Universal behaviour of the $\gamma^*\gamma \rightarrow (\pi^0, \eta, \eta')$ transition form factors

Dmitri Melikhov and Berthold Stech

HEPHY, Austrian Academy of Sciences, Nikolausdorfgasse 18, A-1050 Vienna, Austria
Faculty of Physics, University of Vienna, Boltzmanngasse 5, A-1090 Vienna, Austria
SINP, Moscow State University, 119991 Moscow, Russia
ITP, Heidelberg University, Philosophenweg 16, D-69120, Heidelberg, Germany

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The photon transition form factors of $\pi$, $\eta$ and $\eta'$ are discussed in view of recent measurements. It is shown that the exact axial anomaly sum rule allows a precise comparison of all three form factors at high-$Q^2$ independent of the different structures and distribution amplitudes of the participating pseudoscalar mesons. We conclude: (i) The $\pi\gamma$ form factor reported by Belle is in excellent agreement with the non-strange $I = 0$ component of the $\eta$ and $\eta'$ form factors obtained from the BaBar measurements. (ii) Within errors, the $\pi\gamma$ form factor from Belle is compatible with the asymptotic pQCD behavior, similar to the $\eta$ and $\eta'$ form factors from BaBar. Still, the best fits to the data sets of $\pi\gamma$, $\eta\gamma$, and $\eta'\gamma$ form factors favor a universal small logarithmic rise $Q^2 F_{\pi\gamma}(Q^2) \sim \log(Q^2)$.

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1. INTRODUCTION AND RESULTS

The extensive experimental study of the $\gamma^*\gamma \rightarrow P$ reactions ($P = \pi^0, \eta, \eta'$) attracted much attention from theorists (for recent references see [6–17]). The reason was the report of the BaBar collaboration [4] about a persistent rise of the combination $Q^2 F_{\pi\gamma}(Q^2)$ in the $Q^2$ region from 10 GeV$^2$ to 40 GeV$^2$. The measured form factor surpassed the asymptotic behaviour $Q^2 F_{\pi\gamma}(Q^2) \sim \sqrt{2} f_\pi$, where $f_\pi = 0.131$ GeV, predicted by perturbative QCD (pQCD). Several theoretical investigations [3, 11, 13, 14] indicated, however, that the corresponding increase of the $\pi\gamma$ form factor for large $Q^2$ values is hard to explain. Very recently new experimental information came from the Belle collaboration [15]. The data for $Q^2 F_{\pi\gamma}(Q^2)$ presented by this group show only a very mild (if any) increase in the high-$Q^2$ region.

In this situation a comparison of the $\pi\gamma$ form factor with the ones for $\eta$ and $\eta'$ can be helpful. At first sight, this appears difficult because of the different structure (different quark distribution amplitudes) of these particles. In the present note we show however, that by using the exact anomaly sum rule, the high-$Q^2$ behavior of $\pi$, $\eta$ and $\eta'$ form factors is essentially determined by the well-known lowest order spectral representation of the triangle quark diagram. Therefore, a comparison of the three form factors is possible and can be trusted.

The application of the anomaly sum rule requires to relate the full-QCD spectral densities with the spectral densities obtained from perturbative QCD. This can be achieved by using the concept of duality. This way the anomaly sum rule offers the interesting possibility [10] to calculate the transition form factors without referring to the QCD factorization theorem. No assumptions are needed about the light-cone distribution amplitudes of pseudoscalar mesons with their specific end-point behavior and Gegenbauer coefficients.

The high-$Q^2$ behavior is determined by the high-energy dependence of the spectral density in the corresponding integrals, see Eqs. (10) and (11) below. At high energy—above the resonance region—the spectral density can be very well approximated by perturbation theory. This spectral density is the same for the three form factors. (The effect of the difference between the current masses of strange and non strange quarks should be negligible at high energy). Thus we can conclude that at high-$Q^2$ the functional dependence of these form factors should be the same. A detailed analysis in [14] suggests that the universality may be expected already at $Q^2 \geq 10$ GeV$^2$.

Different decay constants provide for different multiplicity factors but do not affect the slope.

Looking now at the data, the following observations can be made:

(i) The BaBar measurements of the $\eta\gamma$ and $\eta'\gamma$ form factors (Fig. 1) are within errors compatible with pQCD factorization which implies saturation of the combination $Q^2 F_{\eta\gamma}(Q^2)$. Still, the data seem to indicate a very mild (e.g. logarithmic) rise with $Q^2$.

(ii) The large-$Q^2$ behaviour of the form factor $F_{\pi\gamma}(Q^2)$ as observed by BaBar (Fig. 2) is in some conflict with the saturation predicted by QCD factorization. These data suggest an increase of $Q^2 F_{\pi\gamma}(Q^2)$. The rise as seen from the high-$Q^2$ points of the BaBar data is much larger than the rise observed for the $\eta$ and $\eta'$ form factors.

(iii) The Belle [15] measurement of the $\pi\gamma$ form factor Fig. 2 is within errors compatible with pQCD factorization which implies saturation of the combination $Q^2 F_{\pi\gamma}(Q^2)$. Still, these data seem to indicate a logarithmic rise with $Q^2$, but a very mild one like in the $\eta$ and $\eta'$ data.

We now apply our theoretical argument which states that the high-$Q^2$ behavior of the three form factors should be the same. From the observations (i), (ii) (iii) one can already conclude that this property is indeed seen in the data if for $F_{\pi\gamma}$ the Belle data points are used. However, for a more quantitative analyses, the slight logarithmic rise indicated by the data should be taken into account.

To describe the logarithmic rise, the simplest use of the
duality concept—a replacement of the absorptive part of the form factor beyond the resonance region by the lowest-order perturbative QCD spectral density—cannot be maintained: To account in a phenomenological way for higher-order effects, the lowest-order perturbative spectral density will be multiplied by a correction factor $R(s)$ which goes to one for high values of $s$. Here $s$ denotes the square of the energy variable. It turns out that it is sufficient to describe $R(s)$ as a simple function of a single fit parameter $r$: $R(s) = 1 - \frac{s}{s_0}$. The function $R(s)$ starts at the effective threshold relevant for each process. (For technical details see the next Section).\footnote{In principle, $r$ could be $Q^2$-dependent, e.g., $r(Q^2) = \frac{r_0}{1 + aQ^2}$. This choice would restore saturation at large $Q^2$.}

Our simple model for the full set of the form factors thus involves six parameters: the three effective thresholds $s_q^{(I=0,1)}$, $s_s$ for the $I = 0, I = 1$ and $s\bar{s}$ components of the form factors and the three $r$ parameters $r_q^{(I=0,1)}$, $r_s$. The decay constants $f_s$ and $f_q$ and the $\eta - \eta'$ mixing angle $\phi$ are taken from Ref. \cite{20}.

Fig. 1: Form factors $Q^2 F_{\gamma\gamma}(P = \eta, \eta')$ vs $Q^2$: experimental data from Cello and Cleo \cite{1,2} (black dots), BaBar \cite{3} (red squares), and the data borrowed from the time-like region \cite{3} (green triangles). Dashed lines - the results from \cite{14} which obey the factorization theorem at $Q^2 \to \infty$; solid lines - our fits for $r_q^{(I=0)} = r_s = 0.05$ GeV$^2$.

![Form factors Q²Fγγ (P = η, η′) vs Q²](image1)

The form factors of $\eta$ and $\eta'$$^*$

Let us first set $r_s = r_q^{(I=0)} = 0$, as implied by the pQCD factorization theorem. The fit to the existing data yields $s_q^{(I=1)} = 0.67 \pm 0.07$ GeV$^2$ and $s_s = 1.0 \pm 0.03$ GeV$^2$ with $\chi^2/DOF = 72/71$. Strictly speaking, in view of the experimental data, further improvements are unnecessary. Nevertheless, having in mind the BaBar result for the pion form factor, let us allow also nonzero values of

![Form factor Q²Fγγ (P = η, η′) vs Q²](image2)
r setting $r_q^{(l=0)} = r_s$ (at the present accuracy of the data it makes no sense to treat them independently). Then the fitting procedure gives $s_q^{(l=0)} = 0.57 \pm 0.07$ GeV$^2$, $s_s = 0.94 \pm 0.03$ GeV$^2$ and $r_s = r_q^{(l=0)} = 0.05 \pm 0.01$ GeV$^2$ with $\chi^2/DOF = 60/70$. The corresponding results are shown in Fig. 1. Notice that the threshold values obtained by the fits are not far from $s_q^{(l=1)} = 0.56$ GeV$^2$ and $s_s = 0.76$ GeV$^2$ suggested from the relevant elements of the $\eta - \eta'$ mass matrix [24].

The $\pi^0$ form factor

A fit to the data set containing the Cleo and the BaBar results in Fig. 2 leads to a much larger parameter $s_q^{(l=1)} = 0.14 \pm 0.014$ GeV$^2$ than obtained for $r_q^{(l=0)}$ found above. The $\chi$ value is $\chi^2/DOF = 26/30$ and the threshold parameter $s_q^{(l=1)} = 0.32 \pm 0.07$ GeV$^2$. Most important, by setting $r_q^{(l=1)} = 0$ one gets an extremely bad fit with $\chi^2/DOF = 88/31$. Obviously, the BaBar high-$Q^2$ data are not compatible with the asymptotic pQCD result at large $Q^2$ (15-35 GeV$^2$).

In contrast, a fit to the data set containing the Cleo and the Belle data in Fig. 2 is fully compatible with $r_q^{(l=1)} = 0$: for $s_q^{(l=1)} = 0.64 \pm 0.01$ GeV$^2$, one gets an excellent fit to the data with $\chi^2 = 16/29$. No further improvements are necessary. Nevertheless, allowing in addition for a nonzero parameter $r_q^{(l=1)}$ leads to $s_q^{(l=1)} = 0.06 \pm 0.02$ GeV$^2$ and $s_q^{(l=1)} = 0.5 \pm 0.06$ GeV$^2$ with $\chi^2/DOF = 10/28$. Remarkably, this value of $r_q^{(l=1)}$ is equal to $r_q^{(l=0)}$ as obtained from the $\eta$ and $\eta'$ data. Also the effective thresholds for the nonstrange quark sector are very close to each other: $s_q^{(l=1)} \approx s_q^{(l=0)}$.

The shaded region in Fig. 2 corresponds to the variation of $r$ in the range $0.05$ GeV$^2 \leq r \leq 0.17$ GeV$^2$.

To summarize, using the Belle data for the large-$Q^2$ region, the previously puzzling difference between the $(\eta, \eta')\gamma$ and $\pi\gamma$ form factors is no more present: all three processes—after taking particle mixing into account—can be well described by only two effective thresholds and a small universal parameter $r$.

Certainly, more precise measurements are needed to establish a small logarithmic increase of $Q^2 F_{\pi\gamma}(Q^2)$ for high $Q^2$ values indicated by the data and parametrized by a nonzero $r \simeq 0.05$ GeV$^2$.

2. TECHNICAL DETAILS

We now provide some details of our calculation of the $P\gamma$ form factors. For subleties, we refer to [17].

Our starting point is the amplitude

$$
\langle 0|j_\mu^5|\gamma(q_2)\gamma^*(q_1)\rangle = e^2 T_{\mu\alpha\beta}(p|q_1, q_2)\varepsilon_1^\alpha \varepsilon_2^\beta,
$$

$p = q_1 + q_2$.  

(1)

Here $\varepsilon_{1,2}$ denote the photon polarization vectors. This amplitude is considered for $q_1^2 = -Q^2$ and $q_2^2 = 0$. Its general decomposition contains four independent Lorentz structures, but for our purpose only one structure is needed

$$
T_{\mu\alpha\beta}(p|q_1, q_2) = p_\mu \epsilon_{\alpha\beta q_1 q_2} F(p^2, Q^2) + \ldots
$$

(2)

The invariant amplitude $F(p^2, Q^2)$ satisfies the spectral representations in $p^2$ at fixed $Q^2$:

$$
F(p^2, Q^2) = \frac{1}{\pi} \int_0^\infty \frac{ds}{s - p^2} \Delta(s, Q^2),
$$

(3)

where $\Delta(s, Q^2)$ is the physical spectral density and $s_{th}$ denotes the physical threshold. Perturbation theory yields the spectral density as an expansion in powers of $\alpha_s$. The lowest order contribution, $\Delta_{pQCD}^{(0)}(s, Q^2)$, corresponds to the one-loop triangle diagram with the axial current and two vector currents in the vertices [20, 22]

$$
\Delta_{pQCD}^{(0)} = \frac{1}{2\pi} \int_0^\infty \frac{ds}{(s + Q^2)^2} \left[ Q^2 w + 2m^2 \log \frac{1 + w}{1 - w} \right],
$$

(4)

Here $m$ denotes the mass of the quark propagating in the loop. The integral of $\Delta_{pQCD}^{(0)}(s, Q^2)$ from $s = 4m^2$ to infinity is independent of $m$ and $Q^2$ and gives the axial anomaly [23]

$$
\int_0^{4m^2} ds \Delta_{pQCD}^{(0)}(s, Q^2) = \frac{1}{2\pi}.
$$

(5)

According to the Adler-Bardeen theorem [24], radiative corrections to the anomaly vanish: Higher order QCD calculation can change $\Delta_{pQCD}^{(0)}$ but not the integral.

Non-perturbative QCD effects strongly distort $\Delta(s, Q^2)$ compared with $\Delta_{pQCD}^{(0)}(s, Q^2)$ in the low-$s$ region: A meson pole and the hadron continuum are generated. Nevertheless, the integral of the entire absorptive part $\Delta(s, Q^2)$ remains unchanged, still representing the anomaly:

$$
\int_0^\infty ds \Delta(s, Q^2) = \frac{1}{2\pi}.
$$

(6)

For the case of the isovector $\bar{u}u - \bar{d}d$ axial current the spectrum contains the $\pi^0$-meson pole. The physical absorptive part of $F(p^2, Q^2)$ reads

$$
\Delta(s, Q^2) = \pi \delta(s - m^2) \sqrt{2} f_\pi F_{\pi\gamma}(Q^2) + \theta(s - s_{th}) \Delta_{cont}^{(0)}(s, Q^2).
$$

(7)

Here $\Delta_{cont}^{(0)}(s, Q^2)$ denotes the hadron-continuum contribution in the isovector channel. In [17], the $\pi\gamma$ form factor we are interested in appears together with the $\pi$ meson
\( \delta \)-function and the pion decay constant. The anomaly sum rule for \( F_{\pi\gamma}(Q^2) \) then takes the form

\[
F_{\pi\gamma}(Q^2) = \frac{1}{2\sqrt{2}\pi^2f_\pi} \left[ 1 - 2\pi \int_{s_{tah}}^\infty ds \Delta_{cont}^L(s, Q^2) \right]. \tag{8}
\]

For the \( \eta\gamma \) and \( \eta'\gamma \) form factors, one has to consider the isoscalar currents \( \bar{q}q = (\bar{s}u + \bar{d}d)/\sqrt{2} \) and \( ss \), separately. The formulae for \( F_{\pi\eta}(Q^2) \) and \( F_{\pi\eta'}(Q^2) \) are identical to \([8]\) except for the replacements \( \sqrt{2}f_\pi \) by \( f_\eta \) and \( f_{\eta'} \) and \( \Delta_{cont}^L \) by \( \Delta_{cont}^L \) and \( \Delta_{cont}^R \), respectively.

For each channel, the relevant threshold \( s_{th} \) should be used. Taking \( \eta - \eta' \) mixing \([25, 26]\) into account leads to

\[
F_{\pi\eta}(Q^2) = \frac{5}{3\sqrt{2}} F_{\pi\eta}(Q^2) \cos \phi - \frac{1}{3} F_{\pi\eta}(Q^2) \sin \phi,
\]

\[
F_{\pi\eta'}(Q^2) = \frac{5}{3\sqrt{2}} F_{\pi\eta'}(Q^2) \sin \phi + \frac{1}{3} F_{\pi\eta'}(Q^2) \cos \phi. \tag{9}
\]

The \( \eta - \eta' \) mixing angle \( \phi \) is known to be \( \phi \approx 39^\circ \); the decay constants are taken to be \( f_\eta = 1.07f_\pi \), \( f_{\eta'} = 1.36f_\pi \) \([20]\).

According to \([8]\) and \([9]\), the calculation of the \( P\gamma \) form factors requires an Ansatz for the continuum spectral densities \( \Delta_{cont}(s, Q^2) \) for all three cases.

The quark-hadron duality suggests that at large values of \( s \), above the resonance region, the hadron spectral density should be very close to the perturbative QCD spectral density. We therefore use the simple Ansatz

\[
\Delta_{cont}(s, Q^2) = \theta(s - s_{th}) R(s) \Delta_{QCD}^0(s, Q^2),
\]

\( R(s \to \infty) \to 1. \tag{10} \)

It turns out \([17]\) that for the large-\( Q^2 \) behavior of the form factor the behaviour of \( R(s) \) at large \( s \) is essential: e.g. in order to have the logarithmic rise of \( Q^2 F(Q^2) \), \( R(s) \) should contain a \( 1/s \)-correction: For \( R(s) = 1 - r/s \) starting at a finite energy \( s_0, s_0 > s_{th} \), one finds

\[
Q^2 F(Q^2) \sim \frac{Q^2}{Q^2 + s_0} (s_0 - r) + r \log \left( \frac{Q^2 + s_0}{s_0} \right). \tag{11}
\]

Notice that the part of the integral \([8]\) from \( s_{th} \) to \( s_0 \) scales as \( 1/Q^2 \) and is therefore not relevant for the effect discussed here. Thus, neither the details of \( R(s) \) at small \( s \) nor the presence of higher powers of \( 1/s \) affect this behavior. The negative linear term \(-r/s \) in \( R(s) \) is responsible for the logarithmic increase. If absent, the form factor scales as \( F(Q^2) \sim 1/Q^2 \). Evidently, for a universal \( r \) the high-\( Q^2 \) behavior of all three form factors turns out to be the same. For small \( r \), the differences due to different thresholds and decay constants affect the magnitude of the form factors but not their slope.

3. CONCLUSIONS

We revisited the \( P\gamma \) transition form factors, \( P = \pi, \eta, \eta' \), with special emphasis on the new data on the \( \pi\gamma \) form factor reported by Belle \([18]\). Use is made of the exact anomaly sum rule which relates the integral over the hadron spectrum to the axial anomaly. This approach has the advantage that the QCD factorization theorem and the meson distribution amplitudes do not enter the analyses. Thus, the three processes can be easily compared with each other.

- We report that the \( \gamma\gamma^* \to P \) form factors of \( \pi, \eta, \eta' \) are fully compatible with each other—if for the \( \pi\gamma \) form factor the recent Belle data are applied: the parameter \( r \) and the effective threshold used in the description for the \( I = 0 \) nonstrange continuum and the ones for the \( I = 1 \) continuum agree with each other. Thus, the Belle data resolve the puzzle of a qualitatively different behavior of the nonstrange component in \( \eta \) and \( \eta' \) on one hand and of \( \pi \) on the other hand.

- The Belle data for the \( \pi\gamma \) form factor are compatible with the asymptotic pQCD formula indicating that corrections to the asymptotic behavior are small already at \( Q^2 \geq 10 \text{ GeV}^2 \).

- Still, our best fits to the data—for all three processes—suggest a slight increase of the product \( Q^2 F(Q^2) \) at high \( Q^2 \). If confirmed by future experiments, this would put QCD factorization into question and would suggest that the full spectral density of the dispersion representation for the form factor is dual to the lowest order pQCD spectral density only by including an effective \( 1/s \)-correction term.

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