P-Loop Oscillator on Clifford Manifolds and Black Hole Entropy

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Abstract

A new relativity theory, or more concretely an extended relativity theory, actively developed by one of the authors incorporated 3 basic concepts. They are the old idea of Chew about bootstrapping, Not-tale's scale relativity, and enlargement of the conventional time-space by inclusion of noncommutative Clifford manifolds where all p-branes are treated on equal footing. The latter allowed one to write a master action functional. The resulting functional equation is simplified and applied to the p-loop oscillator. Its respective solution is a generalization of the conventional point oscillator. In addition, it exhibits some novel features: an emergence of two explicit scales delineating the asymptotic regimes (Planck scale region and a smooth region of a conventional point oscillator). In the most interesting Planck scale regime, the solution reproduces in an elementary fashion the basic relations of string theory (including string tension quantization). In addition, it is shown that comparing the massive (super) string degeneracy with the p-loop degeneracy one is arriving at the proportionality between the Shannon entropy of a p-loop oscillator in D-dimensional space and the Bekenstein-Hawking entropy of the black hole of a size comparable with a string scale. In conclusion the Regge behavior follows from the solution in an elementary fashion.

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1 Introduction

Recently a new relativity was introduced [1]–[8] with a purpose to develop a viable physical theory describing the quantum "reality" without introducing by hand *a priori* existing background. This theory is based upon 3 main concepts:

1) Chew’s bootstrap idea about how every $p$-brane is made of all the others and how an evolving physical system is able to generate its own background in the process.

2) Nottale’s scale relativity [9]–[10] which adopts the Planck scale $\Lambda = 1.62 \times 10^{-35}$ m as the minimum attainable scale in nature.

3) a generalization of the ordinary space-time (the concept most important for our analysis) by introduction of non-commutative C-spaces leading to full covariance of a quantum mechanical loop equation. This is achieved by extending the concepts of ordinary space-time vectors and tensors to non-commutative Clifford manifolds (it explains the name C-space) where all $p$-branes are unified on the basis of Clifford multivectors. As a result, there exists a one-to-one correspondence between single lines in Clifford manifolds and a nested hierarchy of 0-loop, 1-loop,..., $p$-loop histories in $D$ dimensions ($D=p-1$) encoded in terms of hypermatrices.

The respective master action functional $S\{\Psi[X(\Sigma)]\}$ of quantum field theory in C-space [1, 4] is

$$ S\{\Psi[X(\Sigma)]\} = \int [DX(\Sigma)] \left[ \frac{1}{2} \left( \frac{\delta \Psi}{\delta X} * \frac{\delta \Psi}{\delta X} + E^2 \Psi * \Psi \right) + \frac{1}{3} \Psi * \Psi * \Psi + \frac{2}{4} \Psi * \Psi * \Psi \right]. $$

where $\Sigma$ is an invariant evolution parameter (a generalization of the proper time in special relativity) such that

$$ (d\Sigma)^2 = (d\Omega_{p+1})^2 + \Lambda^{2p}(dx_{\mu}dx^{\mu}) + \Lambda^{2(p-1)}d\sigma_{\mu\nu}d\sigma^{\mu\nu} + ... $$

$$ + (d\sigma_{\mu_1\mu_2...\mu_{p+1}}d\sigma^{\mu_1\mu_2...\mu_{p+1}}) $$

is a Clifford algebra-valued line "living" on the Clifford manifold outside space-time, $\Lambda$ is the Planck scale that allows to combine objects of different dimensionality in Eqs.(2,3) and the multivector $X$ Eq.(3) incorporates both a point history given by the ordinary (vector) coordinates $x_\mu$ and the
holographic projections of the nested family of all $p$-loop histories onto the embedding coordinate spacetime hyperplanes: $\sigma_{\mu\nu},...\sigma_{\mu_1\mu_2...\mu_{p+1}}$. The scalar (from the point of view of ordinary Lorentz transformations but not from the C-space point of view) $\Omega_{p+1}$ is the invariant proper $p+1 = D$-volume associated with a motion of a maximum dimension $p$-loop across the $p+1 = D$-dim target spacetime. Since a Cliffordian multivector with $D$ basis elements (say, $e_1,e_2,...,e_D$) has $2^D$ components, our vector $X$ has also $2^D$ components.

Generally speaking, action (I) generates a master Cantorian (strongly fractal) field theory with a braided Hopf quantum Clifford algebra. This action is unique in a sense that the above algebra selects terms allowed by the action. The quadratic terms are the usual kinetic and mass squared contributions; the cubic terms are the vertex interactions; the quartic terms are the braided scattering of four Clifford lines. In what follows we restrict our attention to a truncated version of the theory by applying it to a linear $p$-loop oscillator. This truncation is characterized by the following 3 simplifications. First, we dropped nonlinear terms in the action, that is the cubic term (corresponding to vertices) and the quartic (braided scattering) term. Secondly, we freeze all the holographic modes and keep only the zero modes which would yield conventional differential equations instead of functional ones. Thirdly, we assume that the metric in C-space is flat.

2 Linear Non-Relativistic $p$-loop Oscillator

We begin this section by properly defining what one means by "relativistic" from the point of view of the new relativity. The complete theory is the master field theory whose action functional admits a noncommutative braided quantum Clifford algebra. As a result of the postulated simplifications, we are performing a reduction of the field theory to an ordinary quantum mechanical theory. It must be kept in mind that fields are not quantized wave functions. For this reason the wave equations that we will be working with refer to a nonrelativistic theory in C-spaces.

Hence, using all these restrictive assumptions, we obtain from the action (I) a C-space $p$-loop wave equation for a linear oscillator

\[
\left\{ -\frac{1}{2\Lambda} \left[ \frac{\partial^2}{\partial x^2} + \Lambda^2 \frac{\partial^2}{(\partial \sigma_{\mu\nu})^2} + \Lambda^4 \frac{\partial^2}{(\partial \sigma_{\mu\nu\rho})^2} + ... + \frac{\partial^{2p+2}}{(\partial \Omega_{p+1})^2} \right] \right\} \Psi = T\Psi
\]

(4)
where \( \frac{\partial^2}{(\partial x_\mu)^2} = g^{\mu\nu} \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x_\nu} \), \( \frac{\partial^2}{(\partial \sigma_{\mu\nu})^2} = G^{\mu\nu\rho\tau} \frac{\partial}{\partial \sigma_\mu} \frac{\partial}{\partial \sigma_\nu} \), etc, \( G^{\mu\nu\rho\tau} \) is some suitably symmetrized product of the two ordinary metric-tensors \( g^{\mu\nu} \).

\( T \) is tension of the spacetime-filling \( p \)-brane, \( D = p + 1 \), \( m_{p+1} \) is the parameter of dimension \( (mass)^{p+1} \), parameter \( L \) (to be defined later) has dimension \( length^{p+1} \) and we use units \( \hbar = 1, c = 1 \). A generalized correspondence principle\(^1\) allows us to introduce the following qualitative correspondence between the parameters \( m_{p+1}, L \), and mass \( m \) and amplitude \( a \) of a point (particle) oscillator:

\[
\begin{align*}
    m_{p+1} \text{("mass")} & \longleftrightarrow m, \\
    L \text{("amplitude")} & \longleftrightarrow a
\end{align*}
\]

We rewrite Eq.(4) in the dimensionless form as follows

\[
\left\{ \frac{\partial^2}{\partial x_\mu^2} + \frac{\partial^2}{\partial \tilde{\sigma}_{\mu\nu}^2} + ... - (\tilde{\Omega}^2 + \tilde{x}_\mu^2 + \tilde{\sigma}_{\mu\nu}^2 + ...) + 2T \right\} \Psi = 0 \quad (5)
\]

where \( T = (T/\sqrt{A} m_{p+1}) \) is the dimensionless tension and \( A \) is a scaling parameter that will be determined below.

\[
\begin{align*}
    \tilde{x}_\mu &= A^{1/4} \frac{\Lambda}{L} x_\mu, \\
    \tilde{\sigma}_{\mu\nu} &= A^{1/4} \sigma_{\mu\nu} \frac{\Lambda^{p-1}}{L}, ..., \tilde{\Omega}_{p+1} &= A^{1/4} \frac{\Omega^{p+1}}{L}
\end{align*}
\]

are the dimensionless arguments, \( \tilde{x}_\mu \) has \( C_D^1 \equiv D \) components, \( \tilde{\sigma}_{\mu\nu} \) has \( C_D^2 \equiv \frac{D!}{(D-2)!2!} \) components, etc.

Inserting the usual Gaussian solution for the ground state into the wave equation (4) we get the value of \( A \):

\[
A \equiv m_{p+1} L^2 / \Omega^{p+1}
\]

Without any loss of generality we can set \( A = 1 \) by absorbing it into \( L \). This will give the following geometric mean relation between the parameters \( L, m_{p+1}, \) and \( \Lambda \)

\[
L^2 = \Lambda^{p+1}/m_{p+1} \Rightarrow \Lambda^{p+1} < L < \frac{1}{m_{p+1}}
\]

meaning that there are three scaling regimes. The scale represented by generalized Compton wavelength \( (1/m_{p+1})^{(1/p+1)} \) will signal a transition from a...
smooth continuum to a fractal (but continuous) geometry. The scale $L$ will signal both a discrete and fractal world (like El Naschie’s Cantorian-fractal spacetime models and $p$-adic quantum mechanics) and $\Lambda$ the quantum gravitational regime.

The dimensionless coordinates then become

$$\tilde{x}_\mu = \sqrt{\Lambda^{p+1} m_{p+1} x_\mu / \Lambda}, \quad \tilde{\sigma}_{\mu\nu} = \sqrt{\Lambda^{p+1} m_{p+1} \sigma_{\mu\nu} / \Lambda^2}, \ldots$$

$$\tilde{\Omega}_{p+1} = \sqrt{\Lambda^{p+1} m_{p+1} \Omega_{p+1} / \Lambda^{p+1}}$$

The dimensionless combination $\Lambda^{p+1} m_{p+1}$ indicating existence of two separate scales: $\Lambda$ and $(1/m_{p+1})^{1/(p+1)}$ obeys the following double inequality:

$$\sqrt{m_{p+1} \Lambda^{p+1}} < 1 < \sqrt{\frac{1}{m_{p+1} \Lambda^{p+1}}} \quad (6)$$

Relations (6) define two asymptotic regions:

1) the ”discrete-fractal” region characterized by $m_{p+1} \Lambda^{p+1} \sim 1$, or the Planck scale regime, and
2) the ”fractal/smooth phase transition ”, or the low energy region characterized by $m_{p+1} \Lambda^{p+1} << 1$.

Since the wave equation (5) is diagonal in its arguments (it is separable) we represent its solution as a product of separate functions of each of the dimensionless arguments $\tilde{x}_\mu, \tilde{\sigma}_{\mu\nu}, \ldots$,

$$\Psi = \prod_i F_i(\tilde{x}_i) \prod_{j<k} F_{jk}(\tilde{\sigma}_{jk}) \ldots \quad (7)$$

Inserting (7) into (5) we get for each of these functions the Whittaker equation:

$$Z'' - (2T - \tilde{y}^2)Z = 0 \quad (8)$$

where $Z$ is any function $F_i, F_{ij}, \ldots, \tilde{y}$ is the respective dimensionless variable $\tilde{x}_\mu, \tilde{\sigma}_{\mu\nu}, \ldots$, and there are all in all $2^D$ such equations. The bounded solution of (8) is expressed in terms of the Hermite polynomials $H_n(\tilde{y})$

$$Z \sim e^{-\tilde{y}^2/2} H_n(\tilde{y}) \quad (9)$$
Therefore the solution to Eq.(5) is

$$\Psi \sim \exp[-(\tilde{x}_\mu^2 + \tilde{\sigma}_{\mu\nu}^2 + ... + \tilde{\Omega}_{p+1}^2)] \prod_i H_n_i(\tilde{x}_i) \prod_{jk} H_{n_{jk}}(\tilde{\sigma}_{jk})... \quad (10)$$

where there are $D$ terms corresponding to $n_1, n_2, ..., n_D$, $D(D - 1)/2$ terms corresponding to $n_{01}, n_{02}, ...$, etc. Thus the total number of terms corresponding to the $N$-th excited state ($N = n_{x1} + n_{x2} + ... + n_{\sigma_{01}} + n_{\sigma_{02}} + ...$) is given by the degree of the Clifford algebra in $D$ dimensions $2^D$.

The respective value of the tension of the $N$-th excited state is

$$T_N = (N + \frac{1}{2} 2^D) m_{p+1} \quad (11)$$

yielding quantization of tension.

Expression (11) is the analog of the respective value of the $N$-th energy state for a point oscillator. The analogy however is not complete. We point out one substantial difference. Since according to a new relativity principle [1]-[8] all the dimensions are treated on equal footing (there are no preferred dimensions) all the modes of the $p$-loop oscillator (center of mass $x_\mu$, holographic modes, $p + 1$ volume) are to be excited collectively. This behavior is in full compliance with the principle of polydimensional invariance by Pezzaglia [12]. As a result, the first excited state is not $N = 1$ (as could be naively expected) but rather $N = 2^D$. Therefore

$$T_1 \to T_{2^D} = \frac{3}{2} (2^D m_{p+1})$$

instead of the familiar $(3/2)m$.

Recalling that $L$ is analogous to the amplitude $a$ and using the analogy between energy $E \sim m_\omega^2 a^2$ and tension $T$, we get $T = m_{p+1} \Omega^2 L^2$. Inserting this expression into Eq.(11) we arrive at the definition of the "frequency" $\Omega$ of the $p$-loop oscillator:

$$\Omega_N = \sqrt{(N + 2^{D-1}) \frac{m_{p+1}}{\Lambda_{p+1}^{p+1}}} \quad (12)$$

where we use $L = \sqrt{\Lambda^{p+1}/m_{p+1}}$.

Having obtained the solution to Eq.(5), we consider in more detail the two limiting cases corresponding to the above defined 1) fractal and 2) smooth regions. The latter (according to the correspondence principle) should be described by the expressions for a point oscillator. In particular, this means
that the analog of the zero slope limit in string theory, the field theory limit, is a collapse of the $p$-loop histories to a point history:

$$\Lambda \to 0, \quad m_{p+1} \to \infty, \quad T \to \infty, \quad \sigma_{\mu\nu}, \sigma_{\mu\nu\rho}, \ldots \to 0, \quad L \to 0.$$ 

and these limits are taken in such a way that the following combination reproduces the standard results of a point-particle oscillator:

$$\tilde{x}_\mu = \frac{x_\mu}{\Lambda} \sqrt{m_{p+1} \Lambda^{p+1}} \to x_\mu / a$$  \hspace{1cm} (13)

where the nonzero parameter $a > \Lambda$ is a finite quantity and is nothing but the amplitude of the usual point-particle oscillator! In string theory, there are two scales, the Planck scale $\Lambda$ and the string scale $l_s > \Lambda$. Without loss of generality we can assign $a \sim l_s$. A large value of $a >> \Lambda$ would correspond to a “macroscopic” string. We shall return to this point when we address the black-hole entropy.

Using Eq.(13) we find $m_{p+1}$ in terms of the other variables:

$$m_{p+1} \to (M_{\text{Planck}})^{p+1} \frac{\Lambda}{a}^2 < (M_{\text{Planck}})^{p+1}$$  \hspace{1cm} (14)

where the Planck mass $M_{\text{Planck}} \equiv 1/\Lambda$. Notice that in the field-theory limit, $\Lambda \to 0$, when the loop histories collapse to a point-history, Eq.(14) yields $m_{p+1} \to \infty$ as could be expected. From Eqs.(11) and (12) follows that in this region

$$T_N \sim (M_{\text{Planck}})^{p+1} \frac{(\Lambda)}{a}^2 < (M_{\text{Planck}})^{p+1}$$

$$\Omega_N \sim (\omega_{\text{Planck}})^{p+1} \frac{\Lambda}{a} < (\omega_{\text{Planck}})^{p+1}$$

$$\omega_{\text{Planck}} = 1/\Lambda$$  \hspace{1cm} (15)

in full agreement with this region’s scales as compared to the Planck scales.

At the other end of the spectrum (discrete-fractal/quantum gravity region) where $m_{p+1} \Lambda^{p+1} \sim 1$ we would witness a collapse of all the scales to only one scale, namely the Planck scale $\Lambda$. In particular, this means that the string scale $a \sim l_s \sim \Lambda$, and the oscillator parameters become

$$\tilde{x}_\mu = \frac{x_\mu}{\Lambda} \sqrt{\Lambda^{p+1} m_{p+1}} \sim \frac{x_\mu}{\Lambda} \quad m_{p+1} \sim \frac{1}{\Lambda^{p+1}} \equiv (M_{\text{Planck}})^{p+1},$$  \hspace{1cm} (16)

The ground state tension is:

$$T_o \sim m_{p+1} \sim \frac{1}{\Lambda^{p+1}}$$
These relations are the familiar relations of string theory. In particular, if we set \( p = 1 \) we get the basic string relation

\[ T \sim \frac{1}{\Lambda T} \equiv \frac{1}{\alpha'} \]

Above we got two asymptotic expression for \( m_{p+1} \)

\[ m_{p+1} = \begin{cases} 
\Lambda^{- (p+1)} (\Lambda/a)^2 & \text{if } \Lambda/a < 1 \\
\Lambda^{- (p+1)} & \text{if } m_{p+1} \Lambda^{p+1} \sim 1, \ a \sim \Lambda
\end{cases} \]

It is suggestive to write \( m_{p+1} \Lambda^{p+1} \) as power series in \((\Lambda/a)^2\) (e.g., cf. analogous procedure in hydrodynamics [13]):

\[ m_{p+1} \Lambda^{p+1} \equiv F(\Lambda/a) = (\Lambda/a)^2[1 + \alpha_1(\Lambda/a)^2 + \alpha_2(\Lambda/a)^4 + ...] \]

where the small coefficients \( \alpha_i \) are such that the series is convergent for \( a \sim \Lambda \).

For example, the dimensionless coordinate \( \tilde{x}_\mu \) given by Eq.(13), becomes in the field theory limit \( \Lambda \to 0 \), after performing a Taylor/binomial expansion of the square root:

\[ \tilde{x}_\mu = \frac{x_\mu}{a} [1 + \frac{\alpha_1}{2} (\Lambda/a)^2 + .....] \to \frac{x_\mu}{a} \]

Notice that \( a \) is a finite nonzero quantity.

If \( p = 1(p + 1 = D = 2) \) then for the ground state \( N = 0 \) Eq.(11) yields the ground energy per unit string length: \( T_{ground} = 2m^2 \) (see footnote\[1\]). Returning to the units \( \hbar \), and introducing \( 1/a = \omega \) (where \( \omega \) is the characteristic frequency) we get (cf.ref \[3\])

\[ \hbar_{eff} = \hbar \sqrt{1 + \alpha_1(\Lambda/a)^2 + \alpha_2(\Lambda/a)^4 + ...} \]

Truncating the series at the second term, we recover the string uncertainty relation \[5\] :

\[ \Delta x \Delta p > \frac{1}{2} |[x, p]| = \frac{\hbar}{2} \Rightarrow \Delta x > \frac{\hbar}{2 \Delta p} + \beta \frac{\Lambda^2}{\hbar} \Delta p \]

\[ ^2 \text{that is for a point oscillator we get } E_{ground} = \hbar \omega / 2 = \sqrt{T_{ground}/8} \]
where $\beta$ is a multiplicative parameter. As $\Lambda \to 0$ one recovers the ordinary Heisenberg uncertainty relations. Interestingly enough, the string uncertainty relation until recently did not have "a proper theoretical framework for the extra term" [14]. On the other hand, this relation has emerged as one of the results of our theoretical model [5].

As a next step we find the degeneracy associated with the $N$-th excited level of the $p$-loop oscillator. The degeneracy $dg(N)$ is equal to the number of partitions of the number $N$ into a set of $2^D$ numbers

$$N = \{ n_{x_1} + n_{x_2} + \ldots + n_{x_D} + n_{\sigma_{\mu\nu}} + n_{\sigma_{\mu\nu\rho}} + \ldots + n_{\Omega_{p+1}} \}.$$ 

This means that there is a collective center of mass excitations, holographic area, volume,...excitations given by the quantum numbers $n_{x_D}; n_{\sigma_{\mu\nu}}; \ldots; n_{\Omega_{p+1}}$ respectively. These collective extended excitations are the true quanta of a background independent quantum gravity. Thus the degeneracy is

$$dg(N) = \frac{\Gamma(2^D + N)}{\Gamma(N + 1) \Gamma(2^D)}$$

where $\Gamma$ is the gamma function.

We compare $dg(N)$ (17) with the asymptotic quantum degeneracy of a massive (super) string state given by Li and Yoneya [16]:

$$dg(n) = \exp \left[ 2\pi \sqrt{n \frac{d_s - 2}{6}} \right]$$

where $d_s$ is the string dimension and $n >> 1$. To this end we equate (18) and degeneracy (17) of the first excited state ($N = 2^D$) of the $p$-loop. This could be justified on physical grounds as follows. One can consider different frames in a new relativity: one frame where an observer sees strings only (with a given degeneracy) and another frame where the same observer sees a collective excitations of points, strings, membranes, $p$-loops, etc. The results pertinent to the degeneracy (represented by a number) should be invariant in any frame.

Solving the resulting equation $dg(N) = dg(n)$ with respect to $\sqrt{n}$ we get

$$\sqrt{n} = \frac{1}{2\pi} \sqrt{\frac{6}{d_s - 2}} \ln \left[ \frac{\Gamma(2^D + 1)}{\Gamma(2^D + 1) \Gamma(2^D)'} \right]$$

The condition $n >> 1$ implies that $D >> 1$ thus simplifying (19). If we set $d_s = 26$ (a bosonic string) and use the asymptotic representation of the logarithm of the gamma function for large values of its argument.
\[ \ln \Gamma(z) = \ln(\sqrt{2\pi}) + (z - 1/2) \ln(z) - z + O(1/z) \]

we obtain the following logarithm of the degeneracy yielding the entropy:

\[ \text{Entropy} = \ln[\text{deg}(n)] \sim \sqrt{n} \approx 2^D \frac{\ln(2)}{2\pi} \sim 2^{D-1} \sim N \quad (20) \]

From Eq. (18) follows that for \( n >> 1 \) the entropy \( = \ln[\text{deg}(n)] \sim \sqrt{n} \). Let us consider a Schwarzschild black hole whose Schwarzschild radius \( R \)

\[ R \sim (GM)^{\frac{1}{d-1}} \]

The black hole mass \( M \) and the string length \( l_s \sim R \) obey the Regge relation

\[ l_s^2 M^2 \sim R^2 M^2 = n \]

which implies that the world sheet area and mass are quantized in Planck units: \( l_s^2 = \sqrt{n} \Lambda^2 \) and \( M^2 = \sqrt{n} M_{\text{Planck}}^2 = \sqrt{n} \Lambda^{-2} \). Li and Yoneya \[16\] obtained the following expression for the Bekenstein-Hawking entropy of a Schwarzschild black hole of a radius \( R \sim l_s \) \[i\] :

\[ S_{BH} \sim \frac{A}{G} \sim \frac{R^{d-2}}{G} \sim (GM)^{\frac{d-2}{d-3}} \]

\[ = G^{\frac{1}{d-3}} M^{\frac{d-2}{d-3}} = RM. \]

From the last two equations Li and Yoneya deduced that the \( (d-2) \)-dimensional horizon area in Planck units was \( S_{BH} \) is

\[ S_{BH} \sim \sqrt{n}. \]

Now taking into account Eq. (20) we obtain

\[ S_{BH} \sim 2^{D-1} \quad (21) \]

This is a rather remarkable fact: the Shannon entropy of a \( p \)-loop oscillator in \( D \)-dimensional space (for a sufficiently large \( D \)), that is a number \( N = 2^D \) (the number of bits representing all the holographic coordinates), is proportional to the Bekenstein-Hawking entropy of a Schwarzschild black hole. For a more rigorous study of the connection between Shannon’s information entropy and the quantum-statistical (thermodynamical) entropy see the work

\[ ^{3} \text{It should be mentioned that the linear relation between the black hole entropy and is justified only for a narrow region of dimensions } D \sim [4, 5]; \text{ in general, this relation loses its linear character.} \]

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\[ \text{10} \]
of Fujikawa \[17\]. Because light is trapped inside, the Black Hole horizon is also an information horizon.

To summarize, expression (20) allows us to easily compare it with the Regge behavior of a string spectrum for large values of \( n \gg 1 \). To this end we associate with each bit of a \( p \)-loop oscillator a fundamental Planck length \( \Lambda \), with area \( \Lambda^2 \), mass \( 1/\Lambda \), etc. The macroscopic string length is of the same order as the Schwarzschild radius, \( R^2 \sim l_s^2 \sim \text{Area}_s = N \times \Lambda^2 \), \( m_s^2 = N \times M_{\text{Planck}}^2 \).

On the other hand, according to (20) \( N \sim \sqrt{n} \) which yields

\[
l_s^2 \sim \sqrt{n} \Lambda^2; \quad m_s^2 \sim \sqrt{n} M_{\text{Planck}}^2
\]

Therefore using the Regge relation between angular momentum and mass-squared, the respective angular momentum \( J \) in units where \( \hbar = 1 \) is of the order:

\[
J = m^2 \times l^2 \sim n M_{\text{Planck}}^2 \Lambda^2 = n
\]

where we use \( M_{\text{Planck}} \Lambda \equiv 1 \) by definition. One can derive the Regge relation more precisely using the Bohr-Sommerfield correspondence principle in quantum mechanics but applied to the string case in question:

\[
\text{Action} = \int P_{\mu\nu} d\Sigma^{\mu\nu} \sim (\text{Tension})(\text{Area}) \sim nh.
\]

For more details of this relationship we refer to [1,3,5] where we have shown that the area-momentum variable \( P_{\mu\nu} \), conjugate to the area tensor \( \Sigma^{\mu\nu} \), obeys the Hamilton-Jacobi equation similar to the point particle case.

Addressing the black holes we still encounter the main remaining question: where does Einstein gravity appear in all of this? The answer lies in the behavior of a self-gravitating gas of loops. This is precisely where the Bohr correspondence limit operates. The large \( n \gg 1 \) limit is similar to the Bohr correspondence for the hydrogen atom (highly excited energy states merge with the continuum) where the product of \( nh \) remains finite when \( n \to \infty; \hbar \to 0 \). As the size of the string gets larger, the \( p \)-loop oscillator begins to resemble a gas of strings, or more precisely a gas of string-bits, a string-polymer. As it gets even larger, the correspondence limit comes into play, and the gas of loops will begin to gravitate. The derivation of Einstein equations for this self-gravitating gas of loops in the large \( n \) limit: the Einstein tensor equals stress energy tensor (with/without a cosmological constant) will be the topic of a future publication.
3 Conclusion

Application of a simplified linearized equation derived from the master action functional of a new (extended) relativity to a $p$-loop oscillator has allowed us to elementary obtain rather interesting results. First of all, the solution explicitly indicates existence of 2 extreme regions characterized by the values of the dimensionless combination $m_{p+1}\Lambda^{p+1}$:

1) the fractal region where $m_{p+1}\Lambda^{p+1} \sim 1$ and 2 scales collapse to one, namely Planck scale $\Lambda$

and

2) the smooth region where $m_{p+1}\Lambda^{p+1} \ll 1$ and we recover the description of the conventional point oscillator. Here 2 scales are present, a characteristic ”length” $a$ that we identified with the string scale $l_s$ and the ubiquitous Planck scale $\Lambda$ ($a > \Lambda$) thus demonstrating explicitly the implied validity of the quantum mechanical solution in the region where $a/\Lambda > 1$.

For a specific case of $p = 1$ (a string) the solution yields (once again in an elementary fashion) one of the basic relations of string theory $T = 1/\alpha'$. In addition, it gives us a string uncertainty relation (this time derived), which in turn is a truncated version of a more general uncertainty relation obtained earlier [5].

Comparing the degeneracy of the first collective state of the p-loop for a very large number of dimensions $D$ with the respective expressions for the massive (super) string theory given by Li and Yoneya [16] we found that the Shannon entropy (which is also in agreement with the logarithm of the degeneracy of states) of a $p$-loop oscillator in $D$-dimensional space (for a sufficiently large $D$), that is a number $N = 2^D$ (the number of bits representing all the holographic coordinates), is proportional to the Bekenstein-Hawking entropy of the Schwarzschild black hole.

The Regge behavior of the string spectrum for large $n >> 1$ also follows from the obtained solution thus indicating its, at least qualitatively correct, character. Thus a study of a simplified model (or ”toy”) problem of a linearized $p$-loop oscillator gave us (with the help of elementary calculations) a wealth of both the well-known relations of string theory (usually obtained with the help of a much more complicated mathematical technique) and some additional relations (the generalized uncertainty relation). This indicates that the approach advocated by a new relativity might be very fruitful, especially if it will be possible to obtain analytic results on the basis of the full master action functional leading to functional nonlinear equations whose study will
involve braided Hopf groups.

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