One-dimensional numerical simulation of cavitation surge in pumping system considering cavity response delay

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Abstract. Cavitation instabilities such as rotating cavitation and cavitation surge often occur in high speed turbopumps. In the present study, numerical simulation of cavitation surge, which numerically solves the one-dimensional momentum and continuity equations with modelled dynamic cavitation characteristics, is conducted. The phase lag of cavity response against the inlet pressure and the suction flow rate variations is modeled in the form of the first-order lag system considering that the frequency of cavitation surge is small. Differently from linear stability analysis made in our previous study, the equations are solved in time domain with retaining some of non-linear terms. The method is validated through the comparison with the stability analysis. Then the effect of phase lags in dynamic cavitation characteristics is examined.

1. Introduction

It is well known that the cavitation instabilities such as rotating cavitation and cavitation surge often occur in high-speed pumps. These cavitation instabilities were found in 1950’s-1970’s (for example, [1]), but they have been still of significant concern in the development of turbopump inducers for modern liquid-propellant rocket engines [2]-[5], since the occurrence of instabilities possibly leads to operational problems in fuel and oxidizer feeding system.

In theoretical point of view, the cavitation instabilities have been discussed based on quasi-steady response characteristics of cavity volume against the suction flow rate and the inlet pressure variations, which are respectively called mass flow gain factor $M$ and cavitation compliance $K$. It was found that the root cause of the instabilities is the positive mass flow gain factor ($M > 0$), the increase of cavity volume against the decrease of flow rate [6], [7]. However, the criterion based on quasi-steady response is not always very accurate for the prediction of the onset point of cavitation surge.

In order to improve the accuracy of onset point prediction, the delay of cavity response was taken into account in a linear stability analysis [11]. The delay has been found both experimentally [8] and theoretically [9], [10]. In some cases phase advance is observed. The stability analysis shows that even small phase lag in the cavitation compliance can drastically stabilize the system, and that inclusion of the delay significantly improves the accuracy of onset point prediction.

The purpose of the present study is to confirm the stabilizing effect of the delay by numerical simulation. The one-dimensional momentum and continuity equations with modeled dynamic cavitation characteristics, are solved numerically in time domain with retaining some of non-linear terms. The method is validated through comparison with the stability analysis. Then it is extended to the case of large amplitude oscillation to discuss about the factors limiting the amplitude.
2. Numerical method

2.1. Analytical model and fundamental equations
We consider a two-dimensional flat-plate cascade with the chord length of $C$, the spacing of $h$ and the stagger angle of $\gamma$ connected to a suction pipe with the finite length of $L_1$ as shown in figure 1. The model is basically similar to that used in our previous study [11]. For simplicity, the flow rate downstream of the cascade is set to be constant. Considering the boundary condition of the constant total pressure at the inlet of the suction pipe, we can express the one-dimensional momentum and continuity equations as follows in the non-dimensional form.

\[ p_{\infty*} - p_{\in*} + \phi_1^2 = \zeta_1 \phi_1^2 = 2L_1^* \frac{d\phi_1}{dt^*} \quad (1) \]

\[ \phi_2 - \phi_1 = \frac{dV_c^*}{dt^*} \quad (2) \]

where the all variables have been normalized by the cascade moving speed $U_T$ which corresponds to the rotating speed of impeller in rotor case, the characteristic time of $h/U_T$ and the characteristic pressure of $\rho U_T^2/2$ ($\rho$ is the density of working liquid); $\phi_1$ and $\phi_2$ are the suction and delivery flow coefficients, $p_{\infty*}$ and $p_{\in*}$ are the normalized upstream total pressure and static pressure at the inlet of cascade, and $\zeta_1$ denotes the loss coefficient of the suction pipeline.

\[ \frac{p_{\infty*}}{p_{\in*}} = \text{const.} \]

\[ \phi_1 \leftrightarrow \phi_2 \quad \text{const.} \]

\[ L_1 \]

\[ \text{Figure 1. Analytical model.} \]

2.2. Modelling of dynamic cavitation characteristics
It is known that the dynamic response of cavity volume shows phase lag against the enforced fluctuations of suction flow rate and inlet pressure [8]-[10]. Considering the fact that the frequency of cavitation surge we focus here is generally small, we assume that the cavity volume responds against the fluctuations in the form of the first order time lag system as follows.

\[ T_M^* \frac{dV_c^*}{dt^*} + V_c^* = -M_0 \phi_1, \quad T_K^* \frac{dV_c^*}{dt^*} + V_c^* = -K_0 p_{\in*} \]

where $M_0$ and $K_0$ are quasi-steady mass flow gain factor and cavitation compliance respectively, and $T_M^*$ and $T_K^*$ are the first order time-lags of the cavity volume change against the suction flow rate and the inlet pressure fluctuations, $\phi_1$ and $p_{\in*}$, respectively. The above two equations are derived against the forced fluctuation of one of $\phi_1$ and $p_{\in*}$ while keeping the other to be constant. Combining the two equations and considering the initial value of $V_c^*(0)$, $\phi_1(0)$ and $p_{\in*}(0)$, we obtain;
\[ T_M^* T_K^* \frac{d^2 V_c^*}{dt^2} + (T_M^* + T_K^*) \frac{dV_c^*}{dt^*} + [V_c^* - V_c^*(0)] \\
= -M_0 \left(T_K^* \frac{d}{dt^*} + 1\right) [\phi_1 - \phi_1(0)] \\
- K_0 \left(T_M^* \frac{d}{dt^*} + 1\right) [p_{in^*} - p_{in^*}(0)] \]  

(3)

By numerical time-integrations of equations (1)-(3), we can obtain \( \phi_1, p_{in^*}\) and \( V_c^*\) in time domain. To do so, the parameters of dynamic cavity response of \( M_0, K_0, T_M^* \) and \( T_K^*\) should be provided. In the present study, we employ the results of the free-streamline theory applied for partial cavitating flow of two-dimensional flat plate cascade [10]. According to the theory, both the mass flow gain factor and the cavitation compliance are functions of \( \sigma/2\alpha\), where \( \sigma\) and \( \alpha\) are the cavitation number and the angle of attack, and therefore \( M_0, K_0, T_M^* \) and \( T_K^*\) can be expressed by

\[ M_0 = M_0(\sigma/2\alpha), \quad K_0 = K_0(\sigma/2\alpha), \quad T_M^* = T_M^*(\sigma/2\alpha), \quad T_K^* = T_K^*(\sigma/2\alpha) \]  

(4)

The cavitation number \( \sigma\) and the angle of attack \( \alpha\) are calculated by the following equations

\[ \sigma = \frac{p_{in^*} - p_v}{\rho W_1^2/2} = \left(\frac{U_T}{W_1}\right)^2 (p_{in^*} - p_v^*) = \frac{1}{1 + \phi_1^2} (p_{in^*} - p_v^*) \]

\[ \alpha = \tan^{-1}(1/\phi_1) - \gamma \]

2.3. Analytical condition

For the test case of simulation, the solidity and the stagger of flat plate cascade are set as \( C/h = 2.0\) and \( \gamma = 79.0^\circ\) respectively. By the free streamline theory, the dynamic response characteristics of cavity volume, i.e. \( M_0, K_0, T_M^* \) and \( T_K^*\), are obtained as shown in figure 2.

![Figure 2](image-url)  

(a) \( M_0 \) and \( K_0 \)  
(b) \( T_M^* \) and \( T_K^* \)

Figure 2. Dynamic response characteristics of cavity volume in two-dimensional flat plate cascade

It is worth noting that the time lag in cavitation compliance \( T_K^*\) is negative roughly in \( 0.2 < \sigma/2\alpha < 1.6\). According to the stability analysis made in our previous study [11], the time lag \( T_K^*\) behaves as a positive damping to the flow system, while the mass flow gain factor \( M_0\) does as a negative damping, and the stability criteria of the system, i.e. the onset condition of cavitation surge, has been
derived as;

\[ \frac{M_0}{K_0} - \frac{T_K^*}{K_0} \geq 2\left(1 + \zeta_1\right)\phi_1 \]

(5)

To illustrate this condition more clearly, we plot the left-hand side (LHS) of this equation against \( \sigma/2\alpha \) by a blue solid curve in figure 3. It is seen that LHS, i.e. the negative damping of the flow system due to cavitation, is larger than RHS in \( \sigma/2\alpha < 3.0 \), suggesting that the onset point is \( \sigma/2\alpha = 3.0 \). Without the delay, the plot of \( M_0/K_0 \) shows that the negative damping occurs even at the largest \( \sigma/2\alpha \) of the plot, showing that the quasi-steady analysis fails to predict the onset point.

![Figure 3](image)

**Figure 3.** Negative damping of flow system due to cavitation, LHS of equation (5) against \( \sigma/2\alpha \)

In the present numerical simulation, the parameters of suction pipe are set as the length of \( L_1^* = 20.0 \) and the loss coefficient of \( \zeta_1 = 10.0 \). We set the initial flow rate of \( \phi_1 = 0.105 \) with the attack angle of \( \alpha = 5.0^\circ \). This condition gives the value of right-hand side of equation (5), friction damping of the flow system, as 2.31 as indicated by a red line in figure 3. We mainly focus on three cases (i)-(iii), listed in table 1. The total damping can be assessed by RHS minus LHS of equation (5), which indicates that cases (i)-(iii) correspond to the total damping with positive, small positive and large negative values respectively.

| Case | \( \sigma/2\alpha \) | \( M_0 \) | \( K_0 \) | \( T_M^* \) | \( T_K^* \) | Total damping RHS-LHS of Eq.(5) |
|------|----------------------|----------|----------|-------------|-------------|-----------------------------|
| (i)  | 4.12                 | 0.646    | 0.051    | 0.851       | 0.769       | 4.72                        |
| (ii) | 2.68                 | 0.810    | 0.074    | 0.884       | 0.692       | 0.72                        |
| (iii)| 1.53                 | 0.673    | 0.038    | 0.601       | -0.180      | -20.14                      |

### 3. Results and discussion

#### 3.1. Stability of flow system

Figure 4 shows the time histories of \( \phi_1, V_c^* \) and \( p_{in}^* \) in cases (i)-(iii). In these simulations, \( M_0, K_0, T_M^* \) and \( T_K^* \) are kept constant as listed in table 1, and the small initial disturbance is given to see the damping characteristics of the variables. In case (i), all variables clearly present damping oscillations. In case (ii), the fluctuations of the variables is gradually decaying. On the other hand, in case (iii) the amplitudes of fluctuations of the variables rapidly increases. These observations agree well with the indication of the total damping of the flow system shown in table 1. In addition, table 2 summarizes the frequencies of the oscillations calculated from figure 4 along with those predicted by the stability analysis [11]. It is
seen that the both frequencies agree well with each other, meaning that the fluctuation characteristics with small amplitude can be well simulated by the present formulations, equations (1)-(3).

![Figure 4. Time histories of $\phi$, $V^*$, and $p_{in}$ in cases (i)-(iii)](image)

| Case | $\sigma / 2\alpha$ | Present simulation | Stability analysis [8] |
|------|-------------------|--------------------|------------------------|
| (i)  | 4.12              | 0.110              | 0.111                  |
| (ii) | 2.68              | 0.090              | 0.0918                 |
| (iii)| 1.53              | 0.112              | 0.114                  |
To see the effect of time lag of cavity volume against the inlet pressure change $T_K^*$, figure 5 shows the comparisons of time histories of the variables between in case (ii) and in case (ii)' with neglected $T_K^*$. It is clear that by neglecting the time lag $T_K^*$ (see dashed curves in the figure), the all variables show the amplifying fluctuations, confirming that the time lag $T_K^*$ has a damping effect against the flow system, which agree well with the onset criteria of cavitation surge expressed by equation (5).

![Figure 5. Comparisons of time histories of $\phi_1$, $V_c^*$ and $p_{in}^*$ between with and without time lag $T_K^*$](image)

**Figure 5.** Comparisons of time histories of $\phi_1$, $V_c^*$ and $p_{in}^*$ between with and without time lag $T_K^*$ (Solid curves: case (ii), and dashed curves: case (ii)' with $T_K^* = 0$)

### 3.2. Nonlinear simulation

In order to see the non-linear effect especially in equation (3), we attempt to continue to calculate case (iii). In this calculation, the cavitation characteristics of $M_0$, $K_0$, $T_M^*$ and $T_K^*$ are updated at each time step by referring to the instantaneous value of $\sigma/2\alpha$. Since the solution with Eq.(3) diverged near cavity collapse, probably due to the small values of $M_0$, $K_0$, $T_M^*$ and $T_K^*$ at large $\sigma/2\alpha$, we replace the equation (3) by the following equation when $V_c^*$ is small.

$$\frac{d^2V_c^*}{dt^2} = kp_g^* = \frac{k'}{V_c^{n*}}$$

(6)

where $p_g^*$ is a normalized partial pressure of non-condensable gas, $n$ is a polytropic index, and $k$ and $k'$ are real constants. The basic idea behind this equation is that non-condensable gas may have significant effect at the collapse of cavity as it is in a single bubble spherical bubble dynamics showing rebounding of bubble.

Figure 6 shows the example of this calculation. We employ iso-thermal change of non-condensable gas ($n=1$). Only when $V_c^*$ is smaller than 0.001, the equation (6) with $k' = 1.0 \times 10^{-5}$ is used instead of equation (3), while we set the values for the constants with no reason. In the figure, we clearly see the occurrence of large amplitude cavitation surge with the waveforms somewhat similar to those observed in experiments. Strong pressure pulse in $p_{in}^*$ with the sudden decrease of flow rate $\phi_1$ occurs at the collapse of cavity volume $V_c^*$. The amplitude of oscillations looks to be set by the conditions that the cavity volume becomes nearly zero at the pressure pulse and that the minimum pressure is limited by the vapor pressure, the latter being caused by the large values of cavitation compliance at smaller values of $\sigma/2\alpha$, as shown in Fig.2(a). Note that the frequency of the fluctuations can be calculated as 0.033 which is far below 0.112 predicted by the linear analysis. Therefore, this kind of nonlinear analysis is necessary to simulate large amplitude cavitation surge for the prediction of amplitude and frequency, although more elaborate examination should be made on the collapsing model of Eq.(6).
4. Conclusions

In the present study, numerical simulation of cavitation surge, which numerically solves the one-dimensional momentum and continuity equations with modelled dynamic cavitation characteristics in time domain, is proposed. The results are compared with the previous stability analysis in the cases with small amplitudes, which shows a good agreement supporting the validity of the present simulation. Also, the phase lag in response of cavity volume against the inlet pressure variation is confirmed to have damping effect of flow system, which also agrees with the onset criteria of cavitation surge proposed by our previous study. Finally, the non-linear simulation of large amplitude cavitation surge is conducted, which seems to fairly well present the actual waveforms of cavitation surge. But for the improvement of the model, more appropriate physics should be considered at the phase of cavity collapse, which remains for our future study.

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