Electromagnetic Waves Described by a Fractional Derivative of Variable and Constant Order with Non Singular Kernel

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Abstract. The concept of differential operator with variable order has attracted attention of many scholars due to their abilities to capture more complexities like anomalous diffusion. While these differential operators are useful in real life, they can only be handled numerically. In this work, we used a newly introduced variable order differential operators that can be used analytically and numerically, has connection with all integral transform to model some interesting mathematical models arising in electromagnetic wave in plasma and dielectric. The differential operators used are non-singular and have the crossover properties therefore the models studied can explain the propagation of the wave in two different layers which cannot be achieved with those differential variable order operators with singular kernels. Using the Laplace transform and its connection with the new differential operator, we derive the exact solution of the models under investigation.

1. Introduction. In the last 3 years, the concepts of fractional differentiation and integration have both witnessed an historical development where the power law kernel was reviewed and replaced by exponential and Mittag-Leffler kernels \([6, 7, 3, 14]\). These developments come from some arguments that were raised about the singularity of the power law around the origin zero. Generally speaking, a finite-time singularity will be observed when one input parameter or variable is time and the result of the output variable blow up toward infinity at a finite time. With no doubt this concept if with great interest in kinematics and in partial differential equations theory. We must note that, in mathematics, the simplest finite-time singularities are those from the power law exponents \(x^{-\alpha}\) (corresponding to the Riemann-Liouville kernel). One can quote some physical example where this singularity can be observed: The tendency of a chalk to bounce when dragged across a blackboard; a coin spun on a flat surface and the bouncing motion of an inelastic ball on a plane.

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However, not all physical problems have finite-space-time singularity as it is argued within the circle of fractional calculus that, power law singularity in Riemann-Liouville derivative is useful to represent such natural occurrence. However, one of the great misleading argument, since finite-time singularities do not necessary occur at the origin, which is the only point of singularity of the Riemann-Liouville power law fractional derivative or at the initial condition. The concept of fractional differentiation with no singular kernel was then suggested to handle such natural occurrence and also to handle those physical problem where the singularities do not occur at the initial conditions. They have been applied in many fields of science with great success [1, 19, 4, 15, 17]. While the concept of fractional derivative with constant order have been in fashion in the last decades, they have been found to be inefficient when dealing with anomalous diffusion therefore, the concept of variable orders was suggested. While these differential operators are excellent suggestion for anomalous diffusion, one will realizes that they cannot be used for analytical purpose. One even go far to ask what is their corresponding integral? Can we prove the fundamental theorem of calculus with these operators? Can we get the connection between these operators and well-known integral transform like Fourier, Laplace, Mellin and Sumudu transform? The answers will be no. Thus, to make these operators useful analytically, Atangana and Koca suggested another kind of derivative with variable order. With these new operators, one can obtain the Laplace, Sumudu, Mellin and Fourier transform. They can be used for analytical purpose and numerical analysis.

In the last past years the concept of fractional differentiation has been revised due to the limitation of existing as they introduce artificial singularities into non-singular models. The artificial singularity come from the definition of the well-Known Riemann-Liouville and Caputo derivative as they are convolution with power law. Some of the limitation of the power law are including its non-ability to capture fading memory and fatigue process which are more abundant in nature, we give an example of runner that start with initial velocity and run after 3 hours, then become very weak at a point that he cannot move again this process cannot be capture by power-law based fractional differential operators. Another weakness lies on the fact that they cannot accurately capture dynamical systems that change within the same interval, we are referencing to the concept of crossover behavior that is observed in many problems in nature. while the new differential operator, based of the Mittag-Leffler kernel has the asymptotic behavior of exponential decay law for earlier time and power law for latter time. This gives the new derivative to capture two different laws within the same interval. Another weakness of the existing derivatives lies on the fact that they cannot provide any statistical setting, as the power law does not provide any Gaussian properties while the new derivative provide a crossover from Gaussian to non-Gaussian without steady state. Any model therefore using the new mathematical operator is able to capture two different waiting time distribution ranging from stretched exponential to the power law. More importantly the existing derivative cannot capture random walk even the well-known Brownian motion, proven in several already published papers, the new derivative capture the random walk as well as the Brownian motion for earlier time.

In this paper, we develop some analytical solutions of some ordinary differential equations arising in electromagnetic, more precisely, we solve the electromagnetic waves in Plasma and Oscillating electric field. We thus structure our paper as follow: In section 1, we present the basic information regarding the generalized
Mittag-Leffler function and the new differential operator with variable order with its properties. Section 2, we consider an analysis of fractional modeling of electromagnetic waves in Plasma under different conditions and finally in section 3 we present an analysis of electromagnetic waves in dielectric using the new differential operator under different instances.

Definition 1.1. The generalized Mittag-Leffler function $E_{\delta,\nu}^\lambda(z)$ is defined as follows ([16])

$$E_{\delta,\nu}^\lambda(z) = \sum_{n=0}^{\infty} \frac{(\lambda)_n z^n}{\Gamma(\delta n + \nu)n!}, \quad (\delta, \nu, \lambda \in \mathbb{C}, \Re(\delta) > 0, \Re(\nu) > 0, \Re(\lambda) > 0),$$

where $(\lambda)_n$ is the Pochhammer symbol

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)}, \quad (\lambda)_0 = 1, \quad (\lambda)_n = \prod_{k=1}^{n} (\lambda + k - 1), \quad n \geq 1.$$

Definition 1.2. Let $f \in H^1(a, b)$ and $0 < \delta \leq 1$, then Atangana-Koca derivative of fractional order in Caputo sense is given as ([5]):

$$A^K_0D_t^\delta f(t) = \frac{1}{g(\delta)} \int_{0}^{t} f'(\tau)E_{\delta,\nu}^\lambda(-g(\delta)(t - \tau)^\delta) \, d\tau,$$

where the function $g(\delta)$ is well-defined such that

$$\lim_{\delta \to 0} \frac{1}{g(\delta)} \int_{0}^{t} f'(\tau)E_{\delta,\nu}^\lambda(-g(\delta)(t - \tau)^\delta) \, d\tau = \int_{0}^{t} \frac{df(\tau)}{dt} \, d\tau = f(t) - f(0).$$

The Laplace transform for the Atangana-Koca fractional derivative (1) is given by

$$\mathcal{L}\{A^K_0D_t^\delta f(t)\}(s) = \frac{1}{g(\delta)}(s\mathcal{L}(\mathcal{F}) - f(0)) \frac{s^{-\delta-1}}{(1 - g(\delta))^\delta},$$

Definition 1.3. Let $g(x) \in C^1[a, b]$ and $f(x)$ a differential function in an open interval $I$. The Atangana-Koca fractional variable order derivative in Caputo sense is given by([2])

$$A^K_0D_t^{\varphi(x)} f(t) = \int_{0}^{t} f'(\tau) \exp(-g(x)(t - \tau)) \, d\tau.$$

The Laplace transform for the Atangana-Koca fractional variable order derivative is given by

$$\mathcal{L}\{A^K_0D_t^{\varphi(x)} f(t)\}(s) = \frac{s\mathcal{L}(\mathcal{F}) - f(0)}{s + g(\delta)}.$$

The m-th derivative of the function $E_{\delta,\nu}^{(m)}(at^\delta)$ with $\delta > 0$ and $\nu > 0$ is given by

$$E_{\delta,\nu}^{(m)}(at^\delta) = \sum_{k=0}^{\infty} \frac{(k + m)!}{k!} \frac{a^k t^{\delta k}}{\Gamma(\delta k + \delta m + \nu)}.$$

The Laplace transform of the function $t^{\delta m + \nu - 1}E_{\delta,\nu}^{(m)}(at^\delta)$ is given as

$$\mathcal{L}\{t^{\delta m + \nu - 1}E_{\delta,\nu}^{(m)}(at^\delta)\}(s) = \frac{m! s^{\delta - \nu}}{(s^\delta - a)^{m+1}}, \quad \delta > 0, \quad \nu > 0, \quad \Re(s) > |a|^\frac{1}{\delta}.$$


For \( \delta \geq \nu, \delta > \lambda, a \in \mathbb{R}, s^{\delta - \nu} > |a| \) and \( |s^\delta + as^n| > |b| \), we have

\[
\mathcal{L} \left\{ \frac{s^\lambda}{s^\delta + as^n + b} \right\} = t^{\delta - \lambda - 1} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-b)^n(-a)^r}{r!(r - \nu) + \delta(n + 1) - \lambda} t^{r(\delta - \nu) + n\delta}. \tag{6}
\]

**Theorem 1.4.** Let \( f \) and \( g \) continuous functions in \([0, \infty)\) of exponential order \( \nu > 0 \) and \( \mathcal{L}\{f(x)\} \) be Laplace transform of \( f(x) \), then

\[
\mathcal{L}\{(f \ast g)(t)\}(s) = \mathcal{L}\{f(t)\}(s)\mathcal{L}\{g(t)\}(s),
\]

where \( (f \ast g)(t) \) represents the convolution of \( f \) and \( g \) is given by

\[
(f \ast g)(t) := \int_0^t f(\tau)g(t - \tau) \, d\tau, \ \forall t \geq 0.
\]

2. **Fractional modeling of electromagnetic waves in Plasma.** Consider fully ionized gas; this gas is a hydrogen plasma, which has the same quantity of electrons and protons. The hydrogen plasma is considered as a uniform slab of plasma of thickness \( L \) in the \( x \) direction and having very large dimensions in the \( y \) and \( z \) dimensions. We take the proton mass to be effectively fixed in a place. Suppose that we displaced the electrons from the protons by a distance \( x \ll L \). An electric field is set up that would exert a force on the electrons, pulling them back to the protons. Letting the electrons go, they would rush back toward the protons, overshoot and an oscillations would be set up with a characteristic frequency \([18]\). We will develop a simple model, i.e. we regard the medium as an assembly of molecular oscillators. If the electrons are displaced in the plasma an electric force(restoring force), \(-kx(t) = -4\pi ne^2x(t)\) is produced in the direction of this equilibrium position. The equation of motion of each electron is therefore

\[
\frac{d^2x(t)}{dt^2} + \frac{4\pi ne^2}{m_e}x(t) = E(t), \tag{7}
\]

where the charge per unit area is \(-neL\), the force per unit area is \( F = -4\pi n^2e^2L_x \), and the mass per unit area is \( nm_eL \). The Equation (7) is the equation of a harmonic oscillator with a frequency

\[
\omega_0 = \sqrt{\frac{4\pi ne^2}{m_e}}. \tag{8}
\]

The Equation (8) gives the expressions for the electron plasma frequency.

2.1. **Zero electric fields.**

**Atangana-Koca fractional order derivative:**

Let us consider (7) via Atangana-Koca fractional derivative (1) in the following way

\[
\frac{1}{\sigma^{2(1-\nu)}} \mathcal{A}K_0D_t^{2\nu}x(t) + \frac{4\pi ne^2}{m_e}x(t) = E(t), 0 < \nu \leq 1. \tag{9}
\]

The auxiliary parameter \( \sigma^{2(1-\nu)} \) is introduced with the finality to be consistent with dimensionality of the fractional differential equation; here \( \sigma \) has dimensions of time(seconds) \([12]\).

Consider \( E(t) = 0, x(0) = x_0(x_0 > 0) \) and \( \dot{x}(0) = 0 \). The Equation (9) may be written as follows

\[
\mathcal{A}K_0D_t^{2\nu}x(t) + \omega^2x(t) = 0, \tag{10}
\]
where
\[ \omega^2 = \frac{4\pi ne^2}{m_e} \sigma^2(1-\nu) = \omega_0^2 \sigma^2(1-\nu), \]
is the fractional electron plasma frequency for different values of \( \nu \) and \( \omega_0^2 = \frac{4\pi ne^2}{m_e} \) is the electron plasma frequency (8).

Applying Laplace transform (1) to (10) and considering \( x(0) = x_0 \) and \( \dot{x}(0) = 0 \), yields
\[ \frac{s^{-2\nu}x(s) - s^{-2\nu-1}x(0)}{g(\nu)^2(1 - g(\nu))^{2\nu}} + \omega^2x(s) = 0. \]

After simplification, we get
\[ \bar{x}(s) = \frac{x_0 s^{-2\nu-1}}{s^{-2\nu} + \Psi^2\omega^2}, \tag{11} \]
where \( \Psi^2 = g(\nu)^2(1 - g(\nu))^{2\nu} \).

Applying the inverse Laplace transform in (11) and using (5), we get
\[ x(t) = x_0 \frac{1}{\Psi^2\omega^2} 2^{2\nu} E_{2\nu,2\nu+1} \left( \frac{-t^{2\nu}}{\Psi^2\omega^2} \right), \]
where \( \Psi^2 = g(\nu)^2(1 - g(\nu))^{2\nu} \).

**Atangana-Koca fractional variable-order derivative:**
Consider (7) via Atangana-Koca fractional variable-order derivative (3) gives
\[ \frac{1}{\sigma^{2(1-g(\xi))}} \mathcal{A}_D^{g(\xi)} x(t) + \frac{4\pi ne^2}{m_e} x(t) = E(t), \tag{12} \]
where \( 0 < g(\xi) \leq 1 \).

Consider \( E(t) = 0, x(0) = x_0(x_0 > 0) \) and \( \dot{x}(0) = 0 \). The Equation (12) may be written as follows
\[ \mathcal{A}_D^{g(\xi)} x(t) + \omega^2 x(t) = 0, \tag{13} \]
where
\[ \omega^2 = \frac{4\pi ne^2}{m_e} \sigma^2(1-g(\xi)) = \omega_0^2 \sigma^2(1-g(\xi)). \]

Applying the Laplace transform (4) to (13), we obtain
\[ \frac{s^2 \bar{x}(s) - sx(0)}{(s + g(\xi))^2} + \omega^2 \bar{x}(s) = 0. \tag{14} \]

Simplifying (14), we have
\[ \bar{u}(s) = \frac{x_0 s}{s^2 + \omega^2(s + g(\xi))^2}, \]
which gives
\[ \bar{x}(s) = \left( \frac{x_0}{1 + \omega^2} \right) \frac{s}{s^2 + \frac{2\omega^2g(\xi)}{1+\omega^2} s + \omega^2g(\xi)^2}. \tag{15} \]

Applying inverse Laplace transform on (15) and using (6), we get
\[ x(t) = \left( \frac{x_0}{1 + \omega^2} \right) \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-b)^n(-a)^r (n+r+1)}{\Gamma(r+2n+1)} t^{r+2n}, \]
where \( a = \frac{2\omega^2g(\xi)}{1+\omega^2} \) and \( b = \frac{\omega^2g(\xi)^2}{1+\omega^2} \).
2.2. Static electric fields.

Atangana-Koca fractional order derivative:

Now we apply a static electric field, \( E(t) = E_0, x(0) = x_0(x_0 > 0) \) and \( \dot{x}(0) = 0 \). The Equation (9) may be written as follows

\[
AK_0 D_t^{2\nu} x(t) + \omega^2 x(t) = \Omega,
\]

where \( \omega^2 = \frac{4\pi ne^2}{m_0} \sigma^2(1-\nu) = \omega_0^2 \sigma^2(1-\nu) \) and \( \Omega = \frac{E_0 \omega^2}{4\pi ne} \). Applying Laplace transform (1) to (16) and considering \( x(0) = x_0 \) and \( \dot{x}(0) = 0 \), yields

\[
\frac{s^{-2\nu} \bar{x}(s) - s^{-2\nu-1} x(0)}{g(\delta)^2 (1 - g(\delta))^{2\delta}} + \omega^2 \bar{x}(s) = \Omega - \frac{1}{s}.
\]

After simplification, we get

\[
\bar{x}(s) = \frac{x_0 s^{-2\delta - 1}}{s^{-2\nu} + \Psi^2 \omega^2} + \frac{\Omega \Psi^2}{s(s^{-2\nu} + \Psi^2 \omega^2)},
\]

where \( \Psi^2 = g(\xi)^2 (1 - g(\xi))^{2\delta} \).

Applying the inverse Laplace transform in (17) and using (5), we get

\[
x(t) = \frac{x_0}{\Psi^2 \omega^2} t^{2\nu} E_{2\nu,2\nu+1} \left( -\frac{\nu^{2\nu}}{\Psi^2 \omega^2} \right) + \frac{\Omega}{\omega^2} E_{2\nu+1,1} \left( -\frac{\nu^{2\nu}}{\Psi^2 \omega^2} \right).
\]

Atangana-Koca fractional variable-order derivative:

The Equation (9) with Atangana-Koca fractional variable-order derivative (3) may be written as follows

\[
AK_0 D_t^{2g(\xi)} x(t) + \omega^2 x(t) = \Omega,
\]

where \( \omega^2 = \frac{4\pi ne^2}{m_0} \sigma^2(1-g(\xi)) = \omega_0^2 \sigma^2(1-g(\xi)) \) and \( \Omega = \frac{E_0 \omega^2}{4\pi ne} \).

Applying the Laplace transform (4) to (18) and using \( x(0) = x_0 \) and \( \dot{x}(0) = 0 \), we obtain

\[
\frac{s^2 \bar{x}(s) - sx(0)}{(s + g(\xi))^2} + \omega^2 \bar{x}(s) = \frac{\Omega}{s}.
\]

Simplifying (19), we have

\[
\bar{x}(s) = \frac{x_0 s}{s^2 + \omega^2 (s + g(\xi))^2} + \frac{(s + g(\xi))^2}{s(s^2 + \omega^2 (s + g(\xi))^2)}.
\]

which gives

\[
\bar{x}(s) = \left[ \frac{u_0 + \Omega}{1 + \omega^2} \right] \frac{s}{s^2 + \frac{2\omega^2 g(\xi)}{1+\omega^2} s + \frac{\omega^2 g(\xi)^2}{1+\omega^2}} + \left( \frac{\Omega}{1 + \omega^2} \right) \frac{2g(\xi)}{s^2 + \frac{2\omega^2 g(\xi)}{1+\omega^2} s + \frac{\omega^2 g(\xi)^2}{1+\omega^2}} + \left( \frac{\Omega}{1 + \omega^2} \right) \frac{g(\xi)^2}{s^2 + \frac{2\omega^2 g(\xi)}{1+\omega^2} s + \frac{\omega^2 g(\xi)^2}{1+\omega^2}}.
\]
Applying inverse Laplace transform on (20) and using (6), we get
\[ x(t) = \left( \frac{x_0 + \Omega}{1 + \omega^2} \right) \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-b)^n(-a)^{r(n+r)} t^{r+2n}}{\Gamma(r+2n+1)} + \left( \frac{2g(\xi)\Omega}{1 + \omega^2} \right) \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-b)^n(-a)^{r(n+r)} t^{r+2n+1}}{\Gamma(r+2n+2)} + \left( \frac{g(\xi)^2\Omega}{1 + \omega^2} \right) \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-b)^n(-a)^{r(n+r)} t^{r+2n+2}}{\Gamma(r+2n+3)}, \]
where \( a = \frac{2\omega^2 g(\xi)}{1+\omega^2} \) and \( b = \frac{\omega^2 g(\xi)^2}{1+\omega^2} \).

2.3. Oscillating electric fields.

Atangana-Koca fractional order derivative:

If the electric field is that of an electromagnetic wave (an oscillating electric field), then \( \mathcal{E}(t) = E_0 e^{i\beta s} \), \( x(0) = x_0 (x_0 > 0) \) and \( \dot{x}(0) = 0 \).

Assuming a damping force \( m_e \chi \frac{dx}{dt} \) that is proportional to the velocity of the electron, the fractional equation of motion with

\[ AK_0 D_t^{2\nu} x(t) + \beta AK_0 D_t^{\nu} x(t) + a^2 x(t) = \left( \frac{e\alpha^2}{k} \right) E_0 e^{i\beta s}, \]

where \( \alpha^2 = \omega^2 = \frac{4\pi n^2 c^2}{m} \), \( \nu = 2(1-\nu) = \omega_0^2 \sigma^2 \), \( k = 4\pi ne \) and \( \beta = \chi \sigma^{1-\nu} \).

Applying the Laplace transform (2) to Equation (21), we obtain

\[ \frac{s^{-2\nu} \bar{x}(s) - s^{-2\nu-1} x(0)}{g(\nu)^2 (1 - g(\nu))^{2\nu}} + \frac{s^{-\nu n} \bar{x}(s) - s^{-\nu n-1} x(0)}{g(\nu) (1 - g(\nu))^\nu} + a^2 \bar{x}(s) = \left( \frac{e\alpha^2}{k} \right) E_0 \left( \frac{s}{s^2 + \eta^2} \right) + \frac{\eta s}{s^2 + \eta^2}, \]

Simplifying Equation (22), we get

\[ \bar{x}(s) = x_0 \frac{s^{-2\nu n-1}}{s^{-2\nu n} + a\beta s^{-n\nu} + a^2 \alpha^2} + x_0 a\beta \frac{s^{-2\nu n} + a\beta s^{-n\nu} + a^2 \alpha^2}{s^{-2\nu n} + a\beta s^{-n\nu} + a^2 \alpha^2} + \frac{e\alpha^2}{k} \frac{s}{s^2 + \eta^2} \frac{s^{-2\nu n} + a\beta s^{-n\nu} + a^2 \alpha^2}{s^{-2\nu n} + a\beta s^{-n\nu} + a^2 \alpha^2} \]

\[ + \frac{i e\alpha^2}{k} \frac{\eta s}{s^2 + \eta^2} \frac{s^{-2\nu n} + a\beta s^{-n\nu} + a^2 \alpha^2}{s^{-2\nu n} + a\beta s^{-n\nu} + a^2 \alpha^2}. \]

Now, we simplify the Equation (23). For this, we write the first term as follows:

\[ x_0 \frac{s^{-2\nu n-1}}{s^{-2\nu n} + a\beta s^{-n\nu} + a^2 \alpha^2} = x_0 \frac{s^{-2\nu n} - 1}{s^{-2\nu n} + a\beta s^{-n\nu} + a^2 \alpha^2} \frac{1}{s^{-2\nu n} + a\beta s^{-n\nu} + a^2 \alpha^2}. \]

Use of identity gives

\[ \frac{1}{1+r} = \sum_{k=0}^{\infty} (-r)^k, \ |r| < 1 \]

and assume that

\[ \frac{a^2 \alpha^2}{s^{2\nu n} + a\beta s^{-n\nu}} < 1, \]
where \(a\) may be written as follows

\[
x_0 \frac{s^{−2ν}−1}{s^{−2ν} + aβs^{−ν} + a^2α^2} = x_0 \frac{1}{aβ} \sum_{k=0}^{∞} \frac{(−a^2α^2)^k}{(αβ)^k} \frac{s^{2nk−1}}{(s^{ν} + 1/αβ)^{k+1}},
\]

(24)

and for Equation (24), we have following expression:

\[
\bar{x}(s) = \frac{x_0}{aβ} \sum_{k=0}^{∞} \frac{(−a^2α^2)^k}{(αβ)^k} \frac{s^{2nk−1}}{(s^{ν} + 1/αβ)^{k+1}} + \frac{x_0}{a} \sum_{k=0}^{∞} \frac{(−a^2α^2)^k}{(αβ)^k} \frac{s^{2nk+ν−1}}{(s^{ν} + 1/αβ)^{k+1}}
\]

\[
+ (\frac{ea^2E_0a}{kβ}) \sum_{k=0}^{∞} \frac{(−a^2α^2)^k}{(αβ)^k} \frac{s^{2nk(k+1)}}{(s^{ν} + 1/αβ)^{k+1}} \frac{s}{s^2 + η^2}
\]

\[
+ i (\frac{ea^2E_0a}{kβ}) \sum_{k=0}^{∞} \frac{(−a^2α^2)^k}{(αβ)^k} \frac{s^{2nk(k+1)}}{(s^{ν} + 1/αβ)^{k+1}} \frac{η}{s^2 + η^2}.
\]

(25)

Applying inverse Laplace transform on (24), we get

\[
\mathcal{L}^{-1}\left\{\frac{s^{−2ν}−1}{s^{−2ν} + aβs^{−ν} + a^2α^2}\right\}
\]

\[
= \frac{x_0}{aβ} \sum_{k=0}^{∞} \frac{(−a^2α^2)^k}{k!(αβ)^k} t^{νv(1−k)} E^{(k)}_{νv,νv(1−2k)+1} \left(−\frac{t^{νv}}{aβ}\right).
\]

Hence, the inverse Laplace transform of (25), we obtain

\[
x(t) = \frac{x_0}{aβ} \sum_{k=0}^{∞} \frac{(−a^2α^2)^k}{k!(αβ)^k} t^{νv(1−k)} E^{(k)}_{νv,νv(1−2k)+1} \left(−\frac{t^{νv}}{aβ}\right)
\]

\[
+ \frac{x_0}{a} \sum_{k=0}^{∞} \frac{(−a^2α^2)^k}{k!(αβ)^k} t^{−ν(k−1)} E^{(k)}_{νv−νv−2νk} \left(−\frac{t^{νv}}{aβ}\right) \cos(η(t−τ))dτ
\]

\[
+ i \left(\frac{ea^2E_0a}{kβ}\right) \int_0^t \sum_{k=0}^{∞} \frac{(−a^2α^2)^k}{k!(αβ)^k} t^{−ν(k−1)−1} E^{(k)}_{νv−νv−2νk} \left(−\frac{t^{νv}}{aβ}\right) \sin(η(t−τ))dτ,
\]

where \(a = g(ν)(1 − g(ν))^ν\).

**Atangana-Koca fractional variable-order derivative:**

The Equation (21) with Atangana-Koca fractional variable-order derivative (3) may be written as follows

\[
AK_αD^2t^α(ξ) x(t) + β AK_αD^2t^α(ξ) x(t) + α^2 x(t) = \left(\frac{ea^2}{k}\right) E_0 e^{iνt},
\]

(26)

where \(α^2 = \omega^2 = \frac{4πne}{m^2} 2^{2(1−g(ξ))} = ω_0^2 2^{2(1−g(ξ))}, k = 4πne\) and \(β = χσ^{1−g(ξ)}\).

Applying the Laplace transform (4) to Equation (26), we obtain

\[
\frac{s^2 \ddot{x}(s) − sx_0}{(s + g(ξ))^2} + β \frac{s \ddot{x}(s) − x_0}{s + g(ξ)} + α^2 x(t) = \left(\frac{ea^2}{k}\right) E_0 \frac{s}{s^2 + η^2} + i \left(\frac{ea^2}{k}\right) E_0 \frac{s}{s^2 + η^2}.
\]

(27)
Simplifying the Equation (27), we have

\[
\ddot{x}(s) = x_0 \frac{s}{s^2 + \beta s(s + g(\xi)) + \alpha^2(s + g(\xi))^2} \\
+ x_0 \beta \frac{s + g(\xi)}{s^2 + \beta s(s + g(\xi)) + \alpha^2(s + g(\xi))^2} \\
+ \left( \frac{\epsilon a^2}{k} \right) E_0 \frac{(s + g(\xi))^2}{s^2 + \beta s(s + g(\xi)) + \alpha^2(s + g(\xi))^2} \frac{s}{s^2 + \eta^2} \\
+ \epsilon i \left( \frac{\epsilon a^2}{k} \right) E_0 \frac{(s + g(\xi))^2}{s^2 + \beta s(s + g(\xi)) + \alpha^2(s + g(\xi))^2} \frac{s}{s^2 + \eta^2}.
\]

From which, we get

\[
\ddot{x}(s) = \frac{x_0}{1 + \beta + \alpha^2} \left( \frac{s}{s^2 + \frac{\beta g(\xi) + 2a^2 g(\xi)}{1 + \beta + \alpha^2} s + \frac{\alpha^2 g(\xi)^2}{1 + \beta + \alpha^2}} \right) \\
+ \frac{x_0 \beta g(\xi)}{1 + \beta + \alpha^2} \left( \frac{s}{s^2 + \frac{\beta g(\xi) + 2a^2 g(\xi)}{1 + \beta + \alpha^2} s + \frac{\alpha^2 g(\xi)^2}{1 + \beta + \alpha^2}} \right) \\
+ \frac{x_0 \beta g(\xi)}{1 + \beta + \alpha^2} \left( \frac{s}{s^2 + \frac{2a^2 g(\xi)}{1 + \beta + \alpha^2} s + \frac{\alpha^2 g(\xi)^2}{1 + \beta + \alpha^2}} \right) \\
+ \left( \frac{\epsilon a^2}{k} \right) E_0 \frac{s^2 + 2sg(\xi) + g(\xi)^2}{s^2 + \beta s(s + g(\xi)) + \alpha^2(s + g(\xi))^2} \frac{s}{s^2 + \eta^2} \\
+ \epsilon i \left( \frac{\epsilon a^2}{k} \right) E_0 \frac{s^2 + 2sg(\xi) + g(\xi)^2}{s^2 + \beta s(s + g(\xi)) + \alpha^2(s + g(\xi))^2} \frac{s}{s^2 + \eta^2}.
\]

Finally, considering the convolution theorem (1.4), the inverse Laplace transform of (28) is given by

\[
x(t) = \frac{x_0 (\beta + 1)}{1 + \beta + \alpha^2} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-b)^n(-a)^r (n+r)}{\Gamma(r+2n+1)} t^{r+2n} \\
+ \frac{x_0 \beta}{1 + \beta + \alpha^2} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-b)^n(-a)^r (n+r)}{\Gamma(r+2n+1)} t^{r+2n+1} \\
+ \left( \frac{\epsilon a^2}{k} \right) E_0 \frac{t}{1 + \beta + \alpha^2} \int_0^t \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-b)^n(-a)^r (n+r)}{\Gamma(r+2n+1)} t^{r+2n-1} \cos(\eta(t-\tau)) d\tau \\
+ \frac{2 \left( \frac{\epsilon a^2}{k} \right) E g(\xi)}{1 + \beta + \alpha^2} \frac{t}{1 + \beta + \alpha^2} \int_0^t \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-b)^n(-a)^r (n+r)}{\Gamma(r+2n+1)} t^{r+2n} \cos(\eta(t-\tau)) d\tau \\
+ \frac{\epsilon a^2}{k} E_0 g(\xi) \frac{t}{1 + \beta + \alpha^2} \int_0^t \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-b)^n(-a)^r (n+r)}{\Gamma(r+2n+2)} t^{r+2n+1} \cos(\eta(t-\tau)) d\tau \\
+ \epsilon i \left( \frac{\epsilon a^2}{k} \right) E_0 \frac{t}{1 + \beta + \alpha^2} \int_0^t \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-b)^n(-a)^r (n+r)}{\Gamma(r+2n+2)} t^{r+2n-1} \cos(\eta(t-\tau)) d\tau.
\]
\[ + 2i \left( \frac{e \alpha^2}{k} \right) \frac{E_0 g(\xi)}{1 + \beta + \alpha^2} \int_0^t \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-b)^n(-a)^r}{\Gamma(r + 2n + 1)} \frac{(n+r)^{n+2}}{r^{r+2n}} \cos(\eta(t - \tau)) \, d\tau \]
\[ + i \left( \frac{e \alpha^2}{k} \right) \frac{E_0 g(\xi)}{1 + \beta + \alpha^2} \int_0^t \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-b)^n(-a)^r}{\Gamma(r + 2n + 2)} \frac{(n+r)^{n+2}}{r^{r+2n+1}} \cos(\eta(t - \tau)) \, d\tau, \]

where \( a = \frac{\beta g(\xi) + 2 \alpha^2 g(\xi)}{1 + \beta + \alpha^2} \) and \( b = \frac{\alpha^2 g(\xi)^2}{1 + \beta + \alpha^2} \).

3. Fractional modeling of electromagnetic waves in dielectric media. In a dielectric media, Maxwell equations take the form [13]

\[ \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \]
\[ \nabla \cdot \vec{B} = 0, \]
\[ \nabla \times \vec{B} = \mu \vec{J} + \mu \epsilon \frac{\partial \vec{E}}{\partial t}, \]
\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}. \]

Taking the curl of the Equation (31) and the vector identity, we have

\[ \nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\nabla \times \frac{\partial \vec{B}}{\partial t}, \]

using the Equation (29), this become

\[ \nabla^2 \vec{E} = \nabla \times \frac{\partial \vec{B}}{\partial t}. \]

Taking the time derivative of (30) gives

\[ \nabla \times \frac{\partial \vec{B}}{\partial t} = \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2}. \]

Combining (33) and (32), the wave propagation in the x direction is written as

\[ \frac{\partial^2 \vec{E}(x, t)}{\partial x^2} - \mu \epsilon \frac{\partial^2 \vec{E}(x, t)}{\partial t^2} = S(x, t), \]

where \( S(x, t) \) is often called the current density source [8, 9, 10].

In order to make the fractional differential equation dimensionally consistent, an alternative procedure for constructing fractional equations was reported in [11]. The proposed alternative is introducing an additional parameter \( \alpha \), which must have dimension of seconds or meter(for the temporal or spatial operator respectively) to be consistent with the dimension of the ordinary operator. To do that, we replace the ordinary time operator by the fractional one as following:

\[ \frac{\partial}{\partial x} \rightarrow \alpha x^{-\alpha} \frac{\partial^\delta}{\partial x^\delta}, \ n - 1 < \delta \leq n, \]

in the spatial case, we can replace the ordinary operator by the fractional spatial operator as follows

\[ \frac{\partial}{\partial t} \rightarrow \alpha t^{-\alpha} \frac{\partial^\nu}{\partial t^\nu}, \ n - 1 < \nu \leq n. \]
where $\alpha_x$ has the dimension of length and $\alpha_t$ has dimensions of time. These parameters characterize the fractional spatial or fractional temporal structures (components that show an intermediate behavior between a system conservative and dissipative),[11] when $\mu$ and $\nu$ are equal to 1, the expression (35) and (36) reduce to ordinary derivative. Considering (35) and (36), the fractional representation of (34) is
\[
\alpha_x^{2\delta -1} \frac{\partial^{2\delta} \vec{E}(x,t)}{\partial x^{2\delta}} - 2^{\nu-1}(\mu\epsilon) \frac{\partial^{2\nu} \vec{E}(x,t)}{\partial x^{2\nu}} = S(x,t),
\]
the order of the derivative to be considered is $0 < \delta, \nu \leq 1$ for the fractional wave equation in space-time domain, respectively.

3.1. Fractional space wave equation in dielectric media. In this section, we will investigate the solutions of the fractional space-time wave equation in dielectric media via fractional derivative of variable and constant order with non singular kernel.

Atangana-Koca fractional order derivative:

Let us consider (37) the spatial fractional wave equation via Atangana-Koca fractional derivative (1) is given by
\[
\text{AK}_{\nu} D_x^{2\delta} (\vec{E}(x,t)) - \mu \epsilon \alpha_x^{2(1-\delta)} \frac{\partial^2 \vec{E}(x,t)}{\partial t^2} = \alpha_x^{2(1-\delta)} S(x,t),
\]
where $S(x,t) = 1$ for $x \geq 0$ and $S(x,t) = 0$ for $x < 0$. Now assuming the following solution
\[
\vec{E}(x,t) = \mathfrak{R}(\hat{E}_0 e^{i\omega t} u(x)),
\]
where $\mathfrak{R}$ is real part, substituting (39) into (38), we have
\[
\text{AK}_{\nu} D_x^{2\delta} (u(x)) + \hat{\theta}^2 u(x) = \alpha_x^{2(1-\delta)},
\]
where $\hat{\theta}^2 = \mu \epsilon \omega^2$ is the wave number and $\hat{\theta} = \theta \alpha_x^{2(1-\delta)}$ is the wave number in presence of fractional space components. The Equation (40) is called fractional Helmholtz equations in Atangana-Koca fractional derivative. For this equation if $\hat{\theta}^2$ has a negative value, then the behavior of $\vec{E}(x,t)$ for the space coordinate grows or decays exponentially, but if $\hat{\theta}^2$ has a positive value, then $\vec{E}(x,t)$ will vary sinusoidally or cosinusoidally for the space coordinate and varies with time in a simple harmonic motion.

Applying Laplace transform (1) to (40) and considering $u(0) = u_0$ and $\dot{u}(0) = 0$, yields
\[
\frac{s^{2\nu-2\delta} \hat{u}(s)}{g(\delta)^2 (1 - g(\delta))^{2\delta}} + \hat{\theta}^2 \hat{u}(s) = \alpha_x^{2(1-\delta)} \frac{1}{s}.
\]
After simplification, we get
\[
\hat{u}(s) = \frac{u_0 s^{2\nu-2\delta-1}}{s^{2\nu-2\delta} + \Psi^2 \hat{\theta}^2} + \frac{\alpha_x^{2(1-\delta)} \Psi^2}{s^{2\nu-2\delta} + \Psi^2 \hat{\theta}^2},
\]
where $\Psi^2 = g(\delta)^2 (1 - g(\delta))^{2\delta}$. Applying the inverse Laplace transform in (41) and using (5), we get
\[
u(x) = \frac{u_0}{\Psi^2 \hat{\theta}^2} x^{2\nu-2\delta} E_{2\nu-2\delta,2\nu+1} \left( -\frac{x^{2\nu}}{\Psi^2 \hat{\theta}^2} \right) + \frac{\alpha_x^{2(1-\delta)}}{\hat{\theta}^2} E_{2\nu-2\delta,1} \left( -\frac{x^{2\nu}}{\Psi^2 \hat{\theta}^2} \right).
Finally, the particular solution of (38) is
\[ \overrightarrow{E}(x, t) = \mathfrak{R} \left[ E_0 e^{i\omega t} \left( \frac{u_0}{\Psi^2 g^2} x^{2n\delta} E_{2n\delta, 2n\delta+1} \left( -x^{2n\delta} \right) + \frac{\alpha^2_{x(1-\delta)}}{\delta^2} E_{2n\delta, 1} \left( -x^{2n\delta} \right) \right) \right], \]
where \( \Psi^2 = g(\delta)^2 (1 - g(\delta))^{2\delta}. \)

**Atangana-Koca fractional variable-order derivative:**

Consider (37) the spatial fractional wave equation Atangana-Koca fractional variable-order derivative (3) is given by
\[ AK_0 D_x 2g(\xi) (\overrightarrow{E}(x, t)) - \mu \epsilon \omega \partial^2 \overrightarrow{E}(x, t) = \alpha_x^{2(1-g(\xi))} S(x, t), \quad (42) \]
where \( 0 < g(\xi) \leq 1, S(x, t) = 1 \) for \( x \geq 0 \) and \( S(x, t) = 0 \) for \( x < 0 \).

Now assuming the following solution
\[ \overrightarrow{E}(x, t) = \mathfrak{R}(E_0 e^{i\omega t} u(x)), \quad (43) \]
where \( \mathfrak{R} \) is real part, substituting (43) into (42), we have
\[ AK_0 D_x 2g(\xi) (u(x)) + \partial^2 u(x) = \alpha_x^{2(1-g(\xi))}, \quad (44) \]
where \( \partial^2 = \mu \epsilon \omega^2 \) and \( \partial^2 = \theta^2 \alpha_x^{2(1-g(\xi))}. \)

Applying the Laplace transform (4) to (44), we obtain
\[ \frac{s^2 \bar{u}(s) - su(0)}{(s + g(\xi))^2} + \partial^2 \bar{u}(s) = \alpha_x^{2(1-g(\xi))} \frac{1}{s}, \quad (45) \]
Simplifying (45), we have
\[ \bar{u}(s) = \frac{u_0 s}{s^2 + \partial^2 (s + g(\xi))^2} + \frac{\alpha_x^{2(1-g(\xi))} (s + g(\xi))^2}{s(s^2 + \partial^2 (s + g(\xi))^2)}, \]
which gives
\[ \bar{u}(s) = \left( \frac{u_0 + \alpha_x^{2(1-g(\xi))}}{1 + \partial^2} \right) \frac{s}{s^2 + \frac{2g(\xi)}{1+2\delta^2} s + \frac{g(\xi)^2}{1+\delta^2}} + \left( \frac{g(\xi)^2}{\alpha_x^{2(1-g(\xi))}} \right) \frac{s}{s^2 + \frac{2g(\xi)}{1+2\delta^2} s + \frac{g(\xi)^2}{1+\delta^2}} + \left( \frac{\alpha_x^{2(1-g(\xi))}}{1 + \partial^2} \right) \frac{s}{s^2 + \frac{2g(\xi)}{1+2\delta^2} s + \frac{g(\xi)^2}{1+\delta^2}}. \quad (46) \]

Applying inverse Laplace transform on (46) and using (6), we get
\[ u(x) = \left( \frac{u_0 + \alpha_x^{2(1-g(\xi))}}{1 + \partial^2} \right) \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-b)^n (-a)^r (n+r)^{r+2n}}{\Gamma(r+n+1)} x^{r+2n} + \left( \frac{2g(\xi)}{\alpha_x^{2(1-g(\xi))}} \right) \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-b)^n (-a)^r (n+r)^{r+2n+1}}{\Gamma(r+n+2)} x^{r+2n+1} + \left( \frac{\partial^2 g(\xi)}{\alpha_x^{2(1-g(\xi))}} \right) \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-b)^n (-a)^r (n+r)^{r+2n+2}}{\Gamma(r+n+3)} x^{r+2n+2}, \]
where \( a = \frac{2g(\xi)}{1+\delta^2} \) and \( b = \frac{\partial^2 g(\xi)^2}{1+\delta^2} \).
Finally the particular solution of (38) is
\[
\vec{E}(x, t) = \mathcal{R}\left[\vec{E}_0 e^{i\omega t} u(x)\right].
\] (47)

3.2. Fractional time wave equation in dielectric media.

**Atangana-Koca fractional order derivative:**

Considering the Equation (37), the temporal fractional wave equation via Atangana-Koca fractional derivative (1) is given by
\[
\mathcal{A}_\nu \mathcal{D}_t^\nu (\vec{E}(x, t)) - \frac{\alpha_t}{\mu\varepsilon} \frac{\partial^2 \vec{E}(x, t)}{\partial x^2} = \frac{\alpha_t}{\mu\varepsilon} S(x, t),
\] (48)

where \( S(x, t) = 1 \) for \( t \geq 0 \) and \( S(x, t) = 0 \) for \( t < 0 \). Now assuming the following solution
\[
\vec{E}(x, t) = \mathcal{R}(\vec{E}_0 e^{i\omega t} u(t)),
\] (49)

substituting (49) into (37), we have
\[
\mathcal{A}_\nu \mathcal{D}_t^\nu (u(t)) + \omega^2 u(t) = -\Omega^2,
\] (50)

where \( \omega^2 = \frac{k^2}{\mu\varepsilon} \alpha_t^2(1-\nu) = \omega_0^2 \alpha_t^2(1-\nu) \) is the fractional relation, \( \omega_0 \) is the natural frequency of the wave and \( \Omega^2 = \frac{1}{\mu\varepsilon} \alpha_t^2(1-\nu) \) is the velocity of the electromagnetic wave considering fractional components.

Applying Laplace transform (2) to (50) and considering \( u(0) = u_0 \) and \( \dot{u}(0) = 0 \) yields the following expression
\[
\mathcal{A}_\nu \mathcal{D}_t^\nu (s) + \omega^2 u(s) = -\Omega^2 \Rightarrow \frac{d}{ds} \bar{u}(s) = \frac{\omega_0^2}{\psi^2 \omega^2} \frac{s^{2\nu-1} - 1}{s^{2\nu} + \frac{1}{\psi^2 \omega^2}}.
\] (51)

Taking the inverse Laplace transform of (51), we get particular solution of the Equation (48) as
\[
\vec{E}(x, t) = \mathcal{R}\left[\vec{E}_0 e^{i\omega_0 x} \left(\frac{u_0}{\psi^2 \omega^2} + \frac{\Omega^2}{\omega^2} \left(1 - \frac{1}{\psi^2 \omega^2}\right)\right)\right],
\]

where \( \psi^2 = g(\nu)^2(1 - g(\nu))^{2\nu} \).

**Atangana-Koca fractional variable-order derivative:**

Consider (37) the temporal fractional wave equation Atangana-Koca fractional variable-order derivative (3) is given by
\[
\mathcal{A}_\nu \mathcal{D}_t^{\nu(\xi)} (\vec{E}(x, t)) - \frac{\alpha_t}{\mu\varepsilon} \frac{\partial^2 \vec{E}(x, t)}{\partial x^2} = -\frac{\alpha_t}{\mu\varepsilon} S(x, t),
\] (52)

where \( 0 < g(\xi) \leq 1 \), \( S(x, t) = 1 \) for \( x \geq 0 \) and \( S(x, t) = 0 \) for \( x < 0 \).

Now assuming the following solution
\[
\vec{E}(x, t) = \mathcal{R}(\vec{E}_0 e^{i\omega_0 x} u(t)),
\] (53)

substituting (53) into (52), we have
\[
\mathcal{A}_\nu \mathcal{D}_t^{\nu(\xi)} (u(t)) + \omega^2 u(t) = -\Omega^2,
\] (54)

where \( \omega^2 = \frac{k^2}{\mu\varepsilon} \alpha_t^2(1-g(\xi)) = \omega_0^2 \alpha_t^2(1-g(\xi)) \) and \( \Omega^2 = \frac{1}{\mu\varepsilon} \alpha_t^2(1-g(\xi)) \).
Applying the Laplace transform (4) to (54) and using \( u(0) = u_0 \) and \( \dot{u}(0) = 0 \), we obtain

\[
\tilde{u}(s) = \left( \frac{u_0 - \Omega^2}{1 + \omega^2} \right) \frac{s}{s^2 + 2\omega^2 g(\xi)} \frac{\Omega^2}{1 + \omega^2} \left( \frac{g(\xi)}{s^2 + 2\omega^2 g(\xi)} \right) \frac{\omega^2 g(\xi)^2}{1 + \omega^2}.
\]

Taking inverse Laplace transform of (55), we obtain

\[
u(t) = \left( \frac{u_0 - \Omega^2}{1 + \omega^2} \right) \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-b)^n(-a)^r (n+r)}{\Gamma(r+2n+1)} (t+2n)^{r+2n}
\]

\[
- \frac{2g(\xi)\Omega^2}{1 + \omega^2} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-b)^n(-a)^r (n+r)}{\Gamma(r+2n+2)} (t+2n+1)^{r+2n+1}
\]

\[
- \frac{\Omega^2 g(\xi)^2}{1 + \omega^2} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-b)^n(-a)^r (n+r)}{\Gamma(r+2n+3)} (t+2n+2)^{r+2n+2},
\]

where \( a = \frac{2\omega^2 g(\xi)}{1 + \omega^2} \) and \( b = \frac{\omega^2 g(\xi)^2}{1 + \omega^2} \).

Finally the particular solution of (54) is

\[
\overline{E}(x, t) = \Re \left[ \overline{E}_0 e^{ikx} \nu(t) \right].
\]

4. Conclusion. We considered in this work modified mathematical models able to portray electromagnetic waves in dielectric and plasma. We considered differential operators with exponential and Mittag-Leffler kernel with variable orders. The motivation behind the choice of these differential operators relies on the fact that they are non-singular and additionally they have mean square displacement able to cross from normal diffusion to confined-diffusion in the case of exponential decay kernel and from normal to sub-diffusion in case of Mittag-Leffler kernel. The Laplace transform and the convolution theorem were used to derive exact solutions to these models for different scenarios. We strongly believe that these new mathematical tools will open doors to new research within the field of variable calculus.

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