MAPPING BETWEEN NONLINEAR SCHÖDINGER EQUATIONS WITH REAL AND COMPLEX POTENTIALS

MARIO SALERNO

Communicated by XXX

Abstract. A mapping between stationary solutions of nonlinear Schrödinger equations with real and complex potentials is constructed and a set of exact solutions with real energies are obtained for a large class of complex potentials. As specific examples we consider the case of the damped dynamics of a quantum harmonic oscillator and the case of dissipative periodic soliton solutions of the nonlinear Schrödinger equation with complex potential.

1. Introduction

Nonlinear wave phenomena with time evolutions governed by non hermitian Hamiltonians are presently attracting a great interest both from the theoretical and from the applicative point of view. The non hermiticity is in general due to the presence of a complex potential in the Hamiltonian accounting for typical dissipative and amplification effects met in classical and quantum contexts [1, 2]. In particular, dissipative solitons [3] of the nonlinear Schrödinger (NLS) equation with periodic complex potentials have been extensively investigated during the past years in connections with the propagation of light in nonlinear optical fibers with periodic modulations of the complex refractive index [4, 5]. Recently similar studies were done for matter wave solitons of Bose-Einstein condensates (BEC) trapped in absorbing optical lattices [6, 7] and in the presence of three body interatomic interactions [8]. In the linear context, the recent discovery [9] that the Schrödinger eigenvalue problem with complex potentials that are invariant under the combined parity and time reversal symmetry (so called $\mathcal{PT}$-potentials), may have fully real spectrum, has raised interest also in view of possible connection with the theory of quantum dissipative systems. Complex potentials with $\mathcal{PT}$-symmetry are presently investigated in nonlinear optics [10, 11] where it has been demonstrated that nonlinear media with linear damping and amplifications that are $\mathcal{PT}$-symmetric can support stable stationary localized and periodic states [12]. Also, quite recently, physical systems with $\mathcal{PT}$-symmetry have been successfully implemented in real experiments [13–15]. Solutions of the NLS equation with a complex potential which
belong to the real part of the spectrum (real energies or real chemical potentials) can exist, however, for generic complex potentials and it is therefore of interest to characterize them in general, independently from the $\mathcal{PT}$-symmetry.

The aim of the present paper is to show how one can systematically construct stationary solutions of the complex nonlinear Schrödinger equation via a mapping between real and complex NLS equations. The problem is formulated in terms of a nonlocal eigenvalue problem which involves only real potentials, whose eigenfunctions and eigenvalues fix amplitudes and energies of the stationary solutions of the complex NLS equation, respectively. The complex potentials and the phases of the solutions are also determined self-consistently through the mapping. To illustrate our approach we discuss the case of the linear Schrödinger equation with complex potentials describing a quantum dissipative oscillator, and the case of the NLS equation with different complex potentials for which we construct periodic dissipative solitons in the form of elliptic functions.

The paper is organized as follows. In section II we introduce model equations and illustrate the mapping used to determine the solutions. In Section III we apply the method to the case the linear Schrödinger equation for a damped harmonic oscillator. In section IV we show how to construct exact solutions of the NLS with periodic complex potentials while in the last section the main results of the paper will be briefly summarized.

2. Model Equations and Mapping

The model equation we consider is the NLS equation with real and complex potentials both of linear and nonlinear types, e.g.

\[ i\psi_t = \frac{1}{2}\psi_{xx} + (V_l(x) + iW_l(x))\psi + (\sigma + V_{nl}(x) + iW_{nl}(x))|\psi|^2\psi. \]  

The case of the linear Schrödinger equation (e.g. \( \sigma = V_{nl} = W_{nl} = 0 \)) can be used as an example of quantum dissipative system. In the nonlinear case the above equation can appear in connection with several interesting phenomena including light propagation in photonic crystals and Bose-Einstein condensates. Due to the possibility of different physical applications we shall keep Eq. (1) in normalized form, looking for stationary solutions of the type

\[ \psi(x, t) = A(x)e^{i\theta(x)}e^{-i\omega t} \]  

(2)
with the amplitude $A(x)$ and phase $\theta(x)$ as real functions. Substituting this expression into Eq. (1), we obtain the system of equations

$$\omega A + \frac{1}{2} A_{xx} - \sigma A^3 - \frac{A}{2} (\theta_x)^2 - V_l A - V_{nl} A^3 = 0 \quad (3)$$

$$\frac{1}{2} A_{xx} + A_x \theta_x - W_l A - W_{nl} A^3 = 0. \quad (4)$$

These equations can be easily separated. In this respect notice that by multiplying Eq. (4) by $A$ and integrating it twice one obtains

$$\frac{1}{2} \theta(x) = B_2 + \int_{-\infty}^x \frac{B_1 + F(y)}{A^2(y)} dy \quad (5)$$

with

$$F(y) = \int_{-\infty}^y \left[ W_l(z) + W_{nl}(z) A^2(z) \right] A^2(z) dz, \quad (6)$$

and $B_1, B_2$ integration constants. By substituting Eq. (5) into Eq. (3) we obtain the following nonlinear eigenvalue problem for the real amplitude $A$:

$$\left\{ -\frac{1}{2} \frac{\partial^2}{\partial x^2} + V_l + (\sigma + V_{nl}) A^2 + 2 \left( \frac{F(x)}{A^2} \right)^2 \right\} A = \omega A \quad (7)$$

where the integration constants $B_1, B_2$, have been fixed to zero for simplicity. Note that for stationary solutions Eq. (7) is completely equivalent to Eq. (1) in the sense that any solution of (7) gives a stationary solution of (1) with the phase fixed by (5). Also note that the dependence on the complex potentials in the eigenvalue problems comes through the function $F$ and for an arbitrary $F(x)$ (e.g. arbitrary complex potentials) the problem can become singular. It is possible, however, to construct potentials $W_l$ and $W_{nl}$ (e.g. functions $F$) so that the solutions of (7) are regular. This establishes a mapping between stationary solutions of the NLS equation with real potentials and stationary solution of Eq. (1) with the phase given by (5). In this respect, one can take $F$ in general to be an analytical function of $A^2$ and derivatives e.g. $F(x) \equiv F(A^2, (A^2)_x, \ldots)$. In the simplest case $F$ can be taken of the form

$$F(x) = \frac{1}{2} C_n A^{n+2}, \quad n = 0, 1, 2\ldots \quad (8)$$

with $C_n$ constants to be determined. Equation (7) then reduces to the following NLS real eigenvalue problem:

$$\left\{ -\frac{1}{2} \frac{\partial^2}{\partial x^2} + V_l + (\sigma + V_{nl}) A^2 + \frac{C_n^2}{2} A^{2n} \right\} A = \omega A \quad (9)$$
which can be solved analytically for particular forms of the potentials \( V_l, V_{nl} \), or numerically with high accuracy (using for example the self-consistent method discussed in [16]) for generic real potentials. In the following we therefore assume that the real amplitudes \( A \) and frequencies \( \omega \) for given \( V_l \) and \( V_{nl} \) are exactly obtained from (9), either analytically or numerically.

On the other hand from Eq. (8) one can characterize the complex potentials which support such solutions. Using Eq. (6) we have indeed that Eq. (8) is satisfied if the amplitude \( A \) is related to \( W_l \) and \( W_{nl} \) by the relation

\[
W_l + W_{nl} A^2 = C_n \left( \frac{n + 2}{2n} \right) \frac{dA^n}{dx}
\]  

(10)

and from Eqs. (5), (8), one gets that the phase is given by

\[
\theta(x) = C_n \int_{-\infty}^x A^n.
\]  

(11)

Note that in this case Eq. (10) allows to relate the constant \( C_n \) to the amplitude of the solution, \( A_0 \), and the amplitudes \( W_{0l}, W_{0nl} \) of the linear and nonlinear complex potentials, respectively. In particular, for the case \( W_{nl} = 0 \) we have that

\[
C_n = \frac{2}{n + 2} \frac{W_{0l}}{A_0^n}, \quad W_{0nl} = 0
\]  

(12)

while for \( W_l = 0 \) one obtains

\[
C_n = \frac{2}{n + 2} \frac{W_{0nl}}{A_0^{n-2}}, \quad W_{0l} = 0.
\]  

(13)

It is worth to note that while the case \( n = 1 \) leads to a pure cubic NLS eigenvalue problem, the case \( n > 1 \) introduces higher order nonlinearities in Eq. (9) which can however be eliminated by redefining the linear real potential as

\[
V_l = \tilde{V}_l - \frac{C_n^2}{2} A^{2n},
\]  

(14)

or the nonlinear real potential as

\[
V_{nl} = \tilde{V}_{nl} - \frac{C_n^2}{2} A^{2n-2}
\]  

(15)

(or a combination of both). Also notice that Eqs. (2), (10) - (13) allow to map solutions of the real eigenvalue problem (9) into solutions of the NLS equation (1) with the corresponding complex potentials determined as in (10). It is clear that this approach can be extended to functions of the of the type

\[
F(x) = \frac{1}{2} \sum_{n=0}^{k} C_n A^{n+2}, \quad k = 0, 1, 2, ...
\]  

(16)
In this case coefficient $C_n$ are self-consistently determined from the real eigenvalue problem

$$\left\{ -\frac{1}{2} \frac{\partial}{\partial x^2} + V_l + (\sigma + V_{nl}) A^2 + \frac{1}{2} \left( \sum_n C_n A^n \right)^2 \right\} A = \omega A \quad (17)$$

and complex potentials and phase are given by

$$W_l + W_{nl} A^2 = \frac{1}{A^2} \frac{dF}{dx} \quad (18)$$

$$\theta(x) = \sum_n C_n \int_{-\infty}^x A^n. \quad (19)$$

Note that the sum in Eq. (16) can include infinite terms and to have a map between real and complex NLS equations it is necessary to subtract higher order nonlinearities from the real linear and nonlinear potentials as done in Eqs. (14)-(15). Finally we remark that if the functions $A_x/A, A_{xx}/A, \ldots$ are bounded, the expression (16) can be further generalized as

$$F(x) = \sum_{n,m} C_{n,m} \frac{d^m A^{n+2}}{d x^m} \quad (20)$$

with $C_{n,m}$ suitable constants and with the complex potentials determined as (18). In all these cases a map between solutions of the real eigenvalue problem (17) and solutions of the NLS equation (1) is constructed.

The mapping guarantees that the constructed solutions always have real $\omega \in \mathbb{R}$ energies (chemical potentials) and may be therefore of physical interest. In the following we illustrate how the mapping works on some specific example.

### 3. A Dissipative Quantum Oscillator

We consider the case of linear Schrödinger equation with $\sigma = 0, V_{nl} = W_{nl} = 0$, and let us restrict to $n = 0$. From Eq. (6) we then have that $F(x) = C_0 A(x)^2/2$ and the eigenvalue problem (9) is the same as for real potential but with shifted eigenvalues $\Omega = \omega - C_0^2/2$. From Eq. (12) we have that $C_0 = W_{0l}$ and from Eq. (12) we get $W_l(x) = C_0 A_x(x)/A(x)$. Finally, the phase of the complex wavefunction is obtained from Eq. (11) as $\theta(x) = W_{0l} x$

We explore the case of a dissipative quantum oscillator by fixing $V_l(x) = \omega_0^2 x^2/2$ and taking the function $A(x)$ as the ground state of the eigenvalue problem (9)

$$A(x) = \frac{1}{\pi^{1/4}} e^{-\frac{x^2}{2}} \quad (21)$$
with eigenvalue \( \omega = \frac{1}{2}(\omega_0 - C_0^2) \). From the mapping we have that the ground state of the complex harmonic oscillator is

\[
\psi_n(x, t) = \frac{1}{\pi^{1/4}} e^{-\frac{\omega_0 x^2}{2}} e^{iW_0(x)} e^{-i\omega t}.
\]

(22)

with the complex potential which support this solution given by

\[
W_l(x) = C_0 \frac{A_x}{A} = -W_0l x.
\]

(23)

Notice that this complex potential, being linear in \( x \), acts as a dissipation for \( x > 0 \) and as a gain for \( x < 0 \) so that the ground state can exist as stationary state thanks to a perfect balance between gain and dissipation. On the other hand one could argue that a shift of the ground state with respect to the origin could make the ground state unstable, i.e. that a possible breaking of balance between damping and gain could lead either to decay or to blow-up of the wavefunction. This, however, is not the case because due to the real part of the potential the wavefunction can be dynamically stable, with its center oscillating inside the parabolic potential, for trap frequencies and slopes of the complex potential properly chosen. In panel (a) of Fig. 1 we show the real and complex part of the potential together with the real and imaginary parts of the ground state while in panels (b) and (c) we depict the dynamics obtained by direct integration of the complex Schrödinger equation both for the rest and for the displaced ground state wavefunction. We see that although the wavefunction is stable under time evolution, its oscillations are undamped, as it is clear from panel (d) of this figure.

A more realistic model of damped oscillations can be obtained by considering the more general form of the function \( F \) in Eq. (20) which allows to have in the complex potential both linear and quadratic damping. More specifically, we take

\[
F(x) = a_1 A(x)^2 + \frac{a_2}{2} \left( \frac{dA(x)}{dx} \right)^2
\]

and fix the real potential as

\[
V_l = \frac{1}{2} \Omega_0^2 x^2 + 4a_1a_2\omega_0 x
\]

(24)

with \( \Omega_0^2 = \omega_0^2(1 - 4a_2^2) \). It can be shown that \( A(x) \) taken as in (21) solves the real eigenvalue problem (9) if \( \omega = \frac{1}{2}(\omega_0 + 4a_1^2) \). The complex potential and the phase are obtained from \( F(x) \) as

\[
W_l(x) = 2a_2\omega_0^2 x^2 - 2a_1\omega_0 x - a_2\omega_0
\]

(25)

and \( \theta(x) = 2a_1 x - a_2 x^2 \).
Figure 1. (a) The real (thick continuous line) and imaginary (thick dashed line) parts of the ground state wavefunction of the harmonic oscillator with complex potential \( \Omega_0 \). The thin dotted and dot-dashed lines refer to the real and imaginary complex potential, respectively. Parameters of the potentials are fixed as \( \omega_0 = 1, W_0l = 0.05 \). (b) Time evolution of the modulo square of the stationary ground state wavefunction. (c) The same as panel (b) but with the initial wavefunction displaced by 0.5 with respect to the center of the parabolic potential. (d) The oscillation of the center of the wavefunction versus time for the dynamics in panel (c).
In panel (a) of Fig. 2 we have depicted the real and complex part of the potential together with the real and imaginary parts of the displaced ground state while in panels (b) and (c) we depict the dynamics obtained by direct integration of the complex Schrodinger equation for the modulo square of the wavefunction and for center of the ground state wavefunction. We see that the wavefunction is stable under time evolution and the oscillations are damped with decay constant $\alpha = 2\omega_0^2 a_2$, correlating with what one would intuitively expected for quantum damped oscillations.

4. Nonlinear Schrödinger Equation with Complex Potentials

4.1. Case n=1

Let us now consider the nonlinear case, starting with the simplest ansatz $8$ with $n = 1$. We fix the nonlinearity to be attractive ($\sigma < 0$) and restrict to linear complex potentials (i.e. $W_{nl} = V_{nl} = 0$) and with linear potential of the form $V_l = V_{0l} \text{cn}^2(x, k)$. In this case the real eigenvalue problem $9$

\[
\left[ -\frac{1}{2} \frac{\partial}{\partial x^2} + V_{0l} \text{cn}(x,k)^2 + (\sigma + \frac{C_1^2}{2}) A^2 \right] A = \omega A. \tag{26}
\]

admits the following exact solutions in terms of elliptic functions

\[ a) \quad A(x) = A_0 \text{cn}(x,k), \quad A_0 = \pm \sqrt{\frac{2(k^2 + V_{0l})}{2|\sigma| - C_1^2}}, \quad \omega = \frac{1 - 2k^2}{2} \tag{27} \]

\[ b) \quad A(x) = A_0 \text{sn}(x,k), \quad A_0 = \pm \sqrt{\frac{2(k^2 + V_{0l})}{C_1^2 - 2|\sigma|}}, \quad \omega = \frac{1 + k^2}{2} + V_{0l}, \tag{28} \]

\[ c) \quad A(x) = A_0 \text{dn}(x,k), \quad A_0 = \pm \sqrt{\frac{2(k^2 + V_{0l})}{2|\sigma| - C_1^2}}, \quad \omega = \frac{k^2}{2} - 1 + V_{0l}(1 - \frac{1}{k^2}). \tag{29} \]
Figure 2. (a) The real (thick continuous line) and imaginary (thick dashed line) parts of the ground state wavefunction of the harmonic oscillator with real and complex potentials fixed as in Eq. (24) and (25), respectively. The wavefunction is displaced by $x_0 = 0.5$ with respect to the center of the parabolic potential and the thin dotted and dot-dashed lines refer to the real and imaginary complex potential, respectively. Parameters of the potentials are $\omega_0 = 0.9$, $a_1 = a_2 = -0.02$. (b) Time evolution of the modulo square of the displaced ground state wavefunction in panel (a). (c) Damped oscillations of the center of the wavefunction versus time for the dynamics in panel (b). Dotted curves refer to the exponential fit $\pm x_0 e^{-\alpha t}$ with $\alpha = 2\omega_0^2 a_2$. 


Similar solutions can be constructed for the case of a repulsive nonlinearity $\sigma > 0$ with linear potentials of the form $V_l = V_0 \text{sn}^2(x,k)$. In this case we have

\begin{align*}
d) & \quad A(x) = A_0 \text{cn}(x,k), \quad A_0 = \pm \sqrt{\frac{2(V_0 - k^2)}{C_1^2 + 2\sigma}}, \\
& \quad \omega = \frac{1 - 2k^2}{2} + V_0l \tag{30}
\end{align*}

\begin{align*}
e) & \quad A(x) = A_0 \text{sn}(x,k), \quad A_0 = \pm \sqrt{\frac{2(k^2 - V_0l)}{C_1^2 + 2\sigma}}, \\
& \quad \omega = \frac{1 + k^2}{2}, \tag{31}
\end{align*}

\begin{align*}
f) & \quad A(x) = A_0 \text{dn}(x,k), \quad A_0 = \pm \frac{1}{k} \sqrt{\frac{V_0l - k^2}{C_1^2 + 2\sigma}}, \\
& \quad \omega = \frac{k^2}{2} - 1 + \frac{V_0l}{k^2}. \tag{32}
\end{align*}

Using the above mapping we can readily construct the stationary solutions of the corresponding complex NLS with linear complex potentials given by

\begin{equation}
W_l = \frac{3}{2} C_1 A_x, \quad C_1 = \frac{2}{3} \frac{W_0l}{A_0} \tag{33}
\end{equation}

and with the phase given by $\theta(x) = C_1 \int_{-\infty}^{x} A(y)dy$. Thus, for example, from the solution a) we get

\begin{align*}
A &= A_0 \text{cn}(x,k), \quad V_l(x) = V_0l \text{cn}^2(x,k), \\
A_0 &= \sqrt{\frac{9(k^2 + V_0l) + 2W_0l^2}{3\sqrt{\sigma}}}, \quad \omega = \frac{1 - 2k^2}{2}, \\
W_l &= -W_0l \text{sn}(x) \text{dn}(x), \\
\theta(x) &= \frac{2W_0l}{3k} \text{arccos}(\text{dn}(x)). \tag{34}
\end{align*}

In similar manner one proceeds with the other solutions above. It is also clear that exact solutions of this type can be constructed also for other types of linear elliptic potentials (we omit them for brevity).

4.2. Case $n=2$

As a further application of the ansatz (8) we consider the case $n = 2$ for which the mappings involves higher order nonlinearities. We assume as before that $V_{nl} =$
$W_{nl} = 0$. In order to balance the quintic nonlinearity in Eq. (9), the potential $V_l$ must be taken as in Eq. (14). We take $\tilde{V}_l = V_{0l} \text{cn}^2(x, k)$ and consider a solution of the form $A(x) = A_0 \text{cn}(x, k)$. One can then check that this is a solution of (9) with

$$V_l(x) = V_{0l} \text{cn}^2(x, k) - \frac{C_2^2}{2} A_0^4 \text{cn}^4(x, k).$$

(35)

if $A_0^2 = V_{0l} + k^2$, $\omega = \frac{1-2k^2}{2}$ From the mapping we then have that $C_2 = \frac{W_{nl}}{2A_0^2}$.

$$W_l(x) = 2C_2 AA = -W_{0l} \text{cn}(x)\text{sn}(x)\text{dn}(x),$$

(36)

and the phase is

$$\theta(x) = x - \frac{x}{k^2} + \frac{1 - k^2 + k^2 \text{cn}^2(x, k)}{k^2 \text{dn}^2(x, k)} E(\text{am}(x, k), k).$$

(37)

As a further example of $n = 2$ we consider the case of pure nonlinear optical lattices, i.e. $V_l = W_l = 0$. Fixing $\tilde{V}_{nl} = 0$ and looking for solutions of the type $A(x) = A_0 \text{cn}(x, k)$, we have from Eq. (15) that that

$$V_{nl}(x) = -\frac{C_2^2}{2} A_0^2 \text{cn}^2(x, k).$$

(38)

with $C_2$ fixed according to Eq. (13) as $C_2 = W_{0nl}/2$. One can easily check that this is indeed a solution of the eigenvalue problem (9) if

$$\omega = \frac{1}{2} - k^2, \quad A_0 = \frac{k}{\sqrt{\sigma}},$$

(39)

(we consider $\sigma < 0$). From the mapping we have that this is also a solution of the NLS with the complex part of the nonlinear potential fixed according to Eq. (10) as

$$W_{nl}(x) = 2C_2 \frac{A_x}{A} = -W_{0nl} \frac{\text{sn}(x, k)\text{dn}(x, k)}{\text{cn}(x, k)}.$$  

(40)

For the cases $n > 2$ one can proceed in similar manner.

4.3. General case

Let us now consider an example with the more general ansatz (16). To this regard we take $F(x) = \frac{1}{2}(C_0 + C_2 A^2)A^2$ and look for solutions of the form $A(x) = A_0 \text{dn}(x, k)$. Let us fix the linear potentials as $\tilde{V}_l = W_l = 0$ and the real nonlinear potential as $V_{nl} = V_{0nl} - \alpha^2 A^2$ with $V_{0nl}$ a constant and with $\alpha_n = C_n^{\frac{1}{2}}, n = 0, 2$
(notice that we fixed all coefficients for \( n \neq 0,2 \), equal to zero). By substituting these expressions into the real eigenvalue problem we find that \( A(x) \) is indeed a solution if
\[
\omega = \alpha_0^2 + \frac{k^2}{2} - 1, \quad \alpha_0 = -\frac{1 + A_0^2(\sigma + V_{0nl})}{2A_0^2\alpha_2}.
\]

Thus, for example, if we fix \( V_{0nl} = 2/k^2 \), \( \alpha_2 = -1/k \) and consider \( \sigma = 1 \) (repulsive interactions), we have
\[
V_{nl} = \frac{2 - \text{dn}^2(x,k)}{k^2} = \frac{1}{k^2} + \text{sn}^2(x,k)).
\]

From Eq. (41) we have
\[
\omega = \sigma - 1 + \frac{1}{A_0^2} + \frac{k^2}{4A_0^4}[(1 + 2\sigma A_0^2 + (\sigma^2 + 2)A_0^4] + \frac{1}{k^2}
\]
\[
\alpha_0 = \frac{1}{k} + \frac{k(1 + \sigma A_0^2)}{2A_0^2},
\]
and from (16) we get the function \( F \) as
\[
F(x) = \frac{1}{\sqrt{2}}(\alpha_0 + \alpha_2 A^2)A^2 = \frac{A_0^2\text{dn}^2(x,k)}{2\sqrt{2}k}\left[2 + \frac{k^2}{A_0^2}(1 + \sigma A_0^2) - 2A_0^2\text{dn}^2(x,k)\right]
\]
Substituting into Eqs. (18) we finally get the complex potential as
\[
W_{nl} = \sqrt{2k}\frac{\text{sn}(x,k)\text{cn}(x,k)}{\text{dn}^3(x,k)} \times \left[2 - \frac{1}{A_0^2} - \frac{k^2}{2A_0^2}(1 + \sigma A_0^2) - 2k^2\text{sn}^2(x,k)\right].
\]

The phase of the solution can be readily obtained from Eq. (19). Notice that in the case \( \sigma = 1, A_0 = 1 \), this solution coincides with the one derived in [18] with a slightly different approach. We remark that the above solutions of the complex NLS equations not only have real energies (chemical potentials) but are also found to be stable (not shown here for brevity) under time evolution.

5. Conclusions

In conclusion we have shown the possibility to construct stationary solutions of the linear and nonlinear Schrodinger equation with complex potentials via a mapping
with stationary solutions of the NLS equation with suitable real potentials. We showed that by means of this mapping it is possible to construct sets of exact solutions with real energies that may be stable under time evolution. As specific example we considered the case of a dissipative quantum oscillator and the case of periodic soliton solutions of the complex nonlinear Schrödinger equations for different types of complex potentials. The presented approach can be applied to other types of linear and nonlinear equations of the NLS type including arbitrary higher order nonlinearities.

We finally remark that a similar approach based on a priori fixing of the solution and a posteriori determination of the complex potential, has been considered also in [17][18], although not in terms of a mapping between stationary solutions of NLS equations.

Acknowledgements

It is a pleasure to dedicate this paper to Professor Vladimir Gerdjikov on the occasion of his 65th birthday in 2012. Partial support from the Ministero dell’ Istruzione, dell’ Università e della Ricerca (MIUR) through a Programma di Ricerca Scientifica di Rilevante Interesse Nazionale (PRIN) 2010-2011 initiative, is acknowledged.

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Mario Salerno
Dipartimento di Fisica “E.R. Caianiello”
Universit`a di Salerno,
84084 Fisciano (SA), Italy
E-mail address: salerno@sa.infn.it