The influence of dissipative effects on dynamic processes in a rotating electrically conductive liquid medium

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Abstract. The present study is concerned with dynamical processes in a rotating layer of electrically conducting incompressible liquid located in a magnetic field which is parallel to the normal vector to the boundary surfaces. We take into account not only the convective terms, but also the diffusion terms in the magnetic field induction equations. This problem, as well as the geophysical hydromagnetics problem, calls for the construction of approximate variants of the principal hydromagnetic equations and rigorous mathematical analysis of these approximate equations. Moreover, advancements in the above problems depend on the approximations to be introduced. To this end, by introducing characteristic scales of variation of the variables in the original equations and estimating the magnitude orders of the terms involved in the equations, one can single out the principal and secondary terms, simplify the equations, and build a model of the process under consideration. By introduction of auxiliary functions, the system of partial differential equations is reduced to one scalar equation. This suggests the conclusion about the analytical structure of magnetohydrodynamic characteristics. From the results obtained it follows that the magnetic field generation in an electrically conducting liquid stems from the instability characterized by the corresponding relations between the gravitation force, the Coriolis force, the magnetic force, and the peculiarities of the relief topography.

1. Introduction

Large-scale motion of electrically conducting fluid has been considered in a number of studies, of which the most well known are the papers [1, 2]. In all these studies, a model constructed in the fast rotation limit was considered; as a result, the inertia forces were neglected in the motion equation.

The papers [3, 4] have been concerned with large-scale motion of electrically conducting viscous liquid confined in a layer between two planes. Note that in [3, 4] no consideration of the inertia force was made and the viscous forces were involved on purpose.

In [5, 6], a mathematical model of dynamics of spatial large-scale motion was constructed in a rotating layer of perfect electrically conducting incompressible liquid of variable depth with due consideration of dissipative effects from the external magnetic field parallel to the layer axis of rotation. The corresponding system of partial differential equations has been reduced to a single equation. An analytical representation of the solution to the perturbation problem in a liquid layer was put forward, which enabled us to construct in an explicit form the solution describing the waves of both small and finite amplitudes in an infinite horizontally extended narrow linear channel.
2. Fundamental equations and boundary conditions

In what follows, we take into account the dissipative effects, namely, the effect of the magnetic field diffusion on its generation was examined in the case when the vector of the external magnetic field is parallel to the normal vector to the liquid layer boundary surface.

Consider the following boundary value problem of magnetic hydrodynamics:

\[
\frac{\partial \mathbf{b}}{\partial t} = \text{rot} \left[ \mathbf{v}, \mathbf{b} \right], \quad \frac{\partial \mathbf{v}}{\partial t} = \text{rot} \left[ \mathbf{b}, \mathbf{v} \right] + \frac{1}{\mu \sigma} \Delta \mathbf{b},
\]

\[
\text{div} \mathbf{b} = 0, \quad \text{div} \mathbf{v} = 0,
\]

\[p(x, y, -h_y) = p_0, \quad b(x, y, -h_y) = b_0, \quad v_z(x, y, -h_y, t) = -\frac{\partial h_y}{\partial t} - v_x \frac{\partial h_y}{\partial x} - v_y \frac{\partial h_y}{\partial y},\]

\[b_x(x, y, -h_y, t) = B_{n0}(x, y, t), \quad b_x \frac{\partial Z}{\partial x} + b_y \frac{\partial Z}{\partial y} + b_z = b_{n0}^{(e)}(x, y, t),\]

where \( \mathbf{b} \) is the magnetic field induction vector, \( \mathbf{v} \) is the liquid velocity in the frame rotating with rate \( \mathbf{\omega} \), \( p \) is the pressure, \( \rho \) is the density, \( b \) is the field magnetic induction vector, \( \mu \) is the magnetic permeability, \( \sigma \) is the electrical conductivity, \( p_0, b_0 \) are constants, \( B_{n0}, b_{n0}^{(e)}, Z \) are given functions, \( h_y \) is the unknown free layer surface.

From the condition that the normal magnetic field component is defined on the surface \( z = -h_y \), we get

\[b_x \frac{\partial h_y}{\partial x} + b_y \frac{\partial h_y}{\partial y} + b_z = b_{n0}(x, y, t), \quad z = -h_y(x, y, t),\]

where \( b_{n0} = B_{n0}(x, y, t) \sqrt{1 + \left( \frac{\partial h_y}{\partial x} \right)^2 + \left( \frac{\partial h_y}{\partial y} \right)^2} \).

After integration of the continuity and solenoidality equations of the magnetic field over the layer depth, the original boundary value problem for long-wave perturbations assumes the form

\[\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + 2 \omega v_y = 2 \omega v_y + g \frac{\partial h_y}{\partial x} + \frac{1}{\mu \rho} \left( b_x \frac{\partial b_x}{\partial x} + b_y \frac{\partial b_y}{\partial y} \right),\]

\[\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} = -2 \omega v_x + g \frac{\partial h_y}{\partial y} + \frac{1}{\mu \rho} \left( b_x \frac{\partial b_x}{\partial x} + b_y \frac{\partial b_y}{\partial y} \right),\]

\[\frac{\partial h_y}{\partial t} + \frac{\partial}{\partial x} [(h_y - Z)v_x] + \frac{\partial}{\partial y} [(h_y - Z)v_y] = 0,\]

\[\frac{\partial}{\partial x} [(h_y - Z)b_x] + \frac{\partial}{\partial y} [(h_y - Z)b_y] = b_{n0}(x, y, t) - b_{n0}^{(e)}(x, y, t),\]

\[\frac{\partial b_x}{\partial t} + v_x \frac{\partial b_x}{\partial x} + v_y \frac{\partial b_x}{\partial y} - b_x \frac{\partial v_x}{\partial x} - b_y \frac{\partial v_x}{\partial y} = \frac{1}{R_m} \Delta^2 b_x,\]

\[\frac{\partial b_y}{\partial t} + v_x \frac{\partial b_y}{\partial x} + v_y \frac{\partial b_y}{\partial y} - b_x \frac{\partial v_y}{\partial x} - b_y \frac{\partial v_y}{\partial y} = \frac{1}{R_m} \Delta^2 b_y,\]

where \( R_m \) is the magnetic Reynolds number.

Let us introduce the full depth function \( H(x, y, t) = H_y(x, y) + \eta(x, y, t) \), where \( H_y(x, y) \) is the thickness of the liquid layer at rest, \( \eta(x, y, t) \) is a small perturbation characterized by the inequality
\( \eta(x, y, t) \ll H_0 \). The solution of the corresponding system will be sought in the form \( \mathbf{v} = \mathbf{v}'(x, y, t) \), \( \mathbf{b} = \mathbf{b}_0 + \mathbf{b}'(x, y, t) \), where \( \mathbf{b}_0 \) describes some stationary uniform background.

Consider the functions \( \tilde{\eta}(x, y, t), \tilde{b}_x(x, y, t), \tilde{b}_y(x, y, t) \) defined by

\[
\eta(x, y, t) = \frac{1}{g} D_y \left( D_y^2 + \alpha^2 \right) \tilde{\eta}(x, y, t), \quad D_y = \frac{\partial}{\partial t}, \quad b_x(x, y, t) = \mu \rho D_x \left( D_x^2 + \alpha^2 \right) \tilde{b}_x(x, y, t), \quad b_y(x, y, t) = \mu \rho D_y \left( D_y^2 + \alpha^2 \right) \tilde{b}_y(x, y, t). \tag{1}
\]

We claim that the above functions \( \tilde{\eta}(x, y, t), \tilde{b}_x(x, y, t), \tilde{b}_y(x, y, t) \) are not unique: if \( \eta_0(x, y, t) \) satisfies (1), then, clearly, (1) is also satisfied by functions of the form

\[
\tilde{\eta}(x, y, t) = \eta_0(x, y, t) + \eta_1(x, y) \cos \alpha + \eta_2(x, y) \sin \alpha,
\]

where \( \eta_j(x, y), j = 1, 3 \) are arbitrary functions. Similarly, the functions \( \tilde{b}_x(x, y, t), \tilde{b}_y(x, y, t) \) can be written as

\[
\tilde{b}_x(x, y, t) = b_x^{(0)}(x, y, t) + b_x^{(1)}(x, y) \cos \alpha + b_x^{(2)}(x, y) \sin \alpha,
\]

\[
\tilde{b}_y(x, y, t) = b_y^{(0)}(x, y, t) + b_y^{(1)}(x, y) \cos \alpha + b_y^{(2)}(x, y) \sin \alpha,
\]

where \( b_x^{(j)}, b_y^{(j)}, j = 1, 3 \) are arbitrary functions of their arguments in the domain under consideration.

Consider the vector

\[
\left( \begin{array}{c} v_x \\ v_y \end{array} \right) = D_x \left( \begin{array}{c} D_x & \alpha \\ \alpha & D_y \end{array} \right) \tilde{\eta}(x, y, t) + DD_y \left( D_x & \alpha \\ -\alpha & D_y \end{array} \right) \tilde{b}_x(x, y, t),
\]

\[
D = \mathbf{b}_0 \frac{\partial}{\partial x} + \mathbf{b}_0 \frac{\partial}{\partial y}.
\]

By integrating this expression, the magnetic field induction components can be written as

\[
\left( \begin{array}{c} b_x \\ b_y \end{array} \right) = D \left( D_x^2 + \alpha^2 \right) \left( \begin{array}{c} D_x & -\alpha \\ \alpha & D_y \end{array} \right) \tilde{\eta}(x, y, t) + DD_y \left( D_x & -\alpha \\ \alpha & D_y \end{array} \right) \tilde{b}_x(x, y, t).
\]

From the resulting analytical representations for the components of velocity and magnetic field one can reduce the original boundary value problem to a single scalar equation, which is a corollary to the solenoidality equation of the magnetic field.

Based on the above, we arrive at the following result. Any sufficiently smooth solution to the small perturbation problem in a layer of perfect incompressible homogeneous electrically conducting rotating layer with due consideration of the magnetic field diffusion and with external magnetic field parallel to the normal vector to the boundary surface can be written as
where the function \( \xi(x, y, t) \) is the solution to the equation

\[
D \left( D_i^2 + \alpha^2 \right) \left[ \left( D_i - \Delta \frac{R_m}{R} \right) D_i - \frac{D_i^2}{\mu \rho} \right] \Delta \xi + \left( \nabla \ln H_0, \nabla \xi \right) = \frac{D \ln H_0, \alpha \left( D_i - \Delta \frac{R_m}{R} \right) \xi}{D(x, y)} +
\]

\[
+ \frac{1}{g(\mu \rho)^2 H_0} \left( D_i^2 + \alpha^2 \right) \left( F^2 + (\alpha D_i^2) \right) \left[ D \xi + \frac{B_{y0}}{2H_0 \sqrt{f}} \left( \nabla(H_0 + Z), \nabla \xi \right) \right] = -\frac{DH_0 + b_0^{(y)}}{(\mu \rho)^2 H_0}.
\]

For \( H_0 = \text{const} \) and \( \nabla Z = \text{const} \), at distances of the order of the wavelength, the solution of the small perturbation problem is expressed in terms of the function \( \xi \),

\( \xi = \text{Re} \left( A \exp \left( i(kx + ly - \sigma t) \right) \right) \).

Moreover, for the frequency of oscillations, and in particular, as \( R_m \to \infty \), we get the expression

\[
\sigma^2 = \frac{1}{2} \left[ 4\omega^2 + \frac{2(b_0^2 + b_{0y}^2)}{\mu \rho} - gH_0 \left( k^2 + l^2 \right) \pm \sqrt{d} \right],
\]

where

\[
d = \frac{1}{2} \left[ gH_0 \left( k^2 + l^2 \right) - 4\omega^2 - \frac{2(b_0^2 + b_{0y}^2)}{\mu \rho} - gH_0 \left( k^2 + l^2 \right) \right]^2 +
\]

\[
+ \frac{4gH_0 \left( k^2 + l^2 \right) b_0^2 + b_{0y}^2}{\mu \rho} - \frac{4(b_0^2 + b_{0y}^2)}{(\mu \rho)^2}.
\]

3. Conclusions

From this expression, a conclusion can be made that, with absent external background field, a steady-state wave regime may exist in the case

\( \omega^2 > \frac{gH_0 \left( k^2 + l^2 \right)}{4} \).

Otherwise, an unstable regime appears and the induced magnetic field can exist on a sufficiently extended time interval.
From the above analysis, a conclusion can be made that the generation of magnetic fields in an electrically conducting liquid is a consequence of an instability characterized by the corresponding relations between the gravity force, the Coriolis force, the magnetic force, and the peculiarities of the relief topography.

Earlier [5], for an external magnetic field parallel to the rotation axis of the layer, it was shown that the magnetic field diffusion contributes to its damping. However, the above analysis shows that the diffusion does not always contribute to the magnetic field damping in the case when the external magnetic field is parallel to the normal vector to the layer boundary surface.

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