Partial differential equations

Self-adjoint and skew-symmetric extensions of the Laplacian with singular Robin boundary condition

Extensions self-adjointes et anti-symétriques du laplacien, avec condition à la frontière de type Robin singulière

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A B S T R A C T

We study the Laplacian in a bounded domain, with a varying Robin boundary condition singular at one point. The associated quadratic form is not semi-bounded from below, and the corresponding Laplacian is not self-adjoint, it has a residual spectrum covering the whole complex plane. We describe its self-adjoint extensions and exhibit a physically relevant skew-symmetric one. We approximate the boundary condition, giving rise to a family of self-adjoint operators, and we describe its spectrum by the method of matched asymptotic expansions. A part of the spectrum acquires a strange behavior when the small perturbation parameter $\varepsilon > 0$ tends to zero, namely it becomes almost periodic in the logarithmic scale $|\ln \varepsilon|$, and in this way “wanders” along the real axis at a speed $O(\varepsilon^{-1})$.

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R É S U M É

Nous étudions le laplacien dans un domaine borné, avec une condition à la frontière de type Robin, variable et singulière en un point. La forme quadratique associée n’est pas bornée inférieurement, et le laplacien correspondant n’est pas self-adjoint ; son spectre résiduel couvre entièrement le plan complexe. Nous décrivons ses extensions self-adjointes et nous en montrons une anti-symétrique, pertinente en physique. Nous approchons la condition de frontière à l’aide d’une famille d’opérateurs self-adjoints et nous décrivons son spectre par la méthode d’appariement des développements asymptotiques. Une partie du spectre adopte un comportement étrange quand le paramètre $\varepsilon > 0$ de petite perturbation tend vers zéro ; précisément, il devient presque périodique en échelle logarithmique $|\log(\varepsilon)|$, et ainsi « erre » le long de l’axe réel à une vitesse $O(\varepsilon^{-1})$.

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1. Description of the singular problem

In a domain $\Omega \subset \mathbb{R}^2$ enveloped by a smooth simple contour $\partial \Omega$, we consider the Laplacian with a Robin-type boundary condition $a\partial_n u - u = 0$. Here, $a$ is a continuous function defined on $\partial \Omega$, and $\partial_n$ denotes the outward normal derivative to $\partial \Omega$.

If $a$ is positive on $\partial \Omega$, the quadratic form $H^2(\Omega) \ni u \mapsto \|\nabla u; L^2(\Omega)\|^2 - a^{-1}u, L^2(\partial \Omega)\|^2$ is naturally associated with this problem and, in view of the compact imbedding $H^1(\Omega) \subset L^2(\partial \Omega)$, this form is semi-bounded and closed, and thus defines a self-adjoint operator with compact resolvent. Therefore, the spectrum is an unbounded sequence of real eigenvalues accumulating at $+\infty$. Note that the first eigenvalue is negative, and goes to $-\infty$ if $a$ is a small positive constant, see [5].

Let $a$ become zero at a point $x_0 \in \partial \Omega$. In this note, we will mainly consider the case where $a$ vanishes at order one, i.e. admits the Taylor formula

$$a(s) = a_0 s + O(s^2), \quad s \to 0 \quad \text{with} \quad a_0 > 0,$$

where $s$ is a curvilinear abscissa starting at $x_0$. For convenience, we denote $b_0 := a_0^{-1}$.

Since we assume $a$ to be continuous, there should exist at least one other point where $a$ vanishes. However, several points of vanishing do not bring any new effect, and we replace the problem by another one: we assume that $\partial \Omega$ is the union of two smooth curves $\Gamma_1$ and $\Gamma_2$ which meet perpendicularly, with $x_0$ in the interior of $\Gamma_1$, and that $a$ vanishes only at $x_0$, according to (1). We complete the Robin boundary condition on $\Gamma_1$ by a Neumann boundary condition on $\Gamma_2$. Therefore, our spectral problem is

$$\begin{cases}
-Du = \lambda u \quad \text{on} \quad \Omega, \\
\partial_n u - u = 0 \quad \text{on} \quad \Gamma_1, \quad \text{and} \quad \partial_n u = 0 \quad \text{on} \quad \Gamma_2.
\end{cases}$$

(2)

The associated quadratic form is defined on $D(q) := \{u \in H^1(\Omega), a^{-\frac{1}{2}}u|_{\Gamma_1} \in L^2(\Gamma_1)\}$ as follows

$$D(q) \ni u \mapsto \int_{\Omega} |\nabla u|^2 \, dx - \int_{\Gamma_1} a^{-1} |u|^2 \, ds.$$  

It is not semi-bounded anymore. Thus, there is no canonical way for defining a self-adjoint operator associated with problem (2). The natural definition becomes the operator $A_0$ acting as $-\Delta$ on the domain

$$D(A_0) := \{u \in D(q), \Delta u \in L^2(\Omega), a\partial_n u - u = 0 \quad \text{on} \quad \Gamma_1, \quad \partial_n u = 0 \quad \text{on} \quad \Gamma_2\}.$$ (3)

Such a problem was studied in [1,6] in a model half-disc, for which the eigenvalue equation had the advantage to decouple in polar coordinates. The authors found that $A_0$ is non-self-adjoint. In [6], they clarified the “paradox” from [1] stating that, for any $\lambda \in \mathbb{C}$, problem (2) has a nontrivial solution, by showing that the spectrum of $A_0^*$ is residual and coincides with the complex plane.

A three-dimensional version of this spectral problem appears also in the modeling of a spinless particle moving in two thin films with a one-contact point, and has been studied in a model domain in [3].

2. Goal and results

In this note, we explain how to find extensions of $A_0$ and give a better understanding of their spectrum, arguing with an asymptotic approach. We also exhibit a relevant skew-symmetric extension using a physical argument.

The domain of $A_0^*$ is

$$D(A_0^*) := \{u \in L^2(\Omega), \Delta u \in L^2(\Omega), a\partial_n u - u = 0 \quad \text{on} \quad \Gamma_1, \quad \partial_n u = 0 \quad \text{on} \quad \Gamma_2\}.$$  

To understand how different $D(A_0^*)$ is from $D(A_0)$, we exhibit two possible singular behaviors for functions in $D(A_0^*)$ at the point $x_0$. Using Kondratiev's theory [4], we investigate a model problem in a half-plane and, as a result, describe $D(A_0^*)$. We deduce, going over the domain $\Omega$, that the deficiency indices of $A_0$ are $(1,1)$, and we classify its self-adjoint extensions using a parametrization $\theta \mapsto e^{i\theta}$ of the unit circle $\mathbb{S}^1 \subset \mathbb{C}$. The description of $A_0^*$ allows us also to introduce a natural skew-symmetric extension of $A_0$ corresponding to a Sommerfeld radiation condition at $x_0$.

Next, we approach our problem by a family of self-adjoint operators by choosing a suitable perturbation of the Robin coefficient $a$. This is done by means of the non-vanishing discontinuous function

$$a_\varepsilon(s) = a_0 \text{sign}(s)\varepsilon + a(s)$$

satisfying $\inf_{\Gamma_1} |a| = \varepsilon$, and we study the discrete spectrum of the associated Robin Laplacian as $\varepsilon \to 0$. Using the method of matched asymptotic expansions, we find that its spectrum is related to the eigenvalues of self-adjoint extensions, with a parameter $\theta$ oscillating in the logarithmic scale as $\varepsilon \to 0$. 

Finally, we describe the differences when the weight function satisfies \( a(s) = a_0 |s| + O(s^2) \) near the singular point, with \( a_0 > 0 \). In particular, the number of singularities of functions in \( D(A_0^+ \nu) \) is now two or four, depending on whether \( a_0 > \frac{2}{3} \) or not.

A similar result has been obtained in [2], where an operator of the type \( \text{div}(\sigma \nabla) \) is considered in a bounded domain, where \( \sigma \) is piecewise constant and changes sign along an interface crossing the boundary.

3. Description of the adjoint operator

In this section, we investigate the following model problem in the half-plane \( \mathbb{R}^+ \): find \( u \in L^2_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}) \) such that

\[
\begin{align*}
- \Delta u &= 0 \quad \text{on } \mathbb{R}^+, \\
\partial_t u - u &= 0 \quad \text{on } \partial \mathbb{R}^+,
\end{align*}
\]

(5)

where \( a \) is the first-order approximation of \( a \) near \( x_0 \): \( a(s) = a_0 s \). Let \( (r, \varphi) \in (0, +\infty) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \) be the associated polar coordinates, the normal derivative reads \( \partial_t u(s, 0) = \pi r^{-1} \partial_r u(r, \pm \frac{\pi}{2}) \). The boundary condition is decoupled: The problem becomes, in polar coordinates:

\[
\begin{align*}
- \partial_r^2 u - r^{-1} \partial_r u - r^{-2} \partial_\varphi^2 u &= 0 \quad \text{on } (0, +\infty) \times (-\frac{\pi}{2}, \frac{\pi}{2}), \\
\forall r > 0: \quad -\partial_\varphi u - u &= 0 \quad \text{at } \varphi = \pm \frac{\pi}{2}.
\end{align*}
\]

The spectrum of the transverse operator \( -\partial_\varphi^2 \) is given by solving the eigenvalue problem:

\[
- g''(\varphi) = \mu g(\varphi), \quad -a_0 g'(\pm \frac{\pi}{2}) - g(\pm \frac{\pi}{2}) = 0.
\]

(6)

Its eigenvalues are \( \{\mu_k, k \geq 0\} := \{-b_0^2\} \cup \{k^2, k = 1, 2, \ldots\} \). The eigenspace associated with \( -b_0^2 \) is generated by \( g_0(\varphi) = e^{-b_0 \varphi} \), and the one associated with \( k^2 \) by

\[
g_k(\varphi) = \sin(k(\varphi + \frac{\pi}{2}))) - k a_0 \cos(k(\varphi + \frac{\pi}{2})).
\]

We introduce two singular solutions to (5):

\[
s^\pm(r, \varphi) = \epsilon r^{\pm i b_0} e^{-b_0 \varphi} \quad \text{with } \epsilon = (2 \sinh(b_0 \pi))^{-1/2}
\]

(7)

where the choice for the normalizing factor \( \epsilon \) will become clear in Proposition 2. Note that \( s^\pm \notin H^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}) \).

Let \( \chi \) be a smooth cut-off function that has a small support and equals one near the point \( x_0 \), and let \( S^\pm \) be the functions deduced in \( \Omega \) from \( s^\pm \): \( S^\pm(x) = \chi(x)s^\pm(r, \theta) \) through local polar coordinates \( \Omega \ni x \mapsto (r, \theta) \in \mathbb{R}^+ \times \mathbb{R} \).

As a consequence of the Kondratiev theorem on asymptotics (see [4] and, e.g., [9, Ch. 3]), we get

Proposition 1. Let \( u \in D(A_0^+ \nu) \), then there exists \( (c_{\text{in}}, c_{\text{out}}) \in \mathbb{C}^2 \) such that

\[
u = c_{\text{in}}(u) S^- + c_{\text{out}}(u) S^+ + \tilde{u}
\]

where \( \tilde{u} \in H^2(\Omega) \cap D(q) \). Moreover, there exists \( C > 0 \) such that, for all \( u \in D(A_0^+ \nu) \), we have

\[
|c_{\text{in}}(u)| + |c_{\text{out}}(u)| + \|\tilde{u}\|_{L^2(\Omega)} \leq C (\|u\|_{L^2(\Omega)} + \|\Delta u\|_{L^2(\Omega)}).
\]

(9)

This decomposition of the operator domain is sufficient to deduce the deficiency indices of the operator. On the one hand, since the operator has real coefficient, \( \dim(\ker(A_0^+ + i)) = \dim(\ker(A_0^+ - i)) \). On the other hand, the standard decomposition together with the last proposition implies \( \dim(\ker(A_0^+ + i)) + \dim(\ker(A_0^+ - i)) = \dim(D(A_0^+)/D(A_0^-)) = 2 \). Therefore, \( \dim(\ker(A_0^+ \pm i)) = 1 \), and the deficiency indices are \((1, 1)\). As a corollary, the spectrum of \( A_0^+ \) covers the whole complex plane.

4. Self-adjoint extensions

Once the domain of the adjoint is explicit, it is standard, see [7,10] and others, to find all self-adjoint extensions of \( A_0 \) by the use of the symplectic form

\[
\psi : (u, v) \mapsto \langle A_0^s v, v \rangle - \langle u, A_0^s v \rangle, \text{ defined on } D(A_0^s).
\]

As a consequence of integration by parts and symplectic algebra, we verify Proposition 2.
Proposition 2. Let $u$ and $v$ in $D(A_0^a)$, written in the form (8). Then

$$
\psi(u, v) = i \left( c_{\text{in}}(u) c_{\text{in}}(v) - c_{\text{out}}(u) c_{\text{out}}(v) \right).
$$

As a consequence of Proposition 2, for any $u \in D(A_0^a)$, we obtain

$$
\psi(u, u) = i |c_{\text{in}}|^2 - |c_{\text{out}}|^2.
$$

Therefore, the self-adjoint extensions are the restrictions of $A_0^a$ onto linear spaces of the functions $u \in D(A_0^a)$, for which $|c_{\text{in}}(u)| = |c_{\text{out}}(u)|$. We conclude with Theorem 3.

Theorem 3. Let $\theta \in \mathbb{R}$, and let $A_0(\theta)$ be the restriction of $A_0^a$ to the domain

$$
D(A_0(\theta)) = \{u \in D(A_0^a), c_{\text{in}}(u) = e^{i\theta} c_{\text{out}}(u)\}.
$$

Then $A_0^a$ is a self-adjoint extension of $A_0$ if and only if there exists $\theta \in \mathbb{R}$ such that $A_0^a = A_0(\theta)$.

Each domain of these extensions has compact injection in $L^2(\Omega)$ because a function $u \in D(A_0(\theta))$ differs from $\tilde{u} \in H^2(\Omega)$ by a linear combination of two functions $S^\pm \in L^2(\Omega)$. Therefore, each of these extensions has compact resolvent. Moreover, it is not semi-bounded from below, and we denote by $(\lambda_k(\theta))_{k \in \mathbb{Z}}$ the increasing sequence of eigenvalues of $A_0(\theta)$.

5. The physical radiation condition and a skew-symmetric extension

In link with the wave equation $-\partial_t^2 W = -\Delta W$, analyzing the propagation of the wave

$$
W^\pm(t, x) = e^{-i\sqrt{\lambda} t} S^\pm(x) = e^{-i\sqrt{\lambda}(t+\lambda^{-1/2} b_0 \ln r)} g_0(\psi),
$$

in the framework of the Sommerfeld or Mandelstam principles, cf. [9, Ch. 5], we can interpret $S^-$ as propagating from $x_0$, whereas $S^+$ would propagate toward $x_0$. Notice that any other radiation principle leads to the same conclusion.

For a fixed $\lambda \in \mathbb{R}$, the scattering theory (cf. [9, Ch. 5]) provides a solution to (2) in the form

$$
\zeta_\lambda = S^+ + e^{i\theta_0} S^- + \tilde{\zeta}_\lambda, \quad \text{with } \tilde{\zeta}_\lambda \in D(q) \cap H^2(\Omega). \quad (10)
$$

This solution is interpreted as the scattering wave initiated with the incident (entering) wave $S^+$, and $e^{i\theta_0}$ is the reflection coefficient, with $|e^{i\theta_0}| = 1$, according to the conservation of energy.

Moreover, a natural skew-symmetric extension $\mathcal{A}_0$ of $A_0$ can be defined in the domain

$$
D(\mathcal{A}_0) = \{u \in D(A_0^a), c_{\text{in}}(u) = 0\}.
$$

This extension corresponds to the natural radiation condition, excluding entering waves.

6. Wandering of the eigenvalues

Assume that $\lambda_k(\theta)$ is a simple eigenvalue of $A_0(\theta)$, and denote by $C_0(S^+ + e^{i\theta} S^-) + \tilde{u}$ an associated eigenfunction normalized in $L^2(\Omega)$. Then standard computations show that $\partial_\theta \lambda_k(\theta) = -|C_0|^2$. Therefore, an eigenvalue $\lambda_k(\cdot)$ is a non-increasing function of $\theta \in \mathbb{R}$ wherever it is simple; moreover, it is decreasing if $C_0 \neq 0$. If $C_0 = 0$ for some $k$ and $\theta$, then the constant eigenvalue $\lambda_k(\theta) = \lambda$ is associated with a trapped mode, that is a non-trivial solution to problem (2) belonging to $D(q) \cap H^2(\Omega)$, which is in $D(A(\theta))$ for any $\theta \in \mathbb{R}$.

The functions $\lambda_k(\cdot)$ are piecewise analytic; moreover, they cannot be all constant, indeed in that case the range of all the eigenvalues $\lambda_k(\theta)$ would be a discrete set, which contradicts the existence of the physical solution in the form (10) for any $\lambda \in \mathbb{R}$.

Therefore, there exists at least one branch $\lambda_k(\cdot)$ which is decreasing where it is regular. This, combined with the periodicity of the spectrum, shows that

$$
\bigcup_{(k, \theta) \in \mathbb{Z} \times \mathbb{R}} \lambda_k(\theta) = \mathbb{R}. \quad (11)
$$
7. The method of matched asymptotic expansions

For \( \varepsilon > 0 \), we recall that \( a_\varepsilon \) was defined in (4) as an approximation of \( a \). Now the quadratic form

\[
u \mapsto \int_\Omega |\nabla u|^2 - \int_{\Gamma_1} a_\varepsilon^{-1}(s)|u|^2\]

is well defined and bounded from below in \( H^1(\Omega) \). We denote by \( A^\varepsilon \) the corresponding self-adjoint operator. The strategy is now to construct quasi-modes for \( A^\varepsilon \) through eigenfunctions of an extension \( A_0(\theta_\varepsilon) \), where \( \theta_\varepsilon \) is to be chosen. The result is given by Theorem 4.

**Theorem 4.** For all \( k \geq 0 \) and for \( \varepsilon \) small enough, there exists \( \theta_\varepsilon \in \mathbb{R} \) such that \( \text{dist} (\lambda_k(\theta_\varepsilon), \sigma(A^\varepsilon)) \to 0 \) as \( \varepsilon \to 0 \). Moreover, the mapping \( \varepsilon \mapsto \theta_\varepsilon \) is periodic with respect to \( \ln \varepsilon \), and

\[
\mathbb{R} = \bigcup_{\varepsilon > 0} \sigma(A^\varepsilon). \tag{12}
\]

The procedure is as follows.

- **Far-field expansion:** outside a fixed neighborhood of \( x_0 \), take a function \( u^\text{out} \) as an eigenfunction of \( A_0(\theta_\varepsilon) \), where \( \theta_\varepsilon \) is to be chosen. Therefore, it behaves near \( x_0 \) as follows:

\[
u^\text{out} : (r, \varphi) \mapsto C(r^{i\theta_0} + e^{i\theta_1}r^{-i\theta_0} + r^{-i\theta_1})^s + \tilde{u}^\text{out},
\]

where \( \tilde{u}^\text{out} \) is regular and small near 0.

- **Near-field expansion.** In local coordinates near \( x_0 \), we perform the scaling \( \chi = \varepsilon \xi \), and considering bounded eigenvalues, we get to solve (5) with \( a(\xi_1) := a_0(\text{sign}(\xi_1) + \xi_1) \). In order to investigate the behavior of solutions to this problem at infinity, we perform the inversion \( \xi \mapsto \eta = |\xi|^{-2}\xi \), which leads to the behavior at the origin \( \eta = 0 \) of (5), but with the weight function \( a_0 : \eta_1 \mapsto \eta_1 + \text{sign}(\eta_1)\eta_1^2 \) in the boundary condition. Near the origin \( \eta = 0 \), we can neglect the part \( \text{sign}(\eta_1)\eta_1^2 \), and according to Kondratiev’s theory [4], there exists a solution to such a problem that behaves as

\[
\eta \mapsto s^- (\eta) + e^{i\theta} s^+ (\eta) + O(|\eta|), \quad \theta \in \mathbb{R} \text{ fixed.}
\]

Therefore, we obtain a solution to (5) with weight \( a \), which produces after rescaling a solution to the Laplace equation in \( \Omega \) that behaves in a neighborhood of \( x_0 \) as

\[
u^\text{in} : (r, \varphi) \mapsto C(e^{i\theta_0}r^{-i\theta_0} + e^{i\theta_1}r^{-i\theta_0} + e^{i\theta_1})^s + \tilde{u}^\text{in}
\]

where \( \tilde{u}^\text{in} \) is decaying outside the neighborhood, and \( \tilde{C} \) is a normalization factor.

- **Matching expansions and conclusion.** Matching the two previous expansions, we obtain:

\[
\theta_\varepsilon = \theta - 2b_0 \ln \varepsilon \pmod{2\pi}. \tag{13}
\]

This formal approach is validated by constructing the quasi-mode from the previous ansätz using cut-off functions: define

\[
u^\text{as} = \chi^\text{in} \nu^\text{in} + \chi^\text{out} \nu^\text{out} - \chi^\text{in} \chi^\text{out} C(s^+ + e^{i\theta_0}s^-),
\]

where \( \chi^\text{in} \) (respectively, \( \chi^\text{out} \)) is localized in a bounded neighborhood of \( x_0 \) (respectively, outside a neighborhood of \( x_0 \) of size \( O(\varepsilon) \)). Evaluating \( (A^\varepsilon - \lambda_k(\theta_\varepsilon))u^\text{as} \), we get that \( \lambda_k(\theta_\varepsilon) \) is close to the spectrum of \( A^\varepsilon \) for \( \varepsilon \) small enough. Note that \( \theta_\varepsilon \) is periodic with respect to \( \ln \varepsilon \) and \( e^{i\theta_0} \) runs over \( S^1 \subset \mathbb{C} \) at the rate \( O(\varepsilon^{-1}) \) as \( \varepsilon \to 0 \). Then, (12) follows from (11).

8. Further questions

When the weight function satisfies \( a(s) = a_0|s| + O(s^2) \), with \( a_0 > 0 \), the situation depends on the parameter \( a_0 \), as described here: The transverse operator in the angular variable in the model half-plane \( \mathbb{R}^2_+ \) is still \( -\partial_\varphi^2 \), but the boundary condition at \( \varphi = \frac{\pi}{2} \) in (6) now becomes \( a_0 \tilde{g}'\left(\frac{\pi}{2}\right) - g\left(\frac{\pi}{2}\right) = 0 \). The negative spectrum of this operator depends on \( a_0 \) as follows.

1° If \( a_0 > \frac{\pi}{2} \), then there is one negative eigenvalue, and the other ones are positive. It produces two oscillatory solutions.
2° If \( \delta_0 = \frac{\pi}{2} \), there is one negative eigenvalue, and null is also an eigenvalue. There are two additional solutions for the model problem, one has the form \( g_0(\varphi) \ln r \), and the other is constant with respect to \( r \).

3° If \( \delta_0 < \frac{\pi}{2} \), then there are two negative eigenvalues. They produce four oscillatory solutions.

Situation 1° can be analyzed exactly in the same way as that we described here. However, situations 2° and 3° are much more different. In particular, the deficiency indices are (2,2), and the self-adjoint extensions are parameterized by two-by-two unitary matrices. The method of the matched asymptotic expansions does not provide an explicit parameter extension as in \([13]\), but a family of unitary matrices depending on \( \varepsilon \), cf \([8]\). This family does not always coincide with the set of all unitary matrices as \( \varepsilon \to 0 \), but it is sufficient for the construction of approximations.

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