Acoustic radiation torque on a particle in a fluid: an angular spectrum based compact expression.

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In this work, we derive a set of compact analytical formulas expressing the three-dimensional acoustic radiation torque (ART) exerted on a particle of arbitrary shape embedded in a fluid and insonified by an arbitrary acoustic field. These compact formula are obtained by expanding the acoustic field as the superposition of plane waves following the angular spectrum method introduced by Sapozhnikov & Bailey [J. Acoust. Soc. Am. 133, 661676 (2013)] for the calculation of the force. This formulation is particularly well suited to determine the ART exerted on a particle when the acoustic field is known in a source plane.

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I. INTRODUCTION

The acoustic radiation torque (ART) exerted by an arbitrary acoustic field on a particle can in general be decomposed into three contributions\footnote{1}: one resulting from the incident wave scattering by the particle, one induced by absorption of the acoustic field by the particle and one resulting from the wave absorption in the viscous boundary layer surrounding the particle\footnote{2,3}. In addition, the particle can also be set in rotation by the so-called Eckart streaming\footnote{4} - a flow resulting from the thermo-viscous absorption of the wave in the bulk of the fluid - in particular when the incident beam is carrying angular momentum\footnote{1,5,6}. The ART can be calculated by transferring the integration of time-averaged stress tensor of angular momentum flux over the particle surface to a far-field spherical surface centered in the mass center of the particle, as first demonstrated by Maidanik\footnote{7}. Based on this idea, Zhang & Marston\footnote{8} derived a compact formula of the axial ART ($T_z$) acting on an axisymmetric object centered on the axis of a cylindrical acoustical vortex beam. They showed that, in this configuration, the scattering contribution vanishes and the ART is proportional to the absorbed energy ($P_{abs}$) with a factor of $M/\omega$ with $M$ the topological charge and $\omega$ the angular frequency, \textit{i.e.}, $T_z = P_{abs}M/\omega$. The theory applies for an elastic sphere in an inviscid fluid, and is also applicable for a sphere embedded in a weakly viscous fluid by modifying the scattering coefficients of the sphere\footnote{1,8}. However, this theory is limited\footnote{9} and cannot address the following situations: (i) non-axisymmetric beams acting on a sphere (e.g. offset incidence of vortex beam on a sphere or oblique incidence on a spheroid and cylinder), (ii) non-axisymmetric objects with respect to the incident direction (e.g., broadside
incidence on a spheroid), and (iii) multiple particles. In 2012, Silva et al.\textsuperscript{10} derived some three-dimensional ART formulas in terms of incident and scattered beam-shape coefficients, which uses a spherical wave expansion to describe the velocity potentials. In this approach, the incident beam shape coefficients are one of the keys factors to calculate the ART. These coefficients are only available analytically for some ideal acoustic beams, e.g., cylindrical Bessel beam of off-axis incidence\textsuperscript{11}. While an arbitrary field can be decomposed into a sum of spherical harmonics, the incident beam shape coefficients can be difficult to determine in practice. Another approach, introduced by Sapozhnikov & Bailey\textsuperscript{12} to calculate the acoustic radiation force, relies on the plane wave decomposition of the acoustic field (angular spectrum method). With this approach, only the knowledge of the field in one plane is required. Hence it can be more easy to set in practice, especially when using planar transducers able to produce complex fields such as acoustical vortices\textsuperscript{13–16}. Here we adopt this approach to derive analytical formula of the ART exerted by an arbitrary acoustic field on a particle of arbitrary shape and size.

II. ANGULAR SPECTRUM BASED ART FORMULAS

In this section, we give a brief overview of the main steps leading to the derivation of the angular spectrum based ART formulas (with the same notations as in Ref.\textsuperscript{9}). The acoustic radiation torque exerted by an acoustic field on a particle can be calculated by transferring the integration of the time-averaged ($\langle \cdot \rangle$) stress tensor of the angular momentum flux over the particle surface to a far-field spherical surface $S_0$\textsuperscript{7,9,10}. Based on the divergence theorem,
the integral expression of the ART is:

\[ T = -\rho_0 \int \int_{S_0} (L) \mathbf{r} \times d\mathbf{S} - \rho_0 \int \int_{S_0} \langle (\mathbf{r} \times \mathbf{u}) \rangle d\mathbf{S}, \quad (1) \]

where \( \langle L \rangle = \langle 1/2 \mathbf{u} \cdot \mathbf{u} - p^2/(2\rho_0 c_0^2) \rangle \) is the time-averaged acoustic Lagrangian, \( \mathbf{r} \) is the field point, \( p \) is the total acoustic pressure field (incident + scattered), \( \mathbf{u} \) is the total acoustic velocity vector, \( \rho_0 \) is the fluid density at rest, \( c_0 \) is the fluid sound speed, and \( d\mathbf{S} = \mathbf{n} \cdot r^2 \sin \theta d\theta d\varphi \) is the differential surface in the far field with \( \mathbf{n} \) the outward unit normal vector. If \( S_0 \) is a sphere whose center coincide with the referential center, then \( \mathbf{r} \times \mathbf{n} = r \mathbf{n} \times \mathbf{n} = 0 \) and the first term in Eq. (1) vanishes. Then, the incident (index ”i”) and scattered (index ”s”) acoustic velocity \( \mathbf{u} \) and pressure \( p \) fields can be described in terms of acoustical potentials \( \Phi \) as:

\[ \mathbf{u}_{i,s} = \nabla \Phi_{i,s} \text{ and } p_{i,s} = i\omega \rho_0 \Phi_{i,s}, \quad (2) \]

with \( i \) is the imaginary unit and \( \omega \) the angular frequency. Hence, the ART expression in Eq. (1) can be written in terms of the incident \( (\Phi_i) \) and scattered \( (\Phi_s) \) velocity potentials as

\[ T = \frac{\rho_0}{2} \text{Im} \left\{ \int \int_{S_0} \left( \frac{\partial \Phi^*_s}{\partial r} \mathbf{L} \Phi_s + \frac{\partial \Phi^*_i}{\partial r} \mathbf{L} \Phi_i + \frac{\partial \Phi^*_s}{\partial r} \mathbf{L} \Phi_s \right) d\mathbf{S} \right\}. \quad (3) \]

where “Im” designates the imaginary part, the star superscript the complex conjugate and \( \mathbf{L} = -i(\mathbf{r} \times \nabla) \) is the angular momentum operator, with its components in the three directions and their recursion relations with normalized spherical harmonics given in detail in Appendix.

Now, assuming that the incident pressure field is known in a plane defined as \( z = 0, \)
\( p_{i \mid z=0} = p_i(x, y, 0) \), the angular spectrum \( S(k_x, k_y) \) of the acoustic field (that is nothing but the 2D spatial Fourier transform of the complex temporal harmonic amplitude of the field
in this plane) reads:

\[
S(k_x, k_y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} p_i(x, y, 0) e^{-i k_x x - i k_y y} dxdy, \tag{4}
\]

with \(x\) and \(y\) the cartesian coordinates in the plane, \(k_x\) and \(k_y\) the wavenumber components in \(x\) and \(y\) directions. Then, the field at any point can be calculated by propagating each plane wave composing the source plane up to the target point \((x, y, z)\):

\[
p_i(x, y, z) = \frac{1}{4\pi^2} \times \int \int_{k_x^2 + k_y^2 \leq k^2} S(k_x, k_y) e^{i k_x x + i k_y y + i \sqrt{k^2 - k_x^2 - k_y^2} z} dk_x dk_y. \tag{5}
\]

where \(k = \omega/c_0\) is the wave number in fluid. In this way, the acoustic field is decomposed into an infinite sum of plane waves and the angular spectrum \(S(k_x, k_y)\) characterizes the relative magnitude of each plane wave. The next step is to solve the scattering problem. For this purpose, this plane wave decomposition must be turned into a spherical wave decomposition, more suitable to solve the scattering problem (see Sapozhnikov & Bailey\(^{12}\)):

\[
p_i = \frac{1}{\pi} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} i^n H_{nm} J_n(kr) Y_{nm}(\theta, \varphi), \tag{6}
\]

where \(Y_{nm}(\theta, \varphi)\) are the spherical harmonics and the \(H_{nm}\) represent the respective weight of each spherical wave:

\[
H_{nm} = \int \int_{k_x^2 + k_y^2 \leq k^2} S(k_x, k_y) Y_{nm}^*(\theta_k, \varphi_k) dk_x dk_y, \tag{7}
\]

with \(\cos \theta_k = [1 - (k_x^2 + k_y^2)/k^2]^{1/2}\) and \(\varphi_k = \arctan(k_y/k_x)\). This expression results from the known decomposition of a plane wave into a sum of spherical waves. Analytical solutions of the scattering problem for spheres embedded in a fluid are known in many cases including
rigid\textsuperscript{17}, elastic\textsuperscript{18} or visco-elastic particles\textsuperscript{1,9}. For non-spherical particles, the scattering problem can be handled with the so-called T-matrix method\textsuperscript{19–24}. Assuming prior knowledge of the scattering coefficients, the scattered field can be written under the form:

\[ p_s = \frac{1}{\pi} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} i^n H_{nm} A_{nm} h_n^{(1)}(kr) Y_{nm}(\theta, \varphi), \]  

(8)

with \( A_{nm} \) the partial wave coefficients which only depend on the index \( n \) for a spherical shape, having \( A_n = (s_n - 1)/2 \) with \( s_n \) the scattering coefficients, and depend on both \( n \) and \( m \) for non-spherical shapes\textsuperscript{9}.

Now the incident and scattered pressure fields are given in terms of the coefficients \( H_{nm} \) based on the angular spectrum method\textsuperscript{12}. The incident (\( \Phi_i \)) and scattered (\( \Phi_s \)) velocity potentials can be easily obtained by using the second equation of Eq. (2), which can then be substituted into Eq. (3). Since the integral is performed on the far field surface \( S_0 \), the asymptotic expressions of Bessel functions can be used [Eqs. (A.1) in appendix], which combined to the recursion relation of Bessel functions [Eqs. (A.2) in Appendix] leads to the following expression of the ART expression in terms of \( H_{nm} \) and \( Y_{nm} \)

\[
T = -\frac{1}{2\pi^2 \rho_0 k^3 c_0} \text{Re} \left\{ \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{n'=0}^{\infty} \sum_{m'=-n'}^{n'} (1 + A^*_n A_{n'm'} H^*_{nm} H_{n'm'} \int_{S_0} \left( Y_{nm}^* Y_{n'm'} \right) \sin \theta d\theta d\varphi \right\}, \]

(9)

The final compact expression of the three-dimensional ART in terms of the \( H_{nm} \) coefficients can be derived by using the recursion and orthogonality relations of the normalized spherical harmonics [see Eqs. (A.3) and (A.4), respectively, in Appendix]:
\[ T_x = -\frac{1}{4\pi^2 \rho_0 k^3 c_0^2} \Re \left\{ \sum_{n=0}^{\infty} \sum_{m=-n+1}^{n} b_{nm} C_{nm} H_{nm}^* H_{n,-1} \right\}, \]  
\text{(10)}

\[ T_y = -\frac{1}{4\pi^2 \rho_0 k^3 c_0^2} \Im \left\{ \sum_{n=0}^{\infty} \sum_{m=-n+1}^{n} b_{nm} C_{nm} H_{nm}^* H_{n,-1} \right\}, \]  
\text{(11)}

\[ T_z = -\frac{1}{2\pi^2 \rho_0 k^3 c_0^2} \Re \left\{ \sum_{n=0}^{\infty} \sum_{m=-n}^{n} m D_{nm} H_{nm}^* H_{nm} \right\}, \]  
\text{(12)}

with \( b_{nm} = \sqrt{(n-m+1)(n+m)} \), \( C_{nm} = A_{nm} + A_{nm}^* + 2A_{nm} A_{nm}^* \), \( D_{nm} = A_{nm} + A_{nm} A_{nm}^* \).

III. BRIEF VALIDATION OF THE ART FORMULAS

To validate the ART expressions obtained with the angular spectrum method (ASM) in the previous section, the torque exerted on an off-axis viscoelastic sphere insonified by a cylindrical Bessel vortex is calculated with Eqs. (10–12) and compared with the results obtained with the multipole expansion method (MEM)\(^\text{11}\) by Gong et al.\(^\text{9}\). In this configuration, analytical expression of the \( H_{nm} \) coefficients is given by:

\[ H_{nm} = 4\pi^2 \omega \rho_0 \Phi_0 \xi_{nm} \]
\[ \times i^{M-m+1} P_n^m (\cos \beta) J_{m-M} (\sigma_0) e^{-ikzz_0} e^{-i(m-M)\varphi_0} \]  
\text{(13)}

where \( \xi_{nm} = [(2n+1)(n-m)!/[4\pi(n+m)!]^{-1/2} \), \( M \) is the topological charge of the Bessel beam, \( \beta \) is the cone angle, \( \sigma_0 = k_r R_0 \) with \( k_r = k \sin \beta \) and \( R_0 = \sqrt{x_0^2 + y_0^2} \), \( x_0, y_0 \) and \( z_0 \) are the offset along the \( x, y \) and \( z \) directions, respectively. The axial component of the wave number is \( k_z = k \cos \beta \), and the original azimuthal angle is \( \varphi_0 = \tan^{-1}(y_0/x_0) \). In the simulations represented on Fig. 1, the topological charge of the cylindrical Bessel beam is \( M = 1 \) with \( \beta = 60^\circ \), the incident frequency is 1 MHz with pressure amplitude 1 MPa, and the particle radius is \( a = 180 \mu m \). The particle is moved off the beam axis [see the schematic
FIG. 1. The 3 projections of the acoustic radiation torque ($T_x$, $T_y$, $T_z$) exerted on an off-axis viscoelastic sphere by a cylindrical Bessel vortex on a viscoelastic PE solid sphere is calculated by the present Angular Spectrum Method (ASM) (equations (10-12)) and compared to results obtained with the Multipole Expansion Method (MEM) (Gong et al.\textsuperscript{9}). (a) Scheme of the simulated configuration. The topological order of the cylindrical Bessel beam is $M = 1$ and the cone angle $\beta = 60^\circ$ (see ref\textsuperscript{25} for more details about acoustical vortices). The particle is moved away from the beam center along $x$ direction of a distance: $x_0 \in [0, 5\lambda]$, with $\lambda$ the wavelength in the fluid. There is no offset along the $y$ direction ($y_0 = 0$) and since cylindrical Bessel are invariant along $z$, the position along this axis does not matter. (b) Figure comparing the values of the 3 projections of the acoustic radiation torque obtained with ASM and MEM, as a function of the particle dimensionless offset $x_0/\lambda$.

in Fig. (1a)] along only $x$ direction with $x_0 \in [0, 5\lambda]$, $y_0 = 0$ and $z_0 = 0$ where $\lambda = 1.5$ mm is the wavelength in water. As observed in Fig. (1b), the calculated three-dimensional ART with the ASM [see eqs. (10-12)] agree well with those by the MEM\textsuperscript{9,26}. Note that since the particle is moved off axis along the $x$ direction, the lateral ART $T_x$ always vanishes
because of the symmetry. For the normal incidence ($x_0 = 0$ and $y_0 = 0$), only the axial ART due to acoustic absorption exists$^{1,8}$. Further confirmation of these formula is under way by comparing directly the analytical expressions of the two formulas.$^{26}$

IV. CONCLUSIONS AND DISCUSSIONS

In this letter, some compact angular spectrum based three-dimensional ART formulas are derived for a single particle immersed in an ideal fluid with no limitation to the particle size, particle shape and beam shape structure. The arbitrary acoustic field is taken as the superposition of plane waves, hence can be expanded based on the angular spectrum method$^{12}$, which is quite practical for finite-aperture real sources. The ART on non-spherical shapes (e.g., spheroid and finite cylinder) can be calculated once the partial wave coefficients $A_{nm}$ are obtained with proper methods, for example, the T-matrix method$^{9,19–24}$. The present theory are still practicable for an absorbing sphere in a viscous fluid if the absorption processes in the particle and viscous layer is accounted in the expression of scattering coefficients$^{1,8}$. In addition, the formulas can be used for multiple objects$^{27,28}$ which are all located inside the chosen far-field spherical shape so that the divergence theorem still holds for the derivation. The ART of particle in experimental sources can be evaluated by the measured acoustic field in the transverse plane as similar as the simulation of acoustic radiation force$^{29}$.

By Combining with the three-dimensional acoustic radiation forces$^{12}$, we can predict the dynamic motions of particles in real acoustic field with six degrees of freedom, i.e., three for translocations and three for spinning motions. It is noteworthy that for a particle located off the axis of a vortex beam, both three-dimensional radiation forces and torques are applied
on the particle so that the particle could rotate around the beam axis (by the azimuthal component of radiation force) and its own center of mass (by the radiation torque). Since particle in a vortex beam could be ejected out of the trap, this work can be used for theoretical guidance on parameters selection of acoustic sources for experimental designs, which can slow down the spinning motions by decreasing the ART, and meanwhile, keep the trapping by the acoustic radiation force.

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APPENDIX:

The far-field asymptotic expressions of the spherical Bessel function and Hankel function of the first kind are, respectively:

\[ j_n(kr) \simeq i^{-(n+1)}e^{ikr}/2kr + i^{n+1}e^{-ikr}/2kr, \]

\[ h_n^{(1)}(kr) \simeq i^{-(n+1)}e^{ikr}/kr. \]

and the recursion relation of the spherical Bessel function is

\[ j'_n(kr) = (n/kr)j_n(kr) - j_{n+1}(kr). \]

where the symbol ' means the derivative with respect to kr. The ladder operators \( L_{\pm}\) has the relationship with the lateral components of the angular momentum operator \( L_{x,y}: \)
$L_{\pm} = L_x \pm iL_y$. The recursion relations of ladder operators $L_{\pm}$ (or axial component of angular momentum operator $L_z$) and normalized spherical harmonics are

\begin{align}
L_+ Y_{nm} &= \sqrt{(n-m)(n+m+1)} Y_{n,m+1}, \\
L_- Y_{nm} &= \sqrt{(n+m)(n-m+1)} Y_{n,m-1}, \\
L_z Y_{nm} &= mY_{nm}.
\end{align}

Finally, the orthogonality relation of the normalized spherical harmonics is

\[ \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta Y^*_{nm}(\theta, \phi) Y_{n'm'}(\theta, \phi) = \delta_{nn'}\delta_{mm'}. \]  

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