Exactly solvable effective mass $D$-dimensional Schrödinger equation for pseudoharmonic and modified Kratzer problems

Sameer M. Ikhdair$^1$ and Ramazan Sever$^2$  

$^1$Department of Physics, Near East University, Nicosia, North Cyprus, Turkey  
$^2$Department of Physics, Middle East Technical University, 06800, Ankara, Turkey  

(Dated: November 6, 2018)

Abstract

We employ the point canonical transformation (PCT) to solve the $D$-dimensional Schrödinger equation with position-dependent effective mass (PDEM) function for two molecular pseudoharmonic and modified Kratzer (Mie-type) potentials. In mapping the transformed exactly solvable $D$-dimensional ($D \geq 2$) Schrödinger equation with constant mass into the effective mass equation by employing a proper transformation, the exact bound state solutions including the energy eigenvalues and corresponding wave functions are derived. The well-known pseudoharmonic and modified Kratzer exact eigenstates of various dimensionality is manifested.

Keywords: Bound states, point canonical transformation, position dependent effective mass function, pseudoharmonic potential, modified Kratzer potential

PACS numbers: 03.65.-w

*E-mail: sikhdaireneu.edu.tr  
†E-mail: severmetu.edu.tr
I. INTRODUCTION

The solution of the Schrödinger equation with position-dependent effective mass (PDEM) for an arbitrary central potential has attracted attention over the past years (cf., e.g. [1] and the references therein). The motivation in this direction arises from considerable applications in the different fields of the material science and condensed matter physics. For instance, such applications in the case of the bound states in quantum system [2], the nonrelativistic Green’s function for quantum systems with the position-dependent mass [3], the Dirac equation with position-dependent mass in the Coulomb field [3], electronic properties of semiconductors [4], $^3He$ cluster [5], quantum dots [6], quantum liquids [7], graded alloys and semiconductor heterostructures [8,9], the dependence of energy gap on magnetic field in semiconductor nano-scale quantum rings [10], the solid state problems with the Dirac equation [11], etc. Almost all of those works mentioned above were focused on obtaining the energy eigenvalues and the potential function for the given quantum system with the PDEM function. The wave functions were either obtained by the solutions to the Schrödinger equation with the constant mass, or a few lower excited states were obtained by acting of the creation operator on the ground state. The effective potentials are the sum of the real potential form and the modification terms emerged from the location dependence of the effective mass [12].

Taking into consideration the PDEM, the main concern, in this work, is in obtaining the energy spectra and/or wavefunctions of the $D$-dimensional Schrödinger equation with a given PDEM for central potentials by the point canonical transformation (PCT) approach [1]. In modern theory, high dimensions are also of interests in many fields [13-25].

In this work, we employ the PCT to solve the $D$-dimensional Schrödinger equation with PDEM for the pseudoharmonic [26-31] and modified Kratzer [32-37] potentials through mapping this wave equation into the well-known exactly solvable $D$-dimensional Schrödinger equation with constant mass for a given PDEM function [1]. Indeed, the PCT approach has enabled us to obtain the exact effective mass bound state solutions including the energy spectrum and corresponding wave functions in any dimension for the exactly solvable classes of quantum molecular potentials.

This work is organized as follows: in section 2, we introduce the methodology. Section 3 is mainly devoted to obtain the exact bound state energy eigenvalue and eigen function solutions of the $D$-dimensional Schrödinger equation with a given PDEM function for two
II. METHODOLOGY

The $D$-dimensional PDEM Schrödinger equation for a central potential $V(r)$ takes the form

$$\nabla^2_D \left( \frac{1}{m(r)} \nabla_D \psi_{l_1 \cdots l_{D-2}}^{(l_{D-1}=l)}(x) \right) + 2 \left[ E - V(r) \right] \psi_{l_1 \cdots l_{D-2}}^{(l_{D-1}=l)}(x) = 0,$$

where the position-dependent mass distribution $m(r)$ is a real function. Here the wave functions $\psi_{l_1 \cdots l_{D-2}}^{(l_{D-1}=l)}(x)$ belong to the energy eigenvalues $E$ and $V(r)$ stands for the $D$-dimensional standard central potential in the configuration space coordinates. Here $r$ represents the $D$-dimensional radius $\left( \sum_{i=1}^{D} x_i^2 \right)^{1/2}$. Atomic units will be used throughout, with $\hbar/2\pi = \hbar = m_0 = e = 1$. Going over to a spherical coordinate system with $D - 1$ angular variables and one radial coordinate we can write [21-25]

$$\psi_{l_1 \cdots l_{D-2}}^{(l)}(x) = r^{-(D-1)/2} R_l(r) Y_{l_1 \cdots l_{D-2}}^{(l)}(\mathbf{x}),$$

where $Y_{l_1 \cdots l_{D-2}}^{(l)}(\mathbf{x})$ represents contribution from the hyperspherical harmonics that arise in higher dimensions with $x$ representing the $D$-dimensional position vector. With the substitutions

$$\nabla^2_D \left( \frac{1}{m(r)} \nabla_D \psi_{l_1 \cdots l_{D-2}}^{(l_{D-1}=l)}(x) \right) = \left( \nabla^2_D \frac{1}{m(r)} \right) \cdot \left( \nabla_D \psi_{l_1 \cdots l_{D-2}}^{(l_{D-1}=l)}(x) \right) + \frac{1}{m(r)} \nabla^2_D \psi_{l_1 \cdots l_{D-2}}^{(l_{D-1}=l)}(x),$$

$$\nabla^2_D = \frac{\partial^2}{\partial r^2} + \frac{(D - 1)}{r} \frac{\partial}{\partial r} - \frac{l(l + D - 2)}{r^2},$$

and the wave functions (2) into Eq. (1) will result in the following time independent $D$-dimensional PDEM $l_D$th partial wave radial Schrödinger wave equation

$$\left\{ \frac{d^2}{dr^2} + m'(r) \left( \frac{D - 1}{2} \frac{1}{r} - \frac{d}{dr} \right) \frac{l_D (l_D + 1)}{r^2} + 2m(r) [E - V(r)] \right\} R_{n,l}(r) = 0,$$

where $m'(r) = dm(r)/dr$ and the transformation $l \to l_D = l + (D - 3)/2$, $D \geq 2$ and $\nabla^2_D$ stand for the $D$-dimensional angular momentum and Laplacian, respectively. Moreover, the $D = 1$ can be obtained through inserting $l = 1$ and 0 or $l_D = 0$ and $-1$. In the present work,
we are concerned in bound states, i.e., \( E < 0 \). We also should be careful about the behavior of the wave function \( R(r) \) near \( r = 0 \) and \( r \to \infty \). It may be mentioned that \( R_i(r) \) should be normalizable [8]. Obviously, it can be shown, most readily, that for a constant mass, i.e., \( m'(r) = 0 \) case, the above equation with potential function \( \tilde{V}(s) \), angular momentum \( \Lambda(l) \) and energy spectrum \( \tilde{E} \) reduces to the usual case of independent-position mass: [19,21-23]:

\[
\left\{ \frac{d^2}{ds^2} - \frac{\Lambda_D(l) (\Lambda_D(l) + 1)}{s^2} + 2 \left[ \tilde{E} - \tilde{V}(s) \right] \right\} \tilde{R}_{n,\Lambda(l)}(s) = 0, \tag{5}
\]

where \( \Lambda_D(l) = \Lambda(l) + (D - 3)/2 \). Consequently, the solutions for a particular central potential \( \tilde{V}(s) \) are the same as long as \( D + 2\Lambda(l) \) remains unaltered. For example, the \( s \)-wave eigensolution \( \tilde{R}_0 \) and energy spectrum \( \tilde{E} \) in four-dimensional solutions are identical to the \( p \)-wave two-dimensional solutions \( \Lambda(l) = 0, D = 4 \longrightarrow \Lambda(l) = 1, D = 2 \). For more detail on inter-dimensional degeneracies the reader may refer to, e.g. [19,23,38,39]. We apply the following point canonical transformation (PCT) \( s \to r \) (i.e., mapping function \( s = q(r) \)) with the substitution of the wave function

\[
\tilde{R}_{n,\Lambda(l)}(s) g^2(r), \tag{6}
\]

into Eq. (5) and after some simple algebra, the transformed Schrödinger equation becomes

\[
\left\{ \frac{d^2}{dr^2} + \left( \frac{2q'}{q} - \frac{q''}{q} \right) \frac{d}{dr} + \left( \frac{g''}{g} - \frac{q''}{q' g} \right) \right. \\
- \Lambda_D(l) (\Lambda_D(l) + 1) \left( \frac{q'}{q} \right)^2 + 2 \left( q' \right)^2 \left[ \tilde{E} - \tilde{V}(q(r)) \right] \left\} \tilde{R}_{n,\Lambda(l)}(r) = 0, \tag{7}
\]

where the primes denote differentiation with respect to \( r \). Further, comparing Eq. (7) with Eq. (4), we find the following PCT transformations:

\[
\frac{1}{2m(r)} \left( \frac{q''}{q'} - 2 \frac{q'}{g} - \frac{m'(r)}{m(r)} \right) = 0, \quad q' = m(r)g^2(r) \tag{8}
\]

\[
V(r) = \frac{(q')^2}{m(r)} \tilde{V}(s), \tag{9}
\]

\[
E_n = \frac{(q')^2}{m(r)} \tilde{E}_n, \tag{10}
\]

and

\[
\Lambda_D(l) (\Lambda_D(l) + 1) \left( \frac{q'}{q} \right)^2 = \frac{l_D (l_D + 1)}{r^2} - \frac{D - 1}{2} \frac{m'(r)}{m(r)} - \frac{1}{2} \left[ F(m(r)) - F(q') \right], \tag{11}
\]

where \( F(x) = \frac{x''}{x} - \frac{3}{2} \left( \frac{x'}{x} \right) \). Therefore, Eq. (6) and Eqs. (8)-(11) can be also found for any potential system with PDEM.
III. APPLICATIONS

We solve the $D$-dimensional PDEM Schrödinger equation exactly for two potentials: the pseudoharmonic potential [26-31] and the modified Kratzer molecular potential [32-37]. The transformation function $g(r)$ will be found for the given effective mass function $m(r) = m_0 r^\lambda$ and the selected PCT function $q(r) = r^\nu$.

A. pseudoharmonic potential

The pseudoharmonic potential is given by [26-31]

$$\tilde{V}(s) = V_e \left( \frac{s}{r_e} - \frac{r_e}{s} \right)^2,$$

as a reference potential where $V_e = \frac{1}{8} \kappa r_e^2$ is the dissociation energy between two atoms in a solid with $\kappa$ is the force constant and $r_e$ is the equilibrium bond length. The eigenvalue problem (5) for $\tilde{V}(s)$ in (12) can be solved analytically to get the exact $D$-dimensional results for energy eigenvalues and wavefunctions of this system (in units in which $m_0 = \hbar = 1$) as [26-30]:

$$\tilde{E}_{n,\Lambda(l)} = -2V_e + \sqrt{\frac{V_e}{2r_e^2}} \left( 4n + 2 + \sqrt{(D + 2\Lambda(l) - 2)^2 + 8V_e^2} \right),$$

and

$$\tilde{R}_{n,\Lambda(l)}(s) = A_{n,l} \sqrt{\frac{(\Lambda(l) + D/2 - 1)^2 + 2V_e r_e^2 + 1}{\Gamma(n + \sqrt{(\Lambda(l) + D/2 - 1)^2 + 2V_e r_e^2 + 1})}} \exp \left( -\frac{1}{2} \sqrt{\frac{2V_e}{r_e^2}} s^2 \right) \times F \left( -n, \sqrt{(\Lambda(l) + D/2 - 1)^2 + 2V_e r_e^2 + 1}; \sqrt{\frac{2V_e}{r_e^2}} s^2 \right),$$

with

$$A_{n,l} = \sqrt{\frac{2 \left( \sqrt{2V_e/r_e} \right)^{\sqrt{(\Lambda(l) + D/2 - 1)^2 + 2V_e r_e^2 + 1}} n!}{\Gamma(n + \sqrt{(\Lambda(l) + D/2 - 1)^2 + 2V_e r_e^2 + 1})}}$$

where $n = 0, 1, 2, \cdots$ and $l = 0, 1, 2, \cdots$ signify the usual radial and angular momentum quantum numbers, respectively and $A_{n,l}$ being the normalization constant. Here $F(-n, k + 1, x)$ is a confluent hypergeometric function. We follow Ref. [1] by taking the power law PDEM function $m(r) = m_0 r^\lambda$ and the PCT function $q(r) = r^\nu$, where $m_0$ is the rest mass and $\lambda$ and $\nu$ are two non-zero real parameters. We consider only the case where $(q')^2/m(r)$
is constant for which \( \nu = 1 + \lambda/2, \lambda \neq 2 \) to avoid position-dependent energy. Inserting them into Eq. (6) and Eqs. (9)-(11), we obtain

\[
V(r) = \frac{m_0}{2} \left( r^{1+\lambda/2} - \frac{r_e^2}{r^{1+\lambda/2}} \right)^2 C^2, \tag{15}
\]

\[
E_{n,l} = \frac{2 + \lambda}{2} \left[ -\eta r_e^2 + 1 + 2n + \sqrt{\Lambda(l) + D/2 - 1} + \eta^2 r_e^4 \right] C, \tag{16}
\]

and

\[
R_{n,l}(r) = a_l (\eta r) (1+\frac{\lambda}{2}) \left( \sqrt{\Lambda(l) + D/2 - 1} + \frac{\lambda}{2} \right) \exp \left( -\frac{\eta^2 r_e^2}{2} \right) F \left( -n, \sqrt{\Lambda(l) + D/2 - 1}; 1; \eta^2 r_e^4 \right), \tag{17}
\]

where \( \eta = \sqrt{\kappa}/2 = 2m_0C/(2 + \lambda) \) with \( C = \nu \eta/m_0 \) is a real potential parameter and \( \Lambda(l) \) is

\[
\Lambda(l) = -\frac{(D - 2)}{2} + \frac{1}{2 + \lambda} \sqrt{(D + 2l - 2)^2 + (2 + \lambda)^2 - 2(2 + \lambda D)}. \tag{18}
\]

The radial wave function (17) must vanish as \( r \to 0 \) and \( r \to \infty \). For the trivial case where \( \lambda = 0 \), i.e., constant mass \((m = m_0)\), we obtain \( \Lambda(l) = l \) from Eq. (18). Hence, we find out that Eqs. (15)-(17) agree with Eqs. (12)-(14). The same procedure leads to the other exact solvable classes belong to various values of parameter \( \lambda \). For example, if we insert \( \lambda = 2 \), the following potential function with its energy spectrum and wave functions are obtained

\[
V(r) = \frac{2\eta^2}{m_0} \left( r^2 - \frac{r_e^2}{r^2} \right)^2, \tag{19}
\]

\[
E_{n,l} = \frac{4\eta}{m_0} \left[ -\eta r_e^2 + 1 + 2n + \sqrt{4M^2 + (2 + \lambda)^2} + \eta^2 r_e^4 \right], \tag{20}
\]

and

\[
R_{n,l}(r) = b_l (\eta r) \sqrt{\Lambda_1(l) + D - 2} \exp \left( -\frac{\eta^2 r_e^2}{2} \right) F \left( -n, \sqrt{\Lambda(l) + D/2 - 1}; 1; \eta^4 \right), \tag{21}
\]

with

\[
\Lambda_1(l) = -\frac{(D - 2)}{2} + \frac{1}{4} \sqrt{(D + 2l - 2)^2 - 4(D - 3)}, \tag{22}
\]

where \( n, l = 0, 1, 2, \ldots \). The pseudoharmonic potential can be treated exactly in three as well as in one and two dimensions. We note the special cases \( D = 1, 2, \) and 3. For \( D = 2 \), with the customary notation \( l = M \) and \( r = \rho \):

\[
E_{n,M} = \frac{\eta \rho^2}{m_0} \left[ -\eta r_e^2 + 1 + 2n + \sqrt{4M^2 + \lambda^2} + \eta^2 r_e^4 \right], \tag{23}
\]

\[
\Lambda_1(l) = -\frac{(D - 2)}{2} + \frac{1}{4} \sqrt{(D + 2l - 2)^2 - 4(D - 3)}, \tag{22}
\]

where \( n, l = 0, 1, 2, \ldots \). The pseudoharmonic potential can be treated exactly in three as well as in one and two dimensions. We note the special cases \( D = 1, 2, \) and 3. For \( D = 2 \), with the customary notation \( l = M \) and \( r = \rho \):

\[
E_{n,M} = \frac{\eta \rho^2}{m_0} \left[ -\eta r_e^2 + 1 + 2n + \sqrt{4M^2 + \lambda^2} + \eta^2 r_e^4 \right], \tag{23}
\]
and
\[
R_{n,M}(\rho) = a_l (\eta \rho)^{(1+\frac{1}{2})} \left( \sqrt{\frac{4M^2 + \lambda^2}{(2+\lambda)^2} + \eta^2 r^2_e + \frac{1}{2}} \right) + \frac{\lambda}{2} \exp \left( -\frac{\eta}{2} \rho^{2+\lambda} \right) F \left( -n, \sqrt{\frac{4M^2 + \lambda^2}{(2+\lambda)^2} + \eta^2 r^2_e + \frac{1}{2}} \right) (n, M = 0, 1, 2, \cdots). \tag{24}
\]

Furthermore, for constant mass case ($\lambda = 0$):
\[
E_{n,M} = \frac{\eta}{m_0} \left[ -\eta r^2_e + 1 + 2n + \sqrt{M^2 + \eta^2 r^2_e} \right], \tag{25}
\]
and
\[
R_{n,M}(r) = a_l (\eta r)^{(1+\frac{1}{2})} \left( \sqrt{\frac{4M^2 + \lambda^2}{(2+\lambda)^2} + \eta^2 r^2_e + \frac{1}{2}} \right) + \frac{\lambda}{2} \exp \left( -\frac{\eta}{2} r^{2+\lambda} \right) F \left( -n, \sqrt{\frac{4M^2 + \lambda^2}{(2+\lambda)^2} + \eta^2 r^2_e + \frac{1}{2}} \right), \tag{26}
\]
which are identical to with those given in Ref. [28]. For $D = 3$ :
\[
E_{n,l} = \frac{\eta \nu^2}{m_0} \left[ -\eta r^2_e + 1 + 2n + \sqrt{(2l + 1)^2 + \lambda (\lambda - 2)} \right], \tag{27}
\]
and
\[
R_{n,l}(r) = a_l (\eta r)^{(1+\frac{1}{2})} \left( \sqrt{\frac{(2l + 1)^2 + \lambda (\lambda - 2)}{(2+l)^2} + \eta^2 r^2_e + \frac{1}{2}} \right) + \frac{\lambda}{2} \exp \left( -\frac{\eta}{2} r^{2+\lambda} \right) \times F \left( -n, \sqrt{\frac{(2l + 1)^2 + \lambda (\lambda - 2)}{(2+l)^2} + \eta^2 r^2_e + \frac{1}{2}} \right), \tag{28}
\]
where $n, l = 0, 1, 2, \cdots$ and inserting $\lambda = 0$:
\[
E_{n,l} = \frac{\eta}{m_0} \left[ -\eta r^2_e + 1 + 2n + \sqrt{(l + 1/2)^2 + \eta^2 r^2_e} \right], \tag{29}
\]
and
\[
R_{n,l}(r) = a_l (\eta r)^{(1+\frac{1}{2})} \left( \sqrt{\frac{(l + 1/2)^2 + \eta^2 r^2_e + \frac{1}{2}}{(2+l)^2}} \right) + \frac{\lambda}{2} \exp \left( -\frac{\eta}{2} r^{2+\lambda} \right) \times F \left( -n, \sqrt{(l + 1/2)^2 + \eta^2 r^2_e + \frac{1}{2}} \right), \tag{30}
\]
which are identical with those given in Refs. [26,27,33]. For $D = 1$ ($s$-wave):
\[
E_n = \frac{\eta \nu^2}{m_0} \left[ -\eta r^2_e + 1 + 2n + \sqrt{\left( \frac{1 + \lambda}{2 + \lambda} \right)^2 + \eta^2 r^2_e} \right], \tag{31}
\]
and
\[
R_n(x) = a_l (\eta x)^{(1+\frac{1}{2})} \left( \sqrt{\left( \frac{1 + \lambda}{2 + \lambda} \right)^2 + \eta^2 r^2_e + \frac{1}{2}} \right) + \frac{\lambda}{2} \exp \left( -\frac{\eta}{2} x^{2+\lambda} \right) \times F \left( -n, \sqrt{\left( \frac{1 + \lambda}{2 + \lambda} \right)^2 + \eta^2 r^2_e + \frac{1}{2}} \right), \tag{32}
\]
and inserting $\lambda = 0$:

$$E_n = \frac{\eta}{m_0} \left[ -\eta r_e^2 + 1 + 2n + \frac{1}{2} \sqrt{1 + 4\eta^2 r_e^4} \right],$$  \hspace{1cm} (33)$$

and

$$R_n(x) = a_l(\eta x)^{\frac{1}{2}}(\sqrt{1+\eta^2 r_e^2}+1) \exp \left( -\frac{\eta}{2} x^2 \right) F \left( -n, \frac{1}{2} \sqrt{1 + 4\eta^2 r_e^4} + 1; \eta x^2 \right),$$  \hspace{1cm} (34)$$

where $n = 0, 1, 2, \ldots$.

**B. modified Kratzer molecular potential**

The modified Kratzer potential is [32-37]

$$\tilde{V}(s) = V_e \left( \frac{s - r_e}{s} \right)^2,$$  \hspace{1cm} (35)$$

as a reference potential. The eigenvalue problem (5) for $\tilde{V}(s)$ in (35) can be solved analytically to get the exact $D$-dimensional results for energy eigenvalues and wavefunctions of this system (in units in which $m_0 = \hbar = 1$) as [32-37]:

$$\tilde{E}_{n,\Lambda(l)} = V_e - \frac{1}{2a^2 \left( 1 + 2n + \sqrt{[D + 2\Lambda(l) - 2]^2 + 8V_e r_e^2} \right)}^2, \quad n = l = 0, 1, 2, \ldots$$  \hspace{1cm} (36)$$

and

$$\tilde{R}_{n,\Lambda(l)}(s) = B_l s^{\frac{1}{2}}(1+\sqrt{[D+2\Lambda(l)-2]^2+8V_e r_e^2}) \exp[-\beta s] \times F \left( -n, 1 + \sqrt{[D + 2\Lambda(l) - 2]^2 + 8V_e r_e^2}; 2ks \right),$$  \hspace{1cm} (37)$$

with $a = 1/(4V_e r_e)$ and $B_l$ is the normalization factor. The wave number $k$ for the modified Kratzer (pseudo-Coulomb) spectrum under consideration is

$$k = \frac{1}{a \left( 1 + 2n + \sqrt{[D + 2\Lambda(l) - 2]^2 + 8V_e r_e^2} \right)}.$$  \hspace{1cm} (38)$$

The PDEM function $m(r) = m_0 r^\lambda$ and the PCT function $q(r) = r^n$ are same as before. Inserting them into Eq. (6) and Eqs. (9)-(11), we obtain

$$V(r) = P \left( \frac{r^{1+\lambda/2} - r_e}{r^{1+\lambda/2}} \right)^2,$$  \hspace{1cm} (39)$$

8
\[
E_{n,l} = P - \frac{32m_0r_e^2P^2}{\left[ (1+2n)(2+\lambda) + \sqrt{(2+\lambda)^2[D+2\Lambda(l)-2]^2 + 32m_0r_e^2P} \right]^2}, \quad (40)
\]

and
\[
R_{n,l}(r) = b_1r^{\frac{1}{2}(1+\frac{\lambda}{2})}\left(1+\sqrt{16M^2+8V_e^2+4\lambda^2}\right)^\frac{1}{2}\exp\left(-\gamma_1r^{1+\frac{\lambda}{2}}\right)
\times F\left(-n,1+\sqrt{16M^2+8V_e^2+4\lambda^2};2\gamma_1r^{1+\frac{\lambda}{2}}\right), \quad (41)
\]

where \(P = \frac{(2+\lambda)^2}{4m_0}V_e\) is a real potential parameter and
\[
\gamma = \frac{16m_0r_eP}{(2+\lambda)^2\left[ (1+2n)(2+\lambda) + \sqrt{(2+\lambda)^2[D+2\Lambda(l)-2]^2 + 32m_0r_e^2P} \right]},
\]

which are identical with those given in Refs. [32,34-36] when \(\lambda\) is set to zero. The new angular momentum \(\Lambda(l)\) is as given in Eq. (18). Particularly, setting \(\lambda = 0\) with \(P = V_e/m_0\) into Eqs. (39)-(41), we recover the constant mass Eqs. (35)-(37). This manifests the generality of our solution for the mass function given by \(m(r) = m_0r^\lambda\). We note the special cases \(D = 1, 2\) and 3. For \(D = 2\), with customary notation \(l = M\), we obtain
\[
E_{n,M} = P - \frac{32m_0r_e^2P^2}{\left[ (1+2n)(2+\lambda) + \sqrt{16M^2+32m_0r_e^2P + 4\lambda^2} \right]^2}, \quad (42)
\]

and
\[
R_{n,M}(r) = b_1r^{\frac{1}{2}(1+\frac{\lambda}{2})}\left(1+\sqrt{16M^2+8V_e^2+4\lambda^2}\right)^\frac{1}{2}\exp\left(-\gamma_1r^{1+\frac{\lambda}{2}}\right)
\times F\left(-n,1+\sqrt{16M^2+8V_e^2+4\lambda^2};2\gamma_1r^{1+\frac{\lambda}{2}}\right), \quad (43)
\]

with
\[
\gamma_1 = \frac{16m_0r_eP}{(2+\lambda)^2\left[ 1 + 2n + \sqrt{16M^2+8V_e^2+4\lambda^2} \right]}, \quad (44)
\]

where \(n, M = 0, 1, 2, \cdots\). Further, when \(\lambda = 0\)
\[
E_{n,M} = P - \frac{8m_0r_e^2P^2}{\left[ 1 + 2n + \sqrt{4M^2+8m_0r_e^2P} \right]^2}, \quad (45)
\]

\[
R_{n,M}(r) = b_1r^{\frac{1}{2}+\sqrt{4M^2+2V_e^2}}\exp\left(-\gamma_1r\right) F\left(-n,1+2\sqrt{4M^2+2V_e^2};2\gamma_1r\right), \quad (46)
\]

where \(P = V_e/m_0\) is a real potential parameter and
\[
\gamma_1 = \frac{4m_0r_eP}{1 + 2n + \sqrt{16M^2+8V_e^2}}, \quad (47)
\]
which are consistent with those given in Ref. [28]. For $D = 3$:

$$E_{n,l} = P - \frac{32m_0r_e^2P^2}{\left[(1 + 2n)(2 + \lambda) + 2\sqrt{(1 + 2l)^2 + 8m_0r_e^2P + \lambda(\lambda - 2)}\right]^2},$$

(48)

and

$$R_{n,l}(r) = b_l F\left(\frac{1 + \lambda}{2}\right) \sqrt{\frac{(1 + 2l)^2 + 8V_e r_e^2}{(2 + \lambda)^2}} \exp\left(-\gamma_2 r^{1 + \frac{1}{2}}\right)$$

\times F\left(-n, 1 + 2n + 2\sqrt{(1 + 2l)^2 + 8V_e r_e^2}, 2\gamma_2 r^{1 + \frac{1}{2}}\right),$$

(49)

$$\gamma_2 = \frac{16m_0r_e P}{(2 + \lambda)^2 \left[1 + 2n + 2\sqrt{(1 + 2l)^2 + 8V_e r_e^2}\right]}.$$

where $n, l = 0, 1, 2, \cdots$. Interestingly for $\lambda = 0$ the above results in (48)-(49) reproduce the well-known three-dimensional relations obtained recently by Refs. [32,34-36,39]. The energy eigenvalues and wave functions are given in the form [32,34-36,40]

$$E_{n,l} = P - \frac{8m_0r_e^2P^2}{\left[1 + 2n + \sqrt{(1 + 2l)^2 + 8m_0r_e^2P}\right]^2},$$

(50)

$$R_{n,l}(r) = b_l \sqrt{8V_e r_e^2 + (2l + 1)^2} \exp\left(-\gamma_2 r\right) F\left(-n, \sqrt{(2l + 1)^2 + 8V_e r_e^2}, 2\gamma_2 r\right),$$

(51)

$$\gamma_2 = \frac{4m_0r_e P}{1 + 2n + (2l + 1)^2 + 8V_e r_e^2}.$$

For $D = 1$ (s-wave):

$$E_n = P - \frac{32m_0r_e^2P^2}{\left[(1 + 2n)(2 + \lambda) + 2\sqrt{(1 + \lambda)^2 + 8m_0r_e^2P}\right]^2},$$

(53)

and

$$R_n(r) = b_l \left(1 + \frac{1}{2}\right) \sqrt{2V_e r_e^2 + \frac{(1 + \lambda)^2}{(2 + \lambda)^2}} \exp\left(-\gamma_3 r^{1 + \frac{1}{2}}\right) F\left(-n, 1 + 2\sqrt{(1 + \lambda)^2 + 2V_e r_e^2}, 2\gamma_3 r^{1 + \frac{1}{2}}\right),$$

(54)

$$\gamma_3 = \frac{16m_0r_e P}{(2 + \lambda)^2 \left[1 + 2n + 2\sqrt{(1 + \lambda)^2 + 2V_e r_e^2}\right]},$$

where $n = 0, 1, 2, \cdots$. When $\lambda = 0$:

$$E_n = P - \frac{8m_0r_e^2P^2}{\left[1 + 2n + \sqrt{1 + 8m_0r_e^2P}\right]^2}.$$

(55)
and
\[ R_n(r) = b_n r^{\frac{1}{2}} \sqrt{8V_e r_e^2 + 1} \exp \left( -\gamma_3 r \right) F \left( -n, 1 + \sqrt{1 + 8V_e r_e^2}, 2\gamma_3 r \right), \quad (56) \]
\[ \gamma_3 = \frac{4m_0r_e P}{1 + 2n + \sqrt{1 + 8V_e r_e^2}}. \]

IV. CONCLUDING REMARKS

In this work, we have studied the exact PDEM Schrödinger equation in \( D \)-dimension for two diatomic molecular potentials, namely, pseudoharmonic potential and modified Kratzer potential. The exactly solvable constant mass \( D \)-dimensional Schrödinger equation has been transformed into the form similar to the effective mass by means of a proper PCT function. The mapping of the resulting transformed equation with the original effective mass equation provide us the required energy spectrum and wave functions for the potential system under consideration. We have applied this methodology to obtain the energy eigenvalues and the corresponding eigenfunctions of the modified Kratzer and the pseudoharmonic potentials. The exact bound state solutions of the constant mass Schrödinger equation for the pseudoharmonic and modified Kratzer problems are recovered for various dimensionality upon inserting \( D = 1, 2 \) and \( \lambda = 0 \).

Acknowledgments

Work partially supported by the Scientific and Technological Research Council of Turkey (TÜBİTAK).
[1] G. Chen and Z.-D. Chen, Phys. Lett. A 331 (2004) 312, and references therein.

[2] A.R. Plastino, A. Puente, M. Casas, F. Garciás, A. Plastino, Revista Mex. Fis. 46 (1) (2000) 78.

[3] A.D. Alhaidari, Int. J. Theor. Phys. 42 (2003) 2999; ibid., Phys. Lett. A 322 (2004) 72.

[4] G. Bastard, Wave Mechanics Applied to Semiconductor Heterostructure, Editions de physique, Les Ulis, France 1988.

[5] M. Barranco, M. Pi, S.M. Gatica, E.S. Hernandez, J. Navarro, Phys. Rev. B 56 (1997) 8997.

[6] L. Serra, E. Lipparini, Europhys. Lett. 40 (1997) 667.

[7] F. Arias de Saavedra, J. Boronati, A. Polis, A. Fabrocini, Phys. Rev. B 50 (1994) 4248.

[8] G.T. Einevoll, P.C. Hemmer, J. Thomson, Phys. Rev. B 42 (1990) 3485.

[9] C. Weisbuch, B. Vinter, Quantum Semiconductor Heterostructures, Academic Press, New York, 1993.

[10] Y.M. Li, H.M. Lu, O. Voskoboynikov, C.P. Lee, S.M. Sze, Surf. Sc. 532 (2003) 811.

[11] R. Renan, M.H. Pacheco, C.A.S. Almeida, J. Phys. A 33 (50) (2000) L509.

[12] B. Gönül, O. Özer, B. Gönül, F. Üzgün, Mod. Phys. Lett. A 17 (37) (2002) 2453; B. Gönül, B. Gönül, D. Tutcu, O. Özer, Mod. Phys. Lett. A 17 (31) (2002) 2057.

[13] C.M. Bender and S. Boettcher, Phys. Rev. D 48 (1993) 4919.

[14] C.M. Bender and K.A. Milton, Phys. Rev. D 50 (1994) 6547.

[15] A. Romeo, J. Math. Phys. 36 (1995) 4005.

[16] S.M. Al-jaber, Nuovo Cimento B 110 (1995) 993.

[17] D.H. Lin, J. Phys. A 30 (1997) 3201.

[18] H. Hoseya, J. Phys. Chem. 101 (1997) 418.

[19] M.M. Neito, Am. J. Phys. 47 (1979) 1067.

[20] S.H. Dong and G.H. Sun, Phys. Lett. A 314 (2003) 261.

[21] S.M. Ikhdair, R. Sever, Int. J. Mod. Phys. A 18 (2003) 4215; ibid., A 19 (2004) 1771; ibid., A 20 (2005) 4035; ibid., A 20 (2005) 6509; ibid., A 21 (2006) 2191; ibid., A 21 (2006) 3989; ibid., A 21 (2006) 6899; ibid., E 17 (2008) 669.

[22] S.M. Ikhdair and R. Sever, Int. J. Mod. Phys. A 21 (2006) 6465.

[23] S.M. Ikhdair and R. Sever, Ann. Phys. (Berlin) 17 (2008) [11] (DOI: 10.1002/andp.200810322).
[24] S.M. Ikhdair, Bound-states of the Klein-Gordon equation with the vector and scalar general Hulthén-type potentials in $D$-dimension, to appear in the Int. J. Mod. Phys. C 20 (1) (2009).

[25] S.M. Ikhdair and R. Sever, Any $l$-state improved quasi-exact analytical solutions of the effective mass Klein-Gordon equation for vector and scalar Hulthén potentials, submitted to Phys. Scr.

[26] S.M. Ikhdair and R. Sever, J. Mol. Struct.:Theochem 806 (2007) 155.

[27] S.M. Ikhdair and R. Sever, Cent. Eur. J. Phys. 6 (2008) 685.

[28] S.M. Ikhdair and R. Sever, Cent. Eur. J. Phys. 5 (2007) 516.

[29] S.M. Ikhdair and R. Sever, arXiv: 0801.4857, to appear in Int. J. Mod. Phys. C 19 (9) (2008).

[30] R. Sever, C. Tezcan, M. Aktaş and Ö. Yeşiltaş, J. Math. Chem. 43 (2007) 845.

[31] M. Sage and J. Goodisman, Am. J. Phys. 53 (1985) 350.

[32] S.M. Ikhdair, Chin. J. Phys. 46 (2008) 291.

[33] S.M. Ikhdair and R. Sever, Cent. Eur. J. Phys. 6 (2008) 697.

[34] S.M. Ikhdair and R. Sever, J. Mol. Struct.:Theochem 855 (2008) 13.

[35] S.M. Ikhdair and R. Sever, Cent. Eur. J. Phys. 6 (2008) 141.

[36] S.M. Ikhdair and R. Sever, Int. J. Mod. Phys. C 19 (2008) 221.

[37] S.M. Ikhdair and R. Sever, DOI: 10-1007/s10910-008-9438-8, to appear in J. Math. Chem.

[38] S.M. Ikhdair and R. Sever, Int. J. Mod. Phys. C 18 (2007) 1571.

[39] H.E. Montgomery, JR., N.A. Aquino and K.D. Sen, Int. J. Quant. Chem. 107 (2007) 798.

[40] C. Berkdemir, A. Berkdemir and J.G. Han, Chem. Phys. Lett. 417 (2006) 326.