ON THE PRIME GRAPH QUESTION FOR INTEGRAL GROUP RINGS OF CONWAY SIMPLE GROUPS

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Abstract. The Prime Graph Question for integral group rings asks if it is true that if the normalized unit group of the integral group ring of a finite group $G$ contains an element of order $pq$, for some primes $p$ and $q$, also $G$ contains an element of that order. We answer this question for the three Conway sporadic simple groups after reducing it to a combinatorial question about Young tableaus and Littlewood-Richardson coefficients. This finishes work of V. Bovdi, A. Konovalov and S. Linton.

1. Introduction

The unit group of the integral group ring $\mathbb{Z}G$ of a finite group $G$ has given rise to many interesting research questions and results, as recorded e.g. in the monographs [Seh93, JdR16]. Many of these questions concern the connection between finite subgroups of units of $\mathbb{Z}G$ and the group base $G$, e.g. the Isomorphism Problem which was open for over 50 years until answered by Hertweck [Her01] or the recently answered Zassenhaus Conjecture [EM 17] which was open for over 40 years. For both questions there turned out to be counterexamples. In this light the weaker versions of the former Zassenhaus Conjecture, which are still open, become more important. One of these versions, the so-called Prime Graph Question, gained some attention after being introduced by Kimmerle in [Kim06]. To formulate it denote by $V(\mathbb{Z}G)$ the group of the so-called normalized units in $\mathbb{Z}G$, i.e. the units whose coefficients sum up to 1.

Prime Graph Question: Let $G$ be a finite group and $p$ and $q$ some primes. If $V(\mathbb{Z}G)$ contains an element of order $pq$, does $G$ contain an element of order $pq$?

The prime graph $\Gamma(G)$ of a group $G$ is a graph whose vertices are the primes appearing as orders of elements in $G$ and two vertices $p$ and $q$ are connected if and only if $G$ contains an element of order $pq$. Hence the Prime Graph Question can also be formulated as: For $G$ a finite group, does $\Gamma(G) = \Gamma(V(\mathbb{Z}G))$ hold?

The Prime Graph Question seems particularly approachable, since, in contrast to other questions on the finite subgroups in $V(\mathbb{Z}G)$, a reduction result is available here: The Prime Graph Question holds for a group $G$, if it holds for all almost simple images of $G$ [KK17, Theorem 2.1]. Recall that a group $A$ is called almost simple if it is isomorphic to a subgroup of the automorphism group of a non-abelian simple group $S$ containing the inner automorphisms of $S$, i.e. $\text{Inn}(S) \cong S \leq A \leq \text{Aut}(S)$. In this case $S$ is called the socle of $A$.

It is known that the Prime Graph Question has a positive answer for almost simple groups with socle isomorphic to a projective special linear group $\text{PSL}(2, p)$ or $\text{PSL}(2, p^2)$ for $p$ a prime [BM17b, Theorem A] or an alternating group of degree up to 17 [BC17]. The question also has a positive answer for groups whose order is divisible by exactly three pairwise different primes [KK17, BM17c] and many almost simple groups whose order is divisible by four pairwise different primes [BM16].

Employing a computer implementation of a method introduced by Luthar and Passi [LP89] and Hertweck [Her07], nowadays known as the HeLP method, in a series of papers between 2007 and 2012 Bovdi, Konovalov and different collaborators of them studied the Prime Graph Question for...
sporadic simple groups [BKS07, BK07a, BK07b, BKL08, BKM08, BK08, BGK09, BK09, BK10, BJK11, BKL11, BK12]. Overall they studied 17 sporadic simple groups and were able to prove the Prime Graph Question for 13 of these groups, an overview of their results can be found in [KK15, Section 5]. Also in [KK15, Corollary 5.4] it was recorded that the Prime Graph Question holds for almost simple groups with a socle isomorphic to one of the 13 sporadic simple groups for which Bovdi, Konovalov et al. proved the Prime Graph Question. The four groups studied by Bovdi, Konovalov et al. for which they were not able to prove the Prime Graph Question were the O’Nan simple group [BGK09] and the three Conway simple groups [BKL11]. Using a method introduced in [BM17c] in this note we obtain a positive answer for the Prime Graph Question for the latter groups. This is the first contribution to the Prime Graph Question for sporadic simple groups since the papers of Bovdi, Konovalov et al.

**Theorem 1.1.** The Prime Graph Question has a positive answer for the sporadic Conway simple groups $Co_3$, $Co_2$ and $Co_1$.

Note that the outer automorphism group of each Conway simple group is trivial, so Theorem 1.1 proves the Prime Graph Question also for every almost simple group whose socle is isomorphic to a Conway simple group.

To describe our main tool let us first introduce some notation. Let $G$ be a finite group, $u \in V(\mathbb{Z}G)$ a unit of order $n$ and $\chi$ an ordinary character of $G$ with corresponding representation $D$. Then we can extend $D$ linearly to $\mathbb{Z}G$ and afterwards restrict it to $V(\mathbb{Z}G)$, obtaining an extension of $D$ and $\chi$. In particular $D(u)$ is then a matrix of finite order dividing $n$. Let $\xi$ be an $n$-th root of unity. Then we denote by $\mu(\xi, u, \chi)$ the multiplicity of $\xi$ as an eigenvalue of $D(u)$. Moreover for an integer $k$ we will denote by $\zeta_k$ a primitive $k$-th root of unity. Then the results given above will follow from an application of the following theorem.

**Theorem 1.2.** Let $G$ be a finite group, $p$ an odd prime and $m$ a positive integer not divisible by $p$. Assume that some $p$-block of $G$ is a Brauer Tree Algebra with ordinary characters $\chi_1, ..., \chi_p$ and with the Brauer tree being a line of the form

$$\chi_1 \chi_2 \chi_3 \chi_{p-2} \chi_{p-1} \chi_p$$

Moreover assume that in the rings of values of $\chi_1, ..., \chi_p$ the prime $p$ is unramified.

Let $u \in V(\mathbb{Z}G)$ be a unit of order $pm$ and let $\xi$ be some $m$-th root of unity. Then the inequality

$$\mu(\xi \cdot \zeta_p, u, \chi_{p-1}) - \mu(\xi \cdot \zeta_p, u, \chi_p) \leq \mu(\xi, u, \chi_1) - \sum_{i=1}^{p-2} (-1)^i \mu(\xi \cdot \zeta_p, u, \chi_i)$$

holds.

This theorem might be regarded as a generalization of [BM16, Proposition 3.2]. The condition on the unit can also be formulated in terms of character values which makes the inequality easier to check by hand, although it becomes much longer then, cf. Lemma 2.3.

Theorem 1.2 will be achieved using the so-called lattice method introduced in [BM17c] and developed further in [BM16]. It is typical for the study of group rings that different fields of mathematics, such as group theory, ring theory, representation theory and number theory are combined to achieve results. In the present paper the main tool is combinatorics, more precisely calculations with Young tableaux and Littlewood-Richardson coefficients, cf. the proof of Theorem 1.2. It is quite plausible that Theorem 1.2 which is the main tool to prove Theorem 1.1 can be applied also to study the Prime Graph Question or related questions for many other groups. Also the methods presented here could be used to prove variations of Theorem 1.2 in particular for other forms of Brauer trees.

We will also need to use the HeLP method to obtain enough restrictions on torsion units which allow the application of Theorem 1.2. We will start by recalling the needed methods in Section 2, develop the necessary combinatorics in Section 3, continue to prove Theorem 1.2 and finally show how this can be applied to prove Theorem 1.1 in Section 4.
2. Preliminaries and methods

A fundamental notion when studying torsion units of integral group rings are so-called partial augmentations. Let $u \in \mathbb{Z}G$ be an element of the form $\sum_{g \in G} z_g g$ and denote by $x^G$ the conjugacy class of an element $x$ in $G$. Then $\varepsilon_x(u) = \sum_{g \in x^G} z_g$ is called the partial augmentation of $u$ at $x$. A fundamental theorem of Hertweck [Her07] Theorem 2.3] states that if $u \in V(\mathbb{Z}G)$ is a unit of order $n$ then $\varepsilon_x(u) \neq 0$ implies that the order of $x$ divides $n$. This implies in particular that the exponents of $G$ and $V(\mathbb{Z}G)$ coincide, a result due originally to Cohn and Livingstone [CL65]. Moreover $\varepsilon_1(u) = 0$ unless $u = 1$ by the Berman-Higman Theorem [LM16 Proposition 1.5.1].

2.1. The HeLP method. Partial augmentations are class functions and as such can be investigated using representation theory. Note that if $\chi$ is an ordinary character of $G$ and $u \in V(\mathbb{Z}G)$ a torsion unit of order $n$ then $\chi(u) = \sum_{g \in \chi} \varepsilon_g(u) \chi(g)$, where the sum runs over the conjugacy classes of $G$. It was shown by Hertweck [Her07] Theorem 3.2] that this also holds for $p$-Brauer characters, if $p$ is not a divisor of $n$ and the sum is understood to run on $p$-regular conjugacy classes of $G$. On the other hand if we are given the partial augmentations of $u$ and its powers we can compute the eigenvalues, including multiplicities, of $u$ by using ordinary or $p$-modular representation of $G$, again assuming $p$ is not dividing $n$. These multiplicities can be expressed in explicit formulas, depending on the partial augmentations of $u$ and its powers, and that the results of these formulas are non-negative integers is the basic idea of the HeLP method, cf. [BM17b, Section 2] for a detailed explanation and [Her07] for the proofs.

Hence there is an algorithmic method which allows to obtain restrictions on the possible partial augmentations of torsion units in $V(\mathbb{Z}G)$, provided we have some knowledge on the characters of $G$. If we apply the HeLP method to the whole character table of $G$ then, for any fixed $n$, we will obtain a finite number of possibilities for the partial augmentations of units of order $n$ in $V(\mathbb{Z}G)$. If it happens that we obtain no possibility for units of some given order $n$, then we know that there exist no units at all in $V(\mathbb{Z}G)$ of order $n$. This is the basic idea in the application of the HeLP method for the study of the Prime Graph Question and in this way Bovdi, Konovalov et al. gave their proofs for 13 sporadic groups. We will also rely on the HeLP method to produce a finite number of possibilities for the partial augmentations of units of certain orders and then Theorem 1.2 will do the rest. We will apply here a computer implementation of the HeLP method as a package [BM17a] in the computer algebra system GAP [GAP17]. This package can also be used to reproduce the results of Bovdi, Konovalov et al.

2.2. The lattice method. The lattice method is another method which makes use of the multiplicities of eigenvalues of a torsion unit $u \in V(\mathbb{Z}G)$ of order $n$ under ordinary representations of $G$. This method was introduced in [BM17c] and the basic idea consists in obtaining restrictions on the simple $kG$-modules when viewed as $k(u)$-modules, where $k$ is a field of characteristic $p$ for a divisor $p$ of $n$. These simple $kG$-modules are the $p$-modular composition factors of ordinary representations of $G$ and studying different ordinary representations with common composition factors can finally produce a contradiction to the existence of $u$.

We will only recall those parts of the lattice method necessary for our means in this article. We will cite the articles [BM17c] [BM16] where this method is explained, but many facts can be found in text books on representation theory. First of all let $G = \langle c \rangle$ be a cyclic group of order $pm$ where $p$ is a prime and $m$ an integer not divisible by $p$. Let $k$ be a field of characteristic $p$ containing a primitive $m$-th root of unity $\zeta$. Then there is, up to isomorphism, exactly $pm$ indecomposable $kG$-modules determined by a pair $(i, j)$ where $1 \leq i \leq p$, $1 \leq j \leq m$ such that a module $M$ corresponding to $(i, j)$ is $i$-dimensional as a $k$-vector space and $e^p$ acts as $\zeta^j$ on $M$. We denote this module by $I_i^j$ and if the action of $e^p$ is clear from the context or if $m = 1$ we simply write $I_i$. The module $I_i^j$ is simple if and only if $i = 1$ and each $I_i^j$ is uniserial [BM17c, Proposition 2.2].

Now assume that $M$ is a $kG$-module, $d$-dimensional as $k$-vector space, such that $e^p$ acts as $\zeta^j$ on $M$ for some fixed $j$. Then by the above we know that $M \cong a_pI_p \oplus a_{p-1}I_{p-1} \oplus \cdots \oplus a_1I_1$ for
certain non-negative integers $a_1, \ldots, a_p$. Moreover $a_p p + a_{p-1} (p-1) + \ldots + a_1 = d$ and hence 

$$\lambda = \left(\frac{p \ldots p}{a_p} \frac{p-1 \ldots 1}{a_{p-1}} \ldots \frac{1 \ldots 1}{a_1}\right)$$

is a partition of $d$. We call $\lambda$ the partition corresponding to $M$.

We recall the necessary background from combinatorics, cf. e.g. [BM97]. A Young diagram corresponding to a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ is a diagram consisting of boxes ordered in rows and columns such that the first row contains $\lambda_1$ boxes, the second row $\lambda_2$ boxes etc. If $\mu = (\mu_1, \ldots, \mu_r)$ is a subpartition of $\lambda$, i.e. $\mu_1 \leq \lambda_1, \ldots, \mu_r \leq \lambda_r$, then the skew diagram corresponding to $\lambda/\mu$ is obtained by erasing from the Young diagram corresponding to $\lambda$, always from left to right, $\mu_1$ boxes in the first row, $\mu_2$ boxes in the second row etc. If we fill a skew diagram with entries from an alphabet, in our case the alphabet always will be the positive integers, a skew diagram becomes a skew tableau $T$. By "filling" we mean writing a letter in every box. $T$ is called semistandard if reading a row from left to right the entries do not decrease and reading a column from top to bottom the entries strictly increase.

For a box $b$ in a skew tableau $T$ we denote by $w(b)$ the word which we obtain reading $T$ from top to bottom and from right to left until $b$, including the entry in $b$. We denote by $w(T)$ the word which we obtain reading all boxes of $T$ in this way, i.e. if $b$ is the lowest box in the first (i.e. most left) column of $T$ then $w(b) = w(T)$. We say that $T$ satisfies the lattice property if for every box $b$ in $T$ the word $w(b)$ contains the letter 1 at least as many times as the letter 2, the letter 2 at least as many times as 3 etc. If the maximal letter of $w(T)$ is 4 and $w(T)$ contains $\nu_1$ times the letter 1, $\nu_2$ times the letter 2 etc. then $\nu = (\nu_1, \ldots, \nu_s)$ is called the content of $T$. Note that if $T$ is a skew tableau satisfying the lattice property then the content of $T$ is a partition. If now $\nu$ is some partition then the Littlewood-Richardson coefficient $c_{\mu,\nu}^\lambda$ is the number of semistandard skew tableaux corresponding to $\lambda/\mu$ which satisfy the lattice property and have content $\nu$. A fact of fundamental importance to us is that the Littlewood-Richardson coefficient $c_{\mu,\nu}^\lambda$ is symmetric in $\mu$ and $\nu$.

These combinatorial objects play a role in the study of torsion units of group rings via the following observation [BM16 Theorem 2.8].

**Theorem 2.1.** Let $C = \langle c \rangle$ be a cyclic group of order $p$, for $p$ a prime, and $k$ a field of characteristic $p$. Let $M, U$ and $Q$ be $kC$-modules with corresponding partitions $\lambda, \mu$ and $\nu$ respectively. Then $M$ contains a submodule $\tilde{U}$ isomorphic to $U$ such that $M/\tilde{U} \cong Q$ if and only if $c_{\mu,\nu}^\lambda \neq 0$.

To pass from multiplicities of eigenvalues under ordinary representations to modules over modular group algebras we will employ the following [BM17c Propositions 2.3, 2.4].

**Proposition 2.2.** Let $C = \langle c \rangle$ be a cyclic group of order $pm$ such that $p$ does not divide $m$. Let $R$ be a complete local ring of characteristic 0 containing a primitive $m$-th root of unity $\xi$ such that $p$ is contained in the maximal ideal of $R$ and not ramified in $R$. Denote by $k$ the residue class field of $R$ and adopt the bar-notatation for reduction modulo the maximal ideal of $R$.

Let $D$ be a representation of $C$ such that the eigenvalues of $D(u)$ in the algebraic closure of the quotient field of $R$, with multiplicities, are $\xi A_1, \xi^2 A_2, \ldots, \xi^m A_m$ for certain multisets $A_i$ consisting of $p$-th roots of unity. Here also $A_i = \emptyset$ is possible. Let $\zeta$ be a non-trivial $p$-th root of unity. Note that since the sum of the eigenvalues of $D(u)$ is an element in $R$ we know for every $i$ that if $A_i$ contains $\zeta$ exactly $r$ times then $A_i$ contains also $\zeta^2, \ldots, \zeta^{p-1}$ exactly $r$ times.

Let $M$ be an $RC$-lattice affording the representation $D$. Then

$$M \cong M_1 \oplus M_2 \oplus \ldots \oplus M_m$$

such that for every $i$ we have: If $A_i$ contains $\zeta$ exactly $r$ times and 1 exactly $s$ times then

$$M_i \cong a I^p_0 \oplus (r-a) I^p_{r-1} \oplus (s-a) I^p_1$$

for some non-negative integer $a \leq \min\{r, s\}$.

For $u \in V(ZG)$ a unit of order $pm$ for which we know the partial augmentations, including those of its powers, these facts allow to derive information on the isomorphism type of simple
$kG$-modules, for $k$ a big enough field of characteristic $p$, from the eigenvalues of $u$ under ordinary representations of $G$, see the proof of Theorem [12] or [BM17c] for a more detailed sketch of the method.

2.3. A reformulation using character values. The multiplicities in Theorem [12] can be computed from the partial augmentations of $u$ and its proper powers. This can be done by hand or using the GAP-package HeLP [BM15], more precisely the command HeLP_MultiplicitiesOfEigenvalues. The theorem can however also be translated in a condition on character values, so that the condition of Theorem 1.2 can be more easily checked without using GAP (or just to look up the character values). This is particularly handy if the characters in question only have integral values and the following lemma provides a way to do this.

**Lemma 2.3.** Let $G$ be a finite group and $u \in V(ZG)$ a unit of order $pq$ where $p$ and $q$ are different primes. Let $\chi$ be a character of $G$ which takes only integral values. Set $\chi(1) = d$, $\chi(u^p) = x$, $\chi(u^q) = y$ and $\chi(u) = z$. Then the following formulas hold.

\[
\begin{align*}
&pq \cdot \mu(1, u, \chi) = d + (q - 1)x + (p - 1)y + (p - 1)(q - 1)z, \\
&pq \cdot \mu(\zeta_p, u, \chi) = d + (q - 1)x - y - (q - 1)z, \\
&pq \cdot \mu(\zeta_q, u, \chi) = d - x + (p - 1)y - (p - 1)z, \\
&pq \cdot \mu(\zeta_{pq}, u, \chi) = d - x - y + z.
\end{align*}
\]

**Proof.** Note that for $i$ not divisible by $p$ we have $\mu(\zeta_p, u, \chi) = \mu(\zeta_p^i, u, \chi)$, for $i$ not divisible by $q$ we have $\mu(\zeta_q, u, \chi) = \mu(\zeta_q^i, u, \chi)$ and for $i$ not divisible by $pq$ also $\mu(\zeta_{pq}, u, \chi) = \mu(\zeta_{pq}^i, u, \chi)$. So

\[
\begin{align*}
\chi(1) &= \mu(1, u, \chi) + (p - 1)\mu(\zeta_p, u, \chi) + (q - 1)\mu(\zeta_q, u, \chi) + (p - 1)(q - 1)\mu(\zeta_{pq}, u, \chi), \\
\chi(u^q) &= \mu(1, u, \chi) - \mu(\zeta_p, u, \chi) + (q - 1)\mu(\zeta_q, u, \chi) - (q - 1)\mu(\zeta_{pq}, u, \chi), \\
\chi(u^p) &= \mu(1, u, \chi) + (p - 1)\mu(\zeta_p, u, \chi) - \mu(\zeta_q, u, \chi) - (p - 1)\mu(\zeta_{pq}, u, \chi), \\
\chi(u) &= \mu(1, u, \chi) - \mu(\zeta_p, u, \chi) - \mu(\zeta_q, u, \chi) + \mu(\zeta_{pq}, u, \chi).
\end{align*}
\]

The lemma now follows by linear transformations.

3. Combinatorics

The main result of this section, which is the main ingredient in the proof of Theorem 1.2 is Proposition [13]. It provides bounds on the possible entries in a semistandard skew tableau satisfying the lattice property for a special form of skew tableaux which are typical for our intended application.

**Definition 3.1.** We call a skew diagram of form $\mathcal{A}$ if it consists of $p$ columns such that the second to $(p - 1)$-th column all end at the same row, say the $m$-th row. This is equivalent to saying that there is a number $m$ such that for a box $b$ in the first to $(p - 2)$-th column and in the $k$-th row, for $k \leq m$, there is always a box to the right of $b$ and moreover the second column contains no box in the $(m + 1)$-th row. Furthermore we also allow the $p$-th column to be empty.

We call the boxes in the first to $(p - 1)$-th column which lie in the $m$-th row and above the body of the skew diagram. The boxes of the first column lying below the $m$-th row are called the tail and the $p$-th column the head of the tableau.

For a semistandard skew tableau $T$ satisfying the lattice property we denote by $\gamma_i(T)$ the number of letters which appear at least $i$ times in $w(T)$.

**Example 3.2.** In Figure 1 the upper skew diagram is of form $\mathcal{A}$ while the lower one is not. In the first skew diagram the boxes in the head are marked with $h$, the boxes of the body with $b$ and the boxes of the tail with $t$.

**Lemma 3.3.** Let $T$ be a semistandard skew tableau of form $\mathcal{A}$ with $p$ columns satisfying the lattice property. Let $b$ be a box in the $k$-th column lying in the body of $T$ with entry $e$ where $1 \leq k \leq p - 1$. Then $w(b)$ contains $e$ at least $(p - k)$ times.
Proof. We will argue by decreasing induction. For \( k = p - 1 \) the claim is clear, since being the entry in \( b \) the letter \( e \) appears in \( w(b) \) at least once. So let \( k < p - 1 \) and let \( e_r \) be the entry in the box \( b_r \) right from \( b \). Note that \( b_r \) exists by the assumption that \( T \) is of form \( A \) and \( b \) a box in the body of \( T \). Then by induction the letter \( e_r \) appears at least \( p - k - 1 \) times in \( w(b_r) \). Since \( e \leq e_r \), since \( T \) is semistandard, and \( w(b_r) \) satisfies the lattice property also \( e \) must appear at least \( p - k - 1 \) times in \( w(b_r) \). Hence \( e \) appears at least \( p - k \) times in \( w(b) \). \( \square \)

Lemma 3.4. Let \( T \) be a semistandard skew tableau of form \( A \) with \( p \) columns satisfying the lattice property. Let \( 1 \leq k \leq p - 2 \) and let \( b \) be a box in the \( k \)-th column of \( T \) inside the body of \( T \) with entry \( e \). Assume that the box right from \( b \) is the \( h \)-th box in the \( (k + 1) \)-th column of \( T \). Then \( e \leq h \).

Proof. Assume that reading from top to bottom and right to left \( b \) is the first box contradicting the claim. In particular, \( e > h \). Let \( b_r \) be the box right from \( b \) with entry \( e_r \). We will show that the \((k + 1)\)-th column contains every letter \( 1, \ldots, e - 1 \). This will imply the lemma since if the \((k + 1)\)-th column contains the letters \( 1, \ldots, e - 1 \) then \( e_r = h < e \), contradicting that \( T \) is a semistandard tableau. We will show also that when \( b_{k+1} \) is a box in the \((k + 1)\)-th column containing an entry \( e_s \) smaller than \( e \) then this entry appears exactly \((p - k)\) times in \( w(b_{k+1}) \) and if there is a box left from \( b_{k+1} \) then the entry of this box is strictly smaller than \( e_s \).

We will argue by decreasing induction. By Lemma 3.3 the word \( w(b) \) contains the letter \( e \) at least \( p - k \) times and so \( w(b) \) also contains every letter smaller than \( e \) at least \( p - k \) times. By assumption \( b \) is the first box violating the lemma, so if there exists a box above \( b \) with entry \( e_a \) then \( e - e_a \geq 2 \). In any case, if such a box exists or not, we have that the \( k \)-th column does not contain the entry \( e - 1 \). Note that \( e - 1 \) is not the entry in a box lying in a row above \( e \) and to the left of \( e \), since this would be a box contradicting the fact that \( T \) is semistandard. Since \( w(b) \) contains \( e - 1 \) at least \( p - k \) times we conclude that every column right from \( b \) must contain \( e - 1 \), in particular the \((k + 1)\)-th column and if \( b_{k+1} \) is the box in the \((k + 1)\)-th column containing \( e - 1 \) then \( e - 1 \) appears exactly \((p - k)\) times in \( w(b_{k+1}) \). Moreover \( b_{k+1} \) lies above \( b_r \) since otherwise \( e - 1 = e_r < e \), which would contradict the fact that \( T \) is semistandard. Since the \( k \)-th column does not contain \( e - 1 \) a box lying left from \( b_{k+1} \), if it exists, must contain an entry smaller than \( e - 1 \).

So assume that \( e_s \leq e - 1 \) is an entry contained in the box \( b_{k+1} \) which lies in the \((k + 1)\)-th column, that \( e_s \) appears exactly \((p - k)\) times in \( w(b_{k+1}) \) and that a box left from \( b_{k+1} \), if existent,
contains an entry smaller than $e_s$. Then $w(b_{k+1})$ contains the letter $e_s$ at least $p - k$ times and so it contains $e_s - 1$ at least $p - k$ times. Since the box above and left from $b_{k+1}$, if existent, has maximal entry $e_s - 2$ we get that $e_s - 1$ must be an entry in the $(k + 1)$-th column, so it must lie in the box $b_{k+1}'$ above $b_{k+1}$. So $e_s - 1$ appears exactly $(p - k)$ times in $w(b_{k+1}')$ and the box left from $b_{k+1}'$, if existent, contains an entry smaller than $e_s - 1$. This finishes the induction step and thus the lemma is proved. \( \square \)

**Proposition 3.5.** Let $T$ be a semistandard skew tableau of form $A$ with $p$ columns satisfying the lattice property. Let $1 \leq k \leq p - 1$ and let $h$ be the height of the $k$-th column inside the body of $T$, i.e. the number of boxes in the $k$-th column of $T$ which lie in the body (so if $k = 1$ we do not count the boxes in the tail).

Then $\gamma_{p-k}(T) \geq h$ and if $k \geq 2$ then $\gamma_{p-k+2}(T) \leq h$.

**Proof.** Let $b$ be the lowest box in the $k$-th column with entry $e$. Then just for the reason that the entries in a column are strictly increasing we have $h \leq c$. By Lemma 3.3, the entry $e$ appears in $w(b)$ at least $(p - k)$ times, so each letter $1, 2, \ldots, h$ appears at least $p - k$ times in $w(T)$ and hence $\gamma_{p-k}(T) \geq h$.

So assume $k \geq 2$ and assume $s = \gamma_{p-k+2}(T) > h$. By Lemma 3.4, the maximal possible entry in the box left from $b$ is $h$. So inside the body an entry bigger than $h$ does not appear in the first to $(k - 1)$-th column. Hence each letter $h + 1, \ldots, s$ must appear in the tail and in the $k$-th column and every column right from the $k$-th column. We will show that the $k$-th column then contains every letter $1, \ldots, s$, which will yield the final contradiction, since clearly the $k$-th column cannot contain more than $h$ entries. The proof is similar to the proof of Lemma 3.1.

Let $b$ be the box in the $k$-th column containing $s$. Then $s$ appears in $w(b)$ exactly $(p - k + 1)$ times, since it appears at least overall $p - k + 2$ times in $w(T)$ and it does not appear in the first to $(k - 1)$-th column inside the body. The box left from $b$, if existent, contains an entry smaller than $h$, so also smaller than $s$, by Lemma 3.3. We will show by decreasing induction that the $k$-th column contains every entry between 1 and $s$ such that if $b_c$ is the box containing $e$ than $e$ appears exactly $(p - k + 1)$ times in $w(b_c)$ and the box left from $b_c$, if existent, contains an entry smaller than $e$. Since this holds for $e = s$ assume that this holds for $e + 1$ and let $b_{c+1}$ be the box in the $k$-th column containing $e + 1$. Since $e + 1$ appears exactly $(p - k + 1)$ times in $w(b_{c+1})$, also $e$ appears at least $p + k - 1$ times in $w(b_{c+1})$. Since the entry in the box which lies above and to the left of $b_{c+1}$ is at most $e - 1$ the box above $b_{c+1}$ must contain $e$. So this box is $b_c$, the letter $e$ appears exactly $(p - k + 1)$ times in $w(b_c)$ and the box to the left of $b_c$ contains an entry smaller than $e$. This finishes the induction step and hence the proof. \( \square \)

4. PROOFS OF MAIN RESULTS

We proceed to prove Theorem 1.2. For the basic facts about Brauer trees we refer to [11,89]. Here we will only need that if $(K, R, k)$ is a $p$-modular system such that the characters labelling the vertices of the Brauer tree are afforded by $RG$-modules then each edge of the Brauer tree corresponds to a simple $kG$-module, each vertex of the Brauer tree corresponds to a simple $RG$-module with the given character and after reducing such a simple $RG$-module modulo the maximal ideal of $R$ the composition factors of this module are those corresponding to the adjacent edges, each with multiplicity 1.

**Proof of Theorem 1.2.** Let $R$ be a complete discrete valuation ring containing a primitive $m$-th root of unity such that $R$ is an algebraic extension of $\mathbb{Z}_p$, representations affording the characters $\chi_1, \ldots, \chi_p$ can be realized over $R$ and $p$ is unramified in $R$. Such a ring $R$ exists by a result of Fong [11,616,619,619]. Let $k$ be the residue class field of $R$. We use the bar-notation to denote reduction modulo the maximal ideal of $R$ also with respect to modules. Let $M$ be a $k$-module. Adapting the notation from skew tableaux we denote by $\gamma_i(M)$ the number of direct indecomposable summands of $M$ of dimension at least $i$.

Let $M_1, \ldots, M_p$ be $RG$-modules corresponding to the characters $\chi_1, \ldots, \chi_p$ and let, as explained in Proposition 2.2, $M_1, \ldots, M_p$ be the direct summands of $\bar{M}_1, \ldots, \bar{M}_p$ respectively on which $u^p$ acts as $\xi^p$. Let $S_1, \ldots, S_{p-1}$ be the simple $kG$-modules corresponding to the edges of the Brauer tree.
in the natural order, i.e. $S'_i$ corresponds to the edge with vertices $\chi_i$ and $\chi_{i+1}$. Let $S_1$, ..., $S_{p-1}$ be the direct summand of $S'_1$, ..., $S'_p$ respectively on which $u^p$ acts as $\xi^p$. We will view $M_i$ and $S_i$ as $k/(u)$-modules, unless explicitly stated otherwise. Let $\lambda_i$ be a partition describing the isomorphism type of $M_i$ and $\mu_i$ the partition describing the isomorphism type of $S_i$. Note that by Proposition 2.2 if $S_j$ is a submodule of $M_i$ then the skew diagram corresponding to $\lambda_i/\mu_j$ is of form $A$ with $p$ columns, since the indecomposable summands of $M_i$ are all of dimension 1, $p-1$ or $p$, i.e. the possible length of rows in a Young diagram corresponding to $\lambda_i$ are 1, $p-1$ and $p$.

Note that if we want to compute the possible isomorphism types of $S_i$ for some $1 < i < p-1$ using Theorem 2.1 we can consider a semistandard skew tableau satisfying the lattice property of shape $\lambda_i/\mu_i-1$ or $\lambda_i+1/\mu_i+1$. Indeed the composition factors of $M_i$ as $kG$-module are $S_{i-1}'$ and $S'_i$. If $S_{i-1}'$ is a submodule of $M_i$ as $kG$-module then $S_i \cong M_i/S_{i-1}'$ and $c_{\mu_i-1,\mu_i}^{\lambda_i} \neq 0$. If on the other hand $S'_i$ is a submodule of $M_i$ as $kG$-module then $c_{\mu_i,\mu_i}^{\lambda_i} \neq 0$, implying $c_{\mu_i-1,\mu_i}^{\lambda_i} \neq 0$ and so we can also consider $\lambda_i/\mu_i-1$. The same arguments apply for $\lambda_{i+1}/\mu_{i+1}$.

We will show that

$$\gamma_{p-2}(S_{p-2}) \geq \mu(\xi \cdot \zeta_p, u, \chi_{p-1}) - \mu(\xi \cdot \zeta_p, u, \chi_p)$$

and also

$$\gamma_{p-2}(S_{p-2}) \leq \mu(\xi, u, \chi_1) - \sum_{i=1}^{p-2} (-1)^i \mu(\xi \cdot \zeta_p, u, \chi_i)$$

which will imply the theorem. First by Theorem 2.1 the isomorphism type of $S_{p-2}$ can be described by a semistandard skew tableau $T$ satisfying the lattice property of shape $\lambda_{p-1}/\mu_{p-1}$. Note that $\mu_{p-1} = \lambda_p$ since $M_p \cong S_p$. By Proposition 3.2, we know that $\gamma_{p-2}(T) = \gamma_{p-2}(S_{p-2})$ is at least as big as the height of the second column of $T$. Now by Proposition 2.2 the number of indecomposable direct summands of dimension at least 2 in $M_{p-1}$, i.e. the number of rows of length at least 2 in a Young diagram corresponding to $\lambda_{p-1}$, is $\mu(\xi \cdot \zeta_p, u, \chi_{p-1})$ and the analogous statement holds for $\lambda_p$. So the height of the second column of $T$ is $\mu(\xi \cdot \zeta_p, u, \chi_{p-1}) - \mu(\xi \cdot \zeta_p, u, \chi_p)$, proving the first claim.

To prove the second claim we will prove by induction on $r$ that if $r$ is odd then

$$\gamma_r(S_r) \leq \mu(\xi, u, \chi_1) - \sum_{i=1}^{r} (-1)^i \mu(\xi \cdot \zeta_p, u, \chi_i)$$

and if $r$ is even then

$$\gamma_{r+1}(S_r) \geq -\mu(\xi, u, \chi_1) + \sum_{i=1}^{r} (-1)^i \mu(\xi \cdot \zeta_p, u, \chi_i).$$

Once this induction reaches $r = p - 2$ we will be done. If $r = 1$ then $\gamma_1(S_1)$ is the number of indecomposable direct summands of $S_1 \cong M_i$ which by Proposition 2.2 is at most $\mu(\xi, u, \chi_1) + \mu(\xi \cdot \zeta_p, u, \chi_1)$.

For the induction step assume that the statement holds for all numbers smaller than $r$. First assume that $r$ is even. Now $S_r$ corresponds to a semistandard skew tableau $T$ satisfying the lattice property of form $\lambda_r/\mu_r-1$. So by Proposition 3.3 the number $\gamma_{r+1}(S_r) = \gamma_{r+1}(T)$ is at least as big as the height of the $(r-1)$-th column in the body of $T$. Since $r \leq p$ this numbers equals the difference of the number of direct indecomposable summands of $M_{r-1}$ which is at least $(r-1)$-dimensional, but not 1-dimensional, and the number of indecomposable direct summands of $S_{r-1}$ of dimension at least $r-1$, i.e. $\gamma_{r-1}(S_{r-1})$. The number of indecomposable direct summands of $M_{r-1}$ which are at least $(r-1)$-dimensional, but not 1-dimensional, is $\mu(\xi \cdot \zeta_p, u, \chi_r)$ by Proposition 2.2. Since by induction $\gamma_{r-1}(S_{r-1}) \leq \mu(\xi, u, \chi_1) - \sum_{i=1}^{r-1} (-1)^i \mu(\xi \cdot \zeta_p, u, \chi_i)$ we obtain that the height of the $(r-1)$-th column in the body of $T$ is at least

$$\mu(\xi \cdot \zeta_p, u, \chi_r) - (\mu(\xi, u, \chi_1) - \sum_{i=1}^{r-1} (-1)^i \mu(\xi \cdot \zeta_p, u, \chi_i)) = -\mu(\xi, u, \chi_1) + \sum_{i=1}^{r-1} (-1)^i \mu(\xi \cdot \zeta_p, u, \chi_i).$$
Now assume that \( r \) is odd, bigger than 2 and smaller than \( p-1 \). Also in this case \( S_r \) corresponds to a semistandard skew tableau \( T \) satisfying the lattice property of form \( \lambda_r / \mu_{r-1} \). So the number 
\[
\gamma_r(S_r) = \gamma_r(T)
\]
is at most as big as the height of the \((p + 2 - r)\)-th column in the body of \( T \) by Proposition 2.2. This is the difference of the number of indecomposable direct summands of dimension at least \( p + 2 - r \) in \( \mathbb{M}_r \), i.e. \( \gamma_{p+2-r}(\mathbb{M}_r) \), and the number of indecomposable direct summands of \( S_{r-1} \) of dimension at least \( p + 2 - r \), i.e. \( \gamma_{p+2-r}(S_{r-1}) = \gamma_{p+1-(r-1)}(S_{r-1}) \). By Proposition 2.2 we know \( \gamma_{p+2-r}(\mathbb{M}_r) = \mu(\xi \cdot \zeta_p, u, \chi_r) \) and by induction \( \gamma_{p+1-(r-1)}(S_{r-1}) \) is at least 
\[
-\mu(\xi, u, \chi_1) + \sum_{i=1}^{r-1} (-1)^i \mu(\xi \cdot \zeta_p, u, \chi_i).
\]
So the height of the \((p + 2 - r)\)-th column of \( T \) is at most
\[
\mu(\xi \cdot \zeta_p, u, \chi_r) - (-\mu(\xi, u, \chi_1) + \sum_{i=1}^{r-1} (-1)^i \mu(\xi \cdot \zeta_p, u, \chi_i)) = \mu(\xi, u, \chi_1) - \sum_{i=1}^{r} (-1)^i \mu(\xi \cdot \zeta_p, u, \chi_i),
\]
finishing the proof of the induction claim.

The proof of the Prime Graph Question for the Conway simple groups now boils down to the application of Theorem 1.2 for various cases described in [BKL11]. The information contained in [BKL11] is not completely sufficient for our purposes, since to compute multiplicities of eigenvalues we do not only need to know the partial augmentations of a unit \( u \in V(ZG) \), but also the partial augmentations of its proper powers. These can be computed using the GAP-package HeLP [BMT16] and we will indicate in all cases we need to consider which characters are sufficient to obtain these possible partial augmentations using the command HeLP_WithGivenOrder. Also in the case of the first Conway group we will use stronger results obtainable by the HeLP method than those given in [BKL11].

We will denote by \( \chi_i \) the \( i \)-th irreducible complex character of a group \( G \) as given in the GAP character table library [Bre12]. We will also use names for conjugacy classes as in [Bre12]. The statements we will need about the \( p \)-blocks of various groups, their defect and their corresponding Brauer trees can all be derived from GAP, but they are also given in [HL89].

**Proof of Theorem 1.2:** We will study the three cases of interest separately.

**Case** \( G = Co_2 \): Assume first that \( G = Co_2 \). By [BKL11] Theorem 1] we only have to consider units of order 35 in \( V(ZG) \). Using \( \chi_2 \) to compute partial augmentations for units of order 5 and \( \chi_2, \chi_3 \) to compute the partial augmentations of units of order 35 we are left with the following possibilities:

\[
(\varepsilon_{5a}(u^7), \varepsilon_{5b}(u^7), \varepsilon_{5a}(u), \varepsilon_{5a}(u), \varepsilon_{7a}(u)) \in \{(-4, 5, 3, 12, -14), (-3, 4, 4, 11, -14)\}.
\]

Note that there is only one class of elements of order 7 in \( G \), so we do not need to consider the partial augmentations of \( u^5 \), since the only class in which a partial augmentation at \( u^5 \) is non-vanishing is \( 7a \).

\( G \) possesses a 5-block of defect 1 whose Brauer tree is a line of form

\[
\chi_5 \quad \chi_{29} \quad \chi_{39} \quad \chi_{35} \quad \chi_{12} \quad \ldots
\]

All irreducible ordinary characters in this block have only integral values and 5 is of course not ramified in \( \mathbb{Z} \). So we can apply Theorem 1.2. We provide the multiplicities of the needed eigenvalues of \( u \), for the two critical distributions of partial augmentations, in Table 1. Here the first entry in each column contains the possible values of \( (\varepsilon_{5a}(u^7), \varepsilon_{5b}(u^7), \varepsilon_{5a}(u), \varepsilon_{5a}(u), \varepsilon_{7a}(u)) \).

So using Theorem 1.2 with \( \xi = 1 \) we would have, e.g. in case \( (\varepsilon_{5a}(u^7), \varepsilon_{5b}(u^7), \varepsilon_{5a}(u), \varepsilon_{5a}(u), \varepsilon_{7a}(u)) = (-4, 5, 3, 12, -14) \) that

\[
\mu(\zeta_5, u, \chi_{35}) - \mu(\zeta_5, u, \chi_{12}) = 4967 \leq \mu(1, u, \chi_5) + \mu(\zeta_5, u, \chi_5) - \mu(\zeta_5, u, \chi_{29}) + \mu(\zeta_5, u, \chi_{39}) = 4945,
\]
a contradiction. The same argument applies to the other possibility of distributions of partial augmentations of \( u \), in which case we get \( 4965 \leq 4944 \).

**Case** \( G = Co_3 \): Let \( G = Co_3 \). By [BKL11] Theorem 2] we only have to consider units of order 35. Using \( \chi_2 \) to compute partial augmentations of units of order 5 and \( \chi_2, \chi_3 \) for order 35 we get

\[
(\varepsilon_{5a}(u^7), \varepsilon_{5b}(u^7), \varepsilon_{5a}(u), \varepsilon_{5a}(u), \varepsilon_{7a}(u)) \in \{(-4, 5, 3, 12, -14), (-3, 4, 4, 11, -14)\}.
\]
Again there is only one class of elements of order 7, so we do not need to consider $u^5$.

Also in this case $G$ possesses a 5-block of defect 1 with Brauer tree of the form

$$
\begin{array}{cccc}
\chi_4 & \chi_24 & \chi_43 & \chi_38 \\
\chi_5 & 33 & 2 & 29 & 3 \\
\chi_{29} & 2119 & 2118 & \\
\chi_{39} & 7029 & 7030 & \\
\chi_{35} & 5071 & 5070 & \\
\chi_{12} & 104 & 105 & \\
\end{array}
$$

Table 1. Multiplicities of eigenvalues for $G = Co_3$ for units of order 35.

All ordinary characters in this block only have integral values. The necessary multiplicities are provided in Table 2.

$$
\begin{array}{cccc}
\chi_4 & \chi_24 & \chi_43 & \chi_38 \\
\chi_5 & 33 & 2 & 29 & 3 \\
\chi_{29} & 3269 & 3268 & \\
\chi_{39} & 13354 & 13355 & \\
\chi_{35} & 11396 & 11395 & \\
\chi_{12} & 1254 & 1255 & \\
\end{array}
$$

Table 2. Multiplicities of eigenvalues for $G = Co_2$ for units of order 35.

In the first case using Theorem 1.2 with $\xi = 1$ we get $10142 \leq 10120$ and in the second case $10140 \leq 10119$.

**Case $G = Co_1$:** Let $G = Co_1$. By [BKL11] Theorem 3] we only have to consider units of order 55 and 65. We will first apply the HeLP method here to obtain stronger results than in [BKL11]. The 2-, 3- and 5-modular Brauer tables are not available in GAP, but the Atlas, also implemented as a GAP-package [WPN+11], contains several representations in these characteristics and we will use a character $\psi$ coming from a 24-dimensional representation of $G$ over $F_2$. The values of $\psi$ on classes of interest can be found in Table 3.

$$
\begin{array}{cccccc}
\psi & 1a & 5a & 5b & 5c & 11a & 13a \\
24 & -6 & 4 & -1 & 2 & -2 & \\
\end{array}
$$

Table 3. A 2-modular Brauer character for $Co_1$.

The results we obtain using the HeLP method are stronger than those in [BKL11], since we include $\psi$. Using $\chi_2, \chi_3$ and $\psi$ we obtain 98 possibilities for the partial augmentations of units of order 5. Then using $\chi_2, \chi_4, \chi_5$ and $\psi$ we can exclude the existence of units of order 65 in $ZG$. For units of order 55 we apply the characters $\chi_2, \chi_3, \chi_4$ and $\psi$. Note that there is only one class of elements of order 11 in $G$.

We get that $(\varepsilon_{5a}(u^{11}), \varepsilon_{5b}(u^{11}), \varepsilon_{5c}(u^{11}), \varepsilon_{5a}(u), \varepsilon_{5b}(u), \varepsilon_{5c}(u), \varepsilon_{11a}(u))$ is one of the four possibilities

$(1, 5, -5, 1, -6, -5, 11), (1, 6, -6, 1, -5, -6, 11), (2, 6, -7, 2, -5, -7, 11), (2, 7, -8, 2, -4, -8, 11)$.

The main 11-block of $G$ has a Brauer tree of the form
Also in this block all ordinary characters only have integral values. The necessary multiplicities are provided in Table 4.

| \chi | \mu(1, u, \chi) | \mu(\zeta_{11}, u, \chi) | \mu(1, u, \chi) | \mu(\zeta_{11}, u, \chi) |
|-----|----------------|------------------------|----------------|------------------------|
| \chi_1 | 1 | 0 | 1 | 0 |
| \chi_{12} | 5668 | 5670 | 5640 | 5640 |
| \chi_{34} | 138600 | 138600 | 138520 | 138520 |
| \chi_{41} | 391385 | 391385 | 391350 | 391350 |
| \chi_{64} | 1929876 | 1929870 | 1929856 | 1929850 |
| \chi_{79} | 4495195 | 4495195 | 4495195 | 4495195 |
| \chi_{85} | 5326485 | 5326495 | 5326570 | 5326580 |
| \chi_{73} | 3734505 | 3734505 | 3734540 | 3734540 |
| \chi_{60} | 1522180 | 1522180 | 1522180 | 1522180 |
| \chi_{38} | 297293 | 297285 | 297303 | 297295 |
| \chi_{14} | 6885 | 6875 | 6880 | 6870 |

Table 4. Multiplicities of eigenvalues for \( G = Co_1 \) for units of order 55.

From the four possibilities we obtain by Theorem 1.2 the inequalities \( 290408 \leq 290389 \), \( 290410 \leq 290391 \), \( 290423 \leq 290304 \) and \( 290425 \leq 290306 \), all of which do not hold.

Remark 4.1. In the proof of Theorem 1.1 to handle the first Conway group we used Theorem 1.2 with \( p = 11 \) instead of \( p = 5 \) for two reasons. One is to show that Theorem 1.2 can also be applied with other primes than 5. The other, more importantly, is that the 11-Brauer trees can be computed using GAP which can not be done as easily with 5-Brauer trees, since the 5-Brauer table is not available in GAP. The 5-Brauer trees are given in [HL89], but the calculations with \( p = 11 \) are easier to verify for a reader who does not have [HL89] available. Using the first Brauer tree given in [HL89, Section 6.22] and Theorem 1.2 with \( p = 5 \) and \( \xi = 1 \) one can also get the same result.

Remark 4.2. Of the 17 sporadic groups studied by Bovdi, Konovalov et al. the only group for which they could not prove the Prime Graph Question, apart from the Conway groups, was the O’Nan simple group [BGK09]. Here it remains to exclude the existence of units of order 33 and 57. Theorem 1.2 can be applied here with \( p = 11 \) to handle units of order 33, but not for order 57. This is not possible since for \( p \in \{3, 19\} \) every \( p \)-block of \( G \) which is a Brauer Tree Algebra contains characters such that \( p \) is ramified in the ring of values of these characters.

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ON THE PRIME GRAPH QUESTION FOR INTEGRAL GROUP RINGS OF CONWAY SIMPLE GROUPS

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