DECOMPOSITIONS OF QUASIRANDOM HYPERGRAPHS INTO HYPERGRAPHS OF BOUNDED DEGREE

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Abstract. We prove that any quasirandom uniform hypergraph $H$ can be approximately decomposed into any collection of bounded degree hypergraphs with almost as many edges. In fact, our results also apply to multipartite hypergraphs and even to the sparse setting when the density of $H$ quickly tends to 0 in terms of the number of vertices of $H$. Our results answer and address questions of Kim, Kühn, Osthus and Tyomkyn; and Glock, Kühn and Osthus as well as Keevash.

The provided approximate decompositions exhibit strong quasirandom properties which is very useful for forthcoming applications. Our results also imply approximate solutions to natural hypergraph versions of long-standing graph decomposition problems, as well as several decomposition results for (quasi)random simplicial complexes into various more elementary simplicial complexes such as triangulations of spheres and other manifolds.

1. Introduction

Decompositions of large mathematical objects into smaller pieces have been an integral part in nearly every field of mathematics. We also find numerous instances of this theme in combinatorics with many connections to other related fields. Potentially the most prominent recent examples are the achievements surrounding the resolution of the Existence conjecture by Keevash [24] and its generalizations by Glock, Kühn, Lo and Osthus as well as Keevash himself [19, 25]. These results include decompositions of quasirandom uniform (multipartite) hypergraphs (the so-called host hypergraphs) into cliques, and more generally, into arbitrary but fixed hypergraphs subject to “obvious” divisibility conditions and the host hypergraphs being sufficiently large.

The analogous question for graphs has been famously solved by Wilson [38, 39, 40]. His results have initiated a very vibrant research area with connections to Latin squares and arrays, geometry, design theory and combinatorial probability theory.

The following three conjectures are a driving force regarding graph decompositions and concern families of sparse graphs whose number of vertices grows with the number of vertices of the host graph; sometimes the sparse graphs even have the same number of vertices as the host graph, a case which turns out to be particularly challenging. Before we state the conjectures, let us introduce some notation. We say a family $\mathcal{H}$ of hypergraphs packs into a hypergraph $G$ if the members of $\mathcal{H}$ can be found edge-disjointly in $G$. The family $\mathcal{H}$ decomposes $G$ if additionally $\sum_{H \in \mathcal{H}} e(H) = e(G)$, where $e(J)$ denotes the number of edges of a hypergraph $J$. We also say there is a decomposition of $G$ into copies of a hypergraph $H$ if the edge set of $G$ can be partitioned into copies of $H$. Given these definitions, we can state the aforementioned conjectures.

(I) (Ringel, 1963) For all $n$ and all trees $T$ on $n+1$ vertices, there is a decomposition of $K_{2n+1}$ into $2n + 1$ copies of $T$.

(II) (Tree packing conjecture – Gyárfás and Lehel, 1976) For all $n$ and all sequences of trees $T_1, \ldots, T_n$ where $T_i$ has $i$ vertices, there is a decomposition of $K_n$ into $T_1, \ldots, T_n$.

(III) (Oberwolfach problem – Ringel, 1967) For all odd $n$ and all 2-regular graphs $F$ on $n$ vertices, there is a decomposition of $K_n$ into copies of $F$.

Until recently (considering that the conjectures were posed in the 1960s and 1970s), no substantial progress had been made, but lately a number of striking and exciting advances have been achieved mostly in conjunction with very elaborated analyses of random processes. This includes the resolution of Conjectures (I) and (II) for bounded degree trees [22], the complete resolutions of Conjecture (I) in [28, 36] and Conjecture (III) in [18, 27], and the resolution of Conjecture (II) for
families of trees with many leaves and a weak restriction concerning the maximum degree [1]. All these results hold for sufficiently large $n$. Preceding these results, there is a collection of approximate decomposition results, that is, where a few edges of the host graph are not covered, under various and quite general conditions, see [3, 9, 13, 29, 35]. The importance of these results should not be underestimated as many of those play a key role in the actual decomposition results.

Although there are numerous (approximate) decomposition results for spanning structures in graphs, the situation for hypergraphs is notably different. There are only a few results concerning various types of Hamilton cycles [7, 15, 16] as well as Keevash’s results concerning factors, that is the vertex-disjoint collection of a hypergraph of fixed size, which follow from results in [25]. The aim of this article is to provide a versatile tool for approximate decompositions of quasirandom hypergraphs into families of spanning bounded degree hypergraphs. A direct consequence of our main result is an asymptotic solution to a hypergraph Oberwolfach problem asked by Glock, Kühn and Osthus [20]; see Section 8.1.

One key feature of our results is their applicability to hypergraphs with vanishing density. This answers a question of Kim, Kühn, Osthus and Tyomkyn in a strong form who asked in [29] for approximate decompositions of sparse quasirandom graphs into bounded degree (spanning) graphs. Previously, a general result for embedding even a single bounded degree hypergraph in a sparse pseudorandom hypergraph was not known to exist – a question that was also remarked upon by Allen, Böttcher, Hán, Kohayakawa and Person in [2, Section 7.5]. See also [21] for sparse embedding results of linear hypergraphs.

In particular for further applications it turned out that (approximate) decompositions of multipartite hypergraphs are highly desirable (to name only two examples, see Kim, Kühn, Osthus and Tyomkyn [29] and Keevash [25] which are for example applied in [18, 22, 27, 28]). In view of this, we provide all our tools also for the multipartite setting and lift the hypergraph blow-up lemma due to Keevash [23], which embeds a single hypergraph, to an approximate decomposition result for the setting of quasirandom hypergraphs. Kim, Kühn, Osthus and Tyomkyn [29] established such a decomposition result for graphs by proving a blow-up lemma for approximate decompositions. Keevash [26] as well as Kim, Kühn, Osthus and Tyomkyn explicitly asked for such a decomposition result for quasirandom hypergraphs.

Whether we have a strong control over the actual (approximate) decomposition, locally and globally, is another decisive factor in the strength of the tool for further applications. Therefore, we make a considerable effort to implement two types of versatile test functions with respect to which the decomposition behaves random-like (for more details we refer the reader to our more technical results in Section 1.2).

Let us start with a simplified version of our main result. We say a hypergraph $G$ is $k$-uniform or a $k$-graph if all edges have size $k$. The neighbourhood $N_G(S)$ of a $(k-1)$-set $S$ of vertices is the set of vertices that form an edge together with $S$. Let $\varepsilon > 0$, $t \in \mathbb{N}$, $d \in (0,1]$ and suppose $G$ has $n$ vertices. We say $G$ is $(\varepsilon, t, d)$-typical if $|\bigcap_{S \subseteq S} N_G(S)| = (1 \pm \varepsilon)d^{|S|}n^t$ for all (non-empty) sets $S$ of $(k-1)$-sets of $V(G)$ with $|S| \leq t$. We refer to the (vertex) degree of a vertex $v$ as the number of edges containing $v$ and let $\Delta(G)$ be the maximum degree in $G$.

The following result is a direct consequence of our main result.

**Theorem 1.1.** For all $\alpha \in (0,1]$, there exist $n_0, t \in \mathbb{N}$ and $\varepsilon > 0$ such that the following holds for all $n \geq n_0$. Suppose $G$ is an $(\varepsilon, t, d)$-typical $k$-graph on $n$ vertices with $k \leq \alpha^{-1}$ as well as $d \geq n^{-\varepsilon}$, and $\mathcal{H}$ is a family of $k$-graphs $H$ on $n$ vertices with $\Delta(H) \leq \alpha^{-1}$ for all $H \in \mathcal{H}$ and $\sum_{H \in \mathcal{H}} e(H) \leq (1 - \alpha)e(G)$. Then $\mathcal{H}$ packs into $G$.

Observe that the binomial random $k$-graph is with high probability $(\varepsilon, t, d)$-typical whenever $k, \varepsilon, t$ are fixed and $d \geq n^{-\varepsilon}$ and consequently with high probability these $k$-graphs can be approximately decomposed into any list of bounded degree hypergraphs with almost as many edges.

1.1. **Applications.** We believe that our main results will be useful for forthcoming applications. We illustrate and discuss some of these applications in the concluding Section 8. Indeed, Theorem 1.1 can be applied to several natural questions on hypergraph decompositions including a

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1Yet, our notion of quasirandomness is more restrictive than hypergraph regularity as used in [23].
question of Glock, Kühn and Osthus, which we consider in Section 8.1. Closely linked to hypergraphs are simplicial complexes which are equivalent to downward closed hypergraphs $H$; that is, whenever $e$ is an edge of $H$, then $H$ contains also all subsets of $e$ as edges. Gowers was one of the first who suggested the investigation of topological analogues of 1-dimensional graph structures in higher dimensions as we may consider a Hamilton cycle in a graph as a spanning 1-dimensional simplicial complex that is homeomorphic to $S^1$ and hence a Hamilton cycle in higher dimensions may be viewed as spanning $k$-dimensional simplicial complex that is homeomorphic to $S^k$. In particular, Linial has considered further questions of this type under the term 'high-dimensional combinatorics' and has achieved several new insights. We discuss implications of Theorem 1.1 to these types of questions in Section 8.2.

1.2. Multipartite graphs and the main result. Let us now turn to the statement of our main result. In fact, it is stated for multipartite hypergraphs, but it easily implies a similar statement for non-partite hypergraphs. We use standard terminology and refer to Section 3.1 for more notational details.

We say that a family/multiset of $k$-graphs $\mathcal{H} = \{H_1, \ldots, H_s\}$ packs into a $k$-graph $G$ if there is a function $\phi : \bigcup_{H \in \mathcal{H}} V(H) \to V(G)$ such that $\phi|_{V(H)}$ is injective and $\phi$ injectively maps edges onto edges. In such a case, we call $\phi$ a packing of $\mathcal{H}$ into $G$. Our general aim is to pack a collection $\mathcal{H}$ of multipartite $k$-graphs into a quasirandom host $k$-graph $G$ having the same multipartite structure, which is captured by a so-called ‘reduced graph’ $R$. For graphs and where $\mathcal{H}$ consists of a single graph only, this result has been proven first by Komlós, Sárközy and Szemerédi [30] and is famously known as the blow-up lemma. In view of this, we say $(H, G, R, \mathcal{X}, \mathcal{V})$ is a blow-up instance of size $(n, k, r)$ if

- $H, G, R$ are $k$-graphs where $V(R) = [r]$;
- $\mathcal{X} = \{X_i\}_{i \in [r]}$ is a vertex partition of $H$ such that $|e \cap X_i| \leq 1$ for all $e \in E(H), i \in [r]$;
- $\mathcal{V} = \{V_i\}_{i \in [r]}$ is a vertex partition of $G$ such that $|V_i| = |X_i| = (1 \pm 1/2)n$ for all $i \in [r]$;
- $H[X_{i_1}, \ldots, X_{i_k}]$ is empty whenever $\{i_1, \ldots, i_k\} \notin E(R)$.

We also refer to $(H, G, R, \mathcal{X}, \mathcal{V})$ as a blow-up instance if $\mathcal{H}$ is a collection of $k$-graphs and $\mathcal{X}$ is a collection of vertex partitions $(X_i^H)_{i \in [r], H \in \mathcal{H}}$ so that $(H, G, R, (X_i^H)_{i \in [r]}, \mathcal{V})$ is a blow-up instance for every $H \in \mathcal{H}$.

Given a blow-up instance $\mathcal{B} = (\mathcal{H}, G, R, \mathcal{X}, \mathcal{V})$ of size $(n, k, r)$, we generalize the notion of typicality to the multipartite setting given by the reduced graph $R$. For finite and disjoint sets $A_1, \ldots, A_r$, we write $\bigsqcup_{i \in [r]} A_i := \{a_1, \ldots, a_r\} : a_i \in A_i$ for all $i \in [r]$. We say $G$ is $(\varepsilon, t, d)$-typical with respect to $R$ if for all $i \in [r]$ and all sets $S \subseteq \bigcup_{e \in E(R)_i} V_j$ with $|S| \leq t$, we have $|V_i \cap \bigcap_{S \in S} N_G(S)| = (1 \pm \varepsilon)de^d|V_i|$. We say the blow-up instance $\mathcal{B}$ is $(\varepsilon, t, d)$-typical if $G$ is $(\varepsilon, t, d)$-typical with respect to $R$ and $|V_i| = |X_i^H| = (1 \pm \varepsilon)n$ for all $H \in \mathcal{H}, i \in [r]$. We say $\mathcal{B}$ is $\Delta$-bounded if $\Delta(R), \Delta(H) \leq \Delta$ for each $H \in \mathcal{H}$.

For the readers' convenience, we first state a simplified version of our main result for multipartite graphs.

**Theorem 1.2.** For all $\alpha \in (0, 1]$, there exist $n_0, t \in \mathbb{N}$ and $\varepsilon > 0$ such that the following holds for all $n \geq n_0$. Suppose $(\mathcal{H}, G, R, \mathcal{X}, \mathcal{V})$ is an $(\varepsilon, t, d)$-typical and $\alpha^{-1}$-bounded blow-up instance of size $(n, k, r)$ with $k \leq \alpha^{-1}$, $r \leq n^\log n$, $d \geq n^{-\varepsilon}$, $|\mathcal{H}| \leq n^{k+1}$, and $\sum_{H \in \mathcal{H}} e_H(X_{i_1}^H, \ldots, X_{i_k}^H) \leq (1 - \alpha)dn^k$ for all $\{i_1, \ldots, i_k\} \in E(R)$. Then there is a packing $\phi$ of $\mathcal{H}$ into $G$ such that $\phi(X_i^H) = V_i$ for all $i \in [r]$ and $H \in \mathcal{H}$.

In numerous applications of the original blow-up lemma for graphs, it has been essential that it provides additional features. In view of this, we make a substantial effort to also include further tools in our results that allow us to control the structure of the packings and will be very useful for future applications. We achieve this with two different types of what we call testers. The first tester is a so-called set tester; with the setting as in Theorem 1.2, we can fix a set $Y \subseteq X_i^H$ and $W \subseteq V_i$ for some $i \in [r]$ and $H \in \mathcal{H}$. Then we find a packing $\phi$ such that $|W \cap \phi(Y)| = |W||Y|/n \pm \alpha n$. Moreover, we can even fix several sets $Y_j$ as above in multiple $k$-graphs $H_j$ in $\mathcal{H}$ and the size of their common intersection with $W$ is as large as we would expect it to be in an idealized random packing.

The second type of tester is a so-called vertex tester. In the simplest form, we fix a vertex $e \in V_i$ with $i \in [r]$ and define a weight function on $\bigcup_{H \in \mathcal{H}} X_i^H$. Then we find a packing such that the
Let us now formally define these two types of testers. Suppose \( B \) is a blow-up instance as above. We say \((W,Y_1,\ldots,Y_m)\) is an \( \ell \)-set tester for \( B \) if \( m \leq \ell \) and there exist \( i \in [r] \) and distinct \( H_1,\ldots,H_m \in \mathcal{H} \) such that \( W \subseteq V_i \) and \( Y_j \subseteq X_{i_j}^H \) for all \( j \in [m] \). We say \((\omega,\epsilon)\) is an \( \ell \)-vertex tester for \( B \) with centres \( \epsilon = \{c_i\}_{i \in I} \) in \( I \subseteq [r] \), if

\[
\begin{align*}
\bullet & \ |I| \leq k - 1 \text{ and } I \subseteq r \text{ for some } r \in E(G), \text{ and } c_i \in V_i \text{ for each } i \in I, \text{ and} \\
\bullet & \ \omega \text{ is a weight function on the } |I| \text{-tuples } \mathcal{X}_{I|} := \bigcup_{H \in \mathcal{H}} \bigcup_{I \subseteq [r]} X_{I|}^H \text{ with } \omega : \mathcal{X}_{I|} \to [0,\ell] \text{ and} \\
\text{whenever } |I| \geq 2, \text{ we have } \supp(\omega) = \omega^{-1}(0,\ell) \subseteq \{x \in \mathcal{X}_{I|} : x = \epsilon \cap \mathcal{X}_{I|} \text{ for some } \epsilon \in \mathcal{H} \}. 
\end{align*}
\]

For an \( \ell \)-tuple function \( \omega : (X_i^1) \to \mathbb{R}_{\geq 0} \) on a finite set \( X \), we define \( \omega(X') := \sum_{S \in (X_i^1)} \omega(S) \) for any \( X' \subseteq X \). The following theorem is our main result.

**Theorem 1.3.** Suppose the assumptions of Theorem 1.2 hold and suppose \( W_{\text{set}}, W_{\text{ver}} \) are sets of \( \alpha^{-1} \)-set testers and \( \alpha^{-1} \)-vertex testers of size at most \( n^{2\log n} \), respectively. Then there is a packing \( \phi \) of \( H \) in \( G \) such that

\[
\begin{align*}
(i) & \ |W \cap \bigcap_{j \in [m]} \phi(Y_j)| = |W|/|Y_1| \cdots |Y_m|/n^m \pm \alpha n \text{ for all } (W,Y_1,\ldots,Y_m) \in W_{\text{set}}; \\
(ii) & \ \omega(\phi^{-1}(\epsilon)) = (1 \pm \alpha)\omega(\mathcal{X}_{I|})/n^{|I|} \pm \alpha n \text{ for all } (\omega,\epsilon) \in W_{\text{ver}} \text{ with centres } \epsilon \text{ in } I. 
\end{align*}
\]

In the same line, we also provide these two types of testers if \( G \) is a (non-multipartite) quasi-random k-graph and the result in fact follows from Theorem 1.3. The definition is adapted in the obvious way. Suppose the vertex set of \( G \) is \( V \) and we aim to pack \( \mathcal{H} \) into \( G \) where the vertex set of \( H \in \mathcal{H} \) is denoted by \( X^H \). For set testers \((W,Y_1,\ldots,Y_m)\), we proceed as above but select \( W \subseteq V \) and \( Y_j \subseteq X_{i_j}^H \). For vertex testers \((\omega,\epsilon)\) with \( \{c_i\}_{i \in I} \) and \( |I| \leq k - 1 \), we also proceed as above but require that \( \{c_i\}_{i \in I} \) is an \( |I| \)-set in \( V \) and \( \omega \) is a function from the union over all \( H \in \mathcal{H} \) of the ordered \( |I| \)-sets of \( X^H \) into \([0,\ell]\), where \( \supp(\omega) \) contains only \( |I| \)-tuples that are contained in an edge of \( H \) if \(|I| \geq 2 \). With this we obtain the following result.

**Theorem 1.4.** For all \( \alpha \in (0,1] \), there exist \( n_0,t \in \mathbb{N} \) and \( \varepsilon > 0 \) such that the following holds for all \( n \geq n_0 \). Suppose \( G \) is an \((\varepsilon,t,d)\)-typical k-graph on \( n \) vertices with \( k \leq \alpha^{-1}, d \geq n^{-\varepsilon} \) and \( \mathcal{H} \) is a family of k-graphs on \( n \) vertices with \( \Delta(H) \leq \alpha^{-1} \) for all \( H \in \mathcal{H} \) and \( |\mathcal{H}| \leq n^s \) such that \( \sum_{H \in \mathcal{H}} e(H) \leq (1 - \alpha)e(G) \). Suppose \( W_{\text{set}}, W_{\text{ver}} \) are sets of \( \alpha^{-1} \)-set testers and \( \alpha^{-1} \)-vertex testers of size at most \( n^{\log n} \), respectively. Then there is a packing \( \phi \) of \( \mathcal{H} \) into \( G \) such that

\[
\begin{align*}
(i) & \ |W \cap \bigcap_{j \in [m]} \phi(Y_j)| = |W|/|Y_1| \cdots |Y_m|/n^m \pm \alpha n \text{ for all } (W,Y_1,\ldots,Y_m) \in W_{\text{set}}; \\
(ii) & \ \omega(\phi^{-1}(\epsilon)) = (1 \pm \alpha)\omega(\bigcup_{H \in \mathcal{H}} V(H))/n^{|I|} \pm \alpha n \text{ for all } (\omega,\epsilon) \in W_{\text{ver}} \text{ with centres } \epsilon \text{ in } I. 
\end{align*}
\]

**2. Proof overview**

In the following we outline our highlevel approach for the proof of our main result in the multipartite setting, that is, assuming we aim to pack k-graphs \( H \in \mathcal{H} \) with vertex partition \((X_1^H,\ldots,X_r^H)\) into \( G \) with vertex partition \((V_1,\ldots,V_r)\). Our general approach is to consider each cluster \( V_i \) in turn and embed simultaneously almost all vertices of \( \bigcup_{H \in \mathcal{H}} X_i^H \) onto \( V_i \). Afterwards we complete the embedding with another procedure.

Let us turn to a more detailed description. At the very beginning, we will use a simple reduction to the case where each \( H \mid X_{i_1}^H,\ldots,X_{i_k}^H \) induces a matching for all \( \{i_1,\ldots,i_k\} \in E(G) \) and \( H \in \mathcal{H} \) by simply splitting the clusters into smaller clusters (see Lemma 3.9). This makes the analysis simpler and cleaner, and we assume this setting from now on.

The proof of the main result consists of two central parts. The first part is an iterative procedure that considers each cluster \( V_i \) of \( G \) in turn and provides a partial packing that embeds almost all vertices of the graphs in \( \mathcal{H} \) onto vertices in \( G \) (at the beginning we fix some suitable ordering as \( r \)
may be even larger than \( n \), in particular, \( r \) is not bounded by a function in terms of \( \varepsilon \). In the \( i \)-th step, we embed almost all vertices of \( \bigcup_{H \in \mathcal{H}} X^H_i \) onto \( V_i \) by finding simultaneously for each \( H \in \mathcal{H} \) an almost perfect matching within a ‘candidacy graph’ \( A^H_i \), which is an auxiliary bipartite graph between \( X^H_i \) and \( V_i \) such that \( xv \in E(A^H_i) \) only if \( v \) is still a suitable image for \( x \) with respect to the embedding obtained in previous steps. Of course, we have to guarantee that this indeed yields an edge-disjoint packing; that is, when we map \( x_1 \in X^H_i \) and \( x_2 \in X^H_i \) on the same vertex \( v \in V_i \), then we have to ensure that for all \( \varepsilon_j \in E(H_j) \) with \( x_j \in \varepsilon_j \), \( j \in [2] \), the \( (k-1) \)-sets \( \varepsilon_1 \setminus \{x_1\} \) and \( \varepsilon_2 \setminus \{x_2\} \) are not embedded onto the same \( (k-1) \)-set (provided they are already embedded).

We achieve this by defining an auxiliary hypergraph \( \mathcal{H}_{aux} \) with respect to the candidacy graphs to which we apply Theorem 3.8 — a result on pseudorandom hypergraph matchings due to Glock and the authors [11] which we state in Section 3.4. There will be a bijection between matchings in \( \mathcal{H}_{aux} \) and valid embeddings of \( \bigcup_{H \in \mathcal{H}} X^H_i \) into \( V_i \). This is one of the main ingredients in the first part of our proof.

Let us give more details for the construction of \( \mathcal{H}_{aux} \). Assume we are in the \( i \)-th step of the partial packing procedure and already found a partial packing \( \phi_0 \) of the initial \( i-1 \) clusters. We define a labelling \( \psi \) on the edges of the candidacy 2-graphs such that for every edge \( xv \in E(A^H_i) \) and \( H \in \mathcal{H} \), the labelling \( \psi(xv) \) contains the set of \( G \)-edges that are used when we extend \( \phi_0 \) by embedding \( x \) onto \( v \). Let us for simplicity assume that only one \( G \)-edge would be covered when we embed \( x \) onto \( v \), say, \( \psi(xv) = g_{xx} \in E(G) \). Since the packing will map multiple vertices of \( \bigcup_{H \in \mathcal{H}} X^H_i \) onto the same vertex \( v \) in \( V_i \), we consider disjoint copies \( (V^i_1 ... V^i_{k-1}) \) of \( V_i \) where the vertex \( v^H \) is the copy of \( v \). For every \( xv \in E(A^H_i) \) and \( H \in \mathcal{H} \), we define the 3-set \( \mathcal{H}_{aux} := \{x, v^H, \psi(xv)\} \) and let \( \mathcal{H}_{aux} \) be the 3-graph with vertex set \( \bigcup_{H \in \mathcal{H}} (X^H_i \cup V_i^H) \) and edge set \( \{\mathcal{H}_{aux} : xv \in E(A^H_i) \} \) for some \( H \in \mathcal{H} \). It is easy to see that there is a one-to-one correspondence between matchings in \( \mathcal{H}_{aux} \) and valid embeddings of \( \bigcup_{H \in \mathcal{H}} X^H_i \) into \( V_i \) such that no \( G \)-edge is used more than once. We used a similar hypergraph construction in our recent proof in [12] of the ‘blow-up lemma for approximate decompositions’.

Of course, whether we can iteratively apply this procedure depends on the choice of the partial packing in each step. Hence, with the aim of avoiding a future failure of the process, we have to maintain several pseudorandom properties throughout the entire process. For instance, we have to ensure that there are many candidates available in each step; in more detail, we guarantee that the updated candidacy graphs after each step remain super-regular even though they naturally become sparser after each embedding step.

Unfortunately, it is not enough to consider only candidacy graphs between pairs of clusters. For clusters indexed by elements in \( I \) where \( |I| \leq k-1 \), we consider candidacy graphs \( A^H_i \) on the clusters \( \bigcup_{I \in \mathcal{I}} (X^H_i \cup V_i) \) with edges of size \( 2|I| \). An edge \( e \in A^H_i \) will then indicate whether the entire set of \( |I| \) vertices in \( e \cap \bigcup_{I \in \mathcal{I}} X^H_i \) can (still) be mapped onto \( e \cap \bigcup_{I \in \mathcal{I}} V_i \). We further discuss the purpose of these candidacy graphs in Section 4.1, where we define them precisely.

The main source yielding a smooth trajectory of our partial packing procedure is the aforementioned Theorem 3.8 from [11], which provides a tool that gives rise to a pseudorandom matching in \( \mathcal{H}_{aux} \) with respect to tuple-weight functions. One difficulty of the first stage of our proof is the careful definition of these tuple-weight functions. For instance, we have to ensure that we can indeed iteratively apply Theorem 3.8. Moreover, we have to guarantee that we can turn the partial packing into a complete one in the second part of our proof. To that end, we define very flexible but complex weight functions on tuples of (hyper)edges of the candidacy graphs that we call edge testers. Dealing with hypergraphs, and especially with hypergraphs with vanishing density, makes it significantly more complex to control the weight of these edge testers during our partial packing procedure than for a similar approach for simple graphs.

One single embedding step is performed by our so-called ‘Approximate Packing Lemma’ (see Section 5). The process where we iteratively apply our Approximate Packing Lemma is described in Section 6 and will provide a partial packing that maps almost all vertices of the graphs in \( \mathcal{H} \) onto vertices in \( G \).

The second part of the proof deals with embedding the remaining vertices and turning the obtained partial packing into a complete one. Our general strategy is to apply a randomized procedure where we unembed several vertices that we already embedded in the first part and find
the desired packing by using a small edge-slice $G_B$ of $G$ put aside at the beginning (that is, we did not use the edges of $G_B$ for the partial packing in the first stage). Of course, we again have to track which vertices are still suitable images during the completion and respect the partial packing of the first part. To that end, for each $H \in \mathcal{H}$ and $i \in [r]$, we track a second type of candidacy graphs $B_i^H$ between $X_i^H$ and $V_i$ with respect to $G_{B_i}$, where $xv \in E(B_i^H)$ only if we could map $x$ onto $v$ during the completion. In fact, we track these candidacy graphs already during the partial packing procedure and carefully control several quantities using our edge testers. In the completion step, we can then apply a randomized matching procedure within the candidacy graphs $B_i^H$ to turn the partial packing into a complete one.

As in many other results that were originally proven for graphs and later lifted to $k$-graphs for $k \geq 3$, we have to overcome numerous difficulties that are specific to hypergraphs. In our case this includes for example the much more complex intersection structure among hyperedges, which in turn complicates the analysis of our partial packing procedure considerably. To this end, several novel ideas are needed.

Let us highlight one obstacle. Suppose we are in the $i$-th step of the iteration where we aim to embed essentially all vertices $\bigcup_{H \in \mathcal{H}} X_i^H$ onto $V_i$, then all $x \in \bigcup_{H \in \mathcal{H}} X_i^H$ have to be grouped according to the edge intersection pattern of all edges that contain $x$ and the edges that intersect these edges with respect to the clusters that have been considered earlier. In this context, we will define the patterns of edges in $H$ in Section 4.2.

As we alluded to earlier, a strong control over the actual packing is of importance for further applications when an entire decomposition is sought. In particular, it is often not enough to control how many vertices of a certain set are mapped to a particular vertex, but how many $(k-1)$-sets of $\mathcal{H}$ are embedded to a particular $(k-1)$-set in $G$. According to the edge intersection pattern of all edges that contain $x$ and the edges that intersect these edges with respect to the clusters that have been considered earlier. In this context, we will define the patterns of edges in $H$ in Section 4.2.

Unfortunately, considering sparse $k$-graphs adds another complexity level to the problem. To be more precise, $o(n)$ and $o(dn)$ no longer mean the same where $d \geq n^{-\varepsilon}$ refers to the density, and thus, terms of size $o(n)$ can no longer be ignored. Essentially at all stages of the proof a substantially more careful analysis is needed to make sure that several quantities are not only $o(n)$ but $o(d^\alpha n)$ because in many natural auxiliary (hyper)graphs considered in our proof vertices have typically $d^m n$ neighbours, where $m \in \mathbb{N}$ grows as we proceed in our procedure.

3. Preliminaries

In this section we clarify notation and collect some important tools that we will frequently use throughout the paper.

3.1. Notation. Let us introduce some general notation and (hyper)graph terminology. Let $\ell \in \mathbb{N}$. We write $[\ell]_0 := \{0\} \cup \{1, 2, \ldots, \ell\}$ and $-\ell := \{-\ell, \ldots, -1\}$, where $\{\}$ := $\emptyset$. We refer to a set of cardinality $\ell$ as an $\ell$-set. For finite and disjoint sets $A_1, \ldots, A_{\ell}$ and $I \subseteq [\ell]$, we write $A_{\ell,I} := \bigcup_{i \in I} A_i$. For a tuple $a = (a_1, \ldots, a_{\ell}) \in \mathbb{R}^\ell$ and $I \subseteq [\ell]$, we write $a_I := (a_i)_{i \in I}$ and $|a| := \sum_{i \in [\ell]} |a_i|$. For a finite set $A$, we write $2^A$ for the powerset of $A$ and $\binom{A}{\ell}$ for the set of all $\ell$-subsets of $A$. For a graph $G$, let $\binom{E(G)}{\ell} := \{e \in E(G) : |e| = \ell\}$ be the set of all matchings of size $\ell$ in $G$. For finite sets $A_1, \ldots, A_{\ell}$, we write $\bigcup_{i \in [\ell]} A_i := \{a_1, \ldots, a_{\ell} : a_i \in A_i \text{ for all } i \in [\ell]\}$, and conversely, whenever we write $\{a_1, \ldots, a_{\ell}\} \in \bigcup_{i \in [\ell]} A_i$, we tacitly assume that $a_i \in A_i$ for all $i \in [\ell]$. For $I \subseteq [\ell]$, we write $A_{\ell,I} := \bigcup_{i \in I} A_i$. Whenever we consider an index set $\{i_1, \ldots, i_{\ell}\} \subseteq \mathbb{N}$, we tacitly assume that $i_1 \leq i_2 \leq \ldots \leq i_{\ell}$. For a real-valued function $f : A \to \mathbb{R}_{\geq 0}$, let $\text{supp}(f) := \{a \in A : f(a) > 0\}$ be its support. For a function $g : A \to B$, let $g(A') := \bigcup_{a \in A'} g(a)$. We often write $a \ll b \ll c$ in our statements meaning that $a, b, c \in (0, 1]$ and there are increasing functions $f, g : (0, 1] \to (0, 1]$ such that whenever $a \leq f(b)$
and $b \leq g(c)$, then the subsequent result holds. We also write $c = a \pm b$ if $a - b \leq c \leq a + b$. For $a \in (0,1]$ and $b \in (0,1]^k$, we write $b \geq a$ whenever $b_i \geq a$ for all $b_i \in b$, $i \in [t]$. Whenever we consider a parameter with letter $d$, usually used for the density, we tacitly assume that $d \in (0,1]$. For the sake of a clearer presentation, we avoid roundings whenever it does not affect the argument.

For a $k$-graph $G$, let $V(G) \setminus E(G)$ denote the vertex set and edge set, respectively. For $X \subseteq V(G)$, let $G[X]$ be the $k$-graph induced on $X$, and for pairwise disjoint subsets $V_1, \ldots, V_k \subseteq V(G)$, let $G[V_1, \ldots, V_k]$ be the $k$-partite subgraph of $G$ induced between $V_1, \ldots, V_k$. Let $e(G)$ denote the number of edges in $G$ and let $e_G(V_1, \ldots, V_k) := e(G[V_1, \ldots, V_k])$. For set $S$ of at most $k-1$ vertices of $G$, we define the neighbourhood of $S$ in $G$ by $N_G(S) := \{g \setminus S : g \in E(G), S \subseteq g\}$ and let $\deg_G(S) := |N_G(S)|$. Note that $N_G(S)$ is a set of $(k-|S|)$-tuples. For $m \in [k-1]$, let $\Delta_m(G) := \max_{S \subseteq \binom{V(G)}{m}} \deg_G(S)$ denote the maximum $m$-degree of $G$. We usually write $\Delta(G)$ instead of $\Delta_1(G)$. Further, we say that $u$ is a $G$-vertex if $u \in V(G)$, and $u$ is a $G$-neighbour of $v \in V(G)$ if $u$ and $v$ are contained in an edge of $G$. If $G$ is a 2-graph, $u$ and $v$ are vertices of $G$, and $S \subseteq V(G)$, we let $N_G[v] := N_G(v) \cup \{v\}$, $N_G(u \setminus v) := N_G(u) \cap N_G(v)$, and $N_G(S) := \left(\bigcup_{v \in S} N_G(v)\right) \setminus S$. We frequently treat collections of (hyper)graphs as the (hyper)graph obtained by taking the disjoint union of all members.

We say a $k$-graph $G$ on $n$ vertices is $(\varepsilon, t, d)$-typical if for all non-empty sets $S \subseteq \binom{V(G)}{k-1}$ with $|S| \leq t$, we have $|\{S \subseteq S : N_G(S) \subseteq (1 \pm \varepsilon)d^{|S|}n\}|$. We often write $N_G(S) := \bigcap_{v \in S} N_G(S)$. Throughout the paper, we usually denote a $(k-1)$-set with the letter $S$, and a set of $(k-1)$-sets with the letter $S$.

For a $k$-graph $G$, we denote by $G_*$ the 2-graph with vertex set $V(G)$ and edge set $\bigcup_{g \in E(G)} \{g\}$. That is, $G_*$ arises from $G$ by replacing each hyperedge in $G$ with a clique of size $k$.

For (hyper)graphs $G$ and $H$, we write $G - H$ to denote the (hyper)graph with vertex set $V(G)$ and edge set $E(G) \setminus E(H)$.

For a graph $G$ and a finite set $\mathcal{E}$, we call $\psi : E(G) \to 2^\mathcal{E}$ an edge set labelling of $G$. A label $\alpha \in \mathcal{E}$ appears on an edge $e$ if $\alpha \in \psi(e)$. Let $||\psi||$ be the maximum number of labels that appear on any edge of $G$. We define the maximum degree $\Delta_\psi(G)$ of $\psi$ as the maximum number of edges of $G$ on which any fixed label appears, and the maximum codegree $\Delta^\circ_\psi(G)$ of $\psi$ as the maximum number of edges of $G$ on which any two fixed labels appear together.

For a graph $G$ and $m \in \mathbb{N}$, let $G^m$ denote the $m$-th power of $G$, that is, the 2-graph which is obtained from $G$ by adding all edges between vertices whose distance in $G$ is at most $m$.

### 3.2. Concentration inequalities

To verify the existence of subgraphs with certain properties we frequently consider random subgraphs and use McDiarmid’s inequality to verify that specific random variables are highly concentrated around their mean.

**Theorem 3.1** (McDiarmid’s inequality, see [34]). Suppose $X_1, \ldots, X_m$ are independent random variables and suppose $b_1, \ldots, b_m \in [0,B]$. Suppose $X$ is a real-valued random variable determined by $X_1, \ldots, X_m$ such that changing the outcome of $X_i$ changes $X$ by at most $b_i$ for all $i \in [m]$. Then, for all $t > 0$, we have

$$P[|X - E[X]| \geq t] \leq 2 \exp\left(-\frac{2t^2}{B\sum_{i=1}^m b_i}\right).$$

We also state the following convenient form of Freedman’s inequality [14].

**Lemma 3.2.** Suppose $X, X_1, \ldots, X_m$ are real-valued random variables with $X = \sum_{i \in [m]} X_i$ such that $|X_i| \leq B$ and $\sum_{i \in [m]} E_i[|X_i|] \leq \mu$, where $E_i[X_i]$ denotes the expectation conditional on any given values of $X_j$ for $j < i$. Then

$$P[|X| > 2\mu] \leq 2 \exp\left(-\frac{\mu}{4B}\right).$$

### 3.3. Graph regularity

In this section we introduce the quasirandom notion of (sparse) $\varepsilon$-regularity for 2-graphs and collect some important results. For a bipartite graph $G$ with vertex partition $(V_1, V_2)$, we define the density of the pair $W_1, W_2$ with $W_i \subseteq V_i$ by $d_G(W_1, W_2) := e_G(W_1, W_2)/|W_1||W_2|$. We say $G$ is $(\varepsilon, d)$-regular if $d_G(W_1, W_2) = (1 \pm \varepsilon)d$ for all $W_i \subseteq V_i$ with $|W_i| \geq \varepsilon|V_i|$, and $G$ is $(\varepsilon, d)$-super-regular if in addition $|N_G(v) \cap V_{3-i}| = (1 \pm \varepsilon)d|V_{3-i}|$ for each $i \in [2]$ and $v \in V_i$.

The following two standard results concern the robustness of $\varepsilon$-regular graphs.

**Fact 3.3.** Suppose $\varepsilon, d \in (0,1]$ and $G$ is an $(\varepsilon, d)$-regular bipartite graph with vertex partition $(A,B)$ and $Y \subseteq B$ with $|Y| \geq \varepsilon|B|$. Then all but at most $2\varepsilon|A| \cdot |Y|$ neighbours in $Y$. 

Theorem 3.5

Combining this with \( \epsilon \) Packing Lemma’ in Section C

4

it appropriate number of common neighbours. It can be proved along the same lines as in \[ 3.4. \]

Pseudorandom hypergraph matchings.

Suppose \( \epsilon \) and \( d \geq n^{-\epsilon} \). Suppose \( G \) is an \( (\epsilon, d) \)-super-regular bipartite graph with vertex partition \((A, B)\), where \( \epsilon^{1/6}n \leq |A|, |B| \leq n \). If \( H \) is a subgraph of \( G \) with \( \Delta(H) \leq \epsilon dn \) and \( X \subseteq A \cup B \) with \( |X| \leq \epsilon dn \), then \( G[A \setminus X, B \setminus X] - H \) is \((\epsilon^{1/3}, d)\)-super-regular.

The following result from [4] is useful to establish \( \epsilon \)-regularity for sparse graphs. For a graph \( G \), let \( C_d(G) \) be the number of 4-cycles in \( G \).

\[ \text{Theorem 3.8} \]

Suppose \( \epsilon \) and \( d \geq n^{-\epsilon} \). Suppose \( G \) is an \( (\epsilon, d) \)-super-regular graph with vertex partition \((A, B)\) and \( |A| = |B| = n \), and for all but at most \( n^{3/4} \) pairs \( \{a, a'\} \) in \( A \), we have \( |N_G(a \land a')| \leq (1 + \epsilon)d^2n \). If \( X \subseteq A \) and \( Y \subseteq B \) with \( n^{3/4 + 3\epsilon} \leq |X|, |Y| \leq \epsilon n \), then \( e_G(X, Y) \leq \epsilon^{1/3}dn \max\{|X|, |Y|\} \).

**Proof.** We have that

\[
\frac{1}{2} \sum_{b \in Y} |N_G(b) \cap X|^2 \leq \sum_{\{a, a'\} \in \binom{Y}{2}} |N_G(a \land a') \cap Y| + \epsilon n^2 \leq \frac{1}{2} (1 + 2\epsilon) |X|^2 d^2 n.
\]

Combining this with \( \frac{1}{2} \sum_{b \in Y} |N_G(b) \cap X|^2 \geq \frac{e_G(X, Y)^2}{2|Y|} \) yields that

\[
e_G(X, Y)^2 \leq (1 + 2\epsilon)d^2 |X|^2 |Y|^n.
\]

Suppose first that \( |Y| \leq |X| \), and suppose for a contradiction that \( e_G(X, Y) \geq \epsilon^{1/3}dn |X| \). By inserting this into (3.1) and solving for \( |Y| \), this implies that \( |Y| \geq \epsilon^{2/3}n/3 \), which is a contradiction to \( |Y| \leq |X| \leq \epsilon n \).

Next, suppose \( |X| \leq |Y| \), and suppose for a contradiction that \( e_G(X, Y) \geq \epsilon^{1/3}dn |Y| \). By inserting this into (3.1) and solving for \( |X| \), this implies that \( |X| \geq \epsilon^{1/3}(|Y|^n)^{1/2} / 2 \). Since \( |Y| \geq |X| \), this yields that \( |Y| \geq \epsilon^{2/3}n/4 \), which is a contradiction to \( |Y| \leq \epsilon n \).

We will also need the following result that is similar to [5, Lemma 2] and guarantees that a (sparse) \((\epsilon, d)\)-super-regular balanced bipartite graph of order \( 2n \) contains a spanning \( m \)-regular subgraph (an \( m \)-factor) for \( m = (1 - 2\epsilon^{1/3})dn \) provided that most pairs of vertices have the appropriate number of common neighbours. It can be proved along the same lines as in [5] by employing Lemma 3.6.

**Lemma 3.7.** Suppose \( 1/n \ll \epsilon \) and \( d \geq n^{-\epsilon} \). Suppose \( G \) is an \((\epsilon, d)\)-super-regular bipartite graph with vertex partition \((A, B)\) where \( |A| = |B| = n \), and suppose that all but at most \( n^{3/2} \) pairs \( \{a, a'\} \) in \( A \) satisfy that \( |N_G(a \land a')| \leq (1 + \epsilon)d^2n \). Then \( G \) contains an \( m \)-factor for \( m = (1 - 2\epsilon^{1/3})dn \).

3.4. Pseudorandom hypergraph matchings. A key ingredient in the proof of our ‘Approximate Packing Lemma’ in Section 5 is the main result from [11] on pseudorandom hypergraph matchings.

For this we need some more notation. Given a finite set \( X \) and an integer \( \ell \in \mathbb{N} \), an \( \ell \)-tuple weight function on \( X \) is a function \( \omega : \binom{X}{\ell} \rightarrow \mathbb{R}_{>0} \). For a subset \( X' \subseteq X \), we then define \( \omega(X') := \sum_{S \subseteq X'} \omega(S) \). For \( m \in [\ell] \), let \( ||\omega||_m := \max_{T \subseteq \binom{X}{\ell}} \sum_{S \subseteq \binom{T}{\ell}} \omega(S) \). Suppose \( H \) is a \( r \)-uniform hypergraph and \( \omega \) is an \( \ell \)-tuple weight function on \( E(H) \). Clearly, if \( M \) is a matching, then a tuple of edges which do not form a matching will never contribute to \( \omega(M) \). We thus say that \( \omega \) is clean if \( \text{supp}(\omega) \subseteq \binom{E(H)}{\ell} \), that is, \( \omega(\mathcal{E}) = 0 \) whenever \( \mathcal{E} \in \binom{E(H)}{\ell} \) is not a matching.

**Theorem 3.8** ([11]). Suppose \( 1/\Delta \ll \delta, 1/r, 1/L, r \geq 2 \), and let \( \epsilon := \delta/50L^2r^2 \). Let \( H \) be an \( r \)-uniform hypergraph with \( \Delta(H) \leq \Delta \) and \( \Delta_2(H) \leq \Delta^{1-\epsilon} \) as well as \( e(H) \leq \exp(\Delta^2) \). Suppose that for each \( \ell \in [L] \), we are given a set \( W_\ell \) of clean \( \ell \)-tuple weight functions on \( E(H) \) of size at most \( \exp(\Delta^2) \) such that \( \omega(E(H)) \geq ||\omega||_m \Delta^{-m \epsilon} \) for all \( \omega \in W_\ell \) and \( m \in [\ell] \). Then there exists a matching \( M \) in \( H \) such that \( \omega(M) = (1 \pm \Delta^{-\epsilon})\omega(E(H))/\Delta^\ell \) for all \( \ell \in [L] \) and \( \omega \in W_\ell \).
3.5. Refining partitions. In this section we provide a useful result to refine the vertex partition of a collection $\mathcal{H}$ of $k$-graphs of bounded degree such that every $H \in \mathcal{H}$ only induces a matching between any $k$-set of the refined partition. For $2$-graphs, a similar approach was already used in [37] by simply applying the classical Hajnal–Szemerédi Theorem. Our result is based on a random procedure which enables us to sufficiently control the weight distribution of weight functions with respect to the refined partition. We used such a procedure already in [12].

Lemma 3.9. Suppose $1/n \ll \varepsilon \ll \beta \ll \alpha, 1/k$ and $r \leq n^{\log n}$. Suppose $\mathcal{H}$ is a collection of at most $n^{2k}$ $k$-graphs. $(X^H_i)_{i \in [r]}$ is a vertex partition of each $H \in \mathcal{H}$, and $\Delta(H) \leq \alpha^{-1}$. Suppose $n/2 \leq |X^H_i| = |X^H_{i,j}| \leq 2n$ for all $H, H' \in \mathcal{H}$ and $i \in [r]$. Suppose $\mathcal{W}$ is a set of at most $n^{5 \log n}$ weight functions $\omega : \mathcal{X}_{i,j} \rightarrow [0, \alpha^{-1}]$ with $I \subseteq [r]$, $|I| \leq k$, and whenever $|I| \geq 2$, we have $\text{supp}(\omega) \subseteq \{x \in \mathcal{X}_{i,j} : x \subseteq e \text{ for some } e \in E(H)\}$. Then for all $H \in \mathcal{H}$ and $i \in [r]$, there exists a partition $(X^H_i)_{j \in [\beta^{-1}]}$ of $X^H_i$ such that

(i) $X^H_i$ is an independent set in $H$ for all $H \in \mathcal{H}$, $i \in [r]$, $j \in [\beta^{-1}]$;
(ii) $|X^H_{i,j}| \leq \ldots \leq |X^H_{i,\beta^{-1}}| \leq |X^H_i| + 1$ for all $H \in \mathcal{H}$, $i \in [r]$;
(iii) $\omega(X^H_i) = \beta \omega(X^H_i) + \beta^{3/2}n$ for all $H \in \mathcal{H}$, $i \in [r]$, $j \in [\beta^{-1}]$, and $\omega \in \mathcal{W}$ with $\omega : \mathcal{X}_i \rightarrow [0, \alpha^{-1}]$;
(iv) $\omega(\bigcup_{H \in \mathcal{H}} \bigcup_{i \in [r]} (X^H_{i,j})) = (1 \pm \varepsilon)\beta^I\omega(\mathcal{X}_{i,j})$ for all $\omega \in \mathcal{W}$ with $\omega : \mathcal{X}_{i,j} \rightarrow [0, \alpha^{-1}]$, $\omega(\mathcal{X}_{i,j}) \geq n^{1+\varepsilon}$, $I = [i_1, \ldots, i_{|J|}] \subseteq [r]$, $|J| \leq k$, and $j_1, \ldots, j_{|J|} \in [\beta^{-1}]$.

We give an analogous statement for the non-multipartite setting, that is, when $r = 1$, where the given weight functions assign weight on vertex tuples. For a finite set $A$ and $\ell \in \mathbb{N}$, let $(A^\ell)_{\prec}$ be the set of all $\ell$-tuples with non-repetitive entries of $A$.

Lemma 3.10. Suppose $1/n \ll \varepsilon \ll \beta \ll \alpha, 1/k$. Suppose $\mathcal{H}$ is a collection of at most $n^{2k}$ $k$-graphs on $n$ vertices with $\Delta(H) \leq \alpha^{-1}$. Suppose $\mathcal{W}$ is a set of at most $n^{5 \log n}$ weight functions $\omega : \bigcup_{H \in \mathcal{H}} (V(H))_{\prec} \rightarrow [0, \alpha^{-1}]$ with $m \in [k-1]$, and whenever $m \geq 2$, we have $\text{supp}(\omega) \subseteq \bigcup_{H \in \mathcal{H}} \{x \in (V(H))_{\prec} : x \subseteq e \text{ for some } e \in E(H)\}$. Then for all $H \in \mathcal{H}$, there exists a partition $(X^H_{i,j})_{j \in [\beta^{-1}]}$ of $V(H)$ such that

(i) $X^H_{i,j}$ is an independent set in $H$ for all $H \in \mathcal{H}$, $j \in [\beta^{-1}]$;
(ii) $|X^H_{i,j}| \leq \ldots \leq |X^H_{i,\beta^{-1}}| \leq |X^H_i| + 1$ for all $H \in \mathcal{H}$;
(iii) $\omega(X^H_{i,j}) = \beta \omega(V(H)) + \beta^{3/2}n$ for all $H \in \mathcal{H}$, $j \in [\beta^{-1}]$, and $\omega \in \mathcal{W}$ with $\omega : \bigcup_{H \in \mathcal{H}} V(H) \rightarrow [0, \alpha^{-1}]$;
(iv) $\omega(\bigcup_{H \in \mathcal{H}} (X^H_{i,j} \times \ldots \times X^H_{i,m})) = (1 \pm \varepsilon)\beta^m\omega(V(H))$ for all $\omega \in \mathcal{W}$ with $\omega : \bigcup_{H \in \mathcal{H}} (V(H))_{\prec} \rightarrow [0, \alpha^{-1}]$, $\omega(V(H)) \geq n^{1+\varepsilon}$, and $\{j_1, \ldots, j_m\} \in [\beta^{-1}]$.

Lemmas 3.9 and 3.10 are very similar to the analogous result for 2-graphs in [12, Lemma 3.7]. The main novelty in our setting is that we allow for weight functions on tuples of vertices. Nevertheless, the proofs follow exactly the same strategy as in the proof our result for 2-graphs. In the following, we therefore just provide a short proof sketch, say, for Lemma 3.9. (A detailed proof can be found in [10].)

We first consider every $H \in \mathcal{H}$ in turn and by randomly partitioning every $X^H_i$ into $\beta^{-1}$ sets, we obtain a partition $(Y^H_{i,j})_{j \in [\beta^{-1}]}$ that essentially satisfies (i)–(iii) with $Y^H_{i,j}$ playing the role of $X^H_{i,j}$. Then we perform a vertex swapping procedure to replace edges in $H_2$ between vertices in $(Y^H_{i,j})_{j \in [\beta^{-1}]}$ and obtain $(Z^H_{i,j})_{j \in [\beta^{-1}]}$ which satisfies (i)–(iii). In the end, for every $H \in \mathcal{H}$, $i \in [r]$, we randomly permute the ordering of $(Z^H_{i,j})_{j \in [\beta^{-1}]}$ to also ensure (iv).

4. Blow-up instances and candidacy graphs

In this section we introduce more notation concerning blow-up instances, which will be useful throughout our packing procedure (in Sections 5 and 6). Let $\mathcal{B} = (\mathcal{H}, G, R, \mathcal{X}, \mathcal{Y})$ be a blow-up instance of size $(n, k, r)$ that is fixed throughout Section 4. Note that the reduced graph $R$ with vertex set $[r]$ gives us a natural ordering of the clusters and we assume this ordering to be fixed (for a different ordering, just relabel the cluster indices). For simplicity, we often write $\mathcal{X}_i := \bigcup_{H \in \mathcal{H}} X^H_i$. 

for \( i \in [r] \) and \( \mathcal{X}_{\cup I} := \bigcup_{H \in \mathcal{H}} (\bigcup_{i \in I} X^H_i) \) for \( I \subseteq [r] \). Further, we call \( I \subseteq [r] \) an index set (of \( \mathcal{B} \)), if \( I \subseteq r \) for some \( r \in E(R) \).

We will introduce some important quantities that we control during our packing procedure. For instance, we track for each edge \( g \in E(G) \) (and for each subset of \( g \)), the set of \( \mathcal{H} \)-edges that still could be mapped onto \( g \) given a function \( \phi \) that already maps vertices of some clusters in \( \mathcal{H} \) onto vertices in \( G \) (see Definition 4.6 in Section 4.4). Similarly, we track for distinct edges \( g, \tilde{g} \in E(G) \), the set of tuples of \( \mathcal{H} \)-edges that still could be mapped together onto \( g \) and \( \tilde{g} \), respectively, with respect to \( \phi \) (see Definition 4.7). To track these quantities, we define edge testers (see Definitions 4.4–4.5) on candidacy graphs (see Definition 4.1) in Sections 4.3 and 4.1, respectively. The definition of these edge testers depends on how the edges in \( \mathcal{H} \) intersect, and to that end, we define patterns (see Definitions 4.2–4.3) in Section 4.2.

4.1. Candidacy graphs. For the purpose of tracking which sets of vertices in \( \mathcal{H} \) are still suitable images for sets of vertices in \( G \), we will consider auxiliary candidacy graphs. To that end, assume we are given \( r_0 \leq r \) and a mapping \( \phi: \bigcup_{H \in \mathcal{H}} \hat{X}^H \cup V_{\cup I} \to V_{[r_0]} \) with \( \hat{X}^H \subseteq X^H_i \) for \( q \in [r_0] \) and \( \phi|V(H) \) is injective for each \( H \in \mathcal{H} \); that is, \( \phi \) already embeds some \( \mathcal{H} \)-vertices onto \( G \)-vertices. We assume \( r_0 \) and \( \phi \) to be fixed throughout the entire Section 4. We define candidacy (hyper)graphs with respect to \( \phi \) and the blow-up instance \( \mathcal{B} = (\mathcal{H}, G, R, X, V) \) such that a (hyper)edge \( \alpha \) of the candidacy graph incorporates the property that the set of \( \mathcal{H} \)-vertices in \( \alpha \) can still be mapped onto the set of \( G \)-vertices in \( \alpha \) with respect to potential \( \mathcal{H} \)-vertices that are already mapped onto \( G \)-vertices by \( \phi \).

**Definition 4.1** (Candidacy graphs \( A^H_\iota(\phi) \)). For all \( H \in \mathcal{H} \) and every index set \( I \subseteq [r] \), let \( A^H_\iota(\phi) \) be the \( 2|I| \)-graph with vertex set \( X^H_{\cup I} \cup V_{\cup I} \) and \( \bigcup_{i \in I} \{x_i, v_i\} \in E(A^H_\iota(\phi)) \) for \( \{x_i, v_i\} \in X^H_i \cup V_i \) if all \( e = e_0 \cup e_m \in E(H[\hat{X}^H_{\cup I}, X^H_{\cup I}]) \) with \( m \in [|I|], \ I_m \in \binom{I}{m}, \ e_o \subseteq (\hat{X}^H_{k \cup I}, m) \), and

\[
(4.1) \quad \phi(e_o) \cup \{v_i\}_{i \in I_m} \in E(G[V_{[r_0 \setminus I]} \cup V_{|I|}]).
\]

We call \( A^H_\iota(\phi) \) the candidacy graph with respect to \( \phi \) and \( G \).

Let us describe Definition 4.1 in words. Suppose first that we are given an index set \( I \subseteq [r] \setminus [r_0] \). Then the set \( I \) contains the indices of clusters whose vertices are not yet embedded by \( \phi: \bigcup_{H \in \mathcal{H}} \hat{X}^H_{\cup I} \to V_{[r_0]} \). If the set of vertices \( \{x_i\}_{i \in I} \in X^H_{\cup I} \) can still be mapped onto \( \{v_i\}_{i \in I} \in V_{\cup I} \), then \( \{v_i\}_{i \in I} \) are still suitable candidates for \( \{x_i\}_{i \in I} \) and we store this information in the candidacy graph \( A^H_\iota(\phi) \) by adding the edge \( \bigcup_{i \in I} \{x_i, v_i\} \in E(A^H_\iota(\phi)) \). Let us spell out what it means that \( \{x_i\}_{i \in I} \) can still be mapped onto \( \{v_i\}_{i \in I} \). It means that all \( H \)-edges \( e \)

- that intersect \( \{x_i\}_{i \in I} \), say in a set of \( m \) vertices \( e_m \) which lie in the clusters with indices \( \{i_1, \ldots, i_m\} \),
- and whose other \( k - m \) vertices \( e_o = e \setminus e_m \) are embedded by \( \phi \),

satisfy (4.1), that is

- the embedding \( \phi_o(e_o) \) together with the vertices of \( \{v_i\}_{i \in I} \) in the clusters with \( \{i_1, \ldots, i_m\} \),
- that is \( \phi(e_o) \cup \{v_{i_1}, v_{i_2}, \ldots, v_{i_m}\} \), forms an edge in \( G \).

We note that we allow \( I \subseteq [r] \) in Definition 4.1 (and not only \( I \subseteq [r] \setminus [r_0] \)) because will use this for \( I \subseteq [r], |I| = 1 \), to track a second type of candidacy graphs between already embedded clusters. At the end of our procedure we will use this second type of candidacy graphs to turn an approximate packing into a complete one.

Let us continue with another comment. Clearly, for \( I = \{i\} \), the candidacy graph \( A^H_i(\phi) \) is a bipartite 2-graph. It is worth pointing out a crucial difference between \( \bigcup_{i \in I} A^H_i(\phi) \) and \( A^H_\iota(\phi) \): If \( \{x_i, v_i\}_{i \in I} \in \bigcup_{I \subseteq [r]} E(A^H_\iota(\phi)) \), then each vertex \( x_i \) on its own can still be mapped onto \( v_i \), whereas if \( \bigcup_{i \in I} \{x_i, v_i\} \in E(A^H_\iota(\phi)) \), then the entire set \( \{x_i\}_{i \in I} \) can still be mapped onto \( \{v_i\}_{i \in I} \).

Let \( A_\iota(\phi) := \bigcup_{H \in \mathcal{H}} A^H_\iota(\phi) \) and let \( A(\phi) \) be the collection of all \( A_\iota(\phi) \) for all index sets \( I \subseteq [r] \setminus [r_0] \). We also refer to subgraphs of \( A^H_\iota(\phi) \) as candidacy graphs.

\[2\]For the sake of readability, we write \( A^H_\iota \) instead of \( A^H_\iota(\phi) \).
To suitably control the candidacy graphs during our approximate packing procedure, it will be important that the neighbourhood $N_{A^H}(x)$ in the candidacy graph $A^H$ of an $H$-vertex $x$ is the intersection of neighbourhoods of $(k-1)$-sets in $G$. To that end, for $\varepsilon > 0$, $q \in \mathbb{N}$, $H \in \mathcal{H}$, $i \in [r]$ and a candidacy graph $A^H_i \subseteq A^H(\phi)$, we say

$$S_x \subseteq \binom{V(H)}{k-1}$$

with $|S_x| \leq q$ such that $N_{A^H_i}(x) = V_i \cap N_G(S_x)$, and every $x \in X^H_i$ is contained in at most $n^{1/4+\varepsilon}$ pairs $\{x, x'\} \in \binom{X^H_i}{2}$ such that $S_x \cap S_{x'} \neq \emptyset$.

We note that during our packing procedure the sets $S_x$ will be uniquely determined and thus, (4.2) is indeed well-defined.

4.2. Patterns. The behaviour of several parameters in our packing procedure depends on the intersection pattern of the edges in $\mathcal{H}$, that is, how edges in $\mathcal{H}$ intersect and overlap. To this end, we associate two vectors in $\mathbb{N}_0^r$ with certain sets of vertices in $H \in \mathcal{H}$ that we call 1-st-pattern and 2nd-pattern. Even though we need the precise definitions of these patterns at certain points throughout the paper, it mostly suffices to remember that every set of vertices and every edge in $\mathcal{H}$ has a unique 1-st- and 2nd-pattern. This allows us to track certain quantities with respect to their patterns. We proceed to the precise definitions of 1-st- and 2nd-patterns.

In order to conveniently define these vectors for a given set of vertices in $H \in \mathcal{H}$, we consider supergraphs of $H$ (namely, for $Z = B$ in Definition 4.2 below). We do this because we define candidacy graphs in Section 4.1, and we will in fact consider two collections $\mathcal{A}$ and $\mathcal{B}$ of candidacy graphs (as mentioned in the proof overview), and thus, we also have to distinguish two types of 1-st-patterns and 2nd-patterns for both $Z \in \{\mathcal{A}, \mathcal{B}\}$. To that end, it is more convenient to imagine that the clusters associated with the candidacy graphs in $\mathcal{B}$ are copies of the original cluster. For all $H \in \mathcal{H}$, $J \subseteq [r]$, and $j \in J$, let $X^H_j :=$ a disjoint copy of $X^H_j$, and let $\pi$ be the bijection that maps a vertex in $X^H_j$ to its copy in $X^H_j$. Let $H_j$ be the supergraph of $H$ with vertex set $V(H_j) := V(H) \cup X^H_{J,j}$ and edge set $E(H_j) := E(H) \cup \{(\pi(x) \cup (e \setminus \{x\}) : e \in E(H), x \in e \cap X^H_{J,j}\}$.

We now define 1-st-patterns and 2nd-patterns and give an illustration in Figure 1. Note that $H_0 - H$ is the empty graph.

**Definition 4.2 (Patterns).** For all $Z \in \{\mathcal{A}, \mathcal{B}\}$, index sets $I \subseteq [r]$, $J \subseteq I$, and all $x = \{x_i\}_{i \in I} \in X^H_I$ for some $H \in \mathcal{H}$, let $x' := \{x_i\}_{i \in I \setminus J} \cup \{\pi(x_i)\}_{i \in J}$, $H_A := H$ and $H_B := H_j - H$. We define the 1-st-pattern $p^1(x, J) \in \mathbb{N}_0^r$ and the 2nd-pattern $p^{2,2nd}(x, J) \in \mathbb{N}_0^r$ as r-tuples where their $\ell$-th entry $p^1(x, J)_{\ell}$ and $p^{2,2nd}(x, J)_{\ell}$ for $\ell \in [r]$ is given by

$$p^1(x, J)_{\ell} := |\{f \in E(H) : (f \cap X^H_j) \setminus \{x_i\}_{i \in I \setminus J} \neq \emptyset, f \setminus X^H_{I[\ell]} \subseteq x', |f \setminus X^H_{I[\ell]}| \geq 2\}|;$$

$$p^{2,2nd}(x, J)_{\ell} := |\{f \in E(H) : f \cap X^H_j \subseteq \{x_i\}_{i \in I \setminus J}, f \setminus X^H_{I[\ell]} \subseteq x', |f \setminus X^H_{I[\ell]}| = 1\}|.$$

Let us describe Definition 4.2 in words. Consider some fixed entry for $\ell \in [r]$. Depending on $Z \in \{\mathcal{A}, \mathcal{B}\}$, we either count edges in $H_A$ or in $H_B$; note that the edges in $H_B = H_j - H$ always contain exactly one copied vertex in $X^H_{J,j}$. Further, $x \in X^H_I$ determines the set $J$ and hence the definitions in (4.3) and (4.4) display no additional dependence on $I$.

We first describe the 1-st-pattern entry $p^1(x, J)_{\ell}$ as defined in (4.3). The condition $(f \cap X^H_j) \setminus \{x_i\}_{i \in I \setminus J} \neq \emptyset$ means that $f$ has a non-empty intersection with $X^H_j$ which does not lie in $\{x_i\}_{i \in I \setminus J}$; in particular, $x, x' \neq f$. The last two conditions in (4.3) mean that all vertices of $f \setminus X^H_{I[\ell]}$ lie in $x'$ and these are at least two vertices. In that sense, $f \cap X^H_j$ is the ‘last’ vertex of $f$ not contained in $x'$.

We now describe the 2nd-pattern entry $p^{2,2nd}(x, J)_{\ell}$ as defined in (4.4). The condition $(f \cap X^H_j) \setminus \{x_i\}_{i \in I \setminus J}$ means that $f$ has a non-empty intersection with $X^H_j$ that lies in $\{x_i\}_{i \in I \setminus J}$; note that we may also count edges $f \in E(H_Z)$ with $f = x'$ if $x' \in E(H_Z)$. The last two conditions in (4.4) mean that $k-1$ vertices of $f$ lie in $X^H_{I[\ell]}$ and the other vertex of $f$ lies in $x'$; note that the last two conditions in (4.4) imply that this must be the copied vertex $f \cap X^H_{J,j}$ if $Z = B$.

We make the following important observation concerning Definition 4.2. We claim that

$$\|p^1(x, J)\| = \|p^{2,2nd}(x, J)\| - 1\{x' \in E(H_Z)\}.$$
and in order to track important quantities during our packing procedure.

We will use weight functions on the edges of the candidacy graphs, which we also call edge testers, in Edge testers. A 

be a collection of candidacy graphs

Further, note that

More generally, we allow to specify whether some vertices of

by checking the conditions in

We start with the definition of

We will also consider a set of vertex tuples that lie in an edge in

To see this is true, let us first assume that

Note that every

that contributes to

has a ‘penultimate’ vertex that lies in

that the conditions in (4.4) are satisfied, and thus

Conversely, every

that contributes to

has a ‘last’ vertex not contained in

if

thus

also contributes to

Hence,

but not to

because of the condition

This implies (4.5). Further, note that

We will also consider a set of vertex tuples that lie in an edge in

Definition 4.3 (\(E_H(p, I, J)\)). For all index sets

, and

, let

More generally, we allow to specify whether some vertices of

lie in clusters with indices in

To this end, for all index sets

, and

, let

Evaluating the function on the edges of the candidacy graphs, we also call edge testers, in order to track important quantities during our packing procedure.

We start with the definition of simple edge testers (Definition 4.4). In Definition 4.5 we introduce more complex edge testers that include simple edge testers. However, because we will frequently use weight functions on the candidacy graphs in form of simple edge testers, we include both definitions for the readers’ convenience.

To that end, given an index set

, a 1st-pattern vector

, a 2nd-pattern vector

, vertices

that we call centres, and an (initial) weight function

, we define a simple edge tester

with respect to

and the candidacy graphs in

in the following Definition 4.4. Ultimately, our aim is to track the

-weight of tuples
in $\mathcal{X}_{\cup J}$ that are mapped onto the centres $c$. We can think of $\omega: E(A_{I\setminus[r]}) \to \mathbb{R}_{\geq 0}$ as an updated weight function that restricts the $\omega_i$-weight that still can be mapped onto the centres $c$ with respect to $\phi: \bigcup_{H \in \mathcal{H}} \widehat{X}^H_{[r]} \to V_{[r]}$ and the candidacy graphs in $\mathcal{A}$. Since supp($\omega_i$) $\subseteq E_H(p, p^{2nd}, I)$, we specify the 1st-pattern and 2nd-pattern of the tuples in supp($\omega_i$) and thus we know exactly how those tuples intersect with edges in $\mathcal{H}$. This will allow us to precisely control the weight of an edge tester during our packing procedure.

**Definition 4.4 (Simple edge tester $(\omega, \omega_i, c, p, p^{2nd})$).** For an index set $I \subseteq [r]$, $\omega: \mathcal{X}_{\cup J} \to [0, s]$ for some $s \in \mathbb{R}_{\geq 0}$ with supp($\omega_i$) $\subseteq E_H(p, p^{2nd}, I)$ for given $p, p^{2nd} \in \mathbb{N}^r_0$, and $c = \{c_i\}_{i \in I} \in \mathcal{V}_{\cup J}$, let $\omega: E(A_{I\setminus[r]}) \to [0, s]$ be defined by

\[
\omega(a) := 1\{c_i \cap V_{[r]} = \{c_i\}_{i \in I\setminus[r]} : \omega_i(x) \}
\]

for all $a \in E(A_{I\setminus[r]})$ and $H \in \mathcal{H}$ where $x = (\phi|_{V(H)})^{-1}(\{c_i\}_{i \in I\setminus[r]} \cup (a \cap X_{\cup J(\cup I\setminus[r])})) \in X^H_{\cup J}$. If no such $x$ exists, we set $\omega(a) := 0$. We say $(\omega, \omega_i, c, p, p^{2nd})$ is a simple $s$-edge tester with respect to $(\omega_i, c, p, p^{2nd}), \phi$ and $\mathcal{A}$.

For the readers’ convenience, let us discuss Definition 4.4 in detail. Suppose we are given an (initial) weight function $\omega_i: \mathcal{X}_{\cup J} \to [0, s]$ with centres $c = \{c_i\}_{i \in I} \in \mathcal{V}_{\cup J}$ for an index set $I \subseteq [r]$ and supp($\omega_i$) $\subseteq E_H(p, p^{2nd}, I)$ with 1st-pattern $p$ and 2nd-pattern $p^{2nd}$. Recall that our aim is to control the $\omega_i$-weight of tuples in $\mathcal{X}_{\cup J}$ that are mapped onto the centres $c$. Therefore, for an edge $a \in E(A_{I\setminus[r]})$, we put the weight $\omega_i(x)$ onto $a$ only if the following are satisfied:

- $a$ contains the centres $\{c_i\}_{i \in I\setminus[r]}$ of the not yet embedded clusters (which is incorporated by the indicator function in (4.6)), and
- $x$ is such that the vertices of $x$ in $X^H_{\cup J(\cup I\setminus[r])}$ are contained in $a$ and $\phi$ maps the vertices in $x \cap X^H_{\cup J(\cup I\setminus[r])}$ onto $\{c_i\}_{i \in I\setminus[r]}$.

Let us now comment on the purpose of more complex edge testers as defined in Definition 4.5. Our partial packing procedure will only provide a packing $\phi: \bigcup_{H \in \mathcal{H}} \widehat{X}^H_{[r]} \to V_{[r]}$ for $[r] \subseteq [r]$ that maps almost all vertices in $\bigcup_{H \in \mathcal{H}} X^H_{\cup I\setminus[r]}$ onto vertices in $V_{[r]}$ and leaves the vertices $\bigcup_{H \in \mathcal{H}} (X^H_{\cup I\setminus[r]} \setminus \widehat{X}^H_{\cup I\setminus[r]})$ unembedded. We will often call the vertices $\bigcup_{H \in \mathcal{H}} (X^H_{\cup I\setminus[r]} \setminus \widehat{X}^H_{\cup I\setminus[r]})$ unembedded by $\phi$ or simply the leftover of the partial packing $\phi$. In the end, we will have to turn such a partial packing into a complete one. Therefore, we will utilize a second collection of candidacy graphs $\mathcal{B}$ during the partial packing that tracks candidates that correspond to edges in $G$ that we reserved in the beginning for the completion step.\(^3\) In order for this to work, we have to take care that the leftover is well-behaved with respect to the candidacy graphs in $\mathcal{B}$. We achieve this by using weight functions on 2-tuples consisting of one hyperedge of an $\mathcal{A}$-candidacy graph and of a collection of edges within the $\mathcal{B}$-candidacy graphs. That is, assume we are initially given a weight function $\omega_i: \mathcal{X}_{\cup J} \to [0, s]$ (that we therefore often call initial weight function) with centres $c = \{c_i\}_{i \in I} \in \mathcal{V}_{\cup J}$ for an index set $I \subseteq [r]$ and recall that our overall aim is to track the $\omega_i$-weight of tuples in $\mathcal{X}_{\cup J}$ that can be mapped onto the centres in $c$. Now, in Definition 4.5 of our general edge testers we allow to specify a set $J \subseteq I$ of indices where we track the $\omega_i$-weight of tuples $x = \{x_i\} \in X^H_{\cup J}$ for all $H \in \mathcal{H}$ such that for each $j \in J$, the vertex $x_j$ can be mapped onto $c_j$ within the candidacy graph $B_j^H \in \mathcal{B}$. That is, if the vertices $\{x_j\}_{j \in J}$ are left unembedded, then they can potentially still be mapped onto $\{c_j\}_{j \in J}$ during the completion process using the candidacy graphs $\mathcal{B}$. Further, we even allow to specify disjoint subsets $J_X$ and $J_Y$ of $J$, where $J_X$ encodes that exactly the vertices $\{x_j\}_{j \in J_X}$ of $x$ are left unembedded, and $J_Y$ encodes whether the tuple $x \in X^H_{\cup J}$ lies in a graph $H$ such that $\phi|_{V(H)}$ leaves the centres $\{c_j\}_{j \in J_Y}$ uncovered.

Assume $\mathcal{B}$ is a fixed collection of candidacy graphs $B_j = \bigcup_{H \in \mathcal{H}} B_j^H \subseteq B_j(\phi)$ for all $j \in [r]$, defined as in Definition 4.1. To make our partial packing procedure more uniform, we will sometimes also treat vertices that are left unembedded by $\phi$ as embedded by some extension $\phi^+: \bigcup_{H \in \mathcal{H}} X^H_{\cup I\setminus[r]} \to V_{[r]}$ of $\phi$ (which only serves as a dummy extension and is not necessarily a packing).

\(^3\)In fact, we partition the edge set of the host graph $G$ into two $k$-graphs $G_A$ and $G_B$, and $\mathcal{A}$ will be a collection of candidacy graphs with respect to $\phi$ and $G_A$, and $\mathcal{B}$ will be a collection of candidacy graphs with respect to $\phi$ and $G_B$. 

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Definition 4.5 \((\text{General} \text{ edge tester } (\omega, \omega, J, J_X, J_V, c, p))\). For an index set \(I \subseteq [r] \), \(J \subseteq I\), disjoint sets \(J_X, J_V \subseteq J\), \(\omega : \mathcal{X}_{IJ} \to [0, s]\) for some \(s \in \mathbb{R}_{>0}\) with \(\text{supp}(\omega) \subseteq E_H(p, I, J)\) for given \(p \in (\mathbb{N}_0)^3\), and \(c = \{c_i\}_{i \in I} \in V_{IJ}, \) let \(\omega : \bigcup_{H \in \mathcal{H}} (E(A^H_{I \setminus [r]}) \cup (\bigcup_{J \in J} E(B^H_J))) \to [0, s] \) be defined by
\[
\omega(\{e, \delta\}) := \sum_{x \in \{e, \delta\} - \text{suitable}} \omega_i(x)
\]
for all \(x \in E(A^H_{I \setminus [r]})\), \(\delta = \{b_j\}_{j \in J} \in \bigcup_{J \in J} E(B^H_J)\) and \(H \in \mathcal{H}\), where we say that \(x \in X^H_{IJ}\) is \(\{e, \delta\}-\text{suitable}\) if
(i) \(\{e \cup b_{JJ}\} \cap X^H_{I \setminus [r]}) = x \cap X^H_{I \setminus [r]})\); 
(ii) \(\{c_i\}_{i \in I \setminus [r]} \subseteq \phi(x \cap \tilde{X}^H_I)\); 
(iii) \(\phi_j(\tilde{X}^H_I)\) for all \(j \in J_Y \cap [r]\); 
(iv) \(x \cap (X^H_{I \setminus [r]} \setminus \tilde{X}^H_I) = x \cap X^H_{I \setminus [r]}\); 
(v) \(\phi^+(x \cap X^H_{I \setminus [r]}) \cap c = \emptyset\).

Note that for each \(\{e, \delta\}\), there is at most one \(\{e, \delta\}-\text{suitable tuple } x\). If no \(\{e, \delta\}-\text{suitable tuple } x\) exists, we set \(\omega(\{e, \delta\}) := 0\). We say \((\omega, \omega, J, J_X, J_V, c, p)\) is an \(s\)-edge tester with respect to \((\omega, J, J_X, J_V, c, p), (\phi, \phi^+), A\) and \(B\).

We often write \(J_{XXV}\) for \(J_X \cup J_V\) if \(J_X\) and \(J_V\) are fixed. Let us comment on Definition 4.5. Suppose we are given an initial weight function \(\omega : \mathcal{X}_{IJ} \to [0, s]\) with centres \(c = \{c_i\}_{i \in I} \in V_{IJ}\) for an index set \(I \subseteq [r]\) and \(J \subseteq I\), and \(\text{supp}(\omega) \subseteq E_H(p, I, J)\). As in the case for simple edge testers, our aim is to control the \(\omega\)-weight of tuples \(x\) in \(\mathcal{X}_{IJ}\) that are mapped onto the centres \(c\). The set \(J \subseteq I\) allows us to specify some vertices \(x \cap X^H_{I \setminus [r]}\) of such tuples \(x\) that are not yet embedded by \(\phi^+\) onto their centres \(\{c_i\}_{i \in J}\) and which will potentially be embedded onto those during the completion process. For the completion, we will use the candidacy graphs \(B^H_J\) in \(B\) and therefore, \((x \cap X^H_I) \cup \{c_j\} = b_j \in E(B^H_J)\) for each \(J\). Furthermore, the sets \(J_X, J_V \subseteq J\) encode the situations that
- only the vertices \(x \cap X^H_{I \setminus [r]}\) of \(x\) are left unembedded by \(\phi\) (see (iv)\(D_{4.5}\)), or
- the centres \(\{c_j\}_{J \in J \cap [r]}\) are uncovered by \(\phi\) (see (iii)\(D_{4.5}\)).

Note that a (general) edge tester \((\omega, \omega, J = \emptyset, J_X = \emptyset, J_V = \emptyset, c, (p, p^{2nd}, 0, 0))\) is equivalent to a simple edge tester \((\omega, c, p, p^{2nd}, \phi)\) with respect to \((\omega, c, p, p^{2nd}, \phi)\) and \(A\).

4.4 Sets of suitable \(\mathcal{H}\)-edges. Next, we define (sub)sets of \(\mathcal{H}\)-edges that we track during our packing procedure. Recall that we consider some fixed \(r \leq r\) and \(\phi : \bigcup_{H \in \mathcal{H}} \tilde{X}^H_{I \setminus [r]} \to V_{I \setminus [r]}\). In Definition 4.6, we define a set \(X_{g, p, p^{2nd}, \phi}(A)\) for every edge \(g \in E(G)\) that contains the sets of vertices that are contained in an \(\mathcal{H}\)-edge with \(1-st\)-pattern \(p\) and \(2nd\)-pattern \(p^{2nd}\), and that still could be mapped together onto \(g\) with respect to \(\phi\) and the candidacy graphs in \(A\). We can track the size of this set \(X_{g, p, p^{2nd}, \phi}(A)\) by using simple edge testers.

Definition 4.6 \((X_{g, p, p^{2nd}, \phi}(A))\). For all \(g = g \cup g \in E(G[V_{I \setminus [r]}])\) for some \(r \in E(R)\) with \(g \in (I \setminus [r]), m \in [k], \), and with \(I := r \setminus [r], |I| = m, g \in V_{I \setminus [r]}\), and for all \(p, p^{2nd} \in \mathbb{N}_0\), let
\[
X_{g, p, p^{2nd}, \phi}(A) := \bigcup_{H \in \mathcal{H}} \left\{x_m \in X^H_{I \setminus [r]} : \begin{array}{l}
x_m \cup g \in E(A^H_I), \\
(\phi|_{V(H)})^{-1}(g) \cup x_m \in E_H(p, p^{2nd}, r) \end{array} \right\}.
\]

Further, for \(\omega : \mathcal{X}_{I \setminus [r]} \to \{0, 1\}\) with \(\omega(x) := 1\{x \in E_H(p, p^{2nd}, r)\}\), we call the simple 1-edge tester \((\omega, g, p, p^{2nd})\) with respect to \((\omega, g, p, p^{2nd}, \phi)\) and \(A\) (as defined in Definition 4.4), the edge tester for \(X_{g, p, p^{2nd}, \phi}(A)\).

Let us describe Definition 4.6 in words. Suppose we are given an edge \(g = g \cup g \in E(G[V_{I \setminus [r]}])\), where \(g \) contains the vertices of \(g\) that lie in clusters that are already embedded by \(\phi\) and \(g\) contains the remaining \(m\) vertices of \(g\) in the not yet embedded clusters with indices \(I = r \setminus [r]\). For each \(H \in \mathcal{H}\), we track the set of vertices \(x_m \in X^H_{I \setminus [r]}\) where
• $x_m$ still could be mapped onto $g_m$ (that is, $x_m \cup g_m \in E(A^H_I)$ in (4.8)), and
• $x_m$ lies in an $H$-edge $e$ with 1st-pattern $p$ and 2nd-pattern $p^{2nd}$ such that if we map $x_m$ onto $g_m$, then $e$ is mapped onto $g$ (that is, $(\phi|_{V(H)})^{-1}(g_x) \cup x_m \in E(H(p,p^{2nd},r))$ in (4.9)).

Further, note that for a simple edge tester $(\omega,\omega, g, p, p^{2nd})$ for $X_{g,p,H^\phi}(A)$ as in Definition 4.6, we have that $\omega(E(A_I)) = |X_{g,p,p^{2nd},\phi}(A)|$ for $I = r \setminus [r_0]$ because the indicator function in (4.6) corresponds to (4.8), and the choice of $x$ in (4.6) corresponds to (4.9) by the definition of $\omega$.

Next, we define in Definition 4.7 a set $E_{g,H,\phi}(A)$ for all distinct $G$-edges $g, h$ with identical $G$-vertex in the last cluster such that $E_{g,H,\phi}(A)$ contains the tuples of $H$-edges $(e, f)$ with identical $H$-vertex in the last cluster, and $e$ and $f$ can still be mapped onto $g$ and $h$ with respect to $\phi$ and the candidacy graphs in $A$. In this case, we ignore the patterns as we only aim for an upper bound on the number of these edges and have some room to spare.

**Definition 4.7 ($E_{g,H,\phi}(A)$).** For edges $g = \{v_{i1}, \ldots, v_{ik}\}$, $h = \{w_{j1}, \ldots, w_{jk}\} \in E(G)$ with $v_{ik} = w_{jk}$ and both $z \in \{g, h\}$, let

$$E_{x,\phi} := E \left( H \left[ \phi^{-1}(x \cap V_0) \cup \bigcup_{i \in [r]} N_A(x \cap V_i) \right] \right);$$

$$E_{g,h,\phi}(A) := \{(e, f) \in E_{g,\phi}(A) \times E_{h,\phi}(A) : e \cap f \cap \partial H_k \neq \emptyset\}.$$  

**5. APPROXIMATE PACKING LEMMA**

In this section we provide our ‘Approximate Packing Lemma’ (Lemma 5.1). Given a blow-up instance $(H, G, R, X, V)$, it allows us for one cluster to embed almost all vertices of $\bigcup_{H \in H} X_H$ into $V_I$, while maintaining crucial properties for future embedding rounds of other clusters. To describe this setup we define a packing instance and collect some more notation.

**5.1. Packing instances.** Our general understanding of a packing instance is as follows. Recall that we will consider the clusters of a blow-up instance one after another. A packing instance arises from a blow-up instance where we have already embedded vertices of some clusters (which is given by a function $\phi_0$) and focuses only on one particular cluster (denoted by $\bigcup_{H \in H} X_H$ and $V_0$) and all clusters that are close to the considered cluster (measured in the reduced graph $R$). We track candidacy graphs as defined in Definition 4.1 and consider a collection of candidacy graphs $A$ between the clusters in $H$ and $G$ that will be used for future embedding rounds. In order to be able to turn a partial packing into a complete one in the end, we do not only track the collection of candidacy graphs in $A$ but also a second collection of candidacy graphs $B$, where the candidacy graphs in $B$ will be used for the completion step. To that end, we also assume that the edges of $G$ are partitioned into two parts $G_A$ and $G_B$ such that the edges in $G_A$ are used for the approximate packing and the edges in $G_B$ are reserved for the completion step. That is, we will think of the graphs in $A$ as candidacy graphs with respect to $\phi_0$ and $G_A$, and of the graphs in $B$ as candidacy graphs with respect to $\phi_0$ and $G_B$.

We make this more precise. Let $n, k, r, r_0 \in N_0$. We say $\mathcal{P} = (H, G_A, G_B, R, A, B, \phi_0, \phi_0)$ is a packing instance of size $(n, k, r, r_0)$ if

• $H$ is a collection of $k$-graphs, $G_A$ and $G_B$ are edge-disjoint $k$-graphs on the same vertex set, and $R$ is a $k$-graph where $V(R) = [r_0] \cup [r_0]$;
• $\{X_H\}_{H \in H} \cup V(R)$ is a vertex partition of $H \in H$ such that $|e \cap X_H| \leq 1$ for all $e \in E(H), H \in \mathcal{H}$;
• $\{V_i\}_{i \in V(R)}$ is a vertex partition of $G_A$ as well as $G_B$;
• $|X_H| = |V_i| = (1 \pm 1/2)n$ for each $i \in V(R)$;
• for all $H \in H$, the hypergraph $H[X_H]$ is a matching if $r \in E(R)$ and empty if $r = (V(R) \setminus E(R))$;
• $A = \bigcup_{H \in H, I \subseteq [r_0]} A_H^I$ is a union of candidacy graphs with respect to $\phi_0$ and $G_A$; in particular, $A_H^I$ is $2|I|$-uniform, and $A_H^I$ is a balanced bipartite 2-graph with vertex partition $(X_H^I, V_i)$ for each $i \in [r_0]$;
• $B = \bigcup_{H \in H, j \in V(R)} B_{jH}$ is a union of candidacy graphs with respect to $\phi_0$ and $G_B$; in particular, $B_{jH}$ is a balanced bipartite 2-graph with vertex partition $(X_H^I, V_j)$ for each $j \in V(R)$;
\[ \phi_0 : \bigcup_{H \in \mathcal{H}} X^H_{-r_0} \rightarrow V_{-r_0} \] with \( X^H_{-r_0} \subseteq X^H_{-} \), \( \phi_0(X^H_{-r_0}) \subseteq V_i \) and \( \phi_0|_{X^H_{-r_0}} \) is injective for all \( H \in \mathcal{H}, i \in -[r_0] \), and \( \phi_0 : \bigcup_{H \in \mathcal{H}} X^H_{-r_0} \rightarrow V_{-r_0} \) is an extension of \( \phi_0 \) with \( \phi_0(X^H_{-}) = V_i \) and \( \phi_0|_{X^H} \) is bijective for all \( H \in \mathcal{H}, i \in -[r_0] \).

For simplicity, we often write \( G := G_A \cup G_B, \mathcal{X}_i := \bigcup_{H \in \mathcal{H}} X^H_{i}, \mathcal{X}_i^{-} := \bigcup_{H \in \mathcal{H}} X^H_{-}, A_I := \bigcup_{H \in \mathcal{H}} \mathcal{A}_H, B_i := \bigcup_{H \in \mathcal{H}} \mathcal{B}_H, \mathcal{X}_0 := \mathcal{X}^{-}_i, \mathcal{X}_0^{-} := \mathcal{X}^{-}_{-r_0}, \) and \( V_0 := V_{-r_0} \) for all \( i \in V(R) \) and index sets \( I \subseteq [r]_0 \). Note that the packing instance \( \mathcal{P} \) naturally corresponds to a blow-up instance \( \langle \mathcal{H}, G, R, \{X^H_i\}_{i \in V(R)}, \mathcal{H} \cup \{V_i\}_{i \in V(R)} \rangle \) of size \((n, k, r + r_0 + 1)\). In particular, we also note the notation of Section 4. For the sake of a better readability, we stick to some conventions:

We will often use the letters \((Z, Z) \in \{(A, A), (B, B)\}\) as many arguments for candidacy graphs \( A_I \) with respect to \( G_A \), and candidacy graphs \( B_i \) with respect to \( G_B \) are the same. Whenever we write \( x \in E(Z_i) \) for some \( i \in [r] \), we tacitly assume that \( x \in \mathcal{X}_i, v \in V_i \). We usually denote edges in \( Z_i \) by \((\text{non-calligraphic})\) letters \( e, f \), and hyperedges in \( A_I \) by \( \sigma \) and a collection of edges from \( \bigcup_{j \in J} E(B_j) \) by \( \delta = \{b_j\}_{j \in J} \), where we allow to slightly abuse the notation and often treat \( \delta \) as \( b_{ij} \). Whenever we write \((v_1, \ldots, v_k) \in E(G_Z)\), we tacitly assume that \( v_i \in V_i \) for all \( i \in [k] \); analogously for \((x_1, \ldots, x_k) \in E(H)\).

The aim of this section is to map almost all vertices of \( \mathcal{X}_0^{-} \) to \( V_0 \) by defining a function \( \sigma : \mathcal{X}_0^{-} \rightarrow V_0 \) in \( A_0 \) (that is, \( x_0 \sigma(x) \in E(A_0) \)) where \( \mathcal{X}_0^{-} \subseteq \mathcal{X}_0 \), while maintaining several properties for the other candidacy graphs. We identify such a function \( \sigma \) with its corresponding edge set \( M(\sigma) \) defined as

\[ M(\sigma) := \{ x_0 \sigma(x) : x \in \mathcal{X}_0^{-}, v \in V_0, \sigma(x) = v \} \]

To incorporate that \( \sigma \) has to be chosen such that each edge in \( G_A \) is used at most once, we define an edge set labelling \( \psi \) with respect to \( \mathcal{P} \) on \( A_0 \) as follows. For every edge \( x \in E(A_0) \), we set

\[ \psi(x) := \{ \phi_0(e \setminus \{x\}) \cup \{v\} : x \in E(H) \text{ with } e \setminus \{x\} \subseteq \mathcal{X}_0^{-} \} \]

We defined the candidacy graphs \( A_0 \) in Definition 4.1 such that \( x_0 \sigma(x) \in E(A_0) \) only if \( \phi_0(e \setminus \{x\}) \cup \{v\} \in E(G\{A\}) \) for each such edge \( e \) as in (5.2). That is, \( \psi(x_0) \) encodes the set of edges in \( G_A \) that are used for the packing when mapping \( x \) onto \( v \). We say

\[ \sigma : \mathcal{X}_0^{-} \rightarrow V_0 \text{ is a conflict-free packing if } \sigma|_{\mathcal{X}_0^{-} \setminus X^H_0} \text{ is injective for all } H \in \mathcal{H} \text{ and } \psi(e) \cap \psi(f) = \emptyset \text{ for all distinct } e, f \in M(\sigma) \text{.} \]

Crucially note that the property that \( \psi(e) \cap \psi(f) = \emptyset \) for all distinct \( e, f \in M(\sigma) \) will guarantee that every edge in \( G_A \) is used at most once. For an illustration, see Figure 2.

Given a conflict-free packing \( \sigma : \mathcal{X}_0^{-} \rightarrow V_0 \) in \( A_0 \), we update the remaining candidacy graphs with respect to \( \sigma \). To account for the vertices in \( \mathcal{X}_0^{-} \setminus \mathcal{X}_0^{-} \) that are left unembded by \( \sigma \), we will consider an extension \( \sigma^+ \) of \( \sigma \) such that \( \sigma^+ \) also maps every vertex \( x_0 \in \mathcal{X}_0^{-} \setminus \mathcal{X}_0^{-} \) to \( V_0 \) and \( \sigma^+|_{\mathcal{X}_0^{-}} \) is injective for all \( H \in \mathcal{H} \) (and hence bijective). We call such a \( \sigma^+ \) a cluster-injective extension of \( \sigma \). The purpose of \( \sigma^+ \) is that \( \sigma^+|_{\mathcal{X}_0^{-} \setminus \mathcal{X}_0^{-}} \) will serve as a ‘dummy’ extension resulting in an easier analysis of the packing process as \( \sigma^+ \) will impose further restriction that culminate in more consistent candidacy graphs. Using Definition 4.1, we will consider the (updated) candidacy graphs \( A^H_0(\phi_0 \cup \sigma^+) \) with respect to \( \phi_0 \cup \sigma^+ \) and \( G_A \) for index sets \( I \subseteq [r] \), as well as the (updated) candidacy graphs \( B^H_i(\phi_0 \cup \sigma^+) \) with respect to \( \phi_0 \cup \sigma^+ \) and \( G_B \) for \( j \in V(R) \).

To track our packing process, we carefully maintain quasirandom properties of the candidacy graphs throughout the entire procedure. Our Approximate Packing Lemma will guarantee that we can find a conflict-free packing that behaves like an idealized random packing with respect to given sets of edge testers (as defined in Definition 4.5), and with respect to weight functions \( \omega : (E(A_0)) \rightarrow [0, s] \) for \( \ell \leq s \) that we will call local testers.

To that end, assume we are given a packing instance \( \mathcal{P} := (\mathcal{H}, G_A, G_B, R, A, B, \phi_0, \phi_0) \) of size \((n, k, r, r_0)\), and \( d = (d_A, d_B, (d_A)_{i \in [r_0]}, (d_B)_{i \in V(R)}) \), and \( q, t \in N \), as well as a set \( W_\text{edge} \) of edge testers. We say \( \mathcal{P} \) is an \((\varepsilon, q, t, d)-\)packing instance with suitable edge testers \( W_\text{edge} \) if \( |X^H_i| = |V_i| = (1 \pm \varepsilon)n \) for all \( i \in V(R) \) and the following properties are satisfied (recall (4.2) and Definitions 4.5–4.7 for (P2)–(P5), respectively, and that we write \( J_X \) for \( J_X \cup J_Y \)).
for all \( i \in V(R) \) and all pairs of disjoint sets \( S_A, S_B \subseteq \bigcup_{r \in E(R)} \{ V \cap r \} \) with \( |S_A \cup S_B| \leq t \), we have \(|V_i \cap N_{G_A}(S_A) \cap N_{G_B}(S_B)| = (1 \pm \varepsilon)d_A|S_A|d_B|S_B|n_i\).

(P2) for all \( H \in \mathcal{H}, i \in [r_0], j \in V(R) \), we have that \( A_i^H \) is \((\varepsilon, d_iA)\)-super-regular and \((\varepsilon, q)\)-well-intersecting with respect to \( G_A \), and \( B_j^H \) is \((\varepsilon, d_jB)\)-super-regular and \((\varepsilon, q)\)-well-intersecting with respect to \( G_B \).

(P3) for every edge tester \((\omega, \varepsilon, j, J_X, J_Y, \varphi, \rho)\), \( A \) and \( B \), with centres \( \varphi \in V_{J_X} \) for \( I \subseteq V(R) \), \( I_{r_0} := (I \cap [r_0]) \setminus J \), and patterns \( p = (\mathbf{p}^A, \mathbf{p}^{A,2nd}, \mathbf{p}^B, \mathbf{p}^{B,2nd}) \in (\mathbb{N}_0^{r_0+1})^4 \), we have that
\[
\omega \left( E(A_{r_0}) \cup \bigsqcup_{j \in J} E(B_j) \right) = \left(1 \{J_XV \cap [-r_0] = \emptyset\} \pm \varepsilon \right) \prod_{Z \in \{A,B\}} \sum_{i \in I_{r_0}} d_{i,A} \prod_{j \in J} d_{j,B} \frac{\omega_{\mathcal{I}}(Z_{[r_0]})}{n} \pm n^\varepsilon;
\]

(P4) for all \( g \in E(G_A[V_{J_Y}]) \) for some \( r \in E(R) \) with \( r \cap [r_0] \neq \emptyset \), and all \( \mathbf{p}^{2nd} \in \mathbb{N}_0^{r_0+r+1} \), the set \( \mathcal{W}_{edge} \) contains the edge tester for \( X_{g,p^{2nd},\varphi} (A) \);

(P5) for all \( g = \{v_1, \ldots, v_k\}, \mathbf{b} = \{w_1, \ldots, w_k\} \in E(G_A) \) with \( v_i = w_i \) and \( I := \{i_1, \ldots, i_k\} \neq \{j_1, \ldots, j_k\} =: J \), we have that
\[
\left| E_{g, A, \varphi} (A) \right| \leq \max \left\{ n^{k-(I \cup J) \cap [-r_0] + \varepsilon}, n^\varepsilon \right\}.
\]

Note that (P1) also implies that \( G_Z \) is \((3\varepsilon, t, d_z)\)-typical with respect to \( R \) for each \( Z \in \{A,B\} \).

Furthermore, we call a function \( \omega = \left( E(A_{r_0}) \right) \rightarrow [0, s] \) for \( \ell \leq s \) a local s-tester (for the \((\varepsilon, q, t, d)\)-packing instance \( \mathcal{P} \)) if \( ||\omega||_{\ell^t} \leq n^{-\ell t + \varepsilon} \) for every \( \ell' \in [\ell] \). We introduce some more notation:

Let \( E_i^{R} := \{ r \in E(R) : \{0,i\} = r \cap [r_0] \} \) and \( b_i := |E_i^R| = \deg_{R}[\{0,i\} \cup \{0,i\}] \) for each \( i \in V(R) \setminus \emptyset \). For \( I \subseteq V(R) \), let \( b_I := \sum_{i \in I \setminus \emptyset} b_i \). For \( i \in [r], j \in V(R) \setminus \{0\} \), let \( d_i^{new} := d_i^{A,B} \) and \( d_j^{new} := d_j^{B,B} \).

Note that for \( i \in V(R) \setminus N_{R_i}[0] \), we have \( b_i = 0 \) and thus \( d_i^{new} = d_{i,Z} \) for \( Z \in \{A,B\} \).

### 5.2. Approximate Packing Lemma

We now state the Approximate Packing Lemma, which is the key tool for the proof of our main result.

**Lemma 5.1** (Approximate Packing Lemma). Let \( 1/n \leq \varepsilon \leq \varepsilon' \leq 1/t \leq 1/k, 1/q, 1/r, 1/(r_0 + 1), 1/s \). Suppose \((\mathcal{H}, G_A, G_B, R, A, B, \varphi, \rho)\) is an \((\varepsilon, q, t, d)\)-packing instance of size \((n, k, r, r_0)\)
with \( \mathbf{d} \geq n^{-\varepsilon} \) and suitable s-edge-testers \( \mathcal{W}_{\text{edge}} \). Suppose further that \( \mathcal{W}_{\text{local}} \) is a set of local s-testers, \( \mathcal{W}_0 \) is a set of tuples \( (\omega, c) \) with \( \omega : \mathcal{X}_0 \to [0, s], c \in \mathcal{V}_0 \), and \( |\mathcal{W}_{\text{edge}}|, |\mathcal{W}_{\text{local}}|, |\mathcal{W}_0| \leq n^{4\log n} \), \( |\mathcal{H}| \leq n^{2k} \), as well as \( e_{\mathcal{H}}(\mathcal{X}_{\mathcal{L}p}) \leq dA_nk \) for all \( r \in E(R) \).

Then there is a conflict-free packing \( \sigma : \mathcal{V}_0^r \to \mathcal{A}_0 \) and a cluster-extensive injection \( \sigma^+ \) of \( \sigma \) such that for all \( H \in \mathcal{H} \), we have \( |\mathcal{X}_0^r \cap X_0^H| \geq (1 - \varepsilon)n \), and for all index sets \( I_A \subseteq [r] \), \( I_B \subset V(R) \) and \( (Z, Z) \in \{ (A, A), (B, B) \} \), there exist spanning subgraphs \( Z_{I_z}^{H,\text{new}} \) of the candidacy graphs \( Z_{I_z}^H(\phi_0 \cup \sigma^+) \) with respect to \( \phi_0 \cup \sigma^+ \) and \( G_Z \) (where \( Z_{I_z}^{\text{new}} := \bigcup_{H \in \mathcal{H}} Z_{I_z}^{H,\text{new}} \) and \( Z_{\text{new}} \) is the collection of all \( Z_{I_z}^{H,\text{new}} \) such that

\[
\begin{aligned}
& \text{(I),L.5.1} \quad Z_{I_z}^{H,\text{new}} \text{ is } (\varepsilon', d_{I_z}^{\text{new}})-\text{super-regular and } (\varepsilon', q + \Delta(R))-\text{well-intersecting with respect to } G_Z \text{ for all } H \in \mathcal{H}, \text{ and all } i \in [r] \text{ if } Z = A, \text{ and all } i \in V(R) \setminus \{0\} \text{ if } Z = B; \\
& \text{(II),L.5.1} \quad \text{for every } \text{(general) } s\text{-edge tester } (\omega, \omega_i, J, J_X, J_V, c, p) \in \mathcal{W}_{\text{edge}} \text{ with respect to } (\omega_r, J, J_X, J_V, c, p), \text{ A and B, with centres } c \in V_{I_0} \text{ for } I \subseteq V(R), \text{ } I_r := (I \cap [r]) \cup J, \text{ } I_r \cup J \neq \emptyset, \text{ and patterns } p = (p^A, p^{A,2nd}, p^B, p^{B,2nd}) \in (\{0, 1\})^4, \text{ the s-edge tester } (\omega_{\text{new}}, \omega_i, J, J_X, J_V, c, p) \text{ defined as in Definition 4.5 with respect to } (\omega, J, J_X, J_V, c, p), \text{ } (\phi_0 \cup \sigma, \phi_0 \cup \sigma^+), \text{ } A_{\text{new}} \text{ and } B_{\text{new}} \text{ satisfies}
\end{aligned}
\]

\[
\omega_{\text{new}}(E(A_{I_0}^r) \cup \bigcup_{j \in J} E(B_{J_z}^{new})) = (1 \{J_X \cap \{0\} = \emptyset\} \pm \varepsilon^2) \sum_{Z \in \{A, B\}} d_Z^{\|p_Z \|_0 - \|p_Z^{\text{2nd}}\|} \prod_{i \in I_r} d_{i_0}^{\text{new}} \prod_{j \in J} d_{j_0}^{\text{new}} \frac{\omega_i(\mathcal{X}^{\text{I}I})}{n^{\|\{0\} \cup \{\emptyset\}\}} \pm n^\varepsilon;
\]

\[
\begin{aligned}
& \text{(III),L.5.1} \quad \omega(M(\sigma)) = (1 + \varepsilon^2)\omega(E(A_0))/(d_{0,A}^n)^{\varepsilon} \pm n^\varepsilon \text{ for every local } s\text{-tester } \omega \in \mathcal{W}_{\text{local}} \text{ with } \omega : (E(A_0))_i \to [0, s]; \\
& \text{(IV),L.5.1} \quad \omega(\{x \in \mathcal{X}_0 \setminus \mathcal{X}_0^{|\sigma} : \sigma(x) = c\}) \leq \omega(\mathcal{X}_0)/n^{1-\varepsilon} + n^\varepsilon \text{ for every } (\omega, c) \in \mathcal{W}_0.
\end{aligned}
\]

Properties (I),L.5.1 and (II),L.5.1 ensure that (P2) and (P3) are also satisfied for the updated candidacy graphs \( A_{\text{new}} \) and \( B_{\text{new}} \), respectively. Property (III),L.5.1 states that \( \sigma \) behaves like a random packing with respect to the local testers, which for instance can be used to establish (P5) for future packing rounds. Property (IV),L.5.1 allows to control the weight on vertices that are not embedded by \( \sigma \) but are nevertheless mapped onto a specific vertex \( c \) by the extension \( \sigma^+ \).

**Proof.** We split the proof into three parts. In Part A we construct an auxiliary supergraph \( H_+ \) of \( H \) by adding some hyperedges to \( H \) for every \( r \in E(R) \) in order to make the packing procedure more uniform. In Part B we construct an auxiliary hypergraph \( H_{\text{aux}} \) for \( A_0 \) such that we can use Theorem 3.8 to find a conflict-free packing in \( A_0 \). In order to be able to apply Theorem 3.8, we exploit (P5) as well as (P2) together with (P3) to control \( \Delta_2(\mathcal{H}_{\text{aux}}) \) and \( \Delta(\mathcal{H}_{\text{aux}}) \), respectively. In Part C we define weight functions and employ the conclusion of Theorem 3.8 to establish (I),L.5.1–(III),L.5.1.

Let \( \Delta := 2(1 + 1/t) \). Note that \( \Delta(R), \Delta(\mathcal{H}), b_i \leq \Delta_R \) for all \( i \in [r] \). For simplicity we write \( d_0 := d_{0,A} \) and \( \varepsilon := \varepsilon^{1/2} \). Further, we choose a new constant \( \Delta \) such that \( \varepsilon' \leq 1/\Delta \leq 1/t \).

**Part A. Construction of \( \mathcal{H}_+ \)**

We construct an auxiliary supergraph \( \mathcal{H}_+ \) of \( H \) by artificially adding some edges to \( H \) for every \( r \in E(R) \) and \( i \in V(R) \setminus \{0\} \). (Recall (5.4) for the definition of \( E_r^R \) and note that it can be that \( r \in E_r^R \cap E_j^R \) for \( i, j \in \{0\} \).) For every \( H \in \mathcal{H} \), we proceed as follows. We obtain \( H_+ \) from \( H \) by adding a minimal number of hyperedges of size \( k \) subject to the conditions that for all \( i \in V(R) \setminus \{0\} \) and \( r \in E(R), \) an \( H_+ \)-edge meets every cluster in \( H \) exactly once, and

\[
\begin{aligned}
& \text{(a)} \quad \deg_{H_+[X^H_{I_0}]}(x_1) \leq 2 \text{ for all } x_1 \in X_1^H, \text{ and } \deg_{H_+[X_{I_0}^F]}(x_0) \leq 2 \text{ for all } x_0 \in X_0^H; \\
& \text{(b)} \quad \text{for all } e \in E(H_+[X^H_{I_0}]), \text{ we have } |\{x \in e : \deg_{H_+[X_{I_0}^F]}(x) = 2\}| \leq 1; \\
& \text{(c)} \quad \text{for all } \{x_0, x_1\} \subseteq X_0^H \cup X_0^H, \text{ if } \{x_0, x_1\} \leq e \text{ for some } e \in E(H_+) \setminus E(H), \text{ then } \{x_0, x_1\} \notin f \text{ for all } f \in E(H).
\end{aligned}
\]

Note that (a)–(c) can be met because \( |X^H| = (1 + \varepsilon)n \) for all \( i \in V(R) \) and \( \Delta(R) \leq \Delta_R \). For all \( i \in V(R) \setminus \{0\} \), let \( H_+^i \) be an arbitrary but fixed k-graph \( H_+ \subseteq H_+^i \subseteq H_+ \) such that for all \( r \in E(R), \)
we have $\deg_{H_i}[X^H_i](x_i) = 1$. Observe that by the construction of $H_+$, we have $\deg_{H_+}[X^H_+(x_i)] = 1$
for all $x_i \in X^H_i$, $i \in [r]$, $r \in R$, and thus $H^+_i = H_+$ for all $i \in [r]$. We make some observations.
(5.5) For all $x \in X^H_i$, $i \in V(R) \setminus \{0\}$, we have $\sum_{r \in R} \deg_{H^+_i}[X^H_+(x)] = b_i$.

By (a), for every $x \in X^H_i$, $i \in V(R) \setminus \{0\}$, there are at most $\Delta_R$ vertices $x' \in X^H_i \setminus \{x\}$
such that $x$ and $x'$ have a common neighbour in $X^H_0$ in $H_+[x_0, X^H_0, X^H_i]$, that is, $x_e \cap e_x' \neq \emptyset$ with $e_y \in E(H_+[x_0, X^H_0, X^H_i])$ and $y \neq e_y$ for both $y \in \{x, x'\}$.

(5.7) If $\{x_0, x_i\}$ lies in an edge of $H_+ - H$, then $\{x_0, x_i\}$ does not lie in an edge of $H$.

(5.8) If $|e \cap X^H_{\{p\}}| \geq 2$, then $e \in E(H)$ for all $e \in E(H_+)$.

We introduce some simpler notation to denote edges in $H_+$ (respectively $H^+_i$) that contain a
vertex $x \in X^H_0$. Let $H_+ := \bigcup_{H \in H} H_+$, and for $i \in V(R) \setminus \{0\}$, let $H^+_i := \bigcup_{H \in H} H^+_i$. For all $x \in \mathcal{X}_0$ and $y \in \mathcal{X}_I$ for some $I \subseteq V(R)$, let

$$E_{x,y} := \{e \in E(H_+|X_0, \mathcal{X}_0, \mathcal{X}_I)| x \in e, e \cap \mathcal{X} \subseteq \{x\} \cup y \cup X_0, e \cap (y \cap \mathcal{X} \neq \emptyset) \neq \emptyset,$$

$$\text{if } e \cap (y \cap \mathcal{X} \not\in \mathcal{I}_i \text{ for some } i \in V(R) \setminus \{0\}, \text{then } e \in E(H^+_i)\}.\]

That is, $E_{x,y}$ contains essentially all $H_+$-edges that contain $x \in \mathcal{X}_0$ and a non-empty subset of $y$
and whose remaining vertices are already embedded and lie in $X_0$. In particular, if $y = y$ is a single
vertex $y \in \mathcal{X}_I$, then $E_{x,y}$ contains all $H^+_I$-edges that contain $x$ and $y$ and whose remaining vertices
lie in $X_0$. Hence, note that by definition of $H_+$ and as observed in (5.5), we have that

$$\left|\bigcup_{x \in \mathcal{X}_0} E_{x,y}\right| = b_i$$

for all $x \in \mathcal{X}_i$, $i \in V(R) \setminus \{0\}$.

Part B. Applying Theorem 3.8

Our strategy is to utilize Theorem 3.8 to find the required conflict-free packing $\sigma$ in $A_0$. To
that end, we will define an auxiliary hypergraph $H_{aux}$ for $A_0$. Let $\psi: E(A_0) \rightarrow 2^E$ be the edge set
labelling with respect to the packing instance as defined in (5.2). For all $H \in \mathcal{H}$, the hypergraph
$H[X^H]$ is a matching if $r \in E(R)$ and empty otherwise, and thus we have that $||\psi(e)|| \leq (\Delta+1) \leq \Delta_R$.

In the following, we may assume that $||\psi(e)|| = \Delta_R$ for all $e \in E(A_0)$ as we may simply add distinct
artificial dummy labels that we ignore afterwards again.

Further, let $(V^H_0)_{H \in \mathcal{H}}$ be disjoint copies of $V_0$, and for all $H \in \mathcal{H}$ and $e = x_0v_0 \in E(A_0^H)$, let
$e^H := x_0v_0^H$ where $v_0^H$ is the copy of $v_0$ in $V_0^H$. Let $H^e := e^H \cup \psi(e)$ for each $e \in E(A_0^H)$, $H \in \mathcal{H}$ and let $H_{aux}$ be the $(\Delta_R+2)$-graph with vertex set $\bigcup_{H \in H}(X_0^H \cup V_0^H) \cup \mathcal{E}$ and edge set $\{H^e : e \in E(A_0)\}$. A key property of the construction of $H_{aux}$ is a bijection between conflict-free pairings $\sigma$ in $A_0$
and matchings $\mathcal{M}$ in $H_{aux}$ by assigning $\sigma$ to $\mathcal{M} = \{\mathcal{H}_e : e \in \mathcal{M}(\sigma)\}$. (Recall that $\mathcal{M}(\sigma)$ is the edge set corresponding to $\sigma$ as defined in (5.1).)

**Step 1. Estimating $\Delta(H_{aux})$ and $\Delta_2(H_{aux})$**

In order to apply Theorem 3.8 to $H_{aux}$, we estimate $\Delta(H_{aux})$ and $\Delta_2(H_{aux})$. We first claim that

$$\Delta(H_{aux}) \leq (1 + \varepsilon^{2/3})d_{H} := \Delta_{aux}.$$ 

Since $A_0^{H}$ is $(\varepsilon, d_0)$-super-regular and $|X_0^{H}| = |V_0| = (1 + \varepsilon)n$ for each $H \in \mathcal{H}$, we have that an appropriate upper bound on $\Delta(H_{aux})$ immediately establishes (5.11). In the following we derive such an upper bound on $\Delta_{aux}(A_0)$ by employing property (P3). For all $r \in E(R)$ with $0 \in r$, $|r \cap \{r_0\}| = k - 1$, and all $g \in E(G_A[V_{1,r}])$ with $g \cap V_0 = \{v_0\}$, note that $\bigcup_{p,p^{2nd}} \mathcal{X}_{g,p} \subseteq A^{p^{2nd}}(A)$ contains by

$$\text{Definition 4.6 all vertices } x_0 \in N_{A_0}(v_0) \text{ (compare with (4.8)) that are contained in an } H \text{-edge that could be mapped onto } g \text{ with respect to } \phi_0 \text{ and } A \text{ (compare with (4.9))}.\]$$

Hence, by the definition of the edge set labelling $\psi$ in (5.2), $g$ appears as a label of $\psi$ on at most $\sum_{p,p^{2nd}} |\mathcal{X}_{g,p} \subseteq A^{p^{2nd}}(A)|$ edges of $A_0$. Note that by Definition 4.3 of $E_{H}(p,p^{2nd},r)$, we obtain

$$\sum_{p,p^{2nd}} |E_{H}(p,p^{2nd},r)| = e_{H}(\mathcal{X}_{p^{2nd}}) \leq d_A n^k,$$

where the last inequality holds by assumption of Lemma 5.1. For all $x_0 \in N_{A_0}(v_0)$ for some $H \in \mathcal{H}$, let $g_{x_0}^{-1} := (\phi_0^{-1}(V(H))^{-1}(g \setminus \{v_0\}) \cup \{x_0\}$. If $g_{x_0}^{-1} \in \mathcal{X}_{g,p} \subseteq A^{p^{2nd}}(A)$ for $p,p^{2nd} \subseteq N_0$, then we have
by (4.9) that \( g^{-1}_{x_0} \in E_H(p, p^{2nd}, r) \), and thus
\[
P^A(g^{-1}_{x_0}, \emptyset) = p, \quad P^{A, 2nd}(g^{-1}_{x_0}, \emptyset) = P^{2nd}, \quad \text{and } \|p\| = \|P^{2nd}\| = 1.
\]
By (P4), the set \( W_{edge} \) contains the (simple) edge tester \((\omega, \omega_i, g, p, p^{2nd}) \) for \( X_{g,p,p^{2nd},\phi_0}(A) \) (as defined in Definition 4.6) with \( \omega(E(A_0)) = |X_{g,p,p^{2nd},\phi_0}(A)| \) and \( \omega_i(X_{\phi_0}) = |E_H(p, p^{2nd}, r)| \).
Hence, by (P3) with \( I = r, J = J_X = J_Y = \emptyset, P^A = p, P^{A, 2nd} = P^{2nd}, P^B = P^{B, 2nd} = 0 \), we obtain
\[
\Delta_\psi(A_0) \leq \sum_{p,p^{2nd}} |X_{g,p,p^{2nd},\phi_0}(A)| \quad \text{(P3)}
\]
\[
\leq \sum_{p,p^{2nd}} \left( 1 + \varepsilon \right) d^{-1}_A d_0 |E_H(p, p^{2nd}, r)| + n^2 \quad \text{(5.12)} \leq \left( 1 + \varepsilon^{2/3} \right) d_0 n.
\]
This establishes (5.11).

Next, we claim that
\[
(5.13) \quad \Delta_2(H_{aux}) \leq n^\varepsilon \leq \Delta_{aux}^{1-\varepsilon^2}.
\]
Note that the codegree in \( H_{aux} \) of two vertices in \( \bigcup_{H \in \mathcal{H}}(X^H_0 \cup V^H_0) \) is at most 1, and similarly, the codegree in \( H_{aux} \) of a vertex in \( \bigcup_{H \in \mathcal{H}}(X^H \cup V^H) \) and a label in \( \mathcal{E} \) is at most 1 because \( \Delta_\psi(A_0) \leq 1 \) for all \( H \in \mathcal{H} \). Hence, an appropriate upper bound on \( \Delta_\psi(A_0) \) establishes (5.13). In the following we derive such an upper bound on \( \Delta_\psi(A_0) \) by employing (P5). For all \( \varphi = \{v_1, \ldots, v_k\}, \mathcal{H} \in \mathcal{H} \), \( H \in \mathcal{H} \), \( g \in (V \cup \hat{H}) \cap \{v_i \} \subseteq V_0 \), and \( \{i_1, \ldots, i_k\} \neq \{j_1, \ldots, j_k\} \), note that \( g \) and \( \hat{H} \) appear together as labels of \( \psi \) on at most \( |E_{g,\varphi,\hat{H}}(A)| \) edges of \( A_0 \). This follows immediately from Definition 4.7 of \( E_{g,\varphi,\hat{H}}(A) \). Note further that \( \{|i_1, \ldots, i_k,j_1, \ldots, j_k\} \cap [-r_0] \geq k \) because \( (g \cup \hat{H}) \cap \{v_i \} \subseteq V_0 \), and \( \{i_1, \ldots, i_k\} \neq \{j_1, \ldots, j_k\} \). Hence, by (P5), we have
\[
|E_{g,\varphi,\hat{H}}(A)| \leq \max \left\{ n^k - |i_1, \ldots, i_k,j_1, \ldots, j_k| \cap [-r_0] + \varepsilon, n^\varepsilon \right\} = n^\varepsilon,
\]
and thus, \( \Delta_\psi^\varepsilon(A_0) \leq n^\varepsilon \), which establishes (5.13).

**Step 2. Applying Theorem 3.8 to \( H_{aux} \)**

Suppose \( W = \bigcup_{\ell \in [\Delta]} W_\ell \) is a set of size at most \( n^4 \log n \) of given weight functions \( \omega \in W_\ell \) for \( \ell \in [\Delta] \) with \( \omega: (E(A_0)) \to [0, \Delta] \) and
\[
(5.14) \quad \|\omega\|_{\ell^k} \leq n^{k-\ell^k + \varepsilon^2} \quad \text{for every } \ell^k \in [\ell].
\]
Note that every weight function \( \omega: (E(A_0)) \to [0, \Delta] \) naturally corresponds to a weight function \( \omega_{H_{aux}}: (E_{H_{aux}}) \to [0, \Delta] \) by defining \( \omega_{H_{aux}}(\{\mathcal{H}_1, \ldots, \mathcal{H}_\ell\}) := \omega(\{e_1, \ldots, e_\ell\}) \). We will explicitly specify \( W \) in Part C, where every weight function \( \omega: (E(A_0)) \to [0, \Delta] \) in \( W_\ell \) for \( \ell \in [\Delta] \) will be defined such that \( \supp(\omega) \subseteq \bigcup_{H \in \mathcal{H}} \left( E_{H_{aux}} \right)^h \) and in particular, such that the corresponding weight function \( \omega_{H_{aux}} \) will also be clean, that is supp(\( \omega_{H_{aux}} \)) \subseteq (E_{H_{aux}})^h = (E_{H_{aux}})^h \). Our main idea is to find a hypergraph packing in \( H_{aux} \) that behaves like a typical random matching with respect to \( \{\omega_{H_{aux}}: \omega \in W\} \) in order to establish (I)\(L_{5.1}-(III)L_{5.1}\).

Suppose \( \ell \in [\Delta] \) and \( \omega \in W_\ell \). If \( \omega(E(A_0)) \geq n^\ell + \varepsilon^2 / 2 \), define \( \tilde{\omega} := \omega \). Otherwise, arbitrarily choose \( \tilde{\omega}: (E(A_0)) \to [0, \Delta] \) such that \( \omega \leq \tilde{\omega}, \tilde{\omega} \) satisfies that supp(\( \tilde{\omega} \)) \subseteq \bigcup_{H \in \mathcal{H}} \left( E_{H_{aux}} \right)^h = \supp(\omega_{H_{aux}}) \subseteq \left( E_{H_{aux}} \right)^h = (E_{H_{aux}})^h = (E_{H_{aux}})^h \), \( \tilde{\omega}(E(A_0)) = n^\varepsilon + \varepsilon^2 / 2 \), and \( \|\tilde{\omega}\|_{\ell^k} \leq n^{-\ell^k + 2\varepsilon^2} \) for all \( \ell^k \in [\ell] \). By (5.11) and (5.13), we can apply Theorem 3.8 (with \( \Delta_{aux}, \varepsilon^2, \Delta_\delta + 2, \Delta, \{\tilde{\omega}_{H_{aux}}: \omega \in W_\ell\} \) playing the roles of the parameters \( \Delta, \delta, r, L, W_\ell \) of Theorem 3.8, respectively) to obtain a matching \( \mathcal{M} \) in \( H_{aux} \) that corresponds to a conflict-free packing \( \varphi: X^0_0 \to V_0 \) in \( A_0 \) with its corresponding edge set \( M(\sigma) \) that satisfies the following properties (where \( \tilde{\varepsilon} = \varepsilon^{1/2} / 2 \)):
\[
(5.15) \quad \omega(M(\sigma)) = (1 + \varepsilon)(1 - \varepsilon^{2/3}) \frac{\omega(E(A_0))}{(d_0 n)^{\tilde{\varepsilon}}} + n^\varepsilon
\]
\[
(5.16) \quad (1 + \varepsilon) \frac{\omega(E(A_0))}{(d_0 n)^{\tilde{\varepsilon}}} + n^\varepsilon \quad \text{for all } \omega \in W_\ell, \ell \in [\Delta].
\]
Part C. Employing weight functions to conclude (I)_{L5.1}–(III)_{L5.1}

Let \( \sigma : \mathcal{A}_0^p \rightarrow V_0 \) be the conflict-free packing in \( \mathcal{A}_0 \) as obtained in Part B and let \( \sigma^+ \) be a cluster-injective extension of \( \sigma \) chosen uniformly and independently at random. We will show that the random \( \sigma^+ \) satisfies with high probability the conclusions of the lemma and thus, there exists a suitable cluster-injective extension \( \sigma^+ \) by picking one such extension deterministically. We may assume that (5.16) holds for a set of weight functions \( W \). Each of these weight functions will only depend on our input parameters. Hence, we could define them right away but for the sake of a cleaner presentation we postpone their definitions to the specific situations when we employ those weight functions to establish (I)_{L5.1}–(III)_{L5.1}. We now define the candidacy graphs \( Z_{Iz}^{H,\text{new}} \leq Z_{Iz}^H (\phi_0 \cup \sigma^+) \) for \( Z \in \{A, B\} \) and all index sets \( I = I_A \subseteq [\nu] \), and \( i = I_B \in V(R) \). If \( I_Z \cap r = \emptyset \) for all \( r \in E(R) \) with \( 0 \in r \), then we set \( Z_{Iz}^{H,\text{new}} := Z_{Iz}^H \). Otherwise, let

\[
A_i^{H,\text{new}} := A_i^H (\phi_0 \cup \sigma^+), \quad \text{and} \quad B_i^{H,\text{new}} := B_i^H (\phi_0 \cup \sigma^+),
\]

with \( A_i^{H+} (\phi_0 \cup \sigma^+) \) defined as in Definition 4.1 with respect to \( H_+ \), \( \phi_0 \cup \sigma^+ \) and \( G_A \), as well as \( B_i^{H+} (\phi_0 \cup \sigma^+) \) defined with respect to \( H_+ \), \( \phi_0 \cup \sigma^+ \) and \( G_B \).

Before we establish (I)_{L5.1}–(III)_{L5.1}, we first estimate \( |\mathcal{Y}_0^\sigma \cap X_0^H| \) for each \( H \in \mathcal{H} \). We define a weight function \( \omega_H : E(A_0^H) \to \{0, 1\} \) for each \( H \in \mathcal{H} \) by \( \omega_H(e) := 1 \{e \in E(A_0^H)\} \), and add \( \omega_H \) to \( W \). Note that \( \omega_H(E(A_0^H)) = (1 + 3\varepsilon) \) because \( A_0^H \) is \( (\varepsilon, d_0) \)-super-regular by (P2). By (5.15), we obtain

\[
|\mathcal{Y}_0^\sigma \cap X_0^H| = \omega_H(M(\sigma)) = (1 \pm \varepsilon)(1 - \varepsilon \cdot 3/2) \cdot \frac{d_0 n^2}{d_0 n} + \varepsilon,
\]

and thus,

\[
(1 - \varepsilon^2/2)n \geq |\mathcal{Y}_0^\sigma \cap X_0^H| \geq (1 - \hat{\varepsilon}) n.
\]

We first prove (II)_{L5.1}, as we can use this for establishing (I)_{L5.1}.

Step 3. Preparation for checking (II)_{L5.1}

We will even show that (II)_{L5.1} holds for edge testers in \( W_{\text{edge}} \cup W_{\text{edge}}' \), where \( W_{\text{edge}}' \) is a set of suitable edge testers satisfying (P3) that we will explicitly specify in Step 11 when establishing (I)_{L5.1}. Throughout Steps 3–10 let \( (\omega, \omega', J, J_X, J_Y, e, \rho) \) be fixed. That is, we fix an index set \( I \subseteq V(R) \), \( J \subseteq I \), disjoint sets \( J_X, J_Y \subseteq J \), and let \( I_{r_0} := (I \cap [\nu] ) \setminus J, I_0 := (I \cap [\nu] ) \setminus J \), \( J_X := J_X \cup J_Y \), and we fix \( \omega : E(A_0^H) \to [0, s], \omega_0 : \mathcal{X}_{IJ} \to [0, s], \rho = (p^A, p^{A_{\text{new}}}, p^B, p^{B_{\text{new}}}) \in (0, r_0^{0, r_0^{+}}) \) with \( p^Z = (p^Z_0, p^Z_{r_0^{+}}) \). By (5.17), we have

\[
\mathcal{Y}_0^\sigma \cap X_0^H = \omega_H(M(\sigma)) = \frac{d_0 n^2}{d_0 n} + \varepsilon,
\]

and thus,

\[
(1 - \varepsilon^2/2)n \geq |\mathcal{Y}_0^\sigma \cap X_0^H| \geq (1 - \hat{\varepsilon}) n.
\]

We consider three different cases depending on whether \( 0 \in (V(R) \setminus I) \cup (J \setminus J_X) \), \( 0 \in I \setminus J \), and \( 0 \in J_X \). Even though we proceed similarly in each of these cases, the effects on (II)_{L5.1} are quite different in each scenario as we try to illustrate in the following. Recall that (II)_{L5.1} ensures that (P3) is also satisfied for the updated candidacy graphs. If \( 0 \in (V(R) \setminus I) \cup (J \setminus J_X) \), we have to update the density factors whereas the magnitude of \( \omega_0(\mathcal{X}_{IJ})/n(I \cap [\nu]) \) in (P3) equals the magnitude of \( \omega_0(\mathcal{X}_{IJ})/n(I \cap [\nu]) \) in (II)_{L5.1}. In contrast, if \( 0 \in I \setminus J \), we additionally have to ensure that the magnitude of \( \omega_0(\mathcal{X}_{IJ})/n(I \cap [\nu]) \) in (P3) will be updated by a factor of \( n^{-1} \) to obtain the magnitude of \( \omega(\mathcal{X}_{IJ})/n(I \cap [\nu]) \) in (II)_{L5.1}. If \( 0 \in J_X \), the magnitudes are again equal, but besides updating the densities we additionally have to consider that \( 0 \in J_X \) and thus (P3) will potentially be updated by the factor \( (1 - \varepsilon^2) \) because otherwise we do not update the weight of the edge tester. Recall that the statement (II)_{L5.1} concerns the weight of the edge tester \( (\omega_{\text{new}}, \omega, J, J_X, J_Y, e, \rho) \) defined as in Definition 4.5 with respect to \( (\omega, J, J_X, J_Y, e, \rho), (\phi_0 \cup \sigma, \phi_0 \cup \sigma^+), A_{\text{new}} \) and \( B_{\text{new}} \).

We collect some common notation that will be used to establish (II)_{L5.1}. Recall that the centres \( c \) are fixed and for all \( H \in \mathcal{H} \) and any \( \delta = (\delta_0, \delta) \in E(A_0^H) \cup \cup \cup E(B_j) \) with \( \delta = (\delta_j)_{j \in J} \), we have \( \omega(\delta_0) > 0 \) only if \( \{c_0\} \in I_{\delta_0 \cup J} \). By Definition 4.5 of an edge tester in (4.7).

(Recall that we allow to treat \( \delta = (\delta_j)_{j \in J} \in \cup \cup \cup E(B_j) \) as \( b_j \).) To that end, for all \( H \in \mathcal{H} \) and \( \delta \in E(A_0^H) \cup \cup \cup E(B_j) \) with \( \{c_0\} \in I_{\delta_0 \cup J} \), let \( y_{\delta} := \{y_i\}_{i \in I_0 \cup J} := (c_0, \delta) \cap V(I_0 \cup J) \).
(α ∪ β) ∩ X′ \mathcal{H}_{I_{r_{0}} \cup J}^\mathcal{H}. Our overall strategy in all three cases is to define for vertices x in some set \mathcal{Y}_{0}^{α, β} \subseteq X_{0}^\mathcal{H} a target set T_{x, α, β} of suitable images for x such that if all x ∈ \mathcal{Y}_{0}^{α, β} are embedded into T_{x, α, β}, then α, β (or \alpha, \beta \setminus \{c_{0}, y_{0}\}) if 0 ∈ I \setminus J) is an element in E(A_{r}^{H, \text{new}}) \cup \bigcup_{j \in J} E(B_{j}^{H, \text{new}}). Hence for all H ∈ \mathcal{H}, x ∈ X_{0}^\mathcal{H} and α, β ∈ E(A_{r}^{H, \text{new}}) \cup \bigcup_{j \in J} E(B_{j}^{H, \text{new}}) with ω(α, β) > 0, we define the following sets. We give more motivation for these definitions in the subsequent paragraph. For an edge e ∈ E(H), let r_{e} ∈ E(R) be such that e ∈ X_{H, r_{e}}.

S_{x, α, β, j} := \{ \phi_{0}(e) \cup \{c_{i} \in r_{e} \cap \mathcal{J}_{x, \{y \}} \}: e ∈ E_{x, y, α, β} \};

S_{x, α, β, j} := \{ S = \phi_{0}(e \setminus \{y \}) \cup \{c_{i} \in r_{e} \cap \mathcal{J}_{x, \{y \}} \}: |S| = k - 1, e ∈ E_{x, y, α, β}, j \in J \setminus \{0\}, y \in y \};

V_{x, α, β} := V_{0} \cup N_{G_{A}}(S_{x, α, β, j}) \cap N_{G_{B}}(S_{x, α, β, j});

T_{x, α, β} := V_{x, α, β} \cap N_{A_{0}^{\mathcal{H}}}(x).

That is, S_{x, α, β, j} and S_{x, α, β, j} are sets of (k - 1)-sets. In general, these (k - 1)-sets consist of the image \phi_{0}(e) of an H_{e}-edge e ∈ E_{x, y, α, β} together with the centers corresponding to the clusters e intersects. The set S_{x, α, β, j} contains all (k - 1)-sets that only intersect with clusters of I_{r}, whereas S_{x, α, β, j} contains (k - 1)-sets that intersect with a cluster of J \setminus \{0\}. Consequently, T_{x, α, β} is the intersection of the A_{0}^{\mathcal{H}}-neighbourhood of x in V_{0} with the common neighborhood V_{x, y, α, β} in G_{A} and G_{B} of all these (k - 1)-sets in S_{x, α, β, j} and S_{x, α, β, j} (see also Figures 3 and 4). Note that S_{x, α, β, j} ∪ S_{x, α, β, j} = ∅ if E_{x, y, α, β} = ∅. Further, since ω(α, β) > 0 and it is required for the edge tester ω that \phi_{0} does not map any vertices \{y \} \in J \setminus \{r_{0}\} onto its centers by (v)D4.5, we have that S_{x, α, β, j} and S_{x, α, β, j} are disjoint sets. We estimate the sizes of T_{x, α, β} and V_{x, α, β} in Step 3.2. Further, for α, β = \{α, β\} ∈ E(A_{r_{0}}) \cup \bigcup_{j \in J} E(B_{j})}, let x_{α} := x \cap X_{H}^\mathcal{H}, x_{β} := x \cap X_{H}^\mathcal{H} (note that x_{α} or x_{β} might be empty), and

(5.19) \mathcal{Y}_{0}^{α, β} := \{ x ∈ X_{H}^\mathcal{H} \setminus \{x_{α}\}: |S_{x, α, β, j} ∪ S_{x, α, β, j}| ≥ 1 \}.

Step 3.1. Weight functions to establish (II)L5.1

We emphasize again that the general strategy for establishing (II)L5.1 is to define tuple weight functions for the edges between the vertices x ∈ \mathcal{Y}_{0}^{α, β} and their corresponding target sets T_{x, α, β}, which we will do in this step depending on the three cases whether 0 ∈ (V(R) \setminus I) \cup (J \setminus J_{XV}), 0 ∈ I \setminus J, and 0 ∈ J_{XV}.

For all H ∈ \mathcal{H}, α, β = \{α, β\} ∈ E(A_{r_{0}}) \cup \bigcup_{j \in J} E(B_{j}) with ω(α, β) > 0, and \mathcal{Y}_{0}^{α, β} as defined in (5.19), we make the following definition. (For notational convenience, we treat \{0\} as ∅ in the following definition.)

(5.20) E_{α, β} := \{ α ∩ (X_{0}^\mathcal{H} \cup V_{0}) \cup \{e \} ∈ E_{X_{0}^\mathcal{H}} \in \bigcup_{0 \in I \setminus J} |\mathcal{Y}_{0}^{α, β}| = \frac{E(A_{r_{0}})}{100} \cup |\mathcal{Y}_{0}^{α, β}| = \{ e \} \in E_{X_{0}^\mathcal{H}} \setminus \{0\}, T_{x, α, β} \} \text{ for all } x ∈ \mathcal{Y}_{0}^{α, β} \}.

Let us explain the definition of E_{α, β}. If 0 ∈ (V(R) \setminus I) \cup (J \setminus J_{XV}), then α ∩ (X_{0}^\mathcal{H} \cup V_{0}) = ∅. Thus, E_{α, β} is the set of clean \mathcal{Y}_{0}^{α, β}]-tuples of edges in E(A_{r_{0}}) between a vertex x ∈ \mathcal{Y}_{0}^{α, β} and its target set T_{x, α, β}. If 0 ∈ I \setminus J, then α ∩ (X_{0}^\mathcal{H} \cup V_{0}) = \{x_{α}, c_{0}\}. (Recall that x_{α} = α ∩ X_{H}^\mathcal{H}. Thus, E_{α, β} is the set of clean \{1 + |\mathcal{Y}_{0}^{α, β}|\}-tuples of edges in E(A_{r_{0}}) where additionally require that the tuple contains the edge x_{α}, c_{0}. If 0 ∈ J_{XV}, we will not make use of the definition of E_{α, β}.

With E_{α, β} we can define the following weight function \omega_{α, β} := \omega(α, β) \cdot \{e \in E_{α, β}\}. The motivation behind this is the following observation for the two cases 0 ∈ (V(R) \setminus I) \cup (J \setminus J_{XV}), and 0 ∈ I \setminus J. We claim that for α, β = \{α, β\} ∈ E(A_{r_{0}}) \cup \bigcup_{j \in J} E(B_{j}) with ω(α, β) > 0, if \omega_{α, β}(M(σ)) > 0 and σ^{+}(x_{β}) ≠ c_{0} if 0 ∈ J, then α, β := \{α ∩ (α ∩ (X_{0}^\mathcal{H} \cup V_{0})), β \} ∈ E(A_{r_{0}}) \cup \bigcup_{j \in J} E(B_{j}^{H, \text{new}}). To see that this is true, note that if \omega_{α, β}(M(σ)) > 0, then the definition of the target sets T_{x, α, β} implies that (4.1) of Definition 4.1 for the updated candidacy graphs A_{r_{0}}^{H, \text{new}} = A_{r_{0}}^{H}(φ_{0} ∪ σ^{+}) and B_{j}^{H, \text{new}} = B_{j}^{H}(φ_{0} ∪ σ^{+}) is satisfied. Hence in this case, for the
edge tester $\omega^{\text{new}}$ as defined in the statement of (II)L5.1, we obtain by (4.7) of Definition 4.5 of $\omega^{\text{new}}$ that $\omega^{\text{new}}(a,b) = \omega(a,b)$ requiring that $\sigma^+(x) \neq c_0$ if $0 \in J$ so that (v)D4.5 of Definition 4.5 is satisfied. Note that $\omega^{\text{new}} = \omega$ if $0 \in (V(R) \setminus I) \cup (J \setminus J_{\text{XY}})$.

In order to ensure that $\sigma^+(x) \neq c_0$ if $0 \in J$, we apply an inclusion-exclusion principle and introduce another weight function $\omega_{a,b}^-$ that accounts for the weight in the case that $0 \in J$ and $\sigma^+(x) = c_0$. To that end, similarly as in (5.20), for all $H \in \mathcal{H}$, $a,b \in \{a,b\} \in E(A^H) \cup \bigcup_{j \in J} E(B^H_j)$ with $\omega(a,b) > 0$, and $\mathcal{F}^\sigma$ as defined in (5.19), we define the following set of edge tuples

$$E^\sigma_{a,b} := \left\{ (x \cap (X_H \cup V_0)) \cup (x_e) \in \mathcal{F}^\sigma \mid x \in \mathcal{F}^\sigma \setminus \{x_e\} \right\}.$$  

Analogously to $\omega_{a,b}$, we define the weight function $\omega_{a,b}^-$ by $\omega_{a,b}^-(e) := \omega(a,b) \cdot 1\{e \in E^\sigma_{a,b}\}$.

The size of the tuple weight functions depends on the cardinality of $\mathcal{F}^\sigma_{a,b}$. To that end, let

$$b^\text{max} := \max\{|\mathcal{F}^\sigma_{a,b}| \mid a,b \in \{a,b\} \in E(A^H) \cup \bigcup_{j \in J} E(B^H_j), \omega(a,b) > 0\},$$

and we will group the tuple weights for all $a,b$ with $\omega(a,b) > 0$ into all possible $(1\{0 \in I \setminus J\} + b)$-tuple weight functions for $b \in [b^\text{max}]$. To that end, for each $b \in [b^\text{max}]$, we set

$$\Omega_b := \left\{ a,b \in \{a,b\} \in E(A^H) \cup \bigcup_{j \in J} E(B^H_j) \mid \omega(a,b) > 0, |\mathcal{F}^\sigma_{a,b}| = b \right\},$$

as well as

$$\omega_b := \sum_{a,b \in \Omega_b} \omega_{a,b}, \quad \text{and} \quad \omega_b^- := \sum_{a,b \in \Omega_b^-} \omega_{a,b}^-.$$  

In the two cases when $0 \in (V(R) \setminus I) \cup (J \setminus J_{\text{XY}})$, and $0 \notin I \setminus J$, we will see that $\sum_{b \in [b^\text{max}]} \omega_b(M(\sigma))$ is the major contribution to $\omega^{\text{new}}(E(A^H) \cup \bigcup_{j \in J} E(B^H_j))$. However, since $\mathcal{F}^\sigma$ is a proper subset of $\mathcal{F}^\sigma_{a,b}$, we additionally need to consider those $a,b \in E(A^H) \cup \bigcup_{j \in J} E(B^H_j)$ for which a relevant vertex in $\mathcal{F}^\sigma_{a,b}$ has not been embedded by $\sigma$ as this might also contribute to the weight of $\omega^{\text{new}}$. This is also the case when $0 \notin J_{\text{XY}}$, because then we require that either $x_e = \emptyset \cap X_0^H$ is not embedded or no $H$-vertex is mapped onto the centre $c_0$.

We collect some notation. For all $a,b \in \{a,b\} \in E(A^H) \cup \bigcup_{j \in J} E(B^H_j)$ with $\omega(a,b) > 0$, let $H_{a,b} \in \mathcal{H}$ be such that $a,b \in \{a,b\} \in E(A^H) \cup \bigcup_{j \in J} E(B^H_j)$, and let $\mathcal{F}^\sigma_{a,b} := \mathcal{F}^\sigma_{a,b} \setminus \mathcal{F}^\sigma$. Recall that $J_X, J_Y \subseteq J$ are disjoint sets and $J_{\text{XY}} = J_X \cup J_Y$. Let $z_e := e$ if $0 \in J_X$, and let $z_e := c_0$ if $0 \in J_Y$. We can now define the following set of edge tuples that we describe in detail below. For $b \in [b^\text{max}]$, $\ell \in [b], m_A, m_B \in [\Delta_0]$, let

$$\Gamma_b(\ell, m_A, m_B) := \left\{ a,b \in \{a,b\} \in \mathcal{F}^\sigma_{a,b} \mid \mathcal{F}^\sigma_{a,b} \cap \mathcal{F}^\sigma, |\mathcal{F}^\sigma_{a,b}| = b, \sigma(x) \in T_{x,a,b} \right\}$$

That is, $\Gamma_b(\ell, m_A, m_B)$ is the set of edges $a,b \in \Omega_b$ such that there exists an $\ell$-set $\mathcal{F}^\sigma_{a,b}$ of vertices in $\mathcal{F}^\sigma_{a,b}$ which are not embedded by $\sigma$, and these $\ell$ vertices in $\mathcal{F}^\sigma_{a,b}$ contribute $m_A$ and $m_B$ many $(k-1)$-sets, and all remaining $b-\ell$ vertices $x \in \mathcal{F}^\sigma_{a,b} \cap \mathcal{F}^\sigma$ are embedded onto their target set $T_{x,a,b}$. Additionally, if $0 \notin I \setminus J$, then we require that $x_e$ is mapped onto $c_0$ by $\sigma$, and if $0 \in J_X$, then we require that $x_e$ is not embedded by $\sigma$, and if $0 \in J_Y$, then we require that no vertex of $X_0^H$ is mapped onto $c_0$ by $\sigma$. Further, let

$$\Gamma^{\text{hit}}_b(\ell, m_A, m_B) := \left\{ a,b \in \Gamma_b(\ell, m_A, m_B) \mid \mathcal{F}^\sigma_{a,b} \cap \mathcal{F}^\sigma = \{x_e\} \right\}.$$

if $x_e \in \mathcal{F}^\sigma_{a,b}$ then $\mathcal{F}^\sigma_{a,b} \cap \mathcal{F}^\sigma = \{x_e\}$.
That is, \( \Gamma_b^{hit}(\ell, m_A, m_B) \subseteq \Gamma_b(\ell, m_A, m_B) \) contains those edges \( a\bar{e} \in \Omega_b \), where the not embedded vertices in \( \mathcal{A}_{\ell,\sigma} \) are nevertheless mapped onto their target set \( V_{x,a\sigma} \) by the random cluster-injective extension \( \sigma^+ \) of \( \sigma \), and \( \sigma^+ \) does not map \( x_\sigma \) onto \( a_0 \). Thus, in such a case the weight of \( a\bar{e} \) will ‘accidentally’ be taken into account in addition to the ‘real’ contribution given by \( \sigma \) (compare also with (5.25) below).

Crucially note that in the two cases when \( 0 \in (V(R) \setminus I) \cup (J \setminus J_{XV}) \) and \( 0 \notin I \cup J \), we know for \( a\bar{e} \in \Omega_b \), that \( a\bar{e}^{new} \in E(\mathcal{A}_{\ell,\sigma}^{new}) \cup \bigcup_{J \in J} E(\mathcal{B}_j^{new}) \) only if \( \sigma^+(x_\sigma) \neq a_0 \) if \( 0 \in J \) and either \( \omega_{a\bar{e}}(\sigma) = \omega(a\bar{e}) > 0 \) or \( a\bar{e} \in \Gamma^H_b(\ell, m_A, m_B) \) for some \( \ell \in [b] \), \( m_A, m_B \in [\Delta]_0 \). (Recall that \( a\bar{e}^{new} = a\bar{e} \) if \( 0 \notin (V(R) \setminus I) \cup (J \setminus J_{XV}) \).) This holds by (4.1) of Definition 4.1 and since we defined \( \mathcal{A}_{\ell,\sigma}^{new} \) and \( \mathcal{B}_j^{new} \) in (5.17) as updated candidacy graphs with respect to \( \phi_0 \cup \sigma^+ \). (Note that since we choose \( \sigma^+ \) as a ‘dummy’ enlargement, we do not require that \( \sigma^+(x) \in N_{A_0}(x) \), which is the reason why \( \sigma^+(x) \in V_{x,a\sigma} \) in (5.24) instead of \( \sigma^+(x) \in T_{x,a\sigma} \).) Hence, for the two cases when \( 0 \notin (V(R) \setminus I) \cup (J \setminus J_{XV}) \), and \( 0 \in I \cup J \), we make the following key observation:

\[
\omega^{new}(E(\mathcal{A}_{\ell,\sigma}^{new}) \cup \bigcup_{J \in J} E(\mathcal{B}_j^{new}))
= \sum_{b \in [b_{\max}]_0} \omega(b(M(\sigma))) - I(0 \in J \setminus J_{XV}) \omega^-(b(M(\sigma))) + \sum_{b \in [b_{\max}]_0, \ell \in [b], m_A, m_B \in [\Delta]_0} \omega(\Gamma^H_b(\ell, m_A, m_B)).
\]

\[\text{(5.25)}\]

In the case that \( 0 \in J_{XV} \), it suffices in view of the statement to establish an upper bound for \( \omega^{new}(E(\mathcal{A}_{\ell,\sigma}^{new}) \cup \bigcup_{J \in J} E(\mathcal{B}_j^{new})) \). Similarly as in (5.25), we have in this case for \( a\bar{e} \in \Omega_b \) and \( b \in [b_{\max}]_0 \) that \( a\bar{e} \in E(\mathcal{A}_{\ell,\sigma}^{new}) \cup \bigcup_{J \in J} E(\mathcal{B}_j^{new}) \) only if \( a\bar{e} \in \Gamma^H_b(\ell, m_A, m_B) \) for some \( \ell \in [b]_0 \), \( m_A, m_B \in [\Delta]_0 \). This holds by (4.1) of Definition 4.1 and since we defined \( \mathcal{A}_{\ell,\sigma}^{new} \) and \( \mathcal{B}_j^{new} \) in (5.17) as updated candidacy graphs with respect to \( \phi_0 \cup \sigma^+ \). Hence, for the case that \( 0 \in J_{XV} \), we make the following key observation:

\[
\omega^{new}(E(\mathcal{A}_{\ell,\sigma}^{new}) \cup \bigcup_{J \in J} E(\mathcal{B}_j^{new})) \leq \sum_{b \in [b_{\max}]_0, \ell \in [b], m_A, m_B \in [\Delta]_0} \omega(\Gamma^H_b(\ell, m_A, m_B)).
\]

\[\text{(5.26)}\]

In Steps 4–7, we will estimate the weight of the contributing terms in (5.25) and (5.26). To do so, we first determine the sizes of the target sets \( T_{x,a\sigma} \) and \( V_{x,a\sigma} \) in the next step.

**Step 3.2.** Size of the target sets \( T_{x,a\sigma} \) and \( V_{x,a\sigma} \)

Let \( b \in [b_{\max}] \) and \( a\bar{e} \in \Omega_b \) be fixed. We observe that

\[
|V_{x,a\sigma}| = (1 \pm 3\varepsilon)d_A^{S_{x,a\sigma},j}d_B^{S_{x,a\sigma},j}n, \quad \text{and} \quad |T_{x,a\sigma}| = (1 \pm 3\varepsilon)d_A^{S_{x,a\sigma},j}d_B^{S_{x,a\sigma},j}d_0n,
\]

\[\text{(5.27)}\]

for \( x \in \mathcal{A}_{\ell,\sigma} \), where we used (P1) and that \( A_H^H \) is \((\varepsilon,d_0)\)-super-regular and \((\varepsilon,q)\)-well-intersecting for each \( H \in \mathcal{H} \). For an illustration of the sets \( S_{x,a\sigma},j,T_{x,a\sigma} \) in the case that \( 0 \notin I \) and \( J = 0 \), see Figure 3.

For \( b_{\ell} = \sum_{i \in I_{\ell}} b_i \) and \( b_J = \sum_{j \in J} b_j \) as defined in (5.4), we claim that

\[
\sum_{x \in \mathcal{A}_{\ell,\sigma}} |S_{x,a\sigma,j}| = b_{\ell} + p_{0} - p_{0}^{A,2nd}; \quad \text{and} \quad \sum_{x \in \mathcal{A}_{\ell,\sigma}} |S_{x,a\sigma,j}| = b_J + p_{0}^{B} - p_{0}^{B,2nd}.
\]

\[\text{(5.28)}\]

We establish the first equation in (5.28); the second one then follows similarly. At this part of the proof it is crucial to refresh Definition 4.2 because we make use of all the details of the pattern definitions. Further, recall that \( y_{a\bar{e}} = \{y_i\}_{i \in I_{R_0} \cup J} := (a\bar{e} \cup \mathcal{U}) \cap \mathcal{X}^H_{\chi(I_{R_0} \cup J)} \) and thus \( x_\sigma = y_0 \) if \( 0 \in I_{R_0} \); otherwise, \( x_\sigma = \emptyset \). Note that in order to compute \( |S_{x,a\sigma,j}| \) it is equivalent to count \( |E_{x,(y)_{i \in I_{\ell}}}| \). By definition of \( \mathcal{H}_+ \), we have that \( |\bigcup_{x \in \mathcal{X}_0} E_{x,y}| = b_i \) for all \( i \in (I_{\ell} \cup J) \setminus \{0\} \) (see (5.10)). That is, there are \( b_{\ell} = \sum_{i \in I_{\ell} \cup J} b_i \) edges \( e \in E_{x,y} \) with \( e \cap \{y_i\}_{i \in I_{\ell}} = 1 \). Out of these \( b_{\ell} \) edges, we claim that \( p_{0}^{A,2nd} \) edges \( e \in E_{x,y} \) satisfy that \( x_\sigma = e \), that is, \( \sum_{i \in I_{\ell}} |E_{x,y}| = p_{0}^{A,2nd} \). For an illustration of these edges in \( \bigcup_{i \in I_{\ell}} E_{x,y} \), see Figure 4. Indeed, since \( \omega(a\bar{e}) > 0 \), we have by Definition 4.4 of an edge tester that \( y_{a\bar{e}} \subseteq (x_\sigma) \) for some \( x_\sigma \in \mathcal{X}_0 \) with \( x_\sigma \in \mathcal{E}_{H}(p,I,J) \). Thus, by Definition 4.3 of \( E_{H}(p,I,J) \), we have that \( p_{0}^{A}(x,J) = p_{0}^{A}(p_{0}^{A})_{i \in V(R)} \) and \( p_{0}^{A,2nd}(x,J) = p_{0}^{A,2nd} = (p_{i}^{A,2nd})_{i \in V(R)} \). Hence, by the definition of a \( 2^{nd} \)-pattern in Definition 4.2 and because \( \mathcal{H}_+ \)
Figure 3. This illustrates the case that \( 0 \notin I \) and for simplicity \( J = \emptyset \). That is, we consider the edge \( a_0 = \{c_1, c_2, c_3, y_1, y_2, y_3\} \in E(A^b) \) with \( y_{a0} = \{y_1, y_2, y_3\} \subset x_I \). Note that the edges \( e_1, e_2, e_3 \in H_+ \) belong to \( E_{x,y_a} \). For the set \( S_{x,a,J} \), we have \( S_{x,a,J} = \{\phi(e_1) \cup \{c_1\}, \phi(e_2) \cup \{c_2, c_3\}, \phi(e_3) \cup \{c_3\}\} \). Accordingly, \( T_{x,a} \) is the intersection in \( V_0 \) of the \( G_A \)-neighbourhoods of these \( (k-1) \)-sets in \( S_{x,a,J} \) and the neighbourhood of \( x \in A^b \). Note that the blue edges \( e_1, e_3 \) in \( H_+ \) satisfy that \( |e_1 \cap y_{a0}| = |e_3 \cap y_{a0}| = 1 \) and thus they do not account for the 1-pattern \( p^4(x_I, \emptyset) \) of \( x_I \) by (4.3) of Definition 4.2. By considering all possible \( x \in X^H_0 \), there are in total \( b_1 = \sum_{i \in I_0} b_i \) many such blue edges in \( H_+ \). Further, note that the grey edge \( e_2 \) satisfies that \( |e_2 \cap y_{a0}| = 2 \) and thus, \( e_2 \) belongs to \( H \) by (5.8) and accounts for the 1-pattern \( p^4(x_I, \emptyset) \) of \( x_I \). Again, by considering all possible \( x \in X^H_0 \), there are in total \( p^4(x_I, \emptyset) \) many such grey edges in \( H_+ \).

Figure 4. This illustrates the case that \( 0 \in I \) and \( y_{a0} = \{y_0, y_1, y_2\} \subset x_I \), and we assume that \( I_{r_0} = \{i_1, i_2\} \) and \( J = \emptyset \). Note that \( y_0 = x_a \). By Definition 4.4 of an edge tester we require that \( x_I \) lies in an H-edge if we assigned positive weight to the tuple \( x_I \). Hence, by the construction of \( H_+ \) (see (5.7)), we have that \( e_1, e_2, e_3 \) are edges in \( H \). Note that the edges \( e_1, e_2, e_3 \) belong to \( E_y \cup E_{y_2} \cup E_{y_0} \). By definition, \( y_0 = x_a \notin \varphi^o_{a0} \) and we thus do not consider the possible target set \( T_{y_0, \emptyset} \) because \( x_0 \) has to be mapped onto the centre \( c_0 \) in \( G \). Hence, the edges \( e_1, e_2, e_3 \) do not account for \( \sum_{x \in x_I} \{S_{x,a,J}\} \). Since \( y_{a0} \subset x_I \), there are \( p^{A,2\text{nd}}(x_I, \emptyset) \) many such blue edges by the definition of the 2 pattern \( p^{A,2\text{nd}}(x_I, \emptyset) \) in Definition 4.2.

is constructed such that each subset \( \{x_{a_j}, y_{j}\} \) only lies in proper edges of \( H \) due to (c) and (5.7), we have \( p^{A,2\text{nd}}_0 = \sum_{i \in I_0} |E_{x_{a_j}, y_{j}}| \). By the definition of \( \varphi^o_{a_0} \) in (5.19) which excludes \( x_{a\delta} \), this accounts for the term \( b_{r_0} - p^{A,2\text{nd}}_0 \) in (5.28). It is worth pointing out that \( p^{A,2\text{nd}}_0 = p^{B,2\text{nd}}_0 = 0 \) if \( 0 \notin I_{r_0} \) because then \( x_{a_\delta} = \emptyset \).

Further, we claim that there are \( p^{A}_0 \) edges \( e \in \bigcup_{x \in x_0} E_{x, \{y_j\}_{i \in I_0}} \) with \( |e \cap \{y_j\}_{i \in I_0}| \geq 2 \) but \( x_{a\delta} \notin e \). Since \( y_{a\delta} \subset x_I \) for some \( x_I \in H \) with \( x_I \in E \), we have that \( p^A(x_I, J) = p^A = (p^A_i)_{i \in V(\varphi)} \). Hence, by the definition of a 1 pattern in Definition 4.2 and by (5.8), there are \( p^B_0 \) edges \( e \in \bigcup_{x \in x_0} E_{x, \{y_j\}_{i \in I_0}} \) with \( |e \cap \{y_j\}_{i \in I_0}| \geq 2 \) because of the last two conditions in (4.3), and all of these edges \( e \) satisfy that \( x_{a\delta} \notin e \) due to the condition \( \iota_f \cap X^H_{ij} \neq \emptyset \in I_0 \) in (4.3). Altogether, this implies (5.28).

Hence, for \( a_\delta \in \Omega_b \), we have by (5.27) and (5.28) that

\[
\prod_{x \in \varphi_{a_\delta}} |T_{x,a_\delta}| = (1 + \tilde{\epsilon})d^B_A + p^B_0 - p^{A,2\text{nd}}_0 \phi^B d^B_A + p^B_0 - p^{B,2\text{nd}}_0 \phi^B d^B_A.
\]
Step 4. Estimating $\sum_{b \in [b_{\max}]} \omega_b(M(\sigma))$ in (5.25)

In this step we estimate the contribution of the first term $\sum_{b \in [b_{\max}]} \omega_b(M(\sigma))$ in (5.25). Throughout this step, let us consider the case that $0 \in (V(R) \setminus I) \cup (J \setminus J_X)$. Note, if at the end of the step we explain how the estimate of $\sum_{b \in [b_{\max}]} \omega_b(M(\sigma))$ changes if $0 \in I \setminus J$. Then, $\omega_0$ is the empty function and thus, $\sum_{b \in [b_{\max}]} \omega_b(M(\sigma)) = \sum_{b \in [b_{\max}]} \omega_b(M(\sigma))$. (That is, $b = 0$ is only relevant if $0 \in I \setminus J$.)

We first consider $\omega_b(M(\sigma))$ for some $b \in [b_{\max}]$. By (5.29) and the definition of $\omega_b$, we have

$$
\omega_b(E(A_0)) = (1 \pm 2\varepsilon)^{d_A^{B_1} + p_0^{A,2nd} - p_0^{B,2nd}} d_B^{B_1} + d_B^{p_0^{B,2nd}} d_0^{n_b} \sum_{\alpha, \beta \in \Omega_b} \omega(\alpha, \beta). 
$$

We verify that $\omega_b$ satisfies (5.14). For all $\{e_1, \ldots, e_b\} \in (E(A_0)_{b'})$, $b' \in [b]$, the number of edges $\{e_{b'+1}, \ldots, e_b\}$ such that $e = \{e_1, \ldots, e_b\} \in (E(A_0))$ with $\omega_b(e) > 0$ is at most $\Delta n^{b-b'}$ (recall that we have chosen $\Delta$ such that $\epsilon' = 1/\Delta < 1/\epsilon < 1/k, 1/r, 1/r_0, 1/s$), implying that $||\omega_b||_V \leq \Delta^2 n^{b-b'} \leq n^{b-b' + \epsilon^2}$. Hence, by adding $\omega_b$ to $\mathcal{V}$, (5.16) implies that

$$
\omega_b(M(\sigma)) = (1 \pm \varepsilon)^{\omega_b(E(A_0))} \pm \varepsilon
$$

(5.30) \hspace{1cm} (1 \pm \varepsilon) = (1 \pm \varepsilon^2) d_A^{B_1} + p_0^{A,2nd} - p_0^{B,2nd} d_B^{B_1} + d_B^{p_0^{B,2nd}} \sum_{\alpha, \beta \in \Omega_b} \omega(\alpha, \beta) \pm \varepsilon.

Finally, observe that

$$
\sum_{b \in [b_{\max}]} \sum_{\alpha, \beta \in \Omega_b} \omega(\alpha, \beta) = \sum_{\alpha, \beta \in E(A_{I_0})} \sum_{\alpha, \beta \in E(B_j)} \omega(\alpha, \beta) = \omega(E(A_{I_0}) \cup \bigcup_{j \in J} E(B_j)),
$$

and thus (5.31) implies that

$$
\sum_{b \in [b_{\max}]} \omega_b(M(\sigma)) = (1 \pm \varepsilon) d_A^{B_1} + p_0^{A,2nd} - p_0^{B,2nd} d_B^{B_1} + d_B^{p_0^{B,2nd}} \omega(E(A_{I_0}) \cup \bigcup_{j \in J} E(B_j)) \pm \varepsilon.
$$

(5.32) \hspace{1cm} \text{which is the desired estimate of } \sum_{b \in [b_{\max}]} \omega_b(M(\sigma)) = \sum_{b \in [b_{\max}]} \omega_b(M(\sigma)) \text{ in the case that } 0 \in (V(R) \setminus I) \cup (J \setminus J_X).

Let us now assume that $0 \in I \setminus J$ and we explain how the estimate of $\sum_{b \in [b_{\max}]} \omega_b(M(\sigma))$ changes. (Note that we allow for $b = 0$. The intuition is that we additionally require that $x_{\alpha}$ is mapped onto $c_0$ which we would expect to happen in an idealized random setting with probability roughly $(d_0n)^{-1}$. In fact, (5.30) is still true in the case that $0 \in I \setminus J$, but note that $\omega_b$ is now a $(1+b)$-tuple weight function which yields an additional factor of $(d_0n)^{-1}$ in (5.31) and thus also in (5.32). Hence, we obtain (5.32) with an additional factor of $(d_0n)^{-1}$ as the desired estimate in the case that $0 \in I \setminus J$.

Step 5. Estimating $\sum_{b \in [b_{\max}]} \omega_b^-(M(\sigma))$ in (5.25) if $0 \in J \setminus J_X$

In this step we establish an upper bound on the contribution of the minuend $\sum_{b \in [b_{\max}]} \omega_b^-(M(\sigma))$ in (5.25) and therefore, suppose that $0 \in J \setminus J_X$.

We first consider $\omega_b^-(M(\sigma))$ for some $b \in [b_{\max}]$. It suffices to establish only a rough upper bound which follows directly by the definition of $\omega_b^-$:

$$
\omega_b^-(E(A_0)) \leq \omega(\alpha, \beta).
$$

(5.33) \hspace{1cm} \text{It is easy to verify that } \omega_b^- \text{ satisfies (5.14). Further, note that } \omega_b^- \text{ is a } (b+1)\text{-tuple weight function. Hence, by adding } \omega_b \text{ to } \mathcal{V}, \text{(5.16) implies that}

$$
\omega_b^-(M(\sigma)) \leq (1 \pm \varepsilon) \omega_b^-(E(A_0)) \pm \varepsilon
$$

(5.34) \hspace{1cm} \text{Finally, observe that}

$$
\sum_{b \in [b_{\max}]} \omega(\alpha, \beta) \leq \omega(E(A_{I_0}) \cup \bigcup_{j \in J} E(B_j)),
$$

(5.35) \hspace{1cm} \text{where the desired estimate in the case that } 0 \in I \setminus J.
and thus (5.34) implies that

\[ \sum_{b \in [b_{\text{max}} \ell]} \omega_b(M(\sigma)) \leq n^{\ell-1} \omega \left( E(A_{I_{\text{ro}}} \cup \bigcup_{j \in J} E(B_j) \right) + n^{2\ell}. \]

Hence, combining (5.32) and (5.35) in the case that \(0 \in J \setminus J_{XV}\), yields that

\[ \sum_{b \in [b_{\text{max}} \ell]} (\omega_b(M(\sigma)) - \omega_b^{-1}(M(\sigma))) = (1 \pm 2\varepsilon(1/3))_{d_A}^{j_A} + p_{0A}^{2n} - p_{0B}^{2n} \]

\[ \omega \left( E(A_{I_{\text{ro}}} \cup \bigcup_{j \in J} E(B_j) \right) \pm 2n^{2\ell}. \]

**Step 6. Estimating \( \omega(\Gamma (\ell, m_A, m_B)) \)**

In this step we derive an upper bound for \( \omega(\Gamma (\ell, m_A, m_B)) \) for fixed \( b \in [b_{\text{max}} \ell], \ell \in [b] \), \( m_A, m_B \in [\Delta] \), and \( \Gamma (\ell, m_A, m_B) \geq \Gamma (\ell, m_A, m_B) \) as defined in (5.23). We will use this bound in the subsequent Step 7 to derive an upper bound for \( \omega(\Gamma (\ell, m_A, m_B)) \) as in (5.25) and (5.26). Throughout this step, let us again consider the case that \(0 \in (V(R) \setminus I) \cup (J \setminus J_{XV})\), and thus \( b, \ell > 0 \). At the end of the step, we explain how the estimate changes if \(0 \in I \setminus J \) or \( 0 \in J_{XV} \).

Our general strategy is based on the following inclusion-exclusion principle. For every \( \alpha \beta \in \Omega_b \), we estimate the \( \omega \)-weight \( \omega(\alpha \beta) \) with tuples \( x \in (\mathcal{X}, \mathcal{E}) \) such that \( b - \ell \) vertices \( x \) of \( x \) are mapped onto their target set \( T_{\alpha \beta} \) and the remaining \( \ell \) vertices \( x_1, \ldots, x_\ell \) of \( x \) satisfy

\[ x \in (\mathcal{X}, \mathcal{E}) \text{ such that } b - \ell \text{ vertices } x \text{ of } x \text{ are mapped onto their target set } T_{\alpha \beta} \text{ and the remaining } \ell \text{ vertices } x_1, \ldots, x_\ell \text{ of } x \text{ satisfy } (*) \]

\[ \sum_{i \in [\ell]} |S_{x_i, \alpha \beta, j}| = m_A, \quad \sum_{i \in [\ell]} |S_{x_i, \alpha \beta, j}| = m_B. \]

Now, subtracting (5.38) from (5.37) yields the \( \omega \)-weight \( \omega(\alpha \beta) \) as in (5.37) with tuples \( x \) where we additionally require that the remaining \( \ell \) vertices \( x_1, \ldots, x_\ell \) of \( x \) are embedded by \( \sigma \) and satisfy (*)

\[ \sum_{i \in [\ell]} |S_{x_i, \alpha \beta, j}| = m_A, \quad \sum_{i \in [\ell]} |S_{x_i, \alpha \beta, j}| = m_B. \]

Hence, summing over the \( \omega \)-weight as in (5.39) for all \( \alpha \beta \in \Omega_b \) yields an upper bound for \( \omega(\Gamma (\ell, m_A, m_B)) \) as defined in (5.23) when \( 0 \in (V(R) \setminus I) \cup (J \setminus J_{XV}) \).

First, in order to estimate (5.37) and (5.38), we define the following sets of tuples of edges in \( A_0 \). For all \( H \in H \) and \( \alpha \beta \in E(A_{I_{\text{ro}}} \cup \bigcup_{j \in J} E(B_j)) \) with \( \alpha \beta \in \Omega_b \), let

\[ E_{\alpha \beta}^{(5.37)} := \bigcup_{x \in \mathcal{X}_{\alpha \beta, x, m_A, m_B}} \left\{ \{e_x\} \in \mathcal{X}_{\alpha \beta, x, m_A, m_B} \in \mathcal{X}^{(5.37)} \right\} \]

\[ E_{\alpha \beta}^{(5.38)} := \bigcup_{x \in \mathcal{X}_{\alpha \beta, x, m_A, m_B}} \left\{ \{e_x\} \in \mathcal{X}_{\alpha \beta, x, m_A, m_B} \in \mathcal{X}^{(5.38)} \right\} \]

Note that the edges in \( E_{\alpha \beta}^{(5.37)} \) and \( E_{\alpha \beta}^{(5.38)} \) correspond to the described situations in (5.37) and (5.38), respectively. We define weight functions \( \omega_{\alpha \beta}^{(5.37)} : (E(A_{I_{\text{ro}}} \cup \bigcup_{j \in J} E(B_j)) \to [0, s] \) and \( \omega_{\alpha \beta}^{(5.38)} : (E(A_{I_{\text{ro}}} \cup \bigcup_{j \in J} E(B_j)) \to [0, s] \) by

\[ \omega_{\alpha \beta}^{(5.37)}(e) := 1 \{ e \in E_{\alpha \beta}^{(5.37)} \} \cdot \omega(\alpha \beta), \text{ and } \omega_{\alpha \beta}^{(5.38)}(e) := 1 \{ e \in E_{\alpha \beta}^{(5.38)} \} \cdot \omega(\alpha \beta). \]

Let

\[ \omega_1^{(5.37)}(e) := \sum_{\alpha \beta \in \Omega_b} \omega_{\alpha \beta}^{(5.37)}(e), \quad \text{and } \omega_1^{(5.38)}(e) := \sum_{\alpha \beta \in \Omega_b} \omega_{\alpha \beta}^{(5.38)}(e). \]
We estimate \( \omega_\Gamma^{(5.37)}(E(A_0)) \) and \( \omega_\Gamma^{(5.38)}(E(A_0)) \). By (5.29) and the definition of \( E_{\alpha,\delta}^{(5.37)} \) and \( E_{\alpha,\delta}^{(5.38)} \) in (5.40) and (5.41), respectively, we obtain

\[
(5.42) \quad \omega_\Gamma^{(5.37)}(E(A_0)) = (1 \pm 2\varepsilon) d_A^{\alpha} p_0^{\alpha} - p_0^{A,2nd} - m_A d_B^{\alpha} p_0^{\alpha} - p_0^{B,2nd} - m_B (d_0 n)^{-\ell} \sum_{\alpha,\delta \in \Omega_b} \omega(\alpha,\delta);
\]

\[
(5.43) \quad \omega_\Gamma^{(5.38)}(E(A_0)) = (1 \pm 2\varepsilon) d_A^{\alpha} p_0^{\alpha} - p_0^{A,2nd} - m_A d_B^{\alpha} p_0^{\alpha} - p_0^{B,2nd} - m_B (d_0 n)^{b} \sum_{\alpha,\delta \in \Omega_b} \omega(\alpha,\delta).
\]

Again, we can add \( \omega_\Gamma^{(5.37)} \) and \( \omega_\Gamma^{(5.38)} \) to \( \mathcal{W} \) and employ property (5.16). This yields that

\[
(5.42) \quad \omega_\Gamma^{(5.37)}(M(\sigma)) = (1 \pm \varepsilon) \frac{\omega_\Gamma^{(5.37)}(E(A_0))}{(d_0 n)^{-\ell}} + n\varepsilon
\]

\[
(5.44) \quad \omega_\Gamma^{(5.38)}(M(\sigma)) = (1 \pm \varepsilon) \frac{\omega_\Gamma^{(5.38)}(E(A_0))}{(d_0 n)^{b}} + n\varepsilon
\]

and

\[
(5.43) \quad \omega_\Gamma^{(5.38)}(M(\sigma)) = (1 \pm \varepsilon) \frac{\omega_\Gamma^{(5.38)}(E(A_0))}{(d_0 n)^{b}} + n\varepsilon.
\]

Finally, as observed in (5.39), subtracting (5.45) from (5.44) gives us an upper bound on \( \omega(\Gamma_b(\ell, m_A, m_B)) \). We obtain

\[
\omega(\Gamma_b(\ell, m_A, m_B)) \leq \omega_\Gamma^{(5.37)}(M(\sigma)) - \omega_\Gamma^{(5.38)}(M(\sigma)) \leq 8\varepsilon d_A^{\alpha} p_0^{\alpha} - p_0^{A,2nd} - m_A d_B^{\alpha} p_0^{\alpha} - p_0^{B,2nd} - m_B (d_0 n)^{-\ell} \sum_{\alpha,\delta \in \Omega_b} \omega(\alpha,\delta) + 2n\varepsilon.
\]

Let us now first assume that \( 0 \in I \setminus J \) and we explain how the estimate on \( \omega(\Gamma_b(\ell, m_A, m_B)) \) changes. If \( 0 \in I \setminus J \), then by the definition of \( \Gamma_b(\ell, m_A, m_B) \) in (5.23), we additionally require that \( x_0 \) is mapped on \( c_0 \) by \( \sigma \) which we would again expect to happen with probability roughly \( (d_0 n)^{-1} \). That is, we have to modify the definitions in (5.40) and (5.41) by additionally adding the edge \( x_0 c_0 \) to the tuples. Again, the estimates for the total weights in (5.42) and (5.43) are still true but we obtain an additional factor of \( (d_0 n)^{-1} \) in (5.44) and (5.45) as the sizes of the tuple functions increased by 1. Thus, we also obtain (5.46) with an additional factor of \( (d_0 n)^{-1} \) which will be our desired estimate in the case that \( 0 \in I \setminus J \).

Finally, let us assume that \( 0 \in J_X \) and we explain how the estimate on \( \omega(\Gamma_b(\ell, m_A, m_B)) \) changes. If \( 0 \in J_X \), then by the definition of \( \Gamma_b(\ell, m_A, m_B) \) in (5.23), we additionally require that either \( x_0 \) is left unembedded by \( \sigma \), or no \( H^{\alpha,\delta} \)-vertex is mapped onto \( c_0 \). This can be achieved by modifying the definition in (5.41) such that the edge tuples are increased by adding the \( A^{\beta} \)-edges \( e_{x_0} \) such that \( e_{x_0} \in e_{x_0}^\sigma \). This ensures that for \( x_0 \in \{x_0, c_0\} \), we either have \( x_0 \) left unembedded by \( \sigma \), or no \( H^{\alpha,\delta} \)-vertex is mapped onto \( c_0 \). The modification adds another factor of \( d_0 n \) to the total weight in (5.43) but also another factor of \( (d_0 n)^{-1} \) to (5.45). Thus, (5.46) will also be our desired estimate in the case that \( 0 \in J_X \).

Step 7. Estimating \( \omega(\Gamma_b^{hit}(\ell, m_A, m_B)) \) in (5.25) and (5.26)

We will use the bounds on \( \omega(\Gamma_b(\ell, m_A, m_B)) \) of Step 6 to derive an upper bound for the \( \omega \)-weight \( \omega(\Gamma_b^{hit}(\ell, m_A, m_B)) \). Let us first assume the two cases that \( 0 \in (V(R) \setminus I) \cup (J \setminus J_X) \), and \( 0 \in J_X \), since both cases yield the same bound on \( \omega(\Gamma_b(\ell, m_A, m_B)) \) in (5.46). The general idea is that we will obtain additional factors \( d_A^{m_A} \) and \( d_B^{m_B} \) in (5.46) when we extend \( \sigma \) to \( \sigma^+ \), that is, when we embed the \( \ell \) unembedded \( H_+ \)-neighbours of each \( \alpha, \delta \) contributing to \( \omega(\Gamma_b(\ell, m_A, m_B)) \). Only if these \( \ell \) vertices are mapped onto their target set \( V_{x,\alpha,\delta} \), then \( \alpha, \delta \) also contributes to \( \omega(\Gamma_b^{hit}(\ell, m_A, m_B)) \); that is, if \( x^+ \in V_{x,\alpha,\delta} \) for all \( x \in \mathcal{P}_0^{x,\alpha,\delta} \setminus \mathcal{P}_0^{x,\alpha,\delta} \). (Recall the definition of \( \Gamma_b^{hit}(\ell, m_A, m_B) \) in (5.24).) This happens roughly with probability \( d_A^{m_A} d_B^{m_B} \). We proceed with the details.

Note that \( \Gamma_b^{hit}(\ell, m_A, m_B) = \Gamma_b(\ell, m_A, m_B) \) for \( \ell = 0 \), and thus we may consider fixed \( b \in [b_{max}], \ell \in [b], m_A, m_B \in [\Delta]_0 \) with \( m_A + m_B > 0 \). Note that we extend \( \sigma|_{V(H)} \) to \( \sigma^+|_{V(H)} \) for
every $H \in \mathcal{H}$ by choosing a bijective mapping of $X_0^H \setminus \mathcal{F}_0^\sigma$ into $V_0 \setminus \sigma(X_0^H \cap \mathcal{F}_0^\sigma)$ uniformly and independently at random. To that end, for $a, b \in E(A_{I_0}^H) \cup \bigcup_{j \in J} E(B_j^H)$, $H \in \mathcal{H}$, and $x \in \mathcal{F}_0^\sigma$, let $V_{x,a,b,\sigma} := V_{x,a,b} \setminus \sigma(X_0^H \cap \mathcal{F}_0^\sigma)$. We first estimate $|V_{x,a,b,\sigma}|$. To that end, let $a, b \in E(A_{I_0}^H) \cup \bigcup_{j \in J} E(B_j^H)$, $H \in \mathcal{H}$, and $x \in \mathcal{F}_0^\sigma$ be fixed. For every $v \in V_{x,a,b,\sigma}$, we define a weight function $\omega_v : E(A_j^H) \rightarrow \{0, 1\}$ by $\omega_v(e) := 1\{v \in e\}$ and let $\omega_{v_{x,a,b}} := \sum_{v \in V_{x,a,b}} \omega_v$. Observe that $\omega_{v_{x,a,b}}(M(\sigma))$ counts the vertices $v \in V_{x,a,b,\sigma}$. Hence,

$$
(5.47) \quad |V_{x,a,b,\sigma}| = |V_{x,a,b}| - \omega_{v_{x,a,b}}(M(\sigma)).
$$

Since $A_0^H$ is $(\varepsilon, d_0)$ super-regular, we have

$$
\omega_{v_{x,a,b}}(E(A_0^H)) = (1 \pm \varepsilon) d_0 n |V_{x,a,b}|.
$$

Adding $\omega_{v_{x,a,b}}$ to $W$ and employing (5.15) yields that

$$
\omega_{v_{x,a,b}}(M(\sigma)) = (1 \pm \varepsilon)(1 - \varepsilon^{2/3}) \frac{\omega_{v_{x,a,b}}(E(A_0^H))}{d_0 n} \leq n^\varepsilon = (1 \pm 5\varepsilon)(1 - \varepsilon^{2/3}) |V_{x,a,b}|.
$$

We conclude that

$$
(5.48) \quad |V_{x,a,b,\sigma}| \leq 2\varepsilon^{2/3} |V_{x,a,b}| \leq 3 \varepsilon^{2/3} d_A s_A s_B s_{x,a,b,\sigma} n.
$$

For $a, b \in \Gamma_b(\ell, m_A, m_B)$, let $\mathcal{F}_0^{\sigma,\varepsilon} := \mathcal{F}_0^\sigma \cap \mathcal{F}_0^{\sigma,\varepsilon}$. By the definition of $\Gamma_b(\ell, m_A, m_B)$ in (5.23), we have

$$
\sum_{x \in \mathcal{F}_0^{\sigma,\varepsilon}} |S_{x,a,b,\sigma}| = m_A, \quad \sum_{x \in \mathcal{F}_0^{\sigma,\varepsilon}} |S_{x,a,b,\sigma,b,\sigma}| = m_B.
$$

Hence, by (5.48) and because $|\mathcal{F}_0^{\sigma,\varepsilon}| = \ell$, we obtain

$$
(5.49) \quad \prod_{x \in \mathcal{F}_0^{\sigma,\varepsilon}} |V_{x,a,b,\sigma}| \leq d_A^{m_A} d_B^{m_B} (3 \varepsilon^{2/3} n)^\ell.
$$

By (5.18), we have that $V_0 \setminus \sigma(X_0^H \cap \mathcal{F}_0^\sigma) \geq \varepsilon^{2/3} n/3$. Now, with (5.49) we obtain that the probability that all $\ell$ vertices $x \in \mathcal{F}_0^{\sigma,\varepsilon}$ are mapped onto their target set $V_{x,a,b,\sigma}$ and if $x \in \mathcal{F}_0^{\sigma,\varepsilon}$ then it is not mapped onto $c_0$ — that is, the probability that $a, b \in \Gamma_b(\ell, m_A, m_B)$ is also contained in $\Gamma^{hit}(\ell, m_A, m_B)$ — is at most

$$
\varepsilon^{2/3} n \cdot (\varepsilon^{2/3} n - 1) \cdots (\varepsilon^{2/3} n - \ell + 1)/3^\ell \leq 10^\ell d_A^{m_A} d_B^{m_B}.
$$

Finally, we can derive an upper bound for the expected value of $\omega(\Gamma^{hit}_b(\ell, m_A, m_B))$.

$$
E \left[ \omega \left( \Gamma^{hit}_b(\ell, m_A, m_B) \right) \right] \leq \omega(\Gamma_b(\ell, m_A, m_B)) 10^\ell d_A^{m_A} d_B^{m_B} \sum_{a, b \in \Omega_b} \omega(a, b) + n^{2\varepsilon}.
$$

By using Theorem 3.1 and a union bound, we can establish concentration with probability, say, at least $1 - e^{-n^\varepsilon}$. Thus, we conclude

$$
\omega \left( \Gamma^{hit}_b(\ell, m_A, m_B) \right) \leq 2 \varepsilon^{1/4} d_A^{p_0^2 - p_0 A_2^{2nd}} d_B^{p_0^2 - p_0 B_2^{2nd}} \sum_{a, b \in \Omega_b} \omega(a, b) + 2n^{2\varepsilon}
$$

for all $b \in [b_{\text{max}}]$, $\ell \in [b]$, $m_A, m_B \in |\Delta_0|$, and $m_A + m_B > 0$. By summing over all values of $b, \ell, m_A, m_B$, we obtain

$$
\sum_{b \in [b_{\text{max}}], \ell \in [b], m_A, m_B \in |\Delta_0|} \omega \left( \Gamma^{hit}_b(\ell, m_A, m_B) \right) \leq 2 \varepsilon^{1/4} d_A^{p_0^2 - p_0 A_2^{2nd}} d_B^{p_0^2 - p_0 B_2^{2nd}} \omega \left( E(A_{I_0}) \cup \bigcup_{j \in J} E(B_j) \right) + n^{3\varepsilon},
$$

which is the desired estimate for the second term in (5.25) for the case that $0 \in (V(R) \setminus I) \cup (J \setminus J_{XV})$, and it is the desired estimate for the right hand side in (5.26) for the case that $0 \in J_{XV}$ (where we also allow $b, \ell = 0$ in the summation).
In the case that $0 \in I \setminus J$, we obtain (5.51) with an additional factor of $(d_0n)^{-1}$ since the estimate on $\omega(\Gamma_b(l, m_A, m_B))$ from the previous step yields an additional factor of $(d_0n)^{-1}$.

**Step 8. Concluding (II)_{L5.1} if $0 \notin (V(R) \setminus I) \cup (J \setminus J_{XY})$**

Finally, we can establish (II)_{L5.1} if $0 \notin (V(R) \setminus I) \cup (J \setminus J_{XY})$ by the estimates derived in the Steps 4 and 7. So, let us assume that $0 \notin (V(R) \setminus I) \cup (J \setminus J_{XY})$. Using (5.32) (respectively (5.36) if $0 \in J \setminus J_{XY}$) and (5.51) in our key observation (5.25) yields that

$$\omega_{\text{new}}(E(A_{i,r}^{\text{new}}) \cup \bigcup_{j \in J} E(B_j^{\text{new}})) = (1 + 3\varepsilon^{1/3})^{d_{i,r} + p_0^{A,2nd} + p_0^{B,2nd} - d_{i,r} + p_0^{A,2nd} + p_0^{B,2nd}} - 2n\varepsilon.$$  

(5.52)

We will use (P3) for $\omega(E(A_{i,r}^{\text{new}}) \cup \bigcup_{j \in J} E(B_j^{\text{new}}))$ in order to obtain (II)_{L5.1} from (5.52). For $0 \notin I$ or $0 \in J \setminus J_{XY}$, we have that

$$d_{i,A}^{\text{new}} = \prod_{i \in I_{r_0}} d_{i,A}^{\text{new}} = \prod_{i \in I_{r}} d_{i,A}^{\text{new}}$$

for $d_{i,A}^{\text{new}}$ defined as in (5.4) because $I_{r_0} = I_r$, and similarly

$$d_{j,B}^{\text{new}} = \prod_{j \in J} d_{j,B}^{\text{new}} = \prod_{j \in J} d_{j,B}^{\text{new}},$$

because by the definition in (5.4), we have $b_0 = 0$ and $d_{i,A}^{\text{new}} = d_{i,B}^{\text{new}} = 1$ and $d_{j,A}^{\text{new}} = d_{j,B}^{\text{new}}$. Hence, altogether using (P3) together with (5.52), we obtain

$$\omega_{\text{new}}(E(A_{i,r}^{\text{new}}) \cup \bigcup_{j \in J} E(B_j^{\text{new}})) = (1 + 3\varepsilon^{1/3})^{d_{i,r} + p_0^{A,2nd} + p_0^{B,2nd} - d_{i,r}} - 2n\varepsilon.$$  

(5.53, 5.54)

which establishes (II)_{L5.1} in the case of $0 \notin I$ or $0 \in J \setminus J_{XY}$.

**Step 9. Concluding (II)_{L5.1} if $0 \in I \setminus J$**

Similar as in Step 8, we can establish (II)_{L5.1} if $0 \in I \setminus J$ by the estimates derived in the Steps 4 and 7. So, let us assume that $0 \in I \setminus J$, and recall that we obtained an additional factor of $(d_0n)^{-1}$ in (5.32) and (5.51). Together with our key observation (5.25) this yields that

$$\omega_{\text{new}}(E(A_{i,r}^{\text{new}}) \cup \bigcup_{j \in J} E(B_j^{\text{new}})) = (1 + 3\varepsilon^{1/3})^{d_{i,r} + p_0^{A,2nd} + p_0^{B,2nd} - d_{i,r}}(d_0n)^{-1}\omega(E(A_{i,r}^{\text{new}}) \cup \bigcup_{j \in J} E(B_j^{\text{new}})) \pm 2n\varepsilon.$$  

(5.55)

We will use (P3) for $\omega(E(A_{i,r}^{\text{new}}) \cup \bigcup_{j \in J} E(B_j^{\text{new}}))$ in order to obtain (II)_{L5.1} from (5.55). For $0 \in I \setminus J$, we have that

$$d_{i,A}^{\text{new}} = \prod_{i \in I_{r_0}} d_{i,A}^{\text{new}} = \prod_{i \in I_{r}} d_{i,A}^{\text{new}}$$

for $d_{i,A}^{\text{new}}$ defined as in (5.4) because $d_0 = d_{0,A}$, and similarly

$$d_{j,B}^{\text{new}} = \prod_{j \in J} d_{j,B}^{\text{new}} = \prod_{j \in J} d_{j,B}^{\text{new}}.$$  

(5.56, 5.57)
Hence, altogether using (P3) together with (5.55), we obtain

\[
\omega^{new} \left( E(A_i^{new}) \cup \bigcup_{j \in J} E(B_j^{new}) \right)
\leq \left( 1 + \varepsilon^{1/3} \right) d_A^{b_i} p_i - p_0^{A,2nd} d_B^{b_j} p_j - p_0^{B,2nd} (d_0 n)^{-1}
\left( \left\{ J_{XV} \cap -[r_{0}] = \emptyset \right\} \pm \varepsilon \right) d_A^{p_A_{-[r_{0}]}} \| p_{A_{-[r_{0}]}} \| d_B^{p_B_{-[r_{0}]}} \| p_{B_{-[r_{0}]}} \|
\prod_{i \in I_{r_0}} d_i A \prod_{j \in J} d_j B \frac{\omega_i(\mathcal{D}_{[j]})}{n^{\|J\cap-[r_{0}]\} / J}} + n^{\varepsilon}
\right) + 2n^{2\varepsilon}
\]  

which establishes (II)_{L,1.5.1} in the case of 0 ∈ I \ J.

**Step 10.** Concluding (II)_{L,1.5.1} if 0 ∈ J_{XV}

For the last case that 0 ∈ J_{XV}, we employ the estimate derived in Step 7 in our key observation 5.26. For 0 ∈ J_{XV}, we have that

\[
\begin{align*}
\omega^{new} \left( E(A_i^{new}) \cup \bigcup_{j \in J} E(B_j^{new}) \right) &\leq \varepsilon^{1/3} d_A^{b_i} p_i - p_0^{A,2nd} d_B^{b_j} p_j - p_0^{B,2nd} (d_0 n)^{-1}
\left( \left\{ J_{XV} \cap -[r_{0}] = \emptyset \right\} \pm \varepsilon \right) d_A^{p_A_{-[r_{0}]}} \| p_{A_{-[r_{0}]}} \| d_B^{p_B_{-[r_{0}]}} \| p_{B_{-[r_{0}]}} \|
\prod_{i \in I_{r_0}} d_i A \prod_{j \in J} d_j B \frac{\omega_i(\mathcal{D}_{[j]})}{n^{\|J\cap-[r_{0}]\} / J}} + n^{\varepsilon}
\right) + 2n^{2\varepsilon}
\end{align*}
\]

This concludes the proof of (II)_{L,1.5.1}.

**Step 11.** Checking (I)_{L,1.5.1}

In order to establish (I)_{L,1.5.1}, we fix H ∈ \mathcal{H}, Z ∈ \{A, B\} and i ∈ [r] if Z = A, i ∈ V(R) \ {0} if Z = B, and we may assume that i ∈ N_{R,0} otherwise Z_i^{H,new} = Z_i^H. We will first show that Z_i^{H,new} as defined in (5.17) is (\varepsilon', d_i^{new})-regular by employing Theorem 3.5. In order to do so, we verify that every vertex in X_H has the appropriate degree in Z_i^{H,new} and that most pairs of vertices in X_H have the appropriate common neighbourhood in Z_i^{H,new}. These properties follow easily due to the typicality of G_Z and because Z_i^H is (\varepsilon, q)-well-intersecting. Finally, we show that also each vertex in V_i has the correct degree in Z_i^{H,new} by employing (II)_{L,1.5.1}. Altogether this will imply that Z_i^{H,new} is (\varepsilon', d_i^{new})-super-regular. Since we basically obtain Z_i^{H,new} from Z_i^H by restricting the neighbourhood of every vertex by b_i ≤ \Delta(R) additional (k-1)-sets in G_Z (see (5.60) and (5.61) below), we will obtain directly from (P1) that Z_i^{H,new} is (\varepsilon', q + \Delta(R))-well-intersecting. We proceed with the details.
For every vertex \( y \in X^H_1 \), let
\begin{equation}
\mathcal{S}_y := \{ \phi_0(x) \cup \sigma^+(x) : e \in E_{x,y}, x \in \mathcal{D}_0 \}
\end{equation}
with \( E_{x,y} \) defined as in (5.9). Note that \( \mathcal{S}_y \subseteq \bigcup_{e \in E(R)} \varepsilon_e \cup V_{j \cup \{i\}} \) and \( |\mathcal{S}_y| = b_i \). Since \( G_Z \) is \((\varepsilon, t, d_Z)\) typical with respect to \( R \) by (P1), and since \( Z^H_i \) is \((\varepsilon, d_i, Z)\)-super-regular and \((\varepsilon, q)\)-well-intersecting by (P2), we conclude by the definition of \( Z^H_{i, \text{new}} \) as in (5.17) that for all \( y \in X^H_i \), we have
\begin{equation}
\deg_{Z^H_{i, \text{new}}}(y) = |N_{Z^H_i}(y) \cap N_{G_Z}(\mathcal{S}_y)| = (1 \pm \varepsilon)d_{i,Z}d_{Z}^2|V_i|.
\end{equation}

Note that (5.61) implies in particular that the density of \( X^H_i \) and \( V_i \) in \( Z^H_{i, \text{new}} \) is \( d_{Z}^2(X^H_i, V_i) = (1 \pm \varepsilon)d_{i,Z}d_{Z}^2 \).

We can proceed similarly as for the conclusion (5.61) and obtain that all but at most \( n^{3/2} \) pairs \( \{y, y'\} \in \binom{X^H_i}{2} \) satisfy
\begin{equation}
|N_{Z^H_{i, \text{new}}}(y \land y')| = (1 \pm \varepsilon)(d_{i,Z}d_{Z}^2)^2|V_i|.
\end{equation}
To see (5.62), note that
\begin{equation}
N_{Z^H_{i, \text{new}}}(y \land y') = N_{Z^H}(y \land y') \cap N_{G_Z}(\mathcal{S}_y) \cap N_{G_Z}(\mathcal{S}_{y'}),
\end{equation}
and for all but at most \( 2\Delta n^2 \) pairs \( \{y, y'\} \in \binom{X^H_i}{2} \), we have \( |\mathcal{S}_y \cup \mathcal{S}_{y'}| = 2b_i \) by (5.6). By employing again (P1) and (P2), we obtain (5.62), where we used (4.2) that for all but at most \( 2n \cdot n^{1/4} \varepsilon \) pairs \( \{y, y'\} \in \binom{X^H_i}{2} \) the sets of \((k-1)\)-sets corresponding to \( N_{Z^H_i}(y) \) and \( N_{Z^H_i}(y') \) are disjoint. Hence, all but at most \( 2n^{5/4} \varepsilon + 2\Delta n \leq n^{3/2} \) pairs \( \{y, y'\} \in \binom{X^H_i}{2} \) satisfy (5.62).

We can now easily derive an upper bound for the number of 4-cycles in \( Z^H_{i, \text{new}} \) by (5.62). To that end, note that every pair \( \{y, y'\} \in \binom{X^H_i}{2} \) together with a pair of common neighbours in \( N_{Z^H_{i, \text{new}}}(y \land y') \) forms a 4-cycle in \( Z^H_{i, \text{new}} \). Hence, by (5.62), the number of 4-cycles in \( Z^H_{i, \text{new}} \) is at most
\begin{equation}
C_4(Z^H_{i, \text{new}}) \leq \frac{|X^H_i|^2}{2} \cdot (1 + 2\varepsilon)(d_{i,Z}d_{Z}^2)^4|V_i|^2 + n^{3/2} \cdot n^2
\leq (1 + 3\varepsilon)(d_{i,Z}d_{Z}^2)^4|X^H_i|^2|V_i|^2.
\end{equation}
Thus, we can apply Theorem 3.5 and obtain that
\begin{equation}
Z^H_{i, \text{new}} \text{ is } (\varepsilon', d_i, Z)\text{-regular.}
\end{equation}

For every \( v \in V_i \), in order to control the degree of \( v \) in \( Z^H_{i, \text{new}} \), we define weight functions \( \omega_v : E(Z^H_i) \rightarrow \{0, 1\} \) by \( \omega_v(x, v) := \mathbb{1}\{v = v\} \), and \( \varepsilon_i : \mathcal{D}_i \rightarrow \{0, 1\} \) by \( \varepsilon_i := \mathbb{1}\{x \in X^H_i\} \), and add the 1-edge tester
\[\omega_v, \varepsilon_i, J = J_Z, J_X = \emptyset, J_Y = \emptyset, e = \{v\}, \rho = \{0, 0, 0, 0\}\]
to \( W_{\text{edge}}^v \) for \( J_A := 0 \) and \( J_B := \{i\} \). This is indeed a (general) 1-edge tester satisfying Definition 4.4. In particular, \( p^{A}(x) = p^{A,2\text{nd}}(x) = p^{B}(x) = p^{B,2\text{nd}}(x) = 0 \) for every \( x \in X^H_i \) by Definition 4.2 because the 1st-pattern and 2nd-pattern of a single vertex is always \( 0 \). Since \( Z^H_i \) is \((\varepsilon, d_i, Z)\)-super-regular, we have that
\[\omega_v(E(Z^H_i)) = (1 \pm \varepsilon)d_{i,Z}|X^H_i|\]
Hence in particular, this general edge tester satisfies (P3). By (II)L5.1, we obtain
\[\deg_{Z^H_{i, \text{new}}}(v) = \omega_v^\text{new}(E(Z^H_{i, \text{new}})) = (1 \pm \varepsilon^2)d_{i,Z}^\text{new}|X^H_i| \pm \varepsilon'.\]
Together with (5.61) and (5.63), this implies that \( Z^H_{i, \text{new}} \) is \((\varepsilon', d_{i,Z}^\text{new})\)-super-regular because \( d_{i,Z}^\text{new} = d_{i,Z}d_{Z}^2 \) (see (5.4)). Further, since the neighbourhood of every vertex \( y \in X^H_i \) in \( Z^H_{i, \text{new}} \) is the intersection of a set \( S^\text{new}_y \) of \((k-1)\)-sets in \( G_Z \) (see (5.61)), and every \( y \in X^H_i \) is contained in at
most $n^{1/4+\varepsilon} + \Delta_R$ pairs $\{y, y'\} \in (X^H_y \rightarrow \frac{X^H_y}{2})$ such that $S^\text{new}_i \cap S^\text{new}_i \neq \emptyset$, we also obtain that $Z^H_i \text{new}$ is $(\varepsilon', q + \Delta(R))$-well-intersecting as defined in (4.2). This establishes (I)L5.1.

**Step 12. Checking (III)L5.1**

For every $\omega \in \mathcal{W}_\text{local}$ with $\omega : (E(A_\omega)) \rightarrow [0, s]$, we add $\omega$ to $W$. Hence, (5.16) yields (III)L5.1.

**Step 13. Checking (IV)L5.1**

In order to establish (IV)L5.1, we fix $(\omega, c) \in \mathcal{W}_0$ with $\omega : \mathcal{X}_0 \rightarrow [0, s]$ and $c \in V_0$. By (5.18), we have that $V_0 \setminus \sigma(X^H_\sigma \cap \mathcal{X}_0^\sigma) \geq \varepsilon^{2/3} n/3$ for every $H \in \mathcal{H}$, and thus, the probability for a vertex $x \in \mathcal{X}_0 \setminus \mathcal{X}_0^\sigma$ to be mapped onto $c$ is at most $3/\varepsilon^{2/3} n$. We therefore expect that $\omega \{x \in \mathcal{X}_0 \setminus \mathcal{X}_0^\sigma : \sigma^+(x) = c\} \leq \omega(\mathcal{X}_0)/n^{1-2\varepsilon}$. By an application of Theorem 3.1 and a union bound, we can establish concentration with probability, say at least $1 - e^{-n^c}$. This establishes (IV)L5.1 and completes the proof of Lemma 5.1.

6. Iterative packing

In this section we essentially prove our main result, Theorem 1.3. We prove the following lemma whose statement is very similar because we only require additionally that for every graph $H \in \mathcal{H}$ and every reduced edge $r \in E(R)$, the graph $H[X^R_r]$ is a matching. This reduction can be achieved by an application of Lemma 3.9 and simplifies several arguments; it is presented in the proof of Theorem 1.3 in Section 7.

**Lemma 6.1.** Let $1/n \ll \varepsilon \ll 1/t \ll \alpha, 1/k$ and $r \leq n^{2\log n}$ as well as $d \geq n^{-\varepsilon}$. Suppose $(\mathcal{H}, G, R, X, V)$ is an $(\varepsilon, t, d)$-typical and $\alpha^{-1}$-bounded blow-up instance of size $(n, k, r)$ and $|\mathcal{H}| \leq n^{\kappa}$. Suppose that $\epsilon_R (\mathcal{X}_r) \leq (1 - \alpha)d n^k$ for all $r \in E(R)$, and $H[X^R_r]$ is a matching if $r \in E(R)$ and empty if $r \in (\mathcal{X}_0) \setminus E(R)$ for each $H \in \mathcal{H}$. Suppose $\mathcal{W}_\text{set}, \mathcal{W}_\text{ver}$ are sets of $\alpha^{-1}$-vertex testers and $\alpha^{-1}$-vertex testers of size at most $n^{3\log n}$, respectively. Then there is a packing $\phi$ of $G$ into $H$ such that

(i) $\phi(X^H_i) = V_i$ for all $i \in [r]$ and $H \in \mathcal{H}$;
(ii) $|W \cap \bigcap_{j \in [\ell]} \phi(Y_j)| = |W||Y_1|\cdots|Y_\ell|/n^l \pm \alpha n$ for all $(W, Y_1, \ldots, Y_\ell) \in \mathcal{W}_\text{set}$;
(iii) $\omega(\phi^{-1}(c)) = (1 \pm \alpha) \omega(\mathcal{X}_I)/n^{|I|} \pm n^\alpha$ for all $(\omega, c) \in \mathcal{W}_\text{ver}$ with centres $c$ in $I$.

**Proof of Lemma 6.1.** We split the proof into five steps. In Step 1, we define a vertex colouring of the reduced graph which will incorporate in which order we consider the clusters in turn. In Step 2, we partition $G$ into two edge-disjoint subgraphs $G_A$ and $G_B$. In Step 3, we introduce candidacy graphs and edge testers that we track for the partial packing in Step 4, where we iteratively apply Lemma 5.1 and consider the clusters in turn with respect to the ordering of the clusters given by the colouring obtained in Step 1. We only use the edges of $G_A$ for the partial packing in Step 4 such that we can complete the packing in Step 5 using the edges of $G_B$.

**Step 1. Notation and colouring of the reduced graph**

We will proceed cluster by cluster in Step 4 to find a function that packs almost all vertices of $H$ into $G$. Since we allow $r$ to grow with $n$ and only require that $r \leq n^{\log n}$, we need to carefully control the growth of the error term. Recall that $R_\ast$ is the 2-graph with vertex set $V(R)$ and edge set $\bigcup_{r \in E(R)} (r)$. Let $c : V(R) \rightarrow [T]$ be a proper vertex colouring of $R^\ast_k$ where $T := k^3 \alpha^{-3}$. The colouring naturally yields an order in which we consider the clusters in turn. To this end, we simply relabel the cluster indices such that the colour values are non-decreasing; that is, $c(1) \leq \cdots \leq c(r)$. Note that the sets $(c^{-1}(j))_{j \in [T]}$ are independent in $R^\ast_k$. We choose new constants $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_T, \mu, \gamma$ such that

$$\varepsilon \ll \varepsilon_0 \ll \varepsilon_1 \ll \cdots \ll \varepsilon_T \ll \mu \ll \gamma \ll 1/t \ll \alpha, 1/k.$$ 

For $i, q \in [r]$ and $I \subseteq [r]$, we define counters $c_i(q), c_i(q), m_i(q)$ (see (6.1)–(6.3) below). Our intuition is the following: If we think of $[q]$ as the indices of clusters that have already been embedded, then $c_i(q)$ is the largest colour of an already embedded cluster in the closed neighbourhood of $i$ in $R_\ast$. That is, $c_i(q)$ is the largest colour that is relevant to $i$ after embedding the first $q$ clusters, and $c_i(q)$ will incorporate how to update the error term.
To be more precise, for \( i, q \in [r] \) and an index set \( I \subseteq [r] \) (that is, \( I \subseteq r \in E(R) \)), let
\[
(6.1) \quad c_i(q) := \max \{0 \cup \{c(j) : j \in N_{R,i} \cap [q]\} \};
\]
\[
(6.2) \quad c_I(q) := \max \{0, c_I(q)\}.
\]

Similarly as \( c_i(q) \), we define \( m_i(q) \) as the number of edges in \( R \) that contain \( i \) and where \( k - 1 \) clusters excluding \( i \) have already been embedded. That is, for \( i, q \in [r] \), let
\[
(6.3) \quad m_i(q) := \{|r \in E(R) : i \in r, |r \cap [q] \cap \{i\}| = k - 1\}.
\]

Further, for all \( i \in [r] \) and index sets \( I \subseteq [r] \), we set \( c_I(0) = c_I(0) = m_i(0) := 0 \). For \( q \in [r] \), recall that \( X_q := \bigcup_{H \in H} X_q^H \), and we set
\[
X_q := \bigcup_{\ell \in [q]} X_{\ell}, \quad V_q := \bigcup_{\ell \in [q]} V_{\ell}.
\]

**Step 2. Partitioning the edges of \( G \)**

In order to reserve an exclusive set of edges for the completion in Step 5, we partition the edges of \( G \) into two subgraphs \( G_A \) and \( G_B \). For each edge \( q \) of \( G \) independently, we add \( q \) to \( G_B \) with probability \( \gamma \) and otherwise to \( G_A \). Let \( d_A := (1 - \gamma)d \) and \( d_B := \gamma d \). Using Chernoff’s inequality and a union bound, we can easily conclude that with probability at least \( 1 - 1/n \) it holds that
\[
\text{for all } i \in [r] \text{ and all pairs of disjoint sets } S_A, S_B \subseteq \bigcup_{r \in E(R) : i \in r} V_{\uplus\{i\}} \text{ with } |S_A \cup S_B| \leq t,
\]
we have \( |V_i \cap N_{G_A}(S_A) \cap N_{G_B}(S_B)| = (1 \pm \varepsilon_0)d_A|S_A|d_B|S_B| n \).

Hence, we may assume that \( G \) is partitioned into \( G_A \) and \( G_B \) such that (6.4) holds. In particular, (6.4) implies that \( G_Z \) is \((2\varepsilon_0, t, d_Z)\)-typical with respect to \( R \) for both \( Z \in \{A, B\} \).

**Step 3. Partial packings, candidacy graphs and edge testers**

For \( q \in [r] \), we call \( \phi : \bigcup_{H \in H, i \in [q]} \hat{X}_H^i \rightarrow V_q \) a \( q \)-partial packing if \( \hat{X}_H^i \subseteq \hat{X}_H^i \) and \( \phi(\hat{X}_H^i) \subseteq V_i \) for all \( H \in H, i \in [q] \) such that \( \phi \) is a packing of \( (H[\hat{X}_H^i \cup \ldots \cup \hat{X}_H^i])_{H \in H} \) into \( G_A[V_q] \). Note that \( \phi|_{\hat{X}_H^i} \) is injective for all \( H \in H \) and \( i \in [q] \). For convenience, we often write
\[
\phi^i : \hat{X}_H^i \rightarrow V_i \text{ a cluster-injective extension of } \phi \text{ if } \phi^i \text{ is an extension of } \phi \text{ such that } \phi^i|_{\hat{X}_H^i} \text{ is injective (and thus bijective)} \text{ and } \phi^i(\hat{X}_H^i) = V_i \text{ for all } H \in H, i \in [q]. \text{ Note that we do not even require that } \phi^i \text{ is an embedding of } H[\hat{X}_H^i \cup \ldots \cup \hat{X}_H^i] \text{ for } H \in H. \]

Suppose \( q \in [r] \) and \( \phi_q : X_q^q \rightarrow V_q \) is a \( q \)-partial packing with a cluster-injective extension \( \phi^q \).

We consider two kinds of candidacy graphs as in Definition 4.1: Candidacy graphs \( A^I_J(\phi_q^+) \) with respect to \( \phi_q^+ \) and \( G_A \) for all index sets \( I \subseteq [r] \setminus [q] \), and candidacy graphs \( B^H_J(\phi_q^+) \) with respect to \( \phi_q^+ \) and \( G_B \) for all \( J \subseteq [r] \). The candidacy graphs \( A^I_J(\phi_q^+) \) will be used to extend the \( q \)-partial packing \( \phi_q \) to a \((q+1)\)-partial packing \( \phi_{q+1} \) via Lemma 5.1 in Step 4, whereas the candidacy graphs \( B^H_J(\phi_q^+) \) will be used for the completion in Step 5.

Given \( q, \phi_q^+ \) and a collection \( A^q \) and \( B^q \) of candidacy graphs \( A^I_J(\phi_q^+) \) and \( B^H_J(\phi_q^+) \) for all index sets \( I \subseteq [r] \setminus [q] \) and \( j \in [r] \), we introduce (general) edge testers as in Definition 4.5 to track several quantities during our packing procedure.

To that end, we first define a set \( W_{\text{initial}} \) of tuples \((\omega, J, J_X, J_Y, c, p)\). We also define a super-set \( W_{\text{hit}} \) of \( W_{\text{ver}} \) containing tuples \((\omega, c)\). For every vertex tester \( (\omega_{\text{ver}}, c) \in W_{\text{ver}} \) as in the assumptions of Lemma 6.1 with centres \( c \in V_{\text{ver}} \) for an index set \( I \subseteq [r] \), and for all \( J \subseteq I \) and pairs of disjoint sets \( J_X, J_Y \subseteq J \), all \( p^A, p^{A,2nd}, p^B, p^{B,2nd,2nd} \in [0, 1)^{r_0} \), we define a tuple \((\omega, J, J_X, J_Y, c, p)\) with \( \omega : \mathcal{F}_J \rightarrow [0, \alpha^{-1}], p := (p^A, p^{A,2nd}, p^B, p^{B,2nd}) \), by
\[
(6.5) \quad \omega(x) := 1\{x \in E_H(p, I, J)\} \cdot \omega_{\text{ver}}(x),
\]
and we add this tuple to \( W_{\text{hit}} \). (Recall Definition 4.3 for \( E_H(p, I, J) \). We also add \((\omega_{\text{ver}}, c) \) to \( W_{\text{hit}} \).

Similarly, for every \( r \in E(R) \), all \( J \subseteq r \) and pairs of disjoint sets \( J_X, J_Y \subseteq J \), all \( g \in V_{\text{ver}} \), and all \( p^A, p^{A,2nd}, p^B, p^{B,2nd,2nd} \in [0, 1)^{r_0} \), we define a tuple \((\omega, J, J_X, J_Y, g, p)\) with \( \omega : \mathcal{F}_J \rightarrow [0, 1], p := (p^A, p^{A,2nd}, p^B, p^{B,2nd}) \), by
\[
(6.6) \quad \omega(x) := 1\{x \in E_H(p, J, J_X, J_Y, g)\},
\]
and we add this tuple to \( W_{\text{initial}} \), and we define a tuple \((\omega, q)\) with \(\omega : \mathcal{X}_{\mathfrak{q}1} \rightarrow \{0, 1\}\) by 
\[ \omega(x) := 1 \{ x \in E(H) \} \] and add \((\omega, q)\) to \( W_{\text{hit}} \).

To control the number of unembedded \(H\)-vertices in one graph \( H \) that could potentially be mapped onto a fixed vertex \( v \) during the completion, we define for all \( j \in [r] \), \( H \in \mathcal{H} \), and \( v \in V_j \), a tuple \((\omega, H) \rightarrow \{ j \}, J \in \{ j \}, J_V = \emptyset, e = \{ v \}, (0, 0, 0, 0)\) with \(\omega, H \rightarrow \mathcal{X}_j \rightarrow \{ 0, 1 \}\) by

\[ \omega, H(x) := 1 \{ x \in X_j^H \}, \]

and we add this tuple to \( W_{\text{initial}} \).

For one single graph \( H \in \mathcal{H} \), we also consider tuples with only two centres. That is, for all \( H \in \mathcal{H} \), \( r \in E(R) \), distinct \( j, J \in r \), \( v \in V_j \), \( w \in V_{Jx} \) and \( p^A = p^{A,2nd} = 0, p^B, p^{B,2nd} \in [\kappa^{-1}]_0 \), we define \( I = J := \{ J, J \}, J \in \{ Jx \}, J_V := \emptyset, \) and a tuple \((\omega, J, J_V, v, w, p)\) with \(\omega, H \rightarrow \mathcal{X}_J \rightarrow \{ 0, 1 \}\), \( p := (p^A, p^{A,2nd}, p^B, p^{B,2nd}) \), by

\[ \omega(x) := 1 \{ x \in E(H, p, I, J, x) \subseteq V(H) \}, \]

and we add this tuple to \( W_{\text{initial}} \).

We now define a set \( W_{\text{edge}} = W_{\text{edge}}(\phi_q, \phi^+, A^q, B^q) \) of edge testers with respect to the elements in \( W_{\text{initial}} \). That is, for every \((\omega, J, J_V, v, w, p) \in W_{\text{initial}} \), let \((\omega, J, J_V, v, w, p) \) be the edge tester with respect to \((\omega, J, J_V, v, w, p) \), \((\phi_q, \phi^+, A^q, B^q) \) as in Definition 4.5, and we add \((\omega_q, J, J_V, v, w, p) \) to \( W_{\text{edge}} \).

Step 4. Induction

We inductively prove that the following statement \( S(q) \) holds for all \( q \in [r]_0 \), which will provide a partial packing of \( \mathcal{H} \rightarrow G_A \).

\( S(q) \). For all \( H, \mathcal{H} \), there exists a \( q \)-partial packing \( \phi_q : \mathcal{X}_{\mathfrak{q}1} \rightarrow V_q \) with \( |\mathcal{X}_{\mathfrak{q}1} \cap X^H_1| \geq (1 - \varepsilon_{\mathcal{X}_{\mathfrak{q}1}}(q))n \) for all \( i \in [q] \), and with a cluster-injective extension \( \phi^+_q \) of \( \phi_q \), and for all index sets \( I \subseteq [r] \setminus [q] \), \( I_B \in [r] \), and \( Z, \mathcal{Z} \in \{ (A, A), (B, B) \} \), there exist subgraphs \( Z^H_{\mathfrak{q}1} \) of the candidacy graphs \( Z^H_{\mathfrak{q}1} \) with respect to \( \phi^+_q \) and \( G_Z \) (where \( Z^H_{\mathfrak{q}1} = \cup_{H \in \mathcal{H}} Z^H_{\mathfrak{q}1} \) and \( Z^q \) is the collection of all \( Z^q \) ) such that

(a) \( Z^H_{\mathfrak{q}1} \) is \((\varepsilon_{\mathcal{X}_{\mathfrak{q}1}}(q), d_{Z^q}^{\varepsilon_{\mathcal{X}_{\mathfrak{q}1}}(q)})\)-super-regular and \((\varepsilon_{\mathcal{X}_{\mathfrak{q}1}}(q), c_{\mathcal{X}_{\mathfrak{q}1}}(q)^{1/2})\)-well-intersecting with respect to \( G_Z \) for all \( i \in [r] \setminus [q] \), \( i_B \in [r] \) and \( Z, \mathcal{Z} \in \{ (A, A), (B, B) \} \);

(b) for every edge tester \((\omega_q, J, J_V, v, w, p) \in W_{\text{edge}}(\phi_q, A^q, B^q) \) with centres \( e \in V_{\mathfrak{q}1} \) for \( I \subseteq [r] \), non-empty \( I_q := (I \setminus [q]) \cup J \), patterns \( p = (p^A, p^{A,2nd}, p^B, p^{B,2nd}) \), and with \( J_{XV} := J_X \cup J_V \), we have that

\[ \omega (E(A^q_{I_q} \cup \bigcup_{J \in J} E(B^q_j))) = \left( \{ J_{XV} \cap [q] = \emptyset \} \pm \varepsilon_{\mathcal{X}_{\mathfrak{q}1}}(q) \right) \prod_{Z \in (A, B)} d_{Z^q_i}^p ||p^2|| \]

\[ = \prod_{i \in I_q \setminus J} d_{m_{\mathfrak{q}1}^i(q)} \prod_{j \in J} d_{m_{\mathfrak{q}1}^j(q)} (\frac{\omega, H_j (\mathcal{X}_{I_q}^j) }{n(|I_q| - |I|}) \pm n^{\varepsilon_{\mathcal{X}_{\mathfrak{q}1}}(q)}); \]

(c) for all \( q = \{ v_i, \ldots, v_k \} \), \( h = \{ w_j, \ldots, w_k \} \in E(G_A) \), \( v_k = w_k, I := \{ i_1, \ldots, i_k \} \neq \{ j_1, \ldots, j_k \} := J \), and \( e^* := \max (\varepsilon_{\mathcal{X}_{\mathfrak{q}1}(q), \varepsilon_{\mathcal{X}_{\mathfrak{q}1}(q)}) \), we have that

\[ |E_{g, \phi, \phi^+}(A^q)| \leq \max \left\{ n^{k-|I\cup J|} n^{\varepsilon_{\mathcal{X}_{\mathfrak{q}1}(q)}}, n^{e^*} \right\}; \]

(d) for all \((\omega, e) \in W_{\text{hit}} \) with \( e \in V_{\mathfrak{q}1} \) and all non-empty \( J \subseteq I \), we have that

\[ \omega \left( \{ x \in \mathcal{X}_{I_q} : x \subseteq e \in \mathcal{X}_{I_q} \} \right) \subseteq \mathcal{X}_{I_q} / n |I_q| \leq \mathcal{X}_{I_q} / n |J| + n \varepsilon_{\mathcal{X}_{\mathfrak{q}1}(q)}; \]

(e) \( |(V_i \cap \phi_q (X^H_i)) \subseteq N_G_B(S)| \leq \varepsilon_T |V_i \cap N_G_B(S)| \) for all \( H, \mathcal{H}, i \in [q] \) and \( S \subseteq \bigcup_{c \in E(R)} u \in V_{\mathfrak{q}1 \setminus (c i)} \) with \( |S| \leq \delta; \)

(f) \( |W \cap \bigcup_{c \in [q]} \phi_q (Y_c)| = |W| \leq \| Y | \leq n^2 \pm \alpha^2 n \) for all \( (W, Y_1, \ldots, Y_t) \in W_{\text{set}} \) with \( W \subseteq I_q \), \( i \in [q] \); and \((g) \omega(\phi_q^{-1}(c)) = (1 \pm \varepsilon_T) \omega(\mathcal{X}_{I_q}) / n \left| I_q \right| \pm n^{\varepsilon_T} \) for all \((\omega, e) \in W_{\text{ver}} \) with centres \( e \in V_{\mathfrak{q}1} \) for \( I_q \subseteq [q] \).
Let us first explain $S(q)(a)–(g)$. Properties $S(q)(a)–(c)$ are used to establish $S(q + 1)$ by applying Lemma 5.1. In particular, these properties will imply that the assumptions (P2), (P3) and (P5) are satisfied, respectively, in order to apply Lemma 5.1. Properties $S(q)(b)$ and (e) can be used to control the leftover for the completion in Step 5. Properties $S(r)(f)$ and (g) will imply the conclusions (ii) and (iii) of Lemma 6.1 as we merely modify the $r$-partial packing $\phi_r$ during the completion in Step 5.

We now inductively prove that $S(q)$ holds for all $q \in [r_0]$. The statement $S(0)$ holds for $\phi_0$ and $\phi_0^p$ being the empty function and $Z_{I_Z}^{H,0}$ being complete multipartite $2|I_Z|$-graphs: Clearly, for all $Z \in \{A, B\}$, and all index sets $I_A \subseteq \{r\}$ and $I_B \subseteq \{r\}$, the candidacy graph $Z_{I_Z}^{H,0}(\phi_0^p)$ is complete $2|I_Z|$-partite. For $S(0)(b)$, consider an edge tester $(w, \omega, J, J_X, J_X, e, c, \rho) \in W^\text{edge}_q$, with centres $c \in V_{\text{dit}}$ and note that, by the definition of an edge tester (see Definition 4.5), we have $\omega(E(A_{\text{dit}}) \cup \bigcup_{j \in J} E(B_j)) = \omega(\mathcal{X}_{\text{dit}})$. For $S(0)(c)$, we observe that for all $q = \{v_1, \ldots, v_k\}, \mathcal{H} = \{w_j, \ldots, w_{\ell_q}\} \subseteq E(G_A)$ with $v_k = w_{\ell_q}$, we have that $|E_{g, h, \phi_0}(A_q)| \leq \alpha - 1 e_H(\mathcal{X}_{i_1}, \ldots, \mathcal{X}_{i_k}) \leq \alpha - 1 n_k \leq n^{k+\varepsilon_0}$ because $e_H(\mathcal{X}_{i_1}) \leq (1 - \alpha) n d_k$ and $\Delta(H) \leq \alpha - 1$ for each $H \in \mathcal{H}$ by assumption. (Recall Definition 4.7 of $E_{g, h, \phi_0}(A_q)$.) $S(0)(d)–(g)$ are vacuously true.

Hence, we assume the truth of $S(q)$ for some $q \in [r - 1]$ and let $\phi_q: \mathcal{X}_{\phi_q} \to V_q, \phi_q^p$, and $A^q$ and $B^q$ be as in $S(q)$. Any function $\sigma: \mathcal{X}_{\phi_{q+1}} \to V_{q+1}$ with $\mathcal{X}_{\phi_{q+1}} \subseteq \mathcal{X}_{q+1}$ naturally extends $\phi_q$ to a function $\phi_{q+1} := \phi_q \cup \sigma$ with $\phi_{q+1}: \mathcal{X}_{\phi_{q+1}} \cup \mathcal{X}_{\phi_{q+1}} \to V_{q+1}$.

We now make a key observation based on the Definition 4.1 of candidacy graphs: By definition of the candidacy graphs $A^q_{\phi_{q+1}} = \bigcup_{H \in \mathcal{H}} A^q_{H, q+1}$ where $A^q_{H, q+1} \subseteq A^q_{q+1}(\phi_{q+1}^p)$, if $\sigma$ is a conflict-free packing in $A^q_{\phi_{q+1}}$ as defined in (5.3), then $\phi_{q+1}$ is a $(q + 1)$-partial packing. See also Figure 2 in Section 5.1.

We aim to apply Lemma 5.1 in order to obtain a conflict-free packing in $A^q_{\phi_{q+1}}$. To this end, we consider subgraphs $H_{q+1}, G_{A, q+1}, G_{B, q+1}, R_{q+1}$ of $\mathcal{H}, G_A, G_B, R$, respectively, that consist only of the ‘relevant’ clusters when finding a conflict-free packing in $A^q_{\phi_{q+1}}$. Note that all relevant clusters lie in $N_{R^q}[q + 1]$. That is, by considering all clusters in $N_{R^q}[q + 1]$, we also account for hyperedges $r \in E(R)$ and all $R$-edges that intersect $r$ where $q + 1 \notin r$ but $r \cap r_{q+1} \neq \emptyset$ for some $r_{q+1} \in E(R)$ with $q + 1 \in r_{q+1}$. Let $Q := [q] \cap N_{R^q}(q + 1)$ and for each $Z \in \{A, B\}$, let

$H_{q+1} := \bigcup_{H \in \mathcal{H}} H \bigg[ \bigcup_{i \in N_{R^q}[q + 1]} X_i^H \bigg]$;

$G_{Z, q+1} := G_Z \bigg[ \bigcup_{i \in N_{R^q}[q + 1]} V_i \bigg]$;

$R_{q+1} := R \big[ N_{R^q}[q + 1] \big]$.

Correspondingly, for $(Z, Z') \in \{(A, A), (B, B), (A, B)\}$, we also define a subset $Z^q[R_{q+1}]$ of $Z^q$. Let $Z^q[R_{q+1}]$ be the collection of all candidacy graphs $Z_{I_Z}^{H, q+1}$ for index sets $I_A \subseteq N_{R^q}[q + 1] \setminus \{q\}$, $I_B \subseteq N_{R^q}[q + 1]$. Following the definition of a packing instance in Section 5.1, we observe that

$\mathcal{P} := (H_{q+1}, G_{A, q+1}, G_{B, q+1}, R_{q+1}, A^q[R_{q+1}], B^q[R_{q+1}], \phi_q|_{\mathcal{X}_{\phi_q}}|_{\mathcal{X}_{\phi_q}})$

is a packing instance of size

$(n, k, |N_{R^q}(q + 1) \setminus \{q\}|, |Q|)$. 

Further, we claim that $\mathcal{P}$ is an $(\varepsilon_{c(q+1)} - 1, (c(q + 1) - 1)^{1/2}, t, d)$-packing instance with suitable edge testers $W^\text{edge}_q$, where

$d = (d_A, d_B, (d^{(c(q+1)-1)}_A)_{i \in N_{R^q}[q+1] \setminus \{q\}}, (d^{(c(q+1))}_B)_{i \in N_{R^q}[q+1]}).$ 

To establish this claim, we first make some important observations. By the definition of $c_l(q)$ in (6.1) and $c_l(q)$ in (6.2), we have:

$\text{(6.9) If } i \in N_{R^q}(q + 1), \text{ then } c(q + 1) = c_l(q + 1) > c_l(q) \text{ for every index set } I \subseteq \{r\} \text{ with } i \in I.$

Note that for the inequality in (6.9) we used that an index set $I$ is contained in some hyperedge $r \in E(R)$, and no vertex of a hyperedge $r$ in $R$ has two neighbours in $R^q$ that are coloured alike.
as we have chosen the vertex colouring as a colouring in $R^3$. In particular, we infer from (6.9) that

$$\text{for all } i \in N_{R^c}(q + 1), \text{ we have } \varepsilon_{c(i(q) + 1) - 1} = \varepsilon_{c(i + 1)} \geq \varepsilon_{c(i)} \text{ and } \varepsilon_{c(i(q) - 1)} = \varepsilon_{c(i + 1)}. \quad \text{For (6.10)}$$

every index set $I \subseteq [r] \setminus [q + 1]$ with $I \cap r \neq \emptyset$ for some $r \in E(R)$ and $q + 1 \in r$, we have

$$\varepsilon_{c(i(q) + 1) - 1} = \varepsilon_{c(i(q) + 1)} \geq \varepsilon_{c(i)} \text{ and } \varepsilon_{c(i(q) - 1)} = \varepsilon_{c(i + 1)}.$$  

Similar, by the definition of $m_i(q)$ in (6.3), we have:

$$m_i(q + 1) = m_i(q) + \left| \{ r \in E(R) : \{ q + 1, i \} = r \cap ([r] \setminus [q] \cup \{ i \}) \right|. \quad \text{If } i \in [r] \setminus N_{R^c}(q + 1), \text{ then } m_i(q + 1) = m_i(q).$$

Hence to see that $\mathcal{P}$ is an $(\varepsilon_{c(i(q) + 1) - 1}, (c(q) + 1) - t)^{1/2}$, $t, d)$-packing instance, note that (P1) follows from (6.4), property (P2) follows from $S(q)(a)$, property (P3) follows from $S(q)(b)$, property (P4) holds by the definition of the edge testers in (6.6), and (P5) follows from $S(q)(c)$.

Observe further that by assumption, we have $|H| \leq n^{2k}$, and $e_{H}(\mathcal{P}_{i|r}) \leq (1 - \alpha)dn^k \leq d_An^k$ for all $r \in E(R(q + 1))$. Hence, we can apply Lemma 5.1 to $\mathcal{P}$ with

| parameter | $n$ | $\varepsilon_{c(i(q) + 1) - 1}$ | $\varepsilon_{c(i(q) + 1)}$ | $t$ | $(c(q) + 1) - t)^{1/2}$ | $s$ | $|N_{R^c}(q + 1) \setminus [q]|$ | $|Q|$ |
|-----------|-----|------------------|------------------|----|------------------|----|------------------|-----|
| plays the role of | $n$ | $\varepsilon$ | $\varepsilon'$ | $t$ | $t$ | $q$ | $r$ | $r_0$ |

and with

- local $\alpha$-testers in $W_{local}$ that we will define explicitly in Steps 4.3–4.7 when establishing $S(q + 1)(c)$–(g);
- tuples $(\omega, c)$ in $W_0$ that we will define explicitly in Step 4.4 when establishing $S(q + 1)(d)$;
- edge testers in $W_{edge}$.

Let $\sigma \in \mathcal{P}_{q + 1} \rightarrow V_{q + 1}$ be the conflict-free packing in $A'_{q + 1}$ obtained from Lemma 5.1 with $|\mathcal{P}_{q + 1} \cap X_{q + 1}^H| \geq (1 - \varepsilon_{c(q) + 1})n$ for all $H \in \mathcal{H}$, which extends $\phi_q \rightarrow \phi_q + \sigma$ with $\phi_q + 1 : X^{\phi_q} \cup \mathcal{P}_{q + 1} \rightarrow V_{q + 1}$, and let $M = M(\sigma)$ be the corresponding edge set to $\sigma$ defined as in (5.1). Further, let $\sigma^+$ be the cluster-injective extension of $\sigma$ obtained from Lemma 5.1. Analogously, $\sigma^+$ extends $\phi_q^+$ to a cluster-injective extension $\phi_q^+ = \phi_q^+ \cup \sigma^+$ of $\phi_q^+$ with $\phi_q^+ : X_{q + 1} \rightarrow V_{q + 1}$.

For all $H \in \mathcal{H}$, $(Z, Z) \in \{(A, A), (B, B)\}$, and all index sets $I_A \subseteq [r] \setminus [q + 1]$, $I_B \subseteq [r]$ with $I_R \cap r \neq \emptyset$ for some $r \in E(R)$ with $q + 1 \in r$, let $Z^H_{I_A} \subseteq Z^H_{I_B}(\phi_q^+ |_{X^\phi_q} \cup \sigma^+) = Z^H_{I_B}(\phi_q^+ \cup \sigma^+)$ be the candidacy graph $Z^H_{new} = Z^H_{I_B} \cap X_{q + 1}^H$ obtained from Lemma 5.1 satisfying (I)_{L5,1}–(III)_{L5,1}.

For all $H \in \mathcal{H}$, $(Z, Z) \in \{(A, A), (B, B)\}$, and all index sets $I_A \subseteq [r] \setminus [q + 1]$, $I_B \subseteq [r]$ with $I_R \cap r \neq \emptyset$ for all $r \in E(R)$ with $q + 1 \in r$, note that $m_i(q) = m_i(q + 1)$ for all $i \in I_Z$ and $Z^H_{I_Z} \cap X_{q + 1}^H = Z^H_{I_B}(\phi^+_{q + 1})$. Thus, in such a case we set $Z^H_{I_Z} = Z^H_{I_B}$. Let $Z^H_{I_Z} = \bigcup_{H \in \mathcal{H}} Z^H_{I_Z}$ and let $Z^{**}$ be the collection of all $Z^H_{I_Z}$ for all index sets $I_A \subseteq [r] \setminus [q + 1]$, $I_B \subseteq [r]$. We will employ Lemma 5.1(I)_{L5,1}–(III)_{L5,1} to establish $S(q + 1)(a)$–(g).

**Step 4.1. Checking $S(q + 1)(a)$**

We fix some $H \in \mathcal{H}$, $Z \in \{A, B\}$, and establish $S(q)(a)$ for the candidacy graph $Z^H_{I_Z}$. For all $i_A \in N_{R^c}(q + 1) \setminus [q]$ and $i_B \in N_{R^c}(q + 1)$, we have by our observation (6.11) for $m_i(q) + 1$ that

$$d_{A_{m}}^{m_z(q + 1)} = d_{Z, z}^{m_z(q + 1)} \in (I)_{L5,1} \text{ as defined in (5.4). Hence with (I)_{L5,1}, (6.9) and (6.11), we obtain that }$$

$$Z^{H_{I_Z}}(\phi^+_{q + 1}, d_{Z, z}^{m_z(q + 1)}) \text{-super-regular and } (\varepsilon_{c_{Z}(q + 1)}, c_{Z}(q + 1))^{1/2}\text{-well-intersecting for all } i_A \in N_{R^c}(q + 1) \setminus [q] \text{ and } i_B \in N_{R^c}(q + 1). \text{ For all } I_A \subseteq [r] \setminus (N_{R^c}(q + 1) \cup [q + 1]) \text{ and } I_B \subseteq [r] \setminus N_{R^c}(q + 1), \text{ we have } m_i(q) = m_i(q + 1) \text{ and } Z^{H_{I_Z}} = Z^{H_{I_B}}. \text{ Hence with } S(q)(a), \text{ we also obtain in this case that } Z^{H_{I_Z}} \text{-super-regular and } (\varepsilon_{c_{Z}(q + 1)}, c_{Z}(q + 1))^{1/2}\text{-well-intersecting. This establishes } S(q + 1)(a).$$

**Step 4.2. Checking $S(q + 1)(b)$**

In order to establish $S(q + 1)(b)$, we fix $(\omega_{q + 1}, \omega_I, J, J_X, J_Y, \mu, \rho) \in W_{edge}^{q + 1}(\phi_q^+, \phi_q^+, A^{q + 1}, B^{q + 1})$ with centres $c \in V_{I(r)}$ for $I \subseteq [r]$, non-empty $I_{r + 1} := (I \setminus [q + 1]) \cup J$, and patterns $\rho = (p^A, p_{2nd}, p^B, p_{2nd})$. 

If $I_{q+1} \cap r = \emptyset$ for all $r \in E(R)$ with $q+1 \in r$, then the conflict-free packing $\sigma$ does not have an effect at all on the considered edge tester; that is, $(\omega_q, \omega_i, J, J_X, J_Y, e, p) = (\omega_{q+1}, \omega_i, J, J_X, J_Y, e, p)$ by Definition 4.5, and thus, $S(q+1)(b)$ holds by $S(q)(b)$. In this case, note in particular that if $p$ is such that $\omega_i(X_{\omega_i}) > 0$, then $\|P_{[q]}^p\| = \|P_{[q+1]}^p\|$ and $\|P_{[q]}^{2nd}\| = \|P_{[q+1]}^{2nd}\|$ since all $Z \in \{A, B\}$ by Definition 4.2 of a 1st-pattern and 2nd-pattern we have for the $(q+1)$th entries that $P_{[q+1]}^{2nd} = 0$ as $I_{q+1} \cap r = \emptyset$ for all $r \in E(R)$ with $q+1 \in r$.

Hence, we may assume that $I_{q+1} \cap r = \emptyset$ for some $r \in E(R)$ with $q+1 \in r$. It is important to note that the edge tester $(\omega_{q+1}, \omega_i, J, J_X, J_Y, e, p) \in \mathcal{W}_{edge}^q(\phi_q, \phi_{q+1}, A^q_1, B^q_1)$ is defined in the same way as the edge tester $(\omega_{q+1}, \omega_i, J, J_X, J_Y, e, p)$ that we obtain from (II)_{L5.1} and, thus, they are identical. As in Step 4.1, for $i \in I_{q+1}$, we have by our observation (6.11) for $m_i(q+1)$ that $d_{v_{i+2}}^s(q+1) = d_{v_{i+2}}^{new}$ for $Z \in \{A, B\}$ and $d_{v_{i+2}}^{new}$ in (I)_{L5.1} as defined in (5.4). Hence with (6.10), (6.11) and (II)_{L5.1}, we obtain that the edge tester $(\omega_{q+1}, \omega_i, J, J_X, J_Y, e, p) \in \mathcal{W}_{edge}^q(\phi_q, \phi_{q+1}, A^q_1, B^q_1)$ with respect to $(\omega_i, J, J_X, J_Y, e, p), \phi_q, \phi_{q+1}, A^q_1$ and $B^q_1$ satisfies $S(q+1)(b)$.

**Step 4.3. Checking $S(q+1)(c)$**

In order to establish $S(q+1)(c)$, we fix $g = \{v_{i_1}, \ldots, v_{i_k}\}, h = \{w_{j_1}, \ldots, w_{j_k}\} \in E(G)$ with $v_{i_k} = w_{j_k}, I := \{i_1, \ldots, i_k\} \neq \{j_1, \ldots, j_k\} =: J$, and $\epsilon_q^g := \max(\epsilon_{c(q+1)}^g, \epsilon_{c(q+1)}^h)$. If $q+1 \notin I \cup J$, we have $(I \cup J) \cap [q] = (I \cup J) \cap [q+1]$, and thus $S(q+1)(c)$ holds by $S(q)(c)$.

Hence, by symmetry, we may assume that $q+1 = i_2$ for some $i \in [k]$. Our strategy is to define a weight function that bounds from above the number of elements in $E_{g, h, \phi_q}(A^q)$ that still can be present in $E_{g, h, \phi_q}(A^{q+1})$ by employing (III)_{L5.1}. For $(e, f) \in E_{g, h, \phi_q}(A^q)$, let $x_{i_k} := e \cap X_{i_k}$, and we define a weight function $\omega_{e, f}: E(A^q_1) \to \{0, 1\}$ by $\omega_{e, f}(xv) := 1_{\{xv = x_{i_k}, n_{x_{i_k}}\}}$. Note that $(e, f) \in E_{g, h, \phi_q}(A^{q+1})$ only if $\omega_{e, f}(M) = 1$ as it is necessary that $\sigma$ embeds $x_{i_k}$ onto $v_{i_k}$. Let $\omega_{g, h} := \sum_{(e, f) \in E_{g, h, \phi_q}(A^q)} \omega_{e, f}$. A key observation is that

$$|E_{g, h, \phi_q}(A^{q+1})| \leq \omega_{g, h}(M).$$

By the definition of $\omega_{g, h}$, we have that

$$\omega_{g, h}(E(A^q_1)) = |E_{g, h, \phi_q}(A^q)|.$$  

By adding $\omega_{g, h}$ to $W_{local}$ and by employing (III)_{L5.1}, we obtain with (6.13) that

$$\omega_{g, h}(M) = (1 + \epsilon_{c(q+1)}^q)(d_{v_{i+2}}^{new}(q+1))^{-1} |E_{g, h, \phi_q}(A^q)| + n\epsilon_{c(q+1)}^{-1}. $$

We further observe that

$$|E_{g, h, \phi_q}(A^q)| \leq \max\left\{n^{k-(I \cup J) \cap [q]} + \epsilon_{c(q+1)}^q, n\epsilon_{c(q+1)}^{-1}\right\},$$

and thus by (6.10), (6.14) and because $d \geq n^{-\epsilon}$, we finally obtain

$$\omega_{g, h}(M) \leq \max\left\{n^{k-(I \cup J) \cap [q]} - 1 + \epsilon_{c(q+1)}, n\epsilon_{c(q+1)}^{-1}\right\}. $$

By our key observation (6.12), this establishes $S(q+1)(c)$.

**Step 4.4. Checking $S(q+1)(d)$**

In order to establish $S(q+1)(d)$, we fix $(\omega, e) \in \mathcal{W}_{hit}$ with $e \in V_{hit}$ and $J \subseteq I$. In view of the statement, we may assume that $q+1 \in J$, otherwise $S(q+1)(d)$ holds by $S(q)(d)$. Our general strategy is to define two weight functions and employ (III)_{L5.1} and (IV)_{L5.1} to derive the desired upper bound.

For $p \in (q, q+1)$, let $W_q := \{x \in X_{\omega_i} : x \in e \text{ for some } e \in E(H), \phi^+_p(x \cap X_{\omega_i}(J \cap [p])) \subseteq e\}$, and let $c := e \cap V_{q+1}$. We define a local tester $\omega_{\sigma} : E(A^{q+1}) \to [0, \alpha^{-1}]$ by

$$\omega_{\sigma}(uv) := \sum_{x \in W_q : u \in x} \mathbb{1}(v = e) \omega(x),$$

and we add $\omega_{\sigma}$ to $W_{local}$. We also define a tuple $(\omega_{\sigma}^+, c)$ with $\omega_{\sigma}^+ : X_{q+1} \to [0, \alpha^{-1}]$ by

$$\omega_{\sigma}^+(u) := \sum_{x \in W_q : u \in x} \omega(x),$$

for all $u \in X_{q+1}$. This establishes $S(q+1)(d)$.
and we add \((\omega_\sigma + c)\) to \(W_0\). We make the following observation

\[(6.15) \quad \omega(W_{q+1}) = \omega_\sigma(M) + \omega_\sigma(\{x \in \mathcal{X}_{q+1} : \mathcal{D}^{+}_{q+1}_x : \sigma^+(x) = c\}).\]

We first employ (III)L5.1 to derive an upper bound on \(\omega_\sigma(M)\)

\[
\omega_\sigma(M) \leq 2(d_A^{m_{q+1}(q)} n)^{-1} \omega_\sigma(E(A^{q+1})) + n^{\varepsilon_{(q+1)}-1}.
\]

Next, we employ (IV)L5.1 to derive an upper bound for the last term of (6.15)

\[
\omega_\sigma(\{x \in \mathcal{X}_{q+1} \setminus \mathcal{D}^{+}_{q+1}_x : \sigma^+(x) = c\}) \leq \omega_\sigma(\mathcal{X}_{q+1})/n^{1-\varepsilon_{(q+1)}-1} + n^{\varepsilon_{(q+1)}-1}.
\]

Finally, plugging (6.16) and (6.17) into (6.15), yields that

\[
\omega(W_{q+1}) \leq \omega_\sigma(\mathcal{X}_{q+1})/n^{1-\varepsilon_{(q+1)}-1} + n^{\varepsilon_{(q+1)}}.
\]

Together with (6.10), this establishes \(S(q+1)(d)\).

**Step 4.5. Checking \(S(q+1)(e)\)**

In order to establish \(S(q+1)(e)\), we fix \(H \in \mathcal{H}\) and \(S \subseteq \bigcup_{\ell \in E(R): q_1+1 \in \ell} V_{\ell}\setminus \{q+1\}\) with \(|S| \leq t\).

Let \(W := V_{q+1} \cap N_G_H(S)\). Our general strategy is to define a weight function that estimates the number of vertices in \(W \cap \sigma(X_{q+1}^H)\) from which we can derive an upper bound for \(|W \setminus \sigma(X_{q+1}^H)|\).

Hence, let \(\omega_W : E(A_{q+1}^q) \rightarrow \{0, 1\}\) be defined by \(\omega_W(e) := 1\{w \in e, w \in W\}\). A key observation is that

\[(6.18) \quad |W \setminus \sigma(X_{q+1}^H)| \leq |W| - \omega_W(M).
\]

By the definition of \(\omega_W\) and because \(A_{q+1}^H(q)\) is \((\varepsilon_{c_{q+1}(q)}, d_A^{m_{q+1}(q)})\)-super-regular for every \(H \in \mathcal{H}\) by \(S(q)(a)\), we have that

\[(6.19) \quad \omega_W(E(A_{q+1}^q)) = (1 \pm 3\varepsilon_{c_{q+1}(q)})d_A^{m_{q+1}(q)}n|W|.
\]

By adding \(\omega_W\) to \(W_{local}\) and by employing (III)L5.1, we obtain with (6.19) that

\[
\omega_W(M) \left((III)_{L5.1}\right) = (1 \pm \varepsilon_{q+1}(q))d_A^{m_{q+1}(q)}n|W|.
\]

Now, this together with (6.18) implies that \(|W \setminus \sigma(X_{q+1}^H)| \leq \varepsilon_T|W|\). Together with \(S(q)(e)\) this establishes \(S(q+1)(e)\).

**Step 4.6. Checking \(S(q+1)(f)\)**

Let \((W, Y_1, \ldots, Y_\ell) \in W_{set}\) be a set tester with \(W \subseteq V_{q+1}\) and \(Y_j \subseteq X_{q+1}^H\) for all \(j \in [\ell]\). We define

\[
E(W, Y_1, \ldots, Y_\ell) := \left\{e_1, \ldots, e_\ell \in \bigcup_{j \in [\ell]} E\left(A_{q+1}^H([W, Y_j]) \cap e_j \neq \emptyset\right) \right\}
\]

and a weight function \(\omega_{(W, Y_1, \ldots, Y_\ell)} : E(A_{q+1}^\ell) \rightarrow \{0, 1\}\) by \(\omega_{(W, Y_1, \ldots, Y_\ell)}(e) := 1\{e \in E(W, Y_1, \ldots, Y_\ell)\}\).

Note that

\[(6.20) \quad \omega_{(W, Y_1, \ldots, Y_\ell)}(M) = |W \cap \bigcap_{j \in [\ell]} \sigma(Y_j)|.
\]

In view of the statement, we may assume that \(|W|, |Y_j| \geq \varepsilon_{c_{q+1}} n\) for all \(j \in [\ell]\). Since \(A_{q+1}^H(q)\) is \((\varepsilon_{c_{q+1}(q)}, d_A^{m_{q+1}(q)})\)-super-regular for every \(H \in \mathcal{H}\) by \(S(q)(a)\), we obtain by Fact 3.3 and because
\[ \ell \leq \alpha^{-1} \] that there are at most \( \varepsilon_{c+1}(q)n \) vertices in \( W \) that do not have \( (1 + \varepsilon_{c+1}(q))d^{m+1}(q)_{\ell} \) neighbours in \( Y_j \) for every \( j \in [\ell] \). Hence, we obtain that
\[ \omega(W_{Y_1,\ldots,Y_\ell}(E(A^q_{\ell+1}))) = |E(W_{Y_1,\ldots,Y_\ell})| = (1 + \varepsilon_{c+1}(q))d^{m+1}(q)_{\ell}W|Y_1| \cdots |Y_\ell|. \]

We check that \( \omega(W_{Y_1,\ldots,Y_\ell}) \) is a local tester: For all \( \{e_1,\ldots,e_L\} \subseteq (E(A^q_{\ell+1}))' \subseteq \ell \), the number of edges \( \{e_{L+1},\ldots,e_L\} \) such that \( e : (e_j,j) \in \ell \) with \( \omega(W_{Y_1,\ldots,Y_\ell}) = 2 \) is at most \( 2n^{L-\varepsilon_{c+1}(q)} \), implying that \( \omega(W_{Y_1,\ldots,Y_\ell}) \) is a local tester and we can add \( \omega(W_{Y_1,\ldots,Y_\ell}) \) to \( W_{local} \). By (III)\ref{5.1}, we conclude that
\[ \omega(W_{Y_1,\ldots,Y_\ell}) = (1 + \varepsilon_{c+1}(q))d^{m+1}(q)_{\ell}W|Y_1| \cdots |Y_\ell| \pm \alpha^2 n, \]
which establishes \( S(q+1)(g) \) by (6.20).

Step 4.7. Checking \( S(q+1)(g) \)

In order to establish \( S(q+1)(g) \), let \( \omega_{ver}, \varepsilon \) be a \( q+1 \)-vertex tester with centres \( e = \{c_i\}_{i \in I} \subseteq V_{\ell-1} \) with \( \ell \subseteq |q+1| \) and \( q+1 \in I \). By (6.5), we defined in particular for all \( p^q, p^{A,2nd} \in [\alpha^{-1}]_{0} \) and \( p := (p^q, p^{A,2nd}, 0, 0) \) a tuple \( (\omega_i, j = 0, j_X = 0, j_Y = 0, e, p) \) with initial weight function \( \omega_i = \omega_{ver} \), corresponding to \( (\omega_{ver}, \varepsilon) \). That is,
\[ \omega_{ver} = \sum_{p : p^q, p^{A,2nd} \in [\alpha^{-1}]_{0}} \omega_{i,p} = \omega_{i,p}(\mathcal{X}_{\ell-1}). \]

Note that \( \omega_{i,p}(\mathcal{X}_{\ell-1}) > 0 \) only if \( p_{\mathcal{X}_{\ell-1}} \) is at most \( |p_{\mathcal{X}_{\ell-1}}| \leq k-1 \). For \( p^q, p^{A,2nd} \in [\alpha^{-1}]_{0} \) and \( p := (p^q, p^{A,2nd}, 0, 0) \) with \( ||p^q|| = ||p^{A,2nd}|| \) set \( \tau_p := (\omega_i, j = 0, j_X = 0, j_Y = 0, e, p) \) the edge tester with respect to \( (\omega_i, j = 0, j_X = 0, j_Y = 0, e, p) \), \( (\phi_q, \phi_{\mathcal{X}_{\ell-1}}) \), \( A^q \) and \( B^q \), which is contained in \( W_{edge} \) and \( \phi_q : E(A^q_{\ell+1}) \rightarrow [0, \alpha^{-1}] \), and let \( q, p := \omega_{ver} \). By \( S(q)(b) \), we obtain
\[ \omega_{q,p}(E(A^q_{\ell+1})) = (1 + \varepsilon_{c+1}(q))d^{m+1}(q)_{\ell}W|Y_1| \cdots |Y_\ell| \pm \alpha^2 n, \]
A key observation is that
\[ \omega_{ver} = \sum_{p : p^q, p^{A,2nd} \in [\alpha^{-1}]_{0}} \omega_{i,p}(\mathcal{X}_{\ell-1}). \]
which follows from the definition of the edge tester \( \tau_p \) for \( p = (p^q, p^{A,2nd}, 0, 0) \) as in Definition 4.5. Note that \( A^q_{\ell+1} = (\varepsilon_{c+1}(q))d^{m+1}(q)_{\ell} \)-super-regular by \( S(q)(a) \). For all \( (p^q, p^{A,2nd} \in [\alpha^{-1}]_{0} \) with \( ||p^q|| = ||p^{A,2nd}|| \) and \( p := (p^q, p^{A,2nd}, 0, 0) \), we add \( \omega_{q,p} : E(A^q_{\ell+1}) \rightarrow [0, \alpha^{-1}] \) to \( W_{local} \) and obtain by (III)\ref{5.1} that
\[ \omega_{q,p}(M) = (1 + \varepsilon_{c+1}(q))d^{m+1}(q)_{\ell}W|Y_1| \cdots |Y_\ell| \pm \alpha^2 n, \]

Together with (6.22) and (6.24), this implies that
\[ \omega_{ver} = (1 + \varepsilon_{c+1}(q))d^{m+1}(q)_{\ell}W|Y_1| \cdots |Y_\ell| \pm \alpha^2 n, \]
which establishes \( S(q+1)(g) \).

Step 5. Completion

Let \( \phi_r : \bigcup_{H \in V_r,i} \mathcal{X}_i \rightarrow V_r \) be an \( r \)-partial packing satisfying \( S(r) \) with \( \varepsilon_T, d_r \)-super-regular and \( \varepsilon_T, t^{2/3} \)-well-intersecting candidacy graphs \( B^H_r := B^H \subseteq B^H_r(\phi_r) \) where \( d_i := d^{m+1}(r) = d^{m+1}(r) \) for all \( i \in [r] \) and \( \mathcal{X}_i^{\phi_r} = \bigcup_{H \in V_r,i} \mathcal{X}_i \). We will apply a random packing procedure in order to complete the partial packing \( \phi_r \) using the edges in \( G_r \). Recall that \( \varepsilon_T, \mu, \gamma \ll 1/t \ll \alpha, 1/k \)
and we often call the vertices $\bigcup_{H \in \mathcal{H},i \in [r]} (X_i^H \setminus \hat{X}_i^H)$ unembedded (by $\phi_r$) or the leftover (of $\phi_r$). Our general strategy is as follows. For every $H \in \mathcal{H}$ in turn, we choose a set $Y_{ij}^H \subseteq \hat{X}_i^H$ for all $i \in [r]$ of size roughly $\mu n$ by selecting every vertex uniformly at random with probability $\mu$ and adding $X_i^H \setminus \hat{X}_i^H$ deterministically. Afterwards we apply a random matching argument to pack $H[Y_{ij}^H]$ into $G_B$, which together with $\phi_r$ yields a complete packing of $H$ into $G_A \cup G_B$. Before we proceed with the details of our random packing procedure in Claim 6 (Steps 5.6–5.11), we verify in Claim 4 (Steps 5.3–5.5) that we can indeed pack a subgraph of one single $H \in \mathcal{H}$ into $G_B$ using another random embedding argument as long as our random packing procedure does not deviate too much from its expected behaviour. To that end, we collect some more notation in Step 5.1, and establish several important leftover conditions in Step 5.2.

### Step 5.1. Notation for the completion

We introduce some more notation. We arbitrarily enumerate the graphs in $\mathcal{H}$ and write $\mathcal{H} = \{H_1, \ldots, H_{|\mathcal{H}|}\}$. For $G^0 \subseteq G_B$ and $B^H_i$ with $H \in \mathcal{H}, i \in [r]$,

(6.25) $let (B^H_i)^{G^0}$ be the subgraph of $B^H_i$ where $N_{(B^H_i)^{G^0}}(x) = V_i \cap N_{G_B \setminus G^0}(S_x)$ for every $x \in X_i^H$, and $S_x \subseteq \binom{V(G)}{k-1}$ is the set such that $N_{B^H_i}(x) = N_{G_B}(S_x)$. Note that $S_x$ exists because $B^H_i$ is $(\varepsilon_T, 1^{2/3})$-well-intersecting. This implies in particular that we removed every edge $xv$ from $B^H_i$ for which there exists an edge $e \in E(H)$ such that $\phi_r(e \setminus \{x\}) \cup \{v\} \in E(G^0)$. We may think of $E(G^0)$ as the edge set in $G_B$ that we have already used in our completion step for packing some other graphs of $\mathcal{H}$ into $G_A \cup G_B$. Consequently, $(B^H_i)^{G^0}$ is the subgraph of the candidacy graph $B^H_i$ that only contains an edge $xv$ if we do not use an edge in $G^0$ when we would map $x$ onto $v$. To count the number of removed edges incident to a vertex $v \in V_i$ in $B^H_i$, we define

(6.26) $\rho_{G^0}^H(v) := \left| \{x \in N_{B^H_i}(v) : S \cup \{v\} \in E(G^0) \text{ for some } S \in S_x \} \right|$, where $S_x \subseteq \binom{V(G)}{k-1}$ is the set such that $N_{B^H_i}(x) = N_{G_B}(S_x)$. Note that

(6.27) $\deg_{B^H_i}(v) - \deg_{(B^H_i)^{G^0}}(v) = \rho_{G^0}^H(v)$ for every $v \in V_i, i \in [r]$.

During our random packing procedure we will guarantee that $\rho_{G^0}^H(v)$ is negligibly small (Step 5.10) and that the probability for a $G_B$-edge to be ‘used’ during the completion is appropriately small (at most $\mu^{3/4}$, see Claim 7). To that end, we control certain conditions for the leftover of $\phi_r$ in the next step.

### Step 5.2. Controlling the leftover for the completion

In this step we make some important observations for the completion. In general, we will employ $S(r)(b)$ multiple times in order to suitably control the leftover, that is, the structure of the vertices that are left unembedded by $\phi_r$.

We start with an observation how to control the number of neighbours of a vertex $v$ in $B^H_j$ that are left unembedded by $\phi_r$.

#### Claim 1. $|N_{B^H_j}(v) \cap (X_j^H \setminus \hat{X}_j^H)| \leq 2\varepsilon_T d_j n$ for all $H \in \mathcal{H}, j \in [r]$ and $v \in V_j$.

**Proof of claim:** Recall that we defined edge testers in (6.7) for all $H \in \mathcal{H}, j \in [r]$ and $v \in V_j$ to count the number of neighbours of $v$ in $B^H_j$ that are left unembedded by $\phi_r$. Let $(\omega, \omega_{e,H}, J = \{j\}, J_X = \{j\}, J_Y = \emptyset, e = \{v\}, (0, 0, 0, 0))$ be the edge tester in $W_{edge}(\phi_r, \phi_r^{\top}, A^r, B^r)$ that we obtain from $S(r)(b)$. Definition 4.5 of this edge tester implies that

$|N_{B^H_j}(v) \cap (X_j^H \setminus \hat{X}_j^H)| = \omega(E(B^H_j)) = \omega(E(B^H_j)) \leq 2\varepsilon_T d_j n$.

This establishes Claim 1.
Next, we control the number of neighbours of a vertex \( v \) in \( B_i^H \) that are embedded but lie in an \( H \)-edge that contains unembedded vertices.

**Claim 2.** \( |N_{B_i^H}(v) \cap \mathcal{X}_i^{\phi_e} \cap N_{H_+}(\mathcal{X}_r \setminus \mathcal{X}_r^{\phi_e})| \leq \varepsilon_T^{1/2}d_in \) for all \( H \in \mathcal{H}, i \in [r] \) and \( v \in V_i \).

**Proof of claim:** Our general strategy is as follows. We do not only consider a single vertex \( v \in V_i \) but also a second vertex \( w \in V_j \) for some \( j \in N_{R_i}(i) \). We can use our defined edge testers and employ \( S(r)(b) \) to count for a fixed vertex \( w \) the number of 2-sets \( \{x_i, x_j\} \) where \( x_i \in N_{B_i^H}(v) \) and \( x_j \in X_j^H \) is left unembedded. Hence, by summing over all possible choices of \( w \), we count all such 2-sets but multiple times. Hence, by a double counting argument, for each fixed \( j \) and all choices of \( w \in V_j \), we can establish an upper bound for \( |N_{B_i^H}(v) \cap \mathcal{X}_i^{\phi_e} \cap N_{H_+}(\mathcal{X}_r \setminus \mathcal{X}_r^{\phi_e})| \). In the end, this implies Claim 2 as there are at most \( k\alpha^{-1} \) choices for \( j \). We proceed with the details.

We fix \( H \in \mathcal{H}, i \in [r], v \in V_i, j \in N_{R_i}(i) \) and consider \( w \in V_j \). For all \( \rho = (0,0,\mathbf{p}^B,\mathbf{p}^{B,2nd}) \), we defined a tuple \( (\omega, j = \{i, j\}, J_X = \emptyset, J_V = \emptyset, \{v, w\}, \rho) \) in (6.8). Let \( (\omega, \{i, j\}, J_X, \{v, w\}, \rho) \) be the edge tester in \( \mathcal{W}_i^{edge}(\phi_r, \phi_r^+, A^r, B^r) \) with respect to \( (\omega, j, J_X, \{v, w\}, \rho) \), \( (\phi_r, \phi_r^+, A^r, B^r) \) that we obtain from \( S(r)(b) \). In order to be able to distinguish these edge testers according to the pattern vector \( \rho \), we write \( \omega_{i,p} := \omega_i \) and \( \omega_p := \omega_v \). Note that \( \sum_{\rho} \omega_{i,p} (X_i^H \cup X_j^H) \leq 2\alpha^{-1}n \) by (6.8). Thus, by Definition 4.5 of an edge tester and by summing over all patterns \( \rho = (0,0,\mathbf{p}^B,\mathbf{p}^{B,2nd}) \) with \( \mathbf{p}^B, \mathbf{p}^{B,2nd} \in \{0,1\}^{\lfloor k\alpha^{-1} \rfloor} \), we can employ conclusion \( S(r)(b) \) to count the tuples \( \{x_i, x_j\} \in X_i^H \cup X_j^H \) for which \( x_i \) is left unembedded, \( x_j \) could still be mapped onto \( \{v, w\} \); that is, \( \{x_i, x_j\} \in E(B_i^H) \cup E(B_j^H) \). Note that we count each such tuple \( \{x_i, x_j\} \) multiple times, namely, for every \( w \in V_j \) such that \( \{x_i, v\}, \{x_j, w\} \in E(B_i^H) \cup E(B_j^H) \). By the Definition 4.1 of the candidacy graphs and because \( B_i^H \) is \( (\varepsilon_T, d_j)-super-regular \), there are \( |N_{B_i^H}(x_j)| = (1 \pm 2\varepsilon_T)d_in \) choices for \( w \in V_j \) such that \( \{x_i, v\}, \{x_j, w\} \in E(B_i^H) \cup E(B_j^H) \). Altogether, this implies that

\[
\left| N_{B_i^H}(v) \cap \mathcal{X}_i^{\phi_e} \cap N_{H_+}(\mathcal{X}_r \setminus \mathcal{X}_r^{\phi_e}) \right| \leq ((1 - 2\varepsilon_T)d_j)\sum_{w \in V_j} \omega_{i,p} (E(B_i^r) \cup E(B_j^r)) \leq (1 - 2\varepsilon_T)d_j)n^{-1} \sum_{w \in V_j} \omega_{i,p} (E(B_i^r) \cup E(B_j^r))
\]

\[
\leq (1 - 2\varepsilon_T)d_j)n^{-1}2n \sum_{\rho} \omega_{i,p} (E(B_i^r) \cup E(B_j^r)) \leq 4d_j^{-1}(2\alpha^{-1} - 2\alpha^{-1})d_jn + n^{\varepsilon_T} \leq \varepsilon_T^{2/3}d_in.
\]

Summing over all \( j \in N_{R_i}(i) \) establishes Claim 2.

The last observation for controlling the leftover concerns the number of \( H \)-edges that contain unembedded vertices and could still be mapped onto an edge \( g \in E(G_B) \). We define the following set in a slightly more general way as we will use this definition again in Step 5.11, where we also consider subsets of edges. For all \( r \in E(R), I \subseteq r, e \in V_{ij}, \) non-empty \( J \subseteq I \), and all pairs of disjoint sets \( J_X, J_V \subseteq J \), let

\[
E_{\phi_r}(e, J, J_X, J_V) := \bigcup_{H \in \mathcal{H}} \left\{ x \in X_H^{ij}: x \subseteq e \text{ for some } e \in E(H), e \cap V_{ij}(I \setminus J) \subseteq \phi_r(x), \phi_r(x) \neq e \subseteq e \cap V_{ij}, \phi_r(\mathcal{X}_r^{ij}) \subseteq x \setminus \mathcal{X}_r^{ij} \right\}
\]

\[
\{x \cap X_j^H, e \cap V_j \in E(B_j^H) \} \right\}
\]

Let us first explain this definition in words for the following more special case. For \( I = r \in E(R), e = g \in E(G_B) \), \( J_X, J \) as above, \( E_{\phi_r}(g, J, J_X, J_V) \) is the set of all \( H \)-edges \( e \) for all \( H \in \mathcal{H} \) such that

\begin{enumerate}
  \item \( e \cap X_j^H, g \cap V_j \) is an edge in \( B_j^H \) for every \( j \in J \),
  \item the \( k - |J| \) vertices \( e \cap X_{ij}(r \setminus J) \) are mapped onto \( g \cap V_{ij}(r \setminus J) \),
  \item \( g \cap V_{ij} \) is a subset of the vertices of \( g \) onto which no \( H \)-vertex is embedded by \( \phi_r \), and
  \item all vertices in \( e \) but the \( |J_X| \) vertices \( e \cap X_{ij}(r \setminus J) \) of \( e \) are embedded by \( \phi_r \).
\end{enumerate}
This means, that if we modified \( \phi_r \) and allowed the vertices \( e \cap X^H_{(J \setminus J_X)} \) to be embedded somewhere else, we could potentially embed \( e \) onto \( g \). Note that for an edge \( g \in E(G_B) \), we naturally have that \( \phi_r(e) \neq g \) because \( \phi_r \) maps \( H \)-edges onto \( G_A \)-edges.

Claim 3. Suppose \( \{E_{\phi_r}(g, J, J_X, J_Y)\} \leq \{1\{J_XV = \emptyset\} + 2\varepsilon_T\gamma^{-1}n^{|J|}\sum_{j \in J}d_j \leq 2|J_XV| + 2n^{|J|} \leq 2n^{|J|} \leq 2n^{|J|} \) for all \( r \in E(R), g \in E(G_B[V_{\{J_r\}^i}]), \) non-empty \( J \subseteq r, \) all pairs of disjoint sets \( J_X, J_Y \subseteq J, \) and \( J_XV = J_X \cup J_Y. \) Note that \( \varepsilon_T \gamma^{-1} \leq \epsilon_T^{1/2} \). Hence \( |E_{\phi_r}(g, J, J_X, J_Y)| \leq \varepsilon_T^{1/2}n^{|J|}\sum_{j \in J}d_j \leq |J_XV| \neq \emptyset. \)

Proof of claim: We fix \( r = I, g, J, J_X \), and \( J_Y \) as in the statement of Claim 3, and recall that in (6.6) we defined a tuple \( (\omega, \lambda, J_X, J_Y, g, p) \) for each \( p \in ([k\alpha^{-1}])^4 \). Let \( (\omega, \lambda, J, J_X, J_Y, g, p) \) be the edge tester in \( W^\text{edge}_{\phi_r}(\phi_r^+, \mathcal{A}^*, B^*) \) with respect to \( (\omega, \lambda, J_X, J_Y, g, p), (\phi_r, \phi_r^+), \mathcal{A}^*, \) and \( B^* \) that we obtain from \( S(r)(A) \). In order to be able to distinguish these edge testers according to the patterns \( p \in ([k\alpha^{-1}])^4 \), we write \( \omega_{i, p} := \omega_i \) and \( \omega_{j, p} := \omega_j \). Note that for \( p = (p^A, p^{A,2nd_a}, p^B, p^{B,2nd}) \) such that \( \omega_{i, p}(\mathcal{X}^{(J)}) \geq 0, \) we have that \( \sum_{i \in I} \omega_{i, p}(\mathcal{Y}^{(J)}) = \omega = 0 \). By Definition 4.5 of an edge tester and by summing over all patterns \( p \in ([k\alpha^{-1}])^4 \), we can utilize our general edge testers to count the edges in \( E_{\phi_r}(g, J, J_X, J_Y) \).

Note that the properties (i),(6.28)-(iv),(6.28) of the definition of \( E_{\phi_r}(g, J, J_X, J_Y) \) in (6.28) correspond to the properties (i),(6.45)-(iv),(6.45) of Definition 4.5 for a general edge tester, respectively. Due to (v),(6.45) of Definition 4.5, the edge tester \( (\omega_{i, p}, \omega_{j, p}, J_X, J_Y, p) \) additionally requires for an element \( e \in E_H(p, I, J) \) that \( \phi_r^+(e \cap \mathcal{X}^{(J)}) \cap g \neq \emptyset. \) That is, elements \( e \in E_{\phi_r}(g, J_X, J_Y, \mathcal{Y}^{(J)}) \) are already embedded by \( \phi_r^+ \) during the partial packing procedure, are not counted by any edge tester. However, our intuition is that the number of such edges only yields a minor order contribution, and in fact, we can employ \( S(r)(d) \) to also account for these edges. We make the following observation

\[
|E_{\phi_r}(g, J_X, J_Y)| \leq \sum_{\{J_XV = \emptyset\}} \omega_{i, p}(\mathcal{Y}^{(J)}) + \sum_{j \in J} \left| \left\{ e \in E_H(\mathcal{X}^{(J)}): \phi_r^+(e \cap \mathcal{X}^{(J)}) \subseteq \emptyset \right\} \right|
\]

We obtain an upper bound on the first term in (6.29) by employing \( S(r)(A) \) as follows

\[
\sum_{\{J_XV = \emptyset\}} \omega_{i, p}(\mathcal{Y}^{(J)}) \leq \sum_{\{J_XV = \emptyset\}} \omega_{i, p}(\mathcal{Y}^{(J)}) \leq \sum_{\{J_XV = \emptyset\}} \omega_{i, p}(\mathcal{Y}^{(J)}) \leq \gamma^{-1}n^{|J|} \sum_{j \in J}d_j \leq \gamma^{-1}n^{|J|} \sum_{j \in J}d_j
\]

where we used that \( \sum_{\{J_XV = \emptyset\}} \omega_{i, p}(\mathcal{Y}^{(J)}) \leq d_A n^k \) and \( d_B n^k \leq \gamma^{-1} \).

An upper bound on the second term in (6.29) can be obtained by employing \( S(r)(d) \)

\[
\sum_{j \in J} \left| \left\{ e \in E_H(\mathcal{X}^{(J)}): \phi_r^+(e \cap \mathcal{X}^{(J)}) \subseteq \emptyset \right\} \right| \leq \sum_{j \in J} \left| \left\{ e \in E_H(\mathcal{X}^{(J)}): \phi_r^+(e \cap \mathcal{X}^{(J)}) \subseteq \emptyset \right\} \right| \leq n^{\gamma^{-1}(1-r)} \sum_{j \in J}d_j, \leq n^{|J|} \sum_{j \in J}d_j
\]

Substituting (6.30) and (6.31) in (6.29) establishes Claim 3.

**Step 5.3. Embedding one \( H \) by a random argument – Claim 4**

We proceed with our argument for embedding one subgraph of a fixed graph \( H \in \mathcal{H}. \) Suppose we are given \( Y_i^H \subseteq X_i^H \) for all \( i \in [r]. \) For all \( r \in E(R) \) and \( i \in [r], \) let

\[
E_r^{bad} := \{ e \in E(H[X_i^H]): |e \cap Y_i^H| \leq 2 \}; \quad \mathcal{E}^{bad} := \bigcup_{r \in E(R)} E_r^{bad}; \quad Y_i^{bad} := \{ e \cap Y_i^H: e \in \mathcal{E}^{bad} \}; \quad Y_i^{good} := Y_i^H \setminus Y_i^{bad}
\]

Claim 4. Suppose \( G^0 \subseteq G_B, H \in \mathcal{H}, \) and \( Y_i^H \subseteq X_i^H = X_i^H \) for all \( i \in [r] \) such that the following hold for all \( i \in [r], \) where \( W_i^H := V_i \setminus \phi_r(\mathcal{X}_i^H \setminus Y_i^H): \)

- An upper bound on the second term in (6.29) can be obtained by employing \( S(r)(d) \)

\[
\sum_{j \in J} \left| \left\{ e \in E_H(\mathcal{X}^{(J)}): \phi_r^+(e \cap \mathcal{X}^{(J)}) \subseteq \emptyset \right\} \right| \leq \sum_{j \in J} \left| \left\{ e \in E_H(\mathcal{X}^{(J)}): \phi_r^+(e \cap \mathcal{X}^{(J)}) \subseteq \emptyset \right\} \right| \leq n^{|J|} \sum_{j \in J}d_j
\]

Substituting (6.30) and (6.31) in (6.29) establishes Claim 3.

**Step 5.3. Embedding one \( H \) by a random argument – Claim 4**

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\[
E_r^{bad} := \{ e \in E(H[X_i^H]): |e \cap Y_i^H| \geq 2 \}; \quad \mathcal{E}^{bad} := \bigcup_{r \in E(R)} E_r^{bad}; \quad Y_i^{bad} := \{ e \cap Y_i^H: e \in \mathcal{E}^{bad} \}; \quad Y_i^{good} := Y_i^H \setminus Y_i^{bad}
\]

Claim 4. Suppose \( G^0 \subseteq G_B, H \in \mathcal{H}, \) and \( Y_i^H \subseteq X_i^H \) for all \( i \in [r] \) such that the following hold for all \( i \in [r], \) where \( W_i^H := V_i \setminus \phi_r(\mathcal{X}_i^H \setminus Y_i^H): \)
(A) $X_i^H \setminus \hat{X}_i^H \subseteq Y_i^H \subseteq X_i^H$ and $|Y_i^H| = |W_i^H| = (1 \pm \varepsilon_T^{1/2})\mu n$;
(B) $|E_{r}^{\text{bad}}| \leq \mu^{3/2}n$ for all $r \in E(R)$;
(C) $|N_{B_i^H}(v) \cap Y_i^\text{bad}| \leq \mu^{3/2}d_i n$ for all $i \in [r]$, $v \in W_i^H$;
(D) $(B_i^H)^{G^0}$ is $(\mu^{1/5}, d_i)$-super-regular for all $i \in [r]$;
(E) $(B_i^H)^{G^0} Y_i^H, W_i^H)$ is $(\mu^{1/6}, d_i)$-super-regular for all $i \in [r]$;
(F) $G_B - G^0$ is $(\mu^{1/2}, t, d_B)$-typical with respect to $R$;
(G) for all $S \subseteq \bigcup_{r \in E(R)} : i_r V_{\cup \{i\}}$ with $|S| \leq t$, we have
$$|W_i^H \cap N_{G_B-G^0}(S)| = (\pm \varepsilon_T^{1/2})\mu |V_i \cap N_{G_B-G^0}(S)|.$$
Then there exists a probability distribution of the embeddings $\phi_H$ of $H := H[Y_{[r]}^H]$ into $\tilde{G} := G_B[W_{[r]}^H] - G^0$ with $\phi_H := \phi_{\tilde{H}} \cup \phi_{|V(H) \setminus V(\tilde{H})|}$ such that
(H) $\phi_H$ is an embedding of $H$ into $G$ where $\phi_H(X_i^H) = V_i$ for all $i \in [r]$;
(J) $\phi_H$ embeds all $H$-edges that contain a vertex in $Y_{[r]}^H$ onto an edge in $G_B - G^0$;
(K) $\mathbb{P}[\phi_H(e \cap X_{[r]}^H) = \{v_i\}_{i \in I}] \leq \prod_{i \in I : e \cap Y_i^H \neq \emptyset} 2(\mu d_i n)^{-1}$
for all $m \in [k]$, index sets $I \in \binom{[r]}{m}$, $\{v_i\}_{i \in I} \in V_{\cup I}$, and $H$-edges $e$ that contain a vertex in $Y_{[r]}^H$.

Proof of claim: We split the proof of Claim 4 into two parts, Step 5.4 and 5.5. In Step 5.4, we greedily embed all the $H$-edges of $\mathcal{E}^{\text{bad}}$ into $G_B - G^0$ by considering the clusters in turn. Afterwards, in Step 5.5, we are left with the $H$-edges in $\mathcal{E}^{\text{good}}$, that is, where only a single vertex is not yet embedded. The assumptions (B) and (C) will guarantee that we only used few edges in $G_B - G^0$ in Step 5.4, such that we merely have to modify our candidacy graphs. Hence, we can easily find a perfect matching in each of these candidacy graphs to embed the $H$-edges of $\mathcal{E}^{\text{good}}$, which will complete the embedding of $H$. This matching procedure can be performed independently for each cluster as the $H$-edges in $\mathcal{E}^{\text{good}}$ only contain a single vertex that is not yet embedded. Clearly, this approach will establish (H), and (J).

In both steps, we will embed the vertices of $H$ by a random process in order to establish (K). We do so by making several random choices sequentially which naturally yields a probability distribution. Since some of these choices lead to instances that do not yield a valid or good embedding, we will discard some of these instances and say the random procedure terminates with failure in these cases. We will show that the proportion of choices with failure instances is exponentially small, that is, the probability that the random procedure terminates with failure is exponentially small in $n$. This allows us to discard some of these choices / instances and to restrict our probability space to the remaining ‘nice’ outcomes. Since the failure probability is exponentially small in $n$, this does not have a significant effect on the probability in (K).

Step 5.4. Proof of Claim 4 - Embedding the $H$-edges in $\mathcal{E}^{\text{bad}}$

Suppose $\phi_{[q]}^{\text{bad}} : Y_{[q]}^{\text{bad}} \cup [q] \rightarrow W_{[q]}^H$ is an injective function. Similarly as we defined candidacy graphs in Definition 4.1, we will also define updated candidacy graphs of $B_i^H$ with respect to $\phi_{[q]}^{\text{bad}}$ for $i \in [r] \setminus [q]$. To that end, for all $i \in [r] \setminus [q]$ and $B \subseteq B_i^H$, let $B(\phi_{[q]}^{\text{bad}})$ be the spanning subgraph of $B$, where we keep the edge $xv$ of $B$ in $B(\phi_{[q]}^{\text{bad}})$ if all $e \in \mathcal{E}^{\text{bad}}$ with $e \cap Y_{\cup [q]}^H = \{x\}$ satisfy that
$$\phi_{|V(H) \setminus V(\tilde{H})|}(e \setminus V(\tilde{H})) \cup \phi_{[q]}^{\text{bad}}(e \cap Y_{\cup [q]}^H) \cup \{v\} \in E(G_B - G^0).$$
Observe that (6.32) is very similar to (4.1) in Definition 4.1. For all $q \in [r]$, $i \in [r] \setminus [q]$, $y \in Y_i^{\text{bad}}$, let
$$b_q(y) := \left\{ e \in \mathcal{E}^{\text{bad}} : e \cap Y_{\cup (i \setminus [q])}^H = \{y\} \right\}.$$
That is, $b_q(y)$ is the number of edges $e$ in $\mathcal{E}^{\text{bad}}$ containing $y$ whose $(k-1)$-set $e \setminus \{y\}$ has already been embedded by $\phi_{|V(H) \setminus V(\tilde{H})|} \cup \phi_{[q]}^{\text{bad}}$.

We make one more observation and fix $y \in Y_{q+1}^{\text{bad}}$. Since $B_{q+1}^H$ is $(\varepsilon_T, d_B^{\log N(q+1)})$-super-regular and $(\varepsilon_T, t^{2/3})$-well-intersecting by $S(r)(a)$, we have that for all $r \in E(R)$ and $e \in E_{\text{bad}}^r$ with
\[ \varepsilon \cap Y_{[r] \cup [q]}^{bad} = \{ y \} \], there exists a \((k - 1)\)-set \( S_r = \phi^+_r(\varepsilon \setminus \{ y \}) \in V_{[r] \setminus \{ q + 1 \}} \), as well as there exists a set \( S_y \) of \((k - 1)\)-sets with \( S_r \subset S_y \) and \( |S_y| = \deg_R(q + 1) \) such that \( N_{B_{q+1}}^H(y) = V_{q+1} \cap N_{G_B}(S_y) \) and \( |N_{B_{q+1}}^H(y)| = (1 + \varepsilon_T)d_B S_y | n \). (See (4.2) for the definition of well-intersecting.) Note that \( \phi^+_r \) embeds \( \varepsilon \setminus \{ y \} \) onto \( S_r \), but \( \phi^+_r|_{V(H) \setminus V(\tilde{H})} \) does not. Hence, \( S_r \) only serves as a ‘dummy’ \((k - 1)\)-set for updating the candidacy graph \( B_{q+1} \) conveniently and to artificially restrict the candidates for \( y \). That is, \( V_{q+1} \cap N_{G_B}(S_y \setminus \{ S_r \}) \) are also suitable candidates where we could embed \( y \), assuming that \( \varepsilon \setminus \{ y \} \) has not been embedded yet. Let \( S_y^{dummy} \) be the set of all these \((k - 1)\)-sets \( S_r \) for \( y \), and note that \( |S_y^{dummy}| = b_q(y) \).

During our process of embedding the \( H \)-edges in \( \varepsilon^{bad} \), we will have to drop this artificial restriction of the candidate sets. To that end, suppose we are given a spanning subgraph \( B \subseteq B_{q+1} \), \( y \in Y_{q+1}^{bad} \), and there exists a set \( S'_y \) of \((k - 1)\)-sets such that we can write
\[ N_B(y) = V_{q+1} \cap N_{G_B - G^o}(S'_y). \]

Then let \( B^{\supset dummy} \) be the spanning supergraph of \( B \) where the neighbourhood of each vertex \( y \in Y_{q+1}^{bad} \) is given by
\[ (6.33) \quad N_{B^{\supset dummy}}(y) = V_{q+1} \cap N_{G_B - G^o}(S'_y \setminus S_y^{dummy}). \]

We inductively prove that the following statement \( C1(q) \) holds for all \( q \in [r]_0 \), which will extend \( \phi^+_q|_{V(H) \setminus V(\tilde{H})} \) by embedding the edges in \( \varepsilon^{bad} \) into \( G_B - G^o \). To that end, we define a set of good pairs of vertices.

For every \( i \in [r] \), let \( Y_i^{good} \subseteq (\varepsilon^{good}) \) be the set containing all pairs \( \{ y, y' \} \) with \( y \in Y_i^{good} \subseteq \bigcup_{r \in E(\tilde{H}) : i \in r} \bar{V}_{[r] \setminus \{ i \}} \) is such that \( N_{H_i}^H(x) = V_i \cap N_{G_B}(S_x) \) for each \( x \in \{ y, y' \} \).

We note for future reference that
\[ (6.35) \quad \left| \left( \frac{Y_i^{good}}{2} \right) \setminus Y_i^{good} \right| \leq 2n \cdot n^{1/4 + \varepsilon_T} \leq n^{4/3}, \]

since \( B^H_i \) is \((\varepsilon_T, t^{2/3})\)-well-intersecting as defined in (4.2).

\( C1(q) \). There exists a probability distribution of the injective functions \( \tilde{\phi}_q^{bad} : Y^{bad}_{[q]} \to W_H \) with \( \phi'_q := \tilde{\phi}_q^{bad} \cup \phi_r|_{V(H) \setminus V(\tilde{H})} \) such that

(1) \( C4 \) \( \phi'_q \) is an embedding of \( H'_q := H[Y^{bad}_{[q]} \cup (V(H) \setminus V(\tilde{H}))] \) into \( G \);

(II) \( C4 \) all edges in \( H[Y^{bad}_{[q]} \cup (V(H) \setminus V(\tilde{H}))] \) that contain a vertex in \( Y^{bad}_{[q]} \) are embedded on an edge in \( G_B - G^o \);

(III) \( C4 \) for every vertex \( x \in Y_i^{good} \) and \( i \in [q] \), there are at most \( \mu d_i n \) vertices \( y \in Y^{bad}_{[q]} \) with \( \phi_q^{bad}(y) \in N_{B_{q+1}}^H(x) \);

(IV) \( C4 \) \( \mathbb{P}[\phi'_q(\varepsilon \cap X_{H_i}^H) = \{ v_1 \}_{i \in I}] \leq \prod_{i \in I : \varepsilon \cap Y_i^{H} \neq \emptyset} 2(\mu d_i n)^{-1} \)

for all \( r \in E(\tilde{H}), m \in [k], I \in (m) \), \( \{ v_1 \}_{i \in I} \in V_{[r]} \) and \( e \in E^{bad} \).

The statement \( C1(0) \) clearly holds for \( \phi_0^{bad} \) being the empty function. Hence, we assume the truth of \( C1(q) \) for some \( q \in [r - 1]_0 \). Our strategy to establish \( C1(q + 1) \) is as follows. We extend the probability space given in \( C1(q) \) by making further random choices. For \( \phi_q^{bad} \) as in \( C1(q) \), we aim to find a matching \( \sigma_{q+1}^{bad} : Y^{bad}_{q+1} \to W_{q+1}^H \) in a suitable candidacy graph between \( Y^{bad}_{q+1} \) and \( W_{q+1}^H \) that extends \( \phi_{q+1}^{bad} \) to \( \tilde{\phi}_{q+1}^{bad} := \phi_q^{bad} \cup \sigma_{q+1}^{bad} \). If we can find such a matching \( \sigma_{q+1}^{bad} \), then \( C1(q + 1)(I) \) \( C4 \) and \( (II) \) \( C4 \) will hold by the definition of this suitable candidacy graph. In particular, we will find \( \sigma_{q+1}^{bad} \) by a random procedure to also ensure (IV) \( C4 \). We will discard an exponentially small proportion of random choices during this procedure in order to satisfy (III) \( C4 \) and to obtain a suitable embedding \( \phi_{q+1}^{bad} \).

Let us describe this suitable candidacy graph. We will choose \( \sigma_{q+1}^{bad} \) randomly in
\[ \tilde{B} := (B_{q+1}^H)^{G^o}(Y_{[q]}^{bad} \supset dummy[Y_{q+1}^{bad}, W_{q+1}^H]). \]
That is, $\tilde{B}$ arises from $B^H_{q+1}$ as follows.

- First, we restrict the candidate sets in $B^H_{q+1}$ to those edges whose corresponding edges in $G_B$ have not been used in $G^\circ$ for packing graphs $H_1, \ldots, H_n$ in previous rounds.
- Second, we restrict the candidate sets with respect to the packing $\phi^q_{bad}$ of the vertices in $Y_{bad}^q$ according to (6.32).
- Third, we drop the restriction of the candidate sets to the dummy $(k-1)$-sets as in (6.33).
- In the end, we consider the induced subgraph of this candidacy graph on the bad vertices $Y_{bad}^q$ and the vertices $W_{H}^q$ that can be used for the completion.

For the sake of a better readability, let $B := (B^H_{q+1})^{G^\circ}$.

To guarantee the existence of $\sigma^q_{bad}$, we will show that the degree of every vertex $y \in Y_{bad}^q$ is sufficiently large in $B$ and also in $\tilde{B}$. Let $y \in Y_{bad}^q$ be fixed. Since $B^H_{q+1}$ is $(\varepsilon_1, t^{2/3})$-well-intersecting by $S(r)(a)$, there exists a set $S_y \subseteq \bigcup_{r \in E(\tilde{B})} q+1 \in E_{H} y \setminus \{q+1\}$ of $(k-1)$-sets with $|S_y| \leq t^{2/3}$ such that $N_{B^H_{q+1}} (y) = V_{q+1} \cap N_{G_B} (S_y)$. Hence, by the definition of $B = (B^H_{q+1})^{G^\circ}$ in (6.25), we have

$$N_B (y) = V_{q+1} \cap N_{G_B} - G^\circ (S_y), \text{ and } \deg_B (y) = (1 \pm \mu^{1/3})d_{q+1}n,$$

because $B$ is $(\mu^{1/5}, d_{q+1})$-super-regular by (D)$_{C4}$. Of course we have to restrict the potential images of $y$ according to the vertices we already embedded by $\phi^q_{bad}$. To this end, let

$$S^q_{y} := \left\{ \phi^q_{y} (\varepsilon \cap Y_{bad}^q) \cap \phi^q_{y} \cap e \mid V_{(H)} \setminus \{H\} : e \in E_{bad}^q, e \cap \varepsilon \cap Y_{bad}^q \cap \mu n \right\},$$

and note that $|S^q_{y}| = b_q(y)$. By the definition of the candidacy graph $B(\phi^q_{bad})$ in (6.32) and by (6.36), we obtain

$$N_B (\phi^q_{bad}) (y) = V_{q+1} \cap N_{G_B} - G^\circ (S_y \cup S^q_{y}),$$

and thus, together with (F)$_{C4}$ and (6.36), we have that

$$\deg_B (\phi^q_{bad}) (y) = (1 \pm \mu^{1/6})d_{q+1}b_q (y)n.$$

Since $|S^q_{y}| = b_q(y)$, this implies

$$\deg_B (\phi^q_{bad}) \cap S^q_{y} = (V_{q+1} \cap N_{G_B} - G^\circ (S_y \cup S^q_{y})) = (1 \pm \mu^{1/6})d_{q+1}n.$$

Now, we obtain

$$N_B (y) = W_{q+1} \cap N_{G_B} - G^\circ (S_y \cap S^q_{y}),$$

and thus, by (G)$_{C4}$ and (6.37), we have that

$$\deg_B (y) = (1 \pm \mu^{1/6})d_{q+1}b_q (y)n.$$

In order to guarantee (IV)$_{C4}$, we find $\sigma^q_{bad}$ via the following random procedure:

- for every vertex $y \in Y_{bad}^q$ in turn, we choose a neighbour in $N_B (y)$ uniformly at random among all neighbours that have not been chosen in previous turns;
- we terminate the random procedure with failure at some step of the procedure, say at the turn of some vertex $y \in Y_{bad}^q$,
  - if we have less than $2(\mu d_{q+1}n)/3$ choices to select an image for $y$ in $N_B (y)$, or
  - if there is a vertex $x \in Y_{good}^q$ such that (III)$_{C4}$ is violated, that is, we mapped in previous turns already more than $\mu^{1/3}d_{q+1}n$ vertices of $Y_{bad}^q$ into $N_{B^H_{q+1}} (x)$.

We show in the following claim that this random procedure terminates with failure only with exponentially small probability. If the procedure does not terminate with failure, we obtain a random $Y_{bad}^q$-saturating matching $\sigma^q_{bad} : Y_{bad}^q \rightarrow W_{q+1}^H$ in $\tilde{B}$, which by definition of the candidacy graph $B^H_{q+1}$ and $\tilde{B}$ implies (C1)$(q+1)$(I)$_{C4}$–(III)$_{C4}$. Further, $\sigma^q_{bad}$ satisfies the following.

$$\text{For all } y \in Y_{bad}^q, w \in W_{q+1}^H, \text{ we have that } \sigma^q_{bad} (y) = w \text{ with probability at most } 2(\mu d_{q+1}n)^{-1}.$$

Hence, the following claim together with (C1)$(q+1)$(IV)$_{C4}$ establishes (C1)$(q+1)$(IV)$_{C4}$. 


Claim 5. The random procedure for computing $\sigma_{q+1}^{bad}$ terminates with failure with probability at most $e^{-n^{1/2}}$.

Proof of claim: Let $Y_{q+1}^{bad} = \{y_1, \ldots, y_m\}$ and we consider every vertex in turn, that is, $y_{\ell+1}$ will be treated after $y_{\ell}$. For all $x \in Y_H^{q+1}$ and $\ell \in [m]$, let $\xi(\ell, x)$ be the random variable that counts the number of covered neighbours so far, that is, the number of vertices $v \in N_{B_H^{q+1}}(x)$ such that $\sigma_{q+1}^{bad}(y) = v$ for some $y \in \{y_i\}_{i \in [\ell]}$. We say the random procedure fails at step $\ell \in [m]$, if $\ell$ is the smallest integer such that

- $\mathbb{P}(\xi(\ell, x) > \frac{1}{2}d_{q+1} \mu n)$

We show that the random procedure fails at some step $\ell \in [m]$ with probability at most $e^{-n^{2/3}}$. A union bound then establishes Claim 5.

We fix $\ell \in [m]$ and a vertex $z \in Y_H^{q+1} \cup \{y_i\}_{i \in [m]} \setminus [\ell]$. By employing (6.38), (6.39) and that $B_H^{q+1}$ is $(\varepsilon_T, t^{2/3})$-well-intersecting, we obtain

$$|N_{B}(y \land z)| = (1 + \frac{1}{\mu})d_{q+1} \mu n$$

for all but at most $n^{1/3}$ vertices $y \in \{y_i\}_{i \in [\ell]}$.

Further, at the turn of each $y_i$, $i \in [\ell]$, we have at least

$$|N_{B}(y_i)| - \mu^{2/3}d_{q+1} \mu n \geq (1 - 2\mu^{1/3})d_{q+1} \mu n$$

choices for the embedding of $y_i$. Hence, by (6.41) and (6.42), the probability that a vertex $y_i$, $i \in [\ell]$ which satisfies (6.41) is mapped into $N_{B}(y)$ is at most

$$\frac{(1 + \mu^{1/3})d_{q+1} \mu n}{(1 - 2\mu^{1/3})d_{q+1} \mu n} \leq 2d_{q+1}.$$

This implies that

$$\mathbb{E}(\xi(\ell, x)) \leq 2d_{q+1} + n^{1/3} \leq 2d_{q+1}Y_{q+1}^{bad} + n^{1/3} \leq 3d_{q+1}^{1/2}n,$$

where we used that $|Y_{q+1}^{bad}| \leq n^{1/3}$ by (B)C4. Hence, Theorem 3.1 implies that $\xi(\ell, x) > \mu^{2/3}d_{q+1} \mu n$ with probability, say, at most $e^{-n^{2/3}}$ for some $z \in Y_{q+1}^{good} \cup \{y_i\}_{i \in [m]} \setminus [\ell]$. Thus, the random procedure fails at step $\ell$ with probability at most $e^{-n^{2/3}}$. A simple union bound completes the proof of Claim 5.

Step 5.5. Proof of Claim 4 - Embedding the $H$-edges in $\mathcal{E}_{\text{good}}$

Let $\phi_r: Y_{\cup [r]}^{\text{good}} \to W_H^{\text{good}} \cup [r]$ and $\phi_r' = \phi_r\mid_{V(H) \setminus V(B)}$ be as in C1(r) obtained in Step 5.4. Recall that for all $i \in [r]$, we have $Y_i^{\text{good}} = Y_i^{H} \setminus Y_i^{bad}$ and let $W_i^{\text{good}} := W_i^{H} \setminus \phi_r'(Y_i^{bad})$, and thus clearly, $|Y_i^{\text{good}}| = |W_i^{\text{good}}|$. For every $i \in [r]$, we aim to embed the vertices $Y_i^{\text{good}}$ onto $W_i^{\text{good}}$ by finding a perfect matching in $(B_i^{H})^{G_{\mathcal{E}}}$.

Let $i \in [r]$ be fixed. By (E)C4, we have that $\tilde{B}_i := (B_i^{H})^{G_{\mathcal{E}}}[Y_i^{H}, W_i^{H}]$ is $(\mu^{1/6}, d_i)$-super-regular. We show that for every vertex in $Y_i^{\text{good}} \cup W_i^{\text{good}}$, we only removed few incident edges when take the subgraph $B_i^{good} := \tilde{B}_i[Y_i^{\text{good}}, W_i^{\text{good}}]$ of $\tilde{B}_i$. For a vertex $v \in W_i^{\text{good}}$, we removed at most $\mu^{3/2}d_i\nu$ incident edges by (C)C4. For a vertex $x \in Y_i^{\text{good}}$, we removed at most $\mu^{3/2}d_i\nu$ incident edges by (III)C4. Hence, by employing Fact 3.4, we obtain that $B_i^{\text{good}}$ is $(\mu^{1/10}, d_i)$-super-regular for every $i \in [r]$.

Our strategy is to apply Lemma 3.7 that allows us to find a regular spanning subgraph of $B_i^{\text{good}}$ from which we can easily take a random perfect matching. In order to satisfy the assumptions of Lemma 3.7, we show that the common neighbourhood of most of the pairs in $Y_i^{\text{good}}$ is also not too large in $B_i^{\text{good}}$. To that end, we fix a pair of vertices $\{y, y'\} \in Y_i^{\text{good}}$ pairs as defined in (6.34) and let $\mathcal{S}_y, \mathcal{S}_{y'}$ be the sets of $(k-1)$-sets such that $\mathcal{N}_{B_i^{H}}(x) = V_i \cap N_{G_B}(\mathcal{S}_x)$ for each $x \in \{y, y'\}$. We have that

$$N_{B_i}(y \land y') = W_i^{H} \cap N_{(B_i^{H})^{G_{\mathcal{E}}}}(y \land y') = W_i^{H} \cap N_{G_B - G_{\mathcal{E}}}(\mathcal{S}_y \cup \mathcal{S}_{y'}).$$
Hence, we obtain
\[
|N_{B_i}^{\text{good}}(y \land y')| \leq |N_{B_i}(y \land y')| \leq (d_B^*)^2 n,
\]
where we used for the last equality that \(S_y \cap S_{y'} = \emptyset\) since \((y, y') \in Y_i^{\text{good}}\pairs\), and thus, \(d_B^{\|S_y \cup S_{y'}\|} = d_B^2\). Hence, (6.35) implies that all but at most \(n^{1/3}\) pairs of vertices in \(Y_i^{\text{good}}\) satisfy (6.44).

Finally, we can apply Lemma 3.7 and obtain a spanning \(\mu_d/2\)-regular subgraph of \(B_i^{\text{good}}\).

In particular, \(B_i^{\text{good}}\) contains \(\mu_d n/2\) edge-disjoint perfect matchings, from which we choose one perfect matching \(\sigma_i^{\text{good}}\): \(Y_i^{\text{good}} \rightarrow W_i^{\text{good}}\) for each \(i \in [r]\) uniformly and independently at random.

We crucially observe:
\[
(6.45) \quad \text{For all } i \in [r], y_i \in Y_i^{\text{good}}, w_i \in W_i^{\text{good}}, \text{ we have that } \sigma_i^{\text{good}}(y_i) = w_i \text{ with probability at least } 2(m\mu_d)^{-1}.
\]

Further, since all the vertices in \(H_s[Y_i^{\text{good}}\cup[r]]\) are isolated, we have that \(\phi^{\text{good}} := \bigcup_{i \in [r]} \sigma_i^{\text{good}}\) is an injective function \(\phi^{\text{good}}: Y_{[r]}^{\text{good}} \rightarrow W_{[r]}^{\text{good}}\) such that \(\phi_i[V(H) \setminus V(\tilde{H})] \cup \phi^{\text{good}}\) is a random packing of \(H[Y_i^{\text{good}} \cup [r]] \cup (V(H) \setminus V(\tilde{H}))\) into \(G\) that embeds all edges in \(H[Y_i^{\text{good}} \cup [r]] \cup (V(H) \setminus V(\tilde{H}))\) that contain a vertex in \(Y_i^{\text{good}}\) on an edge in \(G_B - G^0\). In particular, since we find the perfect matchings in the candidacy graphs \((B_i^{H^0})G^0\) induced on \(Y_i^{\text{good}} \cup W_i^{\text{good}}\) for each \(i \in [r]\), we have for \(\phi^H := \phi_i[V(H) \setminus V(\tilde{H})] \cup \phi^{\text{good}}\) that

- \(\phi^H\) is an embedding of \(H\) into \(G\) where \(\phi^H(X_i^H) = V_i\) for all \(i \in [r]\), which establishes (H)C4;
- all \(H\)-edges that contain a vertex in \(Y_{[r]}^{\text{good}}\) are embedded on an edge in \(G_B - G^0\), which establishes (J)C4;
- for all \(m \in [k]\), index sets \(I \in \binom{[r]}{m}\), \(\{v_i\}_{i \in I} \in V_{[r]}\cup[r]\), and \(H\)-edges \(e\) that contain a vertex in \(Y_{[r]}^{H}\), we have that \(\phi^H(e \cap X_i^{H^0}) = \{v_i\}_{i \in I}\) with probability at most \(\prod_{i \in I}: e \cap Y_i^{H^0} \neq \emptyset 2(m\mu_d)^{-1}\) by \(C1(r)(IV)C4\) and (6.45), which establishes (K)C4.

This completes the proof of Claim 4.

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**Step 5.6. The random packing procedure – Claim 6**

We now proceed to our Random Packing Procedure (RPP). Let \(G_0^0\) be the edgeless graph on \(V(G)\) and let \(\phi^0\) be the empty function. We perform the following random procedure.

**Random Packing Procedure (RPP)**

For \(h = 1, \ldots, |H|\) do:

- Set \(H := H_h\). For all \(i \in [r]\), independently activate every vertex in \(\hat{X}_i^H\) with probability \(\mu\) and let \(Y_i^H\) be the union of \(X_i^H \setminus \hat{X}_i^H\) and all activated vertices in \(\hat{X}_i^H\).
- If the assumptions of Claim 4 are satisfied, apply Claim 4 and obtain a random packing \(\sigma_i^H\) that satisfies (H)C4–(K)C4; otherwise terminate with failure.
- Set \(\phi^H := \phi^{h-1} \cup \phi^H\) and \(G_h^0 := G_{h-1}^0 \cup (\phi^H(H) \cap G_B)\).

**Claim 6.** With probability at least \(1 - 1/n\), the RPP terminates without failure and satisfies conclusion (iii) of Lemma 6.1.

**Proof of claim:** We prove Claim 6 in Steps 5.7–5.11. Our general strategy is to guarantee that we can apply Claim 4 in each turn of the procedure. To that end, we will introduce a collection of random variables that we call *identifiers*. Such an identifier indicates an unlikely event and if this event happens, we say the identifier *detects alarm* and we simply terminate the RPP with failure. This means, if an identifier detects alarm at some turn \(h \in [|H|]\), we terminate the RPP
and deactivate all further identifiers; that is, the probability that they detect alarm is set to 0. We show that the probability that an individual identifier detects alarm is exponentially small in \( n \). In the end, a union bound over all identifiers will imply that with probability at least \( 1 - 1/n \), none of the identifiers will detect alarm and thus, the RPP terminates without failure.

For most of the identifiers, it follows by a standard application of Theorem 3.1 that the probability to detect alarm is exponentially small (in fact, often Chernoff’s inequality suffices). To that end, in the subsequent Steps 5.7–5.11, we often describe only the random variables to which we apply Theorem 3.1.

First, let us observe that \( S(r) \) implies that
\[
(6.46) \quad |X_i^H \setminus \hat{X}_i^H| = |V_i \setminus \phi_r(\hat{X}_i^H)| \leq 2\varepsilon_T n \text{ for all } H \in \mathcal{H}, \ i \in [r].
\]

Further, for \( Y_i^H \) as in the RPP, let \( W_i^H := V_i \setminus \phi_r(\hat{X}_i^H \setminus Y_i^H) \). For convenience, we also call a vertex \( w \in W_i^H \cap \phi_r(\hat{X}_i^H) \) activated by \( H \in \mathcal{H} \). (Recall that we only activate vertices in \( \hat{X}_i^H \).

Step 5.7. Proof of Claim 6 – Establishing \((A)\)\(C_4\)–\((G)\)\(C_4\)

In this step, in order to establish \((A)\)\(C_4\)–\((G)\)\(C_4\) at each turn of the RPP, we consider several random variables for which we individually introduce an identifier that detects alarm if the considered random variable is not within a factor of \((1 \pm \varepsilon)\) of its expectation. As mentioned above, for each identifier a standard application of Theorem 3.1 implies that the probability to detect alarm is exponentially small, say, \( e^{-n^{1/2}} \). Let us only describe the random variables that we consider for establishing \((A)\)\(C_4\)–\((G)\)\(C_4\).

To establish \((A)\)\(C_4\), for each \( H \in \mathcal{H}, \ i \in [r]\), we consider the sum of indicator variables which each indicates whether a vertex is activated in \( X_i^H \). Together with \((6.46)\), this implies \((A)\)\(C_4\).

To establish \((B)\)\(C_4\), for each \( H \in \mathcal{H} \) and \( r \in E(R) \), we consider the random variable that counts how many \( H \)-edges \( e \) lie in \( X_i^H \) where at least two vertices of \( e \) are activated; in view of the statement, we may assume that \( \varepsilon_T(X_i^H) \geq \mu^{1/2} n \). Note that the probability for an edge \( e \) that at least two vertices are activated is at most \( k^2 \mu^2 \). Together with \((6.46)\), this implies \((B)\)\(C_4\).

To establish \((C)\)\(C_4\), for all \( H \in \mathcal{H}, \ i \in [r]\), and \( v \in V_i \), we consider the random variable \( \xi \) that counts how many \( B_i^H \)-neighbours of \( v \) in \( X_i^H \) are activated and lie in an \( H \)-edge \( e \) where a second vertex in \( e \) is either activated or left unembedded. With Claim 2 we have that \( \mathbb{E}[\xi] \leq 2k^2 \mu^2 \alpha^{-1} d_i n + 2\varepsilon_T^{1/2} d_i n \leq \mu^{5/3} d_i n \). Together with Claim 1, this implies \((C)\)\(C_4\).

To establish \((G)\)\(C_4\), for all \( i \in [r]\), \( S \subseteq \bigcup_{r \in E(R); \ i \in r} V_i \setminus \{i\} \), \( |S| \leq t \), we consider the sum of indicator variables which each indicates whether a vertex in \( \phi_r(\hat{X}_i^H) \cap N_{G_B - G_h^3}(S) \) is activated. By \( S(r)(e) \), we further have that
\[
(6.47) \quad \left| \left(V_i \setminus \phi_r(\hat{X}_i^H) \right) \cap N_{G_B}(S) \right| \leq \varepsilon_T |V_i \cap N_{G_B}(S)|.
\]

Altogether, this implies \((G)\)\(C_4\).

In order to establish \((D)\)\(C_4\)–\((F)\)\(C_4\), we claim that if no identifier detected alarm until turn \( h \in \{|\mathcal{H}|\} \), then for all \( i \in [r]\)
\[
(6.48) \quad \left| \bigcup_{S \subseteq S} \left(V_i \cap N_{G_h}(S) \cap N_{G_B}(S) \right) \right| \leq \mu^{2/3} d_i^n n \text{ for all } S \subseteq \bigcup_{r \in E(R); \ i \in r} V_i \setminus \{i\} , \ |S| \leq t ;
\]
\[
(6.49) \quad \rho_{G_h^3}(v) \leq \mu^{2/3} d_i n \text{ for all } H \in \mathcal{H}, \ v \in V_i,
\]
where \( \rho_{G_h^3}(v) \) is defined as in \((6.26)\). We verify \((6.48)\) and \((6.49)\) in the subsequent Steps 5.9 and 5.10, respectively, and first establish \((D)\)\(C_4\)–\((F)\)\(C_4\) assuming \((6.48)\) and \((6.49)\).

To establish \((F)\)\(C_4\), recall that \( G_B \) is \((2\varepsilon_T, t, d_B)\)-typical with respect to \( R \) by \((6.4)\). Hence, we obtain from \((6.48)\) and the definition of typicality that \( G_B - G_h^3 \) is \((\mu^{1/2}, t, d_B)\)-typical with respect to \( R \), which implies \((F)\)\(C_4\).

To establish \((D)\)\(C_4\) for \( H = H_{h+1} \), recall that \( B_i^H \) is \((\varepsilon_T, d_i)\)-super-regular and \((\varepsilon_T, t^{2/3})\)-well-intersecting with respect to \( G_B \) for all \( i \in [r] \). Observe that there exists a set \( S_x \) for every \( x \in X_i^H \) with \( |S_x| \leq t^{2/3} \) and \( N_{B_x^H}(x) = V_i \cap N_{G_B}(S_x) \). By the definition of \( B_i^H \cap G_h^3 \) in \((6.25)\), we have that
$N_{(B^H)^G}^k(x) = V_i \cap N_{G^H-G^k}(S_x)$. This together with (6.48), and (6.49) together with (6.27), implies that we removed at most $2^3/3d_in$ edges incident to every vertex to obtain $(B^H)^G$ from $B^H$. Now Fact 3.4 yields that $(B^H)^G$ is $(\mu^{1/5}, d_i)$-super-regular. This implies (D)C4.

To establish (E)C4 for $H = H_{h+1}$, we exploit that $(B^H)^G$ is $(\mu^{1/5}, d_i)$-super-regular and only have to show that every vertex in $Y_H W^i_H$ has the appropriate degree. To that end, we consider the following random variables for all $x \in X_H^H$, $v \in V_i$, $i \in [r]$. For $x$, we consider the random variable that counts the number of activated vertices by $H$ in $N_{(B^H)^G}^k(x)$, and by employing (6.47) and that $B^H$ is $(\varepsilon, t)$-well-intersecting with respect to $G_B$, we expect that $N_{(B^H)^G}^k(x) \cap W_H^i$ has size $\mu|N_{(B^H)^G}^k(x)| \pm 2\varepsilon_T d_i n$. This yields the appropriate degree for $x$. For $v$, we consider the random variable that counts the number of activated vertices in $N_{(B^H)^G}^k(v)$, and by employing Claim 1, we expect that $N_{(B^H)^G}^k(v) \cap Y^H$ has size $\mu|N_{(B^H)^G}^k(v)| \pm 2\varepsilon_T d_i n$. This yields the appropriate degree for $v$. Altogether this implies (E)C4.

Step 5.8. Probability to use a $G_B$-edge during the completion

We say an edge $g \in E(G_B)$ is used during the RPP if there exists a graph $H_h \in \mathcal{H}$ and an edge $e \in H_h$ such that $\phi^f(e) = g$. In this step we show the following claim that we will apply to establish (6.48) and (6.49) in Steps 5.9 and 5.10.

Claim 7. For every edge $g \in E(G_B)$, the probability that $g$ is used during the RPP is at most $\mu^{3/4}$.

Proof of claim: Let $g \in E(G_B)$ and $r \in E(R)$ with $g = \{v_i\}_{i \in r} \in E(G_B[V_{\cup \mathcal{F}}])$ be fixed. We consider different cases and sets of $H$-edges that could potentially be embedded onto $g$, say in each case with probability at most $\mu^{1/5}$. In the end, a union bound will establish Claim 7. Therefore, for all $m \in [k]$, $J \in \binom{[r]}{m}$, we consider different sets of edges in $H_{\mathcal{F}_{\cup \mathcal{J}}}$ where the $m$ vertices corresponding to the clusters in $J$ are either unembedded or activated and can potentially be mapped onto $\{v_i\}_{i \in J}$, and where the remaining $k - m$ vertices corresponding to the clusters in $r \setminus J$ have already been embedded onto $\{v_i\}_{i \in r \setminus J}$. Let $m \in [k]$ and $J \in \binom{[r]}{m}$ be fixed.

We first consider the set of $H$-edges $e = \{x_i\}_{i \in r}$ for all $H \in \mathcal{H}$ where no vertex of $e$ is left unembedded by $\phi^f$ and for every vertex of $g$, there is an $H$-vertex that is mapped onto $g$. That is, the $m$ vertices $\{x_1\}_{i \in J}$ as well as the $m$ vertices $\{v_i\}_{i \in J}$ are activated, the vertices $\{x_i\}_{i \in J}$ can potentially be mapped onto $\{v_i\}_{i \in J}$, and $\phi^f(\{x_i\}_{i \in J \cup \mathcal{J}}) = \{v_i\}_{i \in J \cup \mathcal{J}}$. To that end, we first consider the set $\tilde{E} := E_{\phi^f}(g, J, J_X = \emptyset, J_Y = \emptyset) \setminus E_{\phi^f}(g, J, J_X = \emptyset, J_Y = \emptyset)$ as defined in (6.28). (Note that we defined this set in (6.28) only in the more convenient way that $g \cap V_{\cup \mathcal{F}} \subseteq g \setminus \phi^f(X_{\cup \mathcal{J}})$, which is the reason why we remove the set $E_{\phi^f}(g, J, J_X = \emptyset, J_Y = \emptyset)$ from the current consideration.)

For an edge $e = \{x_i\}_{i \in r} \in \tilde{E}$, in order that $\{x_i\}_{i \in J}$ can be mapped onto $\{v_i\}_{i \in J}$ during the completion, it must hold that the vertices $\{x_i\}_{i \in J}$ and the vertices $\{v_i\}_{i \in J}$ must become activated. Since $\phi^f(x) \neq g$ and $\phi^f(\{x_i\}_{i \in r \setminus J}) = \{v_i\}_{i \in r \setminus J}$, it holds that $\phi^f(\{x_i\}_{i \in J}) \neq \{v_i\}_{i \in J}$. Hence, the probability that $\{x_i\}_{i \in J}$ and $\{v_i\}_{i \in J}$ become activated is at most $\mu^{m+1}$ because every vertex in $\tilde{X}^H$ for all $i \in [r]$, $H \in \mathcal{H}$ is activated independently with probability $\mu$. Further, by (K)C4, activated vertices $\{x_i\}_{i \in J}$ are mapped onto $\{v_i\}_{i \in J}$ with probability at most $\mu^{m+1} \prod_{i \in J} (\mu d_i n)^{-1}$. Altogether, this implies that the probability that some edge in $\tilde{E}$ is mapped onto $g$ is at most

$$\mu^{m+2} \prod_{i \in J} (\mu d_i n)^{-1}$$

Claim 3

$$\mu^{m+2} \prod_{i \in J} (\mu d_i n)^{-1} \cdot \gamma^{-1} \mu^{m} \prod_{i \in J} d_i \leq \mu^{4/5},$$

where we used for the last inequality that $\mu \ll \gamma$.

Next, we consider the sets $E_{\phi^f}(g, J, J_X, J_Y)$ where $J_X = J_X \cup J_Y$ is non-empty for disjoint $J_X, J_Y \subseteq J$. For an edge $e = \{x_i\}_{i \in r} \in E_{\phi^f}(g, J, J_X, J_Y)$, the vertices $\{x_i\}_{i \in J}$ are mapped onto
\{v_i\}_{i \in J}$ with probability at most $2^m \prod_{i \in J} (\mu d_i n)^{-1}$ by (K)$_{C4}$. This implies that the probability that some edge in $E_{\phi_r}(g, J, J_X = \emptyset, J_Y)$ is mapped onto $g$ is at most

$$2^m \prod_{i \in J} (\mu d_i n)^{-1} |E_{\phi_r}(g, J, J_X, J_Y)|$$

(6.51)

Claim 3

$$2^m \prod_{i \in J} (\mu d_i n)^{-1} \cdot \varepsilon_T^{1/2} n^m \prod_{i \in J} d_i \leq \varepsilon_T^{1/3}.$$

Now, Claim 7 is established by a union bound over all $m \in [k]$, $J \in (\binom{r}{m})$, and all possible sets $J_X, J_Y$ that we considered to be fixed in (6.50) and (6.51).

**Step 5.9. Proof of Claim 6 – Bound in (6.48)**

We use Claim 7 to verify the claimed bound in (6.48). We fix $i \in [r]$, $S \subseteq \bigcup_{v \in V_i} V_{[r] \setminus \{i\}}$ with $|S| \leq t$ and $S \in S$. By Claim 7, each $G_B$-edge is used with probability at most $\mu^{3/4}$ during the RPP, and thus, by an application of Lemma 3.2, we have that $|V_i \cap N_{G_B}(S) \cap N_{G_B}(S)| \leq \mu^{7/10} d_B^{|S|} n$ with probability at least, say, $1 - e^{-n^{3/4}}$. Otherwise we detect alarm. Together with a union bound this implies the claimed bound in (6.48) because $t \cdot \mu^{7/10} d_B^{|S|} n \leq \mu^{2/3} d_B^{|S|} n$.

**Step 5.10. Proof of Claim 6 – Bound for $\rho_{G_B}^H(v)$**

We also use Claim 7 to verify the claimed bound for $\rho_{G_B}^H(v)$ in (6.49). (Recall the definition of $\rho_{G_B}^H(v)$ in (6.26).) For all $H \in \mathcal{H}$, $i \in [r]$ and $v \in V_i$, we have that $v$ has at most $(1 + 2\varepsilon_T d_i n$ neighbours in $B_i^H$. For each such neighbour $x \in N_{B_i^H}(v)$, there exists a set $S_x \subseteq (V(G_i \setminus k-1)$ with $|S_x| \leq t^{3/3}$ such that $N_{B_i^H}(x) = N_{G_B}(S_x)$. Hence, there exist at most $2t^{3/3} d_i n$ edges $g = S \cup \{v\}$ in $G_B$ for some $S \subseteq \bigcup_{x \in N_{B_i^H}(v)} S_x$. By Claim 7, each such $G_B$-edge $g$ is used with probability at most $\mu^{3/4}$ during the entire RPP. We therefore expect that for each $h \in [\mathcal{H}]$ at most $\mu^{3/4} 2t^{3/3} d_i n$ edges incident to $v$ in $B_i^H$ have to be removed when we obtain $(B_i^H)^{G_B}$. Hence, by an application of Lemma 3.2 and a union bound, we obtain that $\rho_{G_B}^H(v) \leq \mu^{2/3} d_i n$ for all $h \in [\mathcal{H}]$, $H \in \mathcal{H}$, $i \in [r]$, $v \in V_i$ with probability at least, say, $1 - e^{-n^{3/4}}$. Otherwise we detect alarm. This implies the claimed bound in (6.49).

**Step 5.11. Proof of Claim 6 – Establishing (iii) of Lemma 6.1**

In order to establish conclusion (iii) of Lemma 6.1, let $(\omega, \phi) \in \mathcal{W}_{\omega_{\phi}}$ with centres $\phi = \{c_i\}_{i \in I}$ in $I$ be fixed. We split the proof into two parts, depending on whether we activate a vertex of a tuple $x \in \mathcal{X}_{\omega_{\phi}}$ that $\phi_r$ already mapped onto $\phi$, or whether a tuple $x \in \mathcal{X}_{\omega_{\phi}} \cap \mathcal{X}_{\phi \phi}$ and $x$ is mapped onto $\phi$ during the completion.

Let us first consider tuples $x \in \mathcal{X}_{\omega_{\phi}}$ that we have already embedded onto $\phi$, that is, $\phi_r(x) = \phi$, and where some vertices of $x$ become activated. We claim that during the entire RPP not too many such tuples become activated. That is, we claim that

$$\omega \left( \bigcup_{H \in \mathcal{H}} \{ x \in X_{\omega_{\phi}}^H : \phi_r(x) = \phi, x \cap Y_{\omega_{\phi}}^H \neq \emptyset \} \right) \leq \mu^{1/2} \omega(\phi_r^{-1}(\phi)) + n^{\varepsilon_T},$$

(6.52)

with high probability. To see (6.52), note that for every $x \in \mathcal{X}_{\omega_{\phi}}$, the probability that $x$ contains an activated vertex is at most $|I|/\mu$. Hence, an application of Theorem 3.1 and a union bound yield (6.52) with probability, say, at least $1 - e^{-n^{\varepsilon_T}}$.

Next, we consider tuples $x \in \mathcal{X}_{\omega_{\phi}}$ that have not been embedded onto $\phi$ by $\phi_r$. To that end, we fix $m \in [k]$ and $J \in \binom{I}{m}$, and a pair of disjoint sets $J_X, J_Y \subseteq J$. We aim to control the $\omega$-weight on the tuples in $E_{\phi_r}(\phi, J, J_X, J_Y)$ as defined in (6.28). Analogously as in Claim 3, we can employ $S(r)(b)$ and (d) for the edge testers that we defined in (6.5). Proceeding as in Claim 3 yields that

$$\omega(E_{\phi_r}(\phi, J, J_X, J_Y)) \leq (1 \{J_X V = \emptyset\} + 2\varepsilon_T) \prod_{i \in J} d_i \omega(\mathcal{X}_{\omega_{\phi}}^J) \frac{n^{1/|J| - |J|}}{n^{1/|J|}} + n^{2\varepsilon_T}.$$

(6.53)
We can now proceed similarly as in the proof of Claim 7 in Step 5.8; that is, we consider different cases and sets of tuples \( x \in \mathcal{X}_J \) that could potentially be embedded onto \( e \), and derive an upper bound on the expected \( \omega \)-weight used in such a case. In the end, a union bound over all these cases gives us an upper bound on the total expected \( \omega \)-weight of tuples that are embedded onto \( e \) during the completion. Therefore, we consider different choices for \( J_X, J_Y \subseteq J \).

For \( x = \{x_i\}_{i \in J} \in E_{\phi_e}(e, J, J_X = \emptyset, J_Y = \emptyset) \setminus E_{\phi_e}(e, J, J_X = \emptyset, J_Y \neq \emptyset) \), in order that \( \{x_i\}_{i \in J} \) can be mapped onto \( \{c_i\}_{i \in J} \) during the RPP, it must hold that the vertices \( \{x_i\}_{i \in J} \) and \( \{c_i\}_{i \in J} \) become activated because \( J_Y = \emptyset \). Since \( \phi_e(\{x_i\}_{i \in J}) \neq \{c_i\}_{i \in J} \), this happens with probability at most \( \mu^{m+1} \). Further, by (K)C4, the activated vertices \( \{x_i\}_{i \in J} \) are mapped onto \( \{c_i\}_{i \in J} \) with probability at most \( 2^{m} \prod_{i \in J} (\mu d_i n)^{-1} \). Altogether, this implies that the expected weight of edges in \( E_{\phi_e}(e, J, J_X = \emptyset, J_Y = \emptyset) \setminus E_{\phi_e}(e, J, J_X = \emptyset, J_Y \neq \emptyset) \) that are mapped onto \( e \) during the RPP is at most

\[
\mu^{m+1} \prod_{i \in J} (\mu d_i n)^{-1} \omega \left( E_{\phi_e}(e, J, J_X = \emptyset, J_Y = \emptyset) \right) \tag{6.53}
\]

\[
\leq \mu^{m+1} \prod_{i \in J} (\mu d_i n)^{-1} \left( 2 \prod_{i \in J} d_i \frac{\omega(\mathcal{X}_J)}{n |I| |J|} + n^{2\varepsilon_T} \right) \leq \mu^{4/5} \frac{\omega(\mathcal{X}_J)}{n |I|} + n^{3\varepsilon_T}.
\]

Next, we consider the set \( E_{\phi_e}(e, J, J_X, J_Y) \) for disjoint but fixed \( J_X, J_Y \subseteq J \) such that \( J_{XV} = J_X \cup J_Y \neq \emptyset \). The vertices \( \{x_i\}_{i \in J} \) are mapped onto \( \{c_i\}_{i \in J} \) with probability at most \( 2^{m} \prod_{i \in J} (\mu d_i n)^{-1} \) by (K)C4. Hence, the expected weight of edges in \( E_{\phi_e}(e, J, J_X, J_Y) \) that are mapped onto \( e \) during the RPP is at most

\[
2^{m} \prod_{i \in J} (\mu d_i n)^{-1} \omega \left( E_{\phi_e}(e, J, J_X, J_Y) \right) \tag{6.53}
\]

\[
\leq \varepsilon_T^{1/2} \frac{\omega(\mathcal{X}_J)}{n |I|} + n^{3\varepsilon_T}.
\]

Altogether, a union bound over all \( m \in [|I|], J \in \binom{[m]}{|I|} \), and all sets \( J_X, J_Y \subseteq J \) that we considered to be fixed in (6.54) and (6.55) together with an application of Lemma 3.2 yields that

\[
\omega \left( \left\{ x \in \mathcal{X}_J : \phi_r(x) \neq e, \phi^{[H]}(x) = e \right\} \right) \leq \mu^{1/2} \frac{\omega(\mathcal{X}_J)}{n |I|} + n^{4\varepsilon_T}
\]

with probability at least, say, \( 1 - e^{-n^{2\varepsilon_T}} \).

Combining this with (6.52) yields

\[
\omega((\phi^{[H]})^{-1}(e)) = (1 \pm \mu^{1/2})\omega(\phi_r^{-1}(e)) \pm \mu^{1/2} \frac{\omega(\mathcal{X}_J)}{n |I|} + 2n^{4\varepsilon_T}
\]

\[
\asymp (1 \pm \alpha) \frac{\omega(\mathcal{X}_J)}{n |I|} \pm n^\alpha.
\]

This establishes conclusion (iii) of Lemma 6.1 and completes the proof of Claim 6.

---

**Step 5.12. Finishing the completion**

As the RPP outputs \( \phi^{[H]} \) with positive probability by Claim 6, we obtain a packing \( \phi := \phi^{[H]} \) of \( \mathcal{H} \) into \( G \) which clearly satisfies conclusions (i) and (iii) of Lemma 6.1. Note that by the construction of \( \phi^{[H]} \), we have that \( \phi|_{X^H} = \phi_r|_{X^H \setminus Y^H} \cup \phi^{H}_i|_{Y^H} \) for all \( H \in \mathcal{H}, i \in [r] \). Since \( |Y^H_i| = (1 \pm 1/2)\mu n \), we therefore merely modified \( \phi_r \) to obtain \( \phi \) and thus, \( S(r)(f) \) easily implies conclusion (ii) of Lemma 6.1. This completes the proof of Lemma 6.1. \( \square \)

7. Proof of the main results

In this section we prove Theorem 1.3 and Theorem 1.4.

**Proof of Theorem 1.3.** Our general approach is as follows. Given a blow-up instance \((\mathcal{H}, G, R, X, Y)\), we refine the vertex partition \( X \) of the graphs in \( \mathcal{H} \) using Lemma 3.9 and we randomly refine the vertex partition \( V \) of \( G \) accordingly. Afterwards we can apply Lemma 6.1 to obtain the required packing of \( \mathcal{H} \) into \( G \).

Suppose \( 1/n \ll \varepsilon \ll 1/t \ll \beta \ll \alpha, 1/k \) for a new parameter \( \beta \). For each \( (W, Y_1, \ldots, Y_m) \in W_{\text{set}} \) with \( W \subseteq V_i, i \in [r] \), and each \( \ell \in [m] \), let \( \omega_{Y_i} : \bigcup_{H \in \mathcal{H}} X^H_i \rightarrow \{0,1\} \) be such that \( \omega_{Y_i}(x) = \)
Let $\mathcal{X}'$ be the $k$-graph with vertex set $[r] \times [\beta^{-1}]$ and edge set
\[ \{(i_\ell, j_\ell) : i_\ell \in [r], j_\ell \in [\beta^{-1}] \}. \]

Note that $\Delta(\mathcal{X}') \leq \alpha^{-1} \beta^{-(k-1)}$ because $\Delta(\mathcal{X}) \leq \alpha^{-1}$. Let $n' := \beta n$.

Employing conclusion (iv) of Lemma 3.9 for the weight functions in $\mathcal{W}_H$ implies for all $\{(i_\ell, j_\ell) : i_\ell \in [r] \}
\in E(\mathcal{X}')$ with $r := \{(i_\ell) : i_\ell \in [r] \} \in E(\mathcal{X})$ that
\[
\sum_{V_i \in \mathcal{V}} e_{\mathcal{H}}(X_{i_1, j_1}, \ldots, X_{i_k, j_k}) = (1 + \varepsilon)\beta^k e_{\mathcal{H}}(\mathcal{X}_{\mathcal{U}_r}) + n^{1+\varepsilon}
\leq (1 + \varepsilon)\beta^k e_{\mathcal{H}}(\mathcal{X}_{\mathcal{U}_r}) + n^{1+\varepsilon} \leq (1 - \alpha/2)dn^k,
\]

because by assumption, $e_{\mathcal{H}}(\mathcal{X}_{\mathcal{U}_r}) \leq (1 - \alpha)dn^{k}$ and $d \geq n^{-\varepsilon}$.

Further, note that Lemma 3.9(iii) implies for each $H \in \mathcal{H}$ that $H[X_{\mathcal{U}_r}]$ is a matching if $r \in E(\mathcal{X}')$ and empty if $r \notin \{(i_\ell) : i_\ell \in [r] \} \in E(\mathcal{X})$.

According to the refinement $\mathcal{X}'$ of $\mathcal{X}$, we claim that there exists a refined partition $\mathcal{Y}' = (V_{i_\ell})_{i_\ell \in [r]} \in [\beta^{-1}]$ of $\mathcal{V}$, where $(V_{i_\ell})_{i_\ell \in [\beta^{-1}]}$ is a partition of $V_i$ for every $i \in [r]$ such that
(a) $|W \cap V_{i_\ell}| = \beta|W| \pm \beta^3/2n$ for all $(W, Y_1, \ldots, Y_m) \in \mathcal{W}_{\mathcal{U}_r}$ and $j \in [\beta^{-1}]$ with $W \in \mathcal{V}$;
(b) $\mathcal{B}' := (\mathcal{H}, G, R', \mathcal{X}', \mathcal{Y}')$ is an $(\varepsilon^{1/2}, t)$-typical, $\beta^{-k}$-bounded blow-up instance of size $(n', k, \beta^{-1}r)$ with $n' = \beta n$.

The existence of such a partition $\mathcal{Y}'$ can be seen by a probabilistic argument. For all $i \in [r]$ and $j \in [\beta^{-1}]$, we take disjoint subset $V_{i_\ell,j}$ of $V_i$ of size exactly $|X_{i_\ell,j}|$ uniformly and independently at random. We analyse the probability that (a) or (b) are not satisfied. To that end, we consider the slightly different random experiment where we assign for all $i \in [r]$, every vertex in $V_i$ uniformly and independently at random to some $V_{i_\ell,j}$ for $j \in [\beta^{-1}]$. For this experiment and a fixed $i \in [r]$, we consider the bad events that (a) or (b) are not satisfied: in such a case, we say the experiment fails (in step i). Standard properties of the multinomial distribution yield that $|V_{i_\ell,j}| = |X_{i_\ell,j}|$ for all $j \in [\beta^{-1}]$, $H \in \mathcal{H}$ with probability at least $\Omega(n^{-\beta})$. Hence, together with Theorem 3.1 and a union bound, this yields that the original experiment fails in step i with probability, say, at most $e^{-n^{1/2}}$. Since in fact we take $V_{i_\ell,j}$ of size exactly $|X_{i_\ell,j}|$, this altogether implies the existence of a refined partition $\mathcal{Y}'$ of $\mathcal{Y}$ satisfying (a) and (b) with positive probability.

We show how to adapt the vertex and set testers from the original blow-up instance to the blow-up instance $\mathcal{B}'$. For each $(W, Y_1, \ldots, Y_m) \in \mathcal{W}_{\mathcal{U}_r}$ and distinct $H_1, \ldots, H_m \in \mathcal{H}$ such that $W \subseteq V_i$ for some $i \in [r]$ and $Y_j \subseteq X_{i_\ell,j}$ for all $\ell \in [m]$, we define $(W_j, Y_1, \ldots, Y_{m,j})$ by setting $W_j := W \cap V_{i_\ell,j}$ and $Y_{i_\ell,j} := Y_i \cap X_{i_\ell,j}$ for all $j \in [\beta^{-1}], \ell \in [m]$. By (a), we conclude that $|W_j| = |W| \pm \beta^{3/2}n$.

Employing conclusion (iii) of Lemma 3.9 for the weight function $\omega_{\mathcal{Y}_j} \in i_{\mathcal{Y}_j}$, we have that
\[
|Y_{i_\ell,j}| = \omega_{\mathcal{Y}_j}(X_{i_\ell,j}) = \omega_{\mathcal{Y}_j}(X_{i_\ell,j}) + \beta^{3/2}n = \beta|Y_i| \pm \beta^{3/2}n.
\]

Let $\mathcal{W}' := \{(W_j, Y_1, \ldots, Y_{m,j}) : j \in [\beta^{-1}], (W, Y_1, \ldots, Y_m) \in \mathcal{W}_{\mathcal{U}_r}\}$.

Hence, we can apply Lemma 6.1 to $\mathcal{B}'$ with set testers $\mathcal{W}'_{\mathcal{U}_r}$ and vertex testers $\mathcal{W}'_{\mathcal{U}_r}$ as follows:

| parameter | $n'$ | $\varepsilon^{1/2}$ | $t$ | $\beta^k$ | $d$ | $r\beta^{-1}$ |
|------------|------|----------------|----|---------|----|-------------|
| plays the role of | $n$ | $\varepsilon$ | $t$ | $\alpha$ | $d$ | $r$ |

This yields a packing of $\mathcal{H}$ into $G$ such that
(I) $\phi(X_{i_\ell,j}) = V_{i_\ell,j}$ for all $i \in [r], j \in [\beta^{-1}], H \in \mathcal{H}$;
(II) $|W_j \cap \bigcap_{\ell \in [m]} \phi(Y_{i_\ell,j})| = |W_j|Y_{i_\ell,j}| \pm \beta^k n'$ for all $(W_j, Y_{i_\ell,j}, \ldots, Y_{m,j}) \in \mathcal{W}'_{\mathcal{U}_r}$;
(III) $\omega'(\phi^{-1}(\varepsilon)) = (1 + \beta^k)\omega(\bigcup_{H \in \mathcal{H}}(\bigcup_{i_\ell} X_{i_\ell,j})) |n^{\alpha}| \pm \beta^k n'$ for all $(\omega', \varepsilon) \in \mathcal{W}'_{\mathcal{U}_r}$ with centres
$e = \{c_{\ell} \}_{i_\ell} \in I \subseteq [r]$ and multi-set $\{j_\ell \}_{i_\ell} \subseteq [\beta^{-1}]$ such that $c_{\ell} \in V_{i_\ell,j}$ for each $i_\ell \in I$. 





For \((W, Y_1, \ldots, Y_m) \in \mathcal{W}_{\text{set}}\), we conclude that
\[
\left| W \cap \bigcap_{\ell \in [m]} \phi(Y_\ell) \right| = \sum_{j \in [\beta^{-1}]} \left| W_j \cap \bigcap_{\ell \in [m]} \phi(Y_{\ell,j}) \right|
\]

\((I),(\ell),(7.1)\)
\[= \sum_{j \in [\beta^{-1}]} \left( \beta^{m+1} \frac{\left| W \right| \left| Y_1 \cdots Y_m \right| + \beta^{1/3}n^{m+1}}{(\beta n)^m} \pm \beta^k n' \right) \]
\[= \left| W \right| \left| Y_1 \cdots Y_m \right| / n^m \pm an. \]

This establishes Theorem 1.3(i).

In order to establish Theorem 1.3(ii), we fix \((\omega, e) \in \mathcal{W}_{\text{ver}}\) with centres \(e = \{c_i\}_{i \in I}\) for \(I \subseteq [r]\) and multiset \(\{j_i\}_{i \in I} \subseteq [\beta^{-1}]\) such that \(c_i \in Y_{i,j_i}\) for each \(i \in I\), and we fix the corresponding tuple \((\omega', e) \in \mathcal{W}'_{\text{ver}}\). We conclude that
\[
\omega(\phi^{-1}(e)) = \omega'(\phi^{-1}(e)) = (1 \pm \beta^k) \frac{\omega' \left( \bigcup_{H \in \mathcal{H}} \left( \bigcup_{i \in I} X_{i,j_i}^H \right) \right)}{n^{\beta^k}} \pm n^{\beta^k}
\]
\[= (1 \pm \beta^k) \frac{\omega' \left( \bigcup_{H \in \mathcal{H}} \left( \bigcup_{i \in I} X_{i,j_i}^H \right) \right)}{n^{\beta^k}} \pm n^{\beta^k} = (1 \pm \alpha) \omega(\mathcal{X}_{\cup I}) / n^{\beta^k} \pm n^\alpha,
\]
where we employed construction (iv) of Lemma 3.9 in the penultimate equation. This establishes Theorem 1.3(ii) and completes the proof. \(\square\)

We now proceed to the proof of Theorem 1.4. The high-level strategy is similar as in the proof of Theorem 1.3. Additionally, we group the hypergraphs in \(\mathcal{H}\) into \(P = \text{polylog} \ n\) many collections of hypergraphs \(\mathcal{H}_1, \ldots, \mathcal{H}_P\) and accordingly we partition the edge set of the host graph \(G\) into \(G_1, \ldots, G_P\) subgraphs. Afterwards, we partition the vertex sets of the graphs in \(\mathcal{H}\) via Lemma 3.10 and randomly partition the vertex set of each \(G_p\) for \(p \in [P]\) accordingly. Then, we can iteratively apply Lemma 6.1 for each \(p \in [P]\) to map \(\mathcal{H}_p\) into \(G_p\). Note that this yields a packing of \(\mathcal{H}\) into \(G\), and considering \(P = \text{polylog} \ n\) partitions enables us to establish conclusions (i) and (ii) of Theorem 1.4.

**Proof of Theorem 1.4.** We set \(P := \log^4 n\) and suppose \(1/n \leq \epsilon \leq 1/t \ll \beta \ll \alpha, 1/k\) for a new parameter \(\beta\).

First, we group the graphs in \(\mathcal{H}\) into \(P\) collections \(\mathcal{H}_1, \ldots, \mathcal{H}_P\) with roughly equally many edges. That is, we claim that there exists a partition of \(\mathcal{H}\) into \(P\) collections of graphs \(\mathcal{H}_1, \ldots, \mathcal{H}_P\) such that each \(H \in \mathcal{H}\) belongs to exactly one \(\mathcal{H}_p\) for \(p \in [P]\), and for each \(p \in [P]\), we have
\[
(7.2) \quad e(\mathcal{H}_p) \leq (1 + \epsilon) P^{-1} e(\mathcal{H}) + n^{1+\epsilon},
\]
and for every \(\omega \in \mathcal{W}_{\text{ver}}\) with \(\omega(V(\mathcal{H})) \geq n^{1+\epsilon}\) and each \(p \in [P]\), we have
\[
(7.3) \quad \omega(V(\mathcal{H}_p)) = (1 + \epsilon) P^{-1} \omega(V(\mathcal{H})).
\]
The existence of \(\mathcal{H}_1, \ldots, \mathcal{H}_P\) can be easily seen by assigning every graph \(H \in \mathcal{H}\) to one collection \(\mathcal{H}_p\) for \(p \in [P]\) uniformly and independently at random.

We now apply Lemma 3.10 to each \(\mathcal{H}_p\). Let \(p \in [P]\) be fixed. For each \((W, Y_1, \ldots, Y_m) \in \mathcal{W}_{\text{set}}\) and each \(\ell \in [m]\) such that \(Y_\ell \subseteq V(H_\ell)\) for \(H_\ell \in \mathcal{H}_p\), let \(\omega_\ell : V(H_\ell) \rightarrow \{0, 1\}\) be such that \(\omega_\ell(x) = 1\{x \in Y_\ell\}\), and let \(W_Y\) be the set containing all those weight functions. Further, let \(\omega_{\text{edges}} : \bigcup_{H \in \mathcal{H}} \left( V(H) \right) / n^{\beta^k} \rightarrow \{0, 1\}\) be defined by \(\omega_{\text{edges}}((x_1, \ldots, x_k)) := 1\{(x_1, \ldots, x_k) \in E(\mathcal{H}_p)\}\), and let \(W_{\mathcal{H}_p}\) be the set containing all those weight functions. We apply Lemma 3.10 to \(\mathcal{H}_p\) with weight functions \(\{\omega: (\omega, e) \in \mathcal{W}_{\text{ver}} \cup W_Y \cup W_{\mathcal{H}_p}\}\). This yields a partition \(\mathcal{X}_p = (X_{\ell,j}^p)_{H \in \mathcal{H}_p, j \in [\beta^{-1}]}\) of \(\mathcal{H}_p\) such that for all \(H \in \mathcal{H}_p\), the partitions \((X_{\ell,j}^p)_{j \in [\beta^{-1}]}\) of \(V(H)\) satisfy the conclusions (i)–(iv) of Lemma 3.10.

For each \(p \in [P]\), let \(R_p\) be the \(k\)-graph with vertex set \([\beta^{-1}]\) and \(r \in \left([\beta^{-1}]\right)^k\) is an edge in \(R_p\) if \(H[X_{\leftarrow j}]\) is non-empty for some \(H \in \mathcal{H}_p\). Clearly, \(\Delta(R_p) \leq \beta^{-k}\). Let \(n' := \beta n\).
Employing conclusion (iv) of Lemma 3.10 for the weight functions in $W_{H_p}$, yields for all $r = \{i_1, \ldots, i_k\} \in E(R_p)$ that
\[
\sum_{H \in \mathcal{H}_p} e_H(X_H^r) = \omega_{edges}(\bigcup_{H \in \mathcal{H}_p} (X_{i_1}^H \times \ldots \times X_{i_k}^H)) \leq (1 + \beta^{1/2})^k \omega_{edges}(V(H_p)) + n^{1+\varepsilon}
\]
(7.4)
\[
= (1 + \beta^{1/2})^k \omega(H_p) + n^{1+\varepsilon} \leq (1 - \alpha/2) P^{-1} dn^k
\]
because by assumption, $e(H) \leq (1 - \alpha) e(G)$.

Now, we want to prepare $G$ accordingly to $H_1, \ldots, H_p$ and their partitions. To that end, we first partition $G$ into $P$ edge-disjoint spanning subgraphs $G_1, \ldots, G_P$ such that $G_p$ is $(\varepsilon^{1/2}, t, P^{-1} d)$-typical for every $p \in [P]$. The existence of $G_1, \ldots, G_P$ can be seen by assigning every edge in $G$ to one subgraph $G_p$ for $p \in [P]$ uniformly and independently at random.

Further, we claim that there exist partitions $V_p = (V_j)_{j \in [\beta^{-1}]}$ of $V(G_p)$ according to the partition $X_p$ of $H_p$, such that for every $p \in [P]$ and $V_p = (V_j)_{j \in [\beta^{-1}]}$, we have
\begin{itemize}
  \item[(a)] $|W \cap V_j| = \beta |W| \pm \beta^{3/2} n$ for all $(W, Y_1, \ldots, Y_m) \in W_{set}$ and $j \in [\beta^{-1}]$.
  \item[(b)] $\mathcal{B}_p := (H_p, G_p, R_p, X_p, V_p)$ is an $(\varepsilon^{1/2}, t, P^{-1} d)$-typical, $\beta - k$-bounded blow-up instance of size $(n', k, \beta^{-1})$ with $n'$, as well as
  \item[(c)] $\sum_{p \in [P]}$ centres collide $\omega(V(H_p)) \leq \beta^{1/2} \omega(V(H))$ for all $(\omega, e) \in W_{even}$ with $\omega(V(H)) \geq n^{1+\varepsilon}$, where we say that the centres $e$ collide (with respect to $V_p$) if $|e \cap V_j| \geq 2$ for some $V_j \in V_p$.
\end{itemize}
The existence of such partitions $V_p$ can be seen by assigning for every $p \in [P]$, every vertex to some $V_j$ for $j \in [\beta^{-1}]$ uniformly and independently at random. Theorem 3.1 and a union bound establish (a) with probability, say, at least $1 - e^{-n^{1/2}}$. For (b), note that standard properties of the multinomial distribution yield for $p \in [P]$ that $|V_j| = |X_j^H|$ for all $j \in [\beta^{-1}]$ and $H \in \mathcal{H}$ with probability at least $\Omega(n^{-\beta^{-1}})$. For (c), note that for $p \in [P]$, the probability that the centres $e$ collide with respect to $V_p$ is at most $k^2 \beta$. By (7.3), we therefore expect that at most $\sum_{p \in [P]} k^2 \beta \omega(V(H_p)) \leq 2k^2 \beta \omega(V(H))$ weight of $\omega$ collides in (c). Since $P$ grows sufficiently fast in terms of $n$, we can establish concentration. That is, Theorem 3.1 and a union bound that yield (c) with probability, say, at least $1 - n^{-\log n}$. Hence, a final union bound yields the existence of these partitions $V_p$ for $p \in [P]$ satisfying (a)–(c) with positive probability.

Next, we iteratively apply Lemma 6.1 to $\mathcal{B}_p$ for $p \in [P]$ which yields a packing $\phi_p$ of $H_p$ into $G_p$. Let us first explain how we adapt the vertex and set testers from the original blow-up instance to the blow-up instance $\mathcal{B}_p$.

For $p \in [P]$, we define
\[
W_{even}(p) := \left\{ (\omega_p, e) : (\omega, e) \in W_{even}, \text{ centres } e = \{c_i\}_{i \in I} \text{ do not collide with respect to } V_p, \right\}
\]
\[
\omega_p := \omega|_{\bigcup_{H \in \mathcal{H}_p, e \in X_j^H}}
\]
where we define $\{i\}_{i \in I} \subseteq [\beta^{-1}]$ as the indices such that $c_i \in V_{\bar{j}}$ for centres $e = \{c_i\}_{i \in I}$ that do not collide with respect to $V_p = (V_j)_{j \in [\beta^{-1}]}$.

For all $j \in [\beta^{-1}]$, $(W, Y_1, \ldots, Y_m) \in W_{set}$ and distinct $H_1, \ldots, H_m \in \mathcal{H}$ such that $Y_\ell \subseteq V(H_\ell)$, we define $Y_{\ell, j} := Y_\ell \cap X_j^H$ for each $\ell \in [m]$. For $j \in [\beta^{-1}], p \in [P]$, we define
\[
W_{set}(j, p) := \left\{ (W_j(p), \{Y_{\ell, j}\}_{\ell \in [m]}, Y_{\ell, j} \subseteq V(H_p)) : (W, Y_1, \ldots, Y_m) \in W_{set}, Y_{\ell[j]} \cap V(H_p) \neq \emptyset \right\},
\]
where we define $W_j(p) \in W_{set}$ recursively by
\[
W_j(p) := W_j(p - 1) \cap \bigcap_{\ell \in [m]} \phi_{p - 1}(Y_{\ell, j})
\]
with $\phi_0$ being the empty function and thus, $W_j(1) = W_j(0) := W \cap V_j$. By (a), we have that $|W \cap V_j| = \beta |W| \pm \beta^{3/2} n$. By employing conclusion (iii) of Lemma 3.10 for the weight function $\omega_{Y_{\ell}} \in W_Y$, we have that
\[
|Y_{\ell, j}| = \omega_{Y_{\ell}}(X_j^H) \Rightarrow \beta \omega_{Y_{\ell}}(V(H_{\ell})) \pm \beta^{3/2} n = \beta |Y_{\ell}| \pm \beta^{3/2} n.
\]
(7.5)
Hence, we iteratively apply Lemma 6.1 for every $p \in [P]$ to $\mathcal{B}_p$ with set testers $\bigcup_{j \in [\beta^{-1}]} \mathcal{W}_{\text{set}}(j, p)$ and vertex testers $\mathcal{W}_{\text{ver}}(p)$ as follows:

| parameter | $n'$ | $\varepsilon^{1/2}$ | $\beta^k$ | $P^{-1}d$ | $\beta^{-1}$ |
|-----------|------|-------------------|----------|----------|----------|
| plays the role of | $n$ | $\varepsilon$ | $t$ | $\alpha$ | $d$ | $r$ |

For every $p \in [P]$, this yields a packing $\phi_p$ of $\mathcal{H}_p$ into $G_p$ such that

(I) $\phi_p(X^H_j) = V_j$ for all $j \in [\beta^{-1}]$, $H \in \mathcal{H}_p$;

(II) $|W_j(p) \cap \bigcap_{\ell \in [m]} Y_{\ell,j} \subseteq V(H_p) \phi_p(Y_{\ell,j})| = |W_j(p) \cap \bigcap_{\ell \in [m]} Y_{\ell,j} \subseteq V(H_p) \phi_p(Y_{\ell,j})| \pm \beta^k n'$ for all $(W_j(p), \{Y_{\ell,j}\}_{\ell \in [m]}: Y_{\ell,j} \subseteq V(H_p)) \in \mathcal{W}_{\text{set}}(j, p), j \in [\beta^{-1}]$;

(III) $\omega_p(\phi_p^{-1}(c))) = (1 \pm \beta^k) \omega_p(\bigcup_{H \in \mathcal{H}_p} (\bigcup_{j \in I} X^H_j)) / n^{|I|} \pm n^{\beta k}$ for all $(\omega_p, c) \in \mathcal{W}_{\text{ver}}(p)$ with centres $c = \{c_i\}_{i \in I}$ that do not collide with respect to $V_p$ and $\{j_i\}_{i \in I} \subseteq [\beta^{-1}]$ such that $c_i \in V_{j_i}$ for each $i \in I$.

Let $\phi := \bigcup_{p \in [P]} \phi_p$ and note that $\phi$ is a packing of $\mathcal{H}$ into $G$.

For $(W, Y_1, \ldots, Y_m) \in \mathcal{W}_{\text{set}}$, we conclude that

$$|W \cap \bigcap_{\ell \in [m]} \phi(Y_{\ell})| = \sum_{j \in [\beta^{-1}]} |W_j(P) \cap \bigcap_{\ell \in [m]} Y_{\ell,j} \subseteq V(H_p) \phi_P(Y_{\ell,j})|$$

(II) $= \sum_{j \in [\beta^{-1}]} (|W_j(0)| |Y_{1,j}| \cdots |Y_{m,j}| / n^m \pm m \beta^k n')$

(a),(7.5) $= |W||Y_1| \cdots |Y_m| / n^m \pm \alpha n$.

This establishes Theorem 1.4(i).

In order to establish Theorem 1.4(ii), we fix $(\omega, c) \in \mathcal{W}_{\text{ver}}$ with centres $c = \{c_i\}_{i \in I}$. Recall that $\text{supp}(\omega) \subseteq E(\mathcal{H})$ for $|I| \geq 2$, and thus, if the centres $c$ collide with respect to one of the partitions $V_p$, then $\omega(\phi_p^{-1}(c)) = 0$. Therefore, we can consider the corresponding tuples $(\omega_p, c) \in \mathcal{W}_{\text{ver}}(p)$ for $p \in [P]$ and conclude that

$$\omega(\phi^{-1}(c)) = \sum_{p \in [P]} \omega(\phi_p^{-1}(c)) = \sum_{p \in [P]: \text{no collision}} \omega_p(\phi_p^{-1}(c))$$

(III) $= \sum_{p \in [P]: \text{no collision}} (1 \pm \beta^k) \omega_p(\bigcup_{H \in \mathcal{H}_p} (\bigcup_{j \in I} X^H_j) / n^{|I|} \pm n^{\beta k})$

(iv) $= \sum_{p \in [P]: \text{no collision}} (1 \pm \beta^k) (1 \pm \beta^{1/2}) |H| \omega(\mathcal{H}_p) / n^{|I|} \pm n^{\beta k})$

(c) $= (1 \pm \alpha) \omega(\mathcal{H}) / n^{|I|} \pm n^\alpha$.

This establishes Theorem 1.4(ii) and completes the proof. \hfill \square

8. Concluding remarks

We conclude with a selection of immediate applications of Theorem 1.1 including an asymptotic solution to a hypergraph Oberwolfach problem asked by Glock, Kühn and Osthus [20]. We first consider several natural hypergraph analogues of graph decomposition questions and then turn to problems concerning simplicial complexes. Along the way we state a few conjectures and open problems.

8.1. Applications to hypergraph decompositions. As we pointed out in the introduction, the Conjectures (I)–(III) intrigued mathematicians for decades. In the following we propose several conjectures of similar spirit for $k$-graphs. Our main results imply approximate versions thereof.

Recall that the Oberwolfach problem in (III) asks for a decomposition of $K_n$ into $(n-1)/2$ copies of a graph on $n$ vertices that is the disjoint union of cycles. There are many definitions for cycles in $k$-graphs and tight cycles are among the most well studied cycles. A $k$-graph is a tight cycle if its
vertex set can be cyclically ordered and the edge set consists of all $k$-sets that appear consecutively in this ordering. We refer to the number of vertices in a tight cycle as its length. One potential version of a hypergraph Oberwolfach problem has recently been asked by Glock, Kühn and Osthus.

**Conjecture 8.1** (Hypergraph Oberwolfach problem; Glock, Kühn and Osthus [20]). Let $k \geq 3$ and suppose $n$ is sufficiently large in terms of $k$ and $k$ divides $\binom{n-1}{k-1}$. Suppose $F$ is a $k$-graph on $n$ vertices that is the disjoint union of tight cycles each of length at least $2k - 1$. Then there is a decomposition of $K_n^{(k)}$ into copies of $F$.

Clearly, Theorem 1.1 yields an approximate solution of Conjecture 8.1.

We think that even an even stronger result is true.

**Conjecture 8.2** (Hypergraph Oberwolfach problem). Let $k \geq 3$ and suppose $n$ is sufficiently large in terms of $k$ and $k$ divides $\binom{n-1}{k-1}$. Suppose $F$ is a $k$-graph on $n$ vertices that is the disjoint union of tight cycles each of length at least $k + 2$. Then there is a decomposition of $K_n^{(k)}$ into copies of $F$.

Observe that Conjecture 8.1 includes the natural generalisation of Walecki's theorem to hypergraphs, namely decompositions into Hamilton cycles. This has already been conjectured by Bailey and Stevens [6] (and when $n$ and $k$ are coprime by Barányi [8] and independently by Katona) and there are a few results that provide approximate decompositions of quasirandom graphs into Hamilton cycles (of various types); see for example [7, 15, 16].

Whenever we allow cycles of length $k + 1$ and the cycle factor consists (essentially) of cycles of length $k + 1$, we suspect that there are more divisibility obstructions present. Hence we pose the following problem.

**Problem 8.3** (Hypergraph Oberwolfach problem). Let $k \geq 3$ and suppose $n$ is sufficiently large in terms of $k$ and $k$ divides $\binom{n-1}{k-1}$. Which disjoint unions of tight cycles whose length add up to $n$ admit a decomposition of $K_n^{(k)}$?

It immediately follows from Theorem 1.1 that the hypergraph Oberwolfach problems are approximately true in a sense that $K_n^{(k)}$ contains $(1 - o(1))\binom{n-1}{k-1}/k$ disjoint copies of $F$ (for any choice of $F$ as above); in fact, we can take any collection of $(1 - o(1))\binom{n-1}{k-1}/k$ cycle factors.

Similarly as for cycles, there is more then one notion for trees in $k$-graphs. Let us stick to the following recursive definition of tree to which we refer as a $k$-tree. A single edge is a $k$-tree. A $k$-tree with $\ell$ edges can be constructed from a $k$-tree with $\ell - 1$ edges $T$ by adding a vertex $v$ and an edge that contains $v$ and a $(k - 1)$-set that is contained in an edge of $T$. For this definition, we propose the following generalisation of Ringel’s conjecture.

**Conjecture 8.4.** Let $k, n \in \mathbb{N} \setminus \{1\}$. Suppose $T$ is a $k$-tree with $n$ edges. Then $K_{kn+k-1}^{(k)}$ admits a decomposition into copies of $T$.

Observe that similarly as for Ringel’s conjecture, the order of the complete graph needs to be at least $kn + k - 1$ if we allow $T$ to be any tree with $n$ edges as the natural generalisation of a star shows. It is an easy exercise to show this conjecture for stars.

There is a conjecture related to Ringel’s conjecture for bipartite graphs due to Graham and Häggkvist stating that $K_{n,n}$ can be decomposed into $n$ copies of any tree with $n$ edges. We propose here the following strengthening.

**Conjecture 8.5.** Suppose $k, n \in \mathbb{N} \setminus \{1\}$ and $T$ is a $k$-tree with $n$ edges. Then there is a decomposition of the complete balanced $k$-partite graph on $kn$ vertices.

The tree packing conjecture has arguably the least obvious strengthening to $k$-graphs and there may be more than one. We propose the following one.

**Conjecture 8.6.** Suppose $k, n \in \mathbb{N} \setminus \{1\}$. Let $T$ be a family of $k$-trees such that $T$ contains $\binom{n-1}{k-2}$ trees with $i$ edges for $i \in [n - k + 1]$. Then $K_n^{(k)}$ admits a decomposition into $T$.

It follows directly from Theorem 1.1 that Conjectures 8.4 and 8.6 are approximately true when restricted to bounded degree trees (and similarly an approximate version of Conjecture 8.5 follows from our main theorem, see Theorem 1.2 in Section 1.2).
8.2. Applications to simplicial complexes. Generalizing long-studied and nowadays classical combinatorial questions to higher dimensions appears to be a challenging but insightful theme. There are several results considering $k$-dimensional permutations and it was Linial and Meshulam [31] who introduced a random model for simplicial complexes whose probability measure is the same as those of a binomial random $(d+1)$-graph; simply add all $d'$-faces for $d' \leq d - 1$ with points in $[n]$ and add every potential $d$-face independently with probability $p$. Recently, Linial and Peled investigated and determined the threshold in $Y_d(n, p)$ for the emergence of a what may considered as an analogue of a giant component [32].

With this topological viewpoint of treating a $k$-graph as a simplicial complex, a cycle in a graph is simply an object homomorphic to $S^1$ and hence a 2-dimensional Hamilton cycle a collection of 2-faces containing every vertex and which is homomorphic to $S^2$. In [33], Luria and Tessler determined the threshold in $Y_2(n, p)$ for the appearance of such a Hamilton cycle (another suitable term may be spanning triangulation of the sphere).

An analogue of Dirac’s theorem was proved by Georgakopoulos, Haslegrave, Narayanan and Montgomery; to be precise, when every pair of vertices is contained in at least $n/3 + o(n)$ edges/2-faces of a 3-graph $G$, then there is a spanning triangulation of the sphere in $G$. This bound remains the same when we replace $S^2$ by any other compact surface without boundary [17].

Instead of only asking for a single triangulation of some surface, we can of course also investigate decompositions into (spanning) triangulations of surfaces. Our results imply that every quasi-random simplicial complex (this in particular includes almost all graphs in $Y_d(n, p)$) can even be decomposed into any list of triangulations of any kind of manifolds provided the density does not vanish too quickly with $n$ and as long as every vertex is contained in at most a bounded number of $d$-faces. The triangulations may even be chosen in advance. Hence a precise statement for 2-complexes is as follows.

**Corollary 8.7.** For all $\alpha > 0$, there exist $n_0, h \in \mathbb{N}$ and $\varepsilon > 0$ such that the following holds for all $n \geq n_0$. Suppose $G$ is an $(\varepsilon, h, d)$-typical 3-graph on $n$ vertices with $d \geq n^{-\varepsilon}$ and $H_1, \ldots, H_\ell$ are spanning triangulations of $S^2$ where every vertex is contained in at most $\alpha^{-1}$ 2-faces and $\ell \leq (1 - \alpha)d n^2/12$. Then $G$ contains edge-disjoint copies of $H_1, \ldots, H_\ell$ such that every 2-face of $G$ is contained in at most one $H_i$.

We omit statements for higher dimensions as they follow in the obvious way from Theorem 1.1. We wonder whether there is always an actual decomposition (subject to certain divisibility conditions). This might be easier than a decomposition into tight Hamilton cycles as the structure of tight Hamilton cycles seems to be more restrictive.

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