On Some Asymptotic Properties and an Almost Sure Approximation of the Normalized Inverse-Gaussian Process

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ABSTRACT

In this paper, we present some asymptotic properties of the normalized inverse-Gaussian process. In particular, when the concentration parameter is large, we establish an analogue of the empirical functional central limit theorem, the strong law of large numbers and the Glivenko-Cantelli theorem for the normalized inverse-Gaussian process and its corresponding quantile process. We also derive a finite sum-representation that converges almost surely to the Ferguson and Klass representation of the normalized inverse-Gaussian process. This almost sure approximation can be used to simulate efficiently the normalized inverse-Gaussian process.

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1 Introduction

The objective of Bayesian nonparametric inference is to place a prior on the space of probability measures. The Dirichlet process, formally introduced in Ferguson (1973), is considered the first celebrated example on this space. Several alternatives of the Dirichlet process have been proposed in the literature. In this paper, we focus on one such prior, namely the normalized inverse-Gaussian process introduced by Lijoi, Mena and Prünster (2005).

We begin by recalling the definition of the normalized inverse-Gaussian distribution. The random vector \((Z_1, \ldots, Z_m)\) is said to have the normalized inverse-Gaussian distribution with parameters \((\gamma_1, \ldots, \gamma_m)\), where \(\gamma_i > 0\) for all \(i\), if it has the joint probability density function

\[
f(z_1, \ldots, z_m) = \frac{e^{\sum_{i=1}^m \gamma_i} \prod_{i=1}^m \gamma_i^{\gamma_i}}{2^{m/2-1} \pi^{m/2}} \times K_{-m/2} \left( \sqrt{\sum_{i=1}^m \frac{\gamma_i^2}{z_i}} \right) \times \left( \sum_{i=1}^m \frac{\gamma_i^2}{z_i} \right)^{-m/4} \times \prod_{i=1}^m z_i^{-3/2} \times I_S(z_1, \ldots, z_m)
\]

where \(K\) is the modified Bessel function of the third type and \(S = \{(z_1, \ldots, z_m) : z_i \geq 0, \sum_{i=1}^m z_i = 1\}\). For more details about the modified Bessel functions consult Abramowitz and Stegun (1972, Chapter 9).
Consider a space $\mathcal{X}$ with a $\sigma$–algebra $\mathcal{A}$ of subsets of $\mathcal{X}$. Let $H$ be a fixed probability measure on $(\mathcal{X}, \mathcal{A})$ and $a$ be a positive number. Following Lijoi, Mena and Prünster (2005), a random probability measure $P_{H,a} = \{P_{H,a}(A)\}_{A \in \mathcal{A}}$ is called a normalized inverse-Gaussian process on $(\mathcal{X}, \mathcal{A})$ with parameters $a$ and $H$, if for any finite measurable partition $A_1, \ldots, A_m$ of $\mathcal{X}$, the joint distribution of the vector $(P_{H,a}(A_1), \ldots, P_{H,a}(A_m))$ has the normalized inverse-Gaussian distribution with parameter $(aH(A_1), \ldots, aH(A_m))$. We assume that if $H(A_i) = 0$, then $P_{H,a}(A_i) = 0$ with probability one. The normalized inverse-Gaussian process with parameters $a$ and $H$ is denoted by $\text{N-IGP}(a, H)$, and we write $P_{H,a} \sim \text{N-IGP}(a, H)$.

One of the basic properties of the normalized inverse-Gaussian process is that for any $A \in \mathcal{A}$,

$$E(P_{H,a}(A)) = H(A) \quad \text{and} \quad \text{Var}(P_{H,a}(A)) = \frac{H(A)(1 - H(A))}{\xi(a)},$$

(1.2)

where here and throughout this paper

$$\xi(a) = \frac{1}{a^2 e^a \Gamma(-2, a)}$$

(1.3)

and $\Gamma(-2, \theta) = \int_0^\infty t^{-3} e^{-t} dt$.

Furthermore, for any two disjoint sets $A_i$ and $A_j \in \mathcal{A}$,

$$E(P_{H,a}(A_i)P_{H,a}(A_j)) = H(A_i)H(A_j)\frac{\xi(a) - 1}{\xi(a)}.$$  

(1.4)

Observe that, for large $a$, $\xi(a) \approx a$ (Abramowitz and Stegun, 1972, Formula 6.5.32, page 263), where we use the notation $f(a) \approx g(a)$ if $\lim_{a \to \infty} f(a)/g(a) = 1$. It follows
from (1.2) that $H$ plays the role of the center of the process, while $a$ can be viewed as the concentration parameter. The larger $a$ is, the more likely it is that the realization of $P_{H,a}$ is close to $H$. Specifically, for any fixed set $A \in \mathcal{A}$ and $\epsilon > 0$, we have $P_{H,a}(A) \overset{p}{\to} H(A)$ as $a \to \infty$ since
\[
\Pr \{|P_{H,a}(A) - H(A)| > \epsilon\} \leq \frac{H(A)(1 - H(A))}{\xi(a)\epsilon^2}.
\]
(1.5)

Similar to the Dirichlet process, a series representation of the normalized inverse-Gaussian process can be easily derived from the Ferguson and Klass representation (1972). Specifically, let $(E_i)_{i \geq 1}$ be a sequence of independent and identically distributed (i.i.d.) random variables with an exponential distribution with mean of 1. Define
\[
\Gamma_i = E_1 + \cdots + E_i.
\]
(1.6)

Let $(\theta_i)_{i \geq 1}$ be a sequence of i.i.d. random variables with values in $\mathcal{X}$ and common distribution $H$, independent of $(\Gamma_i)_{i \geq 1}$. Then the normalized inverse-Gaussian process with parameter $a$ and $H$ can be expressed as a normalized series representation
\[
P_{H,a}(\cdot) = \sum_{i=1}^{\infty} \frac{L^{-1}(\Gamma_i)}{L^{-1}(\Gamma_i)} \delta_{\theta_i}(\cdot),
\]
(1.7)

where
\[
L(x) = \frac{a}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-t/2}t^{-3/2} dt, \quad \text{for } x > 0,
\]
(1.8)

and $\delta_X$ denotes the Dirac measure at $X$ (i.e. $\delta_X(B) = 1$ if $X \in B$ and 0 otherwise). Note that, working with (1.7) is difficult in practice because no closed form for the inverse of the Lévy measure (1.8) exists. Moreover, to determine the random weights in (1.7) an infinite sum must be computed.
This paper is organized as follows. In Section 2, we study the weak convergence of the centered and scaled process

\[ D_{H,a}(\cdot) = \sqrt{\xi(a)} \left( P_{H,a}(\cdot) - H(\cdot) \right), \]

as \( a \to \infty \). Note that, since \( \xi(a) \approx a \), it is possible to replace the normalizing coefficient \( \xi(a) \) by \( a \). Therefore, for simplicity, we focus on the process

\[ D_{H,a}(\cdot) = \sqrt{a} \left( P_{H,a}(\cdot) - H(\cdot) \right). \tag{1.9} \]

In Section 3, we derive the limiting process for the quantile process

\[ Q_{H,a}(\cdot) = \sqrt{a} \left( P_{H,a}^{-1}(\cdot) - H^{-1}(\cdot) \right), \tag{1.10} \]

as \( a \to \infty \), where, in general, the inverse of a distribution function \( F \) is defined by

\[ F^{-1}(t) = \inf \{ x : F(x) \geq t \}, \quad 0 < t < 1. \]

The strong law of large numbers and the Glivenko-Cantelli theorem for the normalized inverse-Gaussian process are discussed in Section 4. In Section 5, we derive a finite sum-representation which converges almost surely to the Ferguson and Klass representation of the normalized inverse-Gaussian process. This new representation provides a simple, yet efficient procedure, to sample the the normalized inverse-Gaussian process.
2 Asymptotic Properties of the Normalized Inverse-Gaussian Process

In this section, we study the weak convergence of the process $D_{H,a}$ defined in (1.9) for large values of $a$. Let $\mathcal{S}$ be a collection of Borel sets in $\mathbb{R}$ and $H$ be a probability measure on $\mathbb{R}$. We recall the definition of a Brownian bridge indexed by $\mathcal{S}$. A Gaussian process $\{B_H(S) : S \in \mathcal{S}\}$ is called a Brownian bridge with parameter measure $H$ if $E[B_H(S)] = 0$ for any $S \in \mathcal{S}$ and

$$Cov(B_H(S_i), B_H(S_j)) = H(S_i \cap S_j) - H(S_i)H(S_j)$$

(2.1)

for any $S_i, S_j \in \mathcal{S}$ (Kim and Bickel, 2003).

The next lemma gives the limiting distribution of the process (1.9) for any finite Borel set $S_1, \ldots, S_m \in \mathcal{S}$, as $a \to \infty$. The proof of the lemma for $m = 2$ is given in the appendix and can be generalized easily to the case of arbitrary $m$.

**Lemma 1.** Let $D_{H,a}$ be defined by (1.9). For any fixed sets $S_1, \ldots, S_m$ in $\mathcal{S}$ we have

$$(D_{H,a}(S_1), D_{H,a}(S_2), \ldots, D_{H,a}(S_m)) \overset{d}{\to} (B_H(S_1), B_H(S_2), \ldots, B_H(S_m)),$$

as $a \to \infty$, where $B_H$ is the Brownian bridge with parameter $H$.

The following theorems show that the process $D_{H,a}$ defined by (1.9) converges to the process $B_H$ on $D[-\infty, \infty]$ with respect to the Skorokhod topology, where $D[-\infty, \infty]$ is the space of cadlag functions (right continuous with left limits) on $[-\infty, \infty]$. Right continuity at $-\infty$ can be achieved by setting $D_{H,a}(-\infty) = D_{H,a}(\infty) = 0$; the left limit
at \( \infty \) also equals zero, the natural value of \( D_{H,a}(\infty) \) and \( D_{H,a}(\infty) \). For more details, consult Pollard (1984, Chapter 5). If \( X \) and \((X_a)_{a>0}\) are random variables with values in a metric space \( M \), we say that \((X_a)_{a}\) converges in distribution to \( X \) as \( a \to \infty \) (and we write \( X_a \overset{d}{\to} X \)) if for any sequence \((a_n)\) converging to \( \infty \), \( X_{a_n} \) converges in distribution to \( X \).

**Theorem 1.** We have, as \( a \to \infty \),

\[
D_{H,a}(\cdot) = \sqrt{a} \left( P_{H,a}(\cdot) - H(\cdot) \right) \overset{d}{\to} B_H(\cdot)
\]

on \( D[-\infty, \infty] \) with respect to Skorokhod topology, where \( B_H \) is the Brownian bridge with parameter measure \( H \).

**Remark 1.** In the proof of Lemma [1], the finite-dimensional convergence is in fact convergence in total variation, which is stronger than convergence in distribution (Billingsley 1999, page 29).

**Remark 2.** Lo (1987) obtained a result similar to that given in Theorem 1 for the Dirichlet process to establish asymptotic validity of the Bayesian bootstrap. An interesting generalization of Lo (1987) to the two-parameter Poisson-Dirichlet process was obtained by James (2008). In both papers, proofs are based on constructing the distributional identities (Proposition 4.1, James, 2008)). Establishing analogous distributional identity for the normalized inverse-Gaussian process does not seem to be trivial.

**Remark 3.** Sethuraman and Tiwari (1982) studied the convergence and tightness of the Dirichlet process as the parameters are allowed to converge in a certain sense. Analogous results for the two-parameter Poisson-Dirichlet process (Pitman and Yor, 1997; Ishwaran and James, 2001) and the normalized inverse-Gaussian process \((P_{H,a})_{a>0}\) follows straightforwardly by applying the technique used in the proof of the tightness part (i.e. condition
3 Asymptotic Properties of the Normalized Inverse-Gaussian Quantile Process

Similar to the frequentist asymptotic theory, in this section we establish large sample theory for the normalized inverse-Gaussian quantile process.

**Corollary 1.** Let $0 < p < q < 1$, and $H$ be a continuous function with positive derivative $h$ on the interval $[H^{-1}(p) - \epsilon, H^{-1}(q) + \epsilon]$ for some $\epsilon > 0$. Let $\lambda$ be the Lebesgue measure on $[0, 1]$. Let $Q_{H,a}$ be the normalized inverse-Gaussian process defined in (1.10). As $a \to \infty$, we have

$$Q_{H,a}(\cdot) \xrightarrow{d} \frac{B_{\lambda}(\cdot)}{h(H^{-1}(\cdot))} = Q(\cdot),$$

in $D[p, q]$. That is, the limiting process is a Gaussian process with zero-mean and covariance function

$$\text{Cov}(Q(s), Q(t)) = \frac{\lambda(s \wedge t) - \lambda(s)\lambda(t)}{h(H^{-1}(s))h(H^{-1}(t))}, \quad s, t \in \mathbb{R}.$$

**Proof.** We only proof part (i). Part (ii) follows similarly. By Theorem 1 (i) the process $\sqrt{a}(P_{H,a} - H)$ converges in distribution to the process $B_H = B_{\lambda}(H) = B_{\lambda} \circ H$. Notice that almost all sample paths of the limiting process are continuous on the interval $[H^{-1}(p) - \epsilon, H^{-1}(q) + \epsilon]$. By Lemma 3.9.23 page 386 of van der Vaart and Wellner (1996), the inverse map $H \mapsto H^{-1}$ is Hadamard tangentially differentiable at $H$ to the
subspace of functions that are continuous on this interval. By the functional delta method (Theorem 3.9.4 page 374 of van der Vaart and Wellner (1996)) we have

\[ Q_{H,a}(\cdot) \overset{d}{\to} - \frac{B_\lambda \circ H \circ H^{-1}(\cdot)}{h(H^{-1}(\cdot))} = \frac{B_\lambda(\cdot)}{h(H^{-1}(\cdot))} \]

in \( D[p, q] \). This completes the proof of the corollary.

Remark 4. A similar result to Corollary 1 for the two-parameter Poisson-Dirichlet process can be obtained by applying Theorem 4.1 and Theorem 4.2 of James (2008).

Remark 5. Similar to Remark 1 of Bickel and Freedman (1981), if \( H^{-1}(0+) > -\infty \) and \( H^{-1}(1) < \infty \) and \( h \) is continuous on \([H^{-1}(0+), H^{-1}(1)]\), the conclusion of the corollary holds in \( D[H^{-1}(0+), H^{-1}(1)] \). For example, if \( H \) is a uniform distribution on \([0, 1]\), then the convergence holds in \( D[0, 1] \).

The next example is a direct application of Corollary 1.

Example 1. In this example we derive the asymptotic distribution for the median and the interquantile range for the normalized inverse-Gaussian process. Let \( Q^1_{H,a}, Q^2_{H,a}, Q^3_{H,a} \) be the first, the second (median) and the third quartiles of \( P_{H,a} \) (i.e. \( P_{H,a}^{-1}(0.25) = Q^1_{H,a}, P_{H,a}^{-1}(0.5) = Q^2_{H,a} \) and \( P_{H,a}^{-1}(0.75) = Q^3_{H,a} \)). Let \( q_1, q_2 \) and \( q_3 \) be the first, the second (median) and the third quartiles of \( H \). From Corollary 1 after some simple calculations, the asymptotic distribution of the median and the interquantile range are given, respectively, by:

\[ \sqrt{a} \left( Q^2_{H,a} - q_2 \right) \overset{d}{\to} N \left( 0, \frac{1}{4h^2(q_2)} \right) \]
and

$$\sqrt{a} (IQR - (q_3 - q_1)) \overset{d}{\to} N \left( 0, \frac{3}{h^2(q_3)} + \frac{3}{h^2(q_1)} - \frac{2}{h(q_1)h(q_3)} \right),$$

where $h = H'$ and $IQR = Q^3_{H,a} - Q^1_{H,a}$. Note that, the asymptotic distributions of the median and the interquantile range for the normalized inverse-Gaussian process coincide with that of the sample median and the sample interquartile range (DasGupta, 2008, page 93).

### 4 Glivenko-Cantelli Theorem for the Normalized Inverse-Gaussian Process

In this section, we show that an analogue of the empirical strong law of large numbers and the empirical Glivenko-Cantelli theorem continue to hold for the normalized inverse-Gaussian process.

**Theorem 2.** Let $P_{H,a} \sim N-IGP(a, H)$. Assume that $a = n^2 c$, for a fixed positive number $c$. Then as $n \to \infty$,

$$P_{H,n^2 c}(A) \overset{a.s.}{\to} H(A),$$

for any measurable subset $A$ of $X$.

**Proof.** For any $\epsilon > 0$, by (4.1), we have

$$\Pr \{|P_{H,n^2 c}(A) - H(A)| > \epsilon\} \leq \frac{H(A)(1 - H(A))\xi(n^2 c)}{\epsilon^2},$$

(4.1)
where $\xi(n^2c)$ is defined by (1.3). Note that

$$\lim_{n \to \infty} \frac{\xi(n^2c)}{1/n^2c} = 1$$

(Abramowitz and Stegun, 1972, Formula 6.5.32, page 263). Since the series $\sum_{n=1}^{\infty} 1/n^2$ converges, it follows by the Limit Comparison Test that the series $\sum_{n=1}^{\infty} \xi(n^2c)$ is also convergent. Thus,

$$\sum_{n=1}^{\infty} \Pr \{|P_{H,n^2c}(A) - H(A)| > \epsilon\} < \infty.$$  

Therefore, by the first Borel-Cantelli Lemma, the proof follows.

The proof of the next theorem follows by arguments similar to that given in the proof of the Glivenko-Cantelli theorem for the empirical process. See, for example, Billingsely (1995, Theorem 20.6).

**Theorem 3.** Let $P_{H,a} \sim N-IGP(a, H)$. Assume that $a = n^2c$, for a fixed positive number $c$. Then

$$\sup_{x \in \mathbb{R}} |P_{H,n^2c}(x) - H(x)| \xrightarrow{a.s.} 0,$$

as $n \to \infty$.

**Remark 6.** Similar to the normalized inverse-Gaussian process, a strong law of large numbers and a Glivenko-Cantelli theorem can also be established for the two-parameter Poisson-Dirichlet process.
5 Monotonically Decreasing Approximation to the Inverse Gaussian Process

In this section, we derive a finite sum representation which converges almost surely to the Ferguson and Klass sum representation of the normalized inverse-Gaussian process. We mimic the approach developed recently by Zarepour and Al Labadi (2012) for the Dirichlet process. Let $X_n$ be a random variable with distribution inverse-Gaussian with parameter $a/n$ and 1 (see equation (3) of Lijoi, Mena and Prünster (2005) for the density of the inverse-Gaussian distribution). Define

$$G_n(x) = \Pr(X_n > x) = \int_x^\infty \frac{a}{n\sqrt{2\pi} t^{-3/2}} \exp \left\{ -\frac{1}{2} \left( \frac{a^2}{n^2 t} + t \right) + \frac{a}{n} \right\} \, dt. \quad (5.1)$$

The following proposition describes properties of $G_n(x)$ that will be used later in the paper.

**Proposition 1.** For $x > 0$, the function $G_n(x)$ defined in (5.1) has the following properties as $n \to \infty$:

(i) $nG_n(x) \to L(x)$,

(ii) $G_n^{-1}\left(\frac{x}{n}\right) \to L^{-1}(x)$,

where $L$ is defined in (2.3).

**Proof.** To prove (i), since $e^{a/n} \leq e^a$ and $\exp \left\{ -\frac{1}{2} \left( \frac{a^2}{n^2 t} + t \right) \right\} \leq 1$, the integrand in $nG_n(x)$ is bounded by $ae^a t^{-3/2}/\sqrt{2\pi}$, which is integrable for any $x > 0$. Hence, the
dominated convergence theorem applies and we have

\[ nG_n(x) = \int_x^\infty \frac{a}{\sqrt{2\pi}} t^{-3/2} e^{-\frac{1}{2}(\frac{a^2}{n} + t)} \frac{a}{n} dt \to \frac{a}{\sqrt{2\pi}} \int_x^\infty e^{-t/2} t^{-3/2} dt = L(x). \]

To prove (ii), notice that the left hand side of (i) is a sequence of monotone functions converging to a continuous monotone function for every \( x > 0 \) (Haan-de and Ferreira 2006, page 5). Thus, (i) is equivalent to \( G_n^{-1}(x/n) \to N^{-1}(x) \). \( \square \)

Proposition 1 gives a simple procedure for an approximate evaluation of both \( L(x) \) and \( L^{-1}(x) \) for any \( x > 0 \). For computational simplicity, a more convenient approximation is presented in the following Corollary. The proof follows straightforwardly by taking \( x = \Gamma_i \) in Proposition 1 and the fact that we have \( \Gamma_{n+1}/n \xrightarrow{a.s.} 1 \) as \( n \to \infty \) (strong law of large numbers).

**Corollary 2.** For a fixed \( i \), as \( n \to \infty \), we have:

\[ L_n^{-1}\left(\frac{\Gamma_i}{\Gamma_{n+1}}\right) \xrightarrow{a.s.} L^{-1}(\Gamma_i). \]

**Remark 7.** The utility of Corollary 2 stems from the fact that all values of \( G_n^{-1}(\Gamma_i/\Gamma_{n+1}) \) are nonzero for \( i \leq n \). This is not the case when working with \( G_n^{-1}(\Gamma_i/n) \).

The following lemma provides a finite sum representation which converges, almost surely, to the Ferguson and Klass (1972) sum-representation for the inverse-Gaussian process. The proof of the lemma is similar to that of Lemma 2 in Zarepour and Al Labadi (2012). Hence, it is omitted.

**Lemma 2.** If \( (\theta_i)_{i \geq 1} \) is a sequence of i.i.d. random variables with common distribution \( H \),
independent of \((\Gamma_i)_{i \geq 1}\), then as \(n \to \infty\)

\[
\sum_{i=1}^{n} G_n^{-1} \left( \frac{\Gamma_i}{\Gamma_{n+1}} \right) \delta_{\theta_i} \overset{a.s.}{\to} \sum_{i=1}^{\infty} L^{-1}(\Gamma_i) \delta_{\theta_i}.
\]  

(5.2)

Here, \(\Gamma_i\), \(N(x)\), and \(G_n(x)\), are defined in (1.6), (1.8), and (5.1), respectively.

By normalizing the finite sum in (5.2), it is possible to obtain a sum representation that converges almost surely to Ferguson and Klass representation of the normalized inverse-Gaussian process. This important result is stated formally in the next theorem.

**Theorem 4.** Let \((\theta_i)_{i \geq 1}\) be a sequence of i.i.d. random variables with values in \(X\) and common distribution \(H\), independent of \((\Gamma_i)_{i \geq 1}\), then as \(n \to \infty\)

\[
P_{\text{new}}^{n,H,a} = \sum_{i=1}^{n} \frac{G_n^{-1} \left( \frac{\Gamma_i}{\Gamma_{n+1}} \right)}{\sum_{i=1}^{n} G_n^{-1} \left( \frac{\Gamma_i}{\Gamma_{n+1}} \right)} \delta_{\theta_i} \overset{a.s.}{\to} P_{H,a} = \sum_{i=1}^{\infty} \frac{L^{-1}(\Gamma_i)}{\sum_{i=1}^{\infty} L^{-1}(\Gamma_i)} \delta_{\theta_i}.
\]  

(5.3)

Here \(\Gamma_i\), \(L(x)\), and \(G_n(x)\), are defined in (1.6), (1.8), and (5.1), respectively.

**Remark 7.** For any \(1 \leq i \leq n\), \(\Gamma_i/\Gamma_{n+1} < \Gamma_{i+1}/\Gamma_{n+1}\) almost surely. Since \(G_n^{-1}\) is a decreasing function, we have \(G_n^{-1}(\Gamma_i/\Gamma_{n+1}) > G_n^{-1}(\Gamma_{i+1}/\Gamma_{n+1})\) almost surely. That is, the weights of the new representation given in Theorem 4 decrease monotonically for any fixed positive integer \(n\). As demonstrated in Zarepour and Al Labadi (2012) in the case of the Dirichlet process, we anticipate that this new representation will yield highly accurate approximations to the normalized inverse-Gaussian process.

**Remark 7.** For \(P_{\text{new}}^{n,H,a}\) of Theorem 4 we can write

\[
P_{\text{new}}^{n,H,a} \overset{d}{=} \sum_{i=1}^{n} p_{i,n} \delta_{\theta_i},
\]  

(5.4)
where \( p_{1,n}, \ldots, p_{n,n} \sim \text{N-IG} \left( a/n, \ldots, a/n \right) \), \( d \) means have the same distribution and the N-IG distribution is given by \((1.1)\). Therefore a similar result to Theorem 2 of Ishwaran and Zarepour for the normalized inverse-Gaussian process follows immediately.

6 Concluding Remarks

The approach used in this paper can be applied to similar processes with tractable finite dimensional distributions. On the other hand, when the finite dimensional distribution is unknown, one may follow the approach of James (2008). However, applying this approach requires constructing distributional identities, which may be difficult in some cases.

One can use the results obtained in this paper to derive asymptotic properties of any Hadamard-differentiable functional of the N-IGP\((a, H)\) as \( a \to \infty \). For different applications in statistics we refer the reader to van der Vaart and Wellner (1996, Section 3.9) and Lo (1987). Moreover, it is possible to extend the results found in this paper to the case when the base measure \( H \) is a multivariate cumulative distribution function. The result of Bickle and Wichura (1972) can be employed in the proof.

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Appendix

Proof of Lemma 1 for $m = 2$: Let $S_1$ and $S_2$ be any two intervals in $\mathbb{R}$. Without loss of generality, we assume that $S_1 \cap S_2 = \emptyset$. The general case when $S_1$ and $S_2$ are not necessarily disjoint follows from the continuous mapping theorem.

Note that

$$(P_{H,a}(S_1), P_{H,a}(S_2), 1 - P_{H,a}(S_1) - P_{H,a}(S_2)) \sim \text{N-IG}(aH(S_1), aH(S_2), a(1 - H(S_1) - H(S_2))),$$

where the N-IG distribution is given by (1.1). For notational simplicity, set $X_i = P_{H,a}(S_i)$, $l_i = H(S_i)$ and $D_i = \sqrt{a} (X_i - l_i)$ for $i = 1, 2$. Thus, the joint density function of $X_1$ and $X_2$ is:

$$f_{X_1,X_2}(x_1, x_2) = \frac{e^a a^3 l_1 l_2 (1 - l_1 - l_2)}{2^{1/2} \pi^{3/2}} \times x_1^{-3/2} x_2^{-3/2} (1 - x_1 - x_2)^{-3/2}$$

$$\times K_{-3/2} \left( a \sqrt{\frac{l_1^2}{x_1} + \frac{l_2^2}{x_2} + \frac{(1 - l_1 - l_2)^2}{1 - x_1 - x_2}} \right)$$

$$\times a^{-3/2} \left( \frac{l_1^2}{x_1} + \frac{l_2^2}{x_2} + \frac{(1 - l_1 - l_2)^2}{1 - x_1 - x_2} \right)^{-3/4}$$

$$= \frac{a^{1/2} e^a l_1 l_2 (1 - l_1 - l_2)}{2^{1/2} \pi^{3/2}} \times x_1^{-3/2} x_2^{-3/2} (1 - x_1 - x_2)^{-3/2}$$

$$\times K_{-3/2} \left( a \sqrt{\frac{l_1^2}{x_1} + \frac{l_2^2}{x_2} + \frac{(1 - l_1 - l_2)^2}{1 - x_1 - x_2}} \right)$$

$$\times \left( \frac{l_1^2}{x_1} + \frac{l_2^2}{x_2} + \frac{(1 - l_1 - l_2)^2}{1 - x_1 - x_2} \right)^{-3/4}.$$

The joint probability density function of $D_1 = \sqrt{a} (X_1 - l_1)$ and $D_2 = \sqrt{a} (X_2 - l_2)$
is:

\[
f_{D_1, D_2}(y_1, y_2) = \frac{a^{1/2}e^{a l_1 l_2 (1 - l_1 - l_2)}}{2^{1/2}\pi^{3/2}} \times \\
\times (y_1/\sqrt{a} + l_1)^{-3/2} (y_2/\sqrt{a} + l_2)^{-3/2} \\
(1 - y_1/\sqrt{a} - l_1 - y_2/\sqrt{a} - l_2)^{-3/2} \\
\times K_{-3/2} \left( a \left( \frac{l_2^2}{y_1/\sqrt{a} + l_1} + \frac{l_1^2}{y_2/\sqrt{a} + l_2} \right) \\
+ \frac{(1 - l_1 - l_2)^2}{1 - y_1/\sqrt{a} - l_1 - y_2/\sqrt{a} - l_2} \right)^{1/2} \\
\times \left( \frac{l_1^2}{y_1/\sqrt{a} + l_1} + \frac{l_2^2}{y_2/\sqrt{a} + l_2} \right) \\
+ \frac{(1 - l_1 - l_2)^2}{1 - y_1/\sqrt{a} - l_1 - y_2/\sqrt{a} - l_2} \right)^{-3/4}.
\]

By Scheffé's theorem (Billingsley 1999, page 29), it is enough to show that:

\[
f_{D_1, D_2}(y_1, y_2) \to f(y_1, y_2) = \frac{1}{2\pi|\Sigma|^{1/2}} \exp \left\{ -(y_1 y_2)\Sigma^{-1}(y_1 y_2)^T/2 \right\},
\]

(7.1)

where \(\Sigma = \begin{bmatrix} l_1 (1 - l_1) & -l_1 l_2 \\ -l_1 l_2 & l_2 (1 - l_2) \end{bmatrix}\).

Since, for large \(z\) and fixed \(\nu\), \(K_\nu(z) \approx \sqrt{\pi/2}z^{-1/2}e^{-z}\) (Abramowitz and Stegun, 1972, Formula 9.7.2, page 378), where we use the notation \(f(z) \approx g(z)\) if \(\lim_{z \to \infty} f(z)/g(z) = \)
1, we get:

\[
\lim_{a \to \infty} f_{D_1, D_2}(y_1, y_2) = \lim_{a \to \infty} \left[ \frac{l_1 l_2 (1 - l_1 - l_2)}{2\pi} \times \right.
\]
\[
\left. \left( \frac{l_1^2}{y_1/\sqrt{a} + l_1} + \frac{l_2^2}{y_2/\sqrt{a} + l_2} + \frac{(1 - l_1 - l_2)^2}{1 - y_1/\sqrt{a} - l_1 - y_2/\sqrt{a} - l_2} \right)^{-1} \times \right.
\]
\[
\left. (y_1/\sqrt{a} + l_1)^{-3/2} (y_2/\sqrt{a} + l_2)^{-3/2} (1 - y_1/\sqrt{a} - l_1 - y_2/\sqrt{a} - l_2)^{-3/2} \times \exp \left( a \left( 1 - \left( \frac{l_1^2}{y_1/\sqrt{a} + l_1} + \frac{l_2^2}{y_2/\sqrt{a} + l_2} \right)^{1/2} \right) \right) \right].
\]

Notice that,

\[
\frac{l_1 l_2 (1 - l_1 - l_2)}{2\pi} \times \left( \frac{l_1^2}{y_1/\sqrt{a} + l_1} + \frac{l_2^2}{y_2/\sqrt{a} + l_2} + \frac{(1 - l_1 - l_2)^2}{1 - y_1/\sqrt{a} - l_1 - y_2/\sqrt{a} - l_2} \right)^{-1}
\]
\[
\times (y_1/\sqrt{a} + l_1)^{-3/2} (y_2/\sqrt{a} + l_2)^{-3/2} (1 - y_1/\sqrt{a} - l_1 - y_2/\sqrt{a} - l_2)^{-3/2}
\]

converges to \(1/\left(2\pi \sqrt{\sigma_{11} \sigma_{22} (1 - \rho_{12}^2)}\right)\), where

\[
\sigma_{11} = l_1 (1 - l_1), \quad \sigma_{22} = l_2 (1 - l_2), \quad \rho_{12} = -\sqrt{\frac{l_1 l_2}{(1 - l_1)(1 - l_2)}}.
\]

To prove the lemma, it remains to show that

\[
a \left( 1 - \left( \frac{l_1^2}{y_1/\sqrt{a} + l_1} + \frac{l_2^2}{y_2/\sqrt{a} + l_2} + \frac{(1 - l_1 - l_2)^2}{1 - y_1/\sqrt{a} - l_1 - y_2/\sqrt{a} - l_2} \right)^{1/2} \right)
\]
converges to
\[-\frac{1}{2(1-\rho_{12}^2)} \left( \left( \frac{y_1}{\sqrt{\sigma_{11}}} \right)^2 + \left( \frac{y_2}{\sqrt{\sigma_{22}}} \right)^2 - 2\rho_{12} \left( \frac{y_1}{\sqrt{\sigma_{11}}} \right) \left( \frac{y_1}{\sqrt{\sigma_{11}}} \right) \right) \].

The last argument follows straightforwardly form the L’Hospital’s rule. □

**Proof of Theorem 1:** Let \((a_n)\) be an arbitrary sequence such that \(a_n \to \infty\). To simplify the notation, in the argument below, we omit writing the index \(n\) of \(a_n\). Assume first that \(H(t) = \lambda(t) = t\) (i.e. \(\lambda\) is the Lebesgue measure on \([0, 1])\). Thus the process (2.2) reduces to

\[ D_{\lambda, a}(t) = \sqrt{a} (P_{\lambda, a}(t) - t). \]

To prove the theorem, we use Lemma 1 and Theorem 13.5 of Billingsley (1999). Therefore, we only need to show that for any \(0 \leq t_1 \leq t \leq t_2 \leq 1\),

\[
E \left[ |D_{\lambda, a}(t) - D_{\lambda, a}(t_1)|^{2\beta} |D_{\lambda, a}(t_2) - D_{\lambda, a}(t)|^{2\beta} \right] \leq |F(t_2) - F(t_1)|^{2a}, \quad (7.2)
\]

for some \(\beta \geq 0\), \(a > 1/2\), and a nondecreasing continuous function \(F\) on \([0, 1]\). Take \(a = \beta = 1\) and \(F(t) = t\). We show that

\[
E \left[ (D_{\lambda, a}(t) - D_{\lambda, a}(t_1))^2 (D_{\lambda, a}(t_2) - D_{\lambda, a}(t))^2 \right] \leq \frac{8a - 1}{a^3} (t_2 - t_1)^2. \quad (7.3)
\]

Observe that
\[ D_{\lambda, a}(t) - D_{\lambda, a}(t_1) = D_{\lambda, a}([t_1, t]) \quad \text{and} \quad D_{\lambda, a}(t_2) - D_{\lambda, a}(t) = D_{\lambda, a}([t, t_2]). \]
Thus, the expectation in the right hand side of (7.3) is equal to

$$
\xi^2(a)E \left[ \{P_{\lambda,a}(t_1, t) - \lambda((t_1, t))\}^2 \{P_{\lambda,a}(t, t_2) - \lambda((t, t_2))\} \right],
$$

where $\lambda((t, t_2]) = t_2 - t$ and $\lambda((t_1, t]) = t - t_1$. Expanding the expression

$$
\{P_{\lambda,a}(t_1, t) - \lambda((t_1, t))\}^2 \{P_{\lambda,a}(t, t_2) - \lambda((t, t_2))\}
$$

(7.5)

gives

$$
P_{\lambda,a}(t_1, t)P_{\lambda,a}(t, t_2) - 2\lambda((t, t_2))P_{\lambda,a}(t_1, t)P_{\lambda,a}(t, t_2)
+ \lambda^2((t, t_2])P_{\lambda,a}(t_1, t) - 2\lambda((t_1, t])P_{\lambda,a}(t_1, t)P_{\lambda,a}(t, t_2)
+ 4\lambda((t_1, t])\lambda((t, t_2])P_{\lambda,a}(t_1, t)P_{\lambda,a}(t, t_2) - 2\lambda((t_1, t])\lambda^2((t, t_2])P_{\lambda,a}(t_1, t)
+ \lambda^2((t_1, t])P_{\lambda,a}(t, t_2) - 2\lambda^2((t_1, t])\lambda((t, t_2])P_{\lambda,a}(t_1, t) + \lambda^2((t_1, t])\lambda^2((t, t_2]).
$$

Using the fact that $P_{\lambda,a}(\cdot)$ is a probability measure and $0 \leq t_1 \leq t \leq t_2 \leq 1$, the expression displayed in (7.5) is less than or equal to

$$
P_{\lambda,a}(t_1, t)P_{\lambda,a}(t, t_2) + \lambda^2((t, t_2])P_{\lambda,a}(t_1, t)
+ 4\lambda((t_1, t])\lambda((t, t_2])P_{\lambda,a}(t_1, t)P_{\lambda,a}(t, t_2) + \lambda^2((t_1, t])P_{\lambda,a}(t, t_2)
+ \lambda^2((t_1, t])\lambda^2((t, t_2])
\leq P_{\lambda,a}(t_1, t)P_{\lambda,a}(t, t_2) + \lambda((t, t_2])P_{\lambda,a}(t_1, t)
+ 4\lambda((t_1, t])\lambda((t, t_2]) + \lambda((t_1, t])P_{\lambda,a}(t, t_2) + \lambda((t, t_2])\lambda((t, t_2]).
By (1.2) and (1.4) we obtain

\[
E \left[ \left\{ P_{\lambda,a}((t_1, t]) - \lambda((t_1, t]) \right\}^2 \{ P_{\lambda,a}((t, t_2]) - \lambda((t, t_2]) \}^2 \right] \leq \frac{8a - 1}{a^3} \times \\
\lambda((t_1, t])\lambda((t, t_2]). \tag{7.6}
\]

Thus, using (7.4) and (7.6), we have

\[
E \left[ (D_{\lambda,a}(t) - D_{\lambda,a}(t_1))^2 (D_{\lambda,a}(t_2) - D_{\lambda,a}(t))^2 \right] = \frac{8a - 1}{a^3} \lambda(t_1, t] \lambda(t_2) \\
= \frac{8a - 1}{a^3} (t - t_1)(t_2 - t) \\
\leq \frac{8a - 1}{a^3} (t_2 - t_1)^2,
\]

for \(0 \leq t_1 \leq t \leq t_2 \leq 1\). This proves the theorem in the case when \(H(t) = t\), i.e. \(H\) is the uniform distribution. Observe that, the quantile function \(H^{-1}(s) = \inf \{ t : H(t) \geq s \}\) has the property: \(H^{-1}(s) \leq t\) if and only if \(s \leq H(t)\). If \(U_i\) is uniformly distributed over \([0, 1]\), then \(H^{-1}(U_i)\) has distribution \(H\). Thus, we can use the representation \(a_i = H^{-1}(U_i)\), where \((U_i)_{i \geq 1}\) is a sequence of i.i.d. random variables with uniform distribution on \([0, 1]\), to have:

\[
P_{H,a}(t) = P_{\lambda,a}(H(t)) \quad \text{and} \quad D_{H,a}(t) = D_{\lambda,a}(H(t)) = D_{\lambda,a} \circ H(t), \quad t \in \mathbb{R},
\]

where \(P_{\lambda,a}\) is the normalized inverse-Gaussian process with concentration parameter \(a\) and Lebesgue base measure \(\lambda\) on \([0, 1]\). From the uniform case, which was already treated, we have \(D_{\lambda,a}(\cdot) = \sqrt{a} (P_{\lambda,a}(\cdot) - \lambda(\cdot)) \overset{d}{\to} B_{\lambda}(\cdot)\). Define \(\Psi : D[0, 1] \to D[-\infty, \infty]\) by \((\Psi x)(t) = x(H(t))\). Since the function \(\Psi\) is uniformly continuous (Billingsley 1999, page 150; Pollard, 1984, page 97), it follows, from the continuous mapping theorem and the fact
that $D_{\lambda,a} \xrightarrow{d} B_{\lambda}$, that $D_{H,a} = \Psi(D_{\lambda,a}) \xrightarrow{d} \Psi(B_{\lambda}) = B_{H}$. This completes the proof of the theorem. \qed