Boundary Conditions, Energies and Gravitational Heat in General Relativity
(a Classical Analysis)

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Abstract

The variation of the energy for a gravitational system is directly defined from the Hamiltonian field equations of General Relativity. When the variation of the energy is written in a covariant form it splits into two (covariant) contributions: one of them is the Komar energy, while the other is the so-called covariant ADM correction term. When specific boundary conditions are analyzed one sees that the Komar energy is related to the gravitational heat while the ADM correction term plays the role of the Helmholtz free energy. These properties allow to establish, inside a classical geometric framework, a formal analogy between gravitation and the laws governing the evolution of a thermodynamical system. The analogy applies to stationary spacetimes admitting multiple causal horizons as well as to AdS Taub–bolt solutions.

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1 Introduction

One of the best–known classical methods to derive the gravitational Hamiltonian from the Hilbert Lagrangian is based on Noether theorem [61, 63]; it relies on the identification of the Hamiltonian density with the Noether charge density associated with a timelike vector field $\xi$. The Hamiltonian is then obtained after integration of the Noether current on a (portion) of a Cauchy hypersurface $\Sigma$ in spacetime. It is also well–known [26, 30, 32, 49, 63] that the Hamiltonian obtained in this way splits into a volume integral on $\Sigma$ (which turns out to be vanishing on shell being proportional to the Hamiltonian constraints) and a surface integral on $\partial \Sigma$ which is nothing but the integral of the Noether superpotential, i.e. the so–called Komar potential [52, 64]. Therefore the gravitational energy, defined as the value on–shell of the Hamiltonian, just reduces to the

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integral of the Komar form. This simple definition of energy suffers however of at least one remarkable drawback. Already when we deal with simple solutions, e.g. asymptotically flat stationary spacetimes, it is well known that Noether techniques allow one to obtain the expected value for the angular momentum but only one–half of the ADM mass. Accordingly, the energy is affected by the so–called anomalous factor \(1/2\). A way to overcome the problem was suggested in \([30, 48, 49]\) (and references therein) and it basically consists in correcting the Komar potential by adding to it a new (covariant) background–dependent term which allows to cure the anomalous factor. The final expression we end up with is nothing but the Noether superpotential associated to the so–called first order covariant Lagrangian \([17, 25, 26, 28, 30, 48, 49]\), obtained from the Hilbert Lagrangian through the addition of a divergence term with an arbitrary non–dynamical background dependence.

The technique of introducing a suitable background dependence into the Lagrangian and the superpotential has been shown to work also for spacetime solutions other than the asymptotically flat ones and for generic spacetime dimensions (see \([18, 22, 23, 24, 26, 49, 50, 55]\)) and this naturally leads to the interpretation of the corresponding Noether energy as the energy of each solution relative to the chosen background.

In this paper, following a thermodynamical–inspired approach \([14, 34, 36, 54]\), we suggest that the anomalous factor of the Komar potential is, actually, by no means “anomalous”. On the contrary what is anomalous is the interpretation of Mass/energy we attribute to it. Indeed the Komar potential turns out to be the right candidate to describe another kind of gravitational energy, which we refer to as the gravitational heat.

The theory of classical thermodynamics can be of great help in understanding this issue. A thermodynamic system is endowed with an internal energy \(E\). Along a (reversible) thermodynamic process the internal energy can be converted into mechanical work \(L\) and into heat \(Q\) according to the first law of thermodynamics \(\delta E = \delta Q - \delta L\). The amount of energy available to do work in a reversible transformation (with initial and final temperature both equal to \(T\)) is the free energy \(F\), while the difference \(E - F = TS \) (\(S = \) entropy) is that part of energy which inevitably transforms into heat; see \([27]\).

In the last thirty years a close relationship between the law of thermodynamics and black hole mechanics has been established, initially only on a formal basis by Bekenstein and Hawking (see \([6, 8, 38]\)) and subsequently on a more physical ground by Hawking’s discovery of the black hole radiation; \([36, 39]\). Therefore, nowadays we have convincing arguments to state that black holes are truly thermodynamic systems (see \([65]\) for a review on the matter). As such, we should be able to attribute to them different kinds of energies, such as the internal energy, free energy and gravitational heat. According to this per-
spective, the gravitational entropy was calculated in \[54\] from the gravitational heat, suitably defined as the difference between the internal energy (obtained via a Brown–York procedure from an action functional with boundary counterterms) and the free energy defined semi–classically as the value on shell of the action functional itself.

In a completely classical (and therefore different) framework we shall give here a geometric definition of internal energy and gravitational heat starting from a unique generating formula which relates the variation of the Hamiltonian with the (pre–)symplectic form of standard General Relativity. Internal energy variations $\delta E$ will be obtained from the variation on–shell of the Hamiltonian. When we rewrite $\delta E$ in a covariant form it splits into two contributions. One of them, namely the variation of the Noether potential, is the only one surviving once Neumann boundary conditions on $\delta E$ are imposed; it then corresponds to the Komar energy. The second contribution entering into $\delta E$ is instead the so–called covariant ADM correction term \([22, 25, 29, 44, 45]\).

When dealing with stationary spacetimes admitting (observer–dependent) horizons (hereafter we shall be more precise on the notion of horizon) the Komar energy turns out to be the gravitational heat $TS$ (for suitable geometrically defined quantities $T$ and $S$) while the covariant ADM correction term is related to the free energy of the system.

The idea to relate somehow entropy with Noether charges is surely not new; see e.g. \([44]\) (see also \([11]\) where the same idea was worked out within path integral methods). We stress however that our definition of entropy differs from the original one due to Wald and it is closest in spirit with Mann’s prescription \([20, 34]\). In this respect we stress that, even if throughout the paper we shall deal with the Noether charge, nowhere in the text we shall make in fact use of Noether theorem.

We also stress that the identification of the Komar energy with the gravitational heat provides just a geometric definition of entropy with no rigorous statistical and/or thermodynamical interpretation (even though it turns out to coincide with its semi–classically formulation based on path–integral techniques; see \([11, 13, 35]\)). Nevertheless it is supported by a great number of applications. Just to mention the most trivial of them let us consider a Schwarzschild black hole of mass $M$. Its temperature $T$ is equal to $\frac{1}{8\pi M}$ while its entropy $S$ (in fundamental units $G = \hbar = c = k_B = 1$) is $4\pi M^2$, namely one–quarter of the horizon area. Hence the gravitational heat $TS$ turns out to be $\frac{4\pi}{M}$ which is exactly the Komar superpotential computed on any surface enclosing the horizon (and this explains the “anomalous” factor $1/2$). This – only apparently – lucky coincidence extends to less trivial black hole solutions and, more generally, to all stationary spacetimes admitting any number of causal horizons \([37, 40, 59]\) (including, beyond stationary black hole horizons, also cosmological and Rindler accelerated horizons, i.e. horizons which are not associated with black holes). Remarkably enough the suggested definition of gravitational heat allows one to simply compute the entropy commonly attributed to Taub–Bolt and Taub-nut solutions \([18, 24, 34, 43, 53, 54]\), whatever the physical meaning is of entropy for these solutions.
According to our point of view entropy turns out to be a geometric quantity the cohomological properties of which can be physically translated into the first law of thermodynamics. In agreement with [24], entropy then arises when an obstruction exists to globally foliating spacetime into surfaces of constant time. In other words, entropy turns out to be closely related to the coordinate singularities of the solution describing spacetime. Being coordinate singularities a signal that there exist regions of inaccessibility for observers which are at rest in the given coordinate frame, entropy eventually turns out to be related to the existence of regions hidden to the observers. This perspective is close in spirit with Padmanabham’s recent and interesting work relating entropy with unobserved degrees of freedom; see [59]. In agreement with Shannon’s original formulation of the theory of information, entropy encodes thence the information content hidden beyond inaccessible regions. Not surprisingly our definition of entropy agrees with the definition given in [59] for static spacetimes, thus endowing Padmanabhan’s physically–motivated construction with a mathematical ground and a much broader domain of applicability.

The paper is organized as follows. Starting from Hamilton’s field equations, in section 2 we derive the relevant expression for the variation $\delta E$ of the energy. We rewrite $\delta E$ into an explicit covariant form and, afterwards, we focus our attention on boundary conditions. Depending on the choice made between Dirichlet or Neumann boundary conditions we obtain different definitions of quasilocal energies, i.e. Dirichlet energy or Neumann energy. The physical interpretation of such boundary conditions is carried out in section 3. It is here recognized that, in spacetimes admitting causal horizons, the Komar energy is related to the gravitational heat, while the ADM correction term plays the role of the free energy. Moreover, the cohomological properties the variational formula for $\delta E$ is endowed with allow one to relate the entropy to homological obstructions in spacetime. We generalize this idea in section 4. We suggest, in complete agreement with [54], that entropy arises whenever an obstruction exists in foliating spacetime into space + time, thus corroborating previous results with an alternative mathematical viewpoint. Moreover, the interplay between boundary conditions and the choice of a reference background is discussed. As a good example, in section 5 we analyse the AdS Taub–bolt solution and we reproduce within our formalism the same numerical values found elsewhere by means of different techniques.

The appendix A deals with spherically symmetric solutions with a cosmological constant (Schwarzschild–deSitter) or with generic external matter fields. This appendix is introduced in order to test on a simple model the main formulae introduced throughout the paper, thus endowing them with a direct physical interpretation. Finally, in appendix B we carry out the analogy between the classical geometric framework developed in this paper and the semi–classical statistical approach based on path integral techniques.
2 The Generating Formula and Boundary Conditions

Let us consider, in a Lorentzian spacetime \( M \) of dimension 4, a region \( D \) that is diffeomorphic to the product \( \Sigma \times \mathbb{R} \) where \( \Sigma \) is a 3–dimensional manifold with a sufficiently regular boundary \( B = \partial \Sigma \). \(^2\) We denote this diffeomorphism by

\[
\psi : \Sigma \times \mathbb{R} \rightarrow D
\]  

(1)

For any \( t \in \mathbb{R} \) a hypersurface \( \Sigma_t \subset D \) is induced by \( \psi \) according to the rule \( \Sigma_t = \psi(\Sigma \times \{t\}) = \psi_t(\Sigma) \) and we require it to be a (portion) of a spacelike Cauchy hypersurface. The set of all \( \Sigma_t \), for \( t \) in \( \mathbb{R} \), defines a foliation of \( D \) labelled by the time parameter \( t \). Moreover, each \( \Sigma_t \) intersects the boundary \( \partial D \) in a 2–dimensional surface \( B_t \) which is diffeomorphic to \( B \), for all \( t \) in \( \mathbb{R} \). The diffeomorphism is established by the map \( \psi_t : B \rightarrow B_t \). Hence the boundary \( B = \partial D \) is a timelike hypersurface globally diffeomorphic to the product manifold \( B \times \mathbb{R} \) and, physically speaking, it describes the histories of the observers located on \( B_t \). The time evolution field \( \xi \) in \( D \) is defined through the (local) rule \( \xi^\mu \nabla_\mu t = 1 \) and, on the boundary \( B \), it is tangent to the boundary itself. We denote by \( u^\mu \) the future directed unit normal to \( \Sigma_t \) and we denote by \( n^\mu \) the outward pointing unit normal of \( B_t \) in \( \Sigma_t \). For simplicity we assume the foliation to be orthogonal so that the vector \( n^\mu \) is also the unit normal of \( B \) in spacetime \( M \). Accordingly, everywhere on \( B \) it holds true that \( u^\mu n^\nu|_B = 0 \).

The evolution vector field can be decomposed as

\[
\xi^\mu = N u^\mu + N^\mu
\]  

(2)

where \( N \) is the lapse and the shift vector \( N^\mu \) is tangent to the hypersurfaces \( \Sigma_t \), i.e. \( N^\mu u_\mu = 0 \) and \( N^\mu n_\mu|_B = 0 \). The metrics induced on \( \Sigma_t \), \( B \) and \( B_t \) by the metric \( g_{\mu\nu} \) are given, respectively, by:

\[
h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu
\]  

(3)

\[
\gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu
\]  

(4)

\[
\sigma_{\mu\nu} = h_{\mu\nu} - n_\mu n_\nu = \gamma_{\mu\nu} + u_\mu u_\nu
\]  

(5)

The metrics (3), (4) and (5) with an index raised through the contravariant metric \( g^{\mu\nu} \) define the projection operators in the corresponding surfaces. We also denote by

\[
K_{\mu\nu} = -h^{\alpha}_\mu \nabla_\alpha u_\nu
\]  

(6)

\[
\Theta_{\mu\nu} = -\gamma^{\alpha}_\mu \nabla_\alpha n_\nu
\]  

(7)

\[
\mathcal{K}_{\mu\nu} = -\sigma^{\alpha}_\mu D_\alpha n_\nu
\]  

(8)

the extrinsic curvatures, respectively, of \( \Sigma_t \) in \( M \), of \( B \) in \( M \) and of \( B_t \) in \( \Sigma_t \). The symbol \( D \) denotes the (metric) covariant derivative on \( \Sigma_t \) compatible with \( h_{\mu\nu} \).

\(^2\)Even though the formalism can be generalized to any spacetime dimension > 2, for the sake of simplicity we assume \( \dim M = 4 \).
We shall denote by \( K = K_{\mu\nu}h^{\mu\nu} \), \( \Theta = \Theta_{\mu\nu}\gamma^{\mu\nu} \) and \( K = K_{\mu\nu}\sigma^{\mu\nu} \) the traces of the appropriate extrinsic curvatures. The momentum \( P^{\mu\nu} \) of the hypersurface \( \Sigma_t \) is defined as:

\[
P^{\mu\nu} = \frac{\sqrt{h}}{2\kappa} (Kh^{\mu\nu} - K_{\mu\nu})
\]

while for the momentum \( \Pi^{\mu\nu} \) of the hypersurface \( B \) we set:

\[
\Pi^{\mu\nu} = -\frac{\sqrt{\gamma}}{2\kappa} (\Theta_{\gamma\mu\nu} - \Theta_{\mu\nu})
\]

Let us consider the projection of Einstein’s equations (in vacuum or with a cosmological constant) into their normal and tangential components relative to the leaves \( \Sigma_t \) of the foliation:

\[
\mathcal{H} = 0 \quad (11)
\]
\[
\mathcal{H}_\alpha = 0 \quad (12)
\]
\[
\mathcal{L}_\xi h_{\mu\nu} = [h_{\mu\nu}] \quad (13)
\]
\[
\mathcal{L}_\xi P^{\mu\nu} = [P^{\mu\nu}] \quad (14)
\]

where \( \mathcal{H}, \mathcal{H}_\alpha \) are the usual Hamiltonian constraints \[57\], \( [h_{\mu\nu}] = \frac{2\sqrt{\kappa}}{\sqrt{\gamma}}(2P^{\mu\nu} - h_{\mu\nu}P) + 2D_{(\mu}N_{\nu)} \) and \( [P^{\mu\nu}] \) shortly denote the right hand side of the Hamilton equations, the detailed expression of which is of no primary interest here and can be found in \[57\]. Notice that (13) is nothing but the definition of the momentum \( P^{\mu\nu} \) conjugated to the spatial metric \( h_{\mu\nu} \), i.e. the Legendre transformation, while equations (11), (12) and (14) correspond, respectively, to Einstein’s equations \( G^{\mu\nu}u_\mu u_\nu, \quad G^{\mu\nu}h_{\alpha\mu}u_\nu \) and \( G^{\alpha\beta}h^\alpha h^\beta \gamma \) written in a first–order formalism in terms of the conjugated fields \( h_{\mu\nu} \) and \( P^{\mu\nu} \).

Let us now denote by \( X = \delta g_{\mu\nu}\frac{\partial}{\partial g_{\mu\nu}} \) a vertical vector field in the configuration bundle of the Lorentzian metrics (in the sequel we shall shortly denote with \( \delta y \) the variation induced on a generic metric–dependent field \( y \) by the variation \( X \) of the metric itself). If we multiply the equations (11)–(14) with, respectively, \( \delta N, \delta N_\alpha, \delta P^{\mu\nu} \) and \( \delta h_{\mu\nu} \), we sum up all the obtained expressions and we suitably rearrange the terms, we finally obtain that the Hamiltonian generating equations (11)–(14) is constrained by the variational formula:

\[
\delta_X H(\xi, \Sigma_t) = \Omega(\Sigma_t, X, \mathcal{L}_\xi g)
\]

where:

\[
\delta_X H(\xi, \Sigma_t) = \int_{\Sigma_t} \left\{ \mathcal{H} \delta N + \mathcal{H}_\alpha \delta N_\alpha + [h_{\mu\nu}]\delta P^{\mu\nu} - [P^{\mu\nu}]\delta h_{\mu\nu} \right\} \text{d}^3 x
\]

denotes the variation of the Hamiltonian along \( X \), while

\[
\Omega(\Sigma_t, X, \mathcal{L}_\xi g) = \int_{\Sigma_t} \omega(X, \mathcal{L}_\xi g) = \int_{\Sigma_t} \left\{ (\mathcal{L}_\xi h_{\mu\nu}) \delta P^{\mu\nu} - (\mathcal{L}_\xi P^{\mu\nu}) \delta h_{\mu\nu} \right\} \text{d}^3 x
\]
is the \((pre-)symplectic form\); see \[16\]. After a lengthy calculation (see, e.g. \[10, 15\]), expression \(16\) can be conveniently rewritten as:

\[
\delta_X H(\xi, \Sigma_t) = \delta_X \int_{\Sigma_t} \{N\mathcal{H} + N^\alpha \mathcal{H}_\alpha\} \, d^3x
\]

\[+ \int_{\partial \Sigma_t} d^2x \left\{ N\delta(\sqrt{\sigma} \epsilon) - N^\alpha \delta(\sqrt{\sigma} j_\alpha) + \frac{N\sqrt{\sigma}}{2} s^{\alpha\beta} \delta \sigma_{\alpha\beta} \right\}
\]

where

\[\epsilon = \frac{1}{\kappa} \kappa,
\]

\[\left( \kappa = \frac{1}{\delta \pi} \right)
\]

\[j_\alpha = -\frac{2}{\sqrt{\sigma}} \sigma_{\alpha\mu} \Pi^{\mu\nu} u_\nu
\]

\[s^{\alpha\beta} = \frac{1}{\kappa} \left[ (n^\mu a_\mu) \sigma^{\alpha\beta} - K \sigma^{\alpha\beta} + \kappa^{\alpha\beta} \right] (a_\mu = u^\nu \nabla_\nu u_\mu)
\]

are, respectively, the \textit{surface quasilocal energy}, the \textit{surface momentum} and the \textit{surface stress tensor} which describe the stress energy–momentum content of the gravitational field inside \(B_t\) (see \[10, 13, 14, 15\] and references quoted therein).

We point out that for non–orthogonal foliations of spacetime, i. e. \(u^\mu n_\mu|_B \neq 0\), an extra term has to be added in \(18\) which corresponds to a symplectic boundary structure; see \[10, 15, 33, 42, 51\].

We define the \textit{variation of the energy} contained in a region \(\Sigma_t\) enclosed by a surface \(B_t\) as the value on–shell of the \textit{variation of the} Hamiltonian. From \(15\) we obtain:

\[
\delta_X E(\xi, B_t) = \int_{B_t} d^2x \left\{ N\delta(\sqrt{\sigma} \epsilon) - N^\alpha \delta(\sqrt{\sigma} j_\alpha) + \frac{N\sqrt{\sigma}}{2} s^{\alpha\beta} \delta \sigma_{\alpha\beta} \right\}
\]

where now the vector field \(X = \delta g^\mu\nu \frac{\partial}{\partial g_{\mu\nu}}\) is meant to be a solution of the linearized field equations or, equivalently, it can be viewed as a vector tangent to the space of solutions (which in turn can be identified with the phase space once the suitable gauge reductions have been performed; see \[16\]). Moreover from \(15\) we have the on–shell relation:

\[
\delta_X E(\xi, \partial \Sigma_t) = \int_{\Sigma_t} \omega(X, \xi g)
\]

Hence, for stationary spacetimes which satisfy vacuum Einstein’s field equations (possibly with a cosmological constant), the 2–form that defines the variation of energy is a closed form. Indeed, the right hand side of \(22\) is vanishing when \(\xi\) is a Killing vector:

\[
\delta_X E(\xi, \partial \Sigma_t) = 0 \quad \text{if} \quad \mathcal{L}_\xi g = 0
\]

We stress that, once we fix \(B_t\) in the expression \(22\), there exist many "energies", depending both on the symmetry vector field \(\xi\) and on the variational
vector field $X$; see, e.g. \cite{1 3 7 14 33 51}. The former vector determines, via its flow parameter, how the observers located on $B_t$ evolve as the “time” flows. The vector field $X$ fixes instead the boundary conditions on the dynamical fields (e.g. Dirichlet or Neumann boundary conditions). Each choice of the pair $(\xi, X)$ gives rise to a particular realization of a physical system endowed hence with its own “energy content”.

**Remark 2.1** We stress that, despite the fact that the Hamiltonian formulation breaks down the covariance of the theory through the spacetime foliation into space+time, covariance can be nevertheless restored at the end. Indeed formula \cite{22} can be conveniently rewritten in an explicit covariant form as follows:

$$\delta_X E(\xi, B_t) = \int_{B_t} \frac{1}{2} U^{\alpha\beta}(\xi, X) ds_{\alpha\beta} (ds_{\alpha\beta} = i_\beta i_\alpha d^4x)$$

where

$$U^{\alpha\beta}(\xi, X) = \delta \left[ \frac{\sqrt{g}}{\kappa} \nabla^{[\beta} \xi^{\alpha]} \right] + \frac{\sqrt{g}}{\kappa} g^{\mu\nu} \delta u^{[\beta}_{\mu\nu} \xi^{\alpha]}$$

with $u^{\beta}_{\mu\nu} = \Gamma^\beta_{\mu\nu} - \delta^\beta_{(\mu} \Gamma^\nu_{\nu)\rho}$. The first term in the right hand side of \cite{26} is nothing but the variation of the Noether superpotential $U^{\text{Kom}}(\xi)$, namely the Komar potential, while the second term is the so-called covariant ADM correction term $U^{\text{CADM}}(\xi, X)$ (see \cite{22 29 31 33}).

Written in terms of variables adapted to the foliation they read, respectively, as follows \cite{25 33}:

$$\delta_X \int_{B_t} U^{\text{Kom}}(\xi) = \frac{1}{2\kappa} \int_{B_t} d^2 x \delta \left[ 2\sqrt{\sigma} u^\mu \Theta^\alpha_{\mu} \xi_{\alpha} \right]$$

and

$$\int_{B_t} U^{\text{CADM}}(\xi, X) = - \int_{B_t} d^2 x \gamma_{\mu\nu} \delta \Pi^{\mu\nu}$$

$$= \frac{1}{2\kappa} \int_{B_t} \sqrt{\sigma} d^2 x N \left[ 2\delta \Theta + \Theta^{\mu\nu} \delta \gamma_{\mu\nu} \right]$$

Taking into account the useful formulae \cite{14}:

$$\Theta_{\mu\nu} = K_{\mu\nu} + (n^\alpha a_\alpha) u_\mu u_\nu + 2\sigma_\alpha u_\nu + 2\sigma_{(\mu} u_{\nu)} K_{\alpha\beta} n^\beta$$

$$\delta \gamma_{\mu\nu} = -(2/N) u_\mu u_\nu \delta N - (2/N) \sigma_{(\mu} u_{\nu)} \delta N^\alpha + \sigma_{(\mu}^\rho \sigma_{\nu)}^\beta \delta \sigma_{\alpha\beta}$$

it easily follows that the sum of \cite{27} and \cite{28} corresponds to \cite{22}.

We just stress that it is indeed the splitting

$$\delta_X E(\xi, B_t) = \delta_X \left\{ \int_{B_t} U^{\text{Kom}}(\xi) \right\} + \int_{B_t} U^{\text{CADM}}(\xi, X)$$

\footnote{We point out that, if we had not assumed the foliation to be orthogonal, there would appear additional terms in the right hand side of formulæ \cite{27} and \cite{28} which however vanish when we assume $\xi$ to be a Killing vector. With the assumption of orthogonal boundaries these extra terms are nevertheless vanishing without any Killing requirement; see \cite{33 15}.}
which will come naturally into play when attempting to describe the gravitational system in terms of thermodynamical variables. Indeed the two contributions in \(31\) will be identified, respectively, with the \(\delta (TS)\) and \(\delta F\) contributions to the internal energy \(E\) (where \(F\) denotes the Helmholtz potential).

We also stress that \(31\) was obtained in \([16, 29, 31, 44]\) in a purely Lagrangian framework as the variation of the (covariantly conserved) charge associated to the generator of symmetries \(\xi\). From the explicit covariance of \(31\) it turns out that the variation of energy depends on the generator of symmetries \(\xi\), namely it depends on the observer, but it does not depend on the coordinate system.

We point out that the expression \(22\) can sometimes be integrated in specific applications, providing us with an explicit value of \(E\); see e.g. section \(5\) and Appendix \(A\). Nevertheless there is no guarantee in general that there really exists a function \(E\) such that its variation fulfills \(22\), even if the infinitesimal quantity \(\delta X E\) is always well defined in our hypotheses.

The problem we address now is to implement boundary conditions which allow formal integration of expression \(22\).

**Dirichlet energy.** We define the Dirichlet quasilocal energy \(\mathcal{E}(\xi, B_t)\) contained in the surface \(B_t\) to be the value obtained from \(22\) by imposing the Dirichlet boundary conditions \(\delta \gamma_{\mu \nu} |_{\mathcal{B}} = 0\). Since \(\delta \gamma_{\mu \nu} |_{\mathcal{B}} = 0\) implies \(\delta N |_{\mathcal{B}} = 0\), \(\delta N^\alpha |_{\mathcal{B}} = 0\) and \(\delta \sigma_{\mu \nu} |_{\mathcal{B}} = 0\), formula \(22\) can be explicitly integrated:

\[
\mathcal{E}(\xi, B_t) = \int_{B_t} d^2 x \sqrt{\sigma} \left( N (\epsilon - \epsilon_0) - N_{\alpha} (j_{\alpha} - j_{\alpha 0}) \right) + \mathcal{E}_0
\]  

(32)

We stress that we are integrating in the space of solutions along a curve of solutions (i.e. the integral curve of \(X\) through the point \(g\)) satisfying the boundary Dirichlet conditions. Therefore the constant of integration \(\mathcal{E}_0\) can be considered as the quasilocal energy associated to a point \(g_0\) in the curve and can be fixed as the reference point (or zero level) for the energy \(33\), namely

\[
\mathcal{E}(\xi, B_t) = \int_{B_t} d^2 x \sqrt{\sigma} \left\{ N (\epsilon - \epsilon_0) - N_{\alpha} (j_{\alpha} - j_{\alpha 0}) \right\}
\]  

(33)

where the subscript 0 refers to the background solution \(g_0\). Comparing \(33\) and \(22\) we obtain the relevant formula

\[
\delta_X E(\xi, B_t) = \delta_X \mathcal{E}(\xi, B_t) + \int_{B_t} \Pi^{\mu \nu} \delta \gamma_{\mu \nu} d^2 x
\]  

(34)

which obviously reduces to \(\delta \llbracket D \rrbracket E = \delta \llbracket D \rrbracket \mathcal{E}\) when we consider variations generated by a vector field \(X\) satisfying the Dirichlet boundary conditions \(\delta \llbracket D \rrbracket \gamma_{\mu \nu} = 0\).

We remark that formula \(32\) is nothing but the value on–shell of the Hamiltonian ensuing from the canonical reduction of the Trace-K Lagrangian \([15, 25, 41, 66]\).
When the observers located on $B_t$ evolve with velocity $\xi^\mu = u^\mu$ (i.e. $N = 1$, $N^\mu = 0$), the Dirichlet energy (33) reduces to the Brown–York quasilocal energy $\mathcal{E}_{BY}(\xi, B_t) = \int_{B_t} d^2 x \sqrt{\sigma} (\epsilon - \epsilon_0)$; see (13). The relationships between the energy (33) and the Brown–York energy have been analysed in [10, 14, 15] (see also [25]). We also point out that there is not a commonly accepted and preferred choice for the (variation of) energy of a gravitational system among $\delta \chi \mathcal{E}(\xi, B_t)$, $\delta \chi \mathcal{E}_{BY}(\xi, B_t)$ or $\delta \chi E(\xi, B_t)$. We nevertheless stress that both the Dirichlet expression $\delta \chi \mathcal{E}(\xi, B_t)$ and the Brown–York definition $\delta \chi \mathcal{E}_{BY}(\xi, B_t)$ are a particular case of the more general prescription (22). In the sequel we shall make use of this latter definition. Indeed, we remark that in all the specific solutions analysed so far (including Kerr–Newman, BTZ, Taub–bolt, isolated horizons solutions) expression (22) always gives rise to the expected numerical value; see [1, 22, 23, 24] and the appendix A.

**Neumann energy.** In deriving expression (32) we have selected the components of the boundary metric $\gamma_{\mu\nu}$ as the symplectic control parameters of the gravitational system [14, 51], while the response variables are the components of the boundary momentum $\Pi_{\mu\nu}$, namely the quasilocal surface energy $\sqrt{\sigma}\epsilon$, the quasilocal surface momentum $\sqrt{\sigma}j_\alpha$ and the surface pressure $\sqrt{\sigma}s_{\alpha\beta}$.

It was argued in [12, 14] that boundary conditions in General Relativity exactly correspond to boundary conditions of a thermodynamical ensemble and that dynamical fields which are conjugated to each other in a symplectic sense, can be also considered as being thermodynamically conjugate. Accordingly, different ensembles can be realized by exchanging the fields that are kept fixed as boundary data with their respective boundary conjugated fields. Under this point of view the variational vector field $X$ turns out to be a variation into the ensemble realized with specified boundary conditions.

In obtaining formula (32) we kept fixed the boundary 3–metric and we obtained the Dirichlet energy. The boundary fields which are symplectically conjugated to the boundary metric are the boundary momenta $\Pi_{\mu\nu}$ defined in (10). Hence we formally describe the gravitational system in another ensemble by taking the momenta as boundary fixed data. We shall now check if the new – Neumann – boundary conditions lead to a mathematically well–defined notion of energy, i.e. we will check whether formula (22) turns out to be integrable under the conditions $\delta \Pi^{\mu\nu}|_B = 0$. That this is indeed the case easily follows by observing that the covariant ADM potential (28) is vanishing if the much weaker condition

$$\gamma_{\mu\nu} \delta \Pi^{\mu\nu}|_B = 0$$

is imposed and, consequently, the contribution to the variation of energy comes entirely from the Komar potential, i.e.

$$\delta_{[N]} E(\xi, B_t) = \delta_{[N]} \int_{B_t} U_{Kom}(\xi)$$

where now $\delta_{[N]}$ refers to variations generated by a vector field $X$ satisfying the (weak) Neumann boundary condition (38). Therefore we end up with a new
energy, say the Neumann quasilocal energy $Q$, defined as

$$Q(\xi, B_t) = \int_{B_t} U_{\text{Kom}} + Q_0$$

$$= \frac{1}{\kappa} \int_{B_t} \sqrt{\sigma} d^2 x \left\{ N n^\mu a_\mu - N^\alpha K_{\alpha\beta} n^\beta \right\} + Q_0$$

(37)

where in the second equality we made use of (2), (27) and (29). The constant of integration $Q_0$ again refers to the energy of a (background) solution $g_0$ lying in the integral curve of $X$ through the point $g$.

Comparing now (37) with (22) we have:

$$\delta_X E(\xi, B_t) = \delta_X Q(\xi, B_t) - \int_{B_t} \gamma_{\mu\nu} \delta \Pi^{\mu\nu} d^2 x$$

(38)

Finally, from (34) and (38) we infer that

$$\delta_X E(\xi, B_t) = \delta_X \left\{ Q(\xi, B_t) - \int_{B_t} \Pi^{\mu\nu} \gamma_{\mu\nu} d^2 x \right\}$$

(39)

i.e., the Dirichlet quasilocal energy $E$ and the Neumann quasilocal energy $Q$ are related by a boundary Legendre transformation exchanging boundary metric with boundary momenta.

We shall now analyse to what extent we can refer to the energy (37) as the physical gravitational heat of a gravitational system.

### 3 Entropy of Causal Horizons

In this section we shall deal with 4–dimensional stationary spacetimes admitting one or more causal horizons; see (37, 46, 59). A causal horizon $H$ is defined as the boundary of the past of a (future inextensible) timelike curve which represents the observer’s worldline. This observer dependent definition encompasses as a special case all (observer independent) standard black hole horizons but it also includes observer dependent horizons such as the cosmological horizon as well as the Rindler horizon (i.e. the horizon tested by an accelerated observer in Minkowski spacetime). In a recent paper (46) the black hole laws of thermodynamics were extended to the more general setting of causal horizons by relying on the notion of local horizon entropy. We claim that the same local notion of entropy can be formulated in terms of the Komar energy (37), extending in this way Wald’s original definition (44) to a broader context.

In the previous section we gave the definition (22) for the (variation of the) internal energy contained in a spacelike region $\Sigma_t$ bounded by a 2–dimensional surface $B_t$. Let us now assume that the spacetime is stationary and that the region $\Sigma_t$ contains the cross section $H$ of a causal horizon $H$. Generalizing Wald’s original definition for stationary black holes, the horizon entropy $S$ can
be heuristically defined for any $H$ as that quantity satisfying the first law of thermodynamics:

$$\delta_X E(\xi, H) = T \delta_X S$$

(40)

If the temperature $T$ can be provided a priori by physical arguments, thence the function $S$ can be truly identified a posteriori with the physical entropy. Namely, expression (40) can be considered as an algorithm to define a quantity $S$ which, by its own definition, is related to the variation of energy in order that the first law holds true.

Let us therefore recall what a physically–motivated definition of temperature could be. In [37] it was shown that causal horizons in stationary spacetimes which satisfy Einstein’s equations in vacuum or with a cosmological constant (and possibly with an electromagnetic field) are necessarily stationary axisymmetric Killing horizons. Let us then identify the vector field $\xi$ with the vector field which coincides on the horizon with the null geodesic generator (if multiple horizons $H_i$ are present, each one is endowed with its own vector $\xi_i$ and two different vectors differ on any horizon for an axial Killing vector, the orbits of which are closed curves of length $2\pi$; see [37, 46]). In analogy with the black hole case, the temperature $T$ of a causal horizon $H$ is identified with the surface gravity, i.e.:

$$T = \frac{|\kappa_H|}{2\pi} = \left| \frac{n^\mu \nabla_\mu (N^2)}{4\pi} \right|_H = \left| \frac{N n^\mu a_\mu}{2\pi} \right|_H$$

(41)

which is constant on $H$ [46] and agrees with the Hawking temperature for black hole horizons [39] or cosmological horizons [37]. For spacetimes with a single horizon, the (global) temperature can be equivalently defined as the inverse of the period $\beta$ of the Euclidean time. Namely, when we approach closely to the horizon along a radial direction (let us suppose that the metric becomes spherically symmetric as it approaches the horizon$^4$) the complexified metric can be written in its Rindler form:

$$g \simeq R^2 \left( \frac{2\pi}{\beta} \right)^2 d\tau^2 + dR^2 + r^2(R)d\Omega$$

(42)

where $\tau = it$ is the complexified time, $d\Omega$ is the metric of the unit sphere and $\beta = 1/T$ is given by (41). Moreover

$$r(R) = \frac{\pi}{\beta} R^2 + r_+$$

(43)

where we have denoted by $r_+$ the radial position of the horizon. In the $\tau - R$ plane, the metric (42) becomes then regular on the horizon $R = 0$ if the complexified time $\tau$ has a period $\beta$; see e.g. [11, 18, 51, 59].

**Remark 3.1** In spacetimes with multiple horizons $H_i$ there is no global notion of time–periodicity. Basically, the neighborhood of each horizon can be covered by a system of coordinates in which the metric is regular on the horizon and $^4$A wider class of metrics will be considered in the next section.
which provides us with a well-defined horizon temperature \( T_i = 1/\beta_i \). However, in the spacetime regions covered by more than one coordinate system, different notions of temperature exist which in general are incompatible among them; see [52]. Despite of this, the definition of temperature still has a local meaning on each single horizon.

**Remark 3.2** Let us now consider a surface \( \Sigma \) of constant time with an outer boundary denoted by \( B \). Let us also suppose that \( \Sigma \) intersects a certain number of horizons \( H_i \) with intersection surfaces denoted by \( H_i \). The (variation of the) internal energy contained in \( \Sigma \) is given by \( \delta X E(\xi, B) \). Since we are dealing with stationary spacetimes and since the surface \( B \) is homological to the union of all \( H_i \) (i.e. \( B \cup_i H_i = \partial \Sigma \) is a homologic boundary) expression (24) holds then true. Therefore the variation of the internal energy receives a contribution from each cross section \( H_i \):

\[
\delta X E(\xi, B) = \sum_i \delta X E(\xi, H_i)
\]

(44)

Each \( H_i \) being endowed with its own energy content and with its own temperature \( T_i \) we expect it features also an entropy contribution \( S_i \) according to the local first law

\[
\delta X E(\xi, H_i) = T_i \delta X S_i
\]

(45)

Basically, for a spacetime with multiple horizons, where a global notion of temperature does not exist (namely, the spacetime does not describe a thermodynamical system in equilibrium), there cannot hold a global first law. Nevertheless many local first laws hold according to (45).

We stress that expression (45) is calculated directly on the horizon independently of what exists outside it. This is a rather desirable property. Indeed even if the whole gravitational system is not in thermal equilibrium it is nevertheless expected that an observer located near the horizon should be able to measure physical properties of the horizon itself without knowledge of spacetime regions far away from him.

We also point out that the expression (45), as it stands, is suitable for applications to a broader class of horizons, e.g. isolated horizons [5], the geometric characterization of which is intrinsically independent on the geometric properties of the spacetime surrounding them. In the isolated horizons framework, the whole spacetime is not even required to be stationary. Rather, it is just assumed that no flow of matter/energy crosses the horizon. Under these conditions the horizon is isolated and it can be considered as a thermodynamical system in equilibrium. Therefore the laws of thermodynamics can be generalized also to this class of horizons; see [5, 9]. In [1] it was indeed shown how the first law can be geometrically reproduced by means of (45).

We finally point out that if we had made use of the definition of internal energy the above construction would have failed to work since \( E(g, H_i) = 0 \) (being both the lapse and the shift vanishing on \( H_i \)).
In the purely Lorentzian sector, we recover the Lorentzian counterpart of expression (42) by considering the class of metrics \([59]\) which are:

i) static in the given reference frame,

ii) have vanishing lapse on some (compact) 2–surface \(H_i\) defined by \(N^2 = 0\) (surfaces corresponding to removable singularities of spacetime) and

iii) non vanishing derivative \(\nabla_{\alpha}N^2 \neq 0\) on \(H_i\).

Near each horizon \(H_i\) we can approximate the lapse function with its Taylor series truncated to the first order, obtaining in this way the metric:

\[
g = -R^2 \left(\frac{2\pi}{\beta}\right)^2 dt^2 + dR^2 + r^2(R)d\Omega \tag{46}
\]

where \(T = 1/\beta\) is given by \([11]\) while \(r(R)\) corresponds to \([13]\). Namely: the zeroes of \(N^2\) determine the horizon radii \(r_+\) while the radial derivatives of \(N^2\) fix the temperatures \(\beta\). When we consider variations \(\delta_Xg\) along a generic vector field \(X\), both parameters \(r_+\) and \(\beta\) are varied.

For the class of metrics \([10]\) the boundary condition \([39]\) calculated on the horizon \(\{R = 0\}\) becomes:

\[
\gamma_{\mu \nu} \delta \Pi^{\mu \nu} \big|_{H} = -\frac{\pi r_+^2}{\beta^2} \delta \beta = \pi r_+^2 \delta T \tag{47}
\]

Therefore, the (weak) Neumann boundary condition \(\gamma_{\mu \nu} \delta \Pi^{\mu \nu} \big|_{H} = 0\) is satisfied if \(\delta \beta \big|_{R=0} = 0\). Namely, the Neumann ensemble is locally realized by the set of metrics admitting a causal horizon inside the surface \(\Sigma\) with a fixed temperature \(T = \frac{1}{\beta}\). Hence, for vector fields \(X\) which describe variation along the Neumann ensemble, from \(\[45]\) we obtain:

\[
\delta_{[N]}(TS) = T\delta_{[N]}(S) = \delta_{[N]}E(\xi, B) = \delta_{[N]} \left( \int_H U_{\text{Kom}}(\xi) \right) \tag{48}
\]

where in the first equality we made use of the property \(\delta_{[N]}(T) = 0\) while the last equality comes from expression \([36]\). The above relation can be formally integrated as:

\[
TS = \int_H U_{\text{Kom}}(\xi) + Q_0 \tag{49}
\]

where \(Q_0\) is a constant of integration and corresponds to the Komar potential calculated for a (background) solution \(g_0\) lying in the integral curve of the vector \(X\). It corresponds to the ground state for the calculation of the gravitational heat \(TS\). From now on we shall set the constant \(Q_0\) equal to zero so that the reference solution corresponds to the one for which the horizon \(H\) shrinks to zero, i.e. the solution for which \(r_+ = 0\) (since our analysis is only local we do not forbid the background solution to have other horizons elsewhere).
It is now easy to verify that the entropy, as defined by (49), corresponds to one–quarter of the horizon area $A_H$. Indeed, when we approach the horizon $H$ the metric assumes its form (46) and expression (37) becomes:

$$
\int_H U_{\text{Kom}} = \frac{1}{\kappa} \int_H \sigma \sqrt{g} d^2 x N n^\mu a_\mu = \frac{1}{4} T \int_H \sqrt{\sigma} d^2 x = T \frac{A_H}{4}
$$

provided that the orientation of the unit normal $n^\mu$ is chosen so that $N n^\mu a_\mu / 2\pi$ turns always to be positive.\footnote{For non compact horizons, e.g. Rindler horizons \cite{59}, formula (50) gives a finite value only if the domain of integration extends up to a finite spatial region. Despite of this, the entropy for unit of area is always a finite constant and it equals 1/4.}

We stress that the definition of entropy as it arises from (49) and (50) is clearly observer–dependent. Indeed, conditions i)–iii) listed above may be rephrased as follows. Given a set of observers evolving with 4–velocity $\xi^\mu$, let us choose an observer–adapted system of coordinates in which the observers themselves are at rest. In this reference frame the metric coordinate singularities correspond to homological boundaries and, according to (50), each boundary gives rise to its own contribution to the total entropy. However, from a physical point of view, the metric coordinate singularities correspond to regions of inaccessibility for the observers (one–way membranes). Entropy turns then out to be associated to the observer unseen degrees of freedom, namely, to the regions which the observers have no physical access to. According to \cite{55} we might be tempted to state that entropy measures the information content beyond the hidden regions and that such an information is related to one–quarter of the area of the surface enclosing the observer–unaccessible regions.

Let us now end this section with some further thermodynamically–inspired consideration. Let us consider again expression (25) and let us evaluate it on the cross section $H_i$ of a causal horizon. According to (49) it can be rewritten as:

$$
\delta_X E(\xi, H_i) = \delta_X (T_i S_i) + \int_{H_i} \frac{\sqrt{g}}{\kappa} g^{\mu\nu} \delta u^{[\beta}_\mu \xi^{\alpha]} ds_{\alpha\beta}
$$

From a thermodynamic point of view, having formally identified $E(\xi, H_i)$ with the internal energy of the gravitational system and the Noether charge with the $TS$ contribution, it is tantamount to formally identify the second term in the right hand side of (51) with the variation of the gravitational free (or Helmholtz) energy $F(\xi, H_i)$:

$$
\delta_X F(\xi, H_i) := \int_{H_i} \frac{\sqrt{g}}{\kappa} g^{\mu\nu} \delta u^{[\beta}_\mu \xi^{\alpha]} ds_{\alpha\beta} = - \int_{H_i} \gamma_{\mu\nu} \delta \Pi^{\mu\nu} d^2 x = \int_{H_i} U_{\text{Cadm}}(\xi, X)
$$

Accordingly, expression (51) turns out to be nothing but the Gibbs–Duhem formula \cite{43, 53}:

$$
\delta_X E(\xi, H_i) = \delta_X (T_i S_i) + \delta_X F(\xi, H_i)
$$
Trying to further enhance the formal analogy between gravitation and thermodynamics we observe that, in the definition (45) of the first law, we are taking into account variations for which no “mechanical work” is done. Namely, all the variation of energy \( \delta E \) among nearby solutions corresponds to “heat” exchange \( T \delta S \). For such (reversible) transformations, thermodynamics predicts that the free energy \( F \) must obey the law \( \frac{\delta F}{\delta T} = -S \). This is exactly in agreement with the identification (52) and with equation (47). For instance, if we consider two nearby Schwarzschild solutions (of mass, respectively, \( M \) and \( M + \delta M \)) the variation of energy \( \delta E \) turns out to be \( \delta E = \delta M \) (see appendix A) and it is equipartitioned into variation of “heat” and variation of “free energy” according to \( \delta M = \delta (M/2) + \delta (M/2) \). This splitting corresponds, from a thermodynamical point of view, to the splitting of \( \delta E = T \delta S \) into \( T \delta S = \delta (TS) - S \delta T \). The free energy contribution \( \delta (M/2) \) therefore corresponds to the \(-S \delta T \) part of the splitting. It does not correspond to mechanical work (no energy can be extracted from the hole in any classical physical process; see e.g. [57]) but it is due to the fact that the transformation is not isothermal (if we change the mass of a Schwarzschild black hole we also change its temperature according to \( T = 1/8\pi M \)). Indeed \(-S \delta T = -4\pi M^2 \delta (1/8\pi M) = \delta (M/2) \).

Further hints corroborating the viability of our definitions will be exhibited in the appendix [57] where the analogies of the formalism developed so far with existing literature on the subject will be analysed.

4 Generalized Entropy

Following the guidelines of [54] and adapting Mann’s ideas to our formalism, we shall now try to generalize the identification of the Noether charge with gravitational heat to a wider class of situations. We nevertheless stress that our approach differs considerably from that of [54]. Even though the latter approach turns out to be more physically motivated we believe that our definitions are better suited to handle specific applications, e.g. the AdS–Taub Bolt solution, where comparable results can be obtained with considerable less efforts.

Let us then consider a stationary spacetime \( M \). Let \( \xi = \partial_t \) be the Killing vector field and let us then denote by \( \cup_i \mathcal{H}_i \) the union of the fixed point sets of \( \xi \). Any \( \mathcal{H}_i \) is an obstruction to foliating spacetime with surfaces of constant time. Therefore we can consider a surface \( \Sigma \) of constant time the boundary of which is formed by an outer boundary \( B \) together with the union of all the cross sections \( H_i \) of \( \mathcal{H}_i \). If a temperature \( T_i \) can be defined for any \( H_i \), following step by step the same arguments outlined in Remark [54] we expect that each \( H_i \) gives rise to a contribution \( T_i \delta S_i \) to the internal energy and therefore a contribution \( S_i \) to the total entropy. Notice however that now our definition encompasses a broader class of solutions, namely the ones with any kind of fixed set points of the Killing vector. For Lorentzian spacetimes, it includes any compact causal horizon but also non compact horizons, such as Rindler horizons or Lorentzian CCBH spacetimes (constant curvature black holes [20]). When dealing with
Euclidean gravity, bolts, nuts as well as Misner strings are allowed to exist. To calculate the contributions to the entropy arising from each singularity $H_i$, it is a common procedure to start from the (local) relation:

$$T_i S_i = E(H_i) - F(H_i)$$

relating temperature and entropy with internal and free energies of each singularity $H_i$. The idea to make use of the above relation dates back to Gibbons and Hawking. They identified the internal energy with the Hamiltonian and the free energy with the gravitational action, as defined in formula (103) of appendix B. In general both the Hamiltonian and the action, when calculated on a given solution, give an infinite result. Therefore one has to introduce a background solution, suitable matched with the original solution, and subtract off from (54) the background action and Hamiltonian. In this way one obtains a finite result which provides us the difference of entropy between the solution and its background.

Another strategy which avoids the problems related to any background fixing procedure has been developed in (54). It consists in modifying the gravitational action, i.e. the gravitational free energy, by suitably adding boundary counter terms coming from the conjectured AdS/CFT correspondence. Gravitational internal energy is then obtained, via a Brown–York procedure, from the modified action functional.

We shall instead face up to the problem following the technique of section 3. First of all let us notice that if we consider the infinitesimal version of (54):

$$\delta(T_i S_i) = \delta E(H_i) - \delta F(H_i)$$

the problems of fixing a suitable background is postponed, in specific applications, up to the end of calculations where the background will resort as a constant of integration. This implies a higher practicality since it is not necessary to select from the very beginning the reference solution and to carry on calculations with it. Indeed, according to our definitions (38) and (52), the relation (55) becomes:

$$\delta(T_i S_i) = \delta \left\{ \int_{H_i} U_{Kom} \right\}$$

which, after integration, leads to the result

$$T_i S_i = \int_{H_i} U_{Kom} + Q_0$$

Now, it is the constant $Q_0$ which encodes the information relative to the background. Different choices of $Q_0$ allow to reproduce many results exhibited elsewhere. For instance, $Q_0$ can be set equal to zero for compact horizons. This choice, see (50), leads indeed to the one–quarter area law for the entropy. Other choices are nevertheless possible and strictly necessary for non compact horizons, where a background subtraction term is necessary to obtain a finite result.
To this end, we recall that, given a variation \( \delta X \) induced by a vector \( X \) in the space of solutions, the background has to belong to the integral curve of \( X \) through the original solution. This means that the given solution has to be, in the suitable sense, continuously deformable into its background, i.e. the solution and its background have to lie, at least, in a path–connected region in the space of solutions. This consideration suggests how the background has to be chosen in many of the applications. The example below will help to clarify this point.

5 AdS Taub–Bolt solution

Let us consider the class of metrics of the kind \[18, 54\]:

\[ g = V(r) \left( d\tau + 2N \cos \theta d\phi \right)^2 + \frac{dr^2}{V(r)} + \left( r^2 - N^2 \right) \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \] \hspace{1cm} (58)

\[ V(r) = \frac{r^2 + N^2 - 2mr + \left( r^4 - 6N^2r^2 - 3N^4 \right) l^{-2}}{r^2 - N^2} \hspace{1cm} m, N \text{ constant} \]

Independently on the parameter \( m \), they are all asymptotically locally AdS \[43\]:

\[ g = V_\infty(r) \left( d\tau + 2N \cos \theta d\phi \right)^2 + \frac{dr^2}{V_\infty(r)} + \left( r^2 - N^2 \right) \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \]

\[ V_\infty(r) = \frac{r^2}{l^2} + \left( 1 - \frac{5N^2}{l^2} \right) + O(r^{-1}) \] \hspace{1cm} (59)

The asymptotic boundary is a squashed \( S^3 \), rather than \( S^1 \times S^3 \) \[43, 56\], and the constant \( N \) parametrizes the squashing. When we apply the definition (22) to the metric (58), variations \( \delta \) are meant inside the class of solutions with asymptotic behavior (59). After some calculations we obtain:

\[ \delta X E(\xi, S_R) = \frac{R^2}{R^2 - N^2} \delta m \] \hspace{1cm} (60)

where \( S_R \) is a surface of constant radius \( R \). In the limit \( R \to \infty \) we have

\[ \delta X E(\xi, S_\infty) = \delta m \] \hspace{1cm} (61)

so that we can refer to the parameter \( m \) as the total Energy/mass of the solution. Notice that, even though the quantity \( \delta X E \) is known to be an homological invariant (see equation (24)) formula (60) depends on the radius. The reason is easily understood.

One of the fixed points set of the \( U(1) \) isometry \( \tau \) is the surface \( r = r_0 \) where \( r_0 \) is a solution of \( V(r_0) = 0 \). The equation \( V(r_0) = 0 \) allows to determine the “mass” parameter \( m \) as a function of \( r_0 \). From (58) we indeed obtain:

\[ m(r_0) = \frac{r_0^4 + (l^2 - 6N^2)r_0^2 + N^2(l^2 - 3N^2)}{2l^2r_0} \] \hspace{1cm} (62)
The fixed set points \( r = r_0 \) is called a "bolt" and its area is equal to
\[
A(r_0) = 8\pi N \frac{3r_0^4 + r_0^2(l^2 - 6N^2) + N^2(3N^2 - l^2)}{r_0 l^2}
\]  
(63)

Moreover, the metric (65) has another coordinate singularity, i.e. another break down of the foliation in constant time surfaces, due to the presence of a Misner string. It is a two dimensional coordinate singularity running along the \( z \)-axis (i.e. \( \theta = 0, \pi \)) from the bolt out to infinity (see [56]). The Misner string was recognized in [56] to be a removable singularity provided we fix the period \( \beta \) of the Euclidean time to be \( \beta = 8\pi N \). In addition, the absence of a conical singularity at the bolt \( r = r_0 \) determines the constraint [43]:
\[
\beta = 8\pi N = \left| \frac{4\pi}{V'(r_0)} \right| 
\]  
(64)

The above equation admits only two solutions, i.e. \( r_0 = r_b^\pm \) and \( r_0 = r_n \), where:
\[
r_b^\pm = \frac{l^2 \pm \sqrt{l^4 - 48N^2l^2 + 144N^4}}{12N}
\]  
(65)
\[
r_n = N
\]  
(66)

for the values \( N \leq \frac{(3\sqrt{7} - \sqrt{6})}{12} \) corresponding to a real square root. This two solutions are known, respectively, as the Taub-bolt and the Taub–NUT metrics.

By substituting \( r_0 = r_n \) into (63), we easily verify that the NUT has zero area.

From the above discussion it follows that a surface enclosing the bolt (or the NUT), i.e. \( S_{(r_0 + \epsilon)} \), where \( r_0 = r_b \) (or, respectively, \( r_0 = r_n \)) is not homologous to spatial infinity. Owing to the presence of the Misner string we also have to consider two cones \( C_1 = \{ \theta = \epsilon, r_0 + \epsilon \leq r < \infty \} \) and \( C_2 = \{ \theta = \pi - \epsilon, r_0 + \epsilon \leq r < \infty \} \) wrapping around the \( z \)-axis from the bolt \( r = r_0 \) up to spatial infinity. Spatial infinity is then homologous to the union \( \Sigma = S_{(r_0 + \epsilon)} \cup C_1 \cup C_2 \). From the homological property we known that each term in \( \Sigma \) gives a contribution to the total energy (61) and, accordingly, to the total entropy.

We can calculate each separate contribution to the entropy following the definition (57). We obtain
\[
TS_{(S_{r_0})} = \lim_{\epsilon \to 0} \int_{r=r_0+\epsilon} U_{Kom} = T \frac{A(r_0)}{4}
\]  
(67)

for the bolt contribution and
\[
TS_{(C_1 \cup C_2)} = \lim_{\epsilon \to 0} \left\{ \int_{\theta = \epsilon} U_{Kom} + \int_{\theta = \pi - \epsilon} U_{Kom} \right\}
\]  
(68)
\[
= \lim_{r \to \infty} \left\{ \frac{N^2}{l^2} + \frac{N^2(3N^2r_0 + r_0^3 - l^2r_0 + m(r_0)l^2)}{l^2(N^2 - r_0^2)} + O(r^{-1}) \right\}
\]

for the Misner string contribution (the details of the calculations follows the ones described in [24, 34] for the asymptotically locally flat case \( l^2 \to \infty \)). The
total entropy $S_{r_0}$ is given by the sum of (67) and (68):

$$S_{r_0} = \frac{A(r_0)}{4} + \beta \cdot \lim_{r \to \infty} \left( \frac{N^2}{l^2} r + \frac{N^2(3N^2r_0 + r_0^3 - l^2r_0 + m(r_0)l^2)}{l^2(N^2 - r_0^2)} + O(r^{-1}) \right)$$

+ $\beta \cdot Q_0$

(69)

where $Q_0$ is the constant of integration; see (49). If we do not take into consideration the constant of integration, the total entropy (69) diverges due to the infinite contribution coming from the Misner string (while it reproduces a finite value in the limit $l^2 \to \infty$). Notice however that the divergent part turns out to be independent on $r_0$. Therefore if we consider the entropy of the Taub–bolt solution relative to the Taub-NUT metric we obtain a finite result and, in the meanwhile, we get rid of the constant of integration which is common for both the solutions. Indeed both Taub–bolt and Taub-NUT can be joined, in the space of solutions, by a curve $\gamma$ the points of which all share the asymptotic form (69). This can be done by just varying $r_0$ from $r_n$ to $r_b$. The constant $Q_0$ corresponds to the Komar integral calculated on an (arbitrarily) fixed point $g_0$ on $\gamma$. The relative entropy of Taub–bolt with respect to the Taub-NUT is then given by:

$$\Delta S = S_{rb} - S_{rn} = \frac{2\pi N}{r_bl^2} \left[ 3r_b^4 + r_b^2(l^2 - 12N^2) + 2r_bN(6N^2 - l^2) + N^2(l^2 - 3N^2) \right]$$

(70)

for both the values $r_b = r_b^\pm$. We remark that the entropy (70) satisfies the first law

$$T \delta(\Delta S) = \delta m(r_b) - \delta m(r_n)$$

(71)

and it also agrees with the results of [18].

6 Conclusions

We finally point out that the thermodynamic analysis we have performed entirely develops from the definition (22) of variation of energy. We remark that this definition, which perfectly agrees with many covariant definitions obtained elsewhere (see, e.g., [17, 29, 31, 44, 45, 51]), was calculated directly from the Hamiltonian equations of motion and therefore it does not depend on the specific gravitational Lagrangian $L$ inside the variational cohomology class $[L]$ we choose to represent the system, where elements of $[L]$ differ each other only for the addition of divergence terms. About this, let us again point out that, even if throughout the paper we have dealt with the Noether charge, nowhere in the text we made use of Noether theorem, which is instead sensitive to the representative chosen inside $[L]$. The splitting of $\delta E$ into $\delta(TS)$ and $\delta F$ depends therefore just on the field content of the theory (i.e., Euler–Lagrange equations).

\[6\text{In particular, diverging terms are eliminated from (69) for any choice of the background } g_0 \text{ in } \gamma \text{ (and none of them corresponds to } Q_0 = 0). \text{ Nevertheless background choices other than the Taub–NUT solution turn out to be, as far as we know, meaningless and rather artificial.}\]
and not on a given Lagrangian description inside \([L]\). Only if we try to formally integrate these functions of state with the appropriate boundary conditions we restore the Lagrangian framework and it can be shown (see appendix [13]) that standard (and widely accepted) results are reproduced in this way. Nevertheless we stress again that in specific applications it is more convenient for calculations to consider just variations of physical observables without performing any a priori integration. In this way complicated calculations related to background choices and background matching conditions are avoided in a first approach. The background will instead come into play just at the end, in a more manageable fashion, as a “constant of integration”.

As future developments, we believe that the ideas suggested in this paper are worth being extended to deSitter and Anti deSitter Einstein–Gauss–Bonnet gravity in order to study the possibility of negative entropy; see [21]. Moreover, the analogy between Noether charge and gravitational heat deserves to be analyzed in processes of collapse of a star into a black hole in order to understand if a sort of gravitational heat can be defined prior to collapse.

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A Spherically Symmetric Solutions

Let us consider a static spherically symmetric solution of the kind:

\[
g = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega
\]

where \(d\Omega = d\theta^2 + \sin(\theta)^2d\phi^2\). The explicit form of \(f(r)\) is dictated by Einstein’s field equations and we shall analyze it later on.

The variation of the energy contained inside a region \(\Sigma_t\) of constant time bounded by a a two–sphere \(S_R\), i.e. \(B_t := \{r = R\}\), turns out to be:

\[
\delta_X E(\xi, S_R) = \int_{S_R} d^2 x N \delta(\sqrt{\sigma} \epsilon) = -\frac{R}{2} \delta f|_R
\]
where \( \xi = \partial_t \). The Noether charge (37), apart from the constant of integration which is set equal to zero throughout this section, becomes:

\[
Q(\xi,S_R) = \frac{R^2}{4} |f'(R)|
\]

having denoted with a prime the derivative with respect to the \( r \)-coordinate. The absolute value arises because when \( f' > 0 \), e.g. on black hole horizons, the normal to the sphere is chosen to point outward, along the direction of increasing \( r \). When \( f' < 0 \), e.g. on cosmological horizons (see below), the unit normal is instead chosen to point inward toward the direction of decreasing \( r \).

Dirichlet boundary conditions for (73) have been extensively analyzed in [13, 14]. We instead point out here that (weak) Neumann boundary condition (35) corresponds to:

\[
- \int_{S_R} \gamma_{\mu\nu}\delta \Pi^{\mu\nu} = -\frac{R^4}{4} \left\{ R \delta f'(R) + 2 \delta f(R) \right\} = 0
\]

It is therefore easy to check that Neumann variations \( \delta_{[N]}Q(\xi,S_R) \) of the Noether charge (74) give rise to the variation of energy (73). No anomalous factor arises in this case!

Now, let us suppose that \( f(r_i) = 0 \) for some \( r_i \), \( (i = 1, 2, \ldots) \) and that \( f'(r_i) \neq 0 \). We define the temperature \( T_i \) of each horizon \( r = r_i \) according to (41), i.e.:

\[
T_i = \left\| \frac{f'(r_i)}{4\pi} \right\|
\]

On the horizon, (74) is equivalent to \( \delta T_i = 0 \) and the Neumann energy (75) becomes:

\[
Q(\xi,H_i) = \left\| \frac{f'(r_i)}{4\pi} \right\| \pi r_i^2 = T_i \frac{A_i}{4} := T_i S_i
\]

where \( A_i \) is the area of the horizon’s cross section. Therefore the above expression has the \( T_i S_i \) form if the entropy is identified with \( S_i := \frac{A_i}{4\pi} \). We stress that so far we have not required the solution to be a vacuum solution (possibly with a cosmological constant). Therefore the result (77) holds true for any spherically symmetric static solution.

Further considerations require to consider on separate grounds the cases in which:

i) the metric (72) is a solution of Einstein’s equations with a (positive) cosmological constant;

ii) the metric is a solution of Einstein’s equations with external matter fields.

i) **Schwarzschild–deSitter solutions**

If we set

\[
f(r) = 1 - \frac{2m}{r} - \frac{r^2}{l^2}
\]

(78)
in the metric (72) we obtain the Schwarzschild–deSitter metric. The cosmolog-
ical constant is \(\Lambda = \frac{3}{l^2}\) and \(m\) is the mass parameter. If \(m = 0\) we recover deSitter spacetime while Schwarzschild solution is obtained for \(\Lambda = 0\).

Provided that \(m < l/\sqrt{27}\) the solution has two positive roots, \(r_+\) and \(r_{++}\) \((r_{++} > r_+)\) corresponding, respectively, to the black hole horizon and the cosmological horizon. From (78) we have:

\[
m = \frac{r_0}{2} \left( 1 - \frac{r_0^2}{l^2} \right)
\]

for both the values \(r_0 = r_+\) and \(r_0 = r_{++}\).

When \(r\) ranges from \(r_+\) up to \(r_{++}\) the vector field \(\xi = \partial_t\) is timelike and it can describe the velocities of observers located between the two horizons (of course this is just one possible choice; see [19] for another normalization of \(\xi\)).

Expression (73) becomes:

\[
\delta X E(\xi, S_R) = \delta m
\]

and does not depend on the radius of integration, as expected from (24).

The temperatures on \(r_+\) and \(r_{++}\) are, respectively:

\[
T_+ = \frac{f'(r_+)}{4\pi} = \frac{1}{4\pi} \left( \frac{1}{r_+} - \frac{3r_+}{l^2} \right)
\]

\[
T_{++} = -\frac{f'(r_{++})}{4\pi} = -\frac{1}{4\pi} \left( \frac{1}{r_{++}} - \frac{3r_{++}}{l^2} \right)
\]

where the minus sign in (82) arises since the surface gravity \(\frac{f'(r_{++})}{2}\) is negative on the cosmological horizon. According to (77) the integral of the Komar charge allows to obtain the one–quarter law for both the horizons \(r_0 = r_+, r_{++}\):

\[
S_0 = \pi r_0^2
\]

Notice from (79), (81) and (83) that, on the black hole horizon, the relation \(\delta m = T_+ \delta S_+\) holds true. Owing to (80) this relation corresponds to the first law

\[
\delta E(\xi, r_+) = \delta m = T_+ \delta S_+
\]

On the cosmological horizon, on the contrary, it holds true the relation \(\delta m = -T_{++} \delta S_{++}\) so that we expect \(\delta E(\xi, r_{++})\) to be equal to \(-\delta m\) if we want to recover the first law for the cosmological horizon, too. This result is only apparently in contrast with (80). Indeed, in obtaining (80) the spacelike unit normal \(n^\mu\) was assumed to point in the direction of increasing \(r\). Nevertheless on the cosmological horizon we have to change the sign, i.e.

\[
\delta X E(\xi, r_{++}) = -\delta m
\]

if we want the unit normal to point toward the observer. In this way the expected value is recovered and the first law is satisfied. Notice that a negative value for the variation of energy contained inside a cosmological horizon is
physically expected; see [19, 62]. Indeed, if a test particle of mass \( \delta m \) is thrown across the cosmological horizon from the interior region, the area of the horizon increases but we expect that the energy enclosed by the horizon should decrease according to (85).

Notice that, even though the absolute value \( |\delta E| = \delta m \) between two nearby Schwarzschild–deSitter solutions is the same when computed on both the horizons, the energies of the horizons \( E(\xi, r_{++}) \) and \( E(\xi, r_+) \) can be nevertheless different from each other (even though, in absolute value, they change for the same amount during the variation). From (84) we obtain:

\[
E(\xi, r_+) = \int_0^{r_+} \frac{\delta m(r)}{\delta r} \, dr = m(r_+) \frac{r_+}{2} \left( 1 - \frac{r_+^2}{l^2} \right) \tag{86}
\]

where the normalization has been reasonably fixed so that the energy vanishes when the horizon mass vanishes, i.e. the background spacetime is the deSitter solution. Notice that, for \( l^2 \to \infty \) the usual mass \( \frac{r_+^2}{2} \) of Schwarzschild black hole is recovered.

According to (85), for the cosmological horizon we obtain:

\[
E(\xi, r_{++}) = -m(r_{++}) + E_0 \tag{87}
\]

In this case different choices are available for the constant of integration. For instance, if we set \( E_0 = 0 \) we fix again the deSitter background. Another possible choice is \( E_0 = \frac{m}{\sqrt{27}} \). In this case \( E(\xi, r_{++}) \to E(\xi, r_+) \) in the Nariai limit \( m \to l/\sqrt{27} \) when \( r_+ \) and \( r_{++} \) approach to each other. We refer the reader to [19] for some deeper insight into the subject (see also [35]).

ii) External matter fields

If we set

\[
f(r) = 1 - \frac{r_+}{r} - \frac{1}{r} \int_{r_+}^r \epsilon(\bar{r}) \bar{r}^2 d\bar{r} \tag{88}
\]

in the metric (72) we obtain a solution of Einstein’s field equations with external matter characterized by a density

\[
\rho(r) = \frac{\epsilon(r)}{8\pi}, \tag{89}
\]

radial pressure

\[
p(r) = \rho(r) \tag{90}
\]

together with angular pressures \( T_{\theta\theta} = T_{\phi\phi} = \frac{G\rho}{2} \), the explicit form of which is of no interest here; see [40, 58]. The function (88) has been normalized [58] in order to have an horizon at \( r_+ \) with temperature:

\[
T_+ = \frac{f'(r_+)}{4\pi} = \frac{1}{4\pi} \left( \frac{1}{r_+} - \epsilon(r_+)r_+ \right) \tag{91}
\]
We however do not exclude that \( f(r) \) can be zero for other values of \( r \). Expression (73) now becomes:

\[
\delta X E(\xi, S_R) = \delta \left\{ \frac{r_+}{2} + \frac{1}{2} \int_{r_+}^{R} e(\bar{r}) \bar{r}^2 d\bar{r} \right\}
\]

(92)

for a generic sphere of radius \( R \) and

\[
\delta X E(\xi, r_+) = \delta \left\{ \frac{r_+}{2} \right\}
\]

(93)

for the horizon.

If we consider variations which keep the radius \( R \) fixed in (92), we easily obtain:

\[
\delta X E(\xi, S_R) = \delta X E(\xi, r_+) - \frac{1}{2} \epsilon(r_+) r_+^2 \delta r_+ \]

(94)

Since from (91) we have:

\[
\delta X E(\xi, S_R) = T_+ \delta S, \quad S = \pi r_+^2
\]

(95)

by comparing (94) and (95) we obtain the first law:

\[
\delta X E(\xi, r_+) = T_+ \delta S + p_+ \delta V_+
\]

(96)

where \( p_+ \) is the radial pressure on the horizon, while \( \delta V_+ = 4\pi r_+^2 \delta r_+ \). Hence \( p_+ \delta V_+ \) is the work term given by the product \( p_+ 4\pi r_+^2 \) of the radial force multiplied by the radial displacement \( \delta r_+ \).

Therefore expression (96) constitutes the generalization of formula (45) in presence of external matter. Namely, the amount of energy to virtually increase the radius of the horizon of a quantity \( \delta r_+ \) is partitioned into a part \( T_+ \delta S \) which comes from the increase of entropy (due to the expansion of the horizon area) and a work term \( p_+ \delta V_+ \) which has to be spent to contrast the external pressure, see (58).

**B  Comparison with the semi–classical approach**

Let us now analyse the interplay between the formalism developed so far and the existing literature on the matter. First of all we stress that the identification of the Komar energy (37) with the gravitational heat \( TS \) ensues from the heuristic definition of entropy given in (45), which a priori has no direct physical ground. The identification is nevertheless in agreement with semi–classical statistical approaches, based on path integral techniques, where entropy is identified by the value of the microcanonical action functional (evaluated on a –complexified–stationary black hole solution; see [11, 14, 45]).
Indeed, let us consider a closed surface $B$ surrounding the horizon $H$ such that $B \cup H$ is the homologic boundary $\partial \Sigma$ of a region $\Sigma$. We can rewrite equation (53) as:

\[
\delta_X (TS) = \delta_X E(\xi, H) - \delta_X F(\xi, H)
\]

where $L = \frac{1}{2\kappa} \sqrt{g} (R - 2\Lambda) \, ds$ is the Hilbert Lagrangian. In the last equality we made use of Stokes’ theorem together with the following relations which hold true on–shell for stationary solutions:

\[
i_{\ell} \delta L = i_{\ell} d \left\{ \frac{\sqrt{2}}{2\kappa} g^{\mu \nu} \delta u_{\mu \nu} \, ds_{\alpha} \right\} = \left( u_{\mu \nu}^{\alpha} = \Gamma_{\mu \nu}^{\alpha} - \delta_{(\mu} \Gamma_{\nu) \rho}^{\rho} \right)
\]

\[
L_{\xi} \left\{ \frac{\sqrt{g}}{2\kappa} g^{\mu \nu} \delta u_{\mu \nu} \, ds_{\alpha} \right\} - di_{\ell} \left\{ \frac{\sqrt{g}}{2\kappa} g^{\mu \nu} \delta u_{\mu \nu} \, ds_{\alpha} \right\}
\]

\[
= -di_{\ell} \left\{ \frac{\sqrt{g}}{2\kappa} g^{\mu \nu} \delta u_{\mu \nu} \, ds_{\alpha} \right\} = dU_{\text{CADM}}(\xi, X)
\]

(98)

From (97) we obtain (apart from a constant of integration):

\[
S = \beta \left\{ \int_B U_{\text{Kom}} + \int_{\Sigma} i_{\ell} |L| \right\}
\]

\[
= \int dt \int_B U_{\text{Kom}} + \int dt \int_{\Sigma} i_{\ell} |L|
\]

\[
= \int dt \wedge U_{\text{Kom}} + \int_{D} L = I_m
\]

(99)

where $I_m$ is the microcanonical action functional for a region $D = \beta \times \Sigma$ of spacetime with outer boundary $B = \beta \times B$ ; see, e.g. [14, 15]. Hence definition (49) together with the relation (53) exactly corresponds to Brown–York’s definition of entropy when stationary black holes are considered.

We also stress that the identification $S = \beta \int_H U_{\text{Kom}}$ exactly agrees with the definition of entropy [59]:

\[
S = \frac{\beta}{\kappa} \int_H \sqrt{\sigma} d^2 x N \, n^\mu a_{\mu}
\]

(100)

\[\text{For notational convenience, from now on, we shall suppress the index } i.\]
which Padhmanabhan has given for static spacetime, as it is can be easily inferred from Padhmanabhan’s definition in which free energy is identified with the Hilbert Lagrangian. Our definition is instead more tied with the Trace-K action functional. By making use of the relations, definition can indeed rewritten as:

$$- \delta_X F(\xi, H) = \int_\Sigma \delta L - \int_B U_{\text{CADM}}(\xi, X)$$  \hspace{1cm} (101)

If we consider the Dirichlet boundary condition \( \delta_X \gamma_{\mu\nu}|_B = 0 \) on the outer boundary \( B \) (which in many applications can be identified with spatial infinity), namely, if we consider the class of metrics which approach a fixed background \( g_0 \) on \( B \), the above expression can be integrated. From we hence obtain:

$$- F(\xi, H) = \int_\Sigma i_\xi L - \frac{1}{\kappa} \int_B \sqrt{\sigma} d^2 x N (\Theta - \Theta_0)$$  \hspace{1cm} (102)

After multiplying the above expression by \( dt \) and integrating over the time interval \( \beta \) we finally obtain the trace-K action functional \( I_g \):

$$I_g := \int_D L - \frac{1}{\kappa} \int_B \sqrt{\sigma} d^2 x N (\Theta - \Theta_0) = -\beta \cdot F(\xi, H)$$  \hspace{1cm} (103)

(which coincides, in the hypotheses we are dealing with, with the first-order covariant Lagrangian for General Relativity; see [25]). We recall that, in a semi-classical path integral approach, \( I_g \) corresponds, in the Euclidean sector, to the logarithm of the thermodynamic partition function; see [14, 36].

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