Non-Hermitian bulk-boundary correspondence and singular behaviors of generalized Brillouin zone

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Abstract
The bulk boundary correspondence, which connects the topological invariant, the continuum band and energies under different boundary conditions, is the core concept in the non-Bloch band theory, in which the generalized Brillouin zone (GBZ), appearing as a closed loop generally, is a fundamental tool to rebuild it. In this work, it can be shown that the recovery of the open boundary energy spectrum by the continuum band remains unchanged even if the GBZ itself shrinks into a point. Contrastively, if the bizarreness of the GBZ occurs, the winding number will become illness. Namely, we find that the bulk boundary correspondence can still be established whereas the GBZ has singularities from the perspective of the energy, but not from the topological invariant. Meanwhile, regardless of the fact that the GBZ comes out with the closed loop, the bulk boundary correspondence cannot be well characterized yet because of the ill-definition of the topological number. Here, the results obtained may be useful for improving the existing non-Bloch band theory.

1. Introduction
Recently, the topological insulators in the Hermitian system have attracted considerable research interest [1–10]. The bulk boundary correspondence is among the central concepts and has been established perfectly using the Bloch band theory [11–15], in which the boundary refers to the energy spectrum originating from the open boundary condition. The bulk contains two physical meanings, one of which is the energy spectrum calculated by the periodic boundary condition, while another one senses the topological invariant of the system calculated in the momentum space. Universally, the physical elucidation of the bulk boundary correspondence is that the energy spectra under the periodic boundary and the one under the open boundary coincide with each other ideally at the thermodynamic limit, except for the zero modes being predicted by the topological invariant.

However, the non-Hermitian topological systems, which introduce some dissipative ingredients, have extended the Frontier of the law of physics and raised many novel phenomena [16–56]. Among the key aspects is the nullities of the bulk boundary correspondence in the non-Hermitian systems [16–21]. Concretely, the spectra under different boundary conditions have an overt distinction [16–23]. To understand the divergence, the references [18–23] introduced the new concepts of the generalized Brillouin zone (GBZ) and biorthogonal set of eigenfunctions. Depending on these, the open boundary energy spectrum can be reproduced by the continuum band, and further, the zero energy modes can be forecasted by the non-Bloch topological invariant as well. Namely, the bulk boundary correspondence in the
Figure 1. Scheme of the topological invariant. (a) In Hermitian case, the topology of the system (trivial, critical or non-trivial) can be obtained when the endpoints of the vector $d(k)$ forms a closed loop, with the wavenumber sweeping across the Brillouin zone [12]. (b) In non-Hermitian, the well-defined winding number necessitates that $R_{\pm}(\beta)$ form a closed loop containing a finite areas when $\beta$ goes around the GBZ, being a closed curve in general. In addition, it should emphasize that a point must have different topological structures with a close loop. In other words, if $R_{\pm}(\beta)$ collapse into a point in some cases, the topological invariant of the system will be different from those described above essentially, i.e. the bulk boundary correspondence may be ambiguous.

non-Hermitian topology is reinstituted from the aspects of the energy and topological number simultaneously. But, based on the current framework of the non-Bloch band theory, the recovery of the bulk boundary correspondence requires that GBZ is a closed curve explicitly [19–23]. Remarkably, we note that a point without any interior area must have different topological structures with a close loop containing some interior areas. Thus, a natural question will arise as how the GBZ collapses into a point and what will happen if the strangeness occurs? Precisely speaking, what effect does the singularities of the GBZ have on the bulk boundary correspondence of the system? In addition, while the GBZ is a closed curve as we expected, will the bulk boundary correspondence hold for granted?

In this work, we show that how the GBZ collapses analytically and numerically, and further, whether the GBZ is a closed curve or not, the energy spectra in an open chain always can be regained by the continuum bands faithfully, which illustrates that the bulk boundary correspondence is still valid from the energy side. On the other hand, the topological invariant will not be well-defined provided that the shrinkage occurs to the GBZ, which demonstrates that the bulk boundary correspondence is illness from the topological invariant aspect. However, it also can be found that whereas the GBZ emerges as a closed curve, the bulk boundary correspondence will not be established definitely. The results we find about the bulk boundary correspondence can be revealed in figure 1. As an illustrative example, a non-Hermitian Su–Schrieffer–Heeger (SSH) model can be constructed since it possesses the structural simplicity and the abundant physical insight concurrently.

The paper is organized as follows. Section 2 provides the generic theoretical model, in which we show how the GBZ becomes a point and the induced imperfect bulk boundary correspondence. Moreover, we find that bulk boundary correspondence is still illness even when the GBZ exists as a closed loop. A concrete non-Hermitian SSH model can be given in section 3 to explain our results numerically and analytically. The conclusions are showed in section 4. In the appendix, we investigate some other examples which reveal the phenomena we find visually.
2. Model and theory

We begin with a one-dimensional tight-binding model, which can be described by [21]

\[
H = \sum_{n} \sum_{i=-N}^{N} \sum_{s,m=1}^{q} t_{i,n,m} C_{n+i}^{s} C_{n,m}^{s},
\]

where \( N \) is the range of hopping amplitude, and \( q \) denotes the degrees of freedom per unit cell. To define the winding number properly, \( q \) should be even number [1–5, 12], here taken to be \( q = 2 \). When \( t_{i,n,m} = t_{i+n,m}^{*} \), the system is Hermitian and the Bloch Hamiltonian \( H(k) \) is

\[
H(k) = \begin{pmatrix} 0 & h(k) \\ h^{*}(k) & 0 \end{pmatrix},
\]

in which the topological invariant can be defined through \( h(k) = d_{n} + id_{p} \) with \( k \) sweeping across the Brillouin zone, as shown in figure 1(a). However, if \( t_{i,n,m} \neq t_{i+n,m}^{*} \), the Brillouin zone must be extended to the GBZ and the topological invariant will be defined as [19, 21]

\[
W = -\frac{\arg R_{+}(\beta) - \arg R_{-}(\beta)}{4\pi},
\]

\( R_{\pm}(\beta) \) are the elements of the non-Bloch Hamiltonian

\[
H(\beta) = \begin{pmatrix} 0 & R_{+}(\beta) \\ R_{-}(\beta) & 0 \end{pmatrix},
\]

where \( H(\beta) \) is the counterpart of the Bloch Hamiltonian \( H(k) \) via replacing \( e^{ik} \rightarrow \beta, k \in \mathbb{C} \).

From equation (3), one can see that the condition that \( R_{\pm}(\beta) \) are the closed loop containing finite areas is necessary for the definition of the topological number [22] (figures 1(b1) and (b2)), which is the essential ingredient characterizing the bulk boundary correspondence. However, we notice that even \( R_{\pm}(\beta) \) remain the shape of closed loop, there may not exist the integer topological invariant to describe the bulk boundary correspondence [24]. The result hints that the bulk boundary correspondence is bizarreness from topological invariant side. Thus, except for the normal case, it is reasonable for the bulk boundary correspondence supporting other situations.

Additionally, the GBZ, which plays an important role in rebuilding the bulk boundary correspondence, is characterized by the pole of the equation \( \det[H(\beta) - E] = \frac{P(\beta)}{\Delta} = 0 \ (p \neq 0) \) [19, 21]. However, we note that, numerically and analytically, the GBZ deforms into a point without any interior area if \( p \) is zero. This phenomenon has a reasonable physical interpretation and engenders unexpected consequences. According to references [19, 21], the GBZ can be visualized as follows. The wave propagates from the right end toward the left, and it will hit the left end and get reflected. Then, the situation that the wave propagates to one end and has no reflection corresponds to that the GBZ becomes a point, i.e. the particle will propagate unidirectionally and accumulate at one end totally. This image also can be verified numerically as we show later (figures 5(a) and 8(b) and (d)). This one-way propagation is the unique nature of non-Hermitian systems due to \( t_{i,n,m} \neq t_{i+n,m}^{*} \) i.e. \( t_{i,n,m} \) is zero while \( t_{i+n,m}^{*} \) survives. In this case, the correspondence \( R_{\pm}(\beta) \) also degenerate to a point on the complex plane (figures 1(b3) and (b4)). Further, any point in the complex plane, relative to the origin, cannot have the concept of encircling [22]. Therefore, the topological invariant should be illness, i.e. the bulk boundary correspondence is destroyed from this point.

On the other hand, the continuum bands can be defined as [19, 21]

\[
E_{GBZ}^{\pm} = R_{\pm}(\beta_{GBZ}) R_{\pm}(\beta_{GBZ}).
\]

By numerous numerical calculations, we find that even though the GBZ reduces into a point, the energy spectra have an overt distinction between the periodic boundary condition and open one. Strikingly, the open boundary energies ever can be reproduced by the continuum bands. In other words, the correctness of the bulk boundary correspondence remains unchanged in this perspective.

Intuitively, the imperfect of the bulk boundary correspondence is induced by the bizarreness of the GBZ. However, through numerical calculations and theoretical analysis, we find that there always exist appropriate parameters in equation (4) holding that the GBZ is a closing loop \( (p \neq 0) \). Meanwhile, one of \( R_{\pm}(\beta) \) remains the function of \( \beta \) while another term is a constant (figures 1(b5) and (b6)). In this case, some features of the system can be concluded as follows. First, the GBZ can be deformed into a unit circle exactly under non-Hermiticity. Second, \( R_{+}(\beta) \) or \( R_{-}(\beta) \) forms a curve with \( \beta \) turning around the GBZ in the counterclockwise direction, but unexpectedly, the curve is unclosed in the complex plane. The statement
3. Non-Hermitian Su–Schrieffer–Heeger model

As a prime example of the preceding discussion, a one-dimensional non-Hermitian SSH system is given as

\[
H = \sum_n \left[ \left( t_1 + \frac{\gamma_1}{2} \right) C_{A,n}^\dagger C_{B,n} + \left( t_1 - \frac{\gamma_1}{2} \right) C_{B,n}^\dagger C_{A,n} \right. \\
\left. + \left( t_2 + \frac{\gamma_2}{2} \right) C_{A,n+1}^\dagger C_{B,n} + \left( t_2 - \frac{\gamma_2}{2} \right) C_{B,n+1}^\dagger C_{A,n} \\
\left. + \left( t_3 + \frac{\gamma_3}{2} \right) C_{A,n}^\dagger C_{B,n+1} + \left( t_3 - \frac{\gamma_3}{2} \right) C_{B,n+1}^\dagger C_{A,n} \right],
\]

(6)

where \( C_{A,n}^\dagger \) (\( C_{A,n} \)) is the creation (annihilation) operator on the sublattices \( A, B \) in the \( n \)th unit cell. \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) represent the non-Hermiticity. \( t_1, t_2 \) and \( t_3 \) characterize the intracell and intercell hoppings.

Based on the Schrödinger equation \( H|\psi\rangle = E|\psi\rangle \), where \( |\psi\rangle = (\psi_{A,1}, \psi_{B,1}, \ldots, \psi_{A,n}, \psi_{B,n}, \ldots)^T \), the eigen equation in the bulk will be written as

\[
\left( t_2 - \frac{\gamma_2}{2} \right) \psi_{B,n} + \left( t_1 + \frac{\gamma_1}{2} \right) \psi_{B,n+1} + \left( t_3 + \frac{\gamma_3}{2} \right) \psi_{B,n} = E\psi_{A,n+1},
\]

(7)

\[
\left( t_3 - \frac{\gamma_3}{2} \right) \psi_{A,n} + \left( t_1 - \frac{\gamma_1}{2} \right) \psi_{A,n+1} + \left( t_2 + \frac{\gamma_2}{2} \right) \psi_{A,n} = E\psi_{B,n+1}.
\]

(8)

The elements of the wavefunction have the form \( \psi_{A,n} = \beta^A \phi_A \) and \( \psi_{B,n} = \beta^B \phi_B \) \([19, 21]\). Then, one can get

\[
\left( t_2 - \frac{\gamma_2}{2} \right) \phi_B + \left( t_1 + \frac{\gamma_1}{2} \right) \beta \phi_B + \left( t_1 + \frac{\gamma_1}{2} \right) \beta^2 \phi_B = E\beta \phi_A,
\]

(9)

\[
\left( t_3 - \frac{\gamma_3}{2} \right) \phi_A + \left( t_1 - \frac{\gamma_1}{2} \right) \beta \phi_A + \left( t_2 + \frac{\gamma_2}{2} \right) \beta^2 \phi_A = E\beta \phi_B.
\]

(10)

Hence, the eigenvalue equation about \( \beta \) can be obtained by multiplying equations (9) and (10), and it is quadratic. Accordingly, the GBZ can be generally determined by \( \beta_2 \) and \( \beta_3 \) satisfying \( |\beta_2| \leq |\beta_1| = |\beta_3| \leq |\beta_4| \), which are the solutions of this eigenvalue equation for a given \( E \).

Next, based on the GBZ, the corresponding topological invariant and continuum bands, and the energy spectra under different boundary conditions, the bulk boundary correspondence will be demonstrated from the signatures of the GBZ.

3.1. The GBZ being a point formed by \( \beta_1 \) and \( \beta_2 \)

For simplicity, \( t_1 = \gamma_1 = 0 \) can be explored firstly. Here, the analytical expressions of the GBZ and the corresponding continuum bands \( E_{\text{GBZ}} \) can be easily achieved.

According to equations (9) and (10), the characteristic equation of \( \beta \) is

\[
\left[ \left( t_1 + \frac{\gamma_1}{2} \right) + \left( t_2 + \frac{\gamma_2}{2} \right) \beta \right] \left[ \left( t_2 - \frac{\gamma_2}{2} \right) + \left( t_1 + \frac{\gamma_1}{2} \right) \beta \right] = E^2 \beta.
\]

(11)

The two solutions have the form \( \beta_{1,2} = \pm \frac{b \pm \sqrt{b^2 - 4ac}}{2a} \) with \( a = \left( t_1 + \frac{\gamma_1}{2} \right)\left( t_2 + \frac{\gamma_2}{2} \right), b = \left( t_1 - \frac{\gamma_1}{2} \right)\left( t_2 + \frac{\gamma_2}{2} \right) + \left( t_2 - \frac{\gamma_2}{2} \right)\left( t_1 + \frac{\gamma_1}{2} \right) - E^2 \) and \( c = \left( t_1 - \frac{\gamma_1}{2} \right)\left( t_2 + \frac{\gamma_2}{2} \right) \). Therefore, the trajectories of \( \beta_1 \) and \( \beta_2 \) satisfying \( |\beta_1| = |\beta_2| \) constitute the GBZ, which implies \( \sqrt{b^2 - 4ac} = -i \eta b, \eta \in \mathbb{R} \) \([57, 58]\). Therefore, the GBZ and continuum bands can be described by

\[
|\beta_{\text{GBZ}}| = \frac{|(t_1 - \frac{\gamma_1}{2})(t_2 - \frac{\gamma_2}{2})|}{|(t_1 + \frac{\gamma_1}{2})(t_2 + \frac{\gamma_2}{2})|},
\]

(12)

and

\[
E_{\text{GBZ}} = \left( t_1 + \frac{\gamma_1}{2} \right)\left( t_1 - \frac{\gamma_1}{2} \right) + \left( t_2 + \frac{\gamma_2}{2} \right)\left( t_2 - \frac{\gamma_2}{2} \right) \pm \sqrt{\frac{4ac}{1 + \eta^2}}.
\]

(13)
Equation (12) analytically shows that the GBZ is a circle with the radius of $|\beta_{GBZ}|$, which is undoubtedly a closed loop in most cases. As shown in figure 2(a), the energy spectrum in open boundary condition composed of the red line and the green dot is deviated from the one in periodic boundary condition (the blue line), but is consistent with the continuum bands in figure 2(b) calculated through $E^{GBZ}$ except for the green dot standing for the two-degenerate zero modes. This deviation originates from the non-Hermitian skin modes \[19, 59\], illustrated in figure 2(c).

On the other hand, the expressions $R_+(\beta)$ can be acquired as

$$R_+(\beta) = t_1 + \frac{\gamma_1}{2} + \left(t_2 - \frac{\gamma_2}{2}\right)\beta^{-1},$$

(14)

$$R_-(\beta) = t_1 - \frac{\gamma_1}{2} + \left(t_2 + \frac{\gamma_2}{2}\right)\beta.$$  

(15)

As shown in figure 2(d), $R_+(\beta)$ encircle the origin once when $\beta$ goes along the GBZ in the counterclockwise way, which means the system is nontrivial and has the zero modes corresponding the green dot in figure 2(a).

The analysis process above shows that the recovery of the bulk boundary correspondence relies on $R_+(\beta)$ being closed curves, which requires that GBZ itself must be a closed loop as well. However, the analytical expression equation (12) shows that if either $t_1$ near to $\frac{\gamma_1}{2}$ or $t_2$ near to $\frac{\gamma_2}{2}$, the GBZ will deform into a point gradually, as indicated schematically in figure 3(a). In addition, the parameter $t_2 = \frac{\gamma_2}{2}$ will simplify equations (14) and (15) into

$$R_+(\beta) = t_1 + \frac{\gamma_1}{2},$$

(16)

and

$$R_-(\beta) = t_1 - \frac{\gamma_1}{2} + 2t_2\beta.$$  

(17)

Clearly, the equation $\det[H(\beta) - E] = R_+(\beta)R_-(\beta) - E^2 = 0$ not exists pole on the complex plane. Based on the result in section 2, the GBZ should be a point, which is in accordance with equation (12) and figure 3(a).

One can see that in figures 3(b) and (c) the open boundary energies cannot be reproduced by the periodic boundary ones yet when $t_2 = \frac{\gamma_2}{2} = 1$. Amusingly, even if the GBZ collapses into a point, the open boundary energy spectra still coincide with the continuum bands $E^{GBZ}$, as shown in figures 3(b) and (d).
However, the winding number is ill-defined because both $R_+ (\beta)$ and $R_- (\beta)$ not emerge as the closed loop, which is attributed to the singularities of the GBZ. Above all, under the condition that the GBZ has strangeness, the bulk boundary correspondence is still valid from the perspective of energy spectrum, but not from the side of topological invariant, i.e. the correctness of the bulk boundary correspondence has been destroyed partially with the GBZ being a point.

3.2. The GBZ being a point formed by $\beta_2$ and $\beta_3$

To further illustrate our results, the case $t_2 = \frac{-\gamma_1}{2}$ can be explored. The characteristic equation of $\beta$ has the form

$$
\left[ \left( t_1 + \frac{\gamma_1}{2} \right) \beta + \left( t_3 + \frac{\gamma_3}{2} \right) \beta^2 \right] \left[ \left( t_1 - \frac{\gamma_1}{2} \right) \beta + 2t_2\beta^2 + \left( t_3 - \frac{\gamma_3}{2} \right) \right] = E^2 \beta^2. \tag{18}
$$

The parameters given ensure that this is a quartic equation. The solutions can be ordered as $|\beta_1| \leq |\beta_2| \leq |\beta_3| \leq |\beta_4|$ for the eigenvalue $E$ and $|\beta_2| = |\beta_3|$ is necessary to recover the bulk boundary correspondence.

It can be found that equation (18) has a constant solution of $E = 0$ and thus the rest of the solutions should be taken precedence. As shown in figures 4(a) and (b), the open boundary energy spectrum matches well with the continuum bands if the zero energy modes is excluded. Furthermore, in figures 4(c) and (d), $R_+ (\beta)$ form the closed loop containing the origin or not, which implies $W = 0$ and $W = 1$ for $t_1 = -0.8$ and $t_1 = 0.1$, respectively. This result also can be confirmed by the zero modes in the open boundary energy spectrum in figure 4(a). Hence, the bulk boundary correspondence can be well revised even one of the solutions is zero for a quartic equation.

Except for $t_2 = \frac{-\gamma_1}{2}$, we assume $t_3 = \frac{-\gamma_1}{2}$ additionally. The characteristic equation (18) can be reduced to

$$
\beta^2 \left[ t_1^2 - \frac{\gamma_1}{4} + 2 \left( t_1 \left( 1 + \frac{\gamma_1}{2} \right) + t_3 \left( 1 - \frac{\gamma_1}{2} \right) \right) \beta + 4t_2t_3\beta^2 - E^2 \right] = 0, \tag{19}
$$

and one can obtain

$$
R_+ (\beta) = t_1 + \frac{\gamma_1}{2} + 2t_3\beta, \tag{20}
$$

$$
R_- (\beta) = t_1 - \frac{\gamma_1}{2} + 2t_2\beta. \tag{21}
$$

Equations (19)–(21) can be understood from different perspectives. First, the quartic equation has multiple roots of $\beta_1 = \beta_2 = 0$, which leads to $|\beta_2| = |\beta_3| = 0$, i.e. the GBZ contracts to a point. Second, similar as above, the equation $\text{det}[H(\beta) - E] = R_+ (\beta)R_- (\beta) - E^2 = 0$ is analytic over the entire complex plane based on the conclusions of section 2. Therefore, the GBZ should be a point. The conclusion of the perspectives matches well with each other.

In figure 5(a), all the skin modes can be displayed. Clearly, when $\gamma_2$ are closing to $2t_3$, the eigenstates will be nudged to the left steadily. This numerical result is in accordance with our analysis in section 2 as well. Accordingly, the continuum bands can be represented as $E^2_{\text{GBZ}} = t_1^2 - \frac{\gamma_1^2}{4}$. In this specified case, the open boundary energy spectrum even can be reconstructed by the continuum bands loyally, but not the periodical boundary energy spectrum, as shown in figures 5(b)–(d). However, owning to the fact that $R_+ (\beta)$ and $R_- (\beta)$ become the constant in this case, the well definition of the topological invariant cannot be ensured. Therefore, if GBZ possesses the bizarre natures, the correctness of the bulk boundary correspondence depends on the point where we interrogate it from.
3.3. The imperfect of bulk boundary correspondence with the GBZ being a closed loop

Intuitively, it seems that the imperfect of the bulk boundary correspondence is induced by the bizarreness of the GBZ. However, we find that even when the GBZ arises with a closed loop exactly, the topological invariant also may be illness, i.e. the bulk boundary correspondence is blurry. We here take the GBZ. However, we find that even when the GBZ arises with a closed loop exactly, the topological correspondence from the energy perspective.

This is a cubic equation of $\beta$. Then, the bulk boundary correspondence is renewed taken for granted as long as the condition $|\beta_2| = |\beta|$ is met. However, if one of the three solutions is $\beta = 0$, by, e.g. $\gamma_3 = \frac{\pi}{2}$, the situation will become subtle,

$$E_{GBZ}^2 = \left( t_1 - \frac{\gamma_1}{2} \right) \left( t_1 + \frac{\gamma_1}{2} \right) \mp \sqrt{\frac{16t_2t_3(t_1 - \frac{\gamma_3}{2})^2}{1 + \eta^2}},$$

$$|\beta| = \sqrt{\frac{t_2}{t_3}},$$

$$R_+ (\beta) = \left( t_1 + \frac{\gamma_1}{2} \right) + 2t_2\beta^{-1} + 2t_3\beta,$$

and

$$R_- (\beta) = t_1 - \frac{\gamma_1}{2}.$$

It can be shown that in figure 6, three different circumstances of open boundary spectra and the continuum energies depending on equation (23) have been revealed. Evidently, the spectra of the open chain still can be regained faithfully using the $E_{GBZ}$, which means the bulk boundary correspondence is reproduced from the energy side.

Equation (24) analytically demonstrates that the GBZ itself is a circle with the radius of $\sqrt{\frac{t_2}{t_3}}$, which is a well-defined closed loop in this situation, and thus implies the rehabilitation of the bulk boundary correspondence from the energy side is reasonable. Interestingly, if $t_2 = t_3$, the GBZ will deform into a unit circle even the system is non-Hermitian. Meanwhile, equations (25) and (26) show that $R_+ (\beta)$ is the function of $\beta$ formally and $R_- (\beta)$ is just a point in the complex plane ($p \neq 0$), which reflects that the GBZ is closed equally based on the results of section 2.

From another perspective, as displayed in figure 7(a), none of $R_+ (\beta)$ remain a closed circle even though the GBZ itself is closed. One collapses into a point, while the other becomes a line segment even $R_+ (\beta)$ is the function of GBZ (equation (25)). In other words, even if the GBZ is an exactly closed loop, the correctness of the bulk boundary correspondence is only partially established. In this situation, the spectral flow from periodic boundary to open one can be explored [57, 58]. In figure 7(b), the black dots are the eigenvalues in open boundary, which is purely imaginary and the imaginary can be explained by max$|E_{GBZ}^2| < 0$ in equation (23). We can see that the periodic boundary spectra (the green circles) flow toward the open boundary ones monotonically and blend into the imaginary axis eventually. Similarly, $R_+ (\beta)$ can be plotted in figure 7(c) using another parameters and clearly, none of them contains a finite area, which reflects the undefinability of the bulk boundary correspondence again. As shown in figure 7(d), the open boundary energy spectra are complex numbers, which can be ensured in terms of min$|E_{GBZ}^2| < 0$ and max$|E_{GBZ}^2| > 0$ in equation (23). In this case, the energy spectra in periodic boundary will evolution into the two black curves gradually.
Figure 6. \( N = 100 \). For (a) and (d), \( \gamma_1 = 0.4, t_2 = -\gamma_2 = 0.25 \) and \( t_3 = -\gamma_3 = 0.05 \). For (b) and (e), \( t_1 = 2, \gamma_1 = 0.4 \) and \( t_3 = -\gamma_3 = 0.05 \). For (c) and (f), \( t_1 = 2, \gamma_4 = 0.4 \) and \( t_2 = -\gamma_2 = 0.25 \). From (a) to (c), these figures show the open boundary spectra under different cases, which is consistent well with the continuum bands ((d)-(f)) based on the equation (23).

Figure 7. The elements of the topological invariant and spectral flow. The common parameters are given by \( N = 50, t_1 = 0.22, \gamma_1 = 0.9, t_2 = -\gamma_2 = 0.25 \). For (a) and (b), \( t_3 = -\gamma_3 = 0.05 \). For (c) and (d), \( t_3 = -\gamma_3 = 0.05 \). For (b) and (d), the green loop represents the energy spectra obtained from the momentum-space Hamiltonian, and the black dots are the open boundary energy spectra. The blue-yellow curves stand for the spectral evolution from the periodic boundary to open ones.

Experimental realization and application. So far, there are many ways that can simulate the topological features, including electric circuits \([60–63]\), cold atoms \([64–69]\), etc. Those artificial systems enjoy a high degree of controllability and thus may be engineered to hold dissipation \([30, 70–77]\), which is a prerequisite description for the non-Hermiticity. Hence, our model can be realized in different types of artificial systems. In addition, we note that this singular properties depend exactly on fine tuning of parameters, which means it is highly sensitive to physical perturbations. Therefore, this property may have an potential applications in sensing \([70, 78]\).

4. Conclusion

In this work, a one dimensional non-Hermitian model has been constructed to investigate the physical properties of the bulk boundary correspondence under the circumstance that the GBZ has bizarre features. We have calculated both the continuum bands and topological invariant relevant to the GBZ and the energy spectra under different boundary conditions. It can be found that the energy bands of the open chain always can be regained by the continuum bands \( E_{\text{GBZ}} \), no matter if the GBZ is a point or not. Oppositely, the singularity of the GBZ will cause the ill-definition of the topological invariant. Accordingly, the bulk boundary correspondence ever can be recovered if we only restrict this concept to band structures, but not relate to the topological number. Counter-intuitively, we also find that the bulk boundary correspondence may retain illness even though the GBZ is a closed loop, since the elements of the Hamiltonian \( R_\pm(\beta) \) is not...
closed on the complex plane. Moreover, those discoveries are an effective supplement to the current non-Bloch band theory.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix

In the main text, the imperfect of bulk boundary correspondence has been explored analytically and numerically in terms of the non-Hermitian SSH model. Here, the properties of bulk boundary correspondence can be elucidated by other systems as well.

For figures 8(a) and (b),
\[
 R_+ (\beta) = \left( t_1 + \frac{\gamma_1}{2} \right) + 2 t_3 \beta + \left( t_4 + \frac{\gamma_4}{2} \right) \beta^2, \tag{27}
\]
\[
 R_- (\beta) = \left( t_1 - \frac{\gamma_1}{2} \right) + 2 t_2 \beta + \left( t_4 - \frac{\gamma_4}{2} \right) \beta^{-2}. \tag{28}
\]
When \( t_4 = \frac{\gamma_4}{2} \), as shown in figure 8(a), the open boundary energies and continuum bands \( E_{GBZ} \) are consistent with each other. On the other hand, with \( t_4 \) closing to \( \frac{\gamma_4}{2} \), the wavefunction develops a tendency to accumulate at one boundary (figure 8(b)).

For figures 8(c) and (d),
\[
 R_+ (\beta) = \left( t_1 + \frac{\gamma_1}{2} \right) + 2 t_3 \beta + 2 t_4 \beta^2 + \left( t_5 - \frac{\gamma_5}{2} \right) \beta^{-2}, \tag{29}
\]
\[
 R_- (\beta) = \left( t_1 - \frac{\gamma_1}{2} \right) + 2 t_2 \beta + \left( t_5 + \frac{\gamma_5}{2} \right) \beta^2. \tag{30}
\]
As displayed in figure 8(c), the energies under open boundary can be reproduced by the continuum bands \( E_{GBZ} \) if \( t_5 = \frac{\gamma_5}{2} \). Analogously, if \( t_5 \) near to \( \frac{\gamma_5}{2} \), the eigenstates have pinned at one termination steadily (figure 8(d)).

For figures 8(e) and (f),
\[
 R_+ (\beta) = \left( t_1 + \frac{\gamma_1}{2} \right) + 2 t_3 \beta^{-1} + 2 t_5 \beta + 2 t_4 \beta^2, \tag{31}
\]
\[ R_-(\beta) = \left(t_1 - \frac{\gamma_1}{2}\right), \]  
(32)
and for figures 8(g) and (h),
\[ R_+(\beta) = \left(t_1 + \frac{\gamma_1}{2}\right) + 2t_2\beta^{-1} + 2t_3\beta + 2t_4\beta^2 + 2t_5\beta^{-2}, \]  
(33)
\[ R_-(\beta) = \left(t_1 - \frac{\gamma_1}{2}\right). \]  
(34)

From figures 8(e)–(h), one can see that the GBZ is closed on the complex plane undoubtedly. However, \( R_+(\beta) \) are not closed, i.e. the bulk boundary correspondence is imperfect.

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