THE SQUARED EIGENFUNCTION SYMMETRIES FOR THE BTL AND CTL HIERARCHIES

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\textbf{ABSTRACT.} In this paper, the squared eigenfunction symmetries for the BTL and CTL hierarchies are explicitly constructed with the suitable modification of the ones for the TL hierarchy, by considering the BTL and CTL constraints. Also the connections with the corresponding additional symmetries are investigated: the squared eigenfunction symmetry generated by the wave function can be viewed as the generating function for the additional symmetries.

\textbf{PACS numbers:} 02.30.Ik

\textbf{Keywords:} squared eigenfunction symmetry, the BTL and CTL hierarchies, additional symmetries.

\section{1. Introduction}

The Toda lattice (TL) equation \cite{1}, as an important integrable system, describes the motion of one-dimensional particles with exponential interaction of neighbors, which plays significant role in physics. The TL hierarchy, which is one of the most important integrable hierarchies, was first introduced by Ueno and Takasaki in \cite{2} to generalize the Toda lattice equations \cite{1} along the work of the KP hierarchy \cite{3}. In \cite{2}, the analogues of the B and C types for the TL hierarchy, i.e. the BTL and CTL hierarchies, are also considered, which are corresponding to infinite dimensional Lie algebras $o(\infty)$ and $sp(\infty)$ respectively. The BTL and CTL hierarchies are also very important in the integrable system just like the TL hierarchy \cite{2}. However, there are few researches on the BTL and CTL hierarchies in literature. So much work can be done for the BTL and CTL hierarchies.

The squared eigenfunction symmetry \cite{4-7}, also called “ghost” symmetry \cite{8}, is a kind of symmetry generated by eigenfunctions and adjoint eigenfunctions in the integrable system. The squared eigenfunction symmetry has many applications in the integrable system. For example, 1) symmetry constraint \cite{5,7,11,14} can be defined by identifying the squared eigenfunction symmetry with the usual flow of the integrable hierarchy; 2) the connection with the additional symmetry \cite{8,15,17}, which is the symmetry depending explicitly on the space and time variables \cite{18-20}; 3) the extended integrable systems \cite{27,28}, which contain the integrable equations with self-consistent sources, can be constructed with the help of the squared eigenfunction symmetry. Recently, the squared eigenfunction symmetries for the BKP hierarchy and the discrete KP hierarchy are systematically developed in \cite{15} and \cite{16}.

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respectively. Also the squared eigenfunction symmetry for the TL hierarchy and its connection with the additional symmetry are investigated in [17]. In this paper, we will concentrate on the construction of the squared eigenfunction symmetry of the BTL and CTL hierarchies.

The squared eigenfunction symmetry of the Toda lattice hierarchy [17] is given in the form of the Kronecker product of the vector eigenfunctions and the vector adjoint eigenfunctions. Because of the BTL and CTL constraints, the squared eigenfunction symmetry can not be defined directly from the one of the TL hierarchy and some modification must be needed. For this, we construct the squared eigenfunction symmetries of the BTL and CTL hierarchies by the suitable combination of the ones for the TL hierarchy. Then the connection with the additional symmetry is investigated: the particular squared eigenfunction symmetries generated by the wave functions can be viewed as the generating functions of the additional symmetries.

This paper is organized in the following way. In Section 2, we recall some basic knowledge about the BTL and CTL hierarchies. Then, we construct the squared eigenfunction symmetry for the BTL hierarchy in Section 3. Next, in Section 4 the squared eigenfunction symmetry for the CTL hierarchy is also investigated. At last, we devote Section 5 to some conclusions and discussions.

2. the BTL and CTL hierarchies

In this section, some basic facts about the BTL and CTL hierarchies are reviewed. One can refer to [2] for more details about the BTL and CTL hierarchies.

Firstly, consider the algebra

$$\mathcal{D} = \{(P_1, P_2) \in \mathfrak{gl}(\mathbb{R}) \times \mathfrak{gl}(\mathbb{R}) \mid (P_1)_{ij} = 0 \text{ for } j - i \gg 0, \ (P_2)_{ij} = 0 \text{ for } i - j \gg 0\},$$

which has the following splitting:

$$\mathcal{D} = \mathcal{D}_+ + \mathcal{D}_-,$$

$$\mathcal{D}_+ = \{(P, P) \in \mathcal{D} \mid (P)_{ij} = 0 \text{ for } |i - j| \gg 0\} = \{(P_1, P_2) \in \mathcal{D} \mid P_1 = P_2\},$$

$$\mathcal{D}_- = \{(P_1, P_2) \in \mathcal{D} \mid (P_1)_{ij} = 0 \text{ for } j \geq i, \ (P_2)_{ij} = 0 \text{ for } i > j\},$$

with \((P_1, P_2) = (P_1, P_2)_+ + (P_1, P_2)_-\) given by

\[
(P_1, P_2)_+ = (P_{1u} + P_{2l}, P_{1u} + P_{2l}), \quad (P_1, P_2)_- = (P_{1l} - P_{2l}, P_{2u} - P_{1u}),
\]

where for a matrix \(P\), \(P_u\) and \(P_l\) denote the upper (including diagonal) and strictly lower triangular parts of \(P\), respectively. For \((P_1, P_2), (Q_1, Q_2) \in \mathcal{D}\), we define

\[
(P_1, P_2)(Q_1, Q_2) = (P_1 Q_1, P_2 Q_2), \quad (P_1, P_2)^{-1} = (P_1^{-1}, P_2^{-1}).
\]

Then the BTL (or CTL) hierarchy is defined in the Lax forms as

\[
\partial_{x_{2n+1}} L = [(L_1^{2n+1}, 0)_+, L] \quad \text{and} \quad \partial_{y_{2n+1}} L = [(0, L_2^{2n+1})_+, L], \quad n = 0, 1, 2, \ldots
\]
where the Lax operator $L$ is given by a pair of infinite matrices

$$L = (L_1, L_2) = \left( \sum_{-\infty < i \leq 1} \text{diag}[a_i^{(1)}(s)] \Lambda^i, \sum_{-1 \leq i < \infty} \text{diag}[a_i^{(2)}(s)] \Lambda^i \right) \in \mathcal{B}$$

(2)

with $\Lambda = (\delta_{j-i,1})_{i,j \in \mathbb{Z}}$, and $a_i^{(k)}(s)$ and $a_i^{(k)}(s)$ depending on $x = (x_1, x_3, x_5, \cdots)$ and $y = (y_1, y_3, y_5, \cdots)$, such that

$$a_1^{(1)}(s) = 1 \quad \text{and} \quad a_{-1}^{(2)}(s) \neq 0 \quad \forall s$$

and satisfies the BTL (or CTL) constraint [2]

$$L^T = -(J, J)L(J^{-1}, J^{-1}) \quad \text{(or} \quad L^T = -(K, K)L(K^{-1}, K^{-1})),$$

(3)

where $J = ((-1)^i \delta_{i+j,0})_{i,j \in \mathbb{Z}}$, $K = \Lambda J$ and $T$ refers to the matrix transpose. The BTL (or CTL) constraint on the components of the Lax operators $L_1$ and $L_2$ is explicitly showed as

$$a_i^{(k)}(s) = (-1)^{i+1}a_i^{(k)}(-s - i) \quad \text{(or} \quad a_i^{(k)}(s) = (-1)^{i+1}a_i^{(k)}(-s - i - 1)), \quad k = 1, 2.$$  

(4)

The Lax equation for the BTL (or CTL) hierarchy can be expressed as a system of equations of the Zakharov-Shabat type:

$$\partial_{x_2n+1}(L_1^{2n+1})_u - \partial_{x_2n+1}(L_2^{2n+1})_u + [(L_1^{2n+1})_u, (L_2^{2n+1})_u] = 0,$$

(5)

$$\partial_{y_2n+1}(L_1^{2n+1})_l - \partial_{y_2n+1}(L_2^{2n+1})_l + [(L_1^{2n+1})_l, (L_2^{2n+1})_l] = 0,$$

(6)

$$\partial_{y_2n+1}(L_1^{2n+1})_u - \partial_{x_2n+1}(L_2^{2n+1})_l + [(L_1^{2n+1})_u, (L_2^{2n+1})_l] = 0, \quad m, n = 0, 1, 2, \cdots$$

(7)

When $m = n = 0$, one can from (7) get the BTL equation

$$\partial_{x_1}a_{-1}^{(2)}(1) = a_{-1}^{(2)}(1)a_{0}^{(1)}(1), \quad \partial_{x_1}a_{-1}^{(2)}(s) = a_{-1}^{(2)}(s)(a_0^{(1)}(s) - a_0^{(1)}(s - 1)) \quad (s \geq 2),$$

$$\partial_{y_1}a_{0}^{(1)}(s) = a_{-1}^{(2)}(s) - a_{-1}^{(2)}(s + 1) \quad (s \geq 1),$$

(8)

and the CTL equation

$$\partial_{x_1}a_{-1}^{(2)}(0) = 2a_{-1}^{(2)}(0)a_0^{(1)}(0), \quad \partial_{x_1}a_{-1}^{(2)}(s) = a_{-1}^{(2)}(s)(a_0^{(1)}(s) - a_0^{(1)}(s - 1)) \quad (s \geq 1),$$

$$\partial_{y_1}a_{0}^{(1)}(s) = a_{-1}^{(2)}(s) - a_{-1}^{(2)}(s + 1) \quad (s \geq 0),$$

(9)

by considering the corresponding constraint [4].

The Lax operator of the BTL (or CTL) hierarchy [11] has the representation

$$L = W(\Lambda, \Lambda^{-1})W^{-1} = S(\Lambda, \Lambda^{-1})S^{-1}$$

(10)

in terms of two pairs of wave operators $W = (W_1, W_2)$ and $S = (S_1, S_2)$, where

$$S_1(x, y) = \sum_{i \geq 0} \text{diag}[c_i(s; x, y)] \Lambda^{-i}, \quad S_2(x, y) = \sum_{i \geq 0} \text{diag}[c'_i(s; x, y)] \Lambda^i$$

(11)

and

$$W_1(x, y) = S_1(x, y)e^{\xi(x, y)}, \quad W_2(x, y) = S_2(x, y)e^{\xi(y, \Lambda^{-1})}$$

(12)
with $c_0(s;x,y) = 1$ and $c_0'(s;x,y) \neq 0$ for any $s$, and $\xi(x,\Lambda^{\pm 1}) = \sum_{n \geq 0} x_{2n+1}\Lambda^{\pm 2n+1}$. Obviously, $W = (W_1, W_2)$ are not uniquely determined, but have the arbitrariness
\[
W_1(x,y) \mapsto W_1(x,y)f^1(\Lambda), \quad W_2(x,y) \mapsto W_2(x,y)f^2(\Lambda).
\]
Here $f^1(\lambda) = \sum_{i \geq 0} f_1^i \lambda^{-i}$ and $f^2(\lambda) = \sum_{i \geq 0} f_2^i \lambda^{i}$ ($f_0^1 = 1$, $f_0^2 \neq 0$) are formal Laurent series with constant scalar coefficients. Under an appropriate choice of $f_i(\lambda)$, $W = (W_1, W_2)$ satisfies
\[
J^{-1}W_i^TJ = W_i^{-1} \text{ for BTL (or $K^{-1}W_i^TK = W_i^{-1}$ for CTL), } i = 1, 2. \tag{13}
\]
The wave operators evolve according to
\[
\begin{align*}
\partial_{x_{2n+1}} S &= -(L_{2n+1}^2, 0)_-S, \quad \partial_{y_{2n+1}} S = -(0, L_{2n+1}^2)_-S, \tag{14}
\partial_{x_{2n+1}} W &= (L_{2n+1}^2, 0)_+W, \quad \partial_{y_{2n+1}} W = (0, L_{2n+1}^2)_+W. \tag{15}
\end{align*}
\]
The vector wave functions $\Psi = (\Psi_1, \Psi_2)$ and the adjoint wave functions $\Psi^* = (\Psi_1^*, \Psi_2^*)$, can also be introduced as
\[
\begin{align*}
\Psi_i(x,y;\lambda) &= (\Psi_i(n;x,y;\lambda))_{n \in \mathbb{Z}} := W_i(x,y)\chi(\lambda), \tag{16}
\Psi_i^*(x,y;\lambda) &= (\Psi_i^*(n;x,y;\lambda))_{n \in \mathbb{Z}} := (W_i(x,y)^{-1})^T\chi^*(\lambda), \tag{17}
\end{align*}
with $\chi(\lambda) = (\lambda^i)_{i \in \mathbb{Z}}$ and $\chi^*(\lambda) = \chi(\lambda^{-1})$, which satisfy the following relations:
\[
L\Psi = (z, z^{-1})\Psi, \quad L^T\Psi^* = (z, z^{-1})\Psi^* \tag{18}
\]
\[
\begin{align*}
\partial_{x_{2n+1}} \Psi &= (L_{2n+1}^2, 0)_+\Psi, \quad \partial_{y_{2n+1}} \Psi = (0, L_{2n+1}^2)_+\Psi, \tag{19}
\partial_{x_{2n+1}} \Psi^* &= -(L_{2n+1}^2, 0)_+^T\Psi^*, \quad \partial_{y_{2n+1}} \Psi^* = -(0, L_{2n+1}^2)_+^T\Psi^*. \tag{20}
\end{align*}
\]
From the BTL (or CTL) constraint \(\text{[13]}\) on the wave operators, the adjoint wave function is connected with the wave function in the following way,
\[
\Psi_i^*(x,y,\lambda) = J\Psi_i(x,y,-\lambda) \quad (\text{or } \Psi_i^*(x,y,\lambda) = \lambda K\Psi_i(x,y,-\lambda)). \tag{21}
\]
If vector functions $q = (q(n;x,y))_{n \in \mathbb{Z}}$ and $r = (r(n;x,y))_{n \in \mathbb{Z}}$ satisfy
\[
\begin{align*}
\partial_{x_{2n+1}} q &= (L_{2n+1}^2)u q, \quad \partial_{y_{2n+1}} q = (L_{2n+1}^2)u q, \tag{22}
\partial_{x_{2n+1}} r &= -(L_{2n+1}^2)^T_u q, \quad \partial_{y_{2n+1}} r = -(L_{2n+1}^2)^T_u q, \tag{22}
\end{align*}
we call them vector eigenfunction and vector adjoint eigenfunction for the BTL (or CTL) hierarchy respectively. Obviously, the wave functions $\Psi_1$ and $\Psi_2$ are eigenfunctions, and the adjoint wave functions $\Psi_1^*$ and $\Psi_2^*$ are the adjoint eigenfunctions. From the BTL (or CTL) constraint \(\text{[13]}\), one can know that
\[
(L_{2n+1}^2)^T_u = -J(L_{2n+1}^2)u J^{-1}, \quad (L_{2n+1}^2)^T_l = -J(L_{2n+1}^2)l J^{-1} \tag{23}
\]
\[
(\text{or } (L_{2n+1}^2)^T_u = -K(L_{2n+1}^2)u K^{-1}, \quad (L_{2n+1}^2)^T_l = -K(L_{2n+1}^2)l K^{-1})
\]
Thus given the vector eigenfunction $q$, $Jq$ (or $Kq$) will be the adjoint eigenfunction for the BTL (or CTL) hierarchy. This fact connected with (3) and (21) shows that in the BTL (or CTL) hierarchy, the adjoint case can be derived directly from the usual case. Therefore, we can only consider the usual case in the study of the BTL (or CTL) hierarchy.

At last, we end this section with the introduction of the additional symmetries of the BTL and CTL hierarchies. The Orlov-Shulman operator [20, 21] is defined as

$$M \equiv (M_1, M_2) = W(\varepsilon, \varepsilon^*)W^{-1},$$

where

$$\varepsilon = \text{diag}[s]\Lambda^{-1}, \quad \varepsilon^* = -\varepsilon^T + \Lambda,$$

satisfying

$$M\Psi = (\partial_+, \partial_+^{-1})\Psi, \quad [L, M] = (1, 1),$$

$$\partial_{x_{2n+1}} M = [(L_1^{2n+1}, 0)_+, M], \quad \partial_{y_{2n+1}} M = [(0, L_2^{2n+1})_+, M].$$

The additional symmetry [29] can be defined by introducing the additional independent variables $x_{m,l}^*$ and $y_{m,l}^*$,

$$\partial_{x_{m,l}^*} W = -(A_{1ml}(M_1, L_1), 0)_- W, \quad \partial_{y_{m,l}^*} W = -(0, A_{2ml}(M_2, L_2))_- W,$$

where $A_{1ml}(M_i, L_i)$ are polynomials in $L_i$ and $M_i$. Denote $A_{ml}(M, L) = (A_{1ml}(M_1, L_1), A_{2ml}(M_2, L_2))$, then

- in BTL case,
  $$A_{ml}(M, L) = M^m L^l - (-1)^l L^{l-1} M^m L;$$

- in CTL case,
  $$A_{ml}(M, L) = M^m L^l - (-1)^l L^l M^m.$$

3. The Squared Eigenfunction Symmetry for the BTL Hierarchies

In this section, we shall construct the squared eigenfunction symmetry for the BTL hierarchy.

Given a couple of vector eigenfunctions $q_1$ and $q_2$, the squared eigenfunction flow of the BTL hierarchy can be defined by its actions on the wave operators,

$$\partial_{\alpha} W_1 = (q_1 \otimes Jq_2 - q_2 \otimes Jq_1)_1 W_1, \quad \partial_{\alpha} W_2 = -(q_1 \otimes Jq_2 - q_2 \otimes Jq_1)_u W_2,$$

where $(A \otimes B)_{ij} = A_i B_j$ for the vectors $A$ and $B$.

According to (10), one can further have the squared eigenfunction flow on the Lax operator

$$\partial_{\alpha} L_1 = [(q_1 \otimes Jq_2 - q_2 \otimes Jq_1)_1, L_1], \quad \partial_{\alpha} L_2 = -[(q_1 \otimes Jq_2 - q_2 \otimes Jq_1)_u, L_2].$$

Next we will show that the definitions above is well-defined: (29) or (30) is consistent with the BTL constraint (3).
Proposition 1. \( \partial_i \alpha \) is consistent with the BTL constraint \( \mathbf{3} \), i.e. \((\partial_i L_i^T) J + (J \partial_i L_i) = 0, i = 1, 2.\)

Proof. Firstly,
\[
J(q_1 \otimes J q_2 - q_2 \otimes J q_1) + (q_1 \otimes J q_2 - q_2 \otimes J q_1)^T J
\]
\[
= J(q_1 \otimes q_2 - q_2 \otimes q_1) + J(q_2 \otimes q_1 - q_1 \otimes q_2) J = 0,
\]
by noting that \( q_1 \otimes J q_2 = (q_1 \otimes q_2) J^T = (q_1 \otimes q_2) J \) and \((q_1 \otimes q_2)^T = q_2 \otimes q_1.\)

Thus \( q_i J = J q_i = 0 \) (see \( \mathbf{2} \)).

Then for \( i = 1 \), from \( \mathbf{3}, \mathbf{30} \),
\[
(\partial_i L_i^T) J + (J \partial_i L_i)
\]
\[
= [(q_1 \otimes J q_2 - q_2 \otimes J q_1)_t, L_1]^T J + J[(q_1 \otimes J q_2 - q_2 \otimes J q_1)_t, L_1]
\]
\[
= -[(q_1 \otimes J q_2 - q_2 \otimes J q_1)_t]^T L_1^T + JL_1(J^{-1}(q_1 \otimes J q_2 - q_2 \otimes J q_1)_t, L_1)
\]
\[
= J((q_1 \otimes J q_2 - q_2 \otimes J q_1)_t, L_1) + JL_1(q_1 \otimes J q_2 - q_2 \otimes J q_1)_t + J[(q_1 \otimes J q_2 - q_2 \otimes J q_1)_t, L_1]
\]
\[
= J[(q_1 \otimes J q_2 - q_2 \otimes J q_1)_t, L_1] + J[(q_1 \otimes J q_2 - q_2 \otimes J q_1)_t, L_1] = 0.
\]

The case \( i = 2 \) can be similarly proved. \( \square \)

Thus \( \partial_i \alpha \) is indeed well-defined. We next will show that this squared eigenfunction flow is indeed a kind of symmetry for the BTL hierarchy, and thus is called the squared eigenfunction symmetry.

Proposition 2.
\[
[\partial_i, \partial_{x_{2n+1}}] = [\partial_i, \partial_{y_{2n+1}}] = 0.
\]

Proof. In fact, according to \( \mathbf{1}, \mathbf{15}, \mathbf{29}, \mathbf{30} \)
\[
[\partial_i, \partial_{x_{2n+1}}] W_1
\]
\[
= \partial_i \left((L_1^{2n+1})_u W_1\right) - \partial_{x_{2n+1}} \left((q_1 \otimes J q_2 - q_2 \otimes J q_1)_t W_1\right)
\]
\[
= \left[(q_1 \otimes J q_2 - q_2 \otimes J q_1)_t, L_1^{2n+1}\right)_u W_1 + \left((L_1^{2n+1})_u(q_1 \otimes J q_2 - q_2 \otimes J q_1)_t W_1\right)
\]
\[
- \left((L_1^{2n+1})_u q_1 \otimes J q_2)_t W_1 + (q_1 \otimes (L_1^{2n+1})_u T_j q_2)_t W_1 + \left((L_1^{2n+1})_u q_2 \otimes J q_1)_t W_1\right)
\]
\[
- (q_2 \otimes (L_1^{2n+1})_u T_j q_1)_t W_1 - (q_1 \otimes J q_2 - q_2 \otimes J q_1)(L_1^{2n+1})_u W_1
\]
\[
= \left((q_1 \otimes J q_2 - q_2 \otimes J q_1)_t, L_1^{2n+1}\right)_u W_1 + \left((L_1^{2n+1})_u(q_1 \otimes J q_2 - q_2 \otimes J q_1)_t W_1\right)
\]
\[
+ \left((q_1 \otimes J q_2 - q_2 \otimes J q_1), (L_1^{2n+1})_u W_1\right)
\]
Lemma 3. For the BTL hierarchy,\nopengroup
\begin{align*}
Y_1(\lambda, \mu) &= \lambda^{-1}(\Psi_1(x, y; \mu) \otimes J\Psi_1(x, y; -\lambda) - \Psi_1(x, y; -\lambda) \otimes J\Psi_1(x, y; \mu)), \\
Y_2(\lambda, \mu) &= \lambda^{-1}(\Psi_2(x, y; \mu^{-1}) \otimes J\Psi_2(x, y; -\lambda) - \Psi_2(x, y; -\lambda^{-1}) \otimes J\Psi_2(x, y; \mu^{-1})).
\end{align*}\n
In [29], there are some mistakes in the corresponding results about \(Y(\lambda, \mu)\) for BTL and CTL (see Proposition 7 and 13 in [29]), and the correct ones should be without “( )\_”.\nopengroup

We denote the squared eigenfunction symmetry generated by \(\Psi_1(x, y; \mu)\) and \(\lambda^{-1}\Psi_1(x, y; -\lambda)\) as \(\partial_{\alpha_1}\), while the one generated by \(-\Psi_2(x, y; \mu^{-1})\) and \(\lambda^{-1}\Psi_2(x, y; -\lambda^{-1})\) as \(\partial_{\alpha_2}\). Then
\begin{align*}
\partial_{\alpha_1}W_1 &= \lambda^{-1}(\Psi_1(x, y; \mu) \otimes J\Psi_1(x, y; -\lambda) - \Psi_1(x, y; -\lambda) \otimes J\Psi_1(x, y; \mu))W_1, \\
\partial_{\alpha_2}W_1 &= -\lambda^{-1}(\Psi_1(x, y; \mu) \otimes J\Psi_1(x, y; -\lambda) - \Psi_1(x, y; -\lambda) \otimes J\Psi_1(x, y; \mu))W_1, \\
\partial_{\alpha_2}W_2 &= -\lambda^{-1}(\Psi_2(x, y; \mu^{-1}) \otimes J\Psi_2(x, y; -\lambda) - \Psi_2(x, y; -\lambda^{-1}) \otimes J\Psi_2(x, y; \mu^{-1}))W_2,
\end{align*}\n
and
\begin{align*}
\partial_{\alpha_2}W_1 &= -\lambda^{-1}(\Psi_2(x, y; \mu^{-1}) \otimes J\Psi_2(x, y; -\lambda) - \Psi_2(x, y; -\lambda^{-1}) \otimes J\Psi_2(x, y; \mu^{-1}))W_1, \\
\partial_{\alpha_2}W_2 &= \lambda^{-1}(\Psi_2(x, y; \mu^{-1}) \otimes J\Psi_2(x, y; -\lambda) - \Psi_2(x, y; -\lambda^{-1}) \otimes J\Psi_2(x, y; \mu^{-1}))W_2.
\end{align*}\n
Further from [26], [34] and [35], we have

**Proposition 4.** The squared eigenfunction symmetries \(\partial_{\alpha_1}\) and \(\partial_{\alpha_2}\) are the generators of the additional symmetries for the BTL hierarchy, that is,
\begin{equation}
\partial_{\alpha_1} = \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{k=-\infty}^{\infty} \lambda^{-k-m-1}\partial_{\alpha_1}^*(m, m+k),
\end{equation}\n
\[ \partial_{\alpha_2} = \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{k=-\infty}^{\infty} \lambda^{-k-m-1} \partial y_{m,m+k}. \quad (41) \]

Thus we have establish the relation between the squared eigenfunction symmetry and the additional symmetry.

4. THE SQUARED EIGENFUNCTION SYMMETRY FOR THE CTL HIERARCHIES

In this section, the squared eigenfunction symmetry for the CTL hierarchy will be given. Similar to the case of the above section, given two eigenfunctions \( q_1 \) and \( q_2 \), one can define the squared eigenfunction flow of the CTL hierarchy by its actions on the wave operators,

\[ \partial_{\alpha} W_1 = (q_1 \otimes K q_2 + q_2 \otimes K q_1) W_1, \quad \partial_{\alpha} W_2 = -(q_1 \otimes K q_2 + q_2 \otimes K q_1) W_2. \quad (42) \]

According to (10), the action of the squared eigenfunction flow on the Lax operator is

\[ \partial_{\alpha} L_1 = [(q_1 \otimes K q_2 + q_2 \otimes K q_1), L_1], \quad \partial_{\alpha} L_2 = -[(q_1 \otimes K q_2 + q_2 \otimes K q_1), L_2]. \quad (43) \]

The next proposition shows that the definitions above is well-defined.

**Proposition 5.** \( \partial_{\alpha} \) is consistent with the CTL constraint (3), i.e. \( (\partial_{\alpha} L_i^T)K + K(\partial_{\alpha} L_i) = 0, i = 1, 2. \)

**Proof.**

\[
K(q_1 \otimes K q_2 + q_2 \otimes K q_1) + (q_1 \otimes K q_2 + q_2 \otimes K q_1)^T K = K(q_1 \otimes q_2 + q_2 \otimes q_1)K^T + K(q_1 \otimes q_2 + q_2 \otimes q_1)K = 0,
\]

by noting that \( K^T = -K \). Then

\[
K(q_1 \otimes K q_2 + q_2 \otimes K q_1) + (q_1 \otimes K q_2 + q_2 \otimes K q_1)^T K = 0, \quad (44)
\]

\[
K(q_1 \otimes K q_2 + q_2 \otimes K q_1) + (q_1 \otimes K q_2 + q_2 \otimes K q_1)^T K = 0, \quad (45)
\]

from the fact if \( KA + A^T K = 0 \), then \( KA_t + A_t^T K = 0 \) and \( KA_u + A_u^T K = 0 \) (see [2]).

The rest of the proof is similarly to the case of the BTL hierarchy. \( \square \)

Thus \( \partial_{\alpha} \) is indeed well-defined. By the same way as the BTL case, one can get the following proposition, which shows that this squared eigenfunction flow is indeed a kind of symmetry for the CTL hierarchy, and thus is called the squared eigenfunction symmetry.

**Proposition 6.**

\[ [\partial_{\alpha}, \partial x_{2n+1}] = [\partial_{\alpha}, \partial y_{2n+1}] = 0. \quad (46) \]
Define the generator of the additional symmetries for the CTL hierarchy as the following double expansions

\[
Y_1(\lambda, \mu) = \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-l-m-1} A_{1m,m+l}(M_1, L_1),
\]

\[
Y_2(\lambda, \mu) = \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-l-m-1} A_{2m,m+l}(M_2, L_2).
\]

This double expansions can be related with the wave functions in the following way [29] by considering [21],

**Lemma 7.** For the CTL hierarchy,

\[
Y_1(\lambda, \mu) = \Psi_1(x, y; \mu) \otimes K \Psi_1(x, y; -\lambda) + \Psi_1(x, y; -\lambda) \otimes K \Psi_1(x, y; \mu), \tag{47}
\]

\[
Y_2(\lambda, \mu) = \lambda^{-2} \Psi_2(x, y; \mu^{-1}) \otimes K \Psi_2(x, y; -\lambda^{-1}) + \mu^{-2} \Psi_2(x, y; -\lambda^{-1}) \otimes K \Psi_2(x, y; \mu^{-1}). \tag{48}
\]

If denote the squared eigenfunction symmetry generated by \(\Psi_1(x, y; \mu)\) and \(\Psi_1(x, y; -\lambda)\) as \(\partial_{\alpha_1}\), then

\[
\partial_{\alpha_1} W_1 = (\Psi_1(x, y; \mu) \otimes K \Psi_1(x, y; -\lambda) + \Psi_1(x, y; -\lambda) \otimes K \Psi_1(x, y; \mu)) W_1, \tag{49}
\]

\[
\partial_{\alpha_1} W_2 = -(\Psi_1(x, y; \mu) \otimes K \Psi_1(x, y; -\lambda) + \Psi_1(x, y; -\lambda) \otimes K \Psi_1(x, y; \mu)) W_2, \tag{50}
\]

And denote the squared eigenfunction symmetry generated by \(-\lambda^{-1} \Psi_2(x, y; \lambda^{-1})\) and \(\lambda^{-1} \Psi_2(x, y; -\lambda^{-1})\) as \(\partial_{\alpha_2}\), that is

\[
\partial_{\alpha_2} W_1 = -\lambda^{-2}(\Psi_2(x, y; \mu^{-1}) \otimes K \Psi_2(x, y; -\lambda^{-1}) + \Psi_2(x, y; -\lambda^{-1}) \otimes K \Psi_2(x, y; \mu^{-1})) W_1, \tag{51}
\]

\[
\partial_{\alpha_2} W_2 = \lambda^{-2}(\Psi_2(x, y; \mu^{-1}) \otimes K \Psi_2(x, y; -\lambda^{-1}) + \Psi_2(x, y; -\lambda^{-1}) \otimes K \Psi_2(x, y; \mu^{-1})) W_2. \tag{52}
\]

From [26], [47] and [48], we have,

**Proposition 8.** The squared eigenfunction symmetries \(\partial_{\alpha_1}\) are the generators of the additional symmetries \(\partial_{\alpha_1} y_{m}\) for the CTL hierarchy, that is,

\[
\partial_{\alpha_1} = \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{k=-\infty}^{\infty} \lambda^{-k-m-1} \partial_{x_{m,m+k}}. \tag{53}
\]

And the relation between the squared eigenfunction symmetry \(\partial_{\alpha_2}\) and the additional symmetries \(\partial_{\alpha_2} y_{m}\) for the CTL hierarchy is as follows:

\[
\partial_{\alpha_2} = \sum_{k=-\infty}^{\infty} \lambda^{-k} \partial_{y_{m,k}}. \tag{54}
\]

**Remark:** Usually, one may think that the result about the CTL hierarchy should be parallel to those about the BTL hierarchy. But here (54) is different from (41). Because of the coefficients of (48), it is difficult to construct the squared eigenfunction symmetry corresponding to the generating function
of the additional symmetries $\partial_{y_{m}^{*}}$. But when $\mu = \lambda$, we can construct $\partial_{\alpha_{2}}$ as (51) and (52). In this case, $\partial_{\alpha_{2}}$ can be viewed as the generating function of $\partial_{y_{0,k}^{*}}$. In order to get the parallel result to the BTL hierarchy, some modifications of (42) and (43) may be needed.

5. CONCLUSIONS AND DISCUSSIONS

The squared eigenfunction symmetries for the BTL and CTL hierarchies are constructed explicitly (see (29), (30), (42) and (43)) in the suitable combination of the ones of the TL hierarchy by considering the BTL and CTL constraint. And the relation with the additional symmetry is also investigated, that is, the squared eigenfunction symmetry can be viewed as the generating function of the additional symmetries when the defined eigenfunctions are the wave functions (see Proposition 4 and 8). These theories are expected to be applied in the study of the symmetry constraints for the BTL and CTL hierarchies and the corresponding additional symmetries.

Acknowledgments

This work is supported by the NSFC (Grant No. 11226196) and “the Fundamental Research Funds for the Central Universities” No. 2012QNA45

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