A hierarchy of commutative Poisson subalgebras for the Sklyanin bracket is proposed. Each of the subalgebras provides a complete set of integrals in involution with respect to the Sklyanin bracket. Using different representations of the bracket, we find some integrable models and a separation of variables for them. The models obtained are deformations of known integrable systems like the Goryachev-Chaplygin top, the Toda lattice and the Heisenberg model.

1 Introduction.

Let us consider a $2 \times 2$ matrix $T(\lambda)$ which depends polynomially on the parameter $\lambda$

$$T(\lambda) = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

whose entries are polynomials of the form

$$A(\lambda) = \lambda^N + A_{N-1} \lambda^{N-1} + \ldots + A_0, \quad B(\lambda) = B_{N-1} \lambda^{N-1} + \ldots + B_0,$$

$$C(\lambda) = C_{N-1} \lambda^{N-1} + \ldots + C_0, \quad D(\lambda) = D_{N-2} \lambda^{N-2} + \ldots + D_0.$$ 

The algebra $\mathfrak{A}_N$ of all functions depending on the coefficients

$$A_0, \ldots, A_{N-1}, \ B_0, \ldots, B_{N-1}, \ C_0, \ldots, C_{N-1}, \ D_0, \ldots, D_{N-2}.$$ 

of the matrix $T(\lambda)$ is a Poisson algebra with respect to the so called Sklyanin bracket. The dimension of generic simplectic leaves for this bracket is equal to $2N$. 

1
The main property of the Sklyanin bracket is that the coefficients of the trace of $T(\lambda)$ generate an $N$-dimensional commutative subalgebra $\mathfrak{A}_N^0$ in $\mathfrak{A}_N$. The generators of $\mathfrak{A}_N^0$ are linear polynomials. In this paper we construct different $N$-dimensional commutative subalgebras $\mathfrak{A}_N^M$, $M \in \mathbb{N}$, in $\mathfrak{A}_N$ generated by polynomials of higher degrees. These subalgebras can be regarded as deformations of the standard trace subalgebra $\mathfrak{A}_N^0$. The generators of $\mathfrak{A}_N^M$ define a set $S_N^M = \{I_i^M, \ldots, I_N^M\}$ of $N$ integrals in involution. The separation variables for all these sets $S_N^M$ coincide with those for $S_N^0$ whereas the separated curves are different.

Using known representations for the Sklyanin bracket and our families $S_N^M$ of integrals in involution we can construct new integrable models. They are deformations of integrable models corresponding to the standard trace integrals. One of the examples is related to a quadratic deformation of the Goryachev-Chaplygin Hamiltonian given by

$$H = J_1^2 + J_2^2 + 4 J_3^2 + 2c_1 x_1 + 2c_2 x_2 + c_3 J_3,$$  \hspace{1cm} (1.4)

where $c_i$ are arbitrary constants. The Lie-Poisson brackets

$$\{J_i, J_j\} = \varepsilon_{ijk} J_k, \quad \{J_i, x_j\} = \varepsilon_{ijk} x_k, \quad \{x_i, x_j\} = 0, \quad i, j, k = 1, 2, 3, \hspace{1cm} (1.5)$$

where $\varepsilon_{ijk}$ is the standard totally skew-symmetric tensor, defines the corresponding equations of motion. These brackets possess two Casimir elements $(x, x)$ and $(x, J)$, where $J = (J_1, J_2, J_3)$, $x = (x_1, x_2, x_3)$ and $(x, y)$ stands for the scalar product in $\mathbb{R}^3$.

On the fixed level $(x, J) = 0$ of the second Casimir element the Hamiltonian (1.4) commutes with an additional cubic integral of motion. This fact ensures the integrability of the Goryachev-Chaplygin Hamiltonian.

The quadratic deformation of the Goryachev-Chaplygin Hamiltonian given by

$$H = I_1 = J_1^2 + J_2^2 + 4 J_3^2 + 2c_1 x_1 + 2c_2 x_2 + c_3 J_3 +$$

$$4(a_1 x_1 + a_2 x_2) J_3 - (a_1^2 + a_2^2) x_3^2 \hspace{1cm} (1.6)$$

also has an additional cubic integral (1.4). If $c_1 = c_2 = c_3 = 0$ then (1.6) gives us a new partially integrable case (i.e. integrable on a special level $(x, J) = 0$ of one of the integrals of motion) for the Kirchhoff problem of motion of a rigid body in the ideal fluid. A similar deformation of the Kowalewski top has been considered in [1, 2].

As an application of our general scheme we present below a separation of variables for this model. The canonical separated variables for the Goryachev-Chaplygin top (1.4) are given by

$$q_{1,2} = J_3 \pm \sqrt{J_1^2 + J_2^2 + J_3^2}, \quad p_{1,2} = \frac{1}{2i} \ln (q_{1,2}(ix_1 - x_2) - (iJ_1 - J_2)x_3). \hspace{1cm} (1.7)$$
If we substitute two pairs of variables (1.7) into the following separated equation

\[
\lambda^3 + a_0 \lambda^2 - I_1 \lambda + I_0 = c_0 \mu - \frac{b_0 \lambda^2(x, x)}{\mu}, \quad \lambda = q_{1,2}, \quad \mu = \exp(2ip_{1,2}),
\]

and solve the couple of linear equations obtained with respect to \( I_1 = H \) and \( I_0 \) we immediately derive the Hamilton function

\[
H_{gch} = J_1^2 + J_2^2 + 4J_3^2 - 2a_0 J_3 - i(c_0 + b_0)x_1 + (c_0 - b_0)x_2.
\]

and additional integrals of motion for (1.9). Substituting the same variables \((p, q)\) into another separated equation

\[
\lambda^3 + a_0 \lambda^2 - I_1 \lambda + I_0 = (c_1 \lambda + c_0)\mu - \frac{(b_1 \lambda + b_0)\lambda^2(x, x)}{\mu},
\]

we get an integrable system with the following Hamiltonian

\[
H = H_{gch} + \left( c_1(iJ_1 - J_2) + b_1(iJ_1 + J_2) \right)J_3 - 2 \left( c_1(ix_1 - x_2) + b_1(ix_1 + x_2) \right)J_3.
\]

After the canonical transformation

\[
x \rightarrow x, \quad J \rightarrow J + Ux, \quad U = \begin{pmatrix} 0 & 0 & -ic_+ \\ 0 & 0 & -c_- \\ ic_+ & c_- & 0 \end{pmatrix}, \quad c_\pm = \frac{b_1 \pm c_1}{2},
\]

the latter Hamiltonian becomes

\[
H = J_1^2 + J_2^2 + 4J_3^2 + 2 \left( i(c_1 + b_1)x_1 - (c_1 - b_1)x_2 + a_0 \right)J_3 + c_1b_1x_3^2
\]

\[
- i(b_0 + c_0 - a_0(c_1 + b_1))x_1 + (c_0 - b_0 - a_0(c_1 - b_1))x_2,
\]

which coincides with (1.6).

In Section 4 the simplest deformations for the Toda lattice and the Heisenberg model are given. We do not know whether physical applications of such deformed models exist.

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## 2 Properties of the Sklyanin bracket

The explicit form of the Sklyanin brackets for the coefficients of the matrix (1.1) can be derived from the following operator definition

\[
\{\mathcal{T}(\lambda), \mathcal{T}(\mu)\} = [r(\lambda - \mu), \mathcal{T}(\lambda)\mathcal{T}(\mu)],
\]

where \( r(x, y) = x + y - xy \).
where we use the standard notations $\frac{1}{2}T(\lambda) = T(\lambda) \otimes Id$, $\frac{3}{2}T(\mu) = Id \otimes T(\mu)$ and

$$r(\lambda - \mu) = \frac{\eta}{\lambda - \mu} \Pi,$$

where

$$\Pi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \eta \in \mathbb{C}. \quad (2.2)$$

**Example 1.** In the simplest case $N = 2$ relation (2.1) is equivalent to

$$\{A_1, A_0\} = 0, \quad \{A_1, B_1\} = \eta B_1, \quad \{A_1, C_1\} = -\eta C_1, \quad \{A_1, B_0\} = \eta B_0, \quad \{A_1, C_0\} = -\eta C_0,$$

$$\{A_0, B_0\} = \eta (B_0 A_1 - A_0 B_1), \quad \{A_0, C_0\} = \eta (A_0 C_1 - C_0 A_1), \quad (2.3)$$

$$\{A_0, D_0\} = \eta (B_0 C_1 - B_1 C_0), \quad \{B_1, C_1\} = \{B_1, B_0\} = 0,$$

$$\{B_0, C_0\} = -\eta D_0, \quad \{B_0, D_0\} = 0, \quad \{B_0, C_1\} = -\eta D_0, \quad \{B_0, C_0\} = -\eta D_0 A_1,$$

$$\{B_0, D_0\} = -\eta B_1 D_0, \quad \{C_1, C_0\} = \{C_1, D_0\} = 0, \quad \{C_0, D_0\} = \eta C_1 D_0.$$  

It was proven in (2) that the coefficients of the determinant

$$d(\lambda) = \det T(\lambda) = A(\lambda) D(\lambda) - B(\lambda) C(\lambda) \quad (2.4)$$

belong to the centre of $\mathfrak{g}$ or, in other words, they are Casimir elements for bracket (2.1). The number of the Casimir functions is $2N - 1$ and therefore we have a $4N - 1$ dimensional Poisson manifold with degenerate Poisson structure (2.1) and $2N$-dimensional generic symplectic leaves.

To bring the Poisson bracket (2.1) into canonical form, a new set of variables

$$d_0, \ldots, d_{2N-2}, \quad q_1, \ldots, q_N, \quad p_1, \ldots, p_N. \quad (2.5)$$

was proposed in [4]. The variables $d_0, \ldots, d_{2N-2}$ are the coefficients of (2.4):

$$d(\lambda) = d_{2N-2} \lambda^{2N-2} + \ldots + d_0;$$

and the variables $q_i$ are zeros of the polynomial $A(\lambda)$:

$$A(\lambda) = \prod_{j=1}^{N} (\lambda - q_j);$$

the variables $p_i$ are defined by

$$p_j = \eta \ln B(q_j).$$

It follows from (2.1) that

$$\{A(\lambda), A(\mu)\} = \{B(\lambda), B(\mu)\} = 0$$

and

$$\{A(\lambda), B(\mu)\} = \frac{\eta}{\lambda - \mu} \left( A(\lambda) B(\mu) - A(\mu) B(\lambda) \right).$$
These relations imply \( \{q_j, q_k\} = \{p_j, p_k\} = 0 \) and \( \{p_j, q_k\} = \delta_{jk} \), respectively. As usual if we fix values of the Casimir elements \( d_j \), then the canonically conjugated variables \( q_i, p_i \) are simplectic coordinates on the corresponding simplectic leaf.

To express the variables (1.3) in terms of (2.5), one can use the formulae

\[
A(\lambda) = \prod_{j=1}^{N} (\lambda - q_j), \quad B(\lambda) = \sum_{j=1}^{N} e^{\eta p_j} \prod_{k \neq j} \left( \frac{\lambda - q_k}{q_j - q_k} \right),
\]

\[
D(\lambda) = \frac{d(\lambda) + B(\lambda)C(\lambda)}{A(\lambda)}, \quad C(\lambda) = \sum_{j=1}^{N} d(q_j) e^{-\eta p_j} \prod_{k \neq j} \left( \frac{\lambda - q_k}{q_j - q_k} \right).
\]

### 3 A construction of commutative subalgebras

Let us introduce the matrix

\[
\tilde{T}(\lambda) = K(\lambda) T(\lambda),
\]

where \( T(\lambda) \) is given by (1.1),

\[
K(\lambda) = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & 0 \end{pmatrix}(\lambda),
\]

whose entries are polynomials of the form

\[
\mathcal{A}(\lambda) = A_M \lambda^M + A_{M-1} \lambda^{M-1} + \cdots + A_0,
\]

\[
\mathcal{B}(\lambda) = b_M \lambda^M + b_{M-1} \lambda^{M-1} + \cdots + b_0, \quad \mathcal{C}(\lambda) = c_M \lambda^M + c_{M-1} \lambda^{M-1} + \cdots + c_0,
\]

and \( b_k, c_k \in \mathbb{C} \) are arbitrary parameters. We require that the trace of the new matrix \( \tilde{T} \) has the form

\[
\text{trace} \tilde{T}(\lambda) = \sum_{i=N}^{N+M} a_{i-N} \lambda^i + \sum_{k=0}^{N-1} I_k \lambda^k, \tag{3.3}
\]

where \( a_i \) are arbitrary fixed constant parameters. It easy to see that both unknown coefficients \( \mathcal{A}_i \) of the polynomial \( \mathcal{A} \) and unknown functions \( I_k \) in (3.3) are uniquely defined from (3.3). Moreover, they are polynomials in variables (1.3) such that all the polynomials \( I_k \) have the same degree \( M + 1 \) and the degree of \( \mathcal{A}_i \) equals \( M - i \).

**Example 1 (continuation).** If \( N = 2 \) and \( M = 1 \) the functions \( \mathcal{A}_i \) and \( I_i \) are given by

\[
\mathcal{A}_1 = a_1, \quad \mathcal{A}_0 = a_0 - a_1 A_1 - b_1 C_1 - c_1 B_1,
\]

\[
I_1 = a_1 (A_0 - A_1^2) + b_1 (C_0 - A_1 C_1) + c_1 (B_0 - A_1 B_1) + a_0 A_1 + b_0 C_1 + c_0 B_1, \tag{3.4}
\]

\[
I_0 = (a_0 - a_1 A_1 - b_1 C_1 - c_1 B_1) A_0 + c_0 B_0 + b_0 C_0. \tag{3.5}
\]

**Theorem 1.** Polynomials \( I_i \) defined by (3.3) commute with each other with respect to the bracket (2.1).
Proof. The explicit form of condition (3.3) is given by

$$A(\lambda)A(\lambda) + B(\lambda)C(\lambda) + C(\lambda)B(\lambda) = \sum_{i=N}^{N+M} a_{i-M}\lambda^k + \sum_{k=0}^{N-1} I_k\lambda^k.$$  

Let us substitute $\lambda = q_j$ into this identity. It follows from the definition $d(\lambda) = AD - BC$ that

$$C(q_j) = -\frac{d(q_j)}{B(q_j)} = -d(q_j)\exp(-\eta p_j).$$

Taking this formula into account we get $N$ linear equations

$$C(q_j)\exp(\eta p_j) - B(q_j)\exp(-\eta p_j) = \sum_{i=0}^{N-1} I_iq_j^i + \sum_{k=N}^{N+M} a_{k-N} q_j^k, \quad j = 1, \ldots, N. \quad (3.6)$$

for $N$ unknowns $I_i$. Following [8], we can easily show that the functions $I_i$ are in involution with respect to (2.1).

Let us rewrite this system in the matrix form

$$\phi(p, q) = S(q)I(p, q) + U(q), \quad (3.7)$$

where

$$\phi_j = C(q_j)\exp(\eta p_j) - B(q_j)\exp(-\eta p_j), \quad U_j = \sum_k a_k q_j^k$$

and

$$S = \begin{pmatrix}
1, & q_1, & \cdots & q_1^{N-1} \\
\vdots & \vdots & \ddots & \vdots \\
1, & q_N, & \cdots & q_N^{N-1}
\end{pmatrix}$$

The solution of (3.7) is given by

$$I(p, q) = W(q) \left( \phi(p, q) - U(q) \right), \quad W(q) = S^{-1}(q). \quad (3.8)$$

The matrix $S$ belongs to the class of so-called Stäckel matrices [8] for which the entries $S_{ij}$ of the $i$-th row are functions depending on $q_i$ only. Differentiating the identity

$$\sum_i W_{mi}S_{il} = \delta_{ml},$$

with respect to $q_j$ and using the main property of the Stäckel matrices, we have

$$\sum_i \frac{\partial W_{mi}}{\partial q_j} S_{il} = -W_{mj} \frac{\partial S_{jl}}{\partial q_j}.\quad (3.9)$$

Substituting $k$ for $m$ and eliminating the right hand side $\partial S_{jl}/\partial q_j$ from the two obtained equations, we find

$$\sum_i \left( W_{jk} \frac{\partial W_{im}}{\partial q_j} - W_{jm} \frac{\partial W_{ik}}{\partial q_j} \right) S_{il} = 0.$$
Since the determinant of the matrix $S$ does not vanish
\[ W_{jk} \frac{\partial W_{im}}{\partial q_j} - W_{jm} \frac{\partial W_{ik}}{\partial q_j} = 0 \] (3.9)
for all $j, k, i, m$.

Calculating the Poisson brackets between $I_k$ and $I_m$, we obtain
\[ \{I_k, I_m\} = \sum_{i,j} \left( W_{jk} \frac{\partial W_{im}}{\partial q_j} - W_{jm} \frac{\partial W_{ik}}{\partial q_j} \right) \frac{\partial \phi_j}{\partial p_j} \left( \phi_j - U_j(q_j) \right). \] (3.10)

It follows from (3.3) that $\{I_k, I_m\} = 0$. The proof is complete.

**Collorary 1.** Variables $p_i, q_i$ are the separated variables for integrals of motion $I_j$ satisfying the separated equations (3.6)-(3.7) whereas variables
\[ \lambda_i = q_i, \quad \mu_i = C(q_i) \exp(\eta p_i) \]
satisfy the characteristic equation $\text{Det}(\tilde{T}(\lambda) - \mu) = 0$. This means that these variables lie on the corresponding equivalent algebraic curves.

**Remark 1.** Theorem 1 for $M = 0$ is a well-known fact. In the case $N = 2, M = 1$ the matrix $\tilde{T}$ with
\[ K(\lambda) = \begin{pmatrix} \lambda + A_0 & b_0 \\ c_0 & 0 \end{pmatrix}, \quad A_0 = a_0 - A_1, \] (3.11)
has been proposed in [2] in a non-factorized form. In the paper [3] this matrix was factorized and generalized to the case of arbitrary $N$. Notice, that our matrix $K$ for $M = 1$ has a more general form
\[ K(\lambda) = \begin{pmatrix} \lambda + A_0 & b_1 \lambda + b_0 \\ c_1 \lambda + c_0 & 0 \end{pmatrix}, \quad A_0 = a_0 - A_1 - b_1 C_1 - c_1 B_1, \]
than (3.11). Parameters $b_1$ and $c_1$ are absolutely essential in constructing new quadratic integrable Hamiltonians.

**Remark 2.** To generalize the condition (3.3) we can assume that
\[ \text{trace } \tilde{T}(\lambda) = \sum_{k \in \varrho} a_k \lambda^k + \sum_{i \in \hat{\varrho}} I_i \lambda^i, \quad a_k \in \mathbb{C}. \] (3.12)
Here $\varrho = \{i_1, \ldots, i_N\}$ is an arbitrary subset in the set of numbers $\{0, 1, \ldots, N + M\}$ and $\hat{\varrho}$ is the corresponding complement such that $\varrho \cup \hat{\varrho} = \{0, 1, \ldots, N + M\}$. But in this case $I_i$ are rational functions of variables (1.3).

**Example 1 (continuation).** If $N = 2$ and $M = 1$ and
\[ \text{trace } \tilde{T}(\lambda) = a_1 \lambda^3 + I_1 \lambda^2 + a_0 \lambda + I_0, \]
these functions are
\[ A_1 = a_1, \quad A_0 = \frac{1}{A_1} \left( a_0 - a_1 A_0 - c_0 B_1 - b_0 C_1 - b_1 C_0 - c_1 B_0 \right), \] (3.13)
\[ I_1 = \frac{1}{A_1} \left( a_0 - a_1 A_0 - A_1^2 \right) - b_1 (C_0 - C_1 A_1) - c_1 (B_0 - B_1 A_1) - c_0 B_1 - b_0 C_1, \]
\[ I_0 = \frac{1}{A_1} \left( A_0 (a_0 - c_1 B_0 - a_1 A_0 - b_1 C_0) + b_0 (C_0 A_1 - A_0 C_1) + c_0 (B_0 A_1 - A_0 B_1) \right). \]
Remark 3. Theorem 1 can be proven in the same way for matrices (1.1) of slightly different structure. For example, we can assume that the entry $D(\lambda)$ of the matrix $T(\lambda)$ has the form

$$D(\lambda) = \lambda^N + D_{N-1}\lambda^{N-1} + D_{N-2}\lambda^{N-2} + \cdots + D_0.$$ 

In this case we have one additional variable $D_{N-1}$ and one additional Casimir function $d_{2N-1} = A_{N-1} + D_{N-1}$.

4 Polynomial deformations of known integrable models

If we identify the Sklyanin bracket (2.1) with a fixed Poisson bracket on some phase space $\mathcal{M}$, then (2.1) can be regarded as an equation for matrix $T(\lambda)$ of the form (1.1) whose entries (1.2) are polynomials in $\lambda$ with coefficients being functions on $\mathcal{M}$. Such a matrix $T(\lambda)$ defines a representation of (2.1) on $\mathcal{M}$.

Using known representations and Theorem 1, we can produce hierarchies of deformations for several integrable models such as the Goryachev-Chaplygin top, the Toda lattice, and the Heisenberg magnetic. According to Collorary 1 integrals of motion for all members of such a hierarchy are separable in the same canonical variables $p_i, q_i$.

4.1 Deformed Goryachev-Chaplygin top

Let us consider the Lie algebra $e^*(3)$ with a natural Lie-Poisson bracket (1.5). The phase space of the Goryachev-Chaplygin top $\mathcal{M}$ is a union of special non-generic coadjoint orbits (symplectic leaves) of $E(3)$ in $e^*(3)$ specified by the fixed value $I_2 = 0$ of the second Casimir operator.

It was observed in [1] that the functions

\begin{align*}
A_1 &= -2J_3, \\
A_0 &= -J_1^2 - J_2^2 - \frac{\varepsilon}{x_3^2}, \\
B_1 &= x_2 + ix_1, \\
B_0 &= -(J_2 + iJ_1)x_3, \\
C_1 &= -x_2 + ix_1, \\
C_0 &= (J_2 - iJ_1)x_3, \\
D_0 &= x_3^2
\end{align*}

satisfy (2.3) if the Poisson bracket is given by (1.1), $I_2 = 0$ and $\eta = -2i$. Formulae (3.4) and (3.5) give us two integrals $I_1$ and $I_0$ such that \{I_1, I_0\} = 0. It easy to verify that (up to constant factors)

$$H_1 = I_1 = a_1(J_1^2 + J_2^2 + 4J_3^2 + \frac{\varepsilon}{x_3^2}) + 2a_0J_3 + b_0(x_2 - ix_1) - c_0(x_2 + ix_1) + b_1(2J_3x_2 - J_2x_3 - 2iJ_3x_1 + iJ_1x_3) - c_1(2J_3x_2 - J_2x_3 + 2iJ_3x_1 - iJ_1x_3)$$

which is equivalent to (1.6). The explicit form of the integral $I_0$ is given by

$$I_0 = 2a_1(J_1^2 + J_2^2)J_3 + a_0(J_1^2 + J_2^2) + b_0(-J_2 + iJ_1)x_3 + c_0(J_2 + iJ_1)x_3$$

$$+ b_1(J_1^2 + J_2^2)(x_2 - ix_1) - c_1(J_1^2 + J_2^2)(x_2 + ix_1) + \frac{\varepsilon}{x_3^2}$$

\begin{align*}
&+ 2a_1J_3 + a_0 + b_1(x_2 - ix_1) - c_1(x_2 + ix_1). \\
\end{align*}

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As examples we also present explicitly one polynomial deformation of higher degree and one rational deformation. Thus if \( M = 2 \) the deformation reads as follows:

\[
H_2 = H_1 + 4a_2 J_3 \left( (J_1^2 + J_2^2 + 2J_3^2) + \varepsilon x_3^{-2} \right) \\
+ b_2 \left( 2iJ_3 x_3 (J_1 + iJ_2) + (J_1^2 + J_2^2 + 4J_3^2 + \varepsilon x_3^{-2})(x_2 - ix_1) \right) \\
- c_2 \left( (2iJ_3 x_3 (J_1 - iJ_2) + (J_1^2 + J_2^2 + 4J_3^2 + \varepsilon x_3^{-2})(x_2 + ix_1) \right),
\]

where \( H_1 \) is defined by (4.1). In the case \( M = 1 \) the simplest rational deformation \( \tilde{H} = H_1 - a_0(2J_3 - 1) \)

\[
\tilde{H} = \frac{H_1 - a_0(2J_3 - 1)}{2J_3},
\]

corresponds to (3.13).

### 4.2 Deformed Toda lattices

The Sklyanin bracket may be identified [3] with the standard bracket in \( \mathcal{M} = \mathbb{R}^{2N} \) with the help of the following ansatz for the matrix \( T(\lambda) \):

\[
T(\lambda) = L_N L_{N-1} \cdots L_1, \quad L_i(\lambda) = \begin{pmatrix} \lambda - p_i & e^{q_i} \\ -e^{-q_i} & 0 \end{pmatrix}, \quad q_i + N = q_i.
\]

Here \( p_i, q_i \) are canonical variables in \( \mathbb{R}^{2N} \).

The matrix \( T(\lambda) \) obeys the Sklyanin bracket (2.1) with \( \eta = 1 \) and describes the periodical Toda lattice [3]. The trace of the matrix \( T(\lambda) \) produces the integrals of motion. The simplest of them are

\[
P = -\sum_{i=1}^{N} p_i, \quad H = -\sum_{i>j}^{N} p_i p_j - \sum_{i=1}^{N} e^{q_{i+1} - q_i}.
\]

The matrix (4.3) has the necessary polynomial structure (1.2) and we may apply Theorem 1 in order to produce deformations of the periodic Toda lattice, polynomial in momenta. For instance, if \( M = 1 \) then the integral of the lowest degree in trace \( \tilde{T}(\lambda) \) is the following quadratic integral

\[
I_{N-1} = -a_0 \sum_{i=1}^{N} p_i - a_1 \left( \sum_{i>j}^{N} p_i p_j + \sum_{i=1}^{N} e^{q_{i+1} - q_i} \right) + e^{q_1} (c_0 + c_1 p_1) - e^{-q_N} (b_0 + b_1 p_N).
\]

### 4.3 Deformed spin chain

Let the phase space \( \mathcal{M} \) be a direct sum of the algebras \( sl^*(2) \) with the following Lie-Poisson bracket

\[
\{ s^i_3, s^i_\pm \} = \pm s^i_\pm, \quad \{ s^i_+, s^i_- \} = 2s^i_3
\]
in the each algebra. The Sklyanin bracket with $\eta = 1$ is identified with (4.6) by

$$T(\lambda) = L_N L_{N-1} \cdots L_1, \quad L_i(\lambda) = \begin{pmatrix} \lambda + s^i_3 & s^-_i \\ s^+_i & \lambda - s^i_3 \end{pmatrix}. $$

It is easy to check that such a product has the polynomial structure described in Remark 3 under the restriction that the Casimir function $d_{2n-1} = A_{N-1} + D_{N+1}$ is equal to zero. Adapting our construction to this case we can construct polynomial deformations for the spin chain. In the simplest case $M = 1$ we obtain the following integrable quadratic Hamiltonian

$$I_{N-1} = H = a_0 S_3 + b_0 S_+ + c_0 S_- + a_1 \left( \sum_{j>i}^N (s^+_i s^-_j + s^+_3 s^+_3) - S^2_3 \right) - 2b_1 \left( S_3 S_+ - \sum_{j>i}^N s^+_3 s^+_j + \frac{1}{2} \sum_{i=1}^N s^+_i s^+_i \right) - 2c_1 \left( S_3 S_- - \sum_{j<i}^N s^-_3 s^-_j + \frac{1}{2} \sum_{i=1}^N s^-_i s^-_i \right),$$

where

$$S_3 = A_{N-1} = \sum s^+_3, \quad S_- = B_{N-1} = \sum s^-_i, \quad S_+ = C_{N-1} = \sum s^+_i.$$

If $b_1 = c_1 = 0$ then $H$ coincides with the general quadratic integral for the standard spin chain.

It is well-known that the Hamiltonians for the Toda lattice and the spin chain derived from the factorised form of operator $T(\lambda)$ have to be transformed to bring them to a local form suitable for applications. We do not know such transformations for the deformed models.

References

[1] V.B. Kuznetsov and A.V. Tsiganov, A special case of Neumann’s system and the Kowalewski-Chaplygin-Goryachev top, J.Phys., v.22, p.L73, 1989.

[2] E.K. Sklyanin, The Goryachev-Chaplygin top and the method of the inverse scattering problem. Differential geometry, Lie groups and mechanics, VI. Zap. Nauchn. Sem. LOMI., v.133, p.236, 1984.

[3] E.K. Sklyanin, The quantum Toda chain., Nonlinear equations in classical and quantum field theory (Meudon/Paris, 1983/1984), 196–233, Lecture Notes in Phys., v.226, Springer, Berlin, 1985.

[4] E.K. Sklyanin, Separation of variables—new trends. Quantum field theory, integrable models and beyond (Kyoto, 1994). Prog. Theor. Phys. (Suppl), v.118, p.35, 1995.
[5] V.V. Sokolov, A new integrable case for the Kirchhoff equation, *Teor.Math.Phys.*, v.128(2), p.31, 2001.

[6] V.V. Sokolov, A generalized Kowalevski Hamiltonian and new integrable cases on $e(3)$ and $so(4)$, Preprint [nlin.SI/0110023], 2001.

[7] V.V. Sokolov, A.V. Tsiganov, On the Lax pairs for the generalized Kowalewski and Goryachev-Chaplygin tops, *Teor.Math.Phys.*, v.(), p., 2002.

[8] P. Stäckel, Über die integration der Hamilton-Jacobischen differentialgeichung mittels separation der variabeln, Habilitationschrift, Halle, 1891.
L.A. Pars, An elementary proof of the Stäkel theorem, *American Math. Monthly*, v.56, p.394, 1949.

[9] A.V. Tsiganov, The Kowalewski top, a new Lax representation. *J.Math.Phys.*, v.38, p.196, 1997.