The average avalanche size in the Manna Model and other models of self-organised criticality

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Abstract. The average avalanche size can be calculated exactly in a number of models of self-organised criticality (SOC). While the calculation is straightforward in one dimension, it is more involved in higher dimensions and further complicated by the presence of different boundary conditions and different forms of external driving. Amplitudes of the leading order are determined analytically and evaluated to obtain analytical references for numerical work. A subtle link exists between the procedure to calculate the average avalanche size and the field theory of SOC.

Key words. Self-organised criticality – random walks – escape time – scaling

PACS. 89.75.Da Systems obeying scaling laws – 89.75.-k Complex systems – 05.65.+b Self-organized systems – 05.70.Jk Critical point phenomena

1 Introduction

The average avalanche size in models of self-organised criticality (SOC) [1] is one of the few observables whose scaling frequently is known exactly even in non-trivial cases. In numerical simulations, it often plays the rôle of a benchmark for convergence [2,3]. In the one-dimensional version of many models, even the amplitude of the average avalanche size is known exactly, as well as the corrections to scaling [4]. In the following, exact results for the average avalanche size are collated and extended to higher dimensions and more complicated boundary conditions.

In a number of conservative SOC models [1,5], particles (height units or slope units) perform a random walk from the point of being added to the system until they leave it. In the Manna model [6] particles move independently from site to site, so that their trajectory is exactly a random walk. In that model, particles are added at (normally randomly and independently chosen) sites by the so-called external drive. If the number of particles at a site exceeds 1, all particles are being redistributed independently from site to site, so that their trajectory is exactly a random walk. In that model, particles are added at (normally randomly and independently chosen) sites by the so-called external drive. If the number of particles at a site exceeds 1, all particles are being redistributed independently and randomly to the nearest neighbours. This process is repeated until no height exceeds the threshold of 1 anymore. Sites that do (and the particles located there) are called active. The totality of redistributions or topplings constitutes an avalanche. A complete separation of the time scales of driving and relaxation is achieved by driving only when no site (or particle) is active. A particle may rest for very long times until being moved again, but while it is moving, it performs a random walk in a time that advances only as long as the particle is active (conditional time scale).

Even in systems where particle movements are not independent, such as the BTW [1] and the Oslo Models [7], where particles are redistributed evenly among nearest neighbours, it has been noted that particles follow random-walker trajectories, because the entire ensemble of possible paths is being generated as sites topple.

It has therefore been noted several times that the average avalanche size in the Manna Model is essentially given by the average escape time of a random walker, e.g. [8,9]. While particles in the Manna Model describe trajectories of a random walker, each of their moves from one site to a neighbouring site is caused by a toppling. In fact, in the Abelian version [10] considered in the following, each toppling causes two particles (2d particles in the BTW and Oslo Models on hypercubic lattices\(^1\)) to move and so the average number of topplings per particle added, which is exactly the average avalanche size, is equal to half the average number of moves each particle makes until its departure from the system.

The number of charges a particle causes during its lifetime (i.e. the number of times a particle arrives at a site until it leaves the system), is exactly equal to the number of moves it makes; while the initial deposition represents a charge, but not a move, the final move (off the system) does not cause a charge.

As opposed to higher moments, the average avalanche size can be calculated because it does not require any in-

\(^{1}\) More generally, in BTW and Oslo Models, \(q\) particles topple, where \(q\) is the coordination number of the lattice in the bulk.
formation about the collective topping of particles. It is merely a matter of stationarity and conservation. The former is important because only at stationarity the average avalanche size can be determined as the number of topplings per particle exiting by averaging over so many avalanches that the vast majority of particles added have left the system. Conservation is important for two reasons. Firstly, particles should not disappear by interaction, which cannot be accounted for in this simple approach. Secondly, each and every toppling must count towards an avalanche.

In the following, the average avalanche size is calculated for hypercubic systems in arbitrary dimensions (but see Section 5). First, it is calculated for a one-dimensional “lattice” with two open boundaries. The result is then generalised to the scaling in arbitrary dimensions. Doing this exactly and on the lattice is a difficult undertaking [9], but the aim of the following is to determine the leading order amplitudes. After taking the continuum limit, they are calculated for a variety of boundary conditions. Some special cases are discussed. Finally, the result is related to some recent field theoretic insights.

2 One dimension

In one dimension, the average number of moves can be calculated fairly easily for a variety of boundary conditions. For brevity, I focus on two open boundaries (i.e. particles leave the system if a toppling site attempts to deposit a particle on an “outside” site). If \( x_0 \) is the site a particle is added to by the external drive, then the average number of moves \( m(x_0; L) \) the particle makes until its departure is given by [11,12]

\[
m(x; L) = 1 + \frac{m(x+1; L) + m(x-1; L)}{2}
\]

where the open boundaries are implemented by imposing \( m(0; L) = m(L+1; L) = 0 \), i.e. a Dirichlet boundary condition. Rearranging terms produces a Poisson equation on the lattice, whose solution is a simple quadratic,

\[
m(x; L) = x(L + 1 - x) .
\]

Summing over the uniform drive (i.e. \( x_0 \) uniformly and randomly taken from \( \{1, \ldots, L\} \)) gives

\[
\overline{m}(L) = \frac{1}{L} \sum_{x=1}^{L} m(x; L) = \frac{(L + 1)(L + 2)}{6}
\]

and thus the expectation of the avalanche size (first moment) is exactly [13]

\[
\langle s \rangle = \frac{1}{2} \overline{m}(L) = \frac{(L + 1)(L + 2)}{12} \propto L^2 .
\]

2.1 Generalisations

In higher dimensions, the scaling \( \langle s \rangle \propto L^2 \) persists, which is of course just the usual escape time of a random walker: It explores the distance \( L \) within \( L^2 \) moves. This argument can be made more rigorous by noting that if the survival probability after \( t \) moves (i.e. the probability of the random walker not having reached an open boundary) is \( \sigma(t, L) \) in one dimension (for the sake of simplicity, this is the probability averaged over the uniform drive), then in higher dimensions \( d \) that probability is simply \( \sigma(t, L)^d \), because of the independence of the \( d \) directions of possible displacement and the hypercubic nature of the boundaries.\(^4\) In the continuum limit, \( t \) is better interpreted as a time, rather than a number of enforced moves. The average residence time in \( d \) dimension, equal to the average time to escape \( \varepsilon_d \), is thus\(^5\)

\[
2 \langle s \rangle_d^n(L) \simeq \varepsilon_d(L) = \int_0^\infty dt \sigma(t, L)^d = \int_0^\infty t \left( \frac{d}{dt} \sigma(t, L)^d \right)
\]

where \(-\frac{d}{dt} \sigma(t, L)^d\) is the probability density of escaping at time \( t \). Its structure reflects the fact that the movement in the \( d \) spatial directions is independent; \(-\frac{d}{dt} \sigma(t, L)\) is the probability density to escape at time \( t \) in one direction, of which there are \( d \) (choices), and \( \sigma(t, L)^d \) is the probability to stay within bounds in the remaining \( d - 1 \) dimensions. Here and in the following, the factor 2 in front of \( \langle s \rangle \) (on the left of Eq. (5)) is retained, acting as a reminder of its origin as the number of particles redistributed in each toppling (the avalanche size \( s \) being measured by the number of topplings). In the BTW and the Oslo Models, that factor 2 has to be replaced by the coordination number of the lattice, \( 2d \) for a hypercubic one with nearest neighbour interaction.

Because \( \sigma(t, L) \) is, by dimensional consistency, bound to be the dimensionless function \( \sigma(t/L^2, 1) \) it follows that \( \varepsilon_d \propto L^2 \), in line with the view that the trajectory of a random walker is essentially a two-dimensional object [14]. Claiming that \( t/L^2 \) is dimensionless means being somewhat cavalier about the dimension of the diffusion constant \( D \), which in the present context relates time and number of moves. If the walker takes, in each time step, one step in any of the \( d \) spatial directions, the variance of its displacement is 1. The diffusion constant, on the other hand, is half the variance of the displacement in each (independent) spatial direction per time, so that \( 2Dd = 1 \) on hypercubic lattices. There is thus a slight conceptual difference between the active particles in the Manne Model on the one hand, which are forced to move to one of their nearest neighbours, and a random walker with a certain

\(^2\)In contrast, the present approach does not allow the calculation of the average avalanche size in the ensemble of avalanches with non-vanishing size.

\(^3\)In the following, when quoting results to leading order the equality sign \( \simeq \) will be used.

\(^4\)If the boundaries are shaped or structured then the survival in one direction depends on the coordinate in the other. Results for that case can be found in [9].

\(^5\)The \( \simeq \) sign applies as \( \varepsilon_d(L)/2 \) is a continuum approximation of \( \langle s \rangle_d^n(L) \), yet \( \varepsilon_d(L) \), in the continuum, itself is calculated exactly.
diffusion constant on the other, which is subject to random motion in each spatial direction independently.

The survival probability can be calculated quite easily, noting that the normalised eigenfunctions of \( \partial_x^2 \) with Dirichlet boundary conditions in one dimension are \( \sqrt{2/L} \sin(q_n x) \) with \( q_n = n \pi / L \), where \( n = 1, 2, \ldots \). With periodic boundary conditions, they are \( \exp(q_n x) \) with \( q_n = 2n \pi / L \) and any integer \( n \), including 0 and negative integers, \( n \in \mathbb{Z} \). As it will turn out below, given the self-adjoint operator \( \partial_x^2 \), it is the presence or absence of the zero mode, i.e. the constant eigenfunction with eigenvalue 0, which decides over conservation or dissipation and the structure of the resulting equation for \( \langle \psi \rangle \).

In one dimension, the probability density function (PDF) of a particle under Brownian Motion started at \( x_0 \) with diffusion constant \( D \) on an interval with open boundaries at 0 and \( L \) is thus

\[
\mathcal{P}(x, t; x_0, L) = \frac{2}{L} \sum_{n=1}^{\infty} \sin(x_0 q_n) \sin(x q_n) e^{-D q_n^2 t} \tag{6}
\]

Since the motion in the different directions is independent, the PDF in higher dimensions is a product of Eq. (6). The expected residence time is given by the integral over time and space, in one dimension

\[
\int_0^L dx \int_0^\infty dt \mathcal{P}(x, t; x_0, L) = \frac{2}{L} \sum_{n=1, odd}^{\infty} \frac{2}{q_n} \sin(x_0 q_n) \frac{1}{D q_n^2} , \tag{7}
\]

where the constraint of \( n = 1, 3, 5, \ldots \) in the sum being odd comes from the integral of \( \sin(x_0 q_n) \), which gives 2/q_n for odd \( n \) and 0 otherwise. For uniform drive \( f_0^L dx_0 (1/L) \), to be replaced by a single source at \( a \), i.e. \( f_0^L dx_0 \delta(a - x_0) \), so that the average avalanche size for a system driven at \( a \) is

\[
\sigma(t, L) = \frac{1}{L} \int_0^L dx_0 \int_0^L dx \mathcal{P}(x, t; x_0, L) = \frac{2}{L^2} \sum_{n=1, odd}^{\infty} \frac{4}{q_n^2} e^{-D q_n^2 t} \tag{8}
\]

and therefore

\[
2 \langle \psi \rangle_{1, a} (L) \simeq \frac{2}{L^2} \sum_{n=1, odd}^{\infty} \frac{4}{q_n^2} \approx \frac{4}{L^2} \sum_{n, m, \ldots = 1, odd} 1 \sin^2 \frac{n \pi a}{L} \sum_n D(q_n^2 + q_m^2 + \ldots) \tag{9}
\]

where the sum runs over \( d \) different indices. Dhar’s result for the lattice in \( d = 2 \) [15, Eq. 21] is recovered by approximating \( \cot(\pi n/(2L + 1)) \approx 1 / \sin(\pi n/(2L + 1)) \approx (2L + 1)/\pi n^2 \) for large \( L \). By comparison with his results it is clear that in general, on hypercubic lattices the confluent singularities in the finite size scaling of \( \langle \psi \rangle \) are \( L^1 \), \( L^0 \) etc.

### 2.2 One dimension again

In the following, a few particular results deriving from Eq. (9) are highlighted. In one dimension,

\[
2 \langle \psi \rangle_{1, a} (L) \approx 2 \frac{L^2}{L^2} \sum_{n=1, odd}^{\infty} \frac{4}{q_n^2} \frac{4 L^2}{\pi^2 n^2} = \frac{L^2}{6} \tag{10}
\]

using \( \sum_{n=1, odd}^{\infty} 1/n^4 = \pi^4/96 \) [16, Secs. 1.471 and 1.647], consistent with Eq. (4).

Sums of this type frequently occur in finite temperature field theory under the label of Matsubara sums [17]. The latter is associated with the technique of representing the sum as one over residues,

\[
2 \sum_{n=1, odd}^{\infty} \frac{1}{n^4} = \frac{1}{2\pi^2} \int_C \frac{1}{z^4} \frac{-i \pi}{1 + \exp(i \pi z)} \tag{11}
\]

where the contour \( C \) (see Figure 1) encircles each (simple) pole of \( -i \pi/(1 + \exp(i \pi z)) \), which are located at \( z = n \) and \( z = -n \) (\( n \) odd; the parity symmetry is the origin of the factor 2 on the left) and have residue 1. Merging the contours for \( z = q_n \) and \( z = -q_n \) and deforming the resulting two contours to enclose the single pole of order 4 at \( z = 0 \) produces the desired result, as the contour has negative orientation and the residue is \(-\pi^4/48\).

It is instructive to attempt to recover Eq. (2), which is twice the avalanche size for a system driven at site \( x = a \). In that case, the uniform drive, \( f_0^L dx_0 (1/L) \), has to be replaced by a single source at \( a \), i.e. \( f_0^L dx_0 \delta(a - x_0) \), so that the average avalanche size for a system driven at \( a \) is

\[
2 \langle \psi \rangle_{1, a} (L) \approx \frac{2}{L^2} \sum_{n=1, odd}^{\infty} \frac{4}{q_n^2} \approx \frac{8La}{\pi^3} \sum_{n=1, odd}^{\infty} \sin \left( \frac{n \pi a}{L} \right) \frac{1}{n^3} \tag{12}
\]

Clearly the terms in the sum contribute significantly less for large \( n \). For small \( n \) and large \( L \), the sin may be approximated by its argument, producing

\[
\frac{L}{a} \sum_{n=1, odd}^{\infty} \sin \left( \frac{n \pi a}{L} \right) \frac{1}{n^3} \approx \pi \sum_{n=1, odd}^{\infty} \frac{1}{n^2} = \frac{\pi^3}{8} \tag{13}
\]

and thus

\[
2 \langle \psi \rangle_{1, a} (L) \simeq \frac{8 La}{\pi^2} \tag{14}
\]

as in Eq. (2). One may be tempted to consider the sum as a Riemann sum with mesh \( 1/L \),

\[
\frac{L}{a} \sum_{n=1, odd}^{\infty} \sin \left( \frac{n \pi a}{L} \right) \frac{1}{n^3} \approx \frac{\pi a}{L} \int_0^\infty du \sin(u) \frac{1}{u^3} \approx \pi \tag{15}
\]

with dummy variable \( u = an \pi / L \). In the last step, the integrand has been approximated by \( 1/u^2 \) valid at small \( u \). In this approximation, the avalanche size is

\[
2 \langle \psi \rangle_{1, a} (L) \approx \frac{8 La}{\pi^2} \tag{16}
\]

a rather poor approximation compared to Eq. (14).
In two dimensions, the same procedures can be followed. For uniform driving, the key sum to perform is

\[
\sum_{n,m=1,\text{odd}}^{\infty} \frac{1}{n^2m^2(n^2+m^2)} = \sum_{m=1,\text{odd}}^{\infty} \frac{\pi^2}{8m^4} - \frac{\pi \tanh(\pi m/2)}{4m^5} = \frac{\pi^6}{768} - \sum_{m=1,\text{odd}}^{\infty} \frac{\pi \tanh(\pi m/2)}{4m^5} \quad (17)
\]

While the author was unable to determine the last sum (relevant literature [18, 16, 19]), it converges extremely quickly because of the large power of \(m\) in the denominator and because \(\tanh\) very quickly approaches 1. With the help of Mathematica [20], one finds

\[
\sum_{m=1,\text{odd}}^{\infty} \frac{\tanh(\pi m/2)}{m^5} = 0.9216754342259668\ldots \quad (18)
\]

to be compared to \(\tanh(\pi/2) = 0.91715233\ldots\) and \(\tanh(\pi/2) + \tanh(3\pi/2)/3^5 = 0.92126688\ldots\). Using the numerical estimate Eq. (18), the average avalanche size in two dimensions with uniform driving in the bulk and open boundaries is

\[
2 \langle s \rangle_2(L) \simeq C_2 \frac{64}{D\pi^6} L^2
\]

with \(C_2 = 0.5279266525115576573\ldots\) and \(D = 1/4\), so that

\[
\langle s \rangle_2(L) \simeq 0.070288507477576867 \ldots L^2.
\]

Extensions of the result above to systems with non-unity aspect ratio \(r = L_x/L_y\) are straightforward. The sum to be performed is

\[
2 \langle s \rangle_2(L_x,L_y) = \frac{64}{D\pi^6} L_x L_y \sum_{n,m=1,\text{odd}}^{\infty} \frac{1}{n^2m^2(rn^2+r^{-1}m^2)}
\]

and thus

\[
2 \langle s \rangle_2(L_x,L_y) = \frac{64}{D\pi^6} L_x L_y \left\{ \frac{\pi^6}{768r} - \sum_{m=1,\text{odd}}^{\infty} \frac{\pi \tanh(\pi mn/2)}{4r^4m^5} \right\}
\]

which according to Eq. (21) is invariant under a change of \(r\) to \(r^{-1}\). While this is not at all obvious in Eq. (22), the sum is somewhat reminiscent of that in [16, Sec 1.471.3]. In the limit of large \(r\), the first term in the curly bracket dominates, producing

\[
2 \langle s \rangle_2(L_x,L_y) = L_y^2/(12D),
\]

as the system is essentially one-dimensional (except for \(D=1/(2d)\), due to the additional degree of freedom). For very small \(r\) the sum may be treated as a Riemann sum.

### 3.1 Mixed boundary conditions

If any of the boundaries is periodically closed or made reflecting, the dimension (i.e. this degree of freedom) effectively disappears from the problem, i.e. the avalanche size is essentially that of a one-dimensional system. The only trace that remains of the “closed dimension” is hidden in the diffusion constant, which is \(D = 1/(2d)\), so that

\[
\langle s \rangle_{2,\text{cyl}} = \frac{L^2}{6} + O(L)
\]

(23)

for cylindrical boundary conditions on square lattices, \(d = 2\). In Eq. (23) the sub-leading terms are indicated as well, because Eq. (4) remains exact, i.e.

\[
\langle s \rangle_{2,\text{cyl}} = \frac{(L+1)(L+2)}{6}
\]

(24)

and obviously in higher dimensions

\[
\langle s \rangle_{d,\text{cyl}} = \frac{(L+1)(L+2)d}{12}
\]

(25)

is the exact expected avalanche size on a hypercubic lattice if only one direction remains open, while \(d - 1\) directions are either periodically closed or reflecting (or, more generally, produce a spectrum containing 0).
The technical reason for the simplicity of the results with cylindrical boundary conditions is the presence of only one sum. The other sums do not occur because the integration over the entire system as well as the integration over uniform drive effectively projects the eigenfunctions of the PDF on a constant, using the scalar product with constant weight. Under that weight $\nabla^2$ is self-adjoint and the constant has eigenvalue 0, provided it is an eigenfunction (which is decided by the boundary conditions). In that case, the $q_0 = 0$ mode is selected in the sum, so that neither any factor $1/q_n$ appears nor a contribution in $1/(q_n^2 + q_m^2 + \ldots)$. The mode with eigenvalue 0 does not decay in time, i.e. it is conserved. A boundary condition that leads to conservation is thus expected to possess such a 0-mode.

The same type of argument therefore applies in higher dimensions. For example, when applying periodic boundary conditions to $d - 2$ directions in hypercubic lattices with $d > 2$, the average avalanche size is essentially that of a two-dimensional system, except for $D$ being changed to $D = 1/(2d)$.

If individual boundaries have mixed conditions, calculations become drastically more complicated. An interesting example is a setup where all boundaries of a two-dimensional lattice are reflecting except for a narrow hole of fixed size $h$ from where all particles added have to escape. Rather counter-intuitively, the scaling of the escape time in $L$ is not very different from the scaling on the open lattice, even when the size of the hole is kept finite and fixed as $L$ is increased. This is surprising, as the particles need to “find the narrow exit” in an increasingly large system — given the presence of an additional length scale (the size of the hole) the scaling of the escape time is no longer determined by dimensional consistency. On the other hand, one may argue that the situation is not much different from a one-dimensional lattice, where the size of the exit remains constant as well.

On the basis of published results on the narrow escape problem [21–23], one finds

$$2 \langle s \rangle_{2,\text{fixed}} \simeq \frac{1}{D} \sum_{n,m=1,\text{odd}}^{\infty} \frac{2}{L} \sin(q_n x_0) \frac{2}{q_n} L \sin(q_m y_0) \frac{2}{q_m} \frac{1}{q_n^2 + q_m^2},$$

(27)

where $\sin(n\pi x_0/L)$ can be approximated by its argument as the terms in the sum vanish at least like $1/n^3$ in large $n$. For small $n$ the resulting sum is divergent in the upper limit, which has to be replaced by the ultraviolet cutoff $L/a$ with lattice spacing $a$,

$$2 \langle s \rangle_{2,\text{fixed}} \approx \frac{16x_0 y_0}{\pi^2 D} \sum_{n,m=1,\text{odd}}^{L/a} \frac{1}{n^2 + m^2}.$$

(28)

The final result hinges on the last sum. One of the summations can be performed beyond the upper cutoff without causing a divergence. The resulting summation involves a term of the form $\tanh(m^2/2m)$, which may be approximated by $1/m$ and thus the sum by $(\pi/8) \ln(L/(2a))$, so that

$$2 \langle s \rangle_{2,\text{fixed}} \approx \frac{2x_0 y_0}{\pi D} \ln(L/(2a)).$$

(29)

The rôle of the upper cutoff becomes clearer in the case $d_0 > 2$, for example fixing the driving position on a three-dimensional lattice. The reason why the expected escape time remains finite even in the thermodynamic limit is because within a finite time the random walker, attempts to travel beyond the finite distance to one of the open boundaries, thus leaving the lattice. Without a finite lattice spacing, the number of “hops” to the open boundary, however, diverges. The difference between thermodynamic and continuum limit is that absolute distances correspond to a fixed number of hops in the former, but not in the latter. From a physical point of view, there is in fact no other difference between the two.

4 Higher dimensions

In higher dimensions the calculation of the relevant sums becomes increasingly computationally demanding. The expected avalanche size for homogeneous drive in a $d$ dimensional hypercubic system with open boundaries generally is according to Eq. (9).

$$2 \langle s \rangle_d (L) \simeq \frac{2dL^2}{\pi^2} \left( \frac{8}{\pi^2} \right)^d C_d$$

(30)

The walk in one dimension can be interpreted as a projection from two dimensions.
where $D = 1/(2d)$ has been used and

$$C_d = \sum_{n_1, n_2, \ldots, n_d=0}^{\infty} \frac{1}{\prod_{i=1}^{d} (2n_i + 1)^2 \sum_{i=1}^{d} (2n_i + 1)^2} .$$

(31)

One of the summations can always be carried out, Eq. (17). Keeping only the two lowest order terms in the resulting sum produces a recurrence relation for $d > 1$,

$$C_d \approx \frac{\pi^2}{8} C_{d-1} - \frac{\pi}{4(d-1)^{(3/2)}} \tanh \left( \frac{\pi}{2} \sqrt{d-1} \right) - \frac{\pi(d-1)}{36(d+7)^{(3/2)}} \tanh \left( \frac{\pi}{2} \sqrt{d+7} \right)$$

(32)

and $C_1 = \pi^4/96$ exactly. Table 1 contains the numerical evaluation of the constants $C_d$ according to Eq. (31) together with the approximation Eq. (32). The amplitude in the last column are well consistent with recent numerical results on the Manna Model [2,3].

5 Arbitrary Adjacency

Eq. (1) points to a more general procedure to calculate the expected number of moves to escape from the lattice. If $|m\rangle$ is a vector whose components $m_i$ are the expected escape times starting from site $i$ and $A$ is the weighted adjacency matrix (closely related to Dhar’s toppling matrix [15], also discussed by Stapleton [12]), proportional to the lattice Laplacian, containing $A_{ii} = -1$ across the diagonal and $A_{ij}$ being the probability of $i$ discharging to $j$ (i.e. $A_{ij} = 1/(2d)$ on hypercubic lattices),\footnote{Because $A$ does not have to be symmetric, the procedure described here covers directed models as well.} then

$$-|1\rangle = A|m\rangle$$

(33)

where $|1\rangle$ is a column of ones. Dissipation at boundary sites is implemented by $\sum_i A_{ij} < 0$, while $\sum_i A_{ij} = 0$ at (conservative) bulk sites. The presence of the non-conservative sites means that $|1\rangle$ is not an eigenvector, in fact $A|1\rangle$ is a vector with components that are 0 for each conservative (bulk) site and negative for all dissipative (boundary) sites. If $|d\rangle$ is a vector whose components $d_i$ are the probability that a particle is deposited at site $i$ by the external drive, with normalisation $\langle d|1\rangle = 1$, then

$$2 \langle s \rangle = \langle d|m\rangle = -\langle d|A^{-1}|1\rangle$$

(34)

provided the inverse $A^{-1}$ of $A$ exists. If $A$’s eigenvectors $|e_i\rangle$ and $|e_i\rangle$ (not necessarily transposed relative to each other, as $A$ may be directed, i.e. not symmetric), with eigenvalues $\lambda_i$ and $\langle e_i|e_j\rangle = \delta_{ij}$, span a subspace containing $\langle d|\rangle$ and $|1\rangle$ respectively, so that

$$\langle d| = \sum_i u_i \langle e_i|$$

(35a)

$$|1\rangle = \sum_i w_i |e_i\rangle ,$$

(35b)

then

$$2 \langle s \rangle = -\sum_i \langle d|e_i\rangle \lambda_i^{-1} \langle e_i|1\rangle = -\sum_i \frac{u_i w_i}{\lambda_i} .$$

(36)

For uniform drive $d_i = 1/N$ in a system with $N$ sites and so $N \langle d\rangle = \langle 1\rangle$ is a row of ones. In that case, if $A$ is symmetric $u_i = w_i/N$ and

$$2 \langle s \rangle = -\frac{1}{N} \sum_i \langle e_i|1\rangle^2 \lambda_i$$

(37)

6 Relation to field theory

There is a subtle but very important link between the calculations performed above and the field theory of the Manna model [24]. Prima facie, it may look accidental that the calculations for the expectation of the escape time of a random walker are identical to those for the expected activity integral. In fact, the bare propagator for the activity at $\omega = 0$ (vanishing frequency, as obtained after Fourier transforming the time domain) is identical to that of the time-dependent PDF of the random walker particle. However, while the former describes the spreading of activity on the microscopic time scale of the Abelian Manna Model [6,10] subject to Poissonian updates (activated random walkers [25]), the latter describes the movement of a particle on the conditional time scale, which advances only when the particle is not stuck on the lattice. Only on that time scale, an actual random walk is performed and the link exists between the number of moves and the residence time.

In the light of the field theory, however, it is clear that the particle movement on the conditional time scale is exactly identical to the spreading of activity: particles moving are active and vice versa. The fact that the average avalanche size can be determined by the considerations presented above means that the bare propagator at $\omega = 0$ is not renormalised at any order. That does not imply that the bare propagator is not renormalised at all, as the statement merely applies to $\omega = 0$. In fact, the time dependence of the propagator is very much expected to be affected by interaction, fluctuations and thus renormalisation, because active particles do not move freely like a random walker, but interact with the particles at rest.

The reason why no renormalisation of the propagator at $\omega = 0$ takes place is the same reason that allows the calculation of the average avalanche size in the first place: Conservation of particles and stationarity: in the stationary state and because of conservation, on average exactly one particle leaves the system per particle added. The number of moves performed by a particle during its residence determines the average avalanche size.

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\[
d \quad C_d \text{ (numerically)} \quad C_d \text{ (approximation Eq. (32))} \quad (\frac{d}{\pi^2})(\frac{8}{\pi^2})^d C_d
\]

| \(d\) | 1.0146780... | 1.0146780... | 0.0833333... |
| 1   | 0.5279266... | 0.5282475... | 0.0702885... |
| 2   | 0.3737684... | 0.3749565... | 0.0650544... |
| 3   | 0.3026980... | 0.3056300... | 0.0529579... |
| 4   | 0.2651000(1)  | 0.2707675... | 0.0469927(4) |
| 5   | 0.205279266... | 0.2146780... | 0.0469927(4) |

Table 1. The constant \(C_d\), Eq. (31), for dimension \(d = 1, 2, \ldots, 5\). The second column shows the numerical evaluation of the sum (with extended double precision, summing up to \(2 \cdot 1000 + 1\) for \(d = 1, 2, 3\), up to \(2 \cdot 500 + 1\) for \(d = 4\) and up to \(2 \cdot 200 + 1\) for \(d = 5\)). Unless an error is stated, the digits shown display convergence. The third column is the recursive approximation Eq. (32). The last column is the amplitude of the leading order \(L^2\) of the average avalanche size, Eq. (30).

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