Pooling designs with surprisingly high degree of error correction in a finite vector space

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\textbf{ABSTRACT}

Pooling designs are standard experimental tools in many biotechnical applications. It is well-known that all famous pooling designs are constructed from mathematical structures by the “containment matrix” method. In particular, Macula’s designs (resp. Ngo and Du’s designs) are constructed by means of the containment relation of subsets (resp. subspaces) in a finite set (resp. vector space). In [J. Guo, K. Wang, A construction of pooling designs with high degree of error correction, J. Combin. Theory Ser. A 118 (2011) 2056–2058], we generalized Macula’s designs and obtained a family of pooling designs with higher degree of error correction. In this paper we consider, as a generalization of Ngo and Du’s designs, \(q\)-analogue of the above designs, and obtain a family of pooling designs with surprisingly high degree of error correction. Our designs and Ngo and Du’s designs have the same numbers of items and pools, but the error-tolerance property of our design is much better than that of Ngo and Du’s designs when the dimension of the space is large enough.

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\section{Introduction}

A group test is applicable to an arbitrary subset of clones with two possible outcomes: a negative outcome indicates that all clones in the subset are negative, and a positive outcome indicates otherwise. A pooling design is a specification of all tests such that they can be performed simultaneously, with the goal being to identify all positive clones with a small number of tests [1–3,7]. A pooling design is usually represented by a binary matrix with columns indexed with items and rows indexed with pools. A cell \((i,j)\) contains a 1-entry if and only if the ith pool contains the jth item. By treating a column as a set of row indices intersecting the column with a 1-entry, we can talk about the union of several columns. A binary matrix is \(s^0\)-\textit{disjunct} if every column has at least \(e+1\) 1-entries not contained in the union of any other \(s\) columns [9]. An \(s^0\)-disjunct matrix is also called \(s\)-disjunct. An \(s^e\)-disjunct matrix is called \textit{fully \(s^e\)-disjunct} if it is not \(s_1^{e_1}\)-disjunct whenever \(s_1 > s\) or \(e_1 > e\). An \(s^e\)-disjunct matrix is \([e/2]\)-error-correcting [4].

For positive integers \(k \leq n\), let \([n] = \{1, 2, \ldots, n\}\) and \(\binom{[n]}{k}\) denote the collection of all \(k\)-subsets of \([n]\).

Macula [8,9] proposed a novel way of constructing disjunct matrices by means of the containment relation of subsets in \([n]\).

\textbf{Definition 1.1} ([8]). For positive integers \(1 \leq d < k < n\), let \(M(d, k, n)\) be the binary matrix with rows indexed with \(\binom{[n]}{d}\) and columns indexed with \(\binom{[n]}{k}\) such that \(M(A, B) = 1\) if and only if \(A \subseteq B\).
D’yachkov et al. [5] discussed the error-correcting property of $M(d, k, n)$.

**Theorem 1.1** ([5]). For positive integers $1 \leq d < k < n$ and $1 \leq s \leq d$, $M(d, k, n)$ is fully $s^1$-disjunct, where $e_1 = \binom{k-s}{d-s} - 1$.

In [6], we generalized Macula’s construction and obtained a family of pooling designs with higher degree of error correction.

**Definition 1.2** ([6]). For positive integers $1 \leq d < k < n$ and $0 \leq i \leq d$, let $M(i; d, k, n)$ be the binary matrix with rows indexed with $\binom{n}{i}$ and columns indexed with $\binom{n}{k}$ such that $M(A, B) = 1$ if and only if $|A \cap B| = i$.

**Theorem 1.2** ([6]). Let $1 \leq s \leq i, [(d + 1)/2] \leq i \leq d < k$ and $n - k - s(k + d - 2i) \geq d - i$. Then:

(i) $M(i; d, k, n)$ is an $s^2$-disjunct matrix, where $e_2 = \binom{k-s}{i-1} \binom{n-k-s(k+d-2i)}{d-i} - 1$;

(ii) for a given $k$, if $i < d$, then $\lim_{n \to \infty} \frac{e_2 + 1}{e_1 + 1} = \infty$.

Let $\mathbb{F}_q$ be a finite field with $q$ elements, where $q$ is a prime power. For a positive integer $n$, let $\mathbb{F}_q^n$ be an $n$-dimensional vector space over $\mathbb{F}_q$. For positive integers $k \leq n$, let $\binom{n}{k}_q$ be the set of all $k$-dimensional subspaces of $\mathbb{F}_q^n$. A matrix representation of a subspace $P$ is a matrix whose rows form a basis for $P$. When there is no danger of confusion, we use the same symbol to denote a subspace and its matrix representation.

Let $m_1$ and $m_2$ be two integers. For brevity we use the Gaussian coefficient

$$\begin{bmatrix} m_2 \\ m_1 \\ q \end{bmatrix} = \frac{m_2!}{(m_2-m_1+1)! (q^1 - 1)} \frac{m_1!}{(m_1)! (q^1 - 1)}.$$

For convenience, we set $\begin{bmatrix} m_2 \\ 0 \\ q \end{bmatrix} = 1$ and $\begin{bmatrix} m_2 \\ m_1 \\ q \end{bmatrix} = 0$ whenever $m_1 < 0$ or $m_2 < m_1$. Then, by Wan [13],

$$\begin{bmatrix} n \\ k \\ q \end{bmatrix} = \frac{n!}{k! (q^1 - 1)}.$$

As a $q$-analogue of Macula’s designs, Ngo and Du [12] constructed a family of disjunct matrices by using the containment relation of subspaces in $\mathbb{F}_q^n$.

**Definition 1.3** ([12]). For positive integers $1 \leq d < k < n$, let $M_q(d, k, n)$ be the binary matrix with rows indexed with $\binom{n}{d}_q$ and columns indexed with $\binom{n}{k}_q$ such that $M_q(A, B) = 1$ if and only if $A \subseteq B$.

D’yachkov et al. [4] discussed the error-tolerance property of $M_q(d, k, n)$.

**Theorem 1.3** ([4]). For positive integers $1 \leq d < k < n$, $k - d \geq 2$ and $1 \leq s \leq q(q^{k-1} - 1)/(q^{k-d} - 1)$, $M_q(d, k, n)$ is $s^1$-disjunct, where $\bar{s}_1 = q^{k-d} \binom{k-1}{d-1}_q - (3 - 1)q^{k-d-1} \binom{k-2}{d-1}_q - 1$. In particular, if $s \leq q + 1$, then $M_q(d, k, n)$ is fully $s^1$-disjunct.

Nan and Guo [10] generalized Ngo and Du’s construction and obtained a family of pooling designs.

**Definition 1.4** ([10]). For positive integers $1 \leq d < k < n$ and $\max\{0, d + k - n\} \leq i \leq d$, let $M_q(i; d, k, n)$ be the binary matrix with rows indexed with $\binom{n}{d}_q$ and columns indexed with $\binom{n}{k}_q$ such that $M_q(A, B) = 1$ if and only if $\dim(A \cap B) = i$.

Note that $M_q(i; d, k, n)$ and $M_q(d, k, n)$ have the same size. In [10], the error-tolerance property of $M_q(i; d, k, n)$ is not well expressed. In this paper, we discuss again the error-tolerance property of $M_q(i; d, k, n)$.

2. The main results

In this section, we discuss the error-tolerance property of $M_q(i; d, k, n)$. We begin with a useful lemma.

**Lemma 2.1.** Suppose $\max\{0, r + m - n\} \leq j \leq r$ and $j \leq m \leq n$. Let $P$ be an $m$-dimensional subspace of $\mathbb{F}_q^n$ and let $W$ be a $j$-dimensional subspace of $P$. Then the number of $r$-dimensional subspaces of $\mathbb{F}_q^n$ intersecting $P$ at $W$ is $f(j, r, n; m) = q^{(r-j)(m-j)} \binom{n-m}{r-j}_q$. Moreover the function $f(j, r, n; m + \alpha)$ about $\alpha$ is decreasing for $0 \leq \alpha \leq n + j - m - r$. 
Proof. Since the general linear group $GL_n(F_q)$ acts transitively on the set of such pairs $(P, W)$, we may assume that

$$P = (I^m) \ 0^{(m,n-m)}, \ W = (I^n) \ 0^{(j,n-j)}.$$

Let $Q$ be any $r$-dimensional subspace of $F_q^n$ satisfying $P \cap Q = W$. Then $Q$ has a matrix representation of the form

$$
\begin{pmatrix}
I^{(j)} & 0^{(j,m-j)} \\
0^{(r-j, i)} & A_2 \\
& A_3
\end{pmatrix},
$$

where $A_2$ is an $(r-j) \times (m-j)$ matrix and $A_3$ is an $(r-j)$-dimensional subspace of $F_q^{m-r}$. Therefore, $f(j, r, n; m) = q^{(r-j)(m-j)} n^{-m}$.

Since

$$f(j, r, n; m) - f(j, r, n; m + 1) = q^{(r-j)(m-j)} \left[ \begin{array}{c} n-m \\ r-j \end{array} \right] q^{(r-j)(m+1-j)} \left[ \begin{array}{c} n-m-1 \\ r-j \end{array} \right] q^{(r-j)(m-j)} \prod_{i=1}^{r-j} (1 - q^{-1}) \geq 0,$$

the desired result follows. □

**Theorem 2.2.** Let $i, d, k, n$ be positive integers with $\lfloor (d + 1)/2 \rfloor \leq i \leq d < k$ and $n - k - s(k + d - 2i) \geq d - i$. If $k - i \geq 2$ and $1 \leq s \leq q^{(k-1)} - 1)/(q^{(i-1)} - 1)$, then the following hold:

(i) $M_q(i; d, k, n)$ is an $F_q$-disjoint matrix, where

$$
\mathcal{E}_2 = q^{(d-j)(k + s(k + d - 2i) - j)} \left[ \begin{array}{c} n-k-s(k+d-2i) \\ d-i \end{array} \right] q^{(k-1)} \left[ \begin{array}{c} k-i \\ i-1 \end{array} \right] - (s-1)q^{k-1} \left[ \begin{array}{c} k-2 \\ i-1 \end{array} \right] - 1;
$$

(ii) for a given $k$, if $i < d$, then $\lim_{n \to \infty} \frac{2+1}{\mathcal{E}_2 + 1} = \infty$.

Proof. (i) Let $B_0, B_1, \ldots, B_\bar{s} \in \left[ \begin{array}{c} n \\ d \end{array} \right]_q$ be any $\bar{s} + 1$ distinct columns of $M_q(i; d, k, n)$. Note that $B_0$ contains $\left[ \begin{array}{c} k \\ i \end{array} \right]_q$ many $i$-dimensional subspaces. To obtain the minimum number of $i$-dimensional subspaces of $B_0$ not contained in $B_j$ for each $1 \leq j \leq \bar{s}$, we may assume that $\dim(B_0 \cap B_j) = k - 1$ and $\dim(B_0 \cap B_j \cap B_l) = k - 2$ for any two distinct $j, l \in \{1, 2, \ldots, \bar{s}\}$.

Then $B_1$ contains $\left[ \begin{array}{c} k-1 \\ i \end{array} \right]_q$ many $i$-dimensional subspaces of $B_0$, while each of $B_2, B_3, \ldots, B_\bar{s}$ contains at most $\left[ \begin{array}{c} k-1 \\ i \end{array} \right]_q - \left[ \begin{array}{c} k-2 \\ i \end{array} \right]_q$ many $i$-dimensional subspaces of $B_0$ not contained in $B_1$. Consequently, the number of $i$-dimensional subspaces of $B_0$ not contained in $B_1, B_2, \ldots, B_\bar{s}$ is at least

$$
\alpha = \left[ \begin{array}{c} k \\ i \end{array} \right] - \left[ \begin{array}{c} k-1 \\ i \end{array} \right] - (s-1) \left[ \begin{array}{c} k-1 \\ i \end{array} \right] - \left[ \begin{array}{c} k-2 \\ i \end{array} \right]_q = q^{k-1} \left[ \begin{array}{c} k-1 \\ i-1 \end{array} \right] - (s-1)q^{k-1} \left[ \begin{array}{c} k-2 \\ i-1 \end{array} \right]_q.
$$

Let $D \in \left[ \begin{array}{c} n \\ d \end{array} \right]_q$ with $\dim(D \cap B_0) = i$. If there exists $j \in \{1, 2, \ldots, \bar{s}\}$ such that $\dim(D \cap B_j) = i$, by $(D \cap B_0) + (D \cap B_j) \subseteq D$, we have

$$
\dim(B_0 \cap B_j) \geq \dim(D \cap B_0) + \dim(D \cap B_j) - \dim((D \cap B_0) + (D \cap B_j)) \geq 2i - d.
$$

Suppose $\dim(B_0 \cap B_j) \geq 2i - d$ for each $j \in \{1, 2, \ldots, \bar{s}\}$. Then

$$
\dim(B_0 + B_1 + \cdots + B_\bar{s}) = \dim(B_0 + B_1 + \cdots + B_{\bar{s}-1}) + \dim(B_\bar{s} - \dim((B_0 + B_1 + \cdots + B_{\bar{s}-1}) \cap B_\bar{s})
$$

$$
\leq \dim(B_0 + B_1 + \cdots + B_{\bar{s}-1}) + \dim(B_\bar{s} - \dim(B_0 \cap B_\bar{s})
$$

$$
\leq \dim(B_0 + B_1 + \cdots + B_{\bar{s}-1}) + k + d - 2i
$$

$$
\leq \dim B_0 + \bar{s}(k + d - 2i)
$$

$$
= k + \bar{s}(k + d - 2i).
$$
Let $P$ be a given $i$-dimensional subspace of $B_0$ not contained in $B_1, B_2, \ldots, B_r$. By Lemma 2.1, the number of $d$-dimensional subspaces $D$ in $\mathbb{F}_q^n$ satisfying $D \cap (B_0 + B_1 + \cdots + B_r) = P$ is at least

$$q^{(d-i)(k+q(d-2i)-i)} \left\lfloor \frac{n-k-\bar{s}(k+d-2i)}{d-i} \right\rfloor_q.$$

Observe that $D \cap B_0 = P$ and $\dim(D \cap B_j) \neq i$ for each $j \in \{1, 2, \ldots, r\}$. Therefore, the number of $d$-dimensional subspaces $D$ in $\mathbb{F}_q^n$ satisfying $\dim(D \cap B_0) = i$ and $\dim(D \cap B_j) \neq i$ for each $j \in \{1, 2, \ldots, r\}$ is at least

$$\alpha q^{(d-i)(k+q(d-2i)-i)} \left\lfloor \frac{n-k-\bar{s}(k+d-2i)}{d-i} \right\rfloor_q.$$

Since $\bar{s}_2 \geq 0$, we have $\alpha > 0$, which implies that

$$\bar{s} \leq \frac{q^{k-i} \left\lfloor \frac{k-1}{i-1} \right\rfloor_q \left( q^{k-1} - 1 \right)}{q^{k-1-i} \left\lfloor \frac{k-2}{i-1} \right\rfloor_q}.$$

Hence, (i) holds.

(ii) is straightforward by (i) and Theorem 1.3. \qed

**Theorem 2.3.** Let $i, d, k, n$ be positive integers with $1 \leq i \leq [(d + 1)/2]$ and $d < k$, $n - (\bar{s} + 1)k \geq d - i$. If $1 \leq \bar{s} \leq q(k-1) - 1)/(q^{k-1} - 1)$, then the following hold:

(i) $M_q(i; d, k, n)$ is an $\bar{s}^2$-disjunct matrix, where

$$\bar{s}_2 = q^{(d-i)(\bar{s}+1)k-i} \left\lfloor \frac{n-(\bar{s}+1)k}{d-i} \right\rfloor_q \left( q^{k-i} \left\lfloor \frac{k-1}{i-1} \right\rfloor_q - (\bar{s}-1)q^{k-1-i} \left\lfloor \frac{k-2}{i-1} \right\rfloor_q \right) - 1;$$

(ii) for a given $k$, $\lim_{n \to \infty} \frac{\bar{s}_2+1}{\bar{s}_2+1} = \infty$.

**Proof.** The proof is similar to that of Theorem 2.2, and will be omitted. \qed

For $q = 2, k = 8, n = 60$, Table 1 shows the disjunct property of our designs and Ngo and Du’s designs for small $i, d, \bar{s}$.

### 3. Concluding remarks

(i) For given positive integers $d < k$, $\lim_{n \to \infty} \frac{\bar{s}_2+1}{\bar{s}_2+1} = 0$. This shows that the test-to-item ratio of $M_q(i; d, k, n)$ is small enough when $n$ is large enough. By Theorems 2.2 and 2.3, our pooling designs are much better than Ngo and Du’s designs when $n$ is large enough.

(ii) Ngo [11] improved the error-tolerance property of $M_q(d, k, n)$ for $\bar{s} \geq q + 2, \bar{s} \geq q + 3$ and $\bar{s} \geq q + 4$, respectively. By a similar method, we also can improve the error-tolerance property of $M_q(i; d, k, n)$ for these cases.

(iii) For positive integers $1 \leq d < k < n$, it seems to be interesting to consider the error-tolerance property of $M_q(0; d, k, n)$.

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References

[1] Y. Cheng, D. Du, Efficient constructions of disjunct matrices with applications to DNA library screening, J. Comput. Biol. 14 (2007) 1208–1216.

[2] Y. Cheng, D. Du, New constructions of one- and two-stage pooling designs, J. Comput. Biol. 15 (2008) 195–205.

[3] D. Du, F.K. Hwang, Pooling Designs and Nonadaptive Group Testing: Important Tools for DNA Sequencing, World Scientific, 2006.

[4] A.G. D’yachkov, F.K. Hwang, A.J. Macula, P.A. Vilenkin, C. Weng, A construction of pooling designs with some happy surprises, J. Comput. Biol. 12 (2005) 1127–1134.

[5] A.G. D’yachkov, A.J. Macula, P.A. Vilenkin, Nonadaptive and trivial two-stage group testing with error-correcting $d^e$-disjunct inclusion matrices, in: Entropy, Search, Complexity, Bolyai Society Mathematical Studied, Vol. 16, Springer, Berlin, 2007, pp. 71–83.

[6] J. Guo, K. Wang, A construction of pooling designs with high degree of error correction, J. Combin. Theory Ser. A 118 (2011) 2056–2058.

[7] T. Huang, C. Weng, Pooling spaces and non-adaptive pooling designs, Discrete Math. 282 (2004) 163–169.

[8] A.J. Macula, A simple construction of $d$-disjunct matrices with certain constant weights, Discrete Math. 162 (1996) 311–312.

[9] A.J. Macula, Error-correcting non-adaptive group testing with $d^e$-disjunct matrices, Discrete Appl. Math. 80 (1997) 217–222.

[10] J. Nan, J. Guo, New error-correcting pooling designs associated with finite vector spaces, J. Comb. Optim. 20 (2010) 96–100.

[11] H. Ngo, On a hyperplane arrangement problem and tighter analysis of an error-tolerant pooling design, J. Comb. Optim. 15 (2008) 61–76.

[12] H. Ngo, D. Du, New constructions of non-adaptive and error-tolerance pooling designs, Discrete Math. 243 (2002) 161–170.

[13] Z. Wan, Geometry of Classical Groups over Finite Fields, second ed., Science Press, Beijing/New York, 2002.