New consistent and asymptotically normal estimators for random graph mixture models

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Abstract: Random graph mixture models are now very popular for modeling real data networks. In these setups, parameter estimation procedures usually rely on variational approximations, either combined with the expectation-maximisation (EM) algorithm or with Bayesian approaches. Despite good results on synthetic data, the validity of the variational approximation is however not established. Moreover, the behavior of the maximum likelihood or of the maximum a posteriori estimators approximated by these procedures is not known in these models, due to the dependency structure on the variables. In this work, we show that in many different affiliation contexts (for binary or weighted graphs), estimators based either on moment equations or on the maximization of some composite likelihood are strongly consistent and \( \sqrt{n} \)-convergent, where \( n \) is the number of nodes. As a consequence, our result establishes that the overall structure of an affiliation model can be caught by the description of the network in terms of its number of triads (order 3 structures) and edges (order 2 structures). We illustrate the efficiency of our method on simulated data and compare its performances with other existing procedures. A data set of cross-citations among economics journals is also analyzed.

Keywords and phrases: composite likelihood, random graph, mixture model, stochastic blockmodel.

1. Introduction

The analysis of network data appears in different scientific fields, such as social sciences, communication networks and many others, including a recent explosion in the field of molecular biology (with the study of metabolic networks, transcriptional regulatory networks and proteins interactions networks). The literature is vast, and we refer for instance to Boccaletti et al. [2006], Goldenberg et al. [2010] and the book by Kolaczyk [2009] for interesting introductions to networks.

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Erdős and Rényi [1959] introduced one of the earliest and most studied random graph model, in which binary random graphs are considered as a set of independent and identically distributed (i.i.d.) Bernoulli edge variables over a fixed set of nodes. This model is however too homogeneous to capture some important features of real networks, such as the presence of 'hubs', namely highly connected nodes. This lack of heterogeneity led to the introduction of mixture versions of the simple Erdős-Rényi model. So-called 'stochastic blockmodels' [Daudin et al., 2008, Frank and Harary, 1982, Holland et al., 1983, Snijders and Nowicki, 1997] were introduced in various forms, primarily in social sciences to study relational data. In this context, the nodes are partitioned into latent groups (blocks) characterizing the relations between nodes. Blockmodeling thus refers to the particular structure of the adjacency matrix of the graph (i.e. the matrix containing the edges indicators). By reordering the nodes with respect to the groups they belong to, this matrix exhibits blocks. Diagonal and off-diagonal blocks respectively represent intra-group and inter-group connections. In case where blocks exhibit the same behaviour within their type (diagonal or off-diagonal), we further obtain what we call an affiliation structure. Affiliation structures are parsimonious in the number of parameters they use and may model a lot of situations. For instance, affiliation models encompass both community structures and disassortative mixing [Newman and Leicht, 2007]. In the first case (community structure) the intra-group connectivities are high while the inter-group connectivities are low. Disassortative mixing rather corresponds to high inter-group connectivities and low intra-group connectivities.

Many networks are or can be weighted (or in other words valued). Those weights are precious additional information on the graph and should be taken into account in their analysis. Well-known examples of weighted networks include airline traffic data between airports, co-authorship networks of scientists [Barrat et al., 2004] or when rather considering the corresponding adjacency matrix, financial correlation matrices [Laloux et al., 1999]. While the two first examples correspond to sparse weighted networks, the last one concerns dense (or complete) weighted graphs. Weighted networks are a way of integrating heterogeneous data and their analysis is thus of primary importance [Newman, 2004]. Community detection (i.e. the problem of finding clusters of nodes with many edges joining vertices of the same cluster and comparatively few edges joining vertices of different clusters) has been widely considered in the context of weighted graphs [see for instance Fortunato, 2010]. While community detection methods are mainly algorithmic, another approach is to rely on generative models and random graphs mixtures. Stochastic blockmodels for analyzing random graphs with non binary relations between nodes have been considered either in the case of a finite number of possible relations [Nowicki and Snijders, 2001] or for more general weighted graphs [Mariadassou et al., 2010]. Our approach builds on these latter references. We also point out the existence of generalized blockmodels for valued networks [Ziberna, 2007, Doreian et al., 2005] which however do not rely on a probabilistic model as we shall do here.

In this article, we will be interested in both binary and weighted random graphs and will focus on mixture models. We mention the existence of an in-
creasing literature on two different related concepts: mixed membership [Airoldi et al., 2008, Erosheva et al., 2004] and overlapping [Latouche et al., 2011a] stochastic blockmodels for binary networks, in which nodes may belong to several classes. However these models are beyond the scope of the present work.

Current estimation procedures in random graph mixture models rely on approximations of the likelihood, which is itself intractable due to the presence of the non observed groups. Either expectation maximization (EM) algorithm [Dempster et al., 1977], or Bayesian approaches are at the core of these strategies. Both rely on the computation of the distribution of the hidden nodes states, conditional on the observed edges variables. However, in the particular case of random graph mixtures, the exact computation of this conditional distribution can not be obtained, due to its non-factorized form. Thus, approximate computations are made, leading to what is called ‘variational’ EM or Bayes strategies [Daudin et al., 2008, Latouche et al., 2011b, Picard et al., 2009, Zanghi et al., 2008]. The major drawback of these methods is their relatively large computational time. Besides, even if these methods exhibit good behavior on simulated data, they suffer from a lack of theoretical support. Indeed, two major features of these procedures still lack understanding. First, the quality of the variational approximation is not known, and this approximation may even prevent convergence to local maxima of the likelihood [Gunawardana and Byrne, 2005]. Second, the consistency of the maximum likelihood or of the maximum a posteriori estimators is still an open question in these models.

Here, we propose simple strategies for estimating the parameters of mixture random graph models, in the particular affiliation case. The methods not only rely on established convergence results, but are also simpler than variational approaches. By focusing on small structures (edges and triads) and treating these as if they were (but never assuming they are) independent, we prove that we may recover the main features of an affiliation model. We adopt strategies based on either solving moment equations or maximizing a composite marginal likelihood. A composite marginal likelihood consists in the product of marginal distributions and may replace the likelihood in models with some dependency structure [see for instance Cox and Reid, 2004, Varin, 2008]. In the weighted random graphs case, our result shows that parameters may be estimated relying on a composite likelihood of univariate marginals. This is not the case for binary random graphs, because parameters of mixtures of univariate Bernoulli distributions are not identifiable. However, parameters of mixtures of 3-variate Bernoulli are identifiable [see Allman et al., 2009, Corollary 5]. Thus, in the binary random graph case, we develop moment or composite likelihood methods based on the marginals of triads, namely the 3 random variables \((X_{ij}, X_{ik}, X_{jk})\) induced by a set of 3 nodes \((i, j, k)\).

Once the convergence of our estimators, let us say \(\hat{\theta}_n\) to \(\theta\), has been established, the next question of interest concerns the order at which the discrepancy \(\hat{\theta}_n - \theta\) converges to zero. We establish asymptotic normality results, thus ob-
taining rates of convergence of our procedures. This is in sharp contrast with existing methods and the first insight on the difficult issue of exhibiting (optimal) rates of convergence for estimation procedures in these random graphs models. Indeed, a still open problem may be stated as follows: what is the parametric rate of convergence when observing \( n(n-1)/2 \) (non independent) random variables over a set of \( n \) nodes, distributed according to a random graph model? Is it \( 1/\sqrt{n} \) or \( 1/n \)? In other words, the issue is whether the observation of these potentially \( n(n-1)/2 \) dependent edges variables over a set of \( n \) nodes enables existence of estimation procedures with rates of convergence of the order \( 1/n \) or rather \( 1/\sqrt{n} \). We obtain here theoretical results with rates of convergence of the order at least \( 1/\sqrt{n} \) (which might not be optimal). Moreover, in the degenerate case where the group proportions are equal, the rates of convergence increase to \( 1/n \). Our simulations seem also to indicate rates of convergence faster than \( 1/\sqrt{n} \), that might be due to degeneracies in the limiting variances of our central limit results, i.e. the fact that these variances might be zero.

The paper is organized as follows. In Section 2, we state the different notations, present the general assumptions of our model as well as the main result: a law of large numbers and a central limit theorem for normalized sums of functions of variables over a \( k \)-tuple of nodes. Section 3 focuses on binary random graphs: after introducing the specific model for binary variables, we present two different estimation procedures. The first one (Section 3.1) relies on moment equations and assumes that the group proportions \( \pi \) are known, while the second one (Section 3.2) is more general and relies on composite likelihood. Section 4 presents the weighted random graph model as well as the parameter estimation procedure, relying also on a composite likelihood approach. While a first part of our work focuses on theoretical results about consistency of the procedures, a second part is dedicated to algorithmic issues as well as experiments. In Section 5, we present the implementation of the estimation procedures. A particular attention is paid to the problem of unraveling the latent structure of the model (Section 5.2). In Section 6, the performances of our procedures are illustrated on synthetic data and we also provide the analysis of a real data example. Finally, all the proofs are postponed to Section 7.

2. Model and main result

Let us first give some notations that will be useful throughout this article. For any \( Q \geq 1 \), let \( S_Q \) denote the simplex \( \{(\pi_1, \ldots, \pi_Q); \pi_i \geq 0; \sum_{i=1}^Q \pi_i = 1\} \) and \( V_Q = \{(v_1, \ldots, v_Q), v_i \in \{0,1\}, \sum_{i=1}^Q v_i = 1\} \). For the sake of simplicity, we only consider in the following undirected graphs with no self-loops. Easy generalizations may be done to handle directed graphs, with or without self-loops.

In this section, we define a general mixture model of random graphs in the following way. First, let \( \{Z_i\}_{1 \leq i \leq n} \) be i.i.d. vectors \( Z_i = (Z_{i1}, \ldots, Z_{iQ}) \in V_Q \), following a multinomial distribution \( \mathcal{M}(1, \pi) \), where \( \pi = (\pi_1, \ldots, \pi_Q) \in S_Q \).
Random variable $Z_i$ indicates to which group (among $Q$ possibilities) node $i$ belongs. These random variables are used to introduce heterogeneity in the random graph model.

Next, the observations $\{X_{ij}\}_{1 \leq i < j \leq n}$ are indexed by the node pairs $\{i, j\}$ and take values in a general normed vector space $\mathcal{X}$ (in the next sections, $\mathcal{X} = \{0, 1\}$ or $\mathbb{N}$ or $\mathbb{R}^l$). We then assume that conditional on the latent classes $\{Z_i\}_{1 \leq i \leq n}$, the random variables $\{X_{ij}\}_{1 \leq i < j \leq n}$ are independent. Moreover, the conditional distribution of $X_{ij}$ depends only on $Z_i, Z_j$ and has finite variance. The model may thus be summarized in the following way

**General model**

\[
\begin{align*}
\cdot & \quad \{Z_i\}_{1 \leq i \leq n} \text{ i.i.d. vectors in } \mathcal{V}_Q, \text{ with distribution } \mathcal{M}(1, \pi), \\
\cdot & \quad \{X_{ij}\}_{1 \leq i < j \leq n} \text{ observations in } \mathcal{X}, \\
\cdot & \quad \mathbb{P}(\{X_{ij}\}_{1 \leq i < j \leq n}|\{Z_i\}_{1 \leq i \leq n}) = \otimes_{1 \leq i < j \leq n} \mathbb{P}(X_{ij}|Z_i, Z_j), \\
\cdot & \quad \mathbb{E}(\|X_{ij}\|^2|Z_i, Z_j) < +\infty.
\end{align*}
\]

(1)

It may be worth noting that the variables $\{X_{ij}\}_{1 \leq i < j \leq n}$ are not independent in general, but we often make use of the fact that sets of non adjacent edges induce independent random variables. More precisely, if $I, J \subset \{1, \ldots, n\}$ with $I \cap J = \emptyset$, then $\{X_{ij}\}_{(i, j) \in I^2}$ and $\{X_{ij}\}_{(i, j) \in J^2}$ are independent.

In the next sections, we will focus on the particular affiliation mixture model, where the conditional distribution of an edge variable $X_{ij}$ only depends on whether the endpoints $i, j$ belong to the same group (i.e. $Z_i = Z_j$). We shall thus refer to the assumption

**Affiliation structure** : $\mathbb{P}(X_{ij}|Z_i, Z_j) = \mathbb{P}(X_{ij}|1_{Z_i = Z_j})$, \hspace{1cm} (2)

where $1_A$ is the indicator function of the set $A$.

Moreover, in the particular case of equal group proportions and affiliation structure, we shall observe some degeneracy phenomenas. These are due to the fact that the distribution becomes invariant under permutation of the specific values of the node groups (see Lemma 1 in Section 7 for more details). For later use, we thus also introduce the equal group proportions setting

**Equal group proportions case** : $\pi_q = 1/Q$ for any $q \in \{1, \ldots, Q\}$. \hspace{1cm} (3)

Let us now motivate the following developments. Under the affiliation structure assumption, the distribution of a single edge follows a two-components mixture of the form

\[X_{ij} \sim \gamma \mathbb{P}(X_{ij}|Z_i = Z_j) + (1 - \gamma) \mathbb{P}(X_{ij}|Z_i \neq Z_j).\]

For weighted random graphs, we shall assume a parametric form for this absolutely continuous conditional distribution, namely $\mathbb{P}(X_{ij}|Z_i = Z_j) = \mathbb{P}_{\theta_{in}}(X_{ij})$ and $\mathbb{P}(X_{ij}|Z_i \neq Z_j) = \mathbb{P}_{\theta_{out}}(X_{ij})$. The vast majority of families of parametric absolutely continuous distributions give finite mixtures whose parameters are identifiable. This is equivalent to saying that $\mathbb{E}[\log(\gamma \mathbb{P}_{\theta_{in}}(X_{12}) + (1 - \gamma) \mathbb{P}_{\theta_{out}}(X_{12}))]$ has a unique maximum at the true parameter value $(\theta_{in}, \theta_{out})$.
Moreover, for any composite of a is at the core of maximum likelihood estimation and motivates the introduction for the limiting distribution \( \Sigma_g \)

which is not the model likelihood as the random variables \( X_{ij} \) are not independent. Its usefulness to estimate the parameters relies on whether the renormalized criterion \( L_X^{\text{comp}}(\theta) \) converges to the expectation \( E[\log(\gamma P_{\theta_{in}}(X_{ij}) + (1 - \gamma)P_{\theta_{out}}(X_{ij}))] \). We shall prove below that the answer is yes and thus, maximizing \( L_X^{\text{comp}}(\theta) \) with respect to \( \theta \) is a good strategy.

In the binary random graph case however, the strategy has to be modified because each random variable \( X_{ij} \) follows a mixture of univariate Bernoulli distributions whose parameters are not identifiable. We thus rather consider mixtures of 3-variate Bernoulli distributions which appear to be sufficient to consistently estimate the parameters.

Thus, we are now interested more generally in the behavior of empirical sums of functions of the random variables induced by a \( k \)-tuple of nodes. These empirical estimators are at the core of the estimation procedures that we shall later consider. To this aim, let us introduce some more notations.

Define the set of nodes \( I = \{1, \ldots, n\} \) and the set of \( k \) distinct nodes \( I_k = \{(i_1, \ldots, i_k) \in I^k; i_j \neq i_l \text{ for any } j \neq l\} \). \( I_k \) is also the set of injective maps from \( \{1, \ldots, k\} \) to \( I = \{1, \ldots, n\} \). For any fixed integer \( k \geq 1 \), and any \( k \)-tuple of nodes \( i = (i_1, \ldots, i_k) \in I_k \), we let \( X_i = (X_{i_1i_2}, X_{i_1i_k}, X_{i_2i_k}, \ldots, X_{i_{k-1}i_k}) \) be the vector of \( p = \binom{k}{2} \) random variables induced by the \( k \)-tuple of nodes \( i \). Moreover, for any \( s \geq 1 \) and any measurable function \( g : \mathcal{X}^p \to \mathbb{R}^s \), we let

\[
\hat{m}_g = \frac{(n - k)!}{n!} \sum_{i \in I_k} g(X_i) \quad \text{and} \quad m_g = E(g(X^{(1, \ldots, k)})).
\]

The next theorem establishes a strong law of large numbers as well as asymptotic normality of the estimator \( \hat{m}_g \). As the random variables \( \{X_{ij}\} \) are not independent, consistency (as well as asymptotic normality) of this empirical estimator is not trivial and has to be established carefully.

**Theorem 1.** Under the assumptions of model (1), for any \( k, s \geq 1 \) and \( p = \binom{k}{2} \) and any measurable function \( g : \mathcal{X}^p \to \mathbb{R}^s \) such that \( E(||g(X^{(1, \ldots, k)})||^2) \) is finite, the estimator \( \hat{m}_g \) is consistent

\[
\hat{m}_g \to m_g \text{ almost surely,}
\]

as well as asymptotically normal \( \sqrt{n}(\hat{m}_g - m_g) \sim_{n \to \infty} \mathcal{N}(0, \Sigma_g) \). If we moreover assume an affiliation structure (2) with equal group proportions (3), then \( \Sigma_g = 0 \) and \( n(\hat{m}_g - m_g) \) converges in distribution as \( n \) tends to infinity.

Let us now give some comments about the previous result. First, an expression for the limiting distribution \( \Sigma_g \) is given in the proof of the theorem. Such
an expression is useful for instance in the construction of confidence intervals. However, although our estimators of the model parameters are derived from estimators of the form $\hat{m}_g$, we did not obtain here simple expressions for their limiting variance from an expression of $\Sigma_g$. Thus, rather than the exact form of the limiting distribution, we are more interested here in rates of convergence.

The theorem states that the convergence of $\hat{m}_g$ to $m_g$ happens with a rate at least $1/\sqrt{n}$. In the case where we consider an affiliation structure with equal group proportions, we prove that the limiting variance is null (i.e. $\Sigma_g = 0$), meaning that $\sqrt{n}(\hat{m}_g - m_g)$ converges in probability to zero. We then further prove that the sequence $n(\hat{m}_g - m_g)$ converges in distribution (to some non-Gaussian limit). Thus in this degenerate case, the convergence of $\hat{m}_g$ happens at the faster rate $1/n$.

We shall see that consistency as well as rates of convergence are preserved in the estimation procedures that we deduce from moment estimators of the form $\hat{m}_g$. To our knowledge, this work is the first one giving some insights about consistency and rates of convergence of estimation procedures in random graphs mixtures models.

In the next sections, we consider two particular instances of the mixture model defined in (1): the binary affiliation model (Section 3) and the weighted affiliation model (Section 4).

3. Binary affiliation model

In the case of binary random graphs, we observe binary random variables $\{X_{ij}\}_{1 \leq i < j \leq n}$ indicating presence (1) or absence (0) of an edge between nodes $i$ and $j$. The latent classes $\{Z_i\}_{1 \leq i \leq n}$ are still distributed as i.i.d. multinomial vectors on $V_Q$. Conditional on these latent classes $\{Z_i\}_{1 \leq i \leq n}$, we assume that $\{X_{ij}\}_{1 \leq i < j \leq n}$ are independent Bernoulli $B(\cdot)$ random variables, with parameters depending on the node groups. More precisely, we restrict our attention to the affiliation structure model (2), where nodes connect differently whether they belong to the same group or not. We let

$$\forall q, \ell \in \{1, \ldots, Q\}, \quad X_{ij} | Z_{iq} Z_{j\ell} = 1 \sim \begin{cases} B(\alpha) & \text{if } q = \ell, \\ B(\beta) & \text{if } q \neq \ell. \end{cases}$$  \hfill (4)

Here, $\alpha$ and $\beta$ respectively are the intra-group and the inter-group connectivities and we let $p_{q\ell} = \alpha 1_{q=\ell} + \beta 1_{q\neq \ell}$, for any $1 \leq q, \ell \leq Q$. In the following, we always assume $\alpha \neq \beta$.

The whole parameter space is given by

$$\Pi = \{ (\pi, \alpha, \beta); \pi \in S_Q \cap (0, 1)^Q, \alpha \in (0, 1), \beta \in (0, 1), \alpha \neq \beta \}.$$  

We will use the notation $b(x, p) = p^x (1 - p)^{1-x}$ (where $x \in \{0, 1\}$ and $p \in [0, 1]$) for Bernoulli density with respect to counting measure. Note that in this setup,
the complete data log-likelihood simply writes

\[ L_{X,Z}(\pi,\alpha,\beta) = \log P_{\pi,\alpha,\beta}(\{X_{ij}\}_{1\leq i<j\leq n}, \{Z_i\}_{1\leq i\leq n}) = \sum_{i=1}^{n} \sum_{q=1}^{Q} Z_{iq} \log \pi_q + \sum_{1\leq i<j\leq n} \sum_{q=1}^{Q} Z_{iq} Z_{jq}\{X_{ij} \log \alpha + (1 - X_{ij}) \log(1 - \alpha)\} + \sum_{1\leq i<j\leq n} \sum_{1 \leq q \neq \ell \leq Q} Z_{iq} Z_{j\ell}\{X_{ij} \log \beta + (1 - X_{ij}) \log(1 - \beta)\}. \] (5)

Figure 1 (left part) displays an example of a binary random graph distributed according to this affiliation model.

### 3.1. Moment estimators in the binary affiliation model with known group proportions

The following approach based on moment equations was initially proposed by Frank and Harary [1982] to estimate the connectivity parameters \(\alpha\) and \(\beta\) (as well as, in some cases, the number of groups \(Q\)). The core idea is simple: the moment equations corresponding to the distribution of a triplet \((X_{ij}, X_{ik}, X_{jk})\) give three equations which can be used to estimate the two parameters \(\alpha\) and \(\beta\), as soon as the group proportions (also appearing in these equations) are known. However, this method has not been thoroughly checked by Frank and Harary and may give rise to multiple solutions. Indeed, these authors never discuss uniqueness of the solutions to the system of (non linear) equations they consider. This point has been partly discussed in Allman et al. [2011] and the estimation procedures proposed here are an echo to the identifiability results obtained there.

The following method applies only when the mixture proportions \(\pi\) are known. We develop in Section 5 an algorithmic procedure that iteratively estimates the group proportions in a first step, and the connectivity parameters \((\alpha, \beta)\) in a second step. This second step uses the method we shall now describe.

First, we let \(s_2 = \sum_q \pi_q^2\) and \(s_3 = \sum_q \pi_q^3\). Then, one easily gets the formulas

\[
\begin{align*}
m_1 &:= \mathbb{E}(X_{ij}) = s_2 \alpha + (1 - s_2) \beta, \\
m_2 &:= \mathbb{E}(X_{ij}X_{ik}) = s_3 \alpha^2 + 2(s_2 - s_3) \alpha \beta + (1 - 2s_2 + s_3) \beta^2, \\
m_3 &:= \mathbb{E}(X_{ij}X_{ik}X_{jk}) = s_3 \alpha^3 + 3(s_2 - s_3) \alpha^2 \beta + (1 - 3s_2 + 2s_3) \beta^3.
\end{align*}
\] (6)

Since any triplet \((X_{ij}, X_{ik}, X_{jk})\) takes finitely many states, its distribution is completely characterized by a finite number of its moments. In the binary affiliation mixture model context, there are in fact only three different moments induced by a triplet distribution. Thus, the previous three moment equations completely characterize the distribution of any triplet \((X_{ij}, X_{ik}, X_{jk})\). Note that
looking at higher order motifs, namely at the distribution of a set of \( p = \binom{k}{2} \) random variables over a set of \( k \) nodes for \( k \geq 4 \) would provide more equations but would also lead to more intricate methods [see for instance Allman et al., 2011].

In the article by Allman et al. [2011], the possible solutions (with respect to \( \alpha \) and \( \beta \)) of this set of moment equations are examined. Their result distinguishes the equal group proportions case \( (\pi_q = 1/Q, \forall 1 \leq q \leq Q) \) where a degeneracy phenomenon takes place.

**Theorem 2.** [Allman et al., 2011]. If \( m_2 \neq m_1^2 \), then the \( \pi_q \)'s are unequal and we can recover the parameters \( \beta \) and \( \alpha \) via the rational formulas

\[
\beta = \frac{(s_3 - s_2 s_3)m_1^3 + (s_2^3 - s_3)m_2 m_1 + (s_3 s_2 - s_2^3)m_3}{(m_2^2 - m_2)(2s_2^3 - 3s_3 s_2 + s_3)} \quad \text{and} \quad \alpha = \frac{m_1 + (s_2 - 1)\beta}{s_2}. \tag{7}
\]

If \( m_2 = m_1^2 \), then the \( \pi_q \)'s are equal and we have

\[
\beta = m_1 + \left( \frac{m_1^3 - m_3}{Q - 1} \right)^{1/3} \quad \text{and} \quad \alpha = Qm_1 + (1 - Q)\beta. \tag{8}
\]

As soon as \( s_2 \) and \( s_3 \) are known, by plugging estimators of the moments \( m_i \) into these equations, we obtain simple estimates for parameters \( \alpha \) and \( \beta \). We thus first introduce empirical moment estimators \( \hat{m}_i \), defined by

\[
\hat{m}_1 = \frac{1}{n(n-1)} \sum_{(i,j) \in I_2} X_{ij}, \quad \hat{m}_2 = \frac{1}{n(n-1)(n-2)} \sum_{(i,j,k) \in I_3} X_{ij}X_{ik},
\]

\[
\hat{m}_3 = \frac{1}{n(n-1)(n-2)} \sum_{(i,j,k) \in I_3} X_{ij}X_{ik}X_{jk}. \tag{9}
\]

Note that those estimators are all of the form \( \hat{m}_g \) for some specific function \( g \). Thus, their consistency is a consequence of Theorem 1. We are then able to prove the following result.

**Theorem 3.** In the binary affiliation model specified by (1) and (4), when the group proportions \( \pi \) are supposed to be known, we have the following results.

i) When the \( \pi_q \)'s are unequal, the estimators \( (\hat{\alpha}, \hat{\beta}) \) defined through (7) where the \( m_i \)'s are replaced by the \( \hat{m}_i \)'s, converge almost surely to \( (\alpha, \beta) \). Moreover, the rate of this convergence is at least \( 1/\sqrt{n} \).

ii) When the \( \pi_q \)'s are equal, the estimators \( (\hat{\alpha}, \hat{\beta}) \) defined through (8) where the \( m_i \)'s are replaced by the \( \hat{m}_i \)'s, converge almost surely to \( (\alpha, \beta) \). Moreover, the rate of this convergence is at least \( 1/n \).

The performances of this method, combined with an iterative procedure to uncover the latent structure and estimate the group proportions are illustrated in Section 6.
3.2. **M-estimators in the binary affiliation model**

We shall now describe another parameter estimation procedure based on M-estimators [see for instance van der Vaart, 1998, Chapter 5], i.e. estimators maximizing some criterion (here, a composite likelihood). This procedure is more direct than the previous moments method developed in Section 3.1, as it does not assume a preliminary knowledge of the group proportions $\pi$.

Let us recall that $X_{12}^{(i,j,k)} = (X_{ij}, X_{ik}, X_{jk})$. The random vectors $X_{12}^{(i,j,k)}$ form a set of non independent, but identically distributed vectors, with distribution does not assume a preliminary knowledge of the group proportions $\pi$.

We emphasize that the distribution (10) is a constrained 3-variate Bernoulli mixture. Parameters identifiability of such a distribution is further discussed below. However we shall already remark that while parameters of mixture models may in general be identified only up to a permutation on the node labels, the constrained form of the mixture (10) has the following consequence: the parameters $\alpha$ and $\beta$ will be exactly recovered as soon as the mixture components are identified from (10) and whatever the labelling of these mixture components. Indeed, among the five unordered components of the mixture, only three of them will be the product of two identical one-dimensional distributions, times a different one. The parameter $\beta$ is then the parameter appearing in exactly two marginals in any of those three components.
Let us consider as our criterion a composite marginal log-likelihood of the observations

\[ L_{\text{compo}}(\pi, \alpha, \beta) = \sum_{(i,j,k) \in I_3} \log P_{\pi,\alpha,\beta}(X^{(i,j,k)}). \]  

(12)

We stress that this quantity is not derived from the marginal of the model complete data likelihood (expressed in (5)) and is simpler. It would be the log-likelihood of the observations if the triplets \( \{X^{(i,j,k)}\}_{(i,j,k) \in I_3} \) were independent, which is obviously not the case. We now define our estimators as

\[ (\hat{\pi}_n, \hat{\alpha}_n, \hat{\beta}_n) = \arg\max_{\pi, \alpha, \beta} L_{\text{compo}}(\pi, \alpha, \beta). \]  

(13)

Note that according to the non uniqueness of group proportions \( \pi \) corresponding to mixture proportions \( \gamma \), the maximum with respect to \( \pi \) in the above equation may not be unique. We also let \( \hat{\gamma}_n \) be defined from \( \hat{\pi}_n \) through (11).

Using Theorem 1, the renormalized criterion (12) converges to a limit. The key point here is that under an identifiability assumption on the model parameters, this limit is a function whose maximum is attained only at the true parameter value \( (\gamma, \alpha, \beta) \). Using classical results from M-estimators [van der Vaart, 1998, Wald, 1949], we can then obtain consistency and asymptotic normality of the estimators defined through (13). We thus need here to assume the identifiability of the model parameters.

Assumption 1. The parameters \( \gamma, \alpha, \beta \) of the model defined by (10) are identifiable. In other words, if there exist \( (\pi, \alpha, \beta) \) and \( (\pi', \alpha', \beta') \) such that for any \( (x, y, z) \in \{0,1\}^3 \) we have

\[ P_{\pi,\alpha,\beta}(X_{12} = x, X_{13} = y, X_{23} = z) = P_{\pi',\alpha',\beta'}(X_{12} = x, X_{13} = y, X_{23} = z), \]

then \( (\gamma, \alpha, \beta) = (\gamma', \alpha', \beta') \), where \( \gamma, \gamma' \) are defined through (11) as functions of \( \pi, \pi' \), respectively.

Let us now give some comments on this assumption. We first mention that identifiability of all the parameters \( (\pi, \alpha, \beta) \) in the model defined by (1) and (4), i.e. relying on the full distribution over \( \bigcup_{n \geq 1} \{0,1\}^{(2)} \) (comprising the marginal distributions of the random graphs over a set of \( n \) nodes, for any value of \( n \)), is a difficult issue, for which only partial results have been obtained in Allman et al. [2011]. Surprisingly, the results under the affiliation assumption are more difficult to obtain than in the non affiliation case. The question here is slightly different and we ask whether a triplet distribution (10) is sufficient to identify only \( \alpha \) and \( \beta \) (as well as the corresponding proportions \( \gamma \)). As already pointed out, the distribution (10) is a constrained distribution from the larger class of 3-variate Bernoulli mixtures. In the case of (unconstrained) finite mixtures of multivariate (or 3-variate) Bernoulli distributions, while the models have been used for decades and were strongly believed to be identifiable [Carreira-Perpiñán and Renals, 2000], the rigorous corresponding result has been established only
very recently and using rather elaborate techniques [see Allman et al., 2009, Corollary 5]. Unfortunately, this latter result does not apply directly here. While this might be hard to establish, we strongly believe that $\gamma, \alpha, \beta$ are identifiable from the distribution (10) and we advocate that from the simulations we performed, it seems a reasonable assumption to make.

In the following, we also restrict our attention to compact parameter spaces, as this greatly simplifies the proofs and is not much restrictive. Generalizations could be done at the cost of technicalities [see for instance van der Vaart, 1998, Chapter 5].

**Assumption 2.** Assume that there exists some $\delta > 0$ such that the parameter space is restricted to $\Pi_\delta = \{ (\pi, \alpha, \beta) \in \Pi; \forall 1 \leq q \leq Q, \pi_q \geq \delta, \alpha \in [\delta, 1 - \delta], \beta \in [\delta, 1 - \delta] \}$.

We are then able to prove the following result.

**Theorem 4.** In the model defined by (1) and (4), under Assumptions 1 and 2, the estimators $(\hat{\gamma}_n, \hat{\alpha}_n, \hat{\beta}_n)$ defined by (13) are consistent, as the sample size $n$ grows to infinity. Moreover, the rate of this convergence is at least $1/\sqrt{n}$ and increases to $1/n$ in the particular case of equal group proportions (3).

Let us now comment this result. We prove that the rate of convergence of our estimators is at least $1/\sqrt{n}$. However, our simulations (see Section 6) seem to exhibit a faster rate, indicating that the limiting covariance matrix of the discrepancy $\sqrt{n}(\hat{\gamma}_n - \gamma, \hat{\alpha}_n - \alpha, \hat{\beta}_n - \beta)$ might be zero, even beyond the case of equal group proportions. Note also that when $Q \leq 3$, a consequence of the above result is that the estimator of the group proportions $\hat{\pi}_n$ defined through $\hat{\gamma}_n$ as the unique solution to the system of equations (11), is also consistent and converges with a rate at least $1/\sqrt{n}$.

As it is always the case for mixture models, the (composite) log-likelihood (12) cannot be computed exactly (except for very small sample sizes). Approximate computation of the estimators in (13) can be done using an EM algorithm [Dempster et al., 1977]. This procedure is presented in Section 5.1. It is known [Wu, 1983] that, under reasonable assumptions, the EM algorithm will give a solution converging to the estimators defined by (13), as the number of iterates grows to infinity.

### 4. Weighted random graphs

In this section, we focus on a particular instance of model (1) for weighted random graphs. The observations are random variables $\{X_{ij}\}_{1 \leq i < j \leq n}$ that are either equal to 0, indicating the absence of an edge between nodes $i$ and $j$, or a non-null real number, indicating the weight of the corresponding edge. We still assume that conditional on the latent structure $\{Z_i\}_{1 \leq i \leq n}$, the random variables $\{X_{ij}\}_{1 \leq i < j \leq n}$ are independent, and the distribution of each $X_{ij}$ only depends
on $Z_i$ and $Z_j$. We now further specify the model by assuming the following form for this distribution

$$\forall q, \ell \in \{1, \ldots, Q\}, \quad X_{ij} | Z_{iq}Z_{j\ell} = 1 \sim p_{q\ell} f(\cdot, \theta_{q\ell}) + (1 - p_{q\ell}) \delta_0(\cdot), \quad (14)$$

where $\{f(\cdot, \theta), \theta \in \Theta\}$ is a parametric family of distributions, $\delta_0$ is the Dirac measure at 0 and $p_{q\ell} \in (0, 1]$ are sparsity parameters. We let $p = \{p_{q\ell}\}$ and $\theta = \{\theta_{q\ell}\}$. The conditional distribution of $X_{ij}$ is thus a mixture of a Dirac distribution at zero accounting for non present edges, with proportion given by the sparsity parameter $p$ (which can be 1 in case of a complete weighted graph) and a parametric distribution with density $f$ that gives the weight of present edges. We focus on two different sparsity structures

- either the sparsity is constant across the graph: $p_{q\ell} = p, \forall 1 \leq q, \ell \leq Q$;
- or the sparsity parameters model an affiliation structure $p_{q\ell} = \alpha_{q=q=\ell} + \beta_{q \neq \ell}$ and we assume $\alpha \neq \beta$.

We moreover assume that we know the sparsity structure type. In any case, the connectivity parameter $\theta$ is assumed to take exactly two different values

$$\forall q, \ell \in \{1, \ldots, Q\}, \theta_{q\ell} = \begin{cases} \theta_{in} & \text{if } q = \ell, \\ \theta_{out} & \text{if } q \neq \ell, \end{cases}$$

with $\theta_{in} \neq \theta_{out}$. For identifiability reasons, we also constrain the parametric family $\{f(\cdot, \theta), \theta \in \Theta\}$ such that any distribution in this set admits a continuous cumulative distribution function (c.d.f.) at zero. Indeed, if this were not the case, it would not be possible to distinguish between a zero weight and an absent edge. Note that this model satisfies the affiliation assumption given by (2). Here, the complete data log-likelihood simply writes

$$L_{X,Z}(\pi, p, \theta) = \log \mathbb{P}_{\pi, p, \theta}(\{X_{ij}\}_{1 \leq i < j \leq n}, \{Z_i\}_{1 \leq i \leq n}) = \sum_{i=1}^{n} \sum_{q=1}^{Q} Z_{iq} \log \pi_q + \sum_{1 \leq i < j \leq n} \sum_{1 \leq q, \ell \leq Q} Z_{iq} Z_{j\ell} \left\{1_{X_{ij} \neq 0}(\log f(X_{ij}, \theta_{q\ell}) + \log p_{q\ell}) + 1_{X_{ij} = 0} \log(1 - p_{q\ell}) \right\}. \quad (15)$$

We now give some examples of parametric families $\{f(\cdot, \theta), \theta \in \Theta\}$ that could be used as weights (or values) on the edges.

**Example 1.** Let $\theta = (\mu, \sigma^2) \in \mathbb{R} \times (0, +\infty)$ and consider $f(\cdot, \theta)$ the density of the Gaussian distribution with mean $\mu$ and variance $\sigma^2$.

**Example 2.** Let $\theta \in (0, +\infty)$ and consider $f(\cdot, \theta)$ the density (with respect to the counting measure) of the Poisson distribution, with parameter $\theta$, truncated at zero. Namely,

$$\forall k \geq 1, \quad f(k, \theta) = \frac{\theta^k}{k!} (e^\theta - 1)^{-1}.$$
Note that in the above example, the Poisson distribution is truncated at zero because, as previously mentioned, it would not be possible to distinguish a zero-valued weight from an absent edge. Sparsity of the graph is modeled through the parameter $p$ only and the density $f(\cdot, \theta)$ concerns weights on present edges.

Figure 1 illustrates the difference between binary and weighted random graph affiliation models. For example the weighted graph of Figure 1 displays no binary affiliation structure: if the weights were truncated using the function $x \rightarrow 1_{x \neq 0}$, we would not obtain that the two groups have different intra-group and inter-group connectivities. This means that classical community clustering algorithms would fail to find any meaningful structure on this type of graph.

To our knowledge, this model has never been proposed in this form in the literature. In particular, the closest form is given in Mariadassou et al. [2010] who do not introduce a possible Dirac mass at zero to enable sparsity of the graph.

Let us now describe our estimation procedure based on $M$-estimators and a composite likelihood criterion. We proceed in two steps and first estimate the sparsity parameter, relying on an induced binary random graph. In a second step, we plug-in this estimator and focus on the connectivity parameters $\theta$ by relying only on the present edges.

**Estimating the sparsity parameter.** Let us first consider the case where $p_{q\ell} = p$. Then, we naturally estimate the sparsity parameter $p$ by

$$\hat{p}_n = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} 1_{X_{ij}=0}.$$ 

The consistency, as well as well as the rate of convergence of this estimator follows from Theorem 1.
In the case where the sparsity parameter rather satisfies $p_{q\ell} = \alpha 1_{q=\ell} + \beta 1_{q\neq \ell}$, with $\alpha \neq \beta$, we rely on the underlying binary random graph (obtained by setting $Y_{ij} = 1_{X_{ij} \neq 0}$) and apply the results of Sections 3.1 or 3.2 to consistently estimate $\alpha$ and $\beta$.

**Estimating the connectivity parameter $\theta$.** The present edges $X_{ij}$ (where $i,j$ are such that $X_{ij} \neq 0$) are non independent random variables, distributed according to a simple univariate mixture model $\sum_{q\ell} \pi_q \pi_\ell p_{q\ell} f(\cdot, \theta_{q\ell})$. For classical distributions $f(\cdot, \theta)$, it is possible to estimate the connectivity parameters $\theta_{q\ell}$ of this univariate mixture directly. In fact, we prove that as soon as the parameters $\{\theta_{q\ell}\}$ are uniquely identified from the mixture $\sum_{q\ell} \pi_q \pi_\ell p_{q\ell} f(\cdot, \theta_{q\ell})$ and for regular parametric families $\{f(\cdot, \theta), \theta \in \Theta\}$, a consequence of Theorem 1 is that maximizing a composite likelihood of the set of present edge variables provides a consistent estimator of the parameters. Let us introduce the needed assumptions.

**Assumption 3.** The parameters of finite mixtures of the family of measures $\mathcal{F} = \{f(\cdot; \theta); \theta \in \Theta\}$ are identifiable (up to label swapping). In other words, for any integer $m \geq 1$,

$$\text{if } \sum_{i=1}^{m} \lambda_i f(\cdot, \theta_i) = \sum_{i=1}^{m} \lambda'_i f(\cdot, \theta'_i) \text{ then } \sum_{i=1}^{m} \delta_{\theta_i}(\cdot) = \sum_{i=1}^{m} \lambda'_i \delta_{\theta'_i}(\cdot).$$

**Example 1, 2 (continued).** Note that both the families of Gaussian and truncated Poisson densities satisfy Assumption 3. More generally, a wide range of parametric families of densities on $\mathbb{R}$ satisfy Assumption 3 [see Section 3.1 in Titterington et al., 1985, for more details].

The next assumption deals with regularity conditions on the model. Note that this assumption could be weakened using the concept of differentiability in quadratic mean [see for instance van der Vaart, 1998].

**Assumption 4.** The functions $\theta \mapsto f(\cdot, \theta)$ are twice continuously differentiable on $\Theta$.

The last assumption is only technical and not very restrictive. It requires the parameter set to be compact and could be weakened at the cost of some technicalities.

**Assumption 5.** Assume that there exists some $\delta > 0$ and some compact subset $\Theta_c \subset \Theta$ such that the parameter space is restricted to the set $\{(\pi, p, \theta); 1 \leq q \leq Q, \pi_q \geq \delta, p \in [\delta, 1-\delta], \theta \in \Theta_c\}$.

Now, each present edge variable $X_{ij}$ such that $X_{ij} \neq 0$ is distributed according to the mixture $\sum_{1 \leq q, \ell \leq Q} \pi_q \pi_\ell p_{q\ell} f(\cdot, \theta_{q\ell})$. As there are only two different
components in this mixture, we express it in the more convenient form

\[ P_{\pi, p, \theta}(X_{ij}) = \left\{ \sum_{q=1}^{Q} \pi_q^2 p_{qq}\right\} f(X_{ij}; \theta_{in}) + \left\{ \sum_{1 \leq q \neq \ell \leq Q} \pi_q \pi_{\ell} p_{q\ell}\right\} f(X_{ij}; \theta_{out}) \]

\[ := \gamma_{in} f(X_{ij}; \theta_{in}) + \gamma_{out} f(X_{ij}; \theta_{out}). \quad (16) \]

We consider a composite log-likelihood of present edges defined by

\[ L^{\text{compo}}(\pi, p, \theta) = \sum_{1 \leq i < j \leq n} \log(\gamma_{in} f(X_{ij}; \theta_{in}) + \gamma_{out} f(X_{ij}; \theta_{out})). \quad (17) \]

We stress that this quantity is not derived from the marginal of the model complete data likelihood (expressed in (15)) and is simpler. We now define estimators as

\[ \hat{\theta}_n = \{ \hat{\theta}_{in}, \hat{\theta}_{out} \} = \arg\max_{\theta} L^{\text{compo}}(\pi, \hat{p}_n, \theta), \quad (18) \]

where \( \hat{p}_n \) is a preliminary step estimator of \( p \). Note that due to the label swapping issue on the hidden states, we estimate the set of values \( \{ \theta_{in}, \theta_{out} \} \) and cannot distinguish \( \theta_{in} \) from \( \theta_{out} \). Section 5.2 further deals with this issue.

We are now able to prove the following theorem.

**Theorem 5.** In the model defined by (1) and (14), under Assumptions 3, 4 and 5, the set of unordered M-estimators \( \hat{\theta}_n = \{ \hat{\theta}_{in}, \hat{\theta}_{out} \} \) defined by (18) is consistent, as the sample size \( n \) grows to infinity. Moreover, the rate of this convergence is at least \( 1/\sqrt{n} \) and increases to \( 1/n \) in the particular case of equal group proportions (3).

The proof mainly relies on the consistency of the normalized criterion (17). This point is a direct consequence of Theorem 1. Then, from the criterion consistency, the identifiability and the regularity assumptions, one can derive the consistency of the corresponding M-estimator from classical theory [van der Vaart, 1998, Wald, 1949].

As already noted in the case of Theorem 4, our result establishes a rate of convergence equal at least to \( 1/\sqrt{n} \). The simulations (Section 6) seem to indicate that the rate may in fact be faster, a phenomenon that may be due to the degeneracy of the limiting variance of \( \sqrt{n}(\hat{\theta}_n - \theta) \).

As for M-estimators in the binary case, we shall approximate this maximum (composite) likelihood estimator using an EM procedure [Dempster et al., 1977] whose convergence properties are well-established [Wu, 1983]. Contrarily the procedure presented in Section 3.2 where we need to adapt the EM framework to our specific model, we rely here on the classical EM algorithm and thus do not recall it.

5. Algorithms

In this section, we provide tools to implement the procedures previously described, as well as a complement on the issue of recovering the latent structure
of a graph.

5.1. EM algorithm with triplets

Let us describe in this section the EM algorithm developed to approximate the estimators defined by (13).

In the following, each set of three nodes \( \{i, j, k\} \) corresponds to an index \( \bar{i} \) ranging over the set \( \{1, \ldots, N\} \), where \( N = n(n-1)(n-2) \) is the total number of triplets. We let \( X^{\bar{i}} = (X^{1\bar{i}}, X^{2\bar{i}}, X^{3\bar{i}}) \) be one of the observed triplets (namely each \( X^{i\bar{i}} \) for \( 1 \leq j \leq 3 \) corresponds to some former random variable \( X^{st} \) for some \( 1 \leq s, t \leq n \)) and \( U_{\bar{i}} \sim M(1, \gamma) \) is the vector encoding the corresponding hidden state. Namely, \( U_{\bar{i}} \in \mathcal{V}_2 \). We also denote by \( \tau_{ik} \) the posterior probability of node triplet \( \bar{i} \) being in state \( k \), conditional on the observation \( X^{\bar{i}} \), namely \( \tau_{ik} = P(U_{ik} = 1|X^{\bar{i}}) \), for \( 1 \leq k \leq 5 \) and \( 1 \leq i \leq N \). Moreover, we encode the fact that, conditional on the 5 different hidden states of \( U \), each coordinate of \( X \) is distributed according either to \( B(\alpha) \) or \( B(\beta) \), using the following notation

\[
\delta_{jk} = (\delta_{jk}^1, \delta_{jk}^2) = (1_{X^{1\bar{i}}|U_{ik} = 1 \sim B(\alpha)}, 1_{X^{2\bar{i}}|U_{ik} = 1 \sim B(\beta)}),
\]

for all \( 1 \leq j \leq 3 \), \( 1 \leq k \leq 5 \) and any \( 1 \leq i \leq N \). Note that \( \delta_{jk} \) is deterministic and that \( \delta_{jk} \in \mathcal{V}_2 \). With these notations, we are in the situation where we consider a composite likelihood (12) of random vectors \( \{X^{\bar{i}}\}^{1 \leq i \leq N} \) from the mixture of 5 different 3-dimensional Bernoulli distributions, the latent classes being the random vectors \( \{U_{ik}\}^{1 \leq i \leq N} \).

The EM-algorithm is intended to iteratively compute and optimize, with respect to \( (\gamma, \alpha, \beta) \) the function

\[
Q((\gamma, \alpha, \beta); (\gamma^{(s)}, \alpha^{(s)}, \beta^{(s)})) = \mathbb{E}_{\gamma^{(s)}, \alpha^{(s)}, \beta^{(s)}} \left[ \log \mathbb{P} \left( \gamma, \alpha, \beta \right) \left( \{U_{ik}\}^{1 \leq i \leq N}, \{X^{\bar{i}}\}^{1 \leq i \leq N} \right) \right],
\]

using the current value of the parameter \( (\gamma^{(s)}, \alpha^{(s)}, \beta^{(s)}) \). If we let \( \tau_{ik}^{(s)} = \mathbb{P}_{\gamma^{(s)}, \alpha^{(s)}, \beta^{(s)}}(U_{ik} = 1|X^{\bar{i}}) \), we can write

\[
Q((\gamma, \alpha, \beta); (\gamma^{(s)}, \alpha^{(s)}, \beta^{(s)})) = \sum_{i=1}^{N} \sum_{k=1}^{5} \tau_{ik}^{(s)} \log \gamma_k + \sum_{i=1}^{N} \sum_{k=1}^{5} \tau_{ik}^{(s)} \sum_{j=1}^{3} \delta_{jk}^1 X^{1\bar{i}} \log \alpha + (1-X^{1\bar{i}}) \log (1-\alpha) + \delta_{jk}^2 \left\{ X^{2\bar{i}} \log \beta + (1-X^{2\bar{i}}) \log (1-\beta) \right\}.
\]

(19)

Starting from an initial value \( (\gamma^{(1)}, \alpha^{(1)}, \beta^{(1)}) \), the EM algorithm proceeds in two iterative steps. At iteration \( s \), the E-step computes the posterior distribution of \( U_{ik} \) conditional on \( X^{\bar{i}} \). Namely,

\[
\tau_{ik}^{(s)} = \frac{\gamma_k^{(s)} \prod_{j=1}^{3} b(X^{1\bar{i}}, \delta_{jk}^1 \alpha^{(s)} + \delta_{jk}^2 \beta^{(s)})}{\sum_{\ell=1}^{5} \gamma_{\ell}^{(s)} \prod_{j=1}^{3} b(X^{1\bar{i}}, \delta_{\ell j}^1 \alpha^{(s)} + \delta_{\ell j}^2 \beta^{(s)})},
\]

where \( b(x, y) = \mathbb{P}(X = x|\gamma = y) \).
for every $1 \leq i \leq N$ and every $1 \leq k \leq 5$. By using (19), we then get the value of $Q((\gamma, \alpha, \beta); (\gamma^{(s)}, \alpha^{(s)}, \beta^{(s)}))$. In the m-step, this quantity is maximized with respect to $(\gamma, \alpha, \beta)$ and the maximizer gives the next value of the parameter $(\gamma^{(s+1)}, \alpha^{(s+1)}, \beta^{(s+1)})$. This step relies on the following equations.

\[
\gamma_k^{(s+1)} = N^{-1} \sum_{i=1}^{N} \tau_{ik}^{(s)}, \quad k = 1, 5, \\
\gamma_k^{(s+1)} = (3N)^{-1} \sum_{i=1}^{N} \tau_{ik}^{(s)} + \tau_{i3}^{(s)} + \tau_{i4}^{(s)}, \quad k = 2, 3, 4, \\
\alpha^{(s+1)} = \left( \sum_{i=1}^{N} \tau_{i1}^{(s)}(X_{i1}^1 + X_{i1}^2 + X_{i1}^3) + \tau_{i2}^{(s)}X_{i1}^3 + \tau_{i3}^{(s)}X_{i1}^2 + \tau_{i4}^{(s)}X_{i1}^1 \right) \\
\beta^{(s+1)} = \left( \sum_{i=1}^{N} 3\tau_{i1}^{(s)} + \tau_{i2}^{(s)} + \tau_{i3}^{(s)} + \tau_{i4}^{(s)} \right)^{-1} \\
\text{subject to } 0 < \alpha < 1, 0 < \beta < 1.
\]

It should be noted that the sum over all the $N$ possible triplets reduces in fact to a sum over 8 different possible patterns for the values of $X^j$. Indeed, the posterior probabilities $\tau_{ik}$ are constant across triplets with the same observed value.

### 5.2. Unraveling the latent structure

The general method we develop in this section aims at recovering the latent structure $\{Z_i\}_{1 \leq i \leq n}$ on the graph nodes. Indeed, the procedures developed in the previous sections only focus on estimating the parameters and do not directly provide an estimate for the node groups.

We rely here on a simple method: we plug-in the estimators obtained from the previous sections in the complete data likelihood of the model (namely the likelihood of the observations $\{X_{ij}\}_{1 \leq i < j \leq n}$ and the latent classes $\{Z_i\}_{1 \leq i \leq n}$). As we do not have estimates of the mixture proportions $\pi$, we simply remove this part from the expression of the complete data likelihood. Then, we simply maximize this criterion (which we call a classification likelihood) with respect to the latent structure $\{Z_i\}_{1 \leq i \leq n}$. In a latter step, we then estimate the unknown proportions $\pi$ by the frequencies observed on the estimated groups $\hat{Z}_i$.

#### Criterion in the binary case.

In this setup, we introduce a criterion $C$, built on the complete data likelihood, where we plugged-in the estimators of $\alpha$ and $\beta$ and removed the dependency on $\pi$. This criterion simply writes

\[
C(\{Z_i\}_{1 \leq i \leq n}) = \sum_{1 \leq i < j \leq n} \sum_{q=1}^{Q} Z_{iq}Z_{jq} \left\{ X_{ij} \log \hat{\alpha} + (1 - X_{ij}) \log(1 - \hat{\alpha}) \right\} \\
+ \sum_{1 \leq i < j \leq n} \sum_{1 \leq q \neq \ell \leq Q} Z_{iq}Z_{j\ell} \left\{ X_{ij} \log \hat{\beta} + (1 - X_{ij}) \log(1 - \hat{\beta}) \right\}.
\]
Criterion in the weighted case. Let us recall that the estimation procedure from Section 4 only recovers the set of unordered values \( \{\theta_{in}, \theta_{out}\} \). As we know these parameters up to permutation only, let \( \{\theta_1, \theta_2\} \) be any label choice for the corresponding estimators. We can consider two different criteria, denoted \( C^{1,2} \) and \( C^{2,1} \), as follows

\[
C^{u,v}(\{Z_i\}_{1 \leq i \leq n}) = \sum_{1 \leq i < j \leq n, 1 \leq q \neq \ell \leq Q} Z_{i\ell}Z_{j\ell} [1_{X_{ij} \neq 0} (\log f(X_{ij}; \hat{\theta}_u) + \log \hat{p}_{q\ell}) + 1_{X_{ij} = 0} \log(1 - \hat{p}_{q\ell})] + \sum_{1 \leq i < j \leq n, 1 \leq q \leq Q} Z_{iq}Z_{jq} [1_{X_{ij} \neq 0} (\log f(X_{ij}; \hat{\theta}_v) + \log \hat{p}_{qq}) + 1_{X_{ij} = 0} \log(1 - \hat{p}_{qq})],
\]

where \( \{u, v\} = \{1, 2\} \). For each of these criteria, we can select the latent structure \( \hat{Z}^{u,v} = (\hat{Z}_1, \ldots, \hat{Z}_n)^{u,v} \) maximizing it. Then, choosing the couple \( (u^*, v^*) \) maximizing the resulting quantity \( C^{u,v}(\hat{Z}^{u,v}) \) seems to be an interesting strategy. We thus finally define our estimated latent structure \( (\hat{Z}_1, \ldots, \hat{Z}_n) \) as \( \hat{Z}^{u^*, v^*} \).

Iterative estimation of the latent structure. In any case (either binary or weighted), we propose to use an iterative procedure to compute the maximum \( \hat{Z} \) of the criterion \( C(\{Z_i\}) \). Starting from an initial value \( Z^{(1)} = (Z_1^{(1)}, \ldots, Z_n^{(1)}) \) of the latent structure, we iterate the following steps. At step \( s \), we (uniformly) choose a node \( i_0 \) and select \( Z^{(s+1)}_{i_0} \) as

\[
Z^{(s+1)}_{i_0} = \arg\max_{1 \leq q \leq Q} C(\{Z_i^{(s)}\}_{i \neq i_0}, Z_{i_0} = q)
\]

while we let \( Z^{(s+1)}_j = Z^{(s)}_j \) for \( j \neq i_0 \). At each time step, we increase the classification likelihood \( C(\{Z_i\}_{1 \leq i \leq n}) \) and thus the procedure eventually converges to a (local) maxima. By using different initial values \( Z^{(1)} = (Z_1^{(1)}, \ldots, Z_n^{(1)}) \), we should finally find the global maxima. Once we estimated the latent groups \( \hat{Z}_i \), we may obtain an estimate of the group proportions \( \hat{\pi} \) by simply taking the corresponding frequencies. The procedure is summarized in Function \texttt{latent.structure}.

**Function latent.structure(graph, parameters)**

input : observed graph, parameter values
output: latent structure and group proportions
Start from latent structure \( \{Z_i\} \)
while convergence is not attained do
  Choose node \( i_0 \)
  Replace \( Z_{i_0} \) with \( \arg\max_q C(\{Z_i\}_{i \neq i_0}, Z_{i_0} = q) \)
Compute group proportions \( \pi \) from \( \{Z_i\} \).
5.3. Complete algorithm description

Algorithm 2 describes the procedures for analyzing binary or weighted random graphs. We introduce a variable 'method' which can take three different values: 'moments' or 'tripletEM' in the binary case and 'weighted' for weighted random graphs. In the weighted case, we moreover use a second variable called 'sparsity' to indicate whether we estimate a global sparsity parameter $p$ ('sparsity=global') or two parameters $\alpha, \beta$ from an affiliation structure ('sparsity=affiliation'). The performances of the procedures proposed in the current section are illustrated in the following one.

 Algorithm 2: Complete algorithm

 if method='moments' then
   Compute $\hat{m}_i$, $i = 1, 2, 3$ from (9)
   // INITIALIZATION
   Start from latent structure $\{Z_i\}$ with proportions $\pi$ and compute $s_2, s_3$
   while convergence is not attained do
     // UPDATE PARAMETERS
     if $\text{abs}(\hat{m}_2 - \hat{m}_1^2) < \epsilon$ then
       Compute $\alpha, \beta$ through (8)
     else
       Compute $\alpha, \beta$ through (7)
     // UPDATE LATENT STRUCTURE
     Apply latent.structure to the current parameter values
   // SPARSITY PARAMETERS
   Transform weights $X_{ij}$ into binary variables $Y_{ij} = 1_{X_{ij} \neq 0}$
   if sparsity='global' then
     Compute $\hat{p} = \frac{2}{n(n-1)} \sum_{i<j} Y_{ij}$
   else
     Estimate $\alpha, \beta$ from EM algorithm with triplets (Section 5.1)
   // CONNECTIVITY PARAMETERS
   Estimate $\{\theta_{in}, \theta_{out}\}$ from EM algorithm with present edges (Section 4)
   // LATENT STRUCTURE
   Apply latent.structure to the parameter values

if method='tripletEM' then
  Estimate $\alpha, \beta$ from EM algorithm with triplets (Section 5.1)
  Apply latent.structure to the parameter values

if method='weighted' then
  // SPARSITY PARAMETERS
  Transform weights $X_{ij}$ into binary variables $Y_{ij} = 1_{X_{ij} \neq 0}$
  if sparsity='global' then
    Compute $\hat{p} = \frac{2}{n(n-1)} \sum_{i<j} Y_{ij}$
  else
    Estimate $\alpha, \beta$ from EM algorithm with triplets (Section 5.1)
  // CONNECTIVITY PARAMETERS
  Estimate $\{\theta_{in}, \theta_{out}\}$ from EM algorithm with present edges (Section 4)
  // LATENT STRUCTURE
  Apply latent.structure to the parameter values
6. Numerical experiments

We carried out a simulation study to examine the bias and variance of the proposed estimators. In the binary affiliation model, we also compared the performance of our proposal with the variational EM (VEM) strategy proposed by Daudin et al. [2008]. Note that Gibbs sampling has already been compared to VEM strategies in Zanghi et al. [2010] and give very similar results. Note also that the weighted affiliation model proposed here is original and we thus cannot compare our results in this case with any other existing implemented method.

6.1. Binary affiliation model

Simulations set-up. In these experiments, we assumed that edges are distributed according to the binary affiliation model described in Section 3. The data were generated in different settings, with the number of groups \( Q \in \{2, 5\} \), the number of vertices \( n \in \{20, 50, 100, 500, 1000\} \). For each of these cases, we created three settings corresponding to models with different ratios of intra and inter-group connectivity parameters (see Table 1). Moreover, we considered two different cases: equal or free group proportions \( \pi \).

In each of these settings, we applied three different methods for estimating the model parameters: the moment method (corresponding to Section 3.1), the triplet EM method (corresponding to Section 3.2) and the variational EM strategy (VEM) proposed by Daudin et al. [2008], that we adapted to constrain it to an affiliation structure. The results for equal or free group proportions were similar and we thus present only the equal group proportions case.

Figure 2 shows the estimated density (over 100 graphs simulations) of the estimators \( \hat{\alpha} \) and \( \hat{\beta} \) for the three algorithms and the three models for graphs with 500 vertices. We see that for a given model the three methods produce estimators with similar densities. In particular, the estimators of \( \alpha \) and \( \beta \) seem to have little or no bias and the variances are of the same order of magnitude for
the three estimation methods. As the behaviour of the estimators of \( \alpha \) and \( \beta \) are comparable over all the simulations, we focus the discussion on the estimation of the parameter \( \alpha \).

![Empirical joint distribution of the estimators \( \hat{\alpha} \) and \( \hat{\beta} \), computed over 100 simulations of graphs with 500 vertices, \( Q = 2 \) groups and equal group proportions. The dotted lines show the true values of \( \alpha \) and \( \beta \).](image)

Figure 3 (top) displays the estimations of \( \alpha \) averaged over 100 graph simulations as a function of the number of graph vertices in log-scale. For all three models, we see that all algorithms do produce unbiased estimation when the number of vertices is large enough. In addition to the asymptotically unbiased estimation, we observe an agreement in the sign of the bias among all algorithms, when the graphs are small. For example, when estimating \( \alpha \) in model 1 where \( (\alpha = 0.3, \beta = 0.03) \), all methods under-estimate \( \alpha \) and over-estimate \( \beta \).

In order to compare the dispersion of all estimators, we consider their empirical standard deviation computed over 100 simulations. Figure 3 (bottom) shows the evolution of the log of the empirical standard deviation of \( \hat{\alpha} \) when the size of the graphs grows from 20 vertices up to 1000 vertices. We see a linear dependence between the log of the graph size and the log standard deviation. The slope of the lines is about \(-1\) which indicates that the standard deviation decreases with rate of the order \(1/n\) (where \(n\) is the number of vertices of the graph). The differences between the intercepts relate to constant factors driving the relations between all rates of convergence. When \( Q = 2 \), we observe very similar intercepts for all methods, both for models 1 and 2. When \( Q = 5 \), VEM appears to converge faster but the order of the standard deviations remain comparable among all estimation methods. For model 3, the moment based
estimations have greater dispersion, but still decrease with the same rate.

We use the adjusted Rand Index [Hubert and Arabie, 1985] to evaluate the agreement between the estimated and the true latent structure. The computa-
tion of the Rand Index is based on a ratio between the number of node pairs belonging to the same and to different classes when considering the actual latent structure versus the estimated one. This index lies between 0 and 1, two identical latent structures having an adjusted Rand Index equal to 1. Figure 4 displays the Rand Index for the three models and five different graph sizes. It appears that the three algorithms allow a reasonable recovery of the latent structure, for models 1 and 2, when the considered graphs have more than 100 vertices. As expected, the larger the number of nodes, the better the recovery of the latent structure we observe. We also notice that our proposed strategy for recovering the latent structure performs as well or better than the variational approach in all cases.

The previous experiments show that the two estimation procedures proposed in this work behave as well or better than the variational based algorithm, both for the parameter estimation and the recovery of the latent structure. Notice also that the moment based method does not depend on any sort of initialization, since it relies on the analytical resolution of a simple system based on triads (order 3 structures).

6.2. Weighted affiliation model

Simulations set-up. In the following experiments, we use a sparsity parameter constant across the graph and non missing edges are distributed according to a Gaussian model as described in Section 4, with different means \( \mu_{\text{in}} \) and \( \mu_{\text{out}} \) and equal variance \( \sigma^2 \). The intricacy of a model is inversely related to the 'distance' between the parameters \( \theta_{\text{in}} \) and \( \theta_{\text{out}} \). We use the Mahalanobis distance \( \Delta = \| (\mu_{\text{in}} - \mu_{\text{out}})/\sigma \| \). Three models are considered with different levels of intricacy: we fix \( \mu_{\text{in}} = 2 \) and \( \mu_{\text{out}} = 1 \), thus \( \Delta = |(\mu_{\text{in}} - \mu_{\text{out}})/\sigma| = 1/\sigma \) which takes the values \( \Delta = 10 \) (model A), \( \Delta = 2 \) (model B) and \( \Delta = 1 \) (model C). We fix the number of groups \( Q = 2 \), equal group proportions and consider different number of vertices \( n \in \{20, 100, 500, 1000\} \).

We computed bias and empirical standard deviations over 100 simulations. As illustrated by Figure 5(a) in the case of \( \hat{\mu}_{\text{in}} \), the method recovers the parameters with no bias, except for model C where a small bias occurs due to the high level of intricacy of the model. Figure 5(b) displays the evolution of the log of the empirical standard deviation of \( \hat{\mu}_{\text{in}} \) when the size of the graphs grows from 20 vertices up to 1000 vertices. As for the binary affiliation model estimators, we observe a linear dependence between the log of the graph size and the log standard deviation, the slope of the lines lying in \([-1/2, -1]\).

Figure 5(b) displays the Rand Index for the three different models (A,B,C) and four different graph sizes. When graphs have more than 100 nodes, recovery of the hidden structure is almost perfect in all situations as previously observed in the binary case.

The previous experiments show that when dealing with weighted affiliation graphs, the estimation of the parameters and of the graph latent structure can be efficiently achieved considering only edges (order 2 structures).
6.3. Cross-citations of economics journals

Let us illustrate the difference between weighted and binary models for graph clustering using a real data example. We consider cross-citations of 42 economics journals over the years 1995-1997 [Pieters and Baumgartner, 2002]. The raw data corresponds to a weighted non symmetric graph where vertices are journals and directed edges the number of citations from one journal to another one. We first take the mean value of citations between each pair of journals (leading to a symmetric adjacency matrix) and work with its normalized Laplacian. Figure 6 displays the affiliation matrices structured according to a partition in four classes. Clustering based on the binary model and on the weighted model (respectively left and right sides of Figure 6) exhibit very different cluster structures. The binary model finds classes which tend to be homogeneous in terms of probability of intra-group and inter-group connections, while the weighted model finds classes which are homogeneous in terms of intra-group and inter-group connection weights. This distinction results in completely different interpretations.

The binary model finds two groups of nodes which are strongly connected within their groups but also with nodes from the other groups. It also exhibit two other smaller classes with low intra-group connectivity and nodes that preferentially link to the first class which plays the role of a reference class. Indeed the first class (top left) found by the binary model is composed by journals with high impact factors: American Economic Review (AER), Econometrica (E), Journal of Economic Literature (JEL), Journal of Economic Perspectives (JEP), Journal of Political Economy (JPE), Quarterly Journal of Economics (QJE), Review of Economic Studies (RES) and Review of Economics and Statistics (RES2).

The result produced by the weighted model shows a main class of strongly interconnected journals and three smaller classes of journals, which weakly cross-cite each other:
Class 1 (health) Health Economics (HE), Journal of Health Economics (JHE);
Class 2 (natural resources) Journal of Agricultural Economics (AJAE), Land Economics (LAE), Journal of Environmental Economics and Management (JEEM);
Class 3 (economic history) Exploration of Economic History (EEH), Journal of Economic History (JEH), Economic History Review (EHR).

Each of these three classes is composed by journals dedicated to similar topics (respectively, health, natural resources and economic history). They preferentially cite journals from the first class which contains journals with less specific topics.

7. Proofs

Proof of Theorem 1. In order to facilitate the reading of the proof, we decompose it into several stages.

Preliminaries. We fix $k, s \geq 1$ and $p = \left(\begin{array}{c} k \\ s \end{array}\right)$. Let us recall that $\mathcal{V}_Q$ is the set of $Q$-size vectors such that for any $v = (v_1, \ldots, v_q) \in \mathcal{V}_Q$, we have $v_i \in \{0, 1\}$ and $\sum_{i=1}^{Q} v_i = 1$. We also let $\mathcal{Q} = \{1, \ldots, Q\}$. We then consider the set

$$
\mathcal{Z} = \left\{ z \in \mathcal{V}_Q^N; \forall q = (q_1, \ldots, q_k) \in \mathcal{Q}^k, \frac{(n - k)!}{n!} n_q := \frac{(n - k)!}{n!} \sum_{i \in \mathcal{I}_k} \prod_{l=1}^{k} z_{i, q_l} \rightarrow_{n \rightarrow \infty} \prod_{l=1}^{k} \pi_{q_l} \right\}.
$$

Moreover, we let $N_\mathcal{Z} = \sum_{z \in \mathcal{Z}} \prod_{l=1}^{k} Z_{i, q_l}$. The strong law of large numbers gives the almost sure convergence, as $n$ tends to infinity, of $[(n-k)!/n!] N_\mathcal{Z}$ to $\prod_{l=1}^{k} \pi_{q_l}$. This implies that $\mathbb{P}(\{Z_n\}_{n \geq 1} \in \mathcal{Z}) = 1$. 
Consistency of $\hat{m}_g$. We first introduce the conditional mean of $g(\mathbb{X}^l)$ given that the hidden groups at position $i$ are given by $q_i$

$$m_g(q) = \mathbb{E}\left(g(\mathbb{X}^l) \mid \prod_{l=1}^{k} Z_{i_{qi}} = 1\right).$$

Using the equalities

$$\forall i \in \mathcal{I}_k, \sum_{q \in Q} q_k \prod_{l=1}^{k} Z_{i_{qi}} = 1 \quad \text{and} \quad m_g = \sum_{q \in Q} \left( \prod_{l=1}^{k} \pi_{q_l} \right) m_g(q),$$

we may write the decomposition

$$\hat{m}_g - m_g = \left( \frac{n-k}{n!} \right) \sum_{q \in Q} \sum_{i \in \mathcal{I}_k} \prod_{l=1}^{k} Z_{i_{qi}} g(\mathbb{X}^l) - \sum_{q \in Q} \left( \prod_{l=1}^{k} \pi_{q_l} \right) m_g(q)$$

$$= \sum_{q \in Q} \left\{ \left( \frac{n-k}{n!} \right) \sum_{i \in \mathcal{I}_k} \left( \prod_{l=1}^{k} Z_{i_{qi}} \right) (g(\mathbb{X}^l) - m_g(q)) + m_g(q) \left( \frac{n-k}{n!} \right) N_q \prod_{l=1}^{k} \pi_{q_l} \right\}.$$

(21)

In order to establish the consistency of $\hat{m}_g$, we rely on a conditioning argument. Let $A$ be the event $\lim \sup_{n \to \infty} |\hat{m}_g - m_g| = 0$. We then have

$$\mathbb{P}(A) = \mathbb{E}[\mathbb{E}(1_A \mid \{Z_n\}_{n \geq 1})].$$

(22)

Now, conditional on $\{Z_n\}_{n \geq 1} = z$, the random variables $\{\mathbb{X}^l, i \in \mathcal{I}_k, \prod_{l=1}^{k} z_{i_{qi}} = 1\}$ form a $n_q$-sample of independent and identically distributed random variables. Letting $B$ be the event

$$\lim \sup_{n \to \infty} \frac{1}{N_q} \sum_{i \in \mathcal{I}_k, \prod_{l=1}^{k} z_{i_{qi}} = 1} (g(\mathbb{X}^l) - m_g(q)) = 0,$$

the strong law of large numbers yields that for any $z \in \mathcal{Z}$,

$$\mathbb{E}(1_B \mid \{Z_n\}_{n \geq 1} = z) = 1.$$

Conditional on $\{Z_n\}_{n \geq 1} = z \in \mathcal{Z}$, we may thus re-write the decomposition (21) as

$$\hat{m}_g - m_g = \sum_{q \in Q} \left\{ \left( \frac{n-k}{n!} \right) n_q \sum_{i \in \mathcal{I}_k, \prod_{l=1}^{k} z_{i_{qi}} = 1} (g(\mathbb{X}^l) - m_g(q)) \right\}$$

$$+ m_g(q) \left( \frac{n-k}{n!} n_q - \prod_{l=1}^{k} \pi_{q_l} \right),$$

which establishes that for any $z \in \mathcal{Z}$, we have $\mathbb{E}(1_A \mid \{Z_n\}_{n \geq 1} = z) = 1$. Coming back to (22), we thus obtain

$$\mathbb{P}( \lim_{n \to \infty} \hat{m}_g = m_g ) = 1.$$
Asymptotic normality of $\hat{m}_q$. Let us now prove a central limit result for $\sqrt{n}(\hat{m}_q - m_q)$. First, the central limit theorem applied to the $Q$-size vector $\sum_{i=1}^{n} (Z_i - \pi) / \sqrt{n}$ gives the following convergence

$$
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (Z_i - \pi) \sim \mathcal{N}(0, \Sigma), \quad \text{as } n \to \infty,
$$

where $\Sigma_{qq} = \pi_q (1 - \pi_q)$ and $\Sigma_{q\ell} = -\pi_q \pi_\ell$ when $q \neq \ell$.

Now, let us consider the second term appearing in the right hand side of (21). To establish a central limit theorem for $N_q$, we decompose the sum of products

$$
\sum_{l \in \mathcal{L}} \prod_{i=1}^{k} Z_{i,q_l} = \sum_{l \in \mathcal{L}} \prod_{i=1}^{k} (Z_{i,q_l} - \pi_{q_l} + \pi_{q_l})
$$

into sums of products of centered terms $(Z_{i,q_l} - \pi_{q_l})$. This leads to

$$
\frac{(n - k)!}{n!} N_q - \sum_{l=1}^{k} \prod_{j \neq l} \pi_{q_j} = \sum_{u=1}^{k} \sum_{L \subset \{1,...,k\}; |L|=u} \frac{(n - u)!}{n!} \left( \prod_{l \in L} \pi_{q_l} \right) \sum_{l \in \mathcal{L}} \prod_{i \in L} (Z_{i,q_l} - \pi_{q_l}),
$$

where $|L|$ denotes the cardinality of the set $L$ and $\mathcal{L}$ denotes the set of injective maps from $L$ to $\mathcal{I} = \{1, \ldots, n\}$. In this expression, the leading term (obtained for singleton sets $L$, i.e. when $u = 1$) gives the rate of convergence in the central limit theorem. In other words,

$$
\sqrt{n} \left( \frac{(n - k)!}{n!} N_q - \sum_{l=1}^{k} \prod_{j \neq l} \pi_{q_j} \right) = \sum_{u=1}^{k} \sum_{L \subset \{1,...,k\}; |L|=u} \frac{\sqrt{n}(n - u)!}{n!} \left( \prod_{l \in L} \pi_{q_l} \right) \sum_{l \in \mathcal{L}} \prod_{i \in L} (Z_{i,q_l} - \pi_{q_l}).
$$

The first term in the right hand side of (24) converges to a linear combination of the coordinates of a $\mathcal{N}(0, \Sigma)$ vector, whereas the second term converges to zero. Indeed, for any value $u \geq 2$ and any set $L$ of cardinality $u$, we may write

$$
\sqrt{n} \frac{(n - u)!}{n!} \sum_{l \in \mathcal{L}} \prod_{i \in L} (Z_{i,q_l} - \pi_{q_l}) = \frac{1}{\sqrt{n}(n - 1) \cdots (n - u + 1)} \sum_{l \in \mathcal{L}} \prod_{i \in L} (Z_{i,q_l} - \pi_{q_l})
$$

which converges to zero. Thus, we get,

$$
\sqrt{n} \sum_{q \in Q^k} m_g(q) \left[ \frac{(n - k)!}{n!} N_q - \prod_{k=1}^{k} \pi_{q_k} \right] = \sum_{q \in Q^k} m_g(q) \left[ \sum_{l=1}^{k} \prod_{j \neq l} \pi_{q_j} \right] \sum_{i=1}^{n} \frac{\sqrt{n}(n - 1) \cdots (n - u + 1)}{\sqrt{n}} \sum_{l \in \mathcal{L}} \prod_{i \in L} (Z_{i,q_l} - \pi_{q_l}) + R_{n,q},
$$

where $R_{n,q} = o_P(1)$ are negligible terms converging in probability to zero, as $n$ tends to infinity. According to (23), we obtain that

$$
\sqrt{n} \sum_{q \in Q^k} m_g(q) \left[ \frac{(n - k)!}{n!} N_q - \prod_{k=1}^{k} \pi_{q_k} \right] \sim_{n \to \infty} \sum_{q \in Q^k} m_g(q) \left[ \sum_{l=1}^{k} \prod_{j \neq l} \pi_{q_j} \right] W_{q_l},
$$
where \( W = (W_1, \ldots, W_Q) \sim \mathcal{N}(0, \Sigma) \).

To obtain a central limit theorem for \( \hat{m}_g \) it now suffices to prove that the first term in the right hand side of (21) is negligible, when scaled by the rate of convergence \( \sqrt{n} \). Indeed, we may write this term as

\[
\tilde{R}_n = \sum_{q \in \mathcal{Q}^k} \left( \frac{(n-k)!}{(n-1)!} \right)^{1/2} \times \left( \frac{(n-k)!}{n!} N_q \right)^{1/2} \times \frac{1}{\sqrt{N_q}} \sum_{l \in \mathcal{I} : \Pi_l = 1} (g(X^l) - m_g(q)),
\]

which satisfies, for any \( k \geq 2 \), any \( \epsilon > 0 \) and any \( z \in \mathcal{Z} \),

\[
\mathbb{P}(|\tilde{R}_n| \geq \epsilon | \{Z_n\}_{n \geq 1} = z) \rightarrow_{n \to \infty} 0.
\]

Using dominated convergence, we also have \( \mathbb{P}(|\tilde{R}_n| \geq \epsilon) \rightarrow_{n \to \infty} 0 \), for any \( \epsilon > 0 \).

Now, going back to (21), we finally obtain

\[
\sqrt{n}(\hat{m}_g - m_g) \sim_{n \to \infty} \sum_{q \in \mathcal{Q}^k} m_g(q) \left[ \sum_{l=1}^k \left( \prod_{j \neq l} \pi_{q_j} \right) W_{q_l} \right] \sim \mathcal{N}(0, \Sigma_g).
\]

**Expression for the limiting variance** \( \Sigma_g \). The computation of the variance \( \Sigma_g \) could be done using the above expression, but this leads to tedious formulas. A simpler expression of the limiting variance is obtained in the following way. We prove that \( \sqrt{n}U_n := \sqrt{n}(\hat{m}_g - m_g) \) has a bounded third order moment. This is sufficient to claim that \( \Sigma_g \) can be obtained as the limiting variance of \( \sqrt{n}U_n \).

First, since non adjacent edges form independent variates, it is easy to see that we have

\[
\mathbb{E}(\|\sqrt{n}U_n\|^3) \leq \left( \frac{(n-k)!}{\sqrt{n}(n-1)!} \right)^3 \sum_{i \cup j \subset [k], |i \cup j| \neq \emptyset} \mathbb{E}(\|g(X^i) - m_g\| \|g(X^j) - m_g\| \|g(X^j) - m_g\|),
\]

where \( i \cap j \) stands for the intersection of \( i \) and \( j \) viewed as index sets (instead of \( k \)-tuples). The above sum contains at most \( k^2n [(n-1) \cdots (n-k+1)]^3 \) terms, which are bounded (there are finitely many of them). Thus this quantity converges to zero as \( n \) tends to infinity. Moreover,

\[
\text{Var}(\sqrt{n}U_n) = \left( \frac{(n-k)!}{\sqrt{n}(n-1)!} \right)^2 \sum_{i \cup j \subset [k], |i \cup j| \neq \emptyset} \text{Cov}(g(X^i), g(X^j)).
\]

The above sum may be decomposed according to the cardinality of the set \( i \cap j \). It is then easy to see that the dominating term is obtained when \( |i \cap j| = 1 \), while the other terms converge to zero. Namely

\[
\text{Var}(\sqrt{n}U_n) = \left( \frac{(n-k)!}{\sqrt{n}(n-1)!} \right)^2 \sum_{i \cup j \subset [k], |i \cup j| = 1} \text{Cov}(g(X^i), g(X^j)) + o(1).
\]

To describe all the possible configurations where \( |i \cap j| = 1 \), we may fix the first index \( i \) to \( (1, \ldots, k) \) and let the second index \( j \) describe the set of indexes.
where some position \( s \) takes one of the values \( \{1, \ldots, k\} \) (corresponding to the intersection \( i \cap j \)) and at any other position, there is some value in \( \{k+1, \ldots, n\} \). For any \( s, t \in \{1, \ldots, k\} \), we thus let \( e^t_s \in \mathcal{I}_k \) satisfying \( e^t_s(s) = t \) and \( e^t_s(j) \in \{k+1, \ldots, n\} \) for any \( j \neq s \). With this notation, we obtain

\[
\Sigma_g = \lim_{n \to \infty} \text{Var}(\sqrt{n}U_n) = \sum_{s=1}^k \sum_{t=1}^k \text{Cov}(g(X^{(1,\ldots,k)}), g(X^{e^t_s})).
\]

Note that in the case of an affiliation structure with equal group proportions, we could prove from this expression that \( \Sigma_g = 0 \) (using for instance the results of Lemma 1 presented below). Anyway this will be a consequence of the following developments.

**The degenerate case.** Let us now finish this proof by considering the specific case where we have an affiliation structure (2) and equal group proportions (3). Coming back to (21), we write \( m_g - m_g = T_1 + T_2 \) where

\[
T_1 = \sum_{q \in \mathcal{Q}^k} \frac{(n-k)!}{n!} \sum_{i \in \mathcal{I}_k} \left( \prod_{l=1}^k Z_{i,q_l} \right) (g(X^l) - m_g(q)),
\]

\[
T_2 = \sum_{q \in \mathcal{Q}^k} m_g(q) \frac{(n-k)!}{n!} N_q - \prod_{i=1}^k \pi_{q_i}.
\]

We first deal with the second term \( T_2 \). According to (24), we have

\[
T_2 = \sum_{q \in \mathcal{Q}^k} m_g(q) \sum_{i=1}^k \frac{1}{Q^{k-i}} \frac{1}{n!} \sum_{i=1}^n (Z_{i,q_i} - \pi_{q_i})
\]

\[
+ \sum_{u=2}^k \sum_{L \subseteq \{1, \ldots, k\}, |L| = u} \frac{(n-u)!}{u!} \frac{1}{Q^{k-u}} \sum_{i \in \mathcal{I}_L, l \in L} \prod_{l \in L} (Z_{i,q_l} - \pi_{q_l}) := T_{2,1} + T_{2,2}.
\]

We now prove that the first term in the right hand side of this equality, namely \( T_{2,1} \) is zero. This result relies on the following lemma, stating that the model is invariant under a permutation of the values of the node groups.

**Lemma 1.** Under the assumptions and notations of Theorem 1, assuming moreover (2) and (3), for any \( \sigma \in \mathcal{S}_{\mathcal{Q}} \) the set of permutations of \( \mathcal{Q} \), we have

\[
\{(Z_i)_{1 \leq i \leq n}, \{X_{ij}\}_{1 \leq i < j \leq n}\} \overset{d}{=} (\{\sigma(Z_i)\}_{1 \leq i \leq n}, \{X_{ij}\}_{1 \leq i < j \leq n}),
\]

where \( \overset{d}{=} \) means equality in distribution. As a consequence, for any value \( q \in \mathcal{Q}^k \), the conditional expectation \( m_g(q) \) is constant along the orbit (induced by \( \mathcal{S}_{\mathcal{Q}} \)) of the point \( q \), i.e. the set of values \( \{m_g(\sigma(q)); \sigma \in \mathcal{S}_{\mathcal{Q}}\} \) is a singleton for any fixed \( q \in \mathcal{Q}^k \).
Indeed, according to (1), (2) and (3), and using that any permutation \( \sigma \) is a one-to-one application, we have

\[
P(\{Z_i\}_{1 \leq i \leq n}, \{X_{ij}\}_{1 \leq i < j \leq n}) = \prod_{i=1}^{n} P(Z_i) \prod_{1 \leq i < j \leq n} P(X_{ij}|1_{Z_i} = Z_j) = \frac{1}{Q^n} \prod_{1 \leq i < j \leq n} P(X_{ij}|1_{\sigma(Z_i)} = \sigma(Z_j)) = \mathbb{P}(\{\sigma(Z_i)\}_{1 \leq i \leq n}, \{X_{ij}\}_{1 \leq i < j \leq n}).
\]

As a consequence, for any \( \sigma \in S_Q \) and any value \( \underline{q} \in Q^k \), the conditional expectation \( m_{\underline{q}}(\sigma(\underline{q})) \) satisfies

\[
m_{\underline{q}}(\sigma(\underline{q})) = \mathbb{E}(g_{\{1, \ldots, k\}}(Z_1, \ldots, Z_k) = \sigma(\underline{q})) = \mathbb{E}(g_{\{1, \ldots, k\}}(\sigma(Z_1), \ldots, \sigma(Z_k)) = \sigma(\underline{q})) = m_{\underline{q}}(\sigma(\underline{q})).
\]

Thus the set of values \( \{m_{\underline{q}}(\sigma(\underline{q})); \sigma \in S_Q\} \) is reduced to a singleton. This finishes the proof of the lemma.

Now, going back to the term \( T_{2,1} \), the set \( Q^k \) may be partitioned into the disjoint union of the orbits induced by \( S_Q \), namely \( Q^k = \cup_{\text{orbit } O} Q \), with \( \underline{q} \rightarrow m_{\underline{q}}(\underline{q}) \) being constant on each orbit \( O \). We let \( m_{\underline{q},O} \) denote the value of the function \( \underline{q} \rightarrow m_{\underline{q}}(\underline{q}) \) on the orbit \( O \). Then we write

\[
T_{2,1} = \frac{1}{nQ^{k-1}} \sum_{\text{orbit } O} m_{\underline{q},O} \sum_{l=1}^{k} \sum_{i=1}^{n} \sum_{\underline{q} \in O} (Z_{iq_l} - \pi_{q_l}).
\]

For each orbit \( O \) and any position \( l \in \{1, \ldots, k\} \), if we fix some \( \underline{q} \in O \), then we argue that \( O \) contains all the points of the form \((q_1, \ldots, q_{l-1}, j, q_{l+1}, \ldots, q_k)\) for any \( 1 \leq j \leq Q \). Indeed, all these points are images of \( \underline{q} \) by the simple transpositions \((q_l j)\). Thus, the sum \( \sum_{\underline{q} \in O} (Z_{iq_l} - \pi_{q_l}) \) contains \( \sum_{\underline{q} \in O} (Z_{iq_l} - \pi_{q_l}) \) which is zero. This proves that \( T_{2,1} = 0 \) and thus

\[
n(m_{\underline{q}} - m_{\underline{q}}) = n(T_1 + T_{2,2}) = \sum_{\underline{q} \in Q^k} \frac{(n-k)!}{(n-1)!} \frac{1}{N_q^{1/2}} \sum_{\underline{q} \in Q^k} \frac{1}{N_q^{1/2}} \sum_{\underline{q} \in Q^k} (g_{\{1, \ldots, k\}} - m_{\underline{q}}(\underline{q}))
+ \frac{1}{Q^{k-2}} \sum_{q, \ell \in Q, q \neq \ell} \frac{1}{(n-1)!} \sum_{1 \leq i \neq j \leq n} (Z_{iq_l} - \pi_q)(Z_{j\ell} - \pi_\ell) + o(1),
\]

where, as in the non degenerate case, we argued that the terms in \( T_{2,2} \) involving sets \( L \) with cardinality \( u \geq 3 \) are negligible. We then obtain that for \( k = 2 \), we have

\[
n(m_{\underline{q}} - m_{\underline{q}}) \sim_{n \to \infty} \frac{1}{Q} \sum_{q, \ell \in Q} V_{q\ell} + \sum_{q, \ell \in Q, q \neq \ell} (W_q W_\ell + \frac{1}{Q^2}),
\]
where for any $1 \leq q, \ell \leq Q$, the random variables $V_{q\ell}$ are independent, with distribution $\mathcal{N}(0, \text{Var}(g(X_{12})|Z_{1q}Z_{2\ell} = 1))$ and $W = (W_1, \ldots, W_Q)$ is independent from the $V_{q\ell}$’s, with distribution $\mathcal{N}(0, \Sigma)$, and in the equal group proportions case $\Sigma$ simplifies to $\Sigma_{q\ell} = -1/Q^2$ when $q \neq \ell$ and $\Sigma_{qq} = (Q - 1)/Q^2$.

In the same way, whenever $k \geq 3$, all the terms appearing in $T_1$ are now negligible and we get

$$n(\hat{m}_g - m_g) \sim_{n \to \infty} \sum_{q,\ell \in Q, q \neq \ell} (W_q W_\ell + \frac{1}{Q^2}).$$

\[\square\]

Proof of Theorem 3. Following the proof of Theorem 1, we can easily write a joint central limit theorem for the triplet $(\hat{m}_1, \hat{m}_2, \hat{m}_3)$. Namely,

$$\sqrt{n} \begin{pmatrix} \hat{m}_1 - m_1 \\ \hat{m}_2 - m_2 \\ \hat{m}_3 - m_3 \end{pmatrix} \sim_{n \to \infty} \mathcal{N}_3(0, V),$$

with some covariance matrix $V$. Thus, we can apply a delta-method [see for instance van der Vaart, 1998, Chapter 3] to the estimators $\hat{\beta} = \phi(\hat{m}_1, \hat{m}_2, \hat{m}_3)$ and $\hat{\alpha} = \psi(\hat{m}_1, \hat{m}_2, \hat{m}_3)$ where the functions $\phi$ and $\psi$ are differentiable. This gives the convergence of the estimators $(\hat{\alpha}, \hat{\beta})$ and guarantees the same rates of convergence for $\hat{m}_i$’s.

\[\square\]

Proof of Theorem 4. Following the classical proof of Wald [1949] [see also van der Vaart, 1998], we may obtain the almost sure convergence of $(\hat{\gamma}_n, \hat{\alpha}_n, \hat{\beta}_n)$ to the true value of the parameter $(\gamma^*, \alpha^*, \beta^*)$, provided the parameter space is compact and the three following assumptions are satisfied:

1) Convergence of the criterion

$$\ell_n(\pi, \alpha, \beta) := \frac{1}{n(n - 1)(n - 2)} \sum_{(i,j,k) \in I_3} \log P_{\pi,\alpha,\beta}(X_{ij}, X_{ik}, X_{jk})$$

$$\to_{n \to \infty} H((\pi, \alpha, \beta); (\pi^*, \alpha^*, \beta^*)) := E_{\pi^*,\alpha^*,\beta^*} \log P_{\pi^*,\alpha^*,\beta^*}(X_{12}, X_{13}, X_{23}),$$

$\mathbb{P}_{\pi^*,\alpha^*,\beta^*}$-almost surely;

2) Identification of the parameter $(\gamma, \alpha, \beta)$

$$H((\pi, \alpha, \beta); (\pi^*, \alpha^*, \beta^*)) \leq H((\pi^*, \alpha^*, \beta^*); (\pi^*, \alpha^*, \beta^*)),$$

with equality if and only if $(\gamma, \alpha, \beta) = (\gamma^*, \alpha^*, \beta^*)$, where $\gamma$ and $\pi$ are related through (10);

3) Uniform equicontinuity of the family of functions $(\pi, \alpha, \beta) \to \ell_n(\pi, \alpha, \beta)$.

Namely, for any $\epsilon > 0$, there exists some $\nu > 0$ such that for all $n \geq 1$ and as soon as $\|((\pi, \alpha, \beta) - (\pi', \alpha', \beta'))\|_{\infty} \leq \nu$, we have $|\ell_n(\pi, \alpha, \beta) - \ell_n(\pi', \alpha', \beta')| \leq \epsilon$. 

Item i) follows from Theorem 1, while ii) follows from Jensen’s inequality and identifiability of the parameters, i.e. Assumption 1. Let us now establish iii).
We fix for the moment some $\nu > 0$ and consider $\eta = (\pi, \alpha, \beta)$ and $\eta' = (\pi', \alpha', \beta')$ such that $||\eta - \eta'||_\infty \leq \nu$. We recall that $(X_{ij}, X_{ik}, X_{jk}) = X^{(i,j,k)}$.

We then write

$$|\log P_\eta(Z_{iq}Z_{jk}Z_{km} = 1, X^{(i,j,k)}) - \log P_{\eta'}(Z_{iq}Z_{jk}Z_{km} = 1, X^{(i,j,k)})|$$

$$\leq |\log \pi_q - \log \pi'_q| + |\log \pi_\ell - \log \pi'_\ell| + |\log \pi_m - \log \pi'_m|$$

$$+ |\log P_\eta(X^{(i,j,k)}|Z_{iq}Z_{jk}Z_{km} = 1) - \log P_{\eta'}(X^{(i,j,k)}|Z_{iq}Z_{jk}Z_{km} = 1)|.$$ 

The second term in the right hand side of this inequality may be bounded as follows

$$|\log P_\eta(X^{(i,j,k)}|Z_{iq}Z_{jk}Z_{km} = 1) - \log P_{\eta'}(X^{(i,j,k)}|Z_{iq}Z_{jk}Z_{km} = 1)|$$

$$\leq 3 \max(||\log \alpha - \log \alpha'||,||\log(1 - \alpha) - \log(1 - \alpha')||,||\log \beta - \log \beta'||,||\log(1 - \beta) - \log(1 - \beta')||).$$

We now make use of the fact that we restricted our attention to the parameter space $\Pi_\delta$, in which all the parameters are lower bounded by $\delta$ (Assumption 2). Moreover, for any $x, y > 0$, we may use $|\log x - \log y| \leq |x - y|/\min(x, y)$. This finally leads to

$$|\log P_\eta(Z_{iq}Z_{jk}Z_{km} = 1, X^{(i,j,k)}) - \log P_{\eta'}(Z_{iq}Z_{jk}Z_{km} = 1, X^{(i,j,k)})| \leq 6\delta^{-1}\nu.$$ 

Now, we obtain

$$P_\eta(X^{(i,j,k)}) = \sum_{q\ell m} P_\eta(Z_{iq}Z_{jk}Z_{km} = 1, X^{(i,j,k)})$$

$$\leq \exp(6\delta^{-1}\nu) \sum_{q\ell m} P_{\eta'}(Z_{iq}Z_{jk}Z_{km} = 1, X^{(i,j,k)}) = \exp(6\delta^{-1}\nu) P_{\eta'}(X^{(i,j,k)}),$$

and thus

$$\log P_\eta(X^{(i,j,k)}) \leq \frac{6\nu}{\delta} + \log P_{\eta'}(X^{(i,j,k)}).$$

As this inequality is symmetric with respect to $\eta$ and $\eta'$, we further obtain

$$|\log P_\eta(X^{(i,j,k)}) - \log P_{\eta'}(X^{(i,j,k)})| \leq \frac{6\nu}{\delta}.$$ 

Finally,

$$|\ell_n(\eta) - \ell_n(\eta')| \leq \frac{1}{n(n-1)(n-2)} \sum_{i,j,k} |\log P_\eta(X^{(i,j,k)}) - \log P_{\eta'}(X^{(i,j,k)})| \leq \frac{6\nu}{\delta},$$

which establishes iii).
To further obtain the rates of convergence of the estimators, one usually proceeds to a Taylor expansion of the derivative $\partial \ell_n(\pi^*, \alpha^*, \beta^*)$ in a vicinity of the estimator $(\hat{\pi}_n, \hat{\alpha}_n, \hat{\beta}_n)$. Let us write

$$0 = \partial \ell_n(\pi_n, \alpha_n, \beta_n) = \partial \ell_n(\pi^*, \alpha^*, \beta^*) + [(\pi_n, \alpha_n, \beta_n) - (\pi^*, \alpha^*, \beta^*)] \partial^2 \ell_n(\pi_n, \alpha_n, \beta_n),$$

where $(\pi_n, \alpha_n, \beta_n)$ is some point between $(\hat{\pi}_n, \hat{\alpha}_n, \hat{\beta}_n)$ and $(\pi^*, \alpha^*, \beta^*)$. Applying Theorem 1 to the quantity $\partial \ell_n(\pi^*, \alpha^*, \beta^*)$, we obtain its almost sure convergence to $E_{\pi^*, \alpha^*, \beta^*}(\partial \log P_{\pi^*, \alpha^*, \beta^*}(X_{12}, X_{13}, X_{23})) = 0$, as well as the asymptotic normality

$$\sqrt{n} \partial \ell_n(\pi^*, \alpha^*, \beta^*) \sim_{n \to \infty} \mathcal{N}(0, J).$$

Now, at a fixed point $(\pi, \alpha, \beta)$, the Hessian matrix $\partial^2 \ell_n(\pi, \alpha, \beta)$ converges from Theorem 1 to $E_{\pi^*, \alpha^*, \beta^*}(\partial^2 \log P_{\pi^*, \alpha^*, \beta^*}(X_{12}, X_{13}, X_{23}))$. Combining the almost sure convergence of $(\hat{\pi}_n, \hat{\alpha}_n, \hat{\beta}_n)$ to $(\pi^*, \alpha^*, \beta^*)$, with uniform equicontinuity of the family of functions $(\pi, \alpha, \beta) \to \partial^2 \ell_n(\pi, \alpha, \beta)$ (the proof is similar to point iii) above and is therefore omitted), we obtain the almost sure convergence

$$\partial^2 \ell_n(\pi_n, \alpha_n, \beta_n) \rightarrow_{n \to \infty} E_{\pi^*, \alpha^*, \beta^*}(\partial^2 \log P_{\pi^*, \alpha^*, \beta^*}(X_{12}, X_{13}, X_{23})) := -K.$$

If the Fisher information matrix $K$ is invertible, we obtain

$$\sqrt{n}[(\hat{\pi}_n, \hat{\alpha}_n, \hat{\beta}_n) - (\pi^*, \alpha^*, \beta^*)] \sim_{n \to \infty} \mathcal{N}(0, K^{-1}J K^{-1}).$$

In this case, the inverse of the limiting variance is known as Godambe information [Varin, 2008]. Its form is due to the fact that $K^{-1} \neq J$ in general, resulting in a loss of efficiency of the estimators. In case where $K$ is not invertible, or when $J = 0$, the rate of convergence of the estimators is faster than $1/\sqrt{n}$. In particular, when the group proportions are equal, we know from Theorem 1 that $n \partial \ell_n(\pi^*, \alpha^*, \beta^*)$ converges in distribution and then the rate of convergence of $(\hat{\pi}_n, \hat{\alpha}_n, \hat{\beta}_n)$ is at least $1/n$.

**Proof of Theorem 5.** The proof follows the scheme described in the proof of Theorem 4. We denote by $(\pi^*, p^*, \theta^*)$ the true value of the parameter and by $P^*$ and $E^*$ the corresponding probability and expectation. First, let us establish the consistency of the normalized composite likelihood (point i)). According to Theorem 1, we have for any fixed value of $(\pi, p, \theta)$,

$$\frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} 1_{X_{ij} \neq 0} \log \mathbb{P}_{\pi, p, \theta}(X_{ij}) \rightarrow_{n \to \infty} E^*(1_{X_{12} \neq 0} \log \mathbb{P}_{\pi, p, \theta}(X_{12}))$$

$$:= H((\pi, p, \theta); (\pi^*, p^*, \theta^*)), \quad P^* \text{ a.s.}$$

Here, we need to deal with the fact that we use a random value for $p$ (a preliminary step estimate) in the definition of $\hat{\theta}$. It is thus necessary to prove that this convergence happens uniformly with respect to $p$. But this is going to be a consequence of point iii) below. Combining this with the almost sure convergence of $\hat{p}_n$ to the true value $p^*$ (this is either a consequence of Theorem 1 when
\( p = p \) is constant, or a consequence of Sections 3.1 and 3.2 when \( p = (\alpha, \beta) \), we get

\[
\frac{2}{n(n-1)} \mathcal{L}_X^{\text{comp}}(\pi, \hat{p}_n, \theta) \xrightarrow{n \to +\infty} H((\pi, p, \theta); (\pi^*, p^*, \theta^*)), \quad \mathbb{P}^* \text{ a.s.}
\]

Moreover, we assumed that \( f(\cdot, \theta) \) has a continuous c.d.f. and the distribution of a present edge is given by (16), so that we have

\[
H((\pi, p, \theta); (\pi^*, p^*, \theta^*)) = \int_x \log(\gamma_{\text{in}} f(x; \theta_{\text{in}}) + \gamma_{\text{out}} f(x; \theta_{\text{out}}))
\]

\[
\times (\gamma_{\text{in}}^* f(x; \theta_{\text{in}}^*) + \gamma_{\text{out}}^* f(x; \theta_{\text{out}}^*)) dx,
\]

where \((\gamma_{\text{in}}, \gamma_{\text{out}})\) as well as \((\gamma_{\text{in}}^*, \gamma_{\text{out}}^*)\) are defined through \((\pi, p)\) and \((\pi^*, p^*)\) respectively. Thus, the difference

\[
H((\pi^*, p^*, \theta^*); (\pi^*, p^*, \theta^*)) - H((\pi, p, \theta); (\pi^*, p^*, \theta^*))
\]

is a Kullback-Leibler divergence between two mixture distributions of the form (16). This entails positivity of this difference. Moreover, Assumption 3 ensures that the difference is zero if and only if

\[
\gamma_{\text{in}} \delta_{\text{in}} + \gamma_{\text{out}} \delta_{\text{out}} = \gamma_{\text{in}}^* \delta_{\text{in}}^* + \gamma_{\text{out}}^* \delta_{\text{out}}^*,
\]

which establishes point \(ii\), up to a permutation on the label parameters \{in, out\}. Finally, the proof of point \(iii\) follows the same lines as in the proof of Theorem 4, and uses the continuity of the map \( \theta \mapsto f(\cdot, \theta) \), which is a consequence of Assumption 4.

To further obtain the rates of convergence of our estimators, we proceed exactly as we did in the proof of Theorem 4. \(\square\)

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