Shy Maps in Topology

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Abstract
There is a concept in digital topology of a shy map. We define an analogous concept in topological spaces: We say a function is shy if it is continuous and the inverse image of every path-connected subset of its image is path-connected. Some basic properties of such maps are presented. For example, every shy map onto a semilocally simply connected space induces a surjection of fundamental groups (but a shy map onto a space that is not semilocally simply connected need not do so).

Key words and phrases: digital topology, fundamental group, wedge

1 Introduction

Shy maps between digital images were introduced in [2] and studied in subsequent papers belonging to the field of digital topology, including [3, 4, 5, 6]. In this paper, we develop an analogous notion of a shy map between topological spaces and study its properties.

Recall that if $F : X \to Y$ is a continuous function of topological spaces such that $F(x_0) = y_0$, then $F$ induces a homomorphism of fundamental groups, $F_* : \Pi_1(X, x_0) \to \Pi_1(Y, y_0)$, defined by $F_*([f]) = [F \circ f]$ for every loop $f : (S^1, s_0) \to (X, x_0)$, where $S^1$ is the unit circle in the Euclidean plane.

A topological space $X$ is semilocally simply connected [8] if for every $x \in X$ there is a neighborhood $N_x$ of $x$ in $X$ such that every loop in $N_x$ is nullhomotopic in $X$.

We let $\mathbb{Z}$ denote the set of integers, and $\mathbb{R}$, the real line.

A digital image is often considered as a graph $(X, \kappa)$, where $X \subset \mathbb{Z}^n$ for some positive integer $n$ and $\kappa$ is an adjacency relation on $X$. A function $f : (X, \kappa) \to (Y, \lambda)$ between digital images is continuous if for every $\kappa$-connected subset $A$ of $X$, $f(A)$ is a $\lambda$-connected subset of $Y$ [10]. A continuous surjection $f : (X, \kappa) \to (Y, \lambda)$ between digital images is called shy [2] if the following hold.

- For all $y \in Y$, $f^{-1}(y)$ is a $\kappa$-connected subset of $X$, and
- for all pairs of $\lambda$-adjacent $y_0, y_1 \in Y$, $f^{-1}(\{y_0, y_1\})$ is a $\kappa$-connected subset of $X$.

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It is shown in [7] that a continuous function between digital images is shy if and only if for every \( \lambda \)-connected subset \( Y' \) of \( Y \), \( f^{-1}(Y') \) is a \( \kappa \)-connected subset of \( X \). Since connectedness for a graph is analogous to what topologists call path-connectedness, we use the following.

**Definition 1.1.** Let \( X \) and \( Y \) be topological spaces and let \( f : X \to Y \). Then \( f \) is shy if \( f \) is continuous and for every path-connected \( Y' \subset f(X) \), \( f^{-1}(Y') \) is a path-connected subset of \( X \). \( \Box \)

Note a shy map in digital topology is defined to be a surjection [2], which we do not require here. This makes the requirement \( Y' \subset f(X) \) of Definition 1.1 noteworthy. For example, the embedding \( f : [0, \pi] \to S^1 \) given by \( f(x) = (\cos x, \sin x) \) is shy according to Definition 1.1. It would not be shy were Definition 1.1 written with the requirement \( Y' \subset f(X) \) replaced by the requirement \( Y' \subset Y \), since, e.g., the arc \( A \) of \( S^1 \) from \((-1, 0)\) to \((1, 0)\) containing \((0, -1)\) is path connected, but \( f^{-1}(A) = \{0, \pi\} \) is not.

Some authors consider digital images as topological spaces rather than as graphs, applying the Khalimsky topology to digital images (see, e.g., [9, 11, 12, 13, 14, 15]). The Khalimsky topology on \( \mathbb{Z} \) takes a basic neighborhood of an integer \( z \) to be \( \{z\} \) if \( z \) is odd; \( \{z - 1, z, z + 1\} \) if \( z \) is even. The quotient map \( q : \mathbb{R} \to \mathbb{Z} \) given by

\[
q(x) = \begin{cases} 
    x & \text{if } x \text{ is an even integer;} \\
    y & \text{if } y \text{ is the unique odd integer such that } |x - y| < 1,
\end{cases}
\]

is easily seen to be a shy map. More generally, the quotient map \( q^n \) that is the \( n \)-fold product of \( q \) as a map from \( \mathbb{R}^n \) (with the Euclidean topology) to \( \mathbb{Z}^n \) (with the Cartesian product topology taken from the Khalimsky topology on \( \mathbb{Z} \)) is a shy map. The shyness of this quotient map is among the reasons why \( \mathbb{R}^n \) is a useful “continuous analog” of \( \mathbb{Z}^n \) and \( \mathbb{Z}^n \) is an interesting “discrete analog” of \( \mathbb{R}^n \).

We mention here that the term path in \( X \) from \( x_0 \) to \( x_1 \) will be used in two senses, as is common practice: It may mean a continuous function \( f : [a, b] \to X \) such that \( f(a) = x_0 \) and \( f(b) = x_1 \), or may mean the image \( f([a, b]) \) of such a function.

## 2 Induced surjection on fundamental group

In [2], it was shown that a digital shy map induces a surjection of digital fundamental groups. In this section, we derive an analogous result for shy maps into semi-locally simply connected spaces.

In this section, we let \( e : [0, 1] \to S^1 \) be defined by \( e(t) = (\cos 2\pi t, \sin 2\pi t) \).

**Lemma 2.1.** ([1], Exercise 4, p. 269) Let \( f, g : [0, 1] \to X \) be paths into the topological space \( X \), such that \( f(0) = g(0) \) and \( f(1) = g(1) \). Let \( h : S^1 \to X \) be defined by

\[
h(e(t)) = \begin{cases} 
g(2t) & \text{if } 0 \leq t \leq 1/2; \\
f(2 - 2t) & \text{if } 1/2 \leq t \leq 1.
\end{cases}
\]
Then \( f \simeq g \) via a homotopy that holds the endpoints fixed if and only if there is a continuous extension of \( h \) to the interior of \( S^1 \). \( \square \)

The following is an immediate consequence of Lemma 2.1.

**Lemma 2.2.** Let \( \alpha, \beta : [0,1] \to Y \) be paths into a topological space \( Y \) such that \( \alpha(0) = \beta(0) \) and \( \alpha(1) = \beta(1) \). Let \( \overline{\alpha}, \overline{\beta} \) be the reverse path of \( \alpha \). If \( \overline{\alpha} \cdot \beta \) is nullhomotopic in \( Y \), then there is a homotopy between \( \alpha \) and \( \beta \) that holds the endpoints fixed. \( \square \)

The main result of this section is the following.

**Theorem 2.3.** Let \( X \) and \( Y \) be topological spaces such that \( Y \) is semilocally simply connected, and let \( F : X \to Y \) be a shy surjection, with \( F(x_0) = y_0 \). Then the induced homomorphism \( F_\ast : \Pi_1(X,x_0) \to \Pi_1(Y,y_0) \), defined by \( F_\ast([f]) = [F \circ f] \), is onto.

**Proof.** Let \( g : (S^1, s_0) \to (Y, y_0) \) be a pointed loop in \( Y \). It suffices to show there is a pointed loop \( f : (S^1, s_0) \to (X, x_0) \) in \( X \) such that \( F \circ f \) is pointed homotopic to \( g \).

For any integer \( n \geq 2 \), we construct a loop \( g_n : S^1 \to Y \) as follows. Without loss of generality, we assume the base point \( s_0 \) of \( S^1 \) is \( e(0) \). Note that \( x_0 \in F^{-1}(g(s_0)) \). For \( i \in \{1, \ldots, n-1\} \) let \( s_i = e(i/n) \) and let \( x_i \in F^{-1}(g(s_i)) \).

For \( i \in \{0, \ldots, n-1\} \), let \( A_i = e([i/n, (i+1)/n]) \) (so \( A_i \) is an arc of \( S^1 \) from \( s_i \) to \( s_{(i+1)} \) mod \( n \)).

Since \( F \) is shy, \( F^{-1}(g(A_i)) \) is path-connected. Moreover, \( \{s_i, s_{(i+1)} \mod n\} \subset A_i \). Therefore, there exists a continuous \( f_i : A_i \to F^{-1}(g(A_i)) \) such that \( f_i(s_i) = x_i \) and \( f_i(s_{(i+1)} \mod n) = x_{(i+1)} \mod n \). Let \( f : S^1 \to X \) be the loop such that \( f|A_i = f_i \) for all \( i \in \{0, \ldots, n-1\} \). Let \( g_n = F \circ f \).

Since \( g(S^1) \) is compact and \( Y \) is semilocally simply connected, there is a finite list \( U_1, \ldots, U_k \) of open sets in \( Y \) that cover \( g(S^1) \) such that each loop into any \( U_j \) is nullhomotopic in \( Y \). On applying Lebesgue’s covering lemma to the open cover \( \{g^{-1}(U_1), \ldots, g^{-1}(U_k)\} \) of \( S^1 \), we see that if \( n \) is sufficiently large then for each \( i \in \{0, \ldots, n-1\} \) the set \( g(A_i) = g(e([i/n, (i+1)/n])) \) is contained in one of the sets \( U_1, \ldots, U_k \). Therefore, we can apply Lemma 2.2 to conclude that for sufficiently large \( n \), \( g \) and \( g_n \) are homotopic in \( Y \) relative to \( \{0,1\} \), whence \( g_n|A_i \) and \( g|A_i \) are homotopic in \( Y \) relative to \( \{h_i(0), h_i(1)\} = \{s_i, s_{(i+1)} \mod n\} \). It follows that, for sufficiently large \( n \), \( g \) and \( g_n = F \circ f \) are homotopic in \( Y \) relative to \( \{s_0\} \). This establishes the assertion. \( \square \)

Theorem 2.3 may fail if \( Y \) is not semilocally simply connected, as shown by the following example.

**Example 2.4.** Let \( H \) be a Hawaiian earring in \( \mathbb{R}^2 \), let \( a \) be the common point of the circles of \( H \), and let \( H' \) be the reflection of \( H \) in the line through \( a \) that is tangent to the circles of \( H \) (so that \( H \cap H' = \{a\} \)). Let \( Y = H \cup H' \), let \( X = (H \times \{0\}) \cup (H' \times \{1\}) \cup (\{a\} \times [0,1]) \subset \mathbb{R}^3 \),
and let \( p : \mathbb{R}^3 \to \mathbb{R}^2 \) be the projection map \( p(x, y, z) = (x, y) \). Then \( p|_X : X \to Y \)
is a shy surjection that does not induce a surjection of fundamental groups: If \( C_n \) and \( C'_n \) respectively denote the \( n \)th largest circles of \( H \) and \( H' \), then there is no loop \( \ell : S^1 \to X \) such that \( p|_X \circ \ell \) is homotopic in \( Y \) to a loop that winds around all the circles of \( Y \) in the order \( C_1, C'_1, C_2, C'_2, C_3, C'_3, \ldots \) \( \square \)

### 3 Operations that preserve shyness

It is shown in [4] that a composition of shy surjections between digital images is shy. The following gives an analogous result for shy maps between topological spaces.

**Theorem 3.1.** Let \( f : X \to Y \) and \( g : Y \to Z \) be shy maps between topological spaces. Suppose \( f \) is a surjection. Then \( g \circ f : X \to Z \) is shy.

**Proof.** It is clear that \( g \circ f \) is continuous. Let \( A \subset g(Y) \) be path-connected. Since \( g \) is shy, \( g^{-1}(A) \) is path-connected. Since \( f \) is a shy surjection, \( f^{-1}(g^{-1}(A)) = (g \circ f)^{-1}(A) \) is path-connected. Therefore, \( g \circ f \) is shy. \( \square \)

The following is suggested by analogous results for shy maps between digital images [4, 5, 6].

**Theorem 3.2.** Let \( f_i : X_i \to Y_i \) be functions between topological spaces, \( 1 \leq i \leq v \). Let \( f = \Pi_{i=1}^v f_i : \Pi_{i=1}^v X_i \to \Pi_{i=1}^v Y_i \) be the product function,

\[
f(x_1, \ldots, x_v) = (f_1(x_1), \ldots, f_v(x_v)) \text{ for } x_i \in X_i.
\]

If \( f \) is shy, then each \( f_i \) is shy.

**Proof.** We use the well known continuity of projection maps \( p_j : \Pi_{i=1}^v Y_i \to Y_i \), defined by \( p_j(y_1, \ldots, y_v) = y_j \).

For some \( x_i \in X_i \), let \( I_j : X_j \to X = \Pi_{i=1}^v X_i \) be the injection defined by

\[
I_j(x) = (x_1, \ldots, x_{j-1}, x, x_{j+1}, \ldots, x_v).
\]

Since \( f \) is shy, \( f \) must be continuous, and it follows that each \( f_i = p_i \circ f \circ I_i \) is continuous.

Let \( A_i \) be a path-connected subset of \( f_i(X_i) \). We must show that \( f_i^{-1}(A_i) \) is a path-connected subset of \( X_i \). Fix \( y_j \in A_j \) for all indices \( j \) and let \( I'_j : Y_j \to Y = \Pi_{i=1}^v Y_i \) be the injection defined by

\[
I'_j(y) = (y_1, \ldots, y_{j-1}, y, y_{j+1}, \ldots, y_v).
\]

Then \( I'_j(A_i) \) is path-connected, since it is homeomorphic to \( A_i \). Therefore, \( f^{-1}(I'_j(A_i)) = \Pi_{j=1}^v f_j^{-1}(p_j^Y(I'_j(A_i))) \), where \( p_j^Y : Y \to Y_j \) is the projection to the \( j \)th coordinate, is path-connected.

Then \( f_i^{-1}(A_i) = p_i(\Pi_{j=1}^v f_j^{-1}(p_j^Y(I'_j(A_i)))) \) is path-connected. This completes the proof. \( \square \)
As the union is if both sets are nonempty. So the union of the two sets is also path-connected.

We see from Lemma 3.3 that if \( f \) is continuous, then \( f \) is shy if \( f \) is continuous and \( Q \) is shy. To see that \( f \) is shy if \( f \) is continuous and \( Q \) is shy, let \( h \) be the function defined by

\[
(f \lor g)(a) = \begin{cases} 
  f(a) & \text{if } a \in X; \\
  g(a) & \text{if } a \in Y.
\end{cases}
\]

The following is suggested by an analogous result for digital images [4].

**Theorem 3.4.** [8] Let \( f : X \to Y \) be a function between topological spaces. Suppose \( X = A \cup B \) where \( A \setminus B \subset \text{Int}A, B \setminus A \subset \text{Int}B \). If \( f|_A \) and \( f|_B \) are continuous, then \( f \) is continuous. \( \square \)

If \( W = X \lor Y \) with \( X \cap Y = \{x_0\} \), \( W' = X' \lor Y' \) with \( X' \cap Y' = \{x_0'\} \), and \( f : (X, x_0) \to (X', x_0') \) and \( g : (Y, x_0) \to (Y', x_0') \) are pointed functions, let \( f \lor g : W \to W' \) be the function defined by

\[
(f \lor g)(a) = \begin{cases} 
  f(a) & \text{if } a \in X; \\
  g(a) & \text{if } a \in Y.
\end{cases}
\]

The proofs in [3, 5, 6] for analogs of the converse of Theorem 3.2 rely on structure that digital images have but that cannot be assumed for topological spaces. It appears that obtaining either a proof or a counterexample for the converse of Theorem 3.2 is a difficult problem.

**Lemma 3.3.** Suppose \( W = A \lor B \), with \( A \setminus B \) and \( B \setminus A \) separated in \( W \). Let \( C \) be a path-connected subset of \( W \) such that \( C \cap A \neq \emptyset \neq C \cap B \). Then the unique point of \( A \cap B \) is in \( C \). Further, each of \( C \cap A \) and \( C \cap B \) is path-connected. \( \square \)

We will use the following “Gluing Rule.”

**Theorem 3.4.** [8] Let \( f : X \to Y \) be a function between topological spaces. Suppose \( X = A \cup B \) where \( A \setminus B \subset \text{Int}A, B \setminus A \subset \text{Int}B \). If \( f|_A \) and \( f|_B \) are continuous, then \( f \) is continuous. \( \square \)

The following is suggested by an analogous result for digital images [4].

**Theorem 3.5.** Let \( W = X \lor Y \), \( W' = X' \lor Y' \), \( f : (X, x_0) \to (X', x_0') \), \( g : (Y, x_0) \to (Y', x_0') \), and \( h = f \lor g : W \to W' \), where \( x_0 \) and \( x_0' \) are the unique points of \( X \cap Y \) and \( X' \cap Y' \), respectively. Assume \( X \setminus Y \) and \( Y \setminus X \) are separated in \( W \), and \( X' \setminus Y' \) and \( Y' \setminus X' \) are separated in \( W' \). Then \( h \) is shy if and only if \( f \) and \( g \) are both shy.

**Proof.** As \( f = h|_X \) and \( g = h|_Y \), and the sets \( X \setminus Y \) and \( Y \setminus X \) are separated, it follows from Theorem 3.3 that \( h = f \lor g \) is continuous if and only if each of \( f \) and \( g \) is continuous.

To see that \( f \) is shy if \( h \) is shy we note that, if \( P' \) is a path-connected subset of \( f(X) \), then \( h^{-1}(P') \) is path-connected and so \( h^{-1}(P') \cap X = f^{-1}(P') \) is path-connected by Lemma 3.3. Similarly, \( g \) is shy if \( h \) is shy.

Suppose each of \( f \) and \( g \) is shy. To prove \( h \) is shy, let \( Q' \) be any path-connected subspace of \( h(X \lor Y) \). We want to show \( h^{-1}(Q') \) is path-connected. We see from Lemma 3.3 that if \( Q' \cap X' \neq \emptyset \neq Q' \cap Y' \), then \( x_0' \in Q' \) and, moreover, \( Q' \cap X' \) and \( Q' \cap Y' \) are path-connected subsets of \( f(X) \) and \( g(Y) \), respectively. This and the shyness of \( f \) and \( g \) imply that each of the sets \( f^{-1}(Q' \cap X') \) and \( g^{-1}(Q' \cap Y') \) is path-connected, and that \( x_0 \) lies in both sets if both sets are nonempty. So the union of the two sets is also path-connected. As the union is \( h^{-1}(Q') \), the proof is complete. \( \square \)
4 Shy maps into $\mathbb{R}$

In this section, we show that shy maps into the reals have special properties. Our results are suggested by analogs for digital images [4].

**Theorem 4.1.** Let $X$ be a connected subset of $\mathbb{R}$ and let $f : X \to \mathbb{R}$ be continuous. Then $f$ is shy if and only if $f$ is monotone non-decreasing or monotone non-increasing.

**Proof.** Suppose $f$ is shy. If $f$ is not monotone, then there exist $a, b, c \in X$ such that $a < b < c$ and either

- $f(a) < f(b)$ and $f(b) > f(c)$, or
- $f(a) > f(b)$ and $f(b) < f(c)$.

In the former case, the continuity of $f$ implies there exist $a' \in [a, b)$ and $c' \in (b, c]$ such that $\{a', c'\} \subset f^{-1}(\{\max\{f(a), f(c)\}\})$ but $b \notin f^{-1}(\{\max\{f(a), f(c)\}\})$. Therefore, $f^{-1}(\{\max\{f(a), f(c)\}\})$ is not path-connected, contrary to the assumption that $f$ is shy. The latter case generates a contradiction similarly. We conclude that $f$ is monotone.

Suppose $f$ is a monotone function. Let $A$ be a path-connected subset of $f(X)$. Let $c, d \in f^{-1}(A)$. Without loss of generality, $c \leq d$ and $f(c) \leq f(d)$.

Since $f$ is continuous and $X$ is connected, for every $x$ such that $c \leq x \leq d$ we have $f(c) \leq f(x) \leq f(d)$. Therefore, $f^{-1}(A)$ contains $[c, d]$, a path from $c$ to $d$. Thus, $f$ is shy. \qed

**Theorem 4.2.** Let $f : S^1 \to \mathbb{R}$ be shy. Then $f$ is a constant function.

**Proof.** Suppose the image of $f$ has distinct points $a, b$, where $a < b$. Then there exist $a' \in f^{-1}(a), b' \in f^{-1}(b)$. There are two distinct arcs $A_0$ and $A_1$ in $S^1$ from $a'$ to $b'$ such that $S^1 = A_0 \cup A_1$ and $A_0 \cap A_1 = \{a', b'\}$. Since $f$ is continuous, there exist $c_0 \in A_0, c_1 \in A_1$ such that $f(c_0) = f(c_1) = (a + b)/2$.

Then $f^{-1}(\{(a + b)/2\})$ is disconnected, contrary to the assumption that $f$ is shy. Therefore, $f$ is a constant function. \qed

**Theorem 4.3.** Let $X$ be a path-connected topological space for which there exists $r \in X$ such that $X \setminus \{r\}$ is not path-connected. Let $f : X \to \mathbb{R}$ be a shy map. Then there are at most 2 path components of $X \setminus \{r\}$ on which $f$ is not identically equal to the constant function with value $f(r)$.

**Proof.** Suppose there exist 2 distinct path components $A, B$ of $X \setminus \{r\}$ on which $f$ is not the constant function with value $f(r)$. Then there exist $a \in A, b \in B$ such that $f(a) \neq f(r) \neq f(b)$.

First, we show that either $f(a) < f(r) < f(b)$ or $f(a) > f(r) > f(b)$. Suppose otherwise.

- Suppose $f(a) < f(r)$ and $f(b) < f(r)$. Without loss of generality, $f(a) \leq f(b) < f(r)$. Since $X$ is pathwise connected and $A$ is a maximal path-connected subset of $X \setminus \{r\}$, it is easy to see that there exists a path $P$...
in \( A \cup \{r\} \) from \( a \) to \( r \). By continuity of \( f \), it is easy to see that there exists \( x_0 \in P \) such that \( f(x_0) = f(b) \). But then \( x_0 \) and \( b \) are in distinct path-components of \( f^{-1}\{\{f(b)\}\} \), contrary to the assumption that \( f \) is shy.

- If \( f(a) > f(r) \) and \( f(b) > f(r) \), then we similarly obtain a contradiction.

We conclude that either \( f(a) < f(r) < f(b) \) or \( f(a) > f(r) > f(b) \).

So if there is a 3rd path-component \( C \) of \( X \{r\} \) on which \( f \) is not identically equal to \( f(r) \), then there exists \( c \in C \) such that either

- \( f(c) < f(r) \), in which case we get a contradiction as in the first case above, since both \( f(c) \) and \( \min\{f(a), f(b)\} \) are less than \( f(r) \); or

- \( f(c) > f(r) \), in which case both \( f(c) \) and \( \max\{f(a), f(b)\} \) are greater than \( f(r) \), so we have a contradiction as in the second case above.

Therefore, there cannot be a 3rd path-component \( C \) of \( X \{r\} \) on which \( f \) is not identically equal to \( f(r) \).

\[ \square \]

5 Concluding remarks

Drawing on the notion of a shy map between digital images \[2\], we have introduced an analogous notion of a shy map between topological spaces. We have shown that shy maps between topological spaces have many properties analogous to those of shy maps between digital images.

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