Note on cyclic sum of certain parametrized multiple series

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Abstract

We prove the cyclic sum formulas for certain two-parameter multiple series. These are new and non-trivial generalizations of the cyclic sum formulas for multiple zeta values and multiple zeta-star values.

Keywords: Cyclic sum; Parametrized multiple series; Multiple zeta value; Multiple zeta-star value; Multiple Hurwitz zeta value

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1 Introduction

In the present paper, we deal with relations for our two-parameter multiple series

\[
\sum_{0 \leq m_1 < \cdots < m_n < \infty} \frac{(\alpha)_{m_1}}{m_1!} \frac{m_n!}{(\alpha)_{m_n}} \left\{ \prod_{i=1}^{n} \frac{1}{(m_i + \alpha)^{a_i} (m_i + \beta)^{b_i}} \right\}, \tag{1}
\]

\[
\sum_{0 \leq m_1 \leq \cdots \leq m_n < \infty} \frac{(\alpha)_{m_1}}{m_1!} \frac{m_n!}{(\alpha)_{m_n}} \left\{ \prod_{i=1}^{n} \frac{1}{(m_i + \alpha)^{a_i} (m_i + \beta)^{b_i}} \right\}, \tag{2}
\]

where \( n \in \mathbb{Z}_{\geq 1} \); \( a_i, b_i \in \mathbb{Z} \) \((i = 1, \ldots, n)\) such that \( a_i + b_i \geq 1 \) \((i = 1, \ldots, n-1)\), \( a_n + b_n \geq 2 \); \( \alpha, \beta \in \mathbb{C} \) such that \( \Re(\alpha) > 0 \), \( \beta \notin \mathbb{Z}_{\leq 0} \). Here \( \mathbb{Z}_{\geq k} := \{ k, k+1, k+ \)}
and $\mathbb{Z}_{\leq k} := \{k, k-1, k-2, \ldots\}$ for $k \in \mathbb{Z}$. The symbol $(a)_m$ denotes the Pochhammer symbol, i.e., $(a)_m = a(a+1) \cdots (a+m-1)$ ($m \in \mathbb{Z}_{\geq 1}$) and $(a)_0 = 1$. The multiple series (1) and (2) are a non-trivial two-parameter extension of the multiple zeta value $\zeta(\{k_i\}_{i=1}^n)$ (MZV for short) and of the multiple zeta-star value $\zeta^*(\{k_i\}_{i=1}^n)$ (MZSV for short), respectively, where

$\zeta(\{k_i\}_{i=1}^n) := \sum_{0 < m_1 < \cdots < m_n < \infty} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}}$

$\zeta^*(\{k_i\}_{i=1}^n) := \sum_{1 \leq m_1 \leq \cdots \leq m_n < \infty} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}}$

$(n \in \mathbb{Z}_{\geq 1}; k_i \in \mathbb{Z}_{\geq 1}$ ($i = 1, \ldots, n - 1$), $k_n \in \mathbb{Z}_{\geq 2}; \{k_i\}_{i=1}^n := k_1, \ldots, k_n$). See Euler [4], Hoffman [8], Zagier [35], which are the pioneer works on MZV and MZSV. The study of this kind of extension was originated by the author in [10]. One of the interesting properties of MZV and MZSV is that they satisfy various interesting relations. Our extensions keep this property. For example, they have the sum and the duality formula, their generalizations (see [10] and [11]) and non-trivial evaluations derived from a hypergeometric identity (see [14]). An advantage of (1) and (2) is that they have a good derivation property on the parameters $\alpha$ and $\beta$, that is, their partial derivatives can be expressed in $\mathbb{Z}$-linear combinations of themselves; see (C1) of Section 3 and the proof of [11, Lemma 2.5]. This allows us to derive many relations from one relation. In Section 3, we shall give such an application of my results. In the present paper, we prove a new class of relations among (1) and (2), the cyclic sum formula (CSF for short). This shows that our multiple series are
the right extensions of MZV and MZSV. Here we put

\begin{align*}
Z_I(\{k_i\}_{i=1}^n; (\alpha, \beta)) &:= \sum_{0 \leq m_1 < \cdots < m_n < \infty} \frac{(\alpha)_{m_1} m_n!}{m_1! (\alpha)_{m_n} (m_1 + \beta)^{k_1}} \left\{ \prod_{i=2}^n \frac{1}{(m_i + \alpha)(m_i + \beta)^{k_i - 1}} \right\}, \\
Z_{II}(\{k_i\}_{i=1}^n; (\alpha, \beta)) &:= \sum_{0 \leq m_1 < \cdots < m_n < \infty} \frac{(\alpha)_{m_1} m_n!}{m_1! (\alpha)_{m_n}} \left\{ \prod_{i=1}^{n-1} \frac{1}{(m_i + \alpha)(m_i + \beta)^{k_i - 1}} \right\} \\
&\quad \times \frac{1}{(m_n + \alpha)^2(m_n + \beta)^{k_n - 2}}, \\
Z_I^*(\{k_i\}_{i=1}^n; (\alpha, \beta)) &:= \sum_{0 \leq m_1 \leq \cdots \leq m_n < \infty} \frac{(\alpha)_{m_1} m_n!}{m_1! (\alpha)_{m_n}} \left\{ \prod_{i=2}^n \frac{1}{(m_i + \alpha)(m_i + \beta)^{k_i - 1}} \right\}, \\
Z(a|b; (\alpha', \beta)) &:= \sum_{m=0}^{\infty} \frac{1}{(m + \alpha')^a(m + \beta)^b},
\end{align*}

where \(k_i \in \mathbb{Z}_{\geq 1} \ (i = 1, \ldots, n-1), \ k_n \in \mathbb{Z}_{\geq 2}; \ a, b \in \mathbb{Z} \) such that \(a + b \geq 2; \) \(\alpha, \alpha', \beta \in \mathbb{C} \) such that \(\text{Re}(\alpha) > 0 \) and \(\alpha', \beta \notin \mathbb{Z}_{\leq 0}. \) The last single series is the case \(n = 1\) of both (1) and (2). Our main results are as follows:

**Theorem 1.** Let \(k_1, \ldots, k_n \in \mathbb{Z}_{\geq 1} \) such that \(k_i \geq 2 \) for some \(i \ (i = 1, \ldots, n). \) Then

(i) (April 2013)

\begin{align*}
\sum_{i=1}^n \sum_{j=0}^{k_i - 2} Z_I(j + 1, k_{i+1}, \ldots, k_n, k_1, \ldots, k_{i-1}, k_i - j; (\alpha, \beta)) \\
= \sum_{i=1}^n Z_{II}(k_{i+1}, \ldots, k_n, k_1, \ldots, k_{i-1}, k_i + 1; (\alpha, \beta)),
\end{align*}
\( (ii) \) \( (October \ 2012) \)

\[
\sum_{i=1}^{n} \sum_{j=0}^{k_i-2} Z^*_i(j+1, k_{i+1}, \ldots, k_n, k_1, \ldots, k_{i-1}, k_i - j; (\alpha, \beta)) \\
= (k - n)Z(n|k - n + 1; (\alpha, \beta)) + nZ(n+1|k - n; (\alpha, \beta))
\]

(4)

for all \( \alpha, \beta \in \mathbb{C} \) such that \( \text{Re}(\alpha) > 0, \beta \notin \mathbb{Z}_{\leq 0} \), where \( k := k_1 + \cdots + k_n \).

Here the empty sum \( \sum_{j=0}^{n-1} \) is regarded as 0.

These relations are new and non-trivial generalizations of CSFs for MZVs (Hoffman and Ohno \[9, \text{p. } 333\]) and MZSVs (Ohno and Wakabayashi \[32\], Theorem 1): the former is the case \( \alpha = \beta = 1 \) of (3) and the latter is that of (4). Another simple specialization \( \alpha = \beta \) of (3) and of (4) give CSFs for one-parameter multiple series proved in \[12\], Theorem 1 (i) and (ii)]; therefore Theorem 1 is a two-parameter extension of the theorem. The relation (4) yields also a new sum formula for (2); see (9) below. It can be seen that the multiple series (1) and (2) have a flavor of multiple Hurwitz zeta values (MHZVs for short). Our results give also relations for MHZVs. In fact, the case \( \alpha = 1 \) of (3), (4) and of (9) give the cyclic sum and the sum formula for the MHZVs

\[
\sum_{0 \leq m_1 \leq \cdots \leq m_n < \infty} \prod_{i=1}^{n} \frac{1}{(m_i + 1)^{a_i}(m_i + \beta)^{b_i}},
\]

where the symbol \( \preceq \) denotes \( < \) or \( \leq \). For other relations for MHZVs, see \[10\], \[14\], \[15\] Note 2. For CSFs and related topics \( (q\text{-analogues etc.}) \), see also \[22\], \[27\], \[28\], \[33\], \[3\], \[29\], \[30\], \[24\]. We shall prove Theorem 1 in Section 2. In Section 3, we shall give applications of Theorem 1 and also pose some problems. The contents of the present paper are results in the author’s research project on parametrized multiple series. See also \[15\] Note 2 (iv) and (v)]. The identities (4), (9) and the identity (3) have been known to be results of the author since October 2012 and April 2013, respectively; see the following Note.
The present paper is a revised version of manuscripts of mine which were submitted to a journal on March 15, 2013 and February 17, 2016. (My manuscript submitted on March 15, 2013 was distributed as a preprint on December 5, 2013.) I proved (4) and (9) in October 2012, and wrote them in an unpublished manuscript in October 2012. I wrote about (4) also in a postdoctoral research plan document submitted to a previous affiliation of mine (Graduate School of Mathematics, Nagoya University, Japan) in February 2013. As regards (3), I proved it in April 2013, and wrote it in an unpublished manuscript on April 9, 2013. As with (4), I wrote about (3) also in a postdoctoral research plan document submitted to my previous affiliation in February 2014. All of the identities (3), (4) and (9) were written in a preprint of mine distributed on February 12, 2015 and in a manuscript of mine submitted to a journal on February 6, 2018 (rejected on June 11, 2018). As regards Section 3, the contents of (C2) and of the former part of (C1) (i.e., the part related to (4)) were written in my manuscript submitted on March 15, 2013 and in its preprint distributed on December 5, 2013. The contents of the latter part of (C1) (i.e., the part related to (3)) were written in my preprint distributed on February 12, 2015 and in my manuscript submitted on February 6, 2018. The contents of (C3) were written in my manuscript submitted on February 17, 2016 (rejected on March 10, 2016). A problem on the multiple series (18) posed in (C3) was written also in a postdoctoral research plan document of mine submitted to my previous affiliation in February 2016. I submitted the present version to many journals in February 2018–February 2022. In this Note and other places of the present paper, I referred to research documents of mine of October 2012–February 2018: unpublished manuscripts, submissions to journals, preprints and postdoctoral research plan documents. I note that all of those research documents are electronic files made in a computer system of my previous affiliation in October 2012–February 2018. (This can be confirmed by asking
people who read the contents of my electronic files in 2012–2018.) In fact, in 2012–March 2018, I used the computer system whenever I wrote research documents.

2 Proof of Theorem 1

Hoffman and Ohno [9], Ohno and Wakabayashi [32] proved CSFs for MZVs and MZSVs by using certain auxiliary multiple series. To prove Theorem 1, we use their method. We introduce the following auxiliary multiple series:

\[ T(\{k_i\}_{i=1}^n; (\alpha, \beta)) \]
\[ := \sum_{0 \leq m_0 < m_1 < \ldots < m_n < \infty} \frac{(\alpha)^{m_0} m_n!}{m_0! (\alpha)^{m_0}} \left\{ \prod_{i=1}^{n} \frac{1}{(m_i + \alpha)(m_i + \beta)^{k_i-1}} \right\} \frac{1}{m_n - m_0}, \]

\[ T^\ast(\{k_i\}_{i=1}^n; (\alpha, \beta)) \]
\[ := \sum_{0 \leq m_0 \leq m_1 \leq \ldots \leq m_n < \infty} \frac{(\alpha)^{m_0} m_n!}{m_0! (\alpha)^{m_0}} \left\{ \prod_{i=1}^{n} \frac{1}{(m_i + \alpha)(m_i + \beta)^{k_i-1}} \right\} \frac{1}{m_n - m_0}. \]

These multiple series converge absolutely for \(k_1, \ldots, k_n \in \mathbb{Z}_{\geq 1}\) such that \(k_i \geq 2\) for some \(i = 1, \ldots, n\) and \(\alpha, \beta \in \mathbb{C}\) such that \(\text{Re}(\alpha) > 0, \beta \notin \mathbb{Z}_{\leq 0}\). For the proof, refer to [12, Proof of Lemma 2.1]. The key to proving CSF in [9] and [32] was a cyclic property of auxiliary multiple series (see [9, Theorem 3.2] and [32, Key Lemma 1]). The above multiple series also have the property, that is, they satisfy the following identities (see also [12, Lemma 2.5]):

\textbf{Lemma 1.} Let \(k_1, \ldots, k_n \in \mathbb{Z}_{\geq 1}\) such that \(k_i \geq 2\) for some \(i = 1, \ldots, n\). Then
\(T(k_1, \ldots, k_n; (\alpha, \beta)) - T(k_n, k_1, \ldots, k_{n-1}; (\alpha, \beta)) = Z_{II}(k_n, k_1, \ldots, k_n-2, k_{n-1} + 1; (\alpha, \beta)) - \sum_{j=0}^{k_n-2} Z_I(j + 1, k_1, \ldots, k_{n-1}, k_n - j; (\alpha, \beta)), \tag{5}\)

\(T^*(k_1, \ldots, k_n; (\alpha, \beta)) - T^*(k_n, k_1, \ldots, k_{n-1}; (\alpha, \beta)) = (k_n - 1)Z(n|k - n + 1; (\alpha, \beta)) + Z(n + 1|k - n; (\alpha, \beta)) - \sum_{j=0}^{k_n-2} Z^*_I(j + 1, k_1, \ldots, k_{n-1}, k_n - j; (\alpha, \beta)) \tag{6}\)

for all \(\alpha, \beta \in \mathbb{C}\) such that \(\text{Re}(\alpha) > 0, \beta \notin \mathbb{Z}_{\leq 0}\), where \(k := k_1 + \cdots + k_n\).

Here the empty sum \(\sum_{j=0}^{k_n-1}\) is regarded as 0.

**Proof.** We prove (6) in the same way as in \cite[Proof of Key Lemma 1]{32} with
Remark 2.4. By using a partial fraction decomposition, we have

\[
\sum_{0 \leq m_0 \leq m_1 \leq \cdots \leq m_n < \infty \atop m_0 \neq m_n} \frac{(\alpha)_{m_0}}{m_0!} \frac{m_n!}{(\alpha)_{m_n}} \frac{1}{(m_0 + \beta)^j} \left\{ \prod_{i=1}^{n-1} \frac{1}{(m_i + \alpha)(m_i + \beta)^{k_i-1}} \right\} \\
\sum_{0 \leq m_0 \leq m_1 \leq \cdots \leq m_n < \infty \atop m_0 \neq m_n} \frac{1}{(m_0 + \alpha)(m_0 + \beta)^{k_n-1}(m_n - m_0)} \\
= \sum_{0 \leq m_0 \leq m_1 \leq \cdots \leq m_n < \infty \atop m_0 \neq m_n} \frac{(\alpha)_{m_0}}{m_0!} \frac{m_n!}{(\alpha)_{m_n}} \frac{1}{(m_0 + \beta)^{j+1}} \left\{ \prod_{i=1}^{n-1} \frac{1}{(m_i + \alpha)(m_i + \beta)^{k_i-1}} \right\} \\
\times \frac{1}{(m_n + \alpha)(m_n + \beta)^{k_n-2} - (m_n - m_0)} \\
\times \left( \frac{1}{m_n - m_0} - \frac{1}{m_n + \beta} \right) \\
= \sum_{0 \leq m_0 \leq m_1 \leq \cdots \leq m_n < \infty \atop m_0 \neq m_n} \frac{(\alpha)_{m_0}}{m_0!} \frac{m_n!}{(\alpha)_{m_n}} \frac{1}{(m_0 + \beta)^{j+1}} \left\{ \prod_{i=1}^{n-1} \frac{1}{(m_i + \alpha)(m_i + \beta)^{k_i-1}} \right\} \\
\times \frac{1}{(m_n + \alpha)(m_n + \beta)^{k_n-2} - (m_n - m_0)} \\
- Z^*_I(j + 1, k_1, \ldots, k_{n-1}, k_n - j; (\alpha, \beta)) \\
+ Z(n|k_1 + \cdots + k_n - n + 1; (\alpha, \beta))
\]

for \( j = 0, \ldots, k_n - 2 \). Further, taking the sum \( \sum_{j=0}^{k_n-2} \) on both sides, we have

\[
T^*(k_1, \ldots, k_n; (\alpha, \beta)) \\
= \sum_{0 \leq m_0 \leq m_1 \leq \cdots \leq m_n < \infty \atop m_0 \neq m_n} \frac{(\alpha)_{m_0}}{m_0!} \frac{m_n!}{(\alpha)_{m_n}} \frac{1}{(m_0 + \beta)^{k_n-1}} \left\{ \prod_{i=1}^{n-1} \frac{1}{(m_i + \alpha)(m_i + \beta)^{k_i-1}} \right\} \\
\times \frac{1}{(m_n + \alpha)(m_n - m_0)} \\
- \sum_{j=0}^{k_n-2} Z^*_I(j + 1, k_1, \ldots, k_{n-1}, k_n - j; (\alpha, \beta)) \\
+ (k_n - 1)Z(n|k_1 + \cdots + k_n - n + 1; (\alpha, \beta)).
\]

(7)
Here we rewrite the first term on the right-hand side of (7) as

\[
\sum_{0 \leq m_0 \leq m_1 \leq \ldots \leq m_n < \infty \atop m_0 \neq m_n} \frac{(\alpha)_{m_0}}{m_0!} \frac{m_n!}{(\alpha)_{m_n}} \frac{1}{(m_0 + \beta)^{k_n-1}} \prod_{i=1}^{n-1} \frac{1}{(m_i + \alpha)(m_i + \beta)^{k_i-1}} \\
\times \frac{1}{(m_n + \alpha)(m_n - m_0)}
\]

\[
= \sum_{0 \leq m_0 \leq m_1 \leq \ldots \leq m_{n-1} < \infty \atop m_0 \neq m_{n-1}} \frac{1}{(m_0 + \alpha)(m_0 + \beta)^{k_n-1}} \prod_{i=1}^{n-1} \frac{1}{(m_i + \alpha)(m_i + \beta)^{k_i-1}} \\
\times \frac{(\alpha)_{m_0 + 1}}{m_0!} \sum_{m_n = m_{n-1}}^{\infty} \frac{m_n!}{(\alpha)_{m_n + 1}} \frac{1}{m_n - m_0}
\]

\[
+ \sum_{0 \leq m_0 = \ldots = m_{n-1} < \infty} \frac{1}{(m_0 + \alpha)(m_0 + \beta)^{k_n-1}} \prod_{i=1}^{n-1} \frac{1}{(m_i + \alpha)(m_i + \beta)^{k_i-1}} \\
\times \frac{(\alpha)_{m_0 + 1}}{m_0!} \sum_{m_n = m_{n-1} + 1}^{\infty} \frac{m_n!}{(\alpha)_{m_n + 1}} \frac{1}{m_n - m_0}.
\]

Using

\[
\frac{(\alpha)_{m+1}}{m!} \sum_{l=0}^{\infty} \frac{l!}{(\alpha)_{l+1}} \frac{1}{l-m} = \frac{n!}{(\alpha)_n} \sum_{l=0}^{m} \frac{(\alpha)_l}{l!} \frac{1}{n-l}
\]

(m, n \in \mathbb{Z} \text{ such that } 0 \leq m < n \text{ and } \alpha \in \mathbb{C} \text{ with Re}(\alpha) > 0; \ [12, \text{Remark 2.4}])
to the inner sums, we have

\[
(8) = \sum_{0 \leq m_0 \leq m_1 \leq \cdots \leq m_{n-1} < \infty} \frac{1}{(m_0 + \alpha)(m_0 + \beta)^{k_1-1}} \left\{ \prod_{i=1}^{n-1} \frac{1}{(m_i + \alpha)(m_i + \beta)^{k_i-1}} \right\} \times \frac{m_{n-1}!}{(\alpha)_{m_{n-1}}} \sum_{l=0}^{m_0} \frac{(\alpha)_l}{l!} \frac{1}{m_{n-1} - l} + \sum_{0 \leq m_0 = \cdots = m_{n-1} < \infty \atop l \neq m_{n-1}} \frac{1}{(m_0 + \alpha)(m_0 + \beta)^{k_1-1}} \left\{ \prod_{i=1}^{n-1} \frac{1}{(m_i + \alpha)(m_i + \beta)^{k_i-1}} \right\} \times \frac{m_{n-1}!}{l! (\alpha)_{m_{n-1}}} \frac{1}{(m_0 + \alpha)(m_0 + \beta)^{k_1-1}} \times \left\{ \prod_{i=1}^{n-1} \frac{1}{(m_i + \alpha)(m_i + \beta)^{k_i-1}} \right\} \frac{1}{m_{n-1} - l} + \sum_{0 \leq m_0 = \cdots = m_{n-1} < \infty \atop l \neq m_{n-1}} \frac{1}{(m_0 + \alpha)(m_0 + \beta)^{k_1-1}} \left\{ \prod_{i=1}^{n-1} \frac{1}{(m_i + \alpha)(m_i + \beta)^{k_i-1}} \right\} \times \frac{(\alpha)_{m_0}}{m_0! (\alpha)_{m_{n-1}+1}} \frac{m_{n-1}!}{m_{n-1}!} = T^*(k_n, k_1, \ldots, k_{n-1}; (\alpha, \beta)) + Z(n + 1|k_1 + \cdots + k_n - n; (\alpha, \beta)).
\]

Hence, substituting this into the right-hand side of (7), we obtain (6). The identity (5) can be proved in the same way as in [9, Section 3] (see also [27] and [28]) with [12, Lemma 2.2]. For more details, refer to [12, Proof of Lemma 2.5]. \[\square\]

**Proof of Theorem 1.** Applying Lemma 1 (i) and (ii) to the index \((k_{i+1}, \ldots, k_n, k_1, \ldots, k_i)\) and taking the sum \(\sum_{i=1}^{n}\) on both sides of the results, we obtain (3) and (4), respectively. \[\square\]

Using Theorem 1 (ii) and Ohno’s argument in [27, p. 4], [28, pp. 138–139]...
(see also [9, Corollary 2.4 and its proof], [32, p. 294], [12, pp. 514–516]), we can prove the following relation:

**Corollary 1** (October 2012). We have

\[
\sum_{\substack{k_1 + \cdots + k_n = k \\
                k_i \in \mathbb{Z}_{\geq 1} (i = 1, \ldots, n-1), \\
                k_n \in \mathbb{Z}_{\geq 2}}}
Z^*_{(\{ k_i \}_{i=1}^n; (\alpha, \beta))} = \binom{k - 2}{n - 1} Z(n - 1|k - n + 1; (\alpha, \beta)) + \binom{k - 2}{n - 2} Z(n|k - n; (\alpha, \beta))
\]

(9)

for all \( k, n \in \mathbb{Z} \) such that \( 0 < n < k \) and \( \alpha, \beta \in \mathbb{C} \) such that \( \text{Re}(\alpha) > 0 \), \( \beta \not\in \mathbb{Z}_{\leq 0} \).

Our relation (9) is a new sum formula for (2); compare (9) with the sum formulas of [14, (R2) of Section 3], [10], [11, Remark 2.4]. Taking \( \alpha = \beta = 1 \) and \( \alpha = \beta \) in (9), we have the sum formula for MZSVs (Granville [7], Hoffman [8], Zagier) and for one-parameter multiple series proved in [12, Corollary 2.8 (ii)], respectively.

### 3 Applications and problems

We give applications of Theorem 1 and also pose some problems. Hereafter we use the following notation:

\[
\{ k_1, \ldots, k_m \}^n := \underbrace{k_1, \ldots, k_m, \ldots, k_1, \ldots, k_m}_{mn},
\]

\[
\{ a_{i_1}, \ldots, a_{m_i} \}_{i=1}^n := a_{i_1}, \ldots, a_{m_1}, \ldots, a_{i_n}, \ldots, a_{m_n}
\]

\( m, n \in \mathbb{Z}_{\geq 1} \); we regard \( \{ k_1, \ldots, k_m \}^0 \) and \( \{ a_{i_1}, \ldots, a_{m_i} \}_{i=1}^0 \) as the empty set \( \emptyset \).

**C1** By partial differentiation, we derive relations among (1) and (2) from the relations (3) and (4). We use some identities for the Pochhammer
symbol \((a)_m\) to calculate the partial derivatives on \(\alpha\). They fairly simplify the calculation. Using

$$
\frac{(\alpha)_m}{(\alpha)_n} \prod_{i=2}^{n} \frac{1}{m_i + \alpha} = \prod_{i=2}^{n} \frac{(\alpha)_{m_i-1}}{(\alpha)_{m_i+1}},
$$

we have

$$
\frac{(-1)^r}{r!} \frac{d^r}{d\alpha^r} \left( \frac{(\alpha)_m}{(\alpha)_n} \prod_{i=2}^{n} \frac{1}{m_i + \alpha} \right)
= \sum_{r_1 + \ldots + r_{n-1} = r, r_i \in \mathbb{Z}_{\geq 0}} \prod_{i=2}^{n} \frac{(\alpha)_{m_i-1}}{(\alpha)_{m_i+1}} \sum_{m_{i-1} \leq l_i \leq \ldots \leq l_{r_i-1} \leq m_i} \frac{1}{(l_1 + \alpha) \cdots (l_{r_i-1} + \alpha)}

\times \frac{(\alpha)_m}{(\alpha)_n} \left( \prod_{i=2}^{n} \frac{1}{m_i + \alpha} \right)

\times \sum_{r_1 + \ldots + r_{n-1} = r, r_i \in \mathbb{Z}_{\geq 0}} \prod_{i=1}^{n-1} \prod_{j=1}^{r_i} \frac{1}{m_{ij} + \alpha}
$$

for \(n \in \mathbb{Z}_{\geq 1}, r \in \mathbb{Z}_{\geq 0}\) and \(m_1, \ldots, m_n \in \mathbb{Z}\) such that \(0 \leq m_1 \leq \cdots \leq m_n\).

These identities give the following expression:

$$
\frac{(-1)^r}{r!} \frac{\partial^r}{\partial\alpha^r} Z_t^r(\{k_i\}_{i=1}^n; (\alpha, \beta)) = \sum_{r_1 + \ldots + r_{n-1} = r, r_i \in \mathbb{Z}_{\geq 0}} Z_t^r(\{k_i, \{1\}_{i=1}^{r_i}\}_{i=1}^{n-1}; k_n; (\alpha, \beta))
$$

for \(r \in \mathbb{Z}_{\geq 0}\). It is obvious that the partial derivative \(\frac{(-1)^r}{r!} \frac{\partial^r}{\partial\alpha^r} Z_t^r(\{k_i\}_{i=1}^n; (\alpha, \beta))\) can also be expressed in a \(\mathbb{Z}\)-linear combination of \(Z_t^r(\{k_i\}_{i=1}^n; (\alpha, \beta))\). Therefore differentiating both sides of (4) gives further relations among (2). In addition, the above examination shows that the partial differentiation on \(\alpha\) and on \(\beta\) give different relations. For instance, we can obtain the following:

For the index \(\{\{1\}_{m-1}^m, 2\}_{n}^n\) \((m, n \in \mathbb{Z}_{\geq 1})\), the relation (4) becomes

$$
Z_t^r(1, \{\{1\}_{m-1}^m, 2\}_{n}^n; (\alpha, \beta)) = Z(mn|n+1; (\alpha, \beta)) + mZ(mn+1|n; (\alpha, \beta))
$$

(12)
\((m, n \in \mathbb{Z}_{\geq 1}, \alpha, \beta \in \mathbb{C} \text{ such that } \text{Re}(\alpha) > 0, \beta \notin \mathbb{Z}_{\leq 0})\). Differentiating both sides of (12) \(r\) times on \(\alpha\) and using (11), we have the relation

\[
\sum_{\substack{r_1 + \cdots + r_n = r \\ r_i \in \mathbb{Z}_{\geq 0}}} Z_t^r(1, \{\{1\}^{m+r_i-1}, 2\}_{i=1}^n; (\alpha, \beta))
\]

\[
= \binom{mn + r - 1}{r} Z(mn + r|n + 1; (\alpha, \beta)) + m \binom{mn + r}{r} Z(mn + r + 1|n; (\alpha, \beta))
\]

for \(r \in \mathbb{Z}_{\geq 0}\). On the other hand, differentiating both sides of (12) \(r\) times on \(\beta\), we have the relation

\[
\sum_{\substack{r_1 + \cdots + r_n + 1 = r \\ r_i \in \mathbb{Z}_{\geq 0}}} Z_t^r(1 + r_1, \{\{1\}^{m-1}, 2 + r_i\}_{i=2}^{n+1}; (\alpha, \beta))
\]

\[
= \binom{n + r}{r} Z(mn|n + r + 1; (\alpha, \beta)) + m \binom{n + r - 1}{r} Z(mn + 1|n + r; (\alpha, \beta))
\]

for \(r \in \mathbb{Z}_{\geq 0}\). We remark that the relation (12) is a generalization of a relation for MZSVs in Ohno and Wakabayashi [32, Examples (b)], Zlobin [36, Theorem 5]. It is worth noting that the case \(m = 1\) of (12) can be derived from the hypergeometric identity of Krattenthaler and Rivoal [25, Proposition 1 (ii)], which is a limiting case of Andrews’s basic hypergeometric identity [2 Theorem 4]: indeed, it can be obtained by taking \(a = \alpha + \beta, c_0 = \alpha, b_i = \alpha, c_i = \beta \ (i = 1, \ldots, s - 1), b_s = \beta, c_s = 1 \ (s \in \mathbb{Z}_{\geq 2}, \alpha, \beta \in \mathbb{C} \text{ such that } \text{Re}(\alpha) > 0, \beta \notin \mathbb{Z}_{\leq 0})\) in [25, Proposition 1 (ii)]. The same way as above can be applied to deriving relations among (1) and (2) from (3). Using

\[
\frac{(\alpha)_{m_1}}{(\alpha)_{m_n}} = \prod_{i=2}^n \frac{(\alpha)_{m_{i-1}}}{(\alpha)_{m_i}}
\]
and the same calculus as in the proof of (10), we have

\[
\frac{(-1)^r \, d^r}{r! \, d \alpha^r} \left( \frac{(\alpha)_{m_1}}{(\alpha)_{m_n}} \right) = \frac{(\alpha)_{m_1}}{(\alpha)_{m_n}} \sum_{r_1 + \cdots + r_{n-1} = r} \sum_{r_i \in \mathbb{Z}_{\geq 0}} \prod_{i=1}^{n-1} \prod_{j=1}^{r_i} \frac{1}{m_{ij} + \alpha}
\]

\[
\prod_{i=1}^{n} \frac{m_{ij}!}{(m_{ij} + \alpha)^{a_i} (m_{ij} + \beta)^{b_i}} \frac{1}{(m_{ij} + \alpha)^{a_i} (m_{ij} + \beta)^{b_i}}
\]

for \( n \in \mathbb{Z}_{\geq 1}, r \in \mathbb{Z}_{\geq 0} \) and \( m_1, \ldots, m_n \in \mathbb{Z} \) such that \( 0 \leq m_1 < \cdots < m_n \). This identity gives the following expression of the partial derivatives on \( \alpha \) of (1):

\[
\frac{(-1)^r \, d^r}{r! \, d \alpha^r} \left( \sum_{0 \leq m_1 < \cdots < m_{n-1} < m_n} \frac{(\alpha)_{m_1}}{m_1!} \frac{m_n!}{(\alpha)_{m_n}} \prod_{i=1}^{n} \frac{1}{(m_i + \alpha)^{a_i} (m_i + \beta)^{b_i}} \right)
\]

\[
\sum_{r_1 + \cdots + r_{n-1} = r} \sum_{r_i \in \mathbb{Z}_{\geq 0}} \prod_{i=1}^{n-1} \prod_{j=1}^{r_i} \frac{1}{m_{ij} + \alpha} \left( \prod_{i=1}^{n} \frac{m_{ij}!}{(m_{ij} + \alpha)^{a_i} (m_{ij} + \beta)^{b_i}} \right) \}
\]

for \( r \in \mathbb{Z}_{\geq 0} \), where

\[
Z_{\{r_i\}_{i=1}^{n-1}}(\{a_i\}_{i=1}^{n} | \{b_i\}_{i=1}^{n}; (\alpha, \beta)) := \sum_{0 \leq m_0 \leq m_1 \leq m_2 \leq \cdots \leq m_{n-1} < m_n < \infty} \frac{(\alpha)_{m_1}}{m_1!} \frac{m_n!}{(\alpha)_{m_n}} \prod_{i=1}^{n-1} \prod_{j=1}^{r_i} \frac{1}{m_{ij} + \alpha} \left( \prod_{i=1}^{n} \frac{1}{(m_i + \alpha)^{a_i} (m_i + \beta)^{b_i}} \right) \}
\]

(13)

where \( n \in \mathbb{Z}_{\geq 1}; r_i \in \mathbb{Z}_{\geq 0} (i = 1, \ldots, n-1) \); the conditions for \( a_i, b_i, \alpha, \beta \) are the same as those in (1). For \( n = 1 \), this multiple series becomes

\[
Z_{\emptyset}(a_1 | b_1; (\alpha, \beta)) = \sum_{0 \leq m_1 < \infty} \frac{1}{(m_1 + \alpha)^{a_1} (m_1 + \beta)^{b_1}}.
\]
where $\emptyset$ is the empty set. (A characteristic of (14) is that its summation
consists of both $<$ and $\leq$. The usefulness of multiple series with such a
summation was shown by Fischler and Rivoal [5], Kawashima [23], Ulanskii
[34].) It is easy to verify that the multiple series (14) can be expressed in a
$\mathbb{Z}$-linear combination of (1) or (2), and that the partial derivative on $\beta$ of (1)
can be expressed in a $\mathbb{Z}$-linear combination of (1). Therefore differentiating
both sides of (3) also gives relations among (1) and (2). For instance, we can
obtain the following: Taking $k_i = 2$ ($i = 1, \ldots, n$) in (3), we have
\begin{equation}
Z_I(1, \{2\}^n; (\alpha, \beta)) = Z_{II}(\{2\}^{n-1}, 3; (\alpha, \beta)) \tag{15}
\end{equation}
for $n \in \mathbb{Z}_{\geq 1}$, $\alpha, \beta \in \mathbb{C}$ such that \text{Re}(\alpha) > 0, $\beta \notin \mathbb{Z}_{\leq 0}$. Differentiating both
sides of (15) $r$ times on $\alpha$ and using (13), we have the relation
\begin{align*}
& \sum_{r_1 + \cdots + r_n = r, s_1 + \cdots + s_n = r} \sum_{r_1, s_i \in \mathbb{Z}_{\geq 0}} Z_{(r_i)_{i=1}^n}(0, \{1 + s_i\}_{i=1}^n \{1\}^{n+1}; (\alpha, \beta)) \\
& = \sum_{r_1 + \cdots + r_n-1 + s_1 + \cdots + s_n = r} (1 + s_n)Z_{(r_i)_{i=1}^{n-1}}(\{1 + s_i\}_{i=1}^{n-1}, 2 + s_n; \{1\}^n; (\alpha, \beta)) \tag{16}
\end{align*}
for $r \in \mathbb{Z}_{\geq 0}$. On the other hand, differentiating both sides of (15) $r$ times on $\beta$, we have the relation
\begin{align*}
& \sum_{r_1 + \cdots + r_n+1 = r, r_i \in \mathbb{Z}_{\geq 0}} Z_I(1 + r_1, \{2 + r_i\}_{i=2}^{n+1}; (\alpha, \beta)) \\
& = \sum_{r_1 + \cdots + r_n = r} Z_{II}(\{2 + r_i\}_{i=1}^{n-1}, 3 + r_n; (\alpha, \beta)) \tag{17}
\end{align*}
for $r \in \mathbb{Z}_{\geq 0}$. As pointed out in a preprint of the author distributed on Febru-
ary 12, 2015, the relation (15) is a generalization of the duality formula for
$\zeta(1, \{2\}^n)$. In fact, it is a new and non-trivial two-parameter generalization; compare (15) with [11, Lemma 2.3].
Problems (December 2018 and 2022). The following problems still defy a solution. I hope that investigating them exposes the reader to new viewpoints. The problems (i), (ii) and (iii) below are on appropriate generalizations and the problem (iv) on appropriate methods.

(i) It is interesting to generalize my duality formula (15) to a full duality formula in appropriate ways and also to more general duality relations among (1). (The research on the duality of (1) has been part of my research project since 2007.) As a note on the latter part of this problem, I point out that my relation (17) is a two-parameter generalization of Ohno’s relation for MZVs [26, Theorem 1] for the index $1, \{2\}^n$ $(n \in \mathbb{Z}_{\geq 1})$. Ohno’s relation is a generalization of the duality formula for MZVs. (My relations (15), (16) and (17) were written in my preprint distributed on February 12, 2015.)

(ii) In [14, Remark 7 (i)], I gave a new proof of Hoffman’s identity $\zeta(\{1\}^k, l + 2) = \zeta(\{1\}^l, k + 2)$ $(k, l \in \mathbb{Z}_{\geq 0}; \frac{3}{11} \text{ Theorem 4.4})$, which is the duality formula for $\zeta(\{1\}^k, l + 2)$. My proof is based on the hypergeometric identities [2, Theorem 4] and [25, Proposition 1 (i)]; therefore it can be regarded as a hypergeometric proof of the duality formula. It is interesting to generalize my proof to a proof of the duality formula for all MZVs in appropriate ways.

(iii) It is interesting to generalize my results in [13], [14], [15] and to extend my ideas in there in appropriate ways, e.g., the identities (29), (30), (28), (58), (27), (57), (66), (67), (70), (71) of [14] and the ideas for their proofs, namely, my identities for

$$
\zeta^*(\{1\}^{k+1}, \{2\}^{l+1}), \zeta^*(k + 2, \{2\}^l), \zeta(\{1\}^k, l + 2), \zeta^*(\{1\}^k, l + 2), \\
\zeta_{k+1,+}^<(\{1\}^k, l + 2; \alpha), \zeta_{k+1,+}^<(\{1\}^k, l + 2; \alpha), \zeta_{k+1,+}^<(\{1\}^{k+1}|\{0\}^k, l + 1; (\alpha, \beta)), \\
\zeta_{k+1,+}^<(\{1\}^{k+1}|\{0\}^k, l + 1; (\alpha, \beta)) \quad (k, l \in \mathbb{Z}_{\geq 0}; \alpha, \beta \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0})
$$

and my ideas for their proofs. (I proved the identities (28), (58), (27), (57) in 2014 and the identities (66), (67), (70), (71) in January 2015. For (29)
(iv) In [14], I studied four-parameter multiple series, (1) and (41) of 
[14]. (The multiple series (1) of [14] is a four-parameter extension of both 
(1) and (2) of the present paper.) I studied relations among them in there 
(see also [17]). It is interesting to study analytic continuations of my four-
parameter multiple series (1) and (41) with complex variables \(a_i, b_i, c_i, d_i, s_i, t_i\) in appropriate ways. (This requires appropriate methods of handling the 
Pochhammer factors of (1) and (41).) This problem grew out of a reading of 
the papers Akiyama, Egami and Tanigawa [1] and Goncharov [6, pp. 25–32], 
where the authors proved analytic continuations of a multiple zeta-function 
and results on its values at non-positive integers. The problem grafts a 
problem on my research objects and on the above authors’s.

Note (2022). (i) I note that a revised manuscript of [12] which contains 
Remark 2.7 was first submitted to the journal on August 4, 2009. This 
is an additional explanation for the addition of Remark 2.7 to [12]. See 
also an explanation at the beginning of [12, Remark 2.7] and my preprint 
\texttt{arXiv:0908.2536v1}.

(ii) In [12, Remark 2.7], I noted a connection between the hypergeometric 
identity [25, Proposition 1 (ii)] and a multiple zeta-star object (the case \(\alpha = \beta\) 
of my multiple series (2)). The connection was noted also in my preprint 
\texttt{arXiv:0908.2536v1}, posted on August 18, 2009. It appears that, after my 
note, some researchers took an interest in the hypergeometric identities [2, 
Theorem 4], [25, Proposition 1] and their proofs. In 2009, based on the 
connection, I began an application of the above hypergeometric identities 
to the study of relations for MZVs, MZSVs and special values of multiple 
polylogarithms. After that, I obtained many results; see [13], [14], [15]. The 
identities (29), (30) and (28), (58) of [14], among others, have been known 
to be results of mine since 2011 and 2014, respectively (see also [13]). (The 
first two identities show the rightness of my observations stated in [13] (A3)
and (A4)]; see [13, Addendum].) I obtained most of the results of [14] and Applications 1 and 2 of [15] in 2013–2015 (see [15, Notes 1 and 2]). I note that most of the contents of [14] and Application 1 of [15] were written in preprints of mine distributed in 2013 and 2015. These preprints were sent to a Japanese zeta-researcher (a JZ for short). To be more precise, an earlier version of [14] and its revised version were sent by e-mail to the JZ in October and December 2013, respectively, and further, a preprint of [14] was also sent by e-mail to the JZ in February 2015. (Application 1 of [15] was written in all these preprints.) These pieces of information about my research activities can be confirmed by asking the JZ. In the preprint sent in February 2015, I wrote the identities (28) and (58) of [14], which are new identities for \( \zeta(\{1\}^k, l + 2) \) and \( \zeta^*(\{1\}^k, l + 2) \). It was already known in 2014 that I discovered these new identities, because information on my discovery was \(* * *\) by \(* * *\) to \(* * *\) in 2014. It appears that my identities are revelations about MZVs and MZSVs. Hence it is interesting to generalize my identities in appropriate ways.

(C2) For \( k_1, \ldots, k_n \in \mathbb{Z}_{\geq 1} \) such that \( k_1 + \cdots + k_n = m + n \) \((m, n \in \mathbb{Z}_{\geq 1})\), the right-hand side of (4) becomes

\[
mZ(n|m + 1; (\alpha, \beta)) + nZ(n + 1|m; (\alpha, \beta)).
\]

This sum is invariant under the replacement \((\alpha, \beta, m, n) \leftrightarrow (\beta, \alpha, n, m)\); therefore the left-hand side of (4) also has the following symmetry:

\[
\sum_{i=1}^{n} \sum_{j=0}^{k_i-2} Z_i^\ast(j + 1, k_{i+1}, \ldots, k_n, k_1, \ldots, k_{i-1}, k_i - j; (\alpha, \beta))
\equiv
\sum_{i=1}^{m} \sum_{j=0}^{l_i-2} Z_i^\ast(j + 1, l_{i+1}, \ldots, l_m, l_1, \ldots, l_{i-1}, l_i - j; (\beta, \alpha))
\]

for any fixed \( m, n \in \mathbb{Z}_{\geq 1} \), all \( k_1, \ldots, k_n, l_1, \ldots, l_m \in \mathbb{Z}_{\geq 1} \) such that \( k_1 + \cdots + k_n = l_1 + \cdots + l_m = m + n \) and \( \alpha, \beta \in \mathbb{C} \) with \( \text{Re}(\alpha), \text{Re}(\beta) > 0 \). An application
of this kind of symmetry of parametrized multiple series was studied also in [11].

(C3) It is interesting to investigate whether CSFs for MZVs and MZSVs can be generalized to three- or more parameter multiple series. In fact, the results will have interesting applications to the study of relations among MZVs and MZSVs. For example, one will be able to derive relations among them from the results by using our method used in (C1) and [14]. (Our method was developed in preprints of [14] distributed in 2013.) We shall pose a problem on this topic. We mainly consider this topic for our three-parameter multiple series

\[ \sum_{0 \leq m_1 \leq \cdots \leq m_n<\infty} \frac{(\alpha)_{m_1} (\beta)_{m_1} m_n! (\gamma)_{m_n}}{m_1! (\gamma)_{m_1} (\alpha)_{m_n} (\beta)_{m_n}} \times \left\{ \prod_{i=1}^{n} \frac{1}{(m_i+\alpha)^{a_i} (m_i+\beta)^{b_i} (m_i+\gamma)^{c_i}} \right\}, \]

where \( n \in \mathbb{Z}_{\geq 1}; a_i, b_i, c_i \in \mathbb{Z} \) (\( i = 1, \ldots, n \)) such that \( a_i + b_i + c_i \geq 1 \) (\( i = 1, \ldots, n-1 \)), \( a_n + b_n + c_n \geq 2 \); \( \alpha, \beta, \gamma \in \mathbb{C}\setminus\mathbb{Z}_{<0} \) such that \( \text{Re}(\alpha+\beta-\gamma) > 0 \); these conditions guarantee the absolute convergence of (18) (see [25, (3.12)]). The multiple series (18) is an extension of (2). The multiple series (18) was discovered by the author in 2014 (before September 17, 2014) by studying the hypergeometric identities of Andrews [2, Theorem 4], Krattenthaler and Rivoal [25, Proposition 1] (see [15, Note 2 (iv)] and [14, Introduction]). Our previous remark [12, Remark 2.7] played an essential role for discovering (18) and its identities. Here we put

\[ Z_1^*\left(\{k_i\}_{i=1}^{n}; (\alpha, \beta, \gamma)\right) := \sum_{0 \leq m_1 \leq \cdots \leq m_n<\infty} \frac{(\alpha)_{m_1} (\beta)_{m_1} m_n! (\gamma)_{m_n}}{m_1! (\gamma)_{m_1} (\alpha)_{m_n} (\beta)_{m_n}} \times \left\{ \prod_{i=2}^{n} \frac{1}{(m_i+\alpha)(m_i+\beta)(m_i+\gamma)^{k_i-2}} \right\}, \]
and
\[
Z(a|b|c; (\alpha', \beta', \gamma')) := \sum_{m=0}^{\infty} \frac{1}{(m+\alpha')(m+\beta')(m+\gamma')^c},
\]
where \(k_i \in \mathbb{Z}_{\geq 1} (i = 1, \ldots, n-1)\), \(k_n \in \mathbb{Z}_{\geq 2}\); \(a, b, c \in \mathbb{Z}\) such that \(a+b+c \geq 2\); \(\alpha, \beta, \gamma \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}\) such that \(\text{Re}(\alpha + \beta - \gamma) > 0\); \(\alpha', \beta', \gamma' \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}\). The above single series is the case \(n = 1\) of (18). The multiple series (19) is an extension of both the multiple series \(Z^*(\{k_i\}_{i=1}^{n}; \alpha)\) of [12] and \(Z^*_I(\{k_i\}_{i=1}^{n}; (\alpha, \beta))\), which are the cases \(\alpha = \beta = \gamma\) and \(\beta = \gamma\) of (19), respectively. CSF for MZSVs can perhaps be regarded as a generalization of the relation
\[
\zeta^*(1, \{2\}^{s-1}) = 2\zeta(2s - 1)
\]
for \(s \in \mathbb{Z}_{\geq 2}\) (see Ohno and Zudilin [31, p. 326], Ohno and Wakabayashi [32]). Indeed, for the index \((1, \{2\}^{s-1})\), it becomes (20). We observed that the same situation occurs also for the one-parameter multiple series \(Z^*(\{k_i\}_{i=1}^{n}; \alpha)\) of [12] and the two-parameter multiple series (2), that is, each of these multiple series satisfies the same relation as (20); see [12, Remark 2.7] and a note on the case \(m = 1\) of (12) stated in (C1). Furthermore, each relation can be generalized to CSF for the multiple series; see [12, Theorem 1.1 (ii)] and Theorem 1. We further studied whether this situation occurs also for more multi-parameter multiple series. As a result, we found that the three-parameter multiple series (19) also satisfies the same relation as (20),
\[
Z^*_I(1, \{2\}^{s-1}; (\alpha, \beta, \gamma)) = Z(s-1|s-1|1; (\alpha, \beta, \gamma)) + Z(s-1|s-1|1; (\alpha, \beta, \alpha+\beta-\gamma))
\]
for \(s \in \mathbb{Z}_{\geq 2}\), \(\alpha, \beta, \gamma \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}\) such that \(\text{Re}(\alpha + \beta - \gamma) > 0\). (For details, see a note at the end of the present part.) This relation gives a generalization of the following three relations: (20), the case \(m = 1\) of (12), and the relation in [12, Remark 2.7], which are the cases \(\alpha = \beta = \gamma = 1\), \(\beta = \gamma\), and \(\alpha = \beta = \gamma\) of (21), respectively. On the basis of the above facts, we pose the problem whether the relation (21) can be generalized to CSF for the three-parameter multiple series (18). (This problem is what the author wrote in
research documents of February 2016.) We note that the relation (21) has an interesting application to the study of relations among MZVs and MZSVs. Indeed, we showed in [14 (R2) of Section 3] that the sum formulas for MZVs and MZSVs can be derived from (21) by partial differentiation. This application gives a new proof of the sum formulas, which is hypergeometric. From these facts, we think that the results of the above problem also have this kind of interesting application. Finally, we give some notes on (18) and (21). Not only (18) but also the four-parameter multiple series of [14 was discovered by the author in 2014 (before September 17, 2014) by studying the hypergeometric identities of Andrews [2, Theorem 4], Krattenthaler and Rivoal [25, Proposition 1] (see [15, Note 2 (iv)] and [14, Introduction]). The results of [14 already show the potentiality of these multiple series (see also [15 Note 2]). Our previous remark [12, Remark 2.7] played an essential role for discovering these multiple series and their identities. The multiple series (18) converges uniformly in any compact subset of the domain \( \{ (\alpha, \beta, \gamma) \in \mathbb{C}^3 : \text{Re}(\alpha), \text{Re}(\beta) > 0, \text{Re}(\alpha + \beta) > \text{Re}(\gamma) > 0 \} \); therefore it is holomorphic in this domain as a function of \((\alpha, \beta, \gamma)\). The relation (21) can be derived from the hypergeometric identity of [25, Proposition 1 (ii)] by taking \( a = \alpha + \beta, c_0 = \alpha + \beta - \gamma, b_i = \alpha, c_i = \beta \) \((i = 1, \ldots, s - 1)\), \( b_s = 1, c_s = \gamma \) \((s \in \mathbb{Z}_{\geq 2}, \alpha, \beta, \gamma \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0} \) such that \( \text{Re}(\alpha + \beta - \gamma) > 0 \)). For another generalization of (20) and a related problem, see [14 Corollary 2.3 (i) and (R1) of Section 3].

**Note (2022).** I give some notes related to my papers [10] and [11]. The study of the multiple series (1) was originated in my master’s thesis [10]. I obtained the results of [10] in 2006. One of the main results is the sum formula for the case \( \alpha = \beta \) of my multiple series (1). In 2006, I talked on the sum formula and its proof at my then academic adviser’s seminar. On February 3, 2007, my master’s thesis passed the final examination. From some facts, I concluded in February 2007 that a certain person had already been aware of the contents of
my master’s thesis [10] at the workshop “Zeta Wakate Kenkyūshūkai” held at Graduate School of Mathematics, Nagoya University, Japan, February 17–18, 2007. (I am writing about this matter in [19].) After my master’s thesis, I generalized my sum formula mentioned above to a large class of relations; see [11, Theorem 1.1]. I talked on my research [11] and related notes (see [18, p. 29]) at Seminar on Analytic Number Theory, Graduate School of Mathematics, Nagoya University, Japan, February 13, 2008. See also [11, Acknowledgments on p. 578]. (The following people were parts of the audience of my talk: Kohji Matsumoto, Yoshio Tanigawa, Takashi Nakamura, Yoshitaka Sasaki.) The title of my talk was “On Ohno’s relation for certain multiple series”. This title was announced both in Japanese and English on the website of Graduate School of Mathematics, Nagoya University, Japan about one week before. In addition, the title was also announced on February 5, 2008 by using a mailing list of the Seminar. My multiple series (1) and (2) and their extensions studied in [14], [18] are highly non-trivial objects. I am writing an expository paper on my previous papers on these multiple series, which is [21].

Corrections to [12]. (i) Page 514, lines 11–13 from the bottom: Change “0 < \text{Re } \alpha < s − 1” into “\text{Re } \alpha > 0” and remove the sentence “The condition 0 < \text{Re } \alpha < s − 1 can be . . . as functions of \alpha.”. (ii) Page 517, line 12 from the bottom: “March 2007” should be “February 3, 2007”.

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