A Note on Type-Two Degenerate Poly-Changhee Polynomials of the Second Kind

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Abstract: In this paper, we first define type-two degenerate poly-Changhee polynomials of the second kind by using modified degenerate polyexponential functions. We derive new identities and relations between type-two degenerate poly-Changhee polynomials of the second kind. Finally, we derive type-two degenerate unipoly-Changhee polynomials of the second kind and discuss some of their identities.

Keywords: modified degenerate polyexponential function; degenerate Changhee polynomials of the second kind; type-two degenerate poly-Changhee polynomials of the second kind; unipoly functions

1. Introduction

As is well known, Changhee polynomials $Ch_n(x)$ are defined by means of the following generating function

$$
\int_{\mathbb{Z}_p} (1+t)^x d_{\mu-1}y = \frac{2}{2+t}(1+t)^x = \sum_{n=0}^{\infty} Ch_n(x) \frac{t^n}{n!}
$$

(1)

(see [1,2]).

In the case when $x = 0$, $Ch_n(0) = Ch_n$ are called Changhee numbers. The Euler polynomials are defined by the following generating function:

$$
\int_{\mathbb{Z}_p} e^{(x+y)t} d_{\mu-1}y = \frac{2}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}
$$

(2)

(see [3]).

When $x = 0$, $E_n(0) = E_n$ are called the Euler numbers. From (1) and (2), we note that

$$
Ch_n(x) = \sum_{l=0}^{n} E_l(x) S_1(n,l),
$$

and

$$
E_n(x) = \sum_{l=0}^{n} Ch_n(x) S_2(n,l), (n \geq 0)
$$

(3)

(see [1]).

For any non-zero $\lambda \in \mathbb{R}$, the degenerate exponential functions are defined by

$$
e^{\lambda t}_\mu(t) = (1+\lambda t)^\frac{1}{\mu}, e^{\lambda t}(t) = e^{\lambda t}_1(t) = (1+\lambda t)^{\frac{1}{\lambda}}.
$$

(4)
Here, we note that
\[ e_\lambda^x(t) = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}, \]
where \((x)_{0,\lambda} = 1, (x)_{n,\lambda} = x(x - \lambda)(x - 2\lambda) \cdots (x - (n - 1)\lambda), (n \geq 1).\

In [7,8], Carlitz considered degenerate Bernoulli polynomials, which are given by
\[ (1 + \lambda t)^\frac{1}{\lambda} - 1 = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{t^n}{n!}. \]

On setting \(x = 0\), \(\beta_{n,\lambda}(0) = \beta_{n,\lambda}\) are called degenerate Bernoulli numbers.

For \(k \in \mathbb{Z}\), the modified degenerate polyexponential function [9] was defined by Kim and Kim to be
\[ Ei_{k,\lambda}(x) = \sum_{n=1}^{\infty} \frac{1}{(n-1)! n^k} (1 \leq |x| < 1). \]

Note that
\[ Ei_{1,\lambda}(x) = \sum_{n=1}^{\infty} \frac{1}{n} (1)_{n,\lambda} x^n = e_\lambda(x) - 1. \]

In [9], Kim et al. introduced degenerate poly-Genocchi polynomials, which are given by
\[ \frac{Ei_{k,\lambda}(\log(1 + \lambda t))}{e_\lambda(t) + 1} \left( \frac{x}{1 + \lambda t} \right)^\frac{k}{\lambda} = \sum_{n=0}^{\infty} c^{(k)}_{n,\lambda}(x) \frac{t^n}{n!} \quad (k \in \mathbb{Z}). \]

In the case when \(x = 0\), \(c^{(k)}_{n,\lambda}(0) = c^{(k)}_{n,\lambda}\) are called degenerate poly-Genocchi numbers.

Let \(\lambda \in \mathbb{C}_p\) with \(|\lambda| \leq 1\). The degenerate Changhee polynomials of the second kind \(Ch_{n,\lambda}(x)\) are defined by
\[ \int_{\mathbb{C}_p} (1 + \lambda \log(1 + \lambda t)) \frac{k}{\lambda} \, d_{p-1}y = \frac{2}{1 + (1 + \lambda \log(1 + t))^\frac{k}{\lambda}} (1 + \lambda \log(1 + \lambda t))^\frac{x}{\lambda} = \sum_{n=0}^{\infty} Ch_{n,\lambda}(x) \frac{t^n}{n!} \]
(see [10]).

When \(x = 0\), \(Ch_{n,\lambda}(0) = Ch_{n,\lambda}\) are called the degenerate Changhee numbers of the second kind.

In [11], the degenerate Daehee polynomials \(D_{n,\lambda}(x)\) are defined by
\[ \frac{\log_{\lambda}(1 + t)}{t} (1 + t)^x = \sum_{n=0}^{\infty} D_{n,\lambda}(x) \frac{t^n}{n!} \quad (\lambda \in \mathbb{R}). \]

For \(x = 0\), \(D_{n,\lambda}(0) = D_{n,\lambda}\) are called degenerate Daehee numbers.

Note that \(\lim_{\lambda \to 0} D_{n,\lambda}(x) = D_{n}(x), (n \geq 0)\) (see [12]).

The degenerate Stirling numbers of the first kind are defined by
\[ \frac{1}{k!} (\log_{\lambda}(1 + t))^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k) \frac{t^n}{n!}, \quad (k \geq 0) \]
(see [6,13–19]).

Note here that \(\lim_{\lambda \to 0} S_{1,\lambda}(n,k) = S_{1}(n,k), \) where \(S_{1}(n,k)\) are the Stirling numbers of the first kind given by
\[ \frac{1}{k!} (\log(1 + t))^k = \sum_{n=k}^{\infty} S_{1}(n,k) \frac{t^n}{n!}, \quad (k \geq 0) \]
The degenerate Stirling numbers of the second kind are given by
\[
\frac{1}{k!} (e^t - 1)^k = \sum_{n=l}^{\infty} S_2,\lambda(n, l) t^n \frac{n!}{n!}
\] (14)
(see [21]).

We note here that \(\lim_{\lambda \to 0} S_2,\lambda(n, k) = S_2(n, k)\), where \(S_2(n, k)\) are the Stirling numbers of the second kind given by
\[
\frac{1}{k!} (e^t - 1)^k = \sum_{n=l}^{\infty} S_2(n, l) t^n \frac{n!}{n!}
\] (15)
(see [1–30]).

In this article, we introduce type-two degenerate poly-Changhee polynomials of the second kind and derive explicit expressions and some identities of those polynomials. In addition, we introduce type-two degenerate unipoly-Changhee polynomials of the second kind and derive explicit multifarious properties.

2. Type-Two Degenerate Poly-Changhee Polynomials of the Second Kind

In this section, we define degenerate Changhee polynomials of the second kind by using the modified degenerate polynexpential function; these are called type-two degenerate poly-Changhee numbers and polynomials of the second kind in the following.

Let \(\lambda \in \mathbb{C}\) and \(k \in \mathbb{Z}\); we consider that the type-two degenerate poly-Changhee polynomials of the second kind are defined by
\[
\frac{2 \text{Ei}_k,\lambda(\log_\lambda(1+t))}{t(1 + (1 + \lambda \log(1+t))^{\frac{\lambda}{\pi}})}(1 + \lambda \log(1+t))^{\frac{\lambda}{\pi}} = \sum_{n=0}^{\infty} Ch^{(k)}_{n,\lambda}(x) t^n \frac{n!}{n!},
\] (16)

In the special case, when \(x = 0\), \(Ch^{(k)}_{n,\lambda}(0) = Ch^{(k)}_{n,\lambda}\) are called type-two degenerate poly-Changhee numbers of the second kind, where \(\log_\lambda(t) = \frac{1}{\pi}(t^\lambda - 1)\) is the compositional inverse of \(e_\lambda(t)\) that satisfies
\[
\log_\lambda(e_\lambda(t)) = e_\lambda(\log_\lambda(t)) = t.
\]

For \(k = 1\) in (16), we get
\[
\frac{2}{1 + (1 + \lambda \log(1+t))^{\frac{\lambda}{\pi}}}(1 + \lambda \log(1+t))^{\frac{\lambda}{\pi}} = \sum_{n=0}^{\infty} Ch^{(k)}_{n,\lambda}(x) t^n \frac{n!}{n!},
\] (17)

where \(Ch^{(k)}_{n,\lambda}(x)\) are called degenerate Changhee polynomials of the second kind (see Equation (10)).

Obviously,
\[
\lim_{\lambda \to 0} \left( \frac{2 \text{Ei}_k(\log_\lambda(1+t))}{t(1 + (1 + \lambda \log(1+t))^{\frac{\lambda}{\pi}})}(1 + \lambda \log(1+t))^{\frac{\lambda}{\pi}} \right) = \sum_{n=0}^{\infty} \lim_{\lambda \to 0} Ch^{(k)}_{n,\lambda}(x) t^n \frac{n!}{n!},
\]
\[
= \frac{2 \text{Ei}_k(\log(1+t))}{t(2 + t)}(1 + t)^x = \sum_{n=0}^{\infty} Ch^{(k)}_{n}(x) t^n \frac{n!}{n!},
\] (18)

where \(Ch^{(k)}_{n}(x)\) are called type-two poly-Changhee polynomials.
By using Equations (7), (10), and (16), we observe that

\[
\sum_{n=0}^{\infty} C_{n,\lambda}^{(k)} \frac{t^n}{n!} = \frac{2E_{k,\lambda}(\log_\lambda (1 + t))}{t(1 + (1 + \lambda \log(1 + t))^{\frac{1}{\lambda}})}
\]

\[
= \frac{2}{1 + (1 + \lambda \log(1 + t))^{\frac{1}{\lambda}}} \sum_{m=0}^{\infty} \frac{1}{(m + 1)^{k-1}} \sum_{l=m+1}^{\infty} S_{l,\lambda}(l, m + 1) \frac{t^l}{l!}
\]

\[
= \left( \sum_{k=0}^{\infty} \frac{C_{n,\lambda}^{(n)}}{n!} \right) \left( \sum_{l=0}^{\infty} \frac{l^k}{l!} \frac{S_{l,\lambda}(l, m + 1)}{l + 1} \frac{t^l}{l!} \right)
\]

\[
L.H.S = \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} C_{n-l,\lambda}^{(l)} \frac{S_{l,\lambda}(l, m + 1)}{l + 1} \frac{t^l}{l!} \right).
\]

Therefore, using (19), we obtain the following theorem.

**Theorem 1.** For \( n \geq 0 \) and \( k \in \mathbb{Z} \), we have

\[
C_{n,\lambda}^{(k)} = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} C_{n-l,\lambda}^{(l)} \frac{S_{l,\lambda}(l, m + 1)}{l + 1} \frac{t^l}{l!}.
\]

**Corollary 1.** For \( n \geq 0 \) and \( k \in \mathbb{Z} \), we have

\[
C_{n,\lambda}^{(1)} = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} C_{n-l,\lambda}^{(l)} \frac{S_{l,\lambda}(l, m + 1)}{l + 1}.
\]

From (16), we observe that

\[
\sum_{n=0}^{\infty} C_{n,\lambda}^{(k)}(x) \frac{t^n}{n!} = \frac{2E_{k,\lambda}(\log_\lambda (1 + t))}{t(1 + (1 + \lambda \log(1 + t))^{\frac{1}{\lambda}})}(1 + \lambda \log(1 + t))^{\frac{1}{\lambda}}
\]

\[
= \left( \sum_{n=0}^{\infty} \frac{C_{n,\lambda}^{(n)}(x)}{n!} \right) \left( \sum_{m=0}^{\infty} \frac{(x)}{m!} (\log(1 + t))^m \right)
\]

\[
= \left( \sum_{n=0}^{\infty} \frac{C_{n,\lambda}^{(n)}(x)}{n!} \right) \left( \sum_{m=0}^{\infty} \frac{S_{l,\lambda}(l, m + 1)}{l + 1} \frac{t^l}{l!} \right)
\]

\[
\sum_{n=0}^{\infty} \frac{C_{n,\lambda}^{(k)}(x)}{n!} = \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} C_{n-l,\lambda}^{(k)}(x) \frac{S_{l,\lambda}(l, m + 1)}{l + 1} \right) \frac{t^n}{n!}.
\]

By comparing the coefficients on both sides of (22), we obtain the following theorem.
Theorem 2. Let $n \geq 0$ and $k \in \mathbb{Z}$. Then, we have

$$Ch_{n,\lambda}^{(k)}(x) = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} Ch_{n-l,\lambda}^{(k)}(x) m_{\lambda} S_{k}(l, m).$$

(23)

In [4], the degenerate Bernoulli polynomials of the second kind are defined by

$$\frac{t}{\log_{\lambda}(1 + t)} (1 + t)^x = \sum_{n=0}^{\infty} b_{n,\lambda}(x) \frac{t^n}{n!}$$

(24)

(see [30]).

For $x = 0$, $b_{n,\lambda}(0) = b_{n,\lambda}$ are called degenerate Bernoulli numbers of the second kind. From (7), we note that

$$\frac{d}{dx} E_{k,\lambda}((\log_{\lambda}(1 + x))) = \frac{d}{dx} \sum_{n=1}^{\infty} \frac{(\log_{\lambda}(1 + x))^n}{(n-1)! n^{k-1}} = \frac{(1 + x)^{\lambda-1}}{\log_{\lambda}(1 + x)} E_{k-1,\lambda}((\log_{\lambda}(1 + x))).$$

(25)

Thus, from (16) and (25), we have

$$\sum_{n=0}^{\infty} Ch_{n,\lambda}^{(k)} \frac{x^n}{n!} = \frac{2}{x(1 + (1 + \lambda \log(1 + x))^{\frac{1}{\lambda}})} \frac{E_{k,\lambda}(\log_{\lambda}(1 + x))}{\lambda}$$

$$= \frac{2}{x(1 + (1 + \lambda \log(1 + x))^{\frac{1}{\lambda}})} \int_{0}^{x} \frac{(1 + t)^{\lambda-1}}{\log_{\lambda}(1 + t)} \int_{0}^{t} \frac{(1 + t)^{\lambda-1}}{\log_{\lambda}(1 + t)} dt \cdots dt$$

$$= \frac{2}{1 + (1 + \lambda \log(1 + x))^{\frac{1}{\lambda}}} \sum_{m=0}^{\infty} \frac{b_{m,\lambda}(\lambda - 1) b_{m+1,\lambda}(\lambda - 1)}{m_{1} + 1} \cdots \frac{b_{m_{k-1},\lambda}(\lambda - 1) x^{m}}{m_{1} + \cdots + m_{k-1} + 1} \cdot \frac{Ch_{n-m,\lambda} x^{n}}{n!}.$$  

(26)

Therefore, using (26), we obtain the following theorem.

Theorem 3. For $n \geq 0$, we have

$$Ch_{n,\lambda}^{(k)} = \frac{1}{2} \sum_{m=0}^{n} \binom{n}{m} \sum_{m_{1} + \cdots + m_{k-1} = m} \frac{m}{m_{1} + \cdots + m_{k-1} + 1} \cdot Ch_{n-m,\lambda}.$$  

(27)

Corollary 2. For $n \geq 0$, we have

$$Ch_{n,\lambda}^{(2)} = \frac{1}{2} \sum_{m=0}^{n} \binom{n}{m} \frac{b_{m,\lambda}(\lambda - 1)}{m + 1} \cdot Ch_{n-m,\lambda}.$$  


Let \( k \geq 1 \) be an integer. For \( s \in \mathbb{C} \), we define the function \( \eta_{k,\lambda}(s) \) as

\[
\eta_{k,\lambda}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{t(1 + (1 + \lambda \log(1 + t))^{1/\lambda})} 2E_{k,\lambda}(\log(1 + t)) \, dt
\]

\[
= \frac{1}{\Gamma(s)} \int_1^\infty \frac{t^{s-1}}{t(1 + (1 + \lambda \log(1 + t))^{1/\lambda})} 2E_{k,\lambda}(\log(1 + t)) \, dt
\]

\[
+ \frac{1}{\Gamma(s)} \int_1^\infty \frac{t^{s-1}}{t(1 + (1 + \lambda \log(1 + t))^{1/\lambda})} 2E_{k,\lambda}(\log(1 + t)) \, dt. \tag{28}
\]

The second integral converges absolutely for any \( s \in \mathbb{C} \), and hence, the second term on the right-hand side vanishes at non-positive integers. That is,

\[
\lim_{s \to -m} \left| \frac{1}{\Gamma(s)} \int_1^\infty \frac{t^{s-1}}{t(1 + (1 + \lambda \log(1 + t))^{1/\lambda})} 2E_{k,\lambda}(\log(1 + t)) \, dt \right| \leq \frac{1}{\Gamma(-m)} M = 0. \tag{29}
\]

On the other hand, for \( \Re(s) > 0 \), the first integral in (29) can be written as

\[
\frac{1}{\Gamma(s)} \sum_{l=0}^\infty \frac{Ch_{k,\lambda}(l)}{l!} \frac{1}{s + l'}
\]

which defines an entire function of \( s \). Thus, we may conclude that \( \eta_{k,\lambda}(s) \) can be continued to an entire function of \( s \).

Further, from (28) and (29), we obtain

\[
\eta_{k,\lambda}(-m) = \lim_{s \to -m} \frac{1}{\Gamma(s)} \int_0^1 \frac{t^{s-1}}{t(1 + (1 + \lambda \log(1 + t))^{1/\lambda})} 2E_{k,\lambda}(\log(1 + t)) \, dt
\]

\[
= \lim_{s \to -m} \frac{1}{\Gamma(s)} \int_0^1 \frac{t^{s-1}}{t} \sum_{l=0}^\infty \frac{Ch_{k,\lambda}(l)}{l!} \, dt = \lim_{s \to -m} \frac{1}{\Gamma(s)} \sum_{l=0}^\infty \frac{Ch_{k,\lambda}(l)}{s + l!}
\]

\[
= \cdots + 0 + \cdots + 0 + \lim_{s \to -m} \frac{1}{\Gamma(s)} \frac{Ch_{k,\lambda}(s)}{s + m} + 0 + 0 + \cdots \tag{30}
\]

\[
= \lim_{s \to -m} \left( \frac{\Gamma(1-s) \sin \pi s}{\pi} \right) \frac{Ch_{k,\lambda}(s)}{s + m} = \Gamma(1 + m) \cos(\pi m) \frac{Ch_{k,\lambda}(s)}{m!} \]

\[
= (-1)^m Ch_{m,\lambda}^{(k)}.
\]

Therefore, using (30), we obtain the following theorem.

**Theorem 4.** Let \( k \geq 1 \) and \( m \in \mathbb{N} \cup \{0\} \), \( s \in \mathbb{C} \); we have

\[
\eta_{k,\lambda}(-m) = (-1)^m Ch_{m,\lambda}^{(k)}.
\]
Theorem 5. Let $k \geq 1$ and $m \in \mathbb{N} \cup \{0\}, s \in \mathbb{C}$; we have

$$\sum_{n=1}^{\infty} \frac{(1)_{m,\lambda} S_{1,\lambda}(n, m)}{m^{k-1}} = \frac{n}{2} \left( C_{n,\lambda}^{(k)} + \sum_{l=0}^{n-1} \sum_{m=0}^{l} \binom{n}{l} (1)_{m,\lambda} S_{1,\lambda}(l, m) C_{n-1-l,\lambda}^{(k)} \right).$$

For $k = 1$ in Theorem 5, we get the following corollary.

Corollary 3. For $m \in \mathbb{N} \cup \{0\}, s \in \mathbb{C}$, we have

$$\sum_{m=1}^{\infty} (1)_{m,\lambda} S_{1,\lambda}(n, m) = \frac{n}{2} \left( C_{n-1,\lambda} + \sum_{l=0}^{n-1} \sum_{m=0}^{l} \binom{n}{l} (1)_{m,\lambda} S_{1,\lambda}(l, m) C_{n-1-l,\lambda} \right).$$

From (16), we note that

$$2E_{k,\lambda}(\log(1 + t)) = t(1 + (1 + \lambda \log(1 + t))^\lambda) \sum_{n=0}^{\infty} C_{n,\lambda}^{(k)} \frac{t^n}{n!}$$

From (16), we note that

$$2E_{k,\lambda}(\log(1 + t)) = t(1 + (1 + \lambda \log(1 + t))^\lambda) \sum_{n=0}^{\infty} C_{n,\lambda}^{(k)} \frac{t^n}{n!}$$

$$= t \left( \sum_{n=0}^{\infty} C_{n,\lambda}^{(k)} \frac{t^n}{n!} \right) \left( 1 + \sum_{m=0}^{\infty} \binom{1}{m} \sum_{l=m}^{\infty} S_{1,\lambda}(l, m) \frac{t^l}{l!} \right)$$

$$= t \left( \sum_{n=0}^{\infty} C_{n,\lambda}^{(k)} \frac{t^n}{n!} \right) \left( 1 + \sum_{l=0}^{\infty} \left( \sum_{m=0}^{l} (1)_{m,\lambda} S_{1,\lambda}(l, m) \right) \frac{t^l}{l!} \right)$$

$$= t \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \binom{n}{l} \left( \sum_{m=0}^{l} (1)_{m,\lambda} S_{1,\lambda}(l, m) C_{n-1-l,\lambda}^{(k)} \right) \frac{t^l}{l!} \right)$$

$$= \sum_{n=1}^{\infty} n \left( \sum_{l=0}^{n} \binom{n}{l} \left( \sum_{m=0}^{l} (1)_{m,\lambda} S_{1,\lambda}(l, m) C_{n-1-l,\lambda}^{(k)} \right) \frac{t^l}{l!} \right).$$

On the other hand,

$$2E_{k,\lambda}(\log(1 + t)) = 2 \sum_{m=1}^{\infty} \frac{(1)_{m,\lambda} (\log(1 + t))_m}{(m-1)! m^k}$$

$$= 2 \sum_{m=1}^{\infty} \frac{(1)_{m,\lambda} (\log(1 + t))_m}{(m-1)! m^k} \frac{m!}{m!}$$

$$= 2 \sum_{m=1}^{\infty} \frac{(1)_{m,\lambda} (\log(1 + t))_m}{m^{k-1}} \frac{\sum_{n=m}^{\infty} n \lambda C_{n,\lambda}^{(k)} \frac{t^n}{n!}}{n!}$$

$$L.H.S = 2 \sum_{n=1}^{\infty} \left( \sum_{m=1}^{n} \frac{(1)_{m,\lambda} S_{1,\lambda}(n, m)}{m^{k-1}} \frac{t^n}{n!} \right).$$

Therefore, using (31) and (32), we obtain the following theorem.
where

\[ \text{Theorem 6.} \]

\[ \text{3. Type-Two Degenerate Unipoly-Changhee Polynomials of the Second Kind} \]

The unipoly function \( u_k(x|p) \) is defined by Kim and Kim to be (see [20]):

\[ u_k(x|p) = \sum_{n=1}^{\infty} \frac{p(n)}{n^k} x^n, \quad (k \in \mathbb{Z}), \]

(34)

where \( p \) is any arithmetic function that is a real or complex valued function defined on the set of positive integers \( \mathbb{N} \).

Moreover,

\[ u_k(x|1) = \sum_{n=1}^{\infty} \frac{x^n}{n^k} = \text{Li}_k(x) \]

(35)

(see [22,23,28]) is the ordinary polylogarithm function.

In this paper, we consider the degenerate unipoly function attached to polynomials \( p(x) \) as follows:

\[ u_{k,\lambda}(x|p) = \sum_{i=1}^{\infty} p(i) \frac{(1)_{i,\lambda} x^i}{i^k}. \]

(36)

It is worth noting that

\[ u_{k,\lambda}\left(x \frac{1}{\Gamma}\right) = \text{Ei}_{k,\lambda}(x) \]

(37)

is the modified degenerate polyexponential function.

By using (36), we define type-two degenerate unipoly-Changhee polynomials of the second kind by

\[ \frac{2u_{k,\lambda}(\log_{\lambda}(1+t)|p)}{t(1 + (1 + \lambda \log(1+t))^\frac{1}{\lambda})} = \left(1 + \lambda \log(1+t)\right)\frac{1}{\lambda} = \sum_{n=0}^{\infty} C^{(k)}_{n,\lambda,p}(x) \frac{t^n}{n!}. \]

(38)

In the case when \( x = 0, C^{(k)}_{n,\lambda,p}(0) = C^{(k)}_{n,\lambda,p} \) are called type-two degenerate unipoly-Changhee numbers of the second kind. Let us take \( p(n) = \frac{1}{\Gamma} \). Then, we have

\[ \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \sum_{m=0}^{l} \frac{n!}{m!} \right) (1)_{m,\lambda} S_1(l, m) C^{(k)}_{n-l,\lambda}(x) \frac{t^n}{n!}. \]

(33)
\[\sum_{n=0}^{\infty} Ch_{n,\lambda,p}^{(k)}(x) \frac{t^n}{n!} = \frac{2u_{k,\lambda}(\log(1+t)|\frac{1}{2}p)}{t(1+(1+\lambda \log(1+t))^\frac{1}{2})} (1+\lambda \log(1+t))^\frac{1}{2}\]
\[= \frac{2}{t(1+(1+\lambda \log(1+t))^\frac{1}{2})} \sum_{m=1}^{\infty} \frac{(\log(1+t))^m}{m^k(m+1)!} (1+\lambda \log(1+t))^\frac{1}{2}\]
\[= \frac{2E \log(1+t)}{t(1+(1+\lambda \log(1+t))^\frac{1}{2})} (1+\lambda \log(1+t))^\frac{1}{2}\]
\[= \sum_{n=0}^{\infty} Ch_{n,\lambda}^{(k)}(x) \frac{t^n}{n!}. \quad (39)\]

Thus, using (39), we have the following theorem.

**Theorem 7.** Let \( n \geq 0 \) and \( k \in \mathbb{Z} \), and let \( \Gamma \) be a Gamma function. Then, we have

\[Ch_{n,\lambda}^{(k)}(x) = Ch_{n,\lambda}(x). \quad (40)\]

From (38), we get

\[\sum_{n=0}^{\infty} Ch_{n,\lambda,p}^{(k)} \frac{t^n}{n!} = \frac{2u_{k,\lambda}(\log(1+t)|\frac{1}{2}p)}{t(1+(1+\lambda \log(1+t))^\frac{1}{2})} \]
\[= \frac{2}{t(1+(1+\lambda \log(1+t))^\frac{1}{2})} \sum_{m=1}^{\infty} \frac{p(m)(1)m_{m,\lambda}(\log(1+t))^m}{m^k(m+1)!} \sum_{l=m+1}^{\infty} S_{1,\lambda}(m + 1, l) \frac{t^l}{l!}\]
\[= \left( \sum_{l=0}^{\infty} Ch_{l,\lambda} \frac{t^l}{l!} \right) \left( \sum_{m=0}^{\infty} \sum_{l=0}^{m} \frac{p(m+1)(1)m_{m+1,\lambda}(m+1)!S_{1,\lambda}(m + 1, l + 1)}{(m+1)^k(l+1)} \frac{t^l}{l!} \right)\]
\[= \sum_{l=0}^{\infty} \left( \sum_{m=0}^{l} \frac{n}{l} \right) \frac{p(m+1)(1)m_{m+1,\lambda}(m+1)!S_{1,\lambda}(m + 1, l + 1)Ch_{n-l,\lambda}}{(m+1)^k(l+1)} \frac{t^l}{l!} \]
\[= \sum_{l=0}^{\infty} \left( \sum_{m=0}^{l} \frac{n}{l} \right) \frac{p(m+1)(1)m_{m+1,\lambda}(m+1)!S_{1,\lambda}(m + 1, l + 1)Ch_{n-l,\lambda}}{(m+1)^k(l+1)} \frac{t^l}{l!}. \quad (41)\]

Therefore, by comparing the coefficients on both sides of (41), we obtain the following theorem.

**Theorem 8.** Let \( n \in \mathbb{N} \) and \( k \in \mathbb{Z} \). Then, we have

\[Ch_{n,\lambda,p}^{(k)} = \sum_{l=0}^{\infty} \sum_{m=0}^{l} \left( \frac{n}{l} \right) \frac{p(m+1)(1)m_{m+1,\lambda}(m+1)!S_{1,\lambda}(m + 1, l + 1)Ch_{n-l,\lambda}}{(m+1)^k(l+1)}. \quad (42)\]

In particular,

\[Ch_{n,\lambda}^{(k)} = Ch_{n,\lambda} = \sum_{l=0}^{\infty} \sum_{m=0}^{l} \left( \frac{n}{l} \right) \frac{S_{1,\lambda}(m + 1, l + 1)Ch_{n-l,\lambda}}{(m+1)^k(l+1)}. \quad (43)\]

From (38), we observe that
\[
\sum_{n=0}^{\infty} C_{n,\lambda}^{(k,p)}(x) \frac{t^n}{n!} = \frac{2\mu_{k,\lambda}(\log_2(1+t)|p)}{t(1 + (1 + \lambda \log(1+t))^{\frac{1}{\gamma}})} (1 + \lambda \log(1+t))^{\frac{1}{\gamma}}
\]
\[
= \frac{2\mu_{k,\lambda}(\log_2(1+t)|p)}{t(1 + (1 + \lambda \log(1+t))^{\frac{1}{\gamma}})} \sum_{m=0}^{\infty} \left( \frac{t}{\lambda} \right)^m \lambda^m (\log(1+\lambda t))^m
\]
\[
= \left( \sum_{n=0}^{\infty} C_{n,\lambda}^{(k)} \frac{t^n}{n!} \right) \left( \sum_{m=0}^{\infty} (x)_{m,\lambda} \sum_{l=m}^{\infty} S_1(l,m) \frac{t^l}{l!} \right)
\]
\[
= \left( \sum_{n=0}^{\infty} C_{n,\lambda}^{(k)} \frac{t^n}{n!} \right) \left( \sum_{l=0}^{\infty} \sum_{m=0}^{l} (n)_{l,m} S_1(l,m) \frac{t^l}{l!} \right)
\]
\[
L.H.S = \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \sum_{m=0}^{l} \left( \frac{n}{l} \right) C_{n-l,\lambda}^{(k)}(x)_{m,\lambda} S_1(l,m) \right) \frac{t^n}{n!}.
\]

From (44), we obtain the following theorem.

Theorem 9. Let \( n \geq 0 \) and \( k \in \mathbb{Z} \). Then, we have

\[
C_{n,\lambda}^{(k)}(x) = \sum_{l=0}^{n} \sum_{m=0}^{l} \left( \frac{n}{l} \right) C_{n-l,\lambda}^{(k)}(x)_{m,\lambda} S_1(l,m).
\]

From (38), we observe that

\[
\sum_{n=0}^{\infty} C_{n,\lambda}^{(k)} \frac{t^n}{n!} = \frac{2\mu_{k,\lambda}(\log_2(1+t)|p)}{t(1 + (1 + \lambda \log(1+t))^{\frac{1}{\gamma}})}
\]
\[
= \frac{2\mu_{k,\lambda}(\log_2(1+t)|p)}{t(1 + (1 + \lambda \log(1+t))^{\frac{1}{\gamma}})} \sum_{m=0}^{\infty} \frac{(m+1)(m+1,\lambda m!)}{(m+1)^k m!} \lambda^m (\log_2(1+t))^{m+1}
\]
\[
= \frac{2\log_2(1+t)}{t(1 + (1 + \lambda \log(1+t))^{\frac{1}{\gamma}})} \sum_{m=0}^{\infty} \frac{(m+1)(m+1,\lambda m!)}{(m+1)^k m!} \lambda^m (\log_2(1+t))^{m}
\]
\[
= \frac{\log_2(1+t)}{t} \frac{2}{1 + (1 + \lambda \log(1+t))^{\frac{1}{\gamma}}} \sum_{m=0}^{\infty} \frac{(m+1)(m+1,\lambda m!)}{(m+1)^k m!} \lambda^m \sum_{l=m}^{\infty} S_1(l,m) \frac{t^l}{l!}
\]
\[
= \left( \sum_{\alpha=0}^{\infty} D_{\alpha,\lambda} \frac{t^\alpha}{\alpha!} \right) \left( \sum_{\beta=0}^{\infty} (\frac{b}{a})^{\beta} \frac{t^\beta}{\beta!} \right) \left( \sum_{\beta=0}^{\infty} \sum_{\alpha=0}^{\beta} \frac{(b)}{a} D_{\beta-a,\lambda} C_{\alpha,\lambda} \frac{t^\beta}{\beta!} \right) \left( \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{(m+1)(m+1,\lambda m!)}{(m+1)^k m!} S_1(l,m) \frac{t^l}{l!} \right)
\]
\[
L.H.S = \sum_{n=0}^{\infty} \sum_{l=0}^{n} \sum_{a=0}^{l} \sum_{m=0}^{l} \left( \frac{n}{l} \right) D_{n-l-a,\lambda} C_{a,\lambda} \frac{p(m+1)(m+1,\lambda m!)}{(m+1)^k m!} S_1(l,m) \frac{t^n}{n!}.
\]

By comparing the coefficients on both sides of (46), we obtain the following theorem.

Theorem 10. Let \( n \geq 0 \) and \( k \in \mathbb{Z} \). Then, we have

\[
C_{n,\lambda}^{(k)} = \sum_{l=0}^{n} \sum_{a=0}^{l} \sum_{m=0}^{l} \left( \frac{n}{l} \right) D_{n-l-a,\lambda} C_{a,\lambda} \frac{p(m+1)(m+1,\lambda m!)}{(m+1)^k m!} S_1(l,m).
\]
4. Conclusions

In this article, we introduced type-two degenerate poly-Changhee polynomials of the second kind and derived some beautiful identities and relations between type-two degenerate poly-Changhee numbers of the second kind and Stirling numbers of first and second kind. In addition, we gave the relation between degenerate Bernoulli polynomials of the second kind and type-two degenerate poly-Changhee numbers of the second kind. Again, we defined type-two degenerate unipoly-Changhee polynomials of the second kind and obtained some properties and relationships of degenerate unipoly-Changhee numbers of the second kind and the Daehee numbers.

Author Contributions: Both authors contributed equally to the manuscript and typed, read, and approved the final manuscript. Both authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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