A DUALITY BETWEEN NON-ARCHIMEDEAN UNIFORM SPACES AND SUBDIRECT POWERS OF FULL CLONES

JOSEPH VAN NAME

Abstract. A uniform space is said to be non-Archimedean if it is generated by equivalence relations. If \( \lambda \) is a cardinal, then a non-Archimedean uniform space \((X, U)\) is \( \lambda \)-totally bounded if each equivalence relation in \( U \) partitions \( X \) into less than \( \lambda \) blocks. If \( A \) is an infinite set, then let \( \Omega(A) \) be the algebra with universe \( A \) and where each \( a \in A \) is a fundamental constant and every finitary function is a fundamental operation. We shall give a duality between complete non-Archimedean \( |A|^+ \)-totally bounded uniform spaces and subdirect powers of \( \Omega(A) \). We shall apply this duality to characterize the algebras dual to supercomplete non-Archimedean uniform spaces.

1. Non-Archimedean Uniform Space Duality

In this paper, we shall assume basic facts about uniform spaces and universal algebra. The reader is referred to [2] or [3] for information about uniform spaces and to [4] for universal algebra. We shall use the entourage definition of uniform spaces, and we shall assume all complete uniform spaces are separated. If \( A \) is an algebra, then we shall write \( V(A) \) for the variety generated by \( A \).

In [4], Marshall Stone constructed a duality between compact totally disconnected spaces and Boolean algebras. This result revolutionized the theory of Boolean algebras since it gives a way to represent Boolean algebras as topological spaces. We shall give an analogous result for uniform spaces.

A uniform space \((X, U)\) is said to be non-Archimedean if \( U \) is generated by equivalence relations. We say that a non-Archimedean uniform space \((X, U)\) is \( \lambda \)-totally bounded if whenever \( E \in U \) is an equivalence relation, then \( E \) partitions \( X \) into less than \( \lambda \) blocks. Clearly, if \((X, U)\) is \( \lambda \)-totally bounded, then each subspace of \( X \) is \( \lambda \)-totally bounded as well.

For this paper, let \( A \) be a fixed infinite set. For each \( a \in A \), let \( \hat{a} \) be a constant symbol. For each \( f : A^n \to A \), let \( \hat{f} \) be an \( n \)-ary function symbol. Let \( \mathcal{F} = \{ \hat{a} | a \in A \} \cup \{ \hat{f} | f : A^n \to A \} \). Let \( \Omega(A) \) be the algebra of type \( \mathcal{F} \) and with universe \( A \) where \( \hat{a}^{\Omega(A)} = a \) for \( a \in A \) and where \( \hat{f}^{\Omega(A)} = f \) for \( f : A^n \to A \). Therefore every \( n \)-ary function on \( A \) is given by a function symbol, so we can regard \( \Omega(A) \) as the full clone of \( A \). We shall now give a duality between subdirect powers of \( \Omega(A) \) and complete non-Archimedean \( |A|^+ \)-totally bounded uniform spaces. With this duality, every complete non-Archimedean uniform space can be represented algebraically simply by letting \( |A| \) be at least as large as every uniform partition.

The algebra \( \Omega(A) \) and the variety \( V(\Omega(A)) \) generated by \( \Omega(A) \) have applications to mathematics besides uniform space duality. For instance, the variety \( V(\Omega(A)) \)
is related to the ultrapower construction and reduced power construction. In fact, one can construct ultrapowers and reduced powers from elements of the variety $V(\Omega(A))$. Also, the first order theory $\text{Th}(\Omega(A))$ of $\Omega(A)$ is appealing since it is generated by the identities in $\Omega(A)$ and a single sentence. In other words, there is a $\phi \in \text{Th}(\Omega(A))$ such that for each $\theta \in \text{Th}(\Omega(A))$, there are identities $I_1, \ldots, I_n \in \text{Th}(\Omega(A))$ such that $(\phi \land I_1 \land \cdots \land I_n) \rightarrow \theta$.

The algebra $\Omega(A)$ serves as an infinite analogue of the two element Boolean algebra $B$ since in $B$ every function can be represented as a combination of the Boolean operations $\wedge, \lor'$. Therefore the variety $V(\Omega(A))$ is analogous to the variety of Boolean algebras. The category of compact totally disconnected spaces is isomorphic to the category of complete non-Archimedean $\mathbb{R}_0$-totally bounded uniform spaces. Therefore it should be possible to reconstruct a duality between compact totally disconnected spaces and the variety of Boolean algebras, but for simplicity we shall only consider the variety $V(\Omega(A))$ when $A$ is infinite.

An algebra $L \in V(\Omega(A))$ shall be called partitionable if there is an injective homomorphism $\phi: L \rightarrow \Omega(A)^I$ for some set $I$. Clearly, the products and subspaces of partitionable algebras are partitionable. Furthermore, each partitionable algebra is isomorphic to a subdirect product of $\Omega(A)$ since each $a \in A$ is a constant in $\Omega(A)$.

Let $Z(L)$ be the collection of all homomorphisms $\phi: L \rightarrow \Omega(A)$. In this paper, the set $A$ will always have the discrete uniformity. Now give $A^\ell$ the product uniformity. Then the topology on $A$ is the discrete topology and the topology on $A^\ell$ is the product topology. Give $Z(L) \subseteq A^\ell$ the subspace uniformity. Then $Z(L)$ is a closed subspace of $A^\ell$ since every convergent net $(\phi_d)_{d \in D}$ in $Z(L)$ converges to some $\phi \in Z(L)$. Thus, since $Z(L)$ is a closed subspace of a complete uniform space, $Z(L)$ is complete.

Let $\ell_1, \ldots, \ell_n \in L$. Then let $E_{\ell_1,\ldots,\ell_n}^\ell$ be the equivalence relation $A^\ell$ where for $r, s \in A^\ell$ we have $(r, s) \in E_{\ell_1,\ldots,\ell_n}^\ell$ if and only if $r(\ell_1) = s(\ell_1), \ldots, r(\ell_n) = s(\ell_n)$. Then the equivalence relations $E_{\ell_1,\ldots,\ell_n}^\ell$ generate the uniformity on $A^\ell$. Take note that each $E_{\ell_1,\ldots,\ell_n}^\ell$ partitions $A^\ell$ into $|A|^n = |A|$ blocks, so the uniform space $A^\ell$ is $|A|^\ell$-totally bounded. Let $E_{\ell_1,\ldots,\ell_n}$ be the restriction of $E_{\ell_1,\ldots,\ell_n}^\ell$ to $Z(L)$. Then the equivalence relations $E_{\ell_1,\ldots,\ell_n}$ generate the uniformity on $Z(L)$. In particular, $Z(L)$ is a $|A|^\ell$-totally bounded non-Archimedean uniform space.

Let $(X, U)$ be a uniform space. Then let $\mathfrak{B}_A(X, U)$ be the collection of all uniformly continuous mappings from $X$ to $A$. Clearly $\mathfrak{B}_A(X, U)$ is a subdirect product of $\Omega(A)$, so $\mathfrak{B}_A(X, U)$ is a partitionable algebra.

If $(X, U)$ is a uniform space, then for each $x \in X$, we have $\pi_x: \mathfrak{B}_A(X, U) \rightarrow \Omega(A)$ be a homomorphism where $\pi_x$ is the projection mapping defined by $\pi_x(f) = f(x)$. Therefore define a mapping $C: (X, U) \rightarrow Z(\mathfrak{B}_A(X, U))$ by $C(x) = \pi_x$. In other words, if $x \in X$, and $f: (X, U) \rightarrow A$ is uniformly continuous, then $C(x)(f) = f(x)$.

If there is any confusion about the space $(X, U)$, then we shall write $C_{(X, U)}$ for the mapping $C$.

Now let $L \in V(\Omega(A))$. If $\ell \in L$, then let $\ell^*: Z(L) \rightarrow A$ be the mapping defined by $\ell^*(\phi) = \phi(\ell)$. We claim that $\ell^*$ is uniformly continuous. Assume that $(\phi, \theta) \in E_{\ell}$. Then $\phi(\ell) = \theta(\ell)$, so $\ell^*(\phi) = \ell^*(\theta)$, and hence $(\ell^*(\phi), \ell^*(\theta)) \in E$ for each equivalence relation $E$ on $A$. Therefore $\ell^*$ is uniformly continuous, so $\ell^* \in \mathfrak{B}_A(Z(L))$. In light of the above discussion, we define a function $\rho: L \rightarrow \mathfrak{B}_A(Z(L))$
by \( \rho(\ell) = \ell^* \). Therefore \( \rho(\ell)(\phi) = \phi(\ell) \) for \( \phi \in Z(\mathcal{L}), \ell \in \mathcal{L} \). We will write \( \rho_\mathcal{L} \) for the mapping \( \rho \) to specify the domain of \( \rho \) in case there may be confusion.

**Exercise 1.1.** If \( f: A^n \to A \) is injective (surjective), then \( \hat{f}: \mathcal{L}^n \to \mathcal{L} \) is injective (surjective) for each \( \mathcal{L} \in V(\Omega(A)) \).

**Theorem 1.2.** The equivalence relations \( \mathcal{E}_\ell \) generate the uniformity on \( Z(\mathcal{L}) \).

**Proof.** Assume that \( \ell_1, \ldots, \ell_n \in \mathcal{L} \). Let \( i: A^n \to A \) be injective. Then \( \hat{i}^\mathcal{L} \) is also injective. Now let \( \ell = i^\mathcal{L}(\ell_1, \ldots, \ell_n) \). Assume \( \phi, \theta \in Z(\mathcal{L}) \) and \( (\phi, \theta) \in \mathcal{E}_\ell \). Then \( \phi(\ell) = \theta(\ell), \phi(i\mathcal{L}(\ell_1, \ldots, \ell_n)) = \theta(i\mathcal{L}(\ell_1, \ldots, \ell_n)) \). Therefore, \( i(\phi(\ell_1), \ldots, \phi(\ell_n)) = i(\theta(\ell_1), \ldots, \theta(\ell_n)) \), so since \( i \) is injective, we have \( \phi(\ell_1) = \theta(\ell_1), \ldots, \phi(\ell_n) = \theta(\ell_n) \), thus \( (\phi, \theta) \in \mathcal{E}_{\ell_1, \ldots, \ell_n} \). In other words, we have \( \mathcal{E}_\ell \subseteq \mathcal{E}_{\ell_1, \ldots, \ell_n} \). Therefore the equivalence relations \( \mathcal{E}_\ell \) generate the uniformity on \( Z(\mathcal{L}) \). \( \Box \)

**Theorem 1.3.** 1. Let \( \mathcal{L} \in V(\Omega(A)) \). Then \( \rho: \mathcal{L} \to \mathfrak{B}_A(Z(\mathcal{L})) \) is a surjective homomorphism, and \( \rho \) is an isomorphism if and only if \( \mathcal{L} \) is partitionable.

2. If \( (X, \mathcal{U}) \) is a uniform space, then the mapping \( \mathcal{C}: (X, \mathcal{U}) \to Z(\mathfrak{B}_A(X, \mathcal{U})) \) is uniformly continuous and \( C''(X) \) is dense in \( Z(\mathfrak{B}_A(X, \mathcal{U})) \). If \( (X, \mathcal{U}) \) is separated and non-Archimedean, then \( \mathcal{C} \) is injective. If \( (X, \mathcal{U}) \) is separated non-Archimedean and \( |A|^{+} \)-totally bounded, then \( \mathcal{C} \) is an embedding. If \( (X, \mathcal{U}) \) is complete non-Archimedean and \( |A|^{+} \)-totally bounded, then \( \mathcal{C} \) is an isomorphism.

**Proof.** 1. If \( \ell \in \mathcal{L} \), then we have \( \rho(\ell) = (\rho(\ell)(\phi))_{\phi \in Z(\mathcal{L})} = (\phi(\ell))_{\phi \in Z(\mathcal{L})} \). Therefore \( \rho \) is a homomorphism since \( \rho \) is a homomorphism in each coordinate.

To prove surjectivity, assume that \( f: Z(\mathcal{L}) \to A \) is uniformly continuous. Then there is an \( \ell \in \mathcal{L} \) where if \( (\phi, \theta) \in \mathcal{E}_\ell \), then \( f(\phi) = f(\theta) \). In other words, if \( \phi(\ell) = \theta(\ell) \), then \( f(\phi) = f(\theta) \). Therefore there is a function \( g: A \to A \) where \( f(\phi) = g(\phi(\ell)) \) whenever \( \phi \in Z(\mathcal{L}) \). Furthermore, we have \( f(\phi) = g(\phi(\ell)) = \phi(g^\mathcal{L}(\ell)) = \rho(g^\mathcal{L}(\ell)) \) for each \( \phi \in Z(\mathcal{L}) \). Therefore \( \rho(g^\mathcal{L}(\ell)) = f \). Thus the mapping \( \rho \) is surjective.

Now assume \( \mathcal{L} \) is partitionable. Then for each pair of distinct \( \ell_1, \ell_2 \in \mathcal{L} \) there is a homomorphism \( \phi: \mathcal{L} \to A \) with \( \rho(\ell_1)(\phi) = \phi(\ell_1) \neq \phi(\ell_2) = \rho(\ell_2)(\phi) \). Therefore \( \rho(\ell_1) \neq \rho(\ell_2) \). We conclude that \( \rho \) is injective. Likewise, if we assume \( \rho \) is an isomorphism, then since \( \mathfrak{B}_A(Z(\mathcal{L})) \) is partitionable, we have \( \mathcal{L} \) be partitionable as well.

2. Since \( \mathcal{C}: (X, \mathcal{U}) \to Z(\mathfrak{B}_A(X, \mathcal{U})) \subseteq A^{\mathfrak{B}_A(X, \mathcal{U})} \), we have \( \mathcal{C} \) be uniformly continuous if and only if \( \mathcal{C} \) be uniformly continuous in every coordinate \( f \in \mathfrak{B}_A(X, \mathcal{U}) \). However, we have \( \mathcal{C}(x) = (\mathcal{C}(x)(f))_{f \in \mathfrak{B}_A(X, \mathcal{U})} = (f(x))_{f \in \mathfrak{B}_A(X, \mathcal{U})} \), so \( \mathcal{C} \) is uniformly continuous.

We shall now show that \( C''(X) \) is dense in \( Z(\mathfrak{B}_A(X, \mathcal{U})) \). The uniformity on \( Z(\mathfrak{B}_A(X, \mathcal{U})) \) is generated by the equivalence relations \( \mathcal{E}_f \) where \( f \in \mathfrak{B}_A(X, \mathcal{U}) \). The blocks in the equivalence relation \( \mathcal{E}_f \) are the nonempty sets of the form \( U_{f,a} = \{ \phi \in Z(\mathfrak{B}_A(X, \mathcal{U})) | \phi(f) = a \} \). Therefore it suffices to show that \( C''(X) \) intersects each non-empty block \( U_{f,a} \).

Now assume that \( U_{f,a} \) is non-empty. Then there is a \( \phi \in Z(\mathfrak{B}_A(X, \mathcal{U})) \) with \( \phi(f) = a \). We claim that \( f(x) = a \) for some \( x \in X \). Therefore, assume that \( f(x) \neq a \) for all \( x \in X \). Let \( i: A \to A \) be a mapping where \( i(a) \neq a \) and \( i(b) = b \) for \( b \neq a \). Then we have \( f = i \circ f = i^{\mathfrak{B}_A(X, \mathcal{U})}(f), \phi(f) = \phi(i^{\mathfrak{B}_A(X, \mathcal{U})}(f)) = i(\phi(f)) \neq a \). Thus, by contrapositive, if \( \phi(f) = a \), then \( f(x) = a \) for some \( x \in X \).
However, we have $C(x)(f) = f(x) = a$, so $C(x) \in U_{f,a}$. Therefore $C''(X)$ is dense in $Z(\mathcal{B}_A(X,U))$.

Now assume that $(X,U)$ is separated and non-Archimedean. Then we shall show that $C$ is injective. Assume that $x, y \in X$, $x \neq y$. Then since $(X,U)$ is separated and non-Archimedean, there is a uniformly continuous function $f : X \rightarrow A$ such that $f(x) \neq f(y)$. Therefore $C(x)(f) = f(x) \neq f(y) = C(y)(f)$, and hence $C(x) \neq C(y)$. We conclude that $C$ is injective.

Now assume that $(X,U)$ is separated, non-Archimedean, and $|A|^+$-totally bounded. Then we shall show that $C$ is an embedding. Assume that $E \in U$ is an equivalence relation. Then since $(X,U)$ is $|A|^+$-totally bounded, there is a function $f : X \rightarrow A$ where $f(x) = f(y)$ if and only if $(x,y) \in E$. Clearly $f$ is uniformly continuous, so $f \in \mathcal{B}_A(X,U)$ and $\mathcal{E}_f$ is an equivalence relation on $Z(\mathcal{B}_A(X,U))$. Now assume that $x, y \in X$. Then $(x,y) \in E$ if and only if $f(x) = f(y)$ if and only if $C(x)(f) = C(y)(f)$ if and only if $(C(x), C(y)) \in \mathcal{E}_f$. Therefore $C$ is an embedding.

If $(X,U)$ is complete, non-Archimedean, and $|A|^+$-totally bounded, then we have $C$ be an embedding, and $Z(\mathcal{B}_A(X,U))$ is the completion of $C''(X)$. However, if $X$ is complete, we have $C''(X) = Z(\mathcal{B}_A(X,U))$. Therefore, in this case, $C$ is a uniform homeomorphism.

Let $\mathcal{L}, \mathcal{M} \in V(\Omega(A))$ and assume that $\phi : \mathcal{L} \rightarrow \mathcal{M}$ is a homomorphism. Then let $Z(\phi) : Z(\mathcal{M}) \rightarrow Z(\mathcal{L})$ be the function defined by $Z(\phi)(\theta) = \theta \circ \phi$ for homomorphisms $\theta : \mathcal{M} \rightarrow A$. One can easily show that the mappings $Z(\phi)$ are uniformly continuous and $Z$ is a functor from the variety $V(\Omega(A))$ to the category of uniform spaces. Now assume that $(X,U), (Y,V)$ are uniform spaces and $f : (X,U) \rightarrow (Y,V)$ is uniformly continuous. Then define a mapping $\mathcal{B}_A(f) : \mathcal{B}_A(Y,V) \rightarrow \mathcal{B}_A(X,U)$ by $\mathcal{B}_A(f)(g) = g \circ f$. Then each $\mathcal{B}_A(f)$ is a homomorphism. Furthermore, $\mathcal{B}_A$ gives a functor from the category of uniform spaces to the variety $V(\Omega(A))$.

**Theorem 1.4.**

1. Let $f : (X,U) \rightarrow (Y,V)$ be uniformly continuous. Then $Z(\mathcal{B}_A(f)) \circ C_{(X,U)} = C_{(Y,V)} \circ f$.

\[
\begin{array}{ccc}
(X,U) & \xrightarrow{f} & (Y,V) \\
\downarrow C & & \downarrow C \\
Z(\mathcal{B}_A(X,U)) & \xrightarrow{Z(\mathcal{B}_A(f))} & Z(\mathcal{B}_A(Y,V))
\end{array}
\]

2. Let $\phi : \mathcal{L} \rightarrow \mathcal{M}$ be a homomorphism. Then we have $\mathcal{B}_A(Z(\phi)) \circ \rho_{\mathcal{L}} = \rho_{\mathcal{M}} \circ \phi$.

\[
\begin{array}{ccc}
\mathcal{L} & \xrightarrow{\phi} & \mathcal{M} \\
\downarrow \rho & & \downarrow \rho \\
\mathcal{B}_A(Z(\mathcal{L})) & \xrightarrow{\mathcal{B}_A(Z(\phi))} & \mathcal{B}_A(Z(\mathcal{M}))
\end{array}
\]

3. The pair of functions $Z(\rho_{\mathcal{L}}) : Z(\mathcal{B}_A(Z(\mathcal{L}))) \rightarrow Z(\mathcal{L})$ and $C_{Z(\mathcal{L})} : Z(\mathcal{L}) \rightarrow Z(\mathcal{B}_A(Z(\mathcal{L})))$ are inverses.

4. The pair of functions $\mathcal{B}_A(C_{(X,U)}) : \mathcal{B}_A(Z(\mathcal{B}_A(X,U))) \rightarrow \mathcal{B}_A(X,U)$ and $\rho_{\mathcal{B}_A(X,U)} : \mathcal{B}_A(X,U) \rightarrow \mathcal{B}_A(Z(\mathcal{B}_A(X,U)))$ are inverses.

**Proof.**

1. Let $x \in X$ and let $g \in \mathcal{B}_A(Y,V)$. Then we have

\[
[(Z(\mathcal{B}_A(f)) \circ C(x))(g) = [Z(\mathcal{B}_A(f))(C(x))](g)
\]

2. Let $\phi : \mathcal{L} \rightarrow \mathcal{M}$ be a homomorphism. Then we have

\[
\mathcal{B}_A(Z(\phi)) \circ \rho_{\mathcal{L}} = \rho_{\mathcal{M}} \circ \phi.
\]

3. The pair of functions $Z(\rho_{\mathcal{L}})$ and $C_{Z(\mathcal{L})}$ are inverses.

4. The pair of functions $\mathcal{B}_A(C_{(X,U)})$ and $\rho_{\mathcal{B}_A(X,U)}$ are inverses.
Thus let $\phi$ be an element with $c$ hence $\hat{A}$.

Let $\ker(A)$.

Proof. $\theta \in Z(A)$.

(2) This proof is analogous to part 1. Let $\ell \in L$ and let $\theta \in Z(M)$. Then we have

$$[(B_A(\phi) \circ \rho(\ell))(\theta) = [B_A(\phi)(\rho(\ell))(\theta)]$$

$$\theta(\ell) = \theta(\phi(\ell)) = \phi(\theta(\ell)) = \rho(\phi(\ell)) = \rho(\theta(\ell)) = \rho(\ell)(\theta).$$

Therefore $\rho \circ \phi = B_A(\phi(\theta)) \circ \rho$.

(3) The uniform space $Z(L)$ is complete, so $C_{Z(L)}$ is a uniform homeomorphism. It therefore suffices to show that $Z(\rho_L) \circ C_{Z(L)} : Z(L) \to Z(L)$ is the identity map. Therefore let $\phi : L \to \Omega(A)$ is a homomorphism and $\ell \in L$. Then we have

$$[Z(\rho_L) \circ C_{Z(L)}(\phi)](\ell) = [Z(\rho_L)(C_{Z(L)}(\phi))](\ell)$$

$$= [C_{Z(L)}(\phi) \circ \rho_L](\ell) = C_{Z(L)}(\phi)(\rho_L(\ell)) = \rho_L(\ell)(\phi) = \phi(\ell).$$

We therefore conclude that $Z(\rho_L) \circ C_{Z(L)}$ is the identity map.

(4) This proof is analogous to 3. Since $B_A(X, U)$ is partitionable, we have $\rho_B(\phi, U)$ be an isomorphism. We therefore need to show that $B_A(C_{U, L}) \circ \rho_B(\phi, U) : B_A(X, U) \to B_A(X, U)$ is the identity map. Thus, assume that $f \in B_A(X, U)$ and $x \in X$. Then

$$[B_A(C_{U, L}) \circ \rho_B(\phi, U)(f)](x) = [B_A(C_{U, L})(\rho_B(\phi, U)(f))](x)$$

$$= (\rho_B(\phi, U)(f) \circ C_{U, L})(x) = \rho_B(\phi, U)(f)(C_{U, L}(x))$$

$$= C_{U, L}(x)(f) = f(x).$$

Therefore $B_A(C_{U, L}) \circ \phi_B(\phi, U)$ is the identity map.

2. A Characterization of non-Archimedean Supercomplete Spaces

A congruence $\theta$ on $L$ is said to be partitionable if $L/\theta$ is partitionable. Let $PC(L)$ denote the collection of all partitional congruences of $L$. One can easily see that $PC(L)$ consists of all congruences of the form $\cap_{\theta \in R} \ker(\theta)$ where $R \subseteq Z(L)$.

Theorem 2.1. Let $L \in V(\Omega(A))$. Let $R \subseteq Z(L)$. Then let $\phi \in Z(L)$. Then $\phi \in R$

if and only if $\cap_{\theta \in R} \ker(\theta) \subseteq \ker(\phi)$.

Proof. $\rightarrow$ Assume $\phi \in R$. Also assume $\ell, m \in L$ and $(\ell, m) \in \cap_{\theta \in R} \ker(\theta)$. Then $\theta(\ell) = \theta(m)$ for $\theta \in R$. Since $\phi \in R$, there is a $\theta \in R$ with $(\phi, \theta) \in E_{\ell, m}$, so $\phi(\ell) = \theta(\ell) = \theta(m) = \phi(m)$. Therefore $(\ell, m) \in \ker(\phi)$. We conclude that $\cap_{\theta \in R} \ker(\theta) \subseteq \ker(\phi)$.

$\leftarrow$ Assume $\cap_{\theta \in R} \ker(\theta) \subseteq \ker(\phi)$. Then let $\ell \in L$ and assume $\phi(\ell) = a$. Let $b \in A$ be an element with $b \neq a$. Let $i : A \to A$ be the map where $i(a) = a$ and $i(c) = b$ for $c \neq a$. Then $\phi(i^2(\ell)) = \phi(i(\phi(\ell))) = a \neq b = \phi(i^2(\ell))$, so $i^2(\ell), b \notin \ker(\phi)$, hence $i^2(\ell), b \notin \ker(\theta)$ for some $\theta \in R$. Therefore $b = \theta(b) \neq \theta(i^2(\ell)) = i(\theta(\ell))$. Thus $\theta(\ell) = a = \phi(\ell)$. Therefore $(\phi, \theta) \in E_{\ell}$. Since $\ell \in L$ is arbitrary, we have $\phi \in R$. 

$\square$
We shall now give a Galois correspondence between closed sets in $Z(\mathcal{L})$ and partitionable congruences in $\mathcal{L}$. Let $f : P(\mathcal{L}^2) \to P(Z(\mathcal{L})), g : P(Z(\mathcal{L})) \to P(\mathcal{L}^2)$ be the mappings where

$$f(R) = \{ \phi \in Z(\mathcal{L})|(a, b) \in \ker(\phi) \text{ for all } (a, b) \in R \} = \{ \phi \in Z(\mathcal{L})|R \subseteq \ker(\phi) \}$$

and where

$$g(S) = \{(a, b) \in \mathcal{L}^2|(a, b) \in \ker(\phi) \text{ for all } \phi \in S \} = \bigcap_{\phi \in S} \ker(\phi).$$

Let $C = g \circ f, D = f \circ g$. Then $C$ and $D$ are closure operators. In other words, we have $C(R) \subseteq C(C(R))$, and if $R \subseteq S$, then $C(R) \subseteq C(S)$ for $R, S \subseteq \mathcal{L}^2$. Let $C^* = \{ R \subseteq \mathcal{L}^2|C(R) = R \} = \{ C(R)|R \subseteq \mathcal{L}^2 \}$ and let $D^* = \{ S \subseteq Z(\mathcal{L})|D(S) = D \} = \{ D(S)|S \subseteq Z(\mathcal{L}) \}$. Let $f^* : C^* \to D^*, g^* : D^* \to C^*$ be the restriction of the functions $f$ and $g$. Then the functions $f^*$ and $g^*$ are inverse functions.

**Theorem 2.2.** The mapping $D$ is the topological closure operator induced by the uniformity on $Z(\mathcal{L})$.

**Proof.** Let $R \subseteq Z(\mathcal{L})$. Then

$$D(R) = f \circ g(R) = f(\bigcap_{\theta \in R} \ker(\theta)) = \{ \phi \in Z(\mathcal{L})|\bigcap_{\theta \in R} \ker(\theta) \subseteq \ker(\phi) \} = \overline{R}.$$

If $(X, \mathcal{U})$ is a uniform space, then let $H(X)$ be the collection of all closed subsets of $X$. Clearly $D^* = H(Z(\mathcal{L}))$ and $C^* = PC(\mathcal{L})$. Therefore we have $f^* : PC(\mathcal{L}) \to H(Z(\mathcal{L}))$ and $g^* : H(Z(\mathcal{L})) \to PC(\mathcal{L})$.

We shall now characterize the partitionable algebras $\mathcal{L}$ where $S(\mathcal{L})$ is supercomplete. For each $E \in \mathcal{U}$, let $\overline{E}$ be the binary relation on $H(X)$ where $(C, D) \in \overline{E}$ if and only if $C \subseteq E[D] = \{ x \in X|(z, x) \in E \text{ for some } z \in D \}$ and $D \subseteq E[C]$. Then the relations $\overline{E}$ generate a uniformity on $H(X)$. Therefore $H(X)$ is a uniform space. With this uniformity, we shall call $H(X)$ the hyperspace of $X$. A separated uniform space $X$ is said to be supercomplete if $H(X)$ is complete.

Take note that if $\mathcal{L}$ is an algebra and $\ell \in \mathcal{L}$, then we have $\phi \in \mathcal{E}_\ell[C]$ if and only if there is some $\theta \in C$ with $(\theta, \phi) \in \mathcal{E}_\ell$. In other words, $\phi \in \mathcal{E}_\ell[C]$ if and only if $\phi(\ell) \in \{ \theta(\ell)|\theta \in C \}$. Therefore $(C, D) \in \overline{\mathcal{E}}$ if and only if $\{ \theta(\ell)|\theta \in C \} = \{ \phi(\ell)|\phi \in D \}$.

**Exercise 2.3.** Every finitely generated algebra $\mathcal{L} \in V(\Omega(A))$ is generated by a single element.

A locally partitionable congruence is a congruence $\theta$ on $\mathcal{L}$ so that whenever $\mathcal{M} \subseteq \mathcal{L}$ is a finitely generated subalgebra, we have $\theta \cap \mathcal{M}^2$ be a partitionable congruence.

Let $LPC(\mathcal{L})$ denote the set of all locally partitionable congruences on $\mathcal{L}$. $LPC(\mathcal{L})$ is closed under arbitrary intersection, so $LPC(\mathcal{L})$ is a complete lattice. Let $FS(\mathcal{L})$ be the collection of all finitely generated subalgebras of $\mathcal{L}$. We shall now give $LPC(\mathcal{L})$ a complete uniformity by representing $LPC(\mathcal{L})$ as an inverse limit.

If $\mathcal{M}, \mathcal{N}$ are finitely generated subalgebras of $\mathcal{L}$ and $\mathcal{M} \subseteq \mathcal{N}$, then define a function $E_{\mathcal{N}, \mathcal{M}} : PC(\mathcal{N}) \to PC(\mathcal{M})$ by letting $E_{\mathcal{N}, \mathcal{M}}(\theta) = \theta \cap \mathcal{M}^2$. One can easily show that $(PC(\mathcal{N}))_{\mathcal{N} \in FS(\mathcal{L})}$ is an inverse system of sets with transitional mappings $E_{\mathcal{N}, \mathcal{M}}$. Let $IL(\mathcal{L})$ be the inverse limit $\lim_{\leftarrow} PC(\mathcal{N})$. Give each $PC(\mathcal{N})$ the
Theorem 2.8. The mappings \( \forall E \) are uniform homeomorphisms.

Let \( \Gamma: LPC(\mathcal{L}) \to IL(\mathcal{L}) \) be the mapping defined by letting \( \Gamma(\theta) = (\theta \cap M^2)_{M \in FS(\mathcal{L})} \). Conversely, define a mapping \( \Delta: IL(\mathcal{L}) \to LPC(\mathcal{L}) \) be the mapping defined by \( \Delta((\theta_M)_{M \in FS(\mathcal{L})}) = \bigcup_M \theta_M \).

Exercise 2.4. The functions \( \Gamma \) and \( \Delta \) are inverses.

Now give \( LPC(\mathcal{L}) \) the uniformity such that the maps \( \Gamma \) and \( \Delta \) are uniform homeomorphisms. Now for each finitely generated subalgebra \( \mathcal{N} \subseteq \mathcal{L} \), let \( \mathcal{F}_\mathcal{N} \) be the equivalence relation on \( LPC(\mathcal{L}) \) where \( (\theta, \psi) \in \mathcal{F}_\mathcal{N} \) if and only if \( \theta \cap \mathcal{N}^2 = \psi \cap \mathcal{N}^2 \).

Exercise 2.5. Let \((X, U)\) be a non-Archimedean uniform space. Let \( N \subseteq \mathbb{B}_A(X, U) \) be a finitely generated subalgebra. Then there is a partition \( P \) such that if \( r: X \to P \) is the function where \( x \in r(x) \) for all \( x \in X \), then \( \mathcal{N} = \{ f \circ r: P \to A \} \).

Theorem 2.6. Let \( \mathcal{L} \) be partitionable. Then \( PC(\mathcal{L}) \) is dense in \( LPC(\mathcal{L}) \).

Exercise 2.7. Assume \( a_i \in A \) for \( i \in I \) and \( b_j \in A \) for \( j \in J \). Then \( \{ a_i | i \in I \} = \{ b_j | j \in J \} \) if and only if for each pair of functions \( \phi, \psi: A \to A \), we have \( \forall i \in I, f(a_i) = g(a_i) \Leftrightarrow \forall j \in J, f(b_j) = g(b_j) \).

Theorem 2.8. The mappings \( f^*: PC(\mathcal{L}) \to H(Z(\mathcal{L})) \) and \( g^*: H(Z(\mathcal{L})) \to PC(\mathcal{L}) \) are uniform homeomorphisms.

Proof. We only need to show that \( g^* \) is a uniform homeomorphism. Since \( Z(\mathcal{L}) \) is generated by equivalence relations \( \mathcal{E}_\mathcal{L} \), the equivalence relations \( \overline{\mathcal{E}_\mathcal{L}} \) generate \( H(Z(\mathcal{L})) \). We have \( (C, D) \in \overline{\mathcal{E}_\mathcal{L}} \) if and only if

\[
\{ \theta(\ell) | \theta \in C \} = \{ \theta(\ell) | \theta \in D \}
\]

if and only if for \( f, g: A \to A \), we have

\[
\forall \phi \in C, f(\phi(\ell)) = g(\phi(\ell)) \Leftrightarrow \forall \phi \in D, f(\phi(\ell)) = g(\phi(\ell))
\]

if and only if for each \( f, g: A \to A \), we have

\[
\forall \phi \in C, \phi(f^C(\ell)) = \phi(g^C(\ell)) \Leftrightarrow \forall \phi \in D, \phi(f^C(\ell)) = \phi(g^C(\ell))
\]

if and only if whenever \( f, g: A \to A \), we have

\[
(f^C(\ell), g^C(\ell)) \in \bigcap_{\phi \in C} \ker(\phi) \Leftrightarrow (f^C(\ell), g^C(\ell)) \in \bigcap_{\phi \in D} \ker(\phi)
\]
if and only if
\[ g^*(C) \cap \langle \ell \rangle^2 = \bigcap_{\phi \in C} \ker(\phi) \cap \langle \ell \rangle^2 = \bigcap_{\phi \in D} \ker(\phi) \cap \langle \ell \rangle^2 = g^*(D) \cap \langle \ell \rangle^2 \]
if and only if \((g^*(C), g^*(D)) \in F_\langle \ell \rangle\). Therefore \(g^*\) is a uniform homeomorphism. □

**Theorem 2.9.** Let \( \mathcal{L} \) be a partitionable algebra. Then \( Z(\mathcal{L}) \) is supercomplete if and only if every locally partitionable congruence on \( \mathcal{L} \) is partitionable.

**Proof.** However, since \( H(Z(\mathcal{L})) \) is uniformly homeomorphic to \( PC(\mathcal{L}) \), we have \( H(Z(\mathcal{L})) \) be complete if and only if \( PC(\mathcal{L}) \) is complete. Since \( PC(\mathcal{L}) \) is a dense subspace of the complete space \( LPC(\mathcal{L}) \), we have \( PC(\mathcal{L}) \) be complete if and only if \( PC(\mathcal{L}) = LPC(\mathcal{L}) \) if and only if each locally partitionable congruence on \( \mathcal{L} \) is partitionable. Therefore \( Z(\mathcal{L}) \) is supercomplete if and only if every locally partitionable congruence on \( \mathcal{L} \) is partitionable. □

**Exercise 2.10.** A partitionable algebra \( \mathcal{L} \) is finitely generated if and only if \( Z(\mathcal{L}) \) is discrete. A partitionable algebra \( \mathcal{L} \) is countably generated if and only if \( Z(\mathcal{L}) \) is uniformizable by a metric.

We shall now prove a purely algebraic result using hyperspaces.

**Corollary 2.11.** If \( \mathcal{L} \) is a countably generated partitionable algebra, then every locally partitionable congruence is partitionable.

**Proof.** If \( \mathcal{L} \) is a countably generated partitionable algebra, then \( Z(\mathcal{L}) \) is uniformizable by a metric. However, in [2] [p. 30], it is shown that every complete metric space is supercomplete. Therefore since \( Z(\mathcal{L}) \) is supercomplete, every locally partitionable congruence is partitionable. □

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DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF SOUTH FLORIDA, 4202 E. FOWLER AVENUE TAMPA, FL 33620, USA

E-mail address: jvanname@mail.usf.edu