The gravity dual of supersymmetric gauge theories on a squashed five-sphere

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We present the gravity dual of large $N$ supersymmetric gauge theories on a squashed five-sphere. The one-parameter family of solutions is constructed in Euclidean Romans $F(4)$ gauged supergravity in six dimensions, and uplifts to massive type $IIA$ supergravity. By renormalizing the theory with appropriate counterterms we evaluate the renormalized on-shell action for the solutions. We also evaluate the large $N$ limit of the gauge theory partition function, and find precise agreement.

I. SUPERSYMMETRIC GAUGE THEORIES ON A SQUASHED FIVE-SPHERE

In [1] supersymmetric gauge theories with general matter content were defined on the $SU(3) \times U(1)$ symmetric squashed five-sphere. The background metric is

$$ds^2_5 = \frac{1}{s^2}(d\tau + C)^2 + d\sigma^2 + \frac{1}{4} \sin^2 \sigma (d\theta^2 + \sin^2 \varphi d\varphi^2) + \frac{1}{4} \cos^2 \sigma \sin^2 \sigma (d\psi + \cos \theta d\varphi)^2,$$  \hspace{1cm} (1)

where $C = -\frac{1}{2} \sin^2 \sigma (d\psi + \cos \theta d\varphi)$ and $s$ is the squashing parameter. The round sphere corresponds to $s = 1$. The theory preserves $3/4$ of the supersymmetry of the round sphere, provided one turns on a background $SU(2)_R$ gauge field

$$A^R = \frac{(1 + \sqrt{1 - s^2})}{s^2} \sqrt{1 - \frac{s^2}{2}} (d\tau + C),$$ \hspace{1cm} (2)

where we have embedded $U(1)_R \subset SU(2)_R$. The background then admits a Killing spinor that solves the Killing spinor equation in [1] and transforms in the 3 of $SU(3)$. The perturbative partition function of the gauge theories was computed in [2] (see also [3]) and the final formula involves triple sine functions, generalizing the double sine functions that appear for squashed three-spheres [4].

A particular class of five-dimensional gauge theories, with gauge group $USp(2N)$ and arising from a $D4 - DB$ system, is expected to have a large $N$ description in terms of massive type $IIA$ supergravity [5, 6]. In [7] the large $N$ limit of the partition function of these theories on the round sphere was computed and successfully compared to the entanglement entropy of the dual warped $AdS_5 \times S^4$ supergravity solution.

One can compute the large $N$ limit of the $USp(2N)$ gauge theory partition function $Z_s$ for the squashed background [8, 9]. The corresponding free energy $F_s = -\log Z_s$ is given by

$$F_s = \frac{1}{27s^2} \left( \frac{3 - \sqrt{1 - s^2}}{1 - \sqrt{1 - s^2}} \right)^3 \mathcal{F}_1.$$ \hspace{1cm} (3)

Here $\mathcal{F}_1$ is the free energy on the round sphere, which scales as $N^{5/2}$ [6, 7]. The computation of [3] involves asymptotic expansions of the triple sine function and standard large $N$ matrix model techniques, and details will appear in [8]. Similarly, we have computed the large $N$ limit of the VEV of a BPS Wilson loop wrapping the $\tau$ circle at $\sigma = 0$, finding

$$\log \langle W \rangle_s = \frac{3 - \sqrt{1 - s^2}}{3(1 + \sqrt{1 - s^2})} \log \langle W \rangle_1,$$ \hspace{1cm} (4)

where $\langle W \rangle_1$ scales as $N^{1/2}$ [3].

In the remainder of this letter we will reproduce (3) and (4) from a dual supergravity computation.

II. EUCLIDEAN ROMANS SUPERGRAVITY

In order to find supergravity duals of the above theories put on general background five-manifolds it is natural to work in the six-dimensional Romans $F(4)$ supergravity theory [10]. The key here is that, as shown in [11], the Romans theory is a consistent truncation of massive type $IIA$ supergravity on $S^4$. In particular, the $AdS_5$ vacuum uplifts to the warped $AdS_5 \times S^4$ solution mentioned above, relevant for the round five-sphere. The bosonic fields consist of the metric, a dilaton $\phi$, a two-form potential $B$, a one-form potential $A_i$, together with an $SO(3) \sim SU(2)$ gauge field $A^i$, $i = 1, 2, 3$. It is convenient to introduce the scalar field $X = \exp(-\phi/2\sqrt{2})$, and define the field strengths $H = dB$, $F = dA + \frac{1}{2} B$, $F^i = dA^i - \frac{1}{2} \epsilon_{ijk} A^j \wedge A^k$, where without loss of generality we have set the gauge coupling to 1. The equations of motion for the Romans theory in Lorentz signature appear in [10, 11]. However, in order to compute the holographic free energy we will work in Euclidean signature. This Wick rotation is not entirely straightforward due to Chern-Simons-type couplings. The Euclidean equations of motion are [8]

$$d(\Delta X^4 \ast H) = \frac{1}{4} F \wedge F + \frac{1}{2} F^i \wedge F^i + \frac{3}{8} X^{-2} \ast F,$$

$$d(X^{-2} \ast F) = -i F \wedge H,$$

$$D(X^{-2} \ast F^i) = -i F^i \wedge H,$$

$$d(X^{-1} \ast dX) = -\left(\frac{1}{4} X^{-6} - \frac{3}{8} X^{-2} + \frac{1}{2} X^2\right) \ast 1 - \frac{1}{8} X^{-2} (F \wedge \ast F + F^i \wedge \ast F^i) + \frac{1}{4} X^4 H \wedge \ast H.$$ \hspace{1cm} (5)
Here $D\omega^i = d\omega^i - \epsilon_{ijk}A^j \wedge \omega^k$ is the $SO(3)$ covariant derivative. Finally, the Einstein equation is

$$R_{\mu\nu} = 4X^{-2}\partial_{\mu}X\partial_{\nu}X + \left(\frac{1}{2}X^{-6} - \frac{3}{2}X^{-2} - \frac{1}{2}X^2\right)g_{\mu\nu} + \frac{1}{4}X^4\left(H_{\mu\nu}^2 - \frac{1}{2}H^2g_{\mu\nu}\right) + \frac{1}{2}X^{-2}\left(F_{\mu\nu}^2 - \frac{1}{8}(F^2)^2g_{\mu\nu}\right),$$

where $F_{\mu\nu}^2 = F_{\mu\rho}F_{\nu\rho}^\ast$, $H_{\mu\nu}^2 = H_{\mu\rho\nu\sigma}H_{\rho\sigma}^\ast$. A solution to the above equations of motion is super-symmetric provided the Killing spinor equation and dilatino equation hold \[8\]:

$$ds^2 = \alpha^2(r)dr^2 + \gamma^2(r)(d\tau + C)^2 + \beta^2(r)\left[d\sigma^2 + \frac{1}{2}\sin^2\sigma(d\theta + \sin^2\theta d\varphi)^2 + \frac{1}{2}\cos^2\sigma\sin^2\sigma(d\psi + \cos^2\psi d\varphi)^2\right],$$

$$B = p(r)dr \wedge (d\tau + C) + \frac{1}{2}q(r)dC,$$

$$A^i = f^i(r)(d\tau + C),$$

where also $X = X(r)$. We have constructed a smooth, supersymmetric, asymptotically locally Euclidean AdS solution to the equations in section \[\text{III}\] which has as conformal boundary the squashed five-sphere background of section \[\text{I}\]. The function $\beta(r)$ can be set to its $AdS_5$ value by using reparametrization invariance, $\beta(r) = 3\sqrt{6}r^2 - 1/\sqrt{2}$. Furthermore, we have performed an $SO(3) \sim SU(2)$ rotation so as to set $f^1(r) = f^2(r) = 0$, and renamed $f^3(r) \equiv f(r)$. Even though we are not able to give a closed expression for the solution, it is possible to give it as an expansion around different limits.

### III. The Solution

The squashed five-sphere background of section \[\text{I}\] has $SU(3) \times U(1)$ symmetry. One expects this symmetry to be preserved by the bulk solution. This leads to the following ansatz for the supergravity fields

$$ds^2 = \alpha^2(r)dr^2 + \gamma^2(r)(d\tau + C)^2 + \beta^2(r)\left[d\sigma^2 + \frac{1}{2}\sin^2\sigma(d\theta + \sin^2\theta d\varphi)^2 + \frac{1}{2}\cos^2\sigma\sin^2\sigma(d\psi + \cos^2\psi d\varphi)^2\right],$$

$$B = p(r)dr \wedge (d\tau + C) + \frac{1}{2}q(r)dC,$$

$$A^i = f^i(r)(d\tau + C),$$

where also $X = X(r)$. We have constructed a smooth, supersymmetric, asymptotically locally Euclidean AdS solution to the equations in section \[\text{III}\] which has as conformal boundary the squashed five-sphere background of section \[\text{I}\]. The function $\beta(r)$ can be set to its $AdS_5$ value by using reparametrization invariance, $\beta(r) = 3\sqrt{6}r^2 - 1/\sqrt{2}$. Furthermore, we have performed an $SO(3) \sim SU(2)$ rotation so as to set $f^1(r) = f^2(r) = 0$, and renamed $f^3(r) \equiv f(r)$. Even though we are not able to give a closed expression for the solution, it is possible to give it as an expansion around different limits.

### A. Expansion around the conformal boundary

Finding the gravity dual to a theory on a prescribed conformal boundary may be regarded as a filling problem in supergravity. As such, it is natural to solve the supergravity equations order by order in an expansion around the boundary at $r = \infty$. We have computed this expansion up to order $O(1/r^9)$. The first terms are given by

$$\alpha(r) = \frac{3}{\sqrt{r}} + \frac{8}{36\sqrt{2}s^2}r^3 + \ldots, \quad \gamma(r) = \frac{3}{s}r + \frac{16 + 7s^2}{12\sqrt{3}s^3}r - \frac{1280 + 1120s^2 + 241s^4}{2592\sqrt{3}s^5}r^3 + \ldots,$$

$$X(r) = 1 + \frac{1 - s^2 - 3\sqrt{1 - s^2}}{54s^2} + \frac{s^2\sqrt{1 - s^2}}{12(1 - s^2 + \sqrt{1 - s^2})}r^3 + \ldots,$$

$$p(r) = -\frac{i\sqrt{3}(s^3 + 3\sqrt{1 - s^2})}{s^3 + \sqrt{1 - s^2}} + \ldots, \quad q(r) = -\frac{3i(s\sqrt{1 - s^2})}{s}r + \frac{\sqrt{\frac{3}{2}4\sqrt{1 - s^2}(5s^2 + 9\sqrt{1 - s^2} - 5)}r}{3s^3} + \ldots,$$

$$f(r) = \frac{1 - s^2 + \sqrt{1 - s^2}}{s^2} + \frac{2(2 + 2s^2 - (2 + s^2)s^2 - 5)}{9s^4}r + \frac{\kappa}{r^3} + \ldots. \quad (10)$$

Notice that the squashing parameter $s$ arises as the boundary value $\lim_{r \to \infty} \gamma(r)/3\sqrt{3}r = s^{-1}$. In the limit $s = 1$ the solution collapses to Euclidean $AdS_5$. The whole solution depends on the single parameter $s$. The extra parameter $\kappa$ is fixed by requiring a non-singular solution at the origin $r = 1/\sqrt{3}$. Alternatively, this can be
computed as an expansion, as is done at the end of next subsection.

B. Expansion around Euclidean AdS

The solution presented in this letter is continuously connected to Euclidean AdS$_6$. Hence it can be given as

\[ \alpha(r) = \frac{3\sqrt{3}}{\sqrt{6r^2 - 1}} \left( -5\sqrt{6} + 330\sqrt{6}r^2 - 3744r^3 + 1620\sqrt{6}r^4 + 8640r^5 - 7560\sqrt{6}r^6 + 5184\sqrt{6}r^7 \right) \delta^2 + \ldots, \]

\[ \gamma(r) = \frac{3\sqrt{6r^2 - 1}}{\sqrt{2}} \left( 55\sqrt{2} - 384\sqrt{3}r + 1080\sqrt{2}r^2 + 768\sqrt{3}r^3 - 5400\sqrt{2}r^4 + 11232\sqrt{2}r^5 - 11664\sqrt{2}r^6 \right) \delta^2 + \ldots, \]

\[ X(r) = 1 - \frac{\sqrt{2}(1 - 2\sqrt{6r} + 6r^2)}{3(6r^2 - 1)^2} \delta + \ldots, \]

\[ p(r) = \frac{18i\sqrt{2}(\sqrt{6} - 16r + 12\sqrt{6}r^2 - 12\sqrt{6}r^4)}{(6r^2 - 1)^3} \delta + \ldots, \]

One can explicitly check that each term of the solution above is non-singular at the origin $r = 1/\sqrt{6}$, giving a regular solution on a manifold $M_6$ with the topology of a six-ball. By comparing the two expansions we find

\[ \frac{3\sqrt{3}}{4} = \delta + \frac{\sqrt{2}}{3} \delta^2 + \frac{113}{36} \delta^3 + \frac{25}{9\sqrt{2}} \delta^4 + \frac{1127}{288} \delta^5 + \frac{35}{9\sqrt{2}} \delta^6 + \ldots \]

which is used in evaluating the on-shell action below.

IV. COMPARISON

The bulk supergravity action of the Romans theory, in Euclidean signature in the gauge $A = 0$, is

\[ S_{\text{bulk}} = \frac{1}{16\pi G_N} \int_{M_6} \left[ R + 4X^{-2}dX \wedge *dX - \left( \frac{2}{3}X^{-6} - \frac{8}{3}X^{-2} - 2X^2 \right) *1 - \frac{1}{2}X^{-2} \left( \frac{2}{3}B \wedge *B + F^i \wedge *F^i \right) - \frac{1}{2}X^4H \wedge *H - iB \wedge \left( \frac{2}{3}B \wedge B + \frac{1}{2}F^i \wedge F^i \right) \right]. \]

Here $G_N$ is the six-dimensional Newton constant. More precisely, we should cut off the manifold $M_6$ at some large constant radius $r = \rho$, and include the Gibbons-Hawking boundary term

\[ S_{\text{GH}} = -\frac{1}{8\pi G_N} \int_{\partial M_6} K \sqrt{\det h} \, d^3x, \]

where $h_{ij}$ is the induced metric on the boundary $\partial M_6 = \{ r = \rho \} \cong S^3$, and $K$ denotes the trace of the second fundamental form. The total action is divergent as one sends $\rho \to \infty$, but may be regularized using holographic renormalization techniques. This leads to the following boundary counterterms \[ S_{\text{ct}} = \frac{1}{8\pi G_N} \int_{\partial M_6} \left\{ \left( \frac{4\sqrt{2}}{3} + \frac{2\sqrt{2}}{3} R(h) - \frac{1}{6\sqrt{2}} \right) \|B\|^2_h + \frac{3}{4\sqrt{2}} R(h)_{ij} R(h)^{ij} - \frac{15}{64\sqrt{2}} \|R(h)\|^2_h \right. \]

\[ + \frac{1}{12\sqrt{2}} \text{Tr}_h B^4 + \frac{5}{8\sqrt{2}} \|d \ast_h B\|_h + \frac{i\sqrt{2}}{3} \|B \wedge B\|_h \]

\[ - \frac{1}{4\sqrt{2}} (B, d \delta_h B) + \frac{i\sqrt{2}}{3} (d \ast_h B \wedge B)_h - \frac{1}{\sqrt{2}} \|d B\|_h \]

\[ + \frac{4\sqrt{2}}{3} (1 - X^2) - \frac{9}{32\sqrt{2}} \|R(h)\|^2_h - \frac{13}{192\sqrt{2}} \|B\|^4_h \sqrt{\det h} \, d^3x \]

\[ - \frac{1}{4\sqrt{2}} B \wedge [d \ast_h B + \frac{\sqrt{3}}{3} B \wedge \delta_h B] - \frac{2}{9} B \wedge [d \ast_h B \wedge B] \right\}. \]

Here $R(h)_{ij} = R(h)_{ij}$ denotes the Ricci tensor of the metric $h_{ij}$, with $R(h)$ the Ricci scalar. The inner product of two $p$-forms $\nu_1, \nu_2$ is defined by $\langle \nu_1, \nu_2 \rangle_h \sqrt{\det h} \, d^3x = \nu_1 \wedge * \nu_2$, which then also defines the square norm via $\|\nu\|^2_h = \langle \nu, \nu \rangle_h$. The adjoint $\delta_h$ of $d$ with respect to $h_{ij}$ acting on the two-form $B$ is $\delta_h B = \ast_h d \ast_h B$, and we have also defined $\text{Tr}_h B^4 \equiv B^i_j B^k_l B^l_i$. Finally, we have defined the $p$-form $(S \circ \nu)_{i_1 \ldots i_p} \equiv S_{[i_1} \nu_{i_2]} \ldots \nu_{i_p]}$, where $S_{ij}$ is any symmetric 2-tensor, and $\nu$ is any $p$-form.

Adding the contributions and taking the cut-off to in-
finity we obtain
\[ S_{\text{bulk}} + S_{\text{GH}} + S_{\text{ct}} = -\frac{27\pi^2}{4G_N} \left( 1 + \frac{8}{3} \delta^2 + \frac{16\sqrt{2}}{27} \delta^3 + \frac{68}{27} \delta^4 \\
+ \frac{28\sqrt{2}}{27} \delta^5 + \frac{32}{27} \delta^6 + \ldots \right) . \] (17)

This should be identified with the holographic free energy. Recalling that \( s^{-1} = 1 + \delta^2 \), this precisely agrees with (3) to sixth order in \( \delta \). It should be straightforward to extend this agreement to higher orders.

The BPS Wilson loop (4) maps to a fundamental string in type \( IIA \) at the “pole” of the internal \( S^4 \) [9]. The string wraps the surface \( \Sigma \) spanned by the \( \tau \) and \( r \) directions at \( \sigma = 0 \). The renormalized string action is
\[ S_{\text{string}} = \int_{\Sigma} \left[ X^{-2} \sqrt{\det \gamma} \, d^2x + iB \right] - \frac{3}{\sqrt{2}} \text{length}(\partial \Sigma) , \] (18)
where \( \gamma_{ab} \) is the induced metric and the second term is a boundary counterterm. We may evaluate this up to sixth order in \( \delta \) for our solution to obtain
\[ S_{\text{string}} = \left( 1 - \frac{4\sqrt{2}}{3} \delta^2 + \frac{8\delta^2}{3} - \frac{5\sqrt{2}\delta^3}{3} + \frac{4\delta^4}{3} \right.
\left. - \frac{7\delta^5}{12\sqrt{2}} + 0 \delta^6 + \ldots \right) S_{\text{string}} |_{\delta=0} , \] (19)
which precisely matches (4).

V. A CONJECTURE

A supersymmetric solution admits an \( SU(2) \) doublet of Killing spinors \( \epsilon_I \). Provided the Killing spinor satisfies a symplectic Majorana condition \( \tilde{C} \epsilon_I = \epsilon_J^T \gamma_{\mu} \epsilon_J \), where \( \gamma \) is the charge conjugation matrix \( (\tilde{C}^{-1} \gamma_{\mu} \tilde{C} = \gamma^*_{\mu}) \), it can be shown [8] that
\[ K_{\mu} = \epsilon_I^T \gamma_{\mu} \epsilon_J \] (20)
is a real Killing vector. For our solution the Killing spinor has 3 integration constants, corresponding to the fact that it is 3/4 BPS, and for an appropriate choice of the Killing spinor in (20) we obtain
\[ K = b_1 \partial_{\varphi_1} + b_2 \partial_{\varphi_2} + b_3 \partial_{\varphi_3} , \] (21)
where
\[ b_1 = 1 + \sqrt{1-s^2} , \quad b_2 = b_3 = 1 - \sqrt{1-s^2} , \] (22)
and \( \varphi_1, \varphi_2, \varphi_3 \) are the standard \( 2\pi \) periodic azimuthal variables \( \varphi_1 = -\tau, \varphi_2 = \tau - \frac{1}{2} (\psi + \varphi), \varphi_3 = \tau - \frac{1}{2} (\psi - \varphi) \). Embedding \( S^5 \subset \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}^2 \). Note that the large \( N \) free energy (23) can then be written as
\[ F = \frac{(b_1 + b_2 + b_3)^3}{27b_1b_2b_3} F_{\text{round}} . \] (23)

It is then natural to conjecture
1. For any supersymmetric supergravity solution with the topology of the six-ball, with at least \( U(1)^3 \) isometry, and for which the Killing vector (20) takes the form (21), the holographic free energy is equal to (23).

2. If we define a supersymmetric gauge theory on the conformal boundary of the background in point 1, the finite \( N \) partition function depends only on \( b_1, b_2, b_3 \).

These conjectures extend to \( 5d/6d \) the results proven for the analogous \( 3d/4d \) context in [12, 13]. Conjecture 1 also extends to the BPS Wilson loop wrapping \( \varphi_i \), at the origin of the perpendicular \( \mathbb{R}^4 \). In this case log \( \langle W \rangle_s = \frac{(b_1 + b_2 + b_3)}{2} \log \langle W \rangle_1 \).

In [9] we construct further families of supersymmetric backgrounds satisfying the conditions of point 1 and verify the conjecture for these cases. These include a supersymmetric solution with the \( SU(2) \) gauge field turned off, with a squashing parameter but for which \( b_1 = 1 \). For this case the free energy does not depend on the squashing parameter, in full agreement with conjecture 1.

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