Abstract

I conjecture\footnote{This note is based on my talk at the Segalfest. Recent discussions with Chris Woodward have been pointing toward a proof of the main statements. However, I prefer to err on the side of caution and preserve their conjectural status.} that index formulas for $K$-theory classes on the moduli of holomorphic $G$-bundles over a compact Riemann surface $\Sigma$ are controlled, in a precise way, by Frobenius algebra deformations of the Verlinde algebra of $G$. The Frobenius algebras in question are twisted $K$-theories of $G$, equivariant under the conjugation action, and the controlling device is the equivariant Gysin map along the “product of commutators” from $G^{2g}$ to $G$. The conjecture is compatible with naive virtual localization of holomorphic bundles, from $G$ to its maximal torus; this follows by localization in twisted $K$-theory.

1. Introduction

Let $G$ be a compact Lie group and let $M$ be the moduli space of flat $G$-bundles on a closed Riemann surface $\Sigma$ of genus $g$. By well-known results of Narasimhan, Seshadri and Ramanathan \cite{NS, R}, this is also the moduli space of stable holomorphic principal bundles over $\Sigma$ for the complexified group $G_C$; as a complex variety, it carries a fundamental class in complex $K$-homology. This paper is concerned with index formulas for vector bundles over $M$. The analogous problem in cohomology—integration formulas over $M$ for top degree polynomials in the tautological generators—has been extensively studied \cite{N}, \cite{K}, \cite{D}, \cite{Th}, \cite{W}, and, for the smooth versions of $M$, the moduli of vector bundles of fixed degree co-prime to the rank, it was completely solved in \cite{JK}. In that situation, the tautological classes generate the rational cohomology ring $H^*(M; \mathbb{Q})$. Knowledge of the integration formula leads to the intersection pairing, and from here, Poincaré duality determines this ring as the quotient of the polynomial ring in the tautological generators by the null ideal of the pairing.

For smooth $M$, index formulas result directly from the Riemann-Roch theorem and the integration formula. This breaks down in the singular case, and no index formula can be so obtained, for groups other than $SU(n)$. Follow-up work \cite{Kie} has extended the results of \cite{JK} to the study of the duality pairing in intersection cohomology, for some of the singular moduli spaces. Whether a useful connection to $K$-theory can be made is not known; so, to that extent, the formulas I propose here are new. But I should point out the novel features of the new approach, even in the smooth case.

More than merely giving numbers, the conjecture posits a structure to these indices; to wit, they are controlled by (finite-dimensional) Frobenius algebras, in the way the Verlinde algebras
control the index of powers of the determinant line bundle $D$. The Frobenius algebras in question are formal deformations of Verlinde algebras. This is best explained by the twisted $K$-theory point of view, [HT1], [HT2], which identifies the Verlinde algebra with a twisted equivariant $K$-theory $\tau K^*_G(G)$. The determinantal twistings appearing in that theorem (cf. §3.12 below) correspond to powers of $D$. Other $K$-theory classes, involving index bundles over $\Sigma$ (see §2.6.iii), relate to higher twistings in $K$-theory, and these effect infinitesimal deformations of the Verlinde algebra. If ordinary (=determinantal) twistings can be represented by gerbes [BCMMS], higher twistings are realized by what could be called virtual gerbes, which generalize gerbes in the way one-dimensional virtual bundles generalize line bundles. In this picture, $K$-theory classes over $M$ of virtual dimension one are automorphisms of virtual gerbes, and arise by comparing two trivializations of a twisting $\tau$ for $K_G(G)$ (§3.12). Relevant examples are the transgressions over $\Sigma$ of delooped twistings for $BG$ (§3.12). All these $K$-classes share with $D$ the property of being “multiplicative in a piece of surface”.

The first formulation (4.14) of the conjecture expresses the index of such a $K$-class over the moduli of $G$-bundles as the partition function for the surface $\Sigma$, in the 2D topological field theory defined by $\tau K^*_G(G)$. This is a sum of powers of the structure constants of the Frobenius algebra, for which explicit formulas can be given (Thm. 4.19). There is a restriction on the allowed twistings, but they are general enough to give a satisfactory set of $K$-classes.

We can reformulate the conjecture in (4.14), (4.21) by encoding part of the Frobenius algebra structure into the product of commutators map $\Pi : G^{2g} \to G$. This map has the virtue of lifting the transgressed twistings for $K_G(G)$ to trivializable ones, which allows one to identify $K^*_G(G^{2g})$ with its $\tau$-twisted version. In particular, we get a class $\tau 1$ in $\tau K^*_G(G^{2g})$. The conjecture then asserts that the index of the $K$-class associated to $\tau$ over the moduli of $G$-bundles equals the Frobenius algebra trace of the Gysin push-forward of $\tau 1$ along $\Pi$.

Having (conjecturally) reduced this index to a map of compact manifolds, ordinary localization methods allow us to express the answer in terms of the maximal torus $T$ and Weyl group of $G$. This reduction to $T$, it turns out, can be interpreted as a virtual localization theorem from the moduli of holomorphic $G_C$-bundles to that of holomorphic $T_C$-bundles. (The word “virtual” reflects the use of the virtual normal bundle, defined by infinitesimal deformations). For this interpretation, however, it turns out that we must employ the moduli $M = M_G, M_T$ of all $G_C$-and $T_C$-bundles, not merely the semi-stable ones. These moduli have the structure of smooth stacks, with an infinite descending stratification by smooth algebraic substacks. Even the simplest case, the Verlinde formula, cannot be reduced to a single integral over the variety of topologically trivial $T_C$-bundles; the correct expression arises only upon summing over all topological $T$-types. (Recall that the non-trivial $T$-types define unstable $G_C$-bundles).

Now is the right time to qualify the advertised statements. The fact that the Verlinde formula, the simplest instance of our conjecture, expresses the indices of positive powers $D$ over $M$ is a fortunate accident. It is an instance of the “quantization commutes with reduction” conjecture of Guillemin and Sternberg [GS], which in this case [12] equates the indices of positive powers of $D$ over $M$ and over the stack $\mathcal{M}$ of all holomorphic $G_C$-bundles. This does not hold for more general $K$-theory classes, for which there will be contributions from the unstable Atiyah-Bott strata, and our deformed Verlinde algebras really control not the index over $M$, but that over $\mathcal{M}$. This incorporates information about the moduli of flat $G$-bundles, and the moduli of flat principal bundles of various subgroups of $G$. In other words, the index information which assembles to a nice structure refers to the stack $\mathcal{M}$ and not to the space $M$.

Hence, the third formulation of the conjecture [GS] expresses the index of any admissible $K$-class (Def. 2.7) over the moduli stack $\mathcal{M}$ of all holomorphic $G_C$-bundles over $\Sigma$ by virtual localization to the stack of holomorphic $T_C$-bundles. This involves integration over the Jacobians,

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*For the expert, we mean $G$-equivariant $B^2 BU_\Sigma$-classes of a point.*
summation over all degrees, leading to a distribution on \( T \), and finally, integration over \( T \) (to extract the invariant part). In \( \S 5 \), these steps are carried out explicitly for the group \( SU(2) \).

However, even if our interest lies in \( M \) (which, our approach suggests, it should not), all is not lost, because a generalization of “quantization commutes with reduction” (first proved in [TZ], for compact symplectic manifolds) asserts, in this case, the equality of indices over \( M \) and \( \mathfrak{M} \), after a large \( D \)-power twist, for the class of bundles we are considering.\(^3\) This follows easily from the methods of [TZ], but there is at present no written account. Because \( M \) is projective algebraic, the index of \( \mathcal{E} \otimes D^{\otimes n} \) over \( M \), for any coherent sheaf \( \mathcal{E} \), is a polynomial in \( n \); so its knowledge for large \( n \) determines it for all \( n \), including, by extrapolation, \( n = 0 \). Thus, the information contained in \( \mathfrak{M} \), which combines index information for the moduli of bundles of subgroups of \( G \), can be disassembled into its constituent parts; the leading contribution, as \( n \to \infty \), comes from \( M \) itself. When \( M \) is smooth, the “\( \mathcal{E} \)-derivatives” of the \( n \to \infty \) asymptotics of the index of \( \mathcal{E} \otimes D^{\otimes n} \) give integration formulas for Chern polynomials of \( \mathcal{E} \) over \( M \); and the author suggests that the Jeffrey-Kirwan residue formulas for the integrals can be recovered in this manner. What one definitely recovers in the large level limit are Witten’s conjectural formulas [W]. Indeed, there is evidence that the relevant field theories are topological limits of Yang-Mills theory coupled to the WZW model (in other words, the \( G/G \) coset model with a Yang-Mills term); this would fit in nicely with the physical argument of [W].

2. The moduli space \( M \), the moduli stack \( \mathfrak{M} \) and admissible \( K \)-classes

In this section, we recall some background material; some of it is logically needed for the main conjecture, but mostly, it sets the stage for my approach to the question. This is anchored in Thm. [2.8]

(2.1) Recall the set-up of [AB]: let \( \mathcal{A} \) be the affine space of smooth connections, and \( \mathcal{G} \) the group of smooth gauge transformations on a fixed smooth principal \( G \)-bundle \( P \) over \( \Sigma \). The \((0,1)\)-component of such a connection defines a \( \partial \)-operator, hence a complex structure on the principal \( G_\mathbb{C} \)-bundle \( P_\mathbb{C} \) associated to \( P \). We can identify \( \mathcal{A} \) with the space of smooth connections of type \((0,1)\) on \( P_\mathbb{C} \); the latter carries an action of the complexification \( \mathcal{G}_\mathbb{C} \) of \( \mathcal{G} \), and the quotient \( \mathcal{A}/\mathcal{G}_\mathbb{C} \) is the set of isomorphism classes of holomorphic principal \( G_\mathbb{C} \)-bundles on \( \Sigma \) with underlying topological bundle \( P_\mathbb{C} \).

(2.2) The space \( \mathcal{A} \) carries a \( \mathcal{G}_\mathbb{C} \)-equivariant stratification, according to the instability type of the holomorphic bundle. The semi-stable bundles define the open subset \( \mathcal{A}^0 \), whose universal Hausdorff quotient by \( \mathcal{G}_\mathbb{C} \) is a projective algebraic variety \( M \), the moduli space of semi-stable holomorphic \( G_\mathbb{C} \)-bundles over \( \Sigma \). The complex structure is descended from that of \( \mathcal{A}^0 \), in the sense that a function on \( M \) is holomorphic in an open subset if and only if its lift to \( \mathcal{A} \) is so. We can restate this, to avoid troubles relating to holomorphy in infinite dimensions. The gauge transformations that are based at one point \( * \in \Sigma \) act freely on \( \mathcal{A}^0 \), with quotient a smooth, finite-dimensional algebraic variety \( M_* \). This is the moduli of semi-stable bundles with a trivialization of the fibre over \( * \). Its algebro-geometric quotient under the residual gauge group \( G_\mathbb{C} \) is \( M \). The other strata \( \mathcal{A}^\xi \) are smooth, locally closed complex submanifolds, of finite codimension; they are labeled by the non-zero dominant co-weights \( \xi \) of \( G \), which give the destabilizing type of the underlying holomorphic \( G_\mathbb{C} \)-bundle. The universal Hausdorff quotient of \( \mathcal{A}^\xi \) can be identified with a moduli space of semi-stable principal bundles under the centralizer \( G_{\langle \xi \rangle} \) of \( \xi \) in \( G_\mathbb{C} \).

(2.3) The stack \( \mathfrak{M} \) of all holomorphic \( G_\mathbb{C} \)-bundles over \( \Sigma \) is the homotopy quotient \( \mathcal{A}/\mathcal{G}_\mathbb{C} \). As such, it seems that we are using new words for an old object, and so it would be, were our interest confined to ordinary cohomology, \( H^\bullet(\mathfrak{M}; \mathbb{Z}) \). However, we will need to discuss its \( K \)-theory, and

\(^3\) An explicit bound for the power can be given, linear in the highest weight.
the index map to \( \mathbb{Z} \). The abstract setting for this type of question is a homotopy category of analytic spaces or algebraic varieties (see e.g. Simpsons work \([S]\), and, with reference to \( \mathfrak{M} \), \([T2]\)). Fortunately, little of that general abstraction is necessary here. It turns out that \( \mathfrak{M} \) is homotopy equivalent to the quotient, by \( G \)-conjugation, of a principal \( OG \)-bundle over \( G^{2g} \). (This is the homotopy fibre of \( \Pi : G^{2g} \to G \); cf. \([F\mathcal{T}\mathcal{H}\mathcal{L}]\)). This follows from Segal's double coset presentation of \( \mathfrak{M} \) \([PS\], Ch. 8\). As a result, there is a sensible topological definition of \( K^*(M) \), which makes it into an inverse limit of finite modules for the representation ring \( R_G \) of \( G \).

(2.4) As in the cohomological setting of Atiyah-Bott, this \( K^*(M) \) can be shown to surject onto the \( \mathcal{G} \)-equivariant \( K\)-theory of \( A^0 \). The latter can be defined as the \( G \)-equivariant version of \( K^*(M_\alpha) \), for the variety \( M_\alpha \) of \([2.2]\). This is as close as we can get to \( K^*(M) \). When the action of \( G \) on \( M_\alpha \) is free, the two groups coincide, but this only happens when \( G = PU(n) \) and the degree of our bundle \( P \) is prime to \( n \). However, some relation between \( K^0_G(M_\alpha) \) and \( M \) always exists. Namely, every holomorphic, \( G \)-equivariant bundle \( E \) over \( M_\alpha \) has an invariant direct image \( q^*_G(E) \), which is a coherent analytic sheaf over \( M \). (This is the sheaf of \( G \)-invariant holomorphic sections along the fibres of the projection \( q : M_\alpha \to M \). The coherent sheaf cohomology groups of \( q^*_G(E) \) are finite dimensional vector spaces, and the alternating sum of their dimensions is our definition of the \( G \)-invariant index of \( E \) over \( M_\alpha \).

(2.5) A similar construction, applied to a stratum \( A^\xi \), allows us to define the \( \mathcal{G} \)-invariant index of holomorphic vector bundles over it. Because \( A \) is stratified by the \( A^\xi \), there is an obvious candidate for the \( \mathcal{G} \)-invariant index of holomorphic vector bundles over \( A \), as a sum over all \( \xi \). (Contributions from the normal bundle must be taken into account; see \([T2]\), §9). This sum may well be infinite. The task, therefore, is to identify a set of admissible \( K \)-theory classes, for which the sum is finite; we then define that sum to be the index over \( \mathfrak{M} \). For a good class of bundles, this can be done, and shown to agree with a more abstract global definition, as the coherent-sheaf cohomology Euler characteristic over the algebraic site of the stack \( \mathfrak{M} \). I shall not prove any of the assertions above, instead will take a low-brow approach and define directly the \( \mathcal{G} \)-equivariant \( K \)-theory classes. When \( G \) is simply connected, they turn out to generate a dense subring of \( K^*(M; \mathbb{Q}) \), rather in the way that a polynomial ring is dense in its power series completion; and their restrictions to the semi-stable part generate \( K^*_\mathcal{G}(M_\alpha) \otimes \mathbb{Q} \). (This can be deduced from the cohomological result of \([AB]\), by using equivariant Chern characters).

(2.6) Note, first, that the pull-back of the bundle \( P_C \) to \( \Sigma \times A \) carries a natural, \( \mathcal{G} \)-equivariant, holomorphic structure, as defined by the \((0,1)\)-part of the universal connection along \( \Sigma \). We might call this the universal bundle on \( \Sigma \times \mathfrak{M} \). A representation \( V \) of \( G \) defines an associated holomorphic, \( \mathcal{G} \)-equivariant vector bundle \( E^*V \) on \( \Sigma \times A \). (We think of \( E \) as the classifying map of the universal bundle to \( BG \)). Let \( \pi : \Sigma \times A \to A \) be the projection, and fix a square root \( K^{1/2} \) of the canonical bundle on \( \Sigma \). We now associate the following objects to \( V \), which we shall call the tautological classes in \( K^*(M) \).

(i) For a point \( x \in \Sigma \), the restriction \( E^*_xV \) of \( E^*V \) to \( \{x\} \times \mathfrak{M} \);

(ii) The index bundle \( \alpha(V) := R^*\pi_*(E^*V \otimes K^{1/2}) \) along \( \Sigma \) over \( A \);

(iii) For any class \( C \in K_1(\Sigma) \), its slant product with \( E^*V \) (the index of \( E^*V \) along a 1-cycle).

Object (i) is an equivariant holomorphic vector bundle, objects (ii) and (iii) are equivariant \( K^0 \) and \( K^1 \) classes over \( A \), the misnomer “bundle” in (ii) notwithstanding. For example, we can represent (ii) by a \( \mathcal{G} \)-equivariant Fredholm complex based on the relative \( \bar{\partial} \)-operator. (The square root of \( K \) leads to the Dirac, rather than \( \bar{\partial} \) index, and the notation \( \alpha(V) \) stems from the Atiyah index map, which we get when \( \Sigma \) is the sphere.) We shall not consider type (iii) objects in this paper, so we refrain from analyzing them further. Note that the topological type of the bundles in (i) is independent of \( x \); we shall indeed see that their index is so as well.
Any reasonable definition of $K^*(M)$ should include the tautological classes, but another distinguished object plays a crucial role:

(iv) The determinant line bundle $D(V) := \text{det} R^* \pi_*(E^*V)$ over $A$.

This is a holomorphic, $\mathbb{C}_G$-equivariant line bundle, or a holomorphic line bundle over $\mathcal{M}$. When $G$ is semi-simple, such line bundles turn out to be classified by their Chern classes in $H^2(M; \mathbb{Z})$ [11]; in the case of $D(V)$, this is the transgression along $\Sigma$ of $c_2(V) \in H^4(BG)$. Not all line bundles are determinants, but they are fractional powers thereof. The convex hull of the $D(V)$ define the semi-positive cone in the group of line bundles; its interior is the positive cone. In particular, when $G$ is simple and simply connected, $H^2(M; \mathbb{Z}) \cong \mathbb{Z}$, and the positive cone consists of positive powers of a single $D$; for $G = SU(n)$, this is $D(\mathbb{C}^n)$, for the standard representation.

2.7 Definition. An admissible class in $K^*(\mathcal{M})$ is a polynomial in the tautological classes and the semi-positive line bundles.

For simply connected $G$, an admissible $K$-class is a finite sum of terms $p_n \otimes D \otimes^n$, where $n \geq 0$ and $p_n$ is a polynomial in the objects (i)-(iii) above. We can actually allow some small negative values of $n$, but the index of such classes turns out to vanish, so little is gained. The following theorem allows our approach to $K^*(M)$ to get off the ground. To simplify the statements, we assume that $G$ is semi-simple.

2.8 Theorem. (i) The coherent sheaf cohomology groups, over the algebraic site of the stack $\mathcal{M}$, of any admissible class $E \in K^*(\mathcal{M})$, are finite-dimensional, and vanish in high degrees.

(ii) The index $\text{Ind} (\mathcal{M}; E)$ over $\mathcal{M}$, defined as the alternating sum of cohomology dimensions, is also expressible as a sum of index contributions over the Atiyah-Bott strata $\mathcal{A}$. Each contribution is the index of a coherent sheaf over a moduli space of semi-stable $G(\xi)$-bundles. These contributions vanish for large $\xi$.

(iii) For sufficiently positive $D$, all $\xi \neq 0$-contributions of $E \otimes D$ to the index vanish.

(iv) Hence, for sufficiently positive $D$, $\text{Ind} (\mathcal{M}; E \otimes D) = \text{Ind} (M; q_G^*(E \otimes D))$.

Proof. (Sketch) For a product of “evaluation bundles” (2.6.i), the results were proved in [11] and [12]. This generalizes immediately to a family of bundles, parametrized by a product of copies of $\Sigma$, and integration along the curves leads to index bundles (2.6.ii). Slant products with odd $K$-homology classes on that product of Riemann surfaces lead to the same conclusion for arbitrary admissible $E$. □

2.9 Remark. As noted in the introduction, the theorem allows us to determine the index of $q_G^*(E)$ over $M$, for any admissible $E$, if index formulas over $\mathcal{M}$ are known. Indeed, suitable positive line bundles $D$ descend to $M$, and so $\text{Ind} (M; q_G^*(E \otimes D^{\otimes n}))$ is a polynomial in $n$. Its value at $n = 0$ can then be determined from large $n$, where it agrees with $\text{Ind} (\mathcal{M}; E \otimes D^{\otimes n})$.

3. Twistings and higher twistings in $K$-theory

We start with some background on twisted $K$-theory and its equivariant versions. The statements are of the “known to experts” kind, but unfortunately, references do not always exist. They will be proved elsewhere.

\footnote{Larger than the negative of the dual Coxeter number}
(3.1) Let $X$ be a compact, connected space. Units in the ring $K^* (X)$, under tensor product, are represented by the virtual vector bundles of dimension $±1$. A distinguished set of units in the 1-dimensional part $GL_1^+$ is the Picard group $Pic(X)$ of topological line bundles; it is isomorphic to $H^2(X; \mathbb{Z})$, by the Chern class. In the other direction, the determinant defines a splitting

$$GL_1^+ (K^* (X)) \cong Pic(X) \times SL_1 (K^* (X)),$$

where the last factor denotes 1-dimensional virtual bundles with trivialized determinant line.

We shall ignore here the twistings coming from the group \{±1\}. The splitting \(3.2\) refines to a decomposition of the spectrum $BU_\otimes$ of 1-dimensional units in the classifying spectrum for complex $K$-theory $\underline{MSL}$; in self-explanatory notation, we have a factorization

$$BU_\otimes \cong K (\mathbb{Z}; 2) \times BSU_\otimes.$$

(3.4) A twisting of complex $K$-theory over $X$ is a principal $BU_\otimes$-bundle over that space. By \(3.3\), this is a pair $\tau = (\delta, \chi)$ consisting of a determinantal twisting $\delta$, which is a $K(\mathbb{Z}; 2)$-principal bundle over $X$, and a higher twisting $\chi$, which is a $BSU_\otimes$-torsor. Twistings are classified, up to isomorphism, by a pair of classes $[\delta] \in H^3(X; \mathbb{Z})$ and $[\chi]$ in the generalized cohomology group $H^1(X; BSU_\otimes)$. This last group has some subtle features over $\mathbb{Z}$; rationally, however, $BSU_\otimes$ is a topological abelian group, isomorphic to $\prod_{n \geq 2} K(\mathbb{Q}; 2n)$ via the logarithm of the Chern character $ch$. We obtain the following.

3.5 Proposition. Twistings of rational $K$-theory over $X$ are classified, up to isomorphism, by the group $\prod_{n > 1} H^{2n+1}(X; \mathbb{Q})$. \(\square\)

3.6 Remark. The usual caveat applies: if $X$ is not a finite complex, we rationalize the coefficients before computing cohomologies.

The twistings in Prop. \(3.5\), of course, are also the twistings for rational cohomology with coefficients in formal Laurent series $\mathbb{Q}((\beta))$ in the Bott element $\beta$ of degree $(-2)$. This is not surprising, as the classifying spectra $BU_\otimes \mathbb{Q}$ and $K(\mathbb{Q}((\beta)), 0)$ for the two theories are equivalent under $ch$. The isomorphism extends naturally to the twisted theories, by a twisted version of the Chern character, as in §3 of [FHT1], where determinantal twistings were considered:

3.7 Proposition. There is a natural isomorphism $\tau ch : \tau K^* (X; \mathbb{Q}) \to \tau H^* (X; \mathbb{Q}((\beta)))$. \(\square\)

3.8 Remark. The strength of the proposition stems from computability of the right-hand side. Let $(A^*, d)$ be a DGA model for the rational homotopy of $X$, $\eta = \eta(3) + \eta(5) + \ldots$ a cocycle representing the twisting, decomposed into graded parts. If we define $\eta' := \beta \eta(0) + \beta^2 \eta(5) + \ldots$, then it turns out that $\tau H^* (X; \mathbb{Q}((\beta)))$ is the cohomology of $A^*((\beta))$ with modified differential $d + \eta' \wedge$. The latter can be computed by a spectral sequence, commencing at $E_2$ with the ordinary $H^* (X; \mathbb{Q}((\beta)))$, and with third differential $\beta \eta(3)$ (cf. [FHT1]).

(3.9) Thus far, the splitting \(3.3\) has not played a conspicuous role: rationally, all twistings can be treated uniformly, with $log ch$ playing the role that $c_1$ plays for determinantal twistings. Things stand differently in the equivariant world, when a compact group $G$ acts on $X$. There are equivariant counterparts to \(3.1\) and \(3.3\), namely, a spectrum $BU_\otimes^G$ of equivariant $K$-theory units, factoring into the equivariant versions of $K(\mathbb{Z}; 2)$ and $BSU_\otimes$. However, the group analogous to \(3.3\), $\prod_{n > 1} H^{2n+1}_G(X; \mathbb{Q})$, no longer classifies twistings for rational equivariant $K$-theory, but only for its augmentation completion. The reason is the comparative dearth of units in the representation ring $R_G$ of $G$, versus its completion.

The easily salvaged part in \(3.2\) is the equivariant Picard group. Realizing twistings in $H^3_G(X; \mathbb{Z})$ by equivariant projective Hilbert bundles allows a construction of the associated
Admissible twistings $\tau$ are classified, up to isomorphism, by a pair of classes $[\delta] \in H^3_G(X; \mathbb{Z})$ and a “higher” class $[\chi(t)]$ in the generalized equivariant cohomology group $[X; B (1 + tBU [[t]])_G]$. One way to define a higher admissible twisting over $G$, equivariant for the conjugation action, uses a (twice deloopable) exponential morphism from $BU^G[[t]]$ to $1 + tBU^G[[t]]$ (taking sums to tensor products). Segal’s theory of $\Gamma$-spaces shows that sufficiently natural exponential operations on the coefficient ring $R_G[[t]]$ define such morphisms. An example is the total symmetric power

$$V(t) \mapsto S_t [t \cdot V(t)] = \sum_{n \geq 0} t^n \cdot S^n [V(t)].$$

(3.13) Allowing rational coefficients, the naive exponential,

$$V(t) \mapsto \exp [t \cdot V(t)] = \sum_{n \geq 0} t^n/n! \cdot V(t)^{\otimes n},$$

(3.14) is more closely related to the previous discussion, in the sense that completing at the augmentation ideal takes us to the ordinary $K$-theory of $BG$, and applying $ch$ to the right leads to the earlier identification of $BSU_\otimes \otimes \mathbb{Q}$ with $\prod_{n \geq 2} K(\mathbb{Q}; 2n)$. Whichever exponential morphism we choose, Bott periodicity permits us to regard $BU^G[[t]]_\oplus$-classes of a point as $B^2BU^G[[t]]_\oplus$-classes, and delooping our morphism produces classes in $B^2(1 + tBU^G[[t]])$ of a point. Transgressing once gives higher twistings for $K_G(G)$. In this paper, we shall pursue the exponential in more detail.

### 4. The Index from Verlinde algebras

(4.1) **The Verlinde algebra and its deformations.** Call a twisting $\tau(t) = (\tau_0, \chi(t))$ non-degenerate if the invariant bilinear form $h$ it defines on $g$ via the restriction of $[\tau_0] \in H^2_G(G)$ to $H^2_G \otimes H^1(T)$, is so; call it positive if this same form is symmetric and positive definite. Integrality of $\tau_0$ implies that $h$ defines an isogeny from $T$ to the Langlands dual torus, with kernel a finite, Weyl-invariant subgroup $F \subset T$. Recall the following result from [FHT1], [FHT2], referring to determinantal twisting $\tau = (\tau_0, 0)$. For simplicity, we restrict to the simply connected case. Let $\sigma$ be the twisting coming from the projective cocycle of the Spin representation of the loop group; it restricts to the dual Coxeter number on $H^3$ for each simple factor.

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5 Even here, we meet a new phenomenon, in that integral twistings are required to define the rational equivariant $K$-theory; for instance, the torsion part of a twisting in $H^2_G(X; \mathbb{Z})$ affects the rational answer.
4.2 Theorem. Let $G$ be simply connected. For a positive determinantal twisting $\tau$, the twisted $K$-theory $\tau K_G^\dim(G)(G)$ is isomorphic to the Verlinde algebra $V_G(\tau - \sigma)$ of $G$, at a shifted level $\tau - \sigma$. It has the structure of an integral Frobenius algebra; as a ring, it is the quotient of $R_G$ by the ideal of representations whose characters vanish at the regular points of $F$. The trace form $\tau Tr : R_G \to \mathbb{Z}$ sends $V \in R_G$ to

$$\tau Tr(V) = \sum_{f \in F^{reg}/W} ch_V(f) \cdot \frac{|\Delta(f)|^2}{|F|},$$

where $ch_V$ is the character of $V$, $\Delta(f)$ is the Weyl denominator and $|F|$ is the order of $F$.

4.4 Remark. (i) The $R_G \otimes \mathbb{C}$-algebra $\tau K_G^\dim(G) \otimes \mathbb{C}$ is supported at the regular conjugacy classes of $G$ which meet $F$, and has one-dimensional fibres. 
(ii) The trace form (4.3) determines the Frobenius algebra: by non-degeneracy, the kernel of the homomorphism from $R_G$ is the null subspace under the bilinear form $(V, W) \mapsto \tau Tr(V \otimes W)$. 
(iii) After complexifying $R_G$, we can represent $\text{Tr}$ by integration against an invariant distribution on $G$. The latter is the sum of $\delta$-functions on the conjugacy classes in (ii), divided by the order of $F$; the factor $|\Delta(f)|^2$ is the volume of the conjugacy class.
(iv) The result holds for connected groups with torsion-free $\pi_1$, although some care must be taken with the ring structure when the adjoint representation does not spin [FHT1]. When $\pi_1$ has torsion or $G$ is disconnected, $\tau K_G^\dim(G)$ is still the Verlinde algebra, but it is larger than the quotient of $R_G$ described in Thm. 4.2 so the trace form on $R_G$ no longer determines $V_G$.

The complexified form of Thm. 4.2 was given a direct proof in [FHT1], using the Chern character to compute the twisted $K$-theory. The trace form was introduced ad hoc, using our knowledge of the Verlinde algebra (§7 of loc. cit.). There is in fact no choice on the matter, and the entire Frobenius structure is determined topologically (see the proof of Thm. 4.9 below).

We now incorporate higher twistings in Theorem 4.2. We tensor with $\mathbb{C}$ for convenience.

4.5 Theorem. Let $G$ be simply connected, $\tau(t) = (\tau_0, \chi(t))$ an admissible twisting with positive determinantal part $\tau_0$. The twisted $K$-theory $\tau K_G^\dim(G) \otimes [[t]]$ is a Frobenius algebra, which is a quotient of $R_G[[t]] \otimes \mathbb{C}$, and a flat deformation of the Verlinde algebra at level $\tau_0$.

4.6 Remark. The use of complex higher twistings forces us to tensor with $\mathbb{C}$. The use of integral twistings of the type (3.13) would lead to a similar result over $\mathbb{Z}$, but as our goal here is an index formula, nothing is lost over $\mathbb{C}$.

Proof. (Idea) All statements follow by computing the Chern character, exactly as in [FHT1]; the completions of $\tau K_G^\dim(G) \otimes \mathbb{C}[[t]]$ at conjugacy classes in $G_\mathbb{C}$ are calculable by spectral sequences as in (3.3). Away from $F$, the $E_2$ term of this sequence is nil. At singular points of $F$, this is not so, but the third differential, which stems from the determinant of the twisting, is exact; so the limit is null again. At regular points of $F$, the same third differential resolves one copy of $\mathbb{C}[[t]]$ in degrees of the parity $\dim G$, and zero otherwise; so the sequence collapses there, and the abutment is a free $\mathbb{C}[[t]]$-module of rank $1$.

(4.7) As explained in (4.3), the Frobenius algebra is completely determined by the trace form $\tau Tr : R_G[[t]] \to \mathbb{C}[[t]]$. At $t = 0$, this is given by an invariant distribution $\varphi_0$ on $G$, specifically, $1/|F|$ times the sum of $\delta$-functions on the regular conjugacy classes meeting $F$. We must describe how this varies with $t$. Recall that the determinantal part $\tau(0) = (\tau_0, 0)$ of the twisting defines an invariant metric $h$ on $\mathfrak{g}$. We will now associate to the unit $\exp(t \cdot V)$ a one-parameter family of conjugation-invariant coordinate changes on the group $G$. More precisely, this is a (formal) path in the complexified (formal) group of automorphisms of the variety $G/G$; or, even more precisely, a formal 1-parameter family of automorphisms of the representation ring $R_G \otimes \mathbb{C}$.
Because $G/G = T/W$, it suffices to describe this on the maximal torus $T$. In flat coordinates $\exp(\xi)$, where $\xi \in \mathfrak{t}$, this is
\[ \xi \mapsto \xi + t \cdot \nabla \left[ ch_V \left( \exp(\xi) \right) \right], \tag{4.8} \]
the gradient being computed with respect to the metric $h$ on $\mathfrak{t}$.

**4.9 Theorem.** The trace form $^\bullet Tr : R_G[[t]] \rightarrow \mathbb{C}[[t]]$ is integration against the invariant distribution $\varphi_t$ on $G_C$, obtained from $\varphi_0$, the distribution associated to $\tau_0 = (\tau_0, 0)$, by the formal family of coordinate changes \([4.8]\) on $G_C$.

**4.10 Remark.** It is no more difficult to give the formula for more general exponential morphisms $V \mapsto \Phi_t(V)$. We assume $\Phi_t$ compatible with the splitting principle, via restriction to the maximal torus; in other words, its value on any line $L$ is a formal power series in $t$, with coefficients Laurent polynomials in $L$. Then, $\log \Phi_t$ extends by linearity to an additive map $t \otimes R_T \rightarrow R_T[[t]]$, and the required change of coordinates is $\xi \mapsto \xi + \log \Phi_t(\nabla ch_V)$, the metric being used to define the gradient. For instance, the symmetric power twisting \([3.13]\) arises from $\Phi_t(L) = (1 - tL)^{-1}$; when $G = S^1$, $ch_V(u) = \sum c_n u^n$, and we take level $h$ for the determinantal part, we get the change of variable $u \mapsto u \cdot \prod (1 - tu^n)^{-nc_n/h}$.

**Proof.** (Sketch) The 2D field theory structure of $^\bullet K_G(G)$ requires the trace form to be the inverse of the bilinear form which is the image of $1 \in K_G(G)$ in $^\bullet K_G(G)^{\otimes 2}$, under the anti-diagonal morphism of spaces $G \rightarrow G \times G$ (and the diagonal inclusion of the acting groups). By localization to the maximal torus $T$, it suffices to check the proposition for tori. (The Euler class of the inclusion $T \subset G$ is responsible for the factor $|\Delta|^2$). Now, the twisting $\tau$ enters the computation of this direct image only via the holonomy representation $\pi_1(T) \rightarrow GL_1(R_T[[t]])$ it defines. Via the metric $h$, the determinantal twisting $\tau_0$ assigns to any $p \in \pi_1(T)$ a weight of $T$, which gives a unit in $R_T$, and this defines the holonomy representation for $\tau_0$. The change of coordinates \([4.8]\) has the precise effect of converting this holonomy representation to the one associated to $\tau$.

**4.11 Index formulas.** In the simply connected case, the class $[\tau_0] \in H^2_G(G; \mathbb{Z})$ determines a unique holomorphic line bundle over $\mathfrak{M}$ (cf. \([2.9]\)) which we call $\mathcal{O}(\tau_0)$. The simplest relation between the Verlinde algebras and indices of bundles over $\mathfrak{M}$ is that $\text{Ind} \left( M; \mathcal{O}(\tau_0 - \sigma) \right)$ is the partition function of the surface $\Sigma$, in the 2D topological field theory defined by $V_G(\tau_0 - \sigma)$. Recall \([1.2]\) that $V_G \otimes \mathbb{C}$ is isomorphic, as an algebra, to a direct sum of copies of $\mathbb{C}$, supported on the regular Weyl orbits in $F$. The traces of the associated projectors are the *structure constants* $\theta_f$ ($f \in F^{\text{reg}}/W$) of the Frobenius algebra; their values here are $|\Delta(f)|^2/|F|$. The partition function of a genus $g$ surface is the sum $\sum \theta_f^{-g}$, which leads to one version of the Verlinde formula
\[ \text{Ind} \left( M; \mathcal{O}(\tau_0 - \sigma) \right) = \sum_{f \in F^{\text{reg}}/W} |\Delta(f)|^{2-2g} \cdot |F|^{g-1}. \tag{4.12} \]

**4.13 Remark.** The downshift by $\sigma$ stems from our use of the $\bar{\partial}$-index; the Dirac index would refer to $\mathcal{O}(\tau_0)$. However, there is no definition of the Dirac index in the singular case, and even less for the stack $\mathfrak{M}$.

The generalization of \([4.12]\) to higher twistings is one form of the main conjecture. Recall the index bundle $\alpha(V)$ over $\mathfrak{M}$ associated to a representation $V$ of $G$, and call $\theta_f(t)$, $f \in F^{\text{reg}}/W$, the structure constants of the Frobenius algebra $^\bullet K_G^{\text{dim}G}(G) \otimes \mathbb{C}[[t]]$ over $\mathbb{C}[[t]]$.

**4.14 Conjecture.** $\text{Ind} \left( \mathfrak{M}; \mathcal{O}(\tau_0 - \sigma) \otimes \exp[t\alpha(V)] \right) = \sum_{f \in F^{\text{reg}}/W} \theta_f(t)^{1-g}$. 

9
4.15 Remark. (i) Expansion in $t$ allows the computation of indices of $\mathcal{O}(\tau_0 - \sigma) \otimes \alpha(V)^{\otimes n}$ from (4.8). More general expressions in the index bundles $\alpha(V)$, for various $V$, are easily obtained by the use of several formal parameters. One can also extend the discussion to include odd tautological generators (2.6(iii)), but we shall not do so here.

(ii) The change in $\theta_f(t)$ is due both to the movement of the point $f$ under the flow, and to the change in the volume form, under the change of coordinates (4.8).

4.16 Transgressed twistings and the product of commutators. Let us move to a more sophisticated version of the conjecture, which incorporates the evaluation bundles (2.6(i)). We refer to [FHT1], §7 for more motivation, in connection to loop group representations. Recall the "product of commutators" map $\Pi : G^{2g} \to G$. If we remove a disk $\Delta$ from $\Sigma$, this map is realized by the restriction to the boundary of flat connections on $\Sigma \setminus \Delta$, based at some boundary point; the conjugation $G$-action forgets the base-point. The homotopy fibre of $\Pi$ is the $\Omega G$-bundle over $G^{2g}$ mentioned in (2.3) the actual fibre over $1 \in G$ is the variety of based flat $G$-bundles on $\Sigma$, and its quotient by $G$-conjugation is $M$.

4.17 The twistings $\tau$ of interest to us are transgressed from $G$-equivariant $B^2BU[[t]]$ classes of a point. For determinantal twistings, we are looking at the transgression from $H^4(BG)$ to $H^2_G(G)$; and, if $G$ is simply connected, all determinantal twistings are so transgressed. In general, transgression is the integration along $S^1$ of the $B^2BU$ class on the universal flat $G$-bundle over $S^1$, which is pulled back by the classifying map of the bundle. The relevant feature of a transgressed twisting $\tau$ is that its (equivariant) pull-back to $G^{2g}$, via $\Pi$, is trivialized by the transgression over $\Sigma - \Delta$. This trivialization gives an isomorphism

$$K^*_G(G^{2g}) \cong \Pi^* \tau K^*_G(G^{2g}),$$

which allows us to define a class $\tau 1 \in \Pi^* \tau K^*_G(G^{2g})$ without ambiguity.

Recall from (2.9) that the homotopy fibre of $\Pi$ over $1 \in G$, when viewed $G$-equivariantly, is represented by the stack $\mathfrak{M}$. Thereon, we have two trivializations of the equivariant twisting $\Pi^* \tau$: one lifted from the base $\{1\}$, and one coming from transgression over $\Sigma \setminus \Delta$. The difference of the two is an element of $K^0(\mathfrak{M})[[t]]$. For the twisting $\tau_0$, this is the line bundle $\mathcal{O}(\tau_0)$; for admissible twistings, it will be a formal power series in $t$, with admissible $K$-class coefficients. For transgressed twistings based on the exponential morphism (3.14), we obtain the exponential $\exp[t\alpha(V)]$ of the index bundle $\alpha(V)$.

4.19 Conjecture. Ind $\mathfrak{M} ; \mathcal{O}(\tau_0 - \sigma) \otimes \exp[t\alpha(V)] = \tau Tr (\Pi; \tau 1) \in \mathbb{C}[[t]]$.

4.20 Remark. Equality of the right-hand sides of (4.14) and (4.19) is part of the definition of the 2D field theory (Frobenius algebra) structure on $\tau K^{\dim G}_G(G)$. The only check there is has been incorporated into Thm. (4.9) which describes the trace map.

The last formulation has the advantage of allowing us to incorporate the evaluation bundles (2.6(i)). Let $W$ be another representation of $G$, and call $[W]$ the image class in $\tau K^{\dim G}_G(G)$.

4.21 Amplification. Ind $\mathfrak{M} ; \mathcal{O}(\tau_0 - \sigma) \otimes \exp[t\alpha(V)] \otimes E^*_W = \tau Tr ([W] \cdot \Pi; \tau 1)$.

5. The index formula by virtual localization

In this section, we explain how the most naive localization procedure, from $G$ to its maximal torus, gives rise to an index formula for admissible $K$-classes, which agrees with Conjecture (1.21). There is an intriguing similarity here with localization methods used by Blau and Thomas [BT] in their path-integral calculations. We emphasize, however, that, in twisted $K$-theory, the localization formula from $G$ to its maximal torus is completely rigorous, and can be applied to
The index of an admissible class over $M$.

(ii) The “one-half” corrects for the double-counting of components in $\mathcal{M}_T$.

5.8 Conjecture. The index of an admissible class over $\mathcal{M}$ is one-half the index of its restriction to $\mathcal{M}_T$, divided by the equivariant $K$-theory Euler class of the conormal bundle $\nu^\ast$. 

5.9 Remark. (i) The index over $\mathcal{M}_T$ is defined as integration over each $J_d$, summation over degrees $d \in \mathbb{Z}$, and, finally, selection of the $T$-invariant part. At the third step, we shall see that the character of the $T$-representation obtained from the first two steps is a distribution over $T$, supported at the regular points of $F$ (see [12]). Miraculously, this corrects the problem which makes Conjecture 5.8 impossible at first sight: the equivariant Euler class of $\nu^\ast$ is singular at the singular conjugacy classes of $G$, so there is no well-defined index contribution over an individual component $J_d$. The sum over $d$ acquires a meaning by extending the resulting distribution by zero, on the singular conjugacy classes.

(ii) The “one-half” corrects for the double-counting of components in $\mathcal{M}_T$, since opposite $T$-bundles induce isomorphic $G$-bundles. In general, we divide by the order of the Weyl group.
We need the Chern character of the equivariant $K$-Euler class of $\nu^*$. The first two terms are sums of line bundles, and they contribute a multiplicative factor of

$$
(1 - u^2 e^{2\eta})^{g-1+2d} (1 - u^{-2} e^{-2\eta})^{g-1-2d} = (-1)^{g-1} (u e^{\eta} - u^{-1} e^{-\eta})^{2g-2} \cdot (u e^{\eta})^{4d}.
$$

Now the log of the Euler class is additive, and we have

$$
4\eta \cdot u^2 e^{2\eta} = \frac{d}{dx} \left( e^{x\eta} \cdot u^2 e^{2\eta} \right) \bigg|_{x=0}
$$

whence the Chern character of the Euler class of the remaining term is the exponential of

$$
-4 \frac{d}{dx} \left[ \log (1 - e^{x\eta} \cdot u^2 e^{2\eta}) + \log (1 - e^{x\eta} \cdot u^{-2} e^{-2\eta}) \right] \bigg|_{x=0} =
$$

$$
= \frac{4 \eta \cdot u^2 e^{2\eta}}{1 - u^2 e^{2\eta}} + \frac{4 \eta \cdot u^{-2} e^{-2\eta}}{1 - u^{-2} e^{-2\eta}} = -4\eta,
$$

and we conclude that the Chern character of the $K$-theory Euler class of $\nu^*$ is

$$
(-1)^{g-1} (u e^{\eta} - u^{-1} e^{-\eta})^{2g-2} \cdot (u e^{\eta})^{4d} \cdot e^{-4\eta}.
$$

We can now write the index formula asserted by Conjecture 5.8. For an admissible $K$-class of the form $D^{\otimes h} \otimes \mathcal{E}$, where $\mathcal{E}$ is a polynomial in the classes (2.6.i–iii), this is predicted to be the $u$-invariant part in the sum

$$
\frac{1}{2} \sum_{d \in \mathbb{Z}} (-1)^{g-1} \cdot \int_{J_d} ch(\mathcal{E}) \cdot \frac{u^{2(h+2)d} \cdot e^{2(h+2)(d-1)\eta}}{(u e^{\eta} - u^{-1} e^{-\eta})^{2g-2}}.
$$

Noting, from (5.3), that $ch(\mathcal{E})$ has a linear $d$-term for each factor $\alpha(V)$, it is now clear, as was promised in Remark (5.9.i), that (5.16) sums, away from the singular points $u = \pm 1$, to a finite linear combination of $\delta$-functions and their derivatives, supported at the roots of unity of order $2(h+2)$. Integration over the torus is somewhat simplified by the formal change of variables $u \mapsto u e^{\eta}$, and the identification $(u - u^{-1})^2 = -|\Delta(u)|^2$ in terms of the Weyl denominator leads to our definitive answer:

$$
\text{Ind}(\mathfrak{g}; D^{\otimes h} \otimes \mathcal{E}) = \frac{1}{4\pi i} \oint \left[ \sum_{d \in \mathbb{Z}} \frac{u^{2(h+2)d}}{\Delta(u)^{2g-2}} \cdot \int_{J_d} ch(\mathcal{E}) \cdot e^{2(h+2)\eta} \right] \frac{du}{u},
$$

where the distribution in brackets is declared to be null at $u = \pm 1$.

**Example 1: Evaluation bundles.** Let $\mathcal{E}$ be an evaluation bundle (2.6.i), $\mathcal{E} = E_x^* V$. Then,

$$
\int_{J_d} ch(\mathcal{E}) \cdot e^{-2(h+2)\zeta} = (2h + 4)^g \cdot ch_V(u),
$$

$$
\sum_{d \in \mathbb{Z}} u^{2(h+2)d} = \frac{1}{2h + 4} \sum_{\zeta \in 2^{h+4} = 1} \delta_{\zeta}(u),
$$

and the index formula (5.17) reduces to the Verlinde formula for an evaluation bundle,

$$
\text{Ind}(\mathfrak{g}; D^{\otimes h} \otimes E_x^* V) = \sum_{\zeta \in 2^{h+4} = 1} \frac{(2h + 4)^{g-1}}{|\Delta(\zeta)|^{2g-2}} \cdot ch_V(\zeta).
$$
(5.22) Example 2: Exponentials of index bundles. Let $V$ be any representation of $G$, with character a Laurent polynomial $f(u) = \sum f_n u^n$. We define $\hat{f}(u) := \sum n \cdot f_n u^n$. For the index bundle $\alpha(V)$, we have

$$\text{ch} \alpha(V) = \text{ch}(R^* \pi_* V) - (g - 1) \cdot \text{ch}(E^*_x V) = d \cdot \hat{f}(ue^\eta) - \eta \cdot \hat{f}(ue^\eta). \quad (5.23)$$

We compute the integral over $J_d$ for insertion in (5.17) (after changing variables $u \mapsto ue^\zeta$):

$$\int_{J_d} \exp \left[ \td \cdot \hat{f}(u) \right] \cdot \exp \left[ - \left( 2h + 4 + t \hat{f}(u) \right) \eta \right] = \left( 2h + 4 + t \hat{f}(u) \right)^g \cdot \exp \left[ \td \cdot \hat{f}(u) \right], \quad (5.24)$$

whereupon the sum in (5.17) becomes again a sum of $\delta$-functions

$$\sum_{d \in \mathbb{Z}} \left[ u \cdot \exp \left( t \hat{f}(u) \left( 2h + 4 \right) \right) \right]^{(2h+4)d} = \sum_{\zeta^{-2h+4} = 1} \delta \left( u \cdot \exp \left( t \hat{f}(u)/(2h+4) \right) \right)^g, \quad (5.25)$$

the denominator arising from the change of variables inside the $\delta$-function. The index formula becomes now a sum over the solutions $\zeta$, with positive imaginary part, of $\zeta^{2h+4} \cdot \exp \left( t \hat{f}(\zeta) \right) = 1$:

$$\text{Ind} \left( \mathfrak{g} \mathfrak{m}; D^\otimes h \otimes \exp [t \alpha(V)] \right) = \sum_{\zeta} \left[ \frac{2h + 4 + t \hat{f}(\zeta)}{|\Delta(\zeta)|^2} \right]^{g-1}. \quad (5.26)$$

This has precisely the form predicted by Conjecture [5.21]. Note that the numerator differs from (5.21) by the volume scaling factor in the coordinate change

$$u \mapsto u_t := u \cdot \exp \left( t \hat{f}(u)/(2h+4) \right).$$

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