Five-body choreography on the algebraic lemniscate is a potential motion

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In a remarkable paper of 2003 by Fujiwara et al. [1], a figure-eight three-body choreography on the algebraic lemniscate by Bernoulli was discovered. Such a choreography was found to be driven by the action of a pairwise potential \( V(r_{ij}) \), depending only on the mutual relative distances \( r_{ij}, i, j = 1, 2, 3 \). In the present Letter we show that two different choreographies of five bodies on the same algebraic lemniscate exist and correspond to solutions of ten coupled Newton equations of motion with a pairwise interaction potential. For each choreography the explicit form of the potential is found and ten constants of motion are presented, thus, it is superintegrable.

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I. INTRODUCTION

In 1993, a remarkable figure-eight trajectory for a choreographic solution of the three-body problem under Newton gravity law was discovered numerically by C. Moore [2]. Later, a rigorous proof of its existence was given by A. Chenciner and R. Montgomery [3]. Since then, many \( N \)-body choreographic trajectories in Newton gravity have been found, all numerically (see e.g. C. Simó [4, 5]). No analytic form of the figure eight trajectory (as well as for the other choreographic trajectories) is known so far, although it can be approximated with high precision by algebraic curves [5].

Notably, the figure-eight trajectory in the three-body Newton problem discovered by Moore looks very much similar to a well known algebraic lemniscate [5]. It seems that such a similarity motivated Fujiwara et al. [1] to study a three-body choreography on the algebraic lemniscate by J. Bernoulli. They found that such a choreographic solution exists being a solution of the Newton equations under the potential \( V \sim 1/4\log r_{ij}^2 - \sqrt{8/3} r_{ij}^2 \) where \( r_{ij} \) is the relative distance between the bodies \( i \) and \( j \) which lie on a common plane [1]. Such potential actually corresponds to a Newtonian gravitational force in two dimensions plus a mutual repulsive force of an inverted harmonic oscillator potential.

In order to study the three body choreography on the lemniscate, the trajectory was parametrized by elliptic Jacobi functions. In particular, in [1] it was found that such a choreography exists only if the period \( \tau \) (or “equivalently” the elliptic modulus of the Jacobi functions) takes a specific, concrete value. Several constants of motion along the trajectory were found indicating that such trajectory is superintegrable. In a subsequent paper [7] the problem of \( N = 5, 7 \)-bodies on the algebraic lemniscate was investigated. In each case, it was found that only for some particular values of the period \( \tau \), for which the center of mass remains fixed, such choreographies are possible: two different values of the period for the 5-body case and three different periods for the 7-body case were found [7]. However, in contrast to the three body case, no potential for which these choreographies are solutions of the Newton equations of motion was found. It was even conjectured in [7] that such choreographies may not satisfy equations of motion under any interaction potential.

In this Letter, we show that the choreography for \( N = 5 \) bodies on the algebraic lemniscate is a solution of ten coupled Newton equations of motion with a certain pairwise potential, \( V = V(r_{ij}) \), depending on the mutual relative distances between the bodies \( r_{ij} (i, j = 1 \ldots 5) \). Such a potential is found explicitly, being in a way (see below), similar to the potential for the 3-body choreography, \( i.e. \) a potential of the form \( V \sim \alpha \log r_{ij}^2 - \beta r_{ij}^2 \) for certain values of the parameters \( \alpha, \beta \). The question about the maximally particular superintegrability of the closed trajectories (Turbiner’s conjecture [8]) for both 3-body and 5-body choreographies, is briefly discussed.

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II. GENERALITIES

The lemniscate of Bernoulli is a planar (8-shaped) algebraic curve of degree 4 satisfying the equation

\[(x^2 + y^2)^2 = c^2(x^2 - y^2),\]

where \(c\) is a parameter related to the homothety of the lemniscate. We parametrize the algebraic lemniscate by using the Jacobi’s elliptic functions \(sn(t, k)\) and \(cn(t, k)\) as

\[x(t) = c \frac{sn(t, k)}{1 + cn^2(t, k)} (\hat{x} + cn(t, k)\hat{y}),\]

with \(\hat{x}\) and \(\hat{y}\) being the Cartesian orthogonal unit vectors on the plane. In (2) \(k\) stands for the elliptic modulus of the Jacobi functions, and the parameter \(t\) is identified with the physical time. Then, parametrization (2) describes a motion on the algebraic lemniscate with period \(\tau(k) = 4K(k)\), where \(K(k)\) is the complete elliptic integral of the first kind:

\[K(k) = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}}.\]  

The period \(\tau(k)\) is defined by the motion on the \(x\)-direction. In \(y\)-direction the motion has a period which is half the period in \(x\). The period \(\tau\) is \(c\) independent. Without loss of generality we can set \(c = 1\) in the discussion which follows. We restrict the study to equal mass bodies \(m_i = 1, i = \ldots N\).

A three body choreography i.e. a periodic motion on a closed orbit, where the bodies chase each other on a common orbit with equal time-spacing, is defined on the lemniscate (following the nomenclature in [1]) by

\[\{x_1(t), x_2(t), x_3(t)\} = \{x(t), x(t + \tau/3), x(t - \tau/3)\},\]

where \(x(t), x(t \pm \tau/3)\) are given in (2). The center of mass should be fixed (at the origin): \(x_{CM} = \sum_{i=1}^{3} x_i(t) = 0\). This condition is satisfied if and only if the elliptic modulus \(k\) takes the value

\[k_0^2 = \frac{2 + \sqrt{3}}{4},\]

(see [1] for details).

In the study carried out in [1] a number of constant quantities for the 3-body choreography was found. In particular, the angular momentum \(\sum_{i=1}^{3} x_i \times v_i = 0\), and the moment of inertia \(\sum_{i=1}^{3} x_i^2 = \sqrt{3}\), are constant. Some of these conserved quantities are global integrals of motion in involution (in the sense of a vanishing Poisson bracket), while some others are conserved quantities only along the lemniscate trajectory. Conserved quantities along special trajectories are called generically “particular integrals” [5].

Among the conserved quantities in the 3-body choreography, it was discovered that the total kinetic energy is a constant of motion \(T = 1/2 \sum_{i=1}^{3} v_i^2 = 3/8\). Thus, for a conservative system it immediately implies that the potential energy is also a constant of motion. This fact is of a crucial importance to find the form of the potential supporting the choreography of 3 and 5 bodies on the lemniscate. Since our approach is essentially different from that used in [1] to find the potential, we analyze first the case of the three-body choreography:

In the case of the three-body choreography found in [1] two momentum-independent conserved quantities were found:

\[I_1 \equiv r_{12}^2 r_{13}^2 r_{23}^2 = \frac{3\sqrt{3}}{2},\]

\[I_2 \equiv r_{12}^2 + r_{13}^2 + r_{23}^2 = 3\sqrt{3},\]

where \(r_{ij}^2 = (x_i - x_j)^2, i, j = 1, 2, 3\) are the relative distances squared between the bodies \(i\) and \(j\). It is evident \(r_{ji}^2 = r_{ij}^2\). The integral \(I_2\), being the sum of squares of all relative distances, is denoted as hyper-radius squared in the space of relative distances. It appears to play an important role in several problems depending on relative distances only, i.e. in translational and rotationally invariant problems.
A natural assumption is that the potential, being a constant of motion, should depend on these momentum-independent conserved quantities, i.e.

\[ V = V(I_1, I_2), \]

and, if we require a pairwise interaction potential, the explicit form is

\[ V = \alpha \log I_1 + \beta I_2. \]

where \( \alpha \) and \( \beta \) are constant parameters which are found by demanding that the Newton equations of motion are satisfied. In \[1\] it was found that \( \alpha = \frac{1}{4}, \beta = -\frac{\sqrt{3}}{2} \). Then, the potential is constant with the value \( V = \frac{1}{4} \log (\frac{3\sqrt{3}}{2}) - \frac{3}{8} \), and the total energy of the three body choreography is

\[ E^{(3)} = T + V = \frac{1}{4} \log (\frac{3\sqrt{3}}{2}) = 0.23869281311055480691. \]

In total six independent conserved quantities depending on coordinates and momenta were found in \[1\]. Most of the conserved quantities found in \[1\] are particular (along the trajectory) integrals of motion. This fact already indicates that the system is particularly superintegrable along the lemniscate; a superintegrable Hamiltonian system is by definition a system with more integrals of motion than degrees of freedom. It is called maximally superintegrable if there exits \( 2n - 1 \) independent integrals of motion for a system with \( n \) degrees of freedom. In the case of three particles on the lemniscate we have (after removing two degrees of freedom corresponding to center of mass) \( n = 4 \) degrees of freedom. If the three body choreography on the lemniscate supports Turbiner’s conjecture (2013), namely that any closed periodic trajectory is particularly (maximally) superintegrable there should be one more independent constant of motion along the trajectory to be discovered. This will be investigated elsewhere.

### III. 5-BODY CHOREOGRAPHY

Now consider a five-body choreography on the algebraic lemniscate defined by

\[ \{x_1(t), x_2(t), x_3(t), x_4(t), x_5(t)\} = \{x(t - 2\tau/5), x(t - \tau/5), x(t), x(t + \tau/5), x(t + 2\tau/5)\}, \]

where \( \tau = 4K(k) \) is the period of the choreography, and the 2-dimensional position vectors \( x(t), x(t \pm \tau/5), x(t \pm 2\tau/5) \) are defined as in \[2\].

The condition that the center of mass of the system is fixed at the origin

\[ X_{CM}(t) = x_1 + x_2 + x_3 + x_4 + x_5 = 0, \]

determines, if exists, the period of the motion, or equivalently the value of the elliptic modulus \( k \) of the Jacobi functions in the parametrization \[2\]. We found two solutions for the elliptic modulus \( k \)

\[ k^2 = \begin{cases} 0.65366041395477321345 \ldots \equiv k_1^2 \\ 0.99764373603161323509 \ldots \equiv k_2^2 \end{cases} \]

c.f. \[2\]. The solutions \[13\] also ensure (a) the conservation of the angular momentum:

\[ L = \sum_{i=1}^{5} x_i \times v_i = 0, \]

and (b) the hyper-radius squared (or the moment of inertia \( \sum x_i^2 \)):

\[ I^{(5)}_{HR} = 5 \sum_{i=1}^{5} x_i^2 = \sum_{i<j}^{5} r_{ij}^2 = \begin{cases} 11.995383205775537457 for k_1 \\ 17.975523091392961251 for k_2 \end{cases} \]

In the search for conserved quantities for the 5-body choreographies on the lemniscate, we found that in addition to the hyper-radius squared \[14\], there exist two more momentum independent constants of motion along the algebraic lemniscate:

\[ I^{(5)}_1 = \begin{cases} r_{12}^2 r_{23}^2 r_{34}^2 r_{45}^2 r_{15}^2 for k_1 \\ r_{13}^2 r_{35}^2 r_{25}^2 r_{24}^2 r_{14}^2 for k_2 \end{cases} \]
The lemniscate and the configuration of five bodies (11) at $t = 0$ and $t = K/5$ for $k_1^2$ (left) and for $k_2^2$ (right), see (13). The position of the center of mass is marked by a bullet and placed at the origin of coordinates. The lines between the bodies indicate the relative distances involved in the conserved quantities $I_1^{(5)}, I_2^{(5)}$ (see the text).

$$I_2^{(5)} = \begin{cases} \frac{r_{12}^2 + r_{23}^2 + r_{34}^2 + r_{45}^2 + r_{15}^2}{2} \text{ for } k_1 \\ \frac{r_{13}^2 + r_{25}^2 + r_{24}^2 + r_{14}^2}{2} \text{ for } k_2 \end{cases}$$

To the best of our knowledge such constants of motion for the 5-body choreographies were not known before. There are no other constants of motion formed by products or sums of some relative distances squared.

The conserved quantities $I_1^{(5)}, I_2^{(5)}$ depend on five (out of ten) different relative distances squared $r_{ij}^2$. Moreover, the subsets of relative distances corresponding to the two possible values of the elliptic modulus $k_{1,2}^2$ are mutually disjoint. Following the nomenclature (11), for the choreography corresponding to the smaller value of the elliptic modulus squared $k_1^2$, the conserved quantities $I_1^{(5)}, I_2^{(5)}$ depend on the relative distances of the nearest-neighbors for $k_1$ and on the relative distances of the next-to-nearest neighbors for $k_2$. So, we can build two sets of relative distances: the set of relative distances between nearest-neighbors and the set of relative distances between next-to-nearest neighbors. Within each set the relative distances oscillate in the same way except for time delays which are multiples of a fifth of the period. Thus, these two sets can be characterized by the amplitude of the oscillations. Representative examples are:

$$(16) \quad [r_{12}^{\min}(k_1) = 0.6866727929990557394, r_{12}^{\max}(k_1) = 1.108730495493916550],$$

$$[r_{13}^{\min}(k_1) = 0.38411684326397186757, r_{13}^{\max}(k_1) = 1.889145301630350845]$$

$$(17) \quad [r_{12}^{\min}(k_2) = 0.26636437357508228993, r_{12}^{\max}(k_2) = 1.791384757193286574],$$

$$[r_{13}^{\min}(k_2) = 0.66719007021760872692, r_{13}^{\max}(k_2) = 1.9952819069627494321].$$

These examples correspond to $k_1^2 = 0.653660413954773213440$ and $k_2^2 = 0.997643736031613235083$. For the smaller value of the elliptic modulus squared $k_1^2$ the relative distances $r_{13}, r_{14}, r_{15}, r_{23}, r_{24}, r_{25}$ oscillate with an amplitude of $0.6866727929990557394$, while for the larger value of the elliptic modulus squared $k_2^2$ the relative distances $r_{12}, r_{14}, r_{15}, r_{23}, r_{24}, r_{25}$ oscillate with an amplitude of $0.997643736031613235083$. Notice that, alternatively to the hyperradius squared (14) we can as well define (as independent) the quantity $I_3^{(5)} = I_{HR}^{(5)} - I_2^{(5)}$ which depends on the relative distances of the next-to-nearest neighbors for $k_1$, and on the relative distances of the nearest-neighbors for $k_2$. So, we can build two sets of relative distances: the set of relative distances between nearest-neighbors and the set of relative distances between next-to-nearest neighbors. Within each set the relative distances oscillate in the same way except for time delays which are multiples of a fifth of the period.
In Fig. 1 we show two representative configurations of the five-body choreography on the lemniscate at \( t = 0 \) and \( t = \frac{K}{5} \), for the case of the elliptic modulus squared \( k^2 = k_1^2 \) (left panel) and for \( k^2 = k_2^2 \) (right panel). For the case \( k_1^2 \) (left panel in Fig. 1): at \( t = 0 \) the relative distances \( r_{15}, r_{24} \) take their maximal values \( (r_{15}^{\text{max}}, r_{24}^{\text{max}}) \) given in (14) while at \( t = K/5 \) the relative distances \( r_{12} \) and \( r_{35} \) take their minimal values \( (r_{12}^{\text{min}}, r_{35}^{\text{min}}) \) also given in (14). On the other side, for the case \( k_2^2 \) (right panel in Fig. 1): at \( t = 0 \) the relative distances \( r_{15}, r_{24} \) take their maximal values \( (r_{15}^{\text{max}}, r_{24}^{\text{max}} = r_{13}^{\text{max}}) \) given in (15) while at \( t = K/5 \) the relative distances \( r_{12} \) and \( r_{35} \) take their minimal values \( (r_{12}^{\text{min}}, r_{35} = r_{13}^{\text{min}}) \) also given in (15).

The key point in the search for the potential is the fact that the total kinetic energy along the lemniscate \( T = \frac{1}{2} \sum_{i=1}^{5} p_i^2 \), is a constant of motion for the two choreographies found (similarly to the case of the three body choreography). It implies, that the potential energy \( V \) is a constant of motion as well. It naturally suggests that the potential can be a function of the momentum-independent constants of motion:

\[
V = V(I_1^{(5)}, I_2^{(5)}, I_{HR}).
\]

If we impose the extra condition of a pairwise nature of the interaction potential, it leads to the explicit form

\[
V = \alpha \ln I_1^{(5)} + a I_2^{(5)} - \beta I_{HR}^{(5)},
\]

where the constants \( \alpha, a, \beta \) are fixed by requiring that the five vectorial Newton equations of motion where the potential is unknown

\[
\frac{d^2}{dt^2} x_i(t) = -\nabla_{x_i} V, \quad i = 1 \ldots 5
\]

are satisfied. It leads to a coupled system of ten first order PDE for the potential \( V \). The solution exists and corresponds to

\[
V = \left\{ \begin{array}{l}
\alpha_1 \left\{ \ln r_{12}^2 + \ln r_{23}^2 + \ln r_{34}^2 + \ln r_{45}^2 + \ln r_{13}^2 \right\} \\
\alpha_2 \left\{ \ln r_{13}^2 + \ln r_{35}^2 + \ln r_{25}^2 + \ln r_{24}^2 + \ln r_{14}^2 \right\} - \beta_{1,2} \sum_{i<j} r_{ij}^2,
\end{array} \right.
\]

with constants (see (19))

\[
\begin{aligned}
&\alpha = 0, \alpha_1 = \frac{1}{4}, \beta_1 = 0.015366041395477321360 \text{ for } k_1 \\
&\alpha = 0, \alpha_2 = \frac{1}{4}, \beta_2 = 0.04976437360316132382 \text{ for } k_2.
\end{aligned}
\]

We immediately notice that the parameters \( \alpha_{1,2} \) take the same value as the parameter \( \alpha \) in the three-body case [9]. The above potentials correspond to two different choreographies of five bodies on the same algebraic lemniscate, with periods: \( r_1 = 8.048777052207468484 \) for \( k_1 \), and \( r_2 = 17.654582260596687373 \) for \( k_2 \). In the form (19) the potential appears to depend on all ten relative distances \( r_{ij} \). However, it is known that on the plane, the motion of five bodies is described, in general, by seven relative distances (in the case of zero angular momentum). The question about how many independent relative distances are needed to describe the 5-body choreographies on the algebraic lemniscate remains to be clarified. This will be investigated elsewhere.

Another question of interest is the notion of superintegrability of the orbit. In the following we collect the constants of motion that we have found for the two choreographies. First of all, in the case of five bodies on the lemniscate we have (after removing two degrees of freedom corresponding to center of mass) \( n = 8 \) degrees of freedom. Let us give the list of conserved quantities (global and particular) found so far,

1. \( L = \sum_{i=1}^{5} \mathbf{x}_i \times \mathbf{v}_i = 0 \), Total Angular Momentum
2. \( E = T + V = \left\{ \begin{array}{l}
0.54804692944384581936 \text{ for } k_1 \\
0.3174790068896754830 \text{ for } k_2
\end{array} \right. \), Total Energy
3. \( I_1^{(5)} = \left\{ \begin{array}{l}
r_{12}^2 r_{23}^2 r_{34}^2 r_{45}^2 r_{15}^2 = 0.26362178303408707110 \text{ for } k_1 \\
r_{13}^2 r_{35}^2 r_{25}^2 r_{24}^2 r_{14}^2 = 30.760801541637359790 \text{ for } k_2
\end{array} \right. \)
4. \( I_2^{(5)} = \left\{ \begin{array}{l}
\frac{1}{r_{12}^2 + r_{23}^2 + r_{34}^2 + r_{45}^2 + r_{15}^2} = 4.051781784548308414 \text{ for } k_1 \\
\frac{1}{r_{13}^2 + r_{35}^2 + r_{25}^2 + r_{24}^2 + r_{14}^2} = 12.515257719766335417 \text{ for } k_2
\end{array} \right. \)
5. $J^{(5)}_{HR} = 5 \sum_{i=1}^{5} x_i^2 = \sum_{i<j} r_{ij}^2 = \begin{cases} 11.99538320577537457 & \text{for } k_1 \\ 17.975523091392961251 & \text{for } k_2 \end{cases}$ Hyper-radius Squared

6. $\sum_{i=1}^{5} \rho_i^{-2} = 9 \sum_{i=1}^{5} J^{(5)}_{HR}$ Sum of Squares of Curvature

7. $T = \frac{1}{2} \sum_{i=1}^{5} v_i^2 = \begin{cases} 1.0656784451054396 & \text{for } k_1 \\ 0.35545935316766729 & \text{for } k_2 \end{cases}$ Kinetic Energy

8. $J_i(k) = v_i^2 + (k^2 - \frac{1}{2}) x_i^2 = \frac{1}{2}$, $i = 1, 2, \ldots 5$

In the list of conserved quantities above, the curvature is defined as $\rho^{-1}(t) = \frac{\|v(t) \times a(t)\|}{|v(t)|^3}$, with $v = \dot{x} \equiv \frac{dx}{dt}$ the velocity, and $a = \ddot{x} \equiv \frac{\frac{d}{dt}v}{\|v\|^2}$ the acceleration. For any point on the lemniscate, the square of the curvature is related to the distance to the origin $\rho^{-2}(t) = 9 x^2(t)$, see [1]. Notice that the constants $J_i(k)$ $i = 1, 2, \ldots 5$ which appear to correspond to the motion, in phase space, of five independent two-dimensional isotropic harmonic oscillators might be only a property of the parametrization of the lemniscate and do not represent true dynamical relations for arbitrary values of the elliptic modulus $k$. Not all above listed conserved quantities are independent. In total we have ten independent conserved quantities and therefore the trajectory is particularly superintegrable. If Turbiner’s conjecture is valid, i.e. that a closed periodic trajectory is maximally particularly superintegrable, then we should have $2n - 1 = 15$ independent constants of motion for the choreographies of five bodies moving along the lemniscate. In such a case, there are more independent constants of motion along the lemniscate to be discovered. This will be investigated elsewhere [9].

IV. CONCLUSION

We studied a choreographic motion of five bodies of unit mass on the (algebraic) lemniscate of Bernoulli parametrized by Jacobi elliptic functions. We found two choreographies with different periods (two specific concrete values for the elliptic modulus) required for the conservation of the center of mass of the system in agreement with the findings of Ref. [2]. We have found explicitly in total ten independent (global and particular) constants of motion associated to these five body choreographies. Thus, the 5-body choreographies on the algebraic lemniscate are superintegrable. In particular, it was shown that the total kinetic energy is a conserved quantity. Two momentum-independent constants of motion depending on five (out of ten) different relative distances $I^{(5)}_1, I^{(5)}_2$ along the trajectory were discovered. Also, the hyper-radius squared $I^{(5)}_{HR}$ in the space of relative distances, which depends on all ten different relative distances, is a constant of motion. We have shown that the potential supporting the choreography of five bodies on the algebraic lemniscate exists and depends on the momentum-independent constants of motion $I^{(5)}_1$ and $I^{(5)}_{HR}$ only. We emphasize that such potential is a solution of ten first order PDE for the potential $V$ given by the Newton equations. For each choreography, the potential is found explicitly being a function of the relative distances with pairwise interactions of the form $V \sim \alpha \log r_{ij}^2 - \beta r_{ij}^2$ for certain values of the parameters $\alpha, \beta$.

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