The neutron-gamma Feynman variance to mean approach: 
gamma detection and total neutron-gamma detection 
(theory and practice)

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Abstract

Two versions of the neutron-gamma variance to mean (Feynman-alpha method or 
Feynman-Y function) formula for either gamma detection only or total neutron-gamma 
detection, respectively, are derived and compared in this paper. The new formulas have 
a particular importance for detectors of either gamma photons or detectors sensitive 
to both neutron and gamma radiation. If applied to a plastic or liquid scintillation 
detector, the total neutron-gamma detection Feynman-Y expression corresponds to a 
situation where no discrimination is made between neutrons and gamma particles. The 
gamma variance to mean formulas are useful when a detector of only gamma radiation 
is used or when working with a combined neutron-gamma detector at high count rates. 
The theoretical derivation is based on the Chapman-Kolmogorov equation with inclu-
sion of general reactions and passage intensities for neutrons and gammas, but with 
the inclusion of prompt reactions only. A one energy group approximation is consid-
red. The comparison of the two different theories is made by using reaction intensities 
obtained in MCNPX simulations with a simplified geometry for two scintillation detec-
tors and a \textsuperscript{252}Cf-source enclosed in a steel container. In addition, the variance to mean 
ratios, neutron, gamma and total neutron-gamma, are evaluated experimentally for a 
weak \textsuperscript{252}Cf neutron-gamma source in a steel container, a \textsuperscript{137}Cs random gamma source 
and a \textsuperscript{22}Na correlated gamma source. Due to the focus being on the possibility of using 
neutron-gamma variance to mean theories for both reactor and safeguards applications, 
we limited the present study to the general analytical expressions for Feynman-Y for-
mulas.

Keywords: variance to mean, Feynman-alpha, Feynman-Y, gamma Feynman-Y, total 
Feynman-Y, fast detection

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1. Introduction

The traditional Feynman-alpha theory for neutrons is used in subcritical reactors, such as during startup with a neutron source, as well as in accelerator-driven systems for reactivity evaluation [3] and in safeguards applications for detection of special nuclear materials and spent fuel assay [4, 5]. In safeguards applications, thermal neutron detection is typically used [4], although a few attempts were made in order to use fast neutron detection for the evaluation of the Feynman-Y function of the variance to mean ratio [6, 7]. The results of Ref. [8] indicate that it is possible to assay a weak $^{252}$Cf source in 100 seconds using a fast detection system (liquid scintillation detectors with pulse shape discrimination) in comparison to 8.5 hours used by a thermal detection system ($^3$He counters).

Thus, it appears at the moment that the performance of the traditional Feynman-alpha method is limited by the need of either using thermal neutron detectors or struggling with methods of discrimination between neutrons and gamma particles in fast scintillation detectors. To address this problem, we suggest here an alternative path, by deriving Feynman-alpha formulas by taking into account the generation and detection of gamma photons. This way, two new versions of the Feynman-alpha theory can be elaborated: one based only on the detection of gamma particles, and another in which both neutrons and gamma photons are detected without identifying which is which, i.e. not performing pulse shape discrimination and only considering the total (neutron and gamma) counts. Having derived the corresponding formulas, they will be investigated both numerically and experimentally.

2. The theoretical background for the neutron-gamma Feynman-alpha theory

2.1. The main concept and assumptions

The neutron-gamma variance to mean (Feynman-alpha) formulas for separate gamma detection and total neutron-gamma detection will be derived by using the Kolmogorov forward approach the same way as described in [1, 9]. In the model that will be used for the derivations we assume that there are neutron and gamma populations: neutrons (denoted as particles of type 1) and gammas (denoted as type 2). Neutrons can undergo the reactions (i) listed below:

- absorption ($i = a$) with no gammas emitted,
- fission ($i = f$) with corresponding gamma emission,
- detection ($i = d$).

Gamma particles can undergo the same reactions (i) as neutrons except fission, i.e. photofission or photonuclear reactions are not considered in the present model.

The assumption behind the model is the same as with the traditional Feynman-alpha theory, i.e. that the medium is infinite and homogeneous with space-independent reaction intensities for absorption of neutrons ($\lambda_{1a}$) and gammas ($\lambda_{2a}$), fission induced
by neutrons ($\lambda_{1f}$) and detection of neutrons ($\lambda_{1d}$) and gammas ($\lambda_{2d}$). Thus, the total transition intensities for neutrons and gammas are denoted as $\lambda_1$ and $\lambda_2$:

$$
\lambda_1 = \lambda_{1a} + \lambda_{1f} + \lambda_{1d}
$$

$$
\lambda_2 = \lambda_{2a} + \lambda_{2d}
$$

(1)

In the model we include a compound Poisson source of neutrons and gammas with emission intensity $S$. The source is assumed to release $m$ neutrons and $n$ gammas in one emission event with the probability distribution $p(m,n)$. For the induced fission reaction, we consider that $k$ neutrons and $l$ gammas are emitted with the probability distribution $f(k,l)$. All fission neutrons are assumed to be prompt, i.e. delayed neutrons as not considered in this work. No delayed generation processes for gamma photons are assumed either.

2.2. The one-group one-point model for separate detection of neutrons and gammas

In order to derive the one-group one-point Feynman-alpha theory for separate detection of neutrons and gammas, let us assume that the source $S$ is switched on at the time $t_0 \leq t$, whereas the separate detection processes for neutrons and gammas both are started at the same fixed time instant $t_d$, where $t_d \leq t$ and $t_d \geq t_0$. Let the random processes $N_1(t)$ and $N_2(t)$ represent the number of neutrons and gammas at the time $t \geq 0$, and $Z_1(t,t_d)$ and $Z_2(t,t_d)$ - the number of neutron and gamma detections in the time interval $[t_d,t]$, respectively. Thus, the joint probability of having $N_1$ neutrons and $N_2$ gammas at time $t$ in the system, and having detected $Z_1$ neutrons and $Z_2$ gammas during the period of time $t - t_d \geq 0$ can be defined as $P(N_1,N_2,Z_1,Z_2,t|t_0)$. By summing up the probabilities of all mutually exclusive events of the particle not having or having a specific reaction within the infinitesimally small time interval $dt$, one obtains the forward Kolmogorov or forward master equation

$$
\frac{\partial P(N_1,N_2,Z_1,Z_2,t)}{\partial t}
$$

$$
= -(\lambda_1 N_1 + \lambda_2 N_2 + S)P(N_1,N_2,Z_1,Z_2,t)
$$

$$
+ \lambda_{1a} \delta(N_1 + 1)P(N_1 + 1,N_2,Z_1,Z_2,t) + \lambda_{2a} \delta(N_2 + 1)P(N_1,N_2 + 1,Z_1,Z_2,t)
$$

$$
+ \lambda_{1f} \sum_{k=0}^{N_1+1} \sum_{l=0}^{N_2} (N_1 + 1 - k) f(k,l) P(N_1 + 1 - k, N_2 - l, Z_1, Z_2, t)
$$

$$
+ \lambda_{1d} \delta(N_1 + 1)P(N_1 + 1,N_2,Z_1 - 1,Z_2,t)
$$

$$
+ \lambda_{2d} \delta(N_2 + 1)P(N_1,N_2 + 1,Z_1 - 1,Z_2 - 1,t)
$$

$$
+ S \sum_m \sum_n p(m,n) P(N_1 - m, N_2 - n, Z_1, Z_2, t)
$$

(2)

where, as mentioned earlier, $f(k,l)$ is the number distribution of neutrons and gammas in a fission event. The initial condition reads as

$$
P(N_1,N_2,Z_1,Z_2,t = t_0 \mid t_0) = \delta_{N_1,0} \delta_{N_2,0} \delta_{Z_1,0} \delta_{Z_2,0}
$$

(3)
Here, as mentioned before, with initial condition for photons in a source and fission event were introduced as

$$G(X,Y,L,W,t) = \sum_{N_1} \sum_{N_2} \sum_{Z_1} \sum_{Z_2} X^{N_1} Y^{N_2} L^{Z_1} W^{Z_2} P(N_1, N_2, Z_1, Z_2, t)$$  \hspace{1cm} (4)$$

with initial condition for $t_0 \leq t$

$$G(X,Y,L,W,t = t_0 | t_0) = 1$$  \hspace{1cm} (5)$$

the following partial differential equation is obtained:

$$\frac{\partial G}{\partial t} = [\lambda_1 a + \lambda_1 dL - \lambda_1 X + q(X,Y) \lambda_{1f}] \frac{\partial G}{\partial X} + [\lambda_2 a + \lambda_2 dW - \lambda_2 Y] \frac{\partial G}{\partial Y} + S[r(X,Y) - 1]G$$  \hspace{1cm} (6)$$

where the generating functions of the number distributions of neutrons and gamma photons in a source and fission event were introduced as

$$q(X,Y) = \sum_{k} \sum_{l} X^k Y^l f(k,l)$$

$$r(X,Y) = \sum_{m} \sum_{n} X^m Y^n p(m,n)$$  \hspace{1cm} (7)$$

Here, as mentioned before, $p(m,n)$ is the probability of having $m$ neutrons and $n$ gammas produced in a source event. For the sake of simplicity, some notations are introduced as follows:

$$\left. \frac{\partial}{\partial X} q(X,Y) \right|_{X=1,Y=1} = \sum_{k} \sum_{l} k f(k,l) = q^{(1,0)}(X,Y)$$  \hspace{1cm} (8)$$

$$\left. \frac{\partial}{\partial Y} q(X,Y) \right|_{X=1,Y=1} = \sum_{k} \sum_{l} l f(k,l) = q^{(0,1)}(X,Y)$$

and

$$\left. \frac{\partial}{\partial X} r(X,Y) \right|_{X=1,Y=1} = \sum_{m} \sum_{n} m p(m,n) = r^{(1,0)}(X,Y)$$  \hspace{1cm} (9)$$

$$\left. \frac{\partial}{\partial Y} r(X,Y) \right|_{X=1,Y=1} = \sum_{m} \sum_{n} n p(m,n) = r^{(0,1)}(X,Y)$$

In a steady subcritical medium with a steady source, a stationary state of the system exists when $t_0 \to -\infty$. For that case the following solutions are obtained for the constant neutron and gamma populations $\bar{N}_1, \bar{N}_2$ and the detection counts $\bar{Z}_1(t), \bar{Z}_2(t)$:

$$\bar{N}_1 = \frac{S r^{(1,0)}(X,Y)}{\lambda_1 - \lambda_{1f} q^{(1,0)}(X,Y)}$$

$$\bar{N}_2 = \frac{S r^{(0,1)}(X,Y)}{\lambda_2} + \frac{S \lambda_{1f} q^{(0,1)}(X,Y)r^{(1,0)}(X,Y)}{\lambda_2(\lambda_1 - \lambda_{1f} q^{(1,0)}(X,Y))}$$  \hspace{1cm} (10)$$

$$Z_1(t) = \lambda_{1d} \bar{N}_1 t$$

$$Z_2(t) = \lambda_{2d} \bar{N}_2 t$$
By introducing the modified second factorial moment of the random variables $a$ and $b$ as follows

$$
\mu_{aa} \equiv \langle a(a-1) \rangle - \langle a \rangle^2 = \sigma_a^2 - \langle a \rangle
$$

$$
\mu_{ab} \equiv \langle ab \rangle - \langle a \rangle \langle b \rangle
$$

(11)

and then taking cross- and auto-derivatives, the following system of differential equations is obtained for the modified second factorial moments $\mu_{XX}, \mu_{XY}, \mu_{YY}$ of the neutron and gamma populations:

$$
\frac{\partial}{\partial t} \mu_{XX} = 2(\lambda_1 q(1,0)(X,Y) - \lambda_1)\mu_{XX} + \lambda_1 q(2,0)(X,Y)\bar{N}_1 + S r(2,0)(X,Y)
$$

$$
\frac{\partial}{\partial t} \mu_{XY} = S r(1,1)(X,Y) + \lambda_1 q(1,1)(X,Y)\bar{N}_1 - \lambda_2 \mu_{XY} + (\lambda_1 q(1,0)(X,Y) - \lambda_1)\mu_{XY} + \lambda_1 q(0,1)(X,Y)\mu_{YY}
$$

$$
\frac{\partial}{\partial t} \mu_{YY} = S r(0,2)(X,Y) - 2\lambda_2 \mu_{YY} + \lambda_1 q(0,2)(X,Y)\bar{N}_1 + 2\lambda_1 q(0,1)(X,Y)\mu_{XY}
$$

(12)

The modified second moments, $\mu_{XL}, \mu_{YL}, \mu_{LL}, \mu_{XW}, \mu_{YW}, \mu_{WW}$ can be found by solving the system of equations:

$$
\frac{\partial}{\partial t} \mu_{XL} = (\lambda_1 q(1,0)(X,Y) - \lambda_1)\mu_{XL} + \lambda_1 d \mu_{XX}
$$

$$
\frac{\partial}{\partial t} \mu_{YL} = -\lambda_2 \mu_{YL} + \lambda_1 q(0,1)(X,Y)\mu_{XL} + \lambda_1 d \mu_{XY}
$$

$$
\frac{\partial}{\partial t} \mu_{LL} = 2\lambda_1 d \mu_{XL}
$$

(13)

$$
\frac{\partial}{\partial t} \mu_{XW} = (\lambda_1 q(1,0)(X,Y) - \lambda_1)\mu_{XW} + \lambda_2 d \mu_{XY}
$$

$$
\frac{\partial}{\partial t} \mu_{YW} = -\lambda_2 \mu_{YW} + \lambda_1 q(0,1)(X,Y)\mu_{XW} + \lambda_2 d \mu_{YY}
$$

$$
\frac{\partial}{\partial t} \mu_{WW} = 2\lambda_2 d \mu_{YW}
$$

(14)

The final expression of Feynman-alpha formulas for gammas is given as below:

$$
\frac{\sigma_{\bar{Z}_2}^2(t)}{Z_2} = 1 + Y_{s1}(1 - \frac{1 - e^{-\omega_{s1}t}}{\omega_{s1}t}) + Y_{s2}(1 - \frac{1 - e^{-\omega_{s2}t}}{\omega_{s2}t})
$$

The two roots $\omega_{s1}$ and $\omega_{s2}$ can be obtained by solving the second order characteristic equation in $\omega$ with known coefficients:

$$
\omega_{s1} = -\lambda_1 q(1,0)(X,Y) + \lambda_1
$$

$$
\omega_{s2} = \lambda_2
$$

(15)
It is interesting to notice that $\omega_{g1}$ is the same for neutrons and gammas. The functions $Y_{g1}$, $Y_{g2}$ in the gamma Feynman-alpha formula (14) are given in the form:

$$-Y_{g1} = \frac{2\lambda_{1d}(\lambda_{1f}\mu_{XY}q^{0,1}(X,Y) - \mu_{YY}(\lambda_{1f}q^{1,0}(X,Y) + \omega_{g1}) + \lambda_{1}\mu_{YY})}{\omega_{g1}(\omega_{g2} - \omega_{g1})N_2}$$

$$-Y_{g2} = \frac{2\lambda_{2d}(\lambda_{1f}\mu_{XY}q^{0,1}(X,Y) - \mu_{YY}(\lambda_{1f}q^{1,0}(X,Y) + \omega_{g2}) + \lambda_{1}\mu_{YY})}{\omega_{g2}(\omega_{g2} - \omega_{g1})N_2}$$

(16)

It can be shown that:

$$Y_{g0} = Y_{g1} + Y_{g2} = \frac{2\lambda_{2d}(\lambda_{1f}\mu_{XY}q^{0,1}(X,Y) - \lambda_{1f}\mu_{YY}q^{1,0}(X,Y) + \lambda_{1}\mu_{YY})}{\omega_{g1}\omega_{g2}N_2}$$

(17)

2.3. The one-group one-point model for total detection of neutrons and gammas

The motivation behind a separate derivation of Feynman-alpha theory for total detection of neutrons and gammas is related to the fact that variance of the total detection of neutrons and gammas cannot be represented as a linear combination of variances of the separate detection of neutrons and gammas because number of detections for neutrons and gamma are not independent variables.

The assumptions below for the one-group one-point Feynman-alpha theory for the total detection of neutrons and gammas are similar to the ones used above for the separate detection of neutrons and gammas with the only difference that $Z(t,t_d)$ represents the number of total neutron and gamma detections in the time interval $[t_d,t]$. Thus, the joint probability of having $N_1$ neutrons and $N_2$ gammas at time $t$, $Z$ neutrons and gammas have been detected during the period of time $t - t_d \geq 0$ can be defined as $P(N_1, N_2, Z, t|t_0)$. By summing up the probabilities of all mutually exclusive events of the particle not having or having a specific reaction within the infinitesimally small time interval $dt$, one obtains the forward-type equation

$$\frac{\partial P(N_1, N_2, Z, t)}{\partial t} = -(\lambda_1 N_1 + \lambda_2 N_2 + S)P(N_1, N_2, Z, t) + \lambda_{1d}(N_1 + 1)P(N_1 + 1, N_2, Z, t) + \lambda_{2d}(N_2 + 1)P(N_1, N_2 + 1, Z, t)$$

$$+ \lambda_{1f} \sum_{k}^{N_1+1} \sum_{l}^{N_2} (N_1 + 1 - k)f(k,l)P(N_1 + 1 - k, N_2 - l, Z, t)$$

$$+ \lambda_{1d}(N_1 + 1)P(N_1 + 1, N_2, Z - 1, t)$$

$$+ \lambda_{2d}(N_2 + 1)P(N_1, N_2 + 1, Z - 1, t)$$

$$+ S \sum_{m}^{N_1} \sum_{n}^{N_2} p(m,n)P(N_1 - m, N_2 - n, Z, t)$$

(18)

with the initial condition

$$P(N_1, N_2, Z, t = t_0 | t_0) = \delta_{N_1,0}\delta_{N_2,0}\delta_{Z,0}$$

(19)

6
Introducing the generating function

\[ G(X, Y, W, t) = \sum_{N_1} \sum_{N_2} \sum_Z X^{N_1} Y^{N_2} W^Z P(N_1, N_2, Z, t) \]  
(20)

with the initial condition for \( t_0 \leq t \)

\[ G(X, Y, W, t = t_0 \ | \ t_0) = 1 \]  
(21)

the following partial differential equation is obtained:

\[
\frac{\partial G}{\partial t} = [\lambda_{1a} + \lambda_{1d} W - \lambda_{1} X + q(X, Y) \lambda_{1f}] \frac{\partial G}{\partial X} + [\lambda_{2a} + \lambda_{2d} W - \lambda_{2} Y] \frac{\partial G}{\partial Y} + S_{r(X, Y)} - |G|
\]  
(22)

The quantities \( \tilde{N}_1, \tilde{N}_2, \tilde{Z} \) are given as follows:

\[ \tilde{N}_1 = \frac{S_r^{(1,0)}(X, Y)}{\lambda_1 - \lambda_{1f} q^{(1,0)}(X, Y)} \]
\[ \tilde{N}_2 = \frac{S(r^{(0,1)}(X, Y))}{\lambda_2} + \frac{S(\lambda_{1f} q^{(0,1)}(X, Y)) \lambda_1 q^{(1,0)}(X, Y)}{\lambda_2 (\lambda_1 - \lambda_{1f} q^{(1,0)}(X, Y))} \]
\[ \tilde{Z} = (\lambda_{1d} \tilde{N}_1 + \lambda_{2d} \tilde{N}_2) t \]  
(23)

The modified second moments, \( \mu_{XX}, \mu_{XY}, \mu_{YY} \) can be found by solving the system of equations in stationary state:

\[
\frac{\partial}{\partial t} \mu_{XX} = 2(\lambda_{1f} q^{(1,0)}(X, Y) - \lambda_{1}) \mu_{XX} + \lambda_{1f} q^{(2,0)}(X, Y) \tilde{Z} + S_{r^{(2,0)}(X, Y)}
\]
\[
\frac{\partial}{\partial t} \mu_{XY} = S^{(1,1)}(X, Y) + \lambda_{1f} q^{(1,1)}(X, Y) \tilde{Z} - \lambda_{2} \mu_{XY} + (\lambda_{1f} q^{(1,0)}(X, Y) - \lambda_{1}) \mu_{XY} + \lambda_{1f} q^{(0,1)}(X, Y) \mu_{YY}
\]
\[
\frac{\partial}{\partial t} \mu_{YY} = S^{(2,0)}(X, Y) - 2\lambda_{2} \mu_{YY} + \lambda_{1f} q^{(0,2)}(X, Y) \tilde{Z} + 2\lambda_{1f} q^{(0,1)}(X, Y) \mu_{XY}
\]  
(24)

The modified second moments, \( \mu_{WX}, \mu_{YW}, \mu_{WW} \) can be found by solving the system of equations:

\[
\frac{\partial}{\partial t} \mu_{WX} = \mu_{WX} (\lambda_{1f} q^{(1,0)}(X, Y) - \lambda_{1}) + \lambda_{1d} \mu_{XX} + \lambda_{2d} \mu_{XY}
\]
\[
\frac{\partial}{\partial t} \mu_{YW} = -\lambda_{2} \mu_{YW} + \lambda_{1f} q^{(0,1)}(X, Y) \mu_{WX} + \lambda_{1d} \mu_{XY} + \lambda_{2d} \mu_{YY}
\]  
(25)

\[
\frac{\partial}{\partial t} \mu_{WW} = 2\lambda_{1d} \mu_{WX} + 2\lambda_{2d} \mu_{YY}
\]

The final expression of Feynman-alpha formulas for gammas is given as below:

\[
\frac{\sigma_{2g}^2(t)}{\tilde{Z}} = 1 + Y_1 (1 - \frac{1 - e^{-\alpha_1 t}}{\alpha_1 t}) + Y_2 (1 - \frac{1 - e^{-\alpha_2 t}}{\alpha_2 t})
\]  
(26)
The two roots $\omega_1$ and $\omega_2$ can be obtained by solving the second order characteristic equation in $\omega$ with known coefficients:

$$\omega_1 = -\lambda_1 f q^{(1,0)}(X,Y) + \lambda_1$$

$$\omega_2 = \lambda_2$$

(27)

It is interesting to notice that $\omega_1$ and $\omega_2$ are the same as in the case of variance to mean formula for only gamma detections.

The functions $Y_{11}$, $Y_{12}$ for the total neutron-gamma Feynman-alpha formula are given in the form

$$Y_{11} = \frac{2\lambda_2 d(-\tilde{\lambda}_1 d(\mu_{X Y}(\lambda_1 f q^{(1,0)}(X,Y) + \omega_1) - \tilde{\lambda}_1 f \mu_{X Y} q^{(0,1)}(X,Y)))}{\omega_1 (\omega_1 - \omega_2)(N_1 \tilde{\lambda}_1 d + N_2 \lambda_2 d)}$$

$$+ \frac{2\lambda_2 d(-\tilde{\lambda}_2 d(\mu_{X Y}(\lambda_1 f q^{(1,0)}(X,Y) + \omega_1) - \tilde{\lambda}_1 f \mu_{X Y} q^{(0,1)}(X,Y)) + \tilde{\lambda}_1 (\tilde{\lambda}_1 d \mu_{X Y} + \lambda_2 d \mu_{Y Y}))}{\omega_1 (\omega_1 - \omega_2)(N_1 \tilde{\lambda}_1 d + N_2 \lambda_2 d)}$$

(28)

$$Y_{12} = \frac{2\lambda_2 d(-\tilde{\lambda}_1 d \mu_{X Y}(\lambda_1 f q^{(1,0)}(X,Y) + \omega_2) - \tilde{\lambda}_1 f \mu_{X Y} q^{(0,1)}(X,Y))}{\omega_2 (\omega_2 - \omega_1)(N_1 \tilde{\lambda}_1 d + N_2 \lambda_2 d)}$$

$$+ \frac{2\lambda_2 d(-\tilde{\lambda}_2 d(\mu_{X Y}(\lambda_1 f q^{(1,0)}(X,Y) + \omega_2) - \tilde{\lambda}_1 f \mu_{X Y} q^{(0,1)}(X,Y)) + \tilde{\lambda}_1 (\tilde{\lambda}_1 d \mu_{X Y} + \lambda_2 d \mu_{Y Y}))}{\omega_2 (\omega_2 - \omega_1)(N_1 \tilde{\lambda}_1 d + N_2 \lambda_2 d)}$$

(29)

It can be shown that

$$Y_{10} = Y_{11} + Y_{12}$$

$$= - \frac{2\lambda_2 d(\tilde{\lambda}_2 d \lambda_1 f \mu_{X Y} q^{(0,1)}(X,Y) - \tilde{\lambda}_2 d \lambda_1 f \mu_{Y Y} q^{(1,0)}(X,Y) + \tilde{\lambda}_1 \tilde{\lambda}_1 d \mu_{X Y} + \lambda_1 \lambda_2 d \mu_{Y Y})}{\omega_1 \omega_2 (N_1 \tilde{\lambda}_1 d + N_2 \lambda_2 d)}$$

$$- \frac{2\lambda_2 d(\tilde{\lambda}_1 d \lambda_1 f \mu_{X Y} q^{(0,1)}(X,Y) - \tilde{\lambda}_1 d \lambda_1 f \mu_{Y Y} q^{(1,0)}(X,Y))}{\omega_1 \omega_2 (N_1 \tilde{\lambda}_1 d + N_2 \lambda_2 d)}$$

(30)

3. Numerical illustration of the new versions of the Feynman-alpha theory and their comparison to the traditional variance to mean ratio for neutrons

In order to compare quantitatively three different versions of the Feynman-alpha theory, namely the traditional one and the two new ones introduced in this paper, MC-NPX simulations were performed in a simplified setup. The setup consists of two EJ-309 liquid scintillation detectors (D76 x 76 mm) and a $^{252}$Cf-source enclosed in a steel container (cylindrical shape) with a thickness of 25 mm, as shown in Figure 1. Each detector is located 175 mm from the source. The comparison of the neutron, gamma and total (neutron and gamma) variance to mean ratios is done by using quantitative values of the transition probabilities and reaction intensities obtained in a way similar to that described in [3, 11, 15]. Gammas from neutron capture are not included either in the simulations or in the theory; they will be considered in further
work. The coefficients obtained in MCNPX simulations and used for building up theoretical variance to mean ratios can be found and easily modified (if needed) in an interactive Mathematica notebook for visualization of gamma and total (neutron-gamma) Feynman-Alpha formula\(^1\).

As shown in Figure 2, the variance to mean ratio for neutrons\(^2\) reaches its asymptotic value in a time range between 100-1000 ns, whereas the variance to mean ratio for gamma detections and total detections result in a plateau starting in the time interval of 1-100 ns. This difference may be explained by two effects, the time of flight of fast neutrons and gammas from the source to the detector and slowing down time in the detector.

It is interesting to notice that the behaviour of the dependence of the variance to mean for the number of gamma and total (neutron and gamma) detections on the detection time are very similar which agrees with theory predictions. Although, an asymptotic value of the Feynman-Y function is an order of magnitude higher for gamma detections in comparison with the total detections. In our view, this observation cannot be taken as granted for a general case, and most probably will vary on a case to case basis.

4. Experimental illustration of the new versions of the Feynman-Y theory and their comparison to the traditional variance to mean ratio for neutrons

4.1. Description of an experimental set-up and procedure

In addition to the numerical evaluation of the newly derived variance to mean formulas for gamma and total (neutron and gamma) detections, these ratios were evaluated experimentally for a weak \(^{252}\)Cf (app.18 kBq) neutron-gamma source (originally, \(^{252}\)Cf ionization chamber detector), \(^{137}\)Cs random gamma source (22 kBq) and \(^{22}\)Na correlated gamma source (1 MBq). An experimental setup consisted of three EJ-309 (D76 x 76 mm) liquid scintillation detectors and one of the three sources, i.e. \(^{252}\)Cf, \(^{137}\)Cs and \(^{22}\)Na, as shown in Figure 3. The detectors were connected to the 8 channel, 12 bit 250 MS/s, VX1720E CAEN digitizer. As a first step of the experimental work, a calibration

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\(^1\)The interactive Mathematica notebook can be downloaded from dx.doi.org/10.13140/
\(^2\)In the case of using EJ-309 liquid scintillation detectors we only consider neutrons with light output higher than 200 keVee
procedure was performed with a $^{137}$Cs source. Pulse height spectra were independently collected during 5 minutes for each detector with a non-overlapping trigger. The results of the calibration are shown in Figure 4. As a result of this procedure, the high voltage bias and the DC offset were adjusted for each detector/channel individually, as follows: Channel 0 (Ch 0, Voltage: 1915 V, DC offset: -38.4), Channel 1 (Ch 1, Voltage: 1830 V, DC offset: -39.9), Channel 3 (Ch 3, Voltage: 1750 V, DC offset: -39.9).

The experimental evaluation of the variance to mean ratios for neutron, gamma and total detections was done with three detectors. Data were collected during 100
seconds using a simultaneous trigger for all detectors. Eventually, the data from the
detector which was located at an angle of 90 degrees to the other two scintillators was
used only for monitoring of the count rate. This was due to a presence of crosstalk
between this detector and the neighbouring ones. Afterwards, the data from the two
remaining detectors were post processed offline to obtain the dependence of the ratio of
the variance to mean of the number of neutron, gamma and total (neutron and gamma)
detections on the detection time (with 0.2 V threshold). This was done the traditional
reactor-physics way \[1, 2\] via evaluating the variance and mean of the numbers of
counts \(N\) in \(k\) consecutive time intervals of length \(T\), as shown below:

\[
\text{Variance}_k(T) = \frac{1}{k-1} \sum_{i=1}^{k} (N_i - \frac{1}{k} \sum_{i=1}^{k} N_i)^2
\]

\[
\text{Mean}_k(T) = \frac{1}{k} \sum_{i=1}^{k} N_i
\]

The uncertainty in the variance to mean ratio was estimated as below \[16\]:

\[
\sigma^2[T] = \frac{(N^2)^2}{(N)^2} \cdot \left( \frac{(N^4 - N^2)}{(N^2)^2 \cdot (k-1)} \right) + \frac{(N^2 - N^2)}{(N^2 \cdot (N \cdot (k-1)))} - 2 \cdot \frac{(N^2 - N^2 \cdot N)}{(N^2 \cdot (N \cdot (k-1)))} + \frac{(N^2 - N^2)}{(k-1)}
\]

In case of collecting data from a \(^{252}\text{Cf}\)-source, pulse shape discrimination was
performed in a way similar to \[17\] by using a charge comparison method \[18\]. In Figure
5 it is seen that the neutrons and gammas are well separated. Thus, for a sufficiently
good separation between neutrons and gammas a discrimination can be achieved by just
setting the PSD parameter threshold equal to 0.115. More accurate separation requires
application of another method for neutron-gamma discrimination, e.g. correlation-based techniques \cite{20} or artificial neural networks \cite{19, 21}.

4.2. Experimental variance to mean ratios for neutron, gamma and total detections

As a first step, an experimental evaluation of variance to mean ratios for gamma detections was performed for three sources $^{252}$Cf, $^{137}$Cs and $^{22}$Na. The results are shown in Figure 6.

As was expected, in case of a Poisson gamma source, such as $^{137}$Cs, the value of the variance of the gamma detections is equal to the mean value. In contrast, $^{22}$Na is a non-Poisson (compound Poisson) source of gamma\footnote{A $^{22}$Na nucleus emits a positron, which due to annihilation leads to emission of two gammas.}, the same way as the $^{252}$Cf source emits multiple gammas. Therefore, the value of the variance is deviating from the mean value of the gamma detections for both sources. It is interesting to notice that the asymptotic value of the variance to mean ratio for the $^{22}$Na-source is higher than for the $^{252}$Cf source, despite the fact that in general there are more gammas emitted in a $^{252}$Cf-source event in comparison to a $^{22}$Na-source event. However, the strength of the $^{22}$Na source is approximately 50 times higher than the strength of $^{252}$Cf.

As the $^{252}$Cf is a source of both neutrons and gammas, the next step of the investigation was related to the evaluation of the variance to mean ratio for neutron, gamma and total\footnote{No neutron-gamma discrimination was applied to the data.} detections. As shown in Figure 7, the asymptotic value of variance to mean ratio for neutron detections is higher than that for gammas and totals. At the same time,
in contrast to the theoretical predictions, the asymptotic value of variance to mean ratio for total detections is overestimated.

In agreement with theoretical predictions the asymptotic value of the variance to mean ratio for neutrons is obtained in a time range between 100-1000 ns, while for gamma detections the plateau in a variance to mean ratio is achieved in a time interval of 1-100 ns. It should be also mentioned that for a case of gamma and total detections, a measurement period of 100 seconds was sufficient to get the results with relatively low uncertainty, while for the case of neutron detections, the uncertainties are significant for comparable measurement times.

5. Conclusion

We have derived two versions of the neutron-gamma variance to mean (Feynman-alpha) formula for separate gamma and total neutron-gamma detections. In this paper we presented the general analytical expressions for the new formulas in order to provide the opportunity of using these new theories for both reactor and safeguards applications. It was found that the variance to mean (Feynman-alpha) formulas for separate gamma and total detections are both obtained in a two-exponential form. Moreover, one of the exponents ($\omega$) is the same for total and gamma detections as for neutron detections. Thus, we can conclude that the variance to mean (Feynman-alpha) formulas for separate gamma and total detections contain the same information on neutron
population and neutron detection characteristics. This is not valid in reverse, i.e. the variance to mean (Feynman-Y) formula for neutron detections does not contain any information on gamma population or detection characteristics.

The results of the numerical and experimental evaluation of the newly derived Feynman-alpha formulas for gamma and total detections show that a variance to mean ratio for neutrons reaches its asymptotic value in a time range between 100-1000 ns, while for gamma and total detections in a time interval of 1-100 ns. The numerical and experimental time dependence of the variance to mean ratio for gamma is very similar to that for total detections, which agrees with theory predictions. A numerical asymptotic value of the Feynman-Y function is an order of magnitude higher for gamma detections as compared to the total detections. Although, the situation is opposite in a case of experimental evaluation where the asymptotic value of Feynman-Y ratio is higher for total detections in comparison with gamma detections In our view, this observation cannot be taken as granted for a general case and most probably will vary on a case by case basis. It is also worth mentioning that for the case of gamma and total detections time of 100 seconds for measurements were sufficient to obtain results with relatively low uncertainty, while for the case of neutron detections much larger uncertainties prevail for comparable measurement times.

Thus, one may conclude that the new formulas for gamma and total neutron-gamma detections have a promise to complement, and in some cases replace the traditional variance to mean (Feynman-Y) formula for neutron detections. The variance to mean (Feynman-Y) formula for total neutron-gamma detections may eliminate the problem
related to discrimination between neutron and gamma particles in scintillation detectors. At the same time the variance to mean (Feynman-alpha) formula for gamma detections has a high potential to be used with detectors of only gamma radiation, which is normally employed in a spent fuel pool.

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