Empirical spectral processes for stationary state space models

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In this paper, we consider function-indexed normalized weighted integrated periodograms for equidistantly sampled multivariate continuous-time state space models which are multivariate continuous-time ARMA processes. Thereby, the sampling distance is fixed and the driving Lévy process has at least a finite fourth moment. Under different assumptions on the function space and the moments of the driving Lévy process we derive a central limit theorem for the function-indexed normalized weighted integrated periodogram. Either the assumption on the function space or the assumption on the existence of moments of the Lévy process is weaker. Furthermore, we show the weak convergence in both the space of continuous functions and in the dual space to a Gaussian process and give an explicit representation of the covariance function. The results can be used to derive the asymptotic behavior of the Whittle estimator and to construct goodness-of-fit test statistics as the Grenander-Rosenblatt statistic and the Cramér-von Mises statistic. We present the exact limit distributions of both statistics and show their performance through a simulation study.

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1. Introduction

In the context of stationary time series, numerous estimators and testing procedures are based on the periodogram, the empirical version of the spectral density (see Brockwell and Davis (1991); Brillinger (1975); Grenander and Rosenblatt (1984); Priestley (1981)). Typical examples for estimators are the Whittle estimator for a parametric model and the empirical estimator of the spectral distribution function. Classical goodness-of-fit tests for the spectral distribution function are the Grenander-Rosenblatt
and the Cramér-von Mises test statistic. They have in common that they have representations as functionals of empirical spectral processes which are based on weighted integrated periodograms. Therefore, to construct confidence bands for these estimators and asymptotic test statistics the asymptotic behavior of the empirical spectral process is required.

In this paper, we derive the asymptotic behavior of the empirical spectral process for a low-frequency sampled $m$-dimensional stationary state space process $Y = (Y_t)_{t \geq 0}$ which is driven by a $d$-dimensional Lévy process $(L_t)_{t \geq 0}$. A Lévy process $L = (L_t)_{t \geq 0}$ is a stochastic process with stationary and independent increments satisfying $L_0 = 0$ almost surely and having continuous in probability sample paths. Then a continuous-time linear state space model with $A \in \mathbb{R}^{r \times r}$, $B \in \mathbb{R}^{r \times d}$, $C \in \mathbb{R}^{m \times r}$ is defined by

$$
\begin{align*}
    dX_t &= AX_t dt + BdL_t, \\
    Y_t &= CX_t, \quad t \geq 0.
\end{align*}
$$

In particular, any multivariate continuous-time ARMA (MCARMA) process has a representation as a state space model (Marquardt and Stelzer (2007)). They are applied in diversified fields as, e.g., in signal processing, systems and control, high frequency financial econometrics and financial mathematics. In particular, a continuous-time state space model sampled discretely is a discrete-time state space model with strong white noise and an ARMA process with a weak white noise (Thornton and Chambers (2017)).

In applications one often observes discrete data although the data are coming from a continuous-time model. However, one is interested to know the model parameters of the background continuous-time model because then the model parameters for different sampling frequencies are known. Therefore, in this paper, we observe the continuous-time process $Y$ at discrete-time points with distance $\Delta > 0$ and define $Y(\Delta) := (Y_k^{(\Delta)})_{k \in \mathbb{N}_0} := (Y_{k\Delta})_{k \in \mathbb{N}_0}$. The empirical version of the spectral density $f_Y^{(\Delta)}(\omega) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \Gamma_Y^{(\Delta)}(h)e^{-i\omega h}$, where $(\Gamma_Y^{(\Delta)}(h))_{h \in \mathbb{Z}}$ is the autocovariance function of $Y(\Delta)$, is the periodogram

$$
I_{n,Y^{(\Delta)}}(\omega) = \frac{1}{2\pi} \sum_{h=-n+1}^{n-1} \Gamma_{n,Y^{(\Delta)}}(h)e^{-i\omega h} = \frac{1}{2\pi n} \left( \sum_{j=1}^{n} Y_j^{(\Delta)} e^{-ij\omega} \right) \left( \sum_{k=1}^{n} Y_k^{(\Delta)} e^{ik\omega} \right)^\top, \quad \omega \in [-\pi, \pi],
$$

where

$$
\Gamma_{n,Y^{(\Delta)}}(h) := \frac{1}{n} \sum_{k=1}^{n-h} Y_{k+h}^{(\Delta)} Y_k^{(\Delta)}^\top \quad \text{and} \quad \Gamma_{n,Y^{(\Delta)}}(-h) := \Gamma_{n,Y^{(\Delta)}}(h)^\top, \quad h = 0, \ldots, n,
$$

is the sample autocovariance function. Then, for a function $g : [-\pi, \pi] \to \mathbb{C}^{m \times m}$ with $\int_{-\pi}^{\pi} \|g(x)\|^2 dx < \infty$ the normalized weighted integrated periodogram is

$$
E_n(g) := \sqrt{n} \int_{-\pi}^{\pi} g(\omega) \left( I_{n,Y^{(\Delta)}}(\omega) - f_Y^{(\Delta)}(\omega) \right) d\omega.
$$

The topic of this paper is the asymptotic behavior of the empirical spectral process $\langle \text{tr}(E_n(g)) \rangle_{g \in \mathcal{G}_m}$ for some function space $\mathcal{G}_m$ of square-integrable functions in $L^2([-\pi, \pi])$ where tr is the abbreviation for trace. The application of the trace has the advantage that multiplication is getting commutative which is not given in the multivariate setting. An important function space is $\mathcal{G}_m^F := \{Id_m \cdot [0,1] : t \in [0,\pi] \}$ where $Id_m$ is the $m \times m$ dimensional identity matrix. Indeed, the process $\langle E_n(g) \rangle_{g \in \mathcal{G}_m^F}$ reflects the deviation of the empirical spectral distribution function $\int_{-\pi}^{\pi} I_{n,Y^{(\Delta)}}(\omega) d\omega$ from
the spectral distribution function $f_Y^{(g)}(\omega) d\omega$. As we will see, and what is well-known for other time series models, the empirical spectral distribution function is a consistent estimator of the spectral distribution function although the periodogram is not a consistent estimator for the spectral density; see Theorem 3.1 in Fasen (2013). The Grenander-Rosenblatt statistic is then $\sup_{t \in [0, \pi]} |\text{tr}(E_n(\text{Id}_m \cdot f_Y^{(g)}(\omega)))|$ and the Cramér-von Mises statistic is $\int_0^\pi |\text{tr}(E_n(\text{Id}_m \cdot f_Y^{(g)}(\omega)))|^2 dt$, respectively, which are functionals of the empirical spectral process. Often, one uses as well the self-normalized versions of them giving $m^{-1/2} \sup_{t \in [0, \pi]} |\text{tr}(E_n(f_Y^{(g)}(\omega) - 1 \cdot f_Y^{(g)}(\omega)))|$ and $m^{-1} \int_0^\pi |\text{tr}(E_n(f_Y^{(g)}(\omega) - 1 \cdot f_Y^{(g)}(\omega)))|^2 dt$, respectively. In these cases the underlying function space of the empirical spectral process is $\mathcal{G}_m := \{f_Y^{(g)}(\omega) - 1 \cdot f_Y^{(g)}(\omega): t \in [0, \pi]\}$. As we show in this paper, the self-normalized versions have the advantage that under the assumption that the driving Lévy process is a Brownian motion, the limit distribution of $(\text{tr}(E_n(g)))_{g \in \mathcal{G}_m}$ is not dependent on the model parameters anymore. The limit process is a time-scaled Brownian motion on $[0, \pi]$. If the assumption that the Lévy process is a Brownian motion fails, the limit process has an additional Gaussian correction term depending on a fourth order cumulant. A further popular example of a function space is $\mathcal{G}_m := \{g_h : h \in \mathbb{Z}\}$ with $g_h(\omega) = e^{i \omega h}$ which models the sample autocovariance function $(\int_{-\pi}^{\pi} g_h(\omega) f_Y^{(g)}(\omega) d\omega)_{h \in \mathbb{Z}} = (\Gamma_{\eta Y^{(g)}}(h))_{h \in \mathbb{Z}}$. The last example we mention is the Whittle function which is the spectral analogue of the likelihood function in the time domain and its derivatives. They have representations as weighted integrated periodograms and thus, their limit behaviors can be derived via empirical spectral processes, see Example 3.10 for further details. These limit behaviors can be used to prove the asymptotic normality of the Whittle estimator. From these examples we already see the broad applications of the empirical spectral process. Further examples are given in the overview paper Dahlhaus and Polonik (2002).

To the best of our knowledge there are not many papers studying the empirical spectral process for multivariate processes. There is to mention Dahlhaus (1988) who investigates the behavior of the empirical spectral process for a wide class of multivariate time series in discrete time with existing moments of all orders and a weak entropy condition for $\mathcal{G}_m$. However, the moment assumption, which is formally an assumption on the cumulant spectrum, is rather strong. The work was extended to univariate locally stationary time series in Dahlhaus and Polonik (2009). For univariate linear processes Mikosch and Norvaiša (1997) consider the empirical spectral process under a stronger entropy condition on the function space $\mathcal{G}_m$, but only assuming a finite fourth moment. Both papers, Dahlhaus (1988) and Mikosch and Norvaiša (1997), show the convergence of the empirical spectral process to a Gaussian process in the space of continuous functions on $\mathcal{G}_m$. But the proof of Mikosch and Norvaiša (1997) is not obvious, see Remark 3.4. In contrast, Bardet et al. (2008) shows the convergence of the empirical spectral process for a wide class of weakly dependent discrete-time processes in the dual space of $\mathcal{G}_m$ under the assumption of a finite fourth moment and a condition on $\mathcal{G}_m$ without using an entropy condition. Empirical spectral processes for $\alpha$-stable linear processes are studied in Can et al. (2010), however, the cases $\alpha \in (0, 1)$ and $\alpha \in (1, 2)$ are handled differently. In general, there is a conflict of goals having a weaker assumption on the function space $\mathcal{G}_m$ and a weaker moment assumption on the driving Lévy process. It seems challenging to get weak assumptions for both the function space $\mathcal{G}_m$ and the driving Lévy process. The special classes $\mathcal{G}_m$ and $\mathcal{G}_m$ resulting in the empirical spectral distribution function and the self-normalized empirical spectral distribution function, respectively, or the sample standardized empirical spectral distribution function for different univariate time series models were examined in several papers. In particular, for short memory linear time series models with finite variance we refer to Anderson (1993), Grenander and Rosenblatt (1984) and with infinite variance to Klüppelberg and Mikosch (1998). Kokoszka and Mikosch (1997) cover both the finite and the infinite variance case for long memory univariate linear time series. A nice survey for linear
time series models is [Mikosch (1998)]. However, apart from Dahlhaus (1988) these papers restrict to one-dimensional models and it seems that even multivariate ARMA models are not covered in the literature yet. But the results of the present paper hold as well for causal multivariate ARMA models driven by a strong white noise (see Remark 3.3).

In the above mentioned literature as well as in the present paper the proofs are based on the independence assumption of the white noise and it seems challenging to relax that assumption to receive results for causal multivariate ARMA models driven by a weak white noise. A reason is that without the independence assumption it is tricky to calculate higher moments. Unfortunately, a continuous-time state space model sampled discretely is only a multivariate ARMA process with a weak white noise and the exact representations of the ARMA parameters are not known. Therefore, it is difficult to use that approach for deriving the asymptotic behavior of the continuous-time state space model sampled discretely.

The paper is structured on the following way. In Section 2, we present preliminaries on discrete-time sampled state space models and on the function spaces \( G_m \) considered in this paper. The main achievements are presented in Section 3: The weak convergence of the empirical spectral process to a Gaussian process in the space of continuous functions on \( G_m \) in Theorem 3.2 and in the dual space of \( G_m \) in Theorem 3.6 and Theorem 3.8, respectively. The covariance function of the Gaussian process has an explicit representation given there. We distinguish different model assumptions allowing either weaker assumptions on \( G_m \) or weaker moment assumptions on the driving Lévy process, respectively.

The applications of these results to construct goodness-of-fit tests for state space models are given in Section 4. In particular, the Grenander-Rosenblatt and the Cramér-von Mises test statistics are further explored and their performance are demonstrated through a simulation study. Finally, Section 5 contains the proofs of the main theorems. The proofs of some auxiliary results are moved to the Appendix.

**Notation**

For some matrix \( A \), \( \text{tr}(A) \) stands for the trace of \( A \) and \( A^\top \) for its transpose. The Kronecker product of some matrices \( A \) and \( B \) is denoted by \( A \otimes B \). We write \( A[S,T] \) for the \((S,T)\)-th component of \( A \) and \( ||A|| = \sqrt{\sum_{S=1}^{r_1} \sum_{T=1}^{r_2} |A[S,T]|^2} = \sqrt{\text{tr}(A^HA)} \) for the Frobenius norm of \( A \in \mathbb{C}^{r_1 \times r_2} \) where \( A^H \) is the adjoint of \( A \). The Frobenius norm can be replaced by any sub-multiplicative matrix norm with minor adaptions. The \( r \)-dimensional identity matrix is written as \( \text{Id}_r \). Throughout the article, we write the \( h \)-th Fourier coefficient of some square integrable function \( g \) on \([-\pi, \pi]\) as \( \hat{g}_h = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\omega) e^{-i\omega h} d\omega \) for \( h \in \mathbb{Z} \) such that \( g(\omega) = \sum_{h=-\infty}^{\infty} \hat{g}_h e^{i\omega h} \) for \( \omega \in [-\pi, \pi] \). For convergence in distribution and convergence in probability, we write \( \overset{D}{\rightharpoonup} \) and \( \overset{P}{\rightharpoonup} \), respectively. We use \( o_P(1) \) and \( O_P(1) \) as shorthand for terms, which converge to 0 in probability and which are tight, respectively. Finally, \( C > 0 \) is a constant which may change from line to line.

2. Preliminaries

2.1. State space models

For the state space model (1.1) we assume the following conditions:

**Assumption A.**
On basis of Brockwell and Davis (1991), Theorem 11.8.3, the spectral density

\[ A \]

Due to Assumption A, where the MA polynomial is

\[ \int \]

The sequence

\[ \text{Let Assumption A hold. Then} \]

Proposition 2.2 (Schlemm and Stelzer (2012), Theorem 3.6).

Remark 2.1.

(a) Consequently, there exists a stationary causal version for the multivariate Ornstein-Uhlenbeck process \( X \) as \( X_t = \int_{-\infty}^{t} e^{A(t-u)} B dL_u \) for \( t \geq 0 \) (cf. Masuda (2004)) and hence, a stationary causal representation for \( Y \) as \( Y_t = \int_{-\infty}^{t} C e^{A(t-u)} B dL_u \) for \( t \geq 0 \). For such a causal representation the assumption that the matrix \( A \) has strictly negative real parts is necessary. In the following, we will always assume that \( Y \) has such a stationary causal representation.

(b) The assumption \( CC^T = \text{Id}_m \) is not restrictive: The MCARMA processes as defined in Marquardt and Stelzer (2007) are state space models and satisfy this condition. Under the assumption of finite second moments, the classes of stationary linear state space models and MCARMA models are equivalent (cf. Schlemm and Stelzer (2012), Corollary 3.4). Thus, for any state space model there exists a representation satisfying \( CC^T = \text{Id}_m \).

If Assumption A holds, the discrete-time sampled process \( Y^{(\Delta)} \) satisfies the following conditions:

Proposition 2.2 (Schlemm and Stelzer (2012), Theorem 3.6).

Let Assumption A hold. Then

\[ Y_k^{(\Delta)} = CX_k^{(\Delta)} \quad \text{and} \quad X_k^{(\Delta)} = e^{A\Delta} X_{k-1}^{(\Delta)} + N_k^{(\Delta)}, \quad k \in \mathbb{N}, \]

where

\[ N_k^{(\Delta)} = \int_{(k-1)\Delta}^{k\Delta} e^{A(k\Delta-u)} B dL_u, \quad k \in \mathbb{Z}. \]

The sequence \( (N_k^{(\Delta)})_{k \in \mathbb{Z}} \) in \( \mathbb{R}^r \) is an i.i.d. sequence with mean zero and covariance matrix \( \Sigma_N^{(\Delta)} = \int_0^\Delta e^{A\Delta} B \Sigma L B^\top e^{A\Delta} du \). Furthermore, \( (Y_k^{(\Delta)})_{k \in \mathbb{N}_0} \) has the vector MA\((\infty)\) representation

\[ Y_k^{(\Delta)} = \sum_{j=0}^{\infty} \Phi_j N_{k-j}^{(\Delta)}, \quad k \in \mathbb{N}_0, \]

where the MA polynomial is

\[ \Phi(x) = \sum_{j=0}^{\infty} \Phi_j x^j = \sum_{j=0}^{\infty} C e^{A\Delta} x^j \quad \text{for} \ x \in \mathbb{C} \ \text{with} \ \|x\| = 1. \quad (2.1) \]

Due to Assumption A the coefficients \( \Phi_j \) in (2.1) are exponentially decreasing which directly implies

\[ \sum_{j=0}^{\infty} h^q \|\Phi_j\| < \infty \quad \text{for} \ q \in \mathbb{N}_0. \quad (2.2) \]

On basis of Brockwell and Davis (1991), Theorem 11.8.3, the spectral density \( f_{Y}^{(\Delta)} \) of \( Y^{(\Delta)} \) is

\[ f_{Y}^{(\Delta)}(\omega) = \frac{1}{2\pi} \Phi(e^{-i\omega}) \Sigma_N^{(\Delta)} \Phi(e^{i\omega})^\top, \quad \omega \in [-\pi, \pi]. \]
2.2. Function spaces

Next, we present the setup for the function spaces $\mathcal{G}_m$ in the definition of the empirical spectral process. Therefore, define

$$\mathcal{H}_m := \{ g : [-\pi, \pi] \to \mathbb{C}^{m \times m} \mid \| g(\cdot) \| \in \mathcal{L}^2([-\pi, \pi]) \},$$

and equip $\mathcal{H}_m$ with the norm

$$\| g \|_2^2 := \frac{1}{2\pi} \int_{-\pi}^{\pi} \| g(x) \|^2 dx \quad \text{for } g \in \mathcal{H}_m,$$

which generates the metric $d_2(f, g) = \| f - g \|_2$ for $f, g \in \mathcal{H}_m$. Suppose $\mathcal{G}_m \subseteq \mathcal{H}_m$. Then, we define for $l \in \mathbb{N}$ the space $\mathcal{F}_{m,l} := \{ F : \mathcal{G}_m \to \mathbb{C}^{l \times l} \mid F \text{ is bounded} \}$ and equip $\mathcal{F}_{m,l}$ with the metric $d_{\mathcal{G}_m}$ generated by the norm

$$\| F \|_{\mathcal{G}_m} := \sup_{g \in \mathcal{G}_m} | F(g) | \quad \text{for } F \in \mathcal{F}_{m,l}.$$

Finally, we define

$$\mathcal{C}(\mathcal{G}_m, \mathcal{C}^{l \times l}) := \{ F \in \mathcal{F}_{m,l} \mid F \text{ is uniformly continuous with respect to the metric } d_{\mathcal{G}_m} \}.$$

The space $(\mathcal{C}(\mathcal{G}_m, \mathcal{C}^{l \times l}), d_{\mathcal{G}_m})$ is complete and hence, a Banach space. Note, a metric space $(\mathcal{G}, d)$ is totally bounded iff its covering numbers

$$N(\varepsilon, \mathcal{G}, d) := \inf \{ \exists g_1, \ldots, g_u \in \mathcal{G} \mid \inf_{i=1}^u d(g_i, g) \leq \varepsilon \ \forall g \in \mathcal{G} \}$$

are finite for every $\varepsilon > 0$. If we assume additionally that $(\mathcal{G}_m, d_2)$ is totally bounded then $(\mathcal{C}(\mathcal{G}_m, \mathcal{C}^{l \times l}), d_{\mathcal{G}_m})$ is a separable Banach space.

For $g \in \mathcal{G}_m$ and $\Phi$ as in (2.1) define the function

$$g^\Phi(\omega) := \Phi(e^{i\omega})^\top g(\omega) \Phi(e^{-i\omega}), \quad \omega \in [-\pi, \pi],$$

with Fourier coefficients

$$\widehat{g^\Phi}_h = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi(e^{i\omega})^\top g(\omega) \Phi(e^{-i\omega}) e^{-ih\omega} d\omega, \quad h \in \mathbb{Z}.$$

Moreover, for $s \geq 0$ we define the space $\mathcal{H}_m^s := \{ g \in \mathcal{H}_m : \| g \|_{\Phi,s} < \infty \}$ with

$$\| g \|_{\Phi,s}^2 := \sum_{h=-\infty}^{\infty} (1 + |h|)^s \| \widehat{g^\Phi}_h \|^2.$$

Then $(\mathcal{H}_m^s, \| \cdot \|_{\Phi,s})$ is a normed space. Indeed, $\| g \|_{\Phi,s} = 0$ implies $\| \widehat{g^\Phi}_h \| = 0$ for all $h \in \mathbb{Z}$. A conclusion of Lemma 2.4 below is then $\| \widehat{g^\Phi}_h \| = 0$ for all $h \in \mathbb{Z}$ and hence, $g$ is zero almost everywhere.

Remark 2.3.

(a) For $s = 0$ we receive with Parseval’s equality

$$\| g \|_{\Phi,0}^2 = \sum_{h=-\infty}^{\infty} \| \widehat{g^\Phi}_h \|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \| g^\Phi(\omega) \|^2 d\omega = \| g \|_2^2.$$
A direct consequence of this lemma and the definition of $\| \cdot \|$ implies

$$\sup_{g \in \mathcal{G}_m} \| g \|_2 < \infty. \text{ Then, } \sup_{\omega \in [-\pi, \pi]} \| \Phi(e^{i\omega}) \| \leq C$$

The next lemma relates $\| \hat{g} \|$ to $\| \hat{g}^\Phi \|$

**Lemma 2.4.**

- **(a)** There exists a constant $C > 0$ such that for any $g \in \mathcal{H}_m$ and $h \in \mathbb{N}_0$:

$$\| \hat{g} \| \leq C \left[ \| \hat{g}^\Phi \| + \| \hat{g}^\Phi_{h-1} \| + \| \hat{g}^\Phi_{h+1} \| \right].$$

- **(b)** Let $g \in \mathcal{H}_m$. Suppose there exist some constants $\lambda, C_1 > 0$ such that $\| \hat{g} \| \leq C_1 e^{-\lambda h}$ for any $h \in \mathbb{N}$. Then, there exist as well a constant $C_2 > 0$ and some $\nu > 0$ such that $\| \hat{g}^\Phi \| \leq C_2 e^{-\nu h}$ for any $h \in \mathbb{N}$.

A direct consequence of this lemma and the definition of $\| \cdot \|_{\Phi,s}$ is the following:

**Corollary 2.5.** Let $\mathcal{G}_m \subseteq \mathcal{H}_m$ and $s \geq 0$. Suppose there exists a constant $C_1 > 0$ such that $\sup_{g \in \mathcal{G}_m} \| g \|_{\Phi,s} \leq C_1$. Then, there exists a constant $C_2 > 0$ such that $\sup_{g \in \mathcal{G}_m} \| g \|_2 \leq C_2$.

3. The functional central limit theorem

The function space $\mathcal{G}_m$ satisfies either of the following assumptions.

**Assumption B.**

Let $\mathcal{G}_m \subseteq \mathcal{H}_m$ be totally bounded, $h \in \mathcal{H}_m$ and $\| g \|_{\Phi,s} < \infty$ for any $g \in \mathcal{G}_m$ and some $s \geq 0$. Suppose that one of the following conditions hold:

- **(B1)** Suppose $s > 1/2$.
- **(B2)** Suppose $s = 0$ and there exists a constant $K_L > 0$ such that the joint cumulant of $BL_1$ satisfies

$$\text{cum}(BL_1[k_1], \ldots, BL_1[k_j]) \leq K_L^j \quad \text{for all } k_1, \ldots, k_j \in \{1, \ldots, r\} \text{ and } j \in \mathbb{N},$$

where $BL_1[k]$ denotes the $k$-th component of the random vector $BL_1$ in $\mathbb{R}^r$. Furthermore, assume that

$$\int_0^1 [\log(N(e, \mathcal{G}_m, d_2))]^2 d\epsilon < \infty.$$

- **(B3)** Suppose $s = 0$. Let the support of $h$ be an interval and $h$ be continuously differentiable in the interior of its support. The function space is defined as

$$\mathcal{G}_m := \{ h(\cdot) \mathbb{1}_{[-\pi,t]}(\cdot) : t \in [-\pi, \pi] \}.$$
Indeed, \((3.1)\) is already satisfied if there exists a constant \(K_L > 0\) such that
\[
\text{cum}(L_1[k_1], \ldots, L_1[k_j]) \leq K_L^j
\]
for all \(k_1, \ldots, k_j \in \{1, \ldots, d\}\) and \(j \in \mathbb{N}\).

In the following, we present some examples for function spaces \(\mathcal{G}_m\) satisfying \((3.2)\).

Example 3.1.

(a) Let \(\mathcal{G}_m\) be defined as in (B3). Since \(\sup_{x \in [-\pi, \pi]} \|h(x)\| < \infty\), it is straightforward to see that the covering numbers satisfy
\[
N\left(\sup_{x \in [-\pi, \pi]} \|h(x)\|/2n, \mathcal{G}_m, d_2\right) \leq n
\]
for any \(n \in \mathbb{N}\).

A direct consequence is that \(N(\epsilon, \mathcal{G}_m, d_2) \leq C \epsilon^{-1}\) for any \(\epsilon > 0\) and \((3.2)\) in (B2) is satisfied. In particular, this space is totally bounded. But in (B2) we have additionally the cumulant condition which is not necessary in (B3). The function spaces in (B3) do not satisfy (B1). Due to Remark 2.5(b) and \(h \in \mathcal{H}_m\), the condition \(\sup_{x \in \mathbb{R}, \varepsilon} \|g\|_{\Phi, 0} = \sup_{x \in [0, \pi]} \|h(\cdot)\|_{\mathcal{I}(0,x)} \leq C \|h\| \leq \mathcal{C} \|h\|_2 < \infty\) is automatically satisfied.

(b) Suppose \(\mathcal{G}_f\) is a Vapnik-Chervonenkis class (VC class) with VC index \(V(\mathcal{G}_f)\), see van der Vaart and Wellner (1996, Section 2.6.2) for a definition, and \(\tilde{g}\) is an envelope with \(\int_{-\pi}^{\pi} \tilde{g}(x)^2 dx < \infty\). Then, due to van der Vaart and Wellner (1996, Theorem 2.6.7) there exists a constant \(C > 0\) such that \(N(\epsilon, \mathcal{G}_f, d_2) \leq C \epsilon^{-V(\mathcal{G}_f) + 1}\) for \(0 < \epsilon < 1\) and hence, \((3.2)\) in (B2) is satisfied.

(c) Let \((\Theta, \tau)\) be a compact metric space and \(\mathcal{G}_m = \{g_\theta \in \mathcal{H}_m : \theta \in \Theta\}\). Suppose the map \(\theta \mapsto g_\theta\) is Hölder continuous with exponent \(b > 0\) and the necessary number of balls to cover \(\Theta\) of radius at most \(\epsilon\) is of order \(\epsilon^{-a}\) for some \(a > 0\). Then \((3.2)\) in (B2) is satisfied because \(N(\epsilon, \mathcal{G}_m, d_2) \leq C \epsilon^{-ab}\).

(d) Suppose \(\mathcal{G}_m\) is a finite-dimensional vector space and an integrable envelope exists. Then, \(\mathcal{G}^{(ij)}_m := \{g_{ij} : g \in \mathcal{G}_m\}\) forms a finite-dimensional vector space of real functions. Then, due to Pollard (1984, Lemma II.28 and Lemma II.25) there exists a \(w_{ij} > 0\) such that \(N(\epsilon, \mathcal{G}^{(ij)}_m, d_2) \leq C \epsilon^{-w_{ij}}\) for any \(\epsilon > 0\). But this implies that the covering numbers \(N(\epsilon, \mathcal{G}_m, d_2)\) behave polynomial which again results in \((3.2)\).

We are able to present the main results of that paper.

Theorem 3.2. Suppose Assumption A and B hold and \(\sup_{g \in \mathcal{G}_m} \|g\|_{\Phi, s} < \infty\). Then, as \(n \to \infty\),
\[
(\text{tr}(E_n(g)))_{g \in \mathcal{G}_m} \overset{p}{\to} (\text{tr}(E(g)))_{g \in \mathcal{G}_m}\ 	ext{in} \ (\mathcal{C}(\mathcal{G}_m, \mathbb{C}), d_{\mathcal{G}_m}),
\]
where
\[
\text{tr}(E(g)) = \text{tr}\left(W'_0 g_0^\Phi\right) + \text{tr}\left(\sum_{h=1}^{\infty} W_h [g_h^\Phi + g_h^\Phi^\top]\right)
\]
and \(W'_0, W_h, h \in \mathbb{N}\), are independent Gaussian random matrices with
\[
\text{vec}(W'_0) \sim \mathcal{N}(0, \mathbb{E}[L_N^{(A)} L_N^{(A)\top} \otimes N_N^{(A)} N_N^{(A)\top} - N_N^{(A)} \otimes N_N^{(A)}]) \quad \text{and} \quad \text{vec}(W_h) \sim \mathcal{N}(0, \Sigma_N^{(A)} \otimes \Sigma_N^{(A)})\ 	ext{for all } h \in \mathbb{N}.
\]

(3.3)
The basic idea is the following. Suppose

$$I_{n,N(\lambda)}(\omega) = \frac{1}{2\pi n} \left( \sum_{j=1}^{n} N_j^{(\lambda)} e^{-ij\omega} \right) \left( \sum_{k=1}^{n} N_k^{(\lambda)} e^{ik\omega} \right)^T$$

is the periodogram and $(E_{n,N(\lambda)}(g^:\Phi))_{g\in\mathcal{G}_n}$, respectively is the empirical spectral process of the i.i.d. sequence $(N_k^{(\lambda)})_{k\in\mathbb{Z}}$ such that

$$\text{tr} \left( E_{n,N(\lambda)}(g^\Phi) \right) = \sqrt{n} \int_{-\pi}^{\pi} \text{tr} \left( g^\Phi(\omega) \left( I_{n,N(\lambda)}(\omega) - \frac{\Sigma^{(\lambda)}}{2\pi} \right) \right) d\omega.$$ 

Since under the trace it is allowed to commute the matrices we obtain

$$\text{tr}(E_{n,N(\lambda)}(g^\Phi)) = \sqrt{n} \int_{-\pi}^{\pi} \text{tr} \left( g(\omega) \left( \Phi(e^{-i\omega})I_{n,N(\lambda)}(\omega)\Phi(e^{i\omega})^\top - I_{Y(\lambda)}(\omega) \right) \right) d\omega$$

and hence, the representation

$$\text{tr}(E_n(g)) = \text{tr}(E_{n,N(\lambda)}(g^\Phi)) + E_{n,R}(g) \quad (3.4)$$

with

$$E_{n,R}(g) := \sqrt{n} \int_{-\pi}^{\pi} \text{tr} \left( g(\omega) \left( I_{n,Y(\lambda)}(\omega) - \Phi(e^{-i\omega})I_{n,N(\lambda)}(\omega)\Phi(e^{i\omega})^\top \right) \right) d\omega,$$

holds. The term $\text{tr}(E_{n,N(\lambda)}(g^\Phi))$ is determining the asymptotic behavior of $\text{tr}(E_n(g))$, whereas $E_{n,R}(g)$ is asymptotically negligible. The details are given in Section 5.

**Remark 3.3.** Suppose $\mathcal{G}_m \subseteq \mathcal{H}_m$ and $(\tilde{N})_{j\in\mathbb{Z}}$ is a strong white noise with finite fourth moment satisfying Assumption B, where in the cumulant condition (B2) the random vector $BL_1$ is replaced by $\tilde{N}_1$. Define then a discrete time MA process of the form

$$\tilde{Y}_k = \sum_{j=0}^{\infty} \tilde{\Phi}_j \tilde{N}_{k-j}, \quad k \in \mathbb{N},$$

where $\sum_{j=0}^{\infty} \tilde{f}^2 ||\tilde{\Phi}_j|| < \infty$. The proof of Theorem 3.2 shows that the results of Theorem 3.2 stay true for this MA process where $\Phi_j$ is replaced by $\tilde{\Phi}_j$ and $N_1^{(\lambda)}$ is replaced by $\tilde{N}_1$, respectively. In particular, every causal multivariate ARMA process driven by a strong white noise has such a representation. In summary, we have derived as well the asymptotic behavior of the empirical spectral process of a causal multivariate ARMA model with a strong white noise.

**Remark 3.4.**

(a) In the case that the driving Lévy process is a Brownian motion it is possible to weaken the entropy condition in (B2); for further details see Dahlhaus (1988), Remark 2.6.

(b) Mikosch and Norvaiša (1997) derive the asymptotic behavior of the empirical spectral process for univariate linear processes. Thereby, they mainly assume a finite fourth moment of the white noise and an entropy condition which is stronger than the entropy condition in (B2). However, the proof of Lemma 5.4 in that paper is based on the assertion that the quadratic form $\hat{Q}_n(\hat{Y}^2)$ is uniformly bounded which is questionable.

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Dahlhaus and Polonik (2009) present a functional central limit theorem for locally stationary univariate time series under the entropy condition (B2) with the cumulant condition replaced by a moment condition. To be more precise for the one-dimensional white noise \((\xi_k)_{k \in \mathbb{N}}\) they assume that \(\mathbb{E}|\xi_1|^k \leq C_k^k\) for any \(k \in \mathbb{N}\) and some constant \(C_k > 0\). This implies that the \(k\)-th cumulant \(\text{cum}_k(\xi_1) \leq k! C_k^k\) for any \(k \in \mathbb{N}\) and some constant \(C_k > 0\) (Saulis and Statulevičius 1991, Lemma 3.1). As the example of the uniform distribution shows, this upper bound is strict and it is not possible to conclude \(\text{cum}_k(\xi_1) \leq C_k^k\) for any \(k \in \mathbb{N}\). Therefore, it is not obvious why in the proof of Lemma 5.7 in Dahlhaus and Polonik (2009), on page 28, the upper bound holds. Thus, we are not using directly Lemma 5.7 of that paper, although it would be obvious why in the proof of Lemma 5.7 in Dahlhaus and Polonik (2009), on page 28, the upper bound holds. However, in the case of light tailed models this is not the case, and we receive additionally a term depending on the fourth cumulant of the white noise.

Next, we derive the autocovariance function of the limit Gaussian process.

**Corollary 3.5.** Let the assumptions of Theorem 3.2 hold. Furthermore, suppose that \(\hat{g}_0^\Phi = g_0^\Phi \) for any \(g \in \mathcal{G}_m\).

(a) Then, \(\text{tr}(E(g)))_{g \in \mathcal{G}_m}\) is a centered Gaussian process with covariance function

\[
\text{Cov}(\text{tr}(E(g_1)), \text{tr}(E(g_2))) = \pi \int_{-\pi}^{\pi} \text{tr} \left( f_Y^{(\Delta)}(\omega)(g_1(\omega) + g_1(-\omega)^\top) H f_Y^{(\Delta)}(\omega)(g_2(\omega) + g_2(-\omega)^\top) \right) d\omega
\]

\[
+ \text{vec} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi(e^{i\omega})^\top g_1(\omega) \Phi(e^{-i\omega}) d\omega \right)^\top \left( \mathbb{E}[N_Y^{(\Delta)} N_Y^{(\Delta)\top} \otimes N_Y^{(\Delta)} N_Y^{(\Delta)\top}] \right. - 3\Sigma_N^{(\Delta)} \otimes \Sigma_N^{(\Delta)}) \text{vec} \left( \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi(e^{i\omega})^\top g_2(\omega) \Phi(e^{-i\omega}) d\omega \right)^H \right).
\]

(b) The representation

\[\text{tr}(E(g)) = \text{tr} \left( W_0^* g_0^\Phi \right) + \text{tr} \left( \sum_{h=-\infty}^{\infty} W_h \left[ g_h^\Phi + g_{-h}^\Phi \right] \right)\]

holds, where \(W_0^*\), \(W_h\), \(h \in \mathbb{Z}\), are independent Gaussian random matrices with

\[
\text{vec}(W_0^*) \sim \mathcal{N}(0, \mathbb{E}[N_Y^{(\Delta)} N_Y^{(\Delta)\top} \otimes N_Y^{(\Delta)} N_Y^{(\Delta)\top}] - 3\Sigma_N^{(\Delta)} \otimes \Sigma_N^{(\Delta)}) \quad \text{and} \quad (3.5)
\]

\[
\text{vec}(W_h) \sim \mathcal{N}(0, \Sigma_N^{(\Delta)} \otimes \Sigma_N^{(\Delta)}), \quad h \in \mathbb{Z}.
\]

(c) Suppose that the driving Lévy process is a \(d\)-dimensional Brownian motion. Then, \(W_0^* = 0_{r \times r}\) a.s. and the covariance function reduces to

\[
\text{Cov}(\text{tr}(E(g_1)), \text{tr}(E(g_2))) = \pi \int_{-\pi}^{\pi} \text{tr} \left( f_Y^{(\Delta)}(\omega)(g_1(\omega)^\top + g_1(-\omega)) f_Y^{(\Delta)}(\omega)(g_2(\omega) + g_2(-\omega)^\top) \right) d\omega.
\]
Remark 3.7. Note, (\[ \text{Theorem 3.6.} \]

The corresponding metric is denoted by \( d_{\Phi,s} \). Of course, for \( F \in \mathcal{G}_m \) the upper bound \( \| F \|_{\Phi,s} \leq \| F \|_{\Phi,s} \) holds. Therefore, it is not surprising that we have weaker assumptions for the convergence in \( (\mathcal{G}_m, d_{\Phi,s}) \) than in \( (\mathcal{G}(\mathcal{G}_m, \mathbb{C}), d_{\Phi,s}) \); In the dual space \( (\mathcal{G}_m^\prime, d_{\Phi,s}^\prime) \), the assumption \( \sup_{g \in \mathcal{G}_m} \| g \|_{\Phi,s} < \infty \) is not required.

**Theorem 3.6.** Suppose Assumption A and B hold, and \( \mathcal{G}_m \) is a Hilbert space. Then, as \( n \to \infty \),

\[
(\text{tr}(E_n(g)))_{g \in \mathcal{G}_m} \overset{\phi}{\to} (\text{tr}(E(g)))_{g \in \mathcal{G}_m} \quad \text{in} \quad (\mathcal{G}_m, d_{\Phi,s}^\prime),
\]

where \( (\text{tr}(E(g)))_{g \in \mathcal{G}_m} \) is defined as in Theorem 3.2

**Remark 3.7.** Note, \( \mathcal{G}_m = \{ h(\cdot) \mathbb{I}_{[\pi, \pi]}(\cdot) : t \in [-\pi, \pi] \} \) in (B3) is not a Hilbert space and hence, this case is not covered in Theorem 3.6.

However, a more general result holds without requiring that \( \mathcal{G}_m \) is totally bounded.

**Theorem 3.8.** For some \( s > 1/2 \), define the set \( \mathcal{H}_m^\alpha := \{ g \in \mathcal{H}_m : \| g \|_{\Phi,s} < \infty \} \). Then,

\[
(\text{tr}(E_n(g)))_{g \in \mathcal{H}_m^\alpha} \overset{\phi}{\to} (\text{tr}(E(g)))_{g \in \mathcal{H}_m^\alpha} \quad \text{in} \quad (\mathcal{H}_m^\alpha, d_{\Phi,s}^\prime).
\]

Indeed, the space \( \mathcal{H}_m^\alpha \) is a Hilbert space with scalar product

\[
\langle g, f \rangle_{\Phi,s} = \sum_{h=-\infty}^{\infty} (1 + |h|)^{2s} \text{tr} \left( \hat{g}_h^{\Phi,s} \hat{f}_h^{\Phi,s} \right) \quad \text{for} \; g, f \in \mathcal{H}_m^\alpha.
\]

**Example 3.9.** Since the autocovariance functions of state space models are exponential decreasing, Lemma 2.4(b) implies that the spectral densities of discretely observed \( m \)-dimensional state space models are in \( \mathcal{H}_m^\alpha \) and in particular, the spectral densities of \( m \)-dimensional ARMA processes are in \( \mathcal{H}_m^\alpha \). Similarly, the inverses of these spectral densities are in \( \mathcal{H}_m^\alpha \) since the spectral densities are rational matrix functions and hence, the inverses are rational matrix functions as well with exponential decreasing Fourier coefficients. The process \( Y^{(\Delta)} \) is an \( m \)-dimensional ARMA process.

**Example 3.10.** The Whittle estimator is a well-known parameter estimator in the frequency domain going back to Whittle \( \{1953\} \), and is well investigated for different time series models in discrete time. The Whittle estimator \( \hat{\theta}_n^{(\lambda)} := \arg \min_{\theta \in \Theta} W_n(\theta) \) is the minimizer of the Whittle function

\[
W_n(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ \text{tr} \left( f_Y^{(\lambda)}(\omega, \theta)^{-1} I_n(\omega) \right) + \log \left( \det \left( f_Y^{(\lambda)}(\omega, \theta) \right) \right) \right], \quad \theta \in \Theta,
\]

where \( f_Y^{(\lambda)}(\omega, \theta) \) is a spectral density for any parameter \( \theta \) in the a parameter space \( \Theta \). Defining

\[
W(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ \text{tr} \left( f_Y^{(\lambda)}(\omega, \theta)^{-1} f_Y^{(\lambda)}(\omega) \right) + \log \left( \det \left( f_Y^{(\lambda)}(\omega, \theta) \right) \right) \right], \quad \theta \in \Theta,
\]

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we receive due to Theorem 3.8 and Example 3.9 under some mild assumptions

\[ \sup_{\varnothing \in \Theta} |W_n(\varnothing) - W(\varnothing)| = n^{-1/2} \sup_{\varnothing \in \Theta} \left| E_n(f_n^{(\Delta)}(\omega, \varnothing))^{-1/2}\right| = o_p(1). \]

Further, it is also possible to derive the asymptotic normality of the Whittle estimator using the asymptotic behavior of the empirical spectral process. The basic ideas of such an approach are given in Bardet et al. (2008), Dahlhaus (1988) and Dahlhaus and Polonik (2002). Fasen-Hartmann and Mayer (2022) prove that the Whittle estimator for state space models with finite fourth moment is a consistent and asymptotically normally distributed estimator without using the empirical spectral process. However, their Whittle function is defined by a sum which approximates the above integral.

4. Goodness-of-fit tests

4.1. Theory

In this section, we investigate the behavior of some goodness-of-fit test statistics which are based on the empirical spectral distribution function.

Theorem 4.1.

Let Assumption A hold and let \( W_0^*, W_h, h \in \mathbb{Z} \), be independent Gaussian random matrices as defined in (3.5) and \((B_t)_{t \in [0,1]}\) be a one-dimensional Brownian motion. Then, the following statements hold:

(a) The Grenander-Rosenblatt statistic satisfies as \( n \to \infty \),

\[
\sqrt{n} \sup_{t \in [0,\pi]} \left| \text{tr} \left( \int_0^t I_n,Y^{(\Delta)}(\omega) - \frac{1}{2\pi} \Phi(e^{-i\omega}) \Sigma_N^{(\Delta)} \Phi(e^{i\omega})^\top d\omega \right) \right| 
\xrightarrow{D} \sup_{t \in [0,\pi]} \left| \text{tr} \left( \frac{W_0^*}{2\pi} \int_0^t \Phi(e^{i\omega})^\top \Phi(e^{-i\omega}) d\omega \right) 
+ \text{tr} \left( \sum_{h=-\infty}^{\infty} \frac{W_h}{2\sqrt{2\pi}} \left( \int_{-t}^t \Phi(e^{i\omega})^\top \Phi(e^{-i\omega}) e^{-ih\omega} d\omega \right) \right) \right| .
\]

If the driving Lévy process is a Brownian motion the limit process reduces to

\[
\sup_{t \in [0,\pi]} \left| \text{tr} \left( \sum_{h=-\infty}^{\infty} \frac{W_h}{2\sqrt{2\pi}} \left( \int_{-t}^t \Phi(e^{i\omega})^\top \Phi(e^{-i\omega}) e^{-ih\omega} d\omega \right) \right) \right| .
\]

(b) The Cramér-von Mises statistic satisfies as \( n \to \infty \),

\[
n \int_0^\pi \left| \text{tr} \left( \int_0^t I_n,Y^{(\Delta)}(\omega) - \frac{1}{2\pi} \Phi(e^{-i\omega}) \Sigma_N^{(\Delta)} \Phi(e^{i\omega})^\top d\omega \right) \right|^2 dt 
\xrightarrow{D} \int_0^\pi \left| \text{tr} \left( \frac{W_0^*}{2\pi} \int_0^t \Phi(e^{i\omega})^\top \Phi(e^{-i\omega}) d\omega \right) 
+ \sum_{h=-\infty}^{\infty} \frac{W_h}{2\sqrt{2\pi}} \left( \int_{-t}^t \Phi(e^{i\omega})^\top \Phi(e^{-i\omega}) e^{-ih\omega} d\omega \right) \right|^2 dt .
\]
If the driving Lévy process is a Brownian motion the limit process reduces to
\[ \int_0^\pi \left[ \text{tr} \left( \sum_{h=-\infty}^{\infty} \frac{W_h}{2\sqrt{2\pi}} \left( \int_{-t}^t \Phi(e^{i\omega})^\top \Phi(e^{-i\omega})e^{-ih\omega}d\omega \right) \right) \right]^2 dt. \]

(c) The self-normalized Grenander-Rosenblatt statistic satisfies as \( n \to \infty, \)
\[ \frac{\sqrt{n}}{\sqrt{m}} \sup_{\omega \in [0,\pi]} \left| \int_0^t \text{tr} \left( I_{n,Y(\omega)}(\omega) f_{Y(\omega)}^{-1}(\omega) \right) d\omega - tm \right| \xrightarrow{\mathcal{D}} \sup_{\omega \in [0,\pi]} \left| \int_0^t \Phi(e^{i\omega})^\top f_{Y(\omega)}^{-1}(\omega) \Phi(e^{-i\omega})d\omega \right| + \sqrt{2\pi}B_{\pi}. \]

If the driving Lévy process is a Brownian motion then the limit distribution is equal to
\[ \sqrt{2\pi} \sup_{t \in [0,1]} |B_t|, \] where \( \sup_{t \in [0,1]} |B_t| \) has distribution
\[ F(x) = \sum_{k=-\infty}^{\infty} (-1)^k \left( \Phi((2k+1)x) - \Phi((2k-1)x) \right), \quad x \geq 0, \] see [Billingstev] (1999), Equation (9.14).

(d) The self-normalized Cramér-von Mises statistic satisfies as \( n \to \infty, \)
\[ \frac{n}{m} \int_0^\pi \left[ \int_0^t \text{tr} \left( I_{n,Y(\omega)}(\omega) f_{Y(\omega)}^{-1}(\omega) \right) d\omega - tm \right]^2 dt \xrightarrow{\mathcal{D}} \int_0^\pi \left[ \int_0^t \Phi(e^{i\omega})^\top f_{Y(\omega)}^{-1}(\omega) \Phi(e^{-i\omega})d\omega \right] + \sqrt{2\pi}B_{\pi}^2 dt. \]

If the driving Lévy process is a Brownian motion the limit distribution reduces to
\( 2\pi \int_0^1 B_t^2 dt. \)

Proof. (a) and (b): Define the set \( \mathcal{G}_m := \{Id_m \mathbb{1}_{[0,1]}(\cdot) : t \in [0,\pi]\}. \) Due to condition (B3) we are allowed to apply Theorem [5.2] such that an application of the continuous mapping theorem results in the statements (a) and (b), respectively.

(c) and (d): Similar arguments as in (a) and (b) with \( \mathcal{G}_m := \{f_{Y(\omega)}^{-1}(\omega) \mathbb{1}_{[0,1]}(\cdot) : t \in [0,\pi]\} \) give the statement. The Brownian motion is popping up because due to Corollary [4.3] the covariance function of the stochastic process
\[ \left( \frac{1}{\sqrt{m}} \text{tr} \left( \sum_{h=-\infty}^{\infty} \frac{W_h}{2\sqrt{2\pi}} \left( \int_{-t}^t \Phi(e^{i\omega})^\top f_{Y(\omega)}^{-1}(\omega) \Phi(e^{-i\omega})e^{ih\omega}d\omega \right) \right) \right) \]
\( t \in [0,\pi] \)
is equal to \( 2\pi \min\{s,t\} \) for \( s,t \in [0,\pi]. \)

\[ \square \]

Remark 4.2. In the case of one-dimensional ARMA processes several spectral goodness-of-fit test statistics were already investigated; see Section 6.2.6 of [Priestley] (1981). The limit distribution of the Grenander-Rosenblatt statistic for ARMA processes with normally distributed white noise is the same as ours (see [Grenander and Rosenblatt] (1984)). Indeed, [Anderson] (1993) uses for linear processes with finite second moments the sample standardized periodogram to estimate the standardized spectral distribution function and obtains as limit the Brownian bridge if the noise is Gaussian. In contrast, the convergence rate and the limit distribution for \( \alpha \)-stable ARMA processes differ, the convergence rate is faster. Instead of the Brownian motion \( \{B_t\}_{t \geq 0} \) an analogue to the Brownian bridge for stable models occurs; for more details see [Klüppelberg and Mikosch] (1996), Section 4.
4.2. Simulations

The simulation study has two major purposes. Firstly, we want to find out if the theoretical results can be observed for finite sample sizes. Therefore, we investigate the behavior of the empirical and limit quantiles of the spectral goodness-of-fit test statistics. Subsequently, we use the quantiles of the limit process to construct some tests. These tests will be applied in different scenarios.

In the following, we focus on the self-normalized versions of the Grenander-Rosenblatt and the Cramér-von Mises statistic, see Theorem 4.1 (c) and (d). We start with investigating their performances in the case of a univariate CARMA(2,1) process with

\[
A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad \Sigma_L = 1,
\]

a bivariate Ornstein-Uhlenbeck process (MCAR(1) process) with

\[
A = \begin{pmatrix} -1/2 & -1/2 \\ 1 & -1 \end{pmatrix} = B, \quad C = Id_2 = \Sigma_L,
\]

and a bivariate CARMA(2,1) process with

\[
A = \begin{pmatrix} -1 & 4 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 4 \\ -1 & -1 \\ 2 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Sigma_L = Id_2.
\]

In all settings, the processes are simulated with an Euler-Maruyama scheme with initial values \( Y(0) = X(0) = 0 \), step size 0.01 and observation distance \( \Delta = 1 \). We take the Brownian motion and the normal-inverse Gaussian process (NIG Lévy process) as driving process. The NIG Lévy process is often used in financial applications as in the modeling of stock returns or stochastic volatility, see Barndorff-Nielsen (1997). The estimation results for finite sample sizes are based on 5000 replicates, whereas the estimation results corresponding to the limit processes are based on 50,000 replicates with the infinite series replaced by a sum consisting of 250 terms. Note that the quantiles for the limit of the Brownian motion driven self-normalized Grenander-Rosenblatt statistic are explicitly known and therefore, have not to be estimated, see (4.1). For each setting, we derive the empirical \( \alpha \)-quantiles for \( \alpha = 0.9, 0.95, 0.975 \) and \( \alpha = 0.99 \). The results of the univariate CARMA(2,1) model are presented in Table 1, those of the bivariate Ornstein-Uhlenbeck model in Table 2 and those of the MCARMA(2,1) model in Table 3, respectively. The quantiles of the limit process in the NIG driven model differ from the associated ones in the Brownian motion driven model due to the additional term corresponding to \( W_0^* \). However, that term is comparatively small so that the difference is unremarkable.

As we can see, the quantiles of the test statistics are really similar to those of the limit processes even for small sample sizes in all settings. However, it should be mentioned that the results become worse when choosing bigger sample sizes. Precisely, when estimating quantiles for \( n = 2500 \) and bigger, the empirical quantiles deviate from the limit quantiles slightly. Since a similar effect does not occur when choosing a remarkably smaller upper integration limit of the Grenander-Rosenblatt statistic as, e.g., \( (\sqrt{n}/\sqrt{m}) \sup_{t \in [0,1/2]} \left| \int_0^t I_n(Y_\omega(t) \omega^{-1}) Y_\omega^{(A)}(\omega^{-1}) d\omega - tm \right| \) we lead the effect back to worse estimation results for frequencies close to \( \pi \).

In the following, we investigate the two test statistics under the hypothesis but as well under some alternatives. We test at the 5% level. Therefore, as hypothesis we use the three models of above but the
The table shows empirical quantiles for the CARMA(2,1) process under different conditions, specifically focusing on the self-normalized Grenander-Rosenblatt and self-normalized Cramér-von Mises statistics for Gaussian and NIG distributions. The data is generated by different processes, and the parameters are chosen to reflect the parametrization in the CARMA(2,1) setting. The estimated quantiles of the limit random variable are denoted as “limit”.

| n  | 90%   | 95%   | 97.5% | 99%   | 90%   | 95%   | 97.5% | 99%   |
|----|-------|-------|-------|-------|-------|-------|-------|-------|
| 50 | 8.1129| 9.4748| 10.2510| 12.2510| 8.1277| 9.5091| 10.7964| 12.6573|
| 100| 8.2701| 9.6119| 10.8399| 11.9292| 8.2646| 9.6339| 10.9753| 12.5016|
| 200| 8.3499| 9.6181| 10.8841| 12.4037| 8.5780| 9.8687| 11.2801| 12.4481|
| 500| 8.5385| 9.8123| 10.9318| 12.3058| 8.5462| 9.9177| 11.1745| 12.2717|
| 1000| 8.6051| 9.9091| 11.1834| 12.4889| 8.5672| 9.8623| 11.0338| 12.4965|
| limit| 8.7067| 9.9583| 11.0970| 12.4712| 8.5691| 9.8042| 10.9474| 12.2893|

| n  | 90%   | 95%   | 99%   | 90%   | 95%   | 97.5% | 99%   |
|----|-------|-------|-------|-------|-------|-------|-------|
| 50 | 72.2949| 102.2965| 130.2198| 178.2448| 72.5537| 102.2986| 137.1271| 197.4643|
| 100| 71.0994| 99.5237| 132.2028| 172.0599| 74.4093| 102.6841| 132.2682| 183.0207|
| 200| 73.8805| 100.7941| 130.0039| 184.5601| 75.3582| 106.9690| 138.9153| 182.5730|
| 500| 75.4037| 105.8723| 135.9596| 176.5912| 75.6696| 107.4858| 142.3017| 178.4640|
| 1000| 74.7563| 104.4687| 137.1544| 172.4098| 75.8411| 105.2057| 133.9230| 176.4671|
| limit| 73.6655| 102.3420| 131.5447| 173.5202| 75.2386| 103.1234| 132.1666| 170.7825|

Table 1: Empirical quantiles of the self-normalized Grenander-Rosenblatt and the self-normalized Cramér-von Mises statistic for the CARMA(2,1) process. The estimated quantiles of the limit random variable are denoted as “limit”.

Data is generated by different processes. Namely, we consider the parametrization in the CARMA(2,1) setting

$$A = \begin{pmatrix} 0 & 1 \\ \vartheta_1 & \vartheta_2 \end{pmatrix}, \quad B = \begin{pmatrix} \vartheta_3 \\ \vartheta_1 + \vartheta_2 \vartheta_3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad \Sigma_L = 1$$

and choose the parameters

(C1) $\vartheta = (-1, -2, 1)$, \quad (C2) $\vartheta = (-1, -2, -3)$,
(C3) $\vartheta = (-2, -3, 5)$, \quad (C4) $\vartheta = (-2, -1, -2)$,
(C5) $\vartheta = (-2, -1, -1)$, \quad (C6) $\vartheta = (-1, -1, -1.5)$,

for the generating processes. In the same way, in the bivariate MCAR(1) setting, we consider the parametrization

$$A = \begin{pmatrix} \vartheta_1 & \vartheta_2 \\ \vartheta_3 & \vartheta_4 \end{pmatrix} = B, \quad C = \text{Id}_2 = \Sigma_L,$$

with

(O1) $\vartheta = (-1, -1/2, 1, -1)$, \quad (O2) $\vartheta = (-1/2, -1, 1, -1)$,
Table 2: Empirical quantiles of the self-normalized Grenander-Rosenblatt and the self-normalized Cramér-von Mises statistic for the bivariate Ornstein-Uhlenbeck process. The estimated quantiles of the limit random variable are denoted as “limit”.

| n   | 90%   | 95%   | 97.5% | 99%   | 90%   | 95%   | 97.5% | 99%   |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|
|     |       |       |       |       |       |       |       |       |
| 50  | 7.8557| 9.1691| 10.2976| 11.5452| 8.1736| 9.4011| 10.4213| 11.8696|
| 100 | 8.0828| 9.2861| 10.4627| 11.9029| 8.3682| 9.6405| 11.0312| 12.3705|
| 200 | 8.3262| 9.6419| 10.7303| 12.2339| 8.5911| 9.9998| 11.3216| 12.6319|
| 500 | 8.5320| 9.7029| 10.8048| 12.3653| 8.6817| 9.8222| 11.1358| 12.8713|
| 1000| 8.6925| 9.9823| 10.9998| 12.4769| 8.7211| 10.0636| 11.3817| 12.8713|
| limit| 8.7067| 9.9583| 11.0970| 12.4712| 8.7211| 10.0636| 11.3817| 12.8713|
|     |       |       |       |       |       |       |       |       |
| 50  | 69.5715| 95.6165| 124.5750| 173.7440| 71.9138| 100.7472| 130.5848| 173.6131|
| 100 | 71.7997| 96.6962| 125.7060| 162.2857| 75.1489| 103.1489| 135.8280| 176.5584|
| 200 | 72.6525| 98.4600| 130.0324| 183.1381| 78.5692| 108.0443| 141.3767| 182.6919|
| 500 | 72.2218| 102.6097| 129.2060| 171.7070| 77.3160| 105.7352| 135.6170| 180.1722|
| 1000| 73.0592| 101.7156| 130.6170| 180.1722| 77.7714| 106.5174| 141.9770| 185.5810|
| limit| 73.8565| 101.6551| 130.2745| 171.7070| 74.3543| 103.2249| 132.9364| 175.0584|

\( (O3) \vartheta = (-1/2, -1/2, 0, -1), \quad (O4) \vartheta = (-1/2, -1/2, 1, -2), \)

and finally, we take

\[
A = \begin{pmatrix} \vartheta_1 & \vartheta_2 & 0 \\ 0 & 0 & 1 \\ \vartheta_3 & \vartheta_4 & \vartheta_5 \end{pmatrix}, \quad B = \begin{pmatrix} \vartheta_1 & \vartheta_2 \\ \vartheta_5 & \vartheta_7 \\ \vartheta_3 + \vartheta_5 \vartheta_6 & \vartheta_4 + \vartheta_5 \vartheta_7 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Sigma_L = \text{Id}_2,
\]

in the MCARMA(2,1) setting with the parameters

\[
(M1) \vartheta = (-2, 1, -3, -1, 1, 1), \quad (M2) \vartheta = (-2, -1, 3, -1, -3, -1, -3),
\]

\[
(M3) \vartheta = (-1, 5, -1, 0, -3, -1, -1), \quad (M4) \vartheta = (-1, 4, -2, 0, -3, -1, -1),
\]

to generate data under different alternatives. The results are presented in Table 4 - Table 6. As suspected, in the correct specified setting, the statistics hold the given level for most sample sizes. Under the alternatives, the statistics reject quite often for moderate sample sizes and detect every alternative with certainty for \( n = 1000 \) and higher. The performances of the self-normalized Grenander-Rosenblatt and Cramér-von Mises statistic seem to be comparable.
Table 3: Empirical quantiles of the self-normalized Grenander-Rosenblatt and the self-normalized Cramér-von Mises statistic for the MCARMA(2,1) process. The estimated quantiles of the limit random variable are denoted as “limit”

5. Proofs

Let $E_{n,N^{(A)}}$ be the empirical spectral process based on the i.i.d. sequence $(N_k^{(A)})_{k \in \mathbb{Z}}$, see (5.1) below for an explicit definition. The aim is to decompose $\text{tr}(E_n(g))$ as in (3.4) in

$$\text{tr}(E_n(g)) = \text{tr}(E_{n,N^{(A)}}(g^\Phi)) + E_{n,R}(g)$$

and show that $E_{n,R}$ is asymptotically negligible. Then, the asymptotic behavior of $\text{tr}(E_n(g))$ is determined by the asymptotic behavior of $\text{tr}(E_{n,N^{(A)}}(g^\Phi))$. Thus, in Section 5.1, we first show the asymptotic behavior of the empirical spectral process $E_{n,N^{(A)}}$ as a kind of special case of Theorem 5.6. Then, in Section 5.2, we derive that the error $E_{n,R}$ which occurs by approximating the original process by the white noise process has no influence on the limit behavior.

### 5.1. The functional central limit theorem for the white noise process

#### 5.1.1. Preliminaries

For $s \geq 0$ we define the space $\mathcal{H}_{s,t}^2 := \{ g \in \mathcal{H}_r \mid \| g \|_{s,t} < \infty \}$ with

$$\| g \|_{s,t}^2 := \| g \|^2_{r,s} := \sum_{h=-\infty}^{\infty} (1 + |h|)^t \| \hat{g}_h \|^2,$$
Table 4: Percentages of rejection of the self-normalized Grenander-Rosenblatt and the self-normalized Cramér-von Mises test statistics of the CARMA(2,1) model based on 5000 replications and the significance level $\alpha = 0.05$. Thereby, “T” stands for the correct specified model whereas “(C1)-(C6)” denote CARMA(2,1) models under the alternative.

which is a normed space. Again, $\|g\|_{s,0} = \|g\|_2$. In the case that $\mathcal{G}_r$ is additionally a vector space, we consider the dual space

$$\mathcal{G}^*_{r} := \{ F : \mathcal{G}_r \to \mathbb{C}^{r \times r} | \text{F is linear} \}$$

with the operator norm

$$\|F\|_{\mathcal{G}^*_{r}} := \sup_{g \in \mathcal{G}_r, \|g\|_s \leq 1} \|F(g)\|$$

for $F \in \mathcal{G}^*_{r}$,

which generates the metric $d^s_{\mathcal{G}^*_{r}}$.

5.1.2. The functional central limit theorem for the white noise process: main results

We introduce the assumptions to derive the asymptotic behavior of the empirical spectral process for the white noise process which correspond to those of Assumption B.

Assumption N. Let $\mathcal{G}_r \subseteq \mathcal{H}_r$, $h \in \mathcal{H}_r$, and suppose $\|g\|_{s,s} < \infty$ for any $g \in \mathcal{G}_r$ and some $s \geq 0$. Suppose that $\mathcal{G}_r$ satisfies either (B1), (B2) or (B3).

Assumption N is not requiring $\mathcal{G}_r$ to be totally bounded, but in (B2) and (B3) this is already satisfied.
Table 5: Percentages of rejection of the self-normalized Grenander-Rosenblatt and the self-normalized Cramér-von Mises test statistics of the bivariate CAR(1) model based on 5000 replications and the significance level $\alpha = 0.05$. Thereby, “T” stands for the correct specified model whereas “(O1)-(O4)” denote CAR(1) models under the alternative.

| $n$ | T  | (O1) | (O2) | (O3) | (O4) |
|-----|-----|------|------|------|------|
| 50  | 3.06| 60.74| 94.00| 61.50| 32.86|
| 100 | 3.64| 86.98| 99.72| 92.12| 55.24|
| 200 | 4.30| 98.96| 100  | 100  | 83.54|
| 500 | 4.03| 100  | 100  | 100  | 99.98|
| 1000| 4.80| 100  | 100  | 100  | 100  |

| $n$ | T  | (O1) | (O2) | (O3) | (O4) |
|-----|-----|------|------|------|------|
| 50  | 4.08| 47.02| 87.98| 55.96| 21.82|
| 100 | 4.30| 73.10| 99.00| 86.78| 36.32|
| 200 | 4.56| 94.68| 100  | 99.26| 62.00|
| 500 | 4.92| 100  | 100  | 100  | 94.42|
| 1000| 4.62| 100  | 100  | 100  | 100  |

Table 5. Percentages of rejection of the self-normalized Grenander-Rosenblatt and the self-normalized Cramér-von Mises test statistics of the bivariate CAR(1) model based on 5000 replications and the significance level $\alpha = 0.05$. Thereby, “T” stands for the correct specified model whereas “(O1)-(O4)” denote CAR(1) models under the alternative.

**Theorem 5.1.** Let Assumption A and N hold. Furthermore, suppose that $\sup_{g \in \mathcal{G}} \|g\|_{\ast, S} < \infty$ and $\mathcal{G}_r$ is totally bounded. Define

$$E_{n,N,(\cdot)}(g) := \sqrt{n} \int_{-\pi}^{\pi} g(\omega) \left( I_{n,N,(\cdot)}(\omega) - \frac{1}{2\pi} \sum_{(\Delta)} \right) d\omega \quad \text{for } g \in \mathcal{G}_r. \quad (5.1)$$

Then, as $n \to \infty$,

$$\left( E_{n,N,(\cdot)}(g) \right)_{g \in \mathcal{G}_r} \overset{\mathcal{D}}{\to} \left( E_N(g) \right)_{g \in \mathcal{G}_r} \quad \text{in } (C(\mathcal{G}_r, C^{\ast, r}) \times d_{\mathcal{G}_r}),$$

where

$$E_N(g) = \hat{g}_0 W'_0 + \sum_{h=1}^{\infty} \left( \hat{g}_h W_h + \hat{g}_{-h} W'_h \right)$$

and $W'_0$ and $(W_h)_{h \in \mathbb{Z}}$ are defined as in Theorem 3.2.

Of course, $\sup_{g \in \mathcal{G}_r} \|g\|_{\ast, S} < \infty$ implies $\sup_{g \in \mathcal{G}_r} \|g\|_2 < \infty$. Similarly, we obtain the following result in the dual space.

**Theorem 5.2.**

Let Assumption A and N hold. Furthermore, suppose that $\mathcal{G}_r$ is a Hilbert space. Then, as $n \to \infty$,

$$\left( E_{n,N,(\cdot)}(g) \right)_{g \in \mathcal{G}_r} \overset{\mathcal{D}}{\to} \left( E_N(g) \right)_{g \in \mathcal{G}_r} \quad \text{in } \left( C^{\ast, r} \times d_{\mathcal{G}_r} \right).$$
Table 6: Percentages of rejection of the self-normalized Grenander-Rosenblatt and the self-normalized Cramér-von Mises test statistics of the MCARMA(2,1) models based on 5000 replications and the significance level \( \alpha = 0.05 \). Thereby, “T” stands for the correct specified model whereas “(M1)-(M4)” denote MCARMA(2,1) models under the alternative.

Note, the assumption of totally boundedness of \( G \) is not necessary. Furthermore, \( G \) as defined in (B3) is not a Hilbert space and hence, not covered in this theorem.

It is well known that a sequence of probability measures in some Banach space converges weakly if it is tight in the weak topology and if the finite dimensional distributions converge. Therefore, we first prove the weak convergence of the finite dimensional distributions of \( E_{\frac{N}{n},N(\Delta)} \).

5.1.3. The functional central limit theorem for the white noise process: Convergence of the finite dimensional distributions

**Lemma 5.3.**

Let Assumption N hold and \( \mathbb{E}\|L_1\|^4 < \infty \). Then, for \( k \in \mathbb{N}, g_1, \ldots, g_k \in \mathcal{H} \) we have

\[
(E_{n,N(\Delta)}(g_1), \ldots, E_{n,N(\Delta)}(g_k)) \stackrel{\mathcal{D}}{\rightarrow} (E_N(g_1), \ldots, E_N(g_k)) \quad \text{in} \quad \mathbb{C}^{r \times k}.
\]

**Proof.** Let \( C_1, \ldots, C_k \in \mathbb{C}^{r \times r} \). Then,

\[
\sum_{j=1}^{k} \text{vec}(C_j^\top) \text{vec}(E_{n,N(\Delta)}(g_j)) = \text{tr} \left( E_{n,N(\Delta)} \left( \sum_{j=1}^{k} C_j g_j \right) \right)
\]

and \( \sum_{j=1}^{k} C_j g_j \in \mathcal{H} \). Thus, it is sufficient to prove that

\[
E_{n,N(\Delta)}(g) \stackrel{\mathcal{D}}{\rightarrow} E_N(g) \quad \text{in} \quad \mathbb{C}^{r \times r} \quad \text{for any} \ g \in \mathcal{H}.
\]
Further, the representation

\[
E_{n,N^{(\Delta)}}(g) = \sqrt{n} \int_{-\pi}^{\pi} \sum_{\ell=-\infty}^{\infty} \hat{g}_\ell e^{i\ell \theta} \left( \frac{1}{2\pi n} \sum_{j=1}^{n} N_j^{(\Delta)} N_k^{(\Delta)\top} e^{-i\ell(j-k)} - \frac{1}{2\pi} \Sigma^{(\Delta)} \right) \, d\omega
\]

\[
= \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \sum_{\ell=-k}^{k} \hat{g}_\ell N_{\ell+k}^{(\Delta)} N_k^{(\Delta)\top} - \sqrt{n} \hat{g}_0 \Sigma^{(\Delta)}
\]

\[
= \sqrt{n} \hat{g}_0 (\Gamma_{n,N^{(\Delta)}}(0) - \Sigma^{(\Delta)}) + \sqrt{n} \left( \sum_{h=1}^{n-1} \hat{g}_h \Gamma_{n,N^{(\Delta)}}(h) + \hat{g}_{-h} \Gamma_{n,N^{(\Delta)}}(h)^\top \right)
\]

(5.2)

holds. We fix an upper bound for \( h \), say \( M \), and apply Lemma 6 in [Fasen-Hartmann and Mayer 2021]. Thus, we have as \( n \to \infty \),

\[
\sqrt{n} \hat{g}_0 (\Gamma_{n,N^{(\Delta)}}(0) - \Sigma^{(\Delta)}) + \sqrt{n} \sum_{h=1}^{M} \left( \hat{g}_h \Gamma_{n,N^{(\Delta)}}(h) + \hat{g}_{-h} \Gamma_{n,N^{(\Delta)}}(h)^\top \right)
\]

\[
\to \hat{g}_0 W_0' + \sum_{h=1}^{M} \left( \hat{g}_h W_h + \hat{g}_{-h} W_h^\top \right).
\]

In view of Proposition 6.3.9 of [Brockwell and Davis 1991], it remains to prove

\[
\lim_{M \to \infty} \limsup_{n \to \infty} \mathbb{P} \left( \left\| \sqrt{n} \sum_{h=M+1}^{n-1} \hat{g}_h \Gamma_{n,N^{(\Delta)}}(h) + \hat{g}_{-h} \Gamma_{n,N^{(\Delta)}}(h)^\top \right\| > \epsilon \right) = 0.
\]

Tschebycheff's inequality leads to

\[
\mathbb{P} \left( \left\| \sqrt{n} \sum_{h=M+1}^{n-1} \hat{g}_h \Gamma_{n,N^{(\Delta)}}(h) + \hat{g}_{-h} \Gamma_{n,N^{(\Delta)}}(h)^\top \right\| > \epsilon \right) \leq \frac{n}{\epsilon^2} \mathbb{E} \left[ \left\| \sum_{h=M+1}^{n-1} \hat{g}_h \Gamma_{n,N^{(\Delta)}}(h) + \hat{g}_{-h} \Gamma_{n,N^{(\Delta)}}(h)^\top \right\|^2 \right].
\]

Since \( (N^{(\Delta)}_k)_{k \in \mathbb{Z}} \) is i.i.d., \( \Gamma_{n,N^{(\Delta)}}(h) \) and \( \Gamma_{n,N^{(\Delta)}}(j) \) are uncorrelated for \( h \neq j \). Due to the use of the Frobenius norm we get

\[
\mathbb{E} \left[ \left\| \sum_{h=M+1}^{n-1} \left( \hat{g}_h \Gamma_{n,N^{(\Delta)}}(h) + \hat{g}_{-h} \Gamma_{n,N^{(\Delta)}}(h)^\top \right) \right\|^2 \right] \leq 2 \sum_{h=M+1}^{n-1} \left( \| \hat{g}_h \|^2 + \| \hat{g}_{-h} \|^2 \right) \mathbb{E} \left[ \| \Gamma_{n,N^{(\Delta)}}(h) \|^2 \right].
\]

On the one hand,

\[
\mathbb{E} \left[ n \| \Gamma_{n,N^{(\Delta)}}(h) \|^2 \right] = \frac{1}{n} \sum_{S,T=1}^{r} \sum_{k=1}^{n-h} \sum_{\ell=1}^{h} \mathbb{E} \left[ (N^{(\Delta)}_{k+h} N^{(\Delta)\top}_{\ell+h} N^{(\Delta)}_{k+\ell}) \right] [S,T] \leq \frac{n-h}{n} C,
\]

where \( C \) is a constant which is independent of \( h \). On the other hand, Parseval’s equality yields then

\[
\lim_{M \to \infty} \limsup_{n \to \infty} \sum_{h=M+1}^{n-1} \left( \| \hat{g}_h \|^2 + \| \hat{g}_{-h} \|^2 \right) \mathbb{E} \left[ \| \Gamma_{n,N^{(\Delta)}}(h) \|^2 \right] = 0,
\]

and the proof is completed. □
5.1.4. The functional central limit theorem for the white noise process: Tightness

To prove Theorem 5.2 and Theorem 5.1 respectively, it remains to show the tightness of \((E_{n,N(n)})_{n \in \mathbb{N}}\). Therefore, it is important that \((\mathcal{C}(\mathcal{G}_r, \mathcal{C}^r \times r), d_{\mathcal{G}_r})\) if \(\mathcal{G}_r\) is totally bounded and \((\mathcal{G}_r^s, d_{\mathcal{G}_r^s})\) are Banach spaces.

In Theorem 5.2 we additionally assumed that \(\mathcal{G}_r\) is a Hilbert space. Thus, \(\mathcal{G}_r^s\) is a Hilbert space as well and the set \(\{\Phi_g : \mathcal{G}_r^s \to \mathcal{C}^r \times r\mid g \in \mathcal{G}_r, \Phi_g(F) = F(g) \quad \forall F \in \mathcal{G}_r^s\}\) is the dual space of \(\mathcal{G}_r^s\). Due to Theorem 2.3 in de Acosta (1970) (cf. Ledoux and Talagrand (2011)) and the convergence of the finite-dimensional distributions, it is then sufficient to show that \((E_{n,N(n)})_{n \in \mathbb{N}}\) is flatly concentrated, i.e., for any \(\varepsilon, \delta > 0\) there exists a finite-dimensional subspace \(L \subset \mathcal{G}_r^s\) with

\[
\inf_{n \in \mathbb{N}} \mathbb{P}(d_{\mathcal{G}_r^s}(E_{n,r}, L) \leq \delta) = \inf_{n \in \mathbb{N}} \mathbb{P}\left(\left\| \text{Pr}_{L}\left(E_{n,N(n)}\right) \right\|_{\mathcal{G}_r^s} < \delta \right) \geq 1 - \varepsilon.
\]

Therefore, we choose a sequence \(L_M \subseteq \mathcal{G}_r^s, M \in \mathbb{N}\), with

\[
\lim_{M \to \infty} \limsup_{n \to \infty} \mathbb{P}\left(\left\| \text{Pr}_{L_M}\left(E_{n,N(n)}\right) \right\|_{\mathcal{G}_r^s} > \delta \right) = 0.
\]

For \(M \in \mathbb{N}\), we define \(L_M \subseteq \mathcal{G}_r^s\) as the linear subspace generated by \((e_h)_{|h| \leq M}\) where \(e_h : \mathcal{G}_r \to \mathcal{C}^r \times r\) is defined as \(e_h(g) = \hat{g}_h\). Then, due to (5.2), we obtain

\[
\left\| \text{Pr}_{L_M}\left(E_{n,N(n)}\right) \right\|_{\mathcal{G}_r^s} \leq \sup_{g \in \mathcal{G}_r, \|g\|_{\mathcal{G}_r} \leq 1} \left\| E_{n,N(n)}(g) \right\|^2 \leq \sup_{g \in \mathcal{G}_r, \|g\|_{\mathcal{G}_r} \leq 1} \left\| \sqrt{n} \sum_{M < |h| \leq n} \hat{g}_h \left(\Gamma_{n,N(n)}(h) - \mathbb{E}[\Gamma_{n,N(n)}(h)]\right) \right\|^2.
\]

Finally, since \(\mathbb{E}[\Gamma_{n,N(n)}(h)] = 0\) for \(h \neq 0\), the sequence \((E_{n,N(n)})_{n \in \mathbb{N}}\) is flatly concentrated in \((\mathcal{G}_r^s, d_{\mathcal{G}_r^s})\) if

\[
\lim_{M \to \infty} \limsup_{n \to \infty} \mathbb{P}\left(\sup_{g \in \mathcal{G}_r, \|g\|_{\mathcal{G}_r} \leq 1} \left\| \sqrt{n} \sum_{M < |h| \leq n} \hat{g}_h \Gamma_{n,N(n)}(h) \right\| > \delta \right) = 0 \quad \forall \delta > 0. \tag{5.3}
\]

As consequence, for the proof of Theorem 5.2 it remains to show (5.3). How is it in Theorem 5.1?

In Theorem 5.1 we additionally assumed that \(\mathcal{G}_r\) is totally bounded such that \((\mathcal{C}(\mathcal{G}_r, \mathcal{C}^r \times r), d_{\mathcal{G}_r})\) is a separable Banach space. Similarly as above a sufficient condition for \((E_{n,N(n)})_{n \in \mathbb{N}}\) to be flatly concentrated in \((\mathcal{C}(\mathcal{G}_r, \mathcal{C}^r \times r), d_{\mathcal{G}_r})\) is

\[
\lim_{M \to \infty} \limsup_{n \to \infty} \mathbb{P}\left(\sup_{g \in \mathcal{G}_r, \|g\|_{\mathcal{G}_r} \leq 1} \left\| \sqrt{n} \sum_{M < |h| \leq n} \hat{g}_h \Gamma_{n,N(n)}(h) \right\| > \delta \right) = 0 \quad \forall \delta > 0. \tag{5.4}
\]

Due to Bharucha-Reid and Römisch (1985), Proposition 2.2, on a separable Banach space tightness is equivalent to flat concentration and uniform boundedness. But under the assumptions of Theorem 5.1 in \((\mathcal{C}(\mathcal{G}_r, \mathcal{C}^r \times r), d_{\mathcal{G}_r})\) the sequence \((E_{n,N(n)})_{n \in \mathbb{N}}\) is uniformly bounded if it is flatly concentrated:

**Lemma 5.4.** Let Assumption A and \(\sup_{g \in \mathcal{G}_r, \|g\|_2 < \infty}\) hold. If Equation (5.4) is satisfied then \((E_{n,N(n)})_{n \in \mathbb{N}}\) is uniformly bounded in \((\mathcal{C}(\mathcal{G}_r, \mathcal{C}^r \times r), d_{\mathcal{G}_r})\).
Proof.

(a) Due to Markov’s inequality we receive

\[
\mathbb{P} \left( \sup_{g \in \mathcal{G}_r} \sqrt{n} \sum_{M < |h| \leq n} \hat{g}_h \Gamma_{n,N(\lambda)}(h) > \delta \right) \\
\leq \frac{1}{\delta^2} \mathbb{E} \left( \sup_{g \in \mathcal{G}_r} \left( \sum_{M < |h| \leq n} \hat{g}_h \Gamma_{n,N(\lambda)}(h) \right)^2 \right) \\
\leq \frac{1}{\delta^2} \sup_{g \in \mathcal{G}_r} \left\| g \right\|^2 \sum_{M < |h| \leq n} (1 + |h|)^{-2s} n \mathbb{E} \left( \sum_{M < |h| \leq n} (1 + |h|)^{-2s} \left\| \Gamma_{n,N(\lambda)}(h) \right\|^2 \right) \\
\leq \frac{1}{\delta^2} \sup_{g \in \mathcal{G}_r} \left\| g \right\|^2 \sum_{M < |h| \leq n} (1 + |h|)^{-2s} \mathbb{E} \left( n \left\| \Gamma_{n,N(\lambda)}(h) \right\|^2 \right) .
\]

Due to Lemma 5.4 and \( \sum_{M < |h|} (1 + |h|)^{-2s} \longrightarrow 0 \) for \( s > 1/2 \), the convergence (5.4) follows.

(b) The proof is analogue to the proof of (a) by replacing \( \sup_{g \in \mathcal{G}_r} \) by \( \sup_{|h| > \delta} \).

\[ \Box \]

Lemma 5.6.

Let Assumption A and Assumption (B1) hold.

(a) Suppose \( \sup_{g \in \mathcal{G}_r} \| g \|_{\infty} < \infty \) and \( \mathcal{G}_r \) is totally bounded. Then, the sequence \( (E_{n,N(\lambda)})_{n \in \mathbb{N}} \) satisfies (5.4), and is flatly concentrated and tight in \( (\mathcal{C}(\mathcal{G}_r, \mathbb{C}^{r \times r}), d_{\mathcal{G}_r}) \).

(b) Suppose \( \mathcal{G}_r \) is a Hilbert space. Then, the sequence \( (E_{n,N(\lambda)})_{n \in \mathbb{N}} \) satisfies (5.3), and is flatly concentrated and tight in \( (\mathcal{C}(\mathcal{G}_r, \mathbb{C}^{r \times r}), d_{\mathcal{G}_r}) \).

For the proof we require some auxiliary lemmas.
Then, there exist some constants $c_1, c_2 > 0$ such that for any $\delta > 0$ and $M \in \mathbb{N}_0$ we have

$$
P\left(\|E_{n,N}^{(M)}(g)\| > \delta\right) \leq c_1 \exp\left(-c_2 \frac{\delta}{\sqrt{\sum_{M < |k| \leq n} \|\hat{g}_k\|^2}}\right) \quad \text{for } g \in \mathcal{H}_r, \ n \in \mathbb{N}.
$$

(5.5)

**Proof.** We prove that Assumption (2.1) (a) of Dahlhaus (1988) is satisfied. Therefore, define the $k$-th order cumulant spectrum $f_{k_1,\ldots,k_j}$ of $\{N_1^{(\Delta)}[k_1],\ldots,N_1^{(\Delta)}[k_j]\}_{k \in \mathbb{N}}$ for $k_1,\ldots,k_j \in \{1,\ldots,r\}$ and $j \in \mathbb{N}$ as in Brillinger (1975), p. 25, and show that there exists some constant $K_f > 0$ such that

$$
|f_{k_1,\ldots,k_j}(\lambda_1,\ldots,\lambda_{j-1})| \leq K_f^j
$$

for all $k_1,\ldots,k_j \in \{1,\ldots,r\}$, $\lambda_1,\ldots,\lambda_{j-1} \in \mathbb{R}$ and $j \in \mathbb{N}$. Since $(N_k^{(\Delta)})_{k \in \mathbb{N}}$ is an i.i.d. sequence, we have

$$
f_{k_1,\ldots,k_j}(\lambda_1,\ldots,\lambda_{j-1}) = \frac{1}{(2\pi)^{j-1}} \text{cum}(N_1^{(\Delta)}[k_1],\ldots,N_1^{(\Delta)}[k_{j-1}],N_1^{(\Delta)}[k_j]).
$$

It remains to show that there exists some constant $K_N > 0$ such that

$$
|\text{cum}(N_1^{(\Delta)}[k_1],\ldots,N_1^{(\Delta)}[k_j])| \leq K_N^j \quad \text{for } k_1,\ldots,k_j \in \{1,\ldots,r\} \text{ and } j \in \mathbb{N}.
$$

Note, $|\text{cum}(BL_1[k_1],\ldots,BL_1[k_j])| \leq K_L^j$ means that the cumulant generating function $C_{BL_1[k_1],\ldots,BL_1[k_j]}$ of $BL_1[k_1],\ldots,BL_1[k_j]$ satisfies

$$
\left|\frac{\partial^j}{\partial u_1 \ldots \partial u_j} C_{BL_1[k_1],\ldots,BL_1[k_j]}(u_1,\ldots,u_j)\right|_{(u_1,\ldots,u_j)=(0,\ldots,0)} \leq K_L^j.
$$

The cumulant generating function $C_{N_1[k_1],\ldots,N_1[k_j]}$ of $N_1^{(\Delta)}[k_1],\ldots,N_1^{(\Delta)}[k_j]$ is

$$
C_{N_1[k_1],\ldots,N_1[k_j]}(u_1,\ldots,u_j) = \log \left(\mathbb{E}\left[\exp\left(i(u_1N_1^{(\Delta)}[k_1] + \ldots + u_jN_1^{(\Delta)}[k_j])\right)\right]\right)
$$

$$
= \int_0^{\Delta} C_{(BL_1)^T,\ldots,(BL_1)^T}(u_1(e_1^T e^{Ax})^T,\ldots,u_j(e_j^T e^{Ax})^T) \ ds,
$$

see Rajput and Rosiński (1989), where $e_k \in \mathbb{R}^r$ denotes the unit vector which is 1 in the $k$-th component and 0 otherwise. Accordingly, we have

$$
\text{cum}(N_1^{(\Delta)}[k_1],\ldots,N_1^{(\Delta)}[k_j]) = \frac{\partial^j}{\partial u_1 \ldots \partial u_j} \int_0^{\Delta} C_{(BL_1)^T,\ldots,(BL_1)^T}(u_1(e_1^T e^{Ax})^T,\ldots,u_j(e_j^T e^{Ax})^T) \ ds\bigg|_{(u_1,\ldots,u_j)=(0,\ldots,0)}.
$$

Interchanging differentiation and integration due to dominated convergence yields

$$
|\text{cum}(N_1^{(\Delta)}[k_1],\ldots,N_1^{(\Delta)}[k_j])| \leq \left(K_0 \max_{i \in [0,\Delta]} \|e^{ix}\|\right)^j (K_L^j)^j,
$$

where $K_0$ is a constant which is independent of $j$ and $k_1,\ldots,k_j$. Therefore, Assumption (2.1) (a) of Dahlhaus (1988) is satisfied and the proof of Lemma 5.7 matches the proof of Lemma 2.3 in Dahlhaus (1988). \hfill \Box

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Lemma 5.8. Let \( \mathcal{Y}_r \subseteq \mathcal{H}_r \) be totally bounded and \( \tilde{\mathcal{Y}}_r \subseteq \mathcal{Y}_r \) with \( \sup_{g \in \tilde{\mathcal{Y}}_r} \| g \|_2 < \infty \). Suppose there exists a constant \( K_L > 0 \) such that the joint cumulant \( \text{cum}(BL_1[k_1], \ldots, BL_1[k_j]) \leq K_L^j \) for all \( k_1, \ldots, k_j \in \{1, \ldots, r\} \) and \( j \in \mathbb{N} \), and

\[
\int_0^1 \left| \log(N(\varepsilon, \mathcal{G}_r, d_2)) \right|^2 d\varepsilon < \infty.
\]

Let \( E_{n,N}^{(M)} \) be defined as in Lemma 5.7. Then, there exists a set \( B_n \) (independent of \( \tilde{\mathcal{Y}}_r \)) with \( \lim_{n \to \infty} P(B_n) = 1 \) and some constants \( c_1, c_2 > 0 \) such that for any \( \delta > 0 \) there exists a \( M_0 \in \mathbb{N} \) with

\[
P \left( \sup_{g \in \tilde{\mathcal{Y}}_r} \| E_{n,N}^{(M)}(g) \| > \delta, B_n \right) \leq c_1 \exp \left( -c_2 \frac{\delta}{\sup_{g \in \tilde{\mathcal{Y}}_r} \sum_{|k| > M} \| g_k \|^2} \right) \quad \forall M \geq M_0, n \in \mathbb{N}.
\]

Proof. The proof goes in the same way as the proof of inequality (28) in Dahlhaus and Polonik (2009) using (5.5) and \( \lim_{M \to \infty} \sup_{g \in \tilde{\mathcal{Y}}_r} \sum_{|k| > M} \| g_k \|^2 = 0. \)

Proof of Lemma 5.6. The proof of (5.4) and (5.3), respectively follow directly from Lemma 5.8 since \( \lim_{n \to \infty} P(B_n) = 1. \)

Lemma 5.9. Let Assumption A and Assumption (B3) hold. The sequence \( (E_{n,N}^{(M)})_{n \in \mathbb{N}} \) is tight in \( (\mathcal{C}(\mathcal{G}_r, C^{[r]}, d_{\mathcal{G}_r}) \).

Proof. Following the lines of the proof of Theorem 3.2 in Klüppelberg and Mikosch (1996), but without using the contraction principle on p. 1875, and using the Markov inequality on p. 1876 with \( \mu = 2 \) instead of Theorem 6.9.4 of Kwapień and Woyczyński (1992), we obtain

\[
\lim_{M \to \infty} \sup_{n \to \infty} P \left( \sup_{|t| \in [-\pi, \pi]} \left\| E_{n,N}^{(M)}(Id_r, 1_{[-\pi, \pi]}(\cdot)) \right\| > \delta \right) = 0
\]

(see as well the arguments in Kokoszka and Mikosch (2000), proof of Theorem 5.1). That results in the flat concentration condition on \( (\mathcal{C}(\mathcal{G}_r^F, C^{[r]}), d_{\mathcal{G}_r^F}) \) with \( \mathcal{G}_r^F := \{ Id_r, 1_{[-\pi, \pi]}(\cdot) : t \in [-\pi, \pi] \} \). Due to the convergence of the finite-dimensional distribution in Lemma 5.3 and the flat concentration on \( (\mathcal{C}(\mathcal{G}_r^F, C^{[r]})) \) we receive the weak convergence as \( n \to \infty \),

\[
(E_{n,N}^{(M)}(g))_{g \in \mathcal{G}_r^F} \xrightarrow{\mathcal{P}} (E_N(g))_{g \in \mathcal{G}_r^F} \quad \text{in } (\mathcal{C}(\mathcal{G}_r^F, C^{[r]}), d_{\mathcal{G}_r^F}).
\]

Since \( h \in \mathcal{H}_r \) is continuously differentiable on the interior of its support, partial integration and the continuous mapping theorem result in the weak convergence as \( n \to \infty \),

\[
(E_{n,N}^{(M)}(g))_{g \in \mathcal{G}_r} \xrightarrow{\mathcal{P}} (E_N(g))_{g \in \mathcal{G}_r} \quad \text{in } (\mathcal{C}(\mathcal{G}_r, C^{[r]}), d_{\mathcal{G}_r}),
\]

and in particular, the tightness.

5.2. Proof of Theorem 3.2

To deduce Theorem 3.6 from Theorem 5.2, we have to check that the error

\[
E_{n,R}(g) = \sqrt{n} \int_{-\pi}^{\pi} \text{tr} \left( g(\omega) \left( I_{n,N}^{(M)}(\omega) - \Phi(e^{-i\omega})I_{n,N}^{(M)}(\omega)\Phi(e^{i\omega}) \right) \right) d\omega,
\]

which is made by approximating the empirical spectral process \( \text{tr}(E_n(g)) \) by the empirical spectral process \( \text{tr}(E_{n,N}^{(M)}(g)) \) is sufficiently small.
Lemma 5.10.
Let Assumption A hold. Let \( \mathcal{G}_m \subseteq \mathcal{H}_m \) be totally bounded.

(a) Suppose \( \sup_{g \in \mathcal{G}_m} \|g\|_2 < \infty \). Then, \( \|E_{n,R}\|_{\mathcal{G}_m} \xrightarrow{p} 0 \) as \( n \to \infty \).

(b) Suppose \( \mathcal{G}_m \) is a Hilbert space. Then, \( \|E_{n,R}\|_{\mathcal{G}_m} \xrightarrow{p} 0 \) as \( n \to \infty \).

Proof.
(a) Define \( R_n(\omega) = I_{n,Y}(\omega) - \Phi(e^{-i\omega})I_{n,Y}(\omega)\Phi(e^{i\omega})^\top \) for \( \omega \in [-\pi, \pi] \). Due to Proposition 2.2 we get

\[
R_n(\omega) = \frac{1}{2\pi n} \left( \sum_{k=1}^{n} \sum_{p=0}^{m} \Phi_p N_{\Delta}^{(k-p)} \right) \left( \sum_{t=0}^{\infty} \Phi_t N_{\Delta}^{(t)} \right)^\top e^{-i(k-\ell)\omega} \]

\[
- \frac{1}{2\pi n} \sum_{k=1}^{n} \sum_{p=0}^{m} \Phi_p N_{\Delta}^{(k)} \left( \sum_{t=0}^{\infty} \Phi_t N_{\Delta}^{(t)} \right)^\top e^{-i(k+p-\ell)\omega} \]

\[
= \frac{1}{2\pi n} \sum_{p=0}^{m} \sum_{t=0}^{\infty} \Phi_p \left( \sum_{k=1}^{n} \sum_{p=0}^{m} \Phi_p N_{\Delta}^{(k-p)} \right) \left( \sum_{t=0}^{\infty} \Phi_t N_{\Delta}^{(t)} \right)^\top \sum_{t=0}^{\infty} \sum_{k=1}^{n} \sum_{p=0}^{m} \Phi_p N_{\Delta}^{(k)} \right) e^{-i(k+p-\ell)\omega} \Phi_t^\top \]

\[
= \sum_{i=1}^{8} R_n^{(i)}(\omega). \]

Thus, we show that

\[
\sup_{g \in \mathcal{G}_m} \left\| \sqrt{n} \int_{-\pi}^{\pi} g(\omega) R_n^{(i)}(\omega) d\omega \right\| \xrightarrow{p} 0, \quad i = 1, \ldots, 8, \]

holds. By symmetry, the proofs for \( i = 6, 7, 8 \) are the same as those for \( i = 2, 3, 4 \), respectively. Note that the proofs for \( i = 4, 5 \) are based on the same ideas as the proof for \( i = 1 \); the case \( i = 3 \) goes very similar to the case \( i = 2 \). Consequently, we only investigate the terms corresponding to \( i = 1, 2 \).

As a first step we consider the case \( i = 1 \):

\[
\sup_{g \in \mathcal{G}_m} \left\| \int_{-\pi}^{\pi} g(\omega) R_n^{(1)}(\omega) d\omega \right\| \]

\[
= \sup_{g \in \mathcal{G}_m} \left\| \sum_{p=0}^{m} \sum_{t=0}^{\infty} \sum_{\ell=1-t}^{\infty} \sum_{u=\ell-1}^{\ell+t-1} \hat{g}_u \Phi_p N_{\Delta}^{(p+u-\ell)} N_{\Delta}^{(t)} \right\| \]

\[
\leq \sup_{g \in \mathcal{G}_m} \sum_{p=0}^{m} \sum_{t=0}^{\infty} \sum_{\ell=1-t}^{\infty} \sum_{u=\ell-1}^{\ell+t-1} \| \Phi_p \| \| \Phi_t \| \| N_{\Delta}^{(p+u-\ell)} \| \| N_{\Delta}^{(t)} \| \| \hat{g}_u \| \]

\[
\leq C \sum_{p=0}^{m} \| \Phi_p \| \sum_{t=0}^{\infty} \sum_{\ell=1-t}^{\infty} \| N_{\Delta}^{(p+u-\ell)} \| \| \Phi_t \| \| N_{\Delta}^{(t)} \| \]

\[
\leq \left( \sum_{p=0}^{m} \| \Phi_p \| (p+1) \right) \sum_{t=0}^{\infty} \sum_{\ell=1-t}^{\infty} \| N_{\Delta}^{(p+u-\ell)} \| \| \Phi_t \| \| N_{\Delta}^{(t)} \| \]

\[
\leq \left( \sum_{p=0}^{m} \| \Phi_p \| (p+1) \sum_{t=0}^{\infty} \sum_{\ell=1-t}^{\infty} \| N_{\Delta}^{(p+u-\ell)} \| \right) \left( \sum_{t=0}^{\infty} \sum_{\ell=1-t}^{\infty} \| N_{\Delta}^{(t)} \| \right) \]

\[
\leq \left( \sum_{p=0}^{m} \| \Phi_p \| (p+1) \right) \left( \sum_{t=0}^{\infty} \sum_{\ell=1-t}^{\infty} \| N_{\Delta}^{(t)} \| \right). \]
where we used that $\sup_{g \in \mathcal{H}_m} \|g\|_2^2 < \infty$. Note that $\sum_{p=0}^{\infty} (1 + p) \|\Phi_p\|$ and $\sum_{r=0}^{\infty} \sqrt{1 + r} \|\Phi_r\|$ are finite due to (2.2), and that due to the strong law of large numbers

$$\frac{1}{1 + p} \sum_{u=1}^{n} \|N_u^{(A)}\| \xrightarrow{a.s.} \mathbb{E}\|N_1^{(A)}\|, \quad \text{as } p \to \infty.$$  

Thus, the right hand side of (5.6) is a.s. finite and

$$\sup_{g \in \mathcal{H}_m} \left| \sqrt{n} \int_{-\pi}^{\pi} g(\omega) R_n^{(1)}(\omega) d\omega \right| \xrightarrow{P} 0, \quad \text{as } n \to \infty.$$  

Next, we investigate $i = 2$: Since the space $(\mathcal{H}_m, d_2)$ is totally bounded, we can approximate the supremum over the potentially uncountable many functions in $\mathcal{H}_m$ by a maximum of finitely many functions. Namely, let $\delta > 0$. Then, there exist a $v \in \mathbb{N}$ and $g_1, \ldots, g_v \in \mathcal{H}_m$ such that $\sup_{g \in \mathcal{H}_m} \min_{j=1, \ldots, v} d_2(g, g_j) < \delta$. Therefore, for fixed $\delta > 0$ and appropriately chosen $g_1, \ldots, g_v$, we can approximate the error term by

$$\sup_{g \in \mathcal{H}_m} \left| \sqrt{n} \int_{-\pi}^{\pi} g(\omega) R_n^{(2)}(\omega) d\omega \right| \leq \sup_{g \in \mathcal{H}_m} \max_{j=1, \ldots, v} \left| \sqrt{n} \int_{-\pi}^{\pi} (g(\omega) - g_j(\omega)) R_n^{(2)}(\omega) d\omega \right| + \max_{j=1, \ldots, v} \left| \sqrt{n} \int_{-\pi}^{\pi} g_j(\omega) R_n^{(2)}(\omega) d\omega \right|$$

$$\leq \sqrt{2\pi \delta} \left( n \int_{-\pi}^{\pi} \left\| R_n^{(2)}(\omega) \right\|^2 d\omega \right)^{1/2} + \max_{j=1, \ldots, v} \left| \sqrt{n} \int_{-\pi}^{\pi} g_j(\omega) R_n^{(2)}(\omega) d\omega \right|.$$  

Since $\delta$ can be chosen arbitrary small, it is sufficient to prove

$$n \int_{-\pi}^{\pi} \left\| R_n^{(2)}(\omega) \right\|^2 d\omega = O_P(1), \quad \text{and}$$

$$\left| \sqrt{n} \int_{-\pi}^{\pi} g(\omega) R_n^{(2)}(\omega) d\omega \right| = o_P(1) \quad \text{for } g \in \mathcal{H}_m.$$

On the one hand, we have

$$\mathbb{E}\left[ n \int_{-\pi}^{\pi} \left\| R_n^{(2)}(\omega) \right\|^2 d\omega \right]$$

$$= \sum_{S,T=1}^{r} \int_{-\pi}^{\pi} \frac{1}{4\pi^2 n} \sum_{p_1,p_2=0}^{\infty} \sum_{k_1,k_2=0}^{\infty} \sum_{l_1,l_2=1}^{\infty} \sum_{\ell=1}^{n} \sum_{\ell' = 1}^{n} \mathbb{E}\left[ \left( \Phi_{p_1} N_{k_1}^{(A)} N_{l_1}^{(A)\top} \Phi_{l_1} \right) [S,T] \right] e^{i\omega_{k_1} + p_1 - l_1 - n - k_2 + p_2 + \ell_2} d\omega$$

$$= \sum_{S,T=1}^{r} \int_{-\pi}^{\pi} \frac{1}{4\pi^2 n} \sum_{p_1,p_2=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=1}^{\infty} \sum_{\ell=1}^{n} \sum_{\ell' = 1}^{n} \mathbb{E}\left[ \left( \Phi_{p_1} N_{k}^{(A)} N_{l}^{(A)\top} \Phi_{l} \right) [S,T] \right] e^{i\omega_{k} + p_1 - n - k - p_2 + \ell_2} d\omega$$

$$+ \sum_{S,T=1}^{r} \int_{-\pi}^{\pi} \frac{1}{4\pi^2 n} \sum_{p_1,p_2=0}^{\infty} \sum_{k_1,k_2=0}^{\infty} \sum_{l_1,l_2=1}^{\infty} \sum_{\ell=1}^{n} \sum_{\ell' = 1}^{n} \mathbb{E}\left[ \left( \Phi_{p_1} \Phi_{l_1} \right) [S,T] \right] e^{i\omega_{k_1} + p_1 - l_1 - n - k_2 + p_2 + \ell_2} d\omega$$
But this term is uniformly bounded due to (2.2) which results in (5.7).

We investigate the terms in (5.9) separately. For the first one, the independency of the sequence \( N^{(\Delta)}_{k} \), \( k \neq l \) and the Cauchy-Schwarz inequality yield

\[
\mathbb{E} \left[ \left( R_{n,1}^{(2)} \right)^2 \right]
\leq \frac{c}{n} \sum_{l=1}^{n} \sum_{t=1}^{n} \mathbb{E} \left[ \left( \int_{-\pi}^{\pi} g(\omega) \Phi_{|t|} |N^{(\Delta)}_{k} N^{(\Delta)^*}_{\ell} e^{-i\omega(k-\ell)} d\omega \right)^2 \right]
\leq \frac{c}{n} \sum_{l=1}^{n} \sum_{t=1}^{n} \left( \sum_{k=1}^{n} \sum_{\ell=1}^{n} \left\| \Phi_{|t|} \right\|^2 \right)^2 \mathbb{E} \left[ \left( \int_{-\pi}^{\pi} g(\omega) \Phi_{|n-\ell-t|} e^{-i\omega(k-\ell)} d\omega \right)^2 \right]
\leq \frac{c}{n} \sum_{l=1}^{n} \sum_{t=1}^{n} \left( \sum_{k=1}^{n} \sum_{\ell=1}^{n} \left\| \Phi_{|t|} \right\|^2 \right)^2 \mathbb{E} \left[ \left( \int_{-\pi}^{\pi} g(\omega) \Phi_{|n-\ell-t|} e^{-i\omega(k-\ell)} d\omega \right)^2 \right] \leq \frac{c}{n} \sum_{l=1}^{n} \sum_{t=1}^{n} \left( \sum_{k=1}^{n} \sum_{\ell=1}^{n} \left\| \Phi_{|t|} \right\|^2 \right)^2 \mathbb{E} \left[ \left( \int_{-\pi}^{\pi} g(\omega) \Phi_{|n-\ell-t|} e^{-i\omega(k-\ell)} d\omega \right)^2 \right].
\]

Finally, due to (2.2) and (2.3) we receive

\[
\mathbb{E} \left[ \left( R_{n,1}^{(2)} \right)^2 \right] \leq \frac{c}{n} \lim_{n \to \infty} 0.
\]
Next, we investigate the second term in (5.9). By the independence of the sequence $N^{(\Delta)}$ similar calculations as above give

$$
\mathbb{E} \left[ \left( R_{n,2}^{(2)} \right)^2 \right] 
\leq \frac{1}{4\pi^2 n} \mathbb{E} \left[ \left\| \sum_{k=1}^{n} \sum_{t=1}^{n} \sum_{\ell=1+\min(k,t)}^{n} \int_{-\pi}^{\pi} g(\omega) \Phi(e^{-i\omega})N^{(\Delta)}_k N^{(\Delta)}_{\ell} \Phi_t^\top e^{-i(\ell - t)\omega} d\omega \right\|^2 \right] 
\leq \frac{1}{4\pi^2 n} \sum_{k=1}^{n} \sum_{t=1}^{n} \sum_{\ell=1+\min(k,t)}^{n} \mathbb{E} \left[ \left\| \int_{-\pi}^{\pi} g(\omega) \Phi(e^{-i\omega})N^{(\Delta)}_k N^{(\Delta)}_{\ell} \Phi_t^\top e^{-i(\ell - t)\omega} d\omega \right\|^2 \right] 
\leq \frac{1}{4\pi^2 n} \sum_{k=1}^{n} \sum_{t=1}^{n} \sum_{\ell=1+\min(k,t)}^{n} \mathbb{E} \left[ \int_{-\pi}^{\pi} \left( g(\omega) \Phi(e^{-i\omega})N^{(\Delta)}_k N^{(\Delta)}_{\ell} \Phi_t^\top \right)^2 [S, T] e^{-i(\ell - t)\omega} d\omega \right].
$$

Furthermore, Cauchy-Schwarz inequality gives the upper bound

$$
\leq \frac{c}{n} \sum_{k=1}^{n} \sum_{t=1}^{n} \left( \sum_{\ell=1+\min(k,t)}^{n} \frac{1}{\tau} \left\| \int_{-\pi}^{\pi} g(\omega) \Phi(e^{-i\omega})e^{-i(\ell - t)\omega} d\omega \right\|^2 \right) \left( \sum_{t=1+\min(k,t)}^{n} \left\| \Phi_t \right\|^2 \right)^2 
\leq \frac{c}{n} \sum_{k=1}^{n} \sum_{t=1}^{n} \left( \sum_{\ell=1+\min(k,t)}^{n} \left\| \int_{-\pi}^{\pi} g(\omega) \Phi(e^{-i\omega})e^{-i(\ell - t)\omega} d\omega \right\| \left\| \Phi_t \right\| \right. 
\leq \frac{c}{n} \sum_{k=1}^{n} \sum_{t=1}^{n} \left( \sum_{\ell=1+\min(k,t)}^{n} \left\| \int_{-\pi}^{\pi} g(\omega) \Phi(e^{-i\omega})e^{-i(\ell - t)\omega} d\omega \right\|^2 \right) \left( \sum_{t=1+\min(k,t)}^{n} \left\| \Phi_t \right\|^2 \right) 
\leq \frac{c}{n} \sum_{k=1}^{n} \sum_{t=1}^{n} \left( \sum_{\ell=1+\min(k,t)}^{n} \left\| \Phi_t \right\|^2 \right) \left( \sum_{t=1+\min(k,t)}^{n} \left\| \phi(\cdot) \Phi(e^{-i\cdot}) \right\|^2 \right) 
\leq \frac{c}{n} \sum_{k=1}^{n} \sum_{t=1}^{n} \left( \sum_{\ell=1+\min(k,t)}^{n} \left\| \Phi_t \right\|^2 \right)^2 \left( \sum_{t=1+\min(k,t)}^{n} \left\| \phi(\cdot) \Phi(e^{-i\cdot}) \right\|^2 \right).
$$

A conclusion of (5.22) and (5.23) is then

$$
\mathbb{E} \left[ \left( R_{n,2}^{(2)} \right)^2 \right] \leq c/n \to 0.
$$

The convergence of $R_{n,2}^{(2)}$ can be proven similarly.

(b) The proof goes analogue to (a) by replacing $\sup_{g \in [0,1]}$ by $\sup_{|g| \leq 1}$. Therefore, we have to take Corollary 2.5 into account implying that there exists a constant $C > 0$ such that $\sup_{|g| \leq 1} \|g\|_2 \leq C$.
Proof of Theorem 3.2
We use the decomposition $\text{tr}(E_n(g)) = \text{tr}(E_{n,N}(g^\Phi)) + E_{n,R}(g)$. A conclusion of Lemma 5.10(a) is that $\|E_{n,R}\|_{\mathcal{G}_n} \xrightarrow{p} 0$ as $n \to \infty$ and hence, it has no influence on the asymptotic behavior of $\text{tr}(E_n(g))$.

It remains to investigate $\text{tr}(E_{n,N}(g^\Phi))$. Therefore, define $\mathcal{G}_r := \{g^\Phi : g \in \mathcal{M}_m \} \subseteq \mathcal{H}_r$. Due to the continuous mapping theorem, it is sufficient to show that $(E_{n,N}(g^\Phi))_{g^\Phi \in \mathcal{G}_r}$ converges weakly in $(\mathcal{C}(\mathcal{G}_r, \mathbb{C}^\infty), d_{\mathcal{G}})$. Indeed, Assumption B for $\mathcal{M}_m$ implies that the set $\mathcal{G}_r$ satisfies the analogue conditions in Assumption N (see Remark 2.3), and as well $\sup_{g^\Phi \in \mathcal{G}_r} \|g^\Phi\|_{\mathcal{G}_r} < \infty$. Consequently, an application of Theorem 5.1 yields

$$(E_{n,N}(g^\Phi))_{g^\Phi \in \mathcal{G}_r} \xrightarrow{\text{D}} (E(g^\Phi))_{g^\Phi \in \mathcal{G}_r} \quad \text{in} \quad (\mathcal{C}(\mathcal{G}_r, \mathbb{C}^\infty), d_{\mathcal{G}_r}),$$

such that we are able to conclude the statement.

\[ \square \]

5.3. Proof of Theorem 3.6

Proof of Theorem 3.6
The proof goes as the proof of Theorem 3.2 using Theorem 5.2 instead of Theorem 5.1 and Lemma 5.10(b) instead of Lemma 5.10(a).

\[ \square \]

5.4. Proof of Theorem 3.8

Proof of Theorem 3.8
Define the Hilbert space $\mathcal{G}_r := \{g^\Phi : g \in \mathcal{M}_m \} \subseteq \mathcal{H}_r$ with scalar product

$$(g^\Phi, f^\Phi)_{\mathcal{G}_r} = \sum_{h=\infty} |h|^{-2} \text{tr} \left( \bar{g}_h \bar{f}_h \right)$$

for $g^\Phi, f^\Phi \in \mathcal{G}_r$. We use as well the representation $\text{tr}(E_n(g)) = \text{tr}(E_{n,N}(g^\Phi)) + E_{n,R}(g)$. Since Assumption (B1) is satisfied, we have by Theorem 5.2 as $n \to \infty$,

$$(\text{tr}(E_{n,N}(g^\Phi)))_{g^\Phi \in \mathcal{G}_r} \xrightarrow{\text{D}} (\text{tr}(E(g^\Phi)))_{g^\Phi \in \mathcal{G}_r} \quad \text{in} \quad (\mathcal{G}_r, d_{\mathcal{G}_r}).$$

If we show that $\sup_{g^\Phi \in \mathcal{G}_r} \|E_{n,R}(g)\| \xrightarrow{p} 0$ the same arguments as in the proof of Theorem 3.2 finish the proof. Indeed, as in the proof of Lemma 5.10, we will show that

$$\sup_{g \in \mathcal{G}_m, \|g\|_{\Phi, \leq 1}} \left\| \sqrt{n} \int_{-\pi}^{\pi} g(\omega) R_n^{(i)}(\omega) d\omega \right\| \xrightarrow{p} 0, \quad i = 1, \ldots, 8,$$

and therefore, it is sufficient to investigate the terms $i = 1, 2$.

The term $i = 1$ can be handled as in Lemma 5.10 because $\sup_{g \in \mathcal{G}_m, \|g\|_{\Phi, \leq 1}} \|g\|_2 < \infty$ due to Corollary 2.5.

Next, we investigate $i = 2$: The Cauchy-Schwarz inequality yields

$$\sup_{g \in \mathcal{G}_m, \|g\|_{\Phi, \leq 1}} \left\| \sqrt{n} \int_{-\pi}^{\pi} g(\omega) R_n^{(2)}(\omega) d\omega \right\|$$
By an application of Lemma 2.4(a) and \( \|g\|_{\Phi, x} \leq 1 \), we obtain \( \sup_{g \in \mathcal{H}_m} \sum_{n=-\infty}^{\infty} (1 + |u|)^{-2s} \|\hat{g}_n\|^2 \leq C \) and hence, the upper bound
\[
\left\| \sqrt{n} \int_{-\pi}^{\pi} g(\omega) R_\alpha^{(2)}(\omega) d\omega \right\|_{\Phi, x} \leq C \left\| \sum_{n=-\infty}^{\infty} (1 + |u|)^{-2s} \|\hat{g}_n\|^2 \right\|^{-1/2} \left( \sum_{n=1}^{\infty} (1 + |u|)^{-2s} \right)^{1/2}
\] (5.10)
follows. Note that \( n^{-1/2} \max_{j=1, \ldots, n} \|N_j^{(\Delta)}\| \xrightarrow{P} 0 \) as \( n \to \infty \), since for any \( \varepsilon > 0 \):
\[
\mathbb{P} \left( \frac{1}{\sqrt{n}} \max_{j=1, \ldots, n} \|N_j^{(\Delta)}\| > \varepsilon \right) \leq 1 - \left( 1 - \frac{\mathbb{E} \|N_1^{(\Delta)}\|^4}{\varepsilon^4 n^2} \right)^n \xrightarrow{n \to \infty} 0.
\]
Then, \( \sum_{n=-\infty}^{\infty} (1 + |u|)^{-2s} < \infty \) for \( s > 1/2 \), \( \sum_{n=0}^{\infty} \|\Phi_i\| < \infty \), \( \frac{1}{2} \sum_{n=1}^{\infty} \|N_j^{(\Delta)}\| = O_E(1) \) and (5.10) yield
\[
\left\| \sqrt{n} \int_{-\pi}^{\pi} g(\omega) R_\alpha^{(2)}(\omega) d\omega \right\|_{\Phi, x} \xrightarrow{P} 0.
\]
The remaining terms \( i = 3, \ldots, 8 \) can be handled similarly. \( \Box \)

A. Appendix

A.1. Proof of Lemma 2.4

Proof of Lemma 2.4

(a) Let \( g \in \mathcal{H}_m \). Note that the Fourier coefficients of \( g^\Phi \) satisfy
\[
\hat{g}_h^\Phi = \sum_{j_1, j_2=0}^{\infty} \Phi_{j_1}^\top \Phi_{j_2} g_{h-j_1+j_2} = \sum_{j_1, j_2=0}^{\infty} e^{A_{j_1} h} C_{j_1}^\top g_{h-j_1+j_2} Ce^{A_{j_2}}, \quad h \in \mathbb{Z}, \tag{A.1}
\]
Therefore, we have the representation
\[
C^\top \widehat{g}_h C = \left[ \sum_{j_1, j_2 = 0}^{\infty} - \sum_{j_1 = 0}^{\infty} \sum_{j_2 = 0}^{\infty} + \sum_{j_1, j_2 = 1}^{\infty} \right] e^{\gamma \Delta j_1} C^\top \widehat{g}_{h - j_1 + j_2} C e^{\gamma \Delta j_2}
\]
\[
= \widehat{g}_h - e^{\gamma \Delta j_1} \widehat{g}_{h - 1} e^{\gamma \Delta} + e^{\gamma \Delta} \widehat{g}_h e^{\gamma \Delta}.
\]

Thus,
\[
\|C^\top \widehat{g}_h C\| \leq C \left[ \|\widehat{g}_h\| + \|\widehat{g}_{h - 1}\| + \|\widehat{g}_h\| \right].
\]

Assumption A says that $CC^\top = I d_m$, hence $\|C^\top \widehat{g}_h C\| = \|\widehat{g}_h\|$ and therefore, the assertion follows.

(b) Due to (A.1) it is sufficient to show that for some $v > 0$ and any $h > 0$
\[
\sum_{j_1, j_2 = 0}^{\infty} ||\Phi_{j_1}|| ||\Phi_{j_2}|| e^{\lambda|j_1 - j_2|} \leq C e^{-vh}.
\]

Let $h > 0$. Since $||\Phi_{j}|| \leq e^{-\mu j}$ for $j \in \mathbb{N}_0$ and some $\mu > 0$, we obtain
\[
\sum_{j_1, j_2 = 0}^{\infty} ||\Phi_{j_1}|| ||\Phi_{j_2}|| e^{\lambda|j_1 - j_2|}
\]
\[
= \sum_{j_2 = 0}^{h} \sum_{j_1 = 0}^{\infty} ||\Phi_{j_1}|| ||\Phi_{j_2}|| e^{-\lambda (j_1 - j_2)} + \sum_{j_1 = 0}^{\infty} \sum_{j_2 = h}^{\infty} ||\Phi_{j_1}|| ||\Phi_{j_2}|| e^{\lambda (j_1 - j_2)}
\]
\[
\leq C e^{-\lambda h} \sum_{j_2 = 0}^{\infty} e^{-(\mu + \lambda) j_2} \left( \frac{1 - e^{-\mu + \lambda}}{1 - e^{-\mu}} \right) + e^{\lambda h} \sum_{j_1 = 0}^{\infty} e^{-(\mu + \lambda) j_1} \left( \frac{e^{\lambda h} - 1}{1 - e^{-\mu}} \right)
\]
\[
\leq C \left( e^{-\lambda h} + e^{-\mu h} \right).
\]

Setting $v = \min\{\lambda, \mu\}$ yields the assertion.

\[\square\]

A.2. Proof of Corollary 3.5

Proof of Corollary 3.5

(a) Define
\[
g_j(\omega) := \phi_j(\omega) + \phi_j(-\omega) = \Phi(\omega) (g_j(\omega) + g_j(-\omega)) \Phi(\omega), \quad \omega \in [-\pi, \pi],
\]
for $j = 1, 2$ with Fourier coefficients $(\overline{g}_j)_h = (\phi^{\top}_j)_h + (\phi^{\top}_j)_{-h}$ for $h \in \mathbb{Z}$. First of all,
\[
\sum_{h = -\infty}^{\infty} \text{tr} \left( \Sigma_N^{(\Delta)} (\overline{g}_1)_h \Sigma_N^{(\Delta)} (\overline{g}_2)_h \right)
\]
\[
= \frac{1}{2\pi} \sum_{h = -\infty}^{\infty} \int_{-\pi}^{\pi} e^{i(h-\ell)(-\omega)} d\omega
\]
Lemma A.1. Let Assumption A and \( \sup_{g \in \mathcal{G}} \|g\|_2 < \infty \) hold.

(a) \( \sup_{t \in \mathbb{Z}} n \mathbb{E} \left[ \left\| \hat{\Gamma}_{n,N(\Delta)}(j) - \mathbb{E} \left[ \Gamma_{n,N(\Delta)}(j) \right] \right\|^2 \right] \leq C. \)

(b) Define for any \( g \in \mathcal{G}_r, M \in \mathbb{N}, \)

\[
\tilde{E}_{n,N(\Delta)}^{(M)}(g) = \sqrt{n} \sum_{h=-M}^{M} \tilde{g}_h \left( \hat{\Gamma}_{n,N(\Delta)}(h) - \mathbb{E} [ \Gamma_{n,N(\Delta)}(h) ] \right).
\]

A.3. Proof of Lemma 5.4

We start with an auxiliary result.

Proof of Lemma 5.4

(a) Define for any \( g \in \mathcal{G}, \)

\[
\text{sup}_{n \in \mathbb{Z}} \mathbb{E} \left[ \left\| \hat{\Gamma}_{n,N(\Delta)}(j) - \mathbb{E} \left[ \Gamma_{n,N(\Delta)}(j) \right] \right\|^2 \right] \leq C.
\]

(b) Define for any \( g \in \mathcal{G}_r, M \in \mathbb{N}, \)

\[
\tilde{E}_{n,N(\Delta)}^{(M)}(g) = \sqrt{n} \sum_{h=-M}^{M} \tilde{g}_h \left( \hat{\Gamma}_{n,N(\Delta)}(h) - \mathbb{E} [ \Gamma_{n,N(\Delta)}(h) ] \right).
\]

Then, since \( (W_h)_{h \in \mathbb{N}} \) is an i.i.d. centered sequence and the sum is well-defined, we obtain

\[
\text{Cov} \left( \text{tr} \left( \sum_{h=1}^{\infty} W_h(g_1^1 \xi_1^1) \right), \text{tr} \left( \sum_{h=1}^{\infty} W_h(g_2^2 \xi_2^2) \right) \right) = \frac{1}{2} \sum_{h=-\infty}^{\infty} \text{tr} \left( \Sigma_{N_1}^{(\Delta)} (g_1^1) \Sigma_{N_2}^{(\Delta)} (g_2^2) \right) - 2 \text{vec} \left( \theta_1^\top \right) ^\top \left( \Sigma_{N_1}^{(\Delta)} \otimes \Sigma_{N_2}^{(\Delta)} \right) \text{vec} \left( \theta_2^\top \right).
\]

Finally, since \( W_0^1 \) is independent from \( (W_h)_{h \in \mathbb{N}} \) we receive

\[
\text{Cov}(\text{tr}(E(g_1)), \text{tr}(E(g_2))) = \frac{1}{2} 2\pi \int_{-\pi}^{\pi} \text{tr} \left( f_Y^{(\Delta)}(\omega)(g_1(\omega) + g_1(-\omega)\xi_1^1) \right) d\omega
\]

\[
= \pi \int_{-\pi}^{\pi} \text{tr} \left( f_Y^{(\Delta)}(\omega)(g_1(\omega) + g_1(-\omega)\xi_1^1) \right) d\omega
\]

\[
+ \text{vec} \left( \theta_1^\top \right) ^\top \left( \Sigma_{N_1}^{(\Delta)} \otimes \Sigma_{N_2}^{(\Delta)} \right) \text{vec} \left( \theta_2^\top \right).
\]

(b) Similar calculations as in (a) yield that the covariance functions coincide.

(c) Follows directly from (a) since for a Gaussian random vector \( N_1^{(\Delta)} \) we have \( \mathbb{E} [ N_1^{(\Delta)} N_1^{(\Delta)\top} ] = 3 \Sigma_{N}^{(\Delta)} \otimes \Sigma_{N}^{(\Delta)}. \)

\[\square\]

A.3. Proof of Lemma 5.4

We start with an auxiliary result.

Lemma A.1. Let Assumption A and \( \sup_{g \in \mathcal{G}} \|g\|_2 < \infty \) hold.

(a) \( \sup_{t \in \mathbb{Z}} n \mathbb{E} \left[ \left\| \hat{\Gamma}_{n,N(\Delta)}(j) - \mathbb{E} \left[ \Gamma_{n,N(\Delta)}(j) \right] \right\|^2 \right] \leq C. \)

(b) Define for any \( g \in \mathcal{G}_r, M \in \mathbb{N}, \)

\[
\tilde{E}_{n,N(\Delta)}^{(M)}(g) = \sqrt{n} \sum_{h=-M}^{M} \tilde{g}_h \left( \hat{\Gamma}_{n,N(\Delta)}(h) - \mathbb{E} [ \Gamma_{n,N(\Delta)}(h) ] \right).
\]
Then, there exists a constant $K > 0$ such that for any $M \in \mathbb{N}$:

$$
\mathbb{E} \left( \sup_{g \in \mathcal{G}} \| \tilde{E}^{(M)}_{n,N^{(\alpha)}}(g) \|^2 \right) < KM.
$$

**Proof.**

(a) On the one hand, we have

$$
n \mathbb{E} \left[ \Gamma_{n,N^{(\alpha)}}(0) - \mathbb{E}[\Gamma_{n,N^{(\alpha)}}(0)] \right] \|^2 = \sum_{S,T=1}^{r} \text{Var} \left( \left( N_{1}^{(\alpha)} N_{1}^{(\alpha)\top} \right) [S,T] \right).
$$

Since $\mathbb{E} \left[ \Gamma_{n,N^{(\alpha)}}(j) \right] = 0$ for $j \neq 0$, we obtain for $j > 0$

$$
n \mathbb{E} \left[ \| \Gamma_{n,N^{(\alpha)}}(j) - \mathbb{E}[\Gamma_{n,N^{(\alpha)}}(j)] \|^2 \right] = \frac{n - j}{n} \sum_{S,T=1}^{r} \mathbb{E} \left[ \left( N_{1}^{(\alpha)} N_{1}^{(\alpha)\top} \right) [S,T]^2 \right] \leq \sum_{S,T=1}^{r} \text{Var} \left( \left( N_{1}^{(\alpha)} N_{2}^{(\alpha)\top} \right) [S,T] \right),
$$

and with similar calculations we obtain the same bound for $j < 0$.

(b) Due to Hölder inequality we receive

$$
\sup_{g \in \mathcal{G}} \left\| \tilde{E}^{(M)}_{n,N^{(\alpha)}}(g) \right\| \leq \sqrt{n} \sup_{g \in \mathcal{G}} \sqrt{\sum_{h=-M}^{M} \| \hat{g}_{h} \|^2 \left( \sum_{h=-M}^{M} \| \tilde{\Gamma}_{n,N^{(\alpha)}}(h) - \mathbb{E}[\tilde{\Gamma}_{n,N^{(\alpha)}}(h)] \|^{2} \right)} \leq K \sqrt{n} \sum_{h=-M}^{M} \| \tilde{\Gamma}_{n,N^{(\alpha)}}(h) - \mathbb{E}[\tilde{\Gamma}_{n,N^{(\alpha)}}(h)] \|^{2}.
$$

Thus, an application of (A.2) gives

$$
\mathbb{E} \left[ \sup_{g \in \mathcal{G}} \left\| \tilde{E}^{(M)}_{n,N^{(\alpha)}}(g) \right\|^{2} \right] \leq 2K^{2}Mn \max_{h=-M,...,M} \mathbb{E} \left\| \tilde{\Gamma}_{n,N^{(\alpha)}}(h) - \mathbb{E}[\tilde{\Gamma}_{n,N^{(\alpha)}}(h)] \right\|^{2} < \infty,
$$

the statement.

Proof of Lemma [5.4]

By Lemma [A.1] there exists a constant $K > 0$ such that

$$
\mathbb{E} \left[ \sup_{g \in \mathcal{G}} \left\| \tilde{E}^{(M)}_{n,N^{(\alpha)}}(g) \right\|^{2} \right] \leq KM \quad \forall M \in \mathbb{N}.
$$

(A.3)

Let $\varepsilon > 0$. Then, due to [5.4] there exists an $M_{0} \in \mathbb{N}$ such that

$$
\limsup_{n \to \infty} \mathbb{P} \left( \sup_{g \in \mathcal{G}} \sqrt{n} \left\| \sum_{|h| \leq n \atop M < |h|} \hat{g}_{h} \tilde{\Gamma}_{n,N^{(\alpha)}}(h) \right\| > \sqrt{\frac{2K}{\varepsilon}} \right) \leq \frac{\varepsilon}{2} \quad \forall M \geq M_{0}.
$$

(A.4)
Define $\delta := \sqrt{8KM_0}/\varepsilon \geq \sqrt{8K}/\varepsilon$. Hence, Markov’s inequality and (A.3) result in

$$
\mathbb{P} \left( \sup_{g \in G_r} \left\| E_{n,N(A)}(g) \right\| > \delta \right) 
\leq \mathbb{P} \left( \sup_{g \in G_r} \sqrt{n} \left\| \sum_{M_0 < |h| \leq n} \hat{g}_h \Gamma_{n,N(A)}(h) \right\| > \delta/2 \right) + \mathbb{P} \left( \sup_{g \in G_r} \left\| E_{n,N(A)}^{(M_0)}(g) \right\| > \delta/2 \right)
$$

Finally, an application of (A.4) yields

$$
\limsup_{n \to \infty} \mathbb{P} \left( \sup_{g \in G_r} \left\| E_{n,N(A)}(g) \right\| > \delta \right) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2},
$$

and hence, the statement follows. \qed

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