Convergence Rates of Training Deep Neural Networks via Alternating Minimization Methods

Jintao Xu ∗ Chenglong Bao † Wenxun Xing ‡

Abstract

Training deep neural networks (DNNs) is an important and challenging optimization problem in machine learning due to its non-convexity and non-separable structure. The alternating minimization (AM) approaches split the composition structure of DNNs and have drawn great interest in the deep learning and optimization communities. In this paper, we propose a unified framework for analyzing the convergence rate of AM-type network training methods. Our analysis is based on the non-monotone $j$-step sufficient decrease conditions and the Kurdyka-Łojasiewicz (KL) property, which relaxes the requirement of designing descent algorithms. We show the detailed local convergence rate if the KL exponent $\theta$ varies in $[0, 1)$. Moreover, the local R-linear convergence is discussed under a stronger $j$-step sufficient decrease condition.

Keywords Deep neural networks training; Alternating minimization; Kurdyka-Łojasiewicz property; Non-monotone $j$-step sufficient decrease; Convergence rate

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1 Introduction

In recent years, deep learning has achieved impressive successes in many areas including computer vision [10, 16], natural language processing [32, 34], and recommender system [11, 13]. For deep neural networks (DNNs) training, the alternating minimization (AM)-type training methods, mainly based on the block coordinate descent (BCD) [30] or the alternating direction method of multipliers (ADMM) [8], have been discussed such as BCD-type algorithms [9, 15, 21, 36, 40, 42] and ADMM-type algorithms [18, 33, 35, 37, 41, 43]. Carreira-Perpiñán and Wang [9], Lau et al. [21], Zeng et al. [40], and Gu et al. [15] designed BCD-type algorithms to train the feedforward neural networks (FNNs) approximately. Additionally, ADMM-type algorithms for FNNs are also proposed by Taylor et al. [33], Wang et al. [35, 37, 41], and Zeng et al. [41]. Furthermore, AM-type training methods are designed to train other neural network models like the convolutional neural networks (CNNs) [15, 16] and the residual networks (ResNets) [20]. Besides, online AM-type training [12] and parallel AM-type training [33, 35] are also implemented. In these methods, auxiliary variables are added for each layer to decouple the nested parameters in DNNs, and the vanishing gradient issue [4, 14] is avoided.

∗Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China.
Email: xujt19@mails.tsinghua.edu.cn
†Yau Mathematical Sciences Center, Tsinghua University, Beijing 100084, China, and Yanqi Lake Beijing Institute of Mathematical Sciences and Applications, Beijing 101408, China.
Email: clbao@mail.tsinghua.edu.cn
‡Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China.
Email: wxing@mail.tsinghua.edu.cn
Let \(\min_{X \in \mathcal{D}} f(X)\) be a DNNs training model, where \(\mathcal{D} \subseteq \mathbb{R}^{m \times n}\), and \(\{X_k\}\) be a sequence generated by an AM-type training algorithm. In this paper, we propose a unified framework for establishing the convergence rate of the objective function value \(\{f(X_k)\}\) generated by various AM-type training algorithms based on a non-monotone \(j\)-step sufficient decrease condition. Specifically, motivated by the algorithm mDLAM in \([56]\), let \(j\) be a positive integer, our analysis imposes a \(j\)-step sufficient decrease condition defined as follows, which relaxes the common descent condition used in \([3]\).

\[A1.\] For a certain \(j \in \mathbb{N}_+\), there exists \(c_1 > 0\) such that
\[
c_1 \text{dist}(0, \partial f(X_{k+j}))^2 \leq f(X_k) - f(X_{k+j}) \tag{1}
\]
for each \(k \geq k_0\).

When \(j = 1\), the above condition can be derived from the sufficient decrease condition H1 and the relative error condition H2 of \([3]\), which is monotone. When \(j \geq 2\), it allows the oscillations of \(\{f(X_k)\}\) during the consecutive \(j\) iterations, which can be classified as a non-monotone method. In Sect. \([5]\) we will show that the existing four AM-type algorithms for training DNNs satisfy \(A1\). Besides, the Kurdyka-Łojasiewicz (KL) property \([2, 23]\) assumption of \(f\), which implies a local sharpness under reparametrization \([3]\), plays a central role in our analysis.

Convergence results have been stated for some AM-type training algorithms \([12, 17, 21, 35, 36, 37, 40, 41]\). From the theoretical perspective, the convergence rate of \(\{X_k\}\) generated by a BCD-type algorithm is analyzed \([21]\), and the convergence of the objective function value \(\{f(X_k)\}\) is proven for BCD-type \([40]\) and ADMM-type algorithms \([35, 41]\). Sample complexity of an AM-type algorithm for ReLU networks is established in \([17]\). Besides, the convergence rate of the expected error of an online AM-type training algorithm is analyzed in \([12]\). However, there are few results about the convergence rate of the objective function value, like that in \([36]\), and we address this question.

The main contribution of this paper is to propose a unified framework based on \(A1\) for theoretically analyzing the convergence rate of the objective function value sequences generated by AM-type training methods. Thanks to the oscillations of the sequence being allowed, a wider range of training methods can be addressed theoretically and uniformly. We establish the local convergence rate in Theorem \(4.1\), which depends on the different values of the Kurdyka-Łojasiewicz (KL) exponent \(\theta \in [0, 1)\) \([2, 23]\) of \(f\). Moreover, if we replace the lower bound in \((1)\) with \(c_2 \text{dist}(0, \partial f(X_{k+j}))^2\), where \(c_2\) and \(\alpha \geq \theta\) are positive constants, we give the local linear convergence in Theorem \(4.2\).

Additionally, the above theoretical results are true for those AM-type training methods with a non-monotone \(j\)-step (\(j \geq 2\)) sufficient decrease condition. In this way, both monotone and non-monotone training algorithms can be handled uniformly.

The rest of the paper is organized as follows. Notations and definitions used throughout this paper are listed in Sect. \([2]\). Four examples are shown in Sect. \([3]\). Estimations of convergence rate are stated in Sect. \([4]\) and we summarize our results in Sect. \([5]\).

## 2 Notations and Definitions

We give notations and definitions that are useful in our analysis.

**Notations.** Throughout this paper, \(\mathbb{R}, \mathbb{R}^m, \mathbb{R}^{m \times n}\), and \(\mathbb{N}_+\) denote the set of real numbers, real \(m\)-dimensional vectors, real \(m \times n\) matrices, and positive integers respectively. \(0\) denotes the matrix of all zeros whose size varies from the context. \(\|\cdot\|\) denotes the Euclidean norm for \(x \in \mathbb{R}^m\) and the Frobenius norm for \(X \in \mathbb{R}^{m \times n}\), respectively. For \(A, B \in \mathbb{R}^{m \times n}\), \(\langle A, B \rangle = \text{tr}(AB^T)\), \(\text{dist}(x, S) := \inf_{s \in S} \|x - s\|\). For any \(u \in \mathbb{R}\), \([u]\) is the greatest integer no larger than \(u\). \(\mathcal{O}(\cdot)\) is the standard big O asymptotic notation.
Definition 2.1. (Fréchet subdifferential [28, 29])
The Fréchet subdifferential of \( f \) at \( X \in \text{dom}(f) \) is the set
\[
\hat{\partial} f(X) = \left\{ G \mid \liminf_{Y \to X} \frac{f(Y) - f(X) - \langle G, Y - X \rangle}{\|Y - X\|} \geq 0 \right\}.
\]

Definition 2.2. (Limiting subdifferential [28, 29])
For each \( X \in \text{dom}(f) \), the limiting subdifferential of \( f \) at \( X \) is the set
\[
\partial f(X) = \left\{ G \mid \exists X_k \to X, f(X_k) \to f(X), G_k \to G, G_k \in \hat{\partial} f(X_k) \right\}.
\]

\[\text{dom}(\partial f) := \{ X \mid \partial f(X) \neq \emptyset \}.\]

Definition 2.3. (Limiting critical point [3])
A point \( X \in \text{dom}(f) \) is called a (limiting) critical point of \( f \) if \( 0 \in \partial f(X) \).

Definition 2.4. (Kurdyka-Łojasiewicz property [2, 23])
A proper lower semicontinuous function \( f \) is said to have the Kurdyka-Łojasiewicz (KL) property with exponent \( \theta \) at \( \hat{X} \in \text{dom}(\partial f) \) if there exist \( c, \tau \in (0, \infty), \theta \in [0, 1) \), and a neighborhood \( U \) of \( \hat{X} \) such that for all \( X \in U \cap \{ X \mid f(\hat{X}) < f(X) < f(\hat{X}) + \tau \} \),
\[
(f(X) - f(\hat{X}))^\theta \leq \text{cdist}(0, \partial f(X)).
\]

We call \( \theta \) as the Kurdyka-Łojasiewicz (KL) exponent at \( \hat{X} \) [23].

KL property is widely used in non-convex optimization [2, 3, 7, 38]. The pioneering work on it is credited to Kurdyka [20] and Łojasiewicz [25, 24]. A large class of functions is proven to satisfy the KL property, for example, real analytic functions [25, 26, 24], functions definable in o-minimal structures [6, 20], uniformly convex functions [7], and subanalytic continuous functions [5]. In the scenario of DNNs training, the linear, polynomial, hyperbolic tangent, and sigmoid activation functions; the squared, logistic, and exponential loss functions; and the squared Frobenius norm regularization terms all satisfy the KL property [40, 41]. More details about the KL property can be seen in [2, 3, 7] and the references therein.

Definition 2.5. (Local convergence)
For a convergent sequence \( \{ X_k \} \) generated by an algorithm \( \mathcal{A} \) with a limit \( X^* \), if the initial point \( X_0 \) is needed to be close enough to \( X^* \), algorithm \( \mathcal{A} \) is said to be local convergence (\( \{ X_k \} \) locally convergent to \( X^* \)).

Definition 2.6. (Root (R)-convergence rate [31])
For any convergent sequence \( \{ X_k \} \) with a limit \( X^* \), \( R_1 := \limsup_{k \to \infty} \| X_k - X^* \|^{-1} \). If \( 0 < R_1 < 1 \), \( \{ X_k \} \) is called Root (R)-linearly convergent. If \( R_1 = 1 \), \( \{ X_k \} \) is called Root (R)-sublinearly convergent.

3 Typical AM-type algorithms
In this section, we present four examples that apply AM-type algorithms for training DNNs. These examples and their numerical results motivate us to construct a unified framework for estimating the convergence rate of the objective function value sequences generated by AM-type DNNs training algorithms.

Example 3.1. (BCD for FNNs [40])
For the feedforward neural networks (FNNs) training, Zeng et al. [40] formulated two optimization models
named two-splitting and three-splitting formulations, and designed BCD-type algorithms for their unconstrained approximations, respectively. For the two-splitting formulation, the objective function is

$$f(X) = \frac{1}{n} \sum_{j=1}^{n} \ell((V_N)_{(j)}, y_j) + \sum_{i=1}^{N} r_i(W_i) + \sum_{i=1}^{N} s_i(V_i)$$

$$+ \frac{\gamma}{2} \sum_{i=1}^{N} \|V_i - \sigma_i(W_i V_{i-1})\|^2,$$  \hspace{1cm} (2)

where $X = (\{W_i\}_{i=1}^{N}, \{V_i\}_{i=1}^{N})$, $\sigma_i$ denotes the activation function of the $i$th layer, $\ell$ is a loss function, $r_i, s_i$ can be seen as regularization terms about $W_i, V_i, i = 1, 2, \ldots, N$, respectively, $(V_N)_{(j)}$ denotes the $j$th column of $V_N$, $j = 1, 2, \ldots, n$, and the last term represents a quadratic penalty for constraints $V_i = \sigma_i(W_i V_{i-1}), i = 1, 2, \ldots, N$. Under the assumptions in the Theorem 1 in [40], (2) satisfies the KL property on any closed set. Moreover, \(\{f(X_k)\}\) generated by the BCD-type algorithm is convergent, and satisfies

$$\frac{a}{b^2} \text{dist}(0, \partial f(X_{k+1}))^2 \leq f(X_k) - f(X_{k+1})$$

for certain $a, b > 0$ (see [40] for the values of $a$ and $b$), which is an A1 with $j = 1$.

And for the three-splitting formulation,

$$f(X) = \frac{1}{n} \sum_{j=1}^{n} \ell((V_N)_{(j)}, y_j) + \sum_{i=1}^{N} r_i(W_i) + \sum_{i=1}^{N} s_i(V_i)$$

$$+ \frac{\gamma}{2} \sum_{i=1}^{N} \|V_i - \sigma_i(U_i)\|^2 + \frac{\gamma}{2} \sum_{i=1}^{N} \|U_i - W_i V_{i-1}\|^2,$$  \hspace{1cm} (3)

where $X = (\{W_i\}_{i=1}^{N}, \{U_i\}_{i=1}^{N}, \{V_i\}_{i=1}^{N})$, and the last two terms are quadratic penalties for constraints $V_i = \sigma_i(U_i)$ and $U_i = W_i V_{i-1}, i = 1, 2, \ldots, N$, respectively. Under the same assumptions, (3) satisfies the KL property on any closed set. Similarly, \(\{f(X_k)\}\) is convergent, and satisfies

$$\frac{a}{b^2} \text{dist}(0, \partial f(X_{k+1}))^2 \leq f(X_k) - f(X_{k+1})$$

for certain $a, b > 0$ (see [40] for the values of $a$ and $b$), which is an A1 with $j = 1$.

**Example 3.2.** (ADMM for FNNs [41])

Zeng et al. [41] considered the augmented Lagrangian function of an FNNs training model and solved it via an ADMM-type algorithm. Technically, they gave

$$f(X) = \frac{1}{2} \|V_N - Y\|^2 + \frac{\lambda}{2} \sum_{i=1}^{N} \|W_i\|^2 + \sum_{i=1}^{N-1} \langle \Lambda_i, \sigma(W_i V_{i-1}) - V_i \rangle$$

$$+ \sum_{i=1}^{N-1} \frac{\beta_i}{2} \|\sigma(W_i V_{i-1}) - V_i\|^2 + \langle \Lambda_N, W_N V_N - V_N \rangle$$

$$+ \frac{\beta_N}{2} \|W_N V_N - V_N\|^2 + \sum_{i=1}^{N} \xi_i \|V_i - V_i\|^2,$$  \hspace{1cm} (4)

where $X = (\{W_i\}_{i=1}^{N}, \{V_i\}_{i=1}^{N}, \{\Lambda_i\}_{i=1}^{N}, \{\nabla_i\}_{i=1}^{N})$ and $\sigma$ denotes the activation function. Suppose that there exist $\chi > 0$ and $k_0 \in \mathbb{N}$ such that $\|V_k^{k-1} - V_k^{k-2}\| \leq \chi \|V_k^k - V_k^{k-1}\|$ for each $k \geq k_0$. Under
the assumptions in the Theorem 7 in \cite{41}, \cite{4} satisfies the KL property, and \{f(X_k)\} generated by the ADMM-type algorithm is convergent. Moreover, there exist \(a, b > 0\) (see \cite{41} for the values of \(a\) and \(b\)) such that
\[
\frac{a}{2b^2(1 + \chi)^2N} \text{dist}(0, \partial f(X_{i+1}))^2 \leq f(X_k) - f(X_{k+1})
\]
for each \(k \geq k_0\), which is an A1 with \(j = 1\).

**Example 3.3. (mDLAM for FNNs \cite{36})**

Wang et al. \cite{36} formulated an FNNs training model and designed an AM-type algorithm called mDLAM to solve it. The objective function of the training model is
\[
f(X) = \ell(v_N, y) + \sum_{i=1}^{N} r_i(W_i) + \frac{\gamma}{2} \sum_{i=1}^{N} \|v_i - W_i u_{i-1}\|^2,
\]
where \(X = (\{W_i\}_{i=1}^{N}, \{u_i\}_{i=1}^{N-1}, \{v_i\}_{i=1}^{N})\). The last term is a quadratic penalty for constraints \(v_i = W_i u_{i-1}, i = 1, 2, \ldots, N\), and the operations of non-linear continuous activation functions \(\sigma_i\) are formulated as inequality constraints \(\sigma_i(v_i) - \epsilon \leq u_i \leq \sigma_i(v_i) + \epsilon, i = 1, 2, \ldots, N-1\). If (5) is real analytic \cite{19}, it satisfies the KL property. Moreover, according to the Lemma 2 and inequality (26) in \cite{36}, \{f(X_k)\} is convergent, and satisfies
\[
\frac{a}{6b^2} \text{dist}(0, \partial f(X_{i+2}))^2 \leq f(X_k) - f(X_{k+2})
\]
for certain \(a, b > 0\) (see \cite{36} for the values of \(a\) and \(b\)), which is a non-monotone 2-step sufficient decrease condition.

**Example 3.4. (BCD for ResNets \cite{40})**

For the residual networks (ResNets) \cite{16} training, Zeng et al. \cite{40} formulated a three-splitting simplified model and its unconstrained approximation. The objective function is
\[
f(X) = \frac{1}{n} \sum_{j=1}^{n} \ell((V_N)_{j}, y_j) + \sum_{i=1}^{N} r_i(W_i) + \sum_{i=1}^{N} s_i(V_i)
\]
\[+ \frac{\gamma}{2} \sum_{i=1}^{N} \|V_i - V_{i-1} - \sigma_i(U_i)\|^2 + \frac{\gamma}{2} \sum_{i=1}^{N} \|U_i - W_i V_{i-1}\|^2,
\]
where \(X = (\{W_i\}_{i=1}^{N}, \{U_i\}_{i=1}^{N}, \{V_i\}_{i=1}^{N})\), and the last two terms represent quadratic penalties for constraints \(V_i - V_{i-1} = \sigma_i(U_i)\) and \(U_i = W_i V_{i-1}, i = 1, 2, \ldots, N\), respectively. Under the assumptions in the Theorem 1 in \cite{40}, \cite{6} satisfies the KL property on any closed set. Moreover, \{f(X_k)\} generated by the BCD-type algorithm is convergent, and satisfies
\[
\frac{a}{3N^2b^2} \text{dist}(0, \partial f(X_{i+1}))^2 \leq f(X_k) - f(X_{k+1})
\]
for certain \(a, b > 0\) (see \cite{40} for the values of \(a\) and \(b\)), which is an A1 with \(j = 1\).

### 4 Theoretical analysis

In this section, we give the following unified convergence rate estimation framework based on A1 and the KL property for AM-type training algorithms.
**Theorem 4.1.** For a proper lower semicontinuous objective function $f$ and a sequence $\{X_k\}$ generated by an AM-type training algorithm, suppose that there exists $\tilde{X} \in \text{dom}(\partial f)$ such that $f$ satisfies the KL property at $\tilde{X}$ with a neighborhood $U_\tilde{X}$ and the KL exponent $\theta$, $f(X_k) \rightarrow f(\tilde{X})$ as $k \rightarrow \infty$, $X_k \in U_\tilde{X}$ for each $k \geq k_0$, and A1 is satisfied. Then the following conclusions hold:

(i) If $\theta = 0$, then \( \{f(X_k)\} \) converges in a finite number of steps;

(ii) If $\theta \in (0, \frac{1}{2})$, then there exist $k_1 \in \mathbb{N}$, $\eta \in (0, 1)$, and $C > 0$ such that $f(X_k) - f(\hat{X}) \leq C\eta^{\frac{k-k_1}{C}} + 1$ for each $k \geq k_1$;

(iii) If $\theta \in (\frac{1}{2}, 1)$, then there exist $k_1 \in \mathbb{N}$ and $C > 0$ such that $f(X_k) - f(\hat{X}) \leq C([\frac{k-k_1}{C}] + 1)^{-\frac{1}{\theta-1}}$ for each $k \geq k_1$.

Additionally, for each accumulation point (if any) $\tilde{X} \in U_\tilde{X}$ of $\{X_k\}$, it is a critical point of $f$ if and only if $f(\tilde{X}) = f(X_k)$.

With the similar arguments as in the proof of Theorem 3 of [22], Theorem 2 of [11] for convergence rate estimation and Theorem 1 of [36] for the analysis of accumulation points, we give the following proofs.

**Lemma 4.1.** Under the assumptions in Theorem 4.1, we have $f(X_k) \geq f(\tilde{X})$, $k \geq k_0$. If $\{f(X_k)\}$ be an infinite sequence, there exists $k_1 \in \mathbb{N}$ such that for each $f(X_k) - f(\tilde{X}) > 0$, $k \geq k_1$,

$$
(f(X_k) - f(\tilde{X}))^{2\theta} \leq \frac{2}{c_1} (f(X_{k-j}) - f(X_k)).
$$

**(Proof.** First of all, the lower-boundedness of $\{f(X_k)\}$ is proven. By A1, $f(X_k) \geq f(X_{k+j}) \geq f(X_{k+2j}) \geq \ldots \geq f(X_{k+nj})$ holds for each $k \geq k_0$. Letting $n \rightarrow \infty$, we have $f(X_k) \geq f(\tilde{X}), k \geq k_0$.

If $\{f(X_k)\}$ be an infinite sequence, there exists a $k_1 \in \mathbb{N}$, $k_1 \geq k_0 + j$ such that $f(X_k) - f(\tilde{X}) < 1$, and for each $f(X_k) - f(\tilde{X}) > 0$, $k \geq k_1$,

$$
(f(X_k) - f(\tilde{X}))^{2\theta} \leq \frac{2}{c_1} \text{dist}(\theta, \partial f(X_k))^2 \\
\leq \frac{2}{c_1} (f(X_{k-j}) - f(X_k)),
$$

where the first inequality follows from the KL property of $f$ at $\hat{X}$, and the second inequality follows from A1. \)

**Lemma 4.2 ([11]).** Suppose that $X_k \rightarrow X$, $G_k \rightarrow G$, and $f(X_k) \rightarrow f(X)$ as $k \rightarrow \infty$, of which $G_k \in \partial f(X_k)$ for each $k$. Then $G \in \partial f(X)$.

**(Proof.** (proof of Theorem 4.1 (i)) is proven by contradiction. If the conclusion is not true, there exists a subsequence $\{k_l\} \subseteq \{k_1, k_1 + 1, \ldots\}$ such that

$$1 \leq c\text{dist}(\theta, \partial f(X_{k_l})).$$

Letting $l \rightarrow \infty$, $1 \leq 0$, a contradiction. Therefore, there exists $k_2 \in \mathbb{N}$ such that $f(X_k) \equiv f(\tilde{X})$ for each $k \geq k_2$.

If $\{f(X_k)\}$ be a finite sequence, (ii) and (iii) hold trivially. Then we only need to prove them in the case of infinite convergence.
When $\theta \in (0, \frac{1}{2}]$, according to (7) in Lemma 4.1,
\[
f(X_k) - f(\bar{X}) \leq (f(X_k) - f(\bar{X}))^{20} \leq \frac{c^2}{c_1} (f(X_{k-j}) - f(X_k))
\]
holds for each $k \geq k_1$, which then implies that
\[
f(X_k) - f(\bar{X}) \leq \frac{c^2}{c_1 + c^2} (f(X_{k-j}) - f(\bar{X})), k \geq k_1.
\]
Hence, we have
\[
f(X_k) - f(\bar{X}) \leq C_1 \left( \frac{c^2}{c_1 + c^2} \right)^{-k_1 + 1}
\]
for each $k \geq k_1$, where
\[
C_1 = \max \left\{ f(X_{k-1}) - f(\bar{X}), f(X_{k-1}) - f(\bar{X}), \ldots, f(X_{k-1}) - f(\bar{X}) \right\}.
\]
It follows from A1 and the infinite convergence assumption of $\{f(X_k)\}$ that $C_1 > 0$. Thus (ii) holds with $C = C_1, \eta = \frac{c^2}{c_1 + c^2}$.

When $\theta \in (\frac{1}{2}, 1)$, given a constant $\omega \in [2, \infty)$, for each $f(X_k) - f(\bar{X}) > 0, k \geq k_1$, if $(f(X_k) - f(\bar{X}))^{-20} \leq \omega(f(X_{k-j}) - f(\bar{X}))^{-20}$, then,
\[
\frac{c_1}{c^2} \leq \omega(f(X_{k-j}) - f(\bar{X}))^{-20} \left( f(X_{k-j}) - f(X_k) \right)^{20} \leq \omega \int_{f(X_{k-j})}^{f(X_k)} x^{-20} \, dx \leq \omega \frac{\omega}{2^\theta - 1} \left( (f(X_k) - f(\bar{X}))^{20+1} - (f(X_{k-j}) - f(\bar{X}))^{20+1} \right),
\]
where the first inequality follows from (7) in Lemma 4.1. Hence
\[
0 < \frac{c_1(2^\theta - 1)}{c^2 \omega} \leq (f(X_k) - f(\bar{X}))^{-20+1} - (f(X_{k-j}) - f(\bar{X}))^{-20+1}.
\]  \hspace{1cm} (8)

If $(f(X_k) - f(\bar{X}))^{-20} > \omega(f(X_{k-j}) - f(\bar{X}))^{-20}$, then,
\[
(f(X_k) - f(\bar{X}))^{-20+1} \geq \omega^{\frac{2^\theta}{2^\theta - 1}} (f(X_{k-j}) - f(\bar{X}))^{-20+1}.
\]
Hence
\[
0 < (\omega^{\frac{2^\theta}{2^\theta - 1}} - 1)C_1^{-20+1} \leq (\omega^{\frac{2^\theta}{2^\theta - 1}} - 1)(f(X_{k-j}) - f(\bar{X}))^{-20+1} \leq (f(X_k) - f(\bar{X}))^{-20+1} - (f(X_{k-j}) - f(\bar{X}))^{-20+1}.
\]  \hspace{1cm} (9)

According to (8) and (9),
\[
(f(X_{k-j}) - f(\bar{X}))^{-20+1} + L \leq (f(X_k) - f(\bar{X}))^{-20+1}
\]
holds for each $f(X_k) - f(\hat{X}) > 0$, $k \geq k_1$, where
\[
L = \min \left\{ \frac{c_1(2\theta - 1)}{c^2\omega}, (\omega \frac{2\theta - 1}{\omega} - 1)C^{\theta - 2\theta + 1} \right\} > 0.
\]
Then we have
\[
f(X_k) - f(\hat{X}) \leq \left( C^{\theta - 2\theta + 1} + L \left( \frac{k - k_1}{j} \right) + 1 \right)^{-\frac{1}{\theta - 2\theta + 1}}\]
\[
\leq L^{-\frac{1}{\theta - 2\theta + 1}} \left( \frac{k - k_1}{j} \right)^{-\frac{1}{\theta - 2\theta + 1}} + 1.
\]
(10)
Clearly, for each $f(X_k) - f(\hat{X}) = 0$, $k \geq k_1$, (10) still holds. Thus we obtain (iii) with $C = L^{-\frac{1}{\theta - 2\theta + 1}}$.

For each accumulation point $\hat{X}$, there exists a subsequence $\{X_{k_l}\}$ such that $\lim_{l \to \infty} X_{k_l} = \hat{X}$. By A1, $\lim_{l \to \infty} \text{dist}(0, \partial f(X_{k_l})) = 0$. For each $k_l$, there exists $G_{k_l} \in \partial f(X_{k_l})$ such that
\[
\text{dist}(0, \partial f(X_{k_l})) \leq ||G_{k_l}|| \leq \text{dist}(0, \partial f(X_{k_l})) + \frac{1}{k_l}.
\]
Letting $l \to \infty$, we have
\[
0 = \lim_{l \to \infty} \text{dist}(0, \partial f(X_{k_l})) \leq \lim_{l \to \infty} ||G_{k_l}|| \leq \lim_{l \to \infty} \left( \text{dist}(0, \partial f(X_{k_l})) + \frac{1}{k_l} \right) = 0.
\]
So $\lim_{l \to \infty} ||G_{k_l}|| = 0$. Without loss of generality, suppose that $G_{k_l} \to \tilde{G}$ as $l \to \infty$. Then $||\tilde{G}|| = \lim_{l \to \infty} ||G_{k_l}|| = 0$, $\tilde{G} = 0$. According to
\[
X_{k_l} \to \hat{X}, G_{k_l} \to 0, \text{and } f(X_{k_l}) \to f(\hat{X}), \text{as } l \to \infty,
\]
and Lemma 4.2, we have $0 \in \partial f(\hat{X})$.

Moreover, if $\hat{X}$ is a critical point, it follows from the Remark 2.5 (d) of [3] that $f(\hat{X}) = f(\hat{X})$. □

Parts (ii) and (iii) of the above theorem implies that $\{f(X_k)\}$ converges at least locally R-linearly and locally R-sublinearly to $f(\hat{X})$, respectively. For each example in Sect. 2, A1 and the KL property are satisfied, and the KL exponent $\theta$ of real analytic function is in $[\frac{1}{2}, 1)$ at a critical point [3, 25]. Furthermore, $\{X_k\}$ is convergent in Examples 1, 2, and 4 [40, 41], so the corresponding $\{f(X_k)\}$ has $O(\eta^k)$ local convergence rate for $\theta = \frac{1}{2}$ and $O(k^{-\theta - \gamma})$ local convergence rate for $\theta \in (\frac{1}{2}, 1)$ by our Theorem 4.1. Besides, if the assumption $\hat{X}_k \in \mathcal{U}_\hat{X}$ for each sufficiently large $k$ is satisfied in Example 3, the aforementioned results also hold, and the Theorem 2 in [35] is a special case of our Theorem 4.1. Moreover, the continuity of $f$ is satisfied in each example in Sect. 3 under certain assumptions [36, 40, 41]. Then, each accumulation point of $\{X_k\}$ is a critical point by our Theorem 4.1 (see [35, 40, 41] for the existence of accumulation points and their properties for each example in Sect. 3).

Moreover, if the following stronger non-monotone $j$-step sufficient decrease condition is satisfied, $\{f(X_k)\}$ converges at least locally R-linearly to $f(\hat{X})$ for each $\theta \in [0, 1)$ as shown in the following Theorem 4.2

A2. For a certain $j \in \mathbb{N}_+$, there exist positive constants $\alpha \in [\theta, \infty)$ and $c_2$ such that
\[
c_2 \text{dist}(0, \partial f(X_{k_j}))^{\frac{\alpha}{2}} \leq f(X_k) - f(X_{k_j})
\]
for each $k \geq k_0$, where $\theta$ is the KL exponent of $f$. 
When $\theta \in (\frac{1}{2}, 1)$, compared with A1, a larger descent of $j$ steps iteration is guaranteed in A2, and it is a more dedicated estimation for $f(X_k) - f(X_{k+j})$.

**Theorem 4.2.** For an objective function $f$ and a sequence $\{X_k\}$, suppose that $f$ satisfies the KL property at $\hat{X}$ with $U_{\hat{X}}$ and $\theta$, $f(X_k) \rightarrow f(\hat{X})$, $X_k \in U_{\hat{X}}$ for each $k \geq k_0$, and A2 is satisfied. Then $\{f(X_k)\}$ has a local convergence rate of $O(\eta^k)$, where $\eta \in (0, 1)$. Additionally, for each accumulation point (if any) $\tilde{X} \in U_{\hat{X}}$ of $\{X_k\}$, it is a critical point if and only if $f(\tilde{X}) = f(\hat{X})$.

**Proof.** When $\theta = 0$, with the similar arguments as in the proof of Theorem 4.1, finite convergence is achieved, and $O(\eta^k)$ complexity bound holds trivially. When $\theta \in (0, 1)$, similarly, we only prove it in the case of infinite convergence. By A2 and the KL property of $f$ at $\hat{X}$, there exists a $k_1 \in \mathbb{N}, k_1 \geq k_0 + j$ such that for each $k \geq k_1$, $\text{dist}(0, \partial f(X_k)) < 1$, and for each $f(X_k) - f(\hat{X}) > 0$,

$$f(X_k) - f(\hat{X}) \leq c^\frac{1}{2} \text{dist}(0, \partial f(X_k))^{\frac{1}{2}} \leq c^\frac{1}{2} \left( f(X_{k-j}) - f(X_k) \right).$$

Then,

$$f(X_k) - f(\hat{X}) \leq \frac{c^\frac{1}{2}}{c^2 + c^\frac{1}{2}} (f(X_{k-j}) - f(\hat{X})).$$

Hence, we have

$$f(X_k) - f(\hat{X}) \leq C_1 \left( \frac{c^\frac{1}{2}}{c^2 + c^\frac{1}{2}} \right)^{\lfloor \frac{k-k_1}{j} \rfloor + 1}$$

for each $k \geq k_1$, where $C_1$ is the same as that in the proof of Theorem 4.1. So a $O(\eta^k)$ local convergence rate is achieved with $\eta = (c^\frac{1}{2}/(c^2 + c^\frac{1}{2}))^{\frac{1}{j}}$. With the similar arguments as in the proof of Theorem 4.1, we obtain the rest of Theorem 4.2.

It is worth noting that although the local R-linear convergence can be achieved in any cases under A2, when $\theta \in (\frac{1}{2}, 1)$, verification whether a training model and algorithm satisfies the stronger decrease condition is a challenging problem [23, 27, 39].

## 5 Conclusions

In this paper, a unified framework is proposed to analyze the convergence rate of the objective function value sequences generated by the AM-type training algorithms. The non-monotone $j$-step sufficient decrease conditions and the KL property play central roles in our analysis. And the requirement of non-increasing property of function value sequence is relaxed in our framework. Based on the squared norm lower bound estimation of the $j$-step descent, three kinds of convergence rates are discussed for different values of the KL exponent $\theta$, respectively. Moreover, if a larger descent is guaranteed, we can improve the convergence rate to $O(\eta^k)$ for $\theta \in (\frac{1}{2}, 1)$.

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