STRONG APPROXIMATION OF NONLINEAR FILTERING FOR MULTISCALE MCKEAN-VLASOV STOCHASTIC SYSTEMS

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Abstract. This work concerns the nonlinear filtering problem of multiscale McKean-Vlasov stochastic systems where the whole systems depend on distributions of fast components. First of all, we prove that the slow component of the original system converges to an average system in the $L^{2p}$ ($p \geq 1$) sense. Moreover, we obtain the strong convergence order for the $L^{2}$ case. Then, given an observation process which depends on the slow component and its distribution, we show that the nonlinear filtering of the slow component and its distribution also converges to that of the average system in the $L^{q}$ ($p \geq 8, 1 \leq q \leq \frac{p}{8}$) sense.

1. Introduction

McKean-Vlasov stochastic differential equations (SDEs for short) are also called distribution-dependent SDEs, or mean-field SDEs. And the difference between McKean-Vlasov SDEs and classical SDEs is that the former depends on the positions and probability distributions of particles. Therefore, McKean-Vlasov SDEs can better describe many models. The study on McKean-Vlasov SDEs was initiated by H. P. McKean [14] who was inspired by Kac’s program in Kinetic Theory. Nowadays, McKean-Vlasov SDEs have been widely applied in many fields, such as biology, game theory, optimal control theory and interacting particle systems. And there are many results about them. Let us recall some works. Sznitman proved the existence and uniqueness of strong solutions to McKean-Vlasov SDEs under global Lipschitz conditions in [22]. Ding and Qiao [1, 2] studied the well-posedness and stability of solutions to McKean-Vlasov SDEs with non-Lipschitz coefficients. Wang investigated the exponential ergodicity of the strong solutions to Landau type McKean-Vlasov SDEs in [24]. Sen and Caines [23] and Liu and Qiao [11] studied nonlinear filtering problems of McKean-Vlasov SDEs with independent noises and correlated noises, respectively.

Besides, multiscale SDEs, or slow-fast systems, are widely used in engineering and science fields (c.f. [4, 5]). The average principle for them was first studied by Khasminskii [9], see [12, 13, 21, 25] (and the references therein) for further generalizations. Here we briefly mention some results related with ours. Liu [13] studied SDEs with two well-separated time scales under Lipschitz conditions and established that the slow part of the original system converges to an average system in the $L^{2}$ sense. Liu, Röckner, Sun and...
Xie considered a class of slow-fast SDEs and proved the convergence in the $L^p$ sense in [12].

Now, multiscale McKean-Vlasov SDEs have also been studied. For example, Röckner, Sun and Xie [21] investigated the following multiscale McKean-Vlasov SDEs: for $T > 0$

$$
\begin{align*}
\begin{cases}
    d\tilde{X}^\varepsilon_t = \tilde{b}_1 \left( \tilde{X}^\varepsilon_t, \mathcal{L}_{X_t}^\varepsilon, \tilde{Z}_t^\varepsilon \right) dt + \tilde{\sigma}_1 \left( \tilde{X}^\varepsilon_t, \mathcal{L}_{X_t}^\varepsilon \right) dB_t, \\
    \tilde{X}_0^\varepsilon = x_0, & 0 \leq t \leq T \\
    d\tilde{Z}_t^\varepsilon = \frac{1}{\varepsilon} \tilde{b}_2 \left( \tilde{X}^\varepsilon_t, \mathcal{L}_{X_t}^\varepsilon, \tilde{Z}_t^\varepsilon \right) dt + \frac{1}{\sqrt{\varepsilon}} \tilde{\sigma}_2 \left( \tilde{X}^\varepsilon_t, \mathcal{L}_{X_t}^\varepsilon, \tilde{Z}_t^\varepsilon \right) dW_t,
\end{cases}
\end{align*}
$$

where $(B_t), (W_t)$ are $n$- and $m$-dimensional standard Brownian motions defined on the complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbb{P})$, respectively, these mappings $\tilde{b}_1 : \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \times \mathbb{R}^m \to \mathbb{R}^n, \tilde{\sigma}_1 : \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \to \mathbb{R}^{n \times n}, \tilde{b}_2 : \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \times \mathbb{R}^m \to \mathbb{R}^m, \tilde{\sigma}_2 : \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \times \mathbb{R}^m \to \mathbb{R}^{m \times m}$ are all Borel measurable, and $\mathcal{L}_{X_t}^\varepsilon$ denotes the distribution of $X^\varepsilon_t$ under $\mathbb{P}$. There they showed that the slow part converges to an average system in the $L^2$ sense. Later, Hong, Li and Liu [16] generalized this result to the infinite dimensional space. Very recently, the first named author [15] also obtain the same result for multiscale multivalued McKean-Vlasov SDEs.

Note that the whole system (1) doesn’t depend on the distribution of the fast component. One improvement is that Gao, Hong and Liu [3] added the distribution of the fast component to the slow component in the system (1), and established the $L^2$ convergence in the infinite dimensional framework. Unfortunately, they deleted the distribution of the slow component in the fast equation. Another improvement is that Xu, Liu, Liu and Miao [25] inserted the distribution of the fast component into the fast part in the system (1), and also prove the $L^2$ convergence.

In this paper, we study multiscale McKean-Vlasov stochastic systems where the whole systems depend on distributions of fast components, and establish an average principle in the $L^{2p}(p \geq 1)$ sense. Concretely speaking, consider the following slow-fast system on $\mathbb{R}^n \times \mathbb{R}^m$

$$
\begin{align*}
\begin{cases}
    dX^\varepsilon_t = b_1(X^\varepsilon_t, \mathcal{L}_{X_t}^\varepsilon, Z^\varepsilon, z_{0,0}, \mathcal{L}_{Z_0}^\varepsilon, \mathcal{L}_{Z^\varepsilon}) dt + \sigma_1(X^\varepsilon_t, \mathcal{L}_{X_t}^\varepsilon) dB_t, \\
    X^\varepsilon_0 = \zeta, & 0 \leq t \leq T \\
    dZ^\varepsilon_t = \frac{1}{\varepsilon} b_2(X^\varepsilon_t, \mathcal{L}_{X_t}^\varepsilon, Z^\varepsilon, z_{0,0}, \mathcal{L}_{Z_0}^\varepsilon, \mathcal{L}_{Z^\varepsilon}) dt + \frac{1}{\sqrt{\varepsilon}} \sigma_2(X^\varepsilon_t, \mathcal{L}_{X_t}^\varepsilon, Z^\varepsilon, z_{0,0}, \mathcal{L}_{Z_0}^\varepsilon, \mathcal{L}_{Z^\varepsilon}) dW_t, \\
    Z^\varepsilon_0 = \xi, & 0 \leq t \leq T \\
    dZ^\varepsilon, z_{0,0,0,0} dt = \frac{1}{\varepsilon} b_2(X^\varepsilon_t, \mathcal{L}_{X_t}^\varepsilon, Z^\varepsilon, z_{0,0,0,0}, \mathcal{L}_{Z_0}^\varepsilon, \mathcal{L}_{Z^\varepsilon}) dt + \frac{1}{\sqrt{\varepsilon}} \sigma_2(X^\varepsilon_t, \mathcal{L}_{X_t}^\varepsilon, Z^\varepsilon, z_{0,0,0,0}, \mathcal{L}_{Z_0}^\varepsilon, \mathcal{L}_{Z^\varepsilon}) dW_t, \\
    Z_0 = z_0, & 0 \leq t \leq T,
\end{cases}
\end{align*}
$$

where these mappings $b_1 : \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \times \mathbb{R}^m \times \mathcal{P}_2(\mathbb{R}^m) \to \mathbb{R}^n, \sigma_1 : \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \to \mathbb{R}^{n \times n}, b_2 : \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \times \mathbb{R}^m \times \mathcal{P}_2(\mathbb{R}^m) \to \mathbb{R}^m, \sigma_2 : \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \times \mathbb{R}^m \times \mathcal{P}_2(\mathbb{R}^m) \to \mathbb{R}^{m \times m}$ are all Borel measurable, and $\zeta, \xi$ are two random variables. To conclude the average principle for the system (2), since two frozen equations are McKean-Vlasov SDEs, their solutions are nonlinear Markov processes, and whether the classical Khasminskii time discretization or the Poisson equation to prove the strong convergence do not work. Therefore, we first linearize the two nonlinear Markov processes (cf. [20]) and show the strong convergence by the modified Khasminskii time discretization method. Moreover, on account of these distributions, we apply a lot of tricks to obtain some estimations.
Next, nonlinear filtering problems mean that people extract some useful information of unobservable phenomena from observable ones, and estimate and predict them. So, the nonlinear filtering theory plays an important role in many areas including stochastic control, financial modeling, speech and image processing, and Bayesian networks \([4, 5, 8, 16, 17, 18]\). And the nonlinear filtering theory of multiscale SDEs is systematically introduced by Kushner in \([10]\). Later more and more results about the nonlinear filtering of multiscale SDEs appear (See \([4, 5, 8, 16, 17, 18]\) and references therein). However there are few results about nonlinear filtering of multiscale McKean-Vlasov SDEs. Hence, we also study the nonlinear filtering problem of them. That is, we define an observation process \(Y_t^\varepsilon\) as follows

\[
Y_t^\varepsilon = V_t + \int_0^t h(X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon})ds,
\]

where \(V\) is a \(l\)-dimensional Brownian motion independent of \(B, W\), and \(h : \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \mapsto \mathbb{R}^l\) is Borel measurable. Then based on the obtained average principle, we establish that the nonlinear filtering of the slow part and its distribution converges to that of the average system in the \(L^q\) (\(p \geq 8, 1 \leq q \leq \frac{p}{2}\)) sense.

The novelty of this paper lies in three folds. The first fold is that the system (2) is more general than that in some known results (cf. \([3, 6, 7, 21, 25]\)). Thus, our result can be applied to many models. The second fold is that we obtain the strong convergence order for the \(L^2\) case, which is important for numerical simulation. The third fold is that we prove the \(L^q\) convergence of nonlinear filtering for multiscale McKean-Vlasov SDEs, which can partly cover some results in \([8, 16, 18]\).

Lastly, we describe our motivation of this paper. Note that in \([25]\), although the multiscale system is general, the average principle is not right. This is because four authors used the Markov property which does not hold for general McKean-Vlasov SDEs. Our first motivation is to correct this mistake. Besides, as far as we know, no average principle for McKean-Vlasov SDEs with two time scales has yet been presented in the \(L^{2p}\) (\(p \geq 1\)) sense. However, people usually need to estimate the higher order moments which possess a good robustness and can be applied in statistics, game theory, finance and other fields. So, it is our second motivation to establish an average principle in the \(L^{2p}\) (\(p \geq 1\)) sense.

The paper proceeds as follows. In Section 2, we introduce some related notations. Then we state main results in Section 3. The proofs of two main theorems are placed in Section 4 and 5 respectively.

The following convention will be used throughout the paper: \(C\) with or without indices will denote different positive constants whose values may change from one place to another.

2. Notations and Assumptions

In this section, we will recall some notations and list all assumptions.

2.1. Notations. In this subsection, we introduce some notations used in the sequel.

Let \(|\cdot|, ||\cdot||\) be the norms of a vector and a matrix, respectively. Let \(\langle \cdot, \cdot \rangle\) be the inner product of vectors on \(\mathbb{R}^n\). \(A^*\) denotes the transpose of the matrix \(A\).

Let \(\mathcal{B}_b(\mathbb{R}^n)\) be the set of all bounded Borel measurable functions on \(\mathbb{R}^n\). Let \(C(\mathbb{R}^n)\) be the set of all functions which are continuous on \(\mathbb{R}^n\). \(C^2(\mathbb{R}^n)\) represents the collection of all functions in \(C(\mathbb{R}^n)\) with continuous derivatives of order up to 2.
Let $\mathcal{B}(\mathbb{R}^n)$ be the Borel $\sigma$-field on $\mathbb{R}^n$. Let $\mathcal{P}(\mathbb{R}^n)$ be the collection of all probability measures on $\mathcal{B}(\mathbb{R}^n)$ with the usual topology of weak convergence. Let $\mathcal{P}_2(\mathbb{R}^n)$ denote the collection of probability measures on $\mathcal{B}(\mathbb{R}^n)$ satisfying:

$$||\mu||^2 := \int_{\mathbb{R}^n} |x|^2 \mu(dx) < \infty.$$  

It is known that $\mathcal{P}_2(\mathbb{R}^n)$ is a Polish space endowed with the $L^2$-Wasserstein distance defined by

$$\mathbb{W}_2(\mu, \nu) := \inf_{\pi \in \Psi(\mu, \nu)} \left( \int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y|^2 \pi(dx, dy) \right)^{\frac{1}{2}}, \quad \mu, \nu \in \mathcal{P}_2(\mathbb{R}^n),$$

where $\Psi(\mu, \nu)$ is the set of all couplings $\pi$ with marginal distributions $\mu$ and $\nu$. Moreover, if $\xi, \zeta$ are two random variables with distributions $\mathcal{L}_\xi, \mathcal{L}_\zeta$ under $\mathbb{P}$, respectively,

$$\mathbb{W}_2(\mathcal{L}_\xi, \mathcal{L}_\zeta) \leq (\mathbb{E}|\xi - \zeta|^2)^{\frac{1}{2}},$$

where $\mathbb{E}$ stands for the expectation with respect to $\mathbb{P}$.

2.2. **Assumptions.** In this subsection, we give out all the assumptions used in the sequel:

(H$_{b_1, \sigma_1}$) There exists a constant $L_{b_1, \sigma_1} > 0$ such that for $x_i \in \mathbb{R}^n, \mu_i \in \mathcal{P}_2(\mathbb{R}^n), z_i \in \mathbb{R}^m, \nu_i \in \mathcal{P}_2(\mathbb{R}^m), i = 1, 2, \nu_i \in \mathcal{P}_2(\mathbb{R}^m)$,

$$|b_1(x_1, \mu_1, z_1, \nu_1) - b_1(x_2, \mu_2, z_2, \nu_2)|^2 + ||\sigma_1(x_1, \mu_1) - \sigma_1(x_2, \mu_2)||^2 \leq L_{b_1, \sigma_1} \left(|x_1 - x_2|^2 + \mathbb{W}_2^2(\mu_1, \mu_2) + |z_1 - z_2|^2 + \mathbb{W}_2^2(\nu_1, \nu_2)\right).$$

(H$_{b_2, \sigma_2}$) There exists a constant $L_{b_2, \sigma_2} > 0$ such that for $x_i \in \mathbb{R}^n, \mu_i \in \mathcal{P}_2(\mathbb{R}^n), z_i \in \mathbb{R}^m, \nu_i \in \mathcal{P}_2(\mathbb{R}^m), i = 1, 2, \nu_i \in \mathcal{P}_2(\mathbb{R}^m)$,

$$|b_2(x_1, \mu_1, z_1, \nu_1) - b_2(x_2, \mu_2, z_2, \nu_2)|^2 + ||\sigma_2(x_1, \mu_1, z_1, \nu_1) - \sigma_2(x_2, \mu_2, z_2, \nu_2)||^2 \leq L_{b_2, \sigma_2} \left(|x_1 - x_2|^2 + \mathbb{W}_2^2(\mu_1, \mu_2) + |z_1 - z_2|^2 + \mathbb{W}_2^2(\nu_1, \nu_2)\right).$$

(H$_{b, 2}$) For some $p \geq 0$, there exist two constants $\beta_1 > 0, \beta_2 > 0$ satisfying $\beta_1 - \beta_2 > (4p + 4)L_{b_2, \sigma_2}$ such that for $x_i \in \mathbb{R}^n, \mu_i \in \mathcal{P}_2(\mathbb{R}^n), z_i \in \mathbb{R}^m, \nu_i \in \mathcal{P}_2(\mathbb{R}^m), i = 1, 2, \nu_i \in \mathcal{P}_2(\mathbb{R}^m)$,

$$2(z_1 - z_2, b_2(x, \mu, z_1, \nu_1) - b_2(x, \mu, z_2, \nu_2)) + (2p + 1)||\sigma_2(x, \mu, z_1, \nu_1) - \sigma_2(x, \mu, z_2, \nu_2)||^2 \leq -\beta_1|z_1 - z_2|^2 + \beta_2 \mathbb{W}_2^2(\nu_1, \nu_2).$$

(H$_h$) $h$ is bounded, and there is a constant $L_h > 0$ such that

$$|h(x_1, \mu) - h(x_2, \mu_2)|^2 \leq L_h(|x_1 - x_2|^2 + \mathbb{W}_2^2(\mu_1, \mu_2)).$$

**Remark 2.1.**

(i) (H$_{b_1, \sigma_1}$) yields that there exists a constant $\tilde{L}_{b_1, \sigma_1} > 0$ such that for $x \in \mathbb{R}^n, \mu \in \mathcal{P}_2(\mathbb{R}^n), z \in \mathbb{R}^m, \nu \in \mathcal{P}_2(\mathbb{R}^m)$

$$|b_1(x, \mu, z, \nu)|^2 + ||\sigma_1(x, \mu)||^2 \leq \tilde{L}_{b_1, \sigma_1}(1 + |x|^2 + ||\mu||^2 + |z|^2 + ||\nu||^2).$$

(ii) (H$_{b_2, \sigma_2}$) implies that there exists a constant $\tilde{L}_{b_2, \sigma_2} > 0$ such that for $x \in \mathbb{R}^n, \mu \in \mathcal{P}_2(\mathbb{R}^n), z \in \mathbb{R}^m, \nu \in \mathcal{P}_2(\mathbb{R}^m)$,

$$|b_2(x, \mu, z, \nu)|^2 + ||\sigma_2(x, \mu, z, \nu)||^2 \leq \tilde{L}_{b_2, \sigma_2}(1 + |x|^2 + ||\mu||^2 + |z|^2 + ||\nu||^2).$$

(iii) By (H$_{b_2, \sigma_2}$) and (H$_{b_2, \sigma_2}$), it holds that for $x \in \mathbb{R}^n, \mu \in \mathcal{P}_2(\mathbb{R}^n), z \in \mathbb{R}^m, \nu \in \mathcal{P}_2(\mathbb{R}^m)$

$$2(z, b_2(x, \mu, z, \nu)) + (2p + 1)||\sigma_2(x, \mu, z, \nu)||^2 \leq -\alpha_1|z|^2 + \alpha_2||\nu||^2 + C(1 + |x|^2 + ||\mu||^2).$$
where \( \alpha_1 := \beta_1 - (2p + 2)L_{b_2, \sigma_2}, \alpha_2 := \beta_2 + (2p + 1)L_{b_2, \sigma_2}, \alpha_1 - \alpha_2 = L_{b_2, \sigma_2} > 0 \) and \( C > 0 \) is a constant.

3. Main results

In this section, we provide main results of the paper.

3.1. The average principle for multiscale McKean-Vlasov SDEs. In this subsection, we state the average principle result for multiscale McKean-Vlasov SDEs.

Let us recall the system (2), i.e.
\[
\begin{align*}
\frac{dX_t^\varepsilon}{dt} &= b_1(X_t^\varepsilon, X_t^\varepsilon, Z_t^z, \xi, \ell_t^\varepsilon, \ell_t^\varepsilon, \ell_t^\varepsilon) + \sigma_1(X_t^\varepsilon, X_t^\varepsilon, Z_t^z, \xi, \ell_t^\varepsilon, \ell_t^\varepsilon, \ell_t^\varepsilon) dB_t, \\
X_0^\varepsilon &= \varrho, \\
\frac{dZ_t^z}{dt} &= \frac{1}{\varepsilon} b_2(X_t^\varepsilon, X_t^\varepsilon, Z_t^z, \xi, \ell_t^\varepsilon) + \sigma_2(X_t^\varepsilon, X_t^\varepsilon, Z_t^z, \xi, \ell_t^\varepsilon) dB_t, \\
Z_0^z &= \xi, \\
\frac{d\ell_t^\varepsilon}{dt} &= \frac{1}{\varepsilon} \mu_0 + \sigma_3(X_t^\varepsilon, X_t^\varepsilon, Z_t^z, \xi, \ell_t^\varepsilon) dB_t, \\
\ell_0^\varepsilon &= \ell_0,
\end{align*}
\]
where \( \mathbb{E}[\varphi]^{2p+2} < \infty, \mathbb{E}[\xi]^{2p+2} < \infty \) (\( p \) is the same to that in (H_{b_2, \sigma_2})). Under (H_{b_1, \sigma_1}) (H_{b_2, \sigma_2}), by [24, Theorem 2.1], the system (2) has a unique strong solution \( (X_t^\varepsilon, Z_t^z, \ell_t^\varepsilon) \).

Next, we take any \( \varepsilon \in \mathbb{R}^n \) and \( \mu \in \mathcal{P}_2(\mathbb{R}^n) \), and fix them. Consider the following SDE:
\[
\begin{align}
\frac{dZ_t^\varepsilon}{dt} &= b_2(X_t^\varepsilon, X_t^\varepsilon, Z_t^z, \xi, \ell_t^\varepsilon) + \sigma_2(X_t^\varepsilon, X_t^\varepsilon, Z_t^z, \xi, \ell_t^\varepsilon) dB_t, \\
Z_0^\varepsilon &= \xi, \\
0 \leq t \leq T.
\end{align}
\]

Under (H_{b_2, \sigma_2}), by [24, Theorem 2.1] we know that the above equation has a unique strong solution \( Z_t^\varepsilon, \xi, \ell_t^\varepsilon \). Moreover, under (H_{b_2, \sigma_2}), by [24, Theorem 3.1], one could obtain that there exists a unique invariant probability measure \( \eta^{\varepsilon, \mu} \) for Eq. (7).

So, we construct an average equation on \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, \mathbb{P}) \) as follows:
\[
\begin{align}
\frac{d\bar{X}_t}{dt} &= \bar{b}_1(\bar{X}_t, \mathcal{L}_{X_t}^\varepsilon) + \sigma_1(\bar{X}_t, \mathcal{L}_{X_t}^\varepsilon) dB_t, \\
\bar{X}_0 &= \varrho,
\end{align}
\]
where \( \bar{b}_1(\bar{x}, \mu) = \int_{\mathbb{R}^n} b_1(x, \mu, z, \nu) \eta^{\varepsilon, \mu} \times \delta_{\eta^{\varepsilon, \mu}}(dz, d\nu) \).

Now, it is the position to state the first main result.

**Theorem 3.1.** Under these assumptions (H_{b_1, \sigma_1}) (H_{b_2, \sigma_2})-(H_{b_2, \sigma_2}), for \( p \geq 1 \), it holds that
\[
\lim_{\varepsilon \to 0} \mathbb{E} \left( \sup_{0 \leq t \leq T} |X_t^\varepsilon - \bar{X}_t|^{2p} \right) = 0,
\]
where \( \bar{X} \) is a solution of Eq. (8). In particular, we have that for any \( 0 < \gamma < 1 \)
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} |X_t^\varepsilon - \bar{X}_t|^{2} \right) \leq C(\varepsilon^{-1} + \varepsilon^{2\gamma} + \varepsilon^{-\gamma}).
\]

The proof of Theorem 3.1 is placed in Section 4.

**Remark 3.2.** For (10), if we take \( \gamma = 1/2 \), it follows that
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} |X_t^\varepsilon - \bar{X}_t|^{2} \right) \leq C_{\varepsilon}^{1/2}.
\]
That is, we obtain the convergence order $1/4$.

3.2. The efficient filtering for multiscale McKean-Vlasov SDEs. In this subsection, we state the efficient filtering result for multiscale McKean-Vlasov SDEs.

Set

$$(\Lambda_t^\varepsilon)^{-1} := \exp \left\{ - \int_0^t h^i(X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}^P) dV_s^i - \frac{1}{2} \int_0^t |h(X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}^P)|^2 ds \right\}.$$ 

Here and hereafter, we use the convention that repeated indices imply summation. Under $(H_h)$, we get that

$$\mathbb{E} \left( \exp \left\{ \frac{1}{2} \int_0^T |h(X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}^P)|^2 ds \right\} \right) < \infty,$$

and furthermore $(\Lambda_t^\varepsilon)^{-1}$ is an exponential martingale under the measure $\mathbb{P}$. Define a probability measure $\mathbb{P}^\varepsilon$ via

$$\frac{d\mathbb{P}^\varepsilon}{d\mathbb{P}} = (\Lambda_T^\varepsilon)^{-1}.$$

Then by the Girsanov theorem, it holds that $Y^\varepsilon$ is a Brownian motion under the probability measure $\mathbb{P}^\varepsilon$.

Define the nonlinear filtering for the state $X_t^\varepsilon$ and the measure $\mathcal{L}_{X_t^\varepsilon}^P$:

$$\rho_t^\varepsilon(F) := \mathbb{E}^{\mathbb{P}^\varepsilon} [F(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon}^P) | \mathcal{F}_t^\varepsilon],$$

$$\pi_t^\varepsilon(F) := \mathbb{E} [F(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon}^P) | \mathcal{F}_t^\varepsilon], \quad F \in \mathcal{B}_b(\mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n)),$$

where $\mathcal{F}_t^\varepsilon = \sigma\{Y_s^\varepsilon : 0 \leq s \leq t\} \vee \mathcal{N}$, and $\mathcal{N}$ denotes the collection of all zero sets under $\mathbb{P}$. Here $\rho_t^\varepsilon(F)$, $\pi_t^\varepsilon(F)$ are called the unnormalized filtering and the normalized filtering of $(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon}^P)$ with respect to $\mathcal{F}_t^\varepsilon$, respectively. By the Kallianpur-Striebel formula, we get the following relationship between $\rho_t^\varepsilon(F)$ and $\pi_t^\varepsilon(F)$:

$$\pi_t^\varepsilon(F) = \frac{\rho_t^\varepsilon(F)}{\rho_t^\varepsilon(1)}.$$

Next, set

$$\Lambda_t^0 := \exp \left\{ \int_0^t h^i(\bar{X}_s, \mathcal{L}_{X_s}^\mathbb{P}) dY_s^i - \frac{1}{2} \int_0^t |h(\bar{X}_s, \mathcal{L}_{X_s}^\mathbb{P})|^2 ds \right\},$$

$$\rho_t^0(F) := \mathbb{E}^{\mathbb{P}^\varepsilon} [F(X_t, \mathcal{L}_{X_t}^\varepsilon) | \mathcal{F}_t^\varepsilon],$$

$$\pi_t^0(F) := \frac{\rho_t^0(F)}{\rho_t^0(1)},$$

and we study the relationship between $\pi_t^\varepsilon$ and $\pi_t^0$. The second main result of the paper is the following theorem.

**Theorem 3.3.** Under these assumptions $(H_{b_1,\sigma_1})$, $(H_{b_2,\sigma_2})$, $(H_{h_1,\sigma_2})$ and $(H_h)$, for $p \geq 8$, the nonlinear filtering of the original system converges to that of the average system in the $L^q$ ($1 \leq q \leq \frac{p}{8}$) sense under $\mathbb{P}$, that is,

$$\lim_{\varepsilon \to 0} \mathbb{E} |\pi_t^\varepsilon(F) - \pi_t^0(F)|^q = 0, \quad F \in C_{b,lip}(\mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n)),$$

where $C_{b,lip}(\mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n))$ stands for the collection of all bounded and Lipschitz continuous functions on $\mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n)$. 

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The proof of Theorem 3.3 is placed in Section 5.

4. Proof of Theorem 3.1

In the section, we prove Theorem 3.1. The proof consists of three parts. In the first part (Subsection 4.1), we segment the time interval $[0, T]$ by the size $\delta$, where $\delta$ is a fixed positive number depending on $\varepsilon$, and introduce three auxiliary processes:

$$
\hat{Z}_{t, \varepsilon} = \xi + \frac{1}{\varepsilon} \int_{0}^{t} b_2(X_{s, \varepsilon}, \mathcal{L}_{X_{s, \varepsilon}}, \hat{Z}_{s, \varepsilon}) ds + \frac{1}{\sqrt{\varepsilon}} \int_{0}^{t} \sigma_2(X_{s, \varepsilon}, \mathcal{L}_{X_{s, \varepsilon}}, \hat{Z}_{s, \varepsilon}) dW_s,
$$

(12)

$$
\hat{Z}_{t, \varepsilon, z, \mathcal{L}_{\hat{Z}_{t, \varepsilon}}} = z_0 + \frac{1}{\varepsilon} \int_{0}^{t} b_2(X_{s, \varepsilon}, \mathcal{L}_{X_{s, \varepsilon}}, \hat{Z}_{s, \varepsilon}) ds + \frac{1}{\sqrt{\varepsilon}} \int_{0}^{t} \sigma_2(X_{s, \varepsilon}, \mathcal{L}_{X_{s, \varepsilon}}, \hat{Z}_{s, \varepsilon}) dW_s,
$$

(13)

$$
\hat{X}_{t} = q + \int_{0}^{t} b_1(X_{s, \varepsilon}, \mathcal{L}_{X_{s, \varepsilon}}, \hat{Z}_{s, \varepsilon}, \mathcal{L}_{\hat{Z}_{s, \varepsilon}}) ds + \int_{0}^{t} \sigma_1(X_{s, \varepsilon}, \mathcal{L}_{X_{s, \varepsilon}}) dB_s,
$$

(14)

where $s(\delta) = \lceil \frac{\varepsilon}{\delta} \rceil \delta$, and $\lceil \cdot \rceil$ denotes the integer part of $\cdot$. Then we estimate $X_{\varepsilon, \xi}, Z_{\varepsilon, \xi}, Z_{\varepsilon, z, \mathcal{L}_{\hat{Z}_{t, \varepsilon}}}, X_{\varepsilon, \xi}, \hat{Z}_{\varepsilon, \xi}, \hat{Z}_{\varepsilon, z, \mathcal{L}_{\hat{Z}_{t, \varepsilon}}}$. In the second part (Subsection 4.2) and the third part (Subsection 4.3), we present some estimates for the frozen equation (7) and the average equation (8), respectively.

4.1. Some estimates for $X_{\varepsilon, \xi}, Z_{\varepsilon, \xi}, Z_{\varepsilon, z, \mathcal{L}_{\hat{Z}_{t, \varepsilon}}}, X_{\varepsilon, \xi}, \hat{Z}_{\varepsilon, \xi}, \hat{Z}_{\varepsilon, z, \mathcal{L}_{\hat{Z}_{t, \varepsilon}}}$.

Lemma 4.1. Under assumptions $(H_{01, \sigma_1}^1)$ $(H_{02, \sigma_2}^2)$ $(H_{03, \sigma_2}^3)$, there exists a constant $C > 0$ such that

$$
\sup_{\varepsilon} \mathbb{E} \left( \sup_{t \in [0, T]} |X_{t, \varepsilon}|^{2p+2} \right) \leq C (1 + \mathbb{E} |q|^{2p+2} + |z_0|^{2p+2} + \mathbb{E} |\xi|^{2p+2}),
$$

$$
\sup_{t \in [0, T]} \mathbb{E} |Z_{t, \varepsilon, \xi}|^{2p+2} \leq C (1 + \mathbb{E} |q|^{2p+2} + |z_0|^{2p+2} + \mathbb{E} |\xi|^{2p+2}),
$$

$$
\sup_{t \in [0, T]} \mathbb{E} |Z_{t, \varepsilon, z, \mathcal{L}_{\hat{Z}_{t, \varepsilon}}}|^{2p+2} \leq C (1 + \mathbb{E} |q|^{2p+2} + |z_0|^{2p+2} + \mathbb{E} |\xi|^{2p+2}).
$$

Proof. For $X_{t, \varepsilon}$, based on the BDG inequality and $(H_{01, \sigma_1}^1)$, we can get

$$
\mathbb{E} \left( \sup_{s \in [0, t]} |X_{s, \varepsilon}|^{2p+2} \right) \leq 3^{2p+1} \mathbb{E} |q|^{2p+2} + 3^{2p+1} \mathbb{E} \left( \sup_{s \in [0, t]} \left| \int_{0}^{s} b_1(X_{r, \varepsilon}, \mathcal{L}_{X_{r, \varepsilon}}, Z_{r, \varepsilon, z, \mathcal{L}_{\hat{Z}_{r, \varepsilon}}}, \mathcal{L}_{\hat{Z}_{r, \varepsilon}}) dr \right|^{2p+2} \right)
$$

$$
+ 3^{2p+1} \mathbb{E} \left( \sup_{s \in [0, t]} \left| \int_{0}^{s} \sigma_1(X_{r, \varepsilon}, \mathcal{L}_{X_{r, \varepsilon}}) dB_r \right|^{2p+2} \right)
$$

$$
\leq 3^{2p+1} \mathbb{E} |q|^{2p+2} + (3t)^{2p+1} \mathbb{E} \left( \int_{0}^{t} b_1(X_{r, \varepsilon}, \mathcal{L}_{X_{r, \varepsilon}}, Z_{r, \varepsilon, z, \mathcal{L}_{\hat{Z}_{r, \varepsilon}}}, \mathcal{L}_{\hat{Z}_{r, \varepsilon}}) dr \right)^{2p+2}
$$

$$
+ 3^{2p+1} t^p \mathbb{E} \left( \int_{0}^{t} \left| \sigma_1(X_{r, \varepsilon}, \mathcal{L}_{X_{r, \varepsilon}}) \right|^2 dr \right)^{2p+2}.
$$
\[
\begin{align*}
\ln & \leq 3^{2p+1} \mathbb{E} \| \varrho \|^{2p+2} + C \mathbb{E} \int_{0}^{t} (1 + |X_{s}^{\varepsilon}| + \| L_{X_{s}^{\varepsilon}}^{P} \| + |Z_{r}^{\varepsilon,20,2\varepsilon_{\xi}}| + \| L_{Z_{r}^{\varepsilon,\xi}}^{P} \|)^{2p+2} dr \\
& \leq C(\mathbb{E} \| \varrho \|^{2p+2} + 1) + C \int_{0}^{t} \mathbb{E} |X_{s}^{\varepsilon}|^{2p+2} dr + C \int_{0}^{t} \mathbb{E} |Z_{r}^{\varepsilon,20,2\varepsilon_{\xi}}|^{2p+2} dr \\
& \quad + C \int_{0}^{t} \mathbb{E} |Z_{r}^{\varepsilon,\xi}|^{2p+2} dr,
\end{align*}
\]
(15)

where \( \| L_{X_{s}^{\varepsilon}}^{P} \|^{2} = \mathbb{E} |X_{s}^{\varepsilon}|^{2} \).

For \( Z_{t}^{\varepsilon,\xi} \), applying the Itô formula to \( |Z_{t}^{\varepsilon,\xi}|^{2p+2} \) and taking the expectation, one could obtain that

\[
\begin{align*}
\mathbb{E} |Z_{t}^{\varepsilon,\xi}|^{2p+2} &= \mathbb{E} |\xi|^{2p+2} + \frac{2p + 2}{\varepsilon} \mathbb{E} \int_{0}^{t} |Z_{s}^{\varepsilon,\xi}|^{2p} (Z_{s}^{\varepsilon,\xi}, b_{2}(X_{s}^{\varepsilon}, L_{X_{s}^{\varepsilon}}^{P}, Z_{s}^{\varepsilon,\xi}, L_{Z_{s}^{\varepsilon,\xi}}^{P})) ds \\
& \quad + \frac{2p(p + 1)}{\varepsilon} \mathbb{E} \int_{0}^{t} |Z_{s}^{\varepsilon,\xi}|^{2p-2} \| \sigma_{2}(X_{s}^{\varepsilon}, L_{X_{s}^{\varepsilon}}^{P}, Z_{s}^{\varepsilon,\xi}, L_{Z_{s}^{\varepsilon,\xi}}^{P}) Z_{s}^{\varepsilon,\xi} \|^{2} ds \\
& \quad + \frac{p + 1}{\varepsilon} \mathbb{E} \int_{0}^{t} |Z_{s}^{\varepsilon,\xi}|^{2p} \| \sigma_{2}(X_{s}^{\varepsilon}, L_{X_{s}^{\varepsilon}}^{P}, Z_{s}^{\varepsilon,\xi}, L_{Z_{s}^{\varepsilon,\xi}}^{P}) \|^{2} ds,
\end{align*}
\]

and

\[
\begin{align*}
\frac{d}{dt} \mathbb{E} |Z_{t}^{\varepsilon,\xi}|^{2p+2} &= \frac{2p + 2}{\varepsilon} \mathbb{E} |Z_{t}^{\varepsilon,\xi}|^{2p} (Z_{t}^{\varepsilon,\xi}, b_{2}(X_{t}^{\varepsilon}, L_{X_{t}^{\varepsilon}}^{P}, Z_{t}^{\varepsilon,\xi}, L_{Z_{t}^{\varepsilon,\xi}}^{P})) \\
& \quad + \frac{2p(p + 1)}{\varepsilon} \mathbb{E} |Z_{t}^{\varepsilon,\xi}|^{2p-2} \| \sigma_{2}(X_{t}^{\varepsilon}, L_{X_{t}^{\varepsilon}}^{P}, Z_{t}^{\varepsilon,\xi}, L_{Z_{t}^{\varepsilon,\xi}}^{P}) Z_{t}^{\varepsilon,\xi} \|^{2} \\
& \quad + \frac{p + 1}{\varepsilon} \mathbb{E} |Z_{t}^{\varepsilon,\xi}|^{2p} \| \sigma_{2}(X_{t}^{\varepsilon}, L_{X_{t}^{\varepsilon}}^{P}, Z_{t}^{\varepsilon,\xi}, L_{Z_{t}^{\varepsilon,\xi}}^{P}) \|^{2} \\
& \leq \frac{2p + 2}{\varepsilon} \mathbb{E} |Z_{t}^{\varepsilon,\xi}|^{2p} (Z_{t}^{\varepsilon,\xi}, b_{2}(X_{t}^{\varepsilon}, L_{X_{t}^{\varepsilon}}^{P}, Z_{t}^{\varepsilon,\xi}, L_{Z_{t}^{\varepsilon,\xi}}^{P})) \\
& \quad + \frac{2p(p + 1)(p + 1)}{\varepsilon} \mathbb{E} |Z_{t}^{\varepsilon,\xi}|^{2p} \| \sigma_{2}(X_{t}^{\varepsilon}, L_{X_{t}^{\varepsilon}}^{P}, Z_{t}^{\varepsilon,\xi}, L_{Z_{t}^{\varepsilon,\xi}}^{P}) \|^{2} \\
& \leq \frac{p + 1}{\varepsilon} \left[ -\alpha_{1} |Z_{t}^{\varepsilon,\xi}|^{2} + \alpha_{2} \| \sigma_{2}(X_{t}^{\varepsilon}, L_{X_{t}^{\varepsilon}}^{P}, Z_{t}^{\varepsilon,\xi}, L_{Z_{t}^{\varepsilon,\xi}}^{P}) \|^{2} + C(1 + |X_{t}^{\varepsilon}|^{2} + \| L_{X_{t}^{\varepsilon}}^{P} \|^{2}) \right] \\
& \quad + C(1 + \mathbb{E} |X_{t}^{\varepsilon}|^{2p+2} + \| L_{X_{t}^{\varepsilon}}^{P} \|^{2p+2}) \\
& \leq \frac{-(\alpha_{1} - \alpha_{2} - L_{b_{2},a_{2}})(p + 1)}{\varepsilon} \mathbb{E} |Z_{t}^{\varepsilon,\xi}|^{2p+2} + \frac{C}{\varepsilon}(\mathbb{E} |X_{t}^{\varepsilon}|^{2p+2} + 1),
\end{align*}
\]

where the above inequality is based on (6). By the comparison theorem, we have that

\[
\begin{align*}
\mathbb{E} |Z_{t}^{\varepsilon,\xi}|^{2p+2} &\leq \mathbb{E} |\xi|^{2p+2} e^{-(\alpha_{1} - \alpha_{2} - L_{b_{2},a_{2}})(p + 1)} + C \int_{0}^{t} e^{-(\alpha_{1} - \alpha_{2} - L_{b_{2},a_{2}})(p + 1)}(\mathbb{E} |X_{s}^{\varepsilon}|^{2p+2} + 1) ds \\
& \leq \mathbb{E} |\xi|^{2p+2} + C \left( \mathbb{E} \left( \sup_{s \in [0,t]} |X_{s}^{\varepsilon}|^{2p+2} \right) + 1 \right).
\end{align*}
\]
(16)
Next, for $Z^\varepsilon,z_0,\mathcal{L}_t^\xi$, by the similar deduction to that for $Z_t^\varepsilon,\xi$, it holds that
\[
\mathbb{E}|Z_t^\varepsilon,z_0,\mathcal{L}_t^\xi|^{2p+2} \leq |z_0|^{2p+2} + \mathbb{E}|\xi|^{2p+2} + C\left(\mathbb{E} \left( \sup_{s \in [0,t]} |X_s^\varepsilon|^{2p+2} \right) + 1 \right),
\] (17)
Inserting (16) (17) in (15), by the Gronwall inequality one can get that
\[
\mathbb{E} \left( \sup_{t \in [0,T]} |X_t^\varepsilon|^{2p+2} \right) \leq C(1 + \mathbb{E}|\varrho|^{2p+2} + |z_0|^{2p+2} + \mathbb{E}|\xi|^{2p+2}),
\]
which together with (16) (17) implies that
\[
\mathbb{E}|Z_t^\varepsilon,z_0,\mathcal{L}_t^\xi|^{2p+2} \leq C(1 + \mathbb{E}|\varrho|^{2p+2} + |z_0|^{2p+2} + \mathbb{E}|\xi|^{2p+2}),
\]
\[
\mathbb{E}|Z_t^\varepsilon,z_0,\mathcal{L}_t^\xi|^{2p+2} \leq C(1 + \mathbb{E}|\varrho|^{2p+2} + |z_0|^{2p+2} + \mathbb{E}|\xi|^{2p+2}).
\]

The proof is complete. \qed

Next, we estimate $\mathbb{E}|X_t^\varepsilon - X_{k\delta}^\varepsilon|^2$ for any $t \in [k\delta, (k+1)\delta)$ and $k = 0, 1, 2, \cdots, \lfloor \frac{T}{\delta} \rfloor - 1$. Note that
\[
X_t^\varepsilon - X_{k\delta}^\varepsilon = \int_{k\delta}^t b_1(X_s^\varepsilon, \mathcal{L}_s^\xi, Z_s^\varepsilon, Z_s^\varepsilon, \mathcal{L}_s^\xi, \mathcal{L}_s^\xi) ds + \int_{k\delta}^t \sigma_1(X_s^\varepsilon, \mathcal{L}_s^\xi) dB_s.
\]
By (4) and the BDG inequality, it holds that
\[
\mathbb{E}|X_t^\varepsilon - X_{k\delta}^\varepsilon|^2 \leq 2\left( \mathbb{E} \left| \int_{k\delta}^t b_1(X_s^\varepsilon, \mathcal{L}_s^\xi, Z_s^\varepsilon, Z_s^\varepsilon, \mathcal{L}_s^\xi, \mathcal{L}_s^\xi) ds \right|^2 + \mathbb{E} \left( \int_{k\delta}^t \sigma_1(X_s^\varepsilon, \mathcal{L}_s^\xi) dB_s \right)^2 \right)
\]
\[
\leq 2\left( \delta \int_{k\delta}^t \mathbb{E} \left| b_1(X_s^\varepsilon, \mathcal{L}_s^\xi, Z_s^\varepsilon, Z_s^\varepsilon, \mathcal{L}_s^\xi, \mathcal{L}_s^\xi) \right|^2 ds + \int_{k\delta}^t \mathbb{E} \left( \sigma_1(X_s^\varepsilon, \mathcal{L}_s^\xi) \right)^2 ds \right)
\]
\[
\leq C(\delta^2 + \delta),
\] (18)
where the last inequality is based on Lemma 4.1

Moreover, by the same deduction to that for $Z_t^\varepsilon,\xi, Z_t^\varepsilon,z_0,\mathcal{L}_t^\xi$ in Lemma 4.1, we obtain the following estimate.

**Lemma 4.2.** Under these assumptions $(H_{b_2,\sigma_2}^1)$-$(H_{b_2,\sigma_2}^2)$, it holds that
\[
\sup_{t \in [0,T]} \mathbb{E}|\hat{Z}_t^\varepsilon,\xi|^2 \leq C(1 + \mathbb{E}|\varrho|^2 + |z_0|^2 + \mathbb{E}|\xi|^2),
\]
\[
\sup_{t \in [0,T]} \mathbb{E}|\hat{Z}_t^\varepsilon,z_0,\mathcal{L}_t^\xi|^2 \leq C(1 + \mathbb{E}|\varrho|^2 + |z_0|^2 + \mathbb{E}|\xi|^2).
\]

**Lemma 4.3.** Suppose $(H_{b_1,\sigma_1}^1)$, $(H_{b_2,\sigma_2}^1)$, $(H_{b_2,\sigma_2}^2)$ hold. Then for $p \geq 1$, there exists a constant $C > 0$ such that
\[
\sup_{t \in [0,T]} \mathbb{E}|Z_t^\varepsilon,\xi - \hat{Z}_t^\varepsilon,\xi|^2 \leq \frac{C}{\beta_1 - \beta_2 - L_{b_2,\sigma_2}}(\delta^2 + \delta),
\]
\[
\sup_{t \in [0,T]} \mathbb{E}|Z_t^\varepsilon,z_0,\mathcal{L}_t^\xi - \hat{Z}_t^\varepsilon,z_0,\mathcal{L}_t^\xi|^2 \leq \frac{C}{\beta_1 - L_{b_2,\sigma_2}}(\delta^2 + \delta).
\]

9
Applying the Itô formula to $|Z_t^\varepsilon - \hat{Z}_t^\varepsilon|^2$ and taking the expectation, by (H$_{b_2,\sigma_2}$) one could obtain that

$$
\mathbb{E}|Z_t^\varepsilon - \hat{Z}_t^\varepsilon|^2
= \frac{1}{\varepsilon} \mathbb{E} \int_0^t 2\langle Z_s^\varepsilon, \hat{Z}_s^\varepsilon, b(X_s^\varepsilon, \mathcal{L}_X^\varepsilon, Z_s^\varepsilon, \mathcal{L}_Z^\varepsilon) \rangle - 2\langle Z_s^\varepsilon, \hat{Z}_s^\varepsilon, b(X_s^\varepsilon, \mathcal{L}_X^\varepsilon, \hat{Z}_s^\varepsilon, \mathcal{L}_Z^\varepsilon) \rangle \rangle \rangle ds
+ \frac{1}{\varepsilon} \mathbb{E} \int_0^t \|\sigma_2(X_s^\varepsilon, \mathcal{L}_X^\varepsilon, Z_s^\varepsilon, \mathcal{L}_Z^\varepsilon) - \sigma_2(X_s^\varepsilon, \mathcal{L}_X^\varepsilon, \hat{Z}_s^\varepsilon, \mathcal{L}_Z^\varepsilon)\|^2 ds
$$

and

$$
\frac{d\mathbb{E}|Z_t^\varepsilon - \hat{Z}_t^\varepsilon|^2}{dt}
\leq \frac{1}{\varepsilon} \mathbb{E} \left[ 2\langle Z_t^\varepsilon - \hat{Z}_t^\varepsilon, b(X_t^\varepsilon, \mathcal{L}_X^\varepsilon, Z_t^\varepsilon, \mathcal{L}_Z^\varepsilon) \rangle - 2\langle Z_t^\varepsilon - \hat{Z}_t^\varepsilon, b(X_t^\varepsilon, \mathcal{L}_X^\varepsilon, \hat{Z}_t^\varepsilon, \mathcal{L}_Z^\varepsilon) \rangle \rangle \rangle \right]
+ (2p + 1)\mathbb{E} \left[ 2\langle Z_t^\varepsilon - \hat{Z}_t^\varepsilon, b(X_t^\varepsilon, \mathcal{L}_X^\varepsilon, \hat{Z}_t^\varepsilon, \mathcal{L}_Z^\varepsilon) \rangle - 2\langle X_t^\varepsilon, \mathcal{L}_X^\varepsilon, \hat{Z}_t^\varepsilon, \mathcal{L}_Z^\varepsilon \rangle \rangle \rangle \right]
\leq \frac{1}{\varepsilon} \mathbb{E} \left[ - (\beta_1 - L_{b_2,\sigma_2}) |Z_t^\varepsilon - \hat{Z}_t^\varepsilon|^2 + \beta_2 \mathbb{E} \mathbb{W}_2^2(\mathcal{L}_Z^\varepsilon, \mathcal{L}_Z^\varepsilon) \right]
+ \frac{C}{\varepsilon} \mathbb{E} \left( |X_t^\varepsilon - X_{t(\delta)}^\varepsilon|^2 \right)
\leq \frac{- (\beta_1 - \beta_2 - L_{b_2,\sigma_2})}{\varepsilon} \mathbb{E}|Z_t^\varepsilon - \hat{Z}_t^\varepsilon|^2 + \frac{C}{\varepsilon} \mathbb{E}|X_t^\varepsilon - X_{t(\delta)}^\varepsilon|^2.
$$

Then it follows from the comparison theorem and (18) that

$$
\mathbb{E}|Z_t^\varepsilon - \hat{Z}_t^\varepsilon|^2 \leq \frac{C}{\varepsilon} (\delta^2 + \delta) \int_0^t e^{- \frac{(\beta_1 - \beta_2 - L_{b_2,\sigma_2})}{\varepsilon}(t-s)} ds \leq \frac{C}{\beta_1 - \beta_2 - L_{b_2,\sigma_2}} (\delta^2 + \delta).
$$

Next, by the similar deduction to the above, we have that

$$
\mathbb{E}|Z_t^{\varepsilon, x_0, \mathcal{L}_X^\varepsilon} - \hat{Z}_t^{\varepsilon, x_0, \mathcal{L}_X^\varepsilon}|^2 \leq \frac{C}{\beta_1 - L_{b_2,\sigma_2}} (\delta^2 + \delta).
$$

The proof is complete. 

Finally, we estimate $\mathbb{E} \sup_{t \in [0,T]} |X_t^\varepsilon - \hat{X}_t^\varepsilon|^2$. By (2) (14), it holds that

$$
\mathbb{E} \sup_{t \in [0,T]} |X_t^\varepsilon - \hat{X}_t^\varepsilon|^2
$$

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Lemma 4.4. Moreover, about \( Z \) \( \in P \), \( \mu \), \( \langle x, \mu, \xi \rangle \), \( \sigma_2(x, \mu, Z^{x, \mu, 0, \xi}_t, L^p_{Z^x, \xi})dW_t, \)
\begin{equation}
\left\{ \begin{array}{l}
dZ_t^{x, \mu, 0, \xi} = b_2(x, \mu, Z_t^{x, \mu, 0, \xi}, L^p_{Z^x, \xi})dt + \sigma_2(x, \mu, Z_t^{x, \mu, 0, \xi}, L^p_{Z^x, \xi})dW_t, \\
Z_0^{x, \mu, 0, \xi} = z_0, \quad 0 \leq t \leq T.
\end{array} \right.
\end{equation}
Under \((H^1_2, \sigma_2)\), we know that the above equation has a unique strong solution \( Z^{x, \mu, 0, \xi} \).
Moreover, about \( Z^{x, \mu, 0, \xi} \), we have the following estimates.

Lemma 4.4. Under these assumptions \((H^1_2, \sigma_2)\), it holds that for any \( t \in [0, T], x \in \mathbb{R}^n, \mu \in P_2(\mathbb{R}^n) \)
\begin{equation}
E|Z_t^{x, \mu, 0, \xi}|^2 \leq |z_0|^2 e^{-\alpha_1 t} + C(1 + |x|^2 + \|\mu\|^2).
\end{equation}
Proof. First of all, we estimate \( Z^{x, \mu, \xi} \). Applying the Itô formula to \( |Z_t^{x, \mu, \xi}|^2 \) and taking the expectation, we get that
\begin{align*}
E|Z_t^{x, \mu, \xi}|^2 &= E|\xi|^2 + 2E \int_0^t \langle Z_s^{x, \mu, \xi}, b_2(x, \mu, Z_s^{x, \mu, \xi}, L^p_{Z^x, \xi}) \rangle ds \\
&\quad + E \int_0^t \|\sigma_2(x, \mu, Z_s^{x, \mu, \xi}, L^p_{Z^x, \xi})\|^2 ds,
\end{align*}
and
\begin{align*}
\frac{d}{dt} E|Z_t^{x, \mu, \xi}|^2 &= E\left( 2 \langle Z_t^{x, \mu, \xi}, b_2(x, \mu, Z_t^{x, \mu, \xi}, L^p_{Z^x, \xi}) \rangle + \|\sigma_2(x, \mu, Z_t^{x, \mu, \xi}, L^p_{Z^x, \xi})\|^2 \right) \\
&\leq E\left( -\alpha_1 |Z_t^{x, \mu, \xi}|^2 + \alpha_2 \|L^p_{Z^x, \xi}\|^2 + C(1 + |x|^2 + \|\mu\|^2) \right) \\
&= -(\alpha_1 - \alpha_2) E|Z_t^{x, \mu, \xi}|^2 + C(1 + |x|^2 + \|\mu\|^2).
\end{align*}
By the comparison theorem, it holds that
\begin{align*}
E|Z_t^{x, \mu, \xi}|^2 &\leq E|\xi|^2 e^{-\alpha_1 t} + C(1 + |x|^2 + \|\mu\|^2) \int_0^t e^{-\alpha_1 (t-s)} ds \\
&\leq E|\xi|^2 e^{-\alpha_1 t} + C(1 + |x|^2 + \|\mu\|^2).
\end{align*}
Next, we deal with \( Z^{x, \mu, 0, \xi} \). By the same deduction to the above, it holds that
\begin{equation}
E|Z_t^{x, \mu, 0, \xi}|^2 \leq |z_0|^2 e^{-\alpha_1 t} + C(1 + |x|^2 + \|\mu\|^2).
\end{equation}
The proof is complete. \( \square \)
Lemma 4.5. Suppose that $(H_{b,\sigma})$ $(H_{b,\sigma}^2)$ hold. Then it holds that for any $x_i \in \mathbb{R}^n, \mu_i \in P_2(\mathbb{R}^n), z_i \in \mathbb{R}^m, \zeta_i \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^m), i = 1, 2,$

$$
\mathbb{E}[Z_{t}^{x_1, \mu_1, \zeta_1, \mathcal{L}_{\zeta_1}^P} - Z_{t}^{x_2, \mu_2, \zeta_2, \mathcal{L}_{\zeta_2}^P}] \\
\leq |z_1 - z_2|^2 e^{-((\beta_1 - L_{b_2, \sigma_2})t) + \mathbb{E}[\zeta_1 - \zeta_2]^2 (e^{\beta_2t} - 1) e^{-((\beta_1 - L_{b_2, \sigma_2})t)}t} \\
+ C(|x_1 - x_2|^2 + \mathbb{W}_2^2(\mu_1, \mu_2)) \frac{1 - e^{-((\beta_1 - L_{b_2, \sigma_2})t)}}{\beta_1 - L_{b_2, \sigma_2}}.
$$

Proof. First of all, we compute $\mathbb{E}[|Z_t^{x_1, \mu_1, \zeta_1} - Z_t^{x_2, \mu_2, \zeta_2}|^2]$. Note that $Z_t^{x_1, \mu_1, \zeta_1}$ and $Z_t^{x_2, \mu_2, \zeta_2}$ solve Eq. (7) with initial values $\zeta_1$ and $\zeta_2$, respectively, i.e.

$$
Z_t^{x_1, \mu_1, \zeta_1} - Z_t^{x_2, \mu_2, \zeta_2} = \zeta_1 - \zeta_2 + \int_0^t \left( b_2(x_1, \mu_1, Z_s^{x_1, \mu_1, \zeta_1}, \mathcal{L}_{Z_s^{x_1, \mu_1, \zeta_1}}^P) - b_2(x_2, \mu_2, Z_s^{x_2, \mu_2, \zeta_2}, \mathcal{L}_{Z_s^{x_2, \mu_2, \zeta_2}}^P) \right) ds \\
+ \int_0^t \left( \sigma_2(x_1, \mu_1, Z_s^{x_1, \mu_1, \zeta_1}, \mathcal{L}_{Z_s^{x_1, \mu_1, \zeta_1}}^P) - \sigma_2(x_2, \mu_2, Z_s^{x_2, \mu_2, \zeta_2}, \mathcal{L}_{Z_s^{x_2, \mu_2, \zeta_2}}^P) \right) dW_s.
$$

Applying the Itô formula to $|Z_t^{x_1, \mu_1, \zeta_1} - Z_t^{x_2, \mu_2, \zeta_2}|^2$ and taking expectation on two sides, we obtain that

$$
\mathbb{E}[|Z_t^{x_1, \mu_1, \zeta_1} - Z_t^{x_2, \mu_2, \zeta_2}|^2] = \mathbb{E}[|\zeta_1 - \zeta_2|^2] + 2 \mathbb{E} \int_0^t (Z_s^{x_1, \mu_1, \zeta_1} - Z_s^{x_2, \mu_2, \zeta_2}, b_2(x_1, \mu_1, Z_s^{x_1, \mu_1, \zeta_1}, \mathcal{L}_{Z_s^{x_1, \mu_1, \zeta_1}}^P) - b_2(x_2, \mu_2, Z_s^{x_2, \mu_2, \zeta_2}, \mathcal{L}_{Z_s^{x_2, \mu_2, \zeta_2}}^P)) ds \\
+ \mathbb{E} \int_0^t \|\sigma_2(x_1, \mu_1, Z_s^{x_1, \mu_1, \zeta_1}, \mathcal{L}_{Z_s^{x_1, \mu_1, \zeta_1}}^P) - \sigma_2(x_2, \mu_2, Z_s^{x_2, \mu_2, \zeta_2}, \mathcal{L}_{Z_s^{x_2, \mu_2, \zeta_2}}^P)\|^2 ds,
$$

and

$$
\frac{d}{dt} \mathbb{E}[|Z_t^{x_1, \mu_1, \zeta_1} - Z_t^{x_2, \mu_2, \zeta_2}|^2] = \mathbb{E} \left( 2(Z_t^{x_1, \mu_1, \zeta_1} - Z_t^{x_2, \mu_2, \zeta_2}, b_2(x_1, \mu_1, Z_t^{x_1, \mu_1, \zeta_1}, \mathcal{L}_{Z_t^{x_1, \mu_1, \zeta_1}}^P) - b_2(x_2, \mu_2, Z_t^{x_2, \mu_2, \zeta_2}, \mathcal{L}_{Z_t^{x_2, \mu_2, \zeta_2}}^P)) \\
+ \mathbb{E}[\|\sigma_2(x_1, \mu_1, Z_t^{x_1, \mu_1, \zeta_1}, \mathcal{L}_{Z_t^{x_1, \mu_1, \zeta_1}}^P) - \sigma_2(x_2, \mu_2, Z_t^{x_2, \mu_2, \zeta_2}, \mathcal{L}_{Z_t^{x_2, \mu_2, \zeta_2}}^P)\|^2] \\
\leq \mathbb{E} \left( 2(Z_t^{x_1, \mu_1, \zeta_1} - Z_t^{x_2, \mu_2, \zeta_2}, b_2(x_1, \mu_1, Z_t^{x_1, \mu_1, \zeta_1}, \mathcal{L}_{Z_t^{x_1, \mu_1, \zeta_1}}^P) - b_2(x_2, \mu_2, Z_t^{x_2, \mu_2, \zeta_2}, \mathcal{L}_{Z_t^{x_2, \mu_2, \zeta_2}}^P) \\
+ (2p + 1) \mathbb{E}[\|\sigma_2(x_1, \mu_1, Z_t^{x_1, \mu_1, \zeta_1}, \mathcal{L}_{Z_t^{x_1, \mu_1, \zeta_1}}^P) - \sigma_2(x_2, \mu_2, Z_t^{x_2, \mu_2, \zeta_2}, \mathcal{L}_{Z_t^{x_2, \mu_2, \zeta_2}}^P)\|^2] \\
+ \mathbb{E}\left( 2(Z_t^{x_1, \mu_1, \zeta_1} - Z_t^{x_2, \mu_2, \zeta_2}, \sigma_2(x_1, \mu_1, Z_t^{x_2, \mu_2, \zeta_2}, \mathcal{L}_{Z_t^{x_2, \mu_2, \zeta_2}}^P) - \sigma_2(x_2, \mu_2, Z_t^{x_2, \mu_2, \zeta_2}, \mathcal{L}_{Z_t^{x_2, \mu_2, \zeta_2}}^P) \\
+ (2p + 1) \mathbb{E}[\|\sigma_2(x_1, \mu_1, Z_t^{x_2, \mu_2, \zeta_2}, \mathcal{L}_{Z_t^{x_2, \mu_2, \zeta_2}}^P) - \sigma_2(x_2, \mu_2, Z_t^{x_2, \mu_2, \zeta_2}, \mathcal{L}_{Z_t^{x_2, \mu_2, \zeta_2}}^P)\|^2] \\
\leq \mathbb{E} \left( (\beta_1 - L_{b_2, \sigma}) |Z_t^{x_1, \mu_1, \zeta_1} - Z_t^{x_2, \mu_2, \zeta_2}|^2 + \beta_2 \mathbb{W}_2^2(\mathcal{L}_{Z_t^{x_1, \mu_1, \zeta_1}}^P, \mathcal{L}_{Z_t^{x_2, \mu_2, \zeta_2}}^P) \\
+ C(|x_1 - x_2|^2 + \mathbb{W}_2^2(\mu_1, \mu_2)) \\
\leq - (\beta_1 - \beta_2 - L_{b_2, \sigma}) \mathbb{E}[|Z_t^{x_1, \mu_1, \zeta_1} - Z_t^{x_2, \mu_2, \zeta_2}|^2 + C(|x_1 - x_2|^2 + \mathbb{W}_2^2(\mu_1, \mu_2)).
$$
By the comparison theorem, it holds that
\[ \mathbb{E}|Z_t^{x_1, \mu_1, \xi_1} - Z_t^{x_2, \mu_2, \xi_2}|^2 \leq \mathbb{E}|\xi_1 - \xi_2|^2 e^{-(\beta_1 - \beta_2 - L_{b_2, \sigma_2}) t} + C(|x_1 - x_2|^2 + \mathbb{W}_2^2(\mu_1, \mu_2)). \]

Next, we investigate \( \mathbb{E}|Z_t^{x_1, \mu_1, z_1, \xi_1} - Z_t^{x_2, \mu_2, z_2, \xi_2}|^2 \). The Itô formula yields
\[
\mathbb{E}|Z_t^{x_1, \mu_1, z_1, \xi_1} - Z_t^{x_2, \mu_2, z_2, \xi_2}|^2 \leq |z_1 - z_2|^2 + \lambda \mathbb{E} \int_0^t e^{\lambda s} |Z_s^{x_1, \mu_1, z_1, \xi_1} - Z_s^{x_2, \mu_2, z_2, \xi_2}|^2 \, ds \\
+ 2E \int_0^t e^{\lambda s} \langle Z_s^{x_1, \mu_1, z_1, \xi_1}, Z_s^{x_2, \mu_2, z_2, \xi_2}, Z_s^{x_1, \mu_1, z_1, \xi_1} \rangle \, ds \\
- b_2(x_2, \mu_2, Z_s^{x_2, \mu_2, z_2, \xi_2}, \xi_2) \rangle \\
+ \mathbb{E} \int_0^t e^{\lambda s} \|\sigma_2(x_1, \mu_1, Z_s^{x_1, \mu_1, z_1, \xi_1}, \xi_2) - \sigma_2(x_2, \mu_2, Z_s^{x_2, \mu_2, z_2, \xi_2}, \xi_2)\|^2 \, ds \\
\leq |z_1 - z_2|^2 + \beta_2 \mathbb{E} \int_0^t e^{\lambda s} e^{-(\beta_1 - \beta_2 - L_{b_2, \sigma_2}) s} ds + C(|x_1 - x_2|^2 + \mathbb{W}_2^2(\mu_1, \mu_2)) \int_0^t e^{\lambda s} ds \\
\leq |z_1 - z_2|^2 + \beta_2 \mathbb{E} \int_0^t e^{\lambda s} e^{-(\beta_1 - \beta_2 - L_{b_2, \sigma_2}) s} ds + C(|x_1 - x_2|^2 + \mathbb{W}_2^2(\mu_1, \mu_2)) \int_0^t e^{\lambda s} ds,
\]
where \( \lambda := \beta_1 - L_{b_2, \sigma_2} \). Then simple calculation implies that
\[
\mathbb{E}|Z_t^{x_1, \mu_1, z_1, \xi_1} - Z_t^{x_2, \mu_2, z_2, \xi_2}|^2 \leq |z_1 - z_2|^2 e^{-(\beta_1 - \beta_2 - L_{b_2, \sigma_2}) t} + \mathbb{E}|\xi_1 - \xi_2|^2 e^{\beta_2 t} - 1 - e^{-(\beta_1 - L_{b_2, \sigma_2}) t} \\
+ C(|x_1 - x_2|^2 + \mathbb{W}_2^2(\mu_1, \mu_2)) \frac{1 - e^{-(\beta_1 - L_{b_2, \sigma_2}) t}}{\beta_1 - L_{b_2, \sigma_2}},
\]
which completes the proof. \( \square \)

**Lemma 4.6.** Suppose that \( (H_{b_1, \sigma_1}^1, H_{b_2, \sigma_2}^1, H_{b_2, \sigma_2}^2) \) hold. Then there exists a constant \( C > 0 \) such that for any \( t \in [0, T] \), \( x \in \mathbb{R}^n \), \( \mu \in \mathcal{P}_2(\mathbb{R}^n) \)
\[
|\mathbb{E}b_1(x, \mu, Z_t^{x, \mu, z_0, \xi}, Z_t^{x, \mu, \xi}) - \tilde{b}_1(x, \mu)|^2 \leq Ce^{-(\beta_1 - L_{b_2, \sigma_2}) t}(\|\xi\| + 1 + |x| + \|\mu\| + |z_0|)^2. \tag{23}
\]

**Proof.** Based on (\( H_{b_1, \sigma_1}^1 \)) and Lemma 4.5, one can obtain that
\[
|\mathbb{E}b_1(x, \mu, Z_t^{x, \mu, z_0, \xi}, Z_t^{x, \mu, \xi}) - \tilde{b}_1(x, \mu)|^2 = |\mathbb{E}b_1(x, \mu, Z_t^{x, \mu, z_0, \xi}, Z_t^{x, \mu, \xi}) - \int_{\mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n)} b_1(x, \mu, y, \nu) \eta^{x, \mu} \times \delta_{y, \nu} (dy, d\nu)|^2
\]
\[
= |\mathbb{E}b_1(x, \mu, Z_t^{x, \mu, y, \nu}, Z_t^{x, \mu, \xi}) \eta^{x, \mu} \times \delta_{y, \nu} (dy, d\nu)|^2.
\]
The proof is complete.

4.3. Some estimates for the average equation (8).

Lemma 4.7. Suppose that \((H_{b_1,σ_1}) (H_{b_2,σ_2}) (H_{b_3,σ_2})\) hold. Then Eq. (8) has a unique strong solution \(X\). Moreover,

\[
\mathbb{E} \left( \sup_{t \in [0, T]} |X_t|^{2p+2} \right) \leq C (1 + \mathbb{E}|\theta|^{2p+2}).
\]  

(24)

Proof. First of all, we justify that for any \(x_i \in \mathbb{R}^n, \mu_i \in \mathcal{P}_2(\mathbb{R}^n), i = 1, 2\)

\[
|\tilde{b}_1(x_1, \mu_1) - \tilde{b}_1(x_2, \mu_2)|^2 \leq L_{\tilde{b}_1} \left( |x_1 - x_2|^2 + \mathbb{W}_2^2(\mu_1, \mu_2) \right),
\]

where \(L_{\tilde{b}_1} > 0\) is a constant. Indeed, by Lemma 4.5, Lemma 4.6 and \((H_{b_1,σ_1})\), it holds that

\[
|\tilde{b}_1(x_1, \mu_1) - \tilde{b}_1(x_2, \mu_2)^2| 
\leq 3|\tilde{b}_1(x_1, \mu_1) - \mathbb{E}b_1(x_1, \mu_1, Z_t^{x_1,μ_1,ζ_0} : \mathcal{L}^p_{Z_t^{x_1,μ_1,ζ_0}})|^2 
\]  

\[+ 3|\tilde{b}_1(x_2, \mu_2) - \mathbb{E}b_1(x_2, \mu_2, Z_t^{x_2,\mu_2,ζ_0} : \mathcal{L}^p_{Z_t^{x_2,\mu_2,ζ_0}})|^2 
\]  

\[+ 3|\mathbb{E}b_1(x_1, \mu_1, Z_t^{x_1,μ_1,ζ_0} : \mathcal{L}^p_{Z_t^{x_1,μ_1,ζ_0}}) - b_1(x_2, \mu_2, Z_t^{x_2,\mu_2,ζ_0} : \mathcal{L}^p_{Z_t^{x_2,\mu_2,ζ_0}})|^2 
\]  

\[\leq C e^{-(\beta_1 - L_{b_2,σ_2})t} (1 + |x_1|^2 + |x_2|^2 + \|\mu_1\|^2 + \|\mu_2\|^2 + |\zeta_0|^2 + \|\mathcal{L}^p_{Z_t^{x_1,μ_1,ζ_0}}\|^2) 
\]  

\[+ C \left( |x_1 - x_2|^2 + \mathbb{W}_2^2(\mu_1, \mu_2) \right),
\]

which implies the required result as \(t \to \infty\). Thus, from [24, Theorem 2.1] it follows that Eq. (8) has a unique strong solution \(X\).

Next, by similar deduction to that for \(X^\varepsilon\) in Lemma 4.1, we have (24). The proof is complete. 

\[\square\]

Lemma 4.8. Suppose that assumptions \((H_{b_1,σ_1}) (H_{b_2,σ_2}) (H_{b_3,σ_2})\) hold. Then there exists a constant \(C > 0\) such that

\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} |\hat{X}_t^\varepsilon - \bar{X}_t|^2 \right) \leq C \left( \frac{\varepsilon}{\delta} + \delta^2 + \delta \right).
\]

Proof. Step 1. We estimate \(\hat{X}_t^\varepsilon - \bar{X}_t\).
Note that
\[
\hat{X}_t^\varepsilon - X_t = \int_0^t \left( b_1(X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}, Z_s^\varepsilon, \mathcal{L}_Z^\varepsilon) - \bar{b}_1(X_s, \mathcal{L}_{X_s}) \right) ds + \int_0^t \left( \sigma_1(X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}) - \sigma_1(X_s, \mathcal{L}_{X_s}) \right) dB_s.
\]

Thus, based on the BDG inequality, we get that
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} |\hat{X}_t^\varepsilon - X_t|^2 \right) \leq 2 \left( \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t \left( b_1(X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}, Z_s^\varepsilon, \mathcal{L}_Z^\varepsilon) - \bar{b}_1(X_s, \mathcal{L}_{X_s}) \right) ds \right|^2 \right) + \mathbb{E} \left( \sup_{0 \leq t \leq T} \left| \int_0^t \left( \sigma_1(X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}) - \sigma_1(X_s, \mathcal{L}_{X_s}) \right) dB_s \right|^2 \right)
\]
\[
\leq 6 \left( \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t \left( b_1(X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}, Z_s^\varepsilon, \mathcal{L}_Z^\varepsilon) - \bar{b}_1(X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}) \right) ds \right|^2 \right) + 6 \left( \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t \left( \bar{b}_1(X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}) - b_1(X_s^\varepsilon, \mathcal{L}_{X_s}) \right) ds \right|^2 \right)
\]
\[
+ 6 \left( \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t \left( \bar{b}_1(X_s^\varepsilon, \mathcal{L}_{X_s}) - \bar{b}_1(X_s, \mathcal{L}_{X_s}) \right) ds \right|^2 \right) + 8 \int_0^T \mathbb{E} \left( \left| \sigma_1(X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}) - \sigma_1(X_s, \mathcal{L}_{X_s}) \right|^2 \right) ds.
\]

Then from the Hölder inequality and \((H_{b_1, \sigma})\), it follows that
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} |\hat{X}_t^\varepsilon - X_t|^2 \right) \leq 6 \left( \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t \left( b_1(X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}, Z_s^\varepsilon, \mathcal{L}_Z^\varepsilon) - \bar{b}_1(X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}) \right) ds \right|^2 \right) + 6T \left( \mathbb{E} \sup_{0 \leq t \leq T} \int_0^t L_{b_1} \left( |X_s^\varepsilon - X_s| + |X_s| \right) ds \right)
\]
\[
+ 6T \left( \mathbb{E} \sup_{0 \leq t \leq T} \int_0^t L_{b_1} \left( |X_s^\varepsilon - \bar{X}_s|^2 \right) ds \right) + 8 \int_0^T \mathbb{E} \left( \left| X_s^\varepsilon - \bar{X}_s \right|^2 \right) ds
\]
\[
\leq 6 \left( \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t \left( b_1(X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}, Z_s^\varepsilon, \mathcal{L}_Z^\varepsilon) - \bar{b}_1(X_s^\varepsilon, \mathcal{L}_{X_s^\varepsilon}) \right) ds \right|^2 \right) + 12TL_{b_1} \int_0^T \mathbb{E} |X_s^\varepsilon - X_s|^2 ds + (12TL_{b_1} + 16L_{b_1, \sigma}) \int_0^T \mathbb{E} |X_s|^2 ds
\]
\[
\begin{align*}
\leq 6 \left( \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t \left( b_1(X_{s(\delta)}^\varepsilon, \mathcal{L}_{X_0}^\varepsilon, \hat{Z}_s^{\varepsilon, z_0, \mathcal{M}_t^\varepsilon}, \mathcal{L}_{\hat{Z}_s}^\varepsilon) - \bar{b}_1(X_{s(\delta)}^\varepsilon, \mathcal{L}_{X_0}^\varepsilon) \right) ds \right|^2 \right) \\
+ 12TLb_1 \int_0^T \mathbb{E} \left| X_s^\varepsilon - X_{s(\delta)}^\varepsilon \right|^2 ds + 2(12TLb_1 + 16Lb_{1, \sigma_1}) \int_0^T \mathbb{E} \left| \hat{X}_s^\varepsilon - \hat{X}_s \right|^2 ds \\
+ 2(12TLb_1 + 16Lb_{1, \sigma_1}) \int_0^T \mathbb{E} \left| \hat{X}_s^\varepsilon - \hat{X}_s \right|^2 ds \\
=: I_1 + I_2 + I_3 + 2(12TLb_1 + 16Lb_{1, \sigma_1}) \int_0^T \mathbb{E} \left| \hat{X}_s^\varepsilon - \hat{X}_s \right|^2 ds.
\end{align*}
\]

By the deduction in Step 2, we know that
\[
I_1 \leq C \left( \frac{\varepsilon}{\delta} + \delta \right).
\] (26)

And (18) (19) imply that
\[
I_2 + I_3 \leq C(\delta^2 + \delta).
\] (27)

Thus, inserting (26), (27) in (25), we obtain that
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} \left| \hat{X}_t^\varepsilon - \hat{X}_t \right|^2 \right) \leq C \left( \frac{\varepsilon}{\delta} + \delta \right) + C(\delta^2 + \delta) + C \int_0^T \mathbb{E} \left( \sup_{0 \leq r \leq s} \left| \hat{X}_r^\varepsilon - \hat{X}_r \right|^2 \right) ds.
\]

By the Gronwall inequality, it holds that
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} \left| \hat{X}_t^\varepsilon - \hat{X}_t \right|^2 \right) \leq C \left( \frac{\varepsilon}{\delta} + \delta^2 + \delta \right).
\]

Step 2. We prove (26).

For any \( t \in [0, T] \), it holds that
\[
I_1 = 6 \left( \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^{t \Delta} \left( b_1(X_{s(\delta)}^\varepsilon, \mathcal{L}_{X_0}^\varepsilon, \hat{Z}_s^{\varepsilon, z_0, \mathcal{M}_t^\varepsilon}, \mathcal{L}_{\hat{Z}_s}^\varepsilon) - \bar{b}_1(X_{s(\delta)}^\varepsilon, \mathcal{L}_{X_0}^\varepsilon) \right) ds \right|^2 \right)
\]
\[
+ \int_{\frac{t}{2} \Delta}^{t \Delta} \left( b_1(X_{s(\delta)}^\varepsilon, \mathcal{L}_{X_0}^\varepsilon, \hat{Z}_s^{\varepsilon, z_0, \mathcal{M}_t^\varepsilon}, \mathcal{L}_{\hat{Z}_s}^\varepsilon) - \bar{b}_1(X_{s(\delta)}^\varepsilon, \mathcal{L}_{X_0}^\varepsilon) \right) ds \right|^2 \right)
\]
\[
\leq 12 \left( \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^{\frac{t}{2} \Delta} \left( b_1(X_{s(\delta)}^\varepsilon, \mathcal{L}_{X_0}^\varepsilon, \hat{Z}_s^{\varepsilon, z_0, \mathcal{M}_t^\varepsilon}, \mathcal{L}_{\hat{Z}_s}^\varepsilon) - \bar{b}_1(X_{s(\delta)}^\varepsilon, \mathcal{L}_{X_0}^\varepsilon) \right) ds \right|^2 \right)
\]
\[
+ 12 \left( \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_{\frac{t}{2} \Delta}^{t \Delta} \left( b_1(X_{s(\delta)}^\varepsilon, \mathcal{L}_{X_0}^\varepsilon, \hat{Z}_s^{\varepsilon, z_0, \mathcal{M}_t^\varepsilon}, \mathcal{L}_{\hat{Z}_s}^\varepsilon) - \bar{b}_1(X_{s(\delta)}^\varepsilon, \mathcal{L}_{X_0}^\varepsilon) \right) ds \right|^2 \right)
\]
\[
=: B_1 + B_2.
\] (28)

Next, we estimate \( B_1 \). Note that
\[
B_1 = 12 \left( \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^{\frac{t}{2} \Delta - 1} \left( b_1(X_{s(\delta)}^\varepsilon, \mathcal{L}_{X_0}^\varepsilon, \hat{Z}_s^{\varepsilon, z_0, \mathcal{M}_t^\varepsilon}, \mathcal{L}_{\hat{Z}_s}^\varepsilon) - \bar{b}_1(X_{s(\delta)}^\varepsilon, \mathcal{L}_{X_0}^\varepsilon) \right) ds \right|^2 \right)
\]
\[
= 12 \mathbb{E} \left( \sup_{0 \leq t \leq T} \left| \sum_{k=0}^{\frac{t}{\Delta}-1} \int_{k \Delta}^{(k+1) \Delta} \left( b_1(X_{s(\delta)}^\varepsilon, \mathcal{L}_{X_0}^\varepsilon, \hat{Z}_s^{\varepsilon, z_0, \mathcal{M}_t^\varepsilon}, \mathcal{L}_{\hat{Z}_s}^\varepsilon) - \bar{b}_1(X_{s(\delta)}^\varepsilon, \mathcal{L}_{X_0}^\varepsilon) \right) ds \right|^2 \right)
\]
Then it holds that
\[ \frac{1}{\delta} \sum_{k=0}^{\lfloor \frac{t}{\delta} \rfloor - 1} \mathbb{E} \left( \left| \int_{k\delta}^{(k+1)\delta} \left( b_t(X_{k\delta}^\varepsilon, \mathcal{L}_{k\delta}^\varepsilon, \hat{Z}^{\varepsilon,20,\xi}_{k\delta} , \mathcal{L}_{\hat{Z}^{\xi,0}}_{k\delta} ) - \bar{b}_t(X_{k\delta}^\varepsilon, \mathcal{L}_{k\delta}^\varepsilon) \right) ds \right|^2 \right) \]
\[ \leq 12 \mathbb{E} \left( \sup_{0 \leq t \leq T} \left\{ \int_{k\delta}^{(k+1)\delta} \left( b_t(X_{k\delta}^\varepsilon, \mathcal{L}_{k\delta}^\varepsilon, \hat{Z}^{\varepsilon,20,\xi}_{k\delta} , \mathcal{L}_{\hat{Z}^{\xi,0}}_{k\delta} ) - \bar{b}_t(X_{k\delta}^\varepsilon, \mathcal{L}_{k\delta}^\varepsilon) \right) ds \right\}^2 \right) \]
\[ \leq 12 \left( \frac{T}{\delta} \right)^2 \sup_{0 \leq t \leq \left\lfloor \frac{T}{\delta} \right\rfloor - 1} \mathbb{E} \left( \left| \int_{k\delta}^{(k+1)\delta} \left( b_t(X_{k\delta}^\varepsilon, \mathcal{L}_{k\delta}^\varepsilon, \hat{Z}^{\varepsilon,20,\xi}_{k\delta} , \mathcal{L}_{\hat{Z}^{\xi,0}}_{k\delta} ) - \bar{b}_t(X_{k\delta}^\varepsilon, \mathcal{L}_{k\delta}^\varepsilon) \right) ds \right|^2 \right) \]
\[ \leq 12 \varepsilon^2 \left( \frac{T}{\delta} \right)^2 \sup_{0 \leq t \leq \left\lfloor \frac{T}{\delta} \right\rfloor - 1} \mathbb{E} \left( \left| \int_{0}^{\delta/\varepsilon} \left( b_t(X_{k\delta}^\varepsilon, \mathcal{L}_{k\delta}^\varepsilon, \hat{Z}^{\varepsilon,20,\xi}_{k\delta} , \mathcal{L}_{\hat{Z}^{\xi,0}}_{k\delta} ) - \bar{b}_t(X_{k\delta}^\varepsilon, \mathcal{L}_{k\delta}^\varepsilon) \right) ds \right|^2 \right) \]
\[ = 12 \varepsilon^2 \left( \frac{T}{\delta} \right)^2 \sup_{0 \leq t \leq \left\lfloor \frac{T}{\delta} \right\rfloor - 1} \mathbb{E} \left( \left| \int_{0}^{\delta/\varepsilon} \left( b_t(X_{k\delta}^\varepsilon, \mathcal{L}_{k\delta}^\varepsilon, \hat{Z}^{\varepsilon,20,\xi}_{k\delta} , \mathcal{L}_{\hat{Z}^{\xi,0}}_{k\delta} ) - \bar{b}_t(X_{k\delta}^\varepsilon, \mathcal{L}_{k\delta}^\varepsilon) \right) ds \right|^2 \right) \]

\[ \Phi(s, r) := \mathbb{E} b_t(X_{k\delta}^\varepsilon, \mathcal{L}_{k\delta}^\varepsilon, \hat{Z}^{\varepsilon,20,\xi}_{k\delta} , \mathcal{L}_{\hat{Z}^{\xi,0}}_{k\delta} ) - \bar{b}_t(X_{k\delta}^\varepsilon, \mathcal{L}_{k\delta}^\varepsilon) \]

In the following, for $0 < r < s \leq \delta/\varepsilon$, set
\[ \Phi(s, r) := \mathbb{E} \left. \left( b_t(X_{k\delta}^\varepsilon, \mathcal{L}_{k\delta}^\varepsilon, \hat{Z}^{\varepsilon,20,\xi}_{k\delta} , \mathcal{L}_{\hat{Z}^{\xi,0}}_{k\delta} ) - \bar{b}_t(X_{k\delta}^\varepsilon, \mathcal{L}_{k\delta}^\varepsilon) \right) ds \right|_{s}^{r} \]

and we estimate $\Phi(s, r)$. For any $s > 0, \vartheta \in L^2(\Omega, \mathcal{F}_s, \mathbb{P}, \mathbb{R}^m), \varsigma \in L^2(\Omega, \mathcal{F}_s, \mathbb{P}, \mathbb{R}^n), \mu \in \mathcal{P}_2(\mathbb{R}^n), z \in \mathbb{R}^m$, we consider two following equations
\[ \hat{Z}_{t,s}^{\varepsilon,\varsigma,\mu,\vartheta} = \vartheta + \frac{1}{\varepsilon} \int_{s}^{t} b_2(\varsigma, \mu, \hat{Z}_{t,s}^{\varepsilon,\varsigma,\mu,\vartheta}, \mathcal{L}_{\hat{Z}_{t,s}^{\varepsilon,\varsigma,\mu,\vartheta}}) d\sigma_t \]
\[ + \frac{1}{\sqrt{\varepsilon}} \int_{s}^{t} \sigma_2(\varsigma, \mu, \hat{Z}_{t,s}^{\varepsilon,\varsigma,\mu,\vartheta}, \mathcal{L}_{\hat{Z}_{t,s}^{\varepsilon,\varsigma,\mu,\vartheta}}) dW_t, \]
\[ \hat{Z}_{t,s}^{\varepsilon,\varsigma,\mu,\vartheta} = \vartheta + \frac{1}{\varepsilon} \int_{s}^{t} b_2(\varsigma, \mu, \hat{Z}_{t,s}^{\varepsilon,\varsigma,\mu,\vartheta}, \mathcal{L}_{\hat{Z}_{t,s}^{\varepsilon,\varsigma,\mu,\vartheta}}) d\sigma_t \]
\[ + \frac{1}{\sqrt{\varepsilon}} \int_{s}^{t} \sigma_2(\varsigma, \mu, \hat{Z}_{t,s}^{\varepsilon,\varsigma,\mu,\vartheta}, \mathcal{L}_{\hat{Z}_{t,s}^{\varepsilon,\varsigma,\mu,\vartheta}}) dW_t, \]

Then it holds that
\[ \hat{Z}_{t,s}^{\varepsilon,\varsigma,\mu,\vartheta} = \hat{Z}_{t,s}^{\varepsilon,\varsigma,\mu,\vartheta}, \hat{Z}_{t,s}^{\varepsilon,\varsigma,\mu,\vartheta} = \hat{Z}_{t,s}^{\varepsilon,\varsigma,\mu,\vartheta} \]
\[ \hat{Z}_{t,s}^{\varepsilon,\varsigma,\mu,\vartheta} = \hat{Z}_{t,s}^{\varepsilon,\varsigma,\mu,\vartheta}, \hat{Z}_{t,s}^{\varepsilon,\varsigma,\mu,\vartheta} = \hat{Z}_{t,s}^{\varepsilon,\varsigma,\mu,\vartheta} \]
\[ t \in [k\delta, (k + 1)\delta], \]
and furthermore

\[
\Phi(s, r) = \mathbb{E}\left[ b_1(X^\varepsilon_{k\delta}, \mathcal{L}^{X^\varepsilon_{k\delta}}, \dot{Z}_{es+k\delta}, \dot{Z}_{es+k\delta}; \mathbb{P}) - \tilde{b}_1(X^\varepsilon_{k\delta}, \mathcal{L}^{X^\varepsilon_{k\delta}}) \right]
\]

\[
b_1(X^\varepsilon_{k\delta}, \mathcal{L}^{X^\varepsilon_{k\delta}}, \dot{Z}_{es+k\delta}, \dot{Z}_{es+k\delta}; \mathbb{P}) - \tilde{b}_1(X^\varepsilon_{k\delta}, \mathcal{L}^{X^\varepsilon_{k\delta}})
\]

Note that \(X^\varepsilon_{k\delta}, \dot{Z}^\varepsilon_{k\delta}\) are \(\mathcal{F}_{k\delta}\)-measurable, and for any \(x \in \mathbb{R}^n\), \(\dot{Z}_t\) is independent of \(\mathcal{F}_{k\delta}\). Thus, we have that

\[
\Phi(s, r) = \mathbb{E}\left[ \left( b_1(x, \mu, \mathcal{L}^{\varepsilon_{k\delta}, x, \mu, z, \nu}, \dot{Z}^\varepsilon_{k\delta}, \dot{Z}^\varepsilon_{k\delta}) - \tilde{b}_1(x, \mu) \right) \left| \mathcal{F}_{k\delta} \right. \right]
\]

Here, we investigate \(\dot{Z}^\varepsilon_{k\delta, x, \mu, z, \nu}\). On one hand, it holds that

\[
\dot{Z}^\varepsilon_{k\delta, x, \mu, z, \nu} = z + \frac{1}{\varepsilon} \int_{k\delta}^{es+k\delta} b_2(x, \mu, \dot{Z}^\varepsilon_{k\delta, x, \mu, z, \nu}, \mathcal{L}^{\varepsilon_{k\delta, x, \mu, z, \nu}})dr
\]

\[
+ \frac{1}{\sqrt{\varepsilon}} \int_{k\delta}^{es+k\delta} \sigma_2(x, \mu, \dot{Z}^\varepsilon_{k\delta, x, \mu, z, \nu}, \mathcal{L}^{\varepsilon_{k\delta, x, \mu, z, \nu}})dW_r
\]

where \(W_u := W_{u+k\delta} - W_{k\delta}\) and \(\tilde{W}_v := \frac{1}{\sqrt{\varepsilon}} \tilde{W}_{\varepsilon v}\) are two \(m\)-dimensional standard Brownian motions. On the other hand, we notice that Eq. (20) is just written as

\[
Z^\varepsilon_{x, \mu, z, \nu} = z + \int_0^s b_2(x, \mu, \mathcal{L}^{\varepsilon_{x, \mu, z, \nu}})dr + \int_0^s \sigma_2(x, \mu, \mathcal{L}^{\varepsilon_{x, \mu, z, \nu}})dW_r,
\]
where \( \zeta \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^m) \), \( \mathcal{L}_{\zeta}^\mathbb{P} = \nu \). That is, for \( s \in [0, \delta/\varepsilon] \), \((\mathcal{L}_{\varepsilon}\mathcal{F}(x, \mu, z, \nu), \mathcal{L}_{\varepsilon, k\delta, x, \mu, z, \nu}^\mathbb{P}) \) and \((\mathcal{L}_{\delta}\mathcal{F}(x, \mu, z, \nu), \mathcal{L}_{\delta, k\delta, x, \mu, z, \nu}^\mathbb{P}) \) have the same distribution. But \((\mathcal{L}_{\delta}\mathcal{F}(x, \mu, z, \nu), \mathcal{L}_{\delta, k\delta, x, \mu, z, \nu}^\mathbb{P}) \) is not a Markov process. Therefore, we need to construct a Markov process based on \((\mathcal{L}_{\delta}\mathcal{F}(x, \mu, z, \nu), \mathcal{L}_{\delta, k\delta, x, \mu, z, \nu}^\mathbb{P}) \).

Let \( C([0, \infty), \mathbb{R}^m) \) be the collection of continuous functions from \([0, \infty)\) to \( \mathbb{R}^m \) with the uniform convergence topology. Set

\[
\tilde{\Omega} := C([0, \infty), \mathbb{R}^m) \times C([0, \infty), \mathcal{P}_2(\mathbb{R}^m)),
\]

\[
\tilde{\mathcal{F}} = \mathcal{B}\left(C([0, \infty), \mathbb{R}^m)\right) \times \mathcal{B}\left(C([0, \infty), \mathcal{P}_2(\mathbb{R}^m))\right),
\]

\[
\tilde{\mathcal{F}}_t = \sigma(M_r, 0 \leq r \leq t), \quad t \geq 0,
\]

where \( M \) is the coordinate process. Then by [20, Theorem 4.11], there exists a unique probability measure \( \tilde{\mathbb{P}}^{x, \mu, z, \nu} \) on \((\tilde{\Omega}, \tilde{\mathcal{F}})\) such that \( M \) is a Markov process with respect to \((\tilde{\mathcal{F}}_t)\) with the transition function \( \{\tilde{\mathbb{P}}^{x, \mu, z, \nu}(t, \nu, \cdot) = \mathcal{L}_{\tilde{x}}^{x, \mu, z, \nu} \times \delta_{\tilde{\mathcal{F}}_t}^{\mathbb{P}} : t \geq 0, (z, \nu) \in \mathbb{R}^m \times \mathcal{P}_2(\mathbb{R}^m)\} \) and \( \mathcal{L}_{\tilde{M}_t}^{x, \mu, z, \nu} = \delta_z \times \delta_{\nu} \). Note that

\[
\mathcal{L}_{\tilde{M}_t}^{x, \mu, z, \nu} := \tilde{\mathbb{P}}^{x, \mu, z, \nu} \circ M_t^{-1} = \int_{\mathbb{R}^m \times \mathcal{P}_2(\mathbb{R}^m)} \mathbb{P}_t^{x, \mu, z, \nu}(z', \nu' ; \cdot) \delta_z \times \delta_{\nu}(dz', d\nu') = \mathcal{L}_{\tilde{x}}^{x, \mu, z, \nu} \times \delta_{\tilde{\mathcal{F}}_t}^{\mathbb{P}}\).

Thus, set

\[
\tilde{\mathbb{P}} := \tilde{\mathbb{P}}^{x, \mu, z, \nu}, \quad (\tilde{\mathcal{L}}_{\tilde{x}}^{x, \mu, z, \nu}, \tilde{\mathcal{L}}_{\tilde{x}}^{x, \mu, z, \nu}) := M_t,
\]

and \( \mathcal{L}_{\tilde{M}_t}^{x, \mu, z, \nu} = \mathcal{L}_{\tilde{x}}^{x, \mu, z, \nu}, \mathcal{L}_{\tilde{x}}^{x, \mu, z, \nu} = \mathcal{L}_{\tilde{x}}^{x, \mu, z, \nu}, \mathcal{L}_{\tilde{x}}^{x, \mu, z, \nu} = \nu \), which implies that

\[
\mathbb{E}\left( b_1(x, \mu, Z_{\varepsilon, k\delta, x, \mu, z, \nu}^\mathbb{P}, \mathcal{L}_{\varepsilon, k\delta, x, \mu, z, \nu}^{\mathbb{P}}) - \tilde{b}_1(x, \mu) , b_1(x, \mu, Z_{\delta, k\delta, x, \mu, z, \nu}^\mathbb{P}, \mathcal{L}_{\delta, k\delta, x, \mu, z, \nu}^{\mathbb{P}}) - \tilde{b}_1(x, \mu) \right) = \mathbb{E}\left( \mathbb{E}\left( b_1(x, \mu, Z_{\tilde{x}}^{x, \mu, z, \nu}^{\mathbb{P}}, \mathcal{L}_{\tilde{x}}^{x, \mu, z, \nu}^{\mathbb{P}}) - \tilde{b}_1(x, \mu) , b_1(x, \mu, Z_{\tilde{x}}^{x, \mu, z, \nu}^{\mathbb{P}}, \mathcal{L}_{\tilde{x}}^{x, \mu, z, \nu}^{\mathbb{P}}) - \tilde{b}_1(x, \mu) \right) \mid \tilde{\mathcal{F}}_t \right)
\]

\[
= \mathbb{E}\left( \mathbb{E}\left( \tilde{b}_1(x, \mu, Z_{\tilde{x}}^{x, \mu, z, \nu}^{\mathbb{P}}) - \tilde{b}_1(x, \mu) , \tilde{b}_1(x, \mu, Z_{\tilde{x}}^{x, \mu, z, \nu}^{\mathbb{P}}) - \tilde{b}_1(x, \mu) \right) \mid \tilde{\mathcal{F}}_t \right) \cdot \mathbb{E}\left( \tilde{b}_1(x, \mu, Z_{\tilde{x}}^{x, \mu, z, \nu}^{\mathbb{P}}) - \tilde{b}_1(x, \mu) \right) \end{equation}

\[
\leq \left( \mathbb{E}\left( \mathbb{E}\left( \tilde{b}_1(x, \mu, Z_{\tilde{x}}^{x, \mu, z, \nu}^{\mathbb{P}}) - \tilde{b}_1(x, \mu) \right)^2 \mid \tilde{\mathcal{F}}_t \right) \right)^{1/2} = \left( \mathbb{E}\left( \tilde{b}_1(x, \mu, Z_{\tilde{x}}^{x, \mu, z, \nu}^{\mathbb{P}}) - \tilde{b}_1(x, \mu) \right)^2 \right)^{1/2}.
\]

Moreover, based on [23, 21] and [22], we obtain that

\[
\left( \mathbb{E}\left( \mathbb{E}\left( \tilde{b}_1(x, \mu, Z_{\tilde{x}}^{x, \mu, z, \nu}^{\mathbb{P}}) - \tilde{b}_1(x, \mu) \right)^2 \mid \tilde{\mathcal{F}}_t \right) \right)^{1/2}
\]
which together with (30) implies (26). The proof is complete.

\[ \Phi(s, r) \leq C e^{-\beta_1 - L_{b_1, a_2}(s - r)/2}. \]

Inserting the above inequality in (29), we get that

\[ B_1 \leq C \left( \frac{\varepsilon}{\delta} \right)^2 \int_0^t \int_t^T C e^{-\beta_1 - L_{b_1, a_2}(s - r)/2} ds dr \leq C \frac{\varepsilon}{\delta}. \] (30)

Next, we estimate \( B_2 \). By (1), Lemma 4.1, 4.2 and the Hölder inequality, one could get that

\[ B_2 \leq 24 \delta \mathbb{E} \sup_{0 \leq t \leq T} \int_0^t \left( |b_1(X_{s(\delta)}^\varepsilon, \mathcal{L}_{X_{s(\delta)}^\varepsilon, \xi}^{\delta})| + |\bar{b}_1(X_{s(\delta)}^\varepsilon, \mathcal{L}_{X_{s(\delta)}^\varepsilon, \xi}^{\delta})| \right) ds \]

\[ \leq C \delta \mathbb{E} \int_0^T \left( 1 + |X_{s(\delta)}^\varepsilon| + \mathcal{L}_{X_{s(\delta)}^\varepsilon}^{\delta} \right) ds \]

which together with (30) implies (26). The proof is complete. \( \square \)

At present, we are ready to prove Theorem 3.1.

**Proof of Theorem 3.1.** Taking \( \delta = \varepsilon^\gamma \), by (19) and Lemma 4.8, we get that

\[ \mathbb{E} \left( \sup_{0 \leq t \leq T} |X_t^\varepsilon - \bar{X}_t|^2 \right) \leq C \left( \varepsilon^{1-\gamma} + \varepsilon^{2\gamma} + \varepsilon^\gamma \right). \]

This is just (10).

Next, by the Chebyshev inequality and (10), it holds that for any \( \theta > 0 \)

\[ \mathbb{P} \left( \sup_{0 \leq t \leq T} |X_t^\varepsilon - \bar{X}_t| > \theta \right) \leq \frac{\mathbb{E} \left( \sup_{0 \leq t \leq T} |X_t^\varepsilon - \bar{X}_t|^2 \right)}{\theta^2} \leq C \left( \varepsilon^{1-\gamma} + \varepsilon^{2\gamma} + \varepsilon^\gamma \right), \]

which implies that

\[ \sup_{0 \leq t \leq T} |X_t^\varepsilon - \bar{X}_t| \to 0, \]
as \( \varepsilon \) tends to 0. Besides, from Lemma \[4.1\] and \[24\] it follows that
\[
\sup_{\varepsilon} \mathbb{E} \sup_{0 \leq t \leq T} |X_t^\varepsilon - \tilde{X}_t|^{2p+2} \leq C(1 + \mathbb{E}|\varrho|^{2p+2} + |z_0|^{2p+2} + \mathbb{E}|\xi|^{2p+2}).
\]
Therefore, by the Vitali convergence theorem one can obtain that
\[
\lim_{\varepsilon \to 0} \mathbb{E} \left( \sup_{0 \leq t \leq T} |X_t^\varepsilon - \tilde{X}_t|^{2p} \right) = 0,
\]
which completes the proof.

5. Proof of Theorem 3.3

In this section, we prove Theorem 3.3. First of all, we prepare an important lemma.

**Lemma 5.1.** Under the assumption \((H_b)\), there exists a constant \( C > 0 \) such that
\[
\mathbb{E}|\rho^0_t(1)|^{-r} \leq \exp\{2r^2 + r + 1)CT/2\}, \quad r > 1.
\]
Since its proof is similar to that of \[16, Lemma 3.6\], we omit it.

Now, we are ready to prove Theorem 3.3.

**Proof of Theorem 3.3. Step 1.** We estimate \( \pi_t^\varepsilon(F) - \pi_t^0(F) \).
Based on the Hölder inequality and these definitions of \( \pi_t^\varepsilon(F) \) and \( \pi_t^0(F) \), we get that
\[
\mathbb{E} |\pi_t^\varepsilon(F) - \pi_t^0(F)|^q = \mathbb{E} \left| \frac{\rho_t^\varepsilon(F) - \rho_t^0(F)}{\rho_t^\varepsilon(1)} - \frac{\pi_t^\varepsilon(F) - \rho_t^0(1)}{\rho_t^\varepsilon(1)} \right|^q
\]
\[
\leq 2^{q-1} \mathbb{E} \left| \frac{\rho_t^\varepsilon(F) - \rho_t^0(F)}{\rho_t^\varepsilon(1)} \right|^q + 2^{q-1} \mathbb{E} \left| \frac{\pi_t^\varepsilon(F) - \rho_t^0(1)}{\rho_t^\varepsilon(1)} \right|^q
\]
\[
\leq 2^{q-1} \left( \mathbb{E} |\rho_t^\varepsilon(F) - \rho_t^0(F)|^{2q} \right)^{\frac{1}{2}} \left( \mathbb{E} |\rho_t^\varepsilon(1)|^{-2q} \right)^{\frac{1}{2}} + 2^{q-1} \|F\|_{C_b,tip} \left( \mathbb{E} |\rho_t^\varepsilon(1)|^{-2q} \right)^{\frac{1}{2}} \left( \mathbb{E} |\rho_t^\varepsilon(1)|^{-2q} \right)^{\frac{1}{2}}
\]
\[
\leq C \left( \mathbb{E} |\rho_t^\varepsilon(F) - \rho_t^0(F)|^{2q} \right)^{\frac{1}{2}} + C \|F\|_{C_b,tip} \left( \mathbb{E} |\rho_t^\varepsilon(1)|^{-2q} \right)^{\frac{1}{2}},
\]
(31)

where \( \|F\|_{C_b,tip} \) denotes the norm of \( F \) in \( C_b,tip(\mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n)) \).

By the deduction in Step 2, it holds that
\[
\lim_{\varepsilon \to 0} \mathbb{E} |\rho_t^\varepsilon(F) - \rho_t^0(F)|^q = 0.
\]
(32)

Inserting (32) in (31), we obtain that
\[
\lim_{\varepsilon \to 0} \mathbb{E} |\pi_t^\varepsilon(F) - \pi_t^0(F)|^q = 0.
\]

**Step 2.** We prove (32).

By the measure transformation and the Hölder inequality, it holds that
\[
\mathbb{E} |\rho_t^\varepsilon(F) - \rho_t^0(F)|^{2q} = \mathbb{E}^{\mathbb{P}^\varepsilon} |\rho_t^\varepsilon(F) - \rho_t^0(F)|^{2q} \leq \left( \mathbb{E}^{\mathbb{P}^\varepsilon} |\pi_t^\varepsilon(F) - \rho_t^0(F)|^4 \right)^{\frac{1}{2}} \left( \mathbb{E}^{\mathbb{P}^\varepsilon} (\Lambda_t^\varepsilon)^2 \right)^{\frac{1}{2}}.
\]
(33)

On one hand, it is not difficult to prove that
\[
(\mathbb{E}^{\mathbb{P}^\varepsilon} (\Lambda_t^\varepsilon)^2)^{\frac{1}{2}} \leq \exp\{CT\}.
\]

On the other hand, from the Jensen inequality, it follows that
\[
\mathbb{E}^{\mathbb{P}^\varepsilon} |\rho_t^\varepsilon(F) - \rho_t^0(F)|^4
\]
\[ A_1 = 2^{4q-1} \mathbb{E}^{\mathbb{P}} \left| F(X_t^\varepsilon, \mathcal{L}_{X_t}^\mathbb{P}) \right| 4q \]
\[ \leq 2^{4q-1} \left( \mathbb{E}^{\mathbb{P}} \left| F(X_t^\varepsilon, \mathcal{L}_{X_t}^\mathbb{P}) - F(\bar{X}_t, \mathcal{L}_{\bar{X}_t}^\mathbb{P}) \right|^{8q} \right)^{\frac{1}{8q}} \left( \mathbb{E}^{\mathbb{P}} \left| \Lambda_t^\varepsilon \right|^{8q} \right)^{\frac{1}{8q}} \]
\[ \leq 2^{4q-1} \left( \mathbb{E}^{\mathbb{P}} \left| F \right|_{C_{lips}(\mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^{n}))}^{d} \left( \mathbb{E}^{\mathbb{P}} \left| X_t^\varepsilon - \bar{X}_t \right|^{2q} + \mathbb{E} \left| X_t^\varepsilon - \bar{X}_t \right|^{2q} \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \left( \mathbb{E}^{\mathbb{P}} \left| \Lambda_t^\varepsilon \right|^{8q} \right)^{\frac{1}{8q}} \]
\[ \leq 2^{4q-1} \left( \mathbb{E}^{\mathbb{P}} \left| F \right|_{C_{lips}(\mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^{n}))}^{d} \left( \mathbb{E}^{\mathbb{P}} \left| X_t^\varepsilon - \bar{X}_t \right|^{8q} + \mathbb{E} \left| X_t^\varepsilon - \bar{X}_t \right|^{8q} \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \left( \mathbb{E}^{\mathbb{P}} \left| \Lambda_t^\varepsilon \right|^{8q} \right)^{\frac{1}{8q}} \]

Then we estimate \( \mathbb{E}^{\mathbb{P}} \left| X_t^\varepsilon - \bar{X}_t \right|^{8q} + \mathbb{E} \left| X_t^\varepsilon - \bar{X}_t \right|^{8q} \). Note that
\[ \mathbb{E}^{\mathbb{P}} \left| X_t^\varepsilon - \bar{X}_t \right|^{8q} + \mathbb{E} \left| X_t^\varepsilon - \bar{X}_t \right|^{8q} = \mathbb{E} \left[ \left| X_t^\varepsilon - \bar{X}_t \right|^{8q} \right] + \mathbb{E} \left| X_t^\varepsilon - \bar{X}_t \right|^{8q} \]
\[ \leq \left( \mathbb{E} \left| X_t^\varepsilon - \bar{X}_t \right|^{16q} \right)^{\frac{1}{2}} \left( \mathbb{E} (\Lambda_T^\varepsilon)^{-2} \right)^{\frac{1}{2}} + \mathbb{E} \left| X_t^\varepsilon - \bar{X}_t \right|^{8q}. \]

By the similar deduction to that in Lemma 5.1, one can get that
\[ \mathbb{E} (\Lambda_T^\varepsilon)^{-2} \leq C, \]
which yields
\[ A_1 \leq 2^{8q-\frac{1}{2}} \left( \mathbb{E}^{\mathbb{P}} \left| F \right|_{C_{lips}(\mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^{n}))}^{d} \right) \left[ C \left( \mathbb{E} \left| X_t^\varepsilon - \bar{X}_t \right|^{16q} \right)^{\frac{1}{2}} + \mathbb{E} \left| X_t^\varepsilon - \bar{X}_t \right|^{8q} \right]. \]
Combining (9) with (35) and taking the limit as $\varepsilon \to 0$, we get that
\[
\lim_{\varepsilon \to 0} A_1 = 0. \tag{36}
\]

For $A_2$, by the Hölder inequality, one can obtain that
\[
A_2 = 2^{4q-1} \mathbb{E}^{\mathbb{P}_t} \left| F(\bar{X}_t, \mathcal{L}_{\bar{X}_t}^\varepsilon) - F(\bar{X}_t, \mathcal{L}_{\bar{X}_t}^0) \right|^{4q} 
\leq 2^{4q-1} \|F\|_{C_{h,lip}(\mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n))}^{4q} \mathbb{E}^{\mathbb{P}_t} \left| \Lambda_t^\varepsilon - \Lambda_t^0 \right|^{4q} 
\leq 2^{4q-1} \|F\|_{C_{h,lip}(\mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n))}^{4q} \mathbb{E} \left| \Lambda_t^\varepsilon - \Lambda_t^0 \right|^{4q} (\Lambda_T^{\varepsilon})^{-1} 
\leq 2^{4q-1} \|F\|_{C_{h,lip}(\mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n))}^{4q} \left( \mathbb{E} \left| \Lambda_t^\varepsilon - \Lambda_t^0 \right|^{8q} \right)^{1/2} \left( \mathbb{E} (\Lambda_T^{\varepsilon})^{-2} \right)^{1/2} 
\leq C \left( \mathbb{E} |\Lambda_t^\varepsilon - \Lambda_t^0|^{8q} \right)^{1/2}.
\]

Next, we observe $|\Lambda_t^\varepsilon - \Lambda_t^0|$. By definitions of $\Lambda_t^\varepsilon$ and $\Lambda_t^0$, it holds that
\[
|\Lambda_t^\varepsilon - \Lambda_t^0| = \left| \exp \left\{ \int_0^t h^i(X_s^\varepsilon, \mathcal{L}_{X_s}^\varepsilon) dV_s^i + \frac{1}{2} \int_0^t [h(X_s^\varepsilon, \mathcal{L}_{X_s}^\varepsilon)]^2 ds \right\} - \exp \left\{ \int_0^t h^i(\bar{X}_s, \mathcal{L}_{\bar{X}_s}^\varepsilon) dV_s^i + \frac{1}{2} \int_0^t [h(\bar{X}_s, \mathcal{L}_{\bar{X}_s}^\varepsilon)]^2 ds \right\} \right| 
= \Lambda_t^0 \cdot \left| \exp \left\{ \int_0^t \left( h(X_s^\varepsilon, \mathcal{L}_{X_s}^\varepsilon) - h(\bar{X}_s, \mathcal{L}_{\bar{X}_s}^\varepsilon) \right) dV_s^i 
+ \frac{1}{2} \int_0^t (|h(X_s^\varepsilon, \mathcal{L}_{X_s}^\varepsilon)|^2 - |h(\bar{X}_s, \mathcal{L}_{\bar{X}_s}^\varepsilon)|^2) ds 
+ \int_0^t (|h(X_s^\varepsilon, \mathcal{L}_{X_s}^\varepsilon)|^2 - h(\bar{X}_s, \mathcal{L}_{\bar{X}_s}^\varepsilon) h(X_s^\varepsilon, \mathcal{L}_{X_s}^\varepsilon) h(\bar{X}_s, \mathcal{L}_{\bar{X}_s}^\varepsilon) \left\} - 1 \right|. \right| 
\tag{37}
\]

Then, we deal with the integral $\int_0^t \left( h(X_s^\varepsilon, \mathcal{L}_{X_s}^\varepsilon) - h(\bar{X}_s, \mathcal{L}_{\bar{X}_s}^\varepsilon) \right) dV_s^i$. From the isometric formula and (H$_h$), it follows that
\[
\mathbb{E} \left| \int_0^t \left( h(X_s^\varepsilon, \mathcal{L}_{X_s}^\varepsilon) - h(\bar{X}_s, \mathcal{L}_{\bar{X}_s}^\varepsilon) \right) dV_s^i \right|^2 
= \mathbb{E} \left( \int_0^t |h(X_s^\varepsilon, \mathcal{L}_{X_s}^\varepsilon) - h(\bar{X}_s, \mathcal{L}_{\bar{X}_s}^\varepsilon)|^2 ds \right) 
\leq \mathbb{E} \left( \int_0^t L_h \left( |X_s^\varepsilon - \bar{X}_s|^2 + \mathbb{W}_2^2(\mathcal{L}_{X_s}^\varepsilon, \mathcal{L}_{\bar{X}_s}^\varepsilon) \right) ds \right) 
\leq C \mathbb{E} \left( \sup_{0 \leq t \leq T} |X_t^\varepsilon - \bar{X}_t| \right),
\]
which implies that
\[
\lim_{\varepsilon \to 0} \mathbb{E} \left| \int_0^t \left( h(X_s^\varepsilon, \mathcal{L}_{X_s}^\varepsilon) - h(\bar{X}_s, \mathcal{L}_{\bar{X}_s}^\varepsilon) \right) dV_s^i \right|^2 = 0,
\]
and then
\[
\lim_{\varepsilon \to 0} \int_0^t \left( h(X_s^\varepsilon, \mathcal{L}_{X_s}^\varepsilon) - h(\bar{X}_s, \mathcal{L}_{\bar{X}_s}^\varepsilon) \right) dV_s^i = 0, \quad \text{a.s.}.
\]
For the integral \( \int_0^t (|h(\bar{X}_s, \mathcal{L}^\varepsilon_{X_s})|^2 - |h(X_s^\varepsilon, \mathcal{L}^\varepsilon_{X_s})|^2) ds \), by (H_h) and the dominated convergence theorem, we obtain that

\[
\lim_{\varepsilon \to 0} \int_0^t \left( |h(\bar{X}_s, \mathcal{L}^\varepsilon_{X_s})|^2 - |h(X_s^\varepsilon, \mathcal{L}^\varepsilon_{X_s})|^2 \right) ds = 0, \quad \text{a.s..}
\]

By the similar deduction to the above equality, one could get

\[
\lim_{\varepsilon \to 0} \int_0^t \left( |h(X_s^\varepsilon, \mathcal{L}^\varepsilon_{X_s})|^2 - h(\bar{X}_s, \mathcal{L}^\varepsilon_{X_s}) h(X_s^\varepsilon, \mathcal{L}^\varepsilon_{X_s}) \right) ds = 0, \quad \text{a.s..}
\]

Thus, by taking the limit on both sides of (37), it holds that

\[
\lim_{\varepsilon \to 0} |\Lambda_t^\varepsilon - \Lambda_t^0| = 0, \quad \text{a.s..}
\]

Also note that

\[
|\Lambda_t^\varepsilon|^{8q} = \exp \left\{ 8q \int_0^t h(X_s^\varepsilon, \mathcal{L}^\varepsilon_{X_s}) dV_s^i + 4q \int_0^t |h(X_s^\varepsilon, \mathcal{L}^\varepsilon_{X_s})|^2 ds \right\} \\
= \exp \left\{ \int_0^t 8qh(X_s^\varepsilon, \mathcal{L}^\varepsilon_{X_s}) dV_s^i - \frac{1}{2} \int_0^t |8qh(X_s^\varepsilon, \mathcal{L}^\varepsilon_{X_s})|^2 ds \right\} \\
\cdot \exp \left\{ (32q^2 + 4q) t \right\}
\]

\[
|\Lambda_t^0|^{8q} \leq \exp \left\{ \int_0^t 8qh(\bar{X}_s, \mathcal{L}^\varepsilon_{X_s}) dV_s^i - \frac{1}{2} \int_0^t |8qh(\bar{X}_s, \mathcal{L}^\varepsilon_{X_s})|^2 ds \right\} \exp\{CT\}.
\]

Thus, by the dominated convergence theorem it holds that

\[
\lim_{\varepsilon \to 0} \Lambda_2 = 0. \quad (38)
\]

Finally, taking the limit on both sides of (34) and inserting (36) and (38) in (34), we have (32). The proof is complete.

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