PERTURBATION THEORY FOR A NON-EQUILIBRIUM STATIONARY
STATE OF A ONE-DIMENSIONAL STOCHASTIC WAVE EQUATION

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Abstract. We address the problem of constructing a non-equilibrium stationary state for a one-dimensional stochastic Klein-Gordon wave equation with non-linearity, using perturbation theory. The linear theory is reviewed, but with the linear equations of motion including an additional potential term which emerges in the renormalization of the perturbation expansion for the state corresponding to the non-linear equations of motion. The potential is the solution to a fixed point equation. Low order terms in the expansion for the two-point function are determined.

1. Introduction

The goal of this article is to introduce a perturbation theory for a putative non-equilibrium stationary state of a one-dimensional stochastic Klein-Gordon wave equation with non-linearity, with the field supported on a ring. The field is weakly coupled to effective thermal baths of differing temperatures that provide both dissipation and noise. The stationary state for the linear dynamics is reviewed, but with the field equations of motion including an additional potential term which arises from a renormalization in the perturbation expansion. The additional potential term, which depends on the difference of temperatures, alters the perturbation and changes the unperturbed state.

The equations for the field \( \phi_t(x) \) and its momentum \( \pi_t(x) \) are given by

\[
\begin{align*}
\partial_t \phi_t(x) &= \pi_t(x) \\
\partial_t \pi_t(x) &= (\partial_x^2 - 1) \phi_t(x) - g(\phi_t(x)) - r(t) \cdot \alpha(x) \\
\end{align*}
\]

We assume that the \( \phi_t \) and \( \pi_t \) are periodic functions in \( x \in [0, 2\pi] \). The non-linearity \( g \) will be a polynomial in the field, particularly \( g(z) = z^3 \). The two components of \( r_t = (r_1(t), r_2(t))^T \in \mathbb{R}^2 \) model the heat baths. The last equations allow energy in the field to dissipate into the baths and serve as a source of energy into the field from the two standard independent Brownian motions \( \omega_i(t), i = 1, 2 \).\( T_1 \) and \( T_2 \) are the bath temperatures.

The \( \alpha_i(x), i = 1, 2 \) are fixed distributions coupling the field to the two thermal baths, and \( \langle \alpha_i \rangle \) is the operation of integrating \( \alpha \) against another function of \( x \), giving out a scalar. The \( \alpha_i \)'s are assumed to satisfy the conditions on their Fourier coefficients \( \{\hat{\alpha}_i(n)\} \): that there exist positive constants \( c_0, c_1, c_2 \) with \( c_1 < c_2 < 1 \) such that \( c_1 \leq |\hat{\alpha}(n) \cdot \hat{\alpha}(n)| \leq c_2(\hat{\alpha}^*(n) \cdot \hat{\alpha}(n)) \leq c_0 \). The bounds ensure that eigenvalues of operators associated with the linear problem are non-degenerate and that certain small denominators (resonances) are manageable. \( \delta \)-functions at antipodal points on the circle are ruled out by this condition, but modifications of them are of course possible.

The Klein-Gordon field can be regarded as a limit of harmonic or anharmonic chain models in the limit with the number of oscillators growing to infinity and with rescaling. There is a
voluminous literature on harmonic and anharmonic chains and their thermodynamics. One of the earliest contributions was that of Rieder, Lebowitz, and Lieb [14], who considered a linear chain of oscillators with heat baths at the ends. They discovered the surprising but now familiar phenomena that the energy current is proportional to the difference of temperatures but independent of the length of the chain, and that the energy density along the chain has a peculiar profile, particularly near its ends. A model consisting of a finite chain of anharmonic oscillators was considered by Eckmann, Pillet, and Rey-Bellet [8], who showed existence of a thermodynamic stationary state with entropy production. An article by Aoki, Lukkarinen and Spohn [1] concerned determination of steady state energy flow through an anharmonic chain of oscillators, with the system modeled via a kinetic theory of colliding phonons. They arrive at a Fourier law for the current and find good agreement with simulations. Their article contains an excellent review of the literature on energy transport in simple models.

For continuum models, as considered here, McKean and Vaninsky established a Gibbs state equilibrium measure for nonlinear wave equations and showed its invariance under a Hamiltonian flow [12]. Stationary states for stochastic non-linear wave equations have been considered by many others, for example, Barbu and Da Prato [2]. Their dynamics, however, include a dissipative \(-\pi(x)\) term in the \(\pi\) equation and a driving cylindrical Weiner process. They show existence and uniqueness of an invariant measure assuming boundedness on the derivative of the non-linear interaction. Gubinelli, Koch, and Oh established local existence in time for a non-linear stochastic wave equation driven by white noise in two dimensions involving a time-dependent (rather than spatially dependent, as here) renormalization [9]. Rey-Bellet and Thomas showed global existence in time for the above equations of motion [1] in spaces of low regularity, \(\phi \in H^s, \ s > 1/3\), for \(g = -G'\), \(G\) a polynomial bounded below, but for smoother \(\alpha\)'s than employed here [15]. Wang and Thomas showed ergodicity for a model with bounded interaction and ultraviolet cut-off [18].

Closer to our problem, but again in a discrete setting, is the work of Bricmont and Kupiainen [6], who consider a three-dimensional system of anharmonic oscillators between two parallel planes where noise and dissipation occur, effectively at different temperatures. Stationarity of a putative measure leads to a system of relations for correlations of the field (BBGKY hierarchy); the system is truncated, leading to an implicit relation for correlation functions. They then find a resulting Fourier law for heat flow. There is presumably a connection between this hierarchy approach and the perturbative approach presented here, which involves an implicit relation for a correlation function. The connection between the approaches remains to be explored.

Section 2 is a review of the linear problem \(g = 0\) [17], but with a potential term \(v = v(x)\) included in the \(\pi\) equation, \(v\) arising in the non-linear perturbation problem. Analytic properties of the field covariance, and especially its dependence on \(v\), are determined. The spectral theory of the Liouville operator associated with the linear equations of motion is described. The spectral properties of this operator are largely determined by the matrix operator of its drift term, \(A_v\). The analysis of \(A_v\), which is not normal, closely follows the self-adjoint case, but with caveats that the eigenprojections of \(A_v\) need to be estimated, and that there is near degeneracy of pairs of eigenvalues causing small-denominator (resonance) issues. Estimates on the eigenvectors and eigenvalues appear in the text as needed, but their proofs are relegated to the appendix.

Section 3 is an introduction to a formal perturbation expansion with \(v^3\) non-linearity in the \(\pi\)-equation of (1), the expansion having a certain 1950-60’s quantum field theory flavor to it. The potential \(v\) makes its appearance by necessity here in a renormalization issue and is shown to be the solution to a fixed-point problem. Only first and second order terms in the expansion are considered.
Much of the analysis of $A_v$ is equivalent to that for a non-Hermitian one dimensional Schrödinger-like operator with complex finite rank interactions; part of the appendix treats this operator.

2. Aspects of the linear problem, $g(\phi) = 0$

2.1. Covariance of the field. Define the matrix operator $A_v$,

$$A_v = \begin{pmatrix}
0 & 1 & 0 & 0 \\
\partial_x^2 - 1 - v(x) & 0 & -\alpha_1(x) & -\alpha_2(x) \\
0 & \langle \alpha_1 \rangle - 1 & 0 \\
0 & \langle \alpha_2 \rangle & 0 & -1
\end{pmatrix},$$

which is simply the linear operator part of the right side of the equations of motion but with an additional continuous potential $v = v(x)$ which will arise in the renormalization issue with $g \neq 0$. The matrix $A_v$ isn't normal, rather it being the sum of a matrix similar to a skew-adjoint operator and a rank 2 symmetric perturbation. It operates on complex valued 4-vectors of the form, $e = (e_\phi(x), e_\pi(x), e_r)^T$; $e_\phi$ and $e_\pi$ are suitably integrable functions in $x$, $e_r \equiv (e_{r_1}, e_{r_2})^T$ has two scalar components. Let

$$\langle f, e \rangle = \langle f, e \rangle_H = \int (f_\phi(x)e_\phi(x) + f_\pi(x)e_\pi(x)) \, dx + f_r \cdot e_r,$$

which will serve both as a pairing of test functions with distributions and as an inner product for a Hilbert space $H$. The matrix $\sqrt{T}$ is defined

$$\sqrt{T} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & \sqrt{T_1} & 0 \\
0 & 0 & \sqrt{T_2}
\end{pmatrix},$$

with $T_1, T_2$ the bath temperatures.

Let $f_n, \lambda_n^* and e_n, \lambda_n$ be eigenvectors and their eigenvalues of $A_v^\dagger$ and $A_v$ respectively, $A_v^\dagger f_n = \lambda_n^* f_n, A_v e_n = \lambda_n e_n$. The mode number $n$ is in the set $\{(n,i) : n \in \mathbb{Z}, i \in \{1, 2\}\}$ or $\{1, 2\}$; the latter labels two exceptional cases of eigenvectors with eigenvalues near $-1$ for small $\alpha$’s, their weight principally on their $r$ components. Define the projections $\{P_n\}$, also $4 \times 4$ matrices, by

$$P_n = \frac{|e_n\rangle\langle f_n|}{\langle f_n, e_n \rangle_H}.$$

Bounds on these operators, their eigenfunction components and eigenvalues uniform in $n$ and $v$ are obtained in the appendix, Lemma 1, the paragraph above it, and Lemma 4, but will be reviewed as needed below. The bounds depend on $v$ but this dependence won’t appear explicitly until variations in $v$ are considered.

For notational convenience, we set

$$\Phi(f) = \langle f, \Phi \rangle$$

with $\Phi_t(x) \equiv (\phi_t(x), \pi_t(x), r_t)^T$ regarded as a column vector. The equations of motion can then be written as

$$d_t \Phi_t(x) = A_v \Phi_t(x) dt + \sqrt{T} d\omega(t)$$
a stochastic differential equation for an infinite-dimensional Ornstein-Uhlenbeck process with solution \( \Phi_t \), a stochastic integral (mild solution, \([7]\)),

\[
\Phi_t = \int_0^t \exp((t-s)A_v)\sqrt{T}d\omega(s) + \exp(tA_v)\Phi_0.
\]

For \( f_n \) an eigenvector of \( A_v^\dagger \) with eigenvalue \( \lambda_n^* \), we have simply that

\[
\Phi_t(f_n) = \int_0^t \exp((t-s)\lambda_n)\langle f_n, \sqrt{T}d\omega(s) \rangle + \exp(t\lambda_n)\Phi_0(f_n).
\]

The process defined by Eq.\((7)\) accommodates a stationary Gaussian measure \( \mu_v \) depending on \( v \). Its mode covariance matrix \( \hat{C}_v(m,n) \), for each \( m,n \) a \( 4 \times 4 \) matrix, is given by

\[
\hat{C}_v(m,n) \equiv E_{\mu_v}[P_m^\dagger \Phi P_n] = -\frac{P_m^\dagger TP_m}{(\lambda_n^* + \lambda_n)}.
\]

\( \hat{C}_v \) depends on \( v \), through its dependence on the eigenfunctions and eigenvalues, but again we do not write their dependence until needed in the non-linear perturbation problem. To see \((10)\), we have from the equation of motion \((7)\) that

\[
d\Phi_t(f_n) = \lambda_n \Phi_t(f_n)dt + \langle f_n, \sqrt{T}d\omega(t) \rangle.
\]

Stationarity of the measure and application of Itô’s lemma gives

\[
\frac{d}{dt}E_{\mu_v}[\Phi_t(f_n^*)\Phi_t(f_n)]|_{t=0} = 0
\]

\[
= (\lambda_n^* + \lambda_n) E_{\mu_v}[\Phi(f_n^*)\Phi(f_n)] + \langle f_n, Tf_m \rangle,
\]

so that

\[
E_{\mu_v}[\Phi(f_n^*)\Phi(f_n)] = -\frac{\langle f_n, Tf_m \rangle}{\lambda_n^* + \lambda_n}.
\]

(This relation can also be obtained by computing the covariance of the integral for \( \Phi_t(f_n) \), Eq.\((9)\), in the limit \( t \to \infty \). It is evident in this latter approach that \( C_v \) is positive definite.) Inserting the definition of the projections \((5)\) completes the derivation of the covariance matrix \((10)\). The denominators in \((10)\) appears dangerous when \( n = m, \lambda_n^* + \lambda_n \sim -|\hat{\alpha}(n)|^2/n^2 \), or when \( \lambda_m \) and \( \lambda_n \) are nearly degenerate, \( \lambda_n^* + \lambda_n \sim \pm |\hat{\alpha}(n)|^2/n \). But in the numerator, only the \( r \)-components of \( f_m \) and \( f_n \) are detected by \( T \) and these components are \( \sim |\hat{\alpha}(m)|/m \) and \( |\hat{\alpha}(n)|/n \) respectively. See the discussion below Eq.\((15)\) concerning the field covariance, as well as Lemma\((3)\) and Eqs.\((34)\) in the appendix for a more refined estimate of the eigenvalues.

The field \( \phi_t(x) \), i.e., the first component of \( \Phi_t \), can be expanded using the \( P_n \)'s in a resolution of the identity,

\[
\phi_t(x) = \sum_n P_{n,\phi(x)} \Phi_t = \sum_n \frac{e_{n,\phi(x)} \Phi_t(f_n)}{\langle f_n, e_n \rangle t}.
\]

\( (P_n \Phi_t \) is a 4-vector, and, \( P_{n,\phi(x)} \Phi_t \) is its \( \phi \)-component at \( x \) and time \( t \), \( \Phi_t(f_n) \) given above in \((9)\).) With this expansion, we can express the space-time covariance just for the field \( \phi \) using the mode
covariance,
\[
C_v(x, y, t) = E_{\mu_v} [\phi(x)\phi_t(y)] = \sum_{m,n} \left[ e_{m,\phi}(x) e^{i\lambda_n} e_{n,\phi}(y) \right] E_{\mu_v} [\Phi(f_m^*)\Phi(f_n)]
\]
\[
= -\sum_{m,n} \left[ \frac{P_{m,n}(y)}{\lambda_m^* + \lambda_n^*} \right] e^{t\lambda_n}.
\]
(15)

\[C_v(x, y, t)\] is bounded and Hölder continuous in its variables, with any index $\gamma < 1$. Moreover this continuity follows by the Kolmogorov continuity theorem [10] that $\phi_t(x)$ is a.s. Hölder continuous with any index $\gamma < 1/2$ in both $x$ and $t$ with respect to the Gaussian measure $\mu_v$.

Showing the convergence of the series (15) for the covariance and its continuity properties in $x$ and $t$ requires detailed estimates on the eigenfunctions and their eigenvalues which are established in the appendix (paragraph preceding Lemma [13] Lemmas [14]). The salient facts are these: (i) There exist constants $\{a_n\}$ uniformly bounded in $n$ such that $F_{n,\phi(x),r} \sim a_n e_{n,\pi}(x)/n^\gamma$, with $\|e_{n,\pi}\|_{L^2} = 1$, $e_{m,\phi} = e_{m,\pi}/\lambda_m$. Moreover, the $e_{n,\pi}$'s are pointwise bounded and almost Lipschitz uniformly in $n$ in the sense that there exists positive constants $a_c$ such that $\sup_x |e_{n,\pi}(x)| \leq a_c \|e_{n,\pi}\|_{L^2}$ and $|e_{n,\pi}(x) - e_{n,\pi}(y)| \leq c_n \|x - y\|^{1/2} \|e_{n,\pi}\|_{L^2}$ for any $\gamma < 1$. (ii) The eigenvalues behave like $\lambda_n \sim \pm |\hat{\alpha}(n)|^2/n^2$ for large $n$ by the non-degeneracy assumption on the $\alpha$'s. The small denominator issue $\lambda_m^* + \lambda_n^* \sim -|\hat{\alpha}(n)|^2/n^2$, $m = n$ in (15) and the nearly degenerate case $\lambda_m^* + \lambda_n \sim \pm i/n$ are mollified by the $1/n^4$ behavior of the numerator. (iii) The estimates here are uniform in the potential $v$, provided $\|v\|_C \leq \varepsilon_v\|\hat{\alpha}\|_{\infty}^2$, for some positive $\varepsilon_v > 0$, ($\|v(x)\|_C \equiv \sup_x |v(x)|$). Similar estimates hold for the time $t$ in the summands of (15). As example, the diagonal terms of the double sum behave as $n^{-2}$, the near diagonal terms as $n^{-3}$, and otherwise the terms are $\sim n^{-2}m^{-2}|n - m|^{-1}$ for $n, m$ large, and absolute convergence follows.

2.2. Semigroup and Generator for $\Phi_t$. Let $F$ be a functional of the field, e.g., a polynomial in $\Phi(f_1), \ldots, \Phi(f_n)$, $f_1, \ldots, f_n$ eigenfunctions of $A^\dagger$ (these polynomials are dense in the $L^2$-Hilbert space of functionals, with $\mu_v$ as measure). The semigroup $e^{tL_v}$ associated with the above process $\Phi_t$ is defined by
\[
e^{tL_v} F(\Phi) = E_{\Phi} [F(\Phi_t)],
\]
where the expectation here is with respect to the Brownian motion with $\Phi_t$ given above [8] and with $\Phi = \Phi_0$. The generator of the semigroup associated with $\Phi_t$ has the expression
\[
(L_v F(\Phi)) = \left\{ \frac{1}{2} \sum_{i=1}^2 \frac{\partial}{\partial x_i} T_i \frac{\partial}{\partial x_i} F(\Phi) + \left\langle A_v \Phi, \frac{\delta}{\delta \Phi} \Gamma(\Phi) \right\rangle \right\},
\]
(17)
where $\frac{\delta}{\delta \Phi}$ $\equiv \left( \frac{\delta}{\delta \lambda_1}, \frac{\delta}{\delta \lambda_2}, \ldots, \frac{\delta}{\delta \lambda_n} \right)^T$ being the functional gradient. $L_v$ is a well-defined operator acting in $L^2$ of $\mu_v$. Polynomials in $\Phi$ of degree $m$ are mapped into polynomials of the same degree by $L_v$ and its semigroup. There is “second quantization” associated with $L_v$. As example, if $f_n$ is an eigenfunction of $A_v$ with eigenvalue $\lambda_n^*$, then $\Phi(f_n)$ is an eigenfunction of $L_v$ with eigenvalue $\lambda_n^*$; $\Phi(f_n)$ is an order 1 Hermite function of the field. Given two eigenfunctions $f_m$ and $f_n$ of $A_v$, the functional $F(\Phi) = \Phi(f_m^*)\Phi(f_n) + \left\langle f_m, T f_n^* \right\rangle/(\lambda_n^* + \lambda_n)$, an order 2 multivariable Hermite function of the field, is also an eigenfunction of $L_v$ with eigenvalue $\lambda_{m,n} = \lambda_m + \lambda_n$. (Note that the denominator $(\lambda_m + \lambda_n)$ has non-zero real part so it does not vanish.) In general, one can construct
higher degree polynomials which are also eigenfunctions of $L_v$ of the form: a monomial of degree $k$, $\Phi(f_{n_1})\Phi(f_{n_2})\cdots\Phi(f_{n_k})$ minus a lower order polynomial, with eigenvalue $\lambda = \lambda_{n_1} + \lambda_{n_2} + \cdots + \lambda_{n_k}$. All of the eigenfunctions constructed are of finite $L^2$-norm with respect to $\mu_v$, as a consequence of the boundedness of the mode covariance and Wick’s theorem. They are complete since the $f_n$’s are. $L_v$ has discrete spectrum. Clearly 0 is an eigenvalue with eigenvector a constant, but it is also an accumulation point, e.g., $\lambda_n, n^* = \lambda_n + \lambda_n^*$ is an eigenvalue which goes to zero, $n \to \infty$. That zero is an accumulation point is the source of small-denominator resonance difficulties.

We remark that the adjoint operator $L_v^\dagger$ defined with respect to $\mu_v$ has the same second derivative part as $L_v$ but the first derivative drift part is different. The dual process is another Ornstein-Uhlenbeck process.

3. Perturbation by a $\phi^3$ non-linearity

The goals of this section are to outline a perturbation expansion, introduce the renormalization issue, and address it at least to second order in a coupling constant $g$. The original equations of motion (11) have the potential $v(x)$ equal to zero. But it will be important to keep $v$ in the present calculations.

3.1. Perturbation expansion. Let $V = \int dx \phi^3(x) \frac{\delta}{\delta \pi(x)}$ be the perturbation, $g$ a coupling constant, and let $\nu_{g,v}$ be the putative invariant measure to equations of motion (11) with $g(\phi(x)) = g\phi^3(x)$ and $-v(x)\phi(x)$ included. Let $L_{g,v} = L_v - gV$ be the associated (Liouville) semigroup generator. Then the formal expansion for $\nu_{g,v}$, (generating functional) is given by

$$\int d\nu_{g,v}(\Phi) F(\Phi) = \int d\mu_v(\Phi) \sum_{m \geq 0} g^m \left( V \frac{Q_v}{L_v} \right)^m F(\Phi)$$

with $Q_v F(\Phi) = F(\Phi) - \int d\mu_v F$, $F$ a functional of the field. The series is obtained by making an ultra-violet cutoff so that 0 is an isolated eigenvalue of $L_v$, writing then the difference $\nu_{g,v} - \mu_v$ as a contour integral over a small circle about 0 with integrand the difference of resolvents, $(z - L_{g,v})^{-1} - (z - L_v)^{-1}$ written via the second resolvent equation (11) and expanding the difference in a formal Born series. Note that the series for $\nu_{g,v}$ has integral 1, since $\mu_v$ does and the facts that $V$ and $Q_v$ (which appears gratuitously) kill constants; $L_{g,v}$ also has an eigenvalue 0.

It is instructive to compute the first order and second order terms in the expansion (18), where the renormalization issue first arises. In preparation, we will represent $V Q_v L_v^{-1}$ operating on a functional as a time integral,

$$V \frac{Q_v}{L_v} F(\Phi) = - \int_0^\infty dt V e^{tL_v} Q_v F(\Phi),$$

ignoring the issue of which $F$’s are in the domain of the operation, (although polynomials in the $\Phi(f_n)$’s certainly are). Since $V$ kills constants we can replace the $Q_v$ simply by the identity, and use the semigroup representation Eqs. (16,8),

$$V \frac{Q_v}{L_v} F(\Phi) = - \int_0^\infty dt V E_\Phi [F(\Phi t)],$$

the expectation $E[\cdot]$ here with respect to Brownian motion.
We define the function $D_v(x, t; z)$,

$$
D_v(x, t; z) = \sum_n e^{\lambda_n} P_n \phi(x) \phi(t) = \sum_n \frac{e_n(x) e^{\lambda_n} f_n(z)}{f_n^*(e_n)}
$$

(21)

which will be ubiquitous in the analysis ($P_n$ is the $\phi, \pi$ entry of the matrix $P_n$). Its manipulations are analogous to normal ordering of creation/annihilation operators. The series is only conditionally convergent, $t > 0$.

For the remainder of the discussion, we confine attention just to the two point function case, $F(\Phi) = \phi(x)\phi(y)$. For this choice, the leading term in the expansion (18) is

$$
g E_{\mu} \left[ V \frac{Q_{\mu}}{E_b} F(\Phi) \right] = 3g \int dt dz C_v(z, y, t) (C_v(z, x, t) D_v(y, y, t) + C_v(z, x, t) D_v(y, z, t))
$$

(22)

by Wick’s theorem. The expression corresponds to the two diagrams shown:

![Diagrams](image)

**Figure 1.** $g^2$-diagrams. The upper loops correspond to the $D_v$ functions, the lower loops to the covariances $C_v$. The horizontal direction is time.

Substituting the mode expansions for $C_v(z, t)$ and $D_v(t; z)$, then doing the $t$ integral, but only retaining the resonant contributions where an $e^{\lambda_n}$ is integrated against $e^{\lambda_n^*}$ and keeping just the diagonal part of the double series for $C_v$, we obtain a series

$$
3g \sum_n \int dz C_v(z, z, 0) \frac{P_{n, \phi(x)} T_{n, \phi(z)}}{(\lambda_n + \lambda_n^*)^2} P_{n, \phi(y), \pi(z)}
$$

(23)

and an identical one with $x$ and $y$ interchanged. The off-diagonal, i.e., non-resonant or nearly resonant, contributions are absolutely summable. The convergence of this sum (23) is somewhat problematic; in terms of superficial degree of divergence [10], the $P_{n, \phi(x), t} \sim 1/n^2$, $P_{n, \phi(x), \pi(z)} \sim 1/n$ and the small denominator $(\lambda_n + \lambda_n^*)^{-2} \sim n^4$. Thus the $n$ term is of degree $-1$ and the sum of degree 0. For this diagram however, and with further analysis of the $e_n \delta$’s and $f_n \delta$’s, there is a fortuitous effective cancellation of the $n$ and $-n$ modes for large $n$, and with this cancellation the sum is finite. The $z$-integral does not enhance the convergence of the sum.

Terms of order $g^2$ in Eq. (18), correspond to the associated diagrams shown, Fig. 2, (and $x$ and $y$ exchanged). For example, the first diagram, the breaching whale, corresponds to the integral,
The diagrams encode the rules for a diagram with an arbitrary number of vertices. Each vertex, other than the terminal vertices $x,y$, has valence $3$ and is the source of a $D$-loop, and $x$ and $y$ each having one unit of valence. A $D$-loop consumes 1 unit of valence only at its right end. A $C$-loop consumes one unit of valence at each of its ends. An order $g$ having one unit of valence. A $D$-loop involves an $m$-loop, and $m+1$ $C$-loops. In effect, $D$ has summands of superficial degree of divergence $-1$, behaving as $n^{-1}$ and $C$ has summands of degree $-2$, behaving as $n^{-2}$ (the diagonal part $m=n$ being of highest degree). A diagram involves an $m+(m+1) = (2m+1)$-fold summation over the modes, counting $C$ as a single sum.

For a $g^m$-diagram there are $m$ time integrals to be performed. Retaining just the most singular resonant contributions results in $m$ constraints among the modes so the total summation becomes $(m+1)$-fold. But each time integral, so constrained to the resonant case introduces a small denominator which can be of degree $+2$. Thus the superficial degree of a diagram with $m$ vertices plus the terminal $x,y$ ones is $D$-degrees $+ C$-degrees $+ small$ denominators $+ summations = -m - 2(m+1) + 2m + (m+1) = -1$.

For the diagrams in Fig. 2, $m=2$; there are $2D$-loops and $3C$-loops, and $2$ time-integrals. For the breaching whale, the first of the diagrams, the time integrals result in the new small denominators $\lambda_n + \lambda_n^*$ and $(\lambda_n + \lambda_n^* + \lambda_{n_2} + \lambda_{n_3}^*)$ (the latter with $\lambda$'s with negative real parts, so the sum is no smaller than $\sim \sum_i |\tilde{a}(n_i)|^2/(n_i)^2$). Each of these small denominators contributes a $+2$ to the degree. The summation is $3 = (m+1)$-fold. The resulting integral

$$-6g^2 \sum_{n, n_1, n_2, n_3} \prod_{i=1}^3 P_{n, \phi(z_i)} T P_{n, \phi(z_i)}^t P_{n, \phi(y), \pi(z_1)} P_{n, \phi(x), \pi(z_2)} dz_1 dz_2 \frac{(\lambda_n + \lambda_n^*)}{(\lambda_n + \lambda_n^*)^3} \frac{(\lambda_n + \lambda_n^* + \lambda_{n_2} + \lambda_{n_3}^*)}{(\lambda_n + \lambda_n^*)^3}$$

is finite.

The third diagram in Fig. 2, (breaching whale with tadpole earrings), has the integral

$$9g^2 \int dt_1 dz_1 dt_2 dz_2 C_v(z_1, z_1, 0) C_v(z_2, z_2, 0) C_v(z_1, z_1, t_1) \times D_v(y, t_1 + t_2; z_1) D_v(x, t_2; z_2).$$

Again, doing the time integrals and retaining just the resonant terms results in the single sum

$$-9g^2 \sum_n \int dz_1 dz_2 C_v(z_1, z_1, 0) C_v(z_2, z_2, 0) \times P_{n, \phi(z_2)} T P_{n, \phi(z_1)}^t P_{n, \phi(y), \pi(z_1)} P_{n, \phi(x), \pi(z_2)}^t$$

Figure 2. $g^2$-diagrams. Terminal vertices $x$ and $y$ can be interchanged.
Including the superficial degree of each of the tadpoles as $-1$, $C_v(z_i, z_i, 0)$ counts as $-1$, the overall degree of the diagram is still $-1$. But the remaining summand above is seen to have degree $0$, the sum has degree $+1$, and indeed the sum is divergent. The fourth diagram in Fig. 2 has integral which also diverges and which does not cancel that of the third. The remedy is to get rid of diagrams with tadpoles.

3.2. Simple renormalization. Diagrams of any order in $g$ with these simple tadpoles are eliminated by a renormalization, an appropriate alteration in the measure $\mu_v$ and shift in the interaction. We begin with the original problem with the linear dynamics having the potential $v(x) = 0$ and generator $L_0$. The idea is to subtract a linear drift term from $L_0$ and add it in as part of the perturbation,

\begin{align}
L_0 &\to L_v \equiv L_0 - \int dx v(x) \phi(x) \frac{\delta}{\delta \pi(x)}, \quad \text{and} \\
- g \int dx &\phi^3(x) \frac{\delta}{\delta \pi(x)} \to - \int dx (g\phi^3(x) - v(x)\phi(x)) \frac{\delta}{\delta \pi(x)}
\end{align}

in such a manner that $g\phi^3(x) - v(x)\phi(x)$ is, for all $x$, a third order Wick polynomial with respect to $\mu_v$, the Gaussian measure invariant under the evolution generated by $L_v$. This is the case if

\begin{equation}
v(x) = 3gE_\mu_0[\phi^3(x)].
\end{equation}

With this choice for $v$, the additional term in the interaction will cancel the self-loops. This *implicit* equation for $v$ has a unique solution for small coupling constant $g$ by the contraction mapping theorem [13]. In outline, and recalling the expansion for the covariance (15), we consider the mapping

\begin{equation}
v \to \tilde{\nu}_v \equiv C_v(x, x, 0) = - \sum_{m,n} \frac{P_{m,\phi(x)}(v)}{\lambda_m(v)} \frac{TP_{n,\phi(x)}(v)}{\lambda_n(v)}.
\end{equation}

Lemma(4) of the appendix provides estimates on the $v$-dependence of the projections and eigenvalues which imply for some $\varepsilon > 0$ and $v$'s in the ball $B = \{v : \|v\|_C \leq \varepsilon \|\tilde{\nu}_v\|^2_{\infty}\}$, the map is Lipschitz in $v$, $\|\tilde{\nu}_v - \tilde{\nu}_{v_1}\|_C \leq C\|v_2 - v_1\|_C$, for some constant $C$. (We know that $\tilde{\nu}_v$ is a continuous function of $x$.) Note particularly that the second line of (ii) of the Lemma shows that the small denominator, $\lambda_n + \lambda_n^*$ moves in a manageable way as a function of $v$. Thus for small enough coupling constant $g$, $v \to 3g\tilde{\nu}_v$ lands in the ball $B$ and is a strict contraction, hence the existence of a fixed point.

The above remarks do not exhaust the problem of tadpoles. For example, one can have an order $g^2$-*composite* tadpole, as shown in Fig. 3, requiring higher order corrections to $v$ and solving a new fixed point problem of higher order in $g$. 
4. Concluding remarks

The authors have investigated selected higher order integrals (order $g^6$ and higher), all of which are seen to be finite. The rank one Brascamp-Lieb inequalities are of utility in estimating these integrals [4, 5, 3]. But a general scheme for showing integrals of arbitrary order are finite, that the perturbation expansion is renormalizable, remains open.

It is interesting to remark that in the case of equilibrium $T = T_1 = T_2$, no renormalization is necessary. In this case, $\mu_v$ is a Gaussian Gibbs measure, with the momentum $\pi$ uncorrelated with $\phi$ or $r$. For the $g^1$-term in the generating functional Eq. (18), we have, letting $V$ operate instead on $\mu_v$,

$$g \int d\mu_v(\Phi) \left( V \frac{Q_v}{L_v} \right) F(\Phi) = \frac{g}{T} \int d\mu_v(\Phi) \left( \int \phi^3(x) \pi(x) dx \right) \frac{Q_v}{L_v} F(\Phi)$$

$$= -\frac{g}{T} \left\langle L_v^\dagger \left( \frac{1}{4} \int \phi^4(x) dx - C \right) \frac{Q_v}{L_v} F(\Phi) \right\rangle_{L^2(\mu_v)}$$

$$= -\frac{g}{T} \left\langle \left( \frac{1}{4} \int \phi^4(x) dx - C \right) , F(\Phi) \right\rangle_{L^2(\mu_v)} \quad (32)$$

the constant $C = E_{\mu_v} \left[ \frac{1}{4} \int \phi^4(x) dx \right]$ chosen so that the expression vanishes if $F$ is a constant. A similar calculation holds for order $m > 1$, resulting indeed in the $\frac{1}{m!} \left( \frac{g}{T} \int \phi^4(x) dx \right)^m$ contribution, with the whole series formally summing to a constant times $\exp \left( -\frac{g}{T} \int \phi^4(x) dx \right)$ as it should. $L_v^\dagger$, the adjoint operator to $L_v$ with respect to $\mu_v$, is simply $L_v$ but with opposite signs for its $\phi$ and $\pi$ drift terms. However, it remains to investigate the diagram integrals at equilibrium $T_1 = T_2$.

The Gaussian measure $\mu_v$ here has a fairly explicit covariance, and it should be possible to identify non-negative Radon-Nikodym factors, i.e., functionals of $\phi$ and $\pi$ and $r$, integrable with respect to $\mu$, to define new measures on the space of fields. Then the issue would be to identify physically compelling processes globally defined in time, modeling non-equilibrium heat flow for which the modified measures are stationary.

It remains to investigate the expected current flow in perturbation theory averaged over $[0, 2\pi]$, i.e.,

$$\frac{1}{2\pi} E_{\nu_v} \left[ \int_0^{2\pi} dx \pi(x) \partial_x \phi(x) \right] =$$

$$\frac{1}{2\pi} \sum_{m,n} \sum \int_0^{2\pi} dx \frac{\epsilon^*_m \pi(x) \partial_x \epsilon_n \phi(x)}{\langle f_m, \epsilon_m \rangle_{H} \langle f_n, \epsilon_n \rangle_{H}} E_{\nu_v} \left[ \Phi(f^*_m) \Phi(f_n) \right]. \quad (33)$$

The integral of $\pi$ against $\partial_x \phi$ is singular and requires refinements of the estimates of lemma [11] in the appendix, particularly on the real and imaginary parts of $\partial_x \epsilon_n \phi(x)$. The scaling of the ring $[0, 2\pi]$ to one of arbitrary size $[0, L]$ should be straightforward. We will address the issue of current flow elsewhere.

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5. Appendix

Using the eigenvalue equation $A_n e_n = \lambda_n e_n$ for the vector $e_n$, one can solve for the components $e_{n,\phi}, e_{n,\pi}$, just in terms of $e_{n,\pi}$, and similarly for $f_n$,

\[
e_n = \left(\frac{(\alpha, e_{n,\pi})}{\lambda_n} \frac{e_{n,\pi}}{\lambda_n + 1}\right) \cdot e_n\]

\[
f_n = \left(\frac{\frac{(\partial^2_x - 1 - v) f_n}{\lambda_n^*} f_n, f_n - \frac{\langle f_n, \alpha \rangle}{\lambda_n + 1}}\right).
\]

The eigenvalue equation for $A_n$ is equivalent to a Schrödinger-like eigenvalue equation, with the eigenvalue appearing implicitly

\[
(\partial^2_x - 1 - v(x)) e_{n,\pi} - \frac{\lambda_n}{\lambda_n + 1} \alpha(x) \cdot \langle \alpha, e_{n,\pi} \rangle = \lambda_n^2 e_{n,\pi}.
\]

The $f_{n,\pi}$ components for the adjoint problem $A_n^* f_n = \lambda_n^* f_n$ are solutions to the complex conjugate of the above, so $f_{n,\pi} = e_{n,\pi}^*$. (Note that there are also two eigenvalues of $A_n$ close to $-1$; their corresponding eigenvectors have most of their weight on their $r$-components, with small $\phi$ and $\pi$ components and do not play an important role in our analysis.)

In order to obtain qualitative information on $e_{n,\pi}$, we first examine eigenfunctions for

\[
h_{\varepsilon} = \partial^2_x - 1 - v - \varepsilon |\alpha| \cdot \langle \alpha, e_{n,\pi} \rangle.
\]

acting in $L^2[0,2\pi]$ with periodic boundary conditions; $v = v(x)$ is real and continuous with bound $|v|_{L^2} \equiv \sup_x |v(x)| \leq \varepsilon_0 \|\alpha\|_{L^p}$, $\varepsilon_0$ prescribed below. The $\varepsilon$ is complex, of modulus less than 1, and ultimately just equal to $\lambda_n/\lambda_n + 1$. The $\alpha$'s are as assumed in the text. This analysis of $h_{\varepsilon}$ largely reviews that in [17], but also provides estimates with $\varepsilon$ needed for the perturbation expansion.

Let $\{\psi_n\}$ be the $L^2$-normalized eigenfunctions for $h_{\varepsilon}$, $h_{\varepsilon} \psi_n = \lambda_n^2 \psi_n$. (The overall sign of $\lambda_n^2$ on the right side is jarring but in keeping with Eq. (35).) We assume that $\|\alpha\|_{L^2}$ is small so that $\lambda_n^2$ is close to $-(n^2 + 1)$, the eigenvalue for $\partial^2_x - 1, |\lambda_n^2 + (n^2 + 1)| \sim \|\alpha\|_{L^2}$, uniformly in $n$. To see this uniformity, suppose first that $v = 0$. Then the two components $(\alpha_i, \psi_n)$, $i = 1, 2$, satisfy

\[
(\alpha, \psi_n) = \varepsilon \langle \alpha, \partial^2_x - 1 - \lambda_n^2 \rangle \cdot \langle \alpha, \psi_n \rangle
\]

The eigenvalues $\{\lambda_n^2\}$ are zeros of a $2 \times 2$ determinant function of $\lambda^2$. Writing the resolvent $(\partial^2_x - 1 - \lambda_n^2)^{-1}$ in its spectral representation, pulling out the $n$ and $-n$ terms, and setting $\Delta \lambda_n^2 = \langle \lambda_n^2 - 1 \rangle - \lambda_n^2$ as the shift in eigenvalue from that of $\partial^2_x - 1$, we have that

\[
\Delta \lambda_n^2 \langle \alpha, \psi_n \rangle = \varepsilon \left(\langle \alpha, P^\alpha_{n,-n}, \alpha \rangle + \Delta \lambda_n^2 \left\langle \alpha, \frac{Q^\alpha_{n,-n}}{(\partial^2_x - 1 - \lambda_n^2)} \alpha \right\rangle \right) \cdot \langle \alpha, \psi_n \rangle
\]

with $P^\alpha_{n,-n}$ projection onto the subspace spanned by $e^{inx}$ and $e^{-inx}, Q^\alpha_{n,-n} = 1 - P^\alpha_{n,-n}$. The $2 \times 2$-matrix in bra-ket inside the parentheses is $O(n^{-\gamma})\|\alpha\|_{L^p}$, $\gamma < 1$ (with constant depending on $\gamma$), by the Estimate [2] below, so that to leading order, $\Delta \lambda_n^2$ is just an eigenvalue of $\varepsilon \langle \alpha, P^\alpha_{n,-n}, \alpha \rangle$ and itself of size $|\varepsilon|\|\alpha(n)|^2$ with corrections $\sim \Delta \lambda_n^2 n^{-\gamma}$. By the non-degenerate condition on the $\alpha$'s, the two eigenvalues of $\langle \alpha, P^\alpha_{n,-n}, \alpha \rangle$ split to the same size as their shift, i.e. there exists a positive constant $c_1$ independent of $n$ such that

\[
|\Delta \lambda_n^2 - \Delta \lambda_n^2| \geq c_1 |\varepsilon|\|\alpha(n)|^2.
\]

If an additional term $v$ suitably small is added to $\partial^2_x - 1, \|v\|_{L^2} \leq |\varepsilon_o|\|\alpha(n)|^2$, for some $\varepsilon_o > 0$. Ineq. (39) still holds, but with a reduced positive constant $c_o$. Again, the $\psi_n$'s will be identified
with the \( e_{n,\varepsilon} \)-components. There will be four eigenvalues for \( A_n \) of "level" \( n, n \neq 0 \), two if \( n = 0 \) and two near \(-1\). In the case \( n \neq 0 \), the eigenvalues are \( \pm i (\sqrt{n^2 + 1} + \frac{1}{2} \Delta \lambda^2_n / \sqrt{n^2 + 1}) \). For \( \varepsilon = \lambda_n / (\lambda_n + 1) \) which has an \( O(1/n) \) imaginary part, \( \lambda_n \) acquires a small negative real part \( \sim -|\hat{\alpha}(n)|^2 / n^2 \).

In the following, we are assuming that the eigenvalues are as above, that there are pairs of eigenvalues \( \lambda^2_n \) and \( \lambda^2_{-n} \) close to \(-n^2 + 1\) and separated by \( \sim |\hat{\alpha}(n)|^2 / n^2 \).

**Lemma 1.** Assume that the eigenfunctions \( \{\psi_n\}_{n \in \mathbb{Z}} \) are \( L^2 \)-normalized. Then the sequences \( \{\{\alpha, \psi_n\}\}_{n \in \mathbb{Z}} \) and \( \{\sup_x |e_n(x)|\}_{n \in \mathbb{Z}} \) are uniformly bounded in \( n \). Moreover, there exists a positive constant \( c \) and a positive \( c_\gamma \) depending on \( \gamma < 1 \) but independent of \( n \) such that

\[
|\psi_n(x) - \psi_n(y)| \leq c (|n||x - y| + c_\gamma |x - y|^\gamma)
\]

for each \( n \). There exists a positive constant \( c' \) such that if \( |3 \varepsilon| \leq c' \Re \varepsilon \) with \( \Re \varepsilon \neq 0 \) (and possibly redefining \( \psi_n^* \) by an overall phase factor), then

\[
\|\psi_n - \psi_n^*\|_{L^2} \leq c' |3 \varepsilon| \left( \frac{1}{|\Re \varepsilon|} + \frac{1}{|n| + 1} \right).
\]

These estimates are also uniform in \( \varepsilon \) provided \( 1/2 \leq |\varepsilon| \leq 1 \) and in the potential \( v \), provided \( \|v\| \leq \varepsilon_0 |\hat{\alpha}|^2_{2 \varepsilon} \), for some positive \( \varepsilon_0 \).

**Remark:** The inequality (41) implies as well, at least for small enough \( 3 \varepsilon / \Re \varepsilon \) that a similar operator bound holds for the difference of projections

\[
\left\| \frac{|\psi_n\rangle \langle \psi_n^*|}{\langle \psi_n^*, \psi_n \rangle} - \frac{|\psi_n\rangle \langle \psi_n|}{\langle \psi_n, \psi_n \rangle} \right\| \leq c' |3 \varepsilon| \left( \frac{1}{|\Re \varepsilon|} + \frac{1}{|n| + 1} \right).
\]

**Proof.** Let \( P_{n, -n}^o \) be as above and \( P_{n,-n} \) projection onto the subspace spanned by \( \psi_n \) and \( \psi_{-n} \) corresponding to the nearly degenerate eigenvalues \( \lambda^2_n, \lambda^2_{-n} \). We assume \( \psi_n \) to be normalized and write it as \( \psi_n = \xi_n^o + (P_{n,-n} - P_{n,-n}^o)\psi_n \) with \( \xi_n^o = P_{n,-n}^o \psi_n \). Note that \( \|\xi_n^o\|_{L^2} \leq \|\psi_n\|_{L^2} = 1 \). The difference of projections is then expressed as a contour integral of resolvents using the second resolvent identity [11]

\[
\psi_n = \xi_n^o - \frac{1}{2\pi i} \oint_{\Gamma_n} dz (\partial^2_x - 1 - z)^{-1} (v + \varepsilon |\alpha| \cdot \langle \alpha \rangle) (h_x - z)^{-1} \psi_n
\]

\[
= \xi_n^o - \sum_{(m: m \neq \pm n)} \frac{\langle \psi_{m}^o, v \psi_n \rangle + \varepsilon \langle \psi_{m}^o, \alpha \rangle \cdot \langle \alpha, \psi_n \rangle}{m^2 + 1 + \lambda^2_n} \psi_{m}^o
\]

\( \{\psi_{m}^o\}_{m \in \mathbb{Z}} \) being the eigenfunctions of \( \partial^2_x - 1 \). Here \( \Gamma_n \) is a loop about \( \lambda^2_n \), its nearby companion \( \lambda^2_{-n} \), and \(-\lambda^2_n + 1 \) (which again is doubly degenerate for \( \partial^2_x - 1, n \neq 0 \), and enclosing no other eigenvalues of \( h_x \)). These eigenvalues are isolated by a distance \( \sim |n| \) from other eigenvalues.

Multiplying through the representation (13) by \( |\alpha| \) gives an implicit estimate for \( |\langle \alpha, \psi_n \rangle| \)

\[
|\langle \alpha, \psi_n \rangle| \leq |\hat{\alpha}(n)| + \sum_{(m: m \neq \pm n)} \frac{|\hat{\alpha}| |m| |v| |c + \varepsilon |\hat{\alpha}| |n|}{m^2 + 1 + \lambda^2_n} |\langle \alpha, \psi_n \rangle|
\]

The coefficient sum in front of \( |\langle \alpha, \psi_n \rangle| \) on the rhs goes as \( \sim c_\gamma |n|^{-\gamma} \) for any \( \gamma < 1 \) for large \( n \) by application of (i) in the Estimate (2) below, since the \( \{\hat{\alpha}(m)\} \) are uniformly bounded and the quotients \( \{m^2 - n^2\} / |m^2 + 1 + \lambda^2_n|, |m| \neq |n| \}_{m,n} \) are uniformly bounded in \( m, n \). This coefficient sum can be made less than 1 in magnitude for large enough \( n \). Thus the implicit bound Ineq. (44).
can be resolved showing that \( \{ \alpha, \psi_n \} \) is uniformly bounded. Moreover, \( |\langle \alpha, \psi_n \rangle - \langle \alpha, \xi^o_n \rangle | \) also goes to zero as \( |n|^{-\gamma}, n \to \infty \). (We are assuming that the small \( n \) terms are finite.)

The same argument again using Eq. (43) shows that the \( \psi_n \)'s are uniformly bounded in \( x \), since the \( \psi_n^o = e^{inx}/\sqrt{2\pi} \) and \( \xi^o_n \)'s are. Note that \( \|\psi_n - \xi^o_n\|_c \) also goes to zero as \( n^{-\gamma}, n \to \infty \). This completes the argument for the first assertion of the lemma.

To show Ineq. (40), we note first that \( |\psi_n^o(x) - \psi_n^o(y)| \leq \min\{||m||x-y||, 1\} \) as well as \( |\xi^o_n(x) - \xi^o_n(y)| \leq \min\{||n||x-y||, 1\} \). Eq. (43) gives the inequality

\[
|\psi_n(x) - \psi_n(y)| \leq |n||x-y| + K_\varepsilon \sum_{\{m : m \neq \pm n\}} \frac{\min\{||m||x-y||, 1\}}{|m^2 + 1 + \lambda^2_n|},
\]

where \( K_\varepsilon = \|\mathcal{A}\|_\infty \|v\|_c + |\varepsilon|\|\mathcal{A}\|_\infty^2 \sup_n |\langle \alpha, \psi_n \rangle| \). We use the uniform boundedness of the quotients \( |m^2 - n^2|/|m^2 + 1 + \lambda^2_n| \) and (ii) in the Estimate (2), with \( \eta = |x-y| \) to establish Ineq. (40) of the lemma.

The last assertion of the lemma follows by showing that \( \psi_n \) and \( \psi^*_n \) are small perturbations of the same eigenfunction for the self-adjoint operator \( \mathcal{H}_R \), where the imaginary part of \( \varepsilon \) has been set to zero. For temporary convenience of notation, let \( \{\psi_n^o, \lambda^o_n\} \) be the normalized eigenvectors and eigenvalues of \( \mathcal{H}_R \). Then as in Eq. (39),

\[
\psi_n = P_n^o \psi_n - i\varepsilon \sum_{\{m : m \neq \pm n\}} \frac{\langle \psi^o_m, \alpha \rangle \cdot \langle \alpha, \psi^o_n \rangle}{-\lambda^2_{o,m} + \lambda^2_n} \psi^o_m,
\]

\( P_n^o \) projection onto the one-dimensional subspace spanned by \( \psi^o_n \). (\( \Gamma_n \) in this situation is a contour just about \( \lambda^2_{o,n} \) and \( \lambda^2_n \); we have a simple perturbation.) The near resonant \( m = -n \) summand has a denominator at least \( c_n \Re\mathcal{A}(n)^2 \) as a consequence of the non-degeneracy assumption, see Ineq. (39), and so this term is bounded in \( L^2 \)-norm by \( |3\varepsilon|/c_n \Re\mathcal{A} \) uniformly in \( n \). The rest of the series is bounded in an \( L^2 \)-sense by \( |c_1|\varepsilon|/|n| \) for a suitable constant \( c_1 \) from the orthogonality of the \( \psi^o_m \)’s. Thus

\[
\|\psi_n - P_n^o \psi_n\|_{L^2} \leq |3\varepsilon| \left( \frac{1}{c_n \Re\mathcal{A}} + \frac{c_1}{|n|} \right).
\]

The function \( \psi^*_n \) satisfies precisely the same estimates; \( \psi_n \) and \( \psi^*_n \) can still be such that \( P_n^o \psi_n \) and \( P_n^o \psi^*_n \) differ by a phase factor, but one can simply choose a phase factor \( e^{i\theta_n} \) so that

\[
\|\psi_n - e^{i\theta_n} \psi^*_n\|_2 \leq 2|3\varepsilon| \left( \frac{1}{c_n \Re\mathcal{A}} + \frac{c_1}{|n|} \right).
\]

All of these estimates are uniform in \( \varepsilon \) provided \( |\varepsilon| \) is bounded away from zero so that \( \lambda^2_{o,n} \) and \( \lambda^2_n \), which depend on \( \varepsilon \), truly separate by \( \sim \|\mathcal{A}(n)\|^2 \). The estimates are uniform in \( v \) for \( \|v\|_c \) much smaller in norm than this same eigenvalue separation.

**Estimate 2.** There exists a finite positive constant \( c_\gamma, \gamma < 1 \) independent of \( n \), such that for \( n \geq 0 \) and for any integer \( n \),

\[
(i) \sum_{m : m \neq \pm n} \frac{1}{|m^2 - n^2|} \leq \frac{c_\gamma}{1 + |n|^{\gamma}},
\]

\[
(ii) \sum_{m : m \neq \pm n} \frac{\min\{||m||\} \varepsilon}{|m^2 - n^2|} \leq c_\gamma \varepsilon^{\gamma}.
\]
Proof. It suffices to prove the estimates for \( n \geq 0 \) and the sums running over \( m \geq 0 \).

(i). The estimate is clear for \( n = 0 \). For \( n > 0 \), we use Hölder’s inequality,

\[
\sum_{m: m \neq n} \frac{1}{|m-n|} \leq \| \chi_{m \neq n} \|_{\ell^p} \| \frac{1}{|m+n|} \|_{\ell^{p/(p-1)}} \leq \frac{c_p}{n^{1/p}}
\]

with \( c_p \) finite for any \( p > 1 \), thus for \( \gamma \equiv 1/p < 1 \).

(ii) The sum is bounded by

\[
\sum_{m: m \neq n, 1 \leq |m-n| \leq \eta} \frac{\eta}{|m-n|} \cdot \left( 1 + \sum_{m: m \neq n, 1/\eta < |m-n|} \frac{1}{|m-n|} \right)
\]

\[
\leq \left\| \chi_{m \neq n} \right\|_{\ell^{p/(p-1)}} \left( \eta \left\| \chi_{|m| \leq 1/\eta} \right\|_{\ell^p} + \left\| \chi_{m > 1/\eta} \right\|_{\ell^p} \right) \leq \frac{c_p \eta^{1-1/p}}{n^{1/p}}
\]

for any \( p > 1 \) or here, with \( \gamma \equiv 1 - 1/p < 1 \). \( \square \)

The projections \( \{ P_n \} = \{ \frac{\chi_n}{c_n} \frac{f_n}{\|f_n\|} \} \), suitably conjugated, are uniformly bounded for \( n \to \infty \). Let \( \Lambda \) be defined by \( \Lambda e = ((-\partial_x^2 + 1)^{1/2} \phi, \pi) \), putting the \( \phi \) and \( \pi \) components of \( e_n \) and \( f_n \) on a comparable \( L^2 \)-norm footing for large \( n \).

**Lemma 3.** There is a positive constant \( c < 1 \) independent of \( n \) such that

\[
c\| \Lambda^{-1} f_n \|_{\mathcal{H}} \leq \| \Lambda e_n \|_{\mathcal{H}} \leq \| \Lambda^{-1} f_n \|_{\mathcal{H}} \| \Lambda e_n \|_{\mathcal{H}}.
\]

The conjugated projections \( \{ \Lambda P_n \Lambda^{-1} \} \) are uniformly bounded.

Proof. Referring to Eq. (53) where the components of \( e_n \) and \( f_n \) are written in terms of their \( \pi \)-components, we have that the maps \( f_{n, \pi} \to (-\partial_x^2 + 1)^{-1/2} f_{n, \phi} \) and \( e_{n, \pi} \to (-\partial_x^2 + 1)^{1/2} e_{n, \phi} \) are uniformly bounded acting in \( L^2[0, 2\pi] \), as are the maps \( f_{n, \pi} \to f_{n, r} \) and \( e_{n, \pi} \to e_{n, r} \), so that the left side of (53) is bounded above by a constant times \( \| f_{n, \pi} \|_{L^2} \| e_{n, \pi} \|_{L^2} \).

The middle term of (53) can also be written in terms of the \( \pi \)-components,

\[
\langle f_n, e_n \rangle_{\mathcal{H}} = 2 \int f_n^*(x) e_n(x) dx + \frac{1}{\lambda_n (1 + \lambda_n)^2} \langle \alpha, f_n \rangle_{\mathcal{H}} \cdot \langle \alpha, e_n \rangle_{\mathcal{H}} = 2 \| f_n, e_n \|_{L^2} \| e_n \|_{L^2} + O(n^{-3})
\]

\[
= 2 c \| f_n, e_n \|_{L^2} \| e_n \|_{L^2} + O(n^{-3})
\]

Here we have used the last inequality (11) of Lemma 1 with the role of \( \varepsilon \) played by \( \varepsilon = \frac{\lambda_n}{\lambda_n + 1} \) with \( |3\varepsilon/\Re \varepsilon| \sim 1/n \). Thus we have a lower bound on the middle term comparable to the left side of (53).

The second inequality of (53) is an application of Cauchy–Schwarz. \( \square \)

The following lemma provides the estimates needed for showing that the mapping \( v \to E_{\mu_v}[\phi^2(x)] \) is Lipschitz continuous in \( v \). We write

\[
P_{m, \phi(x), r}(v) = -\frac{e_{m, \pi}(x) (f_{m, \pi}, \alpha)}{\lambda_m (\lambda_m + 1) (f_{m, \pi}, e_m)_{\mathcal{H}}}
\]

as the \( \phi(x), r \) entry of the matrix \( P_m \) and an analogous expression for \( P_{m, \phi, \pi} \). The eigenfunctions and eigenvalues are of course dependent on \( v \); their dependence will appear explicitly as needed.
Lemma 4. There exists positive constants \( C_\phi, C_\lambda, \) and \( \varepsilon_0 \) independent of \( \mathbf{n} \) such that for two continuous potentials \( v_1 \) and \( v_2 \) with small norms, \( \|v_1\|_C \) and \( \|v_2\|_C \leq \varepsilon_0 \sup_n |\hat{\alpha}(n)|^2 \), then:

\[
\begin{align}
(i) & \quad \left\| P_{\mathbf{n},\phi(x)},r(v_2) - P_{\mathbf{n},\phi(x)},r(v_1) \right\|_C \leq C_\phi \frac{\|v_2 - v_1\|_C}{n^2 + 1} \\
(ii) & \quad |\lambda_n(v_2) - \lambda_n(v_1)| \leq C_\lambda \frac{\|v_2 - v_1\|_C}{|n| + 1}, \\
(iii) & \quad |\Re(\lambda_n(v_2) - \lambda_n(v_1))| \leq C_\lambda \frac{\|v_2 - v_1\|_C}{n^2 + 1}
\end{align}
\]

Proof. (i) By writing the difference of projections for \( A_{v_1} \) and \( A_{v_2} \) as contour integrals using the second resolvent equation, the contour just about the eigenvalues \( \lambda_n(v_1) \) and \( \lambda_n(v_2) \) (the \( v \)'s are \( \pi, \phi \) entries in \( \mathbf{A}_{v} \)), we have

\[
P_{\mathbf{n},\phi(x)},r(v_2) - P_{\mathbf{n},\phi(x)},r(v_1) = - \sum_{m: \ m \neq n} \frac{P_{\mathbf{n},\phi(x)},r(v_2)(v_2 - v_1)}{\lambda_n(v_2) - \lambda_m(v_1)} - \sum_{m: \ m \neq n} \frac{P_{\mathbf{n},\phi(x)},r(v_2)(v_2 - v_1)}{\lambda_n(v_1) - \lambda_m(v_2)}.
\]

The first sum in Eq. (59) equals

\[
\frac{\epsilon_n,\tau(x,v_2)}{\lambda_n(f_n,\epsilon_n)\eta} \sum_{m: \ m \neq n} \frac{(f_n,\pi(v_2), (v_2 - v_1)e_m,\pi)\langle f_m,\pi(v_1), \alpha \rangle}{(\lambda_n(v_2) - \lambda_m(v_1))\lambda_m(v_1)(\lambda_m(v_1) + 1)}
\]

which converges absolutely to a continuous function and is bounded by \( \mathcal{O}(n^{-2})\|v_2 - v_1\|_C \); in particular the nearly resonant small denominator \( |\lambda_n - \lambda_m| \sim |\hat{\alpha}(n)|^2/n \) is mollified by the other factors \( \lambda_n\lambda_m(\lambda_m + 1) \) in the denominator. The other sum in (59) is controlled similarly.

(ii,iii) These estimates are shown by first order perturbation theory, applied to the eigenvalue problem

\[
\left( \partial^2_x - 1 - (v_1 + \tau(v_2(x) - v_1(x))) - \frac{\lambda_n(\tau)}{\lambda_n(\tau) + 1}\langle \alpha \rangle \right) e_{n,\tau}(\tau, x) = \lambda_n^2(\tau)e_{n,\tau}(\tau, x)
\]

The equation for \( d\lambda_n/d\tau \) is

\[
\frac{2\lambda_n(\tau) + \langle f_{n,\pi}(\tau), \alpha \rangle \cdot \langle \alpha, e_{n,\pi}(\tau) \rangle}{2\lambda_n^2(\tau)\langle f_{n,\pi}(\tau), e_{n,\pi}(\tau) \rangle} = \frac{\langle f_{n,\pi}(\tau), (v_2 - v_1)e_{n,\pi}(\tau) \rangle}{\langle e_{n,\pi}(\tau), e_{n,\pi}(\tau) \rangle} + \mathcal{O}(n^{-1})\|v_2 - v_1\|_C.
\]

by Lemma (11), see also the remark Ineq. (12) following it, with \( \varepsilon = \lambda_n/(\lambda_n + 1) \) having an imaginary part \( \mathcal{O}(n^{-1}) \). This shows that

\[
\frac{d\lambda_n(\tau)}{d\tau} = - \frac{\langle e_{n,\pi}(\tau), (v_2 - v_1)e_{n,\pi}(\tau) \rangle}{2\lambda_n(\tau)\langle e_{n,\pi}(\tau), e_{n,\pi}(\tau) \rangle} + \mathcal{O}(n^{-2})\|v_2 - v_1\|_C,
\]

which is \( \mathcal{O}(n^{-1}) \) with real part, \( \mathcal{O}(n^{-2}) \). Integrating this expression in \( \tau \) from 0 to 1 gives (ii) and (iii) of the lemma. \( \square \)
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