τ-CLUSTER MORPHISM CATEGORIES AND PICTURE GROUPS

ERIC J HANSON, KIYOSHI IGUSA

Abstract. τ-cluster morphism categories were introduced by Buan and Marsh as a generalization of cluster morphism categories to τ-tilting finite algebras. In this paper, we show that the classifying space of such a category is a cube complex, generalizing results of Igusa, Orr, Todorov, and Weyman and Igusa. We further show that the fundamental group of this space is isomorphic to a generalized version of the picture group of the algebra, as defined by Igusa. We end this paper by showing that if our algebra is Nakayama, then this space is locally CAT(0), and hence a K(π, 1). We do this by constructing a combinatorial interpretation of the 2-simple minded collections for Nakayama algebras.

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Introduction

Both τ-tilting theory and semi-invariant pictures are active areas of research. In this paper, we show how these topics are related.

In [IOTW16], the second author, Orr, Todorov, and Weyman associate to every hereditary algebra of finite type a finitely presented group, called the picture group. (Semi-invariant) pictures, also

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called scattering diagrams or cluster mutation fans, are of current interest to many researchers. For example, Gross, Hacking, Keel, and Kontsevich and Cheung, Gross, Muller, Musiker, Rupel, Stella, and Williams use scattering diagrams to settle many outstanding conjectures about cluster algebras in [GHKK18] and [CGM+17], and Reading studies the combinatorics of scattering diagrams in [Rea18].

One topic of interest is the computation of the cohomology ring of a picture group. This is difficult to compute directly, so instead, the second author, Orr, Todorov, and Weyman associate a finite CW complex to every hereditary algebra. This CW complex can be realized as the classifying space of the “cluster morphism category” of the algebra, defined in [TT17]. Using a classical result from geometric group theory, this space is then shown to have cohomology isomorphic to that of the picture group in many cases in [TT17], [Igu14].

In this paper, we reestablish the construction of the picture group and the associated CW complex for all $\tau$-tilting finite algebras using $\tau$-tilting theory (see [AIR14], [DIJ17]) and the $\tau$-cluster morphism category (see [BM18a], [BM18b]). $\tau$-tilting theory was introduced as an extension of classical tilting theory better suited for non-hereditary algebras. It has been an area of current research since its introduction. For example, Brüstle, Smith, and Treffinger use $\tau$-tilting theory to describe the wall and chamber structure of finite dimensional algebras in [BST18] and Plamondon shows that for gentle algebras, $\tau$-tilting finiteness and representation finiteness are equivalent in [Pla18].

Our final main result is to recover the equivalence of cohomology in the case that the algebra is Nakayama. Nakayama algebras form a well-studied class of representation-finite, but not necessarily hereditary, algebras. As such, they remain an important class to study. For example, in [MM18], Madsen and Marczinzik give bounds on the global and finitistic dimensions of Nakayama algebras and in [Sen18], Sen shows that the $\varphi$-dimension (as defined in [IT05]) of a Nakayama algebra is always even.

As an application of this work, forthcoming work to appear by the second author, Orr, Todorov, and Weyman use the equivalence established in this paper to compute the cohomology rings of the picture groups of cyclic cluster-tilted algebras of type $D_n$.

**Notation and Terminology.** Let $\Lambda$ be a finite dimensional, basic, connected algebra over an arbitrary field $K$. We denote by $\text{mod}\Lambda$ the category of finitely generated (right) $\Lambda$-modules and by $\text{proj}\Lambda$ the subcategory of projective modules. Throughout this paper, all subcategories will be full and closed under isomorphisms. For $M \in \text{mod}\Lambda$, we denote by $|M|$ the number of (isoclasses of) indecomposable direct summands of $M$. For $M \in \text{mod}\Lambda$, we denote by $\text{add}M$ (resp. $\text{Fac}M, \text{Sub}M$) the subcategory of direct summands (resp. factors, subobjects) of direct sums of $M$. Moreover, $\text{Filt}M$ refers to the subcategory of modules admitting a filtration by the direct summands of $M$.

We denote by $\mathcal{D}^b(\text{mod}\Lambda)$ the bounded derived category of $\text{mod}\Lambda$. The symbol $(-)[1]$ will denote the shift functor in all triangulated categories. We identify $\text{mod}\Lambda$ with the subcategory of $\mathcal{D}^b(\text{mod}\Lambda)$ consisting of stalk complexes centered at zero.

For $\mathcal{M}$ an arbitrary module category, $\tau_{\mathcal{M}}$ (or simply $\tau$ if there is no confusion) will be the Auslander-Reiten translate in $\mathcal{M}$. For an object $X$ in a category $\mathcal{C}$, we define the left-perpendicular category of $X$ as $\perp_X := \{Y \in \mathcal{C} | \text{Hom}(Y, X) = 0\}$. We define the right-perpendicular category, $X^\perp$, dually. If $X \in Y^\perp \cap Y^\perp$ we say that the objects $X$ and $Y$ are $\text{Hom}_{\text{Ext}}$ orthogonal. If in addition $Y[1] \in X^\perp$ and $X[1] \in Y^\perp$, we say $X$ and $Y$ are $\text{Hom}_{\text{Ext}}$ orthogonal. We denote by $\text{ind}(\mathcal{C})$ the category of indecomposable objects of $\mathcal{C}$.

**Organization and Main Results.** The contents of this paper are as follows. In Section [1] we overview the results we will use from $\tau$-tilting theory as well as the construction of the $\tau$-cluster morphism category given in [BM18a].

In Section [2] we recall the definition of a cubical category from [Igu14]. We then prove our first main theorem:
Theorem A (Theorem 2.10). Let $\Lambda$ be $\tau$-tilting finite. Then the $\tau$-cluster morphism category of $\Lambda$ is cubical.

This generalizes a known result from [IT17] in the case that $\Lambda$ is hereditary.

In Section 3, we construct a combinatorial model for the 2-simple minded collections for Nakayama algebras. We use this model to prove that these 2-simple minded collections are given by pairwise compatibility conditions.

Section 4 is devoted to studying the fundamental groups of the classifying spaces of $\tau$-cluster morphism categories. We begin by expanding the definition of the picture group of a representation finite hereditary algebra given in [IOTW16] to $\tau$-tilting finite algebras. We then prove our second main theorem.

Theorem B (Theorem 4.8). Let $\Lambda$ be $\tau$-tilting finite. Then the fundamental group of the classifying space of the $\tau$-cluster morphism category of $\Lambda$ is isomorphic to the picture group of $\Lambda$.

This again generalizes a known result from [IT17] in the case that $\Lambda$ is hereditary. We end this section by using picture groups to construct faithful group functors for Nakayama algebras. This, together with the results of Section 3, allows us to conclude our third main theorem.

Theorem C (Corollary 4.13). Let $\Lambda$ be a Nakayama algebra. Then the classifying space of the cluster morphism complex is a locally $\text{CAT}(0)$ cube complex and hence a $K(\pi,1)$ for the picture group of $\Lambda$.

This generalizes a known result from [Igu14] in the case that $\Lambda$ is a path algebra of type $A_n$ with straight orientation.

1. Background

1.1. $\tau$-Tilting Theory. $\tau$-tilting theory was introduced by Adachi, Iyama, and Reiten [AIR14] as an extension of classical tilting theory better suited for non-hereditary algebras. We recall that a basic object $M \sqcup P[1] \in D^b(\text{mod}\Lambda)$ is called a support $\tau$-rigid pair for $\Lambda$ if $M \in \text{mod}\Lambda$, $P \in \text{proj}\Lambda$, and we have

\[ \text{Hom}(M, \tau M) = 0 = \text{Hom}(P, M). \]

If an addition

\[ |M| + |P| = |\Lambda| \]

then $M \sqcup P[1]$ is called a support $\tau$-tilting pair. We denote by $\text{sr-}\text{rigid}\Lambda$ and $\text{sr-tilt}\Lambda$ the sets of (isoclasses of) support $\tau$-rigid and support $\tau$-tilting pairs for $\Lambda$. We now recall several facts about completing $\tau$-rigid pairs.

Theorem 1.1. Let $M \sqcup P[1] \in \text{sr-}\text{rigid}\Lambda$.

(a) [AIR14, Thm. 2.10] There exists a unique $B \in \text{mod}\Lambda$ such that $M \sqcup B \sqcup P[1]$ is a support $\tau$-tilting pair which satisfies $\text{add}(B \sqcup M) = \text{proj}(\tau M \cap P^\perp)$ and $\tau M \cap P^\perp = \tau(M \sqcup B) \cap P^\perp = \text{Fac} M \sqcup B$. The module $B$ is called the Bongartz complement of $M$ in $P^\perp$.

(b) [AIR14, Thm. 3.8] If $M \sqcup P[1]$ is almost complete, that is $|M| + |P| = |\Lambda| - 1$, then there exist exactly two support $\tau$-tilting pairs containing $M \sqcup P[1]$ as a direct summand. Moreover, these two pairs have the form $M \sqcup B \sqcup P[1]$ and $M \sqcup C \sqcup P[1]$ where $C \in \text{Fac} M \sqcup \text{proj}\Lambda[1]$. Moreover, these two pairs have the form $M \sqcup B \sqcup P[1]$ and $M \sqcup C \sqcup P[1]$ where $C \in \text{Fac} M \sqcup \text{proj}\Lambda[1]$ and $B \in \text{mod}\Lambda$ is as described in (a).

We say that $M \sqcup C \sqcup P[1]$ is the left mutation of $M \sqcup B \sqcup P[1]$ at $B$. This notion of mutation gives the set $\text{sr-tilt}\Lambda$ the structure of a poset by taking the transitive closure of the relation $U < V$ if $U$ is a left mutation of $V$.

We now recall that a subcategory $\mathcal{T} \subset \text{mod}\Lambda$ is called a torsion class if it is closed under extensions and factors. We denote by $\text{tors}\Lambda$ the poset of torsion classes of $\Lambda$ under inclusion. We then have the following.
Theorem 1.2.

(a) \cite{IRTT15} Prop. 1.3 \textit{The poset }\text{tors}_\Lambda\text{ is a lattice. That is:}
- For every \( T, T' \in \text{tors}_\Lambda \), there exists a unique torsion class \( T \lor T' \) such that \( T \lor T' \subset T'' \) whenever \( T, T' \subset T'' \). The torsion class \( T \lor T' \) is called the join of \( T \) and \( T' \).
- For every \( T, T' \in \text{tors}_\Lambda \), there exists a unique torsion class \( T \land T' \) such that \( T'' \subset T \land T' \) whenever \( T'' \subset T \). The torsion class \( T \land T' \) is called the meet of \( T \) and \( T' \).

(b) \cite{AY14} Thm. 2.7, Cor. 2.34 \textit{There is an epimorphism of posets }\text{s}\text{-}\text{tilt}_\Lambda \to \text{tors}_\Lambda\text{ given by }M \sqcup P[1] \to \text{Fac} M.

(c) \cite{DIJ17} Thm. 3.8 \textit{The above morphism is an isomorphism of lattices if and only if }\text{s}\text{-}\text{tilt}_\Lambda\text{ is a finite set. In this case, the algebra }\Lambda\text{ is called }\tau\text{-tilting finite.}

From now on, we assume our algebra \( \Lambda \) is }\tau\text{-tilting finite.

1.2. Semibricks and 2-Simple Minded Collections. We recall that an (indecomposable) object \( S \in \text{mod } \Lambda \) (or more generally \( D^b(\text{mod } \Lambda) \)) is called a \textit{brick} if \( \text{End}(S) \) is a division algebra. A subset \( S \subset \text{mod } \Lambda \) is called a \textit{semibrick} if it consists of pairwise Hom-orthogonal bricks. We also remark that it is customary to use the term semibrick to refer to both the set \( S \) and the object \( \bigsqcup_{S \in S} S \in D^b(\text{mod } \Lambda) \). We denote by \text{brick}_\Lambda \text{ (resp. } \text{sbrick}_\Lambda\text{) the set of isoclasses of bricks (resp. semibricks) in }\text{mod } \Lambda.\text{ Semibricks are closely related to 2-simple minded collections, defined as follows.}

\textbf{Definition 1.3.} A finite collection \( \mathcal{X} \subset D^b(\text{mod } \Lambda) \) is called simple minded if
- Each \( X \in \mathcal{X} \) is a brick.
- \( \text{Hom}(X_i, X_j) = 0 \) for all \( X_i \neq X_j \in \mathcal{X} \).
- \( \text{Hom}(X_i, X_j[m]) = 0 \) for all \( X_i, X_j \in \mathcal{X} \) and \( m < 0 \).
- \( \text{thick}(\mathcal{X}) = D^b(\text{mod } \Lambda) \) containing \( \mathcal{X} \) which is closed under shifts and direct summands.

If in addition \( H^i(X) = 0 \) for all \( X \in \mathcal{X} \) and \( i \neq -1, 0 \), then \( \mathcal{X} \) is called a \textit{2-simple minded collection}. We denote by \( \text{2-smc}_\Lambda \) the set of 2-simple minded collections in \( D^b(\text{mod } \Lambda) \).

\textbf{Example 1.4.} Let \( \Lambda = K(2 \leftarrow 1) \). Then there are five 2-simple minded collections:
\[ \text{2-smc}_\Lambda = \{ S_1 \sqcup P_2, S_1 \sqcup P_1[1], P_1 \sqcup P_2[1], P_2 \sqcup S_1[1], S_1[1] \sqcup P_2[1] \}. \]

We recall the following facts about 2-simple minded collections.

\textbf{Proposition 1.5.} Let \( \mathcal{X} \in \text{2-smc}_\Lambda \). Then

(a) \cite{KY14} Cor. 5.5 \textit{ }|\mathcal{X}| = |\Lambda|.

(b) \cite{BY15} Rmk. 4.11 \textit{Up to isomorphism, }\mathcal{X} = S_p \sqcup S_n[1] \text{ with } S_p, S_n \in \text{sbrick}_\Lambda.

We now recall the notion of mutation for 2-simple minded collections.

\textbf{Definition-Theorem 1.6.} \cite{KY14} Def. 7.5, Prop. 7.6, Lem. 7.8 \textit{Let }\mathcal{X} = S_p \sqcup S_n[1] \in \text{2-smc}_\Lambda. Then there is a left mutation } \mu_S(\mathcal{X}) \text{ at } S \in S_p \text{ given as follows.}
- The module } S \text{ is replaced with } S[1].
- Each module } S' \in S_p \setminus S \text{ is replaced with the cone of } S'[−1] \xrightarrow{g_{S'}} E
  \text{ where the map } g_{S'} \text{ is a minimal left } \text{Filt}_S\text{-approximation.}
- Each shifted module } S'[1] \in S_n[1] \text{ is replaced with the cone of } S' \xrightarrow{g_{S'}} E
  \text{ where the map } g_{S'} \text{ is a minimal left } \text{Filt}_S\text{-approximation.}
In particular, every $S'[-1] \in (S_p \setminus X_i)[-1]$ and every $S'' \in S_n$ admits a minimal left Filt$S$-approximation. Moreover, the map $g_{S''}$ must be either a monomorphism or an epimorphism so that the cone of $g_{S''}$ is isomorphic to either a module or a shifted module.

Under this definition of mutation, 2-smc$\Lambda$ forms a poset. Now recall that a subcategory $W \subset \mod \Lambda$ is called wide if it is closed under extensions, kernels, and cokernels. We denote by wide$\Lambda$ the set of wide subcategories of $\mod \Lambda$. Clearly, wide$\Lambda$ forms a poset under inclusion. Since $\Lambda$ is $\tau$-tilting finite, we have the following.

**Proposition 1.7.**

(a) [Asa17, Thm. 2.3] The maps $X = S_p \sqcup S_n[1] \mapsto S_p$ and $X = S_p \sqcup S_n[1] \mapsto S_n$ are bijections $2$-smc$\Lambda \to \sbrick\Lambda$.

(b) [BY15, Cor. 4.3, Thm. 4.9] There is an isomorphism of lattices $2$-smc$\Lambda \to \tors\Lambda$ given by $X = S_p \sqcup S_n[1] \mapsto \Filt\Fac S_p$.

(c) [Asa17] Prop. 1.26] There is a bijection $2$-smc$\Lambda \to$ wide$\Lambda$ given by $X = S_p \sqcup S_n[1] \mapsto \Filt S_p$.

One of our goals is to define 2-simple minded collections using pairwise compatibility conditions. This leads to the following definition.

**Definition 1.8.** Let $S_p, S_n \in \sbrick\Lambda$. We say that $S_p \sqcup S_n[1]$ is a semibrick pair if $\Hom(S_p, S_n[m]) = 0$ for all $m < 0$. In particular, a semibrick pair $S_p \sqcup S_n[1]$ is a 2-simple minded collection if and only if thick($S_p \sqcup S_n$) = $\Db(\mod \Lambda)$. We say the semibrick pair $S_p \sqcup S_n[1]$ is completable if it is a subset of a 2-simple minded collection.

The following shows that not all semibrick pairs are completable.

**Counterexample 1.9.** Consider the quiver

\[
\begin{array}{c}
1 \\
\uparrow \\
2 \\
\leftarrow \\
3 \\
\end{array}
\]

and let $\Lambda = KQ/\rad^2 KQ$. Then $1 \setminus 2, 2 \setminus 3[1]$ is a semibrick pair, but cannot be completed to a 2-simple minded collection. Indeed, the minimal left-$\Filt_2^1$ approximation $\frac{2}{3} \to \frac{1}{2}$ is neither mono nor epi.

As the only obstruction to completing a semibrick pair we have found is the existence of pairs $S, S' \in \brick\Lambda$ such that $S \sqcup S'[1]$ is a semibrick pair with a minimal left-$\Filt S$ approximation $S' \to E$ which is neither mono nor epi, we propose the following definition and conjecture.

**Definition 1.10.** A semibrick pair $X = S_p \sqcup S_n[1]$ is called mutation compatible if for all $S \in S_p$ and $S' \in S_n$ every minimal left $\Filt S$-approximation $g_{S'} : S' \to E$ is either a monomorphism or an epimorphism.

**Conjecture 1.11.** Every mutation compatible semibrick pair is completable.

We do not have a general proof of Conjecture 1.11, but will prove it in the case that $\Lambda$ is a Nakayama algebra in Section 3.
1.3. **The Category of Buan and Marsh.** We begin with the following result, allowing us to identify wide subcategories of $\Lambda$ with module categories.

**Theorem 1.12.** For $M \sqcup P[1] \in \tau\text{-}\text{rigid} \Lambda$, we denote

$$J(M \sqcup P[1]) := (M \sqcup P)^{\perp} \cap \perp \tau M,$$

which we call the Jasso category of $M \sqcup P[1]$. We then have:

(a) [DIR+18 Thm. 4.12][Jas15 Thm. 3.8] The Jasso category of $M \sqcup P[1]$ is wide and is equivalent to the category $\text{mod} \Lambda'$ for some $\Lambda'$ which satisfies $|M| + |P| + |\Lambda'| = |\Lambda|$.

(b) [DIR+18 Thm. 4.16] Every wide subcategory $W \in \text{wide} \Lambda$ is the Jasso category of some $\tau\text{-}\text{rigid}$ pair.

Now for $W \in \text{wide} \Lambda$, we denote by $\text{st}\text{-}\text{rigid} W$, the set of pairs $M \sqcup P[1]$ such that $P$ is a projective object of $W$ with $\text{Hom}(P, M) = 0 = \text{Hom}(M, \tau W M)$, where $\tau W$ is the Auslander-Reiten translate in $W$. For $M \sqcup P[1] \in \text{st}\text{-}\text{rigid} W$, we define the Jasso category of $M \sqcup P[1]$ in $W$ as

$$J_W(M \sqcup P[1]) := (M \sqcup P)^{\perp} \cap \perp \tau W M \cap W$$

We remark that in general, the projective objects in $W$, the output of $\tau W$, and the Jasso categories in $W$ are different than in $\text{mod} \Lambda$. With this in mind, we recall the following bijections of Buan and Marsh.

**Theorem 1.13.** Let $W \in \text{wide} \Lambda$.

(a) [BM18b Prop. 5.6] Let $M \in \text{ind} (\text{st}\text{-}\text{rigid} W)$ be a module. Then there is a bijection

$$\mathcal{E}^W_M : \{N \sqcup Q[1] \mid M \sqcup N \sqcup Q[1] \in \text{st}\text{-}\text{rigid} W\} \leftrightarrow \text{st}\text{-}\text{rigid} J_W(M)$$

summarized as follows. Decompose $N \sqcup Q[1] = N_1 \sqcup \cdots \sqcup N_n \sqcup Q_1[1] \sqcup \cdots \sqcup Q_q[1]$ into a direct sum of indecomposable objects. Then

$$\mathcal{E}^W_M(N_i) = \frac{N_i}{\text{rad}(M, N_i)}, \text{ if } N_i \notin \text{Fac} M$$

$$\mathcal{E}^W_M(N_i) = B/\text{rad}(M, B)[1] \text{ for some direct summand, } B, \text{ of the Bongartz complement of } M \text{ in } W, \text{ if } N_i \in \text{Fac} M$$

$$\mathcal{E}^W_M(Q_i[1]) = B/\text{rad}(M, B)[1] \text{ for some direct summand, } B, \text{ of the Bongartz complement of } M \text{ in } W$$

and the formula for the bijection extends additively.

(b) [BM18b Prop. 5.10a] Let $P[1] \in \text{ind} (\text{st}\text{-}\text{rigid} W)$ be a shifted projective. Then there is a bijection

$$\mathcal{E}^W_{P[1]} : \{N \sqcup Q[1] \mid N \sqcup P[1] \sqcup Q[1] \in \text{st}\text{-}\text{rigid} W\} \leftrightarrow \text{st}\text{-}\text{rigid} J_W(P[1])$$

summarized as follows. Decompose $N \sqcup Q[1] = N_1 \sqcup \cdots \sqcup N_n \sqcup Q_1[1] \sqcup \cdots \sqcup Q_q[1]$ into a direct sum of indecomposable objects. Then

$$\mathcal{E}^W_{P[1]}(N_i) = N_i$$

$$\mathcal{E}^W_{P[1]}(Q_i[1]) = \mathcal{E}^W_P(Q_i)[1] = Q_i/\text{rad}(P, Q_i)[1]$$

and the formula for the bijection extends additively.

(c) [BM18a Thm. 3.6, 5.9] Let $M \sqcup P[1] \in \text{st}\text{-}\text{rigid} W$. Then there is a bijection

$$\mathcal{E}^W_{M \sqcup P[1]} : \{N \sqcup Q[1] \mid M \sqcup N \sqcup P[1] \sqcup Q[1] \in \text{st}\text{-}\text{rigid} W\} \leftrightarrow \text{st}\text{-}\text{rigid} J_W(M \sqcup P[1])$$

given as follows. Let $U_1 \sqcup \cdots \sqcup U_n$ be any ordered decomposition of $M \sqcup P[1]$ into indecomposable objects. Then define recursively

$$\mathcal{E}^W_{M \sqcup P[1]} = \mathcal{E}^W_{J_W(U_1)}(\mathcal{E}^W_{U_2 \sqcup \cdots \sqcup U_n}(\mathcal{E}^W_{U_1}))$$


These bijections, given explicitly in [BM18a, BM18b], allow for the definition of the \( \tau \)-cluster morphism category of \( \Lambda \).

**Definition 1.14.** [BM18a] The \( \tau \)-cluster morphism category of \( \Lambda \), denoted \( \mathfrak{W}(\Lambda) \), is defined as follows:

- The objects of \( \mathfrak{W}(\Lambda) \) are the wide subcategories of \( \text{mod}\Lambda \).
- For \( W_1, W_2 \in \text{wide}\Lambda \), we define
  \[
  \text{Hom}(W_1, W_2) = \{ [U] \mid U \in s\tau\text{-rigid }W \text{ and } J_{W_1}(U) = W_2 \}.
  \]
  In particular, \( \text{Hom}(W_1, W_2) = 0 \) unless \( W_1 \supset W_2 \).
- For \( [U] \in \text{Hom}(W_1, W_2) \) and \( [V] \in \text{Hom}(W_2, W_3) \), we define
  \[
  [V] \circ [U] = \left[ U \sqcup \left( E_{W_1 U}^{W_1} \right)^{-1}(V) \right]
  \]

We remark that this definition agrees with the definition of the classical cluster morphism category defined for hereditary algebras, given in [IT17].

2. Cubical Categories

Cubical categories were introduced by the second author in [Igu14] as a tool to study the geometry of the classical cluster morphism categories and picture groups of representation-finite hereditary algebras. The goal of this section will be to show that the \( \tau \)-cluster morphism category of any \( \tau \)-tilting finite algebra is cubical, extending the results of [Igu14] and [IT17]. We first recall the definition of a cubical category.

The basic example of a cubical category is the category \( I^k \), whose objects are the subsets of \( \{0, \ldots, k-1\} \) and whose morphisms are inclusions. The general definition also uses the factorization category of a morphism \( A \xrightarrow{f} B \) in a category \( C \). That is, the category Fac(\( f \)) whose objects are factorizations \( A \xrightarrow{g} C \xrightarrow{h} B \) with \( h \circ g = f \) and whose morphisms

\[
(A \xrightarrow{g} C \xrightarrow{h} B) \rightarrow (A \xrightarrow{g'} C' \xrightarrow{h'} B)
\]

are morphisms \( \phi : C \rightarrow C' \) in \( C \) such that \( \phi \circ g = g' \) and \( h = h' \circ \phi \).

**Definition 2.1.** [Igu14] Def. 3.2 A cubical category is a small category \( C \) with the following properties:

(a) Every morphism \( f : A \rightarrow B \) in \( C \) has a rank, \( rkf \), which is a non-negative integer so that \( rk(f \circ g) = rkf + rkg \).

(b) If \( rkf = k \) then the factorization category of \( f \) is isomorphic to the standard \( k \)-cube category: \( \text{Fac}(f) \cong I^k \).

(c) The forgetful functor \( \text{Fac}(f) \rightarrow C \) taking \( A \xrightarrow{g} C \xrightarrow{h} B \) to \( C \) is an embedding. In particular, every morphism of rank \( k \) has \( k \) distinct first factors and \( k \) distinct last factors.

(d) Every morphism of rank \( k \) is determined by its \( k \) first factors.

(e) Every morphism of rank \( k \) is determined by its \( k \) last factors.

In [IT17], exceptional sequences correspond to decompositions of morphisms into irreducible factors. This result applies in the more general context under the broader definition of Buan and Marsh.

**Definition 2.2.** [BM18b] Def. 1.3 Let \( U_1, \ldots, U_t \in \mathcal{D}^b(\text{mod}\Lambda) \) be indecomposable. Then \( (U_1, \ldots, U_t) \) is called a signed \( \tau \)-exceptional sequence if \( U_t \) is support \( \tau \)-rigid for \( \Lambda \) and \( (U_1, \ldots, U_{t-1}) \) is a signed \( \tau \)-exceptional sequence in \( J(U_t) \).
Theorem 2.3. [BM18b, Rmk. 5.12, Thm. 11.8] Let $W$ be a wide subcategory of $\text{mod}\Lambda$. Then there is a bijection

$$\{\text{ordered support } \tau\text{-rigid objects for } W \text{ with } t \text{ indecomposable direct summands}\}$$

$$\downarrow$$

$$\{\tau\text{-exceptional sequences for } W \text{ of length } t\}$$

given by

$$\Psi^W_t(U_1, \ldots, U_t) = (\Psi^W_{t-1}(U_1), \ldots, \Psi^W_{t-1}(U_{t-1}), U_t).$$

Moreover, the map $\Psi^W_t$ induces a bijection

$$\downarrow$$

$$\{\text{Ordered decompositions of } U \in \mathfrak{s}\tau\text{-rigid}\, W \text{ into direct sums of indecomposable objects}\}$$

$$\uparrow$$

$$\{\text{Factorizations of } [U] \text{ into irreducible morphisms in } \mathfrak{W}(\Lambda)\}.$$ 

We now propose the following Lemma.

Lemma 2.4. Let $[U] : W \to W'$ be a morphism in $\mathfrak{W}(\Lambda)$. Then the factorization category $\text{Fac}[U]$ is isomorphic to the cube $\mathcal{I}^n$, where $n = |U|$, and the forgetful functor $\text{Fac}[U] \to \mathfrak{W}(\Lambda)$ is an embedding.

Proof. Let $\{U_1, \ldots, U_n\}$ be the set of indecomposable direct summands of $U$. Then any factorization of $[U]$ has the form

$$W \xrightarrow{[V_1]} J_W(V) \xrightarrow{[V_2]} W'$$

where $V_1 \in \text{add}U$ and $V_2$ is uniquely determined by $V_1$, since $U = V_1 \cup (\mathcal{E}_V^W)^{-1}(V_2)$. Furthermore, there exists a morphism

$$W \xrightarrow{[V'_1]} J_W(V) \xrightarrow{[V'_2]} W'$$

in $\text{Fac}[U]$ if and only if $V_1 \in \text{add}(V'_1)$. Moreover, if such a morphism exists, then $T = \mathcal{E}_V^W(V'_2)$, where $V'_2$ is of the complement of $V_1$ in $V'_2$.

Finally, we need to show that if $V \neq L \in \text{add}(U)$ then $J_W(V) \neq J_W(L)$. Assume to the contrary, and assume without loss of generality that $V$ and $L$ have no common direct summands. Now $\mathcal{E}_L^W(V) \in J_W(L)$ by assumption. We will show that $\mathcal{E}_L^W(V) \notin J_W(V)$.

Write $U = M \sqcup P[1]$. Then recall that $\mathcal{E}_L^W = \mathcal{E}_M^{J_W(P)} \circ \mathcal{E}_P^W$. Now let $N$ be an indecomposable direct summand of $V$.

If $N = Q[1]$ is not a module. Then $\mathcal{E}_P^{J_W(P)}(Q[1]) = Q'[1]$ where $Q'$ is a nonzero quotient of $Q$. We then have $\mathcal{E}_M^{J_W(P)}(Q'[1]) = B[1]$ where $B$ is a quotient of the Bongartz complement of $M$ and there is a map $Q' \to B$. We conclude that $\text{Hom}(N, \mathcal{E}_L^W(N)) \neq 0$. Therefore $\mathcal{E}_L^W(V) \notin N \subseteq J(W)$.

If $N$ is a module, then $\mathcal{E}_L^W(N) = \mathcal{E}_M^{J_W(P)}(N)$. If this is a module, it is a nonzero quotient of $N$ and $\text{Hom}(N, \mathcal{E}_L^W(N)) \neq 0$, so we are done. Otherwise, the image of $N$ is $(B/\text{rad}(M, B))[1]$, where $B$ is the Bongartz complement of $U$ in $W$ and we have a nonzero map $B \to \tau_W N$. Thus since $\text{Hom}(M, \tau_W N) = 0$, the map $B \to \tau_N$ factors through the quotient. We conclude that $\text{Hom}(\mathcal{E}_L^W(N), \tau_W N) \neq 0$. Therefore $\mathcal{E}_L^W(V) \notin \tau(W) \sqcup J_W(V)$. 

In light of Lemma 2.4 and Theorem 2.3, we see that using the rank function $\text{rk}(U) := |U|$, the category $\mathfrak{W}(\Lambda)$ satisfies the first four properties of a cubic category, since the first factors of $[U]$ correspond precisely to the indecomposable direct summands of $U$. 

2.1. Last Morphisms. Our next step is to describe the last factors of morphisms in \( \mathfrak{W}(\Lambda) \). Throughout this section, we denote by last\([U] \) the set of last factors corresponding to the morphism [\( U \)]. We first show we need only consider morphisms with target 0.

**Lemma 2.5.** The following are equivalent for the category \( \mathfrak{W}(\Lambda) \).

- Every morphism is determined by its last factors.
- Every morphism with target 0 is determined by its last factors.

**Proof.** Assume every morphism with target 0 is determined by its last factors. Now let \( \{[U_i] : W_i \to W\} \) be a set of rank one morphisms which form the last factors of at least one morphism \([U] : W' \to W\). Now if \( W = 0 \), then we are done. Otherwise, choose any morphism \([V] : W' \to 0\).

It follows that
\[
\{[E^W_V (U_i)]\} \sqcup \text{last}[V] = \text{last} ([V] \circ [U]).
\]
In particular, \([U] \) is defined uniquely. \(\square\)

For the remainder of this section, we examine the relationship between sets of last factors in \( \mathfrak{W}(\Lambda) \) and 2-simple minded collections. We begin with the following result relating bricks to minimal wide subcategories.

**Theorem 2.6.** \([\text{DIR}^+18 \text{ Thm. 3.3]}, [\text{BCZ}17 \text{ Thm. 1.0.5}]\) There are bijections
\[
\begin{align*}
brick\Lambda & \to \{T \in \text{tors}\Lambda | T \text{ is completely join irreducible}\} \\
brick\Lambda & \to \{W \in \text{wide}\Lambda | 0 \subset W \text{ is a minimal inclusion}\}
\end{align*}
\]
given by \( S \mapsto \text{FiltFac}S \) and \( S \mapsto \text{FiltS} \) respectively. Moreover, we have \( \text{brick(FiltS)} = \{S\} \).

**Lemma 2.7.** Let \( S \in \text{brick}\Lambda \) and let \( W = \text{FiltS} \). Then \( s_{\tau}\)-tilt\(W = \{P_S, P_S[1]\} \) for some \( P_S \in W \). Moreover, the map \( \text{br} : S \mapsto P_S \) is a bijection with inverse \( P_S \mapsto P_S/\text{rad}(P_S, P_S) \).

**Proof.** Since \( 0 \subset W \) is a minimal inclusion, \( W \) contains a unique projective object \( P_S \). Then we have that \( P_S, P_S[1] \in s_{\tau}\)-tilt\(W \). Moreover, as \( 0 \in s_{\tau}\text{-rigid}W \) is almost complete, we conclude that \( s_{\tau}\text{-tilt}W = 2 \) and \( \{P_S, P_S[1]\} = s_{\tau}\text{-tilt}W \).

Now the map \( P_S \mapsto P_S/\text{rad}(P_S, P_S) \) sends \( P_S \) to a brick in \( W \), which must be \( S \) by the previous theorem. We conclude that the map \( \text{br} \) is a bijection. \(\square\)

Now recall that if \( L \) is a lattice and \( v, v' \in L \) with \( v < v' \), then the intervals \([v, v']\) and \((v, v')\) are the sets \( \{v'' \in L | v \leq v'' \leq v'\} \) and \( \{v'' \in L | v < v'' < v\} \). The Hasse quiver of \( L \) is then the quiver with vertex set \( L \) and an arrow \( v \to v' \) if \( v' < v \) and \( (v', v) = 0 \). We remark that in our context, the Hasse quivers \( \text{Hasse}(s_{\tau}\text{-tilt}\Lambda), \text{Hasse}(2\text{-smc}\Lambda), \) and \( \text{Hasse}(\text{tors}\Lambda) \) are the same up to relabeling the vertices.

**Brick labeling** is a way of assigning a brick to each arrow of the quiver \( \text{Hasse}(s_{\tau}\text{-tilt}\Lambda) \) (and the other equivalent Hasse quivers). This labeling has been considered by Demonet, Iyama, Reiten, and Thomas \([\text{DIR}^+18], \text{Barnard, Carroll and Zhu} [\text{BCZ}17], \text{and Asai} \text{Asa}17\). We use only the following facts.

**Definition-Theorem 2.8.**

(a) \([\text{Asa}17 \text{ Def. 1.14}]\) Let \( U \to V \) be an arrow in \( \text{Hasse}(s_{\tau}\text{-tilt}\Lambda) \). Suppose \( U = M \sqcup M' \sqcup P[1] \) with \( M \) indecomposable and \( V \) a (left) mutation of \( U \) at \( M \). Then the brick label assigned to this arrow is the brick \( M/\text{rad}(M \sqcup M', M) \).

(b) \([\text{Asa}17 \text{ Thm. 2.12}]\) Let \( U \in s_{\tau}\text{-tilt}\Lambda \). Let \( \text{Out}(U) \) be the set of bricks labeling arrows \( U \to V \) for some \( V \) in \( \text{Hasse}(s_{\tau}\text{-tilt}\Lambda) \) and let \( \text{In}(U) \) be the set of bricks labeling arrows \( V' \to U \) for some \( V' \) in \( \text{Hasse}(s_{\tau}\text{-tilt}\Lambda) \). Then
\[
\mathcal{X}(U) := \{B | B \in \text{Out}(U)\} \sqcup \{B[1] | B \in \text{In}(U)\}
\]
is a 2-simple minded collection. Moreover, every 2-simple minded collection occurs in this way.

This leads to the main theorem of this section.

**Theorem 2.9.** Let $U \in \s_{t}\text{-}\text{tilt} \Lambda$. Then $\text{br}^{-1}(\text{last}[U]) = \mathcal{X}(U)$, where we define $\text{br}^{-1}(P[1]) := \text{br}^{-1}(P)[1]$. In particular, $\text{br}^{-1} \circ \text{last}$ defines a bijection $\s_{t}\text{-}\text{tilt} \Lambda \rightarrow 2\text{-}\text{smc} \Lambda$.

**Proof.** Decompose $U = M_1 \sqcup \cdots \sqcup M_m \sqcup P_{m+1}[1] \sqcup \cdots \sqcup P_n[1]$ into a direct sum of indecomposable objects. For convenience, we denote $M_i^c = M_i \sqcup \cdots \sqcup M_{m} \sqcup M_{m} \sqcup P_{m+1} \sqcup \cdots \sqcup P_{n}$. Then by Theorem 2.3 we see that

$$\text{last}[U] = \left\{(\mathcal{E}_{M_i^c \sqcup P}[1](M_i)) \right\} \sqcup \left\{(\mathcal{E}_{M_i^c \sqcup P_{m+1}[1]}(P_j[1])) \right\}$$

We will show that these morphisms correspond precisely to the brick labels of $U$. First consider $M_i$ a direct summand of $M$. There are then two subcases to consider.

Case 1: Assume $M_i \notin \text{Fac}M_i^c$. It follows that $M \sqcup P[1]$ has a left mutation at $M_i$ with label

$$S = M_i/\text{rad}(M, M_i).$$

On the other hand, we have

$$\mathcal{E}_{M_i^c \sqcup P}[1](M_i) = \mathcal{E}_{\mathcal{E}_{P}[1](M_i)} \mathcal{E}_{P[1]}(M_i)$$

$$= \mathcal{E}_{\mathcal{E}_{P}[1](M_i)}(M_i)$$

$$= \mathcal{E}_{M_i^c \sqcup P[1]}(M_i).$$

We observe that $S, \mathcal{E}_{M_i^c \sqcup P[1]}(M_i) \in \text{J}(M_i^c \sqcup P[1])$, which has rank one. Thus as $\mathcal{E}_{M_i^c \sqcup P[1]}(M_i)$ is a module, it follows immediately that $\mathcal{E}_{M_i^c \sqcup P[1]}(M_i) = \text{br}(S)$.

Case 2: Assume $M_i \in \text{Fac}M_i^c$. It follows that $\text{Fac}M = \text{Fac}M_i^c$, meaning there exists an indecomposable $M_i'$ such that $M_i^c \sqcup M_i' \sqcup P[1]$ has a right mutation at $M_i$ with label

$$S[1] = (M_i'/\text{rad}(M_i^c \sqcup M_i', M_i'))[1].$$

On the other hand, we have

$$\mathcal{E}_{M_i^c \sqcup P}[1](M_i) = \mathcal{E}_{\mathcal{E}_{P}[1](M_i)} \mathcal{E}_{P[1]}(M_i)$$

$$= \mathcal{E}_{\mathcal{E}_{P}[1](M_i)}(M_i)$$

$$= \mathcal{E}_{M_i^c \sqcup P[1]}(M_i)$$

where $B_i$ is a direct summand of the Bongartz completion of $M_i^c$ in $J(P)$. Now, $M_i^c$ is almost complete support $\tau$-rigid in $J(P)$, which means $B_i$ is precisely the Bongartz complement of $M_i^c$ in $J(P)$. In particular, this means $B_i = M_i'$. We conclude that

$$\mathcal{E}_{M_i^c \sqcup P[1]}(M_i) = (M_i'/\text{rad}(M_i^c, M_i'))[1].$$

As in Case 1, we observe that $S$ is a brick in $J(M_i^c \sqcup P[1])$ and hence $\mathcal{E}_{M_i^c \sqcup P[1]}(M_i) = \text{br}(S[1])$.

Now consider $P_j$ a direct summand of $P$. Then there exists an indecomposable $M_j'$ such that $M \sqcup M_j' \sqcup P_j[1]$ is a right mutation at $P_j[1]$ with label

$$S[1] = (M_j'/\text{rad}(M_j', M_j))[1].$$

On the other hand, we have

$$\mathcal{E}_{M \sqcup P_j[1]}(P_j[1]) = \mathcal{E}_{\mathcal{E}_{P_j[1]}(M)} \mathcal{E}_{P_j[1]}(P_j[1])$$

$$= \mathcal{E}_{M} (P_j/\text{rad}(P_j, P_j)[1])$$

$$= B_j/\text{rad}(M, B_j)[1]$$
where $B_j$ is a direct summand of the Bongartz completion of $M$ in $J(P_j^c)$. As above, we conclude that $B_j = M_j'$ and hence

$$\mathcal{E}_{M_j P_j^c[1]}(P_j[1]) = \left(M_j' / \text{rad}(M, M_j') \right)[1].$$

As in the previous cases, we observe that $S$ is a brick in $J(P_j^c)$ and hence $\mathcal{E}_{M_j P_j^c[1]}(P_j[1]) = \text{br}(S[1]).$

As a corollary, we have proved proving our first main theorem (Theorem A in the introduction).

**Theorem 2.10.** The category $\mathcal{W}(\Lambda)$ is cubical.

2.2. Locally CAT(0) Categories. We now recall the following result based on a classical result of Gromov [Gro87].

**Proposition 2.11.** [Igu14, Prop.3.4, Prop. 3.7] Suppose $C$ is a cubical category. Then the following additional properties are sufficient for the classifying space $BC$ to be locally CAT(0) and thus a $K(\pi, 1)$.

(a) A set of $k$ rank 1 morphisms $f_i : X \to Y_i$ forms the first factors of a rank $k$ morphism if and only if each pair $\{f_i, f_j\}$ forms the set of first factors of a rank 2 morphism. In other words, first factors are given by pairwise compatibility conditions.

(b) A set of $k$ rank 1 morphisms $f_i : X_i \to Y_i$ forms the last factors of a rank $k$ morphisms if and only if each pair $\{f_i, f_j\}$ forms the set of last factors of a rank 2 morphism. In other words, last factors are given by pairwise compatibility conditions.

(c) There is a faithful group functor $g : C \to G$ for some group $G$.

It is shown in [Igu14] that $\mathcal{W}(KQ)$ satisfies the hypotheses of Proposition 2.11 when $Q$ is the quiver $A_n$ with straight orientation. For a general $\mathcal{W}(\Lambda)$, condition (a) follows directly from the definition of a $\tau$-rigid pair. Condition (b) is already known in the case that $\Lambda$ is hereditary [IT17] or gentle [GM16]. In Section 3, we show that condition (b) also holds in the case that $\Lambda$ is Nakayama. It is also known [ASS06, Thm. V.3.2] that a basic connected algebra is Nakayama if and only if its quiver is one of the following.

**Lemma 2.12.** The following are equivalent for the category $\mathcal{W}(\Lambda)$.

- The last factors of all morphisms are given by pairwise compatibility conditions.
- The last factors of all morphisms with target 0 are given by pairwise compatibility conditions.

**Proof.** Assume that the last factors of all morphisms with target 0 are given by pairwise compatibility conditions. Let $\{[U_i] : W_i \to W'\}$ be a set of pairwise compatible (as last factors) morphisms. Choose any morphism $[V] : W' \to 0$. It follows that $\{[\mathcal{E}^W_U(U_i)]\} \sqcup \text{last}[V]$ is a set of pairwise compatible (as last factors) morphisms with target 0. Thus by assumption, it is the set of last morphisms corresponding to some morphism $[V']$. We can then write $[V'] = [V] \circ [U]$. It follows that $\{[U_i]\}$ is the set of last morphisms of $[U]$. □

### 3. 2-Simple Minded Collections for Nakayama Algebras

We recall that $\Lambda$ is called a Nakayama algebra if every indecomposable $\Lambda$-module has a unique composition series. It is well-known [ASS06, Thm. V.3.2] that a basic connected algebra is Nakayama if and only if its quiver is one of the following.

$$n \leftarrow \cdots \leftarrow 2 \leftarrow 1 \quad A_n$$

$$n \leftarrow \cdots \leftarrow 2 \leftarrow 1 \quad \Delta_n$$
For the remainder of this section, it is assumed that \( \Lambda \) is Nakayama. We fix \( n := |\Lambda| \).

Adachi [Ada16] gave a combinatorial interpretation of the \( \tau \)-tilting modules and support \( \tau \)-tilting modules with no projective direct summands for Nakayama algebras. Asai [Asa17] gave an enumeration of the semibricks of Nakayama algebras. In this section, we give a combinatorial interpretation of the 2-simple minded collections of Nakayama algebras and use this to prove the following theorem.

**Theorem 3.1.** Every mutation compatible semibrick pair over a Nakayama algebra is completable. In particular, the 2-simple minded collections for a Nakayama algebra are given by pairwise compatibility conditions.

3.1. **Bricks for Nakayama Algebras.** We begin by recalling the description of indecomposable \( \Lambda \)-modules given in the remarks following [Ada16] Prop 2.2. For \( M \in \text{mod}\Lambda \), we denote the Loewy length of \( M \) by \( l(M) \). Following Adachi’s notation, we denote by \( i_n \) the unique integer \( 1 \leq k \leq n \) such that \( i - k \in n\mathbb{Z} \). We then define

\[
[i,j]_n := \begin{cases} 
[i_n, j_n] & \text{if } i_n \leq j_n \\
[1, j_n] \cup [i_n, n] & \text{if } i_n > j_n 
\end{cases}
\]

and likewise for \((i,j)_n\), etc. For example, \((1,4)_5 = \{2,3\}\) and \((4,1)_5 = \{5\}\).

**Proposition 3.2.** [ASS06 Section V, Ada16 Prop. 2.2] There is a bijection

\[
M : \{(i,j) \in \mathbb{Z}^2 \mid 0 < i \leq n, 0 < j \leq l(P_i)\} \leftrightarrow \text{ind}(\text{mod}\Lambda)
\]

which sends the pair \((i,j)\) to the module \( P_i/\text{rad}^lP_i \). The inverse is given by sending an indecomposable module \( M \) with projective cover \( P_i \) to the pair \((i, l(M))\).

This description leads to the following characterization of nonzero morphisms between indecomposable \( \Lambda \)-modules.

**Lemma 3.3.** [Ada16 Lem. 2.4] The module \( M(i,j) \) has unique composition series corresponding to the composition factors

\[
S_i, S_{(i-1)n}, \ldots, S_{(i-j+1)n}.
\]

In particular, we have

(a) \( \text{Hom}(M(i,j), M(k,l)) \neq 0 \) if and only if \( i \in [k - l + 1, k]_n \) and \( k - l + 1 \in [i - j + 1, i]_n \)

(b) If in addition we have \( (i - j + 1)_n = (k - l + 1)_n \) and \( j \leq l \) then there is a monomorphism \( M(i,j) \rightarrow M(k,l) \).

(c) If in addition we have \( i = k \) and \( j \geq l \) then there is an epimorphism \( M(i,j) \twoheadrightarrow M(k,l) \).

This leads to the following characterization of \( \text{brick}\Lambda \).

**Proposition 3.4.** Let \( M \in \text{ind}(\text{mod}\Lambda) \). Then \( M \) is a brick if and only if \( l(M) \leq n \).

**Proof.** Write \( M = M(i, mn + j) \) where \( m \geq 0 \). If \( m \neq 0 \), then by the previous lemma there is a chain of morphisms

\[
M(i, mn + j) \rightarrow M(i, j) \rightarrow M(i, mn + j)
\]

and therefore \( M \) is not a brick.

If \( m = 0 \), assume there exists a module \( M(k,l) \) such that there is a chain of morphisms

\[
M(i,j) \rightarrow M(k,l) \rightarrow M(i,j)
\]

Then by the previous lemma, we have \( i = k \) and \( j \leq l \leq j \). We conclude that \( M(k,l) = M(i,j) \) and \( M \) is a brick. \( \square \)

As a consequence, we observe the following.
Corollary 3.5. Let $M, N \in \text{brick} \Lambda$. Then $\dim \text{Hom}(M, N) \leq 1$. Moreover, if $\dim \text{Hom}(M, N) = 1$, then every nonzero morphism $M \to N$ is a minimal left $\text{Filt} N$-approximation.

Proof. We prove the result only for the case that $l(N) = n$. The other cases are trivial, as if $l(N) < n$, then $N$ has no self-extensions. Now if the quiver of $\Lambda$ is $A_n$, then $\Lambda \cong KA_n$ and $N \cong P_1$ has no self extensions. Thus we assume without loss of generality that the quiver of $\Lambda$ is $\Delta_n$ and

$$N = \begin{array}{c}
K \leftarrow 1 \leftarrow 1 K \leftarrow 1 K \\
0
\end{array}$$

Thus up to isomorphism, any $E \in \text{Filt} N$ has the form

$$E = \begin{array}{c}
K^m \leftarrow 1 \leftarrow 1 K^m \leftarrow 1 K^m \\
L
\end{array}$$

where $L_{i,j} = \delta_{i-1,j}$ for $i, j \in \{1, \ldots, m\}$. Now in order for $\text{Hom}(M, N) \neq 0$, we see that $M$ must have the form

$$M = \begin{array}{c}
K \leftarrow 1 \leftarrow 1 K \leftarrow 0 \leftarrow 1 \leftarrow 0
\end{array}$$

We now see that $\dim \text{Hom}(M, E) = 1$, with all morphisms of the form

$$\begin{array}{c}
K \leftarrow 1 \leftarrow 1 K \leftarrow 0 \leftarrow 1 \leftarrow 0 \\
f_1 \\
d_1 \\
K^m \leftarrow 1 \leftarrow 1 K^m \leftarrow 1 K^m \leftarrow 1 K^m
\end{array}$$

where $f_1$ is the composition of an automorphism of $K$ and the first inclusion map. In particular, $\dim \text{Hom}(M, N) = 1$ and every nonzero morphism $M \to E$ factors through the inclusion map $N \hookrightarrow E$. □

We are now ready to construct our combinatorial model. Let $D(n, l(1), l(2), \ldots, l(n))$ be the punctured disk $D^2 \setminus \{0\}$ with $n$ marked points on its boundary, labeled counterclockwise $1, 2, \ldots, n$. The value of $l(i)$ is called the length of the marked point labeled $i$. We further assume

1. For all $i \leq n$, we have $l(i) \geq l((i-1)n) - 1$.
2. For all $i < n$, we have $l(i) > 1$.

Under these assumptions, we see that $l(1), \ldots, l(n)$ are the Loewey lengths of the projective modules of some connected Nakayama algebra $\Lambda(l(1), \ldots, l(n))$ whose quiver has $n$ vertices. Conversely, given a Nakayama algebra whose quiver has $n$ vertices, we observe that the Loewey lengths of the projective modules satisfy both (a) and (b). For example, $D(4, 3, 3, 3)$ corresponds to the cyclic cluster-tilted algebra of type $D_4$ and $D(3, 3, 2, 1)$ corresponds to $A_3$ with straight orientation.

We now wish to relate $\text{brick} \Lambda(l(1), \ldots, l(n))$ to certain directed paths in $D(n, l(1), \ldots, l(n))$. We begin with the following definition.

Definition 3.6. A directed path $a : i \to j$ between two marked points of $D(n, l(1), \ldots, l(n))$ is called an arc if

1. $a$ is homotopic to the counterclockwise boundary arc $i \to j$. 

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Under these assumptions, we see that $l(1), \ldots, l(n)$ are the Loewey lengths of the projective modules of some connected Nakayama algebra $\Lambda(l(1), \ldots, l(n))$ whose quiver has $n$ vertices. Conversely, given a Nakayama algebra whose quiver has $n$ vertices, we observe that the Loewey lengths of the projective modules satisfy both (a) and (b). For example, $D(4, 3, 3, 3)$ corresponds to the cyclic cluster-tilted algebra of type $D_4$ and $D(3, 3, 2, 1)$ corresponds to $A_3$ with straight orientation.

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1. $a$ is homotopic to the counterclockwise boundary arc $i \to j$. 

(b) \(a\) does not intersect itself unless \(i = j\), in which case the only intersection occurs at the endpoint.

(c) \(l(i) \geq (j - i)_n\).

We call \(i\) the source of \(a\), denoted \(s(a)\), and \(j\) the target of \(a\), denoted \(t(a)\). We call \((j - i)_n\) the length of \(a\), denoted \(l(a)\), and \([s(a), t(a)]_n\) the support of \(a\), denoted \(\text{supp}(a)\).

Condition (3) can then be rephrased as \(l(s(a)) \geq l(a)\). We denote the set of homotopy classes of arcs of \(D(n, l(1), \ldots, l(n))\) by \(\text{Arc}(n, l(1), \ldots, l(n))\).

The following is clear.

**Proposition 3.7.** There is a bijection \(M : \text{Arc}(n, l(1), \ldots, l(n)) \to \text{brickA}(l(1), \ldots, l(n))\) given by sending an arc \(a\) to the module \(M(s(a), l(a))\).

We now consider the various ways that two distinct arcs can (minimally) intersect and the corresponding morphisms and extensions between modules. As we are only interested in characterizing sets of mutation compatible semibrick pairs, the details regarding extensions are intentionally incomplete. In the diagrams accompanying Lemmas 3.8 and 3.9 the arc \(a_1\) is always shown in blue and the arc \(a_2\) is always shown in orange. We remark that these figures provide only examples, e.g. figures with two arcs each having length \(n\), etc. are not shown.

**Lemma 3.8.** Let \(a_1, a_2 \in \text{Arc}(n, l(1), \ldots, l(n))\) with \(l(a_1) = n\).

(a) If \(a_1\) and \(a_2\) intersect at two different points, then both \(\text{Hom}(M(a_1), M(a_2))\) and \(\text{Hom}(M(a_1), \text{brickA}(a_2))\) are nonzero. Otherwise at least one of these is zero.

(b) If \(a_1\) and \(a_2\) intersect at only one point, then \(\text{Hom}(M(a_1), M(a_2))\) contains an epimorphism if and only if \(s(a_1) = s(a_2)\). In this case, \(\text{Hom}(M(a_2), M(a_1)) = 0 = \text{Ext}(M(a_2), M(a_1))\).

We remark that \(a_1\) appears counterclockwise from \(a_2\).

(c) If \(a_1\) and \(a_2\) intersect at only one point, then \(\text{Hom}(M(a_2), M(a_1))\) contains a monomorphism if and only if \(t(a_1) = t(a_2)\). In this case, \(\text{Hom}(M(a_1), M(a_2)) = 0 = \text{Ext}(M(a_1), M(a_2))\).

We remark that \(a_2\) appears counterclockwise from \(a_1\).

(d) Otherwise \(a_1\) and \(a_2\) do not intersect and are Hom-Ext orthogonal.

Proof. All results regarding morphisms follow directly form Lemma 3.3 and Corollary 3.5. The results regarding extensions are straightforward to verify.

**Lemma 3.9.** Let \(a_1, a_2 \in \text{Arc}(n, l(1), \ldots, l(n))\) with \(l(a_1), l(a_2) < n\).

(a) If \(a_1\) and \(a_2\) do not intersect, then \(M(a_1)\) and \(M(a_2)\) are Hom-Ext orthogonal.

(b) If \(t(a_1) = s(a_2)\) and \(a_1\) does not otherwise intersect \(a_2\) then \(M(a_1)\) and \(M(a_2)\) are Hom orthogonal and \(\text{Ext}(M(a_2), M(a_1)) = 0\). Moreover,

(i) if \(l(a_1) + l(a_2) \leq l(s(a_1))\) then \(\text{Ext}(M(a_1), M(a_2))\) is nonzero and there is an exact sequence

\[M(a_2) \hookrightarrow M(a_1 a_2) \twoheadrightarrow M(a_2).\]

(ii) Otherwise \(M(a_1)\) and \(M(a_2)\) are Hom-Ext orthogonal.

(c) If \(t(a_2) = s(a_1)\) and \(t(a_1) = s(a_2)\) then \(M(a_1)\) and \(M(a_2)\) are Hom orthogonal. Moreover,
(i) if \( l(a_1) + l(a_2) \leq l(s(a_1)) \) then \( \text{Ext}(M(a_1), M(a_2)) \) is non-zero and there is an exact sequence
\[ M(a_2) \hookrightarrow M(a_1a_2) \twoheadrightarrow M(a_2). \]
(ii) if \( l(a_1) + l(a_2) \leq l(s(a_2)) \) then \( \text{Ext}(M(a_2), M(a_1)) \) is non-zero and there is an exact sequence
\[ M(a_1) \hookrightarrow M(a_2a_1) \twoheadrightarrow M(a_1). \]
(iii) otherwise \( M(a_1) \) and \( M(a_2) \) are Hom-Ext orthogonal.
(d) If \( s(a_1) = s(a_2) < t(a_2) < t(a_1) \) and \( a_1, a_2 \) do not otherwise intersect (in this case, \( a_1 \) appears counterclockwise from \( a_2 \)) then \( \text{Hom}(M(a_1), M(a_2)) \) contains a monomorphism and \( \text{Hom}(M(a_2), M(a_1)) = 0 = \text{Ext}(M(a_2), M(a_1)) \).
(e) If \( s(a_2) < s(a_1) < t(a_1) = t(a_2) \) and \( a_1, a_2 \) do not otherwise intersect (in this case, \( a_1 \) appears counterclockwise from \( a_2 \)) then \( \text{Hom}(M(a_1), M(a_2)) \) contains an epimorphism and \( \text{Hom}(M(a_2), M(a_1)) = 0 = \text{Ext}(M(a_2), M(a_1)) \).
(f) If \( a_1 \) and \( a_2 \) intersect once in the interior of the disk then one of \( \text{Hom}(M(a_1), M(a_2)) \) and \( \text{Hom}(M(a_1), M(a_2)) \) is nonempty and contains no monomorphism or epimorphism.
(g) If \( a_1 \) and \( a_2 \) intersect twice in the interior of the disk, then both \( \text{Hom}(M(a_1), M(a_2)) \) and \( \text{Hom}(M(a_2), M(a_1)) \) are nonzero.

**Proof.** All results regarding morphisms follow directly from Lemma 3.8 and Corollary 3.9. The results regarding extensions are straightforward to verify. \( \square \)

We say that an arc \( a_1 \) is **counterclockwise** from an arc \( a_2 \) (and hence that \( a_2 \) is **clockwise** from \( a_1 \)) if \( a_1 \) intersects \( a_2 \) as in Lemma 3.8(c),(d) or Lemma 3.9(d),(e).

### 3.2. Admissible Arc Patterns

Recall that an object \( \mathcal{X} = \mathcal{S}_p \sqcup \mathcal{S}_n[1] \in \mathcal{D}^b(\text{mod}\Lambda) \) is called a mutation compatible semibrick pair if \( \mathcal{S}_p, \mathcal{S}_n \in \text{sbrick}\Lambda, \)

\[ \text{Hom}(\mathcal{S}_p, \mathcal{S}_p[m]) = \text{Hom}(\mathcal{S}_n, \mathcal{S}_n[m]) = \text{Hom}(\mathcal{S}_n, \mathcal{S}_p[m]) = \text{Hom}(\mathcal{S}_p, \mathcal{S}_n[m + 2]) = 0 \]

for all \( m < 0 \), and for every \( S \in \mathcal{S}_p \) and \( S' \in \mathcal{S}_n \) with \( \text{Hom}(S', S) \neq 0 \), every minimal left \( \text{Filt}S' \) approximation \( S' \to E \) is either mono or epi. By translating this definition into the language of arcs, we arrive at the following.

**Definition 3.10.** A collection \( A = A_p \sqcup A_r \) of distinct arcs in \( \text{Arc}(n, l(1), \ldots , l(n)) \) is an **admissible arc pattern** if it can be drawn in \( D(l(1), \ldots , l(n)) \) so that for all \( a_1, a_2 \in A \)

(a) The only intersections between \( a_1 \) and \( a_2 \) are on the boundary of the disk.
(b) If $s(a_1) = s(a_2)$ or $t(a_1) = t(a_2)$ and $a_1$ is counterclockwise from $a_2$, then $a_1 \in A_r$ and $a_2 \in A_g$.

c) If $t(a_1) = s(a_2)$ then at least one of the following holds

(i) $a_1 \in A_r$

(ii) $a_2 \in A_g$

(iii) $l(a_1) + l(a_2) > l(s(a_2))$

We refer to $A_g$ as the set of green arcs and $A_r$ as the set of red arcs.

**Remark 3.11.** It follows immediately from the previous Lemmas that there is a bijection between admissible arc patterns for $D(n, l(1), \ldots, l(n))$ and mutation compatible semibrick pairs for $\Lambda(n, l(1), \ldots, l(n))$ given by

$$A \mapsto \mathcal{X}(a) := \{M(a) | a \in A_g\} \sqcup \{M(a)[1] | a \in A_r\}.$$

Our next task is to determine which admissible arc patterns correspond to 2-simple minded collections.

**Definition 3.12.** An admissible arc pattern is *maximal* if it is not a proper subset of another admissible arc pattern.

**Example 3.13.** A maximal admissible arc pattern for $D(3,3,2,1)$, which corresponds to the algebra $K A_3$ (where $A_3$ is given straight orientation). A complete set of examples for this algebra and for the cyclic cluster-tilted algebra of type $D_4$ can be found in Section 5.

In the above example, we see that

$$\mathcal{X}(A) = \frac{1}{2} \sqcup 2[1] \sqcup 3[1]$$

is a 2-simple minded collection. We now wish to prove that this is the case in general.

**Theorem 3.14.** Let $A$ be an admissible arc pattern for $D(n, l(1), \ldots, l(n))$. Then $\mathcal{X}(A)$ is a 2-simple minded collection if and only if $A$ is maximal. In particular, by Remark 3.11, every 2-simple minded collection occurs in this way.

As admissible arc patterns are given by pairwise compatibility conditions, this result is equivalent to Theorem 3.1. We begin with several results pertaining to maximal arc patterns.

**Definition 3.15.** Let $A$ be an admissible arc pattern for $D(n, l(1), \ldots, l(n))$ and let $A^*$ denote the set of inverse paths to those in $A$. For $v, v'$ marked points on the punctured disk, we say a sequence $(a_1, \ldots, a_k) \in (A \cup A^*)^k$ is an arc path of $A$ from $v$ to $v'$ if

(a) $s(a_1) = v$.

(b) $t(a_k) = v'$.

c) $t(a_j) = s(a_{j+1})$ for $1 \leq j < k$.

d) $a_1 \cdots a_k$ is homotopic to the boundary arc $v \rightarrow v'$.

e) $M(a_1, \ldots, a_k) := M(v \rightarrow v')$ is defined (and is a brick).
We set \( \text{supp}(a_1, \ldots, a_k) := \text{supp}(v \rightarrow v') \). An arc path \((a_1, \ldots, a_k)\) is called \textit{proper} if every subpath \((a_i, \ldots, a_j), 1 \leq i \leq j \leq k\), is either an arc path or the reverse of an arc path. If only conditions (a)-(c) hold, we refer to the sequence of arcs as a \textit{pseudo-arc path}.

We now build toward the main theorem of this section by proving several propositions about proper arc paths.

**Proposition 3.16.** Let \( A \) be a maximal arc pattern for \( D(n, l(1), \ldots, l(n)) \). Then every vertex is the source or target of at least one arc.

**Proof.** Assume there exists a marked point \( j \) which is not the source or target of any arc. Then since \( A \) is maximal, a green arc \( j \rightarrow (j + 1)_n \) must be incompatible with \( A \). Thus there must be a red arc \( R \) with \( t(R) = (j + 1)_n \). Likewise, as a red arc \( (j - 1)_n \rightarrow j \) must be incompatible with \( A \), there must be a green arc \( G \) with \( s(G) = (j - 1)_n \). Now for \( R \) and \( G \) to not cross in the interior of the disk, either \( s(G) = (j + 1)_n \) or \( t(R) = (j - 1)_n \). However, in either case, \( G \) will be counterclockwise from \( R \), a contradiction.

\[
\begin{array}{c}
\bullet (j + 1)_n \\
\bullet (j - 1)_n \\
\times
\end{array}
\]

\[ j \]

**Proposition 3.17.** Let \( A \) be a maximal arc pattern for \( D(n, l(1), \ldots, l(n)) \) and let \( i \) and \( j \) be marked points. Then there exists a pseudo arc path \( i \rightarrow j \).

**Proof.** We say a pair of marked points are \textit{connected} by \( A \) if there is a pseudo arc path between them. By Proposition 3.16 we know that every connected component under this definition contains at least two marked points. Assume for a contradiction there exist two connected components \( C_1 \) and \( C_2 \). We can then choose marked vertices \( i_1, j_1 \in C_1 \) and \( i_2, j_2 \in C_2 \) so that \( C_1 \subset [i_1, j_1]_n, C_2 \subset [i_2, j_2]_n \), and \([i_1, j_1]_n \cap [i_2, j_2]_n = \emptyset\). Indeed, if \( C_2 \cap (i_1, j_1)_n, C_2 \cap (j_1, i_1)_n \neq \emptyset \), then \( A \) contains an arc \( a \) with \( s(a) \in (i_1, j_1)_n \) and \( t(a) \in (j_1, i_1)_n \) or vice versa. This arc will then intersect the pseudo arc path from \( i_1 \) to \( j_1 \), a contradiction. We can further assume that the choice of \( C_2 \) is maximal with respect to the interval \([i_2, j_2]_n\). That is, there is no pseudo arc path \( i_3 \rightarrow j_3 \) with \( i_3 \in (j_1, i_2)_n \) and \( j_3 \in (j_2, i_1)_n \) or vice versa. We now have three cases to consider.

Case 1: Assume there exists a green arc \( G \) with source \( j_1 \). It follows that \( l(k) \geq (i_1 - k)_n \) for all \( k \in [j_1, i_1]_n \). Moreover, there cannot be an arc with source \( j_2 \) or target \( i_2 \), as it would intersect \( G \). Likewise, the only arc that can possibly have target \( i_1 \) is \( G \), as any red arc with target \( i_1 \) would be clockwise from \( G \) or intersect \( G \).

Now, by the maximality of \( A \), there must be a green arc \( G' \) with target \( j_2 \). Otherwise (by the maximality of \([i_2, j_2]_n\)), a red arc \( j_2 \rightarrow i_1 \) would be compatible with \( A \). Now any red arc with source \( s(G') \) would need to have target in \( (s(G'), j_2)_n \), but this would make it clockwise from \( G' \). Thus we can assume the existence a green arc \( G'' \) with target \( s(G') \) as before. By iterating this process, we can assume that \( s(G'') = i_2 \). At this point, there is no way to prevent the compatibility of a red arc \( i_2 \rightarrow i_1 \) with \( A \), a contradiction.
Case 2: Assume there exists an arc \( a \) with source and target in \([i_1, j_1]_n\) such that \([i_2, j_2]_n \subset \text{supp}(a)\). As before, we can assume that \( l(k) \geq (i_1 - k)_n \) for \( k \in [j_1, i_1]_n \). If \( s(a) = j_1 \) and \( a \) is green, then we are in case 1. Thus assume \( s(a) \neq j_1 \) or \( a \) is red. In either case, there cannot be an arc with target \( i_2 \) or source \( j_2 \), as it would intersect \( a \).

We now observe that any green arc with source \( j_1 \) and target in \((j_1, i_1)_n\) would be compatible with \( a \). Thus, by the maximality of \([i_2, j_2]_n\), there must be a red arc \( R \) with source \( i_2 \). Otherwise, a green arc \( j_1 \rightarrow i_2 \) would be compatible with \( A \). Now any green arc with target \( t(R) \) would necessarily be counterclockwise from \( R \). Thus as a green arc \( j_1 \rightarrow t(R) \) must be incompatible with \( A \), there must exist a red arc \( R' \) with source \( t(R) \) as before. By iterating this process, we can assume that \( t(R') = j_2 \). At this point, there is no way to prevent the compatibility of a green arc \( j_1 \rightarrow j_2 \) with \( A \), a contradiction.

Case 3: We now assume there is no arc \( a \) with source and target in \([i_1, j_1]_n\) such that \([i_2, j_2]_n \subset \text{supp}(a)\). By symmetry, we can likewise assume there is no arc \( a \) with source and target in \([i_2, j_2]_n\) such that \([i_1, j_1]_n \subset \text{supp}(a)\). Thus there cannot be an arc with target \( i_2 \), source \( j_2 \) or source \( j_1 \).

As before, we observe that there must be a red arc \( R \) with source \( i_2 \) and \( l(j_1) \geq (t(R) - j)_n \), or a green arc \( j_1 \rightarrow i_2 \) would be compatible with \( A \). Now any green arc with target \( t(R) \) would necessarily be counterclockwise from \( R \). Thus since a green arc \( j_1 \rightarrow t(R) \) must be incompatible with \( A \), there must exist a red arc \( R' \) with source \( t(R) \) and \( l(t(R')) \geq t(R') - j_1 \). By iterating this process, we can assume that \( t(R') = j_2 \). At this point, there is no way to prevent the compatibility of a green arc \( j_1 \rightarrow j_2 \) with \( A \), a contradiction.
We have shown that the marked points of \( D(n,l(1), \ldots, l(n)) \) cannot be partitioned into more than one connected component. This means there exists a pseudo arc path between any two vertices. \( \square \)

**Proposition 3.18.** Let \( A \) be a maximal arc pattern for \( D(n,l(1), \ldots, l(n)) \) and let \( j \) be a marked vertex. Then there exists a proper arc path \( j \to (j+1)_n \).

**Proof.** Clearly, if there is an arc \( j \to (j+1)_n \) then we are done. Thus assume no such arc exists. We first reduce to the case that there exists an arc \( a \in A \) with \([j, j+1]_n \subset \text{supp}(a)\). Indeed, if there exists no such arc, then in particular, no arc has source \( j \) or target \((j+1)_n\). Thus by the maximality of \( A \), there must be a green arc \( G \) with target \( j \) and a red arc \( R \) with source \((j+1)_n\) such that \( l(s(G)) \geq (j+1-l(s(G))_n \) and \( l(j) \geq (t(R) - j)_n \). Otherwise an arc \( j \to (j+1)_n \) would be compatible with \( A \).

Now any green arc with target \( t(R) \) must have source in \((t(R), s(G))_n\), but such an arc would contain \([j, j+1]_n \) in its support. Thus we can assume there is no green arc with target \( t(R) \). Then as a green arc \( j \to t(R) \) must be incompatible with \( A \), there must exist a red arc \( R' \) with source \( t(R) \) such that \( l(j) \geq (t(R') - j)_n \). As before, we can assume there is no green arc with target \( t(R') \) (or it would contain \([j, j+1]_n \) in its support). Thus there exists a red arc \( R'' \) with source \( t(R') \). Iterating this argument, we can assume that \( t(R'') = s(G) \). Again, there can be no green arc with target \( s(G) \). But then there is no way to prevent the compatibility of a red arc \( s(G) \to (j+1)_n \) with \( A \), a contradiction.

![Diagram](image)

Thus let \( a \in A \) be an arc with \([j, j+1]_n \subset \text{supp}(a)\). Then for \( k \in \text{supp}(a) \) we have \( l(k) \geq (t(a) - k)_n \). Now by the previous proposition, there exists a pseudo arc path \((a_1, \ldots, a_k) : t(a) \to (j+1)_n\). Thus there exists some index \( i \leq k \) for which \( s(a_i) = s(a) \) or \( s(a_i) = t(a) \) and \((a_{i+1}, \ldots, a_k)\) remains in the interval \((s(a), t(a))_n\). If \( s(a_i) = s(a) \), then this is a proper arc path \( s(a) \to (j+1)_n \). Otherwise, \( s(a_i) = t(a) \neq s(a) \) and \((a, a_i, \ldots, a_k)\) is a proper arc path \( s(a) \to (j+1)_n \). We denote whichever proper arc path exists by \((a_{j+1}) : s(a) \to (j+1)_n\).

Likewise, there exists a pseudo arc path \((b_1, \ldots, b_{k'}) : s(a) \to j \). Again, there is some index \( i' \leq k' \) for which \( s(b_{i'}) = s(a) \) or \( s(b_{i'}) = t(a) \) and \((b_{i'+1}, \ldots, b_{k'})\) remains in the interval \((s(a), t(a))_n\). If \( s(b_{i'}) = s(a) \), then \((b_{i'}, \ldots, b_{k'})\) is a proper arc path \( s(a) \to j \) and \((b_{i'}^*, \ldots, b_{k'}^*, a_{j+1})\) is a proper arc path \( s(a) \to j \). Otherwise, \( s(b_{i'}) = t(a) \neq s(a) \). In this case, \((a, b_{i'}, \ldots, b_{k'})\) is a proper arc path \( s(a) \to j \) and \((b_{i'}^*, \ldots, b_{k'}^*, a^* a_{j+1})\) is a proper arc path \( j \to (j+1)_n \).
We are now ready to prove Theorem 3.14. Our proof is modeled after that of [GM16, Lem. 8.11] by Garver and McConville.

Proof. It is clear that if \( \mathcal{X}(A) \) is a 2-simple minded collection, then \( A \) is maximal. Thus let \( A \) be a maximal arc pattern and let \( \mathfrak{T} = \text{thick}(\mathcal{X}(A)) \). We will show that \( \mathfrak{T} \) contains all simple modules. As the collection of simple modules forms a 2-simple minded collection, this will imply that \( \mathfrak{T} = \mathcal{D}^b(\text{mod}\Lambda) \).

Let \( S \in \text{brick}\Lambda \) and let \((p_1, \ldots, p_k)\) be a sequence of proper arc paths and reverses of proper arc paths. Then we say \((p_1, \ldots, p_k)\) is an admissible path sequence for \( S \) if
- \( M(|p_j|) \in \mathfrak{T} \) for all \( j \), where by \(|p_j|\) we mean the proper arc path corresponding to \( p_j \).
- Considered as a single arc pseudo arc path, \((p_1p_2\cdots p_k)\) is a proper arc path for \( S \).

We observe that every \( a \in A \) is itself an admissible path sequence for \( M(a) \). Moreover, by Proposition 3.18 every simple module \( S \in \text{mod}\Lambda \) admits an admissible path sequence. We now prove that if \( S \) has an admissible path sequence of length \( k > 1 \), then \( S \) has an admissible path sequence of length \( k - 1 \). In particular, this means \( S \) has an admissible path sequence of length 1 and hence \( S \in \mathfrak{T} \).

Let \((p_1, \ldots, p_k)\) be an admissible path sequence for \( S \) of length \( k > 1 \). Then there exists some index \( 1 < j \leq k \) such that \( \text{supp}(|p_{j-1}|) \neq \text{supp}(|p_j|) \).

If \( \text{supp}(|p_{j-1}|) \cap \text{supp}(|p_j|) = \emptyset \), then there is either an exact sequence

\[ M(|p_{j-1}|) \hookrightarrow M(|p_{j-1}p_j|) \twoheadrightarrow M(|p_j|) \]

or

\[ M(|p_{j-1}|) \hookrightarrow M(|p_{j-1}p_j|) \twoheadrightarrow M(|p_{j-1}|). \]

In either case, \( M(|p_{j-1}p_j|) \in T \) and \((p_1, \ldots, p_{j-2}, p_{j-1}p_j, p_{j+1}, \ldots, p_k)\) is an admissible path sequence for \( S \) of length \( k - 1 \).

If \( \text{supp}(|p_{j-1}|) \cap \text{supp}(|p_j|) \neq \emptyset \), then one of the following holds:

1. \( \text{supp}(|p_{j-1}|) \subset \text{supp}(|p_j|) \) and there is a monomorphism \( f : M(|p_{j-1}|) \hookrightarrow M(|p_j|) \).
2. \( \text{supp}(|p_{j-1}|) \subset \text{supp}(|p_j|) \) and there is a monomorphism \( f : M(|p_j|) \twoheadrightarrow M(|p_{j-1}|) \).
3. \( \text{supp}(|p_j|) \subset \text{supp}(|p_{j-1}|) \) and there is an epimorphism \( f : M(|p_j|) \twoheadrightarrow M(|p_{j-1}|) \).
4. \( \text{supp}(|p_j|) \subset \text{supp}(|p_{j-1}|) \) and there is an epimorphism \( f : M(|p_{j-1}|) \twoheadrightarrow M(|p_j|) \).

In case (1) and (3), we see that \( N := \ker f \in \mathfrak{T} \). Likewise in case (2) and (4), we see that \( N := \text{coker} f \in \mathfrak{T} \). In any case, we then have \( M(|p_{j-1}p_j|) = N \) and \((p_1, \ldots, p_{j-2}, p_{j-1}p_j, p_{j+1}, \ldots, p_k)\) is an admissible path sequence for \( S \) of length \( k - 1 \). □
4. Picture Groups and the Fundamental Group

In the case that $\Lambda$ is hereditary, the picture group of $\Lambda$, denoted $G(\Lambda)$, is defined in [IOTW16]. It is then shown in [IT17] that the fundamental group of $\mathfrak{W}(\Lambda)$ is isomorphic to $G(\Lambda)$. The goal of this section is to extend the definition of the picture group and this result to non-hereditary algebras, and to construct a faithful group functor $\mathfrak{W}(\Lambda) \rightarrow G(\Lambda)$ in the case that $\Lambda$ is a Nakayama algebra.

4.1. Picture Groups. We begin by recalling the definition of the picture group in the hereditary case. We denote by $\alpha_1, \ldots, \alpha_n$ the positive simple roots of a hereditary algebra $\Lambda$. For a root $\alpha$, we denote by $M_\alpha$ the corresponding module.

**Definition-Theorem 4.1.** [IOTW16, Def. 1.1.5, Def. 1.1.8, Thm. 2.2.1] The picture group $G(\Lambda)$ has the following presentation.

- $G(\Lambda)$ has one generator $X_S$ for every brick $S \in \text{brick}\Lambda$.
- For each pair $S_1, S_2$ of Hom-orthogonal bricks such that $\text{Ext}(S_1, S_2) = 0$, write $S_1 = M_\alpha$ and $S_2 = M_\beta$. We then have the relation $X_{S_1}X_{S_2} = \prod_{T \in \text{brick} \left( \text{Filt}(S_1, S_2) \right)} X_T$

where the order of the product is given by the increasing order of the ratio $r/s$ of the decomposition $T = M_{r\alpha + s\beta}$ (going from $0/1$ to $1/0$).

We extend this definition to the non-hereditary case by first recalling several facts about the lattice $\text{tors}\Lambda$. We recall that an interval $[X, Y]$ in a lattice is called a polygon if $(X, Y)$ consists of two disjoint nonempty chains.

**Theorem 4.2.** [DIR+18, Thm. 4.16, Prop. 4.21]

(a) The lattice $\text{tors}\Lambda$ is polygonal. That is

(i) Given arrows $T \rightarrow T_1$ and $T \rightarrow T_2$ in $\text{Hasse}(\text{tors}\Lambda)$, the interval $[T \wedge T_2, T]$ is a polygon.
(ii) Given arrows $T_1 \rightarrow T$ and $T_2 \rightarrow T$ in $\text{Hasse}(\text{tors}\Lambda)$, the interval $[T, T_1 \vee T_2]$ is a polygon.

(b) Every polygon in $\text{Hasse}(\text{tors}\Lambda)$ is of the form

$$\begin{tikzpicture}
\node (S1) at (0,0) {$S_1$};
\node (S2) at (2,0) {$S_2$};
\node (T1) at (0,-1) {$T_1$};
\node (T1') at (2,-1) {$T_1'$};
\node (Tk) at (0,-2) {$T_k$};
\node (S2) at (2,-2) {$S_2$};
\node (FacM) at (1,-3) {$\text{Fac}M$};
\node (J(M) \cap P^\perp) at (1,-4) {$J(\tau M) \cap P^\perp$};
\draw[->] (S1) -- (T1);
\draw[->] (S1) -- (T1');
\draw[->] (S2) -- (Tk);
\draw[->] (S2) -- (T1');
\draw[->] (S2) -- (Tk);
\draw[->] (S2) -- (S1);
\end{tikzpicture}$$

for some $M \sqcup P[1] \in \tau\text{-rigid}\Lambda$ of corank 2.

(c) In the above polygon, we have $J(M \sqcup P[1]) = \text{Filt}(S_1 \sqcup S_2)$. In particular, $J(M \sqcup P[1])$ is a wide subcategory of rank 2.

We observe that there may be multiple torsion classes $\mathcal{T}$ for which $S_1$ and $S_2$ label arrows $\mathcal{T} \rightarrow \mathcal{T}_1$ and $\mathcal{T} \rightarrow \mathcal{T}_2$ (resp. $\mathcal{T}_1 \rightarrow \mathcal{T}$ and $\mathcal{T}_2 \rightarrow \mathcal{T}$). However, as a consequence of (d), the labeling of the polygon $[\mathcal{T}_1 \wedge \mathcal{T}_2, \mathcal{T}]$ (resp. $[\mathcal{T}, \mathcal{T}_1 \vee \mathcal{T}_2]$) depends only on $S_1$ and $S_2$, not on $\mathcal{T}$. Thus we refer to any polygon which occurs in $\text{Hasse}(\text{tors}\Lambda)$ with this labeling as $\mathcal{P}(S_1, S_2)$. We refer to the sequences...
(S₁, T₁, . . . , Tk, S₂) and (S₂, T₁', . . . , Tₖ', S₁) as the sides of the polygon P(S₁, S₂). We now propose the following extension of the definition of the picture group.

**Definition-Theorem 4.3.** The picture group G(Λ) is defined to have the following presentation.

- G(Λ) has one generator Xₜ for every brick t ∈ brick Λ.
- For each polygon P(S₁, S₂), we have the polygon relation

  \[ X_{S₁} X_{T₁} \cdots X_{Tₖ} X_{S₂} = X_{S₂} X_{T₁'} \cdots X_{Tₖ'} X_{S₁} \]

  where (S₁, T₁, . . . , Tk, S₂) and (S₂, T₁', . . . , Tₖ', S₁) are the sides of P(S₁, S₂).

**Proof.** In the case that Λ is a representation-finite hereditary algebra, our relations are the inverse of the relations given in Definition-Theorem 4.1 and hence the groups are isomorphic. Indeed, as in [IOTW16, Example 2.2.2], we need only consider the six cases where Λ = KQ and Q is a Dynkin quiver with two vertices. For example, if

\[ Q = 2 \xleftarrow{(1,2)} 1 \cong B₂ \cong C₂ \]

then brick Λ = \{M₁₀, M₀₁, M₁₁, M₂₁\} and Hasse(tors Λ) consists of two chains:

\[ \text{mod} \Lambda \xrightarrow{M₁₀} \text{FiltFac} M₂₁ \xrightarrow{M₂₁} \text{FiltFac} M₁₁ \xrightarrow{M₁₁} \text{Filt} M₀₁ \xrightarrow{M₀₁} 0 \]

and

\[ \text{mod} \Lambda \xrightarrow{M₀₁} \text{Filt} M₁₀ \xrightarrow{M₁₂} 0. \]

Thus in this case, the two definitions yield isomorphic groups. The other five cases follow from identical reasoning. □

Recall that a sequence (S₁, . . . , Sₖ) of bricks is called a maximal green sequence if it labels a (directed) path mod Λ → 0 in Hasse(tors Λ). We now give three additional presentations of the picture group G(Λ).

**Proposition 4.4.**

(a) G(Λ) is generated by \{Xₜ | t ∈ brick Λ\} with a relation

\[ X_{S₁} \cdots X_{Sₖ} = X_{S₁'} \cdots X_{Sₖ'} \]

if there exist T, T' ∈ tors Λ such that (S₁, . . . , Sₖ) and (S₁', . . . , Sₖ') label (directed) paths T → T' in Hasse(tors Λ).

(b) G(Λ) is generated by \{Xₜ | t ∈ brick Λ\} with a relation

\[ X_{S₁} \cdots X_{Sₖ} = X_{S₁'} \cdots X_{Sₖ'} \]

if (S₁, . . . , Sₖ) and (S₁', . . . , Sₖ') are maximal green sequences for Λ.

(c) G(Λ) is generated by \{Xₜ | t ∈ brick Λ\} ∪ \{gₜ | t ∈ tors Λ\} with a relation

\[ gₜ = Xₜ gₜ', \]

if there is an arrow T S \xrightarrow{T} T' in Hasse(tors Λ) and the relation

\[ g₀ = e. \]

**Proof.** (a) is an immediate consequence of [Rea16, Lem. 9-6.3]. (b) and (c) follow immediately from (a). □
4.2. The Fundamental Group. Our next goal is to show that the fundamental group of $B\mathfrak{W}(\Lambda)$ is isomorphic to the picture group of $\Lambda$. We begin by describing the CW structure of $B\mathfrak{W}(\Lambda)$ using an argument similar to [Igu14, Section 4] and [IT17, Section 4.1]. The following Lemma is equivalent to [BST18, Prop. 3.15].

Lemma 4.5. Let $W$ be a wide subcategory of rank $k \geq 2$. Let $\Sigma(W)$ be the simplicial complex with vertices the (isoclasses of) indecomposable support $\tau$-rigid objects for $W$ so that a set of vertices spans a simplex in $\Sigma(W)$ if and only if their direct sum is support $\tau$-rigid. Then $\Sigma(W)$ is homeomorphic to a $k-1$ sphere and each morphism $[U]: W \to 0$ corresponds to the $k-1$ simplex whose vertices are the first factors of $U$.

Proposition 4.6. The classifying space $B\mathfrak{W}(\Lambda)$ is a $|\Lambda|$-dimensional CW-complex having one cell $e(W)$ of dimension $k$ for every wide subcategory $W$ of rank $k$. The $k$-cell $e(W)$ is the union of the factorization cubes of all morphisms $W \to 0$.

Proof. By the previous Lemma, the forward link $Lk_+(W)$ is a $k-1$ sphere. It follows that the union of factorization cubes corresponding to morphisms $W \to 0$ is homeomorphic to a disk of dimension $k$. We denote this disk by $e(W)$.

We now need only show that $e(W)$ meets lower dimensional cells only along its boundary, but this is clear since lower dimensional cells cannot contain the vertex $W$ which is a vertex of every nonboundary simplex of the cell $e(W)$. See [Igu14, Thm. 4.4] for more details. □

We now recall that the set of rank one wide subcategories is precisely

$$\{\text{Filt} S | S \in \text{brick} \Lambda\}.$$ 

Moreover, for $S \in \text{brick} \Lambda$, there are exactly two morphisms $\text{Filt} S \to 0$, namely $[\text{br}(S)]$ and $[\text{br}(S)[1]]$. It follows that $e(\text{Filt} S)$ is a copy of $S^1$ with generator $[\text{br}(S)[\text{br}(S)[1]]^{-1}$. As the generators of $\pi_1(B\mathfrak{W}(\Lambda))$ correspond to the 1-cells of $B\mathfrak{W}(\Lambda)$, this proves the following proposition:

Proposition 4.7. The fundamental group $\pi_1(B\mathfrak{W}(\Lambda))$ is generated by

$$\{X_S := [\text{br}(S)][\text{br}(S)[1]]^{-1} | S \in \text{brick} \Lambda\}.$$ 

We now need only describe the relations, which are given by the 2 cells. Let $W = \text{Filt}(S_1 \sqcup S_2)$ be a wide subcategory of rank 2. We now observe that for $T, T' \in \text{brick} W$, we have that $[\text{br}(T)]$ and $[\text{br}(T')[1]]$ are compatible as last morphisms if and only if $T \sqcup T'[1] \in 2-\text{smc}W$. This means $e(W)$ and the polygon $P(S_1, S_2)$ have the form shown below, where $e(W)$ gives a homotopy

$$X_{S_1}X_{T_1}\cdots X_{T_k}X_{S_2} \sim X_{S_2}X_{T'_1}\cdots X_{T'_l}X_{S_1}.$$
We have thus proven our second main theorem (Theorem B in the introduction).

**Theorem 4.8.** $\pi_1(\mathcal{B}\mathfrak{W}(\Lambda)) \cong G(\Lambda)$.

### 4.3. Faithful Group Functors for Nakayama Algebras.

The remainder of this section is aimed at constructing a faithful group functor $\mathcal{B}\mathfrak{W}(\Lambda) \to G(\Lambda)$ in the case that $\Lambda$ is Nakayama. One of the key properties used to prove this functor is faithful is that the generator of $G(\Lambda)$ corresponding to each brick in $\Lambda$ is nontrivial. We prove this by defining a morphism of groups $G(\Lambda) \to B^* (\Lambda)$ which does not vanish on any of the generators. The group $B^* (\Lambda)$ is the group of units of the *brick algebra* of $\Lambda$, defined below in Definition-Theorem 4.9. This is a variation of the Hall algebra of $\Lambda$.

As not all results in this section rely on the assumption that $\Lambda$ is Nakayama, we will emphasize precisely where this assumption is used.

**Definition-Theorem 4.9.** Let $\Lambda$ be a Nakayama algebra. Then the *brick algebra* of $\Lambda$ is the free $\mathbb{Z}$-module with basis $\text{brick}\Lambda \cup \{1\}$ and with multiplication given by

\[
\begin{align*}
1 \ast 1 &= 1 \\
1 \ast S &= S \ast 1 = S \quad \text{for all } S \in \text{brick}\Lambda \\
S \ast T &= \begin{cases} 
B(S, T) \text{ if } S \sqcup T \in \text{sbrick}\Lambda, \quad B(S, T) \in \text{brick}\Lambda \text{ and } S \hookrightarrow B \twoheadrightarrow T \text{ is exact} \\
0 &\text{otherwise}
\end{cases}
\end{align*}
\]

This is an associative algebra.

**Proof.** We first observe, by Lemmas 3.8 and 3.9, that for $S = M(a_S), T = M(a_T) \in \text{brick}\Lambda$, we have $S \ast T = 0$ unless the following conditions are met:

1. $t(a_S) = s(a_T)$
2. $l(a_S) + l(a_T) \leq \min(l(s(a_S)), n)$
Moreover, if these conditions are met, then $B(S, T) = M(a_Sa_T)$ is the only brick with an exact sequence $S \to B(S, T) \to T$. We conclude that this multiplication is well-defined.

We now wish to show that this multiplication is associative. Let $S = M(a_S), T = M(a_T), R = M(a_R) \in \text{brick}\Lambda$. Assume first that $S \cdot T = 0$. Now if $T \cdot R = 0$ then we are done. Otherwise, we have

$$l(a_S) + l(a_T a_R) \geq l(a_S) + l(a_T) \geq \min(l(s(a_S)), n).$$

We conclude that $S \cdot (T \cdot R) = 0$.

Likewise, assume that $T \cdot R = 0$ and $S \cdot T \neq 0$. Then we have $t(a_Sa_T) = t(a_T) \neq s(a_R)$ or

$$l(a_Sa_T) + l(a_R) = l(a_S) + l(a_T) + l(a_R) \geq l(a_S) + \min(l(s(a_T)), n) \geq \min(l(s(a_S)), n).$$

We conclude that $(S \cdot T) \cdot R = 0$.

Finally, assume that $S \cdot T \neq 0$ and $T \cdot R \neq 0$. Thus we have $t(a_S) = s(a_T)$ and $t(a_T) = s(a_R)$. Therefore $(S \cdot T) \cdot R = 0$ if and only if $l(a_S) + l(a_T) + l(a_R) > \min(l(s(a_S)), n)$ if and only if $S \cdot (T \cdot R) = 0$. Otherwise, we must have that $(S \cdot T) \cdot R = M(a_Sa_T a_R) = S \cdot (T \cdot R)$. We conclude that $\mathcal{B}(\Lambda)$ is an associative algebra. \qed

We now prove the following Lemmas, which will be critical in what follows.

**Lemma 4.10.** Let $\Lambda$ be an arbitrary $\tau$-tilting finite algebra.

(a) [DIR+18] Thm. 4.12, Prop. 4.13 Let $N \sqcup Q[1] \in s\tau\text{-rigid}\Lambda$. Then there is a label-preserving isomorphism of lattices

$$\mathcal{F}_{N \sqcup Q[1]} : [\text{Fac}N, \perp(\tau N) \cap Q^\perp] \to \text{tors}(J(N \sqcup Q[1]))$$

given by

$$\mathcal{F}_{N \sqcup Q[1]}(T) := J(N \sqcup Q[1]) \cap T.$$

(b) Let $N \sqcup M \sqcup Q[1] \sqcup P[1] \in s\tau\text{-rigid}\Lambda$. Then we have

$$\mathcal{F}_{N \sqcup Q[1]}(\text{Fac}(N \sqcup M)) = \text{Fac}(\mathcal{E}_{N \sqcup Q[1]}(M'))$$

where $M'$ is the direct sum of the indecomposable summands of $M$ sent to modules by $\mathcal{E}_{N \sqcup Q[1]}$.

**Proof.** (b) By definition, we have

$$\mathcal{F}_{N \sqcup Q[1]}(\text{Fac}N \sqcup M) = J(N \sqcup Q[1]) \cap \text{Fac}(N \sqcup M) = N^\perp \cap (\tau N) \cap Q^\perp \cap \text{Fac}(N \sqcup M) = N^\perp \cap (\tau N) \cap Q^\perp \cap \text{Fac}M.$$

Now let $L \in \text{Fac}M$. We observe that $M \in \perp(\tau N)$, so clearly $L \in \perp(\tau N)$ as well. Likewise, $M \in Q^\perp$, so by the definition of projective, $L \in Q^\perp$ as well. Thus we need only show that $N^\perp \cap \text{Fac}M = \text{Fac}(\mathcal{E}_{N \sqcup Q[1]}(M'))$. It is clear that $\text{Fac}(\mathcal{E}_{N \sqcup Q[1]}(M')) \subset N^\perp \cap \text{Fac}M$, so assume $L \in N^\perp \cap \text{Fac}M$. It follows that the quotient map $M \to L$ contains $\text{rad}(N, M)$ in its kernel. Thus this map factors through $\mathcal{E}_{N \sqcup Q[1]}(M')$. \qed

**Lemma 4.11.** Let $\Lambda$ be a Nakayama algebra.

(a) Let $S \neq S' \in \text{brick}\Lambda$. Then $e \neq X_S \neq X_{S'} \in \text{G}(\Lambda)$.

(b) Let $T \neq T' \in \text{tors}\Lambda$. Then $g_T \neq g_{T'} \in \text{G}(\Lambda)$. 
Proof. (a) Define a map \( \phi : G(\Lambda) \to B^*(\Lambda) \) by \( \phi(X_S) = 1 + S \) and \( \phi(X_S^{-1}) = 1 - S \). We wish to show that \( \phi \) is a morphism of groups. This will imply that each \( X_S \) is nontrivial in \( G(\Lambda) \).

First, we observe that for \( S \in \text{brick}\Lambda \), we have
\[
\phi(X_S X_S^{-1}) = (1 + S) * (1 - S) = 1 + S * S = 1
\]
because \( S \) cannot have a self-extension which is a brick. Now let \( S \sqcup S' \) be a semibrick of rank 2. By Lemma 4.10 we work in the wide subcategory \( \text{Filt}(S \sqcup S') \) for the duration of this argument.

Let \( L = (S, T_1, \ldots, T_k, S') \) and \( R = (S', T'_1, \ldots, T'_{k'}, S) \) be the sides of the polygon \( \mathcal{P}(S, S') \). Since \( \Lambda \) is Nakayama, we have \( k, k' \in \{0, 1\} \); however, we work in full generality. Suppose that \( T, T' \) both lie on \( L \), with \( T \) occurring before \( T' \). It follows that \( T' \in \text{FiltFac}(T') \) and \( T \) is torsion-free with respect to \( \text{FiltFac}(T') \). Thus there is no nonzero map \( T' \to T \). In particular, for \( T \neq S \), there is no nonzero map \( T \to S \). This means \( S * S' \in B^*(\Lambda) \) contains no bricks (other than \( S \) and \( S' \)) which occur on \( L \). Likewise, \( S' * S \in B^*(\Lambda) \) contains no bricks (other than \( S \) and \( S' \)) which occur on \( R \). We conclude that
\[
(1 + S) * (1 + S') = 1 + S + S' + \sum_{j=1}^{k'} T'_j
\]
and
\[
(1 + S') * (1 + S) = 1 + S + S' + \sum_{j=1}^{k} T_k.
\]

Again let \( T, T' \) both lie on \( L \). Suppose \( T * T' \neq 0 \), that is, there is a brick \( B \) and an exact sequence \( T' \to B \to T \). Now since \( S \to T' \to B \), we know that \( B \) lies on \( L \). Moreover, since there is a nonzero map \( B \to T \), \( B \) must occur before \( T \). Likewise, since there is a nonzero map \( T' \to B \), \( B \) must occur after \( T' \). In particular, we have that \( T * T' = 0 \) unless \( T = S \) and \( T' = S' \).

The two paragraphs above imply
\[
\phi(X_S X_{T_1} \cdots X_{T_k} X_{S'}) = (1 + S) * \left( \prod_{j=1}^{k} (1 + T'_j) \right) * (1 + S')
\]
\[
= 1 + S + S' + S * S' + \sum_{j=1}^{k} T'_j
\]
\[
= 1 + S + S' + S' * S + \sum_{j=1}^{k'} T''_j
\]
\[
= (1 + S') * \left( \prod_{j=1}^{k'} (1 + T''_j) \right) * (1 + S)
\]
\[
= \phi(X_{S'} X_{T'_1} \cdots X_{T'_{k'}} X_{S})
\]
We conclude that \( \phi \) is a morphism of groups and hence \( e \neq X_S \neq X_{S'} \in G(\Lambda) \) for all \( S, S' \in \text{brick}\Lambda \).

(b) Let \( T, T' \in \text{tors}\Lambda \) and let \( (S_1, \ldots, S_k), (S'_1, \ldots, S'_{k'}) \) label maximal length paths \( T \to T \land T', T' \to T \land T' \) in \( \text{tors}(\Lambda) \). Assume that \( X_{S_1} \cdots X_{S_k} = X_{S'_1} \cdots X_{S'_{k'}} \in G(\Lambda) \) (and hence \( g_T = g_{T'} \)). If \( \max(k, l) > 0 \), let \( S \in \{S_1, \ldots, S_k, S'_1, \ldots, S'_{l}\} \) by of minimal length. By applying the morphism \( \phi \), we then have
\[
(1 + S_1) * \cdots * (1 + S_k) = (1 + S'_1) * \cdots * (1 + S'_{k'})
\]
As \( R * R' \) is either 0 or has length \( l(R) + l(R') \) for all \( R, R' \in \text{brick}\Lambda \), this equality implies that \( S \) must occur in both \( (S_1, \ldots, S_k) \) and \( (S'_1, \ldots, S'_{k'}) \). Thus \( T \land T' \subseteq \text{FiltFac}(\text{brick}(T \land T') \sqcup S) \subset T, T' \), a contradiction. We conclude that \( \max(k, l) = 0 \), and hence \( T = T \land T' = T' \).  \( \square \)
We are now ready to define the faithful group functor.

**Theorem 4.12.** Let $\Lambda$ be any $\tau$-tilting finite algebra which satisfies the conclusion of Lemma 4.11 and let $W, W' \in \text{wide } \Lambda$. For $[M \sqcup P[1]] \in \text{Hom}(W, W')$, let $(S_1, \ldots, S_k)$ label a sequence $\text{Fac} M \to 0$ in $\text{tors} W$. Then there is a faithful functor $F : \mathcal{W}(\Lambda) \to G(\Lambda)$ given by

$$F[M \sqcup P[1]] := X_{S_1} \cdots X_{S_k},$$

where $G(\Lambda)$ is considered as a groupoid with one object.

**Proof.** By Proposition 4.4 and Lemma 4.10, $F$ is well defined and by Lemma 4.11(b), $F$ is faithful, so we need only show that $F$ preserves the composition law. Let

$$W \xrightarrow{[M \sqcup P[1]]} W' \xrightarrow{[M' \sqcup P'[1]]} W''$$

be a sequence of composable morphisms. Let $(S_1, \ldots, S_k)$ label a path $\text{Fac} M \to 0$ in $\text{tors} W$, let $(T_1, \ldots, T_l)$ label a path $\text{Fac} M' \to 0$ in $\text{tors} W'$. It then follows from Lemma 4.10 that $(T_1, \ldots, T_l)$ labels a path

$$\text{Fac}(M \sqcup (E_{M \sqcup P[1]}^{-1}(M))) \to \text{Fac} M$$

in $\text{Hasse}(\text{tors} \Lambda)$. Thus the composition $(T_1, \ldots, T_l, S_1, \ldots, S_k)$ labels a path

$$\text{Fac}(M \sqcup (E_{M \sqcup P[1]}^{-1}(M'))) \to 0$$

in $\text{Hasse}(\text{tors} W)$ We conclude that $F([M' \sqcup P'[1]) \circ [M \sqcup P[1]]) = F[M' \sqcup P'[1]]F[M \sqcup P[1]].$ \hfill$\square$

In particular, Theorem 3.14 and Theorem 4.12 imply the following result (Theorem C in the introduction).

**Corollary 4.13.** For a Nakayama algebra $\Lambda$, the classifying space of the category $\mathcal{W}(\Lambda)$ is a locally $\text{CAT}(0)$ cube complex and thus is a $K(\pi, 1)$ for the group $G(\Lambda)$. 
5. Examples of Theorem 3.14

Example 5.1. Let $A_3$ be given straight orientation. We then see that $KA_3 \cong \Lambda(3, 3, 2, 1)$. The corresponding maximal arc patterns and 2-simple minded collections are then as follows.

| Maximal Arc Pattern | 2-smc | Maximal Arc Pattern | 2-smc |
|---------------------|-------|---------------------|-------|
| ![Arc Pattern 1](image1.png) | $1 \sqcup 2 \sqcup 3$ | ![Arc Pattern 1](image2.png) | $1^1 \sqcup 2^1 \sqcup 3^1$ |
| ![Arc Pattern 2](image3.png) | $1 \sqcup 2 \sqcup 3$ | ![Arc Pattern 2](image4.png) | $1^1 \sqcup 2^1 \sqcup 3^1$ |
| ![Arc Pattern 3](image5.png) | $1 \sqcup 2 \sqcup 3$ | ![Arc Pattern 3](image6.png) | $1^1 \sqcup 2^1 \sqcup 3^1$ |
| ![Arc Pattern 4](image7.png) | $1 \sqcup 2 \sqcup 3$ | ![Arc Pattern 4](image8.png) | $1^1 \sqcup 2^1 \sqcup 3^1$ |
| ![Arc Pattern 5](image9.png) | $1 \sqcup 2 \sqcup 3$ | ![Arc Pattern 5](image10.png) | $1^1 \sqcup 2^1 \sqcup 3^1$ |
| ![Arc Pattern 6](image11.png) | $1 \sqcup 2 \sqcup 3$ | ![Arc Pattern 6](image12.png) | $1^1 \sqcup 2^1 \sqcup 3^1$ |
Example 5.2. Consider the cyclic cluster-tilted algebra of type $D_4$. That is, the algebra $K\Delta_3/\text{rad}^2 K\Delta_3 \cong \Lambda(3, 2, 2, 2)$. The corresponding maximal arc patterns and 2-simple minded collections are given up to cyclic permutation as follows.

| Maximal Arc Pattern | 2-smc | #Perms | Maximal Arc Pattern | 2-smc | #Perms |
|---------------------|-------|--------|---------------------|-------|--------|
| ![Maximal Arc Pattern](image1.png) | $1 \sqcup 2 \sqcup 3 \sqcup 4$ | 1 | ![Maximal Arc Pattern](image2.png) | $1[1] \sqcup 2[1] \sqcup 3[1] \sqcup 4[1]$ | 1 |
| ![Maximal Arc Pattern](image3.png) | $1 \sqcup 2 \sqcup 3 \sqcup 3[1]$ | 4 | ![Maximal Arc Pattern](image4.png) | $4 \sqcup 1[1] \sqcup 2[1] \sqcup 3[1] \sqcup 2$ | 4 |
| ![Maximal Arc Pattern](image5.png) | $1 \sqcup 2 \sqcup 3 \sqcup 4[1]$ | 4 | ![Maximal Arc Pattern](image6.png) | $3 \sqcup 1[1] \sqcup 2[1] \sqcup 3[1]\sqcup 4[1]$ | 4 |
| ![Maximal Arc Pattern](image7.png) | $1 \sqcup 3 \sqcup 3 \sqcup 3[1]$ | 4 | ![Maximal Arc Pattern](image8.png) | $2 \sqcup 1[1] \sqcup 3[1] \sqcup 3[1]\sqcup 2$ | 4 |
| ![Maximal Arc Pattern](image9.png) | $1 \sqcup 3 \sqcup 3[1] \sqcup 4[1]$ | 4 | ![Maximal Arc Pattern](image10.png) | $2 \sqcup 1[1] \sqcup 3[1] \sqcup 3[1]\sqcup 4$ | 4 |
| ![Maximal Arc Pattern](image11.png) | $1 \sqcup 1 \sqcup 3[1] \sqcup 4[1]$ | 4 | ![Maximal Arc Pattern](image12.png) | $3 \sqcup 4 \sqcup 1[1] \sqcup 3[1]$ | 4 |
| ![Maximal Arc Pattern](image13.png) | $1 \sqcup 3 \sqcup 3 \sqcup 4[1]$ | 2 | ![Maximal Arc Pattern](image14.png) | $2 \sqcup 4 \sqcup 1[1] \sqcup 3[1]$ | 2 |
| ![Maximal Arc Pattern](image15.png) | $1 \sqcup 3 \sqcup 3 \sqcup 4[1]$ | 4 | | | |
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REFERENCES

[Ada16] Takahide Adachi. The classification of τ-tilting modules over Nakayama algebras. J. Algebra, 452:227–262, 2016.

[AIR14] Takahide Adachi, Osamu Iyama, and Idun Reiten. τ-tilting theory. Compos. Math., 150(3):415–452, 2014.

[Asa17] Sota Asai. Semibricks. arXiv, 1610.05860, 2017.

[ASS06] Ibrahim Assem, Daniel Simson, and Andrzej Skowroński. Elements of the Representation Theory of Associative Algebras. Cambridge University Press, Cambridge, 2006.

[BCZ17] Emily Barnard, Andrew T. Carroll, and Shijie Zhu. Minimal inclusions of torsion classes. arXiv, 1710.08837, 2017.

[BM18a] Aslak Bakke Buan and Robert J. Marsh. A category of wide subcategories. arXiv, 1802.03812, 2018.

[BM18b] Aslak Bakke Buan and Robert J. Marsh. τ-exceptional sequences. arXiv, 1802.01169, 2018.

[BST18] Thomas Brüstle, David Smith, and Hipolito Treffinger. Wall and chamber structure for finite-dimensional algebras. arXiv, 1805.01880, 2018.

[BY15] Thomas Brüstle and Dong Yang. Ordered exchange graphs. arXiv, 1302.6045, 2015.

[CGM+17] Man Wai Cheung, Mark Gross, Greg Muller, Gregg Musiker, Dylan Rupel, Salvatore Stella, and Harold Williams. The greedy basis equals the theta basis: A rank two haiku. J. Comb. Theory, Series A, 145:150–171, 2017.

[DIJ17] Laurent Demonet, Osamu Iyama, and Gustavus Jasso. τ-tilting finite algebras, bricks, and g-vectors. Int. Math. Res. Notices, rnx135, 2017.

[DIR+18] Laurent Demonet, Osamu Iyama, Nathan Reading, Idun Reiten, and Hugh Thomas. Lattice theory of torsion classes. arXiv, 1711.01785, 2018.

[Gro87] Mikhail Gromov. Hyperbolic groups. In S.M. Gersten, editor, Essays in Group Theory, Math. Sci. Res. Inst. Publ., volume 8, pages 75–263. Springer, New York, 1987.

[Igu14] Kiyoshi Igusa. The category of noncrossing partitions. arXiv, 1411.0196, 2014.

[IOTW16] Kiyoshi Igusa, Kent Orr, Gordana Todorov, and Jerzy Weyman. Picture groups of finite type and cohomology in type An. arXiv, 1609.02636, 2016.

[IIRT15] Osamu Iyama, Idun Reiten, Hugh Thomas, and Gordana Todorov. Lattice structure of torsion classes for path algebras. B. Lond. Math. Soc., 47(4):639–650, 2015.

[IT05] Kiyoshi Igusa and Gordana Todorov. On the finitistic dimension conjecture for Artin algebras. In Fields Inst. Comm., volume 45, pages 201–204. Amer. Math. Soc., Providence RI, 2005.

[IT17] Kiyoshi Igusa and Gordana Todorov. Signed exceptional sequences and the cluster morphism category. arXiv, 1706.02041, 2017.

[Jas15] Gustavo Jasso. Reduction of τ-tilting modules and torsion pairs. Int. Math. Res. Notices, 2015(16):7190–7237, 2015.

[KY14] Steffen Koenig and Dong Yang. Subtitled objects, simple-minded collections, t-structures and co-t-structures for finite-dimensional algebras. Documenta Math., 19:403–438, 2014.

[MM18] Dag Oskar Madsen and René Marczinik. On bounds of homological dimensions in Nakayama algebras. P. Am. Math. Soc. Series B, 5:40–49, 2018.

[Pla18] Pierre-Guy Plamondon. τ-tilting finite gentle algebras are representation finite. arXiv, 1809.06313, 2018.

[Rea16] Nathan Reading. Lattice theory of the poset of regions. In George Grätzer and Friedrich Wehrung, editors, Lattice Theory: Special Topics and Applications, volume 2, chapter 9. Birkhäuser, 2016.

[Rea18] Nathan Reading. A combinatorial approach to scattering diagrams. arXiv, 1806.05094, 2018.

[Sen18] Emre Sen. The ϕ dim of cyclic Nakayama algebras. arXiv, 1806.01449, 2018.