THE THETA DIVISOR AND THREE-MANIFOLD INVARIANTS

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Abstract. In this paper we study an invariant for oriented three-manifolds with 
b₁ > 0, which is defined using Heegaard splittings and the theta divisor of a Rie-
mann surface. The paper is divided into two parts, the first of which gives the 
definition of the invariant, and the second of which identifies it with more classical 
(torsion) invariants of three-manifolds. Its close relationship with Seiberg-Witten 
theory is also addressed.

1. Introduction

Let Y be an oriented three-manifold whose first Betti number b₁(Y) > 0. In this 
paper, we study a topological invariant of Y, which is a function

\[ \theta: \text{Spin}^c(Y) \to \mathbb{Z} \]

on the set of Spin^c structures on Y, defined using Heegaard splittings. Roughly speak-
ing, the invariant measures how the theta divisor of a Riemann surface behaves under 
certain degenerations of the metric which are naturally associated to the Heegaard 
splitting. To facilitate a more precise description, we recall some relevant objects 
associated to Riemann surfaces and then Heegaard splittings.

Fix a Riemannian surface Σ of genus g. We think of the Jacobian J as the space of 
complex line bundles E over Σ of degree g − 1, modulo isomorphism. A generic bundle 
E in J, admits no holomorphic sections. The theta divisor, then, is the locus of line 
bundles which do. Note that the space J is a real 2g-dimensional torus; indeed, a 
spin structure naturally induces an identification between the space J and the torus 
\( H^1(Σ; S^1) \). (Here, we think of the circle \( S^1 \) as \( \mathbb{R}/\mathbb{Z} \).) Moreover, the theta divisor is 
the image of the Abel-Jacobi map

\[ \Theta: \text{Sym}^{g-1}(Σ) \to J, \]

which assigns to a divisor the corresponding holomorphic line bundle.

Now, consider a handlebody \( U \) bounding Σ. Such a handlebody gives rise to a 
canonical g-dimensional torus L(U) in J: L(U) corresponds to the image of \( H^1(U; S^1) \) 
in \( H^1(Σ; S^1) \) via the identification corresponding to a spin structure \( s_0 \) on Σ which

The first author was partially supported by NSF grant number 9971950.
The second author was partially supported by NSF grant number DMS 970435 and a Sloan 
Fellowship.
extends over $U$. (Note, however, that $L(U)$ is independent of the choice of spin structure used in its definition.)

A handlebody $U$ bounding $\Sigma$ can be described using Kirby calculus. $U$ is obtained from $\Sigma$ by first attaching $g$ two-handles along $g$ disjoint simple, closed curves $\{\gamma_1, \ldots, \gamma_g\}$ which are linearly independent in $H_1(\Sigma; \mathbb{Z})$; and then one three-handle. The collection $\{\gamma_1, \ldots, \gamma_g\}$ will be called a complete set of attaching circles for $U$. Since the three-handle is unique, $U$ is determined by a complete set of attaching circles.

A handlebody $U$ gives rise to a class of $U$-allowable metrics on $\Sigma$ (see Definitions 2.2 and 2.6), which correspond to certain degenerations of $\Sigma$. For instance, if $\{\gamma_1, \ldots, \gamma_g\}$ is a complete set of attaching circles for $U$, then any metric which is sufficiently stretched out normal to all of the $\gamma_i$ is $U$-allowable. One special property of a $U$-allowable metric is that the corresponding theta divisor is always disjoint from the subspace $L(U)$ (see Lemma 2.1).

Recall that a genus $g$ Heegaard decomposition of an oriented 3-manifold is a decomposition of $Y = U_0 \cup_\Sigma U_1$ into two handlebodies $U_0$ and $U_1$ which are identified along their boundary, which is a surface $\Sigma$ of genus $g$. Denote by $L_i$ the associated tori $L(U_i)$ in the Jacobian. Fix a one-parameter family $h_t$ of metrics on $\Sigma$ for which $h_0$ is $U_0$-allowable, and $h_1$ is $U_1$-allowable. Then, consider the set of points in $[0, 1] \times [0, 1] \times \text{Sym}^{g-1}(\Sigma)$

$$\{(s, t, D) \mid s \leq t \text{ and } \Theta_{h_s}(D) \in L_0 \text{ and } \Theta_{h_t}(D) \in L_1\}.$$ 

We show that for small, generic perturbations of $L_i$, this set of points is isolated. Moreover, there is a natural map from this set to the set of Spin$^c$ structures on $Y$. Then, $\theta(s)$ is a signed count of the number of points corresponding to the Spin$^c$ structure $s$.

A geometric meaning of this signed count can be given as follows. The one-parameter family of metrics induces a map

$$\Theta: [0, 1] \times \text{Sym}^{g-1}(\Sigma) \to J,$$

by $\Theta(t, D) = \Theta_{h_t}(D)$. The set $\Theta^{-1}(L_0)$ misses the region where $t < \epsilon$, and $\Theta^{-1}(L_1)$ misses the region where $t > 1 - \epsilon$. The tori $L_0$ and $L_1$ can be perturbed slightly to make them disjoint. The invariant $\theta$ then measures the degree to which the preimages under $\Theta$ of these perturbed versions of $L_0$ and $L_1$ are linked. One gets more than a simple linking number – hence the function on Spin$^c(Y)$ – by passing to a suitable covering space of $J$, and looking at the linking numbers between the preimages of the various lifts of $L_0$ and $L_1$. Details are spelled out in Section 2, where the first main result is the following:

**Theorem 1.1.** The invariant

$$\theta: \text{Spin}^c(Y) \to \mathbb{Z}$$
is well-defined; in particular, it does not depend on the metrics, perturbations, and Heegaard decompositions of $Y$.

The invariant $\theta$ also manifestly shares some of the properties of the Seiberg-Witten invariant for three-manifolds.

**Proposition 1.2.** For any given oriented three-manifold $Y$, with $b_1(Y) > 0$ there are only finitely many Spin$^c$ structures for which $\theta(s) \neq 0$. Moreover, $\theta(s) = \theta(\overline{s})$, where the map $s \mapsto \overline{s}$ denotes the natural involution on the set Spin$^c(Y)$. Also, if $-Y$ denotes the oriented manifold obtained by reversing the orientation of $Y$, then

$$\theta_Y(s) = (-1)^{b_1+1}\theta_{-Y}(s).$$

After laying down the basis for the definition of the invariant, we turn to its computation. It will be convenient for us to think of the invariant as an element $\theta \in \mathbb{Z}[\text{Spin}^c(Y)]$, in the usual manner:

$$\theta = \sum_{s \in \text{Spin}^c(Y)} \theta(s)[s],$$

where $\mathbb{Z}[\text{Spin}^c(Y)]$ is to be thought of as a module over the group-ring $\mathbb{Z}[H]$ associated to the group $H = H^2(Y;\mathbb{Z}) \cong H_1(Y;\mathbb{Z})$. In fact, in the computation, we begin by considering a weaker invariant, obtained from $\theta$ by dividing out by the action of the torsion subgroup $\text{Tors}$ of $H^2(Y;\mathbb{Z})$. In keeping with the convention of [15], we underline objects when they are to be viewed modulo the action of the torsion subgroup $\text{Tors}$ so, e.g. $\underline{H}$ and $\underline{\text{Spin}^c(Y)}$ denote the quotients of $H$ and $\text{Spin}^c(Y)$ respectively by the action of $\text{Tors}$. There is an induced invariant

$$\underline{\theta}: \underline{\text{Spin}^c(Y)} \longrightarrow \mathbb{Z},$$

defined by adding the values of $\theta(s)$ for all Spin$^c$ structures in a given orbit. Via the natural identification

(1) $$\underline{\text{Spin}^c(Y)} \cong H,$$

which sends any Spin structure to 0, we can view $\underline{\theta}$ as an element $\underline{\theta} \in \mathbb{Z}[\underline{H}]$.

**Theorem 1.3.** If $b_1(Y) > 1$, then up to sign, $\underline{\theta}$ is equal to the symmetrized Alexander polynomial of $Y$.

**Theorem 1.4.** Suppose $b_1(Y) = 1$, and let $A = a_0 + \sum_{i=1}^k a_i(t^i + t^{-i})$ be the symmetrized Alexander polynomial of $Y$ normalized so that $A(1) = |\text{Tors}H_1(Y;\mathbb{Z})|$. Then,

$$\underline{\theta}(i) = \sum_{j=1}^{\infty} j \cdot a_{|i|+j},$$

(note that we are using the natural identification $\underline{\text{Spin}^c(Y)} \cong \mathbb{Z}$ coming from (1)).
In fact, a closer inspection of the proofs of Theorems 1.3 and 1.4 gives a more refined statement, which identifies the invariant $\theta$ with a torsion invariant $\tau \in \mathbb{Z}[\text{Spin}^c(Y)]$ discovered by Turaev, see [20]. (The element we denote by $\tau$ here is the element of $\mathbb{Z}[\text{Spin}^c(Y)]$ induced from Turaev’s “torsion function” $T$ of §5 from [20].)

**Theorem 1.5.** Suppose $b_1(Y) > 1$. Then the invariant $\theta \in \mathbb{Z}[\text{Spin}^c(Y)]$ agrees, up to possibly translation by two-torsion in $H^2(Y;\mathbb{Z})$ and a sign which depends only on $b_1(Y)$, with the Turaev invariant $\tau$.

When $b_1(Y) = 1$, Turaev’s torsion function depends on a choice of generator of $H$. We recall from Turaev that if one fixes an $H \cong \mathbb{Z}$ and $t$ denotes the positive generator of $H$, then the two torsion functions $T_t$ and $T_{t-1}$ are related by the formula

$$T_{t-1}(s) = T_t(s) - s.$$

Moreover, the support of $T_t$ and $T_{t-1}$ are bounded above and below respectively. Now, one can define a compactly supported torsion function $T'$ which does not depend on a choice of generator by $T'(s) = T_t(s)$ if $s$ is a non-negative multiple of $t$, $T'(s) = T_{t-1}(s)$ otherwise; or equivalently

$$T'(s) = \frac{1}{2}(T_t(s) + T_{t-1}(s) + |s|).$$

Let

$$\tau' = \sum_s T'(s)[s].$$

Our result can then be stated as follows:

**Theorem 1.6.** Suppose $b_1(Y) = 1$. Then the invariant $\theta \in \mathbb{Z}[\text{Spin}^c(Y)]$ agrees, up to possibly translation by two-torsion in $H^2(Y;\mathbb{Z})$, with the Turaev invariant $\tau'$.

The relationship between the invariant $\theta$ and the Seiberg-Witten invariant for three-manifolds can be seen from two different points of view. On the one hand, the invariant arises naturally when studying the Seiberg-Witten equations for Heegaard decompositions; in fact this is how we discovered it. On the other hand, results of Meng-Taubes [15] and Turaev [21], together with our computation, show that the invariant $\theta$ agrees with a numerical invariant obtained from the Seiberg-Witten equations. It is also interesting to compare this with the Morse-theoretic constructions of [11], and also with recent work of Salamon [19].

Throughout the paper, we work with three-manifolds whose first Betti number is positive. In the case where $b_1(Y) = 0$, there is a naturally associated invariant, which is technically more complicated to describe. The reason for this is that, when $b_1(Y) = 0$, the invariant $\theta$ actually depends on the path of metrics used in its definition, so to get a topological quantity, one must correct by a spectral flow correction term. These issues are addressed in [18], where the relationship between this construction and the Casson-Walker invariant is explored.
The present paper is organized as follows. Roughly speaking, it can be divided into two parts: the first of which (Sections 2–4, together with Section 8) defines the invariant, and the second of which (Sections 5–7) calculates it. In Section 2, we describe the metrics on $\Sigma$ induced by the Heegaard decomposition of $Y$, and show that the invariant $\theta(s)$ is independent of choices of metrics, and hence can depend only on the Heegaard decomposition of $Y$ (except for the special case where $Y$ is a rational homology $S^1 \times S^2$ but not an integer homology $S^1 \times S^2$, a case which we return to in Section 8). The results rely on a few technical lemmas about the behaviour of the theta divisor under degenerations of $\Sigma$, which are proved in Section 3. Independence of the Heegaard decomposition, then, amounts to proving “stabilization invariance” of the invariant. This result is proved in Section 4, as a corollary to some results about the behaviour of the theta divisor under degenerations of the metric along homologically inessential curves. The degenerations of Section 4 play an important role in the second part of the paper, as well. The calculation of the invariant depends on a certain perturbation, which involves slightly enlarging the tori $L(U) \subset J$ coming from the handlebodies, to a $g+1$-dimensional torus which intersects the theta divisor even when the metric on $\Sigma$ is $U$-allowable. Another corollary of the results of Section 4, then, is an explicit understanding of the intersection of the theta divisor with these larger tori.

With the technical background in place, we turn to the calculations, which identify the invariant $\theta$ with data of a more directly topological character. In Section 5, we focus on the case when $b_1(Y) > 1$, which is slightly simpler than the calculation when $b_1(Y) = 1$ given in Section 6, as there is more freedom in perturbing the invariant when the second Betti number is large. But the same general idea works in both cases. The close relationship between the topological data obtained and the Alexander polynomial (see Theorems 1.3 and 1.4 above) is explained in Section 7. Indeed, a closer look at the proofs of these results gives the more refined formulations involving Turaev’s torsion invariant (see Theorems 1.5 and 1.6 above), as shown in Subsection 7.1.

A final debt is paid in Section 8, where we address the case of topological invariance in the case where $b_1(Y) = 1$ (with no assumptions on the torsion in $H_1(Y; \mathbb{Z})$). This section should be thought of as an appendix to the first part of the paper, though the proofs are of a slightly different character than those in the rest of the paper (bearing a closer relationship to the issues addressed in [18]).
2. Defining the invariant

The aim of this section is to spell out the details that go into the definition of the invariant \( \theta \) sketched in Section 1, and, indeed to prove that its value depends only on the topology of the Heegaard decomposition of the three-manifold. We complete the proof of Theorem 1.1 in Section 4, where we prove, among other things, that \( \theta \) remains invariant under stabilization.

Fix an oriented three-manifold \( Y \) whose first Betti number \( b_1(Y) > 0 \). The definition of the invariant makes reference to a genus \( g \) Heegaard decomposition of \( Y = U_0 \cup_\Sigma U_1 \); so we discuss some objects naturally associated to such a decomposition.

We give first a convenient definition of the Jacobian of an oriented 2-manifold \( \Sigma \) of genus \( g \), endowed with a Riemannian metric \( h \). Fix a Hermitian line bundle \( E \) over \( \Sigma \) whose Euler number is \( g - 1 \). A Hermitian connection \( A \) over \( E \) is said to have normalized curvature form if its curvature form satisfies

\[
F_A = \frac{1}{2} F_{K(h)},
\]

where \( K(h) \) is the Levi-Civita connection on the canonical bundle for the metric \( h \). Then the Jacobian \( J_h \) is the space of Hermitian connections \( A \) with normalized curvature form, modulo the gauge group of circle-valued functions \( \text{Map}(\Sigma; S^1) \). The group \( H^1(\Sigma; \mathbb{R}) \) acts simply transitively on \( J_h \), with stabilizer \( H^1(\Sigma; \mathbb{Z}) \) and so a point in \( J_h \) gives an identification of \( J_h \) with the \( 2g \)-dimensional torus

\[
J_h \cong \frac{H^1(\Sigma; \mathbb{R})}{H^1(\Sigma; \mathbb{Z})} = H^1(\Sigma; S^1).
\]

Moreover, a spin structure on \( \Sigma \) naturally gives rise to a point in \( J_h \), and hence an identification

\[
J_h \cong H^1(\Sigma; S^1).
\]

When it is clear from the context, we drop the metric \( h \) from the notation for the Jacobian.

We will typically work in a certain cover of the Jacobian which is associated to the Heegaard decomposition. Specifically, the long exact sequence in cohomology for the decomposition induces a (surjective) coboundary map \( \delta : H^1(\Sigma; \mathbb{Z}) \to H^2(Y; \mathbb{Z}) \), whose kernel we denote by \( \Gamma \) (alternatively, this is the subgroup of \( H^1(\Sigma; \mathbb{Z}) \) generated by the image of \( H^1(U_0; \mathbb{Z}) \oplus H^1(U_1; \mathbb{Z}) \) under the obvious inclusion map). We find it convenient, then, to consider the cover \( \tilde{J} \) of \( J \), the space of connections in \( E \) with normalized curvature form, modulo the action by gauge transformations in the kernel of the composite

\[
\text{Map}(\Sigma; S^1) \longrightarrow H^1(\Sigma; \mathbb{Z}) \overset{\delta}{\longrightarrow} H^2(Y; \mathbb{Z})
\]
The space $\tilde{J}$ inherits a natural action of $H^1(\Sigma; \mathbb{R})$, and the action of $H^1(\Sigma; \mathbb{Z})$ on $\tilde{J}$ descends to a free action of

$$\frac{H^1(\Sigma; \mathbb{Z})}{\Gamma} \cong H^2(Y; \mathbb{Z})$$

on $\tilde{J}$, whose quotient is canonically identified with $J$. The condition that $b_1(Y) > 0$ is equivalent to the condition that $\tilde{J}$ is a non-compact space.

Given a metric $h$, there is an “Abel-Jacobi map”

$$\Theta_h : \text{Sym}^{g-1}(\Sigma) \longrightarrow J_h,$$

where $\text{Sym}^{g-1}(\Sigma)$ is the space of effective degree $g - 1$ divisors on $\Sigma$, i.e. $g - 1$-fold symmetric power of $\Sigma$ (note that our conventions are slightly different from those typical in Riemann surface theory, where the Jacobian is often thought of as the group of complex structures on a topologically trivial line bundle, rather than the positive spinor bundle). Given a divisor $D \in \text{Sym}^{g-1}(\Sigma)$, the corresponding connection $\Theta_h(D)$ is characterized by its curvature form (half that of the canonical bundle with metric $h$) and its associated $\bar{\partial}$-operator, which we require to admit a holomorphic section which vanishes exactly at $D$. The image of this map in $J$ is called the theta divisor.

Once again, we find it convenient to work in a lift $\tilde{\text{Sym}}^{g-1}(\Sigma)$ of $\text{Sym}^{g-1}(\Sigma)$. This lift corresponds to the subgroup of $\pi_1(\text{Sym}^{g-1}(\Sigma))$ which is the kernel of the composite

$$\pi_1(\text{Sym}^{g-1}(\Sigma)) \longrightarrow H_1(\text{Sym}^{g-1}(\Sigma)) \xrightarrow{(\Theta_h)_*} H_1(J) \cong H^1(\Sigma; \mathbb{Z}) \xrightarrow{\delta} H^2(Y; \mathbb{Z}),$$

where the first map is the Hurewicz homomorphism. Clearly, $(\Theta_h)_*$ is independent of the Riemannian metric. Thus, we have a map

$$\tilde{\Theta}_h : \tilde{\text{Sym}}^{g-1}(\Sigma) \longrightarrow \tilde{J}$$

which fits into a commutative diagram

$$\begin{array}{ccc}
\tilde{\text{Sym}}^{g-1}(\Sigma) & \xrightarrow{\tilde{\Theta}_h} & \tilde{J} \\
\downarrow & & \downarrow \\
\text{Sym}^{g-1}(\Sigma) & \xrightarrow{\Theta_h} & J.
\end{array}$$

Note that standard Hodge theory gives an identification between the Jacobian and the theta divisor given here with the definitions used in the introduction.

Fix a handlebody $U$ which bounds $\Sigma$, and view the group $H^1(U; \mathbb{R})$ as a subgroup of $H^1(\Sigma; \mathbb{R})$ using the natural inclusion. There is a natural quotient map

$$Q_U : J \longrightarrow \frac{H^1(\Sigma; S^1)}{H^1(U; S^1)} \cong H^2(U, \Sigma; S^1),$$

given as follows. Fix a Spin structure $s_0$ on $U$, and let $p \in J$ be the induced point in the Jacobian. Given any $B \in J$, there is a unique $a \in H^1(\Sigma; S^1)$ so that $B = p + a$;
we define $Q_U(B)$ to be a (modulo $H^1(U; S^1)$). This coset is independent of the spin structure on $U$ since any two spin structures on $U$ differ by a translation by a cohomology class coming from $H^1(U; S^1)$. The torus $L(U)$ defined in the introduction, then, is the preimage $Q_U^{-1}(0)$. Given a point $B \in J$ and a homology class $[\gamma] \in H_1(\Sigma; \mathbb{Z})$ which bounds in $U$ there is a well-defined holonomy, $\text{Hol}_\gamma(B) \in S^1$, which is the Kronecker pairing of $Q_U(B)$ with $[\gamma]$.

For a Heegaard decomposition of $Y$, let $L_0$ and $L_1$ denote the associated tori $L(U_0)$ and $L(U_1)$ in $J$. A Spin$^c$ structure $s$ on $Y$ gives rise to a pair of $g$-dimensional tori $L_0(s)$ and $L_1(s)$ in $\tilde{J}$, up to simultaneous translation by $H^2(Y; \mathbb{Z})$, as follows. Let $s_0$ be a spin structure on $Y$ and let $p$, $L_0$ and $L_1$ be as above. Any Spin$^c$ structure $s$ on $Y$ can be written as $s_0 + \ell$, where $\ell \in H^2(Y; \mathbb{Z})$. Let $\tilde{p}$ be any lift of $p$ to $\tilde{J}$. Then, $L_0(s)$ is the lift of $L_0$ to $\tilde{J}$ which passes through $\tilde{p}$, and $L_1(s)$ is the lift of $L_1$ which passes through $\tilde{p} + \ell$ (the translate of $\tilde{p}$ by the natural action of $H^2(Y; \mathbb{Z})$ on $\tilde{J}$).

Once again, it is easy to see that the subspaces are independent of the spin structure $s_0$. Since the intersection in $H^1(\Sigma; \mathbb{R})$ of the image of $H^1(U_0; \mathbb{R})$ with $\Gamma$ is $H^1(U_0; \mathbb{Z})$, it follows that $L_0(s)$, and similarly $L_1(s)$, are both $g$-dimensional tori embedded in $\tilde{J}$. A torsion Spin$^c$ structure is a Spin$^c$ structure whose associated real cohomology class $\bar{s} = 0$ (this is equivalent to the condition that its first Chern class is torsion).

Note that $s$ is torsion if and only if $L_0(s)$ and $L_1(s)$ intersect. In fact, if $s$ is torsion, then $L_0(s) \cap L_1(s)$ is identified with $H^1(Y; \mathbb{R})/H^1(Y; \mathbb{Z})$.

Having introduced the basic topological objects associated to a Heegaard decomposition, we must flesh out the notion of allowable metrics used in the definition of the invariant $\theta$. The definition corresponds to degenerations of the metric on $\Sigma$. We describe these presently.

Let $\{\gamma_1, \ldots, \gamma_n\}$ be a collection of disjoint simple, closed curves in $\Sigma$. Choose a tubular neighborhood $\nu$ of $\prod_{i=1}^g \gamma_i$, and let $h$ be a metric which extends the product metric over $\nu$ arising naturally from an identification

$$\nu \cong \prod_{i=1}^g [-1, 1] \times S^1.$$ 

Such a metric will be called product-like near the $\gamma_i$. Given such a metric, let $h(T_1, ..., T_n)$ denote the metric obtained by inserting a tube of length $2T_i$ around the curve $\gamma_i$, i.e. $h(T_1, ..., T_n)$ is obtained by attaching

$$\prod_{i=1}^g [-T_i, T_i] \times S^1$$

to $(\Sigma - \nu, h)$ in the obvious manner. The following lemma, whose proof is given in Section 3, describes what happens to the theta divisor as the metric is stretched normal to curves in this way.
Lemma 2.1. Let $U$ be a handlebody with boundary $\Sigma$, and let $\{\gamma_1, ..., \gamma_g\}$ be a complete set of attaching circles for $U$. Then, for any compact set of metrics $\mathcal{H}$ on $\Sigma$ which are product-like near the $\gamma_i$, as the metrics are stretched out normal to the $\gamma_i$, the theta divisor converges as a point set, into

$$\text{Hol}_{\gamma_1}^{-1} \left( \frac{1}{2} \right) \cup ... \cup \text{Hol}_{\gamma_g}^{-1} \left( \frac{1}{2} \right);$$

i.e. given any $\epsilon$, there is a $T_0$ so that for all metrics $h \in \mathcal{H}$, and for all $g$-tuples $(T_1, ..., T_g)$ for which each $T_i > T_0$, we have that

$$\Theta_h(T_1, ..., T_g)(\text{Sym}^{g-1}(\Sigma)) \subset \bigcup_{i=1}^{g} \text{Hol}_{\gamma_i}^{-1} \left( \frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon \right).$$

Note that $L(U)$ could be described as the set of points $B \in J(\Sigma)$ with $\text{Hol}_{\gamma_i}B = 0$ for all $i$. Thus, the above lemma says that for all metrics which are sufficiently stretched out normal to all the $\gamma_i$, the theta divisor misses the torus $L(U)$. Indeed, it allows us to identify a special (path-connected) class of metrics.

Definition 2.2. Let $U$ be a handlebody, and let $\{\gamma_1, ..., \gamma_g\}$ be a complete set of attaching circles for $U$. Fix a metric $k_0$ on $\Sigma$ which is sufficiently stretched out normal to the $\{\gamma_i\}$ according to Lemma 2.1. Another metric $k_1$ on $\Sigma$ is called allowable for $\{\gamma_1, ..., \gamma_g\}$, or simply $\{\gamma_1, ..., \gamma_g\}$-allowable if there is a path $k_t$ connecting $k_0$ to $k_1$, so that

$$\Theta_{k_t}(\text{Sym}^{g-1}(\Sigma)) \cap L(U) = \emptyset.$$ 

In fact, Lemma 2.1 shows that the notion of allowable is independent of the fixed metric $k_0$. It appears, however, to depend on the choice of the $\{\gamma_1, ..., \gamma_g\}$. The following proposition shows that this is not the case: the notion of allowable depends only on the handlebody $U$:

Proposition 2.3. The class of allowable metrics depends only on the handlebody $U$; i.e. if $\{\gamma_1, ..., \gamma_g\}$ and $\{\gamma'_1, ..., \gamma'_g\}$ are any two complete sets of attaching circles for $U$, then a metric is $\{\gamma_1, ..., \gamma_g\}$-allowable if and only if it is $\{\gamma'_1, ..., \gamma'_g\}$-allowable.

The proof relies on the following lemma, whose proof is given in Section 3.

Lemma 2.4. Let $U$ be a handlebody with boundary $\Sigma$, and let $\{\gamma_1, ..., \gamma_g\}$ be a complete set of attaching circles for $U$. Then, for any compact set of metrics $\mathcal{H}$ on $\Sigma$ which are product-like near the $\gamma_i$ for $i = 1, ..., g - 1$, given any $\epsilon > 0$, there is a $T_0$ so that for all $g - 1$ tuples $(T_1, ..., T_{g-1})$ with $T_i \geq T_0$ for all $i$, and all metrics $h \in \mathcal{H}$, we have that

$$\Theta_h^{-1}(\text{Hol}_{\gamma_1}^{-1}(0) \cap ... \cap \text{Hol}_{\gamma_{g-1}}^{-1}(0)) \subset \text{Hol}_{\gamma_g}^{-1} \left( \frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon \right).$$

The above lemma implies the following corollary:
Corollary 2.5. Let $\Sigma$, $\{\gamma_1, \ldots, \gamma_g\}$, and $U$ be as in Lemma 2.1. For any metric $h$ which is product-like around $\{\gamma_1, \ldots, \gamma_{g-1}\}$, there is a constant $T_0$ so that for all collections $T_i \geq T_0$, the metric $h(T_1, \ldots, T_{g-1})$ is $\{\gamma_1, \ldots, \gamma_g\}$-allowable.

Proof. Fix an initial metric $k$ which is product-like along all $\{\gamma_1, \ldots, \gamma_g\}$, and which agrees with $h$ away from a tubular neighborhood of $\gamma_g$. Lemma 2.1 gives us a constant $C_0$ so that for all $g$-tuples $(T_1, \ldots, T_g)$ with $T_i \geq C_0$, $k(T_1, \ldots, T_g)$ is allowable. We can view the metric

$$h_t(T_1, \ldots, T_{g-1}) = th(T_1, \ldots, T_{g-1}) + (1 - t)k(T_1, \ldots, T_{g-1}, C_0)$$

as the result of inserting tubes with parameters $T_1, \ldots, T_{g-1}$ into a one-parameter (compact) family of metrics away from the $\{\gamma_1, \ldots, \gamma_{g-1}\}$. Thus, Lemma 2.4 gives us a number $C_1$ with the property that if all $T_i \geq C_1$, then for all metrics in the path $h_t(T_1, \ldots, T_{g-1})$, the theta divisor misses $L(U)$. Hence, if $T_1, \ldots, T_{g-1} \geq \max(C_0, C_1)$, then $h(T_1, \ldots, T_{g-1})$ is a $\{\gamma_1, \ldots, \gamma_g\}$-allowable metric. □

Proposition 2.3, then, follows easily:

Proof of Proposition 2.3. Fix $U$, and let $\{\gamma_1, \ldots, \gamma_g\}, \{\gamma'_1, \ldots, \gamma'_g\}$ be two complete sets of attaching circles. By standard Kirby calculus [12], we see that it is always possible to move between any two collections $\{\gamma_1, \ldots, \gamma_g\}$ and $\{\gamma'_1, \ldots, \gamma'_g\}$, through a sequence of handle-slides. Since a handle-slide fixes $g - 1$ of the curves, Corollary 2.5 shows that the notion of allowable remains unchanged. □

Proposition 2.3 allows us to refine the earlier definition of allowable metrics:

Definition 2.6. Let $U$ be a handlebody. A metric $k$ on $U$ is called $U$-allowable provided that there is a complete set of attaching circles $\{\gamma_1, \ldots, \gamma_g\}$ for which the metric is allowable.

With this background in place, we now give a definition of $\theta(s)$, where $s \in \text{Spin}^c(Y)$. Fix a smooth path of metrics $\{h_t\}_{t \in [0,1]}$ for which $h_0$ is $U_0$-allowable, and $h_1$ is $U_1$-allowable. Consider the smooth map

$$\Psi: \text{Sym}^{g-1}(\Sigma) \times \{(s, t) \in [0,1] \times [0,1] | s \leq t\} \to \frac{H^1(\Sigma; S^1)}{H^1(U_0; S^1)} \times \frac{H^1(\Sigma; S^1)}{H^1(U_1; S^1)} = \mathbb{T}(Y),$$

defined by

$$\Psi(D, s, t) = Q_0(\Theta_{h_s}(D)) \times Q_1(\Theta_{h_t}(D)),$$

where $Q_i$ denotes the quotient map $Q_{U_i}$, for $i = 0, 1$, and let $M_{\eta_0 \times \eta_1}$ denote the pre-image under $\Psi$ of the point $\eta_0 \times \eta_1 \in \mathbb{T}(Y)$. By Sard’s theorem, for generic $\eta_0 \times \eta_1$, this fiber is a compact, oriented zero-dimensional manifold which misses the locus of points $(D, s, t)$ where $s = t$. Moreover, the points in $M_{\eta_0 \times \eta_1}$ can naturally be partitioned into subsets indexed by the various Spin$^c$ structures on $Y$. Specifically,
for a choice of Spin\(^c\) structure on \(Y\) and corresponding lifts \(L_i(s)\) of \(L_i\) (for \(i = 0, 1\)), there is a subset of \(M_{\eta_0 \times \eta_1}\), denoted \(M_{\eta_0 \times \eta_1}(s)\), corresponding to points
\[
\{(D, s, t) \in \text{Sym}^{g-1}(\Sigma) \times [0, 1] \times [0, 1] | \tilde{\Theta}_{h_s}(D) \in L_0(s) + \eta_0, \tilde{\Theta}_{h_t}(D) \in L_1(s) + \eta_1, s \leq t\}
\]
(where, in the above expression, \(L_i(s) + \eta_i\) denotes the translate of the subset \(L_i(s)\) by \(\eta_i\)). Then, \(\theta_{\eta_0 \times \eta_1}(s)\) is defined to be the signed number of points in this subset. In the next two propositions, we shall see that (given \(h_t\)) there is an open neighborhood \(G\) of zero in \(T(Y)\) with the property that \(\theta_{\eta_0 \times \eta_1}\) is independent of the particular (generic) choice of \(\eta_0 \times \eta_1 \in G\). This is technically somewhat easier when \(b_1(Y) > 1\), so we consider that case first. But before we do that, we pause for a moment to discuss signs.

Since \(\text{Sym}^{g-1}(\Sigma) \times [0, 1] \times [0, 1]\) is naturally oriented, the sign of \(\theta\) is determined by an orientation for the torus
\[
\frac{H^1(\Sigma; S^1)}{H^1(U_0; S^1)} \times \frac{H^1(\Sigma; S^1)}{H^1(U_1; S^1)},
\]
which in turn is determined by an ordering of the attaching circles \(\{\alpha_1, ..., \alpha_g\}\) and \(\{\beta_1, ..., \beta_g\}\). We use an ordering for these attaching circles which is consistent with the orientation of \(H_*(Y)\), arising from Poincaré duality, in the following sense. We explain what this consistence means. The Heegaard decomposition gives a chain complex for \(Y\) with one zero-cell, \(g\) one-cells in one-to-one correspondence with the circles \(\{\alpha_i\}\), \(g\) two-cells which correspond to the \(\{\beta_i\}\), and one three-cell. In general, an orientation for a (finite dimensional) chain complex is canonically equivalent to an orientation for its real homology, since there is a splitting:
\[
\bigoplus_i C_i = \bigoplus_i \left( (\partial B_{i+1}) \oplus H_i \oplus B_i \right),
\]
where \(B_i \subset C_i\) is a vector space which is mapped isomorphically under the boundary homomorphism to the group of boundaries in \(C_{i-1}\), which gives a natural identification of
\[
\bigoplus_i C_i = \left( \bigoplus_i H_i \right) \oplus \left( \bigoplus_i B_i \oplus (\partial B_i) \right)
\]
as oriented vector spaces; and the vector space \(\bigoplus_i (B_i \oplus (\partial B_i))\) is canonically oriented as follows: if \(\{b_j\}\) is a basis for \(\bigoplus_i B_i\), we declare that
\[
\bigoplus_j (b_j \oplus \partial b_j)
\]
is a positive oriented basis for \(\bigoplus_i (B_i \oplus (\partial B_i))\). The orientation on \(Y\) gives a canonical orientation of \(H_0 = C_0, H_3 = C_3\). Thus, in light of the above remarks, the orientation of \(H_*(Y)\) induces an orientation of \(C_1 \oplus C_2\), and hence an ordering of the attaching circles.
Having nailed down the sign, we state the result we have been aiming for, first in the case where \( b_1(Y) > 1 \).

**Proposition 2.7.** When \( b_1(Y) > 1 \), the invariant \( \theta(s) \) depends only on the Heegaard decomposition of \( Y \); i.e. it is independent of the metrics and perturbations used. More precisely, given any one-parameter family \( h_t \) of metrics which connect a \( U_0 \)-allowable metric to a \( U_1 \)-allowable metric, there is an open neighborhood \( G \) of \( 0 \in \mathbb{T}(Y) \) and an integer \( \theta_{h_t}(s) \) with the property that for all generic \( \eta_0 \times \eta_1 \in G \), we have that

\[
\theta_{h_t,\eta_0 \times \eta_1}(s) = \theta_{h_t}(s).
\]

Moreover, if \( h'_t \) is another path of metrics connecting a \( U_0 \)-allowable metric with a \( U_1 \)-allowable metric, then

\[
\theta_{h_t}(s) = \theta_{h'_t}(s).
\]

**Proof.** According to Sard’s theorem, for any generic \( \eta_0 \times \eta_1 \in \mathbb{T}(Y) \), the fiber

\[
M_{h_t,\eta_0 \times \eta_1} = \{(s, t, D) \mid \Theta_{h_s}(D) \in L_0 + \eta_0, \Theta_{h_t}(D) \in L_1 + \eta_1, s \leq t\}
\]

is a compact, canonically oriented, zero-dimensional manifold. We investigate the conditions necessary to show that the fiber misses the boundary of the domain of \( \Psi \).

The set of points \( \eta_0 \times \eta_1 \in \mathbb{T}(Y) \) for which the fiber \( M_{h_t,\eta_0 \times \eta_1} \) does not contain boundary points of the form \((s, 1, D) \) or \((0, t, D) \) is an open set which contains \( 0 \) (since \( h_t \) is \( U_i \)-allowable for \( i = 0, 1 \)). Let \( G \) be a connected neighborhood of \( 0 \) in this set. Moreover, the fiber \( M_{h_t,\eta_0 \times \eta_1} \) cannot contain points of the form \((t, t, D) \) if the spaces \( L_0 + \eta_0 \) and \( L_1 + \eta_1 \) are disjoint; but \((L_0 + \eta_0) \cap (L_1 + \eta_1) \neq \emptyset \) is equivalent to the condition that the image of \( \delta(\eta_0 - \eta_1) = 0 \) in \( H^2(Y; S^1) \). This is a codimension \( b_1(Y) \) sub-torus \( \mathcal{W} \) of \( \mathbb{T}(Y) \), so its complement is dense.

Thus, for a dense set of perturbations \( \eta_0 \times \eta_1 \in G \), the fiber \( M_{\eta_0 \times \eta_1} \) is a smooth submanifold which misses the boundary of the domain of \( \Psi \). Moreover, given two generic perturbations \( \eta_0 \times \eta_1, \eta'_0 \times \eta'_1 \in G \), a generic path in \( G \) misses the locus \( \mathcal{W} \) as well, since it has codimension \( b_1(Y) > 1 \) (this is what distinguishes the case where \( b_1(Y) > 1 \) from the case \( b_1(Y) = 1 \)). By Sard’s theorem, then, a generic such path induces a compact cobordism between \( M_{\eta_0 \times \eta_1} \) and \( M_{\eta'_0 \times \eta'_1} \). By lifting to the covering space \( \widetilde{\Sym}^{g-1}(\Sigma) \), one can easily see that the cobordism respects the partitioning into \( \text{Spin}^c \) structures. Thus, \( G \) has the required property.

Fix \( h_t \) and \( h'_t \). Since the space of \( U \)-allowable metrics is path connected and the space of metrics over \( \Sigma \) is simply-connected, we can connect \( h_t \) and \( h'_t \) by a two-parameter family of metrics with \( k_{t,0} = h_t, k_{t,1} = h'_t, k_{0,t} \) is \( U_0 \)-allowable and \( k_{1,t} \) is \( U_1 \)-allowable. This, together with a small generic perturbation, gives rise to a cobordism between \( M_{\eta_0 \times \eta_1, h_t} \) and \( M_{\eta'_0 \times \eta'_1, h'_t} \) (which once again respects the partitioning into \( \text{Spin}^c \) structures). This completes the proof of the proposition. \( \square \)
The invariance statement in Proposition 2.7 holds when \( b_1(Y) = 1 \) as well, but the argument is more involved. The point is that \( \mathcal{W} \) now separates \( G \) into two components. The above proof shows that if we pick two paths of metrics and two perturbations in the same component, then the invariant remains unchanged. So, from now on, we can drop the path \( h_t \) from the notation for \( \theta \). We must show, then, that the invariant is actually independent of the component. It will be convenient to make use of the involution on the set of Spin\(^c\) structures introduced in Section 1, \( s \mapsto \overline{s} \). This is the map which sends the complex spinor bundle \( W \) of \( s \) to the same underlying real bundle, given its conjugate complex structure (and naturally induced Clifford action). Note that this action fixes those Spin\(^c\) structures which arise from Spin structures, and moreover \( s + \ell = \overline{s} - \ell \), for any \( s \in \text{Spin}^c(Y) \) and \( \ell \in H^2(Y; \mathbb{Z}) \). The invariant is preserved by this involution in the following sense:

**Lemma 2.8.** Let \( Y \) be an oriented three-manifold with \( b_1(Y) > 0 \). Then,

\[
\theta_{\eta_0 \times \eta_1}(s) = \theta_{-\eta_0 \times -\eta_1}(\overline{s})
\]

**Proof.** To see this, we make use of the Serre duality map. Serre duality gives rise to a natural involution on \( J \), with the property that \( q + x = \overline{q} - x \) for any \( q \in J \) and \( x \in H^1(\Sigma; \mathbb{R}) \). For any given metric \( h_t \), this involution preserves the theta divisor (by Serre duality), and fixes the points associated to spin structures. The involution on \( J \) can be lifted to an involution of \( \overline{J} \) which we can assume fixes \( L_0(s) \). The involution then carries \( L_1(s) \) to \( L_1(\overline{s}) \). Unfortunately, this involution is not defined on the symmetric product. Instead, it gives an involution on the set of points in the symmetric product which map injectively to the theta divisor, and more generally, it gives a relation. We write \( D_1 \sim_h D_2 \) if \( \Theta_h(D_1) = \Theta_h(D_2) \).

For any \( y \in [0, 1] \), let \( M_y \) denote the moduli space

\[
M_y = \left\{ (D, s, t) \, | \, \begin{array}{l}
s \leq t, \\
\exists D', D'' \in \text{Sym}^{g-1}(\Sigma), D' \sim_{h_s} D, D'' \sim_{h_s} D, \\
\Theta_{h_s}(D') \in L_0 + \eta_0, \\
\Theta_{h_t}(D'') \in L_1 + \eta_1,
\end{array} \right\},
\]

where \( x = s + y(t - s) \). For generic \( \eta_0, \eta_1, h_t, M_y \) is a smooth cobordism from \( M_0 \) to \( M_1 \). This follows from the fact that the set of points in \( \text{Sym}^{g-1}(\Sigma) \) which do not map injectively into the theta divisor is empty if \( g = 2 \) and has complex codimension 2 for generic \( h \) if \( g > 2 \) (see p. 250 of [1]). As before, \( M_y \) can be partitioned according to Spin\(^c\) structures; let \( M_y(s) \) denote the corresponding set. The same dimension counting also shows that for generic \( h_t, \eta_0, \) and \( \eta_1 \),

\[
M_0(s) = M_{h_t, \eta_0 \times \eta_1}(s) \quad \text{and} \quad M_1(s) = M_{h_t, -\eta_0 \times -\eta_1}(\overline{s}).
\]

This proves the lemma. \( \square \)
If $Y$ is an integral homology $S^1 \times S^2$, the invariance of $\theta$ is an easy consequence of this lemma:

**Proposition 2.9.** If $Y$ is an integral homology $S^1 \times S^2$, then the invariant $\theta(s)$ is independent of metrics and perturbations as well.

**Proof.** If $s$ is not a torsion class, then $L_0(s)$ and $L_1(s)$ are disjoint, so the proof of Proposition 2.7 applies.

For the torsion Spin$^c$ structure $s$ (i.e. the one corresponding to the spin structure), there are a priori two invariants, depending on the sign $\delta(\eta_0 - \eta_1)$ for the perturbation $\eta_0 \times \eta_1$. But Lemma 2.8 guarantees that

$$\theta_{\eta_0 \times \eta_1}(s) = \theta_{-\eta_0 \times -\eta_1}(\tilde{s});$$

and since $s$ comes from the spin structure, $\tilde{s} = s$, while

$$\delta(\eta_0 - \eta_1) = -\delta(-\eta_0 - (-\eta_1)),
$$

so the invariants for both perturbations are equal. \hfill \Box

More generally, we have:

**Proposition 2.10.** If $Y$ is a rational homology $S^1 \times S^2$, then the invariant $\theta(s)$ is independent of metrics and perturbations.

This general case involves a study of spectral flow, and takes us slightly out of the general framework of the preceding discussion (and it is surprisingly more involved than the special case), so we relegate that case to Section 8 (where it is restated as Proposition 8.1).

Our final goal in this section is to prove Proposition 1.2.

**Proof of Proposition 1.2.** The finiteness statement is clear from the fact that the fibers of $\Psi$ are compact.

The statement about involution invariance follows from Lemma 2.8. When we reverse the orientation of $Y$, then the corresponding orientation of $H^*(Y)$ changes by a factor of $(-1)^{b_1+1}$, so the last statement follows from the discussion preceding Proposition 2.7. \hfill \Box

In Section 4, we establish the final step towards proving topological invariance, showing that $\theta(s)$ remains invariant under stabilization. It must be shown that if one begins with $\Sigma$, then the associated invariant agrees with the invariant calculated from the Riemann surface obtained by a connected sum $\Sigma \# (S^1 \times S^1)$, using the handlebodies $U_0 \# (D^2 \times S^1)$ and $U_1 \# (S^1 \times D^2)$. This result fits naturally into the context of a splicing construction, which also proves a technical device which underpins the calculations of the invariants. We defer these results to Section 4, addressing first the lemmas which we have stated thus far without proof.
3. Degenerating metrics

In the definition of $\theta(s)$, we used several facts (Lemmas 2.1 and 2.4) about the behaviour of the theta divisor under certain degenerations of the metric $h$ over $\Sigma$. The aim of this section is to provide proofs of these results. We opt to give analytical proofs, which rely only on elementary properties of the $\overline{\partial}$ operator on cylindrical manifolds. The statements given here, though, can be interpreted in terms of algebraic geometry, where they address degenerations of the theta divisor in a family of curves acquiring nodal singularities. For a related discussion from this point view, see [7]. (This remark pertains also to the discussion in Section 4.)

The proof of both Lemmas 2.1 and 2.4 involve local compactness and then passing to cylindrical end models, as is familiar in gauge theory (though the results here are considerably more elementary than is typical in gauge theory). More precisely, thanks to local compactness, points in the theta divisors for Riemann surfaces undergoing suitable degenerations give rise to points in the $L^2$ theta divisor for the cylindrical end model of $\Sigma - (\gamma_1 \cup \ldots \cup \gamma_n)$, where the $\gamma_i$ are embedded, disjoint, closed curves in $\Sigma$. We then appeal to basic results about the $L^2$ kernel of the Dirac operator. To state these results, we set up some notation.

Let $F^c$ be a compact, oriented two-manifold with $n$ boundary circles, given a product metric in a neighborhood of its boundary. Let $F^+ = F^c \cup_{\partial F^c} \prod_{j=1}^{n} (S^1_{(j)} \times [0, \infty))$ be the associated complete manifold with cylindrical ends, and

$$F = F^c \cup_{\partial F^c} \prod_{j=1}^{n} D_{(j)}$$

be the associated compact Riemann surface; here, $D_{(j)}$ is a copy of the two-dimensional disk with a product metric near its boundary. Conformally, $F$ is obtained from $F^+$ by adding $n$ points “at infinity” $\{p_1, \ldots, p_n\}$, which correspond to the centers of the attached disks in the description of $F$. Fix a spin structure $s_0$ on $F$ with associated spinor bundle $E$ throughout the following discussion (for instance, when $F^c$ has genus zero, this spin structure is uniquely determined). This canonically induces a spin structure on $F^+$, which gives rise to a canonical connection $B_0$ on the spinor bundle $E^+$ over $F^+$ with normalized curvature form. Up to gauge transformations, any other connection with normalized curvature form differs from $B_0$ by a cohomology class in $H^1(F^+; \mathbb{R})$.

Now, let $B$ be any connection on $E^+$ over $F^+$ with normalized curvature form. We can relate the $L^2$-extended $B$-harmonic spinors on $F^+$ with holomorphic data on $E$ over $F$. Following [2], a $B$-harmonic spinor $\Psi$ is said to be an $L^2$-extended section if it is in $L^2_{\text{loc}}$ and over each cylinder $S^1_{(j)} \times [0, \infty)$, we can write the restriction of $\Psi$ as
a sum of an $L^2$ $B$-harmonic section plus a constant (hence $B$-harmonic) section. The holomorphic data over $F$ is obtained by viewing $E$ as a holomorphic vector bundle over $F$ (by using the $\overline{\partial}$-operator associated to the spin-connection coming from $s_0$). Given $p \in F$, let $\mathcal{I}_p$ denote the ideal sheaf at $p$. We have the following result (a related discussion can be found in [2]):

**Proposition 3.1.** Let $B$ be a connection on $E^+$ with normalized curvature form, and write $B = B_0 + i\xi$ for $\xi \in H^1(E^+; \mathbb{R})$. Let

$$\xi_j = \langle \xi, [S^1_{(j)}] \rangle.$$ 

Then, the space of $L^2$-extended holomorphic sections of $E^+$ over $F^+$ is canonically identified with

$$H^0(F, E \otimes \mathcal{I}_{p_1}^{[\xi_1 - \frac{1}{2}]} \otimes \ldots \otimes \mathcal{I}_{p_n}^{[\xi_n - \frac{1}{2}]}).$$

Here, $[x]$ denotes the smallest integer greater than $x$.

**Proof.** Note that the cylindrical-end metric $g_{cyl}$ on $F^+$ is conformal equivalent to the metric $g$ on $F - \{p_1, \ldots, p_n\}$ inherited from $F$. Indeed, we can write

$$g_{cyl} = e^{2\tau} g,$$

where

$$\tau : F^+ \longrightarrow [0, \infty)$$

is a real-valued function, which agrees with the real coordinate projection on the cylindrical ends. The spinor bundles of the two manifolds can be (metrically) identified accordingly, with a change in the Clifford action to reflect the conformal change. With respect to this change, (see [10], bearing in mind that we are in two dimensions), the Dirac operator over $F^+$ can be written:

$$\mathcal{D}_{cyl} = e^{-\frac{1}{2}\tau} \mathcal{D} e^{\frac{1}{2}\tau};$$

i.e., multiplication by $e^{-\frac{1}{2}\tau}$ induces a vector space isomorphism from the $\mathcal{D}$-harmonic spinors to the $\mathcal{D}_{cyl}$-harmonic ones. Moreover, a section of a bundle is in $L^2$ for the cylinder iff $e^\tau \phi$ is in $L^2$ for $F - \{p_1, \ldots, p_n\}$.

From this discussion, it follows that the $L^2$-harmonic spinors on $F^+$ are identified with the space of harmonic spinors over $F - \{p_1, \ldots, p_n\}$ for which $e^\tau \phi$ lies in $L^2$ (for $F - \{p_1, \ldots, p_n\}$). The proposition follows from this, along with some considerations in the neighborhoods of the punctures.

Consider $D - \{0\}$, with the trivial line bundle endowed with a connection $B = d + i\xi d\theta$ – this is the model of the punctured neighborhood of the $p_j \in F$. Under the standard identification $S^1 \times [0, \infty) \cong D - \{0\}$, the function $e^{-t}$ (which is $e^{-\tau}$ over $F^+$) corresponds to the radial coordinate $r$ on the disk. Moreover, multiplication by $e^{\xi t} = r^{-\xi}$ induces an isomorphism from the space of (ordinary) holomorphic functions on $D - \{0\}$ to the space of $\partial_B$-holomorphic functions. Under these correspondences,
a holomorphic function $\phi$ which vanishes to order $k$ corresponds to a $L^2$-harmonic spinor on $F^+$ iff

$$\int |\phi r^{-\xi-\frac{1}{2}}|^2 r dr d\theta \leq C \int r^{2k-2\xi} dr d\theta$$

is bounded, i.e. iff $k > \xi - \frac{1}{2}$. In the borderline case where $\xi \equiv \frac{1}{2}$, the holomorphic functions on $D - \{0\}$ which vanish to order $\xi - \frac{1}{2}$ correspond to harmonic sections over the cylinder whose pointwise norm is bounded, and hence they lie in the space $L^2$ (of the cylinder) extended by constants.

The proposition follows.

Given this proposition, then, we can give a proof of Lemma 2.1.

**Proof of Lemma 2.1.** Suppose that $\{h_i\}_{i=1}^{\infty}$ is a sequence of metrics whose neck-lengths along the $\gamma_i$ all go to infinity, and whose restrictions away from the necks lie in a compact family of metrics on the genus zero surface $F^c = \Sigma - \nu(\gamma_1 \cup \ldots \cup \gamma_g)$. Let $B_i$ be a sequence of connections which lie in the theta divisor of $h_i$. This means that we can find a sequence of non-zero sections $\phi_i$ of $E$ over $\Sigma$ (with metric $h_i$), so that $\nabla_{B_i} \phi_i = 0$. By renormalizing, we can assume without loss of generality that the supremum of $|\phi_i|$ is 1. Since the metric in a neighborhood of the tubes is flat, the supremum is always achieved in the compact piece $F^c \subset \Sigma$. After passing to a subsequence, the $B_i$ converge (locally in $C^\infty$) to a connection $B_\infty$ on $F^+$ with normalized curvature form. In fact, local compactness of holomorphic functions ensures that (after passing to a subsequence) the $\phi_i$ converge locally (in $C^\infty$) to a $B_\infty$-holomorphic section $\phi_\infty$. Once again, the supremum of $|\phi_\infty|$ must be 1, so in particular, $\phi_\infty$ is a non-vanishing, $L^2$-extended, $B_\infty$-holomorphic section. From Proposition 3.1, it then follows that the holonomy of $B_\infty$ around at least one of the boundary circles must be congruent to $\frac{1}{2}$ (modulo $\mathbb{Z}$). Specifically, the spin structure $E_0$ on $F = S^2$ has degree $-1$, so the dimension of the space of $L^2$-bounded, holomorphic sections is calculated by the formula

$$\dim C H^0(F, E_0 \otimes I^{[\xi_1 - \frac{1}{2}]_1} \otimes \ldots \otimes I^{[\xi_g - \frac{1}{2}]_g}) = \min \left(0, \sum_{i=1}^{2g} \left[\xi_i - \frac{1}{2}\right]\right).$$

Note that the holonomies $\xi_i$ all add up to zero, and, since the ends of $F^c$ are naturally come in pairs $S^1_{(i)}$, $S^1_{(i+g)}$ with $\xi_i \equiv -\xi_{i+g} (\mod \mathbb{Z})$, it follows that the dimension is non-zero only if at least one of the holonomies is $\frac{1}{2}$ modulo $\mathbb{Z}$.

Strictly speaking, to apply Proposition 3.1, we note that natural compactification of $F^c$ is a sphere, and the two methods for measuring holonomy – comparing holonomies against any spin structure which extends over $U$ (which is used in the statement of Lemma 2.1) and comparing against the spin structure which extends over the sphere
(which we use in the statement of Proposition 3.1) – coincide. This is obvious from the Kirby calculus description of $U$. 

The proof of Lemma 2.4 is analogous.

**Proof of Lemma 2.4.** As the metric is stretched in a sequence $h_i$, any sequence of points $B_i \in \Theta_h(\text{Sym}^{g-1}(\Sigma)) \cap \text{Hol}_{\gamma_1}^{-1}(0) \cap \ldots \cap \text{Hol}_{\gamma_g}^{-1}(0)$ has a subsequence which converges to a connection $B_\infty$, which now can be viewed as a connection on a torus with cylindrical ends $F^+$. Kernel elements then converge to a section which, according to Proposition 3.1 corresponds naturally to a holomorphic section of a line bundle $E$ over a compact torus $F$. But there is only one spin structure on $F$ which admits harmonic spinors (the trivial bundle), and it corresponds to the spin structure on $F$ which does not bound. Thus, around any curve which bounds in $U$, the difference in the holonomy between this spin structure and any spin structure which bounds in $U$ is $1/2$. 

4. Splicing

The aim of this section is to give a more detailed analysis of the theta divisor, using a splicing construction whose consequences include the stabilization invariance of $\theta(s)$, and a technical result which will be of importance in subsequent sections.

We introduce notation. Fix a pair of Riemann surfaces $\Sigma_i$ for $i = 1, 2$, and let $\Sigma_i^c$ be the complement in $\Sigma_i$ of an open disk centered at $p_i \in \Sigma_i$, endowed with a product-end metric (which we can extend over the disks to obtain the metrics over $\Sigma_i$). Let $\Sigma_i^+$ (for $i = 1, 2$) denote cylindrical-end models of the surfaces,

$$\Sigma_i^+ = \Sigma_i^c \cup (S^1 \times [0, \infty]).$$

Let $\Sigma_1 \# T \Sigma_2$ denote the model for the connected sum of $\Sigma_1$ and $\Sigma_2$, a surface of genus $g = g_1 + g_2$ with a neck length of $2T$; i.e.

$$\Sigma_1 \# T \Sigma_2 = \Sigma_1^c \cup ([T, T] \times S^1) \cup \Sigma_2^c.$$

Fix non-negative integers $k_1$, $k_2$ so that $k_1 + k_2 = g_1 + g_2 - 1$. For all $T$, there is an obvious natural map

$$\gamma_T : \text{Sym}^{k_1}(\Sigma_1^c) \times \text{Sym}^{k_2}(\Sigma_2^c) \to \text{Sym}^{g-1}(\Sigma_1 \# T \Sigma_2).$$

Fix a spin structure $s_0$ over $\Sigma_1 \# \Sigma_2$. This allows us to compare the various Jacobians as $T$ varies; i.e. it gives us identifications:

$$J(\Sigma_1 \# T \Sigma_2) \cong H^1(\Sigma_1 \# T \Sigma_2; S^1).$$

Similarly, we can use the natural extension of this structure over $\Sigma_1$, $\Sigma_2$ to fix identifications

$$J(\Sigma_1) \cong H^1(\Sigma_1; S^1) \quad \text{and} \quad J(\Sigma_2) \cong H^1(\Sigma_2; S^1).$$

To state the splicing result, we use Abel-Jacobi map, thought of as follows. Fix a Riemann surface $\Sigma$ with a basepoint $p$, then

$$\mu^{(k)} : \text{Sym}^{k}(\Sigma) \to J(\Sigma)$$

is the map which takes an effective divisor $D \in \text{Sym}^{k}(\Sigma)$ to the unique connection $A$ with normalized curvature form, which admits a $\partial_A$-meromorphic section $\phi$ whose associated divisor is $D + (g - 1 - k)p$. When $k = g - 1$, then we do not need a base point, and $\mu^{(g-1)}$ agrees with the map $\Theta$ from Section 1.

**Theorem 4.1.** In regions of the symmetric products supported away from the points $p_i$, the composite of $\gamma_T$ with $\Theta_{\Sigma_1 \# \Sigma_2}$ is homotopic to the product of Abel-Jacobi maps. Indeed, for any non-negative $k_1, k_2$ with $k_1 + k_2 = g - 1$, we have that the composite

$$\text{Sym}^{k_1}(\Sigma_1^c) \times \text{Sym}^{k_2}(\Sigma_2^c) \xrightarrow{\gamma_T} \text{Sym}^{g-1}(\Sigma_1 \# T \Sigma_2) \xrightarrow{\Theta_{\Sigma_1 \# \Sigma_2}} H^1(\Sigma_1 \# T \Sigma_2; S^1)$$

converges in the $C^1$ topology, as $T \to \infty$, to the map

$$\text{Sym}^{k_1}(\Sigma_1^c) \times \text{Sym}^{k_2}(\Sigma_2^c) \xrightarrow{\mu_1^{(k_1)} \times \mu_2^{(k_2)}} H^1(\Sigma_1; S^1) \times H^1(\Sigma_2; S^1) \cong H^1(\Sigma_1 \# \Sigma_2; S^1),$$
where $\mu_i$ denotes the Abel-Jacobi map with basepoint $p_i$

$$\mu_i = \mu_i^{(k_i)} : \text{Sym}^{k_i}(\Sigma_i) \longrightarrow H^1(\Sigma_i; S^1).$$

**Remark 4.2.** The seasoned gauge theorist will identify the last vestiges of a “gluing theorem” here. However, the present result is significantly easier than the usual gluing results.

Before giving the proof of Theorem 4.1, we recall the construction of the map

$$\Theta_h : \text{Sym}^{g-1}(\Sigma) \rightarrow H^1(\Sigma; S^1).$$

For a given divisor $D \in \text{Sym}^{g-1}(\Sigma)$, $\Theta_h(D)$ is the unique connection $B$ in the spinor bundle $E$ with normalized curvature form which admits a $\overline{\partial}_B$-holomorphic section whose associated divisor is $D$. To find it, first fix a section $\phi$ of $E$ whose vanishing set is $D$; then find any connection $A$ on $E$ for which $\phi$ is $\overline{\partial}_A$-holomorphic. Now, let $f$ be a function which solves

$$\text{id} \ast df = F_{B_0} - F_A.$$ 

In the above equation (and indeed throughout this section), $B_0$ denotes the connection with normalized curvature form on the spinor bundle induced by the spin structure $s_0$. Then, $A + i \ast df$ will represent $\Theta_h(D)$. The key to Theorem 4.1, then, is to select the initial connection $A$ carefully. Before giving the proof, we name one of the fundamental objects which arises in the construction.

**Definition 4.3.** If $D$ is a divisor of degree $g - 1$ and $(B, \phi)$ is a connection with normalized curvature form for which $\overline{\partial}_B\phi = 0$, and $\phi^{-1}(0) = D$, then we call $(B, \phi)$ a holomorphic pair representative for the divisor $D$. Of course, the gauge equivalence class of $B$ represents $\Theta_h(D)$.

**Proof of Theorem 4.1.** Since one of the $k_i > g_i - 1$, we can assume without loss of generality that $k_2 > g_2 - 1$. Pick a partition of unity $\psi_1, \psi_2$ over $[-2, 2] \times S^1$ subordinate to the cover

$$\{[-2, 1) \times S^1, (-1, 2] \times S^1\}.$$ 

We can transfer this partition of unity (by extending by constants in the obvious way) to a partition of unity on $\Sigma_1 \#_T \Sigma_2$ subordinate to the cover

$$\{\Sigma_1 \cup ([T, 1) \times S^1), ((-1, T] \times S^1) \cup \Sigma_2\}$$ 

(provided that $T > 2$). We denote this partition of unity also by $\{\psi_1, \psi_2\}$ although, technically, it does depend on $T$. However, notice that for all $T$, the $(L^2$ and $C^\infty$) norms of $d\psi_i$ remain constant.

Fix a pair of divisors $D_i \in \text{Sym}^{k_i}(\Sigma_i)$. After deleting the points $p_1$ and $p_2$, we find suitably normalized holomorphic pair representatives for the $D_i$ over the cylindrical-end manifolds; i.e. connections $A_1, A_2$ with fixed curvature form on $\Sigma_1^+, \Sigma_2^+$ and
sections $\phi_1, \phi_2$ whose vanishing locus is $D_1, D_2$ respectively, with asymptotic expansions (with respect to some trivialization of the spinor bundle over the flat cylinder) of the form:

$$\phi_1 = e^{\alpha t} + O(e^{(\alpha-1)t}) \quad \text{and} \quad \phi_2 = e^{-\alpha t} + O(e^{-(\alpha-1)t}),$$

where $\alpha = -g_1 + k_1 + \frac{1}{2} = g_2 - k_2 - \frac{1}{2}$. Note that the leading terms in the asymptotic expansions here have a particularly simple form; this can be arranged by first untwisting the imaginary part of the leading term using a gauge transformation, and then by rescaling the $\phi_i$ by real constants if necessary. Note that the decay rates come from Proposition 3.1: according to that proposition, sections with the prescribed decay for $\phi_i$ correspond to sections of the spinor bundles over $\Sigma_i$ which vanish to order $-g_i + k_i + 1$ at the connected sum point. Starting from these sections $\phi_1, \phi_2$, we will construct for all $T$ holomorphic pair representatives $A_1 \#_T A_2$ for $\gamma_2(D_1, D_2)$, and show that the gauge equivalence classes of $A_1 \#_T A_2$ converge, as $T \to \infty$ to those of $A_1$ and $A_2$.

There is a natural connection with normalized curvature form on $\Sigma_1 \# T \Sigma_2$ induced from $A_1$ and $A_2$, which we write as $A_1 \# A_2$, which is obtained from the identification $J(\Sigma_1 \times \Sigma_2) \cong H^1(\Sigma_1 \# \Sigma_2; S^1) \cong H^1(\Sigma_1; S^1) \times H^1(\Sigma_2; S^1) \cong J(\Sigma_1) \times J(\Sigma_2)$ coming from our fixed spin structure $\mathfrak{s}_0$. Consider, then, the section $\phi_T = \psi_1 \phi_1 + e^{2\alpha T} \psi_2 \phi_2$. For all sufficiently large $T$, this section does not vanish in the neck region $[-1, 1] \times S^1$ (indeed, the restriction of $\phi_T$ to this region is $e^{(\alpha T+\epsilon)} + O(e^{(\alpha-1)T})$). Although $\phi_T$ is not $\overline{\partial}_{A_1 \# A_2}$ holomorphic, it is holomorphic for the $\overline{\partial}$-operator

$$\overline{\partial}_{A_1 \# A_2} = \frac{\overline{\partial} \psi_1 \phi_1}{\phi_T} - e^{2\alpha T} \overline{\partial} \psi_2 \frac{\phi_2}{\phi_T};$$

so if we let $\epsilon$ be the form

$$\epsilon = \text{Im} \left( \frac{\overline{\partial} \psi_1 \phi_1}{\phi_T} + e^{2\alpha T} \overline{\partial} \psi_2 \frac{\phi_2}{\phi_T} \right),$$

then $\phi_T$ is holomorphic for

$$A_3 = A_1 \# A_2 + 2i\epsilon.$$

The connection $A_3$ is a good first approximation to the desired connection corresponding to $\gamma_2$.

The curvature form is normalized once we find $f$ so that $d \ast df = -2d\epsilon$. We would like to show that this does not change the cohomology class by much; i.e. as the tube length $T$ is increased, the cohomology correction $2\epsilon \ast df$ tends to zero (viewed as elements in $H^1(\Sigma_1 \#_T \Sigma_2; \mathbb{R}) \cong H^1(\Sigma_1 \# \Sigma_2; \mathbb{R}) \cong H^1(\Sigma_1^+; \mathbb{R}) \oplus H^1(\Sigma_2^+; \mathbb{R})$). To do this, we find it convenient to use harmonic forms. Let $H_T$ denote the space of harmonic one-forms on $\Sigma_1 \#_T \Sigma_2$,

$$H_T = \{ a \in \Omega^1(\Sigma_1 \#_T \Sigma_2) \mid da = d \ast a = 0 \},$$
and let $\Pi_T$ denote the $L^2$-projection to $\mathcal{H}_T$. By Hodge theory, the map from closed one-forms, given by $z \mapsto [\Pi_T(z)]$, induces the identity map in cohomology. Moreover, it is identically zero on co-closed one-forms (by integration-by-parts). Thus,

$$[\Pi_T(2\epsilon + *df)] = [\Pi_T(2\epsilon)].$$

Note that $\lim_{T \to \infty} \|\epsilon\|_{L^2} = 0$ (indeed, using the fact that $\overline{\partial}(\psi_1 + \psi_2) = 0$, the decay condition in Equation (2) and the expression for $\epsilon$, Equation (3), it is easy to see that $\|\epsilon\|_{L^2} = O(e^{(\alpha - 1)/T})$, so $\lim_{T \to \infty} \Pi_T(\epsilon) = 0$ in $L^2$. The fact that the harmonic projections tend to zero, then, is a consequence of elliptic regularity, as follows:

**Lemma 4.4.** Let $h_T$ be a sequence of harmonic forms on $\Sigma_1 \# \Sigma_2$ whose $L^2$ norm tends to zero. Then the cohomology classes $[h_T]$ tend to zero, as well.

**Proof.** Since the operator $d + d^*$ is translationally invariant in the cylinder $S^1 \times \mathbb{R}$, there is a single constant $C$ which works for all the manifolds $\Sigma_1 \# \Sigma_2$, so that for any form $\phi \in \Lambda^*(\Sigma_1 \# \Sigma_2)$,

$$\|\phi\|_{L^2_{k+1}} \leq C(\|\phi\|_{L^2} + \|(d + d^*)\phi\|_{L^2_k})$$

(see for instance [8]); thus,

$$\|h_T\|_{L^2_k} \leq C(\|h_T\|_{L^2}).$$

This together with the Sobolev lemmas shows that the forms $h_T$ converge to zero in $C^\infty$ over any compact set. But the cohomology class of any of the $h_T$ is determined by its restriction to the (compact) subset $\Sigma_1^c \coprod \Sigma_2^c \subset \Sigma_1 \# \Sigma_2$.

This proves the convergence of $\gamma_T$ in $C^0$. To prove $C^1$ convergence, we argue that any path in the space of divisors over $\Sigma_1^c$ and $\Sigma_2^c$ can be covered by a path in the space of holomorphic pairs whose derivatives satisfy decay conditions analogous to Equation (2). To see this, it helps to consider Fredholm deformation theory for the symmetric product which arises by viewing the latter space as the zeros of a non-linear equation on the cylinder. More precisely, let $Z_i$ be a vector space of compactly-supported forms in $\Sigma_i^c$ which map isomorphically to $H^1(\Sigma_i^c)$. Consider the the map

$$\Omega_{\delta_i}^{0,0}(E_i^+) \times Z_i \longrightarrow \Omega_{\delta_i}^{0,1}(E_i^+),$$

given by $\overline{\partial}_{B_i} \Phi + (ia)^{0,1} \Phi$, with weighted Sobolev topologies on the $\Omega^{0,*}(E_i^+)$, i.e.

$$\|\Phi\|_{\delta_i} = \|e^{-\frac{\delta_i}{T}} \Phi\|_{L^2},$$

where $\delta_i = g_i - 1 - k_i$, and $t$ is a smooth function on $\Sigma_i^+$ which extends the real coordinate function on the cylinder $S^1 \times [0, \infty)$ (the decay rate is chosen to for $\Phi$ to correspond to a holomorphic section of the degree $k_i$ bundle over $\Sigma_i$, according to Proposition 3.1). This is a non-linear, Fredholm map whose zero locus (away from the trivial $\Phi \equiv 0$ solutions) is transversally cut out by the equations. This zero locus,
the space of holomorphic pairs on $\Sigma^+_i$, admits a natural submersion to the symmetric product, given by taking $(A, \Phi)$ to the divisor where $\Phi$ vanishes (this models the quotient by the natural $\mathbb{C}^*$ action on the space of holomorphic pairs). Thus, any tangent vector in $\text{Sym}^k(\Sigma^+_i)$ can be represented by a pair $(a, \phi) \in Z_i \times \Omega^{0,0}(E^+_i)$ where $a$ is compactly supported in $\Sigma^+_i$ and
\[
\phi = Ce^{(-g_i+k_i+\frac{1}{2})t} + O(e^{(-g_i+k_i-\frac{1}{2})t}),
\]
for some constant $C$ (depending on the tangent vector).

Now, consider a pair of smooth paths $D_1(s)$ and $D_2(s)$, and a corresponding paths of holomorphic pairs $(A_1(s), \phi_1(s))$ and $(A_2(s), \phi_2(s))$. Note that the derivative $\frac{d}{ds}(A_1(s) \#_T A_2(s))$, restricted to $\Sigma_i \coprod \Sigma^+_2$, is the differential of $\Theta_{\Sigma_1} \times \Theta_{\Sigma_2}$. To prove the $C^1$ convergence, we must show that the derivative of the error term converges to zero, i.e. writing
\[
[\frac{d}{ds} A_3(s)] = [\frac{d}{ds} A_1(s) \# A_2(s)] + 2i[\Pi_T(\frac{d}{ds} \epsilon(s))],
\]
we must show that $[\Pi_T(\frac{d}{ds} \epsilon)] \to 0$ in $T$. Note first that
\[
\frac{d}{ds} \epsilon(s) = (\overline{\partial} \psi_1) \left( \frac{d\phi_1}{ds} \frac{1}{\phi_T} - \frac{d\phi_T}{ds} \frac{\phi_1}{\phi_T^2} \right) + e^{2\alpha_T} (\overline{\partial} \psi_2) \left( \frac{d\phi_2}{ds} \frac{1}{\phi_T} - \frac{d\phi_T}{ds} \frac{\phi_2}{\phi_T^2} \right)
\]
\[
= (\overline{\partial} \psi_1) \left( \frac{d\phi_1}{ds} \frac{1}{\phi_T} - \left( \psi_1 \frac{d\phi_1}{ds} + e^{2\alpha_T} \psi_2 \frac{d\phi_2}{ds} \right) \frac{\phi_1}{\phi_T^2} \right)
\]
\[
+ e^{2\alpha_T} (\overline{\partial} \psi_2) \left( \frac{d\phi_2}{ds} \frac{1}{\phi_T} - \frac{d\phi_T}{ds} \frac{\phi_2}{\phi_T^2} \right).
\]
It is easy to see from this that the derivative of the error is supported in the region $[-1, 1] \times S^1 \subset \Sigma_i \#_T \Sigma_2$, and it is universally bounded (e.g. in $C^0$) independent of $T$. For example, since
\[
\dot{\phi} = e^{\alpha(t+T)} + O(e^{(\alpha-1)T}) \quad \text{and} \quad \frac{d\phi_2}{ds} = Ce^{-\alpha(t+T)} + O(e^{(-\alpha-1)T})
\]
for $t \in [-1, 1] \times S^1$ (according to Fredholm perturbation theory we discussed above), we see that $|e^{\alpha T}(\overline{\partial} \psi_2) \frac{d\phi_2}{ds} \frac{1}{\phi_T}|$ is bounded above for all $T$. (The other terms follow in a similar manner.) Since, moreover, there is a universal constant $K$ independent of $T$ so that for any harmonic form $h$,
\[
\|h(0)\| \leq Ke^{-T}(\|h(-T)\| + \|h(T)\|)
\]
(this follows from standard asymptotic expansion arguments see for instance [2], [22]), the harmonic projection $\Pi_T(\frac{d}{ds} \epsilon(s))$ converges to zero exponentially. This proves the $C^1$ convergence statement, and concludes the proof of Theorem 4.1. \qed
Theorem 4.1 can be used to prove the “stabilization invariance” of the invariant we are studying. Specifically, in Section 2, we gave a definition of the invariant \( \theta(s) \), which refers to a choice of Heegaard decomposition for \( Y \). In the next proposition, we show that it is independent of that, depending only on the underlying three-manifold.

**Proposition 4.5.** The invariant \( \theta(s) \) is independent of the Heegaard decomposition used in its definition, thus it gives a well-defined topological invariant.

**Proof.** Fix a genus \( g \) Heegaard decomposition of \( Y = U_0 \cup_{\Sigma} U_1 \). There is a “stabilized” genus \( g + 1 \) Heegaard decomposition of \( Y \), corresponding to the natural decomposition

\[
S^3 = (S^1 \times D) \cup_{S^1 \times S^1} (D \times S^1);
\]

i.e. let

\[
U'_0 = U_0 \# (S^1 \times D), \Sigma' = \Sigma \# (S^1 \times S^1) \quad \text{and} \quad U'_1 = U_1 \# (D \times S^1),
\]

and consider the Heegaard decomposition

\[
Y = U'_0 \cup_{\Sigma'} U'_1.
\]

We would like to show that the invariant \( \theta \) associated to the Heegaard decomposition \( U_0 \cup_{\Sigma} U_1 \) agrees with that associated to \( U'_0 \cup_{\Sigma'} U'_1 \), which we will denote \( \theta' \).

Fix a metric on the torus \( S^1 \times S^1 \). We observe that one can find \( U \)-allowable metrics \( h \) on a Riemann surface \( \Sigma \) with the property that for all sufficiently large \( T \), \( h \#_T (S^1 \times S^1) \) are \( U' \)-allowable. To see this, let \( h(T_1) \) denote the metric on \( \Sigma \) which is stretched out along \( g \) of the attaching circles of \( \Sigma \). We show there is a \( T_0 \) so that for all \( T_1, T_2 > T_0 \), \( h(T_1) \#_{T_2} (S^1 \times S^1) \) is \( U' \)-allowable. If this were not the case, we could stretch both tube-lengths simultaneously, and extract a subsequence of spinors, which would converge to a non-zero harmonic spinor either on the punctured \( (S^1 \times S^1) \) (with a cylindrical end attached) – which cannot exist in light of the holonomy constraint coming from \( S^1 \times D \), see Proposition 3.1 – or a harmonic spinor on the genus zero surface with \( g + 1 \) punctures obtained by degenerating the punctured version of \( \Sigma \). This is ruled out by the holonomy constraints at infinity, as in the proof of Lemma 2.4 (the holonomy around the curves corresponding to the attaching circles vanish as in the proof of that lemma; around the curve corresponding to the connected sum neck it vanishes since that curve bounds in \( \Sigma' \)). Thus, for sufficiently large \( T_1 \), the metric \( h(T_1) \) has the desired properties.

In view of this observation, we can find a path of metrics \( h_t \) on \( \Sigma \) to calculate \( \theta \), with the property that for all sufficiently long connected sum tubes, the family of metrics obtained by connecting \( h_t \) with a constant metric on the torus \( F = S^1 \times S^1 \) can be used to calculate \( \theta' \). Choose a point \( p \in \Sigma \). The fiber of the map

\[
\Psi: \text{Sym}^{g-1}(\Sigma) \times [0, 1] \times [0, 1] \to \mathbb{T}(Y)
\]
(used in definition of \( \theta \)) over a generic point \( \eta \in \mathbb{T}(Y) \) misses the submanifold of divisors \( D \in \text{Sym}^{g-1}(\Sigma) \) which contain the point \( p \). In other words, there is a compact region \( K \subset \Sigma - p \) so that \( \Psi^{-1}(\eta) \subset \text{Sym}^{g-1}(K) \times [0, 1] \times [0, 1] \).

Consider the one-parameter family of maps

\[
\Psi'_T : \text{Sym}^{g-1}(\Sigma^\#_TF) \longrightarrow \mathbb{T}'(Y),
\]

used in defining the invariant \( \theta' \) for the Heegaard decomposition \( U_0' \cup_{\Sigma'} U_1' \) (using metrics with length parameter \( T \)). Note that we have an isomorphism

\[
\mathbb{T}'(Y) \cong \mathbb{T}(Y) \times J(F).
\]

Under this isomorphism, the origin corresponds to \( 0 \times s_0 \), where \( s_0 \) is a spin structure on the torus which bounds. Let \( q \in F \) be the pre-image of \( s_0 \) under the Abel-Jacobi map \( \mu^{(1)} : F \longrightarrow J(F) \).

Since \( s_0 \) admits no harmonic spinors, it follows that \( q \in F \) is not the connected sum point. Given a sequence of points \((D_T, s_T, t_T) \in \Psi'^{-1}_T(\eta \times 0)\) with \( T \to \infty \) using compactness on the \( \Sigma \)-side, we obtain a subsequence which converges to a divisor \( D \in \text{Sym}^{g-1}(\Sigma) \) and numbers \((s, t)\), so that \((D, s, t) \in \Psi^{-1}(\eta)\). It follows from our choice of \( \eta \) that the divisor is actually supported in \( \text{Sym}^{g-1}(K) \). Moreover, looking on the \( F \) side, we see that the fiber points must converge to the divisor \( q \). Thus, we see that for all \( T \) sufficiently large, the divisors in the fibers of \( \Psi'^{-1}_T(\eta \times 0) \) are contained in the range of the splicing map

\[
\gamma_T : \text{Sym}^{g-1}(\Sigma^c) \times F^c \longrightarrow \text{Sym}^g(\Sigma^\#_TF),
\]

where \( \Sigma^c \) is a compact set whose interior contains \( K \), and \( F^c \) is some compact subset of the punctured torus (punctured at the connect sum point) which contains \( q \).

But applying Theorem 4.1, we see that the maps

\[
\text{Sym}^{g-1}(\Sigma^c) \times F^c \times [0, 1] \times [0, 1] \longrightarrow \mathbb{T}(Y) \times J(F)
\]

obtained by mapping

\[
(D_1, D_2, s, t) \mapsto \Psi'_T(\gamma_T(D_1, D_2), s, t)
\]

(which we will denote \( \Psi'_T \circ \gamma_T \) in a mild abuse of notation) converge in \( C^1 \) to the map which sends

\[
(D_1, D_2, s, t) \mapsto \Psi(D_1, s, t) \times \mu^{(1)}(D_2).
\]

(Note that \( \mu^{(1)}(D_2) \) does not depend on \( s \) and \( t \) since we are fixing the metric on the torus side.) The preimage of \((\eta, 0)\) under this limiting map is the fiber

\[
(\Psi|_{\text{Sym}^{g-1}(\Sigma^c)})^{-1}(\eta) \times q = \Psi^{-1}(\eta) \times q
\]

(the equality of the two sets follows from the fact that \( K \subset \Sigma^c \)), which is used to calculate \( \theta \). Now the \( C^1 \) convergence, identifies this fiber with the fiber

\[
(\Psi'_T \circ \gamma_T)^{-1}(\eta \times 0),
\]
which is used to calculate $\theta'(g)$ (this is how we chose the subsets $\Sigma_1^c \subset \Sigma_1$ and $F^c \subset F$). Thus, $\theta = \theta'$. Note that our sign conventions are compatible with stabilization, since if $\{\alpha_1, \ldots, \alpha_g\}$, $\{\beta_1, \ldots, \beta_g\}$ are positively ordered for $U_0$ and $U_1$, then $\{\alpha_1, \ldots, \alpha_{g+1}\}$, $\{\beta_{g+1}, \beta_1, \ldots, \beta_g\}$ are positively ordered for $U_0'$ and $U_1'$, since the boundary of the two-cell corresponding to $\beta_{g+1}$ is $-1$ times the boundary of the one-cell corresponding to $\alpha_{g+1}$. \hfill $\Box$

We discuss another consequence of Theorem 4.1, in a case which will prove to be quite useful in the calculations. But first, we must characterize the image of the splicing map, in terms of the connections. We content ourselves with a statement in the case where $k_1 = g_1 - 1$, as this is the only case we need to consider in this paper.

**Proposition 4.6.** Let $V_1 \subset J(\Sigma_1)$, $V_2 \subset J(\Sigma_2)$ be closed subsets of the Jacobians. Suppose that $\Theta_{h^{-1}}(V_1)$ contains no divisors which include the sum point $p_1$, and suppose that $V_2$ contains no points in the theta divisor for $\Sigma_2$. Then, there are compact subsets $\Sigma_i^c \subset \Sigma_i - p_i$ and a real number $T_0 \geq 0$ so that for all $T \geq T_0$, $\Theta_{h_T^{-1}}(V_1 \times V_2)$ lies in the image of the splicing map

$$\gamma_T : \text{Sym}^{g_1 - 1}(\Sigma_1^c) \times \text{Sym}^{g_2}(\Sigma_2) \longrightarrow \text{Sym}^{g_1 - 1}(\Sigma_1 \#_T \Sigma_2).$$

**Proof.** Our hypothesis on $V_1$ gives us a compact set $K_1 \subset \Sigma_1 - p_1$ with the property that $\Theta_{h_T^{-1}}(V_1) \subset \text{Sym}^{g_1 - 1}(K_1)$. Similarly, our hypothesis on $V_2$ gives a compact set $K_2 \subset \Sigma_2 - p_2$ with the property that $(\mu^{(g_2)})^{-1}(V_2) \subset \text{Sym}^{g_2}(K_2)$. We let $\Sigma_i^c$, $\Sigma_2^c$ be any pair of compact sets whose interior contains $K_1$ and $K_2$.

Consider pairs $(A_T, \phi_T)$ over $\Sigma_1 \#_T \Sigma_2$ which correspond to the intersection of the theta-divisor with $V_1 \times V_2$, and which are normalized so that the $L^2$ norms over $\Sigma_1 \#_T \Sigma_2$ of $\phi_T$ is 1. By local compactness, together with the fact that the tube admits no translationally invariant harmonic spinor, any such sequence of pairs $(A_T, \phi_T)$ for tube-lengths $T \to \infty$ must admit a subsequence which converges in $C^\infty$ to an $L^2$ solution $\Phi_1$ and $\Phi_2$ on the two sides $\Sigma_1^c$ and $\Sigma_2^c$, at least one of whose $L^2$ norm is non-zero. By transferring back to $\Sigma_2$ (Proposition 3.1), our assumption on $V_2$ ensures that $\Phi_2 \equiv 0$. By $C^\infty$ convergence, then, the zeros of $\phi_T$ must converge to the zeros of $\Phi_1$ over $\Sigma_1^c$.

Without loss of generality, we might as well assume that all the $A_T$ are of the form $A_1 \#_T A_2$ for fixed $A_1 \in J(\Sigma_1)$, $A_2 \in J(\Sigma_2)$. Note that for each $A_2 \in V_2$, there is a unique $A_2$-holomorphic section $\Phi_2$ over $\Sigma_2$ which, after transferring to $\Sigma_2^c$, admits an asymptotic expansion

$$\Phi_2 = e^{t/2} + O(e^{-t/2})$$

(the growth here corresponds to the pole at $p_2 \in \Sigma$ which we have introduced in our convention for the Abel-Jacobi map). Existence of the section follows from the fact that the $g_2$-fold Abel-Jacobi map has degree one (this is the "Jacobi inversion theorem", see for instance p. 235 of [9]). Uniqueness follows from the fact that a
difference of two such would give an $L^2$ section, showing that $A_2$ actually lies in the theta divisor, which we assumed it could not.

We show the restrictions of $\phi_T$ to the $\Sigma_2$-side come close to approximating $\Phi_2$ or, more precisely, that its zeros converge to those of $\Phi_2$.

Rescale $\phi_T$ so that over $[-1, 1] \times S^1$, it has the form

$$\phi_T = e^{-(T+1)/2} + O(e^{-3(T+1)/2}).$$

Consider the section $\Psi_T = \psi_2(\phi_T - e^{-T/2}\Phi_2)$, viewed as a section of $\Sigma_2^+$ (we can do this, as its support is contained in the support of $\psi_2$). Note that

$$\overline{\mathcal{D}} A_2 \Psi_T = (\overline{\mathcal{D}} \psi_2)(\phi_T - e^{-T/2}\Phi_2).$$

Thus, $\|\overline{\mathcal{D}} A_2 \Psi_T\| = O(e^{-3T/2})$. Since $\overline{\mathcal{D}} A_2$ is Fredholm with index zero (it is a spin connection) and no kernel (it is not in the theta divisor), it has no cokernel, and we can conclude that $\|\Psi_T\|_{L^2(\Sigma_2^+)} \leq Ce^{-3T/2}$ for some constant $C$ independent of $T$.

Since the restriction of $\Psi_T$ to $\Sigma_2^c$ is $A_2$-holomorphic, elliptic regularity on this compact piece shows that the section $\Psi_T = \phi_T - e^{-T/2}\Phi_2$ is bounded by some quantity of order $e^{-3T/2}$. Thus, the zeros of $\phi_T$ in $\Sigma_2^c$ converge to those of $\Phi_2$. \qed

Armed with this proposition, we turn our attention to another important consequence of Theorem 4.1. Let $\Sigma$ be a surface of genus $g$, and let $\{\alpha_1, ..., \alpha_g\}$ be a complete set of attaching circles for a handlebody $U$ bounding $\Sigma$. The holonomy around the first $g - 1$ of the $\alpha_i$ gives a map

$$\text{Hol}_{\alpha_1 \times ... \times \alpha_{g-1}} : J(\Sigma) \longrightarrow \mathbb{T}^{g-1}.$$ 

According to [14] (see also [17], where a related discussion is given), the preimage of a generic point in $\mathbb{T}^{g-1}$ via $\text{Hol}_{\alpha_1 \times ... \times \alpha_{g-1}} \circ \Theta_h$ (for any metric $h$) is homologous to the torus $\alpha_1 \times ... \times \alpha_{g-1} \subset \text{Sym}^{g-1}(\Sigma)$. We would like to find a metric on $\Sigma$ for which these spaces are actually isotopic.

To describe this metric, think of $\Sigma$ as a connected sum of $g - 1$ disjoint tori $F_1, ..., F_{g-1}$ with the remaining torus $F_g$, in such a way that the curve $\alpha_i$ is supported in the torus $F_i$ for $i = 1, ..., g - 1$. Fix a metric $h$ which is product-like along the $g - 1$ connect sum tubes, and let $h(T)$ denote the metric obtained from $h$ by stretching the connect sum tubes by a factor of $T$. (The case where $g = 3$ is illustrated in Figure 4.)

**Corollary 4.7.** Let $\Sigma$ be a surface of genus $g$ viewed as a connected sum of tori as described above, and let $\{\alpha_1, ..., \alpha_g\}$ be a complete set of attaching circles. For any $\eta \in \mathbb{T}^{g-1}$ with the property that $\eta_i \neq 1/2$ for all $i = 1, ..., g - 1$, there is a $T_0$ so that for all $T \geq T_0$ the subset $(\text{Hol}_{\alpha_1 \times ... \times \alpha_{g-1}} \circ \Theta_{h(T)})^{-1}(\eta)$ is isotopic to the torus $\alpha_1 \times ... \times \alpha_{g-1} \subset \text{Sym}^{g-1}(\Sigma)$, where $h(T)$ is the one-parameter family of metrics obtained by stretching the connect sum tubes for the initial metric $h$. 

Proof. We would like to apply a version of Proposition 4.6, with more than one neck (note that the proof works in this context as well). Let $V_1$ be the theta divisor of $F_g$. It contains none of the connect sum points, of course, because it has degree zero. Moreover, the set $\text{Hol}_{\alpha_i}^{-1}(\eta_i)$ misses the theta divisor for $F_i$ for $i = 1, \ldots, g - 1$ (the theta divisor of $F_i$ consists of a single point where the holonomy around $\alpha_i$ is $1/2$). Hence, Proposition 4.6 applies: for all sufficiently long necks, the theta divisor hits $\text{Hol}_{\alpha_1 \times \ldots \times \alpha_{g-1}}^{-1}(\eta)$ in a region corresponding to the splicing map from Theorem 4.1. Thus, the composite of $\Theta$ with the splicing map is $C^1$ close to the map

$$F_1^c \times \ldots \times F_{g-1}^c \longrightarrow H^1(F_1; S^1) \times \ldots \times H^1(F_{g-1}; S^1) \times H^1(F_g; S^1),$$

which is a product of $g - 1$ copies of the Abel-Jacobi map with the inclusion of the point (theta-divisor for the $F_g$). Since the Abel-Jacobi map in this case is a diffeomorphism, the points where the $\alpha_i$-holonomy is trivial forms a smoothly embedded curve. In fact, it is easy to see that this curve is isotopic to $\alpha_i$ (see [14] and also [17]). Also, it is clear that post-composing with evaluation along $\alpha_i$ gives us map to $(S^1)^{g-1}$ with $\eta$ as a regular value, whose fiber is isotopic to $\alpha_1 \times \ldots \times \alpha_{g-1}$. It is easy to see

![Figure 1. Connected sum of $F_1$ and $F_2$ with $F_3$. Attaching circles $\{\alpha_1, \alpha_2, \alpha_3\}$ and connected sum circles $\{\gamma_1, \gamma_2\}$ are included.](image)

that any other $C^1$ close map must have $\eta$ as a regular value, with an isotopic fiber. Thus, the corollary follows from Theorem 4.1. \qed
5. Calculations when $b_1(Y) > 1$

The aim of this section is to prove the following:

**Theorem 5.1.** When $b_1(Y) > 1$, then the polynomial $\theta$ is equal up to sign to the symmetrized Alexander polynomial of $Y$.

In the proof of this theorem, we will naturally meet certain tori in the symmetric product. Given a Heegaard decomposition of $Y$, let $\{\alpha_i\}, \{\beta_i\}$ be complete sets of attaching circles for the two handlebodies. Given any $i$ and $j$, we have tori in $\text{Sym}^{g-1}(\Sigma)$

$$\mathbb{T}_i(\alpha) = \alpha_1 \times \ldots \times \alpha_i \times \ldots \times \alpha_g$$

(where the notation indicates omission of the $i^{th}$ factor) and

$$\mathbb{T}_j(\beta) = \beta_1 \times \ldots \times \beta_j \times \ldots \times \beta_g.$$

We will show that the invariant $\theta$ can be extracted from certain polynomials associated to these tori; these polynomials are defined as follows. Let $\text{Sym}^{g-1}(\Sigma) \to \text{Sym}^{g-1}(\Sigma)$ be the covering space of $\Sigma$ (as in Section 2) corresponding to the kernel of the composite map:

$$\pi_1(\text{Sym}^{g-1}(\Sigma)) \to H_1(\text{Sym}^{g-1}(\Sigma); \mathbb{Z}) \cong H_1(\Sigma; \mathbb{Z}) \to H_1(Y; \mathbb{Z}) \cong H.$$

(Recall that $H$ is by definition $H^2(Y; \mathbb{Z})$.) Thus, $H$ acts freely on $\text{Sym}^{g-1}(\Sigma)$. Let $\tilde{T}_i(\alpha), \tilde{T}_j(\beta)$ be a pair of lifts of $\mathbb{T}_i(\alpha)$ and $\mathbb{T}_j(\beta)$. Note that these lifts are tori, and indeed they map isomorphically to $\mathbb{T}_i(\alpha)$ and $\mathbb{T}_j(\beta)$ respectively. The intersection points of $\mathbb{T}_i(\alpha)$ with $\mathbb{T}_j(\beta)$ correspond to the intersection points of $\tilde{T}_i(\alpha)$ with the various translates under $H$ of the torus $\tilde{T}_j(\beta)$. Then, we define a polynomial (an element of $\mathbb{Z}[H]$) associated to the lifts $\tilde{T}_i(\alpha)$ and $\tilde{T}_j(\beta)$ by the formula

$$C_{i,j} = \sum_{h \in H} \# \left( \tilde{T}_i(\alpha) \cap h\tilde{T}_j(\beta) \right) [h]. \tag{4}$$

(For the intersection numbers here, we use orientations for the $\tilde{T}_i(\alpha)$ and $\tilde{T}_j(\beta)$ induced from orientations of $\mathbb{T}_i(\alpha)$ and $\mathbb{T}_j(\beta)$; we return to a more careful discussion of signs in Section 7.1.) Summing over the action of Tors, we get an induced polynomial $C_{i,j} \in \mathbb{Z}[H]$ (recall that $H = H/\text{Tors}$). In Section 7, we will show that the Alexander polynomial of $Y$ is the greatest common divisor of the $C_{i,j}$ for $i,j = 1, \ldots, g$. Different lifts of the $\mathbb{T}_i(\alpha)$ and $\mathbb{T}_j(\beta)$ give rise to translates of the $C_{i,j}$ and $C_{i,j}$ by elements in $H$.

The main ingredient in the proof of Theorem 5.1 is a perturbation of the invariants, which corresponds to moving the tori $L_0$ and $L_1$. Let $\Lambda_i(\alpha)$ be the space of $B \in J(\Sigma)$ with $\text{Hol}_{\alpha_k} B = 0$ for all $k \neq i$, and similarly let $\Lambda_j(\beta)$ be the space of $B \in J(\Sigma)$ with $\text{Hol}_{\beta_k} B = 0$ for all $k \neq j$. We will move the tori $L_0$ and $L_1$ inside $\Lambda_i(\alpha)$ and
Describe how to calculate $\theta$ in terms of the intersection of the tori $\Lambda_i(\alpha)$ and $\Lambda_j(\beta)$ with the theta divisor. An important point, then, is that we can concretely understand these intersections, for favorable initial metrics. It is with the help of this description, then, that we meet the polynomials described above. But first, we describe how to calculate $\theta$ in terms of the lifts of $\Lambda_i(\alpha)$ and $\Lambda_j(\beta)$. To do this, we discuss the lifts in detail.

There is a lift $\tilde{\delta}: \tilde{J} \rightarrow H^2(Y; \mathbb{R})$ of the coboundary map $H^1(\Sigma; S^1) \rightarrow H^2(Y; S^1)$, which is uniquely specified once we ask that $\tilde{\delta}(L_0(s)) = 0$. With our conventions, then, $\tilde{\delta}(L_1(s)) = \tilde{\delta}$.

To $\alpha_i$, assign an element $\alpha_i^* \in H^1(\Sigma; \mathbb{Z})$, as follows. Let $\gamma_i$ be the core of the $i^{th}$ one-handle in $U_0$ (i.e. this is the oriented curve which intersects only the attaching disk associated to $\alpha_i$, which it intersects positively in a single point), then $\alpha_i^*$ is the Poincaré dual (in $\Sigma$) to a class in $H_1(\Sigma; \mathbb{Z})$ whose image in $H_1(U_0; \mathbb{Z})$ is represented by $\gamma_i$. (The class $\alpha_i^*$ is not uniquely determined by this property, but the our constructions involving $\alpha_i^*$ are independent of its choice.) Note that the class $\mu_i = \delta \alpha_i^* \in H^2(Y; \mathbb{Z})$, is Poincaré dual (in $Y$) to the homology class represented by $\gamma_i$. The element $\beta_i^*$ is defined in the analogous manner, only using $U_1$ instead of $U_0$. We let $\nu_j$ denote $\delta \beta_j^*$. Choose $i$ and $j$ so that $\mu_i$ and $\nu_j$ are not torsion classes; we can find such $i$ and $j$ since $H_1(U_0)$ and $H_1(U_1)$ both surject onto $H_1(Y)$.

By multiplying $\alpha_i$ by $-1$ if necessary, we can assume that $\mu_i$ and $\nu_j$ in $H^2(Y; \mathbb{Z})/\text{Tors}$ are not negative multiples of each other. (In keeping with the conventions introduced in Section 1, we underline objects when viewing them modulo the action of torsion.)

We define subsets $\Lambda_i^0(s), \Lambda_i^+(s) \subset \tilde{J}$ which correspond to all translates of (small perturbations of) $L_0$ and $L_1$ in the directions determined by $\alpha_i^*$ and $\beta_j^*$; more precisely,

$$\Lambda_i^0(s) = L_0(s) + \eta_0 + \mathbb{R}^+ \alpha_i^* \quad \text{and} \quad \Lambda_i^+(s) = L_1(s) + \eta_1 - \mathbb{R}^+ \beta_i^*.$$

Under the map $\tilde{J} \rightarrow J$, the spaces $\Lambda_i^0(s)$ and $\Lambda_i^+(s)$ project to $\Lambda_i(\alpha)$ and $\Lambda_j(\beta)$ respectively.

In the case where $b_1(Y) = 2$, we make use of special allowable metrics:

**Definition 5.2.** Fix an $1/2 > \epsilon > 0$, and let $U$ be a handlebody which bounds $\Sigma$. A metric $h$ on $\Sigma$ is said to be strongly allowable for $\epsilon$ if it is product-like in a neighborhood of $g$ attaching circles $\{\gamma_1, ..., \gamma_g\}$, and any point in the theta divisor for $\Sigma$ must have holonomy around some attaching circle $\gamma_i$ within $\frac{1}{2} \epsilon$ of $\frac{1}{2}$.

Given any $\epsilon > 0$, there exist metrics which are strongly allowable for $\epsilon$ thanks to Lemma 2.1.

**Proposition 5.3.** Let $s$ be any $\text{Spin}^c$ structure on $Y$, and fix $\Lambda_i^+(s)$ for $i = 0, 1$ as above – using classes $\mu_i$ and $\nu_j$ which are not negative multiples of one another. There is an $\epsilon > 0$ with the property that for any metrics $h_0$ and $h_1$ which are $U_0$ and $U_1$-allowable respectively, where $h_0$ is also $\epsilon$-strongly $U_0$-allowable then, we can find
\( \eta_0, \eta_1 \) sufficiently small, with
\[
\theta(s) = \# \left( \tilde{\Theta}_{h_0}^{-1} \left( \Lambda_0^+(s) + \eta_0 \right) \cap \tilde{\Theta}_{h_1}^{-1} \left( \Lambda_1^+(s) + \eta_1 \right) \right).
\]

**Proof.** The proof will rely on the fact that \( \tilde{\delta} \circ \tilde{\Theta}_{h_t} \) has bounded variation along any one-parameter family of metrics. Specifically, let \( h_t \) be a one-parameter family of metrics, and fix a norm on \( \tilde{\Theta} \). Then, there is a constant \( K \) with the property that for any \( D \in \tilde{\text{Sym}}^{-g-1}(\Sigma) \), \( s, t \in [0, 1] \),
\[
(5) \quad |\tilde{\delta} \circ \tilde{\Theta}_{h_s}(D) - \tilde{\delta} \circ \tilde{\Theta}_{h_t}(D)| < K.
\]
This follows immediately from the compactness of \( \tilde{\text{Sym}}^{-g-1}(\Sigma) \), together with the fact that \( \tilde{\delta} \circ \tilde{\Theta} \) is an \( H^2(Y; \mathbb{Z}) \)-equivariant map.

Note that \( \theta(s) \) is calculated by the number of points (counted with signs) in the zero-dimensional submanifold of \( \tilde{\text{Sym}}^{-g-1}(\Sigma) \times [0, 1] \times [0, 1] \)
\[
\{(D, s, t) \mid s \leq t, \tilde{\Theta}_{h_s}(D) \in L_0(s) + \eta_0, \tilde{\Theta}_{h_t}(D) \in L_1(s) + \eta_1\},
\]
a space we denote by \( M(s) \). We construct a cobordism between this space and the points in the intersection stated in the lemma, as follows. We can assume without loss of generality that \( h_t \) is constant between \( [0, 1/4] \) and \( [3/4, 1] \). Moreover, let \( \psi \) be a non-decreasing smooth function on \( [0, 1] \) which is monotone increasing in the range \( [0, 1/4] \), with \( \psi(0) = 0 \) and \( \psi|_{[1/4, 1]} \equiv 1 \). Consider the subspace of \( \tilde{\text{Sym}}^{-g-1}(\Sigma) \times [0, 1] \times [0, 1] \) (which agrees with \( M(s) \) when \( u_1 = u_2 = 0 \)):
\[
M_{u_1, u_2}(s) = \left\{ (D, s, t) \mid \begin{array}{l}
\tilde{\Theta}_{h_s}(D) \in L_0(s) + \eta_0 + u_1 \psi(s) \alpha_i^*, \\
\tilde{\Theta}_{h_t}(D) \in L_1(s) + \eta_1 - u_2 \psi(1 - t) \beta_j^*
\end{array} \right\}.
\]
We argue that for all sufficiently large \( u \),
\[
(6) \quad M_{u, u} = \tilde{\Theta}_{h_0}^{-1} \left( \Lambda_0^+(s) + \eta_0 \right) \cap \tilde{\Theta}_{h_1}^{-1} \left( \Lambda_1^+(s) + \eta_1 \right).
\]
Since \( \mu_1 \) and \( \mu_2 \) are not negative multiples of one another, we see that that as \( u \to \infty \), the distance between the point \( \tilde{\delta}(L_0(s) + \eta_0 + u \alpha_i^*) \) is \( H^2(Y; \mathbb{R}) \) and the ray \( \tilde{\delta}(L_1(s) + \eta_1 - \mathbb{R}^+ \beta_j^*) \) goes to infinity, and similarly the distance between the point \( \tilde{\delta}(L_1(s) + \eta_1 - u \beta_j) \) and the ray \( \tilde{\delta}(L_0(s) + \eta_0 + \mathbb{R}^+ \alpha_i^*) \) goes to infinity. Fix \( u \) large enough that both distances are larger than the constant \( K \) from Inequality (5). This condition ensures that all points \( (D, s, t) \in M_{u, u} \) have \( s \leq 1/4 \) and \( t \geq 3/4 \). Monotonicity of \( \psi \) over \( [0, 1/4] \), and the choice of \( u \) then also ensures that the identification (6) holds.

Thus, Proposition 5.3 is established once we construct a smooth cobordism between \( M_{0,0} \) and \( M_{u,u} \). Consider the spaces obtained by connecting \( M_{0,0} \) to \( M_{u,u} \) by first allowing \( u_1 \) to go from 0 to \( u \) (to connect \( M_{0,0} \) to \( M_{u,0} \)) and then allowing \( u_2 \) to go
from 0 to \( u \) (to connect \( M_{u,0} \) to \( M_{u,u} \)). Since \( h_0 \) and \( h_1 \) are allowable metrics, the 
\( s = 0 \) and \( t = 1 \) boundaries are excluded in this one-parameter family for all small \( \eta_0 \) and \( \eta_1 \). Thus, we get a cobordism between \( M_{0,0} \) and \( M_{u,u} \), provided that the \( M_{u_1,u_2} \) do not hit the \( s = t \) boundary, which is guaranteed if 
\[
(\Lambda_i(\alpha) + \eta_0) \cap (\Lambda_j(\beta) + \eta_1) = \emptyset.
\]
Taking \( \tilde{\delta} \) of both spaces, we get a pair of lines in \( H^1(Y;\mathbb{R}) \), which generically miss each other when \( b_1(Y) > 2 \).

The case where \( b_1(Y) = 2 \) requires a slightly closer investigation. In the first part of the cobordism, where we allow \( u_1 \) to vary in \( M_{u_1,0} \), there are still no \( s = t \) boundaries, as we can arrange for 
\[
(\Lambda_i(\alpha) + \eta_0) \cap (L_1 + \eta_1) = \emptyset
\]
(since, applying \( \tilde{\delta} \), we have a point and a line in a two-space). Now, as \( u_2 \) varies in the \( M_{u,u_2} \), it is easy to see that the only possible \( s = t \) boundaries lie in the range where \( s \leq 1/4 \), by our hypothesis on \( \psi \), and hence they must lie in the set 
\[
\Theta_{h_0}^{-1}(\{(\Lambda_i(\alpha) + \eta_0) \cap (\Lambda_j(\beta) + \eta_1)\}),
\]
since \( h_t \) is constant for \( t \leq 1/4 \). Now, consider the intersection point \( p \) of the induced rays \( \delta(\Lambda_i(\alpha) + \eta_0) \) and \( \delta(\Lambda_j(\beta) + \eta_1) \). Note that as \( h_0 \) is stretched out normal to the attaching disks, the image under \( \delta \) of the intersection 
\[
(\Lambda_i(\alpha) + t\alpha^*_i) \cap \Theta_{h_0}(\text{Sym}^{g-1}(\Sigma))
\]
converges to a discrete subset of the ray \( \mathbb{R}^+\mu \subset H^2(Y;\mathbb{R}) \), consisting of points separated by some distance \( \delta > 0 \) (which depends on the \( \mu_i \) and \( \nu_j \)). If \( p \) misses this discrete set, then if \( h_0 \) is sufficiently stretched out, then all sufficiently small \( \eta_0 \) and \( \eta_1 \) have the property that 
\[
\Theta_{h_0}^{-1}(\{(\Lambda_i(\alpha) + \eta_0) \cap (\Lambda_j(\beta) + \eta_1)\}) = \emptyset.
\]
If, on the other hand, \( p \) lies on the discrete set, then, given any sufficiently small \( 0 < \gamma \), if \( h_0 \) is sufficiently stretched out, then all sufficiently small \( \eta_0 \) and \( \eta_1 \) have the property that 
\[
\Theta_{h_0}^{-1}(\{(\Lambda_i(\alpha) + \eta_0 + \gamma\alpha^*_i) \cap (\Lambda_j(\beta) + \eta_1)\}) = \emptyset.
\]
Thus, in both cases, we have obtained the requisite cobordism. \( \square \)

Moreover, we can describe the intersection appearing in Proposition 5.3, in terms of the tori \( T_i(\alpha) \) and \( T_j(\beta) \) described in the beginning of the section. But first, we state a relevant lemma, whose proof fits naturally into the framework of Section 3.

**Lemma 5.4.** Let \( \Sigma \) be a surface, realized as a connected sum of \( g \) tori as in Corollary 4.7, and let \( \{\alpha_1, ..., \alpha_g\} \) a complete set of attaching circles for the handlebody \( U \) which bounds \( \Sigma \), each of which is disjoint from the separating curves for the connected sum decomposition of \( \Sigma \). Fix a metric \( h \) which is product-like near the \( \alpha_i \) and the
separating curves. Then, there is a $T_0$ so that any metric which is obtained from $h$ by stretching each of the $2g - 1$ curves at least by $T_0$ is $U$-allowable.

**Proof.** Take a weak limit of connections which lie in $L(U)$, as all the $2g - 1$ curves are stretched. Under this limit, the surface degenerates into a collection of genus zero surfaces (with cylindrical ends), whose ends correspond to attaching circles for $U$ or separating curves for $\Sigma$. Thus, the weak limit of connections in $L(U)$ induces a connection over these genus zero surfaces, whose holonomies around all its bounding circles is zero. But none of these support harmonic spinors according to Proposition 3.1, proving the lemma. 

**Proposition 5.5.** There is a $U_0$-allowable metric $h$, for which $\Theta^{-1}_h(\Lambda_i(\alpha) + \eta)$ is isotopic to $T_i(\alpha)$ for all generic, small $\eta$.

**Proof.** Fix a metric $h$ on $\Sigma$ as in Lemma 5.4. Let $h(T)$ denote the metric which is stretched by $T_0$ along the $\alpha_i$ and $T$ along the separating curves. The lemma guarantees that for all $T > T_0$, $h(T)$ is allowable. Now, for all sufficiently large $T$, Corollary 4.7 gives the isotopy of $T_i(\alpha)$ with the subset $\Theta^{-1}_h(\Lambda_i(\alpha))$, where $h = h(T)$.

Putting together Propositions 5.3 and 5.5, we obtain the following topological description of $\theta$.

**Proposition 5.6.** For some Spin$^c$ structure $s \in \text{Spin}^c(Y)$, we have that

$$C_{i,j}[s] = (1 - \mu_i)(1 - \nu_j) \cdot \theta.$$

**Proof.** Let $h_k$ be $U_k$-allowable metrics for $k = 0, 1$. Given $i, j$, we construct a natural element $\tilde{C}_{i,j} \in \mathbb{Z}[\text{Spin}^c(Y)]$ closely related to the $C_{i,j}$ defined in the beginning of this section. The $\tilde{C}_{i,j}$ will be a translate of the following analogue of the $C_{i,j}$, which is assigned to a pair of lifts $\tilde{A}_i$ and $\tilde{B}_j$ (to $\widetilde{\text{Sym}}^{g-1}(\Sigma)$) of the manifolds $A_i = \Theta^{-1}_{h_0}(\Lambda_i(\alpha) + \eta_0)$ and $B_j = \Theta^{-1}_{h_1}(\Lambda_j(\beta) + \eta_1)$:

$$X_{i,j} = \sum_{h \in H} \#(\tilde{B}_j \cap h\tilde{A}_i) [h].$$

To do define the $\tilde{C}_{i,j}$, we must assign a Spin$^c$ structure to each intersection point of $A_i$ with $B_j$. To this end, we assume that $h_0$ and $h_1$ are strongly allowable for some $\epsilon > 0$. If $p \in A_i \cap B_j$, then let $\tilde{p}$ be a lift of $p$. There is a pair of lifts $\tilde{A}_i$ and $\tilde{B}_j$ of $A_i$ and $B_j$ which meet in $\tilde{p}$. Note that there are unique lifts $L_0$ and $L_1$ whose image under $\tilde{\delta}$ lie in an $\epsilon$ neighborhood of $\tilde{p} - \frac{1}{2} \mu_j$ and $\tilde{p} + \frac{1}{2} \nu_j$, respectively. Let
\(G(p)\) denote the Spin\(^c\) structure which corresponds to the difference between these two lifts (i.e. if \(\tilde{L}_0\) and \(\tilde{L}_1\) are the two lifts, then the pair \(L_0(G(p))\) and \(L_1(G(p))\) are translates of \(\tilde{L}_0\) and \(\tilde{L}_1\) by a single cohomology class in \(H^2(Y; \mathbb{Z})\)). By summing over all intersection points which correspond to a given Spin\(^c\) structure (with signs), we obtain the element \(\tilde{C}_{i,j} \in \mathbb{Z}^{[\text{Spin}^c(Y)]}\), which is clearly the translate by some Spin\(^c\) structure of the polynomial \(X_{i,j} \in \mathbb{Z}^{[H]}\) defined above.

It follows from Proposition 5.3 that \(\theta(\mathfrak{s})\) is given by adding up the intersection number of certain lifts of \(A_i\) and \(B_j\). Moreover, the intersection point \(p\) will contribute for each \(k, \ell \geq 0\) in the Spin\(^c\) structure \(G(p) + k \mu_i + \ell \nu_j\) (as those are the Spin\(^c\) structures for which \(p\) lies on the corresponding rays). This proves that

\[
\theta = \tilde{C}_{i,j} \left( \sum_{k=0}^{\infty} \mu_i^k \right) \left( \sum_{\ell=0}^{\infty} \nu_j^\ell \right).
\]

Finally, the proposition is proved once we establish that the \(\tilde{C}_{i,j}\) is a translate of \(C_{i,j}\), as defined in Equation (4). To see this, recall that Proposition 5.5 guarantees that the spaces \(A_i\) and \(B_j\) are isotopic, for suitable choices of allowable metrics \(h_0\) and \(h_1\), to \(T_i(\alpha)\) and \(T_j(\beta)\) respectively. Now the polynomial \(X_{i,j}\), which is clearly a translate of \(\tilde{C}_{i,j}\), depends on the submanifolds \(A_i\) and \(B_j\) only up to isotopy. Thus, the proposition follows.

In particular, we have the following:

**Corollary 5.7.** If \(\mu_i\) and \(\nu_j\) are not negative multiples of one another, then

\[
C_{i,j} = (1 - \mu_i)(1 - \nu_j)\hat{\theta}.
\]

Theorem 5.1 follows from this corollary. First, after handle-slides, we can arrange that all the \(\mu_i\) and \(\nu_j\) are non-zero, so that \(\hat{\theta}\) divides each \(C_{i,j}\) and hence the Alexander polynomial. Furthermore, after additional handleslides we can arrange that \(\mu_1 = \nu_1\), \(\mu_2 = \nu_2\), and \(\mu_1\) and \(\mu_2\) are linearly independent in \(H^2(Y; \mathbb{R})\). This shows that the greatest common divisor of \(C_{1,1}\) and \(C_{2,2}\) is \(\hat{\theta}\), so the latter agrees with the Alexander polynomial.
6. Calculating the invariant when $b_1(Y) = 1$

The aim of this section is to prove the following:

**Theorem 6.1.** Let $A = a_0 + \sum_{i=1}^{k} a_i(T^i + T^{-i})$ be the symmetrized Alexander polynomial of $Y$, normalized so that $A(1) = |\text{Tors} H_1(Y; \mathbb{Z})|$. Then, the Laurent polynomial of $\Theta$ is equal to

$$\Theta = b_0 + \sum_{i=1}^{\infty} b_i(T^i + T^{-i}),$$

where

$$b_i = \sum_{j=1}^{\infty} j \cdot a_{i+j}.$$
for integral homology three-spheres). If the intersection were non-empty, by applying \( \tilde{\delta} \) to both sides, we would get:

\[
\tilde{\delta}(\eta_0) + u_1 \mu_i = \delta + \tilde{\delta}(\eta_1) - u_2 \nu_j.
\]

Thus,

\[
\tilde{\delta}(\eta_0 - \eta_1) + u_1 \mu_i - \delta + u_2 \nu_j = 0,
\]

which is impossible, as it is a sum of four non-negative terms at least one of which (the first) is positive.

\[\square\]

**Remark 6.3.** The hypothesis that \( \delta \) has an opposite sign from \( \mu_i \) and \( \nu_j \) is important. If it is violated, there will be addition correction terms from \( s = t \) boundary components in the cobordism.

We define \( \tilde{C}_{i,j} \) as before.

**Proposition 6.4.** For all \( s \) for which \( \delta \) is a non-negative multiple of \( \mu_i \), the value of \( \theta(s) \) is the coefficient of \([\delta] \) in the Laurent series

\[
\tilde{C}_{i,j} \left( \sum_{k=0}^{\infty} [\mu_i]^k \right) \left( \sum_{\ell=0}^{\infty} [\nu_j]^{\ell} \right).
\]

To finish the proof of Theorem 6.1, after handleslides, we arrange that \( \alpha_g = \beta_g \) is a generator of \( H \). Let \( C \) be the image of \( \tilde{C}_{g,g}[-\alpha_g] \in \mathbb{Z}[H] \). Write

\[
C = \sum_{k \in \mathbb{Z}} d_k T^k,
\]

where \( T \) corresponds to the generator of \( \mathbb{Z}[H] \). Then:

**Proposition 6.5.** Write

\[
\theta = b_0 + \sum b_i (T^i + T^{-i}).
\]

Then,

\[
b_i = \sum_{j=1}^{\infty} j d_{i-j},
\]

and also

\[
b_i = \sum_{j=1}^{\infty} j d_{j-i}.
\]

**Proof.** This is a direct consequence of Proposition 6.4. \( \square \)
Proposition 6.6. \( C \) is the symmetrized Alexander polynomial of \( Y \).

Proof. Proposition 6.5 shows that \( C \) is determined by \( \theta \) and the classes \( \mu_i \) and \( \nu_j \). After a series of handleslides, we can arrange that all \( \mu_i \) and \( \nu_j \) are equal to one, fixed generator of \( H \). It follows from Proposition 6.5 that all \( C_{i,j} \) equal \( C \) up to translation. Since the Alexander polynomial \( A \) (modulo multiplication by \( \pm T^k \)) is the greatest common divisor of the \( C_{i,j} \) (c.f. Proposition 7.1), it follows that \( C \) is a translate of the symmetrized Alexander polynomial. But Proposition 6.5 also shows that \( C \) is symmetric. \( \square \)
7. The Alexander Polynomial

In the calculation of the invariant, we have met certain tori in the symmetric product, to which we associated polynomials $C_{i,j}$ and $C'_{i,j}$. The aim of this section is to relate them to the Alexander polynomial and, in Subsection 7.1, to relate them to Turaev's torsion invariant.

Recall that a Heegaard decomposition of $Y$ and complete sets of attaching circles $\{\alpha_i\}, \{\beta_i\}$ for the two handlebodies naturally give rise to tori, indexed by $i, j = 1, ..., g$

$$T_i(\alpha) = \alpha_1 \times \ldots \times \hat{\alpha_i} \times \ldots \times \alpha_g$$

and

$$T_j(\beta) = \beta_1 \times \ldots \times \hat{\beta_j} \times \ldots \times \beta_g$$

in $\text{Sym}^{g-1}(\Sigma)$. Let $\tilde{\text{Sym}}^{g-1}(\Sigma) \to \text{Sym}^{g-1}(\Sigma)$ be the covering space of $\Sigma$ corresponding to the kernel of the composite map:

$$\pi_1(\text{Sym}^{g-1}(\Sigma)) \to H_1(\Sigma) \to H_1(Y) / \text{Tors} = H.$$

Thus, $H$ acts freely on $\tilde{\text{Sym}}^{g-1}(\Sigma)$. Let $\tilde{T}_i(\alpha), \tilde{T}_j(\beta)$ be a pair of lifts of $T_i(\alpha)$ and $T_j(\beta)$. Note that these lifts are tori, and indeed they map isomorphically to $T_i(\alpha)$ and $T_j(\beta)$ respectively. The intersection points of $T_i(\alpha)$ with $T_j(\beta)$ correspond to the intersection points of $\tilde{T}_i(\alpha)$ with the various translates under $H$ of the torus $\tilde{T}_j(\beta)$.

Then, we define a polynomial (an element of $\mathbb{Z}[H]$) associated to the lifts $\tilde{T}_i(\alpha)$ and $\tilde{T}_j(\beta)$ by the formula

$$C_{i,j} = \sum_{h \in H} \# \left( \tilde{T}_j(\beta) \cap h\tilde{T}_i(\alpha) \right) [h].$$

Note that this agrees with the earlier definition of $C_{i,j}$.

**Proposition 7.1.** The Alexander polynomial of $Y$ is the greatest common divisor of the $C_{i,j}$ for $i = 1, ..., g$.

Before giving the proof, we briefly recall how to calculate the Alexander polynomial from a Heegaard decomposition. The Heegaard decomposition gives rise to a CW complex structure on $Y$, with one zero-cell in $U_0$, $g$ one-cells (one for each handle in $U_0$; i.e. these are obtained by pushing $g$ curves over $\Sigma$ which are dual to the attaching circles $\{\alpha_1, ..., \alpha_g\}$), $g$ two-cells (attached to $\Sigma$ along the attaching circles $\{\beta_1, ..., \beta_g\}$), and one three-cell. Let $\tilde{Y}$ be the maximal free Abelian cover of $Y$, i.e. the one corresponding to the kernel of

$$\pi_1(Y) \to H.$$

This space inherits a natural action of $H = H_1(Y; \mathbb{Z})/\text{Tors}$. Moreover, the lifts of the cells in $Y$ gives and $H$-equivariant CW-complex structure on $\tilde{Y}$; more precisely, choose for each cell in $Y$ a single cell in $\tilde{Y}$ which covers it (this is a fundamental family
of cells in the sense of [20]). Then these cells form a basis of the chain complex $C_\ast(\tilde{Y})$ over the group-ring $\mathbb{Z}[H]$. Thus, we can view the cellular boundary from two-chains to one-chains on $\tilde{Y}$ as a map

$$\partial: (\mathbb{Z}[H])^g \to (\mathbb{Z}[H])^g;$$

i.e. it is a $g \times g$ matrix over $\mathbb{Z}[H]$. Given $i,j$, let $\Delta_{i,j}$ be the determinant of the $(g-1) \times (g-1)$ minor obtained by deleting the $i^{th}$ row and the $j^{th}$ column of this matrix. The Alexander polynomial of $Y$, then, is the greatest common divisor of the $\Delta_{i,j}$.

**Proof of Proposition 7.1.** Let $\tilde{\Sigma}$ denote the cover of $\Sigma$ corresponding to the kernel of the map $\pi_1(\Sigma) \to H$. Note that the space $\tilde{\text{Sym}}^{g-1}_f(\Sigma)$ is the quotient of $\text{Sym}^{g-1}_f(\tilde{\Sigma})$ by the equivalence relation

$$\{x_1, ..., x_{g-1}\} \sim \{h_1 x_1, ..., h_{g-1} x_{g-1}\}$$

for all tuples $(h_1, ..., h_{g-1}) \in H^{g-1}$ with $\sum_{i=1}^{g-1} h_i = 0$.

Now, let $\{a_1, ..., a_g\}$ be the one-cells corresponding to the $\{\alpha_1, ..., \alpha_g\}$, and $\{b_1, ..., b_g\}$ be the two-cells corresponding to $\{\beta_1, ..., \beta_g\}$. Let $\{\tilde{\alpha}_1, ..., \tilde{\alpha}_g\}$ and $\{\tilde{\beta}_1, ..., \tilde{\beta}_g\}$ be lifts of the corresponding attaching circles in $\tilde{\Sigma}$; and $\{\tilde{a}_1, ..., \tilde{a}_g\}$ and $\{\tilde{b}_1, ..., \tilde{b}_g\}$ denote the corresponding lifts of the associated cells in $\tilde{Y}$. Then, the formula for the boundary map is given by

$$\partial \tilde{b}_i = \sum_{j=1}^{g} \left( \sum_{h \in H} \# (\tilde{\beta}_i \cap h \tilde{\alpha}_j) [h] \right) \tilde{a}_j.$$  

From this, then, we can obtain the identification of $C_{i,j} = \Delta_{i,j}$. For notational convenience, we write this out for $i = j = g$, but the general case follows in the same manner:

$$C_{g,g} = (-1)^{g-1} \sum_{h \in H} \# (\tilde{T}_g(\tilde{\beta}) \cap h \tilde{T}_g(\tilde{\alpha})) [h]$$

$$= (-1)^{g-1} \sum_{h \in H} \left( \sum_{h_1 + ... + h_{g-1} = h} \# (\tilde{\beta}_1 \times \times \tilde{\beta}_{g-1} \cap (h_1 \tilde{\alpha}_1 \times ... h_{g-1} \tilde{\alpha}_{g-1})) \right) [h]$$

$$= (-1)^{g-1} \sum_{h_1, ..., h_{g-1}} \# (\tilde{\beta}_1 \times \times \tilde{\beta}_{g-1} \cap (h_1 \tilde{\alpha}_1 \times ... h_{g-1} \tilde{\alpha}_{g-1})) [h_1 + ... + h_{g-1}].$$
Proposition 7.2. gives the following refinement:

Alexander polynomial. The proof of Proposition 7.1, with the underline s removed, product is given by \( (-\beta \prod \sigma) \}\) for each \( S \) where \( i, j \)

Refinements.

7.1. Refinements. We discuss two refinements in the above discussion: signs, and torsion in \( H_1(Y; \mathbb{Z}) \).

If we had used the maximal Abelian cover of \( Y \), which corresponds to the subgroup

\[ \pi_1(Y) \to H_1(Y) = H, \]

we would have obtained the polynomials \( C_{i,j} \) used in the discussion of Section 5. The reduction modulo torsion of these polynomials gives the polynomials \( C_{i,j} \) used for the Alexander polynomial. The proof of Proposition 7.1, with the underlines removed, gives the following refinement:

Proposition 7.2. The polynomial \( C_{i,j} \) is obtained from the \( H \)-equivariant boundary map of the maximal Abelian cover by taking the determinant \( \Delta_{i,j} \) of the \( i \times j \) minor of the boundary map

\[ \partial: (\mathbb{Z}[H]^g \to (\mathbb{Z}[H])^g. \]

This refinement is of interest, as the minor \( \Delta_{i,j} \) appears in the Turaev’s refinement [20] of Milnor torsion [16]. Turaev defines torsion invariant which, for three-manifolds with \( b_1(Y) > 1 \), takes the form of a function \( \tau_Y \in \mathbb{Z}[\text{Spin}^c(Y)] \). Indeed, he shows that the torsion satisfies a formula:

\[ (8) \quad \tau_Y \cdot (1 - \mu_i)(1 - \nu_j) = (-1)^{g+i+j+1} \epsilon \Delta_{i,j}[5] \]

(see Equation (4.1.a) of [20]) for some appropriate sign \( \epsilon = \pm 1 \) and a carefully chosen \( \text{Spin}^c \) structure over \( Y \). Note that we have departed slightly from Turaev’s notation: he defines (for manifolds with \( b_1 > 1 \)) an element \( \tau(Y, s) \in \mathbb{Z}[H] \) which depends on a choice of what he calls an Euler structure, which he shows to be equivalent to a \( \text{Spin}^c \) structure. Then, the element \( \tau_Y \in \mathbb{Z}[\text{Spin}^c(Y)] \) defined by

\[ \tau_Y = \tau(Y, s)[5] \]
is independent of the Spin\(^c\) structure used in its definition. Equivalently, \(\tau_Y\) is given by
\[
\tau_Y = \sum_{s \in \text{Spin}^c(Y)} T(s)[s],
\]
where \(T\) is Turaev’s Torsion function from §5 of [20]. Theorem 1.5 is obtained easily by comparing Equation (8) with Equation (7). However, to make the signs explicit we must make explicit the signs which go into the definition of \(\theta\), then those which go into the relationship between it and the intersection number of the tori from Proposition 5.3.

We now turn to the signs in the identification between \(\theta\) and the intersection numbers of Proposition 5.3 (and Proposition 6.2). The cobordisms between the moduli spaces \(M_{u_1,u_2}\) can be thought of as cobordisms arising from homotopies of \(\Psi, \Psi_{u_1,u_2}(D, s, t) = (\Theta_{h_2}(D) + u_1 \psi(s) \alpha^*_i, \Theta_{h_1}(D) - u_2 \psi(1-t)\beta^*_j).\)

As such, the fibers are seen to be cobordant to the fibers of the map (for large \(u_i\))
\[
\Psi_\infty: \text{Sym}^{g-1}(\Sigma) \times [0,1] \times [0,1] \to S^1_{\alpha_1} \times ... \times S^1_{\alpha_g} \times S^1_{\beta_1} \times ... \times S^1_{\beta_g}
\]
\[
(D, s, t) \mapsto (\Theta_{h_2}(D) + u_1 s \alpha^*_i, \Theta_{h_1}(D) - u_2 (1-t) \beta^*_j).
\]
In turn, these fibers are oriented in the same manner as the fibers of the map
\[
\text{Sym}^{g-1}(\Sigma) \to S^1_{\alpha_1} \times ... \times S^1_{\alpha_g} \times S^1_{\beta_1} \times ... \times S^1_{\beta_g} / S^1_{\alpha_i} \times S^1_{\beta_j}
\]
given by
\[
D \mapsto (\Theta_{h_0}(D), \Theta_{h_1}(D))/S^1_{\alpha_i} \times S^1_{\beta_j}.
\]
The map from
\[
\left(S^1_{\alpha_1} \times ... \times S^1_{\alpha_g} \times S^1_{\beta_1} \times ... \times S^1_{\beta_g}\right) \times \left(S^1_{\alpha_1} \times ... \times S^1_{\alpha_g} \times S^1_{\beta_1} \times ... \times S^1_{\beta_g}\right)
\]
to the quotient torus has degree \((-1)^{i+j+g+1}\). The preimage of composing \(\Theta\) with the map to \(S^1_{\alpha_1} \times ... \times S^1_{\alpha_g} \times S^1_{\beta_1} \times ... \times S^1_{\beta_g}\), obtained by evaluating respective holonomies, is orientation-preserving equivalent to the torus \(T_i(\alpha)\). As a consequence of the above discussion, we obtain the following precise form for the calculation of \(\theta\):
\[
\theta(1-\mu_i)(1-\nu_j) = (-1)^{i+j+g+1}\tilde{C}_{i,j},
\]
where the tori used for \(\tilde{C}_{i,j}\) are oriented as they are written (with respect to some consistent ordering for the attaching circles).

**Proof of Theorem 1.5.** From Proposition 5.6, Equation (8), and Proposition 7.2, it follows that
\[
\theta = x \cdot \tau_Y,
\]
for some class \(x \in H^2(Y; \mathbb{Z})\). Moreover, since both \(\theta\) and \(\tau_Y\) are invariant under conjugation (see Proposition 1.2 for \(\theta\) and §5 of [20] for \(\tau_Y\)), it follows that \(2x = 0\).
To compare signs, note that Turaev uses a slightly different sign conventions. For example, for a chain complex whose $C_1$ is has an oriented basis \{a_1, ..., a_g\} and $C_2$ is oriented by \{b_1, ..., b_h\} with $\delta b_i = 0$ for $i = 1, ..., h$, and $\delta b_j = a_j$ otherwise, using the homology orientation induced by \{\alpha_1, ..., \alpha_h\} and \{\beta_1, ..., \beta_h\}, the sign of the torsion over $\mathbb{R}$ is $(-1)^{(g-h)h}$, so Turaev’s sign-refined torsion has sign $(-1)^{(g-h)h+N(C)} = (-1)^{1+g}$ (here, $(-1)^N(C)$ is defined in [20]). On the other hand, this orientation of $C_1 \oplus C_2$ differs from the orientation induced from our conventions by a sign of

$$(-1)^{\frac{(g-1)(g-2)}{2} + \frac{(h-1)(h-2)}{2}}.$$

Comparing with the sign difference between $\Delta_{i,j}$ and $\tilde{C}_{i,j}$ it follows that

$$\theta = (-1)^{\frac{(h-1)(h-2)}{2}} \tau Y.$$

**Proof of Theorem 1.6.** Similarly to the above, we have that

$$\tilde{C}_{i,j} = (-1)^{1} x \cdot \Delta_{i,j}$$

The formulas relating $\theta$ with $\tilde{C}_{i,j}$ (see Equation (17)) and $T_t$, $T_{t-1}$ with $\Delta_{i,j}$ (see § 4 and 5 of [20]) imply that $T_t(s) = \theta(s + x)$ if $s$, $s + x \geq 0$, and $T_{t-1}(s) = \theta(s + x)$ if $s$, $s + x \leq 0$. Moreover, by Theorem 1.4 and the corresponding relation between $T_t$ with Milnor torsion from [20], it follows that $\tau' = \theta$, where $\tau'$ is induced from $\tau'$ in the usual manner. Since these are non-zero polynomials, it follows that $x = 0$. Now (in view of the discussion of signs given in the previous proof), this implies that $\tau' = x \theta$. Finally, since both $\tau'$ and $\theta$ are symmetric under conjugation, it follows that $2x = 0$. 

$\blacksquare$
8. Wall-Crossing for $\theta$ when $b_1(Y) = 1$

The present section is meant as a technical appendix, where we show that the definition of $\theta(s)$ is independent of the perturbation used in its definition, in the case where $b_1(Y) = 1$. Recall that this was already established in Section 2 for the case where $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}$; this is Proposition 2.9. Indeed, the arguments from that section show that there are at most two values which $\theta_{\eta_0 \times \eta_1}(s)$ can assume (for generic, small $\eta_0 \times \eta_1$), depending on the component in $H_2(Y;\mathbb{R}) - 0$ in which $\delta(\eta_0 - \eta_1)$ lies. Thus, if we fix an identification $H_2(Y;\mathbb{R}) \cong \mathbb{R}$, there are \textit{a priori} two invariants $\theta^\pm(s)$, corresponding to the sign of $\delta(\eta_0 - \eta_1)$ under the identification.

Our goal, then, is to prove the following restatement of Proposition 2.10:

**Proposition 8.1.** When $b_1(Y) = 1$, then the two invariants $\theta^+(s)$ and $\theta^-(s)$ agree.

In essence, this proposition amounts to the calculation of a “wall-crossing formula” much like the sorts of formulae one runs across in gauge theory (see [6]). In the case at hand, we have that the wall-crossing formula is trivial, which is what one expects from the analogy with Seiberg-Witten theory, as the perturbation is “small” (see [15] for a discussion of the three-dimensional Seiberg-Witten invariant).

To prove Proposition 8.1, we explicitly identify the difference, in the following lemma.

**Lemma 8.2.** The difference $\theta^+(s) - \theta^-(s)$ is given by the intersection number

$$\theta^+(s) - \theta^-(s) = \#\{(t,D) \in [0,1] \times \widetilde{\text{Sym}}^{g-1}(\Sigma) | \tilde{\Theta}_h^t(D) \in L_0(s) \cap L_1(s)\}.$$

Strictly speaking, to make sense of this intersection, we must choose a “generic” allowable path of metrics $h_t$, i.e. a path of metrics $h_t$ with the property that $h_0$ and $h_1$ are allowable for $U_0$ and $U_1$ as usual, for which the map $[0,1] \times \text{Sym}^{g-1}(\Sigma) \to J$ given by $(t,D) \mapsto \Theta_{h_t}(D)$ is transversal to the one-manifold $L_0 \cap L_1 \subset J$. We can find such a family, according to the following transversality result, whose proof is given in [18]:

**Theorem 8.3.** If $\Sigma$ is an oriented two-manifold with genus greater than 1, then the $g-1$-fold Abel-Jacobi map

$$\Theta: \text{Met}(\Sigma) \times \text{Sym}^{g-1}(\Sigma) \to J$$

is a submersion. (Here, $\text{Met}(\Sigma)$ denotes the space of all metrics on $\Sigma$.)

In particular, standard transversality theory allows us to conclude:

**Corollary 8.4.** Any smooth path of metrics $h_t$ can be approximated arbitrarily well (in $C^0$) by smooth paths $h'_t$ for which the map $[0,1] \times \text{Sym}^{g-1}(\Sigma) \to J$ given by $(t,D) \mapsto \Theta_{h'_t}(D)$ is transverse to $L_0 \cap L_1 \subset J$. 

\[ \theta^+(s) - \theta^-(s) \]
(Note that the hypothesis that $g > 1$ is not needed in the corollary; for if $g = 0, 1$, then $L_0 \cap L_1$ is automatically disjoint from the image of $\Theta$ for any metric and, in fact, Proposition 8.1 is clear.)

The formulation given in Lemma 8.2 is useful, since we can give the intersection number appearing there an interpretation in terms of the index theory, from which it can be explicitly computed. To this end, we find it convenient to use the notion of spectral flow introduced in [3]: given a one-parameter family of self-adjoint, Fredholm operators, the spectral flow is the intersection number of the (real) spectra of the operators with the zero eigenvalue. We will be interested in the case where the operators are Dirac operators coupled to Spin$^c$ connections with traceless curvature. Specifically, the set $L_0(s) \cap L_1(s)$ is canonically identified with the space of gauge equivalence classes of such connections in the Spin$^c$ structure $s$: it is empty unless $s$ is torsion, in which case it can also be identified with the circle $S^1 = H^1(Y; \mathbb{R})/H^1(Y; \mathbb{Z})$. (A Spin$^c$ connection is a connection on the spinor bundle $W$ of the Spin$^c$ structure and which is compatible with the Levi-Civita connection on the tangent bundle; and the gauge group is the space of circle-valued functions over $Y$.) The crux of the argument, then, is the following:

**Proposition 8.5.** The real spectral flow for the Spin$^c$ Dirac operator around the circle $H^1(Y; \mathbb{R})/H^1(Y; \mathbb{Z})$, thought of as parameterizing equivalence classes of traceless connections $A_t$ in some torsion Spin$^c$ structure $s$, is also calculated by the intersection number (with a factor of two):

$$SF_{S^1}(Y, A_t) = \pm 2\#\{(t, D) \in [0, 1] \times \text{Sym}^{g^{-1}}(\Sigma) | h_t(D) \in L_0(s) \cap L_1(s)\}.$$ 

**Remark 8.6.** The factor of 2 is an artifact of the complex linearity of the Spin$^c$ Dirac operator. Moreover, the sign depends on orientation conventions used.

Proposition 8.1 is an immediate consequence of this spectral flow interpretation, together with the Atiyah-Singer index theorem.

**Proof of Proposition 8.1.** A circle $[A_t]$ of gauge equivalence classes of Spin$^c$ connections in the Spin$^c$ structure $s$ over $Y$ naturally induces a Spin$^c$ structure $\tau$ on $X = S^1 \times Y$, endowed with a (gauge equivalence class of) Spin$^c$ connection whose restriction to the slice $e^{it} \times Y$ is identified with $[A_t]$. According to Atiyah-Patodi-Singer (see [3]), the spectral flow of the Dirac operator around the circle of operators $[A_t]$ is the (real) index of the Dirac operator on $S^1 \times Y$, in the Spin$^c$ structure $\tau$, which, according to the Atiyah-Singer index theorem, is in turn calculated by

$$\text{ind}(D(S^1 \times Y, \tau)) = \frac{c_1(\tau)^2}{4} - \frac{\sigma}{4},$$

where $\sigma$ is the signature of the intersection form of $S^1 \times Y$. In fact, the index vanishes, since the signature $\sigma$ of $S^1 \times Y$ vanishes, and the square of $c_1(\tau)$ is also easily seen to vanish, too, since for any fixed point $p \in S^1$, the restriction of the $c_1(\tau)$ to the
slice \( \{ p \} \times Y \) is \( c_1(\mathfrak{s}) \), which is a torsion class. Thus, in light of Lemma 8.2 and Proposition 8.5 the difference in the invariants must vanish.

We dispense first with the proof of Lemma 8.2, and then return to the more involved Proposition 8.5.

**Proof of Lemma 8.2** Fix a path of perturbations

\[ \varphi: [-1, 1] \longrightarrow G \subset Q_0 \times Q_1 \]

\[ \varphi(t) = (\eta_0(t) \times \eta_1(t)) \]

for which \( \delta(\eta_0(t) - \eta_1(t)) \) is a monotone increasing function of \( t \), which crosses 0 at \( t = 0 \) (here, \( G \) is the neighborhood defined in Proposition 2.7).

According to the transversality result (Theorem 8.3), we can find a one-parameter family of metrics \( h_t \) so that

\[ \Psi: [0, 1] \times [0, 1] \times \text{Sym}^{g-1}(\Sigma) \longrightarrow Q_0 \times Q_1 \]

is transverse to \( \varphi_t \). The set

\[ \Psi^{-1}(\varphi[-1, 1]) \cap \{(s, t, D) | s \leq t\} \]

is a one-dimensional manifold-with-boundary, whose boundary is

\[ \#\partial \Psi^{-1}(\varphi[-1, 1]) = \#\Psi^{-1}(\varphi(1)) \cap \{(s, t, D) | s \leq t\} \]

\[-\#\Psi^{-1}(\varphi(-1)) \cap \{(s, t, D) | s \leq t\} \]

\[-\#\Psi^{-1}(\varphi[-1, 1]) \cap \{(s, s, D)\} \]

The points in these sets are partitioned naturally into Spin\( ^c \) structures. For a fixed Spin\( ^c \) structure \( \mathfrak{s} \), the signed number of points in the first two sets calculates \( \theta^+(\mathfrak{s}) \) and \( \theta^-(\mathfrak{s}) \) respectively while the intersections in the third set all occur at \( \varphi(0) \), and indeed they correspond to

\[ \#\{(t, D) \in [0, 1] \times \text{Sym}^{g-1}(\Sigma) | \widetilde{\Theta}_{h_t}(D) \in L_0(\mathfrak{s}) \cap L_1(\mathfrak{s})\} \]

The lemma follows.

The proof of Proposition 8.5 uses splitting techniques for spectral flow (see [22] and [13]): the spectral flow around the circle \( H^1(Y; \mathbb{R})/H^1(Y; \mathbb{Z}) \) has a contribution from the handlebodies and from the cylinder \( \Sigma \times \mathbb{R} \). Formal properties (reminiscent of the special case of Proposition 2.10 proved in Section 1) show that the contribution from the handlebodies vanishes. The spectral flow on the cylinder is then identified with the intersection number, in a manner akin to Yoshida’s algorithm for calculating the instanton Floer grading [23].

We turn our attention, then, to the splitting of spectral flow. We will consider the spectral flow of the Spin\( ^c \) Dirac operator on various three-manifolds \( Z \), fixing the metric, and varying the Spin\( ^c \) connection \( A \), but keeping its curvature form to be traceless. For a fixed metric and Spin\( ^c \) structure \( \mathfrak{s} \), the set of gauge equivalence
classes of such connections, is analogous to the Jacobian of a Riemann surface: it is (non-canonically) identified with the torus $H^1(Z; S^1)$. If $s$ is actually induced from a spin structure, then this spin structure gives a canonical identification between the two sets.

Over the cylinder $\mathbb{R} \times \Sigma$ given a product metric, a Spin$^c$ structure amounts to a Spin$^c$ structure on $\Sigma$, which in turn corresponds to a line bundle $E$ over $\Sigma$ (by tensoring $E$ with the canonical Spin$^c$ structure on $\Sigma$). Moreover, a Spin$^c$ connection corresponds to a connection on the line bundle $\mathbb{R} \times E$ over $\mathbb{R} \times \Sigma$. With respect to the canonical splitting of the spinor bundle over the cylinder $W = E \otimes (\mathbb{C} \oplus K^{-1}_\Sigma)$, the Dirac operator on the cylinder can be written as

$$\mathcal{D}_A = \frac{\partial}{\partial t} + \sqrt{2}\begin{pmatrix} 0 & \overline{\nabla}_A \\ -\nabla_A & 0 \end{pmatrix},$$

where all derivatives here mean covariant derivatives coupled to $A$ (so that the operator in the second term of the above decomposition is the Spin$^c$ Dirac operator on $\Sigma$). The curvature of the determinant line bundle vanishes iff the corresponding connections on $E$ over $\Sigma$ have normalized curvature form, in the sense of Section 2.

Suppose that $Y$ is a three-manifold with a Heegaard splitting, which we write as

$$Y = U_0 \cup_{\Sigma_0} ([0, 1] \times \Sigma) \cup_{\Sigma_1} U_1,$$

where, of course, the surfaces $\Sigma_0$ and $\Sigma_1$ are topologically identified with $\Sigma$. A path $h_t$ of metrics over $\Sigma$, which is constant near $t = 0$ and $t = 1$, gives rise to a metric on the cylinder $[0, 1] \times \Sigma$, given by the formula $dt^2 + h_t$, which is product-like near the boundary. Fix any metric over $U_0$ (resp. $U_1$) with boundary isometric to $\Sigma$ with metric $h_0$ (resp. $h_1$). Then, these data naturally glue together to give a metric on $Y$.

Suppose $A$ is a Spin$^c$ connection over $Y$ with traceless curvature, and for both $i = 0, 1$, the metric $h_i$ is $U_i$ allowable. Then, the (two-dimensional) Dirac operator on the boundaries of the three pieces of the decomposition of $Y$ of Equation (10) have no kernel. In this general situation, Atiyah-Patodi-Singer (see [2]) show that the restriction of the Dirac operator to the three individual pieces with APS boundary conditions is a Fredholm operator. Thus, if we have a one-parameter family of connections $A_t$ on $Y$ whose curvature has vanishing trace, it makes sense to speak of the spectral flow of the Dirac operators restricted to these three pieces. Indeed, a fairly elementary version of the splitting technology for spectral flow gives the following:

**Proposition 8.7.** Let $Y$ be a three-manifold decomposed (metrically) as in Equation (10), with $U_i$-allowable metrics on $\Sigma_0$ and $\Sigma_1$. Let $[A_i]$ be any closed path of gauge equivalence classes of Spin$^c$ connections over $Y$ with traceless curvature. Then, the spectral flow of the Dirac operator coupled to the $[A_i]$ splits as a sum of the spectral flows of the Dirac operator restricted to the three pieces (with APS boundary conditions):

$$\text{SF}(Y, [A_t]) = \text{SF}(U_0, [A_t|_{U_0}]) + \text{SF}([0, 1] \times \Sigma, [A_t|_{[0, 1] \times \Sigma}]) + \text{SF}(U_1, [A_t|_{U_1}]).$$
The above result is standard (see for example [4], or [5] for a more general result). It is proved by showing that for metrics on $Y$ with sufficiently long cylinders $[-T, T] \times \Sigma$ inserted around $\Sigma_0$ and $\Sigma_1$, the small eigenmodes on $Y$ are approximated by the small eigenmodes of the operators restricted to the individual pieces, under a splicing construction. Since the spectral flow around the circle is independent of these “neck-length” parameters (by the homotopy invariance of spectral flow), we do not need to include them in the above statement of the proposition.

In fact, the only term which contributes in the above decomposition of the spectral flow is the middle term (the one over the cylinder $[0, 1] \times \Sigma$), according to the following result, which is a formal consequence of the conjugation action:

**Lemma 8.8.** Let $U$ be a handlebody which bounds the surface $\Sigma$, equipped with a metric which is product-like near its boundary, where it induces a $U$-allowable metric. The spectral flow of the Dirac operator vanishes around any closed path $[A_t]$ of gauge equivalence classes of Spin$^c$ connections, all of whose curvature is traceless.

**Proof.** Since $H^2(U; \mathbb{Z}) = 0$, there is a unique Spin$^c$ structure over $U$. Moreover, there is a complex-antilinear involution

$$j : W \rightarrow W$$

of the spinor bundle which commutes with Clifford multiplication (actually, this involution exists in much more general contexts, and can be thought of as the basis for the conjugation action on the set of Spin$^c$ structures described in Section 1). It follows that if $B$ is the connection on $W$ coming from a spin structure $s_0$ on $Y$, $a \in \Omega^1(U)$, then

$$D_{B+ia}(j\Psi) = jD_{B-ia}(\Psi).$$

We can express any given closed path $[A_t]$ as $[B + i a_t]$, where $\{a_t\}$ is a one-parameter family of closed one-forms which induces a closed path $[a_t]$ in $H^1(U; S^1)$; and the homotopy invariance of the spectral flow ensures that the spectral flow of the Dirac operator around $[A_t]$ depends only on the free homotopy class of $[a_t] \subset H^1(Y; \mathbb{R})/H^1(Y; \mathbb{Z})$ (in particular, it is independent of the spin structure). Now, the conjugation symmetry gives us that

$$\text{SF}(U, [B + ia_t]) = \text{SF}(U, [B - ia_t]),$$

but these two spectral flows have opposite signs: the path $[-a_t]$ is homotopic to the path $[a_t]$ given the opposite orientation. Thus, the spectral flow around the $[A_t]$ must vanish.

**Proof of Proposition 8.5.**

In view of Proposition 8.7 and Lemma 8.8, the spectral flow over $Y$ is determined by the spectral flow around a loop over the cylinder $[0, 1] \times \Sigma$. So we turn our attention now to the study of the spectral flow over a cylinder. In fact, it is useful to
consider a more general setting – spectral flow along a not necessarily closed path of operators on the cylinder \([0, 1] \times \Sigma\).

Let \((h_t, A_t)_{t \in [0, 1]}\) be a path of metrics over \(\Sigma\) and connections \([A_t] \in J_{h_t}\), both of which are constant near the \(t = 0, 1\) endpoints. We can canonically extend the paths \((h_t, A_t)\) for all \(t \in \mathbb{R}\), so that they remain constant for \(t \leq 0\) and \(t \geq 1\). Suppose, now, that \(A_0\) does not lie in the \(h_0\)-theta divisor and similarly, \(A_1\) does not lie in the \(h_1\)-theta divisor. Then, the associated Dirac operator on \([0, 1] \times \Sigma\) – the one for the metric \(dt^2 + h_t\) and the spin connection obtained by viewing the path of connections \([A_t]\) as a single connection over \([0, 1] \times \Sigma\) – is a Fredholm operator on \(L^2\). Suppose moreover that each pair \((A_t, h_t)\) for \(t \in [0, 1]\) misses the \(h_t\)-theta divisor. Then, if we rescale the family in the \(\mathbb{R}\) direction to move sufficiently slowly, then the Dirac operator on the cylinder \(\mathbb{R} \times \Sigma\) will have no kernel (this follows from a standard adiabatic limit argument, a proof is given in Proposition 5.1 of [18]). Moreover, if we have a two-parameter family:

\[ H : [0, 1] \times [0, 1] \to \text{Met}(\Sigma) \times J, \]

where the boundary misses the theta divisor, then we get a one-parameter family of self-adjoint, Fredholm operators \(D(s)\) indexed by \(s \in [0, 1]\) which we get from \(H(s, t)\) by fixing the \(s\) coordinate and allowing \(t\) to vary. According to the adiabatic limit statement, the spectral flow vanishes if \(H\) always misses the theta divisor. Indeed, by the homotopy invariance of the spectral flow and the connectedness of the theta divisor, the spectral flow depends only on the homological intersection number of \(H\) with the theta divisor. This proves that

\[ \text{SF}(D(s)) = \mu \cdot \#(H \cap \Theta), \]

for some integer \(\mu\), which \textit{a priori} depends only on the genus \(g\) of \(\Sigma\) (which we suppress from the notation wherever it is convenient).

To determine \(\mu\), we consider a simple model case: we construct a two-parameter family of metrics and connections which intersects the theta divisor once, extend it naturally over a three-manifold, calculate the Chern class of the extension, and then compare with the result obtained from the Atiyah-Singer index theorem to calculate the spectral flow. View the surface \(\Sigma\) as a connected sum of \(g\) tori \(F_1, \ldots, F_g\), and let \(\{\alpha_1, \ldots, \alpha_g\}\) be a complete set of attaching circles with \(\alpha_i\) supported in \(F_i\). Also, for \(i = 1, \ldots, g\), let \(\beta_i\) be a simple closed curves in \(F_i\) so that \(\#(\alpha_i \cap \beta_i) = 1\). Consider a two-parameter family

\[ H : [0, 1] \times [0, 1] \to \text{Met}(\Sigma) \times J \]

where the metric \(h\) is held constant (we will say how it is chosen in a moment), and the holonomy of the connection associated to \(H(s, t)\) around \(\alpha_g\) is \(e^{2\pi i t}\) (independent of \(s\)), the holonomy of \(A\) around \(\beta_g\) is \(e^{-2\pi i s}\) and all the other holonomies are trivial (here, the holonomies are measured relative to a spin structure on \(\Sigma\) which extends to the handlebody determined by \(\{\alpha_1, \ldots, \alpha_g\}\)).
For metrics which are sufficiently stretched out along the connected sum curves and the attaching curves \( \{\alpha_i\}_{i=1}^g \), the image of \( H \) intersects the theta divisor transversally in a single point (with some appropriate choice of sign): this fact is closely related to Corollary 4.7. To see this, as in the proof of that proposition, one uses Proposition 4.6 to conclude that (for metrics \( h \) on \( \Sigma \) which are sufficiently stretched out) the intersection is contained in the image of a splicing map, which is \( C^1 \) close to a map

\[
F_1^c \times \ldots \times F_{g-1}^c \longrightarrow H^1(F_1; S^1) \times \ldots \times H^1(F_{g-1}; S^1) \times H^1(F_g; S^1),
\]

which is the degree one Abel-Jacobi map on the first \( g-1 \) torus factors, and constant on the final factor. Requiring the holonomies to be trivial around the \( \alpha_i \) and \( \beta_i \) for \( i = 1, \ldots, g-1 \), is equivalent to restricting to a single point in the domain. Since the degree one Abel-Jacobi map is a diffeomorphism, it follows that \( H \) indeed intersects the theta divisor transversally in a single point.

For each fixed \( s \), the family of connections \( H(s, t) \), where \( t \) varies, canonically extends as a flat connection over (at both \( t = 0 \) and \( t = 1 \)) the handlebody \( U_0 \) obtained by surgeries along the \( \alpha_i \), to give connections \( A_s \) on a line bundle \( L \) over the three manifold \( Y \) obtained as the \( g \)-fold connected sum

\[
Y = (S^1 \times S^2) \# \ldots \# (S^1 \times S^2).
\]

From the construction of \( A_s \), it is clear that its curvature vanishes on all but the final connected summand. Indeed, it is easy to see that the first Chern class \( c_1(L) \) is dual to the two-sphere in that summand. Moreover, the connections at \( s = 0 \) and \( s = 1 \) are gauge equivalent, via a gauge equivalence which extends over \( Y \). Letting \( u \) denote the gauge transformation over \( Y \). Note that the gauge transformation is non-trivial only over the final connected summand of \( Y \), where it gives a map of degree one on its circle \( \beta_g \). Now, the \( A_s \) naturally induce a connection on the line bundle \( M \) over \( S^1 \times Y \) obtained from \([0, 1] \times L \) by identifying \( \{0\} \times L \) with \( \{1\} \times L \) using the gauge transformation \( u \). From what we know about \( L \) and \( u \), it follows easily that the first Chern class of the line bundle \( M \) is Poincaré dual to \( S^1 \times \beta_g \) plus the sphere \( S^2 \) which appears in the final connected summand. Thus, tensoring any spin structure over \( S^1 \times Y \) with \( M \), we obtain a Spin\(^c\) structure \( r \) whose first Chern class is twice the first Chern class of \( M \), so according to the Atiyah-Singer index theorem, the index of the Dirac operator coupled to \( L \) is given by

\[
\text{ind} \mathcal{D}(S^1 \times Y, r) = 2.
\]

Note that this index calculates the spectral flow around the \( S^1 \)-factor, which consists only of the contribution of the cylinder (according to Proposition 8.7 and Lemma 8.8). Since the intersection number of our family with the theta divisor consisted of a single, isolated point, it follows that \( \mu = \pm 2 \) in Equation (11) (in particular, \( \mu \) is independent of the genus \( g \)).
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