Variational formulation for fractional hyperbolic problems in the theory of viscoelasticity

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Abstract. In this article, a theoretical framework for problems involving fractional equations of hyperbolic type arising in the theory of viscoelasticity is presented. Based on the Galerkin method, a variational problem of the fractional viscoelasticity is studied. An appropriate functional setting is introduced in order to establish the existence, uniqueness and a priori estimates for weak solutions. This framework is developed in close concordance with important physical quantities of the theory of viscoelasticity.

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1. Introduction

Viscoelastic materials combine properties of elastic solids and viscous fluids. Elastic materials return to their original configuration when the application of a force ends. However, the deformation of a viscous fluid increases over time when a force is applied. The mathematical modeling of viscoelastic materials is based on the theory of continuum mechanics. For instance, for a linear viscoelastic material with a Kelvin–Voigt constitutive law, the elastic contribution of the stress is proportional to the strain, whereas the viscous part of the stress is proportional to the standard temporal derivative of strain. In [3], a brief review of the viscoelasticity theory with detailed explanations and examples is presented. In [15], various
Theoretical models are analyzed and important practical applications are reviewed (structural systems, marine pipelines, aerospace industry, biomechanics and nanoresonators). These mathematical models are represented by systems of standard partial differential equations that have been extensively studied. For instance, in [6], the existence, uniqueness and stability of the solution of such models are presented.

However, in general, classical mathematical models, such as the ones mentioned above, are not accurate enough for more complex viscoelastic materials such as many polymers [2] and biological tissues [8]. In [2], fractional calculus was considered to construct stress-strain relationships for viscoelastic materials. Also, in [8], viscoelastic properties of human soft tissues were studied considering a Kelvin–Voigt model with a fractional temporal derivative of strain. Both these fractional models were found to provide better approximations of experimental results than classical ones. This agreement with the experiments and the simplicity of the model by introducing very few empirical parameters, make the application of fractional calculus to describe viscoelastic phenomena very attractive. See [9] for more examples of the use of fractional calculus in the theory of viscoelasticity.

1.1. Main result

In 1983, Bagley and Torvik suggested to use fractional derivatives to construct stress-strain relationships for viscoelastic materials. In the early stage, the use of fractional calculus in this context was based on phenomenological arguments [2], but then it was linked to the molecular theory for dilute polymer solutions developed by Rouse [1]. This theoretical basis for the fractional constitutive relations gave confidence in their use to describe accurately the mechanical properties of viscoelastic materials. One of these fractional stress-strain constitutive relations is of Kelvin–Voigt type, which is associated with the following initial boundary value problem of hyperbolic type:

\[
\begin{aligned}
\rho(x) \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x_i} \left[ B_{ij}(x) \frac{\partial}{\partial x_j} C 0D^\alpha u \right] - \frac{\partial}{\partial x_i} \left[ A_{ij}(x) \frac{\partial u}{\partial x_j} \right] &= f(x, t), \\
(x, t) &\in \Omega \times (0, T), \\
u(x, t) &= 0, \quad (x, t) \in \partial \Omega \times (0, T), \\
u(x, 0) &= g(x), \quad \frac{\partial u}{\partial t}(x, 0) = h(x), \quad x \in \Omega,
\end{aligned}
\]  

(1.1)

where summation over repeating indices is assumed. Here, \( \Omega \subset \mathbb{R}^n \) is the domain occupied by the viscoelastic material and \( u \) is the unknown displacement vector field. The material functions are: the mass density \( \rho \), the elastic tensor \( A_{ij} \) and the fractional viscosity tensor \( B_{jk} \) of order \( \alpha \in (0, 1) \). The vector function \( f \) describes an external force acting on the material, and the phenomenon is studied in the time interval \( (0, T) \), with initial displacement an velocity fields \( g \) and \( h \), respectively. The operator \( C 0D^\alpha \) is the Caputo fractional temporal derivative.

In this article, we develop a variational method to study the fractional Kelvin–Voigt model (1.1), providing a basis for a solid mathematical framework to study some important properties of the original model proposed by Bagley and Torvik. These properties include the balance of the kinetic and elastic energies together with the energy dissipated by the fractional viscosity. This variational study is necessary to guarantee the convergence of the Galerkin method and to find error estimates in the Galerkin finite element approximation method to numerically solve (1.1). The main result is the following theorem, which establishes the existence, uniqueness and \( a \ priori \) estimates for weak solutions to (1.1).

**Theorem 1.1.** Under hypothesis \((H.1-H.3)\) in Sect. 5, there exists a unique weak solution to (1.1) and a constant \( C = C(\nu, \rho_0, \Omega, T, \alpha) \) such that

\[
\| u \|_{L^\infty(0, T; H^1_0(\Omega))} + \| u_t \|_{L^\infty(0, T; L^2(\Omega))} + \| u \|_{H^{\alpha/2}_0(0, T; H^1_0(\Omega))} \\
\leq C \left( \| f \|_{L^2(0, T; H^{-1}(\Omega))} + \| g \|_{H^1_0(\Omega)} + \| h \|_{L^2(\Omega)} \right).
\]  

(1.2)
This article is organized as follows. In Sect. 2, the Riemann–Liouville and Caputo fractional derivatives are introduced, which are well documented and their main properties can be found in [10]. In Sect. 3, the $L^2$ theory of the Fourier transform is employed to construct appropriate time-fractional spaces and their variational properties, which are fundamental in the rest of the article. In Sect. 4, the time-fractional properties of these operators are introduced, which are well documented and their main properties can be found in [10]. In Sect. 3, the time-fractional theory of the Fourier transform is employed to construct appropriate time-fractional spaces and their variational properties, which are fundamental in the rest of the article. In Sect. 5, a Galerkin method is implemented to prove Theorem 1.1. Finally, several useful properties of fractional derivatives and spaces used throughout the article are collected in Appendix.

2. Fractional derivatives

In this section, we recall the Riemann–Liouville and Caputo fractional derivatives, two of the most used fractional derivatives to describe viscoelastic materials. We refer to [10] for more details about the properties of these operators.

Let us first recall the usual notation $C^\infty_c(J)$ to represent the set of infinitely differentiable complex valued functions that are compactly supported in the (possibly unbounded) interval $J$. For a (possibly unbounded) interval $(a, b)$ and $\varphi \in C^\infty_c(\mathbb{R})$, the Riemann–Liouville fractional integrals of order $\alpha > 0$ are given by

$$aI^\alpha_\varphi(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{\varphi(s)}{(t-s)^{1+\alpha}} \, ds,$$

$$tI^\alpha_\varphi(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \frac{\varphi(s)}{(s-t)^{1+\alpha}} \, ds,$$

where $\Gamma$ is the Gamma function. These integrals are called the left and right-sided fractional integrals, respectively, and reproduce the well-known Cauchy formula when $\alpha$ is a natural number. To complement the definition, we write $aI^0_\varphi = tI^0_\varphi := I$, where $I$ is the identity operator.

The Riemann–Liouville fractional derivatives of order $\alpha$, with $m - 1 < \alpha \leq m$ for certain $m \in \mathbb{N}$, are defined by

$$aD^\alpha_\varphi(t) := \frac{d^m}{dt^m} \circ aI^{m-\alpha}_\varphi(t), \quad tD^\alpha_\varphi(t) := (-1)^m \frac{d^m}{dt^m} \circ tI^{m-\alpha}_\varphi(t).$$

(2.3)

when $\alpha = m$, an integer, these derivatives coincide, up to a sign, with the usual derivative of order $m$, whereas for non-integer $\alpha$, the left-sided Riemann–Liouville derivative in (2.3) reads

$$aD^\alpha_\varphi(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t \frac{\varphi(s)}{(t-s)^{\alpha+1-m}} \, ds,$$

with a similar expression for the right-sided Riemann–Liouville derivative. To complement the definitions, we assume $aD^0_\varphi = tD^0_\varphi := I$.

On the other hand, by interchanging the operators in (2.3) we obtain the so-called Caputo fractional derivatives of order $\alpha$:

$$C_aD^\alpha_\varphi(t) := aI^{m-\alpha}_\varphi \circ \frac{d^m}{dt^m} \varphi(t), \quad C_tD^\alpha_\varphi(t) := (-1)^m tI^{m-\alpha}_\varphi \circ \frac{d^m}{dt^m} \varphi(t),$$

(2.4)

which coincide, up to a sign, with the usual integer-order derivative when $\alpha = m$, whereas for non-integer $\alpha$, the left-sided Caputo derivative reads

$$C_aD^\alpha_\varphi(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{\varphi^{(m)}(s)}{(t-s)^{\alpha+1-m}} \, ds,$$

with a similar expression for the right-sided Caputo derivative. As previously, we define $C_aD^0_\varphi = C_tD^0_\varphi := I$. 
Remark 2.1. The fractional operators above were defined to be applied to smooth functions, and therefore, all the integrals and derivatives in consideration are well defined. However, such operators can be generalized naturally. For instance, if \( u \in L^1(a, b), -\infty < a < b < +\infty \), then its fractional integrals (2.1) and (2.2) are well defined. Similarly, if \( \alpha \in [0, 1) \) and \( u \) is absolutely continuous in \([a, b]\), then its fractional derivatives (2.3) are also well defined and exist almost everywhere, with

\[
aD_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \left[ \frac{u(a)}{(t-a)^\alpha} + \int_a^t \frac{u'(s)}{(t-s)^\alpha} \, ds \right].
\]  

(2.5)

This identity provides a useful relation between the Riemann–Liouville and Caputo derivatives:

\[
C aD_t^\alpha u(t) = aD_t^\alpha [u(t) - u(a)],
\]

(2.6)

with similar identities for right-sided fractional derivatives and general \( \alpha \geq 0 \). For more details, see Sect. 2.3 of [10]. Observe also that (2.5) and (2.6) imply that

\[
aD_t^\alpha 1 = \frac{1}{\Gamma(1-\alpha)} \frac{1}{(t-a)^\alpha}, \quad C aD_t^\alpha 1 = 0.
\]

(2.7)

Remark 2.2. Concerning (2.7), it is important to mention that Theorem 1.1 is still true if we consider the Riemann–Liouville derivative (instead of the Caputo) in (1.1), but some care has to be taken. If \( g \equiv 0 \) the results remain unchanged, as both operators coincide in this case. On the other hand, if \( g \not\equiv 0 \), the fractional differentiation order \( \alpha \) has to be restricted to the interval \((0, 1/2)\), as the Riemann–Liouville fractional derivative of a time-independent function is nonzero. Indeed, as we implement a variational \( L^2 \)-theory to solve (1.1), it is necessary to restrict the interval of admissible values of \( \alpha \) in order to keep all the expressions in this space. Conversely, the Caputo derivative is zero in time-independent functions, giving it an advantage over the Riemann–Liouville derivative for modeling physical problems.

In order to develop a variational framework to study (1.1), we need to extend the fractional derivatives above to more general functions. To this end, in the next section we will use the \( L^2 \)-theory of the Fourier transform together with an important property of Riemann–Liouville fractional derivatives: Property A.1.

3. Fractional spaces

Motivated by Property A.1, we use the \( L^2 \)-theory of the Fourier transform to extend the fractional Riemann–Liouville derivatives from \( C^\infty_c(\mathbb{R}) \) to a special Hilbert space, suitable to implement a variational framework to study (1.1).

The following definition of fractional spaces by using the Fourier transform is well known, see, for instance, Chapter 15 of [11].

Definition 3.1. Let \( \hat{u} \) denote the Fourier transform of \( u \). For every \( \alpha \geq 0 \), we define the fractional Sobolev space

\[
H^\alpha(\mathbb{R}) = \{ u \in L^2(\mathbb{R}) : |\omega|^\alpha \hat{u} \in L^2(\mathbb{R}) \}.
\]

The fractional space \( H^\alpha(\mathbb{R}) \) is a complex Hilbert space with inner product

\[
(u, v)_\alpha := (u, v) + (|\omega|^\alpha \hat{u}, |\omega|^\alpha \hat{v}),
\]

(3.1)

where \( (u, v) := \int_{-\infty}^\infty u\overline{v} \) is the inner product in \( L^2(\mathbb{R}) \), with the bar denoting the complex conjugate. This inner product generates the norm

\[
\|u\|_\alpha := (\|u\|_{L^2(\mathbb{R})}^2 + |u|_\alpha^2)^{1/2},
\]

(3.2)
where $| \cdot |_{\alpha}$ is the norm

$$|u|_{\alpha} := \| \omega^{\alpha} \hat{u} \|^2_{L^2(\mathbb{R})}.$$  

It is important to remark that these fractional spaces generalize the classical Sobolev spaces $H^m(\mathbb{R})$, for nonnegative integers $m$, in particular $H^0(\mathbb{R}) = L^2(\mathbb{R})$.

The Riemann–Liouville fractional derivatives of order $\alpha$ can be extended from $C_c^\infty(\mathbb{R})$, in (2.3), to every function $u \in H^\alpha(\mathbb{R})$ in the following way:

$$-\infty D_\alpha^\alpha u(t) := \mathcal{F}^{-1}((i \omega)^\alpha \hat{u}(\omega)) \quad \text{and} \quad t D_\alpha^\alpha u(t) := \mathcal{F}^{-1}((-i \omega)^\alpha \hat{u}(\omega)). \quad (3.3)$$

As a consequence of the Plancherel theorem, we can write down the Sobolev norm of $H^\alpha(\mathbb{R})$ in the more classical way

$$\|u\|_\alpha = (\|u\|_{L^2(\mathbb{R})}^2 + \| -\infty D_\alpha^\alpha u \|^2_{L^2(\mathbb{R})})^{1/2}. \quad (3.4)$$

We note that $\| \cdot \|_\alpha$ can also be defined with the right-sided fractional derivative $t D_\alpha^\alpha$, nonetheless both norms are equal.

It is desirable to extend properties from $C_c^\infty(\mathbb{R})$ to $H^\alpha(\mathbb{R})$, and the following result justifies this approach. The proof of this result can be found in Chapter 15 of [11].

**Property 3.1.** The space $C_c^\infty(\mathbb{R})$ is dense in $H^\alpha(\mathbb{R})$.

The fractional space $H^\alpha(\mathbb{R})$ is formed by functions defined in the whole real line. For functions defined in a fixed open interval, it is still desirable to keep important properties of smooth functions in that interval. Therefore, and motivated by Property 3.1, we consider the following definition, see [4,7]. We mention also the recent articles [12,13], where a distributional point of view is used to construct fractional spaces to study Riemann–Liouville fractional integrals (derivatives) and their generalizations.

**Definition 3.2.** Let $\alpha \geq 0$ and the (possibly unbounded) interval $(a, b) \subset \mathbb{R}$. We define the space $H^\alpha_0(a, b)$ as the closure of $C_c^\infty(a, b)$ under the norm $\| \cdot \|_\alpha$.

The fractional space $H^\alpha_0(a, b)$ is a Hilbert subspace of $L^2(a, b)$ with the inner product and norm inherited under the limit of elements in $C_c^\infty(a, b)$: if $u, v \in H^\alpha_0(a, b)$ and $\{ \varphi_j \}_{j=1}^\infty, \{ \psi_j \}_{j=1}^\infty \subset C_c^\infty(a, b)$ are Cauchy sequences with the norm $\| \cdot \|_\alpha$ such that $\varphi_j \rightharpoonup u$ and $\psi_j \rightharpoonup v$ in $L^2(a, b)$, then

$$(u, v)_\alpha := \lim_{j \to \infty} (\varphi_j, \psi_j)_\alpha, \quad \| u \|_\alpha := \lim_{j \to \infty} \| \varphi_j \|_\alpha,$$

where $(\varphi_j, \psi_j)_\alpha$ and $\| \varphi_j \|_\alpha$ are defined in (3.1) and (3.2), respectively. Such inner product and norm are well defined, in the sense that they are independent on the approximating sequence and generalize the corresponding definitions in $C_c^\infty(a, b)$, as can be deduced from the dominated convergence theorem.

Observe also that $H^\alpha_0(a, b) = L^2(a, b)$ and, by Property 3.1, $H^\alpha_0(\mathbb{R}) = H^\alpha(\mathbb{R})$ for every $\alpha \geq 0$.

In this article, we will focus mainly in fractional spaces of order $\alpha \in (0, 1/2)$, this choice will be clear later. For this interval of fractional orders, Definition 3.2 takes a more clear alternative form, allowing to better identify the functions in the fractional space, as we show in the next theorem. In what follows, we denote by $\tilde{u}$ the *continuation of $u$ by zero* outside of $(a, b)$.

**Theorem 3.2.** For every $\alpha \in (0, 1)$ the following properties are equivalent:

1. $u \in H^{\alpha/2}_0(a, b)$;
2. $\tilde{u} \in H^{\alpha/2}(\mathbb{R})$;
3. for every $u \in H^{\alpha/2}_0(a, b)$, there exists $U \in H^{\alpha/2}(\mathbb{R})$ such that the restriction of $U$ to $(a, b)$ is $u$.

**Proof.** It is a simple matter to check that $(i) \Rightarrow (ii) \Rightarrow (iii)$. Let us prove $(ii) \Rightarrow (i)$, as the proof of $(iii) \Rightarrow (ii)$ is similar.

Suppose that $\tilde{u} \in H^{\alpha/2}(\mathbb{R})$. Then, by Property 3.1, there exists a sequence $\{ \varphi_j \}_{j=1}^\infty \subset C_c^\infty(\mathbb{R})$ such that $\varphi_j \rightharpoonup \tilde{u}$ in $H^{\alpha/2}(\mathbb{R})$. Let us now use two standard approximation methods: cutting-off and regularization.
Cutting-off: consider \( \tilde{\varphi}_j := \varphi_j \chi_{(a+1/j, b-1/j)} \), where \( \chi \) is the characteristic function and \( j \) is sufficiently large. We claim that \( \tilde{\varphi}_j \in H^{\alpha/2}(\mathbb{R}) \). Indeed, integration by parts leads to
\[
\tilde{\varphi}_j(\omega) = \frac{i}{\omega} \left( \tilde{\varphi}_j(b)e^{-b\omega i} - \tilde{\varphi}_j(a)e^{-a\omega i} - \int_a^b \varphi_j'(t)e^{-i\omega t} \, dt \right)
= O \left( \frac{1}{|\omega|} \right), \quad \text{as } |\omega| \to \infty.
\]
This estimate, together with the fact that \( \alpha \in (0, 1) \), implies that \( |\omega|^{\alpha/2} \tilde{\varphi}_j \in L^2(\mathbb{R}) \), as expected. Moreover, as \( \varphi_j \to \tilde{u} \) in \( H^{\alpha/2}(\mathbb{R}) \), we can use the dominated convergence theorem to find that
\[
\tilde{\varphi}_j \to \tilde{u} \quad \text{in } H^{\alpha/2}(\mathbb{R}).
\]

Regularization: consider \( \varphi_j^* := \varphi_j \ast \eta_{1/3j} \), where \( \eta_j(t) = j \eta(jt) \) is a regularization sequence:
\[
\eta \in C_c^\infty(-1, 1) \quad \text{with} \quad \int_{-\infty}^\infty \eta(t) \, dt = 1.
\]
By well-known properties of convolution \( * \), for every sufficiently large \( j \), \( \{ \varphi_j^* \}_j \subset C_c^\infty(a, b) \); moreover, by (3.6), \( \{ \varphi_j^* \} \) is a Cauchy sequence with the norm \( \| \cdot \|_{\alpha/2} \) and
\[
\varphi_j^* \to u \quad \text{in } L^2(a, b).
\]
Therefore, we conclude that \( u \in H_0^{\alpha/2}(a, b) \), and the proof is complete. \( \square \)

The following lemma is a consequence of Theorem 3.2, and states that the fractional space \( H_0^{\alpha/2}(a, b) \) contains the classical Sobolev space of integer order:
\[
H^1(a, b) := \{ u \in L^2(a, b) : u' \in L^2(a, b) \}.
\]

**Lemma 3.3.** For every \( \alpha \in (0, 1) \) we have \( H^1(a, b) \subset H_0^{\alpha/2}(a, b) \).

**Proof.** Suppose \( u \in H^1(a, b) \). By proving that \( \tilde{u} \in H^{\alpha/2}(\mathbb{R}) \), then Theorem 3.2 guarantees that \( u \in H^{\alpha/2}(a, b) \). Indeed, note that estimate (3.5), of the proof of Theorem 3.2, is also true for \( \tilde{u} \); and therefore \( |\omega|^{\alpha/2} \tilde{u} \in L^2(\mathbb{R}) \), as expected. \( \square \)

**Remark 3.3.**
\begin{enumerate}
\item Theorem 3.2 guarantees the equivalence of the three main methods to define fractional spaces in an interval, see Chapter 1 in [7].
\item In particular, Lemma 3.3 implies that, for any bounded interval \( (a, b) \), every \( u \in C^1[a, b] \) is such that \( u \in H_0^{\alpha/2}(a, b) \). Therefore, functions in \( H_0^{\alpha/2}(a, b), \alpha \in (0, 1) \), leave no trace on the interval endpoints, so they are not necessarily equal to zero there; see Chapter 16 of [11] for more details.
\end{enumerate}

**Definition 3.4.** For \( \alpha \in (0, 1) \) and \( u \in H_0^{\alpha/2}(a, b) \), we define the fractional Riemann–Liouville derivatives as
\[
aD_t^{\alpha/2}u := \lim_{j \to \infty} -\infty D_t^{\alpha/2} \varphi_j, \quad tD_b^{\alpha/2}u := \lim_{j \to \infty} tD_b^{\alpha/2} \varphi_j,
\] where \( \{ \varphi_j \}_{j=1}^\infty \subset C_c^\infty(a, b) \) is a Cauchy sequence with the norm \( \| \cdot \|_{\alpha/2} \) such that \( \varphi_j \to u \) in \( L^2(a, b) \).

The limits in (3.7) are taken in the \( L^2(\mathbb{R}) \) sense and, therefore, both fractional derivatives belong to this space. By the dominated convergence theorem, these fractional operators are well defined in the sense that they are independent of the approximating sequence.

The following lemma shows that the fractional operators defined by limit processes in (3.7) generalize the classical definition of the fractional derivatives in (2.3), for functions in \( C^1[a, b] \), in a bounded interval \( (a, b) \); recall that by Remark 2.1, the operator in (2.3) is well defined in this space.
Lemma 3.4. Let $\alpha \in (0, 1)$ and let $(a, b)$ be a bounded interval. If $u \in C^1[a, b]$ then $u \in H_0^{\alpha/2}(a, b)$ and both fractional Riemann–Liouville derivatives in (2.3) and (3.7) agree.

Proof. The key observation is that Property A.1 (i) is also true for piecewise differentiable functions with compact support, see Sect. 7.1 of [10]. Therefore, by applying this property to the piecewise differentiable function with compact support $\tilde{u}$, we have

$$-\infty D_t^{\alpha/2} \tilde{u}(\omega) = \mathcal{F}^{-1}((i\omega)^{\alpha/2} \tilde{u}(\omega)),$$

where the fractional derivative here is the classical one in (2.3). On the other hand, Theorem 3.2 implies that $\tilde{u} \in H^{\alpha/2}(\mathbb{R})$ (see also Remark 3.3 (ii)) and by (3.3) we have

$$-\infty D_t^{\alpha/2} \tilde{u}(\omega) = \mathcal{F}^{-1}((i\omega)^{\alpha/2} \tilde{u}(\omega)),$$

where the fractional derivative here is the one in (3.3). We conclude that these fractional derivatives are equal. Moreover, by similar arguments to those in the proof of Theorem 3.2, the fractional derivative in the last equation coincide with the one by the limit in (3.7), and the proof is complete. \qed

From the proof of Theorem 3.2, we observe that the sequence $\{\tilde{\varphi}_j^\alpha\}_{j=1}^\infty \subset C_c^\infty(a, b)$ is such that

$$aD_t^{\alpha/2} \tilde{\varphi}_j^\alpha \to aD_t^{\alpha/2} u, \quad tD_b^{\alpha/2} \tilde{\varphi}_j^\alpha \to tD_b^{\alpha/2} u, \quad \text{in } L^2(\mathbb{R}).$$

These approximations are needed in order to prove energy estimates in Theorem 3.6.

In what follows, we provide some fractional variational formulae that are fundamental to the development of the Galerkin method in Sect. 5.

Theorem 3.5. Let $\alpha \in (0, 1)$ and let $(a, b)$ be a bounded interval.

i If $u, v \in H_0^{\alpha/2}(a, b)$ then

$$\int_a^b aD_t^{\alpha/2} uv \, dt = \int_a^b utD_b^{\alpha/2} v \, dt; \quad (3.8)$$

ii If $u \in C^2[a, b]$ and $v \in H_0^{\alpha/2}(a, b)$ then

$$\int_a^b C aD_t^{\alpha/2} uv \, dt = \int_a^b C aD_t^{\alpha/2} utD_b^{\alpha/2} v \, dt. \quad (3.9)$$

iii If $u, v \in L^2(a, b)$ then

$$\int_a^b aI_t^{\alpha/2} uv \, dt = \int_a^b utI_b^{\alpha/2} v \, dt, \quad (3.10)$$

$$\int_a^b aI_t^{\alpha/2} uv \, dt = \int_a^b aI_t^{\alpha/2} utI_b^{\alpha/2} v \, dt. \quad (3.11)$$

Proof. For $u, v \in C_c^\infty(a, b)$, we provide a proof of (i) in Appendix, see Property A.4. In general, for $u, v \in H_0^{\alpha/2}(a, b)$ we can deduce (i) by a limit argument, as by definition all the elements of this space can be approximate by sequences of smooth functions (recall also Lemma 3.4).

Let us now prove (ii). By (2.6) and the semigroup property (A.8),

$$C aD_t^\alpha u(t) = aD_t^\alpha [u(t) - u(a)] = aD_t^{\alpha/2} aD_t^{\alpha/2} [u(t) - u(a)]. \quad (3.12)$$

We claim that $aD_t^{\alpha/2} [u(t) - u(a)] \in H^1(a, b)$. Indeed, by (2.5) and (2.3),

$$\frac{d}{dt} aD_t^{\alpha/2} [u(t) - u(a)] = aD_t^{\alpha/2} u'(t)$$
\[
= \frac{1}{\Gamma(1-\alpha/2)} \left[ \frac{u'(a)}{(t-a)^{\alpha/2}} + \int_a^t \frac{u''(s)}{(t-s)^{\alpha/2}} ds \right],
\]
and this last term belongs to \( L^2(a,b) \). Therefore, by Lemma 3.3, we can use (3.8) together with (3.12) to deduce (3.9).

The proof of (iii) is similar but using Property A.2 together with (A.6) and (A.7). \( \square \)

**Theorem 3.6.** Let \( \alpha \in (0,1) \). Let \((a, b)\) and \(u\) be a bounded interval and a real-valued function, respectively. Then,

i) if \( u \in H^{\alpha/2}_0(a,b) \),

\[
\|aD_t^{\alpha/2}u\|^2_{L^2(\mathbb{R})} = \frac{1}{\cos(\alpha\pi/2)} \int_a^b aD_t^{\alpha/2}u tD_b^{\alpha/2}u dt; \quad (3.13)
\]

ii) if \( u \in L^2(a,b) \),

\[
\|aI_t^{\alpha/2}u\|^2_{L^2(a,b)} = \frac{1}{\cos(\alpha\pi/2)} \int_a^b aI_t^{\alpha/2}u tI_b^{\alpha/2}u dt, \quad (3.14)
\]

where the Riemann–Liouville fractional integrals are taken in the sense of classical formulae (2.1) and (2.2), see Remark 2.1.

**Proof.** Let us prove (i), the proof of (ii) is similar. First, note that it suffices to prove (3.13) for \( u \in C_c^\infty(a,b) \), as for more general \( u \) we just recall Lemma 3.4 and the comments after its proof. So, in order to prove (3.13) for \( u \in C_c^\infty(a,b) \), we follow [4] and use the Fourier transform; we provide here the details for completeness. Recall the following well-known property of the Fourier transform:

\[
\int_{-\infty}^\infty \hat{u} \hat{v} = \int_{-\infty}^\infty u \hat{v},
\]

with a similar property for the inverse Fourier transform. As \( u \in C_c^\infty(a,b) \), this identity and Property A.1 imply

\[
\int_a^b aD_t^{\alpha/2}utD_b^{\alpha/2}u dt = \int_{-\infty}^\infty -D_t^{\alpha/2}utD_{-\infty}^{\alpha/2}u dt
\]

\[
= \int_{-\infty}^\infty (i\omega)^{\alpha/2} \frac{\hat{u}}{\hat{2}}(i\omega)^{\alpha/2} \frac{\hat{u}}{\hat{2}} e^{-\frac{\alpha\pi i}{2} \text{sgn} \omega} d\omega.
\]

On the other hand, by (A.2)

\[
(-i\omega)^{\alpha/2} = (i\omega)^{\alpha/2} e^{-\frac{\alpha\pi i}{2} \text{sgn} \omega},
\]

and therefore

\[
\int_a^b aD_t^{\alpha/2}utD_b^{\alpha/2}u dt
\]

\[
= \int_{-\infty}^\infty (i\omega)^{\alpha/2} \frac{\hat{u}}{\hat{2}}(i\omega)^{\alpha/2} \frac{\hat{u}}{\hat{2}} \hat{e}^{-\frac{\alpha\pi i}{2} \text{sgn} \omega} d\omega
\]
\[
= \int_{-\infty}^{0} (i\omega)^{\alpha/2} \hat{u}(i\omega)^{\alpha/2} \hat{u} d\omega e^{-\frac{\omega^2}{2}} + \int_{0}^{\infty} (i\omega)^{\alpha/2} \hat{u}(i\omega)^{\alpha/2} \hat{u} d\omega e^{\frac{\omega^2}{2}}
\]

\[
= \cos(\alpha\pi/2) \int_{-\infty}^{\infty} (i\omega)^{\alpha/2} \hat{u}(i\omega)^{\alpha/2} \hat{u} d\omega
\]

\[
+ i \sin(\alpha\pi/2) \left[ \int_{0}^{\infty} (i\omega)^{\alpha/2} \hat{u}(i\omega)^{\alpha/2} \hat{u} d\omega - \int_{-\infty}^{0} (i\omega)^{\alpha/2} \hat{u}(i\omega)^{\alpha/2} \hat{u} d\omega \right]
\]

\[
= \cos(\alpha\pi/2) \int_{-\infty}^{\infty} |\omega|^\alpha |\hat{u}|^2 d\omega,
\]

where the last equality is a consequence of the fact that \( u \) is real-valued, and therefore \( \hat{u}(-\omega) = \hat{u}(\omega) \).

We conclude (3.13) by using Plancherel theorem. □

To finish this section, we show a fractional version of the classical Poincaré inequality.

**Theorem 3.7.** (Fractional Poincaré inequality) There exists \( C > 0 \) such that, for every \( u \in H^{\alpha/2}_0(a, b) \), we have

\[
\|u\|_{L^2(a, b)} \leq C\|aD_t^{\alpha/2}u\|_{L^2(a, b)}.
\]

**Proof.** By Properties A.2 and A.3, we deduce that there exists a positive constant \( C \) such that

\[
\|u\|_{L^2(a, b)} = \|aI^{\alpha/2}aD_t^{\alpha/2}u\|_{L^2(a, b)} \leq C\|aD_t^{\alpha/2}u\|_{L^2(a, b)}
\]

for all \( u \in H^{\alpha/2}_0(a, b) \), as we stated. □

### 4. Space-time fractional spaces

In this section, we present suitable space-time fractional spaces to implement a Galerkin method to solve (1.1) in the next section. This approach is based on classical ideas coming from Fourier analysis, specially the separation of variables method.

We start by recalling some notions of strong measurability and integrability of functions with values in Hilbert spaces. In what follows \( H \) represents a separable complex Hilbert space with inner product \((\cdot, \cdot)\), norm \( \|\cdot\| \) and orthogonal basis \( \{w_k\}_{k=1}^{\infty} \). To fix ideas, we are mainly interested in the Sobolev space \( H^1_0(\Omega) \) of vector fields in \( \Omega \), a bounded domain of \( \mathbb{R}^n \) with Lipschitz boundary (see next section); a possible orthogonal basis for this space can be assembled with the eigenfunctions of the Laplace operator in the domain.

We recall also some well-known facts about measurable functions with values in Hilbert spaces, for more details see Sects. V.4 and V.5 of [14] and Appendix E.5 of [5]. Let \( \alpha \geq 0 \), the (possibly unbounded) interval \((a, b) \subset \mathbb{R} \) and \( H \) be a separable complex Hilbert space.

- A function \( \zeta : (a, b) \to H \) is called *simple* if it has the form
  \[
  \zeta(t) = \sum_{j=1}^{m} e_j \chi_{E_j}(t) w_j, \quad t \in (a, b),
  \]

  where each \( E_j \) is a Lebesgue measurable subset of \((a, b) \) and \( e_j \in \mathbb{C}, \ j = 1, \ldots, m \).

- A function \( u : (a, b) \to H \) is *strongly measurable* if there exists a sequence of simple functions \( \zeta_k : (a, b) \to H \) such that
  \[
  \zeta_k(t) \to u(t) \quad \text{for a.e. } a < t < b.
  \]
iii We define $L^2(a, b; H)$ as the set of strongly measurable functions $u : (a, b) \to H$ such that

$$
\|u\|_{L^2(a, b; H)} := \left( \int_a^b \|u(t)\|^2 \, dt \right)^{1/2} < \infty.
$$

This is a separable complex Hilbert space.

Let us remark that every $u \in L^2(a, b; H)$ can be approximate with smooth (in time) functions of the form

$$
\varsigma(t) = \sum_{j=1}^m \varphi_j(t)w_j,
$$

where $\varphi_j \in C_\infty^\infty(a, b)$, instead of characteristics (as in (4.1)). In what follows, we use $S$ to denote the set of simple functions, that is, functions of the form (4.1) or (4.2).

**Definition 4.1.** For every $\alpha \geq 0$ and $\varsigma \in S$, with

$$
\varsigma(t) = \sum_{j=1}^m \varphi_j(t)w_j,
$$

we define the norm

$$
\|\varsigma\|_{H_0^\alpha(a, b; H)} := \left( \|\varsigma\|^2_{L^2(a, b; H)} + \|0D_\alpha^\varsigma\|^2_{L^2(a, b; H)} \right)^{1/2}
= \left( \sum_{j=1}^m \|\varphi_j\|^2 \|w_j\|^2 \right)^{1/2},
$$

where $\|\varphi_j\|_\alpha$ is the norm considered in (3.2) and (3.4). Let $\alpha \geq 0$, the (possibly unbounded) interval $(a, b) \subset \mathbb{R}$ and $H$ be a separable complex Hilbert space. We define the fractional space $H_0^\alpha(a, b; H)$ as the closure of the set $S$ under the norm $\|\cdot\|_{H_0^\alpha(a, b; H)}$.

The fractional space $H_0^\alpha(a, b; H)$ is a Hilbert subspace of $L^2(a, b; H)$ with the inner product and norm inherited under the limit of elements in $S$. As in the previous section, this space and its inner product and norm are independent of the approximating sequence in $S$ and the orthogonal basis $\{w_k\}_{k=1}^\infty$. This approach is similar to the one considered in Definition 3.2.

As we mentioned in the previous section, we are mainly interested in fractional spaces of order $\alpha \in (0, 1/2)$. For this interval of fractional orders, we have properties analogous to those in Theorem 3.2 and Lemma 3.3; the proof of these properties are similar and we leave the details to the reader. As before, we denote by $\tilde{u}$ the continuation of $u$ by zero outside of $(a, b)$.

**Theorem 4.1.** For every $\alpha \in (0, 1)$ the following properties are equivalent:

i $u \in H_0^{\alpha/2}(a, b; H)$;

ii $\tilde{u} \in H^{\alpha/2}(\mathbb{R}; H)$;

iii there exists $U \in H^{\alpha/2}(\mathbb{R}; H)$ such that its restriction to $(a, b)$ is $u$.

Let us recall the classical time Sobolev space of order one

$$
H^1(a, b; H) := \{ u \in L^2(a, b; H) : u' \in L^2(a, b; H) \},
$$

for more details see Sect. 5.9 of [5].

**Lemma 4.2.** For every $\alpha \in (0, 1)$ we have that $H^1(a, b; H) \subset H_0^{\alpha/2}(a, b; H)$. 
We observe that this lemma implies that, for any bounded interval \((a, b)\), every \(u \in C^1([a, b]; H)\) belongs to the space \(u \in H^{\alpha/2}_0(a, b; H)\). Therefore, the functions in this fractional space leave no trace on the extremes of interval endpoints, so they are not necessarily equal to zero there.

**Definition 4.2.** If \(\alpha \in (0, 1)\) and \(u \in H^{\alpha/2}_0(a, b; H)\), we define the Riemann–Liouville fractional derivatives as

\[
aD^\alpha_t u := \lim_{k \to \infty} -\infty D^\alpha_t \varsigma_k \quad \text{and} \quad tD^\alpha_b u := \lim_{k \to \infty} tD^\alpha_b \varsigma_k,
\]

where \(\{\varsigma_k\}_k^{\infty} \subset S\) is a Cauchy sequence with the norm \(\| \cdot \|_{H^\alpha_0(a, b; H)}\), such that \(\varsigma_k \to u\) in \(L^2(a, b; H)\).

ii If \(\alpha > 0\), \((a, b)\) is a bounded interval and \(u \in L^2(a, b; H)\) we define the Riemann–Liouville fractional integrals

\[
aI^\alpha_t u := \lim_{k \to \infty} -\infty I^\alpha \varsigma_k \quad \text{and} \quad tI^\alpha_b u := \lim_{k \to \infty} tI^\alpha \varsigma_k,
\]

where \(\varsigma_k \to u\) in \(L^2(a, b; H)\).

The limits in (4.4) are taken in the \(L^2(\mathbb{R}; H)\) sense and therefore, both fractional derivatives belong to this space. Moreover, as in the previous section, these fractional operators are well defined in the sense that they do not depend neither on the approximating sequence nor on the orthogonal basis \(\{w_k\}_k^{\infty}\).

On the other hand, by the continuity of the Riemann–Liouville fractional integral operator in the \(L^2(a, b)\) time space, see Property A.2, the space-time operators in (4.5) are also well defined, as they do not depend neither on the approximating sequence nor on the orthogonal basis \(\{w_k\}_k^{\infty}\).

5. Existence, uniqueness and a priori estimates

In this section, we implement a Galerkin method in fractional spaces to prove Theorem 1.1. Also, we prove important energy estimates that, in the end, imply the a priori estimate (1.2).

Let us start with the notion of vector fields. Given a complex functional space \(X\) and a natural number \(n\), we denote by \(X\) the \(n\)-dimensional space of vector fields \(X^n\). In particular, if \(H\) is a complex Hilbert space with inner product \((\cdot, \cdot)\) and norm \(\| \cdot \|\), then \(H\) is a Hilbert space with inner product and norm

\[
(u, v)_H := \sum_{i=1}^{n} (u_i, v_i), \quad \|u\|_H = \left( \sum_{i=1}^{n} |u_i|^2 \right)^{1/2},
\]

respectively, where \(u = (u_1, \ldots, u_n)\) and \(v = (v_1, \ldots, v_n)\) belong to \(H\).

In this section, we consider the fractional initial boundary problem (1.1), where the fractional operator \(C^0D^\alpha_t\) is the Caputo temporal derivative

\[
C^0D^\alpha_t u = 0D^\alpha_t (u - g),
\]

for smooth \(u\) and \(g\), see (2.6). We assume the following hypotheses:

- **H.1** The elastic and fractional viscosity tensors are matrix-valued functions: \(A_{ij} = (A^{kl}_{ij})_{1 \leq k, l \leq n} \in \mathbb{R}^{n \times n^2}\), \(B_{ij} = (B^{kl}_{ij})_{1 \leq k, l \leq n} \in \mathbb{R}^{n \times n^2}\), and they are measurable functions of their arguments. We also assume the symmetry on these tensors:

\[
A^{kl}_{ij} = A_{ij}^{kl} = A^{lk}_{ij} = A^{ji}_{kl}, \quad B^{kl}_{ij} = B_{ij}^{kl} = B^{lk}_{ij} = B_{kl}^{ji} \quad 1 \leq i, j, k, l \leq n.
\]

Additionally, we suppose that \(A\) and \(B\) are uniformly elliptic; that is, there exists \(\nu > 0\) such that for all symmetric matrices \((\eta_{ij}) \in \mathbb{R}^{n \times n}\)

\[

\nu \eta_{ij}^2 \leq A_{ij}(x)\eta_{ij} \quad \text{and}
\]

\[

0 \leq B_{ij}(x) \leq \frac{1}{\nu} \eta_{ij}^2
\]
\[ \nu \eta_{ij}^2 \leq B_{ij}^{kl}(x) \eta_{ij} \eta_{kl} \leq \frac{1}{\nu} \eta_{ij}^2, \]
a.e. \( x \in \Omega. \)

**H.2** The mass density function \( \rho \) is measurable and there exists \( \rho_0 > 0 \) such that
\[ \rho_0 \leq \rho(x) \leq \frac{1}{\rho_0}, \quad \text{a.e. } x \in \Omega. \]

**H.3** \( f \in L^2(0, T; H^{-1}(\Omega)), g \in H_0^1(\Omega) \) and \( h \in L^2(\Omega). \)

Let us recall the usual notation \( H^{-1}(\Omega) \) for the dual of the space \( H_0^1(\Omega) \), where every action of one of its elements \( f \in H^{-1}(\Omega) \) on \( v \in H_0^1(\Omega) \) is represented by \( \langle f, v \rangle \).

We will use a variational method to study (1.1) that requires a suitable notion of weak solution. This notion can be motivated by the fractional integration by parts formula (3.9). Before stating the definition of weak solutions to (1.1), recall the fractional space \( H_0^{\alpha/2}(0, T; H_0^1(\Omega)) \) in Definition 4.1. Let us also recall (5.1) for the inner product of the real Hilbert space of vector fields \( L^2(\Omega) \):
\[ (u, v) := \sum_{i=1}^{n} \int_{\Omega} u_i v_i \, dx, \]
where \( u = (u_1, \ldots, u_n) \), \( u_i \in L^2(\Omega) \), and \( v = (v_1, \ldots, v_n) \), \( v_i \in L^2(\Omega) \), \( i = 1, \ldots, n. \)

**Definition 5.1.** We say that a function \( u \in H_0^{\alpha/2}(0, T; H_0^1(\Omega)) \) is a weak solution of (1.1) if
\begin{enumerate}
  \item \( \frac{\partial u}{\partial t} \in L^2(0, T; L^2(\Omega)); \)
  \item for every vector field \( \varphi \in C^1(\overline{\Omega} \times [0, T]) \), with \( \varphi(x, t) = 0 \) if \( x \in \partial\Omega \) or \( t = T \),
  \[ \int_0^T \left( \rho \frac{\partial u}{\partial t} \cdot \frac{\partial \varphi}{\partial t} \right) \, dt - \int_0^T \left( B_{ij} \left( 0 D_t^{\alpha/2} \frac{\partial u}{\partial x_j}, t D_t^{\alpha/2} \frac{\partial \varphi}{\partial x_i} \right) + \left( A_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial \varphi}{\partial x_i} \right) \right) \, dt \]
  \[ = - \int_0^T \langle f, \varphi \rangle \, dt - \langle h, \varphi(x, 0) \rangle, \]
\end{enumerate}
where, as before, the summation over the repeating indices is assumed and we define, motivated by (2.6),
\[ C 0 D_t^{\alpha/2} \frac{\partial u}{\partial x_j} := 0 D_t^{\alpha/2} \left( \frac{\partial u}{\partial x_j} - \frac{\partial g}{\partial x_j} \right), \]
with the fractional Riemann–Liouville derivative \( 0 D_t^{\alpha/2} \) given by (4.4);
\begin{enumerate}
  \item \( u(0) = g. \)
\end{enumerate}

**Remark 5.2.**
\begin{enumerate}
  \item Under the hypotheses above on \( u \) and \( g \), it can be shown that \( u \in C([0, T]; L^2(\Omega)) \) (see Sect. 5.9.2 of [5]), so the equality (iii) makes sense.
  \item In a classical framework: \( f \in C(\overline{\Omega} \times [0, T]), g, h \in C^1(\overline{\Omega}) \), every classical solution \( u \in C^2(\overline{\Omega} \times [0, T]) \) of (1.1) is also a weak solution. Indeed, by Lemma 4.2 we have \( u \in H_0^{\alpha/2}(0, T; H_0^1(\Omega)). \) Therefore, it suffices to verify (5.2) for every function \( \varphi \in C^1(\overline{\Omega} \times [0, T]) \) with \( \varphi(x, t) = 0 \) if \( x \in \partial\Omega \) or \( t = T. \) This can be done by multiplying the first equation of (1.1) by \( \varphi \) and then integrating by parts in space, in the usual way, and in time with formula (3.9); recall also Lemma 3.4.
  \item By an approximation argument, it is not difficult to show that (5.2) is also true for \( \varphi \in H^1(0, T; H_0^1(\Omega)) \) such that \( \varphi(x, t) = 0 \) for all \( x \in \Omega. \) recall (4.3) for the definition of this space.
\end{enumerate}
5.1. Galerkin approximations

To find weak solutions to (1.1), we use the Galerkin method: to construct solutions of certain finite-dimensional approximations of the problem and then pass to the limit (see Sects. 7.1 and 7.2 of [5].)

Let \( \{ w_k \}_{k=1}^{\infty} \) be an orthogonal basis of \( H^1_0(\Omega) \) that, additionally, is an orthonormal basis of \( L^2(\Omega) \). One of such an orthogonal basis can be assembled with the eigenfunctions of the Laplace operator in the bounded domain \( \Omega \).

Fix a positive integer \( m \) and consider the function

\[
 u_m(t) := \sum_{k=1}^{m} d_m^k(t) w_k. \tag{5.4}
\]

Next, we select the coefficients \( d_m^k(t) \) to make \( u_m \) an approximate solution to (1.1) in the following sense: \( d_m^k(t) \), \( k = 1, \ldots, m \), satisfy the following projection of (1.1) onto the finite-dimensional subspace spanned by \( \{ w_k \}_{k=1}^{m} \): for \( k = 1, \ldots, m \),

\[
 (\rho u_m'', w_k) + \left( B_{ij} C 0 D_t^\alpha \frac{\partial u_m}{\partial x_j}, \frac{\partial w_k}{\partial x_i} \right) + \left( A_{ij} \frac{\partial u_m}{\partial x_j}, \frac{\partial w_k}{\partial x_i} \right) = \langle f, w_k \rangle, \quad t \in (0, T), \tag{5.5}
\]

\[
 d_m^k(0) = (g, w_k), \tag{5.6}
\]

\[
 d_m^k(0) = (h, w_k), \tag{5.7}
\]

where the summation over the indices \( i, j = 1, \ldots, n \) is assumed. This system of ordinary fractional differential equations can be solved by transforming it in a Volterra-type equation to find a unique solution \( u_m \in C^2([0, T]; H^1_0(\Omega)) \) of the form (5.4) that satisfies (5.5)-(5.7), see Property A.6 in Appendix.

5.2. Energy estimates

Before taking \( m \to \infty \) in the Galerkin approximations, we need to control some important norms of the approximate solutions \( u_m \). In fact, we will prove that for every \( m \in \mathbb{N} \), there exists \( C = C(\nu, \rho_0, \Omega, T, \alpha) \) such that

\[
 \max_{0 \leq t \leq T} \left( \| u_m(t) \|_{H^1_0(\Omega)} + \| u'_m(t) \|_{L^2(\Omega)} + \| u_m \|_{H^{\alpha/2}_0(0, T; H^1_0(\Omega))} \right) \leq C \left( \| f \|_{L^2([0, T]; H^{-1}(\Omega))} + \| g \|_{H^1_0(\Omega)} + \| h \|_{L^2(\Omega)} \right), \tag{5.8}
\]

In what follows, \( C \) represents a positive constant that could change from one inequality to the next, but only depends on \( \nu, \rho_0, \Omega, T, \alpha \).

To estimate the first term of the left-hand side of (5.8), we multiply (5.5) by \( d_m' \) and sum \( k = 1, \ldots, m \) to find

\[
 (\rho u_m'', u_m') + \left( B_{ij} C 0 D_t^\alpha \frac{\partial u_m}{\partial x_j}, \frac{\partial u_m'}{\partial x_i} \right) + \left( A_{ij} \frac{\partial u_m}{\partial x_j}, \frac{\partial u_m'}{\partial x_i} \right) = \langle f, u_m' \rangle, \quad t \in (0, T). \tag{5.9}
\]

Observe that this equation can be written in the following way (recall (2.4))

\[
 E'(t) + \left( B_{ij} 0 l_t^{\alpha} \frac{\partial u_m'}{\partial x_j}, \frac{\partial u_m'}{\partial x_i} \right) = \langle f, u_m' \rangle, \tag{5.9}
\]

where \( E(t) \) is the total energy of \( u \) at time \( t \), that is, the sum of the kinetic and elastic energies,

\[
 E(t) := \frac{1}{2} \| \sqrt{\rho} u_m'(t) \|_{L^2(\Omega)}^2 + \frac{1}{2} \left( A_{ij} \frac{\partial u_m}{\partial x_j}(t), \frac{\partial u_m}{\partial x_i}(t) \right). \tag{5.9}
\]
For a given $0 \leq t \leq T$, we integrate (5.9) in the time interval $(0,t)$ to obtain

$$E(t) - E(0) = \int_0^t \langle f, u'_m \rangle ds - \int_0^t \left( B_{ij} 0I_s^{1-\alpha} \frac{\partial u'_m}{\partial x_j}, \frac{\partial u'_m}{\partial x_i} \right) ds. \quad (5.10)$$

This expression can be seen as an energy balance: if we consider the total energy of $u$ as the sum of its kinetic and elastic energies, then the change of this quantity, along the time interval $(0, t)$, is measured by the work done by the external force $f$ minus the energy dissipated by the fractional viscosity, represented by the last term in the right-hand side of the previous equation. The following lemma shows that, in fact, this quantity is nonnegative.

**Lemma 5.1.** For every $t \in [0, T]$ and $u \in L^2(0, T; \mathbf{H}^\alpha_0(\Omega))$ we have

$$\int_0^t \left( B_{ij} 0I_s^{1-\alpha} \frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial x_i} \right) ds \geq 0, \quad (5.11)$$

where $0I_s^{1-\alpha}$ is the Riemann–Liouville fractional integral defined in (4.5).

**Proof.** First observe that, by hypothesis (H.1) and the classical Poincaré inequality,

$$(u, v)_B := \left( B_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_i} \right)$$

generates an equivalent inner product to the usual one of $H^\alpha_0(\Omega)$. Let us select an orthogonal basis \{w_k^B\}_{k=1}^\infty of $H^\alpha_0(\Omega)$ with this new inner product. By (4.5) and the subsequent comments, it suffices to prove (5.11) for $u$ of the form

$$u(x, t) = \sum_{k=1}^m \varphi_k(t)w_k^B,$$

where $\varphi_j \in C_\infty^\infty(a, b)$. In this setting, (5.11) can be written as

$$\sum_{k=1}^m \left( \int_0^t 0I_s^{1-\alpha} \varphi_k(s) \varphi_k(s) ds \right) (w_k^B, w_k^B)_B \geq 0,$$

which is true by (3.11) and (3.14). \qed

The previous lemma together with (5.10), hypotheses (H.1) and (H.2), Cauchy–Schwarz inequality and boundary conditions (5.6) and (5.7) imply that there exists a positive constant $C = C(\nu, \rho_0)$ such that for every $0 \leq t \leq T$ we have

$$\max \left\{ \|u'_m(t)\|_{L^2(\Omega)}, \left\| \frac{\partial u}{\partial x_1}(t) \right\|_{L^2(\Omega)}^2 \right\} \leq C \left( \|f\|_{L^2(0, T; H^{-1}(\Omega))}^2 + \|g\|_{H^\alpha_0(\Omega)}^2 + \|h\|_{L^2(\Omega)}^2 + \int_0^t \|u'_m(s)\|_{L^2(\Omega)}^2 ds \right). \quad (5.12)$$

This estimate and the Grönwall’s inequality imply the existence of a positive constant $C = C(\nu, \rho_0, T)$ such that

$$\max_{0 \leq t \leq T} \|u'_m(t)\|_{L^2(\Omega)} \leq C \left( \|f\|_{L^2(0, T; H^{-1}(\Omega))} + \|g\|_{H^\alpha_0(\Omega)} + \|h\|_{L^2(\Omega)} \right). \quad (5.13)$$

On the other hand, by (5.12) and Poincaré inequality, there exists $C = C(\nu, \rho_0, \Omega, T)$ such that

$$\max_{0 \leq t \leq T} \|u_m(t)\|_{H^\alpha_0(\Omega)} \leq C \left( \|f\|_{L^2(0, T; H^{-1}(\Omega))} + \|g\|_{H^\alpha_0(\Omega)} + \|h\|_{L^2(\Omega)} \right). \quad (5.14)$$
Now, let us estimate the second term in the left-hand side of (5.8). To this end, multiply (5.5) by \( d_m^k - (g, w_k) \) and sum \( k = 1, \ldots, m \), to obtain
\[
(p u_m'' + u_m - g_m) + \left( B_{ij} \partial_t^\alpha \partial_{x_j} (u_m - g_m), \partial_{x_i} (u_m - g_m) \right) + \left( A_{ij} \partial_{x_j} (u_m - g_m), \partial_{x_i} (u_m - g_m) \right) \\
= (f, u_m - g_m), \quad t \in (0, T),
\]
where \( g_m := \sum_{k=1}^m (g, w_k) w_k \), the projection of \( g \) on the finite-dimensional subspace spanned by \( w_1, \ldots, w_m \). We now integrate previous identity in the time-interval \((0, T)\) and then use integration by parts to find that
\[
\int_0^T \left( B_{ij} \partial_t^\alpha \partial_{x_j} (u_m - g_m), \partial_{x_i} (u_m - g_m) \right) dt + \int_0^T \left( A_{ij} \partial_{x_j} (u_m - g_m), \partial_{x_i} (u_m - g_m) \right) dt \\
= -(p u_m''(T), u_m(T) - g_m) + \int_0^T \left( p u_m', u_m' \right) + \left( A_{ij} \partial_{x_j} (u_m - g_m), \partial_{x_i} (u_m - g_m) \right) + (f, u_m - g_m) dt.
\]
Therefore, by the previous equation, hypotheses (H.1) and (H.2), Cauchy–Schwarz inequality and (5.13), (5.14), there exists a positive constant \( C = C(\nu, \rho_0, \Omega, T) \) such that
\[
\int_0^T \left( \partial_t^\alpha \partial_{x_j} (u_m - g_m), \partial_{x_i} (u_m - g_m) \right) dt \\
\leq C \left( \|f\|_{L^2(0, T; H^{-1}(\Omega))}^2 + \|g\|_{H^1_0(\Omega)}^2 \right).
\]
To conclude, observe that previous equation together with (3.9), (3.13) and the fractional Poincaré inequality, Theorem 3.7, imply that there exists a positive constant \( C = C(\nu, \rho_0, \Omega, T) \) such that
\[
\|u_m - g_m\|_{H^\alpha/2(0, T; H^1_0(\Omega))} \leq \frac{C}{\cos(\alpha \pi/2)} \left( \|f\|_{L^2(0, T; H^{-1}(\Omega))} + \|g\|_{H^1_0(\Omega)} + \|h\|_{L^2(\Omega)} \right).
\]
This not only finishes the proof of (5.8), but gives an interesting dependence of the estimate on the fractional order of differentiation \( \alpha \). In fact, observe that previous estimate grows boundlessly as \( \alpha \to 1^- \).

### 5.3. Existence of a weak solution

With the energy estimate (5.8) we can take limit as \( m \to \infty \) in (5.5)-(5.7). Indeed, observe that (5.8) implies the sequences \( \{u_m\}_m \), \( \{u_m\}_m \) and \( \{u_m - g_m\}_m \) are bounded in \( L^2(0, T; H^1_0(\Omega)) \), \( L^2(0, T; L^2(\Omega)) \) and \( H^\alpha/2(0, T; H^1_0(\Omega)) \), respectively. Therefore, there exists a subsequence \( \{u_{m_l}\}_l \) and \( u \in H^\alpha/2(0, T; H^1_0(\Omega)) \), with \( u' \in L^2(0, T; L^2(\Omega)) \), such that
\[
\begin{cases}
  u_{m_l} \rightharpoonup u \in L^2(0, T; H^1_0(\Omega)), \\
  C_0 \partial_t^\alpha/2 u_{m_l} \rightharpoonup C_0 \partial_t^\alpha/2 u \in L^2(0, T; H^1_0(\Omega)) \quad \text{and} \\
  u'_{m_l} \rightharpoonup u' \in L^2(0, T; L^2(\Omega)).
\end{cases}
\]
On the other hand, fix \( N \in \mathbb{N} \) and consider \( \varphi \in C^1([0, T]; H^1_0(\Omega)) \) as follows
\[
\varphi(x, t) = \sum_{k=1}^N d^k(t) w_k,
\]
(5.16)
where \( \{d_k\}_{k=1}^N \subset C^1[0,T] \) and \( d_k(T) = 0 \) for \( k = 1, \ldots, N \). For \( m \geq N \), multiply (5.5) by \( d_k \), sum \( k = 1, \ldots, m \) and then integrate by parts in space and time, as in Remark 5.2(ii), to obtain

\[
\int_0^T \left( \rho \frac{\partial u_{mi}}{\partial t}, \frac{\partial \varphi}{\partial t} \right) dt - \int_0^T \left[ \left( B_{ij} C 0D_t^{\alpha/2} \frac{\partial u_{mi}}{\partial x_j}, tD_T^{\alpha/2} \frac{\partial \varphi}{\partial x_i} \right) + \left( A_{ij} \frac{\partial u_{mi}}{\partial x_j}, \frac{\partial \varphi}{\partial x_i} \right) \right] dt
\]

(5.17)

By taking limit as \( l \to \infty \) and using (5.15) we deduce that \( u \) satisfies (5.2) for every \( \varphi \in C^1([0,T]; H^1_0(\Omega)) \) of the form (5.16). To conclude (5.2) for \( \varphi \in C^1(\Omega \times [0,T]) \), with \( \varphi(x,t) = 0 \) if \( x \in \partial \Omega \) or \( t = T \), we observe that all of such functions can be approximated by functions of the form (5.16) with the norm \( \| \cdot \|_{\text{H}^{\alpha/2}(0,T; H^1_0(\Omega))} \).

Finally, we prove the initial condition \( u(0) = g \). To this end, let us consider (5.17) for any \( \varphi \in C^2([0,T]; H^1_0(\Omega)) \), with \( \varphi(x,T) = \varphi'(x,T) = 0 \) for all \( x \in \Omega \). If we again integrate by parts in space and then take \( l \to \infty \), we deduce that

\[
\int_0^T \left( \rho u, \frac{\partial^2 \varphi}{\partial t^2} \right) dt - \int_0^T \left[ \left( B_{ij} C 0D_t^{\alpha/2} \frac{\partial u}{\partial x_j}, tD_T^{\alpha/2} \frac{\partial \varphi}{\partial x_i} \right) + \left( A_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial \varphi}{\partial x_i} \right) \right] dt
\]

(5.18)

On the other hand, if we integrate by parts (5.2) we have

\[
\int_0^T \left( \rho u, \frac{\partial^2 \varphi}{\partial t^2} \right) dt - \int_0^T \left[ \left( B_{ij} C 0D_t^{\alpha/2} \frac{\partial u}{\partial x_j}, tD_T^{\alpha/2} \frac{\partial \varphi}{\partial x_i} \right) + \left( A_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial \varphi}{\partial x_i} \right) \right] dt
\]

\[
= - \int_0^T (f, \varphi) dt - (h, \varphi(x,0)) + (g, \varphi'(x,0)).
\]

By comparing this expression with (5.18), we conclude that \( u(0) = g \), as \( \varphi \) is arbitrary.

Notice that the solution above satisfies energy estimate (1.2), as a consequence of taking \( m \to \infty \) in (5.8) together with (5.15).

### 5.4. Uniqueness

In order to prove that the solution above is unique, it is suffices to verify that \( u = 0 \) is the unique weak solution of (1.1) with \( f \equiv g \equiv h \equiv 0 \). Suppose then that \( u \) is such a weak solution and fix \( s \in (0,T) \). Let us consider

\[
v(x,t) = \begin{cases} \int_0^s u(x,\tau) d\tau & \text{if } 0 \leq t \leq s, \\ 0 & \text{if } s \leq t \leq T. \end{cases}
\]

We can substitute \( v \) in (5.2), by Remark 5.2 (iii), to find that

\[
\int_0^s \left( \rho \frac{\partial u}{\partial t}, \frac{\partial v}{\partial t} \right) dt - \int_0^s \left[ \left( B_{ij} C 0D_t^{\alpha/2} \frac{\partial u}{\partial x_j}, tD_T^{\alpha/2} \frac{\partial v}{\partial x_i} \right) \right] dt - \int_0^s \left( A_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_i} \right) dt = 0.
\]
Observe that $v_t = -u$ and $v(s) = 0$, so we can transform the previous equation in
\[
\frac{1}{2} \| \sqrt{p}u(s) \|_{L^2(\Omega)}^2 + \int_0^s \left( B_{ij} 0D_t^{\alpha/2} \frac{\partial u}{\partial x_j}, tD_s^{\alpha/2} \frac{\partial v}{\partial x_i} \right) dt + \frac{1}{2} \left( A_{ij} \frac{\partial v}{\partial x_j}(0), \frac{\partial v}{\partial x_i}(0) \right) = 0. \tag{5.19}
\]

The third term of the left-hand side of this equation is nonnegative, by hypothesis (H1); let us see that the second one is also nonnegative. To this end, notice that we can suppose that $u$ is of the form (4.2), as $u \in H_0^{\alpha/2}(0, T; H_0^1(\Omega))$ is a limit of these functions, by Definition 4.1. Then, suppose $u$ have such a form, so we can use (3.9) to obtain
\[
\int_0^s \left( B_{ij} 0D_t^{\alpha/2} \frac{\partial u}{\partial x_j}, tD_s^{\alpha/2} \frac{\partial v}{\partial x_i} \right) dt = \int_0^s \left( B_{ij} \frac{\partial u}{\partial x_j}, tD_s^{\alpha/2} \frac{\partial v}{\partial x_i} \right) dt
\]
\[
= \int_0^s \left( B_{ij} \frac{\partial u}{\partial x_j}, tI_s^{1-\alpha} \frac{\partial u}{\partial x_i} \right) dt,
\]
and this last integral is nonnegative by Lemma 5.1. Therefore, we conclude from (5.19) that $u(s) = 0$, and as $s \in (0, T)$ is arbitrary, it follows that $u \equiv 0$.

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Appendix A. Appendix

In this appendix, we collect several useful properties of fractional operators used throughout the article; we provide some of their proofs for completeness. For more details, see [10]. In what follows $\alpha \in (0, 1)$ and $(a, b)$ is a bounded interval.

Let us start by recalling the Fourier transform and its inverse:
\[
\mathcal{F}(u)(\omega) = \hat{u}(\omega) := \int_{-\infty}^{\infty} u(t) e^{-i\omega t} dt,
\]
\[
\mathcal{F}^{-1}(u)(t) = \check{u}(t) := \int_{-\infty}^{\infty} u(\omega) e^{-i\omega t} d\omega,
\]
where $i = \sqrt{-1}$, and their following well known properties:
\[
\int_{-\infty}^{\infty} \check{uv} = \int_{-\infty}^{\infty} u\check{v}, \quad \check{u}(\omega) = \hat{u}(-\omega). \tag{A.1}
\]

The proof of the following important theorem can be found in Section 7.1 of [10].
**Property A.1.** Given a function $\varphi \in C_c^\infty(\mathbb{R})$, the Fourier transform of its Riemann–Liouville fractional derivatives and integrals satisfy:

i if $\alpha \geq 0$,

$$
\mathcal{F}(-\infty D_t^\alpha \varphi)(\omega) = (i\omega)^\alpha \hat{\varphi}(\omega), \quad \mathcal{F}(tD_\infty^\alpha \varphi)(\omega) = (-i\omega)^\alpha \hat{\varphi}(\omega);
$$

ii if $\alpha \in [0,1)$,

$$
\mathcal{F}(-\infty I_t^{\alpha/2} \varphi)(\omega) = \frac{\hat{\varphi}(\omega)}{(i\omega)^{\alpha/2}}, \quad \mathcal{F}(tD_\infty^{\alpha/2} \varphi)(\omega) = \frac{\hat{\varphi}(\omega)}{(-i\omega)^{\alpha/2}}.
$$

It is important to mention that restricting $\alpha$ to $[0,1)$ in (ii) guarantees that all the expressions remain in $L^2(\mathbb{R})$. We recall that

$$(i\omega)^\alpha = |\omega|^\alpha e^{\frac{\alpha \pi i}{2}} \text{sgn } \omega,$$

where sgn is the sign function.

The proof of the following result can be found in [10], Theorem 2.6.

**Property A.2.** For all $\beta \geq 0$, $aI_t^\beta$ and $tI_b^\beta$ are bounded linear operators in $L^2(a, b)$.

**Property A.3.** (Inverse) For all $u \in H_{0}^{\alpha/2}(a, b)$ we have

$$
aI_t^{\alpha/2} aD_t^{\alpha/2} u = aD_t^{\alpha/2} aI_t^{\alpha/2} u = u, \quad tI_b^{\alpha/2} tD_b^{\alpha/2} u = tD_b^{\alpha/2} tI_b^{\alpha/2} u = u.
$$

**Proof.** Observe that it suffices to prove the result for $u \in C_c^\infty(a, b)$, as by definition all the other elements in $H_{0}^{\alpha/2}(a, b)$ are limits of sequences of these elements (recall also Lemma 3.4 and Property A.2). So let $u \in C_c^\infty(a, b)$. To prove (A.3), we just have to take Fourier transform in both sides of the equations and use Property A.1.

**Property A.4.** (Fractional integration by parts) For every $\varphi, \psi \in C_c^\infty(\mathbb{R})$, we have

$$
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} -\infty D_t^{\alpha/2} \varphi \psi = \int_{-\infty}^{\infty} \varphi tD_\infty^{\alpha/2} \psi, \quad (A.5)
$$

$$
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} -\infty I_t^{\alpha/2} \varphi \psi = \int_{-\infty}^{\infty} \varphi tI_\infty^{\alpha/2} \psi. \quad (A.6)
$$

**Proof.** Let us prove (A.5), the proof of (A.6) is similar. Indeed, by Property A.1 and (A.1) we have

$$
\int_{-\infty}^{\infty} -\infty D_t^{\alpha/2} \varphi \psi = \int_{-\infty}^{\infty} \varphi \left((it)^{\alpha/2} \hat{\psi}\right)^{\wedge} = \int_{-\infty}^{\infty} \varphi \left((-it)^{\alpha/2} \hat{\psi}(-t)\right)^{\vee} = \int_{-\infty}^{\infty} \varphi tD_\infty^{\alpha/2} \psi.
$$

We observe that (A.5) is still true for every $\alpha > 0$.

**Property A.5.** (Semigroup property)

i Given $\beta, \gamma \geq 0$, if $u \in L^1(a, b)$ then

$$
aI_t^\beta aI_t^\gamma u = aI_t^{\beta+\gamma} u, \quad tI_b^\beta tI_b^\gamma u = tI_b^{\beta+\gamma} u. \quad (A.7)
$$
ii If \( u \in C^1[a, b] \) and \( u(a) = 0 \), then
\[
aD_t^{\alpha/2} aD_t^{\alpha/2} u = aD_t^{\alpha} u; \tag{A.8}
\]

Proof. The proof of (i) can be found in Section 2.3 of [10]. To prove (ii) we use the fact that \( u(a) = 0 \) together with (2.5) and (i) to find that
\[
aD_t^{\alpha/2} aD_t^{\alpha/2} u = \frac{d}{dt} aI_t^{1-\alpha/2} aI_t^{1-\alpha/2} u' = aI_t^{1-\alpha} u = aD_t^{\alpha} u. \tag{A.10}
\]

Property A.6. Given \( T > 0 \), vectors \( a, b, c, d \in \mathbb{R}^m \) and a vector field \( f \in L^1(0, T)^m \), there exists a unique vector field \( u \in C^2[0, T]^m \), solution of the fractional Cauchy problem
\[
\begin{cases}
u'' + bC_0D_t^\alpha u + au = f(t), & t \in (0, T), \\ u(0) = c, & u'(0) = d.
\end{cases} \tag{A.9}
\]

Proof. To solve (A.9), we use a classical approach, to transform the equation in a Volterra integral equation of the second kind: indeed, after integrating (A.9) once and using (2.6) we obtain
\[
u' = d + 0I_t f - b0I_t^{1-\alpha}(u - c) - a0I_t u.
\]

After integrating once more and using (A.7), we deduce that
\[
u(t) = c + dt + \frac{bc}{\Gamma(3-\alpha)} t^{2-\alpha} + 0I_t^2 f(t) - b0I_t^{2-\alpha} u(t) - a0I_t^2 u(t), \tag{A.11}
\]
where we have used the identity (see Section 2.5 of [10])
\[
0I_t^\beta 1 = \frac{t^\beta}{\Gamma(\beta+1)}, \quad \beta \geq 0.
\]
The Volterra integral equation (A.11) can be solved by a classical fixed point argument, we omit the details. To conclude, observe that this solution belongs to the space \( C^2[0, T]^m \), by (A.10) and (A.11).

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