Research Article

Optimized Direct Padé and HPM for Solving Equation of Oxygen Diffusion in a Spherical Cell

M. A. Sandoval-Hernandez,1, H. Vazquez-Leal,2 A. Sarmiento-Reyes,1 U. Filobello-Nino,2 F. Castro-Gonzalez,1 A. L. Herrera-May,3,4 R. A. Callejas-Molina,5 and R. Ruiz-Gomez1

1National Institute for Astrophysics, Optics and Electronics, Luis Enrique Erro No. 1, Sta. María Tonantzintla, 72840, Puebla, Mexico
2Facultad de Instrumentación Electrónica, Universidad Veracruzana, Cto. Gonzalo Aguirre Beltrán S/N, 91000, Xalapa, Veracruz, Mexico
3Centro de Investigación de Micro y Nanotecnología, Universidad Veracruzana, Calzada Ruiz Cortines 455, Boca del Río, 94292, Veracruz, Mexico
4Maestría en Ingeniería Aplicada, Facultad de Ingeniería de la Construcción y el Hábitat, Universidad Veracruzana, Calzada Ruiz Cortines 455, 94294, Boca del Río, Veracruz, Mexico
5Instituto Tecnológico de Celaya, Antonio García Cubas Pte No. 600 Esq. Av. Tecnológico, Celaya, 38010, Guanajuato, Mexico

Correspondence should be addressed to M. A. Sandoval-Hernandez; m.sandoval@inaoep.mx

Received 25 January 2018; Accepted 11 July 2018; Published 2 September 2018

Academic Editor: Guang Zhang

Copyright © 2018 M. A. Sandoval-Hernandez et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This work presents the application of homotopy perturbation method (HPM) and Optimized Direct Padé (ODP) to obtain a handy and easily computable approximate solution of the nonlinear differential equation to model the oxygen diffusion in a spherical cell with nonlinear oxygen uptake kinetics. On one hand, the obtained HPM solution is fully symbolic in terms of the coefficients of the equation, allowing us to use the same solution for different values of the maximum reaction rate, the Michaelis constant, and the permeability of the cell membrane. On the other hand, the numerical experiments show the high accuracy of the proposed ODP solution, resulting in $3.58 \times 10^{-4}$ as the lowest absolute relative error (A.R.E.) for a set of coefficients. In addition, a novel technique is proposed to reduce the number of algebraic operations during the process of application of ODP method through the use of the Taylor series, which help to simplify the algebraic expressions used. The powerful process to obtain the solution shows that the Optimized Direct Padé and homotopy perturbation method are suitable methods to use.

1. Introduction

Michaelis-Menten kinetics describes the rate of enzymatic reactions. This model is valid when the concentration of a substrate is higher than the concentration of the enzymes, and for steady state conditions, that is, when the concentration of the complex enzyme-substrate is constant [1]. There are several works about the Michaelis-Menten oxygen uptake kinetics. For example, in [2], the relation between Michaelis-Menten direct and inverse kinetics, chemical kinetics of approximation, and second-order kinetics is presented. Moreover, in [3], the subject of transport phenomena is approached by means of the study of transport of quantity of motion (viscous flow), transport of energy (heat conduction, convection, and radiation), and matter transport (Diffusion) [4, 5]. Consequently, transport phenomena are employed for solving different problems in the area of sciences such as chemistry, biochemistry, soil physics, meteorology, biology, and semiconductors, disciplines in which the use of Bessel functions, differential equations, and Laplace transform is required [6–10]. In [11], the behaviour of dopamine released from a small iontophoresis electrode and its voltammetric detection by a carbon fiber sensor 100pm away is presented as a basis for developing a new paradigm for measuring dopamine kinetics in intact rat neostriatum. In [11], the presented model was derived from the work of diffusion...
in a punctual iontophoretic source where the lineal term of absorption is replaced by a nonlinear expression that describes a Michaelis-Menten kinetic given by a constant \( V_m \), the Michaelis-Menten constant \( K_m \).

In [12], the equation that models oxygen diffusion in a spherical cell, including nonlinear absorption kinetics, is solved by transforming the Lane-Emden equation into its equivalent Volterra integral form, then Adomian decomposition was employed to solve the nonlinear Fredholm-Volterra integral. Furthermore, in [13] a fully symbolic solution was proposed to solve the aforementioned equation by means of a modified Taylor series obtaining an accuracy of \( 1.66 \times 10^{-15} \) as the lowest mean square error without the use of complicated integrals.

There are several methods of solutions for solving nonlinear differential equations such as inverse scattering transformation [14, 15], Darboux transformation [16, 17], bilinear method [18, 19], the tanh-function method [20–22], the variable separation approach [23–25], the symmetry method [26, 27], sine-cosine method [28–30], Adomian decomposition method (ADM) [12, 31–33], and homotopy perturbation method (HPM) [34–36]. HPM is based on the use of power series of \( p \), which transforms a differential equation into a set of linear differential equations.

In [37] was presented a procedure to apply Padé method to find approximate solutions for nonlinear differential equations, which consist in that the solution of a differential equation can be directly expressed as a rational power series of the independent variable as a Padé approximant. From (6) ODP employs a polynomial-like rational expression as the proposal of approximation of the nonlinear differential equation to be solved. In general terms, it works by means of substituting the rational expression in the differential equation and then regroups the powers of the independent variable. It is important to note that due to the rational expression a large amount of algebraic operations is generated. However, this work proposes a Direct Padé (DP) modification oriented to reduce this algebraic operation by means of the Taylor expansion of the rational function.

This paper is organized as follows. In Section 2, a brief description of Michaelis-Menten kinetics and its equation is given. In Section 3, a brief description of Lane-Emden equation is presented. The equation to be solved and its boundary conditions are introduced in Section 4. Section 5 describes in detail Optimized Direct Padé. In Section 6, the HPM solution is presented. Section 7 presents the analysis of solution for the case study by HPM and ODP. In Section 8, its numerical simulations, comparisons, and discussion are presented. Lastly, in Section 9 a brief conclusion of this work is presented.

### 2. Michaelis-Menten Equation

General principles of kinetics in chemical reactions are applicable to catalysed reactions by enzymes in living things. However, this shows a characteristic side which is not observed in nonenzymatic catalyst, the substrate saturation, in terms of enzyme molecules active sites occupation [1]. The study of the effect that substrate concentration has over enzyme activity is not simple task; logically thinking, substrate concentration lowers as the reaction increases. A simplification on kinetics experiments consists in measuring initial velocity \( V_o \). If time is short enough, substrate lowering will be minimal and this could be considered, thus, constant. This behaviour is characteristic of most enzymes and was studied by Michaelis and Menten in 1913 [38]. Figure 1 presents the three phases of enzyme kinetics:

(i) For a low substrate concentration, velocity reaction is directly proportional to substrate concentration (linear relation), first-order kinetics.
(ii) For a high substrate concentration, velocity reaction is practically constant and independent of substrate concentration; kinetics is considered zero order.
(iii) For medium concentrations of substrate, velocity of the process becomes nonlinear and this phase is called mixed kinetics.

The curve which expresses the relationship between substrate concentration and initial velocity has the same shape for most enzymes, a rectangular hyperbola, whose algebraic expression is given by Michaelis-Menten equation (1), which describes the relationship between initial velocity \( V_o \) and substrate concentration, \( S \) [1]:

\[
V_o = \frac{V_{max} [S]}{K_m + [S]},
\]

\( V_{max} \), maximum velocity, and \( K_m \), Michaelis constant, are two characteristic kinetics parameters of every enzyme that can be experimentally determined.

Maximum velocity \( V_{max} \) is obtained when velocity of reaction becomes independent of substrate concentration; when this occurs, velocity reaches a maximum value. This value depends on the available amount of enzyme. Michaelis constant \( K_m \) indicates substrate concentration in which reaction rate is half of maximum velocity. This parameter is independent of enzyme concentration and it is characteristic of every enzyme according to the employed substrate (or substrates). Constant \( K_m \) indicates the affinity that the enzyme has for the substrate, being this higher at a lower value of \( K_m \). For a lower value \( K_m \), the lower the substrate needed to reach half of maximum velocity, thus, the higher the affinity an enzyme has for the substrate.

![Figure 1: Enzyme kinetics: Michaelis-Menten model.](image-url)
3. Lane-Emden Equation

Lane-Emden equation (2) is a basic nonlinear equation arising in the study of stellar structure theory and has

\[ y''(x) + \frac{m}{x} y'(x) + f(x, y) = g(x), \quad 0 < x \leq 1, \quad m \geq 0, \]

(2)

where \( m \) can be chosen to describe a Cartesian geometry, \( m = 0 \); cylindrical, \( m = 1 \); or spherical, \( m = 2 \) [39]. \( f(x, y) \) is a real function and \( g(x) \) is an analytical function. Initial conditions are

\[ y(0) = A, \quad y'(0) = B, \]

(3)

where \( A \) and \( B \) are constants.

Equation (2) is used to model physical and astrophysical phenomena such as theory of stellar structure, spherical cloud of gas thermal behaviour, isothermal gas sphere [40], and transport phenomena [5] as it occurs for the case of particle diffusion in biophysical processes [4] as well in physiological processes.

A model of oxygen diffusion and nonlinear absorption in a sphere was originally proposed and resolved in [41]. Later, the same model was proposed, analysed, and solved in [42]. The equation of the model is a steady state diffusion-reaction equation that represents oxygen transport by means of linear diffusion in a sphere.

4. Equation of Oxygen Diffusion in a Spherical Cell

The differential equation that models the oxygen diffusion in a spherical cell with Michaelis-Menten oxygen uptake kinetics (4) [41–46] is obtained by substituting (1) into (2) and given by

\[ C''(R) + \frac{2}{R} C'(R) - \alpha \frac{C(R)}{K + C(R)} = 0, \quad 0 < R \leq 1, \]

(4)

with boundary conditions

\[ C'(0) = 0, \]

\[ C'(1) + HC(1) = H. \]

(5)

Variables \( C \) and \( R \) represent the oxygen concentration and the radial distance, respectively. Parameters \( \alpha, K, \) and \( H \) are constants that represent the maximum reaction rate, Michaelis constant is the concentration at half saturation and cell membrane permeability, respectively.

When boundary condition \( R = 0 \) is equal to 0, oxygen distribution is symmetrical to the center of the sphere. Boundary condition at \( R = 1 \) determines that oxygen flow through the cell membrane is proportional to \( C'(1) = 1 - C(1) \) differential, which is less than the cell membrane normalized concentration [44, 45].

5. Optimized Direct Padé

The method ODP [37] considers that a nonlinear differential equation can be expressed as follows:

\[ L(u) + N(u) - f(r) = 0, \quad \text{where} \ r \in \Omega \]

(6)

with boundary conditions

\[ B \left(u, \frac{\partial u}{\partial \eta}\right) = 0, \quad \text{where} \ r \in \Gamma, \]

(7)

where \( L \) and \( N \) are the linear and nonlinear operators, respectively; \( f(r) \) is a known analytical function; \( B \) is a boundary operator; \( \Gamma \) is the boundary of the domain \( \Omega \) [47].

\( u_0 \) is the initial condition for (6) that satisfies the boundary condition and \( p \) is the homotopy perturbation parameter. The solution of (6) can be written as

\[ u(\epsilon) = \frac{a^0 x_1 + a^1 x_1 + a^2 x_2 + \cdots + a^m x_m}{1 + b^0 x_1 + b^1 x_1 + b^2 x_2 + \cdots + b^m x_m}, \]

(8)

where \( a_0, a_1, \cdots \) and \( b_0, b_1, \cdots \) are unknowns determined by the method; \( n, m \) are the numerator and denominator order.

There are no systematic criteria to choose the optimal Padé order \( [n/m] \) for a given problem. However, generally a finite number of terms are required to obtain highly precise Padé approximation. The basic procedure for Direct Padé can be described as follows:

(1) Boundary conditions are substituted into (8) to generate an equation for each boundary condition. It is important to note that there exists an algebraic equation for each boundary condition.

(2) \( u(x) \) from (8) is substituted into (6), then regrouping the resulting equation in terms of \( x \) powers. Afterwards, the regrouping procedure includes eliminating denominator terms arising from Padé approximant (8). In this way, the resulting expression is a power series that represent the residual error from (6).

(3) In order to reduce the residual error, from the lowest order, each coefficient from \( x \) powers in the resulting power series is equalized to zero to obtain an algebraic equation in terms of the unknown coefficients of (8).

(4) All the previous steps generate NAES in terms of the unknowns from (8). Finally, NAES are solved to obtain \( a_0, a_1, \cdots \) and \( b_0, b_1, \cdots \).

The disadvantage of the method is that it becomes as difficult as the number of terms that the expression has in the numerator as well in the denominator because it is a rational expression. Even more, the substitution into (8) gets more complicated if higher order derivatives are encountered, mainly because the first derivative of a rational expression lies in terms of another rational expression with a squared denominator and if in the nonlinear equation exist several different \( x \) powers, another rational expression with even more terms is generated. For this reason, a common denominator has to be found, to formulate the system of equations.
that allows finding \( a_i \) and \( b_i \). All this process may be too cumbersome and computationally expensive. In this matter, it is concluded that if the nonlinear equation possesses higher order derivatives of \( x \) (the independent variable) the number of algebraic operations is notoriously increased when \( x \) powers are regrouped to form the equation system to find \( a_i \) and \( b_i \). Thus, in order to avoid the algebraic complexity that implies the substitution of (8) into (6), in this paper we propose a novel strategy to circumvent such issue by obtaining the Taylor expansion of (8).

\[
T(u(x)) = f_0 + \frac{f_1}{1!}x + \frac{f_2}{2!}x^2 + \frac{f_3}{3!}x^3 + \cdots + \frac{f_q}{q!}x^q, \tag{9}
\]

where \( T(\cdot) \) is the Taylor expansion of (8) with respect to \( x \) and \( f_i(a_1, a_2, \ldots, b_0, b_1, \ldots) \) considering \( i = [0, 1, 2, \ldots, n] \), and \( q = m + n - 1 \) is the resulting expansion of (8). The replacement of (8) by (9) decreases enormously the number of algebraic steps when it is substituted into the differential equation obtaining the Taylor expansion of (8). The proposed homotopy perturbation formulation is

\[
H = (1 - p) \left[ K \frac{d^2}{dR^2} C(R) + \frac{d}{dR} C(R) + \frac{2K}{R} \frac{d}{dR} C(R) \right] + p \left[ K \frac{d^2}{dR^2} C(R) + \frac{2K}{R} \frac{d}{dR} C(R) + \frac{2C}{R} \frac{d}{dR} C(R) - \alpha C(R) \right]. \tag{11}
\]

Assuming the solution for (11) is given by a power series of \( p \)

\[
CR = C_0(R) + C_1(R) p + C_2(R) p^2 + \cdots, \tag{12}
\]

substituting (12) into (11), and regrouping in terms of \( p \),

\[
p^0: \frac{d^2}{dR^2} C_0(R) = 0 \quad C_0(0) = c \quad C'_0(0) = 0
\]

\[
p^1: R(K + C_0(R) - 1) \frac{d^2}{dR^2} C_0(R) + R \frac{d^2}{dR^2} C_1(R)
\]

\[
+ (2K + 2C_0(R) - 2) \frac{d}{dR} C_0(R) - R\alpha C_0(R)
\]

\[
+ 2 \frac{d}{dR} C_1(R) = 0, \quad C_1(0) = 0 \quad C'_1(0) = 0,
\] \( \vdots \)

\[
\tag{13}
\]

Substituting the first two system solutions (13) into (12), we obtain

\[
C(R) = A + B + \frac{1}{6} A\alpha R^2. \tag{14}
\]

This is the solution employing HPM, where \( A \) and \( B \) are constants to be determined. By substituting (14) into the second boundary condition of (5), i.e., \( C'(1) + HC(1) = H \), we obtain

\[
\frac{1}{3} A\alpha + HA + \frac{1}{6} H\alpha A + H(B - 1) = 0. \tag{15}
\]

Solving for \( A \) in (15),

\[
A = -\frac{6H(B - 1)}{6H + H\alpha + 2\alpha} \tag{16}
\]

Substituting in (14),

\[
C(R) = B - \frac{6H(B - 1)}{6H + H\alpha + 2\alpha} \frac{H(B - 1)\alpha R^2}{6H + H\alpha + 2\alpha}. \tag{17}
\]

With the end of reducing the residual error, (17) is substituted into (4), at middle point \( R = 0.5 \), between boundaries in order to cancel the error at that point, to obtain for \( B \)

\[
B = \frac{1}{12H\alpha(3H + 8)} \left( 24KH^2 \alpha + 6H^2 \alpha + 144KH^2 \\
+ 3H^2 \alpha^2 + 48KH\alpha + 144H^2 + 14H\alpha^2 + 16\alpha^2 \\
- (4032H^3 \alpha^2 + 448H\alpha^4 - 828H^4 \alpha^2 - 36H^4 \alpha^3 \\
+ 1728H^4 \alpha - 168H^3 \alpha^3 - 192H^2 \alpha^3 + 20736K^2 H^4 \\
+ 9H^4 \alpha^4 - 84H^4 \alpha^4 + 292H^2 \alpha^4 + 2112KH^2 \alpha^3 \\
+ 960KH^3 \alpha^3 + 19008KH^4 \alpha + 2304K^2 H^3 \alpha^2 \\
+ 144KH^4 \alpha^3 - 6912K^2 H^3 \alpha^2 + 2880KH^4 \alpha^2 \\
+ 576K^2 H^4 \alpha^2 + 12672KH^3 \alpha^2 + 1536KH^2 \alpha^3 \\
+ 41472KH^3 \alpha + 2304K^2 H^2 \alpha^2 + 13824KH^2 \alpha^2 \\
+ 13824K^2 H^2 \alpha + 20736H^4 + 256\alpha^4 \\
- 4608H^2 \alpha^2 \right)^{1/2}. \tag{18}
\]

This strategy seeks to reduce the approximation error. Substituting (18) into (17), we obtain the solution in function of \( \alpha \), \( K \), and \( H \).

\[
C(R) = A_1 + A_2 R^2, \tag{19}
\]
where $A_1$ and $A_2$ are given for

\[
A_1 = B - \frac{6H(B - 1)}{6H + Ha + 2\alpha},
\]
\[
A_2 = -\frac{H(B - 1)\alpha}{6H + Ha + 2\alpha}.
\]

(20)

7. **ODP Solution Procedure**

The following rational solution is now proposed for (4):

\[
C(R) = \frac{a_0 + a_1R + a_2R^2}{1 + b_1R + b_2R^2}.
\]

(21)

In order to reduce the amount of algebraic steps, it is proposed in this work, and for the first time, to expand the rational expression (21) and employ the Taylor series of order four:

\[
C(R) = a_i + (a_i - a_0b_1)R + (a_2 - a_0b_2)
\]
\[
+ (-a_1 + a_0b_1)R^2 + ((-a_1 + a_0b_1)b_2
\]
\[
+ (-a_2 + a_0b_2 + b_1a_1 - a_0b_1^2)b_1)R^3
\]
\[
+ ((-a_2 + a_0b_2 + b_1a_1 - a_0b_1^2)b_2
\]
\[
+ (b - 2a_1 - 2b_2a_0b_1 + b_1a_2 - b_1^2a_1 + a_0b_1^3)b_1)R^4,
\]

(22)

differentiating with respect to $R$ and to satisfy the first boundary with $R = 0$ yields

\[
a_1 + a_0b_1 = 0.
\]

(23)

With expansion (22) we have the advantage of eliminating the denominator and thus reducing the amount of algebraic terms when substituting (21) into (4). To satisfy the second boundary condition using (21) and (5)

\[
\frac{a_1 + 2a_2}{1 + b_1 + b_2} + \frac{(a_0 + a_1 + a_2) (b_1 + 2b_2)}{(1 + b_1 + b_2)^2} + \frac{H (a_0 + a_1 + a_2)}{1 + b_1 + b_2} - H = 0.
\]

(24)

Substituting fourth-order Taylor expansion of (21) into (10),

\[
3(2a_2 - 2a_0b_2 + 2(-a_1 + a_0b_1)b_1)K + 3(2a_2 - 2a_0b_2)
\]
\[
+ 2(-a_1 + a_0b_1)b_1)a_0 + 2(a_1 - a_0b_1^2)^2 - a_0 = 0,
\]
\[
(6(-a_1 + a_0b_1)b_2 + 6(-a_2 + a_0b_2 + b_1a_1 - a_0b_1^2)b_1)
\]
\[
\cdot K + 3(2a_2 - 2a_0b_2 + 2(-a_1 + a_0b_1)b_1)(a_1
\]
\[
- a_0b_1) + 6(-a_2 + a_0b_2 + b_1a_1 - a_0b_1^2)b_1)a_0
\]
\[
+ 2(3(-a_1 + a_0b_1)b_2
\]
\[
+ 3(-a_2 + a_0b_2 + b_1a_1 - a_0b_1^2)b_1)K + 2(a_1
\]
\[
- a_0b_1)(a_2 - a_0b_2 + (-a_1 + a_0b_1)b_1)
\]
\[
+ 2(3(-a_1 + a_0b_1)b_2
\]
\[
+ 3(-a_2 + a_0b_2 + b_1a_1 - a_0b_1^2)b_1)a_0 - \alpha(a_1
\]
\[
- a_0b_1) = 0,
\]
\[
12(-a_2 + a_0b_2 + b_1a_1 - a_0b_1^2)b_1
\]
\[
+ 12(b_2a_1 - 2b_2a_0b_1 + b_1a_2 - b_1^2a_1 + a_0b_1^3)b_1)K
\]
\[
+ 3(2a_2 - 2a_0b_2 + 2(-a_1 + a_0b_1)b_1)(a_2 - a_0b_2
\]
\[
+ (-a_1 + a_0b_1)b_1) + (31)
\]
\[
(6(-a_1 + a_0b_1)b_2 + 6(-a_2 + a_0b_2 + b_1a_1 - a_0b_1^2)b_1)
\]
\[
\cdot (a_1 - a_0b_1) + 12(-a_2 + a_0b_2 + b_1a_1 - a_0b_1^2)b_2
\]
\[
+ 12(b_2a_1 - 2b_2a_0b_1 + b_1a_2 - b_1^2a_1 + a_0b_1^3)b_1)a_0
\]
\[
+ 2(4(-a_1 + a_0b_2 + b_1a_1 - a_0b_1^2)b_2
\]
\[
+ 4(b_2a_1 - 2b_2a_0b_1 + b_1a_2 - b_1^2a_1 + a_0b_1^3)b_1)K
\]
\[
+ 2(a_1 - a_0b_1)((-a_1 + a_0b_1)b_2
\]
\[
+ (-a_2 + a_0b_2 + b_1a_1 - a_0b_1^2)b_1)
\]
\[
+ 2(3(-a_1 + a_0b_1)b_2
\]
\[
+ 3(-a_2 + a_0b_2 + b_1a_1 - a_0b_1^2)b_1)(a_1 - a_0b_1)
\]
\[
+ 2(4(-a_1 + a_0b_2 + b_1a_1 - a_0b_1^2)b_2
\]
\[
+ 4(b_2a_1 - 2b_2a_0b_1 + b_1a_2 - b_1^2a_1 + a_0b_1^3)b_1)a_0
\]
\[
- \alpha(a_2 - a_0b_2 + (-a_1 + a_0b_1)b_1) = 0.
\]

(25)

Substituting the values of $\alpha$, $K$, $H$ and solving the simultaneous system of equations given by (23), (24), and (25), the numerical values of $a_0, a_1, a_2, b_1$, and $b_2$ are obtained to satisfy (21).

8. **Numerical Simulations and Discussion**

For all the simulations presented in this work, with the purpose of comparison, Maple 17 numerical routines were employed. For example, numerical values of $\alpha = 2.5, K = 2,$
$H = 4$ were used, with (19) and (21) expressed with the following values:

$$C(R) = 0.8143254327 + 0.123780449 R^2, \quad (26)$$

$$C(R) = 0.8116575042 + 0.0946138264 R^2 \over 1 - 0.03162388156 R^2, \quad (27)$$

In order to verify the precision of the solutions (26) and ODP (27) obtained by HPM, their performance is compared with ADM solutions ($\varphi_2(R)$ and $\varphi_3(R)$) reported in [12], as well against the numerical solution given by Maple 17 numerical routines.

$$\varphi_2(R) = 1 + \frac{\alpha(-2 - H R^2)}{6H(K + 1)}, \quad (28)$$

$$\varphi_3(R) = 1 + \frac{\alpha(-2 - H R^2)}{6H(K + 1)} + \frac{\alpha^2 K (40 + 28H + 7H^2 + (-20H - 10H^2) R^2 + 3H R^4) - 360H^2 (K + 1)^2}{360H^2 (K + 1)^2}, \quad (29)$$

and (13) given by MTSM [13] is

$$\varphi(R) = 1 + \frac{\alpha(-2 - H R^2)}{6H(K + 1)} + \frac{\alpha^2 K (40 + 28H + 7H^2 + (-20H - 10H^2) R^2 + 3H R^4) - 360H^2 (K + 1)^2}{360H^2 (K + 1)^2}, \quad (29)$$

and (13) given by MTSM [13] is

$$\varphi(R) = 1 + \frac{\alpha(-2 - H R^2)}{6H(K + 1)} + \frac{\alpha^2 K (40 + 28H + 7H^2 + (-20H - 10H^2) R^2 + 3H R^4) - 360H^2 (K + 1)^2}{360H^2 (K + 1)^2}, \quad (29)$$

In Table 1 is presented and compared the absolute relative error (A.R.E.) for each approximation according to the values of $\alpha, K, H$, for HPM and ODP solutions, including (28), (29), and (30).

| Case          | ODP          | HPM          | MTSM [13] | $\varphi_2(R)$ [12] | $\varphi_3(R)$ [12] |
|---------------|--------------|--------------|-----------|---------------------|---------------------|
| $\alpha = 2.5, K = 2, H = 4$ | $3.58 \times 10^{-3}$ | $3.21 \times 10^{-3}$ | $6.99 \times 10^{-3}$ | $1.11 \times 10^{-2}$ | $4.54 \times 10^{-3}$ |
| $\alpha = 3.5, K = 2, H = 4$ | $1.00 \times 10^{-3}$ | $6.26 \times 10^{-3}$ | $1.37 \times 10^{-2}$ | $5.16 \times 10^{-3}$ | $5.06 \times 10^{-3}$ |
| $\alpha = 4.5, K = 3, H = 4$ | $8.37 \times 10^{-4}$ | $6.18 \times 10^{-3}$ | $1.34 \times 10^{-2}$ | $4.04 \times 10^{-2}$ | $2.90 \times 10^{-1}$ |
| $\alpha = 8.5, K = 5, H = 10$ | $9.59 \times 10^{-4}$ | $5.90 \times 10^{-3}$ | $1.46 \times 10^{-2}$ | $2.39 \times 10^{-2}$ | $1.66 \times 10^{-3}$ |

Equations (19) and (21) are second-order compact expressions that present a good accuracy at the interval $R \in [0,1]$. Figure 2(a) shows, in a graphical manner, that the obtained solutions with ODP and HPM present a better accuracy than the obtained for (28) and (30), (31), for values of $\alpha = 2.5, K = 2, H = 4$. Additionally, Figures 2(b) and 2(c) show the same comparisons with the values for $\alpha = 3.5, K = 2, H = 4$ and $\alpha = 4.5, K = 3, H = 4$.

As for the previous cases, in Figure 2(d), for higher values of $\alpha = 8.5, K = 5, H = 10$, the obtained solution employing ODP has better accuracy than other solutions. For this case, (32) and (33) describe the algebraic solutions for this value for HPM and ODP, respectively. For the values mentioned before, Figure 2(d) presents the graphics for the solutions employing ODP and HPM compared to the solutions proposed by MTSM and $\varphi_3$.

$$C(R) = 0.7624971978 + 0.1979190019 R^2, \quad (32)$$

$$C(R) = 0.7581194350 + 0.1379309752 R^2 \over 1 - 0.06409108102 R^2, \quad (33)$$

In Table 1 is presented and compared the absolute relative error (A.R.E.) for each approximation according to the values of $\alpha, K, H$, for HPM and ODP solutions, including (28), (29), and (30).

The absolute relative error is defined as the absolute difference of the areas under the curve for the numerical and approximate solutions, divided by the area of the exact solution. The employed integrating method was the Rule of Simpson 1/3.

In all of the presented cases, the A.R.E. obtained with ODP is much lower compared to the Adomian and MTSM methods reported in [12,13] and matches with all the graphics presented in this work.

In the same manner, for the cases ($\alpha = 4.5, K = 3, H = 5$) and ($\alpha = 8.5, K = 5, H = 10$) for HPM it is observed that the relative absolute error is lower than the ones presented in Adomian and MTSM. Besides, the solution obtained by HPM has the advantage of being simpler because it only consists of two terms, compared against (28), (29), and (30) that possess more terms.

Thus, for Optimized Direct Padé and HPM, the proposed results in this work coincide with [12,13,39,48,49]; that is, the oxygen concentration is higher at the substrate and diminishes as it approaches the center of spherical cell, which is due to the oxygen diffusing into the cell; it reacts to give rise to the boundary condition at the center. Likewise, results are congruent with the spherical geometry proposed by Lane-Emden (2).

Finally, it is important to remark that although the precision for ODP tends to be the same as DP, it has the advantage of having a simpler algebraic procedure, avoiding the algebraic complexity of substituting a rational expression that generates more rational expressions with higher powers in the denominator in its case, more numerator terms, making the formulation of the system of equations difficult

Table 1: Absolute relative error comparison.

| Case          | ODP          | HPM          | MTSM [13] | $\varphi_2(R)$ [12] | $\varphi_3(R)$ [12] |
|---------------|--------------|--------------|-----------|---------------------|---------------------|
| $\alpha = 2.5, K = 2, H = 4$ | $3.58 \times 10^{-3}$ | $3.21 \times 10^{-3}$ | $6.99 \times 10^{-3}$ | $1.11 \times 10^{-2}$ | $4.54 \times 10^{-3}$ |
| $\alpha = 3.5, K = 2, H = 4$ | $1.00 \times 10^{-3}$ | $6.26 \times 10^{-3}$ | $1.37 \times 10^{-2}$ | $5.16 \times 10^{-3}$ | $5.06 \times 10^{-3}$ |
| $\alpha = 4.5, K = 3, H = 4$ | $8.37 \times 10^{-4}$ | $6.18 \times 10^{-3}$ | $1.34 \times 10^{-2}$ | $4.04 \times 10^{-2}$ | $2.90 \times 10^{-1}$ |
| $\alpha = 8.5, K = 5, H = 10$ | $9.59 \times 10^{-4}$ | $5.90 \times 10^{-3}$ | $1.46 \times 10^{-2}$ | $2.39 \times 10^{-2}$ | $1.66 \times 10^{-3}$ |
for finding $a_i$ and $b_i$ from the rational expression of the solution.

9. Conclusions

In this article were proposed two compact solutions for Michaelis-Menten oxygen uptake kinetics equation employing HPM and ODP. ODP presents a quotient of two polynomials, one of them of second order divided by a first-order one. On the other hand, HPM presents only two algebraic terms, the independent term and the second degree term. Likewise, the solutions found in this work were compared to those reported by other authors, showing that ODP has a better accuracy. HPM solution shows a good approximation in comparison to (29) obtained by Adomian; despite the fact that it only has two terms, it concluded that the solution is in good agreement. Furthermore, in this paper, we propose to replace the rational expression (8) by its Taylor series in order to reduce the amount of algebraic steps during the application of DP. The graphs presented in this work show

Figure 2: Comparison of HPM (19) and ODP (21) versus numerical solution, (28), (29), and (30).
that the solution given by ODP and HPM tends to converge as we approach $R = 1$ unlike Adomian $\varphi_3(R)$ which tends to diverge by increasing the absolute error. For the case of $\alpha = 8.5, K = 5,$ and $H = 10$ with ODP, the A.R.E. was $9.59 \times 10^{-4}$. Also the HPM and ODP methods were compared against the numerical one obtaining very good approximations. In addition, ODP requires fewer iterations to obtain a more compact and accurate expression than the solution proposed by the Adomian method (29).

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

All the authors express their appreciation to the late Professor Jose Antonio Agustin Perez-Sesma whose contribution to this work was of great significance. Additionally, the authors wish to thank M.TN. César Colocia García and Eng. Francisco Martínez Barrios for their support. The first author is currently a postdoctoral research fellow at Institute for Astrophysics, Optics and Electronics and expresses his gratitude for the support received. He also acknowledges the financial support from the Secretary of Public Education of México (SEP) through grant 2703 E476300.0275652.

References

[1] K. Mathews Christopher, K. E. Van Holde, and G. Ahern Kevin, *Bioquímica*, Pearson Addison Wesley, 2002.

[2] J. Y. Tang, “On the relationships between the Michaelis-Menten kinetics, reverse Michaelis-Menten kinetics, equilibrium chemistry approximation kinetics, and quadratic kinetics,” *Geoscientific Model Development*, vol. 8, no. 12, pp. 3823–3835, 2015.

[3] V. Galvanauskas, V. Grincas, and R. Simutis, “Modeling of Oxygen Supply Profiles in Stirred-Tank Aggregated Stem Cells Cultivation Process,” *International Journal of Biological, Biomolecular, Agricultural, Food and Biotechnological Engineering*, vol. 10, no. 9, pp. 584–587, 2016.

[4] S. I. Rubinow, *Introduction to Mathematical Biology*, John Wiley and Sons, 1975.

[5] R. Byron Bird, E. S. Warren, and N. L. Edwin, “Fenómenos de transporte: un estudio sistemático de los fundamentos del transporte de materia, energía y cantidad de movimiento,” Limusa Noriega Editores, 2006.

[6] L.-M. Warren, C.-S. Julian, and P. Harriot, *Operaciones Unitarias En Ingeniería*, McGraw Hill Interamericana, 2007.

[7] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, 1995.

[8] M. S. Simon and K. K. Ng, *Physics of Semiconductor Devices*, John Wiley & Sons, 2006.

[9] L. Frenzel, *Principles of Electronic Communication Systems*, McGraw-Hill, Inc, 2007.

[10] J. D’azzo and C. H. Houpis, *Feedback Control Systems: Analysis and Synthesis*, McGraw-Hill, 1960.

[11] C. Nicholson, “Interaction between diffusion and Michaelis-Menten uptake of dopamine after iontophoresis in striatum,” *Biophysical Journal*, vol. 68, no. 5, pp. 1699–1715, 1995.

[12] R. Rach, A.-M. Wazwaz, and J.-S. Duan, “A reliable analysis of oxygen diffusion in a spherical cell with nonlinear oxygen uptake kinetics,” *International Journal of Biomathematics*, vol. 7, no. 2, p. 12, 2014.

[13] H. Vazquez-Leal, M. Sandoval-Hernandez, R. Castaneda-Sheissa, U. Filobello-Nino, and A. Sarmiento-Reyes, “Modified Taylor solution of equation of oxygen diffusion in a spherical cell with Michaelis-Menten uptake kinetics,” *International Journal of Applied Mathematical Research*, vol. 4, no. 2, pp. 253–258, 2015.

[14] M. J. Ablowitz, X.-D. Luo, and Z. H. Musslimani, “Inverse scattering transform for the nonlocal nonlinear Schrödinger equation with nonzero boundary conditions,” *Journal of Mathematical Physics*, vol. 59, no. 1, 2018.

[15] R. Beals and R. R. Coifman, “Scattering and inverse scattering for first order systems,” *Communications on Pure and Applied Mathematics*, vol. 37, no. 1, pp. 39–90, 1984.

[16] G. Chaochao, H. Anning, and Z. Zixiang, *Darboux Transformations in Integrable Systems: Theory and their Applications to Geometry*, vol. 26, Springer Science & Business Media, 2006.

[17] P. Adamopoulou, A. Doikou, and G. Papamikos, “Darboux–Bäcklund transformations, dressing & impurities in multi-component NLS,” *Nuclear Physics B*, vol. 918, pp. 91–114, 2017.

[18] P. P. Goldstein, “Hints on the Hirota bilinear method,” *Acta Physica Polonica A*, vol. 112, no. 6, pp. 1171–1184, 2007.

[19] Y. Matsuno, *Bilinear Transformation Method*, Elsevier, 1984.

[20] W. Malfliet, “The tanh method: a tool for solving certain classes of nonlinear evolution and wave equations,” *Journal of Computational and Applied Mathematics*, vol. 164-165, pp. 529–541, 2004.

[21] N. H. Abdel-All, M. A. Abdel-Razek, and A. K. Seddeek, “Expanding the Tanh-Function Method for Solving Nonlinear Equations,” *Applied Mathematics*, vol. 2, no. 9, pp. 1096–1104, 2011.

[22] W. Malfliet and W. Hereman, “The tanh method: I. Exact solutions of nonlinear evolution and wave equations,” *Physica Scripta*, vol. 54, no. 6, pp. 563–568, 1996.

[23] A. V. Kuznetsov and A. A. Avramenko, “The method of separation of variables for solving equations describing molecular-motor-assisted transport of intracellular particles in a dendrite or axon,” *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol. 464, no. 2099, pp. 2867–2886, 2008.

[24] X.-M. Qian, S. Y. Lou, and X.-B. Hu, “Variable separation approach for a differential-difference system: Special Toda equation,” *Journal of Physics A: Mathematical and General*, vol. 37, no. 6, pp. 2401–2411, 2004.

[25] H.-C. Hu, S.-Y. Lou, and Q.-P. Liu, “Darboux Transformation and Variable Separation Approach: The Nizhnik-Novikov-Veselov Equation,” *Chinese Physics Letters*, vol. 20, no. 9, pp. 1413–1415, 2003.

[26] P. E. Hydon, “Introduction to symmetry methods in the solution of differential equations that occur in chemistry and chemical biology,” *International Journal of Quantum Chemistry*, vol. 106, no. 1, pp. 266–277, 2006.
[27] P. E. Hydon and P. E. Hydon, *Symmetry Methods for Differential Equations: A Beginner’s Guide*, vol. 22, Cambridge University Press, 2000.

[28] A. M. Wazwaz, “A sine-cosine method for handling nonlinear wave equations,” *Mathematical and Computer Modelling*, vol. 40, no. 5-6, pp. 499–508, 2004.

[29] A. J. M. Jawad, “The sine-cosine function method for the exact solutions of nonlinear partial differential equations,” *IOSR Journal of Mathematics*, vol. 13, no. 1, pp. 186–191, 2012.

[30] A.-M. Wazwaz, “The tanh method and the sine-cosine method for solving the KP-MEW equation,” *International Journal of Computer Mathematics*, vol. 82, no. 2, pp. 235–246, 2005.

[31] A.-M. Wazwaz, “A new algorithm for calculating Adomian polynomials for nonlinear operators,” *Applied Mathematics and Computation*, vol. 111, no. 1, pp. 33–51, 2000.

[32] G. Adomian, “A review of the decomposition method in applied mathematics,” *Journal of Mathematical Analysis and Applications*, vol. 135, no. 2, pp. 501–544, 1988.

[33] M. Al-Mazmumy, A. Al-Mutairi, and K. Al-Zahrani, “An Efficient Decomposition Method for Solving Bratu’s Boundary Value Problem,” *American Journal of Computational Mathematics*, vol. 07, no. 01, pp. 84–93, 2017.

[34] M. Rabbani and R. Arab, “Extension of some theorems to find solution of nonlinear integral equation and homotopy perturbation method to solve it,” *Mathematical Sciences*, vol. 11, no. 2, pp. 87–94, 2017.

[35] J. W. Schofield, E. A. Gaffney, R. A. Gatenby, and P. K. Maini, “Tumour angiogenesis: The gap between theory and experiments,” *Journal of Theoretical Biology*, vol. 274, no. 1, pp. 97–102, 2011.
