I. INTRODUCTION AND SUMMARY

Proton is known for its longevity, but what is its lifetime? The observed universe is about $10^{10}$ years old, while the proton mean lifetime is experimentally tested to be more than $10^{30} \sim 10^{34}$ years [1–3]. To date, all experiments that attempt to go beyond the Standard Model (SM) [4–6] to observe proton decay predicted by Grand Unified Theories (GUTs) [7–9] have not yet succeeded. This motivates us to ask the following questions and seek their resolutions:

1. Are there alternative routes to test GUT, other than conventionally seeking GUT as an effective field theory that appeared at a higher energy unification?

2. Can the proton be stable with an infinite lifetime? What mechanism protects the proton from decay?

For the first question, Ref. [10–13] recently suggested that instead of only increasing the energy scale from the SM to higher energy looking for the GUT structure (imagine tuning the energy scale along a vertical axis in a phase diagram), we can consider moving from the Standard Model (SM) vacuum to neighbor GUT vacua via quantum phase transitions [14] (imagine tuning the vacuum changing parameters along a horizontal axis in a quantum phase diagram at zero temperature). In Ref. [10–13] viewpoint, the 4d SM and other GUTs are in the same deformation class of quantum field theories [15], labeled by $(G, Z_5)$, the symmetry $G$ and its anomaly $Z_5$: In the limit when internal symmetry is weakly coupled or ungauged, the SM and GUTs could be labeled by an enlarged spacetime-internal symmetry group $G$ and a certain ’t Hooft anomaly [16] of the symmetry $G$. In a modern quantum field theory (QFT) language, ’t Hooft anomaly of the symmetry $G$ in $d + 1$ spacetime is specified via the anomaly inflow [17, 18] by a $(d + 1)d$-symmetric invertible topological quantum field theory (TQFT) denoted as a cobordism invariant $Z_{d+1}$ [19]. Thus, we can tune the SM via quantum phase transitions to the neighbor GUT phases that necessarily allow proton decays. Theoretically, those quantum vacua tuning parameters can be the sign-flipping of the $r$ coefficient of the GUT-Higgs potential $U(\Phi_{\text{GUT}}) = (r(\Phi_{\text{GUT}})^2 + \lambda(\Phi_{\text{GUT}})^4)$ or the sign flipping of the fermion mass.

For the second question, the proton longevity may be protected by subtle symmetries of QFT vacuum (low-energy ground states). Those subtle symmetries, if found, may not be accidental, but be exact — they seem global symmetries at SM energy scales, but they should be dynamical gauge symmetries [20] when approaching the Planck or quantum gravity scales, due to “no global symmetries in quantum gravity” reasoning (see a recent overview [21]). Those subtle symmetries likely are discrete symmetries, subject to the nontrivial check of their anomaly matching or cancellation that we will perform. These discrete symmetries and their anomalies were investigated in the past...
(e.g., \([22-25]\)), they are famous for having potential nonperturbative global anomalies \([26]\) (classified by finite abelian group \(\mathbb{Z}_n\), detectable via large gauge/diffeomorphism transformations that cannot be continuously deformed from the identity), in contrast to the familiar perturbative local anomalies (classified by integer \(\mathbb{Z}\) classes, detectable via infinitesimal gauge/diffeomorphism transformations continuously deformable from the identity, captured by Feynman diagram calculations). These local and global anomalies include gauge, gravitational or mixed gauge-gravitational anomaly types depending on whether their path integral non-invariance is due to gauge or gravitational background fields. But only recently, thanks to the development of cobordism group classification of anomalies, these global anomalies become systematically computable \([19, 27-31]\).

Discrete symmetries and their global anomalies can drastically challenge the paradigm that we used to think of QFT vacuum. For example, if the baryon minus lepton \(B - L\) vector symmetry or more precisely \(X \equiv 5(B - L) - \frac{2}{3}Y\) chiral symmetry \([32, 33]\) (with the integer quantized electroweak hypercharge \(Y\) is a discrete \(Z_{4,X}\) symmetry, although the \(Z_{4,X}\) has no local anomalies, the \(Z_{4,X}\) imposes various global anomaly cancellation conditions \([27, 28, 31, 34]\). In particular, a \(Z_{16}\) class global anomaly of the mixed gauge-gravity type (variation on the \(Z_{4,X}\) gauge field and the Spine space-time diffeomorphism) implies that 15\(N_f\) Weyl fermion SM alone cannot cancel the \(Z_{16}\) global anomaly — its anomaly cancellation requires introducing either the 16th Weyl fermion (the right-handed neutrino), or 4d noninvertible TQFT or interacting conformal field theory (CFT), or 5d invertible TQFT, or breaking the \(Z_{4,X}\) down to fermion parity \(\mathbb{Z}_2^f\), etc \([35-37]\). In other words, rephrasing in terms of quantum phases of QFT language, the QFT vacuum beyond the SM (BSM) could be more quantumly entangled than the Landau-Ginzburg old paradigm. These possible exotic BSM phases are analogous to many exotic quantum phases explored in the contemporary condensed matter community \([38]\). The SM together with those exotic BSM phases constrained by nonperturbative global anomaly cancellations is called Ultra Unification (UU) \([35-37]\).

Another motivation for our present work is expanding the exploration of the deformation class of SM \([12]\). In Ref. \([12]\), we explored the deformation of SM to GUTs. We had included the spacetime symmetry (Spin group), the internal symmetry (Lie algebra \(su(3) \times su(2) \times u(1)_Y\), and four compatible versions Lie groups \(G_{SU_{SM}} = \frac{SU(3) \times SU(2) \times U(1)_Y}{\mathbb{Z}_n, X}\), with \(q = 1, 2, 3, 6\) \([39]\), and a continuous or discrete \(B - L\) like symmetry of the SM. In Ref. \([12]\), we had left the inclusion of the discrete \(B + L\) vector symmetry for future work. (The \(U(1)_{B+L}\) is explicitly broken down to \(Z_{2N_f} \# B+L\) due to the \(SU(2)\) instanton \([40-42]\) by the Adler-Bell-Jackiw (ABJ) anomaly \([43, 44]\), where \(N_f\) is the family number with an extra q-dependent factor denoted as \(\#\), see \([45]\).) Previously we had excluded the \(B + L\) because it is not a symmetry for many GUTs. Now the discrete \(B + L\), allowed in SM but disobeyed by GUTs, gives us the exact opportunity to distinguish the SM from other GUTs. Our present work means to fill this gap left in \([12]\) to include the discrete \(B + L\) symmetry, examining the \(B + L\) anomaly, and its SM deformation class. We will see that the discrete \(B + L\) is a good symmetry for SM and some versions of UU, such that it implies the proton stability in those models. Remarkably, recently Ref. \([46, 47]\) also emphasize that the discrete \(B + L\) can avoid the proton decay.

II. REVISIT THE STANDARD MODEL

Now we revisit the Standard Model (SM) and its symmetry, then explicitly derive the discrete \(B + L\) symmetry. SM is a 4d chiral gauge theory of local Lie algebra \(su(3) \times su(2) \times u(1)_Y\) coupling to \(N_f = 3\) families of 15 or 16 Weyl fermions (written as a left-handed 15 or 16 multiplet \(\psi_L\) in the following representation

\[
(\psi_L)_I = (\bar{d}_R + l_L + q_L + \bar{u}_R + \bar{e}_R)_{11} \oplus \nu_{\alpha_1}, R \nu_{A,R} \sim (\mathbf{3}, \mathbf{1})_2 \oplus (\mathbf{1}, \mathbf{2})_{-3} \oplus (\mathbf{3}, \mathbf{2})_1 \oplus (\mathbf{3}, \mathbf{1})_{-4} \oplus (\mathbf{1}, \mathbf{1})_0 \oplus \nu_{\alpha_1}, R (\mathbf{1}, \mathbf{1})_0
\]

for each family (family index \(I, J = 1, 2, 3\), with \(\psi_L_{11}\) for \(u, d, e\) type, \(\psi_L_{20}\) for \(c, s, \mu\) type, and \(\psi_L_{30}\) for \(t, b, \tau\) type of quarks and leptons) of \(su(3) \times su(2) \times u(1)_Y\). We use \(I = 1, 2, 3\) for \(n_{\nu_{\alpha_1}, R}, n_{\nu_{\alpha_2}, R}, n_{\nu_{\alpha_3}, R} \in \{0, 1\}\) to label either the absence or presence of electron \(e\), muon \(\mu\), or tauon \(\tau\) types of sterile neutrinos. Readers should keep in mind that all unitary internal symmetries of SM belong to this group. The SM lagrangian consists of Yang-Mills terms (their gauge sector indices \(I, J = 1, 2, 3\) for \(u(1), su(2), su(3)\)), a possible theta term for \(su(3)\), the Weyl fermions coupled to Yang-Mills gauge fields, Yukawa-Higgs term, and electroweak Higgs kinetic-potential term:

\[
\mathcal{L}_{SM} = \sum_{I=1,2,3} - \frac{1}{4} F_{\mu, \nu}^a F_{\mu, \nu}^a - \frac{\theta_3}{64\pi^2} g_2^2 \epsilon^{\mu, \nu, \rho, \sigma} F_{\mu, \nu}^a F_{\rho, \sigma}^a + \psi_L^d (i \bar{\sigma}^\mu D_{\mu, A} \psi_L - (\psi_L^d \phi_R \psi_L + \text{h.c.}) + |D_{\mu, A} \phi|^2 - U(\phi),
\]

The \(\mathcal{L}_{SM}\) is a shorthand of \(\mathcal{L}_{SM} = \mathcal{L}_{SM}^d + \mathcal{L}_{SM}^u + \mathcal{L}_{SM}^c = \lambda_{14}^q \bar{q}_{L, A}^q \phi_R d_{R, I}^q + \lambda_{14}^e \bar{e}_{L, A}^e \phi_R u_{R, I}^e + \lambda_{14}^\tau \bar{\nu}_{L, A}^\tau \phi_R \nu_{R, I}^\tau + \text{h.c.}\) with \(a, b\) the \(su(2)\) fundamental’s index, and the h.c. as hermitian conjugate. Diagonalization of Yukawa-Higgs term of quark sector implies that the \(W^\pm\) bosons induces a flavor-changing current mixing between different families, thus we only have a \(U(1)_B\) symmetry for all quarks (instead of an individual \(U(1)\) for each quark family), at least classically.
The diagonalization of Yukawa-Higgs term of lepton sector without neutrino mass term $\mathcal{L}_{YH} = \frac{g}{\sqrt{2}} \partial^\mu \phi^I_L \phi^*_R \nu^\mu_L + \text{h.c.}$ implies that $\mathcal{L}_{SM}$ has individual $U(1)_e, U(1)_\mu, U(1)_\tau$ for each lepton family. However, established experiments say each lepton $U(1)$ is violated, only the total lepton number $U(1)_L$ should be considered, at least classically. Thus, we focus on $U(1)_B$ and $U(1)_L$ transformations, which send

$$
(U(1)_B : \psi_L)_I \rightarrow \left( (e^{-i \alpha_B^L} I_3 \cdot d_R) \oplus (e^{-i \alpha_B^L} I_6 \cdot q_L) \oplus (e^{-i \alpha_B^L} I_3 \cdot \bar{u}_R) \oplus (e^{-i \alpha_B^L} \bar{e}_R) \right)_I \oplus \nu_{\alpha_B^L} \bar{\nu}_{\alpha_B^L} R.
$$

$$
(U(1)_L : \psi_L)_I \rightarrow \left( \bar{d}_R \oplus e^{i \alpha_B^L} I_2 \cdot l_L \oplus q_L \oplus \bar{u}_R \oplus e^{-i \alpha_B^L} \bar{e}_R \right)_I \oplus \left( e^{-i \alpha_L^N} \nu_{\alpha_L^N} \bar{\nu}_{\alpha_L^N} R \right),
$$

with $\alpha_B \in [0, 2\pi \cdot 3)$ and $\alpha_L \in [0, 2\pi)$. The quark's $U(1)_q$ is related to baryon's $U(1)_B$ via $\alpha_q = \alpha_B/3 \in [0, 2\pi)$. Here $l_N$ means a rank-N diagonal identity matrix that can act on the $N$-multiplet.

It is well-known that the $U(1)_B \times U(1)_Y \times U(1)_X=U(1)_B \times U(1)_Y$ of Pati-Salam (PS) models have local anomalies captured by triangle Feynman diagrams with anticommuting coefficients respectively: $2 \cdot 1^2 - 2^2 - (-4)^2 = -18, 2 \cdot (-3)^2 - 6^2 = -18, 1$. Under dynamical gauging electroweak $SU(2) \times U(1)_Y$, the consequential ABJ anomaly implies that the classical continuous $U(1)_B \times U(1)_Y$ symmetry is broken quantum mechanically. Next we check whether any subgroup of the $U(1)_B \times U(1)_L$ still survives under dynamically gauged $G_{SM}$. The Fujikawa path integral method [48] shows that under $U(1)_B$ and $U(1)_L$ transformations with corresponding currents $J_B$ and $J_L$, the path integral $Z$ changes to

$$
\int [D\psi_L][D\psi_L^\dagger] e^{i \left( \int d^4 x (\mathcal{L}_{SM} + \alpha_B (\partial_{\mu} J_B^\mu + \partial_{\mu} (\alpha_L J_L^\mu))) - 18 (\alpha_B + \alpha_L) \nu_{\alpha_B^N} (\alpha_B + \alpha_L) \nu_{\alpha_L^N} \right)}. \tag{4}
$$

Here the instanton numbers $n^{(1)} \equiv \int d^4 x \frac{2}{32\pi^2} \epsilon^{\mu
u\lambda\sigma} \partial_\mu \partial_\nu \partial_\lambda \partial_\sigma F_{1, \mu
u} F_{1, \rho\sigma}$ and $n^{(2)} \equiv \int d^4 x \frac{2}{32\pi^2} \epsilon^{\mu
u\lambda\sigma} \partial_\mu \partial_\nu \partial_\lambda \partial_\sigma F_{2, \mu
u} F_{2, \rho\sigma}$ are quantized in integer $Z$ on spin manifolds.

- $U(1)_B$ symmetry: When $\alpha_B = -\alpha_L$, its Ward identity (the derivative of the partition function $Z$ with respect to $\alpha$ variation $\delta Z/\delta \alpha|_{\alpha=0} = 0$ vanishes) says that $\langle \partial_\mu (J_B^\mu - J_L^\mu) \rangle = 0$. This shows $U(1)_B$ is still a symmetry and is anomaly free under $G_{SM}$.

- $Z_{2N_B} \times Z_{2N_L}$ symmetry: The continuous $U(1)_B$ and $U(1)_L$ are all broken by ABJ anomaly. But when $\alpha_B = \alpha_L$, the quantization of $18 (\alpha_B + \alpha_L) \nu_{\alpha_B^N} (\alpha_B + \alpha_L) \nu_{\alpha_L^N}$ holds when $\alpha_B \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Overall, this shows $Z_{2N_B} \times Z_{2N_L} Z_{2N_B} \times Z_{2N_L}$ is still a symmetry and is anomaly free under $G_{SM}$.

- $U(1)_B \times Z_{2N_B} \times Z_{2N_L}$ symmetry: Since $U(1)_B$ and $Z_{2N_B} \times Z_{2N_L}$ share the fermion parity $Z_2$ that acts on fermions $\psi \rightarrow -\psi$, the precise surviving subgroup (which is not broken under ABJ anomaly) with dynamically gauged $G_{SM}$ is $U(1)_B \times Z_{2N_B} \times Z_{2N_L}$. Hereafter we use a standard notation that $G_1 \times G_2 \equiv (G_1 \times G_2)/G_1$ as modding out their common normal subgroup $G_N$. The above result is not affected by whether the $(\psi_L)_I$ multiplet in (1) includes the 16th Weyl fermion $\bar{\nu}_{\alpha_B^L}$ for each of the $N_f$ families, because $\bar{\nu}_{\alpha_B^L}$ is sterile to $G_{SM}$ gauge forces.

### III. PROTON STABILITY

We have shown $U(1)_B \times U(1)_L$ and $Z_{2N_B} \times Z_{2N_L}$ are good symmetries, anomaly-free with respect to dynamically gauged $G_{SM}$ for the SM. Precisely they send the multiplet $(\psi_L)_I$ to (here $\alpha_B = \alpha_L \in \{0, 2\pi \cdot 3\}$ while $\alpha_B = \alpha_L \in \mathbb{Z}/2\mathbb{Z}$ is discrete):

$$
U(1)_B : \left( (e^{-i \alpha_B^L} I_3 \cdot d_R) \oplus (e^{-i \alpha_B^L} I_6 \cdot q_L) \oplus (e^{-i \alpha_B^L} I_3 \cdot \bar{u}_R) \oplus (e^{-i \alpha_B^L} \bar{e}_R \oplus (e^{-i \alpha_B^L} \bar{e}_R) \right)_I \oplus \nu_{\alpha_B^L} \bar{\nu}_{\alpha_B^L} R.
$$

$$
Z_{2N_B} \times Z_{2N_L} : \left( (e^{-i \alpha_B^L} I_3 \cdot d_R) \oplus (e^{-i \alpha_B^L} I_6 \cdot q_L) \oplus (e^{-i \alpha_B^L} I_3 \cdot \bar{u}_R) \oplus (e^{-i \alpha_B^L} \bar{e}_R) \right)_I \oplus \nu_{\alpha_B^L} \bar{\nu}_{\alpha_B^L} R.
$$

Let us check whether they are familiar GUTs and $\mu$.

1. $U(1)_B$ vector or $U(1)_X$ chiral like symmetry: $U(1)_B$ is a factor of the left-right (LR) $su(3) \times su(2)_L \times su(2)_R$ or $u(1)_{\frac{3}{2} \mathbb{Z} \times \mathbb{Z}}$ model [49]. $U(1)_B$ is part of $SU(4)$ subgroup of Pati-Salam (PS) $su(4) \times su(2)_L \times su(2)_R$ [7]. Thus $U(1)_B$ is not only an anomaly-free symmetry but also already dynamically gauged for LR and PS models. Similarly for the Trinification (Tri) [50–52] with the gauged Lie algebra $su(3) \times su(3)_L \times su(3)_R$, the $U(1)_B$ is contained in the $SU(3)_R$; thus $U(1)_B$ is automatically anomaly-free and dynamically gauged in the Trinification. $U(1)_B$ is not quite correct for Georgi-Glashow $su(5)$, because $U(1)_B$ does not act on the $\psi_L$ reducibly as $\bar{F}$, $10, 1$ multiplets of $su(5)$. But the $U(1)_X$ with $X \equiv 5(B - L) - \frac{2}{3} \tilde{Y}$ [32, 33] fits the role so that $d_R \oplus l_L \oplus (q_L \oplus \bar{u}_R \oplus \bar{e}_R) \oplus \nu_{\alpha_B^L} \bar{\nu}_{\alpha_B^L} R \Rightarrow \mathbb{F} - 3 + 10_1 + 1^5$ in $su(5) \times u(1)_X$. Neither $U(1)_B$ nor $U(1)_X$ is a symmetry for the flipped up (5) model [53], but the $U(1)_X$ with $X_2 \equiv \frac{1}{3} X + \frac{2}{3} \tilde{Y} = (B - L) + \frac{2}{3} \tilde{Y}$ replaces the role so that $\bar{u}_R \oplus l_L \oplus (q_L \oplus d_R \oplus \nu_{\alpha_B^L} \bar{\nu}_{\alpha_B^L} R \Rightarrow \mathbb{F} - 3 + 10_1 + 1^5$ in
su(5) \times u(1)X_2$. Neither $U(1)_{B-L}$, $U(1)_{X}$, nor $U(1)_{X_2}$ is correct for the so(10) GUT [54]. The only sensible $U(1)$ factor allowed for the so(10) GUT is an identical $U(1)$ phase that acts on the 16 that together with Spin(10) forms a $(\text{Spin}(10) \times Z_{4, X} U(1))$ internal symmetry. But this chiral $U(1)$ has a discrete $Z_{4, X} = Z_{4, X_2}$ subgroup (that replaces the role of discrete $B - L$) which is a good symmetry (indeed anomaly-free and gauged) for the so(10) GUT. Finally, $U(1)$ [35–37] requires a discrete $B - L$ or $X$ symmetry to enforce a $Z_{16}$ class global anomaly constraint that can be canceled by replacing the $\bar{v}_R$ with TQFT/CFT exotic phases. See (6) for a summary.

2. $Z_{2N_f, B+L}$ vector symmetry: $Z_{2N_f, B+L}$ is a good global symmetry for LR model, broken down from $U(1)_{B+L}$ by the dynamical $SU(2)_L$ and $SU(2)_R$ instantons. For other models such as PS, $su(5)$, flipped $u(5)$ and so(10), the only compatible subgroup of $U(1)_{B+L}$ is the fermion parity $Z^F_2$. For $U(1)$, we have choices to include the $15N_f$ Weyl fermion SM only but without GUT structure plus exotic TQFT/CFT sectors; those $U(1)$ models can obey the $Z_{2N_f, B+L}$ symmetry. Other $U(1)$ which includes the $su(5)$ GUT allows only the $Z^F_2$ subgroup of $Z_{2N_f, B+L}$. For Trinification, we can even choose a generalized unbroken $U(1)_{B+L}$ global symmetry, which is not only outside the Trinification gauge group but also anomaly-free under that gauge group with Lie algebra $su(3) \times su(3)_L \times su(3)_R$. See (6) for a summary.

| $B - L$ like | $U(1)_{B-L}$ | $U(1)_{B-L}$ | $U(1)_{X}$ | $U(1)_{X_2}$ | $Z_{4, X}$ | $U(1)_{B-L}$ | discrete $B - L$ or $X$ |
|-------------|--------------|--------------|-------------|--------------|-------------|-----------------|-----------------|
| $B + L$ like | $Z_{2N_f, B+L}$ | $Z_{2N_f, B+L}$ | $Z^F_2$ | $Z^F_2$ | $Z^F_2$ | $Z^F_2$ | $U(1)_{B+L}$ | $Z_{2N_f, B+L}$ or $Z^F_2$ |

(6) We check whether these symmetries avoid the proton $p^+$ (or other nucleons like neutron $n$) decay for some dominant channels. These channels are constrained to have the lifetime $\tau$ lower bound around or more than $10^{33}$ years [1–3].

We list down the changes of haryon or lepton number ($\Delta B$ or $\Delta L$):

$$
\begin{array}{ccccccc}
\Delta B & \Delta L & p^+ & \rightarrow & e^+ \pi^0 & p^+ & \rightarrow & \mu^+ \pi^0 & p^+ & \rightarrow & \mu^+ K^0 & p^+ & \rightarrow & e^+ K^0 & n & \rightarrow & e^- K^+
\end{array}
$$

(7) All these processes have $\Delta(B - L) = 0$ and $\Delta(B + L) = -2$, except the last one has $\Delta(B - L) = -2$ and $\Delta(B + L) = 0$.

If $Z_{2N_f, B+L}$ is an exact symmetry of our vacuum, then $\Delta(B + L) = 0 \mod 2N_f = 0 \mod 6$ must hold. Many proton decay channels are thus forbidden. The last channel $n \rightarrow e^- K^+$ is also unlikely because it violates even the discrete $Z_{4, B- L}$ which demands that $\Delta(B - L) = 0 \mod 4$.

Proton decay is forbidden as long as $Z_{2N_f, B+L}$ is an exact symmetry and $N_f > 1$. (Although a single proton decay may be forbidden, the $Z_{2N_f, B+L}$ symmetry still allows $N_f$ protons or baryons together to decay because their $\Delta(B + L) = -2 \times N_f = -6 = 0 \mod 6$ is consistent with $Z_{2N_f, B+L}$ here for $N_f = 3$.) If all experiments support that a proton indeed does not decay at all in our vacuum, then the SM and the $U(1)$ (that contains only SM without GUT) are viable candidates deserving further studies, their generalizations with $Z_{2N_f, B+L}$ symmetry is preferable for future model buildings. For example, anomaly-free discrete gauge symmetries were recently discussed in [55] (see also references therein), which serve to exactly stabilize the proton via the all-orders selection rule.

**IV. COBORDISM AND HIGHER ANOMALIES OF THE STANDARD MODEL WITH $Z_{2N_f, B+L}$ SYMMETRY**

We have checked that $U(1)_{B-L} \times Z^F_2 Z_{2N_f, B+L}$ is free from perturbative local anomaly under dynamical SM gauge group $G_{SM_q} = SU(3) \times SU(2) \times U(1)$. Now we follow the procedure in [12, 31] to check whether the full spacetime-internal symmetry $G$ has any local and global anomalies, including all gauge or gravitational background fields. The spacetime symmetry is Spin group (the double cover of the special orthogonal SO group), and the internal symmetry is $U(1)_{B-L} \times Z^F_2 Z_{2N_f, B+L} \times G_{SM_q}$, or $Z_{4, X} \times Z^F_2 Z_{2N_f, B+L} \times G_{SM_q}$ if the discrete $X$ replacing the continuous $B - L$ symmetry. We focus on $N_f = 3$ thus $Z_{6, B+L} = Z^F_2 \times Z_{3, B+L}$. Given the spacetime-internal $G$, we classify all possible 4d local ($Z$ class) and global ($Z_a$ class) anomalies via computing the 5th cobordism group $TP_5(G)$ [19]:

$$
\text{TP}_5(\text{Spin} \times Z^F_2 U(1)_{B-L} \times Z_{3, B+L} \times G_{SM_q}) = (Z^{11}) \times (Z_9 \times Z^7_3).
$$

(8)

$$
\text{TP}_5(\text{Spin} \times Z^F_2 Z_{4, X} \times Z_{3, B+L} \times G_{SM_q}) = \left\{ \begin{array}{l}
(Z^5 \times Z_2 \times Z^4_2 \times Z_{16}) \times (Z_9 \times Z^4_3), \quad q = 1, 3,

(Z^5 \times Z^2_2 \times Z_4 \times Z_{16}) \times (Z_9 \times Z^3_3), \quad q = 2, 6.
\end{array} \right.
$$

(9)

The anomaly classifications on the right-hand side in the first big bracket (…) are obtained in [12, 31], while those in the second big bracket (…) are new to the literature. The $Z^{11}$ and $Z^5$ classes of local anomalies are familiar to QFT textbook readers [56]. The $Z^2 \times Z^4_2 \times Z_{16}$ classes of global anomalies were characterized in detail before [31]. These
anomalies all canceled, except only $Z^2$ (in $Z^{11}$) classes (involving the cubic pure gauge $U(1)^3_B$ and mixed gauge-gravity $U(1)_B-L$ (gravity)^2 anomalies) and the $Z_{16}$ class (involving the mixed gauge-gravity global anomaly between the $Z_{4,x}$ and the spacetime diffeomorphism), anomalies can have a non-zero coefficient $-N_f + n_{\nu_R} \equiv -N_f + \sum n_{\nu_R}$ if the number of family $N_f$ is distinct from the total number of right-handed neutrinos $n_{\nu_R}$ [12]. The non-zero $Z^2$ and $Z_{16}$ global anomalies are canceled.

The $Z_9 \times Z_3^2$ global anomalies in (8) (which exactly overlaps the $Z_9 \times Z_3^t$ in (9) for the first five generators), involving the discrete $B-L$ background field, are new to the literature: they are generated by cobordism invariants $\Psi_3(B'_3, B_{Z_{3,b+l}}, A'_Z, B_{Z_{3,b+l}} C_1(U(1)_B-L), A'_Z, B_{Z_{3,b+l}} C_2(SU(2)), A'_Z, B_{Z_{3,b+l}} C_2(SU(3)), A'_Z, B_{Z_{3,b+l}} C_1(U(1)_B-L), A'_Z, B_{Z_{3,b+l}} C_2(U(1)_B-L), A'_Z, B_{Z_{3,b+l}} C_3(U(1)_B-L) C_1(U(1)_B-L)).$

Here the generators are written for the $q = 1$ case. Let us explain the notations. Here all cohomology classes are pulled back to the manifold $M$ along the maps given in the definition of cobordism groups, e.g. the $A'_Z, B_{Z_{3,b+l}} \in H^2(BZ_{3,b+l}, Z_3)$ is pulled back to $H^1(M, Z_3).$ The $B'_{Z_{3,b+l}} \equiv (\frac{1}{2} \delta A'_Z, B_{Z_{3,b+l}}$ (mod 3) $\in H^2(BZ_{3,b+l}, Z_3)$ is pulled back to $H^1(M, Z_3).$ The $B'_{Z_{3,b+l}} \equiv (\frac{1}{2} \delta A'_Z, B_{Z_{3,b+l}}$ (mod 3) $\in H^2(BZ_{3,b+l}, Z_3)$ is pulled back to $H^1(M, Z_3).$

Because all these $Z_9 \times Z_3^2$ global anomalies involve internal symmetries (thus gauge anomalies among $B \pm L$, X, and $G_{SM}$) without turning on the spacetime symmetry background (thus they are not mixed gauge-gravitational or gravitational anomalies), as long as these internal symmetries are dynamically gaugeable, we expect all these anomalies are canceled.

Moreover, once $G_{SM}$ is dynamically gauged as in the SM vacuum, we obtain extra 1-form electric and magnetic symmetries, $G_{SM}^2$ and $G_{SM}^m$ kinematically as generalized global symmetries [58] that act on charged objects (1d electric Wilson lines and 1d magnetic 't Hooft lines). By gauging $G_{SM}^c$, we obtain $G_{SM}^{c, m}$ in the dynamical SM gauge theory, for $q = 1, 2, 3, 6$ and we define $q' \equiv 6/q \equiv 3, 2, 1$ as $(n_2, n_3) = (1, 1)$, $(1, 0)$, $(0, 0)$ respectively. We compute the potential 't Hooft anomaly classification of the gauged-$G_{SM}$ SM via removing $G_{SM}$ and including $G_{SM}^c$ and $G_{SM}^m$ into (8) and (9),

$$TP_3(\text{Spin} \times Z_3^2 U(1)_B-L \times Z_{3,b+l} \times Z_6^6/q, U(1)^m_1) = (Z^2 \times Z_6/q) \times (Z_9 \times (Z_3) \times (Z_3)^{m_3}) \times (Z_9 \times (Z_3)^{m_3}).$$

This anomaly classification on the right-hand side in the first big bracket ( . . . ) are obtained recently in [12], while those in the second big bracket ( . . . ) are new to the literature. The $Z^2 q$ local and $Z_6 q$ global anomalies are discussed earlier in (8) and (9), with its anomaly coefficient proportional to $-N_f + n_{\nu_R}$ [12]. The $Z_6/q$ global anomaly is a mixed anomaly between 1-form symmetries of $Z_6^6/q$ and $U(1)^m_1$, which is identified to be non-zero in [12] — which has dynamical constraints on the SM gauge theories. The $(Z_4)^{n_2}$ global anomaly is canceled [12]. The $Z_9$ global anomaly $\Psi_3(B'_3, B_{Z_{3,b+l}})$ and the $(Z_3)^2$ global anomalies $A'_Z, B_{Z_{3,b+l}} C_1(U(1)_B-L)$ and $A'_Z, B_{Z_{3,b+l}} C_1(U(1)_B-L)$ are already canceled zero in the SM earlier in (8). The remaining $(Z_3)^{n_3}$ in (10) (which exactly overlaps the $(Z_3)^{2n_3}$ in (11) for the first two generators) contains three more extra generators (for $n_3 = 1$): $A'_Z, B_{Z_{3,b+l}} C^2_2(U(1)_B-L)$, $A'_Z, B_{Z_{3,b+l}} C_1(U(1)_B-L)$, and $A'_Z, B_{Z_{3,b+l}} C_1(U(1)_B-L) B_{Z_{3,b+l}} C_1(U(1)_B-L)$. The first two of these three terms are mixed anomalies between the 0-symmetry $B + L$ (coupled to 1-form background field $B_{Z_{3,b+l}}$) and the electric 1-symmetry $Z_{3,[1]}$ (coupled to 2-form background field $B_{Z_{3,b+l}}$). It is interesting to check whether these two mixed anomalies occur for SM with $q = 1, 2$ (thus $n_3 = 1$). We can check this mixed anomaly between 0-symmetry and 1-symmetry via the techniques of the analogous anomaly studied previously in [60–65]. If the $Z_3$ class of $A'_Z, B_{Z_{3,b+l}} C^2_2(U(1)_B-L)$ is non-zero, then this means that the $Z_{2N_f}, B + L = Z_6, B + L$ symmetry should be broken down to a $Z^2$ subgroup by the PSU(3) = SU(3)/Z_3 fractional instanton. Note that the instanton number $n(N) = \int d^4x \frac{\pi^2}{12} e^{\mu_\nu_\rho_\sigma}F_{\mu_\nu}F_{\rho_\sigma}$ for SU(N) gauge theory has $n(N) \in Z$ on spin manifolds, while turning on 1-form symmetry $Z_{N,[1]}$ background field can produce PSU(N) fractional instantons such that $n(N) \in Z/N$ on spin manifolds, and $n(N) \in Z/N$ (for odd $N$) or $n(N) \in Z/2N$ (for even $N$) on non-spin manifolds. So if SU(3) instanton was a source of $U(1)_B-L \rightarrow Z_{2N_f}, B + L$ symmetry breaking
in (4), the PSU(3) fractional instanton can further trigger the breaking $Z_{2N_f/3} \rightarrow Z_{2N_f/6}$ symmetry, here the family and color number matching $N_f = N_c = 3$. But it is the SU(2) instanton (not the SU(3) instanton) causing the U(1)$B+L \rightarrow Z_{2N_f/3} \rightarrow Z_{2N_f/6}$ symmetry breaking. So we conclude that the PSU(3) fractional instanton cannot trigger the further breaking even in the presence of $Z_{6/q}[1]$ background field (for $q = 1, 2$). Namely, this suggests no $A_{Z_{3B+L}}(B_{Z_{3}[1]}^{F})^2$ anomaly in the gauged SM.

In comparison, Ref. [45] checked various fractional instanton contributions also showing a negative result on the mixed $(B + L) Z_{6/q}[1]$ anomalies. Ref. [45] found that in the presence of various fractional instantons constructed on a torus with 't Hooft twisted boundary condition [66], the continuous U(1)$B+L$, breaks down to $Z_{2N_f/3} \rightarrow Z_{2N_f/6}$, $Z_{6/q}[1]$, $Z_{2N_f/3}$ for $q = 1, 2, 3, 6$. But the familiar BPST SU(2) instanton [40] already breaks U(1)$B+L$ to $Z_{2N_f/3}$ shown in (4). Since $Z_{2N_f/3} \rightarrow Z_{2N_f/6}$ is not further broken down to its subgroup by fractional instantons, Ref. [45] also found no mixed 't Hooft anomaly between $Z_{2N_f/3}$ and $Z_{6/q}[1]$ symmetry.

Näively, all these so-called 3-torsion $(Z_0 \times (Z_3)^{2})$ and $(Z_0 \times (Z_3)^{2N_f})$ classes correspond to potential global anomalies in the SM between vector symmetries (the vector $B + L$ and the $Z_{3N_f/2}$ from gauging the vector $su(3)$), so there should be no anomalies among vector symmetries. But the $Z_{3N_f/2}$ also partly descends from gauging the chiral u(1)$_Y$. In particular, the last two terms in the $(Z_3)^{2N_f}$, namely $A_{Z_{3B+L}}B_{Z_{3B+L}}B_{Z_{3N_f/2}} \equiv A_{Z_{3B+L}}(3\delta A_{Z_{3B+L}} \mod 3)B_{Z_{3N_f/2}}$, and $A_{Z_{3B+L}}(U(1)B+L)B_{Z_{3N_f/2}}$, anomalies, are subtle anomalies that cannot be checked by the fractional instanton argument above, thus they deserve further future investigations.

Conclusion — We had identified a continuous $B - L$ or discrete $X$, and discrete $B + L$ symmetry in the SM. For $N_f = 3$, we had shown $Z_{2N_f/3} \rightarrow Z_{2N_f/6}$ is free from anomaly with SM gauged Lie group $G_{SM}$ for all four versions of $q = 1, 2, 3, 6$. We also classify all potential mixed anomalies between $Z_{2N_f/3}$ and gravitational background field (from Spin diffeomorphism) or higher symmetries $Z_{6/q}[1] \times U(1)_Y$ of SM in (10) and (11). We show that $Z_{2N_f/3}$ is free from all anomalies in (8) and (9), and it is free from many potential anomalies in (10) and (11) involving higher symmetries as well. Although $Z_{2N_f/3}$ is illegal in many GUTs in (6), $Z_{2N_f/3}$ is a perfectly legal symmetry in the SM and some versions of UU to protect the proton stability. Are there other ways to protect the proton stability in UU? Notice that UU allows a 4d TQFT with discrete gauge forces (e.g., two layers of $Z_2$ gauge theories) constructed out of the symmetry extension [67] (e.g., the first layer 1 $\rightarrow [Z_2] \rightarrow \text{Spin} \times Z_4 \rightarrow \text{Spin} \times Z^F_4$ $Z_4 \rightarrow 1$ and the second layer 1 $\rightarrow [Z_2] \rightarrow \text{Spin} \times Z_8 \rightarrow \text{Spin} \times Z_4 \rightarrow 1$) such that the even class of $Z_{16}$ global anomaly in the original Spin $\times Z^F_4$ $Z_4 \times X$ symmetry is trivialized by pulled back to the extended Spin $\times Z_8$ symmetry [28, 35, 37]. Since the quark and baryon can all carry odd $Z_4 \times X$ charges, it will be interesting to study the impact of the projection symmetry fractionalization of Spin $\times Z^F_4$ $Z_4 \times X$ on the line or surface operators of the 4d discrete gauge TQFT on the proton stability.

In the language of deformation class of quantum field theories [15], our present work together with Ref. [12] suggests that the 4d SM and GUTs can still be in the larger deformation class of QFT by including the $Z_{2N_f/3} \rightarrow Z_{2N_f/6}$ symmetry, except that SM and UU can preserve $Z_{2N_f/3} \rightarrow Z_{2N_f/6}$, but many GUTs break $Z_{2N_f/3}$ explicitly. It will be important to understand deeper the implications of the subtle anomaly cancellation data presented in (8), (9), (10), and (11), so we may use these results to investigate the SM dynamics better.

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