Power calculation for gravitational radiation: oversimplification and the importance of time scale

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A simplified formula for gravitational-radiation power is examined. It is shown to give completely erroneous answers in three situations, making it useless even for rough estimates. It is emphasised that short timescales, as well as fast speeds, make classical approximations to relativistic calculations untenable.

1 Introduction

Gravitational radiation, the vibration of the space-time metric produced by masses in motion, forms one of the accepted predictions of General Relativity. Though not yet directly detected, current projects are producing astrophysically interesting upper bounds and planned instruments may well make the first real observation.\(^1\)

The calculation of gravitational radiation in all but the simplest cases can be difficult and tedious, however. Baker (2006) presented a simplified formula for the gravitational power radiated by an object, or collection of objects, ‘in order to render astrophysical applications more apparent.’ This paper examines that formula to determine its limits of application.

2 Baker’s derivation

The starting point for Baker’s (2006) derivation is the formula for total averaged power radiated by a body, under various simplifying assumptions (the gravitational waves are of small amplitude, the stress-energy tensor can be reduced to the mass density). From Landau & Lifshitz (1975), p. 355, eq. 110.16, this is

\[
- \frac{dE}{dt} = P = \frac{G}{45c^5} |\dddot{D}_{\alpha\beta}|^2
\]

(I have used \(G\) for the gravitational constant instead of Landau & Lifshitz’ \(k\), where the moment of inertia tensor is given by

\[
D_{\alpha\beta} = \int \rho \left(3x_{\alpha}x_{\beta} - \delta_{\alpha\beta}r^2\right) dV,
\]

the integral to be taken over the volume of the body in question, and transformed into a sum when considering a collection of point masses. (This is their equation 110.10, p. 355; I have substituted \(\rho\) for the mass density instead of Landau & Lifshitz’ \(\mu\), since the latter symbol appears with a different meaning below.)

Citing considerations of ‘symmetry’ (which he does not specify), Baker equates \(D_{\alpha\beta}\) with a scalar moment of inertia \(I\), taken to be a mass \(\delta m\) times the square of the radius of gyration \(r\). In taking the third time derivative he obtains

\[
\frac{d^3I}{dt^3} = 2r\delta m \left(\frac{d^3r}{dt^3}\right) + \ldots
\]

(Baker does not say what the missing terms on the right are, nor why he chose to ignore them). He then identifies \(\delta m\) times the third time derivative of \(r\) with the time derivative of force; and finally averages the change in force over some time interval, giving as his formula for power

\[
P = 1.76 \times 10^{-52} \left(2r \frac{\Delta f}{\Delta t}\right)^2
\]

the numerical coefficient chosen to give watts.\(^2\) He then applies his formula to two-body motion and rigid-rod rotation, finding numerical agreement, and considers his formula to have general application.

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\(\text{\(1\)}\) The literature on gravitational radiation is vast and even a brief survey is beyond the scope of this paper. Basic derivations are found in most General Relativity texts, and specific references to Landau & Lifshitz (1975) will appear below. Dietz (2010), to take a recent example, shows the useful application of non-observations to astronomical objects.

\(\text{\(2\)}\) I have not been able to duplicate Baker’s numbers, which he gives in two incompatible equations. In all numerical calculations I will use \(6.1 \times 10^{-55}\) kg\(^{-1}\) m\(^{-2}\) as the coefficient to the moment-of-inertia term in Eq. (1); the difference, large as it is, does not affect my conclusions.
3 Two testing situations

A thorough investigation of Eq. (4) would set out the conditions under which the simplifications and assumptions of its derivation hold good. For present purposes it is sufficient to look at three examples.

First, we follow on from Problem 1, page 356 of Landau & Lifshitz (1975). Two masses, \( m_1 \) and \( m_2 \) are in a circular orbit a distance \( r \) apart. They move with angular frequency \( \omega \) and at time zero lie along the \( x \)-axis, rotating in the \( x, y \) plane. The components of the moment of inertia tensor are

\[
\mathbf{D}_{\alpha\beta} = \mu r^2 \left( 3 \cos^2(\omega t) - 1 \right)
\]

(5)

\[
\mathbf{D}_{xx} = \frac{3}{2} \mu r^2 \cos(2\omega t) + \frac{1}{2} \mu r^2
\]

\[
\mathbf{D}_{yy} = \mu r^2 \left( 3 \sin^2(\omega t) - 1 \right)
\]

\[
\mathbf{D}_{xy} = \mu r^2 \left( 3 \cos(\omega t) \sin(\omega t) \right)
\]

\[
\mathbf{D}_{zz} = -\mu r^2
\]

where \( \mu = m_1 m_2 / (m_1 + m_2) \) is the reduced mass. Next, we add an identical pair of masses, the same distance apart rotating in the same orbit with the same speed, but place them one-quarter of the way around the orbit with respect to the first pair. They add the following terms to the moment of inertia tensor:

\[
\mathbf{D}'_{xx} = \frac{3}{2} \mu r^2 \cos(2\omega t + \pi/2) + \frac{1}{2} \mu r^2
\]

(6)

\[
\mathbf{D}'_{yy} = \mu r^2 \left( 3 \sin^2(\omega t + \pi/2) - 1 \right)
\]

\[
\mathbf{D}'_{xy} = \mu r^2 \left( 3 \cos(\omega t + \pi/2) \sin(\omega t + \pi/2) \right)
\]

\[
\mathbf{D}'_{zz} = -\mu r^2
\]

It is easy to see that \( \mathbf{D}'_{xx} = -\mathbf{D}_{xx}, \mathbf{D}'_{yy} = -\mathbf{D}_{yy}, \mathbf{D}'_{xy} = -\mathbf{D}_{xy} \) and \( \mathbf{D}'_{zz} = \mathbf{D}_{zz} = 0 \). That is, the time derivative (to all orders) of the magnitude of the moment of inertia tensor is identically zero; there is no gravitational radiation. Barker’s formula, Eq. (4), however, predicts double the radiation of the two-body situation.

Next, consider two bodies of mass \( m_1 \) and \( m_2 \) constrained to move along a straight line, which we will identify with the \( z \)-axis. We fix the centre of our coordinate system at their centre of mass. Their distance apart is \( r \). The components of the moment of inertia tensor are

\[
\mathbf{D}_{xx} = -\mu r^2
\]

\[
\mathbf{D}_{yy} = -\mu r^2
\]

\[
\mathbf{D}_{zz} = 2\mu r^2
\]

(7)

with \( \mu \) the reduced mass, as before. (By allowing one of the masses to be much larger than the other we can use these formulae to analyse the motion of a single body.) From these we calculate

\[
|\mathbf{D}_{\alpha\beta}|^2 = 24\mu^2 \left( r^2 \ddot{r}^2 + 2\dot{r} \dot{\ddot{r}} \right)^2
\]

(8)

or, to compare with Barker’s formula,

\[
|\mathbf{D}_{\alpha\beta}|^2 = 24 \left( r \ddot{f} + 2 \dot{f} \dot{f} \right)^2.
\]

(9)

If the force (a sort of reduced force, acting on the reduced mass) is constant, Barker’s formula gives no gravitational radiation; but Eq. (9) shows that there is still some given off.

Barker’s formula has thus been shown to be completely in error both ways, in predicting gravitational radiation when there is none, and predicting none when there is some. Thus it cannot be used even as a rough guide. The examples given are only slightly different from those in Baker (2006) and would be a reasonable first approximation to some astronomical objects (orbits of more than two objects are fairly common, as are linear jets), so his assertion in that paper of the usefulness of his formula in astrophysics is unfounded.

Such a conclusion would not appear to have a big impact on the gravitational-wave community, since Barker’s formula has not been used in astrophysical circles (where more accurate techniques are customary). It has been employed in another context, however, and that forms our third example.

4 Laboratory gravitational waves?

Baker, Li & Li (2006) describe an apparatus intended to generate and detect high-frequency gravitational waves in a laboratory. Only the generation side concerns us here.

Two targets, 20m apart, are hit with high-intensity laser pulses, directed such that they are momentarily accelerated in opposite directions; the authors consider them to emulate, for the duration of the pulse, a two-body orbiting system. The 23TW pulses last for 33.9fs and are repeated ten times per second. Using Eq. (9) the authors calculate that they generate \( 5.5 \times 10^{-15} \) watts of gravitational radiation.

There is one major problem with this analysis. Baker et al. apply their formula for the duration of one laser pulse, \( 3.39 \times 10^{-14} \). Light travels only 10.6\( \mu \)m in this time (something they mention, but without any apparent consideration of its implications). To analyse bodies not in causal contact as if they were part of a Newtonian object, moving in Newtonian ways, is a very questionable procedure. Indeed, it is not clear that the essentially classical definition of a moment of inertia tensor and its time derivatives can be made relativistically meaningful, nor that it would retain its role as

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3 The authors tacitly assume that the pulse changes intensity linearly over its duration, inserting the calculated radiation-pressure force for 23TW and the duration of 33.9 fs in Barker’s formula. It might be more accurate to take the pulse as a square wave, staying at something like its peak intensity for the duration. In principle, the difference is important since the factors enter as squares, and the average of a square is not the square of the average. When I calculate some numbers I will look at both possibilities.
a source of gravitational radiation if it were. And it is
certainly not justified to apply Eq. (1), which is averaged over
a complete period of the gravitational waves, to a tiny frac-
tion of an orbit. By looking at such a short pulse Baker et
al. (2006) have removed themselves from the assumptions
underlying the starting point of Baker’s (2006) derivation,
Eq. (1) and so have no grounds for believing their resulting
numbers.

As far as one can apply Eq. (1) to the apparatus of Baker
et al. one must be restricted to a region in causal contact,
a single laser target. Indeed, not all of that: ordinary matter
is not rigid on this time scale. The impact of the laser pulse
on the rear of a target cannot be known at the front until a
sound wave can traverse the intervening distance. If this pe-
riod is much shorter than the pulse, the whole object cannot
accelerate (as Baker et al. tacitly assume); instead a sound
wave is set ringing through the target.

To clarify the picture it is useful to have some numbers.
Since Baker et al. (2006) give no details about their laser
targets, I will use some nominal values (exact figures are
not important here, as will be clear). For a nominal sound
speed in steel of $5790 \text{ m s}^{-1}$, the pulse of $3.39 \times 10^{-14} \text{ s}$ has
penetrated a distance of $1.96 \times 10^{-10} \text{ m}$ by the time it ends:
a layer of atomic dimensions. Assuming a laser spot size
similar to their quoted detector laser, $1.96 \times 10^{-10} \text{ m}^2$ (not
all of which is in causal contact sideways!), and a density of
steel of $7.9 \times 10^3 \text{ kg m}^{-3}$, we have an accelerated mass of
something like $3.0 \times 10^{-11} \text{ kg}$.

We now turn to Eq. (9) to calculate the gravitational ra-
diation of this accelerated mass. First using the Baker et al.
assumption of a constant change in force,

$$\left| \mathcal{D}_{\alpha\beta} \right|^2 = \frac{98 f^{4/6}}{3 \mu^2}$$

(10)

which gives a radiated power of $2.5 \times 10^{-39} \text{ W}$. If, on the
other hand, we assume a constant force,

$$\left| \mathcal{D}_{\alpha\beta} \right|^2 = 96 f^{4/2} \mu^2$$

(11)

resulting in $7.3 \times 10^{-39} \text{ W}$. The difference between these
numbers and the calculation in Baker et al. (2006) amounts
to twenty-seven orders of magnitude. This difference may
in principle arise either from the simplifications of Baker
(2006) or from the possibility that Eq. (1) is simply inappro-
priate for very short periods of time; in either case, Eq. (4)
must be discarded.

5 Conclusions

A simplified formula for a relativistic effect has been shown
to be completely unreliable, giving infinitely wrong answers
in two instances, and an error of something like twenty-
seven orders of magnitude in a third. The latter number is
not firm, since there is some question as to whether its own
basis is justified; but it is certain that the formula of Baker
(2006) cannot be used. Gravitational waves are not to be
generated in the laboratory in the foreseeable future.

The lesson of this episode is that simplified formulae
must be justified and carefully handled, since it is quite
possible to push them beyond their applicability. In addi-
tion, Newtonian approximations of relativistic effects must
be carefully examined for tacit assumptions that make their
results untenable. In particular, short time scales (as well as
speeds comparable to light) make Newtonian expressions
unreliable.

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4 A true analysis of this interaction of radiation with matter should, of
course, be done in a quantum context.