PUSHOUTS OF EXTENSIONS OF GROUPOIDS BY BUNDLES OF ABELIAN GROUPS

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Abstract. We analyse extensions $\Sigma$ of groupoids $G$ by bundles $A$ of abelian groups. We describe a pushout construction for such extensions, and use it to describe the extension group of a given groupoid $G$ by a given bundle $A$. There is a natural action of $\Sigma$ on the dual of $A$, yielding a corresponding transformation groupoid. The pushout of this transformation groupoid by the natural map from the fibre product of $A$ with its dual to the Cartesian product of the dual with the circle is a twist over the transformation groupoid resulting from the action of $G$ on the dual of $A$. We prove that the full $C^\ast$-algebra of this twist is isomorphic to the full $C^\ast$-algebra of $\Sigma$, and that this isomorphism descends to an isomorphism of reduced algebras. We give a number of examples and applications.

We respectfully dedicate this paper to the memory of Vaughan Jones: Extraordinary mathematician, proud New Zealander, and gracious colleague.

Introduction

There is a significant body of literature regarding the $C^\ast$-algebras of extensions of groupoids by group bundles. The main goal of this paper is to introduce a pushout construction for extensions of groupoids by abelian group bundles and explore its applications.

Specifically, we consider a locally compact Hausdorff groupoid $G$ together with an abelian group bundle $p_A : A \to G^{(0)}$ where $p_A$ a continuous, open map. Then we consider the following notion of an extension that fixes unit spaces, represented by the diagram

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\iota} & \Sigma \\
\downarrow & & \downarrow \\
G^{(0)} & \xleftarrow{P} & \mathcal{G} \end{array}
\]

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here $\Sigma$ is a locally compact Hausdorff groupoid, $p : \Sigma \to \mathcal{G}$ is a continuous open surjective groupoid homomorphism that restricts to a homeomorphism $\Sigma^0 \cong \mathcal{G}(0)$, and $\iota : \mathcal{A} \to \Sigma$ is a groupoid homomorphism that is a homeomorphism onto $\ker(p) = p^{-1}(\mathcal{G}(0))$ in the subspace topology, such that $p \circ \iota = p_A$, $r_\Sigma \circ \iota = r_A = p_A$, $s_\Sigma \circ \iota = s_A = p_A$, $r_G \circ p = r_\Sigma$ and $s_G \circ p = s_\Sigma$.

A fundamental class of such examples are $\mathcal{T}$-groupoids (also called twists) introduced by the second author in [Kum83]. Then $\mathcal{A}$ is the trivial bundle $\mathcal{G}(0) \times \mathcal{T}$ such that $\iota(\sigma, z) = \sigma \iota(s(\sigma), z)$ for all $\sigma \in \Sigma$ and $z \in \mathcal{T}$. These groupoids and their restricted groupoid $C^*$-algebras, $C^*(\mathcal{G}; \Sigma)$, have enjoyed considerable scrutiny [MW92, MW95, Kum83, Kum86]. As usual, in this context we often write $\dot{\sigma}$ in place of $p(\sigma)$.

More recently, we considered more general extensions in [IKSW19] and [IKR+21] as in (†) where it is assumed that $\mathcal{A}$ is endowed with an action of $\mathcal{G}$ and that the extension is compatible in the sense that $\sigma \iota(a) \sigma^{-1} = \iota(\dot{\sigma} \cdot a)$ for all $a \in \mathcal{A}$ and $\sigma \in \Sigma$ such that $p_A(a) = s(\sigma)$.

As a consequence of the main result in [IKR+21], we showed that if $\Sigma$ has a Haar system, then $C^*(\Sigma)$ can be realized as the $C^*$-algebra of a twist. Specifically, the action of $\mathcal{G}$ on $\mathcal{A}$ induces a natural action of $\mathcal{G}$ on $\hat{\mathcal{A}}$ (regarded as a space). We constructed a $\mathcal{T}$-groupoid $\hat{\Sigma}$ of the form

$$
\hat{\Sigma} \xrightarrow{j} \hat{\mathcal{A}} \xrightarrow{\iota} \mathcal{T} \xleftarrow{i} \mathcal{A} \xleftarrow{p} \mathcal{G}.
$$

We proved ([IKR+21, Theorem 3.4]) that $C^*(\Sigma)$ is isomorphic to the restricted $C^*$-algebra $C^*(\mathcal{A} \rtimes \mathcal{G}; \hat{\Sigma})$ of this $\mathcal{T}$-groupoid. (In [IKR+21] the $\mathcal{T}$-groupoid is denoted $\hat{\Sigma}$, but here we use $\hat{\Sigma}$ to avoid possible confusion in our examples.) The $\mathcal{T}$-groupoid $\hat{\Sigma}$ is at the heart of the Mackey obstruction which appears in the classical “Mackey machine” of [Mac58].

The chief motivation for this article is the observation that the $\mathcal{T}$-groupoid $\hat{\Sigma}$ above—which was based on the construction of [MRW96, Proposition 4.3]—is derived from a natural and functorial “pushout” construction based on the second author’s work in [Kum88] for étale groupoids (there called “sheaf groupoids”). Specifically, suppose we are given an extension as in (†), an abelian group bundle $\mathcal{B}$ admitting a $\mathcal{G}$-action, and an equivariant groupoid homomorphism $f : \mathcal{A} \to \mathcal{B}$. Then there is a similar sort of extension

$$
\mathcal{B} \xrightarrow{\iota} f_\Sigma \xleftarrow{p} \mathcal{G} \xrightarrow{\mathcal{G}(0)} f_\Sigma
$$

inducing the given $\mathcal{G}$-action on $\mathcal{B}$. In Theorem 1.5, we show that the construction producing $f_\Sigma$ has good functorial properties that characterize the extension up to a suitable notion of isomorphism. Using these properties, we show in Theorem 2.5 that the collection $T_\mathcal{G}(\mathcal{A})$ of isomorphism classes of extensions of $\mathcal{A}$ by $\mathcal{G}$ forms an abelian group (see also [Tu06, §5.3]).
We close by illustrating how the pushout construction clarifies and interacts with our work in [IKSW19] and [IKR+21]. In Theorem 3.2 we prove that the extension (†) employed in [IKR+21] arises from our pushout construction. Specifically, the natural pairing \((\chi, a) \mapsto \chi(a)\) from \(\hat{A} \ast A\) to \(T\) yields a groupoid homomorphism \(f: \hat{A} \ast A \to \hat{A} \times T\) given by \(f(\chi, a) = (\chi, \chi(a))\) (see Section 3.1). There is a natural action of \(\Sigma\) on \(\hat{A}\) (compatible with that of \(G\) as above) and we prove that \(\tilde{\Sigma} \cong f_* (\hat{A} \times \Sigma)\). This allows us to realise the \(C^\ast\)-algebra of an extension of a groupoid \(G\) by an abelian group bundle \(A\) as the \(C^\ast\)-algebra of a \(T\)-groupoid over the resulting transformation groupoid \(\hat{A} \times G\).

Several consequences flow from this observation. First suppose that \(A\) is an abelian group and that \(A = G(0) \times A\), carrying the action of \(G\) that is trivial in the second coordinate, so that \(\Sigma\) is a generalised twist. Each \(\chi \in \hat{A}\) defines a homomorphism \(f^\chi : A \to T \times G(0)\), so we can form the resulting pushout \(f^\chi_* (\Sigma)\). We prove in Proposition 3.6 that \(C^\ast(\Sigma)\) is the section algebra of an upper-semicontinuous \(C^\ast\)-bundle over \(\hat{A}\) with fibres \(C^\ast(\hat{G}, f^\chi_* (\Sigma))\). When \(A\) is compact, this yields a direct sum decomposition which remains valid for the corresponding reduced \(C^\ast\)-algebras (see Proposition 3.7). In Corollary 3.10 we extend [IKR+21, Theorem 3.4] to the case that \(\Omega\) is a \(T\)-groupoid extension of \(\Sigma\) such that its restriction to \(\iota(A)\) is abelian. When \(G\) is étale, this enables us to generalize [IKR+21, Theorem 4.6] to this case (see Corollary 3.11) thereby providing criteria that guarantee that the natural abelian subalgebra of \(C^\ast_r (\Sigma; \Omega)\) is Cartan (see also [DGN+20, Theorem 5.8] and [DGN20, Theorem 4.6]).

In Subsection 3.2, we consider the case where the extension \(\Sigma\) is determined by an \(A\)-valued 2-cocycle defined on \(G\) and show that the pushout construction is compatible with the natural change of coefficients map on cocycles. We describe the explicit construction of \(\tilde{\Sigma}\) in terms of 2-cocycles at the beginning of Subsection 3.3, and then consider various examples of this construction.

1. Pushouts of Groupoid Extensions

We fix a locally compact Hausdorff groupoid \(G\). In our applications, \(G\) will have a Haar system, but this is not required for the pushout construction itself. However, we do assume that \(G\) has open range and source maps. We call a locally compact abelian group bundle \(p_A : A \to G^{(0)}\) a \(G\)-bundle if \(p_A\) is open and \(G\) acts on the left of \(A\) by automorphisms. For compatibility with [IKSW19]—and other examples we have in mind—we will write the group operations in the fibres of such \(A\) additively.

An extension \(\Sigma\) of \(A\) by \(G\) is determined by a diagram (†) as in the introduction. Recall that \(\Sigma\) is a locally compact Hausdorff groupoid, \(p\) is continuous and open surjection inducing a homeomorphism from \(\Sigma^{(0)}\) onto \(G^{(0)}\), and \(\iota\) is a continuous open injective homomorphism onto \(\ker p = \{ \sigma \in \Sigma : p(\sigma) \in G^{(0)} \}\). We call such an extension compatible if the action of \(G\) on \(A\) induced by conjugation is the given \(G\)-action on \(A\); that is, \(\sigma a \sigma^{-1} = \iota(\sigma \cdot a)\).

Definition 1.1. If \(\Sigma_1\) and \(\Sigma_2\) are both compatible extensions by a locally compact abelian group \(G\)-bundle \(A\), then we say that they are properly isomorphic if there
is a groupoid isomorphism \( f : \Sigma_1 \to \Sigma_2 \) such that the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{i_1} & \Sigma_1 \\
\downarrow{p_1} & & \downarrow{f} \\
\Sigma_2 & \xleftarrow{i_2} & G
\end{array}
\]  

(1.1)

commutes. We let \( T_G(A) \) be the collection of proper isomorphism classes of compatible extensions. We denote the equivalence class of a compatible extension \( \Sigma \) by \([\Sigma]\).

Remark 1.2. The second author considered extensions of this sort for étale groupoids in [Kum88, §2]. In [Tu06, §5.3], Tu denotes this set by \( \text{ext}(G,A) \) and states that it forms an abelian group (see Theorem 2.5 below). Since the openness of \( p_A : A \to G^{(0)} \) implies that \( A \) has a Haar system (see [IKR+21, Lemma 2.1]), it follows that if \( G \) has a Haar system, then we can then equip \( \Sigma \) with a Haar system whenever \([\Sigma] \in T_G(A) \) (see [IKR+21, Lemma 2.6]).

Of course, given \( G \) and a \( G \)-bundle \( A \), we would like to know that \( T_G(A) \) is not empty. To provide a basic example, we follow [Kum88, Definition 2.1].

Example 1.3 (The Semidirect Product). We can build a fundamental compatible extension \( A \triangleleft G \) from the fibred product \( \{(a, \gamma) \in A \times G : p_A(a) = r(\gamma)\} \). We let \( (A \triangleleft G)^{(2)} = \{((a_1, \gamma_1), (a_2, \gamma_2)) : s(\gamma_1) = r(\gamma_2)\} \), and then define

\[
(a_1, \gamma_1)(a_2, \gamma_2) = (a_1 + \gamma_1 \cdot a_2, \gamma_1 \gamma_2) \quad \text{and} \quad (a, \gamma)^{-1} = (-\gamma^{-1} \cdot a, \gamma^{-1}).
\]

Then we can identify the unit space of \( A \triangleleft G \) with \( G^{(0)} \) so that \( r(a, \gamma) = r(\gamma) \) and \( s(a, \gamma) = s(\gamma) \). We can then exhibit \( A \triangleleft G \) as an extension by letting \( \iota(a) = (a, p_A(a)) \), and letting \( p(a, \gamma) = \gamma \). Since

\[
(a', \gamma)(a, p_A(a))(-\gamma^{-1} \cdot a', \gamma^{-1}) = (\gamma \cdot a, p_A(\gamma \cdot a)),
\]

\( A \triangleleft G \) is a compatible extension as required.

Example 1.4. For \( i = 1, 2 \) let \( A_i \) be a locally compact abelian group \( G \)-bundle. Note that \( A_1 \ast A_2 = \{(a, a') : p_{A_1}(a) = p_{A_2}(a')\} \) is also a locally compact abelian group \( G \)-bundle. Let \( \Sigma_i \) be a compatible groupoid extension of \( G \) by \( A_i \). Then as in [Kum88, §2], we may form the fibered product

\[
\Sigma_1 \ast_G \Sigma_2 := \{((\sigma_1, \sigma_2) \in \Sigma_1 \times \Sigma_2 | p_1(\sigma_1) = p_2(\sigma_2)\}.
\]

It is straightforward to check that \( \Sigma_1 \ast_G \Sigma_2 \) is a compatible groupoid extension of \( G \) by \( A_1 \ast A_2 \).

Assume now that \( B \) is another abelian group \( G \)-bundle, and that \( f : A \to B \) is a \( G \)-equivariant map. Following [Kum88, Proposition 2.6], we prove that we can “pushout” \( \Sigma \) in a unique way to an extension of \( G \) by \( B \).

Theorem 1.5 (Pushout Construction). Let \( A \) and \( B \) be locally compact abelian group \( G \)-bundles. Let \( f : A \to B \) be a continuous \( G \)-equivariant map. Assume that \( \Sigma \) is a compatible extension of \( G \) by \( A \). Then there is a compatible extension \( f_* \Sigma \).
of $G$ by $B$ and a homomorphism $f_* : \Sigma \to f_*\Sigma$ such that the following diagram commutes

$$
\begin{array}{ccc}
A & \xrightarrow{\iota} & \Sigma \\
\downarrow{f} & & \downarrow{f_*} \\
B & \xrightarrow{\iota_*} & f_*\Sigma \\
\end{array}
\quad (1.2)
$$

Moreover, $f_*$ and $f_*\Sigma$ are unique up to proper isomorphism in the sense that if $\Sigma'$ is a compatible extension such that the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\iota} & \Sigma \\
\downarrow{f} & & \downarrow{f'} \\
B & \xrightarrow{\iota'} & \Sigma' \\
\end{array}
\quad (1.3)
$$

commutes, then there is a proper isomorphism $g : f_*\Sigma \to \Sigma'$ such that $g \circ f_* = f'$.

**Proof.** Consider the fibred-product groupoid

$$
D := (B \triangleleft G) \times_{\Sigma} \Sigma = \{ ((b,\gamma),\sigma) \in (B \triangleleft G) \times \Sigma : \sigma = \gamma \}
$$
of Example 1.4. Define $\theta : A \to D$ via $\theta(a) = ((-f(a),p_A(a)),\iota(a))$. Since $\iota$ is a homeomorphism onto its closed range, $\theta(A)$ is a closed wide subgroupoid of $D$.

Let $d = ((b,\gamma),\sigma) \in D$. We claim that $d\theta(A) = \theta(A)d$. To see this, note that

$$
d\theta(a) = ((b,\gamma),\sigma)((-f(a),p_A(a)),\iota(a))
= ((b - \gamma \cdot f(a),\gamma),\sigma \iota(a))
= ((-f(\gamma \cdot a) + p_A(\gamma \cdot a),b,\gamma),\iota(\gamma \cdot a)\sigma).
$$

Since $\gamma = \gamma \circ \iota$, we deduce that

$$
d\theta(a) = ((-f(\gamma \cdot a),p_A(\gamma \cdot a),\iota(\gamma \cdot a))(b,\gamma,\sigma)
= \theta(\gamma \cdot a)d.
$$

Let $f_*\Sigma := D/\theta(A)$. As usual, we denote the class of $((b,\sigma),\gamma)$ in $f_*\Sigma$ by $[(b,\sigma),\gamma]$. Then $[(b,\gamma),\iota(a)\sigma] = [(b + f(a),\gamma),\sigma]$. Since $j(A)$ has a Haar system by Remark 1.2, $f_*\Sigma$ is a locally compact Hausdorff groupoid by [IKR+21, Lemma 2.2]. The operations are given by

$$
[(b_1,\gamma_1),\sigma_1][b_2,\gamma_2),\sigma_2] = [(b_1 + \gamma_1 b_2,\gamma_1 \gamma_2),\sigma_1 \sigma_2] \quad \text{and} \quad [(b,\gamma),\sigma]^{-1} = [(-\gamma^{-1} \cdot b,\gamma^{-1}),\sigma^{-1}].
$$

We can identify the unit space with $G^{(0)}$ and then

$$
r([(b,\gamma),\sigma]) = r(\gamma) \quad \text{and} \quad s([(b,\gamma),\sigma]) = s(\gamma).
$$

To see that $f_*\Sigma$ is a compatible extension by $B$, let

$$
\iota_* (b) = [(b,p_B(b)),p_B(b)] \quad \text{and} \quad p_*([(b,\gamma),\sigma]) = \gamma.
$$

It is not hard to verify that this satisfies the algebraic requirements for an extension. The most difficult one might be the inclusion $p_*^{-1}(G^{(0)}) \subseteq \iota_*(B)$ for which we provide an outline of the proof: take $[(b,\gamma),\sigma] \in f_*\Sigma$ such that $p_*([(b,\gamma),\sigma]) = u \in G^{(0)}$. Then $\gamma = u$, giving $\sigma = u$. Since $\Sigma$ is an extension, there exists $a \in A(u)$ such that $\iota(a) = \sigma$. It follows that $[(b,u),\iota(a)] = [(b + f(a),u),u] = \iota_*(b + f(a))$. It
is easy to check that \( b + f(a) \) is independent of the choice of the representative of 
\([((b, \gamma), \sigma)]\).

Since \( \iota_* \) and \( p_* \) are clearly continuous and since \( \iota_* \) is easily seen to be a home-
omorphism onto its range, we just need to see that \( p_* \) is open. For this, we apply
Fell’s Criterion (see [IKR+ 21, Lemma 3.1]). Suppose that \( \gamma_\eta \to \gamma = p_*([(b, \sigma), \gamma]) \).
Since \( p : \Sigma \to \mathcal{G} \) is open, we can pass to a subnet, relabel, and assume that there are
\( \sigma_\eta \to \sigma \) in \( \Sigma \) such that \( \tilde{\sigma}_\eta = \gamma_\eta \). Since \( p_\mathcal{G} \) is open, we can pass to subnet,
relabel, and assume there are \( b_\eta \to b \) in \( \mathcal{B} \) such that \( p_\mathcal{G}(b_\eta) = r(\gamma_\eta) \). Then
\([([b_\eta, \gamma_\eta], \sigma_\eta)] \to [([b, \gamma]), \sigma] \) as required.

The map \( f_* \) is the composition of the embedding of \( \Sigma \) into \( \mathcal{D} \) and the quotient
map \( \mathcal{D} \to \mathcal{D}/\iota(\mathcal{A}) : f_*(\sigma) = \([(0, \iota(\sigma), p(\sigma)), \sigma]) \). Since \( f \) is \( \mathcal{G} \)-equivariant, \( p_\mathcal{G}(f(a)) = p_\mathcal{A}(a) \) and

\[ f_*(\iota(a)) = \<![0, p_\mathcal{A}(a), p_\mathcal{A}(a)] = [[f(a), p_\mathcal{G}(f(a))], p_\mathcal{G}(f(a))] = \iota_*(\iota(a)), \]

and (1.2) commutes as required.

Now let \( \Sigma' \) be an extension as in (1.3). Define \( \tilde{g} : \mathcal{D} \to \Sigma' \) by \( \tilde{g}(b, \chi, \sigma) = \iota'(b)f'(\sigma) \). Since

\[ \iota'(b_1)f'(\sigma_1)\iota'(b_2)f'(\sigma_2) = \iota'(b_1)\iota'(f'(\sigma_1) \cdot b_2)f'(\sigma_1)f'(\sigma_2) \]

and since \( f'(f'(\sigma_1)) = \tilde{\sigma}_1 \), it follows that \( \tilde{g} \) is a groupoid homomorphism. On the
other hand,

\[ \tilde{g}(\theta(a)) = \tilde{g}((-f(a), p_\mathcal{A}(a), \iota(a)) = \iota'(-f(a))f'(\iota(a)) = \iota'(-f(a))\iota'(f(a)) \]

\[ = \iota'(p_\mathcal{A}(a)). \]

Hence \( \tilde{g} \) factors through a homomorphism \( g : f_*\Sigma \to \Sigma' \). Clearly, \( g(\iota_*(b)) = \iota'(b) \)
and \( p' \circ g = p_* \), so \( g \) makes the diagram analogous to (1.11) commute. We have \( g \circ f_* = f' \) by construction.

To see that \( g \) is a proper isomorphism, we still need to see that \( g \) is an isomor-
phism with a continuous inverse.

For this, fix \( a \in \Sigma' \). There exists \( \sigma \in \Sigma \) such that \( p(\sigma) = f'(a) \). Using (1.3),
there exists \( b \) in \( \mathcal{B} \) such that \( \alpha = \iota'(b)f'(\sigma) \). So \( \tilde{g} \), and hence also \( g \), is onto.

Now suppose that \( \iota'(b)f'(\sigma) \) is a unit. Then \( f'(\sigma) = \iota'(-b) \). Hence \( p'(f'(\sigma)) \) is a
unit, and \( \sigma = \iota(a) \) for some \( a \in \mathcal{A} \). But then \( \iota'(-b) = f'(\sigma) = f'(\iota(a)) = \iota'(f(a)) \).
Hence, \( b = -f(a) \). That is,

\[ ((b, p(\sigma)), \sigma) = ((-f(a), p_\mathcal{A}(a)), \iota(a)) \in \theta(\mathcal{A}). \]

Thus \( g \) is injective.

To see that \( g \) is an isomorphism of topological groupoids, it suffices to see that \( g \)
is open. We use Fell’s criterion. Suppose that \( g(\alpha_i) \to g(\alpha) \) where \( \alpha_i = [\alpha, \rho(\sigma_i)], \sigma_i \) \(
and \( \alpha = [\alpha, p(\sigma)], \sigma] \in f_*\Sigma \). Since \( p' \circ g = p_* \), we have \( p(\sigma_i) \to p(\sigma) \).
Since \( p \) is open, we can pass to a subnet, relabel, and assume there exist \( a_i \in \mathcal{A} \)
such that \( \iota(a_i)\sigma_i \to \sigma \). But

\[ \alpha_i = [(-f(a_i), b_i, \rho(\sigma_i), \iota(a_i)\sigma_i)], \]

and then

\[ \iota'(-f(a_i) + b_i)f'(\iota(a_i)\sigma_i) \to \iota'(b)f'(\sigma). \]

It follows that

\[ \iota'(-f(a_i) + b_i) \to \iota'(b). \]
Since \( \iota' \) is a homeomorphism onto its range, \( \alpha_1 \to \alpha \) as required. \( \square \)

**Corollary 1.6.** Let \( A, B \) and \( C \) be locally compact abelian group \( G \)-bundles. Let \( f : A \to B \) and \( g : B \to C \) be continuous \( G \)-equivariant maps. Assume that \( \Sigma \) is a compatible extension of \( G \) by \( A \). Then \( (g \circ f)_* \Sigma \) is properly isomorphic to \( g_*(f_* \Sigma) \).

**Proof.** This follows from the uniqueness of \( (g \circ f)_* \Sigma \) up to proper isomorphism guaranteed by Theorem 1.5. \( \square \)

2. The Extension Group \( T_G(A) \)

As in [Kum88, §2], we can use our pushout construction to introduce a binary operation on \( T_G(A) \). Suppose that \([\Sigma], [\Sigma'] \in T_G(A)\). Define \( \nabla_A : \Sigma \to \Gamma \) by \( \nabla_A(a, a') = a + a' \). Proper isomorphisms \( f : \Sigma \to \Gamma \) and \( f' : \Sigma' \to \Gamma' \) of compatible extensions of \( A \) by \( G \) determine a proper isomorphism \( f_* f'_* \Sigma \to f_* f'_* \Sigma \). The uniqueness assertion of Theorem 1.5 then yields a proper isomorphism \( \nabla_A (\Sigma \ast G \Sigma') \to \nabla_A (\Gamma \ast G \Gamma') \). Hence the formula

\[
[\Sigma] + [\Sigma'] := [\nabla_A (\Sigma \ast G \Sigma')]
\]

(2.1)

is well defined.

**Example 2.1.** Let \([\Sigma] \in T_G(A)\). Let \( A \ast G \) be the semidirect product defined in Example 1.3. Define \( g : (A \ast G) \ast G \Sigma \to \Sigma \) by \( g((a, \sigma), \sigma) = \iota(a) \sigma \). We obtain a commutative diagram

\[
\begin{array}{ccc}
A \ast A & \xrightarrow{\iota \ast \iota} & (A \ast G) \ast G \\
\downarrow \nabla_A & & \downarrow \hat{p} \\
A & \xrightarrow{\iota} & \Sigma
\end{array}
\]

The uniqueness assertion in Theorem 1.5 implies that \( \nabla_A ((A \ast G) \ast G \Sigma) \) is properly isomorphic to \( \Sigma \). In other words, \([A \ast G] + [\Sigma] = [\Sigma] \).

**Example 2.2.** Let \( A \xrightarrow{\iota} \Sigma \xrightarrow{p} G \) be a compatible extension. Then we obtain another compatible extension \( A \xrightarrow{\iota'} \Sigma \xrightarrow{p} G \) by letting \( \iota'(a) = \iota(-a) = \iota(a)^{-1} \). We will write \( \Sigma^{-1} \) for \( \Sigma \) viewed as this alternate extension. Define \( \theta : A \to A \) by \( \theta(a) = -a \). Then \( \theta \) is \( G \)-invariant. Since the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\iota} & \Sigma \\
\downarrow g & & \downarrow p \\
A & \xrightarrow{\iota'} & \Sigma^{-1}
\end{array}
\]

commutes, we can identify \([\theta, \Sigma] \) with \([\Sigma^{-1}] \) by Theorem 1.5.

**Example 2.3.** Take \([\Sigma] \in T_G(A)\) and let \( A \ast G \) be the semidirect product. The map \( g : \Sigma \ast \Sigma^{-1} \to A \ast G \) given by \( g(\sigma, \tau) = (\iota^{-1}(\sigma \tau^{-1}), \sigma) \) is a homomorphism. Since
the diagram

\[
\begin{array}{ccc}
\mathcal{A} \ast \mathcal{A} & \xrightarrow{\ast} & \Sigma \ast \Sigma^{-1} \\
\n & \xrightarrow{g} & \mathcal{G} \\
\mathcal{A} & \xrightarrow{i} & \mathcal{A} \ast \mathcal{G}
\end{array}
\]

commutes, we see that \([\Sigma] + [\Sigma^{-1}] = [\mathcal{A} \ast \mathcal{G}]\) for all \(\Sigma \in T_G(\mathcal{A})\).

Example 2.4. Take \([\Sigma], [\Sigma'] \in T_G(\mathcal{A})\). Let \(f : \Sigma * \mathcal{G} \ast \Sigma' \to \Sigma' * \mathcal{G} \Sigma\) be the flip. Similarly, let \(f : \mathcal{A} * \mathcal{A} \to \mathcal{A} * \mathcal{A}\) be given by \(f(a, a') = (a', a)\). The diagram

\[
\begin{array}{ccc}
\mathcal{A} \ast \mathcal{A} & \xrightarrow{\ast} & \Sigma \ast \Sigma' \\
\n & \xrightarrow{\tilde{f}} & \mathcal{G} \\
\mathcal{A} \ast \mathcal{A} & \xrightarrow{i} & \mathcal{A} \ast \mathcal{G}
\end{array}
\]

commutes. Since \(\nabla^A \circ f = \nabla^A\), it follows from Theorem 1.5 that \([\Sigma] + [\Sigma'] = [\Sigma'] + [\Sigma]\).

In Examples 2.1–2.4, we have proved much of the following theorem, which is patterned on [Kum88, Theorem 2.7].

**Theorem 2.5.** Let \(\mathcal{G}\) be a locally compact groupoid with open range and source maps, and let \(\mathcal{A}\) be a locally compact abelian group \(\mathcal{G}\)-bundle. Then the binary operation \([\Sigma_1, \Sigma_2] \mapsto [\nabla^A(\Sigma_1 * \mathcal{G} \Sigma_2)]\) of (2.1) makes \(T_G(\mathcal{A})\) into an abelian group with neutral element given by the class \([\mathcal{A} \ast \mathcal{G}]\) of the semidirect product of Example 1.3, and \([\Sigma]^{-1} = [\Sigma']^{-1}\) as in Example 2.2. For each continuous \(\mathcal{G}\)-equivariant map \(f : \mathcal{A} \to \mathcal{B}\) of \(\mathcal{G}\)-bundles, define \(T_G(f) : T_G(\mathcal{A}) \to T_G(\mathcal{B})\) to be the induced map: \(T_G(f)[\Sigma] = [f, \Sigma]\). Then \(T_G\) is a functor from the category of \(\mathcal{G}\)-bundles to the category of abelian groups.

**Proof.** By considering diagrams similar to that in Example 2.4, we see that the operation in (2.1) is well-defined and associative. We saw that \([\mathcal{A} \ast \mathcal{G}]\) acts as an identity in Example 2.1 and the statement about inverses follows from Example 2.3. The computation in Example 2.4 shows that \(T_G(\mathcal{A})\) is an abelian group.

By Corollary 1.6 we have \(T_G(f \circ g) = T_G(f) \circ T_G(g)\) if \(f\) and \(g\) are a composable pair of continuous \(\mathcal{G}\)-equivariant maps of \(\mathcal{G}\)-bundles. The proof that \(T_G(f)\) is a group homomorphism follows as in the proof of [Kum88, Theorem 2.7]. □

### 3. Applications and Examples

In this section we consider a unit space fixing extension \(\Sigma\) of \(\mathcal{G}\) by the group bundle \(\mathcal{A}\) as illustrated in the diagram (†) from the introduction. We review the basic details. We assume that all groupoids considered in this section are second-countable locally compact Hausdorff groupoids with Haar systems. The Haar system on \(\Sigma\) is denoted \(\lambda = \{\lambda^u\}_{u \in \Sigma(0)}\) and we further assume that \(p_\mathcal{A} : \mathcal{A} \to \Sigma(0)\) is a bundle of abelian groups that is a closed subgroupoid of \(\Sigma\). It is equipped with a Haar system denoted \(\beta = \{\beta^u\}_{u \in \Sigma(0)}\) and the fibers are denoted \(\mathcal{A}(u)\) for
$u \in \Sigma^{(0)}$. The existence of a Haar system on $\mathcal{A}$ implies that $p\mathcal{A}$ is open. It follows by [IKR$^{+}$21, Lemma 2.6(c)] that there is a Haar system $\alpha = \{\alpha_u\}_{u \in \Sigma^{(0)}}$ on $\mathcal{G}$ such that for all $f \in C_c(\Sigma)$ and $u \in \Sigma^{(0)}$ we have

$$\int_{\Sigma} f(\sigma) \, d\lambda^u(\sigma) = \int_{\mathcal{A}} f(\alpha_u(\sigma)) \, d\alpha^u(\sigma).$$

Moreover, there is a natural action of $\Sigma$, and therefore $\mathcal{G}$, on $\mathcal{A}$.

Note that $\mathcal{G}$ is a continuous, open surjection inducing a homeomorphism from $\Sigma^{(0)}$ onto $\mathcal{G}^{(0)}$, and $\iota: \mathcal{A} \to \Sigma$ is a homeomorphism onto $\ker p$. (Both $p$ and $\iota$ are assumed to be groupoid morphisms).

Recall that if $\Sigma$ is a $\mathcal{T}$-groupoid over $\mathcal{G}$ then

$$C_c(\mathcal{G}; \Sigma) := \{ f \in C_c(\Sigma) : f(t\sigma) = tf(\sigma) \text{ for all } t \in \mathcal{T}, \sigma \in \Sigma \}$$

is a $^*$-algebra under the operations described in [MW92, §2], and $C^*(\mathcal{G}; \Sigma)$ is its closure in the norm obtained by taking the supremum of the operator norm under all $^*$-representations.

We may also view $C_c(\mathcal{G}; \Sigma)$ as compactly supported continuous sections of the one-dimensional Fell line bundle over $\mathcal{G}$ associated to $\Sigma$. One can then construct the associated (right) Hilbert $C_0(\mathcal{G}^{(0)})$-module (see [IKR$^{+}$21, §1.3]) as the completion of $C_c(\mathcal{G}; \Sigma)$ in the norm arising from the $C_0(\mathcal{G}^{(0)})$-valued pre-inner product given by $(f, g) := (f^* + g)|_{\mathcal{G}^{(0)}}$ for all $f, g \in C_c(\mathcal{G}; \Sigma)$. We denote the Hilbert module by $\mathcal{H}(\mathcal{G}; \Sigma)$ and observe that left multiplication induces a natural $^*$-homomorphism $\lambda : C_c(\mathcal{G}; \Sigma) \to \mathcal{L}(\mathcal{H}(\mathcal{G}; \Sigma))$. We may define the reduced norm of an element $f \in C_c(\mathcal{G}; \Sigma)$ to be the operator norm of its image: $\|f\|_r := \|\lambda(f)\|$. Then $C^*_r(\mathcal{G}; \Sigma)$ is the closure of $C_c(\mathcal{G}; \Sigma)$ in the reduced norm.

**Lemma 3.1.** With notation as above, let $F \subset \mathcal{G}^{(0)}$ be a $\mathcal{G}$-invariant clopen subset. Then $F$ is also $\Sigma$-invariant and the reduction $\Sigma|_F$ is a twist over the reduction $\mathcal{G}|_F$. Moreover, the characteristic function of $F$ determines a central multiplier projection $p_F$ such that

$$p_F C^*_r(\mathcal{G}; \Sigma) \cong C^*_r(\mathcal{G}|_F; \Sigma|_F).$$

**Proof.** Observe that $\mathcal{H}(\mathcal{G}; \Sigma)$ decomposes as the direct sum of a Hilbert $C_0(F)$-module and a Hilbert $C_0(F^c)$-module in the following way

$$\mathcal{H}(\mathcal{G}; \Sigma) \cong \mathcal{H}(\mathcal{G}|_F; \Sigma|_F) \oplus \mathcal{H}(\mathcal{G}|_{F^c}; \Sigma|_{F^c}).$$

Note that multiplication by the characteristic function of $F$, which we denote by $p_F$ is the projection onto the first component, that $p_F$ is in the center of the multiplier algebra of $C^*_r(\mathcal{G}; \Sigma)$, and $C_c(\mathcal{G}|_F; \Sigma|_F)$ acts trivially on the second component. Hence the operator norm of $C_c(\mathcal{G}|_F; \Sigma|_F)$ acting on $\mathcal{H}(\mathcal{G}|_F; \Sigma|_F)$ coincides with that of its action on $\mathcal{H}(\mathcal{G}; \Sigma)$.

**3.1. The $\mathcal{T}$-groupoid of an extension.** As noted in the introduction, we want to see that the $\mathcal{T}$-groupoid constructed in [IKR$^{+}$21, §3.1] is an example of the pushout construction of Theorem 1.5. The $C^*$-algebra $C^*(\mathcal{A})$ is abelian and the Gelfand dual of $C^*(\mathcal{A})$ is an abelian group bundle $\tilde{p} : \tilde{\mathcal{A}} \to \mathcal{G}^{(0)} = \Sigma^{(0)}$ with fibers $\tilde{p}^{-1}([u]) \cong \mathcal{A}(u)^\wedge$ (see [MRW96, Corollary 3.4]). Furthermore, since abelian
groups are amenable, it follows from [Wil19, Corollary 5.39] and [Wil07, Proposition C.10] that \( \hat{\rho} \) is open. Therefore we can view \( \hat{A} \) as a right \( G \)-bundle for the natural right action of \( G \) on \( \hat{A} \).

Since \( G \) and \( \Sigma \) both act on \( \hat{A} \), regarded as a topological space fibered over \( \Sigma^{(0)} \), we can form the transformation groupoids \( \hat{A} \times G \) and \( \hat{A} \times \Sigma \). Moreover, \( \hat{A} \rtimes A = \{ (\chi, a) : \hat{\rho}(\chi) = p_\Lambda(a) \} \) is a \( \hat{A} \times G \)-bundle (as well as an \( \hat{A} \times \Sigma \)-bundle).

Defining \( \iota_* : \hat{A} \rtimes A \to \hat{A} \times \Sigma \) by \( \iota_*(\chi, a) = (\chi, a) \) and \( p_* : \hat{A} \times \Sigma \to \hat{A} \times G \) by \( p_*(\chi, \sigma) = (\chi, \hat{\rho}(\sigma)) \), we obtain an extension

\[
\hat{A} \rtimes A \xrightarrow{\iota_*} \hat{A} \times \Sigma \xrightarrow{p_*} \hat{A} \times G.
\]

We defined a \( T \)-groupoid \( \tilde{\Sigma} \) associated to this extension in [IKR + 21, Proposition 3.2] as follows. Define

\[
D = \{ (\chi, z, \sigma) \in \hat{A} \times T \times \Sigma : \hat{\rho}(\chi) = r(\sigma) \}
\]

and let \( H \) be the subgroupoid of \( D \) consisting of triples of the form \( (\chi, \chi(a), a) \) for \( a \in A(\hat{\rho}(\chi)) \). Then \( H \) is a normal subgroupoid of \( D \) and we can form the locally compact Hausdorff groupoid \( \tilde{\Sigma} := D/H \) (we use the notation \( \tilde{\Sigma} \), rather than the notation \( \hat{\Sigma} \) of [IKR + 21], to avoid clashing with classical notational conventions when \( \Sigma \) is a group, for example in Remark 3.3).

**Theorem 3.2.** Let \( \Sigma \) be the extension of \( G \) by the group bundle \( A \) as in the diagram (f) and adopt the notation established above. Let \( f : \hat{A} \rtimes A \to \hat{A} \times T \) be the canonical map given by

\[
f(\chi, a) = (\chi, \chi(a)).
\]

Then \( \tilde{\Sigma} \) is properly isomorphic to the pushout \( f_* (\hat{A} \times \Sigma) \). Moreover,

\[
C^*(\Sigma) \cong C^*(\hat{A} \rtimes G; f_* (\hat{A} \times \Sigma)) \quad \text{and} \quad C^*_r(\Sigma) \cong C^*_r(\hat{A} \rtimes G; f_* (\hat{A} \times \Sigma)).
\]

**Proof.** Theorem 1.5 implies that there is a unique (up to proper isomorphism) extension \( f_* (\hat{A} \times \Sigma) \) of \( \hat{A} \times G \) by \( \hat{A} \times T \) and a twist morphism that is compatible with \( f \). In particular, \( f_* (\hat{A} \times \Sigma) \) is a \( T \)-groupoid. We get a natural map \( g : \hat{A} \times \Sigma \to \tilde{\Sigma} \) given by \( g(\chi, \sigma) = [\chi, 1, \sigma] \), and the diagram

\[
\begin{array}{ccc}
\hat{A} \rtimes A & \xrightarrow{\iota_*} & \hat{A} \times \Sigma \\
\downarrow f & & \downarrow g \\
\hat{A} \times T & \xrightarrow{i} & \tilde{\Sigma}
\end{array}
\]

commutes. The proper isomorphism of \( \tilde{\Sigma} \) with \( f_* (\hat{A} \times \Sigma) \) follows from the uniqueness guaranteed by Theorem 1.5 and the final assertion follows from [IKR + 21, Theorem 3.3].

It follows immediately that if \( \Sigma \) is properly isomorphic to the semidirect product \( A \rtimes G \), then \( [\hat{A} \times \Sigma] = [\hat{A} \rtimes (A \rtimes G)] = [A \rtimes (\hat{A} \times G)] \) and hence \( [\tilde{\Sigma}] \) is trivial. Thus \( C^*(\Sigma) \cong C^*(\hat{A} \times G) \).
Remark 3.3. As mentioned in the introduction, the twist $\hat{\Sigma}$ appearing in Theorem 3.2 is responsible for the Mackey obstruction of the classical normal subgroup analysis of [Mac58]. Indeed, let us apply the theorem when $\Sigma$ is a locally compact group and $\mathcal{A}$ is a closed normal abelian subgroup. Then $\Sigma$ and $\mathcal{G} = \Sigma/\mathcal{A}$ act on $\mathcal{A}$ by conjugation and give right actions on the space of characters $\hat{\mathcal{A}}$. The corresponding twist $\hat{\Sigma}$ is the quotient of the groupoid $(\hat{\mathcal{A}} \times \Sigma) \times \mathcal{T}$ where $(\chi, a\sigma, \theta)$ is identified with $(\chi, \sigma, \theta \chi(a))$ for all $a \in \mathcal{A}$. Let $[\chi, \sigma, \theta]$ be the class of $(\chi, \sigma, \theta)$ in $\hat{\Sigma}$. If $\chi \in \hat{\mathcal{A}}$, then let $\Sigma(\chi)$ and $\mathcal{G}(\chi)$ be the stabilizers at $\chi$ for the actions on $\hat{\mathcal{A}}$, and let $\hat{\Sigma}(\chi)$ be the isotropy group of $\hat{\Sigma}$ at $\chi$. We observe that $\hat{\Sigma}(\chi)$, up to an obvious identification, is the pushout of the group extension

$$
\mathcal{A} \longrightarrow \Sigma(\chi) \longrightarrow \mathcal{G}(\chi)
$$

by the homomorphism $\chi : \mathcal{A} \to \mathcal{T}$. Indeed, this pushout $\chi_*(\Sigma(\chi))$ is the quotient of $\Sigma(\chi) \times \mathcal{T}$ by the equivalence relation identifying $(a\sigma, \theta)$ with $(\sigma, \theta \chi(a))$ for all $a \in \mathcal{A}$. Thus we just identify $[\chi, \sigma, \theta] \in \hat{\Sigma}(\chi)$ with $[\sigma, \theta] \in \chi_*\Sigma(\chi))$. The class of $\Sigma(\chi)$ in $H^2(\mathcal{G}(\chi), \mathcal{T})$ is the classical Mackey obstruction. More precisely, let $L$ be an irreducible unitary representation of $\Sigma$. According to Theorem 3.2, we may view it as a representation of the twisted groupoid $(\hat{\mathcal{A}} \times \mathcal{G}, \hat{\Sigma})$. Its restriction to $\hat{\mathcal{A}}$ defines a measure class which is invariant and ergodic under the action of $\mathcal{G}$. If this measure class is transitive, which will be always the case if $\mathcal{A}$ is regularly embedded, then we have a representation of a twisted transitive measured groupoid $(O \times \mathcal{G}, \hat{\Sigma}|_O)$, where $O \subset \hat{\mathcal{A}}$ is an orbit of the action and $\hat{\Sigma}|_O$ is the reduction of $\hat{\Sigma}$ to $O$. We pick $\chi \in O$. Since the $(\hat{\Sigma}(\chi), \hat{\Sigma}|_O)$-groupoid equivalence $\Sigma^\chi$ compatible with the twists in the sense of [Ren87, Définition 5.3], it implements a bijective correspondence between the unitary representations of $(O \times \mathcal{G}, \hat{\Sigma}|_O)$ and those of $(\mathcal{G}(\chi), \hat{\Sigma}(\chi))$. Therefore $L$ is given by an irreducible unitary representation of the twisted group $(\mathcal{G}(\chi), \hat{\Sigma}(\chi))$.

Example 3.4. Let $H$ be a locally compact abelian group and let $A \subset H$ be a closed subgroup. Then applying the above theorem with $\Sigma = H$ and $\mathcal{A} = A$, we conclude that $\Sigma$ is a bundle of abelian groups over $\Sigma^{(0)} \cong \hat{A}$ where each fiber is an extension of $H/A$ by $\mathcal{T}$. Each of these extensions is abelian because $H$ is abelian (and the action of $H$ on $\mathcal{A}$ is trivial). Hence, each extension is determined by a symmetric $\mathcal{T}$-valued Borel 2-cocycle and any such 2-cocycle is necessarily trivial by [Kle65, Lemma 7.2]. But the twist is not trivial in general: for example, if $H = \mathcal{R}$ and $A = \mathcal{Z} \leq \mathcal{R}$, then triviality of the twist would imply $C^*(\mathcal{R}) \cong C_0(\mathcal{T} \times \mathcal{Z})$, which is nonsense.

Example 3.5 (Generalized Twists). We now consider the case where $A$ is a locally compact abelian group, $\mathcal{A} = \mathcal{G}^{(0)} \times \mathcal{A}$, and $\mathcal{G}$ acts on $\mathcal{A}$ by translation on the first factor. Since this simply gives us a twist when $A = \mathcal{T}$, we will say that $\Sigma$ is a generalized twist in this case. Note that even for twists, $\Sigma$ need not be a trivial extension. Generalized twists were studied in [IKSW19].

View $\hat{\mathcal{A}} := \hat{\mathcal{A}} \times \mathcal{G}^{(0)}$ as a locally compact space. (We put the factor of $\mathcal{G}^{(0)}$ on the right, as a reminder that we are thinking of $\mathcal{A}$ as a space rather than as a group, and to line up with the natural identification of $\hat{\mathcal{A}} \star \mathcal{A}$ with $\hat{\mathcal{A}} \times \mathcal{G}^{(0)} \times \mathcal{A}$, which we make without further comment). Then $\mathcal{G}$ acts on the second factor of $\hat{\mathcal{A}}$. This means we
can replace \( \hat{A} \times G \) and \( \hat{A} \times \Sigma \) with the products \( \tilde{A} \times G \) and \( \tilde{A} \times \Sigma \), respectively. Under these identifications, Equation (3.2) becomes \( f(\chi, u, a) = (\chi, u(\chi(a))) \). Moreover we may assume that the Haar system \( \beta \) on \( A = G^{(0)} \times A \) is constant in the sense that there is a fixed Haar measure \( \mu \) on \( A \) such \( \beta^u = \mu \) for all \( u \in G^{(0)} \).

If \( \chi \in \hat{A} \), then we get a \( G \)-equivariant map \( f^\chi : G^{(0)} \times A \to G^{(0)} \times T \) given by \( f^\chi(u, a) = (u, \chi(\chi(a))) \). Thus we can form the pushout \( f^\chi(\Sigma) \) so that

\[
\begin{array}{ccc}
G^{(0)} \times A & \xrightarrow{\iota} & \Sigma \\
\downarrow f^\chi & & \downarrow p \\
G^{(0)} \times T & \xrightarrow{\iota'} & f^\chi(\Sigma)
\end{array}
\]

commutes. Then \( C^*(G; f^\chi(\Sigma)) \) is the completion of \( C^*_\chi(\Sigma) \) consisting of functions \( g \in C^*_\chi(\Sigma) \) such that \( g(\chi(r(\sigma), a)\sigma) = \chi(\chi(a))g(\sigma) \) with the \( * \)-algebra structure discussed at the beginning of this section.

**Proposition 3.6.** Let \( \Sigma \) be a generalized twist as in Example 3.5. For \( \chi \in \hat{A} \), let \( f^\chi : G^{(0)} \times A \to G^{(0)} \times T \) and \( f^\chi(\Sigma) \) be the \( G \)-equivariant map and \( T \)-groupoid defined above. Then with notation as above,

\[
C^*(\Sigma) \cong C^*(\hat{A} \times G; f^\chi(\hat{A} \times \Sigma))
\]  
(3.3)

and \( C^*(\Sigma) \) is the section algebra of an upper-semicontinuous \( C^* \)-bundle over \( \hat{A} \) with fiber at \( \chi \in \hat{A} \) isomorphic to \( C^*(G; f^\chi(\Sigma)) \).

**Proof.** The isomorphism in (3.3) comes from Theorem 3.2.

The map \( p : \hat{A} \times G^{(0)} \to G^{(0)} \) is continuous and satisfies \( p \circ s = p \circ r \) so that \( f^\chi(\hat{A} \times \Sigma) \) is a groupoid bundle over \( \hat{A} \) as in Appendix A. Hence we can invoke Proposition A.1 to see that \( C^*(\hat{A} \times G; f^\chi(\hat{A} \times \Sigma)) \) is isomorphic to the section algebra of an upper-semicontinuous \( C^* \)-bundle over \( \hat{A} \). Since we can identify \( f^\chi(\hat{A} \times \Sigma)(\chi) \) with \( f^\chi(\Sigma) \) and \( (\hat{A} \times G)(\chi) \) with \( G \), the result follows.

**Proposition 3.7.** With notation as in Example 3.5, suppose that \( A \) compact. Then the dual \( \hat{A} \) is discrete and

\[
C^*(\Sigma) \cong \bigoplus_{\chi \in \hat{A}} C^*(G; f^\chi(\Sigma)) \quad \text{and} \quad C^*_\chi(\Sigma) \cong \bigoplus_{\chi \in \hat{A}} C^*_\chi(G; f^\chi(\Sigma)).
\]

**Proof.** To prove the first isomorphism, note that by Proposition A.1

\[
C^*(\Sigma) \cong C^*(\hat{A} \times G; f^\chi(\hat{A} \times \Sigma))
\]

is a \( \text{ZM}(\hat{A}) \)-algebra. That is, letting \( \text{ZM}(C^*(\Sigma)) \) denote the center of \( M(C^*(\Sigma)) \), there is a \( \sigma \)-unital \( * \)-homomorphism \( \rho : \text{ZM}(\hat{A}) \to \text{ZM}(C^*(\Sigma)) \). Since \( \hat{A} \) is discrete, the images of the characteristic functions of singleton sets under \( \rho \) give rise to a family \( \{q_\chi\}_{\chi \in \hat{A}} \) of mutually orthogonal central projections in \( M(C^*(\Sigma)) \) which sum to unity in the strict topology. Moreover, the summands coincide with the fibers of the upper-semicontinuous \( C^* \)-bundle over \( \hat{A} \) given in Proposition 3.6 and hence

\[
q_\chi C^*(\Sigma)q_\chi = q_\chi C^*(\Sigma) \cong C^*(G; f^\chi(\Sigma))
\]

for all \( \chi \in \hat{A} \).
For the second isomorphism, let \( \pi : C^*(\Sigma) \to C^r_v(\Sigma) \) be the canonical quotient map. An argument like that of the preceding paragraph using the family \( \{\pi(q_\chi)\}_{\chi \in \mathcal{A}} \) of mutually orthogonal central projections in \( M(C^r_v(\Sigma)) \) gives \( C^r_v(\Sigma) \cong \bigoplus_{\chi \in \mathcal{A}} \pi(q_\chi)C^r_v(\Sigma) \). Lemma 3.1 gives \( \pi(q_\chi)C^r_v(\Sigma) \cong C^r_v(G; f^\chi_0(\Sigma)) \), and the result follows.

\[ \square \]

**Remark 3.8.** If \( A = T \) and \( \Sigma \) is a twist, then \( \hat{A} = Z \), and we have \([f^n_\Sigma(\Sigma)] = n[\Sigma]\) for \( n \in Z \). It follows that the central summand corresponding to \( n = 1 \) is isomorphic to \( C^*(G; \Sigma) \) and thus there is central projection \( q = q_1 \in M(C^*(\Sigma)) \) such that

\[
C^*(G; \Sigma) \cong qC^*(\Sigma) \quad \text{and} \quad C^*_v(G; \Sigma) \cong qC^*_v(\Sigma)
\]

Now suppose that \( G = G^{(0)} \) so that \( \Sigma = A \) is itself an abelian group bundle regarded as a groupoid with unit space \( G^{(0)} \) and let \( \Lambda \) be a \( T \)-twist over \( A \). Then since \( \mathcal{A} \) is amenable \( C^*(A; \Lambda) = C^*_v(A; \Lambda) \) (see, for example [SW13, Thm 1]). We shall say that such a twist is *abelian* if \( \Lambda \) is also an abelian group bundle—that is if \( \Lambda(u) \) is abelian for each \( u \in G^{(0)} \). Then \( \Lambda \) is abelian if and only if \( C^*(\Lambda) \) is abelian and in that case \( C^*(\Lambda) \cong C_0(\hat{\Lambda}) \). Arguing as in Example 3.4, we see that such extensions must be pointwise trivial but need not be globally trivial. If \( \Lambda \) is determined by a continuous \( T \)-valued 2-cocycle \( c \), then \( \Lambda \) is abelian if and only if \( c \) is symmetric (cf., [DGN20, Lemma 3.5]). Suppose now that \( \Lambda \) is abelian. For \( n \in Z \), let \( V_n := \{ \chi \in \hat{\Lambda} : \chi(z, u) = z^n \text{ for all } z \in T \text{ and } u \in G^{(0)} \} \). Then \( C^*(\Lambda) \cong C_0(\hat{\Lambda}) \) decomposes as a direct sum with summands of the form \( C_0(V_n) \). Note that each \( V_n \) is clopen. The projection \( q \) in Remark 3.8 may then be identified with the characteristic function of \( U_\Lambda := V_1 \) and hence

\[
C^*(A; \Lambda) \cong qC^*(\Lambda) \cong C_0(U_\Lambda).
\]

See [DGN20, Section 3] for a related construction.

In the case that \( \Lambda \cong T \times \mathcal{A} \) and thus \( \hat{\Lambda} \cong Z \times \hat{\mathcal{A}} \), we have \( U_\Lambda \cong \{1\} \times \hat{\mathcal{A}} \cong \hat{\mathcal{A}} \).

We return now to the more general situation where \( \Sigma \) is a unit space fixing extension of \( G \) by the group bundle \( \mathcal{A} \) as in the diagram (1) from the introduction. Suppose that, in addition, \( \Omega \) is a \( T \)-groupoid extension of \( \Sigma \)

\[
\begin{array}{ccc}
G^{(0)} \times T & \xrightarrow{i} & \Omega \\
\downarrow & & \downarrow \hat{\rho} \\
G^{(0)} & \xleftarrow{\bar{i}} & \Sigma
\end{array}
\]

such that \( \Lambda_\Omega := \bar{\rho}^{-1}(\mathcal{A}) \), its restriction to \( \mathcal{A} \), is an abelian group bundle over \( G^{(0)} \). We may thus regard \( \Omega \) as an extension of \( G \) by \( \Lambda_\Omega \). We assume that \( \mathcal{A}, \Sigma \) and \( G \) are endowed with Haar systems that satisfy (3.1), the Haar system in \( G^{(0)} \times T \) is given by the Haar measure on \( T \), and the Haar system on \( \Omega \) is the one naturally defined by the Haar systems on \( G^{(0)} \times T \) and \( \Sigma \). To declutter notation a little, we write \( \Lambda_\Omega \) for the dual bundle \( (\Lambda_\Omega)^\wedge \).

**Corollary 3.9.** With notation as above let \( f : \check{\Lambda}_\Omega \ast \Lambda_\Omega \to \check{\Lambda}_\Omega \times T \) be given by \( f(\chi, a) = (\chi, \chi(a)) \). Then

\[
C^*(\Omega) \cong C^*(\check{\Lambda}_\Omega \times G; f_*(\Lambda_\Omega \times \Omega)) \quad \text{and} \quad
C^*_v(\Omega) \cong C^*_v(\check{\Lambda}_\Omega \times G; f_*(\Lambda_\Omega \times \Omega)).
\]
Proof. This follows immediately from Remark 3.8, the above discussion, and Theorem 3.2 with $\Lambda_\Omega$ in place of $\mathcal{A}$. □

By arguing as in Remark 3.8 and Corollary 3.9 we may conclude that $C^*(\Sigma; \Omega)$ is isomorphic to the corner associated to the central projection $q_\Omega$ in

$$M(C^*(\hat{\Lambda}_\Omega \rtimes \mathcal{G}; f_*(\hat{\Lambda}_\Omega \rtimes \Omega)))$$

corresponding to the characteristic function of

$$U_\Omega := U_{\Lambda_\Omega} \subset \hat{\Lambda}_\Omega = (\hat{\Lambda}_\Omega \rtimes \mathcal{G})^{(0)}.$$ Observe that $U_\Omega$ is an invariant clopen set under the action of both $\mathcal{G}$ and $\Omega$ and thus both groupoids act on $U_\Omega$.

Corollary 3.10. With notation as above define $g : (\hat{\Lambda}_\Omega \rtimes \mathcal{G})_{U_\Omega} \rightarrow U_\Omega \times T$ by $g(\chi, a) = (\chi, \chi(a))$. Then

$$C^*(\Sigma; \Omega) \cong C^*(U_\Omega \times \mathcal{G}; g_*(U_\Omega \times \Omega)) \text{ and } C^*_r(\Sigma; \Omega) \cong C^*_r(U_\Omega \times \mathcal{G}; g_*(U_\Omega \times \Omega))$$

Proof. Observe that

$$((\hat{\Lambda}_\Omega \rtimes \mathcal{G})_{U_\Omega} \cong U_\Omega \times \mathcal{G} \text{ and } (\hat{\Lambda}_\Omega \times \Omega)_{U_\Omega} \cong U_\Omega \times \Omega.$$ For $(\chi, a) \in U_\Omega \rtimes \Lambda_\Omega \subset \hat{\Lambda}_\Omega \rtimes \Lambda_\Omega$,

$$f(\chi, a) = (\chi, \chi(a)) = g(\chi, a) \in U_\Omega \times T$$

Therefore,

$$(f_*(\hat{\Lambda}_\Omega \times \Omega))_{U_\Omega} \cong g_*(U_\Omega \times \Omega).$$

Hence, by Remark 3.8 and Corollary 3.9

$$C^*(\Sigma; \Omega) \cong g_0C^*(\hat{\Lambda}_\Omega \rtimes \mathcal{G}; f_*(\hat{\Lambda}_\Omega \times \Omega))q_\Omega$$

$$\cong C^*((\hat{\Lambda}_\Omega \times \mathcal{G})_{U_\Omega}; (f_*(\hat{\Lambda}_\Omega \times \Omega))_{U_\Omega})$$

$$\cong C^*(U_\Omega \times \mathcal{G}; g_*(U_\Omega \times \Omega)).$$

The case for the reduced $C^*$-algebras follows by a similar argument. □

Recall that an étale groupoid $\mathcal{G}$ is said to be effective if the interior of the isotropy groupoid is $\mathcal{G}^{(0)}$ and topologically principal if the set of points with trivial isotropy is dense in $\mathcal{G}^{(0)}$. These notions are equivalent if the étale groupoid $\mathcal{G}$ is second countable (see [BCFS14, Lemma 3.1]). The above corollary allows us to generalize [IKR’21, Theorem 4.6] (see also [DGN’20, Theorem 5.8] and [DGN20, Theorem 4.6]).

Corollary 3.11. With notation as above, suppose that $\mathcal{G}$ is étale and that the action groupoid $U_\Omega \rtimes \mathcal{G}$ is second countable and effective. Then the image of $C^*_r(\mathcal{A}, \Lambda_\Omega)$ under the natural embedding into $C^*_r(\Sigma; \Omega)$ is a Cartan subalgebra with Weyl twist $g_*(U_\Omega \times \Omega)$.

Proof. This follows from Corollary 3.10 and [Ren08, Theorem 5.2]. □
Example 3.12. Let $H$ be a discrete abelian group and let $E$ be a $T$-twist over $H$—that is, a central extension by $T$. Since $H$ is discrete, there is a $T$-valued skew-symmetric bicharacter $\varpi$ on $H$ and a set of generating unitaries $\{u_h \mid h \in H\}$ in $C^*(H; E)$ such that for all $g, h \in H$

$$u_g u_h = \varpi(g, h) u_h u_g.$$  

By [Kle65, Lemma 7.2] the extension $E$ is trivial if and only if $\varpi(g, h) = 1$ for all $g, h \in H$. Let $A$ be a subgroup of $H$ which is maximal amongst subgroups on which $\varpi(\cdot, \cdot)$ is identically 1. It is shown in [Kum86, Example 1.12] that the $C^*$-subalgebra $B$ generated by $\{u_a \mid a \in A\}$ is a diagonal subalgebra of $C^*(H; E)$. We now show that this also follows from Corollary 3.11 with $\Sigma := \{g, h\} \forall$ skew-symmetric bicharacter $\varpi$.

Example 3.14. Let $\varphi$ be a continuous normalized $T$-valued 2-cocycle and let $\Sigma_{\varphi}$ be the $T$-twist associated to $\varphi$. Then by Proposition 3.7 and Remark 3.8, and the fact that $\Sigma_{\varphi^n} \cong n \Sigma_{\varphi}$ for all $n \in \mathbb{Z}$, we have

$$C^*(\Sigma_{\varphi}) \cong \bigoplus_{n \in \mathbb{Z}} C^*(\hat{G}; \Sigma_{\varphi^n}).$$

This recovers [BaH14, Theorem 3.2].

Example 3.15 (Transformation groupoids). Let $G$ be a groupoid acting on the right of a locally compact Hausdorff space $X$. Recall that the transformation groupoid $X \rtimes G$ is obtained by endowing the fibered product $X \ast G$ with the groupoid operations $(x, g_1)(x, g_2) = (x, g_1 g_2)$ if $(g_1, g_2) \in G^{(2)}$ and $(x, g)^{-1} = (x, g^{-1})$. 

Assume that \( \varphi : G^{(2)} \to A \) is a 2-cocycle as above. Then one can define a natural 2-cocycle \( \hat{\varphi} : (X \times G)^{(2)} \to X \ast A \) via \( \hat{\varphi}((x, \gamma_1), (x \cdot \gamma_1, \gamma_2)) = (x, \varphi(\gamma_1, \gamma_2)) \). The extension \( \Sigma_{\hat{\varphi}} \) of \( X \times G \) defined by \( \hat{\varphi} \) is isomorphic to the extension \( X \times \Sigma_{\varphi} \), where \( \Sigma_{\varphi} \) is the extension of \( G \) defined by \( \varphi \). To see this, note that \( \Sigma_{\varphi} = \{ ((x, a), (x, \gamma)) : x \in X, a \in A^x, \gamma \in G^x \} \) with the operations

\[
((x, a_1), (x, \gamma_1))((x \cdot \gamma_1, a_2), (x \cdot \gamma_1, \gamma_2)) = ((x, a_1 + \gamma_1 a_2 + \varphi(\gamma_1, \gamma_2)), (x, \gamma_1 \gamma_2))
\]

and

\[
((x, a), (x, \gamma))^{-1} = ((x \cdot \gamma, -\gamma^{-1} a - \varphi(\gamma^{-1}, \gamma)), (x \cdot \gamma, \gamma^{-1})).
\]

On the other hand, \( X \times \Sigma_{\varphi} = \{ (x, (a, \gamma)) : x \in X, a \in A^x, \gamma \in G^x \} \) with the operations

\[
(x, (a_1, \gamma_1))((x \cdot \gamma_1, a_2), (x \cdot \gamma_1, \gamma_2)) = ((x, a_1 + \gamma_1 a_2 + \varphi(\gamma_1, \gamma_2)), (x, \gamma_1 \gamma_2))
\]

and

\[
(x, (a, \gamma))^{-1} = (x \cdot \gamma, (-\gamma^{-1}, a - \varphi(\gamma^{-1}, \gamma), \gamma^{-1})).
\]

Therefore the map \( V : \Sigma_{\hat{\varphi}} \to X \times \Sigma_{\varphi} \) defined by \( V((x, a), (x, \gamma)) = (x, (a, \gamma)) \) is a groupoid isomorphism.

Suppose that \( p_\Sigma : B \to G^{(0)} \) is another abelian \( G \)-bundle and that \( f : A \to B \) is an equivariant map such that \( f|_{A(u)} : A(u) \to B(u) \) is a continuous group homomorphism for all \( u \in G^{(0)} \). There is a \( B \)-valued 2-cocycle \( f_*(\varphi) : G^{(2)} \to B \) given by \( f_*(\varphi)(\gamma_1, \gamma_2) = f(\varphi(\gamma_1, \gamma_2)) \).

**Lemma 3.16.** Let \( \Sigma_{f_*(\varphi)} \) be the extension of \( G \) by \( B \) determined by \( f_*(\varphi) \). Then \( f_*(\varphi) \) is properly isomorphic to \( \Sigma_{f_*(\varphi)} \).

**Proof.** Define \( g : \Sigma_{\varphi} \to \Sigma_{f_*(\varphi)} \) by \( g(a, \gamma) = (f(a), \gamma) \). The diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & \Sigma_{\varphi} \\
\downarrow{g} & & \downarrow{p} \\
B & \xrightarrow{i} & \Sigma_{f_*(\varphi)}
\end{array}
\]

commutes. Therefore the lemma follows from Theorem 1.5. \( \square \)

**3.3. The \( T \)-groupoid defined by a 2-cocycle.** We continue to assume the setting from Section 3.2: \( A \) is an abelian \( G \)-bundle, \( \varphi : G^{(2)} \to A \) is a 2-cocycle, and \( \Sigma_{\varphi} \) is the extension defined by \( \varphi \). Then, as in Example 3.15 there is a 2-cocycle

\[
\hat{\varphi} : (\hat{A} \times G)^{(2)} \to \hat{A} \ast A
\]

defined by

\[
\hat{\varphi}((\chi, \gamma_1), (\chi \cdot \gamma_1, \gamma_2)) = (\chi, \varphi(\gamma_1, \gamma_2))
\]

(3.4)

if \((\gamma_1, \gamma_2) \in G^{(2)} \). Therefore we can identify \( \hat{A} \times \Sigma_{\varphi} \) with \( \Sigma_{\hat{\varphi}} \), the extension of \( \hat{A} \times G \) determined by \( \hat{\varphi} \). Consider the 2-cocycle \( \hat{\varphi} := f_* \hat{\varphi} : (\hat{A} \times G)^{(2)} \to A \times T \) defined via

\[
\hat{\varphi}((\chi, \gamma_1), (\chi, \gamma_2)) = (\chi, \chi(\varphi(\gamma_1, \gamma_2))).
\]

Lemma 3.16 and Theorem 3.2 imply that \( \Sigma_{\varphi} \) is isomorphic to the \( T \)-groupoid defined by \( \hat{\varphi} \) and \( C^*(\Sigma_{\varphi}) \) is isomorphic to \( C^*(\hat{A} \times G; \Sigma_{\hat{\varphi}}) \).
Example 3.17. The following example was studied in [IKSW19]. Let $X$ be a second-countable locally compact Hausdorff space, and $G$ a second-countable locally compact abelian group. Let $\mathcal{G}$ denote the sheaf of germs of continuous $G$-valued functions on $X$, and let $c \in Z^2(\mathcal{G}, \mathcal{G})$ be a normalized Čech two cocycle for some locally finite cover $\mathcal{U} = \{U_i\}_{i \in I}$ of $X$ by precompact open sets. The blow-up groupoid $\mathcal{G}_\mathcal{U}$ with respect to the natural map from $\bigsqcup_i U_i$ into $X$ is

$$
\mathcal{G}_\mathcal{U} = \{(i, x, j) : x \in U_{ij} := U_i \cap U_j\}
$$

with $(i, x, j)(j, x, k) = (i, x, k)$ and $(i, x, j)^{-1} = (j, x, i)$. As noted in [IKSW19, Remark 3.3], the Čech 2-cocycle $c$ defines a groupoid 2-cocycle $\varphi_c : \mathcal{G}_\mathcal{U}^{(2)} \to G$ via

$$
\varphi_c((i, x, j), (j, x, k)) = c_{ijk}(x).
$$

Let $C$ be the extension of $\mathcal{G}_\mathcal{U}$ by the 2-cocycle $\varphi_c$. Define

$$
\hat{\varphi} : ((\hat{G} \times \bigsqcup_i U_i) \rtimes \mathcal{G}_\mathcal{U})^{(2)} \to T \times \hat{G} \times \bigsqcup_i U_i
$$

by

$$
\hat{\varphi}((\tau, (i, x, j)), (\sigma, (j, x, k))) = (\tau(c_{ijk}(x)), \sigma)
$$

for $\tau \in \hat{G}$ and $((i, x, j), (j, x, k)) \in (\mathcal{G}_\mathcal{U})^{(2)}$. Then $\hat{\varphi}$ is a groupoid 2-cocycle, and the pushout groupoid $\Sigma$ is isomorphic to the $T$-groupoid that is the extension of $(\hat{G} \times \bigsqcup_i U_i) \rtimes \mathcal{G}_\mathcal{U}$ defined by $\hat{\varphi}$.

Let $\mathcal{Y} = \{\hat{G} \times U_i\}_{i \in I}$ be the locally finite cover of $\hat{G} \times X$, let $\mathcal{F}$ be the sheaf of germs of continuous $T$-valued functions, and define $\nu^c = \{\nu_{ijk}^c\} \in Z^2(\mathcal{Y}, \mathcal{F})$ by

$$
\nu^c((i, x, j), (j, x, k)) = \tau(c_{ijk}(x)).
$$

Then the 2-cocycle $\hat{\nu} \circ \tau$ is defined by the Čech 2-cocycle $\nu^c \in Z^2(\mathcal{Y}, \mathcal{F})$.

That is, $\nu^c$ is the normalized 2-cocycle considered in [IKSW19, Equation (3.4)]. Hence the generalized Raeburn–Taylor $C^*$-algebra $A(\nu)$ studied in [IKSW19] is isomorphic to the restricted $C^*$-algebra of the $T$-groupoid defined by the 2-cocycle $\nu^c$.

By [IKSW19, Lemma 5.2], $A(\nu)$ is a continuous-trace $C^*$-algebra with spectrum $\hat{G} \times X$ with Dixmier–Douady invariant $\delta(A(\nu)) = [\nu^c]$. For a concrete example, let $G = Z$ and choose a Čech 2-cocycle $c$ associated to any line bundle.

Example 3.18. This example is an expansion of [IKR+21, Example 4.10]. Let $\Gamma = Z$ act on $T$ via rotation by $\alpha \in Q$: $z \cdot k := ze^{2\pi i k \alpha}$. If $\alpha = m/n$ with $m$ and $n$ relatively prime, then $nZ$ fixes the action. We have a short exact sequence of groups

$$
nZ \longrightarrow Z \longrightarrow T 
$$

(3.5)

The action on $T$ leads to an extension of groupoids

$$
nZ \times T \longrightarrow T \times Z \longrightarrow T \times nZ.
$$

(3.6)

Thus, using the notation from the previous section, $A = T \times nZ$, $\Sigma = T \times Z$, and $\mathcal{G} = T \times \mathcal{Z}_n$. The $C^*$-algebra $C^*(T \rtimes \mathcal{Z}_n)$ is the rational rotation $C^*$-algebra $A_n$ (see, for example, [DB84]). The groupoid $\mathcal{D}$ is the cartesian product $T \times T_n \times T \times Z$, where $T_n = T/Z_n$ is the dual of $nZ$. The extension $\overline{\Sigma}$ is the quotient of $\mathcal{D}$ where we identify $(\omega, \chi, z, nl+k)$ with $(\omega, \chi, z, k)$. Therefore the rational rotation
algebra $\mathcal{A}_\alpha$ is the completion of continuous functions $F$ on $T \times T_n \times \mathbb{Z}$ such that
\[ F(\omega, \chi, n \ell + k) = \chi^\ell F(\omega, \chi, k) \text{ for all } \ell \in \mathbb{Z}. \]

The extension $\Sigma$ is properly isomorphic to the one defined by a 2-cocycle. Indeed, let $\sigma = e^{2\pi i \alpha} \in T$ and view $\sigma$ as a character on $T$. Thus we can identify $Z_n$ with $\sigma(\mathbb{Z})$ and then the map $\rho$ in the short exact sequence (3.5) equals $\sigma$. Choose $s \in \mathbb{Z}$ such that $sn = 1 (\bmod n)$. Then the map $\tau : Z_n \to Z$ defined by $\tau(k) = sk$ defines a cross-section of $\rho$. In particular, $Z$ is properly isomorphic to the extension $n\mathbb{Z} \rtimes_\omega \mathbb{Z}_n$ by a two cocycle $\omega : Z_n \rtimes Z_n \to n\mathbb{Z}$ defined by $\tau$. Using the proof of [IKSW19, Proposition A.6], $\omega(\hat{k}_1, \hat{k}_2) = \tau(\hat{k}_1) + \tau(\hat{k}_2) - \tau(\hat{k}_1 + \hat{k}_2)$. A quick computation shows that
\[ \omega(\hat{k}_1, \hat{k}_2) = \begin{cases} 0 & \text{if } \hat{k}_1 + \hat{k}_2 < n \\ n & \text{if } \hat{k}_1 + \hat{k}_2 \geq n, \end{cases} \]
which recovers the 2-cocycle used in Step 2 of the proof of [DB84, Proposition 1].

The map $\bar{\tau} : T \times Z_n \to T \times Z$ defined by $\bar{\tau}(z, k) = (z, \tau(k))$ is a cross-section of the extension of the groupoids (3.6). Hence $T \rtimes Z$ is properly isomorphic to the extension given by the 2-cocycle $\varphi \in Z^2(T \rtimes Z_n, T \rtimes n\mathbb{Z})$ defined by $\varphi((w, k_1), (w \cdot k_1, k_2)) = (w, \omega(k_1, k_2))$. The extension of the 2-cocycle $\varphi$ is $\Sigma_\varphi = T \rtimes n\mathbb{Z} \rtimes \mathbb{Z}_n$ with operations $(w, nl_1, \hat{k}_1)(w \cdot \hat{k}_1, nl_2, \hat{k}_2) = (w, nl_1 + nl_2 + \omega(k_1, k_2), \hat{k}_1 + \hat{k}_2)$ and $(w, nl, \hat{k})^{-1} = (w, -nl - \omega(\hat{k}), -\hat{k})$. Following the proof of [IKSW19, Proposition A.6] the isomorphism between $\Sigma_\varphi$ and $T \rtimes \mathbb{Z}$ is given by $(w, nl, \hat{k}) \mapsto (w, nl + \tau(\hat{k}))$.

We have that $\hat{A} \simeq T_n \times T$ and $\hat{A} \rtimes \mathcal{A} \simeq T_n \times T \rtimes n\mathbb{Z}$. The action of $\mathcal{G} = T \rtimes Z_n$ on $\hat{A}$ is given via $(\chi, w)(\hat{\omega}, \hat{k}) = (\chi, w \cdot \hat{k}) = (\chi, w^\omega \hat{k})$. Therefore we can identify $\hat{A} \rtimes \mathcal{G}$ with $T_n \times T \rtimes Z_n := \{(\chi, w, \hat{k}) \in T_n \times T \times Z_n, \text{ where } (\chi, w, \hat{k}_1)(\chi, w \cdot \hat{k}_1, \hat{k}_2) = (\chi, w, \hat{k}_1 + \hat{k}_2) \text{ and } (\chi, w, \hat{k})^{-1} = (\chi, w \cdot \hat{k}, -\hat{k}) \}$. Thus the 2-cocycle $\hat{\varphi} : (T_n \times T \rtimes Z_n)^{(2)} \to T_n \times T \times n\mathbb{Z}$ of (3.4) is defined by
\[ \hat{\varphi}((\chi, w, \hat{k}_1), (\chi, w \cdot \hat{k}_1, \hat{k}_2)) = (\chi, w, \omega(\hat{k}_1, \hat{k}_2)). \]

By Lemma 3.16, $\hat{\Sigma}$ is properly isomorphic to the extension by the 2-cocycle $\hat{\varphi}$ which is the pushout of $\hat{\varphi}$. Therefore $\hat{\varphi} : (T_n \times T \rtimes Z_n)^{(2)} \to T_n \times T \times T$ is defined by
\[ \hat{\varphi}((\chi, w, \hat{k}_1), (\chi, w \cdot \hat{k}_1, \hat{k}_2)) = (\chi, w, \chi^{\omega(\hat{k}_1, \hat{k}_2)}). \]

Hence the rotation algebra $\mathcal{A}_\alpha$ is isomorphic to $C^*(T_n \times T \rtimes Z_n; \Sigma_{\hat{\varphi}})$. For $\chi \in T_n$, define $\chi_*(\hat{\varphi}) : (T \rtimes Z_n)^{(2)} \to T$ by
\[ \chi_*(\hat{\varphi})((w, k_1), (w \cdot k_1, k_2)) = (w, \chi^{\omega(\hat{k}_1, \hat{k}_2)}). \]

Then Proposition 3.6 implies that $\mathcal{A}_\alpha$ is the section algebra of an upper-semicontinuous $C^*$-bundle over $T_n$ with fiber at $\chi \in T_n$ isomorphic to $C^*(T \rtimes Z_n; \Sigma_{\chi_*(\hat{\varphi})})$. 

Appendix A. Bundles of Twists

Let $\Sigma$ be a twist over $\mathcal{G}$. Alternatively, $\Sigma$ is a $T$-groupoid so that we have the following diagram

$$
\begin{array}{ccc}
\mathcal{G}^{(0)} \times T & \xrightarrow{i} & \Sigma \\
\downarrow & & \downarrow j \\
\mathcal{G}^{(0)} & \xrightarrow{p} & \mathcal{G},
\end{array}
$$

where as usual we have identified $\Sigma^{(0)}$ and $\mathcal{G}^{(0)}$. In particular, if $F \subset \mathcal{G}^{(0)}$ is $\mathcal{G}$-invariant, then it is $\Sigma$-invariant and the reduction $\Sigma|_{F}$ is also a twist over the reduction $\mathcal{G}|_{F}$.

Suppose that $p : \mathcal{G}^{(0)} \to T$ is a continuous map such that $p \circ r = r \circ s$. Then we say that $\Sigma$ is a groupoid bundle over $\mathcal{G}$.

**Proposition A.1.** Suppose that $\mathcal{G}$ is a second countable locally compact Hausdorff groupoid with a Haar system and that $\Sigma$ is a twist over $\mathcal{G}$. If $p : \mathcal{G}^{(0)} \to T$ is a continuous map such that $p \circ r = p \circ s$, then $C^*(\mathcal{G}; \Sigma)$ is a $C_0(T)$-algebra. Let $\Sigma(t)$ be the twist over $\mathcal{G}(t)$ defined above. Then $C^*(\mathcal{G}; \Sigma)$ is (isomorphic to) the section algebra of an upper-semicontinuous $C^*$-bundle over $T$. The fibre $C^*(\mathcal{G}(t); \Sigma(t))$ is isomorphic to $C^*(\mathcal{G}(t); \Sigma(t))$.

**Proof.** Recall that $C^*(\mathcal{G}; \Sigma)$ is the $C^*$-algebra $C^*(\mathcal{G}, \mathcal{B})$ of a Fell bundle $q : \mathcal{B} \to \mathcal{G}$ as described in [MW08, Example 2.9]. Similarly, $C^*(\mathcal{G}(t); \Sigma(t))$ is the $C^*$-algebra $C^*(\mathcal{G}(t), \mathcal{B})$ of $q|_{\mathcal{G}(t)}$. Let $U(t) = \mathcal{G}^{(0)} \setminus p^{-1}(t)$. Using [IW12, Theorem 3.7] (as in [SW13, Lemma 9]), we obtain a short exact sequence

$$
0 \longrightarrow C^*(\mathcal{G}|_{U(t)}, \mathcal{B}) \xrightarrow{i} C^*(\mathcal{G}, \mathcal{B}) \xrightarrow{j} C^*(\mathcal{G}(t), \mathcal{B}) \longrightarrow 0
$$

where $i$ identifies $C^*(\mathcal{G}|_{U(t)}, \mathcal{B})$ with the completion in $C^*(\mathcal{G}, \mathcal{B})$ of the ideal of sections in $\Gamma_c(\mathcal{G}, \mathcal{B})$ that vanish off $\mathcal{G}|_{U(t)}$, and $j$ is given on $\Gamma_c(\mathcal{G}, \mathcal{B})$ by restriction to $p^{-1}(t)$. Now exactly as in [Will19, Proposition 5.37], we see that $C^*(\mathcal{G}, \mathcal{B})$ is a $C_0(T)$-algebra with fibres $C^*(\mathcal{G}(t), \mathcal{B})(t)$ identified with $C^*(\mathcal{G}(t), \mathcal{B})$. \qed

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1 The third author defined groupoid bundles in [Ren15, Definition 3.3] where it is also required that $p$ be open.
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