ENUMERATION OF PERMUTATIONS BY THE PARITY OF DESCENT POSITIONS

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Abstract. Noticing that some recent variations of descent polynomials are special cases of Carlitz and Scoville’s four-variable polynomials, which enumerate permutations by the parity of descent and ascent positions, we prove a $q$-analogue of Carlitz-Scoville’s generating function by counting the inversion number and a type B analogue by enumerating the signed permutations with respect to the parity of descent and ascent positions. As a by-product of our formulas, we obtain a $q$-analogue of Chebikin’s formula for alternating descent polynomials, an alternative proof of Sun’s gamma-positivity of her bivariate Eulerian polynomials and a type B analogue, the latter refines Petersen’s gamma-positivity of the type B Eulerian polynomials.

1. Introduction

In the past few years of this century, several variations and refinements of permutation descent, according to the parity of descent positions, have been studied, see [5, 29, 15, 21, 32, 34, 33, 23, 20, 25]. This paper arose from the observation that some of these...
results are related to a work of Carlitz and Scoville [4] dated back to 1973. For example, Chebikin’s alternating descent polynomial [5] and the bivariate Eulerian polynomials in H. Sun [32] and Y. Sun and Zhai [34] are both special cases of Carlitz-Scoville’s four-variable polynomials enumerating the permutations according to the parity of both descents and ascents. On the other hand, this connection leads immediately to obtain two equivalent simpler versions of Carlitz-Scoville’s generating function. As Carlitz and Scoville’s original proof relies on solving a system of differential equations, this prompted us to find a more conceptuel proof, which led up straightforwardly to a $q$-analogue.

If $\pi$ is a permutation of $[n] := \{1, \ldots, n\}$, an index $i \in [n - 1]$ is a descent position (resp. ascent position) of $\pi$ if $\pi(i) > \pi(i + 1)$ (resp. $\pi(i) < \pi(i + 1)$). Let des $\pi$ (resp. des$_1 \pi$ and des$_0 \pi$) be the number of descents of $\pi$ (resp. at odd and even positions), i.e.,

$$\text{des}_\nu(\pi) = \#\{i \in [n] | \pi(i) > \pi(i + 1) \text{ and } i \equiv \nu \pmod{2}\} \quad (\nu \in \{0, 1\}).$$

The statistics asc $\pi$, asc$_1 \pi$ and asc$_0 \pi$ are defined similarly. For $i \in \{2, 3, \ldots, n - 1\}$, we say $\pi(i)$ is a valley (resp. peak) of $\pi$, if $\pi(i - 1) > \pi(i) < \pi(i + 1)$ (resp. $\pi(i - 1) < \pi(i) > \pi(i + 1)$) and $\pi(i)$ is a double ascent (resp. double descent) of $\pi$, if $\pi(i - 1) < \pi(i) < \pi(i + 1)$ (resp. $\pi(i - 1) > \pi(i) > \pi(i + 1)$). Finally we recall that the inversion number of $\pi$ is

$$\text{inv} \pi = |\{(i, j) | \pi(i) > \pi(j), 1 \leq i < j \leq n\}|.$$

Define the enumerative polynomial of permutations of $\mathfrak{S}_n$ by the parity of ascent and descent positions as

$$P_n(x_0, x_1, y_0, y_1, q) = \sum_{\sigma \in \mathfrak{S}_n} x_0^{\text{asc}_0 \sigma} x_1^{\text{asc}_1 \sigma} y_0^{\text{des}_0 \sigma} y_1^{\text{des}_1 \sigma} q^{\text{inv} \sigma}.$$ 

Recall the following $q$-exponential series

$$\exp_q(x) = \sum_{n \geq 0} \frac{x^n}{n!_q},$$

where $0!_q = 1$ and $n!_q = \prod_{i=1}^{n}(1 + q + \cdots + q^{i-1})$ for $n \geq 1$, and the $q$-trignometric series

$$\cosh_q t = \sum_{n \geq 0} \frac{t^{2n}}{(2n)!_q}, \quad \sinh_q t = \sum_{n \geq 1} \frac{t^{2n-1}}{(2n-1)!_q};$$

$$\cos_q x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!_q}, \quad \sin_q x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!_q}.$$
Theorem 1.1. Let $\alpha = \sqrt{(y_0 - x_0)(y_1 - x_1)}$. Then

$$\sum_{n \geq 1} P_n(x_0, x_1, y_0, y_1, q) \frac{t^n}{n!} = \frac{(x_1 + y_1) \cosh_q(\alpha t) + \alpha \sinh_q(\alpha t) - y_1(\cosh^2_q(\alpha t) - \sinh^2_q(\alpha t)) - x_1}{x_0 x_1 - (x_0 y_1 + x_1 y_0) \cosh_q(\alpha t) + y_0 y_1(\cosh^2_q(\alpha t) - \sinh^2_q(\alpha t))}. \quad (1.1)$$

Remark 1. When $q = 1$ Eq. (1.1) reduces to Carlitz-Scoville’s formula [4, Theorem 3.1]\

$$\sum_{n \geq 1} P_n(x_0, x_1, y_0, y_1, 1) \frac{t^n}{n!} = \frac{(x_1 + y_1) \sum_{n \geq 1} \frac{\beta^{n-1} 2^n}{(2n)!} + \sum_{n \geq 1} \frac{\beta^{n-1} 2^n - 1}{(2n)!}}{1 - (x_0 y_1 + x_1 y_0) \sum_{n \geq 1} \frac{\beta^{n-1} 2^n}{(2n)!}}, \quad (1.2)$$

with $\beta = (y_0 - x_0)(y_1 - x_1)$. For the homogeneous Eulerian polynomials $P_n(y, y, x, x, 1)$, i.e., $\sum_{\sigma \in S_n} x^{\text{des}}_\sigma y^{\text{asc}}_\sigma$, the corresponding formula reads

$$\sum_{n \geq 1} P_n(y, y, x, x, 1) \frac{t^n}{n!} = \frac{e^{xt} - e^{yt}}{xe^{yt} - ye^{xt}}. \quad (1.3)$$

Chen and Fu [6] recently gave a context-free grammar proof of (1.3).

Let $UD_n$ be the set of *up-down* permutations of $12\ldots n$, i.e., permutations $\sigma := \sigma(1)\ldots\sigma(n)$ such that $\sigma(1) < \sigma(2) > \sigma(3) < \ldots$. Obviously

$$P_n(0, 1, 1, 0, q) = \sum_{\sigma \in UD_n} q^{\text{inv}}_\sigma$$

and Eq. (1.1) reduces to a $q$-analogue of André’s classical result (see [31 [16 [18]]) :

$$1 + \sum_{n \geq 1} P_n(0, 1, 1, 0, q) \frac{x^n}{n!} = \frac{1 + \sin_q x}{\cos_q x}. \quad (1.4)$$

For the following two special cases:

$$A_n(x, y, q) := P_n(1, 1, y, x, q) = \sum_{\sigma \in S_n} x^{\text{des}}_1 \sigma y^{\text{des}}_0 \sigma q^{\text{inv}}_\sigma, \quad (1.5a)$$

$$\hat{A}_n(x, y, q) := P_n(y, 1, 1, x, q) = \sum_{\sigma \in S_n} x^{\text{des}}_1 \sigma y^{\text{asc}}_0 \sigma q^{\text{inv}}_\sigma, \quad (1.5b)$$

\(^1\text{Carlitz and Scoville counted a conventional rise at the beginning as position 0 and a conventional descent at the end as position } n \mod 2.\)
we derive from Theorem 1.1 that
\[
\sum_{n \geq 1} A_n(x, y, q) \frac{t^n}{n!} q = \frac{(1 + x) \cosh_q(\alpha) + \alpha \sinh_q(\alpha) - x(\cosh_q^2(\alpha) - \sinh_q^2(\alpha)) - 1}{1 - (x + y) \cosh_q(\alpha) + xy(\cosh_q^2(\alpha) - \sinh_q^2(\alpha))}, \tag{1.6a}
\]
\[
\sum_{n \geq 1} \hat{A}_n(x, y, q) \frac{t^n}{n!} q = \frac{(1 + x) \cosh_q(\alpha) - \alpha \sinh_q(\alpha) - x(\cosh_q^2(\alpha) + \sinh_q^2(\alpha)) - 1}{1 - (x + y) \cosh_q(\alpha) + xy(\cosh_q^2(\alpha) + \sinh_q^2(\alpha))}, \tag{1.6b}
\]
with \(\alpha = \sqrt{(1 - x)(1 - y)}\).

Remark 2. Formulae (1.6a), (1.6b) and (1.6b) are actually equivalent. Indeed, for any \(\sigma \in \mathcal{S}_n\) it is clear that
\[
des_0 \sigma + asc_0 \sigma = \lfloor (n - 1)/2 \rfloor, \tag{1.7a}
\]
\[
des_1 \sigma + asc_1 \sigma = \lceil n/2 \rceil. \tag{1.7b}
\]

Hence the distribution of the quadruple statistics \((asc_0, asc_1, des_0, des_1)\) is equivalent to any pair of the statistics in \(\{des_1, asc_1\} \times \{des_0, asc_0\}\). In particular, we have
\[
\hat{A}_n(x, y, q) = y^{(n - 1)/2} A_n(x, 1/y, q), \tag{1.8}
\]
and
\[
P_n(x_0, x_1, y_0, y_1, q) = x_0^{\lfloor (n - 1)/2 \rfloor} x_1^{n/2} A_n \left( \frac{y_1}{x_1}, \frac{y_0}{x_0}, q \right). \tag{1.9}
\]

The polynomial \(A_n(x, x, q) := \sum_{\sigma \in \mathcal{S}_n} x^{des_\sigma} q^{inv_\sigma}\) is a classical \(q\)-analogue of Eulerian polynomials and Eq. (1.6a) yields Stanley’s formula \(\[30\] [28],
\[
1 + \sum_{n \geq 1} x A_n(x, x, q) \frac{t^n}{n!} q = \frac{1 - x}{1 - x \exp_q((1 - x)t)}, \tag{1.10}
\]
of which another refinement was given in [26].

As a variation of descent, Chebikin [3] introduced the alternating descent set of permutation \(\pi \in \mathcal{S}_n\) by
\[
\hat{D}(\pi) = \{i \in [n - 1]| \pi(i) > \pi(i + 1) \text{ and } i \text{ is odd or } \pi(i) < \pi(i + 1) \text{ and } i \text{ is even}\}.
\]
Hence, the number of alternating descents \(\hat{des}_\pi = |\hat{D}(\pi)|\) equals \(\hat{des}_1 \sigma + \hat{asc}_0 \sigma\) and formula (1.6b) with \(x = y\) and \(q = 1\) reduces to
\[
1 + \sum_{n \geq 1} x \hat{A}_n(x, x, 1) \frac{t^n}{n!} = \frac{1 - x}{1 - x(\sec(1 - x)t + \tan(1 - x)t)}, \tag{1.11}
\]
which is equivalent to [5, Theorem 4.2], see also [15, Eq. (22)]. As Chebikin, being unaware of the work of Carlitz and Scoville, Sun [32] and Sun and Zhai [34] reconsidered
the polynomials $A_n(x, y, 1)$, and a cumbersome formula for (1.6a) is given in [34, Theorem 2.2]. Other proofs of formula (1.11) and generalizations appeared in [29, 15, 20, 25].

As the original proof of (1.2) with $q = 1$ in [4] is not easy (see also the solution of Exercise 4.3.14 in [16]), we shall give a more conceptual proof of (1.6a), which is equivalent to Theorem 1.2, by exploring a sieve method, see [30, 14, 5, 31].

Our second goal is to give a type B analogue of Carlitz and Scoville’s formula, i.e., Theorem 1.1 with $q = 1$. Denote by $B_n$ the collection of type B permutations $\sigma$ of the set $[\pm n] := \{\pm 1, \ldots, \pm n\}$ such that $\sigma(-i) = -\sigma(i)$ for all $i \in [n]$, obviously, $|\sigma| := |\sigma(1)| \cdots |\sigma(n)| \in \mathcal{S}_n$. As usual (see [3, 28]), we always assume that type B permutations are prepended by 0. That is, we identify an element $\sigma \in B_n$ with the word $\sigma(0) \sigma(1) \cdots \sigma(n)$, where $\sigma(0) = 0$. We say that $\sigma \in B_n$ has a descent (resp. ascent) at position $i$, if $\sigma(i) > \sigma(i + 1)$ (resp. $\sigma(i) < \sigma(i + 1)$) for $i \in \{0\} \cup [n - 1]$. By abuse of notation, in this section, we use $\text{des} \sigma$, $\text{des}_1 \sigma$ and $\text{des}_0 \sigma$ to denote the number of descents of $\sigma$ (resp. at odd and even positions). The statistics $\text{asc} \sigma$, $\text{asc}_1 \sigma$ and $\text{asc}_0 \sigma$ are defined similarly for the ascents.

Define the enumerative polynomials

$$B_n(x, y) := \sum_{\sigma \in B_n} x^{\text{des}_1 \sigma} y^{\text{des}_0 \sigma}. \quad (1.12)$$

**Theorem 1.2.** Let $\alpha = \sqrt{(1 - x)(1 - y)}$. Then

$$\sum_{n \geq 1} B_{2n}(x, y) \frac{t^{2n}}{(2n)!} = \frac{(x + y) \cosh(2\alpha t) + (1 - x)(1 - y) \cosh(\alpha t) - (1 + xy)}{(1 + xy) - (x + y) \cosh(2\alpha t)}, \quad (1.13a)$$

$$\sum_{n \geq 1} B_{2n-1}(x, y) \frac{t^{2n-1}}{(2n - 1)!} = \frac{\alpha(1 + y) \sinh(\alpha t) + (1 + xy) - (x + y) \cosh(\alpha t)}{(1 + xy) - (x + y) \cosh(\alpha t)}. \quad (1.13b)$$

**Remark 3.** When $x = y$, the polynomial $B_n(x, x) := \sum_{\sigma \in B_n} x^{\text{des} \sigma}$ is the usual Eulerian polynomial of type B and Theorem 1.2 is equivalent to the known generating function, see [7, Corollary 3.9] or [28, Theorem 13.3],

$$\sum_{n \geq 0} B_n(x, x) \frac{t^n}{n!} = \frac{(x - 1)e^{t(x-1)}}{x - e^{2t(x-1)}}. \quad (1.14)$$

Now, consider the following variant of $B_n(x, y)$

$$\hat{B}_n(x, y) := \sum_{\sigma \in B_n} x^{\text{des}_1 \sigma} y^{\text{asc}_0 \sigma} = y^{[(n+1)/2]} B_n(x, 1/y). \quad (1.15)$$

From Theorem 1.2 we derive plainly the generating function of the latter polynomials.
Theorem 1.3. Let $\alpha = \sqrt{(1 - x)(1 - y)}$. Then
\begin{align}
\sum_{n \geq 1} \hat{B}_{2n}(x, y) \frac{t^{2n}}{(2n)!} &= \frac{(1 + xy) \cos(2\alpha t) - (1 - x)(1 - y) \cos(\alpha t) - (x + y)}{(x + y) - (1 + xy) \cos(2\alpha t)}, \quad (1.16a) \\
\sum_{n \geq 1} \hat{B}_{2n-1}(x, y) \frac{t^{2n-1}}{(2n-1)!} &= \frac{-\alpha(1 + y) \sinh(\alpha t)}{(x + y) - (1 + xy) \cos(2\alpha t)}. \quad (1.16b)
\end{align}

Remark 4. Similar to Chebikin’s alternating descent set of type A (see [5]), we can define the alternating descent set of any $\sigma \in B_n$ by
$$\hat{D}_B(\pi) = \{i \in \{0\} \cup [n-1]|\pi(i) > \pi(i + 1) \text{ if } i \text{ is odd or } \pi(i) < \pi(i + 1) \text{ if } i \text{ is even}\}.$$ 

Let $\hat{\text{des}}_B(\sigma) = |\hat{D}_B(\sigma)|$. Clearly $\hat{B}_n(x, x) = \sum_{\sigma \in B_n} x^{\hat{\text{des}}_B(\sigma)}$, which is the $n$-th alternating Eulerian polynomial of type B in [21], and Theorem 1.3 reduces to the generating function in [23, 0, 24],
$$\sum_{n \geq 0} \hat{B}_n(x, x) \frac{u^n}{n!} = \frac{x - 1}{(x - 1) \cos(u(1 - x)) + (x + 1) \sin(u(1 - x))}. \quad (1.17)$$

Define the general enumerative polynomials of permutations by the parity of the ascent and descent positions:
$$P_n^B(x_0, x_1, y_0, y_1) = \sum_{\sigma \in B_n} x_0^{\text{asc}_0 \sigma} x_1^{\text{asc}_1 \sigma} y_0^{\text{des}_0 \sigma} y_1^{\text{des}_1 \sigma}. \quad (1.18)$$

For any $\sigma \in B_n$ we have
$$\text{des}_0 \sigma + \text{asc}_0 \sigma = \lfloor (n + 1)/2 \rfloor,$$
$$\text{des}_1 \sigma + \text{asc}_1 \sigma = \lfloor n/2 \rfloor. \quad (1.19)$$

Hence the distribution of the quadruple statistics $(\text{asc}_0, \text{asc}_1, \text{des}_0, \text{des}_1)$ is equivalent to any of the four pairs in $\{\text{des}_1, \text{asc}_1\} \times \{\text{des}_0, \text{asc}_0\}$. It follows that
$$P_n^B(x_0, x_1, y_0, y_1) = x_0^{\lfloor (n+1)/2 \rfloor} x_1^{\lfloor n/2 \rfloor} B_n \left( \frac{y_1}{x_1}, \frac{y_0}{x_0} \right). \quad (1.20)$$

We derive plainly the following generating function from Theorem 1.2

Theorem 1.4. We have
\begin{align}
\sum_{n \geq 1} P_{2n}^B(x_0, x_1, y_0, y_1) \frac{t^{2n}}{(2n)!} &= \frac{(x_0 y_1 + x_1 y_0) \sum_{n \geq 0} \frac{\alpha^n (2t)^{2n}}{(2n)!} + \sum_{n \geq 0} \frac{\alpha^{n+1} (2t)^{2n}}{(2n)!} - (x_1 x_0 + y_0 y_1)}{(x_0 x_1 + y_0 y_1) - (y_1 x_0 + x_1 y_0) \sum_{n \geq 0} \frac{\alpha^n (2t)^{2n}}{(2n)!}}, \quad (1.21)
\end{align}
and

\[\sum_{n \geq 1} P^B_{2n-1}(x_0, x_1, y_0, y_1) \frac{t^{2n-1}}{(2n-1)!} = \frac{(y_0^2 - x_0^2)(y_1 - x_1) \sum_{n \geq 0} \frac{\alpha^{n(2n+1)}}{(2n+1)!}}{(x_0x_1 + y_0y_1) - (x_0y_1 + x_1y_0) \sum_{n \geq 0} \frac{\alpha^{n(2n+1)}}{(2n)!}}, \quad (1.22)\]

where \(\alpha = (y_0 - x_0)(y_1 - x_1)\).

In view of (1.15) and (1.20), Theorem 1.2, Theorem 1.3 and Theorem 1.4 are equivalent. We shall give a proof of Theorem 1.2 in the same vein as the proof of (1.6a) with \(q = 1\).

An important feature of Eulerian polynomials is the gamma-nonnegativity [28]. More recently, Sun [33] proved that the bivariate Eulerian polynomials \((1 + y)A_{2n}(x, y, 1)\) and \(A_{2n+1}(x, y, 1)\) are \(\gamma\)-positive (see Theorem 2.3). Our third goal is to derive some symmetric expansion formulae for bivariate polynomials allied to the above four families of bi-Eulerian polynomials. This will be done by applying their generating functions and combinatorics of André permutations [12, 13, 17].

The rest of this paper is organised as follows. We will first study the symmetric and gamma expansions of the two sequences of bi-polynomials as well as their type analogues in Section 2 and postpone the proof of (1.6a) and Theorem 1.2 to Section 3 and Section 4, respectively. We conclude with some open problems in Section 5.

As suggested by a referee, for reader’s convenience, we list the main permutation statistics of this paper in the following table.

| \(\text{des}_0 \pi\) | the number of descents of \(\pi\) at even positions |
| \(\text{des}_1 \pi\) | the number of descents of \(\pi\) at odd positions |
| \(\text{asc}_0 \pi\) | the number of ascents of \(\pi\) at even positions |
| \(\text{asc}_1 \pi\) | the number of ascents of \(\pi\) at odd positions |
| \(\text{inv} \pi\) | the number of inversions of \(\pi\) |
| \(\text{lpk}(\pi)\) | the number of left peaks of \(\pi\), see (2.14) |

**Table 1.** Main statistics of \(\pi \in S_n\)

### 2. Symmetric and Positive Expansions of Bi-Eulerian Polynomials

Define two families of bi-Eulerian polynomials \((\tilde{A}_n(x, y))_{n \geq 1}\) and \((\bar{A}_n(x, y))_{n \geq 1}\) by

\[
\tilde{A}_{2n}(x, y) = (1 + y)A_{2n}(x, y, 1), \quad \tilde{A}_{2n-1}(x, y) = A_{2n-1}(x, y, 1), \quad (2.1a)
\]

\[
\bar{A}_{2n}(x, y) = (1 + y)\tilde{A}_{2n}(x, y, 1), \quad \bar{A}_{2n-1}(x, y) = \tilde{A}_{2n-1}(x, y, 1); \quad (2.1b)
\]

Define two families of bi-Eulerian polynomials \((\tilde{A}_n(x, y))_{n \geq 1}\) and \((\bar{A}_n(x, y))_{n \geq 1}\) by

\[
\tilde{A}_{2n}(x, y) = (1 + y)A_{2n}(x, y, 1), \quad \tilde{A}_{2n-1}(x, y) = A_{2n-1}(x, y, 1), \quad (2.1a)
\]

\[
\bar{A}_{2n}(x, y) = (1 + y)\tilde{A}_{2n}(x, y, 1), \quad \bar{A}_{2n-1}(x, y) = \tilde{A}_{2n-1}(x, y, 1); \quad (2.1b)
\]
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and their type B analogues \( (\tilde{B}_n(x,y))_{n \geq 1} \) and \( (\overline{B}_n(x,y))_{n \geq 1} \) by

\[
\begin{align*}
\tilde{B}_{2n}(x,y) &= B_{2n}(x,y), & \tilde{B}_{2n-1}(x,y) &= (1+y)^{-1}B_{2n-1}(x,y), \\
\overline{B}_{2n}(x,y) &= \tilde{B}_{2n}(x,y), & \overline{B}_{2n-1}(x,y) &= (1+y)^{-1}\tilde{B}_{2n-1}(x,y).
\end{align*}
\]

(2.2a)

(2.2b)

By (1.6a) and (1.6b) (resp. Theorem 1.2 and Theorem 1.3) both polynomials \( \tilde{A}_n(x,y) \) and \( A_n(x,y) \) (resp. \( \tilde{B}_n(x,y) \) and \( B_n(x,y) \)) are symmetric in \( x \) and \( y \).

Recall that a polynomial with real coefficients \( P(x) = \sum_{i=0}^{n} a_i x^i \) is \textit{gamma-positive} (resp. \textit{semi-gamma-positive}) if there are nonnegative numbers \( \gamma_i \) such that \( P(x) = \sum_i \gamma_i x^i (1 + x)^{n-2i} \) (resp. \( P(x) = (1 + x)^{\nu} \sum_i \gamma_i x^i (1 + x^2)^{|n/2|-i} \) with \( \nu = 0 \) or \( 1 \)), see [28] and [22] respectively. It is known that the gamma-positivity is stronger than the semi-gamma-positivity [22].

In this section, we shall first derive the semi-gamma-positive formulae for the bi-Eulerian polynomials \( \tilde{A}_n(x,y), \tilde{A}_n(x,y), \tilde{B}_n(x,y) \) and \( \overline{B}_n(x,y) \) from their generating functions and then apply Hetyei-Reiner’s min-max tree model [17] for permutations to derive the corresponding \( \gamma \)-positive formulae for \( \tilde{A}_n(x,y) \) and \( \tilde{A}_n(x,y) \) as well as their type B analogues by refining Petersen’s proof for the \( \gamma \)-positivity of type B Eulerian polynomials [28].

2.1. Semi-gamma-positivity of bi-Eulerian polynomials. The following generalizes the semi-gamma-positivity of Eulerian polynomials to bi-Eulerian polynomials.

**Theorem 2.1.** Let \( a(n,j) \) (resp. \( \bar{a}(n,j) \)) be the number of permutations in \( S_n \) with \( j \) odd descents and without even descents (resp. ascents) for \( n \geq 1 \) and \( 0 \leq 2j \leq n \). Then

\[
\begin{align*}
\tilde{A}_n(x,y) &= \sum_{j=0}^{\lfloor n/2 \rfloor} a(n,j) (x+y)^j (1+xy)^{\lfloor n/2 \rfloor -j}; \\
\overline{A}_n(x,y) &= \sum_{j=0}^{\lfloor n/2 \rfloor} \bar{a}(n,j) (x+y)^j (1+xy)^{\lfloor n/2 \rfloor -j},
\end{align*}
\]

(2.3a)

(2.3b)

and

\[
\bar{a}(n,j) = a(n, \lfloor n/2 \rfloor - j) \quad \text{for} \quad 0 \leq j \leq \lfloor n/2 \rfloor.
\]

(2.3c)

**Proof.** Let \( \alpha(x,y) = (1-x)(1-y) \). Then

\[
\alpha(x,y) = (1+xy) \cdot \alpha \left( \frac{x+y}{1+xy}, 0 \right).
\]
It follows from (1.6) that
\[ \tilde{A}_n(x, y) = (1 + xy)^{\lfloor n/2 \rfloor} A_n \left( \frac{x + y}{1 + xy}, 0, 1 \right), \] (2.4a)
\[ \overline{A}_n(x, y) = (1 + xy)^{\lfloor n/2 \rfloor} \tilde{A}_n \left( \frac{x + y}{1 + xy}, 0, 1 \right), \] (2.4b)
which are obviously equivalent to (2.3a) and (2.3b), respectively.

Define the completion \( \sigma^c \) of \( \sigma \in S_n \) by \( \sigma^c(i) = n + 1 - \sigma(i) \) for \( 1 \leq i \leq n \). It is clear that the mapping \( \varphi : \sigma \mapsto \sigma^c \) is an involution on \( S_n \) and satisfies \( \text{des}_i \sigma = \text{asc}_i \sigma^c \) for \( i \in \{0, 1\} \).

Thus
\[ (\text{des}_1 \sigma^c, \text{asc}_0 \sigma^c) = (\text{asc}_1 \sigma, \text{des}_0 \sigma) = (|n/2| - \text{des}_1 \sigma, \text{des}_0 \sigma). \]

Eq. (2.3c) follows by restricting \( \varphi \) on the set of permutations in \( S_n \) with \( j \) odd descents and without even descent. \( \square \)

**Remark 5.** The combinatorial interpretation of \( a_{n,j} \) actually follows from the existence of formula (2.3a), which was first conjectured by Sun [32] and then proved by Sun and Zhai [34].

Similarly, we have the following B-analogue of Theorem 2.1.

**Theorem 2.2.** Let \( b(n, j) \) (resp. \( \bar{b}(n, j) \)) be the number of permutations in \( B_n \) with \( j \) odd descents and without even descents (resp. even ascents). Then
\[ \tilde{B}_n(x, y) = \sum_{j=0}^{\lfloor n/2 \rfloor} b(n, j) \left( x + y \right)^j (1 + xy)^{\lfloor n/2 \rfloor - j}, \] (2.5a)
\[ \overline{B}_n(x, y) = \sum_{j=0}^{\lfloor n/2 \rfloor} \bar{b}(n, j) \left( x + y \right)^j (1 + xy)^{\lfloor n/2 \rfloor - j}, \] (2.5b)
and
\[ \bar{b}(n, j) = b(n, \lfloor n/2 \rfloor - j) \quad \text{for} \quad 0 \leq j \leq \lfloor n/2 \rfloor. \] (2.5c)

**Proof.** Let \( \alpha(x, y) = (1 - x)(1 - y) \). Then
\[ \alpha(x, y) = (1 + xy) \cdot \alpha((x + y)/(1 + xy), 0). \]

We derive from Theorem 1.2 and Theorem 1.3 immediately
\[ \tilde{B}_n(x, y) = (1 + xy)^{\lfloor n/2 \rfloor} B_n \left( \frac{x + y}{1 + xy}, 0 \right), \] (2.6a)
\[ \overline{B}_n(x, y) = (1 + xy)^{\lfloor n/2 \rfloor} \tilde{B}_n \left( \frac{x + y}{1 + xy}, 0 \right), \] (2.6b)
which are what (2.5a) and (2.5b) mean.

Consider the negation $\bar{\sigma}$ of $\sigma \in B_n$ by $\bar{\sigma}(i) = -\sigma(i)$ for $1 \leq i \leq n$. It is clear that the mapping $\phi : \sigma \mapsto \bar{\sigma}$ is an involution on $S_n$ and satisfies $\text{des}_i \sigma = \text{asc}_i \bar{\sigma}$ for $i \in \{0, 1\}$. Thus

$$(\text{des}_1 \bar{\sigma}, \text{asc}_0 \bar{\sigma}) = (\text{asc}_1 \sigma, \text{des}_0 \sigma) = (\lfloor n/2 \rfloor - \text{des}_1 \sigma, \text{des}_0 \sigma).$$

Eq. (2.5c) follows by restricting $\phi$ on the set of permutations in $B_n$ with $j$ odd descents and without even descent. □

We note that Eq. (2.3a) does not directly reduce to the known $\gamma$-positivity formula of Eulerian polynomials $A_n(x, x)$ when $x = y$. To derive the latter expansion we shall appeal to the min-max tree representations of permutations due to Hetyei and Reiner [17]. Similarly, to derive gamma-positivity formula of type B Eulerian polynomials $B_n(x, x)$ from Theorem 2.2 we shall appeal to an action on permutations due to Petersen [28].

2.2. Gamma-positivity of bi-Eulerian polynomials of type A. We define the min-max tree $M(w)$ associated to a sequence of distincte integers $w = w_1 \ldots w_n$ as follows.

(1) First, $M(w)$ is a binary tree with vertices labelled $w_1, \ldots, w_n$. Let $i$ be the least integer for which either $w_i = \min \{w_1, w_2, \ldots, w_n\}$ or $w_i = \max \{w_1, w_2, \ldots, w_n\}$. Define $w_i$ to be the root of $M(w)$.

(2) Then recursively define $M(w_1, \ldots, w_{i-1})$ and $M(w_{i+1}, \ldots, w_n)$ to be the left and right subtree of $w_i$, respectively.

Conversely, the left-first order reading of the tree $M(w)$ yields the sequence $w$, see [17, 10] and [31, pp. 57-61].

An interior vertex in $M(w)$ is called a min (resp. max) vertex if it is the minimum (resp. maximum) label among all its descendants. Let $M(w_i)$ (resp. $M_l(w_i), M_r(w_i)$) denote the subtree (resp. the left subtree, the right subtree) of $M(w)$ with root $w_i$.
For $1 \leq i \leq n$, we define the operator $\psi_i$ permuting the labels of $M(w)$ as in the following.

1. If $w_i$ is a min vertex, then replace $w_i$ by the largest element of $M_r(w_i)$, permute the remaining elements of $M_r(w_i)$ such that they keep their same relative orders and all other vertices in $M(w)$ are fixed.

2. If $w_i$ is a max vertex, then replace $w_i$ by the smallest element of $M_r(w_i)$ such that they keep their same relative order, and all other vertices in $M(w)$ are fixed.

An illustration of operator $\psi_2$ is given in Figure 1.

Given a permutation $\pi = \pi(1)\pi(2)\ldots\pi(n)$ of $Y = \{y_1, y_2, \ldots, y_n\}$, which is a set of positive integers. The $\pi(i)$-factorization of $\pi$ is the sequence $(w_1, w_2, \pi(i), w_4, w_5)$, $1 \leq i \leq n$, where

1. the concatenation product $w_1w_2\pi(i)w_4w_5$ is equal to $\pi$;
2. $w_2$ is the longest right factor of $\pi(1)\pi(2)\ldots\pi(i-1)$, all letters of which are greater than $\pi(i)$;
3. $w_4$ is the longest left factor of $\pi(i+1)\pi(i+2)\ldots\pi(n)$, all letters of which are greater than $\pi(i)$.

Note that above any of $w_1$, $w_2$, $w_4$ or $w_5$ may be empty.

**Definition 2.1** (see [13, 10]). A permutation $\pi \in \mathfrak{S}_n$ is an André permutation (of kind I) if $\pi$ has no double descents and ends with ascent, i.e., $\pi(n-1) < \pi(n)$, and if $i \in \{2, \ldots, n\}$ is a valley of $\pi$ and $(w_1, w_2, \pi(i), w_4, w_5)$ is the $\pi(i)$-factorization of $\pi$, then the maximum letter of $w_2w_4$ is in $w_4$.

For example, the André permutations of length 4 are 1234, 1324, 2314, 2134 and 3124.

**Fact 2.2.** The operators $\psi_i$ are commuting involutions acting on $M(w)$ and generate an abelian group $G_w$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{l(w)}$, where $l(w)$ is the number of internal vertices of $M(w)$. Those $\psi_i$ for which $w_i$ is an internal vertex are a minimal set $S_w$ of generators for $G_w$. For any subset $S \subseteq G_w$ we define the HR action $\psi_S$ by $\psi_S(M(w)) = \prod_{i \in S} \psi_i(M(w))$.

For $\pi \in \mathfrak{S}_n$, let $\text{Orb}(\pi)$ be the set of permutations $w$ such that $M(w)$ is in the orbit of $M(\pi)$ under the HR-action. Thus, for any $\pi \in \mathfrak{S}_n$, there is a unique permutation $\pi^A$ in $\text{Orb}(\pi)$ such that all its interior vertices in $M(\pi^A)$ are min vertices.

**Fact 2.3.** A permutation $\pi \in \mathfrak{S}_n$ is an André permutation if and only if all interior vertices of min-max tree $M(\pi)$ are min vertices.

It follows that $\cup_{\pi \in \text{And}_n} \text{Orb}(\pi) = \mathfrak{S}_n$, where $\text{And}_n$ is the set of André permutations in $\mathfrak{S}_n$. Let $\mathfrak{S}_n^*$ (resp. $\text{Orb}^*(\pi)$) be the subset of permutations in $\mathfrak{S}_n$ (resp. $\text{Orb}(\pi)$) which have no even descents. By restriction on the permutations which have only odd-descents.
we have

\[ \mathcal{G}_n^* = \bigcup_{\pi \in \text{And}_n} \text{Orb}^*(\pi). \tag{2.7} \]

For any subset \( S \subseteq [n] \) and André permutation \( \pi \), since all interior vertices of \( M(\pi) \) are min vertices, we have \( \text{des}_S(M(\pi)) \geq \text{des} \pi \).

Let \( \pi \in \text{And}_n \). So all the interior vertices of \( M(\pi) \) are min vertices, if \( \pi(k) \) is a valley of \( \pi \) with the \( \pi(k) \)-factorization \( (w_1, w_2, \pi(k), w_4, w_5) \), then the position of the last letter of \( w_2 \) is a descent position, and the HR action \( \psi_k \) on \( M(\pi) \) will shift the descent position \( k - 1 \) to \( k \), since the vertex in \( M(\pi) \) corresponding to \( \pi(k) \) will be relabelled by the largest letter of its subtree, and all other vertices keep their same relative order. Thus, the HR action \( \psi_S \) with \( S \) being the set of indices of odd-valley-positions in \( \pi \) will evacuate all the even descent positions, and the total number of descents will remain the same, let \( \psi_S(\pi) = \pi' \), clearly \( \pi' \in \text{Orb}^*(\pi) \).

**Fact 2.4.** For \( \pi \in \text{And}_n \) we have

\[ \text{Orb}^*(\pi) = \text{Orb}^*(\pi') = \prod_{i \in S} (1 + \psi_i)\{\pi'\}, \]

where \( S \) is the set of odd ascent positions of \( \pi' \). Moreover, as \( \text{des}_0 \psi_i(\pi') = \text{des}_0(\pi') + 1 \), the following identity holds

\[ \sum_{\sigma \in \text{Orb}^*(\pi)} p^{\text{des} \sigma} = (1 + p)^{[n/2] - \text{des} \pi} P^{\text{des} \pi}. \tag{2.8} \]

Recall that \( a(n, j) \) is the number of permutations in \( \mathcal{G}_n \) with \( j \) odd descents and without even descents.

**Lemma 2.5.** Let \( d(n, j) \) be the number of André permutations in \( \mathcal{G}_n \) with \( j \) descents for \( 0 \leq 2j \leq n \). Then

\[ a(n, j) = \sum_{i=0}^{j} \binom{[n/2] - i}{j - i} d(n, i). \tag{2.9} \]

**Proof.** Applying the above facts

\[ \sum_{\sigma \in \mathcal{G}_n^*} p^{\text{des} \sigma} = \sum_{\pi \in \text{And}_n} \sum_{\sigma \in \text{Orb}^*(\pi)} p^{\text{des} \sigma} \]

\[ = \sum_{\pi \in \text{And}_n} (1 + p)^{[n/2] - \text{des} \pi} P^{\text{des} \pi}. \]

We derive (2.9) by extracting the coefficient of \( p^j \). \( \square \)
Lemma 2.6. If $\bar{d}(n, i)$ is the number of min-max trees on $[n]$ having $i$ max interior vertices with two children, then

$$\bar{d}(n, i) = \sum_{j=i}^{\lfloor \frac{n}{2} \rfloor} \binom{j}{i} d(n, j). \tag{2.10}$$

Proof. If $\pi \in S_n$ is an André permutation, then the number of interior vertices with two children of $M(\pi)$ equals $\text{des}(\pi)$. Any permutation $\pi \in S_n$ such that $M(\pi)$ has $i$ max interior vertices with two children can be obtained from the André permutation $\pi^A$ in $\text{Orb}(\pi)$ by choosing $i$ interior vertices with two children among the interior vertices with two children of $M(\pi^A)$ and then applying HR operator on these $i$ vertices (to transform them into max vertices). Hence, in each orbite of an André min-max tree (i.e., the tree $M(w)$ associated to an André permutation $w$) with $j$ interior vertices having two children, there are $\binom{j}{i}$ min-max trees on $[n]$ having $i$ max interior vertices with two children. The result follows by summing over all the orbits. \qed

Recall that a permutation $w$ of $[n]$ is an André permutation of kind II if, for $1 \leq k \leq n$,

1. the subsequence of the smallest $k$ elements in $w$ has no double descent;
2. the subsequence of the smallest $k$ elements in $w$ ends with an ascent.

The permutation $w$ is called Simsun if it satisfies condition (1), [8, 10, 27]. For example, the five Simsun 3-permutations are: 231, 132, 312, 123, 213 and the five André 4-permutations of the second kind are: 1234, 1423, 3124, 3412, 4123.

Actually, the number of $n$-André permutations and that of $(n - 1)$-simsun permutations are both equal to the Euler number $E_n$, which can be defined by

$$\sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = \sec x + \tan x.$$  

Let $D_n(x)$ (resp. $rs_n(x)$) be the descent polynomial of André permutations (resp. Simsun permutations) of length $n$. By means of generating function argument, Chow and Shiu [8] proved that the descent number is equidistributed over $(n - 1)$-simsun permutations and $n$-André permutations, i.e.,

$$D_n(x) = rs_{n-1}(x) = \sum_{i=0}^{n-1} d(n, i)x^i \quad (n \geq 2) \tag{2.11}$$

with $D_1(x) = 1$.

Combining Theorem 2.1 and Lemma 2.5 we obtain an alternative proof of the following result of H. Sun [33].
Theorem 2.3. Let $d(n,j)$ be the number of André permutations in $S_n$ with $j$ descents for $0 \leq 2j \leq n$ and $\bar{d}(n,i)$ be the number of min-max trees on $n$ vertices having $i$ max interior vertices with two children, then

$$\tilde{A}_n(x,y) = \sum_{j=0}^{\lfloor n/2 \rfloor} d(n,j)(x+y)^j(1+x+y+xy)^{\lfloor \frac{n}{2} \rfloor-j},$$  \hfill (2.12a)$$

$$\tilde{T}_n(x,y) = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \bar{d}(n,i)(x+y)^i(1+x+y+xy)^{\lfloor n/2 \rfloor-i},$$  \hfill (2.12b)$$

and $rs_{n-1}(1+x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \tilde{d}(n,i)x^i$, where $rs_{n-1}(x)$ is the descent polynomial of Simsun permutations.

Proof. Plugging (2.9) in (2.3a) we obtain

$$\tilde{A}_n(x,y) = \sum_{j=0}^{\lfloor n/2 \rfloor} \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{\lfloor n/2 \rfloor - i}{j-i} d(n,i)(x+y)^i(1+x+y+xy)^{\lfloor \frac{n}{2} \rfloor-i-j}$$

$$= \sum_{i=0}^{\lfloor n/2 \rfloor} d(n,i)(x+y)^i \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{\lfloor n/2 \rfloor - i}{j} (x+y)^j(1+x+y+xy)^{\lfloor n/2 \rfloor-i-j}$$

$$= \sum_{i=0}^{\lfloor n/2 \rfloor} d(n,i)(x+y)^i(1+x+y+xy)^{\lfloor n/2 \rfloor-i},$$

which is the right-hand side of (2.12a) upon replacing $i$ by $j$.

By (1.8) and (2.1) we have $\tilde{T}_n(x,y) = y^{\lfloor n/2 \rfloor} \tilde{A}_n(x,1/y)$. Hence

$$\tilde{T}_n(x,y) = \sum_{j=0}^{\lfloor n/2 \rfloor} d(n,j)(1+x+y+xy)^{\lfloor \frac{n}{2} \rfloor-j}.$$  \hfill (2.12b)$$

Now, rewriting $(1+xy)^j$ in the last sum as

$$(1+xy)^j = \sum_{i=0}^{j} (-1)^i \binom{j}{i} (1+x+y+xy)^{j-i}(x+y)^i,$$

we obtain the right-hand side of (2.12b). \hfill \square

Remark 6. Comparing (2.13) with [27, Theorem 2] we notice that $d(n,j)$ is also the number of André permutations of kind II of $[n]$ with $j$ descents. This result is implicit in [12, 13, 10]. If $x = y$, Theorem 2.3 plainly reduces to the classical $\gamma$-formula of Eulerian
polynomials, see [27, Theorem 1],

\[ A_n(x, x) = \sum_{j=0}^{[n/2]} d(n, j) 2^j x^j (1 + x)^{n-1-2j}. \]  

(2.13)

Also, Lin et al. [20, Theorem 1.1] proved the \( x = y \) case of (2.12b).

2.3. Gamma-positivity of bi-Eulerian polynomials of type B. We refine Petersen’s proof of gamma-nonnegativity of type B Eulerian polynomials in [28].

Given a permutation \( u \in S_n \) we denote by \( B(u) \) the set of all permutations \( \omega \in B(u) \) such that \( \omega(i) = \sigma_i u(i) \) with \( \sigma_i \in \{-, +\} \) for \( 1 \leq i \leq n \). Then we have the following observations:

- if \( u(i-1) < u(i) \), then \( \omega(i-1) > \omega(i) \) if and only if \( \sigma_i = - \),
- if \( u(i-1) > u(i) \), then \( \omega(i-1) > \omega(i) \) if and only if \( \sigma_{i-1} = + \).

To put it another way, the sign \( \sigma_j \) controls the descent in position \( j-1 \) if and only if \( j-1 \) is not a descent position of \( u \), and it controls the descent in position \( j \) if and only if \( j \) is a descent position of \( u \).

Consider the example of \( u = 31472865 \). Then there is a descent in position 0 if and only if \( \sigma_1 = - \) while there is a descent in position 1 if and only if \( \sigma_1 = + \). Since \( u(2) = 1 > u(1) = 3 \), the sign \( \sigma_2 \) has no effect whatever on the descent set. With \( u(3) = 4 \), we find that \( \omega(2) > \omega(3) \) if and only if \( \sigma_3 = - \), but that \( \sigma_3 \) does not control whether \( \omega(3) \) is greater than \( \omega(4) \) (\( \sigma_4 \) does that). By considering the sign of each letter in turn.

We summarize the above consideration more precisely in the following

**Observation 2.7.** Let \( u \in S_n \). If \( \omega \in B_n(u) \) with \( \omega(j) = \sigma_j u(j) \), then

- If \( u(j-1) < u(j) > u(j+1) \), then \( \sigma_j \) controls both the descent in position \( j-1 \) and position \( j \). That is, if \( \sigma_j = + \), then in \( \omega, j-1 \) is not a descent position, but \( j \) is a descent position. If \( \sigma_j = - \), then in \( \omega, j-1 \) is a descent position but \( j \) is not. This means, \( \sigma_j \) does not change the number of descents, but it controls the parity of descent position.
- If \( u(j-1) < u(j) < u(j+1) \), then \( \sigma_j \) controls the descent on position \( j-1 \), but no effect on position \( j \). That is, if \( \sigma_j = + \), then \( j-1 \) is not a descent position, if \( \sigma_j = - \), then \( j-1 \) is a descent position.
- If \( u(j-1) > u(j) > u(j+1) \), then \( \sigma_j \) controls the descent on position \( j \), but no effect on position \( j-1 \). That is, if \( \sigma_j = + \), then \( j \) is a descent position, if \( \sigma_j = - \), then \( j \) is not a descent position.
- If \( u(j-1) > u(j) < u(j+1) \), then \( \sigma_j \) has no effect on the descent set.
The number of left peaks of permutation \( u \in \mathfrak{S}_n \) is defined by
\[
\text{lpk}(u) = |\{1 \leq i < n : u(i-1) < u(i) > u(i+1)\}|,
\]
where \( u(0) = 0, u(n+1) = n + 1 \).

**Lemma 2.8.** If \( \omega \in \mathcal{B}_n \) is a permutation with \( j \) odd descents and without even descents, then \( |\omega| \) is a permutation in \( \mathfrak{S}_n \) with \( \text{lpk}(|\omega|) \leq j \).

**Proof.** Since \( \omega \) does not have descents on even positions, we have \( \omega(1) > 0 \) and \( \omega \) does not have double descents. Suppose \( \omega(i) \) is the first valley with \( \sigma_i = - \) and \( \omega(k) \) is the peak closest to \( \omega(i) \) on the right. Then \( \omega(i)\omega(i+1)\ldots\omega(k) \) is an increasing subsequence, and there has no peak in \( |\omega(i)||\omega(i+1)|\ldots|\omega(k)| \). Let \( \omega_0 = \omega(1)\omega(2)\ldots\omega(i-1)|\omega(i)||\omega(i+1)|\ldots|\omega(k)||\omega(k+1)|\ldots|\omega(n) \) then, the difference of peak sets of \( \omega_0 \) and \( \omega \) happens on \( \omega(i-1), |\omega(i)| \) and \( |\omega(k)|, \omega(k+1) \). As it is not possible that both \( \omega(i-1) \) and \( |\omega(i)| \) are peaks in \( \omega_0 \) (but \( \omega(i-1) \) is a peak in \( \omega \)). Since \( |\omega(k)| \geq \omega(k) > \omega(k+1) \), so \( |\omega(k)| \) is the only possible peak candidate of \( |\omega(k)| \) and \( \omega(k+1) \) in \( \omega_0 \) (\( \omega(k) \) is a peak in \( \omega \)). In summary, we have \( \text{lpk}(\omega_0) \leq \text{lpk}(\omega) \). We repeat this process on \( \omega_0 \), finally, we obtain \( \text{lpk}(|\omega|) \leq \text{lpk}(\omega) = j \). \( \square \)

**Lemma 2.9.** Let \( g(n, i) = |\{ u \in \mathfrak{S}_n : \text{lpk}(u) = i \}|. \) Then
\[
b(n, j) = \sum_{i=0}^{j} \binom{\lfloor n/2 \rfloor - i}{j-i} g(n, i) 2^i.
\]

**Proof.** Let \( u \) be a permutation in \( \mathfrak{S}_n \) with \( \text{lpk}(u) = i \leq j \). We can use the following process to transform it to a permutation of \( \mathcal{B}_n \) with \( j \) odd descent and without even descents.

**Process A**

1. Firstly, we sign the \( i \) valleys of \( u \) with either \( - \) or \( + \), which gives \( \omega_1 \).
2. Secondly, in \( \omega_1 \), we sign the peaks at even positions with \( - \), then we obtain \( \omega_2 \) with all the peaks at odd positions (by Remark 2.7).
3. Thirdly, choose a \( j - i \) elements subset \( D \) of \( C := \{1, 3, \ldots, 2\lfloor \frac{n}{2} \rfloor - 1\} \setminus \text{LPK}(\omega_2) \), where \( \text{LPK}(\omega_2) \) is the position set of peaks of \( \omega_2 \). For \( l \in D \), if \( \omega_2(l) \) is a descent then we do nothing with \( \omega_2(l) \), if \( \omega_2(l) \) is an ascent then we sign \( \omega_2(l+1) \) (it must be a double ascent in \( u \)) with \( - \). For \( l \notin D \) but \( l \in C \), if \( \omega_2(l) \) is a descent then we sign \( \omega_2(l) \) (it must be a double descent in \( u \)) with \( - \), if \( \omega_2(l) \) is an ascent, then we do nothing with \( \omega_2(l) \), which gives \( \omega_3 \).
4. Lastly, in \( \omega_3 \) we sign all the double descents at even positions with \( - \), which gives \( \omega_4 \).

By Observation 2.7, we see that \( \omega_4 \) is a permutation in \( \mathcal{B}_n \) with \( j \) odd descents and without even descents.
In this process, no letter in \( u \) is repeatedly signed. And we can see that for a fixed \( u \in \mathfrak{S}_n \) with \( i \) peaks, by Process A, it can produce \( \binom{n/2-i}{j-i} \cdot 2^i \) different permutations in \( \mathcal{B}_n \) with \( j \) odd descents and without even descents. By Lemma 2.8, for \( \omega \in \mathcal{B}_n \) with \( j \) odd descents and without even descents, we have \(|\text{lpk}(\omega)| \leq j \) and by Remark 2.7, the descent positions in \( \omega \) are totally controlled by the signs of peaks, double descents and double ascents of \(|\omega|\), that is \( \omega \) can be constructed by \(|\omega|\) through Process A. This completes the proof. \( \square \)

**Theorem 2.4.** Let \( g(n, j) = |\{ u \in \mathfrak{S}_n : \text{lpk}(u) = j \}| \). Then

\[
\tilde{B}_n(x, y) = \sum_{j=0}^{\lfloor n/2 \rfloor - j} g(n, j) 2^j (x + y)^j (1 + x + y + xy)^{[n/2]-j}, \tag{2.15}
\]

\[
\mathcal{B}_n(x, y) = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \bar{g}(n, j) 2^j (x + y)^j (1 + x + y + xy)^{[n/2]-j} \tag{2.16}
\]

with

\[
\bar{g}(n, j) = \sum_{i=0}^{\lfloor n/2 \rfloor - j} \binom{i+j}{j} g(n, i+j) 2^i. \tag{2.17}
\]

**Proof.** By Theorem 2.2 and Lemma 2.9, we obtain (2.15). To prove (2.16), by (1.15), (2.2a) and (2.2b), we first note

\[
\mathcal{B}_n(x, y) = y^{\lfloor n/2 \rfloor} \tilde{B}_n(x, 1/y).
\]

It follows from (2.15) that

\[
\mathcal{B}_n(x, y) = \sum_{j=0}^{\lfloor n/2 \rfloor} g(n, j) 2^j (1 + xy)^j (1 + x + y + xy)^{[n/2]-j}. \tag{2.18}
\]

The rest of the proof is the same as that of Eq. (2.12b), so it is omitted. \( \square \)

**Remark 7.** When \( x = y \) identity (2.18) reduces to Proposition 10 in [23]. Identity (2.17) is equivalent to the polynomial identity:

\[
\sum_{j=0}^{\lfloor n/2 \rfloor} \bar{g}(n, j) x^j = \sum_{j=0}^{\lfloor n/2 \rfloor} \sum_{i=0}^{\lfloor n/2 \rfloor - j} \binom{i+j}{j} g(n, i+j) 2^i x^j = \sum_{k=0}^{\lfloor n/2 \rfloor} g(n, k) (2 + x)^k. \tag{2.19}
\]
If \( x = y \) identity (2.15) reduces to Petersen’s formula for type B Eulerian polynomial \( B_n(x, x) \), see \([23]\) Theorem 13.5,
\[
B_n(x, x) = \sum_{j=0}^{\lfloor n/2 \rfloor} g(n, j) (4x)^j (1 + x)^{n-2j}, \tag{2.20}
\]
and Eq. (2.16) reduces to Ma et al.’s formula for type B alternating descent polynomials, see \([23]\) Theorem 12
\[
\hat{B}_n(x, x) = \sum_{j=0}^{\lfloor n/2 \rfloor} \tilde{g}(n, j) (-4x)^j (1 + x)^{n-2j}. \tag{2.21}
\]

3. Counting permutations of type A by the parity of descent positions

If \( \sigma = \sigma_1 \cdots \sigma_n \) is a permutation in \( \mathfrak{S}_n \), the descent set \( \text{Des}(\sigma) \) of \( \sigma \) is \( \text{Des}(\sigma) = \{ i : \sigma_i > \sigma_{i+1} \} \subseteq [n-1] \). We denote by \( \text{Des}_0(\sigma) \) (resp. \( \text{Des}_1(\sigma) \)) the set of even (resp. odd) descents of \( \sigma \). For brevity we denote their cardinalities by \( \text{des}_0(\sigma) = |\text{Des}_0(\sigma)| \) and \( \text{des}_1(\sigma) = |\text{Des}_1(\sigma)| \).

Any subset \( S = \{s_1, \ldots, s_k\} \subseteq [n-1] \) can be encoded by the composition \( \text{co}(S) := (s_1, s_2 - s_1, \ldots, s_k - s_{k-1}, n - s_k) \) of \( n \). Clearly this correspondence is a bijection. For any composition \( \lambda = (\lambda_1, \ldots, \lambda_l) \) of \( n \), let \( S_\lambda \) be the subset \( \{\lambda_1, \lambda_1 + \lambda_2, \ldots, \lambda_1 + \cdots + \lambda_{l-1}\} \) of \([n-1]\) and define the \( q \)-multinomial coefficient
\[
\binom{n}{\lambda}_q := \binom{n}{\text{co}(S_\lambda)} \frac{n!_q}{\lambda_1!_q \cdots \lambda_l!_q}.
\]

For any subset \( S \subseteq [n-1] \), let \( \Delta_n(S) := \{ \sigma \in \mathfrak{S}_n \mid \text{Des}(\sigma) \subseteq S \} \) and \( R_n(S) \) be the set of rearrangements of word \( 1^{\lambda_1} \cdots l^{\lambda_l} \), where \( \lambda_i = s_i - s_{i-1} \) for \( i \in [l] \) with \( l = k + 1, s_0 = 0 \) and \( s_l = n \). There is a bijection \( \psi : \sigma \mapsto w \) from \( \Delta_n(S) \) to \( R_n(S) \) defined by \( w(j) = i \) if \( \sigma(j) \in \{\sigma(s_{i-1} + 1), \ldots, \sigma(s_i)\} \) for \( j \in [n] \) and \( i \in [l] \). Clearly the number of inversions of \( w \), i.e., \( |\{i < j \mid w(i) > w(j), i, j \in [n]\}| \), is equal to \( \text{inv}(\sigma) \). By a theorem of MacMahon (see \([2]\) p. 41) we obtain the following known result (see \([31]\) p. 227)).

**Lemma 3.1.** Let \( S = \{s_1, s_2, \ldots, s_k\} \subseteq [n-1] \) and \( \alpha_n(S, q) = \sum_{\sigma \in \Delta_n(S)} q^{\text{inv}(\sigma)} \). Then
\[
\alpha_n(S, q) = \binom{n}{\text{co}(S)}_q.
\]

To prove (1.6a) we need three more lemmas. For convenience, for any subset \( S \subseteq \mathbb{N} \) let \( S_\text{e} = S \cap 2\mathbb{N} \) and \( S_\text{o} = S \cap (2\mathbb{N} + 1) \) be the subsets of even and odd integers of \( S \), respectively. For \( n \in \mathbb{N} \), let \( \text{O}[n] \) (resp. \( \text{E}[n] \)) be the collection of odd (resp. even)}
elements of \([n]\). Consider the polynomial

\[ P_n(x, y, q) := \sum_{S \subseteq [n-1]} \alpha_n(S, q)x^{|S_o|}y^{|S_e|}. \]  

(3.1)

**Lemma 3.2.** For \(n \geq 1\) we have

\[ A_n(x, y, q) = (1 - x)^{\frac{n}{2}}(1 - y)^{\frac{n-1}{2}}P_n\left(\frac{x}{1-x}, \frac{y}{1-y}, q\right). \]  

(3.2)

**Proof.** By Lemma 2.1 we have

\[ P_n(x, y, q) = \sum_{\sigma \in S_n} x^{\text{des}_1(\sigma)}y^{\text{des}_0(\sigma)}q^{\text{inv}(\sigma)} \sum_{S \subseteq [n-1]\setminus D(\sigma)} x^{|S_o|}y^{|S_e|}. \]

as there are \(\lfloor \frac{n}{2} \rfloor\) – \(\text{des}_1(\sigma)\) odd (resp. \(\lfloor \frac{n-1}{2} \rfloor\) – \(\text{des}_0(\sigma)\) even) integers in \([n-1]\setminus D(\sigma)\). In other words, we can write

\[ P_n(x, y, q) = (1 + x)^{\frac{n}{2}}(1 + y)^{\frac{n-1}{2}}A_n\left(\frac{x}{1+x}, \frac{y}{1+y}, q\right), \]

which is equivalent to (3.2). \(\square\)

**Remark 8.** Let \(P_n(x) = \sum_{S \subseteq [n-1]} \alpha_n(S, 1)x^{|S|}\). It is not difficul to see that

\[ P_n(x) = \sum_{k=0}^{n-1} (k + 1)!S(n, k + 1)x^k, \]

where \(S(n, k)\) denotes the Stirling number of the second kind, i.e., the number of ways to partition a set of \(n\) objects into \(k\) non-empty subsets (see [31]). So, when \(x = y\), formula (3.2) reduces to the Frobenius formula, see [11],

\[ A_n(x) = \sum_{k=1}^{n} k!S(n, k)x^{k-1}(1 - x)^{n-k}. \]  

(3.3)

**Lemma 3.3.** We have

\[ B(t, x) := \sum_{n \geq 1} P_{2n}(x, 0, q)\frac{t^{2n}}{(2n)!_q} = \frac{(\cosh_q t - 1)(1 - x(\cosh_q t - 1)) + x \sinh_q^2 t}{1 - x(\cosh_q t - 1)}, \]  

(3.4)

\[ C(t, x) := \sum_{n \geq 1} P_{2n-1}(x, 0, q)\frac{t^{2n-1}}{(2n - 1)!_q} = \frac{\sinh_q t}{1 - x(\cosh_q t - 1)}. \]  

(3.5)
Proof. There is a bijection between the set of compositions \( \gamma = (\gamma_1, \cdots, \gamma_l) \) of \( 2n \) such that \( \gamma_1, \gamma_1 + \gamma_2, \ldots, \gamma_1 + \gamma_2 + \cdots + \gamma_{l-1} \) are odd numbers and the set of subsets \( S_\gamma \) of \( O[2n] \). Hence

\[
\sum_{n \geq 1} P_{2n}(x,0,q) \frac{t^{2n}}{(2n)!q} = \sum_{n \geq 1} \left( \sum_{S \subseteq [2n]} \alpha_{2n}(S,q)x^{|S|} \right) \frac{t^{2n}}{(2n)!q} \\
= \sum_{l \geq 1} \left( \sum_{\gamma} \frac{t^{\gamma_1}}{\gamma_1!q} \cdots \frac{t^{\gamma_l}}{\gamma_l!q} \right) x^{l-1} \\
= \sum_{l \geq 1} \frac{t^{2i}}{2i!q} + x \sum_{l \geq 2} \left( \sum_{i \geq 1} \frac{t^{2i-1}}{(2i-1)!q} \right)^2 \left( x \sum_{i \geq 1} \frac{t^{2i}}{2i!q} \right)^{l-2} \\
= \cosh_q t - 1 + \frac{x \sinh^2_q t}{1 - x(\cosh_q t - 1)},
\]

which gives (3.4).

In the same vein, we have

\[
\sum_{n \geq 1} P_{2n-1}(x,0,q) \frac{t^{2n-1}}{(2n-1)!q} = \sum_{n \geq 1} \sum_{S \subseteq O[2n-1]} \alpha_{2n-1}(S,q)x^{|S|} \frac{t^{2n-1}}{(2n-1)!q} \\
= \sum_{l \geq 1} \left( \sum_{\gamma} \frac{t^{\gamma_1}}{\gamma_1!q} \cdots \frac{t^{\gamma_l}}{\gamma_l!q} \right) x^{l-1} \\
= \sum_{l \geq 1} \left( \sum_{i \geq 1} \frac{t^{2i-1}}{(2i-1)!q} \right) \left( x \sum_{i \geq 1} \frac{t^{2i}}{(2i)!q} \right)^{l-1},
\]

which is clearly equal to (3.5).

Next we generalize (3.4) and (3.5) to the general \( y \).

Lemma 3.4. We have

\[
\sum_{n \geq 1} P_{2n}(x,y,q) \frac{t^{2n}}{(2n)!q} = \frac{B(t,x)}{1 - yB(t,x)}, \tag{3.6}
\]

\[
\sum_{n \geq 1} P_{2n-1}(x,y,q) \frac{t^{2n-1}}{(2n-1)!q} = \frac{C(t,x)}{1 - yB(t,x)}. \tag{3.7}
\]

Proof. Consider

\[
P_n(x,y,q) = \sum_{(\sigma,S)} x^{|S_\sigma|} y^{|S_\sigma|} q^{\inv_{\sigma}} \quad (\sigma \in \mathfrak{S}_n \text{ and } D(\sigma) \subseteq S \subseteq [n - 1]).
\]
There is a bijection between the set of subsets $S$ of $[n-1]$ with fixed even integers $S_e = \{m_1 < \cdots < m_{l-1}\} \subset \mathbb{E}[n-1]$ and the set of sequences of compositions of $m_i - m_{i-1}$ with odd parts for $i \in [l]$ with $m_0 = 0$ and $m_l = n$. Let $co(S_e) = (n_1, \ldots, n_l)$ be the corresponding composition of $n$. Then

$$\sum_{n \geq 1} P_{2n}(x, y, q) \frac{t^{2n}}{(2n)!} = \sum_{l \geq 1} \prod_{i=1}^{l-1} \left[ \sum_{S_i \subseteq \mathbb{O}[2n_i]} \alpha_{2n_i}(S_i, q) x^{|S_i|} \frac{t^{2n_i}}{(2n_i)!} y \right]$$

$$\times \left[ \sum_{S_i \subseteq \mathbb{O}[2n_i]} \alpha_{2n_i}(S_i, q) x^{|S_i|} \frac{t^{2n_i}}{(2n_i)!} \right],$$

which is equal to $\sum_{l \geq 1} y^{l-1} \cdot B(t, x)^l = \frac{B(t, x)}{1-yB(t, x)}$.

Similarly, we have

$$\sum_{n \geq 1} P_{2n-1}(x, y, q) \frac{t^{2n-1}}{(2n-1)!} = \sum_{l \geq 1} \prod_{i=1}^{l-1} \left( \sum_{S_i \subseteq \mathbb{O}[2n_i]} \alpha_{2n_i}(S_i, q) x^{|S_i|} \frac{t^{2n_i}}{(2n_i)!} y \right)$$

$$\times \left( \sum_{S_i \subseteq \mathbb{O}[2n_i]} \alpha_{2n_i}(S_i, q) x^{|S_i|} \frac{t^{2n_i}}{(2n_i)!} \right),$$

which can be written as $\sum_{l \geq 1} y^{l-1} \cdot B(t, x)^{l-1} \cdot C(t, x) = \frac{C(t, x)}{1-yB(t, x)}$. \hfill \qed

We obtain (1.6a) by combining Lemma 3.2, Lemma 3.3 and Lemma 3.4.

4. Counting permutations of type B by the parity of descent positions

Let $B^+_n$ (resp. $B^-_n$) be the subset of permutations in $B_n$ whose first entry is positive (resp. negative). Clearly the doubleton $\{B^-_n, B^+_n\}$ is a partition of $B_n$. Introduce the corresponding enumerative polynomials:

$$B^{-}_n(x, y) = \sum_{\sigma \in B^-_n} x^{\text{des}_1 \sigma} y^{\text{des}_0 \sigma}, \quad B^+_n(x, y) = \sum_{\sigma \in B^+_n} x^{\text{des}_1 \sigma} y^{\text{des}_0 \sigma}.$$

Then $B_n(x, y) = B^-_n(x, y) + B^+_n(x, y)$.

For $\tau \in B_n$, let $\tau^-$ be the permutation in $B_n$ such that $\tau^-(i) = -\tau(i)$ for $i \in [n]$. It is clear that the mapping $\rho : \tau \mapsto \tau^-$ is an involution on $B_n$ such that

$$\text{des}_1 \tau + \text{des}_1 \tau^- = [n/2],$$

$$\text{des}_0 \tau + \text{des}_0 \tau^- = [(n+1)/2].$$

(4.1)

Besides, the restriction of $\rho$ on $B^+_n$ sets up a bijection $\rho : B^+_n \rightarrow B^-_n$, therefore

$$B^-_n(x, y) = x^{[n/2]} y^{[(n+1)/2]} B^+_n(1/x, 1/y).$$

(4.2)
So, we need only to compute the exponential generating functions of $B_n^+(x, y)$.

For $\sigma \in B_n$, we denote by $D(\sigma)$ the set of descents of $\sigma$. If $S$ is a subset of $[n - 1]$ let $\alpha_n^+(S)$ be the number of permutations $\sigma \in B_n^+$ such that $D(\sigma) \subseteq S$. A set composition of set $\Omega$ is an $\ell$-tuple $(\Omega_1, \ldots, \Omega_\ell)$ of subsets of $\Omega$ such that $\{\Omega_1, \ldots, \Omega_\ell\}$ is a set partition of $\Omega$.

**Lemma 4.1.** Let $S = \{s_1 < \cdots < s_k\} \subseteq [n - 1]$, and $s_0 = 0$ and $s_k + 1 = n$. Then

$$\alpha_n^+(S) = \left(\binom{n}{\text{co}(S)}\right)2^{n-s_1}. \quad (4.3)$$

**Proof.** We can construct the permutations $\sigma \in B_n^+$ with $D(\sigma) \subseteq S$ as in the following:

- partition $[n]$ to obtain a set-composition $(\Omega_1, \ldots, \Omega_{k+1})$ of $[n]$ with $|\Omega_i| = s_i$ for $1 \leq i \leq k$ and $|\Omega_{k+1}| = n - s_k$,
- sign the elements in $\Omega_i$ by $\epsilon \in \{-1, 1\}$ for $i = 2, \ldots, k + 1$.
- arrange the elements in each block $\Omega_i$ increasingly.

It is clear that the number of such permutations is

$$\left(\binom{n}{s_1 - s_0, s_2 - s_1, \ldots, s_{k+1} - s_k}\right)2^{n-s_1}.$$

This is the desired formula. \qed

Similar to permutations of type A (see (5.1)), consider the polynomial

$$Q_n^+(x, y) = \sum_{S \subseteq [n]} \alpha_n^+(S) x^{|S_o|} y^{|S_e|}. \quad (4.4)$$

**Lemma 4.2.** We have

$$B_{2n}^+(x, y) = (1 - x)^n(1 - y)^{n-1}Q_{2n}^+ \left(\frac{x}{1 - x}, \frac{y}{1 - y}\right), \quad (4.5)$$

$$B_{2n-1}^+(x, y) = (1 - x)^{n-1}(1 - y)^{n-1}Q_{2n-1}^+ \left(\frac{x}{1 - x}, \frac{y}{1 - y}\right). \quad (4.6)$$

**Proof.** For even index we have

$$Q_{2n}^+(x, y) = \sum_{\sigma \in B_{2n}^+} \sum_{S \subseteq [2n]} \sum_{\text{Des}_0(\sigma) \subseteq S_o \atop \text{Des}_1(\sigma) \subseteq S_e} x^{|S_o|} y^{|S_e|}. \quad (4.7)$$

Now, for any fixed $\sigma \in B_{2n}^+$, writing $T_0 = S_e \setminus \text{Des}_0(\sigma)$ and $T_1 = S_e \setminus \text{Des}_1(\sigma)$, then $|S_e| = \text{des}_0(\sigma) + |T_0|$ and $|S_o| = \text{des}_1(\sigma) + |T_1|$; hence the inner double sum at the right-hand side of (4.7) is a sum over the pairs $(T_0, T_1)$ such that $T_0 \subseteq E[2n]$ and $T_1 \subseteq O[2n]$, and thus equal to

$$y^{\text{des}_0(\sigma)}x^{\text{des}_1(\sigma)}(1 + y)^{n-1-\text{des}_0(\sigma)}(1 + x)^{n-\text{des}_1(\sigma)}. \quad (4.8)$$
Therefore
\[ Q_{2n}^+(x, y) = (1 + x)^n(1 + y)^{n-1}B_{2n}^+(\frac{x}{1+x}, \frac{y}{1+y}), \]
(4.9)
which is equivalent to (4.5).

For odd index, a similar reasoning can be applied with regard to the sum
\[ Q_{2n-1}^+(x, y) = \sum_{\sigma \in B_{2n-1}^+} \sum_{S \subseteq [2n-1]} \sum_{Des_0(\sigma) \subseteq S_e} x^{\left| S_0 \right|} y^{\left| S_e \right|} \]
(4.10)
and leads to the formula
\[ Q_{2n-1}^+(x, y) = (1 + x)^{n-1}(1 + y)^{n-1}B_{2n-1}^+(\frac{x}{1+x}, \frac{y}{1+y}). \]
(4.11)
which is equivalent to (4.6).

\[ \square \]

Lemma 4.3. We have
\[ G := \sum_{n \geq 1} Q_{2n}^+(x, 0) \frac{t^{2n}}{(2n)!} = \cosh(t) - 1 + \frac{x \sinh(t) \sinh(2t)}{1 - x \cosh(2t) - 1}, \]
(4.12)
and
\[ H := \sum_{n \geq 1} \sum_{S \subseteq [2n]} \left( \begin{array}{c} 2n \\ \text{co}(S) \end{array} \right) 2^n x^{\left| S \right|} \frac{t^{2n}}{(2n)!} = \cosh(2t) - 1 + \frac{x \sinh^2(2t)}{1 - x \cosh(2t) - 1}. \]
(4.13)

Proof. By definition, if \( S = \{s_1, s_2, \ldots, s_{l-1}\} \subseteq O[2n] \), let \( \gamma_1 = s_1 \), \( \gamma_i = s_i - s_{i-1} \) for \( i = 2, \ldots, l \) with \( s_l = 2n - 1 \), then \( \gamma_1 \) is odd and \( \gamma_i \) are even for \( i = 2, \ldots, l \). Therefore
\[ G = \sum_{n \geq 1} \sum_{S \subseteq O[2n]} \left( \begin{array}{c} 2n \\ \text{co}(S) \end{array} \right) x^{\left| S \right|} \frac{t^{2n}}{(2n)!} \]
(4.14)
\[ = \sum_{n \geq 1} \left( \sum_{\gamma_1} \frac{1}{\gamma_1!} \cdots \frac{1}{\gamma_l!} x^{l-1} \right) 2^n \gamma_l t^{2n} \]
\[ = \sum_{i \geq 1} \frac{t^{2i}}{(2i)!} + \sum_{i \geq 2} \left( \sum_{i \geq 1} \frac{(t)^{2i-1}}{(2i-1)!} \right) \left( x \sum_{i \geq 1} \frac{(2t)^{2i-1}}{(2i-1)!} \right) \left( \sum_{i \geq 1} \frac{(2t)^{2i}}{(2i)!} \right)^{l-2} \]
\[ = \sum_{i \geq 1} \frac{t^{2i}}{(2i)!} + x \left( \sum_{i \geq 1} \frac{t^{2i-1}}{(2i-1)!} \right) \left( \sum_{i \geq 1} \frac{(2t)^{2i-1}}{(2i-1)!} \right) \left( 1 - x \sum_{i \geq 1} \frac{(2t)^{2i}}{(2i)!} \right)^{l-2}. \]
which is the right-hand side of (4.12). Next,

\[ H = \sum_{n \geq 1} \left( \sum_{\gamma} \frac{1}{\gamma_1!} \cdots \frac{1}{\gamma_l!} x^{l-1} \right) (2t)^{2n} \]  

(4.15)

\[ = \sum_{i \geq 1} \frac{(2t)^{2i}}{(2i)!} + \frac{1}{x} \sum_{l \geq 2} \left( x \sum_{i \geq 1} \frac{(2t)^{2i-1}}{(2i-1)!} \right)^2 \left( x \sum_{i \geq 1} \frac{(2t)^{2i}}{(2i)!} \right)^{l-2} \]

\[ = \sum_{i \geq 1} \frac{(2t)^{2i}}{(2i)!} + x \left( \sum_{i \geq 1} \frac{(2t)^{2i-1}}{(2i-1)!} \right)^2 \frac{1}{1 - x \sum_{i \geq 1} \frac{(2t)^{2i}}{(2i)!}}, \]

which is the right-hand of (4.13). \( \square \)

**Lemma 4.4.** We have

\[ F := \sum_{n \geq 1} \left( \sum_{S \subseteq O[2n-1]} \binom{2n-1}{\text{co}(S)} x^{|S|} \right)^{2n-1} \frac{t^{2n-1}}{(2n-1)!} = \frac{\sinh(2t)}{1 - x(\cosh(2t) - 1)}, \]  

(4.16)

\[ L := \sum_{n \geq 1} \left( \sum_{S \subseteq O[2n-1]} \binom{2n-1}{\text{co}(S)} x^{|S|} \frac{t^{2n-1}}{(2n-1)!} \right) = \frac{\sinh(t)}{1 - x(\cosh(2t) - 1)}. \]  

(4.17)

**Proof.** By definition, if \( S = \{s_1, s_2, \ldots, s_{l-1}\} \subseteq O[2n-1] \), let \( \gamma_1 = s_1, \gamma_i = s_i - s_{i-1} \) with \( s_l = 2n - 1 \), then \( \gamma_1 \) is odd and \( \gamma_i \) are even for \( i = 2, \ldots, l \). Therefore

\[ F = \sum_{n \geq 1} \left( \sum_{\gamma} \frac{1}{\gamma_1!} \cdots \frac{1}{\gamma_l!} x^{l-1} \right) (2t)^{2n-1} \]

\[ = \sum_{i \geq 1} \frac{(2t)^{2i-1}}{(2i-1)!} \left( x \sum_{i \geq 1} \frac{(2t)^{2i}}{(2i)!} \right)^{l-1} \]

\[ = \left( \sum_{i \geq 1} \frac{(2t)^{2i-1}}{(2i-1)!} \right) \frac{1}{1 - x \sum_{i \geq 1} \frac{(2t)^{2i}}{(2i)!}}. \]
which equals the right-hand side of (4.16), besides

\[
L = \sum_{n \geq 1} \left( \sum_{\gamma} \frac{1}{\gamma_1!} \cdots \frac{1}{\gamma_l!} x^{l-1} \right) 2^{2n-1-\gamma_1} t^{2n-1}
\]

\[
= \sum_{l \geq 1} \left( \sum_{i \geq 1} \frac{t^{2i-1}}{(2i-1)!} \right) \left( x \sum_{i \geq 1} \frac{(2i)^{2i}}{(2i)!} \right)^{l-1}
\]

\[
= \left( \sum_{i \geq 1} \frac{t^{2i-1}}{(2i-1)!} \right) \frac{1}{1 - x \sum_{i \geq 1} \frac{(2i)^{2i}}{(2i)!}},
\]

which is equal to the right-hand side of (4.17). □

**Lemma 4.5.** We have

\[
\sum_{n \geq 1} Q^+_{2n}(x, y) \frac{t^{2n}}{(2n)!} = \frac{G}{1 - y H}, \tag{4.18}
\]

\[
\sum_{n \geq 1} Q^+_{2n-1}(x, y) \frac{t^{2n-1}}{(2n-1)!} = L + \frac{y F G}{1 - y H}. \tag{4.19}
\]

**Proof.** The left-hand side of (4.18) is

\[
\sum_{n \geq 1} \left( \sum_{S \subseteq [2n]} \alpha^+_{2n}(S) y^{|S|_2} \right) \frac{t^{2n}}{2n!}
\]

\[
= \sum_{n \geq 1} \sum_{S_1 \subseteq [2m_1]} \binom{2m_1}{\text{co}(S_1)} x^{S_1} \frac{t^{2m_1}}{2m_1!} \prod_{i=2}^l \left[ \sum_{S_i \subseteq O[2m_i]} \binom{2m_i}{\text{co}(S_i)} x^{S_i} \frac{t^{2m_i}}{2m_i!} \right]
\]

\[
= \sum_{l \geq 1} y^{l-1} \cdot G \cdot H^{l-1},
\]

which equals \( \frac{G}{1 - y H} \). The left-hand side of (4.19) is

\[
\sum_{n \geq 1} \left( \sum_{S_1 \subseteq [2n]} \binom{2m_1}{\text{co}(S_1)} x^{S_1} \frac{t^{2m_1}}{2m_1!} \right) \prod_{i=2}^l \left[ \sum_{S_i \subseteq O[2m_i]} \binom{2m_i}{\text{co}(S_i)} x^{S_i} \frac{t^{2m_i}}{2m_i!} \right]
\]

\[
= L + y F \sum_{l \geq 0} (y H)^l,
\]

which equals \( L + \frac{y F G}{1 - y H} \). □
Now, combining Lemma 4.2 and Lemma 4.5 we have

$$\sum_{n \geq 1} B_{2n}^+(x, y) \frac{t^{2n}}{(2n)!} = \frac{(\cosh(at) - 1)(2x \cosh(at) + x + 1)}{1 + xy - (x + y) \cosh(2at)}, \quad (4.20)$$

$$\sum_{n \geq 1} B_{2n-1}^+(x, y) \frac{t^{2n-1}}{(2n - 1)!} = \frac{\sinh(at)(x - 1)(2 \cosh(at)y - y - 1)}{a(xy + 1 - (x + y) \cosh(2at))}, \quad (4.21)$$

with \(a^2 = (1 - x)(1 - y)\). It follows from (4.2) that

$$\sum_{n \geq 1} B_{2n}^-(x, y) \frac{t^{2n}}{(2n)!} = \frac{y(\cosh(at) - 1)(2 \cosh(at) + x + 1)}{1 + xy - (x + y) \cosh(2at)}, \quad (4.22)$$

$$\sum_{n \geq 1} B_{2n-1}^-(x, y) \frac{t^{2n-1}}{(2n - 1)!} = \frac{y \sinh(at)(x - 1)(-2 \cosh(at) + y + 1)}{a(xy + 1 - (x + y) \cosh(2at))}. \quad (4.23)$$

Combining (4.20) with (4.22) and (4.21) with (4.23), we complete the proof of Theorem 1.2.

5. Concluding remarks

In [4] Carlitz and Scoville also considered the more general modulus \(m > 2\) for descents rather than parity, i.e., \(m = 2\). They obtained a general generating function. However, apart from \(m = 2\) the generating function is quite explicit only for certain special cases when \(m = 4\). For the \(q\)-analogue, there are some nice generating functions given by Kurşungöz and Yee [19]. It would be very interesting to have results in this direction.

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References

[1] D. Andrè, Développements de sec \(x\) et de tan \(x\). C.R. Acad. Sci. Paris 88, 965–967 (1879)
[2] G. Andrews, The theory of partitions. Reprint of the 1976 original. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1998.
[3] A. Björner and F. Brenti, Combinatorics of Coxeter groups. Graduate Texts in Mathematics, 231. Springer, New York, 2005. xiv+363 pp.
[4] L. Carlitz and R. Scoville, Enumeration of rises and falls by position. Discrete Math. 5 (1973), 45–59.
[5] D. Chebikin, Variations on Descents and Inversions in Permutations, Electron. J. Combin. 15 (2008), #R132.
[6] W. Y. C. Chen and A. M. Fu, A context-free grammar for the e-positivity of the trivariate second-order Eulerian polynomials. Discrete Math. 345 (2022), no. 1, Paper No. 112661, 9 pp.
[7] C.-O. Chow and I. M. Gessel, On the descent numbers and major indices for the hyperoctahedral group, Adv. Appl. Math. 38, No. 3, 275–301 (2007).
[8] C.-O. Chow and W. C. Shiu, Counting Simsun permutations by descents, Ann. Comb. 15, 625-635 (2011).
[9] M. J. Ding and B. X. Zhu, Stability of combinatorial polynomials and its applications, arXiv:2106.12178 [math.CO].
[10] D. Foata and G.-N. Han, André permutation calculus: a twin Seidel matrix sequence. Sém. Lothar. Combin. 73 (2016), Art. B73e, 54 pp.
[11] D. Foata and M. P. Schützenberger, Théorie géométrique des polynômes eulériens. Lecture Notes in Mathematics, Vol. 138 Spring-Verlag, Berlin-New York 1970 v+94 pp.
[12] D. Foata and M. P. Schützenberger, Nombres d’Euler et permutations alternantes, Manuscript, University of Florida, Gainesville, FL, 1971, available at https://irma.math.unistra.fr/~foata/paper/pub18.pdf.
[13] D. Foata and V. Strehl, Euler numbers and variations of permutations, in: Atti dei Convegni Lincei, vol. 17, Tomo I, 1976, pp. 119-131.
[14] A. M. Garsia, On the "maj" and "inv" q-analogues of Eulerian polynomials, Linear and Multilinear Algebra 8 (1979/80), no. 1, 21–34.
[15] I. M. Gessel and Y. Zhuang, Counting permutations by alternating descents, Electron. J. Combin. 21 (4), 2014, Paper #P4.23.
[16] Ian P. Goulden and David M. Jackson, Combinatorial enumeration. With a foreword by Gian-Carlo Rota. Reprint of the 1983 original. Dover Publications, Inc., Mineola, NY, 2004. xxvi+569 pp.
[17] G. Hetyei and E. Reiner, Permutation trees and variation statistics. European J. Combin. 19 (1998), no. 7, 847–866.
[18] M. Josuat-Vergès, A generalization of Euler numbers to finite Coxeter groups. Ann. Comb. 19 (2015), no. 2, 325–336.
[19] Küşagöz, Kağan and Yee, Ae Ja, Alternating permutations and the mth descents. Discrete Math. 311, No. 22, 2610-2622 (2011).
[20] Z.-C. Lin, S.-M. Ma, D. G. L. Wang and L. Wang, Positivity and divisibility of enumerators of alternating descents. Ramanujan J. 58 (2022), no. 1, 203–228.
[21] S.-M. Ma and Y.-N. Yeh, Enumeration of permutations by number of alternating descents. Discrete Math. 339 (2016), no. 4, 1362–1367.
[22] S.-M. Ma, J. Ma, Y.-N. Yeh, David-Barton type identities and alternating run polynomials. Adv. in Appl. Math. 114 (2020), 101978, 19 pp.
[23] S.-M. Ma, Q. Fang, T. Mansour and Y.-N. Yeh, Alternating Eulerian polynomials and left peak polynomials. Discrete Math. 345 (2022), no. 3, Paper No. 112714, 12 pp.
[24] OEIS Foundation Inc. (2022), The On-Line Encyclopedia of Integer Sequences, Published electronically at http://oeis.org
[25] Q. Q. Pan, A new combinatorial formula for alternating descent polynomials, arXiv preprint arXiv:2207.06212 [math.CO].
[26] Q. Q. Pan and J. Zeng, A q-analogue of generalized Eulerian polynomials with applications. Adv. in Appl. Math. 104 (2019), 85–99.
[27] Q. Q. Pan and J. Zeng, Bränden’s \((p,q)\)-Eulerian polynomials, André permutations and continued fractions. *J. Combin. Theory Ser. A* 181 (2021) 105445.

[28] T. K. Petersen, Eulerian numbers. With a foreword by Richard Stanley. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks] Birkhäuser/Springer, New York, 2015. xviii+456 pp.

[29] J. B. Remmel, Generating functions for alternating descents and alternating major index. *Ann. Comb.* 16 (2012), no. 3, 625–650.

[30] R. P. Stanley, Binomial posets, Möbius inversion, and permutation enumeration. *J. Combinatorial Theory Ser. A* 20 (1976), no. 3, 336–356.

[31] R.P. Stanley, *Enumerative Combinatorics*, vol. 1, second ed., in: Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 2012.

[32] H. Sun, A New Class of Refined Eulerian Polynomials, *J. Integer Seq.* Vol. 21 (2018), Article 18.5.5.

[33] H. Sun, The \(γ\)-positivity of bivariate Eulerian polynomials via the Heteyi-Reiner action. *European J. Combin.* 92 (2021), Paper No. 103166, 8 pp.

[34] Y. Sun and L. Zhai, Some properties of a class of refined Eulerian polynomials. *J. Math. Res. Appl.* 39 (2019), no. 6, 593–602.

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