Separability of symmetric one parameter families of noisy states using conditional quantum relative Tsallis entropy

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The separability range, in the $1: N−1$ partition, of one parameter families of noisy $N$-qubit symmetric states involving W and GHZ states is identified using the conditional quantum relative Tsallis entropy approach. For all $N$, the separability range matches exactly with the range obtained through positive partial transpose criterion. The advantages of using non-commuting version of $q$-conditional relative Tsallis entropy is brought out in these systems, with GHZ having commuting while W has non-commuting features.

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I. INTRODUCTION

It is well known that entropic characterization of separability captures the local versus global disorder of mixed composite states and serves as a convenient tool in identifying the separability range in several one-parameter families of states [1–10]. The quantum versions of more generalized entropies such as Rényi and Tsallis entropies are found to yield better separability range than that obtained through von-Neumann entropy. In fact, positivity of Tsallis conditional entropy is found to capture global vs local disorder in mixed states much better than that through von-Neumann conditional entropy leading to stricter range of separability [3,8].

Making use of the quantum generalization of Rényi relative entropy to the situation when the pair of density matrices are noncommuting, an analogous generalization to Tsallis relative entropy and its conditional version are obtained in Ref. [10]. This generalization was done in anticipation of the fact that the conditional version of generalized Tsallis relative entropy is more effective in identifying separability than its traditional commuting version. The purpose of this paper is to examine this separability range in several one-parameter families of states and serves as a convenient tool in identifying the separability range in several one-parameter families of noisy states [1–10]. The quantum versions of more generalized Tsallis relative entropy is more effective in anticipating the fact that the conditional version of generalized Tsallis relative entropy is more effective in identifying separability than its traditional commuting version. The purpose of this paper is to examine this separability range in several one-parameter families of noisy states involving W and GHZ states is identified using the conditional quantum relative Tsallis entropy approach. For all $N$, the separability range matches exactly with the range obtained through positive partial transpose criterion. The advantages of using non-commuting version of $q$-conditional relative Tsallis entropy is brought out in these systems, with GHZ having commuting while W has non-commuting features.

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The so-called ‘sandwiched’ Tsallis relative entropy [10] is given by

$$\tilde{D}_q^T(\rho||\sigma) = \frac{\text{Tr}\left[\left(\sigma^{1-q} \rho \sigma^{\frac{1}{2q}}\right)^q\right] - 1}{q - 1}$$

quite on the same lines of the definition of generalized version of quantum relative Rényi entropy [11, 12]. It reduces to the traditional relative Tsallis entropy

$$D_q^T(\rho||\sigma) = \frac{\text{Tr}(\rho^{\frac{1}{1-q}}) - 1}{q - 1},$$

when $\sigma$ and $\rho$ commute with each other. It is useful here to note the Lieb-Thirring inequality $\tilde{D}_q^T(\rho||\sigma) \leq D_q^T(\rho||\sigma)$.

The conditional version of $\tilde{D}_q^T(\rho_{AB}||\sigma)$ is defined as [10]

$$\tilde{D}_q^T(\rho_{AB}||\rho_B) = \frac{\hat{Q}_q(\rho_{AB}||\rho_B) - 1}{1 - q}$$

where

$$\hat{Q}_q(\rho_{AB}||\rho_B) = \text{Tr}\left[\left(\rho_{AB}^{\frac{1}{1-q}} \rho_B \rho_{AB} (\rho_A \otimes \rho_B)^{\frac{1}{1-q}}\right)^q\right].$$

In fact, $\tilde{D}_q^T(\rho_{AB}||\rho_B)$ reduces to the Abe-Rajagopal (AR) $q$-conditional Tsallis entropy [3]

$$S_q^T(A||B) = \frac{1}{q - 1} - \frac{\text{Tr}\rho_{AB}}{-\text{Tr}\rho_B} \frac{\text{Tr}\rho_B}{\text{Tr}\rho_B}$$

when the subsystem density matrix $\rho_B$ is a maximally mixed state. The state $\rho_{AB}$ is entangled whenever $S_q^T(A||B) < 0$ [3] and hence the AR criterion has been employed to examine separability of several classes of composite states [3,8]. Though the AR criterion provides a better entropic separability criterion than the one using von-Neumann conditional entropy, it was found necessary to take into consideration the situations where the subsystem density matrix $\rho_B$ is not maximally mixed thereby not commuting with its original density matrix $\rho_{AB}$. This aspect of non-commutativity of subsystem density matrices with their original density matrices in identifying a better entropic separability criterion has been addressed in Ref. [11] using the non-commuting version of Tsallis relative entropy and its conditional version.
As the negative values of the conditional version of sandwiched Tsallis relative entropy (CSTRE) $\tilde{D}_T^q(\rho_{AB}||\rho_B)$ indicate entanglement in the state $\rho_{AB}$, it has been employed as a separability criterion for 3-qubit one-parameter families of noisy W-, GHZ- states in Ref. [10]. Stricter separability range than that obtained through AR $q$-conditional entropy [3] is seen to be achievable using the CSTRE criterion [10]. Also, in the 1 : 2 and 1 : 3 partitions of 3-qubit one-parameter families of noisy W-, GHZ- states, the separability range is seen to match exactly [10] with that through Peres’ PPT criterion [14]. Here we extend this result and show that for all $N$, the separability range obtained through PPT and CSTRE criteria match with each other in the 1 : $N - 1$ partition of the one-parameter families of $N$-qubit W-, GHZ-states.

This paper is organized in three sections: Secion I contains introductory remarks and a recapitulation of the separability criterion formulated using conditional version of sandwiched Tsallis relative entropy. In Section II, we give a general formula, applicable to all $N$, for the separability range of one parameter families of W- and GHZ- states in their 1 : $N - 1$ partitions. Here we also make a distinction between these two families of states and bring out the advantage of the separability criterion using the sandwiched version of Tsallis relative entropy. Section III summarizes the results.

II. ONE PARAMETER FAMILIES OF SYMMETRIC $N$-QUBIT MIXED STATES

The symmetric one parameter family of noisy $N$-qubit mixed states is given by

$$\rho_N(x) = \left(1 - \frac{x}{N + 1}\right) P_N + x|\Phi_N\rangle\langle\Phi_N|$$

with $P_N = \sum_{M=2}^{N-1} \frac{|\Phi_N, M\rangle\langle\Phi_N, M|}{2}$, $M$ is the projector onto the $N + 1$ dimensional maximal multiplicity subspace of the collective angular momentum of $N$-qubits, $|\Phi_N, M\rangle$ being the basis states of this subspace. $|\Phi_N\rangle$ is any pure state belonging to this symmetric subspace.

A. One-parameter family of noisy W states

We have the one-parameter family of noisy W states

$$\rho_N^{(W)}(x) = \left(1 - \frac{x}{N + 1}\right) P_N + x|W_N\rangle\langle W_N|$$

where $|W_N\rangle \equiv \sum_{M=1}^{N-1} \frac{|\Phi_N, M\rangle\langle\Phi_N, M|}{2} - 1$ is one among the basis states of the $N + 1$ dimensional symmetric subspace of collective angular momentum. We recall here that using the AR $q$-conditional entropy [3], the separability range of the 3-qubit state $\rho_3^{(W)}(x)$, in its 1 : 2 partition, is found to be $[0, 0.2]$ while the PPT criterion gives the stricter separability range $[0, 0.1547]$ [8, 10]. In the 1 : 3 partition of the 4-qubit state $\rho_4^{(W)}(x)$ also, the AR-criterion leads to the weaker separability range $[0, 0.1666]$ compared to the range $[0, 0.1123]$ obtained through PPT criterion. An observation of the fact that the single qubit density matrix of $\rho_N^{(W)}(x)$ is not maximally mixed led Rajagopal et al., to make use of the non-commuting version of the Tsallis relative entropy [11, 12] to obtain a better separability range for the case under examination. They proposed the conditional version of the sandwiched relative entropy (CSTRE) $\tilde{D}_T^q(\rho_{AB}||\rho_B)$ and showed that the negative values of $\tilde{D}_T^q(\rho_{AB}||\rho_B)$ indicate entanglement in the state $\rho_{AB}$. Quite in accordance with the expectations, the CSTRE criterion resulted in a better separability range than that through AR-criterion and it even matched with the 1 : 2, 1 : 3 separability ranges of $\rho_3^{(W)}(x), \rho_4^{(W)}(x)$ obtained through PPT criterion.

Having known the strictest possible separability range in the 1 : 2 partition of $\rho_3^{(W)}(x)$ and 1 : 3 partition of $\rho_4^{(W)}(x)$, it would be of interest to obtain the separability range in the 1 : $N - 1$ partition of the $N$-qubit state $\rho_N^{(W)}(x)$ using CSTRE criteria. We again recall here that, using AR-criterion, the 1 : $N - 1$ separability range of the one-parameter family of noisy W states has been obtained to be

$$0 \leq x < \frac{1}{N + 2}$$

in Ref. [8], for any $N \geq 3$. This has been a generalization of their result for $\rho_3^{(W)}(x), \rho_4^{(W)}(x)$, in their respective 1 : $N - 1$ partitions, to $\rho_N^{(W)}(x)$. As we have seen that for 3- and 4-qubit noisy W states the AR-criterion yields a weaker separability range in comparison with that obtained through CSTRE and PPT criteria, our immediate interest is to generalize the CSTRE separability range to $N$-qubit states $\rho_N^{(W)}(x)$, in their 1 : $N - 1$ partition, for any $N \geq 3$. We carry that out in the following.

In order to find the 1 : $N - 1$ separability range of the state $\rho_N^{(W)}(x)$, we need to evaluate the eigenvalues $\lambda_i$ of the ‘sandwiched’ matrix $(I_2 \otimes \rho_B)^{\frac{1}{2}} \rho_N^{(W)}(x) (I_2 \otimes \rho_B)^{\frac{1}{2}}$ so that (See Eq. (4)) $\tilde{Q}_q(\rho_N^{(W)}(x)||\rho_B) = \sum_{i=1}^{N+1} \lambda_i^q$ and

$$\tilde{D}_T^q(\rho_N^{(W)}(x)||\rho_B) = \sum_{i=1}^{N+1} \lambda_i^q - 1 \frac{1}{1 - q}.$$  

Here, as our interest is to find out the 1 : $N - 1$ separability range we have taken the subsystems $A, B$ to correspond respectively to a single qubit and the remaining $N - 1$ qubits (as the state $\rho_N^{(W)}(x)$ is symmetric, it does not matter which qubit we take as subsystem $A$). According to CSTRE criterion, the 1 : $N - 1$ separability range of $\rho_N^{(W)}(x)$ is the range in which the parameter $x$ gives non-negative values for $\tilde{D}_T^q(\rho_N^{(W)}(x)||\rho_B)$, in the limit $q \rightarrow \infty$.

The non-zero eigenvalues $\lambda_i, i = 1, 2, \ldots, N + 1$ being crucial in the evaluation of $\tilde{D}_T^q(\rho_N^{(W)}(x)||\rho_B)$, we exam-
ine the form of these eigenvalues when $N = 3, 4, 5, 6$ to analyze whether a generalization to the case of any $N$ is possible. We explicitly evaluate the eigenvalues $\lambda_i$ of the sandwiched matrix $(I_2 \otimes \rho_B) \frac{1}{N^2} \rho_N^{(W)}(x)(I_2 \otimes \rho_B) \frac{1}{N^2}$ when $N = 3, 4, 5, 6$ and the following table provides the non-zero eigenvalues.

| Number of qubits ($N$) | $(N - 2)$ fold degenerate | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ |
|------------------------|---------------------------|-------------|-------------|-------------|
| $N = 3$                | $(\frac{1-x}{N}) \left(\frac{1-x}{N}\right)^{\frac{1}{N}}$ | $(\frac{1-x}{N}) \left(\frac{1-x}{N}\right)^{\frac{1}{N}}$ | $\left(\frac{1-x}{N}\right) \left(\frac{1-x}{N}\right)^{\frac{1}{N}} \left[1 + \frac{2(1 + x)}{1 - x}\right]$ | $\left(\frac{1-x}{N}\right) \left(\frac{1-x}{N}\right)^{\frac{1}{N}} \left[1 + \frac{2(1 + x)}{1 - x}\right]$ |
| $N = 4$                | $(\frac{1-x}{N}) \left(\frac{1-x}{N}\right)^{\frac{1}{N}}$ | $(\frac{1-x}{N}) \left(\frac{1-x}{N}\right)^{\frac{1}{N}}$ | $\left(\frac{1-x}{N}\right) \left(\frac{1-x}{N}\right)^{\frac{1}{N}} \left[1 + \frac{2(1 + x)}{1 - x}\right]$ | $\left(\frac{1-x}{N}\right) \left(\frac{1-x}{N}\right)^{\frac{1}{N}} \left[1 + \frac{2(1 + x)}{1 - x}\right]$ |
| $N = 5$                | $(\frac{1-x}{N}) \left(\frac{1-x}{N}\right)^{\frac{1}{N}}$ | $(\frac{1-x}{N}) \left(\frac{1-x}{N}\right)^{\frac{1}{N}}$ | $\left(\frac{1-x}{N}\right) \left(\frac{1-x}{N}\right)^{\frac{1}{N}} \left[1 + \frac{2(1 + x)}{1 - x}\right]$ | $\left(\frac{1-x}{N}\right) \left(\frac{1-x}{N}\right)^{\frac{1}{N}} \left[1 + \frac{2(1 + x)}{1 - x}\right]$ |
| $N = 6$                | $(\frac{1-x}{N}) \left(\frac{1-x}{N}\right)^{\frac{1}{N}}$ | $(\frac{1-x}{N}) \left(\frac{1-x}{N}\right)^{\frac{1}{N}}$ | $\left(\frac{1-x}{N}\right) \left(\frac{1-x}{N}\right)^{\frac{1}{N}} \left[1 + \frac{2(1 + x)}{1 - x}\right]$ | $\left(\frac{1-x}{N}\right) \left(\frac{1-x}{N}\right)^{\frac{1}{N}} \left[1 + \frac{2(1 + x)}{1 - x}\right]$ |

It can be readily seen from Table I that, there are only four distinct non-zero eigenvalues for the sandwiched matrix $(I_2 \otimes \rho_B) \frac{1}{N^2} \rho_N^{(W)}(x)(I_2 \otimes \rho_B) \frac{1}{N^2}$. A keen closer look at the eigenvalues $\lambda_i$, $i = 1, 2, 3, 4$ allows us to generalize them to arbitrary $N \geq 3$. We have given the generalized eigenvalues (eigenvalues of the sandwiched matrix for any $N \geq 3$) in the following:

$$
\begin{align*}
\lambda_1 & = \left(\frac{1-x}{N+1}\right) \left(\frac{1-x}{N}\right)^{\frac{1}{N}}, & (N - 2) \text{ fold degenerate;} \\
\lambda_2 & = \left(\frac{1-x}{N+1}\right) \left(\frac{1}{N}\right)^{\frac{1}{N}}, \\
\lambda_3 & = \left(\frac{1-x}{N+1}\right) \left(\frac{1}{N}\right)^{\frac{1}{N}} \left[(N-2)(1-x)^{\frac{1}{N}} + 2(1 + (N-2)x)^{\frac{1}{N}}\right], \\
\lambda_4 & = \left(\frac{1+Nx}{N+1}\right) \left(\frac{1}{N}\right)^{\frac{1}{N}} \left[1 + (N-1)(1 + (N-2)x)^{\frac{1}{N}}\right].
\end{align*}
$$

The separability range obtained through CSTRE, in the $1 : N - 1$ partition, of the state $\rho_N^{(W)}(x)$ for each of the values $N = 3, 4, 5, 6$ allows us to generalize this range to any arbitrary $N$. We obtain

$$
0 \leq x \leq \frac{-N + \sqrt{2N(N - 1)}}{N(N - 2)}
$$

as the $1 : N - 1$ separability range of one parameter family of noisy $W$ states. Note that Eq. (11) is different from the AR result in Eq. (8). Also one can immediately recover the range $(0, 0.1547), (0, 0.1123)$ respectively for the states $\rho_3^{(W)}(x), \rho_4^{(W)}(x)$ using the relation Eq. (11) and this is in accordance with the range obtained using the CSTRE criterion directly for the 3-, 4- qubit states.
\[ \rho_3^{(W)}(x), \rho_4^{(W)}(x) \] in Ref. [10]. Also, we obtain the separability ranges \((0, 0.0883), (0, 0.07275)\) in the \(1:4, 1:5\) partitions respectively for the 5- and 6-qubit states of the one parameter family. We have even verified that these separability ranges (for \(N = 3, 4, 5, 6\)) match with that obtained through PPT criterion. We can thus conjecture that the CSTRE separability range in Eq. (11) for the \(1: N - 1\) partition of the states \(\rho_N^{(W)}(x)\) is also the PPT separability range.

In Fig. 1 we have illustrated how the eigenvalues of the sandwiched matrix given in Eq. (10) allow us to find the \(1:7\) separability range for the one-parameter family of 8-qubit noisy W state \(\rho_8^{(W)}(x)\). Fig. 2 compares the separability range obtained through AR criterion with the one through CSTRE criterion. The non-commutativity of the single qubit marginal with \(\rho_8^{(W)}(x)\) is readily seen to result in a stricter separability range through CSTRE criterion. In fact, we have verified that this separability range matches with that through Peres criterion.

It can be readily seen that as the number of qubits increase, the range in which the state \(\rho_8^{(W)}(x)\) remains separable reduces considerably and for large \(N\) (macroscopic limit) one can expect that a single qubit and its remaining \(N-1\) qubits are entangled for the whole range \(0 \leq x \leq 1\).

**B. One-parameter family of noisy GHZ states**

We now examine the one-parameter family of noisy GHZ states

\[ \rho_N^{(GHZ)}(x) = \left( \frac{1-x}{N+1} \right) P_N + x|GHZ\rangle_N \langle GHZ| \quad (12) \]

in order to find their \(1:N-1\) separability range, using CSTRE criterion. In fact, in Ref [10] it has been shown that for 3- and 4-qubit states \(\rho_N^{(GHZ)}(x)\), their respective \(1:N-1\) separability ranges obtained using CSTRE criterion matched exactly with that through AR and PPT criteria. We now wish to generalize this result to \(N\)-qubit states \(\rho_N^{(GHZ)}(x)\) and our method is similar to the one adopted for one parameter family of noisy W states in the previous section.

The eigenvalues \(\mu_i\) of the sandwiched matrix \((I_2 \otimes \rho_B)^{\frac{1}{2m}} \rho_N^{(GHZ)}(x) (I_2 \otimes \rho_B)^{-\frac{1}{2m}}\) for \(N = 3\) to 6 are given in Table II.

| \(N\) | Eigenvalues \(\mu_i\) |
|-----|------------------|
| 3   |                  |
| 4   |                  |
| 5   |                  |
| 6   |                  |

Here too, there are only four distinct non-zero eigenvalues of the sandwiched matrix, two of which have \(N-3\) and 2-fold degeneracies respectively. It is not very difficult to generalize the eigenvalues \(\mu_i\) given in Table II to any \(N \geq 3\). In fact, we obtain the eigenvalues of the sandwiched matrix \((I_2 \otimes \rho_B)^{\frac{1}{2m}} \rho_N^{(GHZ)}(x) (I_2 \otimes \rho_B)^{-\frac{1}{2m}}\) as in Eq. (13).
TABLE II: The eigenvalues $\mu_i$ of the sandwiched matrix $(I_2 \otimes \rho_B)^{\frac{1}{N}} \rho_N^{(GHZ)}(x)(I_2 \otimes \rho_B)^{\frac{1}{N}}$ for $N = 3, 4, 5, 6$

| Number of | $\mu_1$ | $\mu_2$ | $\mu_3$ | $\mu_4$ |
|-----------|---------|---------|---------|---------|
| qubits ($N$) | (N - 3)-fold degenerate | | | 2-fold degenerate |
| $N = 3$ | $(-\frac{1}{2})$ | $(-\frac{1}{2})$ | $(-\frac{1}{2})$ | $(-\frac{1}{2})$ |
| $N = 4$ | $(-\frac{1}{2})$ | $(-\frac{1}{2})$ | $(-\frac{1}{2})$ | $(-\frac{1}{2})$ |
| $N = 5$ | $(-\frac{1}{2})$ | $(-\frac{1}{2})$ | $(-\frac{1}{2})$ | $(-\frac{1}{2})$ |
| $N = 6$ | $(-\frac{1}{2})$ | $(-\frac{1}{2})$ | $(-\frac{1}{2})$ | $(-\frac{1}{2})$ |

A generalization of the separability ranges obtained in the cases of $N = 3, 4, 5, 6$ lead us to the $1 : N - 1$ separability range for any $N$. We have the $1 : N - 1$ separability range of the state $\rho_N^{(GHZ)}(x)$ using CSTRE criterion as

$$0 \leq x < \frac{2}{N^2 + N + 2}$$

(13)

for large $N$ (macroscopic limit), $x \approx \frac{2}{N^2}$ for $\rho_N^{(GHZ)}(x)$ and $x \approx \frac{2}{N^2 - 1}$ for $\rho_N^{(W)}(x)$. Thus with the increase of $N$, the separability range decreases much faster for one parameter family of GHZ states than for one parameter family of W states.

In fact, the PPT criterion can also be seen to yield the same $1 : N - 1$ separability range for $N = 3, 4, 5, 6$. Therefore, we can conjecture that Eq. (13) gives the PPT separability range in the $1 : N - 1$ partition of the one parameter family of noisy GHZ states $\rho_N^{(GHZ)}(x)$.

Before concluding, we wish to make an important remark: If one considers the conditional version of sandwiched Rényi relative entropy given by $\hat{D}_Q^R(\rho_{AB}||\rho_B) = \log [\tilde{Q}_N(\rho_{AB}||\rho_B)]$ and examine the range of the parameter $x$ where $\hat{D}_Q^R(\rho_{AB}||\rho_B) \geq 0$, the same result as obtained through CSTRE is obtained for both $\rho_N^{(W)}(x)$, $\rho_N^{(GHZ)}(x)$. This implies that Rényi entropy which is additive plays the same role in the identification of the separability range in the one-parameter families of N-qubit states as the non-additive Tsallis entropy.
III. CONCLUSION

In this work, we have obtained the separability range of the one-parameter family of $N$-qubit W and GHZ states in their $1 : N - 1$ partition using the conditional version of sandwiched Tsallis relative entropy (CSTRE). For the one-parameter family of W-states, the CSTRE criterion provides a stricter $1 : N - 1$ separability range when compared to that obtained through AR $q$-conditional entropy approach. The non-commutativity of the single qubit marginal density matrix with the original density matrix of the one-parameter family of $N$-qubit W states is seen to be the reason behind the supremacy of CSTRE criterion over AR criterion. The $1 : N - 1$ separability range, obtained using CSTRE criterion, for the one-parameter family of noisy GHZ states matches with that through AR criterion. This is due to the maximally mixed nature of the single qubit density matrix of the one parameter family of $N$-qubit GHZ states therey implying the commutativity of the marginal with the original density matrix. We have thus established that CSTRE criterion is a non-commuting generalization of the AR-criterion and it matches with the results of AR-criterion in the commuting cases. For both families of states the $1 : N - 1$ separability range obtained using CSTRE criterion matches exactly with that through PPT criterion. It would be of interest to explore the separability ranges of one parameter families of mixed $N$-qubit states in their different partitions using CSTRE criterion and explore how non-commutativity aspect plays a role in finding better separability range.

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