Excluded $t$-factors in Bipartite Graphs: Unified Framework for Nonbipartite Matchings, Restricted 2-matchings, and Matroids

Kenjiro Takazawa

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Abstract

We propose a framework for optimal $t$-matchings excluding the prescribed $t$-factors in bipartite graphs. The proposed framework is a generalization of the nonbipartite matching problem and includes several problems, such as the triangle-free 2-matching, square-free 2-matching, even factor, and arborescence problems. In this paper, we demonstrate a unified understanding of these problems by commonly extending previous important results. We solve our problem under a reasonable assumption, which is sufficiently broad to include the specific problems listed above. We first present a min-max theorem and a combinatorial algorithm for the unweighted version. We then provide a linear programming formulation with dual integrality and a primal-dual algorithm for the weighted version. A key ingredient of the proposed algorithm is a technique to shrink forbidden structures, which corresponds to the techniques of shrinking odd cycles, triangles, squares, and directed cycles in Edmonds’ blossom algorithm, a triangle-free 2-matching algorithm, a square-free 2-matching algorithm, and an arborescence algorithm, respectively.

1 Introduction

Since matching theory [29] was established, a number of generalizations of the matching problem have been proposed, including path-matchings [7], even factors [8, 33, 34], triangle-free 2-matchings [6, 32], square-free 2-matchings [15, 33], $K_{f_t}$-free $t$-matchings [12], $K_{f+1}$-free $t$-matchings [3], 2-matchings covering prescribed edge cuts [4, 21], and $U$-feasible 2-matchings [44]. For most of these generalizations, important results in matching theory can be extended, such as a min-max theorem, polynomial algorithms, and a linear programming formulation with dual integrality. However, while some similar structures are found, in most cases, they have been studied separately and few connections among those similar structures have been identified.

In this paper, we propose a new framework of optimal $t$-matchings excluding prescribed $t$-factors, to demonstrate a unified understanding of these generalizations. The proposed framework includes all of the above generalizations and the arborescence problem. Furthermore, it includes the traveling salesman problem (TSP). This broad coverage implies some intractability of the framework; however we propose a tractable class that includes most of the efficiently solvable classes of the above problems.

Our main contributions are a min-max theorem and a combinatorial polynomial algorithm that commonly extend those for the matching and triangle-free 2-matching problem in nonbipartite graphs, the square-free...
2-matching problem in bipartite graphs, and the arborescence problem in directed graphs. A key ingredient of
the proposed algorithm is a technique to shrink the excluded \( t \)-factors. This technique commonly extends the
techniques used to shrink odd cycles, triangles, squares, and directed cycles in a matching algorithm [10], a
triangle-free 2-matching algorithm [6], a square-free 2-matching algorithms in bipartite graphs [15,33], and
an arborescence algorithm [5,11], respectively. We demonstrate that the proposed framework is tractable in
the class where this shrinking technique works.

1.1 Previous Work

The problems most relevant to our work are the \textit{even factor}, \textit{triangle-free 2-matching}, and \textit{square-free 2-
matching problems}.

1.1.1 Even factor

The even factor problem [8] is a generalization of the nonbipartite matching problem, which admits a further
generalization: the basic/independent even factor problem [8,20] is a common generalization with matroid
intersection. The origin of the even factor problem is the \textit{independent path-matching problem} [7], which is a
common generalization of the nonbipartite matching and matroid intersection problems. In [7], a min-max
theorem, totally dual integral polyhedral description, and polynomial solvability by the ellipsoid method
were presented. These were followed by further analysis of the min-max relation [13] and Edmonds-Gallai
decomposition [36]. A combinatorial approach to the path-matchings was proposed in [37] and completed
by Pap [33], who addressed a further generalization, \textit{the even factor problem} [8].

Here, let \( D = (V,A) \) be a digraph. A subset of arcs \( F \subseteq A \) is called a \textit{path-cycle factor} if it is a
vertex-disjoint collection of directed cycles (dicycles) and directed paths (dipaths). Equivalently, an arc
subset \( F \) is a path-cycle factor if, in the subgraph \( (V,F) \), the indegree and outdegree of every vertex are at
most one. An \textit{even factor} is a path-cycle factor excluding dicycles of odd length (odd dicycles).

While the maximum even factor problem is NP-hard, in \textit{odd-cycle symmetric} digraphs it enjoys min-max
theorems [8,34], the Edmonds-Gallai decomposition [34], and polynomial-time algorithms [8,33]. A
digraph is called \textit{odd-cycle symmetric} if every odd dicycle has its reverse dicycle. Moreover, a maximum-
weight even factor can be found in polynomial time in odd-cycle symmetric weighted digraphs, which are
odd-cycle symmetric digraphs with arc-weight such that the total weight of the arcs in an odd dicycle is
equal to that of its reverse dicycle. The maximum-weight matching problem is straightforwardly reduced to
the maximum-weight even factor problem in odd-cycle symmetric weighted digraphs (see Sect. 2.2.2). The
assumption of odd-cycle symmetry of (weighted) digraphs is supported by its relation to discrete convexity
[27].

The independent even factor problem is a common generalization of the even factor and matroid inter-
section problems. In odd-cycle symmetric digraphs it admits combinatorial polynomial algorithms [8,20]
and a decomposition theorem [20], which extends the Edmonds-Gallai decomposition and the principal
partition for matroid intersection [18,19]. In odd-cycle symmetric weighted digraphs, a linear program with
dual integrality and a combinatorial algorithm for the weighted independent even factor problem have been
presented in [41].

The results are summarized in Table 1. For more details, readers are referred to a survey paper [40].

1.1.2 Restricted \( t \)-matching

The triangle-free 2-matching and square-free 2-matching problems are types of the \textit{restricted 2-matching
problem}, wherein a main objective is to provide a tight relaxation of the TSP.

Here, let \( G = (V,E) \) be a simple undirected graph. For \( v \in V \), let \( \delta(v) \subseteq E \) denote the set of edges incident
to \( v \). For a positive integer \( t \), a vector \( x \in \mathbb{Z}_+^E \) is called a \textit{\( t \)-matching} (resp., \textit{\( t \)-factor}) if \( \sum_{e \in \delta(v)} x(e) \leq t \)
Table 1: Results for path-matchings and even factors. (E), (A), (C) denote the ellipsoid method, an algebraic algorithm, and a combinatorial algorithm, respectively.

| Path-matchings | Independent path-matchings |
|----------------|---------------------------|
| Min-max theorem | Cunningham–Geelen [7] | Cunningham–Geelen [7] |
| Algorithm       | Frank–Szegő [13]         | Iwata–Takazawa [20] |
| Decomposition theorem | Cunningham–Geelen [7] (E) | Cunningham–Geelen [7] (E) |
| LP formulation  | Spille–Szegő [36]        | Iwata–Takazawa [20] |
| Algorithm (Weighted) | Cunningham–Geelen [7] | Cunningham–Geelen [7] (E) |

| Even factors | Independent even factors |
|--------------|--------------------------|
| Min-max theorem | Cunningham–Geelen [8] | Iwata–Takazawa [20] |
| Algorithm     | Pap–Szegő [34]           | Iwata–Takazawa [20] (C) |
| Decomposition theorem | Pap–Szegő [34] (C) | Iwata–Takazawa [20] |
| LP formulation | Király–Makai [22]       | Takazawa [41] |
| Algorithm (Weighted) | Takazawa [38] (C) | Takazawa [41] (C) |

(resp., \( \sum_{e \in \delta(v)} x(e) = t \)) for each \( v \in V \). A 2-matching \( x \) is called triangle-free if it excludes a triple of edges \( (e_1, e_2, e_3) \) such that \( e_1, e_2, \) and \( e_3 \) form a cycle and \( x(e_1) = x(e_2) = x(e_3) = 1 \). For the maximum-weight triangle-free 2-matching problem, a combinatorial algorithm, together with a totally dual integral formulation, is designed [6, 32].

If we only deal with simple 2-matchings \( x \in \{0,1\}^E \), the triangle-free 2-matching problem becomes much more complicated [14]. A vector \( x \in \{0,1\}^E \) is identified by an edge set \( F \subseteq E \) such that \( e \in F \) if and only if \( x(e) = 1 \). That is, an edge set \( F \subseteq E \) is called a simple \( t \)-matching if \( |F \cap \delta(v)| \leq t \) for each \( v \in V \). For a positive integer \( k \), a simple \( 2 \)-matching is called \( C_{\leq k} \)-free if it excludes cycles of length at most \( k \). Finding a maximum simple \( C_{\leq k} \)-free 2-matching is NP-hard for \( k \geq 5 \), and open for \( k = 4 \).

In contrast, the simple \( C_{\leq 4} \)-free 2-matching problem becomes tractable in bipartite graphs. We often refer to simple \( C_{\leq 4} \)-free 2-matching in a bipartite graph as square-free 2-matching. For the square-free 2-matching problem, extensions of the classical matching theory, such as min-max theorems [12, 15, 23, 24], combinatorial algorithms [15, 33], and decomposition theorems [43], have been established.

Further, two generalizations of the square-free 2-matchings have been proposed. Frank [12] introduced a generalization, \( K_{1,t} \)-free \( t \)-matchings in bipartite graphs, and provided a min-max theorem. Another generalization introduced in [44] is \( U \)-feasible 2-matchings. For \( U \subseteq 2^V \), a 2-matching is \( U \)-feasible if it does not contain a 2-factor in \( U \) for each \( U \in \mathcal{U} \). Takazawa [44] presented a min-max theorem, a combinatorial algorithm, and decomposition theorems for the case where each \( U \in \mathcal{U} \) induced a Hamilton-laceable graph [35], by extending the aforementioned theory for square-free 2-matchings in bipartite graphs.

For the weighted case, Király [24] proved that finding a maximum-weight square-free 2-matching is NP-hard (see also [12]). However, Makai [30] presented a linear programming formulation of the weighted \( K_{1,t} \)-free \( t \)-matching problem in bipartite graphs with dual integrality for a special case where the weight is vertex-induced on each \( K_{1,t} \) (see Definition 2.2). Takazawa [39] designed a combinatorial algorithm for this case. The assumption on the weight is supported by discrete convexity in [26], which proved that maximum-weight square-free 2-matchings in bipartite graphs induce an M-concave function on a jump system [31] if and only if the edge weight is vertex-induced on every square.

The aforementioned results on simple restricted \( t \)-matchings are for bipartite graphs. We should also mention another graph class where restricted \( t \)-matchings are tractable, degree-bounded graphs. In subcubic
graphs, optimal 2-matchings excluding the cycles of length three and/or four are tractable \cite{2,16,17,25}. Some of these results are generalized to \( t \)-matchings excluding \( K_{t+1} \) and \( K_t \) in graphs with maximum degree of up to \( t+1 \) \cite{3,28}. In bridgeless cubic graphs, there always exists 2-factors covering all 3- and 4-edge cuts \cite{21}, and found in polynomial time \cite{4}. A minimum-weight 2-factor covering all 3-edge cuts can also be found in polynomial time \cite{4}.

### 1.2 Contribution

It is noteworthy that Pap \cite{33} presented combinatorial algorithms for the even factor and square-free 2-matching problems in the same paper. These algorithms were based on similar techniques to shrink odd cycles and squares, and were improved in complexity by Babenko \cite{1}. However, to the best of our knowledge, there has been no comprehensive theory that considers both algorithm.

In this paper, we discuss \( \mathcal{U} \)-feasible \( t \)-matchings (see Definition \ref{def:U-feasible}). The \( \mathcal{U} \)-feasible \( t \)-matching problem generalizes not only the \( \mathcal{U} \)-feasible 2-matching problem \cite{44} but also all of the aforementioned generalizations of the matching problem, as well as the TSP and the arborescence problem (see Sect. \ref{sect:2.2}). The objective of this paper is to provide a unified understanding of these problems. One example of such an understanding is that \( \mathcal{U} \)-feasibility is a common generalization of the blossom constraint for the nonbipartite matching problem and the subtour elimination constraint for the TSP.

The main contributions of this paper are a min-max theorem and an efficient combinatorial algorithm for the maximum \( \mathcal{U} \)-feasible \( t \)-matching problem in bipartite graphs under a plausible assumption. Note that the \( \mathcal{U} \)-feasible \( t \)-matching problem in bipartite graphs can describe the nonbipartite matching problem. We also remark that it reasonable to impose some assumption in order to obtain a tractable class of the \( \mathcal{U} \)-feasible \( t \)-matching problem. (Recall that it can describe the Hamilton cycle problem.) Indeed, we assume that an expanding technique is always valid for the excluded \( t \)-factors (see Definition \ref{def:U-feasible}). This assumption is sufficiently broad to include the instances reduced from nonbipartite matchings, even factors in odd-cycle symmetric digraphs, triangle-free 2-matchings, square-free 2-matchings, and arborescences. We then show that the \( C_{4k+2} \)-free 2-matching problem, a new class of the restricted 2-matching problem, is contained in our framework. We prove that the \( C_{4k+2} \)-free 2-matching problem under a certain assumption is described as the \( \mathcal{U} \)-feasible 2-matching problem under our assumption, and thus obtain a new class of the restricted 2-matching problem which can be solved efficiently.

The proposed algorithm commonly extend those for nonbipartite matchings, even factors, triangle-free 2-matchings, square-free 2-matchings, and arborescences. Generally, the proposed algorithm runs in \( O(t(|V|^3 \alpha + |V|^2 \beta)) \) time, where \( \alpha \) and \( \beta \) are the time required to check the feasibility of an edge set and expand the shrunk structures, respectively. The complexities \( \alpha \) and \( \beta \) are typically small, i.e., constant or \( O(n) \), in the above specific cases (see Sect. \ref{sect:5.4}).

We further establish a linear programming description with dual integrality and a primal-dual algorithm for the maximum-weight \( \mathcal{U} \)-feasible \( t \)-matching problem in bipartite graphs. The complexity of the algorithm is \( O(t(|V|^3 (|E| + \alpha) + |V|^2 \beta)) \). For the weighted case, we also assume the edge weight to be vertex-induced for each \( U \in \mathcal{U} \). Note that this assumption is also inevitable because the maximum-weight square-free 2-matching problem is NP-hard. To be more precise, our assumption exactly corresponds to the previous assumptions for the maximum-weight even factor and square-free 2-matching problems, both of which are plausible from the discrete convexity perspective \cite{26,27}. This would be an example of a unified understanding of even factors and square-free 2-matchings.

This paper is organized as follows. In Sect. \ref{sect:2} we present a precise definition of the proposed framework. Sect. \ref{sect:3} describes a min-max theorem and a combinatorial algorithm for the maximum \( \mathcal{U} \)-feasible \( t \)-matching problem. In Sect. \ref{sect:4} we extend these results to a linear programming formulation with dual integrality and a primal-dual algorithm for the maximum-weight \( \mathcal{U} \)-feasible \( t \)-matching problem. Conclusions are presented in Sect. \ref{sect:5
2 Our Framework

In this section, we define the proposed framework and explain how the previously mentioned problems are reduced.

2.1 Optimal \( t \)-matching Excluding Prescribed \( t \)-factors

Here, let \( G = (V, E) \) be a simple undirected graph. An edge \( e \) connecting \( u, v \in V \) is denoted by \( \{u, v\} \). If \( G \) is a digraph, then an arc from \( u \) to \( v \) is denoted by \( (u, v) \). For \( X \subseteq V \), let \( G[X] = (X, E[X]) \) denote the subgraph of \( G \) induced by \( X \), i.e., \( E[X] = \{ \{u, v\} \mid u, v \in X, \{u, v\} \in E \} \). Similarly, for \( F \subseteq E \), define \( F[X] = \{ \{u, v\} \mid u, v \in X, \{u, v\} \in F \} \).

Recall that \( \delta(v) \subseteq E \) denotes the set of edges incident to \( v \in V \). For \( F \subseteq E \) and \( v \in V \), let \( \deg_F(v) = |F \cap \delta(v)| \). Then, \( F \) is a \( t \)-matching if \( \deg_F(v) \leq t \) for each \( v \in V \), and a \( t \)-factor if \( \deg_F(v) = t \) for every \( v \in V \).

**Definition 2.1.** For a graph \( G = (V, E) \) and \( \mathcal{U} \subseteq 2^V \), a \( t \)-matching \( F \subseteq E \) is called \( \mathcal{U} \)-feasible if

\[
|F[U]| \leq \left\lfloor \frac{t|U| - 1}{2} \right\rfloor
\]

for each \( U \in \mathcal{U} \).

Equivalently, a \( t \)-matching \( F \) in \( G \) is not \( \mathcal{U} \)-feasible if \( F[U] \) is a \( t \)-factor in \( G[U] \) for some \( U \in \mathcal{U} \). This concept is a further generalization of the \( \mathcal{U} \)-feasible 2-matchings introduced in [44].

In what follows, we consider the maximum \( \mathcal{U} \)-feasible \( t \)-matching problem, whose goal is to find a \( \mathcal{U} \)-feasible \( t \)-matching \( F \) maximizing \( |F| \). We further deal with the maximum-weight \( \mathcal{U} \)-feasible \( t \)-matching problem, in which the objective is to find a \( \mathcal{U} \)-feasible \( t \)-matching \( F \) maximizing \( w(F) = \sum_{e \in F} w(e) \) for a given edge-weight vector \( w \in \mathbb{R}^E \). For a vector \( x \in \mathbb{R}^E \) and \( F \subseteq E \), in general we denote \( x(F) = \sum_{e \in F} x(e) \).

In discussing the weighted version, we assume that \( w \) is vertex-induced on each \( U \in \mathcal{U} \).

**Definition 2.2.** For a graph \( G = (V, E) \), a vertex subset \( U \subseteq V \), and an edge-weight \( w \in \mathbb{R}^E \), \( w \) is called vertex-induced on \( U \) if there exists a function \( \pi_U : U \to \mathbb{R} \) on \( U \) such that \( w(\{u, v\}) = \pi_U(u) + \pi_U(v) \) for each \( \{u, v\} \in E[U] \).

Here, as noted previously, not only the maximum-weight square-free 2-matching problem in bipartite graphs, but also many generalizations in nonbipartite graphs, such as the maximum-weight matching, even factor, and triangle-free 2-matching, and arborescence problems, are reduced to the maximum-weight \( \mathcal{U} \)-feasible \( t \)-matching problem in bipartite graphs under the assumption that \( w \) is vertex-induced on each \( U \in \mathcal{U} \). The reduction is shown in Sect. 2.2.

2.2 Special Cases of \( \mathcal{U} \)-feasible \( t \)-matching in Bipartite Graphs

Here we demonstrate how the problems in the literature are reduced to the \( \mathcal{U} \)-feasible \( t \)-matching problem. In Sect. 2.2.1–2.2.4, we demonstrate reductions to the \( \mathcal{U} \)-feasible \( t \)-matching problem in bipartite graphs, which is the primary focus of this paper. How our algorithm works for those specific cases is described in Sect. 3.4. In Sect. 2.2.3, we show reductions to the \( \mathcal{U} \)-feasible \( t \)-matching problem in nonbipartite graphs. While we do not discuss solvability in nonbipartite graphs in this paper, we show these reductions in order to demonstrate the generality of the proposed framework.
2.2.1 Restricted 2-matchings and Hamilton Cycles in Bipartite Graphs

Let $G = (V, E)$ be a simple bipartite graph. If $t = 2$ and $U = \{ U \subseteq V \mid |U| = 4 \}$, then a $U$-feasible 2-matching in $G$ is exactly a square-free 2-matching in $G$. Generally, a simple $C_{2k}$-free 2-matching in $G$ is exactly a $U$-feasible 2-matching in $G$ where $U = \{ U \subseteq V \mid 1 \leq |U| \leq k \}$. For example, if $U = \{ U \subseteq V \mid 1 \leq |U| \leq |V| - 1 \}$, then the maximum $U$-feasible 2-matching problem includes the Hamilton cycle problem, i.e., if a maximum $U$-feasible 2-matching is of size $|V|$, it is a Hamilton cycle.

Square-free 2-matchings are generalized to $K_{t,t}$-free $t$-matchings in bipartite graphs [12]. A simple $t$-matching is called $K_{t,t}$-free if it does not contain $K_{t,t}$ as a subgraph. A $K_{t,t}$-free $t$-matching in a bipartite graph is exactly a $U$-feasible $t$-matching, where $U = \{ U \subseteq V \mid |U| = 2t \}$.

2.2.2 Matchings and Even Factors in Nonbipartite Graphs

First, we show the reduction of the nonbipartite matching problem to the even factor problem. Then, we present the reduction of the even factor problem to the $U$-feasible $t$-matching problem in bipartite graphs.

Consider the maximum-weight matching problem in a nonbipartite graph $G = (V, E)$ with weight $w \in \mathbb{R}^E$. This can be reduced to the maximum-weight even factor problem in a digraph $D = (V, A)$, where $A = \{ (u, v), (v, u) \mid \{ u, v \} \in E \}$, and an arc-weight $w' \in \mathbb{R}^A$ is defined by $w'(u, v) = w'(v, u) = w(\{ u, v \})$. For a matching $M \subseteq E$ in $G$, it is clear that there exists an even factor $F \subseteq A$ in $D$ with $w'(F) = 2w(M)$. Conversely, for an even factor $F \subseteq A$ in $D$, there exists a matching $M \subseteq E$ with $w(M) \geq w'(F)/2$.

Here, let $D = (V, A)$ and $w \in \mathbb{R}^A$ be an arbitrary instance of the maximum-weight even factor problem (Fig. [1]). Then, define an instance of the maximum-weight $U$-feasible $t$-matching problem as follows. Let $t = 1$. For each $u \in V$, let $u^+$ and $u^-$ be two copies of $u$, and define $\hat{V}^+ = \{ u^+ \mid u \in V \}$ and $\hat{V}^- = \{ u^- \mid u \in V \}$. Let $U \subseteq V$, denote $\hat{U} = \bigcup_{u \in U} \{ u^+, u^- \}$. Now define a bipartite graph $\hat{G} = (\hat{V}, \hat{E})$, $\hat{U} \subseteq 2\hat{V}$, and an edge-weight $\hat{w} \in \mathbb{R}^{\hat{E}}$ by

$$\hat{E} = \{ (u^+, v^-) \mid (u, v) \in A \}, \quad \hat{U} = \{ \hat{U} \mid U \subseteq V, |U| \text{ is odd} \},$$

$$\hat{w}((u^+, v^-)) = w((u, v)) \quad ((u^+, v^-) \in \hat{E}).$$

Note that a 1-matching in $\hat{G}$ corresponds to a path-cycle factor in $D$. For $\hat{U} \in \hat{U}$, a 1-factor in $\hat{G}[\hat{U}]$ corresponds to a vertex-disjoint collection of cycles through $U$ in $D$, which should contain at least one odd cycle. Thus, $\hat{U}$-feasibility of a 1-matching in $\hat{G}$ exactly corresponds to excluding odd cycles in a path-cycle factor in $D$, which results in an even factor.

If $(D, w)$ is odd-cycle symmetric, $\hat{G}[\hat{U}]$ is a symmetric bipartite graph and $\hat{w}$ is vertex-induced on $\hat{U}$ for each $\hat{U} \in \hat{U}$. Thus, the instance constructed in the reduction satisfies our assumption that $\hat{w}$ is vertex-induced on each $\hat{U} \in \hat{U}$.

2.2.3 Triangle-free 2-matchings in Nonbipartite Graphs

Here, let $G = (V, E)$ be an undirected nonbipartite graph and $w \in \mathbb{R}^E$. Now define $\hat{V}^+$, $\hat{V}^-$, and $\hat{U}$ as described in Sect. 2.2.2. Let $t = 1$ and define $\hat{G} = (\hat{V}, \hat{E})$, $\hat{U} \subseteq 2\hat{V}$, and $\hat{w} \in \mathbb{R}^{\hat{E}}$ by

$$\hat{V}^+ = \{ u^+ \mid u \in V \}, \quad \hat{V}^- = \{ v^- \mid v \in V \},$$

$$\hat{E} = \bigcup_{(u,v) \in E} \{ (u^+, v^-), (v^+, u^-) \}, \quad \hat{U} = \{ \hat{U} \mid U \subseteq V, |U| = 3 \},$$

$$\hat{w}(\{ u^+, v^- \}) = \hat{w}(\{ v^+, u^- \}) = w(\{ u, v \}) \quad ((u, v) \in E).$$

It is straightforward that a triangle-free 2-matching $F$ in $G$ corresponds to a $\hat{U}$-feasible 1-matching $\hat{F}$ in $\hat{G}$ such that $w(F) = \hat{w}(\hat{F})$, and vice versa (Fig. [2]).
Figure 1: The maximum even factor problem in $D$ is reduced to the maximum $\hat{U}$-feasible 1-matching problem in $\hat{G}$, where $\hat{U} = \{ \hat{U} \mid U \subseteq V, |U| \text{ is odd} \}$. The set of thick arcs in $D$ is an even factor that corresponds to the set of thick edges in $\hat{G}$, which is a $\hat{U}$-feasible 1-matching.

Figure 2: The maximum triangle-free 2-matching problem in $G$ is reduced to the maximum $\hat{U}$-feasible 1-matching problem in $\hat{G}$, where $\hat{U} = \{ \hat{U} \mid U \subseteq V, |U| = 3 \}$. The set of thick edges in $G$ is a triangle-free 2-matching that corresponds to the set of thick edges in $\hat{G}$, which is a $\hat{U}$-feasible 1-matching.
2.2.4 Matroids and Arborescences

Here, let \( M \) be a matroid with ground set \( V \) and circuit family \( C \subseteq 2^V \). The problem of finding a maximum-weight independent set in \( M \) with respect to \( w \in \mathbb{R}^V \) is described as the maximum-weight \( \hat{U} \)-feasible \( t \)-matching problem in a bipartite graph \( \hat{G} \) as follows. Let \( t = 1 \). Define \( \hat{G} = (\hat{V}, \hat{E}), \hat{U} \subseteq 2^{\hat{V}} \), and \( \hat{w} \in \mathbb{R}^{\hat{E}} \) by

\[
\hat{V}^+ = \{u^+ \mid u \in V\}, \quad \hat{V}^- = \{v^- \mid v \in V\}, \\
\hat{E} = \{\{v^+, v^-\} \mid v \in V\}, \quad \hat{U} = \left\{ \bigcup_{v \in C} \{v^+, v^-\} \mid C \in C \right\}, \\
\hat{w}(\{v^+, v^-\}) = w(v).
\]

Then, it is straightforward that \( I \subseteq V \) is an independent set in \( M \) if and only if the edge set \( \{\{v^+, v^-\} \mid v \in I\} \) is a \( \hat{U} \)-feasible \( 1 \)-matching in \( \hat{G} \).

Arborescences in a digraph are a special case of matroid intersection. Although we do not know how to describe matroid intersection in our framework, the arborescence problem can be reduced to the \( U \)-feasible problem in a bipartite graph as follows. Let \( D = (V, A) \) be a digraph in which we are asked to find a maximum-weight arborescence with respect to an arc weight \( w \in \mathbb{R}^A \). Let \( t = 1 \), and define \( \hat{G} = (\hat{V}, \hat{E}), \hat{U} \subseteq 2^{\hat{V}} \), and \( \hat{w} \in \mathbb{R}^{\hat{E}} \) by

\[
\hat{V}^+ = \{a^+ \mid a \in A\}, \quad \hat{V}^- = \{v^- \mid v \in V\}, \quad \hat{E} = \{\{a^+, v^-\} \mid v \text{ is the head of } a \text{ in } D\}, \\
\hat{U} = \left\{ \{a^+, v^-\} \mid a \in A(C) \cup \{v^- \mid v \in V(C)\} \mid C \text{ is a directed cycle in } D \right\}, \\
\hat{w}(\{a^+, v^-\}) = w(a),
\]

where \( A(C) \) and \( V(C) \) denote the sets of arcs and vertices of a directed cycle \( C \), respectively. Again, it is straightforward that \( A' \subseteq A \) is an arborescence in \( D \) if and only if the edge set

\[
\{\{a^+, v^-\} \mid a \in A', v \text{ is the head of } a \text{ in } D\}
\]

is a \( \hat{U} \)-feasible \( 1 \)-matching in \( \hat{G} \).

2.2.5 Special Cases of Nonbipartite \( U \)-feasible \( t \)-matchings

The simple \( C_{\leq k} \)-free 2-matching problem in a nonbipartite graph \( G = (V, E) \) is exactly the \( U \)-feasible 2-matching problem in \( G \), where \( U = \{U \subseteq V \mid 1 \leq |U| \leq k\} \). For example, if \( U = \{U \subseteq V \mid 1 \leq |U| \leq |V| - 1\} \), then a \( U \)-feasible 2-matching of size \( |V| \) is exactly a Hamilton cycle.

The \( K_{t+1} \)-free \( t \)-matching problem is a generalization of the simple triangle-free 2-matching problem. A simple \( t \)-matching is called \( K_{t+1} \)-free if it does not contain \( K_{t+1} \) as a subgraph. Now a \( K_{t+1} \)-free \( t \)-matching is exactly a \( U \)-feasible \( t \)-matching, where \( U = \{U \subseteq V \mid |U| = t + 1\} \).

A 2-factor covering prescribed edge cuts is also described as a \( U \)-feasible 2-matching. Here, let \( C \subseteq E \) be an edge cut, i.e., \( C \) is an inclusion-minimal edge subset such that deleting \( C \) makes \( G \) disconnected. A 2-factor \( F \) covers \( C \) if \( F \cap C \neq \emptyset \). For a family \( C \) of edge cuts, a 2-factor covering every edge cut in \( C \) is a relaxed concept of Hamilton cycles, i.e., if \( C \) is the family of all edge cuts in \( G \), then a 2-factor covering all edge cuts in \( C \) is a Hamilton cycle.

Now a 2-factor covering every edge cut in \( C \) is described as a \( U \)-feasible 2-factor by putting \( U = \{U \subseteq V \mid \delta(U) \in C\} \), where \( \delta(U) \) denotes \( E[U, V \setminus U] \), i.e., the set of edges connecting \( U \) and \( V \setminus U \). For example, a 2-factor covering all 3- and 4-edge cuts \([4, 21]\) is a \( U \)-feasible 2-factor, where \( U = \{U \subseteq V \mid \delta(U) \text{ is a 3-edge cut or a 4-edge cut}\} \).
3 Maximum $U$-feasible $t$-matching

In this section, we present a min-max theorem and a combinatorial algorithm for the maximum $U$-feasible $t$-matching problem in bipartite graphs. The proposed algorithm commonly extends those for nonbipartite matchings \cite{10}, even factors \cite{33}, triangle-free 2-matchings \cite{6}, square-free 2-matchings \cite{15,33}, and arborescences \cite{5,11}. We begin with a weak duality theorem in Sect. 3.1. The proposed algorithm is described in Sect. 3.2 and its validity is proved in Sect. 3.3 together with the min-max theorem (strong duality theorem). In Sect. 3.4 we demonstrate how the algorithm works in such special cases.

3.1 Weak Duality

Here, let $G = (V, E)$ be an undirected graph and let $U \subseteq 2^V$. For weak duality, $G$ does not need to be bipartite. For $X \subseteq V$, define $U_X \subseteq U$ and $C_X \subseteq X$ by

$$U_X = \{ U \in U \mid U \text{ forms a component in } G[X] \}, \quad C_X = X \setminus \bigcup_{U \in U_X} U.$$ 

Then, the following inequality holds for an arbitrary $U$-feasible $t$-matching $F \subseteq E$ and $X \subseteq V$.

**Lemma 3.1.** Let $G = (V, E)$ be an undirected graph, $U \subseteq 2^V$, and $t$ be a positive integer. For an arbitrary $U$-feasible $t$-matching $F \subseteq E$ and $X \subseteq V$, it holds that

$$|F| \leq t|X| + |E[C_V \setminus X]| + \sum_{U \in U_{V \setminus X}} \left\lfloor \frac{t|U| - 1}{2} \right\rfloor. \quad (4)$$

**Proof.** By counting the number of edges in $F$ incident to $X$, we obtain

$$2|F[X]| + |F[X, V \setminus X]| \leq t|X|. \quad (5)$$

In $G[V \setminus X]$, it holds that

$$|F[V \setminus X]| \leq |E[C_{V \setminus X}]| + \sum_{U \in U_{V \setminus X}} \left\lfloor \frac{t|U| - 1}{2} \right\rfloor. \quad (6)$$

By summing (5) and (6), we obtain

$$2|F[X]| + |F[X, V \setminus X]| + |F[V \setminus X]| \leq t|X| + |E[C_V \setminus X]| + \sum_{U \in U_{V \setminus X}} \left\lfloor \frac{t|U| - 1}{2} \right\rfloor. \quad (7)$$

Since $|F| = |F[X]| + |F[X, V \setminus X]| + |F[V \setminus X]|$ is at most the left-hand side of (7), we obtain (4). \hfill \blacksquare

3.2 Algorithm

Hereafter, we assume that $G$ is bipartite. Let $G = (V, E)$ be a simple undirected bipartite graph. Here, we denote the two color classes of $V$ by $V^+$ and $V^-$. For $X \subseteq V$, denote $X^+ = X \cap V^+$ and $X^- = X \cap V^-$. The endvertices of an edge $e \in E$ in $V^+$ and $V^-$ are denoted by $\partial^+ e$ and $\partial^- e$, respectively.

We begin by describing the shrinking of a forbidden structure $U \in \mathcal{U}$. For concise notation, we denote the input graph as $\hat{G} = (\hat{V}, \hat{E})$ and the graph generated by potential repeated shrinkings as $G = (V, E)$. Consequently, we have $\mathcal{U} \subseteq 2^V$. The solution in hand is denoted by $F \subseteq E$.

Intuitively, shrinking of $U$ involves identifying all vertices in $U^+$ and $U^-$ to obtain new vertices $u^+_U$ and $v^-_U$, respectively, and deleting all edges in $E[U]$. In a shrunk graph $G = (V, E)$, we refer to a vertex $v \in V$ as
a natural vertex if \( v \) is a vertex in the original graph \( \hat{G} \), and as a pseudovertex if it is a newly added vertex when shrinking some \( U \in \mathcal{U} \). We denote the set of natural vertices as \( V_0 \), and the set of pseudovertices as \( V_p \).

For \( X \subseteq \hat{Y} \), define \( X_n = X \cap V_0 \) and \( X_p = \bigcup \{ u^+_U, v^-_U \mid u^+_U, v^-_U \in V_p, U \cap X \neq \emptyset \} \). For \( X \subseteq V \), define \( \hat{X} \subseteq \hat{Y} \) by \( \hat{X} = X_n \cup \{ U^+ \mid u^+_U \in X \cap V_0 \} \cup \{ U^- \mid v^-_U \in X \cap V_0 \} \).

A formal description of shrinking \( U \in \mathcal{U} \) is given as follows.

**Procedure Shrinking** \((U)\). Let \( u^+_U \) and \( v^-_U \) be new vertices, and reset the endvertices of an edge \( e \in E \setminus E[U_n \cup U_p] \) with \( \partial^+ e = u \) and \( \partial^- e = v \) by

\[
\partial^+ e := u^+_U \quad \text{if} \quad u \in U^+_n \cup U^+_p, \\
\partial^- e := v^-_U \quad \text{if} \quad v \in U^-_n \cup U^-_p.
\]

Then, update \( G \) by

\[
V^+ := (V^+ \setminus (U^+_n \cup U^+_p)) \cup \{ u^+_U \}, \quad V^- := (V^- \setminus (U^-_n \cup U^-_p)) \cup \{ v^-_U \}, \quad E := E \setminus E[U].
\]

Finally, \( F := F \cap E \) and return \((G, F)\).

**Procedure Expansion** \((G, F)\). Let \( G := \hat{G} \). For each inclusionwise maximal \( U \in \mathcal{U} \) that is shrunk, we add \( F_U \subseteq \hat{E}[U] \) of \([\lfloor t|U| - 1 \rfloor / 2] \) edges to \( F \) from \( \hat{E}[U] \) for each \( U \in \mathcal{U} \).

In **Procedure Expansion** \((G, F)\), the existence of \( F_U \) is non-trivial. To attain that \( \hat{F} = F \cup \bigcup \{ F_U \mid U \in \mathcal{U} \} \) is a \( \mathcal{U} \)-maximal shrunk set \( \) is a \( t \)-matching in \( \hat{G} \), it should be satisfied for \( F \subseteq E \) and \( F_U \subseteq \hat{E}[U] \) that

\[
\deg_F(u) \leq \begin{cases} 
1 & (u \in V_0), \\
1 & (u \in V_p)
\end{cases} \quad (8)
\]

\[
\deg_{F_U}(u) \begin{cases} 
= t - 1 & (u \text{ is incident to an edge in } F[U, V \setminus U]), \\
\leq t & \text{(otherwise)}.
\end{cases} \quad (9)
\]

To achieve this, we maintain that \( F \) satisfies the degree constraint (8). Moreover, we assume that, for an arbitrary \( F \) with (8), there exists \( F_U \) satisfying \(|F_U| = \lfloor t|U| - 1 \rfloor / 2 \) and (9) for every maximal shrunk set \( U \in \mathcal{U} \). This assumption is formally defined as follows.

**Definition 3.2.** Let \( \hat{G} = (\hat{V}, \hat{E}) \) be a bipartite graph, \( \mathcal{U} \subseteq 2^\hat{V} \), and \( t \) be a positive integer. For pairwise disjoint \( U_1, \ldots, U_l \in \mathcal{U} \), let \( G = (V, E) \) denote the graph obtained from \( \hat{G} \) by executing **Shrink** \((U_1)\), \ldots, **Shrink** \((U_l)\), and let \( F \subseteq E \) be an arbitrary edge set satisfying (8). If there exists \( F_{U_i} \subseteq \hat{E}[U_i] \) satisfying \(|F_{U_i}| = t|U_i| / 2 - 1 \) and (9) for each \( i = 1, \ldots, l \), we say that \((\hat{G}, \mathcal{U}, t)\) admits expansion.

In what follows, we assume that \((\hat{G}, \mathcal{U}, t)\) admits expansion. This is exactly the class of \((\hat{G}, \mathcal{U}, t)\) to which our algorithm is applicable.

This assumption and the degree constraint (8) guarantee that we can always obtain a \( t \)-matching \( \hat{F} = F \cup \bigcup \{ F_U \mid U \in \mathcal{U} \} \) is a maximal shrunk set \( \) in \( G \). Furthermore, we should consider the \( \mathcal{U} \)-feasibility of \( \hat{F} \). We refer to \( F \) in \( G \) as feasible if \( \hat{F} \) is \( \mathcal{U} \)-feasible. If there are several possibilities of \( F_{U_i} \), we say that \( F \) is \( \mathcal{U} \)-feasible if there is at least one \( \mathcal{U} \)-feasible \( \hat{F} \). In other words, \( F \) satisfying (8) is not feasible if, for any possibility of \( \hat{F} \),

\[
|\hat{F}[U']| = \frac{t|U'|}{2}
\]

(10)
Figure 3: In expanding $U \in \mathcal{U}$, $F_1$ is inappropriate because it contains a $t$-factor in $\hat{G}[U']$, while $F_2$ is appropriate.

holds for some $U' \subseteq \mathcal{U}$, and $F$ will have a $t$-factor in $\hat{G}[U']$.

See Fig. 3 for an example. Here, $t = 2$, and we expand $U = \{u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4\}$ by adding $F_U$ of $[(t|U| - 1)/2] = 7$ edges satisfying (9). However, $F_1$ is not $\mathcal{U}$-feasible because it violates (1) for $U' = \{u_3, u_4, u_5, v_3, v_4, v_5, v_6\}$. On the other hand, $F_2$ satisfies (1) for $U'$. Thus, $F$ in $G$ is called feasible, and we select $F_2$ when expanding $U$.

Here, we describe our algorithm in detail. The algorithm begins with $G = \hat{G}$ and an arbitrary $\mathcal{U}$-feasible $t$-matching $F \subseteq \hat{F}$, typically $F = \emptyset$. We first construct an auxiliary digraph.

**Procedure ConstructAuxiliaryDigraph**($G, F$). Construct a digraph $(V, A)$ defined by

$$A = \{(u, v) \mid u \in V^+, v \in V^-; \{u, v\} \in E \setminus F\} \cup \{(v, u) \mid u \in V^+, v \in V^-, \{u, v\} \in F\}.$$  

Define the sets of source vertices $S \subseteq V^+$ and sink vertices $T \subseteq V^-$ by

$$S = \{u \in V_0^+ \mid \deg_F(u) \leq t - 1\} \cup \{u_U^* \in V_0^+ \mid \deg_F(u_U^*) = 0\},$$  

$$T = \{v \in V_0^- \mid \deg_F(v) \leq t - 1\} \cup \{v_U^* \in V_0^- \mid \deg_F(v_U^*) = 0\}.$$  

Then, return $D = (V, A; S, T)$.

Suppose that there exists a directed path $P = (e_1, f_1, \ldots, e_l, f_l, e_{l+1})$ in $D$ from $S$ to $T$. Note that $e_i \in E \setminus F$ ($i = 1, \ldots, l + 1$) and $f_i \in F$ ($i = 1, \ldots, l$). We denote the symmetric difference $(F \setminus P) \cup (P \setminus F)$ of $F$ and $P$ by $F \Delta P$. If $F \Delta P$ is feasible, we execute **Augment**($G, F, P$) below. We then execute **Expand**($G, F$).

**Procedure Augment**($G, F, P$). Let $F := F \Delta P$ and return $F$.

If $F \Delta P$ is not feasible, we apply **Shrink**($U$) after determining a set $U \in \mathcal{U}$ to be shrunk by the following procedure.

**Procedure FindViolatingSet**($G, F, P$). For $i = 1, \ldots, l$, define $F_i = (F \setminus \{f_i, \ldots, f_l\}) \cup \{e_1, \ldots, e_i\}$. Also define $F_0 = F$ and $F_{i+1} = F \Delta P$. Let $i^*$ be the minimum index $i$ such that $F_i$ is not feasible, and let $U \in \mathcal{U}$ satisfy (10) for $F = F_{i^*}$. Then, let $F := F_{i^*-1}$, and return $(F, U)$.

Finally, if $D$ does not have a directed path from $S$ to $T$, we determine the minimizer $X \subseteq \hat{V}$ of (11) as follows.
Algorithm 1 Maximum $\mathcal{U}$-feasible $t$-matching

1: $G \leftarrow \hat{G}$, $F \leftarrow \emptyset$, $D \leftarrow \text{AuxiliaryDigraph}(G, F)$
2: while $D$ has an $S$-$T$ path $P$ do
3: if $F \triangle P$ is feasible then
4: $F \leftarrow \text{Augment}(G, F, P)$
5: $(G, F) \leftarrow \text{Expand}(G, F)$
6: $D \leftarrow \text{AuxiliaryDigraph}(G, F)$
7: else
8: $(F, U) \leftarrow \text{ViolatingSet}(G, F, P)$
9: $(G, F) \leftarrow \text{Shrink}(U)$
10: $D \leftarrow \text{AuxiliaryDigraph}(G, F)$
11: $X \leftarrow \text{Minimizer}(G, F)$, $(G, F) \leftarrow \text{Expand}(G, F)$
12: return $(F, X)$

Procedure FindMinimizer($G, F$). Let $R \subseteq V$ be the set of vertices reachable from $S$, and let $X := (V^+ \setminus R^+) \cup R^-$. If a natural vertex $v \in V^+ \setminus X$ has $t$ edges in $F$ connecting $R^+$ and $v$, then $X := X \cup \{v\}$. If a pseudovertex $v^-_U \in V^- \setminus X$ has one edge in $F$ connecting $R^+$ and $v^-_U$, then $X := X \cup \{v^-_U\}$. Finally, return $X := \hat{X}$.

We then apply Expand($G, F$) and the algorithm terminates by returning $F \subseteq \hat{E}$ and $X \subseteq \hat{V}$.

Now the description of the algorithm is completed. The pseudocode of the algorithm is presented in Algorithm 1. The optimality of $F$ and $X$ is proved in Sect. 3.3. We exhibit how the algorithm works in specific cases such as the square-free 2-matching, even factor, triangle-free 2-matching, and arborescence problems, and discuss the complexity for these cases in Sect. 3.4. Before that, we analyze the complexity of the algorithm for the general case.

Here, let $n = |\hat{V}|$ and $m = |\hat{E}|$. The complexity of the algorithm varies according to the structure of $(G, \mathcal{U}, t)$. Recall that $\alpha$ denotes the time required to determine the feasibility of $F$, and $\beta$ denotes the time required to expand $U$. To be precise, $\alpha$ is the time required to check whether $F$ in a shrunk graph $G$ is feasible, and, if not, find $U \in \mathcal{U}$ for which $F$ satisfies (10).

Between augmentations, we execute Shrink($U$) $O(n)$ times. For one Shrink($U$), we check the feasibility of $F_1, \ldots, F_t$, which requires $O(n\alpha)$ time. We then reconstruct the auxiliary digraph. Here, we should only update the vertices and arcs on the $S$-$T$ path, which takes $O(n)$ time. After augmentation, we expand the shrunk vertex sets, which takes $O(n\beta)$ time in total. Therefore, the complexity for one augmentation is $O(n^2\alpha + n\beta)$. Since augmentation occurs at most $tn/2$ time, the total complexity of the algorithm is $O(t(n^3\alpha + n^2\beta))$.

3.3 Min-max Theorem: Strong Duality

In this section, we strengthen Lemma 3.1 to be a min-max relation and prove the validity of Algorithm 1. We show that the output $(F, X)$ of the algorithm satisfies (4) with equality. This constructively proves the min-max relation for the class of $(G, \mathcal{U}, t)$ that admits expansion.

Theorem 3.3. Let $G = (V, E)$ be a bipartite graph, $\mathcal{U} \subseteq 2^V$, and $t$ be a positive integer such that $(G, \mathcal{U}, t)$ admits expansion. Then, the maximum size of a $\mathcal{U}$-feasible $t$-matching is equal to the minimum of

$$t|X| + |E[C_{V \setminus X}]| + \sum_{U \in \mathcal{U} \setminus X} \left\lfloor \frac{|U|}{2} - 1 \right\rfloor,$$

where $X$ runs over all subsets of $V$. 

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Proof. We denote the output of Algorithm 1 by \((\hat{F}, \hat{X})\). Here, it is sufficient to prove that (5) and (6) hold by equality for \((\hat{F}, \hat{X})\). Since \(X\) is defined based on reachability in the auxiliary digraph \(D\), it is straightforward that \(\hat{F}[\hat{X}] = \emptyset\). Moreover, for every \(v \in \hat{X}\), \(\deg_F(v) = t\) holds; thus, (5) holds by equality. Finally, edges in \(\hat{G}[V \setminus \hat{X}]\) are in \(F\) before the last \(\text{Expand}(G, F)\) or are obtained by expanding pseudovertices \(u_U^-\) and \(v_U^+\), which are isolated vertices in \(G[V \setminus X]\). This means that \(U\) forms a component in \(\hat{G}[\hat{X}]\); thus, the equality in (6) holds.

3.4 Applying our Algorithm to Special Cases

Here, we demonstrate how Algorithm 1 is applied to the special cases of \(C_{\leq k}\)-free 2-matchings, even factors (including nonbipartite matchings), triangle-free 2-matchings, and arborescences. Differences appear in determining feasibility in the shrunk graph and the edges to be added by expansion.

3.4.1 \(C_{\leq k}\)-free 2-matchings in Bipartite Graphs

The case where \(k = 4\), i.e., square-free 2-matchings in a simple bipartite graph, is the most straightforward example. In this case, the family of the shrunk vertex sets never becomes nested, i.e., \(\text{Shrink}(U)\) is always applied to a cycle of length four comprising four natural vertices. Thus, an edge set \(F \subseteq E\) is feasible if and only if \(F\) excludes a cycle of length four, even if the graph is obtained by repeated shrinking. Furthermore, the feasibility of each \(F_i\) (\(i = 1, \ldots, l\)) can be checked in constant time because it is sufficient to determine whether the new edge \(e_i\) added to \(F_i\) is in a square.

When expanding \(U \in \mathcal{U}\), it suffices to choose \(F_U\) consisting of three edges in \(E[U]\) and satisfying (9) for \(t = 2\). This always yields the \(\mathcal{U}\)-feasibility of \(\hat{F}\) and can be performed in constant time for one square.

For the case where \(k \geq 6\), the problem becomes more involved. Suppose that \(\mathcal{U} = \{U \subseteq \hat{V} \mid 1 \leq |U| \leq 6\}\) and we expand \(U \in \mathcal{U}\) with \(|U| = 6\). We then select \(F_U \subseteq E[U]\) with \(|F_U| = 5\) according to (9); however, such \(F\) might not exist. Moreover, even if such \(F_U\) is found, \(F_U\) might contain a cycle of length four, which violates \(\mathcal{U}\)-feasibility.

Such difficulty is inevitable because the simple \(C_{\leq k}\)-free 2-matching problem in bipartite graphs is NP-hard when \(k \geq 6\). Thus, we require an assumption for our algorithm to work. One solution to this difficulty is to impose the connectivity of \(F_U\), i.e., when expanding \(U \in \mathcal{U}\), we require that there always exists \(F_U\) satisfying (9) and that \((U, F_U)\) is connected. If \(t = 2\), this property amounts to the Hamilton-laceability of \(G[U]\) (see [44]).

It is clear that \(K_{1,t}\)-free \(t\)-matchings in bipartite graphs also satisfy this assumption. Under this assumption, \(F \subseteq E\) satisfying (8) is feasible if and only if \(F\) does not contain a \(K_{1,t}\) of natural vertices as a subgraph.

3.4.2 Matchings and Even Factors in Nonbipartite Graphs

Since the nonbipartite matching problem is reduced to the even factor problem, it suffices to discuss only the even factor problem. Here, let \(D = (V, A)\) be an odd-cycle symmetric digraph and define \(\hat{G} = (\hat{V}, \hat{E})\) and \(\mathcal{U}\) by (2). Recall that, if \(D\) is odd-cycle symmetric, then \(\hat{G}[U]\) is a symmetric bipartite graph for each \(U \in \mathcal{U}\). In this case, our algorithm is performed recursively, i.e., a 1-matching \(F \subseteq E\) in a shrunk graph \(G\) is feasible if \(|F[U']| \leq |U'|/2 - 1\) for \(U' = U_0 \cup U_p\) with \(U \in \mathcal{U}\). This can be checked in \(O(n)\) time, and Procedure \(\text{Shrink}(U')\) is executed when a perfect matching in \(G[U']\) is found in our solution. See Fig. 4 for an illustration.

In \(\text{Expand}(G, F)\), we repeat expanding a maximal shrunk vertex set \(U\), where the proper shrunk subsets of \(U\) remain shrunk. We repeat this step until the original graph \(G\) is reconstructed. See Fig. 5 for an illustration of expanding \(U\). Without loss of generality, we can denote the perfect matching in \(G[U']\) by \(\bigcup_{i=1}^{k} \{u_i^+, u_{i+1}^-\}\),
Figure 4: The maximum even factor problem in $D_0$ is reduced to the $\mathcal{U}$-feasible $1$-matching problem in $G$. If we find arc $(u^+_7, u^-_1)$ as an $S$-$T$ path in the auxiliary digraph, we shrink $U = \{u^+_1, \ldots, u^+_7, u^-_1, \ldots, u^-_7\}$ to obtain $G$.

where $k = 2k' + 1$ is odd and $u_{k+1} = u_1$. Furthermore, assume that $u^-_1$ and $u^+_j$ are incident to an edge in $F[U', V \setminus U']$. Now, if $j = 2j' + 1$ is odd, let $F_U' = \bigcup_{i=1}^{j'-1} \{u^+_1, u^-_{i+1}\} \cup \bigcup_{i=j'+1}^{k'} \{\{u^+_2, u^-_{2i+1}\}, \{u^+_2, u^-_{2i}\}\}$. If $j = 2j'$ is even, then let $F_U' = \bigcup_{i=1}^{j'-1} \{u^+_1, u^-_{2i+1}\}, \{u^+_2, u^-_{2i+1}\}\} \cup \bigcup_{i=j}^{k'} \{u^+_1, u^-_{1}\}$. It is straightforward that $\text{EXPAND}(G, F)$ can be performed in $O(n)$ time.

Note that this procedure is possible because $k$ is odd and $G[U']$ is symmetric. We also remark that this procedure corresponds to expanding an odd cycle in an even factor algorithm [33], and expanding an odd cycle in Edmonds’ blossom algorithm [10].

### 3.4.3 Triangle-free 2-matchings

Recall the instance of the $\mathcal{U}$-feasible $t$-matching problem constructed in Sect. 2.2.3. Here, we denote a graph obtained from $\hat{G}$ by repeated shrinkings by $G$. In $G$, a pair of vertices $u \in V^+$ and $v \in V^-$ is referred to as twins if they are copies of the same original vertex or they are pseudovertices added by the same shrinking procedure.

Here, a 1-matching $F \subseteq E$ is infeasible only if it contains a matching of three edges covering three pairs of twins in the original graph $\hat{G}$. In other words, even if $F$ contains a matching of three edges covering three pairs of twins in $G$, it is feasible if the endvertices of those edges are not three pairs of twins in $\hat{G}$.

For example, recall the instance in Fig. 2. In Fig. 6, $G$ is obtained from $\hat{G}$ by shrinking $U = \{u^+_3, u^+_4, u^-_5, u^-_4, u^-_3\}$, and we have a feasible edge set $\{\{u^+_1, u^-_2\}, \{u^+_2, v^-_3\}\}$. If we find an $S$-$T$ path $P_1$ consisting of a single arc resulting from $(u^+_3, u^-_1)$, then $F \Delta P_1$ is feasible and $\text{AUGMENT}(G, F, P)$ and $\text{EXPAND}(G, F)$ follow.

In contrast, in Fig. 7 suppose that we find an $S$-$T$ path $P_2$ consisting of a single arc $(u^+_3, u^-_1)$, then $F \Delta P_2$ is not feasible and $\text{SHRINK}(W)$ follows, where $W = \{u^+_1, u^+_2, u^+_3, u^-_1, u^-_2, u^-_3\}$. Note that the family of vertex sets shrunk by our algorithm corresponds to a triangle cluster in the triangle-free 2-matching algorithm due to Cornuéjols and Pulleyblank [6].

For $\text{EXPAND}(G, F)$, we expand each shrunk $U \in \mathcal{U}$ one by one. We add two edges from $\hat{E}[U]$ to $F$ such that $F$ remains a 1-matching, which always obtains the $\mathcal{U}$-feasibility of the output $F$.

It is clear that the feasibility of $F$ can be determined in $O(n)$ time, and that of $F_i$ in $\text{FIND VIOLATING SET}(G, F, P)$ can be determined in a constant time for each $i = 1, \ldots, l$. In addition, $\text{EXPAND}(G, F)$ needs $O(n)$ time.
Figure 5: Two types of expanding $U$. The set of thick edges in $\hat{G}$ is our $\mathcal{U}$-feasible 1-matching, where the dashed edges are those added when expanding $U$. This $\mathcal{U}$-feasible 1-matching corresponds to the even factor of thick arcs in $D_0$.

Figure 6: If we find an $S$-$T$ path of a single arc resulting from $(u^+, u^-)$ (dotted edge in $G$), we execute $\text{Augment}(G, F, P)$ and $\text{Expand}(G, F)$. The set of thick edges in $\hat{G}$ is the obtained $\mathcal{U}$-feasible 1-matching, where the dashed edges are those added by $\text{Expand}(G, F)$. This $\mathcal{U}$-feasible 1-matching corresponds to a triangle-free 2-matching (indicated by the thick edges) in $\hat{G}_0$. 

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Figure 7: If we find an $S$-$T$ path of a single arc resulting from $(u_3^+, u_1^-)$ (dotted edge in $G$), we execute Shrink($W$), where $W = \{u_1^+, u_3^+, u_4^+, u_5^+, u_5^-, u_3^-, u_2^+, u_2^-\}$. The parallel edges between $u_5^+$ and $u_5^-$ result from $\{u_5^+, u_5^-\}$ and $\{u_3^+, u_1^-\}$. The shrunk triangles $\{u_1, u_2, u_3\}$ and $\{u_3, u_4, u_5\}$ in $G_0$ form a triangle cluster [6].

### 3.4.4 Matroids and Arborescences

Recall the instances constructed in Sect. 2.2.4. If Algorithm 1 is applied to finding a maximum independent set of a matroid, then an $S$-$T$ path always consists of one arc, i.e., the solution is greedily augmented. However, Procedure Shrink($U$) may occur, which is simply the contraction of a circuit for matroids.

If Algorithm 1 is applied to finding an arborescence, then an $S$-$T$ path always consists of one arc and the solution is greedily augmented. Here, Procedure Shrink($U$) corresponds to shrinking a directed cycle.

### 3.5 $C_{4k+2}$-free 2-matchings: New Example in Our Framework

So far, we have viewed that some problems in the literature are described as the $\mathcal{U}$-feasible $t$-matching problem in bipartite graphs under the assumption that $(G, \mathcal{U}, t)$ admits expansion. Here we exhibit a new problem which falls in this framework.

Let $G = (V, E)$ be a simple bipartite graph. A 2-matching $F \subseteq E$ is called $C_{4k+2}$-free if $F$ excludes cycles of length $4k + 2$ for every positive integer $k$. In other words, the length of a cycle in $F$ must be a multiple of four.

Now $C_{4k+2}$-free 2-matchings are described as $\mathcal{U}$-feasible 2-matchings, where

$$\mathcal{U} = \{U \subseteq V \mid |U^+| = |U^-| = 2k + 1 \text{ for some positive integer } k\}.$$  

A solvable class of this form of the $\mathcal{U}$-feasible 2-matching problem is obtained by recalling the instances for even factors: a class where $G[U]$ is a symmetric bipartite graph and all of the twins in $G[U]$ are connected by an edge for every $U \in \mathcal{U}$. To be precise, every $U \in \mathcal{U}$ is described as $U^+ = \{u_1, \ldots, u_{2k+1}\}$ and $U^- = \{v_1, \ldots, v_{2k+1}\}$, where $\{u_i, v_j\} \in E$ if and only if $\{u_j, v_i\} \in E$, and $\{u_i, v_i\} \in E$ for each $i = 1, \ldots, 2k + 1$. Then, it is straightforward to see that $(G, \mathcal{U}, t)$ admits expansion by following the arguments for even factors in Sects. 2.2.2 and 3.4.2.

### 4 Weighted $\mathcal{U}$-feasible $t$-matching

In this section, we extend the min-max theorem and the algorithm presented in Sect. 3 to the maximum-weight $\mathcal{U}$-feasible $t$-matching problem. Recall that $G$ is a simple bipartite graph and $(G, \mathcal{U}, t)$ admits expansion. We further assume that $w$ is vertex-induced on each $U \in \mathcal{U}$, which commonly extends the assumptions for the maximum-weight square-free and even factor problems.
4.1 Linear Program

Here, we describe a linear programming relaxation of the maximum-weight $U$-feasible $t$-matching problem in variable $x \in \mathbb{R}^E$:

\[
\begin{align*}
(P) & \quad \text{maximize} & & \sum_{e \in E} w(e)x(e) & \quad \text{(12)} \\
\text{subject to} & & x(\delta(v)) & \leq t & (v \in V), \quad \text{(13)} \\
& & x(E[U]) & \leq \left\lfloor \frac{|U| - 1}{2} \right\rfloor & (U \in U), \quad \text{(14)} \\
& & 0 & \leq x(e) & \leq 1 & (e \in E). \quad \text{(15)}
\end{align*}
\]

Note that Constraint (14), which describes $U$-feasibility, is a common extension of the blossom constraint for the nonbipartite matching problem ($t = 1$), and the subtour elimination constraints for the TSP ($t = 2$).

Its dual program in variables $p \in \mathbb{R}^V$, $q \in \mathbb{R}^E$, and $r \in \mathbb{R}^U$ is given as follows:

\[
\begin{align*}
(D) & \quad \text{minimize} & & t \sum_{v \in V} p(v) + \sum_{e \in E} q(e) + \sum_{U \in U} \left\lfloor \frac{|U| - 1}{2} \right\rfloor r(U) & \quad \text{(16)} \\
\text{subject to} & & p(u) + p(v) + q(e) & + \sum_{U \in U : e \in E[U]} r(U) \geq w(e) & (e = \{u, v\} \in E), \quad \text{(17)} \\
& & p(v) & \geq 0 & (v \in V), \quad \text{(18)} \\
& & q(e) & \geq 0 & (e \in E), \quad \text{(19)} \\
& & r(U) & \geq 0 & (U \in U). \quad \text{(20)}
\end{align*}
\]

We define $w' \in \mathbb{R}^E$ by

\[
w'(e) = p(u) + p(v) + q(e) + \sum_{U \in U : e \in E[U]} r(U) - w(e) \quad (e = \{u, v\} \in E).
\]

The complementary slackness conditions for (P) and (D) are as follows.

\[
\begin{align*}
x(e) > 0 & \implies w'(e) = 0 & (e \in E), \quad \text{(21)} \\
p(v) > 0 & \implies x(\delta(v)) = 2 & (v \in V), \quad \text{(22)} \\
q(e) > 0 & \implies x(e) = 1 & (e \in E), \quad \text{(23)} \\
r(U) > 0 & \implies x(E[U]) = \left\lfloor \frac{|U| - 1}{2} \right\rfloor & (U \in U). \quad \text{(24)}
\end{align*}
\]

4.2 Primal-Dual Algorithm

In this section, we demonstrate a combinatorial primal-dual algorithm for the maximum-weight $U$-feasible $t$-matching problem in bipartite graphs, where $(G, U, t)$ admits expansion and $w$ is vertex-induced for each $U \in U$.

We maintain primal and dual feasible solutions that satisfy (21), (23), (24), and (22) for $v \in V^-$. The algorithm terminates when (22) is obtained for every $v \in V^+$. Again, we denote the input graph by $\hat{G} = (\hat{V}, \hat{E})$, and the graph in hand, i.e., the graph resulting from possibly repeated shrinkings, by $G = (V, E)$. The variables in the algorithm are $F \subseteq E$, $p \in \mathbb{R}^V$, $q \in \mathbb{R}^E$, and $r \in \mathbb{R}^U$. Note that $p$ and $q$ are always defined on the original vertex and edge sets, respectively.
Initially, we set

\[ F = \emptyset, \quad p(v) = \begin{cases} \max \{ w(e) \mid e \in \delta(v) \} & (v \in V^+), \\ 0 & (v \in V^-), \end{cases} \]
\[ q(e) = 0 \quad (e \in E), \quad r(U) = 0 \quad (U \in \mathcal{U}). \] (25)

The auxiliary digraph \( D \) is constructed as follows. Here, the major differences from Sect. 3.2 are that we only use an edge \( e \) with \( w'(e) = 0 \), and a vertex in \( V^+ \) can become a sink vertex.

**Procedure ConstructAuxiliaryDigraph** \((G, F, p, q, r)\). Here, we define a digraph \((V, A)\) by

\[ A = \{ (\partial^+ e, \partial^- e) \mid e \in E \setminus F, \ w'(e) = 0 \} \cup \{ (\partial^- e, \partial^+ e) \mid e = \{u, v\} \in F\}. \]

The sets of source vertices \( S \subseteq V^+ \) and sink vertices \( T \subseteq V^+ \cup V^- \) are defined by

\[ S = \{ u \in V_n^+ \mid \deg_F(v) \leq t - 1, \ p(u) > 0 \}, \]
\[ \cup \{ u^+_U \in V_p^+ \mid \deg_F(u^+_U) = 0, \ p(u) > 0 \text{ for some } u \in U \} \]
\[ T = \{ v \in V_n^- \mid \deg_F(v) \leq t - 1 \} \cup \{ v^-_U \in V_p^- \mid \deg_F(v^-_U) = 0 \}
\[ \cup \{ u \in V_n^+ \mid \deg_F(u) = t, \ p(u) = 0 \}
\[ \cup \{ u^+_U \in V_p^+ \mid \deg_F(u^+_U) = 1, \ p(u) = 0 \text{ for some } u \in U \}. \]

Return \( D = (V, A; S, T) \).

Suppose that \( D \) has a directed path \( P \) from \( S \) to \( T \), and let \( F' := F \Delta P \). If \( F' \) is feasible, we execute \textsc{Augment}(\( G, F, P \)), which is the same as in Sect. 3.2. Note that, if \( P \) ends in a vertex in \( T \cap V^+ \), then \(|F'| \) does not increase. However, in this case, the number of vertices satisfying (22) increases by one, and we get closer to the termination condition (achieving (22) at every vertex). If \( F' \) is not feasible, we apply \textsc{ViolatingSet}(\( G, F, P \)) as in Sect. 3.2. For the output \( U \) of \textsc{ViolatingSet}(\( G, F, P \)), if \( p(u) = 0 \) holds for some \( u \in U^+ \), then we execute \textsc{Modify}(\( G, F, U \)) below. Otherwise, we apply \textsc{Shrink}(\( U \)) as in Sect. 3.2.

**Procedure Modify** \((G, F, U)\). Let \( u^* \in U^+ \) satisfy \( p(u^*) = 0 \). Then find \( K \subseteq E[U] \) such that

\[ \deg_K(u) = \begin{cases} t & (u \in U_n^+ \setminus \{ u^* \}), \\ t - 1 & (u = u^*), \\ 0 & (u = u^+_U, \ u^* \in U'), \\ \deg_F[U](u) & (u \in U_n^- \cup U_p^-). \end{cases} \]

Here, return \( F := (F \setminus F[U]) \cup K \).

If \( D \) does not have a directed path from \( S \) to \( T \), then update the dual variables \( p, q, \) and \( r \) by procedure \textsc{UpdateDualSolution}(\( G, F, p, q, r \)) described below.
Procedure **UpdateDualSolution**\((G, F, p, q, r)\). Let \(R \subseteq V\) be the set of vertices reachable from \(S\) in the auxiliary digraph \(D\). Then,

\[
\begin{align*}
p(v) & := \begin{cases} 
p(v) - \epsilon & (v \in \hat{R}^+), 
p(v) + \epsilon & (v \in \hat{R}^-), 
p(v) & (v \in \hat{V} \setminus \hat{R}), \end{cases} 
q(e) & := \begin{cases} 
q(e) + \epsilon & (\partial^+ e \in \hat{R}^+, \partial^- e \in \hat{V}^+ \setminus \hat{R}^+), 
q(e) & (v \in \hat{V} \setminus \hat{R}^-), 
\end{cases} 
r(U) & := \begin{cases} 
r(U) + \epsilon & (u^+ \in R^+, v^-_{\hat{U}} \in V^- \setminus R^-), 
r(U) - \epsilon & (u^+_{\hat{U}} \in V^+ \setminus R^+, v^-_{\hat{U}} \in R^-), 
r(U) & (\text{otherwise}), \end{cases}
\end{align*}
\]

where 
\[
\epsilon = \min\{\epsilon_1, \epsilon_2, \epsilon_3\}, \quad \epsilon_1 = \min\{w'((u, v)) | u \in \hat{R}^+, v \in \hat{V} \setminus \hat{R}^-\}, 
\epsilon_2 = \min\{p(u) | u \in \hat{R}^+\}, \quad \epsilon_3 = \min\{r(U) | u^+_{\hat{U}} \in \hat{V}^+ \setminus \hat{R}^+, v^-_{\hat{U}} \in \hat{R}^-\}.
\]

Then return \((p, q, r)\).

Finally, we expand every \(U\) satisfying \(r(U) = 0\) after **Augment**\((G, F, P)\), **Modify**\((G, F, U)\), and **UpdateDualSolution**\((G, F, p, q, r)\). If any \(U' \subseteq U\) satisfies \(r_{U'} > 0\), which implies that \(U'\) had been shrunk before \(U\) was shrunk, then \(U'\) is maintained as shrunk.

Procedure **Expand**\((G, F, r)\). For each shrunk \(U \in \mathcal{U}\) with \(r(U) = 0\), execute the following procedures. Update \(G\) by replacing \(u^+_{\hat{U}}\) and \(v^-_{\hat{U}}\) by the graph induced by \(U_n \cup U_p\) just before **Shrink**\((U)\) is applied. Determine \(F_U \subseteq E[U_n \cup U_p]\) of \([|U_n| + |U_p| - 1]/2\) - 1 edges such that \(F' = F \cup F_U\) can be extended to a \(\mathcal{U}\)-feasible t-matching in \(\hat{G}\). Then return \(F := F'\).

The pseudocode of the maximum-weight \(\mathcal{U}\)-feasible t-matching algorithm is presented in Algorithm 2. For complexity, it follows that one **DualUpdate**\((G, F, p, q, r)\) requires \(O(m)\) time, and it is executed \(O(n^2)\) times. This is the difference from the unweighted version; thus, the total complexity is \(O(n^3(m + \alpha) + n^2\beta))\).

It is clear that the optimal dual solution \((p, q, r)\) found by Algorithm 2 is integer if the edge weight \(w\) is integer. Thus, Algorithm 2 constructively proves the following theorem for the integrality of (P) and (D). This is a common extension of dual integrality theorems for nonbipartite matchings [9], even factors [22], triangle-free 2-matchings [6], square-free 2-matchings [30], and branchings [11].

**Theorem 4.1.** If \((G, \mathcal{U}, t)\) admits expansion and \(w\) is vertex-induced on each \(U \in \mathcal{U}\), then the linear program (P) has an integer optimal solution. Moreover, the linear program (D) also an integer optimal solution such that the number of sets \(U \in \mathcal{U}\) with \(r(U) > 0\) is at most \(n/2\).

5 Conclusion

We have presented a new framework for the optimal \(\mathcal{U}\)-feasible t-matching problem and established a min-max theorem and combinatorial algorithm under the reasonable assumption that \(G\) is bipartite, \((G, \mathcal{U}, t)\) admits expansion, and \(w\) is vertex-induced on each \(U \in \mathcal{U}\). Under this assumption, our problem can describe a number of generalization of the matching problem, such as the matching and triangle-free 2-matching problems in nonbipartite graphs, the square-free 2-matching problem in bipartite graphs, and
Algorithm 2 Maximum-weight \( \mathcal{U} \)-feasible \( t \)-matching

1: Set \( F, p, q, r \) by (25)
2: while Condition (22) is violated do
3: \( D \leftarrow \text{AuxiliaryDigraph}(G, x, p, q, r) \)
4: if \( D \) has an \( S-T \) path \( P \) then
5: if \( F \triangle P \) is \( \mathcal{U} \)-feasible then
6: \( F \leftarrow \text{Augment}(G, F, P) \)
7: \( (G, F) \leftarrow \text{Expand}(G, F, r) \)
8: else
9: \( (F, U) \leftarrow \text{ViolatingSet}(G, F, P) \)
10: if \( p(u) = 0 \) for some \( u \in U \) then
11: \( F \leftarrow \text{Modify}(G, F, U) \)
12: \( (G, F) \leftarrow \text{Expand}(G, F, r) \)
13: else
14: \( (G, F) \leftarrow \text{Shrink}(U) \)
15: else
16: \( (p, q, r) \leftarrow \text{DualUpdate}(G, F, p, q, r) \)
17: \( (G, F) \leftarrow \text{Expand}(G, F, r) \)
18: \( (G, F) \leftarrow \text{Expand}(G, F) \)
19: return \( (F, p, q, r) \)

matroids and arborescences. We have also obtained a new class of the restricted 2-matching problem, the \( C_{4k+2} \)-free 2-matching problem, which can be solved efficiently under a corresponding assumption.

It is noteworthy that the \( \mathcal{U} \)-feasibility is a common generalization of the blossom constraints for the nonbipartite matching problem and the subtour elimination constraints for the TSP. We expect that this unified perspective will provide a new approach to the TSP utilizing matching and matroid theories.

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