Holomorphic matrix models

C. I. Lazaroiu

Institut für Physik
Humboldt Universität zu Berlin
Invalidenstrasse 110, Berlin
Germany
calin@physik.hu-berlin.de

ABSTRACT: This is a study of holomorphic matrix models, the matrix models which underlie the conjecture of Dijkgraaf and Vafa. I first give a systematic description of the holomorphic one-matrix model. After discussing its convergence sectors, I show that certain puzzles related to its perturbative expansion admit a simple resolution in the holomorphic set-up. Constructing a ‘complex’ microcanonical ensemble, I check that the basic requirements of the conjecture (in particular, the special geometry relations involving chemical potentials) hold in the absence of the hermiticity constraint. I also show that planar solutions of the holomorphic model probe the entire moduli space of the associated algebraic curve. Finally, I give a brief discussion of holomorphic ADE models, focusing on the example of the A₂ quiver, for which I extract explicitly the relevant Riemann surface. In this case, use of the holomorphic model is crucial, since the Hermitian approach and its attending regularization would lead to a singular algebraic curve, thus contradicting the requirements of the conjecture. In particular, I show how an appropriate regularization of the holomorphic A₂ model produces the desired smooth Riemann surface in the limit when the regulator is removed, and that this limit can be described as a statistical ensemble of ‘reduced’ holomorphic models.
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Introduction

In the seminal paper [1], Dijkgraaf and Vafa proposed a beautiful conjecture relating matrix models to closed string theory on certain noncompact Calabi-Yau spaces. In its strongest form, this is meant as a relation between the partition function of a certain matrix model and the partition function of Kodaira-Spencer theory [21] on a dual noncompact Calabi-Yau threefold. The large $N$ limit of this relation also leads to a matrix model description of corrections to the Veneziano-Yankielowicz potential of certain $\mathcal{N} = 1$ supersymmetric field theories [1, 14] (moreover, the VY potential itself can be recovered from the non-perturbative part of the large $N$ matrix integral). In fact, there are now independent derivations of the planar version of this conjecture by purely field-theoretic methods [2, 5, 4, 16], as well as first tests of it beyond the large $N$ limit [6, 11]. More recent work on the subject can be found in [15, 8, 5, 17, 18, 13, 25, 26, 27].

As already pointed out in [1], a proper formulation of the conjecture should be given in terms of ‘holomorphic matrix models’, namely some version of matrix models involving ‘contour’ integrals in a space of complex matrices. This is very natural if one remembers that the conjecture was originally derived by considering the worldvolume theory of certain topological B-type branes, which is described by a reduction of holomorphic Chern-Simons theory [7]. Since the latter is formulated without reference to a metric, the proper description of the resulting matrix model should not involve
a hermiticity constraint. This expectation is borne out by the fact that the conjecture describes relations in the chiral ring of the dual field theory [4], and the latter is constrained by holomorphy.

In [1], the authors chose the pragmatic approach of formulating the conjecture in terms of Hermitian matrix models, while pointing out that an appropriate holomorphic formulation should be given. The purpose of the present work is to initiate a systematic study of such holomorphic models, and show how they naturally tie up some loose ends encountered in various investigations and extensions of the conjecture.

If naively taken at face value, the Hermitian formulation leads to various problems, some of which were already encountered (and overcome by a pragmatic prescription) in work aimed at testing it [6]. Among such issues, one can list the following:

(a) Hermitian one-matrix models with generic polynomial potentials of odd degree (and complex coefficients) are ill-defined, since the real part of such potentials is not bounded from below along the real axis. However, such models are naturally required by the conjecture, which is supposed to apply without constraints on the degree of the potential. This issue was encountered in the work of [6], while performing a one-loop test of the conjecture for a cubic potential. As shown in Section 2 and Appendix 2, the pragmatic approach followed in [6] admits a natural justification in the holomorphic setup.

(b) The large \( N \) spectral density \( \omega_0(z) = \text{tr} \frac{1}{z-M} \) of a Hermitian one-matrix model (with matrix \( M \)) can only have cuts along the real axis. This means that the cuts of the hyperelliptic Riemann surface of [1] would be constrained to lie on the real line. In the Hermitian formulation, the conjecture would then imply that the dual Calabi-Yau space is constrained in a similar manner. Moreover, the Hermitian formulation leads to numerous problems in matching parameters and moduli spaces, since a polynomial of degree \( n \) always has \( n \) complex roots, but need not have \( n \) real roots. As clear from the work of [1] (and occasionally pointed out in subsequent work by the same authors), what is needed is a holomorphic extension of the Hermitian matrix model which would allow for an enlarged family of planar limits — namely, such a model should produce solutions whose large \( N \) eigenvalue distributions can be supported along arbitrary one-dimensional curve segments in the complex plane.

In the present note, I show how these and related issues are solved by a direct analysis of holomorphic matrix models, whose construction follows a suggestion already made in [1]. In Sections 1 and 2, I explicitate the non-perturbative definition of such models \(^1\) and study their convergence sectors, thereby refining one side of the conjecture.

\(^1\)I should state from the outset that holomorphic matrix models are not the same as the so-called ‘complex matrix models’ [19] sometimes encountered in the matrix model literature. Rather, they are
I also show (in Appendix 2) that issues like those encountered in [6] admit a natural resolution in this framework (which does recover the prescription used in that paper). Section 3 extracts the loop equations, equations of motion and the planar limit of such models, showing that most of the standard analysis employed for the Hermitian model carries through with certain modifications. Consideration of the large $N$ limit leads to the algebraic curve of [1], which is now unconstrained by any conditions on the location of the cuts. In fact, one can give a ‘reconstruction theorem’, which ensures that the holomorphic model probes the entire moduli space of this algebraic curve. This shows explicitly how such models solve the second issue listed above. Up to some details, the reconstruction result boils down to the well-known relation between the Riemann problem and singular integral equations. From this perspective, holomorphic matrix models give a sort of ‘quantization’ of the classical Riemann problem.

The validity of the conjecture rests crucially on the special geometry relations mentioned in [1], a more detailed justification of which was latter given in [9] (though only for the Hermitian case). To give a clear proof of such relations at the holomorphic level, Section 4 constructs a ‘complex’ microcanonical ensemble by introducing chemical potentials and performing a Legendre transform, thereby obtaining a generating functional which depends on averaged filling fractions; this allows one to show that the special geometry relations of [1] are a direct consequence of standard equations expressing chemical potentials in terms of the microcanonical generating function. In particular, these relations give the finite $N$ analogue of the special geometry constraints.

Section 5 considers holomorphic $ADE$ models, focusing on a detailed analysis of the $A_2$ model. In this case, the problems one encounters by working naively with a Hermitian constraint are considerably more severe than in the one-matrix case. In fact, the Hermitian approach of [22] (with its attending regularization) would lead to a singular curve, which would always be constrained to have a certain number of double points. This would completely miss some of the gauge theory physics extracted in [23, 24] More precisely, such a constraint would force certain filling fractions to be identically zero, thereby contradicting the existence of the gaugino condensates obtained in [23]. As shown in Section 5, this issue is resolved quite naturally in the holomorphic $A_2$ model, by considering a regularization which is natural in that set-up. When removing the regulator, one obtains an ensemble of reduced holomorphic models, whose planar limit allows for non-vanishing values of all filling fractions. It is the large $N$ limit of this ensemble which is relevant for the conjecture of [1].

\footnote{a sort of ‘square roots’ of such models.}
1. Construction of holomorphic one-matrix models and their eigenvalue representation

Following a suggestion made in [1], we start by constructing holomorphic one-matrix models as multidimensional ‘contour’ integrals in a space of complex matrices. The construction is quite similar to that of a Hermitian model, except that the hermiticity constraint is replaced by a more general condition on the eigenvalues. After defining the model and studying its gauge-invariance, we extract an eigenvalue representation by choosing an appropriate multidimensional ‘contour’. With this choice, one obtains an integral expression for the partition function which is formally identical to that of a Hermitian model, except that integration is performed over eigenvalues lying along an open path in the complex plane.

1.1 The partition function

Fixing a positive integer $N$, we let $\text{Mat}_N(\mathbb{C})$ denote the set of all $N \times N$ complex matrices. For any such matrix $M$, we let $p_M(\lambda) = \text{det}(\lambda I - M)$ denote its characteristic polynomial. Define a subset $\mathcal{M}$ of $\text{Mat}_N(\mathbb{C})$ as follows:

$$\mathcal{M} := \{ M \in \text{Mat}_N(\mathbb{C}) \mid p_M(\lambda) \text{ has distinct roots}\} .$$  \hfill (1.1)

This is an open submanifold of $\text{Mat}_N(\mathbb{C})$, consisting of matrices which are diagonalizable by general linear transformations: for every $M$ in $\mathcal{M}$, there exists an $S \in \text{GL}(N, \mathbb{C})$ such that:

$$S M S^{-1} = D := \text{diag}(\lambda_1 \ldots \lambda_N) ,$$  \hfill (1.2)

where $\lambda_j$ are the roots of $p_M(\lambda)$. The later coincide with the eigenvalues of $M$, so that its spectrum is given by:

$$\sigma(M) = \{ \lambda_1 \ldots \lambda_N \} .$$  \hfill (1.3)

Consider a connected, noncompact and boundary-less submanifold $\Gamma$ of $\mathcal{M}$ of real dimension equal to the complex dimension of $\mathcal{M}$:

$$\dim_{\mathbb{R}} \Gamma = \dim_{\mathbb{C}} \mathcal{M} = \dim_{\mathbb{C}} \text{Mat}_N(\mathbb{C}) = N^2 .$$  \hfill (1.4)

Also consider the standard holomorphic symplectic form on $\text{Mat}_N(\mathbb{C})$:

$$w = \bigwedge_{i,j} dM_{ij} .$$  \hfill (1.5)

The sign of the right hand side of course depends on the ordering of pairs $(i,j) \in \{1 \ldots N\} \times \{1 \ldots N\}$, and we shall use the lexicographic ordering:

$$w = dM_{11} \wedge \ldots \wedge dM_{1N} \wedge dM_{21} \wedge \ldots \wedge dM_{2N} \wedge \ldots \wedge dM_{N1} \wedge \ldots \wedge dM_{NN} .$$  \hfill (1.6)
This is implicit in all such expressions encountered below.

Fixing a polynomial \( W(z) = \sum_{m=0}^{n} t_m z^m \) with complex coefficients, we define the \textit{holomorphic one-matrix model} by the partition function:

\[
\tilde{Z}_N(\Gamma, t) = \frac{1}{N} \int_{\Gamma} w e^{-N \text{tr} W(M)},
\]

where \( N \) is a normalization factor to be fixed below.

\section*{1.2 Gauge invariance}

Consider the \( GL(N, \mathbb{C}) \) action \( \tau \) on \( \text{Mat}_N(\mathbb{C}) \) given by similarity transformations:

\[
\tau(S)M := SMS^{-1}, \quad S \in GL(N, \mathbb{C}).
\]

This clearly stabilizes \( \mathcal{M} \); in particular, note that the characteristic polynomial of \( M \) is \( \tau \)-invariant:

\[
p_{\tau(S)M}(\lambda) = p_M(\lambda).
\]

The action \( S(M) = N \text{tr} W(M) \) of our model is obviously invariant with respect to (1.8). The same is true of the holomorphic measure \( w \): since \( \tau(S)_i^j = a_{ij,kl} M_{kl} \), with \( a_{ij,kl} = S_{ik} (S^{-1})^T jl \) (i.e. \( a = S \otimes (S^{-1})^T \)), we have \( \det(a) = 1 \) and:

\[
\tau(S)^*(w) = w.
\]

It follows that the model admits the \( GL(N, \mathbb{C}) \) gauge-invariance (1.8), provided that one chooses the integration manifold \( \Gamma \) such that it is stabilized by the gauge-group action.

\textbf{Observation} Choosing \( S \) to be a permutation matrix (i.e. \( S_{ij} = \delta_{j\sigma(i)} \) with \( \sigma \) a permutation on \( N \) elements), equation (1.10) shows that the holomorphic measure (1.5) is invariant under joint permutations of indices:

\[
dM_{11} \wedge \ldots \wedge dM_{1N} \wedge dM_{21} \wedge \ldots \wedge dM_{2N} \wedge \ldots \wedge dM_{N1} \wedge \ldots \wedge dM_{NN} =
\]

\[
dM_{\sigma(1)\sigma(1)} \wedge \ldots \wedge dM_{\sigma(1)\sigma(N)} \wedge dM_{\sigma(2)\sigma(1)} \wedge \ldots \wedge dM_{\sigma(2)\sigma(N)} \wedge \ldots \wedge dM_{\sigma(N)\sigma(1)} \wedge \ldots \wedge dM_{\sigma(N)\sigma(N)},
\]

a relation which can also be checked directly.

\section*{1.3 Eigenvalue representation}

One can derive an eigenvalue representation of (1.7) as follows. Let \( \gamma : \mathbb{R} \rightarrow \mathbb{C} \) be an open curve in the complex plane, which we assume to be immersed and without self-intersections. We shall take \( \Gamma \) to be the following subset of \( \mathcal{M} \):

\[
\Gamma(\gamma) := \{ M \in \mathcal{M} \mid \sigma(M) \subset \gamma \}.
\]
which obviously satisfies (1.4). We denote \( \tilde{Z}_N(\Gamma(\gamma), t) \) by \( \tilde{Z}_N(\gamma, t) \). Then the integral over gauge-group orbits can be performed as explained in Appendix 1, with the result:

\[
\tilde{Z}_N(\gamma, t) = \frac{1}{N} (-1)^{N^2(N-1)/2} \frac{1}{N!} \text{hv}(H) Z_N(\gamma, t) ,
\]

where \( \text{hv}(H) \) is the ‘holomorphic volume’ of the complex homogeneous space \( H = GL(N, \mathbb{C})/(\mathbb{C}^*)^N \) (see Appendix 1) and

\[
Z_N(\gamma, t) = \int \cdots \int d\lambda_1 \cdots d\lambda_N \prod_{i\neq j} (\lambda_i - \lambda_j) e^{-N \sum_{j=1}^N W(\lambda_j)} .
\]

is the eigenvalue representation of our model. The holomorphic volume \( \text{hv}(H) \) will be discarded (together with the sign prefactors) by choosing \( N = (-1)^{N^2(N-1)/2} \frac{1}{N!} \text{hv}(H) \) in (1.7). Expression (1.14) is formally identical with the eigenvalue representation of the Hermitian matrix model, except that the eigenvalue integral is performed along the contour \( \gamma \) in the complex plane. The pragmatic reader can take (1.14) as the definition of our model.

2. Convergence sectors

It is clear from expression (1.14) that convergence of our partition function depends on the choice of \( \gamma \). In this section, we shall characterize the ‘good’ choices of \( \gamma \) in terms of certain asymptotic sectors of the model, described in terms of cones partitioning the complex plane. Such cones can be identified by performing an elementary analysis of the behavior of the integrand. As we shall see below, this allows us to make non-perturbative sense of models with polynomial potentials of odd degree.

To extract the relevant behavior, let \( z = re^{i\theta} \) with \( r > 0 \) and \( \theta \in \mathbb{R}/2\pi\mathbb{Z} \) and \( t_j = r_j e^{-i\theta_j} \) with \( r_j > 0 \) and \( \theta_j \in \mathbb{R}/2\pi\mathbb{Z} \). Then the potential takes the form:

\[
W(z) = t_0 + \sum_{j=1}^n r_j r^i \cos(j\theta - \theta_j) + i \sum_{j=1}^{\infty} r_j r^i \sin(j\theta - \theta_j) ,
\]

and the behavior of \( |e^{-NW(z)}| = e^{-NReW(z)} \) for \( r \to \infty \) is dictated by the contribution \( r_{j_0} r^{j_0} \cos(j_0 \theta - \theta_{j_0}) \), where \( j_0 = j_0(\theta) \) is the largest \( j \) such that \( r_j \cos(j\theta - \theta_j) \neq 0 \). Namely, \( e^{-NW(z)} \) is exponentially decreasing/increasing depending on whether \( \cos(j_0 \theta - \theta_{j_0}) \) is positive/negative. In the very non-generic case when \( r_j \cos(j\theta - \theta_j) \) vanishes for all \( j = 1 \ldots n \), the quantity \( e^{-NW(z)} \) oscillates indefinitely as \( r \to \infty \).

Let us fix \( \theta \) such that \( \cos(n\theta - \theta_n) \neq 0 \). Then \( e^{-NW(z)} \) is exponentially decreasing as \( r \to \infty \) if and only if \( \cos(n\theta - \theta_n) > 0 \), which gives:

\[
\theta = \frac{\alpha + \theta_n}{n} + \frac{2k}{n} \quad \text{with} \quad k = 0 \ldots n - 1 \quad \text{and} \quad \alpha \in (-\pi/2, \pi/2) .
\]
This defines $n$ angular sectors (=open cones with apex at the origin) in the complex plane, which we denote by $A_k$, $k = 0 \ldots n - 1$. We also define complementary sectors $B_k$ through:

$$\theta = \frac{\alpha + \theta_n}{n} + \pi \frac{2k + 1}{n} \quad \text{with} \quad k = 0 \ldots n - 1 \quad \text{and} \quad \alpha \in (-\pi/2, \pi/2) ; \quad (2.3)$$

these are the sectors for which $\cos(n\theta - \theta_n) < 0$. The sectors $A_k$ and $B_k$ arise consecutively with respect to the trigonometric order and are separated by rays originating at $z = 0$.

To make the integral (1.14) absolutely convergent, we shall require that the curve $\gamma$ asymptotes to some straight lines $d_\pm(t) = \pm \nu_\pm t + \mu_\pm$ for $t \to \pm \infty$, such that the corresponding asymptotes lie inside two of the convergence sectors $A_k$. That is, we require:

$$\exists \lim_{t \to \pm \infty} (\gamma(t) \mp \nu_\pm t - \mu_\pm) = 0 \quad (2.4)$$

$$\exists \lim_{t \to \pm \infty} \gamma'(t) := \pm \nu_\pm .$$

for some $\nu_\pm \in A_{k_\pm}$ and some $\mu_\pm \in \mathbb{C}$. Here $\gamma'(t) = \frac{d\gamma(t)}{dt}$ and $k_\pm \in \{0 \ldots n - 1\}$. Since the integrand of (1.14) is holomorphic, it immediately follows that the partition function is independent of the choice of $\gamma$ as long as this contour has asymptotic behavior (2.4) with $\nu_\pm$ belonging to fixed sectors $A_{k_\pm}$. In particular, the integral does not depend on the precise values of $\nu_\pm$ and $\mu_\pm$, but only on the convergence sectors $A_{k_-}$ and $A_{k_+}$ (figure 1). A complete definition of the model requires that we specify one of the 'phases' $(k_-, k_+)$, together with the potential $W$.

It is also clear that $Z(k_-, k_+, t)$ vanishes if $k_+ = k_-$, and that we have the relation:

$$Z(k_-, k_+, t) = (-1)^N Z(k_+, k_-, t) , \quad (2.5)$$

which results upon reversing the orientation of $\gamma$. Therefore, it suffices to consider the $n(n - 1)/2$ 'phases' indexed by pairs $(k_-, k_+)$ with $k_- > k_+$.

2.1 Scaling symmetry

Let $q$ be a non-vanishing complex number. Writing $z := \frac{z}{q}$, we have:

$$W(z) = W_q(x) , \quad (2.6)$$

where $W_q(x) = \sum_{j=0}^n t_j^{(q)} x^j$, with $t_j^{(q)} := \frac{t_j q^j}{q}$. Performing the change of variables $\lambda := \frac{\mu}{q}$ in (1.14) gives:

$$Z_N(\gamma, t) = \frac{1}{q^{N\pi}} Z_N(\gamma q, t^{(q)}) , \quad (2.7)$$
Figure 1: Convergence sectors of the holomorphic matrix model. We show the case $n = \text{deg}W = 4$, with $\theta_4 = 0$ and a contour belonging to the sector $(k_-, k_+) = (1, 0)$.

where $\gamma_q$ is the path defined through:

$$\gamma_q(t) = q\gamma(t) \quad (2.8)$$

for all real $t$.

For the choice $q_n := r_n^{1/n}e^{-i\theta n}$, the change of variable $z := \frac{z}{q_n}$ gives $t_n^{(q_n)} = 1$, so the transformed potential $W_{q_n}$ has unit leading coefficient. Hence the model depends ‘trivially’ on the parameter $t_n$, and we can set $t_n = 1$ and $\theta_n = 0$ without loss of generality.

Also note that choosing $q := \alpha^k$ with $\alpha = e^{2\pi i/n}$ and $k$ an integer allows one to rotate $\gamma$ by any multiple of $\frac{2\pi}{n}$. Since this does not change the convergence sectors (because $t_n^{(\alpha^k)} = t_n$), it leads to the relation:

$$\mathcal{Z}_N(k + k_-, k + k_+, \{t_m\}) = \alpha^{N^2k}\mathcal{Z}_N(k_-, k_+, \{\alpha^mt_m\}) \quad (2.9)$$

Hence it suffices to consider the ‘phases’ $(k_-, 0)$ with $k_- = 1 \ldots n - 1$.

**Observation:** If one increases the degree of $W$, then the convergence sectors become progressively narrower. Allowing for the case $n = \infty$ (i.e. for an entire function $W$) is often a useful device in the theory of matrix models (the best known example is the matrix model of Kontsevich [12] and its generalizations). In this case, the convergence structure of the holomorphic model can be quite different from that discussed above, and depends on the precise asymptotic behavior of the entire
function $W$. A simple example is provided by the choice $W(z) = e^z$, which gives $|e^{-NW(z)}| = e^{-NRe W(z)} = e^{-Ne^x \cos(y)}$, where $z = x + iy$ with $x, y$ real. Then the convergence sectors are horizontal strips defined by the condition:

$$\cos(y) > 0 \iff y \in \left(\frac{-\pi}{2} + 2\pi k, \frac{\pi}{2} + 2\pi k\right) \; , \; k \in \mathbb{Z} \; . \quad (2.10)$$

Correspondingly, we obtain a convergent partition function by taking $\gamma$ to satisfy (2.4), where now $\nu_\pm > 0$ and $\mu_\pm$ belong to two such bands (figure 2). This example should serve as warning against the idea that one can recover models with power series potentials as naive limits of polynomial models.

![Figure 2: Convergence sectors for an entire potential $W = e^z$, and a good choice of contour for such a model. The filled-in regions are forbidden sectors for $\mu_\pm$.](image)

**2.2 Even and odd degree potentials, and the Hermitian matrix model**

**2.2.1 The case $n = \text{even}$**

In the case $n = \text{even}$, the convergence sectors of the model appear in pairs $A_k, A_{k+n/2}$ which are symmetric with respect to the inversion $z \rightarrow -z$. By the discussion above, we can take $t_n$ positive without loss of generality. Then $\theta_n = 0$ and the two sectors $A_0$ and $A_{n/2}$ lies opposite to each other and are bisected by the positive real axis. Picking $k_- = n/2$ and $k_+ = 0$, we can describe the ‘phase’ $\mathcal{Z}(n/2, 0, t)$ by taking the curve $\gamma$ to coincide with the real line (with its usual orientation). Then the partition function (1.14) reduces to the usual eigenvalue representation of the Hermitian one-matrix model. This gives a partial justification for the formulation used in [1].
2.2.2 The case \( n = \text{odd} \)

In this case, one has an odd number of convergence sectors \( A_k \), whose images under the reflection \( z \to -z \) are the ‘bad’ sectors \( B_{k+[n/2]} \). Picking \( t_n \) to be positive as above, it is clear that the matrix model cannot be defined by choosing \( \gamma = \mathbf{R} \), since the associated integral would diverge; this amounts to the basic observation that the Hermitian matrix model is ill-defined for generic complex polynomial potentials of odd degree. However, one can certainly make sense of the holomorphic model by working with any of the good ‘phases’ \((k_-, k_+)\) — one simply picks a contour \( \gamma \) such that \( \nu_+ \) and \( \nu_- \) belong to some of the convergence sectors \( A_k \).

The fact that the Hermitian model cannot be relevant for odd degree potentials is related to problems observed in [6], which in that paper were avoided by declaring that certain matrices occurring from a perturbative analysis should be anti-Hermitian. A systematic formulation of this idea is to consider the holomorphic matrix model instead. To substantiate this proposal, Appendix 2 shows that the procedure of [6] admits a simple justification in the framework of holomorphic models.

3. Loop equations, equations of motion and the large \( N \) limit

In this section, we study the loop equations and equations of motion of the holomorphic model, as well as its planar limit. As we shall see below, much of the standard fare of Hermitian models can be extended quite directly to the holomorphic case (though there are a few modifications). The most important novel feature of holomorphic matrix models is that they can explore an enlarged set of planar limits, thus probing the entire moduli space of a certain family of algebraic curves. This fact, which is essential for the conjecture of [1], will be established explicitly by proving a reconstruction theorem which associates a planar solution of the model with an arbitrary algebraic curve belonging to this family.

Let \( ds \) be the length element on \( \gamma \) and \( s \) the length coordinate along this curve, centered at some point on \( \gamma \). We let \( \lambda(s) \) denote the parameterization of \( \gamma \) with respect to this coordinate and write \( \lambda_i = \lambda(s_i) \) accordingly. Mimicking usual constructions, we define the spectral density by:

\[
\rho(s) = \frac{1}{N} \sum_{j=1}^{N} \delta(s - s_j), \tag{3.1}
\]

where \( \delta(s - s_j) \) is the delta-function in the coordinate \( s \) (equivalently, the \( \delta \)-function along \( \gamma \) with respect to the measure induced by the length element). Note the normalization condition:

\[
\int_{-\infty}^{\infty} ds \rho(s) = 1. \tag{3.2}
\]
Also consider the resolvent of \( M \):

\[
R(z) := \frac{1}{z - M} ,
\]

and its normalized trace:

\[
\omega(z) = \frac{1}{N} \text{tr} R(z) = \frac{1}{N} \text{tr} \frac{1}{z - M} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{z - \lambda_i} = \int ds \frac{\rho(s)}{z - \lambda(s)} .
\] (3.4)

In the following, we shall need the Sokhotsky-Plemelj formulae:

\[
\lim_{\epsilon \to 0^+} \int_{\gamma} d\lambda \frac{1}{\lambda - \lambda' \pm \epsilon n(\lambda)} = \mathcal{P} \int_{\gamma} d\lambda \frac{1}{\lambda - \lambda'} \mp i\pi \quad \text{for} \quad \lambda \in \gamma ,
\] (3.5)

which we also write symbolically as:

\[
\lim_{\epsilon \to 0^+} \frac{1}{\lambda(s) - \lambda(s') \pm \epsilon n(s)} = \mathcal{P} \frac{1}{\lambda(s) - \lambda(s')} \mp \frac{i\pi}{\lambda(s)} \delta(s - s') .
\] (3.6)

Here \( \mathcal{P} \) stands for the principal value and \( n(s) = i\dot{\lambda}(s) \) is the normal vector field to \( \gamma \) (figure 3).

\[\begin{array}{c}
\text{ Figure 3: } \text{The normal vector field } n(s) = it(s), \text{ and the tangent vector field } t(s) = \dot{\lambda}(s) \text{ of } \gamma. \text{ Note that } |n(s)| = |t(s)| = 1 \text{ since } s \text{ is the length coordinate along } \gamma. \\
\end{array}\]

### 3.1 Loop equations

To extract the loop equations, we follow the method of [10]. For this, start with the identity:

\[
\int_{\gamma} d\lambda_1 \ldots \int_{\gamma} d\lambda_N \sum_{i=1}^{N} \frac{\partial}{\partial \lambda_i} \left[ \prod_{i \neq k} (\lambda_k - \lambda_l) e^{-N \sum_{j=1}^{N} W(\lambda_j)} \frac{1}{z - \lambda_i} \right] = 0 .
\] (3.7)
Performing the partial differentiation, we write this as:
\[
\left\langle \sum_{i=1}^{N} \frac{1}{(z - \lambda_i)^2} - N \sum_{i=1}^{N} \frac{W''(\lambda_i)}{z - \lambda_i} + 2 \sum_{i \neq j} \frac{1}{(\lambda_i - \lambda_j)(z - \lambda_i)} \right\rangle = 0 .
\]  
(3.8)

Using the decomposition:
\[
\frac{1}{(z - \alpha)(z - \beta)} = \frac{1}{\alpha - \beta} \left[ \frac{1}{z - \alpha} - \frac{1}{z - \beta} \right]
\]  
(3.9)

to simplify the last term and combining the result with the first gives:
\[
\left\langle \omega(z)^2 - \frac{1}{N} \sum_{i=1}^{N} \frac{W''(\lambda_i)}{z - \lambda_i} \right\rangle = 0 .
\]  
(3.10)

3.2 Equations of motion

Writing \(\lambda_i = \lambda(s_i)\), the partition function (1.14) becomes:
\[
\mathcal{Z}_N(\gamma, t) = \int ds_1 \ldots \int ds_N \prod_{i=1}^{N} \hat{\lambda}(s_i) \prod_{i \neq j} \left( \lambda(s_i) - \lambda(s_j) \right) e^{-N \sum_{i=1}^{N} W(\lambda(s_i))}
\]
\[
= \int ds_1 \ldots \int ds_N e^{-NS_{eff}(s_1 \ldots s_N)} ,
\]  
(3.13)

where \(\dot{\lambda}(s) := \frac{d\lambda(s)}{ds}\) and:
\[
S_{eff}(s_1 \ldots s_N) = \sum_{j=1}^{N} W(\lambda(s_j)) - \frac{1}{N} \sum_{i \neq j} \ln(\lambda_i - \lambda_j) - \frac{1}{N} \sum_{i=1}^{N} \ln \hat{\lambda}_i .
\]  
(3.14)

Extremizing with respect to \(s_i\) gives the equations of motion:
\[
\frac{2}{N} \sum_{j} \lambda(s_i) - \frac{1}{\lambda(s_i)} = W'(\lambda(s_i)) - \frac{1}{N} \dot{\lambda}(s_i) .
\]  
(3.15)

The prime means that the sum is taken only over \(j \neq i\). The last term is a curvature-induced contribution which is subleading in \(1/N\). It is also easy to check that the equations of motion imply:
\[
\omega(z)^2 - W'(z)\omega(z) + f(z) + \frac{1}{N} \frac{d}{dz}\omega(z) + \frac{1}{N^2} \sum_{i=1}^{N} \frac{\dot{\lambda}(s_i)}{\lambda(s_i)^2} = 0 .
\]  
(3.16)

This ‘operator’ equation holds only for solutions of (3.15), unlike the Ward identity (3.12).
3.3 The large $N$ limit

For any quantity $\phi$, consider the large $N$ expansion:

$$\langle \phi \rangle = \sum_{j=0}^{\infty} \frac{\phi_j}{N^j} ,$$  \hspace{1cm} (3.17)

with coefficients $\phi_0, \phi_1$ etc. In particular, we have $\langle \rho(s) \rangle = \rho_0(s) + O(1/N)$ and $\langle \omega(z) \rangle = \omega_0(z) + O(1/N)$. In the large $N$ limit, the eigenvalues $\lambda_j$ are replaced by the planar spectral density $\rho_0(\lambda)$ supported on the curve $\gamma$. Note that this quantity is complex, since its definition involves averaging with respect to the complex integrand of (1.14). As usual, we have

$$\omega_0(z) = \int ds \frac{\rho_0(s)}{z - \lambda(s)} ,$$  \hspace{1cm} (3.18)

so that $\omega_0$ becomes an analytic function whose cuts $I_\alpha$ are forced to lie on the curve $\gamma$ (figure 4). We shall assume for simplicity that all cuts are of finite length.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.png}
\caption{Cuts of $\omega_0$. We also show a closed contour $\Gamma$ surrounding all cuts and a point $z$ in its exterior.}
\end{figure}

In the planar limit, the average of a product can be replaced by the product of averages and the loop equations (3.12) reduce to the algebraic constraint:

$$\omega_0(z)^2 - W'(z)\omega_0(z) + f_0(z) = 0 ,$$  \hspace{1cm} (3.19)

where

$$f_0(z) = \int ds \rho_0(s) \frac{W'(z) - W'(\lambda(s))}{z - \lambda(s)}$$  \hspace{1cm} (3.20)
is a polynomial of degree \( n - 2 \) with complex coefficients. Since \( \rho(s) \) is normalized by (3.2), equation (3.20) shows that the leading coefficient of \( f_0(z) \) equals \( nt_n \).

In the planar limit, the equations of motion (3.15) become:

\[
2\mathcal{P} \int ds' \frac{\rho_0(s')}{\lambda(s) - \lambda(s')} = W'(\lambda(s)) \quad .
\] (3.21)

The curvature term in (3.15) drops out, since it is subleading in \( 1/N \). The algebraic constraint (3.19) can also be obtained from (3.21) by standard manipulations. Writing:

\[
\omega_0(z) = u_0(z) + \frac{1}{2} W'(z) \quad ,
\] (3.22)

relation (3.19) shows that \( u_0(z) \) is one of the branches of the planar affine curve:

\[
 u^2 - \frac{1}{4} W'(z)^2 + f_0(z) = 0 \quad .
\] (3.23)

Since the branch cuts of \( \omega_0 \) (and thus of \( u_0 \)) must lie along \( \gamma \), it is clear that the polynomial \( f_0 \) in equation (3.23) is constrained by the choice of this curve. However, changing \( \gamma \) without changing its asymptotes allows one to describe any position of the cuts in the complex plane, as long as all these cuts have finite length. This effectively eliminates the constraints that would be present in the Hermitian case (for which \( \gamma \) would be forced to coincide with the real axis).

### 3.4 Reconstruction of a planar solution from the Riemann surface

Given the algebraic curve (3.23), we now show how one can use it to recover an appropriate \( \gamma \) supporting a spectral density \( \rho_0 \) satisfying (3.2), (3.18), (3.20) and the planar equations of motion (3.21). This proves that the holomorphic matrix model is free to explore the whole relevant piece of the moduli space of (3.23), unlike the Hermitian matrix model. In the planar limit, the holomorphic model reduces to the singular integral equation (3.21). Up to some minor details, the existence of a one to one relation between solutions of this equation and members of a family of Riemann surfaces boils down to the well-known relation between the Riemann problem and singular integral equations.

To see this explicitly, assume that one is given a complex degree \( n - 2 \) polynomial \( f_0(z) \), subject only to the constraint that its leading coefficient equals \( nt_n \). Consider the associated curve (3.23). Denoting its branch points by \( a_\alpha, b_\alpha \) (with \( \alpha = 1 \ldots n - 1 \)), choose branch-cuts \( I_\alpha \) connecting \( a_\alpha \) and \( b_\alpha \) (note that we allow \( I_\alpha \) to be curved). This defines two determinations, which we call \( u_0 \) and \( u_1 = -u_0 \). More precisely, \( u_0 \) is the determination which behaves as \( -\frac{1}{2} W'(z) \) for large \( z \). Let us define \( \omega_0 \) by relation (3.22)
and choose a curve $\gamma$ such that $I_\alpha \subset \gamma$ for all $\alpha$. We let $s$ be its length coordinate and $\lambda = \lambda(s)$ the associated parameterization. Choosing the normal $n(s) = i\dot{\lambda}(s)$ (figure 3) we define:

$$
\rho_0(s) := \Lambda(s) \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \left[ \omega_0(\lambda(s) - \epsilon n(s)) - \omega_0(\lambda(s) + \epsilon n(s)) \right]
$$

(3.24)

for every $s$ in $\gamma$. Then $\rho$ vanishes outside of $I_\alpha$ and (3.24) and the Sokhotsky formulae (3.5) imply:

$$
\int ds \frac{\rho_0(s)}{z - \lambda(s)} = \int_{\Gamma} \frac{dx \omega_0(x)}{2\pi i (z - x)} = \omega_0(z)
$$

(3.25)

where $\Gamma$ is a contour surrounding all cuts but not the point $z$ (see figure 4) and the last equality follows by deforming this contour toward infinity to pick the contribution from the residue at $x = z$. This shows that relation (3.18) holds. Using (3.23) and the fact that the leading coefficient of $f_0$ equals $nt_n$ shows that $\omega(z) = 1/z + O(1/z^2)$ for large $|z|$, and combining this with (3.18) shows that $\rho_0$ satisfies the normalization condition (3.2).

Since the cuts $I_\alpha$ connect the branches $u_0$ and $-u_0$, we have:

$$
\lim_{\epsilon \to 0^+} u_0(\lambda + \epsilon n) = \lim_{\epsilon \to 0^+} [-u_0(\lambda - \epsilon n)] , \quad \lambda \in I_\alpha
$$

(3.26)

so that $\lim_{\epsilon \to 0^+} [\omega_0(\lambda + \epsilon n) + \omega_0(\lambda - \epsilon n)] = W'(\lambda)$ for $\lambda \in I_\alpha$. Combining this with equation (3.18) and using the Sokhotsky formulae (3.5) shows that $\rho_0$ satisfies the planar equations of motion (3.21) along the cuts.

To prove (3.20), we use relation (3.18) to compute:

$$
\omega_0(z)^2 = \int ds \int ds' \rho_0(s)\rho_0(s') \frac{1}{(z - \lambda(s))(z - \lambda(s'))} = 2 \int ds \int ds' \rho_0(s)\rho_0(s') \frac{1}{(\lambda(s) - \lambda(s'))(z - \lambda(s))} ,
$$

(3.27)

where we used the identity (3.9) and symmetry of the resulting integrand with respect to the substitution $s \leftrightarrow s'$. Using (3.21) in the right hand side of (3.27) gives:

$$
\omega_0(z)^2 = \int ds \rho_0(z) \frac{W'(\lambda(s))}{z - \lambda(s)} = -\int ds \rho_0(s) \frac{W'(z) - W'(\lambda(s))}{z - \lambda(s)} + W'(z)\omega_0(z).
$$

(3.28)

Comparing with (3.23) and (3.22) shows that equation (3.20) holds.

This construction produces a planar solution of the holomorphic model which recovers any curve of the form (3.23), for an arbitrary choice of degree $n - 2$ polynomial $f_0(z)$ with leading coefficient $nt_n$. Unlike the Hermitian model, the holomorphic model explores the entire family of curves (3.23) in its planar limit. This is a basic pre-requisite for the conjecture of [1].
It is also clear that the precise choice of cuts $I_\alpha$ is irrelevant, as long as they connect the given pairs of branch points $a_\alpha, b_\alpha$. Thus one can use any\(^2\) curve $\gamma$ in this construction, provided that it passes through all the branch points of (3.23).

4. The microcanonical ensemble

The framework of [1] requires that the model obeys certain filling fraction constraints. In this section, I explain how one can impose such constraints on the finite $N$ model\(^3\). The relevant conditions are easiest to formulate by employing a microcanonical ensemble. As we shall see below, the original path integral defines a (grand-)canonical ensemble at zero chemical potentials. This allows one to recover the microcanonical generating function by introducing chemical potentials (which are canonically conjugate to the filling fractions) and then performing a Legendre transform to replace the former by the latter.

4.1 The (grand-)canonical partition function associated with a collection of domains

We start by fixing a collection of domains $D_\alpha$ ($\alpha = 1 \ldots r$) in the complex plane, chosen such that their interiors are mutually disjoint and such that:

$$\bigcup_{\alpha=1}^{r} \overline{D_\alpha} = \mathbb{C}. \quad (4.1)$$

We shall assume that $\gamma$ intersects each closure $\overline{D_\alpha}$ along a single curve segment $\Delta_\alpha$, where $\Delta_\alpha$ lie in ascending order on $\gamma$ with respect to its orientation (figure 5). Condition (4.1) implies:

$$\bigcup_{\alpha=1}^{r} \Delta_\alpha = \gamma. \quad (4.2)$$

For simplicity, we shall take $D_\alpha$ to be infinite strips arranged as in figure 5.

Letting $\chi_\alpha$ denote the characteristic function of $D_\alpha$, consider the matrices $\chi_\alpha(M)$ defined by holomorphic functional calculus:

$$\chi_\alpha(M) = \oint_{\Gamma_\alpha} \frac{dz}{2\pi i} \frac{1}{z - M}, \quad (4.3)$$

\(^2\)Nondegenerate and without self-intersections.

\(^3\)A clear formulation of such constraints beyond the strict large $N$ limit is important in studies of $SO(N)$ and $Sp(N)$ matrix models, whose large $N$ expansion contains terms of order $1/N$ which must be interpreted as contributions to the dual field theory (rather than gravitational contributions, which start at order $1/N^2$). The formulation given below allows one to give clear proofs of relations between the strictly planar and $O(1/N)$ contributions to the (microcanonical) partition function, thereby strengthening arguments like those given in [8]. Similar problems are encountered in more general orientifold models [28].
Figure 5: A choice of strip domains in the complex plane. We also show two of the bounding contours (namely $\Gamma_3$ and $\Gamma_5$). In this example, we take the domains to be infinite strips; this assures us that $\gamma$ and its deformations will cut each such domain along a non-void interval.

with $\Gamma_\alpha$ a (counterclockwise) contour bounding $D_\alpha$ in the complex plane (figure 5). The matrix $\chi_\alpha(M)$ equals the projector on the space spanned by those eigenvectors of $M$ whose eigenvalues lie in $D_\alpha$.

Let $f_\alpha := \frac{1}{N} tr \chi_\alpha(M) = \frac{1}{N} \sum_{j=1}^{N} \chi_\alpha(\lambda_j)$ be the filling fractions for the domains $D_\alpha$. One has:

$$f_\alpha = \int ds \rho(s) \chi_\alpha(\lambda(s)) = \int_{\Gamma_\alpha} \frac{dz}{2\pi i} \omega(z) \ ,$$

(4.4)

since $\omega(z)$ has simple poles with residue $\frac{1}{N}$ at each eigenvalue $\lambda_j$. Relation (4.2) implies $\sum_{\alpha=1}^{r} \chi_\alpha(\lambda) = 1$, so the filling fractions are subject to the constraint:

$$\sum_{\alpha=1}^{r} f_{\alpha} = 1 \ .$$

(4.5)

Picking complex chemical potentials $\mu_\alpha$, we consider the (grand-)canonical ensemble associated with our collection of domains:

$$Z_N(\gamma; t, \mu) = \frac{1}{\mathcal{N}} \int_{\Gamma(\gamma)} dM e^{tr[-NW(M)-N\sum_{\alpha=1}^{r} \mu_\alpha \chi_\alpha(M)]}$$

$$= \int_{\gamma} d\lambda_1 \ldots \int_{\gamma} d\lambda_N \prod_{i \neq j} (\lambda_i - \lambda_j) e^{-N \sum_{j=1}^{N} W(\lambda_j) - N^2 \sum_{\alpha=1}^{r} \mu_\alpha f_\alpha} \ ,$$

(4.6)
which is an analytic function of $\mu_\alpha$. The original partition function results by setting $\mu_\alpha = 0$, and thus it corresponds to a (grand-)canonical ensemble at zero chemical potentials. Notice that the (grand-)canonical partition function can be written:

$$Z_N(\gamma; t, \mu) = \sum_{N_1 + \ldots + N_r = N} \frac{N!}{N_1! \ldots N_r!} e^{-N \sum_{\alpha=1}^{r} \mu_\alpha N_\alpha} Z_{N_1 \ldots N_r}(\gamma; t),$$

where:

$$Z_{N_1 \ldots N_r}(\gamma; t) = \int_\gamma d\lambda_1 \ldots \int_\gamma d\lambda_{N_1} \ldots \int_\gamma d\lambda_{N_1 + \ldots + N_r - 1 + 1} \ldots \int_\gamma d\lambda_N \prod_{i \neq j} (\lambda_i - \lambda_j) e^{-N \sum_{j=1}^{N} W(\lambda_j)}.$$  

**Observation** One can consider a more general version of microcanonical ensemble based on a finite partition of unity, i.e. a finite collection of smooth complex-valued functions $\phi_\alpha(z, \overline{z})$ satisfying the constraint:

$$\sum_\alpha \phi_\alpha = 1.$$  

In certain ways, this is preferable to the approach taken above, since it may be technically desirable to avoid having to deal with characteristic functions. The entire discussion of this section extends easily to this more general set-up. 

Introducing the (grand-)canonical generating function:

$$F_N(\gamma; t, \mu) := -\frac{1}{N^2} \ln Z_N(\gamma; t, \mu),$$

we have the standard relation:

$$\frac{\partial}{\partial \mu_\alpha} F = \langle f_\alpha \rangle.$$  

Here and below, the brackets $\langle \ldots \rangle$ denote the expectation value taken in the (grand-)canonical ensemble.

**4.2 The microcanonical generating function**

Following standard statistical mechanics procedure, we define:

$$S_\alpha := \frac{\partial}{\partial \mu_\alpha} F$$

and perform a Legendre transform to extract the microcanonical generating function:

$$F(\gamma, t, S) := S_\alpha \mu_\alpha(\gamma, t, S) - F(\gamma, t, \mu_\alpha(t, S)).$$
which is an analytic function of the complex variables \( t_\alpha \) and \( S_\alpha \). In this relation, \( \mu_\alpha \) are expressed in terms of \( t \) and \( S \) by solving equations (4.12). The constraint (4.5) and equation (4.11) show that \( S_\alpha \) satisfy:

\[
\sum_{\alpha=1}^{N} S_\alpha = 1 \tag{4.14}
\]

so we can take \( S_1 \ldots S_{r-1} \) to be the independent variables. Then equations (4.12) express \( \mu_\alpha \) as functions of \( t_j \) and these coordinates, and equation (4.13) implies:

\[
\mu_\alpha - \mu_r = \frac{\partial F}{\partial S_\alpha} \quad \text{for} \quad \alpha = 1 \ldots r - 1 \tag{4.15}
\]

This gives the chemical potentials as functions of \( t \) and \( S_\alpha \). Note that \( \mu_\alpha \) are only determined up to a common constant shift; this is due to the constraint (4.14) on \( S_\alpha \). Working with \( F(\gamma, t, S) \) amounts to fixing the expectation values of the filling fractions by imposing the quantum constraint (4.12):

\[
\langle f_\alpha \rangle = \oint_{\gamma_\alpha} \frac{dz}{2\pi i} \langle \omega(z) \rangle = S_\alpha \tag{4.16}
\]

with \( S_\alpha \) treated as fixed parameters. We stress that this condition is only imposed on the expectation values of the filling fractions.

### 4.3 Chemical potentials at large \( N \)

We will now show that the large \( N \) chemical potentials can be expressed as B-type periods of the algebraic curve (3.23), thereby proving that the special geometry relations of [1] hold at the holomorphic matrix level. The argument below is an adaptation of that given in [9], combined with the definition of the microcanonical ensemble given above. In particular, we show that the special geometry relations are simply the large \( N \) limit of the standard equations (4.15). Hence the chemical potentials \( \mu_\alpha \) are the ‘quantum’ (i.e. finite \( N \)) analogues of the B-type periods of [1], while the averaged filling fractions are the ‘quantum’ analogues of the A-type periods. This captures the beautiful intuition of [1].

For this, we start from the expression of the (grand-)canonical generating function in the planar limit:

\[
\mathcal{F}_0(\gamma, t, \mu) = \int dsW(\lambda(s))\rho_0(s) - \mathcal{P} \int ds \int ds' \tilde{K}(\lambda(s), \lambda(s'))\rho_0(s)\rho_0(s') + \sum_{\alpha=1}^{r} \mu_\alpha \int_{I_\alpha} dp_0(s) \tag{4.17}
\]

\footnote{In [1], Dijkgraaf and Vafa gave a beautiful intuitive justification for the special geometry relation \( \Pi_\alpha = \frac{\partial F_0}{\partial S_\alpha} + \text{const} \), where \( F_0 \) is the planar limit of the (microcanonical) generating function while \( S_\alpha \) and \( \Pi_\alpha \) are identified with periods of the curve (3.23). A derivation of this relation (in the context of the Hermitian model) was later given in [9], upon using older results of [20].}
where:
\[
\tilde{K}(\lambda, \lambda') := \ln(\lambda - \lambda') .
\]

(4.18)

In what follows, we shall assume that each cut \(I_\alpha\) lies inside a corresponding domain \(D_\alpha\); in particular, we assume that the number of cuts coincides with the number of domains. With this assumption, we have \(S_\alpha = \langle f_\alpha \rangle = \int_{I_\alpha} d\rho_0(s)\) and the Legendre transform of (4.17) gives the planar limit of the microcanonical generating function:
\[
F_0(\gamma, t, S) = \mathcal{P} \int ds \int ds' \tilde{K}(\lambda(s), \lambda(s'))\rho_0(s)\rho_0(s') - \int ds W(\lambda(s))\rho_0(s) ,
\]
with the constraints:
\[
\int_{I_\alpha} d\rho_0(s) = S_\alpha \quad \text{for} \quad \alpha = 1 \ldots r .
\]

(4.19)

Remember that we allow \(I_\alpha\) to be curved intervals connecting the branch points \(a_\alpha\) and \(b_\alpha\) of the algebraic curve (3.23). In the generic case, none of the cuts is reduced to a double point and one can take \(r = n - 1\).

### 4.3.1 The primitive of \(u_0\) along \(\gamma\)

To extract the large \(N\) chemical potentials, consider the ‘restriction’ of \(u_0\) along \(\gamma\), which we define by:
\[
u_0^p(s) := \frac{1}{2} \lim_{\epsilon \to 0^+} [u_0(\lambda(s) + \epsilon n(s)) + u_0(\lambda(s) - \epsilon n(s))] .
\]

(4.21)

Here \(n(s) = i\dot{\lambda}(s)\).

If \(\lambda(s)\) is a point of \(\gamma\) lying outside the union of \(I_\alpha\), then \(u_0^p(s)\) equals \(u_0(\lambda(s))\), the quantity obtained by substituting \(\lambda(s)\) for \(z\) in the expression:
\[
u_0(z) = \omega_0(z) - \frac{1}{2} W'(z) = \int ds' \frac{\rho_0(s')}{z - \lambda(s')} - \frac{1}{2} W'(z) .
\]

(4.22)

Using (4.22) in the definition of \(u_0^p\) gives:
\[
u_0^p(s) := \mathcal{P} \int ds' \rho_0(s')K(\lambda(s), \lambda(s')) - \frac{1}{2} W'(\lambda(s)) ,
\]
where:
\[
K(\lambda, \lambda') = \frac{1}{\lambda - \lambda'}
\]

(4.24)

is the integral kernel appearing in the planar loop equations and where we used the Sokhotsky identities (3.5).

Consider now the function \(\phi : \gamma \to \mathbb{C}\) defined through:
\[
\phi(s) := 2\mathcal{P} \int ds' \tilde{K}(\lambda(s), \lambda(s'))\rho_0(s') - W(\lambda(s)) .
\]

(4.25)
Noticing that $K(\lambda, \lambda') = \frac{\partial}{\partial \lambda} K(\lambda, \lambda')$, we have:

$$\frac{d}{ds} \phi(s) = 2\dot{\lambda}(s)u_0^\alpha(s) . \tag{4.26}$$

As clear from (4.23), the planar equations of motion (3.21) amount to the requirement that $u_0^\alpha$ vanishes along each of the curve segments $I_\alpha$:

$$u_0^\alpha(s) = 0 \text{ for } \lambda(s) \in I := \cup_{\alpha=1}^{r} I_\alpha . \tag{4.27}$$

This means that $\phi$ is constant along each of these intervals:

$$\phi(s) = \xi_\alpha = \text{ constant on } I_\alpha , \tag{4.28}$$

The jump in the value of $u_0$ between consecutive cuts can be obtained by integrating (4.26):

$$\xi_{\alpha+1} - \xi_\alpha = 2 \int_{b_\alpha}^{a_{\alpha+1}} d\lambda u_0(\lambda) , \tag{4.29}$$

where we used $d\lambda = \dot{\lambda}(s)ds$. This integral is of course taken along $\gamma$.

4.3.2 The planar chemical potentials

Differentiating (4.19) with respect to $S_\alpha$ for some $\alpha < r$ and using relation (4.25) gives:

$$\mu^{(0)}_\alpha - \mu^{(0)}_r = \frac{\partial}{\partial S_\alpha} F_0(\gamma, t, S) = \int_{\cup_{\beta=1}^{r} I_\beta} ds \frac{\partial \rho_0(s)}{\partial S_\alpha} \phi(s) = \xi_\alpha - \xi_r , \tag{4.30}$$

where $\mu^{(0)}_\alpha$ are the planar limits of the chemical potentials. To arrive at the last equality, we used equation (4.28) and noticed that $\int_{I_\beta} \frac{\partial \rho_0(\lambda)}{\partial S_\alpha} = \frac{\partial}{\partial S_\alpha} \int_{I_\beta} d\rho_0(s)$ equals $\delta_{\alpha\beta}$ if $\beta < r$ and $-1$ if $\beta = r$ (by virtue of (4.20) and (4.14)). Relation (4.30) shows that the planar chemical potentials coincide with the quantities $\xi_\alpha$, up to a common additive constant. Using relation (4.29), we obtain:

$$\mu^{(0)}_{\alpha+1} - \mu^{(0)}_\alpha = 2 \int_{b_\alpha}^{a_{\alpha+1}} d\lambda u_0(\lambda) \tag{4.31}$$

As explained above, the quantity $u_0(z)$ has cuts precisely along the intervals $I_\alpha$. Since the other branch of (3.23) is given by $u_1(z) = -u_0(z)$, this allows us to write (4.31) in the form:

$$\mu^{(0)}_\alpha - \mu^{(0)}_{\alpha+1} = \int_{b_\alpha}^{a_{\alpha+1}} dz [u_1(z) - u_0(z)] = \oint_{B_\alpha} dz u(z) , \tag{4.32}$$

22
Consider the cycles $\tilde{B}_\alpha = \sum_{\beta=0}^{r-1} B_\beta$ for all $\alpha = 1 \ldots r - 1$. Then (4.32) implies:

$$\mu^{(0)}_\alpha = \mu^{(0)}_r + \oint_{\tilde{B}_\alpha} dz u(z) \quad \text{for} \quad \alpha = 1 \ldots r - 1 .$$

(4.33)

The quantity $\mu^{(0)}_r$ is undetermined and can be fixed arbitrarily. Following [1], we take $\mu^{(0)}_r = \oint_{B_r} dz u(z)$, where $\Lambda$ is a point close to infinity and $B_r(\Lambda)$ is the cycle described in figure 6. Defining $B_\alpha(\Lambda) = \tilde{B}_\alpha + B_r(\Lambda)$ for all $\alpha = 1 \ldots r - 1$, we obtain:

$$\mu^{(0)}_\alpha = \Pi_\alpha \quad \text{for} \quad \alpha = 1 \ldots r ,$$

(4.34)

with:

$$\Pi_\alpha := \oint_{B_\alpha} dz u(z) .$$

(4.35)

Relation (4.34) shows that the chemical potentials $\mu_\alpha$ are the finite $N$ analogues of the periods $\Pi_\alpha$.

Note also that the filling fractions can be expressed as periods of the meromorphic differential $udz$ over the cycles $A_\alpha$ of figure 6:

$$S_\alpha = \oint_{\gamma_\alpha} \frac{dz}{2\pi i} \omega_0(z) = \oint_{\gamma_\alpha} \frac{dz}{2\pi i} u_0(z) = \oint_{A_\alpha} \frac{dz}{2\pi i} u(z) .$$

(4.36)

In the second equality, we used relation (3.22) and the fact that $W(z)$ is a polynomial. Equation (4.15) now gives the special geometry relation of [1]:

$$\Pi_\alpha - \Pi_r = \frac{\partial}{\partial S_\alpha} F_0 .$$

(4.37)
Note that the quantity in the right hand side is the planar limit of the microcanonical generating function.

It is clear from figure 6 that \( A_\alpha \cap \bar{B}_\alpha = -A_\alpha \cap \bar{B}_{\alpha-1} = +1 \) and \( A_\alpha \cap \bar{B}_\beta = 0 \) if \( \beta \neq \alpha, \alpha - 1 \). This gives \( A_\alpha \cap B_\beta = -B_\alpha \cap A_\beta = \delta_{\alpha\beta} \). Since we also have \( A_\alpha \cap A_\beta = B_\alpha \cap B_\beta = 0 \), it follows that \( A_\alpha, B_\beta \) have canonical intersection form. Hence we have a canonical system of cycles \( A_\alpha, B_\alpha \) with \( \alpha = 1 \ldots r \):

\[
A_\alpha \cap B_\beta = -B_\alpha \cap A_\beta = \delta_{\alpha\beta}, \quad A_\alpha \cap A_\beta = B_\alpha \cap B_\beta = 0 \quad \text{for all} \quad \alpha = 1 \ldots r.
\]

(4.38)

This shows that the properties essential for the conjecture of [1] hold at the level of the holomorphic matrix model.

5. Holomorphic ADE models

In this section, we consider the case of holomorphic ADE models, focusing on the simple example of the holomorphic \( A_2 \) model. As mentioned in the introduction, the Hermitian approach to such models (and its attending regularization, discussed in [22]) leads to various technical problems which would violate the conjecture of [1]. The purpose of this section is to show explicitly how such issues are avoided in the holomorphic set-up, and to provide a reconstruction theorem similar to the one found for the one-matrix case. In particular, we shall extract explicitly the associated Riemann surface (which is expected from the work of of [3]) and show that one must use a certain ‘renormalization’ procedure in order eliminate unwanted constraints on its moduli. In fact, we shall find that the curve expected from the work of [3] and [23] can be obtained by using a certain regulator (which is natural in the holomorphic set-up), though only in the limit where this regulator is removed. As we shall see below, this limit can be described as a statistical ensemble of ‘reduced’ holomorphic models.

5.1 Construction of the models

Consider an ADE quiver diagram with nodes indexed by \( \alpha = 1 \ldots \kappa \) where \( \kappa \) is the rank of the associated simply-laced group. Consider \( N_\alpha \times N_\alpha \) complex matrices \( \Phi^{(\alpha)} \) for each node, and \( N_\alpha \times N_\beta \) complex matrices \( Q^{(\alpha\beta)} \) for each pair of nodes \( \alpha, \beta \) which are connected by an edge (in particular, we have two matrices \( Q^{(\alpha\beta)} \) and \( Q^{(\beta\alpha)} \) for each edge of the quiver). We let \( s_{\alpha\beta} = s_{\beta\alpha} \) be the incidence matrix of the quiver and \( c_{\alpha\beta} = 2\delta_{\alpha\beta} - s_{\alpha\beta} \) be the associated Cartan matrix.

By analogy with [22], we define the holomorphic ADE matrix model associated with this quiver by:

\[
\mathcal{Z} = \int \prod_{\alpha=1}^{\kappa} d\Phi^{(\alpha)} \prod_{\alpha<\beta} [dQ^{(\alpha\beta)} dQ^{(\beta\alpha)}] e^{-N \sum_{\alpha=1}^{\kappa} W_\alpha(\Phi^{(\alpha)}) + W_{\text{int}}(\Phi, Q)},
\]

(5.1)
where \( W_\alpha \) are polynomials of degrees \( n_\alpha \) and:
\[
W_{\text{int}}(\Phi, Q) := \sum_{\alpha < \beta} s_{\alpha \beta} \left[ \text{tr}(Q^{(\alpha \beta)}\Phi^{(\beta)}) - \text{tr}(Q^{(\beta \alpha)}\Phi^{(\alpha)}) \right].
\]
(5.2)

The gauge group is:
\[
G := \prod_{\alpha=1}^\kappa GL(N_\alpha, \mathbb{C}),
\]
(5.3)
with the the obvious action:
\[
(\Phi^{(\alpha)}, Q^{(\alpha \beta)}) \rightarrow (S_\alpha \Phi^{(\alpha)} S^{-1}_\alpha, S_\alpha Q^{(\alpha \beta)} S^{-1}_\beta)
\]
(5.4)
for \( S_\alpha \in GL(N_\alpha, \mathbb{C}) \).

To completely specify the model, one must choose an appropriate integration manifold \( \Gamma \). Before explaining our choice, let us comment on the Hermitian approach [22]. In that case, one takes \( \Gamma \) to consists of matrices \( \Phi^{(\alpha)}, Q^{(\alpha \beta)} \) such that \( \Phi^{(\alpha)} \) are Hermitian and \( Q^{(\beta \alpha)} = (Q^{(\alpha \beta)})^\dagger \) for neighboring nodes \( \alpha \) and \( \beta \). Such a prescription makes sense only if all \( W_\alpha \) have even degree (otherwise, the integral diverges because the absolute value of the integrand explodes when the norm of some \( \Phi^{(\alpha)} \) is large). However, the Hermitian prescription immediately leads to other problems, arising from the integrals over \( Q \). Indeed, it is easy to see that these will bring infinite contributions when some eigenvalue \( \lambda_i^{(\alpha)} \) of \( \Phi^{(\alpha)} \) coincides with some eigenvalue \( \lambda_j^{(\beta)} \) of \( \Phi^{(\beta)} \) for neighboring \( \alpha \) and \( \beta \). In the Hermitian framework, the solution to this problem is to require that the eigenvalues of neighboring \( \Phi^{(\alpha)} \) can never coincide — for example, by taking \( \Phi^{(\alpha)} \) to have alternately negative and positive eigenvalues [22]. It turns out that this prescription would violate the conjecture of [1]. To understand why, consider for simplicity the Hermitian \( A_2 \) model (whose holomorphic version is studied below). The large \( N \) limit of this Hermitian model can be extracted by an argument which is formally identical to that presented in Appendix 3\(^5\), and is governed by an algebraic curve which is a triple cover of the complex plane. Denoting its three branches by \( u_1, u_2 \) and \( u_3 \) (in a convenient enumeration), one finds that cuts connecting \( u_1 \) and \( u_2 \) would correspond precisely to loci where \textit{equal} eigenvalues of \( \Phi^{(1)} \) and \( \Phi^{(2)} \) accumulate — a situation which is forbidden by the regularization prescription used to define the model! Therefore, the regularization prescription of the Hermitian model requires that all such cuts are reduced to double points, which means that the large \( N \) curve must always be singular. Moreover, this violates the requirements of the dual field theory, since it would require that some filling fractions vanish identically, thereby violating the study of gaugino.

---

\(^5\)Except that the third equation in (5.31) used in the appendix never plays any role for the regularization of [22].
condensates performed in [23]. Of course, the nonzero cuts of the resulting curve would also be constrained to alternately lie on the positive and negative halves of the real axis. It should be clear from this discussion that Hermitian ADE models are quite unnatural for the conjecture of [1]. Below, we shall show how a certain regularization prescription of the holomorphic $A_2$ model allows one to obtain an ensemble whose large $N$ limit satisfies the basic requirements of the conjecture. Unlike the Hermitian regularization of [22], the ‘holomorphic’ regulator used below can be removed in a manner which allows us to recover a smooth Riemann surface.

Returning to the holomorphic model, we shall choose the integration manifold $\Gamma$ as follows:

1. Fix contours $\gamma^{(\alpha)}$ in the complex plane such that:
   \[ \gamma^{(\alpha)} \cap \gamma^{(\beta)} = \emptyset \quad \text{for neighboring } \alpha \text{ and } \beta \, . \]  

and such that each $\gamma^{(\alpha)}$ connects two convergence sectors of $W_\alpha$ (as defined in Section 2).

2. Let $\Delta(\gamma)$ be the set of all matrices $(D^{(\alpha)}, Q^{(\alpha\beta)})$ (with $\alpha, \beta = 1 \ldots \kappa$) which satisfy:
   \( a \) $D^{(\alpha)} = \text{diag}(\lambda_1^{(\alpha)} \ldots \lambda_{N_\alpha}^{(\alpha)})$, with distinct $\lambda_1^{(\alpha)} \ldots \lambda_{N_\alpha}^{(\alpha)}$.
   \( b \) $\lambda_j^{(\alpha)}$ lies along $\gamma^{(\alpha)}$ for each $\alpha$ and $j$.
   \( c \) $Q^{(\beta\alpha)}_{ji} = \bar{Q}^{(\alpha\beta)}_{ij}$ (here the bar denotes complex conjugation).

3. Finally, we let $\Gamma$ be the union of $G$-orbits of elements of $\Delta$ under the action (5.4). Gauge-fixing the action (5.1) gives the eigenvalue representation:

\[ Z = \int_{\gamma^{(1)}} d\lambda_1^{(1)} \ldots \int_{\gamma^{(1)}} d\lambda_{N_1}^{(1)} \ldots \int_{\gamma^{(\rho)}} d\lambda_1^{(\rho)} \ldots \int_{\gamma^{(\rho)}} d\lambda_{N_\rho}^{(\rho)} \prod_{\alpha \neq \beta} \frac{(\lambda_1^{(\alpha)} - \lambda_j^{(\alpha)})}{(\lambda_j^{(\alpha)} - \lambda_1^{(\alpha)})} e^{-N \sum_{\rho=1}^{N} \sum_{i=1}^{N_{\rho}} W_\alpha(\lambda_i^{(\alpha)})} \]  

where we dropped constant prefactors and used the identity:

\[ \int_\sigma du \wedge dve^{-(\lambda - \lambda')uv} = - \frac{4\pi i}{\lambda - \lambda'} \]  

with $\lambda \neq \lambda'$ complex and $\sigma$ the contour in $\mathbb{C}^2$ given by:

\[ v = \frac{\bar{u}}{\lambda - \lambda'} \]  

Condition (5.5) acts as a ‘complex’ regulator for the holomorphic model, by preventing common eigenvalues of neighboring $\Phi^{(\alpha)}$ and $\Phi^{(\beta)}$. As in the Hermitian case,

---

6 To be more precise, one has to take into account the action of an appropriate group of permutations on the set $\Delta$. This can be done as in Appendix 1 by working with a fundamental domain of $\Delta$ under this discrete action.
working with the regularized model would therefore not suffice to recover the entire moduli space of planar solutions required by the conjecture of [1]. To eliminate the constraints, the regularization condition (5.5) must be removed by taking the limit of coinciding $\gamma^{(a)}$. This limit can be performed in such a way that the end result is a ‘renormalized’ model which can be described as a statistical ensemble of ‘reduced’ holomorphic models. We now show how this works for the case of holomorphic $A_2$ models.

### 5.2 Example: the holomorphic $A_2$ model

For an $A_2$ quiver, one has $\rho = 2$ and four matrices $\Phi^{(1)}$, $\Phi^{(2)}$, $Q^{(12)}$ and $Q^{(21)}$. The partition function takes the form:

$$Z = \int_{\gamma_1^{(1)}} d\lambda^{(1)}_1 \ldots \int_{\gamma_N^{(1)}} d\lambda^{(1)}_N \int_{\gamma_1^{(2)}} d\lambda^{(2)}_1 \ldots \int_{\gamma_N^{(2)}} d\lambda^{(2)}_N \frac{\prod_{i \neq j} (\lambda^{(1)}_i - \lambda^{(1)}_j) \prod_{i \neq j} (\lambda^{(2)}_i - \lambda^{(2)}_j)}{\prod_{i,j} (\lambda^{(1)}_i - \lambda^{(2)}_j)} e^{-N \sum_{i=1}^{N_1} W_1(\lambda^{(1)}_i) - N \sum_{i=1}^{N_2} W_2(\lambda^{(2)}_i)}$$  

(5.9)

for two disjoint curves $\gamma^{(1)}$ and $\gamma^{(2)}$.

#### 5.2.1 Classical vacua

Extremizing the action:

$$S = N \text{tr} [W_1(\Phi^{(1)}) + W_2(\Phi^{(2)})] + \text{tr} [Q^{(21)} \Phi^{(1)} Q^{(12)} - Q^{(12)} \Phi^{(2)} Q^{(21)}]$$  

(5.10)

gives the equations:

$$NW_1'(\Phi^{(1)}) = -Q^{(12)} Q^{(21)}, \quad NW_2'(\Phi^{(2)}) = Q^{(21)} Q^{(12)},$$  

$$\Phi^{(2)} Q^{(21)} = Q^{(21)} \Phi^{(1)}, \quad \Phi^{(1)} Q^{(12)} = Q^{(12)} \Phi^{(2)}.$$  

(5.11)

Combining these, one easily obtains:

$$W_1'(\Phi^{(1)}) [W_1'(\Phi^{(1)}) + W_2'(\Phi^{(1)})] = 0$$

$$W_2'(\Phi^{(2)}) [W_1'(\Phi^{(2)}) + W_2'(\Phi^{(2)})] = 0.$$  

(5.12)

Assuming that $\Phi^{(1)}$ and $\Phi^{(2)}$ are diagonalizable with eigenvalues $\lambda_i^{(1)}$ and $\lambda_j^{(2)}$, one finds:

$$W_1'(\lambda_i^{(1)}) [W_1'(\lambda_i^{(1)}) + W_2'(\lambda_i^{(1)})] = 0$$

$$W_2'(\lambda_j^{(2)}) [W_1'(\lambda_j^{(2)}) + W_2'(\lambda_j^{(2)})] = 0.$$  

(5.13)

On the other hand, the last row of equations in (5.11) gives:

$$(\lambda_j^{(2)} - \lambda_i^{(1)}) Q_{ji}^{(21)} = 0$$

$$(\lambda_i^{(1)} - \lambda_j^{(2)}) Q_{ij}^{(12)} = 0.$$  

(5.14)
In the generic case when $W_1'$ and $W_2'$ have no common roots, equations (5.13) and (5.14) show that a typical classical vacuum is specified (up to permutations of indices) by choosing:

1. roots $\lambda_1^{(1)} \ldots \lambda_{N_1-k}^{(1)}$ of $W_1'$,
2. roots $\lambda_1^{(2)} \ldots \lambda_{N_2-k}^{(2)}$ of $W_2'$,
3. roots $\mu_1 \ldots \mu_k$ of $W_1' + W_2'$ such that $\lambda_{N_1-k+m}^{(1)} = \lambda_{N_1-k+m}^{(2)} = \mu_m$ for $m = 1 \ldots k$.
4. some nonzero values for $Q_{N_1 - k + m, N_1 - k + m}^{(12)}$ and $Q_{N_1 - k + m, N_2 - k + m}^{(12)}$ for $m = 1 \ldots k$.

In fact, one can set $Q_{N_1 - k + m, N_2 - k + m}^{(12)} = 1$ for all $m = 1 \ldots k$ by using the gauge transformations.

Less generic vacua arise, for example, by allowing some of the $\mu_k$ to coincide. Notice that vacua with $k \neq 0$ (i.e., vacua for which some $\lambda_i^{(1)}$ coincide with some $\lambda_j^{(2)}$) are removed when imposing the condition $\gamma^{(1)} \cap \gamma^{(2)} = \emptyset$ — such vacua will not be visible unless one removes this condition on the contours. We now proceed to remove this regulator, by taking the limit in which the two curves coincide.

### 5.2.2 The limit of coinciding contours

To study the limit when $\gamma^{(1)}$ and $\gamma^{(2)}$ coincide, we let $\gamma^{(2)} = \gamma$, $\gamma^{(1)} = \gamma + \eta m$ (where $n$ is the normal vector field to $\gamma$, chosen as in figure 3 of Section 3) and take the positive quantity $\eta$ to zero (note that this requires $\gamma$ to asymptote to lines lying in the intersection of some convergence sectors of $W_1$ and $W_2$). For small $\eta$, we can use the length coordinate $s$ of $\gamma = \gamma^{(2)}$ also as a parameter along $\gamma^{(1)}$. We then have $\lambda^{(1)}(s) = \lambda(s) + \eta n(s)$, where $\lambda(s)$ is the parameterization of $\gamma$. For $\eta \to 0^+$, the Sokhotsky formulae (3.5) give:

$$\frac{1}{\lambda_i^{(1)} - \lambda_j^{(2)}} = \mathcal{P} \frac{1}{\lambda(s_i^{(1)}) - \lambda(s_j^{(2)})} - \frac{i\pi}{\lambda s_j^{(2)}} \delta(s_i^{(1)} - s_j^{(2)}) \ .$$

(5.15)

Therefore, the denominator in the integrand of (5.9) takes the following form:

$$\lim_{\eta \to 0^+} \frac{1}{\prod_{i,j} (\lambda_i^{(1)} - \lambda_j^{(2)})} = \sum_{k=0}^{\min(N_1, N_2)} \sum_{1 \leq i_1 < \ldots < i_k \leq N_1} \sum_{1 \leq j_1 < \ldots < j_k \leq N_2} \prod_{(i,j) \neq (i_1,j_{\sigma(1)}) \ldots (i_k,j_{\sigma(k)})} \mathcal{P} \frac{1}{\lambda_i^{(1)} - \lambda_j^{(2)}} \ .$$

(5.16)

where $\Sigma_k$ is the group of permutations on $k$ elements. The ‘higher incidence terms’ are terms involving delta-function products of the type $\delta(s_i^{(1)} - s_j^{(2)})\delta(s_i^{(1)} - s_k^{(2)}) = \delta(s_i^{(1)} - s_j^{(2)})\delta(s_j^{(2)} - s_k^{(2)})$ with distinct $i, j, k$. Such terms can be neglected since — as we shall see in a moment — they do not contribute to the final result.
Substituting (5.16) in (5.9) gives:

\[ Z_{\text{lim}}(\gamma) = \sum_{k=0}^{\min(N_1, N_2)} (-i\pi)^k \frac{N_1!N_2!}{k!(N_1-k)!(N_2-k)!} Z_k(\gamma), \]  

(5.17)

where:

\[ Z_k(\gamma) = \int_{\gamma} d\mu_1 \cdots \int_{\gamma} d\mu_k \int_{\gamma} d\lambda_1^{(1)} \cdots \int_{\gamma} d\lambda_{N_1-k}^{(1)} \int_{\gamma} d\lambda_1^{(2)} \cdots \int_{\gamma} d\lambda_{N_2-k}^{(2)} \Delta_k e^{-NS_k}, \]  

(5.18)

with:

\[ \Delta_k := \bar{\Delta}_k \mathcal{P} \frac{1}{\prod_{i=1 \ldots N_1-k}^{j=1 \ldots N_2-k} (\lambda_i^{(1)} - \lambda_j^{(2)})}, \]  

(5.19)

\[ \bar{\Delta}_k = \prod_{i,j=1 \ldots N_1-k}^{i=j=1 \ldots N_2-k} (\lambda_i^{(1)} - \lambda_j^{(1)}) \prod_{i,j=1 \ldots N_2-k}^{i=j=1 \ldots N_2-k} (\lambda_i^{(2)} - \lambda_j^{(2)}) \prod_{i,j=1 \ldots N_1-k}^{i=j=1 \ldots N_1-k} (\mu_i - \mu_j) \]  

(5.20)

and:

\[ S_k := \sum_{j=1}^{N_1-k} W_1(\lambda_j^{(1)}) + \sum_{j=1}^{N_2-k} W_1(\lambda_j^{(2)}) + \sum_{j=1}^{k} [W_1(\mu_j) + W_2(\mu_j)] . \]  

(5.21)

Notice that the ‘higher incidence terms’ of equation (5.16) bring zero contribution to (5.17). This is due to the extra-factors of \( \mu_j - \mu_k \) contributed by the two products in the numerator of the weight factor of (5.9).

**Observation** Consider a quantity \( H(z) \) (which depends on \( \lambda_j^{(a)} \) but is symmetric under separate permutations of \( \lambda_1^{(1)} \ldots \lambda_{N_1}^{(1)} \) and of \( \lambda_1^{(2)} \ldots \lambda_{N_2}^{(2)} \)). Then its average \( \langle H(z) \rangle \) has the following behavior in the limit \( \eta \to 0^+ \):

\[ \lim_{\eta \to 0} \langle H(z) \rangle = \frac{\sum_{k=0}^{\min(N_1, N_2)} (-i\pi)^k \frac{N_1!N_2!}{k!(N_1-k)!(N_2-k)!} Z_k(\langle H(z) \rangle)_k}{\sum_{k=0}^{\min(N_1, N_2)} (-i\pi)^k \frac{N_1!N_2!}{k!(N_1-k)!(N_2-k)!} Z_k(\gamma)} := \langle H(z) \rangle_{\text{lim}}, \]  

(5.22)

where:

\[ \langle H(z) \rangle_k = \frac{1}{Z_k} \int_{\gamma} d\mu_1 \cdots \int_{\gamma} d\mu_k \int_{\gamma} d\lambda_1^{(1)} \cdots \int_{\gamma} d\lambda_{N_1-k}^{(1)} \int_{\gamma} d\lambda_1^{(2)} \cdots \int_{\gamma} d\lambda_{N_2-k}^{(2)} \Delta_k H(z) e^{-NS_k}. \]  

(5.23)

Thus \( \langle H(z) \rangle_{\text{lim}} \) is simply a weighted average taken over the limiting ensemble.
5.2.3 Equations of motion for the limiting ensemble

Let us fix a component $k$ of the limiting ensemble, and work with the model defined by the partition function $Z_k$. Writing $\Delta_k$ as an exponential gives:

$$Z_k(\gamma) = \int d\sigma_1 \ldots \int d\sigma_k \int ds_1^{(1)} \ldots \int ds_{N_1-k}^{(1)} \int ds_1^{(2)} \ldots \int ds_{N_2-k}^{(2)} e^{-N S_k^{eff}} ,$$

where:

$$S_k^{eff} = S_k - \frac{1}{N} \sum_{i,j=1 \ldots N_1-k} \ln(\lambda_i^{(1)} - \lambda_j^{(1)}) - \frac{1}{N} \sum_{i,j=1 \ldots N_2-k} \ln(\lambda_i^{(2)} - \lambda_j^{(2)}) - \frac{1}{N} \sum_{i,j=1 \ldots k} \ln(\mu_i - \mu_j)$$

$$- \frac{1}{N} \sum_{i=1 \ldots N_1-k \atop j=1 \ldots k} \ln(\mu_j - \lambda_i^{(1)}) - \frac{1}{N} \sum_{i=1 \ldots N_2-k \atop j=1 \ldots k} \ln(\lambda_i^{(2)} - \mu_j) + \frac{1}{N} \sum_{i=1 \ldots N_1-k \atop j=1 \ldots N_2-k} \ln(\lambda_i^{(1)} - \lambda_j^{(2)})$$

$$- \frac{1}{N} \sum_{j=1}^{N_1-k} \ln(\lambda_{\sigma_j}^{(1)}) - \frac{1}{N} \sum_{j=1}^{N_2-k} \ln(\lambda_{\sigma_j}^{(2)}) - \frac{1}{N} \sum_{j=1}^{k} \ln(\lambda_{\sigma_j})$$

and we wrote $\lambda_i^{(1)} = \lambda(s_i^{(1)})$, $\lambda_i^{(2)} = \lambda(s_i^{(2)})$ and $\mu_i = \lambda(\sigma_i)$. Extremizing this with respect to $s_i^{(a)}$ and $\sigma_i$ gives the equations of motion:

$$\frac{2}{N} \sum_{j=1}^{N_1-k} \frac{1}{\lambda(s_i^{(1)} - \lambda(s_j^{(1)})} - \frac{1}{N} \sum_{j=1}^{N_2-k} \frac{1}{\lambda(s_i^{(2)} - \lambda(s_j^{(2)})} + \frac{1}{N} \sum_{j=1}^{k} \frac{1}{\lambda(s_i^{(2)} - \lambda(s_j^{(2)})}$$

$$= W'_1(\lambda(s_i^{(1)})) - \frac{1}{N} \frac{\dot{\lambda}(s_i^{(1)})}{\lambda(s_i^{(1)})^2}$$

$$\frac{2}{N} \sum_{j=1}^{N_1-k} \frac{1}{\lambda(s_i^{(2)} - \lambda(s_j^{(2)})} - \frac{1}{N} \sum_{j=1}^{N_2-k} \frac{1}{\lambda(s_i^{(2)} - \lambda(s_j^{(1)})} + \frac{1}{N} \sum_{j=1}^{k} \frac{1}{\lambda(s_i^{(2)} - \lambda(s_j^{(2)})}$$

$$= W'_2(\lambda(s_i^{(2)})) - \frac{1}{N} \frac{\dot{\lambda}(s_i^{(2)})}{\lambda(s_i^{(2)})^2}$$

$$\frac{2}{N} \sum_{j=1}^{k} \frac{1}{\lambda(s_i^{(1)} - \lambda(s_j^{(1)})} + \frac{1}{N} \sum_{j=1}^{N_1-k} \frac{1}{\lambda(s_i^{(1)} - \lambda(s_j^{(1)})} + \frac{1}{N} \sum_{j=1}^{N_2-k} \frac{1}{\lambda(s_i^{(1)} - \lambda(s_j^{(1)})}$$

$$= W'_1(\lambda(\sigma_i)) + W'_2(\lambda(\sigma_i)) - \frac{1}{N} \frac{\dot{\lambda}(\sigma_i)}{\lambda(\sigma_i)^2}$$

Let us introduce the spectral densities:

$$\rho^{(a)}(s) := \frac{1}{N} \sum_{j=1}^{N_1-k} \delta(s - s_j^{(a)}) + \frac{1}{N} \sum_{j=1}^{k} \delta(s - \sigma_j) ,$$

with the normalization:

$$\int ds \rho^{(a)}(s) = \frac{N_0}{N} .$$

We also introduce the traced resolvents:

$$\omega^{(a)}(z) := \frac{1}{N} \sum_{j=1}^{N_1-k} \frac{1}{z - \lambda(s_j^{(a)})} + \frac{1}{N} \sum_{j=1}^{k} \frac{1}{z - \lambda(s_j)} = \int ds \frac{\rho^{(a)}(s)}{z - \lambda(s)} .$$
5.2.4 The large $N$ Riemann surface

For every quantity $\phi$, we define the large $N$ expansion coefficients $\phi_j$ as in equations (3.17). In particular, we have the planar limits $\rho_0^{(a)}$ and:

$$\omega_0^{(a)}(z) = \int ds \frac{\rho_0^{(a)}(s)}{z - \lambda(s)}.$$  \((5.30)\)

In the large $N$ limit, the eigenvalues $\lambda(s_j^{(1)})$, $\lambda(s_j^{(2)})$ and $\lambda(\sigma_k)$ accumulate on curve segments sitting along $\gamma$. The planar limits (5.30) will have cuts along three types of curvilinear intervals:

(1) loci $C_{13}^a$ resulting from a planar distribution of the eigenvalues $\lambda(s_j^{(1)})$ with $j = 1 \ldots N_1 - k$; these will be cuts of $\omega_0^{(1)}$

(2) loci $C_{23}^b$ supporting a distribution of the eigenvalues $\lambda(s_j^{(2)})$ with $j = 1 \ldots N_2 - k$; these give cuts of $\omega_0^{(2)}$

(3) loci $C_{12}^c$ resulting from a distribution of $\lambda(\sigma_j)$ with $j = 1 \ldots k$; they give common cuts of $\omega_0^{(1)}$ and $\omega_0^{(2)}$.

Here $a, b, c$ are indices counting the various occurrences of each type of cut. Note that the third type of locus can only arise from a component of the limiting ensemble for which $\frac{k}{N}$ has a finite limit as $N \to \infty$. Thus cuts of type $C_{12}^c$ can only develop in the large $N$ limit of the ‘renormalized’ model with $\gamma^{(1)} = \gamma^{(2)}$.

The planar limit of the equations of motion (5.26) gives:

$$2 \int ds' \frac{\rho_0^{(1)}(s')}{\lambda(s) - \lambda(s')} - \int ds' \frac{\rho_0^{(2)}(s')}{\lambda(s) - \lambda(s')} = W_1'(\lambda(s)) , \ \lambda(s) \in C_{13}^a$$

$$2 \int ds' \frac{\rho_0^{(2)}(s')}{\lambda(s) - \lambda(s')} - \int ds' \frac{\rho_0^{(1)}(s')}{\lambda(s) - \lambda(s')} = W_2'(\lambda(s)) , \ \lambda(s) \in C_{23}^b$$

$$\int ds' \frac{\rho_0^{(1)}(s')}{\lambda(s) - \lambda(s')} + \int ds' \frac{\rho_0^{(2)}(s')}{\lambda(s) - \lambda(s')} = W_1'(\lambda(s)) + W_2'(\lambda(s)) , \ \lambda(s) \in C_{12}^c.$$  \((5.31)\)

These relations act as large $N$ saddle point equations for the limiting ensemble (5.17). They also represent the ‘quantum’ version of the three branches $W_1'(\lambda) = 0$, $W_2'(\lambda) = 0$ and $W_1'(\lambda) + W_2'(\lambda) = 0$ of the classical equations of motion (5.13). It is also clear that we have:

$$\rho_0^{(1)}(s) = \rho_0^{(2)}(s) , \ \lambda(s) \in C_{12}^c.$$  \((5.32)\)

This is the analogue of the classical relations $\lambda_i^{(1)} = \lambda_i^{(2)}$ on the corresponding branch of the moduli space (see point (3) of Subsection 5.1.1).
As explained in Appendix 3, one can use (5.31) and certain partial fraction decompositions to derive the algebraic constraints:

\[
\begin{align*}
\omega_0^{(1)}(z)^2 - \omega_0^{(1)}(z)\omega_0^{(2)}(z) + \omega_0^{(2)}(z)^2 &= W_1'(z)\omega_0^{(1)}(z) - W_2'(z)\omega_0^{(2)}(z) + f_0^{(1)}(z) + f_0^{(2)}(z) = 0 \\
\omega_0^{(1)}(z)^2 \omega_0^{(2)}(z) - W_1'(z) [\omega_0^{(1)}(z)^2 + f_0^{(1)}(z) - W_1'(z)\omega_0^{(1)}(z)] + g_0^{(1)}(z) - (1 \leftrightarrow 2) &= 0
\end{align*}
\]

where \( f_0^{(a)} \) and \( g_0^{(a)} \) are polynomials defined through:

\[
f_0^{(a)}(z) := \int ds\rho_0^{(a)}(s)\frac{W'_a(z) - W'_a(\lambda(s))}{z - \lambda(s)} \tag{5.34}
\]

and

\[
g_0^{(1)}(z) := \int d\alpha \int d\beta [W'_1(z) - W'_1(\lambda(\alpha))] \frac{\rho_0^{(1)}(\alpha)\rho_0^{(2)}(\beta)}{(\lambda(\alpha) - \lambda(\beta))(z - \lambda(\alpha))} \\
g_0^{(2)}(z) := \int d\alpha \int d\beta [W'_2(z) - W'_2(\lambda(\alpha))] \frac{\rho_0^{(2)}(\alpha)\rho_0^{(1)}(\beta)}{(\lambda(\alpha) - \lambda(\beta))(z - \lambda(\alpha))} . \tag{5.35}
\]

Note that the normalization conditions (5.28) imply:

\[
lcoeff(f_0^{(a)}) = \frac{N_a}{N} \text{deg}(W_a)lcoeff(W_a) . \tag{5.36}
\]

where \( lcoeff(\ldots) \) denotes the leading coefficient. Writing:

\[
\begin{align*}
\omega_0^{(1)}(z) &= u_1(z) + t_1(z) \\
\omega_0^{(2)}(z) &= -u_2(z) + t_2(z) \tag{5.37}
\end{align*}
\]

where:

\[
\begin{align*}
t_1(z) &:= \frac{2W_1'(z) + W_2'(z)}{3} \\
t_2(z) &:= \frac{2W_2'(z) + W_1'(z)}{3} \tag{5.38}
\end{align*}
\]

brings the constraints (5.33) to the form:

\[
\begin{align*}
u_1(z)^2 + u_1(z)u_2(z) + u_2(z)^2 &= p(z) \\
u_1(z)^2u_2(z) + u_1(z)u_2(z)^2 &= -q(z) \tag{5.39}
\end{align*}
\]

where:

\[
\begin{align*}
p(z) &:= t_1(z)^2 - t_1(z)t_2(z) + t_2(z)^2 - f_0^{(1)}(z) - f_0^{(2)}(z) \tag{5.40} \\
q(z) &:= -t_1(z)t_2(z) [t_1(z) - t_2(z)] - t_1(z)f_0^{(2)}(z) + t_2(z)f_0^{(1)}(z) - g_0(z) . \tag{5.41}
\end{align*}
\]
In the last equation, we introduced the polynomial $g_0(z) = g_0^{(1)}(z) - g_0^{(2)}(z)$ which has the following explicit form in terms of matrix model data:

$$
g_0(z) = \int d\alpha \int d\beta [W'_1(z) - W'_1(\lambda(\alpha))] \frac{\rho_0^{(1)}(\alpha)\rho_0^{(2)}(\beta)}{(\lambda(\alpha) - \lambda(\beta))(z - \lambda(\alpha))} - \int d\alpha \int d\beta [W'_2(z) - W'_2(\lambda(\alpha))] \frac{\rho_0^{(2)}(\alpha)\rho_0^{(1)}(\beta)}{(\lambda(\alpha) - \lambda(\beta))(z - \lambda(\alpha))} \quad (5.41)
$$

Defining $u_3(z) := -u_1(z) - u_2(z)$, identities (5.39) can be recognized as the Viete relations of the cubic:

$$u^3 - p(z)u - q(z) = 0 \quad (5.42)$$

when the left hand side is viewed as a polynomial in $u$. This shows that $u_1(z)$, $u_2(z)$ and $u_3(z)$ are the three branches of the affine algebraic curve (5.42). This is the precise form of the curve suggested in [3] (where the explicit form of the polynomials $p, q$ in terms of matrix model data was not given). Note that the left hand side of (5.42) can also be written as:

$$(u-u_1)(u-u_2)(u-u_3) = (u+t_1)(u-t_2)(u-t_1+t_2) + (f_0^{(1)} + f_0^{(2)})u + t_1 f_0^{(2)} - t_2 f_0^{(1)} + g_0 \quad (5.43)$$

Studying the behavior of $u_j$ near the branch cuts of (5.42) allows one to identify these as the loci $C_{a13}$, $C_{b23}$ and $C_{c12}$ where the eigenvalues accumulate; then the jump equations across these cuts can be seen to be equivalent with the planar equations of motion (5.31). Below, we shall use this reasoning in order to give a reconstruction theorem for the holomorphic $A_2$ model, similar to the one we found in Section 3 for the holomorphic one-matrix model.

### 5.2.5 The reconstruction theorem

Let us start with a curve of form (5.42) with $p, q$ given by (5.40), where $f_0^{(a)}$ and $g_0$ are complex polynomials of degree $n - 2$ subject to the constraints (5.36). Given such data, we show how one can construct a curve $\gamma$ and distributions $\rho_0^{(a)}(s)$ along this curve such that relations (5.28), (5.30), (5.31), (5.32) and (5.34), (5.41) hold.

Using expression (5.43), one finds the following asymptotic behavior for large $z$:

$$u_1(z) = -t_1 + \frac{\text{lcoeff}(f_0^{(1)})}{\text{deg}(W_1)\text{lcoeff}(W_1)} + O(1/z^2)$$

$$u_2(z) = +t_2 + \frac{\text{lcoeff}(f_0^{(2)})}{\text{deg}(W_2)\text{lcoeff}(W_2)} + O(1/z^2) \quad (5.44)$$
In particular, we can use these asymptotic forms in order to fix the indexing of branches for (3.23) (i.e. \( u_1 \) is the branch which asymptotes to \(-t_1\), while \( u_2 \) is the branch which asymptotes to \(+t_2\)).

When viewed as a triple cover of the \( z \)-plane, the curve (5.42) has three types of cuts, namely those connecting the pairs of branches \((1, 3)\), \((2, 3)\) and \((1, 2)\). Denote these cuts by \( C^a_{13}, C^b_{23} \) and \( C^c_{12} \). Picking \( \gamma \) to contain all cuts, we let \( \omega^{(1)}_0 \) and \( \omega^{(2)}_0 \) be given by relations (5.37) and define:

\[
\rho_0^{(1)}(s) := \frac{1}{2\pi i} \dot{\lambda}(s) \left[ \omega^{(1)}_0(\lambda(s) - i0) - \omega^{(1)}_0(\lambda(s) + i0) \right] , \quad \text{for } \lambda(s) \in C^a_{13} \\
\rho_0^{(2)}(s) := \frac{1}{2\pi i} \dot{\lambda}(s) \left[ \omega^{(2)}_0(\lambda(s) - i0) - \omega^{(2)}_0(\lambda(s) + i0) \right] , \quad \text{for } \lambda(s) \in C^b_{23} \\
\rho^{(1)}(s) = \rho^{(2)}(s) := \frac{1}{2\pi i} \dot{\lambda}(s) \left[ \omega^{(1)}_0(\lambda(s) - i0) - \omega^{(1)}_0(\lambda(s) + i0) \right] \quad (5.45) \\
= \frac{1}{2\pi i} \dot{\lambda}(s) \left[ \omega^{(2)}_0(\lambda(s) - i0) - \omega^{(2)}_0(\lambda(s) + i0) \right] , \quad \text{for } \lambda(s) \in C^c_{12} .
\]

The very last equality in these relations follows by using (5.37) and the fact that \( u_3 = -u_1 - u_2 \) is continuous across cuts of type \( C_{12} \). In particular, this means that (5.32) holds. Extending \( \rho_0^{(a)} \) by zero to the entire curve \( \gamma \), relations (5.45) and the Sokhotsky formulae (3.5) shows that \( \rho_0^{(a)} \) is continuous across cuts of type \( C_{12} \). Combining this with relations (5.46) and using equations (5.30) and the Sokhotsky formulae (3.5) shows that \( \rho_0^{(a)} \) satisfy the planar equations of motion (5.31).

We next notice that:

\[
\begin{align*}
\rho_0^{(1)}(s) &:= \frac{1}{2\pi i} \dot{\lambda}(s) \left[ \omega^{(1)}_0(\lambda(s) - i0) - \omega^{(1)}_0(\lambda(s) + i0) \right] , \\
\rho_0^{(2)}(s) &:= \frac{1}{2\pi i} \dot{\lambda}(s) \left[ \omega^{(2)}_0(\lambda(s) - i0) - \omega^{(2)}_0(\lambda(s) + i0) \right] , \\
\rho^{(1)}(s) &:= \rho^{(2)}(s) := \frac{1}{2\pi i} \dot{\lambda}(s) \left[ \omega^{(1)}_0(\lambda(s) - i0) - \omega^{(1)}_0(\lambda(s) + i0) \right] .
\end{align*}
\]

For \( (1) \), \( (2) \) and \( (3) \), these cuts by \( \omega^{(1)}_0 \) and \( \omega^{(2)}_0 \) are:

\[
\rho_0^{(a)}(s) := \frac{1}{2\pi i} \dot{\lambda}(s) \left[ \omega^{(a)}_0(\lambda(s) - i0) - \omega^{(a)}_0(\lambda(s) + i0) \right] , \quad \text{for } \lambda(s) \in C^a_{13} \\
\rho_0^{(b)}(s) := \frac{1}{2\pi i} \dot{\lambda}(s) \left[ \omega^{(b)}_0(\lambda(s) - i0) - \omega^{(b)}_0(\lambda(s) + i0) \right] , \quad \text{for } \lambda(s) \in C^b_{23} \\
\rho^{(c)}(s) := \rho^{(c)}(s) := \frac{1}{2\pi i} \dot{\lambda}(s) \left[ \omega^{(c)}_0(\lambda(s) - i0) - \omega^{(c)}_0(\lambda(s) + i0) \right] , \quad \text{for } \lambda(s) \in C^c_{12} .
\]

The last equality in these relations follows by using (5.37) and the fact that \( u_3 = -u_1 - u_2 \) is continuous across cuts of type \( C_{12} \). In particular, this means that (5.32) holds. Extending \( \rho_0^{(a)} \) by zero to the entire curve \( \gamma \), relations (5.45) and the Sokhotsky formulae (3.5) shows that equations (5.30) hold. In turn, these relations, the asymptotic behavior (5.44) and the constraints (5.36) on the leading coefficients imply the normalization conditions (5.28).

We next notice that:

\[
\begin{align*}
\rho_0^{(1)}(s) &:= \frac{1}{2\pi i} \dot{\lambda}(s) \left[ \omega^{(1)}_0(\lambda(s) - i0) - \omega^{(1)}_0(\lambda(s) + i0) \right] , \\
\rho_0^{(2)}(s) &:= \frac{1}{2\pi i} \dot{\lambda}(s) \left[ \omega^{(2)}_0(\lambda(s) - i0) - \omega^{(2)}_0(\lambda(s) + i0) \right] , \\
\rho^{(1)}(s) &:= \rho^{(2)}(s) := \frac{1}{2\pi i} \dot{\lambda}(s) \left[ \omega^{(1)}_0(\lambda(s) - i0) - \omega^{(1)}_0(\lambda(s) + i0) \right] .
\end{align*}
\]

where we used relations (5.37) and (5.38). Since a cut of type \( C_{\alpha\beta} \) connects the branches \( u_\alpha \) and \( u_\beta \), we have the jump equations:

\[
\begin{align*}
u_\alpha(\lambda &+ 0n(\lambda)) = u_\beta(\lambda - 0n(\lambda)) \Rightarrow \\
[u_\alpha(\lambda + 0n(\lambda)) - u_\beta(\lambda + 0n(\lambda))] + [u_\alpha(\lambda - 0n(\lambda)) - u_\beta(\lambda - 0n(\lambda))] &= 0
\end{align*}
\]

along such a cut. Combining this with relations (5.46) and using equations (5.30) and the Sokhotsky formulae (3.5) shows that \( \rho_0^{(a)} \) satisfy the planar equations of motion (5.31).

To prove that (5.34) and (5.41) hold, one can now simply repeat the original argument (found in Appendix 3) leading to the large \( N \) curve (5.42); this is possible since we
already showed that all assumptions of that argument (namely relations (5.30), (5.31), 
(5.32)) hold. This shows that equations (5.39) and (5.40) must hold with polynomials 
f, g defined by relations (5.34) and (5.41). Since by hypothesis the same relations hold 
with the original polynomials \(f_0^{(a)}\) and \(g_0\), it immediately follows that the latter can 
indeed be expressed in the form (5.34) and (5.41). This concludes the proof that the 
‘renormalized’ holomorphic \(A_2\) model indeed probes the whole moduli space of (5.42) 
with the constraints (5.36).

**Observation**  The third ‘branch’ of the planar equations of motion (5.31) is only 
allowed in the ‘renormalized’ model described by the limiting ensemble (5.17). If one 
works with the regularized model instead (the original model for which \(\gamma^{(1)}\) and \(\gamma^{(2)}\) 
are disjoint), then cuts of type \(C_{12}^c\) are not allowed. Indeed, such cuts connect the 
branches \(u_1\) and \(u_2\), and thus are cuts for both \(\omega_0^{(1)}\) and \(\omega_0^{(2)}\), which would require 
\(C_{12}^c \subset \gamma^{(1)} \cap \gamma^{(2)}\), in contradiction with the regularization condition \(\gamma^{(1)} \cap \gamma^{(2)} = \emptyset\). 
Hence the regularized model can only probe that part of the moduli space of (5.42) for 
which all cuts of type \(C_{12}^c\) are reduced to double points. This is similar to what happens 
for the Hermitian \(A_2\) model, as discussed at the beginning of this section. In particular, 
the regularized representation (5.9) cannot capture the entire family of curves (5.42), 
and, in fact, can only describe a subvariety in the space of all allowed polynomials \(f_0^{(a)}\) 
and \(g_0^{(a)}\). To explore the full moduli space, one must consider the limiting ensemble 
(5.17). In the context of the Dijkgraaf-Vafa conjecture, the dual field theory is an 
\(N = 1\) supersymmetric field theory derived from an \(A_2\) quiver (such theories have been 
studied in [23, 24]). This field theory is certainly allowed to explore the branch whose 
classical description is given by \(W'_1(\lambda) + W'_2(\lambda) = 0\). Therefore, consideration of the 
limiting ensemble (5.17) of the holomorphic model is required by the conjecture of [1] 
as applied to the \(A_2\) quiver field theories of [23]. A similar analysis can be performed 
general \(ADE\) models, with the same conclusion.

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**A. Integration over gauge group orbits for the holomorphic one-matrix model**

In this Appendix, we show how the eigenvalue representation (1.14) results from the 
matrix integral (1.7).
\textbf{A.1 Orbit decomposition of } \mathcal{M} \\

Let \( \Delta \) be the set of diagonal \( N \times N \) complex matrices \( D \) with distinct eigenvalues:

\[
\Delta := \{ D = \text{diag}(\lambda_1 \ldots \lambda_N) | \lambda_j \in \mathbb{C} \text{ and } \prod_{i \neq j} (\lambda_i - \lambda_j) \neq 0 \} \quad (A.1)
\]

and fix a fundamental domain \( \Delta_0 \) for the obvious action of the permutation group \( \Sigma_N \) on \( \Delta \). Then relation (1.2) gives the orbit decomposition:

\[
\mathcal{M} = \bigsqcup_{D \in \Delta_0} \mathcal{O}_D \quad (A.2)
\]

where \( \mathcal{O}_D \) is the \( GL(N, \mathbb{C}) \)-orbit of a matrix \( D \in \Delta_0 \) under the similarity action (1.8).

The stabilizer of each \( D \in \Delta_0 \) in \( GL(N, \mathbb{C}) \) is the subgroup \( T_N \approx (\mathbb{C}^*)^N \) consisting of diagonal matrices. Hence each orbit has the form:

\[
\mathcal{O}_D = H := GL(N, \mathbb{C})/T_N \quad (A.3)
\]

where the homogeneous space \( H \) has dimension \( N^2 - N \) (here \( T_N \) acts on \( GL(N, \mathbb{C}) \) from the right, i.e. \( S \rightarrow ST \) for \( S \in GL(N, \mathbb{C}) \) and \( T \in T_N \)). The orbit decomposition (A.2) takes the form:

\[
\mathcal{M} = H \times \Delta_0 \quad . (A.4)
\]

\textbf{A.2 Decomposition of } \( w \) \\

Let:

\[
w_\Delta = \prod_{j=1}^{N} d\lambda_j \quad (A.5)
\]

be the translation-invariant holomorphic volume form on \( \Delta \) and:

\[
w_H = \wedge_{i \neq j} \omega_{ij} \quad (A.6)
\]

be the left-invariant holomorphic volume form on the complex homogeneous space \( H \), where \( \omega = S^{-1} dS \) is the matrix whose elements give a basis of left-invariant holomorphic one-forms on \( GL(N, \mathbb{C}) \).

If \( l_S(S') := SS' \) is the left action of \( GL(N, \mathbb{C}) \) on \( H \), then \( w_H \) satisfies:

\[
(l_S)^* w_H = w_H \quad , \quad S \in GL(N, \mathbb{C}) \quad . (A.7)
\]

Using the projections \( \pi_H \) and \( \pi_\Delta \) of \( \mathcal{M} = H \times \Delta_0 \) onto its two factors, we define a holomorphic \( N^2 \)-form on \( \mathcal{M} \) by:

\[
w_0 := \pi_H^* (w_H) \wedge \pi_\Delta^* (w_\Delta) \quad . (A.8)
\]
For dimension reasons, this must be related to $w$ through:

$$w = f w_0 \ , \quad \text{(A.9)}$$

where $f(M)$ is a holomorphic function on $\mathcal{M}$. Using the left-invariance (A.7) of $w_H$, it is easy to check that $w_0$ is invariant $^7$ under the action (1.8):

$$\tau(S)^*(w_0) = w_0 \ . \quad \text{(A.10)}$$

Using $GL(N, \mathbb{C})$ invariance of $w$ and $w_0$, relation (A.9) implies that $f$ is invariant:

$$f(S M S^{-1}) = f(M) \ . \quad \text{(A.11)}$$

In particular, we have $f(M) = f(D)$ if $S M S^{-1} = D$ with $D$ diagonal. It thus suffices to determine the values of $f$ for diagonal matrices $D$.

For this, we first describe the the cotangent space to $\mathcal{M}$ at the points of $\Delta_0$ (where $\Delta_0$ is viewed as a subset of $\mathcal{M}$). In the vicinity of $\Delta_0$, we can write $M = S D S^{-1} \approx (1 + A) D (1 - A) \approx D + [A, D]$, where $S = e^A \approx 1 + A$ with $A$ an infinitesimal generator of $GL(N, \mathbb{C})$. Therefore, at a point $D$ of $\Delta_0$, we have:

$$dM = dD + [dA, D] \iff dM_{ij} = \delta_{ij}d\lambda_i + (\lambda_j - \lambda_i)dA_{ij} \ . \quad \text{(A.12)}$$

Note that there is no $[A, dD]$ piece in the right hand side of this relation, since we compute the form $dM$ at the point $D$ (where $A = 0$).

Noticing that $w_H = \wedge_{i \neq j} dA_{ij}$ at such a point, relation (A.12) gives the form of $w$ at $M = D$:

$$w = (-1)^{N^2(N-1)/2} \wedge_{i,j} dM_{ij} = (-1)^{N^2(N-1)/2} \left[ \prod_{i \neq j} (\lambda_i - \lambda_j) \right] w_0 \ . \quad \text{(A.13)}$$

Comparing with (A.9), we find that $f$ is given by the usual Vandermonde determinant:

$$f(M) = f(D) = \prod_{i \neq j} (\lambda_i - \lambda_j) \ . \quad \text{(A.14)}$$

Combining this with (A.9) gives the following expression for the holomorphic measure:

$$w = (-1)^{N^2(N-1)/2} \prod_{i \neq j} (\lambda_i - \lambda_j)w_0 \ , \quad \text{(A.15)}$$

where $\lambda_i$ are the eigenvalues of the matrix $M$ at which we evaluate $w$.

$^7$To check this, notice that $\pi_H \circ \tau(S) = l_S \circ \pi_H$. 

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A.3 The eigenvalue representation

The $GL(N,\mathbb{C})$ invariance of the action, relation (A.15) and the decomposition (A.8) allow us to perform the integral over the gauge-group variables in the partition function (1.7):

$$\tilde{Z}_N(\gamma, t) = \frac{1}{N^N(-1)^{N^2(N-1)/2}} hvol(H) \int_{\Delta_0(\gamma)} \prod_{j=1}^N d\lambda_j \prod_{i \neq j} (\lambda_i - \lambda_j) e^{-N \sum_{j=1}^N W(\lambda_j)} \quad (A.16)$$

where $hvol(H) = \int_H \omega_H$ is the ‘holomorphic volume’ of $H$ and:

$$\Delta_0(\gamma) = \{ D = \text{diag}(\lambda_1 \ldots \lambda_N) \in \Delta_0 | \lambda_j \in \gamma, \forall j = 1 \ldots N \} \quad (A.17)$$

Since this relation holds for any choice of fundamental domain $\Delta_0$, we can write (A.16) in the form:

$$\tilde{Z}_N(\gamma, t) = \frac{1}{N^N(-1)^{N^2(N-1)/2}} \frac{1}{N!} hvol(H) Z_N(\gamma, t) \quad , \quad (A.18)$$

where:

$$Z_N(\gamma, t) = \int_{\Delta(\gamma)} \prod_{j=1}^N d\lambda_j \prod_{i \neq j} (\lambda_i - \lambda_j) e^{-N \sum_{j=1}^N W(\lambda_j)} \quad (A.19)$$

with:

$$\Delta(\gamma) = \{ D = \text{diag}(\lambda_1 \ldots \lambda_N) | \lambda_j \in \gamma, \forall j = 1 \ldots N \text{ and } \prod_{i \neq j} (\lambda_i - \lambda_j) \neq 0 \} \quad (A.20)$$

This gives the eigenvalue representation (1.14).

**Observation** When writing the representation (1.14), we have tacitly extended the integral to allow for colliding eigenvalues $\lambda_i = \lambda_j$ (this is certainly allowed, since the integrand of (A.19) is well-behaved there). This amounts to treating non-diagonalizable matrices by a point-splitting prescription, and can be formulated in more detail by working with:

$$\Delta_\epsilon := \{ D = \text{diag}(\lambda_1 \ldots \lambda_N) | \lambda_j \in \mathbb{C} \text{ and } |\lambda_i - \lambda_j| > \epsilon \text{ for } i \neq j \} \quad (A.21)$$

instead of $\Delta$ and with a similar modification $\mathcal{M}_\epsilon$ of $\mathcal{M}$. Then one defines the partition function as the limit $\epsilon \to 0^+$ of the regularized partition function obtained in this manner. It is easy to adapt the derivation given above in order to include explicitly such a regulator. The result, however, is the same as (1.14), because the integrand of (A.19) is well-behaved when eigenvalues come close to each other.
B. Example of the relevance of convergence sectors: the case of a cubic potential

Consider the potential:
\[ W(z) = t_3z^3 + t_2z^2 \tag{B.1} \]

This example appeared in the paper [6], where it was used to carry out a one-loop test of the Dijkgraaf-Vafa conjecture. As in [6], we assume for simplicity that \( t_2 \) and \( t_3 \) are real and positive, and write them as \( t_2 = \frac{m}{2} \) and \( t_3 = \frac{g}{3} \) with positive \( m \) and \( g \). The potential has two local extrema along the real axis, namely a local minimum at \( a_1 = 0 \) and a local maximum at \( a_2 = -\frac{2m}{3g} = -\frac{m}{g} < 0 \). Also note that \( W(z) \) approaches \( \pm\infty \) as \( z \) approaches \( \pm\infty \) along the real axis.

B.1 Summary of the procedure of [6]

The paper [6] follows [1] by formulating a conjecture mapping a one-matrix model based on the potential (B.1) to the closed topological B-model on a certain non-compact Calabi-Yau space. When testing this, [6] performs a perturbative expansion of this model around the two extrema \( a_1 \) and \( a_2 \), thereby re-writing the theory as a two-matrix model for matrices \( M_1, M_2 \) with cubic potentials \( W_1(M_1) \) and \( W_2(M_2) \) and a logarithmic interaction term \( W_{\text{int}}(M_1, M_2) \). Since the point \( a_2 \) is a local maximum for the original potential \( W \), this would produce a negative definite quadratic term in \( W_2(M_2) \), if one starts with a Hermitian matrix model, thereby leading to an ill-defined perturbation expansion. To cure this problem, [6] proposes that one should take \( M_1 \) to be anti-Hermitian and \( M_2 \) to be Hermitian. With this redefinition, the authors of [6] compute the first few perturbative terms and find agreement with the one-loop computation of the Kodaira-Spencer theory of the Calabi-Yau dual. We now show how this pragmatic procedure can be recovered in the holomorphic framework.

B.2 Justification in terms of holomorphic matrix models

Remember that a Hermitian matrix model based on the cubic potential (B.1) is ill-defined. This is due to the exponential increase of the integrand for \( \lambda_j \to -\infty \). On the other hand, the holomorphic matrix model leads to a meaningful integral, provided that one chooses appropriate asymptotic sectors.

Note that our definition (1.7) uses the weight \( \prod_{i \neq j} (\lambda_i - \lambda_j) \) rather than the weight \( \prod_{i < j} (\lambda_i - \lambda_j)^2 \) which is used in [6]. In this appendix, we shall temporarily switch to the representation:
\[ Z_N(\gamma, t) = \int_{\gamma} d\lambda_1 \ldots \int_{\gamma} d\lambda_N \prod_{i < j} (\lambda_i - \lambda_j)^2 e^{-N\sum_{j=1}^{N} W(\lambda_j)} \tag{B.2} \]
which differs from our convention (1.14) by a sign prefactor of \((-1)^{N(N-1)/2}\). This avoids some annoying sign factors when comparing with [6].

Thus we start with the partition function (B.2) for the potential (B.1). Since \(t_3 > 0\), this model has \(\theta_3 = 0\) and the asymptotic sectors shown in figure 7. We claim that the correct partition function relevant for the work of [6] is \(Z(1,0,t)\), associated with the asymptotic sectors \(k_- = 1\) and \(k_+ = 0\). This is convergent by the analysis of Section 2.

In this set-up, the prescription of [6] can be recovered as follows (we shall neglect the gauge group volume prefactors, which are irrelevant here). Since the partition function (B.2) depends only on the asymptotic sectors of \(\gamma\) (namely \(\nu_- \in A_1\) and \(\nu_+ \in A_0\)), we are free to choose this curve as shown in figure 7. This asymptotes to some line \(d_-\) with tangent \(\nu_- \in A_1\) for \(t \to -\infty\), then follows a vertical line through the point \(a_2 = -m/g\), and finally loops back to touch (and then follow) the real axis at some point \(x\) lying in between \(a_2 = -m/g\) and \(a_1 = 0\). Thus an eigenvalue \(\lambda\) sitting on \(\gamma\) will be imaginary if it lies close to \(a_2\) and real if it lies close to \(a_0\), thereby naturally implementing the requirement of [6]. The curve segments along \(\gamma\) around the points \(a_1\) and \(a_2\) for which \(\lambda\) has these properties can be maximized by making the ‘well’ at the bottom of this curve to be very thin (i.e. take \(x\) to be very close to \(a_2\)) and very deep.

\[
\lambda_i^{(0)} = a_1 = 0 \quad \text{for} \quad i = 1 \ldots N_1 \quad , \quad \lambda_j^{(0)} = a_2 = -m/g \quad \text{for} \quad j = N_1 + 1 \ldots N \quad \text{(B.3)}
\]

Figure 7: Convergence sectors for the case \(\deg W = 3, \theta_3 = 0\) and a good choice of contour belonging to the sector \((k_-, k_+) = (1,0)\).

Following [6], we now expand the integral (B.2) (with \(\gamma\) chosen as above) around the saddle point configurations:
with $N_1$ some integer in the set $\{1 \ldots N\}$. Let $N_2 := N - N_1$. This gives:

$$Z(t, \gamma) = \sum_{N_1+N_2=N} \frac{N!}{N_1!N_2!} \int d\mu_1 \ldots \int d\mu_{N_1} \int d\nu_1 \ldots \int d\nu_{N_2} \Delta(\mu, \nu) e^{-NS} \quad (B.4)$$

where we wrote $\lambda_i = \lambda_i^{(0)} + \mu_i$, $\lambda_{N_1+j} = \lambda_{N_1+j}^{(0)} + \nu_j$ and:

$$\Delta(\mu, \nu) = \prod_{1 \leq i_1 < i_2 \leq N_1} (\mu_{i_1} - \mu_{i_2})^2 \prod_{1 \leq j_1 < j_2 \leq N_1} (\nu_{j_1} - \nu_{j_2})^2 \prod_{i=1}^{N_1} \prod_{j=1}^{N_2} (\mu_i - \nu_j + \frac{m}{g})^2$$

$$S = \sum_{i=1}^{N_1} W(\mu_i) - \sum_{j=1}^{N_2} W(-\nu_j) + \frac{m^3}{6g^2} N_2 \quad (B.5)$$

Treating $\mu_i$ and $\nu_j$ as small fluctuations, we naturally have $\mu_i \in i\mathbb{R}$ and $\nu_j \in \mathbb{R}$, since the eigenvalues $\lambda$ lying along $\gamma$ are imaginary in the vicinity of $a_2$ and real in the vicinity of $a_1$. Writing the last term in $\Delta$ as an exponential, one obtains a logarithmic interaction potential\(^8\):

$$W_{\text{int}} = 2N_1 N_2 \ln \frac{m}{g} + 2 \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \ln \left[ 1 + \frac{g}{m} (\mu_i - \nu_j) \right] \quad (B.6)$$

and one-matrix potentials $W_1(\mu) = W(\mu)$ and $W_2(\nu) = -W(-\nu)$, which allows one to write the result as an interacting two-matrix model $[6]$:

$$Z = \sum_{N_1+N_2=N} \frac{N!}{N_1!N_2!} \int_{M_1^+-M_1} \int_{M_2^+-M_2} e^{-N\text{tr}[W_1(M_1)+W_2(M_2)+W_{\text{int}}(M_1,M_2)]} \quad (B.7)$$

where the first integral is over anti-Hermitian matrices while the second integral is over Hermitian matrices. Note that anti-Hermiticity of $M_1$ arises naturally in the holomorphic set-up. This gives a conceptual justification for the procedure of $[6]$. Note also that we have provided a non-perturbative construction of the model involved in that work: it is simply the holomorphic matrix model with potential (B.1), considered in the ‘phase’ $(k_-, k_+) = (1, 0)$.

**C. Derivation of the planar constraints for the $A_2$ model**

Let us show how the non-hyperelliptic Riemann surface expected from the observations of $[3]$ arises in the holomorphic $A_2$ model. For this, we derive two algebraic constraints

\(^8\)The authors of $[6]$ further expand the logarithm as a power series in $\frac{g}{m} (\mu_i - \nu_j)$, a procedure which is justified if $|\frac{g}{m} (\mu_i - \nu_j)| < 1$. 

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which hold in the planar limit, as a consequence of the planar equations of motion (5.31).

To derive the equations of interest, we shall use the partial fraction decompositions:

\[
\frac{1}{(z - u)(z - v)} = \frac{1}{u - v} \left[ \frac{1}{z - u} - \frac{1}{z - v} \right]
\]  

(C.1)

and:

\[
\frac{1}{(z - u)(z - v)(z - w)} = \frac{1}{(u - v)(u - w)z - u} + \frac{1}{(v - u)(v - w)z - v} + \frac{1}{(w - u)(w - v)z - w}.
\]  

(C.2)

**C.1 The first constraint**

Using (C.1), one finds:

\[
\omega_0^{(1)}(z)^2 = 2 \int ds \int ds' \frac{\rho_0^{(1)}(s)\rho_0^{(1)}(s')}{\lambda(s) - \lambda(s')} \frac{1}{z - \lambda(s)}
\]

\[
\omega_0^{(2)}(z)^2 = 2 \int ds \int ds' \frac{\rho_0^{(2)}(s)\rho_0^{(2)}(s')}{\lambda(s) - \lambda(s')} \frac{1}{z - \lambda(s)}
\]

\[
\omega_0^{(1)}(z)\omega_0^{(2)}(z) = \int ds \int ds' \left[ \frac{\rho_0^{(1)}(s)\rho_0^{(2)}(s')}{\lambda(s) - \lambda(s')} \frac{1}{z - \lambda(s)} + \frac{\rho_0^{(2)}(s)\rho_0^{(1)}(s')}{\lambda(s) - \lambda(s')} \frac{1}{z - \lambda(s)} \right]
\]

(C.3)

Combining these equations gives:

\[
\omega_0^{(1)}(z)^2 - \omega_0^{(1)}(z)\omega_0^{(2)}(z) + \omega_0^{(2)}(z)^2 = \int ds \rho_0^{(1)}(s) \frac{W_1'(\lambda(s))}{z - \lambda(s)} + \int ds \rho_0^{(2)}(s) \frac{W_2'(\lambda(s))}{z - \lambda(s)}.
\]

(C.4)

To arrive at this relation, we decomposed the integrals over \( ds \) in (C.3) into the pieces corresponding to the cuts \( C_{13}^a, C_{23}^b \) and \( C_{12}^c \). Then equation (C.4) results upon combining these pieces appropriately and performing the \( s' \) integral by using the planar equations of motion (5.31) and relations (5.32). We next write (C.4) in the form:

\[
\omega_0^{(1)}(z)^2 - \omega_0^{(1)}(z)\omega_0^{(2)}(z) + \omega_0^{(2)}(z)^2 - W_1'(\lambda(z))\omega_0^{(1)}(z) - W_2'(\lambda(z))\omega_0^{(2)}(z) + f_0^{(1)}(z) + f_0^{(2)}(z) = 0,
\]

(C.5)

where we used the planar equations of motion (5.31) and the definition of \( \omega_0^{(\alpha)}(z) \) and we introduced the polynomials:

\[
f_0^{(1)}(z) := \int ds \rho_0^{(1)}(s) \frac{W_1'(z) - W_1'(\lambda(s))}{z - \lambda(s)}
\]

\[
f_0^{(2)}(z) := \int ds \rho_0^{(2)}(s) \frac{W_2'(z) - W_2'(\lambda(s))}{z - \lambda(s)}.
\]

(C.6)
C.2 The second constraint

To derive the second constraint, we use (C.2) to compute:

\[
\omega_0^{(1)}(z)^2 \omega_0^{(2)}(z) = \int d\alpha \int d\beta \int d\gamma \frac{2\rho_0^{(1)}(\alpha)\rho_0^{(1)}(\beta)\rho_0^{(2)}(\gamma)}{(\lambda(\alpha) - \lambda(\beta))(\lambda(\alpha) - \lambda(\gamma))(z - \lambda(\alpha))} + \int d\alpha \int d\beta \int d\gamma \frac{\rho_0^{(2)}(\alpha)\rho_0^{(1)}(\beta)\rho_0^{(1)}(\gamma)}{(\lambda(\alpha) - \lambda(\beta))(\lambda(\alpha) - \lambda(\gamma))(z - \lambda(\alpha))},
\]

where we used redefinitions of \((\alpha, \beta, \gamma)\) by permutations to bring the right hand side to a convenient form. Combining this with the equation obtained by permuting the indices \(1 \leftrightarrow 2\) gives:

\[
\omega_0^{(1)}(z)^2 \omega_0^{(2)}(z) - \omega_0^{(1)}(z) \omega_0^{(2)}(z)^2 = \int d\alpha \int d\beta W'_1(\lambda(\alpha)) \frac{\rho_0^{(1)}(\alpha)\rho_0^{(2)}(\beta)}{(\lambda(\alpha) - \lambda(\beta))(z - \lambda(\alpha))} \quad (1 \leftrightarrow 2),
\]

where we used the planar equations of motion (5.31) to perform the integral over \(\beta\).

Defining the polynomials:

\[
g_0^{(1)}(z) := \int d\alpha \int d\beta [W'_1(z) - W'_1(\lambda(\alpha))] \frac{\rho_0^{(1)}(\alpha)\rho_0^{(2)}(\beta)}{(\lambda(\alpha) - \lambda(\beta))(z - \lambda(\alpha))},
\]

\[
g_0^{(2)}(z) := \int d\alpha \int d\beta [W'_2(z) - W'_2(\lambda(\alpha))] \frac{\rho_0^{(2)}(\alpha)\rho_0^{(1)}(\beta)}{(\lambda(\alpha) - \lambda(\beta))(z - \lambda(\alpha))},
\]

we find:

\[
\omega_0^{(1)}(z)^2 \omega_0^{(2)}(z) - \omega_0^{(1)}(z) \omega_0^{(2)}(z)^2 + g_0^{(1)}(z) - g_0^{(2)}(z) - W'_1(z)U_1(z) + W'_2(z)U_2(z) = 0,
\]

where:

\[
U_1(z) := \int d\alpha \int d\beta \frac{\rho_0^{(1)}(\alpha)\rho_0^{(2)}(\beta)}{(\lambda(\alpha) - \lambda(\beta))(z - \lambda(\alpha))},
\]

\[
U_2(z) := \int d\alpha \int d\beta \frac{\rho_0^{(2)}(\alpha)\rho_0^{(1)}(\beta)}{(\lambda(\alpha) - \lambda(\beta))(z - \lambda(\alpha))}.
\]

Using the equations of motion (5.31), these quantities can be written:

\[
U_1(z) = f_0^{(1)}(z) - W'_1(z)\omega_0^{(1)}(z) + \omega_0^{(1)}(z)^2
\]

\[
U_2(z) = f_0^{(2)}(z) - W'_2(z)\omega_0^{(2)}(z) + \omega_0^{(2)}(z)^2.
\]

Therefore, equation (C.10) becomes:

\[
\omega_0^{(1)}(z)^2 \omega_0^{(2)}(z) - W'_1(z)\omega_0^{(1)}(z)^2 - W'_1(z)\omega_0^{(1)}(z) + f_0^{(1)}(z) + g_0^{(1)}(z) - (1 \leftrightarrow 2) = 0.
\]
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