The Sagnac effect and the Tevatron

A. C. Melissinos
1/15/2011

The rotation of the Earth modifies the path-length of particles moving in opposite directions on the same closed trajectory. We designate the area enclosed by the trajectory by $A$, where the vector indicates the normal to the trajectory plane; the Earth’s angular momentum is $\Omega$. The time difference between the two particles is

$$\delta t = \frac{4 A \cdot \Omega}{v^2}$$

with $v$ the velocity of the particles. This effect, predicted by Sagnac [1] in 1913, was observed by Michelson [2] in 1925, using an optical interferometer. In this case when the two beams merge after a single turn, their relative phase is shifted by

$$\delta \phi = \frac{2\pi}{\lambda} \frac{4 A \cdot \Omega}{c}$$

where $\lambda$ is the wavelength of the light.

Today’s laser gyros consist of a ring cavity in which a lasing medium is inserted. Due to the Earth’s (or the ring’s) rotation the two counter-propagating beams see different cavity lengths and thus have different frequencies. One measures the beat frequency $\delta f$ between the two oppositely circulating beams,

$$\delta f = \frac{4 A \cdot \Omega}{\lambda P}$$

with $P$ the perimeter of the closed trajectory. In modern ring interferometers [3] $\delta f$ can be measured to $\sim 1 \mu$Hz, and thus the Earth rotation frequency to 1 part in $10^7$.

The Sagnac effect should be present in the Tevatron, with the important difference that the trajectory is determined by the machine lattice and the particle energy, and by suitable feedback, the two beams are forced to collide at the intersection regions. Let us consider for the moment single bunches of protons and antiprotons and assume that both follow the same orbit and that their momenta are exactly equal, since this determines the mean orbit radius. If the two bunches are coincident at some point on their trajectory, then after one turn they will be separated at that point by a distance $\delta s$,

$$\delta s = 4|A||\Omega|\sin\phi/c = 2 \times 10^{-6} \text{m},$$

where $\phi = 41.83^\circ$ is the Fermilab latitude, $|A| = \pi \times 10^6 \text{ m}^2$ and $|\Omega| = 2\pi/86,400 = 7.27 \times 10^{-5}$ r/s is the diurnal earth rotation angular frequency. Since the beam rotation frequency is $f \sim 50$ kHz, in one minute the two bunches should be separated by $\sim 6 \text{ m}$.

The problem with this proposal is that it assumes that both bunches are coasting at exactly the same mean radius. The required precision is

$$\frac{\delta R}{R} < \frac{\delta s}{2\pi R} \sim 10^{-9} \frac{1}{\pi}.$$
Even accounting for the momentum compaction in strong focusing machines,
\[
\frac{dp}{p} = Q^2 \frac{dR}{R},
\]
where \(Q\) is the betatron tune \((Q \sim 50)\), the relative momentum difference between the proton and antiproton bunch must be \(\sim 10^{-6}\) which is unattainable. Furthermore, any shift in the bunch longitudinal position affects the rf phase which will restore the centroid of the bunch to the center of the rf bucket. The terrestrial distortion of the machine lattice would affect both bunches equally.

**Demonstration of the Sagnac formula**

Consider a ring of radius \(r\) on a plane surface that is rotating with angular velocity \(\Omega\). A particle of velocity \(v\) is moving on the ring in the direction of rotation. The time required for a complete turn is
\[
t_+ = \frac{2\pi r}{v} + \frac{\Omega r}{v} \quad \text{or} \quad t_+ = \frac{2\pi r}{v - \Omega r}
\]
For a particle moving opposite to the rotation direction
\[
t_- = \frac{2\pi r}{v + \Omega r} \quad \text{and} \quad \delta t = t_+ - t_- = \frac{4\pi r^2 \Omega}{v^2 - (\Omega r)^2}
\]
The projection of \(\Omega\) on the normal to the ring’s plane is \(\Omega \cdot \mathbf{A}\), \(A = \pi r^2\) and we can always assume \(\Omega r \ll v\).

**Relativistic derivation**

We obtain the same result by considering the increase/decrease in the velocity of the \(p/\overline{p}\) bunches. If the linear velocity of the ring is \(\beta_r\), the velocity difference \(d\beta = 2\beta_r/2\gamma^2\), where \(\beta, \gamma\) refer to the \(p/\overline{p}\). It follows that
\[
\frac{dp}{p} = \frac{d\gamma}{\gamma} = \frac{2\beta_r}{c} = \frac{2\Omega \sin \phi R}{c} = \frac{\delta s}{2\pi R},
\]
as in Eq. 9.

**References**

[1] G.Sagnac, C.R.Acad. Sci. (Paris) 157, 708-710 (1913)
[2] A.A.Michelson, H.G.Gale and F.Pearson, Astrophys. J. 61, 137-145 (1925)
[3] R.Anderson, H.R.Bilger and G.E.Stedman, Am.J. Phys 62, 975-985 (1994)