Few-Cycle Optical Pulses in the Gain Media

S V Sazonov
National Research Centre «Kurchatov Institute», 123182 Moscow, Russia
e-mail: sazonov.sergey@gmail.com

Abstract. The propagation analysis of few-cycle self-similar pulses in the gain multilevel media is analyzed. Approximate analytical expressions for the electric field profiles of these pulses are obtained.

1. Introduction
To the present time, the nonlinear optics of few-cycle pulses has been formed as an independent area of research [1 - 4]. The greatest success was achieved in the analysis of soliton and soliton-like propagation modes. Issues related to the propagation of few-cycle pulses in the gain media are much less affected. Perhaps, for the first time this question was raised in Refs. [5, 6]. The authors of these works analyzed the amplification of a few-cycle pulse in an isotropic medium of two-level molecules based on a self-similar solution of the sine-Gordon equation. It was concluded that the signal is amplified not by increasing the density of photons, but by increasing the frequency of each photon. The same question was briefly considered in [7]. Attention was focused on the fact that the area of such a pulse is not zero. Amplification of the pulse is accompanied by its self-compression. This, in turn, leads to a broadening of the pulse spectrum. Therefore, the model of a two-level medium becomes incorrect. In Ref. [8], a model of a multi-level medium was proposed, in which quantum transitions containing one common j-th level are allowed. As a result, a double sine-Gordon equation is obtained for the area of a few-cycle pulse. However, the process of amplification of a pulse in such medium is not fully investigated. The limits of the weak and strong influence of the multi-level nature of the medium were considered.

This paper is devoted to an analytical analysis of the propagation of few-cycle pulses in the gain media consisting of multi-level atoms.

2. Basic Equations
Here we consider situations in which the approximation of sudden perturbations \( \mu \equiv \omega_0 \tau_p \ll 1 \) is valid [5, 6], where \( \omega_0 \) is the characteristic frequency of the considered quantum transitions, \( \tau_p \) is the characteristic temporal scale of the laser pulse. Under this condition, for a pulse propagating along the axis in a medium of two-level atoms, the double sine-Gordon equation is valid [8, 9]

\[
\frac{\partial^2 \theta}{\partial z \partial \tau} = a \sin \theta + b \sin \frac{\theta}{2},
\]
where $\tau = t - z / c$, $t$ is the time, $c$ is the speed of light in vacuum, $a$ and $b$ are the constants characterizing the initial state of the medium (for the gain medium $a > 0$, and the parameter $b$ can be both positive and negative; in the case of a two-level medium we have $b = 0$),

$$\theta = \int_{-\infty}^{\tau} \Omega d\tau' ,$$

(2)

$$\Omega = 2dE / \hbar , \ E \text{ is the pulse electric field, } d = \sqrt{\sum_{k,j} d_{kj}^2} \text{, } d_{kj} \text{ are the dipole moments of the allowed quantum } k \leftrightarrow j \text{ - transitions } k \leftrightarrow j \text{, } \hbar \text{ is the Planck constant.}$$

Let us introduce a self-similar variable $\eta = az \tau$ [10]. Then equation (1) takes the form

$$\eta \frac{d^2 \theta}{d\eta^2} + \frac{d\theta}{d\eta} = \sin \theta + 2q \sin \frac{\theta}{2} ,$$

(3)

where $q = b / 2a$ .

3. Self-similar solutions

For a two-level medium we have $q = 0$. In this case, the self-similar solution of equation (3) is well known [10]. This solution consists of the main maximum of the electric field and the accompanying side oscillations. An approximate solution containing only the main maximum can be obtained by neglecting the first term on the left side of equation (1). Indeed, since $\Omega = \frac{\partial \theta}{\partial \tau} = az \frac{d\theta}{d\eta}$, then near the maximum we have $\frac{d^2 \theta}{d\eta^2} \approx 0$. Then, after integration under condition, we will have [11]

$$\theta = 2 \arctan e^\eta ,$$

(4)

$$\Omega = az \ \text{sech}[az(t - z / c)] .$$

(5)

It is seen from this that the pulse area

$$A = \int_{-\infty}^{\tau} \Omega dt = \theta_{\eta \to +\infty} = \pi .$$

Besides, the amplitude of the pulse increases in proportion to the distance $z$, and its temporal duration decreases: $\tau_p = (az)^{-1}$. Now we proceed similarly with equation (3). Neglecting the term $\eta \frac{d^2 \theta}{d\eta^2}$ in this equation, after integration we will have

$$\eta = -\frac{1}{2(1+q)} \ln\left(\frac{2\sin^2 \frac{\theta}{4}}{2(1-q)}\right) - \frac{1}{2(1-q)} \ln\left(\frac{2\cos^2 \frac{\theta}{4}}{2(1+q)}\right) + \frac{1}{1-q} \ln\left(1 + q^{-1} \cos \frac{\theta}{2}\right) .$$

(6)

Expression (6) describes the profile of the main maximum of the pulse field in an implicit form. From here it is possible to go approximately to the explicit form, noting that the average values
\[ 22 \sin \cos \frac{\sin \theta}{4} = \frac{\cos \theta}{4}. \] Then, having made a replacement \( \cos \frac{\sin \theta}{4} \rightarrow \sin \frac{\sin \theta}{4} \), we find from (6) \( \cos \frac{\theta}{2} = -q \frac{e^\xi - 1}{e^\xi + q} \), where \( \xi = (1 - q^2) \eta = (1 - q^2) a z \tau \). At the opposite replacement \( \sin \frac{\sin \theta}{4} \rightarrow \cos \frac{\sin \theta}{4} \) we have \( \cos \frac{\theta}{2} = -q \frac{e^\xi - 1}{e^\xi - q} \). In the first case, the expression for \( \cos \frac{\theta}{2} \) has no singularity under condition \( q > 0 \) in the denominator. In the second case, the singularity is absent, if \( q < 0 \). Combining both equalities, we write

\[ \cos \frac{\theta}{2} = -q \frac{e^\xi - 1}{e^\xi + |q|}. \] (7)

From here we obtain that \( \cos \frac{\theta}{2} = \frac{q}{|q|} \text{sgn}(q) \) at \( \xi \rightarrow -\infty \). If \( \xi \rightarrow +\infty \), then we have \( \cos \frac{\theta}{2} = \cos \frac{\theta_0}{2} = -q \).

Thus, we conclude that the approximate solution (7) is valid for \(|q| < 1\). Note also that the given asymptotic value of \( \cos \frac{\theta_0}{2} \) zeroes the right-hand sides of Eqs. (1) and (3). Thus, we have for the area of the pulse \( A = 2 \arccos(-q) \). Putting here \( q = 0 \), we obtain \( A = \pi \). As expected, this expression coincides with the area of the pulse in the case of a two-level amplifying medium. Using (7), we find for the electric field of the pulse at the main maximum

\[ \Omega = \frac{2q(1 - q^2)\sqrt{1 + |q| a z}}{(e^{\xi/2} + |q|e^{-\xi/2})(1 - |q|)e^\xi + 2|q|}. \] (8)

The side oscillations accompanying the propagation at the speed of light of the main maximum can be described after linearization of equation (3) near the equilibrium value \( \cos \frac{\theta_0}{2} = -q \). Then we have

\[ \eta \frac{d^2 \psi}{d\eta^2} + \frac{d\psi}{d\eta} = -(1 - q^2) \psi, \]

where \( \psi = \theta - \theta_0 \).

By replacing \( \zeta = 2\sqrt{(1 - q^2)} \eta \) this equation is reduced to the zero order Bessel equation

\[ \frac{d^2 \psi}{d\zeta^2} + \frac{1}{\zeta} \frac{d\psi}{d\zeta} + \psi = 0. \]
As a result, we find the solution
\[ \psi = g J_0 \left( 2 \sqrt{1 - q^2} \eta \right) = g J_0 \left( 2 \sqrt{\xi} \right), \]
where the parameter \( g \) is determined from the condition of continuity of the variable \( \theta \). Hence, for side oscillations of the pulse field profile, we have
\[ \Omega = -g \left( 1 - q^2 \right) az \frac{J_1 \left( 2 \sqrt{\xi} \right)}{\sqrt{\xi}}. \] (9)

Here \( J_0 \) and \( J_1 \) are the Bessel functions of zero and first orders, respectively.

Under condition \( \xi < 0 \) the Bessel function should be replaced by a rapidly decreasing function of McDonald.

The analysis shows that the analytical solution 8), (9) obtained here (only qualitatively coincides with the solution found using numerical simulations with equation (3). Nevertheless, the basic properties of the solution (8) are in good agreement with the numerical solution. For example, this concerns the following equality \( \Omega(-q) = -\Omega(q) \). I.e. when the sign of the parameter \( q \) is changes, the sign of the polarity of the pulse is also changes. Note that more precisely the main maximum of the pulse profile is described by the transcendental solution (6).

From (8) and (9), as well as from the expression for \( \xi \), it can be seen that the characteristic frequency \( \omega \) of a few-cycle pulse has the form \( \omega = (1 - q^2)az \). For the of the pulse we have \( I \sim \Omega^2 \sim \xi^2 \sim \omega^2 \). On the other hand \( I \sim n\hbar\omega \), where \( n \) is the concentration of photons. From here we find that \( \omega \sim \xi \) and \( n \sim z \). Thus, the intensity of a few-cycle pulse (or its energy density) increases due to both an increase in the concentration of photons and an increase in the frequency of photons.

4. Concluding remarks
Thus, in the present work, a theoretical study was conducted of the propagation of a few-cycle electromagnetic pulse in the gain multi-level medium. Under conditions of the approximation of sudden disturbances, the analysis reduces to obtaining self-similar solutions of equation (1). The profile of the electric field of the pulse near the main maximum is given by the approximate expression (8). The side oscillations of the pulse field are described by expression (9).

Note that during the passage of a self-similar pulse of the form (5), the two-level amplifying medium passes into the ground state. This is confirmed by the fact that the area of the pulse \( A = \pi \). If for the considered multi-level medium \( j = 1 \), then the transitions from excited states to ground states are allowed. In this case \( b = 0 \), and Eq. (1) goes into the sine-Gordon equation, which is valid for a two-level medium. In this case, after the passage of a pulse, the atoms are in the ground states.

It is easy to see that in a multi-level gain medium, the atoms do not go into the ground state after the passage of a pulse. There is a population of both the ground and excited states. This is due to the fact that from the \( j \)-th state, transitions are possible both up and down with respect to the energy. Therefore, after the passage of a pulse in a multi-level medium, a certain amount of energy remains. Of course, all this is true, as long as we can neglect the processes of spontaneous relaxation.

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