We present an analysis of wave-mixing in recently developed Josephson Travelling Wave Parametric Amplifiers (JTWPAs). Circuit simulations performed using WRspice show the full behaviour of the JTWPA allowing propagation of all tones. The Coupled Mode Equations (CMEs) containing only pump, signal, and idler propagation are shown to be insufficient to completely capture complex mixing behaviour in the JTWPA. Extension of the CMEs through additional state vectors in the analytic solutions allows closer agreement with WRspice. We consider an ordered framework for the systematic inclusion of extended eigenmodes and make a qualitative comparison with WRspice at each step. The agreement between the two methods validates both approaches and provides insight into the operation of the JTWPA. We show that care should be taken when using the CMEs and propose that WRspice should be used as a design tool for non-linear superconducting circuits such as the JTWPA.

I. INTRODUCTION

A. Josephson Travelling Wave Parametric Amplifiers

Josephson junction (JJ) based parametric amplifiers (JPAs) [1–3] have been used in recent years to provide quantum-limited noise performance for quantum optics experiments [4], single microwave photon detection [5], high-fidelity qubit readout for quantum information technologies [6, 7], as well as producing squeezed states [8]. These microwave, small signal, amplifiers have been shown to exhibit large gain (> 20 dB) [9, 10], and approach the quantum noise limit [11]. Typically these amplifiers have utilised high-Q superconducting resonators which have a limited bandwidth and dynamic range. Removing the resonant architecture and allowing non-linear interactions along a transmission line can increase both the dynamic range and bandwidth [12]. More recently, the Josephson Travelling Wave Parametric Amplifier (JTWPA), based on JJs embedded in a microwave transmission line has been shown to provide large gain without the bandwidth limitation of the JPAs [13–15].

Implementing unbiased JJs along the transmission line leads to a centrosymmetry of the system and four wave mixing (4WM) whereby the signal and idler frequencies are close to the frequency of the pump, \(f_s + f_i = 2f_p\). In this paper we focus primarily on the three wave mixing (3WM) scheme, \(f_s + f_i = f_p\) which shifts the pump frequency away from that of the signal and idler allowing the pump to be filtered more easily from the signal. The 3WM regime also takes advantage of the inherently stronger interactions than the 4WM regime. In this regime the phase modulation effect and the signal gain are controlled independently, the process of which is described in detail by Zorin [16]. To access the 3WM regime rf-SQUIDs are embedded in the transmission line and an externally applied magnetic field modifies the centrosymmetry of the circuit. The circuit as proposed in Ref. [16] is shown in Fig. 1.

II. MODELLING THE JTWPA

A. WRspice Simulations

In order to capture the full behaviour of the JTWPA, we use WRspice to simulate the circuit design shown in Fig. 1. WRspice is a SPICE-like circuit simulator which includes a Josephson junction model [17]. Conventional analytical models describing three-wave mixers consider only three mixing tones, the pump \(f_p\), signal \(f_s\), and idler \(f_i\) [18]. Using WRspice we observe that other mixing tones, especially the harmonics of the pump, are gener-
ated in the JTWPA. In this work we show that generation of other mixing tones has a strong reduction on the signal gain that can be achieved. In WRspice we implement a 2000 cell version of the circuit shown in Fig. 1. The rf-SQUIDs are flux-biased such that we operate in the 3WM regime. A strong (\(I_{p}^{\text{rms}}(0) \approx 1.97 \mu \text{A} \approx -70 \text{dBm}\)), pump current at \(f_p = 12 \text{ GHz}\) and a weak (\(I_{s}^{\text{rms}}(0) \approx 0.07 \mu \text{A} \approx -96 \text{dBm}\)) signal current at \(f_s = 7.2 \text{ GHz}\) are input to the JTWPA at node 0. The values replicate those used as example parameters in the analytical model by Zorin [16]. By performing an FFT of the current entering each node \(n\) we observe the behaviour of all tones propagating along the amplifier. We observe wave mixing processes including generation of the idler tone (\(f_i = 4.8 \text{ GHz}\)) at the difference of the pump and signal tones. This wave mixing derives solely from the non-linear current phase relation of the Josephson junction \(I = I_c \sin(\phi)\) and demonstrates the ability of WRspice to model the non-linear behaviour of the system. Fig. 2 shows a colourmap of the current at each node of the JTWPA as simulated by WRspice. Note that as well as the signal, pump, and idler, we observe significant generation of pump harmonics \(f_{2p}, f_{3p}, f_{4p},\) and \(f_{5p}\). In addition to the pump harmonics, we observe sum-frequency generation associated with the pump and the pump harmonics.

The signal tone amplifies along the JTWPA from an input amplitude of \(I_{s}^{\text{rms}}(0) \approx 0.10 \mu \text{A} \approx 0.19 \mu \text{A}\) representing a signal gain of 5.5 dB. This gain is less than a third of that predicted in Ref. [16] for the same pump and signal input amplitudes and JTWPA length. We show here that the generation of the additional terms seen in the WRspice simulations accounts for most of the reduction in amplifier gain observed in WRspice when compared to the gain expected from the analytical theory described in Ref. [16]. It is therefore clear that for the given circuit parameters additional tones must be taken into account in the analytical theory.

### B. Extension of the Coupled Mode Equations

To allow the analytical theory to capture more of the behaviour demonstrated by the JTWPA simulations we extend the coupled mode equations (CMEs) to include additional tones. The theory extension method is similar to that considered by Chaudhuri et al for the 4WM case [19]. In Table I the conventional theory as presented in Ref. [16] is denoted as ‘CME-1’ and includes the pump, signal and idler tones. Each further CME extension (CME-\(k\)) contains all pump-mediated mixing tones up to and including the \(k\)-th harmonic of the pump. Here we extend up to CME-5. The constituent tones of each CME set are shown in Table I.

The inclusion of tones in the extended CMEs is described in detail below for the case of CME-2 (inclusion of the second harmonic of the pump, \(f_{2p}\), and the pump-mediated sum-frequency generations, \(f_{p+i}\) and \(f_{p+s}\)). We introduce additional propagators \(\partial A_{p+i}/\partial x\), \(\partial A_{p+s}/\partial x\),

| CME-1 | CME-2 | CME-3 | CME-4 | CME-5 |
|-------|-------|-------|-------|-------|
| \(f_1\) | \(f_1\) | \(f_1\) | \(f_1\) | \(f_1\) |
| \(f_s\) | \(f_s\) | \(f_s\) | \(f_s\) | \(f_s\) |
| \(f_p\) | \(f_p\) | \(f_p\) | \(f_p\) | \(f_p\) |
| \(f_{p+i}\) | \(f_{p+i}\) | \(f_{p+i}\) | \(f_{p+i}\) | \(f_{p+i}\) |
| \(f_{p+s}\) | \(f_{p+s}\) | \(f_{p+s}\) | \(f_{p+s}\) | \(f_{p+s}\) |
| \(f_{2p}\) | \(f_{2p}\) | \(f_{2p}\) | \(f_{2p}\) | \(f_{2p}\) |
| \(f_{2p+i}\) | \(f_{2p+i}\) | \(f_{2p+i}\) | \(f_{2p+i}\) | \(f_{2p+i}\) |
| \(f_{2p+s}\) | \(f_{2p+s}\) | \(f_{2p+s}\) | \(f_{2p+s}\) | \(f_{2p+s}\) |
| \(f_{3p}\) | \(f_{3p}\) | \(f_{3p}\) | \(f_{3p}\) | \(f_{3p}\) |
| \(f_{3p+i}\) | \(f_{3p+i}\) | \(f_{3p+i}\) | \(f_{3p+i}\) | \(f_{3p+i}\) |
| \(f_{3p+s}\) | \(f_{3p+s}\) | \(f_{3p+s}\) | \(f_{3p+s}\) | \(f_{3p+s}\) |
| \(f_{4p}\) | \(f_{4p}\) | \(f_{4p}\) | \(f_{4p}\) | \(f_{4p}\) |
| \(f_{4p+i}\) | \(f_{4p+i}\) | \(f_{4p+i}\) | \(f_{4p+i}\) | \(f_{4p+i}\) |
| \(f_{4p+s}\) | \(f_{4p+s}\) | \(f_{4p+s}\) | \(f_{4p+s}\) | \(f_{4p+s}\) |
| \(f_{5p}\) | \(f_{5p}\) | \(f_{5p}\) | \(f_{5p}\) | \(f_{5p}\) |
and $\partial A_{2p}/\partial x$ in the allowed space of states $\Phi$ where,

$$\Phi = \sum_{j=i,s,p+i,p+s,2p} A_j(x) e^{i(k_j x - \omega_j t)} + c.c., \quad (1)$$

where $A_j(x)$ is the amplitude at dimensionless coordinate $x$ along the JTWP of the $j^{th}$ tone in the space of states.

We treat these additional tones as a generated tone in the same way as the idler, that is, $A_{p+i}(0) = A_{p+s}(0) = A_{2p}(0) = A_i(0) = 0$. We then idealise our SQUID embedded transmission line to be purely non-centrosymmetric. This is the 3WM regime, where the coefficient of the cubic non-linearity $\gamma = 0$. We follow the process outlined in Refs.[15] and [16] to obtain the wave equation describing our transmission line of the form,

$$\frac{\partial^2 \Phi}{\partial x^2} - \omega_0^2 \frac{\partial^2 \Phi}{\partial t^2} + \omega_j^2 \frac{\partial^4 \Phi}{\partial x^4 \partial t^2} + \beta \frac{\partial}{\partial x} \left[ \left( \frac{\partial \Phi}{\partial x} \right)^2 \right] + \gamma \frac{\partial}{\partial x} \left[ \left( \frac{\partial \Phi}{\partial x} \right)^3 \right] = 0, \quad (2)$$

where,

$$\omega_0 = \frac{1}{\sqrt{L_s C_0}}, \quad \text{and} \quad \omega_j = \frac{1}{\sqrt{L_s C_j}}.$$

Neglecting all terms proportional to $A_{p+i}$, $A_{p+s}$, and $A_{2p}$, as well as their derivatives shows that we recover the conventional CMEs used to describe the three wave parametric amplification,

$$\frac{dA_i}{dx} = \frac{\beta}{2} \left( k_p k_a A_p A^*_p e^{i(k_p-k_a)x} + k_p k_{p+i} A_{p+i} A^*_p e^{i(k_{p+i}-k_p)x} + k_p k_{p+s} A_{p+s} A^*_p e^{i(k_{p+s}-k_p)x} \right) e^{-ik_i x}, \quad (5)$$

$$\frac{dA_s}{dx} = \frac{\beta}{2} \left( k_p k_i A_p A^*_i e^{i(k_p-k_i)x} + k_p k_{p+i} A_{p+i} A^*_i e^{i(k_{p+i}-k_p)x} + k_p k_{p+s} A_{p+s} A^*_i e^{i(k_{p+s}-k_p)x} \right) e^{-ik_s x}, \quad (6)$$

$$\frac{dA_{p+i}}{dx} = \frac{\beta}{2} \left( -k_p k_i A_p A^*_i e^{i(k_p-k_i)x} + k_p k_{p+i} A_{p+i} A^*_i e^{i(k_{p+i}-k_p)x} + k_p k_{p+s} A_{p+s} A^*_i e^{i(k_{p+s}-k_p)x} \right) e^{-ik_{p+i} x}, \quad (7)$$

$$\frac{dA_{p+s}}{dx} = \frac{\beta}{2} \left( -k_p k_i A_p A^*_i e^{i(k_p-k_i)x} + k_p k_{p+i} A_{p+i} A^*_i e^{i(k_{p+i}-k_p)x} + k_p k_{p+s} A_{p+s} A^*_i e^{i(k_{p+s}-k_p)x} \right) e^{-ik_{p+s} x}, \quad (8)$$

$$\frac{dA_{2p}}{dx} = \frac{\beta}{2} \left( -k_p^2 A^2 e^{i(k_p+k_a)x} - k_{p+i} k_i A_{p+i} A^*_i e^{i(k_{p+i}+k_i)x} - k_{p+s} k_i A_{p+s} A^*_i e^{i(k_{p+s}+k_i)x} \right) e^{-ik_{2p} x}. \quad (9)$$

A similar set of extended equations are constructed for CMEs-3, 4, and 5 (see Appendices A for a full list of the equations). Each set of equations, CME-1 (Eqs. (11) (13)), CME-2 (Eqs. (5) (10)), CME-3 (Eqs. (A1) (A9)), CME-4 (Eqs. (B1) (B12)), and CME-5 (Eqs. (C1) (C15)) are solved numerically using the ode45 function in MATLAB.

and,

$$\beta = \beta_L \frac{1}{2} \sin(\phi_{dc}), \quad \text{and} \quad \beta_L = \frac{2\pi L_g I_c}{\Phi_0},$$

where $\omega_j$ is the plasma frequency, $\omega_0$ is the cutoff frequency, with $L_g$ the geometric inductance of the SQUID loop, $C_0$ capacitive to ground of the line, $C_j$ the junction capacitance, $I_c$ the junction critical current and $\Phi_0$ is the magnetic flux quantum. By assuming the non-linear component of Eq. (2) acts as a perturbation to the super-linear equation,

$$\frac{\partial^2 \Phi}{\partial x^2} - \omega_0^{-2} \frac{\partial^2 \Phi}{\partial t^2} + \omega_j^2 \frac{\partial^4 \Phi}{\partial x^4 \partial t^2} = 0 \quad (3)$$

we take the resulting super-linear dispersion solution,

$$k(\omega) = \frac{\omega}{\omega_0 \sqrt{1 - \omega^2/\omega_j^2}}. \quad (4)$$

and the space of allowed states in Eq. (1) as a trial solution to generate the coupled mode equations CME-2.

For frequencies much lower than the junction plasma frequency $\omega^2/\omega_j^2 \approx 0$, therefore Eq. (4) can be simplified to $k(\omega) \approx \omega/\omega_0$. We now construct a simple set of CMEs including the tones $f_{p+i}$, $f_{p+s}$, and $f_{2p}$ to find,
FIG. 3. Comparison between the extended CME-5 and the WRspice simulations of the pump \( f_p \) and second harmonic of the pump \( f_{2p} \). \( I_p^{\text{rms}}(0) \approx 1.97 \mu A \). The amplitude of both tones measured at each node as simulated in WRspice are well described using CME-5 up to node 250.

C. Comparison of WRspice Simulations and Coupled Mode Equation Solutions

In order to compare the WRspice simulations with the solutions to the Coupled Mode Equations it is necessary to relate the current \( I(n) \) used in WRspice to the amplitude \( A(x) \) used in the CMEs with the following relation,

\[
I_n^{\text{rms}} = |A(x)| \sqrt{\frac{\omega L_g I_c}{2 \beta_L Z}},
\]

where \( Z = \sqrt{L/C_0} \) is the impedance of the line.

To compare the WRspice simulations results with the solutions to the CMEs we focus first on the interaction between the pump \( f_p \) and the second harmonic of the pump \( f_{2p} \). From the WRspice output shown in Fig. 2 it is clear that the \( f_{2p} \) tone is of large amplitude and thus the second harmonic generation of the pump is a dominant mixing mechanism not accounted for in the CME-1 theory. Fig. 3 shows the solution to CME-5 for the \( f_p \) and \( f_{2p} \), tones compared to the WRspice output. The amplitude of both tones is well described by the CME-5 solutions up to node 250, beyond which there is significant disagreement.

There are a number of assumptions made in the original CME-1 theory (and carried through our CME extensions) that are now considered to ensure we are performing WRspice simulations in a regime in which these assumptions are broadly satisfied. The phase of the junction is set by a dc bias of \( \varphi_{dc} = \pi/2 \) in order to operate in a purely non-centrosymmetric regime. The ac phase \( \varphi_{ac} \) is assumed to be small with respect to \( \varphi_{dc} \). Fig. 4 shows that for high pump currents, approaching \( 2 \mu A \), \( \varphi_{ac} \) can no longer be considered to be small in comparison to \( \varphi_{dc} \). In addition, so called optical rectification is absent in the CMEs. Optical rectification is a dc offset generated by all other tones. The consequence of significant optical rectification is a deviation from the optimal \( \varphi_{dc} = \pi/2 \) bias point such that the device no longer operates in the purely non-centrosymmetric regime.

Note also that for such high input pump currents as shown in Fig. 2 for which \( I_p^{\text{rms}}(0) \approx 1.97 \mu A \), pump harmonic generation up to \( f_{2p} \) is observed (not shown in figure). As we only extend the CMEs to CME-5 we choose to reduce the input pump current such that pump harmonics beyond \( f_{2p} \) are insignificant, and that the assumption that \( \varphi_{ac} \) is small compared to \( \varphi_{dc} \) is upheld. Fig. 4 shows that reducing the pump power from \( I_p^{\text{rms}}(0) \approx 1.97 \mu A \) to \( I_p^{\text{rms}}(0) \approx 0.67 \mu A \) reduces the amplitude of \( \varphi_{ac} \), and maintains a bias point \( \varphi_{dc} = 1.57 = \pi/2 \) (non-centrosymmetric regime).

Fig. 5 shows the current of the pump, and the current of the second harmonic of the pump along the JTTPA. The fit of CME-5 to the WRspice data is greatly improved with the pump power reduced to \( I_p^{\text{rms}}(0) \approx 0.67 \mu A \), and remains in agreement over more nodes. Fig. 5 also shows that reducing the number of allowed states in the set of equations (i.e., CME-5 → CME-4 → CME-3 → CME-2 → CME-1) results in an increased deviation of the agreement between the CME solutions and the WRspice output. These results show the risk of reducing the number of tones represented in the CME set. As the number of tones are reduced the behaviour of the
FIG. 5. Pump, and the second harmonic of the pump currents as a function of node number. $I_{p}^{\text{rms}}(0) \approx 0.67 \mu A$. The WRspice simulation of the pump and second harmonic of the pump are shown with dashed lines, and are the same for each panel. The CME-$k$ solutions are shown with solid lines. Each panel shows a decreasing extension of CME. The agreement between the WRspice simulations and the CMEs reduces as the number of included tones in the CME are reduced. (a) CME-5, (b) CME-4, (c) CME-3, (d) CME-2, (e) CME-1.

pump and the second harmonic of the pump are less well described. Indeed, Fig. 5(d) shows no depletion of the pump due to second harmonic generation and that the CME-1 solutions do not capture the behaviour of the pump tone as simulated by WRspice.

D. Effect on Signal Gain

Fig. 6(a) shows the signal current at each node of the JTWPA circuit for the WRspice simulations and each CME extension. With increasing CME extension we see improved agreement between the CME theory and the WRspice simulations. The traditional analytical theory CME-1 predicts a maximum signal at node $n = 1175$ corresponding to a gain of 20 dB. We calculate the JTWPA gain for each CME set from the current measured at this node. (b) Gain measured at $n = 1175$ for each CME extension. The WRspice simulation result is shown with a horizontal dashed line ($G = 8.9$ dB). The measured gain from each CME is reduced as the number of equations in the CME set increases. The gain measured approaches that of the value calculated from the WRspice simulations.

To quantify the reduction in gain observed as the CMEs are extended, we choose the optimal gain node of CME-1 ($n = 1175$) and compare to the other CMEs and the WRspice simulation at this node. Fig. 6(b) shows as the number of terms in the CMEs increase we capture more complex behaviour of the signal as well as the detrimental effect on the gain. WRspice includes all tones propagating along the JTWPA, as noted earlier, and shows an even lower gain than CME-5 at node $n = 1175$.

Fig. 6(a) also shows deamplification of the signal at the beginning of the JTWPA up to approximately node 300. We believe this deamplification is due to conservation of energy and the signal power dispersing into some of the other mixing tones. All tones, with the exception of the pump and the signal, are input to the equations with zero initial amplitude, and thus the power required to generate these tones must initially come from the pump and signal. It is observed that as the number of tones included in the CMEs increases, the number of nodes over which the signal deamplifies increases though the gradient is unchanged.

III. DISCUSSION AND CONCLUSION

Our extension of the CMEs show that CME-1 (including only the pump, signal, and idler) is insufficient to capture the complex behaviour of the JTWPA. As we increase the number of terms in the CMEs we approach the behaviour and gain figures observed in WRspice sim-
and we have discussed possible reasons for this. In or-
merely extended CME-3 by including the third harmonic of the pump, \( f_{3p} \), and the sum-frequency terms \( f_{2p+i} \) and \( f_{2p+s} \). We show in detail all of the terms included in the coupled mode equation forming CME-3.

\[
\frac{dA_p}{dx} = \frac{\beta}{2} \left( -k_1k_2 A_p A_s e^{i(k_p+k_s)x} + k_2k_3 A_2p A_s^* e^{i(k_{2p}+k_{2s})x} + k_{p+s+k_s} A_{p+s} A_s^* e^{i(k_{p+s}+k_s)x} + k_{p+i+k_s} A_p A_{p+i} e^{i(k_{p+i}+k_s)x} + k_{p+i+k} A_{p+i} A_s^* e^{i(k_{p+i}+k_s)x} \right) e^{-ik_p x}
\]

\[
\frac{dA_s}{dx} = \frac{\beta}{2} \left( k_1k_2 A_p A_s e^{i(k_p-k_s)x} + k_2k_3 A_2p A_s^* e^{i(k_{2p}-k_{2s})x} + k_{p+i+k} A_p A_{p+i} e^{i(k_{p+i}-k_s)x} + k_{p+i+k} A_{p+i} A_s^* e^{i(k_{p+i}-k_s)x} \right) e^{-ik_s x}
\]

\[
\frac{dA_{p+i}}{dx} = \frac{\beta}{2} \left( k_1k_2 A_p A_s e^{i(k_{p+i},+k_s)x} + k_2k_3 A_2p A_s^* e^{i(k_{2p+i}-k_{2s})x} + k_{p+i+k} A_p A_{p+i} e^{i(k_{p+i}-k_s)x} + k_{p+i+k} A_{p+i} A_s^* e^{i(k_{p+i}-k_s)x} \right) e^{-ik_{p+i} x}
\]

\[
\frac{dA_{p+s}}{dx} = \frac{\beta}{2} \left( k_1k_2 A_p A_s e^{i(k_{p+s},+k_s)x} + k_2k_3 A_2p A_s^* e^{i(k_{2p+s}-k_{2s})x} + k_{p+i+k} A_p A_{p+i} e^{i(k_{p+i}-k_s)x} + k_{p+i+k} A_{p+i} A_s^* e^{i(k_{p+i}-k_s)x} \right) e^{-ik_{p+s} x}
\]

This work realises a simple, computationally inexpensive, method for extension of the CMEs describing propagators which have been previously neglected and demonstrates the utility of WRspice for simulation of non-linear superconducting circuits, in particular as a design tool for JTWPA.

ACKNOWLEDGMENTS

This project has received funding from the EMPIR programme co-financed by the Participating States and from the European Unions Horizon 2020 research and innovation programme. This work is part of the the Joint Research Project PARAWAVE, and we would like to thank members of the consortium, in particular R. Dolata, M. Khabipov, C. Käßing, and A. B. Zorin for useful discussions on the operation of the JTWPA. The work is partially supported by the UK Department of Business, Energy and Industrial Strategy (BEIS). We thank J. Burnett and J. C. Gallop for critical review of the manuscript.
Appendix B: Extension to CME-4

The penultimate coupled mode equation extension that we present in full is the extension from CME-3 to CME-4 by inclusion of the fourth harmonic of the pump $f_{4p}$, and the sum-frequency terms $f_{3p+i}$ and $f_{3p+s}$. The full list of tones included in CME-4 is shown in Table I. We show below in detail all of the terms included in the coupled mode equation forming CME-4.

\[
\frac{dA_{2p}}{dx} = \frac{\beta}{4} \left( k_{p}^{2} A_{p}^{2} e^{i(2kp)x} - k_{2p} A_{2p} A_{p+i} A_{p+s} e^{i(k_{2p}+k_{p}+k_{i})x} - k_{p+i} k_{s} A_{p+i} A_{s} e^{i(k_{p+i}+k_{s})x} 
+ k_{p} k_{3p} A_{p} A_{3p} e^{i(k_{3p}-k_{p})x} + k_{2p+s} k_{s} A_{2p+s} A_{s} e^{i(k_{2p+s}+k_{s})x} 
+ k_{2p+i} k_{1} A_{2p+i} A_{s} e^{i(k_{2p+i}+k_{1})x} e^{-ik_{2p}x} \right) 
\]

\[
\frac{dA_{p+i}}{dx} = \frac{\beta}{2} \left( -k_{p} k_{i} A_{p} A_{i} e^{i(k_{p}+k_{i})x} + k_{2p} A_{2p} A_{p+i} A_{p+s} e^{i(k_{2p}+k_{p}+k_{i})x} 
+ k_{3p} k_{p+i} A_{3p} A_{p+i} e^{i(k_{3p}+k_{p}+k_{i})x} + k_{2p+s} k_{p+i} A_{2p+s} A_{p+i} e^{i(k_{2p+s}+k_{p}+k_{i})x} e^{-ik_{p+i}x} \right) 
\]

\[
\frac{dA_{3p}}{dx} = \frac{\beta}{2} \left( -k_{p} k_{2p} A_{p} A_{2p} e^{i(k_{p}+k_{2p})x} - k_{p+i} k_{s} A_{p+i} A_{p+s} e^{i(k_{p+i}+k_{p}+k_{s})x} 
- k_{2p+i} A_{2p+i} A_{p+i} e^{i(k_{2p+i}+k_{s}+k_{p})x} - k_{2p+i} A_{2p+i} A_{p+i} e^{i(k_{2p+i}+k_{s}+k_{p})x} e^{-ik_{3p}x} \right) 
\]

\[
\frac{dA_{2p+i}}{dx} = \frac{\beta}{2} \left( -k_{p} k_{2p} A_{p} A_{2p+i} e^{i(k_{p}+k_{2p})x} - k_{p+i} k_{2p} A_{p+i} A_{2p} e^{i(k_{p+i}+k_{2p})x} + k_{3p} k_{i} A_{3p} A_{p+i} e^{i(k_{3p}+k_{i})x} e^{-ik_{2p+i}x} \right) 
\]

\[
\frac{dA_{p+s}}{dx} = \frac{\beta}{2} \left( k_{p} k_{3p} A_{p} A_{3p} e^{i(k_{p}+k_{3p})x} + k_{2p+s} k_{p+s} A_{2p+s} A_{p+s} e^{i(k_{2p+s}+k_{p+s})x} + k_{2p+i} k_{i} A_{2p+i} A_{i} e^{i(k_{2p+i}+k_{i})x} e^{-ik_{p+s}x} \right) 
\]

\[
\frac{dA_{3p+i}}{dx} = \frac{\beta}{2} \left( k_{2p} k_{3p+i} A_{2p} A_{3p+i} e^{i(k_{2p}+k_{3p+i})x} + k_{3p} k_{2p+i} A_{3p+i} A_{2p+i} e^{i(k_{3p}+k_{2p+i})x} + k_{3p} k_{2p+i} A_{3p+i} A_{2p+i} e^{i(k_{3p}+k_{2p+i})x} e^{-ik_{3p+i}x} \right) 
\]

\[
\frac{dA_{3p+s}}{dx} = \frac{\beta}{2} \left( k_{p} k_{3p+s} A_{p} A_{3p+s} e^{i(k_{p}+k_{3p+s})x} + k_{2p+i} k_{3p+i} A_{2p+i} A_{3p+i} e^{i(k_{2p+i}+k_{3p+i})x} + k_{3p} k_{2p+i} A_{3p+i} A_{2p+i} e^{i(k_{3p}+k_{2p+i})x} e^{-ik_{3p+s}x} \right) 
\]

(4A) (5A) (6A) (7A) (8A) (9A)

---

**Table I**

The table shows the list of tones included in CME-4.
\[
\frac{dA_{2p}}{dx} = \beta \left( \frac{k_{2p}^2 A_{2p}^2}{2} e^{i(k_p + k_p)x} - k_p k_{p+s} A_{p+s} e^{i(k_p + k_p)x} - k_{p+i} k_s A_{p+i} e^{i(k_p + k_s)x} \right)
\]
\[
+ k_p k_{p+s} A_{p+s} e^{i(k_p - k_p)x} + k_{2p+s} k_s A_{2p+s} A_{s}^* e^{i(k_{2p+s} - k_p)x} + k_{2p+i} k_{p+i} A_{2p+i} A_{s}^* e^{i(k_{2p+i} - k_p)x} + k_{3p+s} k_{p+s} A_{3p+s+s} A_{s}^* e^{i(k_{3p+s+s} - k_p)x} + k_{3p+i} k_{p+i} A_{3p+i+i} A_{s}^* e^{i(k_{3p+i+i} - k_p)x} \right) e^{-ik_{2p}x}
\]
\[
\frac{dA_{p+s}}{dx} = \beta \left( -k_p k_s A_p A_s e^{i(k_p + k_s)x} + k_{2p} A_{2p} A_{p}^* e^{i(k_{2p} - k_p)x} + k_{3p} k_{p+s} A_{3p+s} A_{s}^* e^{i(k_{3p} - k_{p+s})x} \right)
\]
\[
+ k_{3p} k_{p+s} A_{3p+s} A_{p+s} e^{i(k_{3p} - k_{p+s})x} + k_{2p+i} k_{p+i} A_{2p+i} A_{s}^* e^{i(k_{2p+i} - k_p)x} + k_{4p} k_{2p+i} A_{4p} A_{2p+i} A_{s}^* e^{i(k_{4p} - k_{2p+i})x} + k_{3p} k_{3p+s} A_{3p+s+s} A_{s}^* e^{i(k_{3p+s+s} - k_p)x} \right) e^{-ik_{3p}x}
\]
\[
\frac{dA_{p+i}}{dx} = \beta \left( -k_p k_s A_p A_s e^{i(k_p + k_s)x} + k_{2p} A_{2p} A_{p}^* e^{i(k_{2p} - k_p)x} + k_{3p} k_{p+s} A_{3p+s} A_{s}^* e^{i(k_{3p} - k_{p+s})x} \right)
\]
\[
+ k_{4p} k_{2p+i} A_{4p} A_{2p+i} A_{s}^* e^{i(k_{4p} - k_{2p+i})x} + k_{3p} k_{3p+i} A_{3p+i} A_{s}^* e^{i(k_{3p+i} - k_p)x} \right) e^{-ik_{3p}x}
\]

\[
\frac{dA_{3p}}{dx} = \beta \left( -k_p k_s A_p A_s e^{i(k_p + k_s)x} + k_{2p} A_{2p} A_{p}^* e^{i(k_{2p} - k_p)x} + k_{3p} k_{p+s} A_{3p+s} A_{s}^* e^{i(k_{3p} - k_{p+s})x} \right)
\]
\[
+ k_{4p} k_{2p+i} A_{4p} A_{2p+i} A_{s}^* e^{i(k_{4p} - k_{2p+i})x} + k_{3p} k_{3p+i} A_{3p+i} A_{s}^* e^{i(k_{3p+i} - k_p)x} \right) e^{-ik_{3p}x}
\]
\[
\frac{dA_{4p}}{dx} = \beta \left( -k_p k_{3p} A_{3p} A_{4p} e^{i(k_p + k_{3p})x} - k_{3p} k_{3p+i} A_{3p+i} A_{s}^* e^{i(k_{3p+i} + k_{3p})x} \right)
\]
\[
+ k_{3p} k_{3p+s} A_{3p+s} A_{s}^* e^{i(k_{3p+s} + k_{3p})x} - k_{2p} k_{2p+s} A_{2p+s} A_{s}^* e^{i(k_{2p} + k_{2p+s})x} - k_{3p} k_{3p+i} A_{3p+i} A_{s}^* e^{i(k_{3p+i} + k_{3p})x} \right) e^{-ik_{3p}x}
\]
\[
\frac{dA_{3p+s}}{dx} = \beta \left( -k_p k_{3p} A_{3p} A_{3p+s} e^{i(k_p + k_{3p})x} - k_{3p} k_{3p+i} A_{3p+i} A_{s}^* e^{i(k_{3p+i} + k_{3p})x} \right)
\]
\[
+ k_{3p} k_{3p+s} A_{3p+s} A_{s}^* e^{i(k_{3p+s} + k_{3p})x} - k_{2p} k_{2p+s} A_{2p+s} A_{s}^* e^{i(k_{2p} + k_{2p+s})x} \right) e^{-ik_{3p}x}
\]
\[
\frac{dA_{3p+i}}{dx} = \beta \left( -k_p k_{3p} A_{3p} A_{3p+i} e^{i(k_p + k_{3p})x} - k_{3p} k_{3p+i} A_{3p+i} A_{s}^* e^{i(k_{3p+i} + k_{3p})x} \right)
\]
\[
+ k_{3p} k_{3p+s} A_{3p+s} A_{s}^* e^{i(k_{3p+s} + k_{3p})x} - k_{2p} k_{2p+s} A_{2p+s} A_{s}^* e^{i(k_{2p} + k_{2p+s})x} \right) e^{-ik_{3p}x}
\]
Appendix C: Extension to CME-5

The final coupled mode equation extension that we present in full is the extension from CME-4 to CME-5 by inclusion of the fifth harmonic of the pump $f_{5p}$, and the sum-frequency terms $f_{4p+i}$ and $f_{4p+s}$. The full list of tones included in CME-5 is shown in Table I. We show below in detail all of the terms included in the coupled mode equation forming CME-5.

\[
\frac{dA_i}{dx} = \frac{\beta}{2} \left( k_a k_p A_p A_p^* e^{ik_p k_x} + k_{p+i} k_p A_{p+i} A_{p+i}^* e^{ik_{p+i} k_x} \right)
\]  \hspace{1cm} (C1)

\[
\frac{dA_s}{dx} = \frac{\beta}{2} \left( k_i k_p A_s A_s^* e^{ik_p k_x} + k_{p+s} k_p A_{p+s} A_{p+s}^* e^{ik_{p+s} k_x} \right)
\]  \hspace{1cm} (C2)

\[
\frac{dA_p}{dx} = \frac{\beta}{2} \left( -k_{k_a} k_s A_s A_s^* e^{ik_{k_a} k_x} + k_{p+i} k_i A_{p+i} A_{p+i}^* e^{ik_{p+i} k_x} \right)
\]  \hspace{1cm} (C3)

\[
\frac{dA_{p+i}}{dx} = \frac{\beta}{2} \left( -k_{k_a} k_p A_p A_p^* e^{ik_{k_p} k_x} + k_{2p+i} k_p A_{2p+i} A_{2p+i}^* e^{ik_{2p+i} k_x} \right)
\]  \hspace{1cm} (C4)

\[
\frac{dA_{p+s}}{dx} = \frac{\beta}{2} \left( -k_{k_a} k_p A_p A_p^* e^{ik_{k_p} k_x} + k_{2p+i} k_p A_{2p+i} A_{2p+i}^* e^{ik_{2p+i} k_x} \right)
\]  \hspace{1cm} (C5)
\[
\frac{dA_{2p}}{dx} = \frac{\beta}{2} \left( -k_1k_{p+s}A_{1}A_{p+s}e^{i(k_p+k_{p+s})x} - k_{s}k_{p+i}A_{s}A_{p+i}e^{i(k_s+k_{p+i})x} \\
- \frac{k_2^2A_{2p}^2}{2} e^{i(k_p+k_{p+s})x} + k_{2p+i}e^{i(k_{2p+i}-k_{p+i})x} \\
+ k_{2p+k}\beta A_{2p+s}A_{s}^* e^{i(k_{2p+s}-k_s)x} + k_{3p}k_pA_{3p}A_{p}^* e^{i(k_{3p}-k_p)x} \\
+ k_{3p+i}k_{p+i}A_{3p+i}A_{s}^* e^{i(k_{3p+i}+k_{p+i})x} + k_{3p+s}k_{p+s}A_{3p+s}A_{s}^* e^{i(k_{3p+s}-k_{p+s})x} \\
+ k_{4p}k_{2p+i}A_{4p}A_{2p+i}^* e^{i(k_{4p}-k_{2p+i})x} + k_{4p+i}k_{2p+i}A_{4p+i}A_{2p+i}^* e^{i(k_{4p+i}-k_{2p+i})x} \\
+ k_{4p+s}k_{2p+s}A_{4p+s}A_{2p+s}^* e^{i(k_{4p+s}-k_{2p+s})x} + k_{5p}k_{3p}A_{5p}A_{3p}^* e^{i(k_{5p}-k_{3p})x} \right) e^{-ik_{2p}x}
\]

(C6)

\[
\frac{dA_{2p+i}}{dx} = \frac{\beta}{2} \left( -k_1k_{2p+i}A_{1}A_{2p+i}e^{i(k_p+k_{2p+i})x} - k_{s}k_{p+i}A_{s}A_{2p+i}e^{i(k_s+k_{p+i})x} \\
+ k_{3p}k_{s}A_{3p}A_{s}^* e^{i(k_{3p}-k_s)x} + k_{3p+i}k_{p+i}A_{3p+i}A_{p+i}^* e^{i(k_{3p+i}+k_{p+i})x} \\
+ k_{4p}k_{p+i}A_{4p}A_{p+i}^* e^{i(k_{4p}-k_{p+i})x} + k_{4p+i}k_{p+i}A_{4p+i}A_{p+i}^* e^{i(k_{4p+i}-k_{p+i})x} \\
+ k_{5p}k_{2p+i}A_{5p}A_{2p+i}^* e^{i(k_{5p}-k_{2p+i})x} \right) e^{-ik_{2p+i}x}
\]

(C7)

\[
\frac{dA_{2p+s}}{dx} = \frac{\beta}{2} \left( -k_1k_{p+s}A_{1}A_{2p+s}e^{i(k_p+k_{p+s})x} - k_{s}k_{2p+s}A_{s}A_{2p+s}e^{i(k_s+k_{2p+s})x} \\
- k_{2p+i}k_{p+i}A_{2p+i}A_{p+i}^* e^{i(k_{2p+i}+k_{p+i})x} + k_{3p+i}k_{p+i}A_{3p+i}A_{p+i}^* e^{i(k_{3p+i}+k_{p+i})x} \\
+ k_{4p}k_{2p+i}A_{4p}A_{2p+i}^* e^{i(k_{4p}-k_{2p+i})x} + k_{4p+i}k_{2p+i}A_{4p+i}A_{2p+i}^* e^{i(k_{4p+i}-k_{2p+i})x} \\
+ k_{5p}k_{2p+s}A_{5p}A_{2p+s}^* e^{i(k_{5p}-k_{2p+s})x} \right) e^{-ik_{2p+s}x}
\]

(C8)

\[
\frac{dA_{3p}}{dx} = \frac{\beta}{2} \left( -k_1k_{2p+s}A_{1}A_{2p+s}e^{i(k_p+k_{2p+s})x} - k_{s}k_{2p+s}A_{s}A_{2p+s}e^{i(k_s+k_{2p+s})x} \\
- k_{p+i}k_{p+i}A_{p+i}A_{p+i}^* e^{i(k_{p+i}+k_{p+i})x} + k_{3p+i}k_{p+i}A_{3p+i}A_{p+i}^* e^{i(k_{3p+i}+k_{p+i})x} \\
+ k_{4p}k_{2p+i}A_{4p}A_{2p+i}^* e^{i(k_{4p}-k_{2p+i})x} + k_{4p+i}k_{2p+i}A_{4p+i}A_{2p+i}^* e^{i(k_{4p+i}-k_{2p+i})x} \\
+ k_{5p}k_{2p+s}A_{5p}A_{2p+s}^* e^{i(k_{5p}-k_{2p+s})x} \right) e^{-ik_{3p}x}
\]

(C9)

\[
\frac{dA_{3p+i}}{dx} = \frac{\beta}{2} \left( -k_1k_{3p+i}A_{1}A_{3p+i}e^{i(k_p+k_{3p+i})x} - k_{s}k_{3p+i}A_{s}A_{3p+i}e^{i(k_s+k_{3p+i})x} \\
- k_{p+i}k_{p+i}A_{p+i}A_{p+i}^* e^{i(k_{p+i}+k_{p+i})x} + k_{4p}k_{3p+i}A_{4p}A_{3p+i}^* e^{i(k_{4p}-k_{3p+i})x} \\
+ k_{4p+i}k_{3p+i}A_{4p+i}A_{3p+i}^* e^{i(k_{4p+i}-k_{3p+i})x} + k_{5p}k_{3p+i}A_{5p}A_{3p+i}^* e^{i(k_{5p}-k_{3p+i})x} \right) e^{-ik_{3p+i}x}
\]

(C10)

\[
\frac{dA_{3p+s}}{dx} = \frac{\beta}{2} \left( -k_1k_{3p+s}A_{1}A_{3p+s}e^{i(k_p+k_{3p+s})x} - k_{s}k_{3p+s}A_{s}A_{3p+s}e^{i(k_s+k_{3p+s})x} \\
- k_{p+i}k_{p+i}A_{p+i}A_{p+i}^* e^{i(k_{p+i}+k_{p+i})x} + k_{4p}k_{3p+s}A_{4p}A_{3p+s}^* e^{i(k_{4p}-k_{3p+s})x} \\
+ k_{4p+i}k_{3p+s}A_{4p+i}A_{3p+s}^* e^{i(k_{4p+i}-k_{3p+s})x} + k_{5p}k_{3p+s}A_{5p}A_{3p+s}^* e^{i(k_{5p}-k_{3p+s})x} \right) e^{-ik_{3p+s}x}
\]

(C11)

\[
\frac{dA_{4p}}{dx} = \frac{\beta}{2} \left( -k_1k_{3p+s}A_{1}A_{3p+s}e^{i(k_p+k_{3p+s})x} - k_{s}k_{3p+s}A_{s}A_{3p+s}e^{i(k_s+k_{3p+s})x} \\
- k_{p+i}k_{3p+s}A_{p+i}A_{3p+s}^* e^{i(k_{p+i}+k_{3p+s})x} + k_{3p+i}k_{p+i}A_{3p+i}A_{p+i}^* e^{i(k_{3p+i}+k_{p+i})x} \\
- k_{p+i}k_{2p+i}A_{p+i}A_{2p+i}^* e^{i(k_{p+i}+k_{2p+i})x} - \frac{k_{2p+s}^2A_{2p}^2}{2} e^{i(k_{2p+s}-k_{p+s})x} \\
+ k_{4p+i}k_{4p+i}A_{4p+i}A_{4p+i}^* e^{i(k_{4p+i}+k_{4p+i})x} + k_{3p+i}k_{3p+i}A_{3p+i}A_{3p+i}^* e^{i(k_{3p+i}+k_{3p+i})x} \\
+ k_{5p}k_{5p}A_{5p}A_{5p}^* e^{i(k_{5p}-k_{5p})x} \right) e^{-ik_{4p}x}
\]

(C12)

\[
\frac{dA_{4p+i}}{dx} = \frac{\beta}{2} \left( -k_1k_{4p+i}A_{1}A_{4p+i}e^{i(k_p+k_{4p+i})x} - k_{s}k_{4p+i}A_{s}A_{4p+i}e^{i(k_s+k_{4p+i})x} \\
- k_{p+i}k_{4p+i}A_{p+i}A_{4p+i}^* e^{i(k_{p+i}+k_{4p+i})x} + k_{3p+i}k_{p+i}A_{3p+i}A_{p+i}^* e^{i(k_{3p+i}+k_{p+i})x} \\
+ k_{4p+i}k_{4p+i}A_{4p+i}A_{4p+i}^* e^{i(k_{4p+i}+k_{4p+i})x} + k_{5p}k_{4p+i}A_{5p}A_{4p+i}^* e^{i(k_{5p}-k_{4p+i})x} \right) e^{-ik_{4p+i}x}
\]

(C13)
\[
\frac{dA_{4p+s}}{dx} = \frac{\beta}{2} \left( -k_s k_{4p} A_{4p} A_{4p} e^{i(k_p + k_{4p})x} - k_p k_{3p+s} A_{3p+s} A_{3p+s} e^{i(k_p + k_{3p+s})x} \right. \\
\left. - k_{p+s} k_{3p} A_{p+s} A_{3p} e^{i(k_{p+s} + k_{3p})x} - k_{2p} k_{2p+s} A_{2p+s} A_{2p+s} e^{i(k_{2p} + k_{2p+s})x} \right. \\
\left. + k_{5p} k_{4p} A_{5p} e^{i(k_{5p} - k_{4p})x} \right)e^{-ik_{4p+s}x}
\]

\[
\frac{dA_{2p}}{dx} = \frac{\beta}{2} \left( -k_s k_{4p} A_{4p+s} A_{4p+s} e^{i(k_p + k_{4p+s})x} - k_s k_{4p+s} A_{4p+s} A_{4p+s} e^{i(k_p + k_{4p+s})x} \right. \\
\left. - k_p k_{3p+s} A_{3p+s} A_{3p+s} e^{i(k_p + k_{3p+s})x} - k_{p+s} k_{3p+s} A_{p+s} A_{p+s} e^{i(k_{p+s} + k_{3p+s})x} \right. \\
\left. - k_{2p+s} k_{2p+s} A_{2p+s} A_{2p+s} e^{i(k_{2p+s} + k_{2p+s})x} \right)e^{-ik_{2p+s}x}
\]