Too Global To Be Local: 
Swarm Consensus in Adversarial Settings

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Abstract
Reaching a consensus in a swarm of robots is one of the fundamental problems in swarm robotics, examining the possibility of reaching an agreement within the swarm members. The recently-introduced contamination problem offers a new perspective of the problem, in which swarm members should reach a consensus in spite of the existence of adversarial members that intentionally act to divert the swarm members towards a different consensus. In this paper, we search for a consensus-reaching algorithm under the contamination problem setting by taking a top-down approach: We transform the problem to a centralized two-player game in which each player controls the behavior of a subset of the swarm, trying to force the entire swarm to converge to an agreement on its own value. We define a performance metric for each player's performance, proving a correlation between this metric and the chances of the player to win the game. We then present the globally optimal solution to the game and prove that unfortunately it is unattainable in a distributed setting, due to the challenging characteristics of the swarm members. We therefore examine the problem on a simplified swarm model, and compare the performance of the globally optimal strategy with locally optimal strategies, demonstrating its superiority in rigorous simulation experiments.

Keywords
Swarm Robotics, Consensus Problems

Introduction
Robot swarms use simple local behavioral rules to achieve an emergent behavior over time. The study of swarms gained considerable interest in the scientific community due to its applicability in a variety of areas, such as search and rescue (Chen, Wang and Li (2009); Skinner, Urdahl, Harrington, Balchano, Garcia and Mavris (2018); León, Cardona, Botello and Calderón (2016)), space exploration (Nguyen, Harman and Fairchild (2019); Marco Sabatini and Giovanni B Palmerini (2009); Huan Huang, Leping Yang, Yan-wei Zhu and Yuan-wen Zhang (2014)) and disaster relief (Schurr, Mareck, Tambe, Sceeri, Kasinadunhi and Lewis (2005); Kazi T.A.Siddiqui, David Feil-Seifer, Tianyi Jiang, Sonu Jose, Siming Liu and Sushil Louis (2017); Subramanium Ganesan, Manish Shakya, Aqueel F. Aqueel and Lakshmi M. Nambiar (2011)). As a part of these various applications, swarm members may need to achieve an agreement over a set of variables using local interactions among themselves. This problem is referred to as the consensus problem. Developing consensus-reaching algorithms for swarms has proven to be quite challenging due to the swarm members' limitations in sensing and computation capabilities.

Recently, there has been an interest in the contamination problem (Avrahami and Agmon (2019)) which acts as an extension to the consensus problem. In the contamination problem, swarm members must reach a consensus in spite of the existence of adversarial swarm members that may act to divert the swarm from reaching the desired consensus. The swarm members are divided such that each one can adapt to one of two different states: healthy or contaminated. Each swarm member can change its state, and thus also its behavior, based on external factors by other swarm members and its own internal state. The goal of a consensus protocol in the contamination problem is to guide the swarm towards a desired state.

Past work showed that we can utilize the process of formation creation as a mean to reach consensus (Avrahami and Agmon (2019)). It was shown that gathering swarm members in a geometrical structure named maximal stable cycle (MSC), a clique, can maintain their initial state under certain conditions.

In this paper we examine the contamination problem using a top-down approach, in which we first define the problem as a two player game where each player has full information regarding the swarm members in the game, and has full control of a given group of swarm members that share the same state. We take advantage of the simplified settings of the game to define a measure for the performance of players in the game. We then use this measure to convert the game to an optimization problem. We show that although there is a defined globally optimal solution to the game, it is unattainable in a distributed setting. To overcome this, we perform several relaxations over the initial distributed setting of the problem. Finally, we present simulation results showing that the globally optimal strategy outperforms previously proposed locally optimal strategies for various numbers of swarm members.

The unattainability of the globally optimal solution under the standard distributed setting forces us to focus on the larger space of locally optimal solutions to devise an effective distributed solution to the problem. Since the

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size of the space of locally optimal solutions is directly correlated to the swarm’s population, we can conclude that an efficient method to identify the subset of efficient locally optimal solutions is required to construct scalable distributed solutions to problems in the domain of swarm robotics.

Contributions

In this work, we show a thorough analysis for a problem which had very limited theoretical coverage in the past. Several dynamical aspects of the problem are overlooked in our analysis, as the movement of agents can lead to a continuously changing network of interactions that can impact the ability to reach consensus. Accordingly, previous works chose to tackle this kind of problem under a stochastic perspective (Castellano, Fortunato and Loreto (2009); Valentini, Ferrante and Dorigo (2017)). Since this is the first known theoretical analysis of the problem of reaching swarm consensus in adversarial settings we chose to take a different approach and focus on the inherent properties of the problem in hopes that it would lead us to a series of discoveries which would simplify the initially complicated problem. Indeed, we discovered that the initial problem could be simplified to the problem of forming dense circles while facing adversaries. A problem which was shown to be insurmountable under the current swarm settings. This result means that even while ignoring the dynamical aspects of the problem altogether, a globally optimal solution is unattainable under the current swarm settings. The main contribution of this work is the discovery that there is no attainable globally optimal solution for the problem of swarm consensus under adversarial settings, this puts the spotlight on the problem of finding an effective local optimum solution for the problem. Constructing mechanisms that identify subsets of effective local optima will be of utmost importance in the future since the space of local optima grows exponentially according to the number of agents in the swarm, making most of the local optima result in an overall underperformance of the whole swarm.

Related Work

The contamination problem was motivated from the set of popular problems in distributed computing of reaching a consensus (Olfati-Saber, Fax and Murray (2007)). In networks of agents, a consensus is an agreement regarding a chosen value which depends on the states of all the agents. A consensus protocol is a proposed rule that guides the interaction between an agent and all of its neighbors in the network towards reaching consensus. The instrumental work in paving the way for the development of self-organizing swarms is surveyed in Olfati-Saber, Fax and Murray (2007). A popular approach to solving consensus problems is a graph-based approach. In this approach a consensus reaching algorithm is expressed as an nth-order linear system on a given graph. These algorithms use the algebraic connectivity (Fiedler (1973)) as a measure that quantifies the speed of convergence of consensus algorithms.

The contamination problem is related to the Byzantine-consensus problem (Feldman and Micali (1988)), in which a network of distributed processors should reach an agreement on a value despite byzantine faults which might transpire in the system. In the contamination faults which might transpire in the system. In the contamination problem, though in our problem the processors are mobile, work asynchronously and use no explicit communication mechanisms. To the best of our knowledge, these properties were not considered in the research of the Byzantine-consensus problem.

In another line of research, Valentini, Ferrante, Hamann and Dorigo (2016) examined the speed of reaching an agreement in a swarm of robots with limited sensing capabilities which do not use explicit communication while having major uncertainties in their actuators and sensors. In this work, the majority rule was used for reaching a decision by each individual robot. It was shown that the main deciding factor for the speed of reaching an agreement is the size of the external group that each individual robot can observe. In this research, there are no external robots which act intentionally against the group of robots that has to reach an agreement, as in our case.

The problem of resilient asymptotic consensus (LeBlanc, Zhang, Koutsoukos and Sundaram (2013)) addresses reaching a consensus in a large-scale distributed system while facing misbehaving nodes in the form of adversaries. They defined the problem of reaching asymptotic consensus in the presence of misbehaving nodes given a particular threat model of those presented in Agmon and Peleg (2006) and a scope of threat. A local consensus protocol that is resilient to F adversarial nodes was proposed. While the results of this research have a strong theoretic basis, it assumes that nodes of the network have full knowledge of the network and intentions of the other nodes whereas the agents in the contamination problem do not have full knowledge of the network’s structure.

Saldana, Prorok, Campos and Kumar (2018) presented a formation topology that can be constructed in a fully distributed manner in static networks which guarantees resilient asymptotic consensus facing an unknown malicious agent in the network. The presented topology which is termed triangular robust networks is based on the notion of network robustness presented in LeBlanc, Zhang, Koutsoukos and Sundaram (2013) and has a variety of appealing theoretical properties. While the verification of network robustness was shown to be an NP-hard problem (LeBlanc and Koutsoukos (2013)), triangular network robustness can be verified in polynomial time. Moreover, a triangular robust network can be incrementally expanded in a distributed manner which is competent with distributed robotic systems. The proposed formation can achieve resilient asymptotic consensus in a static networks facing a lone malicious agent, whereas the contamination game requires convergence to consensus in dynamic networks facing a group of malicious agents.

Continuing this line of work, Saldana, Prorok, Sundaram, Campos and Kumar (2017) proposed an approach that provides resilience for networks of agents which are time varying while diverting from the notion of high connectivity rates which are required in the topological measure of network robustness presented in LeBlanc, Zhang, Koutsoukos and Sundaram (2013). The resilient consensus algorithm provided in this work relies on specific topological
properties of the communication graph of agents in the network while in the contamination problem the agents do not have explicit communication capabilities.

Recently, some attention has been given to analysis of the strategies of robotic swarms from a game-theoretic perspective (Givigi Jr and Schwartz (2006, 2007); Douchan, Wolf and Kaminka (2019)). Game theory was discovered to be a valuable tool for controlling behavior in distributed systems, as there are various parallels between the decision making architectures of societal systems which are common in the game theory literature and distributed systems. Particularly, both are comprised of a collection of connected decision making components whose collective behavior depends on the local decisions which are made by the components based on partial information about each other. Finally, a learning process can be used to guide agents towards a solution. This learning process also constitutes a design choice as it is beneficial using a learning process which ensures convergence to a chosen game theoretic solution concept. Game theoretic methods have been used for the purpose of distributed control (Marden and Shamma (2015)). In particular, there have been efforts to use game theory as a tool for modelling cooperative behavior in swarms (Givigi Jr and Schwartz (2006, 2007)) which is the opposite case of the non-cooperative behavior of swarm members in adversarial settings. Douchan, Wolf and Kaminka (2019) proposed a solution method to the problem of spatial coordination by forming a connection between the global utility theoretically reached using the extensive-form game which describes the environment of the robotic swarm and the utility of a single agent in the swarm. This connection is formed using potential games (Monderer and Shapley (1996)) as a tool to aid agents in the swarm in learning optimal actions. It is shown that if a problem in swarm domains can be formalized as a potential game, then agents in the swarm can choose to maximize their own individual payoffs and the system will converge to pure-strategy Nash Equilibrium. This result means that there are cases in which agents in the swarm can be rational and achieve a local optimum of the problem by reaching their own local optimum by choosing an action which gives them maximal expected utility. The question of what should the agents do when the game cannot be represented as a potential game was not answered yet. The contamination problem cannot be represented as a potential game since each agent can only observe a limited area surrounding it which makes it impossible to craft a local utility function that is directly correlated to the global utility of all the members of the swarm. Vamvoudakis and Hespanha (2018) developed a game theoretical solution method to the consensus problem for networked systems with the presence of adversaries. The proposed algorithm enables the agents to reject adversarial input, thereby leading to consensus. Even though they solve the consensus problem in a distributed environment, they do so by rejecting adversarial input completely. In our problem we wish to reach a consensus in spite of the collected adversarial input. Neto and Lima (2005) developed a dynamic programming algorithm to solve a class of stochastic games called two-person zero-sum games and evaluated its performance in the game of robotic soccer. The proposed work intends to model situations of teams with opposing objectives by approaching each team as an augmented agent such that the overall problem reduces to two-person zero-sum games. The appeal in this kind of games is that each equilibria has a similar reward structure which means that all equilibria are interchangeable. The proposed algorithm uses linear programming in order to find the Nash equilibrium in each state of the game which ensures optimal behavior in the worst-case scenario. Despite the effective performance of the algorithm while using a minified model of robotic soccer, the algorithm requires a transition function which cannot be defined in complex swarm environments. Furthermore, the system of equations which are solved in order to find Nash equilibria for each state of the game scales up in the domain of robotic swarms as the approach of regarding a homogeneous swarm as an augmented agent does not perform well in practice.

Contamination Problem as a Two-Player Game

In this section, we formally present the contamination problem and take our first step in our top-down approach by providing a simplified representation for the problem in the form of a two-player game where each player controls a swarm that can be represented by a graph of connected components. We then use this representation to show that the performance of a swarm in the contamination problem is directly correlated to the strength of its constructed connected components. Finally, we present the problem of constructing effective components as an optimization problem.

Preliminaries

The contamination problem can be represented as a time continuous game between two groups of agents that move simultaneously at each time step. We refer to this game as the contamination game. Each player in the contamination game controls a swarm of agents that share the same state. Let $S$ be the group of all the members of the swarm. We assume that all the members of the swarm face the same physical limitations. Each agent has a physical diameter of length $D$, and can observe a limited area surrounding it. Let $S_{\text{min}}$ and $S_{\text{max}}$ denote the minimal and maximal observation radii of each swarm member, respectively. In other words, each agent $a_j \in S$ can be seen by an agent $a_i \in S$ if it lies within a

![Figure 1. The possible relationships between a set of three agents in the contamination game. Each agent is represented in cyan filled circles and the $S_{\text{min}}$ and $S_{\text{max}}$ circles of each agent are colored in red and green, respectively. (i) Agent $a_1$ can clearly see agent $a_2$. (ii) $a_3$ obstructs a part of $a_2$ from $a_1$ and vice versa.](image-url)
distance greater than $S_{\min}$ and smaller than $S_{\max}$ from it. Furthermore, we assume that two agents can observe one another only if we can connect a line between both of their centers without intersecting any other agent (concealing the view) as displayed in Figure 1. Let $O(S_{\min}, S_{\max}, a_i)$ be the observation area of agent $a_i \in S$ based on radii $S_{\min}$ and $S_{\max}$ and physical concealments that may be caused by other agents. In short, we use the notation $O(a_i)$ as we assume that the $S_{\min}$ and $S_{\max}$ have fixed values. We say that an agent $a_i \in S$ can observe another agent $a_j \in S$ if $a_j \in O(a_i)$. Furthermore, each agent is included in its own observation area, i.e., $a_i \in O(a_i) \forall a_i \in S$.

As part of the contamination game, each agent $a_i \in S$ is in either one of two different states: healthy or contaminated, denoted by $s(a_i) \in \{s_H, s_C\}$, respectively. The state of an agent $a_i \in S$ is decided by using a majority rule over the agents that $a_i$ observes. Formally, if we denote the number of agents in an area $A$ by $h(A)$ and similarly the number of contaminated agents in $A$ by $c(A)$, then the state of agent $a_i$ is decided by the following update rule:

$$s(a_i) = \left\{ \begin{array}{ll}
  s_H, & h(O(a_i)) \geq c(O(a_i)) \\
  s_C, & c(O(a_i)) > h(O(a_i))
\end{array} \right.$$

The observation of each agent in $S$ changes through time. Consequently, the state of all agents might change as well. Denote the observation (resp. state) of agent $a_i \in S$ at time $t$ by $O_t(a_i)$ (resp. $s_t(a_i)$). We say that an agent $a_i \in S$ is conquered at time $t + 1$ if $s_{t+1}(a_i) = \{s_H, s_C\} \setminus \{s_t(a_i)\}$. The goal of each agent in the game is to conquer agents of the opposing state, i.e., the goal of healthy agents is to conquer contaminated agents and vice versa.

**The Contamination Game**

The contamination game is a time-continuous two-player game of length $T$ ($T$ is unknown to the players). Each player controls swarm members that share the same state and has full knowledge of the state of all the members of the swarm. In each time-step each agent decides its state based on a majority rule which is applied on its surroundings. The game ends when either one player controls all the swarm members or $T$ time-steps had passed. The goal of each player in the game is to control the majority of swarm members at the end of the game.

Note that since $T$ is unknown to the players, they constantly strive to maximize the number of agents in their swarm.

**Graph Representations**

The observations of agents can be depicted by an observation graph.

**Definition 1.** Let $G_t(S) = (V_t(S), E_t(S))$ be the undirected observation graph of the contamination game of the agents in $S$ where

$$V_t(S) = S$$

$$E_t(S) = \{(a_i, a_j) | a_i \in O(a_j)\}$$

Simply put, each agent in the contamination game is represented as a node in the observation graph and each edge between two nodes represents two agents that can observe one another.

**Figure 2.** An example of a connected components graph containing five connected components of agents.

Furthermore, we can simplify the representation of the contamination game by interpreting it as a group of connected components of agents which share the same state. Each pair of agents that belong to the same connected component necessarily have a path between their nodes in the observation graph.

**Definition 2.** Let $G_t^{cc}(S) = (V_t^{cc}(S), E_t^{cc}(S))$ denote the undirected graph of connected components in the contamination game of the agents in $S$ at time $t$. Each node $v \in V_t^{cc}(S)$ represents a connected component of agents in the observation graph $G_t(S)$ which share the same state. For any pair of components $v_i, v_j \in V_t^{cc}(S)$ with opposing states, there will be an edge in $E_t^{cc}(S)$ if there is any agent in $v_i$ that can observe another agent in $v_j$.

Let $h(V_t^{cc}(S))$ (resp. $c(V_t^{cc}(S))$) be the set of healthy (resp. contaminated) components in the contamination game of the agents in $S$ at time $t$.

Any pair of connected components that are connected by an edge in the connected components graph must be of opposing states since if they share the same state they will be a part of the same connected component. Figure 2 illustrates an example of a connected components graph describing an instance of the contamination game. The healthy and contaminated agents are colored in cyan and red, respectively. The cyan and red edges represent the connections between healthy and contaminated agents, respectively. The black edges are the edges of the connected components graph.

**The WPC Algorithm**

Each swarm in the contamination problem can be described as a set of connected components of agents. Therefore, a swarm that acts optimally in the contamination problem must have strong connected components according to the rules of the problem, that is, connected components that are hard to conquer. Hence, a metric that evaluates the strength of connected components of agents in the contamination problem can be utilized to develop an optimal strategy for swarm members. An intuitive measure for the strength of a connected component would be the minimal number of required agents to conquer it. We start by defining the strength of an individual agent, which is the number of agents it observes that share its own state.

**Definition 3.** Given a connected component of agents $C \in V_t^{cc}(S)$, the connectivity factor of an agent $a_i \in C$ at time $t$, denoted by $c_{f_t}(a_i)$, is defined to be the number of
agents of $C$ that $a_i$ can observe at time $t$, that is
\[
c_{f}t(a_i) = \left\{ \begin{array}{ll}
b_t(O_t(a_i)), & s(a_i) = s_H \\
\epsilon(O_t(a_i)), & s(a_i) = s_C
\end{array} \right.
\]

In order to define the strength of a connected component of agents we first need to inspect the first line of defence which guards the agents in the interior of the structure, which are the agents on the exterior of the structure. We refer to this group of agents as the fence of the component, while each individual agent is referred to as a bare agent. Formally, we define it as follows:

**Definition 4.** Given a connected component of agents $C \in \mathcal{V}^c(V_S)$ an agent $a_i \in C$ is said to be bare at time $t$ if there is some part of its observation area that can be intruded by other agents outside of $C$. Moreover, the fence of $C$ at time $t$, denoted by $F_t(C)$, is the group of bare agents in $C$, or formally
\[
F_t(C) = \{ a_i \in C \mid O_t(a_i) \setminus \bigcup_{a_j \in C \setminus \{ a_i \}} O_t(a_j) \neq \emptyset \}
\]

Figure 3 demonstrates the notion of a bare agent in a connected component of agents. It depicts a connected component of thirteen healthy agents where the bare agents are represented by yellow filled circles whereas the other agents are represented by cyan filled circles.

We wish to be able to measure the strength of each agent on the fence of the connected component, which is measured by the minimal number of required agents to conquer it.

**Definition 5.** The bareness factor of a bare agent $a_i \in S$ at time $t$, denoted by $b_t(a_i)$, is the number of required agents to force $a_i$ to switch its state.

The connectivity factor can be used as a lower bound for the bareness factor. In other words, for each agent $a_i \in S$ we can say that
\[
c_{f}t(a_i) + 1 \leq b_t(a_i)
\]

Initially, it may seem that this bound is the actual value of the bareness factor since if healthy agent $a_i$ observes $c_{f}t(a_i) + 1$ contaminated agents at time $t$ then it will necessarily change its state based on the applied majority rule. However, we must also take into account the fact that the $c_{f}t(a_i) + 1$ contaminated agents must preserve their own contaminated state. Therefore, we need to make sure that the number of healthy agents observed by each contaminated agent is at most the number of contaminated agents observed by it. This can result in scenarios where an opponent must use more than $c_{f}t(a_i) + 1$ agents to conquer $a_i$. As an example, consider the connected component presented in Figure 4. It can easily be seen that all the agents in the connected component are bare, meaning that an opponent can conquer each one of them. Assume the opponent chooses to conquer agent $a_1$.

Consider the area observed by the agent $a_1$. This area can be described as a union of sub-areas such that each sub area is observed by a different subset of agents from $\{a_1, a_2, a_3, a_4\}$. Figure 5 shows the partition of the observation area of $a_1$ into several sub-areas based on the observations of the other agents. The strength of the cyan color of an area is proportional to the number of agents observing this area. The connectivity factor of $a_1$ is 1 since $a_1$ can only observe $a_4$. In order to find the bareness factor of $a_1$ we must devise a way to conquer $a_1$ using the minimal number of agents. We must take into account the partition of the observation area of $a_1$ presented in Figure 5 throughout
this process since if an attacking agent is located in an area which is observed by \( n > 0 \) healthy agents than it must observe at least \( n \) contaminated agents to preserve its state.

While attacking \( a_1 \) we must consider the physical dimensions of the agents. Given a sub area of the observation area of \( a_1 \), denoted by \( O' \subseteq O(a_1) \), we can place a contaminated agent in \( O' \) if it physically fits in \( O' \). In other words, each sub area of \( O(a_1) \) can inhabit a limited number of contaminated agents in a way that they can observe one another. To the purpose of our example, we assume that the the sub area of \( O(a_1) \) which is observed by only \( a_1 \) can inhabit only a single contaminated agent based on the agent’s diameter \( D_r \).

Therefore, we must place an agent in an area of \( O(a_1) \) that is observed by at least two healthy agents. Consequently, this will require placing another agent to preserve the contaminated state of the agents which were previously placed during the attack. This leads to the fact that each possible attack against \( a_1 \) will require more than \( c_f(a_1) + 1 = 2 \) agents, i.e., \( b_1(a_1) > c_f(a_1) + 1 \).

As an example, Figure 6 illustrates a possible attack which conquers \( a_1 \) using four contaminated agents. In this paper we use the aforementioned lower bound as an approximation of the bareness factor of each agent in the connected component. In other words, from now on when we refer to the bareness factor of an agent \( a_i \), at time \( t \) we refer to the value of the lower bound: \( c_f(a_i) + 1 \). From the attacker’s perspective, there are several strategies that can be used when attacking a connected component of agents \( C \in V^{cc}(S) \).

Initially, the attacker can only conquer agents which are on the fence of \( C \) and he can do it by conquering one agent at a time or multiple at once. In most cases, it is infeasible to conquer all of the agents in \( C \) at once. Therefore, an attacker must perform an iterative attack in which he conquers a subset of agents of \( C \) at a time, this subset can even contain all of the agents in \( C \) when we can conquer all of them at once. We refer to an algorithm that performs such an attack as an iterative conquering algorithm, or iterative algorithm, in short.

There is a large variety of possible iterative algorithms that can be used to conquer a connected component of agents. Let \( I \) be the set of all the possible iterative conquering algorithms. We wish to find the iterative algorithm \( A \in I \) which conquers a connected component while allocating the minimal number of agents in the process. The structure of each iterative algorithm is similar. Iterative algorithms mainly differ in the decision of which agents will be conquered at each iteration of the algorithm.

**Definition 6.** The decision rule of an iterative conquering algorithm \( A \in I \), denoted by \( D^A : V^{cc}(S) \times 2^{V(S)} \times \mathbb{Z}^+ \rightarrow 2^{V(S)} \), is a function that receives a connected component of agents, the set of agents that were already conquered by algorithm \( A \) and the current time of the game and returns the set of agents that should be conquered by algorithm \( A \) at its next iteration.

The decision rule embodies the core of the iterative algorithm as it is the function that decides which agents of the connected component will be conquered at each iteration.

Algorithm 1 describes the general form of an iterative conquering algorithm. The algorithm iteratively conquers agents of the connected component and stops when the whole connected component is conquered. At each iteration the algorithm uses its decision rule to decide which subset of agents must be conquered at the current iteration. The algorithm computes the maximal predicted bareness factor of all the agents in the chosen subset, the predictions are made using the \( \text{BFF} \) function which receives an agent and the set of all conquered agents and returns its predicted bareness factor.

If the maximal bareness factor is larger than the total number of agents accumulated thus far, then it means that more
agents are required to conquer the current subset of agents. Otherwise, we simply conquer the subset of agents without allocating any additional agents in the process. Finally, the algorithm returns the number of required agents to conquer all the agents in \( C \).

The execution of an iterative algorithm \( A \in \mathbb{I} \) on a connected component \( C \in V^{cc}(S) \) can be described by a function which maps each iteration to the set of agents conquered during it.

**Definition 7.** Given a connected component \( C \in V^{cc}(S) \) and iterative algorithm \( A \in \mathbb{I} \), the attacking sequence based on \( A \), denoted by \( \phi^A_k : Z \rightarrow 2^n \), maps each iteration of the attack against \( C \) according to \( A \) to the subset of agents of \( C \) that will be conquered in this iteration. Further, let \( \Phi(C) \) be the set of all the possible attacking sequences of \( C \).

In other words, when conquering connected component \( C \in V^{cc}(S) \) at time \( t \) by following attacking sequence \( \phi_C \), we conquer the agents in \( \phi_C(t) \) at iteration \( t \) of our iterative attack.

**Definition 8.** The length of an attacking sequence \( \phi_C \in \Phi(C) \) of connected component \( C \in V^{cc}(S) \), denoted by \( \ell(\phi_C) \), is defined to be the number of steps which are required to conquer \( C \) according to \( \phi_C \). Formally,

\[
\ell(\phi_C) = \max_{i \in \mathbb{Z}} \{ | \phi_C(i) | \neq \emptyset \}
\]

Given a connected component \( C \in V^{cc}(S) \) and an attacking sequence \( \phi_C \in \Phi(C) \), let \( N(\phi_C) \) be the total number of required agents to conquer \( C \) based on attacking sequence \( \phi_C \). It is equivalent to the output of the iterative algorithm that executes the attacking sequence \( \phi_C \), i.e., given iterative algorithm \( A \in \mathbb{I} \) and connected component \( C \in V^{cc}(S) \) we have that \( A(C) = N(\phi_C) \). Given a connected component \( C \in V^{cc}(S) \) and iterative algorithm \( A \in \mathbb{I} \) (resp. attacking sequence \( \phi_C \in \Phi(C) \)), denote by \( c_i^A(C) \) (resp. \( c_i^{\phi_C}(C) \)) the value of \( c \) at the \( i \)-th iteration of \( A \) (resp. \( \phi_C \)) given input \( C \). Similarly, denote by \( r_i^A(C) \) (resp. \( r_i^{\phi_C}(C) \)) the value of \( r \) at the \( i \)-th iteration of \( A \) (resp. \( \phi_C \)) given input \( C \).

**Definition 9.** A singular attacking sequence is an attacking sequence \( \phi_C \in \Phi(C) \) such that \( \ell(\phi_C) = 1 \).
sequence if it produces an iterative attack that conquers a single agent at each iteration. Formally, $\phi_C$ is singular if 

$$|\phi_C(i)| = 1 \forall 0 \leq i \leq \ell(\phi_C)$$

It can be seen that every attacking sequence $\phi_C \in \Phi(C)$ can be transformed into a singular attacking sequence $\phi_C'$. Moreover, we can show that $N(\phi_C) \geq N(\phi_C')$.

**Lemma 1.** Given a connected component $C \in V_{\ell_0}(S)$ at time $t$, each attacking sequence $\phi_C \in \Phi(C)$ which is not singular can be transformed into singular form $\phi_C' \in \Phi(C)$ such that $N(\phi_C) \geq N(\phi_C')$.

**Proof.** First, we describe a procedure that converts a given attacking sequence $\phi_C \in \Phi(C)$ into a singular attacking sequence $\phi_C' \in \Phi(C)$.

The TransformSequence procedure converts a given attacking sequence $\phi_C \in \Phi(C)$ to one of singular form $\phi_C' \in \Phi(C)$ by going over the iterations of the attacking sequence and decomposing each iteration $0 \leq j \leq \ell(\phi_C)$ of $\phi_C$ that conquers several agents into $|\phi_C(j)|$ iterations in $\phi_C'$ while maintaining the iterations that conquer a single agent (lines 6-8). In lines 8-19 of the procedure we predict the bareness factor of each agent in $\phi_C(i)$ such that $N(\phi_C) \geq N(\phi_C')$. Moreover, we can show that $\phi_C(i)$ can be transformed into singular form $\phi_C'$ if it produces an iterative attack that conquers a single agent.

Given attacking sequence $\phi_C \in \Phi(C)$, assume there is at least one iteration $0 \leq i \leq \ell(\phi_C)$ where $|\phi_C(i)| > 1$ and let $0 \leq j \leq \ell(\phi_C)$ be the first such iteration, meaning that $c_{j+1}^{\phi_C} = c_{j}^{\phi_C}(C)$.

Let $\phi_C(j) = \{a_1, \ldots, a_{n+1}\}$ be the agents in $\phi_C(j)$ ordered by their bareness factors, i.e.,

$$u_{\phi_C(C)}^{\phi_C}(C, a_{i_k}) \leq u_{\phi_C(C)}^{\phi_C}(C, a_{i_{k+1}}) \forall 0 \leq k < |\phi_C(j)|$$

For convenience, we will have that $\Delta_{i_k} := \Delta_{i_k}^{\phi_C}(C, a_{i_k})$. Once we conquer the agent $a_{i_k}$ it effects the values of $\Delta_{i_k}$ for any $k < m \leq |\phi_C(j)|$. Let $\alpha_{i_k}$ be the value of $\Delta_{i_k}$ when all the agents $\{a_{i_1}, \ldots, a_{i_{k-1}}\}$ are conquered. We will get the following series:

$$\alpha_{i_1} = \Delta_{i_1}, \alpha_{i_2} = \Delta_{i_2} - \Delta_{i_1} = 1, \ldots, \alpha_{i_n} = \Delta_{i_n} - \sum_{k=1}^{n-1} \alpha_{i_k} - (n-1)$$

Since the agents are ordered based on their bareness factors we know that

$$r_{j_k}^{\phi_C}(C) = r_{j_k}^{\phi_C}(C) + \alpha_{i_n}$$

Whereas using attacking sequence $\phi_C'$ and conquering the agents in $\phi_C(j)$ the required number of agents increases by $\sum_{k=1}^{n-1} \alpha_{i_k}$ which is maximal when $\alpha_{i_k} > 0$ for every $i_k < i_{k+1} \leq i_n$. Therefore, we have that

$$\sum_{k=1}^{n} \alpha_{i_k} = \sum_{k=1}^{n} \Delta_{i_k} - \sum_{m=1}^{n-1} \alpha_{i_m} - (k-1) = \Delta_{i_n} - \sum_{k=1}^{n-1} \alpha_{i_k} - (n-1) + \sum_{k=1}^{n-1} \Delta_{i_k} - \sum_{m=1}^{n-1} \alpha_{i_m} = \Delta_{i_n} - (n-1) + \sum_{k=1}^{n-1} \Delta_{i_k} - \sum_{m=1}^{n-1} \alpha_{i_m} = \Delta_{i_n} - (n-1) < \Delta_{i_n}$$

This means that the required number of agents reduces when we conquer the agents of $\phi_C(j)$ one by one. This applies to each iteration $0 \leq j \leq \ell(\phi_C)$ where $|\phi_C(j)| > 1$. Hence, we get that $N(\phi_C) \geq N(\phi_C')$.

We can infer from Lemma 1 that an iterative algorithm that returns the minimal number of required agents to conquer a connected component of agents must operate based on a singular attacking sequence. Hence, we can narrow down the possible algorithms that we inspect to those that conquer one agent at a time.
To come up with an *optimal* iterative algorithm, one must provide a decision rule which requires in total the minimal number of agents to conquer any component. Inspired by the idiom *“a chain is only as strong as its weakest link”*, we define the following:

**Definition 10.** Given a connected component of agents $C \in V^{\infty}_t(S)$ at time $t$, the *weak point* of $C$ is the bare agent with the minimal bareness factor, i.e.

$$wp(C) = \arg\min_{a_i \in F(C)} \{b_i(a_i)\}$$

Algorithm 2 represents an algorithm that iteratively conquers the weak points of a connected component of agents which we refer to as the Weak Point Conquer (WPC) algorithm.

The WPC algorithm works based on the general iterative conquering algorithm presented in Algorithm 1. The algorithm iterates over the given connected component and conquers the weak point of the connected component as described in the $wpcDr$ variable until the whole component is conquered. Going forward, we refer to the output of the WPC algorithm for the connected component of agents $C \in V^{\infty}_t(S)$, $wpc(C)$, as the WPC value of $C$.

As an example, Figure 7 illustrates an execution of the WPC algorithm on a connected component of eight healthy agents. At each iteration $i$ we conquer the weak point of the component and update $\phi^{wp}_{i+1}(C)$ and $\phi^{wp}_i(C)$ according to (*). We would like to keep track of the agents that are left of the component following $i$ iterations of an iterative algorithm $A$. Given a connected component $C \in V^{\infty}_t(S)$ and iterative algorithm $A \in \Pi$ denote by $C^A_i \subseteq C$ the set of agents that were not conquered after $i$ iterations of algorithm $A$, i.e., $C^A_i = C \setminus H^A_i$. To justify using the WPC algorithm as a measure of the quality of a connected component of agents we have to prove the correctness of the output of the algorithm, i.e., we have to show that the WPC algorithm returns the minimal number of agents which are required to conquer the given connected component as stated in the following lemma:

**Theorem 2.** Given a connected component $C \in V^{\infty}_t(S)$ of $n$ agents, $wpc(C)$ returns the minimal number of agents which are required to conquer $C$.

**Proof.** Assume by contradiction that there is an iterative algorithm $A \in \Pi$ such that $\phi^A_i(C) < \phi^{wp}_i(C)$. This means that there must be an iteration $t$ where WPC and $A$ chose to conquer different agents. Denote the agent chosen by WPC as $a_i$ and the agent chosen by $A$ as $a_j$. Since WPC

**Algorithm 2: WPC($C, t$)**

**Input:** Connected component of agents $C$.

Current time step $t$.

**Output:** Number of required agents to conquer all the agents in $C$ by iteratively conquering its weak points.

1. $wpcDr \leftarrow \{(C, H, t) \rightarrow \arg\min_{a_i \in F(C \cup H)} \{PBF(a_j, h, H)\}\}$
2. return IterativeConqueringAlgorithm($C, t, wpcDr$)

![Figure 7. Example of the execution of the WPC algorithm on a given connected component of healthy agents $C = \{a_1, a_2, \ldots, a_8\}$. Healthy agents are colored in cyan whereas the bare ones are colored in yellow. At each iteration we conquer the bare agent with the minimal bareness factor where ties are broken arbitrarily.](image-url)
always chooses the weak point we know that \( c_f^\text{wpc}(C, a_i) \leq c_f^A(C, a_j) \). This means that \( \Delta^\text{wpc}(C, a_i) \leq \Delta^A(C, a_j) \) which implies that \( r_{t+1}^\text{wpc}(C) \leq r_{t+1}^A(C) \). We can now make the observation that the connectivity factor of all the agents in the connected component can only decrease as we iteratively conquer agents of the connected component, i.e., given iterative algorithm \( \mathcal{A} \in \mathbb{I} \) and connected component \( C \in V_{t+1}^c(S) \) we have that \( c_f^\text{wpc}(C, a_i) \geq c_f^\text{wpc}(C, a_j) \). Consequently, this means that given iterative algorithm \( \mathcal{A} \in \mathbb{I} \) and connected component \( C \in V_{t+1}^c(S) \) then \( \Delta^\text{wpc}(C, a_i) \geq \Delta^\text{wpc}(C, a_j) \) for any agent \( a_i \in S \). Agent \( a_j \) must be conquered by the WPC algorithm at some iteration \( t' > t \). At iteration \( t' \) we have that \( c_f^\text{wpc}(C) > c_f^\text{wpc}(C) = c_i^A(C) \), i.e., WPC has more agents at his disposal at iteration \( t' \) in comparison to the number of agents held by algorithm \( A \) at iteration \( t \). Recall that WPC iteratively conquers the agent with the minimal connectivity factor. From the update rule which is described in \((*)\) we can claim that under the assumption that \( \Delta^\text{wpc}(C, a_i) > \Delta^\text{wpc}(C, a_j) > 0 \) we can say that

\[
\begin{align*}
 r_{t+1}^\text{wpc}(C) = r_t^A(C) + \Delta_t^\text{wpc}(C, a_i) = \\
r_t^\text{wpc}(C) + \Delta_t^\text{wpc}(C, a_j) \geq \Delta_t^\text{wpc}(C, a_j) > 0
\end{align*}
\]

We can now compare the states of algorithms \( \mathcal{A} \) and WPC following iterations \( t \) and \( t' \), we showed that the WPC algorithm has more agents at his disposal and allocated less agents than the number that was allocated by algorithm \( \mathcal{A} \). Furthermore, we can obviously see that \( w_{t+1}^\text{wpc} \leq C_{t+1}^A \) and from the fact that the connectivity factor can only decrease we can say that

\[
\Delta^\text{wpc}_{t+1}(C, a_k) \leq \Delta^A_{t+1}(C, a_k) \quad \forall a_k \in C_{t+1}^\text{wpc}
\]

To conclude, we found that following iteration \( t' \) the WPC algorithm has allocated less agents than algorithm \( \mathcal{A} \) allocated in \( t \) iterations while having more agents at his disposal. Furthermore, there are less agents left to be conquered and their connectivity factors are lower which means that we necessarily must have that \( r_{t+1}^\text{wpc}(C) \leq r_{t+1}^A(C) \) in contradiction.

Theorem 2 shows that the WPC is the sole metric which can be used to quantify the number of agents that are required to conquer a connected component \( C \), this means that \( \text{wpc}(C) \) can be seen as a metric for the resiliency of \( C \). It is important to note that no other metrics should be examined since we have shown that the WPC metric computes the exact minimal number of agents that are required to conquer any component.

**Placing Agents in Connected Components**

We can conclude from the previous subsection that agents in the contamination game should aim to form connected components that have high WPC values. The WPC value of a connected component of agents \( C \in V_{t+1}^c(S) \) does not solely depend on the number of agents in \( C \). The way the agents in \( C \) are placed relative to the center of the connected component and to one another, or \( C \)'s placement, plays a major role in determining the WPC value of \( C \). Formally, we define the placement of a component as follows:

**Definition 11.** Given a connected component \( C \in V_{t+1}^c(S) \) of agents in \( k \)-dimensional space, a placement \( p : [1, n] \mapsto \mathbb{R}^k \) is a mapping between each agent’s index in the component to its position relative to the center of the component. We denote the group of possible placements for connected component \( C \) as \( \mathcal{P}(C) \). We denote the result of applying placement \( p \in \mathcal{P}(C) \) on the agents of \( C \) by \( C_p \), and the current placement of \( C \) by \( pc \).

Figure 8 represents two different placements of a connected component of agents, \( C = \{a_1, a_2, a_3\} \). It illustrates the effect of the placement of agents on the value of \( \text{wpc}(C) \). In the first subfigure the agents in \( C \) are connected in a straight line such that the weak point of the connected component has a connectivity factor of 1 which means that \( w_{\text{pc}}(C) = 1 \). In the second subfigure all the agents in \( C \) are connected to one another. Hence, we have that \( w_{\text{pc}}(C) = 2 \) which is larger than the value of \( w_{\text{pc}}(C) \) in the first subfigure.

As previously mentioned, we search for a placement of agents that maximizes the WPC value of connected components in our swarm, that is, an optimal placement.

**Definition 12.** The optimal placement of a connected component \( C \in V_{t+1}^c(S) \), denoted by \( p^*(C) \), is the placement which maximizes the output of the WPC algorithm for the component, that is,

\[
p^*(C) = \arg \max_{p \in \mathcal{P}(C)} \text{wpc}(C_p)
\]

We refer to the value of \( \text{wpc}(C_{p^*(C)}) \) as the maximal WPC of \( C \).

In previous work (Avrahami and Agmon (2019)), it was shown that for each pair of values of \( S_{\text{min}} \) and \( S_{\text{max}} \) there is a maximal size of a clique of agents, which was referred to as a maximal stable cycle (MSC), we denote this value by \( c(S_{\text{min}}, S_{\text{max}}) \), or \( c \), in short. It can easily be seen that as long as the number of agents in a connected component is less than or equal to \( c \) the optimal placement of the component would be a clique of agents whereas when the

---

**Figure 8.** Example of the effect of the placement of agents on \( \text{wpc}(C) \), where \( C = \{a_1, a_2, a_3\} \) is a connected component of three healthy agents.
The complexity of the problem of finding an optimal placement of $n$ agents in a given area that maximizes the WPC of a fixed fence as the Fixed Maximum WPC (FMWPC) problem. Since we must handle concealments of agents while placing agents in a given area, the effect of placing an agent in a given location $\ell$ in area $A$ can be described by a utility function that receives the observing agent from the fence, locations of agents placed in $A$ and the location in $A$ in which we wish to place an agent. Formally, given a set of agents $C$ and set of locations in two-dimensional space $A$, a utility function for the FMWPC problem, denoted by $u : C \times \mathbb{R}^2 \times 2^A \to \mathbb{R}$, computes the effect of placing an agent in a given location in our two-dimensional space given the locations of the other placed agents in area $A$ and the observing agent from $C$. Formally, the problem is defined as follows:

**Fixed Maximum WPC Problem (FMWPC):**

- **Instance:** A fixed fence of $m$ agents $C = \{a_1, \ldots, a_m\}$, a set of locations in which we can place agents $A$, a number $n$ and a utility function $u$.
- **Objective:** Find the subset of at most $n$ locations from $A$ that maximizes the minimal utility of any agent in $C$, that is,

$$\arg\max_{A' \subseteq A \text{ s.t. } |A'| \leq n} \min_{a_i \in C} \sum_{\ell \in A'} u(a_i, \ell, A')$$

The FMWPC problem is NP-hard, by reduction from the geometric version of the Minimum Membership Set Cover problem (MMSC) (Erlebach and van Leeuwen (2008)). In this problem a set of points must be covered using a given set of circles such that the maximal number of circles covering the same point is minimal.

**Theorem 3.** The FMWPC problem is NP-hard.

**Proof.** We prove this theorem by reduction from the geometric version of the Minimum Membership Set Cover problem. Let $P$ and $R$ be the set of points and circles in an instance of the MMSC problem where the circles have a radius of $r > 0$. Given the instance $(P, R)$ a reduction can be made to an instance of the FMWPC problem using the following set of steps.

First, let $P$ be the locations of the set of agents in $C$. Secondly, let the set of centers of the circles in $R$ be the set of available locations $A$ and set $n$ to be equal to $|R|$. Finally, define our utility function to be the following:

$$u(a_i, \ell, A') = \begin{cases} -1 & \|a_i - \ell\|_k \leq r \\ 0 & \text{Otherwise} \end{cases}$$

That is, the utility of an agent is the negation of the number of agents which are at a distance smaller than $r$ from it. The set of aforementioned transformations results in an instance of the FMWPC problem. It remains to be shown that a solution to the FMWPC problem is equivalent to a solution of the initial instance of the MMSC problem.

In the MMSC problem the *membership* of a point $p \in P$ in a subset of circles $R' \subseteq R$, denoted by $\text{mem}_{R'}(p)$ is defined...
to be the number of circles in $\mathcal{R}'$ containing $p$. That is, if the radii of all the circles in $\mathcal{R}$ are equal to $r$ the membership of a point $p \in \mathcal{P}$ in a given subset of circles $\mathcal{R}' \subseteq \mathcal{R}$ is the number of circles in $\mathcal{R}'$ whose centers are closer to $r$ to $p$.

The objective of the MMSC problem is to find the subset of circles which results in the minimal maximal membership of a point in $\mathcal{P}$ and covers all the points in $\mathcal{P}$, or formally find the following subset of circles:

$$R^* = \arg \min_{\mathcal{R}' \subseteq \mathcal{R}} \max_{p \in \mathcal{P}} \max_{p \in \mathcal{P}} \text{mem}_{\mathcal{R}'}(p)$$

In the case of the aforementioned instance of the FMWPC problem, the objective is to find the subset of locations in $A$ of size at most $|\mathcal{R}|$ such that the minimal utility of any agent in $C$ is maximal. The utility of an agent in $C$ in the reduced instance of the FMWPC problem is equal to the negation of the membership of its matching point in $\mathcal{P}$. Therefore, given a subset of circles $\mathcal{R}'$ and its matching set of locations in $A$ (the centers of these circles), the point with the maximal membership is the location of the agent in $C$ with the minimal utility. Hence, the solution to the reduced instance of the FMWPC is the subset of locations in $A$ which results in the maximal minimal utility, or in other words, the set of circles in $\mathcal{R}$ that result in the maximal maximal membership in the initial instance of the MMSC problem.

This shows that the proposed polynomial reduction results in an instance of the FMWPC problem such that a solution to the reduced instance corresponds to a solution for the initial instance in the MMSC problem which proves our claim.

The hardness of the FMWPC problem directly implies that the problem of finding the optimal placement of $n$ agents in a monotonic connected component is NP-hard. Altogether, Theorem 3 leads to the following corollary:

**Corollary 1.** The problem of finding the optimal placement of $n$ agents in a general connected component is NP-hard.

Since finding the optimal placement among monotonic connected components is NP-hard we can directly conclude that finding the optimal placement among general connected components is NP-hard.

### Bounding the WPC Value

We have shown that finding an optimal placement is NP-hard. This means that constructing an algorithm that finds an optimal placement for a given number of agents is improbable. Nevertheless, finding an upper bound for the WPC value of a connected component can help us come up with an optimal strategy for players in the contamination game.

In this section, we perform a set of steps to compute the upper bound of the WPC value of a connected component of agents. First, we present a tight bound on the number of agents that can be observed by a single agent in the contamination game.

**Lemma 4.** Given an agent $a_i$ that observes a sector of its observation area. The maximal connectivity factor of $a_i$ is achieved by placing agents densely along the longest convex arc of the sector it observes.

**Proof.** Given a pair of agents $a_i$ and $a_j$ it can easily be seen that the closest $a_j$ is to $a_i$ the larger is the observation area it conceals from $a_i$. Figure 10 displays the effect of the distance between two agents on the area that one of them conceals from the other. To maximize the connectivity factor of an agent we must place agents in its observation area while minimizing the portion of its observation area which is concealed from it. Therefore, we can conclude from our earlier observation that to maximize the connectivity factor of an agent we must place agents as furthest as possible from it, that is, we must place agents at a distance of $S_{\text{max}}$ from it along the longest convex arc of the sector.

**Lemma 5.** The maximal connectivity factor of an agent in the contamination game is

$$\frac{2\pi}{\arccos \left(1 - \frac{2D^2}{S_{\text{max}}^2}\right)}$$

**Proof.** We know from Lemma 4 that to maximize the connectivity factor in a sector of the observation area of an agent we must place agents densely along the longest convex arc of the sector. This means that to maximize the connectivity factor of an agent we must place agents densely along the circumference of its observation area, that is, we need to compute the number of agents that can be placed at a distance of $S_{\text{max}}$ from it. Each agent that is placed along the circumference of the circle can be represented as a sector of the circle whose arc can be computed using the cosine rule as depicted in Figure 11. Using the cosine rule we get that the size of the arc is

$$\alpha = \arccos \left(1 - \frac{2D^2}{S_{\text{max}}^2}\right)$$

Hence, the maximal number of agents that can be placed along the circumference of the circle is

$$\left\lfloor \frac{2\pi}{\alpha} \right\rfloor = \left\lfloor \frac{2\pi}{\arccos \left(1 - \frac{2D^2}{S_{\text{max}}^2}\right)} \right\rfloor$$

Since the WPC value of a component is bounded by the maximal connectivity factor of any weak point of a component of agents, the aforementioned bound can be used as a non tight bound for the WPC value of a component of any number of agents.

To achieve a tighter bound we must analyze the properties of the possible observation areas of agents that are weak points of connected components of agents.

**Lemma 6.** The maximal connectivity factor of a weak point of a connected component in the contamination game is bounded by

$$\frac{2\pi}{\arccos \left(1 - \frac{2D^2}{S_{\text{max}}^2}\right)}$$

Figure 10. When an agent $a_i$ observes another agent $a_j$. The further they are from one another the smaller the area that each one of them conceals from the other. In each subfigure the distance between $a_i$ and $a_j$ increases and accordingly the size of the concealed area decreases.
Proof. The fence of a connected component of agents always forms a closed polygon. We showed in Lemma 5 that placing agents along the circumference of the $S_{\text{max}}$ circle maximizes the connectivity factor of an agent. In this case, to maximize the connectivity factor of a bare agent we must place agents along the circumference of the sector of the circle whose angle is the internal angle of the polygon.

Hence, to bound the connectivity factor of the weak point we must have an upper bound for the minimal internal angle in the polygon. Since the polygon is closed it can be concluded that the minimal internal angle cannot exceed $\pi$. Therefore, the connectivity factor of the weak point is bounded by the number of agents that fit in the circumference of the sector of the circle whose angle is $\pi$. Using similar algebra as the one used in Lemma 5 we have that the connectivity factor of the weak point is bounded by

$$\max_{a_i \in C} \left\{ cf(a_i) \right\} \leq 2 \ \text{if } C \in V_{t}^\text{cc}(S) \ \text{such that } |ES(C)| = 1$$

Even though the aforementioned lemma presents a tighter bound for the WPC value than the one presented in Lemma 5 it is still not a tight bound. To tighten the bound we must delve into the properties of observations of bare agents in the contamination game. Assume that there is a component $C \in V_{t}^\text{cc}(S)$ that has a maximal WPC value for any number of agents, that is, for any $C' \in V_{t}^\text{cc}(S)$ of any size we have that $wpc(C') \leq wpc(C)$. We show that this component is monotonic by proving the following lemma:

Lemma 7. If there exists an optimal component $C \in V_{t}^\text{cc}(S)$ it must be monotonic, and furthermore it must be that $|ES(C)| = 1$

Proof. Assume by contradiction there is an optimal component $C \in V_{t}^\text{cc}(S)$ such that $|ES(C)| > 1$. That means that in addition to $wpc(C)$ there is an additional agent $a_i \in C \setminus \{wpc(C)\}$ that forced the WPC algorithm to allocate agents to conquer it. As previously stated, the number of agents allocated by the WPC algorithm can only increase as we iteratively conquer the weak points of the component. Therefore, if $a_i$ forced the WPC algorithm to allocate agents following the loss of $wpc(C)$ it means that the weakpoint of $C \setminus \{wpc(C)\}$ has a higher connectivity factor than that of $wpc(C)$. Hence, we can conclude that $wpc(C) < wpc(C \setminus \{wpc(C)\})$ which contradicts the optimality of $C$. Furthermore, we can conclude that $ES(C) = \{wpc(C)\}$ which means that $|ES(C)| = 1$. □

We can conclude from Lemma 7 the following corollary:

Corollary 2. Given an optimal component $C \in V_{t}^\text{cc}(S)$, the maximal difference in connectivity factors of two bare agents is bounded by 2, that is,

$$\max_{a_i, a_j \in F(C)} \left| cf(a_i) - cf(a_j) \right| \leq 2$$

Proof. According to Lemma 7, if there is an optimal component $C \in V_{t}^\text{cc}(S)$ we have that $|ES(C)| = 1$. This means that besides the first iteration there is no iteration of the WPC algorithm that allocates additional agents. After conquering the first weak point of the component we have $cf(wpc(C)) + 2$ agents at our disposal. Therefore, we can combine this with the fact that $|ES(C)| = 1$ to get that

$$cf(a_i) < cf(wpc(C)) + 2 \ \forall a_i \in C \setminus \{wpc(C)\}$$

The connectivity factor of an agent can decrease by at most one following a single iteration of the WPC algorithm. Therefore, we can conclude that

$$cf(a_i) \leq cf(wpc(C)) + 2 \ \forall a_i \in C$$ □

The aforementioned corollary brings us closer to finding the optimal component which represents a tighter bound for the WPC value of a connected component. We can conclude from it that in an optimal component $C \in V_{t}^\text{cc}(S)$ the WPC value is directly determined based on the minimal connectivity factor of an agent on the fence of the component, that is,

$$wpc(C) = \min_{a_i \in F(C)} cf(a_i) + 1$$

To ease our effort, we wish to focus on a specific subset of components which is defined as follows:

Definition 14. A component $C \in V_{t}^\text{cc}(S)$ will be referred to as a bare component if all of its agents are bare, that is, $F(C) = C$.

Any bare component can be represented as a simple polygon where the locations of the agents represent the points of the polygon. The polygon is simple since if we would assume that there is a hole in the polygon or that it intersects itself it would mean that there is necessarily an agent which is not bare which contradicts the assumption that the component is bare.

Definition 15. A component $C \in V_{t}^\text{cc}(S)$ is called a symmetric component if all of its bare agents observe the exact same image (only change is orientation).

The convex hull of a symmetrical component has the shape of a regular polygon, that is, a simple polygon where all sides and angles are congruent. This notion helps us prove the following lemma:
Lemma 8. The optimal component $C^* \in V^c_i(S)$ must be symmetric.

Proof. The sum of angles of a regular polygon with $n$ sides is $(n - 2) \pi$. In a component $C \in V^c_i(S)$ each agent in the convex hull $a_i \in C$ has an internal angle $\alpha_i$ with a size that is directly correlated to the size of the area that $a_i$ observes inside $C$ (see Figure 12). In a symmetrical component all the agents have the same internal angle $\alpha^* = \frac{2n-2}{n} \pi$, that is, they observe an equal-sized area in the component. We know from Lemma 7 that the WPC value of the optimal component is determined by the lowest connectivity factor of an agent on the fence of the component. Since we are looking at an optimal component we have that

$$wpc(C^*) = \min_{a_i \in F(C^*)} cf(a_i) + 1$$

Assume by contradiction that there is a component $C \in V^c_i(S)$ which is not regular and has a higher WPC value than $C^*$. Since $C$ is not regular there is an agent $a_i \in C$ such that $\alpha_i < \alpha^*$. This means that

$$cf(a_i) < cf(a_j) \quad \forall a_j \in C^*$$

which contradicts the assumption that $wpc(C) > wpc(C^*)$. \qed

Since we view the agents in a two dimensional space, any symmetric component must have a convex hull that has the shape of a regular polygon (we refer to this simply as a circle). In the case of bare components it means that all the agents are necessarily placed along the circumference of a circle. To maximize the connectivity factor of agents along the circle we intuitively need to place agents densely along the circumference of the circle as defined as follows:

Definition 16. A dense circle of radius $r$, denoted by $DC_r \in V^c_i(S)$, is a component of agents which are placed densely along the circumference of a circle of radius $r$.

Figure 13 illustrates an example of a dense circle when $S_{max} = 4$ and $D_R = 0.25$.

Lemma 9. Any dense circle of any radius has a WPC value that is bounded by that of a dense circle with a radius of $\frac{S_{max}}{2}$, that is,

$$wpc(DC_r) \leq wpc(DC_{\frac{S_{max}}{2}}) \quad \forall r \geq 0$$

Proof. In the formation of a dense circle each agent only observes a portion of the circle while there is another portion that is concealed from it. We can show that as the radius of the dense circle increases the number of agents which are concealed from the component increases. Given an agent $a_i$ on the circle, the closest agent it can observe is the one for which the chord between their centers is at a distance smaller than $r - D_r$ from the center of the circle. Figure 14 illustrates a dense circle and the computation of the closest agent observed by an agent in the circle. As seen in the figure we can compute the angle of the circle from which we can observe agents relative to a given agent (in the case of the figure it is the one marked in red). We have that

$$r - D_r = 1 - \frac{D_r}{r} > \frac{x}{r} = \cos \left( \frac{\beta}{2} \right)$$

$$\beta = 2 \text{arccos} \left( \frac{x}{r} \right) > 2 \text{arccos} \left( 1 - \frac{D_r}{r} \right)$$

Hence, two sectors (both clockwise and counterclockwise) of the circle of angles $\beta = 2 \text{arccos} \left( 1 - \frac{D_r}{r} \right)$ cannot be
observed by each agent in the dense circle. According to a previous computation the number of agents that can fit in a sector of a circle that has an angle \( \beta \) is

\[
\frac{\beta}{\arccos (1 - \frac{2D_c}{r^2})} = \frac{2 \arccos (1 - \frac{D_c}{r^2})}{\arccos (1 - \frac{2D_c}{r^2})}
\]

(\star)

Since \( D_r << r \) the term \( \frac{D_c}{r^2} - \frac{2D_c}{r^2} > 0 \) is always positive and increases as long as \( r \) increases. Therefore, the number of agents that are concealed from each agent on the dense circle (described in (\star)) increases as \( r \) increases. There are two distinct cases for the radius of a dense circle. First, there is the case where \( r \leq \frac{S_{max}}{2} \), that is, the whole circle is contained in the observation area of each agent. In this case the each agent that is contained in the concealed sector of the circle has a matching agent on the other side of the circle that can be observed (see Figure 15). Therefore, we must have that

\[
wpc(D_C) \leq wpc(DC_{\text{max}}(S_{max})) \quad \forall r \leq \frac{S_{max}}{2}
\]

Secondly, there is the case where \( r > \frac{S_{max}}{2} \). In this case each agent cannot observe the whole component and the number of closer agents that are concealed keeps increasing. Consequently, we must have that

\[
wpc(D_C) < wpc(DC_{\text{max}}(S_{max})) \quad \forall r > \frac{S_{max}}{2}
\]

Overall, we have that

\[
wpc(D_C) < wpc(DC_{\text{max}}(S_{max})) \quad \forall r \geq 0 \quad \Box
\]

Lemma 9 shows that a dense circle with a radius of \( \frac{S_{max}}{2} \) is the dense circle with the maximal WPC value. Therefore, we refer to it as an optimal dense circle (ODC).

**Corollary 3.** The WPC value of any bare component is bounded by that of the ODC.

Up to this point we have shown that the ODC serves as a tight bound for the WPC value of a bare connected component of agents. The main question that is left unanswered is what happens in the case of components which are not bare? To answer this question we prove the following lemma:

**Lemma 10.** If there is an optimal non-bare component \( C \in V^c(S) \), its WPC value is bounded by that of the ODC, that is,

\[
wpc(C) < wpc(DC_{\text{max}}(S_{max}))
\]

**Proof.** Given a non-bare component \( C \in V^c(S) \) which is optimal, there are two distinct cases depending on the diameter of \( C \), that is, the largest distance between a pair of agents in \( C \). First, there is the case where the diameter of \( C \) is less than \( S_{max} \). In this case we can see that the minimal connectivity factor of an agent on the fence in \( C \) cannot surpass the minimal connectivity factor of an agent in \( DC_{\text{max}}(S_{max}) \) since we have shown in Lemma 8 that any bare component with a diameter lower than \( S_{max} \) cannot have a higher WPC value than that of \( DC_{\text{max}}(S_{max}) \). Furthermore, the non-bare agents that are contained in \( C \) can only decrease the connectivity factor of the bare ones. From Lemma 7 we know that the WPC value of an optimal component is determined by the minimal connectivity factor of an agent in its fence which means that \( wpc(C) < wpc(DC_{\text{max}}(S_{max})) \). Secondly, there is the case in which the diameter of \( C \) exceeds \( S_{max} \). Assume by contradiction that there is an optimal non-bare component with a diameter larger than \( S_{max} \) that has a higher WPC value than the ODC. We have shown in Lemma 9 that

\[
wpc(DC_r) < wpc(DC_{\text{max}}(S_{max})) \quad \forall r > \frac{S_{max}}{2}
\]

Let \( \Delta_r \) be the difference between the WPC values of the ODC and a dense circle of a given radius \( r > 0 \), that is,

\[
\Delta_r = wpc(DC_{\text{max}}(S_{max})) - wpc(DC_r)
\]

Therefore, we must have that each agent on the fence of \( C \) must observe at least \( \Delta_r + 1 \) non-bare agents. We know from Lemma 8 that \( C \) must be symmetric. Since we observe components in two dimensions it means that rotating \( C \) such that one agent on the fence takes the place of another will result in the exact same image of the component that we had previously. This means that besides the bare agents having the shape of a circle, the non-bare ones must have the shape of a circle as well. Due to the symmetry each agent on the fence of the component observes a convex arc of a circle whereas in \( DC_{\text{max}}(S_{max}) \) each bare agent observes a convex arc along its \( S_{max} \) circle. We know from Lemma 5 that the maximal connectivity factor of an agent who observes a sector of a circle is achieved by placing agents densely along the longest convex arc of the circle which makes it impossible that each agent in \( C \) observes more than \( \Delta_r \) agents.

The following theorem directly follows from Lemmas 9 and 10:

**Theorem 11.** The WPC value of any component is bounded by that of the ODC, or in other words, the ODC represents a tight bound for the WPC value of a connected component of agents.
Distributed Strategy for the Contamination Game

We can conclude from the results of the previous sections that the problem of reaching a consensus in the contamination game can be reduced to the problem of forming dense circles while facing adversaries. In this section, we discuss previous attempts towards solving the problem of distributed pattern formation. We then show that forming dense circles in a distributed manner is impossible due to the challenging characteristics of swarm members. Finally, we perform several relaxations to the settings of the contamination problem in order to be able to assess the performance of a strategy that gathers agents in dense circles in simulation. Our problem is an extension to the pattern formation problem (Fujinaga, Yamauchi, Ono, Kijima and Yamashita (2015)) in which a group of robots is required to form a predefined geometric pattern. In our case, a group of agents is required to place themselves on the vertices of a regular polygon to form a dense circle. In the literature, this problem is known as the uniform circle formation problem (Dieudonné and Petit (2008); Dieudonné, Labbani-Igbida and Petit (2008); Flochini, Prencipe and Santoro (2006); Flochini, Prencipe, Santoro and Viglietta (2017); Datta, Dutta, Chaudhuri and Mukhopadhyaya (2013); Jiang, Cao, Wang, Stojmenovic and Bourgeois (2017)). The uniform circle formation problem plays a crucial role in the domain of coordination problems due to the observation by Suzuki and Yamashita (1999) that uniform circles and points are the only patterns formable from any initial configuration under a fully synchronous model. The problem is challenging in the distributed domain due to the fact that the robots have to make independent decisions by their own, while avoiding collisions. Previous attempts at solving the uniform circle formation problem had sets of assumptions which are unfeasible in our work. Flochini, Prencipe, Santoro and Viglietta (2017) provided a constructive proof that the uniform circle formation problem is solvable for any initial configuration. In their work, it was assumed that all robots can observe one another which is obviously not possible under our problem’s settings. Datta, Dutta, Chaudhuri and Mukhopadhyaya (2013) proposed a distributed algorithm for circle formation by a set of oblivious mobile robots where each robot has a unit disk size. Even though it proposes a fully distributed algorithm, this work assumes that robots cannot conceal one another and that the resulting circle is not uniform. Jiang, Cao, Wang, Stojmenovic and Bourgeois (2017) proposes a new approach towards solving the uniform circle formation that constitutes three main phases: consensus on the circle, circle formation and uniform transformation. While the proposed approach showed promising simulation results, it assumes that robots can send messages back and forth between one another which is not possible in our settings. The existence of adversaries in our problem adds another layer of difficulty to the already challenging problem of distributed pattern formation. Pattanayak, Foerster, Mandal and Schmid (2020) presented the problem of distributed pattern formation where robots are susceptible to crash faults, i.e., they stop moving after the crash and never recover. In our work, the robots faults are caused by them switching their own state and adopting an adversarial behavior which is similar to the definition of byzantine faults (Amon and Peleg (2006)) in the fault-tolerant algorithms literature. To our knowledge, no study has been conducted on the problem of distributed pattern formation where the robots are susceptible to byzantine faults.

Impossibility of Coordinated Movement In Dense Circles

As previously mentioned, recent literature which proposed solutions to the distributed uniform circle formation problem assumed that once the robots reach the desired formation they stop moving. One might wonder whether converging to a stationary uniform circle might be a satisfactory solution to the contamination game. We answer this question by proposing the following example: assume there is a contamination game played between two swarms of agents where the healthy swarm is gathered in stationary dense circles while the contaminated agents are spread randomly and are stationary as well. Since there is no movement of any agent in the game it is impossible that either the healthy or the contaminated group will win the game, resulting in a definite tie between the two groups of agents. This simple example shows the importance of moving agents in their constructed formations. Therefore, we aim to develop strategies that do not only gather agents in dense circles but also move them while maintaining their formations. Unfortunately, the following theorem proves that there is no deterministic distributed algorithm that moves robots in a formation of a uniform circle under the swarm settings examined so far.

**Theorem 12.** Given a swarm of anonymous, oblivious robots that do not share the same coordinate system and use no explicit communication, there is no deterministic
algorithm that moves robots in a uniform circle where the robots’ time cycles are fully synchronous.

Proof. Assume by contradiction that there is a deterministic algorithm $A$ that moves the robots in a uniform circle. Take as an example the scenario presented in Figure 16 in which each agent observes the exact same image. Under the assumption that each agent operates based on $A$, each agent in the dense circle will be ordered to move in the same direction relative to its position in the circle. If this implies a movement inwards or outwards of the circle, this will break the formation. If this is a movement on the circumference of the circle, this does not move the formation, which concludes the proof.

Theorem 12 showed that under the settings of the contamination game we cannot propose a distributed strategy that moves agents in a uniform circle even when they are fully synchronous. To overcome this, we perform several relaxations over the initial settings of the contamination game.

**Distributed Strategy for the Simplified Contamination Game**

Following Theorem 12, we transform the settings of the contamination game by allowing the following set of relaxations:

- Agents operate in a random scheduled order in their Look-Compute-Move cycles.
- Agents are no longer anonymous, that is, each agent has its own unique identifier.
- Each agent can internally save its own state throughout the game.
- Agents share the same coordinate system.
- Each agent can communicate with each agent it observes.

We propose a distributed strategy that utilizes each of these relaxations to place agents in uniform circles. We showed in previous subsections that the lack of communication in the initial settings of the contamination game made the development of a distributed strategy for moving agents in uniform circles impossible. Therefore, we propose a distributed strategy that utilizes the assumption that the agents are fully synchronous and the (now possible) communication channels to form uniform circles of agents. Additionally, the proposed communication protocols rely on the fact that each agent has its own unique identifier, that is, there is no anonymity. As a part of our proposed solution, each agent will hold an additional state to its pre-existing healthy or contaminated state which will indicate whether it is a part of a uniform circle or not. We refer to this state as the formation-state of an agent. There are three distinct formation-states for agents in the contamination game. First, there is the single state which is the state of any agent that is not a part of a uniform circle or a group that aims to form a uniform circle. Second, there is the converging state which is the state of an agent which is part of a group of agents that moves together in order to form a uniform circle. Third, there is the circle state which is the state of an agent which is a part of an existing uniform circle. In our proposed solution, each agent adopts a different behavior for each possible formation-state.

**Single Formation-State** In the single formation-state agents aim to form an initial uniform circle of any size. To do so we propose an algorithm that is based on the idea of converging to the self-enclosing circle (SEC) of a group of agents as proposed in Flocchini, Prencipe, Santoro and Viglietta (2017). The strategy of a single formation-state agent is described in broad terms in Algorithm 3.1. We avoid delving into the intricacies of the strategy to make the idea behind it more clear. *.

In each cycle the agent is given the observation that its sensors gathered in its Look phase along with messages that were sent to it from other agents that can observe it with the goal of gathering in a uniform circle as fast as possible. Each message can have a different type where the type differs based on the relationship between the sender and the receiver of the message.

There are three distinct cases for an agent with a single formation-state. First, there is the simple case in which the agent cannot observe any other agent and we have nothing to do besides a simple random walk (lines 35-37). Second, there is the case where an agent can observe a set of single formation-state agents and they form together a clique of

*Full implementation at https://github.com/LiorMoshe/Contamination-Simulator

![Figure 17. The three step communication protocol to gather agents in uniform circles. Assume that the agents are scheduled in the order of their indices, that is, $a_1$ executes before $a_2$ which is followed by $a_3$. (i) Each agent shares its observation with its neighbors. (ii) $a_1$ identifies $\{a_1, a_2, a_3\}$ as a possible uniform circle and proposes it to $a_2$ and $a_3$. $a_2$ and $a_3$ will approve this circle since they do not have better options. (iii) $a_1$ sends an establishment message which notifies that there is an approval of all the members of the circle, agents move to converging formation-state.*
agents. In this case we use a communication protocol that is made of three time-steps whose purpose is to gather the agents in the clique together in a uniform circle as presented in Figure 17. First, each agent shares its observation with its neighboring single formation-state agents (lines 3-5). Then

Algorithm 3.1: SingleStrategy

Input: Current observation of the agent $\text{obs}$. List of messages that were sent to the agent $\text{messages}$

foreach $m \in \text{messages}$ do
  if $m$ is an observation then
    Save agent’s observation.
  else if $m$ is a circle proposal then
    Update $\text{proposal}$ if the given proposal contains a larger circle.
  else if $m$ is a circle approval then
    Save the id of the approving agent.
  else if $m$ is a circle establishment message then
    Change formation-state to CONVERGING.
    Move toward target location in the circle.
  end
end
if formation-state is CONVERGING then
  return
end
clique ← Largest clique based on observations.
if Current agent proposed a circle in the previous time-step then
  if All members of the proposed circle sent approval then
    Send an establishment message to all the approvers.
  else if Some members of the proposed circle sent approval then
    Send a message proposing a new circle which contains the approvers.
end
if $|\text{proposal}| \geq |\text{clique}|$ and $|\text{proposal}| > 1$ then
  Send approval message to the sender of $\text{proposal}$.
  return
end

Algorithm 3.1: SingleStrategy($\text{obs}$, $\text{messages}$)

32 if $|\text{clique}| > 1$ then
33   Send a circle proposal to the members of the clique.
34 else
35   if No agents are observed then
36     Move randomly.
37   else
38     Find the closest agent that has a circle formation-state.
39     Send a message proposing that you join the circle.
40   end
41 end

the first agent that is scheduled in the next time step can compute the maximal sized clique that it has among its neighbors (line 20) and send a proposal to the members of this clique to gather together in a dense circle (lines 32-34). In the following time step the members of the proposed circle will send an approval message to the proposing agent if they do not have a larger clique within their set of neighbors (lines 28-31). Finally, the proposing agent will check that it got the approval of all the agents in the potential circle. If it did it will send a message to all the members of the clique to say that the circle is established. If only a subset of the members of the potential circle approved the message then the agent will compose a new proposal containing only the approving subset (lines 21-26). The third and final case of a single formation-state agent is the case in which it observes only circle formation-state agents. In this case the agent will send a proposal to the member of the existing circle to merge together (lines 37-40). The members of the existing circle decide whether to accept or reject the proposal using a communication protocol which we cover in the upcoming subsection about the circle formation-state. If they approve the proposal both the single formation-state agent and the agents of the existing circle move to converging formation-state and gather in a new uniform circle with a diameter of $S_{max}$.

Converging Formation-State

Agents in converging formation-state interact only with other converging agents that aim to form a uniform circle with them. The strategy for the converging formation-state relies on the simple idea that if each agent will relay information regarding the convergence of the agents it can observe we will get to a point in which each agent knows whether all the members of the uniform circle converged. Notice that this works since all the agents share the same coordinate system with one another. Similarly to the single formation-state, we describe the flow of the converging strategy in broad terms in Algorithm 3.2. Each agent goes over the messages it receives and checks whether there are any messages that contain the convergence state of other agents in its circle (lines 2-5). Then, the agent checks whether it converged to its own target position in the circle in order to determine its action in the current cycle (lines 7-11). Finally, the agent sends to the agents it observes all the information it
Agents in the Circle Formation-State aim to merge together with other uniform circles to strengthen themselves. We describe the strategy for an agent in a circle formation-state broadly in Algorithm 3.3. The strategy of the circle formation-state relies on a similar idea to the one used in the converging formation-state strategy. Each agent passes its knowledge regarding the surroundings of the circle. Similarly to the strategies in other formation-states, the strategy of an agent in a circle formation-state relies on having messages of several types passed between agents in order to coordinate the dense

**Algorithm 3.2: ConvergingStrategy(obs, messages)**

Input : Current observation of the agent obs.
List of messages that were sent to the agent messages.

1. C ← set()
2. foreach m ∈ messages do
   3. if m is of type CONVERGENCE STATE and m.publisher() is in our circle then
      4. Update C based on the content of the message.
   5. end
3. end
4. if Current agent converged to its target then
   5. Add current agent to C.
   6. else
   7. Move toward the agents target location in the circle.
   8. end
9. Send the converged set of agents C in a CONVERGENCE STATE message.
10. if |C| is equal to the size of our circle then
11.   Switch to CIRCLE formation-state.
12. end

Algorithm 3.3: CircleStrategy(obs, messages)

Input : Current observation of the agent obs.
List of messages that were sent to the agent messages

1. mode ← Current circle mode.
2. foreach m ∈ messages do
3. if m is a merge proposal then
4.   Save the given proposal in our state.
5. if m’s circle was previously proposed to by the current agent then
6.   Send approval message to all the agents we observe.
7. Move to CONVERGING formation-state and update target location.
8. end
9. else if m contains exterior info then
10.   Save the given information in m in our state.
11. else if m is a circle publication then
12.   Save the information about the circle in m in our state.
13. else if m is approval of a merge proposal then
14.   Send m to all the agents we observe.
15. Move to CONVERGING formation-state and update target location.
16. return
17. else if m is a random direction and mode is MOVE then
18.   Send m to all the agents we observe.
19. Move in the given random direction.
20. Switch the circle mode.
21. return
22. end
23. if mode is PUBLICIZE then
24.   Send information about our current circle to all observed agents.
25. else if mode is DISCOVERY then
26.   Send exterior info gathered to all the agents we observe.
Figure 18. Usage of PUBLICIZE and DISCOVERY circle modes to help the circle members gain full knowledge regarding the surroundings of the circle in the case of four neighboring dense circles. (i) Each member of each circle sends to its neighbors from other circles information regarding its own circle. (ii) Each agent shares the information it discovered to its circle members.
outcome of the game is directly correlated to the random exploration done by agents. Hence, we use the average final percentage of healthy agents as our evaluation metric for the performance of our strategy. We strive to achieve consistently a percentage higher than 50% since that means that more than half of the agents are healthy at the end of the game. Figure 20 displays the results of our simulations against both of the aforementioned strategies. Note that for each game the initial percentage of healthy agents was 50%, that is, if a game began with 50 healthy agents it means there are a total of 100 agents. It can be seen that the distributed strategy outperformed the potential forces strategy consistently for each number of agents. Further inspection showed that these results are statistically significant (p-value < 0.05) for each number of agents. In the case of the distributed strategy which aims to form cliques, we can see that forming uniform circles has an advantage once we exceed 50 agents. It can be seen that the gap in performance gets larger as the number of agents increases. Furthermore, it can be seen that once we exceed 50 agents the strategy performs better against the distributed strategy that forms cliques in comparison to its performance against itself. Further analysis showed that these results are statistically significant once we exceed 50 agents (p-value < 0.05). When there are less than 60 agents it is harder to form large uniform circles faster than it is to form cliques. Hence, the performance of a strategy that forms uniform circles is nearly identical to the performance of the strategy that forms cliques. Once we exceed 50 agents it is easier to form uniform circles faster which explains why the strategy that forms uniform circles consistently outperforms the strategy that forms cliques.

Conclusions and Future Work

In this work we tackled the robotic swarm contamination game by taking a top-down approach, that is, inspecting theoretical properties of the game from a centralized point of view and using these properties to develop an efficient distributed strategy. A series of theoretical discoveries led us to finding the globally optimal behavior of swarm members in the contamination game which is to gather in dense uniform circles of agents. We have shown that under the current settings of the problem it is impossible to implement this behavior in a distributed manner. Moreover, we covered the performance of an implementation of the aforementioned behavior in a simplified version of the contamination game. This means that an efficient solution to the contamination game under the current settings can only achieve a local optima such as formation of agents in cliques as presented in Avrahami and Agmon (2019). To conclude, our findings show that the contamination game can be directly reduced to the problem of distributed formation of non-stationary uniform circles, that is, a solution to the problem of forming non-stationary uniform circles can be used to design an optimal solution to the contamination game. Going forward, we believe that future work should focus on development of broader frameworks for analyzing games in large populations of players such as in the case of games involving swarm since we can observe that realistic strategies in those games can only achieve a local optimum. Hence, future research directions should focus on deconstructing the large space of local optimum solutions for games involving swarms in a way that would help the research community develop solutions that achieve a local optimum which is guaranteed to be an efficient one.
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