Nonstationary periodic wave regimes on a falling liquid film

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Abstract. The theoretical and numerical study of the periodic regimes of free flowing liquid film is carried out. As a result of the theoretical analysis, a simple three-harmonic model was obtained. The obtained model allowed demonstrating the basic mechanisms of the periodic flow regime. Calculations on the full model, the Nepomnyashchy equation and three harmonic models are performed. The results of the numerical solutions of the Nepomnyashchy equation and new model qualitatively consistent with the results of the solution of the full model are presented.

1. Introduction
Falling film evaporators and crystallizers are especially popular in the chemical and food industry. Heat exchange units based on the phenomenon of laminar film condensation are widely used in refrigeration technology [1-3]. Theoretical foundations of the problem of heat transfer between a plane inclined wall covered by a thin layer of viscous fluid and surrounding gas were laid by Nusselt [4] in 1916. The Nusselt theory takes into account the condensation and evaporation processes and significantly underestimates the experimental data on heat transfer rates due to the presence of waves on the film surface. A series of pioneering works of Kapitza and Kapitza [5] describe numerous wave regimes of a laminar film flow, earlier considered to be trivial. The authors also attempt to explain the additional wave-induced heat transfer rate. Although their simple model provides only a 20% increase in the corresponding coefficient in contrast to its almost 100% increase in experiments. The first realistic theory of a wave influence on the heat transfer of evaporating and condensing film is proposed in [3]. Two different wave regimes are investigated: a quasi-sinusoidal high frequency regime and the regime of “intermediate waves.” The latter is characterized by large humps, separated by rather long relatively plane thin liquid layers. In this case, authors discovered a significant influence of waves on thermal processes. More recent developments usually consider the heat problem to be inseparable from an established hydrodynamic wave regime [6].

Observations of waves under the natural conditions mostly confirm their irregular behaviour, determined by a large number of steady modes and complex transient regimes. It was shown [7] that even in the case of small Reynolds number (flow rate), the periodic solutions of a model equation (Kuramoto-Sivashinsky) demonstrate a chaotic dynamics. To classify and describe the data obtained in both mathematical modeling and experiments, the topological structure of steady-state travelling solutions of the model equation was intensively investigated in [8,9]. Tsvelodub and Trifonov [10] extended these results to the region of moderate Reynolds numbers and provided new data on stability.
of the found regimes. In a series of experimental works by Liu and Gollub [10,11] the basic mechanisms of onset and development of spatio-temporal chaos in wavy film flows were experimentally found. Among them the most significant was the subharmonic instability leading to wavelength doubling. This mechanism is very sensitive to the noise and, therefore, has an irregular character. However, the successive period doubling is not typical of this scenario, and the subharmonic spectral peaks usually do not become dominant [12]. The aim of the present work is to reexamine this mechanism using mathematical modeling to identify its key features.

2. Problem statement and governing equations

The paper [13] provides the governing fluid dynamics equations in a special coordinate system with the film flow area transformed into a strip of constant thickness. It is assumed that the Reynolds number $Re = g h_0^3/3 v^2$ is moderate and the wavelength parameter $\varepsilon = h_0/l_0$ is small (so, that $\varepsilon Re \sim 1$). For convenience the equations take the following dimensionless form:

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left( \frac{Q^2}{h} \right) + \frac{\partial}{\partial \eta} \left( \frac{Q \nu}{h} \right) = -\frac{1}{\varepsilon Re} \frac{\partial^2 Q}{\partial x^2} + \frac{3h}{\varepsilon Re} + \frac{18}{5} h \frac{\partial^3 h}{\partial x^3}$$

$$\frac{\partial h}{\partial t} + \frac{\partial Q}{\partial x} + \frac{\partial \nu}{\partial \eta} = 0$$

(1)

$$Q = V = 0 \text{ at } \eta = -1, \frac{\partial Q}{\partial \eta} = 0, V = 0 \text{ at } \eta = 0.$$

Here, $h(x,t)$ is the dimensionless film thickness, $\eta = y/h(x,t)-1$ is the transformed transverse coordinate, $Q = hu, V = hv$, where $u$ and $v$ are the dimensionless longitudinal and transverse contravariant components of velocity vector. The characteristic scales are: the average thickness $h_0$ and the superficial velocity of a waveless Nusselt flow $u_0 = g h_0^2/3 v$. The scales of the length $l_0$ and time $t_0 = l_0/u_0$ are determined from the relationship:

$$\frac{h_0}{l_0} = \sqrt{\frac{18 \text{ Re}^{5/3}}{5 \text{ } 3^{1/3} \text{ Fi}^{1/3}}}$$

where $\text{Fi} = \sigma^3/\rho^3 g v^4$ is the film number, characterizing liquid properties. This choice provides the value of neutral wave number $\alpha_n \approx 1$.

The equations (1) are solved by the pseudospectral (collocation) method, a detailed description of which is given in [14]. Functions depending on transverse coordinate $\eta$ are expanded in a series on Chebyshev polynomials $T_i$:

$$Q(x,\eta,t) = \sum Q_{2j}(x,t) \left( T_{2j}(\eta) - 1 \right)$$

(2)

$$V(x,\eta,t) = \sum V_{2j+1}(x,t) \left( T_{2j+1}(\eta) - \eta \right)$$

Representation (2) automatically satisfies the boundary conditions on the free surface and the solid wall. Functions that depend on the longitudinal coordinate are expanded into spatial Fourier series:

$$[Q_{2j}, V_{2j+1}, h](x,t) = \sum [Q_{2j,k}, V_{2j+1,k}, h_k](t) e^{ikx}.$$  

(3)

Here $\alpha$ is the wave number of the periodic solution.

The dynamics of periodic perturbations of the free surface of water film in the range of wave numbers $0.41 < \alpha < 0.5$ and Reynolds numbers $1 < Re < 30$ is numerically investigated. Typical scenarios of the evolution of perturbations with small initial amplitude are presented in figure 1. Time dependences of the first two harmonics amplitudes $h_1(t) = |h_{11}(t)| e^{i\phi(t)}, h_2(t) = |h_{21}(t)| e^{i\psi(t)}$ and "phase difference" $\phi(t) = 2\varphi(t) - \psi(t)$, are shown at the value of $\alpha = 0.42$. It is seen that the functions change periodically. During most part of the period, the amplitude of the second harmonic is close to the value that corresponds to the steady-state travelling solution with a double wave number $2\alpha = 0.84$. At the, the amplitude of the first and other odd harmonics grows exponentially. At the
same time, the nonlinear effects are increasing, the principal of which turns out to be the energy transfer from the second to the first harmonic. As a result, the amplitude of the second one passes through null (and its phase immediately changes to π). Soon after that the growth of the first harmonic stops, and from that moment the second harmonic grows for the account of the first harmonic.

In other words, the steady-state travelling regime with the wave number $2\alpha = 0.84$ is unstable with respect to the perturbations of double wavelength ($\alpha = 0.42$). One should expect that the development of this instability will lead to a new quasi-stationary regime with $\alpha = 0.42$. However, after the phase shift the situation changes: all the perturbations of the double wavelength ($\alpha = 0.42$) in the short run (but sufficiently quickly) attenuate, and the initial quasi-sinusoidal wave with $2\alpha = 0.84$ appears again. Such behaviour reminds the phenomenon of intermittency and is undoubtedly interesting to understand.

### 3. Low-dimensional analysis

The above-mentioned dynamic mode is surprisingly stable with respect to the Reynolds number variations. Particularly, the great majority of initial conditions at $Re = 5$ and $0.41 < \alpha < 0.5$ eventually lead to this mode. Film flows with such a small Reynolds number can be well modeled by a relatively simple Nepomnyashchy equation (also known as Kuramoto-Sivashinsky equation) for the deviation of the liquid film height:

$$ H_t + 4HH_x + H_{xx} + H_{xxxx} = 0 $$  \hspace{1cm} (4)

Expanding the solution of the equation (4) into Fourier series, we immediately obtain an infinite system of ordinary differential equations for amplitudes of Fourier harmonics:

$$ H_k - (\alpha^2 k^2 - \alpha^4 k^4)H_k + 2i\alpha k \sum H_r H_{r-k_r} = 0 $$  \hspace{1cm} (5)

Limiting to a rather large but finite number of harmonics we have an initial value problem (5) that can be solved numerically. Firstly, let us suppose that all harmonics except for $H_1, H_{-1}, H_2, H_{-2}, H_3, H_{-3}$ in (5) equal null, and the second harmonic has a constant value, i.e. $H_2 = H_{-2} = const$. At that, one may obtain a low-dimensional approximation of the equations (5):

$$ \dot{H}_1 - (\alpha^2 - \alpha^4)H_1 + 4i\alpha(H_{-2}H_3 + H_{-1}H_2) = 0 $$  \hspace{1cm} (6)

$$ \dot{H}_{-1} - (\alpha^2 - \alpha^4)H_{-1} - 4i\alpha(H_{-3}H_2 + H_{-2}H_1) = 0 $$  \hspace{1cm} (7)

$$ \dot{H}_3 - (9\alpha^2 - 81\alpha^4)H_3 + 12i\alpha H_1 H_2 = 0 $$  \hspace{1cm} (8)

![Figure 1. Evolution of the modulus of harmonics’ amplitudes (left) and phase difference (right) at $\alpha = 0.42$.](image-url)
It is clearly seen that within the interval of wave numbers of interest to us, the modulus of attenuation decrement of $r_3$ much exceeds the growth increment of $r_2$. Thus, one may assume $r_2 = 0$ and the equation (8) gives:

$$H_3 = \frac{4iH_1H_2}{3\alpha - 27\alpha^3}$$  \hspace{1cm} (9)

Substituting this relationship in (6) and converting the real and imaginary parts of $H_2$ to the modulus $|H_2|$ and the phase $\phi = 2\varphi$, we obtain the equations:

$$|H_1| = \left( (\alpha^2 - \alpha^4) + \frac{16H_2^2}{3 - 27\alpha^2} - 4\alpha H_2 \sin\phi \right) |H_1|$$

$$\dot{\phi} = -8\alpha H_2 \cos\phi$$  \hspace{1cm} (10)

Model (10) allows demonstrating the influence of the second harmonics on the evolution of the first one. At constant value of $H_2$ variations of the phase $\phi$ determine the behavior of the modulus of $H_1$: it may exponentially grow or attenuate depending on its value. Besides, the characteristic time of the decay is noticeably lesser than the time of the growth. When the last term in the brackets of (10) has the same sign as the sum of the first two terms (that is always negative), the "effective" decay decrement (at $\sin\phi > 0$) is significantly greater than the growth increment in the opposite case ($\sin\phi < 0$).

There is an example of the calculations by model (10) at the value of $\alpha = 0.45$ shown in figure 2. The evolution of modulus of the first harmonics’ amplitude is given in figure 2a. Time dependence of the phase difference” $\phi(t) = 2\varphi(t)$ is demonstrated in figure 2b.

For this model a similar quasiperiodical regime is set for the most initial conditions within the wave numbers $0.43 < \alpha < 0.5$. Let us analyze this process using the numerical solutions of the original equation (4). An example of such calculations with $N = 10$ harmonics in (5) is given in figure 3 ($\alpha = 0.45$). For this wave number, after a transient processes taking some time depending on initial conditions, the “intermittence” regime establishes. Its features are illustrated in figure 3. The evolution of moduli of the first three harmonics amplitudes is given in figure 3a. The region of interest is shown in figure 3b in a larger scale. The “intermittence” here is even more pronounced than in the solutions to equations (10) (figure 1). The steady-state travelling wave with double wave number $2\alpha = 0.9$ (only the even nonzero harmonics) remains almost unperturbed for a long time (for example, within time range of $140 – 240$). Then, for a short period of time, the flow parameters drastically change (within time range of $240 – 270$) (figure 3a) and again the quasi-stationary regime is set. The situation is, in a
certain sense, analogous to that in figure 2a. The only difference is that \( H_2 \) is not constant now. Characteristic times of the growth and decay periods for \( H_3 \) are, in this case, close to each other. At initial stages of the growth as well as at the final stages of the decay \( H_3 \) is almost equal to \( H_4 \); this is in a good agreement with (9). Only in the vicinity of the maxima of \( H_1 \) this correlation is violated (see figure 3b).

In figures 3c and 3d the time dependences of the phase difference \( \phi(t) = 2\varphi(t) \) are shown on the scales corresponding to figures 3a and 3b. The difference between the dynamics of \( H_1 \) in models (10) and (5) is due to the influence of the first and higher harmonics on \( H_2 \). It is important to note that the decrease of the wave number \( \alpha < 0.43 \) qualitatively changes the dynamics. Now the intermittence regime is absent and after a short transition period (when primarily odd harmonics vanish) the steady mode with double wave number is set and it remains unchanged. In terms of the stability theory, this is due to the fact that the steady-state travelling waves with \( 0.8 < 2\alpha < 0.86 \) are stable with respect to the perturbations of double wavelength. In contrast to this, the regimes with greater \( \alpha \) are unstable [9].

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Figure 3. The moduli of amplitudes of the first three harmonics (a, b) and the phase difference (c, d) obtained by the Nepomnyashchy (Kuramoto-Sivashinsky) equation.
References

[1] Alekseenko S V, Nakoryakov V E and Pokusaev B G 1994 Wave flow of liquid films (New York: Begell House)
[2] Sheid B 2004 Evolution and stability of falling films with thermocapillary effects Ph.D. Thesis Brussels: Universite Libre de Bruxelles
[3] Hirshburg R I, Florschuetz L W 1982 Journal of Heat Transfer 104 (3) 459–464
[4] Nusselt W 1916 Zeitschrift Verein Deutscher Ingenieure 60 541–546
[5] Kapitza P L, Kapitza S P 1965 Collected Papers of P. L. Kapilza, Vol. 2 (Pergamon New York)
[6] Aktershev S P 2010 Thermophysics and Aeromechanics 17(3) 359–370
[7] Sivashinsky G I, Michelson D M 1980 Progress of Theoretical Physics 63(6) 2112–14
[8] Michelson D 1986 Physica D: Nonlinear Phenomena 19(1) 89–111
[9] Tsvelodub O Yu, Trifonov Yu Ya 1989 Physica D: Nonlinear Phenomena 39(2-3) 336–351
[10] Tsvelodub O Yu, Trifonov Yu Ya 1992 Journal of Fluid Mechanics 244 149–169
[11] Liu J, Gollub J P 1993 Physical Review Letters 70(15) 2289
[12] Liu J, Paul J D and Gollub J P 1993 Journal of Fluid Mechanics 250 69–101
[13] Arkhipov D, Vozhakov I, Markovich D, Tsvelodub O 2016 European Journal of Mechanics-B/Fluids 59 52–56
[14] Vozhakov I S, Arkhipov D G and Tsvelodub O Yu 2017 Journal of Physics: Conference Series 899(3) 032025