The spectrum of IIB supergravity on $\text{AdS}_5 \times S^5/\mathbb{Z}_3$
and a $1/N^2$ test of AdS/CFT

Arash Arabi Ardehali, a James T. Liu, a Phillip Szepietowski b

a Michigan Center for Theoretical Physics, Randall Laboratory of Physics, The University of Michigan, Ann Arbor, MI 48109–1040, USA
b Department of Physics, University of Virginia, Box 400714, Charlottesville, VA 22904, USA

E-mail: ardehali@umich.edu, jimliu@umich.edu, pgs8b@virginia.edu

Abstract: We present the complete Kaluza-Klein spectrum resulting from the compactification of IIB supergravity on $S^5/\mathbb{Z}_3$. Knowledge of this spectrum allows us to perform a holographic computation of the difference of central charges $c - a$ of the dual $\text{SU}(N)^3$ quiver gauge theory. We find the numerical value $c - a = 3/16$, in exact agreement with the field theory result.
1 Introduction

As a strong/weak coupling duality, AdS/CFT allows us to gain additional insight into the non-perturbative regime of gauge theories, and in particular supersymmetric Yang-Mills theories. For the same reason, however, it is often a challenge to compare quantities on both sides of the duality. Even when a calculation can be done on both sides, the results do not necessarily agree. A well known example of this is the result for the free energy of $\mathcal{N} = 4$ SYM in going between weak coupling and strong coupling. While both sides of the duality agree on the general behavior $F \sim N^2 T^4$ (as expected from a large $N$ conformal theory), the free energy picks up a $3/4$ factor in going from weak to strong coupling [1, 2].

Of course, there is no reason to expect the free energy to remain independent of the coupling. However, direct comparisons can be made between protected quantities, and there has been much exploration in this direction. In this context, the Weyl anomaly has proven useful in making the connection between field theories and their holographic duals. In four dimensions, the Weyl anomaly manifests itself in the trace of the stress tensor

$$\langle T^\mu_\mu \rangle = \frac{1}{16\pi^2}(cC_{\mu\nu\rho\sigma}^2 - aE_4),$$

where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor, and $E_4 = R_{\mu\nu\rho\sigma}^2 - 4R_{\mu\nu}^2 + R$ is the four-dimensional Euler density. Here there are two central charges, $a$ and $c$, and the former has received much recent attention as the subject of the $a$-theorem in four [3, 4] and possibly higher [5, 6] dimensions.
Consider the case of $\mathcal{N} = 4$ SYM with gauge group $SU(N)$. A one-loop computation in the field theory gives

$$c = a = \frac{N^2 - 1}{4},$$

where $N^2 - 1$ is the dimension of $SU(N)$ and represents the weakly coupled degrees of freedom. The dual theory is given by IIB string theory on $AdS_5 \times S^5$, and the holographic Weyl anomaly, first computed in [7], matches the field theory result at leading order in $N^2$. More generally, for IIB string theory on $AdS_5 \times X^5$, the leading contribution is obtained from the classical gravity sector, and gives [8]

$$c = a = \frac{N^2}{4} \frac{\pi^3}{\text{vol}(X^5)}. \quad (1.3)$$

Our present interest is to extend this comparison between the field theory and the holographic calculation beyond leading order in $N^2$. For the gravitational dual to $\mathcal{N} = 4$ SYM, the $\mathcal{O}(1)$ subtraction that shifts $N^2 \rightarrow N^2 - 1$ was obtained in [9–13] by summing over one-loop corrections due to the Kaluza-Klein tower on $S^5$. Moreover, in theories with reduced supersymmetry there are additional potential sources for subleading contributions to $c$ and $a$. Since $c \sim a \sim N^2$ at leading order, it is often useful to characterize these corrections as a shift in $c - a$. Open string loops, and in particular the inclusion of $D$-branes (such as $D7$ flavor branes) will induce a $1/N$ correction, while closed string loops will induce a $1/N^2$ correction.

In this paper, we focus on IIB string theory on the $S^5/Z_3$ orbifold. This orbifolding reduces the supersymmetry, and as a result the dual quiver gauge theory is $\mathcal{N} = 1$ SYM with gauge group $SU(N)^3$. On the gauge theory side, it is easy to see that the $\mathcal{O}(1)$ contribution to $c - a$ is simply

$$c - a = \frac{3}{16}, \quad (1.4)$$

corresponding to three decoupled $\mathcal{N} = 1$ vector multiplets from the three $U(1)$ factors. Our aim is to reproduce this $3/16$ factor from the gravity side of the duality. Since $Z_3$ acts freely on $S^5$, the $S^5/Z_3$ orbifold has no singularities, and hence requires no brane sources. As a result, the first correction to $c - a$ arises from closed string loops, in agreement with $c - a$ being an $\mathcal{O}(1)$ effect.

There are, in fact, two sources of closed string loop corrections. The first originates from massive string states in the loop, and is most directly encoded by the one-loop $R^4$ term in the type II string effective action in ten dimensions [14–17]. The compactification of the $R^4$ correction on a Sasaki-Einstein manifold $SE_5$ was investigated in [18], and the result is that the five-dimensional action will pick up a $R^2$ term of the form

$$S = \frac{1}{2\kappa_5^2} \int d^5x \left( R + \frac{12}{L^2} + \alpha R_{\mu
u\rho\sigma} R^{\mu
u\rho\sigma} + \cdots \right), \quad (1.5)$$

where the coefficient $\alpha$ may be determined in terms of the data specifying $SE_5$ when written in canonical form as $U(1)$ fibered over a Kahler-Einstein base. The inclusion of this Riemann-
squared correction modifies the holographic Weyl anomaly computation \[19, 20\], and we find
\[ c - a = \frac{\alpha}{8L^2}a_0, \quad \text{where} \quad a_0 = \frac{\pi^2 L^3}{\kappa_5^2}. \]  
(1.6)
(Here \(a_0\) denotes the leading order central charge.) Note, however, that this contribution from massive string loops vanishes in the case of \(S^5/\mathbb{Z}_3\), as the geometry is locally that of \(S^5\), which has \(\alpha = 0\).

Since massive string loops do not contribute to \(c - a\) for the \(S^5/\mathbb{Z}_3\) orbifold, the factor \((1.4)\) must come entirely from the second type of correction. This correction arises at the one-loop level with particles in the Kaluza-Klein tower running in the loop. As mentioned above, this loop correction was previously computed in order to demonstrate the shift \(N^2 \to N^2 - 1\) in both \(a\) and \(c\) for IIB supergravity on \(S^5\). Thus all that is needed here is to repeat the procedure, but this time with the spectrum of IIB supergravity on \(S^5/\mathbb{Z}_3\). We follow the approach of \([11–13]\), and in particular we use the expression for the correction to the leading order Weyl anomaly
\[ \delta (T_\mu^\mu) = -\sum \frac{(E_0 - 2)a_2}{32\pi^2}, \]  
where the sum is over all states in the KK tower. Here \(E_0\) is the lowest energy defining the representation and \(a_2\) is a four-dimensional heat kernel coefficient (with an extra sign for anti-commuting fields). Comparing this with \((1.1)\) gives
\[ c - a = -\frac{1}{2} \sum (E_0 - 2)a_2 \bigg|_{R^2_{\mu\nu\rho\sigma} \text{ term}}, \]  
(1.8)
where, since \(c = a\) at leading order for \(S^5/\mathbb{Z}_3\), the entire contribution to \(c - a\) is from the Kaluza-Klein spectrum.

Our main result is that the sum over Kaluza-Klein modes in \((1.8)\) gives \(3/16\), and hence exactly matches the field theory result. In order to perform the sum, we of course need the KK spectrum on \(S^5/\mathbb{Z}_3\), which may be obtained by \(\mathbb{Z}_3\) projection of the \(S^5\) spectrum. As a bonus, we also elucidate the \(\mathcal{N} = 2\) multiplet structure and shortening conditions of this spectrum.

In section 2, we examine the KK spectrum of IIB supergravity compactified on \(S^5/\mathbb{Z}_3\). Then, in section 3, we compute \(c - a\) by summing over this spectrum and find perfect agreement with the dual quiver gauge theory. Some of the details of constructing the KK spectrum are relegated to appendices.

2 The Kaluza-Klein spectrum of IIB supergravity on \(\text{AdS}_5 \times S^5/\mathbb{Z}_3\)

Type IIB supergravity compactified on \(S^5\) gives rise to \(\mathcal{N} = 8\) gauged supergravity in five dimensions along with an infinite tower of Kaluza-Klein modes. The KK spectrum was worked out in \([21, 22]\) by expanding the linearized ten-dimensional fields in harmonics on the five-sphere. The harmonics fall into representations of the \(\text{SU}(4)\) isometry group of \(S^5\), and
Table 1: The Kaluza-Klein spectrum of IIB supergravity on $\text{AdS}_5 \times S^5$. The representations are labeled $D(E_0, s_1, s_2; l_1, l_2, l_3)$ where $(E_0, s_1, s_2)$ specifies the SO$(2, 4)$ AdS$_5$ representation and $(l_1, l_2, l_3)$ are the Dynkin labels of the SU$(4)$ representation for $S^5$. The labeling of the fields correspond to that of Ref. [21].

The KK levels may be labeled by a single integer $p \geq 2$, which corresponds to the oscillator number in [21]. At each level $p$, the fluctuations assemble into unitary representations of the supergroup SU$(2, 2|4)$. Level $p = 2$ corresponds to the massless $\mathcal{N} = 8$ supergravity multiplet, level $p = 3$ is shortened, and levels $p \geq 4$ are long $\mathcal{N} = 8$ supermultiplets.

The bosonic subgroup of SU$(2, 2|4)$ is SO$(2, 4) \times$ SU$(4)$, where SO$(2, 4)$ is the isometry group of AdS$_5$ and SU$(4)$ is the isometry group of $S^5$. We label AdS$_5$ representations by $D(E_0, s_1, s_2)$, with $E_0$, $s_1$ and $s_2$ the quantum numbers of the lowest state under the maximal compact subgroup SO$(2) \times$ SU$(2) \times$ SU$(2) \simeq$ SO$(2) \times$ SO$(4) \subset$ SO$(2, 4)$. For the KK spectrum on AdS$_5 \times S^5$, since each state in AdS$_5$ transforms under a representation of the $R$-symmetry group SU$(4)$, we also append the Dynkin labels for the SU$(4)$ representation, and label AdS$_5 \times S^5$ representations as $D(E_0, s_1, s_2; l_1, l_2, l_3)$. The KK spectrum is summarized in Table 1.
2.1 The $S^5/Z_3$ orbifold

The $S^5/Z_3$ orbifold is defined by the $Z_3$ action

$$X^i \to e^{2\pi i/3} X^i, \quad i = 1, 2, 3,$$  \hspace{1cm} (2.1)

where the $X^i$ are complex coordinates on the transverse $\mathbb{C}^3$ space to the stack of D3-branes. Since this action is in the center of SU(3), the orbifold preserves $\mathcal{N} = 2$ supersymmetry in five dimensions and gives rise to an $\mathcal{N} = 1$ quiver gauge theory that was investigated in [23–25]. Note that the action (2.1) acts freely away from the origin, so that $S^5/Z_3$ is a lens space. As a result, the KK spectrum on $S^5/Z_3$ is given simply by the subset of states on $S^5$ that are invariant under the $Z_3$ action. Determining the spectrum thus reduces to an exercise in group theory that was initiated in [26, 27]. Here we complete this procedure and highlight the resulting $\mathcal{N} = 2$ supermultiplet structure.

To identify the KK states on $S^5/Z_3$, we decompose SU(4) $\supset$ SU(3) $\times$ U(1), where SU(4) is the isometry group of $S^5$, and the remaining U(1) is the $\mathcal{N} = 2$ R-symmetry. We furthermore normalize the U(1) charge by taking

$$4 \to 3_{1/3} \oplus 1_{-1}.$$  \hspace{1cm} (2.2)

We now observe that the states invariant under (2.1) are those with triality zero. Thus what needs to be done is to take the SU(4) representations in Table 1, branch them into SU(3) $\times$ U(1), and then select the triality zero subset. With the U(1) normalization given in (2.2), it is easy to see that this is equivalent to keeping only states with integer $R$-charge.

Obtaining the KK spectrum on $S^5/Z_3$ is now a straightforward exercise in group theory, and the result is presented in Table 2. The necessary branching rules and details of the notation are given in Appendix A. Since each SU(4) representation branches into an entire series of SU(3) representations, we use the compact notation $D(E_0, s_1, s_2; [a, b]_s)$ where the symbols $[a, b]_s$ are given by

$$[a, b]_s = \bigoplus_l (a + 3l - p, b + 2p - 3l)_{2p-4l+s}. \hspace{1cm} (2.3)$$

(We keep the KK level $p$ implicit for compactness of the notation.) Additional subscripts inside the square brackets will be used to indicate restrictions on the Dynkin labels, and are related to multiplet shortening, as we will see below. For more details, see Appendix A.

2.2 Filling out $\mathcal{N} = 2$ supermultiplets

Since the $S^5/Z_3$ orbifold preserves $\mathcal{N} = 2$ supersymmetry in five dimensions, the KK spectrum of Table 2 ought to fall into representations of the $\mathcal{N} = 2$ superalgebra SU(2, 2$|$1). These representations were constructed in [28, 29] (see also [30]), and may be labeled $D(E_0, s_1, s_2; r)$, where the quantum numbers correspond to those of the lowest energy state. The generic long
represents the Kaluza-Klein spectrum of IIB supergravity on $\text{AdS}_5 \times S^5/\mathbb{Z}_3$. The notation is explained in the text and in Appendix A.

representations are given by

$$D(E_0, s_1, s_2; r) = D(E_0, s_1, s_2; r)$$

$$+ D(E_0 + \frac{1}{2}, s_1 + \frac{1}{2}, s_2; r - 1) + D(E_0 + \frac{1}{2}, s_1, s_2 + \frac{1}{2}; r - 1)$$

$$+ D(E_0 + 1, s_1, s_2; r + 2) + D(E_0 + 1, s_1 \pm \frac{1}{2}, s_2 \pm \frac{1}{2}; r)$$

$$+ D(E_0 + \frac{3}{2}, s_1 + \frac{1}{2}, s_2, r + 1) + D(E_0 + \frac{3}{2}, s_1, s_2 + \frac{1}{2}, r - 1)$$

$$+ D(E_0 + 2, s_1, s_2, r),$$  (2.4)
where the plus/minus signs are uncorrelated. For particular values of the quantum numbers, the long multiplets are truncated to shorter ones. There are three multiplet shortening conditions as follows:

\begin{align*}
\text{conserved :} & \quad E_0 = 2 + s_1 + s_2, \\
\text{chiral (antichiral) :} & \quad E_0 = \frac{3}{2} r \quad (E_0 = -\frac{3}{2} r), \\
\text{semi-long I (semi-long II) :} & \quad E_0 = 2 + 2s_1 - \frac{3}{2} r \quad (E_0 = 2 + 2s_2 + \frac{3}{2} r).
\end{align*}  

(2.5)

All three possibilities will show up in the $S^5/Z_3$ spectrum.

The $\mathcal{N} = 2$ multiplet structure of IIB supergravity compactified on $T^{11,1}$ was highlighted in [31, 32], and subsequently it was demonstrated in [33] that the same pattern of multiplets arise in any Sasaki-Einstein compactification. In the language of [31, 32], the KK spectrum arranges itself into nine families of supermultiplets: Graviton, Gravitino I through IV and Vector I through IV. (Other special multiplets or Betti multiplets may arise depending on topology, but they are not present in the $S^5/Z_3$ spectrum.)

The key to assembling the KK states into $\mathcal{N} = 2$ multiplets is the realization that all states in a given multiplet must transform in the same SU(3) representation. Cursory examination of Table 2 indicates there are nine sets of SU(3) representations:

\[ [0,0], \quad [1,-2], \quad [-2,1], \quad [-4,2], \quad [2,-4], \quad [-1,-1], \quad [-3,0], \quad [0,-3], \quad [-2,-2], \]

(2.6)

where we have suppressed the $R$-charge subscript. However, this is somewhat misleading, as the different symbols are not unique. In particular, the second identity of (A.4) allows us to shift the two labels by three, so that there are only five distinct sets:

\[ [0,0], \quad [1,-2] \sim [-2,1], \quad [-4,2] \sim [-1,-1] \sim [2,-4], \quad [-3,0] \sim [0,-3], \quad [-2,-2]. \]  

(2.7)

By examining the field content transforming in each of these sets, we may arrange the states into the nine general families of [31–33]. We find that the $[0,0]$ states make up Vector Multiplet I, the $[1,-2] \sim [-2,1]$ states fill out Gravitino Multiplets I and III, the $[-4,2] \sim [-1,-1] \sim [2,-4]$ states split up into the Graviton Multiplet, and Vector Multiplets III and IV, the $[-3,0] \sim [0,-3]$ states fill out Gravitino Multiplets II and IV and the $[-2,-2]$ states correspond to Vector Multiplet II. This is summarized in Table 3; for additional information, see Appendix B.

### 2.3 Multiplet shortening

Until now, we have mostly ignored the subscripted constraints in the representation symbols $[a_{>i}, b_{>j}]_s$. What these constraints indicate is that certain representations are absent in the KK spectrum. When arranged in supermultiplets, as in Table 3, the result is that some states are absent within supermultiplets. It is natural to expect that this corresponds to multiplet shortening, and in fact this is exactly what happens. We list the entire set of shortened multiplets in Table 4, and relegate the details to Appendix B.
Table 3. The $\mathcal{N} = 2$ spectrum of IIB supergravity on $S^5/Z_3$. In this table we employ a somewhat unorthodox notation in which the $SU(3)$ representation shorthand $[a, b]_s$ is written explicitly inside of each $\mathcal{N} = 2$ representation as $\mathcal{D}(E_0, s_1, s_2; [a, b]_s)$. This is due to the fact that different terms in the sum (2.3) have different $R$-charges. The notation should be interpreted as a sum of $\mathcal{N} = 2$ multiplets with each term having the appropriate $R$-charge and $SU(3)$ representation as the terms in the sum defined by $[a, b]_s$.

| Supermultiplet | Representation | KK level |
|----------------|----------------|----------|
| Graviton       | $\mathcal{D}(p + 1, 0; [-1, -1]_0)$ | $p \geq 2$ |
| Gravitino I and III | $\mathcal{D}(p + \frac{1}{2}, 0, [1, -2]_1) + \mathcal{D}(p + \frac{1}{2}, 0, [-2, 1]_1)$ | $p \geq 2$ |
| Gravitino II and IV | $\mathcal{D}(p + \frac{3}{2}, 0, [0, -3]_1) + \mathcal{D}(p + \frac{3}{2}, 0, [0, -3]_1)$ | $p \geq 3$ |
| Vector I       | $\mathcal{D}(p, 0, 0; [0, 0]_0)$ | $p \geq 2$ |
| Vector II      | $\mathcal{D}(p + 2, 0, 0; [-2, -2]_1)$ | $p \geq 3$ |
| Vector III and IV | $\mathcal{D}(p + 1, 0, 0; [-1, -1]_2) + \mathcal{D}(p + 1, 0, 0; [-1, -1]_2)$ | $p \geq 2$ |

Table 4. Shortening structure of the $S^5/Z_3$ KK tower. The supermultiplets are given in the conventional notation $\mathcal{D}(E_0, s_1, s_2; r)$ with the $SU(3)$ representation $(l_1, l_2)$ appended. Note that Vector Multiplet II is never shortened.
Of particular interest are the multiplets arising at the non-generic \( p = 2 \) and \( p = 3 \) KK levels. For \( p = 2 \), the shortened multiplets are the Graviton (conserved), Vector I (conserved) and Vectors III and IV (anti-chiral and chiral). This corresponds to the gauged supergravity sector, with the supergravity multiplet coupled to an SU(3) adjoint vector multiplet and the universal hypermultiplet. At the \( p = 3 \) level, we encounter Gravitinos I and II (semi-long) transforming in the adjoint, Gravitinos II and IV (semi-long), and a massive Vector I (chiral and anti-chiral) transforming in the 10 and \( \overline{10} \) of SU(3). There are no long multiplets below \( p = 4 \).

### 2.4 Dual operators with protected dimension

On the CFT side, the shortened multiplets are dual to superfields with protected dimension. After having identified these multiplets, it is a fairly straightforward task to find their dual superfields, building on the earlier results of [27] and with the aid of the similar expressions given for the Klebanov-Witten theory in [31]. The quiver gauge theory consists of SU(\( N \)) gauge superfields \( V_i, i = 1, 2, 3 \), whose field strength superfields we denote by \( W_i^\alpha \), and three triplets of chiral superfields \( A_a, B_a, C_a, a = 1, 2, 3 \), transforming according to the \((N, \bar{N}, 1), (1, N, \bar{N}), (\bar{N}, 1, N)\) representations of the gauge group. Now, the chiral multiplets in Vector Multiplet I, the chiral tensor multiplets in Gravitino Multiplet I and the chiral multiplets in Vector Multiplet IV correspond respectively to the chiral superfields of the form\(^1\)

\[
S^p = Tr \left( (ABC)^{p/3} \right), \quad \Delta^p = p, \quad r = \frac{2}{3}p, \quad p = 3l \geq 3, \quad (2.8)
\]

\[
T^p = Tr \left( W_\alpha (ABC)^{(p-1)/3} \right), \quad \Delta^p = p + 1/2, \quad r = \frac{2}{3}p + \frac{1}{3}, \quad p = 3l + 1 \geq 4, \quad (2.9)
\]

\[
\Phi^p = Tr \left( W^\alpha W_\alpha (ABC)^{(p-2)/3} \right), \quad \Delta^p = p + 1, \quad r = \frac{2}{3}p + \frac{2}{3}, \quad p = 3l + 2 \geq 2. \quad (2.10)
\]

The semi-long multiplets in Graviton Multiplet, Gravitino Multiplet I (SLI), Gravitino Multiplet II (SLII), Gravitino Multiplet I (SLII), Vector Multiplet I and Vector Multiplet IV correspond respectively to the (semi-)conserved superfields of the form

\[
J^p_{\alpha \bar{\alpha}} = Tr \left( J_{\alpha \bar{\alpha}} (ABC)^{(p-2)/3} \right), \quad \Delta^p = p + 1, \quad r = \frac{2}{3}p - \frac{4}{3}, \quad p = 3l + 2 \geq 2, \quad (2.11)
\]

\[
L^p_{1 \bar{\alpha}} = Tr \left( e^V \bar{W}^\alpha e^{-V} (ABC)^{(p-1)/3} \right), \quad \Delta^p = p + 1/2, \quad r = -\frac{2}{3}p + \frac{5}{3}, \quad p = 3l + 1 \geq 4, \quad (2.12)
\]

\[
L^p_{2 \bar{\alpha}} = Tr \left( e^V \bar{W}^\alpha e^{-V} W^2 (ABC)^{(p-3)/3} \right), \quad \Delta^p = p + 3/2, \quad r = -\frac{2}{3}p + 1, \quad p = 3l + 3 \geq 3, \quad (2.13)
\]

\[
L^p_{3 \bar{\alpha}} = Tr \left( W_\alpha (Ae^V \bar{A}e^{-V}) (ABC)^{(p-3)/3} \right), \quad \Delta^p = p + 1/2, \quad r = \frac{2}{3}p - 1, \quad p = 3l + 3 \geq 3, \quad (2.14)
\]

\(^1\)As in [31], while we do not make it explicit, we always mean the symmetrized trace (over the \( a \) indices) and properly inserted field strengths. For example, \( T^4 = Tr \left( W_1^\alpha A_{(a} B_0 C_{c)} + A_{(a} W_2^\beta B_0 C_{c)} + A_{(a} B_0 W_3^\gamma C_{c)} \right) \).
\[ J^p = \text{Tr} \left( J(ABC)^{(p-2)/3} \right), \quad \Delta^p = p, \quad r = \frac{2}{3}p - \frac{4}{3}, \quad p = 3l + 2 \geq 2, \]  
(2.15)

\[ I^p = \text{Tr} \left( JW^2(ABC)^{(p-4)/3} \right), \quad \Delta^p = p + 1, \quad r = \frac{2}{3}p - \frac{2}{3}, \quad p = 3l + 4 \geq 4, \]  
(2.16)

where

\[ J_{\alpha\dot{\alpha}} = W_\alpha e^V \bar{W}_{\dot{\alpha}} e^{-V}, \]  
(2.17)

\[ J = A(e^V \bar{A} e^{-V}). \]  
(2.18)

(Although we have singled out the chiral superfield \( A \), the proper symmetrization over the chiral superfields should be understood.)

Note in particular that CFT operators dual to AdS multiplets at KK level \( p \) have exactly \( p \) superfields in them, similar to the case of the \( S^5 \) compactification. This is to be expected, of course, as \( S^5/\mathbb{Z}_3 \) is simply related to \( S^5 \) by orbifolding. This connection between KK level and the length of the dual operators will provide some insight into the regularization scheme used in the following section.

3 The holographic computation of \( c-a \)

Before proceeding with the computation of \( c-a \), it is worth reviewing the leading order Weyl anomaly for the \( S^5/\mathbb{Z}_3 \) theory. On the gauge theory side, the \( SU(N)^3 \) quiver contains three vector multiplets \( (c = 1/8, a = 3/16) \) in the adjoint and nine chiral multiplets \( (c = 1/24, a = 1/48) \) in bifundamentals. Summing up these contributions then gives

\[ c = \frac{3N^2}{4} - \frac{3}{8}, \quad a = \frac{3N^2}{4} - \frac{9}{16}, \]  
(3.1)

so that \( c-a = 3/16 \). On the gravity side, we use the leading order holographic Weyl anomaly expression (1.3) with \( \text{vol}(S^5/\mathbb{Z}_3) = \text{vol}(S^5)/3 \) to obtain \( c = a = 3N^2/4 \), which agrees with the above at \( \mathcal{O}(N^2) \).

In order to obtain the \( \mathcal{O}(1) \) contribution to \( c-a \), we need to sum over all states in the KK tower according to (1.8). Using the heat kernel coefficients in [34], we tabulate the contribution of individual fields to \( c-a \) in Table 5. Since we are interested in representations of \( \mathcal{N} = 2 \) supersymmetry, we now sum these contributions over each component of the multiplet for vector, gravitino and graviton multiplets. The result is presented in Table 6.

Note in particular that long multiplets do not contribute to \( c-a \), so we only need to sum over the shortened spectrum for \( S^5/\mathbb{Z}_3 \) as given in Table 4. In fact, since equation (1.8) is derived in [13] from relations involving bare masses of the bulk theory, the vanishing contribution of long multiplets (with presumably unprotected masses) is essential for the secured computability of the subleading Weyl anomaly of the boundary theory. Overlooking this subtlety one might have attempted to reproduce the individual central charges \( c \) and \( a \) of the quiver gauge theory following [12], but then the contributions of long multiplets become non-vanishing and a knowledge of renormalized masses of the bulk theory is required.
Graviton $D$ 1 $0$
Gravitino $D$ 0 $0$

Note that this breaks up into three contributions based on $p$

Field | Representation | Contribution to $360(c - a)$
--- | --- | ---
$\phi$ | $D(E_0, 0, 0)$ | $-(E_0 - 2)$
$\lambda$ | $D(E_0, \frac{1}{2}, 0) + D(E_0, 0, \frac{1}{2})$ | $\frac{7}{2}(E_0 - 2)$
$A_\mu$ | $D(3, \frac{1}{2}, \frac{1}{2})$ | 13
$A_{\mu
u}$ | $D(E_0 > 3, \frac{1}{2}, \frac{1}{2})$ | $11(E_0 - 2)$
$\psi_\mu$ | $D(\frac{7}{2}, 1, \frac{1}{2}) + D(\frac{7}{2}, \frac{1}{2}, 1)$ | 173
$h_{\mu\nu}$ | $D(E_0 > \frac{7}{2}, 1, \frac{1}{2}) + D(E_0 > \frac{7}{2}, \frac{1}{2}, 1)$ | $\frac{219}{2}(E_0 - 2)$
$D(E_0 > 4, 1, 1)$ | $-411$

Table 5. The contribution to $360(c - a)$ from fields with spins no higher than two. The massless vector, gravitino and graviton contributions include the appropriate ghost sector.

| Field | Vector | Gravitino | Graviton |
|---|---|---|---|
| $D(E_0, 0, 0; r)$ | $\frac{1}{32}$ | $-\frac{5}{48}(E_0 - \frac{3}{2})$ | $-\frac{5}{27}(E_0 - \frac{3}{2})$
| $D(E_0, \frac{1}{2}, 0; r)$ | $-\frac{1}{96}(E_0 - \frac{3}{2})$ | $\frac{5}{27}(E_0 - \frac{3}{2})$ | $\frac{5}{27}(E_0 - \frac{3}{2})$
| $D(E_0, \frac{1}{2}, \frac{1}{2}; r)$ | $\frac{1}{96}(E_0 - \frac{1}{2})$ | $\frac{5}{27}(E_0 - \frac{1}{2})$ | $\frac{5}{27}(E_0 - \frac{1}{2})$
| $D(E_0, 0, \frac{1}{2}; r)$ | $0$ | $0$ | $0$

Table 6. The contribution to $c - a$ from vector, gravitino and graviton multiplets. Note the vanishing contribution from long multiplets.

Writing out the sum over shortened multiplets for $S^5/\mathbb{Z}_3$, we find

$$c - a = \frac{1}{64} \sum_{p \geq 2} \left\{ \begin{array}{ll}
p(-6p^2 + 3p + 5), & p = 2, 5, 8, \ldots \\
2p(6p^2 - 5), & p = 3, 6, 9, \ldots \\
p(-6p^2 - 3p + 5), & p = 4, 7, 10, \ldots \end{array} \right. \tag{3.2}$$

Note that this breaks up into three contributions based on $p \mod 3$, as one may expect from the nature of the $\mathbb{Z}_3$ orbifold. As in the $S^5$ case treated in [12], this series is divergent, and hence needs to be regulated. The regulation procedure used in [12] is to multiply each term in the sum by $z^p$. The sum then becomes absolutely convergent for $z < 1$. We then analytically continue the result and examine the behavior as $z \to 1$. In the present case, in fact, the result is finite for $z = 1$, and we find $c - a = 3/16$, thus matching the gauge theory result.

Alternatively, we may perform a zeta function regularization. This is complicated somewhat by the fact that the sum splits into three expressions depending on $p \mod 3$. Here it
convenient to introduce the Hurwitz zeta function

\[ \zeta(s, \alpha) = \sum_{n=0}^{\infty} (n + \alpha)^{-s}, \]  

which generalizes the Riemann zeta function

\[ \zeta(s) = \sum_{n=1}^{\infty} n^{-s}. \]  

We then break up the sum (3.2) into three terms, with \( p = 3k + 2 \), \( p = 3k + 3 \) and \( p = 3k + 4 \)

\[
c - a = \frac{3}{64} \sum_{k \geq 0} \left[ -54(k + \frac{2}{3})^3 + 9(k + \frac{2}{3})^2 + 5(k + \frac{2}{3}) + 108(k + 1)^3 - 10(k + 1) \\
- 54(k + \frac{4}{3})^3 - 9(k + \frac{4}{3})^2 + 5(k + \frac{4}{3})^2 \right] \\
= \frac{3}{64} \left[ -54\zeta(-3, \frac{2}{3}) + 9\zeta(-2, \frac{2}{3}) + 5\zeta(-1, \frac{2}{3}) + 108\zeta(-3, 1) - 10\zeta(-1, 1) \\
- 54\zeta(-3, \frac{4}{3}) - 9\zeta(-2, \frac{4}{3}) + 5\zeta(-1, \frac{4}{3}) \right] \\
= \frac{3}{16},
\]  

in perfect agreement with the above.

4 Discussion

We have performed a one-loop test of AdS/CFT by matching the \( O(1) \) contribution to the difference of central charges, \( c - a \), in both the SU(\( N \))^3 quiver gauge theory and its AdS dual. In order to make this comparison, we have explicitly obtained the KK spectrum of IIB supergravity on \( S^5/\mathbb{Z}_3 \). As expected on general grounds, the spectrum may be arranged into nine towers of \( \mathcal{N} = 2 \) supermultiplets in parallel with the \( T^{1,1} \) case.

In addition to \( S^5/\mathbb{Z}_3 \), the \( T^{1,1} \) case was considered in [18], and a prediction was given that Kaluza-Klein loops ought to give a shift in \( c - a \) of 1/12. This would be in addition to the contribution 1/24 that arises from massive string loops. Since the KK spectrum on \( T^{1,1} \) is known [31, 32], it would be informative to see if this prediction pans out. This case is currently under investigation.

Finally, although we have focused on \( c - a \), which is of \( O(1) \), it would be desirable to reproduce either \( c \) or \( a \) directly, as given in (3.1). The difficulty in doing so appears to be twofold. Firstly, one would need to keep subleading terms in the holographic computation of the \( a \) central charge in (1.6). This will involve higher derivative corrections to the volume of \( S^5/\mathbb{Z}_3 \) and the effective AdS radius \( L \). Secondly, for the sum over KK states, while the heat kernel coefficients leading to, say, \( a \) are known, they will depend on Ricci terms that may be shifted around in the one-loop determinants. Thus additional care may be needed to identify the appropriate equations of motion pertaining to the KK tower.
Acknowledgments

The idea to compute the subleading contribution to \(c-a\) for IIB supergravity on \(\text{AdS}_5 \times S^5/\mathbb{Z}_3\) came out of discussions with R. Minasian following the completion of [18]. This work is supported in part by the US Department of Energy under grants DE-SC0007859 and DE-SC0007984.
The representations are given by their Dynkin labels and the U(1) charge is normalized by (2.2).

The KK spectrum on $S^5/\mathbb{Z}_3$ is obtained by projecting the $S^5$ spectrum onto $\mathbb{Z}_3$ invariant states. This is done by first branching the relevant SU(4) representations under SU(4) ⊃ SU(3) × U(1) and then selecting the triality zero representations of SU(3).

Based on the SU(4) representations in Table 1, we need the following branching rules:

\[
\begin{align*}
(0, n, 0) &= \bigoplus_{k=0}^{n} (k, n - k)_{(2n-4k)/3}, \\
(1, n, 0) &= \bigoplus_{k=0}^{n} (k + 1, n - k)_{(2n-4k+1)/3} \bigoplus_{k=0}^{n} (k, n - k)_{(2n-4k-3)/3}, \\
(2, n, 0) &= \bigoplus_{k=0}^{n} (k + 2, n - k)_{(2n-4k+2)/3} \bigoplus_{k=0}^{n} (k + 1, n - k)_{(2n-4k-2)/3} \bigoplus_{k=0}^{n} (k, n - k)_{(2n-4k-6)/3}, \\
(1, n, 1) &= \bigoplus_{k=0}^{n} (k + 1, n - k)_{(2n-4k+4)/3} \bigoplus_{k=0}^{n} (k + 1, n - k + 1)_{(2n-4k)/3} \bigoplus_{k=0}^{n} (k, n - k)_{(2n-4k)/3} \\
&\quad \bigoplus_{k=0}^{n} (k, n - k + 1)_{(2n-4k-4)/3}, \\
(2, n, 1) &= \bigoplus_{k=0}^{n} (k + 2, n - k)_{(2n-4k+5)/3} \bigoplus_{k=0}^{n} (k + 2, n - k + 1)_{(2n-4k+1)/3} \\
&\quad \bigoplus_{k=0}^{n} (k + 1, n - k)_{(2n-4k+1)/3} \bigoplus_{k=0}^{n} (k + 1, n - k + 1)_{(2n-4k-3)/3} \\
&\quad \bigoplus_{k=0}^{n} (k, n - k)_{(2n-4k-3)/3} \bigoplus_{k=0}^{n} (k, n - k + 1)_{(2n-4k-7)/3}, \\
(2, n, 2) &= \bigoplus_{k=0}^{n} (k + 2, n - k)_{(2n-4k+8)/3} \bigoplus_{k=0}^{n} (k + 1, n - k)_{(2n-4k+4)/3} \\
&\quad \bigoplus_{k=0}^{n} (k + 2, n - k + 1)_{(2n-4k+4)/3} \bigoplus_{k=0}^{n} (k + 2, n - k + 2)_{(2n-4k)/3} \\
&\quad \bigoplus_{k=0}^{n} (k + 1, n - k + 1)_{(2n-4k)/3} \bigoplus_{k=0}^{n} (k, n - k)_{(2n-4k)/3} \\
&\quad \bigoplus_{k=0}^{n} (k + 1, n - k + 1)_{(2n-4k-4)/3} \bigoplus_{k=0}^{n} (k, n - k + 1)_{(2n-4k-4)/3} \\
&\quad \bigoplus_{k=0}^{n} (k, n - k + 2)_{(2n-4k-8)/3}. \\
\end{align*}
\]

The representations are given by their Dynkin labels and the U(1) charge is normalized by (2.2).
Since a given SU(3) representation labeled by \((l_1, l_2)\) has triality \(l_1 + 2l_2 \equiv 0 \pmod{3}\), the \(Z_3\) singlet states are those with \(l_1 + 2l_2 \equiv 0 \pmod{3}\), or equivalently \(l_1 \equiv l_2 \pmod{3}\). Note that such states also have integer \(R\)-charge. It is now a straightforward exercise to obtain the \(Z_3\) singlets in the decomposition of the SU(4) representations given in (A.1). For example, the \((0, n, 0)\) representation branches into a sum of SU(3) representations with \(R\)-charge \(2n - 4k\). The requirement that this is an integer gives the condition \(2n - 4k \equiv 0 \pmod{3}\), which is equivalent to \(n + k \equiv 0 \pmod{3}\). We thus let \(k = 3l - n\) with \(l \in \mathbb{Z}\), and find

\[
(0, n, 0) \rightarrow \bigoplus_{l=[n/3]}^{\lfloor 2n/3 \rfloor} (3l - n, 2n - 3l)_{2n-4l},
\]

under the \(Z_3\) projection. The other representations in (A.1) follow a similar pattern.

We find it convenient to introduce a shorthand notation for sets of triality zero representations that show up in the right hand side of expressions such as (A.2). Let

\[
[a, b]_s(n) \equiv \bigoplus_{l=[(n-a)/3]}^{\lfloor (2n+b)/3 \rfloor} (a + 3l - n, b + 2n - 3l)_{2n-4l+s}.
\]

Note that the allowed values of \(l\) in the sum are those for which the Dynkin labels give rise to valid SU(3) representations. In particular, the restriction is that \(l_1 \geq 0 \) and \(l_2 \geq 0\) for a representation labeled by \((l_1, l_2)\). The symbols \([a, b]_s(n)\) satisfy the following relations

\[
[a, b]_s(n - k) = [a + k, b - 2k]_{s-2k}(n),
\]

\[
[a, b]_s(n) = [a + 3k, b - 3k]_{s-4k}(n),
\]

\[
[a, b]_s(n) = [b, a]_{-s}(n),
\]

where the last line corresponds to the conjugate representation. Triality zero states correspond to \(a \equiv b \pmod{3}\).

Because of the small integer offsets such as \(k + 1\) or \(n - k + 1\) that show up in (A.1), in some cases it is necessary to impose a stronger restriction on allowed values of \(l_1\) and \(l_2\). We thus allow for a refinement of the notation in (A.3) by introducing

\[
[a_{>i}, b_{>j}]_s(n).
\]

In particular, the subscripted expressions indicates that the allowed representations are restricted to \(l_1 > i\) and \(l_2 > j\). If the subscript is absent, then the corresponding Dynkin label need only be non-negative.

Finally, this allows us to write the \(Z_3\) singlet content of the branched SU(4) representa-
tions in (A.1) as

\[ (0, n, 0) \rightarrow [0, 0]_0(n), \]

\[ (1, n, 0) \rightarrow [2_{>0}, -1]_{-1}(n) \oplus [0, 0]_{-1}(n), \]

\[ (2, n, 0) \rightarrow [4_{>1}, -2]_{-2}(n) \oplus [2_{>0}, -1]_{-2}(n) \oplus [0, 0]_{-2}(n), \]

\[ (1, n, 1) \rightarrow [2_{>0}, 0]_1(n) \oplus [1_{>0}, 1_{>0}]_0(n) \oplus [0, 0]_0(n) \oplus [1_{>0}, 1_{>0}]_1(n), \]

\[ (2, n, 1) \rightarrow [4_{>1}, -2]_{-1}(n) \oplus [3_{>1}, 0_{>0}]_{-1}(n) \oplus [2_{>0}, -1]_{-1}(n) \oplus [1_{>0}, 1_{>0}]_{-1}(n) \]

\[ \oplus [0, 0]_{-1}(n) \oplus [-1, 2_{>0}]_{-1}(n), \]

\[ (2, n, 2) \rightarrow [4_{>1}, -2]_{0}(n) \oplus [2_{>0}, -1]_0(n) \oplus [3_{>1}, 0_{>0}]_0(n) \oplus [2_{>1}, 2_{>1}]_0(n) \oplus [1_{>0}, 1_{>0}]_0(n) \]

\[ \oplus [0, 0]_{0}(n) \oplus [0_{>0}, 3_{>1}]_0(n) \oplus [-1, 2_{>0}]_0(n) \oplus [-2, 4_{>1}]_0(n). \]

(A.6)

B \( \mathcal{N} = 2 \) multiplet structure

The grouping of the \( S^5/\mathbb{Z}_3 \) KK spectrum shown in Table 2 into \( \mathcal{N} = 2 \) representations proceeds by splitting off one set of SU(3) representations at a time, where the five possible sets are given in (2.7). Starting with \([0, 0]\), we find the fields

\[ \varphi^{(1)}, \ \lambda^{(1)}, \ \varphi^{(2)}, \ A^{(1)}_\mu, \ \lambda^{(3)}, \ \varphi^{(4)}, \]

which is suggestive of a vector multiplet. A more careful consideration of the quantum numbers shows that this in fact fills out Vector Multiplet I, in the language of [31–33]. A similar consideration of the \([-2, -2]\) set gives the fields

\[ \varphi^{(4)}, \ \varphi^{(5)}, \ \varphi^{(6)}, \ \lambda^{(5)}, \ \lambda^{(6)}, \ A^{(3)}_\mu, \]

which fills out Vector Multiplet II. The remaining sets of representations are slightly more challenging to disentangle, as they give rise to a combination of multiplets. The \([1, -2] \sim [-2, 1]\) set corresponds to Gravitino Multiplets I and III, the \([-4, 2] \sim [-1, -1] \sim [2, -4]\) set spits into a Graviton Multiplet and Vector Multiplets III and IV, and the \([-3, 0] \sim [0, -3]\) set corresponds to Gravitino Multiplets II and IV. The results are presented in Tables 7–12, in a similar format as those in [31, 32].

Multiplet shortening in the tables are indicated in the first few columns. Massless (conserved) multiplets are marked by diamonds, chiral multiplets by bullets and semi-long multiplets by stars. Bars on top of the symbols indicates anti-chiral or semi-long II shortening. We have not included the corresponding tables for Gravitino Multiplets III and IV and Vector Multiplet IV, as they are conjugate to Gravitino Multiplets I and II, and Vector Multiplet III. Note that Vector Multiplet II is never shortened.

The SU(3) symbols are defined in Appendix A, and represent a set of triality zero SU(3) representations that show up in the expansion at KK level \( p \). Since all states within a given multiplet transform identically under SU(3), they share a common set of representations generated by \([a, b]\). Note, however, that the \( R \)-charges are shifted as appropriate for different states in the multiplet.
From these tables, we can see the connection between the subscripted restrictions on the Dynkin labels ('> 0' or ' > 1') and multiplet shortening. For example, consider the Graviton Multiplet of Table 7. As indicated in Table 4, this shortens to a massless graviton multiplet at level \( p = 2 \). Since this transforms as a singlet of SU(3), or equivalently as the \((0, 0)\) representation.
representation, all ‘$>$’ subscripted states are absent in this case, leaving only the states marked by diamonds. At levels $p = 3l + 2$, the Graviton Multiplet shortens into semi-long representations at the extremes of the SU(3) sequence, given by $(3l, 0)$ and $(0, 3l)$. In the former semi-long I case, the states in Table 7 with subscript ‘$>$’ in the second position are absent, leaving only the states marked by stars. Further examination of the remaining multiplets demonstrates that the shortening conditions are all consistent with the restrictions on the KK states, as guaranteed by supersymmetry.

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| $(s_1, s_2)$ | $E_0$ | $R$-symm. | Field | SU(3) symbol |
|--------------|-------|------------|--------|---------------|
| $(1/2, 1/2)$ | $p + 3$ | $r$ | $A_{3}^{(3)}$ | $[-2, -2]_{0}$ |
| $(1/2, 0)$   | $p + 5/2$ | $r - 1$ | $\lambda^{(5)}$ | $[-2, -2]_{1}$ |
| $(0, 1/2)$   | $p + 5/2$ | $r + 1$ | $\lambda^{(5)}$ | $[-2, -2]_{1}$ |
| $(0, 1/2)$   | $p + 7/2$ | $r - 1$ | $\lambda^{(6)}$ | $[-2, -2]_{1}$ |
| $(1/2, 0)$   | $p + 7/2$ | $r + 1$ | $\lambda^{(6)}$ | $[-2, -2]_{1}$ |
| $(0, 0)$     | $p + 2$ | $r$ | $\phi^{(4)}$ | $[-2, -2]_{0}$ |
| $(0, 0)$     | $p + 3$ | $r - 2$ | $\phi^{(5)}$ | $[-2, -2]_{2}$ |
| $(0, 0)$     | $p + 3$ | $r + 2$ | $\phi^{(5)}$ | $[-2, -2]_{0}$ |
| $(0, 0)$     | $p + 4$ | $r$ | $\phi^{(6)}$ | $[-2, -2]_{0}$ |

Table 11. Vector Multiplet II, $\mathcal{D}(p + 2, 0, 0; [-2, -2]_{0})$, formed by $[-2, -2]$ representations.

| $(s_1, s_2)$ | $E_0$ | $R$-symm. | Field | SU(3) symbol |
|--------------|-------|------------|--------|---------------|
| * (1/2, 1/2) | $p + 2$ | $r$ | $A_{3}^{(2)}$ | $[-1, -1, 0]_{-2}$ |
| * (0, 1/2)   | $p + 3/2$ | $r + 1$ | $\lambda^{(2)}$ | $[-1, -1]_{1}$ |
| * (1/2, 0)   | $p + 3/2$ | $r - 1$ | $\lambda^{(3)}$ | $[-1, -1, 0]_{-3}$ |
| * (1/2, 0)   | $p + 5/2$ | $r + 1$ | $\lambda^{(4)}$ | $[-1, -1, 0]_{-1}$ |
| * (0, 1/2)   | $p + 5/2$ | $r - 1$ | $\lambda^{(5)}$ | $[-1, -1, 1]_{-3}$ |
| * (0, 0)     | $p + 1$ | $r$ | $\phi^{(2)}$ | $[-1, -1]_{-2}$ |
| * (0, 0)     | $p + 2$ | $r + 2$ | $\phi^{(3)}$ | $[-1, -1]_{0}$ |
| * (0, 0)     | $p + 2$ | $r - 2$ | $\phi^{(4)}$ | $[-1, -1, 1]_{-4}$ |
| * (0, 0)     | $p + 3$ | $r$ | $\phi^{(5)}$ | $[-1, -1, 0]_{-2}$ |

Table 12. Vector Multiplet III, $\mathcal{D}(p + 1, 0, 0; [-1, -1]_{-2})$, formed by $[-1, -1]$ representations.

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