Estimating Vacuum Tunneling Rates

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Abstract

We show that in Euclidean field theories that have bounce solutions, the bounce with the least action is the global minimum of the action in an open space of field configurations. A rigorous upper bound on the minimal bounce action can therefore be obtained by finite numerical methods. This sets a lower bound on the tunneling rate which, fortunately, is often the more interesting and useful bound. We introduce a notion of reduction which allows this bound to be computed with less effort by reducing complicated field theories to simpler ones.

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1 Introduction

Vacuum tunneling in field theories was treated semiclassically in ref. [1]. In the small $\hbar$ limit the tunneling rate per unit volume can be expanded as:

$$\frac{\Gamma}{V} = A \exp \left( \frac{-S[\phi]}{\hbar} \right) \times [1 + O(\hbar)]. \quad (1.1)$$

The exponent is the Euclidean action of the saddle point configuration called the “bounce”. The symbol $\phi$ denotes the entire field configuration at the bounce. The prefactor $A$ has the dimensions of $(\text{mass})^d$ in a $d$ dimensional theory and can be formally written as a ratio of two operator determinants. The tunneling rate is usually estimated by replacing $A$ by $m^d$, where $m$ is a characteristic mass scale of the theory. The error in making this approximation is not significant compared to the error coming from the exponential term which is determined by numerically computing the bounce action $S[\phi]$ (a more accurate estimate may be obtained in some cases, see ref. [2]).

The determination of $\Gamma/V$ is important in extensions of the standard model where the effective potential may have several inequivalent vacua and the realistic and viable vacuum may turn out to be only a local minimum of the potential. In that case, we would be living in a false vacuum now, and our very existence would imply some upper bound on the tunneling rate. If a lower bound on the tunneling rate can be obtained by analytical or numerical methods, then a comparison with the experimental upper bound may place strong constraints on the parameters of the theory. One is therefore often interested in finding an upper bound on the bounce action by analytical or numerical methods.

Recently there has been a renewed interest in considering the tunneling effects in models of particle physics [4, 5, 6, 7, 8]. The effect of such tunneling was considered in ref. [4] to constrain the parameter space of the existing models of dynamical supersymmetry breaking (DSB). Similar tunneling rate computations exist in the literature [3, 4, 5] in the context of the minimal supersymmetric standard model (MSSM), where charge and color breaking (CCB) vacua exist [3]. While analytical estimates of the bounce action can be made in special cases [1, 3], the realistic models of DSB and MSSM often fall in the intractable cases where one must resort to numerical methods. When several fields and coupling constants are involved, the numerical search for the bounce becomes long and time consuming. The search also needs to be repeated many times to map a significant region of the parameter space. New techniques for approximating the bounce action in a short time are therefore welcome.
In this paper we point out the limitations of existing techniques of approximation and indicate a direction for further simplifications and improvements. In particular we show that all finite numerical methods are useful only to find an upper bound on the bounce action. We then develop a notion of “reduction” that provides an intuitive and systematic approach to better techniques for obtaining this bound.

2 The Bounce as a Local Minimum

To begin with, let us consider a theory with \( m \) scalar fields in \( d \) dimensions. The Euclidean action is

\[
S = T + V; \tag{2.2}
\]

where \( T = \sum_{a=1}^{m} \frac{1}{2} \int d^d x \left( \frac{\partial \phi_a}{\partial x^i} \right)^2 \) and \( V = \int d^d x U(\phi_1, \phi_2, \ldots, \phi_m) \) with \( i = 1, 2, \ldots, d \) and \( a = 1, 2, \ldots, m \). Suppose the potential \( U(\phi_1, \phi_2, \ldots, \phi_m) \) has a local minimum \( 2 \) at \( \phi_a = 0 \) and a global minimum at \( \phi_a = \phi_a^t \). Then a bounce solution \( \phi_a(x) \) must possess the following three properties: (i) it approaches the false vacuum \( \phi_a = 0 \) at \( t \to \pm \infty \); (ii) it is a saddle point of the action with a single “direction of instability”; (iii) at the turning point (say at \( t = 0 \)), all the generalized velocities \( \frac{\partial \phi_a}{\partial x^i} \) are zero.

It will be useful to write down the effect of the scale transformation \( (x \to \lambda x) \) on \( T \) and \( V \). In \( d \) dimensions one has: \( T \to T_\lambda = \lambda^{d-2} T \) and \( V \to V_\lambda = \lambda^d V \). Since the bounce is an extremum of the action, we must have, \( \frac{d}{d\lambda} S(\lambda \phi) \big|_{\lambda=1} = 0 \), from which it follows that

\[
(d-2)T(\phi) + (d)V(\phi) = 0. \tag{2.3}
\]

From (2.3) one also finds that \( \frac{d^2}{d\lambda^2} S(\lambda \phi) \big|_{\lambda=1} = (d-2)(d-3)T(\phi) + d(d-1)V(\phi) < 0 \) when \( d > 2 \). Therefore the direction of instability associated with the bounce has a component corresponding to the scale transformations for \( d > 2 \).

In \[10\] it was shown that in \( d > 2 \) dimensional theories with a single real scalar field \( (a = 1) \), the bounce solution with the least action is an \( O(d) \) invariant field configuration with the boundary conditions: \( \frac{d\phi_1(r)}{dr} \big|_{r=0} = 0 \) and \( \phi_1(r) \big|_{r \to \infty} = 0 \), where \( r \) is the radial coordinate in \( R^d \). In this case, the task of finding the bounce action numerically is rather easy. One simply solves the Euclidian equations of motion from some initial point \( \phi_1(0) \) and looks at the limiting value \( \phi_1(\infty) \). For arbitrary values of \( \phi_1(0) \) the value of \( \phi_1(\infty) \) is either an “overshoot” \( (\phi_1(\infty) > 0) \) or an “undershoot” \( (\phi_1(\infty) < 0) \). Since the correct
value of \( \phi_1(0) \) (called the “escape point”) must lie between two trial values which end in an overshoot and an undershoot, the search converges rapidly by bisections.

The task is considerably more complicated in the case of theories with many scalar fields. The bracketing property of the overshoot and the undershoot, which is obviously valid only when the field is a real scalar quantity, is lost. Therefore one can not search for the escape point by integrating the equations of motion. Because the bounce is not a minimum of the action, but a saddle point, a simple minimization of the action would not work either. For instance, the method suggested in ref. [11] of maximizing the action of a field configuration on the lattice with respect to scale transformations and then minimizing it with respect to random variations of the fields is unlikely to converge to a bounce solution because variations that are orthogonal to a scale transformation are hard to identify in an actual lattice computation. A remarkable simplification is however achieved by adopting the method of ref. [7] which consists of adding to the Euclidean action other terms that have the following two properties:

(i) they vanish at the bounce,
(ii) they remove the instability associated with scaling, i.e, the “improved” action with the new terms is minimized with respect to all variations including the scale transformation.

When terms like these are added to the action, a bounce appears as a local minimum (instead of a saddle point) of the improved action. An example of the improved action is

\[
\overline{S}(\phi_a) = S(\phi_a) + \sum_n \alpha_n \Lambda |p_n|, \tag{2.4}
\]

where \( \Lambda = (d-2)T + (d)V \) and \( p_n \) and \( \alpha_n \) are positive numbers for \( n = 1, 2, \ldots \). For \( d > 2 \) the new terms added to the action have the two properties mentioned above [12].

Although the method of improved action reduces the problem to a pure minimization, the bounce is only a local minimum in an infinite dimensional space. In the absence of further information, it is by no means obvious that minimizations by finite numerical methods actually yield a useful result. Typically, the discretization involved in the numerical methods generates spurious minima of the improved action. It is not clear if the result of such a search yields an action that is greater or less than the bounce action. This is a drawback since one is often interested in a rigorous upper bound on the bounce action. Although finite numerical methods will always have some limitations (which we discuss later) we will now show that they can be sufficient to obtain such a rigorous upper bound. The approach suggested by us will also lead to new avenues of simplification and refinement of numerical techniques.
3 The Spherical Bounce as a Global Minimum

Let us define a spherical bounce as an $O(d)$ symmetric field configuration with properties (i) and (ii) of the bounce provided the variations are restricted to preserve the $O(d)$ invariance. Clearly the $O(d)$ invariance of the configuration implies that it has property (iii) of the bounce. Our approach rests on the following two observations about the spherical bounces.

Statement 1: A spherical bounce is also a true bounce.

Statement 2: The spherical bounce with the least action is a global minimum of the action in an open space of field configurations which obey the boundary conditions appropriate for a bounce.

The proof of Statement 2 is simpler for $d > 2$. We will discuss the cases $d = 1, 2$ later. Before we sketch the proofs, let us first point out some immediate gains that result from the knowledge of the space $C_1$. At the heart of our approach is the trivial point that it is always easier to bound a global minimum from above than a local minimum.

There are two general limitations associated with finite numerical methods. Firstly, in multi-scalar theories there may be several bounce solutions. The tunneling rate is determined by the bounce (or bounces) with the least action. There is no numerical method that guarantees convergence to the bounce with the least action. Secondly, any method of discretizing the improved action (2.4) introduces spurious local minima which are local minima of the discretized improved action with respect to a finite dimensional space of variations, but are not local minima when all continuous variations of the fields are considered. These spurious minima may convey no information about any bounce action.

These limitations preclude the possibility of finding the least bounce action to within controllable numerical errors in multi-scalar theories. However, as we have emphasized earlier, in many cases it is useful to find just an upper bound on the least bounce action. The knowledge of the space $C_1$ allows one to approximate the bounce by a field configuration that lies exactly on the space $C_1$ and provides a rigorous upper bound on the least spherical bounce action (which itself bounds from above the least bounce action). As it happens the space $C_1$ is quite simple and it is easy to find field configurations lying on it. We now present the proofs of Statements 1 and 2. The latter also defines the space $C_1$.

\footnote{This approach is closely related to the one suggested by A. Wipf \cite{9}.}
Proof of Statement 1: This statement is in fact a consequence of the principle of symmetric criticality \cite{13}. Writing the action of (2.3) in polar coordinates we have

\[
S = S_1 + S_2,
\]
\[
S_1 = \int d\Omega r^{d-1} dr \left[ \frac{1}{2} \sum_{a=1}^{m} \left( \frac{d\phi_a}{dr} \right)^2 + U(\phi_1, \phi_2, ..., \phi_m) \right],
\]
\[
S_2 = \int d\Omega r^{d-1} dr \left[ \frac{1}{2r^2} \sum_{a=1}^{m} \sum_{i=1}^{d-1} f_i(\Omega)^2 \left( \frac{d\phi_a}{d\theta_i} \right)^2 \right],
\]
(3.5)

where \(\theta_i\) are the angular coordinates and \(f_i\) are the measures corresponding to the angular gradients \(\frac{d\phi_a}{d\theta_i}\). The quantity \(S_2\) is positive definite and is minimized on the space of \(O(d)\) invariant configurations. Therefore a spherical bounce is an extremum with respect to the \(O(d)\) breaking variations too and no new negative eigenvalue is added to the Hessian \(\frac{\partial^2 S}{\partial \phi^2}\) by considering the \(O(d)\) breaking variations. That is, the spherical bounce is also a true bounce. Q.E.D.

Proof of Statement 2 (for \(d > 2\)): Consider the space of all \(O(d)\) invariant field configurations that satisfy the correct boundary conditions and are extremized with respect to the scale transformations \(x \to \lambda x\). Let us call this space \(C\). All points in \(C\) obey (2.3). The space \(C\) naturally splits into two disconnected parts \(C = C_0 \oplus C_1\), where \(C_0\) is the connected space containing the trivial solution \(\phi_a \equiv 0\) where all fields assume their values at the false vacuum at all times. At the trivial solution \(V = 0\), and in a small ball around this solution \(V > 0\) because the false vacuum is a local minimum of energy. On the other hand from (2.3) it is clear that for points in \(C\), \(V \leq 0\). Thus \(C_0\) consists of an isolated point.

Points in \(C_1\) have \(V < 0\) and \(T > 0\). Every nontrivial configuration satisfying (2.3) belongs to \(C_1\). But the the point \(G\) which is the global minimum of the action in \(C_1\) satisfies the correct boundary conditions for the bounce, is a saddle point with a single direction of instability (the direction corresponding to the generator of scale transformations which is “orthogonal” to \(C_1\)) and has \(\frac{\partial S}{\partial \phi_a} = 0\) at \(t = 0\) by the \(O(d)\) invariance. Therefore, by definition, it is the spherical bounce with the least action. Q.E.D.

Comment 1: It is obvious that the action of an arbitrary field configuration in the space \(C\) is necessarily bounded below by either the action of the trivial solution \(C_0\) or the action of a nontrivial solution of the equations of motion lying on \(C_1\). It is a remarkable property of the space \(C\) that it is always the latter. The essential point is this. In a discrete numerical method one allows for variations of the action over a finite dimensional
space only. If scale transformations are included in the variations and \(O(d)\) invariance is imposed throughout the search, then one is assured that the “bounce” obtained by the numerical method lies on the space \(C_1\) and provides the required upper bound.

**Comment 2:** The \(O(d)\) symmetry is indispensible for any numerical method because it reduces the number of variables drastically. However it is redundant from the point of view of defining the space \(C_1\). A milder symmetry like the \(Z_2\) symmetry \(\phi_a(t, x) = \phi_a(-t, x)\) is sufficient to ensure that \(G\) has property (iii) of the bounce. One can redefine the spaces \(C\) and \(C_1\) with the \(O(d)\) symmetry replaced by the \(Z_2\) symmetry. The corresponding point \(G\) is then truly the bounce with the minimum action. This is because, as one can easily show, the point \(G\) is a true bounce and, conversely, every true bounce has the \(Z_2\) symmetry possessed by \(G\). The latter assertion follows from observing that the equations of motion are invariant under \(t \to -t\) and imposing \(\frac{\partial \phi_a}{\partial t} (t = 0) \equiv 0\) implies that any solution \(\bar{\phi}_a\) of the equations of motion also has this symmetry.

**Comment 3:** There is more to gain from the above approach than the insight that numerical methods are useful only to find upper bounds on the bounce action. New and more effective techniques are suggested. This is motivated as follows. Underlying any numerical technique is a discretization of the action. For instance a straightforward discretization of the improved action with \(O(d)\) symmetry is:

\[
\begin{align*}
\overline{S}(\phi) &= T(\phi) + V(\phi) + \alpha |T(\phi) + 2V(\phi)|^2 \\
T(\phi) &= \Omega \Delta^d \sum_{n=1}^{N} \sum_{a=1}^{m} n^3 |\phi^{(n+1)}_a - \phi^{(n)}_a|^2/2\Delta^2 \\
V(\phi) &= \Omega \Delta^d \sum_{n=1}^{N} n^3 U(\phi_1^{(n)}, \phi_2^{(n)}, \ldots \phi_m^{(n)}).
\end{align*}
\]

(3.6)

The radial coordinate in \(R^d\) is discretized into \(N\) points and \(\Omega\) is the solid angle in \(d\) dimensions. The purpose of the discretization is to reduce the space of field variations to a finite dimensional space (to a space \(C_f\) here, which is homomorphic to \(R^{N \times m}\)).

There is no particular reason to believe that the discretization in (3.6) is the most convenient way of reducing the space of all variations to a finite dimensional space. Indeed, as we show below, tunneling in theories with many scalar fields can be extremely difficult to explore with the discretization of (3.6). Since the most significant part of the computations is to make the final point lie on the space \(C_1\) and not how the reduction to some space \(C_f\) is achieved, the natural question to ask is: why not implement the “reduction” of allowed variations at an early stage by reducing the complicated field theory to a simpler
one? The question itself suggests new and systematic ways of improving the numerical techniques.

4 Methods of Reduction

We will seek new methods of reduction. Our search is guided by the technical difficulties encountered with the discretization of (3.6). Let us briefly describe the essential features and the limitations of the reduction characterizing this discretization. We will call this method of reduction method 1.

**Method 1:** With the discretization (3.6) one minimizes the action with the constraint \((d - 2)T + (d)V = 0\). The constraint can be enforced by making the Lagrange multiplier \(\alpha\) large. The result is a discrete trajectory \(\phi_a(n)\) with \(n = 1, 2, \ldots N\), which can be extended to the continuous and piecewise linear function \(\phi_a(r)\) given by \(\phi_a(r) = \left[(1 - r + n)\phi_a(n) + (r - n)\phi_a(n + 1)\right]\) for \(n \leq r < n + 1\) and \(\phi_a(r) = \phi_a(N)\) for \(r > N\). When \(U\) is a polynomial in \(\phi_a\), the functions \(T\) and \(V\) can be calculated to arbitrary accuracy using this function. The condition (2.3) may not be exactly satisfied by \(T\) and \(V\) obtained at this point. But one can always perform a scale transformation to satisfy (2.3) exactly at the end. Thus the method of improved action is successful in bounding the bounce action provided one performs the necessary scale transformation at the end.

In this case the search space for the minimization is \(N \times m\) dimensional, with \(m\) fixed by the theory. In an \(N \times m\) dimensional Euclidean space, if one desires to reach within a distance of \(\delta\) from a minimum by a random iterative search (as suggested in ref.[7]), the number of steps required is about \((\frac{L}{\delta})^{Nm}\), where \(L\) is the maximum step length in any direction. One can improve this by using a multidimensional “greedy” minimization technique such as the Conjugate Gradient Method [14]. The computational time typically grows as some power of \(N \times m\) (depending on the complexity of the function). Also, the search space has many local minima with widely varying actions. Usually, one needs several iterations to arrive at a good one. Clearly the computation becomes harder as the number of fields \(m\) increases. Therefore a method of reduction where the computational complexity and time do not depend on \(m\) becomes desirable for theories with many fields. The alternative method of reduction presented below has precisely this virtue.

**Method 2:** The action is given by (2.2). The false vacuum is at \(\phi_a = 0\). Choose a straight line, passing through the point \(\phi_a = 0\), in the \(m\) dimensional Euclidean space \(R^m\) with axes given by \(\phi_1, \phi_2, \ldots \phi_m\). The fields \(\phi_a\) can be constrained to take values only
on this straight line by putting $\phi_a = y_a \phi$ where $y_a$ are real numbers satisfying $\sum y_a^2 = 1$ and $\phi$ is the reduced scalar field. Then the action is reduced to the reduced action $S[\phi] = \int d^d x \left[ 1/2 (\partial_\phi \phi_a)^2 + U(\phi) \right]$ which is a functional of a single real scalar field and the bounce action in this reduced theory is easily computed by the method of bisections described earlier. Note that the bounce exists if and only if the chosen straight line passes through some point $\phi_a^l$ such that $U(0) > U(\phi_a^l)$. This method is an obvious reduction that intuitively seems to be a correct simplification. However, the relation of the “bounce” obtained in the reduced theory to the true bounce may not be immediately clear. But note that the action is extremized with respect to all variations that do not move the fields $\phi_a(x)$ out of the chosen straight line in field space. The crucial point is that scale transformations are included in these variations. Therefore the solution to the equation of motion satisfies (2.3) and by Statement 2 its action is rigorously an upper bound on the action of the bounce with least action in the full theory. The search time scales like $N \log \left( |\phi_t - \phi_f| / \delta \right)$ when the search is made by the method of undershoots and overshoots, and has no dependence on $m$. The problem of multiple bounces and spurious minima does not usually arise. Also, the result is automatically a point on $C_1$ without the need for adding new terms to the action as in (2.4).

In practice one encounters polynomial potentials in $\phi_a(x)$. It is often possible to find the position of the true vacuum or the saddle point in the potential between the true vacuum and the false vacuum. The lines joining these points to the false vacuum may yield fairly low values for the upper bound on the bounce action. One can do the search over several lines which can be judiciously chosen.

![Fig. 1. Comparison of Methods 1 and 2. The region above the dashed (solid) line is ruled out by method 1 (method 1).](image)

In Figure 1 we show a comparison of the performance of the above two methods in
a realistic DSB model \cite{15} whose vacuum stability has been studied in ref. \cite{4}. For our purpose it is sufficient to consider the scalar fields in the so called messenger sector of the theory which serves to communicate supersymmetry breaking to the fields in the standard model. For a natural range of parameters, the true vacuum in the messenger sector has been shown to be color breaking \cite{4}. The relevant part of the action consists of the complex fields $P, N, S, q$ and $\bar{q}$. The last two fields carry color but the color index can be suppressed in this discussion. Apart from the usual kinetic energy terms, the action has the potential:

$$V = \frac{g^2}{2}(|P|^2 - |N|^2) + (M_2^2 + \lambda_1^2 |S|^2)(|P|^2 + |N|^2)$$

$$+ \kappa^2 |S|^2(|q|^2 + |\bar{q}|^2) + |\kappa q \bar{q}| + \lambda S^2 + \lambda_1 P N |^2 + \alpha^2 (|P|^2 + |N|^2)^2,$$

where $g, \lambda, \lambda_1$ and $\kappa$ are positive coupling constants, $M_2^2 < 0$ and $\alpha^2$ is of the order of $10^{-2}$. A false but phenomenologically viable vacuum exists at $q = \bar{q} = 0$ and $|P| = |N|$ with $S \neq 0$ if $\lambda_1 < (\kappa \lambda_1)/(\kappa + \lambda)$ \cite{4}. We have computed the vacuum tunneling rate using both the methods described above. In Figure 1 we plot our results in the $\lambda, \lambda_1$ plane for $\kappa = 1.0$ and $\alpha^2 = 0.01$. The region below the dotted line is allowed analytically, but the region above the dashed (solid) curve is ruled out by method 2 (method 1) because the lifetime of the false vacuum is shorter than the age of the universe. In this case a larger region in the parameter space is ruled out by using method 1, but the use of method 2, which is much quicker to implement, can vastly reduce the region of parameter space that one must explore with method 1. There may also be theories where method 1 or its variations\footnote{We have suggested a different method of reduction in a related context in ref. \cite{16}.} can provide superior bounds than method 2. A simple variation of method 2 is obtained by replacing the straight line in $R^n$ by a curve. When several intuitive choices for the reduction are available, it is difficult to say which method will provide superior results. However method 2 is always the quickest one and should be used to rule out as much of the parameter space as possible.

5 \hspace{1em} $d = 1, 2$

We would like to remark on the cases $d = 1$ and $d = 2$. Statement 2 is in fact generically true for $d = 2$. To see this consider the deformation of the functional $T$ in $S$ to $T(\eta)$ defined by

$$T(\eta) = \frac{1}{2} \int d^2 x \left( \frac{\partial \phi_a}{\partial x_i} \right)^{2-\eta},$$

(5.8)

\footnote{We have suggested a different method of reduction in a related context in ref. \cite{16}.}
where \( \eta \) is a positive real number. With this deformation the equation (2.3) is modified for \( (d = 2) \) to

\[
\eta T(\eta) + 2V = 0.
\] (5.9)

Statement 2 is true as \( \eta \) approaches zero from the positive direction. Thus \( G \) remains a saddle point in the sense that it is a maximum of the action with respect to some variation orthogonal to \( C_1 \). However the negative eigenvalue of the Hessian \( \frac{\partial^2 S}{\partial \phi_a^2} \) may approach zero as \( \eta \to 0 \). But there is no underlying symmetry to make this situation generic. In other words, the eigenvalues of the Hessian are likely to be of the order of \( m^2 \) where \( m \) is some mass scale in the theory. Unless there is a symmetry to protect its smallness, an eigenvalue can not be made zero without fine tuning the parameters of the theory.

The case \( d = 1 \) can be treated as follows. The equations of motion to be satisfied by the bounce are

\[
\frac{\partial^2 \phi_a}{\partial t^2} = \frac{\partial U}{\partial \phi_a}
\] (5.10)

with the boundary conditions \( \phi_a(\pm \infty) \to 0, \frac{\partial \phi_a}{\partial t}\big|_{t=0} = 0 \). Solutions to (5.10) resemble the motion of a particle of unit mass in the potential \( -U(\phi_a) \). Let us define the surface \( D \) in the \( m \) dimensional space of the “fields” \( \phi_a \) as the surface where \( U = 0 \). The space \( D \) separates as \( D = D_0 \oplus D_1 \) with \( D_0 \) given by the part that is connected to the point \( \phi_a = 0 \). By definition \( D_0 \) and \( D_1 \) are not connected in \( D \). Consider the space of \( Z_2 \) invariant trajectories \( \phi_a(t) = \phi_a(-t) \) with \( \phi_a(0) \in D_1, \phi_a(\infty) \to D_0 \) and satisfying the constraint \( U[\phi_a(t)] \geq 0 \). The action of these trajectories is positive definite and has a global minimum, which, we claim, is generically a bounce.

Let us briefly substantiate the claim which can also be proven along the lines of ref. [17]. \( G \) is a minimum with respect to all allowed variations except variations that move the point \( \phi_a(0) \) out of the surface \( D_1 \). Generically, connected parts in \( D_1 \) are \( m - 1 \) dimensional surfaces. This leaves a single variation that moves \( \phi_a(0) \) out of \( D_1 \). We need to show that there is a direction of instability associated with this variation. The entire trajectory is a solution of the equations of motion. By conservation of the “energy” \( E = \sum_{a=1}^{m} \left[ \frac{1}{2}(\frac{d \phi_a}{d t})^2 \right] - U(\phi_1, \phi_2...\phi_m) \), the total energy is zero on this trajectory and the velocity is zero at the “highest point” \( (\frac{d \phi_a}{d t}(0) = 0) \). This implies that the generator of time translations \( \frac{d \phi_a}{d t} \) has a node at \( t = 0 \). By time translational invariance of the bounce, the eigenvalue of the Hessian \( \frac{\partial^2 S}{\partial \phi_a^2} \) that corresponds to time translations is zero. Therefore there is a nodeless variation with a negative eigenvalue and \( G \) is a bounce.

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