q-Fractional Askey–Wilson Integrals and Related Semigroups of Operators

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December 15, 2020

Abstract

We introduce three one-parameter semigroups of operators and determine their spectra. Two of them are fractional integrals associated with the Askey–Wilson operator. We also study these families as families of positive linear approximation operators. Applications include connection relations and bilinear formulas for the Askey–Wilson polynomials. We also introduce a $q$-Gauss–Weierstrass transform and prove a representation and inversion theorem for it.

2000 Mathematics Subject Classification: Primary: 33D45; 33C45 Secondary: 47D03; 26A33; 41A36.

Filename: IZZFracV3.tex

1 Introduction

The Askey-Wilson operator $D_q$ is a divided difference operator which acts as a lowering (or annihilation) operator on the Askey–Wilson polynomials and many special and limiting cases of them. Several works defined a right inverse to the Askey–Wilson operator, see for example [7, 22, 17]. The idea in these studies was to define $D_q^{-1}$ on a basis for a weighted $L_2$ space then extend it by linearity. On the other hand Hermann Weyl introduced his fractional integral operator of order $\alpha$ as the linear operator whose action on $e^{inx}, n \neq 0$ is $e^{inx}/(in)^\alpha$. He then extended this by linearity, and identified the result as an integral transform. This was done for the Askey-Wilson operator by Ismail in [14] and the kernel was identified explicitly in [17].

The purpose of this work is to introduce a variation on the Ismail-Rahman operators. This variation naturally leads to a semigroup of contraction operators, which we denoted by $\{T_a : a > 0\}$. This semigroup also has the property $D_q T_a = T_{a-1}, a > 1$. We study the applications of this semigroup as well as the semigroup of the adjoint operators $S_a$. It turns

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out that $S_a$ has the same parameter decreasing property with $D_q$ replaced by a new operator $C_q$. Furthermore, we also identify their corresponding infinitesimal generators. The operators $T_a$ are studied as approximation operators, the rate of convergence of $T_a f$ to $f$ on $C[-1,1]$ is found, and their adjoints $S_a$ are investigated as fractional integral operators. We note that there are many $q$-analogues of the standard operators in approximation theory but they all let $q \to 1^-$ as their large parameter tends to $\infty$. Some references are in [27]. The interested reader can find many references simply by searching the internet for “$q$-Bernstein” or “$q$-Szász”, · · · “.

This work initiates a study of fractional powers of inverses of the Askey–Wilson operator, so it is a $q$-analogue of the classical fractional integrals. Fractional integrals and derivatives have appeared in many areas of analysis and some of their applications are in Sneddon’s classic [31]. It must be noted that the existing theory of $q$-fractional calculus deals with powers and inverses of the operator

$$ (D_q f)(x) = \frac{f(x) - f(qx)}{x - qx}, $$

which is radically different from the Askey–Wilson operator defined in (1.7) below. Some references are [1], [2], and [4].

We wish to indicate that this work is just the starting point of a possibly rich theory of $q$-fractional integrals associated with Askey–Wilson type operators. Many problems remain open. For example we have not touched the problem of describing the range of the $q$-fractional integral operators defined in this work as operators on general weighted $L_p$ spaces. Examples of the work on describing the range of the classical fractional integrals are [30], [25], [29]. We described the range only when $T_a$, or its adjoint $S_a$ act on the weighted $L_2$ space of the $q$-Hermite polynomials. In a future work we plan to apply the $q$-fractional integral operators presented here to solve dual series equations. References to similar work for fractional integrals are in [31], see also Askey’s interesting paper [6]. $q$-fractional integrals associated with $D_q$ of (1.1) were used in [5] to solve dual and triple series equations. Another work in progress is to study analogues of Leibniz rule for the fractional powers of the Askey–Wilson operators. Some progress has been made towards the Leibniz rule. The Leibniz rule for fractional powers of $D_q$ is in [1]–[2]. For details, see also [4].

Semigroups usually describe time evolutionary processes. It may be seen from the connection between semigroups and initial value problems, [9]. Given a semigroup $T(t)$ whose infinitesimal generator is $J$, and a function $f$, independent of $t$, the function $(T(t)f)(x)$ solves the initial value problem $Jy(x,t) = \frac{\partial y(x,t)}{\partial t}$, with initial condition $y(x,0) = f(x)$. Therefore each infinitesimal generator we identify leads to a solution of an initial value problem.

We shall follow the standard notations for $q$-series as in [11], [9]. All the results used in this work are in Ismail’s book [15]. We shall also use the Rahman notation, [11],

$$ h(\cos \theta; a_1, \cdots, a_n) := \prod_{j=1}^{n} (a_j e^{i\theta}, a_j e^{-i\theta}; q)_\infty. $$

The continuous $q$-Hermite polynomials play an essential role in the development of the $q$-fractional integral operators $T_a$. Because of this we state few of their properties. They have
the generating function
\[ \sum_{n=0}^{\infty} \frac{H_n(\cos \theta | q)}{(q; q)_n} t^n = \frac{1}{(te^{i\theta}, te^{-i\theta}, q)_{\infty}}. \]  
and the recurrence relation
\[ H_0(x | q) = 1, \quad H_1(x | q) = 2x, \]
\[ 2x H_n(x | q) = H_{n+1}(x | q) + (1 - q^n) H_{n-1}(x | q). \]
The normalized weight function of the continuous \( q \)-Hermite polynomials is given by
\[ w_H(x | q) = \frac{(q, e^{2i\theta}, e^{-2i\theta}; q)_{\infty}}{2\pi \sqrt{1 - x^2}}, \]
then the orthogonality relation is
\[ \int_{-1}^{1} H_m(x | q) H_n(x | q) w_H(x | q) dx = (q; q)_n \delta_{m,n}. \]
Details are in [15], [3], [23].

Section 2 contains some preliminary material. In Section 3 we define the operators \( T_a \) and we show that they form a semigroup of strongly continuous operators. We also identify the infinitesimal generator of \( T_a \). Section 4 evaluates the actions of the Askey–Wilson operator on certain polynomial basis, which are \( q \)-analogues of \( (x + a)^n \), and on \( q \)-exponential functions. In Section 5 we derive several properties of the semigroup generated by \( S_a \), the adjoints of \( T_a \). We also include the inversion formulas for \( T_a \) and \( S_a \). In Section 6 we show that \( T_a \) maps an Askey–Wilson polynomial to an Askey–Wilson polynomial with different parameters. This is used to derive connection relations and bilinear formulas for special Askey–Wilson polynomials. In Section 7 we study the approximation properties of the operators \( T_a \) as maps from \( C[-1, 1] \) to infinitely differentiable functions. We must note that the infinitesimal generator of the semigroup is related to a Voronovskaya type relation. In Section 8, we prove that \( T_a \) are contraction maps. In Section 9 we study a family of operators \( F_a \) initially introduced in [32]. We study their approximation properties as positive linear approximation operators and identify their infinitesimal generator. We also show that they are also essentially fractional powers of an inverse to a divided difference operator \( B \) defined in (9.12), which is not the Askey–Wilson operator. We also indicate a transform for which \( F_a \) acts as a \( q \)-analogue of a multiplier. In Section 10 we briefly mention an analogue of the Gauss–Weierstrass transform and give a representation and inversion formula for our transform. In the last section, Section 11, we apply the fractional integral operators of Section 3 to solve certain dual integral equations following techniques developed earlier by Noble [26] and Sneddon [31].

The Askey-Wilson operator is a linear operator for functions on the real line that is defined via a complex variable \( z \) and its reciprocal \( 1/z \) on the complex plane. For any function \( f(x) \) we write \( x \) as \( x = (z + 1/z)/2 \) and associate it to the function \( \tilde{f}(z) := f(x) \). Then the Askey–Wilson operator is defined by
\[ (D_q f)(x) = \frac{\tilde{f}(q^{1/2}z) - \tilde{f}(q^{-1/2}z)}{(q^{1/2} - q^{-1/2})(z - 1/z)/2}. \]
In the sequel we will assume that the functions are defined on \([-1, 1]\) unless explicitly stated otherwise. In order to define \(D_q\) on such function, say \(f\), we must tacitly assume that \(f\) is defined for \(|z| \leq q^{-1/2}\), with \(x = (z+1/z)/2\). For the brevity we will not always remind readers of this assumption but they must be made aware of it.

We wish to note that the \(q\)-Hermite polynomials play a central part in the defining the semigroups of this paper. When \(q > 1\), with appropriate scaling the polynomials become the \(q^{-1}\)-Hermite polynomials whose analytic properties are radically different, [16]. They are now orthogonal on \(\mathbb{R}\) with respect to infinitely many probability measures. The case \(q > 1\) has a theory that parallels the theory presented here, albeit subtler, and it will be developed in a future work.

2 Preliminaries

Ismail and Rahman [17] showed that the operator \(D_q^{-1}\) defined by

\[
\frac{2q^{1/4}}{(1-q)(q;q)_\infty}(D_q^{-1}f)(\cos \theta) = \int_0^\pi \frac{(-q^{1/4}e^{i\theta}, -q^{1/4}e^{-i\theta}; q^{1/2})_\infty w_H(\cos \phi)f(\cos \phi)}{(-q^{1/4}e^{i\phi}, -q^{1/4}e^{-i\phi}; q^{1/2})_\infty h(\cos \phi; q^{1/2}e^{i\theta}, q^{1/2}e^{-i\theta})} \sin \phi d\phi,
\]

is a right inverse to the Askey–Wilson operator. The original definition in [17] had an additional \(f\) dependent constant term which we decided to drop. What is important is that the strong operator limit \(\lim_{p \to q^-} D_pD_q^{-1}\) is still the identity operator. The Ismail–Rahman approach follows the original approach of Hermann Weyl when he introduced the Weyl fractional integrals, see [31].

We now iterate \(D_q^{-1}f\) and compute \(D_q^{-n}\) for any natural number \(n\). Recall that the Poisson kernel for the continuous \(q\)-Hermite polynomials is [15, Theorem 13.1.6]

\[
\sum_{n=0}^\infty H_n(\cos \theta|q)H_n(\cos \phi|q) \frac{t^n}{(q;q)_n} = \frac{(t^2; q)_\infty}{(te^{i(\theta+\phi)}, te^{i(\theta-\phi)}, te^{-i(\theta+\phi)}, te^{-i(\theta-\phi)}; q)_\infty}.
\]

We shall need the following fact.

Theorem 2.1. Let \(p_n\) be orthonormal with respect to a probability measure \(\mu\) on a compact interval \([a, b]\). If the Poisson kernel \(P_r(x, y) := \sum_{n=0}^\infty p_n(x)p_n(y) r^n\) is \(\geq 0\) for all \(x, y \in \text{supp}(\mu) \subset [a, b]\), then

\[
\lim_{r \to 1^-} \int_a^b P_r(x, y)f(y) d\mu(y) = f(x)
\]

is uniform for all continuous functions on \([a, b]\).

This follows from Korovkin’s theorem, [24]. For completeness we state Korovkin’s theorem. Let \(e_j(x) = x^j\). if \(L_n\) is a sequence of positive linear operators acting on continuous functions on a compact interval \([a, b]\). If \((L_ne_j)(x) \to e_j(x)\) uniformly on \([a, b]\) for \(j = 0, 1, 2\) then \((L_nf)(x) \to f(x)\) uniformly on \([a, b]\) for all continuous functions \(f\).
3 $q$-Fractional Integrals

Motivated by the Ismail-Rahman operator (2.1) we define the operators

\[(T_a f)(\cos \theta) = \frac{(1 - q)^a}{2^a q^{a/4}} (q^a; q)_{\infty} \times \int_0^\pi \left( -q^{1/4} e^{i\phi}, -q^{1/4} e^{-i\phi}; q^{1/2} \right)_{\infty} w_H(\cos \phi) f(\cos \phi) \sin \phi \, d\phi. \]

(3.1)

Theorem 3.1. The operators \(\{T_a : a > 0\}\) have the following properties.

(a) The family \(\{T_a : a > 0\}\) has the property, \(T_a T_b = T_{a+b}\).

(b) On \(C[-1, 1]\), \(T_a\) tends to the identity operator as \(a \to 0^+\).

(c) For \(a > 1\) we have the property \(D_q T_a = T_{a-1}\).

Proof of (a). It is clear that

\[
\left( \frac{2q^{1/4}}{1 - q} \right)^{a+b} \frac{(T_a T_b f)(\cos \theta)}{(-q^{1/4} e^{i\theta}, -q^{1/4} e^{-i\theta}; q^{1/2})_{\infty}} = (q^a, q^b; q)_{\infty} \int_{[0, \pi]^2} \frac{w_H(\cos \phi) f(\cos \psi) \sin \phi \sin \psi}{(-q^{1/4} e^{i\phi}, -q^{1/4} e^{-i\phi}; q^{1/2})_{\infty} h(\cos \psi; q^a/2 e^{i\theta}, q^a/2 e^{-i\theta})} \, d\phi \, d\psi.
\]

The Poisson kernel (2.2) and the completeness of the continuous \(q\)-Hermite polynomials imply

\[
\int_0^\pi (q^a, q^b; q)_{\infty} w_H(\cos \phi) f(\cos \psi) \sin \phi \, d\phi = \frac{(q^a+b; q)_{\infty}}{h(\cos \psi; q^{(a+b)/2} e^{i\theta}, q^{(a+b)/2} e^{-i\theta})}
\]

and the proof of part (a) is complete.

Proof of (b). For continuous functions \(f, f(\cos \theta)/(-q^{1/4} e^{i\theta}, -q^{1/4} e^{-i\theta}; q^{1/2})_{\infty}\) is continuous, hence Theorem 2.1 implies part (b).

The proof of (c) follows by direct computation.

It must be noted that the kernel of the integral operator \(T_a\) becomes quotients of products of theta functions, [33], [28], when \(a = 1\), that is the case of \(D_q^{-1}\) in (2.1). So this case is very special in many ways.

The next theorem describes the eigenvalues of \(T_a\) as an operator on \(L_2[-1, 1, w_H]\).
Remark 3.3. We have the following general comments for 

\[ \lambda \text{ implied by the fact that} \]

The orthogonality and completeness of the continuous q-dimensional, and we conclude that the operator 

\[ T \]

This shows that the only eigenfunctions are 

\[ (−q^{1/4}e^{iθ}, −q^{1/4}e^{−iθ}; q^{1/2})_∞ H_m(\cos θ|q) \]

with eigenvalues

\[ \lambda_m = \frac{(1−q)α}{2^αq^{α/4}} q^{αn/2}. \]  

(3.2)

Proof. It is clear that the action of \( T \) on 

\[ \left(−q^{1/4}e^{iθ}, −q^{1/4}e^{−iθ}; q^{1/2}\right)_∞ H_m(\cos θ|q) \]

given by

\[
\frac{(1−q)α}{2^αq^{α/4}} \left( q^α; q_∞ \right) \int_0^\pi \left(−q^{1/4}e^{iθ}, −q^{1/4}e^{−iθ}; q^{1/2}\right)_∞ w_H(\cos φ|q) H_m(\cos φ|q) \sin φdφ \\
= \frac{(1−q)α}{2^αq^{α/4}} \left(−q^{1/4}e^{iθ}, −q^{1/4}e^{−iθ}; q^{1/2}\right)_∞ \int_{−1}^1 H_m(y|q)w_H(y|q) \sum_{n=0}^∞ H_n(x|q)H_n(y|q) \frac{q^{αn/2}}{(q_4; q)_n} dy \\
= \frac{(1−q)α}{2^αq^{α/4}} q^{αn/2} \left(−q^{1/4}e^{iθ}, −q^{1/4}e^{−iθ}; q^{1/2}\right)_∞ H_m(\cos θ|q),
\]

where we used the orthogonality relation (1.6) and the Poisson kernel (2.2). This shows that 

\[ \left(−q^{1/4}e^{iθ}, −q^{1/4}e^{−iθ}; q^{1/2}\right)_∞ \]

are eigenfunctions.

It is clear that the function on the right-hand side of (2.2) is in \( L_2[−1, 1, w_H] \). Let \( f \) be an eigenfunction with an eigenvalue \( λ \). Since \( 1/\left(−q^{1/4}e^{iθ}, −q^{1/4}e^{−iθ}; q^{1/2}\right)_∞ \) is bounded above and below by positive numbers, then \( f/\left(−q^{1/4}e^{iθ}, −q^{1/4}e^{−iθ}; q^{1/2}\right)_∞ \) is in \( L_2[−1, 1, w_H] \), so we set

\[
\frac{f(\cos θ)}{\left(−q^{1/4}e^{iθ}, −q^{1/4}e^{−iθ}; q^{1/2}\right)_∞} = \sum_{n=0}^∞ f_n H_n(\cos θ|q).
\]

The orthogonality and completeness of the continuous q-Hermite polynomials system, Parseval’s theorem and the Poisson kernel (2.2) show that

\[
\frac{λf(\cos θ)}{\left(−q^{1/4}e^{iθ}, −q^{1/4}e^{−iθ}; q^{1/2}\right)_∞} = \frac{T_α f(\cos θ)}{\left(−q^{1/4}e^{iθ}, −q^{1/4}e^{−iθ}; q^{1/2}\right)_∞} \\
= \frac{(1−q)α}{2^αq^{α/4}} \int_0^\pi \sum_{n=0}^∞ f_n H_n(\cos θ|q) \sum_{k=0}^∞ H_k(\cos θ|q)H_k(\cos φ|q) \frac{q^{αk/2}}{(q_4; q)_k} w_H(\cos φ|q) \sin φdφ \\
= \frac{(1−q)α}{2^αq^{α/4}} \sum_{n=0}^∞ f_n q^{αn/2} H_n(\cos θ|q).
\]

This shows that the only eigenfunctions are 

\[ \left(−q^{1/4}e^{iθ}, −q^{1/4}e^{−iθ}; q^{1/2}\right)_∞ H_m(\cos θ|q) \]

and that the corresponding eigenvalues are given by (3.2). \( \square \)

The operator \( T_α \) is the limit of finite rank operators (its restriction to the span of \( \{H_k(x|q) : 0 \leq k \leq n_1\} \)). Therefore \( T_α \) is compact for any \( α > 0 \) and Hilbert space \( L_2[−1, 1, w_H] \) is infinite dimensional, and we conclude that the operator \( T_α \) is not invertible for any \( α > 0 \). This is also implied by the fact that \( λ_n \to 0 \), hence \( λ = 0 \) is in the spectrum.

Remark 3.3. We have the following general comments for \( T_α \), Similar remarks apply to \( S_α, F_α \) in later sections:
1. $T_a$ is defined for all $\Re(a) > 0$ in \textbf{(3.1)}, and parts (a) and (c) of Theorem 3.1 remain valid, but $T_a$ may not be positive anymore.

2. Let

\begin{equation}
\tag{3.3}
g(\cos \theta) = (-q^{1/4} e^{i\theta}, -q^{1/4} e^{-i\theta}; q^{1/2})<.\end{equation}

For each $f \in L_2[-1, 1, w_H]$, let

\begin{equation}
\tag{3.4}
L f = g f, \quad L^{-1} f = f / g.
\end{equation}

Since both $g, 1/g$ are continuous on $[-1, 1]$, then the operators $L^{-1}, L$, are bounded on $L_2[-1, 1, w_H]$. By Theorem \textbf{3.2} for $\Re(a) > 0$ the operator $L^{-1} T_a L$ has eigen-system $\{\lambda_m, H_m(x|q)\}$, hence it is a trace class operator. Consequently, $T_a : \Re(a) > 0$ are trace class operators since trace class is a two sided ideal.

3. As a matter of fact, $T_a$ can be defined for each $a \in \mathbb{C}$ satisfying $q^{\Re(a)/2+n} \neq 1$ for $n = 0, 1, \ldots$.

The obtained operator is still compact since its kernel is continuous on $[-1, 1] \times [-1, 1]$. As a result of compactness, none of $T_a$s is invertible since the underlying Hilbert space $L_2[-1, 1, w_H]$ is infinitely dimensional. By applying (c) of Theorem \textbf{3.1} we can extend the semigroup law (a) to complex cases as far as all the operators involved exists.

4. Another immediate consequence of Theorem \textbf{3.1} is that $D_q$ is $T_{-1}$, in the sense that the strong operator limit $\lim_{n \to 1^+} D_q T_a$ is the identity operator $I$.

We now identify the infinitesimal generator of $T_a$. From Theorem \textbf{3.2} it follows that

\begin{equation}
\tag{3.5}
T_a (g(x) H_m(x|q)) = \left(\frac{1 - q}{2} q^{(2m-1)/4}\right)^a g(x) H_m(x|q)
\end{equation}

where $g$ is defined in \textbf{(3.3)}. Thus, with $a \to 0$, $I$ the identity operator, we have

\begin{equation}
(T_a - I) g(x) H_m(x|q) = a \log \left(\frac{1 - q}{2} q^{(2m-1)/4}\right) g(x) H_m(x|q) + R_a,
\end{equation}

where the $L_2$-norm of $R_a$ is of $O(a^2 m^2 \|g H_m(\cdot|q)\|)$. Then the linear operator $J = \lim_{a \to 0} a^{-1} [T_a - I]$ is defined on all finite linear combinations of the polynomials $\{g(x) H_n(x|q)\}$. Let $g f \in L_2[-1, 1, w_H]$ with $g$ as in \textbf{(3.3)}, since the $q$-Hermite polynomials form an orthogonal basis, then $g f = \sum_{n=0}^{\infty} c_n H_n(x|q)$. For all $f \in L_2[-1, 1, w_H]$ such that

\begin{equation}
\tag{3.6}
f/g = \sum_{n=0}^{\infty} c_n H_n(x|q),
\end{equation}

\begin{equation}
\tag{3.7}
\sum_{m=0}^{\infty} m^2 |c_m|^2(q; q)_m < \infty,
\end{equation}

it is clear that the infinitesimal generator $J$ is defined by

\begin{equation}
\tag{3.8}
(Jf)(x) = g(x) \sum_{n=0}^{\infty} \log \left(\frac{1 - q}{2} q^{(2m-1)/4}\right) c_m H_m(x|q).
\end{equation}

Now both $g$ and $1/g$ are continuous and uniformly bounded on $[-1, 1]$, hence all $f$ satisfy \textbf{(3.6)} form a dense subspace of $L_2([-1, 1], w_H)$.
4 $q$-Analogues of Powers and Exponential Functions

In [19] and [20] Ismail and Stanton introduced the bases

\[ \phi_n(x) = (q^{1/4}e^{i\theta}, q^{1/4}e^{-i\theta}, q^{1/2})_n = \prod_{k=0}^{n-1} (1 - 2xq^{1/4+k/2} + q^{1/2+k}) , \]

\[ \rho_n(x) = (1 + e^{2i\theta})e^{-in\theta}(-q^{2-n}e^{2i\theta}; q^2)^n, \ n > 0, \ \rho_0(x) := 1, \]

and established $q$-Taylor series expansions of entire functions in these bases.

It is clear that we can replace $n$ by a general parameter $\beta$ in (4.1).

**Theorem 4.1.** The action of $T_a$ on $\phi_\beta(x)$ is given by

\[ T_a \phi_\beta(x) = \frac{(1 - q)^\alpha (q^{\alpha+\beta+1}; q)_\infty}{2^\alpha q^{\alpha/4} (q^{\beta+1}; q)_\infty} \phi_{\alpha+\beta}(x) . \]

**Proof.** It is clear that $T_a \phi_\beta(\cos \theta)$ equals

\[ \times \int_0^\pi \frac{(-q^{1/4}e^{i\theta}, q^{1/4}e^{-i\theta}; q^{1/2})_\infty w_H(\cos \phi | q) \sin \phi d\phi}{(q^{\beta/2+1/4}e^{i\phi}, q^{\beta/2+1/4}e^{-i\phi}; q)^\infty h(\cos \phi; q^\alpha e^{i\theta}, q^\beta e^{-i\theta})} = \frac{(1 - q)^\alpha}{2^\alpha q^{\alpha/4}} (q^{\alpha}; q)_\infty \]

\[ \times \int_0^\pi \frac{(-q^{1/4}e^{i\theta}, q^{1/4}e^{-i\theta}; q^{1/2})_\infty w_H(\cos \phi | q) \sin \phi d\phi}{h(\cos \phi; q^{\beta/2+1/4}, q^{\beta/2+3/4}, q^\beta e^{i\theta}, q^{\alpha/2} e^{-i\theta})} . \]

The integral is an Askey-Wilson integral and using the evaluation [3], [11]

\[ \int_0^\pi \frac{w_H(\cos \phi | q)}{h(\cos \phi; a_1, a_2, a_3, a_4)} \sin \phi d\phi = \frac{(a_1a_2a_3a_4; q)_\infty}{\prod_{1 \leq j < k \leq 4} (a_ja_k; q^2)_\infty} . \]

We note that (4.2) is an analogue of the action if a fractional integral operator on $x^\beta$.

Formula (4.2) is very important because we know how to expand entire functions as

\[ \sum_{n=0}^{\infty} c_n \phi_n(x) . \]

Indeed if $f(x) = \sum_{n=0}^{\infty} f_n \phi_n(x)$, then

\[ T_a \sum_{n=0}^{\infty} f_n \phi_n(x) = \frac{(1 - q)^\alpha}{2^\alpha q^{\alpha/4}} \sum_{n=0}^{\infty} f_n \phi_{\alpha+n}(x) . \]

Recall the definition of the $q$-exponential function $E_q$, which we introduced in [22],

\[ E_q(\cos \theta; \alpha) = \frac{(q^2; q^2)_\infty}{(qa^2; q^2)_\infty} \sum_{n=0}^{\infty} (-i e^{i\theta} q^{(1-n)/2}, -i e^{-i\theta} q^{(1-n)/2}; q) \frac{(-i\alpha)^n}{n (q; q)_n} q^{n^2/4} . \]
In [22], it was shown that,

\[(qt^2; q^2)_\infty \mathcal{E}_q(x; t) = \sum_{n=0}^{\infty} \frac{q^{n^2/4} n}{(q; q)_n} H_n(x \mid q).\]  

We note that \(\mathcal{E}_q(x; t)\) is a \(q\)-analogue of \(e^{xt}\).

**Theorem 4.2.** The operators \(T_n\) have the properties

\[
T_n[(-q^{1/4} e^{i\theta}, -q^{1/4} e^{-i\theta}; q^{1/2})_\infty \mathcal{E}_q(\cos \theta; t)]
\]

\[
= (1 - q)^n(a(q^{a+1/2}; q^2)_\infty (-q^{1/4} e^{i\theta}, -q^{1/4} e^{-i\theta}; q^{1/2})_\infty \mathcal{E}_q(\cos \theta; q^{a/2} t))
\]

and

\[
T_n \left[ \frac{(-q^{1/4} e^{i\theta}, -q^{1/4} e^{-i\theta}; q^{1/2})_\infty}{(te^{i\theta}, te^{-i\theta}; q)_\infty} \right] = \frac{(1 - q)^n(a(q^{a+1/2}; q^2)_\infty (-q^{1/4} e^{i\theta}, -q^{1/4} e^{-i\theta}; q^{1/2})_\infty)}{2^n q^{a/4} (tq^{a+1/2}; q^2)_\infty (tq^{a/2} e^{-i\theta}; q)_\infty}.
\]

We omit the proof, which uses the orthogonality of the continuous \(q\)-Hermite polynomials and the Poisson kernel and is very similar to the proofs in Section 2.

Ismail, Stanton, and Viennot [21] proved that

\[
\int_0^\pi \frac{w_H(\cos \theta) \sin \theta}{h(\cos \theta; a_1, \cdots, a_k)} d\theta,
\]

is essentially the generating function of the crossing numbers of perfect matchings of sets with \(a_1, \cdots, a_k\) as generating function variables. This enables us to evaluate the integrals

\[
T_n \left[ \frac{(-q^{1/4} e^{i\theta}, -q^{1/4} e^{-i\theta}; q^{1/2})_\infty}{(te^{i\theta}, te^{-i\theta}; q)_\infty \cos \phi; a_1, \cdots, a_k} \right]
\]

as a power series in \(a_1, \cdots, a_k\).

Formula \((4.7)\) has the following curious implication,

\[
\int_{-1}^{1} \frac{\mathcal{E}_q(x; t) (T_n f)(x) w_H(x \mid q)}{(-q^{1/4} e^{i\theta}, -q^{1/4} e^{-i\theta}; q^{1/2})_\infty} dx
\]

\[
= \frac{(1 - q)^n(a(q^{a+1/2}; q^2)_\infty)}{2^n q^{a/4} (tq^{a+1/2}; q^2)_\infty} \int_{-1}^{1} \frac{\mathcal{E}_q(x, tq^{a/2}) f(x) w_H(x \mid q)}{(-q^{1/4} e^{i\theta}, -q^{1/4} e^{-i\theta}; q^{1/2})_\infty} dx.
\]

**Proof.** The left-hand side of \((4.10)\) is

\[
\frac{(1 - q)^n(a(q^{a+1/2}; q^2)_\infty)}{2^n q^{a/4}} \int_0^\pi \int_0^\pi \frac{\mathcal{E}_q(\cos \theta; t) w_H(\cos \phi \mid q) \sin \theta \times w_H(\cos \phi \mid q) \sin \phi}{(-q^{1/4} e^{i\theta}, -q^{1/4} e^{-i\theta}; q^{1/2})_\infty \cos \phi; q^{a/2} e^{i\theta}, q^{a/2} e^{-i\theta})} d\phi d\theta
\]

\[
= \int_0^\pi f(\cos \phi) w_H(\cos \phi \mid q) \sin \phi \times (T_n(-q^{1/4} e^{i\theta}, -q^{1/4} e^{-i\theta}; q^{1/2})_\infty \mathcal{E}_q(\cos \theta; t)(\cos \phi) d\phi
\]

\[
= \frac{(1 - q)^n(a(q^{a+1/2}; q^2)_\infty)}{2^n q^{a/4} (tq^{a+1/2}; q^2)_\infty} \int_{-1}^{1} \frac{\mathcal{E}_q(x, tq^{a/2}) f(x) w_H(x \mid q)}{(-q^{1/4} e^{i\theta}, -q^{1/4} e^{-i\theta}; q^{1/2})_\infty} dx.
\]
This completes the proof.

We shall revisit formulas of the type \((5.10)\) in Section 9.

## 5 Adjoint and Inversion

We realized that the Riemann-Liouville and the Weyl fractional integral operators are adjoints with respect to the inner product \((f, g) = \int_0^∞ f(x)g(x) \, dx\).

We define the inner product

\[(f, g) = \int_{-1}^1 f(x)g(x)w_H(x|q)dx.\]

It is easy to see that the adjoint of \(T_a\) is given by

\[
(T_a^* f)(\cos \theta) = \frac{(1 - q)^a}{2^a q^{a^2/4}} (q^a; q)_∞ \times \int_0^π \frac{(-q^{1/4} e^{iφ}, -q^{1/4} e^{-iφ}; q^{1/2})_∞ w_H(\cos φ|q) f(\cos φ) \sin φ dφ}{(-q^{1/4} e^{iθ}, -q^{1/4} e^{-iθ}; q^{1/2})_∞ h(\cos φ; q^{1/2} e^{iθ}, q^{1/2} e^{-iθ}) \sin φ dφ},
\]

where we have applied the symmetry

\[h(\cos \theta; q^{1/2} e^{iφ}, q^{1/2} e^{-iφ}) = h(\cos \phi; q^{1/2} e^{iφ}, q^{1/2} e^{-iφ}).\]

We shall use the operators \(S_a\),

\[
(S_a f)(\cos θ) = \frac{(1 - q)^a}{2^a q^{a^2/4}} (q^a; q)_∞ \times \int_0^π \frac{(-q^{1/4} e^{iφ}, -q^{1/4} e^{-iφ}; q^{1/2})_∞ w_H(\cos φ|q) f(\cos φ) \sin φ dφ}{(-q^{1/4} e^{iθ}, -q^{1/4} e^{-iθ}; q^{1/2})_∞ h(\cos φ; q^{1/2} e^{iθ}, q^{1/2} e^{-iθ}) \sin φ dφ}.
\]

**Theorem 5.1.** The family of operators \(\{S_a : a > 0\}\) has the following properties.

(a) \(\{S_a : a > 0\}\) is a semigroup.

(b) \(S_a\) tends to the identity operator as \(a \to 0^+\).

(c) For \(a > 1\) we have the property \(C_q S_a = S_{a-1}\) where \(C_q\) is defined by

\[
(C_q f)(x) = 2^a \frac{Γ(q^{1/2}z) - z^a Γ(q^{-1/2}z)}{(1 - q)(1 - z^2)z}.
\]

**Proof.** It is clear that \(T_a^*\) and \(T_b^*\) commute since \(T_a\) and \(T_b\) commute. Moreover \(S_a S_b = S_{a+b}\) follows from part (a) of Theorem 3.1 and the property of adjoints. The remaining parts can be proved similar to the proofs of the corresponding parts in Theorem 3.1. □

One can similarly prove Theorem 5.2 below.
Theorem 5.2. The only eigenfunctions of $S_a$ as operator $S_a$ acting on $L_2[-1,1,w_H]$ are the functions $H_m(\cos \theta|q)/(−q^{1/4}e^{i\theta}−q^{1/4}e^{−i\theta};q^{1/2})_\infty$ with eigenvalues

$$\lambda_m = \frac{(1−q)^a}{2^a q^{a/4}} q^{ma/2}.$$  

Moreover the eigenfunctions form a basis for $L_2[-1,1,w_H]$.

Let $g$ be as in (3.3) and for any $f \in L_2[-1,1,w_H]$, let

$$(5.5) \quad (Kf)(\cos \phi) = \frac{f(\cos \phi)}{g^2(\cos \phi)}.$$  

It is clear that $K$ is invertible and

$$(5.6) \quad (K^{-1}f)(\cos \phi) = g^2(\cos \phi) f(\cos \phi).$$  

It is clear that

$$(5.7) \quad S_a = K T_a K^{-1}.$$  

Theorem 5.3. Let $J_t$ and $J_s$ be the infinitesimal generators of $T_a$ and $S_a$, respectively Then

$$(5.8) \quad J_s = K J_t K^{-1}.$$  

Moreover, for any $h \in L_2[-1,1,w_H]$ such that

$$(5.9) \quad gh = \sum_{n=0}^{\infty} c_n H_n(x|q), \quad \text{and} \quad \sum_{n=1}^{\infty} n^2 |c_n|^2 (q;q)_n < \infty,$$

then

$$(5.10) \quad (J_s h)(x) = \sum_{n=0}^{\infty} \log \left( \frac{1−q q^{(2n−1)/4}}{2} \right) c_n \frac{H_n(x|q)}{g(x)}.$$  

Proof. Since $g(x)$, $1/g(x)$ are continuous on $[-1,1]$, then for any $m \in \mathbb{Z}$ we have

$$g^m (L_2[-1,1,w_H]) = L_2[-1,1,w_H].$$  

Then itt is clear that

$$J_s = K J_t K^{-1}.$$  

Now take $f \in L_2[-1,1,w_H]$ where

$$(5.11) \quad (gf)(x) = \sum_{n=0}^{\infty} c_n H_n(x|q), \quad \text{and} \quad \sum_{n=1}^{\infty} n^2 |c_n|^2 (q;q)_n < \infty$$

hold. Since

$$J_t (gf)(x) = g(x) \sum_{n=0}^{\infty} \log \left( \frac{1−q q^{(2n−1)/4}}{2} \right) c_n H_n(x),$$
then
\[ J_s(f/g) = J_t(gf)/g^2 = \sum_{n=0}^{\infty} \log \left( \frac{1 - q}{2} q^{(2n-1)/4} \right) c_n H_n/g. \]

The proof is will be complete after letting
\[ h = f/g, f = gh. \]

**Corollary 5.4.** For \( 0 < q < 1 \) we have
\[
\lim_{y \to \infty} \log q^{-y} \int_0^\infty \left( \frac{2q^{1/4}}{1-q} \right)^a q^{ay} T_a da = I
\]
and
\[
\lim_{y \to \infty} \log q^{-y} \int_0^\infty \left( \frac{2q^{1/4}}{1-q} \right)^a q^{ay} S_a da = I
\]
in strong operator topology.

**Proof.** Since \( \{ (1-q)^a 2^{n/4} q^{ma/2}, g(x) H_m(x|q) \}_{m=0}^\infty \) form a complete eigensystem for \( T_a \), then the system \( \{ (1-q)^a 2^{n/4} q^{ma/2}, H_m(x|q)/g(x) \}_{m=0}^\infty \) is a complete eigensystem for \( S_a \), hence \( f \in L^2[-1,1,w_H] \) implies
\[ f(x) = g(x) \sum_{n=0}^{\infty} c_n H_n(x|q), \quad f(x) = \sum_{n=0}^{\infty} f_n H_n(x|q)/g(x) \]
and
\[ \sum_{n=0}^{\infty} |c_n|^2 < \infty, \quad \sum_{n=0}^{\infty} |f_n|^2 < \infty \]
such that
\[ T_a f = \frac{(1-q)^a}{2^{n/4} q^{ma/2}} g(x) \sum_{n=0}^{\infty} c_n q^{ma/2} H_n(x|q) \]
and
\[ S_a f = \frac{(1-q)^a}{2^{n/4} q^{ma/2}} \sum_{n=0}^{\infty} f_n q^{ma/2} H_n(x|q)/g(x). \]

Then for \( y > 0 \) we have
\[
\int_0^\infty \left( \frac{2q^{1/4}}{1-q} \right)^a q^{ay} (T_a f)(x) da = \frac{g(x)}{\log q^{-1}} \sum_{n=0}^{\infty} \frac{c_n}{y + m/2} H_n(x|q)
\]
and
\[
\int_0^\infty \left( \frac{2q^{1/4}}{1-q} \right)^a q^{ay} (S_a f)(x) da = \frac{1}{\log q^{-1}} \sum_{n=0}^{\infty} \frac{f_n}{y + m/2} H_n(x|q)/g(x).
\]
Then the corollary follows by taking limits. \( \square \)
We next record the inversion formulas for both $T_a$ and $S_a$. Let $\lfloor a \rfloor$ and $\{a\}$ denote the integer and fractional parts of $a$, respectively.

**Theorem 5.5.** The left-inversion formulas of $T_a$ and $S_a$ are given by

(a) $f(x) = (D_q^{\lfloor a \rfloor} T_{1-a}) g(x)$ if $g(x) = (T_a f)(x)$.

(b) $f(x) = (C_q^{\lfloor a \rfloor + 1} S_{1-a}) g(x)$ if $g(x) = (S_a f)(x)$.

**Proof.** It is clear that if $g(x) = (T_a f)(x)$ then

$$[D_q^{\lfloor a \rfloor + 1} T_{1-a} + \epsilon g(x)] = [D_q^{\lfloor a \rfloor + 1} T_{1-a} + \epsilon f(x)]$$

$$= D_q T_{1-\epsilon} f(x),$$

holds for any $\epsilon > 0$. Now let $\epsilon \to 0^+$. The proof for $S_a$ is identical.

## 6 Application to the Askey–Wilson polynomials

We recall that the Askey–Wilson polynomials are, [15],

$$p_n(cos \theta; a, b, c, d) = (ab, ac, ad; q)_n a^{-n}$$

(6.1) $\times_4 \phi_3 \left( q^{-n}, q^{-n+1}abcd; ae^{i\theta}, ae^{-i\theta} \left| q, q \right. \right)$.

**Theorem 6.1.** The operators $T_a$ have the property

$$T_a p_n(\cdot; -q^{1/4}, b, c, d) = (-1)^n \left( -q^{1/4}b, -q^{1/4}c, -q^{1/4}d; q \right)_n$$

$$\times_4 (1-q)^{\lfloor a \rfloor + 1}; q)_\infty$$

(6.2) $\times 2^a q^{a+4n/4}(q;q)_\infty (-q^{1/4+a/2}e^{i\theta}, -q^{1/4+a/2}e^{-i\theta}; q^{1/2})_\infty$

$$\times_5 \phi_4 \left( q^{-n}, q^{-n+3/4}bcd, q, -q^{a/2+1/4}e^{i\theta}, -q^{a/2+1/4}e^{-i\theta} \left| q, q \right. \right)$.

**Proof.** It is clear that

$$T_a p_n(\cdot; -q^{1/4}, b, c, d) = \sum_{k=0}^{n} \left( q^{-n}, q^{-n+3/4}bcd; q \right)_k q^k \int_0^\pi \frac{d\phi}{(q, q^{2+1/4}, -q^{3/4}, q^{a/2}e^{i\theta}, q^{a/2}e^{-i\theta}) 2\pi}.$$

The integral is an Askey–Wilson integral and the above expression simplifies to establish.

The special case when $b = -q^{3/4}$ is very interesting. In this case we have

$$T_a p_n(\cdot; -q^{1/4}, -q^{3/4}, c, d) = (-1)^n \left( q, q^{1/4}c, -q^{1/4}d; q \right)_n$$

(6.3) $\times_4 (1-q)^{\lfloor a \rfloor + 1}; q)_\infty$ $\times 2^a q^{a+4n/4}(q;q)_\infty (-q^{1/4+a/2}e^{i\theta}, -q^{1/4+a/2}e^{-i\theta}; q^{1/2})_\infty$

$$\times_5 \phi_4 \left( q^{-n}, q^ncd, -q^{a/2+1/4}e^{i\theta}, -q^{a/2+1/4}e^{-i\theta} \left| q, q \right. \right).$$
The $\phi_3$ is a multiple of an Askey–Wilson polynomial. Indeed we proved the connection relation
\[ T_a \Phi_n (\cdot; -q^{1/4}, -q^{3/4}, c, d) = \frac{(q; q)_n}{(q^{a+1}; q)_n} q^{an/2} \binom{1 - q}{q} (q^{a+1}; q)_\infty \]
\[ \times \frac{(q^{1/4}, e^{i\theta}, -q^{1/4} e^{-i\theta}; q^{1/2})_\infty}{(q^{1/4+a/2}, e^{i\theta}, -q^{1/4+a/2} e^{-i\theta}; q^{1/2})_\infty} p_n(x; -q^{a/2+1/4}, -q^{3/4+a/2}, cq^{-a/2}, dq^{-a/2}). \]

Our next task is to derive a bilinear formula which follows from the Hilbert-Schmidt decomposition of a symmetric kernel of an integral operator. We follow the technique developed in [13].

We recall the orthogonality relation of the Askey–Wilson polynomial [3], [11], [15]
\[ \int_0^\pi p_m(\cos \theta; t|q) p_n(\cos \theta; t|q) w(\cos \theta; t) d\theta = 2\pi \frac{(q^{a+1}; q)_\infty}{(q^{a}; t|q)_\infty} \prod_{1 \leq j < k \leq 4} (t_j t_k q^n; q)_\infty \delta_{m,n}, \]
where $t = (t_1, t_2, t_3, t_4)$,
\[ w(\cos \theta; t_1, t_2, t_3, t_4) := \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{\prod_{j=1}^4 (t_j e^{i\theta}, t_j e^{-i\theta}; q)_\infty}. \]

Then we define three sequences $\{A_n\}, \{B_n\}$ and $\{C_n\}$. Let
\[ M_n(t_1, t_2, t_3, t_4) = \frac{2\pi (t_1 t_2 t_3 t_4 q^{2n}; q)_\infty (t_1 t_2 t_3 t_4 q^{n-1}; q)_n}{(q^{a+1}; q)_\infty \prod_{1 \leq j < k \leq 4} (t_j t_k q^n; q)_\infty}, \]
\[ A_n = M_n(-q^{a/2+1/4}, -q^{a/2+3/4}, q^{-a/2}, q^{-a/2}), \]
\[ B_n = M_n(-q^{1/4}, -q^{3/4}, c, d), \quad C_n = \frac{(q; q)_n}{(q^{a}; q)_{n+1} (q; q)_\infty} q^{an/2}. \]

Note that (6.4) shows that $T_a$ maps an Askey–Wilson polynomial to another Askey–Wilson polynomial and (6.5) show its orthogonality relation, hence we drive the following the integral equation,
\[ \int_0^\pi w_0(\cos \theta|q) \int_0^\pi w_H(\cos \phi_1|q) p_n(\cos \phi_1 - q^{1/4}, -q^{3/4}, c, d) \sin \phi_1 \]
\[ \times \frac{(q^{1/4} e^{i\phi_1}, -q^{1/4} e^{-i\phi_1}; q^{1/2})_\infty h(\cos \phi_1; q^{a/2} e^{i\theta}, q^{a/2} e^{-i\theta})}{(q^{1/4} e^{i\phi_2}, -q^{1/4} e^{-i\phi_2}; q^{1/2})_\infty h(\cos \phi_2; q^{a/2} e^{i\theta}, q^{a/2} e^{-i\theta})} \sin \phi_2 d\phi_1 d\phi_2 d\theta \]
\[ = A_n C_n^2 \delta_{m,n}, \]
where
\[ w_0(\cos \theta|q) = w(\cos \theta; -q^{a/2+1/4}, -q^{a/2+3/4}, q^{-a/2}, q^{-a/2} d|q) \frac{-q^{1/4+a/2} e^{i\theta}, -q^{1/4+a/2} e^{-i\theta}, q^{1/2}}{\infty}. \]
If we change the order of integration, which leads to

\[
\int_0^\pi \int_0^\pi \frac{w_H(\cos \phi_1|q)p_n(\cos \phi_1| - q^{1/4}, - q^{3/4}, c, d)\sin \phi_1}{(-q^{1/4}e^{i\phi_1}, -q^{1/4}e^{-i\phi_1}; q^{1/2})_{\infty}} \\
\times \frac{w_H(\cos \phi_2|q)p_m(\cos \phi_2| - q^{1/4}, - q^{3/4}, c, d)\sin \phi_2}{(-q^{1/4}e^{i\phi_2}, -q^{1/4}e^{-i\phi_2}; q^{1/2})_{\infty}} \times \frac{w_0(\cos \theta|q)\, d\theta \, d\phi_1 \, d\phi_2}{h(\cos \phi_1; q^{a/2}e^{i\theta}, q^{a/2}e^{-i\theta})h(\cos \phi_2; q^{a/2}e^{i\theta}, q^{a/2}e^{-i\theta})} = A_n C_n^2 \delta_{m,n},
\]

\[
(6.8)
\]

The polynomials \(p_n(x) - q^{a/2+1/4}, - q^{a/2+3/4}, - q^{a/2}c, q^{a/2}d\) are uniquely determined by orthogonality relation \((6.4)\). From \((6.7)\) and uniqueness, we obtain kernel function \(K(\cos \phi_1, \cos \phi_2)\) and it has the following connection relation:

\[
\frac{A_n C_n^2}{B_n} p_n(\cos \phi_2) - q^{1/4}, - q^{3/4}, c, d) = \int_0^\pi K(\cos \phi_1, \cos \phi_2)p_n(\cos \phi_1| - q^{1/4}, - q^{3/4}, c, d)d\phi_1,
\]

where the Kernel function \(K(\cos \phi_1, \cos \phi_2)\) is

\[
K(\cos \phi_1, \cos \phi_2) = \frac{w_H(\cos \phi_1|q)}{(-q^{1/4}e^{i\phi_1}, -q^{1/4}e^{-i\phi_1}; q^{1/2})_{\infty}} \frac{w_H(\cos \phi_2|q)}{(-q^{1/4}e^{i\phi_2}, -q^{1/4}e^{-i\phi_2}; q^{1/2})_{\infty}} \\
\times \frac{\sin \phi_1 \sin \phi_2}{w(\cos \phi_2; - q^{1/4}, - q^{3/4}, c, d)} \times \frac{w_0(\cos \theta|q)}{h(\cos \phi_1; q^{a/2}e^{i\theta}, q^{a/2}e^{-i\theta})h(\cos \phi_2; q^{a/2}e^{i\theta}, q^{a/2}e^{-i\theta})} \, d\theta.
\]

The connection relation \((6.9)\) implies the bilinear formula as follows

\[
\frac{K(\cos \phi_1, \cos \phi_2)}{w(\cos \phi_1| - q^{1/4}, - q^{3/4}, c, d)} = \sum_{n=0}^{\infty} A_n \left(\frac{C_n}{B_n}\right)^2 p_n(\cos \phi_1| - q^{1/4}, - q^{3/4}, c, d) \times p_n(\cos \phi_2| - q^{1/4}, - q^{3/4}, c, d).
\]

\[
(6.11)
\]

Clearly, the left-side of \((6.11)\) is continuous, square integral and symmetric kernel.

## 7 Approximation Operators

We let

\[
e_j(x) = x^j.
\]

Our first result is the following expansion

\[
\frac{1}{(-q^{1/4}e^{i\theta}, -q^{1/4}e^{-i\theta}; q^{1/2})_{\infty}} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n/2}}{(q; q)_n(q; q)_{\infty}} H_n(\zeta|q) H_n(\cos \theta|q),
\]

where \(\zeta = (q^{1/4} + q^{-1/4})/2\).
Proof of (7.2). Let the left-hand side of (7.2) be $\sum_{n=0}^{\infty} c_n H_n(x|q)$. Then

$$c_n(q; q)_n = \int_0^\pi w_H(\cos \theta|q) H_n(\cos \theta|q) \sin \theta d\theta.$$  

In view of the generating function (1.3) we see that

$$\sum_{n=0}^{\infty} c_n t^n = \int_0^\pi \frac{w_H(\cos \theta|q) \sin \theta d\theta}{(-q^{1/4}e^{i\theta}, -q^{1/4}e^{-i\theta}; q^{1/2})_\infty (te^{i\theta}, te^{-i\theta}; q)_\infty}.$$  

The above integral is a special Askey–Wilson integral. Indeed

$$\int_0^\pi \frac{w_H(\cos \theta|q) \sin \theta d\theta}{(-q^{1/4}e^{i\theta}, -q^{1/4}e^{-i\theta}; q^{1/2})_\infty (te^{i\theta}, te^{-i\theta}; q)_\infty} = \frac{(q; q)_\infty}{2\pi} \int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty d\theta}{(-q^{1/4}e^{i\theta}, -q^{1/4}e^{-i\theta}; q^{1/2})_\infty} = \frac{1}{(q; q)_\infty} \sum_{n=0}^{\infty} \frac{(-t\sqrt{q})^n}{(q; q)_n} H_n(\zeta|q),$$

where $\zeta = (q^{1/4} + q^{-1/4})/2$. Therefore

$$\int_0^\pi \frac{w_H(\cos \theta|q) H_n(\cos \theta|q) \sin \theta d\theta}{(-q^{1/4}e^{i\theta}, -q^{1/4}e^{-i\theta}; q^{1/2})_\infty} = (-1)^n \frac{q^{n/2}}{(q; q)_\infty} H_n(\zeta|q),$$

which implies (7.2) and the proof is complete. 

Therefore the action of $T_0$ on the constant function 1 is

$$T_0 e_0(x) = \frac{(1 - q)^a}{2^a q^{a/4}} (-q^{1/4}e^{i\theta}, -q^{1/4}e^{-i\theta}; q^{1/2})_\infty \int_0^\pi \sum_{n=0}^{\infty} \frac{q^{n/2}(-1)^n}{(q; q)_n (q; q)_\infty} H_n(\zeta|q) H_n(\cos \phi|q) \times \sum_{m=0}^{\infty} H_m(\cos \theta|q) H_m(\cos \phi|q) \frac{q^{am/2}}{(q; q)_m} w_H(\cos \phi) \sin \phi d\phi$$

$$= \frac{(1 - q)^a}{2^a q^{a/4}(q; q)_\infty} (-q^{1/4}e^{i\theta}, -q^{1/4}e^{-i\theta}; q^{1/2})_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(a+1)/2}}{(q; q)_n} H_n(\cos \theta|q) H_n(\zeta|q).$$

Therefore we have proved that

$$T_0 e_0(\cos \theta) = \frac{(1 - q)^a (q^{a+1}; q)_\infty}{2^a q^{a/4}(q; q)_\infty} \int_{(-q^{(a+1)/2}e^{i\theta}, -q^{(a+1)/2}e^{-i\theta}; q^{1/2})_\infty} \frac{(-q^{1/4}e^{i\theta}, -q^{1/4}e^{-i\theta}; q^{1/2})_\infty}{(-q^{(a+1)/2}e^{i\theta}, -q^{(a+1)/2}e^{-i\theta}, q^{1/2})_\infty}.$$  

Carlitz [10] proved a bilinear generating function which is equivalent to

$$\int_{(te^{i(\theta+\phi)}, te^{i(\theta-\phi)}; q)_\infty} \frac{\sum_{m=0}^{\infty} H_n(\cos \theta|q) H_{n+m}(\cos \phi|q) \frac{t^m}{(q; q)_n}}{t^{1/4}e^{i\theta}, t^{1/4}e^{-i\theta}; q^{1/2}} \sum_{j=0}^{m} \frac{(t e^{i(\theta+\phi)}, t e^{i(\theta-\phi)}; q)_\infty}{(t^2; q)_j} e^{i\phi(m-2j)}.$$
The special case $m = 1$ is

$$
\sum_{n=0}^{\infty} H_n(\cos \theta | q)H_{n+1}(\cos \phi | q) \frac{t^n}{(q;q)_n} = \frac{2(\cos \phi - t \cos \theta) (qt^2;q)_\infty}{(te^{i(\theta+\phi)}, te^{i(\theta-\phi)}, te^{-i(\theta+\phi)}, te^{-i(\theta-\phi)}; q)_\infty}.
$$

(7.5)

and can be directly proved by applying $D_q$ to the Poisson kernel (2.2) and making use of the lowering relation

$$
D_q H_n(x | q) = 2 (1 - q^n) q^{(1-n)/2} H_{n-1}(x | q),
$$

(7.6)

\cite{15} (13.1.21). Ismail and Simeonov \cite{18} discuss an operational approach to the approach to (7.5).

Ismail and Stanton \cite{20} gave an alternate representation of the right-hand side of (7.4). Their representation is

$$
\sum_{n=0}^{\infty} H_n(\cos \theta | q)H_{n+m}(\cos \phi | q) \frac{t^n}{(q;q)_n} = \frac{(t^2 q^m; q)_\infty}{(te^{i(\theta+\phi)}, te^{i(\theta-\phi)}, te^{i(\phi+\theta)}, te^{-i(\phi-\theta)}; q)_\infty}
\times \sum_{k=0}^{m} \binom{m}{k} q^{(\phi-\theta)} (t e^{i(\phi-\theta)}; q)_k (t e^{i(\phi-\theta)}; q)_{m-k} e^{i(2k-2)\phi}.
$$

(7.7)

This representation shows that the $k$ sum is polynomial in $t$ of degree $m$ while the finite sum in Carlitz’s formula is a polynomial of degree at most $2m$. There is actually a huge amount of cancelations in Carlitz’s formula.

**Theorem 7.1.** The action of $T_a$ on $e_0$ and $e_1$ is given by

$$
T_a e_0(\cos \theta) = \frac{(1 - q)^a(q^{a+1}; q)_\infty}{2^a q^{a/2}(q; q)_\infty} \frac{(-q^{1/4} e^{i\theta}, -q^{1/4} e^{-i\theta}; q^{1/2})_{\infty}}{(-q^{(a+1)/2}/2 e^{i\theta}, -q^{(a+1)/2}/2 e^{-i\theta}; q^{1/2})_{\infty}}
$$

and

$$
T_a e_1(x) = \frac{(1 - q)^a(q^{a+2}; q)_\infty}{2^a q^{a/4}(q; q)_\infty} [x q^{a/2}(1 - q) + \zeta q^{a/2}(q^a - 1)]
\times \frac{(-q^{1/4} e^{i\theta}, -q^{1/4} e^{-i\theta}; q^{1/2})_{\infty}}{(-q^{(a+1)/2}/2 e^{i\theta}, -q^{(a+1)/2}/2 e^{-i\theta}; q^{1/2})_{\infty}},
$$

(7.8)

(7.9)

respectively.

**Proof.** Equation (7.8) is a restatement of (7.3). It is clear from the three term recurrence
We apply (7.5) and conclude that

\[
2 \frac{2^n q^{a/4}}{(1 - q)^a} T_n e_1(\cos \theta) \left( q^{a/4} e^{i\theta}, -q^{1/4} e^{-i\theta}; q^{1/2} \right)_\infty
\]

\[
= \int_0^\pi 2 \cos \phi \sum_{n=0}^\infty \frac{(-1)^n q^{n/2}}{(q; q)_n(q; q)_\infty} H_n(\zeta|q) H_n(\cos \phi|q) w_H(\cos \phi|q)
\times \sum_{m=0}^\infty H_m(\cos \theta|q) H_m(\cos \phi|q) q^{mn/2} \sin \phi \, d\phi
\]

\[
= \sum_{m,n=0}^\infty q^{mn/2} H_m(\cos \theta|q) \frac{(-1)^n q^{n/2} H_n(\zeta|q)}{(q; q)_n(q; q)_m(q; q)_\infty}
\times \int_0^\pi (H_{n+1}(\cos \phi|q) + (1 - q^n) H_{n-1}(\cos \phi|q)) H_m(\cos \phi|q) w_H(\cos \phi|q) \sin \phi \, d\phi.
\]

We now use the orthogonality relation to simplify the last line to

\[
(q; q)_m [\delta_{m,n+1} + (1 - q^n) \delta_{m,n-1}].
\]

We apply (7.5) and conclude that

\[
2 \frac{2^n q^{a/4}}{(1 - q)^a} T_n e_1(\cos \theta) = \frac{2(q^{a+2}; q)_\infty}{(q; q)_\infty} [x q^{n/2} (1 - q) + \zeta q^{1/2} (q^a - 1)]
\times \frac{(-q^{1/4} e^{i\theta}, -q^{1/4} e^{-i\theta}; q^{1/2})_\infty}{(-q^{(a+1)/2} e^{i\theta}, -q^{(a+1)/2} e^{-i\theta}; q^{1/2})_\infty}.
\]

The next step is to find the rate of approximation of a continuous function by $T_a$. To carry this out we will need the following lemmas.

**Lemma 7.2.** The action of $T_a$ on $e_2$ is given by

\[
(T_a e_2)(x) = \frac{(1 - q)^a (q^{a+3}; q)_\infty}{2^n q^{n/4} (q; q)_\infty} \frac{(-q^{1/4} e^{i\theta}, -q^{1/4} e^{-i\theta}; q^{1/2})_\infty}{(-q^{(a+1)/2} e^{i\theta}, -q^{(a+1)/2} e^{-i\theta}; q^{1/2})_\infty}
\]

\[
\{ q^a (1 - q) (1 - q^2)^2 x^2 + \frac{1}{2} (q^a + q) (q^{a/2+1/4} + q^{a/2+/3+4} + q^{a/2+5/4} + q^{a/2+7/4})
\]

\[
-(1 + q) (q^{a/2+1/4} + q^{a/2+3/4} + q^{a/2+5/4} + q^{a/2+7/4}) |x
\]

\[
+ \frac{1}{4} [(1 + q) (q^{1/2} + q + q^a + q^{a+2} + q^{2a+1} + q^{2a+3/2} - 1 - q^{a+1/2} - q^{a+3/2} - q^{2a+2}) + 2 (1 - q^{a+1})(1 - q^{a+2}) - 2 (q^a + q^{a+3})] \}
\]

**Proof.** Note that (1.4) implies

\[
4x^2 H_n(x|q) = H_{n+2}(x|q) + (2 - q^n - q^{n+1}) H_n(x|q) + (1 - q^n)(1 - q^{n-1}) H_{n-2}(x|q),
\]

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hence, using (7.7) for special cases $m = 0$ and $m = 2$, we obtain

$$
4(q; q)_{\infty} \frac{2^a q^{a/4}}{(1 - q)^a} T_a e_2(\cos \theta) = q^a \sum_{n=0}^{\infty} (-1)^n \frac{q^n(a+1)/2}{(q; q)_{n}} H_n(\zeta q) H_{n+2}(x | q)
$$

$$
+ \sum_{n=0}^{\infty} (-1)^n \frac{q^n(a+1)/2}{(q; q)_{n}} [2 - q^n - q^{n+1}] H_n(\zeta q) H_n(x | q)
$$

Thus the above expression is

$$
\begin{align*}
&= 2(q^{a+1}; q)_{\infty} \frac{2^a q^{a/4}}{(1 - q)^a} \left( -q^{a/2+1/4} e^{i\theta}, -q^{a/2+1/4} e^{-i\theta}; q^{1/2} \right)_{\infty} \\
&- (1 + q)(q^{a+3}; q)_{\infty} \left( -q^{a/2+5/4} e^{i\theta}, -q^{a/2+5/4} e^{-i\theta}; q^{1/2} \right)_{\infty} \\
&+ \left( -q^{a/2+1/4} e^{i\theta}, -q^{a/2+1/4} e^{-i\theta}; q^{1/2} \right)_{\infty} [2(q^a + q^{a+3}) \cos 2\theta + 2(q^a + q)(q^{a/2+1/4} + q^{a/2+3/4} + q^{a/2+5/4} + q^{a/2+7/4}) \cos \theta + (1 + q)(q^{1/2} + q + q^a + q^{a+2} + q^{2a+1} + q^{2a+3/2})]. \end{align*}
$$

\[ \square \]

**Lemma 7.3.** We have

$$
T_a((x - y)^2) = O(a),
$$

as $a \to 0$ uniformly in a neighborhood of $a = 0$.

**Proof.** Let $x = \cos \theta$ and $y = \cos \phi$, using (7.8), (7.9) and (7.10), we derive the following formula

$$
x^2 T_a e_0 - 2x T_a e_1 + T_a e_2 = \frac{(1 - q)^a (q^{a+3}; q)_{\infty}}{2^a q^{a/4} (q; q)_{\infty}} \left( -q^{1/4} e^{i\theta}, -q^{1/4} e^{-i\theta}; q^{1/2} \right)_{\infty} \left\{ \begin{array}{c}
((1 - q^{a+1})(1 - q^{a+2}) - 2q^{a/2}(1 - q^{a+2})(1 - q) + q^a (1 - q)(1 - q^2)) x^2 \\
+ \frac{1}{2}(q^a + q)(q^{a/2+1/4} + q^{a/2+3/4} + q^{a/2+5/4} + q^{a/2+7/4}) \\
- \frac{1}{2}(1 + q)(q^{a/2+1/4} + q^{a/2+3/4} + q^{a/2+5/4} + q^{a/2+7/4}) \\
- (1 - q^{a+2})(q^{3/4} + q^{1/4})(q^a - 1) x + \\
\frac{1}{4}(1 + q)(q^{1/2} + q + q^a + q^{a+2} + q^{2a+1} + q^{2a+3/2} - 1 \\
- q^{a+1/2} - q^{a+3/2} - q^{2a+2}) + 2(1 - q^{a+1})(1 - q^{a+2}) - 2(q^a + q^{a+3}).
\end{array} \right\} \tag{7.12}
$$
In fact, the quantity in the square bracket is
\[
[(1 - q^{a+1})(1 - q^{a+2}) - 2q^{a/2}(1 - q^{a+2})(1 - q) + q^{a}(1 - q)(1 - q^{2})]x^2 \\
+ \frac{1}{2}q^{a+2 + 1/4}(1 + q^{1/2})(1 + q)(q^{a} + q - 1 - q^{a+1}) \\
- (1 - q^{a+2})(q^{3/4} + q^{1/4})(q^{a} - 1)]x \\
+ \frac{1}{4}[(1 + q)(q^{1/2} + q)(1 + q^{2a+1/2}) - (1 + q^{a+1/2})(1 + q^{a+3/2}) - q^{a}(1 + q^{2})] \\
+ 2(1 + q^{2a+3}) = x^2(q^2 - q)\log q - \frac{ax}{2}(1 - q)(1 + q)(q^{1/4} + q^{3/4})\log q \\
+ \frac{1}{4}[(1 + q)(-1 - q^{1/2} + 2q + q^{3/2} + q^{2}) - 4q^{2}\log q + O(a^2)].
\]

The next theorem gives the order of the error term in the approximation of a function by $T_a$.

**Theorem 7.4.** Assume that $f$ is twice continuously differentiable. Then

$$
\|T_a f - f\| = O(a).
$$

**Proof.** We write $x = \cos \theta, y = \cos \phi$. Write $f(y) = f(x) + (y - x)f'(x) + (y - x)^2f''(u)$. Using (7.13)–(7.14) we see that for fixed $x$

\[
T_a f(x) + (y - x)f'(x) = [f(x) - xf'(x)]T_a e_0 + f'(x)T_a e_1 \\
= \frac{(1 - q)^a(q^{a+2}; q)_{\infty}}{2^a q^{a/4}(q; q)_{\infty}} \frac{(-q^{1/4}e^{i\theta}, -q^{1/4}e^{-i\theta}; q^{1/2})_{\infty}}{(-q^{(a+1)/2}e^{i\theta}, -q^{(a+1)/2}e^{-i\theta}; q^{1/2})_{\infty}} \\
\times \left[f(x)(1 - q^{a+1}) + (1 - q^{a/2})f'(x)\{x(q^{1+a/2} - 1) - \zeta q^{1/2}(1 + q^{a/2})\}\right].
\]

The quantity in the square bracket is clearly $O(a)$. Assume that $\|f''\| = M$. Then

\[
\|(T_a f)(x) - f(x)\| \leq O(a) + M\|T_a(e_2) - 2xT_a e_1 + e_2(x)T_a(e_0)\| = O(a),
\]

by Lemma 7.3 \qed

\section{8 \ T_a are Contraction Maps}

In this section we prove that the operators $T_a$ are contractions maps.

**Theorem 8.1.** For any $a$ satisfies

$$
(1 - q) < 1,
$$

there exists a positive number $c(q) > 0$ such that for all $a > c(q)$ the operators $T_a$ are contraction operators on $C[-1, 1]$. The condition (8.1) is also necessary if $T_a$ is a contraction operator for any $a > 0$. 

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Proof. It is clear from (3.2) and the fact that the norm is $\geq \lambda_0$ that the condition (8.1) is necessary. It remain to show that (8.1) is sufficient. Note that $T_\alpha$ are positive linear operators, hence we only need to show that $\|T_\alpha e_0\| < 1$, that is $|(T_\alpha e_0)(x)| < 1$ for all $x \in [-1, 1]$ and all $\alpha$ large enough. Observe that

$$\frac{1 + 2xt + t^2}{1 + 2xu + u^2}$$

increases with $x$ for $x \in [-1, 1], 0 \leq u < t \leq 1$. In view of (7.3), the above observation reduces the problem, $\|T_\alpha e_0\| < 1$, to showing that

$$\frac{(1 - q)^a(q^{a+1}; q)_\infty}{2^a q^{a/4}(q; q)_\infty} \frac{(-q^{1/4}; q^{1/2})_\infty^2}{(-q^{(a+1)/2}; q^{1/2})_\infty^2} < 1,$$

for $a > c(q)$, for some $c(q) > 0$. We set

$$h(a) = a \log \left(\frac{1 - q}{2q^{1/4}}\right) + \sum_{n=0}^\infty \log(1 - q^{a+n+1}) - 2 \sum_{n=0}^\infty \log \left(\frac{1 + q^{(a+n+1)/2}}{1 + q^{(a+n+1)/2}}\right).$$

Now

$$h'(a) = \log \left(\frac{1 - q}{2q^{1/4}}\right) - \log q \sum_{n=0}^\infty q^{a+n+1} - \log q \sum_{n=0}^\infty \frac{q^{a+n+1}}{1 + q^{a+n+1}}.$$

Therefore $h''(a) < 0$, so that $h'(a)$ is decreasing. But $h'(a) < 0$ for sufficiently large $a$ and $\lim_{a \to +\infty} h'(a) = +\infty$. Hence there is a $c > 0$ such that $h'(a) < 0$ for all $a > c$. Therefore $h$ is strictly decreasing and the results follows.

\[ \Box \]

9 Another Semigroup

In this section we study the semigroup

$$\int_0^\pi |(q^a; q)_\infty w_H(\cos \phi | q) \frac{f(\cos \phi)}{sin \phi} d\phi | \frac{d\phi}{q^{a/2} e^{i(\phi + \phi)}; q^{a/2} e^{-i(\phi + \phi)}; q^{a/2} e^{i(\phi - \phi)}; q)_\infty}. $$

It is clear from the definition and the orthogonality of the $q$-Hermite polynomial that

$$F_a H_n(x|q) = q^{an/2} H_n(x|q).$$

It must be noted that if we took (9.2) as our starting point, that is we define $F_a$ as a multiplier operator on $L_2[-1, 1, w_H]$ then computed the kernel of $F_a$ as an integral operator we would
have naturally found the definition (9.1). This is similar to the way Weyl originally defined the Weyl fractional integrals, see [34].

The infinitesimal generator can be defined in a way similar to the way we defined the infinitesimal generators for \( T_a \) and \( S_a \).

We now consider \( F_a \) as approximation operators.

**Theorem 9.1.** The operators \( F_a \) have the properties

(a) \((F_a e_0)(x) = 1\),  
(b) \((F_a e_1)(x) = q^{a/2} x\),  
(c) \((F_a e_2)(x) = q^a x^2 + \frac{(1 - q)}{4} (1 - q^a)\).

It is clear from (9.2) that \((F_a H_m(x|q))(x) \rightarrow H_m(x|q), \) uniformly on \([-1,1]\) as \( a \rightarrow 0 \), for \( m = 0, 1, 2 \). Moreover it is clear \( T_a \) is self-adjoint on \( L^2[-1,1,w_H] \).

**Theorem 9.2.** The following statements are true

(a) The operators \( F_a, a \geq 0 \) form a semigroup of positive linear operators defined on \( C[-1,1] \).

(b) \( \lim_{a \rightarrow 0} (F_a f)(x) = f(x), \) for \( f \in C[-1,1] \).

(c) \( \|F_a\| = 1, a \geq 0 \), where the norm is for \( F_a \) acting on \( C[-1,1] \).

(d) The only eigenvalues of \( F_a, a > 0 \), are \( q^{ma/2}, m = 0, 1, \ldots \) with \( H_n \) are the corresponding eigenfunctions.

(e) For every \( a > 0 \), \( F_a \) is a compact operator, hence is not invertible.

**Proof.** The completeness of the \( q \)-Hermite polynomials and the fact that the kernel is the Poisson kernel establish the semigroup property (a). In view of (9.1) we can apply Korovkin’s theorem to establish the \( F_a \) tends to the identity operator as \( a \rightarrow 0^+ \), so (b) holds. Part (c) follows from the positivity of the kernel of \( F_a \), and the fact that \((F_a e_0)(x) = 1\). The proof of parts (d) – (e) are similar to the case of \( T_a \) and are omitted but they use (9.2).

These operators appeared in Suslov’s work [32] where he stated the semigroup property and \( F_a \rightarrow I \) but he assumed that the functions are analytic in the unit disc. Parts (c)-(e) were not mentioned by Suslov.

**Theorem 9.3.** If \( a > 1 \), then the operators \( F_a \) satisfy the commutation relation

\[
(9.3) \quad D_q F_a = q^{a/2} F_a D_q
\]

with

\[
(9.4) \quad F_a D_q f = \frac{4}{(1 - q)(1 - q^{a-1})} \left( F_{a-1} \circ M - q^{(a-1)/2} M \circ F_{a-1} \right) f,
\]

where \((M f)(x) = x \cdot f(x)\). Moreover, \((D_q F_a f)(x) \geq 0\) if \( D_q f \geq 0 \).

The proof uses the raising relation [15]

\[
(9.5) \quad D_q [w_x(x|q) H_n(x|q)] = -\frac{2q^{-n/2}}{1 - q} w_x(x|q) H_{n+1}(x|q)
\]
Proof of Theorem 9.3 It is clear that

\[(D_qF_a f)(x) = \int_{-1}^{1} f(y) w_H(y|q) D_{q,x} \left[ \sum_{n=0}^{\infty} H_n(x|q) H_n(y|q) \frac{q^{an/2}}{(q; q)_n} \right] dy \]

\[= \frac{2}{1-q} \int_{-1}^{1} w_H(y|q) \sum_{n=1}^{\infty} H_{n-1}(x|q) H_n(y|q) \frac{q^{an/2}}{(q; q)_{n-1}} q^{(1-n)/2} f(y) dy \]

\[= \frac{2q^{a/2}}{1-q} \int_{-1}^{1} \sum_{n=0}^{\infty} H_n(x|q) \frac{q^{an/2}}{(q; q)_n} q^{-n/2} w_H(y|q) H_{n+1}(y|q) f(y) dy \]

\[= -q^{a/2} \int_{-1}^{1} f(y) D_{q,y} \left[ w_H(y|q) \sum_{n=0}^{\infty} H_n(x|q) H_n(y|q) \frac{q^{an/2}}{(q; q)_n} \right] dy. \]

We then apply the q-integration by parts formula of [8], see [15, Theorem 16.1], and obtain

\[(D_qF_a f)(x) = q^{a/2} (F_a(D_q f))(x),\]

which proves (9.3).

By direct computation, since with \(y = \cos \phi\), we have

\[ (F_a D_q f)(\cos \theta) = (q^a; q)_\infty \]

\[\times \int_0^\pi (\cos \phi) D_{q,y} \left[ \frac{w_H(\cos \phi|q)}{(q^{a/2} e^{i(\theta+\phi)}, q^{a/2} e^{i(\theta-\phi)}, q^{a/2} e^{-i(\theta+\phi)}, q^{a/2} e^{-i(\theta-\phi)}; q)_\infty} \right] dy \]

\[= \frac{2}{1-q} (q^a; q)_\infty \]

\[\times \int_0^\pi (2y - 2xq^{(a-1)/2}) w_H(y|q) f(y) dy \]

This proves (9.4). Finally (9.3) shows that if \((D_q f)(x)(D_q F_a f)(x) \geq 0\) hence the last assertion follow.

We note that the last statement in Theorem 9.3 namely \((D_q F_a f)(x) \geq 0\) if \(D_q f \geq 0\), is an analogue of a monotonicity preserving property of \(F_a\).

We now define the infinitesimal generator of the semigroup \(F_a\). The continuous \(q\)-Hermite polynomials form a basis for \(L_2[-1, 1, w_H]\), hence it suffices to define the infinitesimal generator \(J\) on the basis. Part (d) of Theorem 9.2 suggests that

\[ J H_m(\cdot|q) = \lim_{a \to 0^+} \frac{q^{ma/2} - 1}{a} H_m(x|q) = \frac{m}{2} \log q H_m(x|q). \]

Thus

\[(9.6) \quad \text{If } f = \sum_{n=0}^{\infty} f_n H_n(\cdot|q), \text{ then } (Jf)(x) := \frac{1}{2} \log q \sum_{n=0}^{\infty} n f_n H_n(x|q). \]

Thus \(J\) is unbounded and is only densely defined on both \(L_2[-1, 1, w_H]\) and \(C[-1, 1]\).
Theorem 9.4. Assume that $f$ is twice continuously differentiable. Then
\[ \| F_a f - f \| = O(a), \]
as $a \to 0^+$. 

The proof is identical to the proof of Theorem 7.4 and uses Theorem 9.1.

We now provide an inversion formula for $F_a$ as operators defined on $L_2[-1, 1, w_H]$. 

Theorem 9.5. Let $a \geq 0$ and $g = \sum_{n=0}^{\infty} q_n H_n(x | q)$. Then $g = F_a f$ with $f \in L_2[-1, 1, w_H]$ if and only if $g = \sum_{n=0}^{\infty} g_n q^{-na/2} H_n(x | q)$. Furthermore, $f = \sum_{n=0}^{\infty} g_n q^{-na/2} H_n(x | q)$.

The theorem follows from Parseval’s theorem and the Riesz-Fischer theorem.

Another inversion formula follows from Theorem 5.5. Recall the notation, from (3.3),
\[ g(\cos \theta) = (-q^{1/4}e^{i\theta}, -q^{1/4}e^{-i\theta}; q^{1/2})_\infty. \]
We then use the fact that
\[ (1 - q)^a \frac{2^{aq^a/4}}{g(q^{1/4}f/g)} = T_a f, \]
and apply part (a) of Theorem 5.5.

We now revisit (4.10). We look for an operator $B_q$ such that $B_q F_a$ is a constant multiple of $F_a - 1$. This will then identify a multiple of $F_a$ as a fractional inverse operator to $B_q$. We found that
\[ (B_q f)(x) = 2^{q^{1/2}/2} \frac{\tilde{f}(q^{1/2}z) - z^2 \tilde{f}(q^{-1/2}z)}{(q - 1)(z^2 - 1)}, \]
will do the job. It is well defined because it is symmetric in $z$ and $1/z$. It is easy to see that
\[ B_q F_a = \frac{2^{q^{1/4}}}{1 - q} F_{a-1}. \]
Therefore we define the new semigroup \{ $G_a : a > 0$ \} by
\[ G_a = \frac{(1 - q)^a}{2^{aq^a/4}} F_{a-1}. \]
Therefore $G_a$ is a semigroup, and is a fractional analogue of an inverse to $B_q$, in the sense that $B_q G_a = G_a - 1$.

We note that $B_q$ also has the representation
\[ (B_q f)(x) = \frac{1}{g(x)} (D_q fg)(x), \]
has the property. This leads us to modify the definition of $F_a$ to Now (4.10) may be stated as
\[ \int_{-1}^{1} E_q(x; t)(G_a f)(x) w_H(x | q) \, dx \]
\[ = \frac{(1 - q)^a}{2^{aq^a/4} \infty} \int_{-1}^{1} E_q(y, tq^{1/2}) f(y) w_H(y | q) dy. \]
This is like a $q$-analogue of a multiplier for the transform whose kernel is $E_q(x; t)w_H(x | q)$. 

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10 A q-Gauss–Weierstrass Transform

The Gauss–Weierstrass transform $W(\lambda)$ is defined by

\[
(W(\lambda)f)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{(t-x)^2/2\lambda} f(t)dt,
\]

(10.1) It is clear that we can rescale to make $\lambda = 1$.

We now define the $q$-analogue of this transform by

\[
(W_q f)(t) = (qt^2; q)_{\infty} \int_{-1}^{1} E_q(x; t)w_H(x|q)f(x)dx
\]

(10.2) The motivation for this is that $w_H$ with rescaling tends to $e^{-x^2/2}$, the weight function for Hermite polynomials, as $q \rightarrow 1^-$. Moreover $E_q(x; t)$ tend to $e^{xt}$. Furthermore $(qt^2; q)_{\infty}$ tends to $e^{-t^2/2}$. There are several $q$-analogues of the exponential function, [11], and they all appear here.

Ismail and Zhang [22] proved that

\[
(qt^2; q)_{\infty} E_q(x; t) = \sum_{n=0}^{\infty} \frac{q^{n^2/4}t^n}{(q; q)_n} H_n(x|q).
\]

(10.3) This shows that $(W_qe_0)(t) = 1$. Note that (10.3) is the analogue of the familiar generating function

\[
\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = \exp(2xt - t^2) = e^{-t^2}e^{2xt}.
\]

We now invert this transform. For $f \in L_2[-1, 1, w_h]$, let $f = \sum_{n=0}^{\infty} f_n H_n(x|q)$. Then Parseval’s formula implies

\[
(W_qf)(t) = \sum_{n=0}^{\infty} f_nq^{n^2/4}t^n.
\]

(10.4) Since $\{f_n\} \in \ell^2$ then $(W_qf)(t)$ is entire and of order zero. Moreover

\[
f_nq^{n^2/4} = \frac{1}{n!} \left. \frac{d^n}{dx^n} (W_qf)(t) \right|_{t=0}.
\]

(10.5) This proves the following theorem.

**Theorem 10.1.** An entire function $g$ is a $q$-Gauss–Weierstrass transform of a a function in $L_2[-1, 1, w_h]$ if and only if $g(x) = \sum_{n=0}^{\infty} g_n x^n$, and $q^{-n^2/4}g_n \in \ell^2$. When this condition is satisfied then $f(x) = \sum_{n=0}^{\infty} q^{-n^2/4}g_n H_n(x|q)$.

We note that an inversion of the Gauss–Weierstrass transform (10.1) also involve an operational representation, see [12].
11 Dual Integral Equations

If we let \( x = \cos \theta \) and \( y = \cos \phi \), then we consider the following dual integral equations

\[
\begin{align*}
\int_0^\pi \int_0^\pi & \left\{ \begin{array}{l}
\left( -q^{1/4}e^{i\theta} - q^{1/4}e^{-i\theta} \right) w_H(\cos \phi q) \psi(\cos \phi) \sin \phi d\phi = F(x) \quad -1 < x < 0, \\
\left( -q^{1/4}e^{i\theta} - q^{1/4}e^{-i\theta} \right) w_H(\cos \phi q) \psi(\cos \phi) \sin \phi d\phi = G(x) \quad 0 < x < 1.
\end{array} \right.
\end{align*}
\]

where \( a > 0 \) and \( b > 0 \). In terms of the \( T_a \) operator, the above equations are

\[
\begin{align*}
T_a \psi &= \left( \frac{1-q}{2q^{1/4}} \right)^a (q^a; q) \infty F \quad -1 < x < 0, \\
T_b \psi &= \left( \frac{1-q}{2q^{1/4}} \right)^b (q^b; q) \infty G \quad 0 < x < 1.
\end{align*}
\]

Next, we follow a nice technique due to Noble in [26]. First we define two functions \( f(x) \) and \( g(x) \)

\[
T_a \psi = \left( \frac{1-q}{2q^{1/4}} \right)^a (q^a; q) \infty f, \quad T_b \psi = \left( \frac{1-q}{2q^{1/4}} \right)^b (q^b; q) \infty g,
\]

then it is easily to know \( f(x) = F(x) \) on \((-1, 0)\) and \( g(x) = G(x) \) on \((0, 1)\). To simplify the writing we shall use the following notation

\[
I_1 = (-1, 0) \quad \text{and} \quad I_2 = (0, 1),
\]

\[
f_1 = F \quad \text{in} \ I_1, \quad f_1 = 0 \quad \text{in} \ I_2, \quad f_2 = f \quad \text{in} \ I_2, \\
g_1 = g \quad \text{in} \ I_1, \quad g_1 = 0 \quad \text{in} \ I_2, \quad g_2 = G \quad \text{in} \ I_2.
\]

Also, \( f_1(x) = F(x) \) on \( I_1 \) and \( g_2(x) = G(x) \) on \( I_2 \). We discuss three cases according to \( a > (\leq) b \), or \( a = b \).

**Case (a):** Assume \( a > b \), then part (a) of Theorem 3.1 yields

\[
\left( \frac{1-q}{2q^{1/4}} \right)^b (q^b; q) \infty T_{a-b} g = \left( \frac{1-q}{2q^{1/4}} \right)^a (q^a; q) \infty f.
\]

The \( f_1(x) \) and \( g_2(x) \) are known, if we evaluate on interval \( I_1 \), then

\[
\left( \frac{1-q}{2q^{1/4}} \right)^b (q^b; q) \infty T_{a-b} g_1 = \left( \frac{1-q}{2q^{1/4}} \right)^a (q^a; q) \infty f_1 - \left( \frac{1-q}{2q^{1/4}} \right)^b (q^b; q) \infty T_{a-b} g_2,
\]

which leads to a integral equations

\[
\int_{-1}^0 g_1(y)K_1(x, y)dy = H(x) \quad \text{on} \ (-1, 0),
\]

where

\[
K_1(x, y) = \left( \frac{1-q}{2q^{1/4}} \right)^a (q^a; q) \infty \times (q^{a-b}; q) \infty
\]

\[
\frac{(-q^{1/4}e^{i\theta}, -q^{1/4}e^{-i\theta}; q^{1/2}) \infty w_H(\cos \phi q)}{(-q^{1/4}e^{i\phi}, -q^{1/4}e^{-i\phi}; q^{1/2}) \infty h(\cos \phi q; q^{(a-b)/2}e^{i\theta}, q^{(a-b)/2}e^{-i\theta})}.
\]

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Next we evaluate the integral on the interval $I$.

\begin{equation}
\psi(11.8) = \left(\frac{1}{2q^{1/4}}\right)^{q^a}q^c f_1 - \left(\frac{1}{2q^{1/4}}\right)^{q^b}q^c f_2.
\end{equation}

If we know $g_1(x)$, then we integrate over $I_2$ and find that

\begin{equation}
\psi(11.9) = \left(\frac{1}{2q^{1/4}}\right)^{q^b}q^c f_1 = \left(\frac{1}{2q^{1/4}}\right)^{q^a}q^c f_2.
\end{equation}

We are now able to evaluate $\psi(x)$ by Theorem 5.5 (a) and (11.3).

**Case (b):** If $a = b$, Theorem 5.5 (a), (11.2) and (11.3) lead to

\begin{equation}
\psi(11.10) = \begin{cases}
\left(\frac{1}{2q^{1/4}}\right)^{q^a}q^c f_1 & -1 < x < 0, \\
\left(\frac{1}{2q^{1/4}}\right)^{q^a}q^c g_2 & 0 < x < 1.
\end{cases}
\end{equation}

Setting

\begin{equation}
h(11.11) = \begin{cases}
\left(\frac{1}{2q^{1/4}}\right)^{q^a}q^c f_1 & -1 < x < 0, \\
\left(\frac{1}{2q^{1/4}}\right)^{q^a}q^c g_2 & 0 < x < 1.
\end{cases}
\end{equation}

Then $\psi(x)$ can be written as

\begin{equation}
\psi(11.12) = \left(\frac{1}{2q^{1/4}}\right)^{q^a}q^c f_1 \begin{array}{c}
\prod_{k=1}^{q^a} q^{1-k} (q^{1-k};q)^c
\end{array}
\end{equation}

\begin{equation}
\times D_q[1] + 1 \int_0^\pi \frac{(-q^{1/4}e^{i\theta}, -q^{1/4}e^{-i\theta};q^{1/2})}{(-q^{1/4}e^{i\phi}, -q^{1/4}e^{-i\phi};q^{1/2})} w_H(\cos(\phi)q) h(\cos(\phi)) \sin(\phi) d\phi.
\end{equation}

**Case (c):** If $a < b$, then we can see that

\begin{equation}
\psi(11.13) = \left(\frac{1}{2q^{1/4}}\right)^{q^b}q^c g = \left(\frac{1}{2q^{1/4}}\right)^{q^a}q^c T_{b-a}f.
\end{equation}

Next we evaluate the integral on the interval $I_2$ and find that

\begin{equation}
\psi(11.14) = \left(\frac{1}{2q^{1/4}}\right)^{q^b}q^c T_{b-a}f_2 = \left(\frac{1}{2q^{1/4}}\right)^{q^b}q^c g_2 = \left(\frac{1}{2q^{1/4}}\right)^{q^a}q^c T_{b-a}f_1.
\end{equation}

It also leads to the integral equation

\begin{equation}
\psi(11.15) \int_0^1 f_2(y) K_2(x,y) dy = P(x) \text{ on } (0,1),
\end{equation}

where

\begin{equation}
K_2(x,y) = \frac{(-q^{1/4}e^{i\theta}, -q^{1/4}e^{-i\theta};q^{1/2})}{(-q^{1/4}e^{i\phi}, -q^{1/4}e^{-i\phi};q^{1/2})} w_H(\cos(\phi)q) h(\cos(\phi);q^{(b-a)/2}e^{i\theta}, q^{(b-a)/2}e^{-i\theta}).
\end{equation}
and

\[(11.17) \quad P(x) = \frac{1 - q}{2q^{1/4}} (q^{1/2}; q) \phi g_2 - \frac{1 - q}{2q^{1/4}} (q^{1/2}; q) T_{b-a} f_1.\]

We then evaluate the integral on the interval \( I_1 \), and conclude that

\[(11.18) \quad \frac{1 - q}{2q^{1/4}} (q^{1/2}; q) g_1 = \frac{1 - q}{2q^{1/4}} (q^{1/2}; q) T_{b-a} (f_1 + f_2).\]

This enables us to find \( \psi(x) \) by using Theorem \( 5.5(a) \) and \( (11.3) \).

If we modify Sneddon’s method \( (31) \), then we also can solve the special case when \( a - b \) is an integer. If \( a - b = 0 \), then it is case (b) that we discuss above.

**Case (d)** if \( a > b \), then define

\[(11.19) \quad m(x) = \begin{cases} \frac{1 - q}{2q^{1/4}} (q^{1/2}; q) D_{q}^{a-b} f_1 & -1 < x < 0, \\ \frac{1 - q}{2q^{1/4}} (q^{1/2}; q) g_2 & 0 < x < 1. \end{cases}\]

It is clear that \( T_{b} \psi(x) = m(x) \), and using Theorem \( 5.5(a) \), we obtain the following integral representation for \( \psi \),

\[(11.20) \quad \psi(x) = \frac{1 - q}{2q^{1/4}} (q^{1/2}; q) \phi \times D_{q}^{[b]+1} \int_{0}^{\pi} \frac{(-q^{1/4} e^{i\theta}, -q^{1/4} e^{-i\theta}; q^{1/2}) \infty w \phi q m(\cos \phi) \sin \phi \d \phi}{(-q^{1/4} e^{i\theta}, -q^{1/4} e^{-i\theta}; q^{1/2}) \infty h(\cos \phi; q^{1/2})/2 e^{i\theta}, q^{1/2})/2 e^{-i\theta})}.\]

**Case (e)** if \( a < b \), then define \( n \) by

\[(11.21) \quad n(x) = \begin{cases} \frac{1 - q}{2q^{1/4}} (q^{1/2}; q) f_1 & -1 < x < 0, \\ \frac{1 - q}{2q^{1/4}} (q^{1/2}; q) g_2 & 0 < x < 1. \end{cases}\]

Thus \( T_{a} \psi(x) = n(x) \), and using Theorem \( 5.5(a) \), we establish the integral representation

\[(11.22) \quad \psi(x) = \frac{1 - q}{2q^{1/4}} (q^{1/2}; q) \phi \times D_{q}^{[a]+1} \int_{0}^{\pi} \frac{(-q^{1/4} e^{i\theta}, -q^{1/4} e^{-i\theta}; q^{1/2}) \infty w \phi q n(\cos \phi) \sin \phi \d \phi}{(-q^{1/4} e^{i\theta}, -q^{1/4} e^{-i\theta}; q^{1/2}) \infty h(\cos \phi; q^{1/2})/2 e^{i\theta}, q^{1/2})/2 e^{-i\theta})}.\]

**Acknowledgments** The first author wishes to thank Zeev Ditzian of the University of Alberta for many enlightening discussions during the preparation of this paper.

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