ENHANCED BRUHAT DECOMPOSITION AND MORSE THEORY

PETR PUSHKAR AND MISHA TYOMKIN

Abstract. Morse function is called strong if all its critical values are pairwise distinct. Given such a function \(f\) and a field \(\mathbb{F}\) Barannikov constructed a pairing of some of the critical points of \(f\), which is now also known as barcode. With every Barannikov pair we naturally associate (up to sign) an element of \(\mathbb{F} \setminus \{0\}\); we call it Bruhat number. The paper is devoted to the study of these Bruhat numbers. We investigate several situations where the product of all these numbers (some being raised to the power \(-1\)) is independent of \(f\) and interpret it as a Reidemeister torsion. We apply our results in the setting of one-parameter Morse theory by proving that generic path of functions must satisfy a certain equation mod 2 (this was initially proven in [ACR05] under additional assumptions).

On the linear-algebraic level our constructions are served by the following variation of a classical Bruhat decomposition for \(GL(\mathbb{F})\). A unitriangular matrix is an upper triangular one with 1’s on the diagonal. Consider all rectangular matrices over \(\mathbb{F}\) up to left and right multiplication by unitriangular ones. Enhanced Bruhat decomposition describes canonical representative in each equivalence class.

Introduction

0.1. The context. In this paper we study a certain invariant of a strong Morse function on a smooth manifold (which is supposed to be closed most of the time). Recall that a function is called Morse if all its critical points are non-degenerate. The function is called strong if all its critical values are pairwise distinct. Morse theory is a classical branch of differential topology: one can extract a lot of topological information about the manifold in terms of a Morse function. On the other hand, Morse functions arise naturally in various situations and their properties are interesting in their own right.

Recall that each critical point of a Morse function carries a number — its index. The very first theorem of Morse theory states that one can find a CW-complex homotopy equivalent to the manifold, which \(k\)-cells correspond to critical points of index \(k\).

Suppose now that one is given not only a strong Morse function \(f\) on a manifold \(M\), but also a field \(\mathbb{F}\). To such a data Barannikov [Bar94] associated a powerful combinatorial structure on the set of critical points of \(f\). Nowadays it is also known as barcode and serves as a well-established tool in applied and symplectic topology. This structure is a pairing of some critical points of neighboring indices. This pairing does depend on the field \(\mathbb{F}\). For example, the number of unpaired critical points of index \(k\) equals to \(\dim\, H_k(M; \mathbb{F})\). Moreover, this pairing relies crucially on the fact the function is strong, i.e. critical points are linearly ordered. If one starts to change a Morse function so that along the way it fails to be strong (i.e. two critical values collide), the pairing changes. We call these pairs Barannikov pairs. We will sketch their definition in the next subsection.

0.2. Bruhat numbers of a single function. To state our results consicely we prefer to speak about oriented strong Morse functions. Roughly, the Morse function \(f\) is called oriented if one has chosen orientation on all the cells in the CW-complex constructed from it. This condition is both technical and minor: one can always orient a Morse function. As usual in topology, this choice only alters certain signs.

Our first result is not a theorem, but rather a construction. Namely with any given oriented Morse function \(f\) on a manifold \(M\) and a field \(\mathbb{F}\) we naturally associate a non-zero number with each Barannikov pair. Here by number we mean an element of \(\mathbb{F}^+ = \mathbb{F} \setminus \{0\}\). We call these numbers “Bruhat numbers” of \(f\) over \(\mathbb{F}\). The reason is that their construction is closely related to the classical Bruhat decomposition for \(GL(\mathbb{F})\). This paper is devoted to the study of these numbers from different perspectives. The first thing to mention is that changing the orientation of \(f\) may only change signs of some Bruhat numbers.

We will now sketch the construction of both Barannikov pairs and Bruhat numbers. Recall that if one chooses a generic Riemannian metric \(g\) on \(M\) then they can consider a Morse complex which counts homology of \(M\) (this is a classical construction, it has nothing to do with a field). It is a chain complex of free abelian groups, formally spanned by critical points (points of index \(k\) are of degree \(k\)). Thus the Morse differential is nothing but an integer matrix: differential of a critical point \(x\) is a linear combination of points of smaller index. The coefficient of the point \(y\) in this linear combination is the number of antigradient flowlines from \(x\) to \(y\), counted with appropriate signs (thus it is non-zero only if \(f(x) > f(y)\)). Since the function is strong, the set of critical points is ordered by increasing of critical values. Next, we note that choosing a different metric \(g'\) results in a different matrix of Morse differential. More precisely, these two matrices differ by a conjugation by...
unitriangular (i.e. triangular with 1’s on the diagonal) one. We treat this unitriangular matrix as that of a change of basis of a complex. As a recollection, the class of a matrix of Morse differential up to conjugation by a unitriangular matrix is a well-defined invariant of $f$ (i.e. doesn’t depend on a metric). The corresponding classification problem is, however, very hard, so we consider the coefficients in $F$. In such a case we prove that every complex is isomorphic, up to unitriangular change of basis, to the direct sum of $0 \to F \to 0$ and $0 \to F \to 0$. Generators of the former (which are themselves critical points) are Barmannik pairs. The corresponding number $\lambda$ is a Bruhat number on a pair.

Weak Morse inequalities state that the number of critical points of index $k$ is greater or equal to $\dim H_k(M; Q)$. Let $\# \text{Cr}(f)$ be the overall number of critical points of $f$. It is easy to show that if $\# \text{Cr}(f) = \sum \dim H_k(M; Q)$ then the Morse differential (w.r.t. any metric) must be identically zero. The next statement is applicable when this is not the case.

**Corollary 0.1.** Let $f$ be an oriented strong Morse function on $M$. Suppose that $\# \text{Cr}(f) > \sum \dim H_k(M; Q)$. Then one can find two critical points $x$ and $y$ of neighboring indices s.t. the number of antigradient flowlines from $x$ to $y$, counted with appropriate signs, is the same for any Riemannian metric. This number is non-zero and equals to some Bruhat number of $f$ over $Q$.

In Subsection 3.7 this statement is derived from Theorem 3.22 which says that the matrix of a Morse differential of $f$ w.r.t. to any metric must obey certain restrictions. These restrictions, in turn, are expressed in terms of Bruhat numbers and Barmannik pairs. They are in the spirit of Bruhat decomposition, see the mentioned subsection for the precise statement and example. Note that in particular we claim that at least one Bruhat number over $Q$ must be integer. For a general Bruhat number this is false, however.

### 0.3. Interplay with the theory of torsions

In Proposition 3.24 we prove that if $F$ is either $Q$ or $F_p$ then one can find a Morse function which has any prescribed number $\lambda \in \mathbb{F}^*$ as one of its Bruhat numbers; the manifold $M$ may be any with $\dim M \geq 4$. Thus one has no control over individual Bruhat number — it may turn out to be any number. We propose, however, to consider the alternating product of all the Bruhat numbers. The word “alternating” here means that in particular we claim that at least one Bruhat number over $Q$ must be integer. For a general Bruhat number this is false, however.

**Theorem 0.2.** Let $f$ be a strong Morse function on $M$ and $F$ be a field. Suppose that $H_k(M) = 0$ for all $0 < k < \dim M$. Then the alternating product of all Bruhat numbers (as an element from $F^*/\pm 1$) is independent of $f$.

This is discussed in Subsections 4.1 and 5.5 in particular we interpret this alternating product as a certain kind torsion. For example, if $M = \mathbb{R}^n$ and $F = Q$ then this product equals to $\pm 2^n/2$, where brackets denote the integral part.

We shall now quickly recall the notion of the Reidemeister torsion of a manifold. This is a purely algebro-topological invariant which itself has nothing to do with Morse theory. Let $\mathbb{Z}[\pi]$ denote the integral group ring of a fundamental group $\pi = \pi_1(M)$. Suppose one is given a map of rings $\rho: \mathbb{Z}[\pi] \to F$ for some field $F$. Then they may considered homology of $M$ with coefficients twisted by $\rho$. If it vanishes, then they may furthermore define the Reidemeister torsion of $M$ w.r.t. $\rho$, which is an element from the quotient group $F^*/\rho(\pm 1)$. This invariant is useful for distinguishing homotopy equivalent but non-homeomorphic manifolds, such as lens spaces.

We will now pour Morse theory into this setting. Suppose that now we are given not only oriented strong Morse function $f$ and a field $F$ but also a map $\rho: \mathbb{Z}[\pi] \to F$. In this case we construct twisted Barmannik pairs and Bruhat numbers. In general, the alternating product of twisted Bruhat numbers may well depend on $F$. The interesting case, however, is when one is able to define the Reidemeister torsion, i.e. when twisted homology vanishes.

**Theorem 0.3.** Let $f$ be an oriented strong Morse function on a manifold $M$, $F$ be a field and $\rho: \mathbb{Z}[\pi] \to F$ be a homomorphism of rings. Suppose that twisted homology vanishes. Then the alternating product of twisted Bruhat numbers of $f$ equals to the Reidemeister torsion of $M$. In particular, it is independent of $f$.

### 0.4. One-parameter Morse theory

Let us now consider not a single strong Morse function but a generic path in the space of all functions on $M$. All but finitely points on this path are themselves strong Morse functions. However, after passing a function which fails to be strong and Morse Barmannik pairs and Bruhat numbers change. Thanks to the genericity assumption on the path it is possible to describe exactly the list of possible changes. For Barmannik pairs this was done already in [Bar94] (see also [CEM06] and pictures in the survey [EH08]). We do the same for
Bruhat numbers on them. In particular, this gives a proof of Theorem 0.2. Moreover, it enables us to prove the theorem of Akhmetev-Cencelj-Repovs [ACR05] in greater generality, which we shall now describe.

Let \{f_t\} be a generic path on functions on \(M\), i.e. \(f_t\) is a function from \(M\) to \(\mathbb{R}\) for each \(t \in [-1, 1]\). To such a path one associates a Cerf diagram. It is a subset of \([-1, 1] \times \mathbb{R}\) consisting of points \((t, x)\) s.t. \(x\) is a critical value of \(f_t\). Practically, it is a set of plane arcs which start and end either at cusps or at vertical lines \(t = \pm 1\). The only possible singularities of a Cerf diagram are cusps and simple transversal self-intersections. By orienting all the functions in a path one may associate a sign with each cusp. The parity of negative cusps is a well-defined invariant of a path \(\{f_t\}\), i.e. it doesn’t depend on the orientations. Another invariant of a path is a number of self-intersections of its Cerf diagram (i.e. the number points \(t_0\) s.t. \(f_{t_0}\) is not strong). In the following statement we consider functions on a cylinder \(M \times [0, 1]\), which is manifold with boundary. Morse theory translates to this setting with ease.

**Corollary 0.4.** Let \(\{f_t\}\) be a generic path of functions on a cylinder \(M \times [0, 1]\) s.t. both \(f_{-1}\) and \(f_1\) have no critical points. Let \(X\) be the number of self-intersections of its Cerf diagram and \(C\) be the number of negative cusps. Then one has
\[
X + C = 0 \pmod{2}.
\]

In [ACR05] this statement was proved under additional assumptions on \(M\). In Subsection 5.6 we derive this corollary from a more general Theorem 5.12.

0.5. Related work. Barannikov pairs were introduced in [Bar94] (see [EH08] for a recent survey). A close idea of construction of Bruhat numbers over \(\mathbb{Q}\) appeared independently in [LNV20]. In [UZ16] authors prove a theorem analogous to Barannikov’s in the setting of complexes over the Novikov ring, which is useful in symplectic topology. Translation of Bruhat numbers to this setting is a current work in progress. Reidemeister torsion in the setting of Morse-Novikov theory is studied in [HL99a; HL99b; Hut02].

0.6. Organization of the paper. The first two sections provide an algebraic foundation for the further Morse-theoretical results. Namely, in Section 1 we do the necessary linear algebra and emphasize connection with the Bruhat decomposition. In Section 2 similar in spirit constructions are presented in realm of homological algebra over a field. Sections 3 to 5 contain (but not exhausted by) results described in, respectively, Subsections 0.2 to 0.4.

The paper contains many constructions and we therefore use a special environment for them. The text after the word “construction” describes input and output. The actual algorithm is placed between signs \(\triangleright\) and \(\triangleright\). By default construction doesn’t involve any choices, i.e. output depends only on the input. If it is not the case, we indicate this explicitly.

0.7. Acknowledgements. P. Pushkar is supported by Russian Foundation for Basic Research under the Grants RFBR 18-01-00461 and supported in part by the Simons Foundation. M. Tyomkin’s research was carried out within the framework of the Basic Research Program at HSE University and funded by the Russian Academic Excellence Project “5-100”. We are grateful to Leonid Rybnikov for fruitful discussions. We thank Andrei Ionov for bringing our attention to the fact that the word “enhanced” has been used since [BB93] for distinguishing, in particular, unitriangular group from upper triangular. Second author thanks Sergey Melikhov for comments on the first version. He also thanks Petr Akhmetev for numerous explanations of the paper [ACR05]. He thanks Anton Ayzenberg for the reference [Cay48].

1. Enhanced vector spaces

In this section we define and study enhanced vector spaces — a notion which we rely on in Section 2. All the constructions lie within the scope of linear algebra. Moreover, our main statement here (Lemma 1.5) may be formulated exclusively in terms of matrices, which is done right below (Lemma 1.2). Later in Section 2 we proceed similarly in the setting of chain complexes over a field.

1.1. Formulation of results. In this subsection we introduce main definitions of this section and formulate the main Lemma 1.3.

We start with the coordinate formulation. Let \(n\) and \(m\) be two natural numbers fixed once and for all throughout this section. Fix also a base field \(\mathbb{F}\), all matrices are assumed to be over it.

Let \(T_n\) be the group of unitriangular matrices, i.e. upper triangular \(n \times n\) matrices with ones on the diagonal. The group \(T_n \times T_m\) acts on the set \(\text{Mat}_{n,m}\) of all \(n \times m\) matrices: \(X \mapsto AXB^{-1}\). Since one has commuting actions of both \(T_n\) and \(T_m\), the orbit space is usually denoted as \(T_n \backslash \text{Mat}_{n,m}/T_m\). Note that two \(n \times m\) matrices lie in the same \(T_n\) orbit if and only if one can be obtained from another by a successive performing of the following elementary operation: add to one row a scalar multiple of another one, provided that the latter is higher than the former. The same goes for \(T_m\) and column operation.
Definition 1.1. An $n \times m$ matrix is called a rook matrix if in every row and in every column there is at most one non-zero entry.

Lemma 1.2. Every orbit in $T_n \setminus \text{Mat}_{s,m}/T_n$ contains exactly one rook matrix.

The classical Bruhat decomposition is obtained from the above statement in two steps: 1) restrict to the case $n = m$ and consider only non-degenerate square matrices, 2) replace $T_n$ by an upper triangular group. Keeping in mind the slightly greater level of generality we propose the term "enhanced Bruhat decomposition" (see, however, [BB93]). Note that rook matrix stores, in particular, the set of elements from $F^*$, in contrast to the matrix of permutation in the classical case. The proof, however, goes along the same lines. See [FH04] for an in-depth discussion. See also [DPST17] for a close in spirit generalization of Bruhat decomposition.

We will now introduce the main notion of the present section. Several basic facts about it will be presented further in Subsections 1.3 and 1.4. Often Bruhat decomposition is proven using inductive arguments. We tried to refrain from those during the course of this section (or, at least, hide them under the carpet of explicit constructions).

Definition 1.3. Let $V$ be a vector space over $F$. An enhancement $\varkappa$ on a vector space $V$ is a choice of two structures:

1) a full flag on $V$, i.e. a sequence of subspaces $0 = V^0 \subset V^1 \subset \ldots \subset V^{\dim V} = V$ s.t. $\dim(V^s/V^{s-1}) = 1$, $s \in \{1, \ldots, \dim V\}$;

2) a non-zero element $\varphi_\varkappa(v)$, in a one-dimensional vector space $V^s/V^{s-1}$, $s \in \{1, \ldots, \dim V\}$.

A vector space $V$ with an enhancement will be called an enhanced vector space and denoted as $(V, \varkappa)$.

Definition 1.4. Let $(V, \varkappa)$ and $(W, \mu)$ be two enhanced vector spaces of the same dimension. We say that they are isomorphic (and use the symbol $\simeq$ for this) if there exist an isomorphism of vector spaces $\varphi: V \rightarrow W$ s.t. 1) $\varphi(V^s) = W^s$; 2) $\varphi_\varkappa(v) = \mu_\varkappa(v)$, where $\varphi_\varkappa: V^s/V^{s-1} \rightarrow W^s/W^{s-1}$ is a map of quotient vector spaces induced by $\varphi$.

By a basis of a finite-dimensional vector space $V$ we will mean a linearly ordered set of generators (zero vector space has empty set as its only basis). Given a basis $v = (v_1, \ldots, v_{\dim V})$ of $V$ one constructs an enhanced vector space $(V, \varkappa(v))$ in the following straightforward way. For $s \in \{1, \ldots, \dim V\}$ set $V^s := \langle v_1, \ldots, v_s \rangle$ and $\varphi_\varkappa(v) := p_\varkappa(v_s)$, where $p_\varkappa: V^s \rightarrow V^s/V^{s-1}$ is a standard projection. By a basis of an enhanced vector space $(V, \varkappa)$ we will mean a basis $v$ of $V$ s.t. $(V, \varkappa) \simeq (V, \varkappa(v))$.

The next lemma is equivalent to Lemma 1.2.

Lemma 1.5. Let $(V, \varkappa)$ and $(W, \mu)$ be two enhanced vector spaces and $A: V \rightarrow W$ be a linear map. There exists a basis $v$ (resp. $w$) of an enhanced vector space $(V, \varkappa)$ (resp. $(W, \mu)$) s.t. the matrix of $A$ in these bases is a rook matrix. Moreover, this rook matrix is uniquely defined.

The change of basis in $(V, \varkappa)$ (resp. $(W, \mu)$) results in multiplication of matrix of $A$ by a matrix from $T_{\dim V}$ (resp. $T_{\dim W}$). So, Lemma 1.5 describes orbits in $T_{\dim W} \setminus \text{Mat}_{\dim V, \dim W}/T_{\dim V}$.

We will stick to the above formulation. It is possible to state the same without appealing to any bases whatsoever; this is done in Subsection 1.3. We call non-zero elements of the mentioned rook matrix “Bruhat numbers” of a map between two enhanced vector spaces.

Remark 1.6. If the field $K$ is an extension of $F$ then one may consider $A$ as a map between vector spaces over $K$. It’s plain to see that rook matrix won’t change after this operation. Indeed, multiplication of matrices only involves additions and multiplications.

Usually what we are given in topological setup is a matrix over $Z$ (note that this is not the case in Subsections 1.2 and 1.3). We then choose some field $F$ and merely consider this matrix over this field. It follows from the previous paragraph that it is enough to consider only $Q$ and $F_p$.

Remark 1.7. In Lemma 1.5 bases $v$ and $w$ themselves need not be unique.

1.2. Construction of a rook matrix. In this subsection we associate a rook matrix to a given map between enhanced vector spaces. In Subsection 1.4 we will show that this is the same matrix as the one addressed in Lemma 1.5.

We will need the following construction as a preliminary step.

Construction 1.8. Let $(V, \varkappa)$ be an enhanced vector space and $A: V \rightarrow W$ be a surjective map of vector spaces. We will now construct an induced enhancement on $W$.

$\varphi$ For $s \in \{1, \ldots, \dim V\}$ define $\varphi_\varkappa$ to be the composition $V^s \xrightarrow{A} W$. 
Take any \( s \) s.t. \( \dim \text{Im} \varphi_s = \dim \text{Im} \varphi_{s+1} + 1 \) and denote this number by \( t \). Set \( W^t \) to be \( \text{Im} \varphi_s \). This defines a full flag on \( W \), i.e., a vector space \( W^t \) for any \( t \in \{1, \ldots, \dim W\} \). We now need to produce an element \( \mu_t \) in the vector space \( W^t/W^{t-1} \); this vector space coincides with \( \text{Im} \varphi_s/\text{Im} \varphi_{s+1} \). Define \( \mu_t \) to be \( \tilde{\varphi}_s(\kappa_t) \), where
\[
\tilde{\varphi}_s: V^s/W^{s-1} \xrightarrow{\sim} \text{Im} \varphi_s/\text{Im} \varphi_{s+1}
\]
is an isomorphism of quotient vector spaces induced by \( \varphi_s \). We have obtained an enhanced vector space \((W, \mu)\).

The following proposition will be used later in Subsection 1.4.

**Proposition 1.9.** Let \((V, \varpi)\) be an enhanced vector space and \( A: V \to W \) be a surjective map of vector spaces. Construction 1.8 produces an enhanced vector space \((W, \mu)\). We claim that for any basis of \((W, \mu)\) one can find a basis of \((V, \varpi)\) s.t. \( A \) maps each basis element to either zero or another basis element.

**Proof.** We continue using notations from Construction 1.8. Let \( w = (w_1, \ldots, w_{\dim W}) \) be the given basis of \((W, \mu)\). Take any \( s \in \{1, \ldots, \dim V\} \). The difference \( \dim \text{Im} \varphi_s - \dim \text{Im} \varphi_{s+1} \) is either zero or one. In the former case set \( v_s \) to be any vector from \( V^s/W^{s-1} \) s.t. its class in \( V^s/W^{s-1} \) coincides with \( \varpi_s \). In the latter case set \( v_s \) to be any preimage of \( w_{\dim \text{Im} \varphi_s} \) under \( A \). It is straightforward to check that the basis \((v_1, \ldots, v_{\dim V})\) satisfies the desired property.

Construction 1.16 and Proposition 1.17 are analogous statements for the case of injective maps. In order to proceed to the construction of a rook matrix we need two more definitions. Fix any \( \varpi_s \). Let \( \lambda \in \mathbb{F}^s := \mathbb{F} \setminus \{0\} \) s.t. \( p(v) = \lambda \varpi_s(v) \), where \( p \) is a projection \( V^{\varpi_s} \to V^{\varpi_s}/V^{\varpi_s-1} \).

**Construction 1.10.** Given a map \( A: V \to W \) between enhanced vector spaces \((V, \varpi)\) and \((W, \mu)\) we will now construct a rook matrix \( R \) of size \((\dim W) \times (\dim V)\).

\( \triangleright \) Fix any \( s \in \{1, \ldots, \dim V\} \). Consider a surjective map \( S: W \to W/A(V^{s-1}) \) and use Construction 1.8 to get an enhanced vector space \((W/A(V^{s-1}), \bar{\mu})\). Consider now an element \( \bar{A}(\varpi_s) = \mu_s \in W/A(V^{s-1}) \), where
\[
\bar{A}: V^s/V^{s-1} \to W/A(V^{s-1})
\]
is a map of quotient vector spaces induced by the restriction \( A|_{V^s} \). If \( \bar{A}(\varpi_s) = 0 \) then the \( s \)-th column of \( R \) is set to be zero. Otherwise, let \( \lambda \) and \( \lambda' \) be, respectively, coefficient and height of \( \bar{A}(\varpi_s) \). Let \( t \in \{1, \ldots, \dim W\} \) be the only number satisfying the condition \( \dim S(W^t) = \dim S(W^{t-1}) + 1 = t' \).

Finally, we set \( R_{t,s} \) to be \( \lambda \) and all the other entries in the \( s \)-th column of \( R \) to be zero.

It is straightforward to check that in every row there is at most one non-zero element, i.e. \( R \) is indeed a rook matrix.

**Remark 1.11.** Informally, the fact that the entry \( R_{t,s} \) of a rook matrix \( R \) is non-zero means that the image of \( \varpi_s \) under \( A \) first appears in the \( t \)-th flag space in \( W \).

### 1.3. Terminological digression.
In this subsection we introduce a bit of terminology which will be useful for understanding the content of Section 2.

Let \( X \) and \( Y \) be two sets and \( \sim \) be an equivalence relation on \( X \). If a map \( g: X \to Y \) is constant on the equivalence classes, then we say that \( g(x) \) is an invariant of some element \( x \in X \).

If, moreover, the induced map \( \tilde{g}: X/\sim \to Y \) is a bijection of sets then we say that \( g(x) \) is a full invariant.

Our next goal is to introduce a certain equivalence relation on the set of maps between fixed enhanced vector spaces. By an automorphism of an enhanced vector space \((V, \varpi)\) we will mean an isomorphism from \((V, \varpi)\) to itself. We say that two maps \( A \) and \( B \) between enhanced vector spaces \((V, \varpi)\) and \((W, \mu)\) are equivalent if there exist an automorphism \( C_1 \) (resp. \( C_2 \)) of \((V, \varpi)\) (resp. \((W, \mu)\)) s.t. \( C_2AC_1 = B \).

**Remark 1.12.** Note that \( A \) and \( B \) are equivalent if and only if there exist bases \( v_a \) and \( v_b \) of \((V, \varpi)\) and bases \( w_a \) and \( w_b \) of \((W, \mu)\) s.t. the matrix of \( A \) in bases \( v_a \) and \( w_a \) coincides with the matrix of \( B \) in bases \( v_b \) and \( w_b \).

Note that Construction 1.10 provides an invariant of a map between enhanced vector spaces considered up to equivalence. This invariant takes values in the set of rook matrices. It will follow from Subsection 1.4 that Lemma 1.3 can be reformulated as follows.

**Lemma 1.13.** Let \((V, \varpi)\) and \((W, \mu)\) be two fixed enhanced vector spaces and \( A: V \to W \) be some linear map. Then the corresponding rook matrix provided by Construction 1.10 is a full invariant of a map considered up to equivalence.

□
1.4. Proof of the main Lemma 7.3. In this subsection we prove Lemma 1.5. First, we prove the partial case when the map in question is an isomorphism. Second, we derive the general statement from it.

First of all, recall the lemma itself.

Lemma 1.5. Let $(V, \kappa)$ and $(W, \mu)$ be two enhanced vector spaces and $A : V \to W$ be a linear map. There exists a basis $v$ (resp. $w$) of an enhanced vector space $(V, \kappa)$ (resp. $(W, \mu)$) s.t. the matrix of $A$ in these bases is a rook matrix. Moreover, this rook matrix is uniquely defined.

See Subsection 4.1 for a context.

We will now deduce uniqueness from the existence. Suppose the map $A$ is represented by a rook matrix $R$ in some bases $v$ and $w$. Then one checks straightforwardly that the Construction 1.10 produces the same matrix $R$ as an output. Therefore $R$ is an invariant of a map $A$ and we’re done. The rest of this subsection is devoted to proving the existence part.

The next proposition is a partial case which will be used later.

Proposition 1.14. Let $(V, \kappa)$ and $(W, \mu)$ be two enhanced vector spaces and $A : V \to W$ be an isomorphism. There exists a basis $v$ (resp. $w$) of an enhanced vector space $(V, \kappa)$ (resp. $(W, \mu)$) s.t. the matrix of $A$ in these bases is a rook matrix.

Proof. By a jump of a function $g : \{0, \ldots, N\} \to \mathbb{Z}_{\geq 0}$, where $N \in \mathbb{Z}_{\geq 0}$ we will mean a number $x > 0$ s.t. $g(x) = g(x - 1) + 1$. Fix any $s \in \{1, \ldots, \dim V\}$. Consider now a function $x \mapsto \dim (A(V^s) \cap W^t)$, for $x \in \{0, \ldots, \dim W\}$ (recall that $\dim V = \dim W$). It has exactly $s$ jumps. Moreover, every jump of a function $x \mapsto \dim (A(V^{s-1}) \cap W^t)$ is also a jump of the function under consideration. Therefore, the latter function has exactly one “new” jump, call it $t$. It follows from the fact that $t$ is a jump that $A(V^s) \cap (W^t \setminus W^{t-1}) \neq \emptyset$; take any element $w$ from this set. It follows from the fact that $t$ is actually a new jump that $A^{-1}(w) \in V^s \setminus V^{s-1}$.

By performing the above operation for all possible $s$ we construct a basis of $W$ and, by taking a preimage, a basis of $V$. The end of the preceding paragraph implies that after appropriate reordering and rescaling these bases are bases of enhanced vector spaces $(V, \kappa)$ and $(W, \mu)$. The statement follows. □

Remark 1.15. This is a known proof of the Bruhat decomposition for $GL_n$ adapted to our enhanced setting.

Construction 1.16. Let $(W, \mu)$ be an enhanced vector space and $A : V \to W$ be an injective map of vector spaces. We will now construct an induced enhancement on $V$.

For $s \in \{1, \ldots, \dim W\}$ define $\varphi_s$ to be the composition

$$V \xrightarrow{A} W \xrightarrow{W/W^s}.$$ 

Take any $s$ s.t. $\dim \ker \varphi_s = \dim \ker \varphi_{s-1} + 1$ (call this number $t$). Set $V^t$ to be $\ker \varphi_s$. This defines a full flag on $V$, i.e. a vector space $V^t$ for any $t \in \{1, \ldots, \dim V\}$. We now need to produce an element $x_\kappa$ in the vector space $V^t/W^{t-1}$, which coincides with $\ker \varphi_s/\ker \varphi_{s-1}$. By identifying $V$ with $\im A$, we say that $\ker \varphi_s \subset W^s$. Define $x_\kappa$ to be $\alpha^{-1}(\mu_s)$, where

$$\alpha : \ker \varphi_s/\ker \varphi_{s-1} \xrightarrow{\cong} W^s/W^{s-1}$$

is an isomorphism of quotient vector spaces induced by the mentioned inclusion. We have obtained an enhanced vector space $(V, \kappa)$.

Proposition 1.17. Let $(W, \mu)$ be an enhanced vector space and $A : V \to W$ be an injective map of vector spaces. Construction 1.16 produces an enhanced vector space $(V, \kappa)$. We claim that for any basis of $(V, \kappa)$ one can find a basis of $(W, \mu)$ s.t. $A$ maps each basis element to another basis element.

Proof. We continue using notations from Construction 1.16. Let $v = (v_1, \ldots, v_{\dim V})$ be a given basis of $(V, \kappa)$. Take any $s \in \{1, \ldots, \dim W\}$. The difference $\dim \ker \varphi_s - \dim \ker \varphi_{s-1}$ is either zero or one. In the former case set $w_s$ to be any vector from $W^s \setminus W^{s-1}$ s.t. its class in $W^s/W^{s-1}$ coincides with $\mu_s$ in the latter case set $w_s$ to be $A(v_{\dim \ker \varphi_{s-1}})$. It is straightforward to check that the basis $(w_1, \ldots, v_{\dim W})$ satisfies the desired property. □

Proof of Lemma 7.3. Uniqueness is shown in the beginning of the present subsection. To show the existence, consider the composition of three maps:

$$V \to V/\ker A \xrightarrow{\cong} \im A \to W.$$ 

Induce enhancement on $V/\ker A$ from $V$ via Construction 1.8 and on $\im A$ from $W$ via Construction 1.16. Apply Proposition 1.14 to the middle map to obtain bases $\tilde{v}$ and $\tilde{w}$ of its source and target respectively. Apply now Proposition 1.17 to the basis $\tilde{v}$ to get a basis $v$ of $V$. Apply Proposition 1.17 to the basis $\tilde{w}$ to get a basis $w$ of $W$. By construction, bases $v$ and $w$ are the desired ones. □
1.5. On a matrix of a map between enhanced vector spaces. In this subsection we give several properties of a matrix of a map between enhanced vector spaces, written in appropriate basis.

Construction 1.18. Let $R$ be a rook $n \times m$ matrix. We will now define a subset $\mathcal{T}(R)$ of a set $\text{Mat}_{n,m}$ of all $n \times m$ matrices.

$\triangleright$ Let $M \in \text{Mat}_{n,m}$ be a matrix. We say that its entry $M_{i,j}$ is covered if there exists a pair of indices $(i',j')$ s.t. the following two conditions hold:

1) $R_{i',j'} \neq 0$,
2) $(i < i' \text{ AND } j > j')$ OR $(i \leq i' \text{ AND } j > j')$.

The matrix $M$ is said to be in $\mathcal{T}(R)$ if the the following two conditions hold:

1) if the entry $M_{i,j}$ is not covered and $R_{i,j} = 0$ then it equals to zero,
2) if the entry $M_{i,j}$ is not covered and $R_{i,j} \neq 0$ then $M_{i,j} = R_{i,j}$.

Here is an example, for $F = \mathbb{Q}$, of the matrix $R$ and the general form of a matrix $M$ from the set $\mathcal{T}(R)$:

$$
R = \begin{pmatrix}
0 & 0 & 4 \\
3 & 0 & 0 \\
0 & 0 & 0 \\
0 & 2 & 0
\end{pmatrix}, \quad M = \begin{pmatrix}
* & * & * \\
3 & * & * \\
0 & * & * \\
0 & 2 & *
\end{pmatrix}.
$$

In the case of classical Bruhat decomposition analogous set is nothing but a Bruhat cell. The next proposition is straightforward and well-known in the classical case.

Proposition 1.19. Let $(V, \varpi)$ and $(W, \mu)$ be two enhanced vector spaces and $A: V \to W$ be some linear map. Let also $v$ (resp. $w$) be some basis of enhanced vector space $(V, \varpi)$ (resp. $(W, \mu)$). Then the matrix of $A$ in the bases $v$ and $w$ belongs to $\mathcal{T}(R)$, where $R$ is a rook matrix from Lemma 1.18.

The next statement links properties of integral matrix (considered up to unitriangular change of basis) and its enhanced Bruhat decomposition over $\mathbb{Q}$. Let $\text{Mat}_{n,m}(\mathbb{Z})$ be the set of all $n \times m$ matrices over $\mathbb{Z}$. In what follows we will sometimes view it as a subset of matrices over $\mathbb{F} = \mathbb{Q}$ without mentioning this explicitly. Let also $T_n(\mathbb{Z})$ be the group of unitriangular matrices over $\mathbb{Z}$. The group $T_n(\mathbb{Z}) \times T_m(\mathbb{Z})$ acts on the set $\text{Mat}_{n,m}(\mathbb{Z})$. The next proposition follows from Proposition 1.19.

Proposition 1.20. Consider $M \in \text{Mat}_{n,m}(\mathbb{Z})$. Then any element $M'$ from the orbit $T_n(\mathbb{Z}) \cdot M \cdot T_m(\mathbb{Z})$ lies in the set $\mathcal{T}(R)$, where $R$ is a rook matrix over $\mathbb{F} = \mathbb{Q}$ associated with $M$.

As a corollary one gets that at least one non-zero entry of $R$ is integer. It also follows that there is at least one pair of indices $(i, j)$ s.t. the entry $M_{i,j}$ is the same for any $M'$ from the mentioned orbit.

1.6. Geometric approach to enhancements, dual enhancement and enhancements on subspaces and quotient vector space.

We recall that for an affine subspace $A$ in a vector space $V$ a set $\{a - b | a, b \in A\}$ is a vector subspace of $V$ associated with $A$, it is called the direction of $A$. We will denote this vector subspace by $\text{dir} A$.

Definition 1.21. An enhancement of a vector space $V$ is a subset of $V$, which we will also call $\varpi$, such that: $\varpi$ is a disjoint union of affine subspaces $A^i$ ($A^i \cap A^j = \emptyset$ for $i \neq j$) of dimension $i - 1$ for $i \in \{1, \ldots, \text{dim} V\}$ and such that $0 \notin \varpi$ and $\text{dir} A^{i+1} = \text{span}(A^i)$ for each $i \in \{1, \ldots, \text{dim} V\}$, where $A^{\text{dim} V + 1}$ equals to $V$.

The collection $A^1, \ldots, A^{\text{dim} V}$ of affine subspaces is uniquely recovered from the enhancement $\varpi$. For the corresponding algebraically defined enhancement $\varpi$ flag spaces are $V^i = \text{span}(A^i) = \text{dir} A^{i+1}$ and non-zero elements $\varpi^*$ are images of $A^i$ under natural quotient maps $V^i \to V^i/V^{i-1}$. The full flag $V^1 \subset \ldots \subset V^{\text{dim} V} = V$ of directions of enhancement we will denote by $\text{dir} \varpi$.

Dual vector space $V^*$ to an enhanced vector space $(V, \varpi)$ is naturally enhanced as well: a dual enhancement $\varpi^*$ on $V^*$ is, by definition, a union of the following spaces

$$
\text{Uni}(A^i) = \{ p \in V^* | p(A^i) = 1 \}.
$$

Note that $\text{dim} \text{Uni}(A^i) = \text{dim} V - i - 1$. Note also that dual flag to $0 = \text{dir} A^1 \subset \ldots \subset \text{dir} A^{\text{dim} V} \subset V$, which by definition consists of annihilator spaces to flag spaces

$$
0 = \text{Ann}(V) \subset \text{Ann}(\text{dir} A^{\text{dim} V - 1}) \subset \ldots \subset \text{Ann}(\text{dir} A^1) \subset V^*.
$$

coinsides with

$$
0 = \text{dir} \text{Uni}(A^{\text{dim} V}) \subset \text{dir} \text{Uni}(A^{\text{dim} V - 1}) \subset \ldots \subset \text{dir} \text{Uni}(A^1) \subset V^*.
$$
If \( \kappa = \kappa(v) \), where \( v = (v_1, \ldots, v_n) \) is ordered basis of \( V \), then \( \kappa^* = \kappa(g_n, \ldots, g_1) \) for a dual basis \( \{g_1, \ldots, g_n\} \) of \( V^* \) (such that \( g_i(v_j) = \delta_{ij} \)).

If \( L \subseteq V \) is a linear subspace of an enhanced vector space \( (V, \kappa) \) then one can easily show that the set \( \kappa_1 = \kappa \cap L \) is an enhancement on the vector space \( L \). Hence, for an injective linear map \( i: N \to V \) the preimage \( i^*(\kappa) \) is also an enhancement, and we denote it by \( i^* \kappa \).

A quotient vector space \( V/L \) of an enhanced vector space \( (V, \kappa) \) by a subspace \( L \) is also enhanced with an enhancement \( \kappa_{V/L} \) defined by the following construction. For a natural quotient map \( \psi: V \to V/L \) the dual map \( \psi^*: (V/L)^* \to V^* \) is an injection. The vector space \( V^* \) is enhanced with the dual enhancement \( \kappa^* \), hence one can consider the enhancement \( (\psi^*)^* \kappa \) on \( V/L^* \). The dual enhancement \( ((\psi^*)^* \kappa)^* \) is an enhancement on the vector space \( (V/L)^* \) which we will identify with \( V/L \) and the corresponding enhancement denote by \( \kappa_{V/L} \) and we call it as enhancement \( \kappa \) quotient by \( L \).

1.7. Construction of enhancement from flag and enhancement. We say that two enhancements \( \kappa_1 \) and \( \kappa_2 \) (on the same vector space) are parallel iff flags \( \kappa_1 \) and \( \kappa_2 \) coincide. Note that a set of all enhancements parallel to a given one is naturally a torus \( (\mathbb{R} \setminus \{0\})^{\dim V} \). Namely, for every pair \( (\kappa_1, \kappa_2) \) of parallel enhancements we construct an ordered collection of non-zero numbers \( \Lambda(\kappa_1, \kappa_2) = (\lambda_1(\kappa_1, \kappa_2), \ldots, \lambda_{\dim V}(\kappa_1, \kappa_2)) \), where each number \( \lambda_i = \lambda_i(\kappa_1, \kappa_2) \) is given by the relation \( \kappa_1^i = \lambda_i \kappa_2^i \).

The main construction of this section is the following. We will correspond to pair \((f, \kappa)\) consisting of a full flag \( F = W^1 \subseteq \ldots \subseteq W^{\dim V} \) on \( V \) and an enhancement \( \kappa \) on \( V \) an enhancement \( \chi = \chi(f, \kappa) \) on \( V \) such that \( \dir \chi = f \). It is given by the following inductive procedure, we will construct affine spaces \( B^1 \subseteq W^1, B^2 \subseteq W^2, \ldots \) step by step. Denote by \( A^1, \ldots, A^{\dim L} \) a collection of affine subspaces, generating \( \kappa \). Let \( A^1 = \text{Uni}(A^{\dim V}_{L^*}) \). Let us construct a point \( B^1 \) - geometrically it is the intersection of \( \kappa \) and \( W \), and we define it in the following way: consider the smallest \( k_1 \) such that \( \tilde{A}^k|_{W^1} \neq 0 \) and let
\[
B^1 = \{x \in W^1| p(x) = 1 \forall p \in \tilde{A}^k|_{W^1}\}.
\]
To construct \( B^2 \) we take smallest \( k_2 \) such that \( \tilde{A}^k \wedge \tilde{A}^k|_{W^2} \neq 0 \) (exterior product of sets is a set of all exterior products of elements from initial sets)
\[
B^2 = \{x \in W^2| p(x) = 1 \forall p \in \tilde{A}^k|_{W^2} \cup \tilde{A}^k|_{W^2} : p|_{B^1} = 0\}.
\]
To construct \( B^{s+1} \) we take smallest \( k_{s+1} \) such that \( \tilde{A}^k \wedge \tilde{A}^k \wedge \ldots \wedge \tilde{A}^k|_{W^{s+1}} \neq 0 \) and
\[
B^{s+1} = \{x \in W^{s+1}| p(x) = 1 \forall p \in \tilde{A}^k|_{W^{s+1}} \cup \tilde{A}^k|_{W^{s+1}} \cup \ldots \cup \tilde{A}^k|_{W^{s+1}} : p|_{B^1 \cup B^2 \cup \ldots \cup B^s} = 0\}.
\]
Each \( B^s \) is an affine space of the dimension \( s - 1 \) and the set \( B^1 \cup \ldots \cup B^{\dim V} \) is the desired enhancement \( \chi(f, \kappa) \). Also we get a permutation \( \sigma(f, \kappa) = (k_1, \ldots, k_{\dim V}) \).

Remark 1.22. If \( \dir \kappa = f \) then \( \chi(f, \kappa) = \kappa \). If the field \( \mathbb{F} \) is \( \mathbb{R} \) or \( \mathbb{C} \) and \( \dim V > 1 \) then the map
\[
(f, \kappa) \mapsto \chi(f, \kappa)
\]
is discontinuous, even functions \( \lambda_i(\kappa_1, \chi(\dir \kappa_1, \kappa_2)) \) are discontinuous, but their product
\[
\lambda_1(\kappa_1, \chi(\dir \kappa_1, \kappa_2)) \cdots \lambda_{\dim V}(\kappa_1, \chi(\dir \kappa_1, \kappa_2))
\]
is a continuous function – its value is the determinant of any isomorphism \( I \) such that \( I(\kappa_1^i) = \kappa_2^i \) for any \( s \).

1.8. Enhancements and grassmanians and determinant. Suppose \( V \) is a vector space with two enhancements \( \kappa, \chi \) then for every subspace \( L \subseteq V \) we get enhancements \( \kappa_L, \chi_L \) on \( L \) and we can construct numbers \( \lambda_1(\kappa_L, \chi_L), \ldots, \lambda_{\dim V}(\kappa_L, \chi_L) \) and permutation \( \sigma(\dir \kappa_L, \chi_L) \) Consider product
\[
d(\kappa_L, \chi_L) = (-1)^{|\sigma(\dir \kappa_L, \chi_L)|} \lambda_1(\kappa_L, \chi_L) \cdots \lambda_{\dim V}(\kappa_L, \chi_L).
\]
By construction this function has only non-zero values. This function has meaning of ordinary determinant in the following sense. Consider vector space \( V \) with ordered basis \( v \) and vector space \( U \) with ordered basis \( u \). Then vector space \( V \oplus U \) is endowed by ordered basis \( (v, u) \) and by ordered basis \( (u, v) \) and hence one can construct enhancements \( \kappa, \chi \) and \( \kappa, \chi \). Then for any isomorphism \( A: V \to U \) its determinant equals to \( d_{\kappa(\kappa(\chi^v, \chi^u)))} \) for \( L \subseteq V \oplus U \) being a graph of \( A \). If the map \( A: V \to U \) is not an isomorphism we still get a non-zero number and for 0 operator this number is 1.
2. Enhanced complexes

In this section we define and study enhanced complexes — an algebraic object which will carry a certain information about a strong Morse function (see Subsection 3.2). All the constructions lie within the scope of homological algebra of chain complexes over a field. They are similar in spirit to those in Section 3. The purpose is that enhanced complex is a useful algebraic container that stores some information about a strong Morse function, see Section 3.

2.1. Definition of an enhanced complex.

In this subsection we define the object of study of this section.

Definition 2.1. Let $C$ be a (chain) complex of vector spaces,
$$C_{n+1} = 0 \rightarrow C_n \xrightarrow{d_n} \ldots \xrightarrow{d_0} C_0 \rightarrow 0 = C_{-1}.$$  

An enhancement $\varkappa$ on a complex $C$ is an enhancement $(C_*, \varkappa)$ on a vector space $C_* := \oplus_{i=0}^{\infty} C_i$ satisfying the condition that each $C_i$ is a subcomplex of $C$ (we will therefore write $C'$ instead of $C_i^\varkappa$ in order to stress the structure of a complex). A complex with an enhancement will be called an enhanced complex and denoted as $(C, \varkappa)$.

We call the number $n$ the dimension of $C$ and denote it by $\dim C$.

Remark 2.2. 1. Recall that the aforementioned condition amounts to the following two:
   1) $C'$ is decomposed into the direct sum of graded components $\oplus_k C_k^\varkappa$ s.t. $C_k^\varkappa \subset C_k$ for $k \in \{0, \ldots, n\}$; 2) $\varkappa(C_i^\varkappa) \subset C_{i-1}^\varkappa$ for $k \in \{1, \ldots, n\}$.

2. By Construction 1.16 the vector space $C_k^\varkappa$ is also enhanced.

Remark 2.3. For an enhanced complex $(C, \varkappa)$ the set $\{1, \ldots, \dim C_*\}$ is $\mathbb{Z}_{\geq 0}$-graded: the degree $\deg s$ of $s$ is given by the only degree in which the complex $C'_{\varkappa}/C_{\varkappa}$ is non-zero.

Definition 2.4. Let $(C, \varkappa)$ and $(D, \mu)$ be two enhanced complexes with $\dim C_* = \dim D_*$. We say that they are isomorphic (and use the symbol $\simeq$ for this) if there exist an isomorphism of complexes $\varphi: C \xrightarrow{\sim} D$ s.t. the induced map $C_* \xrightarrow{\varphi} D_*$ is the isomorphism on enhanced vector spaces.

Remark 2.5. As we will show in Section 3 given a strong Morse function and suitable orientations one can construct an enhanced complex, which is well-defined up to isomorphism.

2.2. Enhancement on $H_*(C)$. In this subsection we construct enhancement on a homology of a certain class of complexes, which includes enhanced ones.

For any complex $C$ denote by $H_*(C)$ the direct sum $\oplus n H_{\cdot}(C)$. Let $(C, \varkappa)$ be an enhanced complex. Then for any $s$ the vector space $H_*(C_{\varkappa}, C_{\varkappa}^{s+1})$ is one-dimensional with a preferred generator of degree $\deg s$ given by a class of relative chain $\varkappa_C \in C_{\varkappa}^{s+1}/C_{\varkappa}^{s+1}$.

Construction 2.6. Let $C$ be a filtered (possibly infinite-dimensional) complex over a field $F$, $0 = C^0 \subset \ldots \subset C^N = C$. Suppose that for any $s$ vector space $H_*(C_{\varkappa}, C_{\varkappa}^{s+1})$ is one-dimensional with a chosen generator $h_{i_s}$. We will now construct an enhancement on a homology vector space $H_*(C)$.  

First, we will construct a filtration on $H_*(C)$. For $s \in \{0, \ldots, N\}$ let $i_s: H_*(C_{\varkappa}) \rightarrow H_*(C)$ be a map induced by inclusion. Define the subset $H_{\cdot}$ (which stands for homology) of the set $\{1, \ldots, N\}$ to be the set of all s.s.t. $\dim H_{i_s} = \dim \ker i_{s-1} + 1$. Let $s_0$ be the $i_0$th element of $H$ (counting from 1) and $s_0$ be zero. The sequence of subspaces $0 = \ker i_{s_0} \subset \ker i_{s_1} \subset \ldots \subset \ker i_{s_{N+1}} = H_{\cdot}$ is a full flag on $H_{\cdot}(C)$ (follows from considering the exact sequence of a pair $(C_{\varkappa}, C_{\varkappa}^{s+1})$ for all $s$). To complete the construction of enhancement we will now produce an element from $\ker i_{s_{i_{s_0}}}/\ker i_{s_{i_{s_0}}-1}$ for a given $i \in \{1, \ldots, \dim H_{\cdot}(C)\}$. We denote $h_s$ by $s$ for convenience. Let $\deg s$ denote the degree in which the graded vector space $H_*(C_{\varkappa}, C_{\varkappa}^{s+1})$ is non-zero (this notation is coherent with the case when $C$ is an enhanced complex).

Consider the following diagram, with horizontal line being a portion of a long exact sequence of a pair $(C_{\varkappa}, C_{\varkappa}^{s+1})$:

$$
\begin{array}{ccc}
H_{\deg s}(C_{\varkappa}^{s+1}) & \xrightarrow{\partial} & H_{\deg s}(C_{\varkappa}^{s+1}) \\
\downarrow & & \downarrow \\
H_1(C) & \xrightarrow{\cup} & H_0(C)
\end{array}
$$

We write $i_s$ both for a map $H_1(C_{\varkappa}) \rightarrow H_1(C)$ and for its restriction to $H_{\deg s}(C_{\varkappa}^{s+1})$. It follows from the definition of $H$ that $\dim \ker C_{\varkappa}^{s+1} = 1$, therefore $\ker p_s$ is a proper subspace of $H_{\deg s}(C_{\varkappa}^{s+1})$, which in turn implies that $p_s$ is surjective. Denote by $p_s^{-1}(h_s)$ any preimage of $h_s$; it is defined up to elements from $\ker p_s \simeq \ker i_s$. Finally, the desired element is a class of $i_s(p_s^{-1}(h_s))$ in the quotient space $\ker i_s/\ker i_{s-1}$ (mind that $\ker i_{s-1} = \ker i_{s-1}$); it is well-defined.

Remark 2.7. Note that to a flag space of $H_1(C)$ of dimension $d$ one can associate a number $s \in \{1, \ldots, N\}$ as the unique solution of equation $\dim \ker i_s = \dim \ker i_{s-1} + 1 = d$. In other words, this is the smallest $s$ such that given flag space is contained in the image of $i_s$.  

<
If one is given a non-zero vector \( v \) in the enhanced vector space \( H_s(C) \) then they may consider the flag space of least possible dimension containing \( v \) (its dimension is \( \ell(v) \)). Combining this with the previous paragraph one can associate a number \( s \) with a vector \( v \). We will make use of this association in Subsection 2.3.

For a detailed treatment of the mentioned long exact sequence see [Lau15]. Without taking \( x_s \) into account it was first considered in [LNV20]. This preferred generator appeared independently in [LNV20]. We denote obtained enhancement as \((H_s(C), x_s)\). Specializing the above discussion to a fixed degree \( k \) (and thus having \( \deg s_i = k \)) we get an enhanced vector space \((H_s(C), x_k)\); one may check that this is the same enhancement as the one induced by inclusion \( H_{k}(C) \hookrightarrow H_{s}(C) \) via Construction 1.16. Note that the above procedure also gives enhancement on \( H_{s}(C^*) \) for any \( s \) (it will be crucial in what follows).

**Remark 2.8.** The reason for the chosen level of generality is that one may take the input to be a complex of singular chains on a manifold equipped with a Morse function. The filtration is then given by the sublevel sets. See Subsection 3.5, where B-data (described below in Subsection 2.3) is extracted from the function this way.

**Remark 2.9.** In the case when \( C \) is an enhanced complex one may check that the following alternative construction produces the same enhancement on \( H_s(C) \). Consider the map \( \partial : C_s \to C_{s+1} \). Induce enhancements on \( \ker \partial \) and \( \text{im} \partial \) via Construction 1.16. Induce enhancement on \( \ker \partial/ \text{im} \partial \) via Construction 1.8.

### 2.3. B-data.

In this subsection we introduce a certain data extracted from an enhanced complex. This data is invariant under isomorphisms. In Subsection 2.4 we show that this data is in fact a full invariant of an enhanced complex considered up to isomorphism. The letter \( B \) stands simultaneously for Barannikov, Bruhat and barcode.

**Remark 2.10.** The extraction of B-data is done in the same way for the (more general) case of a complex \( C \) which satisfies conditions of Construction 2.6 (compare Remark 2.8). Indeed, one has to merely replace the symbol \( [x_s] \) with \( h_s \). In this case, however, we don’t claim that this data is a full invariant; we don’t even define any equivalence relation on the set of such complexes.

We will now describe the B-data. It consists of several parts:

i) A non-negative integer \( N \) along with a \( \mathbb{Z}_{\geq 0} \)-grading on a set \( \{1, \ldots, N\} \), denoted by \( \deg \);

ii) Decomposition of \( \{1, \ldots, N\} \) into the union of three disjoint sets \( U, I, H \) (these letters stand for upper, lower and homological, for the reasons described below);

iii) Bijection \( b : U \stackrel{1-1}{\longrightarrow} L \) of degree \( -1 \) w.r.t. the grading. Map \( b \) must satisfy \( b(s) < s \);

iv) A function \( \lambda : U \longrightarrow \mathbb{F}^* \). We write \( \lambda_s \) for its value on \( s \in U \).

We call the image of \( \lambda \) “Bruhat numbers” of an enhanced complex (see the proof of Theorem 2.16 for the explanation). Two numbers \( s \) and \( b(s) \) are said to form a Barannikov pair (or simply a pair). It’s convenient to think of each Bruhat number as a full invariant of an enhanced complex considered up to isomorphism. The letter \( B \) stands simultaneously for Barannikov, Bruhat and barcode.

**Remark 2.11.** The extraction of B-data is done in the same way for the (more general) case of a complex \( C \) which satisfies conditions of Construction 2.6 (compare Remark 2.8). Indeed, one has to merely replace the symbol \( [x_s] \) with \( h_s \). In this case, however, we don’t claim that this data is a full invariant; we don’t even define any equivalence relation on the set of such complexes.

![Figure 1](image-url)

**Figure 1**

```
\begin{align*}
R_1 &= \begin{pmatrix} 0 & 0 & 6 & 0 \end{pmatrix}, \\
R_2 &= \begin{pmatrix} 0 & 0 & 0 & 4 \\
3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \end{pmatrix}.
\end{align*}
```

We will now show how to extract such a data from an enhanced complex \((C, x)\). In future we will refer to it as a B-data of \((C, x)\) and use the same letters \( N, U, L, H, b \) and \( \lambda \) for its ingredients when necessary. We continue using notations introduced in Subsections 2.1 and 2.2. Let \( \delta : H_{\deg s}(C^*, C^{s-1}) \rightarrow H_{\deg s-1}(C^{s-1}) \) be the connecting homomorphism. To begin with, set \( N \) to be
the number of filtration components (counting from zero) and set the grading on \{1, \ldots, N\} to be the one defined in Remark 2.3. Now for each \( s \in \{1, \ldots, N\} \) s.t. \( \delta([\kappa_j]) \neq 0 \) do the following.

1. Put \( s \in U \).
2. For any \( t \in \{1, \ldots, s-1\} \) let \( t_*^{-1}: H_{\deg s}(C^t) \to H_{\deg s}(C^{s-1}) \) be the map induced by inclusion.
   Now choose the only \( t \) s.t. \( \dim \text{Im} \ t_*^{-1} = \dim \text{Im} \ t_1^{-1} + 1 = \text{ht}(\delta([\kappa_j])) \) (the function \( \text{ht}(\cdot) \) here is taken w.r.t. the enhancement on \( H_{\deg s-1}(C^{s-1}) \) constructed from \( C^{s-1} \)). See Remark 2.7 for an informal meaning of such a \( t \). Put this \( t \) in \( L \).
3. Set \( \lambda(s) := \text{cf}(\delta([\kappa_j])) \) w.r.t. the same enhancement.
4. Set \( b(s) := t \).

Using diagram chasing similar to that in Construction 2.6 one verifies that such an operation is well-defined in a sense that, first, each number will be put somewhere at most once and, second, that \( b \) enjoys desired properties, see [Lau15]. Those numbers in \( \{1, \ldots, N\} \) which were not put anywhere by this any operation are put in \( H \). The extraction of B-data \([\lambda], [\nu] \) is over, it’s plain to see that it’s invariant under isomorphism.

Remark 2.11. Here is another way to extract a B-data out of an enhanced complex. Apply Lemma 2.5 to a differential \( \partial_k : C_k \to C_{k-1} \) viewed as a map between enhanced vector spaces (see Remark 2.2). This would give a sequence \( \{R_k\} \) of rook matrices and thus we’re done. Although this way is shorter, we find it less instructive.

Yet another approach is to use spectral sequence of a filtered (actually, enhanced) complex.

Again, arguing as in Construction 2.6 one can show that \# \( \{ s \in H \mid \deg s = k \} = \dim H_k(C) \). We stress that everything except \( \lambda \) was essentially constructed in [Bar94], while homological language was first used in [BLN13]. A close idea of construction of Bruhat numbers over \( \mathbb{Q} \) appeared independently in [LNV20].

Remark 2.12. Consider once again the portion of a long exact sequence of a pair \((C^s, C^{s-1})\):

\[
H_{\deg s}(C^s) \xrightarrow{p_*} H_{\deg s}(C^s, C^{s-1}) \xrightarrow{\delta} H_{\deg s-1}(C^{s-1}).
\]

Since the middle term is one-dimensional, there are two cases possible:

1. The map \( p_* \) is surjective whilst \( \delta \) is zero. This case was used in Construction 2.6.
2. The map \( \delta \) is injective whilst \( p_* \) is zero. This case was used in the extraction of B-data.

Remark 2.13. Let \( s \) and \( t \) be some elements from \( \{1, \ldots, N\} \). They form a Barannikov pair (i.e. \( b(s) = t \)) if and only if the following equations hold:

\[
\dim H_*(C^{s-1}, C^t) = \dim H_*(C^s, C^{s-1}) = \dim H_*(C^{s-1}, C^{t-1}) - 1 = \dim H_*(C^s, C^t) - 1.
\]

This can be proven either by diagram chasing similar to that in Construction 2.6 or by Theorem 2.16 (see [CP05] and also [LNV13]).

We will now take coordinate viewpoint, which will be useful in formulation of the classificational Theorem 2.20 in Subsection 2.4. By a basis of a chain complex \( C \) we will mean a basis \( \{c_1, \ldots, c_{\dim C_*}\} \) of a vector space \( C_* \) s.t. each \( c_s \) belongs to some \( C_k \), where \( k \) depends on \( s \).

By a basis of an enhanced complex \((C, \kappa)\) we will mean a basis \( c \) of a chain complex \((C, \partial)\) s.t. \( (C_* , \kappa) \simeq (C_* , \kappa(c)) \) (see Definition 2.1). Every enhanced complex can be equipped with a basis.

Vice versa, given a basis \( c \) of a chain complex \( C \) s.t. the span \( \langle c_1, \ldots, c_s \rangle \) is a subcomplex (for each \( s \in \{1, \ldots, \dim C_*\} \)) one may construct an enhanced complex \((C, \kappa(c))\) by declaring \( (C_* , \kappa) := (C_* , \kappa(c)) \).

Remark 2.14. Note that two enhanced complexes \((C, \kappa)\) and \((D, \mu)\) are isomorphic if and only if the following holds:

1. \( \dim C_* = \dim D_* \) and the gradings on \( \{1, \ldots, \dim C_*\} \) constructed from \((C, \kappa)\) and \((D, \mu)\) coincide (see Remark 2.3).
2. There exist bases \( c \) of \((C, \kappa)\) and \( d \) of \((D, \mu)\) s.t. corresponding two matrices of differentials coincide (compare Remark 1.12).

Definition 2.15. Let \((C, \kappa)\) be an enhanced complex and \( c \) be its basis. We call \( c \) a Barannikov basis if the matrix of differential \( \partial_k : C_k \to C_{k-1} \) in the basis \( c \) is a rook matrix (for any \( k \)).

The set of rook matrices \( \{R_k\} \) of differentials \( \partial_k \) doesn’t depend on a particular choice of a Barannikov basis. Indeed, this set is precisely the B-data, which is an invariant of an enhanced complex. The purpose of Subsection 2.4 is to show that every enhanced complex admits a Barannikov basis. This will imply that B-data is a full invariant of an enhanced complex considered up to isomorphism.

2.4. Classification of enhanced complexes. In this subsection we prove classificational...
Theorem 2.16. For every enhanced complex $(C, \varkappa)$ there exists a Barannikov basis $c$. Moreover, the matrix of $\partial$ is the same for any Barannikov basis.

Remark 2.17. 1. Barannikov basis itself need not be unique (compare Remark 1.7).
2. Put differently, one may say that B-data is a full invariant of an enhanced complex considered up to isomorphism (see Subsection 1.3).
3. It is profitable to have a Barannikov basis at hand, since the complex takes the simplest form possible and becomes tractable.
4. The case when the complex is not enhanced, but only filtered, was proven in [Bar94]. See also [Mel00; Thi90; Thi97] for an “ungraded” setting where a single upper triangular matrix is considered.

For a matrix $X$ we denote by $X_{•,j}$ its $j$th column and by $X_{i,•}$ its $i$th row.

Proof of Theorem 2.16. Uniqueness follows from the existence and the fact that B-data is invariant under isomorphisms (which is shown in Subsection 2.3). The rest is devoted to proving the existence.

Fix any $k \in \{0, \ldots, \dim C\}$. The differential $\partial_k : C_k \to C_{k-1}$ is a map of enhanced vector spaces $(C_k, \varkappa_k)$ and $(C_{k-1}, \varkappa_{k-1})$ (see Remark 2.2). Therefore, Lemma 1.5 produces, in particular, a rook matrix $X$ and a basis $x$ of $(C_k, \varkappa_k)$. Analogously, applying Lemma 1.5 to $\partial_{k+1}$ one obtains a rook matrix $Y$ and another basis of $(C_k, \varkappa_k)$, call it $y$. Construct now a third basis $v$ of $(C_k, \varkappa_k)$ as follows.

Fix $s \in \{1, \ldots, \dim C_k\}$. Obviously at least one of three cases listed below holds. On the other hand, it follows from $\partial^2 = 0$ and proof of Lemma 1.5 that all three are mutually exclusive. So, we define $v_s$ depending on which of them holds.

1. $X_{•,s} \neq 0$. Set $v_s := x_s$.
2. $Y_{•,s} \neq 0$. Set $v_s := y_s$.
3. Both $X_{•,s}$ and $Y_{•,s}$ are zero. One is free to take either $x_s$ or $y_s$ as $v_s$.

We have constructed a basis of $(C_k, \varkappa_k)$ for each $k$. The last step is to construct a basis $c$ of $(C, \varkappa)$. Take any $s \in \{1, \ldots, \dim C\}$. Let $v$ be a constructed basis of $(C_{\deg v}, \varkappa_{\deg v})$. Define $c_s$ to be $v_{\dim C_{\deg v}}$ (see Definition 2.1). Finally, by construction $c$ is a Barannikov basis.

Remark 2.18. Three mentioned cases correspond respectively to the fact that $s$ belongs to 1) $U$, 2) $L$, 3) $H$.

2.5. Z-enhanced complexes. In this subsection we introduce a certain analogue of an enhanced complex, which is itself a complex of free abelian groups.

Definition 2.19. Let $C$ be a (chain) complex of free abelian groups

$$C_{n+1} = 0 \to C_n \xrightarrow{\partial_n} \ldots \xrightarrow{\partial_2} C_0 \to 0 = C_{-1}.$$ 

A $\mathbb{Z}$-enhancement $\varkappa$ on a complex $C$ is a choice of the following two structures.

1. A filtration

$$0 = c^0 \subset \ldots \subset c^r C_s = C$$

of $C$ by subcomplexes s.t. for each $s \in \{1, \ldots, \rk C_s\}$ the quotient complex $C^s/C^{s-1}$ is isomorphic to $\mathbb{Z}$ concentrated in one degree.

2. A generator of $C^s/C^{s-1} \cong \mathbb{Z}$.

A complex with a $\mathbb{Z}$-enhancement will be called a $\mathbb{Z}$-enhanced complex and denoted as $(C, \varkappa)$.

The following notions and statements go in exactly the same manner as in the honest enhanced case.

1. The definition of an isomorphism between two $\mathbb{Z}$-enhanced complexes.
2. The definition of a basis of a $\mathbb{Z}$-enhanced complex (recall that by a basis we always mean a linearly ordered set of generators). Matrix of differential $\partial_k$ in any basis is obviously integral, yet it will play important role in the end of this subsection.
3. Every $\mathbb{Z}$-enhanced complex can be equipped with a basis.
4. Let $c = (c_1, \ldots, c_{\rk C_s})$ be a basis of a complex of free abelian groups s.t. 1) for each $s$ the span $(c_1, \ldots, c_s)$ is a subcomplex, 2) the induced filtration satisfies the condition 1) from Definition 2.1. Then one can construct a $\mathbb{Z}$-enhanced complex $(C, \varkappa(c))$.

It follows directly from the definitions that if $(C, \varkappa)$ is a $\mathbb{Z}$-enhanced complex then $C \otimes \mathbb{F}$ is an enhanced complex over $\mathbb{F}$. We denote it by $(C \otimes \mathbb{F}, \varkappa)$.

Remark 2.20. An oriented strong Morse function on a manifold naturally gives rise to a $\mathbb{Z}$-enhanced complex. However, classifying such complexes up to isomorphism is a transcendently hard problem. So, following [Bar94], we proceed by tensoring the given complex by $\mathbb{F}$ for various fields. See Subsection 3.3.
Remark 2.11. Recall also that in Subsection 1.5 we associated a subset $T$ of matrices with a rook matrix $R$. The statement now follows from Proposition 2.21.

Proposition 2.21. Let $c$ be any basis of a $Z$-enhanced complex $(C, \varpi)$. Then the matrix of differential $\partial_l$ in this basis belongs to the set $T(R_k)$, where $R_k$ is a rook matrix from the $B$-data of $(C \otimes Q, \varpi)$.

Proof. Let $D$ be a matrix of differential $\partial_l$ in some basis $c$ of $(C, \varpi)$. After choosing another basis $c'$ the matrix $D$ gets multiplied by a unitriangular matrices (over $Z$) from the left and from the right. The statement now follows from Proposition 1.20. □

See Subsection 1.5 for an example, which may be treated as a complex concentrated in two degrees. By a degree of a pair we will mean degree of its lower point. A Bannikov pair is called short if there are no pairs of the same degree that lie inside it. Formally, $(s, t)$ is a short pair if there is no pair $(s', t')$ of the same degree s.t. $s < s' < t' < t$.

Corollary 2.22. Let $(C, \varpi)$ be a $Z$-enhanced complex. Bruhat number of $(C \otimes Q, \varpi)$ on any short pair is integer.

Proof. Take any short pair of degree, say, $k - 1$. Let $R_k$ be a rook matrix from the $B$-data of $(C \otimes Q, \varpi)$. Short pairs correspond precisely to those non-zero entries of $R_k$ which are not covered (in the terminology of Subsection 1.5). The statement now follows from Proposition 2.21. □

The next statement is a mere combination of the previous two.

Corollary 2.23. Let $(C, \varpi)$ be $Z$-enhanced complex and let $(s, t)$ be a short Bannikov pair of $(C \otimes Q, \varpi)$. Let also $c$ be any basis of $(C, \varpi)$. Then element $c_s$ appears in the differential of $c_t$ with the coefficient equal to Bruhat number on a pair $(s, t)$. In particular, this coefficient doesn’t depend on $c$.

2.6. Bruhat numbers over the rationals. In this subsection we state several facts about interplay between $Z$-enhanced complex $(C, \varpi)$ and enhanced complex $(C \otimes Q, \varpi)$. The proofs will be given elsewhere.

For an abelian group $G$ we denote by $\#G$ its order and by $\text{Tors} G$ its torsion subgroup.

Proposition 2.24. Let $(C, \varpi)$ be a $Z$-enhanced complex. Let also $s$ and $t$ ($s > t$, both from $\{1, \ldots, \text{rk} C_s\}$) be a Bannikov pair of enhanced complex $(C \otimes Q, \varpi)$ with Bruhat number $\lambda \in Q^*$. One then has

$$\pm \lambda = \frac{\# \text{Tors} H_s(C^s, C^{s-1})}{\# \text{Tors} H_s(C^{s-1}, C^s)} = \frac{\# \text{Tors} H_s(C^{s-1}, C^s)}{\# \text{Tors} H_s(C^{s-1}, C^s)} = \frac{\# \text{Tors} H_s(C^{s-1}, C^s)}{\# \text{Tors} H_s(C^{s-1}, C^s)}.
$$

In Section 3 we will interpret Bruhat numbers as a certain kind of torsion. From this viewpoint the given formula is of type “torsion=torsion”. For its close relative, see [Tur01, Theorem 4.7], proven in weaker generality by Milnor [Mil69]. See also [Cha17] for similar in spirit statement in symplectic topology. We stress out that we place no acyclicity condition on a complex $C$.

Proposition 2.25. Let $(C, \varpi)$ be a $Z$-enhanced complex. Then the following are equivalent:

1) $\text{Tors} H_s(C^s, C_t) = 0$ for all $s > t$ (both from $\{1, \ldots, \text{rk} C_s\}$).
2) the Bruhat numbers of $(C \otimes Q, \varpi)$ equal to $\pm 1$,
3) the $Z$-enhanced complex $(C, \varpi)$ is isomorphic (in the sense of item 1) in the list from Subsection 2.3 to the direct sum of complexes of two forms: 1) $0 \rightarrow \mathbb{Z} \rightarrow 0$, and 2) $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 0$.

2.7. Taking B-data commutes with taking sub- and quotient complexes. Given an enhanced complex $(C, \varpi)$ and two integers $0 \leq l \leq m \leq \text{dim} C$, consider a quotient complex $C^m/C^l$; it inherits an enhancement. The goal of this subsection is to provide a recipe on how to express B-data of this complex in terms of the initial one. To this aim we, first, describe this recipe and, second, prove that it is correct.

Let us fix the notations first. Aforementioned enhanced complex will be denoted as $(C^m, \varpi)[m]$. Let $(U, L, H, b, \lambda)$ be a B-data associated to $(C, \varpi)$. Given $l$ and $m$ we will now define another B-data $(U', L', H', b', \lambda')$.

Set $U' := \{s \in U | l < s \leq m, b(s) > l\} - l$ (by convention, subtracting an integer $l$ from a subset of integers yields another subset formed by differences with $l$ of each element individually),

$L' := \{s \in L | l < s \leq m, b(s) = l\} - l$, $H' := \{1, \ldots, m - l\} \setminus (U' \cup L')$. Define the grading on $U'$ via its injection into $U$. Proceed similarly for $L'$ and $H'$. For $s \in U'$ define 1) $b'(s) := b(s + l) - l$, 2) $\lambda'(s) := \lambda(s + l)$.
Proposition 2.26. Let \((\mathcal{C}, \varpi)\) be an enhanced complex and \((U, L, H, b, \lambda)\) be its B-data. For a given \(0 \leq l \leq m \leq \dim \mathcal{C}_\bullet\) the B-data of \((\mathcal{C}^\mu_l, \varpi^m_l)\) coincides with the data \((U^l, L^l, H^l, b^l, \lambda^l)\) constructed above.

**Proof.** Take Barannikov basis of \((\mathcal{C}, \varpi)\) which exists by Theorem 2.16. Its elements with indices from \(l + 1\) to \(m\), when mapped to a \((l, m)\)-slice, again form a Barannikov basis. The statement follows. \(\square\)

Remark 2.27. Informally, the B-data of \((\mathcal{C}^\mu_l, \varpi^m_l)\) is obtained from that of \((\mathcal{C}, \varpi)\) by the simplest procedure possible: one has to cut it from below and above at the given levels \(l\) and \(m\).

Remark 2.28. Although usage of Theorem 2.16 makes the proof shorter, it is still possible to prove the above statement directly from the definitions.

2.8. Torsion of a chain complex. We begin this subsection by recalling Milnor’s [Mil66] definition of torsion of a chain complex, closely following [Tur01]. After this we define torsion of an enhanced complex and calculate it in terms of B-data.

Let \(v = (v_1, \ldots, v_{\dim V})\) and \(v' = (v'_1, \ldots, v'_{\dim V})\) be two bases of \(V\). Denote by \([v'/v] \in \mathbb{P}\) the determinant of a transition matrix from \(v\) to \(v'\). We call two bases \(v\) and \(v'\) equivalent if \([v'/v] = 1\).

Let us now be given an exact triple of vector spaces \(0 \to U \to V \to W \to 0\) along with bases \(u\) and \(w\) of \(U\) and \(W\) respectively. Construct a basis \(uw\) of \(V\) as follows. For a vector \(w_i \in W\) set \(\pi^{-1}(w_i) \in V\) to be any lift w.r.t. \(\pi\). Now set \(uw := ([\pi^1(v_1), \ldots, \pi^1(v_{\dim U})], \pi^{-1}(w_1), \ldots, \pi^{-1}(w_{\dim W}))\).

Equivalence class of \(uw\) is independent of chosen lifts of \(w_i\).

Remark 2.29. Recall that for a chain complex \((\mathcal{C}, \partial)\) one defines boundaries \(B_k\) to be \(\text{Im} \partial_{k+1}\) and cycles \(Z_k\) to be \(\text{Ker} \partial_k\). One then has two exact triples:

1. \(0 \to B_k \to Z_k \to H_k \to 0\)
2. \(0 \to Z_k \to C_k \to B_{k-1} \to 0\).

Let us now be given bases \(c_k\) of \(C_k\) and \(b_k\) of \(H_k\) (for all admissible \(k\)). Choose any basis \(b_k\) of \(B_k\). Construct, first, a basis \(b_kb_k\) of \(Z_k\) via triple 1), and, second, a basis \(b_kb_kb_{k-1}\) of \(C_k\) via triple 2).

Define the torsion of \(\mathcal{C}\) to be

\[
\tau(\mathcal{C}) := \prod_{i=0}^{n} [b_khb_{k-1}/c_i]^{(-1)^{i+1}} \in \mathbb{P}^*.
\]

It is straightforward to show that \(\tau(\mathcal{C})\) depends only on \((\mathcal{C}, \partial)\), equivalence class of \(c_k\) and that of \(b_k\) (see [Tur01]).

Remark 2.29. If one replaces basis \(c_k\) with \(c'_k\) for some particular \(k\) then the torsion gets multiplied by \([c'_k/c_k]^{(-1)^{k+1}}\).

Any two bases of a given enhanced vector space \((V, \varpi)\) are equivalent. Analogously, for any two bases \(c\) and \(c'\) of an enhanced complex \((\mathcal{C}, \varpi)\) one has equivalence between \(c\) and \(c'\) where by \(c\) we mean (here and further) an ordered subset of \(c\) corresponding to a basis of \((\mathcal{C}_k, \varpi_k)\) (see Remark 2.22).

Let us now assemble all the pieces together. Let an enhanced complex \((\mathcal{C}, \varpi)\) be given. Choose any basis \(c\) of \((\mathcal{C}, \varpi)\). Recall that by Subsection 2.2 we have an enhanced vector space \((H_k(\mathcal{C}), \varpi_k)\) for each \(k\). Choose any basis \(h_k\) of \((H_k(\mathcal{C}), \varpi_k)\). Define the torsion of an enhanced complex \((\mathcal{C}, \varpi)\) to be the torsion of \(\mathcal{C}\) w.r.t. bases \(c\) and \(h_k\); denote it by \(\tau(\mathcal{C}, \varpi)\). This number is well-defined since equivalence classes of both \(c\) and \(h_k\) are well-defined.

Remark 2.30. We stress out that \(\tau(\mathcal{C}, \varpi)\) depends only on the enhancement on \(H_k(\mathcal{C})\). Bruhat numbers, however, depend on enhancement on \(H_k(C)\) for various \(s\).

Remark 2.31. Some kind of interplay between filtration and torsion also appears in [GKZ98, Appendix A].

Construction 2.32. Let \((N, U, L, H, b)\) be a part of a B-data. We will now construct a permutation \(\sigma\) of \(N\) elements.

Note that \(b\) doesn’t have anything to do with \(b_k\) from the definition of torsion. For a fixed \(k\), the set \(U\) determines a subset of a set \(\{1, \ldots, \#\{s \in \{1, \ldots, N\} \mid \deg s = k\}\}\), call it \(U_k\). Define \(L_k\) and \(H_k\) similarly; the map \(b\) determines a bijection \(b_k : U_k \to L_{k-1}\). We will now define a permutation \(\sigma_k\) on \(\#\{s \in \{1, \ldots, N\} \mid \deg s = k\}\) elements by writing integers in a row. First, write down elements of \(L_k\) in increasing order. Second, write down elements of \(H_k\) also in increasing order. Third, write down elements of \(U_k\), but this time in the order of increasing of \(b_k(s)\), for \(s \in U_k\).

For two permutations \(\sigma\) and \(\pi\) of length \(l\) and \(m\) their (direct) sum \(\sigma + \pi\) is defined as a permutation of \(l + m\) elements acting as \(\sigma\) on the first \(l\) elements and as \(\pi\) on the last \(m\) elements. We define \(\sigma\) to be the sum \(\sigma_0 + \ldots + \sigma_n\), where \(n = \max_{s \in \{1, \ldots, N\}} \deg s\).

The sign of a permutation \(\sigma\) will be denoted as \((-1)^\sigma\).
Proposition 2.33. Let \((C, \pi)\) be an enhanced complex and \((U, \lambda)\) be a part of its B-data. Let also \(\sigma\) be the permutation from Construction 2.32. We then have
\[
\tau(C, \pi) = (-1)^{p} \prod_{s \in U} \lambda(s)^{(-1)^{p} \deg s}.
\]

Proof. Since any two bases of an enhanced vector space are equivalent, we may calculate \(\tau(C, \pi)\) in some Barannikov basis \(c\), which exists by Theorem 2.16. We continue using notations introduced in the beginning of this subsection. Choose basis \(h_{k}\) to be \(c_{s}\) for all \(s \in H\). Similarly, choose basis \(b_{k}\) to be \(c_{s}\) for all \(s \in L\) (the linear order on both bases is induced from that on \(c\)). These choices yield a right-hand side by the very definitions. \(\square\)

3. Morse theory

In the first part of this section we introduce, after neccesary preparations, a construction which associates an enhanced complex over \(\mathbb{F}\) with a strong Morse function (see Subsection 3.3). This justifies a thorough study of enhanced complexes in the previous section. We then proceed to discuss various properties of Bruhat numbers of a given strong Morse function. The majority of results translates readily to the setting of discrete Morse theory in a sense of Forman [For98], as in the smooth case, the strongness assumption on a function is crucial and must be satisfied.

3.1. Setup. In this subsection we recall basic notions of Morse theory and fix appropriate notations, setting the stage for our results.

Let \(M\) be a smooth closed manifold fixed once and for all throughout this section. Recall that a smooth function \(f: M \to \mathbb{R}\) is called Morse if all its critical points are non-degenerate. A smooth function is called strong if all its critical values are pairwise distinct. Fix a strong Morse function \(f\) on \(M\) once and for all throughout this section. For \(a \in \mathbb{R}\) the subspace \(M^{a} := \{x \in M | f(x) \leq a\}\) is called a sublevel set.

Morse’s idea was to track how the homotopy type of \(M^{a}\) changes while \(a\) grows from \(-\infty\) to \(+\infty\). This is performed by investigating the critical points of \(f\), the set of which is denoted by \(\text{Cr}(f) \subset M\).

Since \(f\) is strong those are in bijection with critical values of \(f\) (this set is finite because of the compactness of \(M\)). Keeping this bijection in mind, we will freely switch between points and values without mentioning this explicitly. The set \(\text{Cr}(f)\) is \(\mathbb{Z}_{\geq 0}\)-graded by index of a critical point, the degree of \(c \in \text{Cr}(f)\) is denoted by \(\deg c\). Though it’s more natural to say “index of critical point”, we will mostly say “degree” in order to be consistent with Section 2. The set of all critical points of degree \(k\) is denoted by \(\text{Cr}_{k}(f)\). Note that the set \(\text{Cr}(f)\) is also naturally linearly ordered; we denote by \(c_{s} \in \text{Cr}(f)\) for \(s \in \{1, \ldots, \#\text{Cr}(f)\}\) its \(s\)th element w.r.t. this order. By \(\varepsilon\) we will mean a sufficiently small positive real number.

It follows from foundational results of Morse theory, which we recall in Subsection 3.4, that for \(c \in \text{Cr}(f)\) one has \(H_{\text{deg}(c)}(M^{\langle c \rangle + \varepsilon}, M^{\langle c \rangle - \varepsilon}; \mathbb{Z}) \simeq \mathbb{Z}\). We say that a critical point is oriented if the generator of this free abelian group of rank one is chosen. A strong Morse function is called oriented if all its critical points are oriented (see Subsection 3.3 for a discussion).

It will be convenient to fix the set of real numbers \(r_{s}\) for \(s \in \{0, \ldots, \#\text{Cr}(f)\}\) s.t.
\[
0 < f(c_{1}) < r_{1} < \ldots < f(c_{\#\text{Cr}(f)}) < r_{\#\text{Cr}(f)}.
\]
Such numbers are called regular values.

Fix a field \(\mathbb{F}\) once and for all. All the chain complexes and homologies are assumed to be over \(\mathbb{F}\) unless stated otherwise. If the group of coefficients is given explicitly, it goes after a semicolon, e.g. \(H_{2}(M; \mathbb{Z})\).

3.2. B-data associated with a strong Morse function. In this subsection we present and discuss the

Construction 3.1. Let \(f\) be an oriented strong Morse function on \(M\) and \(\mathbb{F}\) be a field. We will now construct a B-data.

\(\triangleright\) Since \(\text{Cr}(f)\) is linearly ordered, it is in natural bijection with \(\{1, \ldots, N\}\); define the grading on the latter by that on the former. The manifold \(M\) is filtered, as a topological space, by subspaces \(\mathcal{D} = M^{r_{0}} \subset M^{r_{1}} \subset \ldots \subset M^{r_{N}} = M\).

Moreover, for each \(s \in \{1, \ldots, N\}\) the space \(M^{r_{s}}\) is homotopy equivalent to \(M^{r_{s-1}}\) with a single cell of dimension \(\deg s\) attached. Since \(f\) is oriented, the one-dimensional vector space \(H_{\text{deg}(s)}(M^{r_{s}}, M^{r_{s-1}}) \simeq H_{\text{deg}(s)}(M^{\langle r_{s} \rangle}, M^{\langle r_{s-1} \rangle}; \mathbb{F}) \simeq \mathbb{F}\) has a preferred basis \(o \otimes 1\), where \(o\) is a generator of \(H_{\text{deg}(s)}(M^{r_{s}}, M^{r_{s-1}}; \mathbb{Z}) \simeq \mathbb{Z}\). Therefore, the complex of singular chains on \(M\) (with coefficients in \(\mathbb{F}\)) satisfies conditions of Construction 2.6 and we are able to extract a B-data as in Subsection 2.3 (see Remark 2.10).

\(\triangleright\)

In particular, we have just constructed enhancement on a homology vector space \(H_{s}(M)\) as well as on \(H_{s}(M^{r_{s}})\) for all \(s\). We call the image of \(\lambda\) “Bruhat numbers” of oriented Morse function; the same goes for the Barannikov pairs (or, briefly, pairs). Informally, Construction 3.1 decomposes
some critical points (equivalently, values) of \( f \) into Barannikov pairs. Moreover, it associates a Bruhat number (i.e., an element of \( \mathbb{F} \)) with each pair (see the beginning of the Subsection 2.3). Points \( c_s \in \text{Cr}(f) \) s.t. \( s \in \mathcal{H} \) are called homological critical points. The number of homological points of index \( k \) equals to \( \dim H_k(M) \) (see Subsection 2.3). Analogously, points from \( U \) (resp. \( L \)) are called upper (resp. lower). We stress out that we haven’t yet considered any finite-dimensional approximation of filtered complex of singular chains on \( M \); we will do so in Subsections 4.3 and 4.5.

**Remark 3.2.** Changing the orientation of some critical point \( c_s \in \text{Cr}(f) \) alters the B-data as follows. First, the decomposition into pairs stays the same. Second, if \( c_s \) is homological, then the whole B-data stays the same. Otherwise, if \( c_s \) belongs to some pair, then the Bruhat number on this pair gets multiplied by \(-1\). Therefore, canonically we can associate Bruhat numbers to pairs only up to a sign.

**Remark 3.3.** The original Barannikov’s construction [Bar94] produces the same set of pairs. To see this, one should combine Remark 2.17 and results from Subsection 3.5. See also [EH08] for a topological data analysis perspective.

We conclude this subsection by several remarks. The construction of homological critical points goes back to Lyusternik and Shnirelman. Note that this construction implies weak Morse inequalities: \( \# \text{Cr}(f) \geq \dim H_0(M; \mathbb{F}) \) for any field \( \mathbb{F} \). See also [Vit92] for the innovative fruitful applications of similar ideas in symplectic topology.

**Remark 2.13** translates to topological setting as follows. The condition for two critical values \( a \) and \( b \) to form a Barannikov pair is equivalent to

\[
\dim H_0(M^{a-e}, M^{b+e}) = \dim H_0(M^{a+e}, M^{b-e}) = \dim H_0(M^{a-e}, M^{b-e}) - 1 = \dim H_0(M^{a+e}, M^{b+e}) - 1.
\]

See [CP05] and also [LNV13].

### 3.3. A digression on oriented Morse functions

In this subsection, we give an alternative definition of an oriented Morse function (see Subsection 3.1).

Let \( Q \) be a quadratic form on a vector space \( V \) over \( \mathbb{R} \). Consider a vector subspace \( L \subset V \) of maximal dimension (this dimension is called negative inertia index of \( Q \)) such that the restriction \( Q_{|L} \) is negative-definite. It’s easy to see that the space \( \{L\} \) of all such vector subspaces is contractible. Therefore, since any covering of a contractible space is trivial, the space of all such oriented subspaces has two contractible components. Note that a particular choice of a component determines an orientation of \( L \) (as a vector space).

Now, given a Morse function \( f \) and any its critical point \( p \), one may consider the vector space \( T_pM \) and a Hessian \( \text{Hess}_p(f) \) on it. Function \( f \) is called oriented if, for any critical \( p \) one of two mentioned components is chosen. One may check that this definition coincides with the one given in Subsection 3.4.

### 3.4. CW-complex associated with a strong Morse function

In this subsection, we briefly recall the classical results from Morse theory, following [Mil63], in order to fix the notations needed further in Subsection 3.5.

**Theorem 3.4.** Let \( f \) be an oriented strong Morse function on a closed manifold \( M \). Then there exists a CW-complex \( K \) s.t. the following holds:

1. \( M \) is simple homotopy equivalent to \( K \),
2. cells of \( K \) are in bijection with critical points of \( f \). Moreover, dimension of a cell equals to the index of a critical point.

**Remark 3.5.** The CW-complex \( K \) need not be unique (see Construction 3.11), but its simple homotopy type obviously is. Roughly speaking, the purpose of Subsection 3.5 is to encode the information which can be extracted uniquely from \( f \) in algebraic terms. Note that Morse theory originated before CW-complexes were invented, see [MB34].

**Remark 3.6.** Orientations of cells in \( K \) may be naturally chosen by invoking orientation of \( f \), see Construction 3.5.

**Remark 3.7.** The fact that the mentioned homotopy equivalence (as well as all the others in this subsection) is actually simple is folklore. Although not stated explicitly in [Mil63], it follows readily from the proof given there. We will need this fact in Section 3 for statements involving (Reidemeister) torsion. We will denote general homotopy equivalence by \( \simeq \) and, whenever we want to emphasize that it is simple, we write \( \simeq_* \).

The key ingredient in the proof of Theorem 3.4 is the following construction (we continue using notations introduced in Subsection 3.1). For a topological space \( X \), by \( X \cup_\varphi e^k \) we mean \( X \) with a \( k \)-cell attached along \( \varphi \).
Construction 3.8. For \( s \in \{1, \ldots, \# \mathrm{Cr}(f)\} \) let \( r_s \) and \( r_{s-1} \) be two corresponding regular values of \( f \) and let \( k = \deg s \). We will now recall the construction of a continuous map \( \varphi : S^{k-1} \to M^{r_{s-1}} \) s.t. \( M^{r_s} \cong M^{r_{s-1}} \cup e^k \). NB: this construction involves some choices.

Remark 3.9. We stress out that it’s not claimed that the homotopy class of a map \( \varphi \) satisfying the property that \( M^{r_s} \cong M^{r_{s-1}} \cup e^k \) is unique. This assertion is only true for the map \( \varphi \) constructed by the recipe given below.

\( \triangleright \) We will only sketch the argument, for details see [Mil63]. The space \( M^{r_{s-1}} \) is a smooth manifold with boundary \( f^{-1}(r_{s-1}) \). Choose an antigradient-like vector field on \( M \). Its flow produces a smooth map \( S^{k-1} \to f^{-1}(r_{s-1}) \subset M^{r_s} \), where the source is viewed as a small sphere around critical point \( e_s \). For any gradient-like vector field the resulting map will differ by an isotopy. This way one gets an embedding \( M^{r_{s-1}} \cup e^k \to M^{r_s} \). It is then shown to be a simple homotopy equivalence.

Note that the sphere \( S^{k-1} \) is naturally oriented since any maximal vector subspace in \( T_e M \) on which Hessian restricts as a negative-definite form is oriented (see Subsection 3.3). Therefore the cell \( e^k \) is naturally oriented too.

The next proposition is obvious.

Proposition 3.10. Although the map \( \varphi \) from Construction 3.8 depends on some choices made along the way, its homotopy class is uniquely defined.

In order to get to Theorem 3.4 one then proceeds with

Construction 3.11. Let \( s, r_s, r_{s-1} \) and \( k \) be as in Construction 3.8. Suppose that \( M^{r_{s-1}} \cong K \), where \( K \) is some CW-complex. We will now recall the construction of a cellular map \( \psi : S^{k-1} \to K \) s.t. \( M^{r_s} \cong K \cup e^k \) (note that r.h.s. is again a CW-complex). NB: this construction involves some choices.

\( \triangleright \) Apply cellular approximation theorem to the map \( \varphi \) from Construction 3.8

Again, the next proposition is obvious.

Proposition 3.12. Although the map \( \varphi \) from Construction 3.11 depends on some choices made along the way, its homotopy class is uniquely defined.

3.5. Enhanced complex associated with a strong Morse function.

In this subsection we discuss the following two statements.

Construction 3.13. Let \( f \) be an oriented strong Morse function on \( M \) and \( \mathbb{F} \) be a field. We will now construct an enhanced complex \((\mathcal{C}, \varpi)\). NB: this construction involves some choices.

Proposition 3.14. Although Construction 3.13 depends on some choices made along the way, the isomorphism class of an enhanced complex \((\mathcal{C}, \varpi)\) is uniquely defined.

We will first describe Construction 3.13.

\( \triangleright \) Take any CW-complex \( K \) constructed by the virtue of Theorem 3.4. Its cells are naturally linearly ordered by the order of critical values of \( f \). Moreover, the first \( s \) cells form a CW-subcomplex, simple homotopy equivalent to \( M^{r_s} \). Consider an algebraic complex \( \mathcal{C} \) (of free abelian groups) associated with \( K \). It has a preferred (ordered) basis \( c \) given by cells (mind that they carry natural orientation since \( f \) is oriented, see Construction 3.8). Thus one has a \( \mathbb{Z} \)-enhanced complex \((\mathcal{C}, \varpi(c))\).

The desired enhanced complex is now taken to be \((\mathcal{C} \otimes \mathbb{F}, \varpi(c))\) (see Subsections 2.3 and 2.5).

\( \langle \rangle \)

Remark 3.15. Construction 3.13 produces not only an enhanced complex, but also its basis. The matrix of differential, however, does depend on the choices made (practically, a cellular approximation from Construction 3.11). From this viewpoint, Proposition 3.14 says that for any two choices, the corresponding matrices of differential are conjugate by an unitriangular (i.e. triangular with ones on the diagonal) base change. This can be pushed to the full proof, see Remark 3.17.

Proof of Proposition 3.14. By the classificational Theorem 2.16 it suffices to prove that B-data associated to \((\mathcal{C}, \varpi)\) is uniquely defined (see Remark 2.17). We will do that by identifying it with a B-data constructed from \( f \) in Subsection 3.2. We denote the complex of singular chains (over \( \mathbb{F} \)) mentioned there by \( \mathcal{C}_{\text{sing}} \) and by \( \mathcal{C}_{\text{sing}}^* \) we mean the subcomplex corresponding to subspace \( M^{r_s} \). Recall that since \( f \) is oriented the generator of \( H_{\text{deg}s}(\mathcal{C}_{\text{sing}}, \mathcal{C}_{\text{sing}}^{0}) \approx \mathbb{F} \) is chosen (see Subsection 3.2).

First of all, the two gradings on the set \( \{1, \ldots, \# \mathrm{Cr}(f)\} \) coincide since they are defined in terms of indices of critical points of \( f \). Next, for each \( s \) homology vector spaces \( H_0(\mathcal{C}^*) \) and \( H_0(\mathcal{C}_{\text{sing}}^*) \) are naturally isomorphic (for all \( k \)) since both of them compute \( H_0(\mathcal{C}^{r_k}) \). Moreover, their filtrations are the same (w.r.t. the mentioned isomorphism) since they are defined topologically in terms of inclusions \( M^{r_k} \hookrightarrow M^{r_s} \) for various \( k \). Finally, the chosen generators in \( H_{\text{deg}s}(\mathcal{C}_{\text{sing}}, \mathcal{C}_{\text{sing}}^{0}) \) and in
Remark 3.16. It follows from Subsection 3.4 that B-data associated \((C, \infty)\) coincides with the one constructed in Subsection 3.2. It follows from Remark 1.6 that it is enough to consider only fields \(Q\) and \(F_p\) (note that this is not the case in Subsections 1.2 and 1.3).

Remark 3.17. It’s possible to prove Proposition 3.14 somewhat more explicitly without appealing to Theorems 2.10. We will now sketch the argument. Consider two approximations \(\psi_1\) and \(\psi_2\) from Construction 3.11. Since they are homotopic, corresponding algebraic complexes associated to CW-complexes differ by a change of basis. This change of basis is unimodular by construction. Arguing inductively on the number of cells, one obtains a unimodular change of basis which turns one chain complex into another. This means precisely that two corresponding enhanced complexes are isomorphic.

Remark 3.18. B-data stays unaltered if one replaces \(f\) with \(\varphi \circ f \circ \psi\), where \(\psi : M \to M\) is any diffeomorphism and \(\varphi : \mathbb{R} \to \mathbb{R}\) is a diffeomorphism preserving an orientation. Consequently, B-data stays the same under the continuous deformation of \(f\) in the class of oriented strong Morse functions on \(M\) (see Subsection 5.1).

3.6. Invariant of a map between two manifolds equipped with Morse functions. In this subsection we discuss the following construction.

Construction 3.19. Let \(M_1\) and \(M_2\) be two manifolds equipped with oriented strong Morse functions \(f_1\) and \(f_2\) respectively. Let also \(l : M_1 \to M_2\) be a continuous map. We will now construct a rook matrix \(R_k\) of size \(\dim H_k(M_2) \times \dim H_k(M_1)\) (for any \(k\)).

By Subsection 3.2 the function \(f_1\) (resp. \(f_2\)) gives rise to an enhancement on a homology vector space \(H_k(M_2)\) (resp. \(H_k(M_2)\)). Consider the induced map \(l^* : H_k(M_1) \to H_k(M_2)\) and plug it into Lemma 1.5 to get the desired rook matrix.

Without choosing a particular orientations of strong Morse functions the non-zero entries of \(R_k\) are defined only up to a sign. Note that we didn’t make use of any complexes whatsoever.

3.7. Morse complex. In this subsection we describe how Morse complex fits into our setting of enhanced complexes. In particular, we give a certain description of a matrix of Morse differential (w.r.t. any Riemannian metric) in terms of B-data.

Let \(g\) be a generic Riemannian metric on \(M\) and \(f\) be an oriented strong Morse function. Then one can define a Morse complex \(\mathcal{M}(f, g)\) whose integral homology is naturally isomorphic to that of \(M\). It’s a complex of free abelian groups, formally generated by critical points. In this basis the differential (mapping \(k\)-chains to \((k-1)\)-chains) becomes a matrix \((\partial_{ij})\). The matrix element \(\partial_{ij}\) equals to the number of antigradient flowlines from \(j^{th}\) critical point of index \(k\) to \(i^{th}\) critical point of index \(k - 1\), counted with appropriate signs. For brevity, we say that \(i^{th}\) critical point “appears in the differential” of \(j^{th}\) critical point with coefficient \(\partial_{ij}\).

Remark 3.20. As we saw in Subsection 3.4 in order to construct a complex generated by critical points out of a function \(f\) one has to make a choice of cellular approximation. On the other hand, in order to construct a Morse complex one has to choose a Riemannian metric. These two choices are actually very close to each other in the following sense.

Given a function \(f\) one can construct a handle decomposition s.t. each handle of index \(k\) corresponds to some critical point of index \(k\). The metric \(g\) specifies the way to modify this decomposition, such that each handle of index \(k\) is attached to the union of handles of smaller index. This, in turn, allows one define the handle complex, which is precisely the Morse complex. On the other hand, one can collapse handles to cells and obtain a CW-complex.

In our terms, one has obtained a \(Z\)-enhanced complex \((\mathcal{M}(f, g), \varpi(c))\), where \(c\) is a basis consisting of critical points (see Subsection 2.5). It follows directly that B-data of \((\mathcal{M}(f, g) \otimes \mathbb{F}, \varpi(c))\) coincides with that of \(f\) constructed in Subsection 3.2. Therefore, by Theorem 2.16 enhanced complex \((\mathcal{M}(f, g) \otimes \mathbb{F}, \varpi(c))\) is isomorphic to \((C, \infty)\) from Subsection 3.5. So, practically, one may use any option to extract B-data from a given \(f\): either a complex of singular chains, or a CW-complex or a Morse complex.

Remark 3.21. Analogously to Remark 3.15 the matrix of Morse differential does depend on Riemannian metric, while the isomorphism class of enhanced complex \((\mathcal{M}(f, g) \otimes \mathbb{F}, \varpi(c))\) doesn’t. The same goes for \(Z\)-enhanced complex \((\mathcal{M}(f, g), \varpi(c))\).

For a B-data associated to \(f\) let \(R_k\) be the corresponding rook matrix of size \(\text{Cr}_{k-1}(f) \times \text{Cr}_k(f)\) (see Subsection 2.3). In other words, non-zero elements of \(R_k\) equal to the Bruhat numbers on Bannikov pairs of points of degrees \(k\) and \(k - 1\). The next theorem follows readily from Proposition 2.21.
**Theorem 3.22.** Let $f$ be an oriented strong Morse function on a manifold $M$. Let also $R_k$ be the rook matrix associated to $f$ over $Q$ (for $k \in \{1, \ldots, \dim M\}$). Then the matrix of Morse differential $\partial_k$ w.r.t. any Riemannian metric $g$ belongs to the set $T(R_k)$.

For example suppose that $f$ has a B-data as depicted in Figure 1 and $k = 2$. Then the corresponding rook matrix and general form of a matrix of a second Morse differential $P$ are

$$R_2 = \begin{pmatrix} 0 & 0 & 4 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \\ 0 & 2 & * \end{pmatrix}.$$

Recall that weak Morse inequalities state that $\# \text{Cr}_k(f) \geq \dim H_k(M; F)$ (for any field $F$). It is easy to see that if $\# \text{Cr}_k(f) = \sum_k \dim H_k(M; Q)$ then the Morse differential (w.r.t. any metric) must be identically zero. The next corollary is applicable when this is not the case.

**Corollary 0.1.** Let $f$ be an oriented strong Morse function on $M$. Suppose that $\# \text{Cr}_k(f) > \sum_k \dim H_k(M; Q)$. Then one can find two critical points $x$ and $y$ of neighboring indices s.t. the number of antigradient flowlines from $x$ to $y$, counted with appropriate signs, is the same for any Riemannian metric. This number is non-zero and equals to some Bruhat number of $f$ over $Q$.

**Proof.** By assumption there is at least one Bannikov pair of $f$ over $Q$. Take any short one. The statement now follows from Corollary 2.23.

**Remark 3.23.** The first part of Corollary 0.1 can be proven without appealing to any Bannikov pairs and Bruhat numbers whatsoever. Indeed, if one unwraps all the definitions and constructions involved, they arrive at a proof which is based on techniques like exact sequences of a pair.

### 3.8. A few examples and properties.

In this subsection we quickly give several introductory examples and properties of Bruhat numbers.

1. Let $f$ be a function on $\mathbb{R}^{2n}$ which is a decent of $x_1^2 + 2x_2^2 + \ldots + (n + 1)x_{n+1}^2$ defined on a unit sphere $S^{2n} \subset \mathbb{R}^{n+1}$. It has $(n + 1)$ critical points of all possible indices from 0 to $n$ (ordered by increasing of index). If $\text{char} \mathbb{F} = 2$ then all of them are homological. Otherwise, $(2k)^{\text{th}}$ and $(2k - 1)^{\text{th}}$ critical points form a Bannikov pair with Bruhat number $\pm 2$ (for any $k \in \{1, \ldots, [n/2]\}$, where brackets denote the integral part). This is seen readily from any description of B-data, given either in Subsection 3.2 or in Subsection 3.3. See Figure 2 for an example for $n = 6$.

2. The proof of the next statement uses one-parameter Morse theory and therefore postponed till Subsection 5.2.

**Proposition 3.24.** Let $F$ be either $\mathbb{Q}$ or $\mathbb{F}_p$ and $\lambda \in \mathbb{F}^*$ be any non-zero number. Let also $M$ be any closed manifold s.t. $\dim M \geq 4$. Then one can find an oriented strong Morse function $f$ on $M$ which has $\lambda$ as one of its Bruhat numbers.

In particular, Bruhat number over $F = Q$ may well be non-integer.

3. Recall that a Bannikov pair is called short if there are no pairs of the same degree lying inside it, see Subsection 2.5. The next statement follows directly from Corollary 2.22.

**Proposition 3.25.** Over $F = Q$ Bruhat number on any short pair is integer.

In particular, if there are Bannikov pairs over $Q$ at all, at least one of them must carry integer Bruhat number.

4. The first statement from Subsection 2.6 translates straightforwardly to the topological setting via Subsection 3.5. In this setting it expresses Bruhat numbers of oriented strong Morse function over $Q$ in terms of torsion in (relative) integral homology of various sublevel sets.

### 3.9. Poincare duality.

In this subsection prove the following

**Proposition 3.26.** Let $M$ be closed and orientable and $f$ be an oriented strong Morse function on it. Let also $F$ be a field. Then B-data for $-f$ is B-data for $f$ turned upside down. Bruhat numbers on pairs remain the same.

We need to first make some comments on the formulation. Since $M$ is orientable and $f$ is oriented, the strong Morse function $-f$ is also naturally oriented. Each critical point of $f$ of index $k$ is also a critical point of $-f$ of index $n - k$, where $n = \dim M$. By turning the B-data upside down we mean, formally speaking, precomposing all its ingredients with the automorphism of the set $\{1, \ldots, \# \text{Cr}(f)\}$ given by $s \mapsto \# \text{Cr}(f) - s$. Under this operation upper and lower critical
points swap their roles (while the pairing remains the same). Note that the classical Poincare duality over the field \( \dim H_k(M; \mathbb{F}) = \dim H_{n-k}(M; \mathbb{F}) \) follows immediately from Proposition 3.26.

Indeed, homological points remain homological after the involution, but their indices change to complementary ones. The proof, however, goes along the classical lines.

**Proof of Proposition 3.26.** Choose a generic Riemannian metric \( g \) on \( M \). Matrix of differential in a Morse complex \( M(-f, g) \) is obtained from that in \( M(f, g) \) by transposition. Therefore, the same goes for the matrices of differentials in the two corresponding based enhanced complexes. Consider now any unitriangular matrix \( P \) which maps the basis \( c \) (of critical points) of the first complex to a Brannikov basis. If follows from the formula \((P_{k-1}D_kP_k)^T = P_k^T D_k^T P_{k-1}^T\) that transposed matrix \( P_k^T \) (which is unitriangular w.r.t. reversed order) maps the initial basis of the second complex to the Brannikov one. Moreover, the matrices of differentials after these two changes of bases still differ by a transposition. The statement follows. \( \square \)

3.10. On pairs of extremal degrees. Recall that by a degree of a pair we mean degree of its lower point. In this subsection we prove

**Proposition 3.27.** Let \( f \) be a strong Morse function on \( M \) and \( F \) be a field.

1) The set of pairs of degree \( 0 \) is independent of \( F \). Bruhat number on any such pair is \( \pm 1 \).

2) Suppose that \( M \) is orientable. Then the set of pairs of degree \( \dim M - 1 \) is again independent of \( F \). Bruhat number on any such pair is again \( \pm 1 \).

**Proof.** Fix any \( s \in \{1, \ldots, \# \text{Cr}(f)\} \) s.t. \( c_s \in U \) (i.e. \( c_s \) is an upper point in a pair) and \( \deg c_s = 1 \). We need to prove that \( \lambda(s) = \pm 1 \). Recall that \( \{r_s\} \) is a set of regular values of \( f \) and let \( d = \dim H_{\lambda^{-1}}(M^{\lambda^{-1}}) \). Let also \( i_1 < \ldots < i_d \) be the ordered sequence of numbers which comprise the set \( \{t \in H^{-1} \text{deg } t = 0\} \), where \( H^{-1} \) is the set \( H \) from the Brannikov data for the function \( f|_{M^{\lambda^{-1}}} \). In other words, \( \{c_{i_1}, \ldots, c_{i_d}\} \) is the ordered set of homological points of degree zero for \( f|_{M^{\lambda^{-1}}} \). It follows from Construction 2.6 that the sequence \( ([c_{i_1}, \ldots, c_{i_d}]) \) is a basis (up to signs, which are irrelevant to the statement) of an enhanced vector space \( H_0(M^{\lambda^{-1}}) \) (square brackets denote taking the homology class).

On the other hand, the connecting homomorphism \( \delta : H_0(M^{\lambda^{-1}}) \to H_0(M^{\lambda^{-1}}) \) maps the chosen generator, represented by an oriented segment, to \([a] - [b] \), where \( a \) and \( b \) are the endpoints. The first statement now follows. The second one is obtained from it via the Poincare duality (Proposition 3.26). \( \square \)

**Remark 3.28.** Orientability assumption in the second statement of Proposition 3.27 is crucial. Indeed, the conclusion fails already for \( M = \mathbb{RP}^2 \) and \( F = \mathbb{Q} \). See Subsection 3.8.

**Remark 3.29.** Since the set of pairs from Proposition 3.27 is independent of the field \( F \) one is tempted to find an alternative definition which doesn’t involve \( F \). Indeed, it can be proven that the mentioned set may be recovered from the Kronrod-Reeb graph \( \mathbb{AK}45 \mathbb{AK}46 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathbb{BGW}14 \mathb
By results from Subsection 3.4 starting from \( f \) one can construct a (non-unique) CW-complex \( X \) which is simple-homotopy equivalent to \( M \). So, from the viewpoint of torsion theory \( X \) and \( M \) are the same (see Subsection 4.2). Next, in Subsection 3.5 we used \( X \) to construct a (unique up to isomorphism) enhanced complex \((C, x)\). Further, in Subsection 2.8 we studied the torsion (defined in a classical way) of any enhanced complex. By the very definitions it is a topological torsion of \( X \) (and, therefore, \( M \)) w.r.t. to a certain basis in homology (see Remark 4.2). Finally, Proposition 2.33 identified the torsion (defined in a classical way) with the formula from Definition 4.1.

**Remark 4.2.** Here is a more concise way of saying the same. Take enhancement on a homology \( H_\bullet(X) \) given by \( f \). Take any basis of this enhancement. Any two such bases differ by a unitriangular matrix. Therefore, they are equivalent (in the sense of Subsection 2.8). Thus the torsion of \( M \) w.r.t. any chosen basis of enhancement on \( H_\bullet(M) \) is the same. This is precisely \( \tau \).

**Remark 4.3.** In this section we will actually be interested only in the number \( \pm \tau \in \mathbb{F}^*/\pm 1 \). So the reader may temporarily disregard the permutation \( \sigma \) and orientation of \( f \). The sign will be important later in Section 5.

**Theorem 0.2.** Let \( f \) be a strong Morse function on \( M \) and \( \mathbb{F} \) be a field. Suppose that \( H_n(M) = 0 \) for all \( 0 < k < \dim M \). Then the alternating product of all Bruhat numbers (as an element from \( \mathbb{F}^*/\pm 1 \)) is independent of \( f \).

The proof requires bifurcation analysis and therefore postponed till Subsection 5.5. For example, taking \( M \) to be \( \mathbb{RP}^n \) one sees that \( \tau(f, \mathbb{Q}) = \pm 2^{n/2} \), where brackets denote integral part. Indeed, one has to calculate such a \( \tau \) for some particular Morse function on \( \mathbb{RP}^n \). They do so for a standard one from Subsection 5.8.

**Remark 4.4.** Since in Theorem 0.2 the number \( \pm \tau(f, \mathbb{F}) \) turns out to be independent of \( f \) one is tempted to give an alternative definition which doesn’t involve \( f \). We will now sketch the construction. Recall that the torsion is defined whenever there is a chosen basis in homology (or, at least, an equivalence class of such a basis). Usually in topology there’s no canonical choice of such a basis, so one studies torsion in the setting where there’s no homology at all. However, in \( H_0(M) \) and \( H_{\dim M}(M) \) there is an obvious distinguished basis. The number \( \pm \tau(f, \mathbb{F}) \) is the torsion w.r.t. to it.

4.2. Reidemeister torsion: recallment. In this subsection we briefly recall the notion of Reidemeister torsion (see [Tur01] for details).

For a topological space \( X \) let \( \pi \) be its fundamental group and let \( \tilde{X} \to X \) be a universal covering. Choose a CW-decomposition of \( X \) and denote the corresponding algebraic complex of free abelian groups by \( C \). Consider the lift of the CW-structure to \( \tilde{X} \). By the virtue of an action of \( \pi \) on its cells one construct a complex of free (right) \( \mathbb{Z}[\pi] \)-modules \( \tilde{C} \) (recall that \( \mathbb{Z}[\pi] \) is an integral group ring of \( \pi \)). The rank of \( k \)-chains still equals to the number of \( k \)-cells in \( X \). Moreover, one can choose a basis in \( \tilde{C} \) by choosing any lift of each cell. Therefore, each element of this basis is defined up to multiplication by elements of \( \pi \).

Let now \( \rho : \mathbb{Z}[\pi] \to \mathbb{F} \) be a ring homomorphism. This gives \( \mathbb{F} \) the structure of a left \( \mathbb{Z}[\pi] \)-module by the formula \( r \cdot x = \rho(r)x \), where \( r \in \mathbb{Z}[\pi] \), \( x \in \mathbb{F} \). On the other hand, \( \mathbb{F} \) is also a right module over itself. Putting all this together one now considers \( \tilde{C} \otimes_{\mathbb{Z}[\pi]} \mathbb{F} \), which is a complex of vector spaces over \( \mathbb{F} \). Moreover, this complex carries a basis, each element of which is defined up to multiplication by elements of \( \rho(\pi) \) (which is a multiplicative subgroup of \( \mathbb{F}^* \)). The homology of \( \tilde{C} \otimes_{\mathbb{Z}[\pi]} \mathbb{F} \) is called \( \rho \)-twisted homology of \( X \) and denoted here by \( H_\bullet(X ; \rho) \). If one is given an equivalence class of its basis (in a sense of Subsection 2.8), then they can consider torsion \( \tau(X, \rho) \) of the complex of twisted chains, which lives in \( \mathbb{F}^*/\rho(\pm 1) \). The sign ambiguity is due to several things: 1) there is no canonical orientation of cells of \( X \), 2) there is also no canonical linear order on them, 3) there is no canonical CW-decomposition after all. If twisted homology vanishes, \( \tau(X, \rho) \) is called the Reidemeister torsion of \( X \) w.r.t. \( \rho \). It is known to be independent of CW-decomposition and to be stable under simple homotopy equivalences. The theorem of Chapman [Cha74] states that homeomorphism is a simple homotopy equivalence.

**Remark 4.5.** Practically, passing from \( \tilde{C} \) to \( \tilde{C} \otimes_{\mathbb{Z}[\pi]} \mathbb{F} \) means the following. Write the differential of \( \tilde{C} \) in some basis (for example, in the basis of lifts of cells) to get a matrix with coefficients in \( \mathbb{Z}[\pi] \). Replace each coefficient \( x \) with \( \rho(x) \) and consider the resulting matrix as a map between vector spaces over \( \mathbb{F} \).

**Remark 4.6.** Traditionally torsion theory is said to be born in the beginning of 20th century by the virtue of works of Reidemeister and Franz. We note that, quite surprisingly, the work of Cayley [Cay48] contains some of the first torsion-theoretic ideas while staying in the purely algebraic setting.
4.3. Reidemeister torsion and Bruhat numbers. In this subsection we introduce the notion of a twisted $B$-data and show that the alternating product of its Bruhat numbers equals to the Reidemeister torsion, whenever the latter is defined (see Theorem 0.3). In particular, this product is independent of a function.

Let $G$ be a multiplicative subgroup of $\mathbb{F}^*$ and $V$ be a vector space over $\mathbb{F}$. By $V/G$ we will denote a set, which is a quotient of $V$ by the natural action of $G$.

Definition 4.7. An enhancement up to $G$ on a vector space $V$ is a choice of two structures:

1) a full flag on $V$, i.e. a sequence of subspaces $\emptyset = V^0 \subset V^1 \subset \ldots \subset V^{\dim V} = V$ s.t.
   \[
   \dim(V^i/V^{i-1}) = 1, \ s \in \{1, \ldots, \dim V\};
   \]
2) a non-zero element $\pi_s$ in a quotient set $(V^s/V^{s-1})/G$, $s \in \{1, \ldots, \dim V\}$.

Enhancement up to $G$ is still denoted by $(V, \pi)$.

Definitions of isomorphism between two such vector spaces as well of the complex enhanced up to $G$ go exactly in the same manner as in the usual case. Moreover, all the major statements from Sections 1 and 2 translate readily to this new setting, with the only following exception. The non-zero elements of rook matrix from Section 1 are only defined up to multiplication by elements from $G$. Consequently, Bruhat numbers of a complex enhanced up to $G$ now live in the quotient set $\mathbb{F}^*/G$.

If $f$ is a strong Morse function and $\mathbb{F}$ is a field of characteristic not two, then $f$ defines a complex enhanced up to $\mathbb{Z}_2 = \{\pm 1\} \subset \mathbb{F}^*$ (see Remark 3.2).

Construction 4.8. Let $f$ be an oriented strong Morse function on a manifold $M$, $\mathbb{F}$ be a field and $\rho: \mathbb{Z}[\pi] \to \mathbb{F}$ be a homomorphism of rings. We will now construct an isomorphism class of a complex enhanced up to $\rho(\pi)$.

$\triangleright$ Apply Construction 3.11 to $f$ to get a CW-complex $X$, simple homotopy equivalent to $M$. Lift the CW-structure to the universal cover $\tilde{X}$, choose any preimage of each cell and consider the corresponding algebraic complex $\tilde{C}$ of free $\mathbb{Z}[\pi]$-modules. Its basis is defined up to action of $\pi$ and naturally ordered. Moreover, the matrix of differential is upper triangular w.r.t. this order, since so is differential in $\tilde{C}$. In particular, the span of the first $s$ basis elements is a subcomplex (for $s \in \{1, \ldots, \# \text{Cr}(f)\}$).

Consider now the complex $\tilde{C} \otimes_{\mathbb{Z}} \mathbb{F}$. It inherits a linearly ordered basis which is defined up to an action of $\rho(\pi)$. The desired complex enhanced up to $\rho(\pi)$ is now taken to be the one associated with this basis (see the end of the Subsection 2.3).

It remains to prove that obtained complex is well-defined up to an isomorphism. Indeed, let $X'$ be another CW-complex obtained by the virtue of Construction 3.11. Matrices of cellular differentials (in any degree) in these two complexes are conjugated by a unitriangular matrix over $\mathbb{Z}$ (see Remark 3.17). Therefore, the matrices of differentials in $\tilde{C}$ and $\tilde{C}'$ are conjugated by a triangular matrix with elements from $\pi$ on the diagonal. The desired statement follows.

Consequently, given a data as in the Construction 4.8 one can construct Barannikov pairs and Bruhat numbers, which are elements of $\mathbb{F}^*/\rho(\pi)$ (without choosing a particular orientation of $f$ these numbers live in $\mathbb{F}^*/(\pm \rho)$). To emphasize the presence of $\rho$ we say “twisted Barannikov pairs” and “twisted Bruhat numbers”. One then defines torsion $\tau(f, \rho)$ of $f$ exactly as in Definition 1.1. Again, as in Subsection 4.2 Proposition 2.33 justifies the name. In short, the alternating product of twisted Bruhat numbers of $f$ equals to the torsion of $M$ w.r.t. to a certain basis of the vector space $H_\rho(M; \rho)$ defined by $f$. Generally, this basis and, consequently, $\tau(f, \rho)$ may well depend on $f$; this basis is even non-uniquely defined, but this arbitrariness doesn’t affect $\tau(f, \rho)$.

However, combining all of the above with Subsection 4.2 one gets

Theorem 0.3. Let $f$ be an oriented strong Morse function on a manifold $M$, $\mathbb{F}$ be a field and $\rho: \mathbb{Z}[\pi] \to \mathbb{F}$ be a homomorphism of rings. Suppose that twisted homology vanishes. Then the alternating product of twisted Bruhat numbers of $f$ equals to the Reidemeister torsion of $M$. In particular, it is independent of $f$.

5. One-parameter Morse theory

One-parameter Morse theory deals with generic paths (in other words, one-parameter families) in the space of all smooth functions on $M$. The endpoints of a generic path are strong Morse functions — this is essentially the statement that strong Morse functions form an open dense subspace. However, finitely many points of such a path may fail to be either strong or Morse functions. It is exactly at these points where the B-data associated with a strong Morse function changes. In this section we describe how exactly these changes look like (see Subsection 5.3). This allows to prove some statements from the previous sections (see Subsection 5.5). On the other
hand, this also enables us to reprove a theorem of Akhmetev-Cencelj-Repovs [ACR05] in greater
generality (see Subsection 5.6).

5.1. Generalities on one-parameter Morse theory. In this subsection we recall foundations of
one-parameter Morse theory, initiated by Cerf [Cer70].

A path in the space of functions on $M$ is practically a map $F: M \times [-1, 1] \to \mathbb{R}$. Let $t$ be a
coordinate along $[-1, 1]$, which we occasionally refer to as “time”. Define the function $f_1: M \to \mathbb{R}$
by $f_1(x) := F(x, t)$. For convenience we will write $\{f_1\}$ instead of $F$. By a point of a path $\{f_t\}$ we
will mean a function $f_{t_0}$ for some particular $t_0 \in [-1, 1]$. Fix a generic path $\{f_t\}$ once and for all
throughout this section (see [Cer70] for the precise definition of genericity). Its endpoints $f_{-1}$ and $f_1$
are strong Morse functions on $M$. Moreover, the same holds for all but finitely many points of
$\{f_t\}$. The rest of this subsection is devoted to describing what changes may occur to a function
at these points.

We first introduce a couple of definitions. One says that two strong Morse functions $f$ and $g$ are
isotopic if there exists a diffeomorphism $\phi: (M, f) \to (M, g)$ s.t.

$g = \phi \circ f \circ \psi$. Roughly, isotopic functions represent the same object from the viewpoint of Morse
theory (see Remark 3.18). Analogously, two paths $\{f_t\}$ and $\{g_t\}$ are said to be equivalent if there
exists an isotopy $\{\phi_t\}: \mathbb{R} \times [-1, 1] \to \mathbb{R}$ s.t.

$g_t = \phi_t \circ f_t \circ \psi_t$ ($\phi_0 = \text{id}$, $\psi_0 = \text{id}$).

It is a folklore result that if path $\{f_t\}$ consists only of strong Morse functions then it is equivalent
to a constant path. See [GWW] for a rigourous proof. The following description of changes of a
strong Morse function along a general generic path is to be understood as description of a certain
explicit representative in the equivalence class of a path in question.

We will depict paths of functions in the following manner. The Cerf diagram of a path $\{f_t\}$ is a
subset of $[-1, 1] \times \mathbb{R}$ consisting of points $(t, x)$ s.t. $x$ is a critical value of $f_t$. Topologically it is a
set of (possibly self-intersecting and non-closed) curves in the plane.

As proven in [Cer70] in a generic one-parameter family there are two possible changes of isotopy
class of a strong Morse function, which we call events. (Since there are only finitely many of them
anyway, we assume for convenience that $f_t$ is strong Morse for all $t$ except for a single value $t = 0$.)

1) The function $f_0$ is strong, but non-Morse. This case is given by the local formula

$f_1(x_1, \ldots, x_n) = x_1^3 \pm tx_1 + Q(x_2, \ldots, x_n),$

where $(x_1, \ldots, x_n)$ is some small coordinate neighborhood around the non-Morse critical point
of $f_0$ (outside of a bit bigger neighborhood the function doesn’t change at all) and $Q$ is a non-degenerate quadratic form. At the moment $t = 0$ the birth/death (depending on the sign) of two points of neighboring indices happens (the lower index is the index of $Q$). On a Cerf diagram this corresponds to a (left or right) cusp. This event is called
birth/death event.

2) The function $f_0$ is Morse, but not strong. This happens when two critical values collide. Outside of small neighborhoods around two corresponding critical points the function doesn’t change at all; moreover, critical points themselves don’t move along the path. On the Cerf diagram this corresponds to a transversal self-intersection; in a sense a pair of critical values is swapped. The space on non-strong functions is sometimes called Maxwell stratum. So, we call this event a Maxwell event.

Remark 5.1. Note that in both cases there are exactly two distinguished critical points: either
two newborn/about to die points or a couple with swapping critical values. We will refer to them as
“critical points involved in event”. They go one straight after another in the natural linear
order. All the other critical points of $f_{-1}$ and of $f_1$ are in natural bijection. We will always keep
this bijection in mind without mentioning it explicitly.

Now we may describe a Cerf diagram a bit more precisely: it is a set of plane arcs (smooth in
the interior) whose endpoints are either at cusps or has $t$ coordinate equal to $\pm 1$. These arcs don’t
have vertical tangencies and may self-intersect. To each arc an integer number is assigned, namely
the index of any critical point on it (mind that for a generic point in a path critical points are in
bijection with critical values). From the viewpoint of Theorem 5.4 each arc corresponds to a cell
in the CW-complex obtained via $f_1$. Birth/death event translates to birth/death of two cells of
neighboring dimensions s.t. one appears in the cellular differential of another one with coefficient
$\pm 1$. Note that this is exactly the building block of the simple homotopy equivalence.

5.2. Any Bruhat number is realizable. In this subsection we prove

Proposition 3.24. Let $F$ be either $\mathbb{Q}$ or $\mathbb{F}_p$ and $\lambda \in F^*$ be any non-zero number. Let also $M$ be
any closed manifold s.t. $\dim M \geq 4$. Then one can find an oriented strong Morse function $f$ on
$M$ which has $\lambda$ as one of its Bruhat numbers.

The plan would be to construct a generic path of functions on $M$ which starts with any strong
Morse function and ends with the one satisfying the desired property. The tools for constructing
such a path were essentially developed by Smale and restated in Morse-theoretical terms by Milnor in 
\cite{Mil65}, which is our main reference here. In the following three statements one is given a strong
Morse function \( f_{-1} \) on a manifold \( M \), equipped with a generic Riemannian metric \( g \). Recall from
Subsection 3.3 that \( g \) gives rise to a Morse complex, formally spanned by critical points. We write \( \partial_k \)
differential of \( f \), which maps \( k \)-chains to \((k-1)\)-chains. The following two operations alter
the function, but doesn’t change the metric.

**Proposition 5.2.** Given any \( k \in \{0, \ldots, \dim M - 1 \} \), one can find a generic path \( \{f_t\} \) which
contains a single event, namely a birth event. The indices of newborn points are \( k \) and \( k+1 \) and
they lie in the small neighborhood of any regular point of \( f_{-1} \) chosen in advance. Moreover, if \( c_k \)
and \( c_l \) are two critical points not involved in the event, then one appears in the Morse differential
of another with the same coefficient for \( f_{-1} \) and \( f_1 \).

**Proposition 5.3.** Suppose that \( c_k \) and \( c_{k+1} \) are two neighboring critical points of \( f_{-1} \) of the
same index \( k \). Then one can find a generic path \( \{f_t\} \) which contains a single event, namely a Maxwell
event; its swapping points are \( c_k \) and \( c_{k+1} \). Moreover, the matrix of Morse differential \( \partial_k \)
(resp. \( \partial_{k+1} \)) for \( f_1 \) equals to that for \( f_{-1} \) multiplied on the right (resp. left) by a transposition \( (s, s+1) \).

The next operation is called handle sliding. It alters the metric, but doesn’t change the function.

**Proposition 5.4.** Suppose that \( c_k \) and \( c_{k+1} \) are two neighboring critical points of \( f_{-1} \) of the
same index \( k \), \( k \in \{2, \ldots, \dim M - 2 \} \). Suppose also that points \( c_k \) and \( c_{k+1} \) lie in the same connected
component of \( f_{-1}^{-1}(\{c_k\} \pm \varepsilon_{f_{-1}} \{c_k\} \pm \varepsilon_{f_{-1}}) \). Then one can find a new metric \( g' \) s.t. new matrix
of Morse differential \( \partial_k \) (resp. \( \partial_{k+1} \)) equals to the old one \( \partial_k \) (resp. \( \partial_{k+1} \)) multiplied on the right
(resp. left) by the base change mapping \( c_k \) to \( c_k \pm c_{k+1} \).

**Proof of Proposition 3.24.** Take any oriented strong Morse function \( f_0 \) on \( M \) and any Riemannian
metric \( g_0 \). We now begin changing \( f_0 \) (in the small neighborhood of its regular point) and \( g_0 \).
After any birth event we orient two newborn points s.t. the coefficient of one in the differential of
another would be \( +1 \) (there are two such choices). First of all, fix any \( k \in \{2, \ldots, \dim M - 2 \} \) and
any \( l \in \mathbb{Z} \setminus 0 \) s.t. the class of \( l \) in \( F \) is \( \lambda \).

1. Apply Proposition 5.2 to introduce two critical points \( a \) and \( b \) of indices \( k \) and \( k-1 \)
respectively; call the result \( f_1 \).
2. Apply it once again to introduce two new points \( a \) and \( b \) again of indices \( a \) and \( b \) s.t. \( a \) and \( b \) satisfy hypothesis of Proposition 5.4. The submatrix
of Morse differential which takes into account only mentioned four points now looks like
\[
\begin{pmatrix}
p & 1 \\
1 & q
\end{pmatrix}
\]
for some integers \( p \) and \( q \). Although we don’t need it, it follows from the
equality \( \mathbb{H}_c(f_t^{-1}(\{a\} \pm \varepsilon, f_t^{-1}(b) \pm \varepsilon)); \mathbb{Z} \) = 0 that these numbers must satisfy
\( pq - 1 = \pm 1 \).
3. Apply Proposition 5.4 to \( a \) and \( c \) sufficiently many times so as to obtain the submatrix of
differential which looks like
\[
\begin{pmatrix}
p' & 1 \\
1 & l
\end{pmatrix}
\]
for some \( p' \in \mathbb{Z} \). Call the new metric \( g_1 \).
4. Apply Proposition 5.3 to \( a \) and \( c \) to finally get the submatrix
\[
\begin{pmatrix}
l & p'
\end{pmatrix}
\]
. Call the result \( f_2 \).

It then follows straightforwardly that \( a \) and \( b \) form a Baramnikov pair (i.e. the bifurcation
was non-trivial) with Bruhat number \( \lambda \). Note that points \( a \) and \( b \) also in pair and the

**Remark 5.5.** Note that after performing the set of operations from the proof of Proposition 3.24
the alternating product of Bruhat numbers doesn’t change (up to sign; see Subsection 4.2). It is not
surprising since indeed it sometimes doesn’t depend on the function at all. However, we will
now sketch the construction of a strong Morse function on \( \mathbb{C}P^2 \) s.t. it has only one Baramnikov
and the corresponding Bruhat number is \( \lambda \) for any \( \lambda \) from either \( \mathbb{Q} \) or \( \mathbb{R} \).

Consider first a standart strong Morse function on \( \mathbb{C}P^2 \) which has 3 critical points \( a, b \) and \( c \)
of (indices 0, 2 and 4 respectively). Apply Proposition 5.2 to introduce a pair of points \( d \) and \( e \)
of (indices 2 and 1) between (in the sense of natural linear order) \( a \) and \( b \). Apply Proposition 5.4
enough many times s.t. \( e \) would appear in the differential of \( b \) with coefficient \( l \), where \( |l| = \lambda \).
Finally, apply Proposition 5.3 to points \( b \) and \( d \) (the bifurcation will neccessarily be non-trivial).
The resulting function will have points \( b \) and \( e \) paired with Bruhat number \( \lambda \).

5.3. B-data in families. In this subsection we start describing how B-data behaves along the
generic path of functions. In Subsection 5.4 we finish this description.

First of all, we will orient all the functions in the path in the following way. Pick up a generic

point on some arc of the Cerf diagram. It corresponds to a critical point of some \( f_t \); orient it.
Extend this orientation by continuity to all the critical points lying the same arc (excluding the

---

**Proof of Proposition 3.24.** Take any oriented strong Morse function \( f_0 \) on \( M \) and any Riemannian
metric \( g_0 \). We now begin changing \( f_0 \) (in the small neighborhood of its regular point) and \( g_0 \).
After any birth event we orient two newborn points s.t. the coefficient of one in the differential of
another would be \( +1 \) (there are two such choices). First of all, fix any \( k \in \{2, \ldots, \dim M - 2 \} \) and
any \( l \in \mathbb{Z} \setminus 0 \) s.t. the class of \( l \) in \( F \) is \( \lambda \).

1. Apply Proposition 5.2 to introduce two critical points \( a \) and \( b \) of indices \( k \) and \( k-1 \)
respectively; call the result \( f_1 \).
2. Apply it once again to introduce two new points \( a \) and \( b \) again of indices \( a \) and \( b \) s.t. \( a \) and \( b \) satisfy hypothesis of Proposition 5.4. The submatrix
of Morse differential which takes into account only mentioned four points now looks like
\[
\begin{pmatrix}
p & 1 \\
1 & q
\end{pmatrix}
\]
for some integers \( p \) and \( q \). Although we don’t need it, it follows from the
equality \( \mathbb{H}_c(f_t^{-1}(\{a\} \pm \varepsilon, f_t^{-1}(b) \pm \varepsilon)); \mathbb{Z} \) = 0 that these numbers must satisfy
\( pq - 1 = \pm 1 \).
3. Apply Proposition 5.4 to \( a \) and \( c \) sufficiently many times so as to obtain the submatrix of
differential which looks like
\[
\begin{pmatrix}
p' & 1 \\
1 & l
\end{pmatrix}
\]
for some \( p' \in \mathbb{Z} \). Call the new metric \( g_1 \).
4. Apply Proposition 5.3 to \( a \) and \( c \) to finally get the submatrix
\[
\begin{pmatrix}
l & p'
\end{pmatrix}
\]
. Call the result \( f_2 \).

It then follows straightforwardly that \( a \) and \( b \) form a Baramnikov pair (i.e. the bifurcation
was non-trivial) with Bruhat number \( \lambda \). Note that points \( a \) and \( b \) also in pair and the
}
cusps). Apply this procedure to all the arcs. This recipe allows us to orient all the points in the path \( \{f_t\} \) by making only finite number of binary choices, namely \( 2^l \) where \( l \) is the number of arcs. We use the term “orientation of an arc” for short.

Recall that we have to fix a field \( \mathbb{F} \) in order to define B-data. Next, if the path \( \{f_t\} \) consists of only strong Morse functions, then this data stays the same for all the time, see Remark 3.18 and Subsection 5.1.

We use the term “bifurcations” for the description of the way B-data changes after two events from Subsection 5.1. Disregarding the Bruhat numbers, this description was presented already in [Bar94] (see [Lau15] for a different proof). See also the paper [CEM06] and pictures in the survey [EH08]. Thus our job is to determine how Bruhat numbers change along the way (see Remark 3.3). In the case of birth/death event we restrict ourselves to birth one for brevity (death one is obtained from birth one by reversing the time).

Before turning to formal statements, let us describe the basic general properties of bifurcations. Recall that in the path \( \{f_t\} \) all functions are strong Morse except for \( f_0 \). Consider the following subset of critical points of \( f_1 \): 1) two points involved in event (i.e. either two newborn points or a couple with swapping critical values, see Remark 5.1). 2) the points paired with those, if any. We will refer to points from this subset as ones “involved in the bifurcation”. The reason behind this name is that all the other points retain their original pairs (in a sense of bijection from Remark 5.1). Moreover, the Bruhat numbers on these pairs remain the same. Thanks to all these facts, we are able to use pictorial format for describing the bifurcations. Namely, we will only depict points which are involved in the bifurcation (there are at most four of them).

**Proposition 5.6.** After the birth event a Barannikov pair of two newborn critical points appears; its Bruhat number is \( \pm 1 \). All the other pairs and Bruhat numbers remain unaltered. See Figure 3.

**Proof.** Recall that by the preceding discussion in the present subsection it suffices to track down only the Bruhat numbers. The first statement follows from the description given in Subsection 5.1.

Note that \( \# \text{Cr}(f_1) = \# \text{Cr}(f_{-1}) + 2 \); denote the non-Morse critical point of \( f_0 \) by \( p \). Recall that outside of a small neighborhood of \( p \) the function \( f_1 \) doesn’t change along the path. Thus all critical points of \( f_{-1} \) are also critical for \( f_1 \). Let \( c_{s+1} \) and \( c_s \) be two newborn critical points of \( f_1 \). Let \( r_0 < \ldots < r_{\# \text{Cr}(f_1)} \) be regular values of \( f_1 \). The sequence \( r_0 < \ldots < r_{s-1} < r_{s+2} < r_{\# \text{Cr}(f_1)} \) is the set of regular values of \( f_{-1} \), so any sublevel set of \( f_{-1} \) is also a sublevel set for \( f_1 \). Then one arrives at the following diagram.

\[
\begin{array}{cccccccc}
M^c_0 & \subset & \ldots & \subset & M^{r_{s-1}} & \subset & \ldots & \subset & M^{r_{s+2}} \\
M^{c_0} & \subset & \ldots & \subset & M^c_s & \subset & \ldots & \subset & M^{c_{s-1}}
\end{array}
\]

Here equality sign denotes set-theoretical equality of two subspaces of \( M \) and \( \cong \) denotes a homeomorphism. The latter takes place since attaching two cells as in Subsection 5.1 doesn’t change the homeomorphism type of a space. It now follows that Construction 3.1 produces the same B-data for \( f_{-1} \) and \( f_1 \) except for the above-mentioned newborn pair.

**Remark 5.7.** The sign in \( \pm 1 \) depends on the chosen orientations of arcs (see beginning of this subsection). See Subsection 5.6 for a theorem where it plays important role.

**Figure 3.** Birth of two critical points.

5.4. Maxwell event. In this subsection we consider the second type of event, namely self-intersection of a Cerf diagram (in other words, Maxwell event). This finishes the description of bifurcations of B-data in families started in Subsection 5.3.

Let us fix the notations first. Let \( c_{s+1} \) and \( c_s \) be two critical points of \( f_{-1} \) participating in the bifurcation. Recall from Subsection 5.1 that \( \text{Cr}(f_1) \) coincides, as an ordered subset of \( M \), with \( \text{Cr}(f_{-1}) \) with the order of \( c_{s+1} \) and \( c_s \) reversed. Let \( r_0 < \ldots < r_{\# \text{Cr}(f_{-1})} \) be regular values of \( f_{-1} \) and \( f_1 \).

**Proposition 5.8.** After the Maxwell event two types of bifurcations possible.
1) Trivial bifurcation. After it points $c_{s+1}$ and $c_s$ keep their initial pairs (if any) and Bruhat numbers on them. All combinatorial variants of juxtaposition of points are possible. The values $\deg c_{s+1}$ and $\deg c_s$ may be any.

2) Non-trivial bifurcation. The necessary condition is $\deg c_{s+1} = \deg c_s$. The list of five possible variants is given in Figure 4.

All the points not participating in the bifurcation keep their initial pairs (if any) and Bruhat numbers on them.

Figure 4. Non-trivial bifurcations at the self-intersection of a Cerf diagram

Remark 5.9. As seen on Figure 4 pairing and Bruhat numbers may well change after the non-trivial bifurcation. Note that the mentioned list of five cases doesn’t exhaust all possible combinatorial variants of juxtaposition of points.

Proof of Proposition 5.8. As in Proposition 5.6 it suffices to track down only Bruhat numbers. Let $r_0 < \ldots < r_{\# C_r(f_1)}$ be regular values of $f_1$. These values are also regular for $f_1$. Moreover, all but one sublevel sets of $f_1$ coincide. The only exception is the sublevel set for the regular value $r_s$; so, we write $M_{r_s} = \{x \in M | f_1(x) \leq r_s\}$ and $\tilde{M}_{r_s} = \{x \in M | f_1(x) \leq r_s\}$. One arrives at the following diagram.

The case of trivial bifurcation and the very last statement of Proposition 5.8 now follow directly by unwrapping Construction 3.1. We now need to show how Bruhat numbers change after cases 1-3 of non-trivial bifurcation (see Figure 4; two other don’t place any restriction on these numbers). By the last statement it suffices to prove that $\tau(f_1) = -\tau(f_1)$ (see Subsection 4.1). Since the number $\tau$ admits torsion-theoretic interpretation, it depends only on two things: equivalence class basis $c$ of an enhanced complex and equivalence class of basis of homology. The latter is uniquely determined by the enhancement on $H_\bullet(M)$ induced by a function. Since both points involved in the event are not homological, this enhancement is the same for $f_1$ and $f_1$.

We will now investigate how basis $c$ changes after the bifurcation. Choose any CW-approximation (or any Riemannian metric) for $f_1$ in the sense of Subsection 3.4 (or, respectively, Subsection 3.7). The resulting CW-complex (or, respectively, Morse complex) also serves as a CW-complex associated with $f_1$ with the only difference that $s$th and $(s+1)$th generators got swapped in the linear order. The determinant of a transposition matrix is $-1$ and the formula $\tau(f_1) = -\tau(f_1)$ now follows from Remark 2.29. □

5.5. Manifolds with almost no $F$-homology. In this subsection we prove the following theorem, see Subsection 4.1 for a context.

Theorem 0.2. Let $f$ be a strong Morse function on $M$ and $F$ be a field. Suppose that $H_k(M) = 0$ for all $0 < k < \dim M$. Then the alternating product of all Bruhat numbers (as an element from $F^*/\pm 1$) is independent of $f$.

Proof. Take two strong Morse functions on $M$ and connect them by a generic path. The plan is to use Propositions 5.6 and 5.8 to prove that $\pm \tau$ doesn’t change after any bifurcation along the
path. The case of birth/death event obviously follows from Proposition 5.6. The rest is devoted to dealing with the Maxwell event.

It follows from Proposition 5.8 that if both points involved in Maxwell event are paired (i.e. not homological) then the number ±τ stays the same after the bifurcation. On the other hand, by assumption any homological point is of degree either zero or dim M. In the former case, the first part of Proposition 3.27 implies that ±τ again stays the same (regardless of whether the bifurcation is trivial or not). The rest is devoted to the latter case.

If M is non-orientable then f may have a homological point of degree dim M only if char F = 2, but this case makes the initial statement trivial (see Remark 3.16). If M is orientable we make use of the second part of Proposition 3.27 in the same way as earlier.

5.6. A theorem of Akhmetiev-Cencelj-Repovs. In this subsection we apply our methods to reprove the theorem of Akhmetiev-Cencelj-Repovs [ACR05] in greater generality. Roughly it says that several numerical invariants of a generic path in the space of strong Morse functions satisfy a certain equation mod 2.

First of all we need to pass to a bit more general setting. Cobordism is a manifold M with boundary ∂M0 ⊔ ∂M1. By a Morse function f on a cobordism (M, ∂M0, ∂M1) we will mean a function f : M → [0, 1] with only non-degenerate critical points, all in the interior M, s.t. f−1(0) = ∂M0 and f−1(1) = ∂M1. Strongness property is defined in the same manner as in the closed case. All the results from Subsections 3.4 and 3.7 generalize readily to this setting, see [Mil65]. The only thing to mention here is that now construction from Subsection 3.4 attaches cells one-by-one starting from ∂M0. As a consequence, one obtains a CW-decomposition of M/∂M0, not M itself. Thus, all the various constructed complexes count relative homology H1(M, ∂M0). All the results about enhanced complexes also translate readily. Trivial cobordism attaches cells one-by-one starting from ∂M0.

We will now introduce mentioned invariants of a generic path \{f_t\}. The first one is the number of self-intersections of the Cerf diagram (or, in our terminology, the number of Maxwell events), denote it X. To get to the second one recall that in Subsection 3.3 we described the procedure of orienting the arcs of a Cerf diagram, which outputs an orientation of each strong Morse function in a path. After orienting the arcs somehow one can assign a sign to each cusp of a Cerf diagram (i.e. birth/death event) as follows. Let t0 be a point of birth (resp. death) event. Pick any value t0 > s0 (resp. t0 < s0) s.t. all functions between t0 and t0 (s0 excluded) are strong Morse. Denote by cs+1 and cs two bornorph (resp. about to die) critical points of fts. It follows from classical results recalled in Subsection 3.4 that cellular differential of cs+1 contains cs with coefficient either 1 or −1 regardless of choices made (essentially, cellular approximation). Using another language, one may say the same about the Morse differential w.r.t. to any Riemannian metric (see Subsection 3.7).

The sign of a cusp is now defined as the sign of this number. Let C be the number of negative cusps. Changing orientation of an arc changes the sign of each cusp which serves as this arc’s endpoint (obviously, there are at most two such cusps). Therefore, if both f−1 and f1 have no critical points, then the parity of C is a well-defined invariant of a path \{f_t\}. We are now turning to a corollary which is easier to state compared to the main theorem. For example, it doesn’t appeal to any field or Bruhat numbers whatsoever, thus remaining entirely in the realm of Cerf theory.

Corollary 0.4. Let \{f_t\} be a generic path of functions on a cylinder M × [0, 1] s.t. both f−1 and f1 have no critical points. Let X be the number of self-intersections of its Cerf diagram and C be the number of negative cusps. Then one has

\[ X + C = 0 \pmod{2}. \]

Remark 5.10. In [ACR05] Corollary 0.4 was proved using two different methods, both requiring additional assumptions on M. The first method is based on h-principle and requires M to be stably parallelizable and simply-connected. The second one is based on parametric Morse theory and requires the dimension of M to be at least five. This method is geometrical and relies on strong results of Smale akin to those in Subsection 5.2. These results, in turn, only valid if the dimension of M is big enough and lead, for example, to the famous h-cobordism theorem. On the other hand, our approach of using enhanced complexes, after deducing all the necessary generality in Subsection 5.3 is entirely combinatorial.

We will now give one more definition in order to state the main theorem of this subsection. Fix a field F s.t. char F ≠ 2 and consider the set of Barannikov pairs of some oriented strong Morse function f. We say that two pairs overlap if they overlap when viewed as segments on the real line. Formally, two pairs (s1, s2) and (t1, t2) overlap if either s1 < t1 < s2 < t2 or t1 < s1 < t2 < s2.

Let O be the number of all overlapping (unordered) pairs of Barannikov pairs (we stress out that we place no restrictions on the degrees of points). Define now

\[ τ'(f, F) := (-1)^O \prod_{s ∈ U} λ(s)^{(−1)^{deg s}} ∈ F^*. \]
Remark 5.11. In words, this is the alternating product of all the Bruhat numbers times the sign depending on the parity of $O$. This sign is different from the one from Definition 4.1. As usual, we drop the ingredients of $\tau'$ when they are understood.

Let now $(M, \partial M_1, \partial M_2)$ be any cobordism s.t. relative homology $H_*(M, \partial M_1)$ vanishes (recall that we take coefficients to be in $F$ by default). (For example, one may take an h-cobordism.) Then Theorem 0.2 (in the aforementioned setting) implies that $\pm \tau'(f)$ in independent of $f$.

We are now ready to state the main theorem of this subsection, which we do in multiplicative notation. Recall that $X$ is the number of self-intersections of the Cerf diagram and $C$ is the number of its negative cusps.

Theorem 5.12. Let $F$ be a field and $(M, \partial M_1, \partial M_2)$ be a cobordism s.t. $H_*(M, \partial M_1) = 0$. Let also $\{f_t\}$ be a (somehow oriented) generic path of functions on it. Then one has

$$\frac{\tau'(f_1)}{\tau'(f_{-1})}(-1)^X(-1)^C = 1.$$ 

Proof. The plan is to track down when the number $\tau'(f_t)$ changes its sign as $t$ varies from $-1$ to $1$. It suffices to prove that it does so exactly after $t$ passes 1) a self-intersection of the Cerf diagram, 2) a negative cusp (either left or right). Note that by assumption there are no homological points, i.e. all of the points are paired.

We will now turn to the first case. Suppose that the bifurcation is non-trivial. Then the decomposition of critical points into pairs remains the same, thus the number $O$ remains unaltered. It then follows directly from Proposition 5.6 that the alternating product of Bruhat numbers change its sign. Suppose now that the bifurcation is trivial. This time the set of Bruhat numbers remains the same, but the number $O$ increases or decreases by one (depending on the justaposition of pairs involved in the bifurcation).

Since $\tau(f, F) = 1$ for any $f$ without critical points at all (and for any $F$), the Corollary 0.4 follows straightforwardly by taking any $F$ s.t. char $F \neq 2$. 

□
A. Cayley. “On the Theory of Elimination”. In: Cambridge and Dublin Math. Journal 3 (1848), pp. 116–120.

Marston Morse and George Booth Van Schaack. “The Critical Point Theory Under General Boundary Conditions”. In: Annals of Mathematics 35.3 (1934), pp. 545–571.

G.M. Adelson-Velski and A.S. Kronrod. “About level sets of continuous functions with partial derivatives”. In: Dokl. Akad. Nauk SSSR 49 (1945), pp. 239–241.

G. Reeb. “Sur les points singuliers d’une forme de Pfaff complètement intégrable ou d’une fonction numérique”. In: C. R. Acad. Sci. Paris 222 (1946), pp. 847–849.

John Milnor. Morse theory. Vol. 51. Princeton university press, 1963.

John Milnor. Lectures on the h-cobordism theorem. Vol. 2258. Princeton university press, 1965.

John Milnor. “Whitehead torsion”. In: Bulletin of the American Mathematical Society 72.3 (1966), pp. 358–426.

Jean Cerf. “La stratification naturelle des espaces de fonctions différentiables réelles et le théorème de la pseudo-isotopie”. In: Publications Mathématiques de l’IHÉS 39 (1970), pp. 5–173.

T. A. Chapman. “Topological Invariance of Whitehead Torsion”. In: American Journal of Mathematics 96.3 (1974), pp. 488–497.

Philip Thiéssse. Upper triangular similarity of upper triangular matrices. Rep. 9092/A, Econometric Inst., Erasmus Univ., Rotterdam. 1990.

Claude Viterbo. “Symplectic topology as the geometry of generating functions”. In: Mathematische Annalen 292 (1992), pp. 685–710.

A. Beilinson and J. Bernstein. “A Proof of Jantzen Conjectures”. In: Advances in Soviet Mathematics 16 (1993).

Serguei Barannikov. “The Framed Morse complex and its invariants”. In: Advances in Soviet Mathematics 22 (1994), pp. 93–115.

Philip Thiéssse. “Upper triangular similarity of upper triangular matrices”. In: Linear Algebra and its Applications 260 (1997), pp. 119–149. ISBN: 0024-3795.

Robin Forman. “Morse Theory for Cell Complexes”. In: Adv. in Math. 134 (1998), pp. 90–145.

Israel Gelfand, Mikhail Kapranov, and Andrei Zelevinsky. Discriminants, resultants, and multidimensional determinants. Lectures in Mathematics. ETH Zürich. Birkhäuser Boston, 1998.

Michael Hutchings and Yi-Jen Lee. “Circle-valued Morse theory and Reidemeister torsion”. In: Geometry and Topology 3 (1999), pp. 369–396.

Michael Hutchings and Yi-Jen Lee. “Circle-valued Morse theory, Reidemeister torsion, and Seiberg-Witten invariants of 3-manifolds”. In: Topology 38 (1999), pp. 861–888.

Anna Melnikov. “B-Orbits in Solutions to the Equation $X^2 = 0$ in Triangular Matrices”. In: Journal of Algebra 223.1 (2000), pp. 101–108. ISBN: 0021-8693.

Vladimir Turaev. Introduction to Combinatorial Torsions. Lectures in Mathematics. ETH Zürich. Birkhäuser Basel, 2001. ISBN: 9783764364038.

Michael Hutchings. “Reidemeister torsion in generalized Morse theory”. In: Forum Mathematicum 14 (2002), pp. 209–244.

William Fulton and Joe Harris. Representation Theory: A First Course. Graduate Texts in Mathematics. Springer, New York, NY, 2004.

“Some Algebraic Properties of Cerf Diagrams of One-Parameter Function Families”. In: Functional Analysis and Its Applications 39 (2005).

Yu. V. Chekanov and P. E. Pushkar’. “Combinatorics of fronts of Legendrian links and the Arnold 4-conjectures”. In: Russian Mathematical Surveys 60.1 (2005), pp. 95–149.

David Cohen-Steiner, Herbert Edelsbrunner, and Dmitriy Morozov. “Vines and vineyards by updating persistence in linear time”. In: Symposium on Computational Geometry. ACM, 2006, pp. 119–126.

Herbert Edelsbrunner and John Harer. “Persistent homology - A survey”. In: Surveys on Discrete and Computational Geometry 453 (2008).

Dorian Le Peutrec, Francis Nier, and Claude Viterbo. “Precise Arrhenius Law for p-forms: The Witten Laplacian and Morse-Barannikov Complex”. In: Annales Henri Poincare 14 (2013), pp. 567–610.

Ulrich Bauer, Xiaoyin Ge, and Yusu Wang. “Measuring Distance between Reeb Graphs”. In: Proceedings of the Thirtieth Annual Symposium on Computational Geometry. SOCG’14. Association for Computing Machinery, 2014, pp. 464–473. ISBN: 9781450325943.
[Mne14] Pavel Mnev. Lecture notes on torsions. 2014. arXiv: [1406.3705 [math.AT]]

[Lau15] Francois Laudenbach. On an article by S. A. Barannikov. 2015. arXiv: [1509.03490 [math.GT]]

[UZ16] Michael Usher and Jun Zhang. “Persistent homology and Floer–Novikov theory”. In: Geometry & Topology 20 (2016).

[Cha17] Francois Charette. “Quantum Reidemeister torsion, open Gromov-Witten invariants and a spectral sequence of Oh”. In: International Mathematics Research Notices 2019 (2017), pp. 2483–2518.

[DPS17] Jean-Guillaume Dumas, Clement Pernet, and Ziad Sultan. “Fast computation of the rank profile matrix and the generalized Bruhat decomposition”. In: Journal of Symbolic Computation 83 (2017). Special issue on the conference ISSAC 2015: Symbolic computation and computer algebra, pp. 187–210. ISSN: 0747-7171.

[LNV20] Dorian Le Peutrec, Francis Nier, and Claude Viterbo. Bar codes of persistent cohomology and Arrhenius law for p-forms. 2020. arXiv: [2002.06949 [math.AP]]

[GWW] David Gay, Katrin Wehrheim, and Chris Woodward. Connected Cerf theory.