SHORT GEODESIC LOOPS AND $L^p$ NORMS OF EIGENFUNCTIONS ON LARGE GENUS RANDOM SURFACES

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Abstract. We give upper bounds for $L^p$ norms of eigenfunctions of the Laplacian on compact hyperbolic surfaces in terms of a parameter depending on the growth rate of the number of short geodesic loops passing through a point. When the genus $g \to +\infty$, we show that random hyperbolic surfaces $X$ with respect to the Weil-Petersson volume have almost surely at most one such loop of length less than $c \log g$ for small enough $c > 0$. This allows us to deduce that the $L^p$ norms of $L^2$ normalised eigenfunctions on $X$ are bounded by $\frac{1}{\sqrt{\log g}}$ almost surely in the large genus limit for any $p > 2 + \varepsilon$ for $\varepsilon > 0$ depending on the spectral gap $\lambda_1(X)$ of $X$.

1. Introduction

1.1. Background and main result. In the setting of a compact $n$-dimensional Riemannian manifold $(M, g)$, a deep understanding of the shape and asymptotics of eigenfunctions of the Laplacian is intimately linked to underlying geometric properties of the space itself. One means to realise this connection is through studying the $L^p$ norms of the eigenfunctions themselves. Indeed, as an example, primitive estimates show that the multiplicity of the eigenvalues are influenced by the sup norms of the eigenfunctions as well as the volume of the space through

$$m(\lambda) \leq \text{Vol}(M) \left( \max_{x \in M} \{ |\psi(x)| : \Delta \psi = \lambda \psi, \|\psi\|_2 = 1 \} \right)^2,$$

where $m(\lambda)$ is the multiplicity of the eigenvalue $\lambda$ (see for example the proof of Proposition 2.1 in [19]).

Eigenfunctions of the Laplacian feature prominently in quantum mechanics since they are precisely the states for which the probability measures $|\psi(x, t)|^2 d\text{Vol}_M(x)$ are constants, where $\psi(x, t)$ is the free quantum evolution of a wavefunction $\psi(x)$. In this setting, a widely studied problem is to understand the properties of the eigenfunctions in the high energy, or large eigenvalue, limit, aiming to recover some characteristics of the classical dynamics, for example in the study of Quantum (Unique) Ergodicity [43, 45, 16, 37, 31, 23].

In the large eigenvalue aspect, Sogge’s [41] seminal work identified the link between the growth of $L^p$ norms of eigenfunctions and their $L^2$ norms in terms of their eigenvalue. In particular, if $\Delta \psi = \lambda \psi$ then

$$\|\psi\|_p \lesssim_M \lambda^{\sigma(n,p)} \|\psi\|_2,$$

where $\sigma(n,p)$ is the best constant depending on the dimension $n$ and the exponent $p$.

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where

$$\sigma(n, p) = \begin{cases} \frac{n}{2} \left( \frac{1}{2} - \frac{1}{p} \right) - \frac{1}{4}, & \text{if } 2 \leq p \leq \frac{2(n+1)}{n-1}, \\ \frac{n-1}{4} \left( \frac{1}{2} - \frac{1}{p} \right), & \text{if } 2(n+1) \leq p \leq \infty \end{cases}$$

Here we use $\lesssim$ to denote that the quantity is bounded up to a constant and subscripts are used to indicate specific dependencies of such a constant. These bounds are sharp on the sphere and attained for high $p$ by zonal spherical harmonics (concentration of the mass around a point) and for low $p$ by Gaussian beams (concentration along closed geodesics). However, in the case of manifolds of non-positive curvature (or without conjugate points), Bérard [7] had previously obtained a logarithmic improvement of the sup norm. This was more recently extended to values of $p > \frac{2(n+1)}{n-1}$ by Hassell and Tacy [24].

The implied constant in the Sogge bound was investigated by Donnelly [19] to reveal that the underlying geometry again plays an important role. More specifically, the constant depends upon bounds on the injectivity radius and the sectional curvature of the manifold.

In this paper we restrict our attention to hyperbolic surfaces and investigate the influence of the geometry on $L^p$ norms. Rather than seeking bounds in terms of eigenvalues, we focus on their dependence on the growth rate of short geodesic loops (see (1.1) below). Our goal is to understand this geometric connection with random hyperbolic surfaces, using integration tools on the moduli space developed by Mirzakhani [32, 33, 34, 36]. In [34], Mirzakhani initiated a theory of large genus random surfaces with respect to the Weil-Petersson volume (see Section 5 for background). An important success of these methods was the proof by Mirzakhani and Petri [35] that the length of short geodesics on random surfaces follow a Poisson distribution in the large genus limit. From there, it is natural to try to connect the behaviour of closed geodesics to the spectrum of random surfaces via Selberg’s theory (see for example [8] for background on Selberg’s trace formula). We present in this paper one of the first attempts at such a connection between the geometry of random surfaces and eigenfunctions of the Laplacian.

A central motivation in our work is to understand the delocalisation properties of eigenfunctions on large volume manifolds. In a recent article [30], Le Masson and Sahlsten proposed a version of quantum ergodicity for hyperbolic surfaces of large genus. The theorem is a delocalisation result analogous to the quantum ergodicity theorem of Šnirel’man [43], Zelditch [45] and Colin de Verdière [16], but valid in the large volume limit, and for eigenfunctions in a bounded spectral interval. This result was inspired by corresponding theorems on regular graphs [2, 13], viewed as discrete analogues of hyperbolic surfaces. We will follow a similar heuristic to push the graph methods of Brooks and Le Masson [12] to the continuous setting. This deterministic aspect will be combined with new estimates on short geodesics of random surfaces, to obtain bounds on $L^p$ norms for random surfaces. Recently, there has been major breakthroughs in the study of eigenvectors on random regular graphs, from optimal sup-norm bounds [6] to the proof of their Gaussian behaviour [4]. Our hope is to have provided a stepping stone towards the adaptation of these more advanced results.

Before we state our main theorem, let us define the model of random surfaces we are considering. For any $g \geq 2$, we denote by $\mathcal{M}_g$ the moduli space of compact hyperbolic surfaces of genus $g$. It can be seen as a quotient $\mathcal{M}_g = T_g/\text{Mod}_g$ of the Teichmüller space $T_g$ by the mapping class group $\text{Mod}_g$ (see Section 5 for definitions). The Teichmüller space $T_g$ is equipped with a symplectic form $\omega_g$ called the Weil-Petersson form that is invariant under the action of $\text{Mod}_g$. The associated volume form then descends to the quotient $\mathcal{M}_g$, which is of finite total volume. Denoting by $\text{Vol}_{wp}(A)$ the Weil-Petersson volume of a measurable

\footnote{Note that by the Gauss-Bonnet theorem the genus $g$ and the volume of a hyperbolic surface $|X|$ are related by the formula $|X| = 2\pi(2g - 2)$, and are therefore equivalent parameters.}
set \( A \subset \mathcal{M}_g \), we obtain the probability measure

\[
P_g(A) = \frac{\text{Vol}_{wp}(A)}{\text{Vol}_{wp}(\mathcal{M}_g)}.
\]

One of the remarkable achievements of Mirzakhani was to compute the Weil-Petersson volume of \( \mathcal{M}_g \), and more generally of the moduli spaces of surfaces with boundaries and punctures, making it possible to estimate such probabilities. Note that an alternative model of random surfaces has been developed by Brooks and Makover [11]. These two models are not equivalent and we will only work here with the Weil-Petersson model.

The main theorem we prove is the following.

**Theorem 1.1.** Let \( X \) be a random compact hyperbolic surface of genus \( g \) chosen according to the probability \( \mathbb{P}_g \). We define \( \beta \in [0, \frac{1}{2}) \) such that the smallest non-zero eigenvalue of the Laplacian on \( X \) is at least \( \frac{1}{4} - \beta^2 \). Then there exists \( c > 0 \) such that with probability tending to 1 when \( g \to +\infty \) we have the following bounds. For an eigenfunction \( \psi_\lambda \) of the Laplacian with eigenvalue \( \lambda \geq \frac{1}{4} \),

\[
\| \psi_\lambda \|_p \lesssim_{p, \lambda, c} \frac{1}{\sqrt{\log(g)}} \| \psi_\lambda \|_2,
\]

for any \( 2 + 4\beta < p \leq \infty \). Moreover, if \( \psi_\lambda \) is an eigenfunction of the Laplacian with eigenvalue \( \lambda \in [0, \frac{1}{4} - \varepsilon) \) for some \( \varepsilon > 0 \) then

\[
\| \psi_\lambda \|_p \lesssim \left( g^{\varepsilon \sqrt{\pi} - 1} \right)^{1 - \frac{2}{p}} \| \psi_\lambda \|_2,
\]

for any \( 2 < p \leq \infty \). More precisely, the rate of convergence of the probability we obtain is \( 1 - O \left( g^{-\frac{1}{2} + \delta c} \right) \), where \( \delta > 0 \) is an implicit but universal constant.

Note the different behaviour between the two parts of the spectrum: \([1/4, +\infty)\), to which we will often refer as the *tempered spectrum*, and \((0, 1/4)\) called the *untempered spectrum*.

Our second theorem is a result on short geodesic loops on random surfaces, that appears crucially in the proof of Theorem 1.1 but can in itself be of interest. For any \( X \in \mathcal{M}_g \), let us denote by \( N_L(X, x) \) the number of primitive geodesic loops \( \gamma \) (not necessarily simple) of length \( \ell_X(\gamma) \leq L \) passing through a point \( x \in X \), and

\[
N_L(X) = \sup_{x \in X} N_L(X, x).
\]

**Theorem 1.2.** There exists \( \delta > 0 \) such that for all \( c > 0 \)

\[
P_g(X \in \mathcal{M}_g : N_{c \log g}(X) > 1) = O \left( g^{-\frac{1}{2} + \delta c} \right),
\]

and therefore for \( c \) small enough this probability tends to 0 when \( g \to +\infty \).

The previous theorem says that, almost surely when \( g \to +\infty \), at any point of a random surface there is no more than one geodesic loop of length less than \( c \log g \) passing through this point. This implies in particular that if there is one, this loop is necessarily simple.

Theorem 1.1 relies on Theorem 1.2 together with a deterministic theorem about \( L^p \) norms. Let \( X = \Gamma \backslash \mathbb{H} \) be a compact hyperbolic surface with fundamental domain \( D \) for which there exists an \( R = R(X) \geq 0 \) such that for all \( z, w \in D \),

\[
|\{ \gamma \in \Gamma | d(z, \gamma w) \leq r \}| \lesssim_\delta e^{\delta r} \quad \text{for any } \delta > 0 \text{ and } r \leq R.
\]

(1.1)

In other words, this condition is asking that there exist an \( R \geq 0 \) such that

\[
N_r(X) \lesssim_\delta e^{\delta r} \quad \text{for any } \delta > 0 \text{ and } r \leq R.
\]

(1.2)

Our deterministic result is then stated as follows.
Theorem 1.3. Suppose that $X = \Gamma \backslash \mathbb{H}$ is a compact hyperbolic surface whose smallest non-zero eigenvalue of the Laplacian is at least $\frac{1}{4} - \beta^2$ for some $\beta \in [0, \frac{1}{2})$. For an eigenfunction $\psi_\lambda$ of the Laplacian with eigenvalue $\lambda \geq \frac{1}{4}$, we have that

$$\|\psi_\lambda\|_p \lesssim_{p, \lambda} \frac{1}{\sqrt{R}} \|\psi_\lambda\|_2,$$

for any $2 + 4\beta < p \leq \infty$. Moreover, if $\psi_\lambda$ is an eigenfunction of the Laplacian with eigenvalue $\lambda \in [0, \frac{1}{4} - \epsilon)$ for some $\epsilon > 0$ then for any $\delta > 0$

$$\|\psi_\lambda\|_p \lesssim_\delta \frac{1}{(e^{(1-\delta)\sqrt{R}} - 1)^{1-\frac{1}{p}}} \|\psi_\lambda\|_2,$$

for $2 < p \leq \infty$. Throughout, $R$ is given by (1.1).

Remark 1.4. A radius $R$ can always be found, since (1.1) holds trivially by taking $R$ as the injectivity radius of the surface. In this case we obtain the decay rate in terms of this geometric quantity. On the other hand, by noticing that the number of fundamental domains intersecting a ball of radius $R$ is greater than $e^{R}/g$ we see that we have necessarily $R \leq \log(g)$.

1.2. Further remarks and perspectives. The point of view of random surfaces for the study of eigenfunctions that we are developing opens some new and interesting perspectives.

Optimal bounds. One can ask what is the best bound on $L^p$ norms that can be obtained in the large genus limit. Clearly for any function $\psi : X \to \mathbb{R}$

$$\|\psi\|_\infty \geq \frac{\|\psi\|_2}{\sqrt{|X|}} \quad (1.3)$$

with equality if and only if $\psi$ is constant almost everywhere. Eigenfunctions of non-zero eigenvalue are not constant and therefore some correction is required.

On large random regular graphs, the following was proved by Bauerschmidt, Huang and Yau [6, Theorem 1.2]. Let $G_{N,d}$ be the set of random regular graphs of degree $d$ on $N$ vertices. We put the uniform probability measure on $G_{N,d}$. There exists $d_0$ very large but fixed such that for $d \geq d_0$ and with probability tending to 1 when $N \to +\infty$, any eigenvector $v$ in the tempered spectrum satisfies

$$\|v\|_\infty \lesssim \frac{(\log N)^{\alpha}}{\sqrt{N}} \|v\|_2,$$

for some $\alpha > 0$ depending on the distance of the eigenvector from the boundaries of the tempered spectrum.

Inspired by this graph result we can formulate the following conjecture.

Conjecture 1.5. Let $X$ be a compact hyperbolic surface of genus $g$ chosen uniformly at random with respect to the Weil-Petersson volume. Then for any $\epsilon > 0$ and any eigenfunction $\psi_\lambda$ with eigenvalue $\lambda \in \left(\frac{1}{4} + \epsilon, +\infty\right)$ we have

$$\|\psi_\lambda\|_\infty \lesssim \frac{(\log g)^{\alpha(\epsilon)}}{\sqrt{g}} \|\psi_\lambda\|_2$$

for some function $\alpha(\epsilon) > 0$ of $\epsilon$, with probability tending to 1 when $g \to +\infty$.

Such a result on the sup norm would give a strong form of delocalisation. In particular it would prevent concentration of eigenfunctions on sets of volume less than $g/\log(g)^{2\alpha}$. 
**Arithmetic surfaces.** In the compact arithmetic setting, and for a Hecke eigenfunction $\psi_\lambda$, stronger bounds exist both in terms of the eigenvalue, due to Iwaniec and Sarnak [28], and in terms of the genus (or more precisely the congruence level), due to Saha and Hue-Saha [39, 25]. The bounds for $\psi$ with $\Delta \psi = \lambda \psi$ in the eigenvalue aspect is

$$\|\psi\|_\infty \lesssim \varepsilon, g^{5/24+\varepsilon} \|\psi\|_2,$$

for any $\varepsilon > 0$. In the level aspect the bound is more complex and depends on the arithmetic properties of the level but it has a power decay in terms of the genus of the form

$$\|\psi\|_\infty \lesssim \lambda \ g^{-\alpha} \|\psi\|_2$$

for some exponent $\alpha > 0$. Note that similar level aspect bounds have been obtained previously in the non-compact case of congruence covers of the modular surface [9, 22].

**Hybrid bounds.** The bounds we obtain depend implicitly on the eigenvalue. It would be interesting to have an explicit dependence both in terms of eigenvalue and genus. Such hybrid bounds were obtained in the arithmetic setting for Maass cusp forms by Templier and Saha [42, 38]. Developing such a theory on random surfaces could for example allow one to improve eigenvalue bounds for a positive measure set of surfaces. Alternatively, in a similar way as the work of Baurerschmidt, Huang and Yau [6] requires graphs of very large degree, we could expect that Conjecture 1.5 could be easier to approach if we assume the eigenvalue $\lambda$ to be large.

**Multiplicities.** As we have observed at the beginning of the introduction, the sup norm of an $L^2$ normalised eigenfunction $\psi_\lambda$ with eigenvalue $\lambda$ can be linked to the multiplicity of $\lambda$ by

$$\frac{m(\lambda)}{g} \lesssim \|\psi_\lambda\|_2^2. \quad (1.4)$$

Through this inequality our result is connected to the problem of limit multiplicities in representation theory initiated by DeGeorge and Wallach [17, 18]. Bounds for multiplicities in arithmetic settings have been studied by Sarnak and Xue [40]. Recently [1], it was proved that for a general Benjamini-Schramm converging sequence of compact hyperbolic surfaces $(X_n)$ with associated genus $g_n \to +\infty$, for any $\lambda > 0$ the ratio $m(\lambda)/g \to 0$ when $g_n \to +\infty$. Note that a sequence of random compact hyperbolic surfaces of increasing genus converges in the sense of Benjamini-Schramm to the hyperbolic plane almost surely ([34, Section 4.4]). In this case our theorem provides a rate via (1.4).

**Corollary 1.6.** Let $X$ be a random compact hyperbolic surface of genus $g$ chosen according to the probability $\mathbb{P}_g$. Denote by $m(\lambda)$ the multiplicity of an eigenvalue $\lambda \in (0, +\infty)$. Then there exists $c > 0$ such that the following bounds are satisfied with probability tending to 1 when $g \to +\infty$.

$$\frac{m(\lambda)}{g} \lesssim \frac{1}{\log g}$$

for tempered eigenvalues $\lambda \in (\frac{1}{4}, +\infty)$ and

$$\frac{m(\lambda)}{g} \lesssim \frac{1}{g^{2\varepsilon} \varepsilon},$$

for untempered eigenvalues $\lambda \in (0, \frac{1}{4} - \varepsilon)$.

Here the constant $c > 0$ determines the length $c \log(g)$ of closed geodesic loops that we can control (see Theorem 1.2). In our case $c$ can be very small and is not explicit. To make it explicit and optimise it, we would need a more careful analysis of the product in Lemma 6.3, which in turns requires more precise estimates than the ones in [36].
Optimal spectral gap. In the case of untempered eigenvalues, we expect that the multiplicity \( m(\lambda) \) tends to 0 when \( g \to +\infty \), implying that almost surely the spectral gap is optimal in the large genus limit (see [44, Section 10.4]). This can be seen as a random surfaces analogue of Selberg’s \( \frac{1}{4} \) conjecture. It is likely that a more quantitative understanding of Theorem 1.2 — and therefore a more explicit constant \( c \) — is required to prove such a result on random surfaces (see for example how such properties on short loops are used to prove an analogous theorem on regular graphs [10]). However, improving the sup norm bound for untempered eigenfunctions can only give at best \( m(\lambda) \leq 1 \) by an inequality such as (1.4), due to the absolute lower bound on sup norms (1.3). On the other hand, an optimal spectral gap theorem for random surfaces would improve Theorem 1.1 by extending the validity of the bound down to \( p > 2 \).

1.3. Outline of the article. Aside from the introduction, this article consists of five other sections organised as follows.

1. Section 2: An overview of the preliminaries of the harmonic analysis used in the proof of the deterministic results.
2. Section 3: The proof of Theorem 1.3 in the case of a hyperbolic surface with optimal spectral gap.
3. Section 4: The proof of Theorem 1.3 in the case of a hyperbolic surface with an arbitrary spectral gap.
4. Section 5: An overview of the preliminaries of the Teichmüller and random surface theory used in the proof of the probabilistic results.
5. Section 6: The proofs of Theorem 1.2 and Theorem 1.1.

2. Harmonic Analysis on Hyperbolic Surfaces

In this section, we introduce some background necessary for our investigation. Much of what is found here is standard and we refer to Katok [29] for the background on hyperbolic geometry and Bergeron [8] and Iwaniec [27] for the background on invariant integral operators and the Selberg transform.

We will work with the Poincaré upper half-plane as a model for the hyperbolic plane

\[ \mathbb{H} = \{ z = x + iy \in \mathbb{C} : y > 0 \} \]

which is equipped with the standard hyperbolic Riemannian metric

\[ ds^2 = \frac{dx^2 + dy^2}{y^2}. \]

The distance between two points \( z, z' \in \mathbb{H} \) with respect to the metric is denoted by \( d(z, z') \) and the associated hyperbolic volume is given by

\[ d\mu(z) = \frac{dx \, dy}{y^2}. \]

We identify the group of orientation-preserving isometries of \( \mathbb{H} \) with the projective special linear group \( \text{PSL}(2, \mathbb{R}) \), which contains the \( 2 \times 2 \) matrices, with real entries, that have determinant 1 modulo \( \pm I_2 \), where \( I_2 \) the \( 2 \times 2 \) identity matrix. The group acts transitively on points \( z \in \mathbb{H} \) via Möbius transformations

\[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}, \]

where \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{R}). \)

A hyperbolic surface can be seen as a quotient \( X = \Gamma \backslash \mathbb{H} \), where \( \Gamma \leq \text{PSL}(2, \mathbb{R}) \) is a fixed point free Fuchsian group. In other words, \( \Gamma \) is a fixed point free discrete subgroup of \( \text{PSL}(2, \mathbb{R}). \) Denote by \( D \subseteq \mathbb{H} \) a fundamental domain associated with \( \Gamma \). The Riemannian
metric on $\mathbb{H}$ is then naturally inherited by the quotient in the standard way as a Riemannian manifold quotient since the group acts isometrically.

The *injectivity radius* on the surface $X = \Gamma\backslash\mathbb{H}$ at a point $z$ is defined as

$$\text{InjRad}_X(z) = \frac{1}{2} \inf \{ d(z, \gamma z) : \gamma \in \Gamma \setminus \{\pm \text{id}\} \}$$

and this gives the largest $R > 0$ such that the ball $B_X(z, R)$ is isometric to a ball of radius $R$ in the hyperbolic plane. In the case when the surface $X$ is compact, there exists a universal positive lower bound for the injectivity radius at each of the points. This allows for the injectivity radius of a compact surface $X$ to be defined as

$$\text{InjRad}(X) = \inf_{z \in X} \text{InjRad}_X(z) > 0.$$

We say that a bounded measurable kernel $K: \mathbb{H} \times \mathbb{H} \to \mathbb{C}$ is invariant under the diagonal action of $\Gamma$ if for any $\gamma \in \Gamma$ we have

$$K(\gamma \cdot z, \gamma \cdot w) = K(z, w), \quad (z, w) \in \mathbb{H} \times \mathbb{H}.$$ Such kernels are also referred to as point-pair invariant.

A radial kernel $k: [0, +\infty) \to \mathbb{C}$ is a bounded, measurable, function. Given such a kernel, the mapping $K: \mathbb{H} \times \mathbb{H} \to \mathbb{C}$ given by

$$(z, w) \mapsto k(d(z, w))$$

is an invariant kernel for $(z, w) \in \mathbb{H} \times \mathbb{H}$. Conversely, an invariant kernel gives rise to a radial kernel in an obvious way and so the two can be identified.

To construct an invariant integral operator on the surface $\Gamma\backslash\mathbb{H}$, we firstly note that functions on $X$ are naturally identified with $\Gamma$-periodic functions on a fundamental domain $D \subseteq \mathbb{H}$. Given an invariant kernel $K$, we then define an associated automorphic kernel on $D \times D$ by

$$K_\Gamma(z, w) = \sum_{\gamma \in \Gamma} K(z, \gamma w).$$

This summation converges if one imposes some decay condition on the kernel $K$, such as the existence of some $\delta > 0$ such that

$$|K(\varrho)| = O\left(e^{-(1+\delta)\varrho}\right).$$

With this, we may define an associated invariant integral operator $A$ on the surface $X$ by

$$Af(z) = \int_D \sum_{\gamma \in \Gamma} K(z, \gamma w)f(w) \, d\mu(w)$$

for any $\Gamma$-invariant function $f$ and $z \in D$.

The importance of the radial operators is derived from their connection to the Laplacian. The Laplacian $\Delta$ on $\mathbb{H}$ is given in coordinates $z = x + iy$ by the differential operator

$$\Delta = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right),$$

and since the Laplacian commutes with isometries it can be considered as a differential operator on any hyperbolic surface $\Gamma\backslash\mathbb{H}$.

In the case that $X = \Gamma\backslash\mathbb{H}$ is a compact surface, the spectrum of the Laplacian on $X$ denoted by $\sigma_X(\Delta)$ is discrete and contained in the interval $[0, \infty)$. Moreover, the eigenfunctions corresponding to the eigenvalue 0 are all constant functions and thus in particular, the corresponding eigenspace is one dimensional. From the general theory of the Laplacian on compact Riemannian manifolds, there exists a sequence $0 = \lambda_0 \leq \lambda_1 \leq \ldots \to \infty$ and an orthonormal basis $\{\psi_{\lambda_i}\}_{i \geq 0}$ of $L^2(\Gamma\backslash\mathbb{H}) \cong L^2(D)$ such that

$$\Delta \psi_{\lambda_i} = \lambda_i \psi_{\lambda_i},$$
that is, $\psi_{\lambda_i}$ is an eigenfunction corresponding to the eigenvalue $\lambda_i$. In the case of a hyperbolic surface, it is instructive to partition the spectrum into two parts: the \textit{tempered spectrum} which corresponds to the portion of the spectrum inside $[\frac{1}{4}, \infty)$ and the \textit{untempered spectrum} corresponding to $[0, \frac{1}{4})$. When a surface has $\sigma_X(\Delta) \subseteq \{0\} \cup [\frac{1}{4}, \infty)$, we say that it has \textit{optimal spectral gap}.

We recall that any eigenfunction of the Laplacian is also an eigenfunction of an invariant integral operator. The corresponding eigenvalue of the integral operator is determined by the \textit{Selberg transform} $S(k)$ of the radial kernel $k: [0, \infty) \to \mathbb{C}$, which is defined as the Fourier transform

$$S(k)(r) = h(r) = \int_{-\infty}^{+\infty} e^{i r u} g(u) \, du$$

of the function

$$g(u) = \sqrt{2} \int_{|u|}^{+\infty} \frac{k(\varrho) \sinh \varrho}{\sqrt{\cosh \varrho - \cosh u}} \, d\varrho.$$ 

More precisely, we recall the following result which can be found in [8] or [27].

\begin{theorem}[(8, Sections 3.3, 3.4) or (27, Theorem 1.14)] Let $X = \Gamma \backslash \mathbb{H}$ be a hyperbolic surface and $k: [0, \infty) \to \mathbb{C}$ a radial kernel. Suppose that $\psi_\lambda$ is an eigenfunction of the Laplacian with eigenvalue $\lambda = s_\lambda^2 + \frac{1}{4}$ for $s_\lambda \in \mathbb{C}$. Then $\psi_\lambda$ is an eigenfunction of the convolution operator $A$ with invariant kernel $k$ and

$$(A\psi_\lambda)(z) = \int_{\mathbb{H}} k(d(z, w))\psi_\lambda(w) \, d\mu(w) = h(s_\lambda)\psi_\lambda(z),$$

where $h(s_\lambda) = S(k)(s_\lambda)$.

Given a suitable function $h$, one can also recover a radial kernel $k$ by taking an inverse Selberg transform:

$$g(u) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-isu} h(s) \, ds$$

and then

$$k(\varrho) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{g'(u)}{\sqrt{\cosh u - \cosh \varrho}} \, du.$$ 

3. Deterministic Bounds for Surfaces with Optimal Spectral Gap

Consider a compact hyperbolic surface $X = \Gamma \backslash \mathbb{H}$ with $D \subseteq \mathbb{H}$ a fundamental domain of $X$ and assume the short closed geodesics condition (1.1) holds for some $R \geq 0$ on $X$. In this section, we additionally assume that $X$ has an optimal spectral gap so that $\sigma_X(\Delta) \subseteq \{0\} \cup [\frac{1}{4}, \infty)$. In this case, by letting $\lambda = s_\lambda^2 + \frac{1}{4}$ be the parametrisation of the eigenvalue $\lambda$ of the Laplacian as described in Section 2, then $s_\lambda$ is either in $[0, \infty)$ or is equal to $\frac{1}{2}i$, with the latter case occurring when $\lambda = 0$.

The extra assumption on the surfaces here provides for slightly stronger results as emphasised in Theorem 1.3. Moreover, the crux of the methodology that we use to prove the result can be demonstrated without the additional technicalities that are brought about by the small eigenvalues. In fact, for this reason we also defer the proof of the result for untempered eigenfunctions to Section 4 and focus solely on the tempered portion of the spectrum.

3.1. Outline of the proof. The $L^p$ norm bounds for tempered eigenfunctions in Theorem 1.3 is proven through the following methodology. We will use Selberg’s theory to build a convolution operator $W_{R, \lambda}$ satisfying on the spectral side

$$\|W_{R, \lambda} \psi_\lambda\|_p \gtrsim R \|\psi_\lambda\|_p$$
for any eigenfunction $\psi_\lambda$ of eigenvalue $\lambda \geq \frac{1}{4}$, and on the geometric side
\[ \| W_{R, \lambda} \|_{L^2(X) \to L^p(X)} \lesssim \sqrt{R}, \]
with $R$ being given by (1.1). The latter inequality will be obtained via a $TT^*$ argument:
\[ \| T \|_{L^2(X) \to L^p(X)} = \| TT^* \|_{L^q(X) \to L^p(X)}. \]

For this purpose:
1. We firstly define via the inverse Selberg transform a family of operators $P_t$ that can be seen as a smoothened version of the wave cosine kernel $\cos(t\sqrt{\Delta})$ and which will be used as a building block for our operator $W_{R, \lambda}$.
2. Preparing for the $TT^*$ argument, we next prove a linearisation formula of the type
   \[ P_t P_s^* = \frac{1}{2} (Q_{t+s} + Q_{|t-s|}), \]
   where $Q_t$ is a family of operators studied previously by Brooks and Lindenstrauss [14]. This is done looking at the spectral action of the operators via the Selberg transform. (Lemma 3.1)
3. We use relevant bounds obtained in [14] (reproduced in Lemma 3.2) to bound the operator norms of $Q_t$ for $t \leq R$, where $R$ is given by (1.1). (Lemma 3.3)
4. The operator $W_{R, \lambda}$ is then defined roughly as
   \[ W_{T, \lambda} = \int_0^R \cos(s\lambda t) P_t dt. \]
5. We realise the $TT^*$ argument to finally bound $\| W_{R, \lambda} \|_{L^2(X) \to L^p(X)}$, and combine this with a lower bound on the spectral action of $W_{R, \lambda}$ to obtain our deterministic result. (Lemma 3.4 and Theorem 3.5)

3.2. Proof of Theorem 1.3 for optimal spectral gap surfaces. We begin by constructing a family of integral operators to analyse the eigenfunctions of the Laplacian. To this end, we define for $t \geq 0$ and $r$ in the same range as the $s_\lambda \in \mathbb{C}$, the functions $j_t$ given by
\[ j_t(r) = \sqrt{\cosh(\pi r)} \int e^{irv} \ell_t(\vartheta) \sinh(\vartheta) \sqrt{\cosh(\vartheta) - \cosh(v)} d\vartheta dv. \]

Using the Selberg transform, one may associate to $j_t$ a radial kernel $\ell_t(z, w) = \ell_t(d(z, w))$ for an integral operator $P_t$ acting on functions of $\mathbb{H}$ given by
\[ P_t f(z) = \int_\mathbb{H} \ell_t(z, w) f(w) d\mu(w). \]

The kernel $\ell_t$ is in fact real valued, which can be seen by the fact that the Selberg transform of the complex conjugate of $\ell_t$ coincides with $j_t$, since $j_t$ is real valued for the specified $r$. Indeed,
\[ \ell_t(r) = \int_{-\infty}^{\infty} e^{irv} \int |v| \ell_t(\vartheta) \sinh(\vartheta) \sqrt{\cosh(\vartheta) - \cosh(v)} d\vartheta dv. \]

Next, consider $P_t$ as an operator from $L^2(\mathbb{H}) \to L^p(\mathbb{H})$ for some $p \geq 2$. The adjoint of $P_t$ then maps $P_t^*: L^q(\mathbb{H}) \to L^2(\mathbb{H})$ where $q$ is the conjugate index of $p$, and is given by
\[ P_t^* f(z) = \int_\mathbb{H} \ell_t(d(z, w)) f(w) d\mu(w). \]
since the kernel $\ell_t$ is real and symmetric in $z$ and $w$. These operators may then be defined on the surface via the fundamental domain and using the automorphic kernel formed from the group $\Gamma$ generating the surface $X$ as described in the previous section to give

$$P_t f(z) = \int_D \sum_{\gamma \in \Gamma} \ell_t(z, \gamma w) f(w) d\mu(w),$$

and

$$P_t^* f(z) = \int_D \sum_{\gamma \in \Gamma} \ell_t(z, \gamma w) f(w) d\mu(w).$$

In Section 4 we will see that the desired result in fact holds trivially for the constant eigenfunctions, thus we will only use $P_t$ to analyse the eigenfunctions corresponding to the top of the spectrum. Thus, when testing our operator against an arbitrary function, we will remove the component of the function corresponding to the zero eigenspace. To this end, we then define the operator $\Pi$:

$$L^q(X) \to L^p(X)$$

by

$$f \mapsto f - \int_D f(z) d\mu(z) =: \bar{f},$$

where $\int_D f(z) d\mu(z) = 1/\text{Vol}(D) \int_D f(x) d\mu(z).$

Next we begin to understand the pertinent properties of the operators $P_t$. One crucial property that they possess is a linearisation formula under composition with their adjoint.

**Lemma 3.1.** The integral kernel of the composition operator $P_t P_t^*: L^q(X) \to L^p(X)$ for $t, s \geq 0$ is given by

$$\frac{1}{2} \left( k_{t+s} + k_{|t-s|} \right),$$

where $k_t$ is the associated radial kernel through the Selberg transform with the function

$$h_t(r) = \frac{\cos(rt)}{\cosh(\pi r/2)}.$$

In particular, if $Q_t: L^q(X) \to L^p(X)$ is the associated integral operator for the kernel $k_t$, then

$$P_t P_t^* \Pi = \frac{1}{2} \left( Q_{t+s} \Pi + Q_{|t-s|} \Pi \right). \quad (3.1)$$

**Proof.** This is essentially a consequence of trigonometric relations of the cosine function. Notice firstly that the kernel of $P_t P_t^*$ on $\mathbb{H}$ is given by the convolution kernel

$$m_{t,s}(z, w) = \int_{\mathbb{H}} \ell_t(z, w') \ell_s(w, w') d\mu(w'),$$

which is itself a radial kernel by invariance of the measure $d\mu$ under isometries. Let $M_{t,s}(d(z, w)) = m_{t,s}(z, w)$ denote the associated function on $\mathbb{H}$ that generates $m_{t,s}$. By Theorem 2.1, for an eigenfunction $\psi$ of the Laplacian on $\mathbb{H}$ with corresponding eigenvalue $\lambda = \frac{1}{4} + r^2$, for $r$ the eigenvalue parameter from before, we obtain

$$P_t P_t^* \psi = S(M_{t,s})(r) \psi,$$

where $S(M_{t,s})$ denotes the Selberg transform of the function $M_{t,s}$. On the other hand, by applying each of the operators in turn,

$$P_t P_t^* \psi = j_t(r) j_s(r) \psi,$$
and hence
\[ j_t(r) j_s(r) = S(M_t,s)(r). \]

Notice then for real \( r \), that one has
\[ j_t(r) j_s(r) = \frac{\cos(r(t + s))}{2\cosh(\frac{\pi r}{2})} + \frac{\cos(r|t - s|)}{2\cosh(\frac{\pi r}{2})} = \frac{1}{2}(h_{t+s}(r) + h_{|t-s|}(r)). \]

Similarly, when \( r = bi \) for \( b \in [0, \frac{1}{2}] \), we obtain
\[ j_t(r) j_s(r) = \frac{\cosh(b(t + s))}{2\cos(\frac{\pi b}{2})} + \frac{\cosh(b(t - s))}{2\cos(\frac{\pi b}{2})} = \frac{1}{2}(h_{t+s}(r) + h_{|t-s|}(r)). \]

By applying the inverse Selberg transform, it follows that
\[ m_{t,s} = \frac{1}{2}(k_{t+s} + k_{|t-s|}), \]
where \( k_t \) is as given in the statement of the lemma, and therefore
\[ P_t P_s^* = \frac{1}{2}(Q_{t+s} + Q_{|t-s|}). \]

By composing with \( \Pi \), we obtain (3.1).

We remark that the function \( h_t(r) \) in the previous lemma is precisely the Selberg transform considered by Brooks and Lindenstrauss [14]. It was introduced previously in the article of Iwaniec and Sarnak [28], where its Fourier transform was used to define a kernel to obtain sup norm bounds of eigenfunctions of the Laplacian on arithmetic surfaces. Thus much is already known regarding estimates on the kernel induced by this function through the Selberg transform. Indeed, Brooks and Lindenstrauss [14] have obtained the following bounds that are crucial in our investigation.

**Lemma 3.2 (Brooks and Lindenstrauss [14]).** With \( k_t \) as above denoting the kernel associated via the Selberg transform with the function \( h_t \), we have the following estimates. A sup norm bound of
\[ \| k_t \|_\infty \lesssim e^{-t/2}, \]
and rapid decay outside a ball of radius \( 4t \) of the type
\[ \int_4^{\infty} |k(\varrho)| \sinh(\varrho) d\varrho \lesssim e^{-t}. \]

Next we consider the operator \( Q_t \) as defined in Lemma 3.1. We combine the bounds of Lemma 3.2 with (1.1) to obtain suitable bounds on the operator norm of \( Q_t \Pi \) in terms of the parameter \( t \).

**Lemma 3.3.** Suppose that \( Q_t \) and \( \Pi \) are defined as above. For \( t \leq \frac{R}{4} \), with \( R \) as in (1.1), one may bound the \( L^p(\mathcal{X}) \to L^p(\mathcal{X}) \) operator norm by
\[ \| Q_t \Pi \|_{L^p(\mathcal{X}) \to L^p(\mathcal{X})} \lesssim e^{-\alpha_p t}, \]
where \( \alpha_p \) can be chosen to equal \((\frac{1}{2} - \delta)(1 - \frac{3}{p})\) for any \( \delta > 0 \).

**Proof.** We will proceed by interpolation, first calculating the norm \( \| Q_t \Pi \|_{L^1(\mathcal{X}) \to L^\infty(\mathcal{X})} \). We have by the definition of the automorphic kernel integral operator that
\[ \| Q_t \|_{L^1(\mathcal{X}) \to L^\infty(\mathcal{X})} \leq \sup_{z,w \in D, \gamma \in \Gamma} \sum_{\gamma \in \Gamma} |k_t(d(z, \gamma w))|. \]
This summation can then be split into two parts corresponding to propagation at times shorter and longer than $4t$. Indeed, for any $z, w \in D$,
\[
\sum_{\gamma \in \Gamma} |k_t(d(z, \gamma w))| \leq \sum_{\gamma \in \Gamma} |k_t(d(z, \gamma w))|1_{[0,4t]}(d(z, \gamma w)) + \int_{4t}^{\infty} |k_t(\rho)| e^\rho \, d\rho,
\]
with the latter integral arising from the fact that $|\{ \gamma \in \Gamma : m \leq d(z, \gamma w) \leq m+1 \}| = O(e^m)$, by a simple counting argument of the number of fundamental domains intersecting a ball of radius $m$.

By Lemma 3.2, we then obtain
\[
\sum_{\gamma \in \Gamma} |k_t(d(z, \gamma w))| \lesssim |\{ \gamma \in \Gamma | d(z, \gamma w) \leq 4t \}| e^{-t/2} + e^{-t}.
\]
Assumption (1.1) then asserts for $t \leq R/4$ that we have
\[
|\{ \gamma \in \Gamma | d(z, \gamma w) \leq 4t \}| \lesssim e^{\delta t},
\]
for any $\delta > 0$. This yields
\[
\sum_{\gamma \in \Gamma} |k_t(d(z, \gamma w))| \lesssim e^{-t(\frac{1}{2} - \delta)},
\]
and hence
\[
\|Q_t \|_{L^1(X) \rightarrow L^\infty(X)} \lesssim e^{-t(\frac{1}{2} - \delta)}.
\]
Incorporating the operator $\Pi$, we then obtain
\[
\|Q_t \Pi f\|_{\infty} = \|Q_t (f - \bar{f})\|_{\infty}
\leq \|Q_t\|_{L^1 \rightarrow L^\infty}\|f\|_1 + \|\bar{f}\|_1
\lesssim e^{-t(\frac{1}{2} - \delta)}\|f\|_1,
\]
and thus we get the bound
\[
\|Q_t \Pi\|_{L^1 \rightarrow L^\infty} \lesssim e^{-t(\frac{1}{2} - \delta)}.
\]
Next we calculate the $L^2(X) \rightarrow L^2(X)$ norm. We note that the operators $Q_t$ and $\Pi$ acting on $L^2(X)$ to $L^2(X)$ are both self-adjoint. Indeed, the former has a real and symmetric kernel and the latter is a projection. In addition, the operators $Q_t$ and $\Pi$ commute with each other since for any $f \in L^2(X)$,
\[
\Pi Q_t f(z) = Q_t f(z) - \frac{1}{\text{Vol}(D)} \int_D Q_t f(w) \, d\mu(w)
= Q_t f(z) - \frac{1}{\text{Vol}(D)} \int_D f(w') \int_D \sum_{\gamma \in \Gamma} k_t(w, \gamma w') d\mu(w) d\mu(w')
= Q_t f(z) - h_t \left( \frac{1}{2i} \right) \bar{f}
= Q_t (f - \bar{f})(z)
= Q_t \Pi f(z).
\]
This means that $Q_t \Pi$ is a self-adjoint operator from $L^2(X)$ to $L^2(X)$ and its norm is equal to its spectral radius. It follows from the projection operator, Theorem 2.1 and the fact that $X$ has optimal spectral gap, the norm is given by
\[
\|Q_t \Pi\|_{L^2(X) \rightarrow L^2(X)} = \sup_{r \in [0, \infty)} |h_t(r)| \leq 1.
\]
Finally, we apply the Riesz-Thorin interpolation theorem to get the desired bound.
\[\square\]
We now construct an operator specific to an eigenvalue \( \lambda \geq \frac{1}{4} \) of the Laplacian of \( X \). To do this, we wish to combine our propagators \( P_t \) along values of \( t \) for which the bounds obtained in Lemma 3.3 are valid. In doing so, we are able to exhibit the dependence upon the parameter \( R \) of the surface. To this end, fix \( T \leq \frac{1}{8} R \) and let \( W_{T,\lambda} : L^2(X) \to L^p(X) \) to be the operator defined for any \( p \geq 2 \) by

\[
W_{T,\lambda} f(z) = \int_0^T \cos(s\lambda t) P_t \Pi f(z) \, dt, \tag{3.4}
\]

where \( s_\lambda \) is the spectral parameter in the parametrisation \( \lambda = s_\lambda^2 + \frac{1}{4} \) of the eigenvalue.

To calculate the \( L^2(X) \to L^p(X) \) operator norm we will employ a \( TT^* \) argument, that is we use the fact that

\[
\|W_{T,\lambda}\|_{L^2(X)\to L^p(X)}^2 = \|W_{T,\lambda} W_{T,\lambda}^*\|_{L^q(X)\to L^p(X)},
\]

where \( q \) is the conjugate index of \( p \).

**Lemma 3.4.** Let \( \lambda \geq \frac{1}{4} \) be an eigenvalue of \( \Delta \) on \( X \) and fix \( T \leq \frac{1}{8} R \) where \( R \) is as in (1.1). If \( W_{T,\lambda} \) is defined as in (3.4), then

\[
\|W_{T,\lambda}\|_{L^2(X)\to L^p(X)} \lesssim_p \sqrt{T}.
\]

**Proof.** We compute through an application of Minkowski’s integral inequality that

\[
\|W_{T,\lambda} W_{T,\lambda}^*\|_{L^q(X)\to L^p(X)} = \left\| \int_0^T \int_0^T \cos(s\lambda t) \cos(s\lambda s) P_t \Pi P_s^* \, ds \, dt \right\|_{L^q(X)\to L^p(X)}
\]

\[
\leq \int_0^T \int_0^T \|P_t \Pi P_s^*\|_{L^q(X)\to L^p(X)} \, ds \, dt.
\]

It thus suffices to consider the norm \( \|P_t \Pi P_s^*\|_{L^q(X)\to L^p(X)} \).

Notice that by a similar argument to that used in the proof of Lemma 3.3, we can see that \( \Pi \) commutes with the adjoint \( P_s^* \). We then use Lemma 3.1 to deduce that

\[
\|P_t \Pi P_s^*\|_{L^q(X)\to L^p(X)} \lesssim \|Q_{t+s} \Pi\|_{L^q(X)\to L^p(X)} + \|Q_{t-s} \Pi\|_{L^q(X)\to L^p(X)}.
\]

Now since \( t + s \leq 2T \leq \frac{1}{4} R \), it follows from Lemma 3.3 that

\[
\|P_t \Pi P_s^*\|_{L^q(X)\to L^p(X)} \lesssim e^{-\alpha_p (t+s)} + e^{-\alpha_p |t-s|}
\]

\[
\lesssim e^{-\alpha_p |t-s|}.
\]

Taking \( \delta \) sufficiently small so that \( \alpha_p > 0 \) one may substitute this bound back into the integral to obtain

\[
\|W_{T,\lambda} W_{T,\lambda}^*\|_{L^q(X)\to L^p(X)} \lesssim \int_0^T \int_0^T e^{-\alpha_p |t-s|} \, ds \, dt
\]

\[
= \int_0^T \int_0^t e^{-\alpha_p (t-s)} \, ds \, dt + \int_0^T \int_t^T e^{-\alpha_p (s-t)} \, ds \, dt
\]

\[
\lesssim_p T.
\]

The bound

\[
\|W_{T,\lambda}\|_{L^2(X)\to L^p(X)} \lesssim_p \sqrt{T}
\]

is then immediate. \( \square \)

With this upper bound, we turn to examining the spectral action of \( W_{T,\lambda} \) on an eigenfunction with eigenvalue \( \lambda \). For this, we use the explicit form of the Selberg transform to obtain our desired result.
Theorem 3.5. Suppose that $X$ is a compact hyperbolic surface such that $\sigma_X(\Delta) \subseteq \{0\} \cup \left[\frac{1}{4}, \infty\right)$. If $\psi_\lambda$ is an eigenfunction of $\Delta$ with eigenvalue $\lambda \geq \frac{1}{4}$, then
\[ \|\psi_\lambda\|_p \lesssim \lambda, \frac{1}{\sqrt{R}} \|\psi_\lambda\|_2, \]
where $R$ is given by the assumption on short geodesic in (1.1). In particular, the bound always holds when $R = \text{InjRad}(X)$.

Proof. We consider the action of the test operator $W_{T,\lambda}$, given by (3.4), on $\psi_\lambda$, but with $T = \frac{1}{8}R$. By Lemma 3.4, one immediately obtains
\[ \|W_{T,\lambda}\psi_\lambda\|_p \leq \|W_{T,\lambda}\|_{L^2(X) \to L^p(X)} \|\psi_\lambda\|_2 \lesssim_p \sqrt{T}\|\psi_\lambda\|_2. \]
On the other hand, applying Theorem 2.1 provides that
\[ \|W_{T,\lambda}\psi_\lambda\|_p = \frac{1}{\sqrt{\cosh\left(\frac{\pi\lambda}{2}\right)}} \int_0^T \cos^2(s\lambda t) \|\psi_\lambda\|_p \lesssim \lambda, T\|\psi_\lambda\|_p. \]
Dividing through then gives
\[ \|\psi_\lambda\|_p \lesssim \lambda, \frac{1}{\sqrt{R}} \|\psi_\lambda\|_2. \]

\[ \square \]

4. Deterministic Bounds for Surfaces with an Arbitrary Spectral Gap

We now consider the case where the spectrum of the Laplacian on the compact hyperbolic surface $X$ takes values in the full range $[0, \infty)$. To deal with this, we utilise two separate methods for the eigenfunctions belonging to the different parts of the spectrum.

For the untempered spectrum, we demonstrate a far stronger bound on the norms of eigenfunctions than previously obtained in the optimal spectral gap case above. Indeed, we show that the norm has some exponential decay in the parameter $R$ given by (1.1). This is carried out via a rescaled ball averaging operator of functions on the surface, which was previously used by Le Masson and Sahlsten [30]. The pertinent information required here is the spectral action of this operator on eigenfunctions, which is given through the Selberg transform.

For the portion of the spectrum lying above $\frac{1}{4}$, we may use an identical technique to the optimal spectral gap case to obtain the relevant bounds. However, due to the introduction of eigenfunctions in the untempered spectrum the result is weakened slightly and is only valid for values of $p$ bounded below by a function dependent on the spectral gap of the surface. We begin by providing an outline of the proof.

4.1. Outline of proof. The methodology for the proof is similar to that in the optimal spectral gap case, so we emphasise the main differences.

(1) Firstly we show the stronger exponential decay result for the $L^p$ norms of the untempered portion of the spectrum. This is done via a rescaled averaging operator over hyperbolic balls on the surface to obtain the $L^\infty$ norm and then a simple interpolation of this with the trivial $L^2$ norm bound provides the result for general $p > 2$. (Theorem 4.1 and Corollary 4.2)

(2) For the tempered portion of the spectrum, we utilise the same method as in Section 3. The main difference is that the existence of untempered eigenfunctions, other than constants, put restrictions upon the values of $p$ for which the bounds are valid dependent upon the spectral gap. These come from a technicality in the computation of the $L^2 \to L^2$ operator norm of the propagation operator since the convolution operator eigenvalue for untempered eigenfunctions of the Laplacian exhibits exponential growth in the propagation parameter. (Theorem 4.3)
4.2. Untempered eigenfunctions deterministic bound proof. We start by defining the required ball averaging operator on the surface. Let \( (B_t)_{t \geq 0} \) denote the family of operators

\[
B_t f(z) = \frac{1}{\sqrt{\cosh(t)}} \int_{B(z,t)} f(w) d\mu(w),
\]

acting on appropriate functions of \( \mathbb{H} \). We pass this to an operator on the surface \( X = \Gamma \backslash \mathbb{H} \) by considering functions defined upon a fundamental domain \( D \) and using the automorphic kernel, so that

\[
B_t f(z) = \frac{1}{\sqrt{\cosh(t)}} \int_D \sum_{\gamma \in \Gamma} 1_{\{d(z,\gamma w) < t\}} f(w) d\mu(w).
\] (4.1)

It then follows immediately that the kernel of this operator is induced by the function

\[
k_t(\varrho) = \frac{1}{\sqrt{\cosh(t)}} 1_{\{\varrho < t\}},
\]

whose Selberg transform is given by

\[
S(k_t)(r) = 4\sqrt{2} \int_0^t \cos(ru) \sqrt{1 - \frac{\cosh(u)}{\cosh(t)}} \, du.
\]

We now prove the required bounds in order to deduce our desired result for eigenfunctions in the untempered spectrum. In doing so, we complete the result for the optimal spectral gap case in the previous section, since we then have the required bound for the constant eigenfunctions. We initially prove a slightly stronger result than required, namely that a real linear combination of real-valued Laplacian eigenfunctions corresponding to eigenvalues in the untempered spectrum have strong sup norm decay. The case of an arbitrary untempered eigenfunction follows immediately from a simplification of the proof of this result.

**Theorem 4.1.** Let \( \varepsilon > 0 \) and suppose that

\[
f = \sum_{j=1}^n \alpha_j \psi_j
\]

is a finite real linear combination of mutually orthogonal untempered real-valued eigenfunctions \( \{\psi_j\}_{j=1}^n \) of the Laplacian with corresponding eigenvalues \( \{\lambda_j\}_{j=1}^n \subseteq [0, \frac{1}{4} - \varepsilon) \). Then for any \( \delta > 0 \)

\[
\|f\|_\infty \lesssim \frac{1}{e^{(\sqrt{\varepsilon} - \delta)|R| - 1}} \|f\|_2
\]

where \( R \) is given by (1.1).

**Proof.** The eigenfunctions are smooth so it follows that \( f \) is smooth. The compactness of the surface gives that there exists \( x \in D \) such that \( |f(x)| = \|f\|_\infty \). Without loss of generality we can assume that \( f(x) > 0 \). Moreover, we can assume that each \( \alpha_j \psi_j(x) > 0 \), since removing negative terms increases the value of \( f(x) \) (and hence \( \|f\|_\infty \), albeit now potentially attained at a point different to \( x \)) whilst decreasing \( \|f\|_2 \) by orthogonality,

\[
\left\| \sum_{j \in I \subseteq \{1, \ldots, n\}} \alpha_j \psi_j \right\|_2^2 = \sum_{j \in I \subseteq \{1, \ldots, n\}} \|\alpha_j \psi_j\|_2^2 \leq \sum_{j=1}^n \|\alpha_j \psi_j\|_2^2 = \left\| \sum_{j=1}^n \alpha_j \psi_j \right\|_2^2.
\]

We now consider the ball-averaging operators defined in (4.1) for radii \( t \leq R \), where \( R \) is given by the surface assumption (1.1). The fact \( t \leq R \) means that by definition the number of terms in the summation in the automorphic kernel of \( B_t \) is bounded by \( e^{|R|t} \) for any \( \gamma > 0 \).
We now use the Selberg transform of the associated kernel function of $B_t$ to analyse the action of $B_t$ on $f$ about the point $x$,

$$B_t f(x) = \sum_{j=1}^{n} S(k_t)(s_{\lambda_j} i) \alpha_j \psi_j(x), \quad (4.2)$$

where $s_{\lambda_j} \in [\sqrt{\varepsilon}, \frac{1}{2}]$ is the eigenvalue parameter of $\lambda_j$, where $\lambda_j = \frac{1}{4} - s_{\lambda_j}^2$. We now demonstrate that the values of $S(k_t)(s_{\lambda_j} i)$ are in fact non-negative and bounded below for large enough $t$ (and hence large enough $R$ in (1.1)) by an exponentially growing term. Notice that

$$S(k_t)(s_{\lambda_j} i) \approx \int_0^t \cos(s_{\lambda_j} u) \sqrt{1 - \frac{\cosh(u)}{\cosh(t)}} \, du$$

and hence the values are non-negative. Moreover, we may use the fact that the term underneath the square root is bounded by 1 and hence the integral is bounded below by the integral without the square root which can be explicitly calculated. Thus,

$$S(k_t)(s_{\lambda_j} i) \approx \int_0^t \cos(s_{\lambda_j} u) \sqrt{1 - \frac{\cosh(u)}{\cosh(t)}} \, du$$

It is easy to see that this expression increases for all values of $t$ in the parameter $s_{\lambda_j}$ and hence we may bound this expression below with $s_{\lambda_j}$ replaced by $\sqrt{\varepsilon}$. In addition, one may observe for large enough $t$ that this expression is bounded below by $\sinh(\sqrt{\varepsilon} t)$ and in fact the size of $t$ required for this bound may be taken to be uniform in $\varepsilon$ (it suffices to take $t \geq 3$). Thus for $t \geq 3$ and each $j = 1, \ldots, n$,

$$S(k_t)(s_{\lambda_j} i) \gtrsim \sinh(\sqrt{\varepsilon} t).$$

In particular, using the non-negativity of each term in (4.2), we thus obtain for $t \geq 3$ that

$$B_t f(x) \gtrsim \sinh(\sqrt{\varepsilon} t) f(x) = \sinh(\sqrt{\varepsilon} t) \|f\|_{\infty}.$$  

Conversely, notice for $t \leq R$ that there is at most $e^{\delta t}$ non-zero terms in the summation for the automorphic kernel of $B_t$ for any $\delta > 0$, and hence we have

$$\left\| \sum_{\gamma \in \Gamma} \frac{1_{\{d(x, \gamma \cdot) \leq t\}}}{\sqrt{\cosh(t)}} \right\|_2^2 \leq \int_D \sum_{\gamma \in \Gamma} \frac{1_{\{d(x, \gamma w \cdot) \leq t\}}}{\cosh(t)} \, d\mu(w) \leq \delta \frac{e^{\delta t}}{\cosh(t)} \text{ Vol(Ball of radius } t) \lesssim \delta e^{\delta t}. $$
By the Cauchy-Schwarz inequality, we obtain for all $\delta > 0$ that

$$|B_t f(x)| = \left| \int_D \sum_{\gamma \in \Gamma} \frac{1_{d(x,\gamma) \leq t}}{\cosh(t)} f(w) \, d\mu(w) \right|$$

$$\leq \left| \sum_{\gamma \in \Gamma} \frac{1_{d(x,\gamma) \leq t}}{\cosh(t)} \right| \|f\|_2$$

$$\lesssim \delta e^{\delta t} \|f\|_2.$$

We may then combine the two inequalities so that for any $\delta > 0$ and $3 \leq t \leq R$,

$$\|f\|_\infty \lesssim \frac{e^{\delta t}}{\sinh(\sqrt{\varepsilon} t)} \|f\|_2.$$

It follows that

$$\|f\|_\infty \lesssim \frac{1}{e(\sqrt{\varepsilon} - \delta) R - 1} \|f\|_2,$$

and taking $t = R$ then gives the result. 

By using the same argument as in the above proof and applying an interpolation argument on the norms, we obtain the desired eigenfunction bound for any eigenfunctions corresponding to an eigenvalue in the untempered spectrum.

**Corollary 4.2.** Suppose that $\psi_\lambda$ is an eigenfunction of the Laplacian for the surface $X$ with eigenvalue $\lambda \in \left[0, \frac{1}{4}\right)$. Then for any $\varepsilon > 0$ for which $\lambda \in \left[0, \frac{1}{4} - \varepsilon\right)$ we have

$$\|\psi_\lambda\|_p \lesssim \delta \left( \frac{\lambda}{e(\sqrt{\varepsilon} - \delta) R - 1})^{\frac{1}{2} - \frac{2}{p}} \right) \|\psi_\lambda\|_2,$$

for any $\delta > 0$, where $R$ is given by (1.1).

**Proof.** Once again, by compactness of $D$ there exists some $x \in D$ for which $|\psi_\lambda(x)| = \|\psi_\lambda\|_\infty$. Using the ball averaging operator then gives that

$$|B_t \psi_\lambda(x)| = |S(k_t)(s_\lambda)| |\psi_\lambda(x)|.$$

For $t \geq 3$, we obtain as in Theorem 4.1 that

$$|B_t \psi_\lambda(x)| \geq \sinh(\sqrt{\varepsilon} t) |\psi_\lambda|_\infty.$$

Analysing the upper bound as before then results in

$$\|\psi_\lambda\|_\infty \lesssim \delta \left( \frac{\lambda}{e(\sqrt{\varepsilon} - \delta) R - 1})^{\frac{1}{2} - \frac{2}{p}} \right) \|\psi_\lambda\|_2,$$

for any $\delta > 0$. We now use interpolation to see that

$$\|\psi_\lambda\|_p \leq \|\psi_\lambda\|_2 \|\psi_\lambda\|_\infty^{\frac{2}{p}} \lesssim \delta \left( \frac{\lambda}{e(\sqrt{\varepsilon} - \delta) R - 1})^{\frac{1}{2} - \frac{2}{p}} \right) \|\psi_\lambda\|_2.$$

\[\square\]
4.3. Proof of Theorem 1.3. For the tempered eigenfunctions, we can use the same method as in the optimal spectral gap case. The smaller spectral gap associated with the surface weakens the values of $p$ for which the result holds, however at worst we obtain that the bounds are valid for at least $p > 4$.

**Theorem 4.3.** Suppose that $X$ is a compact hyperbolic surface whose smallest non-zero eigenvalue of the Laplacian is at least $\frac{1}{2} - \beta^2$ for some $\beta \in [0, \frac{1}{2})$. For a tempered eigenfunction $\psi_\lambda$ of the Laplacian with eigenvalue $\lambda \geq \frac{1}{4}$, we have the following bound

$$\|\psi_\lambda\|_p \lesssim_{p, \lambda} \frac{1}{\sqrt{R}} \|\psi_\lambda\|_2,$$

for any $2 + 4\beta < p \leq \infty$, with $R$ as in (1.1).

**Proof.** We utilise the operator $W_{T, \lambda}$ as defined by (3.4). As in Lemma 3.4, the calculation of the $L^2(X) \to L^p(X)$ norm of $W_{T, \lambda}$ is reduced to computing the operator norms

$$\|Q_t \Pi\|_{L^1(X) \to L^{\infty}(X)} \quad \text{and} \quad \|Q_t \Pi\|_{L^2(X) \to L^2(X)},$$

where $Q_t$ is the operator defined in Lemma 3.1. Using the same argument as in Lemma 3.3, with $t \leq R/4$ we obtain that

$$\|Q_t \Pi\|_{L^1(X) \to L^{\infty}(X)} \lesssim e^{-t(\frac{1}{2} - \delta)},$$

for any $\delta > 0$. For the $L^2(X) \to L^2(X)$ norm, we notice that in this case there is an exponential growth in the spectral radius. Indeed, we now have

$$\|Q_t \Pi\|_{L^2(X) \to L^2(X)} = \sup_{\delta \in [0, \infty), \, \alpha \in [0, \beta]} \frac{|\cos(rt)|}{\cosh(\pi r/2)} \leq e^{\beta t}.$$

Applying the Riesz-Thorin Interpolation Theorem, we then obtain for the conjugate exponent $q$ of $p$ and any $\delta > 0$

$$\|Q_t \Pi\|_{L^q(X) \to L^p(X)} \lesssim e^{-t(\frac{1}{2} - \delta - \frac{1}{p} + \frac{2\beta - \beta^2}{p})}.$$

When $p > 2 + 4\beta$ (assuming $\delta < 1$), the norm exhibits exponential decay. Since this is true for all $0 < \delta < 1$, it follows that there is exponential decay whenever $p > 2 + 4\beta$ and in this case, we can show as in Lemma 3.4 that

$$\|W_{T, \lambda}\|_{L^2(X) \to L^p(X)} \lesssim p \sqrt{T}.$$

Since the spectral action of $W_{T, \lambda}$ on $\psi_\lambda$ is identical to that considered in Theorem 3.5, we also recover the lower bound

$$\|W_{T, \lambda} \psi_\lambda\|_p \gtrsim T \|\psi_\lambda\|_p.$$

Combining these two estimates gives the desired result. \qed

Theorem 1.3 is then obtained by combining Theorem 3.5, Corollary 4.2 and Theorem 4.3.

5. Teichmüller Theory and Random Surfaces

This section gathers much of the background required and notation utilised when formulating and working with probabilistic statements on surfaces in this paper. Further details on the foundational material on Teichmüller theory, geodesics and mapping class groups can be found in [26], [15] and [20].

Let $g, n \geq 0$ be integers. We will denote by $\Sigma_{g, n}$ a surface of genus $g$ with $n$ boundary components; if $n = 0$ this is simply written as $\Sigma_g$. Given the $n$ boundary components, one can associate a length vector $L = (L_1, \ldots, L_n) \in \mathbb{R}_{\geq 0}^n$ to the surface such that the $i^{th}$
boundary component has length $L_i$. If $L_i = 0$, then the component is thought of as a cusp or marked point on the surface.

The Teichmüller space of signature $(g, n)$ and length vector $L \in \mathbb{R}^n_{\geq 0}$ is defined to be the space

$$T_{g,n}(L) = \left\{ (X, f): \text{geodesic boundary components with lengths corresponding to } L \text{ and } f: \Sigma_{g,n} \to X \text{ is a diffeomorphism} \right\} / \sim,$$

where $\sim$ is the equivalence relation defined by $(X, f) \sim (Y, g)$ if and only if there exists a biholomorphism $h: X \to Y$ for which

$$g^{-1} \circ h \circ f: \Sigma_{g,n} \to \Sigma_{g,n}$$

is isotopic to the identity or equivalently, if $g \circ f^{-1}: X \to Y$ is isotopic to an isometry. In an element $[X, f]$, the mapping $f$ is called a marking on $X$. This space can be endowed with a natural complex analytic structure for which it is independent of the base surface $\Sigma_{g,n}$ up to biholomorphism. For notation, when $L$ is the zero vector we denote $T_{g,n} = T_{g,n}(0, \ldots, 0)$ and when there are no boundary components we simply write $T_g$ for $T_{g,0}$.

There exists a natural group action on the Teichmüller space that acts by changing the marking. The group is called the mapping class group $\text{Mod}_{g,n}(\Sigma_{g,n})$ and is defined as the collection of orientation-preserving diffeomorphisms of $\Sigma_{g,n}$ that fix the boundary components setwise identified up to isotopy to the identity mapping. If $[\varphi] \in \text{Mod}_{g,n}(\Sigma_{g,n})$ then the action on an element $[X, f] \in T_{g,n}(L)$ is given by

$$[\varphi] \cdot [X, f] = [X, f \circ \varphi^{-1}].$$

The moduli space $M_{g,n}(L)$ is then the space obtained through identification of points in the Teichmüller space up to the mapping class group action. That is,

$$M_{g,n}(L) = T_{g,n}(L)/\text{Mod}_{g,n}(\Sigma_{g,n}).$$

As with the Teichmüller space, we use the shorthand notation $M_{g,n} = M_{g,n}(0, \ldots, 0)$ and $M_g = M_{g,0}$.

As well as a group action, there is an associated symplectic form on $T_{g,n}(L)$ called the Weil-Petersson form denoted by $\omega_{g,n}$ which is invariant under the action of the mapping class group (see Goldman [21]). Due to this invariance, the form passes also to the moduli space and hence provides a volume form on $M_{g,n}(L)$ called the Weil-Petersson volume

$$\frac{\wedge^{3g+n-3}\omega_{g,n}}{(3g+n-3)!}.$$

In particular, we write

$$V_{g,n}(L) = \int_{M_{g,n}(L)} \frac{\wedge^{3g+n-3}\omega_{g,n}}{(3g+n-3)!},$$

for the volume of $M_{g,n}(L)$ and use the shorthand notation $V_{g,n} = V_{g,n}(0, \ldots, 0)$ and $V_g = V_{g,0}$.

Some particularly important results concerning volumes of moduli spaces that will be made use of here are from Mirzakhani [34] and Mirzakhani and Zograf [36] and we reproduce them for the convenience of the reader. The first allows one to relate the volumes $V_{g,n}(L)$ to $V_{g,n}$.

**Lemma 5.1** (Mirzakhani [34, Equation 3.7]). Given any $g, n \in \mathbb{N}$ and $L \in \mathbb{R}^n_{\geq 0}$,

$$V_{g,n}(2L) \leq e^{|L|} V_{g,n},$$

where $|L| = L_1 + \cdots + L_n$.

The second result shows a relationship between volumes with different genus and boundary components. For $g \to \infty$, the relation is asymptotically sharp.
Lemma 5.2 (Mirzakhani [34, Equation 3.20]). Given \( g, n \in \mathbb{N} \cup \{0\} \) with \( 2g - 2 + n \geq 0 \) and \( 0 \leq i \leq n/2 \),
\[
V_{g,n} \lesssim V_{g+i,n-2i},
\]
where the implied constant is independent of \( g, n \) and \( i \).

The last volume estimate result we need provides growth estimates for moduli space volumes in the large genus limit.

Theorem 5.3 (Mirzakhani and Zograf [36, Theorem 1.2]). There exists a universal constant \( C \in (0, \infty) \) such that for any given \( k \geq 1 \) and \( n \geq 0 \),
\[
V_{g,n} = C \sqrt{g} (2g - 3 + n)! (4\pi^2)^{2g-3+n} \left( 1 + O\left( \frac{1}{g} \right) \right),
\]
as \( g \to \infty \). In particular,
\[
V_g = C \sqrt{g} (2g - 3)! (4\pi^2)^{2g-3} \left( 1 + O\left( \frac{1}{g} \right) \right),
\]
as \( g \to \infty \).

Notice in particular that the volume of the moduli space is finite and hence there is a probability measure on the moduli space called the Weil-Petersson probability measure, \( \mathbb{P}_{g,n} \). If \( A \subseteq \mathcal{M}_{g,n} \) we will write
\[
\mathbb{P}_{g,n}(A) = \frac{1}{V_{g,n}} \int_{\mathcal{M}_{g,n}} 1_A(X) dX,
\]
where we use \( dX \) as shorthand for the Weil-Petersson volume measure and \( X \) for an element of the moduli space. Moreover, one can determine the expectation of a measurable function \( F : \mathcal{M}_{g,n} \to \mathbb{R} \) by
\[
\mathbb{E}_{g,n}(F) = \frac{1}{V_{g,n}} \int_{\mathcal{M}_{g,n}} F(X) dX.
\]

An extremely useful result that we will use for calculating integrals of certain functions over moduli space will be Mirzakhani’s integral formula. For this, we need to introduce the notion of cutting open a surface along a system of curves.

To this end, recall that in the free homotopy class of a simple closed curve on a hyperbolic surface, there exists a unique simple closed geodesic minimising length amongst all curves in the homotopy class. When we consider a simple closed curve, we will always be considering the free homotopy class or simple closed geodesic representative in this class. If \( \Gamma = \{ \gamma_1, \ldots, \gamma_k \} \) is a \( k \)-tuple of such curves on \( \Sigma_g \) then one may consider the (possibly disconnected) surface obtained from cutting \( \Sigma_g \) along the curves in \( \Gamma \). This is most naturally done by removing a regular neighbourhood of \( \Gamma \), denoted \( N(\Gamma) \), from \( \Sigma_{g,n} \); that is, a metric neighbourhood about each curve that retracts back to the curve. Suppose that \( \Sigma_g \setminus N(\Gamma) = \bigsqcup_{i=1}^q \Sigma_{g_i,n_i}(L_i) \) where
\[
\sum_{i=1}^q n_i = 2k,
\]
and the \( L_i \in \mathbb{R}_{\geq 0}^n \) partition the vector \( L = (\ell(\gamma_1), \ell(\gamma_1), \ldots, \ell(\gamma_k), \ell(\gamma_k)) \) with \( \ell(\gamma_i) \) being the length of the geodesic \( \gamma_i \). We then write
\[
V_g(\Gamma, L) = \prod_{i=1}^q V_{g_i,n_i}(L_i).
\]
For Mirzakhani’s integral formula, one considers the integral of so-called geometric functions on the moduli space. These are defined from a multicurve such as $\Gamma$ above. Indeed, let $F : \mathbb{R}^+_k \to \mathbb{R}$ be a function, then define $F^\Gamma : \mathcal{M}_g \to \mathbb{R}$ by

$$F^\Gamma(X) = \sum_{(\alpha_1, \ldots, \alpha_k) \in \text{Mod}(\Sigma_g) \cdot \Gamma} F(\ell_X(\gamma_1), \ldots, \ell_X(\gamma_k)),$$

where $\ell_X(\gamma_i)$ is the length of the simple closed geodesic in the free homotopy class of the image of $\gamma_i$ under the marking on $X$. Mirzakhani’s integral formula is then stated as follows.

**Theorem 5.4** (Mirzakhani [32, Theorem 7.1]). Given a $k$-tuple of simple closed curves $\Gamma$ and a function $F : \mathbb{R}^+_k \to \mathbb{R}$, one has

$$\int_{\mathcal{M}_g} F^\Gamma(X) dX = C_\Gamma \int_{\mathbb{R}^+_k} F(x) V_g(\Gamma, (x_1, x_1, \ldots, x_k, x_k)) x_1 \cdots x_k \prod_{i=1}^k dx_i,$$

where $C_\Gamma$ is a constant dependent upon the number of symmetries of the multicurve $\Gamma$ and the number of handles $\Gamma$ separates from $\Sigma_g$.

### 6. Short Geodesic Loops on Random Surfaces

Consider a compact hyperbolic surface $X = \mathbb{H}/\Gamma$ with a fundamental domain $D$. Recall that the assumption on the surfaces we considered in (1.1) was the existence of an $R \geq 0$ such that for all $z, w \in D$

$$|\{ \gamma \in \Gamma : d(z, \gamma w) < r \}| \lesssim e^{\delta r}, \quad \text{for all } \delta > 0 \text{ and } r \leq R.$$

In this section, we demonstrate that one can take $R = c \log(g)$ for sufficiently small $c$ independent of the genus $g$ such that the assumption is satisfied asymptotically almost surely as $g \to \infty$. For this, we show the sufficient condition that about any point $z \in D$ there is at most one geodesic curve passing through $z$ that projects to a closed geodesic loop on the surface with length at most $c \log(g)$ almost surely as $g \to \infty$; that is, we show Theorem 1.2. An outline of the proof of this result is given as follows.

#### 6.1. Outline of the proof.

Theorem 1.2 is proven using contradiction via the following methodology.

1. Given two closed geodesics of length at most $c \log(g)$ on a hyperbolic surface $X$ of genus $g$ passing through the same point, we determine an upper bound on the number of self-intersections that these two curves can have between one another and with themselves. (Lemma 6.2)

2. The bound determined then provides an upper bound on the number of components in a multicurve obtained by taking a regular neighbourhood of the original curves and we show that for large enough genus $g$, this multicurve is separating and has total length at most $4c \log(g)$. (Lemma 6.2)

3. We next prove an estimate on the order of growth on the sum of the products of the volumes of moduli spaces given by cutting along a multicurve as above over all possible configurations of subsurface genera that such a multicurve could cut into given the number of components in the curve. (Lemma 6.3)

4. Using this estimate, we show that asymptotically as $g \to \infty$ such a multicurve almost surely does not exist on the surface. This is done by computing an upper bound on the number of separating multicurves with a bounded number of components (computed by (2)) with length at most $4c \log(g)$ that can exist on a surface and showing it asymptotically tends to zero as $g \to \infty$. (Theorem 6.4)

5. One can then conclude that on almost all surfaces there is at most one closed geodesic loop of length at most $c \log(g)$ based at a point asymptotically as $g \to \infty$. 


6.2. **Bounds on** $N_{c \log(g)}(X)$ **are sufficient for condition** \( (1.1) \). Before starting the proof, we provide a simple argument demonstrating that showing condition \( (1.2) \) holds is sufficient for \( (1.1) \) and hence provides the probabilistic result that we seek.

**Lemma 6.1.** If for all \( z \in \mathbb{H} \) one can show that
\[
P_g(X = \mathbb{H}/\Gamma \in M_g : |\{ \gamma \in \Gamma \setminus \{id\} : d(z, \gamma z) < 2r\}| > 2) \to 0,
\]
as \( g \to \infty \), then it follows that for all \( z, w \in \mathbb{H} \) that
\[
P_g(X = \mathbb{H}/\Gamma \in M_g : |\{ \gamma \in \Gamma \setminus \{id\} : d(z, \gamma w) < r\}| > 3) \to 0,
\]
In particular, condition \( (1.2) \) implies \( (1.1) \).

**Proof.** Fix \( z, w \in \mathbb{H} \) and suppose that \( \gamma_i \) for \( i = 1, \ldots, 4 \) are distinct non-identity elements of \( \Gamma \) such that
\[
d(z, \gamma_i w) < r,
\]
for each \( i \). Notice then that the elements \( \gamma_j \gamma_1^{-1} \) are each distinct for \( j = 2, 3, 4 \) in \( \Gamma \) and are not the identity. Moreover, for each \( j = 2, 3, 4 \) we have
\[
d(\gamma_1 w, (\gamma_j \gamma_1^{-1})(\gamma_1 w)) \leq d(\gamma_1 w, z) + d(\gamma_j w, z) < 2r.
\]
This means that there are at least three elements in the set
\[
\{ \gamma \in \Gamma \setminus \{id\} : d(\gamma_1 w, \gamma \gamma_1 w) < 2r\},
\]
which by assumption asymptotically has probability tending to zero in the \( g \to \infty \) limit and so the result holds. \( \square \)

6.3. **Proving Theorems 1.1 and 1.2.** Armed with Lemma 6.1, we now work toward showing that there exists a sufficiently small \( c > 0 \) independent of \( g \) for which
\[
|\{ \gamma \in \Gamma : d(z, \gamma z) < r\}| \leq 2, \text{ whenever } r \leq c \log(g),
\]
almost surely asymptotically as \( g \to \infty \). In other words, we aim to show that there is at most one geodesic loop on the surface that passes through the point \( z \) with length at most \( c \log(g) \) for almost every surface of genus \( g \) as \( g \to \infty \).

To this end, we begin by demonstrating that if there were two such curves, then asymptotically as \( g \to \infty \) they would generate a multicurve on the surface consisting of simple closed geodesics of total length at most \( 4c \log(g) \) that together separate the surface when cut along. This result is in fact an improvement on the result of Mirzakhani and Petri [35, Proposition 4.5] where dependence upon the lengths of curves on the genus was not implicitly accounted for.

**Lemma 6.2.** Suppose that \( \alpha \) and \( \beta \) are curves in the surface \( \Sigma_g \) intersecting at least one point whose lengths are bounded by \( c \log(g) \) for \( c < \frac{1}{2} \) then, there exists a separating multicurve \( \Gamma \) on \( \Sigma_g \) consisting of \( O(g^{2c}) \) simple closed geodesics whose total length is bounded by \( 4c \log(g) \) for \( g \) sufficiently large.

**Proof.** The proof is similar to that of [35, Proposition 4.5] and so we only sketch the portions that require further considerations. The crux of the argument is to determine the number of intersections between \( \alpha, \beta \) and themselves to understand the Euler characteristic of the subsurface obtained by taking a regular neighbourhood (neighbourhood in the sense of the underlying metric on the surface such that the boundary retracts back to the set) of \( \alpha \cup \beta \) which will be a collection of disjoint simple closed geodesics and then cutting \( \Sigma_g \) along this multicurve and seeing that it is separating.

It is shown in [5, Theorem 1.2] that for a closed geodesic \( \gamma \) on a hyperbolic surface of length \( L \), the total number of self intersections \( \gamma \) can have is bounded by \( 2e^{2L} \). Thus for \( \alpha \) and \( \beta \) with length at most \( c \log(g) \) the total number of self intersections of either curve with
itsl is given by $2g^{2c}$. The argument to determine the number of intersections between $\alpha$ and $\beta$ is then shown as in [35, Proposition 4.5] and can be bounded by

$$
\left( \frac{c\log(g)}{\log(\coth(\frac{c\log(g)}{4}))} \right)^2.
$$

It can easily be seen that

$$\frac{1}{\log(\coth(x))} = O(e^{2x}),$$

and hence for $x = \frac{c}{4}\log(g)$ we have that the total number of intersections $I$ between the curves is $O(g^{2c})$.

Now, consider a regular neighbourhood of $\alpha \cup \beta$ in $\Sigma_g$. The boundary of this neighbourhood will be a collection of disjoint simple closed curves and we consider the multicurve $\Gamma$ consisting of the simple closed geodesics that are freely homotopic to the boundary curves (throwing away any such repeated curves). By construction when taking the neighbourhood of the set, each boundary component will be homotopic to simple closed segments of $\alpha \cup \beta$ with each such segment appearing exactly twice (the portion of the neighbourhood either side of the union of the curves). Since the geodesics in the free homotopy classes are length minimising, their total sum must then be at most twice the total sum of the curves $\alpha$ and $\beta$ from this double counting and so the total length of the multicurve constructed is bounded by $4c\log(g)$.

![Diagram of the formation of the subsurface $\Sigma_{g',n'}$ from the regular neighbourhood of $\alpha \cup \beta$ in $\Sigma_g$.](image)

**Figure 1.** Some possibilities of the formation of the subsurface $\Sigma_{g',n'}$ from the regular neighbourhood of $\alpha \cup \beta$ in $\Sigma_g$. One begins by taking the regular neighbourhood of the union $\alpha \cup \beta$ and homotoping the boundary components to their geodesic representations (middle figure). Then one cuts along these geodesics to obtain the subsurface $\Sigma_{g',n'}$ (bottom figure).

It is clear then by construction that $\alpha$ and $\beta$ are together filling curves for the subsurface constructed by the regular neighbourhood. Thus, if $(g',n')$ is the signature of this subsurface $\Sigma_{g',n'}$ we have by [3, Lemma 2.1] that

$$2g' + n' - 2 \leq I - 1.$$ 

If also $\alpha$ and $\beta$ filled the surface $\Sigma_g$ then by the same argument one would have that

$$2g - 2 \leq I - 1,$$

which for $g$ sufficiently large is not possible since $I = O(g^{2c}) = o(g)$, and so $\Gamma$ is separating when $g$ is large enough. Two such possibilities of how this multicurve could separate the
surface are given in Figure 1. The number of components in \( \Gamma \) is given by \( n' \) which from the inequality \( n' \leq 2g' + n' \leq I + 1 \) is seen to be \( O(g^{2e}) \).

So with this result, from two non-identity elements in the set \( \{ \gamma \in \Gamma : d(z, \gamma z) < r \} \) we obtain for large enough genus a separating multicurve with total length at most \( 4c \log(g) \) consisting of disjoint simple closed geodesics. To understand how such a multicurve can cut \( \Sigma_g \), we require an estimate on the product of volumes of moduli spaces of the subsurfaces obtained from cutting along the multicurve when the lengths of the curves can depend on the genus. In particular, we wish to see how the sum of such products can grow over all possible genera configurations on the subsurfaces from a given number of boundary components on each subsurface. The starting point for this is the relation between different volumes given in Mirzakhani [34, Lemma 3.2] which is reproduced here in Lemma 5.2 and the growth estimate on volumes of moduli spaces from Mirzakhani and Zograf [36] stated in Theorem 5.3. Through carefully analysing the product we obtain the following volume growth estimate.

**Lemma 6.3.** Suppose that \( q, k(g), n_1(g), \ldots, n_q(g) \in \mathbb{N} \) with \( q \leq k(g) + 1 \) and \( \sum_{i=1}^{q} n_i(g) = 2k(g) \), then

\[
\sum \prod_{\{g_i\}} q V_{g_i, n_i(g)} = O \left( \frac{V g^2k(g) D(k(g) \sqrt{k(g)})}{g^{\frac{3}{2}(q-1)}} \right),
\]

as \( g \to \infty \) where \( k(g) = O(g^{2e}) \) and the sum is over all multisets \( \{g_i\}_{i=1}^{q} \subseteq \mathbb{N} \) such that \( \sum_{i=1}^{q} g_i = g + q - k(g) - 1 \) and \( 2g_i - 3 + n_i \geq 0 \) for all \( i = 1, \ldots, q \) and \( D \) is some universal constant independent of all of the parameters.

**Proof.** By Lemma 5.2, one has

\[
V_{g_i, n_i(g)} \leq \begin{cases} \sqrt{V_{g_i+n_i(g)/2,0}} & \text{for } n_i(g) \text{ even}, \\ \sqrt{V_{g_i+n_i(g)/2-1/2,1}} & \text{for } n_i(g) \text{ odd}. \end{cases}
\]

In either case, by Theorem 5.3

\[
V_{g_i, n_i(g)} = C(2g_i + n_i(g) - 3)! \left( \frac{4\pi^2}{e} \right)^{2g_i+n_i(g) - 3} \left( \frac{1}{\sqrt{g_i}} \right)^{2g_i - 3} \left( 1 + O \left( \frac{1}{g_i + n_i(g)/2} \right) \right).
\]

This latter remainder term can be bounded by some \( C' \) independent of \( g_i \) and \( n_i(g) \) and so we have

\[
\frac{1}{V_g} \sum \prod_{\{g_i\}} q V_{g_i, n_i(g)} = D(k(g)) \sum_{\{g_i\}} \prod_{i=1}^{q} \frac{(2g_i - 3 + n_i(g))/(4\pi^2) \left( \frac{e}{4\pi^2} \right)^{2g_i-3+n_i(g)}}{\max \{ 1, \sqrt{g_i} \}} \left( \frac{1}{\sqrt{g}} \right)^{2g - 3} \left( \frac{4\pi^2}{e} \right)^{2g - 3} \left( \frac{1}{\sqrt{g_i}} \right)^{2g - \frac{5}{2}} \prod_{i=1}^{q} \max \{ 1, \sqrt{g_i} \}.
\]

for some constant \( D \) independent of the \( n_i, g, k \) and \( q \). To tackle the factorial terms, we use Stirling’s approximation to infer that \( n! \asymp \sqrt{n} \left( \frac{n}{e} \right)^n \) so that the summand is bounded up to a constant uniform in \( g, q \), the \( n_i(g) \) and \( k(g) \) by

\[
\sqrt{g} \prod_{i=1}^{q} \frac{(2g_i - 3 + n_i(g))/(4\pi^2) \left( \frac{e}{4\pi^2} \right)^{2g_i-3+n_i(g)}}{\max \{ 1, \sqrt{g_i} \}} \leq 1,
\]

since \( q \geq 2 \). Next,

\[
\frac{\sqrt{g}}{\prod_{i=1}^{q} \max \{ 1, \sqrt{g_i} \}} = \frac{\sqrt{g} (1 - q + \sum_{i=1}^{q} g_i)}{\prod_{i=1}^{q} \max \{ 1, \sqrt{g_i} \}} = O(\sqrt{k(g)}).
\]
Lastly, one can observe that
\[
\prod_{i=1}^{q} (2g_i - 3 + n_i(g))^{2g_i - \frac{5}{2} + n_i(g)} \leq \prod_{i=1}^{q} \left( 2g_i - \frac{5}{2} + n_i(g) \right)^{2g_i - \frac{5}{2} + n_i(g)}.
\]
Thus up to a constant independent of \(k(g), g\) and \(q\) the sum of the products is bounded by
\[
\sqrt{k(g)} \sum_{\{g_i\}} \prod_{i=1}^{q} \frac{(2g_i - \frac{5}{2} + n_i(g))^{2g_i - \frac{5}{2} + n_i(g)}}{(2g - 3)^{2g - \frac{5}{2}}}.
\]
One can split the summation over the multi-sets over the sum of the number of non-zero terms and the second is an upper bound on the number of ways of choosing the \(j\) non-zero terms and \(j\) non-zero terms such that they sum to \(2g - 2 - \frac{q}{2}\) (given by the Euler characteristic constraint in the hypothesis of the result). Note that this second term is bounded up to a constant by \((2g - 3)^{j-1}\). Now, the maximal term in the interior sum is given when \(j - 1\) of the terms are 1 and the remaining non-zero term is \(2g - 2 - \frac{q}{2} - (j - 1)\). This gives an upper bound of
\[
\frac{(2g - 2 - \frac{q}{2} - (j - 1))^{2g - 2 - \frac{q}{2} - (j - 1)}}{(2g - 3)^{2g - \frac{5}{2}}} \leq \frac{(2g - 3)^{2g - \frac{5}{2} - \frac{1}{2}(q - 1) - (j - 1)}}{(2g - 3)^{2g - \frac{5}{2}}}
\leq \frac{1}{(2g - 3)^{\frac{1}{2}(q - 1) + (j - 1)}}.
\]
This means that
\[
\sqrt{k(g)} \sum_{\{g_i\}} \prod_{i=1}^{q} \frac{(2g_i - \frac{5}{2} + n_i(g))^{2g_i - \frac{5}{2} + n_i(g)}}{(2g - 3)^{2g - \frac{5}{2}}} \leq \sqrt{k(g)} \sum_{j=1}^{q} \frac{\binom{q}{j}}{(2g - 3)^{\frac{1}{2}(q - 1)}}
= O \left( \frac{\sqrt{k(g)} 2^q}{g^{\frac{1}{2}(q-1)}} \right)
= O \left( \frac{\sqrt{k(g)} 2^{K(g)}}{g^{\frac{1}{2}(q-1)}} \right).
\]
We now show that a separating multicurve as in Lemma 6.2 existing on a surface \(\Sigma_g\) tends to zero in the Weil-Petersson probability asymptotically as \(g \to \infty\). For this, we let \(K(g)\) denote the number of components in the multicurve so that by Lemma 6.2, \(K(g) = O(g^{2c})\) for \(c < \frac{1}{2}\).
Theorem 6.4. Choosing $c$ sufficiently small independently of $g$ and $K(g) = O(g^{2c})$, we have that

$$
\mathbb{P}_g\left(X \in \mathcal{M}_g : \text{most } K(g) \text{ disjoint simple closed curve components of total length at most } 4c \log g.\right) \rightarrow 0,
$$
as $g \rightarrow \infty$. In fact there exists a universal constant $\kappa > 0$ such that this probability is a $O(g^{-\frac{1}{2} + \kappa c})$, where the implied constant is independent of $c$.

Proof. Suppose that $\Gamma$ is a separating multicurve and let $k(\Gamma)$ denote the number curve components in $\Gamma$. We bound the probability by the sum over all mapping class orbits of multicurves $[\Gamma]$ of multicurves with $k(\Gamma) = k$ for $k = 1, \ldots, K(g)$ and then sum over each multicurve in each orbit. This decomposition will in particular count a multicurve of $k$ components $k!$ times (since any permutation of the curves in the multicurve is also in this sum). To account for this overcounting, for each $k$ we divide through by $k!$. Note also that the number of such curves in the sums is finite since the number of simple closed geodesics on the surface is also finite. If we denote $L = 4c \log(g)$ then the probability we wish to compute is bounded by

$$
\frac{1}{V_g} \sum_{k=1}^{K(g)} \sum_{[\Gamma]} \frac{1}{k!} \int_{\mathcal{M}_g} \sum_{\gamma \in [\Gamma]} 1_{[0,L]}(\ell_{\gamma_1}(X) + \cdots + \ell_{\gamma_k}(X)) dX.
$$

Here $\gamma_i$ denotes the $i^{th}$ component of the multicurve $\Gamma'$ in the mapping class group orbit of $\Gamma$. We next separate the sum over $[\Gamma]$ to account for the number of connected components that such a curve splits the surface into. If we denote by $q(\Gamma)$ the number of connected components in $\Sigma_g \setminus \Gamma$, then $q(\Gamma)$ will range between 2 and $k + 1$ since the multicurve is separating. The above expression is then equal to

$$
\frac{1}{V_g} \sum_{k=1}^{K(g)} \sum_{q=2}^{k+1} \frac{1}{k!} \int_{\mathcal{M}_g} \sum_{\gamma \in [\Gamma']} 1_{[0,L]}(\ell_{\gamma_1}(X) + \cdots + \ell_{\gamma_k}(X)) dX.
$$

Notice that the interior is a geometric function and so we can use Mirzakhani’s integral formula given by Theorem 5.4 to obtain

$$
\frac{1}{V_g} \sum_{k=1}^{K(g)} \sum_{q=2}^{k+1} \frac{C_{\Gamma}}{k!} \int_{\mathbb{R}_+^k} 1_{[0,L]}(x_1 + \cdots + x_k) x_1 \cdots x_k V_g(\Gamma, x) \prod_{i=1}^k dx_i,
$$

where $C_{\Gamma}$ depends on the number of handles that $\Gamma$ cuts off of $X$ and the symmetries of the curve and may be bounded independently of $\Gamma$ by 1, and $V_g(\Gamma, x)$ is the volume of the moduli space of the cut surface $\Sigma_g \setminus \Gamma$ with boundary component lengths given by $x = (x_1, x_1, \ldots, x_k, x_k)$. Suppose that $\Sigma_g \setminus \Gamma = \bigsqcup_{i=1}^q \Sigma_{g, n_i}(x^i)$ where the $x^i$ partition $x$ so that $x^i$ is of length $n_i$. By the additivity of the Euler characteristic, we have that

$$
2g - 2 = \sum_{i=1}^q 2g_i - 2q + 2k.
$$

In particular, using the volume estimate of Lemma 5.1,

$$
V_g(\Gamma, x) = \prod_{i=1}^q V_{g_i, n_i}(x^i) \leq \prod_{i=1}^q e^{\frac{1}{2} \|x^i\|} V_{g_i, n_i} = e^{x_1 + \cdots + x_k} \prod_{i=1}^q V_{g_i, n_i},
$$
where $|x^i|$ denotes the sum of the components of $x^i$. We then carefully analyse the integral so that
\[
\frac{1}{V_g} \sum_{k=1}^{K(g)} \sum_{q=2}^{k+1} \sum_{[\Gamma]} C_{k} k! \prod_{i=1}^{\frac{q}{k}} \int_{\mathbb{R}_{\geq 0}^k} 1_{[0,L]}(x_1 + \cdots + x_k) e^{x_1 + \cdots + x_k} x_1 \cdots x_k \prod_{i=1}^{k} d x_i
\]
\[
\leq \frac{1}{V_g} \sum_{k=1}^{K(g)} \sum_{q=2}^{k+1} \sum_{[\Gamma]} 1_{[0,L]}(x_1 + \cdots + x_k) x_1 \cdots x_k \prod_{i=1}^{k} d x_i
\]
\[
\leq \frac{1}{V_g} \sum_{k=1}^{K(g)} \sum_{q=2}^{k+1} \sum_{[\Gamma]} 1_{[0,L]}(x_1 + \cdots + x_k) x_1 \cdots x_k \prod_{i=1}^{k} d x_i
\]

with the factor $L^{2k} k^{-k}$ arising from the fact that the maximum of $x_1 \cdots x_k$ subject to $\sum_{i=1}^{k} x_i = L$ arises when each $x_i$ is equal to $LK^{-1}$ and then the measure of the set $\sum_{i=1}^{k} x_i = L$ is bounded by $L^k$.

Notice that the sum over $[\Gamma]$ with $q(\Gamma) = q$ and $k(\Gamma) = k$ may be re-written as the sum over the pairs $\{(g_i, n_i)\}_{i=1}^{q}$ for which
\[
\sum_{i=1}^{q} g_i = g + q - k - 1, \text{ and } \sum_{i=1}^{q} n_i = 2k,
\]
of the original summand multiplied by the number of mapping class orbits of multicurves that cut $\Sigma_g$ in this way. Since two multicurves are in the same mapping class orbit if and only if their dual multicurve graphs are the same (i.e. they glue back the surfaces in the same way) the number of mapping class orbits of curves cutting into $q$ connected components with $k$ curve components in the same topological way (same topological decomposition of the surface) is bounded by the number of ways of glueing the surfaces $\Sigma_{g_i, n_i}$ to obtain a homeomorphism of $\Sigma_g$ which is bounded by $q^k$. This gives the following upper bound for the probability (up to a constant independent of $g$)
\[
\frac{1}{V_g} \sum_{k=1}^{K(g)} \sum_{q=2}^{k+1} \sum_{\{(g_i, n_i)\}_{i=1}^{q} : \sum_{i=1}^{q} n_i = 2k} \frac{1}{k!} \prod_{i=1}^{q} V_{g_i, n_i} L^{2k} e^{L^{-k} k^{-k}}.
\]

By Lemma 6.3, we have that
\[
\sum_{\{(g_i, n_i) : \sum_{i=1}^{q} g_i = g + q - k - 1\}} \prod_{i=1}^{q} V_{g_i, n_i} = O \left( \frac{V_g 2^k D^k \sqrt{\ell}}{g^{\frac{1}{2}} (q-1)} \right),
\]
with the implied constant being independent of $g$, $k$ and the $n_i$. This gives the probability up to a constant as
\[
\sum_{k=1}^{K(g)} e^L \frac{L^{2k} k^{k+1}}{k!} \sum_{q=2}^{K(g)} \sum_{\{(g_i, n_i) : \sum_{i=1}^{q} n_i = 2k\}} \frac{q^k 2^k D^k \sqrt{\ell}}{g^{\frac{1}{2}} (q-1)}.
\]

\[\text{For } k \text{ boundary components, we choose the connected component among } q \text{ of them to which we glue it, giving } q^0. \text{ This only gives an upper bound and could be optimised.}\]
Now the number of ways to choose \( q \) non-negative integers with sum equal to \( 2k \) is bounded by \((2k)^q \lesssim 2^k k^q\). Using this together with Stirling’s approximation \( k! \gtrsim k^{k+1/2}e^{-k} \) we obtain an upper bound (up to a constant) of the form

\[
\sum_{k=1}^{K(g)} e^{k} \left( \frac{L}{k} \right)^{2k} 2^k D^k \sqrt{g} e^k \sum_{q=2}^{k+1} k^q q^k g^{-q^2/2}.
\]

One may see that the maximum of the summand in \( q \) is attained at

\[
q' = \frac{k}{\frac{1}{2} \log(g) - \log(k)}.
\]

Observe that such a value of \( q \) is at least 2 only when \( k \geq 2W(\sqrt{g}/2) \) (assuming \( g \) is sufficiently large such that \( 2W(\sqrt{g}/2) \geq 1 \), where \( W \) is the inverse function of \( f(x) = xe^x \) (or Lambert \( W \)-function). For \( g \) large enough, we also have that

\[
\frac{k}{\frac{1}{2} \log(g) - \log(k)} \leq k + 1,
\]

whenever \( k \leq K(g) \). Indeed, this inequality is equivalent to

\[
ke^{k+1} \leq \sqrt{g},
\]

and since the left hand side is increasing in \( k \), it suffices to check that the inequality holds when \( k = K(g) \leq cg^{2\alpha} \) for some constant \( \alpha \) independent of \( g \). Notice that the exponential term is bounded by \( c \) and so we have that the inequality holds whenever \( g^{2\alpha - 2c} \geq \alpha e \). For \( c < \frac{1}{10} \) say, this means that the inequality holds whenever \( g \geq \alpha^{10} e^{10} \).

Thus for \( g \) sufficiently large we can decompose the sum over \( k \) into the sum from \( k = 1 \) to \( k = \lfloor 2W(\sqrt{g}/2) \rfloor \) for which the sum over \( q \) takes maximum value at \( q = 2 \) and from \( k = \lfloor 2W(\sqrt{g}/2) \rfloor + 1 \) for which the sum over \( q \) takes maximum value at the stationary point \( q' \) defined by (6.1). Bounding by the maximum term, multiplying by the number of terms in the equation and substituting back in \( L = 4c \log(g) \) we are left with the sums

\[
g^{4c-\frac{1}{2}} \sum_{k=1}^{\lfloor 2W(\sqrt{g}/2) \rfloor} e^{k} k^3 2^k D^k \left( \frac{4c \log g}{k} \right)^{2k} + g^{\frac{1}{2} + 4c} \sum_{k=\lfloor 2W(\sqrt{g}/2) \rfloor + 1}^{K(g)} ke^{k+1} 2^k D^k \left( \frac{4c \log g}{k} \right)^{2k} \left( \frac{k}{\sqrt{g}} \right)^{q'} (q')^k.
\]

We can then show that both of these terms tend to zero as \( g \to \infty \). Indeed for the first sum notice that for \( g \) large enough we have that \( 2W(\sqrt{g}/2) \lesssim \log(g) \) and so we can bound the \( k^3 \) term by \( \log(g)^3 \). Moreover, by Stirling’s approximation \((2k)! \asymp (2k)^{2k+\frac{1}{2}} / e^{-2k} \), so we have that

\[
\frac{1}{k^{2k}} \lesssim e^{-2k^2/2k} (2k)^{2k+\frac{1}{2}} e^{-2k},
\]

and thus the first sum is bounded by

\[
(\log(g))^3 g^{4c-\frac{1}{2}} \sum_{k=1}^{\infty} \frac{(16D)^{\frac{1}{2}} e^{-\frac{1}{2} c \log(g)}}{(2k)!} = (\log(g))^3 g^{4c-\frac{1}{2}} \cosh(16e^{-\frac{1}{2} D^\frac{1}{2}} c \log(g)) \lesssim (\log(g))^3 \frac{g^{4c(1+4c-\frac{1}{2} D^\frac{1}{2})}}{g^2},
\]

which goes to zero when \( c \) is sufficiently small.
For the second sum, notice that $k \leq K(g) \leq \alpha g^{2c}$ and so
\[
\left( \frac{k}{\sqrt{g}} \right)^{q'} \leq \alpha^q g^{(2c-\frac{1}{2})q'} \leq \alpha^k g^{(2c-\frac{1}{2})} \leq \alpha^k g^{4c-1},
\]
where this last inequality comes precisely from the fact that all of the $k$ in the range of this sum are such that $q' \geq 2$ by construction, and the power of the $g$ is negative since $c$ is sufficiently small. Moreover,
\[
\frac{(4c \log(g))^{2k}(q')^k}{k^{2k}} = \left( \frac{4c \log(g)}{k} \right)^{2k} \left( \frac{1}{k} \right)^{2k} \left( \frac{1}{\log(g)} \right)^k \leq \left( \frac{16c^2 \log(g)}{k} \beta \log(g)^k \right)^k,
\]
where $\beta = \frac{2}{1-2c} > 1$ for sufficiently small $c$ is independent of $k$ and comes from bounding the second term in the denominator below by $(1 - 2c)^k$. By Stirling’s approximation we get for the previous expression
\[
k^{\frac{1}{2}} e^{-k \left( \frac{16c^2 \beta \log(g)^k}{k^2} \right)}.
\]
Finally, using the fact that the $k \leq \alpha g^{2c}$ we have that this second sum is bounded above by
\[
\alpha^q g^{7c-\frac{1}{2}} \sum_{k=\lceil 2W(\sqrt{g}/2) \rceil + 1}^{K(g)} \frac{(16Dc^2 \beta \log(g))^k}{k!} \leq \alpha^q g^{c(11+8\beta)-\frac{1}{2}}
\]
and by choosing $c$ sufficiently small it goes to zero as $g \to \infty$.

As an immediate corollary of Lemma 6.1 and Theorem 6.4 we obtain the probabilistic estimate Theorem 1.2. Combining this with Theorem 1.3 then gives Theorem 1.1 as desired.

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