On the entropy production of time series with unidirectional linearity

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Abstract

There are non-Gaussian time series that admit a causal linear autoregressive moving average (ARMA) model when regressing the future on the past, but not when regressing the past on the future. The reason is that, in the latter case, the regression residuals are only uncorrelated but not statistically independent of the future. In previous work, we have experimentally verified that many empirical time series indeed show such a time inversion asymmetry.

For various physical systems, it is known that time-inversion asymmetries are linked to the thermodynamic entropy production in non-equilibrium states. Here we show that such a link also exists for the above unidirectional linearity.

We study the dynamical evolution of a physical toy system with linear coupling to an infinite environment and show that the linearity of the dynamics is inherited to the forward-time conditional probabilities, but not to the backward-time conditionals. The reason for this asymmetry between past and future is that the environment permanently provides particles that are in a product state before they interact with the system, but show statistical dependencies afterwards.

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From a coarse-grained perspective, the interaction thus generates entropy. We quantitatively relate the strength of the non-linearity of the backward conditionals to the minimal amount of entropy generation.

1 Unidirectional linearity in time series

To study the implications and the different versions of the thermodynamic arrow of time has attracted interest of theoretical physicists and philosophers since a long time \[1, 2, 3, 4, 5, 6, 7\]. More specifically, it is the question how the difference between time reversibility of microscopic physical dynamics is consistent with the existence of irreversible processes on the macroscopic level. The most prominent examples of irreversibilities (e.g. heat always flows from the hot to the cold reservoir, never vice versa, every kind of energy can be converted into heat, but not vice versa) can directly be explained by the fact that the processes generate entropy and their inverted counterpart is therefore forbidden by the second law.

Here we describe an asymmetry between past and future whose connection to the second law is more subtle. An extensive analysis of more than 1000 time series \[8\] showed that there are many cases where the statistics could be better explained by a linear autoregressive model from the past to the future and only few cases where regressing the past on the future yields a better model \[9, 8\]. In the context of non-equilibrium thermodynamics it has been shown for various physical models (e.g. \[10, 11\], and also in a more abstract setting \[12\]) that statistical asymmetries between past and future can be related to thermodynamic entropy production.

This paper is in the same spirit, but we will try to use only those assumptions about the underlying physical system that are necessary to make the case and try to simplify the argument as much as possible. The ingredients are (1) a system interacting with an environment consisting of infinitely many subsystems that are initially in a product state, each system having an abstract vector space as phase space, (2) linear volume preserving dynamical equations for the joint system. We will not refer to any other ingredients from physics, like energy levels, thermal Gibbs states, etc. Of course, this raises the question of how to define entropy production. Here, we interpret the generation of dependencies among an increasing number of particles this way.

To describe the model more precisely, we start with preliminary remarks
on statistical dependencies. First we introduce the following terminology.

**Definition 1 (linear models)**

The joint distribution $P_{X,Y}$ of two real-valued random variables $X$ and $Y$ is said to admit a linear model $X \to Y$ with additive noise (linear model, for short) if $Y$ can be written as

$$Y := \alpha X + \epsilon$$

with a structure coefficient $\alpha \in \mathbb{R}$ and a noise term $\epsilon$ that is statistically independent of $X$ ($X \perp \epsilon$, for short).

It should be emphasized that statistical independence between two random variables $Z, W$ is defined by factorizing probabilities

$$P_{Z,W} = P_Z \otimes P_W,$$

instead of the weaker condition of uncorrelatedness, which is defined by factorizing expectations:

$$\mathbb{E}(ZW) = \mathbb{E}(Z)\mathbb{E}(W).$$

(1)

Uncorrelatedness between $X$ and $\epsilon$ is automatically satisfied if $\alpha$ is chosen to minimize the least square error.

Except for the trivial case of independence, $P_{X,Y}$ can only admit linear models in both directions at the same time if it is bivariate Gaussian. This can be shown using the theorem of Darmois Skitovich [13], which we rephrase now because it will also be used later.

**Lemma 1 (Theorem of Darmois & Skitovich)**

Let $Y_1, Y_2, \ldots, Y_k$ be statistically independent random variables and the two linear combinations

$$W_1 := \sum_{j=1}^{k} \beta_j^{(1)} Y_j$$

$$W_2 := \sum_{j=1}^{k} \beta_j^{(2)} Y_j$$

be independent. Then all $Y_j$ with $\beta_j^{(1)} \beta_j^{(2)} \neq 0$ are Gaussian.
In the context of causal inference from statistical data, it has been proposed to consider the direction of the linear model as the causal direction \[14, 15\]. In \[8\] we have shown that the same idea can be used to solve the following binary classification problem: Given numbers \(X_1, X_2, X_3, \ldots\) that are known to be the values of an empirical time series in their correct or in their time reversed order. Decide whether \(X_1, X_2, X_3, \ldots\) or \(\ldots, X_3, X_2, X_1\) is the correct order. Certainly, this problem is less relevant than the problem of inferring causality since our experiment required to artificially blur the true direction even though it was actually known. The motivation for our study was to test causal inference principles by applying them to this artificial problem.

To explain our “time direction inference rule” we first introduce an important class of stochastic processes \[16\]:

**Definition 2 (ARMA models)**

We call a time series \((X_t)_{t \in \mathbb{Z}}\) an autoregressive moving average process of order \((p, q)\) if it is weakly stationary and there is an iid noise \(\epsilon_t\) with mean zero such that

\[
X_t = \sum_{i=1}^{p} \phi_i X_{t-i} + \sum_{j=1}^{q} \theta_j \epsilon_{t-j} + \epsilon_t \quad \forall t \in \mathbb{Z}.
\]

For \(q = 0\) the process reduces to an autoregressive process and for \(p = 0\) to a moving average process. The short-hand notations are ARMA\((p,q)\), AR\((p)\), and MA\((q)\). The first and the second sums are called the AR-part and the MA-part, respectively.

The process is called causal\(^1\) if

\[
\epsilon_t \perp\!\!\!\perp X_{t-i} \quad \forall i > 0.
\]

Note that a process is called weakly stationary if the mean \(\mathbb{E}(X_t)\) and second order moments \(\mathbb{E}(X_t X_{t+h})\) are constant in time \[16\]. In \[8\] we have shown the following theorem:

**Theorem 1 (non-invertibility of non-Gaussian processes)**

If \((X_t)_{t \in \mathbb{Z}}\) is a causal ARMA process with non-vanishing AR-part, then \((X_{-t})_{t \in \mathbb{Z}}\) is a causal ARMA process if and only if \((X_t)\) is a Gaussian process.

\(^1\)[16] chooses a different definition, but we have argued in \[8\] that it is equivalent to ours.
In particular, a process with long-tailed distributions like e.g. Cauchy can only be causal in one direction (provided that it has an AR-part). In [8] we have postulated that whenever a time series has a causal ARMA model in one direction but not the other the former is likely to be the true one, but some remarks on the practical implementation need to be made: Testing condition (2) yields p-values for the hypothesis of independence. The performance of our inference method depends heavily on how these p-values are used to decide whether a linear model is accepted for one and only one of the directions. Our rule depends on two parameters \( \alpha \) and \( \delta \), the significance level and the gap, respectively. We say that an ARMA model is accepted for one direction but not the other if the p-value for the direction under consideration is above \( \alpha \) and it is below \( \alpha \) for the converse direction and, moreover, the gap is at least \( \delta \). By choosing a small value \( \alpha \) and a large value \( \delta \) one gets fewer decisions but also the fraction of wrong classifications decreases. On 1180 empirical time series from EEGs [8] we were able to classify around 82\% correctly when the parameters are set to yields decisions for about 4\% of the time series. When decisions were made for a larger fraction of time series, the number of correct answers still significantly exceeded chance level. Qualitatively similar results were obtained for 200 time series from different areas, like finance, physics, transportation, crime, production of goods, demography, economy, Neuroscience, and agriculture [9].

2 Physical toy model

Here we describe a physical model that suggests that the observed asymmetry is an implication of generally accepted asymmetries between past and future. We assume that the values \( X_t \) as observables of a classical physical system.\(^2\) For our toy model, we use only two properties of physical models that we consider decisive for the argument:

1. The state of a system is a point in some phase-space \( S \) that is a sub-manifold of \( \mathbb{R}^n \).
2. The dynamical evolution of an isolated system is given by a family \( M_t \) of volume-preserving bijections on \( S \).

Due to Liouville’s Theorem, this holds for the dynamics of all Hamiltonian

\(^2\)Of course, such an embedding is hard to imagine for time series from stock markets, for instance. However, other time series, e.g., EEG-data, are closer related to physical observables.
systems, other dynamical maps can only be obtained by restricting the joint evolution of a composed system to one of its components.

For simplicity, we restrict the attention to an AR(1) process:

\[ X_t = \phi X_{t-1} + \epsilon_t \]  

(3)

We will now interpret \( X_t \) as a physical observable of a system \( S^{(0)} \), whose state is changed by interacting with its environment. The latter consists of an infinite collection of subsystems \( S^{(j)} \) with \( j \in \mathbb{Z} \setminus \{0\} \). Each subsystem is described by the real-valued observable \( Z^{(j)} \). Its value at time \( t \) is denoted by \( Z^{(j)}_t \), hence \( X_t = Z^{(0)}_t \), but we will keep the notation \( X_t \) whenever its special status among the variables should be emphasized.

Then we define a joint time evolution by

\[ Z^{(0)}_{t+1} = \gamma_{11} Z^{(0)}_t + \gamma_{12} Z^{(-1)}_t \]  

(4)

\[ Z^{(1)}_{t+1} = \gamma_{21} Z^{(0)}_t + \gamma_{22} Z^{(-1)}_t \]  

(5)

\[ Z^{(j)}_{t+1} = Z^{(j-1)}_t \quad \text{for } j \neq 0, 1. \]  

(6)

The dynamics thus is a concatenation of the map \( \Gamma \) on \( \mathbb{R}^2 \), given by the entries \( \gamma_{kl} \), with a shift propagating the state of subsystem \( S^{(j)} \) to \( S^{(j+1)} \).

The environment may be thought of as a beam of particles that approaches site \( S^{(0)} \), interacts with it, and disappears to infinity; we have discretized the propagation only to make it compatible with the discrete stochastic process. The interaction is given by \( \Gamma \). The phase space of the systems \( S^{(j)} \) may be larger than one-dimensional, but we assume that the variables \( Z^{(j)}_t \) define the observables that are relevant for the interaction. To ensure conservation of volume in the entire phase space, \( \Gamma \) needs to be volume-preserving, i.e. \(|\det(\Gamma)| = 1\). Since our model should be interpreted as the discretization of a continuous time process we assume \( \Gamma \in SL(2) \).

One checks easily that the above dynamical system generates for \( t > 0 \) the causal AR(1)-process

\[ X_t = \gamma_{11} X_{t-1} + \epsilon_t \quad \text{with} \quad \epsilon_t := \gamma_{12} Z^{(0)}_t , \]

if we impose the initial conditions

\[ Z^{(j)}_0 \text{ i.i.d. with some distribution } Q \]  

(7)

Actually, it would be sufficient to impose independence only for the non-positive \( j \), but later it will be convenient to include also positive values \( j \) and
assume that the whole ARMA process has a starting time \( t = 0 \). This will make it easier to track the increase of dependencies over time. The fact that every \( Z_j^{(i)} \) is drawn from the same distribution \( Q \) ensures that the process \((X_t)_{t \in \mathbb{N}}\) is stationary.

We will now show that, under generic conditions, the dynamics creates statistical dependencies between the subsystems. We will later see that this is the reason why the time-inverted version of the above scenario would not be a reasonable physical model for the process \((X_{-t})\). We need the following Lemma:

**Lemma 2 (dependencies from sequences of adjacent operations)**

Let \( \Gamma \in \text{SL}(2) \) have non-diagonal and diagonal entries. Denote by \( \Gamma_{l,l+1}^{(n)} \) the embedding into the two-dimensional subspaces of \( \mathbb{R}^n \) that correspond to consecutive components \( l, l+1 \) with \( l = 0, \ldots, n-1 \), i.e.,

\[
\Gamma_{l,l+1}^{(n)} := 1_{l-1} \oplus \Gamma \oplus 1_{n-l-1},
\]

where \( 1_m \) denotes the identity matrix in \( m \) dimensions. Let \( P \) be a non-Gaussian distribution on \( \mathbb{R}^n \). Then the application of

\[
\Gamma_{0,1}^{(n)} \circ \Gamma_{2,3}^{(n)} \circ \cdots \circ \Gamma_{n-2,n-1}^{(n)}
\]

to \( \mathbb{R}^n \) turns the product distribution \( P^\otimes n \) into a non-product distribution.

**Proof:** Due to Lemma 1, \( \Gamma_{n-2,n-1}^{(n)} \) generates dependencies between the last and the second last component. Since none of the other operations acts on the last component, the dependence between the last component and the joint system given by the remaining \( n - 2 \) components, is preserved. \( \square \)

To apply Lemma 2 to our system, it is sufficient to focus on the region of the chain on which the dependencies have been generated after the time \( t \) under consideration. It is given by

\[
S^{0 \ldots t} := S^{(0)} \times S^{(1)} \times \cdots \times S^{(t)}.
\]

Its state is given by the variable transformation

\[
(Z_t^{(0)}, Z_t^{(1)}, \ldots, Z_t^{(t)}) = (\Gamma_{0,1}^{(t+1)} \circ \Gamma_{1,2}^{(t+1)} \circ \cdots \circ \Gamma_{t-1,t}^{(t+1)}) (Z_0^{(-t)}, \ldots, Z_0^{(0)}),
\]

and all the other sites are still jointly independent and independent of the region \( \mathcal{S} \). If the relation between \( X_t \) and \( X_{t+1} \) is non-trivial (i.e., neither deterministic nor independent) \( \Gamma \) must have diagonal and non-diagonal entries, which implies that \( \mathcal{S} \) is not a product state.
The following argument shows that the dependencies between the outgoing particles is closely linked to the irreversibility of the scenario: The fact that the time evolution generates a causal AR(1)-process is ensured by independence of $Z_t^{(0)}, Z_t^{(-1)}, Z_t^{(-2)}, \ldots$ describing the incoming particles. If the variables $Z_t^{(1)}, Z_t^{(2)}, \ldots$ are also independent we can run the process backwards to induce the causal AR(1)-process $(X_t)$. However, by virtue of Theorem [1] this is only possible for $(X_t)$ Gaussian.

Summarizing the essential part of the argument, the joint distribution $P_{X_t, X_{t+1}}$ has a linear model from $X_t$ to $X_{t+1}$ but not vice versa because the incoming particles are jointly independent but the outgoing particles are dependent. Now we show a quantitative relation between the non-linearity in backward time direction and the generated dependencies. To this end, we measure the strength of the statistical dependencies of the joint system as follows. If a system consists of finitely many subsystems its multi-information is defined by

$$I(Y_1, \ldots, Y_k) := \sum_{j=1}^{k} H(Y_j) - H(Y_1, \ldots, Y_k).$$

Here, $H(.)$ is the differential Shannon entropy [17]

$$H(Y_1, \ldots, Y_n) := -\int p(y_1, \ldots, y_n) \log p(y_1, \ldots, y_n) dy_1, \cdots dy_n,$$

where $p(y_1, \ldots, y_n)$ denotes the joint probability density of the random variables $Y_1, \ldots, Y_n$. For $k = 2$, the multi-information coincides with the usual mutual information $I(Y_1 : Y_2)$.

For our infinite system we define multi-information as follows:

**Definition 3 (multi-information)**

The multi-information of the joint system of all $S^{(j)}$ at time $t$ is defined by

$$I(t) := \lim_{m \to \infty} I_{-m,\ldots,-m}(t),$$

whenever the limit exists.

Its increase in time can easily be computed:

\[ \text{\ldots} \]
Lemma 3 (multi-information as pairwise information)
Let the initial state of $S^{-\infty...\infty}$ satisfy the conditions (7). Then the multi-information generated by the process in eqs. (4) to (6) with $\Gamma \in SL(2)$ satisfies:

$$I(t) - I(t-1) = I(Z_t^{(0)} : Z_t^{(1)}) \quad \forall t \geq 0.$$  

Proof: We consider the state of the system $S^0...t$ at time $t$ that we had obtained if the interaction would have been inactive (i.e., $\Gamma = 1$) during the last time step. It is described by the transformed variables

$$(\tilde{Z}_t^{(0)}, \ldots, \tilde{Z}_t^{(t)}) := (\Gamma_{1,2}^{(t+1)} \circ \Gamma_{2,3}^{(t+1)} \circ \cdots \circ \Gamma_{t-1,t}^{(t+1)}) (Z_0^{(-t)}, \ldots, Z_0^{(0)}).$$

(10)

Their multi-information coincides with $I(t-1)$ because the shift part of the dynamics is irrelevant.

The true state of system $S^0...t$ at time $t$ is then given by additionally applying $\Gamma_{0,1}^{(t)}$ to eq. (10). The increase of multi-information caused by applying $\Gamma$ to system $S^{(0)}$ and $S^{(1)}$ can be computed as follows. Clearly, the joint entropy of the system $S^{0...t}$ remains constant. Hence the only change of multi-information is due to the change of the marginal entropies of $S^{(0)}$ and $S^{(1)}$. Since $\Gamma_{0,1}^{(t+1)}$ also preserves the joint entropy of system $S^{0,1}$, the increase of the marginal entropies coincides with the pairwise mutual information created between $S^{(0)}$ and $S^{(1)}$. Hence,

$$I(t) - I(t-1) = I(Z_t^{(0)} : Z_t^{(1)}),$$

where we have used the fact that the state of all systems $S^{(j)}$ with $j > 0$ is only shifted. □

To show the link between the amount of generated dependencies and the non-linearity of the backward process, we measure the latter as follows.

Definition 4 (measuring non-linearity of joint distributions)
Let $L$ be the set of joint distributions $R_{X,Y}$ that admit a linear model from $X$ to $Y$. Set

$$D(P_{X,Y}||L) := \inf_{R_{X,Y} \in L} D(P_{X,Y}||R_{X,Y}),$$

where $D$ denotes the relative entropy distance [17] and the infimum is taken over all distributions in $L$.

Then we have:
Theorem 2 (non-linearity of backwards model and multi-inf.)
Let \((X_t)\) be a causal AR(1)-process and \(I(t)\) the multi-information of all the “particles” in the toy model given by eqs. (4) to (6). Then,
\[ I(t) - I(t - 1) \geq D(P_{X_t,X_{t-1}} \| L) . \]

Proof: Assume \(X_t\) and \(X_{t-1}\) are neither linear dependent nor statistically independent because otherwise the bound becomes trivial since we had \(P_{X_t,X_{t-1}} \in L\). The idea of the proof is the following: we figure out how much the joint distribution of \(X_t\) and \(X_{t-1}\) has to be modified to admit a linear model from \(X_t\) to \(X_{t-1}\). To obtain a linear model only from \(X_t\) to \(X_{t-1}\) by reversing the physical toy model it is sufficient to replace \(S^{(1)}\) at time \(t\) with a system that is independent of the remaining ones. More precisely, we replace the joint distribution \(P\) of all \(Z^{(j)}_t\) by the unique distribution \(\tilde{P}\) for which \(Z^{(1)}_t\) and the remaining variables are independent but the marginal distribution to \(Z^{(1)}_t\) and the rest coincide with \(P\), i.e.,
\[ \tilde{P} := P_{Z^{(1)}_t} \otimes P_{Z^{(-1)}_t,Z^{(0)}_t,Z^{(2)}_t,Z^{(2)}_t,\ldots} . \]

Then we check how this changes the joint distribution of \(X_t\) and \(X_{t-1}\). The inverse dynamics \(t \mapsto t - 1\) is given by
\[
\begin{align*}
Z^{(0)}_{t-1} &= \tilde{\gamma}_{11}Z^{(1)}_t + \tilde{\gamma}_{12}Z^{(0)}_t \\
Z^{(-1)}_{t-1} &= \tilde{\gamma}_{21}Z^{(1)}_t + \tilde{\gamma}_{22}Z^{(0)}_t \\
Z^{(j)}_{t-1} &= Z^{(j-1)}_t \quad \text{for } j \neq 0, -1 ,
\end{align*}
\]
where \(\tilde{\gamma}_{kl}\) denote the entries of \(\Gamma^{-1}\).

Since \(X_t = Z^{(0)}_t\) and
\[
X_{t-1} = \tilde{\gamma}_{11}Z^{(0)}_t + \tilde{\gamma}_{12}Z^{(1)}_t ,
\]
which is implied by eq. (11), the pair \((Z^{(0)}_t,Z^{(1)}_t)\) and \((X_t,X_{t-1})\) span the same probability space (note that both coefficients in eq. (14) are non-zero because we have excluded the cases of linear dependency and statistical independence). Hence \(\tilde{P}_{Z^{(0)}_t,Z^{(1)}_t}\) induces by variable transformation a distribution \(\tilde{P}_{X_t,X_{t-1}}\) satisfying
\[
D(P_{X_t,X_{t-1}} \| \tilde{P}_{X_t,X_{t-1}}) = D(P_{Z^{(0)}_t,Z^{(1)}_t} \| \tilde{P}_{Z^{(0)}_t,Z^{(1)}_t}) .
\]
The left hand side is an upper bound for the distance of $P_{X_{t-1},X_t}$ to a linear model from $X_t$ to $X_{t-1}$ because $\tilde{P}_{X_t,X_{t-1}}$ admits such a model. The right hand side coincides with the mutual information between $Z_t^{(1)}$ and $X_t = Z_t^{(0)}$ (since mutual information is known to be the relative entropy distance to the product of marginal distributions [17]), which is exactly the multi-information generated in step $(t-1) \mapsto t$ due to Lemma [3]. □

If $X_t$ is Gaussian, the stochastic process can be obtained without generation of multi-information: If $C$ denotes the covariance matrix of the pair $(X_t, Z_t^{(-1)})$, which is diagonal by assumption (because the variables are independent and identically distributed), then the generation of multi-information is zero if and only if $\Gamma^T C \Gamma$ is diagonal. The easiest case is that $\Gamma$ rotates the space $\mathbb{R}^2$ by some angle $\alpha$. Even though this dynamics leaves the entire joint state of the system invariant, it can induce any stationary AR(1)-process. This is because then $|\phi|^2 \leq 1$ in eq. (3) and we can thus write

$$X_{t+1} = \cos \alpha X_t + \epsilon_t$$

with $\epsilon_t := \sin \alpha Z_t^{(0)}$.

Note that Gaussian processes can also be realized by a system that does generate multi-information. For instance,

$$\Gamma := \begin{pmatrix} \cos \alpha & \sin \alpha \\ 0 & \cos^{-1} \alpha \end{pmatrix}.$$

induces the same process $(X_t)$ as a rotation by the angle $\alpha$, but induces dependent outgoing particles because $\Gamma^T \Gamma$ is non-diagonal. This shows that the correspondence between entropy production and time-inversion asymmetry of $(X_t)$ can only consist of lower bounds.

### 3 Interpretation

We first discuss the interpretation of the Gaussian case. To show an even closer link to thermodynamics, we recall that Gaussian distributions often occur in the context of thermal equilibrium states. For instance, the variable position and momentum of a harmonic oscillator are Gaussian distributed in thermal equilibrium. Hence we interpret the case of the isotropic Gaussian as thermal equilibrium dynamics. The fact that the joint distribution $P_{X_t,X_{t+1}}$
coincides with $P_{X_t X_{t-1}}$ is exactly the symmetry imposed by the well-known detailed-balance condition \cite{18} that holds for every Gibbs state.

In order to interpret the scenario in the non-Gaussian case as entropy production, we note that the sum over the marginal entropies of the subsystems increase linearly in time. The fact that the joint Shannon entropy remains constant loses more and more its practical relevance since it requires complex joint operations to undo the dependencies. From a coarse-grained point of view, the entropy increases in every step.

In our experiments we found several examples of time series that could better fit with a causal ARMA model from the future to the past than vice versa, even though this was only a minority of those for which a decision was made. Of course, there is no contradiction to the second law if this is the case. To avoid such misconclusions we discuss which assumptions could be violated to generate time series that admit non-Gaussian ARMA models in the \textit{wrong} direction.

To this end, we list the requirements which jointly make the time-inverted scenario of the above dynamics extremely unlikely:

1. The “incoming particles” (which correspond to the outgoing ones in the original scenario) and $S^{(0)}$ had to be statistically dependent\footnote{This indicates that they have already been interacting earlier, cf. Reichenbach’s principle of the common cause \cite{1}, which is meanwhile one of the cornerstones of causal inference}.

2. The coupling between $S^{(0)}$ and the incoming particles must be chosen such that it exactly removes the incoming dependencies. There is nothing wrong with \textit{dependent} particles approaching $S^{(0)}$, and a coupling that destroys dependencies between the particles and $S^{(0)}$ by creating additional dependencies with a third party. However, removing dependencies in a \textit{closed} system requires transformations that are specifically adapted to the kind of dependencies that are present. In other words, the coupling between $S^{(0)}$ and the incoming particles had to be one of the “few” linear maps $\tilde{\Gamma} \in SL(2)$ needed for undoing the operation that created the incoming dependencies.

We want to be more explicit about the last item and recall that the joint state of $S^0, \ldots, t$ at time $t$ is given by

$$
(\Gamma_{0,1}^{(t+1)} \circ \Gamma_{1,2}^{(t+1)} \circ \cdots \circ \Gamma_{t-1,t}^{(t+1)}) Q^{\otimes (t+1)}.
$$
We now run the time inverted dynamics (11)–(12) (starting from $t$ and ending at 0) to this input using some arbitrary $\hat{\Gamma} \in SL(2)$. The state of $S^{-t,\ldots,0}$ then reads

\[
\left(\hat{\Gamma}_0^{(t+1)} \circ \hat{\Gamma}_1^{(t+1)} \circ \cdots \circ \hat{\Gamma}_{t-1}^{(t+1)}\right)Q^{(t+1)},
\]

where we have defined

\[
\hat{\Gamma} := \hat{\Gamma} \circ \Gamma.
\]

Due to Lemma 2 this can only be a product state if $\hat{\Gamma}$ has only diagonal or only off-diagonal entries (or if $Q$ is Gaussian). This shows that the dependencies can only be resolved by $\hat{\Gamma}$ if it is adjusted to the specific form of the dependencies of the incoming particles.

This kind of mutual adjustment between mechanism and incoming state is unlikely. Similar arguments have been used in causal inference recently [19, 20]. According to the language used there, the incoming state and the coupling share algorithmic information, which indicates that the incoming state and the coupling have not been chosen independently.

To generate a process $(X_t)_{t \in \mathbb{Z}}$ that admits a linear model in backward direction thus requires a different class of dynamical models. For instance, the joint dynamics could be non-linear.

4 Conclusions and discussion

We have discussed time series that admit a causal ARMA model in forward direction but requires non-linear transitions in backward directions to remain causal. Since previous experiments verified that some empirical time series indeed show this asymmetry, we have presented a model that relates it to the thermodynamic arrow of time.

To this end, we have presented a toy model of a physical system coupled to an infinite environment where we linked the asymmetry to the thermodynamical entropy production.

The essential point is that the linearity of the joint dynamics is inherited to the forward but not to the backward conditionals. Of course, not every physical dynamics is linear. Nevertheless, the result suggests that simplicity of the laws of nature is inherited only to the forward time conditionals. Since

\footnote{Note that the thermodynamic relevance of algorithmic information has also been pointed out in [21].}
stochastic processes usually describes the state of a system that strongly interacts with its environment there is no simple entropy criterion to distinguish between the true and the wrong time direction. Hence, more subtle asymmetries as the ones described here are required.

The asymmetries fit to observations in [22] discussing physical interacting models of a causal relation between two random variables $X$ (cause) and $Y$ (effect), where $P(Y|X)$ was simple and $P(X|Y)$ complex, which has been used in recent causal inference methods [23, 24]. It should be emphasized that such kind of reasoning cannot be justified by referring to Occam’s Razor only, i.e., the principle to prefer simple models if possible. The point that deserves our attention is to justify that Occam’s Razor should be applied to causal conditionals $P(\text{effect}|\text{cause})$ instead of non-causal conditionals like $P(\text{cause}|\text{effect})$. Studying these asymmetries for time-series highlights the relation to commonly accepted asymmetries between past and future.

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