Algorithmic Game Theory

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Part I

Price of anarchy
Congestion games
Congestion games

- A network in which every edge has its own latency function.
- Traffic follows the optimal path

\[ L(x) = 1 \]

\[ L(x) = x \]

**Figure 1:** The Pigou network
A traffic of rate (slightly less than) $1$
Every driver will follow the lower road
The expected latency is $1$
But if the traffic splits, the expected latency is
$$\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$$
The Price of Anarchy is $4/3$!
Braess’ paradox

- Traffic of rate 1 will prefer the path (1, 3, 2, 4) with latency 2
- If the traffic splits equally between the upper and lower path, the expected latency drops to 3/2
- The Price of Anarchy is 4/3
- Notice the “paradox”: The removal of road (3, 2) improves the traffic conditions for everybody!
Nash (Wardrop) equilibria

- Fix a network with latency functions on its edges and traffic rates between its nodes.
- At a Wardrop or Nash equilibrium every bit of flow follows a path of minimum latency.
- Networks with continuous latency functions always have a Wardrop equilibrium.
Price of anarchy

The Price of Anarchy of a game is

\[
P_{oA} = \frac{\text{cost of worst Nash equilibrium}}{\text{socially-optimal cost}}
\]

Similar notion: Price of Stability in which we consider the best Nash equilibrium.

**Theorem (Roughgarden, Tardos, 2001)**

*Every continuous congestion game with linear latency functions has Price of Anarchy at most 4/3.*

**Theorem (Roughgarden, Tardos)**

*For arbitrary continuous latency functions, the Nash equilibrium is no worse that the optimum of the traffic scaled by a factor of 2.*
Finite vs continuous games

- Finite (atomic) games have a finite number of players.
- Continuous (non-atomic) games have infinitely many players.

Parallel issue: Games vs Markets

- Traffic conditions do not really change if one driver changes her behavior.
- Similarly, prices in markets do not change if one buyer changes her behavior.
Finite congestion games

- Each player has a source and destination and wants to establish a path between them.
- The cost of each edge depends on the number (instead of the set) of players who use it.
- In the example below, every edge has cost proportional to the number of players using it.
- Player 1 goes from 1 to 5; player 2 goes from 2 to 5; both have two strategies.
Finite congestion games

Figure 2: Prisoners’ dilemma as congestion game: D (top), C (bottom)

Figure 3: El Farol Bar as congestion game: Going (top), staying (bottom)
Finite congestion games

**Theorem (Rosenthal, 1972)**

*Every finite congestion game with increasing linear costs has a pure Nash equilibrium.*

**Theorem**

*The Price of Anarchy of finite congestion games is 5/2.*

- Finite games have much higher Price of Anarchy than infinite games
- Why? Because of the power of individual player to affect the values
Part II

Mechanisms
Mechanisms = Algorithms + Incentives

- Internet routing (interactions between ISPs)
- Sponsored search
- Online auctions (e.g. Ebay)
- P2P (e.g. free-riders)
- ...
Mechanism design

Mechanisms as algorithms
- Given an objective, design a **game** whose equilibrium optimizes the objective.

Objectives Usually we want to optimize one of the following:
- Revenue (sum of payments)
- Social welfare (sum of player values)
- Other (for example, minmax)
**Setting:**

- We want to sell an object to n bidders (buyers).
- Each bidder has a value $v_i$ for the object, which is known only to him/her.
- **Objective:** Social welfare, equivalent to “give the item to the bidder with the highest value”.
**Single-item auction**

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- We want to sell an object to $n$ bidders (buyers).
- Each bidder has a value $v_i$ for the object, which is known only to him/her.
- **Objective:** Social welfare, equivalent to “give the item to the bidder with the highest value”.

**Features:**
- Incomplete information: only the bidders know their values
- Money may be used as an incentive. But, money may not be part of the objective.
- Direct revelation: The bidders declare all their values at the beginning.
**Single-item auction**

**Auctions for maximizing welfare:**

- Each bidder declares a value $\tilde{v}_i$, not necessarily equal to the true value $v_i$. 

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![Auction scene](image)
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- **First-price auction:** The bidder pays her bid.
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- **First-price auction**: The bidder pays her bid.

- The first-price auction is not truthful.
Vickrey auction: The bidder pays only the second highest bid.
Single-item auction

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Proposition
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Why is the Vickrey auction truthful?
- The payment depends only on the values of the other players
- The allocation is monotone: increasing the declared value makes it more likely to get the item
**Sponsored search auctions**

**GSP: Generalized Second-Price auction**

- Order the bids

\[ v_1 \geq v_2 \geq \cdots \geq v_n \]
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- Give the top slot to first bidder for a price of \( p_1 = v_2 \)

GSP is not truthful! Its Price of Anarchy is \( \phi \approx 1.618 \).
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- Give the second slot to second bidder for a price of \( p_2 = v_3 \), etc
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### Social choice - voting

**Voting problem:** Aggregating preferences

**Mechanisms without money**

| Voter   | Candidate 1 | Candidate 2 | Candidate 3 |
|---------|-------------|-------------|-------------|
| Voter 1 | 1           | 2           | 3           |
| Voter 2 | 2           | 1           | 3           |
| Voter 3 | 3           | 2           | 1           |
| Voter 4 | 2           | 3           | 1           |
Gibbard-Shatterwaite theorem

| Voter 1 | Candidate 1 | Candidate 2 | Candidate 3 |
|---------|-------------|-------------|-------------|
| Voter 2 | 2           | 1           | 3           |
| Voter 3 | 3           | 2           | 1           |
| Voter 4 | 2           | 3           | 1           |

- Many voting schemes: Borda, plurality, ...
- All can be manipulated

Theorem (Gibbard-Shatterwaite, 1975)

Only *dictatorial* voting systems for three or more candidates are truthful.

Similar to Arrow’s impossibility theorem.
Mechanisms with payments for general domains

|        | Outcome 1 | Outcome 2 | Outcome 3 |
|--------|-----------|-----------|-----------|
| Bidder 1 | 1         | 5         | 10        |
| Bidder 2 | 2         | 8         | 5         |
| Bidder 3 | 4         | 6         | 4         |
| Bidder 4 | 4         | 8         | 10        |

**Setting:** The numbers indicate how much bidders are willing to pay for the outcomes

**Goal:** Select the most desirable outcome (social welfare)

**Main obstacle:** Bidders may lie about their values
The VCG mechanism

- Selects the outcome which maximizes social welfare
- Each player pays her value, but she gets a discount equal to the increase of the global objective because of her participation.

In the example, Outcome 3 is selected
- The social welfare is $10 + 5 + 4 + 10 = 29$
- Without bidder 1, the social welfare becomes $8 + 6 + 8 = 22$
- Bidder 1 gets a discount of $29 - 22 = 7$ and pays only $10 - 7 = 3$

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|-------|-----------|-----------|-----------|
| Bidder 1 | 1         | 5         | 10        |
| Bidder 2 | 2         | 8         | 5         |
| Bidder 3 | 4         | 6         | 4         |
| Bidder 4 | 4         | 8         | 10        |
Truthfulness

Theorem

The VCG mechanism is truthful.

- VCG for a single item auction is the Vickrey (second-price) auction

|       | Outcome 1 | Outcome 2 | Outcome 3 |
|-------|-----------|-----------|-----------|
| Bidder 1 | 9         | 0         | 0         |
| Bidder 2 | 0         | 12        | 0         |
| Bidder 3 | 0         | 0         | 5         |

Player 2 gets the item and gets a discount of $12 - 9 = 3$. She pays, $12 - 3 = 9$, the second price.
VCG for the shortest-path problem

Buying edges to build a shortest path

- VCG selects a shortest path $P$: $P = (1, 2, 3, 4)$
- To compute the payment of an edge $e$ on the path $P$:
  - We remove $e$ and compute a shortest path $P_e$
  - The payment for edge $e$ is
    \[ p_e = v_e + \text{length of } P_e - \text{length of } P \]

For example,
- for edge $(1, 2)$, $P_e = (1, 3, 4)$. The payment is $1 + 7 - 6 = 2$
Roberts’ theorem

**Theorem (Roberts, 1979)**

*For general domains with three or more outcomes, only the VCG and its variants (affine maximizers) are truthful.*

- This is a devastating theorem, similar to the Gibbard-Shatterwaite theorem: No general mechanisms for objectives other than the social welfare.
- Major open problem: understanding the power of mechanisms for restricted domains. For example, combinatorial auctions (selling many items).
Minmax objective

- Scheduling problem: allocate a set of tasks to a set of selfish workers, each with its own skills
- How to allocate the tasks? how much to pay the workers?

**Theorem**

No mechanism can find an optimal solution; not even a 2.618-approximate solution.
Cake-cutting: How to cut a cake for \( n \) kids that have different preferences?

- Fairness for 2: I cut, you choose
- Fairness for many? Beautiful results, many open problems
HOMO SAPIENS?

**Ultimatum Game:** Two players will split $100 as follows: The first player proposes a split and the second player accepts or rejects. If he accepts, the players get the proposed shares, otherwise they both receive nothing.
**Homo sapiens?**

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**Traveler’s dilemma:** Each player proposes an amount between $2 and $100. If they agree, each player gets the proposed amount. If they disagree, they both get the minimum value, but the player who proposed the minimum value gets also a bonus of $2.
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TRAVELER’S DILEMMA: Each player proposes an amount between $2 and $100. If they agree, each player gets the proposed amount. If they disagree, they both get the minimum value, but the player who proposed the minimum value gets also a bonus of $2.

- Are we rational? Are we selfish?
- At what level?
  - at the level of genes, organisms, families, communities, species?
Tragedy of commons