Crossed homomorphisms and low dimensional representations of mapping class groups of surfaces

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Abstract

We continue the study of low dimensional linear representations of mapping class groups of surfaces initiated by Franks–Handel and Korkmaz. We consider $(2g + 1)$-dimensional complex linear representations of the pure mapping class groups of compact orientable surfaces of genus $g$. We give a complete classification of such representations for $g \geq 7$ up to conjugation, in terms of certain twisted 1-cohomology groups of the mapping class groups. A new ingredient is to use the computation of a related twisted 1-cohomology group by Morita. The classification result implies in particular that there are no irreducible linear representations of dimension $2g + 1$ for $g \geq 7$, which marks a contrast with the case $g = 2$.

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1 Introduction

Linear representations are fundamental objects associated with the mapping class group of a surface. However, the whole picture of them, even in low dimensions, are not understood very well. Until the appearance of the Topological Quantum Field Theory around 1990, the symplectic representation was almost the only known linear representation, whereas the residually finiteness ([7], [2]) had assured the existence of other linear representations. Even after the TQFT, the existence of lower dimensional representations which does not factor through the permutation group of punctures continued to remain unknown. We note, if the surface has \( r > 1 \) punctures, that the mapping class group usually considered has a surjection to the symmetric group of degree \( r \) which is induced by the permutation of the punctures, and therefore the totality of linear representations should contain the representation theory of symmetric group and would be rather complicated.

In 2011, Franks–Handel [6] and then Korkmaz [12] considered the pure version of the mapping class group of a surface of genus \( g \geq 1 \), the group of those mapping classes with trivial permutation of the punctures, and showed that there exist in fact no nontrivial complex linear representations of dimensions less than \( 2g \), with exceptions of abelian representations which occur only in the case of \( g \leq 2 \). This result was soon succeeded by Korkmaz [13], who showed up to conjugation that the symplectic representation is the only nontrivial complex linear representation of dimension \( 2g \) in the case of \( g \geq 3 \).

In this paper, following these works of Franks–Handel and Korkmaz, we consider the complex linear representations of dimension \( 2g + 1 \). As was studied by Trapp [21], such a representation can be constructed from a crossed homomorphism of the mapping class group with values in the 1-homology group of the closed surface of the same genus. Our main result asserts, in the case of \( g \geq 7 \), that all nontrivial representations of dimension \( 2g + 1 \) can be obtained in this way up to dual (Theorem 1.6). By using this main result, in this introduction, we give a complete classification of the \((2g+1)\)-dimensional complex linear representations for \( g \geq 7 \). It follows in particular that there are no irreducible representations of dimension \( 2g + 1 \). The classification is described in terms of certain twisted 1-cohomology group of the (pure) mapping class group. A new ingredient to
prove the main result is to use the computation of a related twisted 1-cohomology group by Morita [16].

1.1 Statement of the classification result

Setting

Let \( S = S^p_{g,r} \) denote a connected compact oriented surface of genus \( g \geq 1 \) with \( p \geq 0 \) boundary components and \( r \geq 0 \) punctures in interior. Here, a puncture is meant by a marked point. The mapping class group of \( S \), denoted by \( \text{Mod}(S) \), is defined to be the group of the isotopy classes of the orientation preserving homeomorphisms of \( S \) which preserve the marked points and the boundary, both pointwise. Here, the isotopies are assumed to preserve the marked points and the boundary, both pointwise. An \( n \)-dimensional linear representation of \( \text{Mod}(S) \) is simply a group homomorphism \( \text{Mod}(S) \rightarrow \text{GL}(n, \mathbb{C}) \). The two \( n \)-dimensional linear representations \( \phi_1 \) and \( \phi_2 : \text{Mod}(S) \rightarrow \text{GL}(n, \mathbb{C}) \) are said to be conjugate if there exists some \( A \in \text{GL}(n, \mathbb{C}) \) such that \( \phi_1(f) = A\phi_2(f)A^{-1} \) for each \( f \in \text{Mod}(S) \). Let \( X \) be the set of conjugacy classes of all \((2g + 1)\)-dimensional linear representations of \( \text{Mod}(S) \). We denote by \( X_0 \) the set \( X \) with the class of the trivial representation removed.

Symplectic representation \( \rho_0 \)

Let \( \bar{S} \) denote the closed surface obtained from \( S \) by gluing a 2-disk along each boundary component of \( S \) and forgetting all the punctures. We denote by \( H \) the homology group \( H_1(\bar{S}; \mathbb{Z}) \), which is a free abelian group of rank \( 2g \), and by \( H_\mathbb{C} \) the homology group \( H_1(\bar{S}; \mathbb{C}) \) with coefficients in \( \mathbb{C} \). Note that \( H_\mathbb{C} \) is canonically isomorphic to \( H \otimes \mathbb{Z}_\mathbb{C} \) so that we may consider as \( H \subseteq H_\mathbb{C} \).

The inclusion \( S \hookrightarrow \bar{S} \) induces the homomorphism \( \text{Mod}(S) \rightarrow \text{Mod}(\bar{S}) \) by extending mapping classes of \( S \) with the identity on \( \bar{S} \setminus S \) so that the natural action of \( \text{Mod}(\bar{S}) \) on \( H_\mathbb{C} \) induces a group homomorphism \( \rho_0 : \text{Mod}(S) \rightarrow \text{GL}(H_\mathbb{C}) \). We call \( \rho_0 \) the symplectic representation of \( \text{Mod}(S) \). We use the same symbol to denote the matrix form of this representation as \( \rho_0 : \text{Mod}(S) \rightarrow \text{GL}(2g, \mathbb{C}) \) with respect to a certain basis for \( H_\mathbb{C} \), which we often do not specify explicitly. This would not make confusion since we are concerned with linear representations only up to conjugation. We consider \( H_\mathbb{C} \), as well as \( \mathbb{C}^{2g} \), a left \( \text{Mod}(S) \)-module via \( \rho_0 \).

Semidirect product and linear representation

A crossed homomorphism of \( \text{Mod}(S) \), with values in a left \( \text{Mod}(S) \)-module \( M \), is a mapping \( k : \text{Mod}(S) \rightarrow M \) which satisfies

\[
k(fg) = k(f) + fk(g) \quad \text{for } f, g \in \text{Mod}(S).
\]

(1.1)

The crossed homomorphism \( k \) is said to be principal if there exists some \( m \in M \) such that \( k(f) = fm - m \) for each \( f \in \text{Mod}(S) \). All the crossed homomorphisms of \( \text{Mod}(S) \) with values in \( M \) naturally form an additive group, and its quotient by the subgroup consisting of all the principal crossed homomorphisms is isomorphic to the first cohomology group of \( \text{Mod}(S) \) with twisted coefficients in \( M \), which we denote by \( H^1(\text{Mod}(S); M) \).
In order to construct a \((2g+1)\)-dimensional linear representation from a crossed homomorphism, we consider the Mod\((S)\)-module \(H_\mathbb{C} = \mathbb{C}^{2g}\), via the symplectic representation. For the natural left action of \(\text{GL}(2g, \mathbb{C})\) on \(\mathbb{C}^{2g}\), we denote the corresponding semidirect product by \(\mathbb{C}^{2g} \rtimes \text{GL}(2g, \mathbb{C})\). The correspondence

\[
(z, A) \mapsto \begin{pmatrix} A & z \\ 0 & 1 \end{pmatrix} \quad (z \in \mathbb{C}^{2g}, A \in \text{GL}(2g, \mathbb{C}))
\]

defines an injective homomorphism \(i : \mathbb{C}^{2g} \rtimes \text{GL}(2g, \mathbb{C}) \to \text{GL}(2g + 1, \mathbb{C})\). Given a crossed homomorphism \(c : \text{Mod}(S) \to \mathbb{C}^{2g}\), with values in the left \(\text{Mod}(S)\)-module \(H_\mathbb{C} = \mathbb{C}^{2g}\), the correspondence

\[
f \in \text{Mod}(S) \mapsto (c(f), \rho_0(f))
\]

defines a homomorphism \(\text{Mod}(S) \to \mathbb{C}^{2g} \rtimes \text{GL}(2g, \mathbb{C})\). Composing this homomorphism with \(i\), one obtains a nontrivial \((2g + 1)\)-dimensional linear representation

\[
\phi_c : \text{Mod}(S) \to \text{GL}(2g + 1, \mathbb{C}).
\]

It will turn out that its conjugacy class \([\phi_c] \in X_0\) depends only on its cohomology class in \(H^1(\text{Mod}(S); H_\mathbb{C})\), and furthermore, does not change under scalar multiplication by a nonzero complex number in the cohomology group (Lemma \[1.4\]). Therefore, the correspondence which sends the crossed homomorphism \(c\) to \([\phi_c] \in X_0\) descends to a mapping

\[
\sigma : H^1(\text{Mod}(S); H_\mathbb{C})/\mathbb{C}^\times \to X_0
\]

where \(\mathbb{C}^\times\) denotes \(\mathbb{C} \setminus \{0\}\).

Now let \(\iota : X_0 \to X_0\) be the involution which sends the conjugacy class of any linear representation \(\phi : \text{Mod}(S) \to \text{GL}(2g + 1, \mathbb{C})\) to the class of its dual representation

\[
\phi^* : \text{Mod}(S) \to \text{GL}(2g + 1, \mathbb{C}), \quad \phi^*(f) = (\iota^*(\phi(f)))^{-1} \quad \text{for} \ f \in \text{Mod}(S).
\]

Let \(\overline{\sigma} : H^1(\text{Mod}(S); H_\mathbb{C})/\mathbb{C}^\times \to X_0/\langle \iota \rangle\) denote the composition of \(\sigma\) with the quotient mapping \(X_0 \to X_0/\langle \iota \rangle\). We denote by \(\text{Fix}(\iota)\) the set of fixed points of \(\iota\). The following classification result states that we can identify \(H^1(\text{Mod}(S); H_\mathbb{C})/\mathbb{C}^\times\) with almost the exact half of \(X_0\) via \(\sigma\), if \(g\) is sufficiently large:

**Theorem 1.1.** Let \(g \geq 7\).

1. The mapping \(\overline{\sigma} : H^1(\text{Mod}(S); H_\mathbb{C})/\mathbb{C}^\times \to X_0/\langle \iota \rangle\) is bijective.

2. The set \(\text{Fix}(\iota)\) consists of a single element represented by the direct sum of the symplectic representation and the 1-dimensional trivial representation.

**Remark 1.2.** (1) As a consequence, for two crossed homomorphisms \(c\) and \(c'\), we can easily see, once a basis of \(H_\mathbb{C}\) is fixed arbitrarily, that the representations \(\phi_c\) and \(\phi_{c'}\) represent the same class in \(X_0\) if and only if they are conjugate by an element in \(i(\mathbb{C}^{2g} \rtimes \text{GL}(2g, \mathbb{C}))\).

(2) For the case \(r = 0\) and \(p = 1\), the computation by Morita \[16\] implies

\[
\#H^1(\text{Mod}(S); H_\mathbb{C})/\mathbb{C}^\times = 2
\]
Therefore, at least for \( g \geq 7 \), we can conclude that the two \((2g+1)\)-dimensional complex linear representations, the one constructed by Trapp [21], and the other by Matsumoto–Nishino–Yano [15] are conjugate up to dual. It might be interesting to point out that both the representations seem to have their origins in Iwahori–Hecke algebra of Artin group.

(3) The above Theorem 1.1 does not hold for \((g, r, p) = (2, 0, 0)\). In fact, while Morita’s computation (ibid) shows \( H^1(\text{Mod}(S); H_C) = 0 \) in this case, the Jones representation of genus 2 ([9], [10]), a modification of an Iwahori–Hecke algebra representation of the 6-strand braid group, gives a family of infinitely many non-conjugate irreducible 5-dimensional complex linear representations of \( \text{Mod}(S) \). It is not known whether the theorem holds for \( g = 3-6 \) or not.

The following is an obvious corollary to Theorem 1.1:

**Corollary 1.3.** Let \( g \geq 7 \). Then \( \text{Mod}(S) \) has no irreducible complex linear representations of dimension \( 2g + 1 \).

Conversely, this corollary, together with results by Franks–Handel and Korkmaz, implies the surjectivity of \( \sigma \) immediately (Theorem 1.6). However, our argument for the corollary simultaneously implies the surjectivity of \( \sigma \), the proof of which occupies the most part of this paper. It might be worthwhile to pursue simpler arguments for the corollary.

### 1.2 Proof of Theorem 1.1

Suppose \( g \geq 1 \) for the moment. We first check that the mapping \( \sigma \) is well-defined. Recall that for a crossed homomorphism \( c : \text{Mod}(S) \to H_C = \mathbb{C}^{2g} \) via the symplectic representation \( \rho_0 \), the linear representation \( \phi_c : \text{Mod}(S) \to \text{GL}(2g + 1, \mathbb{C}) \) is defined by

\[
\phi_c(f) = \begin{pmatrix} \rho_0(f) & c(f) \\ 0 & 1 \end{pmatrix} \quad \text{for } f \in \text{Mod}(S).
\]  

(1) The conjugacy class \([\phi_c] \in X_0\) depends only on the cohomology class of \( c \) in \( H^1(\text{Mod}(S); H_C)\).

(2) For any \( z \in \mathbb{C}^\times \), \([\phi_{zc}] = [\phi_c] \) in \( X_0\).

**Proof.** (1) Let \( c' : \text{Mod}(S) \to \mathbb{C}^{2g} \) be another crossed homomorphism representing \([c] \in H^1(\text{Mod}(S); \mathbb{C})\). We can then choose \( w_0 \in \mathbb{C}^{2g} \) so that

\[
c(f) - c'(f) = \rho_0(f)w_0 - w_0
\]

for each \( f \in \text{Mod}(S) \). Putting \( A := \begin{pmatrix} I & w_0 \\ 0 & 1 \end{pmatrix} \) where \( I \) denotes the identity matrix, a direct computation implies

\[
A\phi_c(f)A^{-1} = \phi_{c'}(f) \quad \text{for each } f \in \text{Mod}(S).
\]
This shows $[\phi_c] = [\phi_c]$.

(2) For any $z \in \mathbb{C}^*$, let $A := \begin{pmatrix} zI & 0 \\ 0 & 1 \end{pmatrix}$. Then for each $f \in \text{Mod}(S)$, a direct computation implies

$$A\phi_c(f)A^{-1} = \phi_{zc}(f).$$

Hence we have $[\phi_{zc}] = [\phi_c]$ in $X_0$. \hfill $\Box$

In view of this lemma, the following is well-defined.

**Definition 1.5.** We define $\sigma : H^1(\text{Mod}(S); H_C)/\mathbb{C}^* \to X_0$ as the mapping which sends the class represented by a crossed homomorphism $c : \text{Mod}(S) \to H_C = \mathbb{C}^{2g}$ to $[\phi_c] \in X_0$. We also define $\bar{\sigma} : H^1(\text{Mod}(S); H_C)/\mathbb{C}^* \to X_0/\langle t \rangle$ as the composition of $\sigma$ with the quotient mapping $X_0 \to X_0/\langle t \rangle$ where $t$ denotes the involution induced by taking the dual representation.

We remark that $\sigma$, and hence $\bar{\sigma}$, are independent of the identification $H_C = \mathbb{C}^{2g}$, i.e., the choice of a basis of $H_C$.

To prove Theorem 1.1, the most crucial is the following.

**Theorem 1.6.** For $g \geq 7$, the mapping $\bar{\sigma}$ is surjective.

The assumption $g \geq 7$ for Theorem 1.1 is necessary to apply this theorem. In particular, the injectivity of $\bar{\sigma}$ holds for $g \geq 1$ as shown below. The proof of this theorem occupies most of the remaining sections.

We proceed to complete the proof of Theorem 1.1 assuming Theorem 1.6.

Suppose $g \geq 1$ again. We first check $\bar{\sigma}$ is injective. Any element of $H^1(\text{Mod}(S); H_C)$ is represented by a crossed homomorphism $c : \text{Mod}(S) \to H_C = \mathbb{C}^{2g}$. We denote its representing class in $H^1(\text{Mod}(S); H_C)$ and $H^1(\text{Mod}(S); H_C)/\mathbb{C}^*$ by $[c]$ and $[\bar{c}]$, respectively.

Let $c_1$, $c_2 : \text{Mod}(S) \to H_C = \mathbb{C}^{2g}$ be two crossed homomorphisms which satisfy $\bar{\sigma}(c_1) = \bar{\sigma}(\bar{c}_2)$. We then have either $[\phi_{c_1}] = [\phi_{c_2}]$ or $[\phi_{c_1}] = \iota([\phi_{c_2}])$.

In the case $[\phi_{c_1}] = [\phi_{c_2}]$, choose $A \in \text{GL}(2g + 1, \mathbb{C})$ so that $\phi_{c_2}(f) = A\phi_{c_1}(f)A^{-1}$ for each $f \in \text{Mod}(S)$. This equality is equivalent to $\phi_{c_2}(f)A = A\phi_{c_1}(f)$, and if we take a block decomposition of $A$ as $A = \begin{pmatrix} A_0 & w \\ s & x \end{pmatrix}$ with $A_0$ a $2g \times 2g$ matrix and $x \in \mathbb{C}$, it becomes

$$\begin{pmatrix} \rho_0(f)x_0 + c_2(f)s & \rho_0(f)x + c_2(f)x \\ s & x \end{pmatrix} = \begin{pmatrix} A_0\rho_0(f) & A_0c_1(f) + w \\ s\rho_0(f) & sc_1(f) + x \end{pmatrix}$$

for each $f \in \text{Mod}(S)$. In view of the lower left block, we have

$$s = s\rho_0(f).$$

Namely, $s$ is a $\text{Mod}(S)$-homomorphism of $H_C$ to the trivial $\text{Mod}(S)$-module $\mathbb{C}$. Therefore, by the irreducibility of the symplectic representation $\rho_0$, $s = 0$ (c.f. Remark 2.9). We now have $A = \begin{pmatrix} A_0 & w \\ 0 & x \end{pmatrix}$ and hence $A_0 \in \text{GL}(2g, \mathbb{C})$. Then the upper left block of (1.3) becomes $\rho_0(f)A_0 = A_0\rho_0(f)$, which means $A_0$ is a $\text{Mod}(S)$-endomorphism of $H_C$. 

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Therefore, Schur’s lemma and the irreducibility of $\rho_0$ imply $A_0 = zI$ for some $z \in \mathbb{C}^\times$. Now the upper right block of (1.3) becomes

$$\rho_0(f)w + c_2(f)x = zc_1(f) + w.$$  

Hence, for each $f \in \text{Mod}(S)$, we have

$$c_1(f) = (x/z)c_2(f) + \rho_0(f)(w/z) - (w/z),$$

which shows $[c_1] = [c_2]$.

In the case $[\phi_{c_1}] = \iota([\phi_{c_2}])$, the representation $\phi_{c_1}$ has both an invariant subspace of dimension 2$g$ and an invariant subspace of dimension 1, since the dual of $\phi_{c_2}$ has an invariant 1-dimensional subspace. Then the irreducibility of $\rho_0$ implies that $\phi_{c_1}$ is conjugate to the direct sum of $\rho_0$ and a 1-dimensional linear representation $\varepsilon'$, so that we have $[\phi_{c_1}] = [\rho_0 \oplus \varepsilon']$. By considering the determinants of the representations on both sides of this formula, we see $\varepsilon'$ is actually the trivial representation $\varepsilon$, since for each $f \in \text{Mod}(S)$ it can be easily seen $\det \rho_0(f) = 1$, and then by (1.2), $\det \phi_{c_1}(f) = 1$. Hence we have $[\phi_{c_1}] = [\rho_0 \oplus \varepsilon] = [\rho_0]$. We then have $[c_1] = 0$ by the argument of the previous case. Since $\iota$ is an involution of $X_0$, the same argument implies $[c_2] = 0$. In particular, we have $[c_1] = [c_2]$. This proves the injectivity of $\bar{\sigma}$ for $g \geq 1$.

Next, we prove the second part of Theorem 1.1. We clearly have $[\rho_0 \oplus \varepsilon] \in \text{Fix}(\iota)$, since both $\rho_0$ and $\varepsilon$ are self-dual. Conversely, let $\phi : \text{Mod}(S) \to \text{GL}(2g+1, \mathbb{C})$ be a representative of a fixed point of $\iota$, and now suppose $g \geq 7$. Then by Theorem 1.6, we have

$$[\phi] = [\phi_c] = \iota([\phi_c])$$

for some crossed homomorphism $c : \text{Mod}(S) \to H\mathbb{C}$. Then the argument above implies again $[c] = 0$, and hence $[\phi] = [\rho_0 \oplus \varepsilon]$.

This completes the proof of Theorem 1.1 assuming Theorem 1.6.

1.3 Outline of paper

The rest of this paper is essentially devoted to the proof of Theorem 1.6 and is organized as follows. In Section 2, we review some fundamental results we need in later sections. We then start with the analysis of eigenvalue and eigenspace of the image $\phi(t_a)$ of the Dehn twist along any nonseparating simple closed curve $a$ under a nontrivial $(2g+1)$-dimensional linear representation $\phi : \text{Mod}(S) \to \text{GL}(2g+1, \mathbb{C})$. In section 3, we review and slightly generalize results of Korkmaz [13] to show that $\phi(t_a)$ has a unique eigenvalue 1, and to provide a certain restriction of the dimension of the corresponding eigenspace. In Section 4, we give a further restriction on the eigenspace with the assistance of a certain twisted 1-cohomology group of the mapping class group of a surface and see that the Jordan form of $\phi(t_a)$ is uniquely determined. We then apply this result to a finite generating set of $\text{Mod}(S)$, which consists of Dehn twists along nonseparating simple closed curves and is given in Theorem 2.6, to complete the proof of Theorem 1.6. An algebraic characterization for images of the generators is presented in Section 5 and is used to prove Theorem 1.6 in Section 6. The characterization for the images consists of a previously known theorem by Korkmaz [13] (Theorem 5.3) and our new Theorems.

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The proofs of the latter two theorems, which depend only on the braid and commuting relations among the generators, require rather long computational argument and are postponed to Section 7. Section 8 provides a remark on a straightforward proof of Korkmaz’s classification theorem [13] of 2g-dimensional linear representations. Finally in Section 9, we discuss a generalization toward higher dimensional linear representations.

Notation

For a simple closed curve $a$ on $S$, the Dehn twist along $a$ is denoted by $t_a$. By a Dehn twist, we always mean the right-handed one. For a linear representation $\phi : \text{Mod}(S) \to \text{GL}(2g+1, \mathbb{C})$, we denote by $L_a$ the image of the Dehn twist $\phi(t_a)$. For an eigenvalue $\lambda$ of $L_a$, its eigenspace is denoted by $E^a_\lambda$. For a square matrix $M$, the multiplicity of $\lambda$ in the characteristic polynomial of $M$ is denoted by $\lambda_\#(M)$. We will omit the symbol $M$ and simply write $\lambda_\#$ if what it means is clear from the context.

For square matrices $M_1, M_2, \ldots, M_k$, $\text{diag}(M_1, M_2, \ldots, M_k)$ denotes the block diagonal matrix with block diagonals $M_1, M_2, \ldots, M_k$. The identity matrix of degree $n$ is denoted by $I_n$. We consider an element of $\mathbb{C}^m$ as a column vector.

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2 Preliminaries

In this section, we collect the previous results on or related to low dimensional complex linear representations of the mapping class group $\text{Mod}(S)$. We also recall some generality result on mapping class groups. More technical results will be recalled in later sections when necessary.

2.1 Statements of Franks–Handel and Korkmaz’s works

We begin with the precise statement of the classification results by Franks–Handel and Korkmaz. We note the first homology group $H_1(\text{Mod}(S); \mathbb{Z})$ is the abelianization of the group $\text{Mod}(S)$.

**Theorem 2.1** (Franks–Handel [6] and Korkmaz [12]). Let $g \geq 1$ and $n \leq 2g - 1$. Then any linear representation $\phi : \text{Mod}(S) \to \text{GL}(n, \mathbb{C})$ factors through the abelianization map $\text{Mod}(S) \to H_1(\text{Mod}(S); \mathbb{Z})$. In particular, if $g \geq 3$, then $\phi$ is trivial. (c.f. Theorem 2.4.)

**Theorem 2.2** (Korkmaz [13]). Let $g \geq 3$. Then, any nontrivial representation $\phi : \text{Mod}(S) \to \text{GL}(2g, \mathbb{C})$ is conjugate to the symplectic representation $\rho_0$. 

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2.2 Generalities on mapping class groups

We refer to Farb–Margalit [5] as a basic reference.

**Theorem 2.3** ([14], Theorem 1.2). Let \( g \geq 1 \), and let \( a \) and \( b \) be two nonseparating simple closed curves on \( S \). Then, there exists a sequence \( a = a_1, a_2, \ldots, a_k = b \)
of nonseparating simple closed curves on \( S \) such that each consecutive pair \( a_i \) and \( a_{i+1} \) intersect transversely at a single point.

The first homology group \( H_1(\text{Mod}(S); \mathbb{Z}) \) is given by Korkmaz [11] as follows.

**Theorem 2.4.** For \( g \geq 1 \),

\[
H_1(\text{Mod}(S); \mathbb{Z}) \cong \begin{cases} 
\mathbb{Z}/12\mathbb{Z} & \text{if } (g, p) = (1, 0); \\
\mathbb{Z}^p & \text{if } g = 1 \text{ and } p \geq 1; \\
\mathbb{Z}/10\mathbb{Z} & \text{if } g = 2; \\
0 & \text{otherwise.}
\end{cases}
\]

For a simple closed curve \( a \) on \( S \), the right-handed Dehn twist along \( a \) is denoted by \( t_a \). We collect here some useful relations among Dehn twists.

**Theorem 2.5.** (1) For any simple closed curve \( a \subset S \) and \( f \in \text{Mod}(S) \),

\[
ft_a f^{-1} = t_{f(a)}.
\]

(2) (commuting relation) For any two disjoint simple closed curves \( a \) and \( b \subset S \),

\[
t_a t_b = t_b t_a.
\]

(3) (braid relation) If two simple closed curves \( a \) and \( b \subset S \) intersect transversely at a single point,

\[
t_a t_b t_a = t_b t_a t_b.
\]

(4) (lantern relation) Suppose \( S = S_{0,0} \). Let \( a, b, c, d \) be the boundary curves of \( S \), and let \( x, y, z \) be the simple closed curves on \( S \) depicted in Figure 1. Then

\[
t_a t_b t_c t_d = t_x t_y t_z.
\]

The following explicit generating set of \( \text{Mod}(S) \) can be derived from the Lickorish generators of \( \text{Mod}(\bar{S}) \) by repeated applications of the Birman exact sequence as well as the star relation, for instance (c.f. [5]):

**Theorem 2.6.** Let \( g \geq 2 \). The group \( \text{Mod}(S) \) is generated by the Dehn twists along the following nonseparating simple closed curves depicted in Figures 2 and 3:

- \( a_1, b_1, a_2, b_2, \ldots, a_g, b_g; c_1, c_2, \ldots, c_{g-1}; \)
- \( e_1, e_2, \ldots, e_p; f_1, f_2, \ldots, f_r. \)

We remark that these generators are actually excessive, but are convenient for our purpose.
Figure 1: Lantern relation

Figure 2: Explicit generators (1)

Figure 3: Explicit generators (2)
2.3 Triviality of representation

The following triviality criterion for a linear representation was proved by Korkmaz [13] using Theorems 2.1 and 2.4 together with the nilpotency of the group of upper triangular unipotent matrices.

**Theorem 2.7** ([13], Lemma 4.8). Let \( g \geq 3 \), and \( \psi : \text{Mod}(S) \to \text{GL}(m, \mathbb{C}) \) be an arbitrary homomorphism with \( m \geq 1 \). Suppose there exists a flag

\[
0 = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_k = \mathbb{C}^m
\]

which is \( \text{Mod}(S) \)-invariant under the linear action \( \psi \) such that \( \dim W_{i+1}/W_i \leq 2g - 1 \) for \( 0 \leq i \leq k - 1 \). Then \( \psi \) is trivial.

**Remark 2.8.** In case of \( g = 2 \), Theorem 2.7 does not hold true, but becomes true if the consequence is changed to that the image of \( \psi \) is an abelian group (ibid). This is due to the less known fact that the commutator subgroup of \( \text{Mod}(S) \) is perfect for \( g \geq 2 \) ([14, Theorem 4.2]).

**Remark 2.9.** Theorem 2.7 together with Remark 2.8 implies immediately that the symplectic representation \( \rho_0 \) is irreducible, for \( g \geq 2 \).

3 Eigenvalue and eigenspace

In this section, following the argument of Korkmaz [13], we determine the eigenvalue of the image of the Dehn twist along a nonseparating simple closed curve under any \((2g + 1)\)-dimensional representation, and give a lower bound for the dimension of the corresponding eigenspace.

In general, it is known that Theorem 2.4 implies such an eigenvalue under any \((2g + 1)\)-dimensional representation must be a root of unity if \( g \geq 3 \) (cf [1] and [4]). In low dimensional case, however, the eigenvalue suffers from further restriction while the additional assumption on \( g \) is necessary.

We begin with recalling necessary results by Korkmaz.

3.1 General results from [13]

Let \( \phi : \text{Mod}(S) \to \text{GL}(m, \mathbb{C}) \) be a homomorphism. For a simple closed curve \( a \subset S \), the image \( \phi(t_a) \) is denoted by \( L_a \). For an eigenvalue \( \lambda \) of \( L_a \), the eigenspace of \( L_a \) corresponding to \( \lambda \) is denoted by \( E^a_\lambda \).

**Theorem 3.1** ([12], Lemma 4.2). Let \( a, b, c, d \) be nonseparating simple closed curves on \( S \) such that \( f(c) = a \) and \( f(d) = b \) for some \( f \in \text{Mod}(S) \). Suppose \( \lambda \) is an eigenvalue of \( L_a = \phi(t_a) \). Then \( E^a_\lambda = E^b_\lambda \) if and only if \( E^c_\lambda = E^d_\lambda \).

The proof follows from Theorem 2.5 (1) and the basic fact that the eigenspace of the conjugation of a linear mapping by a linear isomorphism coincides with the image of the eigenspace of the linear mapping under the linear isomorphism.

The next result follows from Theorems 2.3 and 3.1 together with the fact that \( \text{Mod}(S) \) is generated by Dehn twists along nonseparating simple closed curves if \( g \geq 2 \).
Theorem 3.2 ([13], Lemma 4.3). Let $g \geq 2$ and $\phi : \text{Mod}(S) \to \text{GL}(m, \mathbb{C})$ be a homomorphism. Suppose that $a$ and $b$ are two nonseparating simple closed curves on $S$ which intersect transversely at a single point. If $E^a_\lambda = E^b_\lambda$ for an eigenvalue $\lambda$ of $L_a = \phi(t_a)$, then $E^a_\lambda$ is invariant under the $\text{Mod}(S)$-action via $\phi$.

A careful analysis using Theorem 2.1 and some linear algebra implies the following.

Theorem 3.3 ([13], Lemma 4.5). Let $g \geq 3$ and $\phi : \text{Mod}(S) \to \text{GL}(m, \mathbb{C})$ be a homomorphism. Suppose $a$ is a nonseparating simple closed curve on $S$. If $\mu$ is an eigenvalue of $L_a = \phi(t_a)$ with multiplicity $\mu_\# \leq 2g - 3$, then $\mu = 1$ and the dimension of $E^a_\mu$ coincides with $\mu_\#$.

This theorem immediately implies the following:

Theorem 3.4 (c.f. [13], Corollary 4.6). Let $g \geq 3$ and $\phi : \text{Mod}(S) \to \text{GL}(m, \mathbb{C})$ be a homomorphism. If $m \leq 4g - 4$, then $L_a = \phi(t_a)$ has at most two eigenvalues.

### 3.2 The case of dimension $2g + 1$

We now consider $(2g + 1)$-dimensional representations. The following is an analogue of Korkmaz [13, Lemma 5.1],

Lemma 3.5. Let $g \geq 5$, and $\phi : \text{Mod}(S) \to \text{GL}(2g + 1, \mathbb{C})$ be an arbitrary homomorphism. Let $a$ be a nonseparating simple closed curve on $S$, and $\lambda$ be an eigenvalue of $L_a = \phi(t_a)$.

Let $\lambda_\#$ denote the multiplicity of $\lambda$ in the characteristic polynomial of $L_a$. If $\lambda_\# \geq 4$, then $\dim(E^a_\lambda) \geq 2g - 1$; in particular, it holds $\lambda_\# \geq 2g - 1$.

Proof of Lemma 3.5 We choose a non-separating simple closed curve $b$ on $S$ which intersects with $a$ transversely at a single point. We consider a regular neighbourhood of $a \cup b$ in the interior of $S$, and denote the closure of its complement in $S$ by $R$. Since $R$ and $S \setminus R$ both have genus at least one, the inclusion map $R \hookrightarrow S$ induces an injective homomorphism $\text{Mod}(R) \to \text{Mod}(S)$, and via this, we may consider $\text{Mod}(R)$ as a subgroup of $\text{Mod}(S)$ (see Paris–Rolfsen [18]).

Now assume $\dim(E^a_\lambda) \leq 2g - 3$. We set

$$W := \begin{cases} 
\ker(L_a - \lambda I)^{\lambda_\#} & \text{if } \lambda_\# \leq 2g - 3 \\
\ker((L_a - \lambda I)^4) & \text{if } \lambda_\# \geq 2g - 2 \text{ and } \dim(E^a_\lambda) = 1 \\
\ker((L_a - \lambda I)^3) & \text{if } \lambda_\# \geq 2g - 2 \text{ and } \dim(E^a_\lambda) = 2 \\
\ker((L_a - \lambda I)^2) & \text{if } \lambda_\# \geq 2g - 2 \text{ and } \dim(E^a_\lambda) = 3 \\
E^a_\lambda & \text{if } \lambda_\# \geq 2g - 2 \text{ and } 4 \leq \dim(E^a_\lambda) \leq 2g - 3
\end{cases}$$

Then, since any element of $\text{Mod}(R)$ commutes with $t_a$, $W$ is $\text{Mod}(R)$–invariant, and since $g \geq 5$, its dimension satisfies

$$4 \leq \dim(W) \leq 2(g - 1) - 1.$$ 

Since $g - 1 \geq 3$, we may apply Theorem 2.7 for the $\text{Mod}(R)$–invariant flag $0 \subset W \subset \mathbb{C}^{2g+1}$ to see $\phi(\text{Mod}(R))$ is a trivial group. Since $t_a$ is conjugate to a Dehn twist contained in $\text{Mod}(R)$, we have $L_a = I$, which contradicts to the assumption $\dim(E^a_\lambda) \leq 2g - 3$. We hence have $\dim(E^a_\lambda) \geq 2g - 2.$
Next, assume \( \dim (E^a_\lambda) = 2g - 2 \). If \( E^a_\lambda \neq E^b_\lambda \), we have \( 2g - 5 \leq \dim (E^a_\lambda \cap E^b_\lambda) \leq 2g - 3 \). On the other hand, \( E^a_\lambda \cap E^b_\lambda \) is clearly Mod(\( R \))-invariant. We then consider the Mod(\( R \))-invariant flag

\[
0 \subset E^a_\lambda \cap E^b_\lambda \subset \mathbb{C}^{2g+1}.
\]

Since \( g \geq 5 \), we have \( \dim (E^a_\lambda \cap E^b_\lambda) \leq 2g - 3 \) and \( \dim (\mathbb{C}^{2g+1}/E^a_\lambda \cap E^b_\lambda) \leq 6 \leq 2g - 3 \). Therefore, Theorem 2.7 implies \( \phi(\text{Mod}(R)) \) is trivial, which contradicts to \( \dim (E^a_\lambda) = 2g - 2 \).

If \( E^a_\lambda = E^b_\lambda \), then \( E^a_\lambda \) is Mod(\( S \))-invariant by Theorem 3.2. We can therefore apply Theorem 2.7 to the flag \( 0 \subset E^a_\lambda \subset \mathbb{C}^{2g+1} \) to see \( \phi \) is trivial. This contradicts the assumption \( \dim (E^a_\lambda) = 2g - 2 \).

This proves \( \dim (E^a_\lambda) \geq 2g - 1 \).

The following is a slight generalization of Korkmaz [13, Lemma 5.2], and tells that an eigenvalue of \( L^a \) must be 1 if its eigenspace has a small codimension:

**Lemma 3.6.** Let \( \phi : \text{Mod}(S) \to \text{GL}(m, \mathbb{C}) \) be an arbitrary homomorphism. Let \( a \) be a nonseparating simple closed curve on \( S \), and \( \lambda \) be an eigenvalue of \( L^a = \phi(t^a) \). Let \( s = m - \dim (E^a_\lambda) \) where \( E^a_\lambda \) denotes the corresponding eigenspace.

If \( g \geq 3 \) and \( m > 7s \), then \( \lambda = 1 \).

**Proof.** Since \( g \geq 3 \), we can apply Theorem 2.5 (4) to choose nonseparating simple closed curves \( c_1 = a, c_2, \ldots, c_7 \) on \( S \) so that they satisfy the lantern relation

\[
t_{c_1} t_{c_2} t_{c_3} t_{c_4} = t_{c_5} t_{c_6} t_{c_7}.
\]

For each \( i = 1, 2, \ldots, 7 \), we can choose \( \psi_i : \mathbb{C}^m \to \mathbb{C}^s \) so that \( \ker \psi_i = E^c_i \) with rank \( \psi_i = s \). Then we see \( \bigcap_{i=1}^7 E^c_i = \ker (\Psi := \bigoplus_{i=1}^7 \psi_i : \mathbb{C}^m \to \bigoplus_{i=1}^7 \mathbb{C}^s) \). Therefore, we have

\[
\dim \left( \bigcap_{i=1}^7 E^c_i \right) = m - \text{rank } \Psi \geq m - 7s > 0.
\]

We may thus choose \( v_0 \in \bigcap_{i=1}^7 E^c_i \) with \( v_0 \neq 0 \). We multiply the images under \( \phi \) of the left- and right-hand sides of (3.1) with \( v_0 \) to obtain

\[
\lambda^4 v_0 = \lambda^3 v_0.
\]

Since \( L^a \) is nonsingular, we have \( \lambda = 1 \). \( \square \)

Now we are ready to prove the following:

**Theorem 3.7.** Suppose \( g \geq 7 \). Let \( a \) be a non-separating simple closed curve on \( S \). For an arbitrary homomorphism \( \phi : \text{Mod}(S) \to \text{GL}(2g + 1, \mathbb{C}) \), the image \( L^a = \phi(t^a) \) of the Dehn twist along \( a \) has exactly one eigenvalue 1. Furthermore, the eigenspace \( E^a_1 \) has dimension at least \( 2g - 1 \).
Proof. We first observe the eigenvalue of $L_a$ is unique. Since $g \geq 3$ and $2g + 1 \leq 4g - 4$, the application of Theorem 3.3 implies the number of eigenvalues of $L_a$ is at most two. Assume that $L_a$ has distinct eigenvalues $\mu$ and $\lambda$ with $\mu_# \leq \lambda_#$. Here, $\mu_#$ and $\lambda_#$ denote the multiplicities of $\mu$ and $\lambda$, respectively, in the characteristic polynomial of $L_a$. Since $\mu_# + \lambda_# = 2g + 1$, we see $\mu_# \leq g$ and $\lambda_# \geq g + 1$. In particular, since $\mu_# \leq 2g - 3$, by Theorem 3.3 we have $\mu = 1$ and $\dim (E_\mu^a) = \mu_#$.

On the other hand, we see $\lambda_# \geq 4$ since $\mu_# \geq g + 1$, and also $g \geq 5$ by assumption. Therefore, Lemma 3.5 implies $\dim (E_\lambda^a) \geq 2g - 1$. This shows that the codimension $s = 2g + 1 - \dim (E_\lambda^a)$ of $E_\lambda^a$ satisfies $s \leq 2$. Together with the assumption $g \geq 7$, we have $2g + 1 = 15 > 14 \geq 7s$.

Hence, we have $2g + 1 > 7s$, and therefore Lemma 3.6 implies $\lambda = 1$, which contradicts to $\lambda \neq \mu$. This proves the uniqueness of the eigenvalue of $L_a$.

Now, let $\lambda$ be the unique eigenvalue of $L_a$. Since $\lambda_# = 2g + 1 \geq 4$, Lemma 3.5 implies $\dim (E_\lambda^a) \geq 2g - 1$ again, we hence have $\lambda = 1$ by Lemma 3.6. This completes the proof. 

4 Further restriction of eigenspace

In this section, we show that the twisted cohomology group $H^1(\text{Mod}(S^1_{g,0}); H_C)$ gives a further restriction of $\dim (E_\lambda^a)$ in Theorem 3.7. We first review necessary computation results. For generalities on cohomology of groups, we refer to [3].

4.1 Twisted 1-cohomology

Recall that $S = S_{g,r}^p$ denotes the oriented surface of genus $g$ with $p$ boundary components and $r$ punctures. We denote $H = H_1(\bar{S}; \mathbb{Z})$ and $H_C = H_1(\bar{S}; \mathbb{C})$ where $\bar{S}$ denotes the closed surface of genus $g$ obtained from $S$ by by gluing a 2-ball to each boundary component and forgetting the punctures. The homology group $H_1(\bar{S})$ with coefficients in any abelian group is naturally a left $\text{Mod}(S)$-module via the natural surjection $\text{Mod}(S) \to \text{Mod}(\bar{S})$. In this setting, we first describe $H^1(\text{Mod}(S); H_C)$ for general $p$, $r$.

Theorem 4.1. Let $g \geq 2$. For $p$, $r \geq 0$, $H^1(\text{Mod}(S); H_C) \cong \mathbb{C}^{p+r}$

where $\mathbb{C}^0$ denotes 0.

This theorem is due to Morita [16] for the case $p+r \leq 1$, and is due to Hain [8, Proposition 5.2] for the case $g \geq 3$ and general $p$, $r \geq 0$. See also Putman [19, Theorem 3.2] for the case $g \geq 3$. The computation with the full generality on $(g, p, r)$ was communicated to the author by Sato [20].
We remark that the twisted coefficients used by Morita and Hain in their computations are slightly different from \( H_c \): Morita computed \( H^1(\text{Mod}(S); H) \cong \mathbb{Z}^{p+r} \) and Hain computed \( H^1(\text{Mod}(S); H_1(\tilde{S}; \mathbb{Q})) \cong \mathbb{Q}^{p+r} \) in the indicated range of \((g, p, r)\). The cohomology with coefficients in \( H_c \) can be obtained from these computations as follows.

For any \( g \geq 2 \) and \( p, r \geq 0 \), the homology group \( H_0(\text{Mod}(S); H) \) is by definition the coinvariant of the \( \text{Mod}(S) \)-module \( H \), and can be easily seen to be zero. Therefore, a standard argument of the universal coefficient theorem implies, for \( k = \mathbb{Z}, \mathbb{Q}, \mathbb{C} \),

\[
H^1(\text{Mod}(S); H_1(\tilde{S}; k)) \cong \text{Hom}_\mathbb{Z}(H_1(\text{Mod}(S); H), k).
\]

Since \( \text{Mod}(S) \) is finitely generated, so is \( H_1(\text{Mod}(S); H) \). We hence have

\[
H^1(\text{Mod}(S); H_c) \cong H^1(\text{Mod}(S); H) \otimes_\mathbb{Z} \mathbb{C} \\
\cong H^1(\text{Mod}(S); H_1(\tilde{S}; \mathbb{Q})) \otimes_\mathbb{Q} \mathbb{C},
\]

which implies Theorem 4.1 except for the case \( g = 2 \) and \( p + r > 1 \).

In addition to the computation mentioned above, Morita combinatorially defined a crossed homomorphism

\[
k_0 : \text{Mod}(S^1_{g,0}) \rightarrow H
\]

which represents a generator of \( H^1(\text{Mod}(S^1_{g,0}); H) \). This crossed homomorphism has the property \( k_0(t_l) = 0 \) for a certain nonseparating simple closed curve \( l \) on \( S^1_{g,0} \) (see [17], especially the proof of Proposition 6.15 therein). This particular property implies the following.

**Theorem 4.2.** Let \( R \) be a compact connected oriented surface of genus at least 2 with nonempty connected boundary and no punctures. Also let \( \tilde{R} \) denote the closed surface obtained from \( R \) by gluing a 2-disk along the boundary. Suppose

\[
c : \text{Mod}(R) \rightarrow H_1(\tilde{R}; \mathbb{C})
\]

is an arbitrary crossed homomorphism. Let \( d \) be a nonseparating simple closed curve on \( R \). Let \( \tilde{d} \) denote the oriented curve obtained from \( d \) by choosing an arbitrary orientation, and \([\tilde{d}]\) denote its representing class in \( H_1(\tilde{R}; \mathbb{C}) \). Then

\[
c(t_d) = z[\tilde{d}]
\]

for some \( z \in \mathbb{C} \).

**Proof.** Let \( k_0 : \text{Mod}(R) \rightarrow H_1(\tilde{R}; \mathbb{Z}) \) be Morita’s crossed homomorphism above for \( R \). We may consider \( k_0 \) represents a generator of \( H^1(\text{Mod}(R); H_1(\tilde{R}; \mathbb{C})) \cong \mathbb{C} \) via the inclusion \( H_1(\tilde{R}; \mathbb{Z}) \hookrightarrow H_1(\tilde{R}; \mathbb{C}) \cong H_1(\tilde{R}; \mathbb{Z}) \otimes_\mathbb{Z} \mathbb{C} \). Hence, there exist \( A \in \mathbb{C} \) and \( x \in H_1(\tilde{R}; \mathbb{C}) \) such that

\[
c(f) = Ak_0(f) + f_*x - x \quad (4.1)
\]

for each \( f \in \text{Mod}(R) \). As mentioned above, there exists a nonseparating simple closed curve \( l \) on \( R \) such that \( k_0(t_l) = 0 \). By the classification theorem for surfaces, there exists \( \varphi \in \text{Mod}(R) \) such that \( \varphi(l) = d \). Then we have \( t_d = \varphi t_l \varphi^{-1} \) by Theorem 2.5. On the
other hand, the property of crossed homomorphism \( [1.1] \) implies \( k_0(f^{-1}) = -f^{-1}k_0(f) \) for any \( f \in \text{Mod}(R) \), and hence

\[
\begin{align*}
k_0(t_a) &= k_0(\varphi t_i \varphi^{-1}) = k_0(\varphi) + \varphi k_0(t_i \varphi^{-1}) \\
&= k_0(\varphi) + \varphi_*(k_0(t_i) + (t_i)_* k_0(\varphi^{-1})) \\
&= k_0(\varphi) + (\varphi t_i)_* k_0(\varphi^{-1}) = k_0(\varphi) - (\varphi t_i \varphi^{-1})_* k_0(\varphi) \\
&= k_0(\varphi) - (t_d)_* k_0(\varphi).
\end{align*}
\]

In view of \( [4.1] \), we then have

\[
c(t_d) = (t_d)_*(x - Ak_0(\varphi)) - (x - Ak_0(\varphi)).
\]

As is well-known, the action of \( t_d \) on \( H_1(\bar{R}; \mathbb{C}) \) is given by

\[
(t_d)_* x = x + (\langle \vec{d}, x \rangle \vec{d}) \quad \text{for } x \in H_1(\bar{R}; \mathbb{C})
\]

where \( \langle \cdot, \cdot \rangle \) denotes the algebraic intersection form on \( H_1(\bar{R}; \mathbb{C}) \). We then conclude

\[
c(t_d) = \langle [\vec{d}], x - Ak_0(\varphi) \rangle \vec{d} = z \vec{d}
\]

with \( z = \langle [\vec{d}], x - Ak_0(\varphi) \rangle \in \mathbb{C} \). This completes the proof of Theorem \( 4.2 \).

\[
\begin{proof}
\end{proof}
\]

4.2 The restriction of \( \dim (E_1^n) \)

We can now prove the following.

**Theorem 4.3.** Suppose \( g \geq 7 \). Let \( a \) be a nonseparating simple closed curve on \( S \). For any non-trivial homomorphism \( \phi : \text{Mod}(S) \to \text{GL}(2g + 1, \mathbb{C}) \), the image \( L_a = \phi(t_a) \) has a unique eigenvalue \( 1 \), and \( \dim (E_1^n) = 2g \).

\[
\begin{proof}
\end{proof}
\]

We divide into two subcases.

(I) **The case** \( E_1^n = E_1^b \). By Theorem \( 3.2 \) \( E_1^n \) is a \( \text{Mod}(S) \)-invariant \((2g - 1)\)-dimensional subspace of \( \mathbb{C}^{2g+1} \). Therefore, we have a \( \text{Mod}(S) \)-invariant flag \( 0 \subset E_1^n \subset \mathbb{C}^{2g+1} \), whose all successive quotients have dimensions less than or equal to \( 2g - 1 \). Therefore, since \( g \geq 3 \), Theorem \( 2.7 \) implies \( \phi \) is trivial, which contradicts to \( \dim (E_1^n) = 2g - 1 \). Let \( b \) be a non-separating simple closed curve which intersects \( a \) transversely at a single point. We divide into two cases according to whether \( E_1^n \) coincides with \( E_1^b \) or not.

(II) **The case** \( E_1^n \neq E_1^b \). We choose a compact connected subsurface \( R \) of \( S \) so that \( R \) is disjoint from \( a \cup b \) and has genus \( g - 1 \), nonempty connected boundary, and no punctures, as schematically depicted in Figure \( 4 \). We may consider \( \text{Mod}(R) \) as a subgroup of \( \text{Mod}(S) \) \((c.f. \ [15])\). Since the Dehn twists \( t_a \) and \( t_b \) commute with any element of \( \text{Mod}(R) \), both \( E_1^n \) and \( E_1^b \) are \( \text{Mod}(R) \)-invariant, and so is \( E_1^n \cap E_1^b \). Its dimension satisfies

\[
2g - 3 \leq \dim (E_1^n \cap E_1^b) \leq 2g - 2.
\]

We divide into two subcases.
(i) The case $\dim (E^a_1 \cap E^b_1) = 2g - 3$. In the $\text{Mod}(R)$-invariant flag $0 \subset E^a_1 \cap E^b_1 \subset \mathbb{C}^{2g+1}$, we see the dimensions of the successive quotients are less than or equal to $2(g - 1) - 1$, and $g - 1 \geq 3$. Therefore, the restriction of $\phi$ to $\text{Mod}(R)$ is trivial by Theorem 2.7. Since $t_a$ is conjugate to an element of $\text{Mod}(R)$, we have $L_a = \phi(t_a) = I$. This contradicts to the assumption $\dim (E^a_1) = 2g - 1$.

(ii) The case $\dim (E^a_1 \cap E^b_1) = 2g - 2$:

In the $\text{Mod}(R)$-invariant flag $0 \subset E^a_1 \cap E^b_1 \subset \mathbb{C}^{2g+1}$, we see $\dim \left( \mathbb{C}^{2g+1}/(E^a_1 \cap E^b_1) \right) = 3 \leq 2g - 3$, and hence the induced action of $\text{Mod}(R)$ on $\mathbb{C}^{2g+1}/(E^a_1 \cap E^b_1)$ is trivial by Theorem 2.1. On the other hand, since $g - 1 \geq 3$, Theorem 2.2 implies that the action of $\text{Mod}(R)$ on $E^a_1 \cap E^b_1$ via $\phi$ is either (A) trivial, or (B) conjugate to the symplectic representation of $\text{Mod}(R)$, which we denote by $\rho^R_0 : \text{Mod}(R) \to \text{GL}(H_1(\bar{R}; \mathbb{C}))$. Here, we denote by $\bar{R}$ the closed surface obtained from $R$ by gluing a 2-disk along the boundary of $R$.

In case A), we choose an arbitrary basis of $E^a_1 \cap E^b_1$ and extend it arbitrarily to a basis $\alpha$ of $\mathbb{C}^{2g+1}$. For each $f \in \text{Mod}(R)$, we have

$$\phi(f) = \begin{pmatrix} I_{2g-2} & * \\ 0 & I_3 \end{pmatrix}$$

according to $\alpha$. This implies that $\phi(\text{Mod}(R))$ is an abelian group. Therefore, $\phi(\text{Mod}(R))$ is trivial by Theorem 2.4. As before, this implies $L_a = \phi(t_a) = I$ and contradicts to $\dim (E^a_1) = 2g - 1$.

In case B), we may choose an isomorphism $u : E^a_1 \cap E^b_1 \to H_1(\bar{R}; \mathbb{C})$ such that

$$u(\phi(f)v) = f_* u(v) \quad (f \in \text{Mod}(R), v \in E^a_1 \cap E^b_1).$$

Here, $f_*$ denotes the natural action of $f$ on $H_1(\bar{R}; \mathbb{C})$.

We now fix a basis of $\mathbb{C}^{2g+1}$ extending an appropriate basis of $E^a_1 \cap E^b_1$. Then, under the identification of $E^a_1 \cap E^b_1$ with $H_1(\bar{R}; \mathbb{C})$ via $u$, the image of $f \in \text{Mod}(R)$ under $\phi$ has the form

$$\phi(f) = \begin{pmatrix} \rho^R_0(f) & w_1 & w_2 & w_3 \\ 0 & I_3 \end{pmatrix} \quad (w_1, w_2, w_3 \in H_1(\bar{R}; \mathbb{C})).$$

For another $f' \in \text{Mod}(R)$ with

$$\phi(f') = \begin{pmatrix} \rho^R_0(f') & w'_1 & w'_2 & w'_3 \\ 0 & I_3 \end{pmatrix} \quad (w'_1, w'_2, w'_3 \in H_1(\bar{R}; \mathbb{C})),
\]
we have
\[
\phi(f f') = \left( \begin{array}{ccc}
\rho_{R}^{R}(f f') & w_{1} + f_{*}w'_{1} & w_{2} + f_{*}w'_{2} \\
0 & w_{3} + f_{*}w'_{3} & I_{3}
\end{array} \right).
\]

This formula shows for \( i = 1, 2, \) and \( 3 \), that the correspondence \( f \in \text{Mod}(R) \mapsto w_{i} \) defines a crossed homomorphism
\[
c_{i} : \text{Mod}(R) \to H_{1}(\overline{R}; \mathbb{C}).
\]

Now let \( d \) be a non-separating simple closed curve on \( R \). We fix an orientation of \( d \) and denote its representing homology class by \([\overline{d}] \in H_{1}(\overline{R}; \mathbb{C})\). Then, by Theorem 4.2, there exists a complex number \( z_{i} \) for each \( i \) such that \( c_{i}(t_{d}) = z_{i}[\overline{d}] \). On the other hand, the action of \( t_{d} \) on \( H_{1}(\overline{R}; \mathbb{C}) \) is given by (4.2). Consequently, we have rank \( (\phi(t_{d}) - I) \) = 1. Since \( t_{a} \) is conjugate to \( t_{d} \) in \( \text{Mod}(S) \), we may conclude rank \( (L_{a} - I) \) = 1, which contradicts the assumption \( \text{dim}(E_{a}^{i}) = 2g - 1 \).

We may now conclude \( \text{dim}(E_{a}^{i}) \neq 2g - 1 \). This completes the proof of Theorem 4.3.

\section{5 The images of generators of Mod(S)}

Let \( \phi : \text{Mod}(S) \to \text{GL}(2g + 1, \mathbb{C}) \) be any nontrivial homomorphism and \( a \) be any non-separating simple closed curve on \( S \). Theorem 4.3 in the previous section shows the uniqueness of the Jordan form of \( \phi(t_{a}) \) for \( g \geq 7 \). To prove our goal Theorem 1.6, we have only to show that this restriction on \( \phi(t_{a}) \), together with a certain nontriviality assumption, forces the image of a set of generators of \( \text{Mod}(S) \) into the expected forms, with respect to some basis of \( \mathbb{C}^{2g+1} \). As a set of generators, we use the one given by Theorem 2.6.

Among the generators of \( \text{Mod}(S) \), the Dehn twists along the curves
\[
\bullet \ a_{1}, \ b_{1}, a_{2}, b_{2}, \ldots, a_{g}, b_{g}; \ c_{1}, c_{2}, \ldots, c_{g-1};
\]
\[
\bullet \ e_{1}, e_{2}, \ldots, e_{p}; \ f_{1}, f_{2}, \ldots, f_{r}
\]
in Figures 2 and 3, the images of \( t_{a_{i}}s' \) and \( t_{b_{i}}s' \) have already been considered by Korkmaz. To state his result, we need to set up some notation.

We use the symbol \( U \) and \( \widehat{U} \) to denote the \( 2 \times 2 \) matrices:
\[
U = \begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}, \quad \widehat{U} = \begin{pmatrix}
1 & 0 \\
-1 & 1
\end{pmatrix}.
\]

\textbf{Definition 5.1.} For \( i = 1, 2, \ldots, g \), we define the \( 2g \times 2g \) matrices \( A_{i} \) and \( B_{i} \) to be the block diagonal matrices
\[
A_{i} = \text{diag}(I_{2}, I_{2}, \ldots, U, I_{2}, \ldots, I_{2}),
\]
\[
B_{i} = \text{diag}(I_{2}, I_{2}, \ldots, \widehat{U}, I_{2}, \ldots, I_{2}),
\]
where \( U \) and \( \widehat{U} \) are in the \( i \)th entry.
We note that $A_i$ and $B_i$ are the images of the Dehn twists $t_{a_i}$ and $t_{b_i}$, respectively, under the symplectic representation $\rho_0$ with respect to the following basis $\{x_i, y_i\}$ of $H_1(\bar{S}; \mathbb{C})$:

**Definition 5.2.** We define the basis $\{x_1, y_1, \ldots, x_g, y_g\}$ of $H_C = H_1(\bar{S}; \mathbb{C})$ as follows. Let $x_i$ and $y_i$ be the oriented curves on $S$ depicted in Figure 5. For each $i$, the homology classes $x_i$ and $y_i$ in $H_C$ are defined to be the classes represented by the images of the oriented curves denoted by the same symbols under the inclusion $S \hookrightarrow \bar{S}$.

Now the result of Korkmaz can be stated as follows.

**Theorem 5.3** ([13], Lemma 4.7). Let $g \geq 1$, $m \geq 2g$ and let $\phi : \text{Mod}(S) \to \text{GL}(m, \mathbb{C})$ be a homomorphism. Let $a$ be any nonseparating simple closed curve on $S$. Suppose that the Jordan form of $L_a = \phi(t_a)$ is given by $\begin{pmatrix} U & 0 \\ 0 & I_{m-2g} \end{pmatrix}$. Suppose also that there exists a nonseparating simple closed curve $b \subset S$ intersecting $a$ transversely at a single point such that $E_1^a \neq E_1^b$. Then there exists a basis of $\mathbb{C}^m$ with respect to which

\[
L_{a_i} = \begin{pmatrix} A_i & 0 \\ 0 & I_{m-2g} \end{pmatrix} \quad \text{and} \quad L_{b_i} = \begin{pmatrix} B_i & 0 \\ 0 & I_{m-2g} \end{pmatrix}
\]

for $i = 1, 2, \ldots, g$.

Next, we describe two theorems which will control the images of the remaining generators of $\text{Mod}(S)$ under any nontrivial $(2g + 1)$-dimensional representation. We need to set up some further notation.

**Definition 5.4.** We set

\[
L = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

and

\[
C_k = \text{diag}(I_{2k-2}, L, I_{2g-2k-2}) \in \text{GL}(2g, \mathbb{C})
\]

for $k = 1, 2, \ldots, g - 1$.

We note $C_k$ is the image of $t_{c_k}$ under the symplectic representation $\rho_0$ with respect to the basis $\{x_i, y_i\}$.

The following theorem will provide the control of the images of $t_{c_k}s'$.
Theorem 5.5. Let \( g \geq 2 \), and let
\[
\tilde{A}_i = \begin{pmatrix} A_i & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{B}_i = \begin{pmatrix} B_i & 0 \\ 0 & 1 \end{pmatrix} \in \text{GL}(2g+1, \mathbb{C})
\]
for \( 1 \leq i \leq g \). For each \( k \) with \( 1 \leq k \leq g-1 \), suppose \( \tilde{X}_k \in \text{GL}(2g+1, \mathbb{C}) \) satisfies the following conditions (i)-(iv):

(i) \( \tilde{X}_k \) has exactly one eigenvalue 1.

(ii) \( \tilde{X}_k \tilde{A}_i = \tilde{A}_i \tilde{X}_k \) for \( i = 1, 2, \ldots, g \).

(iii) If \( g \geq 3 \), then \( \tilde{X}_k \tilde{B}_j = \tilde{B}_j \tilde{X}_k \) for each \( j \) satisfying \( 1 \leq j \leq g \) with \( j \neq k, k+1 \).

(iv) \( \tilde{X}_k \tilde{B}_j \tilde{X}_k = \tilde{B}_j \tilde{X}_k \tilde{B}_j \) for \( j = k, k+1 \).

Then, there exist nonzero complex numbers \( p_1, p_2, \ldots, p_{g-1} \) such that for
\[
P = \text{diag}(I_2, p_1I_2, p_2I_2, \ldots, p_{g-1}I_2) \in \text{GL}(2g, \mathbb{C}) \quad \text{and} \quad \tilde{P} = \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix},
\]
it holds \( \tilde{P}^{-1} \tilde{A}_i \tilde{P} = \tilde{A}_i \) and \( \tilde{P}^{-1} \tilde{B}_i \tilde{P} = \tilde{B}_i \) for each \( i = 1, 2, \ldots, g \), and furthermore, it also holds for each \( k = 1, 2, \ldots, g-1 \),
\[
\tilde{P}^{-1} \tilde{X}_k \tilde{P} = \begin{pmatrix} C_k & w_k \\ s_k & 1 \end{pmatrix} \tag{5.2}
\]
where \( w_k, s_k \in \mathbb{C}^{2g} \) with either \( w_k = 0 \) or \( s_k = 0 \).

Remark 5.6. Conversely, if each \( \tilde{X}_k \) is given as the right-hand side of (5.2), it is easy to see the conditions (i)-(iv) are satisfied.

The next theorem will provide the control of the images of the rest of the generators.

Theorem 5.7. Let \( g \geq 2 \), and \( \tilde{A}_i, \tilde{B}_i \) as in Theorem 5.5. Suppose the matrix \( \tilde{F} \in \text{GL}(2g+1, \mathbb{C}) \) satisfies the following conditions (i)-(iv):

(i) \( \tilde{F} \) has exactly one eigenvalue 1,

(ii) \( \tilde{F} \tilde{A}_i = \tilde{A}_i \tilde{F} \) for \( 1 \leq i \leq g \).

(iii) \( \tilde{F} \tilde{B}_j = \tilde{B}_j \tilde{F} \) for \( 2 \leq j \leq g \).

(iv) \( \tilde{F} \tilde{B}_1 \tilde{F} = \tilde{B}_1 \tilde{F} \tilde{B}_1 \).

Then, \( \tilde{F} = \begin{pmatrix} A_1 & w \\ s & 1 \end{pmatrix} \) where \( w, s \in \mathbb{C}^{2g} \) with either \( w = 0 \) or \( s = 0 \).

Conversely, it is clear that \( \tilde{F} \) given as the consequence of the theorem satisfies all the conditions (i)-(iv) in the theorem.

Remark 5.8. The above conditions (i)-(iv) for \( \tilde{F} \) in Theorem 5.7 resemble but do not coincide with the conditions for \( \tilde{X}_1 \) in Theorem 5.5. In fact, due to the difference, the consequence of the former theorem does not need to take conjugation of \( \tilde{F} \) unlike the latter theorem.

The proofs of Theorems 5.5 and 5.7 are straightforward matrix computation, which are elementary but rather long. Therefore, we postpone them to Section 7.
6 A dichotomy of representations

In this section, we combine the results in previous sections to complete the proof of Theorem 1.6. To do this, the following dichotomy result is crucial.

Theorem 6.1. Assume $g \geq 7$. Let $\phi : \text{Mod}(S) \to \text{GL}(2g + 1, \mathbb{C})$ be any nontrivial linear representation. Then, with respect to some basis of $\mathbb{C}^{2g+1}$, one of the following holds:

(A) For each $f \in \text{Mod}(S)$, $\phi(f)$ has the form
\[
\begin{pmatrix}
F & w \\
0 & 1
\end{pmatrix}
\quad (F \in \text{GL}(2g, \mathbb{C}), \ w \in \mathbb{C}^{2g}).
\]

(B) For each $f \in \text{Mod}(S)$, $\phi(f)$ has the form
\[
\begin{pmatrix}
F & 0 \\
s & 1
\end{pmatrix}
\quad (F \in \text{GL}(2g, \mathbb{C}), \ s \in \mathbb{C}^{2g}).
\]

6.1 Proof of Theorem 6.1

Assume $g \geq 7$. Let $\phi : \text{Mod}(S) \to \text{GL}(2g + 1, \mathbb{C})$ be an arbitrary non-trivial homomorphism. We choose a nonseparating simple closed curve $a$ on $S$, and set $L_a := \phi(t_a)$. By Theorem 4.3, $L_a$ has a unique eigenvalue 1, and it holds $\dim(E^a_1) = 2g$. We choose a nonseparating simple closed curve $b$ which intersects $a$ transversely at a single point. We first observe $E^a_1 \neq E^b_1$. Indeed, if $E^a_1 = E^b_1$, then Theorems 2.3 and 3.1 imply $E^a_1 = E^a_1$ for any nonseparating simple closed curve $x$ on $S$. Since the Dehn twists along such $x$s' generate $\text{Mod}(S)$, $E^a_1$ is $\text{Mod}(S)$-invariant via $\phi$, and the action of $\text{Mod}(S)$ on $E^a_1$ is trivial. We may now change the basis of $\mathbb{C}^{2g+1}$ so that its first $2g$ elements form a basis of $E^a_1$ to obtain $\phi(f) = \begin{pmatrix} I & * \\ 0 & 1 \end{pmatrix}$ for each $f \in \text{Mod}(S)$. This shows $\text{Im} \phi$ is an abelian group. We then see $\text{Im} \phi$ is trivial by Theorem 2.4. This contradicts to $\dim(E^a_1) = 2g$, and therefore, we have $E^a_1 \neq E^b_1$.

Now, to complete the proof of Theorem 6.1, we have only to prove the theorem for $f$ in the fixed generating set of $\text{Mod}(S)$ given by Theorem 2.6: the Dehn twists along the nonseparating simple closed curves

$$a_1, b_1, \ldots, a_g, b_g; c_1, c_2, \ldots, c_{g-1};$$
$$e_1, e_2, \ldots, e_p; f_1, f_2, \ldots, f_r$$

depicted in Figures 2 and 3. As usual, we denote $L_c = \phi(t_c)$ for a simple closed curve $c$ on $S$.

Since $\dim(E^a_1) = 2g$, the Jordan form of $L_a$ is given by
\[
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & I_{2g-1}
\end{pmatrix}.
\]

Therefore, we can apply Theorem 5.3 to obtain
\[
L_{a_i} = \begin{pmatrix} A_i & 0 \\ 0 & 1 \end{pmatrix}, \quad L_{b_i} = \begin{pmatrix} B_i & 0 \\ 0 & 1 \end{pmatrix}
\]
for $i = 1, 2, \ldots, g$. 

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after changing the basis of $\mathbb{C}^{2g+1}$ appropriately. Here, the matrices $A_i$ and $B_i$ are the ones given in Definition 5.1. Then by putting $X_k = L_{c_k}$ for each $k = 1, 2, \ldots, g - 1$, we can apply Theorem 5.5 to obtain

$$L_{c_k} = \begin{pmatrix} C_k & w_k \\ t_{s_k} & 1 \end{pmatrix}$$

for each $k$

after changing the basis of $\mathbb{C}^{2g+1}$ further, with $L_{a_i}$ and $L_{b_i}$ unchanged. Here, the matrix $C_k$ is the one given in Definition 5.4 and $w_k, s_k \in \mathbb{C}^{2g}$ with either $w_k = 0$ or $s_k = 0$.

Now we note $c_i$ and $c_j$ are disjoint for any $i \neq j$, and therefore $t_{c_i}$ and $t_{c_j}$, and hence $L_{c_i}$ and $L_{c_j}$, are commutative. By the same reason, the matrices $C_i$ and $C_j$ are also commutative since they coincide with the images of the Dehn twists $t_{c_i}$ and $t_{c_j}$, respectively, under $\rho_0$ with respect to the basis $\{x_1, y_1, \ldots, x_g, y_g\}$ given in Definition 5.2. This implies either

$$w_1 = w_2 = \cdots = w_{g-1} = 0$$

or

$$s_1 = s_2 = \cdots = s_{g-1} = 0.$$

Indeed, for $X, Y \in \text{GL}(2g, \mathbb{C})$ and column vectors $w, s \in \mathbb{C}^{2g}$, we see

$$\begin{pmatrix} X & w \\ 0 & 1 \end{pmatrix} \begin{pmatrix} Y & 0 \\ 1 & s \end{pmatrix} = \begin{pmatrix} X Y + w \, s' & w \\ 1 & s \end{pmatrix},$$

$$\begin{pmatrix} Y & 0 \\ 1 & s \end{pmatrix} \begin{pmatrix} X & w \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} Y X & Y w \\ s X & s w + 1 \end{pmatrix}.$$

Hence, if $X$ and $Y$ are commutative, comparing the upper left blocks of these two matrix multiplications, we see the two matrices $\begin{pmatrix} X & w \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} Y & 0 \\ 1 & s \end{pmatrix}$ are commutative only if the $2g \times 2g$ matrix $w \, s'$ is equal to the zero matrix, and in that case, it holds either $w = 0$ or $s = 0$.

Therefore, we see that either all of $L_{c_i} s'$ are of type (A), or all of $L_{c_i} s'$ are of type (B).

Next, choose any simple closed curve $e \in \{e_1, e_2, \ldots, e_p, f_1, f_2, \ldots, f_r\}$, and set $L_e := \phi(t_e)$. Then $e$ intersects $b_1$ transversely at a single point and is disjoint from $a_1, a_2, \ldots, a_g$ and $b_2, b_3, \ldots, b_g$. Hence $t_e$ and $t_{b_1}$ satisfy the braid relation, and $t_e$ commutes with $t_{a_1}, t_{a_2}, \ldots, t_{a_g}$ and $t_{b_2}, t_{b_3}, \ldots, t_{b_g}$. Therefore, we can apply Theorem 5.7 with $\tilde{F} = L_e$ to obtain either (A) $L_e = \begin{pmatrix} A_1 & w \\ 0 & 1 \end{pmatrix}$ with some $w \in \mathbb{C}^{2g}$, or (B) $L_e = \begin{pmatrix} A_1 & 0 \\ 1 & s \end{pmatrix}$ for some $s \in \mathbb{C}^{2g}$. Furthermore, since $e_1, e_2, \ldots, e_p$ and $f_1, f_2, \ldots, f_r$ are pairwise disjoint, the images of Dehn twists along them are all commutative. Therefore, by the previous argument above, the types (A) or (B) for the images of these Dehn twists are all the same and do not depend on the choice of $e$.

Finally, since $e$ is disjoint from any $c_i$, $L_e$ commutes with $L_{c_i}$. Therefore, the type (A) or (B) for $L_e$ coincides with that for $L_{c_i} s'$. This completes the proof of Theorem 6.1.
6.2 Proof of Theorem 1.6

We now prove Theorem 1.6. Let \( \phi : \text{Mod}(S) \to \text{GL}(2g+1, \mathbb{C}) \) be any nontrivial linear representation. We need to prove that either \([\phi] = \sigma([\phi_c])\) or \(\iota([\phi]) = \sigma([\phi_c])\) for some crossed homomorphism \(c : \text{Mod}(S) \to H_C = \mathbb{C}^{2g}\). By Theorem 6.1 we have only to consider the cases (A) and (B) in the theorem.

The case (A)

For each \(f \in \text{Mod}(S)\), \(\phi(f)\) has the form

\[
\phi(f) = \begin{pmatrix} F & w \\ 0 & 1 \end{pmatrix}
\]

with \(F \in \text{GL}(2g, \mathbb{C})\) and \(w \in \mathbb{C}^{2g}\).

The correspondence \( f \mapsto F \) defines a linear representation \(\tilde{\phi} : \text{Mod}(S) \to \text{GL}(2g, \mathbb{C})\). By Theorem 2.2 \(\tilde{\phi}\) is trivial or conjugate to the symplectic representation \(\rho_0\).

If \(\tilde{\phi}\) is trivial, then \(\text{Im} \phi\) is abelian, and since \(g \geq 3\), \(\phi\) is trivial by Theorem 2.4, which contradicts to the assumption. Therefore, we see \(\tilde{\phi}\) is conjugate to \(\rho_0\).

By changing the basis of \(\mathbb{C}^{2g+1}\) if necessary, we may assume \(\tilde{\phi}\) coincides with the matrix form of \(\rho_0\) with respect to the basis \(\{x_1, y_1, \ldots, x_g, y_g\}\) of \(H_C\). Then we may consider \(w \in H_C\), and the correspondence \(f \in \text{Mod}(S) \mapsto w\) defines a crossed homomorphism

\(c : \text{Mod}(S) \to H_C\)

with values in \(H_C\). Namely, it holds

\(c(f_1f_2) = c(f_1) + \rho_0(f_1)c(f_2)\) \quad \((f_1, f_2 \in \text{Mod}(S))\).

Therefore, we have \([\phi] = [\phi_c] = \sigma([c])\) in \(X_0\).

The case (B)

For each \(f \in \text{Mod}(S)\), \(\phi(f)\) has the form

\[
\phi(f) = \begin{pmatrix} F & 0 \\ s & 1 \end{pmatrix}
\]

with \(F \in \text{GL}(2g, \mathbb{C})\) and \(s \in \mathbb{C}^{2g}\).

Let \(\phi^* : \text{Mod}(S) \to \text{GL}(2g+1, \mathbb{C})\) denote the dual representation of \(\phi\) defined by

\(\phi^*(f) = \overline{\phi(f)^{-1}}\) for each \(f \in \text{Mod}(S)\).

Then \(\phi^*\) is clearly a representation of Type (A), and hence we can apply the previous argument to obtain \([\phi^*] = \sigma([c])\) for some crossed homomorphism \(c\). In other words, we have \(\iota([\phi]) = \sigma([c])\) in \(X_0\). This completes the proof of Theorem 1.6.

7 Braid and commuting relations in matrices

In this section we prove Theorems 5.5 and 5.7. We first recall necessary results of Korkmaz, which was originally used for proving Theorem 2.2.

\(\square\)
7.1 Preliminary from [13]

The next theorem follows from the irreducibility of the symplectic representation \( \rho \) for \( g = 1 \) together with Schur’s lemma, or alternatively, can be verified by straightforward computation.

**Theorem 7.1 (13, Lemma 2.2).** Let \( X, Y \) and \( Z \) be \( 2 \times k \), \( k \times 2 \) and \( 2 \times 2 \) matrices with entries in \( \mathbb{C} \), respectively.

1. If \( UX = X \) and \( UX = X \), then \( X = 0 \).
2. If \( YU = Y \) and \( YU = Y \), then \( Y = 0 \).
3. If \( ZU = UZ \) and \( ZU = UZ \), then \( Z = \alpha I_2 \) for some \( \alpha \in \mathbb{C} \).

This theorem can be generalized as follows by induction on \( g \).

**Theorem 7.2 (13, Lemma 2.3).** Let \( X, Y, Z \) be matrices with entries in \( \mathbb{C} \) such that the multiplications given below are all defined.

1. If \( A_i X = X \) and \( B_i X = X \) for \( 1 \leq i \leq g \), then \( X = 0 \).
2. If \( YA_i = Y \) and \( YB_i = Y \) for \( 1 \leq i \leq g \), then \( Y = 0 \).
3. If \( ZA_i = A_i Z \) and \( ZB_i = B_i Z \) for \( 1 \leq i \leq g \), then \( Z = \text{diag}(\alpha_1 I_2, \alpha_2 I_2, \ldots, \alpha_g I_2) \) for some \( \alpha_1, \alpha_2, \ldots, \alpha_g \in \mathbb{C} \).

We remark that Theorem 7.2 does not assume any of \( X, Y \) and \( Z \) represents a \( \text{Mod}(S) \)-homomorphism, unlike Theorem 7.1. In fact, the assumption of Theorem 7.1 implies, for instance, \( Z \) represents a \( \text{Mod}(S_{1,0}) \)-endomorphism of \( H_C \) because \( U \) and \( \hat{U} \) are, respectively, the images of the Dehn twists \( t_a \) and \( t_b \), which generate \( \text{Mod}(S_{1,0}) \), under the symplectic representation with respect to the basis \( \{x_1, y_1\} \) given in Definition 5.2. Therefore the consequence of the theorem follows from Schur’s lemma. On the other hand, the assumption of Theorem 7.2, for \( g \geq 2 \), does not imply \( Z \) represents a \( \text{Mod}(S_g) \)-endomorphism of \( H_C \) since otherwise the consequence of the theorem could be strengthened to \( Z = aI \) rather than a block diagonal matrix, which is not necessarily true. The difference between the two theorems is due to the fact that the Dehn twists \( t_{a_1}, t_{a_2}, \ldots, t_{a_g} \) and \( t_{b_1}, t_{b_2}, \ldots, t_{b_g} \) are not sufficient to generate \( \text{Mod}(S_{g,0}) \) if \( g \geq 2 \).

7.2 A key lemma

A key step to prove Theorem 5.5 is the following, which is to characterize the matrix satisfying the conditions for \( \tilde{X}_1 \) in Theorem 5.5.

**Lemma 7.3.** Let \( g \geq 2 \), \( m \geq 1 \), and \( \tilde{X} \in \text{GL}(2g + m, \mathbb{C}) \). Let

\[
\tilde{A}_i = \begin{pmatrix} A_i & 0 \\ 0 & I_m \end{pmatrix} \quad \text{and} \quad \tilde{B}_i = \begin{pmatrix} B_i & 0 \\ 0 & I_m \end{pmatrix}
\]

for \( 1 \leq i \leq g \).

Suppose \( \tilde{X} \) satisfies the following conditions (i)-(iv).
(i) $\tilde{X}$ has a unique eigenvalue 1.

(ii) $X\tilde{A}_i = \tilde{A}_i\tilde{X}$ for $1 \leq i \leq g$.

(iii) If $g \geq 3$, then $\tilde{X}\tilde{B}_j = \tilde{B}_j\tilde{X}$ for $3 \leq j \leq g$.

(iv) $X\tilde{B}_j\tilde{X} = \tilde{B}_j\tilde{X}\tilde{B}_j$ for $j = 1, 2$.

Then, there exists a nonzero complex number $p$ such that for $P = \text{diag}(I_2, pI_2, I_{2g-4})$ and $\tilde{P} = \begin{pmatrix} P & 0 \\ 0 & I_m \end{pmatrix}$, it holds

$$\tilde{P}^{-1}\tilde{X}\tilde{P} = \begin{pmatrix} C_1 & W_1 \\ S_1 & T \end{pmatrix}$$

where $C_1$ and $T$ are, respectively, $2g \times 2g$ and $m \times m$ matrices, and

$$C_1 = \begin{pmatrix} L & 0 \\ 0 & I_{2g-4} \end{pmatrix} \text{ with } L \text{ the } 4 \times 4 \text{ matrix given in Definition 5.4}.$$

$$W_1 = {}^t(\begin{array}{cccc} w & 0 & -w & 0 \\ 0 & 0 & \cdots & 0 \end{array}), \quad S_1 = (\begin{array}{cccc} 0 & s & 0 & -s \\ 0 & 0 & \cdots & 0 \end{array})$$

with $w, s, 0 \in \mathbb{C}^m$; $^tws = 0$; $^twt = 0$; $^tws = 0$; and $T^2 - T = s^t w$.

To prove the lemma, we first observe:

**Lemma 7.4.** Let $Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, d \in \mathbb{C}$.

(1) If $YU = UY$, then $a = d$ and $c = 0$ so that $Y = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$.

(2) If $YU = UY = Y$, then $a = d = c = 0$ so that $Y = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$.

**Proof.** Straightforward.

Proof of Lemma 7.3

We write $\tilde{X} = \begin{pmatrix} X & W \\ S & T \end{pmatrix}$ where $X$ and $T$ are, respectively, $2g \times 2g$ and $m \times m$ matrices. By a straightforward computation, we see the condition (ii) implies

$$XA_i = A_iX,$$

$$W = A_iW,$$

$$SA_i = S$$

for $1 \leq i \leq g$. 

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In case \( g \geq 3 \), the condition (iii) similarly implies
\[
\begin{align*}
XB_j & = B_j X, \\
B_j W & = W, \\
SB_j & = S
\end{align*}
\] (7.4)
for \( 3 \leq j \leq g \).

The condition (iv) implies
\[
\begin{align*}
XB_j X + WS & = B_j XB_j, \\
XB_j W + WT & = B_j W, \\
SB_j X + TS & = SB_j, \\
SB_j W + T^2 & = T
\end{align*}
\] (7.7)
for \( j = 1, 2 \).

In view of (7.3) and (7.6), we have

- \( S(A_i - I_{2g}) = 0 \) for \( 1 \leq i \leq g \),
- \( S(B_j - I_{2g}) = 0 \) for \( 3 \leq j \leq g \) if \( g \geq 3 \).

Therefore, we can easily see that all entries of \( S \) are zero except in the second and the fourth columns. Thus, we may write
\[
S = \begin{pmatrix} 0 & s_2 & 0 & s_4 & 0 & \cdots & 0 \end{pmatrix} \quad \text{where } s_2, s_4, 0 \in \mathbb{C}^m. (7.11)
\]

Similarly, in view of (7.2) and (7.5), we can see all the entries of \( W \) are zero except in the first and the third rows. We may thus write
\[
W = \begin{pmatrix} w_1 & 0 & w_3 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad \text{where } w_1, w_3, 0 \in \mathbb{C}^m. (7.12)
\]

We then have
\[
WS = \begin{pmatrix} 0 & w_1 s_2 & 0 & w_1 s_4 \\
0 & 0 & 0 & 0 \\
0 & w_3 s_2 & 0 & w_3 s_4 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{pmatrix}
\] (7.13)
where the upper left block is a \( 4 \times 4 \) matrix.

Next, we consider the form of \( X \). Suppose \( g \geq 3 \) for the moment, and we write
\[
X = \begin{pmatrix} X_0 & W_0 \\ S_0 & T_0 \end{pmatrix}
\]
where \( X_0 \) and \( T_0 \) are, respectively, \( 4 \times 4 \) and \((2g - 4) \times (2g - 4)\) matrices. For \( 3 \leq i \leq g \), we set \((2g - 4) \times (2g - 4)\) matrices \( \bar{A}_i \) and \( \bar{B}_i \) by
\[
A_i = \begin{pmatrix} I_4 & 0 \\ 0 & \bar{A}_i \end{pmatrix}, \quad B_i = \begin{pmatrix} I_4 & 0 \\ 0 & \bar{B}_i \end{pmatrix}.
\]

Then the equalities (7.1) for \( i \geq 3 \) and (7.4) imply
• $W_0 \bar{A}_i = W_0 = W_0 \bar{B}_i$, 
• $S_0 = \bar{A}_i S_0 = \bar{B}_i S_0$, 
• $T_0 \bar{A}_i = \bar{A}_i T_0$ and $T_0 \bar{B}_i = \bar{B}_i T_0$

for $3 \leq i \leq g$. We can then apply Theorem 7.2 to obtain $W_0 = 0$, $S_0 = 0$, and $T_0 = \text{diag}(\alpha_1 I_2, \alpha_2 I_2, \ldots, \alpha_{g-2} I_2)$ for some $\alpha_1, \alpha_2, \ldots, \alpha_{g-2} \in \mathbb{C}$. In view of the form of $S_0$ (7.11), we see each $\alpha_i$ is an eigenvalue of $\bar{X}$, and therefore, $\alpha_i = 1$ by the condition (i). In case $g = 2$, we may simply set $X_0 = X$. In short, we conclude for $g \geq 2$ that

$$X = \begin{pmatrix} X_0 & 0 \\ 0 & I_{2g-4} \end{pmatrix}$$

(7.14)

where $X_0$ is a $4 \times 4$ matrix.

We next write $X_0 = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$ where $X_1$ and $X_4$ are $2 \times 2$ matrices. Then the equality (7.1) for $i = 1, 2$ implies

• $X_1 U = UX_1$, 
• $X_2 U = X_2 = UX_2$, 
• $X_3 U = X_3 = UX_3$, 
• $X_4 U = UX_4$.

We can therefore apply Lemma 7.4 to obtain

$$X_0 = \begin{pmatrix} a & b & 0 & \alpha \\ 0 & a & 0 & 0 \\ 0 & \beta & c & d \\ 0 & 0 & 0 & c \end{pmatrix}$$

for some $\alpha, \beta, a, b, c, d \in \mathbb{C}$.

In particular, in view of the first and the third columns of $X_0$ together with (7.14) and (7.11), we see $a$ and $c$ are both eigenvalues of $\bar{X}$, and therefore, we have $a = c = 1$ by the condition (i). We may thus write

$$X_0 = \begin{pmatrix} I_2 + b(U - I_2) & \alpha(U - I_2) \\ \beta(U - I_2) & I_2 + d(U - I_2) \end{pmatrix}.$$ 

(7.15)

Next, we consider the equality (7.7). We set the $4 \times 4$ matrix $B_j'$ as

$$B_j' = \begin{pmatrix} B_j' & 0 \\ 0 & I_{2g-4} \end{pmatrix}$$

for $j = 1, 2$.

In view of (7.14) and (7.13), the equality (7.7) implies

$$X_0 B_j' X_0 + (WS)_{1,1} = B_j' X_0 B_j'$$

for $j = 1, 2$ (7.16)
where \((WS)_{1,1}\) denotes the upper left block of \(WS\) in \((\ref{7.13})\), and can be written as

\[
(WS)_{1,1} = \begin{pmatrix}
  w_1 s_2 (U - I_2) & w_1 s_4 (U - I_2) \\
  w_3 s_2 (U - I_2) & w_3 s_4 (U - I_2)
\end{pmatrix}.
\]

Then, together with \((\ref{7.15})\) and obvious equalities \((U - I_2)^2 = (\hat{U} - I_2)^2 = 0\), \((U - I_2)\hat{U}(U - I_2) = (-1)(U - I_2)\), and \((\hat{U} - I_2)U(\hat{U} - I_2) = (-1)(U - I_2)\), \((\ref{7.16})\) for \(j = 1\) implies the following:

\[
\hat{U} + b\hat{U}(U - I_2) + b(U - I_2)\hat{U} - b^2(U - I_2) + w_1 s_2(U - I_2) = \alpha\hat{U} + (1 - b)\alpha(U - I_2) + w_1 s_4(U - I_2) = \alpha\hat{U} (U - I_2),
\]

\[
\beta(U - I_2)\hat{U} + \beta(1 - b)(U - I_2) + w_3 s_2(U - I_2) = \beta(U - I_2)\hat{U},
\]

\[
I_2 + (2d - \alpha\beta)(U - I_2) + w_3 s_4(U - I_2) = I_2 + d(U - I_2).
\]

Similarly, the lower right block of \((\ref{7.16})\) for \(j = 2\) implies

\[
\hat{U} + d\hat{U}(U - I_2) + d(U - I_2)\hat{U} - d^2(U - I_2) + w_3 s_4(U - I_2) = \hat{U}^2 + d\hat{U}(U - I_2)\hat{U}.
\]

By straightforward computations of the \((2,1)\)-entries of \((\ref{7.17})\) and \((\ref{7.21})\), we obtain \(b = 1\) and \(d = 1\), respectively. Then, straightforward computations of the \((1,2)\)-entries of \((\ref{7.17})\), \((\ref{7.18})\), \((\ref{7.19})\), and \((\ref{7.21})\) imply in turn \(w_1 s_2\), \(w_1 s_4\), \(w_3 s_2\), and \(w_3 s_4\) are all zero. This means simply

\[
WS = 0.
\]

Furthermore, in view of the \((1,2)\)-entry of \((\ref{7.20})\), we obtain \(\alpha\beta = 1\). Therefore, we have

\[
X = \begin{pmatrix}
  1 & 1 & 0 & \alpha \\
  0 & 1 & 0 & 0 \\
  0 & 1 & 0 & 1 \\
  \alpha & 0 & 0 & 1 \\
  0 & I_{2g - 4}
\end{pmatrix}
\]

and

\[
\tilde{X} = \begin{pmatrix}
  X \\
  S \\
  T
\end{pmatrix}
\]

where \(W\) and \(S\) are as \((\ref{7.12})\) and \((\ref{7.11})\) with \(WS = 0\).

Now, we can see by tedious but straightforward computation that the equalities \((\ref{7.8})\), \((\ref{7.9})\), and \((\ref{7.10})\) are, respectively, equivalent to

\[
w_3 = \frac{1}{\alpha} w_1 \quad \text{and} \quad \mathbf{w}_1 T = \mathbf{w}_1, \quad (\ref{7.23})
\]

\[
s_4 = \alpha s_2 \quad \text{and} \quad Ts_2 = s_2, \quad (\ref{7.24})
\]

and

\[
s_2 \mathbf{w}_1 = T^2 - T = s_4 \mathbf{w}_3. \quad (\ref{7.25})
\]
Finally, let \( p = -\frac{1}{a} \), \( P = \text{diag}(I_2, pI_2, I_{2g-4}) \), and \( \bar{P} = \begin{pmatrix} P & 0 \\ 0 & I_m \end{pmatrix} \). Also, let \( s = s_2 \) and \( w = w_1 \). Then a direct computation implies
\[
\bar{P}^{-1}X\bar{P} = \begin{pmatrix} C_1 & W_1 \\ S_1 & T \end{pmatrix}.
\]
where
\[
W_1 = P^{-1}W = ^t( w & 0 & -w & 0 & 0 & \cdots & 0), \\
S_1 = SP = ( 0 & s & 0 & -s & 0 & \cdots & 0).
\]
Furthermore, by (7.22) and (7.23)-(7.25), we have
\[
^tws = 0, \quad ^twT = ^sw, T s = s, \quad \text{and} \quad T^2 - T = sw.
\]
This completes the proof of Lemma 7.3.

**Remark 7.5.** In general, the conclusion of Lemma 7.3 cannot be strengthened to “either \( W_1 = 0 \) or \( S_1 = 0 \).” In fact, a counterexample is given for \( m = 2 \) by
\[
\bar{X} = \begin{pmatrix} C_1 & W_1 \\ S_1 & T \end{pmatrix} \quad \text{with} \quad T = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix},
\]
and
\[
W_1 = ^t( w & 0 & -w & 0 & 0 & \cdots & 0), \quad S_1 = ( 0 & s & 0 & -s & 0 & \cdots & 0)
\]
where \( w = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) and \( s = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \).

### 7.3 Proof of Theorem 5.5

We first observe that the conclusion of Lemma 7.3 can be strengthened if \( m = 1 \). Here, we denote
\[
\tilde{A}_i = \begin{pmatrix} A_i & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{B}_i = \begin{pmatrix} B_i & 0 \\ 0 & 1 \end{pmatrix} \in \text{GL}(2g+1, \mathbb{C})
\]
for \( i = 1, 2, \ldots, g \).

**Theorem 7.6.** Let \( g \geq 2 \). Suppose \( \tilde{X} \in \text{GL}(2g+1, \mathbb{C}) \) satisfies the following conditions (i)-(iv).

(i) \( \tilde{X} \) has a unique eigenvalue 1.

(ii) \( \tilde{X} \tilde{A}_i = \tilde{A}_i \tilde{X} \) for \( 1 \leq i \leq g \).

(iii) If \( g \geq 3 \), then \( \tilde{X} \tilde{B}_j = \tilde{B}_j \tilde{X} \) for \( 3 \leq j \leq g \).

(iv) \( \tilde{X} \tilde{B}_j \tilde{X} = \tilde{B}_j \tilde{X} \tilde{B}_j \) for \( j = 1, 2 \).
Then there exists a nonzero complex number \( p \) such that for \( P = \text{diag}(I_2, pI_2, I_{2g-4}) \) and \( \tilde{P} = \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} \), it holds

\[
\tilde{P}^{-1}X\tilde{P} = \begin{pmatrix} C_1 & w \\ t & s \end{pmatrix}
\]

where \( w, s \in \mathbb{C}^g \) with either \( w = 0 \) or \( s = 0 \), and the entries of \( w \) and \( s \) are all zero except for the first through fourth rows.

**Proof.** By Lemma 7.3 with \( m = 1 \), there exists a nonzero complex number \( p \) such that for \( P \) and \( P \) as in the theorem, it holds

\[
\tilde{P}^{-1}X\tilde{P} = \begin{pmatrix} C_1 & w \\ t & s \end{pmatrix}
\]

where \( w, s \in \mathbb{C}^g \) and \( t \in \mathbb{C} \) with the properties given in the lemma, among which

\[
\begin{align*}
\begin{pmatrix} w \\ t \end{pmatrix} &= \begin{pmatrix} w & 0 & -w & 0 & 0 & \cdots & 0 \\ t & 0 & s & 0 & -s & 0 & \cdots & 0 \end{pmatrix}, \\
\begin{pmatrix} s \\ t \end{pmatrix} &= \begin{pmatrix} 0 & s & 0 & -s & 0 & \cdots & 0 \end{pmatrix}
\end{align*}
\]

for some \( w, s \in \mathbb{C} \) with \( ws = 0 \). This implies either \( w = 0 \) or \( s = 0 \). We then have \( t = 1 \) by considering the determinant of \( \tilde{P}^{-1}X\tilde{P} \). This completes the proof. \( \square \)

We note in case \( g = 2 \), Theorem 7.6 implies Theorem 5.5 by putting \( X = X_1 \). Hereafter in this subsection, we assume \( g \geq 3 \), and the index \( i \) is considered modulo \( g \). We consider a certain periodic homeomorphism of the closed surface of \( \bar{S} \). Recall that \( \bar{S} \) is obtained from the surface \( S \) by gluing a 2-disk to each boundary component and forgetting all the punctures. We configure \( \bar{S} \) as in Figure 6 as well as simple closed curves \( a_i, b_i, c_k \) for \( 1 \leq i \leq g \) and \( 1 \leq k \leq g - 1 \). We denote by \( r \in \text{Mod}(\bar{S}) \) the mapping class represented by the counter-clockwise \( 1/g \)-rotation around the center. We see

\[
r(a_i) = a_{i+1}, \quad r(b_i) = b_{i+1}, \quad \text{and} \quad r(c_k) = c_{k+1}
\]

for \( 1 \leq i \leq g \) and \( 1 \leq k \leq g - 2 \). We now denote the symplectic representation of \( \text{Mod}(\bar{S}) \) by \( \bar{\rho}_0 : \text{Mod}(\bar{S}) \to \text{GL}(2g, \mathbb{C}) \). Let \( G = \bar{\rho}_0(r) \) with respect to the basis \( \{ x_i, y_i \} \) as in Definition 5.2 with \( S \) replaced by \( \bar{S} \) where the curves \( x_i \) and \( y_i \) are also reconfigured in an obvious manner. One can easily see \( G = \begin{pmatrix} 0 & I_2 \\ I_{2g-2} & 0 \end{pmatrix} \). Also, let \( \tilde{G} = \begin{pmatrix} G & 0 \\ 0 & 1 \end{pmatrix} \). We note

\[
A_i = \bar{\rho}_0(t_{a_i}), \quad B_i = \bar{\rho}_0(t_{b_i}), \quad \text{and} \quad C_k = \bar{\rho}_0(t_{c_k})
\]

for \( 1 \leq i \leq g \) and for \( 1 \leq k \leq g - 1 \) with respect to the same basis. Therefore, by making use of Theorem 2.5 (1), we see for instance

\[
G^{-1}A_i G = \bar{\rho}_0(r^{-1} t_{a_i} r) = \bar{\rho}_0(t_{a_{i+1}}) = \bar{\rho}_0(t_{a_{i-1}}) = A_{i-1} \quad \text{for} \quad 1 \leq i \leq g.
\]

Combining similar computations, we obtain for \( 1 \leq i \leq g \) and \( 2 \leq k \leq g - 1 \),

\[
G^{-1}A_i G = A_{i-1}, \quad G^{-1}B_i G = B_{i-1}, \quad \text{and} \quad G^{-1}C_k G = C_{k-1}
\]

as well as

\[
\tilde{G}^{-1}A_i \tilde{G} = A_{i-1}, \quad \tilde{G}^{-1}B_i \tilde{G} = B_{i-1}.
\]

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We can now begin the proof of Theorem 5.5 for \( g \geq 3 \). Suppose the matrices \( \tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_{g-1} \in \text{GL}(2g+1, \mathbb{C}) \) satisfy the conditions (i)-(iv) in the theorem.

We first take \( \tilde{X} = \tilde{X}_1 \) and apply Theorem 7.6 to obtain \( p_1 \in \mathbb{C}^\times \), \( P_1 = \text{diag}(I_2, p_1 I_2, I_{2g-4}) \), and \( \tilde{P}_1 = \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} \) so that

\[
\tilde{P}_1^{-1} \tilde{X}_1 \tilde{P}_1 = \begin{pmatrix} C_i & w_1 \\ s_i & 1 \end{pmatrix}
\]

where \( w_1, s_1 \in \mathbb{C}^{2g} \) with either \( w_1 = 0 \) or \( s_1 = 0 \), and the entries of \( w_1 \) and \( s_1 \) are all zero except for the first through fourth rows.

Suppose next that nonzero complex numbers \( p_i \) for \( 1 \leq i \leq k \) with \( 1 \leq k < g - 1 \) are provided so that for

\[
P_i = \text{diag}(I_{2i}, p_i I_2, I_{2g-2i-2}) ,
\]

\[
Q_k = P_1 P_2 \cdots P_k , \text{ and } \tilde{Q}_k = \begin{pmatrix} Q_k & 0 \\ 0 & 1 \end{pmatrix} ,
\]

\[
\tilde{Q}_k^{-1} \tilde{X}_i \tilde{Q}_k = \begin{pmatrix} C_k & w_i \\ s_i & 1 \end{pmatrix} \text{ for } 1 \leq i \leq k
\]

where \( w_i, s_i \in \mathbb{C}^{2g} \) with either \( w_i = 0 \) or \( s_i = 0 \), and the entries of \( w_i \) and \( s_i \) are all zero except for the \((2i-1)\)st through \((2i+2)\)nd rows. Under this assumption, we seek \( p_{k+1} \in \mathbb{C}^\times \) which produces appropriate \( P_{k+1}, Q_{k+1}, \) and \( \tilde{Q}_{k+1} \).

We first observe that all of \( A_i \) and \( B_i \) are block diagonal matrices with each diagonal block a \( 2 \times 2 \) matrix. Therefore, all of \( A_i \) and \( B_i \) commute with \( P_1, P_2, \ldots, P_k \). We hence have

\[
\tilde{Q}_k^{-1} \tilde{A}_i \tilde{Q}_k = \tilde{A}_i \text{ and } \tilde{Q}_k^{-1} \tilde{B}_i \tilde{Q}_k = \tilde{B}_i \text{ for } 1 \leq i \leq g.
\]
Now, let $\tilde{X}_{k+1} = \tilde{Q}_{k+1}^{-1}\tilde{X}_{k+1} \tilde{Q}_k$. Then by taking the conjugation of the conditions (ii)-(iv) by $\tilde{Q}_k$, we see that $\tilde{X}_{k+1}$ satisfies the same conditions (i)-(iv) for $\tilde{X}_{k+1}$. We set further

$$\tilde{X} = (\tilde{G}^k)^{-1} \tilde{X}_{k+1} \tilde{G}^k.$$ 

Since $$(\tilde{G}^k)^{-1} \tilde{B}_{k+1} \tilde{G}^k = \tilde{B}_1$$ and $$(\tilde{G}^k)^{-1} \tilde{B}_{k+2} \tilde{G}^k = \tilde{B}_2$$ in particular, the matrix $\tilde{X}$ satisfies the assumption of Theorem 7.6. Therefore, we can apply Theorem 7.6 to obtain $p_{k+1} \in \mathbb{C}^\times$ so that for $P = \text{diag}(I_2, p_{k+1} I_2, I_{2g-4})$ and $\tilde{P} = \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}$, 

$$\tilde{P}^{-1} (\tilde{G}^k)^{-1} \tilde{X}_{k+1} \tilde{G}^k \tilde{P} = \begin{pmatrix} C_1 & w'_{k+1} \\ \text{s}'_{k+1} & 1 \end{pmatrix}$$

where $w'_{k+1}$, $s'_{k+1} \in \mathbb{C}^{2g}$ with either $w'_{k+1} = 0$, or $s'_{k+1} = 0$, and the entries of $w'_{k+1}$ and $s'_{k+1}$ are all zero except for the first through fourth rows. On the other hand, it is easy to see

$$G_k P (G^k)^{-1} = \text{diag}(I_{2k+2}, p_{k+1} I_2, I_{2g-2k-4}),$$

which we denote by $P_{k+1}$. We further set

$$\tilde{P}_{k+1} = \begin{pmatrix} P_{k+1} & 0 \\ 0 & 1 \end{pmatrix}, \text{ and } \tilde{Q}_{k+1} = \begin{pmatrix} Q_{k+1} & 0 \\ 0 & 1 \end{pmatrix}.$$

We then compute

$$\tilde{Q}^{-1}_{k+1} \tilde{X}_{k+1} \tilde{Q}_{k+1} = \tilde{P}^{-1}_{k+1} \tilde{X}_{k+1} \tilde{P}_{k+1} = \tilde{G}^k \left( \tilde{P}^{-1} (\tilde{G}^k)^{-1} \tilde{X}_{k+1} \tilde{G}^k \tilde{P} \right) (\tilde{G}^k)^{-1}$$

$$= \begin{pmatrix} G_k & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} C_1 & w'_{k+1} \\ \text{s}'_{k+1} & 1 \end{pmatrix} \begin{pmatrix} (G^k)^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} G_k C_1 (G^k)^{-1} & G_k w'_{k+1} \\ (G_k \text{s}'_{k+1} (G^k)^{-1} & 1 \end{pmatrix} \begin{pmatrix} C_{k+1} & G_k w'_{k+1} \\ \text{s}'_{k+1} & 1 \end{pmatrix}.$$ 

Here, we used an obvious relation $^t G = G^{-1}$ in the last equality.

Now, we set

$$w_{k+1} = G_k w'_{k+1} \text{ and } s_{k+1} = G_k \text{s}'_{k+1},$$

so that we have

$$\tilde{Q}^{-1}_{k+1} \tilde{X}_{k+1} \tilde{Q}_{k+1} = \begin{pmatrix} C_{k+1} & w_{k+1} \\ \text{s}_{k+1} & 1 \end{pmatrix}$$

where $w_{k+1}$, $s_{k+1} \in \mathbb{C}^{2g}$ with either $w_{k+1} = 0$ or $s_{k+1} = 0$. It is easy to see that the entries of $w_{k+1}$ and $s_{k+1}$ are all zero except for the $(2k + 1)st$ through $(2k + 4)th$ rows. Finally, for $1 \leq i \leq k$, we note we may write

$$C_i = \begin{pmatrix} C_i \\ 0 \end{pmatrix}, \text{ and } P_{k+1} = \begin{pmatrix} I_{2k+2} & 0 \\ 0 & \tilde{P}_{k+1} \end{pmatrix}$$

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for some $$(2k + 2) \times (2k + 2)$$ matrix $$\tilde{C}_i$$ and $$(2g - 2k - 2) \times (2g - 2k - 2)$$ matrix $$\tilde{P}_{k+1}$$.

We hence have

$$P_{k+1}^{-1}C_iP_{k+1} = C_i \quad \text{for } 1 \leq i \leq k.$$ 

This implies for $$1 \leq i \leq k$$ that

$$\tilde{Q}_{k+1}^{-1}X_i\tilde{Q}_{k+1} = \tilde{P}_{k+1}^{-1}(\tilde{Q}_k^{-1}X_i\tilde{Q}_k)\tilde{P}_{k+1} = \left( \begin{array}{c} C_i \\ s_iP_{k+1}^{-1} \end{array} \right) \cdot \left( \begin{array}{c} 1 \\ \frac{P_{k+1}^{-1}w_i}{1} \end{array} \right).$$

For $$i \leq k$$, since the entries of $$w_i$$ are all zero except for the first $$2k + 2$$ rows, we can easily see $$P_{k+1}^{-1}w_i = w_i$$. Similarly, we see for $$i \leq k$$,

$$\left( \begin{array}{c} 1 \\ \frac{s_iP_{k+1}^{-1}}{1} \end{array} \right) = \left( \begin{array}{c} 1 \\ \frac{s_i}{1} \end{array} \right).$$

We can now conclude

$$\tilde{Q}_{k+1}^{-1}X_i\tilde{Q}_{k+1} = \left( \begin{array}{c} C_i \\ s_i \end{array} \right) \quad \text{for } 1 \leq i \leq k + 1$$

where $$w_i, s_i \in \mathbb{C}^{2g}$$ with either $$w_i = 0$$ or $$s_i = 0$$, and the entries of $$w_i$$ and $$s_i$$ are all zero except for the $$(2i - 1)$$st through $$(2i + 2)$$nd rows.

Now, we apply this process repeatedly starting from $$k = 1$$ to $$k = g - 2$$. We then obtain nonzero complex numbers $$p_1, p_2, \ldots, p_{g-1}$$ so that for $$P = Q_{g-1} = \text{diag}(I_2, p_1I_2, \ldots, p_{g-1}I_2)$$ and $$\tilde{P} = \left( \begin{array}{cc} P & 0 \\ 0 & 1 \end{array} \right)$$ have the desired property

$$\tilde{P}_{k+1}^{-1}X_k\tilde{P}_{k+1} = \left( \begin{array}{c} C_k \\ s_k \end{array} \right) \quad \text{for } 1 \leq k \leq g - 1$$

where $$w_k, s_k \in \mathbb{C}^{2g}$$ with either $$w_k = 0$$ or $$s_k = 0$$. This completes the proof of Theorem \ref{thm:5.5}. \hfill \Box

7.4 Proof of Theorem \ref{thm:5.7}

We first prove an analogue of Lemma \ref{lem:7.3}.

**Lemma 7.7.** Let $$g \geq 2$$, $$m \geq 1$$, and $$\tilde{F} \in \text{GL}(2g + m, \mathbb{C})$$. Let

$$\tilde{A}_i = \left( \begin{array}{cc} A_i & 0 \\ 0 & I_m \end{array} \right) \quad \text{and} \quad \tilde{B}_i = \left( \begin{array}{cc} B_i & 0 \\ 0 & I_m \end{array} \right) \quad \text{for } 1 \leq i \leq g.$$ 

Suppose $$\tilde{F}$$ satisfies the following conditions (i)-(iv).

(i) $$\tilde{F}$$ has a unique eigenvalue 1.

(ii) $$\tilde{F}\tilde{A}_i = \tilde{A}_i\tilde{F}$$ for $$1 \leq i \leq g$$.

(iii) $$\tilde{F}\tilde{B}_j = \tilde{B}_j\tilde{F}$$ for $$2 \leq j \leq g$$.

(iv) $$\tilde{F}\tilde{B}_1\tilde{F} = \tilde{B}_1\tilde{F}\tilde{B}_1$$.
Then, it holds
\[ \tilde{F} = \begin{pmatrix} A_1 & W \\ S & T \end{pmatrix} \]
where \( T \) is an \( m \times m \) matrix, and
\[ W = ^t(w \ 0 \ 0 \ 0 \ 0 \ \cdots \ 0), \quad S = (0 \ s \ 0 \ 0 \ 0 \ \cdots \ 0) \]
with \( w, s, 0 \in \mathbb{C}^m; \ 1^tws = 0; \ 1^twT = 1^tw, Ts = s; \) and \( T^2 - T = s 1^tw. \)

**Proof of Lemma 7.7**

The proof is basically the same as Lemma 7.3. We write \( \tilde{F} = \begin{pmatrix} X & W \\ S & T \end{pmatrix} \) where \( X \) and \( T \) are, respectively, \( 2g \times 2g \) and \( m \times m \) matrices. Then the conditions (ii) to (iv), in turn, imply
- \( (7.1)-(7.3) \) for \( 1 \leq i \leq g; \)
- \( (7.4)-(7.6) \) for \( 2 \leq j \leq g; \)
- \( (7.7)-(7.10) \) for \( j = 1. \)

By the same argument for Lemma 7.3, by making use of \( (7.3), (7.6) \) and \( (7.2), (7.5), \) respectively, we obtain
\[ S = (0 \ s \ 0 \ 0 \ 0 \ \cdots \ 0), \quad W = ^t(w \ 0 \ 0 \ 0 \ 0 \ \cdots \ 0) \]
for some \( s, w \in \mathbb{C}^m \) where \( 0 \in \mathbb{C}^m. \) Here, we note the entries in the fourth column of \( S \) and the third row of \( W \) must be zero because the condition (iii) is assumed for \( j \geq 2 \) rather than \( j \geq 3. \)

Next, by the same argument for Lemma 7.3 again, by using \( (7.1), \) and \( (7.4) \) for \( j \geq 3 \) if \( g \geq 3, \) we obtain
\[ X = \begin{pmatrix} X_0 & 0 \\ 0 & I_{2g-4} \end{pmatrix} \]
where \( X_0 \) is the \( 4 \times 4 \) matrix given by \( (7.15) \) for some \( \alpha, \beta, b, d \in \mathbb{C}. \) Furthermore, a direct computation shows that the equality \( (7.4) \) for \( j = 2 \) implies \( \alpha = \beta = d = 0. \)

Therefore, we may write
\[ X_0 = \begin{pmatrix} I_2 + b(U - I_2) & 0 \\ 0 & I_{2g-4} \end{pmatrix} \]
with \( b \in \mathbb{C}. \)

Then, a straightforward computation shows that the equality \( (7.7) \) implies for \( X_1 = I_2 + b(U - I_2), \)
\[ X_1 \hat{U}X_0 + \begin{pmatrix} 1^twS \\ 0 \end{pmatrix} = \hat{U}X_1\hat{U}. \]

By computing the matrix entries of the both sides, we can easily see \( b = 1 \) and \( 1^tws = 0. \)

In particular, we have \( X = A_1. \)

Finally, the equalities \( (7.8)-(7.10) \) for \( j = 1 \) can be rewritten as
\[ A_1 B_1 W + W T = B_1 W, \]
\[ SB_1 A_1 + TS = SB_1, \]
\[ SB_1 W + T^2 = T. \]

These imply, in turn,
\[ {^t}w T = {^t}w, \quad Ts = s, \quad \text{and} \quad s {^t}w = T^2 - T. \]

This completes the proof of Lemma 7.7. \( \Box \)

We can now prove Theorem 5.7. Suppose \( \widetilde{F} \in \text{GL}(2g + 1, \mathbb{C}) \) satisfies the assumption of Theorem 5.7. Then we can apply Lemma 7.7 with \( m = 1 \) to obtain
\[ \widetilde{F} = \begin{pmatrix} A_1 & w \\ t^s & t \end{pmatrix} \]

where \( w, s \in \mathbb{C}^{2g} \) and \( t \in \mathbb{C} \) with the property given in the lemma, among which
\[ w = {^t}(w_1 0 0 \cdots 0), \]
\[ s = {^t}(0 s_2 0 \cdots 0) \]

for some \( w_1, s_2 \in \mathbb{C} \) with \( w_1 s_2 = 0 \). This implies either \( w = 0 \) or \( s = 0 \). We then have \( t = 1 \) from \( \det \widetilde{F} = 1 \). This completes the proof of Theorem 5.7. \( \Box \)

8 A straightforward proof of Theorem 2.2

The proof of Theorem 2.2 given in [13] seems rather implicit in its final step, in the sense that it assumes without any reference to the literature that any injective homomorphism \( f : \text{Sp}(2g, \mathbb{Z}) \to \text{GL}(2g, \mathbb{C}) \) is conjugate to the inclusion \( \text{Sp}(2g, \mathbb{Z}) \hookrightarrow \text{GL}(2g, \mathbb{C}) \) if it satisfies \( f(A_i) = A_i \) and \( f(B_i) = B_i \) for \( 1 \leq i \leq g \) where \( \text{Sp}(2g, \mathbb{Z}) \) is considered as the subgroup of \( \text{GL}(2g, \mathbb{C}) \) generated by \( A_1, A_2, \ldots, A_g, B_1, B_2, \ldots, B_g \) given in Definition 5.1 and \( C_1, C_2, \ldots, C_{g-1} \) given in Definition 5.4. One can avoid this and give a straightforward proof of Theorem 2.2 by showing the following \( 2g \)-dimensional analogue of Theorem 5.5 which is actually almost the same as proving the assumption just mentioned.

Proposition 8.1. Let \( g \geq 2 \). For each \( k \) with \( 1 \leq k \leq g - 1 \), suppose \( X_k \in \text{GL}(2g, \mathbb{C}) \) satisfies the following conditions (i)-(iv).

(i) \( X_k \) has exactly one eigenvalue 1.
(ii) \( X_k A_i = A_i X_k \) for \( i = 1, 2, \ldots, g \).
(iii) If \( g \geq 3 \), then \( X_k B_j = B_j X_k \) for each \( j \) satisfying \( 1 \leq j \leq g \) and \( j \neq k, k + 1 \).
(iv) \( X_k B_j X_k = B_j X_k B_j \) for \( j = k, k + 1 \).
Then, there exist nonzero complex numbers $p_2, p_3, \ldots, p_9$ such that for

$$P = \operatorname{diag}(I_2, p_2I_2, p_3I_2, \ldots, p_9I_2) \in \mathrm{GL}(2g, \mathbb{C}),$$

it holds $P^{-1}A_iP = A_i$ and $P^{-1}B_iP = B_i$ for each $i = 1, 2, \ldots, g$, and furthermore, it also holds for each $k = 1, 2, \ldots, g - 1$, $P^{-1}X_kP = C_k$.

This proposition does not follow directly from Theorem \ref{thm:original-theorem} but can be proved by the same line of arguments, which is simpler because of lower degree of matrix.

After completing the work in this paper, the author was informed that Korkmaz revised and combined his two papers \cite{12} and \cite{13} into a single paper with the same title as \cite{12}, which contained the proof of the assumption mentioned above.

\section{Appendix}

In this appendix, we slightly generalize Lemma \ref{lem:existence} which originated from Korkmaz \cite{13} as mentioned before. Rather complicated arguments below, together with Remark \ref{rem:app-remark}, might suggest the limitation of the approach adopted in this paper. We first show:

\begin{Lemma} \label{lem:appendix-lemma}
Let $g \geq 3$ and $m \geq 0$. Suppose

$$\phi : \operatorname{Mod}(S) \to \mathrm{GL}(2g + m, \mathbb{C})$$

is an arbitrary homomorphism. Let $a$ be a non-separating simple closed curve on $S$, and $t_a$ the right-handed Dehn twist along $a$. Let $\lambda$ be an eigenvalue of $L_a = \phi(t_a)$.

If $\lambda_\# \geq m + 3$, then $\dim(E_\lambda^g) \geq 2g - 2$. In particular, $\lambda_\# \geq 2g - 2$. Furthermore, if $g \geq m + 4$, then $\dim(E_\lambda^g) \geq 2g - 1$.

We remark that the case $m = 0$ is nothing but Korkmaz \cite{13} Lemma 5.1, and the case $m = 1$ corresponds to Lemma \ref{lem:existence}. If $m \geq 1$, the last consequence of the lemma can be strengthened to $\dim(E_\lambda^g) \geq 2g$, as we show in Proposition \ref{prop:appendix-proposition}.

\end{Lemma}

\begin{proof}[Proof of Lemma \ref{lem:appendix-lemma}]
We first show $\dim(E_\lambda^g) \geq 2g - 2$. To do so, assume to the contrary that $\dim(E_\lambda^g) \leq 2g - 3$.

Consider a regular neighborhood of $a$, take the closure of its complement, and denote it by $S_a$. Note the genus of $S_a$ is $g - 1 \geq 2$. The inclusion $S_a \hookrightarrow S$ induces a homomorphism $\operatorname{Mod}(S_a) \to \operatorname{Mod}(S)$. Consider its composition with $\phi$ and denote it by the same symbol as $\phi : \operatorname{Mod}(S_a) \to \mathrm{GL}(2g + m, \mathbb{C})$. For each $i$ with $1 \leq i \leq \lambda_\#$, we set $W_i := \ker((L_a - \lambda I)^i)$. We also set $W_0 = 0$ and $W^{\lambda_\# + 1} = \mathbb{C}^{2g + m}$. Then, since the elements of $\operatorname{Mod}(S_a)$ commute with $t_a$, we obtain an $\operatorname{Mod}(S_a)$-invariant flag

$$0 = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_\lambda \subset W^{\lambda_\# + 1} = \mathbb{C}^{2g + m}.$$ 

The dimension of each successive quotient $W_{i+1}/W_i$ ($0 \leq i \leq \lambda_\# - 1$) is equal to the number of the Jordan blocks for $L_a$ with eigenvalue $\lambda$ and with degree $\geq i + 1$, which is at most the number of all the Jordan blocks with eigenvalue $\lambda$. We thus have $\dim(W_{i+1}/W_i) \leq \dim(E_\lambda^g)$. Therefore, we have $\dim(W_{i+1}/W_i) \leq 2g - 3$. Furthermore, by the assumption $\lambda_\# \geq m + 3$, we have $\dim(W^{\lambda_\# + 1}/W^{\lambda_\#}) = 2g + m - \lambda_\# \leq 2g - 3$.

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Then by Theorem 2.7 together with Remark 2.8, \( \phi(\text{Mod}(S_a)) \) is an abelian group. Hence \( \phi \) is trivial on the commutator subgroup of \( \text{Mod}(S_a) \).

Next, choose a simple closed curve \( a' \) on \( S_a \) so that \( a' \) is isotopic to \( a \) in \( S \). Since the genus of \( S_a \) is at least two, we may choose simple closed curves \( b, c, d, x, y, z \) on \( S_a \) which satisfy the lantern relation

\[
t_{a'}t_{b'}t_{c'}t_{d'} = t_xt_yt_z.
\]

Then we may write \( t_{a'} = (t_xt_b^{-1})(t_yt_c^{-1})(t_zt_d^{-1}) \), and hence \( t_{a'} \) is contained in the commutator subgroup of \( \text{Mod}(S_a) \). Hence we have \( \phi(t_{a'}) = I \). Since \( t_{a'} = t_a \) in \( \text{Mod}(S) \), we have \( L_a = I \), which contradicts to \( \dim(E^a_\lambda) \leq 2g - 3 \). This shows \( \dim(E^a_\lambda) \geq 2g - 2 \).

Next, we prove the latter part of the lemma. Suppose \( g \geq m + 4 \). Assume \( \dim(E^a_\lambda) = 2g - 2 \) on the contrary.

Let \( b \) be a non-separating simple closed curve on \( S \) which intersects with \( a \) transversely at a single point. Consider a regular neighbourhood of \( a \cup b \) in the interior of \( S \), and denote the closure of its complement in \( S \) by \( R \), whose genus is \( g - 1 \). We divide into two cases according to whether \( E^a_\lambda \) coincides with \( E^b_\lambda \) or not.

(I) The case \( E^a_\lambda = E^b_\lambda \). Since \( g \geq 2 \), \( \text{Mod}(S) \) is generated by Dehn twists along non-separating simple closed curves. Therefore, by Theorem 3.2 \( E^a_\lambda \) is \( \text{Mod}(S) \)-invariant and has dimension \( 2g - 2 \). We also see \( \dim(C^{2g+m}/E^a_\lambda) = 2g + m - (2g - 2) = m + 2 \leq g - 2 < 2g - 1 \). Therefore, we can apply Theorem 2.7 to the \( \text{Mod}(S) \)-invariant flag \( 0 \subset E^a_\lambda \subset C^{2g+m} \) to see \( \phi \) is trivial. This contradicts to \( \dim(E^a_\lambda) = 2g - 2 \).

(II) The case \( E^a_\lambda \neq E^b_\lambda \). Observe \( E^a_\lambda \cap E^b_\lambda \) is \( \text{Mod}(R) \)-invariant and its dimension satisfies

\[
2g - (m + 4) \leq \dim(E^a_\lambda \cap E^b_\lambda) \leq 2g - 3.
\]

Therefore, we have a \( \text{Mod}(R) \)-invariant flag

\[
0 \subset E^a_\lambda \cap E^b_\lambda \subset C^{2g+m}
\]

which satisfies \( \dim(E^a_\lambda \cap E^b_\lambda) \leq 2g - 3 \) and \( \dim(C^{2g+m}/(E^a_\lambda \cap E^b_\lambda)) \leq 2g - 3 \). Then by Theorem 2.7 \( \phi(\text{Mod}(R)) \) is trivial, since \( g \geq 3 \). Since \( t_a \) is conjugate to an element of \( \text{Mod}(R) \) in \( \text{Mod}(S) \), we have \( L_a = I \). This contradicts to \( \dim(E^a_\lambda) = 2g - 2 \).

This completes the proof. \( \square \)

Finally, we show that the lower bound for \( \dim(E^a_\lambda) \) can be improved by 1, if \( m \geq 1 \).

**Proposition 9.2.** Suppose the assumption of Lemma 9.1 with \( g \geq m + 4 \). If \( m \geq 1 \), then it holds \( \dim(E^a_\lambda) \geq 2g \).

**Proof.** By Lemma 9.1, it is sufficient to confirm \( \dim(E^a_\lambda) \neq 2g - 1 \). Assume to the contrary that \( \dim(E^a_\lambda) = 2g - 1 \). Let \( b \) be a non-separating simple closed curve which intersects with \( a \) transversely at a single point. Choose a separating simple closed curve \( c_0 \) such that \( c_0 \) bounds a compact surface \( R \) of genus \( g - 1 \) with connected boundary and with no punctures so that the two curves \( a \) and \( b \) are contained in the complement of \( R \). Since the genera of both \( R \) and its complement are at least 1, the inclusion \( R \hookrightarrow S \) induces an injective homomorphism \( \text{Mod}(R) \rightarrow \text{Mod}(S) \) (I), via which we consider \( \text{Mod}(R) \) as a subgroup of \( \text{Mod}(S) \). We divide into two cases according to whether \( E^a_\lambda = E^b_\lambda \) or not.
I. The case $E^a_\lambda = E^b_\lambda$. Since $g \geq 2$, Mod($S$) is generated by Dehn twists along non-separating simple closed curves. Therefore, $E^a_\lambda$ is Mod($S$)-invariant by Theorem 3.2 and has dimension $2g - 1$. We also have $\dim (\mathbb{C}^{2g+m}/E^a_\lambda) = m + 1 < 2g - 1$. Hence we can apply Theorem 2.7 to the Mod($S$)-invariant flag

$$0 \subset E^a_\lambda \subset \mathbb{C}^{2g+m}$$

to see that $\phi$ is trivial, since $g \geq 3$. This contradicts to $\dim (E^a_\lambda) = 2g - 1$.

II. The case $E^a_\lambda \neq E^b_\lambda$. Since the elements of Mod($R$) commute with both $t_a$ and $t_b$, $E^a_\lambda \cap E^b_\lambda$ is Mod($R$)-invariant, and its dimension satisfies $\dim (E^a_\lambda \cap E^b_\lambda) \leq 2g - 2$.

If $\dim (E^a_\lambda \cap E^b_\lambda) < 2g - 2$, then, since $\dim (E^a_\lambda \cap E^b_\lambda) \leq 2g - 3$ and

$$\dim (\mathbb{C}^{2g+m}/(E^a_\lambda \cap E^b_\lambda)) \leq 2m + 2 \leq 2g - 3,$$

we can apply Theorem 2.7 to the Mod($R$)-invariant flag $0 \subset E^a_\lambda \cap E^b_\lambda \subset \mathbb{C}^{2g+m}$ to see that $\phi$ is trivial on Mod($R$) since the genus of $R$ is at least 3. Since $t_a$ is conjugate to an element of Mod($R$), $L_a = I$, which contradicts to $\dim (E^a_\lambda) = 2g - 1$. Therefore, we have $\dim (E^a_\lambda \cap E^b_\lambda) = 2g - 2$.

Next, since $\dim (\mathbb{C}^{2g+m}/(E^a_\lambda \cap E^b_\lambda)) = m + 2 \leq 2g - 3$, the action of Mod($R$) on $\mathbb{C}^{2g+m}/(E^a_\lambda \cap E^b_\lambda)$ induced by $\phi$ is trivial by Theorem 2.1. On the other hand, since the genus of $R$ is at least 3, Theorem 2.2 implies that the action of Mod($R$) via $\phi$ on $E^a_\lambda \cap E^b_\lambda$ is either trivial or conjugate to the symplectic representation $\rho^R_0 : \text{Mod}(R) \to \text{GL}(H_1(R; \mathbb{C}))$ where $\bar{R}$ denotes the closed surface obtained from $R$ by gluing a 2-disk along its boundary. If the action is trivial, we may take any basis of $E^a_\lambda \cap E^b_\lambda$ and extend it arbitrarily to a basis of $\mathbb{C}^{2g+m}$, according to which we have

$$\phi(f) = \begin{pmatrix} I_{2g-2} & * \\ 0 & I_{m+2} \end{pmatrix}$$

for each $f \in \text{Mod}(R)$. Therefore, $\phi(\text{Mod}(R))$ is an abelian group. On the other hand, Mod($R$) is perfect since the genus of $R$ is at least 3. Hence $\phi(\text{Mod}(R))$ is trivial. Again, since $t_a$ is conjugate in Mod($S$) to an element of Mod($R$), we have $L_a = I$, which contradicts to $\dim (E^a_\lambda) = 2g - 1$. Therefore, the only possible case is that the action of Mod($R$) on $E^a_\lambda \cap E^b_\lambda$ is conjugate to the symplectic representation $\rho^R_0$.

In this case, since $H_1(\bar{R}; \mathbb{C}) = H_1(R; \mathbb{C})$, we may choose an isomorphism $u : E^a_\lambda \cap E^b_\lambda \to H_1(R; \mathbb{C})$ such that

$$u(\phi(f) \cdot v) = f_* u(v) \quad (f \in \text{Mod}(R), \; v \in E^a_\lambda \cap E^b_\lambda).$$

Here, $f_*$ denotes the natural action of $f$ on $H_1(R; \mathbb{C})$.

We now fix a basis of $\mathbb{C}^{2g+m}$ extending an arbitrary basis of $E^a_\lambda \cap E^b_\lambda$. Then, under the identification of $E^a_\lambda \cap E^b_\lambda$ with $H_1(R; \mathbb{C})$ via $u$, the image of $f \in \text{Mod}(R)$ under $\phi$ has the form

$$\phi(f) = \begin{pmatrix} \rho^R_0(f) & w_1 & w_2 & \cdots & w_{m+2} \\ 0 & I_{m+2} \end{pmatrix} \quad (w_1, w_2, \ldots, w_{m+2} \in H_1(R; \mathbb{C})).$$

For another $f' \in \text{Mod}(R)$ with

$$\phi(f') = \begin{pmatrix} \rho^R_0(f') & w'_1 & w'_2 & \cdots & w'_{m+2} \\ 0 & I_{m+2} \end{pmatrix} \quad (w'_1, w'_2, \ldots, w'_{m+2} \in H_1(R; \mathbb{C})),$$

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we have
\[ \phi(ff') = \begin{pmatrix} p_R^R(ff') \\ 0 \end{pmatrix} \begin{pmatrix} w_1 + f_*w'_1 & w_2 + f_*w'_2 & \cdots & w_{m+2} + f_*w'_{m+2} \\ \end{pmatrix}(m+2) \]

This formula shows for each \( i \) that the correspondence \( f \in \text{Mod}(R) \mapsto w_i \) defines a crossed homomorphism
\[ c_i : \text{Mod}(R) \to H_1(R; \mathbb{C}). \]

Now let \( d \) be a non-separating simple closed curve on \( R \). We fix an orientation of \( d \) and denote its representing homology class by \([\tilde{d}] \in H_1(R; \mathbb{C})\). Then by Theorem 4.2 there exists a complex number \( z_i \) for each \( i \) such that \( c_i(t_d) = z_i[\tilde{d}] \). On the other hand, the action of \( t_d \) is given by
\[ (t_d)_*x = x + \langle [\tilde{d}], x \rangle [\tilde{d}] \quad \text{for} \quad x \in H_1(R; \mathbb{C}) \]
where \( \langle \cdot, \cdot \rangle \) denotes the algebraic intersection form on \( H_1(R; \mathbb{C}) \). Therefore, we have
\[ \text{rank} \left( \phi(t_d) - I \right) = 1. \]

Since \( t_a \) is conjugate to \( t_d \) in \( \text{Mod}(S) \), we can conclude \( \text{rank} \left( L_a - I \right) = 1 \), which contradicts the assumption \( \dim \left( E_a^\lambda \right) = 2g - 1 \).

This completes the proof of Proposition 9.2. \( \Box \)

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