Abstract: We found a new formulation to the Euler-Lagrange equation of the Willmore functional for immersed surfaces in $\mathbb{R}^m$. This new formulation of Willmore equation appears to be of divergence form, moreover, the non-linearities are made of jacobians. Additionally to that, if $\vec{H}$ denotes the mean curvature vector of the surface, this new form writes $\mathcal{L}\vec{H} = 0$ where $\mathcal{L}$ is a well defined locally invertible self-adjoint operator. These 3 facts have numerous consequences in the analysis of Willmore surfaces. One first consequence is that the long standing open problem to give a meaning to the Willmore Euler-Lagrange equation for immersions having only $L^2$ bounded second fundamental form is now solved. We then establish the regularity of weak $W^{2,p}$--Willmore surfaces for any $p$ for which the Gauss map is continuous : $p > 2$. This is based on the proof of an $\epsilon$--regularity result for weak Willmore surfaces. We establish then a weak compactness result for Willmore surfaces of energy less than $8\pi - \delta$ for every $\delta > 0$. This theorem is based on a point removability result we prove for Willmore surfaces in $\mathbb{R}^m$. This result extends to arbitrary codimension the main result in [KS3] established for surfaces in $\mathbb{R}^3$. Finally, we deduce from this point removability result the strong compactness, modulo the Möbius group action, of Willmore tori below the energy level $8\pi - \delta$ in dimensions 3 and 4. The dimension 3 case was already solved in [KS3].

I Introduction

Weak formulations of PDE offer not only the possibility to enlarge the class of solutions to the space of singular solutions but also provide a flexible setting in which the analysis of smooth solutions becomes much more efficient. This is the idea that we want to illustrate in this paper by introducing this new weak formulation of Willmore surfaces.

For a given oriented surface $\Sigma$ and a smooth positive immersion $\Phi$ of $\Sigma$ into the Euclidian space $\mathbb{R}^m$, for some $m \geq 3$, we introduce first the Gauss map $\vec{n}$ from $\Sigma$ into $Gr_{m-2}(\mathbb{R}^m)$, the grassmanian of oriented $m-2$--planes of $\mathbb{R}^m$, which to every point $x$ in $\Sigma$ assigns the unit $m-2$--plane $N_{\Phi(x)}\Phi(\Sigma)$ orthogonal to the oriented tangent space $T_{\Phi(x)}\Phi(\Sigma)$. This map $\vec{n}$ from $\Sigma$ into $Gr_{m-2}(\mathbb{R}^m)$ defines a projection map $\pi_{\vec{n}} :$ for every vector $\xi$ in $T_{\Phi(x)}(\mathbb{R}^m)$ $\pi_{\vec{n}}(\xi)$ is the orthogonal projection of $\xi$ onto $N_{\Phi(x)}\Phi(\Sigma)$. Let then $\vec{B}_x$ be the second fundamental form of the immersion $\Phi$ of $\Sigma$. $\vec{B}_x$ is a symmetric bilinear form on $T_x\Sigma$ with values into $N_{\Phi(x)}\Phi(\Sigma)$. $\vec{B}_x$ is given

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by $\vec{B}_x = \pi_{\vec{n}} \circ d^2\Phi$. By the mean of the ambiant scalar product in $\mathbb{R}^m$, which induces a metric $g$ on $\Sigma$, we define the trace of $\vec{B}_x$, $\text{tr}(\vec{B}_x)$, which is a vector in $N_{\Phi(x)}\Phi(\Sigma)$ given by $\text{tr}(\vec{B}_x) = \vec{B}_x(e_1, e_1) + \vec{B}_x(e_2, e_2)$ where $(e_1, e_2)$ is an arbitrary orthonormal basis of $T_x\Sigma$. The mean curvature vector $\vec{H}(x)$ at $x$ of the immersion by $\Phi$ of $\Sigma$ is with these notations the vector in $N_{\Phi(x)}\Phi(\Sigma)$ given by

$$\vec{H}(x) = \frac{1}{2} \text{tr}(\vec{B}_x) \cdot.$$

In the case where $m = 3$, $\vec{H}(x)$ is the product of the mean value $H = 1/2(\kappa_1 + \kappa_2)$ of the principal curvatures $\kappa_1$, $\kappa_2$ of the surface at $\Phi(x)$ by $\vec{n}$, the unit normal vector.

The so called Willmore Functional is then the following Lagrangian

$$W(\Phi(\Sigma)) = \int_{\Sigma} |\vec{H}|^2 \, d\text{vol}_g \quad (1.1)$$

where $d\text{vol}_g$ is the area form of the metric $g$ induced on $\Phi(\Sigma)$ by the canonical metric on $\mathbb{R}^m$.

This lagrangian has been apparently first considered in the early 20th century in various works by Thomsen [Tho], Schadow and a bit later by Blaschke [Bla]. It has been reintroduced and more systematically studied in the framework of the conformal geometry of surfaces in space by Willmore in 1965 [Wil]. Beyond conformal geometry this lagrangian plays an important role in various areas in science such as molecular biology, where it has been considered as a surface energy for lipid bilayers known as Helfrich Model [Hef], such as non-linear elasticity in solid mechanics where it arises as limiting energy in thin plate theory (see [FJM] for instance) or even in general relativity where the lagrangian $(1.1)$ is the main term in the so called Hawking quasi local mass (see [Haw], [HI])...etc. One of the reason for the genericity of this lagrangian is maybe the property discovered by White [Whi] for $m = 3$ and proved by B.Y. Chen [Che] for arbitrary $m$ which says that the functional remains unchanged under the action of a conformal diffeomorphism of $\mathbb{R}^m$ (and even under conformal changes of metric of the ambient space).

We are interested in this work in the critical points of $(1.1)$ for perturbations of the form $\Phi + t\phi$ where $\phi$ is an arbitrary smooth map from $\Sigma$ into $\mathbb{R}^m$. These critical points are the so called Willmore surfaces. Because of the invariance of the lagrangian under the action of conformal transformations of $\mathbb{R}^m$, images of Willmore surfaces by such conformal transformations are still Willmore. Examples of Willmore surfaces are minimal surfaces for which $\vec{H} \equiv 0$ and which realize then absolute minimum of $W$. Willmore surfaces satisfy an Euler-Lagrange equation discovered by Willmore for $m = 3$ (though it was apparently known by it’s predecessors on the subject Thomsen, Schadow and Blaschke in the twenties) and was established in it’s full generality, for arbitrary $m$, by Weiner in [Wei]. Before presenting the equation we need the following notations: Consider for every vector $\vec{w}$ in $N_{\Phi(x)}\Phi(\Sigma)$ the symmetric endomorphism $A_\vec{w}$ of $T_x\Sigma$ satisfying for every pair of vectors $\vec{X}$ and $\vec{Y}$ in $T_x\Sigma$ the identity $g(A_\vec{w}(\vec{X}), \vec{Y}) = B_x(\vec{X}, \vec{Y}) \cdot \vec{w}$, where $\cdot$ denotes the standard scalar product in $\mathbb{R}^m$. The map which to $\vec{w}$ assigns the symmetric endomorphism $A_\vec{w}$ of $T_x\Sigma$ for the scalar product $g$ is an homomorphism that we denote $A_\vec{w}$ from
$N_{\Phi(x)} \Phi(\Sigma)$ into $S_g \Sigma_x$, the linear space of symmetric endomorphisms from $T_x \Sigma$ with respect to $g$. Denote $\tilde{A}_x$ the endomorphism of $N_{\Phi(x)} \Phi(\Sigma)$ obtained by composing the transpose $^t A_x$ of $A_x : A_x = ^t A_x \circ A_x$. Let $(\tilde{e}_1, \tilde{e}_2)$ be an orthonormal basis of $T_x \Sigma$ and let $\tilde{L}$ be a vector in $N_{\Phi(x)} \Phi(\Sigma)$, we have that

$$\tilde{A}_x (\tilde{L}) = \sum_{i,j} \overline{B}(\tilde{e}_i, \tilde{e}_j) \overline{B}(\tilde{e}_i, \tilde{e}_j) \cdot \tilde{L}$$

With these notations, $\Phi$ is a smooth Willmore immersion if and only if it solves the following Euler-Lagrange equation

$$\Delta_{\perp} \overline{H} - 2|\overline{H}|^2 \overline{H} + \tilde{A}(\overline{H}) = 0$$

where $\Delta_{\perp}$ is the negative covariant laplacian for the connection $D$ in the normal bundle $N \Phi(\Sigma)$ to $\Phi(\Sigma)$ issued from the ambient scalar product in $\mathbb{R}^m$ : for every section $\sigma$ of $N \Phi(\Sigma)$ one has $D \overline{X} \sigma := \pi_n (\sigma \ast \overline{X})$. In the particular case when $m = 3$, the mean curvature vector $\overline{H}$ is oriented along the unit normal to $\Phi(\Sigma)$, $\overline{H} = H \overline{n}$, and (1.2) is equivalent to the following equation satisfied by the mean curvature function $H$ :

$$\Delta_g H + 2H (|H|^2 - K) = 0$$

where $\Delta_g$ is the negative laplace operator for the induced metric $g$ on $\Phi(\Sigma)$ and $K$ is the scalar curvature of $(\Sigma, g)$.

Despite their elegant aspects equations (1.2) and (1.3) offer challenging mathematical difficulties. First of all one has to observe that the highest order term $\Delta_{\perp} \overline{H}$ for (1.2) or $\Delta_g H$ for (1.3) is non-linear since the metric $g$ defining the Laplace operator depends on the variable immersion $\Phi$. Another difficulty comes from the fact that the Euler-Lagrange equations (1.2), (1.3) are a-priori non compatible with the Lagrangian (1.1) in the following sense. Making the minimal regularity assumption which ensures that the Lagrangian (1.1) is finite - the second fundamental form $\overline{B}$ is $L^2$ on $\Phi(\Sigma)$ - is not enough in order for the non-linearities in the equations (1.2) or (1.3) to have a distributional meaning : the expression $|\overline{H}|^2 \overline{H}$ requires at least that $\overline{H}$ is in $L^3$ and not only in $L^2$... etc.

Recently the author proved in [Ri1] that that any Euler Lagrange equation of any 2-dimensional conformally invariant lagrangian with quadratic growth (such as the harmonic map equations into riemannian submanifolds or such as the prescribed mean curvature equation) can be written in divergence form. This divergence form has numerous consequences for the analysis of this equation. It permits, in particular, to extend the set of solutions to subspaces of distribution with very low regularity requirement. It seems that the analysis developed in [Ri1] could be extended to other conformally invariant equations such as the harmonic map equations into Lorentzian manifolds. Granting this observation together with the correspondance established by Bryant [Bry] between Willmore surfaces in $\mathbb{R}^3$ and harmonic maps into the Minkowski sphere $S^{3,1}$ in $\mathbb{R}^{4,1}$, the author found not really the technic but at least a strong encouragement for looking for a divergence form to the Willmore Euler-Lagrange equation (1.1).

Our first main result in the present work is the following

**Theorem I.1** *Willmore Euler-Lagrange Equation (1.2) is equivalent to*

$$d \left( *_g d \overline{H} - 3 *_g \pi_n (d \overline{H}) \right) + d \left( \overline{H} \wedge d \overline{n} \right) = 0$$

(1.4)
where *g* is the Hodge operator on *Σ* associated to the induced metric *g* and where we make the implicit identification, by the mean of the standard volume form of *R*^m^, between *m* − 1-vectors and vectors in *R*^m^ (i.e. *H* ∧ *d̂n* is then a 1 form on *Σ* with values into *R*^m^).

Assuming the immersion *Φ* is conformal from the flat disc *D*^2^ = *Σ* into *R*^m^, then *Φ* is Willmore if and only if

\[ \Delta *H* - 3 \text{div}(\pi_\Sigma(\nabla *H*)) - \text{div}\left( *H* ∧ \nabla^\perp *n* \right) = 0 \]  \hspace{1cm} (1.5)

where the operators \( \Delta \), \( \text{div} \) and \( \nabla \) are taken with respect to the flat metric on *D*^2^ (\( \Delta = \partial^2_{x_1} + \partial^2_{x_2} \), \( \text{div} X = \text{tr} \circ \nabla \) and \( \nabla = (\partial_{x_1}, \partial_{x_2}) \)). The operator \( \nabla^\perp \) denotes the rotation by \( \pi/2 \) of \( \nabla \) : \( \nabla^\perp := (-\partial_{x_2}, \partial_{x_1}) \).

This justifies then the following terminology : for a given map \( \bar{n} \) from *D*^2^ into \( G_{m-2}(\mathbb{R}^m) \) we shall denote \( L_{\bar{n}} \) and call it the Willmore operator the operator which to a function \( \bar{v} \) from *D*^2^ into *R*^m^ assigns

\[ L_{\bar{n}}\bar{v} := \Delta \bar{v} - 3 \text{div}(\pi_\Sigma(\nabla \bar{v})) - \text{div}(\bar{v} ∧ \nabla \bar{n}) \]  \hspace{1cm} (1.6)

Although it was not difficult to check it came as a nice surprise that this elliptic operator is self-adjoint : for any choice of map \( \bar{n} \) in \( W^{1,2}(D^2, G_{m-2}(\mathbb{R}^m)) \) and for any choice of compactly supported maps \( \bar{v} \) and \( \bar{w} \) from *D*^2^ into *R*^m^ we have

\[ \int_{D^2} \bar{v} \cdot L_{\bar{n}}\bar{w} = \int_{D^2} L_{\bar{n}}\bar{v} \cdot \bar{w} \]  \hspace{1cm} (1.7)

Another important information is the following

\[ \pi_\Sigma(\bar{v}) := (\bar{v} \cdot \bar{n}(x)) \cdot \bar{n}(x) \]

where \( \cdot \) is the contraction operators between \( p^- \) and \( q^- \) vectors producing \( |p^- q^-| \) vectors in *R*^m^.

Observe that

\[ \text{div}(\pi_\Sigma(\nabla \bar{v})) = \Delta \left[ ((\bar{v} \cdot \bar{n}(x)) \cdot \bar{n}(x)) - (\bar{v} \cdot \nabla \bar{n}(x)) \cdot \bar{n}(x) - (\bar{v} \cdot \bar{n}(x)) \cdot \nabla \bar{n}(x) \right] \]  \hspace{1cm} (1.8)

Thus, assuming now that the unit \( m - 2 \)-vector \( \bar{n} \) is in \( W^{1,2} \), the distribution \( L_{\bar{n}}\bar{v} \) given by (1.4) is well defined for an arbitrary choice of \( \bar{v} \) in \( L^2(D^2) \). This shows that the Euler-Lagrange equation in the form (1.4) or (1.5) is compatible with the Lagrangian (1.4). Indeed the equation has a distributional sense under the least possible regularity requirement for the immersion \( \Phi(\Sigma) \) : This minimal requirement is for the Gauss map to be in \( W^{1,2} \) on \( \Sigma \) with respect to the induced metric.

We assume first that \( \Phi(\Sigma) \) is included in a graph \( G_f \) of a map \( f \) from *D*^2^ into *R*^m−2^. In order to ensure that the Gauss map to that graph is \( W^{1,2} \) on the graph, we are making the minimal regularity requirements on \( f \) which are \( f \in W^{2,2} \) for \( m = 3 \) in one hand and \( f \in W^{1,\infty} \cap W^{2,2} \) for \( m \geq 3 \) in the other hand.

We now define the notion of Weak Willmore Graph. To that purpose it will be convenient to introduce locally a conformal parametrization of the graph \( G_f \). The local existence of such a parametrization is given in [MS] for \( m = 3 \) and \( f \in W^{2,2}(D^2) \) and by theorem 5.1.1 in [Hel] for \( m > 3 \) and \( f \in W^{1,\infty} \cap W^{2,2} \).
Let $f$ be a $W^{2,2}$ function from $D^2$ into $\mathbb{R}$ for $m = 3$, respectively let $f$ be a Lipschitz and $W^{2,2}$ map from $D^2$ into $\mathbb{R}^{m-2}$ for $m > 3$, we say that the graph $G_f$ of $f$ is a Weak Willmore Graph when, in a conformal parametrization $\Phi$ from $D^2$ into $G_f$, the $L^2$ mean curvature vector $\vec{H}$ and the $W^{1,2}$ Gauss map $\vec{n}$ solve the equation.

$$\mathcal{L}_\vec{n} \vec{H} = \Delta \vec{H} - 3 \text{div}(\pi_\vec{n}(\nabla \vec{H})) - \text{div} \left( \vec{H} \wedge \nabla \vec{n} \right) = 0 \quad \text{in } \mathcal{D}'(D^2) \quad (1.9)$$

\[\square\]

Defining $W^{2,2}$ Willmore immersion is a-priori problematic for the following reasons. Let $\Phi$ be a $W^{2,p}$ map from a closed surface into $\mathbb{R}^m$. The assumption that $\Phi$ is an immersion means that $|\partial_{x_1}\Phi \wedge \partial_{x_2}\Phi|/|\nabla \Phi|^2 > 0$ a.e. Assuming that $p > 2$ implies that $\nabla \Phi$ is continuous and therefore the immersion assumption gives a positive lower bound on $\Sigma$ to the function $|\partial_{x_1}\Phi \wedge \partial_{x_2}\Phi|/|\nabla \Phi|^2$. As a consequence we obtain that $\vec{n}$ is also in $W^{1,p}$. Whereas this was a-priori not true for $p = 2$. The assumption that $p > 2$ implies moreover that $\Phi(\Sigma)$ is made of a finite union of $W^{2,p}$ graphs because of the continuity of the Gauss map in that case. Hence, using definition I.1, the assumption that the immersion $\Phi$ is in $W^{2,p}$ for some $p > 2$ permits to define $W^{2,p}$ Willmore immersions. Note that this extends to the situation where we only assume that the second derivatives of $\Phi$ are in the Lorentz space $L^{2,1}$. This assumption on $\Phi$, which is weaker than the $W^{2,p}$ assumption for $p > 2$, still ensures the continuity of the Gauss map $\vec{n}$ and that the second fundamental form of $\Phi(\Sigma)$ is in $L^2$ for the induced metric.

It is a "border line assumption" in that respect. $\nabla^2 \Phi \in L^{2,1}(\Sigma)$ means that for some metric on $\Sigma$

$$\int_0^{+\infty} \left| \{ x \in \Sigma : |\nabla^2 \Phi(x)| \geq \lambda \} \right|^{1/2} < +\infty \quad (1.10)$$

where $|\cdot|$ denotes the measure associated to the choice of metric we made on $\Sigma$. An introduction to Lorentz spaces can be found in [Ta2].

The second main result in the present work is the following

**Theorem I.2** Let $\Sigma$ be a closed surface and let $\Phi$ be an immersion of $\Sigma$ in some euclidian space $\mathbb{R}^m$ with 2 derivatives in the Lorentz space $L^{2,1}$. Assume $\Phi$ is Willmore. Then $\Phi(\Sigma)$ is the image of a real analytic immersion. \[\square\]

Note that since any $W^{2,p}$ map for $p > 2$ has two derivatives in $L^{2,1}$, theorem I.2 holds for $W^{2,p}$ Willmore immersions for $p > 2$.

The previous theorem is a direct consequence of the following $\epsilon$--regularity result. This result was previously established by Kuwert and Schaeztle in [KS1] for strong solutions: in the case where the surface is already assumed to be smooth.

**Theorem I.3** [$\epsilon$--regularity for Weak Willmore Graphs.] Let $m$ be an integer larger than 2. There exists $\epsilon > 0$ such that the following holds. Let $f$ be a function from the unit 2-dimensional disk $D^2$ into $\mathbb{R}^{m-2}$ with second derivatives in the Lorentz space $L^{2,1}$. Assume that the graph of $f$, $G_f \subset \mathbb{R}^m$, is Willmore (according to definition I.1). Denote $\vec{n}$ the Gauss map to this graph
which is a map from \( G_f \) into the Grassmannian of unit \( m - 2 \)-vectors of \( \mathbb{R}^m \). Then, under the following small energy assumption

\[
\int_{G_f} |\nabla \vec{n}|^2 \, d\text{vol}_{G_f} \leq \varepsilon ,
\]

where the metric on \( G_f \) is the one induced by the flat metric of \( \mathbb{R}^m \), we have that the intersection of \( G_f \) with the cylinder \( G_f(1/2) := G_f \cap B_{1/2}(0) \times \mathbb{R}^{m-2} \) is \( C^\infty \) and we have for every \( k \in \mathbb{N} \)

\[
\|\nabla^k \vec{n}\|_{L^\infty(G_f(1/2))} \leq C_k \int_{G_f} |\nabla \vec{n}|^2 \, d\text{vol}_{G_f} ,
\]

where \( C_k \) are universal constants. □

An interesting problem is to study whether this result extends or not under the weaker assumption that \( f \) is in \( W^{2,2} \) for \( m = 3 \) or respectively in \( W^{2,2} \cap W^{1,\infty} \) for \( m > 3 \).

Another fundamental question is to describe the "boundary" of the Moduli spaces of closed Willmore surfaces of given genus and bounded Willmore energy. In other words one aims to describe the limiting behavior of sequences of Willmore surfaces \( S_n \) with fixed topology and bounded Willmore energy. Modulo the action of the Moebius group of conformal transformations of \( \mathbb{R}^m \), which preserves Willmore Lagrangian, and therefore Willmore equation (I.4), we can always fix the area of each \( S_n \) to be equal to 1. Now using Federer Fleming argument we can extract a subsequence to that sequence such that the current of integration on \( S_n \) converges for the Flat topology to some limiting integral current \( S \) (see [Fe] for the terminology of integral currents). Since \( S_0 \) has a uniformly bounded Willmore energy and a fixed topology, the \( L^2 \) norm for the induced metric of it's second fundamental form and hence the \( W^{1,2} \) norm on the surface of the Gauss map are bounded. Applying then theorem [L3] and a classical argument of concentration compactness we then deduce that \( S_n \) converges, in a suitable parametrization, in the \( C^k \) topology to \( S \) outside finitely many points \( \{p_1, \ldots, p_k\} \). This strong convergence implies \( S \) is a smooth Willmore surface a-priori outside these points. The question to know whether these singular points are so called "removable" or not is then fundamental. In the case where \( S \) is a graph in a neighborhood of \( p_k \) the regularity of \( S \) about \( p_k \) is given by following result which extends to arbitrary codimensions the main result of [KSS].

**Theorem I.4 [Point removability for Willmore graphs.]** Let \( f \) be a continuous function from \( B_1^2(0) \) into \( \mathbb{R}^{m-2} \). Assume that \( f \) realizes a \( W^{2,p} \)-Willmore graph over \( D^2 \setminus \{0\} \) for some \( p > 2 \) and that the \( W^{1,2} \) energy of the Gauss map on \( G_f \setminus \{(0, f(0))\} \) is bounded. Then the graph of \( f \) over the whole disk \( D^2 \) is a \( C^{1,\alpha} \) submanifold of \( \mathbb{R}^m \) for every \( \alpha < 1 \). Moreover, if \( \widetilde{H} \) denotes the mean curvature vector of the graph, there exists a constant vector \( \widetilde{H}_0 \) such that \( \widetilde{H}(x) - \widetilde{H}_0 \log|x-x_0| \) is a \( C^{\beta,\alpha} \) function on the graph, where \( x_0 = (0, f(0)) \) and \( |x-x_0| \) denotes the distance in the graph between \( x \) and \( x_0 \). If \( \widetilde{H}_0 = 0 \) then \( G_f \) is an analytic Willmore graph. □

Granting this result for \( m = 3 \) Kuwert and Schätzle were able to establish the fact that \( S \) is a smooth Willmore surface in \( \mathbb{R}^3 \) under the assumption that
the Willmore energy of the $S_n$ is less than $8\pi - \delta$ for any fixed $\delta > 0$. This
last fact ensures that $S$ will be a graph about each $p_i$, $i = 1 \cdots k$ and that the
residues $\tilde{H}_0$ will be equal to 0 at each $p_i$. The arguments to prove that, under
the assumption that $W(S_n) < 8\pi - \delta$, $S$ is a graph about each $p_i$ and that the
residues $\tilde{H}_0$ at each $p_i$ are equal to 0 can be found page 344 of [KS3] and are
not specific to the codimension 1. Therefore combining them with our point
removability result, theorem [I.3] with these arguments we can now state our
last main result

**Theorem I.5 [Weak compactness of Willmore Surfaces below $8\pi$]** Let $m$ be an arbitrary integer larger than 2. Let $\delta > 0$. Consider $S_n \subset \mathbb{R}^m$ to be a sequence of smooth closed Willmore embeddings with uniformly bounded
topology, area equal to one and Willmore energy $W(S_n)$ bounded by $8\pi - \delta$. Assume that $S_n$ converges weakly as varifolds to some limit $S$ which realizes a
non zero current. Then $S$ is a smooth Willmore embedding. \hfill $\square$

Observe that the assumption that $S_n$ is a smooth Willmore immersion com-
bined with the fact that $W(S_n) < 8\pi$ implies that $S_n$ is an embedding due to a
result by Li and Yau [LY].

Finally, combining again arguments in [KS3] (pages 350-351) together with
our point removability result and a theorem by Montiel in [Mon] which in p ar-

**Theorem I.6 [Strong compactness of Willmore torii below $8\pi$]** Let $m = 3$ or $m = 4$. Let $\delta > 0$ arbitrary. The space of Willmore embedded torii in $\mathbb{R}^m$
having Willmore energy less that $8\pi - \delta$ is compact up to Möbius transformations
under smooth convergence of compactly contained surfaces in $\mathbb{R}^m$. \hfill $\square$

This extends to $m = 4$ theorem 5.3 of [KS3] where the above statement was
proved for $m = 3$.

A crucial role in the analysis of Willmore surfaces that we develop in this
paper is played by the following observation. We present it for $m = 3$. Consider
an $L^2$ map $\bar{w}$ from $D^2$ into $\mathbb{R}^3$. Assume it is in the kernel of the Willmore
operator $\mathcal{L}_{\bar{w}}$ for some map $\bar{w}$ in $W^{1,2}(D^2, S^2) : \mathcal{L}_{\bar{w}} \bar{w} = 0$. Introduce the following Hodge decomposition

$$\nabla \bar{w} - 3\pi_N(\nabla \bar{w}) = \nabla \tilde{A} + \nabla^1 \tilde{B}$$

(I.13)

for the boundary condition $\tilde{A} = 0$ on $\partial D^2$ for instance. Denote $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$ the
canonical basis of $\mathbb{R}^3$. Then $\tilde{A}$ and $\tilde{B} = \sum_{i=1}^3 B_i \tilde{e}_i$ solve the following equations

$$\Delta \tilde{A} = \nabla \tilde{H} \wedge \nabla^1 \tilde{u} = \partial_y \tilde{H} \wedge \partial_x \tilde{u} - \partial_x \tilde{H} \wedge \partial_y \tilde{u}$$

$$\Delta B_i = 3 \text{div}(\pi_N(\nabla^1 \bar{w})) = 3 \sum_{j=1}^3 \nabla(n_i, n_j) \cdot \nabla^1 w_j$$

(I.14)

$$= 3 \sum_{j=1}^3 \partial_y(n_i, n_j) \partial_x w_j - \partial_x(n_i, n_j) \partial_y w_j$$

\begin{align*}
\Delta \tilde{A} &= \nabla \tilde{H} \wedge \nabla^1 \tilde{u} \\
\Delta B_i &= 3 \text{div}(\pi_N(\nabla^1 \bar{w})) = 3 \sum_{j=1}^3 \nabla(n_i, n_j) \cdot \nabla^1 w_j \\
&= 3 \sum_{j=1}^3 \partial_y(n_i, n_j) \partial_x w_j - \partial_x(n_i, n_j) \partial_y w_j
\end{align*}
where \( \vec{w} = \sum_{i=1}^{3} w_i \vec{\epsilon}_i \) and \( \vec{n} = \sum_{i=1}^{3} n_i \vec{\epsilon}_i \). The striking fact in the system (I.14) is that all the non-linearities are linear combinations of jacobians. This special algebraic structure plays a special role in geometric analysis. This was probably first discovered by Wente in [Wen]. An in-depth description of this is given in the book of Hélein [Hel]. This role is illustrated by the so called Wente type estimates (see section 3.1 of [Hel]) which are used intensively in the present work.

Final Remarks:

i) The analysis we are developing in this work should give the direction for a new proof to Simon’s result [Si2] on the existence of embedded energy minimizing Willmore Tori in \( \mathbb{R}^m \) for every \( m \geq 3 \).

ii) Our approach should be very useful in the study of the Willmore Flow initiated in the work of Kuwert and Schätzle [KS1], [KS2], [KS3] (see also [Sim]).

iii) Observe that for a Willmore surface the Hodge decomposition (I.13) applied to the mean curvature vector \( \vec{H} \) gives the following system.

\[
\begin{align*}
\Delta \vec{A} &= \nabla \vec{H} \wedge \nabla \perp \vec{n} \\
\Delta \vec{B} &= -3 \nabla H \cdot \nabla \perp \vec{n}
\end{align*}
\]  

(I.15)

Since \( |\nabla \vec{A}|^2 + |\nabla \vec{B}|^2 = 4 |\nabla H|^2 + |H|^2 |\nabla \vec{n}|^2 \), Wente estimates, with optimal Wente constants, applied to (I.15) should give interesting lower bounds to the Willmore energy of Willmore immersions of Tori.

iv) Starting from the conservation law (I.4) it is maybe possible to extend the notion of Willmore surfaces to varifolds with \( L^2 \)–bounded generalized mean curvature (see [Si1] for this last notion). The need to extend Willmore surfaces to a larger class of objects seems as natural as the extension of the notion of minimal surfaces to minimal varifolds is.

The paper is organised as follows. In section 2 we compute the conservation law satisfied by Willmore surfaces (proof of theorem I.1). In section 3 we give a proof of the \( \varepsilon \)–regularity for Willmore graphs, theorem I.3. In section 4 we give a proof of the point removability result for Willmore graphs, theorem I.4. A large part of the paper is devoted to the appendix in which we study, outside the context of Willmore surfaces, various properties of the Willmore operator \( \mathcal{L}_n \).

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II Conservation laws for Willmore surfaces

In this section the operators \( \nabla, \text{div}, \Delta \) will be taken with respect to the flat metric on the unit 2-dimensional disk \( D^2 = \{ z \in \mathbb{C} ; |z| < 1 \} \) Let \( \phi \) be a
smooth conformal embedding of the unit disk \( D^2 \) into \( \mathbb{R}^n \). Denote \( \Sigma = \Phi(D^2) \) and 
\[
e^\lambda = \left| \frac{\partial \Phi}{\partial x_1} \right| = \left| \frac{\partial \Phi}{\partial x_2} \right|
\]
Because of the topology of \( D^2 \), the normal bundle to \( \Sigma \) is trivial and there exists therefore a smooth maps \( \vec{n}(z) = (\vec{n}_1(z), \cdots, \vec{n}_{m-2}(z)) \) from \( D^2 \) into the orthonormal \( m - 2 \) frames in \( \mathbb{R}^m \) satisfying 
\[
(\vec{n}_1(z), \cdots, \vec{n}_{m-2}(z)) \quad \text{is a positive orthonormal basis to } N_{\Phi(z)} \Sigma,
\]
where \( N_{\Phi(z)} \Sigma = (T_{\Phi(z)} \Sigma)^\perp \) is the orthonormal \( m - 2 \) plane to the tangent plane \( T_{\Phi(z)} \Sigma \) of \( \Sigma \) at \( \Phi(z) \). We denote by \( (\vec{e}_1, \vec{e}_2) \) the orthonormal basis of \( T_{\Phi(z)} \Sigma \) given by 
\[
\vec{e}_i = e^{-\lambda} \frac{\partial \Phi}{\partial x_i}.
\]
With these notations the second fundamental form \( h \) which is a symmetric \( 2 \)-form on \( T_{\Phi(z)} \Sigma \) into \( N_{\Phi(z)} \Sigma \) is given by 
\[
h = \sum_{\alpha,i,j} h_{\alpha}^{ij} \vec{n}_\alpha \otimes (\vec{e}_i)^* \otimes (\vec{e}_j)^* \quad \text{with} \quad h_{\alpha}^{ij} = -e^{-\lambda} \left( \frac{\partial \vec{n}_\alpha}{\partial x_i}, \vec{e}_j \right) \tag{II.16}
\]
In particular the mean curvature vector \( \vec{H} \) is given by 
\[
\vec{H} = \sum_{\alpha=1}^{m-2} H_\alpha \vec{n}_\alpha = \frac{1}{2} \sum_{\alpha=1}^{m-2} (h_{11}^\alpha + h_{22}^\alpha) \vec{n}_\alpha \tag{II.17}
\]
Let \( \vec{n} \) be the \( m - 2 \) vector of \( \mathbb{R}^m \) given by \( \vec{n} = \vec{n}_1 \wedge \cdots \wedge \vec{n}_2 \). We identify vectors and \( m - 1 \)-vectors in \( \mathbb{R}^m \) in the usual way. Hence we have for instance 
\[
\vec{n} \wedge \vec{e}_1 = (-1)^{m-1} \vec{e}_2 \quad \text{and} \quad \vec{n} \wedge \vec{e}_2 = (-1)^{m-2} \vec{e}_1 \tag{II.18}
\]
Since \( \vec{e}_1, \vec{e}_2, \vec{n}_1, \cdots, \vec{n}_{m-2} \) is a basis of \( T_{\Phi(z)} \mathbb{R}^m \), we can write for every \( \alpha = 1 \cdots m - 2 \) 
\[
\nabla \vec{n}_\alpha = \sum_{\beta=1}^{m-2} (\nabla \vec{n}_\alpha, \vec{n}_\beta) \vec{n}_\beta + \sum_{i=1}^{2} (\nabla \vec{n}_\alpha, \vec{e}_i) \vec{e}_i
\]
and consequently 
\[
\vec{n} \wedge \nabla \perp \vec{n}_\alpha = (-1)^{m-1} \left( \nabla \perp \vec{n}_\alpha, \vec{e}_1 \right) \vec{e}_2 + (-1)^{m-2} \left( \nabla \perp \vec{n}_\alpha, \vec{e}_2 \right) \vec{e}_1 \tag{II.19}
\]
Using the symmetry of the second fundamental form (i.e. \( h_{ij}^\alpha = h_{ji}^\alpha \)) and the conformality of \( \Phi \) we have 
\[
\left( \frac{\partial \vec{n}_\alpha}{\partial x_1}, \vec{e}_2 \right) = \left( \frac{\partial \vec{n}_\alpha}{\partial x_2}, \vec{e}_1 \right) \tag{II.20}
\]
Thus, combining (II.19) and (II.20), we have
\[ \vec{n} \wedge \nabla^\perp \vec{n} \alpha \]
\[ = (-1)^{m-1} \begin{pmatrix} -\frac{\partial \vec{n}_\alpha}{\partial x_1} \cdot \vec{e}_2 \\ \frac{\partial \vec{n}_\alpha}{\partial y} \cdot \vec{e}_1 \end{pmatrix} \vec{e}_2 + (-1)^{m-1} \begin{pmatrix} -\frac{\partial \vec{n}_\alpha}{\partial x_2} \cdot \vec{e}_2 \\ \frac{\partial \vec{n}_\alpha}{\partial y} \cdot \vec{e}_1 \end{pmatrix} \vec{e}_1 \]
\[ + (-1)^{m-1} \left[ \begin{pmatrix} -\frac{\partial \vec{n}_\alpha}{\partial x_1} \cdot \vec{e}_1 \\ \frac{\partial \vec{n}_\alpha}{\partial x_2} \cdot \vec{e}_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \vec{e}_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \vec{e}_2 \right] \] (II.21)

This implies
\[ \nabla \vec{n}_\alpha + (-1)^{m-1} \vec{n} \wedge \nabla^\perp \vec{n} \alpha \]
\[ = \sum_{\beta=1}^{m-2} (\nabla \vec{n}_\alpha, \vec{n}_\beta) \cdot \vec{n}_\beta - 2e^\lambda H^\alpha \left[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \vec{e}_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \vec{e}_2 \right] \] (II.22)

Since \( \partial_\alpha \Phi = e^\lambda \vec{e}_1 \) and \( \partial_\beta \Phi = e^\lambda \vec{e}_2 \) we finally get the following important identity
\[ \nabla \vec{n}_\alpha + (-1)^{m-1} \vec{n} \wedge \nabla^\perp \vec{n} \alpha = \sum_{\beta=1}^{m-2} (\nabla \vec{n}_\alpha, \vec{n}_\beta) \cdot \vec{n}_\beta - 2H^\alpha \nabla \Phi \] (II.23)

Following the "Coulomb Gauge extraction Method" presented in the proof of lemma 4.1.3 of [Hel] we can choose a trivialization \((\vec{n}_1 \cdots \vec{n}_{m-2})\) of the orthonormal frame bundle associated to our trivial bundle \(N\Sigma\) satisfying
\[ \forall 1 \leq \alpha, \beta \leq m-2 \quad \text{div}(\nabla \vec{n}_\alpha, \vec{n}_\beta) = 0 \] (II.24)

Combining (II.23) and (II.24) we obtain in one hand
\[ \text{div} \left( \nabla \vec{n}_\alpha + (-1)^{m-1} \vec{n} \wedge \nabla^\perp \vec{n} \alpha \right) \]
\[ = \sum_{\beta=1}^{m-2} (\nabla \vec{n}_\alpha, \vec{n}_\beta) \cdot \nabla \vec{n}_\beta - 2\nabla H^\alpha \cdot \nabla \Phi - 2H^\alpha \Delta \Phi \] (II.25)

In the other hand a classical elementary computation gives
\[ \Delta \Phi = 2e^{2\lambda} \vec{H} \] (II.26)

Therefore combining the two last identities we obtain
\[ \text{div} \left( \nabla \vec{n}_\alpha + (-1)^{m-1} \vec{n} \wedge \nabla^\perp \vec{n} \alpha \right) \]
\[ = \sum_{\beta=1}^{m-2} (\nabla \vec{n}_\alpha, \vec{n}_\beta) \cdot \nabla \vec{n}_\beta - 2\nabla H^\alpha \cdot \nabla \vec{\phi} - 4e^{2\lambda} H^\alpha \vec{H} \] (II.27)
Multiplying this identity by $H^\alpha$, summing over $\alpha$ between 1 and $m - 2$ and using the expression of $H^\alpha \nabla \Phi$ given by (II.23) we obtain

\[
\sum_{\alpha=1}^{m-2} H^\alpha \operatorname{div} (\nabla \vec{n}_\alpha + (-1)^{m-1} \vec{n} \wedge \nabla^\perp \vec{n}_\alpha) - \sum_{\alpha=1}^{m-2} \nabla H^\alpha \cdot \nabla \vec{n}_\alpha
\]

\[
-(-1)^{m-1} \sum_{\alpha=1}^{m-2} \nabla H^\alpha \cdot \vec{n} \wedge \nabla^\perp \vec{n}_\alpha
\]

\[
= \sum_{\alpha,\beta=1}^{m-2} H^\alpha (\nabla \vec{n}_\alpha, \vec{n}_\beta) \cdot \nabla \vec{n}_\beta - \sum_{\alpha,\beta=1}^{m-2} \nabla H^\alpha \cdot (\nabla \vec{n}_\alpha, \vec{n}_\beta) \vec{n}_\beta - 4 e^{2\lambda} H^\alpha \vec{H} .
\]  

(II.28)

For a purpose that will be clear later we rewrite equation (II.28) in the following way

\[
\sum_{\alpha=1}^{m-2} H^\alpha \Delta \vec{n}_\alpha - \sum_{\alpha=1}^{m-2} \nabla H^\alpha \cdot \nabla \vec{n}_\alpha - (-1)^{m-1} \sum_{\alpha=1}^{m-2} \operatorname{div} (H^\alpha \vec{n} \wedge \nabla^\perp \vec{n}_\alpha)
\]

\[
= \sum_{\alpha,\beta=1}^{m-2} H^\alpha (\nabla \vec{n}_\alpha, \vec{n}_\beta) \cdot \nabla \vec{n}_\beta - \sum_{\alpha,\beta=1}^{m-2} \nabla H^\alpha \cdot (\nabla \vec{n}_\alpha, \vec{n}_\beta) \vec{n}_\beta - 2(-1)^{m-1} \sum_{\alpha=1}^{m-2} H^\alpha \operatorname{div} (\vec{n} \wedge \nabla^\perp \vec{n}_\alpha) - 4 e^{2\lambda} H^\alpha \vec{H} .
\]  

(II.29)

We shall give now an expression of $(-1)^{m-1} \sum_{\alpha=1}^{m-2} H^\alpha \operatorname{div} (\vec{n} \wedge \nabla^\perp \vec{n}_\alpha)$ which will be useful in the sequel. Using (II.19) we have in one hand

\[
(-1)^{m-1} \operatorname{div} (\vec{n} \wedge \nabla^\perp \vec{n}_\alpha)
\]

\[
= (\nabla^\perp \vec{n}_\alpha, \nabla \vec{e}_1) \vec{e}_2 - (\nabla^\perp \vec{n}_\alpha, \nabla \vec{e}_2) \vec{e}_1
\]

\[
+ (\nabla^\perp \vec{n}_\alpha, \vec{e}_1) \nabla \vec{e}_2 - (\nabla^\perp \vec{n}_\alpha, \vec{e}_2) \nabla \vec{e}_1
\]

\[
= \sum_{\beta=1}^{m-2} (\nabla^\perp \vec{n}_\alpha, \vec{n}_\beta) \left[ (\vec{n}_\beta, \nabla \vec{e}_1) \vec{e}_2 - (\vec{n}_\beta, \nabla \vec{e}_2) \vec{e}_1 \right]
\]

\[
+ \sum_{\beta=1}^{m-2} \left[ (\nabla^\perp \vec{n}_\alpha, \vec{e}_1) (\nabla \vec{e}_2, \vec{n}_\beta) - (\nabla^\perp \vec{n}_\alpha, \vec{e}_2) (\nabla \vec{e}_1, \vec{n}_\beta) \right] \vec{n}_\beta .
\]  

(II.30)
In the other hand, using the symmetry of \( h \),

\[
\begin{align*}
(\nabla^\perp \tilde{n}_\alpha, \tilde{e}_1) (\nabla \tilde{e}_2, \tilde{n}_\beta) - (\nabla^\perp \tilde{n}_\alpha, \tilde{e}_2) (\nabla \tilde{e}_1, \tilde{n}_\beta) &= \\
= \left(\frac{\partial \tilde{e}_2}{\partial x_1} \tilde{n}_\alpha - \frac{\partial \tilde{e}_1}{\partial x_2} \tilde{n}_\alpha \right) \\
&+ \left(\frac{\partial \tilde{e}_2}{\partial x_2} \tilde{n}_\alpha - \frac{\partial \tilde{e}_1}{\partial x_1} \tilde{n}_\alpha \right) - \\
&= e^{2\lambda} \left[ h_{12}^\alpha h_{12}^\beta - h_{22}^\alpha h_{11}^\beta + h_{11}^\alpha h_{22}^\beta \right] \\
&= e^{2\lambda} \sum_{i,j} h_{ij}^\alpha h_{ij}^\beta - 4e^{2\lambda} H^\alpha H^\beta .
\end{align*}
\]

Combining (II.30) and (II.31) we obtain

\[
\begin{align*}
(-1)^{m-1} & \sum_{\alpha=1}^{m-2} H^\alpha \div (\tilde{n} \wedge \nabla^\perp \tilde{n}_\alpha) \\
&= \sum_{\alpha,\beta=1}^{m-2} H^\alpha (\nabla^\perp \tilde{n}_\alpha, \tilde{n}_\beta) \left[(\tilde{n}_\beta, \nabla \tilde{e}_1) \tilde{e}_2 - (\tilde{n}_\beta, \nabla \tilde{e}_2) \tilde{e}_1 \right] \\
&+ e^{2\lambda} \sum_{\alpha,\beta,i,j} h_{ij}^\alpha h_{ij}^\beta - 4e^{2\lambda} |\tilde{H}|^2 \tilde{H} .
\end{align*}
\]

That we rewrite in the following form

\[
\begin{align*}
-2(-1)^{m-1} & \sum_{\alpha=1}^{m-2} H^\alpha \div (\tilde{n} \wedge \nabla^\perp \tilde{n}_\alpha) - 4e^{2\lambda} |\tilde{H}|^2 \tilde{H} \\
&= -2 \sum_{\alpha,\beta=1}^{m-2} H^\alpha (\nabla^\perp \tilde{n}_\alpha, \tilde{n}_\beta) \left[(\tilde{n}_\beta, \nabla \tilde{e}_1) \tilde{e}_2 - (\tilde{n}_\beta, \nabla \tilde{e}_2) \tilde{e}_1 \right] \\
&- 2e^{2\lambda} \sum_{\alpha,\beta,i,j} h_{ij}^\alpha h_{ij}^\beta + 4e^{2\lambda} |\tilde{H}|^2 \tilde{H} .
\end{align*}
\]

\( \tilde{H} \) is a section of \( N\Sigma \). By definition the covariant (negative) laplacian of \( \tilde{H} \) for the connection given by the orthogonal projection (w.r.t. the standard scalar product in \( \mathbb{R}^m \)) on the fibers is given by

\[
e^{2\lambda} \Delta_{\mathcal{L}} \tilde{H} := \pi_{\tilde{c}} \div (\pi_{\tilde{c}}(\nabla \tilde{H})) ,
\]

where \( \pi_{\tilde{c}} \) is the orthogonal projection on the fibers of \( N\Sigma \). Using (II.24) we have that

\[
e^{2\lambda} \Delta_{\mathcal{L}} \tilde{H} := \pi_{\tilde{c}} \div (\nabla H^\alpha \tilde{n}_\alpha + H^\alpha (\nabla \tilde{n}_\alpha, \tilde{n}_\beta) \tilde{n}_\beta) \\
= \sum_{\alpha} \Delta H^\alpha \tilde{n}_\alpha + 2 \sum_{\alpha,\beta} \nabla H^\alpha \cdot (\nabla \tilde{n}_\alpha, \tilde{n}_\beta) \tilde{n}_\beta \\
+ \sum_{\alpha,\beta,\gamma} H^\alpha (\nabla \tilde{n}_\alpha, \tilde{n}_\beta) (\nabla \tilde{n}_\beta, \tilde{n}_\gamma) \tilde{n}_\gamma .
\]
Assuming now that our embedding is Willmore, it is equivalent to assume that \( \vec{H} \) satisfies (I.2) (see [Wei]) which means with the present notations that \( h \) satisfies
\[
\Delta \vec{H} + \sum_{i,j,\alpha,\beta} h_{ij}^\alpha h_{ij}^\beta \vec{H}_\alpha - 2|\vec{H}|^2 \vec{H} = 0 , \tag{II.35}
\]
Thus, for a Willmore embedding the following identity holds
\[
-2(-1)^{m-1} \sum_{\alpha=1}^{m-2} H^\alpha \cdot \nabla H^\alpha - 4e^{2\lambda} |\vec{H}|^2 \vec{H}
= -2 \sum_{\alpha,\beta=1}^{m-2} H^\alpha (\nabla \vec{H}_\alpha, \vec{H}_\beta) \left[ (\vec{H}_\beta, \nabla \vec{e}_1) \vec{e}_2 - (\vec{H}_\beta, \nabla \vec{e}_2) \vec{e}_1 \right]
+ 2e^{2\lambda} \Delta \vec{H}
\tag{II.36}
\]
Combining (II.29), (II.34) and (II.36) we obtain that our conformal embedding \( \Phi \) is Willmore if and only if the following identity holds
\[
\sum_{\alpha=1}^{m-2} H^\alpha \Delta \vec{H}_\alpha - \sum_{\alpha=1}^{m-2} \nabla H^\alpha \cdot \nabla \vec{H}_\alpha - (-1)^{m-1} \sum_{\alpha=1}^{m-2} \text{div} \left( H^\alpha \nabla \vec{H}_\alpha \right)
= \sum_{\alpha,\beta=1}^{m-2} H^\alpha (\nabla \vec{H}_\alpha, \vec{H}_\beta) \cdot \nabla \vec{H}_\beta - \sum_{\alpha,\beta=1}^{m-2} \nabla H^\alpha \cdot (\nabla \vec{H}_\alpha, \vec{H}_\beta)
-2 \sum_{\alpha,\beta=1}^{m-2} H^\alpha (\nabla \vec{H}_\alpha, \vec{H}_\beta) \left[ (\vec{H}_\beta, \nabla \vec{e}_1) \vec{e}_2 - (\vec{H}_\beta, \nabla \vec{e}_2) \vec{e}_1 \right]
+ 2 \sum_{\alpha} \Delta H^\alpha \vec{H}_\alpha + 4 \sum_{\alpha,\beta} \nabla H^\alpha \cdot (\nabla \vec{H}_\alpha, \vec{H}_\beta)
+ 2 \sum_{\alpha,\beta,\gamma} H^\alpha (\nabla \vec{H}_\alpha, \vec{H}_\beta) (\nabla \vec{H}_\beta, \vec{H}_\gamma) \vec{H}_\gamma
\tag{II.37}
\]
We prove now that the previous identity (II.37) can be written in divergence form. First we have

\[
(\nabla \cdot \vec{n}_\alpha, \vec{n}_\beta) \left[(\vec{n}_\beta, \nabla e_1) \ e_2 - (\vec{n}_\beta, \nabla e_2) \ e_1\right] = \left[\partial_{\vec{n}_\alpha} \frac{\partial \vec{n}_\alpha}{\partial x_1}, \vec{n}_\beta\right] \left(\partial_{\vec{n}_\beta} e_1 \ e_2 - \left(\partial_{\vec{n}_\beta} e_2 \right) \ e_1\right)
\]

\[
= -2 \left(\partial_{\vec{n}_\alpha} \frac{\partial \vec{n}_\alpha}{\partial x_2}, \vec{n}_\beta\right) \left(\partial_{\vec{n}_\beta} \frac{\partial \vec{n}_\beta}{\partial x_1} \ e_2\right) - \left(\partial_{\vec{n}_\alpha} \frac{\partial \vec{n}_\alpha}{\partial x_1}, \vec{n}_\beta\right) \left(\partial_{\vec{n}_\beta} \frac{\partial \vec{n}_\beta}{\partial x_2} \ e_2\right) + 2 \left(\partial_{\vec{n}_\alpha} \frac{\partial \vec{n}_\alpha}{\partial x_1}, \vec{n}_\beta\right) \left(\partial_{\vec{n}_\beta} \frac{\partial \vec{n}_\beta}{\partial x_2} \ e_1\right)
\]

\[
= \sum_{\alpha, \beta} H^\alpha H^\beta \left(\partial_{\vec{n}_\alpha} \frac{\partial \vec{n}_\alpha}{\partial x_1}, \vec{n}_\beta\right) \left(\partial_{\vec{n}_\beta} \frac{\partial \vec{n}_\beta}{\partial x_2} \ e_1\right) - \sum_{\alpha, \beta} H^\alpha \left(\nabla \cdot \vec{n}_\alpha, \vec{n}_\beta\right) \cdot \nabla \Phi = 0\]  

(II.40)

Thus we have

\[
-2 \sum_{\alpha, \beta=1}^{m-2} H^\alpha H^\beta \left(\nabla \cdot \vec{n}_\alpha, \vec{n}_\beta\right) \cdot \nabla \Phi = 2 \sum_{\alpha, \beta, \gamma=1}^{m-2} H^\alpha H^\beta \left(\nabla \cdot \vec{n}_\alpha, \vec{n}_\beta\right) \cdot \nabla \Phi\]  

(II.41)

Since \((\nabla \cdot \vec{n}_\alpha, \vec{n}_\beta) = -(\nabla \vec{n}_\beta, \vec{n}_\alpha)\), we deduce that

\[
\sum_{\alpha, \beta=1}^{m-2} H^\alpha H^\beta \left(\nabla \cdot \vec{n}_\alpha, \vec{n}_\beta\right) \cdot \nabla \Phi \equiv 0\]  

and we finally get that

\[
-2 \sum_{\alpha, \beta=1}^{m-2} H^\alpha H^\beta \left(\nabla \cdot \vec{n}_\alpha, \vec{n}_\beta\right) \cdot \nabla \Phi
\]

\[
+2 \sum_{\alpha, \beta, \gamma=1}^{m-2} H^\alpha H^\beta \left(\nabla \cdot \vec{n}_\alpha, \vec{n}_\beta\right) \cdot \nabla \vec{n}_\gamma
\]

\[
= 2 \sum_{\alpha, \beta=1}^{m-2} H^\alpha H^\beta \left(\nabla \cdot \vec{n}_\alpha, \vec{n}_\beta\right) \cdot \nabla \vec{n}_\beta
\]
Substituting this identity in (II.37) we obtain that the conformal embedding $\Phi$ is Willmore if and only if

$$m - 2 \sum_{\alpha=1}^{m-2} \nabla^2 \bar{n}_\alpha - \sum_{\alpha=1}^{m-2} \nabla H^\alpha \cdot \nabla \bar{n}_\alpha = (-1)^{m-1} \sum_{\alpha=1}^{m-2} \text{div} \left( H^\alpha \bar{n} \wedge \nabla^\perp \bar{n}_\alpha \right)$$

$$= 3 \sum_{\alpha,\beta=1}^{m-2} H^\alpha (\nabla \bar{n}_\alpha, \bar{n}_\beta) \cdot \nabla \bar{n}_\beta + 2 \sum_{\alpha=1}^{m-2} \Delta H^\alpha \bar{n}_\alpha$$

(II.42)

Using the condition (II.24) satisfied by our special choice of trivialization of the normal bundle, we rewrite the previous identity in the following way

$$\sum_{\alpha=1}^{m-2} H^\alpha \nabla^2 \bar{n}_\alpha - \sum_{\alpha=1}^{m-2} \nabla H^\alpha \cdot \nabla \bar{n}_\alpha = (-1)^{m-1} \sum_{\alpha=1}^{m-2} \text{div} \left( H^\alpha \bar{n} \wedge \nabla^\perp \bar{n}_\alpha \right)$$

$$= 3 \text{div} \left[ \sum_{\alpha,\beta=1}^{m-2} H^\alpha (\nabla \bar{n}_\alpha, \bar{n}_\beta) \bar{n}_\beta \right] + 2 \sum_{\alpha=1}^{m-2} \Delta H^\alpha \bar{n}_\alpha \quad (II.43)$$

Observe in one hand that

$$- (-1)^{m-1} \sum_{\alpha=1}^{m-2} H^\alpha \bar{n} \wedge \nabla^\perp \bar{n}_\alpha = \sum_{\alpha=1}^{m-2} H^\alpha \nabla^\perp \bar{n}_\alpha \wedge \bar{n} = \nabla^\perp \bar{H} \wedge \bar{n}, \quad (II.44)$$

and in the other hand that

$$\sum_{\alpha,\beta=1}^{m-2} H^\alpha (\nabla \bar{n}_\alpha, \bar{n}_\beta) \bar{n}_\beta = \sum_{\beta=1}^{m-2} (\nabla \bar{H}, \bar{n}_\beta) \bar{n}_\beta \quad . \quad (II.45)$$

Using these 2 observations the Willmore equation (II.43) becomes

$$\Delta \bar{H} - 3 \sum_{\alpha=1}^{m-2} \nabla H^\alpha \cdot \nabla \bar{n}_\alpha - 3 \sum_{\alpha=1}^{m-2} \Delta H^\alpha \bar{n}_\alpha + \text{div} \left( \nabla^\perp \bar{H} \wedge \bar{n} \right)$$

$$= 3 \text{div} \left[ \sum_{\beta=1}^{m-2} (\nabla \bar{H}, \bar{n}_\beta) \bar{n}_\beta \right] \ . \quad (II.46)$$

Observe now that

$$\sum_{\alpha=1}^{m-2} \nabla H^\alpha \bar{n}_\alpha + \sum_{\beta=1}^{m-2} (\nabla \bar{H}, \bar{n}_\beta) \bar{n}_\beta$$

$$= \nabla \bar{H} - \sum_{\alpha=1}^{m-2} H^\alpha \nabla \bar{n}_\alpha + \sum_{\alpha,\beta=1}^{m-2} H^\alpha (\nabla \bar{n}_\alpha, \bar{n}_\beta) \bar{n}_\beta$$

$$= \nabla \bar{H} - \sum_{\alpha=1}^{m-2} H^\alpha (\nabla \bar{n}_\alpha, \bar{e}_1) \bar{e}_1 - \sum_{\alpha=1}^{m-2} H^\alpha (\nabla \bar{n}_\alpha, \bar{e}_2) \bar{e}_2$$

$$= \nabla \bar{H} - (\nabla \bar{H}, \bar{e}_1) \bar{e}_1 - (\nabla \bar{H}, \bar{e}_2) \bar{e}_2 = \pi_{\bar{n}}(\nabla \bar{H}) \quad (II.47)$$
So finally we have proved that the Willmore equation can be written in the following form

\[ \Delta \vec{H} - 3 \text{div}(\pi \vec{n}(\nabla \vec{H})) + \text{div} \left( \nabla^\perp \vec{H} \wedge \vec{n} \right) = 0 \]  

(II.48)

### III $\epsilon$–regularity for Weak Willmore graphs

This section is devoted to the proof of the $\epsilon$–regularity theorem I.3.

Let then $f$ be a function from $D^2$ into $\mathbb{R}^{m-2}$ with 2 derivatives in the Lorentz space $L^{2,1}$ realizing a Willmore graph. We consider the bilipschitz conformal parametrization from $D^2$ into $G_f$ given by theorem 5.1.1 of [Hei]. Call this parametrization $\Phi$. According to definition I.1 this means that the mean curvature $\vec{H}$ satisfies the equation (I.9). Denote by $\chi$ a smooth cut-off function equal to 1 on $D_{\frac{1}{2}}$ and compactly supported in $D_2$. Since $L^2(\vec{n} \nabla^\perp \vec{H}) = 0$, we have

\[ L^2(\vec{n} \chi \vec{H}) = 2 \text{div}(\nabla \chi \vec{H}) - \vec{H} \Delta \chi - 6 \text{div}(\pi \vec{n}(\vec{H}) \nabla \chi) \]

(III.49)

Let

\[ g_1 = 2 \text{div}(\nabla \chi \vec{H}) - \vec{H} \Delta \chi - 6 \text{div}(\pi \vec{n}(\vec{H}) \nabla \chi) \]

(III.50)

and

\[ g_2 = 3(\vec{H} \cdot \nabla \vec{n}) \cdot \vec{n} \nabla \chi + 3(\vec{H} \cdot \vec{n}) \cdot \nabla \vec{n} \nabla \chi - \vec{H} \wedge \nabla^\perp \vec{n} \nabla \chi \]

(III.51)

We have

\[ \|g_1\|_{L^2(D^2)} \leq C \int_{D^2 \setminus D_{\frac{1}{2}}} |\vec{H}|^2 \]

(III.52)

and

\[ \|g_2\|_{L^1(D^2)} \leq C \int_{D^2 \setminus D_{\frac{1}{2}}} |\vec{H}| |\nabla \vec{n}| \]

(III.53)

Denote $\vec{v}_1$ the solution of (A.39) given by lemma A.1 for $\vec{g} = \vec{g}_1$ and let $\vec{v}_2$ be the solution of (A.30) given by lemma A.3 for $\vec{g} = \vec{g}_2$. We have in particular

\[ \|\nabla \vec{v}_1\|_{L^2} \leq C \left[ \int_{D^2 \setminus D_{\frac{1}{2}}} |\vec{H}|^2 \right]^{\frac{1}{2}} \]

(III.54)

and

\[ \|\nabla \vec{v}_2\|_{L^2(D^2)} \leq C \int_{D^2 \setminus D_{\frac{1}{2}}} |\vec{H}| |\nabla \vec{n}| \leq C \varepsilon^2 \left[ \int_{D^2 \setminus D_{\frac{1}{2}}} |\vec{H}|^2 \right]^{\frac{1}{2}} \]

(III.55)

$\vec{v} := \chi \vec{H} - \vec{v}_1 - \vec{v}_2$ is in $L^2(D^2)$ and satisfies $L^2(\vec{v}) = 0$ we also observe that since $\chi$ is compactly supported in $D^2$, $\nabla \vec{v}$ is a sum of a compactly supported
distribution in the interior of $D^2$ and a $L^{2,\infty}$ function. The trace of $\tilde{v}$ on $\partial D^2$ is therefore well defined and equals zero. Assuming $\nabla \tilde{n}$ is in the Lorentz space $L^{2,1}(D^2)$, we are now in position to apply lemma A.3 and we deduce that $\tilde{v}$ is identically 0. Thus we have proved that $\nabla (\chi \tilde{H})$ is $L^{2,\infty}$ and

$$
\| \nabla (\chi \tilde{H}) \|_{L^{2,\infty}(D^2)} \leq C \left[ \int_{D^2 \setminus D_{1/2}^1} \tilde{H}^2 \right]^{\frac{1}{2}}. \tag{III.56}
$$

We have $\mathcal{L}_1(\chi \tilde{H}) = \mathcal{L}_1(\tilde{H}) = 0$ on $D_{1/2}^1$. We proceed to the following Hodge decomposition of $\nabla \tilde{H} - 3\tilde{n}(\nabla \tilde{H})$ on $D_{1/2}^1$: $\nabla \tilde{H} - 3\tilde{n}(\nabla \tilde{H}) = \nabla C + \nabla D + \tilde{r}$ with the boundary conditions $C = 0$ on $\partial D_{1/2}^1$ and $\partial D/\partial \nu = 0$ on $D_{1/2}^1$ and where $\tilde{r}$ is harmonic. The following equations then holds in one hand

$$
\begin{cases}
\Delta C = \text{div}(\tilde{H} \wedge \nabla \tilde{n}) & \text{in } D_{1/2}^1 \\
C = 0 & \text{on } \partial D_{1/2}^1
\end{cases} \tag{III.57}
$$

In the other hand

$$
\begin{cases}
\Delta D = 3 \text{div}(\tilde{n}(\nabla \tilde{H})) & \text{in } D_{1/2}^1 \\
\frac{\partial D}{\partial \nu} = 0 & \text{on } \partial D_{1/2}^1
\end{cases} \tag{III.58}
$$

Using now the right-hand-sides of (III.57, III.58) are Jacobians of $\tilde{H}$ and $\tilde{n}$, since $\nabla \tilde{n} \in L^2(D_{1/2}^1)$ and $\nabla \tilde{H} \in L^{2,\infty}(D_{1/2}^1)$, we have, using the Wente estimate A.30,

$$
\| \nabla C \|_{L^2(D_{1/2}^1)} + \| \nabla D \|_{L^2(D_{1/2}^1)} \leq C \| \nabla \tilde{n} \|_{L^2(D_{1/2}^1)} \| \nabla \tilde{H} \|_{L^{2,\infty}(D_{1/2}^1)} \tag{III.59}
$$

Since $\tilde{r}$ is harmonic we have that $\| \tilde{r} \|_{L^2(D_{1/4}^1)} \leq C \| \tilde{r} \|_{L^{2,\infty}(D_{1/2}^1)}$. Combining this last inequality together with (III.59), and the fact that $|\nabla \tilde{H} - 3\tilde{n}(\nabla \tilde{H})| \geq |\nabla \tilde{H}|$, we have established in particular that

$$
\| \nabla \tilde{H} \|_{L^2(D_{1/4}^1)} \leq C \| \nabla \tilde{n} \|_{L^2(D_{1/2}^1)} \| \nabla \tilde{H} \|_{L^{2,\infty}(D_{1/2}^1)} + \| \nabla \tilde{H} \|_{L^{2,\infty}(D_{1/2}^1)} \tag{III.60}
$$

Using one more time the same Hodge decomposition but on $D_{1/4}^1$ instead of $D_{1/2}^1$, and replacing Wente’s estimate A.30 by Wente’s inequality (3.47) in theorem 3.4.1 of [Hel] (which was originally obtained by Luc Tartar in [Ta1]) and arguing similarly as before we get

$$
\| \nabla \tilde{H} \|_{L^{2,1}(D_{1/4}^1)} \leq C \| \nabla \tilde{n} \|_{L^2(D_{1/2}^1)} \| \nabla \tilde{H} \|_{L^2(D_{1/4}^1)} + \| \nabla \tilde{H} \|_{L^{2,\infty}(D_{1/4}^1)} \tag{III.61}
$$

Using now equation (II.24), since $\nabla \lambda \in L^{2,1}(D^2)$ (see [Hel]), we have that $\nabla (\Delta \tilde{\phi})$ is in $L^{2,1}(D_{1/8}^1)$ and this implies that $\nabla \tilde{n}$ is in $L^\infty(D_{1/8}^1)$. Moreover, combining (III.56), (III.60) and (III.61) we obtain

$$
\| \nabla \tilde{n} \|_{L^\infty(D_{1/8}^1)} \leq C \int_{D^2} |\nabla \tilde{n}|^2, \tag{III.62}
$$

and theorem I.3 is proved.

$\square$
IV Point removability for Willmore graphs.

This section is devoted to the proof of theorem I.4

Under the assumptions of the theorem we shall consider the Lipshitz conformal parametrization $\Phi$ from $D^2$ into $G_1$, the graph of $f$, obtained by following the arguments of [KS3] (pages 332-334) which are based on the use of Huber's result on conformal parametrizations of complete surfaces in $\mathbb{R}^m$ [Hub] together with the estimates given by the work of Müller and Sverák [MS]. The preimage of $(0, f(0))$ by $\Phi$ being in the inside of $D^2$ we can always assume (after a possible composition by a Moebius transformation of $D^2$) that $\Phi^{-1}((0, f(0)) = 0$. Using theorem (3) we have that $\Phi$ is $C^\infty$ in $D^2 \setminus \{0\}$, moreover, from (1.12), there exist a positive function $\delta(r)$ going to zero as $r$ goes to zero such that

$$\forall x \in D^2 \setminus \{0\} \quad |x| |\nabla \tilde{n}(x)| + |x|^2 |\nabla^2 \tilde{n}| \leq \delta(|x|) \quad .$$  (IV.63)

The distances are taken with respect to the flat metric on $D^2$, it is however equivalent to the distance for the induced metric in the graph due to the estimates in [MS].

A positive number $\varepsilon$ being given, we can restrict to a smaller disk and dilate to ensure that

$$\|x| |\nabla \tilde{n}(x)||^2 + \int_{D^2} |\nabla \tilde{n}|^2 \leq \varepsilon \quad .$$  (IV.64)

Since $\tilde{H}$ is in $L^2(D^2)$ the distribution $L_{\tilde{r}}\tilde{H}$ makes sense in $D'(D^2)$. Moreover, since $\tilde{\Phi}$ is Willmore in $D^2 \setminus \{0\}$, the distribution $L_{\tilde{r}}\tilde{H}$ is supported in zero and is therefore a finite linear combination of derivatives of the Dirac mass at the origin. Since $L_{\tilde{r}}\tilde{H}$ is a sum of an $H^{-2}$ distribution and derivatives of $L^1$ functions it can only be proportional to the Dirac mass centered at the origin itself:

$$L_{\tilde{r}}\tilde{H} = \tilde{c}_0 \tilde{\delta}_0 \quad .$$  (IV.65)

In anticipation to the result we introduce the constant $\tilde{H}_0$ satisfying $\tilde{c}_0 = -4\pi \tilde{H}_0$. Let $\tilde{L}$ the solution to the following problem given by lemma A.3

$$\begin{cases}
L_{\tilde{r}}\tilde{L} = -4\pi \tilde{H}_0 \quad \text{in } D^2 \quad , \\
\tilde{L} = 0 \quad \text{on } \partial D^2 \quad .
\end{cases}$$  (IV.66)

We have $\nabla \tilde{L} \in L^{2,\infty}$. Since $\tilde{n}$ is smooth in $D^2 \setminus \{0\}$ and satisfy $\|x|^k |\nabla^k \tilde{n}|_{L^\infty(D^2)} < +\infty$, we can apply lemma A.9 in each annulus $D^2_1 \setminus D^2_2 \setminus \cdots \subset D^2_1 \setminus \{0\}$ to deduce that $\tilde{L}$ is smooth in $D^2 \setminus \{0\}$ and that

$$\sup_{x \in D^2} |x| |\nabla \tilde{L}(x)| < +\infty \quad .$$  (IV.67)

Like in the previous section we introduce the cut-off function $\chi$ equals to 1 on $D^2_{1/2}$ and compactly supported in $D^2$. We introduce $\tilde{g}_1$ and $\tilde{g}_2$ like in (III.50) and (III.51) and consider $\tilde{v}_1$ and $\tilde{v}_2$ the solution respectively of (A.3) given by lemma A.1 for $\tilde{g} = \tilde{g}_1$ and the solution of (A.30) given by lemma A.3 for $\tilde{g} = \tilde{g}_2$. $\tilde{v}_1$ and $\tilde{v}_2$ satisfy (III.54) and (III.55). Therefore we have in particular that $\nabla \tilde{v}_1$ and $\nabla \tilde{v}_2$ are in $L^{2,\infty}$ and, like for $\tilde{L}$, since $L_{\tilde{r}}\tilde{v}_i$ equals 0 on $D^2_{1/2}$ and since $\tilde{g}_1$ and $\tilde{g}_2$ are smooth, we have for $i = 1, 2$

$$\sup_{x \in D^2} |x| |\nabla \tilde{v}_i(x)| < +\infty \quad .$$  (IV.68)
Denote $\vec{w} := \vec{H} - \vec{v}_1 - \vec{v}_2 - \vec{L}$. It is an $L^2$ solution to $L_\varpi \vec{w} = 0$ which is smooth in $D^2 \setminus \{0\}$ and equal to 0 on $\partial D^2$. We claim that $\vec{w}$ is identically 0 on $D^2$.

For $r > 0$ we denote $\chi_r(x) = \chi(x/r)$. Let $\vec{\phi}_i$ be a sequence of normalised eigenvectors of $L_\varpi$ in $W^{1,2}_0(D^2, \mathbb{R}^m)$ and forming an Hilbert orthonormal Basis of $L^2(D^2, \mathbb{R}^m)$ (the existence of such a Basis is obtained by combining the result of Lemma A.1 and Hilbert-Schmidt theorem). Denote $\lambda_i$ the corresponding eigenvalues. From lemma A.5 again we know that $\lambda_i \neq 0$. We have

$$
\int_{D^2} (1 - \chi_r) \vec{w} \cdot \vec{\phi}_i = \frac{1}{\lambda_i} \int_{D^2} (1 - \chi_r) \vec{w} \cdot L_\varpi \vec{\phi}_i
$$

$$
\int_{D^2} \nabla \chi_r \vec{w} \cdot \left[ \nabla \vec{\phi}_i - 3\pi_\varpi(\nabla \vec{\phi}_i) - \vec{\phi}_i \right]_\varpi \nabla \vec{n}
$$

$$
= \frac{1}{\lambda_i} \int_{D^2} \nabla \chi_r \vec{w} \cdot \left[ \nabla \vec{\phi}_i - 3\pi_\varpi(\nabla \vec{\phi}_i) - \vec{\phi}_i \right]_\varpi \nabla \vec{n} \tag{IV.69}
$$

$$
= \frac{1}{\lambda_i} \int_{D^2} (1 - \chi_r) \vec{w} \cdot \left[ \nabla \vec{\phi}_i - 3\pi_\varpi(\nabla \vec{\phi}_i) \right]_\varpi \nabla \vec{n} + \frac{1}{\lambda_i} \int_{D^2} \nabla \chi_r \vec{w} \cdot \left[ \nabla \vec{\phi}_i - 3\pi_\varpi(\nabla \vec{\phi}_i) \right]_\varpi \nabla \vec{n}
$$

$$
= \frac{1}{\lambda_i} \int_{D^2} \nabla \chi_r \vec{w} \cdot \left[ \nabla \vec{\phi}_i - 3\pi_\varpi(\nabla \vec{\phi}_i) \right]_\varpi \nabla \vec{n} = 1
$$

Since in the sense of distribution $L_\varpi \vec{w} = 0$, we have

$$
\int_{D^2} \nabla \chi_r \vec{w} \cdot \left[ \nabla \vec{\phi}_i - 3\pi_\varpi(\nabla \vec{\phi}_i) \right]_\varpi \nabla \vec{n} = 0 \tag{IV.70}
$$

Thus, in the last term of (IV.69), we can substract to $\vec{\phi}_i$ the vector $\vec{c}_{r,i}$ which is the average of $\vec{\phi}_i$ on $D^2 \setminus D_{r/2}$, and we get

$$
\int_{D^2} (1 - \chi_r) \vec{w} \cdot \vec{\phi}_i = \frac{1}{\lambda_i} \int_{D^2} \nabla \chi_r \vec{w} \cdot \left[ \nabla \vec{\phi}_i - 3\pi_\varpi(\nabla \vec{\phi}_i) \right]_\varpi \nabla \vec{n}
$$

$$
= \frac{1}{\lambda_i} \int_{D^2} \nabla \chi_r \vec{w} \cdot \left[ \nabla \vec{\phi}_i - 3\pi_\varpi(\nabla \vec{\phi}_i) \right]_\varpi \nabla \vec{n} - \frac{1}{\lambda_i} \int_{D^2} \nabla \chi_r \vec{w} \cdot \left[ \nabla \vec{\phi}_i - 3\pi_\varpi(\nabla \vec{\phi}_i) \right]_\varpi \nabla \vec{n} \cdot (\vec{\phi}_i - \vec{c}_{r,i})
$$

Denote $\nu(r) = \sup_{r < |x| < r^2} |\nabla \vec{w}|(x) + |x| |\vec{w}|(x)$. With this notation we control the right-hand-side in the following way. In one hand, using Cauchy-Schwartz,

$$
\left| \frac{1}{\lambda_i} \int_{D^2} \nabla \chi_r \vec{w} \cdot \left[ \nabla \vec{\phi}_i - 3\pi_\varpi(\nabla \vec{\phi}_i) \right]_\varpi \nabla \vec{n} \right| \leq C_i \int_{D^2 \setminus D_{r/2}} \frac{\nu(r)}{r^2} |\nabla \vec{\phi}_i| \tag{IV.72}
$$

Using lemma A.5 and the fact that $\nu(r)$ converges to zero as $r$ converges to zero (This is obtained combining (IV.63), (IV.67) and (IV.68)) we have that the left-hand-side of (IV.72) converges to 0 as $r$ goes to 0. In the other hand, we
have, using Cauchy-Schwartz and Poincaré inequalities
\[
\left| -\frac{1}{\lambda_t} \int_{D^2} \nabla \chi_r \left[ \nabla \bar{w} - 3\pi_t(\nabla \bar{w}) - \bar{w} \wedge \nabla^\perp \bar{n} \right] \cdot (\tilde{\phi}_i - \tilde{c}_{r,i}) \right|
\leq C_1 \int_{D^2 \setminus D^2_{1/2}} \frac{\nu(r) + \nu(r) \delta(r)}{r^3} \left| \tilde{\phi}_i - \tilde{c}_{r,i} \right|
\leq C_1 \left[ \nu(r) + \nu(r) \delta(r) \right] \frac{1}{r^2} \left[ \int_{D^2 \setminus D^2_{1/2}} \left| \tilde{\phi}_i - \tilde{c}_{r,i} \right|^2 \right]^{1/2}
\leq C_1 \left[ \nu(r) + \nu(r) \delta(r) \right] \left[ \int_{D^2 \setminus D^2_{1/2}} \frac{|\nabla \tilde{\phi}_i|^2}{|x|^2} \right]^{1/2}
\] (IV.73)
Again using lemma $A.5$ and the fact that $\nu(r)$ and $\delta(r)$ converge to zero as $r$ converges to zero we obtain that the left-hand-side of (IV.73) also converges to 0 as $r$ goes to 0. So finally combining (IV.71), (IV.72) and (IV.73) we have that $\int_{D^2} (1 - \chi_r) \bar{w} \cdot \tilde{\phi}_i$ converges to zero as $r$ goes to zero which implies that $\int_{D^2} \bar{w} \cdot \tilde{\phi}_i = 0$. Since this holds for every $i$ and since $\tilde{\phi}_i$ realizes an orthonormal basis of $L^2$ we obtain that $\bar{w}$ is identically zero and hence
\[
\bar{H} = \bar{L} + \bar{v}_1 + \bar{v}_2
\] (IV.74)
Because of lemma $A.9$ since $L_\varphi \bar{v}_i$ equals to zero on $D^2_{1/2}$, we deduce that $\bar{v}_1$ and $\bar{v}_2$ are smooth on $D^2_{1/2}$. Thus it remains to study the asymptotic expansion of $\bar{L}$ at the origin. First we observe that since $\nabla \bar{L} \in L^{2,\infty}$, we have that $\nabla \bar{H} \in L^{2,\infty}$ and using (IV.26) we deduce that
\[
\Delta \nabla \Phi = 4e^{2\lambda} \bar{H} \nabla \lambda + 2e^{2\lambda} \nabla \bar{H} \in \cap_{q < 2} L^p
\] (IV.75)
Since $e^\lambda = |\nabla \Phi|$ we deduce from (IV.76) that $\nabla e^\lambda \in L^q$ for every $q < +\infty$. Bootstraping this information again in (IV.76) we deduce that $\Delta \nabla \tilde{\phi} \in L^{2,\infty}$ which implies in particular that $\nabla^2 \tilde{n}$ is in $L^{2,\infty}$. From [Hel] (proof of theorem 5.1.1) we see that the Coulomb framing $(\bar{e}_1, \bar{e}_2)$ has the same regularity as $\tilde{n}$: $\nabla^2 \bar{e}_i \in L^{2,\infty}$. In particular this implies that $\bar{e}_i \in C^{0,\alpha}$ for every $0 < \alpha < 1$. We claim now that $e_i(0) \cdot \bar{H}_0 = 0$. We have, using the fact that $\bar{H} \cdot \bar{e} \equiv 0$,
\[
-4\pi \bar{e}_i(0) \cdot \bar{H}_0 \delta_0 = e_i \cdot \text{div}(\nabla \bar{H} - 3\pi_t(\nabla \bar{H}) - \bar{H} \wedge \nabla^\perp \bar{n})
= \text{div}(e_i \cdot \nabla \bar{H} - 3\bar{e}_i \cdot \bar{H} \wedge \nabla^\perp \bar{n}) - \nabla \bar{e}_i \cdot \left[ \nabla \bar{H} - 3\pi_t(\nabla \bar{H}) - \bar{H} \wedge \nabla^\perp \bar{n} \right]
= \text{div}( - \bar{H} \cdot \nabla \bar{e}_i - 3\bar{e}_i \cdot \bar{H} \wedge \nabla^\perp \bar{n}) - \nabla \bar{e}_i \cdot \left[ \nabla \bar{H} - 3\pi_t(\nabla \bar{H}) - \bar{H} \wedge \nabla^\perp \bar{n} \right]
\] (IV.76)
Observe now that the right-hand-side of (IV.76) is an $L^p$ function for some $p > 1$ and that this function should be proportional to the Dirac mass at the origin. This implies that the coefficient $4\pi e_i(0) \cdot \bar{H}_0$ is zero. Let now $\bar{H} := \bar{L} - \bar{H}_0 \log |x|$. We have
\[
L_\varphi \bar{H} = -3\text{div}(\pi_t(\bar{H}_0) \nabla \log |x|) - \nabla \log |x| \bar{H}_0 \wedge \nabla^\perp \bar{n}
\] (IV.77)
Since \( \pi_{\tilde{R}}(\tilde{H}_0) = (\tilde{H}_0 \cdot \tilde{e}_1) \tilde{e}_1 + (\tilde{H}_0 \cdot \tilde{e}_2) \tilde{e}_2 = (\tilde{H}_0 \cdot (\tilde{e}_1 - \tilde{e}_1(0))) \tilde{e}_1 + (\tilde{H}_0 \cdot (\tilde{e}_2 - \tilde{e}_2(0))) \tilde{e}_2 \) and since \( \tilde{e}_i \) are in \( C^{0,\alpha} \) for every \( \alpha < 1 \) we have then \( r^{-1}\pi_{\tilde{R}}(\tilde{H}_0) \in L^p \) for every \( p < +\infty \). Thus we have proved that \( L_{\tilde{R}} \tilde{R} \in W^{-1,p} \) for every \( p < +\infty \). Arguing like in the proof of lemma A.1 we have that \( \tilde{R} \in \cap_{p<+\infty} W^{1,p} \). Hence we have proved that \( \tilde{H} - \tilde{H}_0 \log |x| \) is in \( C^{0,\alpha} \) for every \( \alpha < 1 \) and this finishes the proof of theorem 1.3.

□

A Appendix

Lemma A.1 There exists \( \varepsilon_0 > 0 \) such that for every \( 0 < \varepsilon < \varepsilon_0 \), there exists a constant \( C > 0 \) independent of \( \varepsilon \) such that the following holds. Let \( \tilde{n} \) from \( D^2 \) into the space of unit \( m-2 \)-vectors in \( \mathbb{R}^m \) satisfying

\[
\int_{D^2} |\nabla \tilde{n}|^2 \, dx \leq \varepsilon . \tag{A.1}
\]

Let \( \tilde{g} \) be an arbitrary distribution in the Sobolev space \( H^{-1}(D^2, \mathbb{R}^m) \) dual to \( W_0^{1,2}(D^2, \mathbb{R}^m) \), then there exists a unique map \( \tilde{v} \) in \( W_0^{1,2}(D^2, \mathbb{R}^m) \) satisfying

\[
\begin{cases}
\Delta \tilde{v} - 3 \text{div}(\pi_{\tilde{n}}(\nabla \tilde{v})) - \text{div}(\tilde{v} \wedge \nabla \tilde{n}) = \tilde{g} & \text{in } D^2 \\
\tilde{v} = 0 & \text{on } \partial D^2 ,
\end{cases} \tag{A.2}
\]

and

\[
\int_{D^2} |\nabla \tilde{v}|^2 \leq C \| \tilde{g} \|_{H^{-1}}^2 . \tag{A.3}
\]

Moreover the operator \( L_{\pi_{\tilde{R}}}^{-1} \) which to \( \tilde{g} \) assigns \( \tilde{v} \) satisfying (A.2) is selfadjoint and compact from \( L^2(D^2, \mathbb{R}^m) \) into itself.

□

Before proving lemma A.1 we shall prove first the following.

Lemma A.2 There exists \( \varepsilon_0 > 0 \) such that for every \( 0 < \varepsilon < \varepsilon_0 \), there exists a constant \( C > 0 \) independent of \( \varepsilon \) such that the following holds. Let \( \tilde{n} \) from \( D^2 \) into the space of unit \( m-2 \)-vectors in \( \mathbb{R}^m \) satisfying

\[
\int_{D^2} |\nabla \tilde{n}|^2 \, dx \leq \varepsilon . \tag{A.4}
\]

Let \( \tilde{g} \) be an arbitrary distribution in the Sobolev space \( H^{-1}(D^2, \mathbb{R}^m) \) dual to \( W_0^{1,2}(D^2, \mathbb{R}^m) \), then there exists a unique map \( \tilde{v} \) in \( W_0^{1,2}(D^2, \mathbb{R}^m) \) satisfying

\[
\begin{cases}
\Delta \tilde{v} - 3 \text{div}(\pi_{\tilde{n}}(\nabla \tilde{v})) = \tilde{g} & \text{in } D^2 \\
\tilde{v} = 0 & \text{on } \partial D^2 ,
\end{cases} \tag{A.5}
\]

and

\[
\int_{D^2} |\nabla \tilde{v}|^2 \leq C \| \tilde{g} \|_{H^{-1}}^2 . \tag{A.6}
\]

□
Proof of lemma A.2

First of all we show that under the assumption (A.4) of the lemma, the following implication holds for every $\vec{C}$ in $W^{1,2}(D^2, \mathbb{R}^m)$

$$
\left\{ \begin{array}{ll}
\Delta \vec{C} - 3 \text{div}(\pi_{\vec{n}}(\nabla \vec{C})) = 0 & \text{in } D^2 \\
\vec{C} = 0 & \text{on } \partial D^2 
\end{array} \right. \quad \Rightarrow \quad \vec{C} \equiv 0 \quad (A.7)
$$

The proof of (A.7) goes as follows. Since
\[ \text{div}(\nabla \vec{C} - 3 \pi_{\vec{n}}(\nabla \vec{C})) = 0, \]
from Poincaré lemma, there exists $\vec{D}$ in $W^{1,2}(D^2, \mathbb{R}^m)$ satisfying
\[ \nabla \perp \vec{D} = \nabla \vec{C} - 3 \pi_{\vec{n}}(\nabla \vec{C}). \quad (A.8) \]

This implies in particular that $\vec{D}$ is a $W^{1,2}$ solution of the following equation

\[
\left\{ \begin{array}{ll}
\Delta \vec{D} = 3 \sum_{k=1}^{m} \nabla \perp C^k \cdot \nabla (e^k_1 \vec{e}_1) + 3 \sum_{k=1}^{m} \nabla \perp C^k \cdot \nabla (e^k_2 \vec{e}_2) & \text{in } D^2 \\
\frac{\partial \vec{D}}{\partial \nu} = 0 & \text{on } \partial D^2 
\end{array} \right. \quad (A.9)
\]

where $C^k$ are the coordinates of $\vec{C}$ in the canonical basis of $\mathbb{R}^m$, $(\vec{e}_1, \vec{e}_2)$ is an orthonormal basis of the 2-dimensional subspace defined by $\vec{n}$ given by lemma 5.1.4 in [Hel]. From this lemma we have in particular the existence of a constant $C$ such that
\[ \int_{D^2} \left| \nabla \vec{e}_1 \right|^2 + \left| \nabla \vec{e}_2 \right|^2 \, dx \leq C \int_{D^2} \left| \nabla \vec{n} \right|^2 \, dx \quad . \quad (A.10) \]

Using now the Neuman boundary condition version of Wente’s lemma given in [Hel] (lemma 3.1.2) we have,
\[
\int_{D^2} \left| \nabla \vec{D} \right|^2 \, dx \leq C_1 \left[ \int_{D^2} \left| \nabla \vec{e}_1 \right|^2 + \left| \nabla \vec{e}_2 \right|^2 \, dx \right] \int_{D^2} \left| \nabla \vec{C} \right|^2 \, dx \\
\leq C_1 \varepsilon \int_{D^2} \left| \nabla \vec{C} \right|^2 \, dx 
\]

(the boundary condition studied in [Hel] lemma 3.1.2 is the Dirichlet condition but modulo a slight modification the $W^{1,2}$ estimate can be obtained also for the Neuman boundary condition by classical arguments in elliptic theory)

Observe that (A.8) implies that
\[
\left| \nabla \vec{D} \right|^2 = \left| \pi_T(\nabla \vec{C}) \right|^2 + 4 \left| \pi_{\vec{n}}(\nabla \vec{C}) \right|^2 \geq \left| \nabla \vec{C} \right|^2, \quad (A.12)
\]

where $\pi_T$ denotes the orthogonal projection on the 2-plane in $\mathbb{R}^m$ defined by $\vec{n}$. Combining then (A.11) and (A.12) we obtain, for $\varepsilon < 1/(2C_1)$, that $\vec{C} \equiv 0$ which proves the implication (A.7).

Let now $\vec{g}$ in $H^{-1}(D^2, \mathbb{R}^m)$ and let $\vec{B}$ solving

\[
\left\{ \begin{array}{ll}
\Delta \vec{B} = \vec{g} & \text{in } D^2 \\
\vec{B} = 0 & \text{on } \partial D^2 
\end{array} \right. \quad (A.13)
\]
We claim that there exists \((\vec{A}, \vec{F})\) solutions to

\[
\begin{align*}
\text{div} \vec{F} &= \text{div} \left( \pi_T(\nabla^\perp \vec{A}) - \frac{1}{2} \pi_R(\nabla^\perp A) \right) \quad \text{in } D^2 \\
\text{curl} \vec{F} &= -\text{curl} \left( \pi_T(\nabla \vec{B}) - \frac{1}{2} \pi_R(\nabla \vec{B}) \right) \quad \text{in } D^2 \\
\vec{F} \cdot \nu &= 0 \quad \text{on } \partial D^2
\end{align*}
\]

(A.14)

where \(\vec{A}\) is the \text{curl}–part in the Hodge decomposition of \(\pi_T(\vec{F}) - 2 \pi_R(\vec{F})\) given by:

\[
\begin{align*}
- \Delta \vec{A} &= \text{curl} \left( \pi_T(\vec{F}) - 2 \pi_R(\vec{F}) \right) \quad \text{in } D^2 \\
\vec{A} &= 0 \quad \text{on } \partial D^2
\end{align*}
\]

(A.15)

(remark that \(\vec{F}(x)\) is an element of \(\mathbb{R}^2 \otimes \mathbb{R}^m\) and that \(\vec{F} \cdot \nu \in \mathbb{R}^m\) in (A.14) has to do with the scalar product between the unit exterior normal to \(\partial D^2\) and the \(\mathbb{R}^2\) part in \(\vec{F}\). The existence of a solution \((\vec{A}, \vec{F})\) of the system (A.14), (A.15) is again a consequence of Wente’s estimate: we write in one hand

\[
\text{div} \left( \pi_T(\nabla^\perp \vec{A}) - \frac{1}{2} \pi_R(\nabla^\perp A) \right) = \frac{3}{2} m \sum_{k=1}^{m} \nabla^\perp A^k \cdot \nabla(e_1^k \vec{e}_1) + \frac{3}{2} m \sum_{k=1}^{m} \nabla^\perp A^k \cdot \nabla(e_2^k \vec{e}_2)
\]

(A.16)

and in the other hand

\[
\text{curl} \left( \pi_T(\nabla \vec{B}) - \frac{1}{2} \pi_R(\nabla \vec{B}) \right) = \frac{3}{2} \sum_{k=1}^{m} \nabla B^k \cdot \nabla^\perp (e_1^k \vec{e}_1) + \frac{3}{2} \sum_{k=1}^{m} \nabla B^k \cdot \nabla^\perp (e_2^k \vec{e}_2)
\]

(A.17)

where \(A^k\) and \(B^k\) are the coordinates of respectively \(\vec{A}\) and \(\vec{B}\). Therefore, using Wente’s estimate, we have the following a-priori inequalities

\[
\int_{D^2} |\vec{F}|^2 \leq C_2 \left[ \int_{D^2} |
abla \vec{e}_1|^2 + |
abla \vec{e}_2|^2 dx \right] \int_{D^2} |\nabla \vec{A}|^2 + |\nabla \vec{B}|^2 dx \\
\leq C_2 \epsilon \int_{D^2} |\nabla \vec{A}|^2 + |\nabla \vec{B}|^2 dx 
\]

(A.18)

From (A.15) and standard elliptic estimates we have

\[
\int_{D^2} |\nabla \vec{A}|^2 dx \leq C_3 \int_{D^2} |\vec{F}|^2 dx 
\]

(A.19)

Thus for \(C_3 C_2 \epsilon < 1/2\), a standard fixed point argument gives the existence and uniqueness of \((\vec{A}, \vec{F})\) satisfying (A.14) and (A.15). Since

\[
\text{div} \left( \vec{F} - \left( \pi_T(\nabla^\perp \vec{A}) - \frac{1}{2} \pi_R(\nabla^\perp A) \right) \right) = 0 
\]

there exists \(\vec{C}\) in \(W^{1,2}_0(D^2, \mathbb{R}^m)\) satisfying

\[
\vec{F} - \left( \pi_T(\nabla^\perp \vec{A}) - \frac{1}{2} \pi_R(\nabla^\perp A) \right) = \nabla^\perp \vec{C} 
\]

(A.20)
Then we deduce that
\[ \pi_T(\vec{F}) - 2\pi_\vec{n}(\vec{F}) = \nabla^\perp \vec{A} + \pi_T(\nabla^\perp \vec{C}) - 2\pi_\vec{n}(\nabla^\perp \vec{C}) \ . \quad (A.21) \]

Taking the curl of this identity together with (A.15) implies that \( \vec{C} \) solves the following equation
\[
\begin{cases}
\Delta \vec{C} - 3 \text{div}(\pi_\vec{n}(\nabla \vec{C})) = -\text{curl}(\pi_T(\nabla^\perp \vec{C}) - 2\pi_\vec{n}(\nabla^\perp \vec{C})) = 0 \quad \text{in } D^2 \\
\vec{C} = 0 \quad \text{on } \partial D^2 .
\end{cases}
\]
(A.22)

implies then that \( \vec{C} \equiv 0 \) therefore
\[ \pi_T(\vec{F}) - 2\pi_\vec{n}(\vec{F}) = \nabla^\perp \vec{A} . \quad (A.23) \]

From (A.14) there exists \( \vec{v} \) in \( W^{1,2}_0(D^2, \mathbb{R}^m) \) satisfying
\[ \vec{F} = -
\pi_T(\nabla \vec{B}) + \frac{1}{2} \pi_\vec{n}(\nabla \vec{B}) + \nabla \vec{v} \quad (A.24) \]

Combining (A.23) and (A.24) we obtain that
\[ \nabla^\perp \vec{A} = -\nabla \vec{B} + \pi_T(\nabla \vec{v}) - 2\pi_\vec{n}(\nabla \vec{v}) . \quad (A.25) \]

Comparing this identity with (A.13) we deduce that \( \vec{v} \) solves (A.5) and from (A.7) we know that this is the unique solution. (A.6) follows from (A.18) and (A.19). This completes then the proof of lemma A.2.

\[ \square \]

**Proof of Lemma A.1**

Denote \( \Delta_0^{-1} \) the continuous isomorphism from \( H^{-1}(D^2, \mathbb{R}^m) \) into \( W^{1,2}_0(D^2, \mathbb{R}^m) \) which to a distribution \( \vec{g} \) in \( H^{-1}(D^2, \mathbb{R}^m) \) assigns the solution \( \vec{v} \) of
\[
\begin{cases}
\Delta \vec{v} = \vec{g} \quad \text{in } D^2 \\
\vec{v} = 0 \quad \text{on } \partial D^2 .
\end{cases}
\]
(A.26)

We have seen in lemma A.2 that the operator \( A_\vec{n}\vec{v} := \Delta \vec{v} - 3 \text{div}(\pi_\vec{n}(\nabla \vec{u})) \)

is a continuous isomorphism from \( W^{1,2}_0(D^2, \mathbb{R}^m) \) into \( H^{-1}(D^2, \mathbb{R}^m) \) and the norm of \( A_\vec{n} \) and \( A_\vec{n}^{-1} \) are independent of \( \vec{n} \) satisfying (A.4) for \( \varepsilon < \varepsilon_0 \) where \( \varepsilon_0 \) is universal given by lemma A.2. Our aim is to show the invertibility of the operator
\[ \Delta_0^{-1} A_\vec{n}(\vec{v}) = \Delta_0^{-1} \text{div}(\vec{v} \wedge \nabla^\perp \vec{n}) \]

from \( W^{1,2}_0(D^2, \mathbb{R}^m) \) into itself, with norms independent of \( \vec{n} \) satisfying (A.4) for \( \varepsilon < \varepsilon_0 \). This is a direct consequence of the invertibility of \( \Delta_0^{-1} A_\vec{n} \) from \( W^{1,2}_0(D^2, \mathbb{R}^m) \) into itself and the fact that by the mean of Wente estimate (theorem 3.1.2 of [Hel]) the operator \( \Delta_0^{-1} \text{div}(\vec{v} \wedge \nabla^\perp \vec{n}) \) satisfies for every \( \vec{v} \) in \( W^{1,2}_0(D^2, \mathbb{R}^m) \)

\[ \| \Delta_0^{-1} \text{div}(\vec{v} \wedge \nabla^\perp \vec{n}) \|_{W^{1,2}} \leq C \int_{D^2} |\nabla \vec{v}|^2 dx \int_{D^2} |\nabla \vec{n}|^2 dx \leq C \varepsilon \| \vec{v} \|^2_{W^{1,2}} \quad (A.27) \]

We have then proved the first statement of lemma A.1 and it remains to show the compactness and the self-adjointness of \( L_\vec{n}^{-1} \) from \( L^2 \) into itself. Compactness
is clear since \( L^{-1}_n \) sends \( H^{-1}(D^2; \mathbb{R}^m) \) into \( W_0^{1,2} \) which is a compact subspace of \( L^2 \). Let \( \vec{g} \) and \( \vec{h} \) in \( L^2(D^2, \mathbb{R}^m) \) (we can take them first smooth). Denote \( \vec{v} := L^{-1}_n(\vec{g}) \) and \( \vec{w} := L^{-1}_n(\vec{h}) \) we have

\[
\int_{D^2} \vec{g} \cdot L^{-1}_n(\vec{h}) \, dx
= \int_{D^2} \Delta \vec{v} \cdot \vec{w} - 3 \text{div}(\pi_n(\nabla \vec{v})) \cdot \vec{w} - \text{div} (\vec{v} \wedge \nabla \perp \vec{n}) \cdot \vec{w} \, dx
= \int_{D^2} \vec{v} \cdot \Delta \vec{w} + 3\pi_n(\nabla \vec{v}) \cdot \nabla \vec{w} + (\vec{v} \wedge \nabla \perp \vec{n}) \cdot \nabla \vec{w} \, dx
= \int_{D^2} \vec{v} \cdot \Delta \vec{w} + 3\vec{v} \cdot \pi_n(\nabla \vec{w}) - \vec{v} \cdot (\nabla \vec{w} \wedge \nabla \perp \vec{n})
= \int_{D^2} \vec{v} \cdot \Delta \vec{w} - 3\vec{v} \cdot \text{div}(\pi_n(\nabla \vec{w})) - \vec{v} \cdot \text{div}(\vec{w} \wedge \nabla \perp \vec{n})
= \int_{D^2} \vec{v} \cdot \vec{h} = \int_{D^2} L^{-1}_n(\vec{g}) \cdot \vec{h} .
\]

This shows the self-adjointness of \( L^{-1}_n \) and lemma A.1 is proved. \( \qed \)

We now extend the two previous lemma to \( L^1 \) datas. We have first

**Lemma A.3** There exists \( \varepsilon_0 > 0 \) such that for every \( 0 < \varepsilon < \varepsilon_0 \), there exists a constant \( C > 0 \) independent of \( \varepsilon \) such that the following holds. Let \( \vec{n} \) from \( D^2 \) into the space of unit \( m - 2 \)-vectors in \( \mathbb{R}^m \) satisfying

\[
\int_{D^2} |\nabla \vec{n}|^2 \, dx \leq \varepsilon .
\]  

Let \( \vec{g} \) be an arbitrary map in \( L^1(D^2, \mathbb{R}^m) \), then there exists a unique map \( \vec{v} \) with \( \nabla \vec{v} \) in \( L^{2,\infty}(D^2, \mathbb{R}^2 \otimes \mathbb{R}^m) \) satisfying

\[
\begin{aligned}
\Delta \vec{v} - 3 \text{div}(\pi_n(\nabla \vec{v})) - \text{div} (\vec{v} \wedge \nabla \perp \vec{n}) &= \vec{g} & \text{in } D^2 \\
\vec{v} &= 0 & \text{on } \partial D^2 ,
\end{aligned}
\]

and

\[
\|
\nabla \vec{v}
\|_{L^{2,\infty}(D^2)} \leq C \| \vec{g} \|_{L^1(D^2)} .
\]

Before proving lemma A.3, we shall prove first the following.

**Lemma A.4** There exists \( \varepsilon_0 > 0 \) such that for every \( 0 < \varepsilon < \varepsilon_0 \), there exists a constant \( C > 0 \) independent of \( \varepsilon \) such that the following holds. Let \( \vec{n} \) from \( D^2 \) into the space of unit \( m - 2 \)-vectors in \( \mathbb{R}^m \) satisfying

\[
\int_{D^2} |\nabla \vec{n}|^2 \, dx \leq \varepsilon .
\]
Let \( \tilde{g} \) be an arbitrary map in \( L^1(D^2, \mathbb{R}^m) \), then there exists a unique map \( \tilde{v} \) with \( \nabla \tilde{v} \) in \( L^{2,\infty}(D^2, \mathbb{R}^2 \otimes \mathbb{R}^m) \) satisfying

\[
\begin{align*}
\Delta \tilde{v} - 3 \text{div}(\pi_{\tilde{n}}(\nabla \tilde{v})) &= \tilde{g} & \text{in } D^2 \\
\tilde{v} &= 0 & \text{on } \partial D^2 ,
\end{align*}
\]  

(A.33)

and

\[
\|\nabla \tilde{v}\|_{L^{2,\infty}(D^2)} \leq C \|\tilde{g}\|_{L^1(D^2)} .
\]  

(A.34)

**Proof of lemma A.3.** We first recall the following generalization of Wente’s estimate established in [Hel] (theorem 3.4.5). Let \( a \) and \( b \) be two functions on \( D^2 \) with \( \nabla a \in L^{2,\infty}(D^2) \) and \( \nabla b \in L^2(D^2) \), then there is a unique solution \( \varphi \) in \( W_0^{1,2}(D^2, \mathbb{R}) \) satisfying

\[
\begin{align*}
\Delta \varphi &= \nabla a \cdot \nabla b & \text{in } D^2 \\
\varphi &= 0 & \text{on } \partial D^2 ,
\end{align*}
\]  

(A.35)

and the following estimate holds: there exists a positive constant \( C \) independent of \( a \) and \( b \) such that

\[
\|\nabla \varphi\|_{L^2(D^2)} \leq C \|\nabla a\|_{L^{2,\infty}(D^2)} \|\nabla b\|_{L^2(D^2)} .
\]  

(A.36)

The proof of lemma A.3 follows then step by step the proof of lemma A.2 by replacing the original Wente estimate by the estimate (A.36) and the \( L^2 \) norms of \( \nabla \tilde{C}, \nabla \tilde{B}, \nabla \tilde{A}, \tilde{F} \) and \( \nabla \tilde{v} \) replaced by their \( L^{2,\infty} \) norms. □

**Proof of lemma A.4.**

Let \( \tilde{g} \) be in \( L^1(D^2, \mathbb{R}^m) \). Denote again \( A_{\tilde{n}} \tilde{v} := \Delta \tilde{v} - 3 \text{div}(\pi_{\tilde{n}}(\nabla \tilde{v})) \). Using lemma A.3 we first get the existence of \( \tilde{v}_0 \) with \( \nabla \tilde{v}_0 \in L^{2,\infty} \) satisfying

\[
\begin{align*}
A_{\tilde{n}} \tilde{v}_0 &= \tilde{g} & \text{in } D^2 \\
\tilde{v}_0 &= 0 & \text{on } \partial D^2
\end{align*}
\]  

(A.37)

Then we construct by induction the following sequence \( \tilde{v}_k \) with \( \tilde{v}_0 \) given by (A.37) and \( \tilde{v}_k \) for \( k \geq 1 \) will be the element of \( W_0^{1,2}(D^2, \mathbb{R}^m) \) solving

\[
\Delta_{\tilde{n}}^{-1} A_{\tilde{n}}(\tilde{v}_k) = \Delta_{\tilde{n}}^{-1} \text{div}(\tilde{v}_{k-1} \wedge \nabla \tilde{n})
\]  

(A.38)

where \( \Delta_{\tilde{n}}^{-1} \) is the operator introduced in the proof of lemma A.1 by (A.26).

This problem admits indeed a solution for the following reason: let \( \tilde{v}_{k-1} \) being given, with the assumption that \( \nabla \tilde{v}_{k-1} \) in \( L^{2,\infty} \), using (A.36) we have

\[
\|\Delta_{\tilde{n}}^{-1} \text{div}(\tilde{v}_{k-1} \wedge \nabla \tilde{n})\|_{W^{1,2}} \leq C \|\nabla \tilde{v}_{k-1}\|_{L^{2,\infty}} \|\nabla \tilde{n}\|_{L^2} .
\]  

(A.39)

Moreover we have seen in the proof of lemma A.1 using lemma A.2 that \( \Delta_{\tilde{n}}^{-1} A_{\tilde{n}} \) is a continuous isomorphism of \( W_0^{1,2} \). Combining (A.39) and this last fact we get the existence and uniqueness of \( \tilde{v}_k \), once \( \tilde{v}_{k-1} \) is known, and the following inequality holds

\[
\|\nabla \tilde{v}_k\|_{L^{2,\infty}} \leq \|\nabla \tilde{v}_0\|_{L^2} \leq C \|\nabla \tilde{v}_{k-1}\|_{L^{2,\infty}} \|\nabla \tilde{n}\|_{L^2} .
\]  

(A.40)

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Then the following inequality holds
\[ C > a \text{ constant} \]
Thus, under the assumption that \( C\|\nabla \bar{n}\|_{L^2} < 1/2 \) the series \( \sum_{k=0}^{\infty} \bar{v}_k \) converges to a limit \( \bar{v} = \sum_{k=0}^{\infty} \bar{v}_k \) solving (A.30) and (A.31) holds. The uniqueness of \( \bar{v} \) follows from the uniqueness of the solution of \( A_n(\bar{\bar{v}}) = \bar{g} \) for arbitrary \( \bar{g} \in L^1 \) established in lemma A.4 and the previous considerations with \( C\|\nabla \bar{n}\|_{L^2} \) being small enough. □

**Lemma A.5** There exists \( \varepsilon_0 > 0 \) such that for every \( 0 < \varepsilon < \varepsilon_0 \), there exists a constant \( C > 0 \) independent of \( \varepsilon \) such that the following holds. Let \( \bar{n} \) from \( D^2 \) into the space of unit \( m - 2 \)-vectors in \( \mathbb{R}^m \) satisfying
\[
\|x| |\nabla \bar{n}(x)\|_{L^\infty(\mathcal{D}^2)} + \int_{\mathcal{D}^2} |\nabla \bar{n}|^2 \, dx \leq \varepsilon. \tag{A.41}
\]

Let \( \bar{g} \) be an arbitrary function in \( L^2(\mathcal{D}^2, \mathbb{R}^m) \), let \( \bar{v} \) be the unique map in \( W_0^{1,2}(\mathcal{D}^2, \mathbb{R}^m) \) given by lemma A.3 satisfying
\[
\begin{cases}
\Delta \bar{v} - 3 \text{div}(\pi_n(\nabla \bar{v})) - \text{div}(\bar{v} \wedge \nabla \bar{n}) = \bar{g} & \text{in } \mathcal{D}^2 \\
\bar{v} = 0 & \text{on } \partial \mathcal{D}^2 ,
\end{cases} \tag{A.42}
\]
and denote \( \bar{v}_0 \) and \( \bar{v}_\perp \) the maps from \( \mathcal{D}^2 \) into \( \mathbb{R}^m \) given by
\[
\bar{v}_0(x) = \frac{1}{2\pi |x|} \int_{\partial B_{|x|}(0)} \bar{v} \quad \text{and} \quad \bar{v}_\perp = \bar{v} - \bar{v}_0. \tag{A.43}
\]

Then the following inequality holds
\[
\|\nabla \bar{v}_0\|_{L^\infty(\mathcal{D}^2)} + \int_{\mathcal{D}^2} \frac{|\nabla \bar{v}_\perp|^2}{|x|^2} + \int_{\mathcal{D}^2} |\nabla^2 \bar{v}_\perp|^2 \leq C \int_{\mathcal{D}^2} |\bar{g}|^2 \tag{A.44}
\]
Before proving lemma A.5 we need to establish the following intermediate result.

**Lemma A.6** Let \( a \) and \( b \) be two functions respectively in \( W^{2,2}(\mathcal{D}^2, \mathbb{R}) \) and \( W^{1,2}(\mathcal{D}^2, \mathbb{R}) \) such that \( b \in C^1(\mathcal{D}^2 \setminus \{0\}) \) and
\[
\sup_{x \in \mathcal{D}^2 \setminus \{0\}} |x| |\nabla b(x)| < +\infty.
\]
Let \( \varphi \) be the solution in \( W^{1,2} \) of
\[
\begin{cases}
\Delta \varphi = \frac{\partial a}{\partial x_1} \frac{\partial b}{\partial x_2} - \frac{\partial a}{\partial x_2} \frac{\partial b}{\partial x_1} & \text{in } \mathcal{D}^2 \\
\varphi = 0 & \text{on } \partial \mathcal{D}^2
\end{cases} \tag{A.45}
\]
Denote \( \varphi_0 \) and \( \varphi_\perp \) the functions on \( \mathcal{D}^2 \) given by
\[
\varphi_0(x) = \frac{1}{2\pi |x|} \int_{\partial B_{|x|}(0)} \varphi \quad \text{and} \quad \varphi_\perp = \varphi - \varphi_0. \tag{A.46}
\]
Then the following inequality holds
\[
\|\nabla \varphi_0\|_{L^\infty(\mathcal{D}^2)}^2 + \int_{\mathcal{D}^2} \frac{|\nabla \varphi_\perp|^2}{|x|^2} + \int_{\mathcal{D}^2} |\nabla^2 \varphi_\perp|^2 \leq C \left[ \|\nabla b\|_{L^\infty}^2 + \int_{\mathcal{D}^2} |\nabla b|^2 \right] \left[ \int_{\mathcal{D}^2} \frac{|\nabla a_\perp|^2}{|x|^2} + \|\nabla a_0\|_{L^\infty}^2 \right]. \tag{A.47}
\]
Proof of lemma [A.6] Since $\varphi_0$ is the first term in the Fourier decomposition of $\varphi$ (for the angle variable), we have that

$$\Delta \varphi_0 = \frac{\partial a_0}{\partial x_1} \frac{\partial b_0}{\partial x_2} - \frac{\partial a_0}{\partial x_2} \frac{\partial b_0}{\partial x_1} + \left( \frac{\partial a_\perp}{\partial x_1} \frac{\partial b_\perp}{\partial x_2} - \frac{\partial a_\perp}{\partial x_2} \frac{\partial b_\perp}{\partial x_1} \right)_0$$  \hspace{1cm} \text{(A.48)}$$

Indeed it is clear that $\frac{\partial a_\perp}{\partial x_1} \frac{\partial b_0}{\partial x_2} - \frac{\partial a_\perp}{\partial x_2} \frac{\partial b_0}{\partial x_1}$ as well as $\frac{\partial a_\perp}{\partial x_1} \frac{\partial b_\perp}{\partial x_2} - \frac{\partial a_\perp}{\partial x_2} \frac{\partial b_\perp}{\partial x_1}$ have no 0–component. This comes from the fact that $a_0$ and $b_0$ depends only on $|x|$ which imply the identities

$$\frac{\partial a_\perp}{\partial x_1} \frac{\partial b_0}{\partial x_2} - \frac{\partial a_\perp}{\partial x_2} \frac{\partial b_0}{\partial x_1} = \frac{1}{r} \frac{\partial a_\perp}{\partial \theta} \dot{b}_0(r)$$

$$\frac{\partial a_\perp}{\partial x_1} \frac{\partial b_\perp}{\partial x_2} - \frac{\partial a_\perp}{\partial x_2} \frac{\partial b_\perp}{\partial x_1} = \dot{a}_0(r) \left( \frac{1}{r} \frac{\partial b_\perp}{\partial \theta} \right).$$  \hspace{1cm} \text{(A.49)}$$

It is also clear that, since both $a_0$ and $b_0$ only depends on $|x|$ that the first jacobien in (A.48) is zero. Thus we have, using one more time (A.49),

$$\ddot{\varphi}_0 + \frac{\dot{\varphi}_0}{r} = \Delta \varphi_0 = \left( \frac{\partial a_\perp}{\partial x_1} \frac{\partial b_\perp}{\partial x_2} - \frac{\partial a_\perp}{\partial x_2} \frac{\partial b_\perp}{\partial x_1} \right)_0$$

$$= \left( \frac{\partial a_\perp}{\partial x_1} \frac{\partial b_\perp}{\partial x_2} - \frac{\partial a_\perp}{\partial x_2} \frac{\partial b_\perp}{\partial x_1} \right)_0$$  \hspace{1cm} \text{(A.50)}$$

Denoting $h(r)$ the right-hand side of (A.50), we then have that $\dot{\varphi}_0 = r^{-1} \int_0^r h(s) s \, ds$. Therefore $|\dot{\varphi}_0|$ can be bounded in the following way :

$$|\dot{\varphi}_0| \leq \frac{1}{2\pi r} \int_{B_r} |\nabla a_\perp| |\nabla b|$$

$$\leq C ||x|| |\nabla b(x)|| \left[ \int_{B_r} \frac{|\nabla a_\perp|^2}{|x|^2} \right]^\frac{1}{2}. $$  \hspace{1cm} \text{(A.51)}$$

Regarding $\varphi_\perp$ we have :

$$\Delta \varphi_\perp = \left( \frac{\partial a}{\partial x_1} \frac{\partial b}{\partial x_2} - \frac{\partial a}{\partial x_2} \frac{\partial b}{\partial x_1} \right)_\perp$$

$$= \left( \frac{\partial a_0}{\partial x_1} \frac{\partial b}{\partial x_2} - \frac{\partial a_0}{\partial x_2} \frac{\partial b}{\partial x_1} \right)_\perp + \left( \frac{\partial a_\perp}{\partial x_1} \frac{\partial b}{\partial x_2} - \frac{\partial a_\perp}{\partial x_2} \frac{\partial b}{\partial x_1} \right)_\perp$$  \hspace{1cm} \text{(A.52)}$$

In one hand we have

$$\int_{D^2} \left[ \left( \frac{\partial a_0}{\partial x_1} \frac{\partial b}{\partial x_2} - \frac{\partial a_0}{\partial x_2} \frac{\partial b}{\partial x_1} \right)_\perp \right]^2 \leq \int_{D^2} \left| \frac{\partial a_0}{\partial x_1} \frac{\partial b}{\partial x_2} - \frac{\partial a_0}{\partial x_2} \frac{\partial b}{\partial x_1} \right|^2 \leq C \|\nabla a_0\|_\infty \int_{D^2} |\nabla b|^2.$$  \hspace{1cm} \text{(A.53)}$$
In the other hand we have

\[
\frac{1}{2} \left\| \frac{\partial a}{\partial x_1} \frac{\partial b}{\partial x_2} - \frac{\partial a}{\partial x_2} \frac{\partial b}{\partial x_1} \right\|^2 \leq C \|x| \| \nabla b(x) \|_\infty \int_{D^2} \frac{|\nabla a|^2}{|x|^2} \tag{A.54}
\]

Combining (A.51), (A.53) and (A.54) we get (A.47) and lemma A.6 is proved.

\[ \square \]

**Proof of lemma A.5.** Let \( \vec{A}, \vec{B} \) and \( \vec{C} \) be the solutions respectively of

\[
\begin{align*}
\Delta \vec{A} &= \vec{g} \quad \text{in } D^2 \\
\vec{A} &= 0 \quad \text{on } \partial D^2
\end{align*}
\]

\[
\begin{align*}
\Delta \vec{B} &= \text{div}(\vec{v} \wedge \nabla ^\perp \vec{n}) \quad \text{in } D^2 \\
\vec{B} &= 0 \quad \text{on } \partial D^2
\end{align*}
\]

and

\[
\begin{align*}
\Delta \vec{C} &= 3 \text{div}(\pi \vec{n}(\nabla ^\perp \vec{v})) \quad \text{in } D^2 \\
\partial \vec{C} / \partial \nu &= 0 \quad \text{on } \partial D^2
\end{align*}
\]

It is clear that \( \vec{v} = \vec{A} + \vec{B} + \vec{C} \). Using standard elliptic estimates for equation (A.55) and applying lemma A.6 to equation (A.56) and the Neuman-boundary condition version of lemma A.6 to equation (A.57) we obtain

\[
\|\nabla \vec{A}_0\|_\infty^2 + \int_{D^2} \frac{|\nabla \vec{A}|^2}{|x|^2} + \int_{D^2} |\nabla^2 \vec{A}|^2 \leq C \int_{D^2} |\vec{g}|^2 ,
\]

\[
\|\nabla \vec{B}_0\|_\infty^2 + \int_{D^2} \frac{|\nabla \vec{B}|^2}{|x|^2} + \int_{D^2} |\nabla^2 \vec{B}_\perp|^2 \leq C \|\vec{v}_\perp\|_\infty^2 \tag{A.58}
\]

and

\[
\|\nabla \vec{C}_0\|_\infty^2 + \int_{D^2} \frac{|\nabla \vec{C}|^2}{|x|^2} + \int_{D^2} |\nabla^2 \vec{C}_\perp|^2 \leq C \|\vec{v}_\perp\|_\infty^2 + \|\vec{v}_0\|_\infty^2 \tag{A.59}
\]

where we have used the fact (like in (A.9)) that \( \text{div}(\pi \vec{n}(\nabla ^\perp \vec{v})) \) is a jacobian of the form \( -\sum_{k,i} \nabla ^\perp \vec{v}_k \cdot \nabla (e_i \vec{e}_k) \) and \( (\vec{e}_1, \vec{e}_2) \) is an orthonormal frame generating the 2-plane \( \vec{n} \) and given by lemma 5.1.4 of [Hel]. Combining (A.58), (A.59), (A.60) and choosing \( \varepsilon \) small enough, we get (A.47) and lemma A.6 is proved.

**Lemma A.7** There exists \( \varepsilon_0 > 0 \) such that for every \( 0 < \varepsilon < \varepsilon_0 \) the following holds. Let \( \vec{n} \) be a \( W^{1,2} \) map from \( D^2 \) into the space of unit \( m - 2 \)-vectors in \( \mathbb{R}^m \) satisfying

\[
\int_{D^2} |\nabla \vec{n}|^2 \, dx \leq \varepsilon . \quad \tag{A.61}
\]
Let $\vec{\phi}$ be a $W^{1,2}$ eigenvector of $L$ i.e. solution to the equation
\[
\begin{cases}
\Delta \vec{\phi} - 3 \text{div}(\pi_n(\nabla \vec{\phi})) - \text{div} \left( \vec{\phi} \wedge \nabla \perp \vec{n} \right) = \lambda \vec{\phi} & \text{in } D^2 \\
\vec{\phi} = 0 & \text{on } \partial D^2,
\end{cases}
\] (A.62)
for some constant $\lambda \in \mathbb{R}$. Then, assuming furthermore that the gradient of $\vec{n}$ is in the Lorentz space $L^{2,1}$, we have that $\vec{\phi}$ is Lipschitz with second derivatives in $L^{2,1}$.

\begin{proof}[Proof of lemma A.7]
First we prove that $\vec{\phi}$ is in $W^{1,p}(D^2)$ for every $1 \leq p < +\infty$. Let $2 < p < +\infty$ and denote $q$ the constant in $(1,2)$ given by $1/p = 1/q - 1/2$ in such a way that $W^{1,q}_0(D^2)$ embeds in $L^p$. We denote $\vec{g} := \lambda \vec{\phi} \in L^q(D^2)$ and we follow step by step the proof of lemma A.2 starting from A.13 replacing the assumption that $\vec{g}$ is in $H^{-1}$ by the hypothesis that $\vec{g}$ is in $L^q$. Precisely we first observe, using classical elliptic estimate that
\[
\|\nabla \vec{B}\|_{L^p(D^2)} \leq C \|\vec{g}\|_{L^q(D^2)}
\] (A.63)
Replacing Wente’s inequalities by classical $L^q$ bound for Calderon-Zygmund operators we obtain the following a-priori estimate
\[
\|\vec{F}\|_{L^p(D^2)} \leq C \|\vec{F}\|_{W^{1,q}} \leq C \left[ \int_{D^2} |\nabla A|^q + |\nabla e|^q + |\nabla B|^q \right]^{1/q}
\leq C \|\nabla A\|_p \|\nabla e\|_2 + \|\nabla B\| \|\nabla e\|_2
\leq C \varepsilon \left[ \|\nabla A\|_p \|\nabla B\|_p \right].
\] (A.64)
From this a-priori estimate, like in the proof of lemma A.2 we get the existence of $\vec{v}$ solving $L\vec{v} = \vec{g}$ which this time in $W^{1,p}_0$ for any $p > 2$. Because of the uniqueness given by lemma A.2 they all coincide with $\vec{\phi}$ and we have then proved that $\vec{\phi} \in \cap_{p<+\infty} W^{1,p}_0(D^2)$.

Let $x_0$ be a point in the interior of $D^2$. For any $\varepsilon > 0$ we can find a radius $\rho > 0$ such that the $L^{2,1}$ Lorentz norm of $\nabla \vec{n}$ on $B_\rho(x_0)$ is less than $\varepsilon$
\[
\|\nabla \vec{n}\|_{L^{2,1}(B_\rho(x_0))} \leq \varepsilon.
\] (A.65)
Consider a smooth cut-off function $\chi$ equals to 1 on $B_{1/2}(0)$ and equals to 0 outside $B_1(0) = D^2$. Denote $\vec{w}$ the function on the two dimensional disk equals to
\[
\vec{w}(x) = \vec{\phi}(\rho x + x_0) \chi(x).
\]
In view of the computations (III.49), (III.50), (III.51), since $\vec{\phi} \in W^{1,p}(D^2)$ for every $p < +\infty$, we have the existence of $\vec{k}$ in $L^{2,1}(D^2)$ such that
\[
\begin{cases}
L\vec{w} = \vec{k} & \text{in } D^2 \\
\vec{w} = 0 & \text{on } \partial D^2.
\end{cases}
\] (A.66)
We proceed now to the following Hodge decomposition of $\nabla \vec{w} - 3\pi_{\vec{n}}(\nabla \vec{w})$ on $D^2$

$\nabla \vec{w} - 3\pi_{\vec{n}}(\nabla \vec{w}) = \nabla \vec{C} + \nabla^\perp \vec{D}$ with the boundary conditions $C = 0$ on $\partial D^2$ and $\partial D/\partial \nu = 0$ on $D^2$. The following equations then holds In one hand

\[
\begin{aligned}
\Delta \vec{C} &= \text{div}(\vec{w} \wedge \nabla^\perp \vec{n}) + \vec{k} \quad \text{in } D^2 \\
\vec{C} &= 0 \quad \text{on } \partial D^2
\end{aligned}
\]  

(A.67)

In the other hand

\[
\begin{aligned}
\Delta \vec{D} &= 3\text{div}(\pi_{\vec{n}}(\nabla^\perp \vec{w})) \quad \text{in } D^2 \\
\frac{\partial \vec{D}}{\partial \nu} &= 0 \quad \text{on } \partial D^2
\end{aligned}
\]  

(A.68)

Using now the fact that the space of $L^2$ functions on $D^2$ having first derivatives in $L^{2,1}$ embeds in $L^\infty$, we get the a-priori estimates

\[
\|\nabla \vec{C}\|_{L^\infty(D^2)} + \|\nabla^2 \vec{C}\|_{L^2,1(D^2)} \leq C\|\Delta \vec{C}\|_{L^2,1(D^2)}
\]  

(A.69)

\[
\begin{aligned}
&\leq C \left[ \|k\|_{L^2,1(D^2)} + \|\nabla \vec{n}\|_{L^2,1(D^2)} \|\nabla \vec{w}\|_{L^\infty} \right] \\
&\leq C \frac{\|\nabla \vec{n}\|_{L^2,1(D^2)}}{\|\nabla \vec{w}\|_{L^\infty}} & \text{and}
\end{aligned}
\]

(A.70)

where we have used the fact that $|\text{div}(\pi_{\vec{n}}(\nabla \vec{w}))| \leq C|\nabla \vec{n}| \|\nabla \vec{w}\|$. Thus for $\varepsilon$ having been choosed small enough in (A.65), following the construction of lemma A.2, we get the existence of a lipschitz solution of (A.66) with second derivatives in $L^{2,1}$. From the uniqueness of lemma A.2, we obtain that this solution coincides with $\vec{w}$ and we have then established that $\vec{\phi}(px + x_0)\chi(x)$ is Lipschitz with second derivatives in $L^{2,1}$.

A similar argument including the boundary condition $\vec{C} = 0$ could be carried out for any point $x_0$ on the boundary of $D^2$. We then obtain that $\vec{\phi}$ is Lipschitz with second derivatives in $L^{2,1}$ and lemma A.7 is proved.

**Lemma A.8** There exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ the following holds. Let $\vec{n}$ be a $W^{1,2}$ map from $D^2$ into the space of unit $m - 2$-vectors in $\mathbb{R}^m$ satisfying

\[
\int_{D^2} |\nabla \vec{n}|^2 \, dx \leq \varepsilon .
\]  

(A.71)

Let $\vec{v}$ be a function in $L^2(D^2)$ such that $\nabla \vec{v}$ is a sum of a compactly supported distribution in the open disk and a function in $L^{2,\infty}(D^2)$ (in such a way that the trace of $\vec{v}$ on $\partial D^2$ is well defined). Assume $\nabla \vec{n}$ is in the Lorentz space $L^{2,1}(D^2)$ and that $\vec{v}$ satisfies in the distributional sense the system

\[
\begin{aligned}
\Delta \vec{v} - 3 \text{div}(\pi_{\vec{n}}(\nabla \vec{v})) - \text{div} (\vec{v} \wedge \nabla^\perp \vec{n}) &= 0 \quad \text{in } D^2 \\
\vec{v} &= 0 \quad \text{on } \partial D^2
\end{aligned}
\]  

(A.72)

Then $\vec{v}$ is identically $0$ in $D^2$. □
Proof of lemma [A.8]. We consider a smoothing \( \vec{v}_\delta \) of \( \vec{v} \) obtained by making convolution with a function with support shrinking as one get close to the boundary in order to ensure that \( \nabla \vec{v}_\delta \in L^{2,\infty}(D^2) \), that \( \vec{v}_\delta = \vec{v} \) in a neighborhood of \( \partial D^2 \) and that \( \vec{v}_\delta \to \vec{v} \) in \( L^2(D^2, \mathbb{R}^m) \). Let \( \vec{\phi}_i \) be a sequence of normalised eigenvectors of \( \mathcal{L}_{\vec{\alpha}} \) in \( W^{1,2}(D^2, \mathbb{R}^m) \) and forming an Hilbert orthonormal Basis of \( L^2(D^2, \mathbb{R}^m) \) (the existence of such a Basis is obtained by combining the result of Lemma [A.1] and Hilbert–Schmidt theorem). Denote \( \lambda_i \) the corresponding eigenvalues. From lemma [A.1] again we know that \( \lambda_i \neq 0 \). We have that

\[
\int_{D^2} \vec{v}_\delta \cdot \vec{\phi}_i = \lambda_i^{-1} \int_{D^2} \vec{v}_\delta \cdot \mathcal{L}_{\vec{\alpha}} \vec{\phi}_i .
\]

(A.73)

From lemma [A.7] \( \vec{\phi}_i \) is Lipschitz and \( \nabla^2 \vec{\phi}_i \in L^{2,\infty} \). Therefore, since \( \nabla \vec{v}_\delta \in L^{2,\infty} \) and since both \( \vec{\phi}_i \) and \( \vec{v}_\delta \) are 0 on \( \partial D^2 \), we have clearly that for any \( p > 2 \)

\[
\int_{D^2} \vec{v}_\delta \cdot \mathcal{L}_{\vec{\alpha}} \vec{\phi}_i = \left\langle \mathcal{L}_{\vec{\alpha}} \vec{v}_\delta, \vec{\phi}_i \right\rangle_{W^{-1,p},W^{1,p}}
\]

(A.74)

Observe now that \( \Delta \vec{v}_\delta \) converges to \( \Delta \vec{v} \) in \( H^{-2}(D^2) \) which is dual to \( W^{2,2}(D^2) \), that \( \text{div}(\pi_{\alpha}(\nabla \vec{v}_\delta)) \) converges to \( \text{div}(\pi_{\alpha}(\nabla \vec{v})) \) in \( W^{-1,1} \oplus H^{-2}(D^2) \) and that \( \text{div}(\vec{v}_\delta \wedge \nabla^2 \vec{v}) \) converges to \( W^{-1,1}(D^2) \). So using that \( \vec{\phi}_i \) is in \( W^{1,\infty} \cap W^{2,2}(D^2) \) and interpreting the duality in the right-hand-side of (A.74) by the mean of these norms we can pass in the limit as \( \delta \) goes to zero and we have that

\[
\int_{D^2} \vec{v} \cdot \mathcal{L}_{\vec{\alpha}} \vec{\phi}_i = \left\langle \mathcal{L}_{\vec{\alpha}} \vec{v}, \vec{\phi}_i \right\rangle_{H^{-2,\infty},W^{1,\infty}}
\]

(A.75)

Combining (A.72), (A.73) and (A.75) we obtain that for every \( i \int_{D^2} \vec{v} \cdot \vec{\phi}_i = 0 \) and since \( (\vec{\phi}_i) \) realizes an Hilbert base of \( L^2 \) we deduce that \( \vec{v} \) is identically zero on \( L^2 \) which ends the proof of lemma [A.8]. \( \square \)

Lemma A.9 There exists \( \varepsilon_0 > 0 \) such that for every \( 0 < \varepsilon < \varepsilon_0 \), there exists a constant \( C > 0 \) independent of \( \varepsilon \) such that the following holds. Let \( \vec{n} \) be a lipschitz map from \( D^2 \) into the space of unit \( m-2 \)-vectors in \( \mathbb{R}^m \) satisfying

\[
\int_{D^2} |\nabla \vec{n}|^2 \, dx \leq \varepsilon ,
\]

(A.76)

Let \( \vec{g} \) be a map in \( L^p(D^2) \) for some \( p > 1 \). Let \( \vec{v} \) be an \( L^2 \) solution of

\[
\Delta \vec{v} - 3 \text{div}(\pi_{\alpha}(\nabla \vec{v})) - \text{div} (\vec{v} \wedge \nabla^2 \vec{n}) = 0 \quad \text{in } D^2 .
\]

(A.77)

Then we have

\[
\| \nabla^2 \vec{v} \|_{L^p(D^2)} \leq C \left[ \| \vec{g} \|_{L^p(D^2)} + \| \vec{v} \|_{L^2(D^2)} \right] ,
\]

(A.78)

where \( D^{1/2}_{1/2} \) is the disk centered at 0 with radius 1/2. \( \square \)

The proof of this result goes following the same line as the one followed in some of the previous lemma.

Remark: The assumption that \( \vec{v} \) is in \( L^2 \) is not optimal in the previous lemma for the result to be true. This is however sufficient for the use we are making of it.
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