1. PROOFS OF PROPOSITIONS AND THEOREM 1

1.1. Conditions

We first list some of the conditions needed in the proofs of our asymptotic results.

(C1) The distortion functions satisfy \( \phi(u) > 0 \) and \( \psi(u) > 0 \) for all \( u \in [U_L, U_R] \), where \([U_L, U_R]\) denotes the compact support of \( U \). Moreover, the distortion functions \( \phi(u) \) and \( \psi(u) \) have third order continuous derivatives. The density function \( f_U(u) \) of the random variable \( U \) is bounded away from 0 and satisfies the Lipschitz condition of order 1 on \([U_L, U_R]\).

(C2) The density of \( X \), \( f_X(x) \) is bounded away from zero for all \( x \in [X_L, X_R] \), where \([X_L, X_R]\) denotes the compact support of \( X \). The functions \( g(x) \) and \( E[Y^2|X = x] \) have third order continuous derivatives on \([X_L, X_R]\).

(C3) The kernel function \( K(\cdot) \) is a univariate bounded continuous and symmetric density function about zero, satisfying that \( \int |t|^j K(t) dt < \infty \) for \( j = 1, 2, 3, 4 \), \( \mu_2 = \int t^2 K(t) dt \neq 0 \) and \( \mu_4 = \int t^4 K(t) dt \neq 0 \), moreover, \( \mu_{K^2} = \int K^2(t) dt > 0 \) and \( \mu_{K^2,2} = \int t^2 K^2(t) dt > 0 \). The second order derivative of \( K(\cdot) \) is bounded on \( \mathbb{R} \), satisfying a Lipschitz condition.

1.2. A technical lemma

Lemma 1. Suppose \( E(T|U = u) = m(u) \) and its derivatives up to the second order are bounded for all \( u \in \Omega \), where \( \Omega \) is defined in condition (A1). \( E[T^3] < \infty \) and \( \sup_u \int |t|^2 f(u, t) dt < \infty \) for any \( s > 2 \), where \( f(u, t) \) is the joint density of \( U \) and \( T \). Let \( (U_i, T_i), i = 1, 2, \ldots, n \) be independent and identically distributed (i.i.d.) samples from \((U, T)\). If (A1)-(A3) hold, and \( n^{2s-1}h \rightarrow \infty \) for \( s < 1 - \frac{3}{2} \), then

\[
\sup_{u \in \Omega} \left| \frac{1}{n} \sum_{i=1}^{n} K_h(U_i - u) T_i - f_U(u)m(u) - \frac{1}{2} \int [f_U(u)m(u)]'' u^2h^2 \right| = O(\tau_{n, h}), \text{a.s.}
\]

where \( \mu_2 = \int K(u)u^2 du \), \( \tau_{n, h} = h^3 + \sqrt{\log n/(nh)} \).

Proof. Lemma 1 can be immediately proven using the result obtained in Mack and Silverman (1982), see also in Lemma 7.1 of Fan and Huang (2005). \( \blacksquare \)
Lemma 2. Suppose that the Conditions (C1)-(C4) hold. Let $M(w)$ be a continuous function satisfying $EM^2(u) < \infty$. Then,

$$n^{-1} \sum_{i=1}^{n} \left( \hat{Y}_i - Y_i \right) M(W_i) = n^{-1} \sum_{i=1}^{n} \left( |\hat{Y}_i| - |Y_i| \right) \frac{E[YM(W)]}{E(|Y|)} + o_P(n^{-1/2})$$

$$n^{-1} \sum_{i=1}^{n} \left( \hat{X}_i - X_i \right) M(W_i) = n^{-1} \sum_{i=1}^{n} \left( |\hat{X}_i| - |X_i| \right) \frac{E[XM(W)]}{E(|X|)} + o_P(n^{-1/2}).$$

Proof. Lemma 2 is the result of Lemma B.2 in Zhang et al. (2012), see also in Zhao and Xie (2018).

1.3. Proof of Propositions 1-2

In the following, Step 1-Step 3 give the detailed proof of asymptotic expression of $\hat{g}(x) - g(x)$. Step 4 gives the detailed proof of asymptotic expression of $\hat{g}'(x) - g'(x)$. Step 5 gives the detailed proof of asymptotic expression of $\hat{E}(\phi^2(U)) - E(\phi^2(U))$. Step 6 is the detailed proof of asymptotic expression of $\hat{\sigma}_1^2 - \sigma_1^2$, and Step 7 is the detailed proof of asymptotic expression of $\hat{\sigma}_2^2(x) - \sigma_2^2(x)$.

Proof. Step (1). In this step, we first analyse $\hat{M}_{n,h}(x) = \frac{1}{nh_2} \sum_{i=1}^{n} \left( \frac{\hat{X}_i - x}{h_2} \right) \delta \left( \frac{\hat{X}_i - x}{h_2} \right) \hat{Y}_i$, $\delta = 0, 1$. Note that $|\hat{Y}| - E(|Y|) = O_P(n^{-1/2})$ and $|\hat{X}| - E(|X|) = O_P(n^{-1/2})$. Using the asymptotic results of local linear estimators (Fan and Gijbels; 1996), we have

$$\hat{\phi}(u) - \phi(u) = \frac{S_{n2}(u)V_{n0,|\hat{Y}|}(u) - S_{n1}(u)V_{n1,|\hat{Y}|}(u)}{S_{n2}(u)S_{n0}(u) - [S_{n1}(u)]^2} - \phi(u) \tag{A.1}$$

$$= \frac{1}{nh_1 f_U(u) E(|Y|)} \sum_{i=1}^{n} K \left( \frac{U_i - u}{h_1} \right) \left[ |\hat{Y}_i| - \phi(U_i) E(|Y|) \right] + \frac{\mu h_1^2}{2} \phi''(u) + O_P(h_1^2 + (nh_1)^{-1/2} + O_P(n^{-1/2}),$$

$$\hat{\psi}(u) - \psi(u) = \frac{S_{n2}(u)V_{n0,|\hat{X}|}(u) - S_{n1}(u)V_{n1,|\hat{X}|}(u)}{S_{n2}(u)S_{n0}(u) - [S_{n1}(u)]^2} - \psi(u) \tag{A.2}$$

$$= \frac{1}{nh_1 f_U(u) E(|X|)} \sum_{i=1}^{n} K \left( \frac{U_i - u}{h_1} \right) \left[ |\hat{X}_i| - \psi(U_i) E(|X|) \right] + \frac{\nu h_1^2}{2} \psi''(u) + O_P(h_1^2 + (nh_1)^{-1/2} + O_P(n^{-1/2}),$$

For $\delta = 0$, using (A.2), Taylor expansion entails that

$$\hat{M}_{n,0}(x) = \frac{1}{nh_2} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_2} \right) \hat{Y}_i + \frac{1}{nh_2} \sum_{i=1}^{n} \left( \frac{X_i - x}{h_2} \right) K' \left( \frac{X_i - x}{h_2} \right) \hat{Y}_i \tag{A.3}$$

$$+ O_P \left( h_1^4 + \frac{\log n}{nh_1} \right)$$

$$= \frac{1}{nh_2} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_2} \right) \hat{Y}_i + A_{n1}(x) + O_P \left( h_1^4 + \frac{\log n}{nh_1} \right).$$
Using Lemma 1, we have

\[ A_{n1}(x) = -\frac{1}{nh_2^n} \sum_{i=1}^{n} \left( \frac{\tilde{\psi}(U_i) - \psi(U_i)}{\psi(U_i)} \right) K' \left( \frac{X_i - x}{h_2} \right) X_i \tilde{Y}_i \]

\[ + \frac{1}{nh_2^n} \sum_{i=1}^{n} \left( \frac{[\tilde{\psi}(U_i) - \psi(U_i)]^2}{\psi^2(U_i)} \right) K' \left( \frac{X_i - x}{h_2} \right) X_i \tilde{Y}_i + O_P \left( h_2^{-1} \left( h_2^3 + \sqrt{\log n/nh_1} \right)^3 \right) \]

\[ = A_{n1,1}(x) + A_{n1,2}(x) + o_P \left( h_2^2 + (nh_2)^{-1/2} \right). \]

Using the condition (C3), we have \( \int K'(t) dt = 0, \int tK'(t) dt = -1, \int t^2K'(t) dt = 0 \) and \( \int t^3K'(t) dt = -3\mu_2. \)

Recalling that \( E(Y|X = x_0) = g(x_0), \) for any constant \( x_0 \in \mathcal{X}, \) we have

\[ E \left[ \frac{1}{h_2} K' \left( \frac{X - x_0}{h_2} \right) Xg(X) \right] = \int (x_0 + th_2)g(x_0 + th_2)f_X(x_0 + th_2)K'(t)dt \]

\[ = -h_2 \{ g(x_0)f_X(x_0) + x_0[g(x_0)f_X(x_0)]' \} \]

\[ -h_2^3 \mu_2 \left\{ \frac{3}{2} [g(x_0)f_X(x_0)]'' + \frac{x_0}{2}[g(x_0)f_X(x_0)]''' \right\} + O(h_2^5) \]

\[ \overset{\text{def}}{=} -h_2 s_1(x_0) - h_2^3 s_2(x_0) + O(h_2^5). \]

Using (A.2), Lemma 1 and the U-statistic’s property (Serfling; 1980) we have

\[ A_{n1,1}(x) = \frac{\mu_2 h_2^2}{2} E \left( \frac{\phi(U)\psi''(U)}{\psi(U)} \right) s_1(x) + h_2^3 s_2(x) + O(h_2^5) + O_P \left( h_2^3 h_2^{-1} \sqrt{\log n/nh_1} \right). \] (A.6)

Let \( \Delta_i = \frac{|X_i| - E(|X_i|)\psi(U_i)}{E(|X_i|)} \), then we have

\[ A_{n1,2}(x) = \frac{1}{nh_2^n} \sum_{s=1}^{n} \sum_{t=1}^{n} \sum_{i=1}^{n} \tilde{Y}_i X_i \left( \frac{X_i - x}{h_2} \right) K \left( \frac{U_s - U_t}{h_1} \right) K \left( \frac{U_t - U_i}{h_1} \right) \Delta_i \Delta_t \] (A.7)

\[ + \frac{\mu_2 h_2^2}{nh_2} \sum_{s=1}^{n} \sum_{i=1}^{n} \tilde{Y}_i X_i \left( \frac{X_i - x}{h_2} \right) K \left( \frac{U_s - U_i}{h_1} \right) \Delta_s \]

\[ + \frac{\mu_2 h_2^2}{nh_2} \sum_{i=1}^{n} \frac{\psi''(U_i) s_2(X_i)}{\psi^2(U_i)} K' \left( \frac{X_i - x}{h_2} \right) + o_P(h_2^2 + (nh_2)^{-1/2}). \]

Similar to (A.5)-(A.6), the third term in (A.7) is of order \( O_P(h_2^2). \) The U-statistic method in Serfling (1980) entails that the second term of (A.7) is of order \( O_P(n^{-1/2}h_2^2). \) The second order of the projection of U-statistic entails that the first term is \( O_P(\frac{1}{nh_1}) = O_P((nh_2)^{-1/2}) as \frac{h_2}{nh_1} \to 0. \) Consequently, we have that \( A_{n1,2}(x) = o_P((nh_2)^{-1/2}). \)

As \( h_2^2 h_2^{-1} \{ \log n \} \to 0, h_2^4 h_2^{-2} \to 0 \) and \( \frac{h_2^{1/2} \log n}{nh_1} \to 0, \) we have

\[ \tilde{M}_{n0,y}(x) = \frac{1}{nh_2} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_2} \right) \tilde{Y}_i + \frac{\mu_2 h_2^2}{2} E \left( \frac{\phi(U)\psi''(U)}{\psi(U)} \right) s_1(x) \] (A.8)

\[ + o_P \left( h_2^2 + (nh_2)^{-1/2} \right) \]

\[ \overset{\text{def}}{=} M_{n0,y}(x) + \frac{\mu_2 h_2^2}{2} E \left( \frac{\phi(U)\psi''(U)}{\psi(U)} \right) s_1(x) + o_P \left( h_2^2 + (nh_2)^{-1/2} \right). \]
For $\delta = 1$, using (A.2), Taylor expansion entails that

$$
\hat{M}_{n_1,Y}(x) = \frac{1}{nh_2} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_2} \right) \left( \frac{X_i - x}{h_2} \right) \tilde{Y}_i
+ \frac{1}{nh_2} \sum_{i=1}^{n} \left( \frac{X_i - X_i}{h_2} \right) \left\{ K' \left( \frac{X_i - x}{h_2} \right) \left( \frac{X_i - x}{h_2} \right) + K \left( \frac{X_i - x}{h_2} \right) \right\} \tilde{Y}_i
+ O_P \left( h_1^4 + \frac{\log n}{n h_1} \right)
$$

(A.9)

Moreover,

$$
\hat{B}_{n_1}(x) = \frac{1}{nh_2} \sum_{i=1}^{n} \left( \frac{X_i - X_i}{h_2} \right) \left\{ K' \left( \frac{X_i - x}{h_2} \right) \left( \frac{X_i - x}{h_2} \right) + K \left( \frac{X_i - x}{h_2} \right) \right\} \tilde{Y}_i
+ O_P \left( h_2^2 + (nh_2)^{-1/2} \right)
$$

(A.10)

Using (A.4), (A.10) and (A.11), we have

$$
\hat{B}_{n_1}(x) = \frac{h_2^2}{2} E \left( \frac{\phi(U) \psi''(U)}{\psi(U)} \right) s_3(x) h_2 + O_P(n^{-1/2}h_2) + O_P(h_2^2 + (nh_2)^{-1/2}).
$$

(A.12)

Consequently, together with (A.9), (A.10) and (A.12), we have

$$
\hat{M}_{n_1,Y}(x) = \frac{1}{nh_2} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_2} \right) \left( \frac{X_i - x}{h_2} \right) \tilde{Y}_i
+ \frac{\mu_2 h_1^2}{2} E \left( \frac{\phi(U) \psi''(U)}{\psi(U)} \right) s_3(x) h_2 + O_P(h_2^2 + (nh_2)^{-1/2})
$$

(A.13)

Step (2). In this step, we analyse $\hat{Q}_{uw}(x) = \frac{1}{nh_2} \sum_{i=1}^{n} \left( \frac{X_i - x}{h_2} \right)^w K \left( \frac{X_i - x}{h_2} \right), w = 0, 1, 2$. 


For \( w = 0 \), similar to (A.8) and (A.13), we have

\[
\hat{Q}_{n0}(x) = \frac{1}{nh_2} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_2} \right) \left( \frac{X_i - x}{h_2} \right) + \frac{1}{nh_2} \sum_{i=1}^{n} K' \left( \frac{X_i - x}{h_2} \right) \left( \frac{\hat{X}_i - X_i}{h_2} \right) + o_P(h_2^2 + (nh_2)^{-1/2})
\]

\[
= \frac{1}{nh_2} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_2} \right) + \mu_2 h_2^2 \frac{E}{2} \left( \frac{\psi''(U)}{\psi(U)} \right) (xf'_X(x) + f_X(x)) + o_P(h_2^2 + (nh_2)^{-1/2})
\]  

\[
\def \hat{Q}_{n0}(x) + \frac{\mu_2 h_2^2}{2} E \left( \frac{\psi''(U)}{\psi(U)} \right) s_4(x) + o_P(h_2^2 + (nh_2)^{-1/2}).
\]  

For \( w = 1 \), we have

\[
\hat{Q}_{n1}(x) = \frac{1}{nh_2} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_2} \right) \left( \frac{X_i - x}{h_2} \right) + \frac{1}{nh_2} \sum_{i=1}^{n} \left\{ K' \left( \frac{X_i - x}{h_2} \right) \left( \frac{X_i - x}{h_2} \right) + K \left( \frac{X_i - x}{h_2} \right) \right\} \left( \hat{X}_i - X_i \right)
\]

\[
+ o_P(h_2^2 + (nh_2)^{-1/2})
\]

\[
= \frac{1}{nh_2} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_2} \right) \left( \frac{X_i - x}{h_2} \right) + C_{n1}(x) + o_P(h_2^2 + (nh_2)^{-1/2})
\]  

Similar to (A.10), we have

\[
C_{n1}(x) = \frac{\mu_2 h_2^2}{2} E \left( \frac{\psi''(U)}{\psi(U)} \right) (xf'_X(x) + 2f_X(x)) + o_P(h_2^2 + (nh_2)^{-1/2})
\]

\[
= \frac{\mu_2 h_2^2}{2} E \left( \frac{\psi''(U)}{\psi(U)} \right) s_5(x) h_2 + o_P(h_2^2 + (nh_2)^{-1/2}).
\]  

Thus, we have

\[
\hat{Q}_{n1}(x) = \frac{1}{nh_2} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_2} \right) \left( \frac{X_i - x}{h_2} \right) + \frac{\mu_2 h_2^2}{2} E \left( \frac{\psi''(U)}{\psi(U)} \right) s_5(x) h_2 + o_P(h_2^2 + (nh_2)^{-1/2})
\]

\[
\def \hat{Q}_{n1}(x) + \frac{\mu_2 h_2^2}{2} E \left( \frac{\psi''(U)}{\psi(U)} \right) s_5(x) h_2 + o_P(h_2^2 + (nh_2)^{-1/2}).
\]  

For \( w = 2 \), we have

\[
\hat{Q}_{n2}(x) = \frac{1}{nh_2} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_2} \right) \left( \frac{X_i - x}{h_2} \right)^2
\]

\[
+ \frac{1}{nh_2} \sum_{i=1}^{n} \left\{ K' \left( \frac{X_i - x}{h_2} \right) \left( \frac{X_i - x}{h_2} \right)^2 + 2K \left( \frac{X_i - x}{h_2} \right) \left( \frac{X_i - x}{h_2} \right) \right\} \left( \frac{\hat{X}_i - X_i}{h_2} \right)
\]

\[
+ o_P(h_2^2 + (nh_2)^{-1/2})
\]

\[
= \frac{1}{nh_2} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_2} \right) \left( \frac{X_i - x}{h_2} \right)^2 + D_{n2}(x) + o_P(h_2^2 + (nh_2)^{-1/2}).
\]
Similar to (A.10) and (A.16), we have

$$D_{n1}(x) = \frac{\mu h^2}{2} E \left( \frac{\psi''(U)}{\psi(U)} \right) \mu_2(xf'_X(x) + f_X(x)) + o_P(h^2_2 + (nh_2)^{-1/2})$$  \hspace{1cm} (A.19)

$$= \frac{\mu^2 h^2}{2} E \left( \frac{\psi''(U)}{\psi(U)} \right) s_4(x) + o_P(h^2_2 + (nh_2)^{-1/2}).$$

Then, we have

$$Q_{n2}(x) = \frac{1}{nh_2} \sum_{i=1}^n K \left( \frac{X_i - x}{h_2} \right) \left( \frac{X_i - x}{h_2} \right)^2$$  \hspace{1cm} (A.20)

$$+ \frac{\mu^2 h^2}{2} E \left( \frac{\psi''(U)}{\psi(U)} \right) s_4(x) + o_P(h^2_2 + (nh_2)^{-1/2})$$

$$= Q_{n2}(x) + \frac{\mu^2 h^2}{2} E \left( \frac{\psi''(U)}{\psi(U)} \right) s_4(x) + o_P(h^2_2 + (nh_2)^{-1/2}).$$

**Step 3.** Using these asymptotic expressions obtained in Step 1 and Step 2, we have

$$\hat{g}(x) = \frac{Q_{n2}(x)M_{n0, \hat{Y}}(x) - Q_{n1}(x)M_{n1, \hat{Y}}(x)}{Q_{n2}(x)Q_{n0}(x) - \left[ Q_{n1}(x) \right]^2}. \hspace{1cm} (A.21)$$

Using Lemma 1, we have

$$\hat{M}_{n0, \hat{Y}}(x) - \hat{Q}_{n0}(x)g(x)$$  \hspace{1cm} (A.22)

$$= M_{n0, \hat{Y}}(x) - Q_{n0}(x)g(x)$$

$$+ \frac{\mu^2 h^2}{2} \left\{ E \left( \frac{\psi''(U)}{\psi(U)} \right) s_1(x) - E \left( \frac{\psi''(U)}{\psi(U)} \right) s_4(x)g(x) \right\} + o_P \left( h^2_2 + (nh_2)^{-1/2} \right)$$

$$= \frac{1}{nh_2} \sum_{i=1}^n K \left( \frac{X_i - x}{h_2} \right) \left[ Y_i - g(X_i) \right] + \frac{1}{nh_2} \sum_{i=1}^n K \left( \frac{X_i - x}{h_2} \right) \left[ g(X_i) - g(x) \right]$$

$$+ \frac{\mu^2 h^2}{2} \left\{ E \left( \frac{\psi''(U)}{\psi(U)} \right) s_1(x) - E \left( \frac{\psi''(U)}{\psi(U)} \right) s_4(x)g(x) \right\} + o_P \left( h^2_2 + (nh_2)^{-1/2} \right)$$

$$= \frac{1}{nh_2} \sum_{i=1}^n K \left( \frac{X_i - x}{h_2} \right) \left[ Y_i - g(X_i) \right] + h^2_2 \mu_2 \left[ g'(x)f'_X(x) + \frac{1}{2} g''(x)f_X(x) \right]$$

$$+ \frac{\mu^2 h^2}{2} \left\{ E \left( \frac{\psi''(U)}{\psi(U)} \right) s_1(x) - E \left( \frac{\psi''(U)}{\psi(U)} \right) s_4(x)g(x) \right\}$$

$$+ O_P(h^2_2) + o_P \left( h^2_2 + (nh_2)^{-1/2} \right).$$

Using (A.20) and Lemma 1, we have

$$\hat{Q}_{n2}(x) = \mu_2 f_X(x) + \frac{h^2}{2} f''_X(x) \int u^4 K(u)du + \frac{\mu h^2}{2} E \left( \frac{\psi''(U)}{\psi(U)} \right) s_4(x) \hspace{1cm} (A.23)$$

$$+ o_P(h^2_2 + (nh_2)^{-1/2}).$$

Together with (A.22) and (A.23), we have

$$\hat{Q}_{n2}(x) [ M_{n0, \hat{Y}}(x) - \hat{Q}_{n0}(x)g(x) ]$$  \hspace{1cm} (A.24)

$$= \frac{\mu_2 f_X(x)}{nh_2} \sum_{i=1}^n K \left( \frac{X_i - x}{h_2} \right) \left[ Y_i - g(X_i) \right] + h^2_2 \mu_2 f_X(x) \left[ g'(x)f'_X(x) + \frac{1}{2} g''(x)f_X(x) \right]$$

$$+ \frac{\mu^2 h^2}{2} \left\{ E \left( \frac{\psi''(U)}{\psi(U)} \right) s_1(x) - E \left( \frac{\psi''(U)}{\psi(U)} \right) s_4(x)g(x) \right\}$$

$$+ o_P(h^2_2 + (nh_2)^{-1/2}).$$
Moreover, similar to (A.22), we have
\[
\hat{M}_{n_1, \hat{Y}}(x) - \hat{Q}_{n_1}(x)g(x) = \hat{M}_{n_1, \hat{Y}}(x) - Q_{n_1}(x)g(x)
\]
\[
= M_{n_1, \hat{Y}}(x) - Q_{n_1}(x)g(x)
\]
\[
= \frac{\mu_2 h_2^3}{2} \left\{ E \left( \frac{\phi(U)\psi''(U)}{\psi(U)} \right) s_3(x) - E \left( \frac{\psi''(U)}{\psi(U)} \right) s_5(x)g(x) \right\} + o_P(h_2^4 + (nh_2)^{-1/2})
\]
\[
= \frac{1}{nh_2} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_2} \right) \left[ \hat{Y}_i - g(X_i) \right]
\]
\[
= g'(x) f_X(x) \mu_2 h_2 + \frac{\mu_2 h_2^3}{2} \left\{ E \left( \frac{\phi(U)\psi''(U)}{\psi(U)} \right) s_3(x) - E \left( \frac{\psi''(U)}{\psi(U)} \right) s_5(x)g(x) \right\}
\]
\[
= o_P(h_2^3 + (nh_2)^{-1/2}).
\]
Together with (A.17) and (A.25), we have
\[
\hat{Q}_{n_1}(x) \left[ \hat{M}_{n_1, \hat{Y}}(x) - \hat{Q}_{n_1}(x)g(x) \right]
\]
\[
= g'(x) f_X(x) f_X(x) \mu_2 h_2^2 + o_P(h_2^4 h_2^2 + h_2^3) + o_P(h_2^2 + (nh_2)^{-1/2}).
\]
Consequently, using (A.24) and (A.26), we have
\[
\hat{Q}_{n_2}(x) \left[ \hat{M}_{n_1, \hat{Y}}(x) - \hat{Q}_{n_1}(x)g(x) \right] - \hat{Q}_{n_1}(x) \left[ \hat{M}_{n_1, \hat{Y}}(x) - \hat{Q}_{n_1}(x)g(x) \right]
\]
\[
= \frac{\mu_2 f_X(x)}{nh_2} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_2} \right) \left[ \hat{Y}_i - g(X_i) \right] + \frac{\mu_2 h_2^3}{2} f_X(x) g''(x)
\]
\[
+ \frac{\mu_2^2 h_2^2}{2} f_X(x) \left\{ E \left( \frac{\phi(U)\psi''(U)}{\psi(U)} \right) s_1(x) - E \left( \frac{\psi''(U)}{\psi(U)} \right) s_4(x)g(x) \right\}
\]
\[
+ o_P(h_2^3 + (nh_2)^{-1/2}).
\]
Similar to (A.27), we have
\[
\hat{Q}_{n_2}(x) \hat{Q}_{n_0}(x) = \left[ \hat{Q}_{n_1}(x) \right]^2
\]
\[
= \mu_2 f_X(x) + \frac{h_2^3}{2} f_X(x) f_X(x) + f_X(x) f_X(x) \int u^2 K(u)du - \frac{h_2^4}{2} \int f_X'(x)^4 u^2 K(u)du + \frac{\mu_2^2 h_2^2}{2} f_X(x) g''(x)
\]
\[
+ \mu_2^2 h_2^2 E \left( \frac{\phi(U)\psi''(U)}{\psi(U)} \right) s_4(x) f_X(x) + o_P(h_2^3 + (nh_2)^{-1/2})
\]
\[
= \mu_2 f_X(x)
\]
Together with (A.27) and (A.28), we have
\[
\hat{Q}_{n_2}(x) \left[ \hat{M}_{n_0, \hat{Y}}(x) - \hat{Q}_{n_0}(x)g(x) \right] - \hat{Q}_{n_1}(x) \left[ \hat{M}_{n_1, \hat{Y}}(x) - \hat{Q}_{n_1}(x)g(x) \right]
\]
\[
= \frac{\mu_2 f_X(x)}{nh_2} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_2} \right) \left[ \hat{Y}_i - g(X_i) \right] + \frac{\mu_2 h_2^3}{2} f_X(x) g''(x)
\]
\[
+ \frac{\mu_2^2 h_2^2}{2} f_X(x) \left\{ E \left( \frac{\phi(U)\psi''(U)}{\psi(U)} \right) s_1(x) - E \left( \frac{\psi''(U)}{\psi(U)} \right) s_4(x)g(x) \right\}
\]
\[
+ o_P(h_2^3 + (nh_2)^{-1/2}).
\]
Recalling the definitions of $s_1(x)$ and $s_4(x)$ in (A.5) and (A.14), we have $\frac{s_1(x)}{f_X(x)} = g(x) + xg'(x) + xg(x)\frac{f_X(x)}{f_X(x)}$. 7
\[
\frac{s_n(x)}{f_X(x)} = 1 + x f'_X(x).
\]
Thus, from (A.29), we have
\[
\sqrt{nh_2} \left( \hat{g}(x) - g(x) - \frac{\mu_2 h_2^2}{2} g''(x) - \frac{\mu_3 h_2^2}{2} E \left( \frac{\phi(U)\psi''(U)}{\psi(U)} \right) m_1(x) - E \left( \frac{\psi''(U)}{\psi(U)} \right) m_2(x) \right) \\
= \frac{1}{\sqrt{nh_2 f(x)}} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_2} \right) \left[ \hat{Y}_i - g(X_i) \right] + o_P(1)
\]
(A.30)
\[
\leadsto N \left( 0, \mu_{K^2} \left\{ \Var(\phi(U)) E(Y^2|X = x) + \Var(Y|X = x) \right\} \right) ,
\]
where \( \mu_{K^2} = \int K^2(u) du \), \( m_1(x) = g(x) + x g'(x) + x g(x) \frac{f(x)}{f_X(x)} \), \( m_2(x) = g(x) + x g(x) \frac{f'(x)}{f_X(x)} \).

**Step 4.** In the following, we define \( \mu_d = \int u^d K(u) du \) for \( d = 2m, \ m = 1, 2, \ldots \). For the estimator of \( \hat{g}'(x) \), we have
\[
\hat{h}_2 \left[ \hat{g}'(x) - g'(x) \right] = Q_{n_0} \left( \hat{M}_{n_1, \hat{X}}(x) - 2 Q_{n_2} h_2 g'(x) \right) - Q_{n_1} \left( \hat{M}_{n_0, \hat{X}}(x) - 2 Q_{n_1} h_2 g'(x) \right) \left[ Q_{n_2}(x) Q_{n_0} - \left[ Q_{n_1}(x) \right]^2 \right] .
\]
(A.31)

Similar to (A.22), we have
\[
\hat{M}_{n_0, \hat{X}}(x) - Q_{n_1}(x) h_2 g'(x) \\
= M_{n_0, \hat{X}}(x) - Q_{n_1}(x) h_2 g'(x) \left( \mu_2 h_2^2 \frac{E \left( \phi(U)\psi''(U) \psi(U) \right) s_1(x) - \mu_3 h_2^2 E \left( \frac{\psi''(U)}{\psi(U)} \right) s_5(x) h_2^2 g'(x) + o_P(h_2^2 + (nh_2)^{-1/2}) \right) \\
+ \frac{\mu_2 h_2^2}{2} E \left( \frac{\phi(U)\psi''(U)}{\psi(U)} \right) s_1(x) - \mu_2 h_2^2 E \left( \frac{\psi''(U)}{\psi(U)} \right) s_5(x) h_2^2 g'(x) + o_P(h_2^2 + (nh_2)^{-1/2}) \\
= 1 \frac{1}{nh_2} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_2} \right) \left[ \hat{Y}_i - g(X_i) \right] + \frac{1}{nh_2} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_2} \right) \left[ g(X_i) - (X_i - x) g'(x) \right] \\
+ \frac{\mu_2 h_2^2}{2} E \left( \frac{\phi(U)\psi''(U)}{\psi(U)} \right) s_1(x) - \mu_2 h_2^2 E \left( \frac{\psi''(U)}{\psi(U)} \right) s_5(x) h_2^2 g'(x) + o_P(h_2^2 + (nh_2)^{-1/2}) \\
+ g''(x) f_X(x) + \frac{\mu_2 h_2^2}{12} \left[ 3 g''(x) f''_X(x) + 2 g'''(x) f'_X(x) \right] + \frac{\mu_2 h_2^6}{36} g''''(x) f''_X(x) \\
+ \frac{\mu_2 h_2^2}{2} E \left( \frac{\phi(U)\psi''(U)}{\psi(U)} \right) s_1(x) - \mu_2 h_2^2 E \left( \frac{\psi''(U)}{\psi(U)} \right) s_5(x) h_2^2 g'(x) + o_P(h_2^2 + (nh_2)^{-1/2}) .
\]
From (A.17), we have
\[
\hat{Q}_{n_1}(x) \\
= f'_X(x) \mu_2 h_2 + \frac{\mu_4}{6} f''_X(x) h_2^2 + \frac{\mu_2 h_2^2}{2} E \left( \frac{\psi''(U)}{\psi(U)} \right) s_5(x) h_2 + o_P(h_2^2 + (nh_2)^{-1/2}) + O_P(h_2^2) .
\]
(A.33)
From (A.32)-(A.33), we have

\[
\hat{Q}_{n1}(x) \left[ \hat{M}_{n0, \hat{Y}}(x) - \hat{Q}_{n1}(x) h_{2g'}(x) \right]
= \frac{f_X(x)\mu_2}{n} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_2} \right) \left[ \hat{Y}_i - g(X_i) \right] + f_X(x)f_X(x)g(x)\mu_2 h_2
+ \frac{\mu_3^2 h_3^2}{2} f_X(x)[g(x)f''_X(x) + g''(x)f_X(x)] + \frac{\mu_4}{6} f_X(x)f_X(x)g(x)h_3^2
+ \frac{\mu_2 h_3^2}{2} E \left( \frac{\phi(U)\psi''(U)}{\psi(U)} \right) s_3(x) + \frac{\mu_2 h_3^2 h_2}{2} f_X(x)g(x)E \left( \frac{\psi''(U)}{\psi(U)} \right) s_5(x)
+ o_p(h_3^2 + (nh_2)^{-1/2}).
\]

Similar to (A.25), we have

\[
\hat{M}_{n1, \hat{Y}}(x) - \hat{Q}_{n2}(x) h_{2g'}(x)
= M_{n1, \hat{Y}}(x) - Q_{n2}(x) h_{2g'}(x)
\]

\[
= M_{n1, \hat{Y}}(x) - Q_{n2}(x) h_{2g'}(x)
+ \frac{\mu_2 h_3^2}{2} E \left( \frac{\phi(U)\psi''(U)}{\psi(U)} \right) s_3(x)h_2 - \frac{\mu_3^2 h_3^2}{2} E \left( \frac{\psi''(U)}{\psi(U)} \right) s_4(x)h_2g'(x) + o_p(h_3^2 + (nh_2)^{-1/2})
\]

\[
= \frac{1}{nh_2} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_2} \right) \left( \frac{X_i - x}{h_2} \right) \left[ \hat{Y}_i - g(X_i) \right]
+ \frac{1}{nh_2} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_2} \right) \left( \frac{X_i - x}{h_2} \right) [g(X_i) - (X_i - x)g'(x)]
+ \frac{\mu_2 h_3^2}{2} E \left( \frac{\phi(U)\psi''(U)}{\psi(U)} \right) s_3(x)h_2 - \frac{\mu_3^2 h_3^2}{2} E \left( \frac{\psi''(U)}{\psi(U)} \right) s_4(x)h_2g'(x) + o_p(h_3^2 + (nh_2)^{-1/2})
\]

\[
= \frac{1}{nh_2} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_2} \right) \left( \frac{X_i - x}{h_2} \right) \left[ \hat{Y}_i - g(X_i) \right]
+ g(x)f_X(x)\mu_2 h_2 + \frac{1}{6} g(x)f_X(x)\mu_4 h_2^2 + \frac{1}{2} g''(x)f_X(x)\mu_4 h_2^2 + \frac{1}{6} g'''(x)f_X(x)\mu_4 h_2^2
+ \frac{\mu_2 h_3^2 h_2}{2} E \left( \frac{\phi(U)\psi''(U)}{\psi(U)} \right) s_3(x) - \frac{\mu_3^2 h_3^2 h_2}{2} E \left( \frac{\psi''(U)}{\psi(U)} \right) s_4(x)g'(x) + o_p(h_3^2 + (nh_2)^{-1/2}).
\]

Using (A.14), we have

\[
\hat{Q}_{n0}(x) = f_X(x) + \frac{\mu_2}{2} f_X''(x)h_2^2 + \frac{\mu_2 h_2^2}{2} E \left( \frac{\psi''(U)}{\psi(U)} \right) s_4(x) + o_p(h_3^2 + (nh_2)^{-1/2}).
\]

Together with (A.35)-(A.36)

\[
\hat{Q}_{n0}(x) \left[ \hat{M}_{n1, \hat{Y}}(x) - \hat{Q}_{n2}(x) h_{2g'}(x) \right]
= \frac{f_X(x)}{nh_2} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_2} \right) \left( \frac{X_i - x}{h_2} \right) \left[ \hat{Y}_i - g(X_i) \right] + g(x)f_X(x)f_X(x)\mu_2 h_2

+ \mu_4 h_2 f_X(x) \left[ \frac{1}{6} g(x)f_X''(x) + \frac{1}{2} g''(x)f_X'(x) + \frac{1}{6} g'''(x)f_X(x) \right]
+ \frac{\mu_2 h_3^2 h_2}{2} f_X(x)E \left( \frac{\phi(U)\psi''(U)}{\psi(U)} \right) s_3(x) - \frac{\mu_3^2 h_3^2 h_2}{2} f_X(x)E \left( \frac{\psi''(U)}{\psi(U)} \right) s_4(x)g'(x)

+ \frac{\mu_2}{2} g(x)f_X''(x)h_2^2 + \frac{\mu_2 h_2^2 h_2}{2} E \left( \frac{\psi''(U)}{\psi(U)} \right) s_4(x)g'(x) + o_p(h_3^2 + (nh_2)^{-1/2}).
\]
From (A.34) and (A.37), we have

\[
\dot{\bar{Q}}_{n0}(x) \left[ \bar{M}_{n1}\bar{\gamma}(x) - \bar{Q}_{n2}(x)h_2g'(x) \right] - \dot{\bar{Q}}_{n1}(x) \left[ \bar{M}_{n0}\bar{\gamma}(x) - \bar{Q}_{n1}(x)h_2g'(x) \right] = \frac{f_X(x)}{n h_2} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_2} \right) \left( \frac{X_i - x}{h_2} \right) \left[ \bar{Y}_i - g(X_i) \right] + \left[ \frac{\mu_4 - \mu_2^2}{2} \bar{f}'_X(x)f_X(x)g''(x) + \frac{\mu_4}{6} g'''(x)f\_X^2(x) \right] h_2^3 + \frac{\mu_2 h_2^2 h_2}{2} E \left( \frac{\bar{\phi}(U) \bar{\psi}''(U)}{\bar{\psi}(U)} \right) \left[ \bar{f}'_X(x)f_X(x) - \mu_2 s_4(x)g'(x)f_X(x) + s_5(x)g(x)f_X(x) \right] + o_P(h_2^3 + (nh_2)^{-1/2}).
\]

(A.38)

Note that

\[
v_1(x) \overset{\text{def}}{=} \frac{1}{\mu_2 f\_X^2(x)} \left\{ f_X(x)s_1(x) - f\_X^2(x)s_1(x) \mu_2 \right\} = x \left[ g''(x) + \frac{g''(x)}{f_X(x)} - \frac{g(x)f\_X'(x)}{f_X^2(x)} \right] + \frac{g(x)f\_X'(x)}{f_X(x)} + 2g'(x)
\]

(A.39)

\[
v_2(x) \overset{\text{def}}{=} \frac{1}{\mu_2 f\_X^2(x)} \left\{ \mu_2 s_4(x)g(x)f_X(x) - \mu_2 s_4(x)g'(x)f_X(x) - s_5(x)g(x)f_X(x) \right\} = x \left[ \frac{g(x)f\_X'(x)}{f_X(x)} - \frac{g(x)f\_X'(x)}{f_X^2(x)} - \frac{g(x)f\_X(x)}{f_X(x)} \right] - \frac{g(x)f\_X(x)}{f_X(x)} - g'(x).
\]

(A.40)

From (A.28), we have

\[
\dot{\bar{Q}}_{n2}(x)\bar{Q}_{n0}(x) - \left[ \bar{Q}_{n1}(x) \right]^2 \rightarrow^P \mu_2 f\_X^2(x).
\]

Together with (A.31),(A.38)-(A.40), we have

\[
h_2 \left[ g'(x) - g'(x) \right] = \frac{\dot{\bar{Q}}_{n0}(x) \left[ \bar{M}_{n1}\bar{\gamma}(x) - \bar{Q}_{n2}(x)h_2g'(x) \right] - \dot{\bar{Q}}_{n1}(x) \left[ \bar{M}_{n0}\bar{\gamma}(x) - \bar{Q}_{n1}(x)h_2g'(x) \right]}{\dot{\bar{Q}}_{n2}(x)\bar{Q}_{n0}(x) - \left[ \bar{Q}_{n1}(x) \right]^2} = \frac{1}{nh_2 f_X(x) \mu_2} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_2} \right) \left( \frac{X_i - x}{h_2} \right) \left[ \bar{Y}_i - g(X_i) \right] + \left[ \frac{\mu_4 - \mu_2^2}{2} \bar{f}'_X(x)f_X(x)g''(x) + \frac{\mu_4}{6} g'''(x)f\_X^2(x) \right] h_2^3 + \frac{\mu_2 h_2^2 h_2}{2} E \left( \frac{\bar{\phi}(U) \bar{\psi}''(U)}{\bar{\psi}(U)} \right) v_1(x) + \frac{\mu_2 h_2^2 h_2}{2} E \left( \frac{\bar{\phi}(U) \bar{\psi}''(U)}{\bar{\psi}(U)} \right) v_2(x) + o_P(h_2^3 + (nh_2)^{-1/2}).
\]

(A.41)
Let \( \mu_{K^2,t} = \int t^2 K^2(t)dt \). From (A.41), we have

\[
\sqrt{nh^2} \left\{ \hat{g}'(x) - g'(x) - \left[ \frac{\mu_4 - \mu_4^2 f_X(x)g''(x)}{2\mu_2} + \frac{\mu_4\hat{g}''(x)}{\hat{\mu}_2} \right] h^2 \right\}
\]

\[
= \frac{1}{\sqrt{nh^2f_X(x)\mu_2}} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h^2} \right) \left( \frac{X_i - x}{h^2} \right) \left[ \hat{g}(x) - g(x) \right] + o_p(1)
\]

\[
\rightarrow N \left( 0, \mu_{K^2,t} \left\{ \text{Var}(\hat{g}(U))E(Y^2|X = x) + \text{Var}(Y|X = x) \right\} \right).
\]

**Step 5.** Using (A.1) and U-statistic method in Serfling (1980), as \( \frac{\log^2 n}{nh^4} \rightarrow 0 \) and \( nh^4 \rightarrow 0 \), we have

\[
\hat{E}[\phi^2(U)] = \frac{1}{n} \sum_{i=1}^{n} \phi^2(U_i) + \frac{2}{E([Y])} \sum_{i=1}^{n} \phi^2(U_i) \left[ |Y_i| - E([Y]) \right] + o_p(n^{-1/2}).
\]

From (A.43), we have

\[
\sqrt{n} \left( \hat{E}[\phi^2(U)] - E[\phi^2(U)] \right)
\]

\[
= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ \phi^2(U_i) - E[\phi^2(U_i)] \right\} + \frac{2}{\sqrt{n}E([Y])} \sum_{i=1}^{n} \phi^2(U_i) \left[ |Y_i| - E([Y]) \right] + o_p(1)
\]

\[
= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ \phi^2(U_i) \left[ \frac{2|Y_i|}{E([Y])} - 1 \right] - E[\phi^2(U_i)] \right\} + o_p(1)
\]

\[
\rightarrow N \left( 0, \text{Var} \left( \phi^2(U) \left[ \frac{2|Y|}{E([Y])} - 1 \right] \right) \right).
\]

**Step 6.** Using Lemma 2, we have

\[
\frac{1}{n} \sum_{i=1}^{n} (\hat{X}_i - X_i) = \frac{1}{n} \sum_{i=1}^{n} \left( |\hat{X}_i| - |X_i| \right) \frac{E(X)}{E(|X|)} + o_p(n^{-1/2})
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} (\hat{X}_i - X_i)(X_i - E(X)) = \frac{1}{n} \sum_{i=1}^{n} \left( |\hat{X}_i| - |X_i| \right) \frac{E(X(X - E(X)))}{E(|X|)} + o_p(n^{-1/2})
\]

Using (A.2) and Lemma 1, we have

\[
\frac{1}{n} \sum_{i=1}^{n} (\hat{X}_i - X_i)^2 = \frac{1}{n} \sum_{i=1}^{n} \frac{\hat{X}_i^2}{\hat{\psi}(U_i)} + o_p \left( h^4 + \frac{\log n}{nh^4} \right)
\]

\[
= O_p \left( h^4 + \frac{\log n}{nh^4} \right) = o_p(n^{-1/2}).
\]
Together with (A.45)-(A.47), \( \frac{1}{n} \sum_{i=1}^{n} \bar{X}_i - E(X) = O_P(n^{-1/2}) \) and \( \frac{1}{n} \sum_{i=1}^{n} X_i - E(X) = O_P(n^{-1/2}) \), we have

\[
\hat{\sigma}_1^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - E(X))^2 + \frac{1}{n} \sum_{i=1}^{n} (\bar{X}_i - X_i)^2 + \left( E(X) - \bar{X} \right)^2 \\
+ 2 \sum_{i=1}^{n} (\bar{X}_i - X_i) \left( E(X) - \bar{X} \right) + \frac{2}{n} \sum_{i=1}^{n} (\bar{X}_i - X_i)(X_i - E(X)) \\
+ \frac{2}{n} \sum_{i=1}^{n} (X_i - E(X)) \left( E(X) - \bar{X} \right) \\
= \frac{1}{n} \sum_{i=1}^{n} (X_i - E(X))^2 + \frac{2}{n} \sum_{i=1}^{n} (|\bar{X}_i| - |X_i|) \frac{\text{Var}(X)}{E(|X|)} + o_P(n^{-1/2}).
\]

From (A.48), we have

\[
\sqrt{n} (\hat{\sigma}_1^2 - \sigma_1^2) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} [(X_i - E(X))^2 - \sigma_1^2] \\
+ \frac{2}{n} \sum_{i=1}^{n} (|\bar{X}_i| - |X_i|) \frac{\text{Var}(X)}{E(|X|)} + o_P(1) \\
\xrightarrow{L} N \left( 0, \text{Var} \left( (X - E(X))^2 + (\psi(U) - 1) \frac{2|X|\text{Var}(X)}{E(|X|)} \right) \right).
\]

**Step 7.** In this step, we analyse the asymptotic expression of \( \hat{\gamma}_2(x) \). Recalling the definition of \( \hat{\gamma}_2(x) \), we have

\[
\hat{\gamma}_2(x) = \frac{\hat{Q}_{n2}(x) \hat{M}_{n0, \bar{Y}_2}(x) - \hat{Q}_{n1}(x) \hat{M}_{n1, \bar{Y}_2}(x)}{\hat{Q}_{n2}(x) \hat{Q}_{n0}(x) - \left( \hat{Q}_{n1}(x) \right)^2}.
\]

We now analyse \( \hat{M}_{n0, \bar{Y}_2}(x) = \frac{1}{nh_2} \sum_{i=1}^{n} \left( \frac{\bar{X}_i - x}{h_2} \right)^{\delta} K \left( \frac{\bar{X}_i - x}{h_2} \right) \bar{Y}_i^2, \delta = 0, 1. \)

For \( \delta = 0 \), similar to (A.8), we have

\[
\hat{M}_{n0, \bar{Y}_2}(x) = \frac{1}{nh_2} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_2} \right) \bar{Y}_i^2 + \frac{1}{nh_2} \sum_{i=1}^{n} \left( \frac{\bar{X}_i - X_i}{h_2} \right) K' \left( \frac{X_i - x}{h_2} \right) \bar{Y}_i^2 \\
+ O_P \left( h_1^2 + \frac{\log n}{nh_1} \right) \\
= \frac{1}{nh_2} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_2} \right) \bar{Y}_i^2 + \epsilon_{n1}(x) + o_P(h_1^2 + (nh_1)^{-1/2}).
\]

In the following, we define

\[
g_{Y^2}(x) = E(Y^2 | X = x).
\]

Similar to (A.4)-(A.7), we have

\[
\epsilon_{n1}(x) = -\frac{1}{nh_2} \sum_{i=1}^{n} \left( \frac{\hat{\psi}^2(U_i) - \psi(U_i)}{\psi(U_i)} \right) K' \left( \frac{X_i - x}{h_2} \right) X_i \bar{Y}_i^2 + o_P(h_1^2 + (nh_1)^{-1/2}) \\
= \frac{\mu_2 h_1^2}{2} E \left( \frac{\phi^2(U) \psi''(U)}{\psi(U)} \right) \{g_{Y^2}(x)f_X(x) + x[g_{Y^2}(x)f_X(x)]'\} + o_P(h_1^2 + (nh_1)^{-1/2}) \\
= \frac{\mu_2 h_1^2}{2} E \left( \frac{\phi^2(U) \psi''(U)}{\psi(U)} \right) s_6(x) + o_P(h_1^2 + (nh_1)^{-1/2}).
\]
Thus, we have
\[
\hat{M}_{n_0 \tilde{Y}_2}(x) = \frac{1}{nh_2} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_2} \right) \hat{Y}_i^2 + \frac{\mu_2 h_2^2}{2} E \left( \frac{\phi^2(U) \psi''(U)}{\psi(U)} \right) s_0(x) + o_P(h_2^2 + (nh_2)^{-1/2}).
\] (A.53)

For $\delta = 1$, similar to (A.13), we have
\[
\hat{M}_{n_1 \tilde{Y}_2}(x) = \frac{1}{nh_2} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_2} \right) \hat{Y}_i^2 + \frac{\mu_2 h_2^2}{2} E \left( \frac{\phi^2(U) \psi''(U)}{\psi(U)} \right) s_7(x) + o_P(h_2^2 + (nh_2)^{-1/2}).
\] (A.54)

Similar to (A.11)-(A.12), we have
\[
F_{n_1}(x) = \frac{\mu_2 h_2^2}{2} E \left( \frac{\phi^2(U) \psi''(U)}{\psi(U)} \right) \left\{ x[g_{\tilde{Y}_2}(x) f_X(x)]'' + 2[g_{\tilde{Y}_2}(x) f_X(x)]' \right\} + o_P(h_2^2 + (nh_2)^{-1/2})
\] (A.55)

def\[\mu_2 h_2^2}{2} E \left( \frac{\phi^2(U) \psi''(U)}{\psi(U)} \right) s_7(x) + o_P(h_2^2 + (nh_2)^{-1/2}).
\]

Thus, we have
\[
\hat{M}_{n_1 \tilde{Y}_2}(x) = \frac{1}{nh_2} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_2} \right) \hat{Y}_i^2 + \frac{\mu_2 h_2^2}{2} E \left( \frac{\phi^2(U) \psi''(U)}{\psi(U)} \right) s_7(x) + o_P(h_2^2 + (nh_2)^{-1/2}).
\] (A.56)

Note that $g_{\tilde{Y}_2}(x) = E[\phi^2(U)] g_{Y_2}(x)$. Using (A.14) and (A.53), we have
\[
\hat{M}_{n_0 \tilde{Y}_2}(x) - \hat{Q}_{n_0}(x) g_{\tilde{Y}_2}(x)
\] (A.57)
Together with (A.23),

\[
\dot{Q}_{n2}(x) \left\{ \hat{M}_{n0,y^2}(x) - \dot{Q}_{n0}(x)g_{y^2}(x) \right\} = \frac{\mu_2 f_X(x)}{nh_2} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_2} \right) \left[ \hat{Y}_i^2 - E[\phi^2(U)]g_{y^2}(X_i) \right] 
+ \hat{h}_2^2 \mu_2^2 E[\phi^2(U)] f_X(x) \left[ g_{y^2}(x) f_X(x) + \frac{1}{2} g_{y^2}''(x) f_X(x) \right] 
+ \frac{\mu_2^2 \hat{h}_2^2 f_X(x)}{2} \left\{ E \left( \frac{\phi^2(U)^2}{\psi(U)} \right) s_6(x) - E[\phi^2(U)] E \left( \frac{\psi''(U)}{\psi(U)} \right) s_4(x) g_{y^2}(x) \right\} 
+ o_P(h_2^2 + (nh_2)^{-1/2}).
\]

Using (A.25) and (A.56), we have

\[
\dot{M}_{n1,y^2}(x) - \dot{Q}_{n1}(x)g_{y^2}(x) = \frac{1}{nh_2} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_2} \right) \left( \frac{X_i - x}{h_2} \right) \hat{Y}_i^2 - Q_{n1}(x)g_{y^2}(x) 
+ \frac{\mu_2 \hat{h}_2^2}{2} \left\{ E \left( \frac{\phi^2(U)}{\psi(U)} \right) s_7(x) - E[\phi^2(U)] E \left( \frac{\psi''(U)}{\psi(U)} \right) s_5(x) g_{y^2}(x) \right\} + o_P(h_2^2 + (nh_2)^{-1/2}) 
+ \frac{\mu_2 \hat{h}_2^2}{2} \left\{ E \left( \frac{\phi^2(U)}{\psi(U)} \right) s_7(x) - E[\phi^2(U)] E \left( \frac{\psi''(U)}{\psi(U)} \right) s_5(x) g_{y^2}(x) \right\} 
+ O_P(h_2^2) + o_P(h_2^2 + (nh_2)^{-1/2}).
\]

Together with (A.17), (A.25) and (A.33), we have

\[
\dot{Q}_{n1}(x) \left[ \hat{M}_{n1,y^2}(x) - \dot{Q}_{n1}(x)g(x) \right] = E[\phi^2(U)] g_{y^2}(x) f_X(x) f_X(x) \mu_2^2 \hat{h}_2^2 + O_P(h_2^2 + (nh_2)^{-1/2}) + o_P(h_2^2 + (nh_2)^{-1/2}).
\]

Consequently, using (A.58) and (A.60), we have

\[
\dot{Q}_{n2}(x) \left[ \hat{M}_{n0,y^2}(x) - \dot{Q}_{n0}(x)g_{y^2}(x) \right] - \dot{Q}_{n1}(x) \left[ \hat{M}_{n1,y^2}(x) - \dot{Q}_{n1}(x)g_{y^2}(x) \right] = \frac{\mu_2 f_X(x)}{nh_2} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_2} \right) \left[ \hat{Y}_i^2 - E[\phi^2(U)]g_{y^2}(X_i) \right] + E[\phi^2(U)] \mu_2^2 \hat{h}_2^2 f_X^2(x) g_{y^2}(x) 
+ \frac{\mu_2^2 \hat{h}_2^2 f_X(x)}{2} \left\{ E \left( \frac{\phi^2(U)^2}{\psi(U)} \right) s_6(x) - E[\phi^2(U)] E \left( \frac{\psi''(U)}{\psi(U)} \right) s_4(x) g_{y^2}(x) \right\} 
+ o_P(h_2^2 + (nh_2)^{-1/2}).
\]
Using (A.28) and (A.61), we have

\[ \frac{1}{nh_{fX}(x)} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_2} \right) \left[ \hat{Y}_i^2 - E[\phi^2(U)] \hat{g}_Y(x_i) - \frac{\mu_2 h^2 E[\phi^2(U)]}{2} \hat{g}_Y'(x_i) \right] + \frac{\mu_2 h^2}{2f_X(x)} \left[ E \left( \frac{\phi^2(U)}{\psi(U)} \right) s_0(x) - 2E[\phi^2(U)]E \left( \frac{\phi(U)}{\psi(U)} \right) s_1(x)g(x) \right] + a_P(h^2 + (nh_2)^{-1/2}). \]

From (A.44), we have \( E[\phi^2(U)] = O_P(n^{-1/2}) = a_P((nh_2)^{-1/2}). \) Thus, using (A.29)-(A.30), (A.62), we have

\[ \hat{g}_Y'(x) - E[\phi^2(U)] \hat{g}_Y'(x) \]

\[ = \hat{g}_Y'(x) - g_Y'(x) - E[\phi^2(U)] \left[ \hat{g}_Y'(x) - g_Y'(x) \right] + [g_Y'(x) - E[\phi^2(U)]g_Y'(x)] + a_P((nh_2)^{-1/2}). \]

Together with \( E[\phi^2(U)] \xrightarrow{P} E[\phi^2(U)] \), and \( \sigma^2(x) = \frac{g_Y'^2(x)}{f_X(x)^2} - g^2(x) \), we have

\[ \frac{1}{nh_{fX}(x)} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_2} \right) \left[ \frac{\hat{Y}_i^2}{E[\phi^2(U)]} - 2g(x) \hat{Y}_i - g_Y(x) + 2g(x)g_Y(x) \right] \]

\[ + \frac{\mu_2 h^2}{2f_X(x)} \left[ g_Y'(x) - 2g(x)g_Y''(x) + \frac{\mu_2 h^2}{2f_X(x)} E \left( \frac{\phi''(U)}{\psi(U)} \right) s_4(x) \left[ 2g^2(x) - g_Y(x) \right] \right] \]

\[ + \frac{\mu_2 h^2}{2f_X(x)E[\phi^2(U)]} \left[ E \left( \frac{\phi^2(U)}{\psi(U)} \right) s_0(x) - 2E[\phi^2(U)]E \left( \frac{\phi(U)}{\psi(U)} \right) s_1(x)g(x) \right] \]

\[ + a_P((nh_2)^{-1/2}). \]

From \( s_4(x) \) defined in (A.5) and (A.52), we have \( m_3(x) = \frac{s_4(x)}{f_X(x)^2} = g_Y^2(x) + xg_Y'(x) + xg_Y(x) \frac{f_X'(x)}{f_X(x)} \). Thus, from (A.29), we have

\[ \sqrt{nh_2} \left( \hat{g}_Y^2(x) - g_Y^2(x) - \frac{\mu_2 h^2}{2} [g_Y'(x) - 2g(x)g_Y''(x)] \right) \]

\[ - \frac{\mu_2 h^2}{2f_X(x)} E \left( \frac{\phi''(U)}{\psi(U)} \right) s_4(x) \left[ 2g^2(x) - g_Y(x) \right] \]

\[ - \frac{\mu_2 h^2}{2f_X(x)E[\phi^2(U)]} \left[ E \left( \frac{\phi^2(U)}{\psi(U)} \right) s_0(x) - 2E[\phi^2(U)]E \left( \frac{\phi(U)}{\psi(U)} \right) s_1(x)g(x) \right] \]

\[ = \frac{1}{\sqrt{nh_{fX}(x)}} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_2} \right) \left[ \frac{\hat{Y}_i^2}{E[\phi^2(U)]} - 2g(x) \hat{Y}_i - g_Y(x) + 2g(x)g_Y(x) \right] + a_P(1). \]
1.4. Proof of Theorem 1

Proof. To give the asymptotic expression of \( \hat{\rho}(x) \), it is equivalent to present the asymptotic expression of \( \hat{\rho}^2(x) \). Note that

\[
\hat{\rho}^2(x) - \rho^2(x) = \frac{\sigma_1^2 [g'(x)]^2}{\sigma_1^2 [g'(x)]^2 + \sigma^2(x)} - \frac{\sigma_2^2 [g'(x)]^2}{\sigma_2^2 [g'(x)]^2 + \sigma^2(x)}
\]

(A.66)

\[
= \frac{\sigma_1^2 [g'(x)]^2 - \sigma_2^2 [g'(x)]^2}{\sigma_1^2 [g'(x)]^2 + \sigma^2(x)} \frac{\sigma_2^2 [g'(x)]^2}{\sigma_2^2 [g'(x)]^2 + \sigma^2(x)}
\]

\[
= \frac{\sigma_1^2 [g'(x)]^2 - \sigma_2^2 [g'(x)]^2}{\sigma_1^2 [g'(x)]^2 + \sigma^2(x)} \frac{\sigma_2^2 [g'(x)]^2}{\sigma_2^2 [g'(x)]^2 + \sigma^2(x)}
\]

(A.68)

Using (A.49) and (A.64), we have \( \sigma_1^2 - \sigma_1^2 = O_P(n^{-1/2}) \), and

\[
h_2 R_{n3}(x) = O_P(h_2 n^{-1/2}), h_2 R_{n4}(x) = O_P(h_2 n^{-1/2}) + O_P(h_2^3 + (nh_2)^{-1/2}) = o_P(h_2^3 + (nh_2)^{-1/2}).
\]

(A.67)

From (A.42), (A.49) and (A.64), we have \( \hat{g}'(x) \xrightarrow{p} g'(x), \sigma_1^2 \xrightarrow{p} \sigma_1^2, \sigma_2^2(x) \xrightarrow{p} \sigma^2(x) \), and then

\[
h_2 R_{n1}(x) = \frac{\sigma_1^2 \sigma_2^2(x)}{\sigma_1^2 [g'(x)]^2 + \sigma_2^2(x)} h_2 \left\{ [g'(x)]^2 - [g'(x)]^2 \right\} + o_P(h_2^3 + (nh_2)^{-1/2})
\]

(A.68)

\[
= \frac{2 \sigma_1^2 \sigma_2^2(x) g'(x)}{\sigma_1^2 [g'(x)]^2 + \sigma_2^2(x)} h_2 \left\{ g'(x) - [g'(x)]^2 \right\} + o_P(h_2^3 + (nh_2)^{-1/2})
\]

\[
\def \zeta(x) h_2 \left\{ g'(x) - [g'(x)]^2 \right\} + o_P(h_2^3 + (nh_2)^{-1/2})
\]

(A.69)

\[
= \frac{\zeta(x)}{nh_2^{f(x)}(x) \mu_2} \sum_{i=1}^{n} K \left( \frac{X_i - x}{h_2} \right) \left( \frac{X_i - x}{h_2} \right) \left[ \hat{y}_i - g(X_i) \right]
\]

\[
+ \frac{\zeta(x)}{nh_2^{f(x)}(x) \mu_2} \left[ \frac{\mu_4 - \mu_3^2 f''(x)g''(x)}{2 \mu_2} + \frac{\mu_4}{6 \mu_2} g'''(x) \right] h_2^3 + \frac{\mu_2 h_2^2}{2} E \left( \frac{\psi''(U)}{\psi(U)} \right) v_1(x) \zeta(x)
\]

\[
+ \frac{\mu_2 h_2^2}{2} E \left( \frac{\psi''(U)}{\psi(U)} \right) v_2(x) \zeta(x) + o_P(h_2^3 + (nh_2)^{-1/2})
\]

\[
h_2 R_{n2}(x) = \frac{\sigma_1^2 [g'(x)]^2}{\sigma_1^2 [g'(x)]^2 + \sigma_2^2(x)} h_2 \left\{ \sigma_2^2(x) - [g'(x)]^2 \right\} + o_P(h_2^3 + (nh_2)^{-1/2})
\]

(A.69)

\[
= \frac{\mu_2 \pi(x) h_2^3}{2} \left[ g(y_2) - 2g(x) g'(x) + s_2 \pi(x) h_2^2 E \left( \frac{\psi''(U)}{\psi(U)} \right) s_4(x) \left[ 2g(x) - g(y_2) \right] + o_P(h_2^3 + (nh_2)^{-1/2})
\]

\[
+ \frac{\mu_2 \pi(x) h_2^2}{2f(x)} E \left[ \phi''(U) \right] \left[ E \left( \frac{\phi''(U) \psi''(U)}{\psi(U)} \right) s_6(x) - 2E[\phi''(U)] E \left( \frac{\phi(U) \psi''(U)}{\psi(U)} \right) s_1(x) g(x) \right]
\]

\[
+ o_P(h_2^3 + (nh_2)^{-1/2}).
\]
Together with (A.66)-(A.69), we have

\[
\begin{align*}
    h_2 \left[ \hat{\rho}^2(x) - \rho^2(x) \right] &= h_2 R_{n1}(x) - h_2 R_{n2}(x) + o_P(h_2^3 + (nh_2)^{-1/2}) \\
    &= \frac{\zeta(x)}{nh_2 f_X(x) \mu_2} \sum_{i=1}^n K \left( \frac{X_i - x}{h_2} \right) \left( \frac{X_i - x}{h_2} \right) \left[ \tilde{Y}_i - g(X_i) \right] \\
    &+ \left[ \frac{\mu_1 - \mu_2^2}{2} \frac{f_X'(x)g''(x)\zeta(x)}{f_X(x)} \right] + \frac{\mu_4}{6\mu_2} g'''(x)\zeta(x) \left[ \frac{g''(x) - 2g(x)g''(x)}{2} \right] h_2^3 \\
    &+ \frac{\mu_2 h_2^4 h_2^2}{2} E \left( \frac{\phi(U)\psi''(U)}{\psi(U)} \right) \left[ \frac{v_1(x)\zeta(x)}{f_X(x)} + \frac{2s_1(x)g(x)\pi(x)}{f_X(x)} \right] \\
    &+ \frac{\mu_2 h_2^4 h_2^2}{2} E \left( \frac{\phi(U)\psi''(U)}{\psi(U)} \right) \left[ \frac{v_2(x)\zeta(x) - s_4(x)\pi(x)\frac{2g^2(x) - g_Y^2(x)}{f_X(x)}}{f_X(x)} \right] \\
    &- \frac{\mu_2 h_2^4 h_2^2}{2} E \left( \frac{\phi(U)\psi''(U)}{\psi(U)} \right) \left[ \frac{s_6(x)\pi(x)}{f_X(x)E[\phi^2(U)]} \right] + o_P(h_2^3 + (nh_2)^{-1/2}).
\end{align*}
\]

Using asymptotic expression (A.70) and the equation \( h_2 (\hat{\rho}(x) - \rho(x)) = \frac{1}{\hat{\rho}(x) + \rho(x)} h_2 \left[ \hat{\rho}^2(x) - \rho^2(x) \right] \), we complete the proof of Theorem 1.

\[\square\]

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