Non-linear mode coupling and the growth of perturbations in $\Lambda$CDM

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Abstract: Cosmic structures at small non-linear scales $k > L \sim 0.2h$ Mpc$^{-1}$ have an impact on the longer (quasi-)linear wavelengths with $k < L$ via non-linear UV-IR mode coupling. We evaluate this effect for a $\Lambda$CDM universe applying the effective fluid method of Baumann, Nicolis, Senatore and Zaldarriaga [1]. For $k < L$ the $\Lambda$CDM growth function for the density contrast is found to receive a scale dependent correction and an effective anisotropic stress sources a shift between the two gravitational potentials, setting $\phi - \psi \neq 0$. Since such a situation is generically considered as a signature of modified gravity and/or dark energy, these effects should be taken into account before any conclusions on the dark sector are drawn from the interpretation of future observations.
1. Introduction

The standard ΛCDM model of the universe is characterized by a specific evolution of
the density contrast $\delta \rho / \rho \equiv \delta \propto a D(a)$ i.e by a specific growth function $D(a)$ for
the matter perturbations, and by the equality of the two gravitational potentials $\phi$ and $\psi$ as
is implied by the lack of anisotropic stress in the CDM. In contrast, evolving dark energy,
or a modification of Einstein gravity, would generically give rise to an anisotropic stress
as well as to modifications in the Poisson equation [2], which would manifest themselves
in a different functional form for $D(a)$. Probing the growth function for different values of
scale factors $a$ would thus be invaluable to determine the dark energy related cosmological
parameters. It is for this reason that the three-dimensional mapping of the structure of
the universe by future weak lensing surveys such as EUCLID [3] is very much expected to
significantly constrain dark energy and modified gravity models.

At a practical level the determination of the growth function proceeds from the as-
sumption that the universe can be considered as a collection of ideal fluids with some small
perturbations. In this setting, an accurate (scale independent) parametrization for $D$ reads
[4]

$$D(a) = \exp \left[ \int_0^a d\ln \bar{a} (\Omega_m(\bar{a})^{\gamma} - 1) \right]$$ (1.1)

where $\Omega_m(a) = H_0^2 \Omega_m a^{-3} / H(a)^2$ and $\gamma$ ($= \frac{6}{11}$ for ΛCDM) depends e.g. on the equation
of state parameter of the dark energy.\(^1\) However, structure formation itself feeds back on
the stress-energy and hence on the effective behavior of the cosmological fluids. As has
been pointed out in [1], this effect can be studied by integrating out the short-wavelength
perturbations, obtaining an effective theory for the long-wavelength universe that has an
equation of state different from the homogeneous background and, moreover, is no longer
strictly ideal but is also characterized by a viscosity parameter. Very roughly, one can
think that this effect is caused by the motion of small-scale lumps of matter and the tidal
effects of longer wavelength perturbations on this motion. However, one should also note
that the scales that have virialized can actually be shown to decouple completely from the
large-scale dynamics at all orders in the post-Newtonian expansion, apart of course from
contributing a correction to the energy density [1]; in effect, for a distant observer virialized
systems behave as if they were point particles.

In the present paper we follow this approach and study the effective cosmological fluid
of the long-wavelength perturbations in a pure ΛCDM universe. We point out that the
naive expectations for e.g. the growth function in the linear regime are violated at some
level due to the effective pressure and anisotropic stress generated from the non-linear mode
coupling between short and long wavelengths. This effect is generic and scale-dependent
and should therefore be taken into account before any conclusions about the properties of
dark energy are drawn from the interpretation of observations. On the other hand, it could
also be used as a consistency check of the concordance model.

\(^1\)It is interesting to note that scale dependent growth functions are physically more realistic for dark
energy models [5].
We proceed with our investigation as follows: in the next section we briefly review the ingredients of the effective fluid approach to the non-linear coupling of long to short scales [1] that we need to derive our results. Using the methods described in section 2, we compute in section 3 the evolution of perturbations in this effective fluid and the corresponding corrections to the linear ΛCDM growth function, as well as the so-called gravitational slip: the divergence of the two gravitational potentials $\phi - \psi$. In the process we also provide an exact analytic expression for the linear ΛCDM growth function which, to our knowledge, has not appeared in the literature previously. We summarize and conclude in section 4.

2. The Effective fluid

As matter perturbations in the universe grow under the influence of gravity there finally comes a point for a given scale where the density contrasts become larger than unity. At this point naive perturbation theory breaks down and other techniques need to be used in order to follow the further evolution of perturbations, eventually having to resort to numerical simulations. In our universe scales with $k > k_{nl} \sim 0.2h$ Mpc$^{-1}$ have entered the non-linear regime.

Apart from the difficulty of following the evolution of non-linear perturbations, another issue is the inevitable coupling between different scales which is absent in the linear regime. In particular long and short scales influence each other and it is expected that the formation of non-linear perturbations might have an impact on larger, linear scales through UV-IR coupling. It has even been suggested that even the observed acceleration might be due to such non-linear effects, and this possibility has recently spurred a lively debate in the literature [6]. Regardless of whether such an explanation for the observed acceleration is feasible, the UV-IR coupling is always present and should be quantified, especially given the accuracy of forthcoming observations which promise to map the universe and its history with unprecedented accuracy.

The authors of [1] address this non-linear mode coupling in a way that is simple and physically intuitive. By smoothing out short scale non-linearities they are led to an effective approach for longer wavelengths where the CDM fluid on scales $k < k_{nl}$ is replaced by an effective fluid which, unlike the underlying CDM matter content, possesses anisotropic stress and pressure. In more detail, the basis of this approach is an effective energy momentum tensor, including the effects of gravity, which is smoothed out on a scale $L \leq k_{nl}$. To second order in the derivatives of the gravitational potential and peculiar velocities, this procedure adds to the rhs of the Einstein equations the terms

\[ \tau_{00}^L = -[\rho v_s^i v_s^i]_L - \frac{[\partial_i \phi_s \partial_i \phi_s]_L - 4[\phi_s \partial^2 \phi_s]_L}{8\pi G a^2} + O \left( \frac{(\partial u_l)^2}{L^2}, \frac{(\partial \phi_l)^2}{L^2} \right) \]  
\[ \tau_{ij}^L = [\rho v_s^i v_s^j]_L - \frac{[\partial_i \phi_s \partial_j \phi_s]_L - 2[\partial_i \phi_s \partial_j \phi_s]_L}{8\pi G a^2} + O \left( \frac{(\partial u_l)^2}{L^2}, \frac{(\partial \phi_l)^2}{L^2} \right). \]  

(2.1)  
(2.2)

The subscripts $l$ and $s$ refer to “long” modes with $k < L$ and “short” modes with $k > L$.
respectively. By the notation \([X]_L\) we mean

\[ [X]_L(x) \equiv \int d^3x' W_L(|x - x'|)X(x'), \tag{2.3} \]

where \(W_L(|x - x'|)\) is a window function that eliminates modes with \(k > L\). Note that since we are only considering terms linear in metric perturbations (but not their spatial derivatives) and quadratic in velocities, the expressions in \(\tau_{\mu\nu}\) coincide to this order with proper volume averages. For \(k \ll L\) the third terms on the rhs of (2.1) and (2.2) are suppressed. Note also that the expressions for (2.1) and (2.2) contain short wavelength modes with \(k > L\) but the Fourier modes of \([\tau^0_0]_L\) and \([\tau^i_j]_L\) themselves are only defined for \(k < L\). For the discussion that follows it is the spatial part \([\tau^i_j]_L\) that will be most important.

According to [1], the effective stress tensor can be expressed as

\[ [\tau^i_j]_L(x) = \langle [\tau^i_j]_L \rangle + \Delta \tau^i_j(x) + \alpha^i_j(x), \tag{2.4} \]

with the first and second terms admitting an effective description to lowest order in \((k/k_{NL})^2\) as the stress tensor of an imperfect fluid with pressure and anisotropic stress

\[ \langle [\tau^i_j]_L \rangle + \Delta \tau^i_j = (P + \Delta P) \delta^i_j + \Sigma^i_j, \tag{2.5} \]

with \(\Sigma^i_i = 0\). In particular, the pressure terms can be written as

\[ P \equiv \frac{1}{3} \langle [\tau^i_i]_L \rangle = w \rho, \tag{2.6} \]

\[ \Delta P = c^2_s \rho \delta \tag{2.7} \]

where \(w\) and \(c^2_s\) are the equation of state parameter and the sound speed squared, while the scalar anisotropic stress is given as

\[ \Sigma^i_j = \eta \left( \frac{k_i k_j}{k^2} - \frac{1}{3} \delta^i_j \right) k^2 v, \tag{2.8} \]

where \(\eta > 0\) is the coefficient of shear viscosity. Note that

\[ \langle [\tau^i_j]_L \rangle = 0, \tag{2.9} \]

where \(\hat{\tau}^i_j = \tau^i_j - \delta^i_j/3 \tau^i_i\), so there is no zero mode for the anisotropic stress. It is important to stress at this point that although the universe only contains CDM, the effective description on long wavelengths involves both pressure and shear viscosity. It is customary to define the anisotropy scalar \(\sigma\) as

\[ \sigma \equiv -\frac{1}{\rho} \frac{k_i k_j}{k^2} \Sigma^i_j = -\frac{2}{3} \eta k^2 v \equiv c^2_{vis} \frac{-k^2 v}{\mathcal{H}}, \tag{2.10} \]

where we have defined the dimensionless parameter \(c^2_{vis} \equiv 2\eta \mathcal{H}/3\rho\).

The possibility of expressing the effective stress-energy tensor in terms of long wavelength perturbations such as in eqs (2.7) and (2.8) is a manifestation of non-linear mode
coupling relating the short wavelength modes in the definition of $[\tau^i_j]_L$ to modes with $k < L$. The effective fluid is then described by two parameters $c_s^2$ and $c_{vis}^2$ which can be determined by using the definition (2.2) via

$$c_s^2 = \frac{1}{3\rho} \frac{\langle [\tau^i_j]_L \delta_i \rangle}{\langle \delta_i \delta_i \rangle}$$  \hspace{1cm} (2.11)$$

$$c_{vis}^2 = -\frac{H k_i k_j}{\rho} \frac{\langle [\hat{\tau}^i_j]_L \nabla \cdot u_i \rangle}{\langle \nabla \cdot u_i \nabla \cdot u_i \rangle}.$$  \hspace{1cm} (2.12)$$

The tidal effects of long wavelength gravitational perturbations on shorter wavelength modes produce correlations of the long wavelength $\delta$ and $\nu$ to the short wavelength modes in $[\tau^i_j]_L$, making $c_s^2$ and $c_{vis}^2$ different from zero.

Finally, there will also be a stochastic part, given by the $\alpha^i_j$ term. It encodes deviations from the average value that are uncorrelated to long wavelength variables and acts as an external source. The variance of these terms can be obtained via

$$\langle \alpha^i_j \alpha^k_l \rangle = \langle [\tau^i_j]_L [\tau^k_l]_L \rangle - \langle [\tau^i_j]_L \rangle \langle [\tau^k_l]_L \rangle - \langle \Delta \tau^i_j \Delta \tau^k_l \rangle.$$  \hspace{1cm} (2.13)$$

Summarizing, we see that the UV-IR coupling can be described by the use of an effective theory where the short scale fluctuations act as an effective *imperfect* fluid on longer wavelengths. The Einstein linearized equations with $\Delta \tau^i_j + \alpha^i_j$ added to the rhs can then be used to follow the linear response of long wavelength perturbations to short wavelength non-linearities. The effective fluid is characterized by two parameters: $c_s^2$ and $c_{vis}^2$. The correlators on the rhs of (2.11) and (2.12) can be determined from small-scale N-body simulations using the definitions (2.1) and (2.2) for $[\tau^0_0]_L$ and $[\tau^i_j]_L$. Once these parameters are determined, no further reference to the short scale dynamics is needed. Alternatively, the parameters of the effective fluid could be considered as free parameters to be fitted from observations - see section 3.

### 3. Perturbations of the effective fluid in $\Lambda$CDM

#### 3.1 Density contrast and gravitational slip

We can now proceed to evaluate the effect of short wavelength fluctuations on longer wavelengths using the approach outlined above. We start with the Einstein equations relating the gravitational potentials with the components of the effective energy momentum tensor. The 00 and 0$i$ equations give

$$k^2 \phi + 3\mathcal{H} \left( \dot{\phi} + \mathcal{H} \psi \right) = -\frac{3}{2} H_0^2 \frac{\Omega_m}{a} \delta$$  \hspace{1cm} (3.1)$$

$$\dot{\phi} + \mathcal{H} \psi = -\frac{3}{2} H_0^2 \frac{\Omega_m}{a} (1 + w) \nu ,$$  \hspace{1cm} (3.2)$$

while the traceless part of the $i j$ equation gives

$$k^2 (\phi - \psi) = \frac{9}{2} H_0^2 \frac{\Omega_m}{a} (1 + w) \sigma .$$  \hspace{1cm} (3.3)$$
We can further use the energy-momentum conservation law \( \nabla_\mu T^{\mu \nu} = 0 \) which results in
\[
\dot{\delta} = (1 + w) \left( k^2 v + 3 \dot{\phi} \right) - 3 H \left( \frac{\delta P}{\rho_m} - w \delta \right),
\] (3.4)
\[
\dot{v} = - \left( 1 - 3w + \frac{\ddot{w}}{1 + w} \right) H v - \frac{1}{1 + w} \frac{\delta P}{\rho_m} + \sigma - \psi
\] (3.5)
with \( v \) the peculiar velocity potential: \( u_i = ik_i v \).

According to the discussion in the previous section we can express the pressure perturbation and the anisotropic stress of the effective fluid in terms of long wavelength perturbations \( \delta \) and \( v \) and a stochastic part uncorrelated with them
\[
\delta P = c_s^2 \rho_m \delta + c_s^2 \rho_m \alpha_1
\] (3.6)
\[
\sigma = c_{vis}^2 \frac{-k^2 v}{H} + c_{vis}^2 \alpha_2 \simeq c_{vis}^2 \delta + c_{vis}^2 \alpha_2,
\] (3.7)
where we have used the linear perturbation theory relation \( \delta \simeq -k^2 v/H \) for the (long wavelength) velocity. The quantities \( \alpha_1 \) and \( \alpha_2 \) encode the (dimensionless) stochastic fluctuations of \( \delta P \) and \( \sigma \). We stress that we ignore here the non-linearities of the long wavelength perturbations\(^2\).

We can now obtain an equation for the evolution of the long wavelength density contrast \( \delta \). To simplify the result we make the following assumptions:

1. We are interested in following perturbations on scales smaller than the horizon so that \( k^2 > H^2 \).

2. We will also assume that \( c_s^2 \) and \( c_{vis}^2 \) are small enough such that \( c^2 \frac{k^2}{H^2} < 1 \). This requires \( c^2 < 10^{-5} \left( \frac{k_{nl}}{L} \right)^2 \) where \( L \) is the smoothing scale we use to define the long wavelength sector and \( k_{nl} \sim 0.2 \) h Mpc\(^{-1} \) is the scale of non-linearity.

3. The time scales of evolution on scales \( k > L \) is assumed not to be much faster than cosmological time scales, ie \( \frac{d}{d\eta} (c\alpha) \sim \mathcal{H} \alpha \) and that

4. the timescale for evolution for \( w \) is similar \( \frac{dw}{d\eta} \sim \mathcal{H} w \).

Assumptions 2), 3) and 4) can of course be checked once the parameters have been calculated from first principles.

Given the above we obtain for the density perturbation to leading order in \( c^2 \)
\[
\ddot{\delta} + \mathcal{H} \dot{\delta} - \frac{3}{2} H_0^2 \frac{\Omega_m}{a} \delta = -k^2 c_s^2 \delta - k^2 \left( c_s^2 \alpha_1 - c_{vis}^2 \alpha_2 \right),
\] (3.8)
which reduces to the standard equation when short-scale non-linearities are ignored ie \( c_s^2 \rightarrow 0 \) and \( c_{vis}^2 \rightarrow 0 \). It is useful to express the above in terms of the scale factor \( a \). Using \( \frac{d}{d\eta} = a H \frac{da}{d\eta} \) we have for \( \delta(a) \)
\[
\frac{d^2}{da^2} \delta + ( \frac{d}{da} \ln \mathcal{H} + \frac{2}{a} ) \frac{d}{da} \delta - \frac{3}{2} \left( \frac{H_0}{H} \right)^2 \frac{\Omega_m}{a^3} \delta = - \frac{1}{a^2} \frac{k^2}{\mathcal{H}^2} \left( c_s^2 \delta + c_s^2 \alpha_1 - c_{vis}^2 \alpha_2 \right).
\] (3.9)

\(^2\)We could have included terms quadratic in the gradients of the potentials and the velocities, studying non-linearities on long wavelengths via second order perturbation theory.
We denote the solution in the absence of mode coupling (rhs is taken to be zero) by \( \tilde{\delta} \)

\[
\frac{d^2}{da^2} \tilde{\delta} + \left( \frac{d}{da} \ln \mathcal{H} + \frac{2}{a} \right) \frac{d}{da} \tilde{\delta} - \frac{3}{2} \left( \frac{H_0}{\mathcal{H}} \right)^2 \frac{\Omega_m}{a^3} \tilde{\delta} = 0 . \tag{3.10}
\]

Then, to leading order in \( c^2 \), the solution of the long wavelength density contrast in the presence of small scale non-linearities is

\[
\delta(a) = \tilde{\delta}(a) - \int_{a_{\text{in}}}^{\infty} dx \frac{k^2}{x^2 \mathcal{H}(x)^2} \left( c_s^2 \tilde{\delta}(x) + c_s^2 \alpha_1 - c_s^2 \alpha_2 \right) , \tag{3.11}
\]

where \( G(a, x) \) is the appropriate Green function satisfying

\[
\frac{d^2}{da^2} G(a, x) + \left( \frac{d}{da} \ln \mathcal{H} + \frac{2}{a} \right) \frac{d}{da} G(a, x) - \frac{3}{2} \left( \frac{H_0}{\mathcal{H}} \right)^2 \frac{\Omega_m}{a^3} G(a, x) = \delta_D(a - x) . \tag{3.12}
\]

The solution to (3.10) that grows like \( \tilde{\delta} \propto a \) at early times (during matter domination) reads

\[
\tilde{\delta}(a) = \frac{\tilde{\delta}_{\text{in}}}{a_{\text{in}}} \frac{5 H_0^2 \Omega_m}{2} \frac{\mathcal{H}(a)}{a} \int_{0}^{a} \frac{dx}{\mathcal{H}(x)^3} , \tag{3.13}
\]

where \( a_{\text{in}} \) is the scale factor at which we start the computation and \( \tilde{\delta}_{\text{in}} \) is the density contrast at that time. In a \( \Lambda \)CDM universe we have

\[
\mathcal{H} = H_0 a \left( \frac{\Omega_m}{a^3} + \Omega_\Lambda \right)^{1/2} , \tag{3.14}
\]

and the integral can be performed to give

\[
\int_{0}^{a} \frac{du}{u^3 \left( \frac{\Omega_m}{a^3} + \Omega_\Lambda \right)^{3/2}} = \frac{2}{5 \Omega_m^{3/2}} a^{5/2} F(a) , \tag{3.15}
\]

where we have defined \( F \)

\[
F(a) \equiv 2 F_1 \left( \frac{3}{2} ; \frac{5}{6} ; \frac{11}{6} ; -\frac{\Omega_\Lambda}{\Omega_m} a^3 \right) \tag{3.16}
\]

with \( 2 F_1 \) the hypergeometric function. Thus,

\[
\tilde{\delta}(a) = \frac{\tilde{\delta}_{\text{in}}}{a_{\text{in}}} \left( 1 + \frac{\Omega_\Lambda}{\Omega_m} a^3 \right)^{1/2} a F(a) , \tag{3.17}
\]

and the \( \Lambda \)CDM growth function for the density contrast is

\[
D(a) = \left( 1 + \frac{\Omega_\Lambda}{\Omega_m} a^3 \right)^{1/2} F(a) . \tag{3.18}
\]

The Green function for the problem (satisfying homogeneous initial conditions at \( a = 0 \)) is found to be

\[
G(a, x) = \Theta(a - x) \frac{2}{5 H_0^3 \Omega_m^{3/2}} x^2 \mathcal{H}(x) \frac{\mathcal{H}(a)}{a} \left( a^{5/2} F(a) - x^{5/2} F(x) \right) . \tag{3.19}
\]
Using the above, we can now obtain for the solution for the density contrast from (3.11)

\[ \delta(a) = \left( 1 - \frac{k^2 c_s^2}{H_0^2} E_1(a) \right) \tilde{\delta}(a) - \frac{k^2 c_s^2}{H_0^2} S(a, \alpha), \] (3.20)

where

\[ E_1(a) = \frac{2}{5 \Omega_m a} \int_0^a dx \left( 1 + \frac{\Omega_\Lambda}{\Omega_m} a^3 \right)^{1/2} x F(x) \left( 1 - \frac{x}{a} \right)^{5/2} \frac{F(x)}{F(a)} \frac{c_s^2(a)}{c_s^2} \] (3.21)

and

\[ S(a, \alpha) = \frac{2}{5 \Omega_m a^2 F(a)} \left( 1 + \frac{\Omega_\Lambda}{\Omega_m} a^3 \right)^{1/2} \int_0^a dx \left( 1 - \frac{x}{a} \right)^{5/2} \frac{F(x)}{F(a)} \frac{c_s^2(a)}{c_s^2} \alpha, \] (3.22)

with \( \alpha \equiv \alpha_1 - \frac{c_{vis}^2}{c_s^2} \alpha_2 \) and \( c_s^2 \) the speed of sound today. For the difference of the two gravitational potentials we get

\[ -k^2 (\phi - \psi) = \frac{45}{4} H_0^2 c_{vis}^2 \left( \frac{1 - \frac{3}{5} \left( 1 + \frac{\Omega_\Lambda}{\Omega_m} a^3 \right)^{1/2} F(a)}{1 + \frac{\Omega_\Lambda}{\Omega_m} a^3} \right)^{3/2} a F(a) \tilde{\delta}(a) + \frac{9}{2} H_0^2 \Omega_m c_{vis}^2 \alpha_2. \] (3.23)

### 3.2 The matter powerspectrum and weak lensing

In the previous subsection we saw how the impact of UV-IR coupling on the growth of perturbation can be described in terms of an effective quasi-linear theory on scales larger than \( L \). The main results where equations (3.20) for the density contrast and (3.23) for the gravitational slip. From (3.20) we immediately obtain

\[ \langle \delta_k \delta_p \rangle = \left( 1 - 2 \frac{k^2 c_s^2}{H_0^2} E_1(a) \right) \langle \tilde{\delta}_k \tilde{\delta}_p \rangle - \frac{k^2 c_s^2}{H_0^2} \frac{p^2 c_s^2}{H_0^2} \langle \tilde{S}_k \tilde{S}_p \rangle, \] (3.24)
were we have used \( -k^2 \phi \simeq \frac{3}{2a} \Omega_m H_0^2 \delta \). To simplify our formulae we will make the assumption that

\[
\langle S_k S_p \rangle = (2\pi)^3 \delta(k + p) \beta(a, k) P_3(k),
\]

(3.25)

where \( P_3(k) \) is the density contrast power-spectrum: \( \langle \delta_k \delta_p \rangle = (2\pi)^3 \delta(k + p) P_3(k) \). In particular this entails that different modes of the stochastic fields are uncorrelated for different momentum magnitudes, at least to this order in \( c^2 \). Given this assumption we can write

\[
\langle \delta_k \delta_p \rangle = \left( 1 - \frac{2}{H_0^2} E_1(a) - \frac{k^4 c_s^4}{H_0^4} \beta(a, k) \right) \langle \delta_k \delta_p \rangle,
\]

(3.26)

where

\[
\beta(a, k) \equiv \frac{\langle S_k S_{-k} \rangle}{\langle \delta_k \delta_{-k} \rangle}
\]

(3.27)

denotes the ratio of the stochastic to the density power-spectra. Using

\[
c_s^4 \rho_m (\alpha_k) \simeq \frac{1}{9} \left( \langle \tau^2 \rangle_L - \langle (\tau^2) \rangle_{-k} \right)
\]

(3.28)

we obtain

\[
c_s^4 \beta \simeq E_2(a) \frac{k_{eq}^2}{k T(k)^2 \gamma_2}
\]

(3.29)

where

\[
E_2(a) = a^4 F(1)^2 \int_0^a dx \left( 1 - \left( \frac{x}{a} \right)^{5/2} \frac{F(x)}{F(a)} \right)^2 \frac{c_s^4(a)}{c_s^4},
\]

(3.30)

\[
\gamma_2 \simeq 1.4 \times 10^{-3} \int \frac{dq}{k_{eq}^2 q^2} \frac{H_0^2 P_3(q)^2}{\delta_H^2},
\]

(3.31)

and \( T(k) \) is the transfer function for \( \Lambda \)CDM (see eg. [7] eq. 6.5.12). To obtain (3.29) we used the fact that since \( k \) is in the linear regime \( P_{lin}(k) = 2 \pi^2 \delta_H^2 \frac{k}{H_0} T(k)^2 \frac{D(a)^2}{D_T^2} \) and \( \delta_H \) is the perturbation amplitude at the current horizon scale: \( \delta_H = 4.9 \times 10^{-5} \) [8].

We have now reached our main conclusion. We see that we obtain a scale dependent correction to the pure \( \Lambda \)CDM growth function

\[
D(a) \to Q(k, a)^{1/2} D(a) \equiv \left( 1 - \frac{2}{H_0^2} c_s^2 E_1(a) - \frac{k^3}{T(k)^2 H_0^2} \frac{k_{eq}}{a} \gamma_2 E_2(a) \right)^{1/2} D(a).
\]

(3.32)

This may have interesting ramifications for weak lensing observations, which probe the potential \( k_i k_j p m ((\phi + \psi)_k (\phi + \psi)_p) \) [9]. Because of the UV-IR coupling, this will be also modified from its pure \( \Lambda \)CDM form. Defining \( \psi = \eta \phi + \lambda \) we obtain from (3.23)

\[
\eta = -\frac{15}{2} c_{vis}^2 \frac{1 - \frac{3}{5} \left( \frac{\Omega_k}{\Omega_m} a^3 \right)^{1/2} F(a)}{\left( 1 + \frac{3}{5} \frac{\Omega_k}{\Omega_m} a^3 \right)^{3/2} F(a)}
\]

(3.33)

and

\[
\lambda = \frac{9}{2} \frac{\Omega_m H_0^2}{a} c_{vis}^2 \alpha_2(k).
\]

(3.34)
The weak lensing potential can then be calculated to be

\[ k_i k_j p_i p_m \langle (\phi + \psi)_k (\phi + \psi)_p \rangle = \left[ 4 \left( 1 + \frac{n}{2} \right) Q(k, a) + c_{\text{vis}}^4 \beta_2(k) \right] \]

\[ \times \left( \frac{3}{2} H_0^2 \Omega_m a \right) \frac{k_i k_j p_i p_m}{k^2 p^2} \langle \tilde{\delta}_k \tilde{\delta}_p \rangle , \]

(3.35)

with \( \beta_2(k) \equiv \langle \alpha_{2k} \alpha_{2-k} \rangle / \langle \tilde{\delta}_k \tilde{\delta}_{-k} \rangle \sim \gamma_2(k) \). We can ignore terms that are not enhanced by factors of \( k \), obtaining

\[ k_i k_j p_i p_m \langle (\phi + \psi)_k (\phi + \psi)_p \rangle \approx 4 Q(k, a) \left( \frac{3}{2} H_0^2 \Omega_m a \right) \frac{k_i k_j p_i p_m}{k^2 p^2} \langle \tilde{\delta}_k \tilde{\delta}_p \rangle . \]

(3.36)

In our case the weak lensing potential directly probes the modified growth function with contributions from the effective anisotropic stress being subdominant.

An accurate determination of the magnitude of the parameters \( c_s^2 \) and \( \gamma_2 \) controlling the correction \( Q - 1 \) is beyond the scope of this paper since it would require calculations in the non-linear regime in two distinct ways: 1) Determination of \( \gamma_2 \) requires knowledge of the non-linear powerspectrum. 2) The effective sound speed \( c_s^2 \) is a distinctly non-linear phenomenon which at lowest order depends on the three-mode coupling between one long and two short-wavelength modes - see (2.11) which is zero in linear theory. Therefore, it is more straightforward to consider that a scale-dependent correction to the \( \Lambda \text{CDM} \) growth function due to non-linear mode coupling would have the form

\[ Q(k) = \left( 1 - 2 \frac{k^2}{H_0^2} \mu_1 - \frac{k^3}{T(k)2 H_0^3} k_{\text{eq}} \mu_2 \right) , \]

(3.37)

where \( \mu_1 \) and \( \mu_2 \) are parameters to be fitted by observation.

However, before closing this section let us make the following observations. We expect that as non-linearities grow, so will the sound speed \( c_s^2 \). We can thus obtain an upper bound on the value of the functions \( E1(a) \) and \( E2(a) \) today (\( a = 1 \)) by setting \( c_s^2(a)/c_s^2 \to 1 \). This gives

\[ E1(a = 1) \leq 0.26 , \quad E2(a = 1) \leq 0.1 \]

(3.38)

Furthermore, simply extrapolating the linear theory powerspectrum into the non-linear regime to calculate \( \gamma_2 \) and using 2nd order perturbation theory [10] to evaluate \( c_s^2 \) we obtain

\[ c_s^2 \sim \frac{\delta_H^2}{2} \frac{k_{\text{eq}}^2}{H_0^4} \int d\kappa \kappa T^2(\kappa) \simeq \mathcal{O}(1) \times 4.2 \times 10^{-6} , \]

(3.39)

\[ \gamma_2 \sim \frac{\delta_H^2}{2} \int d\kappa T^4(\kappa) \sim 10^{-13} \]

(3.40)

We thus see that at the non-linear scale \( k_{\text{nl}} \sim 0.2 \text{ h Mpc}^{-1} \) linear theory extrapolations predict corrections to the growth function of the order of 10% and 1% respectively for the two correction terms in \( Q \) (3.32), with the second term growing faster than the first with \( k \).

We stress again that these estimates are based on linear and second order extrapolations.
in the highly non-linear regime. However, they could be considered indicative of a possibly measurable effect towards the end of the linear regime, accessible through forthcoming combinations of weak lensing and deep redshift surveys such as EUCLID [3].

4. Summary and Discussion

We have considered the influence of short wavelength matter fluctuations on longer wavelengths through non-linear mode coupling in ΛCDM, applying the method of [1] in this case. Integrating out the short scales gives rise to an effective long-wavelength fluid coupled to gravity that has slightly different properties as compared to the pure ΛCDM. The effective fluid is viscous and exhibits a small pressure as well as an anisotropic stress, all of which are in part correlated with the longer wavelength perturbations and in part purely stochastic.

We have discussed the nature of these corrections and derived their contributions to the growth function and the weak lensing potential of standard ΛCDM cosmology. In the process we also provided an analytic expression for the growth function of pure ΛCDM which, to our knowledge has not appeared in the literature before, equation (3.18). The contribution from the induced anisotropic stress turns out to be negligible for present observational purposes, but the modification of the growth function $Q$, which is scale-dependent, could be of importance. It contains a term that scales as $k^2$ and is due to the perturbation of the effective pressure, while a $k^3/T(k)^2$ term appears because of the stochastic nature of small scale fluctuations that are uncorrelated to long wavelength variables and effectively act as an external source in the evolution equation of the long-wavelength perturbation. The corrections are characterized by two parameters, $c_2^2$ and $\gamma_2$, which can in principle be calculated but, perhaps more realistically, can also be fitted from observations. We extrapolated perturbation theory to obtain estimates for these parameters which should more appropriately be considered as lower limits; a more precise estimation would require one to go beyond perturbation theory into the non-linear regime of structure formation.

We would like to end by noting that searches for modifications of gravity and/or dynamical dark energy focus on the modifications of the growth function compared to its ΛCDM form as well as a non-zero difference in the two gravitational potentials. However, as we have discussed here, such modifications arise at some level even in standard ΛCDM because of the non-linearity of gravity which couples different scales. In this case, the difference in the two potentials seems too small to be observationally relevant but scale dependent corrections to the growth function might be detectable. These effects are always present and should be taken into account in future surveys before any conclusions are drawn with pertaining to the dark sector. Furthermore, such effects could be used as consistency checks of the standard cosmological model, provided that a more accurate determination of the parameters of the effective fluid is achieved.

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