Momentum Lattice Simulation on a Small Lattice Using Stochastic Quantization

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Abstract

We have studied the scalar $\phi^4$-model in the symmetric phase and the non-compact $U(1)$ gauge theory on a momentum lattice using the Langevin equation for generating configurations. In the $\phi^4$-model we have analyzed the renormalized mass and in the $U(1)$-model we have analyzed the Wilson loop operator. We used a second order algorithm for solving the Langevin equation, and we looked for the convergence rate of the method. We studied the stochastic time needed to generate equilibrium configurations and compared first and second order schemes for both models.
1 Introduction

One major problem in lattice gauge theory has been the increased amount of computer time and memory needed to do simulations near a critical point. The so-called critical slowing down can be countered, e.g., with the Fourier acceleration method \cite{1}. Many other methods have been developed to accelerate convergence of the calculation, e.g., the cluster algorithm \cite{2}, suitable for Ising-type systems. When we studied the scalar $\phi^4$-model near the critical point on a momentum lattice \cite{3}, it turned out that quite reasonable results for the renormalized mass, wave function renormalization, etc., could be obtained on relatively small lattices ($3^4—7^4$). In this note we want to report a more detailed numerical analysis of the convergence behavior on a momentum lattice.

2 Momentum Space

Momentum space has several interesting advantages over coordinate space:

- The kinetic energy of the action is local.
- One can implement Fourier acceleration to fight critical slowing down.
- At the critical point, the correlation length goes to infinity. The behavior at $x \to \infty$ in coordinate space corresponds to $k \to 0$ in momentum space.
- The correlation function in momentum space behaves as $1/(m_R^2 + k^2)$ near the critical point. At the critical point, $m_R \to 0$, but one can stay away from the pole by choosing some $k^2 \neq 0$.
- Although the action in momentum space is non-local, one can use a fast Fourier transform to switch to coordinate space, where the interaction is local.

3 Stochastic Quantization

We have used stochastic quantization, which is a direct method to go from the continuum formulation to practical algorithms for numerical simulations. We have studied the $\phi_{3+1}^4$ model and the non-compact $U(1)_{3+1}$ gauge model. Stochastic quantization was introduced by Parisi and Wu \cite{4}, and it has been applied by many authors (for a review see Damgaard and H"uffel \cite{5,6}). The idea is to consider the Euclidean quantum field as the equilibrium state of a statistical system coupled to a heat reservoir. The evolution of this statistical system is described by the Langevin equation, which
reads in momentum space,

\[
\frac{\partial \hat{\phi}(k, \tau)}{\partial \tau} = -\frac{\delta S[\hat{\phi}]}{\delta \hat{\phi}(-k, \tau)} + \hat{\eta}(k, \tau),
\]

where \(S[\hat{\phi}]\) is the action in momentum space and \(\hat{\eta}(k, \tau)\) is a field of Gaussian noise. When \(\tau \to \infty\) the statistical system is “equal” to the Euclidean field; \(\tau\) was introduced as fictitious time to evolve the system. Our goal is to solve the Langevin equation many times to generate a good representation of the canonical ensemble. Then we perform the canonical averaging of the desired observable.

### 4 Numerical Point of View

The expectation value of an observable is defined by

\[
\overline{O[\hat{\phi}]} = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau d\tau' O[\hat{\phi}(\tau')].
\]

Because we suppose that our system is ergodic, the expectation value can be computed over the configuration space,

\[
\langle O[\hat{\phi}] \rangle = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} O[\hat{\phi}_n].
\]

So one must solve the Langevin equation \(N\) times in parallel, which can be done using a parallel processor.

The Langevin equation is a first order differential equation that we integrate up to a final value \(\tau_{\text{final}}\), by increments of \(\Delta \tau\). It can be solved using a standard method like Euler’s method where the truncation term is \(O(\Delta \tau^2)\), or a higher order scheme like Heun’s method (second order Runge-Kutta) where the truncation term is \(O(\Delta \tau^3)\). Batrouni et al. have shown that a higher order integration scheme has many advantages, and our experience agrees with theirs. For the same level of error, Heun’s method is faster; it takes fewer “time” steps. For the same “time” step, it was 46% slower for the \(\phi^4\)-model, but in the case of the non-compact \(U(1)\)-model it was only 12% slower; in the case of the \(\phi^4\)-model, the second order scheme required the evaluation of two additional fast Fourier transforms, but in the case of the \(U(1)\)-model only a few more additions and multiplications were required. In Fig.[1] we display for the \(\phi^4\)-model the behavior of the renormalized mass \(m_R\) in the neighborhood of the critical point. We have computed in Fig.[2] the renormalized mass \(m_R\) as a function of \(\tau_{\text{final}}\) for large \(\Delta \tau\) on a lattice of \(3^4\) points. This corresponds to a coupling parameter \(\kappa = 0.12025\), while the critical point is at \(\kappa_c = 0.1257(1)\). Stability of the solution is obtained for a value of \(\tau_{\text{final}} \approx 12.5\) by Heun’s method. For larger values of \(\tau_{\text{final}}\), the
accumulation of error becomes important and fluctuations can appear. We can see
that the value of $m_R$ drops beyond $\tau_{\text{final}} \approx 25$ in Fig.[2]. It is easier to determine the
stability region in the second order method.

For the non-compact $U(1)$-model we have evaluated the expectation value
$\langle \sum_{\mu,\nu=1}^4 F_{\mu \nu} F_{\mu \nu} \rangle$, where $F_{\mu \nu}$ is the field tensor. We get very good agreement with
the known exact value; an error of less than 1% can be obtained easily. In momentum
space $\langle F_{\mu \nu} F_{\mu \nu} \rangle$ is a local observable.

We also computed the expectation value $W(I, J)$ of the Wilson loop observable,
which corresponds to the creation of a quark-antiquark pair, the propagation of this
“meson” and the destruction of the pair ($I \times J$ being the size of the rectangular
loop measured in units of the lattice spacing). It is a non-local observable that can be
computed analytically in the $U(1)$-model. It seems to be more sensitive to fluctuations
than $\langle F_{\mu \nu} F_{\mu \nu} \rangle$. When integrating the Langevin equation, if we continue to iterate too
long after the equilibrium is reached, the field of noise $\hat{\eta}(k, \tau)$ creates some fluctuations
in the solution. These fluctuations are more important for non-local observables,
which are difficult to evaluate on small lattices. In Figs. [3] and [4], we computed the
expectation value of the Wilson loop operator for different sizes of loops and for two
different “time” steps on a lattice of $6^4$ points. We can see the fluctuation introduced
by the noise field $\hat{\eta}(k, \tau)$. In Fig.[3] the “time” step was $\Delta \tau = \tau_{\text{final}}/100$, and in
Fig.[4] the “time” step was $\Delta \tau = \tau_{\text{final}}/150$. The important difference is the number
of “time” steps. When we compare the results of Figs. [3,4] we notice a difference
which seems to exceed the error $O(\Delta \tau^2)$ due to finite $\Delta \tau$. We can reduce fluctuations
by increasing the number of configurations of fields, but we can also average several
values of $\tau$ up to $\tau_{\text{final}}$. Although we have not used the last method, we think it might
reduce fluctuations.

Stochastic quantization allows us to choose the value of $\tau_{\text{final}}$. The propagator for
the free scalar theory in Euclidean momentum space for finite $\tau_{\text{final}}$ is

$$\langle \hat{\phi}(k) \hat{\phi}(k') \rangle = \delta(k + k') \frac{1 - \exp(-2\tau_{\text{final}}(k^2 + m^2))}{k^2 + m^2}$$,

compared with the Euclidean Feynman propagator,

$$\langle \hat{\phi}(k) \hat{\phi}(k') \rangle = \delta(k + k') \frac{1}{k^2 + m^2}$$.

We can see that one must choose $2\tau_{\text{final}}(k^2 + m^2)$ large enough to make the exponential term negligible. For the non-compact $U(1)$-model in momentum space, we are free to choose a value for each lattice point (Fourier acceleration). The point $k = 0$
turns out to be dangerous in the $U(1)$-model because there is no interaction between
lattice sites. For the $\phi^4$-model that does not seem to be the case; the convergence
rate seems to be a little slower near the critical point, but does not differ very much.
5 Conclusion

Our study of two models shows the feasibility of doing lattice simulations on a small momentum lattice. The second order scheme was by far more appropriate because of its greater accuracy and speed. The difference in the results due to variation of $\Delta \tau$ encountered during the simulation of the $U(1)$-model for the Wilson loop operator does not seem to pose a major problem. We believe that the averaging over several values of $\tau$ up to $\tau_{\text{final}}$ can increase the numerical stability. This kind of behavior has not been seen in the $\phi^4$-model. The knowledge of the stability region is very important for simulations of theories like $SU(2)$ and $SU(3)$. For these models, the time needed for generating equilibrium configurations is much larger than for the $U(1)$-model.

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Figure Captions

**Fig. 1** Renormalized mass $m_R$ of the $\phi^4$-model versus coupling parameter $\kappa$. The semi-analytical results are taken from Lüscher and Weisz Ref. [7]. The full curve is a fit to Lüscher and Weisz's data to guide the eye.

**Fig. 2** Renormalized mass $m_R$ of the $\phi^4$-model versus stochastic time $\tau_{final}$.

**Fig. 3** Wilson loop $W(I, J)$ of the $U(1)$-model versus coupling constant $g$; $\Delta \tau = \tau_{final}/100$.

**Fig. 4** Same as Fig.[3] but $\Delta \tau = \tau_{final}/150$. 
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