)
\]
We write down the coefficients of the polynomial \( P(f_1, h) \) in a way we need them
\[
a_1 = \frac{1}{3\alpha_1}, \quad a_2 = 0,
\]
\[
b_1 = \frac{6}{\alpha_1} (A_1 + B_1\alpha_1), \quad b_2 = 0,
\]
\[
c_1 = -36\alpha_1 B_1^2 + \frac{36}{\alpha_1} (A_1 - B_1\alpha_1)^2, \quad c_2 = 0,
\]
\[
d_1 = -216\alpha_1 B_1^2 (A_1 - B_1\alpha_1) + \frac{72}{\alpha_1} (A_1 - B_1\alpha_1)^3 + (12)^2 \alpha_1^2 B_1^3,
\]
\[
d_2 = -(12)^3 \alpha_1^2.
\]
Now, we explore the necessary conditions for integrability as they are given in Appendix A2. The first necessary condition from Theorem 5 – I, \( a_1 = -4/n(n+1) \) for some \( n \in \mathbb{N} \) gives
that $\alpha_1 = \frac{n(n+1)}{12}$ for some $n \in \mathbb{N}$. Recall that we have to deal only with the cases
\begin{equation}
n = 1 \quad (\alpha_1 = \frac{1}{6}), \quad n = 2 \quad (\alpha_1 = \frac{1}{2}), \quad \text{and} \quad n = 3 \quad (\alpha_1 = 1) \tag{38}
\end{equation}
because of Proposition 1.

Next, we proceed with the cases of condition II.

The case II1, $m = 1$ gives that $\alpha_1 = \frac{1}{16}$, while the condition $b_1 = 0$ amounts to
\begin{equation}
\frac{A_1}{B_1} = \alpha_1 = \frac{1}{16} \left( \frac{A}{B} = \frac{1}{16} \right). \tag{39}
\end{equation}

The case II2, $m = 2$ gives $\alpha_1 = \frac{5}{16}$, but for values of $\alpha_1$ different from \{1, 1/2, 1/6, 1/16\} we have non-integrability again due to the results from Step 1.

The case II3, $m = 3$ does not occur here since $c_2 = 0$ and $16a_1d_2 + 3b_1c_2 = 0$ yield $a_1d_2 = 0$, which is not possible.

The case II, $m > 3$ is also not possible due to $d_2 \neq 0$ and the assumptions $A_1 \neq 0$ and $B_1 \neq 0$.

Finally, the case III does not occur, because after expressing all necessary conditions we get $B_1 = 0$ which is a contradiction to our assumption.

Similarly, (35) can be written as
\begin{equation}
\xi''_2 - f_2(\tau, h)\xi_1 = 0
\end{equation}
with $f_2(\tau, h) = 12\alpha_1\varphi(\tau) + 6(C_1 - B_1\alpha_1)$. The polynomial $P(f_2, h)$ is exactly as (37) with $A_1$ replaced by $C_1$. The necessary conditions I, II, II, and III give nothing new. In the case II where $m = 1$ and $\alpha_1 = 1/16$ we obtain from the condition $b_1 = 0$ that
\begin{equation}
\frac{C_1}{B_1} = \alpha_1 = \frac{1}{16} \left( \frac{C}{B} = \frac{1}{16} \right). \tag{40}
\end{equation}

This condition, together with (39), results in
\begin{equation}
A = C, \quad A = \frac{1}{16}, \quad \alpha = \gamma, \quad \frac{\alpha}{\beta} = \frac{1}{16}, \tag{41}
\end{equation}
but we know that in this case Hamiltonian (2) is integrable.

Summarizing, we get a more definitive answer for the case $\alpha_1 = 1/16$: together with already established condition $\alpha_1 = \gamma_1(\alpha = \gamma)$ Hamiltonian (2) is necessarily integrable if $A_1 = C_1(A = C)$, but this gives the already known integrable case (iii) from the Introduction.

**Step 3. Higher variational equations**

Further, it remains to deal with cases (38). In any of these cases the Galois group of variational equation is abelian (see Appendix A.2). Hence, we need to study the Galois groups of higher variational equations. As it was explained in Section 2, to write them we put
\begin{align}
q_1 &= 0 + \varepsilon \xi_1^{(1)} + \varepsilon^2 \xi_1^{(2)} + \varepsilon^3 \xi_1^{(3)} + \ldots, \\
q_2 &= 0 + \varepsilon \xi_2^{(1)} + \varepsilon^2 \xi_2^{(2)} + \varepsilon^3 \xi_2^{(3)} + \ldots, \tag{42} \\
q_3 &= q_3^0 + \varepsilon \xi_3^{(1)} + \varepsilon^2 \xi_3^{(2)} + \varepsilon^3 \xi_3^{(3)} + \ldots
\end{align}
and substitute these expressions into (32). Comparing the terms with the same order in $\varepsilon$, we obtain the variational equations up to order three.
We are going to write these higher variational equations as second-order linear equations with respect to the independent variable $\tau$. The first variational equation (VE$_1$) is nothing but (34)–(36)

\[
(\xi_1^{(1)})'' - [12\alpha_1 \wp(\tau) + 6(A_1 - B_1\alpha_1)]\xi_1^{(1)} = 0,
\]
\[
(\xi_2^{(1)})'' - [12\alpha_1 \wp(\tau) + 6(C_1 - B_1\alpha_1)]\xi_2^{(1)} = 0,
\]
\[
(\xi_3^{(1)})'' - 12\wp(\tau)\xi_3^{(1)} = 0.
\]

For the second variational equation (VE$_2$) we get

\[
(\xi_1^{(2)})'' - [12\alpha_1 \wp(\tau) + 6(A_1 - B_1\alpha_1)]\xi_1^{(2)} = W_1^{(2)},
\]
\[
(\xi_2^{(2)})'' - [12\alpha_1 \wp(\tau) + 6(C_1 - B_1\alpha_1)]\xi_2^{(2)} = W_2^{(2)},
\]
\[
(\xi_3^{(2)})'' - 12\wp(\tau)\xi_3^{(2)} = W_3^{(2)},
\]

where

\[
W_j^{(2)} = 12\alpha_1 \xi_j^{(1)} \xi_3^{(1)}, \quad j = 1, 2, \quad W_3^{(2)} = 6\alpha_1 [ (\xi_1^{(1)})^2 + (\xi_2^{(1)})^2 ] + 6(\xi_3^{(1)})^2.
\]

The third variational equation (VE$_3$) is

\[
(\xi_1^{(3)})'' - [12\alpha_1 \wp(\tau) + 6(A_1 - B_1\alpha_1)]\xi_1^{(3)} = W_1^{(3)},
\]
\[
(\xi_2^{(3)})'' - [12\alpha_1 \wp(\tau) + 6(C_1 - B_1\alpha_1)]\xi_2^{(3)} = W_2^{(3)},
\]
\[
(\xi_3^{(3)})'' - 12\wp(\tau)\xi_3^{(3)} = W_3^{(3)}.
\]

Here

\[
W_j^{(3)} = 12\alpha_1 (\xi_j^{(1)} \xi_3^{(2)} + \xi_j^{(2)} \xi_3^{(1)}), \quad j = 1, 2.
\]

We do not need $W_3^{(3)}$, but it is calculated to be $W_3^{(3)} = 12\alpha_1 (\xi_1^{(1)} \xi_2^{(2)} + \xi_2^{(1)} \xi_2^{(2)}) + 12\xi_3^{(1)} \xi_3^{(2)}$. Keeping our notation from Section 2 we have

\[
\begin{align*}
    r_2 &= (0, W_1^{(2)}, 0, W_2^{(2)}, 0, W_3^{(2)}), \quad r_3 &= (0, W_1^{(3)}, 0, W_2^{(3)}, 0, W_3^{(3)}).
\end{align*}
\]

To prove non-integrability of (31) we have to show that the identity component of the Galois group of (VE$_k$, $k = 2, 3$) is not abelian. To get this let us note that the variational equations have a singular point at $\tau = 0$. Then it is enough to obtain a logarithm around the singular point, or equivalently, a residue different from zero in some integrand (see [10]). Now we are going to calculate the local solutions around $\tau = 0$ and to show that a logarithmic term appears in the solutions of (VE$_3$). Notice that all expansions below are convergent in some vicinity of $\tau = 0$ [20].

Let $\xi_{j,1}^{(1)}, \xi_{j,1}^{(1)}$, $j = 1, 2, 3$ be two linearly independent solutions of (43)–(45) with unit Wronskian. Then, the fundamental matrix $Y(\tau)$ of (VE$_1$) and its inverse have the block-diagonal form

\[
Y(\tau) = \begin{pmatrix}
Y_1 & 0 & 0 \\
0 & Y_2 & 0 \\
0 & 0 & Y_3
\end{pmatrix}, \quad Y^{-1}(\tau) = \begin{pmatrix}
Y_1^{-1} & 0 & 0 \\
0 & Y_2^{-1} & 0 \\
0 & 0 & Y_3^{-1}
\end{pmatrix},
\]

where

\[
Y_j = \begin{pmatrix}
\xi_{j,1}^{(1)} & \xi_{j,2}^{(1)} \\
\xi_{j,1}^{(1)' - \xi_{j,1}^{(1)}} & \xi_{j,2}^{(1)'} - \xi_{j,1}^{(1)'}
\end{pmatrix}, \quad Y_j^{-1} = \begin{pmatrix}
\xi_{j,2}^{(1)'} - \xi_{j,1}^{(1)} & -\xi_{j,2}^{(1)'} \\
-\xi_{j,1}^{(1)'} & \xi_{j,1}^{(1)'}
\end{pmatrix}.
\]
Now, we return to the integrability analysis of cases (38).

**Step 4. Non-integrability of the case** $n = 3, \quad \alpha_1 = 1$.

One can find the following expansions of the solutions of (VE1) (43), (44) and (45) with $\alpha_1 = 1$.

\[\xi^{(1)}_{1,1} = \frac{1}{\tau^3} \left[ 1 - \frac{3}{5} (A_1 - B_1) \tau^2 + \frac{3}{10} \left( A_1^2 - 2A_1B_1 + \frac{B_1^2}{2} \right) \tau^4 + \ldots \right],\]
\[\xi^{(1)}_{1,2} = \frac{\tau^4}{7} \left[ 1 + \frac{1}{3} (A_1 - B_1) \tau^2 + \ldots \right].\]  

Similarly, we have
\[\xi^{(1)}_{2,1} = \frac{1}{\tau^3} \left[ 1 - \frac{3}{5} (C_1 - B_1) \tau^2 + \frac{3}{10} \left( C_1^2 - 2C_1B_1 + \frac{B_1^2}{2} \right) \tau^4 + \ldots \right],\]
\[\xi^{(1)}_{2,2} = \frac{\tau^4}{7} \left[ 1 + \frac{1}{3} (C_1 - B_1) \tau^2 + \ldots \right]\]

and also,
\[\xi^{(1)}_{3,1} = \frac{1}{\tau^3} - \frac{g_2}{20} \tau - \frac{g_3}{14} \tau^3 + \frac{g_2^2}{400} \tau^5 + \ldots, \quad \xi^{(1)}_{3,2} = \frac{\tau^4}{7} \left[ 1 + \frac{3}{220} g_2 \tau^4 + \ldots \right].\]  

There are no logarithmic terms in the expansions around $\tau = 0$ of the local solutions of (VE2) and in what follows we write down the expansions of their local solutions only in the cases we will use further.

For the first equation of (VE2) with a specific choice of the right-hand terms
\[\left( \xi^{(2)}_{1,j} \right)'' - [12 \alpha_1 \wp(\tau) + 6(A_1 - B_1)] \xi^{(2)}_{1,j} = 12 \xi^{(1)}_{1,2} \xi^{(1)}_{3,1}\]
we find the following expansions around $\tau = 0$
\[\xi^{(2)}_{1,j} = \xi^{(1)}_{1,j} + \tau^3 \left[ - \frac{2}{7} - \frac{1}{7} (A_1 - B_1) \tau^2 + \ldots \right], \quad j = 1, 2.\]  

Similarly, for the second equation
\[\left( \xi^{(2)}_{2,j} \right)'' - [12 \alpha_1 \wp(\tau) + 6(C_1 - B_1)] \xi^{(2)}_{2,j} = 12 \xi^{(1)}_{2,2} \xi^{(1)}_{3,1}\]
we obtain
\[\xi^{(2)}_{2,j} = \xi^{(1)}_{2,j} + \tau^3 \left[ - \frac{2}{7} - \frac{1}{7} (C_1 - B_1) \tau^2 + \ldots \right], \quad j = 1, 2.\]  

Finally, for the third equation in (VE2) with the following choice of the right-hand terms
\[\left( \xi^{(2)}_{3,j} \right)'' - 12 \wp(\tau) \xi^{(2)}_{3,j} = 6 \left[ (\xi^{(1)}_{1,j})^2 + (\xi^{(1)}_{2,j})^2 + (\xi^{(1)}_{3,j})^2 \right]\]
we get the expansions
\[\xi^{(2)}_{3,j} = \xi^{(1)}_{3,j} + \frac{1}{\tau^4} \left[ \frac{9}{4} + \frac{6}{5} (C_1 + A_1 - 2B_1) \tau^2 + K_4 \tau^4 + \ldots \right], \quad j = 1, 2.\]  

where $K_4$ is
\[K_4 := -\frac{339}{400} B_1^2 - \frac{12}{25} C_1^2 + \frac{24}{25} A_1 B_1 + \frac{24}{25} C_1 B_1 - \frac{12}{25} A_1^2.\]
Now we will study the local solutions of (VE₃). Consider the first equation with the following specific representatives on the right-hand side.

\[(\xi^{(3)}_1)^{\prime\prime} - [12\alpha_1 \wp(\tau) + 6(A_1 - B_1)]\xi^{(3)}_1 = 12 \left(\xi^{(1)}_{1,1} \xi^{(2)}_{3,2} + \xi^{(2)}_{1,2} \xi^{(1)}_{3,2}\right) = W^{(3)}_1. \quad (66)\]

Denote

\[ (v^{(3)}_{1,1}, v^{(3)}_{1,2}) = Y^{-1}_1 \begin{pmatrix} 0 \\ W^{(3)}_1 \end{pmatrix}. \]

Then, for \(v^{(3)}_{1,1} = -12\xi^{(1)}_{1,2} [\xi^{(1)}_{1,1} \xi^{(2)}_{3,2} + \xi^{(2)}_{1,2} \xi^{(1)}_{3,2}]\) we obtain

\[ \text{Res}_{\tau=0} v^{(3)}_{1,1} = -\frac{36}{35} (A_1 + 2C_1 - B_1). \quad (67)\]

Similar computations for (51)

\[(\xi^{(3)}_2)^{\prime\prime} - [12\alpha_1 \wp(\tau) + 6(C_1 - B_1)]\xi^{(3)}_2 = 12 \left(\xi^{(1)}_{2,1} \xi^{(2)}_{3,2} + \xi^{(2)}_{1,2} \xi^{(1)}_{3,2}\right) = W^{(3)}_2 \quad (68)\]

give

\[ (v^{(3)}_{2,1}, v^{(3)}_{2,2}) = Y^{-1}_2 \begin{pmatrix} 0 \\ W^{(3)}_2 \end{pmatrix} \]

and

\[ \text{Res}_{\tau=0} v^{(3)}_{2,1} = -\frac{36}{35} (C_1 + 2A_1 - B_1). \quad (69)\]

Suppose for a moment that both residues (67) and (69) are zero, that is,

\[ A_1 + 2C_1 - 3B_1 = 0, \quad C_1 + 2A_1 - 3B_1 = 0. \]

The only solution of this system is \(A_1 = B_1 = C_1 (A = B = C)\), but for these values of the parameters we recover the already known integrable case (i) from the Introduction.

On the other hand, if \(A_1 \neq B_1 \neq C_1\), then at least one of the above residues is non-trivial. Thus, we have obtained a nonzero residue at \(\tau = 0\), which implies a logarithm in the solutions of (VE₃). Then, the Galois group of (VE₃) is solvable, but not abelian. Hence, the non-integrability of Hamiltonian system (32) in this case follows from Theorem 3.

**Step 5. Non-integrability of the case \(n = 2, \ \alpha_1 = \frac{1}{2}\).**

Here we will show that for \(\alpha_1 = 1/2\) Hamiltonian system (32) is not integrable, whatever the other parameters are. The same line of considerations as in the previous step is taken.

One can find the following expansions of the solutions of (VE₁) (43), (44) and (45) with \(\alpha_1 = \frac{1}{2}\).

\[ \xi^{(1)}_{1,1} = \frac{1}{\tau^2} \left[1 - \left(A_1 - \frac{B_1}{2}\right) \tau^2 + \frac{3}{2} \left(A_1^2 - A_1 B_1 + \frac{B_1^2}{10}\right) \tau^4 + \ldots\right], \]

\[ \xi^{(1)}_{1,2} = \frac{\tau^3}{5} \left[1 + \frac{3}{7} \left(A_1 - \frac{B_1}{2}\right) \tau^2 + \ldots\right]. \quad (70)\]

To get the expansions of the solutions of (44) we only need to replace \(A_1\) with \(C_1\).

\[ \xi^{(1)}_{2,1} = \frac{1}{\tau^2} \left[1 - \left(C_1 - \frac{B_1}{2}\right) \tau^2 + \frac{3}{2} \left(C_1^2 - C_1 B_1 + \frac{B_1^2}{10}\right) \tau^4 + \ldots\right], \]
\[ \xi^{(1)}_{2,2} = \frac{\tau^3}{5} \left[ 1 + \frac{3}{7} \left( C_1 - \frac{B_1}{2} \right) \tau^2 + \ldots \right]. \]  

(71)

Notice that we already have the expansion for \( \xi^{(1)}_3 \) in (59).

There are no logarithmic terms in the expansions around \( \tau = 0 \) of the local solutions of (VE2). We calculate the expansions of the solutions of (VE2) only in the cases we need in the sequel.

For the first equation of (VE2) with a specific choice of the right-hand terms

\[ \left( \xi^{(2)}_1 \right)'' - \left[ 6 \alpha_1 \wp (\tau) + 6 \left( A_1 - \frac{B_1}{2} \right) \right] \xi^{(2)}_1 = 6 \xi^{(1)}_{1,1} \xi^{(1)}_{3,2} \]  

(72)

we find the following expansions

\[ \xi^{(2)}_{1,j} = \xi^{(1)}_{1,j} + \tau^4 \left[ \frac{1}{7} + \frac{1}{14} \left( A_1 - B_1/2 \right) \tau^2 + \ldots \right], \quad j = 1, 2. \]  

(73)

Similarly, for the second equation

\[ \left( \xi^{(2)}_2 \right)'' - \left[ 6 \alpha_1 \wp (\tau) + 6 \left( C_1 - \frac{B_1}{2} \right) \right] \xi^{(2)}_2 = 6 \xi^{(1)}_{1,1} \xi^{(1)}_{3,2} \]  

(74)

we obtain

\[ \xi^{(2)}_{2,j} = \xi^{(1)}_{2,j} + \tau^4 \left[ \frac{1}{7} + \frac{1}{14} \left( C_1 - B_1/2 \right) \tau^2 + \ldots \right], \quad j = 1, 2. \]  

(75)

Finally, for the third equation in (VE2) with the following choice of the right-hand terms

\[ \left( \xi^{(2)}_3 \right)'' - 12 \wp (\tau) \xi^{(2)}_3 = 3 \left[ \left( \xi^{(1)}_{1,1} \right)^2 + \left( \xi^{(1)}_{2,1} \right)^2 \right] + 6 \left( \xi^{(1)}_{3,2} \right)^2 \]  

(76)

we get the expansions

\[ \xi^{(2)}_{3,j} = \xi^{(1)}_{3,j} + \frac{1}{\tau^2} \left[ -1 + \frac{1}{2} \left( A_1 - B_1 + C_1 \right) \tau^2 + \ldots \right], \quad j = 1, 2. \]  

(77)

It is enough for our purposes to consider only the first equation of (VE3) with the following choice of the right-hand terms.

\[ \left( \xi^{(3)}_1 \right)'' - \left[ 6 \alpha_1 \wp (\tau) + 6 \left( A_1 - \frac{B_1}{2} \right) \right] \xi^{(3)}_1 = 6 \left[ \xi^{(1)}_{1,2} \xi^{(2)}_{3,3} + \xi^{(2)}_{1,2} \xi^{(1)}_{3,1} \right] = W^{(3)}_1. \]  

(78)

Denoting again

\[ (v^{(3)}_{1,1}, v^{(3)}_{1,2}) = Y_{-1}^{-1} \begin{pmatrix} 0 \\ W^{(3)}_1 \end{pmatrix} \]

we have, this time for \( v^{(3)}_{1,2} = 6 \xi^{(1)}_{1,1} \left[ \xi^{(1)}_{1,2} \xi^{(2)}_{3,2} + \xi^{(2)}_{1,2} \xi^{(1)}_{3,1} \right] \)

\[ \text{Res}_{\tau=0} v^{(3)}_{1,2} = -\frac{12}{35} \neq 0. \]  

(79)
Thus, we have a nonzero residue at $\tau = 0$, which implies a logarithm in the solutions of \((VE_3)\). Then, the Galois group of \((VE_3)\) is solvable, but not abelian. Again, the non-integrability of Hamiltonian system \((32)\) in this case follows from Theorem 3.

**Step 6. Integrability of the case $n = 1$, $\alpha_1 = \frac{1}{6}$.**

As we pointed out in the Introduction this integrable case has a multi-dimensional generalization. Here we merely present the commuting integrals in our three-dimensional case for $A, B, C$ arbitrary.

With $\gamma = \alpha$, $\beta = 6\alpha$ and after rescaling $t$ Hamiltonian \((2)\) reads

$$
H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \frac{1}{2}(\tilde{A}q_1^2 + \tilde{C}q_2^2 + \tilde{B}q_3^2) + (q_1^2 + q_2^2)q_3 + 2q_3^3,
$$

where $\tilde{A} = A/\alpha$, $\tilde{B} = B/\alpha$, $\tilde{C} = C/\alpha$. Then the corresponding first integrals are (see, e.g., [5,6])

$$
G_1 = \frac{(q_2p_1 - q_1p_2)^2}{\tilde{A} - \tilde{C}} - q_1^4 - q_1^2q_2^2 - 4p_3q_1p_1 + (\tilde{B} - 4\tilde{A} + 4q_3)(p_1^2 + \tilde{A}q_1^2),
$$

$$
G_2 = -\frac{(q_2p_1 - q_1p_2)^2}{\tilde{A} - \tilde{C}} - q_2^4 - q_1^2q_2^2 - 4p_3q_2p_2 + (\tilde{B} - 4\tilde{C} + 4q_3)(p_2^2 + \tilde{C}q_2^2).
$$

This completes the proof of Theorem 1. \(\square\)

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### A Some applications of the Morales-Ramis theory

**A.1 Homogeneous potentials**

In this part we follow mainly the treatment in [10], which is a generalization of a non-integrability theorem by Yoshida [16] for the case $n = 2$ based on Ziglin’s theorem.

Consider a Hamiltonian system with $n$ - degrees of freedom ($n \geq 2$) governed by a natural Hamiltonian

$$
H = T + V = \frac{1}{2}(y_1^2 + \ldots + y_n^2) + V(x_1, x_2, \ldots, x_n),
$$

where $V$ is a homogeneous function of degree $k \neq 0$.

Taking advantage of homogeneity, it is possible to obtain a particular solution of the form

$$
\Gamma : x_j = u(t)c_j, \quad y_j = \dot{u}(t)c_j, \quad j = 1, \ldots, n,
$$

where $u(t)$ is a solution of the hyperelliptic equation $\dot{u}^2 = \frac{2}{k}(1 - u^k)$ and $c = (c_1, \ldots, c_n)$ is a solution of the nonlinear system

$$
c = V'(c).
$$

Such solutions are called Darboux points. Then the (VE) along $\Gamma$ is given by

$$
\ddot{\xi} = -u(t)^{k-2}V''(c)\xi.
$$
Since $V''(e)$ is diagonalizable, we can write (85) as a direct sum of second-order equations

$$\ddot{\xi}_j = -u(t)^{k-2}\lambda_j \dot{\xi}_j, \quad j = 1, 2, \ldots, n,$$

(86)

where $\lambda_j$ are the eigenvalues of $V''(e)$ (usually called Yoshida coefficients).

In order to get (NVE) we rule out the equation corresponding to $\lambda_n = k - 1$

$$\ddot{\eta} = -u(t)^{k-2}\text{diag}(\lambda_1, \ldots, \lambda_{n-1})\eta,$$

(87)

with $\eta = (\xi_1, \ldots, \xi_{n-1})$.

Further, we change the independent variable by $x := (u(t))^k$ and obtain a system of hypergeometric differential equations

$$x(1-x)\frac{d^2\xi_j}{dx^2} + \left(\frac{k-1}{k} - \frac{3k-2}{2k}x\right)\frac{d\xi_j}{dx} + \frac{\lambda_j}{2k}\xi_j = 0, \quad j = 1, \ldots, n - 1. \quad (88)$$

It is clear that the identity component of the Galois group of (87) is solvable (abelian) if, and only if, each one of the identity components of the Galois groups of hypergeometric equations (88) is solvable (abelian).

Finally, exploring the Galois groups of hypergeometric equations (88) the following result is obtained.

**Theorem 4** (Theorem 5.1 [10]) If the Hamiltonian system with Hamiltonian (82) is completely integrable with holomorphic or meromorphic first integrals, then each pair $(k, \lambda_j)$ belongs to one of the following lists

| 1 | $(k, s + s(s-1)k/2)$ |
|---|------------------|
| 2 | $(2, \mathbb{C})$ |
| 3 | $(-2, \mathbb{C})$ |
| 4 | $(-5, \frac{49}{24} - \frac{1}{12} (4 + 10s)^2)$ |
| 5 | $(-5, \frac{49}{24} - \frac{1}{12} (4 + 10s)^2)$ |
| 6 | $(-5, \frac{49}{24} - \frac{1}{12} (4 + 10s)^2)$ |
| 7 | $(-3, \frac{25}{24} - \frac{1}{24} (2 + 6s)^2)$ |
| 8 | $(-3, \frac{25}{24} - \frac{1}{24} (2 + 6s)^2)$ |
| 9 | $(-3, \frac{25}{24} - \frac{1}{24} (2 + 6s)^2)$ |
| 10 | $(-3, \frac{25}{24} - \frac{1}{24} (2 + 6s)^2)$ |
| 11 | $(-3, \frac{25}{24} - \frac{1}{24} (2 + 6s)^2)$ |
| 12 | $(-3, \frac{25}{24} - \frac{1}{24} (2 + 6s)^2)$ |
| 13 | $(-3, \frac{25}{24} - \frac{1}{24} (2 + 6s)^2)$ |
| 14 | $(-3, \frac{25}{24} - \frac{1}{24} (2 + 6s)^2)$ |
| 15 | $(4, -\frac{1}{8} + \frac{1}{8} (\frac{4}{3} + 4s)^2)$ |
| 16 | $(4, -\frac{1}{8} + \frac{1}{8} (\frac{4}{3} + 4s)^2)$ |
| 17 | $(5, -\frac{9}{16} + \frac{1}{16} (4 + 10s)^2)$ |
| 18 | $(5, -\frac{9}{16} + \frac{1}{16} (4 + 10s)^2)$ |

where $s$ is an arbitrary integer and $\mathbb{C}$ stands for an arbitrary complex number.

The above result is extended by Maciejewski, Przybilska and Yoshida in several ways. We will use the following

**Theorem 5** (Theorem 1.3 [19]) Assume that the Hamiltonian system defined by Hamiltonian (82) with a homogeneous potential $V$ of degree $k \in \mathbb{Z} \setminus \{0\}$ satisfies the following conditions:

1. there exists a nonzero $e \in \mathbb{C}^n$ such that $e = V'(e)$, and
2. matrix $V''(e)$ is diagonalizable with eigenvalues $\lambda_1, \ldots, \lambda_{n-1}, \lambda_n = k - 1$;
3. the system admits an additional first integral $F$, which is meromorphic in a connected neighborhood $U$ of $\Gamma$. 

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Then either

(i) there exist $1 \leq r < n$ such that the pair $(k, \lambda_r)$ belongs to the table in Theorem 4, or

(ii) there exist $1 \leq i < j < n$ such that

$$\frac{1}{2k} \sqrt{(k-2)^2 + 8k\lambda_i} = \frac{1}{2k} \sqrt{(k-2)^2 + 8k\lambda_j + l},$$

for some $l \in \mathbb{Z}$.

A.2 Necessary conditions for integrability of Hamiltonian systems which have (NVE) of Lamé type

Here we recall some facts concerning the integrability of Hamiltonian systems with two degrees of freedom, an invariant plane and which (NVE) is of Lamé type. More details can be found in [10,21]. In our case the (NVE) splits into two equations of Lamé type, and therefore, these arguments can be applied.

Classically the Lamé equation is written in the form

$$\ddot{\xi} - n(n+1)\wp(t)\dot{\xi} + B = 0,$$

(90)

where $\wp(t)$ is the Weierstrass function with invariants $g_2$ and $g_3$, satisfying

$$\dot{\wp}^2 = 4\wp^3 - g_2\wp - g_3 \text{ with } \Delta = g_3^3 - 27g_2^2 \neq 0.$$

The known (mutually exclusive) cases of closed-form solutions of (90) are:

(i) The Lamé and Hermite solutions. In this case $n \in \mathbb{Z}$ and $g_2, g_3, B$ are arbitrary parameters;

(ii) The Brioschi-Halphen-Crawford solutions. Here $m := n + 1/2 \in \mathbb{N}$ and the parameters $g_2, g_3, B$ must satisfy an algebraic equation.

(iii) The Baldassarri solutions. Now $n + 1/2 \in \frac{1}{3}\mathbb{Z} \cup \frac{1}{4}\mathbb{Z} \cup \frac{1}{5}\mathbb{Z} \setminus \mathbb{Z}$ with additional algebraic relations between the other parameters.

Note that in the case (i) the identity component of the Galois group $G^0$ is of the form

$$\left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)$$

and in the cases (ii) and (iii) $G^0 = id$ (G is finite). And these are the all cases when the Lamé equation is integrable.

Now consider a natural two-degrees-of-freedom Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) + V(q_1, q_2),$$

(91)

$q_j(t) \in \mathbb{C}, p_j(t) = \dot{q}_j, j = 1, 2$. We assume that there exists a family of solutions of the form

$$\Gamma_h : q_2 = p_2 = 0, \quad q_1 = q_1(t, h), \quad p_1(t, h) = \dot{q}_1(t, h)$$

and $q_1(t, h)$ is a solution of

$$\frac{1}{2}q_1^2 + \wp(q_1) = h, \quad h \in \mathbb{R}.$$

The (NVE) along $\Gamma_h$ is

$$\ddot{\xi} - f(t, h)\dot{\xi} = 0,$$

(92)

where $f(t, h) = f(q_1(t, h))$ is such that (92) is of type (90).
In [10, 21] the type of the potentials $V$ with this property is obtained as well as the necessary conditions for the integrability of the Hamiltonian systems with Hamiltonian (91). In order to formulate the result we need certain additional quantities.

Since $f(t, h)$ depends linearly on $\varphi(t)$, then $\dot{f}^2$ is a cubic polynomial in $f$, depending also in $h$, namely

$$\dot{f}^2 := P(f, h) = P_1(f) + hP_2(f).$$

(93)

The following coefficients are introduced

$$P(f, h) = (a_1 + ha_2)f^3 + (b_1 + hb_2)f^2 + (c_1 + hc_2)f + (d_1 + hd_2).$$

(94)

Now we are ready to give the corresponding result. Note that the following theorem gives necessary conditions only from the analysis of the first variational equation.

**Theorem 6** (Theorem 6.2 [10]). Assume that a natural Hamiltonian system has (NVE) of Lamé type, associated to the family of solutions $\Gamma_h$, lying on the plane $q_2 = 0$ and parameterized by the energy $h$. Then, a necessary conditions for integrability are that the related polynomials $P_1$ and $P_2$ satisfy $a_2 = 0$, and one of the following conditions holds:

I. $a_1 = \frac{4}{n(n+1)}$ for some $n \in \mathbb{N}$;

II. $a_1 = \frac{16}{4m^2-1}$ for some $m \in \mathbb{N}$. Then, assuming the conjecture above is true, one should have $b_2 = 0$ and we should be in one of the following cases:

1. $m = 1$ and $b_1 = 0$,
2. $m = 2$ and $c_1 = 0$, $16a_1c_1 + 3b_1^2 = 0$,
3. $m = 3$ and $16a_1d_2 + 11b_1c_2 = 0$, $1024a_1^2d_1 + 704a_1b_1c_1 + 45b_1^3 = 0$,
4. $m > 3$. Then, we should have $b_1 = 0$ and, furthermore, either $c_1 = c_2 = 0$ if $m$ is congruent with 1, 2, 4 or 5 modulo 6, or $d_1 = d_2 = 0$ if $m$ is odd;

III. $a_1 = \frac{4}{n(n+1)}$ with $n + 1/2 \in \frac{1}{2} \mathbb{Z} \cup \frac{1}{4} \mathbb{Z} \cup \frac{1}{8} \mathbb{Z} \setminus \mathbb{Z}$, $b_2 = 0$ and either $c_2 = 0$, $b_1^2 - 3a_1c_1 = 0$ or $c_2b_1 - 3a_1d_2 = 0$, $2b_1^3 - 9a_1b_1c_1 + 27a_1^2d_1 = 0$.

It is clear that the condition I. in the above theorem gives the Lamé and Hermite solutions (i), the condition II.---the Brioschi-Halphen-Crawford solutions (ii), and the condition III.---the Baldassarri solutions (iii).

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