On local solutions of the Ramanujan equation and their connection formulae

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Abstract

We show connection formulae of local solutions of the Ramanujan equation between the origin and the infinity. These solutions are given by the Ramanujan function, the $q$-Airy function and the divergent basic hypergeometric series $2\varphi_0(0;0;-q,x)$. We use two different $q$-Borel-Laplace transformations to obtain our connection formulae.

1 Introduction

In this paper, we show two essentially different connection formulae of some basic hypergeometric series between the origin and the infinity. In 1846, E. Heine [5] introduced the basic hypergeometric series $2\varphi_1(a,b;c;q,x)$ as follows:

$$2\varphi_1(a,b;c;q,x) := \sum_{n\geq0} \frac{(a,b;q)_n}{(c;q)_n(q;q)_n} x^n, \quad c \notin q^{-\mathbb{N}}. \quad (1)$$

Here, $(a;q)_n$ is the $q$-shifted factorial;

$$(a;q)_n := \begin{cases} 1, & n = 0, \\ (1-a)(1-aq)\ldots(1-aq^{n-1}), & n \geq 1, \end{cases} \quad (a_1,a_2,\ldots,a_m;q)_\infty := (a_1;q)_\infty(a_2;q)_\infty\ldots(a_m;q)_\infty.$$

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The $q$-shifted factorial $(a; q)_n$ is a $q$-analogue of the shifted factorial $(\alpha)_n$:

$$(\alpha)_n := \begin{cases} 1, & n = 0, \\ \alpha(\alpha + 1) \ldots \{\alpha + (n - 1)\}, & n \geq 1. \end{cases}$$

The basic hypergeometric series (1) is a $q$-analogue of the hypergeometric series $2F_1(\alpha, \beta; \gamma, z)$:

$$2F_1(\alpha, \beta; \gamma, z) := \sum_{n \geq 0} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!} z^n. \quad (2)$$

This series (2) has the following famous degeneration diagram

![Degeneration Diagram](image)

Recently, Y. Ohyama [11] shows that there exists “the digeneration diagram” of Heine’s series (1) as follows:

$$2\varphi_1(a, b; c; z) \rightarrow 1\varphi_1(a; c; z) \rightarrow 1\varphi_1(a; 0; z) \rightarrow J^{(1)}_\nu, J^{(2)}_\nu \rightarrow q\text{-Airy} \rightarrow Ramanujan$$

We remark that there exist three different $q$-Bessel functions $J^{(j)}_\nu$, $j = 1, 2, 3$ [2] and two $q$-analogues of the Airy function. In this point, this diagram is essentially different from the diagram (3).

Ismail has pointed out that the Ramanujan function is one of $q$-analogues of the Airy function [6]. The Ramanujan function appears in the third identity on p.57 of Ramanujan’s “Lost notebook” [12] as follows (with $x$ replaced by $q$):

$$A_q(-a) = \sum_{n \geq 0} \frac{a^n q^{n^2}}{(q; q)_n} = \prod_{n \geq 1} \left( 1 + \frac{aq^{2n-1}}{1 - q^n y_1 - q^{2n} y_2 - q^{3n} y_3 - \cdots} \right)$$
where
\[
y_1 = \frac{1}{(1-q)\psi^2(q)},
\]
\[
y_2 = 0,
\]
\[
y_3 = \frac{q+q^3}{(1-q)(1-q^2)(1-q^3)\psi^2(q)} - \sum_{n \geq 0} \frac{(2n+1)q^{2n+1}}{1-q^{2n+1}} \left( \frac{1-q}{1-q^2(1-q^3)} \psi^6(q) \right),
\]
\[
y_4 = y_1 y_3,
\]
\[
\psi(q) = \sum_{n \geq 0} q^{n(n+1)} \frac{1}{(q^2; q^2)_n} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}.
\]

To be precise, the Ramanujan function is given by
\[
A_q(x) := \sum_{n \geq 0} \frac{q^{n^2}}{(q; q)_n} (-x)^n.
\]
This function satisfies the following second order linear q-difference equation;
\[
qxu(q^2 x) - u(qx) + u(x) = 0.
\)
(4)
The equation (4) has another solution which is given by a divergent series
\[
\theta_q(x)_{2\varphi_0} \left( 0, 0; -q; -\frac{x}{q} \right) = \theta_q(x) \sum_{n \geq 0} \frac{1}{(q; q)_n} \left\{ (-1)^n q^{\frac{n(n-1)}{2}} \right\}^{-1} \left( -\frac{x}{q} \right)^n.
\]
Here, \( \theta_q(\cdot) \) is the theta function of Jacobi (see the section 2).
An asymptotic formula for the Ramanujan function is obtained by M. E. H. Ismail and C. Zhang as follows [7];
\[
A_q(x) = \frac{(qx, q/x; q^2)_\infty}{(q; q^2)_\infty} \varphi_1 \left( 0; q; q^2, \frac{q^2}{x} \right) \frac{q(q^2 x, 1/x; q^2)_\infty}{(1-q)(q; q^2)_\infty} \varphi_1 \left( 0; q^3; q^2, \frac{q^3}{x} \right).
\)
(5)
From the viewpoint of connection problems on q-difference equations, we can regard the formula (5) as one of connection formulae of the Ramanujan function.
The other $q$-analogue of the Airy function is known as the $q$-Airy function $Ai_q(\cdot)$. The $q$-Airy function has found in the study of the second $q$-Painlevé equation [4]. The function $Ai_q(\cdot)$ is defined by

$$Ai_q(x) := \sum_{n \geq 0} \frac{1}{(-q; q)_n(q; q)_n} \left\{ (-1)^n q^{\frac{n(n-1)}{2}} \right\} (-x)^n$$

and satisfies the following $q$-difference equation

$$u(q^2x) + xu(qx) - u(x) = 0. \quad (6)$$

The other solution of the equation (6) around the origin is given by

$$u(x) = \frac{\theta_q(q^2x)}{\theta_q(-q^2x)} Ai_q(-x).$$

Ismail also has pointed out the Ramanujan function and the $q$-Airy function are different. But the relation between them has not known. In the section 3, we give the connection formula between these functions with using the $q$-Borel-Laplace transformations of the second kind.

**Theorem** For any $x \in \mathbb{C}^*$, we have

$$A_{q^2}\left(-\frac{q^3}{x^2}\right) = \frac{1}{(q, -1; q)_\infty} \left\{ \theta \left(\frac{x}{q}\right) Ai_q(-x) + \theta \left(-\frac{x}{q}\right) Ai_q(x) \right\}. $$

Connection problems on linear $q$-difference equations between the origin and the infinity are studied by G. D. Birkhoff [11]. The first example of the connection formula was found by G. N. Watson [14] in 1912. This formula is known as “Watson’s formula for $2\varphi_1(a, b; c; q, x)$” as follows [2];

$$2\varphi_1(a, b; c; q; x) = \frac{(b, c/a; q)_\infty(ax, q/ax; q)_\infty(c, b/a; q)_\infty(x, q/x; q)_\infty}{(c, b/a; q)_\infty(bx, q/bx; q)_\infty(c, a/b; q)_\infty(x, q/x; q)_\infty} 2\varphi_1(a, aq/c; aq/b; q; cq/abx)$$

+ \frac{(a, c/b; q)_\infty(bx, q/bx; q)_\infty(c, a/b; q)_\infty(x, q/x; q)_\infty}{(c, a/b; q)_\infty(x, q/x; q)_\infty} 2\varphi_1(b, bq/c; bq/a; q; cq/abx). \quad (7)$$

But other connection formulae had not found for a long time. Recently, C. Zhang gives connection formulae for some confluent type basic hypergeometric series [15, 16, 17]. In [16], Zhang gives a connection formula of Jackson’s first and second $q$-Bessel function $J^{(j)}_\nu(x; q), (j = 1, 2)$;

$$J^{(1)}_\nu(x; q) := \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \left(\frac{x}{2}\right)^\nu \sum_{n \geq 0} \frac{1}{(q^{\nu+1}; q)_n} \left(-\frac{x^2}{4}\right)^n.$$
and
\[ J^{(2)}_\nu(x; q) := \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \left( \frac{x}{2} \right)^\nu \sum_{n \geq 0} \frac{q^n}{(q^{\nu+1}; q)_n} \left(-\frac{q^n x^2}{4}\right)^n \]
with using the \( q \)-Borel-Laplace transformations of the second kind \( B^-_q \) and \( L^-_q \). These transformations are defined for a formal power series \( f(x) = \sum_{n \geq 0} a_n x^n \), \( a_0 = 1 \) as follow:

1. The \( q \)-Borel transformation of the second kind is
\[ (B^-_q f)(\xi) := \sum_{n \geq 0} a_n q^{-\frac{n(n-1)}{2}} \xi^n \quad (=: g(\xi)). \]

2. The \( q \)-Laplace transformation of the second kind is
\[ (L^-_q g)(x) := \frac{1}{2\pi i} \int_{|\xi|=r} g(\xi) \theta_q \left( \frac{x}{\xi} \right) \frac{d\xi}{\xi}, \]
where \( r > 0 \) is enough small number.

In [9] and [10], we obtained connection formulae of the Hahn-Exton \( q \)-Bessel function
\[ J^{(3)}_\nu(x; q) := \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} x^\nu \sum_{n \geq 0} \frac{q^n}{(q^{\nu+1}; q)_n} \left(-x^2\right)^n \]
and the \( q \)-confluent type basic hypergeometric function
\[ _1\phi_1(a; b; q, x) := \sum_{n \geq 0} \frac{(a; q)_n}{(b; q)_n(q; q)_n} (-1)^n q^{\frac{n(n-1)}{2}} x^n \]
by these transformations. In section 3, we use these transformations to obtain connection formula between the Ramanujan function and the \( q \)-Airy function.

On the other hand, the \( q \)-Borel-Laplace transformations of the first kind are defined for a formal power series as follow:

1. The \( q \)-Borel transformation of the first kind is
\[ (B^+_q f)(\xi) := \sum_{n \geq 0} a_n q^{\frac{n(n-1)}{2}} \xi^n \quad (=: \varphi(\xi)). \]
2. The $q$-Laplace transformation of the first kind is

\[
(L_q^\pm \varphi) (x) := \frac{1}{1 - q} \int_0^{\infty} \frac{\varphi(\xi)}{\theta_q \left( \frac{\xi}{x} \right)} \frac{d_q \xi}{\xi} = \sum_{n \in \mathbb{Z}} \frac{\varphi(\lambda q^n)}{\theta_q \left( \frac{\lambda q^n}{x} \right)},
\]

here, this transformation is given by Jackson’s $q$-integral [2].

These two different types of $q$-Borel-Laplace transformations are introduced by J. Sauloy [13] and studied by C. Zhang. We remark that each $q$-Borel transformation is formal inverse of each $q$-Laplace transformation, i.e.,

\[
L_q^\pm \circ B_q^\pm f = f.
\]

The application of the $q$-Borel-Laplace transformations of the first kind is found in [15, 17]. Zhang gives the connection formula of the divergent basic hypergeometric series $2\varphi_0(a,b; -; q, x)$ as follows;

**Theorem (Zhang, [15])** For any $x \in \mathbb{C}^*$, we have

\[
2f_0(a, b; \lambda, q, x) = (b; q)_{\infty} \frac{\theta_q(a) \theta_q(qax/\lambda)}{(b/a; q)_{\infty} \theta_q(\lambda) \theta_q(\lambda/x)} 2\varphi_1 \left( a, b; \lambda, q, x \right) + (a; q)_{\infty} \frac{\theta_q(b) \theta_q(qbx/\lambda)}{(a/b; q)_{\infty} \theta_q(\lambda) \theta_q(\lambda/x)} 2\varphi_1 \left( b, a; \lambda, q, x \right)
\]

where $\lambda \in \mathbb{C}^* \setminus \{-q^n; n \in \mathbb{Z}\}$.

Here, $2f_0(a, b; \lambda, q, x)$ in the left-hand side is the $q$-Borel-Laplace transform of the function $2\varphi_0(a, b; -; q, x)$. But other application of this method (of the first kind) has not known. In the section 4, we show the connection formula of the divergent series

\[
2\varphi_0(a, b; -; q, x) = \sum_{n \geq 0} \frac{1}{(q; q)_n} \left\{ (-1)^n q^{n(n-1)} \right\}^{-1} x^n.
\]

This formula is given by the following theorem;

**Theorem** For any $x \in \mathbb{C}^* \setminus [-\lambda; q]$,

\[
\theta_q(x)2f_0 \left( 0, 0; -; q, x/q \right) = (q; q)_{\infty} \frac{\theta_q(x)\theta_q^2 \left( -\frac{\lambda^2}{x} \right)}{\theta_q \left( -\frac{\lambda}{q} \right) \theta_q \left( \frac{\lambda}{x} \right)} 2\varphi_1 \left( 0; q, q^2, q^2 / x \right) + (q; q)_{\infty} \frac{\theta_q(x)\theta_q^2 \left( -\frac{\lambda^2}{x} \right) \lambda}{1 - q \theta_q \left( -\frac{\lambda}{q} \right) \theta_q \left( \frac{\lambda}{x} \right)} x \varphi_1 \left( 0; q^3, q^2, q^3 / x \right).
\]
2 Basic notations

In this section, we review our notations. We assume that $q \in \mathbb{C}^*$ satisfies $0 < |q| < 1$. The $q$-shifted operator $\sigma_q$ is given by $\sigma_q f(x) = f(qx)$. For any fixed $\lambda \in \mathbb{C}^* \setminus q\mathbb{Z}$, the set $[\lambda; q]$-spiral is $[\lambda; q] := \lambda q^\mathbb{Z} = \{\lambda q^k; k \in \mathbb{Z}\}$. The (generalized) basic hypergeometric series $r\varphi_s(a_1, \ldots, a_r; b_1, \ldots, b_s; q, x)$ is

$$r\varphi_s(a_1, \ldots, a_r; b_1, \ldots, b_s; q, x) := \sum_{n \geq 0} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s; q)_n} \frac{(-1)^n q^{n(n-1)/2}}{x^n}. $$

This series has radius of convergence $\infty$, 1 or 0 according to whether $r - s < 1$, $r - s = 1$ or $r - s > 1$ (see [2] for further details). In connection problems, the theta function of Jacobi is important. This function is defined by

$$\theta_q(x) := \sum_{n \in \mathbb{Z}} q^{n(n-1)/2} x^n, \quad x \in \mathbb{C}^*. $$

We denote $\theta_q(\cdot)$ or more shortly $\theta(\cdot)$. The theta function has the following properties:

1. Jacobi’s triple product identity is

$$\theta_q(x) = \left(q, -x, -\frac{q}{x}; q\right)_\infty. $$

2. The $q$-difference equation which the theta function satisfies;

$$\theta_q(q^k x) = q^{-n(n-1)/2} x^{-k} \theta_q(x), \quad \forall k \in \mathbb{Z}. $$

3. The inversion formula;

$$\theta_q \left( \frac{1}{x} \right) = \frac{1}{x} \theta_q(x). $$

We remark that the function $\theta(-\lambda x)/\theta(\lambda x)$, $\lambda \in \mathbb{C}^*$ satisfies a $q$-difference equation

$$u(qx) = -u(x)$$

which is also satisfied by the function $u(x) = e^{\pi i \left( \frac{\log x}{\log q} \right)}$. 
3 Two types of the $q$-analogue of the Airy function and the connection formula

There are two different $q$-analogue of the Airy function. One is called the Ramanujan function which appears in [12]. Ismail [6] pointed out that the Ramanujan function can be considered as a $q$-analogue of the Airy function. The other one is called the $q$-Airy function which is obtained by K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta and Y. Yamada [8]. In this section, we see the properties of these functions. We explain the reason why they are called $q$-analogue of the Airy function and we show $q$-difference equations which they satisfy.

3.1 The Ramanujan function $A_q(x)$

The Ramanujan function appears in Ramanujan’s “Lost notebook” [12]. Ismail has pointed out that the Ramanujan function can be considered as a $q$-analogue of the Airy function. The Ramanujan function is defined by following convergent series;

$$A_q(x) := \sum_{n \geq 0} \frac{q^{n^2}}{(q; q)_n} (-x)^n = _0\varphi_1 (-; 0; q, -qx).$$

In the theory of ordinary differential equations, the term Plancherel-Rotach asymptotics refers to asymptotics around the largest and smallest zeros. With $x = \sqrt{2n + 1} - 2^{\frac{1}{3}} 3^{\frac{1}{3}} n \frac{1}{t}$ and for $t \in \mathbb{C}$, the Plancherel-Rotach asymptotic formula for Hermite polynomials $H_n(x)$ is

$$\lim_{n \to +\infty} e^{-\frac{x^2}{2}} \frac{e^{-x^2}}{3^{\frac{1}{3}} \pi^{\frac{2}{3}} 2^{\frac{1}{3}} n^{\frac{1}{3}} \sqrt{n!}} H_n(x) = \text{Ai}(t). \tag{8}$$

In [6], Ismail shows the $q$-analogue of (8);

Proposition 1. One can get

$$\lim_{n \to \infty} q^{n^2} \frac{1}{t^n} h_n(\sinh \xi_n | q) = A_q \left( \frac{1}{t^2} \right)$$

where $e^{\xi_n} = tq^{-\frac{n}{2}}$. 

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Here, \( h_n(\cdot|q) \) is the \( q \)-Hermite polynomial. In this sense, we can deal with the Ramanujan function \( A_q(x) \) as a \( q \)-analogue of the Airy function. The Ramanujan function satisfies the following \( q \)-difference equation:

\[
(qxσ^2_q - σ_q + 1) u(x) = 0. \tag{9}
\]

**Remark 1.** We remark that another solution of the equation (9) is given by

\[
u(x) = \theta(x_2\varphi_0(0, 0; -q, -x/q).
\]

Here,

\[
2\varphi_0 \left( 0, 0; -q, \frac{-x}{q} \right) = \sum_{n \geq 0} \frac{1}{(q; q)_n} \left\{ (-1)^n q^{\frac{n(n-1)}{2}} \right\}^{-1} \left( -\frac{x}{q} \right)^n
\]

is a divergent series.

### 3.2 The \( q \)-Airy function \( \text{Ai}_q(x) \)

The \( q \)-Airy function is found by K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta and Y. Yamada [8], in their study of the \( q \)-Painlevé equations. This function is the special solution of the second \( q \)-Painlevé equations and given by the following series

\[
\text{Ai}_q(x) := \sum_{n \geq 0} \frac{1}{(-q, q; q)_n} \left\{ (-1)^n q^{\frac{n(n-1)}{2}} \right\} (-x)^n = \varphi_1(0; -q; q, -x).
\]

T. Hamamoto, K. Kajiwara, N. S. Witte [4] proved following asymptotic expansions;

**Proposition 2.** With \( q = e^{-\frac{s^2}{\delta^2}}, x = -2ie^{-\frac{s^3}{\delta^2}} \) as \( \delta \to 0 \),

\[
\varphi_1(0; -q; q, qx) = 2\pi i \delta^{-\frac{1}{2}} e^{-\left( \frac{s^2}{3\delta} \right)} \ln 2 + \left( \frac{s^3}{5\delta} \right) s \left[ \text{Ai} \left( se^{-\frac{s}{2}} \right) + O(\delta^2) \right],
\]

\[
\varphi_1(0; -q; q, qx) = 2\pi i \delta^{-\frac{1}{2}} e^{-\left( \frac{s^2}{3\delta} \right)} \ln 2 - \left( \frac{s^3}{5\delta} \right) s \left[ \text{Ai} \left( se^{-\frac{s}{2}} \right) + O(\delta^2) \right]
\]

for \( s \) in any compact domain of \( \mathbb{C} \).
Here, $\text{Ai}(\cdot)$ is the Airy function. From this proposition, we can regard the $q$-Airy function as a $q$-analogue of the Airy function.

We can easily check out that the $q$-Airy function satisfies the second order linear $q$-difference equation
\[
\left(\sigma_q^2 + x\sigma_q - 1\right) u(x) = 0.
\]
(10)

Another solution of the equation (10) is given by
\[
u(x) = e^{\pi i \left(\log_q x\right)} \varphi_1(0; -q; q, x) = e^{\pi i \left(\log_q x\right)} \text{Ai}_q(-x).
\]

3.3 Covering transformations

We define a covering transformation of a second order linear $q$-difference equation.

Definition 1. For a $q$-difference equation
\[
a(x)u(q^2 x) + b(x)u(q x) + c(x)u(x) = 0,
\]
(11)

we define the covering transformation as follows
\[
t^2 := x, \quad v(t) := u(t^2), \quad p := \sqrt{q}.
\]

The covering transform of the equation (11) is given by
\[
a(t^2)v(p^2 t) + b(t^2)v(pt) + c(t^2)v(t) = 0.
\]

By the covering transformation, the equation
\[
\left(K \cdot x\sigma_q^2 - \sigma_q + 1\right) u(x) = 0
\]
is transformed to
\[
\left(K \cdot t^2\sigma_p^2 - \sigma_p + 1\right) v(t) = 0,
\]
(12)

where $K$ is a fixed constant in $\mathbb{C}^*$. 

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3.4 The q-Airy equation around the infinity

We consider the behavior of the equation (10) around the infinity. We set \( x = 1/t \) and \( z(t) = u(1/t) \). Then \( z(t) \) satisfies

\[
\left(-\sigma_q^2 + \frac{1}{q^2} \sigma_q + 1\right) z(t) = 0.
\]

We set \( \mathcal{E}(t) = 1/\theta(-q^2 t) \) and \( f(t) = \sum_{n \geq 0} a_n t^n; \ a_0 = 1 \). We assume that \( z(t) \) can be described as

\[
z(t) = \mathcal{E}(t) f(t) = \frac{1}{\theta(-q^2 t)} \left( \sum_{n \geq 0} a_n t^n \right).
\]

The function \( \mathcal{E}(t) \) has the following property:

**Lemma 1.** For any \( t \in \mathbb{C}^* \),

\[
\sigma_q \mathcal{E}(t) = -q^2 t \mathcal{E}(t), \quad \sigma_q^2 \mathcal{E}(t) = q^5 t^2 \mathcal{E}(t).
\]

From this lemma, \( f(t) \) satisfies the following equation

\[
\left(-q^5 t^2 \sigma_q^2 - \sigma_q + 1\right) f(t) = 0.
\]

(13)

Since (13) is the same as (12) for \( K = -q^5 \), we obtain

\[
f(t) = \varphi_1(-; 0; q^2, q^5 t^2) = A_{q^5}(-q^3 t^2).
\]

We show a connection formula for \( f(t) \). In order to obtain a connection formula, we need the \( q \)-Borel transformation and the \( q \)-Laplace transformation following Zhang [16].

3.5 The \( q \)-Borel transformation and the \( q \)-Laplace transformation

**Definition 2.** For \( f(t) = \sum_{n \geq 0} a_n t^n \), the \( q \)-Borel transformation is defined by

\[
g(\tau) = (B_q^{-1} f)(\tau) := \sum_{n \geq 0} a_n q^{-n(n-1)/2} \tau^n,
\]

and the \( q \)-Laplace transformation is given by

\[
(L_q^{-1} g)(t) := \frac{1}{2\pi i} \int_{|\tau|=r} g(\tau) \theta \left( \frac{t}{\tau} \right) \frac{d\tau}{\tau}, \quad 0 < r < \frac{1}{|q^2|}.
\]
The $q$-Borel transformation can be considered as a formal inverse of the $q$-Laplace transformation.

**Lemma 2.** For any entire function $f$,
\[ \mathcal{L}_q^{-1} \circ B_q^{-1} f = f. \]

**Proof.** We can prove this lemma calculating residues of the $q$-Laplace transformation around the origin. \qed

The $q$-Borel transformation has following operational relation;

**Lemma 3.** For any $l, m \in \mathbb{Z}_{\geq 0}$,
\[ B_q^{-1}(t^m \sigma_q^l) = q^{-\frac{m(m-1)}{2}} \tau^m \sigma_q^{l-m} B_q^{-1}. \]

### 3.6 The connection formula of the $q$-Airy function

Applying the $q$-Borel transformation in 3.5 to the equation (12) and using lemma 3, we obtain the first order $q$-difference equation
\[ g(q\tau) = (1 + q^2\tau)(1 - q^2\tau)g(\tau). \]

Since $g(0) = 1$, $g(\tau)$ is given by an infinite product
\[ g(\tau) = \frac{1}{(-q^2 \tau; q)_{\infty}(q^2 \tau; q)_{\infty}} \]

which has single poles at
\[ \{ \tau; \tau = \pm q^{-2-k}, \forall k \in \mathbb{Z}_{\geq 0} \}. \]

By Cauchy’s residue theorem, the $q$-Laplace transform of $g(\tau)$ is
\[ f(t) = \frac{1}{2\pi i} \int_{|\tau|=r} g(\tau) \theta \left( \frac{t}{\tau} \right) \frac{d\tau}{\tau} = \sum_{k \geq 0} \text{Res} \left\{ g(\tau) \theta \left( \frac{t}{\tau} \right) \frac{1}{\tau}; \tau = -q^{-2-k} \right\} - \sum_{k \geq 0} \text{Res} \left\{ g(\tau) \theta \left( \frac{t}{\tau} \right) \frac{1}{\tau}; \tau = q^{-2-k} \right\} \]

where $0 < r < r_0 := 1/|q^2|$. We can calculate the residue from lemma 4.

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Lemma 4. For any $k \in \mathbb{N}, \lambda \in \mathbb{C}^*$, one can get:

1. \[
\text{Res} \left\{ \frac{1}{(\tau/\lambda; q)_{\infty}} \tau^{1-k}\tau = \lambda q^{-k} \right\} = \frac{(-1)^{k+1}q^{\frac{k(k+1)}{2}}}{(q; q)_k(q; q)_{\infty}},
\]

2. \[
\frac{1}{(\lambda q^{-k}; q)_{\infty}} = \frac{(-\lambda)^{-k}q^{\frac{k(k+1)}{2}}}{(\lambda; q)_{\infty} (q/\lambda; q)_k}, \quad \lambda \notin q\mathbb{Z}.
\]

Summing up all of residues, we obtain

\[
f(t) = \frac{\theta(q^2t)}{(q, -1; q)_{\infty}} \phi_1 \left( 0, -q; \frac{1}{t} \right) + \frac{\theta(-q^2t)}{(q, -1; q)_{\infty}} \phi_1 \left( 0, -q; -\frac{1}{t} \right).
\]

We obtain a connection formula for $z(t) = \mathcal{E}(t)f(t)$. Finally, we acquire the following connection formula between the Ramanujan function and the $q$-Airy function.

Theorem 1. For any $x \in \mathbb{C}^*$,

\[
A_q \left( -\frac{q^3}{x^2} \right) = \frac{1}{(q, -1; q)_{\infty}} \left\{ \theta \left( \frac{x}{q} \right) A_i q(-x) + \theta \left( -\frac{x}{q} \right) A_i q(x) \right\}.
\]

Here, both $A_q(x)$ and $A_i q(x)$ are defined by convergent series on whole of the complex plain. The connection formula above is valid for any $x \in \mathbb{C}^*$.

4 Connection formula of the divergent series

$2\phi_0(0, 0; -; q, \cdot)$

In this section, we show a connection formula of the divergent series $2\phi_0$. This series appears in the second solution of the Ramanujan equation (9). At first, we review two $q$-exponential functions to consider our connection formula.

4.1 Two different $q$-exponential functions

In this section, we review two different $q$-exponential functions from the viewpoint of the connection problems. One of the $q$-exponential function $e_q(x)$ is given by

\[
e_q(x) := \phi_0(0; -; q, x) = \sum_{n \geq 0} \frac{x^n}{(q; q)_n}.
\]
The other $q$-exponential function $E_q(x)$ is

$$E_q(x) := 0 \varphi_0(-; -; q, -x) = \sum_{n \geq 0} \frac{q^{n(n-1)/2}}{(q; q)_n} x^n.$$ 

The function $e_q(x)$ satisfies the following first order $q$-difference equation

$$\{\sigma_q - (1 - x)\} u(x) = 0$$

and $E_q(x)$ satisfies

$$\{(1 + x)\sigma_q - 1\} u(x) = 0.$$ 

The limit $q \to 1 - 0$ converges the exponential function

$$\lim_{q \to 1 - 0} e_q(x(1 - q)) = \lim_{q \to 1 - 0} E_q(x(1 - q)) = e^x.$$ 

In this sense, these functions considered as $q$-analogues of the exponential function. It is known that there exists the relation between these functions:

$$e_q(x) E_q(-x) = 1, \quad e_{q^{-1}}(x) = E_q(-qx).$$

But another relation has not known. We show the connection formula between them and give alternate representation of $e_q(\cdot)$.

### 4.2 The connection formula and alternate representation

At first, we show the following connection formula between $e_q(\cdot)$ and $E_q(\cdot)$.

**Theorem 2.** For any $x \in \mathbb{C}^* \setminus [1; q]$,

$$e_q(x) = \frac{(q; q)_\infty}{\theta_q(-x)} E_q \left( -\frac{q}{x} \right)$$

where $|x| < 1$.

**Proof.** The function $e_q(x)$ and $E_q(x)$ have infinite product as follows:

$$e_q(x) = \frac{1}{(x; q)_\infty}, \quad |x| < 1$$
and
\[ E_q(x) = (-x; q)_\infty. \]

We remark that \( e_q(x) \) can be described as
\[
e_q(x) = \frac{1}{\theta_q(-x)} \left( q, \frac{q}{x} ; q \right)_\infty = \frac{(q; q)_\infty}{\theta_q(-x)} E_q \left( -\frac{q}{x} \right)
\]
where \(|x| < 1\). We obtain the conclusion.

Therefore, these \( q \)-exponential functions are related by the connection formula between the origin and the infinity. If we replace \( x \) by \( x/q \), we obtain the following lemma. This is useful to consider the connection problem in the last section.

**Lemma 5.** For any \( x \in \mathbb{C}^* \setminus [1; q] \), the function \( e_q(x/q) \) has the following alternate representation.
\[
e_q \left( \frac{x}{q} \right) = \frac{(q; q)_\infty}{\theta_q \left( -\frac{x}{q} \right)} 0\varphi_1 \left( -; q; q^2 \frac{q^5}{x^2} \right) - \frac{(q; q)_\infty}{\theta_q \left( -\frac{x}{q} \right)} \frac{q^2}{(1 - q)x} 0\varphi_1 \left( -; q^3; q^2 \frac{q^7}{x^2} \right).
\]

**Proof.** From theorem 2,
\[
1\varphi_0 \left( 0; -; q, \frac{x}{q} \right) = \frac{(q; q)_\infty}{\theta_q \left( -\frac{x}{q} \right)} E_q \left( -\frac{q^2}{x} \right) = \frac{(q; q)_\infty}{\theta_q \left( -\frac{x}{q} \right)} 0\varphi_0 \left( -; q^3; q^2 \frac{q^7}{x^2} \right).
\]

Here,
\[
0\varphi_0 \left( -; q^2 \frac{q^2}{x} \right) = \sum_{k \geq 0} \frac{1}{(q; q)_k} (-1)^k q^{\frac{k(k+1)}{2}} \left( \frac{q^2}{x} \right)^k
\]
and we remark that \((a; q)_{2k} = (a, aq; q^2)_k \) [2]. By separating the terms with even and odd \( k \geq 0 \), we obtain the conclusion.

**4.3 The connection formula of the series** \( 2\varphi_0(0, 0; -; q, \cdot) \)

The aim of this section is to give a proof for the following theorem;
Theorem 3. For any $x \in \mathbb{C}^* \setminus [-\lambda; q]$,

$$
\theta_q(x) \varphi_0 \left( 0, 0; -; q, -\frac{x}{q} \right) = (q; q)_\infty \theta_q(x) \theta_q \left( \frac{\lambda^2}{x q} \right) \frac{1}{\theta_q \left( -\frac{\lambda q}{x} \right)} \varphi_1 \left( 0; q; q^2, \frac{q^2}{x} \right)
$$

\[ + \frac{(q; q)_\infty}{1 - q} \theta_q \left( \frac{\lambda q}{x} \right) \theta_q \left( \frac{\lambda}{x} \right) \lambda \varphi_1 \left( 0; q^3, q^2, \frac{q^3}{x} \right). \]

We define the $q$-Borel-Laplace transformations of the first kind to obtain the connection formula between the origin and the infinity.

**Definition 3.** For any analytic function $f(x)$, the $q$-Borel transformation of the first kind $B_q^+$ is

$$
(B_q^+ f)(\xi) := \sum_{n \geq 0} a_n q^{\frac{n(n-1)}{2}} \xi^n =: \varphi(\xi),
$$

the $q$-Laplace transformation of the first kind $L_q^+$ is

$$
(L_q^+ \varphi)(x) := \sum_{n \in \mathbb{Z}} \theta_q \left( \frac{\lambda q^n}{x} \right). \varphi \left( \lambda q^n \right).
$$

We remark that the $q$-Borel transformation $B_q^+$ is formal inverse of the $q$-Laplace transformation $L_q^+$ as follows;

**Lemma 6.** For any entire function $f(x)$, we have

$$
L_q^+ \circ B_q^+ f = f.
$$

We give the proof of theorem 3.

**Proof.** We apply the $q$-Borel transformation $B_q^+$ to the divergent series $v(x) = 2 \varphi_0 (0, 0; -; q, -x/q)$. We obtain

$$
(B_q^+ v)(\xi) = \varphi_0 \left( 0; -; q, \frac{\xi}{q} \right) =: \varphi(\xi).
$$

From lemma 5

$$
\varphi(\xi) = \frac{(q; q)_\infty}{\theta_q \left( -\xi q \right)} \varphi_1 \left( -; q; q^2, \frac{q^5}{\xi^2} \right) - \frac{(q; q)_\infty}{\theta_q \left( -\xi q \right)} \frac{q^2}{(1 - q) \xi} \varphi_1 \left( -; q^3, q^2, \frac{q^3}{\xi^2} \right).
$$
where $|\xi/q| < 1$.

We apply the $q$-Laplace transformation $\mathcal{L}_q^+$ to $\varphi(\xi)$:

\[
(\mathcal{L}_q^+ \varphi)(x) = \sum_{n \in \mathbb{Z}} \varphi(\lambda q^n) = \frac{1}{\theta_q\left(-\frac{\lambda}{q}\right)} \theta_q\left(\frac{\lambda q^n}{x}\right) \frac{1}{\theta_q\left(\frac{\lambda q}{x}\right)} \theta_q\left(\frac{\lambda q^n}{x}\right) \sum_{n \in \mathbb{Z}} (q^2)^{(n-m)(n-m-1)/2} \left(-\frac{\lambda^2}{qx}\right)^{n-m}
\]

\[
\cdot \sum_{m \geq 0} (-1)^m (q^2)^{m(m-1)/2} (q^3, q^2; q^2)_m \left(q^3 x\right)^m.
\]

Therefore,

\[
2f_0 \left(0, 0; -; q, -\frac{x}{q}\right) = \mathcal{L}_q^+ \circ \mathcal{B}_q^+ 2\varphi_0 \left(0, 0; -; q, -\frac{x}{q}\right)
\]

\[
= (q; q)_\infty \frac{\theta_q^2 \left(-\frac{\lambda^2}{qx}\right) \theta_q\left(\frac{\lambda}{x}\right)}{\theta_q\left(-\frac{\lambda}{q}\right) \theta_q\left(\frac{\lambda}{x}\right)} \varphi_1 \left(0; q, q^2, q^2\right) + \frac{(q; q)_\infty}{1-q} \frac{\theta_q^2 \left(-\frac{\lambda^2}{x}\right)}{\theta_q\left(-\frac{\lambda}{q}\right) \theta_q\left(\frac{\lambda}{x}\right)} \varphi_1 \left(0; q^3; q^2, q^3\right).
\]

We obtain the conclusion.

\[\square\]

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