On the Existence of Supersingular Curves
Of Given Genus

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Introduction

In this note we shall show that there exist supersingular curves for every positive genus in characteristic 2. Recall that an irreducible smooth algebraic curve \( C \) over an algebraically closed field \( \mathbb{F} \) of characteristic \( p > 0 \) is called supersingular if its jacobian is isogenous to a product of supersingular elliptic curves. An elliptic curve is called supersingular if it does not have points of order \( p \) over \( \mathbb{F} \). It is not clear a priori that there exist such curves for every genus. Indeed, note that in the moduli space \( A_g \otimes \mathbb{F}_p \) of principally polarized abelian varieties the locus of supersingular abelian varieties has dimension \([g^2/4]\) (cf. [O, L-O]), while the locus of jacobians has dimension \(3g - 3\) for \( g > 1 \). Therefore, as far as dimensions are concerned there is no reason why these loci should intersect for \( g \geq 9 \).

In this paper we construct for every integer \( g > 0 \) a supersingular curve of genus \( g \) over the field \( \mathbb{F}_2 \). In particular this shows that for every \( g > 0 \) there exists an irreducible curve of genus \( g \) whose jacobian is isogenous to a product of elliptic curves. We refer to [E-S] for related questions in characteristic 0. We do our construction by taking a suitable fibre product of Artin-Schreier curves. This construction is inspired by coding theory, where the introduction of generalized Hamming weights led us to consider such products, cf. [G-V 2].

More generally, we are able to construct in characteristic \( p \) a supersingular curve over \( \mathbb{F}_p \) of any genus \( g \) whose \( p \)-adic expansion consists of the digits 0 and \((p-1)/2\) only. We can also count on how many moduli the construction depends.

§1 Fibre products of Artin-Schreier curves.

Let \( \mathbb{F} \) be a fixed algebraic closure of the prime field \( \mathbb{F}_2 \). We consider a finite dimensional \( \mathbb{F}_2 \)-linear subspace \( \mathcal{L} \) of the function field \( \mathbb{F}(x) \). Define the operator \( \varphi \) on \( \mathbb{F}(x) \) by \( \varphi(f) = f^2 + f \). We shall assume that \( \mathcal{L} \cap \varphi(\mathbb{F}(x)) = \{0\} \).

To an element \( f \in \mathcal{L} - \{0\} \) we associate the complete non-singular Artin-Schreier curve \( C_f \) with affine equation

\[
y^2 + y = f.
\]

Choose a basis \( f_1, \ldots, f_k \) of \( \mathcal{L} \) and let \( \phi_i : C_{f_i} \to \mathbb{P}^1 \) be the morphism given by the inclusion \( \mathbb{F}(x) \subset \mathbb{F}(x, y) \). Then we define a curve \( C^\mathcal{L} \) by

\[
C^\mathcal{L} = \text{Normalization of } (C_{f_1} \times \ldots \times C_{f_k}),
\]

where the product means the fibre product taken with respect to the morphisms \( \phi_i \). Up to \( \mathbb{F}(x) \)-isomorphism the curve \( C^\mathcal{L} \) is independent of the chosen basis of \( \mathcal{L} \).
In the following we need some properties of the curve $C^{\mathcal{L}}$; the reader can find a proof in [G-V 2].

\textbf{(1.1) Proposition.} i) The jacobian of $C^{\mathcal{L}}$ decomposes up to isogeny as

$$\text{Jac}(C^{\mathcal{L}}) \sim \prod_{f \in \mathcal{L} - \{0\}} \text{Jac}(C_f)$$

and therefore the genus $g(C^{\mathcal{L}})$ can be expressed as

$$g(C^{\mathcal{L}}) = \sum_{f \in \mathcal{L} - \{0\}} g(C_f)$$

in terms of the genera of the $C_f$.

\textbf{(1.2) Corollary.} Suppose that for all $f \in \mathcal{L} - \{0\}$ the curve $C_f$ is supersingular or rational. Then the fibre product $C^{\mathcal{L}}$ is supersingular or rational.

As ingredients for our fibre product we shall use special Artin-Schreier curves. We consider for $h \geq 1$ the vector space $\mathcal{R}_h$ of 2-linearized polynomials

$$\{R = \sum_{i=0}^{h} a_i x^{2^i} : a_i \in \mathbb{F}\}$$

and define

$$\mathcal{R}_h^* = \{R \in \mathcal{R}_h : a_h \neq 0\}.$$ 

We proved in [G-V 1] the following result.

\textbf{(1.3) Proposition.} The Artin-Schreier curve $C_R$ with affine equation $y^2 + y = xR(x)$ for $R \in \mathcal{R}_h^*$ is a (hyperelliptic) supersingular curve of genus $2^{h-1}$.

\section*{§2 The Construction.}

In this section we describe how to construct a curve of a given genus in characteristic 2. Here the construction is done over a finite extension of the prime field. In Section 3 we shall show that we can find such a curve over the prime field $\mathbb{F}_2$.

\textbf{(2.1) Theorem.} Let $\mathbb{F}$ be an algebraic closure of $\mathbb{F}_2$. For every positive genus $g$ there exists a supersingular curve over $\mathbb{F}$.

\textbf{Proof.} Take $g > 0$ and write $g$ as a dyadic expansion in the form

$$g = 2^{s_1}(1 + \ldots + 2^{r_1}) + 2^{s_2}(1 + \ldots + 2^{r_2}) + \ldots + 2^{s_t}(1 + \ldots + 2^{r_t}),$$

(1)
where \( s_i, r_i \in \mathbb{Z}_{\geq 0} \) and \( s_i \geq s_{i-1} + r_{i-1} + 2 \) for \( i = 2, \ldots, t \). We now choose for \( i = 1, \ldots, t \) a \( \mathbb{F}_2 \)-linear subspace \( L_i \) of \( \mathbb{F}(x) \) contained in \( \mathcal{R}_{u_i}^* \cup \{0\} \) with \( u_i = (s_i + 1) - \sum_{j=1}^{i-1}(r_j + 1) \) and \( \dim(L_i) = r_i + 1 \). We put \( \mathcal{L} = \bigoplus_{i=1}^{t}(xL_i) \). It follows directly from Propositions 1 and 2 that \( C^L \) is supersingular and that since \( u_i + 1 \geq u_i + 1 \) for \( 1 \leq i \leq t - 1 \) the genus satisfies

\[
g(C^L) = \sum_{f \in \mathcal{L} - \{0\}} g(C_f)
= \sum_{i=1}^{t} 2^{u_i - 1} \cdot 2^{\sum_{j=1}^{i-1}(r_j + 1)}(2^{r_i + 1} - 1)
= \sum_{i=1}^{t} 2^{s_i}(2^{r_i + 1} - 1).
\]

This last expression yields the expression for \( g \) in (1), hence \( g(C^L) = g \). □

From the preceding proof we conclude that there exists supersingular curves of genus \( g > 0 \) already over the field \( \mathbb{F}_{2^m} \) with \( m = \max_{1 \leq i \leq t}(r_i + 1) \), where the \( r_i \) occur in the expansion (1).

Example. We construct a supersingular curve of genus 30. We write 30 = 2(1+2+2^2+2^3) and this tells us that \( t = 1 \), \( s_1 = 1 \) and \( r_1 = 3 \). So our curve is defined over the finite field \( \mathbb{F}_{16} \). We set \( \mathbb{F}_{16} = \mathbb{F}_2(\alpha) \) with \( \alpha^4 + \alpha + 1 = 0 \). Let \( L \subset \mathcal{R}_{u_1}^* \cup \{0\} = \mathcal{R}_{s}^* \cup \{0\} \) be the 4-dimensional space generated by \( x^4, \alpha x^4, \alpha^2 x^4 \) and \( \alpha^3 x^4 \). Then \( \mathcal{L} = xL \) and \( C^L \) is the desired supersingular curve of genus 30. Its function field \( \mathcal{F} = \mathbb{F}_{16}(x, y_0, y_1, y_2, y_3) \) with \( y_i^2 + y_i = \alpha^i x^5 \) is a Galois extension of \( \mathbb{F}_{16}(x) \) of degree 16. Consider the element \( y = \sum_{i=0}^{3} \alpha^i y_i \). Then for all non-trivial \( \sigma \in \text{Gal}(\mathcal{F}/\mathbb{F}_{16}(x)) \) we have \( \sigma(y) \neq y \), hence \( \mathcal{F} = \mathbb{F}_{16}(x, y) \). We obtain

\[
y^{16} + y = \sum_{i=0}^{3} \alpha^i (y_i^{16} + y_i)
= \sum_{i=0}^{3} \alpha^i (\alpha^8 x^{40} + \alpha^{4i} x^{20} + \alpha^{2i} x^{10} + \alpha^i x^5)
= \alpha^6 x^{40} + x^{20} + \alpha^{12} x^{10} + \alpha^9 x^5.
\]

Thus we have found a supersingular curve of genus 30 over the field \( \mathbb{F}_{16} \) with affine equation

\[
y^{16} + y = \alpha^6 x^{40} + x^{20} + \alpha^{12} x^{10} + \alpha^9 x^5.
\]

§3 Equations for fibre products

The preceding example suggests to study curves of the following type. We consider curves \( C = C_{S, R} \) defined by an equation

\[
S(y) = xR_1(x) + (xR_2(x))^2 + \ldots + (xR_n(x))^{2^{n-1}},
\]
where $S \in \mathbb{F}[y]$ is a 2-linearized polynomial $S = y^{2^n} + A_{n-1}y^{2^{n-1}} + \ldots + A_0y$ with $A_0 \neq 0$ and where the $R_i \in \mathbb{F}[x]$ for $i = 1, \ldots, n$ are also 2-linearized polynomials (not all 0). We shall assume for a moment that this equation defines an irreducible curve. Consider the $\mathbb{F}_2$-vector space

$$\Sigma = \{ \sigma \in \mathbb{F} : S(\sigma) = 0 \}.$$  

An element $\sigma \in \Sigma$ acts on $C$ via $y \mapsto y + \sigma$. Thus the curve $C$ is a Galois covering of $\mathbb{P}^1$ with Galois group of type $\Sigma \cong (\mathbb{Z}/2\mathbb{Z})^n$. A $\mathbb{F}_2$-linear subspace $\Sigma'$ of $\Sigma$ of codimension 1 defines an irreducible quotient curve $C/\Sigma'$. If $\sigma \in \Sigma - \Sigma'$ then the linear subspace $\Sigma'$ corresponds to a splitting of $S = B(B + B(\sigma))$, (3)

where $B$ is the 2-linearized monic polynomial of degree $2^{n-1}$ in $\mathbb{F}[y]$ with zero set $\Sigma'$. Note that $B(\sigma) \in \mathbb{F}^*$ is independent of the choice of $\sigma \in \Sigma - \Sigma'$. If we put

$$B = y^{2^{n-1}} + B_{n-2}y^{2^{n-2}} + \ldots + B_0y,$$  

and $\beta = B(\sigma)$ then by comparing coefficients, (3) is equivalent to the system of equations

$$\begin{align*}
\beta B_0 + 0 &= A_0, \\
B_{i-1}^2 + \beta B_i &= A_i & \text{for } i = 1, \ldots, n-2, \\
B_{n-2}^2 + \beta &= A_{n-1}.
\end{align*}$$  

(5)

The compatibility of (5) comes down to the equation

$$\sum_{j=1}^{n} \frac{A_{n-j}^{2^{j-1} - 1}}{\beta^{2^{j-1} - 1}} = 1 \text{ or } \beta^{2^n} + \sum_{j=1}^{n} A_{n-j}^{2^{j-1}} \beta^{2^n - (2^j - 1)} = 0.$$  

(6)

Observe that $\alpha = \beta^{-1}$ satisfies a linearized equation, namely

$$A_0^{2^{n-1}} \alpha^{2^n} + A_1^{2^{n-2}} \alpha^{2^{n-1}} + \ldots + A_{n-1} \alpha^2 + \alpha = 0.$$  

(7)

Define the $\mathbb{F}_2$-vector space

$$A = \{ \alpha \in \mathbb{F} : \alpha \text{ satisfies (7)} \}.$$  

The elements of $A - \{0\}$ parametrize the hyperplanes of $\Sigma'$ of $\Sigma$. The hyperplane corresponding to $\alpha \in A - \{0\}$ will be denoted by $\Sigma_\alpha$. Moreover, we set

$$T = xR_1 + (xR_2)^2 + \ldots + (xR_n)^{2^{n-1}}.$$  

(3.1) Lemma. Each quotient curve $C_\alpha := C/\Sigma_\alpha$ with $\alpha \in A - \{0\}$ is of the form

$$w^2 + w = \alpha^2 T,$$  

(8)

where $w = \alpha B$ with $B$ corresponding to $\Sigma_\alpha$.

Proof. One checks that $w$ is invariant under $y \mapsto y + \sigma$ with $\sigma \in \Sigma_\alpha$. Substitution of (3) in (2) yields (8). □
Corollary. The curve in (8) is $\mathbb{F}[x]$-isomorphic to

$$w^2 + w = \alpha^2 x R_1 + \alpha x R_2 + \alpha^{2^{-1}} x R_3 + \ldots + \alpha^{2^{-(n-2)}} x R_n.$$  \hfill (9)

and therefore it is supersingular if not rational.

\textbf{Proof.} Consider the $\mathbb{F}[x]$-isomorphism

$$w \mapsto w + \sum_{i=2}^{n} \sum_{j=0}^{i-2} (\alpha^{2^{-i}} x R_i)^{2^j}.$$  

and apply Proposition (1.3). $\square$

Proposition. If the curve $C$ defined by (2) is irreducible then it is a fibre product which is supersingular if not rational. Its jacobian is up to isogeny the product of the jacobians of the curves given in (8) with $\alpha \in A - \{0\}$.

\textbf{Proof.} Choose a $\mathbb{F}_2$-basis of $A$, say $\alpha_1, \ldots, \alpha_n$. The curves $C_\alpha = C/\Sigma_\alpha$ are quotients of $C$, hence $C$ admits a morphism $\phi : C \rightarrow C_{\alpha_1} \times \ldots \times C_{\alpha_n}$, the fibre product of the $C_{\alpha_i}$ with respect to the (canonical) maps $C_{\alpha_i} \rightarrow \mathbb{P}^1$. Since the $\alpha_i$ are $\mathbb{F}_2$-independent, Galois theory and Lemma (3.1) yield that the function fields of the curves $C_{\alpha_1} \times \ldots \times C_{\alpha_j}$ and $C_{\alpha_{j+1}}$ are linearly disjoint for $j = 1, \ldots, n - 1$. So the fibre product $C_{\alpha_1} \times \ldots \times C_{\alpha_n}$ is a covering of degree $2^n$ of $\mathbb{P}^1$. Since $C$ is also a covering of $\mathbb{P}^1$ of degree $2^n$ it follows that $\phi$ is an isomorphism. The Proposition now follows from Proposition (1.1) and from Corollary (3.2). $\square$

The condition that the curve $C$ be irreducible is given in the following Lemma.

Lemma. The curve defined by (2) is irreducible if and only if the $n$-dimensional $\mathbb{F}_2$-vector space $\mathcal{L}$ of functions $\alpha^2 T$ with $\alpha \in A$ satisfies $\mathcal{L} \cap \varphi(\mathbb{F}(x)) = \{0\}$.

\textbf{Proof.} The implication “$\Rightarrow$” follows from Proposition 3.3. As to the implication “$\Leftarrow$”, we use the theory of Artin-Schreier extensions (see [B], Ch. V, §11). According to that theory the compositum of the function fields $\mathbb{F}(x, w_\alpha)$ with $w_\alpha = \alpha B$ satisfying (8) has degree

$$\#(\mathcal{L}/\mathcal{L} \cap \varphi(\mathbb{F}(x))) = 2^n$$

over $\mathbb{F}(x)$. Comparison with the degree of $y$ in (2) shows the irreducibility. $\square$

Theorem. For every integer $g > 0$ there exists a supersingular curve of genus $g$ over the prime field $\mathbb{F}_2$.

\textbf{Proof.} We construct a supersingular curve of the form (2) with prescribed genus $g > 0$. Recall the binary expansion of $g$ given in (1)

$$g = 2^{s_1}(1 + \ldots + 2^{r_1}) + 2^{s_2}(1 + \ldots + 2^{r_2}) + \ldots + 2^{s_t}(1 + \ldots + 2^{r_t})$$
where \( s_i, r_i \in \mathbb{Z}_{\geq 0} \) and \( s_i \geq s_{i-1} + r_{i-1} + 2 \) for \( i = 2, \ldots, t \). By \( w \) we denote the binary weight \( w = \sum_{i=1}^{t} (r_i + 1) \) of \( g \). First we determine the LHS \( S(y) \in \mathbb{F}_2[y] \) of (2) and the \( w \)-dimensional \( \mathbb{F}_2 \)-vector space \( A \).

We start with \( F_0(x) = x \) and we construct inductively a sequence of \( \mathbb{F}_2 \)-linearized polynomials \( F_i \in \mathbb{F}_2[x] \) for \( i = 1, \ldots, t \) as follows. We set

\[
F_i = (F_{i-1})^{2r_i+1} + F_{i-1}.
\]

Obviously, \( F_{i-1} \) divides \( F_i \) for \( i = 1, \ldots, t \).

Let \( S(y) = F_t(y) \in \mathbb{F}_2[y] \). It has degree \( 2^w \). Furthermore we define \( \mathbb{F}_2 \)-linear spaces

\[
A^{(i)} = \{ \alpha \in \mathbb{F} : F_i(\alpha) = 0 \} \quad \text{for} \quad i = 1, \ldots, t.
\]

We set \( A = A^{(t)} \). By the divisibility property of the \( F_i \) the subspaces \( A^{(i)} \) form a flag in \( A \).

Now we consider for \( 1 \leq i \leq t - 1 \) the polynomials

\[
F_i(\alpha, x) = F_i(\alpha)x^{2h_i+1} \in \mathbb{F}_2[\alpha, x],
\]

where the \( h_i = s_{i+1} - w + 1 + \sum_{j=i+1}^{t} (r_j + 1) \) form a monotonically increasing sequence. We define for \( j = 0, \ldots, w - 1 \) polynomials \( xR_{w-j}(x) \in \mathbb{F}_2[x] \) with \( R_{w-j} \) 2-linearized by writing

\[
\sum_{i=1}^{t-1} F_i(\alpha, x) = \sum_{j=0}^{w-1} xR_{w-j}(x)\alpha^{2^j}
\]

Here \( xR_{w-j} \) is the sum (possibly empty) of those monomials \( x^{2h_i+1} \) occuring in the polynomials \( F_i(\alpha, x) \) which have the monomial \( \alpha^{2^j} \) as coefficient.

For \( \alpha \in A - \{0\} \) the curves \( C_\alpha \) with equation (9) can be written as

\[
w^2 + w = F_{t-1}(\alpha)x^{2h_t+1} + \ldots + F_0(\alpha)x^{2h_1+1}
\]

(10)(after we have converted the coefficients \( \alpha^2, \alpha, \ldots, \alpha^{2-(w-2)} \) to \( \alpha^{2w-1}, \alpha^{2w-2}, \ldots, \alpha \)). For the \( 2^{w-(r_t+1)(2r_t+1) - 1} \) values of \( \alpha \in A - A^{(t-1)} \) the irreducible Artin-Schreier curve \( C_\alpha \) with equation (10) has genus \( 2^{w-(r_t+1)(2r_t+1)} \) and these curves \( C_\alpha \) contribute \( 2^{2^t(1+2+\ldots+2^{r_t})} \) to the genus of the fibre product (2). The curves \( C_\alpha \) with \( \alpha \in A^{(t-1)} - A^{(t-2)} \) contribute \( 2^{s_{t-1}(1+2+\ldots+2^{r_{t-1}})} \) to the genus. Continuing in this way we see that the supersingular curve over \( \mathbb{F}_2 \) with affine equation

\[
S(y) = xR_1 + (xR_2)^2 + \ldots + (xR_w)^{2^{w-1}}
\]

has the prescribed genus. □

**Example.** Take \( g = 221 = 1 + 2^2(1 + 2 + 4) + 2^6(1 + 2) \). We have \( s_1 = 0, s_2 = 2, s_3 = 6; r_1 = 0, r_2 = 2, r_3 = 1 \) and \( w = 6 \). We find

\[
F_0(x) = x, \quad F_1(x) = x^2 + x, \quad F_2(x) = x^{16} + x^8 + x^2 + x,
\]

\[
F_3(x) = x^{64} + x^{32} + x^{16} + x^4 + x^2 + x.
\]
The space $A$ equals $\{ \alpha \in F : F_3(\alpha) = 0 \}$. For $i = 0, \ldots, 2$ the polynomials $F_i(\alpha, x)$ are

$$F_0(\alpha, x) = \alpha x^3, \quad F_1(\alpha, x) = (\alpha^2 + \alpha)x^5, \quad F_2(\alpha, x) = (\alpha^{16} + \alpha^8 + \alpha^2 + \alpha)x^9.$$ 

From the identity

$$\sum_{i=0}^{2} F_i(\alpha, x) = \sum_{j=0}^{w-1} xR_{w-j}(x)\alpha^{2^j}$$

we get

$$xR_6 = x^9 + x^5 + x^3, \quad xR_5 = x^9 + x^5, \quad xR_3 = x^9, \quad xR_2 = x^9, \quad xR_1 = xR_4 = 0.$$ 

This gives a supersingular curve of genus 221 defined by $F_3(y) = \sum_{k=1}^{6} (xR_k)^{2^{k-1}}$ i.e. by the equation

$$y^{64} + y^{32} + y^{16} + y^4 + y^2 + y = x^{288} + x^{160} + x^{144} + x^{96} + x^{80} + x^{36} + x^{18}.$$ 

§4 Number of moduli

Here we count the number of moduli for our families. In the investigation of the curves $y^2 + y = xR(x)$ for $R \in \mathcal{R}_h^*$ in [G-V 1] the polynomial

$$E_{h,R}(x) = R(x)^{2^h} + \sum_{i=0}^{h} (a_i x)^{2^{h-i}}$$

of degree $2^{2h}$ played an important role. We define the radical $\overline{W}_R$ of $R$ as the subspace of $F$ formed by the elements satisfying the equation $E_{h,R}(x) = 0$.

(4.1) Proposition. Let $h \geq 2$ and let $R = \sum_{i=0}^{h} a_i x^{2^i}$ and $R' = \sum_{i=0}^{h} a'_i x^{2^i}$ be elements of $\mathcal{R}_h^*$. Then the curves $C_R$ and $C_{R'}$ are isomorphic over $F$ if and only if there exists a $\rho \in F^*$ such that $a'_i = a_i \rho^{2^i+1}$ for $i = 1, \ldots, h$.

Proof. Since both $C_R$ and $C_{R'}$ are hyperelliptic curves an isomorphism $\alpha$ induces an isomorphism $\alpha' : \mathbb{P}^1 \to \mathbb{P}^1$ which fixes the (unique) branch point $\infty$ and is of the form $x \mapsto \lambda x + \mu$ with $\lambda, \mu \in F, \lambda \neq 0$. Let $\overline{W}_R$ (resp. $\overline{W}_{R'}$) be the radical of $R$ (resp. of $R'$). Then by [G-V 1] we have $\lambda^{-1}\overline{W}_R = \overline{W}_{R'}$. This implies that $E_{h,R}(\lambda X) = c_\lambda E_{h,R'}(X)$. By writing $X^{2h} E_{h,R}(X) = \sum_{i}(U_i + U_i^2)$ with $U_i = a_i^{2^{h-i}} X^{2h+2^{h-i}}$ we see that

$$X^{2h} c_\lambda a_i^{2^{h-i}} = X^{2h+2^{h-i}} a_i^{2^{h-i}} \quad \text{for } i \geq 1$$

and

$$\lambda^{2h} c_\lambda \in F^*_{2i} \quad \text{for all } i \geq 1 \quad \text{with } a_i \neq 0.$$
Relation (12) implies $\lambda^2 c_\lambda \in \mathbb{F}^*_2 d$ with $d = \gcd \{ i \geq 1 : a_i \neq 0 \}$. There exists an element $\eta \in \mathbb{F}^*_2 d$ such that we can write

$$
\lambda^2 c_\lambda = \eta(2^i+1)2^{h-i} \quad \text{for} \quad i \geq 1 \quad \text{with} \quad a_i \neq 0.
$$

Substituting (13) in (11) we find

$$
a'_i = (\lambda/\eta)^{2^{i+1}} a_i \quad \text{for} \quad i = 1, \ldots, h.
$$

Conversely, the relation $a'_i = \rho^{2^i+1} a_i$ for $i = 1, \ldots, h$ shows that $\rho x R(\rho x) = x R'(x) + (\rho^2 a_0 + a'_0) x^2$. Since for fixed $R \in \mathcal{R}^*$ and varying $a \in \mathbb{F}$ the curves $C_{R+ax^2}$ are mutually isomorphic over $\mathbb{F}$ we conclude $C_R \cong C_{R'}$. \(\square\)

**Remark.** The conclusion of the Lemma still holds for $h = 1$ if we restrict to isomorphisms which are isomorphisms of Artin-Schreier coverings of $\mathbb{P}^1$ of degree 2.

**4.2 Corollary.** Let $n \geq 1$. The intersection of the supersingular locus with the hyperelliptic locus in the moduli space $\mathcal{M}_{2^n} \otimes \mathbb{F}_2$ of curves of genus $g = 2^n$ has dimension $\geq n$.

Furthermore, consider two $n$-dimensional $\mathbb{F}_2$-subspaces $L$ and $L'$ of polynomials $R = \sum_{i=1}^h a_i x^{2^i} \in \mathbb{F}[x]$ with $a_1 \neq 0$ if $R \neq 0$. Let $\mathcal{L} = x L$ and $\mathcal{L}' = x L'$ and set $C = C_{\mathcal{L}}$ and $C' = C_{\mathcal{L}'}$. We then have:

**4.3 Lemma.** The curves $C$ and $C'$ are isomorphic as Galois covers of type $(\mathbb{Z}/2\mathbb{Z})^n$ of $\mathbb{P}^1$ if and only if there exists a $\rho \in \mathbb{F}^*$ such that under $x \mapsto \rho x$ the space $\mathcal{L}$ is transformed into $\mathcal{L}'$.

**Proof.** If the curves $C$ and $C'$ are isomorphic as Galois covers of $\mathbb{P}^1$ then the corresponding quotient curves $C_R$ and $C_{R'}$ of genus $> 0$ (for $R \in L$) are isomorphic as covers of $\mathbb{P}^1$. By Lemma (4.1) and the subsequent remark this happens only if there is a $\rho \in \mathbb{F}^*$ such that the transformation $x \mapsto \rho x$ transforms $x R$ into $x R'$ for $R \in L$. \(\square\)

**4.4 Proposition.** Let $g \geq 2$ be written as in (1). Then the supersingular locus in $\mathcal{M}_g \otimes \mathbb{F}_2$ has dimension $\geq \sum_{i=1}^h (r_i + 1)u_i - 1$, where $u_i = (s_i + 1) - \sum_{j=i}^{i-1} (r_j + 1)$.

**Proof.** We consider curves of the form $C_{\mathcal{L}}$ of genus $g \geq 2$ with $\mathcal{L} = \bigoplus x L_i$ as in the proof of Theorem (2.1). Let $m = \sum_{i=1}^h (r_i + 1)u_i$. Then the $m$ coefficients of the polynomials in a basis of $\mathcal{L}$ which is a union of the bases of the $(r_i + 1)$-dimensional summands $x L_i$ in $\mathcal{R}^* u_i$ define an open subset $Q$ of affine $m$-space $\mathbb{A}_F^m$. Take the $(m-1)$-dimensional quotient of $Q$ under the action of $\mathbb{F}^*$. A given curve $C$ of genus $> 1$ can be written only in finitely many ways as a Galois cover of $\mathbb{P}^1$ since $\# \text{Aut}(C)(\mathbb{F}) < \infty$. Then Lemma (4.3) implies that the natural morphism $Q \to \mathcal{M}_g \otimes \mathbb{F}_2$ is quasi-finite (onto its image). This proves our result. \(\square\)

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