RESEARCH ON THE COMPOSITION CENTER
OF A CLASS OF RIGID DIFFERENTIAL
SYSTEMS

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Abstract In this paper, we answer the question: under what conditions a class of rigid differential systems have a composition center. We give the sufficient and necessary conditions for these systems to have a center at origin point. At the same time, we give the formula of focal values and the highest order of fine focus.

Keywords Uniformly isochronous center, rigid system, composition conjecture, composition center, center conditions.

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1. Introduction

Consider the system

\[
\begin{cases}
x' = -y + x(P_1(x, y) + P_2(x, y) + \ldots + P_n(x, y)) = -y + xP, \\
y' = x + y(P_1(x, y) + P_2(x, y) + \ldots + P_n(x, y)) = x + yP,
\end{cases}
\]

(1.1)

where \( P_k(x, y) = \sum_{i+j=k} p_{ij} x^i y^j \), \( p_{ij} \) are real numbers. The system in polar coordinates becomes

\[ r' = P_1 r^2 + P_2 r^3 + \ldots + P_n r^{n+1}, \quad \theta' = 1. \]

This system is called a rigid system [3] because the derivative of the angular variable is constant. It is clear that the origin is the only critical point and if it is a center then it is a uniformly isochronous center [12]. In [9, 16], the authors have proved that a planar polynomial differential system of degree \( n+1 \) has a center at the origin of coordinates, then this center is uniform isochronous if and only if by doing a linear change of variables and a scaling of time it can be written as (1.1). The interest in the uniform isochronous centers has attracted people’s attention since the 17th century. So far, there are many people who have strong interest in this problem and have achieved fruitful results [2,8,9,12,16]. In [1,2] the authors have used techniques based on normal forms and commutation and have proved that the rigid system (1.1), in the cases: \( P = P_1 + P_n \) or \( P = P_2 + P_{2m} \) or \( P = P_1 + P_2 + P_3 + P_4 \), it has a center if and only if it is reversible. In [20],

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the author calculated by computer to get the center condition for this system with $P = P_k + P_{2k}$ ($k = 2, 3, 4, 5$). In [14,15], the numbers of limit cycles of (1.1) have been discussed. In [18,19], some new methods have been used to studied the center problem of this system.

In this paper, we consider the following rigid system

$$\begin{align*}
  x' &= -y + x(P_1(x, y) + P_3(x, y) + P_7(x, y)), \\
  y' &= x + y(P_1(x, y) + P_3(x, y) + P_7(x, y)).
\end{align*}$$

(1.2)

By [5,6], this system has a center at $(0,0)$ if and only if all solutions $r(\theta)$ of periodic differential equation

$$\frac{dr}{d\theta} = r(P_1(\cos \theta, \sin \theta)r + P_3(\cos \theta, \sin \theta)r^3 + P_7(\cos \theta, \sin \theta)r^7)$$

(1.3)

near the solution $r = 0$ are periodic, $r(0) = r(2\pi)$. In such case it is said that equation (1.3) has a center at $r = 0$.

As we known, the derivation of conditions for a center is a difficult and long-standing problem in the theory of nonlinear differential equations, however due to complexity of the problem necessary and sufficient conditions are known only for a very few families of polynomial systems [4, 13, 17]. In [5, 6] the authors introduce a simple condition called Composition Conditions, which ensures that the Abel equation

$$\frac{dr}{d\theta} = A(\theta)r^2 + B(\theta)r^3$$

(1.4)

has a center. Roughly speaking the composition condition says that the primitives of the functions $A$ and $B$ depend functionally on a new $2\pi$-periodic function. When an Abel equation has a center because $A$ and $B$ satisfy the composition condition we will say that the equation has a Composition Center [6].

The Composition Conjecture is that the composition condition is not only the sufficient but also necessary condition for a center. This conjecture first appeared in [7] with classes of coefficients which are polynomial functions in $t$, or trigonometric polynomials. A counterexample was presented in [5] to demonstrate that the conjecture is not true. To find the restrictive conditions under which the composition conjecture is true, this is an open problem which has attracted during the last years a wide interest. In [5] the author has proved that for a family of cubic system the composition conjecture is valid. [21, 22] the author used the different method from [1, 2] to prove that for system (1.1) with $P = P_1 + P_{2n}$ and $P = P_1 + P_{2m}$, the composition conjecture is true. The authors in paper [6, 10, 11] give the sufficient and necessary conditions for the $r = 0$ of the Abel equation (1.4) to be a composition center.

In this paper, we find out all the restrictive conditions under which the origin point of (1.3) is a composition center. At the same time, we give the sufficient and necessary conditions for equation (1.2) to have a center at origin point by using a different method from [1, 2, 20]. These center conditions are more succinct and beautiful than those calculated by computer.

2. Several Lemmas

Alwash and Lloyd [5–7] proved the following statement.
Lemma 2.1. If there exists a differentiable function $u$ of period $2\pi$ such that

$$\dot{A}_1(\theta) = u'(\theta)A_1(u(\theta)), \quad \dot{A}_2(\theta) = u'(\theta)\dot{A}_2(u(\theta))$$

for some continuous functions $\dot{A}_1$ and $\dot{A}_2$, then the Abel differential equation

$$\frac{dr}{d\theta} = \dot{A}_1(\theta)r^2 + \dot{A}_2(\theta)r^3$$

has a center at $r \equiv 0$.

The condition in Lemma 2.1 is called the Composition Condition. This is a sufficient but not a necessary condition for $r = 0$ to be a center [7, 10].

The following statement presents a generalization of Lemma 2.1.

Lemma 2.2. If there exists a differentiable function $u$ of period $2\pi$ such that

$$\dot{A}_i(\theta) = u'(\theta)\dot{A}_i(u), \quad (i = 1, 2, \ldots, n) \quad (2.1)$$

for some continuous functions $\dot{A}_i (i = 1, 2, \ldots, n)$, then the differential equation

$$\frac{dr}{d\theta} = r \sum_{i=1}^{n} \dot{A}_i(\theta)r^i$$

has a center at $r = 0$.

Denote:

$$P_k = \sum_{i+j=k} p_{ij} \cos^i \theta \sin^j \theta, \quad P_k = \int_0^\theta P_k d\theta, \quad C_k = \frac{k!}{i!(k-i)!}.$$ 

Lemma 2.3. If $p_{10}^2 + p_{01}^2 \neq 0$ and

$$\int_0^{2\pi} P^{2i+1}_1 P_3 d\theta = 0, \quad (i = 0, 1), \quad \int_0^{2\pi} P^{2j+1}_1 P_7 d\theta = 0, \quad (j = 0, 1, 2, 3),$$

then

$$P_3 = P_1(\lambda_1 + 2\lambda_2 \dot{P}_1 + 3\lambda_3 \dot{P}_1^2), \quad P_7 = P_1(\mu_1 + 2\mu_2 \dot{P}_1 + \ldots + 7\mu_7 \dot{P}_1^6),$$

where $\mu_j (j = 1, 2, \ldots, 7)$ are real numbers and

$$\lambda_1 = \frac{1}{(P_{10}^2 + P_{01}^2)^3} (p_{30}p_{10}(p_{10}^4 + 3p_{01}^4) + p_{21}p_{01}(p_{10}^2 - p_{01}^2) \dot{P}_1^2 + 2p_{12}p_{01}^2p_{10}(p_{10}^2 - p_{01}^2) + 4p_{03}p_{01}^3P_{10}^2),$$

$$\lambda_2 = -3\lambda_3p_{01}, \quad \lambda_3 = -\frac{1}{3(P_{10}^2 + P_{01}^2)^3} ((p_{30} - p_{12})(p_{10}^3 - 3p_{10}p_{01}^2) + (p_{03} - p_{21})(p_{01}^3 - 3p_{01}p_{10}^2)).$$

Proof. Denote

$$a = \frac{p_{10}}{\sqrt{p_{10}^2 + p_{01}^2}}, \quad b = \frac{p_{01}}{\sqrt{p_{10}^2 + p_{01}^2}}.$$
\[ U = a \cos \theta + b \sin \theta, \quad V = a \sin \theta - b \cos \theta, \]

then
\[ P_1 = \sqrt{p_{10}^2 + p_{01}^2} U, \quad \tilde{P}_1 = \sqrt{p_{10}^2 + p_{01}^2} V + p_{01}, \]
\[ a^2 + b^2 = 1, \quad U^2 + V^2 = 1, \]
\[ \int_0^{2\pi} U^2 d\theta = \int_0^{2\pi} V^2 d\theta = \pi, \]
\[ \int_0^{2\pi} U^{2k} d\theta = \int_0^{2\pi} V^{2k} d\theta = \frac{(2k - 1)!!}{(2k)!!} 2\pi. \]

Obviously, \( P_3 \) and \( P_7 \) can be rewritten in the following forms
\[ P_3 = m_1 U + m_3 U^3 + n_1 V + n_3 V^3, \]
\[ P_7 = s_1 U + s_3 U^3 + s_5 U^5 + s_7 U^7 + t_1 V + t_3 V^3 + t_5 V^5 + t_7 V^7, \]
where \( n_i (i = 1, 3) \), \( s_j, t_j (j = 1, 3, 5, 7) \) are real numbers and
\[ m_1 = 3p_{30} a b^2 + p_{21} (b^3 - 2a^2 b) + p_{12} (a^3 - 2ab^2) + 3p_{03} a^2 b, \]
\[ m_3 = (p_{30} - p_{12}) (a^3 - 3ab^2) + (p_{03} - p_{21}) (b^3 - 3a^2 b). \]
By \( \int_0^{2\pi} \tilde{P}_1^{2i+1} P_3 d\theta = 0, \ (i = 0, 1, \) we get
\[ \int_0^{2\pi} (n_1 V^2 + n_3 V^4) d\theta = 0, \]
\[ \int_0^{2\pi} (n_1 V^4 + n_3 V^6) d\theta = 0, \]
i.e.,
\[ n_1 + \frac{3}{4} n_3 = 0, \quad \frac{3}{4} n_1 + \frac{5}{8} n_3 = 0, \]
solving these equations we get \( n_1 = n_3 = 0 \), so,
\[ P_3 = m_1 U + m_3 U^3 = P_1 (\lambda_1 + 2\lambda_2 \tilde{P}_1 + 3\lambda_3 \tilde{P}_1^3), \]
where \( \lambda_1 = \frac{m_1 + m_3}{\sqrt{p_{10}^2 + p_{01}^2}}, \lambda_2 = -3\lambda_3 p_{01}, \lambda_3 = -\frac{m_3}{3(\rho_{03} + \rho_{01})} \).
By
\[ \int_0^{2\pi} \tilde{P}_1^{2j+1} P_7 d\theta = 0, \ (j = 0, 1, 2, 3), \]
we get
\[ \int_0^{2\pi} (t_1 V^2 + t_3 V^4 + t_5 V^6 + t_7 V^8) d\theta = 0, \]
\[ \int_0^{2\pi} (t_1 V^4 + t_3 V^6 + t_5 V^8 + t_7 V^{10}) d\theta = 0, \]
\[ \int_0^{2\pi} (t_1 V^6 + t_3 V^8 + t_5 V^{10} + t_7 V^{12}) d\theta = 0, \]
\[ \int_0^{2\pi} (t_1 V^8 + t_3 V^{10} + t_5 V^{12} + t_7 V^{14}) d\theta = 0, \]
is satisfied. Then 
\[ \begin{pmatrix} 1 & 3 & 5 & 35 & \frac{512}{7} \\ \frac{3}{4} & \frac{5}{8} & \frac{35}{64} & \frac{63}{128} \\ \frac{5}{8} & \frac{35}{64} & \frac{63}{128} & \frac{211}{576} \\ \frac{35}{64} & \frac{63}{128} & \frac{231}{576} & \frac{129}{1024} \end{pmatrix} \begin{pmatrix} t_1 \\ t_3 \\ t_5 \\ t_7 \end{pmatrix} = 0, \]

since the value of the determinant of the coefficient matrix of the above equations is not equal to zero, so 
\[ t_1 = t_3 = t_5 = t_7 = 0 \]
and
\[ P_7 = s_1 U + s_3 U^3 + s_5 U^5 + s_7 U^7 = P_1(\mu_1 + 2\mu_2 P_1 + \ldots + 7\mu_7 P_1^6). \]

\[ \square \]

3. Main results

Consider equation
\[ \frac{dr}{d\theta} = r(P_1 r + P_2 r^3 + P_7 r^7), \]  
where \( P_k = \sum_{i+j=k} p_{ij} \cos^i \theta \sin^j \theta, p_{ij} \) (\( i, j = 0, 1, 2, ..., k, k = 1, 3, 7 \)) are real numbers and \( p_{10}^2 + p_{01}^2 \neq 0. \)

**Theorem 3.1.** Suppose that one of the following conditions:

1. \( (14 + \lambda_3)(35 + 23\lambda_3)(330 + 703\lambda_3 + 65\lambda_3^2) \neq 0; \)
2. \( 14 + \lambda_3 = 0, p_{01} \neq 0; \)
3. \( 14 + \lambda_3 = 0, p_{01} = 0, b_3 = 0; \)
4. \( 35 + 23\lambda_3 = 0, p_{01} \neq 0; \)
5. \( 35 + 23\lambda_3 = 0, p_{01} = 0, b_5 = 0; \)
6. \( 330 + 703\lambda_3 + 65\lambda_3^2 = 0, p_{01} \neq 0; \)
7. \( 330 + 703\lambda_3 + 65\lambda_3^2 = 0, p_{01} = 0, b_7 = 0, \)
is satisfied. Then \( r = 0 \) is a center of (3.1) if and only if

\[ \int_0^{2\pi} P_1^{2i+1} P_3 d\theta = 0, \quad (i = 0, 1), \quad \int_0^{2\pi} P_1^{2j+1} P_7 d\theta = 0, \quad (j = 0, 1, 2, 3). \]  

I.e.,

\[ p_{10}p_{21} - p_{01}p_{12} + 3p_{10}p_{03} - 3p_{01}p_{30} = 0; \]  
\[ p_{30}p_{01}^3 - p_{21}p_{01}^2 p_{10} + p_{12}p_{01}p_{10}^2 - p_{03}p_{10}^3 = 0; \]  
\[ p_{10}(5p_{61} + 3p_{43} + 5p_{25} + 35p_{07}) - p_{01}(5p_{16} + 3p_{34} + 5p_{52} + 35p_{70}) = 0; \]  
\[ p_1^3(3p_{61} + 3p_{43} + 7p_{25} + 63p_{07}) - 3p_{01}^2 p_{01}(3p_{52} + 3p_{34} + 7p_{70} + 7p_{16}) \]  
\[ + 3p_{10}p_{01}^2(3p_{25} + 3p_{43} + 7p_{07} + 7p_{61}) - p_{01}^3(3p_{16} + 3p_{34} + 7p_{52} + 63p_{70}) = 0; \]  
\[ p_1^5(5p_{61} + 7p_{43} + 21p_{25} + 231p_{07}) - 5p_{10}^4 p_{01}(5p_{52} + 7p_{70} + 7p_{34} + 21p_{16}) \]  
\[ + 10p_{10}^3 p_{01}(5p_{43} + 7p_{61} + 7p_{25} + 21p_{07}) - 10p_{10}^2 p_{01}^3(5p_{34} + 7p_{16} + 7p_{52} + 21p_{70}) = 0; \]  
\[ + 5p_{10} p_{01}^4(5p_{25} + 7p_{07} + 7p_{43} + 21p_{16}) - p_{01}^5(5p_{16} + 7p_{34} + 21p_{52} + 231p_{70}) = 0; \]  
\[ p_{70} p_{01} - p_{61} p_{61} p_{10} + 5p_{52} p_{01} p_{10} - 34 p_{01} p_{10}^4 + p_{34} p_{01} p_{10}^4 - 2p_{52} p_{01} p_{10}^5 \]  
\[ + p_{16} p_{01} p_{10}^7 - p_{07} p_{10} = 0. \]  

Moreover, this center is a composition center and uniformly isochronous center. Where \( \lambda_3 \) is the same as it is in Lemma2.3, \( b_i = \frac{1}{n} \int_0^{2\pi} P_i \sin i\vartheta d\vartheta, \) (\( i = 3, 5, 7 \)).

**Proof. Necessity:** Let \( r(\theta, c) \) be the solution of (3.1) such that \( r(0, c) = c(0 < c \ll 1) \). We write

\[
r(\theta, c) = c \sum_{n=0}^{\infty} a_n(\theta)c^n,
\]

where \( a_0(0) = 1 \) and \( a_n(0) = 0 \) for \( n \geq 1 \). The origin of (3.1) is a center if and only if \( r(\theta + 2\pi, c) = r(\theta, c) \), i.e., \( a_0(2\pi) = 1, a_n(2\pi) = 0 (n = 1, 2, 3,...) \) \([10, 12]\).

Substituting \( r(\theta, c) \) into (3.1) we obtain

\[
\sum_{n=0}^{\infty} a'_n(\theta)c^n = cP_1(\sum_{n=0}^{\infty} a_n(\theta)c^n)^2 + c^3P_3(\sum_{n=0}^{\infty} a_n(\theta)c^n)^4 + c^7P_7(\sum_{n=0}^{\infty} a_n(\theta)c^n)^8. \tag{3.9}
\]

Equating the corresponding coefficients of \( c^n \) of (3.9), we get

\[
\begin{align*}
a'_0(\theta) &= 0, a_0(0) = 1, \\
a'_1(\theta) &= P_1a_0^2, a_1(0) = 0, \\
a'_2(\theta) &= 2a_0a_1P_1, a_2(0) = 0, \\
a'_3(\theta) &= (2a_0a_2 + a_1^2)P_1 + a_3^4P_3, a_3(0) = 0, \\
a'_4(\theta) &= (2a_0a_3 + 2a_1a_2)P_1 + 4a_0^3a_1P_3, a_4(0) = 0,
\end{align*}
\]

Solving these equations we obtain

\[
a_0 = 1, a_1 = \bar{P}_1, a_2 = \bar{P}_1^2, a_3 = \bar{P}_1^3 + \gamma_0, a_4 = \bar{P}_1^4 + \gamma_1,
\]

where

\[
\gamma_0 = \bar{P}_3, \quad \gamma_1 = 2\bar{P}_1\bar{P}_3 + 2\bar{P}_1\bar{P}_3.
\tag{3.10}
\]

By this we see that \( a_i(2\pi) = 0, (i = 0, 1, 2, 3) \) and from \( a_4(2\pi) = 0 \) implies that

\[
\int_0^{2\pi} \bar{P}_1\bar{P}_3d\theta = 0. \tag{3.11}
\]

Denote:

\[
\phi = f + \gamma c^3 + \alpha c^6 + \beta c^7 + \delta c^{10},
\]

where

\[
f = \sum_{i=0}^{\infty} \bar{P}_1^ic^i, \gamma = \sum_{i=0}^{\infty} \gamma_ic^i, \alpha = \sum_{i=0}^{\infty} \alpha_ic^i, \beta = \sum_{i=0}^{\infty} \beta_ic^i, \delta = \sum_{i=0}^{\infty} \delta_ic^i.
\]

Thus

\[
\begin{align*}
\phi^2 &= f^2 + 2f\gamma c^3 + (\gamma^2 + 2\alpha c^6) + 2f\beta c^7 + 2f\gamma c^9 + 2(f\delta + \gamma\beta)c^{10} + \alpha^2c^{12} + 2(\gamma\delta + \alpha\beta)c^{13} + \beta^2c^{14} + ..., \tag{3.12}
\end{align*}
\]

\[
\phi^4 = f^4 + 4f^3\gamma c^3 + (6f^2\gamma^2 + 4f^3\alpha)c^6 + 4f^3\beta c^7 + (4f\gamma^3 + 12f^2\gamma\alpha)c^9 + (4f^3\delta + 12f^2\gamma\beta)c^{10} + (\gamma^4 + 12f\gamma^2\alpha + 6f^2\alpha^2)c^{12} + ..., \tag{3.13}
\]

\[
\phi^8 = f^8 + 8f^7\gamma c^3 + (28f^6\gamma^2 + 8f^7\alpha)c^6 + 8f^7\beta c^7 + ..., \tag{3.14}
\]

Composition center of rigid differential...
\[ f^m = \sum_{k=0}^{\infty} C_{m+k-1}^{n-1} \bar{P}_1^k c^k. \] (3.15)

Substituting \( r = \phi c = (f + \gamma c^3 + \alpha c^6 + \beta c^7 + \delta c^{10})c \) into (3.9) and using (3.12)–(3.15) we get
\[ a'_5 = P_1(5 \bar{P}_1^5 + 2 \sum_{i+j=1} \bar{P}_1^i \gamma_j) + P_3 C_6^3 \bar{P}_1, \]
solving this equation we obtain
\[ a_5 = \bar{P}_1^5 + \gamma_2, \] (3.16)
where
\[ \gamma_2 = 3 \bar{P}_1^2 \bar{P}_3 + 4 \bar{P}_1 \bar{P}_1 \bar{P}_3 + 3 \bar{P}_1 \bar{P}_3. \] (3.17)

By (3.11) and (3.16) and (3.17) we have \( a_5(2\pi) = 0 \).

Applying (3.9) and (3.12)–(3.15) we obtain
\[ a'_6 = P_1(6 \bar{P}_1^6 + 2 \sum_{i+j=2} \bar{P}_1^i \gamma_j) + P_3 (C_6^3 \bar{P}_1^3 + C_4^4 \gamma_0), \]
solving this equation we get
\[ a_6 = \bar{P}_1^6 + \gamma_3 + \alpha_0, \] (3.18)
where
\[ \gamma_3 = 4 \bar{P}_1^3 \bar{P}_3 + 6 \bar{P}_1^2 \bar{P}_3 \bar{P}_3 + 6 \bar{P}_1 \bar{P}_1 \bar{P}_3 \bar{P}_3 + 4 \bar{P}_1 \bar{P}_3, \] (3.19)

By (3.11) and (3.18) and (3.19) we see that if \( a_6(2\pi) = 0 \), then
\[ \int_0^{2\pi} \bar{P}_1^3 P_3 d\theta = 0. \] (3.20)

Using (3.11) and (3.20) and Lemma2.3 we get
\[ P_3 = P_1(\lambda_1 + 2 \lambda_2 \bar{P}_1 + 3 \lambda_3 \bar{P}_1^2), \] (3.21)
where \( \lambda_i (i = 1, 2, 3) \) are the same as they are in Lemma2.3. Therefore,
\[ \bar{P}_3 = \lambda_3 \bar{P}_1^3 + \lambda_2 \bar{P}_1^2 + \lambda_1 \bar{P}_1. \] (3.22)

By (3.21) and (3.22) we see that \( \gamma_k (k = 0, 1, 2, 3) \) are the polynomials with respect to \( \bar{P}_1 \) of degree \( k + 3 \), \( \alpha_0 \) is a polynomial on \( \bar{P}_1 \) of degree 6, so they are \( 2\pi \) periodic functions.

Applying (3.9) and (3.12)–(3.15) we get
\[ a_{7+k} = \bar{P}_1^{3+k} + \gamma_{4+k} + \alpha_{1+k} + \beta_k, \] (k = 0, 1, 2),
where
\[ \gamma_k = \sum_{j=0}^{k} (k + 1 - j)(1 + j) \bar{P}_1^{k-j} \bar{P}_1 \bar{P}_3, \] (3.24)
By these relations and (3.11) and (3.20)-(3.22) we see that periodic functions, thus from 

\[ a_{\gamma+k}(2\pi) = 0, \quad (k = 0, 1, 2) \] 

imply that

\[ \int_0^{2\pi} \bar{P}_1 P_\gamma d\theta = 0. \] 

(3.26)

Applying (3.9) and (3.12)–(3.15) we obtain

\[ a_{10+k} = \bar{P}_1^{10+k} + \gamma_{7+k} + \alpha_{4+k} + \beta_{3+k} + \delta_k, \quad (k = 0, 1, 2, \ldots, 5), \] 

(3.27)

where \( \gamma_k, \beta_k \) are expressed by (3.24) and (3.25), respectively, \( \alpha_k \) is the polynomial on \( P_1 \) of degree \( 6+k \), \( \delta_k \) \( (k = 0, 1, 2, \ldots, 5) \) are the solutions of the following equations:

\[ \delta_0' = 4P_3\beta_0 + 8P_7\gamma_0, \]
\[ \delta_1' = 2P_1(\delta_0' + \gamma_0\beta_0) + 4P_3 \sum_{i+j=1} C_2^{i+j} \bar{P}_1^i P_1^j + 8P_7 \sum_{i+j=1} C_3^{i+j} P_1^j, \]
\[ \delta_2' = 2P_1 \sum_{i+j=1} (\bar{P}_1^i \delta_j + \gamma_i \beta_j) + 4P_3 \sum_{i+j=2} C_2^{i+j} \bar{P}_1^i P_1^j + 8P_7 \sum_{i+j=2} C_3^{i+j} P_1^j, \]
\[ \delta_3' = 2P_1 \sum_{i+j=2} (\bar{P}_1^i \delta_j + \gamma_i \beta_j) + 4P_3 \sum_{i+j=3} C_2^{i+j} \bar{P}_1^i P_1^j + 8P_7 \sum_{i+j=3} C_3^{i+j} P_1^j, \]
\[ + \bar{P}_7(8 \sum_{i+j=3} C_6^{i+j} \bar{P}_1^{i+j} + 28\gamma_0^2 + 8\alpha_0), \]
\[ \delta_4' = 2P_1 \sum_{i+j=3} (\bar{P}_1^i \delta_j + \gamma_i \beta_j) + 8P_7 \sum_{i+j=4} C_6^{i+j} \bar{P}_1^{i+j}, \]
\[ + 3 \sum_{i+j+l=1} (i+1) \bar{P}_1^j \gamma_{i+l} P_7(8 \sum_{i+j=4} C_6^{i+j} \bar{P}_1^{i+j} + 28 \sum_{i+j+l=2} C_5^{i+j} \bar{P}_1^i \sum_{i+j+l=2} \gamma_{i+j+l}) \]
\[ + 8 \sum_{i+j+l=2} C_5^{i+j} \bar{P}_1^i \alpha_{i+j+l} + 8\beta_0), \]
\[ \delta_5' = 2P_1 \sum_{i+j=4} (\bar{P}_1^i \delta_j + \gamma_i \beta_j) + \sum_{i+j=5} (\gamma_i \delta_j + \alpha_i \beta_j) + \frac{1}{2} \beta_0^2 + P_3(4 \sum_{i+j=5} C_2^{i+j} \bar{P}_1^i \beta_j \]
\[ + 4 \sum_{i+j=2} C_2^{i+j} \bar{P}_1^i \delta_j + 12 \sum_{i+j+l=2} (i+1) \bar{P}_1^j \gamma_{i+j+l} P_7(8 \sum_{i+j=5} C_6^{i+j} \bar{P}_1^j \gamma_j \]
\[ + 28 \sum_{i+j+l=2} C_5^{i+j} \bar{P}_1^i \gamma_{i+j+l} + 8 \sum_{i+j=2} C_6^{i+j} \bar{P}_1^i \alpha_{i+j} + 8 \sum_{i+j=1} C_6^{i+j} \bar{P}_1^i \beta_j). \]
Solving these equations we get

\[ \delta_0 = 4P_4P_7 + 4P_7P_7; \]

\[ \delta_1 = 8P_1P_3P_7 + 24P_1P_3P_7 + 10P_1P_3P_7 + 10P_1P_3P_7 + 30P_1P_3P_7 + 6P_1P_3P_7; \]

\[ \delta_2 = 12P_1P_3P_7 + 84P_3P_1P_7 + 60P_1P_3P_7 + 60P_1P_3P_7 + 18P_1P_3P_7; \]

\[ + 24P_1P_7P_1P_3 + 60P_1P_7P_3P_7 + 12P_1P_7P_3P_7 + 18P_7P_1P_7P_3 \]

\[ + 48P_1P_7P_1P_7 + 6P_7P_1P_3 + 126P_7P_3P_7; \]

\[ \delta_3 = 28P_1P_3P_7 + 16P_1P_3P_7 + 108P_1P_3P_7 + 42P_1P_3P_7 + 90P_1P_3P_7 \]

\[ + 18P_1P_3P_1P_7 + 210P_1P_3P_3P_7 + 144P_1P_3P_5P_7 + 252P_1P_3P_3P_7 \]

\[ + 42P_1P_3P_3P_7 + 96P_1P_3P_3P_7 + 12P_1P_3P_3P_7 + 224P_3P_3P_3 \]

\[ + 14P_3P_3P_7 + 16P_3P_7P_7 + 28P_3P_7P_7 + 108P_3P_7P_7 + 210P_3P_7P_7 \]

\[ + 4P_3P_7P_7 + 392P_3P_7P_7 + 210P_3P_7P_7 + 54P_7P_7P_7 \]

\[ + 14P_7P_7P_7; \]

\[ \delta_4 = 40P_1P_3P_7 + 20P_1P_3P_7 + 16P_1P_3P_7 + 64P_1P_3P_7 + 120P_1P_3P_7 \]

\[ + 24P_1P_3P_1P_7 + 378P_1P_3P_1P_7 + 252P_1P_3P_3P_7 + 378P_1P_3P_3P_7 \]

\[ + 72P_1P_3P_3P_7 + 144P_1P_3P_5P_7 + 18P_1P_3P_7P_7 + 54P_1P_3P_3P_7 \]

\[ + 40P_1P_3P_3P_7 + 40P_1P_3P_3P_7 + 64P_1P_3P_3P_7 + 252P_1P_3P_3P_7 \]

\[ + 504P_1P_3P_3P_7 + 8P_1P_3P_3P_7 + 784P_1P_3P_3P_7 + 420P_1P_3P_3P_7 \]

\[ + 108P_1P_3P_3P_7 + 28P_1P_3P_3P_7 + 84P_1P_3P_3P_7 + 504P_1P_3P_3P_7 \]

\[ + 120P_1P_3P_3P_7 + 24P_1P_3P_3P_7 + 16P_1P_3P_3P_7 + 40P_1P_3P_3P_7 + 4P_3P_3P_3 \]

\[ + 16P_1P_3P_3P_7 + 378P_1P_3P_3P_7 + 560P_1P_3P_3P_7 + 10P_3P_3P_3P_7 \]

\[ + 40P_1P_3P_3P_7 + 252P_1P_3P_3P_7 + 672P_1P_3P_3P_7 + 48P_3P_3P_3P_7 \]

\[ + 1008P_1P_3P_3P_7 + 120P_1P_3P_3P_7; \]

\[ \delta_5 = 54P_1P_3P_3P_7 + 24P_1P_3P_3P_7 + 240P_1P_3P_3P_7 + 90P_1P_3P_3P_7 + 150P_1P_3P_3P_7 \]

\[ + 30P_1P_3P_3P_7 + 588P_1P_3P_3P_7 + 384P_1P_3P_3P_7 + 504P_1P_3P_3P_7 \]

\[ + 108P_1P_3P_3P_7 + 192P_1P_3P_3P_7 + 24P_1P_3P_3P_7 + 1008P_1P_3P_3P_7 \]

\[ + 81P_1P_3P_3P_7 + 72P_1P_3P_3P_7 + 108P_1P_3P_3P_7 + 432P_1P_3P_3P_7 \]

\[ 882P_1P_3P_3P_7 + 12P_1P_3P_3P_7 + 1176P_1P_3P_3P_7 + 630P_1P_3P_3P_7 \]

\[ + 162P_1P_3P_3P_7 + 42P_1P_3P_3P_7 + 240P_1P_3P_3P_7 + 120P_1P_3P_3P_7 \]

\[ + 180P_1P_3P_3P_7 + 300P_1P_3P_3P_7 + 60P_1P_3P_3P_7 + 36P_1P_3P_3P_7 \]

\[ + 90P_1P_3P_3P_7 + 9P_1P_3P_3P_7 + 384P_1P_3P_3P_7 + 882P_1P_3P_3P_7 \]

\[ + 1344P_1P_3P_3P_7 + 96P_1P_3P_3P_7 + 96P_1P_3P_3P_7 + 504P_1P_3P_3P_7 \]
Using (3.26) and (3.35) and (3.29) we see that

\[ + 1344\bar{P}_1 P_1^3 P_7 P_3 P_5 + 96\bar{P}_1 P_3 P_7 P_1 P_3 + 2016\bar{P}_1 P_1^3 P_3 P_7 + 240\bar{P}_1 P_1 P_2 P_7 P_1 + 54 P_1^2 P_1 P_7 P_1 + 240 P_1^2 P_3 P_7 P_1 + 588 P_1^2 P_1 P_7 P_3 + 1008 P_1^2 P_3 P_7 P_3 + 1260\bar{P}_1 P_3 P_1 P_7 P_3 + 1008\bar{P}_1^2 P_1 P_7 P_3 + 72 P_1^2 P_3 P_7 P_3 + 294 P_1^2 P_1 P_7 P_3 + 480\bar{P}_1 P_3 P_1 P_7 P_3 + 384\bar{P}_1^2 P_1 P_7 P_3 + 162 P_1^2 P_3 P_7 P_3 + 90\bar{P}_1 P_3^2 P_7 P_3 + 120 P_1 P_3 P_7 P_3 + 60\bar{P}_1 P_3 P_1 P_7 P_3 + 108\bar{P}_1^2 P_1 P_7 P_3 + 192\bar{P}_1 P_3 P_1 P_7 P_3 + 24 P_1^2 P_1 P_7 P_3 + 504 P_1^2 P_3 P_7 P_3 + 48 P_1 P_1 P_7 P_3 + 6 P_1 P_3 P_7 P_3 - 6 P_3^2 P_3 P_7 + 30 P_1^2 P_3 P_1 P_7 + 252 P_1^2 P_3 P_7 P_3 + 840 P_1^2 P_3 P_7 P_3 + 1764 P_1 P_3 P_7 P_3 + \]

\[ + 2268 P_1^2 P_3 P_7 P_3 + 168 P_1^2 P_3 P_7 P_3 + 567 P_1^2 P_1 P_7 P_3 - 372 P_1 P_3 P_3 P_3 P_1 + \]

\[ + 42 P_1^2 P_3 P_7 + 804 P_1 P_3 P_7 P_3 P_3 - 114 P_1^2 P_3 P_3 P_3 P_1 = \frac{\lambda_3}{280} (1184850 + 310617\lambda_3) P_1^8 P_1^2 + \frac{\lambda_2}{35} (139058 + 75104\lambda_3) P_1^7 P_7 + ..... (3.33) \]

By (3.11) and (3.20) and Lemma 2.3 we see that \( \gamma_i \) and \( \alpha_j \) are the polynomials functions with respect to \( P_1 \) and they are \( 2\pi \)-periodic. Thus, by (3.27) we see that if \( a_{10+k}(2\pi) = 0 \), then \( \beta_{3+k}(2\pi) + \delta_k(2\pi) = 0 \), \( (k = 0, 1, ..., 5) \). By \( a_{10}(2\pi) = 0 \) and (3.28) we get \( \beta_3(2\pi) + \delta_0(2\pi) = 0 \), i.e.,

\[ 56 \int_0^{2\pi} P_1^3 P_7 d\theta + 4 \int_0^{2\pi} \bar{P}_3 P_7 d\theta = 4(14 + \lambda_3) \int_0^{2\pi} P_1^3 P_7 d\theta = 0. \quad (3.34) \]

**Case 1.** If \( (14 + \lambda_3)(35 + 23\lambda_3)(330 + 703\lambda_3 + 65\lambda_3^2) \neq 0 \). By (3.34) we have

\[ \int_0^{2\pi} P_1^3 P_7 d\theta = 0. \quad (3.35) \]

Using (3.26) and (3.35) and (3.29) we see that \( a_{11}(2\pi) = 0 \). Using (3.30) we see that if \( a_{12}(2\pi) = 0 \), then

\[ \beta_5(2\pi) + \delta_2(2\pi) = \int_0^{2\pi} (252\bar{P}_1^5 P_7 + 48 P_1 P_1 P_7 P_1 P_3 + 6 P_1 P_1 P_7 P_3 + 126 P_1^2 P_7 P_3) d\theta = \frac{36}{5} (35 + 23\lambda_3) \int_0^{2\pi} \bar{P}_1^5 P_7 d\theta = 0, \quad (3.36) \]

by the hypothesis, it implies

\[ \int_0^{2\pi} \bar{P}_1^5 P_7 d\theta = 0. \quad (3.37) \]

By (3.26) and (3.35) (3.37) we see that \( a_{13}(2\pi) = 0 \). Applying (3.32) we see that if \( a_{14}(2\pi) = 0 \), then

\[ \beta_7(2\pi) + \delta_4(2\pi) = \int_0^{2\pi} (792\bar{P}_1^7 P_7 + 48 P_1 P_1 P_7 P_1 P_3 + 252\bar{P}_1^2 P_7 P_1^2 P_3) d\theta \]

\[ + \int_0^{2\pi} (672\bar{P}_1^2 P_7 P_1 P_3 + 48 P_1 P_1 P_7 P_1 P_3 + 1008\bar{P}_1^2 P_7 P_3 + 120\bar{P}_1 P_1^2 P_7 P_3) d\theta \]
by the hypothesis, it implies

\[
\frac{12}{5} (330 + 703\lambda_3 + 65\lambda_3^2) \int_0^{2\pi} \tilde{P}_1^2 P_7 d\theta = 0,
\]

(3.38)

by the hypothesis, it implies

\[
\int_0^{2\pi} \tilde{P}_1^2 P_7 d\theta = 0.
\]

(3.39)

In summary, by (3.11) and (3.20) and (3.26), (3.35),(3.37) and (3.39) imply that the condition (3.2) is the necessary for the origin to be a center of equation (3.1).

**Case 2.** If \(14 + \lambda_3 = 0, p_{01} \neq 0\), then \(\lambda_2 = -3\lambda_3p_{01} \neq 0\) and (3.34) is an identity, by this we see that the identity (3.35) is valid. Similar to case 2, we know that the condition (3.2) is the necessary for the origin to be a center of equation (3.1).

**Case 3.** If \(14 + \lambda_3 = 0, p_{01} = 0, b_3 = 0\), then \(p_{10} \neq 0, P_1 = p_{10} \cos \theta, \tilde{P}_1 = p_{10} \sin \theta\). By (3.26) we get

\[
\int_0^{2\pi} \tilde{P}_1^3 P_7 d\theta = 0
\]

i.e., the identity (3.35) is valid. Similar to case 2, we know that the condition (3.2) is the necessary for the origin to be a center of equation (3.1).

**Case 4.** If \(35 + 23\lambda_3 = 0, p_{01} \neq 0\), then \(\lambda_2 = -3\lambda_3p_{01} \neq 0\) and the identity (3.35) is valid, (3.36) is an identity, and \(a_{11}(\theta)\) and \(a_{12}(\theta)\) are \(2\pi\)-periodic functions, by this we see that the coefficient function \(2(4\tilde{P}_1 P_7 + 56\tilde{P}_1 P_7)\) of \(\bar{P}_1\) in the formula \(\beta_0(\theta) + \delta_3(\theta)\) is \(2\pi\)-periodic and from \(a_{13}(2\pi) = 0\) follows that

\[
\beta_0(2\pi) + \delta_3(2\pi)
\]

\[
= \int_0^{2\pi} (4\tilde{P}_1^3 P_7 + 392\tilde{P}_1^3 P_7 + 210\tilde{P}_1^2 P_7 P_3 + 54\tilde{P}_1 P_7 P_3^2 P_3 + 14P_3^2 P_7) d\theta
\]

\[
= \lambda_2 (28\lambda_3 + \frac{2803}{5}) \int_0^{2\pi} \tilde{P}_1^5 P_7 d\theta = 0,
\]

as \(\lambda_3 = -\frac{35}{23}, \lambda_2 = -3\lambda_3 p_{01} \neq 0\), from above follows that

\[
\int_0^{2\pi} \tilde{P}_1^5 P_7 d\theta = 0.
\]
i.e., the identity (3.37) is valid. Similar to case 1, we can get (3.39). Thus the condition (3.2) is the necessary for the origin to be a center of equation (3.1).

**Case 5.** If $35 + 23\lambda_3 = 0, p_{01} = 0, b_5 = 0$, then $p_{10} \neq 0, P_1 = p_{10} \cos \theta, \bar{P}_1 = p_{10} \sin \theta$. By (3.26) and (3.35) we get $\int_0^{2\pi} \sin \theta P_7 d\theta = 0, \int_0^{2\pi} \sin 3\theta P_7 d\theta = 0$, thus

$$\int_0^{2\pi} \bar{P}_1^5 P_7 d\theta = \frac{P_{10}^5}{16} \int_0^{2\pi} (10 \sin \theta - 5 \sin 3\theta + 5\sin 5\theta) P_7 d\theta = 0,$$

i.e., the identity (3.37) is valid. Similar to case 4, we get (3.39). Thus the condition (3.2) is the necessary for the origin to be a center.

**Case 6.** If $330 + 703\lambda_3 + 65\lambda_3^2 = 0, p_{01} \neq 0, (14 + \lambda_3)(35 + 23\lambda_3) \neq 0$ and $139058 + 75104\lambda_3 \neq 0, (3.35)$ and (3.37) are valid and (3.38) is an identity, if $a_{14}(\theta)$ is $2\pi$-periodic function, by this we see that the coefficient $2(792P_1^2P_7 + 48P_1^2P_7^2P_3 + 252P_1^4 P_7^2 P_3 + 672P_1^6 P_7 P_3 + 48P_1^7 P_7 P_3 + 1008P_1^7 P_7 P_3 + 120P_1^7 P_3^2 P_7)$ of $\bar{P}_1$ in the formula $\beta_6(\theta) + \delta_5(\theta)$ is $2\pi$-periodic, by this from $a_{15}(2\pi) = 0$ follows that

$$\beta_6(2\pi) + \delta_5(2\pi) = \int_0^{2\pi} (6\bar{P}_1 P_7 P_7 - 6\bar{P}_1^5 P_3 P_7 + 30\bar{P}_1^4 P_3^2 P_7 \bar{P}_1 + 252\bar{P}_1^4 P_5 P_7 P_1^2$$

$$+ 840\bar{P}_1^3 P_5^3 P_7 \bar{P}_1^3 + 1764\bar{P}_1^2 P_5^2 P_7 \bar{P}_1^4 + 2268\bar{P}_1^5 P_3 P_7 + 168\bar{P}_1^4 P_3^2 P_7 \bar{P}_1 + 567\bar{P}_1^4 P_3^2 P_7^2$$

$$- 372\bar{P}_1^4 P_3 P_7 P_1^2 + 42\bar{P}_1^2 P_3^2 P_7 + 804\bar{P}_1 P_3 P_7 \bar{P}_1 P_3 - 114\bar{P}_1^2 P_3^2 P_7 \bar{P}_1 P_3) d\theta$$

$$= \lambda_2(139058 + 75104\lambda_3) \int_0^{2\pi} \bar{P}_1^5 P_7 d\theta = 0,$$

thus the identity (3.39) is valid. Therefore, the condition (3.2) is the necessary for the origin to be a center of equation (3.1).

**Case 7.** If $330 + 703\lambda_3 + 65\lambda_3^2 = 0, p_{01} = 0, b_7 = 0$, then $p_{10} \neq 0, P_1 = p_{10} \cos \theta, \bar{P}_1 = p_{10} \sin \theta$. By (3.26) and (3.35) and (3.37) we get $\int_0^{2\pi} \sin \theta P_7 d\theta = 0, \int_0^{2\pi} \sin 3\theta P_7 d\theta = 0, \int_0^{2\pi} \sin 5\theta P_7 d\theta = 0$, thus

$$\int_0^{2\pi} \bar{P}_1^5 P_7 d\theta = \frac{P_{10}^5}{64} \int_0^{2\pi} (35 \sin \theta - 21 \sin 3\theta + 7 \sin 5\theta - 7 \sin 7\theta) P_7 d\theta = 0,$$

i.e., the identity (3.39) is valid. Thus the condition (3.2) is the necessary for the origin to be a center of equation (3.1).

**Sufficiency:** Now, we show that the condition (3.2) is also sufficient for $r = 0$ to be a center.

By Lemma 2.3, using (3.2) we have

$$P_3 = P_1(\lambda_1 + 2\lambda_2 \bar{P}_1 + 3\lambda_3 \bar{P}_1^2), P_7 = \bar{P}_1 \sum_{k=1}^{7} k\mu_k \bar{P}_1^{k-1},$$

where $\lambda_i, \mu_j$ are real numbers. As $P_1, \bar{P}_1$ are $2\pi$-periodic functions, by Lemma 2.2, $r = 0$ is a center and composition center of (3.1).

In summary, the present theorem has been proved.

Obviously, by this theorem implies the following result.
Corollary 3.1. Suppose that $\lambda_3 \geq 0$. Then $r = 0$ is a center of (3.1), if and only if
\[ \int_0^{2\pi} P_i^{2i+1} P_3 d\theta = 0, \quad (i = 0, 1), \quad \int_0^{2\pi} \tilde{P}_i^{2j+1} P_7 d\theta = 0, \quad (j = 0, 1, 2, 3). \]
Moreover, this center is a composition center. (3.3)–(3.8) are all the focus values of system (1.2).

Remark 3.1. By Theorem 3.1, the focal values of system (1.2) is a constant multiple of six definite integrals (3.2) and the highest order of the fine focus is seven.

Theorem 3.2. For equation (3.1), if one of the following conditions:
1. $14 + 3\lambda_3 = 0, p_{01} = 0, b_3 \neq 0$;
2. $35 + 23\lambda_3 = 0, p_{01} = 0, b_5 \neq 0$;
3. $330 + 703\lambda_3 + 65\lambda_3^2 = 0, p_{01} = 0, b_7 \neq 0$,
is satisfied, then the origin point of (3.1) can't be a composition center, where $\lambda_3, b_i (i = 3, 5, 7)$ are the same as they are in Theorem 3.1.

Proof. Now we only prove that if the first condition of the present theorem is satisfied, then the origin point of (3.1) can't be a composition center. Conversely, assuming that $14 + 3\lambda_3 = 0, p_{01} = 0, b_3 \neq 0$ and $r = 0$ is a composition center, then $p_{10} \neq 0$ and $\tilde{P}_1 = p_{10} \cos \theta, \tilde{P}_3 = p_{10} \sin \theta$,
\[
\int_0^{2\pi} \tilde{P}_1 P_3 d\theta = 0, \quad \int_0^{2\pi} \tilde{P}_3 P_3 d\theta = 0,
\]
by Lemma 2.3
\[ P_4 = P_1 \left( \frac{p_{10}}{p_{10}^3 - p_{12} - p_1^2} \right), \]
i.e., $P_4$ and $P_3$ satisfy the composition conditions (2.1) with $u = \tilde{P}_1$ (if taking $u = \cos \theta - \sin \theta$, it is a not a $2\pi$-periodic function and does not meet the requirements of the Lemma 2.2). As $r = 0$ is a composition center of (3.1),
\[ P_7 = P_1 \psi(\tilde{P}_1) = p_{10} \cos \theta \psi(p_{10} \sin \theta) \]
and
\[ b_3 = \frac{1}{\pi} \int_0^{2\pi} P_7 \sin 3\theta d\theta = \frac{1}{\pi} \int_0^{2\pi} P_1 \cos \theta \psi(p_{10} \sin \theta)(3 \sin \theta - 4 \sin^3 \theta) d\theta = 0, \]
it is contradict with $b_3 \neq 0$. Therefore, if the first condition of the present theorem is satisfied, then $r = 0$ can't be a composition center of (3.1). Similarly, in the other cases, the present conclusions are valid.

Example 3.1. The system
\[
x' = -y + x(x + y)(d_1 + d_4 (x^2 - 4xy + y^2)) + d_2 x^6 + (-d_2 + d_3) x^5 y \\
+ (d_2 - d_3 + d_4) x^4 y^2 + (-d_2 + d_3 - d_4 + d_5) x^3 y^3 \\
+ (d_2 - d_3 + d_4) x^2 y^4 + (-d_2 + d_3) x y^5 + d_2 y^6), \\
y' = x + y (x + y)(d_1 + d_4 (x^2 - 4xy + y^2)) + d_2 x^6 + (-d_2 + d_3) x^5 y \\
+ (d_2 - d_3 + d_4) x^4 y^2 + (-d_2 + d_3 - d_4 + d_5) x^3 y^3 \\
+ (d_2 - d_3 + d_4) x^2 y^4 + (-d_2 + d_3) x y^5 + d_2 y^6)
\]
has a composition center at \((0,0)\), where, \(d_1 \neq 0, d_i (i = 1, 2, ..., 5)\) are arbitrary numbers. In this example \(\lambda_3 = \frac{2}{\sqrt{7}} > 0\) and the conditions of Theorem 3.1 are satisfied.

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