BLACK HOLE ENTROPY
FROM
BPS CLOSED STRING

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Abstract

By using BPS closed string, the entropy is calculated of the extremal five dimensional black hole consisting of Dirichlet onebranes, Dirichlet fivebranes and Kaluza-Klein momentum in the flat background approximation. In our formulations we consider two kinds of BPS closed strings with or without a winding number. In the former case heavy excitation modes of closed strings are used to derive the entropy. In the latter case we have no oscillator modes and consider collective motion of such massless closed strings. The entropy is given by the number of the ways how we divide the Kaluza-Klein momentum among the massless closed strings. In both cases the black hole entropy is the same as the Bekenstein-Hawking entropy. We argue that the collective modes of closed strings without winding is equivalent to a single closed string with winding. We propose that the two closed string pictures are connected with the open string pictures by the modular transformation.
1 Introduction

The recent breakthrough of string theory makes it possible to calculate the Bekenstein-Hawking entropy of black holes microscopically. The black holes are classical solution of supergravity with branes and the entropies are originated from brane fluctuations expressed in terms of open strings with Dirichlet boundary condition.

Up to now we have two approaches to calculate microscopic entropy of black holes. One is that we consider a \( \sigma \)-model of the low energy effective theory of the open string system for counting the black hole microstates. This one brings us the first success of the microscopic calculations. The other is that we consider the collective motion of massless open strings and count the ways to divide a Kaluza-Klein momentum among individual massless modes, giving the microscopic entropy.

In this paper we establish another approach by using closed strings which are exchanged by Dirichlet branes. We show that the black hole entropy from the closed string picture precisely agrees with the Bekenstein-Hawking entropy. We consider a five-dimensional extremal black hole in the type IIB supergravity which is made of intersecting Dirichlet onebranes and fivebranes and of Kaluza-Klein momentum along the direction of the intersections. The original ten-dimensional spacetime is compactified on a fivetorus on which the branes are wrapping. These branes exchange closed strings with a winding number. In the low energy limit \( \alpha' \to 0 \) the world sheets are squeezed so that any massive oscillator modes contribute to the black hole entropy. The modular transformation enable us to change this closed string picture to the above mentioned \( \sigma \)-model picture of open string. The system of closed strings will be the \( \sigma \)-model of the low energy open string effective theory.

Further, the closed string with a winding number can be regarded as a solitonic state which consists of infinite number of closed strings with no winding number. The system of these closed strings with no winding number is directly described by the type IIB supergravity. We show that the collective motion of the closed strings have the same physical degrees of freedom as a single closed string with winding number. Then, we have again the correct Bekenstein-Hawking entropy. These closed string pictures and their interrelation with the open string pictures will provide us with a deep understanding of the relation between macroscopic black hole and its microstates.

This paper is organized as follows: First, we briefly review the black hole which is considered in this paper in section 2. We give the metric of the black hole and the...
macroscopic black hole entropy. The supersymmetry of the system is also discussed in the presence of branes. In section 3 we consider the BPS closed string system with winding number, and show that we obtain the black hole entropy which is precisely the same as the macroscopic entropy. In section 4 we discuss the relation between the BPS closed string system with a winding number and the \( \sigma \)-model of the low energy effective open string theory. We discuss the relation by using the modular transformation, and we find that an exotic Neumann boundary condition is needed. We derive the central charge and the target space of the closed string system. We conclude that the \( \sigma \)-model is identical with the BPS closed string system in the low energy limit. In section 5 we also derive the black hole entropy by using infinite number of closed string with no winding. We can regard the single closed string with a unit winding number as a solitonic state that consists of infinite number of closed strings with no winding. We discuss the relation between the closed string system with no winding and the massless open string system mentioned above. These system are found to be the same. We find that the formulation of the open string system should be modified in order to obtain correct results. The last section is devoted to the discussions.

## 2 Macroscopic Black Hole Entropy

We work in ten-dimensional Minkowski space with the time coordinate \( x^0 \) and the space coordinates \( x^1, \cdots, x^9 \). The coordinates \( x^5, \cdots, x^9 \) are compactified on fivetorus \( T^5 \) with their radii \( R_5, \cdots, R_9 \). Let us first consider \( Q_1 \) Dirichlet onebranes, \( Q_5 \) Dirichlet fivebranes and \( Q_K \) Kaluza-Klein momentum. The Dirichlet onebranes are located at \( x^1 = x^2 = \cdots = x^8 = 0 \). The Dirichlet fivebranes are located at \( x^1 = x^2 = \cdots = x^4 = 0 \). We put the Kaluza-Klein momentum along the direction \( x^9 \).

The five dimensional extremal black hole is a solution of the type IIB supergravity. The dilaton field and the ten-dimensional string metric are\(^{[13]}\)

\[
\begin{align*}
e^{2\phi} &= H_1 H_5^{-1}, \\
ds^2 &= H_1^{1/2} H_5^{1/2} \left[ H_1^{-1} H_5^{-1} (-dx_0^2 + dx_5^2 + K(dx_0 - dx_9)^2) + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + H_5^{-1}(dx_5^2 + dx_6^2 + dx_7^2 + dx_8^2) \right], \\
B_{09}^{RR} &= \frac{1}{2}(H_1^{-1} - 1), \\
DB_{ijk}^{RR} &= \frac{1}{2} \epsilon_{ijkl} \partial_9 H_5, \quad i, j, k, l = 1, \cdots, 4.
\end{align*}
\] (2.1)
The quantities $H_1$, $H_5$ and $K$ are harmonic functions in the four-dimensional space $(x^1, x^2, x^3, x^4)$ such that

$$H_1 = 1 + G_5 \frac{4M_{D1}Q_1}{\pi r^2}, \quad H_5 = 1 + G_5 \frac{4M_{D5}Q_5}{\pi r^2}, \quad K = G_5 \frac{4M_{KK}Q_K}{\pi r^2},$$

(2.2)

where $G_5 = \pi g^2 \alpha'/4(R_5 R_6 R_7 R_8 R_9)$ is the five-dimensional Newton constant and $r^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$. $M_{D1}$, $M_{D5}$ and $M_{KK}$ are the masses of branes in the modified Einstein frame, which are

$$M_{D1} = \frac{R_9}{g \alpha'}, \quad M_{D5} = \frac{R_5 R_6 R_7 R_8 R_9}{g \alpha'}, \quad M_{KK} = 1/R_9.$$  

(2.3)

Because of the BPS property the mass of the extreme black hole is

$$M_{BH} = Q_1 M_{D1} + Q_5 M_{D5} + Q_K M_{KK}.$$  

(2.4)

In order to calculate the horizon area of the black hole the metric is sent to the Einstein frame, and the Bekenstein-Hawking entropy is

$$S_{BH} = 2\pi \sqrt{Q_K Q_1 Q_5}.$$  

(2.5)

Let $Q_L$ and $Q_R$ be the type IIB supercharges of positive chirality generated by left- and right-moving closed strings, respectively. At the spatial infinity the system is invariant under linear combinations $\epsilon_L Q_L + \epsilon_R Q_R$ of supersymmetries with $\epsilon_L$ and $\epsilon_R$ are covariantly constant spinors such that

$$\epsilon_L = \Gamma^0 \Gamma^9 \epsilon_R = \Gamma^0 \Gamma^5 \Gamma^6 \Gamma^7 \Gamma^8 \Gamma^9 \epsilon_R = \Gamma^0 \Gamma^9 \epsilon_L.$$  

(2.6)

This is derived from the Killing spinor condition

$$0 = \delta \psi_\mu = \frac{1}{\kappa} (\partial_\mu + \frac{1}{4} \omega_\mu^{rs} \gamma_{rs}) \varepsilon + \frac{i e^{-\phi}}{96} (H^{RR}_{\nu\rho\lambda} \gamma^\nu_{\mu} - 9 H^{RR}_{\nu\rho\lambda} \gamma^\rho_{\mu}) \varepsilon^*$$

(2.7)

in the Schwarz’s notation[15] and $H^{RR} = dB^{RR}$. The solution has the tensor product structure

$$\epsilon_L = \epsilon_R = \xi^+_{SO(1,1)} \xi^+_{SO(4)} \xi^+_{SO(4)},$$

(2.8)

where + means the positive chirality. $\xi^+_{SO(1,1)}$ has one component and the two $\xi^+_{SO(4)}$ have two components so that 4 out of the original 32 supersymmetries are survived in the system. The other supersymmetries become nilpotent.

The condition (2.6) tells us about the number of manifest supersymmetry in the low energy. What is necessary condition for considering the black hole entropy is that in the
near horizon geometry $BTZ \times S^3 \times T^4$, where $BTZ$ is the BTZ black hole geometry [16]. So we work in the near horizon geometry. However, we make use of the flat space approximation. For the purpose of counting the black hole microstates we expect that the flat space approximation will not change the number of states, although of course other important detailed structure will change.

3 Black Hole Entropy from BPS Closed Strings

In this section we calculate the black hole entropy in a microscopic point of view. We consider the above extremal black hole (2.1). Recent developments for calculating black hole entropy are based on the fact that fluctuations of branes are expressed by open strings with Dirichlet boundary condition. In ref. [10] what is responsible for the entropy of the black hole is the microscopic degrees of freedom which stem from an ensemble of massless open strings with one end attached to a onebrane and the other to a fivebrane. There are two approaches to calculate the microscopic entropy. One is that we consider a $\sigma$-model of the low energy effective open string theory. The other is that the entropy is given by the partition number which is derived from counting the ways how the Kaluza-Klein momentum are distributed among the massless open string states.

In this paper we consider a different microscopic picture from the above two to calculate the black hole entropy. We are interested in counting the number of BPS closed string states. We take the weak string coupling limit $g \to 0$. We use the closed strings exchanged between the Dirichlet onebranes and fivebranes.

For simplicity we start the discussion with the BPS closed string exchanged by single onebrane and single fivebrane. Suppose the closed string is wrapping once around the intersection between the onebrane and fivebrane. The closed string have a unit winding number in the $x^9$ direction. We might think that the BPS closed string with winding could not be exchanged between the onebranes and the fivebranes. Fortunately, such closed string is exchanged, which is discussed in subsection 3.2. The winding string can really propagate with a winding number $p = \oint \gamma^{-1} \partial \gamma$ in their notation. If the space-time is really flat, such winding string would not be exchanged. We put the Kaluza-Klein momentum $n/R_9$ in the $x^9$ direction besides winding mode. These two data determine

* The discussion is given in the subsection 3.1 why there is only the closed sting with the unit winding number.
the zeromode of the closed string. The zeromode part of the $X^9$ coordinate is

$$X^9 = x^9 + \alpha' \left( \frac{n}{R_9} + \frac{R_9 w}{\alpha'} \right) (\tau + \sigma) + \alpha' \left( \frac{n}{R_9} - \frac{R_9 w}{\alpha'} \right) (\tau - \sigma) + \cdots$$  \hspace{1cm} (3.1)

with $w = 1$ in the flat space approximation. The Virasoro generators,

$$L_0 = \frac{\alpha'}{4} \left( p_\mu^2 + \left( \frac{n}{R_9} + \frac{R_9}{\alpha'} \right)^2 \right) + N, \quad \tilde{L}_0 = \frac{\alpha'}{4} \left( p_\mu^2 + \left( \frac{n}{R_9} - \frac{R_9}{\alpha'} \right)^2 \right) + \tilde{N},$$  \hspace{1cm} (3.2)

vanish where $N$ and $\tilde{N}$ are right- and left-moving number operator, respectively and $\mu = 0, 1, 2, 3, 4$. We should note that the authors\cite{17} write down the spacetime Virasoro generators in the NSR formalism. For the later convenience we deal with the Green-Schwarz formalism for the type IIB superstring and we adopt the $(x^0, x^9)$ light-cone gauge.\footnote{We use the notations in the text book\cite{18}.}

So, the fermionic coordinates belong to the spinor representation of $Spin(9,1)$ Lorentz group. Let us consider the BPS string states with the five-dimensional mass $M = n/R_9 + R_9/\alpha'$. According to the Virasoro condition the right-moving oscillator modes freeze out, and the left-moving degrees of freedom survive; $N = 0$ and $\tilde{N} = N + n$. This means that the right-mover supercharges are nilpotent; $Q_R \approx 0$. When we consider the BPS mass in the flat space approximation, we have the contribution $R_9/\alpha'$ from the winding mode. However, what we would like to consider is the near horizon geometry $BTZ \times S^3 \times T^4$, and then the extremal condition should be $M = n/R_9$ (see for example \cite{19}). In the flat space approximation this condition will be achieved by the small radius limit $R_9 \to 0$.

Further, the BPS condition $\Gamma^0 \Gamma^9 \epsilon_L = \epsilon_L$ restricts the left-mover supercharges. This reduces the $16 = 8_S + 8_C$ of the original fermionic coordinates to $8_S$. This restriction, however, makes no effect to the physical fermionic degrees of freedom since the restricted $8_C$ is already excluded by the light-cone gauge $\Gamma^+ \theta_L = 0$.

We should notice that if we did not take any further BPS condition into account, we had the $8_V$ bosonic and $8_S$ fermionic string coordinates in the left-moving physical degrees of freedom. As will be seen, the $8_V$ and $8_S$ are in fact twice as large as the correct degrees of freedom of the BPS string with $32/8=4$ supersymmetries. We shall return to this point later, but we tentatively take $8_V$ and $8_S$ for a while.

Suppose that we put a onebrane and a fivebrane with a distance $L \neq 0$. In general, the propagation of a closed string can be decomposed as a superposition of the propagations of particles with any spin. If the closed string propagates in the distance $L$, not all such
particles propagates in the distance $L$. Only particles contribute this process with mass at most $M < 1/L$. This means that we have a cutoff $N_0 \sim 1/L$ in the level of the closed string oscillator modes. This point becomes important to explain the black hole entropy in terms of closed string picture. We will conclude that what is responsible for the black hole entropy is such a closed string that propagates not through a non-zero distance but through the zero distance. Namely, it is the contact interaction that the closed string induces between the onebrane and the fivebrane at their intersection. Let us consider this point. Let us count the degeneracy of states $d_n$ for the closed string propagating in the distance $L$. The massive modes with mass heavier than $1/L$ decouple in the process. In this situation the degeneracy of states $d_n$ is calculated from

$$\sum_{n=0}^{\infty} d_n q^n = \text{tr} q^N \big|_{M<1/L} = 16 \prod_{n=1}^{N_0} \left( \frac{1+q^n}{1-q^n} \right)^8. \quad (3.3)$$

This shows that the degeneracy of states $d_n$ is suppressed for a large enough value of the level $n$. So we will find it hard to produce the Bekenstein-Hawking entropy (2.5) even if we consider the ensemble of such strings. The true situation is not this case: Let us send $L \to 0$ or equivalently $N_0 \to \infty$. Since the Dirichlet onebrane and fivebrane have the intersection, they have the contact interaction at the intersection exchanging the closed string. Now, in turn, the value of $d_n$ is monotonically increasing as the value of $n$ increases, since any heavier modes contribute to $d_n$. As a result, contact interaction is needed to produce arbitrary large number of degeneracy of states. Then, we concentrate on the closed strings which generate contact interaction.

Next, let us consider the remaining BPS condition. Let us count the number of states of a single closed string. In the first a few level we have $\text{tr} q^N = 16 + 256q + 2304q^2 + \cdots$. These number of states are nothing but those required by the $1/4$ of the 32 original supersymmetries. However, the correct number of supersymmetries is just the $1/8$ of the 32 in the present system. The resolution is very simple to reduce more half of 8 supersymmetries. Of course, anomaly free set of physical superstring is $8_V + 8_S$. But the supersymmetry restriction (2.6) tells us that the relevant degrees of freedom of the low energy system is more half BPS states of $8_V + 8_S$. The bosonic and fermionic oscillator modes $\alpha_n^i$ and $S_n^a$ should belong to a representation of the present 4 supersymmetries. This means that they have four degrees of freedom for each level:

$$\alpha_n^i, \quad S_n^a, \quad i, a = 1, \cdots, 4. \quad (3.4)$$

Especially, if we consider the system at infinity (or flat space approximation) the fermionic
coordinates become the spinor with the same index structure of eq. (2.6). Now, the correct physical degrees of freedom read

\[ \text{tr} \ q^{\tilde{N}} = 8 \prod_{n=1}^{\infty} \left( \frac{1 + q^n}{1 - q^n} \right)^4 = 8 + 64q + 320q^2 + \cdots. \]  

(3.5)

Next, we would like to consider the statistical ensemble, especially the grand canonical ensemble, of such closed strings. When we work with open strings, we consider the process that open strings do not create and annihilate and then it is enough to consider them as a canonical or micro-canonical ensemble. In the case of the closed strings they are creating and annihilating. Then, we consider them as a grand canonical ensemble. One might think that it was strange to count the number of states of such an intermediate closed string. Rather, this way of counting can be seen in anywhere. For example, similar situation is found in the system of photons in a black box. The number of them does not conserve and they create and annihilate on a wall of the box. They are also in intermediate states in a sense.

Suppose there are \( Q_1 \) Dirichlet onebranes and \( Q_5 \) Dirichlet fivebranes. Since the two kinds of branes have \( Q_1Q_5 \) intersections, we have \( Q_1Q_5 \) closed strings. Each closed string has a Kaluza-Klein momentum \( n_i \) with a given total Kaluza-Klein momentum; \( Q_K = \sum_{i=1}^{Q_1Q_5} n_i \). We have a degeneracy of dividing the total number \( Q_K \) into the positive integers \( n_i (i = 1, 2, \cdots, Q_1Q_5) \) of the individual closed strings. Taking this degeneracy into account, we obtain

\[ \sum_{n=0}^{\infty} d_n q^n = \left( \text{tr} \ q^{\tilde{N}} \right)^{Q_1Q_5} \approx \exp \left( -\frac{\pi^2 c}{6 \ln q} \right), \quad (q \rightarrow 1), \]  

(3.6)

where we have a central charge \( c = 4(1 + 1/2)Q_1Q_5 \). In the leading order a simple saddle point evaluation allows us to obtain

\[ d_n = \exp \left( 2\pi \sqrt{\frac{mc}{6}} \right). \]  

(3.7)

The degeneracy of states (3.7) with \( n = Q_K \) and the center \( c = 4(1 + 1/2)Q_1Q_5 \) precisely produce the Bekenstein-Hawking entropy (2.5).

Finally, we consider the mass of the present black hole. Adding up the masses of onebranes, fivebranes and BPS strings, we have the total mass

\[ M_{BH} = Q_1M_{D1} + Q_5M_{D5} + Q_KM_{KK}. \]  

(3.8)

\[ \text{\textsuperscript{\dagger}}\text{Instead of considering the present case in which all onebranes and fivebranes have unit winding number, we may consider another cases in which, for example, single onebrane has winding number } Q_1 \text{ and single fivebrane has winding number } Q_5. \text{ This is discussed in the subsection 3.1.} \]
We have to care about a contribution from the winding mode of the BPS strings. In the flat space the black hole mass has the contribution $R_9 Q_1 Q_5 / \alpha'$ from the winding mode. However, the near horizon geometry do not produce such a contribution as is already explained. In any case if we consider the weak string coupling limit $g \to 0$ and the small radius limit $R_9 \to 0$, the mass of winding mode becomes negligible compared with the masses of onebrane, fivebrane and Kaluza-Klein momentum.

### 3.1 Units of the winding number

In this subsection we discuss that there are several choices of the units of winding numbers and why there are only the closed strings with the unit winding number.

In this paper we consider $Q_1$ onebranes and $Q_5$ fivebranes with winding number one around the $x^9$ direction with radius $R_9$. Instead of this case there are many other possibilities giving the same result. Let us suppose that there are $q_1$ onebranes with a winding number $w_1$ and $q_5$ fivebranes with a winding number $w_5$. Then, the total Ramond-Ramond charges are $Q_i = w_i q_i$ ($i = 1, 5$), respectively, for onebranes and fivebranes. We assume that $w_1$ and $w_5$ are relatively prime number. In the following discussion we concentrate on a specific onebrane and a specific fivebrane.

Let us consider the process that a closed string is emitted from the onebrane and is absorbed into the fivebrane. Since the closed string has to wrap around the onebrane integral times, the closed string has to have winding number in unit of $w_1$; $w = w_1 n$ where $n \in \mathbb{Z}$. On the other hand, the same argument should hold for the fivebrane, and thus the closed string also has to have winding number in unit of $w_5$. In order to satisfy the above two conditions, we find the closed string winding number $w = w_1 w_5 m$ where $m \in \mathbb{Z}$.

In this situation we put a Kaluza-Klein momentum $p_9 = n / R_9$ on the closed string. The BPS mass of the closed string is $M = n / R_9 + mw_1 w_5 R_9 / \alpha'$. Since the mass is linear in $n$ and $m$, this closed string can split into $m$ pieces with unit winding number $w_1 w_5$ and mass $M_i = n_i / R_9 + w_1 w_5 R_9 / \alpha'$ where $n_1 + n_2 + \cdots + n_m = n$. Then, it will be enough to consider the closed string with unit winding number $w_1 w_5$.

In this case the Virasoro condition of the total system is $N = 0$ and $\tilde{N} = w_1 w_5 Q_K$. Since we are considering $q_1 q_5$ such closed string, the total central charge is $c = 6 q_1 q_5$ and the entropy is $S = 2\pi \sqrt{N c / 6} = 2\pi \sqrt{Q_1 Q_5 Q_K}$. This final formula does not depend on the ways how we divide $Q_i$ into two integers $q_i$ and $w_i$. 
3.2 BPS property and exchanging winding strings

In this subsection we show that in the flat background approximation the BPS closed string with a winding number can be exchanged by the onebranes and the fivebranes in the weak string coupling and small radius limits; $g \rightarrow 0$ and $R_9 \rightarrow 0$. In fact it is seen that the number of global supersymmetries is the same as that of the system without a winding number. We notice that winding string indeed can be exchanged in the near horizon geometry\cite{17}.

Since the ten-dimensional spacetime is compactified by a fivetorus to a five-dimensional spacetime, onebranes and fivebranes are looked as a point particle sitting at the origin seen from the five-dimensional spacetime. The winding number of the fundamental string is the conserved charge of the gauge symmetry stemming from the NS-NS two-form field. Suppose that the onebranes and fivebranes exchange a closed string with a winding number. No matter how onebranes and fivebranes change their NS-NS charge from zero, the total NS-NS charge at the origin is conserved and is zero. Then, we have no NS-NS gauge field emerging from the origin. Then, the supergravity solution of this system is identical with the ordinary one for Dirichlet onebrane, Dirichlet fivebrane and Kaluza-Klein momentum. This is an important fact to explain why the black hole has a huge number of degeneracy of its microstates.

Next, let us see that in the flat background approximation it is possible to exchange winding string and the four global supersymmetries survive. Since we are taking the weak string coupling limit $g \rightarrow 0$, the charge lattice of NS-NS and R-R two form fields collapses. We are also taking the small radius limit $R_9 \rightarrow 0$. The mass of the winding string is negligible compared with the mass of the branes. Thus, branes can emit and absorb winding string without energy. Then, after emitting and absorbing winding string, the system is still BPS state. In order to make sure this point we show two discussions to find that we have four global supersymmetries. First, we start the discussion with open string with Dirichlet boundary condition. According to the standard discussion\cite{1} the open string has four global supersymmetries. Now, suppose that this open string propagates around the compact $x^9$ space making a world sheet which can be interpreted as that of closed string with a winding number. This closed string is nothing but what we are dealing with. Then, we have four global supersymmetries. This result also can be obtained using supergravity. As is recognized above, the supergravity solution of the system with intermediating winding strings is identical to that without winding strings.
The authors[17] also considered the winding string with winding number, \( p = \oint \gamma^{-1} \partial \gamma \) in their notation, in the same background \( AdS_3 \times S^3 \times T^4 \). These two solution provide us with the same condition for the global supersymmetry; \( \delta \psi_\mu = 0 \), and thus the two system have the same global supersymmetry.

4 Ensemble of BPS Closed Strings as a \( \sigma \)-model

In this section let us discuss about the relation between open and closed string pictures. Before doing this we recall that the brane fluctuations are expressed in terms of open strings with Dirichlet boundary condition.[10] We will find it convenient to briefly recall the quantization of branes by open strings. There are three types of open strings which are connecting two onebranes, two fivebranes and a onebrane at one end and a fivebrane at the other end. We call these (1,1), (5,5) and (1,5) strings, respectively. In the following we consider onebrane worldvolume gauge theory, although fivebrane worldvolume theory will give the same results. There are \( Q_1 Q_5 \) different (1,5) strings since we have \( Q_1 \) onebranes and \( Q_5 \) fivebranes. The (1,5) strings form a matter with the bi-fundamental representation in the \( U(Q_1) \) and \( U(Q_5) \) gauge theories. The degrees of freedom of the (1,1) and (5,5) strings are dropped by \( D \)-flatness and gauge fixing conditions.[20] The relevant degrees of freedom come from (1,5) strings. In summary, the physical fluctuations of onebranes and fivebranes are described by \( Q_1 Q_5 \) (1,5) strings in the onebrane worldvolume gauge theory.

Now let us start the discussion with the simplest case \( Q_1 = Q_5 = 1 \). The relevant (1,5) open string has the boundaries on the onebrane at one end and on the fivebrane at the other end. As increasing the world sheet time \( \tau \) from 0 to \( \pi \), the boundary point of the (1,5) open string on the onebrane sweeps along the onebrane and the boundary point on the other end draws a copy of a circle on the fivebrane. Interchanging the parameters \( \sigma \rightarrow \tau \) and \( \tau \rightarrow -\sigma \), the open string world sheet can be interpreted as the world sheet of a BPS closed string.[1] Notice that the time \( \tau \) of the closed string is corresponding to \( \sigma \) of a open string and vise versa. The argument by Polchinski[1] should be naturally extended to the case with both the Kaluza-Klein and winding modes. Namely, the closed string world sheet is rewritten in terms of the above (1,5) open string by the modular transformation. We know that in terms of the closed string picture the onebrane and the

\footnote{The cylinder amplitude has modulus. In the following discussion, we use the particular value of the modulus.}
fivebrane become the boundary states of a closed string at \( \tau = 0, \pi \) with winding number \( w = 1 \) in the \( x^9 \) direction. Then, corresponding to this fact, the open string should have a winding number around its \( \tau \) direction. The resultant world sheet is topologically a copy of an annulus. For definiteness we consider that the inner boundary circle of the annulus is on the onebrane and the outer one is on the fivebrane.

We think that this closed/open string correspondence needs more explanation: The closed/open correspondence by the modular transformation should be persisted not in only the oscillator modes but also in the zeromodes. For the purpose of explaining, we begin with a BPS closed string in the flat background approximation. If the BPS closed string has both the Kaluza-Klein momentum \( n/R_9 \) and the winding number \( R_9 w/\alpha' \), the corresponding open string also does in some sense. Here \( n, w = 0, \pm 1, \pm 2, \cdots \). We impose the boundary condition of BPS closed string and interchanging with \( \tau \to -\sigma \) and \( \sigma \to \tau \). The resultant open string coordinate is

\[
X^9 = x^9 + \alpha' \left( \frac{n}{R_9} + \frac{R_9 w}{\alpha'} \right) (-\sigma + \tau) + \alpha' \left( \frac{n}{R_9} - \frac{R_9 w}{\alpha'} \right) (-\sigma - \tau) + \cdots. \tag{4.1}
\]

This open string has, in fact, an exotic Neumann boundary condition. How should we modify the conventional argument of the open string boundary condition? Let us consider this point. Let \( S_0 \) be the Polyakov action for the open string. The variation of the action is

\[
\delta S_0 = -\frac{1}{2\pi \alpha'} \int d^2 \sigma \delta X_\mu \partial^\alpha \partial_\alpha X^\mu + \frac{1}{2\pi \alpha'} \int d\tau \left[ \delta X^\mu \partial_\sigma X_\mu \right]_{\sigma=\pi}. \tag{4.2}
\]

The first term gives the equation of motion. Substituting the solution \((4.1)\) into eq. \((4.2)\), however, the second term does not annihilate. To save the situation we have to consider that the open string with winding number should have a interaction with the boundary. All we have to do is to find such an interaction that cancels the contribution from the boundary in eq. \((4.2)\). The desirable interaction is

\[
S_{\text{int}} = \frac{n}{\pi R_9} \int d\tau \left. X^9 \right|_{\sigma=\pi}. \tag{4.3}
\]

Now, the total action \( S = S_0 + S_{\text{int}} \) has the correct variation. We conclude that the closed string Kaluza-Klein momentum is interpreted as a winding mode of the open string with the exotic Neumann boundary condition.

We explain why there is a one-to-one relation between the degrees of freedom of the relative motion of the D1, D5 branes and that of a single closed string. For counting the relative degrees of freedom of D1-D5 brane system, we consider not many closed strings
but a single closed string. We would like to discuss this point. It is well known that
the relative motion between the onebrane and the fivebrane is described by a single open
string stretching between them. In our picture, the worldsheet of this open string can be
interpreted as that of a closed string. Then, the relative motion corresponds to the single
closed string. We also consider the above discussion in terms of the closed string picture
in the following way: Simultaneously D-brane does not emits or absorbs many closed
string because of the fact that D-branes have a unit Ramond-Ramond charge. Suppose
the branes exchange \( m \neq 1 \) closed strings. Notice that, as shown in the subsection 3.1,
even though we consider the single closed string with winding number \( m \) decays into \( m \)
closed strings with unit winding number. So, it is enough to consider closed strings with
unit winding number. Each closed string with unit winding number contribute to the
evaluation of the Ramond-Ramond charge, as was demonstrated by Polchinski[1], by a
unit charge. Then, the net Ramond-Ramond charge is \( m \), which contradicts the fact that
brane has a unit Ramond-Ramond charge. Therefore, we consider a single closed string
with unit winding number.

Here we come back to the discussion. We generalize the above closed/open corre-
spondence to the cases when we have many onebranes and fivebranes \( Q_1, Q_5 > 1 \). In
this case the open strings have Chan-Paton factor. This factor indicates which onebrane
and fivebrane the open strings connect with. In terms of closed string language, we can
say that the Chan-Paton factor is interpreted as some kind of index which onebrane and
fivebrane the closed string begins to propagate from and terminates at. Then, we have
\( Q_1Q_5 \) closed string world sheets. In conclusion, the situation is summarized as follows:
The Chan-Paton factor is not a dynamical object but we have \( Q_1Q_5 \) world sheets. So, we
have \( Q_1Q_5 \) closed string world sheets which are topologically a copy of an annulus.

Now, let us consider the low energy limit \( \alpha' \rightarrow 0 \). The limit \( \alpha' \rightarrow 0 \) forces the area
of the world sheets to be vanishing by a familiar energetic argument. Namely, the world
sheet is squeezed. The world sheets become a copy of a circle, and does not shrink to a
point since it has a unit winding number. Since open string boundaries are restricted on
the onebranes and the fivebranes, the world sheets are restricted on their intersections.
In terms of open string language, in the \( \alpha' \rightarrow 0 \) limit the lowest lying modes dominate in
the spectrum, which are hypermultiplet matter in a gauge field theory.[21, 22] While in
terms of closed string language massive modes does not decouple because of the reason
explained in section 3. Squeezed worldsheets of a closed string becomes a worldline of
a hypermultiplet matter of a (1,5) string in the onebrane worldvolume gauge theory.
Considering the physical degrees of freedom in the BPS closed string system as in section 3, we have a center
\[ c = 4 \left( 1 + \frac{1}{2} \right) Q_1 Q_5. \] (4.4)

Finally, we show that the target space of the closed string system is \( (T^4)^{Q_1 Q_5} / S_{Q_1 Q_5} \) where \( S_{Q_1 Q_5} \) is the permutation group on \( Q_1 Q_5 \) objects. Since the world sheet of BPS closed string is interpreted by that of open string, through the modular transformation, it is sufficient to start with considering using open string. Suppose the onebranes are fluctuating in the \((x^5, x^6, x^7, x^8)\) directions. Let \( \phi_i \ (i = 5, 6, 7, 8) \) be the position of a onebrane in the \((x^5, x^6, x^7, x^8)\) directions. The position in the non-compact directions is irrelevant here. During the onebrane fluctuations the \((1,5)\) open strings do not acquire mass from the string tension, since the fivebranes have the worldvolume along the \((x^5, x^6, x^7, x^8)\) directions. Namely, the onebranes and the fivebranes are always intersecting during the fluctuations. In order to describe such fluctuations we consider the vertex operator for the position of onebrane which is \( V = \phi_i(X) \partial_\sigma X^i \ (i = 5, 6, 7, 8) \). This shows that the massless fluctuations \( \delta \phi_i \) of onebranes are driven by the open string coordinates \( X^i \ (i = 5, 6, 7, 8) \). This result is easily interpreted in terms of closed string language: The brane fluctuations preserving contact closed string interaction is driven by the closed string coordinates \( X^i \ (i = 5, 6, 7, 8) \). Then, the target space of a single closed string is described by \( X^i \ (i = 5, 6, 7, 8) \) which is a position in the fourtorus \( T^4 \). Considering all the closed strings, the total target space would be \( (T^4)^{Q_1 Q_5} \). Since we have multiplicity in the onebrane and fivebrane intersections as well as self-intersections the target space should be divided by the symmetry factor \( S_{Q_1 Q_5} \). Then, we have the desired target space
\[ (T^4)^{Q_1 Q_5} / S_{Q_1 Q_5}. \] (4.5)

In conclusion, the central charge (4.4) and target space (4.5) precisely agree with the result of the \( \sigma \)-model from open string. The present closed string system is identical to the \( \sigma \)-model of the low energy effective open string system.

5 Black Hole Entropy from Massless Closed Strings

In this section, we re-derive the black hole entropy microscopically by using the infinite number of closed strings with no winding.

For simplicity we start with considering a single closed string with no winding and a onebrane and a fivebrane. The closed string is exchanged between the onebrane and the
fivebrane with the contact interaction. We put Kaluza-Klein momentum $n/R_9$ along the $x^9$ direction on this closed string where $n$ is a positive integer. Then, the closed string is a massless BPS state with $p_0 = n/R_9$. In the $\alpha' \to 0$ limit, the worldsheet shrink to a worldline since there is no winding number. The relevant degrees of freedom are center-of-mass motions. Then, the closed string has no oscillator modes; $N = \tilde{N} = 0$. The situation is the same as that in ref. [11] if their open string is replaced to the closed string. The number of the supercharges is the same as that in eq. (2.6), and we have four supercharges. The four supersymmetries implies that we have four bosonic and fermionic ground states, denoted by $|n, i\rangle$ and $|n, \dot{a}\rangle$ ($i, \dot{a} = 1, \cdots, 4$), respectively. For a while, we consider the number of brane intersections as $n^\# = 1$.

Next, we consider the ensemble of the above strings. Let $N_{n, i}^b$ and $N_{n, \dot{a}}^f$ be the number of the bosonic $|n, i\rangle$ and fermionic $|n, \dot{a}\rangle$ states, respectively. The total Kaluza-Klein momentum is

$$\sum_{n=1}^{\infty} \sum_{i=1}^{4n^\#} \frac{n}{R_9} \left( N_{n, i}^b + N_{n, \dot{a}}^f \right),$$

which should be identical to $Q_K/R_9$. There are many ways to divide the total Kaluza-Klein momentum $Q_K/R_9$ among the individual massless BPS states. The number of the ways is

$$d_{Q_K} = 8^{n^\#} \left( \prod_{n=1}^{\infty} \prod_{i=1}^{4n^\#} \sum_{N_{n, i}^b = 0}^{\infty} \sum_{N_{n, \dot{a}}^f = 0}^{\infty} \right) \delta \left( Q_K, \sum_{n=1}^{\infty} \sum_{i=1}^{4n^\#} n \left( N_{n, i}^b + N_{n, \dot{a}}^f \right) \right),$$

where $\delta(m, n)$ is the Kronecker delta. Here we multiply by the degeneracy $8^{n^\#}$ of the ground states with no Kaluza-Klein momentum; $|0, i\rangle$ and $|0, \dot{a}\rangle$. We notice that in the above equation we have infinite number of summations over all integer variables $N_{n, i}^b$ and $N_{n, \dot{a}}^f$ ($n = 1, 2, \cdots, \infty$ and $i, \dot{a} = 1, \cdots, 4$). Then, we obtain

$$\sum_{Q_K = 0} d_{Q_K} q^{Q_K} = 8^{n^\#} \prod_{n=1}^{\infty} \prod_{i=1}^{4n^\#} \left( \sum_{N_{n, i}^b = 0}^{\infty} \sum_{N_{n, \dot{a}}^f = 0}^{\infty} q^n \left( N_{n, i}^b + N_{n, \dot{a}}^f \right) \right)$$

$$= 8^{n^\#} \prod_{n=1}^{\infty} \left( \frac{1 + q^n}{1 - q^n} \right)^{4n^\#}.$$  

This result shows that the infinite number of closed strings with no winding have the same degrees of freedom as that, in eq. (3.5), of the single closed string with unit winding number.

\*In the section, the word “massless” is used in ten dimensions.
Now, we consider the general case $Q_1, Q_5 > 1$. There are $Q_1 Q_5$ intersections, and we have exactly the same closed string system as the above one in each intersection. In this case the total number of states $d_{Q_K}$ is given by eq. (5.2) with $n^2 = Q_1 Q_5$. The number of states $d_{Q_K}$ in the large $Q_K$ limit is easily calculated from eq. (5.3) with $n^2 = Q_1 Q_5$, and we obtain the entropy

$$S = \ln d_{Q_K} = 2\pi \sqrt{\frac{Q_K c}{6}},$$  \quad (5.4)$$

where $c = 4(1 + 1/2)Q_1 Q_5$. We derive the black hole entropy microscopically by using the infinite number of closed strings with no winding. The entropy is precisely the same as that calculated from eq. (5.6) with $c = 6Q_1 Q_5$ which was derived from the closed string system with a winding number in section 3.

In the above calculations we find three facts: As seen from eq. (5.3) the two closed string systems with winding and without winding have exactly the same degrees of freedom. The two systems give the same microscopic entropy of the black hole. Let us consider the small radius limit $R_9 \to 0$. Even in the flat background approximation the two closed string system have the same five-dimensional mass $M = Q_K/R_9$ in generic values of $Q_1$ and $Q_5$. Thus, we can regard the single closed string with a unit winding number as a solitonic state that consists of infinite number of closed strings with no winding.

Next, we interpret the present picture of closed strings with no winding as that of open strings. Interchanging the parameters $\sigma \to -\tau$ and $\tau \to \sigma$, the closed string world sheet is converted into the open string world sheet as explained in section 4. The zeromode part of the resultant open string $X^9$ coordinate is

$$X^9 = x^9 + 2\alpha' \frac{n}{R_9} \sigma + \cdots.$$  \quad (5.5)$$

In this open string picture, the original Kaluza-Klein momentum $n/R_9$ looks as if it appears as a winding number along the $x^9$ direction with the radius $R'_9 = \alpha'/R_9$. If the radius of the $x^9$ direction is $R'_9$, the open string will satisfy the periodic boundary condition $X^9(\sigma + \pi) = X^9(\sigma) + 2\pi n R'_9$ and will looks like a closed string. What the original Kaluza-Klein momentum is a good quantum number is explained by the fact that such “closed string” has the winding number $n$ along the $x^9$ direction with the radius $R'_9$. On the other hand, this open string has no winding along the $x^9$ direction, since it is related to the closed string with no winding by the modular transformation.

Finally, we comment on the work in refs. [10, 11] using the open string picture men-
tioned in the beginning of section 3. The zeromode part of the $X^9$ coordinate of open strings in refs. [10, 11] is

$$X^9 = x^9 + 2\alpha' \frac{n}{R_9} \tau + \cdots. \quad (5.6)$$

Since the $x^9$ direction is compactified with the radius $R_9$, we need a periodic boundary condition on the string coordinate $(5.6)$. However, the string coordinate jumps by $2\pi n\alpha'/R_9$ when one circles along the $\tau$ direction in the world sheet. This condition contradicts the periodic boundary condition. Further, without winding along the $x^9$ direction, it is seemed that the Kaluza-Klein momentum of the open strings cannot be a good quantum number. Therefore, we should modify eq. $(5.6)$ to eq. $(5.5)$.

6 Discussions

First, let us discuss the relations between several approaches to calculate the black hole entropy. Let us consider the BPS closed string system with no winding number. In the flat background, the presence of branes leads to a situation that the worldsheet boundary may be regarded as a kind of non-vanishing tadpoles of closed string. If we sum over all tadpoles, we can recover the true vacuum with vanishing tadpole. The condition of vanishing tadpole is satisfied by the conformal invariance of the system. Namely, the system of the BPS closed string with no winding and with branes is described by type IIB supergravity. In conclusion, the dynamics of BPS closed strings with no winding is governed by the conventional type IIB supergravity. We notice that we do not take the limit $\alpha' \to 0$. This simple picture provides us with the equivalence between the microscopic and macroscopic black hole entropies. Further, as discussed in the present paper, the closed string with winding number is a solitonic state of the system of the BPS closed strings with no winding. Then, the three theories should be mutually equivalent. The BPS closed strings with a winding number has no such correspondence as that between massless BPS strings and supergravity, or rather it becomes a kind of $\sigma$-model from open string system proposed by ref. [3].

Finally, let us discuss, in non-extremal case, the expression of the entropy

$$S = 2\pi(\sqrt{n_L} + \sqrt{n_R})(\sqrt{Q_1} + \sqrt{\overline{Q}_1})(\sqrt{Q_5} + \sqrt{\overline{Q}_5}), \quad (6.1)$$

taking the discussion in ref. [3] into consideration. If we change branes to its anti-branes or left-moving Kaluza-Klein momentum to right-moving one in the extremal system

\[\text{Here we rewrite the } X^5 \text{ in refs. [10] into our notation } X^9.\]
\[ (2.1) \], we have eight kinds of BPS states. If we consider the non-extremal system with \( Q_1, Q_5, \bar{Q}_1, \bar{Q}_5, n_L, n_R \neq 0 \), the relevant string coordinate is given by the eight individual BPS string coordinates. Then, the entropy formula (6.1) is looked as if it is saturated by the contributions from the eight individual minimal BPS states. We have the correct black hole mass formula

\[
M_{BH} = (Q_1 + Q_1)M_{D1} + (Q_5 + Q_5)M_{D5} + (n_L + n_R)M_{KK}
\]

(6.2)
in contrast with the discussion in ref. [9].

To simplify the discussion let us first consider the case in which we have eight supersymmetries. If we include Kaluza-Klein momentum in the left and right sectors, the number of supersymmetries increases by twice compared with the extremal case in the present paper. While we already have the four supersymmetries \( 4_L \) in the left-moving sectors, the four supersymmetries \( 4_R \) are restored in the right-moving sectors; \( 8 = 4_L + 4_R \). Each BPS state of left-mover and right-mover contains four bosonic and fermionic oscillator modes as explained in section 3. Let us call this BPS state minimal BPS state. Fortunately, the multiplets of this eight supersymmetries are BPS states of the original 32 IIB supersymmetries. Then, thanks to the BPS property, we can carry out a reliable calculation perturbatively for deriving the entropy. In the lowest string coupling order the left and right sectors are calculated independently. Namely, the entropy is saturated by the two individual minimal BPS states, and we obtain

\[
S = 2\pi(\sqrt{n_L} + \sqrt{n_R})(\sqrt{Q_1Q_5}).
\]

(6.3)
This result is converted, by \( U \)-dualities [24, 10, 9], to the other two cases in which we have anti-onebranes or anti-fivebranes in addition to onebranes, fivebranes and left-moving Kaluza-Klein momentum. Then, we can carry out reliable calculations for deriving the three entropies (6.3) with both \( (n_L, Q_1, Q_5) \) and \( (n_R, Q_1, Q_5) \) cyclically permuted. The permutations are induced by \( U \)-duality. Moreover, we can generalize the system to that with one-, five-branes and their anti-branes and left-moving Kaluza-Klein momentum. This system is still BPS state of the original IIB string theory. Taking 4 individual minimal BPS states into account, we obtain the entropy

\[
S = 2\pi\sqrt{n_L}(\sqrt{Q_1} + \sqrt{\bar{Q}_1})(\sqrt{Q_5} + \sqrt{\bar{Q}_5}).
\]

(6.4)
This entropy can be converted to other two cases, as the above (6.3) case, by \( U \)-duality.
What we learn in the above is that we have eight mutually independent minimal BPS states in the weak string coupling limit $g \rightarrow 0$. Since the number of BPS states hopefully does not change under the variation of the value of the string coupling constant, we microscopically obtain the result (6.1).

Acknowledgments

The work was supported in part by the Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists.

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