SYMMETRY UNDER $\alpha \to \alpha + 1$ IS FORBIDDEN
BY HELICITY CONSERVATION

by

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Abstract

The question as to whether helicity conservation in spin one-half Aharonov Bohm scattering is sufficient in itself to determine uniquely the form of the spinor wave function near the origin is examined. Although it is found that a one parameter family of solutions is compatible with this conservation law, there must nonetheless be singular solutions which break the symmetry $\alpha \to \alpha + 1$ required for an anyon interpretation. Thus the free parameter which occurs does not allow one to eliminate the singular solutions even though it does in principle mean that they can be transferred at will between the spin up and spin down configurations.
I. Introduction

The study of spin one-half Aharonov Bohm (AB) scattering has led to a growing realization that spin can play a decisive role in this phenomenon. Gerbert\textsuperscript{1} examined this problem from a somewhat mathematical point of view, namely the method of self-adjoint extensions. He considered the spin up case and noted in the one sector which allowed a normalizable but singular solution that it was possible in principle to have a one parameter family of solutions. The value of this parameter prescribes the relative contributions of the regular and irregular solutions (i.e., $J_{±|m+α|}$ respectively). This problem was subsequently considered by this author\textsuperscript{2} who sought to reformulate the problem as a limit of a fully realizable physical configuration. It consisted in distributing the magnetic flux throughout a cylinder of radius $R$ which was allowed to go to zero at the end of the calculation. In contrast to ref. 1 both the spin up and spin down cases were considered, thereby allowing one to obtain results for polarized beam experiments.

The results of ref. 2 can be summarized without detailed mathematics. It was shown under quite general conditions that for a configuration in which the magnetic moment interaction is attractive (repulsive) the solution consists entirely of a singular (nonsingular) function. Not surprisingly, this result agreed with that of Gerbert provided that a specific value was chosen for his free parameter. Similar results were obtained by Alford et al.\textsuperscript{3} for a single spin projection and under somewhat more restrictive conditions.

There has been considerable reluctance to accept such results without reservation since they are not reconcilable to the anyon view. This is easily seen by a simple argument. The singular solutions can occur only when $αs < 0$ where $s$ is twice the spin projection and $α$ is the flux parameter. In this case the sign of the phase shift is precisely reversed\textsuperscript{2} relative to the spinless result. Since the condition $αs < 0$ is not invariant under the anyonic displacement $α → α + 1$, one clearly has a serious clash between these results and the anyon interpretation.

To bolster the latter approach various things have been proposed. One of these\textsuperscript{4} sought
to use a highly singular nongauge potential to keep the particle wave function away from the \( r < R \) region so that its magnetic moment interaction could not affect the solution. This, of course, can be done and is a perfectly comprehensible quantum mechanical result although one must do the calculation carefully\(^5\) if Klein’s paradox is to be avoided.

Another attempt to circumvent the singular solutions issue makes use of an appeal to helicity conservation. It is well known that helicity is conserved for a Dirac particle in a time independent magnetic field. Indeed, the solution of ref. 2 has been shown\(^6\) to be fully compatible with this principle. On the other hand it is not unreasonable to ask (as has recently been done\(^7\)) whether the converse is true. Namely, does helicity conservation uniquely imply the solution of ref. 2? That work (i.e., ref. 7) concluded that there is in fact a one parameter family of solutions (much as in ref. 1 for the case of a single spin component) and that the symmetry \( \alpha \rightarrow \alpha + 1 \) could thereby be retained.

This question is reexamined in the present work. Before presenting the details it may be well to note at the outset that the conclusion of ref. 7 concerning \( \alpha \rightarrow \alpha + 1 \) is certainly and obviously incorrect. If it were true, then each spin component would have exactly the same scattering amplitude which would, of course, be equal to the standard spinless AB amplitude. That clearly implies an absence of spin rotation during the scattering process, and that helicity cannot be conserved.

In the following section the spin one-half AB problem is briefly reviewed and a two parameter family of solutions derived with no assumptions made concerning helicity conservation. The subsequent section uses a direct calculation of a cross section in a hypothetical scattering experiment to infer the existence of a relation between these two parameters when helicity conservation is required. A concluding section summarizes the results obtained and offers some general comments on the spin one-half AB problem.
II. The General Scattering Amplitude

One starts with the Dirac equation in two spatial dimensions

\[ E\psi = [M\beta + \beta\gamma \cdot \Pi] \psi \]  \hspace{1cm} (2.1)

where \( \Pi_i = -i\partial_i - eA_i \) and

\[ eA_i = \alpha\epsilon_{ij}r_j/r^2 \ . \]

Since both spin up and spin down components are to be included, a convenient choice for the Dirac matrices is given by

\[ \beta = \sigma_3 \]
\[ \beta\gamma_i = (\sigma_1, s\sigma_2) \]  \hspace{1cm} (2.2)

where \( s = \pm 1 \) (for spin up and spin down) and \( \sigma_i \) are the usual Pauli matrices. When it is desired to display (2.1) in four-dimensional form, \( s \) should be replaced by \( \rho_3 \) (namely, the third Pauli matrix which satisfies \( [\rho_3, \sigma_i] = 0 \)). In order to be able to interpret in a physical context the plausibility of the results of this study the second order form of (2.1) is quite useful. One finds

\[ (E^2 - M^2)\psi = \left[ \Pi^2 + \alpha s\sigma_3 \frac{1}{r} \delta(r) \right] \psi \]

or (in cylindrical coordinates)

\[ \left[ \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \left( \frac{\partial}{\partial \phi} + i\alpha \right)^2 + k^2 - \alpha s\sigma_3 \frac{1}{r} \delta(r) \right] \psi = 0 \]

where

\[ k^2 \equiv E^2 - M^2 \ . \]

It should be noted that with the choice (2.2) for the matrix \( \beta \) it is readily seen that the physical (or large component) is \( \psi_1 \). Thus upon expanding \( \psi_1 \) as

\[ \psi_1 = \sum_{-\infty}^{\infty} g_m(r)e^{im\phi} \]
it follows that

\[
\left\{ \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + k^2 - \frac{(m + \alpha)^2}{r^2} - \alpha s \frac{1}{r} \delta(r) \right\} g_m(r) = 0 \quad .
\] (2.3)

Clearly the delta function term in (2.3) can be interpreted as a potential which is repulsive (attractive) for \( \alpha s \) greater (less) than zero. Thus the result of ref. 2 which found a strictly regular (irregular) solution near the origin in the \( R \to 0 \) limit in those cases was quite reasonable. In the current context, however, one proceeds in the spirit of refs. 1 and 7.

Since the solutions of (2.3) must be normalizable at the origin, it follows that the allowable solutions must be \( J_{|m+\alpha|}(kr) \) except possibly, when \( |m + \alpha| < 1 \). Restricting attention to this latter case one has as the most general solution for the angular momentum state \( j = m + \frac{1}{2} s \)

\[
\psi_j = e^{-i(N+\frac{1}{2})\phi} e^{-is\phi/2} \left[ A_s J_s(\beta - \frac{1}{2}) - \frac{1}{2} + B_s J_{s}^{\frac{3}{2} - s}(-\beta - \frac{1}{2}) \right]
\] (2.4)

where \( \alpha \equiv N + \beta \) with \( N \) the largest integer in \( \alpha \). Since this implies that \( |\beta - \frac{1}{2}| \leq \frac{1}{2} \), it follows that \( A_s \) is the coefficient of the singular term and \( B_s \) the coefficient of the regular one. It is not difficult to verify that by leaving the ratio

\[
B_s/A_s \equiv \tan \mu_s \quad \left( -\frac{\pi}{2} \leq \mu_s \leq \frac{\pi}{2} \right)
\]

arbitrary and unspecified one is accommodating all the results of the self-adjoint extension approach.

One now follows standard scattering theory. Upon equating the coefficients of \( e^{-ikr} \) terms in (2.4) and a plane wave of the form \( e^{-ikr\cos\phi} \) (i.e., incident from the right), one evaluates \( A_s \). The scattering amplitude is then the difference between the coefficients of the asymptotic limits of the \( e^{ikr/r^\frac{1}{2}} \) term in (2.4) and the plane wave. One thus obtains for this amplitude

\[
f_j = (2\pi ik)^{-\frac{1}{2}} e^{i(\pi - \phi)[N+\frac{1}{2}+\frac{s}{2}]} \left\{ e^{2i\delta_s} - 1 \right\}
\] (2.5)
where
\[
e^{2i\delta_s} = e^{-i\pi[N + \frac{1}{2} + s/2]} \frac{\exp\left\{ -i\frac{\pi}{2} \left[ s\left( \beta - \frac{1}{2} \right) - \frac{1}{2} \right] \right\} + \tan \mu_s \exp\left\{ i\frac{\pi}{2} \left[ s\left( \beta - \frac{1}{2} \right) - \frac{1}{2} \right] \right\}}{\exp\left\{ i\frac{\pi}{2} \left[ s\left( \beta - \frac{1}{2} \right) - \frac{1}{2} \right] \right\} + \tan \mu_s \exp\left\{ -i\frac{\pi}{2} \left[ s\left( \beta - \frac{1}{2} \right) - \frac{1}{2} \right] \right\}} . \tag{2.6}
\]

It is worth emphasizing that one has a two parameter \((\mu_+ \text{ and } \mu_-)\) family of solutions. Using the step function \(\theta(x) \equiv \frac{1}{2} \left( 1 + \frac{x}{|x|} \right)\) contact with results of ref. 2 is made by taking \(\mu_s = \frac{\pi}{2} \theta(\alpha s)\) which is seen to imply for (2.5) the form
\[
e^{2i\delta_s} = e^{i\pi|\alpha|}
\]
which is (remarkably) independent of the spin parameter \(s\). This means that insofar as spin is concerned the entire amplitude is described by the factor \(\exp\left\{ i\left( \pi - \phi \right)s/2 \right\}\). This is in fact the matrix appropriate to a rotation by \(\pi - \phi\) (i.e., the scattering angle) which is necessary to yield a solution consistent with helicity conservation. As yet unresolved is the question as to whether there are other values of \(\mu_s\) which are consistent with this conservation law. It is to this issue that attention is now directed.

### III. The Helicity Constraint

In order to determine the constraints placed upon the amplitude (2.5) when helicity conservation is required it is useful to consider an idealized experiment. For the sake of simplicity it is prescribed by requiring that the incoming beam be filtered in such a way as to leave only the orbital angular momentum for which \(|m + \alpha| < 1\). The incoming beam is assumed to be totally polarized along the direction of the unit vector \(n\) in the plane and the detector is set up to accept only events along a second direction \(n'\), also in the scattering plane.

A convenient tool for this purpose is the projection operator
\[
P_n = \frac{1}{2} \left( 1 + \rho \cdot n \right)
\]
where \(\rho_1\) and \(\rho_2\) are the Pauli matrices which act in the spin space of the system. This allows one to write for the cross section
\[
\sigma = \text{Tr} \ P_n^* f P_n f^* . \tag{3.1}
\]
Now helicity conservation must imply that all scattering events which arrive at the detector will be counted provided that \( \mathbf{n}' \) is a vector which is rotated by an angle of \( \pi - \phi \) (i.e., the scattering angle) relative to \( \mathbf{n} \). On the other hand the factor

\[
\exp[i(\pi - \phi)]\rho_3/2
\]

which occurs in the scattering amplitude is easily seen to have only the effect of “rotating back” this same vector. Thus (3.1) becomes for this choice

\[2\pi k\sigma = \text{Tr} \ P_\mathbf{n} \left[ e^{2i\delta_s} - 1 \right] P_\mathbf{n} \left[ e^{-2i\delta_s} - 1 \right].\] (3.2)

Even more useful now is to invoke a detector which accepts only helicity violating (i.e., spin flip) events. This has the effect of replacing one of the \( P_\mathbf{n} \)'s in (3.2) by \( P_{-\mathbf{n}} \). One completes the exercise by setting

\[e^{2i\delta_s} - 1 = a + b\rho_3\]

where \( a \) and \( b \) are to be determined from Eq. (2.6) and by requiring that no helicity violating events occur. One finds that

\[2\pi k\sigma = |b|^2\]

for this process which thus imposes the requirement that the matrix \( e^{2i\delta_s} \) is proportional to the unit matrix. It is worth noting that the choice \( \mu_+ = \mu_- \) is not compatible with this condition. If it were then one would have the possibility of having regular solutions for both spin projections and thus at least one choice that would satisfy the anyon symmetry \( \alpha \rightarrow \alpha + 1 \). The only solution for arbitrary flux is \( \mu_+ = \mu_- + \pi/2 \). Upon adopting this condition and setting \( \mu_+ = \mu \) Eq. (2.5) becomes

\[f_j = (2\pi ik)^{-\frac{1}{2}} e^{i(\pi - \phi)[N+\frac{1}{2}+s]/2} \left[ e^{2i\delta} - 1 \right]\]

where

\[\tan \delta = \frac{-1 + (-1)^{N+1} \tan \mu}{1 + (-1)^{N+1} \tan \mu \tan \frac{\alpha \pi}{2}}.\]
The choice $\mu = \frac{\pi}{2} \theta(\alpha)$ corresponds to the case considered in ref. 2. Since it was based on a physical limiting process one calls it the physical scattering amplitude $f_p$. Upon summing over all partial waves it is seen to have the form

$$f_p = \left( \frac{i}{2\pi k} \right)^{1/2} \frac{\sin \pi |\alpha|}{\cos \phi / 2} e^{i(\pi-\phi)[N+\frac{1}{2}+s/2]} e^{i\phi \epsilon(\alpha)/2} .$$

where $\epsilon(\alpha)$ is the alternating function. The opposite choice $\mu = \frac{\pi}{2} \theta(-\alpha)$ corresponds to the “antiphysical” scattering amplitude

$$f_{\bar{p}} = -\left( \frac{i}{2\pi k} \right)^{1/2} \frac{\sin \pi |\alpha|}{\cos \phi / 2} e^{i(\pi-\phi)[N+\frac{1}{2}+s/2]} e^{-i\phi \epsilon(\alpha)/2} .$$

It is “antiphysical” in the sense that a repulsive delta function interaction corresponds to a singular wave function while an attractive one implies a regular solution. It is of some interest to observe, however, that $f_p$ and $f_{\bar{p}}$ imply that all experiments (even those using polarized beams) cannot distinguish between these two amplitudes. This could only be done if an interference using a non-AB interaction term could be made sensitive to the $e^{\pm i\phi \epsilon(\alpha)/2}$ factor. Finally, it is to be noted that in the most general case the total scattering amplitude can also readily be obtained with the result

$$f = -(2\pi ki)^{-\frac{1}{2}} \left[ \frac{\cos(\phi/2 - \pi \alpha)}{\cos \phi / 2} \right] e^{i(\pi-\phi)[N+\frac{1}{2}+s/2]}$$

where $\delta$ is given by (3.3).

**IV. Conclusion**

It is well to discuss with some care the precise sense in which the $\alpha \to \alpha + 1$ anyon symmetry fails in the context of this study. One should keep in mind that that symmetry is a consequence of the fact that the partial wave differential equation depends in the spinless case only on the combination $(m + \alpha)^2$. As has been realized, however, the corresponding spin one-half second order differential equation has a delta function potential which is essentially a Zeeman interaction. Its coefficient is proportional to $\alpha s$, a term which clearly
breaks the $\alpha \to \alpha + 1$ symmetry. This leads in the physical $R$ limiting model of ref. 2 to the existence of a singular solution when $\alpha s < 0$. From the work of refs. 1 and 7 one knows, of course, that if one follows the self-adjoint extension method any single spin component can (at least mathematically) be required to have only a regular solution. In the absence of a helicity conservation principle this can even be done to both components. Once helicity conservation is invoked one is free to constrain only one of the two spin values.

The model of ref. 2 yields an amplitude $f_p$ which had its origin in a solution which was regular (irregular) for $\alpha s > 0 (\alpha s < 0)$. However, in the self-adjoint extension approach one can exactly reverse this situation to obtain $f_p$ which came from a wave function which (oddly) has greater concentration at the origin for a repulsive Zeeman interaction than for an attractive one. However, in neither case is the singular solution avoided. It is merely shifted at will from one spin component to another.

It has been shown in some detail in ref. 7 that helicity conservation does not preclude a one parameter family of extensions. That result has also emerged in the current study from a somewhat different perspective. However the claim of ref. 7 that the symmetry $\alpha \to \alpha + 1$ could thus be preserved in the self-adjoint extension approach has been seen to be both mathematically and physically untenable. (In fact no precise argument for this conclusion is spelled out in that work). In view of the considerable interest in the anyonic interpretation in recent years is is of some importance that this crucial matter not remain uncorrected.

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