Stability of branes trapped by $d$-dimensional black holes

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Abstract

The system of extended objects interplaying with a black hole describes or mimics various gravitational phenomena. In this brief paper, we report the results of stability analysis of codimension-one Dirac-Nambu-Goto branes rest at the equatorial plane of $d$-dimensional spherical black holes, including the Schwarzschild and Schwarzschild-(anti-)de Sitter black holes. For the Schwarzschild and Schwarzschild-anti-de Sitter backgrounds the stability of branes is shown analytically by means of a deformation technique. In contrast, for the Schwarzschild-de Sitter background we demonstrate with the help of numerics that the brane is unstable (only) against the $s$-wave sector of perturbations.

1 Introduction

The system of an extended object such as a string and a brane interplaying with a black hole appears in various contexts of gravitational physics. As a first stage of this direction, one may customarily neglect the thickness and the backreaction of brane into the geometry, in which case the motion of brane is governed by Dirac-Nambu-Goto action [1]. In spite of this test brane approximation, the system, which will be referred to as a brane–black-hole system, is able to provide a rich variety of physically intriguing results. Prime examples are the cosmological domain wall interacting with a...
primordial black hole, a domain wall pierced by a black hole [2], and a probe Dq-brane interacting with background Dp-branes, and so on. The final topic quoted above has attracted much interest from the viewpoint of the gauge/gravity duality, according to which Dp/Dq-branes describe a certain kind of phase transitions in gauge theories [3], and its motion corresponds to the motion of quarks, mesons and baryons in the quark-gluon plasma [4]. It is also an interesting observation that the brane–black-hole system captures several aspects of the ‘merger’ of black holes appearing in the black-hole black-string system in Kaluza-Klein spaces [5, 6]. Besides these applications, the minimal surfaces, i.e., the minimizer of the Dirac-Nambu-Goto action, is used as a technical tool to prove some important theorems in general relativity such as the positive mass theorem [7].

In every context of the above applications of brane–black-hole system, a natural question arises as to which equilibrium state is most stable or what is the final state of dynamics such as gravitational scattering and capturing processes. The answer, of course, will depend on both the symmetries of the background black-hole spacetime and the boundary conditions of the brane at the asymptotic regions (infinity and horizons). As a specific example, let us consider a spherically symmetric static black hole as a background geometry. A naïve expectation is that the final state of the dynamics of an isolated brane–black-hole system is the configuration in which the brane is trapped on the equatorial plane of the black hole, because the equatorial plane, which has the highest symmetries, will always be an extremizer of the action and attractive force of the black hole seems to put such a brane at the bottom of a concave potential.

As we will see soon, however, the problem turns out to be more subtle practically than one expects as above. To assess the linear stability of the brane at the equatorial plane, one is forced to evaluate a one-dimensional Schrödinger equation as occurred in a perturbative stability analysis of black hole. As known well, a careful treatment is required to conclude the stability even for higher dimensional static black holes possessing maximally symmetric horizons [8, 9]. In particular, the presence of a cosmological constant and a charge/momentum makes the potentials in the Schrödinger equation non-trivial and complexity increases as dimensionality becomes higher. The intricacy of the effective potentials is the major obstacle for the analytic study of stability.

In addition, we should remind some facts known in the mathematical studies of minimal surfaces in flat Euclidean space $E^n$, which has a long history since the pioneering works by Young and Laplace. The plane in $E^3$, which may be regarded as a counterpart of the brane on the equatorial plane in our setting, is the only globally well defined single valued graph of the minimal surface equation, which is known as the Bernstein theorem, and its expected extension to general dimension is known as the Bernstein conjecture. Although the spacetime dimensionality appears to play no rôle to prevent this conjecture, this is not the case [10]. The minimal surface need not be smooth necessarily in arbitrary dimensions, and there exists a rather explicit example called minimal cone for $n \geq 8$, which is curved and even singular at the apex. Quite recently, it was pointed out that this failure of the Bernstein conjecture and the existence of minimal cones had involved a variety of gravitational and non-gravitational systems [11], including the stable (local) cone solution of brane in the brane–black-hole system [6].

As far as the authors know, an initial attempt to address the stability of brane trapped on the
equatorial plane of a black hole was made by Higaki et al. [12], and this appears to be the only study in the literature. They discussed the 4-dimensional Reissner-Nordström-de Sitter black and found a conclusive result that the electromagnetic charge and positive cosmological constant tend to destabilize the brane. As mentioned above, however, the stability of minimal surfaces/branes in general dimensions has non-trivial features, and the results of perturbation analysis would shed light on this issue. Motivated by this and recent growing interest in multidimensional dynamics of gravity, this paper explores the stability of branes trapped on their equatorial planes of $d$-dimensional black holes ($d \geq 4$), including the case of asymptotically anti-de Sitter black hole while switching off the electromagnetic field for simplicity.

The organization of this paper is as follows. In the next section, we briefly review the covariant perturbation method of a test brane. In Sec. 3 we specify the background spacetime and reduce the covariant equation to a single eigenvalue problem in radial direction. In Sec. 4 the (in-)stability of brane is examined. The final section is devoted to discussions. In Appendix A a prescription to give the boundary condition of perturbations is given. We use the geometrical unit $c = G_d = 1$, where $G_d$ is the $d$-dimensional gravitational constant, and $(−, +, +, . . . )$ sign convention. Our notation of the cosmological constant is that the Einstein equation in $d$-dimension without a matter content takes the form of $R_{\mu\nu} = 2\Lambda g_{\mu\nu} / (d − 2)$, and we rewrite $\Lambda = \epsilon(d − 2)(d − 1)/(2b^2)$ where $\epsilon = 0, \pm 1$ and $b (> 0)$ is a length scale.

2 Covariant Perturbation Method

Let us consider a timelike hypersurface embedded in the background spacetime $(M, g_{\mu\nu})$ as $x^\mu = X^\mu(\xi^a)$, where $x^\mu (\mu, \nu = 0, 1, \ldots, d − 1)$ and $\xi^a (a, b = 0, 1, \ldots, d − 2)$ are the coordinates of the spacetime and the hypersurface, respectively. The Dirac-Nambu-Goto action is

$$I = −\tau \int d^{d−1}\xi \sqrt{−\gamma}, \quad (1)$$

where $\tau$ is the tension of the brane and $\gamma$ is the determinant of induced metric given by

$$\gamma_{ab} := g_{\mu\nu} \partial_a X^\mu \partial_b X^\nu. \quad (2)$$

The equation of motion derived from the action is expressed as the vanishing trace of extrinsic curvature,

$$\gamma^{ab} K_{ab} = 0. \quad (3)$$

Specifically, the brane motion is described by minimal surfaces.

The perturbation of a background brane equilibrium can be thought of as a test scalar field on the background brane regarded as a $(d − 1)$-dimensional submanifold $(\Sigma, \gamma_{ab})$. A systematic covariant treatment was given in [13]. The quantitative measure of brane bending is expressed by projecting the variation of the brane $\delta X^\mu$ into normal direction $n^\mu$ of the background equilibrium,

$$\Phi := n_\mu \delta X^\mu. \quad (4)$$
Varying the equation of motion (3), it turns out that the linear perturbation equations simplify to a scalar field equation on the hypersurface,

\[ \Box^{(\gamma)} \Phi - \left( R^{(\gamma)} - h_{\mu\nu} R^{\mu\nu} \right) \Phi = 0, \tag{5} \]

where \( h_{\mu\nu} := g_{\mu\nu} - n_\mu n_\nu \) is the projection tensor onto the worldsheet. \( \Box^{(\gamma)} = (\gamma)^{-1/2} \partial_a [(\gamma)^{1/2} \gamma^{ab} \partial_b] \) and \( R^{(\gamma)} \) are the d’Alembertian operator and the Ricci tensor associated with the induced metric \( \gamma_{ab} \), respectively.

3 Reduction to an Eigenvalue Problem

As a background spacetime, we consider the static, spherically symmetric \( d \)-dimensional spacetimes \( (d \geq 4) \) whose line element is

\[ g_{\mu\nu} dx^\mu dx^\nu = -f(r) dt^2 + f(r)^{-1} dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\Omega_{d-3}^2 \right), \tag{6} \]

where \( d\Omega_{d-3}^2 \) is the line element of a unit \( (d-3) \)-sphere. Let us consider the Dirac-Nambu-Goto \( (d-1) \)-dimensional brane \( (\Sigma, \gamma_{ab}) \) with an \( O(d - 2) \) invariance embedded in the background (6). Denoting the brane configuration by \( \theta = \theta(r) \), the induced metric reads

\[ \gamma_{ab} dx^a dx^b = -f(r) dt^2 + \left[ f(r)^{-1} + r^2 \theta'^2(r) \right] dr^2 + r^2 \sin^2 \theta \, d\Omega_{d-3}^2. \tag{7} \]

Without writing down the explicit form of Dirac-Nambu-Goto equation of motion, one might deduce that the equatorial plane \( (\theta = \pi/2) \) always extremalizes the action [since the equatorial plane is geodesic in the \( (r, \theta) \)-plane]. This can be easily checked as follows. For the present brane configuration, the Dirac-Nambu-Goto action computes to give

\[ I = -\tau \Omega_{d-3}(t_2 - t_1) \int dr \, (r \sin \theta)^{d-3} \sqrt{1 + f r^2 \theta'^2}, \tag{8} \]

where \( \Omega_{d-3} \) is the volume of a unit \( (d - 3) \)-sphere, and \( (t_2 - t_1) \) is a time interval. Then, varying this action, one obtains an equation of motion,

\[ \theta'' + \frac{r}{2} \left[ 2(d-2)f + rf' \right] \theta'^3 - \frac{d-3}{\tan \theta} \theta'^2 + \left( \frac{d-1}{r} + \frac{f'}{f} \right) \theta' - \frac{d-3}{r^2 f \tan \theta} = 0. \tag{9} \]

Thus, \( \theta(r) \equiv \pi/2 \) obviously solves this equation, as we desired to show. Hereafter, we shall concentrate on the brane located at the equatorial plane and consider its linear perturbations.

The variables in equation (5) are separable by setting \( \Phi = r^{-(d-3)/2} \chi(r)e^{-i \omega t} Y(\Omega) \) and introducing the tortoise coordinate \( dr_* = dr/f \). Here, \( Y(\Omega) \) is a spherical harmonic on a \( (d - 3) \)-sphere (see, e.g., [8]) satisfying

\[ \left[ \Delta^{(d-3)} + \ell (\ell + d - 4) \right] Y = 0, \quad \ell = 0, 1, 2, \ldots, \tag{10} \]
where $\Delta^{(d-3)}$ is the Laplace-Beltrami operator on the unit $(d-3)$-sphere and $\ell$ is the multipole index. It then follows that the wave equation reduces to the Schrödinger form,

$$-\frac{d^2 \chi}{d r^2} + V(r) \chi = \omega^2 \chi ,$$

with an effective potential,

$$V(r) = \frac{f}{r^2} \left[\ell(\ell + d - 4) - (d - 3) + \frac{(d - 1)(d - 3)}{4} f + \frac{d - 1}{2} r f'\right].$$

We have hitherto nowhere used Einstein’s equations to derive the perturbation equations. From now on, to make the discussion reasonably focused, we specialize to the cases where the background spacetime satisfies the vacuum Einstein equation with or without a cosmological constant, $R_{\mu\nu} = \epsilon(d - 1) g_{\mu\nu} / b^2$, where $\epsilon = 0, \pm 1$ and $b (> 0)$ is a curvature radius. That is, we consider the case where $f(r)$ takes the form,

$$f(r) = 1 - \left(\frac{r_0}{r}\right)^{d - 3} - \epsilon \left(\frac{r_0}{r}\right)^2 ,$$

where $r_0 (> 0)$ is a constant corresponding to a mass of the black hole. The case of $\epsilon = 0$ and $\epsilon = +1(-1)$, respectively, corresponds to the $d$-dimensional Schwarzschild black hole and Schwarzschild-dS(AdS) black hole. For the simplest Schwarzschild case, we first remark that the range of $r^*$ is $-\infty < r^* < \infty$ and hence the outside wedge of a black hole is globally hyperbolic, indicating that the dynamics is uniquely determined by the initial data. The parallel will be inferred for the Schwarzschild-dS case if we recognize $r^* \to \infty$ at the cosmological horizon $r_c$ instead of infinity, where $r_c$ is the larger root of $f(r) = 0$. In contrast to these two cases, the range of $r^*$ is incomplete, which we take $-\infty < r^* < 0$, for the Schwarzschild-AdS, implying that the outside region of a black hole ceases to be globally hyperbolic. This distinguished property demands a special care for the dynamics of $\chi$ in the $\epsilon = -1$ case.

For the metric (13), the potential is written explicitly as

$$V(r) = \frac{f}{4 r^2} \left[4\ell(\ell + d - 4) + (d - 5)(d - 3) + (d - 3)(d - 1) \left(\frac{r_0}{r}\right)^{d - 3} - \epsilon (d^2 - 1) \left(\frac{r_0}{r}\right)^2\right].$$

Inspecting (14), one finds a universal property that the potential $V$ is bounded below and vanishes at the event horizon $r_s \to -\infty$, independent of the sign of the cosmological constant. However, behaviors of the potential $V$ away from the event horizon is very sensitive to $\epsilon$. For $\epsilon \geq 0$, the potential vanishes at $r_s \to \infty$ (infinity for $\epsilon = 0$ and cosmological horizon for $\epsilon = +1$). A distinct feature of a positive cosmological constant is that it will give a negative contribution to the potential [the last term in equation (14)]. By virtue of this, $V$ approaches to zero from below as $r_s \to \infty$ and causes an instability, as we will show later. Whereas, the potential $V$ diverges at infinity in the case of $\epsilon = -1$. This illustrates that the negative cosmological constant acts as a ‘confining box.’ In either case, the potential $V$ fails to be positive-definite in general.

In the rest of this paper, we examine the stability of the brane for each background.

\[1\] More precisely, the metric function (13) for $\epsilon = 1$ describes a Schwarzschild-dS black hole provided $r_0$ satisfies

$$r_0^{d - 3} \leq 2 b^{d - 3}(d - 3)^{(d - 3)/2}(d - 1)^{- (d - 1)/2}.$$
4 Stability Analysis

In order to conclude the linear stability, we must show that the Schrödinger equation $A\chi = \omega^2 \chi$ [equation (11)] admits no normalizable negative mode solutions $\omega^2 < 0$, where we have defined $A := -d^2/dr^2 + V$. The operator $A$ is elliptic and identified as a Hamiltonian operator on a Hilbert space of square-integrable functions $L^2(r_*, dr_*)$ on a static timeslice. To establish the stability, it is necessary to show that $A$ is a positive, self-adjoint operator with $L^2(r_*, dr_*)$-norm.

Before embarking on the stability analysis, let us digress here and first focus on the dynamics of a scalar field $\chi$ in this background. Since the outside wedge is globally hyperbolic for the Schwarzschild(-dS) spacetime, the dynamics of $\chi$ is well-posed in this region. Since the potential $V$ vanishes as $r_* \to \pm \infty$, the only possible boundary condition to obtain a normalizable solution is the Dirichlet boundary condition $\chi = 0$ at $r_* \to \pm \infty$ (see Appendix A). Accordingly, the self-adjoint extension $A_E$ of $A$ is unique, that is to say, the Hamiltonian $A$ is essentially self-adjoint. Hence all that remains to solve for the stability is to show the positivity of $A_E$. This is a standard prescription to pursue the dynamics in globally hyperbolic spacetimes.

For the Schwarzschild-AdS case, on the other hand, the domain of outer communication is no longer globally hyperbolic. This means that the ordinary Cauchy evolution determines a solution of evolution equations only within the domain of influence for a given initial data slice. Nevertheless, this difficulty can be cured for a static spacetime and it is always possible to find a sensible dynamics—and this is essentially the unique recipe for defining dynamics under quite reasonable conditions—beyond the domain of dependence of initial data slice [15]. To define a unitary dynamics throughout the non-globally hyperbolic static spacetime, we need a self-adjoint extension of $A$, which may or may not be unique. As is well known, choosing a self-adjoint extension is equivalent to choosing a boundary condition.

Let us determine possible boundary conditions for $\epsilon = -1$. Since the potential vanishes at the horizon, the normalizability singles out a unique boundary condition therein. Henceforth, our primary concern is the boundary condition at infinity. Near infinity $r_* \sim -b^2/r \sim 0$, $A$ behaves as

$$A \sim -\frac{d^2}{dr_*^2} + \frac{d^2 - 1}{4r_*^2},$$

(15)

this yields the asymptotic solution,

$$\chi \sim C_1 (r_*^{(d+1)/2} + \cdots) + C_2 (r_*^{-(d-1)/2} + \cdots).$$

(16)

The normalizability requires $C_2 = 0$, viz, $V$ is in the limit point case at infinity. Therefore, the self-adjoint extension of $A$ is found to be unique even for $\epsilon = -1$. This observation is central to the stability discussion developed below.

Given a self-adjoint extension $A_E$ of $A$, let us next move on to the issue on the positivity of $A_E$. If the potential $V$ happens to be positive, $A_E$ corresponds to a Friedrichs extension and the stability against linear perturbation immediately follows [14]. As we have seen, however, the potential function $V$, equation (12), is not positive-definite, so it is far from obvious whether $A_E$ has positive eigenvalues.
To see the positivity of $A_E$, it is of great benefit to employ the idea proposed by Ishibashi and Kodama \[9\]. Following their algorithm, the potential in the Schrödinger equation can be ‘deformed’ in such a way that a newly deformed potential is positive-definite. Let $S$ denote an arbitrary function of $r$ and

$$\hat{D} = \frac{d}{dr} - S$$

be a derivative operator. Straightforward calculation shows that

$$\langle \chi, A\chi \rangle_{L^2} = - \left[ \bar{\chi} \hat{D}\chi \right]_{\text{boundary}} + \int dr_\ast \left( |\hat{D}\chi|^2 + V_S |\chi|^2 \right),$$

and

$$V_S = V - f \frac{d}{dr} S - S^2.$$ (18)

It deserves to remark that this expression is formal because there is in general no guarantee that the boundary term and the integrand are both finite. If one can show that the boundary term vanishes and if one can find an appropriate $S$ that makes $V_S$ positive, the Hamiltonian operator $A$ turns out to be a positive operator, which has at least one positive self-adjoint extension corresponding to the Friedrichs extension. Utilizing this formula, we are thereby able to demonstrate the positivity of $A_E$, \textit{i.e.}, the stability of test brane perturbations.

4.1 Schwarzschild(-AdS) Background

Let us begin by the analysis of $\epsilon \leq 0$. As we have shown that the boundary condition is necessarily of Dirichlet-type, it then follows that the boundary term in equation (18) indeed drops off and $V_S \geq 0$ implies a stable dynamics. We can accomplish this by the choice

$$S = \frac{d - 3}{2r} f,$$ (20)

giving rise to a positive potential for any $\ell \geq 0$,

$$V_S = \frac{f}{r^2} \left[ \ell(\ell + d - 4) - \epsilon(d - 1) \left( \frac{r}{b} \right)^2 \right].$$ (21)

Then, the test brane perturbation in Schwarzschild(-AdS) black hole is stable for any $d \geq 4$ dimensions.\footnote{One can show that the string motion in the background of BTZ black hole ($d = 3, \epsilon = -1$) \[16\] is also stable.}

4.2 Schwarzschild-dS Background

According to \[12\], the four-dimensional Schwarzschild-dS black hole exhibits an instability only for the $\ell = 0$ mode. We now show that this result continues to be true for all $d \geq 4$.\footnote{One can show that the string motion in the background of BTZ black hole ($d = 3, \epsilon = -1$) \[16\] is also stable.}
As the domain surrounded by event and cosmological horizons is globally hyperbolic, equation (18) receives no boundary contribution. We find that

\[
S = \frac{d-1}{2r} f
\]

leads to the deformed potential

\[
V_S = \frac{(\ell - 1)(\ell + d - 3)f}{r^2}.
\]

So the system is stable in \( d \geq 4 \) dimensions against for all \( \ell \geq 1 \) modes. A possible unstable mode exists only for the \( s \)-wave. \(^3\)

We will next show numerically that the \( s \)-wave actually excites an unstable mode. To perform numerics, we have to make dimensionless quantities. Here, the normalization by \( r_0 \) seems not so useful because it is a measure of the mass, which has the dimensional dependence as \( r_0^{d-3} \sim G^d M \). To compare black holes in different spacetime dimensions, it is more advantageous, instead of \( r_0 \) and \( b \), to use \( r_c \) and \( r_h \), which are, respectively, the radii of the cosmological horizon and the event horizon. Simple calculations show that they are related by

\[
\frac{r_0^{d-3}}{r_c^{d-3}} = \frac{r_h^{d-3} - r_c^{d-3}}{r_c^{d-1} - r_h^{d-1}}, \quad b^{-2} = \frac{r_c^{d-3} - r_h^{d-3}}{r_c^{d-1} - r_h^{d-1}}.
\]

One may recognize immediately that the background becomes the Schwarzschild black hole as the ratio \( r_c/r_h \) tends to infinity.

We show the values of growth rate \( \sigma := \text{Im}(\omega) \) of perturbations in typical dimensions in Table 1. \(^4\) When the ratio \( r_c/r_h \) is fixed, the growth rate \( \sigma r_h \) becomes higher as the dimension gets higher. The brane on the equatorial plane of the higher dimensional Schwarzschild-de Sitter black hole is more unstable than that of the lower dimensional one with the identical value of ratio \( r_c/r_h \). It can be understood by the form of potential in each dimension. When the spacetime dimension is fixed, the growth rate \( \sigma r_h \) becomes lower as the ratio \( r_c/r_h \) gets bigger. In other words, smaller cosmological constant reduces the instability of the system. This tendency is compatible with the fact that the system in the case of the Schwarzschild black hole background is stable.

To sum up, we found that the positive cosmological constant strengthens the instability of branes on the equatorial plane for the arbitrary dimensional Schwarzschild-de Sitter black holes. We can also say that the higher dimensional background has a destabilization effect of the system.

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\(^3\)It is worthwhile to emphasize that equation (23) can be derived from equation (12) without any reference to the particular form of \( f \). Thus, all the spherically symmetric static black-hole type backgrounds of the form \(^3\) admit a stable brane dynamics for \( \ell \geq 1 \) modes.

\(^4\)The proof of \( \omega \) being pure imaginary is relegated to Appendix. The numerical results presented in Table 1 were obtained by two independent shooting calculations. However, the numerical credibility becomes worse as the dimensionality increases, \( d \geq 10 \), in particular for large \( r_c/r_h \). It would be interesting to apply the matched asymptotic expansion method for the large-\( d \) limit \(^4\), by which one may be able to determine the growth rate analytically.
Table 1: Numerical values of the growth rate $\sigma = \text{Im}(\omega)$ of the $\ell = 0$ perturbation of brane in the Schwarzschild-de Sitter background. The horizon radius $r_h$ is set to unity.

| $r_c$ | $d = 4$       | $d = 5$       | $d = 6$       | $d = 10$      | $d = 11$      |
|------|---------------|---------------|---------------|---------------|---------------|
| 2    | $1.332 \times 10^{-1}$ | $2.429 \times 10^{-1}$ | $3.277 \times 10^{-1}$ | $4.762 \times 10^{-1}$ | $4.865 \times 10^{-1}$ |
| 20   | $4.435 \times 10^{-2}$ | $4.967 \times 10^{-2}$ | $4.997 \times 10^{-2}$ | $5.00 \times 10^{-2}$  | $5.00 \times 10^{-2}$  |
| 100  | $9.750 \times 10^{-3}$ | $9.996 \times 10^{-3}$ | $1.000 \times 10^{-2}$ | $1.00 \times 10^{-2}$  | $1.00 \times 10^{-2}$  |

5 Discussions

We have seen that the branes rest at the equatorial plane of the $d$-dimensional ($d \geq 4$) Schwarzschild and Schwarzschild-AdS black holes are stable for the linear perturbations, while for the Schwarzschild-dS background the brane is unstable against the $s$-wave perturbation. The covariant perturbation equation of brane is translated into the eigenvalue problem in the Schrödinger form after the separation of variables. Then, the stabilities were proved by showing the positivity of the Hamiltonian operator by means of a deformation technique. In the asymptotically de Sitter case, the positivity of Hamiltonian operator only for the non-spherical perturbations ($\ell \geq 1$) was shown, which means that one has to solve the eigenvalue problem explicitly to pursue the behavior of spherical mode. To this end, we numerically solved the equation for the spherical perturbation and showed that the mode is an unstable one in every dimension, growing exponentially in time. One can recognize this coming from the fact that large $\ell$ has the stabilizing effect. The dangerous modes originate from the lower angular eigenvalues since the effective potential becomes higher as $\ell$ increases. This result is reminiscent of the fact that the Gregory-Laflamme instability of black branes exists only in the $s$-wave sector [18].

For definiteness of our argument, we have limited our consideration to a neutral black hole as a background black hole. The inclusion of electromagnetic charge is straightforward. The electromagnetic charge will leave a negative imprint on the effective potential (12) as in an analogous fashion in four dimensions [12], where an instability was explicitly shown. Thus there seems no compelling reason to stabilize the brane motion in the charged case. As we said, this instability occurs only for the $s$-wave in any spacetimes of the form (6) such as a black hole in different gravitational theories.

In this paper, we have ignored the backreaction of brane self-gravity and the thickness of brane, both of which cannot be negligible in general situations. If one takes into account the backreaction, it is known that the domain wall exhibits repulsive gravitational fields [19]. Regarding the thick branes interplaying with black holes, several equilibrium solutions were constructed numerically or analytically [20, 21]. The introduction of the thickness is known to result in the expulsion of branes/domain walls by an extremal/near-extremal black hole. Thus, it would be interesting to see how the competition between the attractive and repulsive forces appearing in more realistic modelings of brane–black-hole system affects the stability of the brane examined in this paper.

While we have restricted ourselves to the linear stability of brane and found, for example, the stability in the Schwarzschild background, it is known that a certain sort of non-linear perturbations to the Dirac-Nambu-Goto brane in the Schwarzschild background, which mimic the recoil of black hole...
due to the Hawking emission, evolves and results in a reconnection of the bent brane, corresponding to the eventual escape of black hole into the bulk \[22\]. It is noted that our results are not inconsistent with those in \[22\] since the asymptotic boundary conditions are different. In a similar vein, the existence of a critical escape velocity of the black hole from the brane was investigated in \[23\]. Interestingly, the existence of critical escape velocity depends on the number of codimensions. Although we have been concerned only with the codimension-one branes in this paper, it would be interesting to investigate how the number of codimensions affects the stability of the brane, which can be examined with the covariant perturbation methods developed in \[24\].

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**A Boundary Conditions**

Here, we assemble several basic facts about the behaviors of physical perturbations in the case of \(\epsilon \geq 0\), which are used in this paper and might be helpful for readers.

For the Schwarzschild case (\(\epsilon = 0\)), \(r_* \to -\infty\) as \(r \to r_h\) (\(r = r_h\) is the event horizon) and \(r_* \to +\infty\) as \(r \to +\infty\). For the Schwarzschild-dS case (\(\epsilon = +1\)), \(r_* \to -\infty\) as \(r \to r_h\) again and \(r_* \to +\infty\) as \(r \to r_c\) (\(r = r_c\) is the cosmological horizon). One can see that the potential (14) vanishes as \(r_* \to \pm \infty\). Thus, the linearly independent asymptotic solutions are the plane waves, \(e^{\pm i\omega r_*}\). Since we assume the time dependence of \(\chi \propto e^{-i\omega t}\), two modes \(e^{-i\omega r_*}\) and \(e^{i\omega r_*}\) correspond to the ingoing and outgoing waves, respectively. Hence, we impose the following boundary conditions on a physical mode, \(\chi \to e^{\pm i\omega r_*}\) as \(r_* \to \pm \infty\).

As far as an unstable mode, corresponding to \(\text{Im}(\omega) > 0\), is concerned, we can show that \(\omega\) is pure imaginary as follows. Multiplying the complex conjugate of \(\chi\), \(\bar{\chi}\), with equation (11), and integrating it, we have

\[
(\chi, A\chi)_{L^2} = \int_{-\infty}^{+\infty} \left( \frac{d\chi}{dr_*} \right)^2 + V|\chi|^2 \, dr_* - \left[ \frac{\chi}{dr_*} \right]_{-\infty}^{+\infty} = \omega^2 \int_{-\infty}^{+\infty} |\chi|^2 \, dr_*.
\]

(25)

Since the surface term in equation (25) vanishes due to the exponential suppression at infinity, the all integrals in equation (25) are real. Thus, \(\omega^2\) must be real, and furthermore \(\omega\) must be pure imaginary [since we are looking for the unstable mode, \(\text{Im}(\omega) > 0\)]. Therefore, we can write as \(\omega = i\sigma\) (\(\sigma > 0\)) and equation (25) reads

\[
-\sigma^2 = \int_{-\infty}^{+\infty} \left( \left| \frac{d\chi}{dr_*} \right|^2 + V|\chi|^2 \right) \, dr_* / \int_{-\infty}^{+\infty} |\chi|^2 \, dr_*.
\]

(26)

What we can see from this equation is that if the potential \(V(r)\) is positive definite, the right hand side is positive and equation (26) makes no sense, implying the absence of unstable mode. In terms
of \( \sigma \), the asymptotic boundary conditions mentioned above are rewritten as \( \chi \rightarrow e^{\pm \sigma r_*} \) as \( r_* \rightarrow \mp \infty \), which are used in the numerical analysis.

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