ON A POISSON’S EQUATION ARISING FROM MAGNETISM

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Abstract. We review the proof of existence and uniqueness of the Poisson’s equation 
\[ \Delta u + \text{div} m = 0 \]
whenever \( m \) is a unit \( L^2 \)-vector field on \( \mathbb{R}^3 \) with compact support; by 
standard linear potential theory we deduce also the \( H^1 \)-regularity of the unique weak 
solution.

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1. Introduction

In the standard theory of ferromagnetic materials is usually considered an energy, called 
magnetostatic, which is the energy of the magnetostatic field set up by the magnetization 
vector field \( m \). It turns out that the magnetostatic energy takes the form 
\[ \int |\nabla u|^2 \, dx \]
where the scalar potential \( u: \mathbb{R}^3 \rightarrow \mathbb{R} \) satisfies the following equation arising from Maxwell’s 
equations:
\[ \text{div}(\nabla u + m\chi_{\Omega}) = 0, \quad \text{on} \ \mathbb{R}^3, \]  
(1.1)
being \( \Omega \) an open and bounded domain in \( \mathbb{R}^3 \), which represents the region occupied by a 
ferromagnetic material, and \( \chi_{\Omega} \) its characteristic function, that is \( \chi_{\Omega} = 1 \) on \( \Omega \) and 0 otherwise 
in \( \mathbb{R}^3 \); for more details on equation (1.1) see [2], [5] and [6]. Without loss of generality, since 
we will not vary the temperature, which is related with the variation of \( |m| \), we will consider 
vector fields \( m: \Omega \rightarrow S^2 \), being \( S^2 \) the boundary of the unit ball in \( \mathbb{R}^3 \). Replacing \( m\chi_{\Omega} \) with 
\( m: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \), equation (1.1) takes the form
\[ \Delta u + \text{div} m = 0, \quad \text{on} \ \mathbb{R}^3, \]  
with \( |m| = \chi_{\Omega} \).
(1.2)
There is a huge literature on the Poisson’s type equation (1.2); we just mention a very recent 
application in the context of micromagnetics materials: such an equation has been considered 
in [3] and [4] where an homogenization procedure of a more complete energy functional for 
polycrystalline magnetic materials has been investigated. In order to solve equation (1.2) we
have to introduce its weak formulation, that is
\[ \int (\nabla u + m) \cdot \nabla \varphi \, dx = 0, \quad \forall \varphi \in C^\infty_c(\mathbb{R}^3). \] (1.3)
In this short note we will explain how the proof of existence and uniqueness of the solution of equation (1.3) in a suitable space of Sobolev-type works; moreover, we will find, exploiting the standard tools coming from the linear potential theory, the explicit form of the solution from which, in particular, it will descend more regularity of such a solution: more precisely, the unique weak solution turns out to be \( H^1(\mathbb{R}^3) \), and such a regularity has been stated in [5], but without proof.

2. Some preliminaries of potential theory

We now recall some well known results coming from potential theory; for details we refer to [7]. Let \( n \geq 1 \) be an integer and, for each \( f : \mathbb{R}^n \to \mathbb{R} \) measurable and for each \( \alpha > 0 \), let \( I_\alpha \) be the Riesz potential given by
\[ I_\alpha(f)(x) := c(n, \alpha) \int \frac{f(y)}{|x-y|^{n-\alpha}} \, dy, \quad x \in \mathbb{R}^n, \] (2.1)
for a suitable positive constant \( c(n, \alpha) \). It turns out that if \( \alpha, \beta > 0 \) and \( \alpha + \beta < n \) then for any \( f \in \mathcal{S}(\mathbb{R}^n) \), being \( \mathcal{S}(\mathbb{R}^n) \) the Schwartz space on \( \mathbb{R}^n \),
\[ I_\alpha(I_\beta(f)) = I_{\alpha+\beta}(f). \] (2.2)
Let \( \alpha \in (0, n) \) and \( 1 \leq p < +\infty \) with
\[ \frac{1}{p} - \frac{\alpha}{n} < 1. \]
First of all, it turns out that if, more generally, \( f \in L^p(\mathbb{R}^n) \) then the integral on the right hand-side of (2.1) converges absolutely for almost any \( x \in \mathbb{R}^n \). Moreover, if in particular
\[ \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} \]
then
\[ I_\alpha : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n) \] (2.3)
is linear and continuous. Strictly related with Riesz potentials is the notion of Riesz transform: for any \( f \in L^p(\mathbb{R}^n) \), with \( 1 \leq p < +\infty \), and for any \( j = 1, \ldots, n \), we let
\[ R_j(f)(x) := c(n) \lim_{\varepsilon \to 0^+} \int_{|y| < \varepsilon} \frac{x_j - y_j}{|x-y|^{n+1}} f(y) \, dy, \quad x \in \mathbb{R}^n, \]
whenever the limit exists; \( c(n) \) is a suitable positive constant. It turns out that
\[ R_j : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n) \]
is linear and continuous; furthermore, we have the following fundamental relation between the first order Riesz potential $I_1$ and the Riesz transform:

$$R_j(f) = -\partial_j I_1(f), \quad (2.4)$$

for any $j = 1, \ldots, n$.

## 3. Existence and uniqueness

Let $f \in L^2(\mathbb{R}^3; \mathbb{R}^3)$. We first investigate existence and uniqueness of weak solutions of $\Delta u + \text{div } f = 0$ on $\mathbb{R}^3$, following, for instance, [1]. Let $\mathcal{E}: \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ be the fundamental solution of the Laplace operator on $\mathbb{R}^3$, i.e.

$$\mathcal{E}(x) := -\frac{1}{4\pi|x|}.$$ 

Moreover, let

$$\mathcal{P}(f)(x) := -\int \nabla \mathcal{E}(x - y) \cdot f(y) \, dy, \quad x \in \mathbb{R}^3.$$ 

**Theorem 3.1.** Let $H := \{ u \in L^6(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3; \mathbb{R}^3) \}$. Then $\mathcal{P}(f)$ is the unique weak solution in $H$ of the equation $\Delta u + \text{div } f = 0$ on $\mathbb{R}^3$.

**Proof.** We divide the proof in some steps.

**Step 1.** First of all we claim that $\mathcal{P}(f) \in H$. For, let $g \in L^2(\mathbb{R}^3)$ and, for any $i = 1, 2, 3$ we let

$$\mathcal{P}_i(g)(x) := -\int \partial_i \mathcal{E}(x - y)g(y) \, dy, \quad x \in \mathbb{R}^3.$$ 

Since

$$\partial_i \mathcal{E} = c \frac{x_i}{|x|^3}$$

then

$$|\mathcal{P}_i(g)(x)| \leq c \int \frac{|g(y)|}{|x - y|^2} \, dy = I_1(|g|)$$

and $\mathcal{P}_i(g) \in L^6(\mathbb{R}^3)$ from (2.3), being $g \in L^2(\mathbb{R}^3)$. In order to prove that $\partial_j \mathcal{P}_i(g) \in L^2(\mathbb{R}^3)$, let us choose a sequence $(g_h)_{h \in \mathbb{N}}$ in $C_c^\infty(\mathbb{R}^3)$ with $g_h \to g$ strongly in $L^2(\mathbb{R}^3)$. Using the explicit form of $\mathcal{E}$, we immediately get

$$|\mathcal{P}_i(g_h) - \mathcal{P}(g)| \leq c |I_1(g_h - g)|, \quad \text{on } \mathbb{R}^3.$$ 

Hence, by the continuity of $I_1 : L^2(\mathbb{R}^3) \to L^6(\mathbb{R}^3)$, we get $\mathcal{P}_i(g_h) \to \mathcal{P}_i(g)$, strongly in $L^6(\mathbb{R}^3)$. Now, for any $h \in \mathbb{N}$ we have

$$\mathcal{P}_i(g_h)(x) = \int \mathcal{E}(x - y)\partial_i g_h(y) \, dy = \bar{c} I_2(\partial_i g_h)(x), \quad \bar{c} \neq 0.$$
Therefore, using (2.4) and (2.2) we easily get
\[ R_j R_i (g_h) = \partial_j I_2 (\partial_i g_h) = \frac{1}{c} \partial_j \mathcal{P}_i (g_h). \]
By the continuity of the Riesz transform we deduce that \( ||\partial_j \mathcal{P}_i (g_h)|| \leq c ||g_h|| \). Thus \( \partial_j \mathcal{P}_i (g_h) \rightharpoonup u_{ij} \), for some \( u_{ij} \in L^2 (\mathbb{R}^3) \). Passing to the limit as \( h \to +\infty \) in
\[ \int \partial_j \mathcal{P}_i (g_h) \varphi dx = -\int \mathcal{P}_i (g_h) \partial_j \varphi dx \]
we deduce that \( u_{ij} = \partial_j \mathcal{P}_i (g) \) which means that \( \nabla \mathcal{P}_i (g) \in L^2 (\mathbb{R}^3; \mathbb{R}^3) \). In order to conclude it is sufficient to notice that
\[ \mathcal{P} (f) = \sum_{i=1}^{3} \mathcal{P}_i (f^{(i)}) \]
being \( f^{(i)} \), for \( i = 1, 2, 3 \), the components of \( f \).

**Step 2.** Now we prove that \( \mathcal{P} (f) \) is a weak solution of the equation \( \Delta u + \text{div} f = 0 \), that is
\[ \int (\nabla \mathcal{P} (f) + f) \cdot \nabla \varphi dx = 0, \quad \forall \varphi \in C_c^\infty (\mathbb{R}^3). \quad (3.1) \]
Let \((f_h)_{h \in \mathbb{N}}\) be a sequence in \( C_c^\infty (\mathbb{R}^3; \mathbb{R}^3) \) with \( f_h \to f \) strongly in \( L^2 (\mathbb{R}^3; \mathbb{R}^3) \). First of all we have, integrating by parts,
\[ -\mathcal{P} (f_h) (x) = \int \mathcal{E} (x - y) \text{div} f_h (y) dy \]
and therefore, since \( \mathcal{E} \) is the fundamental solution of the Laplace operator on \( \mathbb{R}^3 \),
\[ -\int \mathcal{P} (f_h) \Delta \varphi dx = \int \text{div} f_h \varphi dx = -\int f_h \cdot \nabla \varphi dx. \]
Passing to the limit as \( h \to +\infty \) we obtain
\[ -\int \mathcal{P} (f) \Delta \varphi dx = -\int f \cdot \nabla \varphi dx \]
which means, since \( \mathcal{P} (f) \in H \),
\[ \int \nabla \mathcal{P} (f) \cdot \nabla \varphi dx = -\int f \cdot \nabla \varphi dx. \]
Therefore we get (3.1).

**Step 3.** In order to conclude the proof, we have to show that the weak solution in \( H \) is unique. If \( u_1, u_2 \in H \) satisfy
\[ \int (\nabla u + f) \cdot \nabla \varphi dx = 0, \quad \forall \varphi \in C_c^\infty (\mathbb{R}^3) \]
then \( w := u_2 - u_1 \) satisfies
\[ \int \nabla w \cdot \nabla \varphi dx = 0, \quad \forall \varphi \in C_c^\infty (\mathbb{R}^3). \]
Now, if we choose \( \varphi_h \to w \) in \( H \) then passing to the limit as \( h \to +\infty \),
\[
0 = \int \nabla w \cdot \nabla \varphi_h \, dx \to \int |\nabla w|^2 \, dx
\]
from which we get \( w \) constant, and since \( w \in H \), we deduce that \( w = 0 \), and thus \( u_1 = u_2 \),
which yields the conclusion. \( \square \)

We are ready to prove the existence and uniqueness result for the equation (1.3).

**Corollary 3.2.** Let \( \Omega \) be an open and bounded domain in \( \mathbb{R}^3 \) and let \( \mathbf{m} \in L^2(\mathbb{R}^3;\mathbb{R}^3) \) with \( |\mathbf{m}| = \chi_\Omega \). Then the equation \( \Delta u + \text{div} \mathbf{m} = 0 \) on \( \mathbb{R}^3 \) admits a unique weak solution \( u \in H^1(\mathbb{R}^3) \).

**Proof.** Taking into account Theorem 3.1, it is sufficient to prove that \( P(\mathbf{m}) \in L^2(\mathbb{R}^3) \). Using the very definition of \( P(\mathbf{m}) \) and \( E \), we have, since \( |\mathbf{m}| = \chi_\Omega \), \( |P(\mathbf{m})| \leq cI_1(\chi_\Omega) \) and \( I_1(\chi_\Omega) \in L^2(\mathbb{R}^3) \) since \( \chi_\Omega \in L^\infty(\Omega) \) and \( \Omega \) is bounded; this yields the conclusion. \( \square \)

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