Extra States and Symmetries in $D < 2$ Closed String Theory

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We show that there is $(p - 1)(p' - 1)$ dimensional semi-relative BRST cohomology at each non-positive ghost number in the $(p, p')$ minimal conformal field theory coupled to two dimensional quantum gravity. These closed string states are related to currents and symmetry charges of ‘exotic’ ghost number. We investigate the symmetry structure generated by the most conventional currents (those of vanishing total ghost number), and make a conjecture about the extended algebra which results from incorporating the currents at negative ghost numbers.

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1. Introduction

Our understanding of $D \leq 2$ string theories has been greatly improved in the past two years by the realization that the double-scaling limit of matrix models can be used to probe these string theories to all orders in perturbation theory (and sometimes beyond)[1][2][3], and by the concurrent realization that these models are closely related to topological field theories [4][5]. However, we still lack a complete understanding of how one can obtain the remarkable results of matrix models working directly in a continuum (Liouville theory $\times$ conformal matter) formulation.

A focus of recent studies has been the role of the ‘extra’ physical states called special states, first discovered and studied in the $c = 1$ matrix model in [6][7][8] and later uncovered by Polyakov in the continuum version of this theory [9]. These states (and others, which we will also call special states) have since been found rigorously in the BRST cohomology of the continuum theory [10][11]. They play a crucial role in our understanding of the correspondence between the continuum theory and the $c = 1$ matrix model. In particular, some of the special states are responsible for generating the $W_\infty$ symmetry of the theory [12][13]. In addition, one can see the analogue of the free fermion structure of the matrix models by analyzing the ‘ground ring’ formed by the operators of vanishing ghost number (the ghost number increases by one in going from states to operators; in this paper we use the convention that the $\text{SL}(2,\mathbb{C})$ invariant vacuum has ghost number -1). The $W_\infty$ symmetry leads to powerful non-linear Ward Identities which can be used to greatly simplify computations in this theory [14][15][16], and the fact that the tachyon states of the theory form modules of the ground ring also leads to certain identities the correlation functions must satisfy [17][18][19].

While the special states have proven to be of great utility in the $D = 2$ theory, their presence in the $D < 2$ theories [20][21] (minimal models coupled to 2D quantum gravity) has not yet been fully understood and exploited. In this paper, we will show that there are infinitely more ‘extra’ physical states than those which have been widely discussed to date, in the semi-relative cohomology of the $D < 2$ theories (a similar observation has been made regarding the $D = 2$ theory by Witten and Zwiebach [15]). Some of these states correspond to standard currents (of total ghost number zero$^1$), which generate symmetries of the theories. In this paper we determine the symmetry algebra of the $(p,p')$ models.

\(^1\) Actually, since as we will see these currents have non-vanishing chiral ghost numbers which sum to zero, they aren’t quite like conventional currents
by which we mean 2-d gravity coupled to \((p, p')\) minimal matter, with \(p > p'\) and \(p, p'\) relatively prime). We also provide a preliminary exploration of the action of the associated conserved charges on the physical states. We find that the charges of total ghost number zero act trivially on the regular tachyon states (dressed minimal CFT primary fields) of these models, but expect that the full algebra of exotic symmetries will act non-trivially on the infinite tower of states discovered in \([20]\).

2. Formalism

2.1. Physical State Conditions in (Bosonic) String Theory

Since the new string states we find are in the semi–relative cohomology, and satisfy slightly weaker conditions than the ‘normal’ physical state conditions one is used to from critical string theory, we take a moment to review here how this set of conditions arises in the operator formalism. We will follow the treatment of Distler and Nelson \([21]\). For more details on the operator formalism see also \([22]\)\([23]\)\([24]\)\([25]\).

Denote by \(\mathcal{M}_{g,n}\) the space of conformally distinct smooth Riemann surfaces with \(g\) handles and \(n\) marked points, and by \(\mathcal{P}_{g,n}\) the space of such surfaces where in addition we have chosen a local coordinate \(z_i\) centered around each of the marked points. Then a conformal field theory associates to each element \(X \in \mathcal{P}_{g,1}\) a vector \(\langle X \rangle\) in a Hilbert space \(\mathcal{H}\). One can think of obtaining this state by performing a functional integral over all of the Riemann surface except for an excised disk around the marked point, yielding a wavefunctional of the boundary conditions one chooses to insert on the disk: If one inserts boundary conditions corresponding to the state \(\langle \psi \rangle \in \mathcal{H}\), then the resulting complex number is \(\langle X | \psi \rangle\). Similarly, to an element of \(\mathcal{P}_{g,N}\) a conformal field theory associates a vector in the \(N\)-fold tensor product \(\mathcal{H}^\otimes N\) (actually, more precisely one gets a vector in \(\mathcal{H}^\otimes Q \otimes \mathcal{H}^{\otimes N-Q}\) where \(Q \leq N\) is determined by the orientation chosen for the boundaries of the disks surrounding the punctures; this needn’t concern us here).

Given any \(N\) vectors \(\langle \psi_1 \rangle, \ldots, \langle \psi_N \rangle \in \mathcal{H}\), we wish to construct a differential form of degree \(3g - 3 + N\) on \(\mathcal{M}_{g,N}\), which we can then integrate over \(\mathcal{M}_{g,N}\) to give us the genus \(g\) \(N\)-point function of the operators \(\psi_1, \ldots, \psi_N\) associated with the vectors. First, we will construct such a form on \(\mathcal{P}_{g,N}\), by specifying its action on any \(3g - 3 + N\) tangent vectors to \(\mathcal{P}_{g,N}\) at some point \(Y\). Take an \(N\)-tuple of vector fields \(\vec{v} = (v_1, \ldots, v_N)\) (one \(v_i\) in the neighborhood of each of the punctures) which do not belong to the ‘Borel subalgebra’
$B(Y, z_1, \ldots, z_n)$, where the $z_i$ are local coordinates vanishing at the punctures $P_i$.\(^2\) These together correspond to a nonzero tangent vector $V$ to $\mathcal{P}_{g,N}$ at $Y$. Define:

$$b[\vec{v}] = \sum_{i=1}^{N} \oint b_{zz}^{(i)}(z) v_i^z(z) dz$$ \hspace{1cm} (2.1)

where $b_{zz}^{(i)}$ is the spin-2 anti-commuting ghost field of the $bc$ system introduced in string theory to fix diffeomorphism invariance, and acts on the $i$-th copy of $\mathcal{H}$. Then we define the differential form associated to $|\psi_1\rangle \otimes \ldots \otimes |\psi_N\rangle$ by:

$$\tilde{\Phi}(V_1, \ldots, V_{3g-3+N}, \bar{V}_1, \ldots, \bar{V}_{3g-3+N}) \equiv$$

$$\langle Y | b[\vec{v}_1] \ldots b[\vec{v}_{3g-3+N}] \bar{b}[\bar{\vec{v}}_1] \ldots \bar{b}[\bar{\vec{v}}_{3g-3+N}] |\psi_1\rangle \otimes \ldots \otimes |\psi_N\rangle$$ \hspace{1cm} (2.2)

Here, $\bar{b}$ is the anti-holomorphic counterpart to $b$, and the $\vec{v}$'s correspond to the conjugates $\bar{V}_i$ to the $V_i$.

Now, (2.2) does give us a form on $\mathcal{P}_{g,N}$; what we really want is a form on $\mathcal{M}_{g,N}$. The simplest possibility is that $\tilde{\Phi}$ is simply the lift of the desired form $\Phi$ on $\mathcal{M}_{g,N}$:

$$\tilde{\Phi} = \pi^* \Phi$$ \hspace{1cm} (2.3)

where $\pi : \mathcal{P}_{g,N} \to \mathcal{M}_{g,N}$ is the projection map for the $\mathcal{P}_{g,N}$ bundle over $\mathcal{M}_{g,N}$. This is the case exactly when each of the states satisfies:

$$L_n |\psi\rangle = b_n |\psi\rangle = \bar{L}_n |\psi\rangle = \bar{b}_n |\psi\rangle = 0, \hspace{1cm} n \geq 0$$ \hspace{1cm} (2.4)

where the $b_n$ and the $L_n$ are defined in the normal way as the Laurent coefficients of the $b$ ghost field and the holomorphic stress-energy tensor, and likewise for their counterparts $\bar{b}_n$ and $\bar{L}_n$. These are the most familiar physical state conditions from critical string theory, termed in [23] the ‘Strong Physical State Conditions.’ We can get weaker physical state conditions by realizing that (2.3) is not really necessary for us to reconstruct a form $\Phi$ on $\mathcal{M}_{g,N}$ from $\tilde{\Phi}$.

Choose a section $\sigma : \mathcal{M}_{g,N} \to \mathcal{P}_{g,N}$. Then we can imagine pulling back $\tilde{\Phi}$ using this section:

$$\Phi = \sigma^* \tilde{\Phi}.$$ \hspace{1cm} (2.5)

\(^2\) Elements of $B$ are sets of $\vec{v}_i$ which are all the restriction of a single vector field $\vec{v}$ holomorphic on $Y \setminus \{P_1, \ldots, P_n\}$. 


While the resultant $\Phi$ obviously depends on the section $\sigma$ chosen, it turns out that if $\tilde{\Phi}$ is a closed differential form, then $\Phi$ in (2.5) yields a well-defined cohomology class irrespective of which section we choose. Hence, its integral (modulo fixing conditions at the boundaries of $\mathcal{M}_{g,N}$) will be well defined.

In fact, $\tilde{\Phi}$ is indeed a closed form whenever the states involved in its definition in (2.2) obey a weaker condition than (2.4), namely that

$$ (Q + \bar{Q})|\psi\rangle = 0 \quad (2.6) $$

where $Q$ and $\bar{Q}$ are the holomorphic and anti-holomorphic BRST operators.

However, another complication arises: There is in general an obstruction to choosing a global section $\sigma$ (the bundle $\pi : \mathcal{P}_{g,N} \to \mathcal{M}_{g,N}$ is nontrivial). What we can do is find sections which are well defined up to constant phase jumps across coordinate patch overlaps. If we can construct $\tilde{\Phi}$ so that it is insensitive to such jumps, then again $\Phi$ defined via (2.5) will be acceptable. The necessary conditions for this to occur are:

$$ (Q + \bar{Q})|\psi\rangle = (L_0 - \bar{L}_0)|\psi\rangle = (b_0 - \bar{b}_0)|\psi\rangle = 0. \quad (2.7) $$

Actually, we will only consider states which are annihilated by both $Q$ and $\bar{Q}$ and which are in the nullspaces of both $L_0$ and $\bar{L}_0$. With this proviso, the conditions (2.4) correspond to states of the closed string which are in the chiral relative BRST cohomology (they are annihilated by $b_0$ and $\bar{b}_0$): The chiral halves would be bona fide open string states. The conditions (2.7) correspond to states of the closed string which are in the semi−relative cohomology.

2.2. Symmetries and Descent Equations

Let us briefly recall the formalism for currents conserved up to BRST commutators, following [15]. We assume we are working in the context of a two dimensional quantum field theory, in local complex coordinates on some Riemann surface. A strictly conserved current corresponds to a one form

$$ \Omega^1 = J_x dz - J_{\bar{x}} d\bar{z} \quad (2.8) $$

which is closed,

$$ d\Omega^1 = 0. \quad (2.9) $$
One then obtains a conserved charge by integrating this one form around a contour, the charge being conserved in the sense that this integral gives the same result for homologous contours [we will throughout use the convention that \[ \frac{1}{2\pi i} \oint \frac{dz}{z} = -\frac{1}{2\pi i} \oint \frac{d\bar{z}}{\bar{z}} = 1 \].

However, in BRST cohomology \( Q_{TOT} = Q + \bar{Q} \) commutators are trivial, so in BRST quantization a well-defined charge on the Hilbert space of physical states need only be invariant up to \( Q_{TOT} \) commutators when evaluated on homologous cycles. This corresponds to the case that \( \Omega^1 \) is only closed up to a \( Q_{TOT} \) commutator of some two form \( \Omega^2 \):

\[
d\Omega^1 = \{Q_{TOT}, \Omega^2\}. \tag{2.10}
\]

As discussed in [15], when \( \Omega^2 \neq 0 \), the Ward Identity corresponding to the symmetry generated by this current can be nonlinear.

Of course, in order for the conserved charge to map physical states to physical states, it must commute with the BRST operator \( Q_{TOT} \). This occurs if there is a zero form \( \Omega^0 \) such that:

\[
d\Omega^0 = \{Q_{TOT}, \Omega^1\}. \tag{2.11}
\]

It is then the case that \( \{Q_{TOT}, \Omega^0\} = 0 \) automatically.

Thus, the whole hierarchy of descent equations is given by:

\[
0 = \{Q_{TOT}, \Omega^0\}
\]

\[
d\Omega^0 = \{Q_{TOT}, \Omega^1\} \tag{2.12}
\]

\[
d\Omega^1 = \{Q_{TOT}, \Omega^2\}
\]

These equations imply that \( \Omega^1 \) is a conserved current in the BRST formalism. So in particular, finding symmetries of the theory is equivalent to finding zero forms \( \Omega^0 \) which give non-trivial results for \( \Omega^1 \) when plugged into the descent equations.

The descent equations can be re-written in terms of states instead of operators, with the results that for a given \( |\Omega^0\rangle \)

\[
|\Omega^1_z\rangle = b_{-1}|\Omega^0\rangle
\]

\[
|\Omega^1_{\bar{z}}\rangle = \bar{b}_{-1}|\Omega^0\rangle \tag{2.13}
\]

\[
|\Omega^2\rangle = b_{-1}\bar{b}_{-1}|\Omega^0\rangle
\]

Hence, to find conventional symmetries of the closed string theory, which correspond to charges which have ghost number zero, one must find BRST invariant states \( |\Omega^0\rangle \) in the semi-relative cohomology at ghost number 0. Then the corresponding operator \( \Omega^1 \) obtained from the descent equations will also have vanishing ghost number, as will the charge obtained from its contour integral.
3. Extra States in Semi-relative Cohomology

3.1. Proof of Existence of Extra States

In order to clarify the discussion below, we introduce some notation. We will denote by $H^n_{\text{rel}}(k)$ the chiral relative BRST cohomology of the $(p, p')$ minimal model coupled to Liouville theory at ghost number $n$ built on the Liouville Fock space with momentum $k$. $H^n_{\text{abs}}(k)$ will denote the corresponding absolute BRST cohomology.

One can find the analogue of the ground ring in the $D < 2$ string theories by examining the appropriate cohomology of the $(p, p')$ minimal model coupled to quantum gravity. This has been done by Kutasov, Martinec, and Seiberg [17], who find that in the $(p, p')$ model the ground ring consists of operators $R_{n, n'}$ with $1 \leq n \leq p - 1$ and $1 \leq n' \leq p' - 1$. This ring is the ring generated by $X = R_{2,1}$ and $Y = R_{1,2}$:

$$R_{n, n'} = X^{n-1} Y^{n'-1}$$

(3.1)

(the relations $X^{p-1} = Y^{p'-1} = 0$ hold).

Let us denote by $O_{n, n'}$ the holomorphic chiral half of $R_{n, n'}$. If we call the Liouville momentum of the operator $O_{n, n'}$ $p_{n, n'}$, then we see that in the notation introduced above, these ring elements correspond to nontrivial cohomology classes $|O_{n, n'}\rangle \in H^{(-1)}_{\text{rel}}(p_{n, n'})$ in the chiral BRST cohomology. In fact, these are the only such classes in the relative cohomology at their respective momenta and ghost number -1. Now, recall the result that:

$$H^{(n)}_{\text{abs}}(p) \simeq H^{(n)}_{\text{rel}}(p) \oplus c_0 H^{(n-1)}_{\text{rel}}(p)$$

(3.2)

Since we know that $H^{(0)}_{\text{rel}}(p_{n, n'}) = \emptyset$ for all $n, n'$, it follows that

$$\dim H^{(0)}_{\text{abs}}(p_{n, n'}) = 1.$$  

(3.3)

Call a representative of the non-trivial cohomology class $|W_{n, n'}\rangle$. We will now show that

$$\Omega_{n, n'} = W_{n, n'} \bar{O}_{n, n'} + O_{n, n'} \bar{W}_{n, n'}$$

(3.4)

is the zero-form component of a current, in the formalism described in section 2.2. In order to do this, we must prove that $|\Omega_{n, n'}\rangle$ is in the semi-relative BRST cohomology and that $b_{-1}|\Omega_{n, n'}\rangle \neq 0$.

Consider the state $b_0|W_{n, n'}\rangle$. We know this does not vanish, because there is no relative cohomology with momentum $p_{n, n'}$ at ghost number 0. However, $Q(b_0|W_{n, n'}\rangle) = 0$, 

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since \([Q,b_0] = L_0\) and \(|W_{n,n'}\rangle\) is annihilated by both \(Q\) and \(L_0\). In addition, \(b_0|W_{n,n'}\rangle\) is obviously annihilated by \(b_0\), so it is in the relative cohomology at ghost number -1. Since \(|O_{n,n'}\rangle\) is the only nontrivial relative cohomology at ghost number -1 with the right momentum, we must have:

\[
b_0|W_{n,n'}\rangle = k|O_{n,n'}\rangle, \ k \neq 0.
\] (3.5)

But then consider the state in the full closed string cohomology given by:

\[
|\Omega_{n,n'}\rangle = |O_{n,n'}\rangle \otimes |\bar{W}_{n,n'}\rangle + |W_{n,n'}\rangle \otimes |\bar{O}_{n,n'}\rangle.
\] (3.6)

This is clearly annihilated by \(Q_{TOT}\) and \(b_0 - \bar{b}_0\), so it is in the semi-relative cohomology of the closed string theory. Hence, the operator \(\Omega_{n,n'}\) given in (3.4) corresponding to this state is the zero form component of a current if \(b_{-1}|\Omega_{n,n'}\rangle \neq 0\). We shall prove that this is the case in section 3.2, by making a connection with previous work in the \(c=1\) theory which allows us to give a more explicit construction of these states.

3.2. Comparison to \(c=1\) Model

We might expect these currents to correspond in a simple way to some subalgebra of the infinite symmetry algebra present in the \(c=1\) theory. This is in fact the case.

Recall that in the \(c=1\) theory, the symmetries include a \(W_\infty\) algebra but also include an infinite number of ‘internal’ symmetries, one for each ground ring element. If we define following [26]

\[
a = [Q, \phi] = c \partial \phi + \sqrt{2} \partial c
\] (3.7)

then the new chiral ghost number one operators in the absolute cohomology of the \(c=1\) model were given by

\[
aO(0) = \frac{1}{2\pi i} \oint \frac{dz}{z} a(z) \ O(0)
\] (3.8)

where \(O\) is an element of the \(c=1\) chiral ground ring [13]. While it might seem that these operators are BRST trivial from (3.7), this is not so strictly speaking because \(\phi\) is not in the usual space of conformal fields. The new symmetry generators then had zero form pieces corresponding to the new semi-relative cohomology classes \((a + \bar{a})(O\bar{O})\).

Because the Liouville sector remains intact in the minimal models coupled to gravity, one can still construct the operator \(a\) and this construction of the new semi-relative
cohomology should generalize to all of the minimal models. Hence, we find that our new operators \( \Omega_{n,n'} \) are given more explicitly by the formula

\[
\Omega_{n,n'} = (a + \bar{a})R_{n,n'}.
\]  

(3.9)

This allows us to fix the constant \( k \) in equation (3.5) to be \( \sqrt{2} \). It is also now a trivial matter to check the remaining condition which insures that \( \Omega_{n,n'} \) generates a non-trivial symmetry, namely that \( b_{-1}|\Omega_{n,n'}\rangle \neq 0 \). From equations (3.7), (3.9) we see with a little bit of work that the symmetries are in fact nontrivial (in this sense, at least).

Although we have exhibited that the \( (p,p') \) model still has a Lie algebra of symmetries with \( (p - 1)(p' - 1) \) generators, one may wonder what has become of the rest of the vast symmetry structure of the \( c=1 \) theory. Indeed, if one views the \( (p,p') \) models as SO(2,\( \mathbb{C} \)) rotations of the \( c=1 \) model, one might expect the \( W_\infty \) structure to survive \cite{27} \cite{28}. Nevertheless, it follows from the mathematical analysis \cite{11} that in the \( (p,p') \) model there are no potential generators of conventional (vanishing ghost number) symmetries other than those given above.

### 3.3. Generalization to Other Ghost Numbers

It is clear that the construction of sections 3.1 and 3.2 will generalize to other ghost numbers as follows. Assume we have some momentum \( p \) and ghost number \( n \) such that:

\[
\dim H_{\text{rel}}^{(n)}(p) = 1. 
\]  

(3.10)

Suppose a representative of this cohomology class is \( |A_n^p\rangle \). Then as discussed in section 3.2, we can construct a new element of the semi-relative cohomology at ghost number \( n+1 \):

\[
(a + \bar{a})A_n^p\bar{A}_n^p(0)|0\rangle.
\]  

(3.11)

These states will correspond to ‘symmetries’ of non-standard ghost number. The associated conserved charges will map ‘normal’ physical states to physical states of exotic ghost number.
4. Algebra of the Symmetries

4.1. Role of New Symmetries in $D=2$ Case

Before proceeding to determine the symmetry structure of the $D < 2$ theories, it will be helpful to recall certain facts revealed in the analysis of \[15\] about the analogues of our new symmetry generators in the $D=2$ case.

In the (compactified) $c=1$ model, the chiral ground ring is generated by two operators $x$ and $y$, while the anti-chiral ground ring is generated by their counterparts $x'$ and $y'$. We can combine these chiral components to form four closed-string operators:

\[ a_1 = xx' \quad a_2 = yy' \quad a_3 = xy' \quad a_4 = yx'. \]  

In \[12\][15\] the special states are interpreted in terms of the differential geometry of the quadric cone $Q$ formed by the ring generators subject to the ‘fermi-surface’ relation

\[ a_1 a_2 - a_3 a_4 = \mu \]  

where $\mu$ is the cosmological constant in the Liouville theory. In this context, the new symmetry currents derived from the zero form $(a + \bar{a})O$ (with $O$ a ground ring element) correspond to vector fields

\[ \text{Currents from } (a + \bar{a})O \sim f S, \quad S \equiv x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - x' \frac{\partial}{\partial x'} - y' \frac{\partial}{\partial y'}. \]  

Here, $f$ is some function on $Q$ which must have the proper quantum numbers to represent the symmetry generator in question.

We will use this information about the $D=2$ theory to help us determine the algebra of the new symmetries in $D < 2$.

4.2. Symmetry Algebra in $D < 2$

The dimension 1, ghost number zero currents $J_{n,n'}$ in the closed string theory (the one-form components in the terminology of section 2.2) are defined using the descent equations applied to the zero forms $\Omega$ given in equation (3.9):

\[ \Omega_{n,n'} \rightarrow J_{n,n'} \text{ via descent equations.} \]
We know from the explicit construction of the ground ring generators \[17\] that

\[
p_{n,n'} = -\left[(n - 1) + \left(\frac{p}{p'}\right)(n' - 1)\right] \frac{\gamma}{2} \quad \gamma = \sqrt{\frac{2p'}{p}}
\]  

(4.5)

is the Liouville momentum of \(J_{n,n'}\). Hence, by momentum conservation alone it is clear that the symmetry algebra must be of the form

\[
[J_{m,n}, J_{m',n'}] = F_{m+n+1,n+n-1} J_{m+m-1,n+n-1}
\]

(4.6)

where it is understood that \(J_{a,b} \equiv 0\) unless \(1 \leq a \leq p - 1\) and \(1 \leq b \leq p' - 1\). Since explicit forms of these operators are not available (however see \[29\] \[30\] for recent progress in constructing explicit representatives of BRST cohomology classes for \(c < 1\) matter coupled to gravity), we must find some means other than direct computation of determining the non-vanishing structure constants. This is where the interpretation of section 4.1 will be useful.

Just as in the D=2 theory, we can think of the ring in the \(D < 2\) case as having four chiral generators in terms of which \(X\) and \(Y\) are defined:

\[
X = R_{2,1} = xx' \quad Y = R_{1,2} = yy'.
\]

(4.7)

Of course, in view of the conditions in the \((p,p')\) model that \(X^{p-1} = Y^{p'-1} = 0\) we should also set

\[
x^{p-1} = x'^{p-1} = y^{p'-1} = y'^{p'-1} = 0.
\]

(4.8)

Now, by analogy with the D=2 theory we expect the new symmetries should act on the states of the \(D < 2\) models as polynomials in the new ring generators multiplied by the analogue of \(S\) (defined in equation (4.3)). Thus, we make the identification

\[
J_{n,n'} = f_{n,n'} S
\]

(4.9)

with the understanding that the variables \(x, x', y, y'\) in the definition of \(S\) are now to be interpreted as defined above for the \((p,p')\) model. Here, \(f_{n,n'}\) is a function with the same quantum numbers as the operator on the left hand side. As it is a polynomial function of the ring generators, we gather that \(f\) must have the form

\[
f_{n,n'} \sim X^{n-1}Y^{n'-1} = (xx')^{n-1}(yy')^{n'-1}.
\]

(4.10)

Hence, up to normalization we can find the current algebra by taking the commutators of the associated vector fields

\[
[J_{n,n'}, J_{m,m'}] = [f_{n,n'}S, f_{m,m'}S]
\]

(4.11)

with the result that

\[
[J_{n,n'}, J_{m,m'}] \sim (n + n' - m - m')J_{n+m-1,n'+m'-1}
\]

(4.12)

with the same understanding as after equation (4.6) about the indices on the currents.
4.3. Conjectured Algebra of Exotic Symmetries

As discussed in section 3.3, our construction of closed string symmetries will generalize to all cases where there is isolated chiral relative cohomology at a given ghost number and momentum. It is known from the mathematical analysis \[20\][11] that there is in fact \((p - 1)(p' - 1)\) dimensional relative cohomology at all negative ghost numbers in the \((p, p')\) model. If we denote the dimension one currents obtained from this cohomology at ghost number \(k\) as \(J^k_{n,n'}\), with \(1 \leq n \leq p - 1, 1 \leq n' \leq p' - 1\), then the natural conjecture for the extended symmetry algebra is simply

\[
[J^k_{n,n'}, J^j_{m,m'}] \sim (n + n' - m - m')J^{j+k}_{n+m-1,n'+m'-1} \tag{4.13}
\]

with \(j, k \leq 0\), and again with the implicit understanding that \(J \equiv 0\) if its indices do not lie in the standard allowed range for the \((p, p')\) model.

5. Action of Conventional Charges on States

5.1. States and Currents in \(D < 2\)

It is obvious from the form of \(S\), equation (4.3), that the most conventional symmetries we have constructed (those with vanishing total ghost number) annihilate the ring states. In fact, their associated charges also annihilate the normal ‘tachyon’ states obtained by dressing the primary fields of the minimal models.

For the \((p, p')\) model, the spectrum of tachyon states is

\[
T_{n,n'} = c\bar{c}O_{n,n'}e^{\left[1 + \frac{p-n}{p'}-\frac{p-n'}{p}\right]\phi} \tag{5.1}
\]

(with \(1 \leq n \leq p - 1, 1 \leq n' \leq p' - 1\)) where \(O_{n,n'}\) is a matter primary field with \(O_{n,n'} = O_{p-n,n'-n'}\) (so we identify \(T_{n,n'} = T_{p-n,n'-n'}\)). Of course, there are also \((p - 1)(p' - 1)\) special states at every negative value of the ghost number for which we have no such convenient representation.

To actually construct the charges

\[
Q_{n,n'} = \frac{1}{2\pi i} \oint J_{n,n'} \tag{5.2}
\]

associated with \(J_{n,n'}\), we first use the descent equations to get a more explicit form of \(J_{n,n'}\). If we define:

\[
\partial O_{n,n'} = \{Q_{TOT}, Z_{n,n'}\}, \quad \partial(aO_{n,n'}) = \{Q_{TOT}, Y_{n,n'}\} \tag{5.3}
\]

then the descent equations tell us that

\[
J_{n,n'} = (Y_{n,n'}\bar{O}_{n,n'} + Z_{n,n'}\bar{aO}_{n,n'})dz + (O_{n,n'}\bar{Y}_{n,n'} - aO_{n,n'}\bar{Z}_{n,n'})d\bar{z}. \tag{5.4}
\]

Using this form of \(J_{n,n'}\), we will argue that \(Q_{n,n'}\) must annihilate the tachyon states (5.1).
5.2. Action of Conventional Charges on Tachyon States

From (5.4) it follows immediately that

\[ Q_{n,n'} = \oint \frac{dz}{2\pi i} (Y_{n,n'} \bar{O}_{n,n'} + Z_{n,n'} \bar{a} \bar{O}_{n,n'}) + \oint \frac{dz}{2\pi i} (O_{n,n'} \bar{Y}_{n,n'} - aO_{n,n'} \bar{Z}_{n,n'}) \] (5.5)

are the conserved charges.

Notice that although \( Q_{n,n'} \) does indeed have vanishing total ghost number, this arises in a peculiar way: The terms containing \( Y \) and \( O \) have left-right ghost numbers \((0,0)\), but the terms containing \( Z \) and \( aO \) have left-right ghost numbers \((-1,1)\) and \((1,-1)\). While in the D=2 theory these types of operators can map special tachyons to the new states which contain parts of ghost number \((1,-1)\) and \((-1,1)\) (such states do exist in the D=2 theory, see [15]), there are no such states in \( D < 2 \). Hence, it follows immediately that the pieces in \( Q_{n,n'} \) of this type act trivially on the tachyon states in \( D < 2 \). So we need only consider the action of the terms containing \( Y \bar{O} \) and \( O \bar{Y} \).

However, the result of applying \( Q_{n,n'} \) to a tachyon state must be another tachyon state (there are no other candidates in \( D < 2 \)) and hence must be left-right symmetric. Recalling our conventions for contour integrals (see section 2.2), it is evident that any left-right symmetric pieces which emerge from applying the \( Y \bar{O} \) and \( O \bar{Y} \) terms to a tachyon state will exactly cancel each other. Hence, the charges \( Q_{n,n'} \) annihilate the tachyon states (5.1).

5.3. Are the Charges Physically Non-trivial?

Since we have seen that the conventional charges annihilate the ring states and the tachyon states, one might wonder if they are physically relevant. At any rate, it seems unlikely that the full extended symmetry algebra discussed in section 4.3 will act trivially on the entire tower of special states obtained in [20] [11], but we have no convincing evidence regarding this matter. If it does, one would have essentially another infinite tower of states (corresponding to semi-relative cohomology classes) in the \( D < 2 \) closed string theories which are effectively decoupled from the conventional states in the relative cohomology.
6. Conclusion

There are several striking features of the $c < 1$ matrix models which we would like to understand from the continuum perspective. In particular, the fact that the square root of the partition function $Z$ is a $\tau$ function of the KdV hierarchy $[31]$, and the emergence of an infinite number of Virasoro and W-constraints which also uniquely determine $Z$ $[32]$, are still mysterious from the perspective of Liouville theory coupled to minimal matter (however see $[17]$ for some interesting conjectures relating the ground ring to Douglas’ operators $P,Q$ $[31]$).

It seems plausible that the incorporation of all of the ‘exotic’ currents found here into an extended symmetry algebra, as discussed briefly in section 4.3, might shed some light on these issues. However, the role of these symmetries in the $D < 2$ theory is not yet well understood. We have argued that the most conventional ones act trivially on the ring and tachyon states, but the interesting question of the action of the full symmetry algebra on the entire tower of states in the $(p,p')$ model remains open.

To extract a deeper understanding of string theory from the matrix models, it is important that we reproduce as much of their structure from the continuum Liouville × Matter viewpoint as we can. Uncovering the KdV/Virasoro structure in the continuum formulation will help us to understand the ‘miracles’ of the matrix models in terms of conventional string theory, and is certainly a problem which warrants further study.

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