Gradient-type estimates for the dynamic $\phi^4_2$-model

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Abstract.

We prove gradient bounds for the Markov semigroup of the dynamic $\phi^4_2$-model on a torus of fixed size $L > 0$. For sufficiently large mass $m > 0$ these estimates imply exponential contraction of the Markov semigroup. Our method is based on pathwise estimates of the linearized equation. To compensate the lack of exponential integrability of the stochastic drivers we use a stopping time argument and the strong Markov property in the spirit of Cass–Litterer–Lyons [7].

Following the classical approach of Bakry-Émery, as a corollary we prove a Poincaré/spectral gap inequality for the $\phi^4_2$-measure of sufficiently large mass $m > 0$ with almost optimal carré du champ.

Keywords: Gradient estimates, stopping time argument, spectral gap inequality, singular SPDEs.

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1 Introduction

We consider the dynamic $\phi^4_2$-model on the torus $\mathbb{T}^2 = \mathbb{R}^2/L\mathbb{Z}^2$ of fixed size $L > 0$ given by

\[
\begin{cases}
\epsilon_t - \Delta u - m u = -u^3 + 3\omega u + \sqrt{2} \xi \\
u|_{t=0} = f,
\end{cases}
\]

(1.1)

where $m > 0$ is a positive mass, $\xi$ denotes space-time white noise and $f$ is a suitable initial condition. The infinite counter term $+3\omega u$ on the r.h.s. of (1.1) is reminiscent of renormalization (see Section 2 below) since the SPDE is singular due to the roughness of $\xi$.

This model serves as a toy example in the stochastic quantization of Euclidean quantum field theories. It describes the natural reversible dynamics of the $\phi^4_2$-measure formally given by

\[
v(du) = \frac{1}{Z} \exp\left\{ -\int_{\mathbb{T}^2} dx \left( \frac{\epsilon}{2} |\nabla u(x)|^2 + \frac{1}{4} |u(x)|^4 - \frac{3}{2} |\omega u(x)|^2 \right) \right\} du.
\]

(1.2)
The construction of \((1.2)\) was one of the first achievements in quantum field theory and goes back to Nelson \cite{28}. Alternatively, Parisi and Wu in \cite{31} proposed the use of \((1.1)\) in order to construct and sample via MCMC methods the measure \((1.2)\). A first attempt to implement this approach was made by Da Prato and Debussche in \cite{9}. Later, along with the development of regularity structures \cite{16} and paracontrolled calculus \cite{15}, \((1.1)\) was studied extensively by many authors, see for example \cite{27, 34, 33, 35, 17, 26, 13, 14}. These results justified rigorously the connection of the singular dynamics and the measure in the sense of Parisi and Wu.

In the current work we study the regularization properties of the Markov semigroup \(\{P_t\}_{t \geq 0}\) associated to \((1.1)\) (see \((7.1)\) below for the definition) through gradient-type estimates. Gradient-type estimates of Markov semigroups are important in the study of functional inequalities, e.g. spectral gap (or infinite dimensional Poincaré) and log-Sobolev inequalities, and transportation inequalities (see for example \cite{22, 3, 4, 8}). These estimates usually require some convexity assumption, see for example \cite{8} Property (H.C.K.), p. 232 and p. 235. In the case of \((1.1)\) convexity is destroyed by the presence of the infinite counter term \(-\frac{3}{2}\omega\mu\) and at first glance it is unclear whether any type of such estimates can be derived. The argument we present here allows us to prove the following gradient estimate for the semigroup \(\{P_t\}_{t \geq 0}\).

**Theorem 1.1.** Let \(\{P_t\}_{t \geq 0}\) be the Markov semigroup associated to \((1.1)\) and \(\kappa \in (0,1)\). For every \(q > 1\) and \(\epsilon < 1 - \kappa\) there exists \(m_\epsilon \equiv m_\epsilon(\epsilon, q, L) > 0\) such that

\[
\left\|DP_t F(f)\right\|_{L^2} \leq C(t \wedge 1)^{-\frac{\kappa - \epsilon}{2}} e^{-(m-m_\epsilon)t} \left(P_t \left\|DF\right\|_{H^{\kappa}_x}^q \right)^{\frac{1}{q}},
\]

for every cylindrical functional \(F\), \(t > 0\), \(f \in C^\infty\) and an implicit constant \(C \equiv C(\epsilon, \kappa, q, L) < \infty\) which is uniform in \(f\) and \(m\). In the case \(\kappa = 0\) the estimate holds for \(\epsilon = 0\) and a universal constant \(C\) which is independent of \(L\).

Replacing \(L^q\)-norm on the r.h.s. by an \(L^1\)-norm yields the strong gradient estimate \cite[Theorem 3.2.4]{4}. The main difference is that the strong gradient estimate implies the log-Sobolev inequality, while \((1.3)\) the (weaker) spectral gap inequality (see for example \cite{8} Section 1 and \cite{4} Sections 4 and 5). Note that in contrast to the classical literature here we insist on a gradient estimate where the r.h.s. depends on the \(H^{\kappa}_x\)-norm, allowing for \(\kappa\) arbitrarily close to 1. This is almost in line with the behaviour of the Gaussian free field in dimension 2 where the *carré du champ* is given by the \(H^{\kappa}_x\)-inner product or, equivalently, its *Cameron-Martin space* is given by \(H^1_x\). As an immediate consequence \((1.3)\) implies exponential contraction for \(m > m_\epsilon\) in the following sense,

\[
\sup_{\|h\|_{L^2} \leq 1} \sup_{\|f\|_{H^{\kappa}_x} \leq 1} |P_t F(f+h) - P_t F(f)| \leq C(t \wedge 1)^{-\frac{\kappa - \epsilon}{2}} e^{-(m-m_\epsilon)t},
\]

where the second supremum is taken over all cylindrical functionals \(F\).

In recent years gradient-type estimates of the form \((1.3)\) have seen a rise in popularity. Starting with the work of Bakry–Émery \cite{3} it has become a vast research topic to relate these estimates to lower bounds of the Ricci curvature of the associated manifold. Since the interpretation of the heat flow on a manifold as a formal gradient flow with respect to the entropy on the Wasserstein space \cite{29}, the notion of displacement convexity of the entropy is also closely related to lower bounds of the Ricci curvature \cite{30}. This relationship can be associated to exponential contraction of the heat flow with respect to the Wasserstein metric which in our case corresponds to \((1.4)\). Indeed, in \cite{37} it has been shown that in the finite-dimensional case all these notions are equivalent. In the infinite-dimensional setting we, for example, refer to \cite{11}.

In order to prove \((1.3)\) we study the linearized equation

\[
\begin{align*}
(\partial_t - \Delta + m)J^l_0 h &= -3(u^2 - \infty)J^l_0 h \quad \text{on } \mathbb{R}_{>0} \times \mathbb{T}^2, \\
J^l_0 h|_{t=0} &= h,
\end{align*}
\]
for suitable initial condition $h$. In the absence of the counter term one easily obtains a contraction estimate for any $m > 0$ of the form

$$
||\mathbb{F}^t_{0,t}h||^2_{L^2_x} \leq e^{-2mt}||h||^2_{L^2_x},
$$

(1.6)

which in turn implies the strong gradient estimate, see for example [21] Lemma 2.1 where the same dynamics are considered in the 1-dimensional setting on the whole space $[1]$. To deal with the counter term we appeal to the Da Prato–Debussche decomposition (see Section 2 below), understanding $u^2 = \infty$ as

$$
u^2 = -\infty = \nu^2 + 2\nu \mathbb{1} + \mathbb{V} + c_{1,\infty},
$$

(1.7)

where $\mathbb{1}$ is the solution to the stochastic heat equation (2.1) with zero initial data, $\mathbb{V}$ its second Wick power defined in (2.1) and $c_{1,\infty}$ the constant defined in (2.5)\footnote{Using a post-processing of (1.6) as in Proposition 3.6 below one can upgrade the $L^2$-estimate to an $H^s$-estimate for $s \in [0,1]$ in the case of the torus.}. The idea is to treat the lower order terms in (1.7), namely $2\nu \mathbb{1} + \mathbb{V} + c_{1,\infty}$, as drift terms and absorb them to the mass $m$. Due to the lack of the required exponential integrability, in order to obtain a meaningful gradient estimate we restart the noise every time the Wick powers exceed a certain barrier using a stopping time argument in the spirit of Cass–Litterer–Lions [7] for rough differential equations (see Section 3.2 below). This argument allows us to bypass the problem of exponential integrability of the Wick powers. Instead, we need to study the exponential integrability of the counting process $N(t)$ of the number of restarts to reach time $t$ which due to the strong Markov property has exponential tails (see Proposition 3.3 below). A crucial ingredient to our approach is the “coming down from infinity”), which ensures that the estimates on

$$
\mathbf{E}[||\mathbb{F}^t_{0,t}||^p_{L^2_x \rightarrow L^p_x}] \leq Ce^{-(m-m_\star)t},
$$

for some $m_\star > 0$ and $C < \infty$ uniformly in $f$, see Proposition 3.4. Using a simple post-processing we can upgrade the above estimate to

$$
\mathbf{E}[||\mathbb{F}^t_{0,t}||^p_{L^2_x \rightarrow H^s_x}] \leq C(t \wedge 1)^{-\frac{s}{2}} e^{-(m-m_\star)t},
$$

(1.8)

see Proposition 3.6.

As we already mentioned earlier, the motivation to study gradient-type estimates for Markov semigroups comes from applications on functional inequalities. As a consequence of (1.3) we derive a spectral gap inequality for the Markov semigroup $\{P_t\}_{t \geq 0}$ based on the celebrated method of Bakry–Émery. Due to the presence of the $H^s_x$-norm for $\kappa$ arbitrarily close to 1 the carré du champ is almost optimal when compared to the small scale behaviour of the Gaussian free field in 2-dimensions on a torus of fixed size $L > 0$ (which plays the role of an infra-red cutoff).

**Theorem 1.2.** Under the assumptions of Theorem 1.1 the following spectral gap inequality holds

$$
P_t F^2(f) - (P_t F(f))^2 \leq C \int_0^t (s \wedge 1)^{-\kappa - \kappa} e^{-2(m-m_\star)s} ds P_t ||FG||^2_{H^{-\kappa}_x} \quad \nu\text{-a.s. in } f,
$$

1The constant $c_{1,\infty}$ appears due to the fact that we insist on using Wick powers of $\mathbb{1}$ which at time $t = 0$ vanish. This is just a technical convenience but not necessary in our approach.
for every cylindrical functional $F$, $t > 0$ and implicit constant $C \equiv C(\varepsilon, \kappa, L) < \infty$ which is uniform in $f$ and $m$. In the case $\kappa = 0$ the estimate holds for $\varepsilon = 0$ and a universal constant $C$ which is independent of $L$.

Let us mention that a spectral gap-type inequality for the Markov semigroup generated by $\mathcal{L}$ has already been obtained in [35] in the total variational norm in $C^{\omega}$ based on a combination of the strong Feller property, a support theorem and the “coming down from infinity” property. The same holds in dimension 3 based on the results from [18, 19, 26]. Although the total variational norm is stronger than any Wasserstein metric, the results in [35] do not provide an estimate w.r.t. the $L^2$-derivative.

Using the ergodicity of $P_t$, see for example [35, Corollary 6.6], as a corollary we prove a spectral gap inequality for the $\phi^4_3$-measure for large masses $m > m_*$.

**Corollary 1.3.** Under the statement of Theorem 1.2 and the additional assumption $m > m_*$, the $\phi^4_3$-measure satisfies the spectral gap inequality

$$E_\nu F^2 - (E_\nu F)^2 \leq C \frac{1}{(m-m_*)^{\frac{2}{3}} \wedge (m-m_*)} E_\nu \|DF\|^2_{H^{x,\kappa}}.$$  \hfill (1.9)

for every cylindrical functional $F$, where for $\kappa = 0$ the estimate holds for $\varepsilon = 0$.

**Remark 1.4.** We emphasize that in order to obtain (1.9) we need to choose $m$ large enough and, in particular, $m > m_*$, to ensure that the spectral gap constant does not blow-up in the limit $t \to \infty$. This is a technical restriction of the method presented here and it is rather unnatural in the case of the torus. On the other hand, such a condition would be natural in the whole plane regime, provided that the dependence of the implicit constant $C$ and the mass $m_*$ on $L$ can be eliminated. As we already stated in Theorem 1.2, $C$ does not depend on $L$ for $\kappa = 0$ and it would be interesting to investigate whether the dependence of $m_*$ on $L$ can be eliminated as well to allow for a large scale analysis. At first sight this seems possible using suitable weighted norms (in the spirit of [27, 13]), but it is rather unclear whether one can derive meaningful estimates in this direction.

Spectral gap inequalities are a convenient tool which quantifies ergodicity. When it comes to applications beyond the study of long time behavior, they have been used in the context of stochastic homogenization [12] to obtain stochastic estimates on the corrector. In a similar spirit, spectral gap inequalities can be used as a tool in deriving stochastic estimates in the context of singular SPDEs [23] (see also [20] Section 5) for a simpler example).

While completing this work, a relevant work [5] appeared, which derives log-Sobolev inequalities for the $\phi^4$-measure in dimensions 2 and 3 with carré du champs given by the $L^2$-norm. More precisely, the authors study approximations of the measure with ultraviolet and infra-red cutoffs and derive lower and upper bounds on the log-Sobolev constant independent of the cutoffs. Their approach is based on the machinery developed in [6] in combination with correlation inequalities. Although these results are optimal in the large scale regime and they imply the spectral gap inequality, the techniques presented here are more appropriate in the small scale regime.

### 1.1 Notation

For $\beta \in \mathbb{R}$ we set $C^{\beta} := B^{\beta}_{\infty,\infty}(\mathbb{T}^2)$ and the corresponding norm is denoted by $\|\cdot\|_\beta$. The space of arbitrarily smooth functions is accordingly denoted by $C^\infty$. We set $L^p_g := L^p(\mathbb{T}^2)$ and $\|\cdot\|_{L^p_g}$ for the corresponding norm. Similarly, we use the same notation for $L^2_{1,\alpha} := L^2(\mathbb{R}_+ \times \mathbb{T}^2)$ and $H^\alpha := H^\alpha(\mathbb{T}^2)$. Note that we have $L^p_g = B^{0,p}_{\infty,\infty}(\mathbb{T}^2)$ and $H^\alpha = B^{\alpha}_{2,2}(\mathbb{T}^2)$. The space $FC^\infty_h$ denotes all cylindrical functions, i.e. for a distribution $u$ we have $F \in FC^\infty_h$ if there exists $n \in \mathbb{N}$, $F \in C^\infty_h(\mathbb{R}^n)$ and $h_i \in C^\infty(\mathbb{T}^2)$ for $i = 1, \ldots, n$ such that $F(u) = F(u(h_1), \ldots, u(h_n))$ where we write $u(h) := \int_{\mathbb{T}^2} u(x)h(x)dx$ for the natural pairing. Moreover, $a \wedge b := \min\{a, b\}$. 


1.2 Outline

In Section 2 we recall the Da Prato–Debussche ansatz for (1.1) including the construction and regularity of the Wick powers. In Section 3 we outline the ideas needed in order to prove our main theorem. This includes the $L^2$-energy estimate of the solution to the linearized equation (1.5), the stopping time argument that we employ in order to bypass the problem of exponential integrability of the Wick powers and finally the upgrade to an $H^κ$-estimate. In Section 4 we prove the spectral gap inequalities Theorem 1.2 and Corollary 1.3. In Section 5 we include the intermediate proofs of the $L^2$- and $H^κ$-estimate. Finally, in the appendix we gather some auxiliary results which are partially known in the literature.

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2 General framework

We denote by $0_{0,t}$ the solution to the stochastic heat equation
\[
\begin{aligned}
(\partial_t - \Delta + m) I &= \sqrt{2} \xi \quad \text{on } \mathbb{R}_{>0} \times T^2 \\
I|_{t=0} &= 0,
\end{aligned}
\] (2.1)
which is explicitly given by
\[
0_{0,t}(\varphi) = \sqrt{2} \xi (\mathbb{1}_{0,t}) H_{t-} \ast \varphi
\]
for all sufficiently nice test functions $\varphi : T^2 \to \mathbb{R}$ where $(t,x) \mapsto H_t(x)$ denotes the heat kernel associated with the operator $(\partial_t - \Delta + m)$. We also denote by $\mathcal{W}_{0,t}$ and $\mathcal{W}_{0,t}$ its second and third Wick powers defined as the limits
\[
\mathcal{W}_{0,t} := \lim_{\delta \searrow 0} \left( 1_{0,t} \right)^2 - c_{0,t}^{(\delta)}, \quad \mathcal{W}_{0,t} := \lim_{\delta \searrow 0} \left( 1_{0,t} \right)^3 - 3c_{0,t}^{(\delta)} 1_{0,t},
\] (2.2)

where $c_{0,t}^{(\delta)} = \mathbb{E} \left( 1_{0,t}^{(\delta)} (0) \right)^2$, $\delta$ denotes some space mollification and the convergence takes place in $C^{-\alpha}$ for every $\alpha > 0$. For simplicity, we write $\mathcal{W}_{0,t}$, $k = 1, 2, 3$, to denote the collection of $\mathcal{W}_{0,t}$, $\mathcal{W}_{0,t}$, $\mathcal{W}_{0,t}$. We are only interested in the analytical properties of the Wick powers $\mathcal{W}_{0,t}$, $k = 1, 2, 3$, given by the next proposition.

Proposition 2.1. Let $T > 0$. For any $k = 1, 2, 3$, $\alpha > 0$ and $p < \infty$ we have
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left\| \mathcal{W}_{0,t} \right\|^{-\alpha}_p \right]^\frac{1}{\alpha} \leq C
\] (2.3)

where the constant $C \equiv C(L,T,\alpha,p)$ does not depend on $m$, vanishes for $T \searrow 0$ and grows at most polynomially in $T$.

We postpone the proof of this proposition in the appendix, Section C, where we present an alternative argument in the spirit of [20 Section 5] and [23] using the fact that the white noise $\xi$ satisfies a spectral gap inequality. Note that we stress the independence of the constant $C$ on $m$, which allows us to ensure that $m_*$ in Theorem 1.1 is independent of $m$ (in particular, $\theta$ in Proposition 3.3 can be chosen independently of $m$).
We interpret the solution $u$ of (1.1) using the Da Prato–Debussche decomposition [9], namely, we define $u_{0,t} := 1_{0,t} + v_{0,t}$, where
\[
\begin{aligned}
& \left\{ \left[ \partial_t - \Delta + m \right] v_{0,t} = -v_{0,t}^3 - 3v_{0,t}^2 1_{0,t} - 3v_{0,t} \nabla v_{0,t} - \nabla v_{0,t} + 3c_{t,\omega} \left( 1_{0,t} + v_{0,t} \right) \right\}, \\
& v_{0,t} = f,
\end{aligned}
\]
where $f \in C^{-\alpha_0}$ for $\alpha_0 > 0$ sufficiently small. Let us remark on the constant $c_{t,\omega}$ which appears on the r.h.s. of (2.4). This is due to the fact that we renormalize the Wick powers via time dependent constants in order for them to vanish at time $t = 0$, although renormalization on the level of the dynamics is done via a time-independent constant $c^{(\delta)}_{0,\omega}$ to ensure that the resulting Markov processes is homogeneous in time. In the limit $\delta \downarrow 0$ the difference between the two constants leads to
\[
c_{t,\omega} := 2 \int_t^{\infty} H_2(s) \, ds \lesssim t^{-\frac{\beta}{2}}
\]
for every $\beta > 0$. A crucial ingredient that we use in the sequel is the “coming down from infinity” for the solution $v_{0,t}$ to (2.4) which we include in the appendix, Section D. We refer the reader to [9, 27, 35] for details on the global well-posedness of (2.4).

For $t \geq s$ we also consider the restarted processes $\nabla v_{s,t}$, $k = 1, 2, 3$ which are defined via the solution to
\[
\begin{aligned}
& \left\{ \left[ \partial_t - \Delta + m \right] 1_{s,t} = \sqrt{\delta} \xi \right\} \text{ on } \mathbb{R}_{>0} \times \mathbb{T}^2, \\
& 1_{s,t}|_{t=s} = 0,
\end{aligned}
\]
and respectively via (2.2) with $1_{(s)}^{(\delta)}$ replaced by $1_{s,t}^{(\delta)}$. Note that $\{ \nabla v_{s,t} \}_{t \geq s}$ and $\{ \nabla v_{s,s} \}_{t \geq s}$ are equal in law and $\{ \nabla v_{s,s} \}_{t \geq s}$ is independent of $\{ \nabla v_{0,t} \}_{t \in [0,s]}$.

Similarly, we consider $v_{s,t}$ which is defined as the solution to
\[
\begin{aligned}
& \left\{ \left[ \partial_t - \Delta + m \right] v_{s,t} = -v_{s,t}^3 - 3v_{s,t}^2 1_{s,t} - 3v_{s,t} \nabla v_{s,t} - \nabla v_{s,t} + 3c_{t,\omega} \left( 1_{s,t} + v_{s,t} \right) \right\}, \\
& v_{s,t}|_{t=s} = u_s.
\end{aligned}
\]
Note that all pathwise and stochastic estimates for $\nabla v_{s,t}$, $k = 1, 2, 3$, and $v_{0,t}$ extend to $\nabla v_{s,t}$, $k = 1, 2, 3$, and $v_{s,t}$. Especially, due to the “coming down from infinity” property pathwise estimates on $v_{s,t}$ do not depend on $u_s$.

### 3 Strategy of the proof

In this section we want to give an outline of the proof of Theorem 1.1. By [35] Theorem 4.2] for $f \in C^{-\alpha_0}$ we know that $\left\{ u^t_{0,t} \right\}_{t \geq 0}$ is a Markov process with $u^t_{0,t}|_{t=0} = f$. In particular, for $F \in \mathcal{FC}_b$ the operator
\[
P_t F(f) := E \left[ F(u^t_{0,t}) \right]
\]
yields a one-parameter semigroup. We denote by $D$ the $L^2$–derivative, i.e. we have
\[
D F(u) = \sum_{i=1}^n \partial_i F(u(h_1), \ldots, u(h_n)) h_i.
\]
The implicit function theorem implies that the map 
\[ f \mapsto v^f \]
is differentiable and for any \( h \in C^\infty \) it holds that \( J_{0,t}^f h := v_{0,t}(f).h \) is a (mild) solution of the equation
\[
\left\{ (\partial_t - \Delta + m)J_{0,t}^f h = -3 \left( \left( v_{0,t}^f \right)^2 + 2v_{0,t}^f \mathbf{1}_{0,t} + \nabla v_{0,t} \right) J_{0,t}^f h + 3c_t \omega J_{0,t}^f h \right\}
\]
\[ J_{0,0}^f h = h. \]

For a proof we refer to Section \( \text{[F]} \).

By definition, we have \( u_{0,t}^f = \mathbf{1}_{0,t} + v_{0,t}^f \) and since \( \mathbf{1}_{0,t} \) does not depend on the initial condition we can conclude that also \( f \mapsto u^f \) is differentiable, i.e. there exists \( u^f(f) = v^f(f) : X \to Y \) (cf. Section \( \text{[F]} \) for the definition of the function spaces \( X \) and \( Y \)) such that
\[ u^{f+h} - u^f - u^f(f).h = (f^{f+h} - f^f - v^f(f).h) = o(\|h\|_{C^\infty}). \]

Thus we can compute for any \( t \geq 0 \) and \( F \in FC^\infty_0 \) using a simple Taylor expansion
\[
P_t F(f + h) - P_t F(f) = E \left[ F(u_{0,t}^{f+h}) - F(u_{0,t}^f) \right]
= E \left[ \sum_{i=1}^{n} \partial_i F(u_{0,t}^f(h_1), \ldots, u_{0,t}^f(h_n)) (u_{0,t}^{f+h}(h_i) - u_{0,t}^f(h_i)) \right] + o(\|h\|_{C^\infty})
= E \left[ \sum_{i=1}^{n} \partial_i F(u_{0,t}^f(h_1), \ldots, u_{0,t}^f(h_n)) (v_{0,t}^f, f(h_i), h_i)_{L^2} \right] + o(\|h\|_{C^\infty}).
\]

This shows that \( f \mapsto P_t F(f) \) is differentiable and we have
\[
(P_t F)'(f).h = E \left[ \left( DF(u_{0,t}^f), v_{0,t}^f(h) \right)_{L^2} \right] = E \left[ \left( DF(u_{0,t}^f), J_{0,t}^f h \right)_{L^2} \right].
\]

Moreover, by Proposition \( \text{[3.6]} \) we see that \( (P_t F)'(f) : L^2_2 \to \mathbb{R} \) is a bounded linear functional\( ^3 \)

and thus there exists \( DP_t F(f) \in L^2_2 \) such that
\[ (P_t F)'(f).h = \int_{\mathbb{R}^2} DR F(f)(x) h(x) dx \]

and in particular
\[ \|DP_t F(f)\|_{L^2_2} = \sup_{\|h\|_{L^2} \leq 1} \left| (P_t F)'(f).h \right|. \]

By \( \text{[3.2]}, \text{[3.3]} \) and the Hölder’s inequality in probability for any \( \kappa \geq 0 \)
\[ \|DP_t F(f)\|_{L^2_2} = \sup_{h \in C^\kappa, \|h\|_{L^2_2} \leq 1} \left| (P_t F)'(f).h \right|
\leq E \left[ \left\| DF(u_{0,t}^f) \right\|_{H^\kappa_2}^{q} \right]^{\frac{1}{q}} \left( \sup_{h \in C^\kappa, \|h\|_{L^2_2} \leq 1} E \left[ \left\| J_{0,t}^f h \right\|_{H^\kappa_2}^{p} \right] \right) \frac{1}{p}
= \left( P_t \left\| DF \right\|_{H^\kappa_2} \right)^{\frac{1}{q}} \left( \sup_{h \in C^\kappa, \|h\|_{L^2_2} \leq 1} E \left[ \left\| J_{0,t}^f h \right\|_{H^\kappa_2}^{p} \right] \right) \frac{1}{p} \quad (3.4) \]

\( ^3 \)more specifically the extended operator initially defined on the dense subspace \( C^\infty \)
where in the first line we used that \( C^\infty \) is dense in \( L^2_x \). Hence, in order to prove Theorem 1.1 we have to estimate the quantity

\[
\left( \sup_{h \in C^\infty, \|h\|_{L^2_x} \leq 1} E \left[ \left\| f_{0,t} \right\|_{H^p_x}^p \right] \right)^{\frac{1}{p}} \tag{3.5}
\]

uniformly in the initial condition \( f \in C^{\infty} \). In order to do this, we will proceed in three steps. The first step is to prove an \( L^2_x \)-energy estimate with the drawback that the implicit constant is random and moreover it is not clear that it is integrable. The second step – which is our core argument – shows that this constant is indeed integrable and moreover uniformly in the initial condition. The third step is a post-processing from \( L^2_x \) to \( H^k_x \) for any \( k < 1 \).

Before we embark in discussing our intermediate results, let us give the proof of our main theorem.

**Proof of Theorem 1.1.** For \( \kappa \in (0, 1) \) the assertion follows from combining (3.4) and Proposition 3.6 below, which provides an estimate on (3.5). For \( k = 0 \) we apply Proposition 3.4. \( \square \)

### 3.1 \( L^2_x \)-energy estimate

For \( s \leq t \) and \( h \in C^\infty \) we define \( J_{s,t} h \) as the solution to the equation

\[
\left\{ \begin{array}{l}
(\partial_t - \Delta + m) J_{s,t} h = -3 \left( v_{s,t}^2 + 2 v_{s,t} 1_{s,t} + \nabla v_{s,t} \right) J_{s,t} h + 3 c_{t-s,0} J_{s,t} h, \\
J_{s,t} h|_{t=s} = h.
\end{array} \right. \tag{3.6}
\]

In order to ease notation we will suppress the dependence on the initial condition but we will always assume that \( v_{s,t}|_{t=s} = u_t^f \).

The first step towards bounding (3.5) is a standard energy estimate in order to bound the \( L^2_x \)-norm of \( J_{s,t} h \) with respect to the \( L^2_x \)-norm of \( h \). From now on, all proofs are postponed to Section 5.

**Proposition 3.1.** For all \( s \leq t' \leq t \), \( m > 0 \) we have

\[
\left\| J_{s,t} h \right\|_{L^2_x}^2 + \int_{t'}^t e^{-2m(t-r)+2 \int_{r}^{t'} g(s,r) \, ds} \| \nabla J_{s,t} h \|_{L^2_x}^2 \, dr + \int_{t'}^t e^{-2m(t-r)+2 \int_{r}^{t'} g(s,r) \, ds} \| v_{s,t} J_{s,t} h \|_{L^2_x}^2 \, dr \\
\leq e^{-2m(t-t')} + 2 \int_{t'}^t g(s,r) \, dr \left\| J_{s,t} h \right\|_{L^2_x}^2,
\tag{3.7}
\]

where

\[
g(s,t) := c \left( \left\| 1_{s,t} \right\|_{-\frac{1}{2}}^2 + \left\| \nabla 1_{s,t} \right\|_{-\frac{1}{2}}^2 + \left\| \nabla v_{s,t} \right\|_{-\frac{1}{2}}^2 + \left\| v_{s,t} \right\|_{-\frac{1}{2}}^2 \right) \left( 1_{s,t} \right)_{-\alpha}^2 + \left\| \nabla v_{s,t} \right\|_{-\alpha}^2 + c_{t-s,0} \right)
\]

for some deterministic constant \( c \equiv c(\alpha) < \infty \). In particular, we have

\[
\left\| J_{s,t} h \right\|_{L^2_x}^2 \leq e^{-2m(t-s)+2 \int_{s}^{t} g(s,r) \, dr} \left\| h \right\|_{L^2_x}^2.
\tag{3.8}
\]

There are some important things we want to remark concerning Proposition 3.1. The first remark is that if it were not for the singular nature and the renormalization procedure involved the error term \( g \) would be zero and hence we would have a clean energy estimate. The second is that in order to prove Theorem 1.1 with an \( L^2_x \)-norm on the r.h.s. it is enough to consider (3.8) but since our goal is to achieve an \( H^k_x \)-estimate it is crucial to use the additional information...
coming from (3.7), namely, the estimate on the gradient of $J_s^x h$ and the product $v_{x,r} J_s^x h$. The last and most important thing we want to remark makes the bridge to our next section. Notice that by Fernique’s theorem the quantity $\|1_{s,t}\|_\alpha$ in $g(s,t)$ has Gaussian moments, whereas $\|\mathcal{N}_{x,t}\|_\alpha$ has only exponential moments. Therefore, the pre-factor on the r.h.s. of (3.7) fails to be stochastically integrable. To overcome this problem we appeal to a stopping time argument, which we explain in the next section.

### 3.2 Stopping time argument and $L^2_x$-estimate

In order to bypass the issue of integrability of $e^{\int_0^t g(s,r)dr}$ we appeal to probabilistic arguments inspired by [7]. More precisely, we restart the Wick powers $\mathcal{N}_k, k = 1, 2, 3$, each time they exceed a certain barrier. This allows us to replace $g(s,t)$ by a deterministic constant times a counting processes $\mathcal{N}(t)$, see (3.11) below. By choosing the length of the time interval small enough we can ensure the exponential integrability of the counting process $\mathcal{N}(t)$, see Proposition 3.3. The drawback is the exponential factor $e^{\theta t}$ appearing in Theorem 1.1.

We define the stopping time

$$\tilde{\tau}_1 := \inf \left\{ t \geq 0 : \sup_{k=1,2,3} \| \mathcal{N}_{0,t} \|_\alpha \geq \eta \right\}$$

and for $\theta \in (0,1)$ we set

$$\tau_1 := \tilde{\tau}_1 \wedge \theta.$$

The value of $\eta \equiv \eta(\alpha,L)$ will be fixed via

$$\sup_{\theta \in (0,1]} \mathbb{P}(\tilde{\tau}_1 \leq \theta) < \frac{1}{4}. \quad (3.9)$$

This is possible due to Markov’s inequality, (2.3) and the fact that $\theta < 1$ since

$$\mathbb{P}(\tilde{\tau}_1 \leq \theta) \leq \mathbb{P} \left( \sup_{k=1,2,3} \sup_{t \leq \theta} \| \mathcal{N}_{0,t} \|_\alpha \geq \eta \right) \leq \mathbb{P} \left( \sup_{k=1,2,3} \sup_{t \leq 1} \| \mathcal{N}_{t-} \|_\alpha \geq \eta \right) < \frac{1}{4}.$$

We inductively define a sequence of stopping times for $n > 1$ via

$$\tilde{\tau}_n := \inf \left\{ t \geq \tau_{n-1} : \sup_{k=1,2,3} \| \mathcal{N}_{t-1,t} \|_\alpha \geq \eta \right\},$$

where $\mathcal{N}_{t-1,t}$ denotes the process at time $t$ restarted at time $s$, and

$$\tau_n := \tau_{n-1} + (\tilde{\tau}_n - \tilde{\tau}_{n-1}) \wedge \theta.$$

Furthermore, we define the standard filtration of $\sigma$-algebras for $t > 0$

$$\mathcal{F}_t := \sigma(\xi(h) : h \in L^2_{x,t}, \text{supp} h \subset (0,t) \times \mathbb{T}^2).$$

We notice that since $\sigma(1_{0,\lambda t}) \subset \mathcal{F}_t$ and the process $1_{\tau,t+}$ is independent of $1_{0,\lambda t}$ (cf. [35, Proposition 2.3]), by the strong Markov property for any stopping time $\tau$ the process $1_{\tau,t+}$ is independent of $\mathcal{F}_\tau$. Since $\sigma(\mathcal{N}_{0,t-}) \subset \mathcal{F}_t$, we have that for any $n \geq 1$

$$\tilde{\tau}_n - \tilde{\tau}_{n-1} \text{ is independent of } \mathcal{F}_{\tilde{\tau}_{n-1}}.$$
and thus
\[ \nabla^* \tau_n, \tau_{n+1} \text{ is independent of } \mathcal{F}_{\tau_n}. \] (3.10)

Let \( t \leq \tau_1 \). By the definition of \( \tau_1 \) we know that \( \| \nabla^* \alpha \|_{t, \alpha} < \eta \) for all \( k = 1, 2, 3 \). Then by Proposition 3.1 and Lemma D.3 for any \( \epsilon > 0 \) we have that
\[
\| J_{0, h} \|_{L_2^2}^2 \leq e^{-2m+2c \inf(f(0,r))} \| h \|_{L_2^2}^2.
\]
for some \( c = c(\alpha, \eta) < \infty \). For \( \tau_{n-1} \leq t \leq \tau_n \) we have by Proposition 3.1 in the same manner
\[
\| J_{\tau_{n-1}, h} \|_{L_2^2}^2 \leq e^{-2m(t-\tau_{n-1})+2c \inf(f(0,r))} \| J_{\tau_{n-2}, \tau_{n-1}, h} \|_{L_2^2}^2.
\]
and thus by induction we get that
\[
\| J_{0, h} \|_{L_2^2}^2 \leq e^{-2m+2c \inf(f(0,r))} \| h \|_{L_2^2}^2.
\]
From now on we set \( \gamma := \frac{1-\alpha(1+2\epsilon)}{1+\alpha} \). By introducing the following counting process
\[
N(t) := \inf\{ n \geq 1 : \tau_n \geq t \}
\]
we furthermore estimate using \( \tau_i - \tau_{i-1} \leq \theta \) for any \( t \geq 0 \)
\[
\| J_{0, h} \|_{L_2^2}^2 \leq e^{-2m+2c(1+2\epsilon)N(t)} \| h \|_{L_2^2}^2.
\]
(3.12)

**Remark 3.2.** Although we suppressed the dependence on the initial condition \( f \) to ease the notation, we should also point out that our estimates do not depend \( f \). This is possible because of the “coming down from infinity” property (cf. Section D), which allows us to ensure that the gradient estimate in Theorem 3.1 is uniform in \( f \).

The above procedure boils the problem of estimating \( J_{0, h} \) to showing exponential moment for \( N(t) \). Since the sequence \( \{ \tau_n \}_{n \geq 1} \) has independent increments we can expect this provided we choose \( \theta \) small enough. This is the content of the next proposition, which is in the core of our argument, therefore we present the proof here.

**Proposition 3.3.** Let \( c = c(\alpha, \eta) > 0 \) as in (3.12). For all \( p \geq 1 \) there exists \( \theta_0 \equiv \theta_0(\alpha, p, \eta) \in (0, 1) \) which is independent of \( m \) such that for all \( \theta \leq \theta_0 \) and \( t \geq 0 \)
\[
E\left[ e^{p \theta N(t)} \right] \leq C e^{2h2^{p-1}},
\]
where \( C \) is a universal constant uniform in \( L \) and \( m \).

\[ \textit{at least if conditioned onto } \mathcal{F}_{\tau_{n-1}} \]
Proof. Let \( n \geq 1 \). The Markov inequality and \((3.10)\) yield
\[
\mathbb{P}(N(t) \geq n) = \mathbb{P} (\tau_n \leq t) = \mathbb{P} \left( \sum_{k=1}^{n} (\tau_k - \tau_{k-1}) \leq t \right) = \mathbb{P} \left( e^{-\frac{2\ln 2}{\theta} \sum_{k=1}^{n} (\tau_k - \tau_{k-1})} \geq e^{-\frac{2\ln 2}{\theta} t} \right)
\leq e^{\frac{2\ln 2}{\theta}} \mathbb{E} \left[ e^{-\frac{2\ln 2}{\theta} \sum_{k=1}^{n} (\tau_k - \tau_{k-1})} \right] = e^{\frac{2\ln 2}{\theta}} \left( \mathbb{E} \left[ e^{-\frac{2\ln 2}{\theta} \tau_1} \right] \right)^n.
\]
(3.13)
Moreover, we estimate
\[
\mathbb{E} \left[ e^{-\frac{2\ln 2}{\theta} \tau_1} \right] \leq e^{-2\ln 2 + \mathbb{P}(\tilde{\tau}_1 \leq 0)} \leq \frac{1}{4} \mathbb{P}(\tilde{\tau}_1 \leq \theta).
\]
which combined with \((3.9)\) yields
\[
\mathbb{E} \left[ e^{-\frac{2\ln 2}{\theta} \tau_1} \right] \leq \frac{1}{2},
\]
Finally, we have by \((3.13)\) that
\[
\mathbb{P}(N(t) \geq n) \leq 2^{-n} e^{\frac{2\ln 2}{\theta} t}
\]
and the claim follows by choosing \( \theta_0 \) small enough such that \( \theta_0^2 < \frac{\ln 2}{2\theta} \).
\(\square\)

As an immediate consequence of Proposition \(3.3\) and \((3.12)\) we obtain the following \(L^2\)-estimate.

**Proposition 3.4.** For every \( p \geq 1 \) there exists \( m_\ast \equiv m_\ast(\alpha, p, L) > 0 \) such that for every \( t \geq 0 \),
\[
\mathbb{E} \left[ \left\| J_{0,t} \right\|_{L^2_{H_x}}^p \right]^{\frac{1}{p}} \leq C e^{-(m-m_\ast)t},
\]
for some universal constant \( C < \infty \) which is uniform in \( m \) and \( L \).

### 3.3 Upgrade from \(L^2\) to \(H^k_x\)

In this section we upgrade the \(L^2\)-estimate in Proposition \(3.4\) to an \(H^k_x\)-estimate.

The first step is to post-process Proposition \(3.1\) using \((3.12)\).

**Corollary 3.5.** For every \( t' \leq t \) we have that
\[
\left\| J_{0,t'} \right\|_{L^2_{H_x}}^{2} + \int_{t'}^{t} e^{-2m(t-s)} \left\| \nabla J_{0,s} h \right\|_{L^2_{H_x}}^{2} ds + \int_{t'}^{t} e^{-2m(t-s)} \left\| v_{0,s} h \right\|_{L^2_{H_x}}^{2} ds \leq e^{-2m \left( e^{2\theta N(t')} + \int_{t'}^{t} e^{2\theta N(s)} g(0,s) ds \right)} \left\| h \right\|_{L^2_{H_x}}^{2}.
\]
(3.14)

We can now upgrade Proposition \(3.4\) to \(H^k_x\).

**Proposition 3.6.** Let \( \kappa \in (0,1) \) and \( p \geq 1 \). For every \( \alpha < \frac{1-\kappa}{\kappa} \) there exists \( m_\ast \equiv m_\ast(\alpha, p, L) > 0 \) such that
\[
\mathbb{E} \left[ \left\| J_{0,t} \right\|_{L^2_{H_x}}^p \right]^{\frac{1}{p}} \leq C(t \wedge 1)^{-\frac{1+5\kappa}{2}} e^{-(m-m_\ast)t},
\]
for some constant \( C \equiv C(p, \alpha, \kappa, L) < \infty \) which is uniform in \( f \).

Here we need \( \kappa < 1 \) to ensure the integrability of the exponent when \( t \searrow 0 \). Moreover, we again crucially used the “coming down from infinity” property that ensures that the bound does not depend on the initial data \( f \).
4 Spectral gap inequalities

In this section we give our main application of the gradient estimate Theorem 1.1. At the core of the argument lies the celebrated method of Bakry and Émery (cf. [2], [30]) to prove log-Sobolev inequalities as well as spectral gap inequalities.

Theorem 4.1. Let $\Delta \lambda = e^{-\psi} dx$ be a probability measure on a smooth and flat manifold $M$ such that $\psi \in C^2(M)$ and $D^2 \psi \geq \rho I$ for some $\rho > 0$. Then $\Delta \lambda$ satisfies a log-Sobolev inequality with constant $\rho$.

Hence by the convexity of the potential it is natural to expect that (1.1) even satisfies a log-Sobolev inequality but due to the singular nature of the equation we are only able to prove a spectral gap inequality. At this point we want to mention that in [22] it was shown that (1.1) does satisfy a log-Sobolev inequality when $d = \frac{n}{2}$ with respect to $L^{2}_{\lambda}$. In the following we also want to point out how the required renormalization procedure obstructs us from proving an log-Sobolev inequality. The first step is to show the following identity (cf. [4, p. 131, (3.1.21)]), the proof of which can be found in the appendix, Section E.

Proposition 4.2. The following identity holds for every $t > 0$ and $F \in \mathcal{F}C_{b}$,

$$P_{t}F^{2}(f) - (P_{t}F(f))^{2} = 2 \int_{0}^{t} P_{t-s} \left( \|DP_{s}F\|_{L^{2}_{\lambda}}^{2} \right) (f) ds \text{ v.a.s. in } f. \quad (4.1)$$

We are now in position to prove Theorem 1.2 and Corollary 1.3.

Proof of Theorem 1.2 and Corollary 1.3. We apply Theorem 1.1 combined with (4.1) and the fact that $P_{t}$ is a Markov semigroup yielding

$$P_{t}(F^{2}) - (P_{t}F)^{2} \lesssim \int_{0}^{t} (s \wedge 1)^{-\kappa} e^{-2(m-m_{s})s} ds \|D_{1}F\|_{L^{2}_{\lambda}}^{2}.$$ 

Finally, choosing $m > m_{s}$ and noting that

$$\int_{0}^{\infty} (s \wedge 1)^{-\kappa} e^{-2(m-m_{s})s} ds \lesssim \frac{1}{(m-m_{s})^{1-\kappa}} \lor \frac{1}{m-m_{s}},$$

we appeal to ergodicity (cf. [35, p. 1241, Corollary 6.6]) letting $t \not\to \infty$ to obtain (1.9).

5 Proof of intermediate statements

In this section we collect the proofs of the intermediate statements missing from the previous section.

Proof of Proposition 7.1. Testing the equation (3.6) with $J_{s} h$ yields

$$\frac{1}{2} \left( \|J_{s} h\|_{L^{2}_{\lambda}}^{2} + \|\nabla J_{s} h\|_{L^{2}_{\lambda}}^{2} + m \|J_{s} h\|_{L^{2}_{\lambda}}^{2} + 3 \|\nabla(J_{s} h)\|_{L^{2}_{\lambda}}^{2} \right) = -6 \left( 1_{s, t}, v_{s, t}(J_{s} h)^{2} \right)_{L^{2}_{\lambda}} - 3 \left( \nabla_{x, t} (J_{s} h)^{2} \right)_{L^{2}_{\lambda}} + 3 c_{t-x, \infty} \|J_{s} h\|_{L^{2}_{\lambda}}^{2}. \quad (5.1)$$

We start by estimating $\left( 1_{s, t}, v_{s, t}(J_{s} h)^{2} \right)_{L^{2}_{\lambda}}$. To this end, we apply [35, Proposition A.8] to get

$$\left( 1_{s, t}, v_{s, t}(J_{s} h)^{2} \right)_{L^{2}_{\lambda}} \lesssim \left( 1_{s, t}, \left\| v_{s, t}(J_{s} h)^{2} \right\|_{L^{2}_{\lambda}} \right).$$

and the equation does not require any renormalization
and then use Proposition A.9 in [35] such that we end up with
\[ \left\| v_{x,t}(J_s,h)^2 \right\| \leq \left\| v_{x,t}(J_s,h)^2 \right\|_{L^1_t}^{1-\alpha} \left\| \nabla \left( v_{x,t}(J_s,h)^2 \right) \right\|_{L^1_t}^{\alpha} + \left\| v_{x,t}(J_s,h)^2 \right\|_{L^1_t}. \]

Moreover, the Cauchy–Schwarz inequality and the chain rule yield
\[ \left\| v_{x,t}(J_s,h)^2 \right\|_{L^1_t}^{1-\alpha} \left\| \nabla \left( v_{x,t}(J_s,h)^2 \right) \right\|_{L^1_t}^{\alpha} + \left\| v_{x,t}(J_s,h)^2 \right\|_{L^1_t} \]
\[ \lesssim \left\| J_{s,t}h \right\|_{L^2_t}^{1-\alpha} \left\| v_{x,t}(J_s,h) \right\|_{L^2_t}^{1-\alpha} \left\| (J_s,h)^2 \nabla v_{x,t} + 2v_{x,t}(J_s,h)\nabla J_{s,t}h \right\|_{L^1_t}^{\alpha} + \left\| J_{s,t}h \right\|_{L^2_t} \left\| v_{x,t}(J_s,h) \right\|_{L^2_t}. \]

The Cauchy–Schwarz inequality again implies
\[ \left\| (J_s,h)^2 \nabla v_{x,t} + 2v_{x,t}(J_s,h)\nabla J_{s,t}h \right\|_{L^1_t}^{\alpha} \lesssim \left\| \nabla v_{x,t} \right\|_{L^2_t}^{\alpha} \left\| J_{s,t}h \right\|_{L^2_t}^{2\alpha} + 2\alpha \left\| v_{x,t}(J_s,h) \right\|_{L^2_t}^{\alpha} \left\| \nabla J_{s,t}h \right\|_{L^2_t}^{\alpha}. \]

Hence we have shown that
\[ \left\| v_{x,t}(J_s,h)^2 \right\|_{L^1_t} \lesssim \left\| \nabla v_{x,t} \right\|_{L^2_t}^{\alpha} \left\| v_{x,t}(J_s,h) \right\|_{L^2_t}^{1-\alpha} \left\| J_{s,t}h \right\|_{L^2_t}^{1-\alpha} \left\| \nabla J_{s,t}h \right\|_{L^2_t}^{\alpha} + \left\| J_{s,t}h \right\|_{L^2_t} \left\| v_{x,t}(J_s,h) \right\|_{L^2_t}. \]

Then we have by Young’s inequality for some \( \lambda > 0 \) to be chosen later
\[ I_3 = \left\| v_{x,t} \right\|_{L^2_t} \left\| J_{s,t}h \right\|_{L^2_t} \left\| v_{x,t}(J_s,h) \right\|_{L^2_t} \leq \frac{1}{2\lambda} \left\| v_{x,t} \right\|_{L^2_t}^2 + \frac{\lambda}{2} \left\| J_{s,t}h \right\|_{L^2_t}^2 \]

and
\[ I_1 = \left\| v_{x,t} \right\|_{L^2_t} \left\| J_{s,t}h \right\|_{L^2_t} \left\| v_{x,t}(J_s,h) \right\|_{L^2_t} \left\| \nabla J_{s,t}h \right\|_{L^2_t}^{\alpha} \]
\[ \leq \frac{1 + \alpha}{2\lambda} \left\| v_{x,t} \right\|_{L^2_t} \left\| J_{s,t}h \right\|_{L^2_t} \left\| v_{x,t}(J_s,h) \right\|_{L^2_t} \left\| \nabla J_{s,t}h \right\|_{L^2_t} \]
\[ \leq \frac{1 + \alpha}{2\lambda} \left\| v_{x,t} \right\|_{L^2_t} \left\| J_{s,t}h \right\|_{L^2_t} + \frac{\alpha}{2} \left\| v_{x,t}(J_s,h) \right\|_{L^2_t} \]

as well as
\[ I_2 = \left\| v_{x,t} \right\|_{L^2_t} \left\| J_{s,t}h \right\|_{L^2_t} \left\| v_{x,t}(J_s,h) \right\|_{L^2_t} \left\| \nabla J_{s,t}h \right\|_{L^2_t} \]
\[ \leq \frac{1}{2\lambda} \left\| v_{x,t} \right\|_{L^2_t} \left\| J_{s,t}h \right\|_{L^2_t} \left\| v_{x,t}(J_s,h) \right\|_{L^2_t} \left\| \nabla J_{s,t}h \right\|_{L^2_t} \]
\[ \leq \frac{1 + \alpha}{2\lambda} \left\| v_{x,t} \right\|_{L^2_t} \left\| J_{s,t}h \right\|_{L^2_t} + \frac{\alpha}{2} \left\| v_{x,t}(J_s,h) \right\|_{L^2_t} \]

For the second term on the right hand side of (5.1) we proceed similarly. First of all, Proposition A.8 in [35] yields
\[ \left\| \nabla v_{x,t}(J_s,h)^2 \right\|_{L^1_t} \leq \left\| \nabla v_{x,t} \right\|_{L^2_t} \left\| (J_s,h)^2 \right\|_{L^2_t}^{2(1-\alpha)} + \left\| J_{s,t}h \right\|_{L^2_t}^2 \]

and hence the Cauchy–Schwarz inequality combined with Young’s inequality with the same \( \lambda > 0 \) as before yields
\[ \left\| \nabla v_{x,t}(J_s,h)^2 \right\|_{L^1_t} \leq 2\alpha \frac{2 - \alpha}{2\lambda} \left\| \nabla v_{x,t} \right\|_{L^2_t}^{2(1-\alpha)} + \frac{\alpha \lambda}{2} \left\| \nabla J_{s,t}h \right\|_{L^2_t}^2 + \left\| \nabla v_{x,t} \right\|_{L^2_t} \left\| J_{s,t}h \right\|_{L^2_t}^2. \]

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Then we set

\[ g(s, t) := \frac{1}{2\lambda} \| l_{s,t} \|^2 - \frac{1}{\alpha} \left( \frac{1}{2} \lambda \right) \| l_{s,t} \| ^ \frac{3}{2} + \frac{1}{\alpha} \| \nabla l_{s,t} \|^2 + \frac{1}{\alpha} \| l_{s,t} \| ^ \frac{3}{2} + \| \nabla l_{s,t} \| \| l_{s,t} \| ^ \frac{3}{2} \]

By choosing \( \lambda \) small enough, we can absorb some of the terms into \( \| \nabla J_{s,t}h \|^2 \) respectively \( \| v_{s,t}(J_{s,t}h) \|^2 \) into the right hand side and we end up with the estimate

\[ \frac{1}{2} \partial_t \| J_{s,t}h \|^2 + m \| J_{s,t}h \|^2 + \frac{1}{2} \| \nabla J_{s,t}h \|^2 + \frac{1}{2} \| v_{s,t}(J_{s,t}h) \|^2 \leq g(s,t) \| J_{s,t}h \|^2. \]  

(5.2)

Then the chain rule combined with (5.3) yields

\[ \partial_t \left( e^{2mt - 2 \int_0^t g(s,r)dr} \| J_{s,t}h \|^2 \right) + e^{2mt - 2 \int_0^t g(s,r)dr} \| \nabla J_{s,t}h \|^2 + e^{2mt - 2 \int_0^t g(s,r)dr} \| v_{s,t}(J_{s,t}h) \|^2 \leq 0. \]

(5.3)

Integrating (5.3) from \( t' \) to \( t \) we end up with

\[ \| J_{s,t}h \|^2 + \int_{t'}^t e^{-2m(t-t')+2 \int_0^{t'} g(s,r)dr} \| \nabla J_{s,t}h \|^2 \| v_{s,t}(J_{s,t}h) \|^2 \leq e^{-2m(t-t')} \| J_{s,t}h \|^2. \]

\[ \square \]

**Proof of Corollary 5.5** The estimate (5.2) yields for \( s = 0 \)

\[ \partial_t \left( e^{2mt} \| J_{0,t}h \|^2 \right) + e^{2mt} \left( \| \nabla J_{0,t}h \|^2 + \| \nabla J_{0,t}h \|^2 \right) \leq 2e^{2mt} g(0,t) \| J_{0,t}h \|^2. \]

Then we integrate from \( t' \) to \( t \) to obtain

\[ \| J_{0,t}h \|^2 + \int_{t'}^t e^{-2m(t-t')} \left( \| \nabla J_{0,s}h \|^2 + \| \nabla J_{0,s}h \|^2 \right) ds \leq e^{-2m(t-t')} \| J_{0,t}h \|^2 + 2 \int_{t'}^t e^{-2m(t-s)} g(0,s) \| J_{0,s}h \|^2 ds. \]

Applying (3.12) to \( \| J_{0,t}h \|^2 \) respectively to \( \| J_{0,t}h \|^2 \) yields the assertion. \( \square \)

**Proof of Proposition 4.4** First of all, Duhamel’s formula yields

\[ J_{0,t}h = S_{t/2} J_{0,1/2} h - 3 \int_{1/2}^t S_{s-1/2} \left\{ \left( \frac{2}{3} v_{0,s} + 2v_{0,s} \right) \delta_{0,s} + \nabla v_{0,s} - c_{s,m} \right\} J_{0,s}h \] ds

\[ = I_1 + I_2 + I_3 + I_4 + I_5. \]

Then we estimate \( I_1 \) according to

\[ \left\| S_{1/2} J_{0,1/2} h \right\| \leq \left( \| 0 \| \right) \leq \left( \| 0 \| \right) \leq (t \land 1)^{-\frac{d}{2}} e^{-m(t-s)} \left\| J_{0,1/2} h \right\| \leq \left( \| 0 \| \right) \leq (t \land 1)^{-\frac{d}{2}} e^{-m(t-s)} \left\| J_{0,1/2} h \right\|. \]

For \( I_2 \) we further estimate

\[ \int_{1/2}^t \left\| S_{s-1/2} \left( \frac{2}{3} v_{0,s} J_{0,s}h \right) \right\| ds \leq \left( \| 0 \| \right) \leq \left( \| 0 \| \right) \leq (t \land 1)^{-\frac{d}{2}} e^{-m(t-s)} \left\| v_{0,s} J_{0,s}h \right\| \] ds.
\[
\lesssim \int_{t}^{s} ((t-s) \wedge 1)^{-\frac{\alpha}{2}} \|v_{0,s}\|_{L_{2}^{\gamma}} e^{-m(t-s)} \|v_{0,s} J_{0,s} h\|_{L_{2}^{\gamma}} ds
\]
\[
\lesssim \left( \int_{t}^{s} ((t-s) \wedge 1)^{-\alpha} \|v_{0,s}\|_{L_{2}^{\gamma}} ds \right)^{\frac{1}{2}} \left( \int_{t}^{s} e^{-2m(t-s)} \|v_{0,s} J_{0,s} h\|_{L_{2}^{\gamma}}^2 ds \right)^{\frac{1}{2}}
\]
\[
\lesssim \left( \int_{t}^{s} ((t-s) \wedge 1)^{-\alpha} \|v_{0,s}\|_{L_{2}^{\gamma}}^2 ds \right)^{\frac{1}{2}} \lesssim \left( \int_{t}^{s} ((t-s) \wedge 1)^{-\alpha} \|v_{0,s}\|_{L_{2}^{\gamma}}^2 ds \right)^{\frac{1}{2}} e^{-m\frac{2\alpha}{\theta}(s)^{\frac{1}{2}}} \left( e^{2\alpha\theta N(s)^{\frac{1}{2}}} + \int_{t}^{s} e^{2\alpha\theta N(s)} g(0,s) ds \right)^{\frac{1}{2}} \|h\|_{L_{2}^{\gamma}},
\] where we used again Hölder’s inequality in the third step.

Estimating \( I_3 \) yields
\[
\int_{t}^{s} \|S_{t-s}(v_{0,s} 1_{0,s} J_{0,s} h)\|_{B_{2,\infty}^{\theta}} ds \lesssim \int_{t}^{s} \left( ((t-s) \wedge 1)^{-\frac{\alpha}{2}} \|v_{0,s}\|_{L_{2}^{\gamma}} e^{-m(t-s)} \|v_{0,s} J_{0,s} h\|_{B_{2,\infty}^{\theta}} ds \right)
\]
\[
\lesssim \left( \int_{t}^{s} ((t-s) \wedge 1)^{-\alpha} \|v_{0,s}\|_{L_{2}^{\gamma}}^2 ds \right)^{\frac{1}{2}} \left( \int_{t}^{s} e^{-2m(t-s)} \|v_{0,s} J_{0,s} h\|_{B_{2,\infty}^{\theta}}^2 ds \right)^{\frac{1}{2}}
\]
\[
\lesssim \left( \int_{t}^{s} ((t-s) \wedge 1)^{-\alpha} \|v_{0,s}\|_{L_{2}^{\gamma}}^2 ds \right)^{\frac{1}{2}} \times e^{-m\frac{2\alpha}{\theta}(s)^{\frac{1}{2}}} \left( e^{2\alpha\theta N(s)^{\frac{1}{2}}} + \int_{t}^{s} e^{2\alpha\theta N(s)} g(0,s) ds \right)^{\frac{1}{2}} \|h\|_{L_{2}^{\gamma}},
\]
using Hölder’s inequality in the third step.

The term \( I_4 \) is estimated via
\[
\int_{t}^{s} \|S_{t-s}(\nabla v_{0,s} J_{0,s} h)\|_{B_{2,\infty}^{\theta}} ds \lesssim \int_{t}^{s} \left( ((t-s) \wedge 1)^{-\alpha} \|\nabla v_{0,s}\|_{B_{2,\infty}^{\theta}} \|J_{0,s} h\|_{B_{2,\infty}^{\theta}} ds \right)
\]
\[
\lesssim \left( \int_{t}^{s} ((t-s) \wedge 1)^{-\alpha} \|\nabla v_{0,s}\|_{B_{2,\infty}^{\theta}}^2 ds \right)^{\frac{1}{2}} \left( \int_{t}^{s} e^{-2m(t-s)} \|J_{0,s} h\|_{B_{2,\infty}^{\theta}}^2 ds \right)^{\frac{1}{2}}
\]
\[
\lesssim \left( \int_{t}^{s} ((t-s) \wedge 1)^{-\alpha} \|\nabla v_{0,s}\|_{B_{2,\infty}^{\theta}}^2 ds \right)^{\frac{1}{2}} \times e^{-m\frac{2\alpha}{\theta}(s)^{\frac{1}{2}}} \left( e^{2\alpha\theta N(s)^{\frac{1}{2}}} + \int_{t}^{s} e^{2\alpha\theta N(s)} g(0,s) ds \right)^{\frac{1}{2}} \|h\|_{L_{2}^{\gamma}},
\]
where again we have used Hölder’s inequality in the third step.

Finally, we estimate \( I_5 \)
\[
\int_{t}^{s} \|S_{t-s}C_{0,s} h\|_{B_{2,\infty}^{\theta}} ds \lesssim \int_{t}^{s} s^{-\frac{\theta}{2}} e^{-m(t-s)} \|J_{0,s} h\|_{B_{2,\infty}^{\theta}} ds
\]
\[
\lesssim \left( \int_0^t s^{-\beta} ds \right)^{\frac{1}{2}} \left( \int_0^t e^{-2m(t-s)} \|J_0h\|_{L_2^2}^2 ds \right)^{\frac{1}{2}} \\
\approx \left( \int_0^t s^{-\beta} ds \right)^{\frac{1}{2}} e^{-mt} \left( \int_0^t e^{2\kappa T N(t)^{\frac{1}{2}}} g(0,s) ds \right)^{\frac{1}{2}} \|h\|_{L_2^2},
\]

where we have used Hölder’s inequality in the second step.

Then we use monotonicity of \( t \to N(t) \) to infer
\[
\int_0^t e^{2\kappa T N(t)} g(0,s) ds \leq e^{2\kappa T N(t)} \int_0^t g(0,s) ds
\]
which all in all yields
\[
\|J_0h\|_{L_2^2} \lesssim e^{-mt + C\theta T N(t)} \left( 1 + \int_0^t g(0,s) ds \right) \|h\|_{L_2^2} \\
\times \left( (t \wedge 1)^{\frac{1}{2}} + \left( \int_0^t ((t-s) \wedge 1)^{-\kappa} \|v_{0,s}\|_{L_2^2}^2 ds \right)^{\frac{1}{2}} \right. \\
+ \sup_{0 \leq s \leq t} \|1_{0,s}\|^{-\alpha} \left( \int_0^t ((t-s) \wedge 1)^{-\kappa-\alpha} \|v_{0,s}\|_{L_2^2}^2 ds \right)^{\frac{1}{2}} \\
+ \sup_{0 \leq s \leq t} \|v_{0,s}\|^{-\alpha} \left( \int_0^t ((t-s) \wedge 1)^{-\kappa-\alpha} ds \right)^{\frac{1}{2}} + \left( \int_0^t s^{-\beta} ds \right)^{\frac{1}{2}} 
\]

Using the definition of \( g \) we see that
\[
\int_0^t g(0,s) ds \lesssim t \sup_{0 \leq s \leq t} \|1_{0,s}\|^{-\alpha} + \sup_{0 \leq s \leq t} \|1_{0,s}\|^{-\alpha} \int_0^t \|v_{0,s}\|_{L_2^2}^2 ds + t \sup_{0 \leq s \leq t} \|1_{0,s}\|^{-\alpha} \\
+ t \sup_{0 \leq s \leq t} \|v_{0,s}\|^{-\alpha} + t^{1-\beta}
\]
and for any \( p \geq 1 \) we can estimate
\[
\int_0^t E \left[ \|v_{0,s}\|_{L_2^2}^{2\alpha p} \right]^{\frac{1}{p}} \lesssim \int_0^t s^{-\frac{(1+\beta)2\alpha}{2\alpha-p}} ds \lesssim t^{1-\frac{(1+\beta)2\alpha}{2\alpha-p}}.
\]

Moreover, by Proposition 2.1 for every \( p < \infty \) there exists \( r > 0 \) such that
\[
E \left[ \sup_{0 \leq s \leq t} \|1_{0,s}\|_{-\alpha}^p \right]^{\frac{1}{p}} \lesssim (1+t)^r, \quad (5.4)
\]
\[
E \left[ \sup_{0 \leq s \leq t} \|v_{0,s}\|_{-\alpha}^p \right]^{\frac{1}{p}} \lesssim (1+t)^r. \quad (5.5)
\]

Any positive power of \( t \) can brutally be bounded by \( C_\sigma e^{\sigma t} \) for \( \sigma > 0 \), thus we have for any \( p \geq 1 \)
\[
E \left[ \left( \int_0^t g(0,s) ds \right)^p \right]^{\frac{1}{p}} \lesssim e^{\sigma t}
\]
where we implicitly used Hölder’s inequality in expectation.

Also, again for any \( p \geq 1 \) we have
\[
\left( \int_0^t ((t-s) \wedge 1)^{-\kappa} E \left[ \|v_{0,s}\|_{L_2^2}^p \right]^{\frac{1}{p}} ds \right)^{\frac{1}{2}} \lesssim \left( \int_0^t ((t-s) \wedge 1)^{-\kappa-1-\alpha} ds \right)^{\frac{1}{2}} \lesssim (t \wedge 1)^{-\kappa-\alpha}.
\]
and similarly
\[
\left( \int_{t}^{t+s} ((t-s) \wedge 1)^{-\kappa - \alpha} E \left[ \|v_{0,s}\|^2_{2\alpha} \right] \right)^{\frac{1}{2}} \lesssim (t \wedge 1)^{-\kappa + \frac{k \sigma}{2}}.
\]
Dividing by \( \|h\|_{L^2} \), taking the supremum and using (5.4), (5.5) and (3.3), we conclude that
\[
E \left[ \|J_{0,t}\|^p_{L^2_{\alpha} \rightarrow H^\kappa_{\alpha, \kappa, L}} \right] \lesssim (t \wedge 1)^{-\kappa + \frac{k \sigma}{2}} e^{-\left( m - 10 \sigma - \frac{1}{10} \right)t},
\]
where we have used H"older’s inequality in probability repeatedly.

\[\square\]

**Appendix**

**A Estimate on the renormalization constant**

**Proposition A.1.** The following estimate holds for any \( \beta \in (0, 1) \) and \( t > 0 \),
\[
c_{t, \infty} = 2 \int_{t}^{\infty} ds H_{2\alpha}(s) \lesssim \beta t^{-\frac{\beta}{2}}.
\]

**Proof.** By a simple computation in Fourier space we have that
\[
c_{t, \infty} = 2 \sum_{k \in \mathbb{Z}^2} e^{-\ell(m+|k|^2)}.
\]
Noticing that \( e^{-\ell(m+|k|^2)} \lesssim \frac{r^{\beta}}{(m+|k|^2)^{\frac{\beta}{2}}} \) for any \( \beta \in (0, 1) \) we get the assertion since the sum \( \sum_{k \neq 0} \frac{1}{|k|^{\beta}} \) is finite. \[\square\]

**B Besov-norm estimates**

**Lemma B.1** ([36, p. 308, (A.2)]). Let \( \alpha \leq \beta \) and \( p, q \geq 1 \), then we have
\[
\|f\|_{B^\alpha_p, q} \leq \|f\|_{B^\beta_p, q}.
\]

**Lemma B.2** ([36, p. 309, Proposition A.5]). Let \( \alpha \leq \beta \) and \( p, q \geq 1 \), then it holds that
\[
\|S_t f\|_{B^\beta_p, q} \lesssim e^{-\ell m} (t \wedge 1)^{-\frac{\alpha - \beta}{2q}} \|f\|_{B^\alpha_p, q}
\]
where \( S_t \) denotes the semigroup generated by \( \Delta - m \) for \( m \geq 0 \).

**Lemma B.3** ([36, p. 309, Proposition A.6]). Let \( \alpha \geq 0 \) and \( p, q \geq 1 \), then
\[
\|fg\|_{B^\alpha_p, q} \leq \|f\|_{B^\alpha_p, q} \|g\|_{B^\alpha_p, q}
\]
where \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \) as well as \( \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \).

**Lemma B.4** ([36, p. 309, Proposition A.7]). Let \( \alpha < 0 \) and \( \beta > 0 \) such that \( \alpha + \beta > 0 \) and \( p, q \geq 1 \), then
\[
\|fg\|_{B^\alpha_p, q} \leq \|f\|_{B^\alpha_p, q} \|g\|_{B^\alpha_p, q}
\]
where \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \) and \( \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \).
C Stochastic estimates

In this section we provide an alternative argument for the stochastic estimates in Proposition 2.1 using the spectral gap inequality (C.1) for the noise $\xi$ in the spirit of [20] Section 5 and [23].

Let $F$ be cylindrical in $\xi$, i.e. there is $n \in \mathbb{N}$, $F \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$ and $h_1, \ldots , h_n \in L^2_{\mathbb{P}}$ such that $F(\xi) = F(\xi(h_1), \ldots , \xi(h_n))$. Since $\xi$ is Gaussian it satisfies the following spectral gap inequality (cf. [10, p. 652, Proposition 4.1])

$$E \left[ |F(\xi) - E[F(\xi)]|^2 \right] \leq E \left[ \left\| \frac{\partial}{\partial \xi} F(\xi) \right\|_{L^2_{\mathbb{P}}}^2 \right]$$

(C.1)

where $\frac{\partial}{\partial \xi}$ denotes the Malliavin derivative with respect to the noise $\xi$. This in turn can be used to construct the singular products as follows.

Proof of Proposition 2.1 For simplicity we assume that the noise $\xi$ is smooth. By the spectral gap inequality (C.1) we know that for nice enough functionals $\Pi(\xi)$ and $p \geq 2$ there holds

$$E^\mathbb{P} \left[ |\Pi(\xi) - E\Pi(\xi)|^p \right] \lesssim_p E^\mathbb{P} \left[ \left\| \frac{\partial}{\partial \xi} \Pi(\xi) \right\|_{L^2_{\mathbb{P}}}^p \right].$$

By duality, an estimate of the form

$$E^\mathbb{P} \left| \left\langle \frac{\partial}{\partial \xi} \Pi(\xi) , (\delta \xi) \right\rangle \right| \leq C E^\mathbb{P} \left\| (\delta \xi) \right\|_{L^q_{\mathbb{P}}}^q$$

(C.2)

for any $\delta \xi : \Omega \to L^2_{\mathbb{P}}[\mathbb{R}]$ where $q \in [1, 2]$ is the dual exponent of $p$, implies

$$E^\mathbb{P} \left\| \frac{\partial}{\partial \xi} \Pi(\xi) \right\|_{L^q_{\mathbb{P}}}^p \leq C.$$

For $t > 0$ and $x \in \mathbb{T}^2$ we consider $\Pi_t(\xi) \in \left\{ 1_{0,t}(x), \nabla_0 t(x), \nabla_0^2 t(x) \right\}$, where

$$\nabla_0 t(x) := 1_{0,t}(x) - c_{0,t}, \quad \nabla_0^2 t(x) := 1_{0,t}(x) - 3c_{0,t} 1_{0,t}(x),$$

for $c_{0,t} = E[1_{0,t}(0)]^2$. We treat $\Pi_t(\xi) \equiv \Pi_t[\xi](x)$ as a functional of $\xi$ and aim to prove the following stochastic estimates (replacing $x$ by 0 using stationarity) which are uniform in $m$,

$$E^\mathbb{P} \left| \Pi_t(0) \right| \lesssim \lambda^{-\alpha} \sqrt{t}^{\alpha},$$

(C.3)

and

$$E^\mathbb{P} \left| (\Pi_{t+r} - \Pi_t)(0) \right| \lesssim \lambda^{-\alpha} \sqrt{t+r}^{\alpha} \sqrt{t}^{\alpha},$$

(C.4)

for every $\alpha \in (0, \frac{1}{\Pi(\mathbb{T}^2)})$, where $[\Pi] = 1, 2, 3$ for $\Pi = \mathbb{1}, \nabla, \nabla^2$ respectively and $(\cdot)_{\lambda}$ denotes convolution with a suitable semigroup $\psi_{\lambda}$. By a Kolmogorov-type continuity criterion, see for [27] Lemma 10], we then obtain (2.3). It is important to stress the uniformity of our estimates in $m$ which allows us to ensure that $m$, in Theorem 1.1 does not depend on $m$. This will be obvious in what follows except (C.17) where one should pay attention on how the power on $\sqrt{r}$ is chosen.

For $\delta \xi \in L^q_{\mathbb{P}}[\mathbb{R}^2]$ we let $\delta 1_{0,t}(x) := \frac{\partial}{\partial \xi} 1_{0,t}(\delta \xi) = \int_0^t ds H_{t-s} \ast \delta \xi(s, x)$ and consider $\delta \Pi_t \in \left\{ \delta 1_{0,t}, \delta 1_{0,t} \delta 1_{0,t}, \delta 1_{0,t} 1_{0,t} \right\}$. As in [23], in order to prove (C.3) and (C.4) we appeal to duality and derive the following estimates for the Malliavin derivative of $\Pi_t$,

$$E^\mathbb{P} \left| \delta \Pi_t(0) \right| \lesssim \lambda^{-\alpha} \sqrt{t}^{\alpha},$$

(C.5)

\footnote{where $\Omega$ denotes the underline probability space}

\footnote{or equivalently $\theta$ in Proposition 2.1 does not depend on $m$}
\[
E^{\frac{1}{q}} \left| (\delta \Pi_{r} - \delta \Pi_{r})(0) \right|^{q'} \lesssim \lambda^{-\|2i\|_{\alpha}} \sqrt{r^2 + r^2} \|E^{\frac{1}{q}} \left| \delta \xi \right|^{q'} \|_{L_{q}^{2}}, \quad (C.6)
\]
for all \( q' < q < 2 \). Note that in \( C.5 \) and \( C.6 \) we ask for an estimate of the \( L_{q}^{2} \)-norm by the \( L_{q}^{2}, L_{q}^{2} \)-norm which is stronger than the \( L_{q}^{2} \)-norm for \( q < 2 \), therefore implying the dual estimate \( C.2 \). As in \( [23] \) estimating the \( L_{q}^{2} \)-norm for all \( q' < q < 2 \) allows us to proceed inductively, namely, in order to derive the dual estimate for \( \delta^{*} V_{0,l} \) we need the stronger estimate on \( \delta 1_{0,l} \) and similarly for \( \delta^{*} V_{0,l} \).

To this end, we denote by \( \bar{w} \) the \( L_{q}^{2}, L_{q}^{2} \)-norm on the r.h.s. of \( C.5 \) and \( C.6 \) and introduce another scaling parameter \( \Lambda \), coming from \( \langle \cdot \rangle_{\Lambda} \). We estimate commutators of the form
\[
\langle [\delta \Pi, \langle \cdot \rangle_{\Lambda}] \rangle_{\Lambda}(0) = \int dx \psi_{\Lambda}(-x) \int dy \psi_{\Lambda}(y) (\delta \Pi(x-y) - \delta \Pi(x)) \Pi_{\Lambda}(x-y).
\]
Using the Cauchy–Schwarz inequality in the \( x \)-variable we have\[8\]
\[
E^{\frac{1}{q'}} \left| \left( \delta 1_{0,l}, \langle \cdot \rangle_{\Lambda} \right) \Pi_{\Lambda}(0) \right|^{q'} = E^{\frac{1}{q'}} \left| \int dx \psi_{\Lambda}(-x) \int dy \psi_{\Lambda}(y) \left| \delta \Pi(x-y) - \delta \Pi(x) \right| \Pi_{\Lambda}(x-y) \right|^{q'} \leq \int dx |\psi_{\Lambda}(y)||\psi_{\Lambda}(y)| E^{\frac{1}{q'}} \||\delta \Pi(x-y) - \delta \Pi(x)||^{q'} \|_{L_{q}^{2}} \left\| E^{\frac{1}{q'}} \left| \Pi_{\Lambda}(0) \right|^{p} \right\|_{L_{p}^{1}}. \quad (C.7)
\]
For \( C.5 \) we let \( \delta \Pi = \delta 1_{l} \), and \( \Pi \in \{ 1_{0,l}, V_{0,l} \} \). Using the interpolation inequality Lemma \( C.1 \) and the Cauchy–Schwarz inequality in the \( s \)-variable we see that
\[
\left\| E^{\frac{1}{q'}} \left| \delta 1_{0,l}(x-y) - \delta 1_{0,l}(x) \right|^{q'} \right\|_{L_{q}^{2}} \leq \int_{0}^{t} ds \int dz |H_{t-s}(z-y) - H_{t-s}(z)| E^{\frac{1}{q'}} \left| \delta \xi \right|^{q'} \|_{L_{q}^{2}} \leq |y|^{1-\alpha} \left( \int_{0}^{t} ds e^{-2m(t-s)(t-s)^{-1+\alpha}} \right)^{\frac{1}{q}} w \leq |y|^{1-\alpha} \sqrt{t} \frac{w}{\alpha}, \quad (C.8)
\]
for all \( \alpha \in (0,1) \) uniformly in \( m \). Combining \( C.7 \) and \( C.8 \) yields
\[
E^{\frac{1}{q'}} \left| \left( \delta 1_{0,l}, \langle \cdot \rangle_{\Lambda} \right) \Pi_{\Lambda}(0) \right|^{q'} \lesssim \Lambda^{-1} \lambda^{1-\alpha} \sqrt{t} E^{\frac{1}{q}} \left| \Pi_{\Lambda}(0) \right|^{p} \frac{w}{\alpha}. \quad (C.9)
\]
Using \( C.3 \) and the dyadic summation identity
\[
\langle [\delta \Pi, \langle \cdot \rangle_{\Lambda}] \rangle_{\Lambda} = \sum_{k \geq 1} \langle [\delta \Pi, \langle \cdot \rangle_{\Lambda}] \rangle_{\Lambda+k-2\Lambda},
\]
we obtain via \( C.9 \)
\[
E^{\frac{1}{q'}} \left| \left( \delta 1_{0,l}, \langle \cdot \rangle_{\Lambda} \right) \Pi_{\Lambda}(0) \right|^{q'} \lesssim \Lambda^{-1} \lambda^{1-\|2i\|_{\alpha}} \sqrt{t} \frac{w}{\alpha}.
\]
A simple post-processing of the last estimate choosing \( \Lambda \sim \lambda \) gives
\[
E^{\frac{1}{q'}} \left| (\delta 1_{0,l})_{\lambda}(0) \right|^{q'} \lesssim \lambda^{-\|2i\|_{\alpha}} \sqrt{t} \frac{w}{\alpha}, \quad (C.10)
\]
\[8\]Here \( p \geq 2 \) satisfies \( \frac{1}{q'} = \frac{1}{q} + \frac{1}{p} \).
therefore yielding (C.5).

For (C.6) we write \( \delta 1_{0,t}, \Pi_{t+r} - \delta 1_{0,t}, \Pi_t = \delta 1_{0,t}, (\Pi_{t+r} - \Pi_t) + \Pi_t (\delta 1_{0,t+r} - \delta 1_{0,t}) \) and use (C.7) for the pairs \( \Pi_\delta = \delta 1_{0,t+r} - \Pi_t \) and \( \delta 1_{0,t+r} - \Pi_t = \Pi = \Pi_t \). For the first pair we apply (C.9) to get

\[
E_\frac{r}{\tau} \left| \left( \left[ \delta 1_{0,t+r}, (\cdot) \right] \Pi_{t+r} - \Pi_t \right)_{\Lambda} \right|_{\Lambda}^q \lesssim \Lambda^{-1} \lambda^{1+\alpha} \sqrt{t + r^\alpha} E_\frac{r}{\tau} \left| \Pi_{t+r} - \Pi_t \right|_{\Lambda}^0 |p|_\pi.
\]

Plugging in (C.4) for \( \Pi_t \in \{ 0, t, \infty \} \) and proceeding as for (C.10) yields

\[
E_\frac{r}{\tau} \left| \left( \delta 1_{0,t+r}, (\cdot) \right) \Pi_{t} \right|_{\Lambda} (0) \left| \right|^q \lesssim \lambda^{-|\Pi_{t+r}|}\alpha \sqrt{t + r^{(\Pi_{t+r})\alpha}} |p|_\pi. \quad \text{(C.11)}
\]

For the second pair, abbreviating \( \delta 1_{\Pi_{t+r}} := \delta 1_{0,t+r} - \delta 1_{0,t} \), (C.7) implies

\[
E_\frac{r}{\tau} \left| \left( \delta 1_{0,t+r} - \delta 1_{0,t}, (\cdot) \right) \Pi_{t} \right|_{\Lambda} (0) \left| \right|^q \lesssim \int dx |\psi_\lambda (y)||\psi_\lambda|_{L^2} \left| E_\frac{r}{\tau} \left| \delta 1_{\Pi_{t+r}}, (y-x) - \delta 1_{\Pi_{t+r}}, (x) \right| \right|_{L^2}^{q} E_\frac{r}{\tau} \left| \Pi_{t} (0) \right|_{p}^{q}.
\]

We use the following estimate

\[
\left| E_\frac{r}{\tau} \left| \delta 1_{\Pi_{t+r}}, (y-x) - \delta 1_{\Pi_{t+r}}, (x) \right| \right|_{L^2}^{q} \lesssim |y|^{1-2\alpha} \lambda \sqrt{t + r^{\alpha}} |p|_\pi \text{ (C.13)}
\]

for every \( \alpha \in (0, \frac{1}{2}) \), which itself is an interpolation of the two estimates

\[
\left| E_\frac{r}{\tau} \left| \delta 1_{\Pi_{t+r}}, (y-x) - \delta 1_{\Pi_{t+r}}, (x) \right| \right|_{L^2}^{q} \lesssim |y|^{1-\beta} \sqrt{t + r^{\beta}} |p|_\pi, \quad \text{(C.14)}
\]

\[
\left| E_\frac{r}{\tau} \left| \delta 1_{\Pi_{t+r}}, (y-x) - \delta 1_{\Pi_{t+r}}, (x) \right| \right|_{L^2}^{q} \lesssim |r|^{1-\beta} \sqrt{t + r^{\beta}} |p|_\pi, \quad \text{(C.15)}
\]

for every \( \beta \in (0, 1) \). Estimate (C.14) follows along the same lines as (C.8) using the triangle inequality. For (C.13) using again the triangle inequality, translation invariance and the semigroup property in the form

\[
\delta 1_{0,t+r}(x) = \int dz e^{-mr} \tilde{H}_r (z) \delta 1_{0,t}(x-z) + \int_{t+r}^{t+r} ds H_{t-s} \delta 1_{t+s}(x, t+r) \]

where \( \tilde{H}_r \) stands for the massless heat kernel, we observe

\[
\left| E_\frac{r}{\tau} \left| \delta 1_{\Pi_{t+r}}, (y-x) - \delta 1_{\Pi_{t+r}}, (x) \right| \right|_{L^2}^{q} \lesssim \left| E_\frac{r}{\tau} \left| \delta 1_{0,t+r}, (-z) - \delta 1_{0,t}, \right| \right|_{L^2}^{q} + \left| e^{-mr} - 1 \right| \left| E_\frac{r}{\tau} \left| 1_{0,t+r}, \right| \right|_{L^2}^{q} + \left| E_\frac{r}{\tau} \left| 1_{0,t}, \right| \right|_{L^2}^{q} =: I_1 + I_2 + I_3.
\]

For \( I_1 \) using (C.8) we obtain

\[
\int dz H_r (z) \left| E_\frac{r}{\tau} \left| 1_{0,t+r}, (-z) - \delta 1_{0,t}, \right| \right|_{L^2}^{q} \lesssim \sqrt{r^{1-\beta}} \sqrt{r^\beta} |p|_\pi \lesssim \sqrt{r^{1-\beta}} \sqrt{t + r^{\beta}} |p|_\pi.
\]

\text{using $\beta = \alpha$ and $\frac{1}{1+\alpha}$}
To estimate $I_2$ we use Young’s inequality for convolution, the Cauchy–Schwarz inequality in the $s$-variable and the Hölder’s inequality again in the $s$-variable to treat the integral of the exponential yielding
\[
\left\| E^{\frac{t}{2}} |\delta 1_{0,t}|^q \right\|_{L^2_{x} L^{q}_{t}} \leq \int_0^t ds \left\| H_{t-s} \right\|_{L^1_{x}} \left\| E^{\frac{t}{2}} |\delta \xi|^q \right\|_{L^2_{x}} \leq \left( \int_0^t ds e^{-2m(t-s)} \right)^{\frac{1}{2}} \left( \int_0^t ds \left\| E^{\frac{t}{2}} |\delta \xi|^q \right\|_{L^2_{x}}^2 \right)^{\frac{1}{2}} \lesssim \frac{1}{\sqrt{m^{1-\beta}}} \sqrt{r^{\beta}} \mathcal{W} \tag{C.16}
\]
f for every $\beta \in [0,1)$. This in turn implies the estimate
\[
|e^{-mr} - 1| \left\| E^{\frac{t}{2}} |\delta 1_{0,t}|^q \right\|_{L^2_{x}} \lesssim |e^{-mr} - 1| \frac{1}{\sqrt{m^{1-\beta}}} \sqrt{r^{\beta}} \mathcal{W} \lesssim \sqrt{r^{1-\beta}} \sqrt{r^\beta} \mathcal{W}, \tag{C.17}
\]
where the implicit constant is uniform in $m$. To estimate $I_3$ we use (C.16) for $\beta = 0$ and a change of variables in $s$ which leads to
\[
\left\| E^{\frac{t}{2}} |\delta 1_{t+r}|^q \right\|_{L^2_{x}} \lesssim \sqrt{r} \mathcal{W} \lesssim \sqrt{r^{1-\beta}} \sqrt{r^\beta} \mathcal{W}.
\]
In total, (C.12) and (C.13) imply the estimate
\[
E^{\frac{t}{2}} \left| \left( \delta 1_{0,t+r} - \delta 1_{0,t} \right) \Pi_{\lambda} \right| \Lambda (0)^q \lesssim \Lambda^{-1} \lambda^{1-2\alpha} \sqrt{r^\alpha} \sqrt{r + r^\alpha} E^{\frac{t}{2}} |\Pi_{\lambda}(0)|^p \mathcal{W}.
\]
Plugging in (C.3) for $\Pi_{l} \in \{ 1_{0,t}, \nabla 1_{0,t} \}$ and proceeding as in (C.11) gives
\[
E^{\frac{t}{2}} \left| \left( \delta 1_{0,t+r} - \delta 1_{0,t} \right) \Pi_{\lambda} \right| \Lambda (0)^q \lesssim \lambda^{-(|\Pi|+2)\alpha} \sqrt{r^\alpha} \sqrt{r + r^{(|\Pi|+1)\alpha}} \mathcal{W}. \tag{C.18}
\]
Combining (C.11) and (C.18) implies (C.6).

Lemma C.1. For all $\alpha \in (0,1)$ the following estimate holds
\[
\int dz |H_{t-s}(z-y) - H_{t-s}(z)| \lesssim e^{-m(t-s)} |y|^\alpha \sqrt{t-s}^{-\alpha}.
\]

Proof. Interpolating the two estimates
\[
\int dz |H_{t-s}(z-y) - H_{t-s}(z)| \leq 2 \| H_{t-s} \|_{L^1_x} = 2e^{-m(t-s)}
\]
and
\[
\int dz |H_{t-s}(z-y) - H_{t-s}(z)| \leq \| \nabla H_{t-s} \|_{L^1_x} |y| \leq e^{-m(t-s)} \sqrt{t-s}^{-1} |y|
\]
yields the assertion.

D Estimates on the remainder

Lemma D.1. Let $\alpha > 0$ be sufficiently small. For every $p < \infty$
\[
\sup_{t \leq 1} \left\| v_{0,t} \right\|_{L^p_x} \leq C,
\]
where $C$ depends polynomially on $\sup_{t \leq 1} \| \nabla^k v_{0,t} \|_{C^{-\alpha}}$ for $k = 1, 2, 3$ and is uniform in the initial condition $f$. In particular, $C$ has finite moments of every order.
Proof. Follows from [35 Proposition 3.7]. The constant $c_{t, \infty}$ in Proposition [A.1] can be absorbed into the terms $\nabla_{0,t}$ and $\nabla_{t, 0}$, which together with Proposition [A.1] yield
\[
\sup_{t \leq 1} \| \nabla_{0,t} - c_{t, \infty} \| \leq \sup_{t \leq 1} \| \nabla_{0,t} \|_{C - \alpha}, \quad \sup_{t \leq 1} \| \nabla_{t, 0} - 3c_{t, \infty} I_{0,t} \| \leq \max_{k = 1, 3} \| \nabla_{0,t} \|_{C - \alpha},
\]
for any $\alpha' > 0$, allowing us to apply [35 Proposition 3.7].

Lemma D.2. Let $\alpha > 0$ be sufficiently small. Then for every $\kappa > 0$ sufficiently small the following estimate holds
\[
\sup_{t \leq 1} t^{\kappa + \varepsilon} \| \nabla_{0,t} \|_{\kappa} \leq C,
\]
where $C$ depends polynomially on $\sup_{t \leq 1} \| \nabla_{0,t} \|_{C - \alpha}$ for $k = 1, 2, 3$ and is uniform in the initial condition $f$.

Proof. The statement follows essentially from the proof of Lemma 5.1 for $s = \frac{1}{2}$ in [36] working with $I_{0,t}, \nabla_{0,t} - c_{t, \infty}, \nabla_{t, 0} - 3c_{t, \infty} I_{0,t}$ as we explained in the proof of Lemma D.1. The terms $I_0$ and $I_1$ in the notation of [36] proof of Lemma 5.1 can be ignored.

Lemma D.3. Let $\alpha > 0$ be sufficiently small. Then for any $\varepsilon > 0$ the following estimate holds
\[
\sup_{t \leq 1} t^{1 + \varepsilon} \| \nabla_{0,t} \|_{L^\infty} \leq C,
\]
where $C$ depends polynomially on $\sup_{t \leq 1} \| \nabla_{0,t} \|_{C - \alpha}$ for $k = 1, 2, 3$ and is uniform in the initial condition $f$.

Proof. To ease the notation we set $\eta := \max_{k = 1, 2, 3} \sup_{t \leq 1} \| \nabla_{0,t} \|_{C - \alpha}$. By Duhamel’s formula, we have
\[
\| \nabla_{0,t} \|_{L^\infty} \leq \| \nabla H_{t, \varepsilon} * \nabla_{0, \varepsilon} \|_{L^\infty} + \sum_{k = 0}^{3} \int_{\varepsilon}^{t} \| \nabla H_{t' - r} * (\nabla_{0,t'} \nabla_{0,t'}) \|_{L^\infty} dr
\]
\[
+ 3 \int_{\varepsilon}^{t} c_{r, \varepsilon} \| \nabla H_{t' - r} * (I_{r,t'} + \nabla_{0,t'}) \|_{L^\infty} dr.
\]
Note in the following that $D_{0, \varepsilon}(\mathbb{T}^d) = C^0 \rightarrow L^\infty$ continuously for any $\varepsilon > 0$. Then, first of all, by Young’s inequality and (D.1), we have
\[
\| \nabla H_{t, \varepsilon} * \nabla_{0, \varepsilon} \|_{L^\infty} \leq \| \nabla H_{t, \varepsilon} \|_{L^{\infty}} \| \nabla_{0, \varepsilon} \|_{L^\infty} \leq t^{\frac{1}{2}} t^{-\frac{1}{4}} t^{-\frac{1}{2}} = t^{-1 - \varepsilon}
\]
for $p$ large enough. In the same vain, using (D.1) and $p$ large enough yields
\[
\int_{\varepsilon}^{t} \| \nabla H_{t' - r} \|_{L^\infty} dr \leq \int_{\varepsilon}^{t} \| \nabla H_{t' - r} \|_{L^{p'}} \| \nabla_{0, \varepsilon} \|_{L^p} dr
\]
\[
\leq \int_{\varepsilon}^{t} (t - r)^{-\frac{1}{2}} r^{-\frac{3}{2}} dr \leq t^{-1 - \varepsilon}.
\]
Moreover, using the semigroup property of the heat kernel and Young’s inequality again we note
\[
\int_{\varepsilon}^{t} \| \nabla H_{t' - r} * \nabla_{0, \varepsilon} \|_{L^\infty} dr = \int_{\varepsilon}^{t} \| \nabla H_{t' - r} * \nabla_{0, \varepsilon} \|_{L^\infty} dr
\]
\[
\leq \int_{\varepsilon}^{t} \| \nabla H_{t' - r} \|_{L^\infty} \| \nabla_{0, \varepsilon} \|_{L^\infty} dr.
\]

Moreover, by Lemma B.2 we conclude
\[
\int_0^t \left\| \nabla H_{t-r} \ast \nabla \eta \right\|^2_{L^2_d} \, dr \lesssim \int_0^t (t-r)^{-\frac{1}{2}} (t-r)^{-a-\varepsilon} \, dr \lesssim \eta t^{-\frac{1}{2}a-\varepsilon}.
\]
Similarly, using (D.2) and Lemma B.2 we end up with
\[
\int_0^t \left\| \nabla H_{t-r} \ast \left( v_{0,r} \nabla V_{0,r} \right) \right\|^2_{L^2_d} \, dr \lesssim \int_0^t (t-r)^{-\frac{1}{2}} (t-r)^{-a-\varepsilon} \left\| v_{0,r} \right\|_{L^2_d} \left\| \nabla V_{0,r} \right\|_{L^2_d} \, dr \\
\lesssim \eta \int_0^t (t-r)^{-\frac{1}{2}} (t-r)^{-a-\varepsilon} r^{-1-2\alpha} \, dr \lesssim \eta t^{-\frac{1}{2}3a-\varepsilon}
\]
and in the same vain
\[
\int_0^t \left\| \nabla H_{t-r} \ast \left( v_{0,r} \nabla V_{0,r} \right) \right\|^2_{L^2_d} \, dr \lesssim \int_0^t (t-r)^{-\frac{1}{2}} (t-r)^{-a-\varepsilon} \left\| v_{0,r} \right\|_{L^2_d} \left\| \nabla V_{0,r} \right\|_{L^2_d} \, dr \\
\lesssim \eta \int_0^t (t-r)^{-\frac{1}{2}} (t-r)^{-a-\varepsilon} r^{-2\alpha} \, dr \lesssim \eta t^{-2\alpha-2\varepsilon}.
\]
Finally, using Lemma B.2, (D.2) and (A.1) we get
\[
\int_0^t \left\| \mathbf{1}_{(0,r]} + v_{0,r} \right\|^2_{L^2_d} \, dr \lesssim \int_0^t (t-r)^{-\frac{1}{2}} (t-r)^{-a-\varepsilon} r^{-\gamma} \left\| v_{0,r} \right\|_{L^2_d} \left\| \mathbf{1}_{(0,r]} \right\|_{L^2_d} \, dr \lesssim \eta t^{-2\alpha-2\varepsilon}.
\]

\[\square\]

## E Proof of the Bakry–Émery identity

In [34] it was proved that
\[
\mathcal{E}(F,F) := \int_{S(U^2)} \|DF\|^2_{L^2_d} \, dv
\]
where $F \in \mathcal{F}C^\infty_b$ is closable (see also [11]) and the closure gives rise to a quasi-regular Dirichlet form (cf. [24]), hence to a generator $\mathcal{L}$ with domain $D(\mathcal{L}) \subset D(\mathcal{E})$ such that
\[
\mathcal{E}(F,F) = -\int_{S(U^2)} F \mathcal{L} F \, dv.
\]
We denote by $\{\mathcal{P}_t\}_{t \geq 0}$ the associated semi-group. Then, by [34] Theorem 3.13], we infer that $\mathcal{P}_t F = \mathcal{P}_t F$ $\nu$-almost surely for all $F \in \mathcal{F}C^\infty_b$ and hence by continuity in time they are indistinguishable (see also [24] p. 67]). Moreover, by [34] Theorem 3.7] (and the discussion thereafter) $K := C_c(U^2) \subset L^2(U^2)$ is a dense and linear subspace consisting of $\nu$-admissible elements. Hence assumptions (C.1), (C.2) and (C.3) of [11] Section 4] are fulfilled. Moreover, $f \mapsto \mathcal{P}_t F(f)$ is quasi-continuous for any $F \in \mathcal{F}C^\infty_b$. Now we can prove Proposition 4.2.

**Proof of Proposition 4.2** Following [22] Proof of Theorem 1.1] we prove the $\nu$-a.s. identity
\[
\frac{d}{ds} \mathcal{P}_{t-s} (\mathcal{P}_s F)^2 = -2 \mathcal{P}_{t-s} \left( \|DF\|^2_{L^2_d} \right).
\]
and use the same notation. Let $0 \leq r_1, r_2 \leq t$ and define $H(r_1, r_2) := \mathcal{P}_{t-r_1} (\mathcal{P}_{r_2} F)^2$. 

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By [1] p. 364, Theorem 4.3] and since \( P_r F \in D(\mathcal{E}) \) it holds that
\[
P_r F(u'_t) - P_r F(f) = \int_0^t \mathcal{L}(P_r F)(u'_t) \, ds + M_r
\]
where \( M \) is a continuous martingale.

Moreover, by [1] p. 365, Proposition 4.5] the quadratic variation of \( M \) is given by
\[
(M)_r = \int_0^r \|DP_r F(u'_t)\|^2_{L_2} \, ds
\]
Then by Itô’s formula [32, p. 222, Theorem 3.3] we compute
\[
(P_r F)^2(u'_t) = (P_r F)^2(f) + 2 \int_0^t P_r F(u'_t) \, dM_s + 2 \int_0^t P_r F(u'_t) \mathcal{L}(P_r F)(u'_t) \, ds + 2 \int_0^t \|DP_r F(u'_t)\|^2_{L_2} \, ds
\]
and hence
\[
P_{r_1-r_2}(P_r F)^2(f) = (P_r F)^2(f) + 2 \int_0^{r_1-r_2} P_r(F \mathcal{L}P_r F)(f) \, ds + 2 \int_0^{r_1-r_2} \|DP_r F\|^2_{L_2} (f) \, ds.
\]
Then we see that
\[
\frac{\partial}{\partial r_1} P_{r_1-r_2}(P_r F)^2(f) = -2P_{r_1-r_2}(P_r F \mathcal{L}P_r F)(f) - 2P_{r_1-r_2}\|DP_r F\|^2_{L_2} (f)
\]
and on the other hand we have
\[
\frac{\partial}{\partial r_2} P_{r_1-r_2}(P_r F)^2(f) = 2P_{r_1-r_2}(P_r F \mathcal{L}P_r F)(f).
\]
Continuity follows in the same vain as in [23 Proof of Theorem 1.1]. Finally, we have
\[
\frac{d}{ds} P_{r-r_2}(P_r F)^2 = \frac{\partial}{\partial r_1} P_{r_1-r_2}(P_r F)^2(f) \bigg|_{r_1=r_2=s} + \frac{\partial}{\partial r_2} P_{r_1-r_2}(P_r F)^2(f) \bigg|_{r_1=r_2=s}
\]
\[
= -2P_{r_1-r_2}\|DP_r F\|^2_{L_2} (f).
\]
Integrating from 0 to \( t \) proves the claim.

\[\square\]

F Differentiability with respect to the initial data

We set
\[
G(f,v)(t) := S(t)f + \int_0^t S(t-s)F(v_s) \, ds - v_t
\]
where \( F(v_t) := -\left(v^3 + 3v^2\mathbf{1}_I + 3v\gamma \mathbf{1}_I + \mathbf{1}_I - c_{t,m}(v_t + \mathbf{1}_I)\right) \). By [33] Theorem 3.9] there exist fixed parameters \( \gamma, \beta > 0 \) such that for any \( f^* \in C^{-\infty}_0 \) and \( T > 0 \) we can find a unique solution \( v^* \) to (2.4] satisfying \( G(f^*, v^*)(t) = 0 \), for every \( 0 \leq t \leq T \), and \( \sup_{0 \leq t \leq T} \|v^*_t\|_B < \infty \). We define
\[
X := \left\{ f \in C^{-\infty}_0 : \|f\|_{-\infty} \leq R \right\}, \quad Y := \left\{ v: [0,T] \to C^\beta : \sup_{0 \leq t \leq T} t^\beta \|v_t\|_B \leq 1 \right\}
\]
for some $T^*$ to be chosen below. Then again by [35, Theorem 3.9] we know that $G(f^*, v^*)$ is Frechét-differentiable and we have

$$G_v(f^*, v^*) = \int_0^t S(t-s) (F'(v_s) \delta v_s) ds =: (K - Id) \delta v_t$$

where $F'(v_s) \delta v_s := -3 (v_s^2 + 2v_s I_t + V_t - \epsilon_t, v_s) \delta v_s$. A simple calculation shows that

$$\| K \delta v_t \|_\beta \lesssim \int_0^t (t-s)^{-\alpha/2} \| \delta v_s \|_\beta ds \lesssim (T \wedge T^*)^{1-\alpha/2} \sup_{0 \leq t \leq T^*} t^\gamma \| \delta v_t \|_\beta.$$ 

Choosing $T^*$ small enough such that the r.h.s. above is strictly smaller than 1 we get by the Neumann-series criterion that $G_v(f^*, v^*): Y \to Y$ is a bijection. Hence by [38, Theorem 4.E] we get that $f \mapsto v^f$ is differentiable and its derivative in $h$ is a mild solution to (3.6) on $(0, T \wedge T^*)$. Concatenating this argument to cover the whole time interval $(0, T)$ proves the assertion.

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