Transformation optics for thermoelectric flow

Wencong Shi, Troy Stedman and Lilia M Woods

Department of Physics, University of South Florida, Tampa, FL 33620, United States of America

1. Author to whom any correspondence should be addressed.

E-mail: lmwoods@usf.edu

Keywords: thermoelectricity, thermodynamics, metamaterials

Abstract

Transformation optics (TO) is a powerful technique for manipulating diffusive transport, such as heat and electricity. While most studies have focused on individual heat and electrical flows, in many situations thermoelectric effects captured via the Seebeck coefficient may need to be considered. Here we apply a unified description of TO to thermoelectricity within the framework of thermodynamics and demonstrate that thermoelectric flow can be cloaked, diffused, rotated, or concentrated. Metamaterial composites using bilayer components with specified transport properties are presented as a means of realizing these effects in practice. The proposed thermoelectric cloak, diffuser, rotator, and concentrator are independent of the particular boundary conditions and can also operate in decoupled electric or heat modes.

1. Introduction

Unprecedented opportunities to manipulate electromagnetic fields and various types of transport have been discovered recently by utilizing metamaterials (MMs) capable of achieving cloaking, rotating, and concentrating effects [1–4]. Transformation optics (TO), a technique that maps fields into a new coordinate system by requiring that the governing physical equations remain invariant [5–9], can be used to design the MMs. As a result of the coordinate transformations, the MMs properties are highly anisotropic and inhomogeneous, capable of changing fields and currents in a prescribed way. This method has proven to be effective in achieving a negative index of refraction [5], negative magnetic permeability and electromagnetic cloaks [6, 7], and high resolution imaging devices [8, 9]. Significant interest has also been directed towards manipulating diffusive transport, such as acoustic flow [10–12], heat and electricity [13], and matter waves [14]. Requiring invariance of the underlying governing equations under specific coordinate transformations can also be achieved in diffusive phenomena.

The success of the original TO technique beyond its traditional application in electromagnetism shows that this approach is quite general. Heat and particle diffusion equations governed by energy and particle conservation, along with respective continuity equations, remain invariant under coordinate transformations. MM designs based on the principles of TO have been achieved in the laboratory [2, 3]. Bifunctional cloaks capable of guiding decoupled types of flows, such as heat and electric currents, have also been demonstrated [15]. A single MM designed to have cloaking properties, for example, of two independent phenomena can be especially useful for applications where different degrees of control are needed.

Nevertheless, most of the applications in diffusive transport have been on individual or decoupled fluxes and fields. For example, thermal MMs have been designed either for cloaking, rotating, or concentrating heat flux [13, 16, 17]. Similarly, applying TO to Ohm’s law has been successful in manipulating DC currents as well [18]. The bifunctional MMs, on the other hand, are designed to control independent heat (captured by Fourier’s law) and electric (captured by Ohm’s law) currents [19]. In many materials, however, thermoelectric (TE) effects that account for coupling between thermal and electric flows cannot be neglected [20]. Specifically, in systems with a significant Seebeck coefficient, the diffusive transport cannot be described by the Fourier and Ohm’s laws, and TE effects have to be taken into account. Recent work has shown that the thermodynamic governing equations for TE transport are invariant under coordinate transformations and by utilizing TO, MMs for TE cloaking can...
be constructed [21]. This development is a significant step towards accessing a multi-physics range of action, such that the proposed TE cloak can operate as a thermal cloak (under a thermal gradient only), an electric cloak (under a potential gradient only), and under general TE boundary conditions.

In this work, TO techniques are applied to steady state TE transport using a general thermodynamic description, where the heat-electric coupling via the Seebeck coefficient is taken into account explicitly in the governing laws of charge and energy conservation. We broaden the applicability of this powerful technique to TE phenomena and achieve effects beyond cloaking. Using specific coordinate transformations, diffusing, rotating, and concentrating TE flow are demonstrated. The highly anisotropic and inhomogeneous materials that produce these effects are simulated with a MM design based on laminate composites. The mapped properties are also strongly dependent on the properties of the original material. Having a starting material with anisotropic characteristics may lead to more options for control. In this work, we not only broaden the applicability of TO to coupled diffusive transport phenomena to realize several unusual effects, but we also broaden the theory of MM design. This work presents new perspectives for manipulating TE phenomena as well as realizing a multi-domain range of applications.

2. Theoretical concepts and basic equations

The production of a charge current from heat flow and the production of heat flow due to a voltage difference are basic TE phenomena, termed Peltier and Seebeck effects, respectively. These reciprocal phenomena can be described in the framework of thermodynamics, which captures the charge and energy conservation laws in a unified manner [22]. These governing equations are typically considered within the Onsager-de Groot–Callen theory in which linear relations between the charge and heat flows due to electrochemical and temperature gradients. The coefficients in the constitutive equations correspond to the properties of the involved materials. Specifically, \( \bar{\sigma} \) is the conductivity, \( \bar{k} \) is the thermal conductivity, and \( \bar{S} \) is the Seebeck coefficient, which can be Cartesian tensors in general. Also, \( \bar{S}^T \) is the transpose counterpart of \( \bar{S} \), while \( \bar{\sigma} = \bar{\sigma}^T \) and \( \bar{k} = \bar{k}^T \) as required by Onsager-de Groot–Callen theory. One notes that when \( \bar{S} = 0 \) in equations (2), the coupling between the electric and thermal transport is not present and the classic Ohm’s law, \( \bar{J} = -\bar{\sigma} \cdot \nabla \mu \), and Fourier’s law, \( \bar{J}_0 = -\bar{k} \cdot \nabla T \), are recovered.

TO can now be applied to this type of diffusive transport. This includes a diffeomorphism that maps a coordinate system \( r = (x^1, x^2, x^3) \) to a coordinate system \( r' = (x'^1, x'^2, x'^3) \) with an identity map on the boundary of some region \( \Omega \). Given that \( r \) and \( r' \) are related by a smooth and invertible function \( r'(r) \), the invertible jacobian matrix \( \bar{A} \) has elements \( A_{ij} = \frac{\partial x'^i}{\partial x^j} \). The governing equations are required to remain invariant under such a diffeomorphism, which results in transformed fields and materials properties. The mathematical expressions for TO applied to thermoelectricity are obtained as [21]

\[
J'(r') = \frac{\bar{A}}{|\bar{A}|} \cdot J(r), \quad J'_0(r') = \frac{\bar{A}}{|\bar{A}|} \cdot J_0(r),
\]

\[
\nabla \mu' = \bar{A}^{-T} \cdot \nabla \mu, \quad \nabla T' = \bar{A}^{-T} \cdot \nabla T,
\]

\[
\bar{\sigma}' = \frac{\bar{\sigma} \cdot \bar{A}^T}{|\bar{A}|}, \quad \bar{k}' = \frac{\bar{\sigma} \cdot \bar{A}^T \cdot \bar{A}}{|\bar{A}|}, \quad \bar{S}' = \bar{A}^{-T} \cdot \bar{S} \cdot \bar{A}^T,
\]

where \( J'(r') \) and \( J'_0(r') \) are current and heat flows in the new coordinate system \( r' = (x'^1, x'^2, x'^3) \), and \( \nabla \mu', \nabla T', \sigma', \kappa', \) and \( \bar{S}' \) are the transformed gradients and properties, respectively. \( \bar{A}^{-T} \) is the transpose of \( \bar{A} \), and \( \bar{A}^{-T} = (\bar{A}^T)^{-1} \). We point out that the above expressions are obtained by assuming that the transport properties are temperature independent. From equation (5) one finds that, in general, the transformed \( \sigma' \), \( \kappa' \), and \( \bar{S}' \) are inhomogeneous and anisotropic. One further notes that this inhomogeneity and anisotropy is present even if the initial \( \bar{\sigma}, \bar{k}, \) and \( \bar{S} \) are homogeneous and isotropic. Equation (5) further show that in this case the Seebeck coefficient does not change upon the transformation and retains its initial homogeneous and isotropic value. Equations (3–5) indicate that a given coordinate transformation changes the physical properties
and currents while preserving the form of the governing equations of thermoelectricity (equation (1)). Therefore, manipulating the TE transport in a desired way can be achieved by specifying an appropriate diffeomorphism and the electric and thermal currents will ‘follow’ accordingly.

Here we consider different types of 2D coordinate transformations, summarized in Table 1 (left column), with which various effects are achieved. These include cloaking, diffusing, concentrating and rotating the coupled electric and heat currents. Circular and square shapes are specified for the cloaking diffeomorphism. The circular cloak is represented by a \( 0 < r < R_2 \) region, where the diffeomorphism blows up the center point to a circle of radius \( R_1 \). The square cloak diffeomorphism blows up the center point of a square of side \( L_2 \) into a square of side \( L_1 \). The diffusing diffeomorphism is the expansion of \( 0 < r < R_2 \) into \( 0 < r < R_2 \) and compression of \( R_1 < r < R_2 \) into \( R_1 < r < R_1 \) regions. The concentrating diffeomorphism is somewhat opposite of the diffusing diffeomorphism. Here, a circle of radius \( R_2 \) is shrunk to a circle of radius \( R_1 \) while simultaneously stretching a concentric annulus of inner radius \( R_2 \) and outer radius \( R_1 \). Thus, the currents within \( r < R_1 \) will be concentrated in the \( r < R_1 \) region accordingly. The rotating diffeomorphism, on the other hand, rotates the \( r < R_1 \) region at a given angle \( \theta_0 \), while the \( R_1 < r < R_2 \) region is twisted by an angle \( \theta = \theta_0 \) at \( r = R_1 \), matching the rotation within \( r < R_1 \), while the boundary at \( r = R_1 \) is fixed.

We further examine in more detail the case of homogeneous materials for which \( \sigma = \sigma \hat{T} \), \( \kappa = \kappa \hat{I} \), and \( \hat{S} = \hat{S} \hat{T} \) with \( \hat{T} \) being the identity tensor. From equation (5) one notes that the transformed conductivity and thermal conductivity are \( \sigma' = \sigma - \frac{\hat{A}}{|\hat{A}|} \) and \( \kappa' = \frac{\kappa}{|\hat{A}|} \), respectively, while the Seebeck coefficient remains the same \( \hat{S} = \hat{S} \). For the diffeomorphisms we consider in Table 1, one can further represent the transformed properties of the original homogenous and isotropic medium, such that

\[
\begin{align*}
\sigma' &= \sigma R(\theta') T(r') R^t(\theta), \\
\kappa' &= \kappa R(\theta') T(r') R^t(\theta), \\
\sigma' &= \sigma T(x', y'), \quad \sigma' = \sigma T(x', y'), \\
\hat{S}' &= \hat{S}' = \hat{S}.
\end{align*}
\]

For coordinate transformations with circular regions, \( T(r') \) is a diagonal matrix giving the radial and azimuthal responses determined by the coordinate transformation, while \( R(\theta') = \begin{pmatrix} \cos \theta' & -\sin \theta' \\ \sin \theta' & \cos \theta' \end{pmatrix} \) is the matrix for a rotation by angle \( \theta' \). For square regions, the elements of the connecting matrix \( T \) depend on \( (x', y') \). In Table 1 (right column), we summarize the types of considered diffeomorphisms in terms of their \( r'(r) \) relations and the corresponding \( T \) matrices. Note that \( R(\theta') \) has the same form in all cases concerning circular regions.

### 3. Simulations of cloaking, rotating, and concentrating of thermoelectric flow

After establishing the TO mathematical basis of manipulating the TE flow, we are in a position to examine how the coupled electric and thermal currents change when the diffeomorphisms from Table 1 are applied. As discussed earlier, the conductivity and thermal conductivity of the transformed region are highly anisotropic even if the original material is isotropic and homogeneous. From the cloaking \( T_{cl}(r') \), one finds that the radial response \( \sigma'_{r'rr'} \rightarrow 0 \) and the azimuthal response \( \sigma'_{\varphi'\varphi'} \rightarrow \infty \) as \( r' \rightarrow R_1 \), thus electric and thermal currents cannot penetrate the \( r' < R_1 \) region and the \( r' < R_1 \) region is shielded. A similar situation occurs for the square cloak, for which \( \sigma'_{x'x'} \rightarrow 0 \) and \( \sigma'_{y'y'} \rightarrow \infty \) as \( x' \rightarrow L_1 \) so the traversing currents cannot enter the square region. For the diffuser, we find that the behavior of the conductivity and thermal conductivity is similar to the circular cloak although the extreme values of the radial and azimuthal components are not present. Nevertheless, \( \sigma'_{\varphi'\varphi'} \gg \sigma'_{r'rr'} \) and \( \kappa'_{\varphi'\varphi'} \gg \kappa'_{r'rr'} \) at \( r' = R_1 \), which assures that current of a lesser density is able to enter the \( r' < R_1 \) region and thereby produces diffuse flow.

On the other hand, \( T_{im}(r') \) shows that the electrical conductivity components have the property that \( \sigma'_{r'rr'}(r' = R_2) \gg \sigma'_{\varphi'\varphi'}(r' = R_2) \). Thus the electric current tends to go radially instead of azimuthally as \( r' \rightarrow R_2 \). Even if \( J \) is concentrated toward the center, the current will flow in a horizontal path without any distortion in the \( r' < R_1 \) region. The thermal current exhibits similar behavior since the thermal conductivity transforms similarly to the electrical conductivity (equation (6)). For the rotator, we find that \( \sigma'_{\varphi'\varphi'} \gg \sigma'_{r'rr'} \gg \kappa'_{\varphi'\varphi'} \gg \kappa'_{r'rr'} \) at \( r' \rightarrow R_1 \) and the TE flow spirals towards the center of the \( r' < R_1 \) region until it has rotated by \( \theta_0 \) at \( r' = R_1 \). The values of \( \sigma'_{r'rr'} \rightarrow 0 \) and \( \sigma'_{\varphi'\varphi'} \rightarrow \infty \) as \( r' = R_1 \) are the same as the ones before the transformation is applied, thus the currents travel in a direction dictated by \( \theta_0 \) without further distortion. Let us also emphasize that shielding, concentrating, rotating, or diffusing of TE flow become possible if the Seebeck coefficient remains the same everywhere.

In Figure 1 we show simulations based on the finite element COMSOL Multiphysics package for the electric and thermal currents under square cloaking, diffusing, concentrating, and rotating transformations. Details of
Table 1. The mathematical and graphical representations of the considered diffeomorphisms for a cloak (circular and square), diffuser, concentrator, and rotator are given in the left column. The corresponding transformation matrices (see equations 6) for an initially homogenous medium are shown in the right column. Here we define $\alpha_{ij} = R_i - R_j$. The regions where the transformations apply are also denoted in parenthesis in the second column.

| Type of diffeomorphism | Coordinate transformation of a homogeneous material |
|-----------------------|--------------------------------------------------|
| Circular cloak        | $r' = \frac{\alpha_{12} r + \alpha_{12} R_0}{\alpha_{12}}$, $0 < r < R_1$ |
|                       | $r' = r$, $r > R_3$ |
|                       | $\theta' = \theta$ for all $r$ |
|                       | $T_d(r') = \begin{pmatrix} r' - R_3 \\ 0 \\ 0 \\ r' - R_1 \end{pmatrix}$ (grey region) |
| Square cloak          | $x' = \frac{L_1 - L_2 x + L_1}{L_2}$, $0 < x < L_2$ |
|                       | $y' = y \left( \frac{L_1 - L_2}{L_2} + \frac{L_1}{x} \right)$, $|y| < x$ |
| Diffuser              | $r' = \frac{R_4}{R_1}$, $0 < r < R_4$ |
|                       | $r' = \frac{\alpha_{31} r + \alpha_{32} R_0}{\alpha_{31}}$, $R_2 < r < R_3$ |
|                       | $r' = r$, $r > R_3$ |
|                       | $\theta' = \theta$, for all $r$ |
|                       | $T_{dp}(r') = \begin{pmatrix} \alpha_{31} r' \\ \alpha_{32} R_3 + \alpha_{32} r' \alpha_{31} r' + 1 \end{pmatrix}$ (grey region) |
| Concentrator          | $r' = \frac{R_2}{R_0}$, $0 < r < R_2$ |
|                       | $r' = \frac{\alpha_{32} r + \alpha_{32} R_0}{\alpha_{32}}$, $R_2 < r < R_3$ |
|                       | $r' = r$, $r > R_3$ |
|                       | $\theta' = \theta$ for all $r$ |
|                       | $T_{dc}(r') = \begin{pmatrix} \alpha_{32} r' \\ \alpha_{32} R_3 + \alpha_{32} r' \alpha_{32} r' + 1 \end{pmatrix}$ (grey region) |
| Rotator               | $\theta' = 0 + \phi_0$, $0 < r < R_1$ |
|                       | $\theta' = 0 + \phi_0(r - R_3)$, $R_1 < r < R_2$ |
|                       | $\theta' = 0$, $r > R_3$ |
|                       | $r' = r$ for all $\theta$ |
|                       | $T_{rot}(r') = \begin{pmatrix} \theta_0^2 r'^2 - \zeta + 1 \\ 0 \\ 0 \end{pmatrix}$ (white annular region) |
|                       | $\zeta = \sqrt{\theta_0^2 r'^2 (\theta_0^2 r'^2 + 4 \alpha_{12}^2)}$ |
Figure 1. Simulations for an ideal TE (a) square cloak, (b) diffuser, (c) concentrator, and (d) rotator. The material properties in all of these cases are homogeneous and isotropic where \( \sigma = 100 \frac{S}{m}, \kappa = 1 \frac{W}{m \cdot K}, S = 10^{-3} \frac{V}{K} \). Simulations for ideal anisotropic TE (e) square cloak, (g) concentrator, and (f) rotator where \( \sigma_{xx} = 10\sigma_{yy} = 10\sigma_{xy} = 10\sigma_{yx} = \frac{1}{2} \), \( \kappa_{xx} = 10\kappa_{yy} = 10\kappa_{xy} = 10\kappa_{yx} = 1 \frac{W}{m \cdot K}, S_{xx} = 10S_{yy} = 10S_{xy} = 10S_{yx} = 10^{-3} \frac{V}{K} \). The dimensions of the simulations are listed in detail in method. The black curves and arrows correspond to the thermal equipotentials and currents, respectively. The blue curves and arrows correspond to the electric equipotentials and current, respectively. The background color, specified by the color bars, represents the steady state temperature profile (color bars given in K). The voltage profile has a similar behavior and is not shown.

The calculations and the applied TE boundary conditions are given in Methods. The results for the TE transport for an initially homogeneous and isotropic material (figures 1(a)–(d)) illustrate that the transformed regions are not affected by any heat or electric currents in the exterior region where the transformations are absent. After reaching a steady state, constant temperature and potential profiles of \((T_1 + T_2)/2\) and \((V_1 + V_2)/2\) respectively in the cloaked region are achieved (figure 1(a)), where \(T_1, V_1\) \((T_2, V_2)\) are the temperature and electric potential at the left (right) edges of the simulation. Figures 1(a)–(d) further shows that the electric and thermal currents in the outermost regions \([x^l], |y^l| > L_2\) for the square cloak, \(r^l > R_2\) for the rotator, \(r^l > R_3\) for the diffuser and concentrator) are the same as those in the material before the transformations were applied. For the diffuser shown in figure 1(b), the currents that enter the \(r^l < R_2\) region become less dense compared to the original flow. The results displayed in figure 1(c) for the TE concentrator show that stronger TE currents are experienced in the \(r^l < R_1\) region. Also, an object in the \(r^l < R_1\) region for the TE rotator in figure 1(d) will detect electric and heat currents in a different direction than the original flow as specified by the angle \(\theta_0\).

One of the advantages of TO is that these effects are independent of the specific properties of the original medium provided equations (3–5) are satisfied. In figures 1(e)–(h) we illustrate TE cloaking (square case), diffusing, concentrating, and rotating when the original medium is anisotropic. We find that the isotherms and equipotentials are different in the \(r^l > R_2\) regions compared to the isotropic cases, while the \(r < R_1\) regions have the same behavior for the corresponding cases in figures 1(a)–(d) confirming that TE cloaking, rotating, and concentrating are achieved.

It is further important to note that the performance of the TE cloak, diffuser, concentrator, and rotator rely on the simultaneous satisfaction of equations (1) and (2) due to the coupled electric and thermal transport upon the application of temperature and potential gradients. It is clear that if the Seebeck coefficient is neglected, then we recover the independent electric and thermal flows captured by Ohm’s law \(J = -\sigma \cdot \nabla \mu\) and Fourier’s law \(J_\theta = -\kappa \cdot \nabla T\), respectively. Since \(\nabla^r\) and \(\nabla^l\) from equations (5) preserve the invariance of these decoupled equations, the transformed systems can operate as a heat cloak, diffuser, concentrator, or rotator when only a temperature gradient is applied and as an electric cloak, concentrator, or rotator when only a potential difference is applied.

It is also interesting to consider how the TE flow changes when some of the conditions imposed by the applied diffeomorphisms (equations (5), table 1) are not satisfied. In figure 2 we show results for an anisotropic original medium with off-diagonal electrical conductivity, thermal conductivity, and Seebeck coefficient in various situations. For example, we find that when \(S' = 10S\), TE cloaking is not achieved since the isotherms and equipotentials are distorted in the region \(r^l > R_2\) (figure 2(a)) when compared to the ideal TE anisotropic
Using the results for the transport properties in equations (1)–(4), we summarize these results for the transformed transport properties for each layer. In table 2, we show the distorted isotherms and equipotentials at the boundary, where the identity mapping as required by TO is clearly not maintained when compared to figures 1(g), 1(h).

4. Laminate thermoelectric MMs

The practical implementation of manipulating TE transport according to a given diffeomorphism requires materials with prescribed highly anisotropic and inhomogeneous properties, which are not found in nature. Therefore, MMs are used to approximately achieve the desired effects. Here we propose that the effects of TE cloaking, diffusing, concentrating, and rotating can be obtained with composites based on bilayered constituents. Given that the original material is isotropic and homogeneous and the layer thickness of these components are small compared to the diameter of the cloak, their effective properties follow basic rules for elements being connected in series and in parallel [2, 17, 23]. Specifically, the circular MM TE cloak and diffuser are composed of concentric A, B bilayers whose radial and azimuthal transport components are given by

\[
\sigma'_{rr} = \frac{2\sigma_A\sigma_B}{\sigma_A + \sigma_B}, \quad \sigma'_{\theta\theta} = \frac{\sigma_A + \sigma_B}{2}, \quad \kappa'_{rr} = \frac{2\kappa_A\kappa_B}{\kappa_A + \kappa_B}, \quad \kappa'_{\theta\theta} = \frac{\kappa_A + \kappa_B}{2},
\]

\[
\sigma'_{rr}S'_{rr} = \frac{2\sigma_A S_A + \sigma_B S_B}{\sigma_A S_A + \sigma_B S_B}, \quad \sigma'_{\theta\theta}S'_{\theta\theta} = \frac{\sigma_A S_A + \sigma_B S_B}{2}.
\]

Similar relations are found for the square cloak where \(\sigma'_{rr}, \sigma'_{\theta\theta}, \kappa'_{rr}, \kappa'_{\theta\theta}, S'_{rr}, S'_{\theta\theta}\) in the above expressions are substituted with \(\sigma'_{x'x'}, \sigma'_{y'y'}, \kappa'_{x'x'}, \kappa'_{y'y'}, S'_{x'x'}, S'_{y'y'}\), respectively. The layered transport properties of the TE concentrator and rotator MMs can also be found similarly with

\[
\sigma'_{rr} = \frac{\sigma_A + \sigma_B}{2}, \quad \sigma'_{\theta\theta} = \frac{2\sigma_A\sigma_B}{\sigma_A + \sigma_B}, \quad \kappa'_{rr} = \frac{\kappa_A + \kappa_B}{2}, \quad \kappa'_{\theta\theta} = \frac{2\kappa_A\kappa_B}{\kappa_A + \kappa_B},
\]

\[
\sigma'_{rr}S'_{rr} = \frac{\sigma_A S_A + \sigma_B S_B}{2}, \quad \sigma'_{\theta\theta}S'_{\theta\theta} = \frac{2\sigma_A S_A + \sigma_B S_B}{\sigma_A S_A + \sigma_B S_B}.
\]

Using the material response properties for each case, one can then solve for the material properties of the A and B layers. In table 2, we summarize these results for the transformed transport properties for each diffeomorphism. These expressions show that the extreme values at the different boundaries (as discussed in section 3) are approximately achieved in each case.

We construct the MM composites based on bilayer constituents having an equal thickness. The bilayered MMs are sketched in table 2 for each case, where definitions of several characteristic parameters are also given. Using the results for the transport properties in equations (7–10) and taking the index \(n = \{1, 2, ..., N\}\), the conductivities for the concentric layers comprising the TE circular cloak are taken to be
Table 2. Bilayered metamaterials for circular and square TE cloaks, a diffuser, a concentrator, and a rotator applied to thermoelectric transport. Here $\alpha_{A(R)} = R_i - R_j$ and $r'_{A(R)}$ is the distance from the origin to the corresponding layer.

| Cloak and diffuser |
|-------------------|
| $\sigma_{A(R)} = P_{A(R)}(r'_{A(R)}) \sigma; S_{A(R)} = S$ or $S_{A(R)} = P_{A(R)}(r'_{A(R)}) \sigma$ (cloak) |
| $\sigma_{A(R)} = N_{A(R)}(r'_{A(R)}) \sigma; S_{A(R)} = S$ or $S_{A(R)} = N_{A(R)}(r'_{A(R)}) \sigma$ (diffuser) |

| Concentrator |
|----------------|
| $\sigma_{A(R)} = D_{A(R)}(r'_{A(R)}) \sigma; S_{A(R)} = S$ or $S_{A(R)} = D_{A(R)}(r'_{A(R)}) \sigma$ |

| Square cloak |
|----------------|
| $\sigma_{A(R)} = Q_{A(R)}(x'_{A(R)}, y'_{A(R)}) \sigma; S_{A(R)} = S$ or $S_{A(R)} = Q_{A(R)}(x'_{A(R)}, y'_{A(R)}) \sigma$ |

| Rotator |
|----------------|
| $\sigma_{A(R)} = T_{A(R)} \left( \frac{R_i + R_j}{2} \right) \sigma; S_{A(R)} = S$ or $S_{A(R)} = T_{A(R)} \left( \frac{R_i + R_j}{2} \right) \sigma$ |

$\sigma_{A(R)} = \sigma P_{A} (R_i + (2n - 1)\bar{d})$ and $\sigma_{B(R)} = \sigma P_{B} (R_i + 2n\bar{d})$ with thickness $\bar{d} = \frac{R_2 - R_1}{2N}$ (as done in [21]), while the layer conductivities for the diffuser are taken as $\sigma_{A(R)} = \sigma N_{A} (R_i + (2n - 1)\bar{d})$ and $\sigma_{B(R)} = \sigma N_{B} (R_i + 2n\bar{d})$ with thickness $\bar{d} = \frac{R_2 - R_1}{2N}$. The square cloak is composed of bilayers whose properties are dependent on both the $x$ and $y$ variables, but the conductivity varies little along the edge of a layer. With this in mind and because of the square symmetry, we assign the layer conductivity to be the average conductivity sampled along the two corners and center of a layer edge. For example, using the right edge, we take

$$\sigma_{A(R)} = \sigma P_{A} \left( \frac{R_i + R_j}{2} \right) \sigma; S_{A(R)} = S$$ or $S_{A(R)} = P_{A(R)} \left( \frac{R_i + R_j}{2} \right) \sigma$
the bilayers for the TE rotator are not concentric since the coordinate transformation deforms the region azimuthally. The thermal conductivities for the $A_n$ and $B_n$ layers have similar expressions with $s$ being substituted by $k$ and the Seebeck coefficient is the same everywhere in all cases. Interchanging $\pm$ with $\mp$ in table 2 corresponds to swapping the $A$ and $B$ components.

The simulations for the MM square cloak, concentrator, rotator, and diffuser for various numbers of layers and sectors are given in figure 3. These results show that the bilayered composites with transport properties taken as discussed above achieve the effects of cloaking, concentrating, rotating, or diffusing TE flow only approximately when compared with the ideal situations in figure 1. Nevertheless, the performance of the different MMs improves as the number of layers and sectors increases. For smaller $N$ (figure 3(a)–(d)), some isotherms and equipotentials become distorted and nonuniform around the different boundaries, but as $N$ increases these undesired effects reduce. Reducing the thickness of each layer also leads to improved performance of the TE composites.

The MMs can further be evaluated quantitatively by using the standard deviations of temperature and potential differences between the ideal system and the MMs [21, 24], defined respectively as

$$\sigma_{STD}^T = \sqrt{\frac{1}{X} \sum_i (\Delta T_i^M - \Delta T_i^M)^2}$$

$$\sigma_{STD}^V = \sqrt{\frac{1}{X} \sum_i (\Delta V_i^M - \Delta V_i^M)^2}$$

and calculated on a grid of $X$ points. Here $\Delta T_i^M = T_i^M - T_i$ and $\Delta V_i^M = \frac{1}{X} \sum_i \Delta T_i^M$, where $T_i^M$ and $T_i$ are the temperatures at the $i$th grid point with and without the MM, respectively. Similar notation applies for the standard deviation of the potential differences. Figure 4 shows the results for the temperature difference standard deviation for MMs constructed with a different numbers of bilayer components. We find that $\sigma_{STD}^T$ decreases as $N$ is increased for the square cloak and diffuser (figures 4(a), (d)). For the concentrator, increasing $N$ and the number of sectors leads to a reduced $\sigma_{STD}^T$ (figure 4(b)), although the effect of the number of layers is stronger. A similar sector number - $\sigma_{STD}^T$ trend is found for the TE rotator as expected (figure 4(c)). It is further important to note that the transport properties of the bilayered components of MM composites have largely different values, which may be difficult for practical implementations. This large deviation can reduced by increasing the number of MM components.

5. Methods

The simulations for the steady state of equations (2) are performed using finite element analysis as implemented in the COMSOL MULTIPHYSICS package. We have also used the built in ‘Electric Currents’ and ‘Coefficient from PDE’ interfaces to solve the governing and constitutive equations for each diffeomorphism. The initial material has a geometry of a 2D square with $L \times L = 0.1 \text{ m} \times 0.1 \text{ m}$, where $L$ is the length of the square side. The following dimensions are used in the simulations presented in this paper: original material block—$L \times L = 0.1 \text{ m} \times$
The boundary conditions are taken such that the left side has temperature \( T_1 = 285 \text{ K} \), the right side has temperature \( T_3 = 300 \text{ K} \), the top and bottom edges are insulated. The right end has an outward normal current density \( 1 \text{ A m}^{-2} \), while the chemical potential is maintained as constant throughout the calculations.

6. Conclusions

In conclusion, our work shows that the TO techniques successfully applied to many areas such as optics, acoustics, and thermal conduction can also be used in thermoelectricity. Since the governing equations of TE transport are invariant under coordinate transformations, effects such as cloaking, diffusing, concentrating, and rotating of the coupled electric and thermal flows can be achieved in a localized region. This approach is quite general as it can be applied to inhomogeneous materials and is independent of the boundary conditions and regions shape. The specific coordinate transformation requires highly anisotropic and inhomogeneous materials with extreme properties, which are not readily found in nature. Here we have shown that MMs based on bilayer laminates can be constructed using basic circuit theory to achieve targeted functionalities for thermoelectricity. A particular strength of our work is that it gives explicit forms of various transformations (which can be used in other areas) combined with clear routes of MM construction.

TO applied in thermoelectricity offers a novel potential to control TE transport as well as to access multiphysics domains within a single device. Manipulating thermal and electric flows is of great importance to the electronics industry and TO in thermoelectricity offers new opportunities for control. In addition to its generality in terms of materials and boundary conditions, with this method one could design a single device capable of directing thermal and electric currents coupled via the Seebeck coefficient as well as thermal and electric currents independently under appropriate boundary conditions. Therefore, depending on the need for components and parts of electronics devices, we suggest that a greater flexibility in terms of direction and functionality can be achieved. Additionally, we envision that TO techniques and the routes of MM construction may be modified in order to conceive schemes of separating thermal from electric current flows in TE materials. Being able to manipulate thermal and electric currents when coupling via the Seebeck coefficient is present may be a promising route for designing TE devices with enhanced efficiency for converting heat to electricity.

Acknowledgments

We acknowledge financial support from the US National Science Foundation under Grant No. DMR-1400957. The use of the University of South Florida Research Computing facilities is also acknowledged.

ORCID iDs

Lilia M Woods © https://orcid.org/0000-0002-9872-1847

References

[1] Leonhardt U and Philbin T G 2009 Transformation optics and the geometry of light Prog. Opt. 53 69–152
[2] Kadic M, Bückmann T, Schittny R and Wegener M 2013 Metamaterials beyond electromagnetism Rep. Prog. Phys. 76 126501
[3] Engheta N and Ziolkowski R W 2006 Metamaterials: Physics and Engineering Explorations (New York: Wiley)
[4] Pendry J B, Schurig D and Smith D R 2006 Controlling electromagnetic fields Science 312 1780
[5] Soukoulis C M, Linden S and Wegener M 2007 Negative refractive index at optical wavelengths Science 315 47
[6] Schurig D, Mock J J, Justice B I, Cummer S A, Pendry J B, Starr A F and Smith D R 2006 Metamaterial electromagnetic cloak at microwave frequencies Science 315 977
[7] Cai W, Chettiar U K, Kildishev A V and Shalaev V M 2007 Optical cloaking with metamaterials Nat. Photon. 1 224
[8] Liu Z, Lee H, Xiong Y, Sun C and Zhang X 2007 Far-field optical hyperlens magnifying sub-diffraction-limited objects Science 315 1686
[9] Vakil A and Engheta N 2011 Transformation optics using graphene Science 332 1291
[10] Milton G W, Briane M and Willis J R 2006 On cloaking for elasticity and physical equations with a transformation invariant form New J. Phys. 8 248
[11] Cummer S A and Schurig D 2007 One path to acoustic cloaking New J. Phys. 9 45
[12] Greenleaf A, Kuryley Y, Lassas M and Uhlmann G 2008 Isotropic transformation optics: approximate acoustic and quantum cloaking New J. Phys. 10 115024
[13] Guenneau S, Amra C and Veynante D 2012 Transformation thermodynamics: cloaking and concentrating heat flux Opt. Express 20 8207
[14] Zhang S, Genov D A, Sun C and Zhang X 2008 Cloaking of matter waves Phys. Rev. Lett. 100 123002
[15] Han T, Bai X, Gao D, Thong J T L, Li B and Qiu C W 2014 Experimental demonstration of a bilayer thermal cloak Phys. Rev. Lett. 112 054302
[16] Schittny R, Kadic M, Guenneau S and Wegener M 2013 Experiments on transformation thermodynamics: molding the flow of heat Phys. Rev. Lett. 110 195901
[17] Han T, Yuan T, Li B and Qiu C W 2013 Homogeneous thermal cloak with constant conductivity and tunable heat localization Sci. Rep. 3 1593
[18] Wang Q H, Kalantar-Zadeh K, Kis A, Coleman J N and Strano M S 2012 Electronics and optoelectronics of two-dimensional transition metal dichalcogenides Nat. Nanotechnol. 7 699
[19] Moccia M, Castaldi G, Savo S, Sato Y and Galdi V 2014 Independent manipulation of heat and electrical current via bifunctional metamaterials Phys. Rev. X 4 021025
[20] Goupil C 2016 Continuum Theory and Modeling of Thermoelectric Elements (New York: Wiley)
[21] Stedman T and Woods L M 2017 Cloaking of thermoelectric transport Sci. Rep. 7 6988
[22] MacDonald D K C 2006 Thermoelectricity: an introduction to the principles (New York: Dover)
[23] Goupil C, Seifert W, Zabrocki K, Müller E and Snyder G J 2011 Thermodynamics of thermoelectric phenomena and applications Entropy 13 1481
[24] Hu R, Hu J-Y, Wu R-K, Xie B, Yu X-J and Luo X-B 2016 Examination of the thermal cloaking effectiveness with layered engineering materials Chin. Phys. Lett. 33 044401