Abstract

We construct the family of spin chain Hamiltonians, which have affine $U_{qg}$ quantum group symmetry. Their eigenvalues coincide with the eigenvalues of the usual spin chain Hamiltonians which have non-affine $U_{qg_0}$ quantum group symmetry, but have the degeneracy of levels, corresponding to affine $U_{qg}$. The space of states of these chains are formed by the tensor product of the fully reducible representations.
1 Introduction

Quantum group symmetry plays great role in integrable statistical models \cite{1,2,3} and conformal field theory \cite{4,5,6}.

It is well known that many integrable Hamiltonians have quantum group symmetry. For example, \(XXZ\) Heisenberg Hamiltonian with particular boundary terms \cite{5} is \(U_q\sl_2\)-invariant. Infinite \(XXZ\) spin chain has larger symmetry: affine \(U_q\hat{\sl}_2\) \cite{7}. Single spin site of most considered Hamiltonians form irreducible representation of Lie algebra or its quantum deformation.

Here we construct family of spin chain Hamiltonians, which have affine quantum group symmetry. The space of states of these chains are formed by the tensor product of the fully reducible representations. We show that the model, considered in \cite{8}, which corresponds to some generalization of the Habbard Hamiltonian in the strong repulsion limit, is a particular case of our general construction. The affine quantum group symmetry leads to high degeneracy of energy levels.

The energy levels of these spin chains are formed on the states, constructed from highest weight vectors of quantum group representations. In particular cases the restriction of considered spin chain on these states gives rise to Heisenberg spin chain or Haldane Shastry long range interaction spin chain.

It is difficult in a moment to name a set of physical problems, with which the constructed Hamiltonians directly related (besides mentioned). However it is essential to point out that affine symmetries appear in 2D physics when matter fields interact with gravity (in a noncritical string theory).

2 Definitions

Let us recall the definition of quantum Kac-Moody group \(U_q g\). It is generated by the generators \(e_i, f_i, h_i\) satisfying the relations

\[
[h_i, e_j] = c_{ij} e_j \quad [h_i, f_j] = -c_{ij} f_j
\]
\[
[e_i, f_j] = \delta_{ij} [h]_q
\]

and \(q\)-deformed Serre relations, which we don’t write here. Here \(q\) is a deformation parameter, \([x]_q := (x^q - x^{-q})/(q - q^{-1})\), \(c_{ij}\) is a Cartan matrix of corresponding Kac-Moody algebra \(g\).
On $U_qg$ there is a Hopf algebra structure:
\[
\Delta(e_i) = k_i \otimes e_i + e_i \otimes k_i^{-1} \quad \Delta(k_i^{\pm 1}) = k_i^{\pm 1} \otimes k_i^{\mp 1}
\]
\[
\Delta(f_i) = k_i \otimes f_i + f_i \otimes k_i^{-1}
\]
where $k_i := q^{h_i}$. This comultiplication can be extended to $L$-fold tensor product by
\[
\Delta^{L-1}(e_i) = \sum_{i=1}^{L} k_i \otimes \ldots \otimes k_i \otimes \underbrace{e_i \otimes k_i^{-1} \otimes \ldots \otimes k_i^{-1}}_{L}
\]
\[
\Delta^{L-1}(f_i) = \sum_{i=1}^{L} k_i \otimes \ldots \otimes k_i \otimes \underbrace{f_i \otimes k_i^{-1} \otimes \ldots \otimes k_i^{-1}}_{L}
\]
\[
\Delta^{L-1}(k_i^{\pm 1}) = k_i^{\pm 1} \otimes \ldots \otimes k_i^{\mp 1}
\]

Let $g$ be an affine algebra and $g_0$ is the underlying finite algebra: $g = \hat{g}_0$. Then for any complex $x$ there is the $q$-deformation of loop homomorphism $\rho_x : U_qg \to U_qg_0$, which is given by
\[
\rho_x(e_0) = x f_\theta \quad \rho_x(f_0) = x^{-1} e_\theta \quad \rho_x(h_0) = -h_\theta
\]
\[
\rho_x(e_i) = e_i \quad \rho_x(f_i) = f_i \quad \rho_x(h_i) = h_i, \quad (1)
\]
where $i = 1 \ldots n$ and $\theta$ is a maximal root of $U_qg$. Using $\rho_x$ one can construct the spectral parameter dependent representation of $U_qg$ from the representation of $U_qg_0$.

Let $V_1(x_1)$ and $V_2(x_2)$ are constructed in such way irreducible finite dimensional representations of $U_qg$ with parameters $x_1$ and $x_2$ correspondingly. The $U_qg$-representations on $V_1(x_1) \otimes V_2(x_2)$ constructed by means of $\Delta$ and $\bar{\Delta}$ are both irreducible, in general, and equivalent:

\[
R(x_1, x_2)\Delta(g) = \bar{\Delta}(g)R(x_1, x_2), \quad g \in U_qg
\]
(2)

The $R$-matrix $R(x_1, x_2)$ depends only on $x_1/x_2$ and is a Boltzmann weight of some integrable statistic mechanical system.
3 Quantum group invariant Hamiltonians for reducible representations

Let $V = \oplus_{i=1}^{N} V_{\lambda_i}$ is a direct sum of finite dimensional irreducible representations of $U_qg$. We denote by $V(x_1, \ldots, x_N)$ corresponding affine $U_qg$ representation with spectral parameters $x_i$:

$$V(x_1, \ldots, x_N) = \oplus_{i=1}^{N} V_{\lambda_i}(x_i)$$

We consider the intertwining operator

$$H(x_1, \ldots, x_N) : V(x_1, \ldots, x_N) \otimes V(x_1, \ldots, x_N) \rightarrow V(x_1, \ldots, x_N) \otimes V(x_1, \ldots, x_N),$$

$$[H(x_1, \ldots, x_N), \Delta(a)] = 0, \text{ for all } a \in U_qg.$$ If $V = V_{\lambda}$ consists of one irreducible component then $H$ is a multiple of identity, because the tensor product is irreducible in this case. To carry out the general case let us gather all equivalent irreps together:

$$V(x_1, \ldots, x_N) = \bigoplus_{i} N_{\lambda_i} \hat{\otimes} V_{\lambda_i}(x_i),$$

where all $V_{\lambda_i}(x_i)$ are nonequivalent and $N_{\lambda_i} \simeq \mathbb{C}^{n_i}$ have a dimension equal to the multiplicity of $V_{\lambda_i}(x_i)$ in $V(x_1, \ldots, x_N)$. By the hat over the tensor product we mean that $U_qg$ doesn’t act on $N_{\lambda_i} \hat{\otimes} V_{\lambda_i}(x_i)$ by means of $\Delta$ but acts as $id \otimes g$.

So, we have:

$$V(x_1, \ldots, x_N) \otimes V(x_1, \ldots, x_N) = (\bigoplus_{i} N_{\lambda_i} \hat{\otimes} V_{\lambda_i}(x_i)) \bigotimes (\bigoplus_{i} N_{\lambda_i} \hat{\otimes} V_{\lambda_i}(x_i))$$

$$= \bigoplus_{i,j} N_{\lambda_i} \hat{\otimes} N_{\lambda_j} \hat{\otimes} \left(V_{\lambda_i}(x_i) \otimes V_{\lambda_j}(x_j)\right) \quad (3)$$

Now, $V_{\lambda_i}(x_i) \otimes V_{\lambda_j}(x_j)$ is equivalent only to itself and to $V_{\lambda_j}(x_j) \otimes V_{\lambda_i}(x_i)$ (for $i \neq j$) by the operator $\hat{R}(x_i/x_j) = PR(x_i/x_j)$, where $P$ is tensor product permutation: $P(v_1 \otimes v_2) = v_2 \otimes v_1$. So, the commutant $H(x_1, \ldots, x_N)$ of $U_qg$ on $V(x_1, \ldots, x_N) \otimes V(x_1, \ldots, x_N)$ has the following form:

\[\text{The U_qg-equivalence of V_{\lambda_i}(x_i) requires that the spectral parameters x_i and highest weights } \lambda_i \text{ are the same.}\]
\[
H\big|_{\bigoplus_{\lambda_{ij}} N_{\lambda_i} \otimes N_{\lambda_j} \otimes V_{\lambda_i} \otimes V_{\lambda_j}} = A_{ij} \otimes \text{id}V_{\lambda_i} \otimes V_{\lambda_j} + B_{ij} \otimes \hat{R}V_{\lambda_i} \otimes V_{\lambda_j} (x_i/x_j)
\] (4)

where \( A_{ij} \) and \( B_{ij} \) are any operators on \( N_{\lambda_i} \otimes N_{\lambda_j} \).

Let us consider some particular cases of this general construction.

1. Let \( V(x) = V(x, x) = V_\lambda(x) \oplus V_\lambda(x) \). The second term in (4) is absent in this case and \( H \) has factorized form:

\[
H = A \otimes \text{id}V_{\lambda} \otimes V_{\lambda}, \quad A = a_{\alpha\delta}
\]

where \( \alpha, \beta, \gamma, \delta = \pm \) are indexes, corresponding to each \( V_\lambda \).

2. Let now \( V(x_1, x_2) = V_{\lambda_1}(x_1) \oplus V_{\lambda_2}(x_2) \) \((V_\lambda(x_i) \text{ are mutually nonequivalent})\). Then \( H \) acquires the following form

\[
H(x_1, x_2) = \begin{pmatrix}
    a \cdot \text{id} & 0 & 0 & 0 \\
    0 & c \cdot \text{id} & d \cdot R_{21}(x_2/x_1) & 0 \\
    0 & e \cdot R_{12}(x_1/x_2) & f \cdot \text{id} & 0 \\
    0 & 0 & 0 & g \cdot \text{id}
\end{pmatrix}
\] (5)

Here we used \( R_{21} = \sum_i a_i \otimes b_i \) for \( R_{12} = \sum_i a_i \oplus b_i \). Note, that we can normalize \( R \)-matrices to satisfy the unitarity condition \( R_{12}(z)R_{21}(z^{-1}) = \text{id} \). This leads to

\[
H(x_1, x_2)^2 = \text{id} \otimes \text{id}
\] (6)

3. If we choose \( g = sl(2) \) and \( V = V_f \oplus V_0 \oplus V_0 \oplus \ldots \oplus V_0 \), where \( V_f \) is fundamental representation of \( U_q sl_2 \) and \( V_0 \) is trivial one dimensional representation of one, one can obtain the Hamiltonian, corresponding to a strong repulsion limit of some generalization of Hubbard model, considered in [8]. The representation (5.13) there is a \( U_q sl_2 \)-representation on \( V \).

Following [8] from the operator \( H \) the following Hamiltonian acting on \( W = V \otimes L \) can be constructed: \footnote{Here and in the following we omit the dependence on \( x_i \)}

\[
\hat{H} = \sum_{i=1}^{L-1} H_{ii+1}
\] (7)
Here and in the following for the operator $X = \sum_i x_i \otimes y_i$ on $V \otimes V$ we denote by $X_{ij}$ its action on $W$ defined by

$$X_{ij} = \sum_l \text{id} \otimes \ldots \otimes \text{id} \otimes x_i \otimes \text{id} \otimes \ldots \otimes \text{id} \otimes y_l \otimes \text{id} \otimes \ldots \otimes \text{id} \quad (8)$$

By the construction, $\bar{H}$ is quantum group invariant:

$$[\bar{H}, \Delta^{L-1}(g)] = 0 \quad \forall g \in U_q g$$

Let $V^0$ is the linear space, spanned by the highest weight vectors in $V$: $V^0 := \bigoplus_{i=1}^N v^0_{\lambda_i}$, where $v_{\lambda_i} \in V_{\lambda_i}$ is a highest weight vector, and $W^0 := V^0 \otimes^L$. The space $W^0$ is $\bar{H}$-invariant. This follows from the intertwining property of $\bar{H}$. For general $q$, $W$ is $U_q g$-irreducible module so the action of $U_q g$ on $W^0$ generate all $W$. So, the energy levels of $\bar{H}$ are highly degenerate.

First, one can consider $\bar{H}$ on the space $W^0$ and determine (if it is possible) the energy levels and corresponding eigenvectors there. Then performing the quantum group on each eigenvector of some energy level one can obtain the whole eigenspace for this level. Moreover, the space $W^0$ itself is a direct sum of $\bar{H}$-invariant spaces, each is spanned by the tensor products of fixed number highest weight vectors from each equivalence class of irreps:

$$W^0 = \bigoplus_{p_1, \ldots, p_M} W^0_{p_1, \ldots, p_M}$$

$$W^0_{p_1, \ldots, p_M} := \left\{ \bigoplus C v^0_{\lambda_1} \otimes \ldots \otimes v^0_{\lambda_L} \mid \#((\lambda_i, x_i)) \in \{(\lambda_1, x_1), \ldots, (\lambda_N, x_N)\} = p_i \right\}$$

The $\bar{H}$-invariance of $W^0_{p_1, \ldots, p_M}$ follows again from the definition of $\bar{H}$ as an intertwining operator. The energy levels are now determined on these spaces. Note that the dimension of $W^0_{p_1, \ldots, p_M}$ is

$$\binom{L}{p_1 \ldots p_M}$$

Every Hamiltonian eigenvector $w_0 \in W^0_{p_1, \ldots, p_M}$ gives rise to a $U_q g$-representation space of dimension

$$\prod_{k=1}^M (\dim V_{\lambda_k})^{p_k} \quad (9)$$
This is the degeneracy level of its energy value. In the particular case when all $V_{\lambda}$ are equivalent, the degeneracy level is $(\text{dim} V_{\lambda})^L$. Note that

$$\dim W = \sum_{p_1, \ldots, p_M} \left( \frac{L}{p_1 \ldots p_N} \right) \prod_{k=1}^{N} (\text{dim} V_{\lambda_k})^{p_k} = \left( \sum_{k=1}^{N} N_{\lambda_k} \text{dim} V_{\lambda_k} \right)^L$$

as it must be.

For example, if we choose two equivalent representations (the first case above), then $\dim V^0 = 2$ and there is one term in decomposition (9). $H$ now is the most general action on $V^0 \otimes V^0$. As a particular case, the $XYZ$ Hamiltonian in the magnetic field can be obtained. This case is most trivial because the degeneracy of all energy levels is the same. So, for the statistical sum $Z_H(\beta) = \sum_n \exp(-\beta E_n)$ we have

$$Z_H(\beta) = (\text{dim} V_{\lambda})^L Z_{XYZ}(\beta)$$

Let us choose

$$a = g = e = d = 1 \quad c = f = 0$$

for the second example. Then the restriction of $\tilde{H}$ on $W^0$ coincides with the Bethe $XXX$ spin chain

$$\tilde{H}|_{W^0} = H_{XXX} = \sum_i P_{i+1} = \frac{1}{2} \sum_i (1 + \sigma_i^z \sigma_{i+1}^z)$$

The space $W^0_{p_1 p_2}$, $p_1 + p_2 = L$ corresponds to all states with the same $s_z = p_1/2$ value of spin projection $S^z = 1/2 \sum_i \sigma_i^z$. If we return to $\tilde{H}$ the energy level degeneracy of each eigenstate with the same spin projection is multiplies by $(\text{dim} V_{\lambda_1})^{2s_z} (\text{dim} V_{\lambda_2})^{L-2s_z}$.

4 Generalization to long range interaction spin chains

Let us consider now the generalization of above construction in case of long range interacting Hamiltonians.
Recall the Haldane-Shastry spin chain is given by

\[ H_{HS} = \sum_{i<j} \frac{1}{d_{i-j}^2} P_{ij}, \quad (12) \]

Here the spins take values in the fundamental representation of \( \mathfrak{sl}_n \). It is well known that the Hamiltonian \((12)\) is integrable if \(d_i\) has one of the following values

\[ d_j = \begin{cases} 
  j, \quad & \text{rational case} \\
  (1/\alpha) \sinh(\alpha j), \alpha \in \mathbb{R}, \quad & \text{hyperbolic case} \\
  (L/\pi) \sin(\pi j/L), \quad & \text{trigonometric case}
\end{cases} \quad (13) \]

The trigonometric model is defined on periodic chain and the sum in \((12)\) is performed over \(1 \leq i, j \leq L\). Rational and hyperbolic models are defined on infinite chain.

One can try to generalize the Hamiltonian \((12)\) for the reducible spin representations by

\[ \tilde{H}_{HS} = \sum_{i<j} \frac{1}{d_{i-j}^2} H_{ij}, \quad (14) \]

where \(H\) is taken for the case \((10)\) of second example in the previous section. But it is easy to see that it isn’t invariant with respect to quantum group. This is because the equation

\[ \hat{R}_{ij}(x_1, x_2) \Delta^{L-1}(g) = \Delta^{L-1}(g) \hat{R}_{ij}(x_1, x_2), \quad g \in U_q g \quad (15) \]

is valid only for \(i = j \pm 1\).

To overcome this difficulty let us substitute instead of \(H_{ij}\) the operator

\[ F_{[ij]} = G_{[ij]} H_{j-i}^{-1} G_{[ij]}^{-1}, \quad \text{where} \quad G_{[ij]} = H_{ii+1} H_{i+1i+2} \cdots H_{j-2j-1} \quad (16) \]

The ’nonlocal’ term like \(F_{[ij]}\) appeared as a boundary term in the construction of quantum group invariant and in some sense periodic spin chains \([12, 13]\).

Note that it follows from \((5,10,6)\) that \(H_{ii+1}\) satisfy

\[ H_{i-1}^{-1} H_{ii+1} H_{i-1} = H_{ii+1} H_{i-1} H_{ii+1} \quad H_{ii+1}^2 = 1 \]

\(^5\) Note that \(F_{[ij]}\) and \(G_{[ij]}\) act nontrivially on all indexes \(i, i+1, \ldots, j\). So we include them into bracket to not confuse with the definition \([3]\).
This is a realization of permutation algebra. In contrast to standard realization by $P_{ij}$, the relation

$$P_{i-1i}P_{ii+1}P_{i-1i} = P_{i-1i+1}$$

isn’t fulfilled. The restriction of $H_{ii+1}$ on the highest weight space $W_0$ coincides with $P_{ii+1}$. Also it is easy to see from (16) that

$$F_{[ij]}|W_0 = P_{ij}$$

So, the spin chain defined by

$$\tilde{H}_{HS} = \sum_{i<j} \frac{1}{d_i^2} F_{[ij]},$$

(17)

is quantum group invariant and its restriction on the space $W^0$ it coincides with the Haldane-Shastry spin chain (12). The energy levels of $\tilde{H}_{HS}$ coincides with the levels of (12). The degeneracy degree with respect to the later is defined by (14).

## 5 Acknowledgement

One of us (A.S.) acknowledge the Institute of Theoretical Physics in Bern for hospitality and especially H.Leutwyler for many interesting discussions.

This work was supported by Schweizerischer Nationalfonds and Grant 211-5291 YPI of the German Bundesministerium fur Forschung und Technologie.

## References

[1] M.Jimbo. A $q$-difference analogue of $U(g)$ and Yang-Baxter equation. *Lett. Math. Phys.*, 10:63, 1985.

[2] M.Jimbo. A $q$-analog of $U(gl(N + 1))$; Hecke algebra and the Yang-Baxter equation. *Lett. Math. Phys.*, 11:247, 1986.

[3] V.G. Drinfeld. Quantum groups. In *ICM proceedings*, pages 798–820, New-York: Berkeley, 1986.
[4] G.Moore and N.Reshetikhin. A comment on quantum group symmetry in conformal field theory. *Nucl. Phys.*, B328:557, 1989.

[5] V. Pasquier and H. Saleur. Common structures between finite systems and conformal field theories through quantum groups. *Nucl. Phys.*, B330:523, 1990.

[6] G.Sierra C.Gomes. The quantum group symmetry of rational conformal field theories. *Nucl. Phys.*, B352:791–828, 1991.

[7] B.Davies O.Foda M.Jimbo T.Miwa and A.Nakayashiki. Diagonalization of the XXZ Hamiltonian by vertex operators. *Commun. Math. Phys.*, 151:89–153, 1993.

[8] V.Rittenberg F.Alcaraz, D.Arnaudon and M.Scheunert. Hubbard-like models in the infinite repulsion limit and finite-dimensional representations of the affine algebra $U_q(\hat{sl}(2))$. preprint CERN-TH6935/93, 1993.

[9] F.D.M.Haldane. *Phys. Rev. Lett.*, 60:635, 1988.

[10] B.S.Shastry. *Phys. Rev. Lett.*, 60:639, 1988.

[11] V.I.Inozemtsev. *J.Stat. Phys.*, 59:1143, 1990.

[12] P.P.Martin. *Potts models and related problems in statistical mechanics*. Word Scientific, Singapore, 1991.

[13] P.Prester H.Grosse, S.Pallua and E.Raschhofer. On a quantum group invariant spin chain with nonlocal boundary condition. *J.Phys.A: Math.Gen*, 27:4761, 1994.