EINSTEIN-DE BROGLIE RELATIONS ON THE LATTICE

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Historically the starting point of wave mechanics is the Planck and Einstein-de Broglie relations for the energy and momentum of a particle, where the momentum is connected to the group velocity of the wave packet. We translate the arguments given by de Broglie to the case of a wave defined on the grid points of a space-time lattice and explore the physical consequences such as integral period, wave length, discrete energy, momentum and rest mass.

1 Einstein-de Broglie relations: continuous case

After Einstein applied the Planck formula $E = h\nu$ (quantization of the energy for the orbits of the harmonic oscillator) to the energy of the light waves in the photo-electric effect, de Broglie generalized this expression to relativistic momentum of a massive or massless particle.

Bascially de Broglie arguments [1] are bases on the transformations properties of the frequency and wave number of a plane wave and the transformations of the energy and the relativistic momentum of a particle.

Let us consider a plane wave with the wave normal \( \vec{n} \) in the xy-plane of a system \( S \) with angular frequency \( \omega \), wave vector \( \vec{k} \) and phase velocity \( v_\varphi \). It is described by a wave function

$$\psi(\vec{x}, t) = A \cos(\omega t - \vec{k} \cdot \vec{x}) \quad (1)$$

In a coordinate system \( S' \) moving in the direction of the x-axis with the velocity \( v \) relative to \( S \) the wave function will be described by

$$\psi(\vec{x}', t') = A \cos(w' t' - \vec{k}' \cdot \vec{x}') \quad (2)$$

Since the argument of both functions should be the same it follows: by elementary calculations:

$$w' = \frac{\omega (1 - \vec{v} \cdot \vec{n}) / v_\varphi}{(1 - v^2/c^2)^{1/2}}$$

$$\vec{k}' = \vec{k} + \vec{\nu} \left[ \frac{\vec{\nu} \cdot \vec{k}}{v^2} \right] \left\{ 1 - (1 - v^2/c^2)^{1/2} \right\} - v^2 k v_\varphi / c^2 \frac{(1 - v^2/c^2)^{1/2}}{c}$$

1
\[ k' = k \frac{\left( 1 - \frac{v^2}{c^2} + \frac{v^2v_\phi^2}{c^4} + \frac{(\vec{v} \cdot \vec{n})^2}{c^4} - \frac{2(\vec{v} \cdot \vec{n})v_\phi}{c^4} \right)^{1/2}}{\left( 1 - v^2/c^2 \right)^{1/2}} \]  

where \( k \equiv |\vec{k}| \) is the wave number.

Suppose a particle of energy \( E \) and relativistic momentum \( \vec{p} \) is moving with respect to a coordinate system \( S \) with velocity \( \vec{u} \). In a coordinate system \( S' \) moving in the direction of the x-axis with the velocity \( v \) relative to \( S \), the particle will be described by the energy \( E' \) and the relativistic momentum \( \vec{p}' \), which are related to the old coordinates by

\[
E' = E \frac{1 - \vec{v} \cdot \vec{u}/c^2}{\left( 1 - v^2/c^2 \right)^{1/2}}
\]

\[
\vec{p}' = \vec{p} + \frac{\vec{v}}{v^2} \left( \vec{v} \cdot \vec{p} \right) \left\{ 1 - \left( 1 - v^2/c^2 \right)^{1/2} \right\} - \frac{v^2 \vec{p}}{u} 
\]

\[
p' = \left\{ p^2 \left( 1 - \frac{v^2}{c^2} \right) + p^2 \frac{v_\phi^2}{c^2} + \frac{(\vec{v} \cdot \vec{n})^2}{c^4} - \frac{2p(\vec{v} \cdot \vec{n})}{u} \right\}^{1/2} 
\]

\[
(1 - v^2/c^2)^{1/2}
\]

Comparison of formulas (3) and (4) leads to the conclusion that \( w, \vec{k} \) transform in the same way as \( E, \vec{p} \) provided \( \vec{k} \) and \( \vec{p} \) are parallel and the phase velocity \( v_\phi \) is related to the velocity of the particle \( u \) by the expression [2]

\[ v_\phi = c^2/u \]  

Following Einstein’s hypothesis that the energy should be proportional to the frequency of a light quanta,

\[ E = \hbar w \]  

de Broglie made the assumption that for a particle there is an associate wave satisfying

\[ E = \hbar w, \quad p = \hbar \vec{k} \]  

Since the phase velocity of the wave \( v_\phi \) does not correspond to the velocity of the particle, de Broglie suggested that there is a wave packet associated with the particle, consisting of a superposition of waves with different wave vectors \( \vec{k} \) and amplitudes \( \hat{\psi} \left( \vec{k} \right) \)

\[
\psi (\vec{x}, t) = \int_{-\infty}^{\infty} d^3k \hat{\psi} \left( \vec{k} \right) \exp i \left\{ w \left( \vec{k} \right) t - \vec{k} \cdot \vec{x} \right\} 
\]

\[ 2 \]
If we suppose that the momentum vari is very little around a fixed value $\vec{k}_0$, namely, $|\vec{k} - \vec{k}_0| \leq \Delta k$, then the function $w(\vec{k})$ can be expanded around $w_0 \equiv w(\vec{k}_0)$. Easy calculations gives:

$$
\psi(\vec{x}, t) = \exp \left\{ i \left( w_0 t - \vec{k}_0 \cdot \vec{x} \right) \Delta k \right\} \int_{\Delta k} d^3k \exp \left\{ i (w'_0 t - x) \Delta k \right\}
$$

This wave represent a packet with phase velocity $v_p = w_0/k_0$ and group velocity $v_g = w'_0 \equiv dw/dk(k_0)$.

From the Einstein-de Broglie relations follows:

$$
v_p = \frac{w}{k} = \frac{E}{p} = \frac{c^2}{u} \quad (9)
$$

$$
v_g = \frac{d w}{d k} = \frac{d E}{d p} = \frac{p c^2}{E} = u \quad (10)
$$

where in the last equation we have used $E = \left(p^2 c^2 + m_0^2 c^4\right)^{1/2}$. There fore we have $v_p = c^2/v_g$ in agreement with de Broglie assumption about the wave-packet.

The Einstein-de Broglie relation were used to write the wave function associated to a particle

$$
\psi(\vec{x}, t) = \exp \left\{ i \left( E t - \vec{p} \cdot \vec{x} \right) \right\} \right/ \hbar \quad (11)
$$

If the energy and relativistic momentum are connected by $E^2 - p^2 c^2 = m_0^2 c^4$ the wave function satisfies

$$
\left( -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \Delta \right) \psi(\vec{x}, t) = \frac{m_0^2 c^4}{\hbar^2} \psi(\vec{x}, t) \quad (12)
$$

2 Einstein-de Broglie relations: discrete case

If we introduce the assumption os a discrete space-time [3] we must have

$$
t = n \tau \quad , \quad \vec{x} = \vec{j} \varepsilon \quad , \quad n, j_1, j_2, j_3 \in Z
$$

$$
T = N \tau \quad , \quad \lambda = M \varepsilon \quad , \quad N, M \in Z
$$

$$
w = \frac{2\pi}{N \tau} \quad , \quad k = \frac{2\pi}{M \varepsilon} \quad , \quad \frac{1}{N}, \frac{1}{M} \in Q
$$

where $\varepsilon, \tau$ are the fundamental length and time.
From these quantities one constructs the discrete wave functions (for simplicity we use only one spacial coordinate):

$$\psi(x,t) = \exp\left\{2\pi i \left(\frac{n}{N} - \frac{j}{M}\right)\right\}$$

(14)

which is periodic in $n, j$ with period $N$ and wave length $M$.

We introduce another wave function

$$\psi(x,t) = \left(\frac{1 + \frac{1}{2}i\frac{2\pi}{N}}{1 - \frac{1}{2}i\frac{2\pi}{N}}\right)^n \left(\frac{1 - \frac{1}{2}i\frac{2\pi}{M}}{1 + \frac{1}{2}i\frac{2\pi}{M}}\right)^j$$

(15)

This is a hot periodic function in $n$ or $j$, but is quasi-periodic in the sense that in the limit $n \to \infty$, $j \to \infty$, $\tau \to 0$, $\varepsilon \to 0$, $n\tau = t$, $j\varepsilon = x$,

$$\psi(x,t) \to \exp\ i2\pi (wt - kx).$$

which is periodic in $t$ and $x$.

The arguments leading to de Broglie relations are translated into the discrete language. The integral Lorentz transformations are factorized with the help of Kac generators [4]

$$S_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$S_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad S_4 = \begin{pmatrix} 2 & 1 & 1 & 1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix}$$

in such a way that an element of the complete Lorentz group

$$L = P_1^{\alpha}P_2^{\beta}P_3^{\gamma}S_4 \ P_1^{\delta}P_2^{\epsilon}P_3^{\eta}S_4 \ldots S_3S_2S_1$$

where $P_1 = S_1S_2S_3S_4S_1$, $P_2 = S_2S_3S_2$, $P_3 = S_3$; $\alpha, \beta, \gamma, \delta, \epsilon, \eta, \rho, \sigma, \tau = 0, 1$.

The energy and the relativistic momentum are written

$$E = \frac{m_0c^2(c\Delta t)}{(c\Delta t)^2 - (\Delta x)^2}^{1/2}, \quad P = \frac{m_0c\Delta x}{(c\Delta t)^2 - (\Delta x)^2}^{1/2}$$

(16)
hence
\[
\frac{pc^2}{E} = \frac{\Delta x}{\Delta t} = u
\] (17)

As in the continuous case, if \( v_\varphi = c^2/u \) \((E, p)\) transform in the same way as \((w, k)\), hence
\[
E = \hbar w \quad , \quad p = \hbar k
\] (18)

The identification of \( v_g = u \) is made by the superposition of two wave functions of slightly different wave length and period \([5]\)
\[
\psi(x, t) = \cos 2\pi \left( \frac{t}{T} - \frac{x}{\lambda} \right) + \cos 2\pi \left( \frac{t}{T'} - \frac{x}{\lambda'} \right)
\]

To this wave it corresponds a phase velocity
\[
v_\varphi = \frac{\frac{1}{T} + \frac{1}{T'}}{\frac{1}{\lambda} + \frac{1}{\lambda'}} = \frac{\Delta w}{\Delta k}
\] (19)

and a group velocity
\[
v_g = \frac{\frac{1}{T} - \frac{1}{T'}}{\frac{1}{\lambda} - \frac{1}{\lambda'}} = \frac{\Delta w}{\Delta k}
\] (20)

where \( \Delta \) and \( \tilde{\Delta} \) are the difference and average operator

From (16) (17) and (18) we have
\[
v_\varphi = \frac{\tilde{\Delta} w}{\Delta k} = \frac{\tilde{\Delta} E}{\Delta p} = \frac{c^2}{u}
\] (21)
\[
v_g = \frac{\Delta w}{\Delta k} = \frac{\Delta E}{\Delta p} = u
\] (22)

To prove the last equality in (22) we take the expression \( E^2/c^2 - p^2 = m_0^2 c^2 \) and apply the total difference to both sides. From the definition \( \Delta f \equiv f(x + \Delta x) - f(x) \) we get \( \Delta (E^2) = (E + \Delta E)^2 - E^2 \) and \( \Delta (p^2) = (p + \Delta p)^2 - p^2 \). Therefore
\[
\left\{ 2E \Delta E + (\Delta E)^2 \right\} / c^2 - 2p \tilde{\Delta} p - (\Delta p)^2 = 0
\] (23)

The difference of momentum in two consecutive events of the particle \( \Delta p_\mu \) is also a 4-vector. In order to calculate the invariant \( (\Delta E)^2/c^2 - (\Delta p)^2 \) we
take an inertial system such that $\Delta p = 0$ which means that the momentum is the same for two consecutive events, and the velocity and energy are the same. Hence $\Delta E = 0$, so that in an arbitrary inertial frame.

$$(\Delta E)^2/c^2 - (\Delta p)^2 = 0$$

Inserting this result in (23) and using (21) we obtain

$$\Delta E = u\Delta p$$

To prove the last equality of (21) we apply the identity $\Delta (fg) = \Delta f \Delta g + \Delta f \Delta g$ to the expression $E^2/c^2 - p^2 = m_0^2c^2$, as in the continuous case.

3 Physical consequences

If we accept the assumption of a discrete space and time as a consequence of the interaction of fundamental entities [6] we may conceive a vibration on this network, similar to the waves propagating on a discrete string. The plane waves satisfy the properties of section 2 therefore, we can talk of phase velocity and group velocity of the packet. Those properties can be associated to a particle whose structure is attached to the wave packet, but with experimental data given by the Einstein-de Broglie relations.

In particular we have the following physical consequences:

i) the frequency and the wave number are discrete, because the period and wave length are integral multiple of fundamental time and length

ii) the energy and relativistic momentum are discrete due to the Einstein-de Broglie relations

$$m_0c^2\Delta t \left\{ (c\Delta t)^2 - (\Delta x)^2 \right\}^{1/2} = \frac{h}{NT}, \quad N \text{ integer}$$

$$m_0c\Delta x \left\{ (c\Delta t)^2 - (\Delta x)^2 \right\}^{1/2} = \frac{h}{M\varepsilon}, \quad M \text{ integer}$$

iii) in the rest system

$$m_0 = \frac{h}{c^2 N\tau}$$

we have a discrete mass spectrum.

iv) the wave equation on the lattice reads [7]:

$$\left( -\frac{1}{c^2\tau^2} \Delta_n \nabla_n \Delta_j \nabla_j + \frac{1}{\varepsilon^2} \Delta_j \nabla_j \Delta_n \nabla_n \right) \psi(x,t) = \frac{m_0^2c^2}{\hbar^2} \Delta_j \nabla_j \Delta_n \nabla_n \psi(x,t)$$
where the solutions are given by (14) or (15) provided the dispersion relations are satisfied

$$\frac{1}{c^2} \frac{1}{\tau^2} \tan^2 \frac{\pi}{N} - \frac{4}{\varepsilon^2} \tan^2 \frac{\pi}{M} = \frac{m_0^2 c^2}{\hbar^2}$$

in the first case and

$$\frac{1}{c^2} \left( \frac{1}{N \tau} \right)^2 - \left( \frac{1}{M \varepsilon} \right)^2 = \frac{m_0^2 c^2}{\hbar^2}$$

in the second case.

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