A TOPOLOGY ON POINTS ON STACKS

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Abstract. For a variety over certain topological rings $R$, like $\mathbb{Z}_p$ or $\mathbb{C}$, there is a well-studied way to topologize the $R$-points on the variety. In this paper, we generalize this definition to algebraic stacks. For an algebraic stack $X$ over many topological rings $R$, we define a topology on the isomorphism classes of $R$-points of $X$. We prove expected properties of the resulting topological spaces including functoriality. Then, we extend the definition to the case when $R$ is the ring of adeles of some global field. Finally, we use this last definition to strengthen the local-global compatibility for stacky curves of Bhargava–Poonen to a strong approximation result.

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Date: May 21, 2020.

2010 Mathematics Subject Classification. Primary 14G20; Secondary 14A20, 11R56.

Key words and phrases. Stacks, Local Fields, Adeles, Non-archimedean geometry.

The work in this thesis was partly supported by Simons Foundation grants #402472 (to Bjorn Poonen) and #550033, and by National Science Foundation grant DMS-1601946.
1. Introduction

To study a variety $X$ defined over $\mathbb{Q}$, it is often useful to study the base change of this variety to the various completions of $\mathbb{Q}$. One benefit of this is that for any place $p$ of $\mathbb{Q}$, the points $X(\mathbb{Q}_p)$ can be endowed with the structure of a topological space.

In this paper, we will generalize this definition to the case of stacks. When $X$ is an algebraic stack over $\mathbb{Q}_p$ we define a natural topology on the isomorphisms classes of $\mathbb{Q}_p$ points, which we denote by $X(\mathbb{Q}_p)$. We in fact give a define a topology more generally: for a large class of topological rings $R$ and for finite type algebraic stacks $X$ over $R$, we define a natural topology on $X(R)$.

The idea for the definition comes from the fact that for $X$ and $Y$ two varieties over $\mathbb{Q}_p$ and $X \to Y$ a smooth morphism, the map $X(\mathbb{Q}_p) \to Y(\mathbb{Q}_p)$ is open, and in particular a quotient onto its image. If one assumes that a smooth map of stacks should have the same property one arrives at the definition given in Section 5.

In order for the definition to make sense, we establish in Section 6 that for any algebraic stack $X$ over $R$ and $x \in X(R)$, there is a smooth cover $Z \to X$ such $x$ lifts to a point of $Z(R)$. To do this we introduce “cocycle” spaces. If we are given a smooth cover $X \to X$ by a scheme and we wish to define a map $T \to X$ for a scheme $T$, one strategy is to find a flat cover $T' \to T$ and to give a map $T' \to Z$ and certain descent data; these cocycle spaces parameterize such maps $T' \to Z$ together with descent data.

In Section 9, we prove expected functorialities of these spaces.

In Section 10 and 11, we prove basic properties of $X(R)$ for particularly well-behaved topological rings. In particular we establish that smooth maps between stacks induce open maps on $\mathbb{R}$-points, that for separated algebraic stacks $X$ over $\mathbb{Q}_p$, $X(\mathbb{Q}_p)$ is Hausdorff, and that for proper Deligne-Mumford stacks $X$ with finite diagonal over $\mathbb{Q}_p$, $X(\mathbb{Q}_p)$ is furthermore compact.

Finally, in Section 13 we explain how if $A$ is a ring of adeles of a global field and $X$ a finitely presented stack over $A$ we can define a topology on $X(A)$. We conclude this section by giving an application of the theory, strengthening a result of Bhargava and Poonen (definitions can be found in Section 13).

**Theorem 1.0.1** (Strong approximation for stacky curves). Let $k$ be a number field. Let $X$ be a stacky curve over $k$ of genus less than $1/2$. Let $k'$ be obtained by removing one place from the adeles of $K$. Then

$$X(k) \to X(A')$$

has dense image.

A related topological space associated to a algebraic stack over topological field has been defined by Ulirsch in [Uli17]. The major difference between the topological space defined there and the one in this paper is that the space defined in [Uli17] identifies two points if they become isomorphic after a non-archimedean field extension. No such identification happens in this paper.

2. Background

The theory of algebraic spaces and algebraic stacks begins with the theory of descent. We begin by giving a brief review of this theory.

Let $R$ be a ring. Let $S$ be an $R$-algebra. For an $R$-module $M$, we may extend scalars $M \otimes_R S$ to obtain an $S$-module. Set $S_2 = S \otimes_R S$, and note there are two $R$-algebra maps $i_1, i_2 : S \to S \otimes_R S$ given by $i_1(s) = s \otimes 1$ and $i_2(s) = 1 \otimes s$. There is a canonical isomorphism $(M \otimes_R S) \otimes_{S,i_1} (S \otimes_R S)$.
\( S \rightarrow (M \otimes_R S) \otimes_{i_2, S} (S \otimes_R S) \) coming from the fact that both \((M \otimes_R S) \otimes_{S, i_1} (S \otimes_R S)\) and \((M \otimes_R S) \otimes_{S, i_2} (S \otimes_R S)\) are canonically isomorphic to \(M \otimes_R (S \otimes_R S)\).

Thus an \( S \)-module \( N \) can only potentially be obtained from an extension of scalars from an \( R \)-module if there is an isomorphism \( N \otimes_{S, i_1} (S \otimes_R S) \rightarrow N \otimes_{S, i_2} (S \otimes_R S) \).

With this in mind, we define:

**Definition 2.0.1.** Let \( R \) be a ring and \( S \) an \( R \)-algebra. An \( S \)-module with descent datum to \( R \) is a pair \((N, \psi)\) of an \( S \)-module \( N \) and an isomorphism \( \psi : N \otimes_{S, i_1} (S \otimes_R S) \rightarrow N \otimes_{S, i_2} (S \otimes_R S) \).

Using these we define a category of modules with descent data.

**Definition 2.0.2.** Let \( R \) be a ring and \( S \) an \( R \)-algebra. Let \( \text{Mod}_{\text{Desc}}_{R \rightarrow S} \) denote the category of \( S \)-modules with descent datum to \( R \), where the objects are \( S \)-modules with descent datum to \( R \), and a morphism \((N_1, \psi_1) \rightarrow (N_2, \psi_2)\) is \( S \)-module morphism \( \phi : N_1 \rightarrow N_2 \) such the following diagram is commutative:

\[
\begin{array}{c}
N_1 \otimes_{S, i_1} (S \otimes_R S) \quad \psi_1 \quad N_1 \otimes_{S, i_2} (S \otimes_R S) \\
\Downarrow \phi \quad \Downarrow \phi
\end{array}
\]

\[
N_2 \otimes_{S, i_1} (S \otimes_R S) \quad \psi_1 \quad N_2 \otimes_{S, i_2} (S \otimes_R S)
\]

If \( \text{Mod}_R \) is the category of \( R \)-modules, we have already seen there is a natural functor \( \text{Mod}_R \rightarrow \text{Mod}_{\text{Desc}}_{R \rightarrow S} \).

**Theorem 2.0.3** ([Faithfully flat descent,[Sta18, Tag 032M]]). If \( R \rightarrow S \) is a faithfully flat map of rings, \( \text{Mod}_R \rightarrow \text{Mod}_{\text{Desc}}_{R \rightarrow S} \) is an equivalence of categories.

Furthermore, we can describe a quasi-inverse. Begin with \((N, \psi) \in \text{Mod}_{\text{Desc}}_{R \rightarrow S} \). We have two maps \( N \rightarrow N \otimes_{S, i_2} (S \otimes_R S) \): the first is the natural map onto the first factor and the second is the composite \( N \rightarrow N \otimes_{S, i_1} (S \otimes_R S) \xrightarrow{\psi} N \otimes_{S, i_2} (S \otimes_R S) \). We associate the \( R \)-module \( \text{Eq}(N \Rightarrow N \otimes_{S, i_2} (S \otimes_R S)) \).

2.1. **Sheaves.** We now turn our attention to sheaves. First let \( \mathcal{C} = \text{Alg}_R^{op} \) be the opposite category to the category of \( R \)-algebras. For an \( R \)-algebra \( S \), let \( \text{Spec} \ S \) denote the associated object in \( \mathcal{C} \).

We make \( \mathcal{C} \) a site by saying that \( \text{Spec} \ S_2 \rightarrow \text{Spec} \ S_1 \) is a covering if the associated ring map \( S_1 \rightarrow S_2 \) is faithfully flat. To an \( R \) module \( M \), we can associate a presheaf on this site denote by \( \tilde{M} \) where \( \text{(Spec)}(\tilde{M}) = M \otimes_R S \). Then Theorem 2.0.3 implies that \( \tilde{M} \) is a sheaf.

Now let \( S \) be a scheme. Let \( S_{\text{fpf}} \) be the fppf site over \( S \): the objects are \( S \)-schemes, morphisms are scheme morphisms over \( S \), and coverings are faithfully flat and locally finitely presented morphisms. For any \( S \)-scheme \( X \), \( F_X(Y) = \text{Hom}_S(Y, X) \) defines a presheaf \( F_X \) on \( S_{\text{fpf}} \). Theorem 2.0.3 implies:

**Theorem 2.1.1** ([Sta18, Tag 023Q]). The presheaf \( F_X \) is a sheaf on \( S_{\text{fpf}} \).

Let us expand upon what it means for \( F_X \) to be a sheaf. Let \( T \) be another \( X \)-scheme, and let \( T' \rightarrow T \) be a faithfully flat cover. The question we seek to answer is: given a map \( T' \rightarrow X \), when is there a map \( T \rightarrow X \) making the following commute:

\[
\begin{array}{c}
T' \\
\downarrow
\end{array} \quad \begin{array}{c}
\rightarrow
\end{array} \quad \begin{array}{c}
T \rightarrow X
\end{array}
\]
That $F_X$ is a sheaf implies that $F_X(T)$ is the equalizer in the sequence $F_X(T) \to F_X(T') \rightrightarrows F_X(T \times_T T')$. Then this gives that a map $T' \to X$ comes from a map $T \to X$ if and only if two compositions $T' \times_T T' \to X$ are equal.

We give another consequence of the fact that $F_X$ is a sheaf. Let $X' \to X$ be a faithfully flat cover. Let $T$ be an $S$-scheme with a map $T \to X$. Define $T' = T \times_X X'$; then we get an associated map $T' \to X'$. The two composites $T' \times_T T' \to X'$ define a map $T' \times_T T' \to X \times_X X'$.

This map makes the following diagram commute (where the vertical arrows are projections):

\[
\begin{array}{ccc}
T' \times_T T' & \longrightarrow & X' \times_X X' \\
\downarrow & & \downarrow \\
T' & \longrightarrow & X'.
\end{array}
\tag{2.1.1}
\]

Again we can go backwards: another way we can specify a map $T \to X$ is to choose an fppf cover $T' \to X'$ and a map $T' \times_T T' \to X \times_X X'$ making Equation 2.1.1 commute.

That this defines a map $T \to X$ again follows from the fact that $F_X$ is a sheaf. This strategy will inform our method of defining maps to stacks.

2.2. Algebraic spaces and stacks. Let $X$ be an algebraic space over $S$. As algebraic spaces are quotients of schemes by étale equivalence relations, there is a scheme $Y$ with surjective étale map to $X$. If we let $R = Y \times_X Y$, then $R$ is a subscheme of $Y \times Y$ whose two projections $R \rightrightarrows Y$ are étale. Specifying the data $R \rightrightarrows Y$ specifies $X$ as the quotient of the sheaf $F_Y$ by the equivalence relation defined by $R$.

If $T$ is an $S$-scheme, one way of giving a map $T \to X$ is to find an étale cover $T' \to T$ with map $T' \to Y$ and a map $T' \times_T T' \to Y \times_X Y$ making the following diagram commute:

\[
\begin{array}{ccc}
T' \times_T T' & \longrightarrow & R \\
\downarrow & & \downarrow \\
T' & \longrightarrow & Y.
\end{array}
\]

Thus we can define a map $T \to X$ using only maps between schemes.

Now we turn to algebraic stacks. Let $\mathcal{X}$ be an algebraic stack over $S$. Then there is a scheme $X$ with a smooth cover $\pi : X \to \mathcal{X}$. Then $R = X \times_X Y$ is an algebraic space with two projections $R \rightrightarrows X$.

Let $T$ be an $S$-scheme. We may begin to give a map $f : T \to \mathcal{X}$ similarly as before, by giving a smooth cover $T' \to T$, a map $T' \to X$, and a map $f : T' \times_T T' \to X \times_X X$ making the following diagram commute:

\[
\begin{array}{ccc}
T' \times_T T' & \longrightarrow & R \\
\downarrow & & \downarrow \\
T' & \longrightarrow & X.
\end{array}
\]

However, for stacks this is not sufficient.

The map $T' \times_T T' \to X \times_X X$ defines a map $(T' \times_T T') \times_S (T' \times_T T') \to (X \times_X X) \times_S (X \times_X X)$, which we may restrict to a map $(T' \times T') \times_Y (T' \times T') \to (X \times_X X) \times_S (X \times_X X)$, and the commutativity of the last diagram guarantees the image lands in $(X \times_X X) \times_X (X \times_X X)$. Thus this defines a map $(T' \times T') \times_T (T' \times T') \to (X \times_X X) \times_X (X \times_X X)$, but this is canonically isomorphic to a morphism $T' \times_T T' \times T' \to X \times_X X \times_X X$. 
The condition for the maps $T' \to X$ and $T' \times_T T' \to R$ to define a map $T \to X$ is for the following diagram to commute (where the vertical arrows are the projections):

$$
\begin{array}{ccc}
T' \times_T T' & \longrightarrow & X \\
\downarrow & & \downarrow \\
T' & \longrightarrow & X
\end{array}
$$

**Remark 2.2.1.** For stacks the map $T' \times_T T' \to R$ is additional information, whereas for algebraic spaces the question is only whether or not one exists.

### 3. Conventions and notations

Throughout the text all topological rings are assumed to be Hausdorff. When we work over a ring $R$, all algebraic spaces, schemes, and stacks are finite-type over $R$.

For $S' \to S$ a map of schemes and $\mathcal{F}$ an fppf sheaf on the category of $S'$-schemes, denote by $\text{Res}_{S'/S} \mathcal{F}$ the sheaf such that for a $S$-scheme $T$ we have $(\text{Res}_{S'/S} \mathcal{F})(T) := \mathcal{F}(T \times_S S')$. If $\mathcal{F}$ is an fppf sheaf on the category of $S$-schemes and $\mathcal{F}_S$, its pullback to the category for $S'$-schemes, by abuse of notation we write $\text{Res}_{S'/S} \mathcal{F}$ for $\text{Res}_{S'/S} \mathcal{F}_S$. For a $S$-scheme, $X$, we will identify $X$ with its associated fppf sheaf, allowing us to write $\text{Res}_{S'/S} X$. Note that the sheaves obtained in this way are not necessarily representable.

### 4. Topologies on points of varieties

In this section $R$ is a topological ring. For a large class of $R$, we will explain how for any finite-type $R$-scheme $X$ we may define a topology on $X(R)$. The topology will satisfy the following axioms:

1. If $X = \mathbb{A}^1_R$, then $X(R) = R$.
2. If $X \to Y$ is a morphism of finite-type $R$-schemes, then $X(R) \to Y(R)$ is continuous.
3. If $X$ is a finite-type $R$-scheme and $Y \to X$ is a closed immersion, then $Y(R) \subseteq X(R)$ is a closed subset with the subspace topology.
4. If $X$ and $Y$ are two finite-type $R$-schemes, then $(X \times_R Y)(R) = X(R) \times Y(R)$.
5. If $X$ is a finite-type $R$-scheme, and $Y \to X$ is an open immersion, then $Y(R) \subseteq X(R)$ is an open subset with the subspace topology.

**Definition 4.0.1.** Let $R$ be a topological ring. A topology $X(R)$ for every finite-type $R$-scheme $X$ is called a **excellent topologization of $R$-points of finite-type $R$-schemes** if (1) – (5) hold.

Let $R$ be a topological ring. A topology $X(R)$ for every finite-type $R$-scheme $X$ is called a **good topologization of $R$-points of finite-type $R$-schemes** if (1) – (4) hold.

**Definition 4.0.2.** If $R$ is topological ring and we give $R^\times$ the subset topology, then we say $R$ is **continuously invertible** if the map $R^\times \to R^\times$ given by $r \mapsto 1/r$ is continuous.

This condition implies that if $f \in R[x_1, \ldots, x_n]$ and $U \subseteq R^n$ is such that $f(U) \subseteq R^\times$, then the map $U \to R$ given by $1/f(x_1, \ldots, x_n)$ is continuous.

By local topological ring, we mean a topological ring that is local as a ring. We will first explain how if $R$ is a continuously invertible topological ring, then $R$ has a unique excellent topologization of $R$-points of finite-type $R$-schemes.

First if $X = \mathbb{A}^n$ is affine space, then the space $X(R) = R^n$ we give product topology, which is required by properties (1) and (4). If $X \subseteq \mathbb{A}^n$ is a closed subscheme of affine space, we define the
topology as the subspace topology \( X(R) \subseteq \mathbb{R}^n \), which is required by property (3). We must show this is independent of the affine embedding.

Suppose \( X \) has two embeddings into affine spaces, \( i_1 : X \hookrightarrow \mathbb{A}^n \) and \( i_2 : X \hookrightarrow \mathbb{A}^m \). Then there is a morphism \( r_1 : \mathbb{A}^n \to \mathbb{A}^m \) such that \( r_1 \circ i_1 = i_2 \). Similarly, there is a morphism \( r_2 : \mathbb{A}^m \to \mathbb{A}^n \) such that \( r_2 \circ i_2 = i_1 \). As polynomial maps are continuous, \( r_1 \) and \( r_2 \) induce continuous maps between the two topologies on \( X(R) \) defined by the two embeddings. Thus the topologies must be the same.

Next if \( X = \text{Spec} \ A \) for an finitely generated \( R \)-algebra \( A \), and \( f \in A \) and \( U = \text{Spec} \ A[1/f] \subseteq X \) is a distinguished open, we would like to see that the topology on \( U(R) \) induced as a subspace of \( X(R) \) is same as the topology on \( U(R) \) when \( U \) is viewed as an affine scheme. Choose a closed embedding \( X \hookrightarrow \mathbb{A}^n \) and let \( x_1, \ldots, x_n \) be the coordinates on \( \mathbb{A}^n \). Let \( U_{\text{sub}} \) be the set \( U(R) \) equipped with the subspace topology as a subspace \( U(R) \subseteq X(R) \subseteq \mathbb{R}^n \). Let \( U_{\text{aff}} \) be the set \( U(R) \) equipped with the topology of \( U \) as an affine scheme. Now we check that \( U \subseteq \mathbb{A}^{n+1} \) as a closed subspace since \( U \cong \text{Spec} \ A[x_{n+1}]/(x_{n+1} f - 1) \). The map \( (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 1/(f(x_1, \ldots, x_n))) \) is defined on an open subset of \( \mathbb{A}^n \) mapping to \( \mathbb{A}^{n+1} \) and restricts to a continuous map \( U_{\text{sub}} \to U_{\text{aff}} \). The projection onto the first \( n \) coordinates gives a map \( \mathbb{A}^{n+1} \to \mathbb{A}^n \) which gives a continuous map \( U_{\text{aff}} \to U_{\text{sub}} \). Thus both must have the same topologies.

Let \( X \) be such a finite-type scheme. First note any \( R \)-point of \( X \) is contained in an affine subset of \( X \). On each affine \( U \subseteq X \), give \( U(R) \) topology as an affine subset; this is required by (5)s. We must check that for two affines, \( U, V \), the topologies agree on \( U \cap V \). But \( U \cap V \) is covered by opens which are distinguished in both \( U \) and \( V \). On these distinguished opens, we have shown that the topology induced from \( U(R) \) and \( V(R) \) match the topology given by viewing these distinguished opens as affine varieties; thus the topologies match on the overlap, so these match we have defined a topology on \( X(R) \). It is straightforward to check properties (1) – (5).

Now we consider a more general class of rings. Let \( I \) be an index set, and for each \( i \in I \) let \( R_i \) be a continuously invertible topological local ring. Let \( R = \prod_{i \in I} R_i \) be the product. We will describe how to give a good topologization on the \( R \)-points of finite-type \( R \)-schemes for this class of \( R \). For any cofinite subset \( J \subseteq I \), we set \( R_J = \prod_{i \in J} R_i \). Then \( R = (\prod_{i \in \setminus J} R_i) \times R_J \).

As \( R_i \) is a continuously invertible topological local ring, we have already described a topologization of the \( R_i \)-points of any finite-type \( R_i \)-scheme. Let \( X \) be a finite-type \( R \)-scheme. For any cofinite \( J \subseteq I \), and open \( U \subseteq \prod_{i \in \setminus J} X(R_i) \), consider \( U \times X(R_J) \subseteq (\prod_{i \in \setminus J} X(R_i)) \times X(R_J) = X(R) \); we define the topology on \( X(R) \) to have these subsets as a basis.

We check properties (1) and (2). First if \( X = \mathbb{A}^1 \), then \( X(R) = R \) as sets. A basis for the topology is given by sets of the form \( U \times R_J \) for cofinite subsets \( J \subseteq I \) and open \( U \subseteq \prod_{i \in \setminus J} \mathbb{A}^1(R_i) \); this is precisely the topology on \( R \) as a direct product so (1) is established.

Let \( f : X \to Y \) be a morphism of finite-type \( R \)-schemes. Let \( J \subseteq I \) a cofinite subset and let \( U \subseteq \prod_{i \in \setminus J} Y(R_i) \) be an open. Let \( g : \prod_{i \in \setminus J} X(R_i) \to \prod_{i \in \setminus J} Y(R_i) \) be induced from \( f \); this is continuous as each \( R_i \) is continuously invertible and local and we have checked the topologization is excellent for these rings. Therefore, \( g^{-1}(U) \) is open in \( \prod_{i \in \setminus J} X(R_i) \). We then have that \( f^{-1}(U \times Y(R_J)) = g^{-1}(U) \times X(R_J) \) and this is open in \( X(R) \) as it is in the described basis of opens. Therefore \( X(R) \to Y(R) \) is continuous, so we have (2).

To check property (3), let \( X \to Y \) be a closed immersion of finite-type \( R \)-schemes. We first check that image is a closed set. The image is the intersection over \( j \in I \) of \( X(R_J) \times \prod_{i \in I \setminus j} Y(R_i) \), which are each closed, so the image is closed. Now we check that \( X(R) \to Y(R) \) is a topological immersion. For any \( i \), \( X(R_i) \to Y(R_i) \) is a closed immersion, thus a subset of \( X(R_i) \) is open if and only if it is the restriction of an open subset of \( Y(R_i) \). For any cofinite \( J \) and open subsets
Let \( R \) be a local ring with the property that smooth maps of \( \text{finite-type}, \text{quasi-separated} \)-schemes defined above has the smooth quotient property. Thus identifying \( (X \times_R Y) \) and \( X \) \( R \)-points by hypothesis. Therefore \( f(W) = f(W) \times Y(R_j) \) is open. The set \( f(U) \) is a union of these \( f(V) \) and is thus open as claimed. 

6. Definitions

Definition 5.0.1. Let \( S \) be a scheme and \( T \) be \( S \)-scheme. Let \( n \geq 0 \)

If \( S \) is the spectrum of field \( k \), then \( T = \text{Spec} \, R \) for some \( R \), and we define the degree of \( T \) over \( S \) to be \( \dim_k \, R \), the dimension of \( R \) as a \( k \)-vector space.

For general scheme \( S \), and for any \( n \geq 0 \), we say that \( T \to S \) has degree less than or equal to \( n \) if for each \( s \in S \), if \( k(s) \) is the residue field at \( s \), then \( T \times_S \text{Spec} \, k(s) \) has degree less than or equal to \( n \).

If \( R \to R' \) is a morphism of rings such that \( \text{Spec} \, R' \to \text{Spec} \, R \) is quasi-finite, we say that \( R \to R' \) has degree less than or equal to \( n \) if \( \text{Spec} \, R' \to \text{Spec} \, R \) has degree less than or equal to \( n \).

Remark 5.0.2. If \( T \to S \) is a quasi-finite and representable map of algebraic stacks, then for every field \( k \) with map \( \text{Spec} \, k \to S \), the pullback \( T \times_S \text{Spec} \, k \) is a scheme. In this we can extend the definition of having degree less than or equal to \( n \) to this situation.
The map $T \to S$ is of degree less than or equal to $n$ if for all fields $k$ and all morphisms $\text{Spec} k \to S$, $T \times_S \text{Spec} k \to \text{Spec} k$ is of degree less than or equal to $n$.

**Definition 5.0.3.** Let $R$ be a ring. We say that $R$ is sufficiently disconnected

1. Finitely generated projective modules over $R$ are free.
2. For any faithfully flat étale $R \to R'$, there exists an $R'$-algebra $R''$ such that $R''$ is finite étale over $R$.

**Lemma 5.0.4.** Let $R$ be a sufficiently disconnected ring. For any $n \geq 0$ and any faithfully flat étale $R \to R'$ of degree less than or equal to $n$, there exists an $R'$-algebra $R''$ such that $R''$ is finite étale and free over $R$ of rank $n!$.

**Proof.** Given $R \to R'$, let $S$ be finite étale $R'$-algebra given by Definition 5.0.3.

Let $T \subseteq S$ be the image of $R'$. Applying Lemma 14.0.6, we conclude that $T$ is étale over $R$. Since $R'$ and $S$ are étale over $R$ they are finitely presented over $R$, so there is a noetherian subring over which $R'$ and $S$ are defined; $T$ is then defined too over that noetherian subring, and since it is a subalgebra of the module finite $S$, we conclude that $T$ too is finitely presented as an $R$-module. As $T$ is both finitely presented and flat over $R$ as an $R$-module, $T$ is a projective $R$-module. Using property (1) of sufficiently disconnected, we see that $T$ is moreover free as an $R$-module. Since $T$ is a quotient of $R'$ it must have rank $m$ less than or equal to $n$. Let $R'' = T^{n!/m}$. Then the diagonal defines an $R$-algebra homomorphism $T \to R''$ and the composite $R' \to T \to R''$ makes $R''$ an $R'$-algebra. Finally, by construction $R''$ is finite étale and free of rank $n!$ over $R$. □

The following is a scheme-theoretic version of Lemma 5.0.4.

**Lemma 5.0.5.** Let $R$ be a sufficiently disconnected ring. Let $U \to V$ be a separated, surjective étale map of algebraic stacks over $R$ which is representable and of degree less than or equal to $n$. Then for any $v \in V(R)$, there is a finite étale $R$-algebra $S$ of rank $n!$ and $u \in U(S)$ such that $u$ and $v$ have the same image in $V(S)$.

**Proof.** Consider the pullback $U \times_{V,v} \text{Spec} R \to \text{Spec} R$. This is a separated algebraic space that surjective étale over $\text{Spec} R$. If we find a finite étale $R$-algebra $S$ of rank $n!$ and $u' \in (U \times_{V,v} \text{Spec} R)(S)$, then if we denote the composite $\text{Spec} S \to U \times_{V,v} \text{Spec} R \to U$ by $u$, $u$ has the desired properties. In this way we reduce to the case when $U$ is a separated algebraic space and $V = \text{Spec} R$.

Now let $W \to U$ be an étale cover by an affine scheme. Then $W = \text{Spec} R'$ for some étale $R$-algebra $R'$. Applying Definition 5.0.3 gives an $R'$-algebra $R''$ which is finite étale over $R$. By Remark 14.0.7, the image of $\text{Spec} R'' \to U$ is finite étale over $R$, and thus an affine $R$-scheme. This image is of the form $\text{Spec} S'$ for some $R$-algebra $S'$. We have a map $u' : \text{Spec} S' \to U$ by construction. Now we may take a finite partition of $\text{Spec} R$ by open sets such that over each open set in the partition $\text{Spec} S'$ has constant degree. It suffices to prove the result separately over each open in the partition. Thus passing to an open set in this partition, we may assume that $S'$ is finite étale of some fixed rank $m$ over $R$. Set $S = (S')^{n!/m}$. The diagonal defines a map $S' \to S$, we have a composite map $\text{Spec} S \to \text{Spec} S' \to U$ which we denote by $u$. Then $S$ and $u$ have the desired properties. □

**Proposition 5.0.6.** Complete noetherian local rings are sufficiently disconnected.

Before we prove this, we prove an intermediary lemma.

**Lemma 5.0.7.** Let $R$ be a complete noetherian local ring. Let $R'$ be an étale $R$-algebra. Then $R'$ is of the form $T \times S$ where $T$ is finite étale over $R$ and $S$ is étale but not faithfully flat over $R$. 


Proof. Let \( m \) be the maximal ideal of \( R \). Let \( \widehat{R}' \) be the \( m \)-adic completion of \( R' \).

Because \( R \to R' \) is étale, \( R'/m^sR' \) is étale over \( R/m^sR \). Furthermore, \( R'/m^sR' \) is finite over \( R/m^sR \) as it is quasi-finite over an artinian local ring. Thus, we have that \( \widehat{R}' \) is finite étale over \( R = \widehat{R} \). Set \( T = \widehat{R}' \).

Now as \( R' \to \widehat{R}' \) is surjective, by Nakayama’s lemma we conclude that \( R' \to T \) is surjective. By Lemma 14.0.6, Spec \( \widehat{R}' \to \text{Spec } R' \) has clopen image. Therefore, we may write \( R' = \widehat{R}' \times S \) for some (necessarily étale) \( R \)-algebra \( S \). By construction \( R'/mR' \to T/mT \) is surjective, so \( S/mS = 0 \), so as claimed \( S \) is not faithfully flat over \( R \). \( \square \)

Proof of Proposition 5.0.6. Let \( R \) be a complete noetherian local ring with maximal ideal \( m \). Property (1) of Definition 5.0.3 is clear, so we prove property (2).

Let \( R' \) be an étale faithfully flat \( R \)-algebra. We will find an \( R' \)-algebra \( R'' \) such that \( R \to R'' \) is finite étale.

By Lemma 5.0.7, \( R' = T \times S \) for \( T \) finite étale over \( R \). We take \( R'' = T \). \( \square \)

Definition 5.0.8. Let \( R \) be a topological ring. We say \( R \) is essentially analytic if \( R \) is a local, sufficiently disconnected, continuously invertible topological ring such that every étale map \( X \to Y \) of finite-type \( R \)-schemes induces a local homeomorphism \( X(R) \to Y(R) \).

Remark 5.0.9. Essentially analytic rings include \( \mathbb{Z}_p, \mathbb{Q}_p, \mathbb{R}, \) and \( \mathbb{C} \).

For these rings, the fact that they are essentially analytic comes from the inverse function theorem.

Here is the main definition of the paper.

Definition 5.0.10. Let \( R \) be an essentially disconnected, continuously invertible topological ring. Let \( \mathcal{X} \) be an algebraic \( R \)-stack. We topologize \( \mathcal{X}(R) \) in the following way:

For each smooth cover \( Z \to \mathcal{X} \) from an \( R \)-scheme \( Z \), let \( \mathcal{X}(R)_Z \) be the image of \( Z(R) \) in \( \mathcal{X}(R) \) given the quotient topology.

Let \( \mathcal{C}_\mathcal{X} \) be the category of \( R \)-schemes \( Z \) together with a smooth cover \( Z \to \mathcal{X} \), and maps from \( Z_1 \to \mathcal{X} \) to \( Z_2 \to \mathcal{X} \) are smooth maps \( Z_1 \to Z_2 \) making the following diagram commute:

\[
\begin{array}{ccc}
Z_1 & \to & Z_2 \\
\downarrow & & \downarrow \\
\mathcal{X} & & \\
\end{array}
\]

Then let \( \mathcal{X}(R) \) be given the topology of \( \text{colim}_{Z \in \mathcal{C}_\mathcal{X}} \mathcal{X}(R)_Z \).

For this definition to make sense, we will need to see that every \( x \in \mathcal{X}(R) \) is in \( \mathcal{X}(R)_Z \) for some \( Z \). Theorem 7.0.7 establishes this.

Remark 5.0.11. Note that in Definition 5.0.10, we get the same colimit if we restrict to morphisms of the form \( Z_1 \to Z_2 \), for smooth cover \( Z_2 \to \mathcal{X} \). In this way, if convenient, we may assume that the colimit is over a filtered category.

6. Cocycle spaces

In this section \( R \) is a ring and all objects are finite-type over \( R \). Additionally throughout this section, \( T \) will be an \( R \)-scheme, \( T' \to T \) will be an fppf cover, \( \mathcal{X} \) will be an algebraic stack over \( T \), and \( Z \) an algebraic space over \( T \), and \( Z \to \mathcal{X} \) will be a smooth cover. We briefly recall Section 2.2.

Let us be given maps \( T' \to Z \) and \( T' \times_T T' \to Z \times_{\mathcal{X}} Z \) making the following diagram commute:
\[ T' \times_T T' \longrightarrow Z \times_Z Z \]
\[ T' \longrightarrow Z. \]  
\hspace{1cm} \text{(6.0.1)}

From this data we get a map \( T' \times_T T' \times_T T' \to Z \times_Z Z \times_Z Z \)
If the diagram
\[ T' \times_T T' \times_T T' \longrightarrow Z \times_Z Z \times_Z Z \]
\[ \begin{array}{c}
\pi_{23} \\
\pi_{23}
\end{array} \]
\[ T' \times_T T' \longrightarrow Z \times_Z Z \]
\hspace{1cm} \text{(6.0.2)}

commutes, we get an associated map \( T' \to X \).

Now a map \( T' \to Z \) is the same data as a \( T \)-point of \( \text{Res}_{T'/T}Z \). A map \( T' \times_T T' \to Z \times_Z Z \) is the same as the data of a \( T \)-point of \( \text{Res}_{T' \times_T T'/T}(Z \times_Z Z) \). There are two maps \( \pi_1, \pi_2 : \text{Res}_{T'/T}Z \to \text{Res}_{T' \times_T T'/T}(Z \times_Z Z) \) coming from the two projections \( T' \times_T T' \to T' \). There are also two maps \( \pi_1, \pi_2 : \text{Res}_{T' \times_T T'/T}(Z \times_Z Z) \to \text{Res}_{T' \times_T T'/T}Z \) coming respectively from the two projections \( Z \times_Z Z \to Z \).

Finally, the data of \( T' \to Z \) and \( T' \times T' \to Z \times_Z Z \) making Diagram 6.0.1 commute can thus be rephrased as the data of a \( T \)-point of the limit, \( P_{T', Z \to X} \), of the following diagram:

\[ \begin{array}{c}
\pi_1 \\
\pi_2
\end{array} \]
\[ \text{Res}_{T'/T}Z \quad \pi_1 \]
\[ \text{Res}_{T'/T}Z \times_Z Z \]
\[ \pi_2 \quad \text{Res}_{T' \times_T T'/T}Z \\
\pi_2 \quad \text{Res}_{T' \times_T T'/T}Z \]
\hspace{1cm} \text{(6.0.3)}

We get maps \( d_u, d_t : P_{T', Z \to X} \to \text{Res}_{T' \times_T T' \times_T T'}(Z \times_Z Z) \) corresponding to the upper and lower paths of Diagram 6.0.2. Let \( \mathcal{C}_{T', Z \to X} \) be the equalizer of

\[ P_{T', Z \to X} \quad d_u \\
\quad d_t \]
\[ \text{Res}_{T' \times_T T' \times_T T'}Z \times_Z Z. \]

As it stands this \( \mathcal{C}_{T', Z \to X} \) is only a sheaf. If it is representable by a scheme or algebraic space, we will refer to \( \mathcal{C}_{T', Z \to X} \) as a cocycle space.

Descent gives us a map \( \mathcal{C}_{T', Z \to X} \to X \) (see Section 2.2).

**Proposition 6.0.1.** Let \( \mathcal{Y} \to X \) be a map of algebraic stacks over \( T \). If \( Z \times_X \mathcal{Y} \) is an algebraic space, then there is a canonical isomorphism \( \mathcal{C}_{T', Z \to X \times_X \mathcal{Y}} \cong \mathcal{C}_{T', (Z \times_X \mathcal{Y}) - \mathcal{Y}} \).

The reason for assuming that \( Z \times_X \mathcal{Y} \) is an algebraic space is that this is the only context in which we have a definition of \( \mathcal{C}_{T', (Z \times_X \mathcal{Y}) - \mathcal{Y}} \).

**Proof.** We will describe a map in both directions. As the construction of the cocycle space is functorial in \( T \) it will suffice to give what this map does on \( T \)-points.

First let us begin with constructing the map \( \mathcal{C}_{T', Z \to X \times_X \mathcal{Y}} \to \mathcal{C}_{T', (Z \times_X \mathcal{Y}) - \mathcal{Y}} \). We will describe how a \( T \)-point of \( \mathcal{C}_{T', Z \to X \times_X \mathcal{Y}} \) gives a \( T \)-point of \( \mathcal{C}_{T', (Z \times_X \mathcal{Y}) - \mathcal{Y}} \).

The data of a \( T \)-point of \( \mathcal{C}_{T', Z \to X \times_X \mathcal{Y}} \) is the same as the following data:
(1) Maps $T' \to Z$ and $T' \times_T T' \to Z \times_Z Z$ making Diagrams 6.0.1 and 6.0.2 commute. This then defines a map $f_1 : T \to \mathcal{X}$ making the following diagram commute:

$$
\begin{array}{c}
T' & \longrightarrow & Z \\
\downarrow & & \downarrow \\
T & \underset{f_1}{\longrightarrow} & \mathcal{X}.
\end{array}
$$

(2) A map $T \to \mathcal{Y}$. This defines a composite $T \to \mathcal{Y} \to \mathcal{X}$, which we denote by $f_2 : T \to \mathcal{X}$.

(3) An isomorphism $f_1 \cong f_2$.

The isomorphism $f_1 \cong f_2$ by functoriality defines an isomorphism between the composite $T' \to Z \to \mathcal{X}$ and the composite $T' \to T \to \mathcal{Y} \to \mathcal{X}$. The map $T' \to Z$, the composite $T' \to T \to \mathcal{Y}$, and this isomorphism together define a map $T' \to Z \times_Z \mathcal{Y}$.

Similarly, the isomorphism $f_1 \cong f_2$ by functoriality defines an isomorphism between the composite $T' \times_T T' \to Z \times_Z Z \to \mathcal{X}$ and the composite $T' \times_T T' \to T \to \mathcal{Y} \to \mathcal{X}$. The map $T' \times_T T' \to Z \times_Z Z$, the composite $T' \times_T T' \to T \to \mathcal{Y}$, and this isomorphism define a map $T' \times_T T' \to (Z \times_Z Z \times_Z \mathcal{Y})$. But $(Z \times_Z Z \times_Z \mathcal{Y})$ is canonically isomorphic to $(Z \times_Z \mathcal{Y}) \times_{\mathcal{Y}} (Z \times_Z \mathcal{Y})$. Thus we get a map $T' \times_T T' \to (Z \times_Z \mathcal{Y}) \times_{\mathcal{Y}} (Z \times_Z \mathcal{Y})$.

The commutativity of Diagram 6.0.2 implies that the analogous diagram produced by the maps $T' \to Z \times_Z \mathcal{Y}$ and $T' \times_T T' \to (Z \times_Z \mathcal{Y}) \times_{\mathcal{Y}} (Z \times_Z \mathcal{Y})$ also commutes. This is precisely the data needed to define a map $T \to \mathcal{C}_{T', (Z \times_Z \mathcal{Y})} \to \mathcal{Y}$. So we have produced the map $\mathcal{C}_{T', (Z \times_Z \mathcal{Y})} \to \mathcal{Y}$.

We now describe the map $\mathcal{C}_{T', (Z \times_Z \mathcal{Y})} \to \mathcal{C}_{T', Z \times_Z \mathcal{Y}}$. Again, we explain how a $T$-point of $\mathcal{C}_{T', (Z \times_Z \mathcal{Y})} \to \mathcal{Y}$ gives a $T$-point of $\mathcal{C}_{T', Z \times_Z \mathcal{Y}}$.

A $T$-point of $\mathcal{C}_{T', (Z \times_Z \mathcal{Y})} \to \mathcal{Y}$ is the following data:

(1) Map $T' \to Z \times_Z \mathcal{Y}$
(2) Map $T' \times_T T' \to (Z \times_Z \mathcal{Y}) \times_{\mathcal{Y}} (Z \times_Z \mathcal{Y}) \cong (Z \times_Z Z) \times_Z \mathcal{Y}$,

making the analogue of Diagram 6.0.1 and Diagram 6.0.2 commute.

By composing with the appropriate projections, we get:

(1) Map $T' \to Z$
(2) Map $T' \times_T T' \to Z \times_Z Z$

making Diagram 6.0.1 and Diagram 6.0.2 commute. This defines a map $T \to \mathcal{C}_{T', Z \times_Z \mathcal{Y}}$. Let $f_1$ be the composite $T \to \mathcal{C}_{T', Z \times_Z \mathcal{Y}} \to \mathcal{X}$. We can then identify the composite $T' \to Z \times_Z \mathcal{Y} \to Z \to \mathcal{X}$ with $T' \to T \underset{f_1}{\longrightarrow} \mathcal{X}$ and the composite $T' \times_T T' \to (Z \times_Z \mathcal{Y}) \times_{\mathcal{Y}} (Z \times_Z \mathcal{Y}) \to Z \times_Z Z \to \mathcal{X}$ with $T' \times_T T' \to T \underset{f_1}{\longrightarrow} \mathcal{X}$.

Now form the composition $T' \to Z \times_Z \mathcal{Y} \to \mathcal{Y}$. The fact that the $Z \times_Z \mathcal{Y} \to \mathcal{Y}$ version of Diagram 6.0.1 commutes defines an isomorphism between the two composites $T' \times_T T' \to \mathcal{Y}$. The fact that the $Z \times_Z \mathcal{Y} \to \mathcal{Y}$ version of Diagram 6.0.2 commutes defines a “cocycle” compatibility between the induced isomorphisms between three composites $T' \times_T T' \times_T T' \to T' \to \mathcal{Y}$. Descent from this defines a map $f_2 : T \to \mathcal{Y}$.

Furthermore, descent theory gives us canonical isomorphisms between the composites $T' \to Z \times_Z \mathcal{Y} \to \mathcal{Y}$ and $T' \to T \underset{f_1}{\longrightarrow} \mathcal{Y}$, and also between the composites $T' \times_T T' \to (Z \times_Z \mathcal{Y}) \times_{\mathcal{Y}} (Z \times_Z \mathcal{Y}) \to \mathcal{Y}$ and $T' \times_T T' \to T \underset{f_1}{\longrightarrow} \mathcal{Y}$. Thus we can and do assume that these isomorphic composites are equal.
(Also note that since $Z \times_X \mathcal{Y}$ is an algebraic space, there is at most one isomorphism between any two maps to $Z \times_X \mathcal{Y}$.)

Next, the map $T' \to Z \times_X \mathcal{Y}$ gives an isomorphism between the composites $T' \to T \xrightarrow{f_1} \mathfrak{X}$ and $T' \to T \xrightarrow{f_2} \mathfrak{Y}$. Likewise, the map $T' \times_T T' \to (Z \times_X \mathcal{Y}) \times_{\mathcal{Y}} (Z \times_X \mathcal{Y}) \cong (Z \times_X Z) \times_X \mathcal{Y}$ gives an isomorphism between the composites $T' \times_T T' \to T \xrightarrow{f_1} \mathfrak{X}$ and $T' \times_T T' \to T \xrightarrow{f_2} \mathfrak{X}$.

The commutativity of the $Z \times_X \mathcal{Y} \to \mathcal{Y}$ version of Diagram 6.0.1 implies that isomorphism between the composites $T' \times_T T' \to T \xrightarrow{f_1} \mathfrak{X}$ and $T' \times_T T' \to T \xrightarrow{f_2} \mathfrak{X}$ is induced from an isomorphism of the composites $T' \to T \xrightarrow{f_1} \mathfrak{X}$ and $T' \to T \xrightarrow{f_2} \mathfrak{X}$ by functoriality from either projection. Then descent (or the stack property) implies that these identifications must be induced by functoriality from an isomorphism $f_1 \cong f_2$.

The data of $T \to \mathcal{C}_{T', Z \to \mathfrak{X}}$ and $T \to \mathcal{Y}$ and isomorphism $f_1 \cong f_2$, together define a map $T \to \mathcal{C}_{T', Z \to \mathfrak{X}} \times_X \mathcal{Y}$. Thus we have described the map $\mathcal{C}_{T', Z \times_X \mathcal{Y} \to \mathcal{Y}} \to \mathcal{C}_{T', Z \to \mathfrak{X}} \times_X \mathcal{Y}$.

These maps are inverses. □

7. Representability of cocycle spaces

In this section $R$ is a ring and all objects are finite-type over $R$. Additionally throughout this section, $T$ will be an $R$-scheme, but now $T' \to T$ will be a finite étale cover. Still $\mathfrak{X}$ will be an algebraic stack over $T$, and $Z$ an algebraic space over $T$, and $Z \to \mathfrak{X}$ will be a smooth cover.

**Lemma 7.0.1.** Assume $\mathfrak{X}$ is an algebraic space, $Z$ is relatively affine over $T$. For any $t \in T$, if $k(t)$ denotes the residue field at $t$, any finite set of points in $(Z \times_X Z) \times_T \text{Spec } k(t)$ is contained in an affine subset of $(Z \times_X Z) \times_T \text{Spec } k(t)$.

**Proof.** By base changing $Z \to T$ to $\text{Spec } k(t)$, we may reduce to the case when $T$ is the spectrum of a field and $t \in T$ is the unique point and $Z$ is affine. Thus we have to show that $Z \times_X Z$ has the property that any finite collection of points is contained in an affine.

The diagonal $\mathfrak{X} \xrightarrow{\Delta} \mathfrak{X} \times_T \mathfrak{X}$ is a locally closed immersion, which implies that $Z \times_X Z$ is locally closed in $Z \times_T Z$. On the other hand $Z$ is affine, so $Z \times_T Z$ is affine, and therefore that $Z \times_X Z$ is quasi-affine over a field. Finite-type quasi-affine schemes over fields have the property that any finite collection of points is contained in an affine, as quasi-projective varieties over fields have this property and quasi-affine varieties are quasi-projective. □

**Proposition 7.0.2.** If $\mathfrak{X}$ is an algebraic space and $Z$ is a relatively affine $T$-scheme, Then $\mathcal{C}_{T', Z \to \mathfrak{X}}$ is representable by a scheme.

**Proof.** Note that $T' \times_T T'$ and $T' \times_T T' \times_T T'$ are also both finite and locally free over $T$. The sheaf $\mathcal{C}_{T', Z \to \mathfrak{X}}$ is constructed as a limit of various restriction of scalars of $Z$ and $Z \times_X Z$. These restrictions of scalars are schemes by Lemma 7.0.1 and [BLR90][Theorem 7.6.4]. Limits of schemes are schemes, so $\mathcal{C}_{T', Z \to \mathfrak{X}}$ is a scheme. □

**Lemma 7.0.3.** Recall we assumed $T' \to T$ is finite étale. Let $X'$ be an algebraic space over $T'$. Then $\text{Res}_{T'/T} X'$ is an algebraic space over $T$.

**Proof.** The question is local on $T$, so we can assume that $T' \to T$ is free of a fixed rank $n$. There exists a finite étale cover $T'' \to T$ such that $T'' \times_T T' \cong (T'')^n$ (the fiber product of $n$ copies of $T'$ over $T$) as a $T$-scheme.
Now \((\text{Res}_{T'/T} X') \times_T T'' \to \text{Res}_{T'/T} X')\) is representable, surjective, and étale. Furthermore, \(\text{Res}_{T'/T} X' \times_T T'' \cong (X' \times_T T'')^n\) (the fiber product of \(n\) copies of \(X'\) over \(T'\)) as functor on \(T\)-schemes. Therefore, \(\text{Res}_{T'/T} X'\) is an étale quotient of an algebraic space, so is an algebraic space.

Proposition 7.0.4. If \(\mathcal{X}\) is an algebraic stack then \(\mathcal{C}_{T', Z \to X}\) is representable by an algebraic space.

Proof. By Lemma 7.0.3 all spaces used to construct \(\mathcal{C}_{T', Z \to X}\) are algebraic spaces. As the limit of algebraic spaces are algebraic spaces, \(\mathcal{C}_{T', Z \to X}\) is an algebraic space.

Proposition 7.0.5. Suppose that \(T' \to T\) is finite étale. If \(\mathcal{X}\) is a scheme, then the following diagram is cartesian:

\[
\begin{array}{c}
\mathcal{C}_{T', Z \to X} \\
\downarrow \\
\mathcal{X} \\
\downarrow \\
\text{Res}_{T'/T} \mathcal{X}
\end{array}
\]

Proof. In the case when \(\mathcal{X}\) is a scheme and we have a map \(T' \to Z\), there is at most one map \(T' \times_T T' \to Z \times_X Z\) making Diagram 6.0.1 commute. Additionally, once we find such a map Diagram 6.0.2 automatically commutes.

Furthermore, we can find a map \(T' \times_T T' \to Z \times_X Z\) making Diagram 6.0.1 commute precisely there is a morphism \(T \to \mathcal{X}\) making

\[
\begin{array}{c}
\mathcal{T}' \\
\downarrow \\
\mathcal{T}
\end{array}
\]

commute.

Thus a point of \(\mathcal{C}_{T', Z \to X}\) is a \(T\)-point of \(\mathcal{X}\) and a \(T'\)-point of \(Z\) having the same image in \(\mathcal{X}(T')\). This precisely says the desired diagram is cartesian.

Proposition 7.0.6. Suppose that \(T' \to T\) is finite étale. Then the map \(\mathcal{C}_{T', Z \to X} \to \mathcal{X}\) is a smooth cover.

Proof. First suppose that \(\mathcal{X}\) is a scheme. Proposition 7.0.5 expresses \(\mathcal{C}_{T', Z \to X} \to \mathcal{X}\) as a pullback of \(\text{Res}_{T'/T} Z \to \text{Res}_{T'/T} \mathcal{X}\). As \(Z \to \mathcal{X}\) is a smooth cover, so is \(\text{Res}_{T'/T} Z \to \text{Res}_{T'/T} \mathcal{X}\) is a smooth cover, and therefore \(\mathcal{C}_{T', Z \to X} \to \mathcal{X}\) is a smooth cover.

In general, we pullback \(\mathcal{C}_{T', Z \to X} \to \mathcal{X}\) along a smooth cover \(X \to \mathcal{X}\) from scheme \(X\). Proposition 7.0.5 says that \(\mathcal{C}_{T', Z \to X} \times_X X \to X\) is isomorphic to \(\mathcal{C}_{T', Z \times_X X \to X} \to X\) which is a smooth cover by the scheme case. Thus we conclude that \(\mathcal{C}_{T', Z \to X} \to \mathcal{X}\) is a smooth cover.

The following is proved via a different method by M. Bhargava and B. Poonen in the case when \(R\) is a noetherian local ring [BP19]

Theorem 7.0.7. Assume \(R\) be a sufficiently disconnected topological ring, and let \(T = \text{Spec } R\). Let \(\mathcal{X}\) be an algebraic stack over \(T\), and let \(T \to \mathcal{X}\) be a section. Then there is a \(T\)-scheme \(Y\) with smooth cover \(Y \to \mathcal{X}\) and map \(T \to Y\) making the following diagram commute:

\[
\begin{array}{c}
Y \\
\downarrow \\
\mathcal{X}
\end{array}
\]
Proof. Let $X \to \mathfrak{X}$ be any separated smooth cover from by scheme. Let $Z = X \times_X T$.

Now $Z \to T$ is smooth, so there is an étale surjective morphism $T_1 \to T$ such that $Z \times_T T_1 \to T_1$ has a section. As $R$ is sufficiently disconnected by Lemma 5.0.5 this implies there exists $T' \to T_1$ such that the composite $T' \to T_1 \to T$ is surjective finite étale map. The induced map $T' \to T_1 \to Z$ makes $T' \to Z$ making the following diagram commute

$$
\begin{array}{ccc}
T' & \to & Z \\
\downarrow & & \downarrow \\
T & \to & Z
\end{array}
$$

Now from the maps $T \to \mathfrak{X}$ and $T' \to Z$, we can construct $T' \times_T T' \to Z \times_T Z$ (which component-wise is the map $T' \to Z$), and this clearly makes Diagram 6.0.1 and Diagram 6.0.2 commute. This gives us a map $T \to \mathcal{E}_{T',Z \to \mathfrak{X}}$. Now if $\mathcal{E}_{T',Z \to \mathfrak{X}}$ is only an algebraic space, we run the same procedure again lifting $T$ to a smooth cover of $\mathcal{E}_{T',Z \to \mathfrak{X}}$ by a scheme keeping in mind Proposition 7.0.2. This proves the proposition.

### Proposition 7.0.8

Let $T = \text{Spec } R$, so $\mathfrak{X}$ is a finite-type $R$-stack. There are smooth covers $\pi_N : Z_N \to \mathfrak{X}$ by schemes for $N \geq 1$, such that $\bigcup_N \pi_N(Z_N(R)) = \mathfrak{X}(R)$ and additionally, for any sufficiently disconnected $R$-algebra $R'$, $\bigcup_N \pi_N(Z_N(R')) = \mathfrak{X}(R')$.

Additionally if $\mathfrak{X}$ is a Deligne-Mumford stack, there is a single smooth cover $\pi : Z \to \mathfrak{X}$ be a scheme such that $\pi(Z(R)) = \mathfrak{X}(R)$ and furthermore for any sufficiently disconnected $R$-algebra $R'$, $\pi(Z(R')) = \mathfrak{X}(R')$.

Proof. Let $X \to \mathfrak{X}$ be any smooth cover by a scheme. Let $E_N$ be the $R$-scheme that parametrizes étale $R$-algebras which are free as $R$-modules of rank $N$ and are equipped with an $R$-basis. This is a smooth scheme. Let $F_N \to E_N$ be the universal étale cover of rank $N$. Set $Z'_N = \mathcal{E}_{F_N,E_N \times_R \mathfrak{X} \to \mathfrak{X}}$. By Prop 7.0.4, the fiber of $Z'_N \to E_N$ over a point corresponding to a free étale $R$-algebra $S$ equipped with a basis as an $R$-module is $\mathcal{E}_{S,\mathfrak{X} \to \mathfrak{X}}$. By Lemma 5.0.5, for $x \in \mathfrak{X}(R)$ there is a finite free étale $R$-algebra $S$, of some rank $N \geq 1$, such that $x$ lifts to a point of $X(S)$. Such an $x$ thus lifts to an $R$-point of $Z'_N$. By Prop 7.0.4, the same property holds for any sufficiently disconnected $R$-algebra $R'$ and $x \in \mathfrak{X}(R')$. The $Z'_N$ exhibit the desired properties, except they are algebraic spaces and not necessarily schemes.

Assume for a moment we have proved the results in the second paragraph of the statement for algebraic spaces. Then we can find a cover $Z_N \to Z'_N$ such that for any sufficiently disconnected $R$-algebra $R'$, $Z_N(R') \to Z'_N(R')$ is surjective. Because $\bigcup_N Z'_N(R') = \mathfrak{X}(R')$, we also have $\bigcup_N Z_N(R') = \mathfrak{X}(R')$.

We now prove the results in the second paragraph of the statement. Assume that $\mathfrak{X}$ is a Deligne-Mumford stack and let $X \to \mathfrak{X}$ be a separated étale cover of degree less than or equal to $n$ for some $n \geq 1$ by an affine scheme. Applying Lemma 5.0.5 to $X \to \mathfrak{X}$ for any $x \in \mathfrak{X}(R)$ there must be a free étale $R$-algebra $S$ of rank $n$! and $x' \in \mathfrak{X}(S)$ such that $x$ and $x'$ have the same image in $\mathfrak{X}(S)$. In fact the same conclusion holds for any sufficiently disconnected $R$-algebra $R'$ and any $x \in \mathfrak{X}(R')$ (by applying Lemma 5.0.5 to $X \otimes_R R' \to \mathfrak{X} \otimes_R R'$). This implies that for any such $R'$, $Z'_n(R') \to \mathfrak{X}(R')$ is surjective. If $\mathfrak{X}$ is an algebraic space, then by Proposition 7.0.2 $Z'_n$ is a scheme and we take $Z = Z'_n$.

If $\mathfrak{X}$ is just a Deligne-Mumford stack, $Z'_n$ is only an algebraic stack. By the now completed algebraic space case, we may find a cover $Z \to Z'_n$ by a scheme such that for every any sufficiently disconnected $R$-algebra $R'$, $Z(R') \to Z'_n(R')$ is surjective. The composite $Z \to Z'_n \to \mathfrak{X}$ then has the desired properties.

\qed
Proposition 8.0.3. Let $U \to X$ smooth covers of $X$ that are homeomorphic to any other, so every morphism in the colimit is a homeomorphism. We conclude $X$ is a quotient of $U(R)$ by the smooth quotient property. Therefore, the natural continuous map $\pi_U(U(R)) \to \pi_V(V(R))$ is a homeomorphism.

Proof. Note that as sets $\pi_U(U(R)) \to \pi_V(V(R))$ is just the identity $X(R) \to X(R)$.

Set $Z = U \times_X V$, and consider the diagram

$$
\begin{array}{ccc}
Z & \longrightarrow & U \\
\downarrow & & \downarrow \pi_U \\
V & \longrightarrow & X.
\end{array}
$$

The top and left arrow are smooth morphisms of schemes, and the map $Z(R) \to U(R)$ is surjective and thus a quotient by the smooth quotient property. Therefore, $\pi_U(U(R))$ is a quotient of $Z(R)$. Similarly, $Z(R) \to V(R)$ is surjective thus also a quotient, so the composite $Z(R) \to V(R) \to \pi_V(V(R))$ is a quotient.

Therefore, $\pi_U(U(R))$ and $\pi_V(V(R))$ are both quotients of $Z(R)$: both spaces are the underlying set $X(R)$ with the topology given as a quotient of $Z(R)$ by the map $Z(R) \to X(R)$. Therefore, the topologies on both $\pi_U(U(R))$ and $\pi_V(V(R))$ match, so the natural morphism $\pi_U(U(R)) \to \pi_V(V(R))$ must be a homeomorphism.

Proposition 8.0.2. Let $X$ be an algebraic space over $R$, and $\pi_U : U \to X$ be a smooth cover by a scheme such that $U(R) \to X(R)$ is surjective. Then when $X(R)$ is given the topology as in Definition 5.0.10, $U(R) \to X(R)$ is a quotient map of topological spaces.

Proof. Recall that in Definition 5.0.10, we define $X(R)$ as a certain colimit over the category of smooth covers of $X$ by schemes. If $V$ is any smooth cover of $X$ by a scheme, then $U \coprod V$ is another smooth cover. Thus the smooth covers of $X$ which receive an $X$-morphism from $U$ are final in this category. Therefore, $X(R) = \colim_{V \to X} \pi_V(V(R))$ (with notation as in Definition 5.0.10) where the colimit is now over smooth covers of $X$ which receive an $X$-morphism from $U$.

By Proposition 8.0.1, for each smooth cover $V \to X$ by a scheme with an $X$-morphism $U \to V$, the associated map $\pi_U(U(R)) \to \pi_V(V(R))$ is a homeomorphism. Therefore, every such $\pi_V(V(R))$ is homeomorphic to any other, so every morphism in the colimit is a homeomorphism. We conclude that $\pi_U(U(R)) \to X(R)$ must therefore be a homeomorphism. As $\pi_U(U(R))$ is by definition a quotient of $U(R)$, $X(R)$ has the topology as a quotient of $U(R)$.

Proposition 8.0.3. Let $f : X \to Y$ be a morphism of algebraic spaces over $R$. Then $X(R) \to Y(R)$ is continuous. Moreover, if $f$ is smooth and $X(R) \to Y(R)$ is surjective, then $X(R) \to Y(R)$ is a quotient.
Proof. Let us first do the case of general \( f \). By Proposition 7.0.8, there exists a smooth cover \( U \to Y \) by a scheme such that \( U(R) \to Y(R) \) is surjective. Set \( Z = X \times_Y U \), and let \( W \to Z \) be a smooth cover by a scheme such that \( W(R) \to Z(R) \) is surjective again by Proposition 7.0.8. Then \( W \to X \) is a smooth cover of \( X \) by a scheme such that \( W(R) \to X(R) \) is surjective. By Proposition 8.0.2, \( W(R) \to X(R) \) is a quotient map. Then by Proposition 14.0.3, we conclude that \( X(R) \to Y(R) \) is continuous.

Now assume \( f \) is smooth and \( X(R) \to Y(R) \) is surjective. This implies that \( Z(R) \to U(R) \) is surjective and thus \( W(R) \to U(R) \) is surjective and thus a quotient by the smooth quotient property. The smooth quotient property implies that \( W(R) \to U(R) \) is a quotient. Thus Proposition 14.0.4 implies that \( X(R) \to Y(R) \) must be a quotient as desired. \( \square \)

9. Functorialities

In this section, \( R \) is a sufficiently disconnected topological ring together a good topologization of \( R \)-points of finite-type \( R \)-schemes with the smooth quotient property.

Lemma 9.0.1. Let \( X \) be an algebraic space over \( R \), \( \mathfrak{Y} \) be an algebraic stack over \( R \), and \( X \to \mathfrak{Y} \) be a map over \( R \). Then there exists a smooth covering \( Y \to \mathfrak{Y} \) by a scheme such that the canonical map \( (X \times_{\mathfrak{Y}} Y)(R) \to X(R) \) is a surjective quotient.

Proof. It suffices to find a \( Y \) such that \( (X \times_{\mathfrak{Y}} Y)(R) \to X(R) \) is surjective on \( R \)-points by Proposition 8.0.3. Remark 7.0.9, explains the existence of such a \( Y \). \( \square \)

Proposition 9.0.2. Let \( X \) be an algebraic space over \( R \), \( \mathfrak{Y} \) be an algebraic stack over \( R \), and \( X \to \mathfrak{Y} \) be a map over \( R \). Then the natural map \( X(R) \to \mathfrak{Y}(R) \) is continuous.

Proof. We choose \( \pi : Y \to \mathfrak{Y} \) as in Lemma 9.0.1 with \( Y \) a scheme. Let \( \pi(Y(R)) \subseteq \mathfrak{Y}(R) \) given the quotient topology from \( Y(R) \).

Consider the following diagram:

\[
\begin{array}{ccc}
X \times_{\mathfrak{Y}} Y & \longrightarrow & Y \\
\downarrow & & \downarrow \pi \\
X & \longrightarrow & \mathfrak{Y}.
\end{array}
\]

Taking \( R \)-points, we get a commutative diagram of sets

\[
\begin{array}{ccc}
(X \times_{\mathfrak{Y}} Y)(R) & \longrightarrow & Y(R) \\
\downarrow & & \downarrow \pi \\
X(R) & \longrightarrow & \mathfrak{Y}(Y(R)).
\end{array}
\]

The right, left, and top maps are continuous. The left map is a quotient by Lemma 9.0.1. Thus Proposition 14.0.3 implies that \( X(R) \to \pi(Y(R)) \) is continuous. The map \( \pi(Y(R)) \to \mathfrak{Y}(R) \) is continuous by construction of the topology on \( \mathfrak{Y}(R) \). Therefore the composite \( X(R) \to \mathfrak{Y}(R) \) is continuous. \( \square \)

Theorem 9.0.3. Let \( \mathfrak{X} \) and \( \mathfrak{Y} \) be stacks over \( R \) with a \( R \)-morphism \( \mathfrak{X} \to \mathfrak{Y} \), then the natural map of sets \( \mathfrak{X}(R) \to \mathfrak{Y}(R) \) is continuous.
Proof. Let $\pi : X \to \mathfrak{X}$ be a smooth cover by a scheme. Let $\pi(X(R))$ denote the set $\pi(X(R))$ with the quotient topology from $X(R)$. By Proposition 9.0.2, the natural maps $X(R) \to \mathfrak{Y}(R)$ are continuous. Furthermore they factor through $X(R) \to \pi(X(R))$. As $\mathfrak{X}(R) = \text{colim}_{\pi:X \to \mathfrak{X}} \pi(X(R))$, by definition, we thus conclude the map $\mathfrak{X}(R) \to \mathfrak{Y}(R)$ is continuous. □

**Proposition 9.0.4.** Let $R'$ (in addition to $R$) be a sufficiently disconnected ring together with a good topologization of the $R'$-points on finite-type $R'$-schemes with the smooth quotient property. Let $R \to R'$ be a continuous ring homomorphism, and assume furthermore, that $X(R) \to X(R')$ is continuous for all finite-type $R$-schemes $X$. For each algebraic stack $\mathfrak{X}$ over $R$, the natural map $\mathfrak{X}(R) \to \mathfrak{X}_{R'}(R')$ is continuous.

Proof. Let $\pi : Z \to \mathfrak{X}$ be a smooth cover by an $R$-scheme. Give $\pi(Z(R))$ the quotient topology from $Z(R)$ and $\pi(Z(R'))$ the quotient topology from $Z(R')$. There is a commutative diagram (as sets for now):

$$
\begin{array}{ccc}
Z(R) & \longrightarrow & Z(R') \\
\downarrow & & \downarrow \\
\pi(Z(R)) & \longrightarrow & \pi(Z(R')).
\end{array}
$$

The two vertical arrows are quotient maps, and the top arrow is continuous. This implies the bottom map is continuous by Proposition 14.0.3.

Taking a colimit over $Z$, we conclude that the map

$$
\text{colim}_{\pi:Z \to \mathfrak{X}} \pi(Z(R)) \to \text{colim}_{\pi:Z' \to \mathfrak{X} \times_{\text{Spec } R} \text{Spec } R'} \pi(Z'(R'))
$$

is continuous. This precisely says that $\mathfrak{X}(R) \to \mathfrak{X}(R')$ is continuous. □

**Proposition 9.0.5.** Let $X$ be an algebraic space over $R$ and and $\mathfrak{X}$ an algebraic stack over $R$. Let $X \to \mathfrak{X}$ be a smooth covering such that $X(R) \to \mathfrak{X}(R)$ is surjective. Then $X(R) \to \mathfrak{X}(R)$ is a quotient map.

Proof. Let $\pi : U \to \mathfrak{X}$ be a smooth cover by a scheme such that $U(R) \to \mathfrak{X}(R)$ is surjective. At least one exists as we can can take $U$ to be a cover of $X$ such that $U(R) \to X(R)$ is surjective using Proposition 7.0.8. In any case, then $U \times_X X \to X$ is surjective on $R$-points so is a quotient map by Proposition 8.0.3. Similarly, $U \times_X X \to U$ is a quotient on $R$-point. Therefore, topologizing $\mathfrak{X}(R)$ as a quotient of $U$ is the same as topologizing it as a quotient of $X$. Smooth covers of $\mathfrak{X}$ that are surjective on $R$-points are final among all smooth covers (as long as one exists). The definition topology on $\mathfrak{X}(R)$ is as a colimit over quotients of $U(R)$ where $U \to \mathfrak{X}$ is a smooth cover and by the last sentence we can restrict ourselves to covers such that $U(R) \to \mathfrak{X}(R)$ is surjective. We conclude that $\mathfrak{X}(R)$ must have the topology as a quotient of $X(R)$. □

10. Algebraic spaces over essentially analytic rings

In this section the ring $R$ is an essentially analytic topological ring.

**Proposition 10.0.1.** Let $X$ be an algebraic space over $R$, and let $\pi : U \to X$ be an étale map from a scheme $U$. Equip $\pi(U(R))$ with topology as a quotient of $U(R)$. Then the map $U(R) \to \pi(U(R))$ is a local homeomorphism. In particular it is open.
Proof. Consider the following cartesian square:

\[
\begin{array}{ccc}
U \times_X U & \xrightarrow{\pi_1} & U \\
\downarrow \pi_2 & & \downarrow \pi \\
U & \xrightarrow{\pi} & X.
\end{array}
\]

As the two arrows to \(X\) are étale, the two arrows from \(U \times_X U\) are étale. This leads to a commutative diagram.

\[
\begin{array}{ccc}
(U \times_X U)(R) & \xrightarrow{\pi_1} & U(R) \\
\downarrow \pi_2 & & \downarrow \pi \\
U(R) & \xrightarrow{\pi} & \pi(U(R)).
\end{array}
\]

We first prove that \(U(R) \to \pi(U(R))\) is open. Let \(V \subseteq U(R)\) be an open subset. Since \(R\) is essentially analytic and each \(\pi_i\) is an étale morphism of schemes, they induce local homeomorphisms on \(R\) points. Thus each \(\pi_i\) induce open maps on \(R\)-points. Therefore, \(\pi_2(\pi_1^{-1}(V))\) is open. Now since \((U \times_X U)(R) = U(R) \times_{X(R)} U(R)\) as sets, a set theoretic calculation shows that \(\pi^{-1}(\pi(\pi_2(\pi_1^{-1}(V)))) = \pi_2(\pi_1^{-1}(V))\). By the definition of the quotient topology, this implies that \(\pi(\pi_2(\pi_1^{-1}(V)))\) is open, and this is equal to \(\pi(V)\). Therefore, \(\pi(V)\) is open in \(\pi(U(R))\). Since \(V\) was arbitrary, this proves that \(U(R) \to \pi(U(R))\) is an open map.

Now consider the map \(U \times_X U \xrightarrow{\pi_2} U\). The map \(u \mapsto (u, u)\) provides a section. As section of an étale morphism is an open immersion, the diagonal \(\Delta \subseteq (U \times_X U)(R)\) is open.

As \(U \times_X U \to U \times_R U\) is a locally closed immersion, \((U \times_X U)(R)\) has the subspace topology inherited from \(U(R) \times U(R)\). In particular, the topology on \((U \times_X U)(R)\) is generated by the restriction of opens of the form \(V \times W\) for \(V, W \subseteq U(R)\) open.

Let us now check that \(U(R) \to \pi(U(R))\) is a local homeomorphism at each point of \(U(R)\). Let \(y \in U(R)\). As the topology at \((y, y) \in (U \times_X U)(R)\) is generated by opens of the form \((W \times W) \cap (U \times_X U)(R)\) for \(W \subseteq U(R)\) open and \(\Delta\) is open, there exists an open \(W \subseteq U(R)\) containing \(y\) such that \((W \times W) \cap (U \times_X U)(R)\) is open.

For any two \(w, w' \in W\), if \(\pi(w) = \pi(w')\), then \((w, w') \in U(R) \times_{X(R)} U(R) \subseteq U(R) \times U(R),\) but also \((w, w') \in (W \times W) \cap (U \times_X U)(R) \subseteq \Delta,\) so \(w = w'\). Thus the map \(W \to \pi(U(R))\) is injective. We have already concluded that \(W \to \pi(U(R))\) is open. Finally, as \(U(R) \to \pi(U(R))\) is a quotient map, we can conclude that \(W\) is an homeomorphism onto its image, concluding the proof.

**Proposition 10.0.2.** Let \(X\) be an algebraic space over \(R\), and let \(\pi : Z \to X\) be a smooth cover by a scheme. Given \(\pi(Z(R))\) the quotient topology from the topology of \(Z(R)\). Then the map \(Z(R) \to \pi(Z(R))\) is open.

**Proof.** Consider the following diagram:

\[
\begin{array}{ccc}
Z \times_X Z & \xrightarrow{\pi_1} & Z \\
\downarrow \pi_2 & & \downarrow \pi \\
Z & \xrightarrow{\pi} & X,
\end{array}
\]

which leads to the diagram
Proof. We will first consider the case when \( \pi \) is a morphism onto an open subset. \( \pi \) is open. Then the topologies defined on \( X \) and \( Z(\pi) \) are identical. Let \( U = \pi(\pi^{-1}(U)) \), which as the property that \( \pi^{-1}(\pi(U)) = U \). This implies \( \pi(U) \) is open, but \( \pi(U) = \pi(U) \), completing the proof. \( \square \)

**Proposition 10.0.3.** Let \( X \) be an algebraic space over \( R \). Let \( U \) and \( V \) schemes with smooth covers \( \pi_U : U \to X \) and \( \pi_V : V \to X \), and \( f : U \to V \) be an \( X \)-morphism. Let \( \pi_{UV}(U(R)) \) be the image of \( U(R) \) in \( X(R) \) with the quotient topology and similarly for \( \pi_V(V(R)) \). Then, inclusion \( \pi_{UV}(U(R)) \to \pi_V(V(R)) \) is a homeomorphism onto an open subset.

**Proof.** We will first consider the case when \( V = U \sqcup W \) where \( W \to X \) is also a smooth cover. The continuous injection \( \pi_{UV}(U(R)) \to \pi_V(V(R)) \) is open by Proposition 10.0.2. Therefore since open continuous injection are homeomorphisms onto open subsets, \( \pi_{UV}(U(R)) \to \pi_V(V(R)) \) is homeomorphism onto an open subset.

Now consider the general case. Let the map \( U \sqcup V \to X \) be denoted by \( d \). The last case says that the topologies defined on \( \pi_{UV}(U(R)) \) and \( \pi_V(V(R)) \) map are open in that defined by \( d((U \sqcup V)(R)) \). Thus \( \pi_{UV}(U(R)) \to \pi_V(V(R)) \) must be a homeomorphism onto an open subset. \( \square \)

**Proposition 10.0.4.** Let \( X \) and \( Y \) be two algebraic spaces over \( R \) and \( f : X \to Y \) be a smooth map. Then \( X(R) \to Y(R) \) is open.

**Proof.** We will show that for any \( x \in X(R) \), there is an open subset \( W \subseteq X(R) \) containing \( x \) such that the map \( W \to Y(R) \) is open. This is clearly sufficient.

Let \( y = f(x) \in Y(R) \). Let \( U \) be a scheme with an étale cover \( \pi : U \to Y \) such that \( y \in \pi(U(R)) \). Let \( u \in U(R) \) be such that \( \pi(u) = y \). Also, let \( V \) be a scheme with an étale cover \( \pi : V \to U \times_Y X \) such that \( (u, x) \in p(V(R)) \). Now the image of \( V(R) \) in \( X \) is open and contains \( x \); call this open \( W \). Additionally, \( V \to U \) is smooth, so the image of \( V(R) \) in \( U(R) \) is open. As \( U \to Y \) is étale, \( U(R) \to Y(R) \) is a local homeomorphism, the image of the composite \( V(R) \to U(R) \to Y(R) \) is open. The image of this composite is the same as \( f(W) \) and additionally contains \( y \) as \( W \) contains \( x \). Thus \( W \) is the desired open, completing the proof. \( \square \)

**Proposition 10.0.5.** Let \( X \to Y \) be an étale map of algebraic spaces over \( R \). Then \( X(R) \to Y(R) \) is a local homeomorphism.

**Proof.** Let \( x \in X(R) \) and let \( U \to X \) be an étale map such that there is a point \( u \in U(R) \) mapping to \( x \). Then \( U \to X \) and \( U \to Y \) are étale, so \( U(R) \to X(R) \) and \( U(R) \to Y(R) \) are local homeomorphisms, so \( X(R) \to Y(R) \) is a local homeomorphism when restricted to the image of \( U(R) \). This is an open subset and contains \( x \), so \( X(R) \to Y(R) \) is a local homeomorphism at \( x \). As \( x \) was arbitrary this completes the proof. \( \square \)

**Proposition 10.0.6.** Let \( X \) be a separated algebraic space over \( R \). Then the topological space \( X(R) \) is Hausdorff.
Proposition 10.0.7. In addition to being essentially analytic assume that \( R \) is a local field or the valuation ring of a local field. Let \( X \to Y \) be a proper morphism of separated algebraic spaces over \( R \). Then \( X(R) \to Y(R) \) is a proper morphism of topological spaces.

Proof. Chow’s lemma for algebraic spaces ([Sta18, Tag 088P]) implies that there is an algebraic space \( X' \) which is a closed sub-algebraic space of \( \mathbb{P}^n_X \) with a surjective map \( X' \to X \). Now since \( X' \to X \) is surjective, to prove \( X \to Y \) is proper, it suffices to prove that \( X' \to X \) and \( X' \to Y \) are proper. Then \( X' \) is also a closed sub-algebraic space of \( \mathbb{P}^n_X \). It is thus also a closed subspace of \( \mathbb{P}^n_Y \). It therefore suffices to prove that \( \mathbb{P}^n_X \to X \) and \( \mathbb{P}^n_Y \to Y \) induce proper maps on \( R \)-points. Thus to prove the proposition, it suffices to prove the statement: if \( Y \) is an algebraic space and \( X = \mathbb{P}^n_Y \), then \( X(R) \to Y(R) \) is a proper map.

Let \( U \to Y \) be an étale cover by a scheme such that \( U(R) \to Y(R) \) is surjective, which is possible by Proposition 7.0.8. By Proposition 10.0.5, \( U(R) \to Y(R) \) is a local homeomorphism. So the conclusion for \( \mathbb{P}^n_Y \to Y \) follows from that of \( \mathbb{P}^n_U \to U \). For the \( R \) as in the statement, \( \mathbb{P}^n_U(R) \to U(R) \) is proper, establishing the proposition. \( \square \)

11. Algebraic stacks in the essentially analytic case

In this this section \( R \) is essentially analytic in addition to being sufficiently disconnected.

Proposition 11.0.1. Let \( \mathcal{X} \) be an algebraic stack over \( R \), and let \( \pi : Z \to \mathcal{X} \) be a smooth cover by a scheme. Let \( \pi(Z(R)) \) denote the image of \( Z(R) \) under \( \pi \) viewed as a set. Topologize \( \pi(Z(R)) \) by giving it the quotient topology viewing \( \pi(Z(R)) \) as a quotient of \( Z(R) \). Then the map \( Z(R) \to \pi(Z(R)) \) is open.
Proof. The diagram:

\[
\begin{array}{ccc}
  Z \times X & \xrightarrow{\pi_1} & Z \\
  \downarrow^{\pi_2} & \downarrow^\pi & \\
  Z & \xrightarrow{\pi} & X
\end{array}
\]

leads to the diagram of topological spaces:

\[
\begin{array}{ccc}
  (Z \times X)(R) & \xrightarrow{\pi_1} & Z(R) \\
  \downarrow^{\pi_2} & \downarrow^\pi & \\
  Z(R) & \xrightarrow{\pi} & \pi(Z(R)).
\end{array}
\]

Let \( U \subseteq Z(R) \) be open. We must show that \( \pi(U) \subseteq \pi(Z(R)) \) is open. Now \( \pi_1^{-1}(U) \) is open as \( \pi_1 \) is continuous, and the fact that \( \pi_2 \) is smooth implies \( \pi_2(\pi_1^{-1}(U)) \) is open by Proposition 10.0.2. Let \( W = \pi_2(\pi_1^{-1}(U)) \), which as the property that \( \pi^{-1}(\pi(W)) = W \). This implies \( \pi(W) \) is open, but \( \pi(W) = \pi(U) \), completing the proof. □

Proposition 11.0.2. Let \( \mathfrak{X} \) be an algebraic stack over \( R \). Let \( \pi_X : X \to \mathfrak{X} \) and \( \pi_Y : Y \to \mathfrak{X} \) be smooth covers by schemes \( X \) and \( Y \), with a map \( f : X \to Y \) such that \( \pi_X \cong \pi_Y \circ f \). Let \( \pi_X(X(R)) \) denote the image of \( X(R) \) in \( \mathfrak{X}(R) \) (as a set) topologized with the quotient topology. Similarly, define \( \pi_Y(Y(R)) \). Then the inclusion \( \pi_X(X(R)) \to \pi_Y(Y(R)) \) is a homeomorphism onto an open subset.

Proof. We will first consider the case when \( Y = X \sqcup Z \) where \( Z \to \mathfrak{X} \) is also a smooth cover. The continuous injection \( \pi_X(U(R)) \to \pi_Y(V(R)) \) is open by Proposition 11.0.1. Therefore since open continuous injection are homeomorphisms onto open subsets, \( \pi_X(X(R)) \to \pi_Y(Y(R)) \) is a homeomorphism onto an open subset.

Now consider the general case. Let the map \( X \sqcup Y \to X \) be denoted by \( d \). The last case says that the topologies defined on \( \pi_X(X(R)) \) and \( \pi_Y(Y(R)) \) are homeomorphisms onto open subsets of \( d((U \sqcup V)(R)) \). Thus \( \pi_X(X(R)) \to \pi_Y(Y(R)) \) must be a homeomorphism onto an open subset. □

Proposition 11.0.3. Let \( \mathfrak{X} \) be an algebraic stack over \( R \). Let \( \pi : Z \to \mathfrak{X} \) be a smooth cover by a scheme. Let \( \mathfrak{X}(R) \) be given the topology as in Definition 5.0.10. Then \( Z(R) \to \mathfrak{X}(R) \) is open.

Proof. This follows immediately from the previous proposition. □

Theorem 11.0.4. Let \( f : \mathfrak{X} \to \mathcal{Y} \) be a smooth morphism of algebraic stacks over \( R \). Then the induced map \( f : \mathfrak{X}(R) \to \mathcal{Y}(R) \) is open.

Proof. Let \( x \in \mathfrak{X}(R) \). We can choose smooth covers \( \pi : U \to \mathfrak{X} \) to \( \rho : V \to \mathcal{Y} \) by schemes such that there exists \( u \in U(R) \) mapping to \( x \) and such that there is a smooth map \( g : U \to V \) making the following diagram commute:

\[
\begin{array}{ccc}
  U & \xrightarrow{g} & V \\
  \downarrow^\pi & \downarrow^\rho & \\
  \mathfrak{X} & \xrightarrow{f} & \mathcal{Y}.
\end{array}
\]
The morphism $U \to \mathcal{Y}_X V$ is smooth, but $\mathcal{Y}_X V \to V$ is also smooth, so $U \to V$ is smooth thus $U(R) \to V(R)$ is an open map, thus a quotient onto its image. The sets $\pi(U(R))$ and $\rho(V(R))$ have the topologies as quotients of $U(R)$ and $V(R)$.

Now take an open $W \subseteq X(R)$ containing $u$. We may shrink $W$ so that $W \subseteq \pi(U(R))$. Let $W' = \pi^{-1}(W) \subseteq U(R))$. As we have seen all maps are open except maybe the bottom one, $\rho(g(W')) = f(W)$ is open. Thus $f$ is open.

**Proposition 11.0.5.** Let $R$ be a local field or the valuation ring of a local field. Let $\mathcal{X}$ be a separated algebraic stack over $R$. Then $\mathcal{X}(R)$ is a Hausdorff topological space.

**Proof.** The proof is very similar to the proof of Proposition 10.0.6

To show that $\mathcal{X}(R)$ is Hausdorff, we will show that the diagonal $\Delta(X(R)) \subseteq \mathcal{X}(R) \times \mathcal{X}(R)$ is closed.

Let $x, y \in \mathcal{X}(R)$. We will show that in some open near $(x, y) \in \mathcal{X}(R) \times \mathcal{X}(R)$, the restriction of the diagonal map is a closed immersion. By Theorem 7.0.7, we may find a smooth cover $\pi : W \to \mathcal{X} \times \mathcal{X}$ by a scheme such that $(x, y)$ lifts to some $w \in W(R)$.

Consider the pullback square

$$
\begin{array}{ccc}
Z & \xrightarrow{\delta} & W \\
\downarrow & & \downarrow \pi \\
\mathcal{X} & \xrightarrow{\Delta} & \mathcal{X} \times \mathcal{X}.
\end{array}
$$

By the separated hypothesis, $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is a proper morphism, so $Z \to W$ is. Therefore $\delta(Z(R)) \to W(R)$ is a proper map of topological spaces by Proposition 10.0.7.

Additionally, it is easily checked $\pi^{-1}(\Delta(Z(R))) = \Delta(Z(R))$. As $\pi$ is open by Theorem 11.0.4 we have that $\pi(W(R))$ is open. Additionally, as $\pi$ is open it is a quotient map onto its image. The definition of the quotient topology gives that $\pi(\delta(Z(R)))$ is closed in $\pi(W(R))$. But $\pi(\delta(Z(R))) = \Delta(X(R)) \cap \pi(W(R))$. As $\pi(W(R))$ is an open containing $(x, y)$, $\Delta(X(R))$ is a closed near $(x, y)$. As $\Delta(X(R))$ is the image of the diagonal $\mathcal{X}(R) \to \mathcal{X}(R) \times \mathcal{X}(R)$, we conclude this diagonal is closed near $(x, y)$. As $(x, y)$ is arbitrary, we conclude the diagonal in $\mathcal{X}(R) \times \mathcal{X}(R)$ is closed immersion so $\mathcal{X}$ is Hausdorff.

**Lemma 11.0.6.** Let $R$ be a local field of characteristic zero. Let $\mathcal{X} \to \mathcal{Y}$ be a surjective map of finite-type Deligne-Mumford stacks over $R$. Then there exists a finite extension $R'$ of $R$ such that each point in $y \in \mathcal{Y}(R)$ lifts to point in $\mathcal{Y}(R')$.

**Proof.** As there are only finitely many extensions of $R$ of a given degree, it suffices to prove that there is a number $n$ such that each $y \in \mathcal{Y}(R)$ there is a finite extension $R'$ of $R$ of degree less than or equal to $n$, such that $y$ lifts to some point of $\mathcal{X}(R')$.

If $\mathcal{X} \to \mathcal{Y}$ is étale, then the degree must be bounded and the result of the lemma holds. If $X \to \mathcal{X}$ is any smooth cover, if the result of the lemma holds for $X \to \mathcal{Y}$ it holds for $\mathcal{X} \to \mathcal{Y}$. In this way we pass to the case when $\mathcal{X}$ is an algebraic space.

Now if the $Y \to \mathcal{Y}$ is a surjective morphism for which the result of the lemma holds, and it holds for $\mathcal{X} \times \mathcal{Y} \to Y$, then it holds for $\mathcal{X} \to \mathcal{Y}$. As the result of the lemma holds for étale covers, we may thus replace $\mathcal{Y}$ by a scheme. We have reduced to the case when $\mathcal{X}$ is an algebraic space and $\mathcal{Y}$ is a scheme. But we take any surjective map $X \to \mathcal{X}$ by a scheme, and again it suffices to show the result of the lemma holds for $X \to \mathcal{Y}$.

Therefore, we must prove the result of the lemma holds for $X \to Y$ a surjective map of schemes. In this case, there is an open subset $U \subseteq Y$ over which $X \to Y$ is smooth. Smooth maps have étale
sections. Therefore, over the smooth locus the result of the lemma follows from the étale case. We then reduce to \( Y \setminus U \) and lemma then follows from noetherian induction. \( \square \)

**Proposition 11.0.7.** Let \( R \) be a local field of characteristic zero. Let \( \mathcal{X} \) be a proper Deligne-Mumford stack with finite diagonal over \( R \). Then \( \mathcal{X}(R) \) is a compact topological space.

**Proof.** By Chow’s lemma (see [Ols16]), we may find a surjective map \( X \to \mathcal{X} \) where \( X \) is a projective scheme over \( R \).

By Lemma 11.0.6 there is a finite extension \( R' \) of \( R \) such that each \( R \)-point of \( \mathcal{X} \) lifts to an \( R' \)-point of \( X \). Then let \( Z \) be the pullback in the following diagram:

\[
\begin{array}{ccc}
Z & \longrightarrow & \text{Res}_{R'/R} X \\
\downarrow & & \downarrow \\
\mathcal{X} & \longrightarrow & \text{Res}_{R'/R} \mathcal{X}.
\end{array}
\]

As \( X \) is projective, the top right is a scheme.

Then \( Z \) is proper over \( \mathcal{X} \) as it is the pullback of a proper map. As \( X \) is proper, this implies that \( Z \) is proper. Therefore, \( Z(R) \) is compact, and as \( Z(R) \to X(R) \) is surjective and \( X(R) \) is Hausdorff by Proposition 11.0.5, we conclude that \( \mathcal{X}(R) \) is compact. \( \square \)

**Lemma 11.0.8.** If \( R \) and \( R' \) are two continuously invertible, sufficiently disconnected, topological local rings and \( R \to R' \) is a continuous ring homomorphism, then for any finite-type \( R \)-scheme \( X \), \( X(R) \to X(R') \) is continuous.

**Proof.** As \( R \) and \( R' \) are both local, any \( R \)-point of \( X \) is contained in an affine open. In this way we reduce to the case that \( X \) is affine using property (5) of Definition 4.0.1. Next, we embed \( X \) in \( \mathbb{A}^n_R \) for some \( R \), and reduce to the case when \( X = \mathbb{A}^n_R \) using property (3) of Definition 4.0.1. Then the fact that \( \mathbb{A}^n_R(R) \to \mathbb{A}^n_R(R') \) is continuous follows from properties (1) and (4) of Definition 4.0.1. \( \square \)

**Theorem 11.0.9.** Let \( R' \) be an essentially analytic topological ring. Let \( R \) be an open subring of \( R' \) that is also essentially analytic. Let \( \mathcal{X} \) be an algebraic stack over \( R \). Then \( \mathcal{X}(R) \to \mathcal{X}(R') \) is an open map.

If additionally \( R \) is a complete discrete valuation ring with the usual topology given by the maximal ideal, and \( R' \) is its fraction field and \( \mathcal{X} \) is separated, then \( \mathcal{X}(R) \to \mathcal{X}(R') \) is an open embedding.

**Proof.** By Lemma 11.0.8, Proposition 9.0.4 applies, so \( \mathcal{X}(R) \to \mathcal{X}(R') \) is continuous.

For every \( N \geq 1 \), let \( \pi_N : Z_N \to \mathcal{X} \) be as in Proposition 7.0.8. For every \( x \in \mathcal{X}(R) \) there exists an \( N \) such that \( x \) lifts to a point \( z \in Z_N(R) \), and for every \( x \in \mathcal{X}(R') \) there exists an \( N \) such that \( x \) lifts to a point \( z \in Z_N(R') \).

Theorem 11.0.4 implies that \( \mathcal{X}(R) = \bigcup \pi_N(Z_N(R)) \) where each \( \pi_N(Z_N(R)) \) is an open subset, and also that \( \mathcal{X}(R') = \bigcup \pi_N(Z_N(R')) \).

Now consider the commutative diagram (as sets first)

\[
\begin{array}{ccc}
Z_N(R) & \longrightarrow & Z_N(R') \\
\downarrow & & \downarrow \\
\pi_N(Z_N(R)) & \longrightarrow & \pi_N(Z_N(R')).
\end{array}
\]
The vertical arrows are quotients and are open by Theorem 11.0.4, and the top arrow is open. Thus the bottom arrow must be an open. As the union of open maps is open, $\mathfrak{x}(R) \to \mathfrak{x}(R')$ is an open inclusion.

If $R$ is a complete discrete valuation ring and $R'$ its fraction field and $\mathfrak{x}$ is separated, then $\mathfrak{x}(R) \to \mathfrak{x}(R')$ is injective by the valuative criterion for separatedness. In this case, $\mathfrak{x}(R) \to \mathfrak{x}(R')$ is an injective open map, so must be an open immersion. □

12. Products of local rings

Let $I$ be an index set, and let $R = \prod_{i \in I} R_i$ be a product of rings $R_i$. For any $i \in I$, let $e_i \in R$ be the element which is 1 in the $i$th position and 0 in all others. For $J \subseteq I$, let $e^J$ be the element of $R$ whose $i$th component is 1 if $i \in J$ and 0 otherwise, and let $R^J = \prod_{i \in J} R_i = R[1/e^J]$. Let $U^J = \text{Spec } R[1/e^J] \subseteq \text{Spec } R$. For $J_1, J_2 \subseteq I$, $U^{J_1 \cap J_2} = U^{J_1} \cap U^{J_2}$ as $\text{Spec } R[1/e^{J_1}] \cap \text{Spec } R[1/e^{J_2}] = \text{Spec } R[1/e^{J_1} e^{J_2}] = \text{Spec } R[1/e^{J_1 \cap J_2}]$. Similarly, $U^{J_1} \cup U^{J_2} = U^{J_1 \cup J_2}$. In particular, this implies that if $J_1$ and $J_2$ are disjoint, $U^{J_1 \cap J_2} = \emptyset$, and if $J_1, \ldots, J_k$ is a partition of $I$, $\text{Spec } R = U^{J_1} \bigcup \cdots \bigcup U^{J_k}$. We will use the notation and facts presented in this paragraph throughout the section.

**Proposition 12.0.1.** Let $I$ be an index set and $\{k_i\}_{i \in I}$ be a set of fields indexed by $I$. Let $R = \prod_{i \in I} k_i$. Then any open cover of $\text{Spec } R$ can be refined to a disjoint open cover, and furthermore the opens in that disjoint open cover may be taken to be of the form $\text{Spec } R[1/e^J] \subseteq \text{Spec } R$ for $J \subseteq I$.

**Proof.** Let $\{U_j\}$ be an open cover of $\text{Spec } R$. By refining the cover, we may assume each $U_j$ equals $\text{Spec } R[1/f_j]$ for some $f_j \in R$. As $\text{Spec } R$ is quasi-compact, we may assume that the cover is finite.

Now we claim each element of $R$ is the product of a unit and an idempotent. Indeed let $r \in R$ and let $r_i \in k_i$ be its $i$th component. Let $J = \{i \in I : r_i \neq 0\}$. Let $s_i = r_i$ for $i \in J$ and $s_i = 1$ for $i \notin J$. Then the element of $R$ which is $s_i^{-1}$ in the $i$th position is an inverse of $s$, so $s$ is a unit. Furthermore, $r = se^J$, so $r$ has the claimed form.

We conclude from this that there are $J_j \subseteq I$ such that $f_j = u_j e^{J_j}$ where $u_j$ is a unit. Then $\text{Spec } R[1/f_j] = \text{Spec } R[1/e^{J_j}]$.

As the $\text{Spec } R[1/e^{J_j}]$ cover $\text{Spec } R$, $\bigcup J_j = I$. We refine the finite union $\bigcup J_j = I$ to a finite partition of $I$, $\prod_k K_k = I$. Then $\prod_k \text{Spec } R[1/e^{K_k}]$ is a pairwise disjoint open cover refining the original cover. □

**Proposition 12.0.2.** Let $I$ be an index set, and let $\{\mathcal{O}_v\}_{v \in I}$ be a set of local rings indexed by $I$. Let $R = \prod_{v \in I} \mathcal{O}_v$ and let $X = \text{Spec } R$. Then any open cover of $X$ can be refined to a disjoint open cover, and furthermore the opens in that disjoint open cover may be taken to be of the form $\text{Spec } R[1/e^J] \subseteq \text{Spec } R$ for $J \subseteq I$.

**Proof.** For each $v$, let $m_v$ be the maximal ideal of $\mathcal{O}_v$ and let $k_v$ be the residue field. Let $M = \{(r_v)_{v \in I} : \text{ for all } r_v \in m_v\}$.

Let $\{U_j\}$ be a cover of $\text{Spec } R$. As $\text{Spec } R$ is quasi-compact, we can refine this cover to a finite cover. We can refine the cover further so that each $U_j = \text{Spec } R[1/f^j]$ for $f^j \in I$.

Let $J_j = \{v \in I : f^j_v \in \mathcal{O}_v^\times\}$. Because $f^j_v | e^{J_v}$, $\text{Spec } R[1/e^{J_v}] \subseteq \text{Spec } R[1/f^j_v]$. Note that the image of $f^j$ in $R/M = \prod_{v \in J} k_v$ is equal to the image of $e^{J_v}$ in $R/M$ times a unit. This means that the $U_j$ cover $\text{Spec } R/M$ if and only if the $\text{Spec } R[1/e^{J_v}]$ do. But by the preparation paragraph at the beginning of this section, this implies $\bigcup J_j = I$. Again, by that paragraph this
implies that \( \bigcup_j \text{Spec } R[1/e^{j_i}] = \text{Spec } R \), and thus \( \bigcup_j \text{Spec } R[1/e^{j_i}] \) is a refinement of our original cover.

Now again, let \( \{K_k\} \) be a finite partition of \( I \) which refines \( \{J^i\} \). Then \( \{\text{Spec } R[1/e^{K_k}]\} \) is a refinement of \( \{\text{Spec } R[1/e^{J^i}]\} \), which is a refinement of our original cover. But now as the \( K_k \) are disjoint, the cover by the \( \text{Spec } R[1/e^{K_k}] \) is a disjoint open cover and thus is the desired cover. \( \square \)

**Lemma 12.0.3.** Let \( \{R_i\}_{i \in I} \) be a collection of complete discrete valuation rings and let \( N \geq 1 \) be an integer. For each \( i \), let \( S_i \) be a finite free étale \( R_i \)-algebra of rank less than or equal to \( N \). Set \( R = \prod_i R_i \) and \( S = \prod_i S_i \). Then \( S \) is a finite étale \( R \)-algebra.

**Proof.** For any \( i \in I \), \( S_i \) is the direct product of finitely many discrete valuation rings. To \( i \) we may attach a multiset \( M_i \) whose elements are the ranks over \( R_i \) of the discrete valuation rings whose product is \( S_i \). As the rank of \( S_i \) is bounded, there are only finitely many possible such multisets.

Thus we find a finite partition of \( I \) such that every subset \( J \) in the partition has the property that the \( M_i \) are the same for all \( i \in J \). As the conclusion of the lemma is local on \( \text{Spec } R \), we may thus pass to an open subset of \( \text{Spec } R \) corresponding to one subset of the partition. In this way we may assume that \( M_i \) is constant for \( i \in I \). Let \( M = M_i \) for any \( i \in I \).

Now let \( m_1, \ldots, m_k \) be an enumeration of the elements of \( M \). For any \( i \in I \), \( S_i \) may be written \( \prod_{j=1}^{k_i} S_{i,j} \) where \( S_{i,j} \) is a free étale \( R_i \)-algebra of rank \( m_j \). Therefore, \( \prod_i S_i = \prod_i \prod_j S_{i,j} = \prod_j \prod_i S_{i,j} \). So to prove that \( S \) is finite étale over \( R \), it suffices to show that the \( \prod_i S_{i,j} \) are finite étale over \( R \). Thus we replace \( S_i \) with \( S_{i,j} \) to assume that each \( S_i \) is a discrete valuation of some fixed rank \( m \) over \( R_i \).

Now \( S_i \) must be an unramified extension of \( R_i \) of degree \( m \), and by the principal element theorem for such extensions, we have that \( S_i \cong R_i[x]/f_i(x) \) where \( f_i(x) \) is a monic polynomial of degree \( m \). Let \( f(x) \in R[x] \) be the monic degree \( m \) polynomial whose image in \( R[x] \) is \( f_i(x) \) for all \( i \) (its coefficients are in the \( i \)th component are given by the coefficients of \( f_i \)). There is a canonical map \( R[x]/f(x) \to \prod_i R_i[x]/f_i(x) \) which is evidently both injective and surjective. Thus \( S_i \cong R[x]/f(x) \).

As each \( S_i \) is étale, \( f_i(x) \) and \( f_j(x) \) are coprime, so there are polynomials \( a_i(x) \) of degree less than \( m \) such that \( a_i(x)f_j'(x) \equiv 1 \mod f_i(x) \). Let \( a(x) \in R[x] \) be the polynomial of degree less than \( m \) whose image in each \( R_i[x] \) is \( a_i(x) \). Then the image of \( a(x)f'(x) \) in \( R[x]/f(x) \) is a divisor of \( \prod_i R_i[x]/f_i(x) \) of degree 1. Thus \( f(x) \) and \( f'(x) \) are coprime, so \( R[x]/f(x) \) is an étale \( R \)-algebra, and \( S = R[x]/f(x) \). \( \square \)

**Lemma 12.0.4.** Let \( R \) be a product of local rings. Any finitely-generated projective module over \( R \) of constant rank is free.

**Proof.** Let \( M \) be the projective module over \( R \). The module \( M \) corresponds to a locally free sheaf \( \tilde{M} \) on \( \text{Spec } R \). As \( M \) is projective As there is an open cover of \( \text{Spec } R \) such that the restriction of \( \tilde{M} \) is free as \( \tilde{M} \) is locally free, Proposition 12.0.2 implies there is a finite disjoint open cover such that the restriction of \( \tilde{M} \) is free when restricted to this cover. But as the cover is disjoint and the rank of the sheaf is constant, this means that \( \tilde{M} \) must be free, which implies that \( M \) \( \square \)

**Proposition 12.0.5.** Let \( R \) be a product of fields and complete discrete valuation rings.

**Proof.** Lemma 12.0.4 already establishes property (1) of sufficiently disconnected, so are left to establish property (2). Let \( R' \) be a faithfully flat étale \( R \)-algebra. We will find a \( R' \)-algebra that is finite étale as an \( R \)-algebra.

For any module \( M \) over \( R \), we will denote by \( \tilde{M} \) its corresponding quasi-coherent sheaf on \( \text{Spec } R \). Write \( R = \prod_{i \in I} R_i \) where \( I \) is an index set and each \( R_i \) is either a field or a complete discrete valuation ring. For each \( i \), let \( R'_i = R' \otimes_R R_i \). By Lemma 5.0.7 each \( R'_i \) is of the form \( S_i \times T_i \) where
$S_i$ is finite étale over $R_i$ and $T_i$ is étale but not faithfully flat over $R_i$. As $R \to R'$ is faithfully flat so is $R_i \to R'_i$. Therefore, each $S_i$ is nonzero. Let $m_i$ be the degree of $S_i$ over $R_i$. Let $S = \prod_{i \in J} S_i$. Note that $S$ is finite and locally free over $R$, because it is the product of free $R_i$-modules of bounded rank. Finally, by Lemma 12.0.3, $S$ is finite étale over $R$. So now $S$ is free and finite étale over $R$ of rank $m$. Then $S$ is the desired $R'$-algebra.

**Remark 12.0.6.** Let $R$ be a product of fields and complete discrete valuations. Then $R$ is sufficiently disconnected by Proposition 12.0.5, and the good topologization of $R$-points on finite-type $R$-schemes as described in Section 4 has the smooth quotient property by Proposition 4.0.5, we may topologize $\mathfrak{X}(R)$ for any finite-type algebraic stack $\mathfrak{X}$ over $R$ and the results of this paper apply to the topologization.

**Lemma 12.0.7.** Let $R = \prod_i R_i$ be a product of fields and complete discrete valuation rings. Let $\mathfrak{X}$ be a quasi-separated Deligne-Mumford stack over $R$. Then $\mathfrak{X}(R) = \prod_i \mathfrak{X}(R_i)$ as topological spaces.

**Proof.** We first assume that $\mathfrak{X}$ is an algebraic space.

By Proposition 7.0.8 we may choose a smooth cover $Z \to \mathfrak{X}$ by a separated scheme such that $Z(R) \to \mathfrak{X}(R)$ is surjective. Let $T = Z \times_{\mathfrak{X}} Z$. We may describe $\mathfrak{X}(R)$ as the coequalizer of the two projections $T(R) \rightrightarrows Z(R)$ by Proposition 9.0.5. Similarly we may describe $\mathfrak{X}(R_i)$ as the coequalizer of $T_i(R_i) \rightrightarrows Z_i(R_i)$. By Remark 4.0.3 $T(R) = \prod_i T_i(R_i)$ and $\prod_i Z_i(R_i) = Z$. Therefore, we may also describe $\mathfrak{X}(R)$ may also be described as the coequalizer of $\prod_i T_i(R_i) \rightrightarrows \prod_i Z_i(R_i)$, but as coequalizers commute with products this implies that $\mathfrak{X}(R) = \prod_i \mathfrak{X}(R_i)$.

The proof when $\mathfrak{X}$ is a Deligne-Mumford stack is the same, but $Z \times_{\mathfrak{X}} Z = T$ is only an algebraic space and we use the algebraic space case to deduce that $T(R) = \prod_i T(R_i)$.

**Proposition 12.0.8.** Let $S$ be the direct limit of sufficiently disconnected rings. Then $S$ is sufficiently disconnected.

**Proof.** Write $S = \lim_{\rightarrow} T_i$, where each $T_i$ is sufficiently disconnected. We first check property (1) of sufficiently disconnected. Let $M$ be a finitely generated projective module over $S$. As finitely generated projective modules are finitely presented, there must exist an $i$ and finitely presented projective module $M_i$ over $T_i$ such that $M \cong M_i \otimes_{T_i} S$. As $T_i$ is sufficiently disconnected, $M_i$ is free, so $M$ is free.

Now we check property (2). Let $R'$ be a faithfully flat étale $S$-algebra. Then as $R'$ is a finitely presented $S$-algebra, there is an $i$ and a faithfully flat étale $T_i$-algebra $R'_i$ such that $R' \cong R'_i \otimes_{T_i} S$. As $T_i$ is sufficiently disconnected, there is a $R'_i$-algebra $R''_i$ that is finite étale as a $T_i$-algebra. Then $R'' = R''_i \otimes_{T_i} S$ is a $R'$-algebra that is finite étale as an $S$-algebra. Thus we conclude that $S$ is sufficiently disconnected.

13. **Stacks over the adeles**

In this section $k$ will be a global field. For any place, $v$, of $k$, $k_v$ will denote the completion at $v$, and if $v$ is nonarchimedean $\mathcal{O}_v$ will denote the valuation ring of $k_v$. Through the section $J$ will be a set of places of $k$ and $R = \prod_{v \in J} (k_v, \mathcal{O}_v)$. For any finite set of places $J \subseteq I$, we will let $R_J$ denote $\prod_{v \in J} k_v \times \prod_{v \in I \setminus J} \mathcal{O}_v$.

If $\mathfrak{Y}$ is an algebraic stack over $R_J$, then by Remark 12.0.6 we may topologize $\mathfrak{Y}(R_J)$. If $J' \supseteq J$, we have a natural open inclusion $R_J \to R_{J'}$ which by Lemma 11.0.8 induces a natural map $\mathfrak{Y}(R_J) \to \mathfrak{Y}(R_{J'})$. Now assume furthermore that $\mathfrak{Y}$ is quasi-separated. By Lemma 12.0.7, $\mathfrak{Y}(R_J) \cong$
We have \( \prod_{\ell \in J} \mathcal{Y}(k_{\ell}) \times \prod_{\ell \in I \setminus J} \mathcal{Y}(\mathcal{O}_{\ell}) \), and similarly \( \mathcal{Y}(R_{J'}) \cong \prod_{\ell \in J'} \mathcal{Y}(k_{\ell}) \times \prod_{\ell \in I \setminus J} \mathcal{Y}(\mathcal{O}_{\ell}) \). Then by Theorem 11.0.9, \( \mathcal{Y}(\mathcal{O}_{\ell}) \) is open in \( \mathcal{Y}(k_{\ell}) \). This implies \( \mathcal{Y}(R_{J}) \) is open in \( \mathcal{Y}(R_{J'}) \).

If \( \mathcal{X} \) is any finitely presented stack over \( R \), there is a finite \( J \subseteq I \) and stack \( \mathcal{Y} \) over \( R_{J} \) such that \( \mathcal{X} \cong \mathcal{Y} \times_{\text{Spec } R} \text{Spec } R \). Furthermore, as \( \mathcal{X} \) is finitely presented, \( \mathcal{X}(R) = \varinjlim_{J' \supseteq J} \mathcal{Y}(R_{J'}) \) where the colimit is over finite subsets of \( I \) containing \( J \) (see [LMB00][Proposition 4.18])

**Definition 13.0.1.** For an algebraic \( R \)-stack \( \mathcal{X} \) of finite presentation, there exists a finite subset \( J \subseteq I \) and \( R_{J} \)-scheme \( \mathcal{Y} \) such that \( \mathcal{X} \cong \mathcal{Y} \times_{\text{Spec } R} \text{Spec } R \). We define \( \mathcal{X}(R) = \varprojlim_{J' \supseteq J} \mathcal{Y}(R_{J'}) \) as a topological space.

**Proposition 13.0.2.** Let \( \mathcal{X} \) be a separated Deligne-Mumford stack of finite presentation over \( R \), then

\[
\mathcal{X}(R) = \prod_{\ell}(\mathcal{X}(k_{\ell}), \mathcal{X}(\mathcal{O}_{\ell}))
\]

as topological spaces.

**Proof.** Let \( J \subseteq I \) be a finite subset such that there exists \( \mathcal{Y} \) an algebraic stack over \( R_{J} \) with \( \mathcal{X} \cong \mathcal{Y} \times_{R_{J}} R \).

Using Proposition 7.0.8 find \( Z \to \mathcal{Y} \) a smooth cover by a separated scheme such that, \( Z(R_{J'}) \to \mathcal{Y}(R_{J'}) \) is surjective for all \( J' \supseteq J \).

For such \( J' \) by Lemma 12.0.7, \( \mathcal{Y}(R_{J'}) = \prod_{\ell \in J'} \mathcal{Y}(k_{\ell}) \times \prod_{\ell \not\in I \setminus J} \mathcal{Y}(\mathcal{O}_{\ell}) \) as topological spaces. Note that \( \mathcal{Y}(k_{\ell}) = \mathcal{X}(k_{\ell}) \) and \( \mathcal{Y}(\mathcal{O}_{\ell}) = \mathcal{X}(\mathcal{O}_{\ell}) \). Therefore, \( \mathcal{Y}(R_{J'}) = \prod_{\ell \in J'} \mathcal{X}(k_{\ell}) \times \prod_{\ell \not\in I \setminus J} \mathcal{X}(\mathcal{O}_{\ell}) \). Taking a colimit over \( J' \) yields the result. \( \square \)

**Definition 13.0.3.** Let \( k \) be a field of characteristic zero. A stacky curve over \( k \) is an algebraic stack over \( k \) which is smooth, irreducible, 1-dimensional, and Deligne-Mumford and such that there is a dense Zariski open \( U \subseteq X \) such that \( U \) is a scheme.

**Definition 13.0.4.** Let \( k \) be a field of characteristic 0, and let \( \mathcal{X} \) be a stacky curve over \( k \). Let \( \mathcal{X}_{\text{coarse}} \) be the coarse moduli space and let \( \mathcal{X} \to \mathcal{X}_{\text{coarse}} \) be the natural morphism. Let \( P_{1}, \ldots, P_{n} \in \mathcal{X}_{\text{coarse}}(k) \) over which \( \mathcal{X} \to \mathcal{X}_{\text{coarse}} \) is not an isomorphism. For each \( 1 \leq i \leq n \) let \( e_{i} \) be the order of the stabilizer over \( P_{i} \). Then we define the Euler characteristic of \( \mathcal{X} \) to be

\[
\chi(\mathcal{X}) = \chi(\mathcal{X}_{\text{coarse}}) - n + \sum_{i} \frac{1}{e_{i}}
\]

Define the genus of \( \mathcal{X} \) by

\[
g(\mathcal{X}) = \frac{2 - \chi(\mathcal{X})}{2}.
\]

**Lemma 13.0.5.** Let \( \mathcal{X} \) be a stacky curve over a essentially analytic field \( k \) of characteristic 0. Let \( U \subseteq \mathcal{X} \) be a dense open substack which is a scheme. Then \( U(k) \) is dense in \( \mathcal{X}(k) \).

**Proof.** Set \( X = \mathcal{X}_{\text{coarse}} \). Note that \( U \to X \) is an open inclusion. Let \( C \) be the complement of \( U \) in \( X \) with the reduced scheme structure; this is a finite scheme. By Proposition 7.0.8, we may find an étale cover \( f : Z \to X \) by a scheme, such that \( Z(k) \to X(k) \) is surjective. Note that \( Z \) must also be 1-dimensional and smooth.

Let \( x \in \mathcal{X}(k) \), and let \( V \subseteq \mathcal{X}(k) \) be any open subset containing \( x \). The goal is to show that \( V \cap U(k) \) is nonempty. Let \( z \in Z(k) \) be any preimage of \( x \).

Let \( Z' \subseteq Z \) be the connected component of \( Z \) containing \( z \). We will first show that composite \( Z' \to X \) is nonconstant. Assume by sake of contradiction that it is constant. Because \( Z' \to X \) is constant, \( Z' \) must map to a unique point of \( X \), and furthermore since \( Z'(k) \) is nonempty, the point
in the image of $Z'$ must have residue field $k$. Let $p \in X(\overline{k})$ be a geometric point localized at the image of $Z'$. As $X$ is a coarse moduli space, $X(\overline{k}) \to X(\overline{k})$ is bijective, so we may lift $p \in X(\overline{k})$ to some map $q : \text{Spec } \overline{k} \to X$. Set $T = Z' \times_{X,q} \text{Spec } \overline{k}$. Consider the map $T(\overline{k}) \to Z'(\overline{k})$. This must be surjective as every point in $Z'(\overline{k})$ maps to $p \in X(\overline{k})$ and therefore must have image in $X$ isomorphic to $q$. Therefore, $T(\overline{k})$ is infinite. On the other hand, $T$ is étale over $\text{Spec } \overline{k}$, so $T(\overline{k})$ is finite. This is a contradiction, so $Z' \to X$ must be nonconstant.

As $k$ is essentially analytic, near $z \in Z'(k)$ the space $Z'(k)$ is homeomorphic to an open subset of $k$. In particular any nonempty open $W \subseteq Z'(k)$ has infinitely many points and thus is Zariski dense in $Z'$. If the open $W$ under the composite $Z'(k) \to X(k) \to X(k)$ lands in $C(k)$, then as $W$ is Zariski dense in $Z'$, $Z' \to X(k)$ maps to $p \in X(k)$ and therefore must have image in $X$ isomorphic to $q$. Therefore, $T(\overline{k})$ is infinite. On the other hand, $T$ is étale over $\text{Spec } \overline{k}$, so $T(\overline{k})$ is finite. This is a contradiction, so $Z' \to X$ must be nonconstant.

Theorem 13.0.6. Let $k$ be a number field. Let $R$ be obtained from the ring of adeles of $k$ by removing the factor corresponding to one place. Let $\mathcal{X}$ be a stacky curve over $k$ of genus less than $1/2$. Then $\mathcal{X}(k)$ is dense in $\mathcal{X}(R)$.

Proof. In the notation of the section, $I$ is all but one place of $k$.

We may assume $\mathcal{X}(R)$ is nonempty. By [BP19], the genus hypothesis on $\mathcal{X}$ implies that $\mathcal{X}_{\text{coarse}} \cong \mathbb{P}^1_k$ and the map $\mathcal{X} \to \mathcal{X}_{\text{coarse}}$ is an isomorphism away from one point, which we assume to be $\infty$. We must show that any nonempty open $V \subseteq \mathcal{X}(R')$ contains a point of $\mathcal{X}(k)$.

There is a finite set of places $S$ of $k$ such that $\mathcal{X}$ descends to an $\mathcal{O}_{k,S}$-stack $\mathcal{X}'$. We can enlarge $S$ so that $\mathcal{X}'_{\text{coarse}} \cong \mathbb{P}^1_{\mathcal{O}_{k,S}}$. We view $\mathbb{A}^1_{\mathcal{O}_{k,S}}$ as an open substack in $\mathcal{X}'$. For any $S'$ a set of places, let $R^{S'} = \prod_{v \in S'}(k_v, \mathcal{O}_v)$.

Now there must be a finite set of places $S' \supseteq S$ and nonempty open subsets $V_v \subseteq \mathcal{X}'(k_v)$ for $v \in S'$ such that $V \supseteq \prod_{v \in S'} V_v \times \mathcal{X}(R^{S'}) \subseteq \prod_v \mathcal{X}(k_v) = \mathcal{X}(R)$. By Lemma 13.0.5, we can shrink each $V_v$ to be nonempty have and the property that $V_v \subseteq \mathbb{A}^1(k_v)$. Then $V$ contains $\prod_{v \in S'} V_v \times \mathbb{A}^1(R^{S'}) \subseteq \mathbb{A}^1(R)$, and strong approximation for the adeles guarantees that this has a $k$-point. This completes the proof. □
14. Appendix

Lemma 14.0.1. Let $I$ be an index set. For each $i \in I$, let $X_i$ be a topological space and $Z_i \subseteq X_i$ a subspace.

Then, the natural map $\prod_{i \in I} Z_i \to \prod_{i \in I} X_i$ is a homeomorphism onto its image where we give each product the product topology.

Proof. Let $T$ be the image of $\prod_{i \in I} Z_i$ in $\prod_{i \in I} X_i$. The map $\prod_{i \in I} Z_i \to T$ is bijective. Thus the topology on $T$ is coarser than that on $\prod_{i \in I} Z_i$. Therefore, we must show that any open $U \subseteq \prod_{i \in I} Z_i$ has open image in $T$.

For each $i \in I$, we choose an open $U_i \subseteq Z_i$ such that $U_i = Z_i$ for all but finitely many $i$. The set $\prod_{i \in I} U_i$ is open in $\prod_{i \in I} Z_i$, and opens of this type form a basis for the topology on $\prod_{i \in I} Z_i$. Therefore, it suffices to show that for any open of $\prod_{i \in I} Z_i$ of this form has open image in $T$.

Let us choose such an open and thus such $U_i$. As $Z_i$ is a subspace of $X_i$, we may find open $V_i \subseteq X_i$ such that $V_i \cap Z_i = U_i$; if $U_i = Z_i$, we may take $V_i = X_i$. For each $i \in I$, choose such a $V_i$ while choosing $V_i = X_i$ if $U_i = Z_i$. Then $V = \prod_{i \in I} V_i$ is an open subset of $\prod_{i \in I} X_i$. Furthermore, $V \cap T$ is the image of $\prod_{i \in I} U_i$. Thus the image of $\prod_{i \in I} U_i$ in $T$ is open as desired. \hfill \Box

Lemma 14.0.2. Let $I$ be an index set. For each $i \in I$, let $X_i \to Y_i$ be a map of topological spaces which is a quotient onto its image.

Then the natural map $\prod_{i \in I} X_i \to \prod_{i \in I} Y_i$ is a quotient onto its image.

Proof. For each $i$, let $R_i = X_i \times_{Y_i} X_i$. Then $Y_i$ is the coequalizer of $R_i \rightrightarrows X_i$ where the maps are the projections. As products commute with connected colimits, we conclude that $\prod_{i \in I} R_i \rightrightarrows \prod_{i \in I} X_i$, and thus $\prod_{i \in I} Y_i$ is a quotient of $\prod_{i \in I} X_i$. \hfill \Box

Proposition 14.0.3. Let $X, Y, Z$, and $W$ be topological spaces. Let

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\scriptstyle s} & & \downarrow{\scriptstyle g} \\
Z & \xrightarrow{t} & W
\end{array}
$$

be a commutative diagram of the underlying sets. If $f$, $g$, and $s$ are continuous and if furthermore, $s$ is a quotient map, then $t$ is continuous.

Proof. Let $U \subseteq W$ be an open subset. Let $U' = f^{-1}(g^{-1}(U))$. As $f$ and $g$ are continuous, $U' \subseteq X$ is open.

By the commutativity of the diagram, we also have $U' = s^{-1}(t^{-1}(U))$. Therefore, $U' = s^{-1}(s(U'))$. By the definition of the quotient topology, as $U'$ is open and $U' = s^{-1}(s(U'))$, $s(U') \subseteq Z$ is open. But as $s$ is surjective, $s(U') = s(s^{-1}(t^{-1}(U))) = t^{-1}(U)$. Therefore, $t^{-1}(U)$ is open. As $U$ was arbitrary, we conclude that $t$ is continuous as desired. \hfill \Box

Proposition 14.0.4. Let $X, Y, Z$, and $W$ be topological spaces. Let

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\scriptstyle s} & & \downarrow{\scriptstyle g} \\
Z & \xrightarrow{t} & W
\end{array}
$$

be commutative diagram of topological spaces. If $f$ and $g$ are quotients, then so is $t$.

Proof. As $f$ and $g$ are quotients, so is $g \circ f$. This means that $W$ has the finest topology such that $X \to W$ is continuous.
If $t$ is not a quotient, we may put a finer topology on $W$ so that $Z \to W$ is continuous. If this were the case, then that finer topology on $W$ would make the composite $X \to Z \to W$ be continuous. However, $W$ already has the finest topology making that composite continuous. We conclude $t$ is a quotient.

Proposition 14.0.5. Let $U$ and $V$ be topological spaces and $f : U \to V$ a continuous map.

Let $I$ be an index set and let $\{U_i\}_{i \in I}$ be an open cover of $U$. If for every $i$, $U_i \to V$ is a quotient onto its image, then $U \to V$ is a quotient onto its image.

Proof. Let $Z$ be the image of $U$ in $V$, and let $Z_i$ be the image of $U_i$ for each $i \in I$. To show that $U \to V$ is a quotient onto $Z$, we must show that if $W \subseteq U$ is an open such that $f^{-1}(f(W)) = W$, then $f(W)$ is open in $Z$.

Lemma 14.0.6. Let

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{g} & & \downarrow{h} \\
Z & & Z
\end{array}
$$

be a commutative diagram of schemes. If $g$ is finite étale and $h$ is separated and étale, then $f$ is étale and the scheme theoretic image of $f$ is finite étale over $Z$.

Therefore, if $R$ is a ring, $R'$ an étale $R$-algebra, and $S$ a $R'$-algebra which is finite étale as an $R$-algebra, the image $T$ of $R'$ in $S$ is finite étale over $R$.

Proof. Consider $X \times Y$ let the projection to $X$ be $\pi_1$ and the projection to $Y$ be $\pi_2$. Note that $\pi_1$ and $\pi_2$ are pullbacks of $h$ and $g$ respectively, so are étale.

The graph $\Gamma_f$ provides a section to $\pi_1$, and thus as a section of an étale map $X \xrightarrow{\Gamma_f} X \times_Z Y$ has open image. The composite $X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{\pi_2} Y$ is $f$, and the first map is an open inclusion and the second étale, so we conclude that $f$ is étale.

Now $f$ has open image because it is étale, and it has closed image because $X$ is finite. Therefore, $f$ is surjective onto a clopen subset of $Y$. This clopen subset must be the scheme theoretic image of $Y$ and since $Y \to Z$ is étale and the scheme theoretic image of $X$ is open, we conclude that the restriction of $h$ to the scheme theoretic image of $X$ is étale.

Remark 14.0.7. Lemma 14.0.6 holds when $Y$ is only an algebraic space.
Acknowledgments

I would to thank Bjorn Poonen for suggesting this problem and for his guidance.

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