THE BASIC GEOMETRY OF WITT VECTORS, I
THE AFFINE CASE

JAMES BORGER

Abstract. We give a concrete description of the category of étale algebras over the ring of Witt vectors of a given finite length with entries in an arbitrary ring. We do this not only for the classical $p$-typical and big Witt vector functors but also for variants of these functors which are in a certain sense their analogues over arbitrary local and global fields. The basic theory of these generalized Witt vectors is developed from the point of view of commuting Frobenius lifts and their universal properties, which is a new approach even for the classical Witt vectors. The larger purpose of this paper is to provide the affine foundations for the algebraic geometry of generalized Witt schemes and arithmetic jet spaces. So the basics here are developed somewhat fully, with an eye toward future applications.

Introduction

Witt vector functors are certain functors $W$ from the category of rings (always commutative) to itself, and as with any ring, $W(A)$ can be viewed as an object of algebraic geometry, for any given ring $A$. In practice, however, rings $W(A)$ of Witt vectors are viewed as formal algebraic constructions and, to my knowledge, never as genuinely geometric objects. Indeed, I am not aware of a truly geometric description of the spectrum of any nontrivial ring of Witt vectors in the literature.

There is a reason for this. The usual way of analyzing a ring such as $W(A)$ geometrically would be to find a transparent presentation of it over some ring which is already geometrically comprehensible, like a field or the ring of integers. An equivalent approach, more or less, would be to describe $\text{Hom}(W(A), -)$, the functor of points of $W(A)$, in a transparent way. But I am confident that no such description exists. The reason for this lies in the definition of Witt vector functors.

Let us consider the usual, $p$-typical Witt vectors, where $p$ is a fixed prime number. (One of the main points of this paper is to work with generalized Witt vectors, such as the big Witt vectors, but we can ignore them for now.) Traditionally $W(A)$ is defined to be the set $A \times A \times \cdots$ together with exotic addition and multiplication laws, which are given in terms of some implicitly defined polynomials. While these formulas have a certain remote beauty, they do not bring us closer to any direct, geometric understanding of $W(A)$.

To achieve this, it is important to have a defining universal property of rings of Witt vectors. It is as follows. The ring $W(A)$ is an example of a ring $B$ equipped with a ring map $B \to A$ and with a ring endomorphism $\psi_p$ which reduces to the Frobenius map $x \mapsto x^p$ modulo the ideal $pB$. In fact, $W(A)$ is universal with respect to this structure: if for the moment $A$ is $p$-torsion free, then so is $W(A)$, and any $p$-torsion-free ring $B$ equipped with the structure above admits a unique map to $W(A)$ compatible with this data. When $A$ is allowed to have nontrivial
p-torsion, the universal property is subtler, but it is determined by the universal property above and some purely category-theoretic concepts (Kan extensions and comonads, see [13]). So in a certain precise sense, the property above is the defining property of rings of Witt vectors.

The point is that this defining property is for ring maps into $W(A)$, rather than maps out of it. In other words, $W(A)$ is most naturally described not as a quotient of a big coproduct—a presentation—but as a subring of a big product. In particular, we should not expect to be able to say much about its functor of points $\text{Hom}(W(A), -)$, and we should not expect to be able to analyze it in a simple way using the most basic methods of algebraic geometry, simply because they are adapted to rings given by presentations. Instead, as with other quotient constructions in the category of schemes, the natural way to understand it is with descent theory. I hope a glance at figure 1 will be enough to convince the reader of this. One purpose of this paper, then, is to show that from such a point of view, we can understand the geometry of rings of Witt vectors in a way that is natural and largely isolates explicit computations with polynomials.

In fact, the approach we take here and in the sequel [4] will be broader in four independent ways. First, we will extend the Witt functors and our geometric analysis from affine schemes to arbitrary schemes and even to arbitrary algebraic spaces. Second, we will do the same for the arithmetic jet space functors, which are the right adjoints of the functors on schemes induced by Witt vector functors. They are algebraic versions of Beuym’s formal $p$-jet space functor [7][8], which in turn is a certain canonical lift of the Greenberg transform [11][12] to characteristic zero. Third, we will not restrict to the $p$-typical Witt vectors, as above, but will consider commuting Frobenius lifts at arbitrary families of primes. The most important example is the big Witt vector functor mentioned above, which is where we work with all primes simultaneously. Indeed, the reason these two papers exist is to provide a solid foundation for future work on the big de Rham–Witt cohomology of schemes of finite type over $\mathbb{Z}$ (and, more generally, for future work on $\Lambda$-algebraic geometry, which can be thought of as a globalization of the theory $\Lambda$-rings, in the sense of Grothendieck’s Riemann–Roch theory). Fourth, and finally, we will consider not just lifts of the $p$-th power Frobenius maps to $\mathbb{Z}$-algebras, but lifts of higher-power Frobenius maps to algebras over rings $R$ of integers in general global fields. The reason for the interest in this is that I expect such Witt vector rings to shed light on motives with complex multiplication by $R$. For example, Witt vectors relative to the ring of integers $R$ of a local field have been used by Drinfeld [10], Proposition 1.1; they also appear Hazewinkel’s work [18], (18.6.13). In [4], we will even generalize further to allow Spec $R$ to be a nonaffine curve over a finite field $F$.

Most of this will be done in the second paper [4]. Our purpose here is to set up the foundations we need in the affine setting; scheme theory will not appear at all.

Here is an overview of the contents of the paper. Sections 1–6 define our generalized Witt vectors. In our approach, Frobenius lifts are central and Witt polynomials are just a computational device. This is new even for the traditional big and $p$-typical Witt vectors.

Section 1 defines our generalized Witt vector functors. For any Dedekind domain $R$ and any family $E$ of maximal ideals with finite residue field, we define the $E$-typical Witt vector functor $W_{R,E}$ from the category of $R$-algebras to itself. The $R$-algebra $W_{R,E}(A)$ comes with a map $W_{R,E}(A) \to A$ and with commuting endomorphisms $\psi_m (m \in E)$ such that each $\psi_m$ lifts the Frobenius map $x \mapsto x^{[R:m]}$ modulo the ideal generated by $m$; as above, it is the universal object with this structure. (Again, there are some subtleties if $A$ is not flat over $R$, but they are easily handled with some category theory.) We also define the left adjoint of $W_{R,E}$,
Figure 1. As a topological space, Spec $W_1(A)$ (traditionally denoted $W_2(A)$) is two copies of Spec $A$ glued along Spec $A/pA$. This is also true as schemes if we assume that $A$ is $p$-torsion free and we glue transversely and with a Frobenius twist, as indicated.

which we denote $\Lambda_{R,E} \circ -$, and discuss how these functors are related to earlier approaches.

Section 2 defines truncated versions of these two functors and gives some of their basic properties.

Sections 3–5 give techniques for reducing statements about these two functors from the case where $R$ and $E$ are general to the case where $E$ consists of a single principal ideal in a (possibly different) ring $R$, where the functors can be analyzed explicitly using generalizations of the classical Witt polynomials. The definition in terms of Frobenius lifts is better for general theoretical purposes, but the Witt polynomials are at times a useful tool. In general, they exist only locally on $R$.

Section 3 treats the case where $E$ consists of a single principal ideal. This is where the generalized Witt polynomials occur.

Section 4 gives an explicit description of the Witt functor when $E$ consists of a single ideal $m$, not necessarily principal. The point is that because $m$ is locally principal, we can reduce to the case of section 3 by some basic localization arguments.

Section 5 then explains how to build up the functors $W_{R,E}$ and $\Lambda_{R,E} \circ -$ in the general case from the single-prime case of section 4. By a limiting process, we reduce to the case where $E$ is finite, in which case the functors are compositions of functors in the single-prime case.

Section 6 gives a number of basic ring-theoretic facts about Witt vectors which will be needed later.

Sections 7–9 prove the main results, which relate Witt vector functors and étale maps. For simplicity of exposition, let us restrict to the usual $p$-typical Witt vectors, for some prime number $p$. For any ring $A$ and any integer $n \geq 1$, we have the diagram

\[ W_n(A) \xrightarrow{\alpha_n} W_{n-1}(A) \times A \xrightarrow{s_{opr_2}} A/p^n A \]
where $\alpha_n$, $s$, and $t$ are defined in terms of the usual Witt components as follows:

$\alpha_n: (a_0, \ldots, a_n) \mapsto (a_0, \ldots, a_{n-1}, a_0 p^n + p a_1 p^{n-1} + \cdots + p^n a_n)$

$s: (a_0, \ldots, a_{n-1}) \mapsto (a_0 p^n + \cdots + p^n a_n) \bmod p^n A$

$t: a \mapsto a \bmod p^n A$.

(Note that we shift the usual indexing by 1; so our $W_n$ is traditionally denoted $W_{n+1}$. This is preferable for several reasons, as discussed in 2.4.) Figure 1 depicts the induced diagram of schemes when $n = 1$.

Now let $C$ denote the category in which an object is a triple $(B, \varphi)$, where $B$ is an étale $(W_{n-1}(A) \times A)$-algebra and $\varphi$ is an isomorphism of $A/p^n A$-algebras

$A/p^n A \otimes_{\text{topr}_1} B \xrightarrow{\varphi} A/p^n A \otimes_{\text{copr}_1} B$

and where a morphism $(B_1, \varphi_1) \to (B_2, \varphi_2)$ is a $(W_{n-1}(A) \times A)$-algebra map $f: B_1 \to B_2$ such that

$\varphi_2 \circ (A/p^n A \otimes_{\text{topr}_1} f) = (A/p^n A \otimes_{\text{copr}_1} f) \circ \varphi_1$.

In other words, $C$ is the category of algebras equipped with gluing data relative to the diagram (0.0.1), or equivalently $C$ is the weak fiber product of the category of étale $W_{n-1}(A)$-algebras and the category étale $A$-algebras over the category of étale $A/p^n A$-algebras.

**Theorem A.** The base-change functor from category of étale $W_n(A)$-algebras to $C$ is an equivalence. If $A$ is $p$-torsion free, then a quasi-inverse is given by sending $(B, \varphi)$ to the equalizer of the two maps $B \xrightarrow{\varphi \circ (1 \otimes \text{id}_B)} A/p^n A \otimes_{\text{copr}_1} B$.

The first statement can be expressed succinctly in geometric terms; it says that the map $\alpha_n$ satisfies effective descent for étale algebras and that descent data is equivalent to gluing data with respect to the diagram (0.0.1). Using the theorem, it is possible to access the category of étale $W_n(A)$-algebras by induction. This is still true when $E$ consists of more than one ideal, but now because Witt vector functors are iterates of those in the single-prime case.

Section 9 proves van der Kallen’s theorem [26], (2.4), for our generalized Witt vector functors:

**Theorem B.** Let $f: A \to B$ be an étale morphism of $R$-algebras. Then the map $\text{W}_{R,E,n}(f): \text{W}_{R,E,n}(A) \to \text{W}_{R,E,n}(B)$ of $R$-algebras is étale.

This theorem is fundamental in extending Witt constructions to the global case and will be used often in [4]. Van der Kallen’s argument, which has an infinitesimal flavor, could be extended to our generalized Witt vectors with only minor modifications. Instead we deduce theorem B from theorem A, and so our argument has a globally geometric flavor. (Note that, until recently, van der Kallen’s paper [26] had escaped the notice of many workers in de Rham–Witt theory, to whom theorem B was unknown even for the $p$-typical Witt vectors.)

Last, I would like to thank Amnon Neeman for some helpful discussions on some technical points and Lance Gurney for some comments on earlier versions of this paper.

**Contents**

Introduction

1. Generalized Witt vectors and $\Lambda$-rings
1. Generalized Witt vectors and $\Lambda$-rings

The theory in this paper is developed for generalized versions of usual Witt vectors and $\Lambda$-rings, and the purpose of this section is to define them. It is largely an expansion in more concrete terms of Borger–Wieland [5], or rather of the parts about Witt vectors and $\Lambda$-rings. The approach here will allow us to avoid much of the abstract language of operations on rings, as first introduced in Tall–Wraith [25].

For the traditional approach to defining $\Lambda$-rings and the $p$-typical and big Witt vectors, see §1 of chapter IX of Bourbaki [6] and especially the exercises for that section. One can also see Witt’s original paper [28] on the $p$-typical Witt vectors (reprinted in [29]) and his unpublished notes on the big Witt vectors [29], pp. 157–163.

1.1. Supramaximal ideals. Let us say that an ideal $\mathfrak{m}$ of a ring $R$ is supramaximal if either

(a) $R/\mathfrak{m}$ is a finite field, $R_{\mathfrak{m}}$ is a discrete valuation ring, and $\mathfrak{m}$ is finitely presented as an $R$-module, or

(b) $\mathfrak{m}$ is the unit ideal.

By far the most important example is a maximal ideal with finite residue field in a Dedekind domain. The reason we allow the unit ideal is only so that a supramaximal ideal remains supramaximal after any localization.

Note that a supramaximal ideal $\mathfrak{m}$ is invertible as an $R$-module. Indeed, locally at $\mathfrak{m}$ it is the maximal ideal of a discrete valuation ring, and away from $\mathfrak{m}$ it is the unit ideal.

1.2. General notation. Fix a ring $R$ and a family $(\mathfrak{m}_\alpha)_{\alpha \in E}$ of pairwise coprime supramaximal ideals of $R$ indexed by a set $E$. Note that because the unit ideal is coprime to itself, it can be repeated any number of times; otherwise the ideals $\mathfrak{m}_\alpha$ are distinct. For each $\alpha \in E$, let $q_\alpha$ denote the cardinality of $R/\mathfrak{m}_\alpha$. We will often abusively speak of $\mathfrak{m}_\alpha$ rather than $\alpha$ as being an element of $E$, especially when $\mathfrak{m}_\alpha$ is maximal, in which case it comes from a unique $\alpha \in E$. 

Figure 2. Dependence between sections
Let \(R[1/E]\) be the \(R\)-algebra whose spectrum is the complement of \(E\) in \(\text{Spec } R\). It is the universal \(R\)-algebra in which every \(m_\alpha\) becomes the unit ideal. It also has the more concrete description
\[
R[1/E] = \bigotimes_{\alpha \in E} R[1/m_\alpha],
\]
where the tensor product is over \(R\) and \(R[1/m_\alpha]\) is defined to be the coequalizer of the maps
\[
\text{Sym}(R) \longrightarrow \text{Sym}(m_\alpha^{-1})
\]
of symmetric algebras, where \(m_\alpha^{-1}\) is the dual of \(m_\alpha\), one of the maps is \(\text{Sym}\) applied to the canonical map \(R \to m_\alpha^{-1}\), and the other is the map induced by the \(R\)-module map \(R \to \text{Sym}(m_\alpha^{-1})\) that sends 1 in \(R\) to the element 1 in \(\text{Sym}(m_\alpha^{-1})\) in degree zero.

Finally, we write \(N\) for \(\{0, 1, 2, \ldots\}\). When we refer to its monoid structure, we will always mean under addition.

1.3. Examples. All foundational phenomena occur already when \(R = \mathbb{Z}\). So little understanding will be lost if the reader restricts to this case. It also covers the most important applications: big Witt vectors, \(p\)-typical Witt vectors, and \(A\)-rings. (See \([1.15, 1.17]\) below.) There are interesting applications to other finitely generated Dedekind domains \((\mathbb{F}_p[t], \mathbb{Z}[i], \text{and so on})\), but this paper is devoted to foundations.

1.4. \(E\)-flat \(R\)-modules. Let us say that an \(R\)-module \(M\) is \(E\)-flat if for all maximal ideals \(m\) in \(E\), the following equivalent conditions are satisfied:

(a) \(R_m \otimes_R M\) is a flat \(R_m\)-module,

(b) the map \(m \otimes_R M \to M\) is injective.

The equivalence of these two can be seen as follows. Condition (b) is equivalent to the statement \(\text{Tor}^1_R(R/m, M) = 0\), which is equivalent to
\[
\text{Tor}^1_R(R/m, R_m \otimes_R M) = 0.
\]
Since \(R_m\) is a discrete valuation ring, this is equivalent to the \(R_m\)-module \(R_m \otimes_R M\) being torsion free and hence flat.

We say an \(R\)-algebra is \(E\)-flat if its underlying \(R\)-module is.

1.5. Proposition. Any product of \(E\)-flat \(R\)-modules is \(E\)-flat, and any sub-\(R\)-module of an \(E\)-flat \(R\)-module is \(E\)-flat.

Proof. We will use condition (b) above. Let \((M_i)_{i \in I}\) be a family of \(E\)-flat \(R\)-modules. We want to show that for each maximal ideal \(m\) in \(E\), the composition
\[
m \otimes \prod_i M_i \longrightarrow \prod_i m \otimes M_i \longrightarrow \prod_i M_i
\]
is injective. Because each \(M_i\) is \(E\)-flat, the right-hand map is injective, and so it is enough to show the left-hand map is injective.

Since \(m\) is assumed to be finitely presented as an \(R\)-module, we can express it as a cokernel of a map \(N' \to N\) of finite free \(R\)-modules. Then we have the following diagram with exact rows
\[
\begin{array}{ccccc}
N' \otimes_R \prod_i M_i & \longrightarrow & N \otimes_R \prod_i M_i & \longrightarrow & m \otimes_R \prod_i M_i & \longrightarrow & 0 \\
\downarrow \sim & & \downarrow \sim & & \downarrow & \\
\prod_i N' \otimes_R M_i & \longrightarrow & \prod_i N \otimes_R M_i & \longrightarrow & \prod_i m \otimes_R M_i & \longrightarrow & 0.
\end{array}
\]
The left two vertical maps are isomorphisms because \(N'\) and \(N\) are finite free. Therefore the rightmost vertical map is an injection (and even an isomorphism).
Now suppose $M'$ is a sub-$R$-module of an $E$-flat $R$-module $M$. Since $m$ is an invertible $R$-module, $m \otimes R M'$ maps injectively to $m \otimes R M$. Since $M$ is $E$-flat, $m \otimes R M'$ further maps injectively to $M'$, and hence to $M'$. □

1.6. $Ψ$-rings. Let $A$ be an $R$-algebra. Let us define a $Ψ_{R,E}$-action, or a $Ψ_{R,E}$-ring structure, on $A$ to be a commuting family of $R$-algebra endomorphisms $ψ_α$ indexed by $α \in E$. This is the same as an action of the monoid $N^{(E)} = \bigoplus_E N$ on $A$. For any element $n \in N^{(E)}$, we will also write $ψ_n$ for the endomorphism of $A$ induced by $n$. A morphism of $Ψ_{R,E}$-rings is defined to be an $N^{(E)}$-equivariant morphism of rings.

The free $Ψ_{R,E}$-ring on one generator $e$ is $Ψ_{R,E} = R[e] \otimes_R N^{(E)}$, where $N^{(E)}$ acts on $Ψ_{R,E}$ through its action on itself in the exponent. In particular, $Ψ_{R,E}$ is freely generated as an $R$-algebra by the elements $ψ_n(e)$, where $n \in N^{(E)}$. Then it is natural to write $ψ_n = ψ_n(e) \in Ψ_{R,E}$ and $ψ_α = ψ_{b_α} \in Ψ_{R,E}$, where $b_α \in N^{(E)}$ denotes the $α$-th standard basis vector, and $e = ψ_0 \in Ψ_{R,E}$ for the identity operator.

In the language of plethystic algebra [5], we can interpret $Ψ_{R,E}$ as the free $R$-plethory $R[ψ_α | α \in E]$ on the $R$-algebra endomorphisms $ψ_α$. Then a $Ψ_{R,E}$-action in the sense above is the same as a $Ψ_{R,E}$-action in the sense of abstract plethystic algebra. In particular, $Ψ_{R,E}$ can be viewed as the ring of natural unary operations on $Ψ_{R,E}$-rings, as in [118] below.

1.7. $E$-flat $Λ$-rings. Let $A$ be an $R$-algebra which is $E$-flat. Define a $Λ_{R,E}$-action, or a $Λ_{R,E}$-ring structure, on $A$ to be a $Ψ_{R,E}$-action with the following Frobenius lift property: for all $α \in E$, the endomorphism $id \otimes ψ_α$ of $R/m_α \otimes_R A$ agrees with the Frobenius map $x \mapsto x^{p^α}$. A morphism of $E$-flat $Λ_{R,E}$-rings is simply defined to be a morphism of the underlying $Ψ_{R,E}$-rings. Let us denote this category by $\text{Ring}_{Λ_{R,E}}^E$.

1.8. The ghost ring. Since an action of $Ψ_{R,E}$ on an $R$-algebra $A$ is the same as an action (in the category of $R$-algebras) of the monoid $N^{(E)}$, the forgetful functor from the category of $Ψ_{R,E}$-rings to that of $R$-algebras has a right adjoint given by

$$A \mapsto \prod_{N^{(E)}} A = A^{N^{(E)}},$$

where $N^{(E)}$ acts on $A^{N^{(E)}}$ through its action on itself in the exponent. (This is a general fact about monoid actions in any category with products.) In particular, for $α \in A^{N^{(E)}}$ and $n, n' \in N^{(E)}$, the $n$-th component of $ψ_{n'}(α)$ is the $(n + n')$-th component of $α$.

One might call $A^{N^{(E)}}$ the cofree $Ψ_{R,E}$-ring on the $R$-algebra $A$. It has traditionally been called the ring of ghost components or ghost vectors. By [117] it is $E$-flat if $A$ is.

When $|E| = 1$, there is the possibility of confusing the ghost ring $A^N$, which has the product ring structure, with the usual ring $A^N$ of Witt components (see [35]), which has an exotic ring structure. To prevent this, we will use angle brackets $⟨a_0, a_1, \ldots⟩$ for elements of the ghost ring.

1.9. Witt vectors of $E$-flat rings. Let us now construct the functor $W^E_{R,E}$. We will show in [119] that it is the right adjoint of the forgetful functor from the category of $E$-flat $Λ_{R,E}$-rings to that of $E$-flat $R$-algebras. (Further, the flatness will be removed in [113].)

Let $A$ be an $E$-flat $R$-algebra. Let $U_0(A)$ denote the cofree $Ψ_{R,E}$-ring $A^{N^{(E)}}$. For any $i \geq 0$, let

$$U_{i+1}(A) = \{ b \in U_i(A) \mid ψ_α(b) - b^{p^α} \in m_α U_i(A) \text{ for all } α \in E \}.$$
This is a sub-$R$-algebra of $A^N(E)$. Indeed, it is the intersection over $\alpha \in E$ of the equalizers of pairs of $R$-algebra maps

$$U_i(A) \to R/m_\alpha \otimes_R U_i(A)$$

given by $x \mapsto 1 \otimes \psi_\alpha(x)$ and by $x \mapsto (1 \otimes x)^q_\alpha$.

Now define

$$(1.9.1) \quad W^n_{R,E}(A) = \bigcap_{i \geq 0} U_i(A).$$

This is the ring of $E$-typical Witt vectors with entries in $A$. It is a sub-$R$-algebra of $A^N(E)$. Observe that $W^n_{R,E}(A) = A^{N(E)}$ if $A$ is an $R[1/E]$-algebra.

1.10. Proposition. \begin{enumerate}
    \item $W^n_{R,E}(A)$ is a sub-$\Psi_{R,E}$-ring of $A^{N(E)}$.
    \item This $\Psi_{R,E}$-ring structure on $W^n_{R,E}(A)$ is a $\Lambda_{R,E}$-ring structure.
    \item The induced functor $A \mapsto W^n_{R,E}(A)$ from $E$-flat $R$-algebras to $E$-flat $\Lambda_{R,E}$-rings is the right adjoint of the forgetful functor.
\end{enumerate}

Proof. (a): Let us first show by induction that each $U_i(A)$ is a sub-$\Psi_{R,E}$-ring of $A^{N(E)}$. For $i = 0$, we have $U_0(A) = A^{N(E)}$, and so it is clear. For $i \geq 1$, we use the description of $U_{i+1}(A)$ as the intersection of the equalizers of the pairs of ring maps

$$U_i(A) \to R/m_\alpha \otimes_R U_i(A)$$

given in [1.9]. Observe that both these ring maps become $\Psi_{R,E}$-ring maps if we give $R/m_\alpha \otimes_R U_i(A)$ a $\Psi_{R,E}$-action by defining $\psi_\beta : a \otimes x \mapsto a \otimes \psi_\beta(x)$, for all $\beta \in E$. Since limits of $\Psi_{R,E}$-rings exist and their underlying rings agree with the limits taken in the category of rings, $U_{i+1}(A)$ is a sub-$\Psi_{R,E}$-ring of $A^{N(E)}$. Therefore $W^n_{R,E}(A)$, the intersection of the $U_i(A)$, is also a sub-$\Psi_{R,E}$-ring of $A^{N(E)}$.

(b): Suppose $\alpha \in E$ and $x \in W^n_{R,E}(A)$. For any $i \geq 0$, because $x \in U_{i+1}(A)$, we have

$$\psi_\alpha(x) - x^{q_\alpha} \in m_\alpha U_i(A).$$

Therefore we have

$$\psi_\alpha(x) - x^{q_\alpha} \in \bigcap_{i \geq 0} m_\alpha U_i(A).$$

On the other hand, because $m$ is a finitely generated ideal, we have

$$\bigcap_{i \geq 0} m_\alpha U_i(A) = m_\alpha \bigcap_{i \geq 0} U_i(A) = m_\alpha W^n_{R,E}(A),$$

and hence

$$\psi_\alpha(x) - x^{q_\alpha} \in m_\alpha W^n_{R,E}(A),$$

which is what it means for the $\Psi_{R,E}$-structure to be a $\Lambda_{R,E}$-structure, by definition.

(c): Let $A$ be an $E$-flat $R$-algebra, let $B$ be an $E$-flat $\Lambda_{R,E}$-ring, and let $\gamma : B \to A$ be an $R$-algebra map. By the cofree property of $A^{N(E)}$, there is a unique $\Psi_{R,E}$-ring map $\gamma : B \to A^{N(E)}$ lifting $\bar{\gamma}$. We now only need to show that the image of $\gamma$ is contained in $W^n_{R,E}(A)$. By induction, it is enough to show that if $\text{im}(\gamma) \subseteq U_i(A)$, then $\text{im}(\gamma) \subseteq U_{i+1}(A)$.

Let $b$ be an element of $B$. Then for each $\alpha \in E$, we have

$$\psi_\alpha(\gamma(b)) - \gamma(b)^{q_\alpha} = \gamma(\psi_\alpha(b) - b^{q_\alpha}) \in \gamma(m_\alpha B) \subseteq m_\alpha \text{im}(\gamma) \subseteq m_\alpha U_i(A).$$

Therefore, by definition of $U_{i+1}(A)$, the element $\gamma(b)$ lies in $U_{i+1}(A)$. $\square$
1.11. Exercises. Let $R = \mathbb{Z}$. If $E$ consists of the single ideal $p\mathbb{Z}$, then $W^i_\mathbb{Z}(\mathbb{Z})$ agrees with the subring of the ghost ring $\mathbb{Z}^N$ consisting of vectors $a = \langle a_0, a_1, \ldots \rangle$ that satisfy
\[
a_n \equiv a_{n+1} \text{ mod } p^{n+1}
\]
for all $n \geq 0$. In particular, the elements are $p$-adic Cauchy sequences and the rule $a \mapsto \lim_{n \to \infty} a_n$ defines a surjective ring map $W^i_\mathbb{Z}(\mathbb{Z}) \to \mathbb{Z}_p$.

We can go a step further with $W^i_\mathbb{Z}(\mathbb{Z}_p)$. Let $I$ denote the ideal $p\mathbb{Z}_p \times p^2\mathbb{Z}_p \times \cdots$ in $\mathbb{Z}_p^N$. Then $W^i_\mathbb{Z}(\mathbb{Z}_p)$ is isomorphic to the ring $\mathbb{Z}_p \oplus I$ with multiplication defined by the formula $(x, y)(x', y') = (xx', xy' + yx' + yy')$.

Now suppose that $E$ consists of all the maximal ideals of $\mathbb{Z}$, and identify $\mathbb{N}^E$ with the set of positive integers, by unique factorization. Then $W^i_\mathbb{Z}(\mathbb{Z})$ consists of the ghost vectors $\langle a_1, a_2, \ldots \rangle$ that satisfy
\[
a_j \equiv a_{p^j} \text{ mod } p^{1 + \text{ord}(j)}
\]
for all $j \geq 1$ and all primes $p$.

1.12. Representing $W^i$. Let us construct a flat $R$-algebra $\Lambda_{R,E}$ representing the functor $W^i_R$. First we will construct objects $\Lambda_{R,E}$ representing the functors $U_i$. For $i = 0$, it is clear: $U_0$ is represented by $\Lambda_{R,E}^0 = \Psi_{R,E}$. Now assume $\Lambda_{R,E}^i$ has been constructed and that it is a sub-$R$-algebra of $R[1/E] \otimes_R \Psi_{R,E}$ satisfying
\[
R[1/E] \otimes_R \Lambda_{R,E}^i = R[1/E] \otimes_R \Psi_{R,E}.
\]
Then let $\Lambda_{R,E}^{i+1}$ denote the sub-$\Lambda_{R,E}$-algebra of $R[1/E] \otimes_R \Psi_{R,E}$ generated by all elements $\pi^* \otimes (\psi_\alpha(f) - f^{p_\alpha})$, where $\pi^* \in \kappa_\alpha^{-1} \subseteq R[1/E]$, $f \in \Lambda_{R,E}^i$, and $\alpha \in E$. Then $\Lambda_{R,E}^i$ is flat over $R$. Indeed, it is $E$-flat because it is a sub-$R$-algebra of $R[1/E] \otimes_R \Psi_{R,E}$, and it is flat away from $E$ because $R[1/E] \otimes_R \Lambda_{R,E}^i$ agrees with the free $R[1/E]$-algebra $R[1/E] \otimes_R \Psi_{R,E}$. It also clearly represents $U_i$.

Finally, we set
\[
\Lambda_{R,E} = \bigcup_{i \geq 0} \Lambda_{R,E}^i \subseteq R[1/E] \otimes_R \Psi_{R,E}.
\]
It is flat over $R$ because it is a colimit of flat $R$-algebras, and it represents $W^i_R$ because each $\Lambda_{R,E}^i$ represents $U_i$.

For example, if $E = E' \cup E''$, where $E''$ consists of only copies of the unit ideal, then $\Lambda_{E,E'}$ agrees with the group algebra $\Lambda_{E,E'}[\mathbb{N}^{E''}]$.

1.13. Definition of $W$ in general. We extend the functor $W^i_{R,E}$ to the category of all $R$-algebras by taking its left Kan extension. (See Borceux [2, 3.7, for example, for the general theory of Kan extensions.) More precisely, let $\mathcal{R}_{R,E}$ denote the full subcategory of $\mathcal{R}$ consisting of the $E$-flat $R$-algebras, and let
\[
i: \mathcal{R}_{R,E} \longrightarrow \mathcal{R}_R
\]
denote the canonical embedding. Then $\iota \circ W^i_{R,E}$ has a left Kan extension $W_{R,E}$ along $\iota$:
\[
\begin{array}{ccc}
\mathcal{R}_{R,E} & \longrightarrow & \mathcal{R}_R \\
W^i_{R,E} & \downarrow & W_{R,E} \\
\mathcal{R}_{R,E} & \longrightarrow & \mathcal{R}_R
\end{array}
\]
Indeed, because the original functor $W^i_{R,E}$ is representable (or rather its underlying set-valued functor is), the functor represented by $i(\Lambda_{R,E})$ is its left $R$. For any $A \in \mathcal{R}_R$, let us call the ring $W_{R,E}(A)$ the ring of $E$-typical Witt vectors with entries in $A$. 

\[\text{THE BASIC GEOMETRY OF WITT VECTORS, I} 9\]
Since $i$ is full and faithful, $W_{R,E}$ agrees with $W^R_{R,E}$ on $E$-flat $R$-algebras:

\[(1.13.1) \quad W_{R,E} \circ i \sim \sim i \circ W^R_{R,E}.\]

1.14. **Ghost map $w$.** The ghost map

\[w: W_{R,E}(A) \rightarrow \prod_{\mathbb{N}(E)} A\]

is the natural map induced by the universal property of Kan extensions applied to the inclusion maps $W^R_{R,E}(A) \rightarrow \prod_{\mathbb{N}(E)} A$, which are functorial in $A$. Equivalently, it is the morphism of functors induced by the map

\[\Psi_{R,E} = \Lambda^0_{R,E} \rightarrow \Lambda_{R,E}\]

of representing objects. When $A$ is $E$-flat, it is harmless to identify $w$ with the inclusion map.

1.15. **Example: $p$-typical and big Witt vectors.** Suppose $R$ is $\mathbb{Z}$. If $E$ consists of the single ideal $p\mathbb{Z}$, then $W$ agrees with the classical $p$-typical Witt vector functor [28]. Indeed, for $p$-torsion free rings $A$, this follows from Cartier’s lemma, which says that the traditionally defined $p$-typical Witt vector functor restricted to the category of $p$-torsion-free rings has the same universal property as $W^R$. (See Bourbaki [6], IX.44, exercise 14 or Lazard [22], VII§4.) Therefore, they are isomorphic functors.

For $A$ general, one just observes that that the traditional functor is represented by the ring $\mathbb{Z}[x_0, x_1, \ldots]$, which is $p$-torsion free, and so it is the left Kan extension of its restriction to the category of $p$-torsion-free rings. Therefore it agrees with $W$ as defined here.

Another proof of this is given in [25]. It makes a direct connection with the traditional Witt components, rather than going through the universal property.

Suppose instead that $E$ is the family of all maximal ideals of $\mathbb{Z}$. Then $W$ agrees with the classical big Witt vector functor. As above, this can be shown by reducing to the torsion-free case and then citing the analogue of Cartier’s lemma. (Which version of Cartier’s lemma depends on how we define the classical big Witt vector functor. If we use generalized Witt polynomials, then we need Bourbaki [6], IX.55, exercise 41b. If it is defined as the cofree $\lambda$-ring functor, as in Grothendieck [13], then we need Wilkerson’s theorem [27], proposition 1.2.)

Finally, we will see below (3.5) that when $R$ is a complete discrete valuation ring and $E$ consists of the maximal ideal of $R$, then $W$ agrees with Hazewinkel’s ramified Witt vector functor [18], (18.6.13).

1.16. **Comonad structure on $W$.** The functor $W^R: \text{Ring}_R \rightarrow \text{Ring}_R$ is naturally a comonad, being the composition of a functor (the forgetful one) with its right adjoint, and this comonad structure prolongs naturally to $W_{R,E}$. The reason for this can be expressed in two ways—in terms of Kan extensions or in terms of representing objects.

One way is to invoke the general fact that $W_{R,E}$, as the Kan extension of the comonad $W^R_{R,E}$, has a natural comonad structure. This uses (1.13.1) and the fullness and faithfulness of $i$. The other way is to translate the structure on $W^R$ of being a comonad into a structure on its representing object $\Lambda_{R,E}$. One then observes that this is exactly the same structure for the underlying $R$-algebra $i(\Lambda_{R,E})$ to represent a comonad on $\text{Ring}_R$. (This is called an $R$-plethory structure in [5].)
1.17. Λ-rings. The category $\text{Ring}_{\Lambda, R, E}$ of $\Lambda_{R, E}$-rings is by definition the category of coalgebras for the comonad $W_{R, E}$, that is, the category of $R$-algebras equipped with a coaction of the comonad $W_{R, E}$. Since $W_{R, E}$ extends $W_{R, E}$, a $\Lambda_{R, E}$-ring structure on an $E$-flat $R$-algebra $A$ is the same as a commuting family of Frobenius lifts $\psi_\alpha$.

When $R = \mathbb{Z}$ and $E$ is the family of all maximal ideals of $\mathbb{Z}$, then a $\Lambda$-ring is the same as a $\lambda$-ring in the sense of Grothendieck’s Riemann–Roch theory [13] (and originally called a “special $\lambda$-ring”). In the $E$-flat case, this is Wilkerson’s theorem ([27], proposition 1.2). The proof is an exercise in symmetric functions, but the deeper meaning eludes me. The general case follows from the $E$-flat case by category theory, as in [1.15].

1.18. Free $\Lambda$-rings and $\Lambda \odot -$ . Since $W_{R, E}$ is a representable comonad on $\text{Ring}_R$, the forgetful functor from the category of $\Lambda_{R, E}$-rings to the category of $R$-algebras has a left adjoint denoted $\Lambda_{R, E} \odot -$. This follows either from the adjoint functor theorem in category theory (3.3.3 of Borceux [2]), or by simply writing down the adjoint in terms of generators and relations, as in 1.3 of Borger–Wieland [5].

The second approach involves the $R$-plethory structure on $\Lambda_{R, E}$, and is similar to the description of tensor products, free differential rings, and so on in terms of generators and relations.

The functor $\Lambda_{R, E} \odot -$, viewed as an endofunctor on the category of $R$-algebras, is naturally a monad, simply because it is the left adjoint of the monad $W_{R, E}$. Further, the category of algebras for this monad is naturally equivalent to $\text{Ring}_{\Lambda_{R, E}}$.

We can interpret $\Lambda_{R, E}$ as the set natural operations on $\Lambda_{R, E}$-rings. Indeed, a $\Lambda_{R, E}$-ring structure on a ring $A$, induces map $A \to W_{R, E}(A)$ and hence a set map

$$\Lambda_{R, E} \times A \to \Lambda_{R, E} \times W_{R, E}(A) = \Lambda_{R, E} \times \text{Hom}_R(\Lambda_{R, E}, A) \to A,$$

which is functorial in $A$. All natural operations on $\Lambda_{R, E}$-rings arise in this way. See Borger–Wieland [5] for an abstract account from this point of view.

1.19. Remark: identity-based approaches. It is possible to express the approach to natural operations more concretely by using universal identities rather than the language of category theory. (See Buium, Buium–Simanca, and Joyal [7] [10] [20] [21] for example.) In this subsection, I will say something about that point of view and its relation to the category-theoretic one, but it will not be used elsewhere in this paper.

First suppose that for each $\alpha \in E$, the ideal $\mathfrak{m}_\alpha$ is generated by a single element $\pi_\alpha$. For any $\Lambda_{R, E}$-ring $A$ and any element $a \in A$, there exists an element $\delta_\alpha(a) \in A$ such that

$$\psi_\alpha(a) = a^{q_\alpha} + \pi_\alpha \delta_\alpha(a).$$

If we now assume that $A$ is $E$-flat, then the element $\delta_\alpha(a)$ is uniquely determined by this equation, and therefore $\delta_\alpha$ defines an operator on $A$:

$$\delta_\alpha(a) = \frac{\psi_\alpha(a) - a^{q_\alpha}}{\pi_\alpha}.$$

Observe that if the integer $q_\alpha$ maps to 0 in $R$, for example when $R$ is a ring of integers in a function field, then $\delta_\alpha$ is additive; but otherwise it essentially never is. (Also note that $\delta_\alpha$ is the same as the operator $\theta_{\pi_\alpha, 1}$ defined in [3.1] below.)

Conversely, if $A$ is an $E$-flat $R$-algebra, equipped with with operators $\delta_\alpha$, then there is at most one $\Lambda_{R, E}$-ring structure on $A$ whose $\delta_\alpha$-operators are the given ones. To say when such a $\Lambda_{R, E}$-ring structure exists, we only need to express in
This offers another point of view on the difference between a $\Lambda_{R,E}$ given by operators of the point of view of universal algebra (and hence so is a $\Lambda_{R,E}$). The structure of a commuting family of Frobenius lifts does not have this property because of the existential quantifier hidden in the word identities, as above. The structure of a commuting family of Frobenius lifts, but it is still equivalent to having a $\Lambda_{R,E}$-structure. The point of all this, then, is that if we no longer require $A$ to be $E$-flat, a $\delta_{R,E}$-structure is generally stronger than having a commuting family of Frobenius lifts, but it is still equivalent to having a $A$-structure. This offers another point of view on the difference between a $A$-structure and a commuting family of Frobenius lifts. $\delta_{R,E}$-structure is well behaved from the point of view of universal algebra (and hence so is a $A$-structure) because it is given by operators $\delta_{\alpha}$ whose effect on the ring structure is described by universal identities, as above. The structure of a commuting family of Frobenius lifts does not have this property because of the existential quantifier hidden in the word lift.

Let me sketch the equivalence between $\delta_{R,E}$-structures and $A$-structures. For $E$-flat $R$-algebras $A$, it was explained above. For general $A$, the equivalence can be shown by checking that the cofree $\delta_{R,E}$-ring functor is represented by an $E$-flat $R$-algebra (in fact, a free one). It therefore agrees with the left Kan extension of its restriction to the category of $E$-flat algebras, and hence agrees with $W_{R,E}$.

We can extend the identity-based approach to the case where the ideals $m_{\alpha}$ are not principal, but then we would need operators

$$\delta_{\alpha,\pi_{\alpha}}(x) = \pi_{\alpha}^*(\psi_{\alpha}(x) - x^{q})$$

for every element $\pi_{\alpha}^* \in m_{\alpha}^{-1}$, or at least for those in a chosen generating set of $m_{\alpha}^{-1}$, and we would then need more axioms relating them. A particularly convenient generating set of $m_{\alpha}^{-1}$ is one of the form $\{1, \pi_{\alpha}^*\}$, which always exists. Further, for each $\alpha \in E$, it is enough to use the operators $\psi_{\alpha}$ and $\delta_{\alpha,\pi_{\alpha}}$ instead of $\delta_{\alpha,1}$ and $\delta_{\alpha,\pi_{\alpha}}$, because $\delta_{\alpha,1}$ can be expressed in terms of $\psi_{\alpha}$, by (1.19.7). Therefore if we fix elements $\pi_{\alpha}^* \in m_{\alpha}^{-1}$ which are $R$-module generators modulo 1, the relations needed for the generating set $\bigcup_{\alpha \in E} \{\psi_{\alpha}, \delta_{\alpha,\pi_{\alpha}}\}$ of operators are those in (1.19.1)–(1.19.6) but one needs to make the following changes for each $\alpha \in E$: replace each occurrence of $\pi_{\alpha}^{-1}$ with $\pi_{\alpha}^*$, and add axioms that $\psi_{\alpha}$ is an $R$-algebra homomorphism, that $\psi_{\alpha}$ commutes with all $\psi_{\alpha'}$ and all $\delta_{\alpha',\pi_{\alpha'}}$, and that (1.19.7) holds.

As remarked above, when $R$ is an $\mathbb{F}_p$-algebra for some prime number $p$, the polynomials $C_{\alpha}(x,y)$ are zero and so the axioms above simplify considerably. In
particular, the operators $\delta_\alpha$ are additive, and so, it is possible to describe a $\Lambda_{R,E}$-structure using a cocommutative twisted bialgebra, the additive bialgebra of the plethory $\Lambda_{R,E}$. (See Borger–Wieland [5], sections 2 and 10.)

1.20. Localization of the ring $R$ of scalars. Let $R'$ be an $E$-flat $R$-algebra such that the structure map $R \to R'$ is an epimorphism of rings. (For example, the map $\text{Spec } R' \to \text{Spec } R$ could be an open immersion.) Then the family $(m_\alpha')_{\alpha \in E}$ induces a family $(m'_\alpha)_{\alpha \in E}$ of ideals of $R'$, where $m'_\alpha = m_\alpha R'$. By the assumptions on $R'$, each $m'_\alpha$ is supramaximal. Let us write $E' = E$ and use the notation $E'$ for the index set of the $m'_\alpha$.

Let us construct an isomorphism:

\[(1.20.1) \quad R' \otimes_R \Lambda_{R,E} \xrightarrow{\sim} \Lambda_{R',E'} .\]

The category $\text{Ring}_{\Lambda_{R',E'}}$ (see [1.7]) is a subcategory of the category of $\text{Ring}_{\Lambda_{R,E}}$.

Indeed, any object $A' \in \text{Ring}_{\Lambda_{R',E'}}$ is an $R$-algebra with endomorphisms $\psi_\alpha'$, for each $\alpha \in E$. These endomorphisms are again commuting Frobenius lifts, simply because $A'/m'_\alpha A' = A'/m_\alpha A'$. Since $A'$ is $E'$-flat (and by the assumptions on $R'$), $A'$ is $E$-flat. Therefore, it can be viewed as a $\Lambda_{R,E}$-ring.

Further, $\text{Ring}_{\Lambda_{R',E'}}$ agrees with the subcategory of $\text{Ring}_{\Lambda_{R,E}}$ consisting of objects $A$ whose structure map $R \to A$ factors through $R'$, necessarily uniquely. Now consider the underlying-set functor on this category. From the definition of $\text{Ring}_{\Lambda_{R',E'}}$, this functor is represented by the right-hand side of (1.20.1), and from the second description, it is represented by the left-hand side. Let (1.20.1) be the induced isomorphism on representing objects. It sends an element $r' \otimes f$ to $r' f$.

The isomorphism of represented functors which is induced by (1.20.1) gives natural maps

\[(1.20.2) \quad W_{R',E}(A') \xrightarrow{\sim} W_{R,E}(A'),\]

for $R'$-algebras $A'$.

Finally, let us show that for any $R'$-algebra $B'$, the following canonical map is an isomorphism:

\[(1.20.3) \quad \Lambda_{R,E} \otimes B' \xrightarrow{\sim} \Lambda_{R',E'} \otimes B'.\]

It is enough to show that for any $R'$-algebra $A'$, the induced map

$$\text{Hom}_{R'}(\Lambda_{R,E} \otimes B', A') \to \text{Hom}_R(\Lambda_{R,E} \otimes B', A')$$

is a bijection. Since $\text{Ring}_R$ is a full subcategory of $\text{Ring}_{\Lambda_{R,E}}$, the right-hand side agrees with $\text{Hom}_R(\Lambda_{R,E} \otimes B', A')$, and so the map above is an isomorphism by (1.20.2).

1.21. Teichmüller lifts. Let $A$ be an $R$-algebra, let $A^\circ$ denote the commutative monoid of all elements of $A$ under multiplication, and let $R[A^\circ]$ denote the monoid algebra on $A^\circ$. Then for each $\alpha \in E$, the monoid endomorphism $a \mapsto a^{q_\alpha}$ of $A^\circ$ induces an $R$-algebra endomorphism $\psi_\alpha$ of $R[A^\circ]$ which reduces to the $q_\alpha$-th power map modulo $m_\alpha$. Since $R[A^\circ]$ is free as an $R$-module, it is flat. And since the various $\psi_\alpha$ commute with each other, they provide $R[A^\circ]$ with a $\Lambda_{R,E}$-structure. Combined with the $R$-algebra map $R[A^\circ] \to A$ given by the counit of the evident adjunction, this gives, by the right-adjoint property of $W_{R,E}$, a $\Lambda_{R,E}$-ring map $t: R[A^\circ] \to W_{R,E}(A)$. We write the composite monoid map

$$A^\circ \xrightarrow{\text{unit}} R[A^\circ] \xrightarrow{t} W_{R,E}(A)^\circ$$

as simply $a \mapsto [a]$. It is a section of the $R$-algebra map $w_0: W_{R,E}(A) \to A$ and is easily seen to be functorial in $A$. The element $[a]$ is called the Teichmüller lift of $a$. 
2. Grading and truncations

2.1. Grading and truncations. The plethory $\Psi_{R,E}$ is $\otimes$-graded by the monoid $\mathbb{N}^{(E)}$:

$$\Psi_{R,E} = \bigotimes_{\alpha \in E} \otimes_{i \in \mathbb{N}} R[\psi^\alpha_i] = \bigotimes_{n \in \mathbb{N}^{(E)}} R[\psi_n] = R[\psi_n | n \in \mathbb{N}^{(E)}].$$

(There is also a useful $\oplus$-grading on the underlying $R$-algebra given by $\deg \psi^\alpha_n = q^\alpha_n$. There the graded pieces are $R$-modules, whereas above they are $R$-algebras.) Then for each $n \in \mathbb{Z}^{(E)}$, written $\alpha \mapsto n_{\alpha}$, we have in the language of [5] a sub-$R$-$R$-biring

$$\Psi_{R,E,n} = \bigotimes_{\alpha \in E} \otimes_{0 \leq i \leq n_{\alpha}} R[\psi^\alpha_i] = \bigotimes_{n' \leq n} R[\psi_{n'}] = R[\psi_{n'} | n' \leq n],$$

where for two elements $n, n'$ of $\mathbb{N}^{(E)}$, we write $n \leq n'$ if for all $\alpha \in E$ we have $n_{\alpha} \leq n'_{\alpha}$.

Define a similar filtration on $\Lambda_{R,E}$ by

$$\Lambda_{R,E,n} = \Lambda_{R,E} \cap (R[1/E] \otimes_R \Psi_{R,E,n}).$$

2.2. Proposition. (a) For each $n \in \mathbb{N}^{(E)}$, the $R$-algebra $\Lambda_{R,E,n}$ admits a unique $R$-$R$-biring structure such that the map $\Lambda_{R,E,n} \to \Lambda_{R,E}$ is an $R$-$R$-biring map.

(b) For each $m, n \in \mathbb{N}^{(E)}$, we have

$$\Lambda_{R,E,m} \circ \Lambda_{R,E,n} \subseteq \Lambda_{R,E,m+n},$$

where $\circ$ denotes the composition map

$$\Lambda_{R,E} \times \Lambda_{R,E} \to \Lambda_{R,E}.$$ 

Proof. (a): Write $\Lambda = \Lambda_{R,E}$, $\Lambda_n = \Lambda_{R,E,n}$, and so on. Let us consider uniqueness first. Because $\Lambda_n \subseteq \Lambda$, all the biring structure maps but the coaddition and comultiplication are determined. To show coaddition and comultiplication are determined, it is enough to show $\Lambda_n \otimes_R \Lambda_n \subseteq \Lambda \otimes_R \Lambda$. Since both sides of this are $E$-flat, it is enough to show it after base change to $R[1/E]$, where it becomes the obvious containment $\Psi_n \otimes_R \Psi_n \subseteq \Psi \otimes_R \Psi$.

Now consider existence. Since $\Lambda_n \subseteq \Lambda$, it is enough to construct the coaddition and comultiplication maps. Let $\Delta$ denote the coaddition (resp. comultiplication) map on $R[1/E] \otimes_R \Psi$. It sends $\psi_\alpha$ to $\psi_\alpha \otimes 1 + 1 \otimes \psi_\alpha$ (resp. $\psi_\alpha \otimes \psi_\alpha$). Then we have

$$\Delta(R[1/E] \otimes_R \Psi_n) \subseteq R[1/E] \otimes_R \Psi_n \otimes_R \Psi_n,$$

and hence

$$\Delta(\Lambda_n) \subseteq \Lambda \otimes_R \Psi^2 \cap (R[1/E] \otimes_R \Psi^2) = \Lambda \otimes_R \Psi.$$

Thus $\Delta$ restricts to a map $\Delta^+$ (resp. $\Delta^-$) from $\Lambda_n$ to $\Lambda_n \otimes_R \Lambda_n$. To check that this data satisfies the biring axioms, it is enough, by flatness, to do so after base change to $R[1/E]$, where they become the biring axioms on $R[1/E] \otimes_R \Psi_n$.

(b): Combine the definition [21.1] with the inclusion

$$(R[1/E] \otimes_R \Psi_m) \circ (R[1/E] \otimes_R \Psi_n) \subseteq (R[1/E] \otimes_R \Psi_{m+n})$$

and the inclusion $\Lambda_m \circ \Lambda_n \subseteq \Lambda$. \qed
2.3. Witt vectors of finite length. We can express \(2.2\) in terms of functors rather than the objects that represent them. (I will leave it to the reader to express the hypotheses and the uniqueness statements in terms of functors.)

Let \(W_{R,E,n}\) (or often just \(W_n\)) denote the \(R\)-algebra-valued functor represented by the biring \(\Lambda_{R,E,n}\):

\[
W_{R,E,n}(A) = \text{Hom}_R(\Lambda_{R,E,n}, A).
\]

We then have

\[
W_{R,E}(A) = \lim_n W_{R,E,n}(A).
\]

(Note that in most applications, it is better to view \(W_{R,E}(A)\) as a pro-ring than to actually take the limit. If we preferred topological rings to pro-rings, we could take the limit and endow it with the natural pro-discrete topology.) We will see below that all the maps in this projective system are surjective.

As before, we will often drop some of the subscripts \(R\) and \(E\) of \(W\) when the meaning is clear.

Write

\[
[0, n] = \{i \in \mathbb{N}^E \mid i \leq n\}.
\]

The (truncated) ghost map

\[
w_{\leq n} : W_{R,E,n}(A) \rightarrow A^{[0,n]},
\]

is the one induced by the inclusion \(\Psi_{R,E,n} \subseteq \Lambda_{R,E,n}\) of birings. For any \(i \in [0, n]\), the composition \(w_{\leq n}\) with the projection onto the \(i\)-th factor gives another natural map

\[
w_i : W_{R,E,n}(A) \rightarrow A.
\]

Last, let us note that the containment \((2.2.1)\) induces co-plethysm maps

\[
W_{R,E,m+n}(A) \rightarrow W_{R,E,n}(W_{R,E,m}(A)).
\]

All of the maps above are functorial in \(A\).

Finally, observe that for any element \(f \in \Lambda_{R,E,n}\) the natural \(\Lambda_{R,E}\)-ring operation

\[
f : W_{R,E}(A) \rightarrow W_{R,E}(A)
\]

descends to a map \(f : W_{R,E,m+n}(A) \rightarrow W_{R,E,m}(A)\). Indeed, it is the composition

\[
W_{R,E,m+n}(A) \xrightarrow{(2.3.5)} W_{R,E,n}(W_{R,E,m}(A)) \equiv \text{Hom}(\Lambda_{R,E,n}, W_{R,E,m}(A))
\]

\[
\xrightarrow{-(f)} W_{R,E,m}(A),
\]

where \(-(f)\) denote the map that evaluates at \(f\). Particularly important is the example \(f = \psi_n\), where the induced map

\[
\psi_n : W_{R,E,m+n}(A) \rightarrow W_{R,E,m}(A)
\]

is a ring homomorphism.

2.4. Remark: traditional versus normalized indexing. Consider the \(p\)-typical Witt vectors, where \(R = \mathbb{Z}\) and \(E\) consists of the single ideal \(p\mathbb{Z}\). Let \(W'_n\) denote Witt’s functor, as defined in [28]. So, for example, \(W'_n(\mathbb{F}_p) = \mathbb{Z}/p^n\mathbb{Z}\). In [3.3], we will construct an isomorphism \(W'_n \cong W_n+1\). Thus, up to a normalization of indices, our truncated Witt functors agree with Witt’s.

The reason for the normalization is to make the indexing behave well under plethysm. By \((2.2.1)\) and \((2.3.3)\), the index set has the structure of a commutative monoid, and so it is preferable to use an index set with a familiar monoid structure. If we were to insist on agreement with Witt’s indexing, we would have to replace the sum \(m + n\) in \((2.2.1)\) and \((2.3.3)\) with \(m + n - (1,1,\ldots)\), where this would be
computed in the product group \( \mathbb{Z}^E \). The reason why this has not come up in earlier work is that the plethysm structure has traditionally been used only through the Frobenius maps \( \psi_\alpha \). In other words, only the shift operator on the indexing set was used. Thus the distinction between \( \mathbb{N} \) and \( \mathbb{Z}_{\geq 1} \) was not so important because the shift operator \( n \mapsto n + 1 \) is written the same way on both. But making this a monoid isomorphism would involve the unwelcome addition law \( m + n - 1 \) on \( \mathbb{Z}_{\geq 1} \).

Things are different with the big Witt vectors, where \( R \) is \( \mathbb{Z} \) and \( E \) consists of all maximal ideals. They are also traditionally indexed by the positive integers (1.15), but in this case, they are used multiplicatively rather than additively. In particular, the monoid structure that is required is the obvious one; so the traditional indexing is in agreement with the normalized one: the big Witt ring \( W_{p^n}(A) \) (using multiplicative indexing) is naturally isomorphic to our \( p \)-typical ring \( W_n(A) \) and to Witt’s \( W_{n+1}(A) \).

2.5. Localization of the ring \( R \) of scalars. Let \( R' \) be an \( E \)-flat \( R \)-algebra such that the structure map \( R \to R' \) is an epimorphism of rings, as in 1.20.

Then for each \( n \in \mathbb{N}^E \), we have

\[
R' \otimes_R \Lambda_{R,E,n} = R' \otimes_R (\Lambda_{R,E} \cap (R[1/E] \otimes_R \Psi_{R,E,n})) \\
\cong (R' \otimes_R \Lambda_{R,E}) \cap (R'[1/E] \otimes_{R'} \Psi_{R',E',n}).
\]

(We only need to check that the displayed map is an isomorphism along \( E \), in which case it is true because \( R' \) is \( E \)-flat over \( R \).) By (1.20.1), this gives an isomorphism of \( R' \)-algebras

\[
R' \otimes_R \Lambda_{R,E,n} \cong \Lambda_{R',E',n}.
\]

The induced isomorphism of represented functors gives natural maps

\[
W_{R',E',n}(A') \cong W_{R,E,n}(A'),
\]

for \( R' \)-algebras \( A' \). If \( A \) is an \( R \)-algebra, the inverse of this map induces a map

\[
R' \otimes_R W_{R,E,n}(A) \longrightarrow W_{R',E',n}(R' \otimes_R A)
\]

We will see in 6.1 that this is an isomorphism.

As with (1.20.3), the map (2.5.2) induces an isomorphism

\[
\Lambda_{R,E,n} \circ B' \cong \Lambda_{R',E',n} \circ B',
\]

for any \( R' \)-algebra \( B' \).

2.6. Proposition. Let \( A \) be an \( E \)-flat \( R \)-algebra. Then the ghost map

\[
w_{\leq n} : W_{R,E,n}(A) \longrightarrow A^{[0,n]}
\]

is injective. If \( A \) is an \( R[1/E] \)-algebra, it is an isomorphism.

Recall that the analogous facts for infinite-length Witt vectors are also true, either by construction (1.9) or by the universal property (1.10).

**Proof.** Observe that if every ideal in \( E \) is the unit ideal, then \( \Lambda_{R,E} = \Psi_{R,E} \), and hence \( \Lambda_{R,E,n} = \Psi_{R,E,n} \). The statement about \( R[1/E] \)-algebras then follows from (2.5.1). The statement about \( E \)-flat \( R \)-algebras follows by considering the injection \( A \to R[1/E] \otimes_R A \) and applying the previous case to \( R[1/E] \otimes_R A \). \( \square \)
3. Principal single-prime case

For this section, we will restrict to the case where $E$ consists of one ideal $m$ generated by an element $\pi$. Our purpose is to extend the classical components of Witt vectors from the $p$-typical context (where $R$ is $\mathbb{Z}$ and $E$ consists of the single ideal $p\mathbb{Z}$) to this slightly more general one. The reason for this is that the Witt components are well-suited to calculation. In the following sections, we will see how to use them, together with 4.1, 5.4, and 6.1, to draw conclusions when $E$ is general.

In fact, the usual arguments and definitions in the classical theory of Witt vectors carry over as long as one modifies the usual Witt polynomials by replacing every $p$ in an exponent with $q_m$, and every $p$ in a coefficient with $\pi$. Some things, such as the Verschiebung operator, depend on the choice of $\pi$, and others do not, such as the Verschiebung filtration.

Let $n$ denote an element of $\mathbb{N}$. Let us abbreviate $\Lambda = \Lambda_{R,E}$, $\Lambda_n = \Lambda_{R,E,n}$, $q = q_m$, $\psi = \psi_m$, and so on.

3.1. $\theta$ operators. Define elements $\theta_{\pi,0}, \theta_{\pi,1}, \ldots$ of $R[1/\pi] \otimes_R \Lambda = R[1/\pi] \otimes_R \Psi_m$ recursively by the generalized Witt polynomials

$$\psi^n = \theta_{\pi,0}^n + \pi \theta_{\pi,1}^n + \cdots + \pi^n \theta_{\pi,n}.$$

(Note that the exponent on the left means iterated composition, while the exponents on the right mean usual exponentiation, iterated multiplication.) By 1.18, we can view the elements $\theta_{\pi,i}$ as natural operators on $\Lambda_{R[1/\pi],m}$-rings. We will often write $\theta_i = \theta_{\pi,i}$ when $\pi$ is clear.

3.2. Lemma. We have

$$\psi \circ \theta_{\pi,n} = \theta_{\pi,n}^q + \pi \theta_{\pi,n+1} + \pi P(\theta_{\pi,0}, \ldots, \theta_{\pi,n-1}),$$

for some polynomial $P(\theta_{\pi,0}, \ldots, \theta_{\pi,n-1})$ with coefficients in $R$.

Proof. It is clear for $n = 0$. For $n \geq 1$, we will use induction. Recall the general implication

$$x \equiv y \mod m \implies x^{q^j} \equiv y^{q^j} \mod m^{j+1},$$

for $j \geq 1$, which itself is easily proved by induction. Together with the formula (3.2.1) for $\psi \circ \theta_{\pi,i}$ with $i < n$, this implies

$$\psi \circ \psi^n = \sum_{i=0}^{n} \pi^i (\psi \circ \theta_i)^{q^{n-i}} \equiv \pi^n \psi \circ \theta_n + \sum_{i=0}^{n-1} \pi^i (\theta_i q^{n-i}) \mod m^{n+1} R[\theta_0, \ldots, \theta_{n-1}]$$

When this is combined with the defining formula (3.1.1) for $\psi^{(n+1)}$, we have

$$\pi^n \psi \circ \theta_n \equiv \pi^n \theta_{\pi,q^n} + \pi^{n+1} \theta_{n+1} \mod m^{n+1} R[\theta_0, \ldots, \theta_{n-1}].$$

Dividing by $\pi^n$ completes the proof. □

3.3. Proposition. The elements $\theta_{\pi,0}, \theta_{\pi,1}, \ldots$ of $R[1/\pi] \otimes_R \Lambda$ lie in $\Lambda$, and they generate $\Lambda$ freely as an $R$-algebra. Further, the elements $\theta_{\pi,0}, \ldots, \theta_{\pi,n}$ lie in $\Lambda_n$, and they generate $\Lambda_n$ freely as an $R$-algebra.

This is essentially Witt’s theorem 1 [28].
Proof. By induction, the elements $\theta_0, \ldots, \theta_n$ generate the same sub-$R[1/\pi]$-algebra of $R[1/\pi] \otimes_R A$ as $\psi^{\circ 0}, \ldots, \psi^{\circ n}$, and are hence algebraically independent over $R[1/\pi]$. Since $R \subseteq R[1/\pi]$, they are also algebraically independent over $R$.

Let $B_n$ be the sub-$R$-algebra of $R[1/\pi] \otimes_R A$ generated by $\theta_0, \ldots, \theta_n$, and let $B = \bigcup_n B_n$. To show $\Lambda \supseteq B$, we may assume by induction that $\Lambda \supseteq B_n$ and then show $\Lambda \supseteq B_{n+1}$. By \ref{3.2} and because $\Lambda$ is a $\Lambda$-ring, we have

$$\pi \theta_{n+1} \in (\psi \circ \theta_n - \theta_0^q) + m\Lambda_n \subseteq m\Lambda.$$  

Dividing by $\pi$, we have $\theta_{n+1} \in \Lambda$, and hence $\Lambda \supseteq B_n[\theta_{n+1}] = B_{n+1}$.

On the other hand, by \ref{3.2} again, we have

$$\psi \circ \theta_n \equiv \theta_0^q \mod mB_{n+1}$$

for all $n$. Therefore $B$, since it is generated by the $\theta_n$, is a sub-$\Lambda$-ring of $R[1/\pi] \otimes_R A$. It follows that $B \supseteq \Lambda \circ e = \Lambda$ and hence that $B = \Lambda$.

Lastly, the equality $\Lambda_n = B_n$ follows immediately from the above:

$$\Lambda_n = \Lambda \cap \Psi_{R[1/\pi], m, n} = B \cap \Psi_{R[1/\pi], m, n}$$

$$= R[\theta_0, \ldots, \theta_n][\theta_0, \ldots, \theta_n]$$

$$= R[\theta_0, \ldots, \theta_n] = B_n.$$  

$\square$

3.4. Example: Presentations of $\Lambda_n \circ A$. Using \ref{5.3} we can turn a presentation of a ring $A$ into a presentation of $\Lambda_n \circ A$. Let us illustrate this in the $p$-typical case, when $R = \mathbb{Z}$ and $\pi = p$. Then we have

$$\Lambda_n \circ \mathbb{Z}[x] \cong \Lambda_n = \mathbb{Z}[\theta_0, \ldots, \theta_n],$$

where $\theta_k$ is short for $\theta_{p,k}$, which corresponds to the element $\theta_{p,k}(x) = \theta_{p,k} \circ x$.

Because $\Lambda_n \circ$ preserves coproducts and coequalizers, we have

\begin{equation}
\Lambda_n \circ (\mathbb{Z}[x_1, \ldots, x_r]/(f_1, \ldots, f_s)) = \mathbb{Z}[(\theta_i(x_j))/((\theta_i(f_k))],
\end{equation}

where $0 \leq i \leq n$, $1 \leq j \leq r$, and $1 \leq k \leq s$. Here each expression $\theta_i(x_j)$ is a single free variable, and $\theta_i(f_k)$ is understood to be the polynomial in the variables $\theta_i(x_j)$ that results from expanding $\theta_i(f_k)$ using the sum and product laws for $\theta_i$. Because $\Lambda_n \circ$ preserves filtered colimits, we can give a similar presentation of $\Lambda_n \circ A$ for any ring $A$. Similarly, we can take the colimit over $n$ to get a presentation for $\Lambda \circ A$.

In the $E$-typical case, where $E$ is finite, one can write down a presentation by iterating \ref{3.3}, according to \ref{5.3} below. We can pass from the case where $E$ is finite to the case where it is arbitrary by taking colimits, as in \ref{5.1}.

The method above is not particular to the $\theta$ operators—it works for any subset of $\Lambda_n$ that generates it freely as a ring. For example, we can use the $\delta$ operators of \ref{1.19}. Let $\delta^i \in \Lambda$ denote the $i$-th iterate of $\delta_x$. Then the elements $\delta^0, \ldots, \delta^n$ lie in $\Lambda_n$ and freely generate it as an $R$-algebra. (As in \ref{3.3} this follows by induction, but in this case, there are no subtle congruences to check.) Therefore we have

\begin{equation}
\Lambda_n \circ (\mathbb{Z}[x_1, \ldots, x_r]/(f_1, \ldots, f_s)) = \mathbb{Z}[(\delta^i(x_j))/((\delta^i(f_k))],
\end{equation}

where $0 \leq i \leq n$, $1 \leq j \leq r$, and $1 \leq k \leq s$. We interpret the expressions $\delta^i(x_j)$ and $\delta^i(f_k)$ as above. The general $E$-typical case is handled similarly. (See Buium–Simanca \cite{9}, proof of proposition 2.12.)
3.5. Witt components. It follows from (3.3) that, given $\pi$, we have a bijection
\[
W_m(A) \xrightarrow{\sim} A \times A \times \cdots,
\]
which sends a map $f: A \to A$ to the sequence $(f(\theta_{\pi,0}), f(\theta_{\pi,1}), \ldots)$. To make the dependence on $\pi$ explicit, we will often write $(x_0, x_1, \ldots)_\pi$ for the image of $(x_0, x_1, \ldots)$ under the inverse of this map. If $R = \mathbb{Z}$ and $\pi = p$, then this identifies $W_m(A)$ with the ring of $p$-typical Witt vectors as defined traditionally. Similarly, when $R$ is a complete discrete valuation ring, we get an identification of $W_m(A)$ with Hazewinkel’s ring of ramified Witt vectors $W_{q,\infty}(A)$. (See [18], (18.6.13), (25.3.17), and (25.3.26)(i).) We call the $x_i$ the Witt components (relative to $\pi$) of the element $(x_0, \ldots)_\pi \in W(A)$.

Similarly, using the free generating set $\theta_{\pi,0}, \ldots, \theta_{\pi,n}$ of $\Lambda_n$, we have a bijection
\[
W_{m,n}(A) \xrightarrow{\sim} A^{[0,n]}.
\]
As above, we will write $(x_0, \ldots, x_n)_\pi$ for the image of $(x_0, \ldots, x_n)$ under the inverse of this map. This identifies $W_{m,n}(A)$ with the traditionally defined ring of $p$-typical Witt vectors of length $n + 1$. (For remarks on the $+1$ shift, see [23].)

Note that the Witt components do not depend on the choice of $\pi$ in a simple, multilinear way. For example, if $u$ is an invertible element of $R$ and we have
\[
(x_0, x_1, \ldots)_\pi = (y_0, y_1, \ldots)_u\pi,
\]
then we have
\[
x_0 = y_0, \quad x_1 = uy_1, \quad x_2 = u^2y_2 + \pi^{-1}(u - u^2)y_1^2, \quad \ldots.
\]

As in (3.3), we could use the free generating set $\delta^0, \delta^1, \ldots$ of $\Lambda$ instead of $\theta_0, \theta_1, \ldots$. This would give a different bijection between $W_m(A)$ and the set $A^\Lambda$, and hence an $R$-algebra structure on the set $A^\Lambda$ which isomorphic to Witt’s but not equal to it. The truncated versions agree up to $A \times A$, but differ after that. This is simply because $\delta^0 = \theta_0$ and $\delta^1 = \theta_1$, but $\delta^2 \neq \theta_2$. (See Joyal [21], p. 179.)

3.6. The ghost principle. It follows from the descriptions (3.5.1) and (3.5.2) that $W_m$ and $W_{m,n}$ preserve surjectivity. On the other hand, every $R$-algebra is a quotient of an $m$-flat $R$-algebra (even a free one). Therefore to prove any functorial identity involving rings of Witt vectors when $m$ is principal, it is enough to restrict to the $m$-flat case. Further, any $m$-flat $R$-algebra $A$ is contained in an $R[1/m]$-algebra, such as $R[1/m] \otimes_R A$. Since $W_m$ and $W_{m,n}$ preserve injectivity, being representable, it is even enough to check functorial identities on $R[1/m]$-algebras $A$, in which case rings of Witt vectors agree with the much more tractable rings of ghost components.

3.7. Verschiebung. For any $R$-algebra $A$ define an operator $V_\pi$, called the Verschiebung (relative to $\pi$), on $W_m(A)$ by
\[
V_\pi((y_0, y_1, \ldots)_\pi) = (0, y_0, y_1, \ldots)_\pi.
\]
This is clearly functorial in $A$. With respect to the ghost components, it satisfies
\[
V_\pi((z_0, z_1, \ldots)) = (0, \pi z_0, \pi z_1, \ldots).
\]
Using this formula, we see $V_\pi$ extends to a map on ghost rings. It is clearly $R$-linear, and hence by the ghost principle, so is the operator $V_\pi$ of (3.7.1).

3.8. Example. $W_n(R) = W_{R,m,n}(R)$ has a presentation
\[
R[x_1, \ldots, x_n]/(x_i x_j - \pi^{i-j} x_j \mid 1 \leq i < j \leq n),
\]
where the element $x_i$ corresponds to $V_\pi^i(1)$.
3.9. Teichmüller lifts. Under the composite map

\[ A \xrightarrow{a \mapsto [a]} W(A) \xrightarrow{w} A \times A \times \cdots \]

(see [1,21]), the image of \( a \) is \( \langle a, a^q, a^{q^2}, \ldots \rangle \). It follows from the ghost principle that

\[ [a] = \langle a, 0, 0, \ldots \rangle_\pi \in W(A). \]

Multiplication by Teichmüller lifts also has a simple description in terms of Witt components:

\[ [a](\ldots, b_i, \ldots)_\pi = (\ldots, a^q b_i, \ldots)_\pi. \]

Again, this follows from the ghost principle.

4. General single-prime case

Assume \( E \) consists of a single ideal \( m \), possibly not principal. Let \( n \) denote an element of \( N \). Let us continue to write \( W = W_{R,E} \) and \( W_n = W_{R,E,n} \).

Let \( K_m \) denote \( R_m[1/m] \). If \( m \) is the unit ideal, we understand \( R_m \), and hence \( K_m \), to be the zero ring. Otherwise, \( R_m \) is a discrete valuation ring and \( K_m \) is its fraction field. In particular, \( m \) becomes principal in \( R[1/m], R_m \), and \( K_m \). So the following proposition allows us to describe \( W_{R,n}(A) \) for general \( R \) in terms of \( W_{R,n}(A) \) when \( R \) is principal, and hence in terms of Witt components.

4.1. Proposition. For any \( R \)-algebra \( A \), the ring \( W_n(A) \) is the equalizer of the two maps

\[ W_{R[1/m],E,n}(R[1/m] \otimes_R A) \times W_{R_m,E,n}(R_m \otimes_R A) \xrightarrow{\pi_1 \times \pi_2} W_{K_m,E,n}(K_m \otimes_R A) \]

induced by projection onto the two factors and the bifunctoriality of \( W_{-,-}(\cdot) \).

Proof. The evident diagram

\[ R \xrightarrow{\cdot 1} R[1/m] \times R_m \xrightarrow{\pi_1 \times \pi_2} K_m \]

is an equalizer diagram. Since \( K_m \) is \( m \)-flat, so is any sub-\( R \)-module of \( K_m \). It follows that for any \( R \)-algebra \( A \), the diagram

\[ A \xrightarrow{\cdot (1) \otimes_R A} (R[1/m] \times R_m) \otimes_R A \xrightarrow{\pi_1 \otimes \pi_2} K_m \otimes_R A \]

is an equalizer diagram. Since \( W_{R,E,n} \) is representable, it preserves equalizers, and so the induced diagram (writing \( W_n = W_{R,E,n} \))

\[ W_n(A) \xrightarrow{\cdot (1) \otimes_R A} W_n(R[1/m] \otimes_R A) \times W_n(R_m \otimes_R A) \xrightarrow{\pi_1 \otimes \pi_2} W_n(K_m \otimes_R A) \]

is also an equalizer diagram. Then (4.5) completes the proof. \( \square \)

4.2. Verschiebung in general. We can define Verschiebung maps

\[ V^j : m^j \otimes_R W(A) \rightarrow W(A), \]

when \( m \) is possibly non-principal by removing the dependence of \( V_\pi \) on the choice of \( \pi \). To define it, it is enough, by [11] to restrict to the case where \( m \) is principal, as long as our construction is functorial in \( A \) and \( R \). So, choose a generator \( \pi \in m \) and define

\[ V^j_a(a \pi^j \otimes y) = a V^j_\pi(y), \]

for all \( a \in R, y \in W(A) \). It is well defined because \( V_\pi \) is \( R \)-linear. On ghost components it satisfies

\[ V^j(x \otimes \langle z_0, z_1, \ldots \rangle) = \langle 0, \ldots, 0, xz_0, xz_1, \ldots \rangle. \]
where the number of leading zeros is \( j \). In particular, it is independent of the choice of \( \pi \), by the ghost principle.

If we write \( W(A)_{(j)} \) for \( W(A) \) viewed as a \( W(A) \)-algebra by way of the map \( \psi_j : W(A) \to W(A) \), then the map

\[(4.2.3) \quad V^j : m^j \otimes_R W(A)_{(j)} \longrightarrow W(A),\]

is \( W(A) \)-linear, as is easily checked using the ghost principle. Expressed as a formula, it says

\[(4.2.4) \quad V^j(x \otimes y\psi_j(z)) = V^j(x \otimes y)z.\]

In particular, the image \( V^j W(A) \) of \( V^j \) is an ideal of \( W(A) \).

Let us also record the identities

\[(4.2.5) \quad \psi_j(V^j(x \otimes y)) = xy\]

and

\[(4.2.6) \quad V^j(x \otimes y)V^j(x' \otimes y') = xV^j(x' \otimes yy') \in m^jV^jW(A).\]

Again, one checks these using the ghost principle.

Finally, for any \( n \in \mathbb{N} \), the map \( V^j \) descends to a map

\[(4.2.7) \quad V^j : m^j \otimes_R W_n(A)_{(j)} \longrightarrow W_{n+j}(A),\]

and the obvious analogues of the identities above hold here.

### 4.3. Remark

We can define Verschiebung maps even if we no longer assume there is only one ideal in \( E \). For any \( j \in \mathbb{N}^E \), let \( J \) denote the product ideal \( \prod_{i} m_i^{\alpha_i} \). Then \( V^j \) would be a map \( J \otimes_R W(A) \to W(A) \). The identities above, suitably interpreted, continue to hold. We will not need this multi-prime version.

### 4.4. Proposition

The sequence

\[(4.4.1) \quad 0 \longrightarrow m^j \otimes_R W_n(A)_{(j)} \xrightarrow{V^j} W_{n+j}(A) \longrightarrow W_j(A) \longrightarrow 0\]

is exact.

**Proof.** First consider the case where \( m \) is principal. Let \( \pi \in m \) be a generator. Using \( \textbf{6.7} \), it is clear that \( V^j \) is injective and that its image is the set of Witt vectors whose Witt components (relative \( \pi \)) are 0 in positions 0 to \( j-1 \). By \( \textbf{8.6} \), the pre-image of 0 under the map \( W_{n+j}(A) \to W_j(A) \) is the same subset, and the map \( W_{n+j}(A) \to W_j(A) \) is surjective.

Now consider the general case. Augment diagram \( \textbf{4.4.1} \) in the obvious sense by expressing each term of \( \textbf{4.4.1} \) as an equalizer as in \( \textbf{4.1} \) Here we use that \( m \) is \( R \)-flat. It then follows from the principal case and the snake lemma that \( \textbf{4.4.1} \) is left exact.

It remains to prove that the map \( W_{n+j}(A) \to W_j(A) \) is surjective. By induction, we can assume \( n = 1 \). By \( \textbf{11} \), for any \( i \in \mathbb{N} \) we have

\[W_i(A) = W_{R_m,i}(R_m \otimes_R A) \times_{W_{R_m,i}(R_m \otimes_R A)} W_{R[1/m]_i,R[R[1/m] \otimes_R A]}\]

Now let \( \pi \) denote a generator of the maximal ideal of \( R_m \), and suppose two elements

\[y = (y_0, \ldots, y_j) \in W_{R_m,j}(R_m \otimes_R A),\]
\[z = (z_0, \ldots, z_j) \in (R[1/m] \otimes_R A)^{j+1} = W_{R[1/m],j,R[R[1/m] \otimes_R A]}\]

have the same image in \( W_{R_m,j}(K_m \otimes_R A) \). To lift the corresponding element of \( W_j(A) \) to \( W_{j+1}(A) \), we need to find elements

\[y_{j+1} \in R_m \otimes_R A \quad \text{and} \quad z_{j+1} \in R[1/m] \otimes_R A\]
such that in $K_m \otimes_R A$, we have
\[(4.4.2) \quad y_0^{j+1} + \cdots + \pi^{j+1}y_{j+1} = z_{j+1}.\]
So, choose an element $z_{j+1} \in A$ whose image under the surjection
\[A \rightarrow A/(mA)^{j+1} = R_m/(mR_m)^{j+1} \otimes_R A\]
agrees with the image of $y_0^{j+1} + \cdots + \pi^{j}y_j$. It follows that the element
\[y_0^{j+1} + \cdots + \pi^{j}y_j - 1 \otimes z_{j+1} \in R_m \otimes_R A\]
lies in $\pi^{j+1}(R_m \otimes_R A)$. It thus equals $\pi^{j+1}y_{j+1}$ for some element $y_{j+1} \in R_m \otimes_R A$. And so $y_{j+1}$ and $z_{j+1}$ satisfy \[(4.4.2).\]

4.5. Corollary. For any $R$-algebra $A$, we have
\[(4.5.1) \quad \bigoplus_{i \in [0,n]} m^i \otimes_R A_{(i)} \xrightarrow{\sim} \text{gr}_V W_n(A),\]
where $A_{(i)}$ denotes $A$ viewed as a $W_n(A)$-module via the ring map $w_i : W_n(A) \rightarrow A$.

4.6. Reduced ghost components. We can define infinitely many ghost components for Witt vectors of finite length $n$ if we are willing to settle for answers modulo $m^{n+1}$.

First assume $m$ is generated by $\pi$. By examining the Witt polynomials \[(3.1.1),\]
we can see that for any $i \geq 0$, the composite
\[W(A)^{\pi^i} \rightarrow A \rightarrow A/m^{n+1}A\]
vanes on $V^{n+1}W(A)$. It therefore factors through $W_n(A)$, giving a map $\bar{\psi}_i$ from $W_n(A)$ to $A/m^{n+1}A$.

When $m$ is not assumed to be principal, we define $\bar{\psi}_i$ by localizing at $m$:
\[W_n(A) \rightarrow W_{m,n}(R_m \otimes_R A) \xrightarrow{\psi_i} (R_m \otimes_R A)/m^{n+1}(R_m \otimes_R A) = A/m^{n+1}A,\]
where the middle map is $\bar{\psi}_i$ as just constructed in the principal case. We call the composition
\[(4.6.1) \quad W_n(A) \xrightarrow{\psi_i} A/m^{n+1}A\]
the $i$-th reduced ghost component.

5. Multiple-prime case

The purpose of this section is to give two results about reducing the family $E$ (of \[(1.2)\]) to simpler families. The first reduces from the case where $E$ is arbitrary to the case where it is finite, and the second reduces from the case where it is finite to the case where it has a single element.

5.1. Proposition. The canonical maps
\[(5.1.1) \quad \text{colim}_{E'} \Lambda_{E'} \rightarrow \Lambda_E,\]
\[(5.1.2) \quad \text{colim}_{E'} \Lambda_{E',n'} \rightarrow \Lambda_{E,n},\]
are isomorphisms. Here $E'$ runs over the finite subfamilies of $E$, and $n'$ is the restriction to $E'$ of a given element $n \in \mathbb{N}(E)$.

Proof. Consider \[(5.1.1)\] first. Since each map $\Lambda_{E'} \rightarrow \Lambda_E$ is an injection, \[(5.1.1)\] is an injection. Therefore, since $\Lambda_E$ is freely generated as a $\Lambda_E$-ring by the element $e = \psi_0$, it is enough to show the sub-$\Psi_E$-ring $\text{colim}_{E'} \Lambda_{E'}$ of $\Lambda_E$ is a sub-$\Lambda_E$-ring. Since it is flat, we only need to check the Frobenius lift property. So, suppose $m \in E$. For any element $x$ of the colimit, there is a finite family $E''$ such that $x \in \Lambda_{E''}$ and $m \in E''$. But $\Lambda_{E''}$ is a $\Lambda_{E''}$-ring. So we have $\psi_m(x) \equiv x^{m}$ modulo
are isomorphisms, where
\[ (5.3.2) \]
\[ n \in \text{colim}_E \Lambda_E. \]

Proof. It is enough to show each map becomes an isomorphism after base change orator
\[ \Psi \]
holds for the colimit ring.

5.2. Corollary. For any \( R \)-algebra \( A \), the canonical maps
\[ W_E(A) \to \lim_{E'} W_{E'}(A), \]
\[ W_{E,n}(A) \to \lim_{E'} W_{E',n'}(A) \]
are isomorphisms, where \( E' \), \( n \), and \( n' \) are as in \([5.1]\).

5.3. Proposition. Let \( E' \cup E'' \) be a partition of \( E \). Then the canonical maps
\[ \Lambda_{E'} \otimes_R \Lambda_{E''} \to \Lambda_E \]
\[ \Lambda_{E',n'} \otimes_R \Lambda_{E'',n''} \to \Lambda_{E,n} \]
are isomorphisms, where \( n' \) and \( n'' \) denote the restrictions to \( E' \) and \( E'' \) of a given element \( n \in \mathbf{N}(E) \).

Proof. It is enough to show each map becomes an isomorphism after base change to \( R[1/E'] \) and \( R[1/E''] \). So, by \([1.20.1]\), we can assume every element in either \( E' \) or \( E'' \) is the unit ideal. In the second case, we have
\[ \Lambda_{E'} \otimes_R \Lambda_{E''} = \Lambda_{E'} \otimes_R R[\mathbf{N}(E'')] = \Lambda_{E'}[\mathbf{N}(E'')] = \Lambda_E \]
The argument for \([5.3.2]\) is the same, but we replace the generating set \( \mathbf{N}(E'') \) with \([0,n'']\).

Now suppose every element in \( E' \) is the unit ideal. Then a \( \Lambda_{E'} \)-ring is the same as a \( \Psi_{E'} \)-ring. So we have
\[ \Lambda_{E'} \otimes_R \Lambda_{E''} = \Lambda_{E''}[\mathbf{N}(E')] = \Lambda_E. \]
For \([5.3.2]\), replace \( \mathbf{N}(E') \) with \([0,n']\), as above.

5.4. Corollary. Let \( E' \cup E'' \) be a partition of \( E \). Then for any \( R \)-algebra \( A \), the canonical maps
\[ W_E(A) \to W_{E'}(W_{E'}(A)) \]
\[ W_{E,n}(A) \to W_{E',n'}(W_{E',n'}(A)) \]
are isomorphisms, where \( n, n', n'' \) are as in \([5.3]\).

5.5. Remark. By the results above, it is safe to say that expressions such as
\[ \Lambda_{m_1} \otimes_R \cdots \otimes_R \Lambda_{m_r} \text{ and } W_{m_1} \cdots \cdots W_{m_1}(A) \]
are independent of the ordering of the \( m_i \), assuming the \( m_i \) are pairwise coprime. (Note that it is not generally true that \( P \otimes P' \cong P' \otimes P \) for plethories \( P \) and \( P' \). See \([5.2.8]\).)

If we ask that the expressions in \([5.5.1]\) be independent only up to isomorphism, then it is not even necessary that the \( m_i \in E \) be pairwise coprime \([1.2]\). But invariance up to isomorphism is not a such a useful property, and most of the time coprimality really is necessary. For example, we could look at rings with more than one Frobenius lift at a single maximal ideal, but we would not be able to reduce to the case of a single Frobenius lift. Indeed, if \( E \) consists of a single maximal ideal \( m \), the two endomorphisms \( \psi_{WW(A)} \) and \( W(\psi_{W(A)}) \) of \( WW(A) \) commute, and the
first is clearly a Frobenius lift, but the second is generally not. Therefore \(W W(A)\)
cannot be the cofree ring with two commuting Frobenius lifts at \(m\).

In fact, I believe this is the only place where we use the coprimality assumption directly. The rest of our results depend on it only through \(\ref{5.3}\). Although I know of no applications, it would be interesting to know whether the abstract set up of this paper, and then the main results, hold when we allow more than one Frobenius lift at each maximal ideal.

6. Basic affine properties

This section provides some basic results about the commutative algebra of Witt vectors. They are the minimal results needed to be able to prove the main theorems of part I and to set up the global theory in part II. There are other basic results that could have been included here, but which I have put off to \(\ref{4}\), where they are proved for all algebraic spaces.

We continue with the notation of \(\ref{1.2}\). Fix an element \(n \in N(E)\), and write \(W_n = W_{R,E,n} \) and so on, for short. By \(\ref{5.2}\) we may assume that \(E\) agrees with the support of \(n\), and in particular that it is finite.

6.1. Proposition. Let \(R'\) be an \(E\)-flat \(R\)-algebra such that the structure map \(R \to R'\) is a ring epimorphism (as in \(\ref{1.20}\)). Then the composition

\[
R' \otimes_R W_n(A) \xrightarrow{\text{id}_n \otimes V^1} R' \otimes_R W_n(A) \xrightarrow{\text{\ref{2.5.3}}} W_{R',E',n}(R' \otimes_R A)
\]

is an isomorphism, where \(E'\) is as in \(\ref{1.20}\).

Proof. We may assume by \(\ref{5.4}\) that \(E\) consists of a single ideal \(m\). Using \(\ref{1.1}\) and the flatness of \(R'\) over \(R\), we are reduced to showing that the functors \(W_R[1/m],n,\)
\(W_{R,a,n},\) and \(W_{K,a,n}\) commute with the functor \(R' \otimes_R -\). In particular, we may assume that \(m\) is a principal ideal.

The result is clear for \(n = 0\), because \(W_0\) is the identity functor. So assume \(n \geq 1\). By \(\ref{4.4}\), we have the following map of exact sequences

\[
\begin{array}{ccc}
0 & \to & R' \otimes_R m \otimes_R W_n(A) \\
\downarrow & & \downarrow \\
0 & \to & m \otimes_R W_n(A)
\end{array}
\]

\[
\begin{array}{ccc}
0 & \to & R' \otimes_R W_n(A) \\
\downarrow & & \downarrow \\
0 & \to & R' \otimes_R A
\end{array}
\]

where the vertical maps are given by \(\text{\ref{2.5.3}}\). By induction the leftmost vertical arrow is an isomorphism. Therefore the inner one is, too. \(\square\)

6.2. Proposition. For any ideal \(I\) in an \(R\)-algebra \(A\), let \(W_n(I)\) denote the kernel of the canonical map \(W_n(A) \to W_n(A/I)\). Then we have

\[
W_n(I)W_n(J) \subseteq W_n(IJ)
\]

for any ideals \(I, J\) in \(A\).

Proof. Let us first show that we may assume \(E\) consists of a single ideal \(m\). In doing this, it will be convenient to prove the following equivalent form of the statement: if \(IJ \subseteq K\), where \(K\) is an ideal in \(A\), then \(W_n(I)W_n(J) \subseteq W_n(K)\). Suppose \(E = E'\setminus \{m\}\). Let \(n'\) be the restriction of \(n\) to \(E\). Let \(I' = W_{E',n'}(I)\), \(J' = W_{E',n'}(J)\), and \(K' = W_{E',n'}(K)\). By \(\ref{5.4}\) we have \(W_{E,n} = W_{m,n_n} \circ W_{E',n'}\), and hence \(W_{E,n}(I) = W_{m,n_n}(I')\) and so on. By induction, we have \(I'J' \subseteq K'\), and then applying the result in the single-ideal case gives

\[
W_{E,n}(I)W_{E,n}(J) = W_{m,n_n}(I')W_{m,n_n}(J') = W_{m,n_n}(K') = W_{E,n}(K).
\]
So we will assume $E = \{m\}$ and drop $E$ from the notation.

By 6.1 the statement is Zariski local on $R$, and so we may assume the ideal $m$ is generated by some element $\pi$. We will work with Witt components relative to $\pi$.

We need to show that for any elements $x = (x_0, \ldots, x_n) \in W_n(I)$ and $y = (y_0, \ldots, y_n) \in W_n(J)$, the product $xy$ is in $W_n(IJ)$. So it is sufficient to show this in the universal case, where $A$ is the free polynomial algebra $R[x_0, y_0, \ldots, x_n, y_n]$, $I$ is the ideal $(x_0, \ldots, x_n)$, and $J$ is the ideal $(y_0, \ldots, y_n)$.

Consider the following commutative diagram

\[
\begin{array}{ccc}
W_n(A) & \xrightarrow{w_{[0,n]}} & A^{[0,n]} \\
\downarrow & & \downarrow \\
W_n(A/IJ) & \xrightarrow{w_{[0,n]}} & (A/IJ)^{[0,n]}.
\end{array}
\]

We want to show that the image of $xy$ in $W_n(A/IJ)$ is zero. Since $A/IJ$ is flat (even free) over $R$, the lower map $w_{[0,n]}$ is injective, and so it is enough to show the image of $xy$ in $(A/IJ)^{[0,n]}$ is zero. But by the naturality of the ghost map, we have $w_{[0,n]}(x) \in I^{[0,n]}$ and $w_{[0,n]}(y) \in J^{[0,n]}$. Therefore $w_{[0,n]}(xy)$ lies in $(IJ)^{[0,n]}$, which maps to zero in $(A/IJ)^{[0,n]}$. □

6.3. Remark. Although the proof of 6.2 given above uses some properties specific to Witt vector functors, the result is true for any representable ring-valued functor. See Borger–Wieland [5], 5.5.

6.4. Corollary. Let $I$ be an ideal in an $R$-algebra $A$. If $I^m = 0$, then $W_n(I)^m = 0$.

6.5. Proposition. Let $\varphi: A \to B$ be a map of $R$-algebras. If it is surjective, then so is the map $W_n(\varphi): W_n(A) \to W_n(B)$.

Proof. By 6.1 we may assume $E$ consists of one ideal $m$. Since surjectivity can be checked Zariski locally on $R$, it is enough by 6.1 to assume $m$ is principal. Then using the Witt components, we can identify the set map underlying $W_n(\varphi)$ with the map $\varphi^{[0,n]}: A^{[0,n]} \to B^{[0,n]}$, which is clearly surjective. □

6.6. Corollary. If $\varphi: A \to B$ is surjective, then

\[
W_n(A \times_B A) \xrightarrow{W_n(\text{pr}_1)} W_n(A) \xrightarrow{W_n(\varphi)} W_n(B)
\]

is a coequalizer diagram.

Proof. The functor $W_n$ is representable, and hence commutes with limits. (See 2.3) Therefore $W_n(A \times_B A)$ agrees with $W_n(A) \times_{W_n(B)} W_n(A)$, which is an equivalence relation on $W_n(A)$, the quotient by which is the image of $W_n(\varphi)$. By 6.5 this is all of $W_n(B)$. □

6.7. Remark. This result is particularly appealing when $A$ is $E$-flat and $B$ is not. Then we can describe $W_n(B)$ in terms of $W_n(A)$ and $W_n(A \times_B A)$, which are directly accessible because $A$ and $A \times_B A$ are $E$-flat.

6.8. Proposition. Suppose $E$ consists of one ideal $m$, and let $A$ be an $R$-algebra. For any $i \geq 0$, the map $\text{Spec}(\text{id} \otimes \tilde{w}_i)$ of schemes induced by the ring map

\[
\text{id} \otimes \tilde{w}_i: R/m \otimes_R W_n(A) \to R/m \otimes_R A/m^{n+1}
\]

is a universal homeomorphism. For $i = 0$, it is a closed immersion defined by a square-zero ideal.
Proof. Consider the diagram

\[
\begin{array}{ccc}
R/m \otimes_R W_n(A) & \xrightarrow{id \otimes w_i} & R/m \otimes_R A/m^{n+1}A \\
\downarrow id \otimes w_0 & & \downarrow r \otimes a + ra \\
A/mA & \xrightarrow{x \mapsto x^q} & A/mA.
\end{array}
\]

To show it commutes, it is enough to assume \( m \) is principal, generated by \( \pi \). Then \( \pi \) is principal, generated by \( \pi \), and so we are reduced to showing

\[ w_i(a) = a_0^q + \pi a_1^q + \cdots + \pi^t a_t \equiv a_0^q \mod mA, \]

for any element \( a = (a_0, a_1, \ldots) \in W(A) \).

Therefore, \( id \otimes w_0 \) is the composition of a map whose kernel is a nil ideal and power of the Frobenius map. The scheme maps induced by both of these are universal homeomorphisms.

Now let us show that \( id \otimes w_0 \) (which equals \( id \otimes w_0 \)) is a surjection with square-zero kernel. The map \( id \otimes w_0 \) is surjective by [1,21] or [1,4]. So let us show the square of its kernel is zero. By [1,3], the kernel of the map \( W_n(A) \to R/m \otimes_R A \) is the ideal \( V^1W_n(A) + mW_n(A) \). Hence it is enough to show \( (V^1W_n(A))^2 \subseteq mW_n(A) \). This follows from [1,22].

6.9. Proposition. Let \((B_i)_{i \in I}\) be a family of \( A \)-algebras such that the induced map \( \coprod \mathrm{Spec} \ B_i \to \mathrm{Spec} \ A \) is surjective. Then the induced map \( \coprod \mathrm{Spec} \ W_n(B_i) \to \mathrm{Spec} \ W_n(A) \) is also surjective.

Proof. By [5,4] it is enough to assume \( E \) consists of one ideal \( m \). Further, it is enough to show surjectivity after base change to \( R[1/m] \) and to \( R/m \). For \( R[1/m] \), it follows from [6,1] and the equality \( W_n(C) = C^{[0,n]} \), when \( m \) is the unit ideal. Now consider base change to \( R/m \). By [6,8] the ring \( W_n(A)/mW_n(A) \) is a nilpotent extension of \( A/mA \), and likewise for each \( B_i \), and so we are reduced to showing that \( \coprod \mathrm{Spec} B_i/mB_i \to \mathrm{Spec} A/mA \) is surjective. This is true since base change distributes over disjoint unions and preserves surjectivity.

6.10. Proposition. The \( R \)-algebra \( \Lambda_{R,E,n} \) is finitely presented, and the functor \( W_n \) preserves filtered colimits of \( R \)-algebras.

Proof. Since \( W_n \) is represented by \( \Lambda_{R,E,n} \), the two statements to be proved are equivalent. By [5,3] we may assume \( E \) consists of a single ideal \( m \). By EGA IV 2.7.1 [13], the first statement can be verified fpqc locally on \( R \), and in particular after base change to \( R[1/m] \) and to \( R/m \). Therefore by [2,5,1], we can assume \( m \) is generated by a single element \( \pi \). But by [3,3] the \( R \)-algebra \( \Lambda_{R,E,n} \) is generated by the finite set \( \theta_{\pi,0}, \ldots, \theta_{\pi,n} \).

7. Some general descent

The purpose of this section is to record some facts about descent of étale algebras which we will use to prove the main theorem [9,2]. The results are general in that they mention nothing about Witt vectors or anything else in this paper. It would be reasonable to skip this section and refer back to it only as needed.

More precisely, we do the following. First, we set up some language and notation for descent. This is essentially a repetition of parts of Grothendieck’s TDTE I [17]. (It could not be anything but.) Second, we prove an abstract result [7,10] relating gluing data and descent data for certain simple gluing constructions. Third, we recall Grothendieck’s theorem [7,11] on integral descent of étale maps. Finally, we prove [7,12] which will provide the outline of the proof of the main theorem [9,2].
Aside from the language of descent, only these three results will be referred to outside this section.

Language

7.1. Fibered categories. Let $C$ be a category with fibered products. Let $E$ be a category fibered over $C$. (See [17], A.1.1, or [1], VI.6.1.) For any object $S$ of $C$, let $E_S$ denote the fiber of $E$ over $S$. Let us say that a map $q: T \to S$ in $C$ is an $E$-equivalence if $q^*: E_S \to E_T$ is an equivalence of categories, and let us say that $q$ is a universal $E$-equivalence if for any map $S' \to S$ in $C$, the base change $q': S' \times_S T \to S'$ is an $E$-equivalence.

For the applications in the next section, the reader can take $C = \text{the category of affine schemes},$ $E = \text{the fibered category over } C \text{ where } E_S \text{ is the category of affine étale } S \text{-schemes and the functors } q^* \text{ are given by base change.}$ (7.1.1)

Then any closed immersion defined by a nil ideal is a universal $E$-equivalence (EGA IV 18.1.2 [16]).

7.2. Composition notation. Let $S$ be an object of $C$, and let $C_{S \times S}$ denote the category of objects over $S \times S$. That is, an object of $C_{S \times S}$ is a pair $(T, \pi_T)$, where $T$ is an object of $C$ and $\pi_T$ is a map $T \to S \times S$, called its structure map; a morphism is a morphism in $C$ commuting with the maps to $S \times S$. For such an object, let $\pi_{T,1}, \pi_{T,2}$ denote the composition of the structure map $T \to S \times S$ with the projections $\text{pr}_1, \text{pr}_2: S \times S \to S$. ($\pi_{T,1}$ is the ‘source,’ and $\pi_{T,2}$ is the ‘target’.)

We will often abusively leave $\pi_T$ implicit and say that $T$ is an object of $C$.

Let $1_S$ denote the object $(S, \Delta)$ of $C_{S \times S}$, where $\Delta : S \to S \times S$ is the diagonal map.

Given two objects $T, U \in C_{S \times S}$, define $TU \in C_{S \times S}$ as follows. As an object of $C$, it is the fibered product

\[
    TU \xrightarrow{\pi_{T,1}} T \xrightarrow{\pi_{T,2}} S. 
\]

We give $TU$ the structure of an object of $C_{S \times S}$ with the map

\[
    TU = T \times_S U \xrightarrow{(\pi_{T,1} \circ \text{pr}_1, \pi_{T,2} \circ \text{pr}_2)} S \times S. 
\]

7.3. Category objects and equivalence relations. A category object over $S$ is an object $R \in C_{S \times S}$ together with maps

\[
    e_R: 1_S \to R, \quad c_R: RR \to R 
\]

in $C_{S \times S}$ (called identity and composition) satisfying the usual identity and associativity axioms in the definition of a category.

A morphism $f: R \to R'$ of such category objects defined to be a morphism in $C_{S \times S}$ satisfying the functor axioms, that is, such that

\[
    f \circ e_R = e_R \circ f \quad \text{and} \quad c_R \circ f = f \circ c_R, 
\]

where $ff$ denotes the map $RR \to R'R'$ induced by $f$.

A category-object structure on a subobject $R \subseteq S \times S$ is a property of $R$ in that when it exists, it is unique. One might say that $R$ is a reflexive transitive relation.
on $S$. We say $R$ is an equivalence relation on $S$ if, in addition, the endomorphism $(\text{pr}_2, \text{pr}_1)$ of $S \times S$ that switches the two factors restricts to a map

$$s: R \to R$$

(which is of course unique when it exists).

7.4. Pre-actions (gluing data). Let $T$ be an object of $C_{S \times S}$. A pre-action of $T$ on an object $X \in E_S$ is defined to be an isomorphism

$$(7.4.1) \quad \varphi: \pi_{T,2}^* (X) \cong \pi_{T,1}^* (X)$$

in $E_T$. A pre-action is also called a gluing datum on $X$ relative to the pair of maps $(\pi_{T,1}, \pi_{T,2})$. (Actually, Grothendieck [17], A.1.4, calls $\varphi^{-1}$ the gluing datum.) Let $\text{PreAct}(T, X)$ denote the set of pre-actions of $T$ on $X$. Any map $T \to T'$ in $C_{S \times S}$ naturally induces a map

$$\text{PreAct}(T', X) \to \text{PreAct}(T, X).$$

If $f: X \to X'$ is a morphism in $E_S$ between objects $X, X'$ with pre-actions $\varphi, \varphi'$, then we say $f$ is $T$-equivariant if the following diagram commutes:

In this way, the objects of $E_S$ equipped with a pre-action of $T$ form a category.

7.5. Actions. Now let $R$ be a category object over $S$. An action of $R$ on $X$ is defined to be a pre-action $\varphi$ of $R$ on $X$ such that the diagram

$$e^* \pi_{R,2}^* (X) \xrightarrow{e^*(\varphi)} e^* \pi_{R,1}^* (X)$$

and the diagram

$$e^* \pi_{R,2}^* (X) \xrightarrow{c^*(\varphi)} e^* \pi_{R,1}^* (X)$$

and

$$c_\varphi^* \pi_{R,2}^* (X) \xrightarrow{\text{pr}_1^* (\varphi)} \text{pr}_1^* \pi_{R,2}^* (X),$$

commute. Here, $\text{pr}_1$ and $\text{pr}_2$ denote the projections $RR \to R$ onto the first and second factors, and the arrows represented by equality signs are the canonical structure maps of the fibered category $E$ (notated $c_{f,g}$ in [17], A.1.1(ii)). We will often use the following more succinct, if slightly abusive, expressions:

$$(7.5.1) \quad e^*(\varphi) = \text{id}_X, \quad c^*(\varphi) = (\text{pr}_1^* \varphi) \circ (\text{pr}_2^* \varphi).$$
Let \( \text{Act}(R, X) \) denote the set of actions of \( R \) on \( X \). A \( R \to R' \) is a morphism of category objects induces a map
\[
\text{Act}(R', X) \longrightarrow \text{Act}(R, X)
\]
in the obvious way.

Last, note that if \( R \) is an equivalence relation, then the diagram
\[
\begin{array}{ccc}
\pi_{R,2}(X) & \xrightarrow{s^*} & \pi_{R,1}(X) \\
\downarrow & & \downarrow \\
\pi_{R,1}(X) & \xrightarrow{\varphi^{-1}} & \pi_{R,2}.
\end{array}
\]
commutes. This follows immediately from (7.5.1). The abbreviated version is (7.5.2)
\[
s^*(\varphi) = \varphi^{-1}.
\]

7.6. Descent data. Let \( q: S' \to S \) be a map in \( \mathcal{C} \), and put
\[
R(S'/S) = S' \times_S S'.
\]
View \( R(S'/S) \) as an object in \( \mathcal{C}_{S' \times S'} \) by taking \( \pi_{R(S'/S)} \) to be the evident monomorphism
\[
R(S'/S) = S' \times_S S' \hookrightarrow S' \times S'
\]
Then \( R(S'/S) \) is an equivalence relation on \( S' \). An action \( \varphi \) of \( R(S'/S) \) on an object \( X' \) of \( \mathcal{E}_{S'} \) is also called a descent datum on \( X' \) from \( S' \) to \( S \). (Again, it is actually \( \varphi^{-1} \) that is called the descent datum in [17].) We might call \( R(S'/S) \) the descent, or Galois, groupoid of the map \( q: S' \to S \).

Because the two compositions \( R(S'/S) = S' \times_S S' \hookrightarrow S' \to S \) are equal, for any object \( X \in \mathcal{E}_S \), the object \( q^*(X) \) of \( \mathcal{E}_{S'} \) has a canonical pre-action of \( R(S'/S) \), and it is easy to check that this is an action. We say that \( q \) is a descent map for the fibered category \( \mathcal{E} \) if the functor from \( \mathcal{E}_S \) to the category of objects of \( \mathcal{E}_{S'} \) with an \( R \)-action is fully faithful. We say it is an effective descent map if it is an equivalence.

7.7. When gluing data is descent data. Now suppose we have a diagram
\[
(7.7.1) \quad S'' \longrightarrow S' \longrightarrow S
\]
in \( \mathcal{C} \) such that the two compositions \( S'' \hookrightarrow S \) are equal. The universal property of products gives a map
\[
S'' \longrightarrow S' \times_S S' = R(S'/S).
\]
For any object \( X' \in \mathcal{E}_{S'} \), this map induces a function
\[
\text{Act}(R(S'/S), X) \longrightarrow \text{PreAct}(S'', X).
\]
Let us say that gluing data on \( X' \) is descent data relative to the diagram (7.7.1) when this map is a bijection.

Gluing two objects

Here we spell out in perhaps excessive detail some basic facts about equivalence relations on disjoint unions which are \( \mathcal{E} \)-trivial (though not necessarily trivial) on each factor.

From now on, let \( \mathcal{C} \) denote the category of affine schemes, schemes, or algebraic spaces. (We only need some weak hypotheses on coproducts in \( \mathcal{C} \), but let us not bother to determine which ones we need.)
7.8. Equivalence relations on a disjoint union. Suppose $S$ is a coproduct $S_a + S_b$ of two objects $S_a, S_b \in C$. (We use the symbols $a, b$ to index the summands only to emphasize their distinction from the symbols 1, 2 that index the factors in the product $S \times S$.) Let $R$ be an equivalence relation on $S$, and let $R_{ij}$ denote $R \times_{S \times S} (S_i \times S_j)$, for any $i, j \in \{a, b\}$. Let $\pi_{R_{ij}, 1}$ denote the evident composition

$$R_{ij} = R \times_{S \times S} (S_i \times S_j) \xrightarrow{pr_1} S_i$$

and $\pi_{R_{ij}, 2}$ the analogous map $R_{ij} \rightarrow S_j$. We will sometimes view $R_{ij}$ as an object of $C_{S \times S}$ using the induced map $R_{ij} \rightarrow S_i \times S_j \rightarrow S \times S$.

Let $e_i : S_i \rightarrow R_{ii}$ and $e_{ijk} : R_{ij} \times_{R_{ik}} R_{jk} \rightarrow R_{ik}$ and $s_{ij} : R_{ij} \rightarrow R_{ji}$ denote the evident restrictions of $e$ and $c$ and $s$.

7.9. Actions over a disjoint union. For any object $X$ over $S$, write $X_a = S_a \times_S X$ and $X_b = S_b \times_S X$.

For any pre-action

$$(7.9.1) \quad \varphi : \pi^*_{R, 2}X \rightarrow \pi^*_{R, 1}X,$$

of $R$ on $X$, let us write $\varphi_{ij}$ for the restriction of $\varphi$ to $R_{ij}$. In order for this pre-action to be an action, it is necessary and sufficient that for all $i, j, k \in \{a, b\}$ we have

$$(7.9.2) \quad e^*_i(\varphi_{ii}) = \text{id}_X, \quad \text{and}$$
$$(7.9.3) \quad e^*_{ijk}(\varphi_{ik}) = pr_1^*(\varphi_{ij}) \circ pr_2^*(\varphi_{jk}).$$

This is just a restatement of (7.5.1), summand by summand. In that case, (7.5.2) becomes

$$(7.9.4) \quad s^*_{ij}(\varphi_{ji}) = \varphi_{ij}^{-1}.$$ 

7.10. Proposition. Let $R$ be an equivalence relation on $S = S_a + S_b$ such that for $i = a, b$, the map $e_i : S_i \rightarrow R_{ii}$ is a universal $E$-equivalence. Then for any object $X \in E_S$, the map

$$\text{Act}(R, X) \xrightarrow{\varphi \mapsto \varphi_{ba}} \text{PreAct}(R_{ba}, X)$$

is a bijection.

Proof. Let us first show injectivity. Let $\varphi$ and $\varphi'$ be actions of $R$ on $X$ such that $\varphi_{ba} = \varphi'_{ba}$. We need to show that this implies $\varphi_{ij} = \varphi'_{ij}$ for all $i, j \in \{a, b\}$. Consider each case separately. For $ij = ba$, it is true by assumption. When $ij = ab$, equation (7.9.4) and the given equality $\varphi_{ba} = \varphi'_{ba}$, imply

$$\varphi_{ab} = s_{ba}(\varphi_{ba})^{-1} = s_{ba}(\varphi'_{ba})^{-1} = \varphi'_{ab}.$$ 

When $i = j$, since $e_i$ is an $E$-equivalence, it is enough to show $e^*_i(\varphi_{ii}) = e^*_i(\varphi'_{ii})$. But by (7.9.2), we have

$$e^*_i(\varphi_{ii}) = \text{id}_X, \quad e^*_i(\varphi'_{ii}).$$

Therefore $\varphi = \varphi'$, which proves injectivity.

Now consider surjectivity. Let $\varphi_{ba}$ be a pre-action of $R_{ba}$ on $X$. Define

$$(7.10.1) \quad \varphi_{ab} = s_{ab}(\varphi_{ba})^{-1}$$

and for $i = a, b$ define $\varphi_{ii}$ to be the map such that

$$(7.10.2) \quad e^*_i(\varphi_{ii}) = \text{id}_X,$$

which exists and is unique because $e_i$ is an $E$-equivalence. We need to check that the pre-action $\varphi = \varphi_{aa} + \varphi_{ab} + \varphi_{ba} + \varphi_{bb}$ of $R$ on $X$ is actually an action. To do this, we will verify the relations (7.9.2) and (7.9.3).

The identity axiom (7.9.2) holds because it is the defining property (7.10.2) of $\varphi_{ii}$. 
Now consider the associativity axiom (7.9.3) for the various possibilities for \( ijk \).

Since \( i, j, k \in \{a, b\} \), two of \( i, j, k \) must be equal.

If \( i = j \), the composition

\[
R_{ijk} \xrightarrow{pr_i^{-1}} S_{jj} R_{ijk} \xrightarrow{e_j \times \text{id}} R_{jj} R_{jk}
\]

is an \( E \)-equivalence, because it is a base change of the universal \( E \)-equivalence \( e_j \).

Therefore it is enough to show

\[
f^* c_{ijk}^*(\varphi_{jk}) = f^* pr_i^1(\varphi_{jj}) \circ f^* pr_j^2(\varphi_{jk}).
\]

By the equality \( pr_1 \circ f = e_j \circ \pi_{R_{jk},1} \) and (7.10.2), we have

\[
f^* pr_i^1(\varphi_{jj}) = \pi_{R_{jk},1}^1 \circ e_j^* \circ \pi_{X_{ij},1}(\varphi_{jj}) = \pi_{X_{ij},1} = \text{id}.
\]

On the other hand, by \( c_{jjk} \circ f = \text{id}_{R_{jk}} = pr_2 \circ f \), we have \( f^* c_{jjk}^*(\varphi_{jk}) = f^* pr_2(\varphi_{jk}) \).

Equation (7.10.3) then follows.

The case \( j = k \) is similar to the case \( i = j \). (Or apply \( s \) to the case \( i = j \).)

Last, suppose \( i = k \). The following diagram is easily checked to be cartesian:

(7.10.4)

\[
\begin{array}{ccc}
R_{ij} & \xrightarrow{(\text{id}_{R_{ij}}, s_{ij})} & R_{ij} R_{ji} \\
\pi_{R_{ij},1} & \downarrow & \downarrow e_{ij} \\
S_{ij} & \xrightarrow{c_{ij}} & R_{ii}.
\end{array}
\]

(This is just another expression of the existence and uniqueness of inverses in a groupoid.) Since \( e_i \) is a universal \( E \)-equivalence, \( (\text{id}_{R_{ij}}, s_{ij}) \) is an \( E \)-equivalence. So it is enough to show axiom (7.9.3) after applying \( (\text{id}_{R_{ij}}, s_{ij})^* \), that is, to show

(7.10.5)

\[
(\text{id}_{R_{ij}}, s_{ij})^* c_{ijj}^*(\varphi_{ii}) = (\text{id}_{R_{ij}}, s_{ij})^* \pi_{R_{ij},1}^1(\varphi_{ij}) \circ (\text{id}_{R_{ij}}, s_{ij})^* pr_2(\varphi_{ji}).
\]

By the commutativity of (7.10.3) and (7.10.2), we have

\[
(\text{id}_{R_{ij}}, s_{ij})^* c_{ijj}^*(\varphi_{ii}) = \pi_{R_{ij},1}^1(\varphi_{ij}) = \pi_{R_{ij},1}(\text{id}_{X_{ij}}) = \text{id}.
\]

Combining this with the equation \( \varphi_{ji} = s_{ji}^*(\varphi_{ij})^{-1} \), equation (7.10.5) reduces to

\[
(\text{id}_{R_{ij}}, s_{ij})^* \pi_{R_{ij},1}^1(\varphi_{ij}) = (\text{id}_{R_{ij}, s_{ij}})^* pr_2 s_{ji}(\varphi_{ij}).
\]

But this holds because we have

\[
pr_1 \circ (\text{id}_{R_{ij}}, s_{ij}) = \text{id}_{R_{ij}} = s_{ij} \circ s_{ij} = s_{ij} \circ pr_2 \circ (\text{id}_{R_{ij}}, s_{ij}).
\]

Therefore the equations in (7.9.3) hold for all \( i, j, k \), and so the pre-action is an action. \(\square\)

Grothendieck’s theorem

7.11. Theorem. Every surjective integral map \( Y \to X \) of affine schemes is an effective descent map for the fibered category \( E \) over \( C \) of (7.1.7).

Observe that it is not necessary to assume the corresponding map of rings is injective.

This theorem is proven in SGA 1 IX 4.7 [1], up to two details. First, the proof there only covers morphisms \( Y \to X \) which are also finite and of finite presentation; and second, the statement there has no affineness in the assumptions or in the conclusion. The first point can be handled by a standard limiting argument (or one can apply Theorem 5.17 plus Remark 2.5(1b) in D. Rydh [24]). The second point can be handled with Chevalley’s theorem; the form most convenient here
would the final one, Rydh’s Theorem (8.1), which is free from noetherianness, separatedness, finiteness, and scheme-theoretic assumptions.

**Gluing and descent of étale algebras**

**7.12. Proposition.** Consider a diagram of rings

\[
\begin{array}{ccc}
B & \xrightarrow{d} & B' \\
\downarrow{e} & & \downarrow{h_1} \\
A & \xrightarrow{f} & A'
\end{array}
\]

such that \( h_i \circ e' = e'' \circ g_i \), for \( i = 1, 2 \). Also assume the following properties are satisfied:

(a) the two parallel right-hand squares are cocartesian,
(b) both rows are equalizer diagrams,
(c) relative to the lower row, gluing data on any étale \( A' \)-algebra is descent data,
(d) \( f \) satisfies effective descent for the fibered category of étale algebras, and
(e) \( e' \) is étale.

Then \( e \) is étale and the left-hand square is cocartesian.

**Note** that when we use the language of descent in the category of rings (as in (c) and (d)), we understand that it refers to the corresponding statements in the opposite category.

**Proof.** Property (a) equips the étale \( A' \)-algebra \( B' \) with gluing data \( \varphi \) relative to \((g_1, g_2)\). Indeed, take \( \varphi \) to be the composition

\[
\begin{array}{ccc}
A'' & \xrightarrow{\sim} & B'' \\
\downarrow{\sim} & & \downarrow{\sim} \\
B' & \xrightarrow{h_1} & B''
\end{array}
\]

By property (c), this gluing data comes from unique descent data relative to \( f \).

Therefore by (d) and (e), the \( A' \)-algebra \( B' \) descends to an étale \( A \)-algebra \( C \).

Now apply the functor \( C \otimes_A - \) to the lower row of diagram (7.12.1). By (a) and the definition of descent, the result can be identified with the sequence

\[
\begin{array}{ccc}
C & \xrightarrow{h_1} & B''
\end{array}
\]

This sequence is also an equalizer diagram, because the lower row of (7.12.1) is an equalizer diagram, by (b), and because \( C \) is étale over \( A \) and hence flat. Again by (b), the upper row of (7.12.1) is an equalizer diagram, and so we have \( C = B \).

Therefore, \( B \) is an étale \( A \)-algebra and the left-hand square is cocartesian.

\[\square\]

**8. Affine ghost descent, single-prime case**

We return to the notation of [12]. Suppose \( E \) consists of a single maximal ideal \( m \), and fix an integer \( n \geq 1 \). Let \( A \) be an \( R \)-algebra, and let \( \alpha_n \) denote the map

\[
(8.0.2) \quad W_n(A) \xrightarrow{\alpha_n} W_{n-1}(A) \times A
\]

given by the canonical projection on the factor \( W_{n-1}(A) \) and the \( n \)-th ghost component \( w_n \) on the factor \( A \). Let \( I_n(A) \) denote the kernel of \( \alpha_n \). For example, if \( m \) is generated by \( \pi \), then in terms of the Witt components, we have

\[
(8.0.3) \quad I_n(A) = \{(0, \ldots, 0, a)_{\pi} \in A^{[0,n]} | \pi^a a = 0\}.
\]

**8.1. Proposition.** We have the following:

(a) \( \alpha_n \) is an integral ring homomorphism.
(b) The kernel $I_n(A)$ of $\alpha_n$ is a square-zero ideal.

(c) If $A$ is $m$-flat, then $\alpha_n$ is injective.

(d) The diagram

$$
\begin{array}{ccc}
W_{n-1}(A) & \xrightarrow{w_n} & A/m^nA \\
\uparrow & & \uparrow \\
W_n(A) & \xrightarrow{w_n} & A,
\end{array}
$$

where the vertical maps are the canonical ones, is cocartesian.

(e) View $A$ as a $W_n(A)$-algebra by the map $w_n: W_n(A) \to A$. Then every element of the kernel of the multiplication map

$$
A \otimes_{W_n(A)} A \to A
$$

is nilpotent.

(f) In the diagram

$$
(8.1.1) \quad W_n(A) \xrightarrow{\alpha_n} W_{n-1}(A) \times A \xrightarrow{\overline{w_n \circ pr_2}} A/m^nA,
$$

where $\overline{pr_2}$ denotes the reduction of $pr_2$ modulo $m^n$, the image of $\alpha_n$ agrees with the equalizer of $\overline{w_n \circ pr_1}$ and $\overline{pr_2}$.

Proof. (a): It is enough to show that each factor of $W_n(A)$ is integral over $W_n(A)$. The first factor is a quotient ring, and hence integral. Now consider an element $a \in A$. Then $a^{q^n}$ is the image in $A$ of the Teichmüller lift $[a] \in W_n(A)$. (See 1.21) Therefore the second factor is also integral over $W_n(A)$.

(b): It suffices to show this after base change to $R[1/m] \times R_m$. Therefore, by 6.1 we may assume $m$ is generated by a single element $\pi$. Then an element of the kernel of $\alpha_n$ will be of the form $V^n[\pi]^n[a] = (0, \ldots, 0, a)_{\pi}$, where $\pi^n a = 0$. On the other hand, by 1.2.6 we have

$$
(V^n[\pi]^n[a]) (V^n[\pi]^n[b]) = \pi^n V^n[\pi][ab] = (0, \ldots, 0, \pi^n ab)_{\pi} = 0.
$$

(c): We have $(w_{\leq n-1} \times \text{id}_A) \circ \alpha_n = w_{\leq n}$. Since $A$ is $m$-flat, the map $w_{\leq n}$ is injective (2.4), and hence so is $\alpha_n$.

(d): As above, it is enough by 6.1 to assume $m$ is generated by a single element $\pi$. Then we have

$$
A \otimes_{W_n(A)} W_{n-1}(A) = A \otimes_{W_n(A)} W_n(A)/V^nW_n(A) = A/\overline{w_n(V^nW_n(A))}A.
$$

Examining the Witt polynomials $8.1.1$ shows $w_n(V^nW_n(A)) = \pi^n A$.

(e): Again, by 6.1 we may assume $m$ is generated by a single element $\pi$. To show every element $x \in I$ is nilpotent, it is enough to restrict $x$ to a set of generators. Therefore it is enough to show $(1 \otimes a - a \otimes 1)^{q^n} = 0$ for every element $a \in A$.

Now suppose that, for $j = 0, \ldots, q^n$, we could show

$$
(8.1.2) \quad \binom{q^n}{j} a^j \in \text{im}(w_n).
$$

Then we would have

$$
(1 \otimes a - a \otimes 1)^{q^n} = \sum (-1)^j \binom{q^n}{j} a^j \otimes a^{q^n-j}
$$

$$
= \sum (-1)^j \otimes \binom{q^n}{j} a^j a^{q^n-j} = (1 \otimes a - 1 \otimes a)^{q^n} = 0,
$$

which would complete the proof. So let us show (8.1.2).
Let \( f = \operatorname{ord}_p(q) \) and \( i = \operatorname{ord}_p(j) \). Then we have
\[
\operatorname{ord}_p \left( q^n \right) = \operatorname{ord}_p \left( q^n j^{-1} \right) + \operatorname{ord}_p \left( \frac{q^n - 1}{j - 1} \right) \geq nf - i.
\]

It follows that \( \left( q^n \right)^j a^j \) is an \( R \)-linear multiple of \( \pi^{nf-i}a^j \). Since \( w_n \) is an \( R \)-algebra map, it is therefore enough to show
\[
\pi^{nf-i}a^j \in \operatorname{im}(w_n).
\]

Now, for \( b \in A \) and \( s = 0, \ldots, n \), we have \( \pi^{-s}b^s = w_n(V_{\pi^{-s}}[b]) \), and therefore \( \pi^{-s}b^s \) is in the image of \( w_n \). So to show \( (8.3.1) \), it is enough to find an integer \( s \) and an element \( b \in A \) such that \( \pi^{-s}b^s \) is an \( R \)-linear divisor of \( \pi^{nf-i}a^j \). In particular, it is sufficient for \( b \) and \( s \) to satisfy \( b^s = a^j \) and \( n - s \leq nf - i \).

Take \( s \) to be the greatest integer at most \( if^{-1} \). Then we have \( q^s \mid j \); so if we set \( b = a^j/q^s \in A \), we have \( b^s = a^j \). It remains to show \( n - s \leq nf - i \). This is equivalent to \( n - if^{-1} \leq nf - i \), which is in turn equivalent to \( (1 - f)(n - if^{-1}) \leq 0 \). And this holds because \( 1 - f \leq 0 \) and \( n - if^{-1} \geq 0 \). (Recall that \( j \leq q^n \).) This completes the proof of (e).

\( \square \): As above, we may assume that \( \mathfrak{m} \) can be generated by a single element \( \pi \).

For any element \( a = (a_0, \ldots, a_n) \in W_n(A) \), we have
\[
\alpha_n(a) = \left( (a_0, \ldots, a_{n-1}), a_0^n + \cdots + \pi^{n-1}a_{n-1}^q + \pi^n a_n \right).
\]

Therefore an element \( ((a_0, \ldots, a_{n-1}), b) \in W_{n-1}(A) \times A \) lies in the image of \( \alpha_n \) if and only if
\[
a_0^n + \cdots + \pi^{n-1}a_{n-1}^q \equiv b \mod \mathfrak{m}^n A,
\]
which is exactly what we needed to show. \( \square \)

**8.2. Corollary.** For any \( R \)-algebra \( A \), the ghost map
\[
w_{\leq n} : W_n(A) \longrightarrow A^{[0, n]}
\]
is integral, and its kernel \( J \) satisfies \( J^{2^n} = 0 \).

**Proof.** By \( (8.1) \) and induction on \( n \). \( \square \)

**8.3. Theorem.** (a) The map \( \alpha_n \) is an effective descent map for the fibered category of étale algebras.

(b) Relative to the diagram
\[
W_n(A) \xrightarrow{\alpha_n} W_{n-1}(A) \times A \xrightarrow{\wedge_n, \operatorname{opr}} A/\mathfrak{m}^n A,
\]

gluing data on any étale \( W_{n-1}(A) \times A \)-algebra is descent data \( (7.7) \).

(c) If \( A \) is \( \mathfrak{m} \)-flat, then for any \( A' \)-algebra \( B' \) equipped with gluing data \( \varphi \), the descended \( A' \)-algebra is the subring \( B \) of \( B' \) on which the following diagram commutes:

\[
\begin{array}{ccc}
A/\mathfrak{m}^n A \otimes_{w_n, \operatorname{opr}} B' & \xrightarrow{1 \otimes \operatorname{id}_{B'}} & B' \\
\varphi \downarrow & & \downarrow \\
A/\mathfrak{m} A \otimes_{\operatorname{pr}_2} B' & \xrightarrow{1 \otimes \operatorname{id}_{B'}} & A/\mathfrak{m} A \otimes_{\operatorname{pr}_2} B'.
\end{array}
\]
Proof. (a): This follows from Grothendieck’s theorem (7.11) and (8.1)–(1). (b): We will use (7.10) where \( C \) and \( E \) are as in (7.1.1). In the notation of (7.13) put \( S_n = \text{Spec} W_n(A) \) and \( S_0 = \text{Spec} A \). Let \( \Gamma \) be the equivalence relation \( S \times_{\text{Spec} W_n(A)} S \) on \( S \). By (8.1), we have \( \Gamma_{ba} = \text{Spec} A/m^n A \). The map \( e_a \) is an isomorphism because \( W_n - 1(A) \) is a quotient ring of \( W_n(A) \). The map \( e_b \) is a nil immersion, by (8.1), and hence is an \( E \)-equivalence. Thus we can apply (7.10) which says that a \( \Gamma \)-action is the same as a \( \Gamma_{ba} \)-pre-action. In other words, gluing data is descent data. (c): This will follow from (7.12) once we verify the hypotheses. (7.12) (a)–(b) are clear; (7.12) (c) follows from (b) above; (7.12) (d) follows from (a) above; and (7.12) (e) follows from the definition of \( B \), for the top row of (7.12.1), and from (8.1) and (1), for the bottom row. \( \square \)

8.4. Remark. For any ring \( C \), let \( \text{EtAlg}_C \) denote the category of \( \text{étale} \) \( C \)-algebras. Then another way of expressing part (b) of this theorem is that the induced functor \( \text{EtAlg}_{W_n(A)} \rightarrow \text{EtAlg}_{W_n-1(A) \times \text{EtAlg}_{A/m^n A}} \text{EtAlg}_A \) is an equivalence. (Of course, the fibered product of categories is taken in the weak sense.) In particular, we can prove things about \( \text{étale} \) \( W_n(A) \)-algebras by induction on \( n \). This is the main technique in the proof of 9.2. But it also seems interesting in its own right and will probably have applications beyond the present paper.

8.5. Remark. If we let \( \bar{W}_n(A) \) denote the image of \( \alpha_n \), then the induced diagram

\[
\begin{array}{ccc}
\bar{W}_n(A) & \longrightarrow & W_{n-1}(A) \times A \\
& \nearrow \bar{w}_n \circ \text{pr}_1 & \downarrow \text{pr}_2 \\
& A/m^n A & \\
\end{array}
\]

satisfies all the conclusions of the theorem above, regardless of whether \( A \) is \( m \)-flat.

Indeed, it is an equalizer diagram by (8.1) and the definition of \( \bar{W}_n(A) \); it is an effective descent map by (8.3) and (8.1) (c). Last, because \( \bar{W}_n(A) \) is the image of \( \alpha_n \), gluing (resp. descent) data relative to \( \bar{W}_n(A) \) agrees with gluing (resp. descent) data relative to \( W_n(A) \). In particular, gluing data relative to \( \bar{W}_n(A) \) is descent data.

9. \( W \) and \( \text{étale} \) morphisms

We continue with the notation of (1) but \( E \) is no longer required to consist of one maximal ideal.

9.1. Lemma. Consider a commutative square of affine schemes (or any schemes)

\[
\begin{array}{ccc}
X & \xrightarrow{g} & X' \\
\downarrow f & & \downarrow f' \\
Y & \xrightarrow{h} & Y', \\
\end{array}
\]

and let \( U \) be an open subscheme of \( Y \). Suppose the following hold

(a) \( f \) and \( f' \) are \( \text{étale} \),
(b) the square above becomes cartesian after the base change \( U \times_Y - \),
(c) \( g \) and \( h \) become surjective and universally injective after the base change \( (Y - U) \times_Y - \).

Then the square above is cartesian.

Proof. Let \( e \) denote the induced map \( (g, f'): X' \rightarrow X \times_Y Y' \). It is enough to show \( e \) is \( \text{étale} \), surjective, and universally injective (EGA IV (17.9.1) (10)). The composition of \( e \) with \( \text{pr}_2: X \times_Y Y' \rightarrow Y' \) is \( f' \). Because \( f \) is \( \text{étale} \), so is its base
change $\text{pr}_2$. Combining this with the étaleness of $f'$ implies that $e$ is étale (EGA IV 17.3.4 [18]).

The surjectivity and universal injectivity of $e$ can be checked after base change over $Y$ to $U$ and to $Y - U$. By assumption $e$ becomes an isomorphism after base change to $U$. In particular, it becomes surjective and universally injective.

Let $e, g, h$ denote the maps $e, g, h$ pulled back from $Y'$ to $Y - U$. Let $h'$ denote the base change of $h$ from $Y$ to $X$. Then, as above, we have $g = h' \circ e$. Since $h$ is universally injective, so is $h'$. Combining this with the fact that $g$ is universally injective, implies that $e$ is as well (EGA I 3.5.6–7 [14]). Finally $e$ is surjective since $h'$ is injective and $g$ is surjective. \hfill \Box

9.2. Theorem. For any étale map $\varphi : A \to B$ and any element $n \in \mathbf{N}^{(E)}$, the induced map $W_n(\varphi) : W_n(A) \to W_n(B)$ is étale.

Proof. By [5.2], it is enough to assume $E$ consists of a single maximal ideal $m$. Also, it will simplify notation if we assume $m$ is principal, generated by an element $\pi$.

We may do this by by [6.1] and because it is enough to show étaleness after applying $R_m \otimes R -$ and $R[1/m] \otimes R -$.

We will reason by induction on $n$, the case $n = 0$ being clear because $W_0$ is the identity functor. So from now on, assume $n \geq 1$.

Let $W_n(A)$ denote the image of $\alpha_n : W_n(A) \to W_{n-1}(A) \times A$, and let $\bar{\alpha}_n$ denote the induced injection $W_n(A) \to W_{n-1}(A) \times A$. Define $W_n(B)$ and $\bar{\alpha}_n$ for $B$ similarly.

Step 1: $W_n(B)$ is étale over $W_n(A)$.

To do this, it suffices to verify conditions (a)–(e) of [7.12] for the following diagram

\[
\begin{array}{ccc}
W_n(B) & \xrightarrow{\bar{\alpha}_n} & W_{n-1}(B) \times B \\
\downarrow & & \downarrow \\
W_n(A) & \xrightarrow{\bar{\alpha}_n} & W_{n-1}(A) \times A
\end{array}
\]

where the vertical maps are induced by $\varphi$ and functoriality. We know [7.12] (a) holds by induction. Conditions [7.12] (c)–(d) hold by [8.3] (or [8.5]). Condition [7.12] (e) was shown already in [8.1] (or [8.8]). Now consider [7.12] (b). It is clear that the square of $\text{pr}_2$ maps is cocartesian. So, all that remains is to check that the square of $\bar{w}_n \circ \text{pr}_1$ maps is cocartesian. By induction, $W_{n-1}(B)$ is étale over $W_{n-1}(A)$, and so this follows from [9.1] which we can apply by [6.1] and [8.8].

Step 2: $W_n(B)$ is étale over $W_n(A)$.

By [8.11], the kernel $I_n(A)$ of the map $\alpha_n : W_n(A) \to W_n(A)$ has square zero. Therefore by EGA IV 18.1.2 [18], there is an étale $W_n(A)$-algebra $C$ and an isomorphism $f : C \otimes_{W_n(A)} W_n(A) \to W_n(B)$. Now consider the square

\[
\begin{array}{ccc}
C & \xrightarrow{d} & W_n(B) \\
\downarrow & & \downarrow \\
W_n(A) & \xrightarrow{d} & W_n(B)
\end{array}
\]

where the upper map is the one induced by $f$ and where $d$ will soon be defined. By [8.11], the kernel $I_n(B)$ of the right-hand map has square zero. Therefore since $C$ is étale over $W_n(A)$, there exists a unique map $d$ making the diagram commute. Let us now show that $d$ is an isomorphism.
Because \( C \) is \( \text{étale} \) and hence flat over \( W_n(A) \), we have a commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & I_n(B) & \rightarrow & W_n(B) & \rightarrow & W_n(B) & \rightarrow & 0 \\
& & \uparrow e & & \downarrow d & & \downarrow f & & \\
0 & \rightarrow & C \otimes_{W_n(A)} I_n(A) & \rightarrow & C & \rightarrow & C \otimes_{W_n(A)} W_n(A) & \rightarrow & 0.
\end{array}
\]

So to show \( d \) is an isomorphism, it is enough to show \( e \) is an isomorphism. Because \( I_n(A) \) is a square-zero ideal, the action of \( W_n(A) \) on it factors through \( W_n(A) \). Therefore, \( e \) factors as follows:

\[
C \otimes_{W_n(A)} I_n(A) = C \otimes_{W_n(A)} W_n(A) \otimes_{W_n(A)} I_n(A)
\]

\[
\cong W_n(B) \otimes_{W_n(A)} I_n(A) \xrightarrow{g} I_n(B),
\]

Since \( f \) is an isomorphism, it is enough to show \( g \) is an isomorphism.

Using the description (8.0.3) of \( I_n \), the map \( g \) can be extended to the following commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & I_n(B) & \rightarrow & B & \xrightarrow{\pi^n} & \pi^n B & \rightarrow & 0 \\
& & \downarrow g & & \downarrow (\text{pr}_2 \circ \alpha_n) \cdot \varphi & & \downarrow (\text{pr}_2 \circ \alpha_n) \cdot \varphi & & \\
0 & \rightarrow & W_n(B) \otimes I_n(A) & \rightarrow & \bar{W}_n(B) \otimes A & \xrightarrow{\pi^n} & \bar{W}_n(B) \otimes \pi^n A & \rightarrow & 0
\end{array}
\]

where \( \otimes \) denotes \( \otimes_{W_n(A)} \), for short. Therefore it is enough to show the right two vertical maps are isomorphisms, and to do this, it is enough to show the right-hand square in the diagram

\[
\begin{array}{ccc}
W_n(B) & \rightarrow & \bar{W}_n(B) \\
\downarrow & & \downarrow \text{pr}_2 \circ \alpha_n \\
W_n(A) & \rightarrow & \bar{W}_n(A)
\end{array}
\]

is cocartesian. We will do this by applying (6.1) with \( U = \text{Spec} R[1/\mathfrak{m}] \otimes_R W_n(A) \).

By step 1, condition (9.1(a)) holds. Now consider conditions (9.1(b)–(c)). By 8.3(b), the horizontal maps in the left-hand square have square-zero kernel. In particular, the scheme maps they induce are universal homeomorphisms. And by 6.1 they become isomorphisms after applying \( R[1/\mathfrak{m}] \otimes_R \). Therefore it is enough to show (9.1(b)–(c)) hold for the perimeter of the diagram above. In this case, (9.1(b)) follows from (6.1) and (9.1(c)) follows from (6.8). \( \square \)

**9.3. Corollary.** Let \( B \) be an \( \text{étale} \) \( A \)-algebra, and let \( C \) be any \( A \)-algebra. Then for any \( n \in \mathbb{N}^{(E)} \), the induced diagram

\[
\begin{array}{ccc}
W_n(B) & \rightarrow & W_n(B \otimes_A C) \\
\downarrow & & \downarrow \\
W_n(A) & \rightarrow & W_n(C)
\end{array}
\]

is cocartesian.

**Proof.** By (5.4) we can assume \( E \) consists of a single ideal \( \mathfrak{m} \). The proof will be completed by (6.1) once we check its hypotheses are satisfied. Condition (a) of (9.1) holds by (9.2) condition (b) holds by (6.1) and (2.6) and condition (c) holds by (6.8). \( \square \)
9.4. $W_n$ does not generally commute with coproducts. Almost anything is an example. For instance, with the $p$-typical Witt vectors, $W_1(A @Z A)$ is not isomorphic to $W_1(A) @W_1(Z) W_1(A)$, when $A$ is $F_p[x]$ or $Z[x]$.

9.5. Other truncation sets for the big Witt vectors. In the case where $R = Z$, another system of truncations of $W$ has been studied. See Hesselholt–Madsen [19], subsection 4.1, for example. This system in this paper is less general than it but is cofinal to it. Given a finite set $T$ of positive integers closed under divisibility, they define an endofunctor $W_T$ of the category of rings. Such functors also preserve étale maps. Indeed, it is enough to show that the base change to $Z[1/T]$ and to $Z(p)$, for each prime $p \in T$, is étale. Applying the identity $W_T(A[1/p]) = W_T(A[1/p])$, which can be established by looking at the graded pieces of the Verschiebung filtration, it is enough to consider $Z[1/T]$-algebras and $Z(p)$-algebras. In the either case, $W_T(A)$ is simply a product of $p$-typical Witt rings $W_n(A)$ for various primes $p$ and lengths $n$ (see [19], (4.1.10)), in which case the result follows from [12] or van der Kallen’s original theorem [20], (2.4).

References

[1] Revêtements étales et groupe fondamental (SGA 1). Documents Mathématiques (Paris), 3. Société Mathématique de France, Paris, 2003. Séminaire de géométrie algébrique du Bois Marie 1960–61. Directed by A. Grothendieck, With two papers by M. Raynaud, Updated and annotated reprint of the 1971 original [Lecture Notes in Math., 224, Springer, Berlin].

[2] Francis Borceux. Handbook of categorical algebra, 1, volume 50 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1994. Basic category theory.

[3] Francis Borceux. Handbook of categorical algebra, 2, volume 51 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1994. Categories and structures.

[4] James Borger. Basic geometry of witt vectors, II: Spaces. To appear.

[5] James Borger and Ben Wieland. Plethystic algebra. Adv. Math., 194(2):246–283, 2005.

[6] N. Bourbaki. Éléments de mathématique. Algèbre commutative. Chapitres 8 et 9. Springer, Berlin, 2006. Reprint of the 1983 original.

[7] Alexandru Buium. Geometry of $p$-jets. Duke Math. J., 82(2):349–367, 1996.

[8] Alexandru Buium. Arithmetic differential equations, volume 118 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2005.

[9] Alexandru Buium and Santiago R. Simanca. Arithmetic Laplacians. Adv. Math., 220(1):246–277, 2009.

[10] V. G. Drinfeld. Coverings of $p$-adic symmetric domains. Funkcional. Anal. i Priloˇ zen., 10(2):29–40, 1976.

[11] Marvin J. Greenberg. Schemata over local rings. Ann. of Math. (2), 73:624–648, 1961.

[12] Marvin J. Greenberg. Schemata over local rings. II. Ann. of Math. (2), 78:256–266, 1963.

[13] Alexander Grothendieck. La théorie des classes de Chern. Bull. Soc. Math. France, 86:137–154, 1958.

[14] Alexander Grothendieck. Éléments de géométrie algébrique. I. Le langage des schémas. Inst. Hautes Études Sci. Publ. Math., (4):228, 1960.

[15] Alexander Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II. Inst. Hautes Études Sci. Publ. Math., (24):231, 1965.

[16] Alexander Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV. Inst. Hautes Études Sci. Publ. Math., (32):361, 1967.

[17] Alexander Grothendieck. Technique de descente et théorèmes d’existence en géométrie algébrique. I. Généralités. Descente par morphismes fidèlement plats. In Séminaire Bour- baki, Vol. 5, pages Exp. No. 190, 299–327. Soc. Math. France, Paris, 1995.

[18] Michel Hazewinkel. Formal groups and applications, volume 78 of Pure and Applied Mathematics. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1978.

[19] Lars Hesselholt and Ib Madsen. Cyclic polytopes and the $K$-theory of truncated polynomial algebras. Invent. Math., 130(1):73–97, 1997.

[20] André Joyal. $\delta$-anneaux et $\lambda$-anneaux. C. R. Math. Rep. Acad. Sci. Canada, 7(4):227–232, 1985.
[21] André Joyal. δ-anneaux et vecteurs de Witt. C. R. Math. Rep. Acad. Sci. Canada, 7(3):177–182, 1985.
[22] Michel Lazard. *Commutative formal groups*. Springer-Verlag, Berlin, 1975. Lecture Notes in Mathematics, Vol. 443.
[23] D. Rydh. Noetherian approximation of algebraic spaces and stacks. arXiv:0904.0227v1.
[24] D. Rydh. Submersions and effective descent of étale morphisms. *Bull. Soc. Math. France*. To appear.
[25] D. O. Tall and G. C. Wraith. Representable functors and operations on rings. *Proc. London Math. Soc. (3)*, 20:619–643, 1970.
[26] Wilberd van der Kallen. Descent for the K-theory of polynomial rings. *Math. Z.*, 191(3):405–415, 1986.
[27] Clarence Wilkerson. Lambda-rings, binomial domains, and vector bundles over C P(∞). *Comm. Algebra*, 10(3):311–328, 1982.
[28] Ernst Witt. Zyklische Körper und Algebren der Charakteristik p vom Grad pⁿ. Struktur diskret bewerteter perfekten Körper mit vollkommenem Restklassen-körper der charakteristik p. *J. Réine Angew. Math.*, (176), 1937.
[29] Ernst Witt. *Collected papers. Gesammelte Abhandlungen*. Springer-Verlag, Berlin, 1998. With an essay by Günter Harder on Witt vectors, Edited and with a preface in English and German by Ina Kersten.

*Australian National University*

*E-mail address: james.borger@anu.edu.au*