An Integrated Panel Data Approach to Modelling Economic Growth

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Abstract

Empirical growth analysis is plagued with three problems — variable selection, parameter heterogeneity and cross-sectional dependence — which are addressed independently from each other in most studies. This study is to propose an integrated framework that allows for parameter heterogeneity and cross-sectional error dependence, while simultaneously performing variable selection. We derive the asymptotic properties of the estimator, and apply the framework to a dataset of 89 countries over the period from 1960 to 2014. Our results support the “optimistic” conclusion of Sala-I-Martin (1997), and also reveal some cross-country patterns not found previously.

Keywords: Cross-Sectional Dependence, Growth Regressions, Parameter Heterogeneity, Variable Selection

JEL classification: C23, O47

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1 Introduction

Following the seminal works of Kormendi and Meguire (1985) and Barro (1991), a vast amount of studies in the empirical growth literature have attempted to identify salient determinants of economic growth. A main tool used by these studies is “cross-country growth regressions” — that is, to regress observed GDP growth on a plethora of possible explanatory variables that could possibly affect growth across countries. Excellent surveys of these studies and their role in the broader context of economic growth theory are provided in Durlauf and Quah (1999), Temple (1999) and Durlauf et al. (2005).

Despite the vast amount of research, the literature has identified a number of problems with conventional growth regressions, among which three deserve particular attention. The first problem is determining what variables to be included in growth regressions. This problem arises because of the nature of growth theories: although a plethora of growth theories have been proposed to identify factors that affect growth, these theories are open-ended in the sense that the validity of one causal theory of growth does not imply the falsity of another (Brock and Durlauf, 2001). In words of Durlauf et al. (2008), “a given body of candidate growth theories defines a space of possible models rather than a single specification”. From an empirical perspective, this problem stems from the fact that the number of potential explanatory variables is large (over 140 identified in Durlauf et al., 2005) relative to the number of countries with enough data availability, rendering the all-inclusive regression computationally infeasible (Sala-I-Martin et al., 2004; Durlauf et al., 2005). In dealing with this problem some studies have resorted to simply “trying” combinations of variables which could be potentially important determinants of growth and report the results of their preferred specification. However, as noted by Leamer (1983) and Sala-I-Martin et al. (2004) such “data-mining” could lead to spurious inference.

The second problem with conventional growth analysis is that most empirical growth studies assume that the parameters of growth regressions are identical across countries. This assumption complies with the classical Solow model (Mankiw et al., 1992), which assumes that all countries share an identical aggregate Cobb-Douglas production function. However, an increasing number of studies (e.g., Durlauf and Johnson, 1995; Durlauf et al., 2001; Salimans, 2012) have suggested that the parameters are heterogeneous across countries. These studies, though using different econometric methods, all suggest that the assumption of a single linear growth model that applies to all countries is inappropriate. For example, Durlauf and Johnson (1995) employ a regression tree analysis to show that a cross-sectional regression using the Summers and Heston (1991) data appears to provide support for several distinct regimes in which aggregate production functions vary among
countries according to their level of development, while Durlauf et al. (2001), employing a varying coefficient growth model, also find strong evidence of parameter heterogeneity across countries.

The third problem is that few studies in the empirical growth literature allow for cross-sectional dependence of individual countries. Panel data econometrics has recently seen an increasing interest in models with unobserved time-varying heterogeneity caused by latent common shocks influencing all units, possibly to a different degree. This type of heterogeneity introduces cross-sectional dependence to individual countries, which, when neglected, can lead to biased estimates and spurious inference (Pesaran, 2006; Bai, 2009). In the context of cross-country growth analysis, the problem of cross-sectional dependence seems particularly salient due to the omnipresence of common global shocks (such as global financial crises and world oil price shocks) that affect all countries through trade and financial linkages (Chudik et al., 2017). Durlauf and Quah (1999) discuss the possibility of cross-sectional dependence in a Lucas (1993) growth model with human capital spillovers. They find that these spillovers markedly change the dynamics of convergence and the authors call for the modelling of cross-country interactions in empirical convergence analysis.

The three aforementioned problems have received more or less individual attention in the growth literature. For example, Durlauf et al. (2001) address the problem of parameter heterogeneity using a varying coefficient growth model, but do not deal with the problems of variable selection and cross-sectional dependence; both Sala-I-Martin et al. (2004) and Moral-Benito (2012) select growth determinants using Bayesian averaging, but do not account for parameter heterogeneity and cross-sectional dependence.

The main goal of this study is to propose an integrated framework that is capable of dealing with parameter heterogeneity and cross-sectional dependence, while simultaneously performing variable selection. Specifically, parameter heterogeneity is allowed for by permitting the coefficients to vary across countries according to a country’s initial conditions, while cross-sectional dependence is accounted for via a factor structure. We then propose a least absolute shrinkage and selection operator (LASSO) estimator to select growth determinants, establish the associated asymptotic results, and further verify our asymptotic results through extensive simulations, which constitutes another contribution of this paper. We apply this framework to a new dataset of 89 countries over the period 1960-2014. Our findings broadly support the more “optimistic” conclusion of Sala-I-Martin (1997), that is, some variables are important regressors for explaining cross-country growth patterns. Moreover, our empirical results also provide support to some important hypotheses in the growth literature, e.g., “middle income trap hypothesis”, “natural resources curse
hypothesis”, “religion works via belief, not practice”, etc.

The rest of the paper is organized as follows. Section 2 explains how to extend the canonical cross-country growth regression to account for the aforementioned issues. Section 3 describes a procedure for estimating the extended growth regression model, and presents the associated asymptotic properties. Section 4 describes the data. The empirical results are presented in Section 5. Section 6 presents several robustness checks. Section 7 concludes. Preliminary lemmas, proofs of the main theorems and Monte Carlo simulations, together with auxiliary tables and figures, are presented in the supplementary Appendix A. The proofs of the preliminary lemmas are presented in the supplementary Appendix B, which can be found at https://ssrn.com/abstract=3348229.

2 A Varying Coefficient Growth Regression Model with Factor Structure and Sparsity

A generic representation of the canonical cross-country growth regression is

\[ y_{it} = x_{it}'\beta_0 + e_{it}, \]  

(2.1)

where \( i = 1, 2, \ldots, N \) index countries; \( t = 1, 2, \ldots, T \) index time; \( y_{it} \) is the rate of economic growth; \( x_{it} \) represents a set of observable explanatory variables, including those originally suggested by Solow as well as other growth theories, and \( e_{it} \) is an error term. Equation (2.1) represents the baseline for much of growth econometrics.

However, as discussed in the introduction, (2.1) is based on two problematic assumptions. First, it assumes that the parameters (i.e., \( \beta_0 \)) are homogeneous across all countries. Second, it assumes that there is no cross-sectional dependence across countries. To relax the two assumptions, in what follows we extend the conventional cross-country growth regression of (2.1) in two ways. In Section 2.1 we allow for parameter heterogeneity by allowing \( \beta_0 \) to vary across countries according to a country’s initial conditions. In Section 2.2 we introduce cross-sectional dependence into the model by means of a factor structure.

2.1 Parameter Heterogeneity

Following Durlauf et al. (2001), we allow for parameter heterogeneity by generalizing (2.1) into a varying coefficient model:

\[ y_{it} = x_{it}'\beta_0(z_{it}) + e_{it}, \]  

(2.2)
where \( z_{it} \) can be interpreted as some measure of “development” (or “initial condition”) of a country, and \( \beta_0(z) = (\beta_{01}(z), \ldots, \beta_{0p}(z))^\prime \) is a vector of smooth functions that map the scalar index variable \( z_{it} \) into a set of country-specific parameters.

This generalization in (2.2) provides a framework within which one can bridge the gap between cross-country regression models and new growth theories. For instance, if one believes that initial GDP per capita causally affects a country’s production technology and growth as in Durlauf et al. (2001), then initial GDP per capita can be introduced as a “development” index. As pointed out by Durlauf and Johnson (1995), (2.2) is compatible both with a model in which economies pass through distinct phases of development towards a unique steady state as well one in which multiple steady states exist.

### 2.2 Cross-Sectional Error Dependence

Having accounted for parameter heterogeneity, we next introduce the cross-sectional dependence of error terms into (2.2) using a factor structure:

\[
e_{it} = \gamma_0 f_{0t} + \epsilon_{it},
\]

(2.3)

where \( f_{0t} \) is an \( r \times 1 \) vector of unobservable common factors, \( \gamma_0 \) is an \( r \times 1 \) vector of factor loadings that capture country-specific responses to the common shocks, and \( \epsilon_{it} \) is the idiosyncratic error term. These factors can capture unobservable impacts like world oil price shocks, global financial crises, recessions in major advanced economies, local spillover effects along channels determined by shared culture heritage, geographic proximity, economic/social interaction, and so forth (Chudik et al., 2011). Moreover, the components of the factor structure are allowed to drive both economic growth and explanatory variables, thus partially accounting for potential endogeneity of explanatory variables, which is neglected by the traditional approaches to causal interpretation of cross-country empirical analysis.

### 2.3 The Varying Coefficient Growth Regression Model with Factor Structure and Sparsity

Substituting (2.3) into (2.2) yields the following growth regression model that allows for parameter heterogeneity and cross-sectional dependence:

\[
y_{it} = x_{it}^\prime \beta_0(z_{it}) + \gamma_0 f_{0t} + \epsilon_{it},
\]

(2.4)
which extends the local Solow growth model investigated in Durlauf et al. (2001) into a panel data context with interactive fixed effects (or factor structure). From an econometric perspective, (2.4) extends the panel data model with interactive fixed effects in Pesaran (2006) and Bai (2009) into a varying coefficient context, which is naturally motivated by the relevant empirical literature in economic growth.

In addition to parameter heterogeneity and cross-sectional dependence, we are also interested in another issue that is prominent in the empirical growth literature — variable selection. This issue is important because (1) the dimension of $x_{it}$ can be very large; and (2) not all elements of $x_{it}$ drive economic growth. In other words, for those regressors not driving economic growth, it is reasonable to assume that their associated coefficients are zero, which is called “sparsity” in the literature of high dimensional econometrics.

In order to formally introduce the sparsity to the model (2.4), we assume that there exists an unknown set $\mathcal{A}^\dagger \subseteq \{1, \ldots, p\}$ satisfying that $E[|\beta_{0j}(z_{it})|^2] = 0$ if and only if $j \in \mathcal{A}^\dagger$. For notational simplicity, we assume $\mathcal{A}^\dagger = \{p^* + 1, \ldots, p\}$ for an unknown integer $p^*$ satisfying $1 \leq p^* < p$. Further, let $\mathcal{A}^* = \{1, \ldots, p^*\}$, $x^*_{it} = (x_{it,1}, \ldots, x_{it,p^*})'$, and $\beta^*_0(z) = (\beta_{01}(z), \ldots, \beta_{0p^*}(z))'$. Throughout this study, we always define the variables or functions corresponding to the sets $\mathcal{A}^*$ and $\mathcal{A}^\dagger$ with super-indices $^*$ and $^\dagger$ respectively. Thus, identifying growth determinants is equivalent to distinguishing $\mathcal{A}^*$ and $\mathcal{A}^\dagger$, which will be achieved by a LASSO estimator presented in the following section. Finally, regarding the dimension of regressors, we consider two cases where (1) $p$ is fixed, and (2) $p$ diverges as the sample size increases. We refer to them as the low dimensional (LD) case and the high dimensional (HD) case, respectively. In terms of econometric methodology, both cases with the sparsity setting have not been studied in the literature to the best of our knowledge.

We emphasize that as discussed in the introduction, failure to perform variable selection may result in spurious inference, failure to allow parameters to differ across countries is inconsistent with the increasing body of research that find cross-country parameters heterogeneity, and failure to account for cross-sectional dependence can lead to biased estimates and spurious inference. These possible consequences thus necessitate an integrated approach to simultaneously addressing the three issues. In the following section, we introduce a LASSO estimator that is designed specifically for performing variable selection on the extended growth regression model in (2.4).
3 Estimation

In this section, we propose a procedure to estimate model (2.4) and derive the associated asymptotic properties. Specifically, we first use a sieve method to approximate the coefficient functions of the growth regression model in (2.4), and then propose a LASSO estimator to select the significant variables. When estimating the growth regression model, we employ the principle component analysis (PCA) technique to estimate the unobservable factor structure.

Before proceeding further, we introduce some notations that will be used throughout this section. Let \( Y_t = (y_{t1}, \ldots, y_{tT})', X_t = (x_{t1}, \ldots, x_{tT})', Z_t = (z_{t1}, \ldots, z_{tT})', \varepsilon_t = (\varepsilon_{t1}, \ldots, \varepsilon_{tT})', F_0 = (f_{01}, \ldots, f_{0T})', \) and \( \Gamma_0 = (\gamma_{01}, \ldots, \gamma_{0N})'. \) \( \| \cdot \| \) denotes the Euclidean norm of a vector or the Frobenius norm of a matrix; for a square matrix \( W, \) let \( \eta_{\text{min}}(W) \) and \( \eta_{\text{max}}(W) \) stand for the minimum and maximum eigenvalues of \( W \) respectively; \( M_W = I_T - P_W \) denotes the orthogonal projection matrix generated by matrix \( W, \) where \( P_W = W(W'W)^{-1}W'. \) \( W \) is a matrix with full column rank.

As mentioned above, we first approximate the unknown coefficient functions of the growth regression model in (2.4), using the sieve method. Let all elements of \( \beta_0(z) \) belong to a Hilbert space \( L^2(\mathcal{R}, \pi(w)) = \{ g \mid \int_\mathcal{R} g^2(w)\pi(w)dw < \infty \}, \) where \( \pi(\cdot) \) is a known weight function. The use of the density \( \pi(w) \) makes the Hilbert space sufficiently large to include all kinds of functions that are of interest to economists and econometricians such as linear, power, and polynomial functions.

We first introduce some important properties of the Hilbert space \( L^2(\mathcal{R}, \pi(w)) \) and its associated orthonormal systems. Define an inner product \( \langle f, g \rangle = \int_\mathcal{R} f(w)g(w)\pi(w)dw \) for \( f, g \in L^2(\mathcal{R}, \pi(w)) \) that induces norm \( \| f \|_{L^2} = \sqrt{\langle f, f \rangle} \). Suppose that there exists a complete orthogonal function sequence \{\( h_j(w) \mid j \geq 0 \)\} in the space \( L^2(\mathcal{R}, \pi(w)) \) satisfying that \( \langle h_i, h_j \rangle = \delta_{ij} \) (the Kronecker delta), and that \{\( h_j(w) \mid j \geq 0 \)\} is uniformly bounded in the sense that \( \sup_{w \in \mathcal{R}} \sup_{j \geq 0} |h_j(w)\pi^{1/2}(w)| < \infty. \) The existence of the complete orthogonal sequence is guaranteed by Theorem 5.4.7 in Dudley (2003, p. 169), while its uniform boundedness is fulfilled in two situations that are mostly used in the literature (Newey, 1997 and Dong and Linton, 2018). With regard to the support \( \mathcal{R}, \) it can be bounded or unbounded. When \( \mathcal{R} \) is a compact interval on the real line, conventional orthonormal sequences such as Fourier series or polynomial sequence can be used. When \( \mathcal{R} \) is unbounded such as \( \mathcal{R} = (-\infty, +\infty) \) with \( \pi(w) \equiv \exp(-w^2/2) \) (or \( \pi(w) \equiv \exp(-w^2) \)), the sequence of probabilists’ (or physicists’) Hermite polynomials is an orthogonal basis in \( L^2(\mathcal{R}, \pi(w)) \) that satisfies the uniform boundedness. As for the choice of \( L^2(\mathcal{R}, \pi(w)) \), one needs to
take into account several factors such as how large he expects the function to be and what type of regression function he would like to employ in the space. The baseline is that $\mathcal{R}$ should cover the range of $[\min\{z_{it}\}, \max\{z_{it}\}]$ in practice. We refer interested readers to Chen (2007) and Dong and Linton (2018) for more relevant discussions.

For $\forall g \in L^2(\mathcal{R}, \pi(w))$, we can then expand $g(w)$ as follows: $g(w) = \sum_{j=0}^{\infty} c_j h_j(w)$, where $c_j = \langle g, h_j \rangle$. By the Parseval equality, $\|g\|_{L^2}^2 = \sum_{j=0}^{\infty} c_j^2$, implying the attenuation of the coefficients. Define the partial sum $g_m(w) := \sum_{j=0}^{m-1} c_j h_j(w)$ and the residue $\delta_g(w) := \sum_{j=m}^{\infty} c_j h_j(w)$ for $m \geq 1$. It is easy to see that the convergence of $g_m(w)$ to $g(w)$ as $m \to \infty$ holds in the norm sense and even in the point-wise sense when the function is smooth. Throughout this study, for any vector of functions $G(\cdot) = (g_1(\cdot), \ldots, g_{dG}(\cdot))^\prime$, its norm is defined as $\|G\|_{L^2} = (\sum_{\ell=1}^{d_G} \|g_\ell\|_{L^2}^2)^{1/2}$.

Applying the above expansion to the coefficient functions of our growth regression model (i.e., (2.4)) yields

$$\beta_{0\ell}(z) = C_{\beta_0}^\prime H_{m\ell}(z) + \delta_{\beta_0}(z)$$

for any functional component $\beta_{0\ell}(z)$ with $\ell = 1, \ldots, p$, where $C_{\beta_0} = (c_{\ell,0}, \ldots, c_{\ell,m\ell-1})'$, $H_{m\ell}(z) = (h_0(z), \ldots, h_{m\ell-1}(z))'$, $\delta_{\beta_0}(z) = \sum_{j=m\ell}^{\infty} c_{\ell,j} h_j(w)$, and $c_{\ell,j} = \langle \beta_{0\ell}, h_j \rangle$ for $j \geq 0$. Letting $m_\ell$ for $\ell = 1, \ldots, p$ be the same value $^1 m$ allows us to write

$$\beta_0(z) = C_{\beta_0} H_m(z) + \Delta_m(z) := \beta_{0,m}(z) + \Delta_m(z),$$

where

$$C_{\beta_0} = \begin{pmatrix} c_{1,0}, \ldots, c_{1,m-1} \\ \vdots \\ c_{p^*,0}, \ldots, c_{p^*,m-1} \\ 0_{(p-p^*) \times m} \end{pmatrix} = \begin{pmatrix} C_{\beta_0}^* \\ 0_{(p-p^*) \times m} \end{pmatrix},$$

$$\Delta_m(z) = \begin{pmatrix} \delta_{\beta_01}(z) \\ \vdots \\ \delta_{\beta_{p^*1}}(z) \\ 0_{(p-p^*) \times 1} \end{pmatrix} = \begin{pmatrix} \Delta_m^*(z) \\ 0_{(p-p^*) \times 1} \end{pmatrix}.$$  

$^1$The truncation parameter is set to be the same value for all components of $\beta_0(z)$ due to the following reasons. (1) The truncation parameters are chosen by the researcher. As long as they are within a reasonable range, the theoretical results presented in this section would carry through. (2) Using the same truncation parameter can substantially simplify the notations as shown in (3.1). (3) Using the same truncation parameter can speed up the computational process. See Section A.2 of the supplementary Appendix A for details. We will further explain how $m$ is chosen in practice in Section 5.
Thus, the first $p^*$ elements of $\beta_0(z)$ can be expressed as $\beta_0^*(z) = \beta_{0,m}^*(z) + \Delta_m(z)$, where $\beta_{0,m}^*(z) = C_{\beta_0}^* H_m(z)$.

Having approximated the coefficient functions, we then move on to estimate the growth regression model (2.4). Substituting (3.1) into (2.4) yields the following regression model in matrix form:

$$Y_i \approx \phi_i[\beta_{0,m}] + F_0 \gamma_{0i} + \mathcal{E}_i = \mathcal{Z}_i \text{vec}(C_{\beta_0}) + F_0 \gamma_{0i} + \mathcal{E}_i$$

where $\phi_i[\beta] = (x_{i1}'\beta(z_{i1}), \ldots, x_{iT}'\beta(z_{iT}))'$ for any $p \times 1$ vector of functions $\beta(z)$, $\mathcal{Z}_i = (\mathcal{Z}_i, \ldots, \mathcal{Z}_{iT})'$ and $\mathcal{Z}_{it} = H_m(z_{it}) \otimes x_{it}$. The approximation sign ($\approx$) in the above equation is due to the omission of the truncation residual.

If $F_0 \gamma_{0i}$ were known, we could simply perform the group LASSO technique of Yuan and Lin (2006) on

$$Y_i - F_0 \gamma_{0i} \approx \mathcal{Z}_i \text{vec}(C_{\beta_0}) + \mathcal{E}_i$$

to estimate $C_{\beta_0}$ and investigate the sparsity. Because both $F_0$ and $\gamma_{0i}$ are unknown, we project out the factor structure to obtain

$$M_{F_0} Y_i \approx M_{F_0} \mathcal{Z}_i \text{vec}(C_{\beta_0}) + M_{F_0} \mathcal{E}_i,$$

where $M_{F_0}$ is a projection matrix defined above.

Our objective function can then be defined as

$$Q_\lambda(C_\beta, F) = \sum_{i=1}^{N} (Y_i - \phi_i[\beta_m])' M_F (Y_i - \phi_i[\beta_m]) + \sum_{j=1}^{p} \lambda_j \|C_{\beta,j}\|, \quad (3.2)$$

where $\beta_m(w) = C_\beta H_m(w)$, $C_{\beta,j}$ stands for the $j^{th}$ row of $C_\beta$, and $\lambda = (\lambda_1, \ldots, \lambda_p)'$ is the vector including the weight parameters of the coefficient functions, and is to be determined by data. The first term on the right hand side of (3.2) is commonly used when estimating a panel data model with a factor structure (e.g., $S_{NT}(\beta, F)$ of Bai, 2009). The second term is in the spirit of the group LASSO technique initially proposed by Yuan and Lin (2006) and subsequently extended by Wang and Xia (2009) for the nonparametric setting. Moreover, the penalty term in (3.2) is equivalent to performing the hard-thresholding rule of the lasso literature on the terms $\|C_{\beta,j}\|$ for $j = 1, \ldots, p$. Thus, the resulting estimator in (3.3) below is a thresholding rule, which automatically sets small estimated coefficient functions to zero (i.e., penalizes small estimated coefficients) to reduce model complexity (see the Sparsity property mentioned in Fan and Li (2001) for details).
The estimators of $C_{\beta_0}$ and $F_0$ that work for both the LD and HD cases can be readily obtained as

$$
(\widehat{C}_{\beta}, \widehat{F}) = \arg\min_{C_{\beta}, F \in D_F} Q_{\lambda}(C_{\beta}, F),
$$

where $D_F = \{ F \mid F_{\|,r} = I_r \}$. The numerical implementation of (3.3) is presented in Section A.2 of the supplementary Appendix A to conserve space of the main text. In what follows, we always partition $\hat{A}^\tau$ of the supplementary Appendix A to conserve space of the main text. In what follows, we always partition $\hat{C}_{\beta}$, according to $A^\tau$ and $A^\dagger$, as $\hat{C}_{\beta} = (\hat{C}_{\beta}, \hat{C}_{\beta}^\dagger)^\prime$ wherever necessary.

At this point, it is convenient to state some fundamental assumptions that are needed for the derivation of the asymptotic results for both the LD and HD cases.

**Assumption 1.**

1. Let $\mathcal{F}_-^\infty$ and $\mathcal{F}_+^\infty$ denote the $\sigma$-algebras generated by $\{(x_t, z_t, \varepsilon_t, f_0t) \mid t \leq 0\}$ and $\{(x_t, z_t, \varepsilon_t, f_0t) \mid t \geq \tau\}$ respectively, where $x_t = (x_{1t}, \ldots, x_{NT})^\prime$, $z_t = (z_{1t}, \ldots, z_{NT})^\prime$, $\varepsilon_t = (\varepsilon_{1t}, \ldots, \varepsilon_{NT})^\prime$. Let $\alpha(\tau) = \sup_{A \in \mathcal{F}_-^\infty, B \in \mathcal{F}_+^\infty} |\Pr(A) \Pr(B) - \Pr(AB)|$ be the mixing coefficient.

   (a) $\{X_t, Z_t, E_t, \gamma_{0t}\}$ is identically distributed over $i$. $\{x_t, z_t, \varepsilon_t, f_0t\}$ is strictly stationary and $\alpha$-mixing such that for some $\nu_1 > 0$, $E[\|\varepsilon_{1t}\| + \|\varepsilon_{11}\|^{4+\nu_1}] < \infty$, and the mixing coefficient satisfies $\sum_{t=1}^\infty |\alpha(t)|^{\nu_1/(2+\nu_1)} < \infty$.

   (b) $E[\varepsilon_{11}] = 0$, $E[\varepsilon_{11}^2] = \sigma_\varepsilon^2$, and $\{\varepsilon_{it}\}$ is independent of the other variables. Let $E[\varepsilon_{it} \varepsilon_{jt}] = \sigma_{ij}$ for $i \neq j$, $\sum_{i \neq j} |\sigma_{ij}| = O(N)$, and $\sum_{i,j=1}^N \sum_{t,s=1}^T |E[\varepsilon_{it} \varepsilon_{jt}]| = O(NT)$.

2. Let $\|T^\dagger F_0 - \Sigma_f\| = O_P \left(\frac{1}{\sqrt{T}}\right)$ and $\|T^\dagger \Gamma_0^\dagger \Gamma_0 - \Sigma_\gamma\| = O_P \left(\frac{1}{\sqrt{N}}\right)$, where $\Sigma_f$ and $\Sigma_\gamma$ are deterministic and positive definite. Moreover, $E[\|f_0\|^4] < \infty$ and $E[\|\gamma_{01}\|^4] < \infty$.

**Assumption 2.**

Let $\Omega(F) = \frac{1}{NT} \{ \Omega_1(F) - \Omega_2(F) \Omega_3^{-1} \Omega_2(F) \}$, where $\Omega_1(F) = \sum_{i=1}^N Z_i^\prime M_F Z_i$, $\Omega_2(F) = \sum_{i=1}^N \gamma_{0i} \otimes (M_F Z_i)$, and $\Omega_3 = \Gamma_0^\dagger \Gamma_0 \otimes I_T$.

1. Assume all elements of $\beta_0(z)$ belong to $L^2(\mathcal{R}, \pi(w)) = \{ g \mid \int_\mathcal{R} g^2(w) \pi(w) dw < \infty \}$, and there exists a constant $\mu$ such that $\|\Delta_m(z)\|_{L^2} = O(m^{-\mu/2})$, where $\pi(\cdot)$ is a known probability weight function. Suppose that $\sup_{w \in \mathcal{R}} f_z(w)/\pi(w) < \infty$, where $f_z(w)$ is the density function of $z_it$. Let $\eta_{\max}(\frac{1}{NT} \sum_{i=1}^N Z_i^\prime Z_i^\prime) < \infty$ with a probability approaching one.

2. Suppose $\inf_{F \in D_F} \eta_{\min}(\Omega(F)) > 0$. 

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Assumption 1 is standard in the literature. The mixing conditions are similar to Assumption C of Bai (2009) and Assumption 3.4 of Fan et al. (2016).

The order of \( \| \Delta_m(z) \|_{L^2} = O(m^{-\mu/2}) \) of Assumption 2.1 requires smoothness of the coefficient functions, and is the same as Assumption 3 of Newey (1997), wherein this requirement is discussed in details. The condition on \( \eta_{\text{max}}(\frac{1}{NT} \sum_{i=1}^{N} Z_i^t Z_i^t) < \infty \) is similar to Assumption 3.1 of Fan et al. (2016). To give an example, consider a special case where \( f(z(w)) = \pi(w) \), {\( z_{it} \)} and {\( x_{it} \)} are mutually independent, \( E[x_{it}] = 0 \) and \( E[x_{it} x_{it}'] = I_p \). In this case, it is easy to see that \( \eta_{\text{max}}(\frac{1}{NT} \sum_{i=1}^{N} Z_i^t Z_i^t) < \infty \) holds true for both of the LD and HD cases by some standard analysis. The restriction on the density of \( z_{it} \) of Assumption 2.1 gives 

\[
E[h_j^2(z_{it})] = \int h_j^2(w)\pi(w)\frac{f_z(w)}{\pi(w)} dw \leq O(1) \int h_j^2(w)\pi(w) dw < \infty,
\]

which holds true uniformly in \( j \) and is equivalent to Assumption 3.3.ii of Fan et al. (2016).

Assumption 2.2 ensures that the estimators of (3.3) are well defined, and is equivalent to Assumption A of Bai (2009). To justify this assumption, we follow Lam and Yao (2012) and assume that \( \frac{\Gamma_0^t \Gamma_0}{N} = I_r \). With this latter assumption, \( \Omega(F) \) reduces to 

\[
\Omega(F) = \frac{1}{NT} \sum_{i=1}^{N} Z_i^t M_F Z_i - \frac{1}{N^2T} \sum_{i,j=1}^{N} \gamma_{0i} \gamma_{0j} Z_i^t M_F Z_j.
\]

If we further assume \( \{\gamma_{0i}\} \) is independent of \( \{Z_i\} \) for simplicity, then the second term in the above equation becomes negligible due to \( \frac{\Gamma_0^t \Gamma_0}{N} = I_r \). Thus, \( \Omega(F) \) can be simplified as \( \frac{1}{NT} \sum_{i=1}^{N} Z_i^t M_F Z_i \), which is achievable provided that there are enough variations in \( Z_i \) (i.e., variations in \( z_{it} \) and \( x_{it} \)) as explained in Bai (2009, p. 1241). Moreover, if we further let \( E[Z_i] = 0 \) (e.g., \( E[x_{it}] = 0 \), and assume that \( \{x_{it}\} \) and \( \{z_{it}\} \) are independent of each other), \( \Omega(F) \) further reduces to \( \frac{1}{NT} \sum_{i=1}^{N} Z_i^t Z_i \) by a proof similar to (2) of Lemma A.2. In this case, it becomes even more straightforward to justify Assumption 2.2.

### 3.1 Low Dimensional Case

Given the above setting, we first investigate the asymptotic properties of the estimator in (3.3) for the LD case (i.e., \( p \) is fixed). In order to identify \( \mathcal{A}^* \) and \( \mathcal{A}^\dagger \) and then to establish the asymptotic properties, we impose the following assumptions.

**Assumption 3.**

1. \( \frac{m^2}{T} \to 0 \) and \( \frac{\lambda_{\text{max}}}{N^{\frac{\mu}{2}}T} \to 0 \), where \( \lambda_{\text{max}} = \max\{\lambda_1, \ldots, \lambda_p^*\} \).
2. $N \rightarrow \kappa_0$ and $\frac{\lambda_{\text{min}}}{m \frac{1}{2} N^* T} \rightarrow \kappa_1$, where $\lambda_{\text{min}} = \min\{\lambda_{p^*+1}, \ldots, \lambda_p\}$, $0 \leq \kappa_0 < \infty$, and $\kappa_1 > 0$.

The conditions of Assumption 3, though seemingly complicated, can be justified. For example, let $N = \lfloor T b_0 \rfloor$, $m = \lfloor T b_1 \rfloor$, $\lambda_{\text{max}} = T b_2$, and $\lambda_{\text{min}} = T b_3$, where $\lfloor a \rfloor$ means the largest integer part of a real number $a$. Then Assumption 3 essentially requires that $0 < b_0 \leq 1$, $0 < b_1 < \frac{1}{2}$, $b_2 < \frac{3}{4} b_0 + 1$ and $b_3 \geq \frac{b_2}{2} + \frac{7 b_0}{8} + 1$. As for the choice of the weight parameters, $\lambda_j$’s, we propose a data driven method in Section A.2 of the Supplementary Appendix A.

With Assumption 3, the following theorem holds when $N \rightarrow \kappa_0$ with $0 \leq \kappa_0 < \infty$.

**Theorem 3.1.** Let Assumptions 1-3.1 hold. As $(N, T) \rightarrow (\infty, \infty)$,

1. $\|\hat{\beta}_m - \beta_0\|_2 = o_P(1)$.

Additionally, let Assumption 3.2 hold.

2. $\Pr(\|\hat{C}_* \beta_m^* - \beta_0^*\|_2 = 0) \rightarrow 1$;

3. $\|\hat{\beta}_m^* - \beta_0^*\|_2 = O_P \left( \sqrt{\frac{m}{N T}} + m^{-\frac{3}{4}} + \frac{m \lambda_{\text{max}}}{N T} \right)$, where $\hat{\beta}_m^*(z) = \hat{C}_* \beta_m(z)$.

The first two results of Theorem 3.1 mean that we can distinguish between $\mathcal{A}^*$ and $\mathcal{A}^1$; while the third result of Theorem 3.1 provides the rate of convergence for the coefficient functions of the variables which truly drive economic growth. Moreover, provided that $\lambda_{\text{max}}$ is chosen properly, the third result of Theorem 3.1 can be further simplified as $\|\hat{\beta}_m^* - \beta_0^*\|_2 = O_P \left( \sqrt{\frac{m}{N T}} + m^{-\frac{3}{4}} \right)$, which is the usual rate of convergence in sieve method based regressions.

To select the optimal weight parameters, we propose the following BIC type criterion:

$$\text{BIC}_\lambda = \ln \text{RSS}_\lambda + \text{df}_\lambda \Upsilon_{N T},$$

where $\text{RSS}_\lambda = \frac{1}{N T} \sum_{i=1}^N (Y_i - \phi_1[\hat{\beta}_m^\lambda])' M_{\hat{F}_\lambda}(Y_i - \phi_1[\hat{\beta}_m^\lambda]), \hat{\beta}_m^\lambda(z) = \hat{C}_\beta^\lambda H_m(z)$, $(\hat{C}_\beta^\lambda, \hat{F}_\lambda)$ are obtained by implementing (3.3) using $\lambda$ as the weight vector, and $\text{df}_\lambda$ is the number of nonzero coefficient functions identified using $\hat{C}_\beta^\lambda$.

To apply the BIC in (3.4), we still need to choose $\Upsilon_{N T}$. After examining the rates associated with (3.4), we require $\Upsilon_{N T}$ to satisfy the following condition

$$\Upsilon_{N T} \rightarrow 0 \text{ and } \Upsilon_{N T} \sqrt{N} \rightarrow \kappa_2 > 0, \text{ where } \kappa_2 \text{ is a large constant or } \infty.$$  

(3.5)

Two values for $\Upsilon_{N T}$ that are commonly used in the literature and that also satisfy (3.5) are $\ln \frac{N}{\sqrt{N}}$ and $\frac{N}{\sqrt{N}}$ for some given $a \in (0, \frac{1}{4})$. In Section A.2 of the supplementary Appendix
A, we consider several forms for $\Upsilon_{NT}$, and investigate the finite sample performances for each of them through Monte Carlo simulations.

We select $\lambda$ by

$$\hat{\lambda} = \arg\min_{\lambda} \text{BIC}_\lambda. \quad (3.6)$$

Further letting $S_{\hat{\lambda}} = \{ j | \| \hat{C}_{\beta,j} \| > 0, 1 \leq j \leq p \}$ represent the set of relevant variables identified using $\hat{C}_{\beta,j}$, then the following result follows.

**Theorem 3.2.** Let (3.5) and Assumptions 1-3 hold. Then $\Pr(S_{\hat{\lambda}} = A^*) \to 1$ as $(N, T) \to (\infty, \infty)$.

Theorem 3.2 means all zero coefficient functions can be identified. In other words, all variables not driving economic growth can be identified and thus removed from the growth regression. With those variables removed, we can proceed to perform the post-selection estimation using the remaining model.

To complete the discussion on the LD case, we propose the following assumption and state the asymptotic normality associated with (3.3).

**Assumption 4.**

1. Suppose that for $t \geq s$, $E[f_{0t}f_{0s} | X_{Nt}] = a_{ts}$, and $\sum_{t=1}^{T} \sum_{s=1}^{t} |a_{ts}| = O(T)$, where $X_{Nt} := \{ (x_{1t}, z_{1t}), \ldots, (x_{Nt}, z_{Nt}) \}$. Moreover, $\frac{NT}{m \nu} \to 0$, $\frac{mN}{T} \to 0$, $\frac{T}{N^2} \to 0$ and $rac{m\lambda_{max}}{\sqrt{NT}} \to 0$.

2. Let $\Sigma^*_z = E[Z_{11}^*Z_{11}^{*\prime}]$ and $\Omega_* = \lim_{(N,T) \to (\infty,\infty)} E[\Psi_1 \Psi_1^{\prime}]$ for $\forall z \in V_z$, where

   \[
   \Psi_1 = \sqrt{\frac{NT}{m}} [H^*_m(z) \otimes I_p] \Sigma^*_z \Sigma^*_z^{-1} : \frac{1}{NT} \sum_{i=1}^{N} \left\{ Z_{1i} - \frac{1}{N} \sum_{j=1}^{N} Z_{1i}^* \gamma_{0j} \Sigma_{\gamma}^{-1} \gamma_{0j} \right\} \psi_i,
   \]

   \[
   \Psi_2 = I_{mp} + \Sigma^*_z^{-1} E[Z_{11}^* \gamma_{01}] \Sigma_{\gamma}^{-1} E[\gamma_{01} Z_{11}^*], \quad \text{and} \quad Z_{1i}^* = H_m(z_{1i}) \otimes x_{1i}^*.
   \]

   Suppose that for $\forall z \in V_z$, as $(N,T) \to (\infty, \infty)$, $\Psi_1 \to_D N(0, \Omega_*)$.

Note that Assumption 4.1 is in the spirit of Connor et al. (2012, Eq. 3 and Eq. 20). Assumption 4.2 is equivalent to Assumption E of Bai (2009). To conserve space, we provide a detailed justification of the last part of this assumption in the supplementary Appendix A.

**Theorem 3.3.** Let Assumptions 1-4 hold. For $\forall z \in V_z$, as $(N, T) \to (\infty, \infty)$,

$$\sqrt{\frac{NT}{m}} (\hat{\beta}_m^*(z) - \beta_0^*(z)) \to_D N(0, \Omega_*), \text{ where } \hat{\beta}_m^*(z) = \hat{C}^*_m H_m(z).$$

12
It is worth emphasizing that (i) deriving the rates of convergence in Theorem 3.1 does not require Assumption 4; (ii) the asymptotic distribution in Theorem 3.3 also applies to the post-selection estimation conditional on that no regressors associated with $\mathcal{A}^*$ are removed. The associated proof is self-evident and thus is omitted here. Recently, more robust inferences on post-selection procedure have been developed, but most (if not all) of the studies (e.g., Dezeure et al., 2015; Hyun et al., 2018) focus on parametric models with i.i.d. data. It remains an open question as to how to extend these recent studies to a nonparametric panel data setting that allows for cross-sectional dependence and time series autocorrelation.

### 3.2 High Dimensional Case

In this subsection, we allow the dimension of $x_{it}$ to diverge as the sample size increases (i.e., $p^* \to \infty$ and $p \to \infty$). The following assumptions are crucial for establishing asymptotic properties for the HD case.

**Assumption 5.**

1. $\|E\|_{sp} = O_P(\max\{\sqrt{N}, \sqrt{T}\})$, where $\|\cdot\|_{sp}$ denotes the spectral norm of a matrix and $E = (E_1, \ldots, E_N)'$;

2. $p^* \lambda_{\max} \sqrt{\xi_{NT}} \to 0$, $(\frac{\xi_{NT} + mp}{NT} + p^* m^{-p}) \sqrt{\xi_{NT}} \to \kappa_2$, $\frac{\xi_{NT} \lambda_{\min}}{NT} \to \kappa_3$, where $\xi_{NT} = \min\{N, T\}$, $0 \leq \kappa_2 < \infty$ and $\kappa_3 > 0$.

Assumption 5.1 is identical to Assumption A.1.iii of Su et al. (2015). Assumption 5.2 can be verified in exactly the same way as shown under Assumption 3. Again, the data driven algorithm of Section A.2 of the supplementary Appendix A can help us bypass the complexity of Assumption 5.2 in practice.

With regard to the selection of weight parameters, we still use the BIC criterion defined in (3.4) and select $\lambda$ by $\hat{\lambda} = \arg\min_\lambda \text{BIC}_\lambda$ but with a different restriction on the penalty term.

$$\Upsilon_{NT} \to 0 \text{ and } \Upsilon_{NT} \xi_{NT}^{1/8} \to \kappa_5 > 0, \text{ where } \kappa_5 \text{ is a large constant or } \infty. \quad (3.7)$$

A natural choice of $\Upsilon_{NT}$ is $\frac{\ln \xi_{NT}}{\xi_{NT}^{1/8}}$, where $\xi_{NT}$ is defined in Assumption 5. Again, in Section A.2 of the supplementary Appendix A, we consider three forms of $\Upsilon_{NT}$, and implement numerical simulations to examine their finite sample performances.

Given the above setting, the following theorem holds.

**Theorem 3.4.** Let Assumptions 1, 2 and 5 hold. As $(N, T) \to (\infty, \infty)$,
1. \( \Pr(\| \hat{C}_\beta \| = 0) \rightarrow 1 \).

2. Suppose that \( (p^* m)^2 T \rightarrow 0 \), \( p^* m^{-\mu} \rightarrow 0 \), and \( \frac{N}{T} \rightarrow \kappa_0 < \infty \). Then \( \| \hat{\beta}_m - \beta^*_0 \|_{L^2} = O_P \left( \sqrt{\frac{p^* m N}{T}} + \sqrt{p^* m^{-\mu}} + \lambda_{\max} \frac{p^* m N}{T} \right) \).

3. Suppose that (3.7) holds. Then \( \Pr(S_{\hat{\lambda}} = A^*) \rightarrow 1 \).

It is worth emphasizing that if one is interested in consistent estimation with the purpose of distinguishing between \( A^* \) and \( A^\dagger \) only, the conditions imposed in the second result of Theorem 3.4 are not needed. Moreover, provided that \( \lambda_{\max}^* \) is chosen properly, the second condition of Theorem 3.4 reduces to: \( \| \hat{\beta}_m^* - \beta^*_0 \|_{L^2} = O_P \left( \sqrt{\frac{p^* m N}{T}} + \sqrt{p^* m^{-\mu}} \right) \), where the leading term \( \sqrt{\frac{p^* m N}{T}} \) is as expected, and the rate \( \sqrt{p^* m^{-\mu}} \) is due to the fact that the truncation residual \( x_{it}' \Delta_m(z) \) is increasing with \( p^* \) in the HD setting.

Note that all central limit theorems established so far only apply to the finite dimensional case. For the HD case, one cannot derive any asymptotic normality for estimates of interest unless some transformation is further employed (see Huang et al., 2008 for examples using i.i.d. data). However, performing a similar transformation in the presence of weak cross-section dependence and serial correlation is mathematically more involved, and thus is beyond the scope of this study.

In summary, in either case (LD or HD), when \( \Pr(S_{\hat{\lambda}} = A^*) \rightarrow 1 \), all zero coefficient functions can be identified. In the growth regression context, this is equivalent to saying that all variables not driving economic growth can be identified and thus removed from the growth regression. Moreover, the varying coefficients can be recovered using the sieve method, and the factor structure can be estimated by the PCA technique. Thus, all the three aforementioned issues that are prominent in the empirical growth literature (i.e., variable selection, parameter heterogeneity, and cross-sectional dependence) can be addressed simultaneously within a single integrated framework. Before moving on to the empirical analysis, we next describe the data employed in this study.

4 Data

Of the many variables that have been found to be significantly correlated with growth in the literature, we select 61 variables using the following criteria. The first derives from our aim of obtaining comparable results with the existing literature, and the second comes from the fact that we need to work with a balanced panel. With these restrictions, the total size of our data set becomes 62 variables (including the dependent variable, the growth rate of
per capita GDP) for 88 countries covering the period 1960 to 2014. The data set contains countries in different stages of development and with a wide geographic dispersion. The explanatory variables cover a wide range of different factors, including data on economic development, social issues, health, geography, politics, education and more. The variable names, their means, and standard deviations are presented\textsuperscript{2} in Table 1. Table 2 provides a list of the included countries.

A common practice in the literature is to take a five-year simple moving average of both dependent and independent variables\textsuperscript{3}. This technique has the advantages of reducing the potential effects of short-term fluctuations and maintaining a high number of time series observations. Despite these advantages, this technique may still suffer from reverse causality or simultaneity, because causality between regressors and growth could go the other way as well or some regressors and growth may be simultaneously determined (e.g., Bils and Klenow, 2000). To mitigate this problem, we deviate from the common practice by measuring dependent and independent variables differently. Specifically, while the dependent variable is measured as a five-year moving average of economic growth, all explanatory variables are measured at the beginning of each five year period, with the exception of the variables related to war, geography, and terms of trade\textsuperscript{4} (Salimans, 2012). These latter explanatory variables are expected to be truly independent of contemporaneous economic growth, and thus also are measured as five-year moving averages (as with the dependent variable). This treatment further alleviates endogeneity, which is already mitigated by the use of multi-factor error structure as discussed in Section 2.

Given that recent literature on economic growth emphasizes the importance of institutional quality (Acemoglu et al., 2008; Acemoglu et al., 2019), we briefly elaborate on the data on institutional quality used in this paper. Three sets of variables are widely used in the literature to measure institutional quality. The first set is the survey indicators of institutional quality from the International Country Risk Guide (Hall and Jones, 1999; Acemoglu et al., 2001). The second set is the “polity2” variable from the Polity IV dataset (Acemoglu et al., 2008, 2019). It measures the degree of constraints on politicians

\textsuperscript{2}We provide detailed data sources in the supplementary Appendix A.

\textsuperscript{3}Another popular method of looking at annual data in empirical growth literature is to use averaged five-year period data. But, as is stressed by Soto (2003) and Attanasio et al. (2000), the use of n-year averages is not suitable because of the lost of information that it implies. In addition, Soto (2003) and Attanasio et al. (2000) point out that attempting to use data on averaged five-year periods severely limits the number of observations to draw from in the data.

\textsuperscript{4}Specifically, these variables include: fraction spent in war (each five-year period); number of war participation (each five-year period); number of revolutions (each five-year period); coups d’état and coup attempts within (each five-year period); time of independence; East Asian dummy; African dummy; European dummy; Latin American dummy; British colony dummy; Spanish colony dummy; landlocked country dummy; percentage of land area in Koeppen-Geiger tropics; percentage of land area within 100 km of ice-free coast; terms of trade; and terms of trade growth.
and politically connected elites through five indicators: intensity of political competition, regulation of political participation, competitiveness of executive recruitment, openness of executive recruitment, and the constraints it places on its chief executive. The third set includes the “civil liberties” and “political rights” from Freedom House (Barro, 1998; Sala-I-Martin, 1997; Acemoglu et al., 2019). The civil liberties index is made up of four subcategories: freedom of expression and belief, associational and organizational rights, rule of law, and personal autonomy and individual rights, while the political rights index is composed of three subcategories: electoral process, political pluralism and participation, and functioning of government. In this paper, we choose not to use the first set of variables, because it only dates back to 1984 (much later than the beginning year of our study period) and also is not available for many of our sample countries. With regard to the other two sets of institutional quality variables, we mainly rely on the third one, while using the second one as a robustness check.

5 Empirical Results

In this section, we present results obtained from the varying coefficient growth regression model with factor structure and sparsity.

5.1 Choices of the Number of Factors, the Development Index and the Truncation Parameter

In Section 3, we assume that the number of factors \( r \) is known. In practice, \( r \) is unknown and has to be estimated. The main tool for estimating the number of factors of large dimensional datasets is the use of information criterion. In view of the fact that there are 59 observable explanatory variables in our case, we follow Ando and Bai (2017) to choose the number of factors by minimizing the following criterion function:

\[
\text{PIC}(r) = \hat{\sigma}^2 \cdot \left( 1 + r \cdot \frac{NT}{NT} \log(NT) \right),
\]

where \( \hat{\sigma}^2 = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \left( y_{it} - x_{it}' \hat{\beta}_m - \hat{f}_t \hat{\gamma}_i \right)^2 \), and for \( \forall r, \hat{\beta}_m, \hat{f}_t \) and \( \hat{\gamma}_i \) are the corresponding estimates obtained using the approach presented in Section 3.

Given the range of the index variable, there are many basis functions we can use (e.g., Dong and Linton, 2018). The choice of the basis functions is equivalent to the choice of

\[^5\text{We also use secondary sources to resolve ambiguous cases or those without data coverage in Freedom House. The secondary sources are the dichotomous measures by Cheibub et al. (2010) and Boix et al. (2013). See Acemoglu et al. (2019).}\]
the kernel functions in the literature of kernel regression. In practice, the results are not really sensitive to the choice of the basis functions, so long as \( R \) defined in the beginning of Section 3 can cover \([\min\{z_{it}\}, \max\{z_{it}\}]\). In this study, we adopt the Hermite functions studied in Dong and Linton (2018), and also provide some intensive simulation studies in the supplementary Appendix A of this paper.

We now turn to the choice of the development index \( z_{it} \). In Section 2, we have specified a varying coefficient growth regression model capable of capturing parameter heterogeneity by means of a development index. Of the possible development indices, output and human capital are believed to be the most important ones in previous studies (e.g., Durlauf and Johnson, 1995; Salimans, 2012). Following those studies, we consider four alternative development indices\(^6\): (1) log initial GDP per capita, (2) initial primary schooling enrolment rate, (3) initial secondary schooling enrolment, and (4) initial higher education enrolment rate.

When choosing among the four alternative development indices and the truncation parameter \( m \), we use the root mean squared error (RMSE) which is consistent with the criterion function used in estimation. Specifically, for each development index and each truncation parameter, we first choose the number of factors and select the regressors. Then we run post selection regression by letting the weight parameters be zero to calculate RMSE. The results are summarized in Table 4. For the purpose of comparison, we also consider the constant coefficient growth regression model with interactive fixed effects (where the coefficient \( \beta_0 \) is a vector of constant), and the varying coefficient (with log initial GDP per capita as the development index) panel data model with only fixed effects. Table 4 shows that the varying coefficient panel data model with log initial GDP per capita as the development index has the smallest RMSEs in general, and thus fits the data best.

Having selected log initial GDP per capita as the development index, we next move on to a more subtle question, i.e., the choice of the truncation parameter. As explained in the simulation study of Section A.2 of the supplementary Appendix A, choosing the optimal \( m \) is theoretically challenging when both the factor structure and variable selection procedure get involved. In practice, we would like to pick an \( m \), which can explain the model as much as possible. Therefore, we turn to look at what is left unexplained under each choice of \( m \). Given \( x_{it} \) explains what are observed, we turn to consider the cumulative variation of the residuals explained by the factors with different truncation parameters. The results are summarized in Table 5. As can be seen, when \( m = 3, 4, 5 \), the number of factors identified by data is 4, 6, 6, respectively. For \( m = 4, 5 \), most variation of the residuals is explained,

\(^6\)More discussions on the index variable are provided in Section A.1.3 of the supplementary Appendix A.
but \( m = 4 \) gives what is unexplained the lowest proportion.

In summary, the model with six factors and log initial GDP per capita as the development index (i.e., \( r = 6, \ m = 4 \) and \( z = \log \) initial GDP per capita) receives the most support from the data. Hence, in what follows we concentrate on the results obtained from this model.

5.2 Estimates of the Common Factors and Loadings

Figure 1 and Figure 2 plot the estimates of the factors identified above and their corresponding loadings respectively. The former shows that all the common factors vary considerably over time with the exception of the first factor which exhibits a relatively small amount of variation during the sample period, while the latter shows that all the factor loadings vary substantially across countries.

5.3 Estimates of the Coefficient Functions of Selected Variables

5.3.1 General Findings

We have identified a number of robust growth determinants (variables), thus broadly supporting the “optimistic” conclusion of Sala-I-Martin (1997), Fernández et al. (2001) and Sala-I-Martin et al. (2004), that is, some variables are important regressors for explaining cross-country growth patterns. Specifically, we have identified 31 “robust” growth determinants, including initial GDP per capita, institutional quality (civil liberties), human capital, trade, and macroeconomic policies, as well as natural resources, geographical characteristics, colonial origin, religion, and war-related variables. Since the coefficients in our specification are functions of log initial GDP per capita, we report in Table 3 the estimated coefficient function for each of the 31 robust growth determinants at \( \ln(\text{initial GDP per capita}) = 3.98 \) (minimum), 5, 6, 7, 8, and 8.81 (maximum), together with their associated 95% bootstrapped confidence intervals (CI)\(^7\). To offer a more detailed look at these coefficient functions, we also plot each of them against log initial GDP per capita in Figure A.2 of the supplementary Appendix A.

\(^7\)Note that these confidence intervals need to be interpreted carefully. As well understood, one cannot establish the confidence intervals for the estimates under HD case unless certain transformation is further employed (e.g., Huang et al., 2008). However, if one regards 31 (the number of selected variables) as a relatively small number, then one can treat our regression as a LD case and employ the same bootstrap procedure as in Su et al. (2015). In order to ensure the validity of the bootstrap procedure, stronger assumptions on the error terms are needed. For example, one can employ the martingale difference type of assumptions (see Assumption A.4 of Su et al., 2015), or simply assume that the error terms are i.i.d. over both \( i \) and \( t \). Generally speaking, when the error term exhibits both cross-sectional and serial correlation, the bootstrap results are not reliable or incorrect.
Although our results broadly support the conclusion of previous studies, we note three differences in results between this and previous studies. First, our set of robust growth determinants differs from those identified in previous studies, in spite of many overlaps between them. Specifically, some variables appear to be robust in our study but not in previous studies (such as secondary school enrolment rate and terms of trade growth) or vice versa (such as primary school enrolment rate and fraction of mining in GDP). There are three possible reasons for this difference: (1) we use a different model (i.e., we allow for parameter heterogeneity and cross-sectional dependence); (2) we use a different variable selection method (i.e., we use a LASSO method while most previous studies use Bayesian averaging methods); and (3) we use a different dataset (i.e., our dataset spans a longer time period and covers a slightly different set of countries).

Second, our estimated coefficients vary considerably across countries according to their level of development, while those in most previous studies are identical across countries. Specifically, some of our estimated coefficients have the same sign but different values across different levels of initial GDP per capita (such as civil liberty, terms of trade growth, and percentage of land area in tropics), while others not only have different signs but also different magnitudes across different levels of development (such as consumption share of government, life expectancy, military expenditure, and OPEC dummy). These results suggest that it is inappropriate to apply growth regressions with homogeneous parameters.

Third, our estimated coefficients reveal some cross-country patterns not found in previous studies. Taking the coefficient of initial GDP per capita for example, while it is negative for all other countries, it is positive for countries with GDP per capita between $1,780 and $2,117 in 1960 U.S. dollars (between $13,166 and $15,665 in 2010 U.S. dollars). This finding is in line with the “middle income trap” hypothesis suggesting that some developing countries get stuck at middle-income levels and fail to advance into high-income countries (Eichengreen et al., 2014). To give another example, our estimated coefficient of oil reserve starts out negative, increases monotonically with GDP per capita, and then eventually becomes positive for economies with initial GDP per capita above $2,175 in 1960 U.S. dollars ($16,094 in 2010 U.S. dollars). This latter finding is consistent with previous studies (e.g., Leite and Weidmann, 1999) that stresses the role of economic institutions in determining the effects of natural resources on growth. Specifically, these studies suggest that in developed economies where economic institutions are generally well-developed, natural resources tend to promote economic growth, whereas in developing economies where economic institutions are generally weak, natural resources tend to hamper economic growth. We will discuss these two examples in more details below.
5.3.2 Specific Findings

Due to space limitations, we will not discuss all of the 31 robust growth determinants in details but, instead, will concentrate on some of the robust growth determinants that have received relatively more attention in the literature, such as initial GDP per capita, price for investment goods, human capital, natural resources, trade, religion, and institutional quality.

Figure 3.1 presents our estimate of the coefficient of initial GDP per capita. This figure reveals three findings. First, this coefficient is negative for most GDP per capita levels, largely supporting the conditional convergence hypothesis. Second, the coefficient has an inverse U-shaped relationship with initial GDP per capita. This finding is consistent with Durlauf et al. (2001) who finds that the coefficient of initial GDP per capita do not exhibit any sort of monotonicity with respect to level of development. It is also in line with Salimans (2012) who finds that coefficient of initial GDP per capita first increases with level of development up to a point and then declines afterwards. Third, the coefficient is positive for countries with GDP per capita between $319 and $1,812 in 1960 U.S. dollars (or between $2,554 and $14,510 in 2014 U.S. dollars), suggesting that these middle-income countries have failed to catch up with more developed countries. As an example, we find two countries in our sample (South Africa and Colombia) have never been able leave the “middle-income range” over the entire sample period since their GDP per capita fell into this range at the beginning of the sample period (i.e., 1960). This finding is consistent with the “middle income trap hypothesis”, which postulates a dim chance for middle income countries to advance to high-income status (e.g., Eichengreen et al., 2014).

Figure 3.2 shows the coefficient of price for investment goods. Three findings emerge from this figure. First, this coefficient is negative for countries with initial GDP per capita up to $543 in 1960 U.S. dollars (or $4,348 in 2014 U.S. dollars), suggesting that for these countries a relative low price of investment goods in the first year of each five-year period is strongly and positively related to subsequent income growth. This finding is not surprising because a low investment price stimulates investment (including investment in machinery and equipment), which further spurs economic growth (De Long and Summers, 1991, 1992). Second, the coefficient falls in absolute value as initial GDP per capita increases, meaning that the marginal effect of investment price on growth is stronger for poor countries than for rich countries. This latter finding is consistent with Temple (1999) who finds that the growth-spurring effects of investment is greater for developing countries. Such a finding may be because of the all-encompassing nature of the equipment and machinery category in the data. It is likely that purchase of any type of machinery will be of assistance in
developing countries whereas growth in developed countries may be fostered more by the purchase of innovative equipment that is subsumed within the equipment and machinery category (ab Iorwerth, 2005). Third, for countries with initial GDP per capita above $543 in 1960 U.S. dollars (or $4,348 in 2014 U.S. dollars) the estimated coefficient of investment goods price is positive but insignificant, because the associated confidence intervals contain zero. This suggests that investment goods price has no growth effects for these countries. A possible reason is that the data from the Penn World Table is disaggregated enough to distinguish price of equipment investment, which has strong growth effects, and price of other forms of investment, which have little growth effects (De Long and Summers, 1991).

Figure 3.3 shows the coefficient of secondary schooling enrolment. Two findings stand out from this figure. First, for most low income countries secondary schooling enrolment is either statistically or economically insignificant. This finding can be possibly explained by Bils and Klenow (2000)’s argument on reverse causality (i.e., from growth to education). Specifically, Bils and Klenow (2000) find that the impact of schooling on growth can explain a small fraction of the empirical relationship between education and growth, while reverse causality (i.e., from growth to education) possibly accounts for most of the relationship. In our case where reverse causality and omitted variables are partially removed by the use of lagged independent variables and multi-factor structure, the coefficient of secondary schooling is likely to mainly reflect the impact of secondary schooling on growth instead of the reverse. This may in turn suggest that the impact of secondary schooling on growth may indeed be quite small for low income countries. A possible reason for this small impact is the low quality of secondary education in these countries that fails to translate additional years of secondary schooling into better cognitive skills (Hanushek and Wößmann, 2012). The second finding that emerges from Figure 3.3 is that for most high income countries secondary schooling enrolment has a positive, strong, and statistically significant effect on growth. One plausible reason for this positive strong effect is that high income countries in general have much better quality of secondary education (Hanushek and Wößmann, 2012). The strong impact of higher education may also be partially explained by the argument of Bils and Klenow (2000). Specifically, although reverse causality is partially mitigated in this study by using lagged secondary schooling enrolment, it cannot be completely ruled out, thereby causing the coefficient of secondary schooling enrolment for high income countries to appear stronger than it actually is.

Figure 3.4 shows that the coefficient of higher education enrolment. Three findings emerge. First, this coefficient is lower than the coefficient of secondary schooling enrolment at any level of development. This finding is consistent with the concavity argument
which suggests that labour market returns are characterized by a concave relationship with education that implies decreasing returns to additional years of school (Psacharopoulos and Sanyal, 1982; Psacharopoulos and Patrinos, 2004). Second, the coefficient of higher education enrolment is lower for low income countries than for high income countries. One likely reason is the low quality of higher education in low income countries that fails to translate additional years of higher education into an increase in human capital. For example, studies find that in Middle East and North Africa countries public higher education institutions tend to focus on the production of credentials rather than the mix of skills demanded in a competitive private-sector-led economy (Psacharopoulos and Patrinos, 2018; Assaad, 2014). Another possible reason is that a high proportion of higher education graduates in low income countries choose to work in the public sector\(^8\) because this sector provides a fertile ground for engaging in rent-seeking activities, which yield very high private returns but negative social returns (Pritchett, 2001; Owusu, 2005, 2006). A third finding that emerges from Figure 3.4 is that the coefficient of higher education enrolment is negative for almost all countries with the exception of middle income countries. This finding is consistent with Salimans (2012) who, using a Bayesian model averaging method that allows for parameter heterogeneity, finds that the effect of higher education is negative for 90% of his sample countries and positive for the rest. It is also broadly consistent with Sala-I-Martin et al. (2004) who, using a Bayesian model averaging method that does not allow for parameter heterogeneity, finds a negative effect of higher education. While the negative effect of higher education for low income countries is understandable as explained above, that for high income countries is counter-intuitive and may reflect the crudity of the higher education enrolment measure. Specifically, “higher education” is not clearly defined as to what types of schools (e.g., community colleges, vocational schools, trade schools, liberal arts colleges, and universities) are included and what forms of training and skills acquisition are fostered (Becker, 1992). In addition, as is well known in the empirical growth literature, years of schooling is a crude measure of schooling experience, as it ignores the influence of student–teacher ratios, teacher experience, peer grouping, curricula, and other factors that are believed to influence the amount of knowledge and skills embodied in individuals during the years of schooling (Becker, 1992).

Figure 3.5 shows the coefficient of oil reserve. Two findings from this figure are noteworthy. First, this coefficient is negative for countries with initial GDP per capita below $2,373 in 1960 U.S. dollars (or $19,002 in 2014 U.S. dollars). This finding is consistent with the “natural resources curse hypothesis” (e.g., Sachs and Warner, 2001) and can be

\(^8\)For example, according to Darvas et al. (2017), in Sub-Saharan countries approximately 50% of higher education graduates were employed by the public sector in 2011. This figure is even higher prior to 2011.
explained by the rent-seeking behaviour of countries with large endowments of natural resources. Second, this coefficient increases monotonically with GDP per capita and eventually becomes positive for economies with initial GDP per capita above $2,373 in 1960 U.S. dollars (or $19,002 in 2014 U.S. dollars). This latter finding is line with recent studies (e.g., Leite and Weidmann, 1999) that suggest that the contribution of natural resources to a country’s economy does not take place in isolation, but rather in the overall context of the country’s economic management and institutions. Put differently, it is the quality and competency of these policies and institutions that determines whether natural resources can promote economic growth. Specifically, in developed economies where economic institutions are generally well-developed, natural resources tend to promote economic growth; whereas in developing economies where economic institutions are generally weak, natural resources tend to hamper economic growth.

Figure 3.6 presents the coefficient of terms of trade growth. This coefficient is positive for all countries, suggesting that growth tends to be faster in countries where terms of trade growth is higher. This finding is consistent with previous studies (Bleaney and Greenaway, 2001) that find that an improvement in terms of trade leads to higher levels of investment and hence long-run economic growth. In addition, this figure shows that this coefficient increases with GDP per capita, indicating that the marginal effect of terms of trade growth is larger in richer countries than in poorer ones. This latter finding is consistent with recent studies (e.g., Blattman et al., 2007) that suggest that higher volatility in terms of trade reduces investment and hence growth because of aversion to risk. Therefore, terms of trade instability is likely to have a smaller negative impact on rich countries because these countries have more sophisticated institutions and markets and thus are more likely to have cheaper ways to insure against price volatility than poor countries.

Here we note that as in Sala-I-Martin et al. (2004), trade openness (defined as exports plus imports as a share of GDP) is insignificant, presumably reflecting the crudity of this measure, and perhaps the distinction between opening to international trade generating a one-time step increase in income as factors are reallocated according to comparative advantage versus an ongoing growth impact associated with greater openness.

Figure 3.7-3.9 present the coefficients of fraction of Christian, Muslim, and Jewish respectively. These coefficients are negative at all levels of development or nearly all levels of development. This finding is consistent with Barro and McCleary (2005) who find that religion works via belief, not practice. They argue that higher church attendance uses up time and resources and eventually runs into diminishing returns. The “religion sector”, as they call it, can consume more than it yields.
We are also very interested in the two institutional quality variables (i.e., civil liberties and political rights). From Figure A.2 of the supplementary Appendix A, we see that civil liberties is a significant growth determinant while political rights is not. This finding is consistent with Acemoglu et al. (2019) who finds that among the components of democratic institutions that matter for growth, civil liberties is the most important. For readers’ convenience, we reproduce the estimated coefficient of civil liberties in Figure 3.10. As can be seen, this coefficient is positive for nearly all level of GDP per capita, thus being consistent with Acemoglu et al. (2019) who find that democratic institutions are not only good for developed economies but also for low income economies. To investigate the robustness of this result, we re-estimated our model using an alternative measure of institutional quality — the “polity2” variable from the Polity IV dataset. Our results indicate that “polity2” is still identified as a significant growth determinant. In addition, Figure A.3 shows that the estimated coefficient of “polity2” is still statistically non-negative for most of the sample countries, confirming that institutions are important for growth. Here we note that the magnitude of the estimated coefficient of “polity2” is much smaller than that of civil liberties. The finding is consistent with that of BenYishay and Betancourt (2010) who find that civil liberties (i.e., individual freedoms and rights) dominates constraints on politicians in every possible comparison in terms of predictive performance and statistical significance.

6 Robustness Check

In order to further confirm the above empirical results, we conduct a number of robustness checks. First, we examine the sensitivity of our results by considering alternative choices of $\Upsilon_{NT}$. Second, we test for autocorrelation and cross-sectional dependence in residuals in order to investigate the validity of our bootstrap procedure. Finally, we consider longer lags for initial GDP per capita to assess if the above findings carry through.

6.1 Different Choices of $\Upsilon_{NT}$

We consider three alternative choices for $\Upsilon_{NT}$: (1) $\frac{\ln \xi_{NT}}{\sqrt{\xi_{NT}}}$, (2) $\frac{\ln(N+T)}{\sqrt{N+T}}$, (3) $\frac{10}{\sqrt{\xi_{NT}}}$. Note that these three choices are taken from the HD case in our simulation studies presented in supplementary Appendix A, where the three choices of $\Upsilon_{NT}$ have been proved to be reasonable. Our results show that the variables selected using each of the three alternative $\Upsilon_{NT}$’s, their associated coefficients, and the number of factors are identical to those presented in Section 5, indicating that our empirical results are robust to different choices of $\Upsilon_{NT}$.
6.2 Testing for Autocorrelation and Cross-Sectional Dependence in Residuals

As discussed in the footnote of Section 5.3.1, the validity of the bootstrap procedure used for generating the confidence intervals for the coefficient functions hinges on some assumptions concerning the error terms. For example, Su et al. (2015) employ the martingale difference type of assumptions (see Assumption A.4 of their paper). Alternatively, one can assume that the error terms are cross-sectionally independent. Put differently, as long as we can show that correlation along either of the two dimensions (the time and cross-sectional dimensions) is negligible, the confidence intervals produced using the bootstrap procedure are reliable if one regards the number of selected variables (i.e., 31) as relatively small.

Specifically, we conduct two tests: (1) the Ljung-Box Q-Test to test for autocorrelation in each of the time series \( \{\hat{\varepsilon}_1, \ldots, \hat{\varepsilon}_{iT}\} \); and (2) a cross-sectional dependence (CD) test to examine cross-sectional dependence in residuals. The test statistic of this latter test is of the following form:

\[
CD = \sqrt{\frac{2T}{N(N-1)}} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \rho_{ij},
\]

where \( \rho_{ij} \) measures the correlation between \( \{\hat{\varepsilon}_1, \ldots, \hat{\varepsilon}_{iT}\} \) and \( \{\hat{\varepsilon}_j, \ldots, \hat{\varepsilon}_{jT}\} \). See Juodis and Reese (2019) for more details on the CD test.

Our results from the Ljung-Box Q-Test show that 50.56% individual countries reject the null hypothesis of no autocorrelation at 1% significance level, while our statistic from the CD test is 0.2876, thus failing to reject the null that there is no cross-sectional independence. These results suggest that correlation along the cross-sectional dimension is negligible due to the use of the factor structure, thus ensuring the validity of the bootstrap procedure used in our empirical study.

6.3 Initial GDP Per Capita with Longer Lags

In Section 5 the dependent variable is measured as a five-year moving average of economic growth, while the explanatory variables (including initial GDP per capita) are measured at the beginning of each five-year period with the exception of the variables related to war, geography, and terms of trade (Salimans, 2012). In other words, in Section 5 initial GDP per capita is lagged by three years. However, as is well known in the literature, initial GDP per capita may have a much longer effect on GDP growth. Therefore, in the subsection
we check the robustness of our empirical results by using longer lags for initial GDP per capita. Specifically, we lag initial GDP per capita by 20 years whenever it is possible. For the first nineteen periods initial GDP per capita lagged by 20 years is not available, we use the initial GDP per capita in 1960. We also estimate the model using GDP per capita lagged by 15 and 25 years respectively, and the results are qualitatively similar to those reported below. Thus, in what follows we concentrate on results of the lag length of 20.

As shown in Figure A.4 of the supplementary Appendix A, our findings using lag length of 20 are, in general, consistent with those presented in Section 5. Specifically, a comparison of this figure and Figure A.2 reveals that most of the variables selected here coincide with those selected in Section 5, implying that most of the selected variables are robust to the choice of lag. More specifically, there are 21 variables that are selected in both cases. These variables include initial GDP per capita, institutional quality (civil liberties), high school enrolment, terms of trade growth, etc. In addition, we find that these variables exhibit cross-country patterns very similar to those obtained in Section 5. For example, the coefficient of initial GDP per capita is negative for all countries with the exception of middle income countries, confirming the “middle-income trap hypothesis” discussed above. To give another example, the coefficient of civil liberties is positive for nearly all countries, confirming that institutional quality is important for growth.

7 Conclusion

A rigorous cross-country growth regression analysis should simultaneously account for three major problems identified in the literature — variable selection, parameter heterogeneity, and cross-sectional dependence. Though these three problems have received individual attention, little or no research has sought to integrate them into a single, comprehensive framework. The purpose of this study is to fill this void by proposing a new, integrated framework that is capable of dealing with parameter heterogeneity and cross-sectional dependence, while simultaneously performing variable selection. Specifically, parameter heterogeneity is allowed for by means of a varying coefficient growth regression model, while cross-sectional dependence is introduced into the model via a multi-factor structure. For simplicity, we refer to the resulting growth regression model as the “varying coefficient growth regression model with factor structure and sparsity”. We then propose a LASSO estimator that is capable of performing variable selection on this model. In addition, we have established the associated asymptotic results for this estimator and further investigate the performance of the estimator by conducting extensive simulations.
We apply the above framework to a new data set that covers 89 countries over the period from 1960 to 2014. We have identified 31 robust growth determinants, providing evidentiary support for the canonical neoclassical growth variables; i.e., initial income, investment, and population growth, as well as macroeconomic policies, geography, institutions, religion and ethnic fractionalization. Moreover, we find that all the coefficients of the robust growth determinants vary considerably across countries according to their level of development, which reveals some interesting cross-country patterns not found in previously.

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### Table 1: Definitions of All Variables in the Regression

| Variables | Description | Formula | Mean  | Std   |
|-----------|-------------|---------|-------|-------|
| **EG**    | Economic growth rate | ln(rgdpo_t/rgdpo_{t-1}) | 0.0363 | 0.0649 |
| log(GPC)  | log GDP per capita | 6.0642 | 0.9861 |
| csh_g     | Government consumption share | 0.2074 | 0.1163 |
| Openness  | Openness measure | csh_x + csh_m | -0.0322 | 0.1274 |
| IP        | Investment price, i.e., price level of capital formation | ln(pop_t/pop_{t-1}) | 0.0197 | 0.0127 |
| PGR       | Population growth rates | 0.7396 | 0.2170 |
| Sch_P     | Primary school enrolment | 0.4762 | 0.3204 |
| Sch_S     | Secondary school enrolment | 0.1336 | 0.1581 |
| Sch_H     | Higher education School Enrolment | 0.5932 | 0.1102 |
| LE        | Life expectancy | 0.0399 | 0.0296 |
| PESS      | Public education spending share in GDP | GFCF - GFCF_PS | 0.0713 | 0.0501 |
| PIS       | Public investment share | Exports_OM + Exports_ARM | 0.9789 | 0.2085 |
| Land      | Land area (sq. km / 1,000,000) | 0.8719 | 2.1518 |
| Mining    | Fraction GDP in mining | Mining | 0.0733 | 0.0874 |
| Fertility | Fertility rate, total (births per woman) | 4.7412 | 1.9724 |
| Military  | Military expenditure share in GDP | 0.0295 | 0.0302 |
| PCS       | Public consumption share | GGFCE - PESS - Military | 0.0809 | 0.0501 |
| Malaria   | Malaria prevalence: Incidence of malaria (per 1,000 population at risk) | 155.1712 | 245.5145 |
| Inflation | Inflation rate | 0.0794 | 0.0487 |
| Political | Political rights | 4.2100 | 1.9429 |
| Civil     | Civil liberties | 4.1496 | 1.6628 |
| Cap       | Degree of capitalism | 3.2697 | 1.7275 |
| Trade     | Terms of trade | 1.3475 | 2.2807 |
| Tra_Gro   | Terms of trade growth | 0.0073 | 0.0974 |
| Locked    | Landlocked country dummy (1, yes; 0, no) | 0.2967 | 0.4438 |
| Ind_Year  | Time of independence | 1.4382 | 1.0382 |
| kgatr     | Percentage of land area in Koeppen-Geiger tropics | 0.4017 | 0.4205 |
| kgptr     | Percentage of population in Koeppen-Geiger tropics | 0.3915 | 0.4217 |
| lcr100km  | Percentage of land area within 100 km of ice-free coast | 0.3788 | 0.3640 |
| pop100cr  | Ratio of population within 100 km of ice-free coast/navigable river to total population | 0.4520 | 0.3728 |
| cen_lat   | Latitude of country centroid | 0.1522 | 0.2197 |
| Bri_Col   | British colony dummy (1, yes; 0, no) | 0.2584 | 0.4378 |
| Spa_Col   | Spanish colony dummy (1, yes; 0, no) | 0.1910 | 0.3931 |
| Variable   | Description                                                                 | Coefficient 1 | Coefficient 2 |
|------------|------------------------------------------------------------------------------|---------------|---------------|
| Oil_OPEC   | Oil-producing country dummy (1, yes; 0, no)                                  | 0.0674        | 0.2508        |
| Gas        | Proved reserves (cubic meters / 10^12)                                      | 1.3789        | 6.1997        |
| Oil        | Proved reserves (bbl / 10^9)                                                | 4.5521        | 19.1378       |
| Chris      | Percentage of Christian                                                     | 0.5369        | 0.3807        |
| Mus        | Percentage of Muslim                                                        | 0.3046        | 0.3825        |
| Hin        | Percentage of Hindu                                                         | 0.0263        | 0.1211        |
| Bud        | Percentage of Buddhist                                                      | 0.0410        | 0.1541        |
| Fol        | Percentage of Folk religion                                                 | 0.0284        | 0.0628        |
| Oth        | Percentage of other religion                                                | 0.0037        | 0.0064        |
| Jew        | Percentage of Jewish                                                        | 0.0019        | 0.0027        |
| GS         | Government spending share of GDP                                            | 0.1501        | 0.0686        |
| Distortion | Real exchange rate distortions                                              | 129.6824      | 35.8479       |
| OO         | Outward orientation                                                        | -2.7398       | 0.7542        |
| SIL        | Ethnolinguistic fractionalization                                           | 0.4886        | 0.3127        |
| ESP        | English-speaking population in percentage                                   | 0.1762        | 0.2692        |
| EA         | East Asian dummy                                                            | 0.0225        | 0.1482        |
| AF         | African dummy                                                               | 0.4270        | 0.4947        |
| EU         | European dummy                                                              | 0.1124        | 0.3158        |
| LA         | Latin American dummy                                                        | 0.1573        | 0.3641        |
| WarFrac    | Fraction spent in war (1960-2014)                                           | 0.3265        | 0.4343        |
| NoWars     | No. of war participation (1960-2014)                                        | 0.8028        | 1.2609        |
| Coup       | coups d’etat and coup attempts within (1960-2014)                          | 0.1870        | 0.4964        |
| Revolution | Number of revolutions (1960-2014)                                           | 0.1941        | 0.5038        |
| Pop_Dens   | Population density/1000                                                     | 0.0812        | 0.1135        |
| WorkIR     | Growth rate of work force                                                   | ln(WP_t/WP_{t-1}) | 0.0210 | 0.0132 |
| Polity2    | Combined polity score                                                       | 0.1132        | 6.5051        |

rgdpo — Size of economy (GDP in million)

pop — Population (in million)

csh_x — Share of merchandise exports

csh_m — Share of merchandise imports

WP — Fraction population of work force (1-A65-U15)

A65 — Fraction population over 65 years old

U15 — Fraction population under 15 years old

GFCF — Gross fixed capital formation

GFCF_PS — Gross fixed capital formation, private sector

Exports_OM — Percentage of ores and metals exports

Exports_ARM — Percentage of agricultural raw materials exports

GGFCE — General government final consumption expenditure share in GDP
Table 2: Sample Countries and Their Associated ISO 3166-1 alpha-3 Codes

| Country          | ISO Code | Name                                               |
|------------------|----------|----------------------------------------------------|
| AGO              | HND      | Honduras                                           |
| ALB              | HRV      | Albania                                            |
| ARM              | HTI      | Armenia                                            |
| AZE              | IND      | Azerbaijan                                         |
| BDI              | IRN      | Burundi                                            |
| BEN              | JAM      | Benin                                              |
| BFA              | JOR      | Burkina Faso                                       |
| BGD              | JPN      | Bangladesh                                         |
| BGR              | KAZ      | Bulgaria                                           |
| BLR              | KEN      | Belarus                                            |
| BRA              | KGZ      | Brazil                                             |
| BWA              | KHM      | Botswana                                           |
| CAF              | LAO      | Central African Republic                           |
| CIV              | LBN      | Côte d’Ivoire                                      |
| CMR              | LKA      | Cameroon                                           |
| COG              | LSO      | Congo                                              |
| COL              | MDA      | Colombia                                           |
| DOM              | MDG      | Dominican Republic                                 |
| DZA              | MEX      | Algeria                                            |
| ECU              | MKD      | Ecuador                                            |
| EGY              | MLI      | Egypt                                              |
| ETH              | MNG      | Ethiopia                                           |
| GAB              | MOZ      | Gabon                                              |
| GBR              | MWI      | United Kingdom                                     |
| GEO              | MYS      | Georgia                                            |
| GHA              | NAM      | Ghana                                              |
| GIN              | NER      | Guinea                                             |
| GMB              | NIC      | Gambia                                             |
| GNB              | NPL      | Guinea-Bissau                                       |
| GTM              | OMN      | Guatemala                                          |
| AGO              | HND      | Angola                                             |
| ALB              | HRV      | Albania                                            |
| ARM              | HTI      | Armenia                                            |
| AZE              | IND      | Azerbaijan                                         |
| BDI              | IRN      | Burundi                                            |
| BEN              | JAM      | Benin                                              |
| BFA              | JOR      | Burkina Faso                                       |
| BGD              | JPN      | Bangladesh                                         |
| BGR              | KAZ      | Bulgaria                                           |
| BLR              | KEN      | Belarus                                            |
| BRA              | KGZ      | Brazil                                             |
| BWA              | KHM      | Botswana                                           |
| CAF              | LAO      | Central African Republic                           |
| CIV              | LBN      | Côte d’Ivoire                                      |
| CMR              | LKA      | Cameroon                                           |
| COG              | LSO      | Congo                                              |
| COL              | MDA      | Colombia                                           |
| DOM              | MDG      | Dominican Republic                                 |
| DZA              | MEX      | Algeria                                            |
| ECU              | MKD      | Ecuador                                            |
| EGY              | MLI      | Egypt                                              |
| ETH              | MNG      | Ethiopia                                           |
| GAB              | MOZ      | Gabon                                              |
| GBR              | MWI      | United Kingdom                                     |
| GEO              | MYS      | Georgia                                            |
| GHA              | NAM      | Ghana                                              |
| GIN              | NER      | Guinea                                             |
| GMB              | NIC      | Gambia                                             |
| GNB              | NPL      | Guinea-Bissau                                       |
| GTM              | OMN      | Guatemala                                          |

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Table 3: Estimates of Coefficients at log(GPC)=3.98, 5, 6, 7, 8 and 8.81

| log(GPC) | log(GPC)=3.98 | log(GPC)=5  | log(GPC)=6  | log(GPC)=7  | log(GPC)=8  | log(GPC)=8.81 |
|---------|-------------|-------------|-------------|-------------|-------------|---------------|
| log(GPC) | -0.0766     | -0.0447     | -0.0117     | 0.0255      | -0.0531     | -0.2055       |
|         | (-0.1145, -0.0429) | (-0.0618, -0.0305) | (-0.0038, 0.0249) | (0.0080, 0.0469) | (-0.0886, -0.0034) | (-0.2883, -0.0898) |
| csh_g   | 0.3341      | 0.1483      | -0.0587     | -0.2143     | -0.0531     | -0.2055       |
|         | (0.1077, 0.5534) | (0.0706, 0.2248) | (-0.1110, -0.0107) | (-0.3141, -0.1454) | (-0.5782, 0.0082) | (-0.9416, 0.4733) |
| IP      | -0.1284     | -0.0545     | -0.0093     | 0.0160      | 0.0282      | 0.0325        |
|         | (-0.1971, -0.0621) | (-0.0686, -0.0422) | (-0.0191, 0.0059) | (0.0046, 0.0406) | (-0.0146, 0.0638) | (-0.0895, 0.1209) |
| PGR     | 3.2821      | 0.5353      | 0.8485      | 1.5689      | 0.6502      | -2.2853       |
|         | (1.7325, 4.7072) | (0.0426, 0.9486) | (0.4054, 1.2542) | (1.1371, 1.9781) | (-0.1228, 1.3620) | (-3.5441, -0.7206) |
| School_S| 0.1322      | 0.0823      | 0.2220      | 0.0326      | -0.2066     | -0.1808       |
|         | (0.0108, 0.1495) | (-0.0078, -0.0801) | (-0.0548, 0.0177) | (0.1989, 0.4201) | (0.5495, 1.1719) | (-0.3986, 0.1401) |
| School_H| -1.5869     | -0.1175     | 0.2220      | 0.0326      | -0.2066     | -0.1808       |
|         | (-2.4454, -0.7631) | (-0.3214, 0.1119) | (0.1562, 0.3020) | (-0.0157, 0.0921) | (-0.2810, -0.1039) | (-0.3986, 0.1401) |
| LE      | 0.3148      | 0.2935      | -0.0597     | -0.2370     | 0.1455      | 1.0373        |
|         | (-0.0171, 0.6403) | (0.1811, 0.3938) | (-0.1739, 0.0389) | (-0.3918, -0.0990) | (-0.2875, 0.5411) | (0.0000, 1.9922) |
| PESS    | -2.4990     | -0.5737     | -0.2476     | -0.0158     | 0.3070      | 0.7260        |
|         | (-2.9450, 0.0107) | (-0.7969, -0.3513) | (-0.4210, -0.0944) | (-0.2163, 0.1716) | (-0.6249, 1.3302) | (-1.5776, 3.4530) |
| Military| -1.1748     | -0.1436     | -0.0428     | -0.0020     | 0.6417      | 1.9197        |
|         | (-2.4253, -0.0639) | (-0.3588, 0.1047) | (-0.1667, 0.0719) | (-0.2609, 0.1609) | (0.0134, 1.1085) | (0.1982, 3.4369) |
| Inflation| -0.0055     | 0.0006      | -0.0011     | -0.0031     | 0.0007      | 0.0107        |
|         | (-0.0094, -0.0012) | (0.0000, 0.0012) | (-0.0016, -0.0005) | (-0.0039, -0.0023) | (-0.0043, 0.0048) | (-0.0020, 0.0209) |
| Civil   | 0.0183      | 0.0050      | -0.0004     | 0.0033      | 0.0166      | 0.0340        |
|         | (0.0048, 0.0308) | (0.0011, 0.0085) | (-0.0027, 0.0024) | (0.0000, 0.0064) | (0.0027, 0.0282) | (-0.0036, 0.0661) |
| Tra_Gro | 0.0009      | 0.0304      | 0.0048      | 0.0018      | 0.0796      | 0.2279        |
|         | (0.0117, 0.0458) | (0.0117, 0.0458) | (-0.0162, 0.0252) | (-0.0248, 0.0351) | (-0.0062, 0.1727) | (-0.0124, 0.4799) |
| kgatr   | -0.3099     | -0.0815     | -0.0763     | -0.1422     | -0.1597     | -0.0776       |
|         | (-0.4367, -0.1693) | (-0.1281, -0.0391) | (-0.1161, -0.0447) | (-0.1887, -0.0969) | (-0.2602, -0.0333) | (-0.3270, 0.2396) |
| lcr100km| 0.6793      | 0.1930      | 0.0511      | -0.0621     | -0.3850     | -0.9133       |
|         | (0.5005, 0.8498) | (0.1396, 0.2461) | (0.0127, 0.0940) | (-0.1305, -0.0005) | (-0.5977, -0.2168) | (-1.4617, -0.5189) |
| cen_lat | 0.5894      | 0.0293      | 0.0360      | 0.1693      | 0.0873      | -0.3058       |
|         | (0.3011, 0.8198) | (-0.0723, 0.1294) | (-0.0315, 0.1114) | (0.0985, 0.2655) | (-0.0526, 0.2459) | (-0.6344, 0.0789) |
| Spa_Col | -0.5174     | -0.1564     | -0.0204     | 0.0548      | 0.1945      | 0.4167        |

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|                |            |            |            |            |            |            |
|----------------|------------|------------|------------|------------|------------|------------|
| Oil_OPEC       | (-0.6748, -0.3222) | (-0.2111, -0.1080) | (-0.0503, 0.0091) | (0.0171, 0.0950) | (0.1040, 0.3131) | (0.1867, 0.6800) |
| Oil            | -0.1254    | -0.0460    | -0.0947    | -0.0815    | 0.1358     | 0.5247     |
| (-1.0835, 0.6246) | (-0.2325, 0.1181) | (-0.1496, -0.0423) | (-0.1383, -0.0344) | (-0.0124, 0.2413) | (0.1090, 0.8785) | 0.0057     |
| Chris          | 0.2689     | 0.1112     | -0.0056    | 0.1262     | -0.2818    | -0.4469    |
| (0.1164, 0.4267) | (0.0650, 0.1711) | (-0.0535, 0.0461) | (-0.2001, -0.0629) | (-0.4824, -0.1333) | (-0.8908, -0.0694) | -0.5692    |
| Mus            | -0.0736    | 0.0130     | -0.0292    | 0.1685     | -0.3744    | -0.5692    |
| (-0.1864, 0.0735) | (-0.0383, 0.0599) | (-0.0774, 0.0195) | (-0.2344, -0.1004) | (-0.5710, -0.2335) | (-1.0085, -0.2474) | 51.3537    |
| Oth            | -3.7386    | 0.6890     | -1.1133    | 2.9016     | 21.6126    | 51.3537    |
| (-13.0771, 4.1928) | (-3.0584, 0.7782) | (-3.0584, 0.7782) | (0.2919, 6.3889) | (11.6269, 32.3911) | (27.3474, 75.2005) | 36         |
| Jew            | -32.4452   | -10.5477   | -2.0163    | -0.9826    | -2.7502    | -3.6114    |
| (-55.7335, -10.0679) | (-18.6022, -3.8609) | (-7.6269, 3.6243) | (-6.9105, 4.0461) | (-12.1471, 5.8912) | (-27.3730, 18.2231) | 36         |
| GS             | -0.2516    | -0.2079    | -0.0476    | -0.0738    | -0.5101    | -1.2583    |
| (-0.5709, 0.0529) | (-0.3047, -0.1254) | (-0.1142, 0.0215) | (-0.1676, 0.0172) | (-0.8230, -0.1881) | (-2.1023, -0.4396) | 0.0577     |
| Distortion     | 0.0231     | 0.0884     | 0.1085     | 0.0482     | -0.1148    | -0.3280    |
| (-0.0449, 0.1048) | (0.0580, 0.1187) | (0.0701, 0.1435) | (-0.0064, 0.1025) | (-0.2793, 0.0061) | (-0.6718, -0.0148) | -1192537   |
| OO             | 1.0569     | 4.1811     | 5.1496     | 2.3016     | 5.4384     | 15.5641    |
| (-2.1588, 4.9579) | (2.7557, 5.6164) | (3.3263, 6.8047) | (-2.2707, 4.851) | (-13.1848, 3.040) | (-31.9253, -0.7429) | 36         |
| ESP            | 0.3819     | 0.1039     | 0.0581     | 0.1010     | 0.1193     | 0.0577     |
| (0.2151, 0.5483) | (0.0575, 0.1465) | (0.0214, 0.0975) | (0.0494, 0.1631) | (-0.0059, 0.2319) | (-0.2260, 0.2967) | 36         |
| EA             | 0.2924     | 0.0841     | 0.0263     | -0.1256    | -0.5515    | -1.1753    |
| (-0.1574, 0.7975) | (-0.0589, 0.2081) | (-0.0668, 0.1185) | (-0.2975, 0.0293) | (-1.0307, -0.1357) | (-2.2484, -0.2191) | 1.2174     |
| EU             | -1.0237    | -0.2206    | -0.0275    | 0.0738     | 0.4780     | 1.2174     |
| (-1.7313, -0.3070) | (-0.3896, -0.0438) | (-0.0797, 0.0197) | (0.0169, 0.1225) | (0.3182, 0.6845) | (0.7527, 1.7964) | 36         |
| WarFrac        | -0.0044    | -0.0035    | -0.0012    | -0.0130    | 0.0058     | 0.0411     |
| (-0.0372, 0.0351) | (-0.0121, 0.0072) | (-0.0179, -0.0040) | (-0.0208, -0.0036) | (-0.0157, 0.0235) | (-0.0227, 0.0882) | 0.0251     |
| Coup           | -0.0190    | -0.0004    | -0.0028    | -0.0070    | 0.0016     | 0.0251     |
| (-0.0287, -0.0089) | (-0.0031, 0.0020) | (-0.0051, -0.0005) | (-0.0102, -0.0038) | (-0.0086, 0.0113) | (-0.0005, 0.0502) | (-0.0034, 0.0464) | 36         |
| Revolution     | -0.0052    | 0.0044     | -0.0016    | 0.0066     | 0.0021     | 0.0252     |
| (-0.0198, 0.0090) | (0.0007, 0.0075) | (-0.0042, 0.0011) | (-0.0106, -0.0020) | (-0.0054, 0.0084) | (-0.0034, 0.0464) | 36         |
Figure 1: Estimates of Common Factors

\( \hat{f}_{t,j} \) stands for the estimate of the \( j \)th factor, where \( j = 1 \ldots, 6 \).

Figure 2: Estimates of Factor Loadings

\( \hat{\gamma}_{t,j} \) stands for the estimate of the \( j \)th factor loading, where \( j = 1 \ldots, 6 \).
Table 4: Comparison among Different Models

| z of Varying Coefficient Model | FE   | CC   |
|--------------------------------|------|------|
| ln(GPC)                       |      |      |
| School P                      | 0.0199 | 0.0222 |
| School S                      | 0.0217 | 0.0277 |
| School H                      | 0.0277 | 0.0346 |
| m = 4                          | 0.0170 | 0.0202 |
| m = 5                          | 0.0160 | 0.0179 |
|                                | 0.0184 | 0.0168 |
|                                | 0.0295 | 0.0220 |

1. “FE” refers to the varying coefficient (with ln(GPC) as the development index) panel data model with fixed effects
2. “CC” refers to the constant coefficient panel data model with interactive fixed effects, where the coefficients are constant and six factors are selected.

Table 5: Cumulative Variation of the Residuals Explained by the Factors with Different Truncation Parameters

| No. Factors | 1   | 2   | 3   | 4   | 5   | 6   |
|-------------|-----|-----|-----|-----|-----|-----|
| m = 3       | 67.43% | 82.83% | 89.65% | 93.45% |      |      |
| m = 4       | 87.86% | 94.87% | 96.65% | 98.10% | 98.71% | 99.01% |
| m = 5       | 71.78% | 88.84% | 93.77% | 95.70% | 97.17% | 97.99% |

Figure 3: Estimates of Selected Coefficient Functions
Appendix A is divided into four sections. Section A.1 discusses some possible extensions, and also provides justification to Assumption 4.2 of the paper. Section A.2 provides a numerical algorithm, and then examines the asymptotic results of Section 3 through several simulations. Section A.3 presents the preliminary lemmas and the proofs of the main theorems. In Section A.4, we provide auxiliary tables and figures of the empirical study.

Recall that in the main text, we have let $\xi_{NT} = \min\{N, T\}$, $\|\cdot\|_{sp}$ be the spectral norm of a matrix, and $\lfloor a \rfloor$ stand for the largest integer part of a real number $a$. Here we further define some notations, which will be used throughout this file. Let $\phi_i^\ast[\beta^\ast] = (x_{i1}^\ast \beta^\ast(z_{i1}), \ldots, x_{iT}^\ast \beta^\ast(z_{iT}))'$, and $\phi_i^\dagger[\beta^\dagger] = (x_{i1}^\dagger \beta^\dagger(z_{i1}), \ldots, x_{iT}^\dagger \beta^\dagger(z_{iT}))'$, where $\beta^\ast(\cdot)$ and $\beta^\dagger(\cdot)$ are $p^* \times 1$ and $(p - p^*) \times 1$ respectively. Moreover, $\text{diag}\{A_1, \ldots, A_k\}$ means constructing block diagonal matrix from matrices (or scalars) $A_1, \ldots, A_k$. $a \lor b$ means $\max\{a, b\}$.

A.1 Extra Discussions

In this section, we discuss three possible extensions of our model.

A.1.1 On Time Trends

As mentioned in the introduction of the main text, our primary focus is on proposing an integrated framework to tackle the three issues of variable selection, parameter heterogeneity, and cross-sectional dependence in the context of cross-country growth regressions. Here, we would like to briefly discuss the issue of time trend that has found limited attention in the empirical growth literature (Eberhardt and Teal, 2011) but can be partially solved under our setting.

As pointed out by Eberhardt and Teal (2011), any macro production function is likely to contain at least some countries with certain time trends in the input and output variables, and these time-series properties need to be taken into account in the empirical analysis. The question in fact has been addressed to some extent by recent developments in econometrics. We provide two examples below.

Though macro variables are likely to have certain time trends, they do not have to be as strong as polynomial terms like $t$ or $t^2$. More often than not, it can simply be captured by a structure like

$$x_{it} = A_id_{it} + B'_if_{it} + u_{it}, \quad (A.1.1)$$

where $f_{it}$ is the same as that of (2.4) of the main text, $d_{it}$ includes other unobservable common shocks, both $A_i$ and $B_i$ are the unknown factor loadings, and $u_{it}$ stands for an error term. This structure has been well discussed in Pesaran (2006) and Kapetanios et al. (2011).

In the second case, as in Pedroni (2007) we assume that different countries have different types of time trends. This case has been partially discussed in Chen et al. (2012b) and Gao et al. (2019). In particular, Gao et al. (2019) allow the regressors $x_{it}$ to have the following form
\[ x_{it} = g_i(t/T) + v_i + u_{it}, \quad (A.1.2) \]

where \( g_i(\cdot) \) is a trending function and varies across \( i \), \( v_i \) is an individual effect, and \( u_{it} \) stands for an error term. Using (A.1.2), \( g_i(t/T) \) can mimic different types of time trends for each individual country.

For either case of (A.1.1) and (A.1.2), our methodology including the estimator and its asymptotic results remains valid with some minor modifications regarding the proof.

**A.1.2 On Modelling**

As suggested by one referee, one can extend our model to allow for non-linearity in \( x_{it} \):

\[ y_{it} = \sum_{j=1}^{p} \beta_{0j}(z)g_{0j}(x_{it}) + \gamma_{0i}f_{it} + \varepsilon_{it}. \]

There are two possibilities regarding \( g_{0j}(\cdot) \)'s: (1) \( g_{0j}(\cdot) \)'s are known, and (2) \( g_{0j}(\cdot) \)'s are unknown.

For case 1, since \( g_{0j}(\cdot) \)'s are parametrically known, we may simply define \( x_{it,j}^* = g_{0j}(x_{it}) \) to obtain the following model:

\[ y_{it} = \sum_{j=1}^{p} \beta_{0j}(z_{it})x_{it,j}^* + \gamma_{0i}f_{it} + \varepsilon_{it}. \]

This new model then becomes a special case of our growth regression model defined in (2.4). Therefore, our methodology outlined in the main text still apply to this new model.

For case 2, we consider a simple model where \( \beta_{0j}(z) \)'s are unknown constant

\[ y_{it} = \sum_{j=1}^{p} \beta_{0j}g_{0j}(x_{it}) + \gamma_{0i}f_{it} + \varepsilon_{it}. \]

The above model is in fact an extension of Connor et al. (2012), where no factor structure is included. A key question of studying such a model is how to impose an identification restriction to separate \( \beta_{0j} \) and \( g_{0j}(\cdot) \). This issue has been partially investigated by Fan et al. (2016) and Dong et al. (2020) using the sieve method. However, due to the complexity of such a model, these studies only consider the LD case without an unobservable factor structure. A more general framework that nests both the LD and HD cases with factor structure is even more complex, and therefore we will leave it for future research.

**A.1.3 On Index Variable**

In theory, \( z_{it} \) can be a vector and thus include all the index variables simultaneously. However, as well understood in the literature of nonparametric regression, when the dimension of \( z_{it} \) increases, the numerical accuracy decreases dramatically in practice, which is commonly referred to as “the curse of dimensionality”. That is why many studies (e.g., Connor et al., 2012; Vogt, 2012; Dong and Linton, 2018) use additive forms when they could use more general models. Under the varying coefficient setting, the curse of dimensionality becomes even more severe for finite sample studies, because the dimension of \( Z_{it} = H_m(z_{it}) \otimes x_{it} \) is \( pm \). In addition, our treatment of \( z_{it} \) as a scalar is also consistent with the literature of economic growth (e.g., Durlauf et al., 2001).
A.1.4 Justification of Assumption 4.2

To justify this condition, we again use the assumption \( \Gamma_t^0 \Gamma_0 \) = \( I_r \) employed by Lam and Yao (2012) for simplicity. With this assumption, \( \Psi_1 \) reduces to

\[
\Psi_1 = \sqrt{\frac{1}{m} \left[ H_m'(z) \otimes I_p^* \right] \Sigma_2^{-1} \cdot \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} Z_i' \mathcal{E}_i,}
\]

in which \( \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} Z_i' \mathcal{E}_i \) is of a standard form of the sieve–based regression. To further derive an asymptotic distribution based on the above form of \( \Psi_1 \), we outline two methods below.

1. As with the development of Lemma A.1 of Chen et al. (2012a), we can use the large and small block techniques commonly used in time series econometrics to justify the Lindeberg-Feller condition. Note that the same technique is employed by Dong et al. (2015) in proving their Theorem 3.1 in the context of a stationary panel data model.

2. Alternatively, we can adopt the martingale difference condition on the error terms, and verify Lemma B.1 of Chen et al. (2012b). The same technique is also employed by Chen et al. (2012b) in proving their Theorem 3.2, in a stationary panel data setting and by Dong et al. (2019) in proving their Theorem 2.2 in a nonstationary panel data setting.

Both methods would entail a lengthy derivation, even when we use a simplified version of \( \Psi_1 \) by imposing \( \Gamma_t^0 \Gamma_0 \) = \( I_r \). As our paper is already very long, we adopt a type of high–level form in Assumption 4.2 to keep our paper focused and concise.

A.2 Numerical Studies

In this section, we examine the finite sample performance of the methodology of Section 3 using simulations. Firstly, we describe the numerical algorithm which will be used throughout the numerical studies of this paper.

- The following procedure essentially combines two algorithms discussed in Bai (2009) and Wang and Xia (2009) together. For each given \( \lambda = (\lambda_1, \ldots, \lambda_p)' \), the estimates can be obtained using the following iteration procedure. Let \( \hat{C}_\beta^{(n)} \) and \( \hat{F}^{(n)} \) be the estimates obtained from the \( n^{th} \geq 1 \) iteration. Then, for the \( (n + 1)^{th} \) iteration, the estimates are obtained as

Sub-step 1: \[ \text{vec}(\hat{C}_\beta^{(n+1)}) = \left( \sum_{i=1}^{N} Z_i' M F^{(n)} \right) Z_i + \frac{D_{m,p}^{(n)}}{2} \sum_{i=1}^{N} Z_i' M F^{(n)} Y_i, \]

Sub-step 2: \[ \frac{1}{NT} \sum_{i=1}^{N} \left( Y_i - \phi_i \left[ \hat{\beta}_m^{(n+1)} \right] \right) \left( Y_i - \phi_i \left[ \hat{\beta}_m^{(n+1)} \right] \right)' \hat{F}^{(n+1)} = \hat{F}^{(n+1)} V_{NT}, \]

where \( D_{m,p} = I_m \otimes \text{diag}\left\{ \frac{\lambda_1}{\|C_{\beta,1}\|}, \ldots, \frac{\lambda_p}{\|C_{\beta,p}\|} \right\} \); and \( V_{NT} \) is a diagonal matrix with the diagonal being the \( r \) largest eigenvalues of

\[
\frac{1}{NT} \sum_{i=1}^{N} \left( Y_i - \phi_i \left[ \hat{\beta}_m^{(n+1)} \right] \right) \left( Y_i - \phi_i \left[ \hat{\beta}_m^{(n+1)} \right] \right)' \]
arranged in descending order. We stop the iteration when the estimates reach certain threshold, say \( \|\hat{C}_\beta^{(n+1)} - \hat{C}_\beta^{(n)}\| \leq \epsilon \). To start the above iteration, we randomly generate \( \bar{F}^{(0)} \), where each element of \( \bar{F}^{(0)} \) follows \( N(0,1) \).

When choosing the optimal \( \lambda \), we follow Wang and Xia (2009) to simplify it as follows:

\[
\lambda = \nu \left( \|\hat{C}_{\beta,1}\|^{-1}, \ldots, \|\hat{C}_{\beta,p}\|^{-1} \right)'
\]

where \( \nu \) is a scalar, and \( \hat{C}_{\beta,j} \) stands for the \( j^{th} \) row of the unregularized estimator \( \hat{C}_\beta \) (i.e., implementing (3.3) of the main text with \( \lambda = 0_{p \times 1} \)). The idea for choosing \( \lambda \) now becomes straightforward. The unregularized estimator \( \hat{C}_\beta \) is a consistent estimator, and provides information on how likely each row of \( C_{\beta_0} \) is a zero row. In other words, smaller \( \|\hat{C}_{\beta,j}\| \) implies that the \( j^{th} \) row of \( C_{\beta_0} \) is more likely to be zero and hence suggests a larger regularizer on \( \|C_{\beta,j}\| \). Then the selection on the vector \( \lambda \) reduces to the selection on the scalar \( \nu \). Finally, we consider the possible value of \( \nu \) over a sufficiently large interval of the real line. The optimal \( \nu \) is chosen by minimizing the BIC type criteria proposed in the main text.

We now start introducing our data generating process, and the relevant statistics to be calculated. Consider the model (2.4) of the main text. For the factor structure, let \( \mathbf{f}_{it} \sim \text{i.i.d. } N(0_{p \times 1}, I_r) \) and \( \gamma_{it} \sim \text{i.i.d. } N(0.5 \cdot 1_{r \times 1}, I_r) \). In order to generate the regressors and univariate index variable, we firstly generate \( v_{it} = 0.5 v_{i,t-1} + \xi_{it} \), where \( \xi_{it} \sim \text{i.i.d. } N(0_{p \times 1}, I_p) \). Then let \( x_{it} = v_{it} + |\gamma_{it}| \mathbf{f}_{it} \), and \( z_{it} = |v_{it}| + \text{i.i.d. } N(0,1) \), where \( v_{it} \) stands for the first element of \( v_{it} \). By doing so, we generate certain correlation between the regressors and the factor structure, and also introduce some correlation between \( z_{it} \) and \( x_{it} \). The error terms are generated as \( \varepsilon_t = 0.5 \varepsilon_{t-1} + \zeta_t \) in which \( \zeta_t \sim \text{i.i.d. } N(0_{N \times 1}, \Sigma_\zeta) \) and \( \Sigma_\zeta = \{0.5^{|i-j|}\}_{N \times N} \), so that the weak cross-sectional dependence among individuals, and serial correlation over time dimension are generated. For both LD and HD cases, the rest of the settings are as follows:

- **LD:** \( p^* = 2 \), \( p = 5 \), \( r = 3 \), \( \beta_{01}(z) = \exp(-z^2/2) + 0.4 \), and \( \beta_{02}(z) = z \cdot \exp(-z^2/2) + 0.7 \). Let \( \Upsilon_{NT} = \frac{\ln N}{\sqrt{N}} \), or \( \frac{\ln(N + T)}{\sqrt{N + T}} \), or \( \frac{\sqrt{N}}{\sqrt{N}} \).
- **HD:** \( p^* = 2 \cdot \lfloor (1.2(NT)^{1/6}) \rfloor \), \( p = \min\{N,T\} \), \( r = 3 \). For \( j = 1, \ldots, p^* \), \( \beta_{0j}(z) = \exp(-z^2/2) + 0.4 \) when \( j \) is odd, and \( \beta_{0j}(z) = z \cdot \exp(-z^2/2) + 0.7 \) when \( j \) is even. Let \( \Upsilon_{NT} = \frac{\ln N_T}{\sqrt{\xi_N T}} \), or \( \frac{\ln(N + T)}{\sqrt{N + T}} \), or \( \frac{\sqrt{N}}{\sqrt{N_T}} \), where \( \xi_{NT} = \min\{N,T\} \).

As shown in the main text, the number of truncation parameter should increase with the sample size, so we follow Dong and Linton (2018) to let the truncation parameter\(^1\) be \( m = \lfloor 1.2(NT)^{1/6} \rfloor \), and adopt the Hermite functions as the basis functions for simplicity\(^2\). Finally, we repeat the above procedure 1000 times, and let \( N \in \{40, 80, 120\} \) and \( T \in \{40, 80, 120\} \).

For each dataset generated, we implement variable selection first. After identifying \( A^* \) and \( A^T \), we implement post-selection estimation (i.e., remove the irrelevant regressors and then implement (3.3) of the main

\(^1\)Note that the optimal choice of \( m \) may not be the optimal one, but the two designs satisfy all the requirements of our assumptions. Although the optimal choice of truncation parameter and the optimal bandwidth selection have been solved for some cross-sectional models and time series models (e.g., Gao, 2007; Hall et al., 2007) under low dimensional cases, it is well understood that the question is still open even for the nonparametric panel data model with fixed effects (cf., Chen et al., 2012b; Su and Jin, 2012). The question is even more daunting when the factor structure and variable selection procedure get involved.

\(^2\)The choice of basis functions is equivalent to the choice on the kernel function in the literature of kernel regression. The results are not sensitive to different choices given \( \mathcal{R} \) defined in the beginning of Section 3 covers \( \min\{z_{it}\}, \max\{z_{it}\} \) in practice.
text with $\lambda = 0_{p \times 1}$) for the coefficient functions. Moreover, we calculate the following statistics to evaluate the finite sample performance.

- To evaluate the sensitivity of variable selection procedure, we report two percentages for each choice of $\Upsilon_{NT}$ under each design: (1) the percentage of missed true regressors (i.e., false negative rate, FNR) of 1000 replications; and (2) the percentage of falsely selected noise regressors (i.e., false positive rate, FPR) of 1000 replications.

- To evaluate the estimates on the coefficient functions, we take $\beta_{01}(\cdot)$ as an example. For the $j^{th}$ replication, we obtain $\hat{\beta}_{1j}(z)$ for all $z$ (given it is not identified as 0; otherwise, we record 0 as the estimate). For all $z$, we calculate $\hat{\beta}_1(z) = \frac{1}{1000} \sum_{j=1}^{1000} \hat{\beta}_{1j}(z)$, and also record the 95% confidence bands based on $\{\hat{\beta}_{1j}(z)|j = 1, \ldots, 1000\}$. We plot these values over a certain range of $z$. The values of $\beta_{01}(z)$ are plotted in solid black line, the values of $\hat{\beta}_1(z)$ are plotted in red dotted line, and the associated 95% confidence bands are plotted in blue dashed curves.

We notice that the FNR and FPR produced by the three choices of $\Upsilon_{NT}$ for both the LD and HD cases are almost identical. So, we focus on the FNR and FPR associated with the first choice (i.e., $\ln N \sqrt{N}$ and $\ln \xi_{NT} \sqrt{\xi_{NT}}$) for each case in Table A.1 below. It is clear that our method proposed in Section 3 works well, as both FNR and FPR are either 0 or very close to 0. It is not surprising that FPR has smaller values while the ratio of $T/N$ (or $T$) is small, as we set $p = \min\{N,T\}$. It is worth mentioning that although FPR is slightly higher than zero for the HD case, slightly over selecting the regressors will still yield consistent estimation.

Table A.1: FNR & FPR

|       | $N \setminus T$ | FNR | FPR |
|-------|-----------------|-----|-----|
|       | 40   | 80  | 120 | 40 | 80 | 120 |
| LD ($\Upsilon_{NT} = \ln N \sqrt{N}$) | | | | | | |
| 40   | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% |
| 80   | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% |
| 120  | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% |
| HD ($\Upsilon_{NT} = \ln \xi_{NT} \sqrt{\xi_{NT}}$) | | | | | | |
| 40   | 0.00% | 0.00% | 0.00% | 14.60% | 2.40% | 2.20% |
| 80   | 0.00% | 0.00% | 0.00% | 3.80% | 4.80% | 4.10% |
| 120  | 0.00% | 0.00% | 0.00% | 3.10% | 4.00% | 4.60% |

We plot $\beta_{01}(z)$ and $\beta_{02}(z)$ on $[-1,2]$ in Figure A.1, as the majority of $z_i$’s lie in this range. Due to similarity, we do not report the estimates of the rest of the coefficient functions. It is easy to see that as the sample size increases, the 95% confidence bands become much narrower and the mean estimate approaches the true curve. Also, as expected, estimates from the HD cases have wider 95% confidence bands, thus being less accurate than those from the LD case.

### A.3 Proofs

Before proving the main theorems, we present the following preliminary lemmas.
Figure A.1: $\beta_{01}(z)$ and $\beta_{02}(z)$ of both LD and HD cases. Specifically, top left panel is for $\beta_{01}(z)$ of LD case; top right panel is for $\beta_{02}(z)$ of LD case; bottom left panel is for $\beta_{01}(z)$ of HD case; bottom right panel is for $\beta_{02}(z)$ of HD case.
Lemma A.1. Consider two non-singular symmetric matrices $A, B$ with the same dimensions $k \times k$. Suppose that their minimum eigenvalues satisfy that $\eta_{\min}(A) > 0$ and $\eta_{\min}(B) > 0$ uniformly in $k$. Then $\|A^{-1} - B^{-1}\| \leq \eta_{\min}^{-1}(A) \cdot \eta_{\min}^{-1}(B) \|A - B\|$.

Lemma A.2. Let Assumptions 1 and 2 hold. As $(N,T) \rightarrow (\infty, \infty)$,

1. $\frac{1}{NT}E'\xi = O_P\left(\frac{1}{\sqrt{N}}\right)$ and $\frac{1}{\sqrt{NT}}E'\xi = O_P\left(\frac{1}{\sqrt{NT}}\right)$, where $E = (\xi_1, \ldots, \xi_N)'$,

2. $\sup_{F \in D_F} \frac{1}{NT} \sum_{i=1}^{N} E_i'P_F E_i = O_P\left(\frac{1}{\sqrt{N}}\right)$,

3. $\sup_{F \in D_F} \left| \frac{1}{NT} \sum_{i=1}^{N} \gamma_0' F_0' M_F E_i \right| = O_P\left(\frac{1}{\sqrt{NT}}\right)$,

4. $\sup_{\|C_\beta\| \leq M, F \in D_F} \left| \frac{1}{NT} \sum_{i=1}^{N} (\phi_i[\beta_{0,m}] - \phi_i[\beta_m])' M_F E_i \right| = O_P\left(\frac{1}{\sqrt{N}}\right) + O_P\left(\frac{1}{\sqrt{T}}\right)$,

5. $\sup_{F \in D_F} \left| \frac{1}{NT} \sum_{i=1}^{N} \phi_i[\Delta_m]' M_F \phi_i[\Delta_m] \right| = O_P\left(\frac{1}{m^\mu}\right)$,

6. $\sup_{F \in D_F} \left| \frac{1}{NT} \sum_{i=1}^{N} \phi_i[\Delta_m]' M_F F_0 \gamma_0 \right| = O_P\left(\frac{1}{m^{\frac{r}{2}}}\right)$,

7. $\sup_{\|C_\beta\| \leq M, F \in D_F} \left| \frac{1}{NT} \sum_{i=1}^{N} \phi_i[\Delta_m]' M_F \{\phi_i[\beta_m] - \phi_i[\beta_{0,m}]\} \right| = O_P\left(\frac{1}{m^{\frac{r}{2}}}\right)$,

where $M$ is a sufficiently large constant.

Let $\Pi_{NT}^{-1} = V_N (F_0^T F_0 / T)^{-1} (T_0^T T_0 / N)^{-1}$, where $V_N$ is a diagonal matrix with the diagonal being the $r$ largest eigenvalues of

$$\frac{1}{NT} \sum_{i=1}^{N} \left( Y_i - \phi_i[\hat{\beta}_m] \right) \left( Y_i - \phi_i[\hat{\beta}_m] \right)'$$

arranged in descending order.

Lemma A.3. Let Assumptions 1, 2 and 3 hold. As $(N,T) \rightarrow (\infty, \infty)$,

1. $\|\hat{\beta}_m - \beta_0\|_{L^2} = O_P(1)$;

2. $\|P_{\hat{F}} - P_{F_0}\| = O_P(1)$;

3. $V_N \rightarrow V$, where $V$ is an $r \times r$ diagonal matrix consisting of the eigenvalues of $\Sigma_f \Sigma_\gamma$;

4. $\frac{1}{\sqrt{T}} \|\tilde{F} \Pi_{NT}^{-1} - F_0\| = O_P\left(\|\hat{\beta}_m - \beta_0\|_{L^2}\right) + O_P\left(\frac{1}{\sqrt{N}}\right) + O_P\left(\frac{1}{\sqrt{T}}\right)$;

5. $\left\| \frac{1}{T} \tilde{F}' (F - F_0 \Pi_{NT}) \right\| = O_P\left(\|\hat{\beta}_m - \beta_0\|_{L^2}\right) + O_P\left(\frac{1}{T}\right) + O_P\left(\frac{1}{N}\right)$;

6. $\|P_{\hat{F}} - P_{F_0}\|^2 = O_P\left(\|\hat{\beta}_m - \beta_0\|_{L^2}\right) + O_P\left(\frac{1}{T}\right) + O_P\left(\frac{1}{N}\right)$.

Lemma A.4. Let Assumptions 1-3 hold. As $(N,T) \rightarrow (\infty, \infty)$,
1. \( \Pr(\|\hat{C}_\beta^{t}\| = 0) \to 1; \)
2. \( \|\hat{C}_\beta - C_0^*\| = O_P \left( \sqrt{\frac{\pi}{NT}} + m^{-\frac{d}{2}} + \frac{m\lambda_{\max}^T}{N} \right). \)

**Lemma A.5.** Let Assumptions 1, 2 and 5 hold. As \((N,T) \to (\infty, \infty),\)

1. \[ \sup_{\|C_\beta\| \leq a_0 \sqrt{p}, F \in \mathcal{D}_F} \left| \frac{1}{NT} \sum_{i=1}^N \phi_i[\hat{\beta}_{0,m}] - \phi_i[\beta_{m}] M_F E_i \right| = O_P \left( \sqrt{\frac{p(\xi_{NT} + mp)}{NT}} \right); \]
2. \[ \sup_{F \in \mathcal{D}_F} \left| \frac{1}{NT} \sum_{i=1}^N \phi_i[\Delta_m] M_F \phi_i[\Delta_m] \right| = O_P(p^* m^{-\mu}); \]
3. \[ \sup_{F \in \mathcal{D}_F} \left| \frac{1}{NT} \sum_{i=1}^N \phi_i[\Delta_m] M_F F_0 \gamma_{0k} \right| = O_P(\sqrt{p^* m^{-\frac{d}{2}}}); \]
4. \[ \sup_{F \in \mathcal{D}_F} \left| \frac{1}{NT} \sum_{i=1}^N \phi_i[\Delta_m] M_F E_i \right| = O_P(\sqrt{p^* m^{-\frac{d}{2}}}); \]
5. \[ \sup_{\|C_\beta\| \leq a_0 \sqrt{p}, F \in \mathcal{D}_F} \left| \frac{1}{NT} \sum_{i=1}^N \phi_i[\Delta_m] M_F \{ \phi_i[\beta_{m}] - \phi_i[\beta_{0,m}] \} \right| = O_P(\sqrt{p^* p^* m^{-\frac{d}{2}}}), \]

where \(a_0\) is a sufficiently large constant.

**Lemma A.6.** Let Assumptions 1, 2 and 5 hold. As \((N,T) \to (\infty, \infty),\)

1. \( \|\hat{\beta}_m - \beta_0\|_2 = o_P(1); \)
2. \( \|P_{\hat{F}} - P_0\| = o_P(1). \)

Additionally, suppose that \(p^* m^{-\mu} \to 0.\)

3. \( V_{NT} \to^{P} V, \) where \( V \) is an \( r \times r \) diagonal matrix consisting of the eigenvalues of \( \Sigma_f \Sigma_g; \)
4. \[ \frac{1}{\sqrt{T}} \|\hat{F} \Pi_{NT}^{-1} - F_0\| = O_P(\|\hat{\beta}_m - \beta_0\|_2) + O_P \left( \frac{1}{\sqrt{N}} \right) + O_P \left( \frac{1}{\sqrt{T}} \right); \]
5. Suppose further that \( \frac{(p^*)^2}{T} \to 0, \) and \( \frac{N}{T} \to \kappa_0 < \infty. \) Then
\[ \|\hat{\beta}_m - \beta_0\|_2 = O_P \left( \sqrt{\frac{p^* m}{NT}} + \sqrt{p^* m^{-\mu}} + \lambda_{\max}^T \frac{p^* m}{NT} \right). \]

**Proof of Theorem 3.1:**

(1). The first result follows from (1) of Lemma A.3.

(2)-(3). The second and third results follow from Lemma A.4.

**Proof of Theorem 3.2**

The procedure is similar to that given for (3) of Theorem 3.4 below, so we leave the details in the supplementary Appendix B.

**Proof of Theorem 3.3:**

Based on the development of Lemma A.4, the definition of \(\hat{\beta}_m^*\) and (3.1), we can write for \(\forall z \in V_z,\)
\[ \sqrt{\frac{NT}{m}} \left( \hat{\beta}_m^*(z) - \beta_0^*(z) \right) \]
where the second equality follows from Assumption 2.1 and the condition \( \frac{NT}{m^{NT}} \to 0 \); the third equality follows from the above development on \( \text{vec}(\hat{C}_{\beta}^* - \hat{C}_{\beta_0}^*) \) and \( \text{vec}(\hat{C}_{\beta_0}^* - \hat{C}_{\beta_0}^*) \), and the fact that \( \Sigma_{Z,f}^{-1} \) reduces to \( \Sigma_{Z}^{-1} \) using Assumption 4; and the fourth equality follows from the proof of (2) of Lemma A.4, \( \frac{mN}{\sqrt{NT}} \to 0 \), and \( \frac{m\lambda_{\max}}{\sqrt{NT}} \to 0 \).

We next consider \( \Lambda_1 \) by starting with \( \frac{1}{NT} \sum_{i=1}^{N} Z_i^* M_F \mathcal{E}_i \).

\[
\frac{1}{NT} \sum_{i=1}^{N} Z_i^{*F} M_F \mathcal{E}_i = \frac{1}{NT} \sum_{i=1}^{N} Z_i^{*F} M_{F_0} \mathcal{E}_i + \frac{1}{NT} \sum_{i=1}^{N} Z_i^{*F} (M_F - M_{F_0}) \mathcal{E}_i
\]

\[
= \frac{1}{NT} \sum_{i=1}^{N} Z_i^{*F} M_{F_0} \mathcal{E}_i - \frac{1}{NT} \sum_{i=1}^{N} Z_i^{*F} (P_F - P_{F_0}) \mathcal{E}_i
\]

\[
:= D_1 - D_2.
\]

Firstly, we shall show \( \left\| \frac{1}{NT} [H_{m}(z) \otimes I_{P^{*}}] A_{1NT}^{-1} \Sigma_{Z}^{-1} D_2 \right\| = o_p(1) \). Let \( Z_i^* \) be the \( j \)th column of \( Z_i^* \), and let \( Z_{i,t,j}^* \) be the \( t \)th element of \( Z_{i,j}^* \). Write

\[
D_2 = \frac{1}{NT} \sum_{i=1}^{N} Z_i^{*F} \left( \frac{\hat{F}^{*F}}{T} - P_{F_0} \right) \mathcal{E}_i
\]

\[
= \frac{1}{NT} \sum_{i=1}^{N} Z_i^{*F} \left( \frac{\hat{F} - F_0 \Pi_{NT}}{T} \right) \Pi_{NT} F_0 \mathcal{E}_i + \frac{1}{NT} \sum_{i=1}^{N} Z_i^{*F} \left( \frac{\hat{F} - F_0 \Pi_{NT}}{T} \right) (\hat{F} - F_0 \Pi_{NT})^T \mathcal{E}_i
\]

\[
+ \frac{1}{NT} \sum_{i=1}^{N} Z_i^{*F} F_0 \Pi_{NT} \left( \frac{\hat{F} - F_0 \Pi_{NT}}{T} \right) \mathcal{E}_i + \frac{1}{NT} \sum_{i=1}^{N} Z_i^{*F} F_0 \left( \Pi_{NT} \Pi_{NT}^T - (F_0/T - F_0/T)^{-1} F_0^T \right) \mathcal{E}_i
\]

\[
:= D_{21} + D_{22} + D_{23} + D_{24},
\]

where the definitions of \( D_{21} \) to \( D_{24} \) are obvious.

In what follows, we let \( D_{2\ell,j} \) be the \( j \)th row of \( D_{2\ell} \) for \( \ell = 1, 2, 3, 4 \). Thus, for \( D_{21} \), consider

\[
\|D_{21,j}\| = \left\| \frac{1}{NT} \sum_{i=1}^{N} Z_{i,j}^{*F} \left( \frac{\hat{F} - F_0 \Pi_{NT}}{T} \right) \Pi_{NT} F_0 \mathcal{E}_i \right\|
\]

\[
\leq \left\| \frac{1}{NT} \sum_{i=1}^{N} (\mathcal{E}_i F_0) \otimes Z_{i,j}^{*F} \right\| \cdot \left\| \frac{1}{\sqrt{T}} \text{vec} \left[ (\hat{F} - F_0 \Pi_{NT}) \Pi_{NT}^T \right] \right\|
\]

\[
= O_p \left( \frac{1}{\sqrt{NT}} \right) \frac{1}{\sqrt{T}} \|\hat{F} - F_0 \Pi_{NT}\|.
\]
Summing up over \( j \) for \( D_{21,j} \), we obtain that \( \|D_{21}\| = O_P \left( \sqrt{\frac{m}{NT}} \right) \frac{1}{\sqrt{T}} \|\hat{F} - F_0\Pi_{NT}\| \).

For \( D_{22} \), write

\[
\|D_{22,j}\| = \left\| \frac{1}{NT} \sum_{i=1}^{N} Z_{i,j}^* (\hat{F} - F_0\Pi_{NT}) T (\hat{F} - F_0\Pi_{NT})' E_i \right\| \\
\leq \left\| \frac{1}{NT} \sum_{i=1}^{N} E_i' \otimes Z_{i,j}^* \right\| \cdot \left\| \frac{1}{T} \text{vec} \left[(\hat{F} - F_0\Pi_{NT})(\hat{F} - F_0\Pi_{NT})'\right] \right\| \\
= O_P \left( \frac{1}{\sqrt{N}} \right) \frac{1}{\sqrt{T}} \|\hat{F} - F_0\Pi_{NT}\|^2.
\]

Summing \( D_{22,j} \) up over \( j \), we obtain that \( \|D_{22}\| = O_P \left( \sqrt{\frac{m}{N}} \right) \frac{1}{\sqrt{T}} \|\hat{F} - F_0\Pi_{NT}\|^2 \).

For \( D_{23} \), write

\[
\|D_{23,j}\| = \left\| \frac{1}{NT} \sum_{i=1}^{N} Z_{i,j}^* F_0 \sqrt{T} (\hat{F} - F_0\Pi_{NT})' \right\| \\
\leq \left\| \frac{1}{NT} \sum_{i=1}^{N} E_i' \otimes Z_{i,j}^* \right\| \cdot \left\| \frac{1}{\sqrt{T}} \text{vec} \left[\Pi_{NT}(\hat{F} - F_0\Pi_{NT})'\right] \right\|.
\]

Note that

\[
E \left\| \frac{1}{NT} \sum_{i=1}^{N} E_i' \otimes Z_{i,j}^* F_0 \right\|^2 = \frac{1}{N^2T^3} E \left\| \sum_{i=1}^{N} E_i' \otimes \sum_{t=1}^{T} Z_{it,j}^* f_{0t} \right\|^2 = \frac{1}{N^2T^3} \sum_{s=1}^{T} E \left\| \sum_{i=1}^{N} E_i' \otimes \sum_{t=1}^{T} Z_{it,j}^* f_{0t} \right\|^2
\]

\[
= \frac{2}{N^2T^2} \sum_{l_1 > l_2}^{N} \sum_{l_1 = 1}^{l_2} \sum_{l_1 = 1}^{l_2} E \left[ Z_{l_1 t_1,j}^* Z_{l_2 t_2,j}^* E \left[ \|f_{0t}\|^2 | X_{Nt_1} \right] \right] \sigma_{l_1 l_2}
\]

\[
= \frac{2}{N^2T^2} \sum_{l_1 > l_2}^{N} \sum_{l_1 = 1}^{l_2} \sum_{l_1 = 1}^{l_2} E \left[ Z_{l_1 t_1,j}^* Z_{l_2 t_2,j}^* a_{l_1 l_2} \sigma_{l_1 l_2} \right]
\]

\[
\leq O(1) \frac{2}{N^2T^2} \sum_{l_1 > l_2}^{N} \sum_{l_1 = 1}^{l_2} \sum_{l_1 = 1}^{l_2} a_{l_1 l_2} \sigma_{l_1 l_2} = O(1) \frac{1}{NT},
\]

where the fourth equality follows from Assumption 1.2; the sixth equality follows from Assumption 4; and the seventh equality follows from both Assumptions 1.1 and 4.1.

Thus, \( \|D_{23,j}\| = O_P \left( \frac{1}{\sqrt{NT}} \right) \frac{1}{\sqrt{T}} \|\hat{F} - F_0\Pi_{NT}\| \). Summing \( D_{23,j} \) up over \( j \), we obtain that \( \|D_{23}\| = O_P \left( \sqrt{\frac{m}{NT}} \right) \frac{1}{\sqrt{T}} \|\hat{F} - F_0\Pi_{NT}\| \).

Similarly, write for \( D_{24} \).
\[ \|D_{24,j}\| = \left\| \frac{1}{NT} \sum_{i=1}^{N} Z_{i,j}^{*} F_0 [\Pi_{NT}^2 - (F_0^T F_0 / T)^{-1}] F_0^T \mathcal{E}_i \right\| \leq \left\| \frac{1}{NT} \sum_{i=1}^{N} (\mathcal{E}_i^* F_0) \otimes \frac{Z_{i,j}^{*} F_0}{T} \right\| \cdot \| \Pi_{NT}^2 - (F_0^T F_0 / T)^{-1} \| = O_P \left( \frac{1}{\sqrt{NT}} \right) \| \Pi_{NT}^2 - (F_0^T F_0 / T)^{-1} \|. \]

Summing \( D_{24,j} \) up over \( j \), we obtain that \( \|D_{24}\| = O_P \left( \sqrt{\frac{NT}{m}} \right) \| \Pi_{NT}^2 - (F_0^T F_0 / T)^{-1} \| \), where \( \| \Pi_{NT}^2 - (F_0^T F_0 / T)^{-1} \| = O_P(1) \) by the development for the fourth result of Lemma A.3.

Based on the analyses of \( D_{21} \) to \( D_{24} \), we obtain
\[ \sqrt{\frac{NT}{m}} \|D_2\| = O_P(1) \frac{1}{\sqrt{T}} \| \mathcal{F} - F_0 \Pi_{NT} \| + O_P(1) \| \Pi_{NT}^2 - (F_0^T F_0 / T)^{-1} \| \]
\[ + O_P(1) \sqrt{T} \cdot \frac{1}{T} \| \mathcal{F} - F_0 \Pi_{NT} \|^2 \]

which further gives \( \sqrt{\frac{NT}{m}} \left[ \mathcal{H}_m(z) \otimes I_{d_z} \right] A_{11}^{-1} \Sigma_Z^{-1} \| D_2 \| = O_P(1) \) by Lemma A.3 and the condition \( \frac{T}{NT} \to 0 \).

Similarly, we obtain
\[ \left\| \sqrt{\frac{NT}{m}} \left[ \mathcal{H}_m(z) \otimes I_{p_F} \right] A_{11}^{-1} \Sigma_Z^{-1} \left[ 1 \frac{1}{NT} \sum_{i=1}^{N} A_{3,i} \mathcal{E}_i \right. \right. \]
\[ \left. \left. - \sqrt{\frac{NT}{m}} \left[ \mathcal{H}_m(z) \otimes I_{p_F} \right] A_{11}^{-1} \Sigma_Z^{-1} \left[ 1 \frac{1}{NT} \sum_{i=1}^{N} \tilde{A}_{3,i} \mathcal{E}_i \right. \right. \right. \]
\[ \left. \left. \right. \right. \left. \right. \right. \left. \right. \right. \right. \right. \right. \right. \right. \right. \left. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \r
Thus, for a large constant $A$, $\hat{C}_\beta$ lies in a ball $\left\{ C_\beta \mid \|C_\beta - C_{\beta_0}\| \leq Ah_{NT}^{1/2} \right\}$ with probability approaching 1. According to $\hat{C}_\beta$ and $\hat{C}_\beta^\dagger$, construct $C_\beta$ and $U$ as $C_\beta = (C_\beta', C_\beta')'$ and $U = (U', U')'$, where $C_\beta' = C_{\beta_0} + h_{NT}^{1/2}U^*$ and $C_\beta^\dagger = C_{\beta_0} + h_{NT}^{1/2}U^T U^{1/2}$ with $\|U\|^2 \leq A^2$. Further define

$$V_\lambda(U^*, U^T, F) = \frac{1}{NT} Q_\lambda(C_\beta, F) = \frac{1}{NT} Q_\lambda((C_{\beta_0} + h_{NT}^{1/2}U^*, h_{NT}^{1/2}U^{1/2}), F),$$

(A.3.2)

so $\hat{C}_\beta$ and $\hat{C}_\beta^\dagger$ can be obtained by minimizing $V_\lambda(U^*, U^T, F)$ over $\|U\|^2 \leq A^2$ except on an event with probability converging to 0.

It suffices to show that for any $U = (U^*, U^T)$ with $\|U\|^2 \leq A^2$ and $\|U^T\| > 0$, $V_\lambda(U^*, U^T, F) > V_{NT}(U^*, 0_{(p-p^*)\times m}, F)$ with a probability converging to 1 regardless of the value of $F$. Recall that some notations used below have been defined in the beginning of the supplementary file. Further denote $\beta_m(\cdot) = C_\beta^\dagger H_m(\cdot)$. Then write

$$V_\lambda(U^*, U^T, F) - V_{NT}(U^*, 0_{(p-p^*)\times m}, F)$$

$$= \frac{1}{NT} \sum_{i=1}^N \Delta \phi_i^*[\beta_m^*] M_F \Delta \phi_i^*[\beta_m^*] + \frac{1}{NT} \sum_{i=1}^N \Delta \phi_i^*[\beta_m^*]' M_F \Delta \phi_i^*[\beta_m^*]'$$

$$+ \frac{1}{NT} \sum_{i=1}^N \gamma_0' F_0' M_F F_0 \gamma_0 + \frac{1}{NT} \sum_{i=1}^N \mathcal{E}_i' M_i E_i$$

$$+ \frac{2}{NT} \sum_{i=1}^N \Delta \phi_i^*[\beta_m^*]' M_F E_i + \frac{2}{NT} \sum_{i=1}^N \Delta \phi_i^*[\beta_m^*]' M_F E_i + \frac{2}{NT} \sum_{i=1}^N \gamma_0' F_0' M_F E_i$$

$$+ \frac{2}{NT} \sum_{i=1}^N \Delta \phi_i^*[\beta_m^*]' M_F F_0 \gamma_0 + \frac{2}{NT} \sum_{i=1}^N \Delta \phi_i^*[\beta_m^*]' M_F F_0 \gamma_0$$

$$+ \frac{2}{NT} \sum_{i=1}^N \Delta \phi_i^*[\beta_m^*]' M_F \phi_i^*[\beta_m^*]' + \sum_{j=1}^{p^*} \frac{\lambda_j}{NT} \|C_{\beta,j}\| + \sum_{j=p^*+1}^{p} \frac{\lambda_j}{NT} \|C_{\beta,j}\|$$

$$= B_{1NT} + 2B_{2NT} + 2B_{3NT} + 2B_{4NT} + B_{5NT},$$

where the definitions of $B_{jNT}$ for $j = 1, \ldots, 5$ are obvious.

For $B_{1NT} + 2B_{2NT}$, write

$$B_{1NT} + 2B_{2NT} \geq \frac{1}{NT} \sum_{i=1}^N \Delta \phi_i^*[\beta_m^*]' M_F \Delta \phi_i^*[\beta_m^*]'$$

$$- \frac{1}{NT} \sum_{i=1}^N \Delta \phi_i^*[\beta_m^*]' M_F \Delta \phi_i^*[\beta_m^*]' - \frac{1}{NT} \sum_{i=1}^N \phi_i^*[\beta_m^*]' M_F \phi_i^*[\beta_m^*]'$$
\[
\begin{align*}
&\geq -\frac{1}{NT} \sum_{i=1}^{N} \Delta \phi_i^* [\beta_m^*]' M_F \Delta \phi_i^* [\beta_m^*] \\
&\geq -h_{NT} \frac{1}{NT} \sum_{i=1}^{N} U^* Z_i^* M_F Z_i^* U^* \\
&\geq -h_{NT} \rho_1 \| U^* \|^2 \geq -h_{NT} \rho_1 A^2
\end{align*}
\]

where the third inequality follows by construction.

For \( B_{3NT} + B_{4NT} \), it is easy to know that

\[
|B_{3NT} + B_{4NT}| \leq \left\{ \frac{1}{NT} \sum_{i=1}^{N} \| \Delta \phi_i^* [\beta_m^*] \|^2 \right\}^{1/2} \cdot \left\{ \frac{1}{NT} \sum_{i=1}^{N} E_i^* M_F E_i \right\}^{1/2} + \left\{ \frac{1}{NT} \sum_{i=1}^{N} \| \Delta \phi_i^* [\beta_m^*] \|^2 \right\}^{1/2} \cdot \left\{ \frac{1}{NT} \sum_{i=1}^{N} \gamma_0 F_0^* M_F F_0 \gamma_0 \right\}^{1/2} = O_p(1) h_{NT}^{1/2}.
\]

For \( B_{5NT} \), we have

\[
B_{5NT} = \sum_{j=p^*+1}^{p} \frac{\lambda_j}{NT} \| C_{\beta,j} \| \geq \frac{\lambda_{\min}^{\dagger}}{NT} \left( \sum_{j=p^*+1}^{p} \| C_{\beta,j} \|^2 \right)^{1/2} = \frac{\lambda_{\min}^{\dagger}}{NT} \| U \|.
\]

In view of the above development and the condition \( \frac{\lambda_{\min}^{\dagger}}{NT h_{NT}} \rightarrow \kappa_3 \) with \( \kappa_3 \) being sufficiently large, the first result follows.

(2). This result follows from (5) of Lemma (A.6).

(3). We define three mutually exclusive sets \( A^- \), \( A^0 = \{ \lambda \in \mathbb{R}^p : S_\lambda = A^* \} \) and \( A^+ = \{ \lambda \in \mathbb{R}^p : S_\lambda \supset A^* \} \) according to whether the model \( S_\lambda \) is under fitted, correctly fitted, or over fitted respectively. Suppose that there is a sequence \( \{ \lambda_{NT} \} \) that satisfies the conditions required by the first result of this theorem. Let \( (\hat{C}^{\lambda_{NT}}, \hat{F}^{\lambda_{NT}}) \) denote the estimator obtained by implementing (3.3) using \( \lambda_{NT} \).

Case 1: Under-fitted model. Without loss of generality, we assume that only one variable is missing, so suppose that the first \( p^* - 1 \) rows of \( \hat{C}_\beta \) are obtained from the under-fitted model and the \( p^{\text{th}} \) row of \( \hat{C}_\beta \) is a 0 row. Moreover, let \( \text{RSS}_0 = \frac{1}{NT} \sum_{i=1}^{N} \left( Y_i - \phi_i[\beta_{0,m}] \right)' M_{F_0} (Y_i - \phi_i[\beta_{0,m}]) \).

We then write

\[
\text{RSS}_\lambda - \text{RSS}_0 = \frac{1}{NT} \sum_{i=1}^{N} \left( Y_i - \phi_i[\beta_{0,m}] \right)' M_{F_{\lambda}} (Y_i - \phi_i[\beta_{0,m}]) - \frac{1}{NT} \sum_{i=1}^{N} \left( Y_i - \phi_i[\beta_{0,m}] \right)' M_{F_0} (Y_i - \phi_i[\beta_{0,m}]) \geq \rho_1 \| C_{\beta_0,p^*} \|^2 \geq \frac{\rho_1}{2} \| \beta_{0,p^*} \|^2 \| \tilde{Z}_1 \|^2 > 0,
\]

where the first inequality follows from the development of Lemma A.6. Similarly, we have

\[
\text{RSS}_{\lambda_{NT}} - \text{RSS}_0 = \text{vec}(C_{\beta_0} - \bar{C}_\beta^{\lambda_{NT}})' \frac{1}{NT} \sum_{i=1}^{N} Z_i^* M_{F_{\lambda_{NT}}} Z_i \text{vec}(C_{\beta_0} - \bar{C}_\beta^{\lambda_{NT}})
\]

Electronic copy available at: https://ssrn.com/abstract=3348229
By the first result of this theorem, we know that \( \Pr(\text{df}_\lambda) \) is the unpenalized estimator under the model determined by \( \hat{C}_\beta \) and recall that \( \hat{C}_\lambda \) determines a model \( S_\lambda \). Under such a model \( S_\lambda \), we can define another unpenalized estimator as

\[
(\hat{C}_\beta, \hat{F}) = \underset{C_\beta, F}{\arg \min} \frac{1}{NT} \sum_{i=1}^{N} (Y_i - \phi_i[\beta_m])' M_F(Y_i - \phi_i[\beta_m]) 
\]

subject to \( \|C_\beta\| \leq a_0 \sqrt{p} \) and \( F \in D_F \), where, for \( j = 1, \ldots, p \), \( \|C_{\beta,j}\| = 0 \) with \( \forall j \notin S_\lambda \). In other words, \((\hat{C}_\beta, \hat{F})\) is the unpenalized estimator under the model determined by \( \hat{C}_\beta \). By definition, we obtain immediately that \( \text{RSS}_S \geq \text{RSS}_{S_\lambda} \), where \( \text{RSS}_{S_\lambda} = \frac{1}{NT} \sum_{i=1}^{N} (Y_i - \phi_i[\beta_m])' M_F(Y_i - \phi_i[\beta_m]) \).

Write

\[
\ln \text{RSS}_{S_\lambda} - \ln \text{RSS}_{\lambda NT} = \ln \left( 1 + \frac{\text{RSS}_{S_\lambda} - \text{RSS}_{\lambda NT}}{\text{RSS}_{\lambda NT}} \right) \geq -\frac{\text{RSS}_{S_\lambda} - \text{RSS}_{\lambda NT}}{\text{RSS}_{\lambda NT}}.
\]

In view of the proof of Lemma A.6, it is easy to see that \( \text{RSS}_{\lambda NT} \) converges to a positive constant. As for \( \text{RSS}_{S_\lambda} - \text{RSS}_{\lambda NT} \), we obtain

\[
\text{RSS}_{S_\lambda} - \text{RSS}_{\lambda NT} = \frac{1}{NT} \sum_{i=1}^{N} (Y_i - \phi_i[\hat{\beta}_m])' M_F(Y_i - \phi_i[\hat{\beta}_m]) - \frac{1}{NT} \sum_{i=1}^{N} (Y_i - \phi_i[\lambda^\ast])' M_{\lambda^\ast}(Y_i - \phi_i[\lambda^\ast NT]).
\]

Using the development of Lemma A.6, it is not hard to see \( |\text{RSS}_{S_\lambda} - \text{RSS}_{\lambda NT}| \leq O_P(1)h_{NT}^{1/2} \). Thus, we can further write

\[
\ln \text{RSS}_{S_\lambda} - \ln \text{RSS}_{\lambda NT} \geq -\frac{\text{RSS}_{S_\lambda} - \text{RSS}_{\lambda NT}}{\text{RSS}_{\lambda NT}} \geq -\left| O_P(1)h_{NT}^{1/2} \right|.
\]

We then write

\[
\inf_{\lambda \in A^+} \text{BIC}_\lambda - \text{BIC}_{\lambda NT} = \inf_{\lambda \in A^+} \ln \text{RSS}_{S_\lambda} - \ln \text{RSS}_{\lambda NT} + (\text{df}_\lambda - \text{df}_{\lambda NT}) \Upsilon_{NT}.
\]

By the first result of this theorem, we know that \( \Pr(\text{df}_{\lambda NT} = p^\ast) \to 1 \). Since \( \lambda \in A^+ \), we must have \( \Pr(\text{df}_\lambda \geq p^\ast + 1) \to 1 \). Given \( \Upsilon_{NT} h_{NT}^{-1/2} \to \infty \), it is clear \( \Pr(\inf_{\lambda \in A^+} \text{BIC}_\lambda > \text{BIC}_{\lambda NT}) \to 1 \).

Combining Cases 1 and 2, we obtain that \( \Pr(\inf_{\lambda \in A^\ast \cup A^+} \text{BIC}_\lambda > \text{BIC}_{\lambda NT}) \to 1 \). This further indicates that \( \Pr(S_\lambda = A^\ast) \to 1 \). The proof of is now complete.

\[\blacksquare\]
As discussed in the context of Figure 1 in the main text, these confidence intervals need to be interpreted with caution. As well understood, one cannot establish the confidence intervals for the estimates under HD case unless certain transformation is further employed (e.g., Huang et al., 2008; Dong et al., 2017). However, if one regards 31 as a relatively small number after selection, we can then employ a procedure similar to the relevant literature by considering our regression under LD framework. In order to ensure the validity of the bootstrap procedure, stronger assumptions on the error terms are needed. For example, one can employ the martingale difference type of assumptions (see Assumption A.4 of Su et al., 2015), or simply assume that the error terms are i.i.d. over both $i$ and $t$. Generally speaking, when the error term exhibits both cross-sectional and serial correlation, the bootstrap results are not reliable or incorrect.
Figure A.3: Estimates of Coefficient Functions of All Selected Variables (Using Polity2)
Figure A.4: Estimates of Coefficient Functions of All Selected Variables (Using 20 years lag)
Figure A.5: Estimates of Coefficient Functions of All Selected Variables (Using A Fixed Effects Model)

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Supplementary Appendix B to

“An Integrated Panel Data Approach to Modelling Economic Growth”

(NOT for publication)

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This appendix provides the proofs omitted in the supplementary Appendix A.

Proofs of LD Case

Proof of Lemma A.1:

Write

\[ \|A^{-1} - B^{-1}\| = \|B^{-1} (B - A) A^{-1}\| = \|\text{vec} \left( B^{-1} (B - A) A^{-1} \right) \| \]

\[ = \| (A^{-1} \otimes B^{-1}) \text{vec} (B - A) \| \leq \eta_{\text{min}}^{-1} (A \otimes B) \| \text{vec} (B - A) \| \]

\[ = \eta_{\text{min}}^{-1} (A) \cdot \eta_{\text{min}}^{-1} (B) \| A - B \|. \]

The proof is then complete.

Proof of Lemma A.2:

(1). Firstly, write

\[ \frac{1}{N^2 T^2} E \| \mathcal{E}^t \mathcal{E}^t \| = E \left\| \frac{1}{N T} \sum_{i=1}^{N} \mathcal{E}^t_i \mathcal{E}^t_i \right\| = \frac{1}{N^2 T^2} \sum_{t=1}^{T} \sum_{s=1}^{T} \left( \sum_{i=1}^{N} E[\varepsilon_{it}^2 \varepsilon_{is}^2] + \sum_{i \neq j} E[(\varepsilon_{it} \varepsilon_{jt} - \sigma_{ij})(\varepsilon_{is} \varepsilon_{js} - \sigma_{ij})] + \sum_{i \neq j} \sigma_{ij}^2 \right) \]

\[ = \frac{1}{N^2 T^2} \sum_{t=1}^{T} \left( \sum_{i=1}^{N} E[\varepsilon_{it}^4] + \sum_{i \neq j} E[(\varepsilon_{it} \varepsilon_{jt} - \sigma_{ij})^2] \right) \]

\[ + \frac{1}{N^2 T^2} \sum_{t \neq s} \left( \sum_{i=1}^{N} E[\varepsilon_{it}^2 \varepsilon_{is}^2] + \sum_{i \neq j} E[(\varepsilon_{it} \varepsilon_{jt} - \sigma_{ij})(\varepsilon_{is} \varepsilon_{js} - \sigma_{ij})] \right) + \frac{1}{N^2} \sum_{i \neq j} \sigma_{ij}^2 \]

\[ = O(1) \frac{1}{N} + O(1) \frac{1}{T}, \quad (B.1) \]

where the fifth equality follows from using the mixing condition on \( \varepsilon_{it} \varepsilon_{jt} \) across \( t \). Thus, \( \frac{1}{N T} \| \mathcal{E}^t \mathcal{E}^t \| = O_P \left( \frac{1}{\sqrt{N}} \right) + O_P \left( \frac{1}{\sqrt{T}} \right) \).

Secondly, note that

\[ E \left\| \frac{1}{NT} \mathcal{E} \mathcal{E}^t \right\|^2 = \left\{ E \left[ \frac{1}{NT} \mathcal{E}^t \mathcal{E} \right] \right\}^2 \]

\[ = \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{N^2 T^2} \sum_{t=1}^{T} \sum_{s=1}^{T} E[\varepsilon_{it} \varepsilon_{jt} \varepsilon_{is} \varepsilon_{js}] \]

Electronic copy available at: https://ssrn.com/abstract=3348229
\[
\sum_{t=1}^{T} \sum_{s=1}^{T} \frac{1}{N^2T^2} \left( \sum_{i=1}^{N} E[x_{it}^2] + \sum_{i \neq j} E[x_{it}x_{tj}] \right) = O \left( \frac{1}{N} \right) + O \left( \frac{1}{T} \right),
\]
where the last step follows from (B.1). Thus, \( \frac{1}{N^2T} \|\mathcal{E}E'\| = O_P \left( \frac{1}{\sqrt{N}} \right) + O_P \left( \frac{1}{\sqrt{T}} \right). \]

(2). Write

\[
\sup_{F \in D_F} \frac{1}{NT} \sum_{i=1}^{N} E'_i P_F E_i = \sup_{F \in D_F} \frac{1}{NT} \text{tr} (P_F E E') \leq \sup_{F \in D_F} \frac{r}{NT} ||P_F||_{sp} ||E'E'||_{sp}
\]

\[
= O_P \left( \frac{1}{\sqrt{N}} \right) + O_P \left( \frac{1}{\sqrt{T}} \right),
\]
where the inequality follows from the fact that \( |\text{tr} (A)| \leq \text{rank} (A) \|A\|_{sp} \); and the second equality follows from (1) of this lemma.

(3). Write

\[
\sup_{F \in D_F} \frac{1}{NT} \gamma_0' \gamma_0 F_0' M_F E_i = \sup_{F \in D_F} \frac{1}{NT} \text{tr} (F_0' M_F E'_0) \leq \sup_{F \in D_F} \frac{r}{NT} ||F_0||_{sp} ||M_F||_{sp} ||E'_0||_{sp} = \sup_{F \in D_F} \frac{r}{NT} ||F_0||_{sp} ||E'_0||_{sp}^{1/2}
\]

\[
= \sup_{F \in D_F} \frac{r}{NT} ||F_0||_{sp} ||E'_0||_{sp} \left( \frac{1}{NT} ||E'E'|| \right)^{1/2} = O_P \left( \frac{1}{\sqrt{N}} ||E'E'|| \right)^{1/2}
\]

\[
= O_P \left( \frac{1}{\sqrt{N}} \right) + O_P \left( \frac{1}{\sqrt{T}} \right),
\]
where the first inequality follows from the fact that \( |\text{tr} (A)| \leq \text{rank} (A) \|A\|_{sp} \); the second equality follows from Fact 5.10.18 of Bernstein (2005); and the last equality follows from (1) of this lemma.

(4). Write

\[
\frac{1}{NT} \sum_{i=1}^{N} (\phi_i[\beta_{0,m}] - \phi_i[\beta_m])' M_F E_i
\]

\[
= \frac{1}{NT} \sum_{i=1}^{N} (\phi_i[\beta_{0,m}] - \phi_i[\beta_m])' E_i + \frac{1}{NT} \sum_{i=1}^{N} (\phi_i[\beta_{0,m}] - \phi_i[\beta_m])' P_F E_i
\]

\[
:= \Lambda_1 + \Lambda_2.
\]

For \( \Lambda_1 \), write

\[
\sup_{\|C_\beta\| \leq M} \frac{1}{NT} \sum_{i=1}^{N} (\phi_i[\beta_{0,m}] - \phi_i[\beta_m])' E_i \leq \sup_{\|C_\beta\| \leq M} \|\text{vec}(C_{\beta_0} - C_\beta)\| \cdot \left( \frac{1}{NT} \sum_{i=1}^{N} Z_i' E_i \right)
\]

\[
= \sup_{\|C_\beta\| \leq M} \|C_{\beta_0} - C_\beta\| \cdot O_P \left( \sqrt{\frac{m}{NT}} \right) = O_P \left( \sqrt{\frac{m}{NT}} \right),
\]
where the first equality follows from some standard analysis on the term \( \frac{1}{NT} \sum_{i=1}^{N} Z_i' E_i \) using Assumption 1.1; and the last equality follows from \( \|C_\beta\| \leq M \).

In order to consider \( \Lambda_2 \), let \( \Delta b = (\phi_1[\beta_{0,m}] - \phi_1[\beta_m], \ldots, \phi_N[\beta_{0,m}] - \phi_N[\beta_m]) \). Note that

\[
\sup_{\|C_\beta\| \leq M} \frac{1}{NT} ||\Delta b||^2 = \sup_{\|C_\beta\| \leq M} \frac{1}{NT} \sum_{i=1}^{N} (C_\beta - C_{\beta_0})' Z_i' Z_i (C_\beta - C_{\beta_0})
\]
\[ \leq O_P(1) \sup_{\|C_\beta\| \leq M} \|C_\beta - C_{\beta_0}\|^2 = O_P(1), \quad \text{(B.2)} \]

where the inequality follows from Assumptions 2.1, and \( \|1/NT \sum_{i=1}^N \mathcal{Z}_i' \mathcal{Z}_i' - \Sigma_2 \| = o_P(1) \) by some standard analysis using Assumption 1.1; and the last equality follows from \( \|C_\beta\| \leq M \).

Then we are able to write
\[
\sup_{\|C_\beta\| \leq M, F \in D_F} \left\| \frac{1}{NT} \sum_{i=1}^N (\phi_i[\beta_{0,m}] - \phi_i[\beta_m])' P_F E_i \right\| = \sup_{\|C_\beta\| \leq M, F \in D_F} \left\| \frac{1}{NT} \text{tr} (P_F E' \Delta b') \right\|
\]
\[
\leq \frac{r}{NT} \sup_{\|C_\beta\| \leq M, F \in D_F} \|P_F E' \Delta b\|_{sp} \leq \sup_{\|C_\beta\| \leq M, F \in D_F} \frac{r}{NT} \|P_F\|_{sp} \|E\|_{sp} \|\Delta b\|_{sp}
\]
\[
= \frac{1}{NT} \|E\|_{sp} \left( \frac{1}{NT} \|E'\| \right)^{1/2} \left( \frac{1}{\sqrt{NT}} \|\Delta b\| \right) = O_P \left( \frac{1}{\sqrt{N}} \right) + O_P \left( \frac{1}{\sqrt{T}} \right),
\]

where the second equality follows from Fact 5.10.18 of Bernstein (2005); and the last step follows from (1) of this lemma and (B.2).

Based on the above development on \( \Lambda_1 \) and \( \Lambda_2 \), the result follows.

(5). Write
\[
\sup_{F \in D_F} \left\| \frac{1}{NT} \sum_{i=1}^N \phi_i[\Delta_m] M_F \phi_i[\Delta_m] \right\| \leq \left\| \frac{1}{NT} \sum_{i=1}^N \phi_i[\Delta_m] \phi_i[\Delta_m] \right\|.
\]

Note that
\[
\frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T E \|\phi_i[\Delta_m]\|^2 = E\|x_i \Delta_m(z_{it})\|^2 \leq E\|x_i\|^2 \cdot E\|\Delta_m(z_{it})\|^2 \leq O(1) \int_{\mathbb{R}} \|\Delta_m(w)\|^2 \pi(w) \cdot \rho(w) dw = \|\Delta_m(w)\|^2_{L_2} = O(m^{-\mu}),
\]

where the second and third equalities follow from Assumption 2.1. Then the result follows.

(6). Write
\[
\sup_{F \in D_F} \left\| \frac{1}{NT} \sum_{i=1}^N \phi_i[\Delta_m] M_F F_0 \gamma_0 \right\| = \sup_{F \in D_F} \left\| \frac{1}{NT} \text{tr} (M_F F_0 \Gamma_0 \Delta') \right\|
\]
\[
\leq \frac{r}{NT} \sup_{F \in D_F} \|M_F\|_{sp} \|F_0\|_{sp} \|\Gamma_0\|_{sp} \|\Delta\|_{sp} = O_P(m^{-\frac{\mu}{2}}),
\]

where \( \Delta = (\phi_1[\Delta_m], \ldots, \phi_N[\Delta_m]) \); and the second equality follows from \( \frac{1}{NT} \|\Delta\|^2 = O_P(m^{-\mu}) \) as in (5) of this lemma.

(7). Write
\[
\sup_{\|C_\beta\| \leq M, F \in D_F} \left\| \frac{1}{NT} \sum_{i=1}^N \phi_i[\Delta_m] M_F \{\phi_i[\beta_m] - \phi_i[\beta_{0,m}]\} \right\|
\]
\[
\leq \left\{ \frac{1}{NT} \sum_{i=1}^N \|\phi_i[\Delta_m]\|^2 \right\}^{1/2} \sup_{\|C_\beta\| \leq M} \left\{ \frac{1}{NT} \sum_{i=1}^N \|\phi_i[\beta_m] - \phi_i[\beta_{0,m}]\|^2 \right\}^{1/2} = O_P(m^{-\frac{\mu}{2}}),
\]

where the last equality follows from (5) of this lemma. The proof is now complete.
Proof of Lemma A.3:

(1). For notational simplicity, let $\Delta \phi_i[\beta_m] = \phi_i[\beta_{0,m}] - \phi_i[\beta_m]$. Let $\xi_F = \text{vec} (M_F F_0)$, $A_{1F} = \frac{1}{NT} \sum_{i=1}^N Z_i^t M_F Z_i$, $A_2 = \frac{1}{NT} \left( \Gamma_0' \Gamma_0 \right) \otimes I_T$, and $A_{3F} = \frac{1}{NT} \sum_{i=1}^N \gamma_i \otimes (M_F Z_i)$, where $Z_i$ has been defined in Assumption 2. By the definition of (3.3) and Lemma A.2, we have

$$0 \geq \frac{1}{NT} Q_\lambda(\hat{C}_\beta, \hat{F}) - \frac{1}{NT} Q_\lambda(C_{\beta_0}, F_0)$$

$$= \frac{1}{NT} \sum_{i=1}^N \left( \Delta \phi_i[\beta_m] + F_0 \gamma_{0i} \right)' M_F \left( \Delta \phi_i[\beta_m] + F_0 \gamma_{0i} \right) + \frac{1}{NT} \sum_{i=1}^N \varepsilon_i'M_F \varepsilon_i$$

$$+ \frac{2}{NT} \sum_{i=1}^N \left( \Delta \phi_i[\beta_m] + F_0 \gamma_{0i} \right)' M_F \varepsilon_i + \sum_{j=1}^p \frac{\lambda_j}{NT} \| \hat{C}_{\beta,j} \|$$

$$- \frac{1}{NT} \sum_{i=1}^N \left( \phi_i[\Delta_m] + \varepsilon_i \right)' M_F_0 \left( \phi_i[\Delta_m] + \varepsilon_i \right) - \sum_{j=1}^{p^*} \frac{\lambda_j}{NT} \| C_{\beta_0,j} \|$$

$$= \frac{1}{NT} \sum_{i=1}^N \left( \Delta \phi_i[\beta_m] + F_0 \gamma_{0i} \right)' M_F \left( \Delta \phi_i[\beta_m] + F_0 \gamma_{0i} \right)$$

$$+ \sum_{j=1}^p \frac{\lambda_j}{NT} \| \hat{C}_{\beta,j} \| - \sum_{j=1}^{p^*} \frac{\lambda_j}{NT} \| C_{\beta_0,j} \| + O_P \left( \frac{1}{\sqrt{\xi NT}} + m^{-\frac{3}{2}} \right)$$

$$= \text{vec}(C_{\beta_0} - \hat{C}_\beta)' \frac{1}{NT} \sum_{i=1}^N Z_i^t M_F Z_i \text{vec}(C_{\beta_0} - \hat{C}_\beta) + \frac{1}{NT} \text{tr} \left( M_F \Gamma_0' \Gamma_0 F_0 \hat{F} \right)$$

$$+ 2 \text{vec}(C_{\beta_0} - \hat{C}_\beta)' \frac{1}{NT} \sum_{i=1}^N Z_i^t M_F F_0 \gamma_{0i} + \sum_{j=1}^p \frac{\lambda_j}{NT} \| \hat{C}_{\beta,j} \| - \sum_{j=1}^{p^*} \frac{\lambda_j}{NT} \| C_{\beta_0,j} \|$$

$$+ O_P \left( \frac{1}{\sqrt{\xi NT}} + m^{-\frac{3}{2}} \right),$$

where the second equality follows from Lemma A.2. Thus, we can further write

$$\sum_{j=1}^{p^*} \frac{\lambda_j}{NT} \| C_{\beta_0,j} \| \geq \text{vec}(C_{\beta_0} - \hat{C}_\beta)' \frac{1}{NT} \sum_{i=1}^N Z_i^t M_F Z_i \text{vec}(C_{\beta_0} - \hat{C}_\beta) + \frac{1}{NT} \text{tr} \left( M_F \Gamma_0' \Gamma_0 F_0 \hat{F} \right)$$

$$+ 2 \text{vec}(C_{\beta_0} - \hat{C}_\beta)' \frac{1}{NT} \sum_{i=1}^N Z_i^t M_F F_0 \gamma_{0i} + O_P \left( \frac{1}{\sqrt{\xi NT}} + m^{-\frac{3}{2}} \right)$$

$$\geq \text{vec}(C_{\beta_0} - \hat{C}_\beta)' \left( A_{1F}' - A_{3F}' A_2^{-1} A_{3F} \right) \text{vec}(C_{\beta_0} - \hat{C}_\beta)$$

$$+ [\xi_\hat{F} + \text{vec}(C_{\beta_0} - \hat{C}_\beta)' A_{3F}' A_2^{-1}] A_2 [\xi_\hat{F} + A_2^{-1} A_{3F} \text{vec}(C_{\beta_0} - \hat{C}_\beta)]$$

$$+ O_P \left( \frac{1}{\sqrt{\xi NT}} + m^{-\frac{3}{2}} \right)$$

$$\geq O_P(1) \| C_{\beta_0} - \hat{C}_\beta \|^2 + O_P \left( \frac{1}{\sqrt{\xi NT}} + m^{-\frac{3}{2}} \right). \quad (B.3)$$

Till now, we can conclude that

$$\| C_{\beta_0} - \hat{C}_\beta \|^2 = O_P \left( \frac{1}{\sqrt{\xi NT}} + m^{-\frac{3}{2}} + \frac{\lambda_{\text{max}}}{NT} \right) = o_P(1), \quad (B.4)$$

where the second equality follows from Assumption 3.

(2). By (B.3) and (B.4), we can further obtain that

$$o_P(1) \geq \frac{1}{NT} \text{tr} \left[ \left( F_0' M_F F_0 \right) \left( \Gamma_0' \Gamma_0 \right) \right] + o_P(1),$$
so \( \frac{1}{N^2} \text{tr} \left( (F_0' M \hat{F}_0 - F_0' M F_0) (\Gamma_0' \Gamma_0) \right) = o_P(1) \). As in Bai (2009, p. 1265), we can further conclude that \( \frac{1}{T} \text{tr} \left( F_0' M \hat{F}_0 - F_0' M F_0 \right) = o_P(1) \), \( \|P_{\hat{F}} - P_{F_0}\| = o_P(1) \), and \( \frac{1}{T} \hat{F}' F_0 \) is invertible with probability approaching one. Thus, the second result of this lemma follows.

(3). Minimizing the objective function (3.2) of the main text with respect to \( F \) does not involve the penalty term. Thus, following the same arguments as in Bai (2009, p. 1236), the estimate \( \hat{F} \) of (3.3) is obtained by

\[
\frac{1}{N^T} \sum_{i=1}^{N} \left( Y_i - \phi_1[\hat{\beta}_m] \right) \left( Y_i - \phi_1[\hat{\beta}_m] \right)' \hat{F} = \hat{F} V_{NT},
\]

where \( V_{NT} \) is a diagonal matrix with the diagonal being the \( r \) largest eigenvalues of

\[
\frac{1}{N^T} \sum_{i=1}^{N} \left( Y_i - \phi_1[\hat{\beta}_m] \right) \left( Y_i - \phi_1[\hat{\beta}_m] \right)'
\]

arranged in descending order.

We now consider \( V_{NT} \) and write

\[
\hat{F} V_{NT} = \left[ \frac{1}{N^T} \sum_{i=1}^{N} \left( Y_i - \phi_1[\hat{\beta}_m] \right) \left( Y_i - \phi_1[\hat{\beta}_m] \right)' \right] \hat{F}
\]

\[
= \left[ \frac{1}{N^T} \sum_{i=1}^{N} \left( \phi_i[\beta_0] + F_0 \gamma_0 + \epsilon_i - \phi_1[\hat{\beta}_m] \right) \left( \phi_i[\beta_0] + F_0 \gamma_0 + \epsilon_i - \phi_1[\hat{\beta}_m] \right) \right] \hat{F}
\]

\[
= \frac{1}{N^T} \sum_{i=1}^{N} \left( \phi_i[\beta_0] - \phi_1[\hat{\beta}_m] \right) \left( \phi_i[\beta_0] - \phi_1[\hat{\beta}_m] \right)' \hat{F}
\]

\[
+ \frac{1}{N^T} \sum_{i=1}^{N} \left( \phi_i[\beta_0] - \phi_1[\hat{\beta}_m] \right) (F_0 \gamma_0)' \hat{F} + \frac{1}{N^T} \sum_{i=1}^{N} (F_0 \gamma_0) \left( \phi_i[\beta_0] - \phi_1[\hat{\beta}_m] \right)' \hat{F}
\]

\[
+ \frac{1}{N^T} \sum_{i=1}^{N} \left( \phi_i[\beta_0] - \phi_1[\hat{\beta}_m] \right) \epsilon_i' \hat{F} + \frac{1}{N^T} \sum_{i=1}^{N} \epsilon_i \left( \phi_i[\beta_0] - \phi_1[\hat{\beta}_m] \right)' \hat{F}
\]

\[
+ \frac{1}{N^T} \sum_{i=1}^{N} \epsilon_i \epsilon_i' \hat{F} + \frac{1}{N^T} \sum_{i=1}^{N} F_0 \gamma_0 \epsilon_i' \hat{F} + \frac{1}{N^T} \sum_{i=1}^{N} \epsilon_i \gamma_0 F_0' \hat{F} + \frac{1}{N^T} \sum_{i=1}^{N} F_0 \gamma_0 \gamma_0' F_0' \hat{F}
\]

\[
:= I_{1NT}(\hat{\beta}_m, \hat{F}) + \cdots + I_{5NT}(\hat{\beta}_m, \hat{F}) + I_{6NT}(\hat{F}) + \cdots + I_{9NT}(\hat{F}),
\]

where the definitions of \( I_{1NT}(\beta, F) \) to \( I_{5NT}(\beta, F) \) and \( I_{6NT}(F) \) to \( I_{9NT}(F) \) should be obvious.

Note that \( I_{9NT}(\hat{F}) = F_0'(\Gamma_0' \Gamma_0/N)/(F_0' \hat{F})/T \). Thus, we can write

\[
\hat{F} V_{NT} - F_0'(\Gamma_0' \Gamma_0/N)/(F_0' \hat{F})/T
\]

\[
= I_{1NT}(\hat{\beta}_m, \hat{F}) + \cdots + I_{5NT}(\hat{\beta}_m, \hat{F}) + I_{6NT}(\hat{F}) + \cdots + I_{8NT}(\hat{F}). \tag{B.5}
\]

Right multiplying each side of (B.5) by \((F_0' \hat{F})/T)^{-1}(\Gamma_0' \Gamma_0/N)^{-1}, \), we obtain

\[
\hat{F} V_{NT}(F_0' \hat{F})/T)^{-1}(\Gamma_0' \Gamma_0/N)^{-1} - F_0 \]

\[
= \left[ I_{1NT}(\hat{\beta}_m, \hat{F}) + \cdots + I_{8NT}(\hat{F}) \right](F_0' \hat{F})/T)^{-1}(\Gamma_0' \Gamma_0/N)^{-1}. \tag{B.6}
\]

Below, we examine each term on the right hand side of (B.6) and show that \( V_{NT} \) is non-singular. Write

\[
\frac{1}{\sqrt{T}} \left\| \hat{F} V_{NT}(F_0' \hat{F})/T)^{-1}(\Gamma_0' \Gamma_0/N)^{-1} - F_0 \right\|
\]
\[ \leq \frac{1}{\sqrt{T}} \left[ \|I_{1NT}(\hat{\beta}_m, \hat{F})\| + \cdots + \|I_{8NT}(\hat{F})\| \right] \cdot (F_0' \hat{F}/T)^{-1}(G_0'G_0/N)^{-1}. \]

We know \((F_0' \hat{F}/T)^{-1} = O_P(1)\) and \((G_0'G_0/N)^{-1} = O_P(1)\), so focus on \(\frac{1}{\sqrt{T}} \|I_{jNT}(\hat{\beta}_m, \hat{F})\|\) with \(j = 1, 2, \ldots, 5\) and \(\frac{1}{\sqrt{T}} \|I_{jNT}(\hat{F})\|\) with \(j = 6, 7, 8\) below.

For \(I_{1NT}(\hat{\beta}_m, \hat{F})\), we have

\[ \frac{1}{\sqrt{T}} \|I_{1NT}(\hat{\beta}_m, \hat{F})\| \leq \frac{\sqrt{T}}{NT} \sum_{i=1}^{N} \left\| \phi_i[\beta_0] - \phi_i[\hat{\beta}_m] \right\|^2 \]

\[ \leq \frac{\sqrt{T}}{NT} \sum_{i=1}^{N} \left\| \phi_i[\beta_0,m] - \phi_i[\hat{\beta}_m] \right\|^2 + \frac{\sqrt{T}}{NT} \sum_{i=1}^{N} \left\| \Delta_m \right\|^2 \]

\[ = \text{vec}(C_{\beta_0} - \hat{C}_\beta)' \frac{\sqrt{T}}{NT} \sum_{i=1}^{N} Z_i'Z_i \text{vec}(C_{\beta_0} - \hat{C}_\beta) + O_P(m^{-\mu}) \]

\[ = O_P(\|C_{\beta_0} - \hat{C}_\beta\|^2) + O_P(m^{-\mu}) = O_P(\|\hat{\beta}_m - \beta_0\|_{L^2}), \]

where the first and second equalities follow from Assumption 2.1.

For \(I_{2NT}(\hat{\beta}_m, \hat{F})\), write

\[ \frac{1}{\sqrt{T}} \|I_{2NT}(\hat{\beta}_m, \hat{F})\| \leq \frac{\sqrt{T}}{NT} \sum_{i=1}^{N} \left\| \phi_i[\beta_0] - \phi_i[\hat{\beta}_m] \right\|^2 \left\{ \frac{1}{NT} \sum_{i=1}^{N} \|F_0'0_i\|^2 \right\}^{1/2} \]

\[ = O_P(\|\hat{\beta}_m - \beta_0\|_{L^2}), \]

where the second inequality follows from Cauchy-Schwarz inequality; and the last line follows from the same arguments given for \(I_{1NT}(\hat{\beta}_m, \hat{F})\) and the fact that \(\frac{1}{NT} \sum_{i=1}^{N} \|F_0'0_i\|^2 = O_P(1)\). Similarly, we have \(\frac{1}{\sqrt{T}} \|I_{jNT}(\hat{\beta}_m, \hat{F})\| = O_P(\|\hat{\beta}_m - \beta_0\|_{L^2})\) for \(j = 3, 4, 5\). By (1) of Lemma A.2 and \(\frac{1}{\sqrt{T}} \|\hat{F}\| = O(1)\), we also obtain \(\frac{1}{\sqrt{T}} \|I_{6NT}(\hat{F})\| = O_P \left( \frac{1}{\sqrt{N}} \right) \) + \(O_P \left( \frac{1}{\sqrt{T}} \right) \).

For \(I_{7NT}(\hat{F})\) and \(I_{8NT}(\hat{F})\), write

\[ E \left[ \frac{1}{NT} \sum_{i=1}^{N} F_0'0_i \epsilon_i' \right]^2 = \sum_{t=1}^{T} \sum_{s=1}^{T} \frac{1}{N^2T^2} \sum_{i=1}^{N} \sum_{j=1}^{N} E[f_0'0_i \epsilon_{i,s} f_0'0_j \epsilon_{j,s}] \]

\[ = O(1) \sum_{t=1}^{T} \sum_{s=1}^{T} \frac{1}{N^2T^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \left\{ E[f_0'0_i \epsilon_{i,s}]^4 E[f_0'0_j \epsilon_{j,s}]^4 \right\}^{1/4} |\sigma_{ij}| \]

\[ \leq O(1) \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} |\sigma_{ij}| = O \left( \frac{1}{N} \right), \]

where the first inequality follows from Cauchy-Schwarz inequality and Assumption 1. We then can conclude that \(\frac{1}{\sqrt{T}} \|I_{7NT}(\hat{F})\| = \frac{1}{\sqrt{T}} \|I_{8NT}(\hat{F})\| = O_P \left( \frac{1}{\sqrt{N}} \right) \).

Based on the above analysis and by left multiplying (B.5) by \(\hat{F}'/T\), we obtain

\[ V_{NT} - (\hat{F}' F_0/T)(G_0'G_0/N)(F_0' \hat{F}/T) = \frac{1}{T} \hat{F}' \left[ I_{1NT}(\hat{\beta}_m, \hat{F}) + \cdots + I_{8NT}(\hat{F}) \right] = o_P(1). \]
Thus, \( V_{NT} = (\hat{F}^t F_0 / T)(\Gamma_0^T \Gamma_0 / N)(F_0^t \hat{F} / T) + o_P(1) \). When proving the second result of this lemma, we have shown that \( F_0^t \hat{F} / T \) is non-singular with probability approaching one, which implies that \( V_{NT} \) is invertible with probability approaching one. We now left multiply (B.5) by \( F_0 / T \) to obtain

\[
(F_0^t \hat{F} / T)V_{NT} = (F_0^t F_0 / T)(\Gamma_0^T \Gamma_0 / N)(F_0^t \hat{F} / T) + o_P(1)
\]

based on the above analysis. It shows that the columns of \( F_0^t \hat{F} / T \) are the (non-normalized) eigenvectors of the matrix \((F_0 F_0 / T)(\Gamma_0 \Gamma_0 / N)\), and \( V_{NT} \) consists of the eigenvalues of the same matrix (in the limit). Thus, the result follows.

(4). According to the above analysis, (B.6) can be summarized by

\[
\frac{1}{\sqrt{T}} \| \hat{F} \Pi_{NT}^1 - F_0 \| = O_P(\| \hat{\beta}_m - \beta_0 \|_{L^2}) + O_P \left( \frac{1}{\sqrt{N}} \right) + O_P \left( \frac{1}{\sqrt{T}} \right).
\]

(5). According to (B.6),

\[
\frac{1}{T} F_0^t (\hat{F} - F_0 \Pi_{NT}) = \frac{1}{T} F_0^t \left[ I_{1NT}(\hat{\beta}_m, \hat{F}) + \cdots + I_{8NT}(\hat{F}) \right] V_{NT}^{-1}.
\]

Note that \( V_{NT}^{-1} = O_P(1) \), so we focus on \( \frac{1}{T} F_0^t \left[ I_{1NT}(\hat{\beta}_m, \hat{F}) + \cdots + I_{8NT}(\hat{F}) \right] \) below. By the proof given for the first result of this lemma, it is easy to show that

\[
\left\| \frac{1}{T} F_0^t \left[ I_{1NT}(\hat{\beta}_m, \hat{F}) + \cdots + I_{5NT}(\hat{\beta}_m, \hat{F}) \right] \right\| = O_P(\| \hat{\beta}_m - \beta_0 \|_{L^2}).
\]

We now consider \( \left\| \frac{1}{T} F_0^t I_{6NT}(\hat{F}) \right\| \). Firstly, note that \( \frac{1}{NT} \sum_{i=1}^{N} \| F_0^t \xi_i \|^2 = O_P(1) \). Secondly, for \( \frac{1}{NT} \sum_{i=1}^{N} \| \xi_i^t \hat{F} \|^2 \), we have

\[
\frac{1}{NT} \sum_{i=1}^{N} \| \xi_i^t \hat{F} \|^2 \leq 2 \frac{1}{NT} \sum_{i=1}^{N} \| \xi_i^t F_0 \Pi_{NT} \|^2 + 2 \frac{1}{NT} \sum_{i=1}^{N} \| \xi_i^t (\hat{F} - F_0 \Pi_{NT}) \|^2
\]

\[
= 2 \frac{1}{NT} \sum_{i=1}^{N} \| \xi_i^t F_0 \Pi_{NT} \|^2 + 2 \frac{1}{NT} \sum_{i=1}^{N} \left\{ \xi_i^t (\hat{F} - F_0 \Pi_{NT}) \left( \hat{F} - F_0 \Pi_{NT} \right)^t \right\} \xi_i
\]

\[
\leq O_P(1) + O(1) \frac{1}{N} \| \xi_i^t \xi_i \| \frac{1}{T} \| \hat{F} - F_0 \Pi_{NT} \|^2,
\]

where \( \xi_i \) has been defined in Lemma A.2. In connection with (1) of Lemma A.2 and (2) of this lemma, it gives that

\[
\left\| \frac{1}{T} F_0^t I_{6NT}(\hat{F}) \right\| \leq \frac{1}{T} \left( \frac{1}{NT} \sum_{i=1}^{N} \| F_0^t \xi_i \|^2 \right)^{1/2} \left( \frac{1}{NT} \sum_{i=1}^{N} \| \xi_i^t \hat{F} \|^2 \right)^{1/2}
\]

\[
= O_P(1) + O_P \left( \frac{1}{\sqrt{T}} \right) \left\{ \frac{1}{NT} \| \xi_i^t \xi_i \| \frac{1}{T} \| \hat{F} - F_0 \Pi_{NT} \|^2 \right\}^{1/2}
\]

\[
= O_P(1) + O_P \left( \frac{1}{\sqrt{T}} \right) O_P \left( \frac{1}{\sqrt{N}} + \frac{1}{\sqrt{T}} \right) O_P \left( \| \hat{\beta}_m - \beta_0 \|_{L^2} + \frac{1}{\sqrt{N}} + \frac{1}{\sqrt{T}} \right)
\]

\[
= O_P(1) \left\{ \frac{1}{T} + \frac{\| \hat{\beta}_m - \beta_0 \|_{L^2} }{\sqrt{T} \sqrt{N}} + \frac{1}{\sqrt{T} \sqrt{N^3}} \right\}
\]

\[
\leq o_P(1) \| \hat{\beta}_m - \beta_0 \|_{L^2} + O_P(1) \frac{1}{T} + O_P(1) \frac{1}{T} \sqrt{N^3}.
\]
where the second equality follows from (1) of Lemma A.2 and the second result of this lemma.

For $\|\frac{1}{T} F'_0 I_{NT} (\hat{F})\|$, we have

$$
\begin{align*}
\|\frac{1}{T} F'_0 I_{NT} (\hat{F})\| & \leq \|\frac{1}{T} F'_0 F_0\| \cdot \|\frac{1}{NT} \sum_{i=1}^{N} \gamma_0 \mathcal{E}'_i (\hat{F} - F_0 \Pi_{NT})\| \\
& \quad + \|\frac{1}{T} F'_0 F_0\| \cdot \|\frac{1}{NT} \sum_{i=1}^{N} \gamma_0 \mathcal{E}'_i F_0 \Pi_{NT}\|
\end{align*}
$$

By Assumption 1.2, $\|\frac{1}{T} F'_0 F_0\| = O_P(1)$. By the first two results of this lemma, we have $\|\Pi_{NT}\| = O_P(1)$ and $\frac{1}{\sqrt{N}} \|\hat{F} - F_0 \Pi_{NT}\| = O_P(\|\hat{\beta}_m - \beta_0\|_{L2}) + O_P \left( \frac{1}{\sqrt{N}} \right) + O_P \left( \frac{1}{\sqrt{T}} \right)$. Therefore, we focus on $\|\frac{1}{N \sqrt{T}} \sum_{i=1}^{N} \gamma_0 \mathcal{E}'_i\|$ and $\|\frac{1}{NT} \sum_{i=1}^{N} \gamma_0 \mathcal{E}'_i F_0\|$ below. Write

$$
E \left[ \frac{1}{N \sqrt{T}} \sum_{i=1}^{N} \gamma_0 \mathcal{E}'_i \right]^2 = \frac{1}{N^2 T} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t=1}^{T} E[\gamma_0 \gamma_0_j] E[\varepsilon_u \varepsilon_{j,t}]
$$

$$
\leq O(1) \frac{1}{N^2 T} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t=1}^{T} |E[\varepsilon_u \varepsilon_{j,t}]| = O(1) \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} |\sigma_{ij}| = O \left( \frac{1}{N} \right) \tag{B.7}
$$

and using Assumption 1, it is easy to show that

$$
E \left[ \frac{1}{NT} \sum_{i=1}^{N} \gamma_0 \mathcal{E}'_i F_0 \right]^2 = O \left( \frac{1}{NT} \right), \tag{B.8}
$$

which immediately yields

$$
\begin{align*}
\|\frac{1}{T} F'_0 I_{NT} (\hat{F})\| & = O_P \left( \|\hat{\beta}_m - \beta_0\|_{L2} \cdot \frac{1}{\sqrt{N}} \right) + O_P \left( \frac{1}{\sqrt{N}} \right) + O_P \left( \frac{1}{\sqrt{T}} \right) \\
& \leq O_P \left( \|\hat{\beta}_m - \beta_0\|_{L2}^2 \right) + O_P \left( \frac{1}{N} \right) + O_P \left( \frac{1}{T} \right).
\end{align*}
$$

Similarly, $\|\frac{1}{T} F'_0 I_{SN} (\hat{F})\| = O_P \left( \|\hat{\beta}_m - \beta_0\|_{L2} \right) + O_P \left( \frac{1}{N} \right) + O_P \left( \frac{1}{T} \right)$.

Based on the above analysis, we have

$$
\|\frac{1}{T} F'_0 (\hat{F} - F_0 \Pi_{NT})\| = O_P(\|\hat{\beta}_m - \beta_0\|_{L2}) + O_P \left( \frac{1}{N} \right) + O_P \left( \frac{1}{T} \right), \tag{B.9}
$$

which further indicates

$$
\begin{align*}
\|\frac{1}{T} \hat{F}' (\hat{F} - F_0 \Pi_{NT})\| & = \|\frac{1}{T} (\hat{F} - F_0 \Pi_{NT} + F_0 \Pi_{NT})' (\hat{F} - F_0 \Pi_{NT})\| \\
& \leq \|\frac{1}{T} (\hat{F} - F_0 \Pi_{NT})' (\hat{F} - F_0 \Pi_{NT})\| + \|\Pi_{NT}\| \cdot \|\frac{1}{T} F'_0 (\hat{F} - F_0 \Pi_{NT})\| \\
& = O_P(\|\hat{\beta}_m - \beta_0\|_{L2}) + O_P \left( \frac{1}{N} \right) + O_P \left( \frac{1}{T} \right). \tag{B.10}
\end{align*}
$$

(6). Note (B.9) and (B.10) can be respectively expressed as
\[
\frac{1}{T} F_0' \hat{F} - \frac{1}{T} F_0' F_0 \Pi_{NT} = O_P(\|\hat{\beta}_m - \beta_0\|_{L^2}) + O_P\left(\frac{1}{N}\right) + O_P\left(\frac{1}{T}\right) 
\]
and
\[
I_r - \frac{1}{T} F_0' F_0 \Pi_{NT} = O_P(\|\hat{\beta}_m - \beta_0\|_{L^2}) + O_P\left(\frac{1}{N}\right) + O_P\left(\frac{1}{T}\right),
\]
which further give
\[
\frac{1}{T} \Pi_{NT}' F_0' \hat{F} - \frac{1}{T} \Pi_{NT}' F_0' F_0 \Pi_{NT} = O_P(\|\hat{\beta}_m - \beta_0\|_{L^2}) + O_P\left(\frac{1}{N}\right) + O_P\left(\frac{1}{T}\right)
\]
and
\[
I_r - \frac{1}{T} \Pi_{NT}' F_0' \hat{F} = O_P(\|\hat{\beta}_m - \beta_0\|_{L^2}) + O_P\left(\frac{1}{N}\right) + O_P\left(\frac{1}{T}\right). 
\]
Summing up the above two equations yields
\[
I_r - \frac{1}{T} \Pi_{NT}' F_0' F_0 \Pi_{NT} = O_P(\|\hat{\beta}_m - \beta_0\|_{L^2}) + O_P\left(\frac{1}{N}\right) + O_P\left(\frac{1}{T}\right). 
\]
(B.11)

Note that it is easy to show that
\[
\| P_\hat{F} - P_{F_0} \|^2 = \text{tr} \left[ (P_\hat{F} - P_{F_0})^2 \right] = \text{tr} \left[ P_\hat{F} - P_\hat{F} P_{F_0} P_{F_0} P_\hat{F} + P_{F_0} \right]
\]
\[
= \text{tr}[I_r] - 2\text{tr} \left[ P_\hat{F} P_{F_0} \right] + \text{tr}[I_r] = 2\text{tr} \left[ I_r - \hat{F}' P_{F_0} \hat{F}/T \right] 
\]
and, when proving this lemma, we have shown that
\[
\frac{F_0' \hat{F}}{T} = \frac{F_0' F_0}{T} \Pi_{NT} + O_P(\|\hat{\beta}_m - \beta_0\|_{L^2}) + O_P\left(\frac{1}{N}\right) + O_P\left(\frac{1}{T}\right). 
\]
Therefore, we can write
\[
\hat{F}' P_{F_0} \hat{F}/T = \Pi_{NT}' \left( \frac{F_0' F_0}{T} \right) \Pi_{NT} + O_P(\|\hat{\beta}_m - \beta_0\|_{L^2}) + O_P\left(\frac{1}{N}\right) + O_P\left(\frac{1}{T}\right). 
\]
In connection with (B.11), we then obtain that
\[
\hat{F}' P_{F_0} \hat{F}/T = I_r + O_P(\|\hat{\beta}_m - \beta_0\|_{L^2}) + O_P\left(\frac{1}{N}\right) + O_P\left(\frac{1}{T}\right),
\]
which completes the proof.

\textbf{Proof of Lemma A.4:}

(1). For simplicity, we show that \( \Pr(\|\hat{C}_{\beta,p}\| = 0) \rightarrow 1 \) only. The proofs for \( \|\hat{C}_{\beta,j}\| \) with \( j = p^* + 1, \ldots, p - 1 \) are the same. By (B.4) and Assumption 3, we can conclude that
\[
\|C_{\beta_0} - \hat{C}_\beta\| = O_P\left(\frac{1}{\sqrt{N_T}}\right). 
\]
(B.12)

If \( \|\hat{C}_{\beta,p}\| \neq 0 \), the following equation must hold:
\[
0 = \frac{\partial}{\partial C_{\beta,p}} Q_\lambda(C_\beta, F)\big|_{(C,F)=(\hat{C}_\beta,F)} = -2B_1 + B_2, 
\]
(B.13)
where $B_1 = \sum_{i=1}^{N} Z_{ip}^t M_{\bar{F}}(Y_i - \phi_i \hat{\beta}_m)$, $Z_{ip} = (x_{i1,p} H_m(z_{i1}), \ldots, x_{iT,p} H_m(z_{iT}))'$ and $B_2 = \frac{\lambda_p}{\|C_{\beta,p}\|} \hat{C}_{\beta,p}$. For $B_1$, write
\[
\frac{1}{NT} B_1 = \frac{1}{NT} \sum_{i=1}^{N} Z_{ip}^t M_{\bar{F}}(\phi_i \hat{\beta}_m - \phi_i \hat{\beta}_m + F_0 \gamma_i + \xi_i)
\]

In view of (B.12) and the development of Lemma A.3, it is easy to know $\frac{\|\xi_{\sqrt{NT} \sqrt{m}} \|}{\|\xi_{\sqrt{m}NT} \|}$ is a constant $\kappa_1$ by Assumption 3. Therefore, $\Pr(\|B_1\| < \|B_2\|) \rightarrow 1$, which implies that, with a probability tending to 1, (B.13) does not hold. The above analysis implies that $\hat{C}_{\beta,p}$ must be located at a place where the objective function $Q_X(C_{\beta}, F)$ is not differentiable with respect to $C_{\beta,p}$. Since $Q_X(C_{\beta}, F)$ is not differentiable with respect to $C_{\beta,p}$ only at the origin, we immediately obtain that $\Pr(\|\hat{C}_{\beta,p}\| = 0) \rightarrow 1$. Similarly, we can show $\Pr(\|\hat{C}_{\beta,j}\| = 0) \rightarrow 1$ with $j = p^* + 1, \ldots, p - 1$. The proof of the first result is complete.

(2). Note that (B.12) only gives a slow rate. Below, we aim to improve this rate. Having proved $\Pr(\|\hat{C}_{\beta}^1\| = 0) \rightarrow 1$, we delete the corresponding rows of $\hat{C}_\beta$ and $x_{i,j}$ for $j = p^* + 1, \ldots, p$ from the objective function. Following the same arguments as in Bai (2009, p. 1236), the estimator $\hat{C}_\beta$ given by (3.3) can be written as
\[
\text{vec}(\hat{C}_\beta) = \left( \sum_{i=1}^{N} Z_i^t M_{\bar{F}} Z_i + \frac{D_{m,p^*}}{2} \right)^{-1} \sum_{i=1}^{N} Z_i^t M_{\bar{F}} Y_i,
\]
where $Z_i^* = (Z_{i1}, \ldots, Z_{iT})'$ and $D_{m,p^*} = I_m \otimes \text{diag} \left\{ \frac{\lambda_1}{\|C_{\beta,1}\|}, \ldots, \frac{\lambda_{p^*}}{\|C_{\beta,p^*}\|} \right\}$. Correspondingly, we denote
\[
\text{vec}(\hat{C}_\beta^2) = \left( \sum_{i=1}^{N} Z_i^t M_{\bar{F}} Z_i \right)^{-1} \sum_{i=1}^{N} Z_i^t M_{\bar{F}} Y_i.
\]
Thus, we can write
\[
\hat{C}_\beta^* - C_{\beta_0} = (\hat{C}_\beta^* - \hat{C}_\beta^2) + (\hat{C}_\beta^2 - C_{\beta_0}).
\]
Below, we investigate the terms on the right hand side of the above equation.

Firstly, consider $\hat{C}_\beta^* - \hat{C}_\beta^2$, and write
\[
\text{vec}(\hat{C}_\beta) - \text{vec}(\hat{C}_\beta^2) = \left\{ \left( \sum_{i=1}^{N} Z_i^t M_{\bar{F}} Z_i + \frac{D_{m,p^*}}{2} \right)^{-1} - \left( \sum_{i=1}^{N} Z_i^t M_{\bar{F}} Z_i \right)^{-1} \right\} \sum_{i=1}^{N} Z_i^t M_{\bar{F}} Y_i.
\]
By Lemma A.1 and Assumption 2, we just need to consider the next term in order to get the difference between $\hat{C}_\beta^*$ and $\hat{C}_\beta^2$.
\[
\left\| \frac{1}{NT} \sum_{i=1}^{N} Z_i^t M_{\bar{F}} Z_i + \frac{D_{m,p^*}}{2NT} - \frac{1}{NT} \sum_{i=1}^{N} Z_i^t M_{\bar{F}} Z_i \right\| \frac{D_{m,p^*}}{2NT} = O \left( \frac{\sqrt{m} \lambda_{\max}}{\sqrt{NT}} \right).
\] (B.14)

Moreover, it is easy to know $\frac{1}{NT} \sum_{i=1}^{N} Z_i^t M_{\bar{F}} Y_i = O_P(\sqrt{m})$, which in connection with (B.14) indicates $\|\hat{C}_\beta^* - \hat{C}_\beta^2\| = O_P \left( \frac{m \lambda_{\max}}{\sqrt{NT}} \right).

We now focus on $\hat{C}_\beta^2 - C_{\beta_0}$, and write
\[
\text{vec}(\hat{C}_\beta^2) - \text{vec}(C_{\beta_0}) = \left[ \sum_{i=1}^{N} Z_i^t M_{\bar{F}} Z_i \right]^{-1} \sum_{i=1}^{N} Z_i^t M_{\bar{F}} \xi_i
\]
where the definitions of $\Lambda_1$-$\Lambda_3$ should be obvious. Note

$$
\frac{1}{NT} \sum_{i=1}^{N} Z_i^t M_F Z_i^t = \frac{1}{NT} \sum_{i=1}^{N} Z_i^t M_F \Sigma_{Z_i}^t \cdot (1 + o_P(1)) = \Sigma_{Z_i}^t \cdot (1 + o_P(1)),
$$

where $\Sigma_{Z_i}^t = E[Z_i^t Z_i^t] - E[Z_i^t f_0]\Sigma_f^{-1} E[f_0 Z_i^t]$. Similar to (A.5) of Su and Jin (2012), we obtain $\|\Lambda_3\| = O_P(m^{-\frac{1}{2}})$. In the following, we focus on studying $\Lambda_2$ at first, and then turn to $\Lambda_1$.

In the rest proofs of this lemma, we always let $\Xi_{NT} = (F_0' F/T)^{-1}(\Gamma_0' \Gamma_0/N)^{-1}$ for simplicity, and we have shown $\|\Xi_{NT}\| = O_P(1)$ in the proof of Lemma A.3. Recall that we have denoted $\Pi_{NT}$ and $V_{NT}$ in Lemma A.3, so $\Pi_{NT}^{-1} = V_{NT} \Xi_{NT}$. Then we start our investigation on $\Lambda_2$, and write

$$
\frac{1}{NT} \sum_{i=1}^{N} Z_i^t M_F F_0 \gamma_{0i} = -\frac{1}{NT} \sum_{i=1}^{N} Z_i^t M_F \left( \hat{F} \Pi_{NT}^{-1} - F_0 \right) \gamma_{0i}
= -\frac{1}{NT} \sum_{i=1}^{N} Z_i^t M_F \left[ I_{1NT}(\hat{\beta}_m^*, \hat{F}) + \cdots + I_{8NT}(\hat{F}) \right] \Xi_{NT} \gamma_{0i}
= -(J_{1NT} + \cdots + J_{8NT}),
$$

where the second equality follows from (B.6): $I_{1NT}(\beta, F)$ to $I_{8NT}(F)$ have been defined in the proof of Lemma A.3 but excluding $x_{it,j}$ for $j = p^* + 1, \ldots, p$; and the definitions of $J_{1NT}$ to $J_{8NT}$ should be obvious. In view of the decomposition of $J_{2NT}$ below, it is easy to know that $\|J_{1NT}\| = o_P(\|\hat{C}_\beta - C_{\beta_0}\|)$. Thus, we start from $J_{2NT}$ and write

$$
J_{2NT} = \frac{1}{NT} \sum_{i=1}^{N} Z_i^t M_F I_{2NT}(\hat{\beta}_m^*, \hat{F}) \Xi_{NT} \gamma_{0i}
= \frac{1}{NT} \sum_{i=1}^{N} Z_i^t M_F \frac{1}{NT} \sum_{i=1}^{N} \left( \phi_j^*[\hat{\beta}_m^*] - \phi_j^*[\hat{\beta}_m^*] \right) (F_0 \gamma_{0j})' \hat{F} \Xi_{NT} \gamma_{0i}
+ \frac{1}{NT} \sum_{i=1}^{N} Z_i^t M_F \frac{1}{NT} \sum_{i=1}^{N} \phi_j^*[\Delta_m^*] (F_0 \gamma_{0j})' \hat{F} \Xi_{NT} \gamma_{0i}
= \frac{1}{N^2T} \sum_{i=1}^{N} \sum_{j=1}^{N} Z_i^t M_F Z_j \gamma_{0j} \left( F_0' \hat{F} \right) \left( F_0' \hat{F} \right)^{-1} \left( \Gamma_0' \Gamma_0 \right)^{-1} \gamma_{0i} \left[ \text{vec}(\hat{C}_\beta) - \text{vec}(C_{\beta_0}) \right]
+ \frac{1}{NT} \sum_{i=1}^{N} Z_i^t M_F \frac{1}{NT} \sum_{i=1}^{N} \phi_j^*[\Delta_m^*] (F_0 \gamma_{0j})' \hat{F} \Xi_{NT} \gamma_{0i}
= \frac{1}{N^2T} \sum_{i=1}^{N} \sum_{j=1}^{N} Z_i^t M_F Z_j \gamma_{0j} \left( \Gamma_0' \Gamma_0 \right)^{-1} \gamma_{0i} \left[ \text{vec}(\hat{C}_\beta) - \text{vec}(C_{\beta_0}) \right]
+ \frac{1}{NT} \sum_{i=1}^{N} Z_i^t M_F \frac{1}{NT} \sum_{i=1}^{N} \phi_j^*[\Delta_m^*] (F_0 \gamma_{0j})' \hat{F} \Xi_{NT} \gamma_{0i}
:= J_{2NT,1} + J_{2NT,2}.
$$
By the procedure similar to (A.5) of Su and Jin (2012), we know
\[
\left\| \sum_{i=1}^{N} Z_i^{*\prime} M_{\tilde{F}} Z_i^{*} \right\|^{-1} = O_P(m^{-\frac{1}{2}}),
\]
so negligible. We will further study \( J_{2NT,1} \) later.

For \( J_{3NT} \), write
\[
J_{3NT} = \frac{1}{NT} \sum_{i=1}^{N} Z_i^{*\prime} M_{\tilde{F}} I_{3NT}(\tilde{\beta}_m, \tilde{F}) \Xi_{NT}\gamma_0 \ni
\]
\[
= \frac{1}{NT} \sum_{i=1}^{N} Z_i^{*\prime} M_{\tilde{F}} \frac{1}{NT} \sum_{j=1}^{N} F_0 \gamma_0 (\phi_j^{*}[\beta_0^*] - \phi_j^{*}[\tilde{\beta}_m^*])' \tilde{F} \Xi_{NT}\gamma_0 \ni
\]
\[
= \frac{1}{NT} \sum_{i=1}^{N} Z_i^{*\prime} M_{\tilde{F}} (\tilde{F} \Pi_{NT}^{-1} - F_0) \frac{1}{NT} \sum_{j=1}^{N} \gamma_0 (\phi_j^{*}[\beta_0^*] - \phi_j^{*}[\tilde{\beta}_m^*])' \tilde{F} \Xi_{NT}\gamma_0 \ni
\]
\[
:= \frac{1}{NT} \sum_{i=1}^{N} Z_i^{*\prime} M_{\tilde{F}} J_{3NT,i},
\]
where \( J_{3NT,i} = (\tilde{F} \Pi_{NT}^{-1} - F_0) \frac{1}{NT} \sum_{j=1}^{N} \gamma_0 (\phi_j^{*}[\beta_0^*] - \phi_j^{*}[\tilde{\beta}_m^*])' \tilde{F} \Xi_{NT}\gamma_0 \). Below, we are going to show that
\[
\left\| \sum_{i=1}^{N} Z_i^{*\prime} M_{\tilde{F}} Z_i^{*} \right\|^{-1} = o_P(\| \tilde{C}_\beta - C_{\beta_0} \|) \quad (B.15)
\]
By the procedure similar to (A.5) of Su and Jin (2012), we focus on \( \frac{1}{NT} \sum_{i=1}^{N} \| J_{3NT,i} \|^2 \).

\[
\frac{1}{NT} \sum_{i=1}^{N} \| J_{3NT,i} \|^2 \leq \frac{1}{NT} \sum_{i=1}^{N} \| \tilde{F} \Pi_{NT}^{-1} - F_0 \|^2 \left\| \frac{1}{NT} \sum_{j=1}^{N} \gamma_0 (\phi_j^{*}[\beta_0^*] - \phi_j^{*}[\tilde{\beta}_m^*])' \tilde{F} \Xi_{NT}\gamma_0 \ni \right\|^2
\]
\[
= O_P(1) \frac{1}{T} \| \tilde{F} \Pi_{NT}^{-1} - F_0 \|^2 \left( \frac{1}{N \sqrt{T}} \sum_{j=1}^{N} \| \phi_j^{*}[\beta_0^*] - \phi_j^{*}[\tilde{\beta}_m^*] \| \right)^2
\]
\[
= O_P(1) \frac{1}{T} \| \tilde{F} \Pi_{NT}^{-1} - F_0 \|^2 \left( \frac{1}{N} \sum_{j=1}^{N} \left\{ \frac{1}{T} \| \phi_j^{*}[\beta_0^*] - \phi_j^{*}[\tilde{\beta}_m^*] \|^2 \right\}_{1/2} \right)^2
\]
\[
= o_P(\| \tilde{C}_\beta - C_{\beta_0} \|),
\]
where the second inequality follows from \( \Xi_{NT} = O_P(1) \) and \( \frac{1}{\sqrt{T}} \| \tilde{F} \| = O(1) \); and the last equality follows from \( \frac{1}{\sqrt{T}} \| \tilde{F} \Pi_{NT}^{-1} - F_0 \| = o_P(1) \). Thus, we can conclude that (B.15) holds.

For \( J_{4NT} \), write
\[
J_{4NT} = \frac{1}{NT} \sum_{i=1}^{N} Z_i^{*\prime} M_{\tilde{F}} I_{4NT}(\tilde{\beta}_m, \tilde{F}) \Xi_{NT}\gamma_0 \ni
\]
\[
\leq \frac{1}{NT^2 \Delta_0} \sum_{i=1}^{N} \sum_{j=1}^{N} Z_i^{*\prime} M_{\tilde{F}} \left( \phi_j^{*}[\beta_0^*] - \phi_j^{*}[\tilde{\beta}_m^*] \right) \epsilon_j^{*} F_0 \Xi_{NT}\gamma_0 \ni
\]
\[
+ \frac{1}{NT^2 \Delta_0} \sum_{i=1}^{N} \sum_{j=1}^{N} Z_i^{*\prime} M_{\tilde{F}} \phi_j^{*}[\Delta_m^*] \epsilon_j^{*} F_0 \Xi_{NT}\gamma_0 \ni
\]
\[
+ \frac{1}{NT^2 \Delta_0} \sum_{i=1}^{N} \sum_{j=1}^{N} Z_i^{*\prime} M_{\tilde{F}} \left( \phi_j^{*}[\beta_0^*] - \phi_j^{*}[\tilde{\beta}_m^*] \right) \left[ \epsilon_j^{*}(\tilde{F} - F_0 \Pi_{NT}) \right] \Xi_{NT}\gamma_0 \ni
\]
\[ J_{4NT,1} + J_{4NT,2} + J_{4NT,3}. \]

For \( J_{4NT,1} \), write
\[
\| J_{4NT,1} \| = \left\| \frac{1}{N^2T^2} \sum_{i=1}^{N} \sum_{j=1}^{N} Z_i^{*T} M_F \left( \phi_i^{[\beta_{0,m}]} - \phi_i^{[\tilde{\beta}_m]} \right) \mathcal{E}_j F_0 \Pi_{NT} \Xi_{NT} \gamma_0 i \right\|
\]
\[
= \frac{1}{N^2T^2} \left\| \sum_{i=1}^{N} \sum_{j=1}^{N} Z_i^{*T} M_F Z_j \text{ vec}(\tilde{C}_\beta - C_{\beta_0}) \mathcal{E}_j F_0 \Pi_{NT} \Xi_{NT} \gamma_0 i \right\|
\]
\[
\leq \frac{1}{N^2T^2} \left\| \sum_{i=1}^{N} \sum_{j=1}^{N} Z_i^{*T} M_F Z_j \text{ vec}(\tilde{C}_\beta - C_{\beta_0}) \right\| \cdot \left\| \text{ vec}(\tilde{C}_\beta - C_{\beta_0}) \right\|
\]
\[
\leq O_P(1) \frac{1}{NT} \sum_{i=1}^{N} \| Z_i^{*T} M_F \| \| \gamma_0 i \| \cdot \frac{1}{N} \sum_{j=1}^{N} \| Z_j \| \cdot \frac{1}{T} \| \mathcal{E}_j F_0 \| \cdot \| \text{ vec}(\tilde{C}_\beta - C_{\beta_0}) \|
\]
\[
\leq \frac{1}{T} O_P(\sqrt{mT}) \cdot O_P(\sqrt{mT}) \cdot O_P(T^{-1/2}) \cdot \| \text{ vec}(\tilde{C}_\beta - C_{\beta_0}) \|
\]
\[
= o_P(\| \tilde{C}_\beta - C_{\beta_0} \|),
\]
where the last line follows from \( \frac{m}{T^2} \to 0 \). Thus, \( \| J_{4NT,1} \| \) is negligible. Similarly, we can show both \( \| J_{4NT,2} \| \) and \( \| J_{4NT,3} \| \) are negligible by taking \( \frac{1}{T} \| \phi_i^{[\Delta_m]} \|^2 = O(m^{-1/2}) \) and \( \frac{1}{\sqrt{T}} \| \hat{F} \Pi_{NT} - F_0 \| = o_P(1) \) into account, respectively. Analogous to the derivations of \( J_{3NT,3} \) and \( J_{3NT,1} \), we can obtain that \( \| J_{3NT} \| \) is negligible.

Below, we take a careful look at \( J_{6NT} \). According to Assumption 1, let \( \Omega_e = E[\mathcal{E}_i \mathcal{E}_i^T] \), which is a deterministic matrix uniformly in \( i \). Thus, write
\[
J_{6NT} = \frac{1}{NT} \sum_{i=1}^{N} \sum_{j=1}^{N} Z_i^{*T} M_F \frac{1}{NT} \mathcal{E}_j \mathcal{E}_j^T \hat{F} \Xi_{NT} \gamma_0 i
\]
\[
= \frac{1}{NT^2} \sum_{i=1}^{N} Z_i^{*T} M_F \Omega_e \hat{F} \Xi_{NT} \gamma_0 i
\]
\[
+ \frac{1}{NT} \sum_{i=1}^{N} Z_i^{*T} M_F \frac{1}{NT} \sum_{j=1}^{N} (\mathcal{E}_j \mathcal{E}_j^T - \Omega_e) \hat{F} \Xi_{NT} \gamma_0 i
\]
\[
:= J_{6NT,1} + J_{6NT,2}.
\]
We focus on \( J_{6NT,2} \) at first.
\[
J_{6NT,2} = \frac{1}{N^2T^2} \sum_{i=1}^{N} \sum_{j=1}^{N} Z_i^{*T} (\mathcal{E}_j \mathcal{E}_j^T - \Omega_e) \hat{F} \Xi_{NT} \gamma_0 i
\]
\[
+ \frac{1}{N^2T^2} \sum_{i=1}^{N} \sum_{j=1}^{N} Z_i^{*T} P_F (\mathcal{E}_j \mathcal{E}_j^T - \Omega_e) \hat{F} \Xi_{NT} \gamma_0 i
\]
\[
:= J_{6NT,21} + J_{6NT,22}.
\]
Further decompose \( J_{6NT,21} \) as
\[
J_{6NT,21} = \frac{1}{N^2T^2} \sum_{i=1}^{N} \sum_{j=1}^{N} Z_i^{*T} (\mathcal{E}_j \mathcal{E}_j^T - \Omega_e) F_0 \Pi_{NT} \Xi_{NT} \gamma_0 i
\]
\[
+ \frac{1}{N^2T^2} \sum_{i=1}^{N} \sum_{j=1}^{N} Z_i^{*T} (\mathcal{E}_j \mathcal{E}_j^T - \Omega_e) (\hat{F} - F_0 \Pi_{NT}) \Xi_{NT} \gamma_0 i
\]
Then by a development similar to Jiang et al. (2017, pp. 30-31), we obtain that \( \|J_{6NT,21}\| = o_P \left( \frac{\sqrt{N}}{T} \right) \). Similarly, \( \|J_{6NT,22}\| = o_P \left( \frac{\sqrt{N}}{NT} \right) \). Therefore, we obtain \( \|J_{6NT,2}\| = o_P \left( \frac{\sqrt{N}}{NT} \right) \).

We will consider \( J_{6NT,1} \) together with \( J_{2NT,1} \) and \( J_{8NT} \) later on. Then we only have one term \( J_{7NT} \) left to consider.

\[
J_{7NT} = \frac{1}{NT} \sum_{i=1}^{N} Z_{i}^{*} M_{F} \frac{1}{NT} \sum_{j=1}^{N} F_{0} \gamma_{0j}^{1} E_{j}^{\prime} \tilde{F}_{\gamma} \Xi \gamma_{0i}.
\]

Notice that

\[
\frac{1}{NT} \sum_{j=1}^{N} \gamma_{0j} E_{j}^{\prime} \tilde{F} = \frac{1}{NT} \sum_{j=1}^{N} \gamma_{0j} E_{j}^{\prime} F_{0} + \frac{1}{NT} \sum_{j=1}^{N} \gamma_{0j} E_{j}^{\prime} (F_{0} - \hat{F} \Pi_{NT}^{-1})
\]

\[
= O_P \left( \frac{1}{\sqrt{NT}} \right) + \left\| \frac{1}{N^{1/2}} \sum_{j=1}^{N} \gamma_{0j} E_{j}^{\prime} \right\|_{\frac{1}{\sqrt{T}}} \left\| F_{0} - \hat{F} \Pi_{NT}^{-1} \right\|
\]

\[
= O_P \left( \frac{1}{\sqrt{NT}} \right) + O_P \left( \frac{1}{\sqrt{N}} \right) \frac{1}{\sqrt{T}} \left\| F_{0} - \hat{F} \Pi_{NT}^{-1} \right\|
\]

where the second equality follows from (B.8); and the third equality follows from (B.7). Following the arguments given for \( J_{6} \) of Bai (2009, pp. 1271-1272), it is easy to show that \( \|J_{7NT}\| = o_P \left( \frac{\sqrt{N}}{NT} \right) + o_P(\|\hat{C}_{\beta} - C_{\beta_{0}}\|) \).

Based on the above analyses, we have

\[
\text{vec}(\hat{C}_{\beta}^{*}) - \text{vec}(C_{\beta_{0}}^{*}) + \sum_{j=1}^{n} J_{2NT,1} \cdot (1 + o_P(1))
\]

\[
= \sum_{j=1}^{n} \left\{ \frac{1}{NT} \sum_{i=1}^{N} Z_{i}^{*} M_{F} E_{i} - J_{6NT,1} - J_{8NT} \right\} \cdot (1 + o_P(1))
\]

\[
= \sum_{j=1}^{n} \left\{ \frac{1}{NT} \sum_{i=1}^{N} Z_{i}^{*} M_{F} - \frac{1}{N} \sum_{j=1}^{N} Z_{j}^{*} M_{F} \gamma_{0j}^{1} (\hat{\gamma}_{0j}^{1} / N)^{-1} \gamma_{0i} \right\} E_{i} \cdot (1 + o_P(1))
\]

\[
- \sum_{j=1}^{n} \cdot J_{6NT,1} \cdot (1 + o_P(1)).
\]

Further organise the above equation, we have

\[
\text{vec}(\hat{C}_{\beta}^{*}) - \text{vec}(C_{\beta_{0}}^{*}) = A_{1NT}^{-1} \sum_{j=1}^{n} \left\{ \frac{1}{NT} \sum_{i=1}^{N} Z_{i}^{*} M_{F} - A_{3,i} \right\} E_{i} \cdot (1 + o_P(1))
\]

\[
- A_{1NT}^{-1} \sum_{j=1}^{n} \cdot J_{6NT,1} \cdot (1 + o_P(1)),
\]

where

\[
A_{1NT} = I_{m_{p}^{*}} + \sum_{j=1}^{n} A_{2NT} \cdot (1 + o_P(1)),
\]

\[
A_{2NT} = \frac{1}{N^{2T}} \sum_{i=1}^{N} \sum_{j=1}^{N} Z_{i}^{*} M_{F} Z_{j}^{*} \gamma_{0j}^{1} (\hat{\gamma}_{0j}^{1} / N)^{-1} \gamma_{0i}.
\]

\[
A_{3,i} = \frac{1}{N} \sum_{j=1}^{N} Z_{j}^{*} M_{F} \gamma_{0j}^{1} (\hat{\gamma}_{0j}^{1} / N)^{-1} \gamma_{0i}.
\]
Note that
\[
\sqrt{\frac{NT}{m}} J_{6NT,1} = \frac{1}{(mN)^{1/2}} \sum_{i=1}^{N} Z_i' M F \Omega e \hat{F} \Xi_{NT} \gamma_{0i} \\
= \frac{\sqrt{N}}{\sqrt{mT}} \cdot \frac{1}{NT} \sum_{i=1}^{N} Z_i' M F \Omega e \hat{F} \Xi_{NT} \gamma_{0}(v_i) = O_P \left( \frac{N}{T} \right) = O_P(1).
\]
where the last equality follows from Assumption 3. Thus, we obtain \(\|J_{6NT,1}\| = O_P \left( \frac{\sqrt{mN}}{NT} \right)\). Moreover, it is easy to show \(\frac{1}{NT} \sum_{i=1}^{N} \{ Z_i' M F + A_{3,i} \} E_i = O_P \left( \frac{\sqrt{mN}}{NT} \right)\). Based on the above development, the proof is complete.

**Proof of Theorem 3.2:**

Recall that we have denoted the set \(A^*\). Before proceeding further, we introduce some variables to facilitate the development. For an arbitrary model \(S\), we say it is under-fitted if it misses at least one variable with a nonzero coefficient; it is over-fitted if \(S\) not only includes all relevant variables but also includes at least one redundant regressor (i.e., \(A^* \subset S\) but \(A^* \neq S\)). Then, according to whether the model \(S_\lambda\) is under fitted, correctly fitted, or over fitted, we create three mutually exclusive sets \(A^-\), \(A^0 = \{ \lambda \in \mathbb{R}^p : S_\lambda = A^* \}\) and \(A^+ = \{ \lambda \in \mathbb{R}^p : S_\lambda \supset A^* , S_\lambda \neq A^* \}\). Suppose that there is a sequence \(\{ \lambda_{NT} \}\) that ensures the conditions required by Lemma A.4. Let \(\hat{\lambda}^{\lambda_{NT}}, \hat{F}^{\lambda_{NT}}\) denote the estimator obtained by implementing (3.3) using \(\lambda_{NT}\).

**Case 1: Under-fitted model.** Without loss of generality, we assume that only one variable is missing, and suppose that the first \(p^* - 1\) rows of \(\hat{C}_\beta^{\lambda}\) are obtained from the under-fitted model and the \(p^{th}\) row of \(\hat{C}_\beta^{\lambda}\) is a 0 row. Moreover, let \(\text{RSS}_0 = \frac{1}{NT} \sum_{i=1}^{N} (Y_i - \phi_i[\beta_{0,m}])' M F_0 (Y_i - \phi_i[\beta_{0,m}])\).

We then write
\[
\text{RSS}_\lambda - \text{RSS}_0 = \frac{1}{NT} \sum_{i=1}^{N} (Y_i - \phi_i[\hat{\beta}_{m}])' M F_\lambda (Y_i - \phi_i[\hat{\beta}_{m}]) \\
- \frac{1}{NT} \sum_{i=1}^{N} (Y_i - \phi_i[\beta_{0,m}])' M F_0 (Y_i - \phi_i[\beta_{0,m}]) \\
\geq \rho_1 \|C_{\beta_{0, p^*}}\|^2 > \frac{\rho_1}{2} \| \beta_{0, p^*} \|_2^2 > 0,
\]
where the first inequality follows from the development given for (B.3) of Appendix B.

Again, using the development given for (B.3) of Appendix B, we have
\[
\text{RSS}_{\lambda NT} - \text{RSS}_0 = \text{vec}(C_{\beta_0} - \hat{C}_\beta^{\lambda_{NT}})' \frac{1}{NT} \sum_{i=1}^{N} Z_i' M F_{\lambda_{NT}} Z_i \text{vec}(C_{\beta_0} - \hat{C}_\beta^{\lambda_{NT}}) \\
+ \frac{1}{NT} \text{tr} (M F_{\lambda_{NT}} F_0 \Gamma_0 \Gamma_0 F_0' M F_{\lambda_{NT}}) \\
+ 2 \text{vec}(C_{\beta_0} - \hat{C}_\beta^{\lambda_{NT}})' \frac{1}{NT} \sum_{i=1}^{N} Z_i' M F_{\lambda_{NT}} F_0 \gamma_{0i} + o_P(1) = o_P(1),
\]
where the second equality follows from the development of Lemma A.3.

Thus, we can conclude that \(\Pr (\inf_{\lambda \in A^-} \text{BIC}_\lambda > \text{BIC}_{\lambda_{NT}}) \rightarrow 1\).

**Case 2: Over-fitted model.** Consider \(\forall \lambda \in A^+\) and recall that \(\hat{C}_\beta^{\lambda}\) determines a model \(S_\lambda\). Under such a model \(S_\lambda\), we can define another unpenalized estimator as

\[\text{Under-fitted case allows for including redundant regressor.}\]
\[
(\hat{\beta}, \hat{F}) = \arg\min_{\beta \in \mathcal{A}} \frac{1}{NT} \sum_{i=1}^{N} (Y_i - \phi_i[\beta_m])' M_F(Y_i - \phi_i[\beta_m])
\]

subject to \(F \in \mathcal{D}_F\), where \(\|C_{\beta,j}\| = 0 \text{ with } \forall j \notin S_{\lambda}\). In other words, \((\hat{\beta}, \hat{F})\) is the unpenalized estimator under the model determined by \(\hat{\beta}.\) By definition, we obtain immediately that RSS_{\lambda} \geq RSS_{\lambda_T}, \text{ where } RSS_{\lambda} = \frac{1}{NT} \sum_{i=1}^{N} (Y_i - \phi_i[\hat{\beta}_m])' M_F(Y_i - \phi_i[\hat{\beta}_m]).

Write
\[
\ln RSS_{\lambda} - \ln RSS_{\lambda_T} = \ln \left(1 + \frac{RSS_{\lambda} - RSS_{\lambda_T}}{RSS_{\lambda_T}}\right) \geq - \frac{RSS_{\lambda} - RSS_{\lambda_T}}{RSS_{\lambda_T}}.
\]

In view of the proof of Lemma A.3, it is easy to see that RSS_{\lambda_T} converges to a positive constant. With regard to \(RSS_{\lambda} - RSS_{\lambda_T}\), we have
\[
RSS_{\lambda} - RSS_{\lambda_T} = \frac{1}{NT} \sum_{i=1}^{N} (Y_i - \phi_i[\hat{\beta}_m])' M_F(Y_i - \phi_i[\hat{\beta}_m])
\]
\[
- \frac{1}{NT} \sum_{i=1}^{N} (Y_i - \phi_i[\hat{\beta}_m])' M_{F\lambda_T}(Y_i - \phi_i[\hat{\beta}_m]).
\]

By Lemmas A.2 and A.4, it is not hard to see \(|RSS_{\lambda} - RSS_{\lambda_T}| \leq O_P(1) \frac{\sqrt{N}}{\sqrt{N}},\) so we can further write
\[
\ln RSS_{\lambda} - \ln RSS_{\lambda_T} \geq - \frac{RSS_{\lambda} - RSS_{\lambda_T}}{RSS_{\lambda_T}} \geq - \left|O_P(1) \frac{1}{\sqrt{N}}\right|.
\]

We then write
\[
\inf_{\lambda \in A^+} BIC_{\lambda} - BIC_{\lambda_T} = \inf_{\lambda \in A^+} \ln RSS_{\lambda} - \ln RSS_{\lambda_T} + (df_{\lambda} - df_{\lambda_T}) \Upsilon_{NT}.
\]

By Lemma A.4, we know that \(\Pr(df_{\lambda_T} = p^*) \to 1\). Since \(\lambda \in A^+\), we must have that \(\Pr(df_{\lambda} \geq p^* + 1) \to 1\). Then it is clear that \(\Pr(\inf_{\lambda \in A^+} BIC_{\lambda} > BIC_{\lambda_T}) \to 1\) by the requirement on \(\Upsilon_{NT}\).

Combining Cases 1 and 2, we obtain that \(\Pr(\inf_{\lambda \in A^+} BIC_{\lambda} > BIC_{\lambda_T}) \to 1\). This further indicates that \(\Pr(S_{\lambda} = A^+) \to 1\). The proof is now complete. \(\blacksquare\)

**Proofs of HD Case**

Note that under the HD setting, the elements of \(\beta_m(z)\) belonging to \(L^2(\mathcal{R}, \pi(w))\) indicates that \(\|C_{\beta}\| \leq a_0 \sqrt{p}\) with \(a_0\) being a large constant. We will be repeatedly using this fact below.

**Proof of Lemma A.5:**

(1). Write
\[
\frac{1}{NT} \sum_{i=1}^{N} (\phi_i[\hat{\beta}_0,m] - \phi_i[\hat{\beta}_m])' M_F\xi_i
\]
\[
= \frac{1}{NT} \sum_{i=1}^{N} (\phi_i[\hat{\beta}_0,m] - \phi_i[\hat{\beta}_m])' \xi_i + \frac{1}{NT} \sum_{i=1}^{N} (\phi_i[\hat{\beta}_0,m] - \phi_i[\hat{\beta}_m])' P_F\xi_i
\]
\[
:= \Lambda_1 + \Lambda_2.
\]
For $\Lambda_1$, write

$$
\sup_{\|C_\beta\| \leq a_0 \sqrt{p}} \left| \frac{1}{NT} \sum_{i=1}^{N} (\phi_i[\beta_{0,m}] - \phi_i[\beta_m])' \mathcal{E}_i \right| \leq \left\| \frac{1}{NT} \sum_{i=1}^{N} z'_i \mathcal{E}_i \right\| \cdot \sup_{\|C_\beta\| \leq a_0 \sqrt{p}} \|\text{vec}(C_{\beta_0} - C_\beta)\|
$$

$$
= O \left( \sqrt{\frac{mp}{NT}} \right) \cdot \sup_{\|C_\beta\| \leq a_0 \sqrt{p}} \|C_{\beta_0} - C_\beta\| = O_F \left( \sqrt{\frac{p^2}{NT}} \right)
$$

where the first equality follows from some standard analysis on the term $\frac{1}{NT} \sum_{i=1}^{N} z'_i \mathcal{E}_i$ using Assumption 1.1; and the last equality follows from $\|C_\beta\| \leq a_0 \sqrt{p}$.

In order to consider $\Lambda_2$, let $\Delta b = (\phi_1[\beta_{0,m}] - \phi_1[\beta_m], \ldots, \phi_N[\beta_{0,m}] - \phi_N[\beta_m])$, and note that

$$
\sup_{\|C_\beta\| \leq a_0 \sqrt{p}} \frac{1}{NT} \|\Delta b\|^2 = \sup_{\|C_\beta\| \leq a_0 \sqrt{p}} \frac{1}{N} \sum_{i=1}^{N} (C_\beta - C_{\beta_0})' z'_i z_i (C_\beta - C_{\beta_0})
$$

$$
\leq O_F(1) \sup_{\|C_\beta\| \leq a_0 \sqrt{p}} \|C_\beta - C_{\beta_0}\|^2 = O_F(p),
$$

(B.16)

where the inequality follows from Assumption 2.1.

Then we are able to write

$$
\sup_{\|C_\beta\| \leq a_0 \sqrt{p}, F \in D_F} \left| \frac{1}{NT} \sum_{i=1}^{N} (\phi_i[\beta_{0,m}] - \phi_i[\beta_m])' P_F \mathcal{E}_i \right|
$$

$$
= \sup_{\|C_\beta\| \leq a_0 \sqrt{p}, F \in D_F} \left| \frac{1}{NT} \text{tr} (P_F \mathcal{E}' \Delta b') \right| \leq \frac{r}{NT} \sup_{\|C_\beta\| \leq a_0 \sqrt{p}, F \in D_F} \|P_F \mathcal{E}' \Delta b'\|_{sp}
$$

$$
\leq \sup_{\|C_\beta\| \leq a_0 \sqrt{p}, F \in D_F} \frac{r}{NT} \|P_F\|_{sp} \|\mathcal{E}\|_{sp} \|\Delta b\|_{sp} = O_F \left( \sqrt{\frac{p^2}{NT}} \right),
$$

where the last equality follows from Assumption 5.1 and (B.16).

Based on the above development, the result follows.

(2). Write

$$
\sup_{F \in D_F} \left| \frac{1}{NT} \sum_{i=1}^{N} \phi_i [\Delta_m]' M_F \phi_i [\Delta_m] \right| \leq \left| \frac{1}{NT} \sum_{i=1}^{N} \phi_i [\Delta_m]' \phi_i [\Delta_m] \right|
$$

$$
= \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \|x^*_it\|^2 \|\Delta^*_m(z^*_it)\|^2 = O_F \left( p^* m^{-\mu} \right),
$$

(B.17)

where the last equality follows from Assumption 2.1 and $E\|x^*_it\|^2 = O(p^*)$.

(3). Write

$$
\sup_{F \in D_F} \left| \frac{1}{NT} \sum_{i=1}^{N} \phi_i [\Delta_m]' M_F F_0 \gamma_0 \right| = \sup_{F \in D_F} \left| \frac{1}{NT} \text{tr} (M_F F_0 \Gamma_0') \Delta \right|
$$

$$
\leq \frac{r}{NT} \sup_{F \in D_F} \|M_F\|_{sp} \|F_0\|_{sp} \|\Gamma_0\|_{sp} \|\Delta\|_{sp} = O_F \left( \sqrt{p^* m^{-\mu}} \right),
$$

where $\Delta = (\phi_1(\Delta_m), \ldots, \phi_N(\Delta_m))$; and the second equality follows from that $\frac{1}{NT} \|\Delta\|^2 = O_F \left( p^* m^{-\mu} \right)$ as in (2) of this lemma.

(4). Similar to the proof for (3) of this lemma, the result follows.

(5). Write
where the equality follows from (B.16) and (B.17). Then the proof is complete.

Proof of Lemma A.6:

(1) Still, let \( \Delta \phi_1[\beta_m] = \phi_1[\beta_{0,m}] - \phi_1[\beta_m] \), \( \xi_F = \text{vec}(MF_0) \), \( A_{1,F} = \frac{1}{NT} \sum_{i=1}^{N} Z_i'M_F Z_i \), \( A_2 = \frac{1}{NT} (\Gamma'_0 \Gamma_0) \otimes I_T \), and \( A_{3,F} = \frac{1}{NT} \sum_{i=1}^{N} \gamma_0_i \otimes (MF_0 Z_i) \). By the definition of (3.3) and Lemma A.5, we have

\[
0 \geq \frac{1}{NT} Q_{\lambda}(\hat{C}_\beta, \hat{F}) - \frac{1}{NT} Q_{\lambda}(C_{\beta_0}, F_0) = \frac{1}{NT} \sum_{i=1}^{N} \left( \Delta \phi_1[\beta_m] + F_0 \gamma_0_i \right)' M_{\hat{F}} \left( \Delta \phi_1[\beta_m] + F_0 \gamma_0_i \right) + \frac{1}{NT} \sum_{i=1}^{N} \mathcal{E}_i' M_{\hat{F}} \mathcal{E}_i + \frac{2}{NT} \sum_{i=1}^{N} \left( \Delta \phi_1[\beta_m] + F_0 \gamma_0_i \right)' M_{\hat{F}} \mathcal{E}_i + \sum_{j=1}^{p} \frac{\lambda_j}{NT} \| \hat{C}_{\beta,j} \| \sum_{j=1}^{p} \frac{\lambda_j}{NT} \| \hat{C}_{\beta_0,j} \| - \sum_{j=1}^{p^*} \frac{\lambda_j}{NT} \| \hat{C}_{\beta,j} \| + O_P \left( \sqrt{\frac{p(\xi_{NT} + mp)}{NT}} + \sqrt{pp^*m^{-\frac{3}{2}}} \right) - \sum_{j=1}^{p^*} \frac{\lambda_j}{NT} \| \hat{C}_{\beta_0,j} \| = \text{vec}(C_{\beta_0} - \hat{C}_\beta)' \frac{1}{NT} \sum_{i=1}^{N} Z_i'M_{\hat{F}} Z_i \text{vec}(C_{\beta_0} - \hat{C}_\beta) + \frac{1}{NT} \text{tr} \left( M_{\hat{F}} F_0 \Gamma'_0 \Gamma_0 F_0'M_{\hat{F}} \right) + 2 \text{vec}(C_{\beta_0} - \hat{C}_\beta)' \frac{1}{NT} \sum_{i=1}^{N} Z_i'M_{\hat{F}} F_0 \gamma_0_i + \sum_{j=1}^{p} \frac{\lambda_j}{NT} \| \hat{C}_{\beta,j} \| - \sum_{j=1}^{p^*} \frac{\lambda_j}{NT} \| \hat{C}_{\beta_0,j} \| + O_P \left( \sqrt{\frac{p(\xi_{NT} + mp)}{NT}} + \sqrt{pp^*m^{-\frac{3}{2}}} \right),
\]

where the second equality follows from (1) of Lemma A.2, and Lemma A.5. Thus, we can further write

\[
\sum_{j=1}^{p^*} \frac{\lambda_j}{NT} \| \hat{C}_{\beta_0,j} \| \geq \text{vec}(C_{\beta_0} - \hat{C}_\beta)' \frac{1}{NT} \sum_{i=1}^{N} Z_i'M_{\hat{F}} Z_i \text{vec}(C_{\beta_0} - \hat{C}_\beta) + \frac{1}{NT} \text{tr} \left( M_{\hat{F}} F_0 \Gamma'_0 \Gamma_0 F_0'M_{\hat{F}} \right) + 2 \text{vec}(C_{\beta_0} - \hat{C}_\beta)' \frac{1}{NT} \sum_{i=1}^{N} Z_i'M_{\hat{F}} F_0 \gamma_0_i + O_P \left( \sqrt{\frac{p(\xi_{NT} + mp)}{NT}} + \sqrt{pp^*m^{-\frac{3}{2}}} \right) \geq \text{vec}(C_{\beta_0} - \hat{C}_\beta)' \left( A_{1,F} - A_{3,F} A_{2}^{-1} A_{3,F} \right) \text{vec}(C_{\beta_0} - \hat{C}_\beta) + [\xi_{\hat{F}} + \text{vec}(C_{\beta_0} - \hat{C}_\beta)' A_{3,F} A_{2}^{-1}] A_2 [\xi_{\hat{F}} + A_{2}^{-1} A_{3,F} \text{vec}(C_{\beta_0} - \hat{C}_\beta)] + O_P \left( \sqrt{\frac{p(\xi_{NT} + mp)}{NT}} + \sqrt{pp^*m^{-\frac{3}{2}}} \right).
\]
\[
\geq O_P(1) ||C_{\beta_0} - \hat{C}_\beta||^2 + O_P \left( \frac{1}{\sqrt{\xi_{NT}}} + \sqrt{\frac{p(\xi_{NT} + mp)}{NT}} + \sqrt{pp^*m^{-\frac{3}{2}}} \right).
\]  

(B.18)

Till now, we can conclude that
\[
||C_{\beta_0} - \hat{C}_\beta||^2 = O_P \left( \frac{1}{\sqrt{\xi_{NT}}} + \sqrt{\frac{p(\xi_{NT} + mp)}{NT}} + \sqrt{pp^*m^{-\frac{3}{2}}} + \frac{p^*\lambda_{\max}}{NT} \right) = o_P(1),
\]  

(B.19)

where the second equality follows from Assumption 5.2.

(2). By (B.18) and (B.19), we can further write that
\[
o_P(1) \geq \frac{1}{NT} \text{tr} \left[ (F_0'M_{\hat{F}}F_0) \ (\Gamma_0'\Gamma_0) \right] + o_P(1),
\]

so \(\frac{1}{NT} \text{tr} \left[ (F_0'M_{\hat{F}}F_0) \ (\Gamma_0'\Gamma_0) \right] = o_P(1). As in Bai (2009, p. 1265), we can further conclude that \(\frac{1}{T} \text{tr} \ (F_0'M_{\hat{F}}F_0) = o_P(1), \|P_{\hat{F}} - P_{F_0}\| = o_P(1),\) and \(\frac{1}{T} \hat{F}'F_0\) is invertible with probability approaching one. Thus, the second result of this theorem follows.

(3)-(4). The results follow by the procedure similar to (3) and (4) of Lemma A.3 noting that \(p^*m \to 0,\) so the details are omitted.

(5). The notations used below are the same as the LD case, but some having diverging dimension because of \(p^* \to \infty.\) For example,
\[
\frac{1}{NT} \sum_{i=1}^{N} \phi_i^s[\Delta_m] = O_P(p^*m^{-\mu}) \quad \text{and} \quad Z_i^s = O_P(\sqrt{p^*m}),
\]

which will be used repeatedly below.

With the first two results of this lemma in hand, we are then able to establish the first result of Theorem 3.3 of the main text. Thus, we delete the corresponding rows of \(\hat{C}_\beta\) and \(x_{it,j}\) for \(j = p^* + 1, \ldots, p\) from the objective function. Note that (A.3.1) only gives a slow rate. Below, we aim to improve this rate under the HD case.

The estimator \(\hat{C}_{\beta}^s\) given by (3.3) can be written as
\[
\text{vec}(\hat{C}_{\beta}^s) = \left( \sum_{i=1}^{N} Z_i^s' M_{\hat{F}} Z_i^s + \frac{D_{m,p^*}}{2} \right)^{-1} \sum_{i=1}^{N} Z_i^s' M_{\hat{F}} Y_i,
\]

where \(Z_i^s = (Z_{i1}^s, \ldots, Z_{iT}^s)'\) and \(D_{m,p^*} = I_m \otimes \text{diag} \left\{ \frac{\lambda_1}{\|C_{\beta,1}\|}, \ldots, \frac{\lambda_{p^*}}{\|C_{\beta,p^*}\|} \right\}.\) Correspondingly, we denote that
\[
\text{vec}(\hat{C}_{\beta}^s) = \left( \sum_{i=1}^{N} Z_i^s' M_{\hat{F}} Z_i^s \right)^{-1} \sum_{i=1}^{N} Z_i^s' M_{\hat{F}} Y_i.
\]

Thus, we can write
\[
\hat{C}_{\beta}^s - C_{\beta_0}^s = (\hat{C}_{\beta}^s - \hat{C}_\beta^s) + (\hat{C}_\beta^s - C_{\beta_0}^s).
\]

Note that
\[
\text{vec}(\hat{C}_{\beta}^s) - \text{vec}(\hat{C}_\beta^s) = \left( \sum_{i=1}^{N} Z_i^s' M_{\hat{F}} Z_i^s + \frac{D_{m,p^*}}{2} \right)^{-1} - \left( \sum_{i=1}^{N} Z_i^s' M_{\hat{F}} Z_i^s \right)^{-1} \sum_{i=1}^{N} Z_i^s' M_{\hat{F}} Y_i.
\]
By Lemma A.1 and Assumption 2, we just need to consider the next term in order to get the difference between \( \hat{C}_\beta \) and \( \hat{C}_\beta^* \).

\[
\begin{align*}
\frac{1}{NT} \sum_{i=1}^{N} Z_i^{\prime \prime} M_{\hat{F}} Z_i^* + \frac{D_{m,p^*}}{2NT} - \frac{1}{NT} \sum_{i=1}^{N} Z_i^{\prime \prime} M_{\hat{F}} Z_i^* &= \frac{D_{m,p^*}}{2NT} \text{ in (B.20).}
\end{align*}
\]

Moreover, it is easy to know \( \| \frac{1}{NT} \sum_{i=1}^{N} Z_i^{\prime \prime} M_{\hat{F}} Y_i \| = O_P(\sqrt{mp^*}) \), which in connection with (B.20) indicates \( \| \hat{C}_\beta^* - \hat{C}_\beta^* \| = O_P \left( \frac{mp^* \lambda_{\max}^*}{NT} \right) \).

We now focus on \( \hat{C}_\beta^* - C_{\beta_0}^* \), and write

\[
\begin{align*}
\text{vec}(\hat{C}_\beta^*) - \text{vec}(C_{\beta_0}^*) &= \left[ \sum_{i=1}^{N} Z_i^{\prime \prime} M_{\hat{F}} Z_i^* \right]^{-1} \sum_{i=1}^{N} Z_i^{\prime \prime} M_{\hat{F}} \mathcal{E}_i \\
&+ \left[ \sum_{i=1}^{N} Z_i^{\prime \prime} M_{\hat{F}} Z_i^* \right]^{-1} \sum_{i=1}^{N} Z_i^{\prime \prime} M_{\hat{F}} F_0 \gamma_{0i} \\
&+ \left[ \sum_{i=1}^{N} Z_i^{\prime \prime} M_{\hat{F}} Z_i^* \right]^{-1} \sum_{i=1}^{N} Z_i^{\prime \prime} M_{\hat{F}} \phi_i^{*} \Delta_{m}^*
\end{align*}
\]

where the definitions of \( \Lambda_1-\Lambda_3 \) should be obvious. Note

\[
\frac{1}{NT} \sum_{i=1}^{N} Z_i^{\prime \prime} M_{\hat{F}} Z_i^* = \frac{1}{NT} \sum_{i=1}^{N} Z_i^{\prime \prime} M_{F_0} Z_i^* \cdot (1 + o_P(1)) = \Sigma_{Z,F}^* \gamma_{0i} + \sum_{i=1}^{N} Z_i^{\prime \prime} M_{\hat{F}} F_0 \gamma_{0i}
\]

where \( \Sigma_{Z,F}^* = E[|Z_{i1}^{\prime \prime} Z_{i2}^*|^2] - E[|Z_{i1}^{\prime \prime} F_0 \gamma_{0i}] \). Similar to (A.5) of Su and Jin (2012), we obtain \( \| \Delta_{m}^* \| = O_P \left( \sqrt{mp^*} \right) \). In the following, we focus on studying \( \Lambda_2 \) at first, and then turn to \( \Lambda_1 \).

In the rest proofs of this lemma, we always let \( \Xi_{NT} = (F_0^\prime \hat{F}/T)^{-1}(I_0^\prime \Gamma_0/N)^{-1} \) for simplicity. Recall that we have denoted \( \Pi_{NT} \) and \( V_{NT} \) in Lemma A.3, so \( \Pi_{NT}^2 = V_{NT} \Xi_{NT} \). Then we start our investigation on \( \Lambda_2 \), and write

\[
\begin{align*}
\frac{1}{NT} \sum_{i=1}^{N} Z_i^{\prime \prime} M_{\hat{F}} F_0 \gamma_{0i} &= \frac{1}{NT} \sum_{i=1}^{N} Z_i^{\prime \prime} M_{\hat{F}} (\hat{F} \Pi_{NT}^{-1} - F_0) \gamma_{0i} \\
&= \frac{1}{NT} \sum_{i=1}^{N} Z_i^{\prime \prime} M_{\hat{F}} \left[I_{1NT}(\hat{\beta}_m^*, \hat{F}) + \cdots + I_{8NT}(\hat{F}) \right] \Xi_{NT} \gamma_{0i} \\
&:= J_{1NT} + \cdots + J_{8NT},
\end{align*}
\]

where the second equality follows from (B.6); \( I_{1NT}(\hat{\beta}, F) \) to \( I_{8NT}(F) \) have been defined in the proof of Lemma A.3 but excluding \( x_{it, j} \) for \( j = p^* + 1, \ldots, p \); and the definitions of \( J_{1NT} \) to \( J_{8NT} \) should be obvious. In view of the decomposition of \( J_{2NT} \) below, it is easy to know that \( \| J_{1NT} \| = o_P(\| \hat{C}_\beta^* - C_{\beta_0}^* \|) \). Thus, we start from \( J_{2NT} \) and write

\[
\begin{align*}
J_{2NT} &= \frac{1}{NT} \sum_{i=1}^{N} Z_i^{\prime \prime} M_{\hat{F}} I_{2NT}(\hat{\beta}_m^*, \hat{F}) \Xi_{NT} \gamma_{0i} \\
&= \frac{1}{NT} \sum_{i=1}^{N} Z_i^{\prime \prime} M_{\hat{F}} \frac{1}{NT} \sum_{j=1}^{N} \left( \phi_j^{*} \Delta_{\beta_0, m}^* - \phi_j^{*} \Delta_{\beta_m}^* \right) \gamma_{0i} \Xi_{NT} \gamma_{0i}
\end{align*}
\]
By the procedure similar to (A.5) of Su and Jin (2012), we focus on

\[
\frac{1}{NT} \sum_{i=1}^{N} \sum_{j=1}^{N} \phi_j^*[\Delta_m^*] (F_0 \gamma_{0j})' \vec{F} \Xi_{NT} \gamma_{0i}
\]

where \( \gamma_{0i} \approx 0 \). We will further study \( J_{2NT,1} \) later.

For \( J_{3NT} \), write

\[
J_{3NT} = \frac{1}{NT} \sum_{i=1}^{N} \sum_{j=1}^{N} \phi_j^*[\Delta_m^*] (F_0 \gamma_{0j})' \vec{F} \Xi_{NT} \gamma_{0i}
\]

\[
= \frac{1}{NT} \sum_{i=1}^{N} Z_i^* M_{\vec{F}} \left( \frac{\Gamma_0^*}{N} \right)^{-1} \gamma_{0i} \left[ \vec{C}_\beta - \vec{C}_\beta \right]
\]

\[
+ \frac{1}{NT} \sum_{i=1}^{N} \phi_j^*[\Delta_m^*] (F_0 \gamma_{0j})' \vec{F} \Xi_{NT} \gamma_{0i}
\]

where \( J_{3NT,i} = (\vec{F} \Xi_{NT} \gamma_{0i} - F_0) \frac{1}{NT} \sum_{j=1}^{N} \gamma_{0j} (\phi_j^*[\beta_0^*] - \phi_j^*[\beta_m^*])' \vec{F} \Xi_{NT} \gamma_{0i} \). Below, we are going to show that

\[
\left\| \left[ \sum_{i=1}^{N} Z_i^* M_{\vec{F}} Z_i^* \right]^{-1} \right\|_{NT J_{3NT}} = o_P(\|\vec{C}_\beta - C_{\beta_0}\|)
\]

(B.21)

By the procedure similar to (A.5) of Su and Jin (2012), we focus on \( \frac{1}{NT} \sum_{i=1}^{N} \|J_{3NT,i}\|^2 \).

\[
\frac{1}{NT} \sum_{i=1}^{N} \|J_{3NT,i}\|^2 \leq \frac{1}{NT} \sum_{i=1}^{N} \|\vec{F} \Xi_{NT} \gamma_{0i} - F_0\|^2 \left( \frac{1}{NT} \sum_{j=1}^{N} \gamma_{0j} \phi_j^*[\beta_0^*] - \phi_j^*[\beta_m^*] \right)^2 \left\| \vec{C}_\beta - C_{\beta_0} \right\|^2
\]

\[
\leq O_P(1) \left( \frac{1}{NT} \sum_{j=1}^{N} \left\| \phi_j^*[\beta_0^*] - \phi_j^*[\beta_m^*] \right\|^2 \right)^{1/2}
\]

\[
= o_P(\|\vec{C}_\beta - C_{\beta_0}\|^2),
\]

Electronic copy available at: https://ssrn.com/abstract=3348229
where the second inequality follows from \( \Xi_{NT} = O_P(1) \) and \( \frac{1}{\sqrt{T}} \| \hat{F} \| = O(1) \); and the last equality follows from \( \frac{1}{\sqrt{T}} \| \hat{F} \| - F_0 \| = o_P(1) \). Thus, we can conclude that (B.21) holds.

For \( J_{NT} \), write

\[
J_{NT} = \frac{1}{NT} \sum_{i=1}^{N} Z_i^T M_{\hat{F}} I_{NT}(\hat{\beta}_m, \hat{F}) \Xi_{NT} \gamma_{0i}
\]

\[
\leq \frac{1}{N^2T^2} \sum_{i=1}^{N} \sum_{j=1}^{N} Z_i^T M_{\hat{F}} \left( \phi_j^* [\hat{\beta}_{0,m}] - \phi_j^* [\hat{\beta}_m^*] \right) \mathcal{E}_j^T F_0 \Pi_{NT} \Xi_{NT} \gamma_{0i}
\]

\[
+ \frac{1}{N^2T^2} \sum_{i=1}^{N} \sum_{j=1}^{N} Z_i^T M_{\hat{F}} \phi_j^* [\Delta_m^*] \mathcal{E}_j^T F_0 \Pi_{NT} \Xi_{NT} \gamma_{0i}
\]

\[
+ \frac{1}{N^2T^2} \sum_{i=1}^{N} \sum_{j=1}^{N} Z_i^T M_{\hat{F}} \left( \phi_j^* [\hat{\beta}_0^*] - \phi_j^* [\hat{\beta}_m^*] \right) \left\| \mathcal{E}_j^T (\hat{F} - F_0 \Pi_{NT}) \right\| \Xi_{NT} \gamma_{0i}
\]

\[
:= J_{NT,1} + J_{NT,2} + J_{NT,3}.
\]

For \( J_{NT,1} \), write

\[
\| J_{NT,1} \| = \left\| \frac{1}{N^2T^2} \sum_{i=1}^{N} \sum_{j=1}^{N} Z_i^T M_{\hat{F}} \left( \phi_j^* [\hat{\beta}_{0,m}] - \phi_j^* [\hat{\beta}_m^*] \right) \mathcal{E}_j^T F_0 \Pi_{NT} \Xi_{NT} \gamma_{0i} \right\|
\]

\[
= \left\| \frac{1}{N^2T^2} \sum_{i=1}^{N} \sum_{j=1}^{N} Z_i^T M_{\hat{F}} Z_j^T \text{vec}(\hat{C}_\beta - C_{\beta_0}) \mathcal{E}_j^T F_0 \Pi_{NT} \Xi_{NT} \gamma_{0i} \right\|
\]

\[
\leq \left\| \frac{1}{N^2T^2} \sum_{i=1}^{N} \sum_{j=1}^{N} Z_i^T M_{\hat{F}} Z_j^T \mathcal{E}_j^T F_0 \Pi_{NT} \Xi_{NT} \gamma_{0i} \right\| \cdot \left\| \text{vec}(\hat{C}_\beta - C_{\beta_0}) \right\|
\]

\[
\leq O_P(1) \frac{1}{NT} \sum_{i=1}^{N} \left\| Z_i^T M_{\hat{F}} \right\| \| \gamma_{0i} \| \cdot \frac{1}{N} \sum_{j=1}^{N} \left\| Z_j^T \right\| \frac{1}{T} \| \mathcal{E}_j^T F_0 \| \cdot \left\| \text{vec}(\hat{C}_\beta - C_{\beta_0}) \right\|
\]

\[
\leq \frac{1}{T} O_P(\sqrt{mp^* T}) \cdot O_P(\sqrt{mp^* T}) \cdot O_P(T^{-1/2}) \cdot \left\| \text{vec}(\hat{C}_\beta - C_{\beta_0}) \right\|
\]

\[
= o_P(\| \Xi_{NT} \|),
\]

where the last line follows from \( \frac{(mp^*)^2}{T} \to 0 \). Thus, \( \| J_{NT,1} \| \) is negligible. Similarly, we can show both \( \| J_{NT,2} \| \) and \( \| J_{NT,3} \| \) are negligible by taking \( \frac{1}{T} \| \phi_j^* [\Delta_m^*] \|^2 = O(p^* m^{-p}) \) and \( \frac{1}{\sqrt{T}} \| \hat{F} \| - F_0 \| = o_P(1) \) into account, respectively. Analogous to the derivations of \( J_{NT} \) and \( J_{NT} \), we can obtain that \( \| J_{NT} \| \) is negligible.

Below, we take a careful look at \( J_{6NT} \). According to Assumption 1, let \( \Omega_e = E[\mathcal{E}_j \mathcal{E}_j^T] \), which is a deterministic matrix uniformly in \( i \). Thus, write

\[
J_{6NT} = \frac{1}{NT} \sum_{i=1}^{N} Z_i^T M_{\hat{F}} \frac{1}{NT} \sum_{j=1}^{N} \mathcal{E}_j \mathcal{E}_j^T \hat{F} \Xi_{NT} \gamma_{0i}
\]

\[
= \frac{1}{NT^2} \sum_{i=1}^{N} Z_i^T M_{\hat{F}} \Omega_e \hat{F} \Xi_{NT} \gamma_{0i}
\]

\[
+ \frac{1}{NT} \sum_{i=1}^{N} Z_i^T M_{\hat{F}} \frac{1}{NT} \sum_{j=1}^{N} (\mathcal{E}_j \mathcal{E}_j^T - \Omega_e) \hat{F} \Xi_{NT} \gamma_{0i}
\]

\[
:= J_{6NT,1} + J_{6NT,2}.
\]

We focus on \( J_{6NT,2} \) at first.
Similarly, given for further decompose $J_{6NT,21}$ as

$$ J_{6NT,21} = \frac{1}{N^2T^2} \sum_{i=1}^{N} \sum_{j=1}^{N} Z_i^{ij} (E_j^{ij} - \Omega_e) F_0 \Pi_{NT} \Xi_{NT} \gamma_{0i} $$

$$ + \frac{1}{N^2T^2} \sum_{i=1}^{N} \sum_{j=1}^{N} Z_i^{ij} P_{E} (E_j^{ij} - \Omega_e) (\hat{F} - F_0 \Pi_{NT}) \Xi_{NT} \gamma_{0i} $$

$$:= J_{6NT,211} + J_{6NT,212}. $$

Then by a development similar to Jiang et al. (2017, pp. 30-31), we obtain that $\|J_{6NT,21}\| = o_P \left( \sqrt{\frac{\nu_m}{NT}} \right)$. Similarly, $\|J_{6NT,22}\| = o_P \left( \sqrt{\frac{\nu_m}{NT}} \right)$. Thus, we obtain $\|J_{6NT,2}\| = o_P \left( \sqrt{\frac{\nu_m}{NT}} \right)$.

We will consider $J_{6NT,1}$ together with $J_{2NT,1}$ and $J_{8NT}$ later on. Then we only have one term $J_{7NT}$ left to consider.

$$ J_{7NT} = \frac{1}{NT} \sum_{i=1}^{N} Z_i^{ij} M_{E} \frac{1}{NT} \sum_{j=1}^{N} F_0 \gamma_{0j} E_j^{ij} \hat{F} \Xi_{NT} \gamma_{0i} $$

$$ = \frac{1}{NT} \sum_{i=1}^{N} Z_i^{ij} M_{E} (F_0 - \hat{F} \Pi_{NT}^{-1}) \frac{1}{NT} \sum_{j=1}^{N} \gamma_{0j} E_j^{ij} \hat{F} \Xi_{NT} \gamma_{0i}. $$

Notice that

$$ \frac{1}{NT} \sum_{j=1}^{N} \gamma_{0j} E_j^{ij} \hat{F} = \frac{1}{NT} \sum_{j=1}^{N} \gamma_{0j} E_j^{ij} F_0 + \frac{1}{NT} \sum_{j=1}^{N} \gamma_{0j} E_j^{ij} (F_0 - \hat{F} \Pi_{NT}^{-1}) $$

$$ = O_P \left( \frac{1}{\sqrt{NT}} \right) + \left\| \frac{1}{N\sqrt{T}} \sum_{j=1}^{N} \gamma_{0j} E_j^{ij} \right\| \frac{1}{\sqrt{T}} \|F_0 - \hat{F} \Pi_{NT}^{-1}\| $$

$$ = O_P \left( \frac{1}{\sqrt{NT}} \right) + O_P \left( \frac{1}{\sqrt{N}} \right) \frac{1}{\sqrt{T}} \|F_0 - \hat{F} \Pi_{NT}^{-1}\|, $$

where the second equality follows from (B.8); and the third equality follows from (B.7). Following the arguments given for $J_6$ of Bai (2009, pp. 1271-1272), it is easy to show that $\|J_{7NT}\| = o_P \left( \sqrt{\frac{\nu_m}{NT}} \right) + o_P(\|C_\beta - C_{\beta_0}\|)$.

Based on the above analyses, we have

$$ \text{vec}(\hat{C}_\beta^2) - \text{vec}(C_{\beta_0}^2) - \Sigma_{2,f}^{-1} J_{2NT,1} \cdot (1 + o_P(1)) $$

$$ = \Sigma_{2,f}^{-1} \left\{ \frac{1}{NT} \sum_{i=1}^{N} Z_i^{ij} M_{E} E_i + J_{6NT,1} + J_{8NT} \right\} \cdot (1 + o_P(1)) $$

$$ = \Sigma_{2,f}^{-1} \cdot \frac{1}{NT} \sum_{i=1}^{N} \left\{ Z_i^{ij} M_{E} + \frac{1}{N} \sum_{j=1}^{N} Z_j^{ij} M_{F} \gamma_{0j} (\Gamma_0 \Gamma_0/N)^{-1} \gamma_{0i} \right\} E_i \cdot (1 + o_P(1)) $$

$$ + \Sigma_{2,f}^{-1} \cdot J_{6NT,1} \cdot (1 + o_P(1)). $$
Further organise the above equation, we have

\[
\text{vec}(\hat{C}_3^\beta) - \text{vec}(C_3^*) = A_{1NT}^{-1} \Sigma_{Z, f}^{-1} \cdot \frac{1}{NT} \sum_{i=1}^{N} \{ Z_i^{*'} M_{\hat{F}} + A_{3,i} \} E_i \cdot (1 + o_P(1)) \\
+ A_{1NT}^{-1} \Sigma_{Z, f}^{-1} \cdot J_{6NT,1} \cdot (1 + o_P(1)),
\]

where

\[
A_{1NT} = I_{mp^*} - \Sigma_{Z, f}^{-1} A_{2NT} \cdot (1 + o_P(1)),
\]

\[
A_{2NT} = \frac{1}{N^2T} \sum_{i=1}^{N} \sum_{j=1}^{N} Z_i^{*'} M_{\hat{F}} Z_j^{*} \gamma_0 \left( \frac{\Gamma_0 \Gamma_0}{N} \right)^{-1} \gamma_0,
\]

\[
A_{3,i} = \frac{1}{N} \sum_{j=1}^{N} Z_j^{*'} M_{\hat{F}} \gamma_0 \left( \frac{\Gamma_0 \Gamma_0}{N} \right)^{-1} \gamma_0.
\]

(B.22)

Note that

\[
\sqrt{\frac{NT}{mp^*}} J_{6NT,1} = \frac{1}{(mN)T^2} \sum_{i=1}^{N} Z_i^{*'} M_{\hat{F}} \Omega_\varepsilon \hat{F} \Xi_{NT} \gamma_0
\]

\[
= \frac{\sqrt{N}}{\sqrt{mp^*T}} \cdot \frac{1}{NT} \sum_{i=1}^{N} Z_i^{*'} M_{\hat{F}} \Omega_\varepsilon \hat{F} \Xi_{NT} \gamma_0(v_i) = O_P \left( \frac{N}{T} \right) = O_P(1).
\]

where the last equality follows from the assumption in the body of this theorem. Thus, we obtain \( \| J_{6NT,1} \| = O_P \left( \sqrt{\frac{m}{NT}} \right) \). Moreover, it is easy to show \( \frac{1}{NT} \sum_{i=1}^{N} \{ Z_i^{*'} M_{\hat{F}} + A_{3,i} \} E_i = O_P \left( \sqrt{\frac{mp^*}{NT}} \right) \).

Based on the above development, the proof is complete.

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