Long-Run Average Sustainable Harvesting Policies: Near Optimality*

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Abstract

This paper develops near-optimal sustainable harvesting strategies for the predator in a predator-prey system. The objective function is of long-run average per unit time type. To date, ecological systems under environmental noise are usually modeled as stochastic differential equations driven by a Brownian motion. Recognizing that the formulation using a Brownian motion is only an idealization, in this paper, it is assumed that the environment is subject to disturbances characterized by a jump process with rapid jump rates. Under broad conditions, it is shown that the systems under consideration can be approximated by a controlled diffusion system. Based on the limit diffusion system, control policies of the original systems are constructed. Such an approach enables us to develop sustainable harvesting policies leading to near optimality. To treat the underlying problems, one of the main difficulties is due to the long-run average objective function. This in turn, requires the handling of a number of issues related to ergodicity. New approaches are developed to obtain the tightness of the underlying processes based on the population dynamic systems.

Keywords. Sustainability; near-optimal strategy; harvesting policy; long-run-average control; ergodicity.

Subject Classification. 60H10, 92D25, 93E20.

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\section{Introduction}

This work encompasses the study of controlled predator-prey systems. The control process is devoted to harvesting activities, which is one of the central issues in bio-economics. It has been widely recognized that it may not be a good idea to consider only maximizing short-term benefits focusing purely on harvesting. Although over-harvesting in a short period may maximize the short-term economic benefits, it breaks the balance between harvesting and its ecological implications. Thus simple minded policies may lead to detrimental after effect. As a result, it is crucially important to pay attention not to render exceedingly harmful decision to the environment. This situation has been observed in some optimal harvesting models with finite-time yield or discounted yield; see, e.g., \cite{1, 2, 20, 25, 26}, among others.

In contrast, ecologists and bio-economists emphasize the importance of sustainable harvest in both biological conservation and long-term economic benefits; see \cite{3, 9, 22}. They introduce the concept of maximum sustainable yield, which is the largest yield (or catch) that can be taken from a species’ stock over an infinite horizon. Their findings indicate that it is more reasonable to maximize the yield in such a way that a species is sustainable and not in danger leading to extinction of the species. Inspired by the idea of using maximum sustainable yield, we pay special attentions to sustainability, biodiversity, biological conservation, and long-term economic benefits, and consider long-term horizon optimal strategies in this paper. In lieu of discounted profit, we examine objective functions that are long-run average per unit time type. As was alluded to, the papers \cite{1, 2, 20, 25, 26} concentrated on finite time horizon problems as well as long term objective function under discounting. However, there seems to be not much effort devoted to long-run average criteria for the harvesting problem to the best of our knowledge. Discounted objective pays more attention to the current performance, whereas it is certainly necessary to examine the performance when the future is as important. This is particularly the case when we take sustainability and long-term economic benefits into consideration. We consider a long-run-average optimal harvesting problem for a predator-prey model subject to random perturbations, in which only the predator takes harvesting action. This type of optimal harvesting problems have been studied by some authors; see for example, \cite{18, 19}. However, harvesting efforts in these papers are confined to constant-harvesting strategies only, which are usually far from optimal for a larger and more realistic class of harvesting strategies. In contrast to the discounted criteria, the long-run average criteria are much more difficult to handle. One of the main difficulties is due to the long-run average cost criteria. To treat long-run average objective, one has to handle a number of delicate issues that are related to ergodicity.

To date, ecological systems under environmental noise are usually modeled by stochastic differential equations driven by a Brownian motion. An important aspect of our work is concerned with what if the noise is not of Brownian motion type. An innovation of the current paper is the use of wideband noise. It has been widely recognized that Brownian motion is only an idealized formulation or suitable limits of systems in the real world. To be more realistic, we would better assume that the environment is subject to disturbances
characterized by a jump process with rapid jump rates. This jump process can be modeled by the so-called wideband noise. Motivated by the approach in [17], we consider a Lotka-Volterra predator-prey model with wideband noise and harvesting in this paper. Denote by $X^\varepsilon(t)$ and $Y^\varepsilon(t)$ the sizes of the prey and the predator, respectively. The system of interest is of the form

\[
\begin{align*}
\frac{dX^\varepsilon(t)}{dt} &= X^\varepsilon(t) \left[ a_1 - b_1 X^\varepsilon(t) - c_1 Y^\varepsilon(t) \right] dt + \frac{1}{\varepsilon} X^\varepsilon(t) r_1(\xi^\varepsilon(t)) dt \\
\frac{dY^\varepsilon(t)}{dt} &= Y^\varepsilon(t) \left[ a_2 - h(Y^\varepsilon(t)) u(t) - b_2 Y^\varepsilon(t) + c_2 X^\varepsilon(t) \right] dt + \frac{1}{\varepsilon} Y^\varepsilon(t) r_2(\xi^\varepsilon(t)) dt,
\end{align*}
\]

where $\varepsilon$ is a small parameter, $\xi(t)$ is an ergodic, time-homogeneous, Markov-Feller process, and $\xi^\varepsilon(t) = \xi\left(\frac{t}{\varepsilon^2}\right)$, $a_i, b_i, c_i$, $i = 1, 2$ are positive constants, and $u(t)$ represent the harvesting effort at time $t$ while $h(\cdot) : \mathbb{R}_+ \mapsto [0, 1]$ indicates the effectiveness of harvesting, which is assumed to be dependent of the population of the predator. Thus, the amount of harvested biomass in a short period of time $\Delta t$ is $Y^\varepsilon(t) h(Y^\varepsilon(t)) u(t) \Delta t$. Let $\Phi(\cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be the revenue function that provides the economic value as a function of harvested biomass. The time-average harvested value over an interval $[0, T]$ is

\[
\frac{1}{T} \int_0^T \Phi\left(h(Y^\varepsilon(t)) Y^\varepsilon(t) u(t)\right) dt.
\]

Our goal is to

\[
\text{maximize } \liminf_{T \to \infty} \frac{1}{T} \int_0^T \Phi\left(h(Y^\varepsilon(t)) Y^\varepsilon(t) u(t)\right) dt \text{ a.s.}
\]

In our set up, the harvesting strategy (the control) is only for the predator, $Y^\varepsilon(\cdot)$, which is also assumed in many papers (e.g., [5, 6, 10, 27]). The rational is that the predator has main impacts on the system, whereas the economic influence of the prey is not as significant. In addition, the prey may be too small or too passive to catch. Thus we focus on the situation when the control is in $Y^\varepsilon$ equation only.

Because of the complexity of the model, developing optimal strategies for the controlled system (1.1) and (1.2), are usually difficult. Nevertheless, one may wish to construct policies based on the limit system. A natural question arises: Can optimal or near-optimal harvesting strategies for the diffusion model be near optimal harvesting strategies for the wideband-width model when $\varepsilon$ is sufficiently small? In a finite horizon, nearly optimal controls for systems under wideband noise perturbations were developed in the work of Kushner and Ruggaldier [17]. As was noted in their paper, that the original systems subject to wideband noise perturbations are rather difficult to handle; there may be additional difficulties if the systems are non-Markovian. For infinite horizon problems, it was assumed in [17] that the slow and fast components are jointly Markovian. By working with the associated probability measures, under suitable conditions, the authors established that there is a limit system being a controlled diffusion process. Using the optimal or near-optimal controls of the limit systems, one constructs controls for the original systems and show the controls are nearly optimal. Inspired by their work, we aim to develop near-optimal policies in this paper in an infinite horizon. We focus on objective functions being long-run average per unit time type. By assuming the perturbing noise being Markovian, we develop near-optimal harvesting
strategies (near-optimal controls). In contrast to optimal controls in a finite horizon, to show that the approximation works over an infinite time interval as in our setting, the ergodicity and the existence of the invariant measure have to be established. In this paper, we first show that there exists an optimal harvesting strategy for the limit controlled diffusion. Then, we show that using near-optimal control of the limit diffusion system in the original system leads to near-optimal controls of the original system.

We note that in [17], nonlinear systems were treated so a number of assumptions were posed for such wideband noise driven systems in a general setting. In contrast, we have specific systems to work with thus we can no longer posing general conditions as in the aforementioned paper. Instead, we need to start from scratch. In fact, conditions (C1)-(C4) posed in [17, Section 7, p. 310] include the existence of $\delta$-optimal control, the existence of the associate invariant measure, tightness of the state process, and the value function under certain admissible class and the value function under stationary admissible relaxed controls being equal. Because the problems were formulated in a general setting, these conditions are abstract and are used as sufficient conditions to obtain near-optimal controls for wide-band noise systems. In contrast, for the system that we are dealing with, it is rather difficult to verify these conditions. Some sufficient conditions were also proposed in [17, Conditions (D1)-(D4)] by means of a perturbed Lyapunov function method. These conditions were given to verify conditions (C1)-(C4). Nevertheless, verifying conditions (D1)-(D4) in [17] is still a difficult task for our model. To begin with, it is difficult to find appropriate Lyapunov functions verifying conditions (D1)-(D4). To overcome the difficulty, we propose a new approach rather than finding a function $V$ satisfying the conditions (D1)-(D4) in [17]. More precisely, by analyzing the dynamics of the limit controlled diffusion when the population of the species is low, we obtain the tightness of probability measures of the controlled diffusion process. Then, using the above as a bridge, probabilistic arguments enable us to prove the tightness of probability measure of the controlled process perturbed by wideband noise. Moreover, we use stochastic analysis to carry out the desired estimates. The analysis itself is new and interest in its own right. Therefore, the problem arises in control and optimization, but our solution methods are mainly probabilistic.

The rest of the paper is organized as follows. In Section 2 we formulate the problem and identify the limit diffusion system. The main results are given in Section 3 while their proofs are provided in Section 4. Section 5 is devoted to some remarks and possible generalizations. Finally, we prove some auxiliary results in an appendix.

## 2 Formulation

We work with a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ satisfying the usual condition. Denote $\mathbb{R}_+^2 = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ and $\mathbb{R}_+^{2,0} = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$. To simplify notations, we denote $z = (x, y), \tilde{z} = (\tilde{x}, \tilde{y}), Z(t) = (X(t), Y(t)), Z^\varepsilon(t) = (X^\varepsilon(t), Y^\varepsilon(t))$. We assume that harvest efforts can be represented by a number in a finite
interval $\mathcal{M} := [0, M]$. Suppose $\xi(t)$ is a pure jump Markov-Feller process taking values in a compact metric space $\mathcal{S}$. Suppose its generator is given by

$$Q\phi(w) = q(w) \int_{\mathcal{S}} \Lambda(w, d\tilde{w})\phi(\tilde{w}) - q(w)\phi(w)$$

where $q(\cdot)$ is continuous on $\mathcal{S}$ and $\Lambda(w, \cdot)$ is a probability measure on $\mathcal{S}$ for each $w$. Suppose that $\xi(t)$ is uniformly geometric ergodic, that is

$$\|P(t, w, \cdot) - \overline{P}(\cdot)\|_{TV} \leq C_0 \exp(-\gamma_0 t), \text{ for any } t \geq 0, w \in \mathcal{S}, \quad (2.1)$$

where $\overline{P}(\cdot)$ is a probability measure in $\mathcal{S}$ and $C_0, \gamma_0$ are some positive constants. Clearly $\overline{P}(\cdot)$ is an invariant probability measure of $\{\xi(t)\}$. Let $\chi(w, \cdot) = \int_0^\infty [P(t, w, \cdot) - \overline{P}(\cdot)] dt$. It is well known that if $\phi(w)$ is a continuous function on $\mathcal{S}$ satisfying $\int_{\mathcal{S}} \phi(w)\overline{P}(dw) = 0$ then

$$\psi(w) := \int_{\mathcal{S}} \chi(w, d\tilde{w})\phi(\tilde{w}) \text{ satisfying } Q\psi(w) = -\phi(w). \quad (2.2)$$

Note that $\psi(\cdot)$ is well defined thanks to the exponential decay in $\text{(2.1)}$. Suppose that $r_i(\cdot)$ is bounded in $\mathcal{S}$, and $\int_{\mathcal{S}} r_i(w)\overline{P}(dw) = 0$, $i = 1, 2. \quad (2.3)$

Let $A = (a_{ij})_{2 \times 2}$ with

$$a_{ij} = \int_{\mathcal{S}} \int_{\mathcal{S}} \chi(w, d\tilde{w})\overline{P}(dw) [r_i(w)r_j(\tilde{w}) + r_j(w)r_i(\tilde{w})].$$

We suppose that $A$ is positive definite with square root $(\sigma_{ij})_{2 \times 2}$. Consider the diffusion

$$\begin{cases}
\frac{dX(t)}{dt} = X(t)\left[\overline{\sigma}_1 - b_1 X(t) - c_1 Y(t)\right] dt + X(t)(\sigma_{11} dW_1(t) + \sigma_{12} dW_2(t)) \\
\frac{dY(t)}{dt} = Y(t)\left[\overline{\sigma}_2 - h(Y(t)) u(t) - b_2 Y(t) + c_2 X(t)\right] dt + Y(t)(\sigma_{12} dW_1(t) + \sigma_{22} dW_2(t)),
\end{cases} \quad (2.4)$$

where $\overline{\sigma}_1 = a_1 + \frac{a_{11}}{2} = a_1 + \frac{\sigma_{11}^2 + \sigma_{12}^2}{2}, \overline{\sigma}_2 = a_2 + \frac{a_{22}}{2} = a_2 + \frac{\sigma_{22}^2 + \sigma_{12}^2}{2}, W_1, W_2$ are two independent Brownian motions.

We suppose that the function $\Phi(\cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$ represents the yield that is Lipschitz in its argument satisfying $\Phi(0) = 0$. That is, the yield is zero if we harvest nothing. If we want to maximize the average amount of the species harvested, then $\Phi(y) = y$. If we want to maximize the average money earned, $\Phi(y)$ should have a “saturated” form, such as $\Phi(y) = \frac{y}{c + y}$. We assume the effectiveness $h(\cdot) : \mathbb{R}_+ \mapsto [0, 1]$ is an increasing function and $h(0) = 0$. This stems from that the effectiveness increases as the density of the species increases.

Let $PM^\varepsilon$ be the class of functions $v : \mathbb{R}_+^2 \times \mathcal{S} \mapsto \mathcal{M}$ such that under the feedback control $u(t) = v(Z^\varepsilon(t))$ there exists a solution process to \text{(1.1)}, which is a Markov-Feller process. For $v \in PM^\varepsilon$, define

$$J^\varepsilon(v) := \lim_{T \to \infty} \frac{1}{T} \int_0^T \Phi\left(h(Y^\varepsilon(t)Y^\varepsilon(t))v(Y^\varepsilon(t))\right) dt \text{ a.s.}$$
For the wideband noise system, it is difficult to find an optimal control, that is, a control \( v^* \in \text{PM}^\varepsilon \) satisfying

\[
\mathcal{J}^\varepsilon = \sup_{v \in \text{PM}^\varepsilon} \{J^\varepsilon(v)\}.
\]

Thus, our goal is to find a near-optimal control \( v \in \text{PM}^\varepsilon \) using the limit diffusion system. To do that, we broaden the class of controls by use of the “relaxed controls”.

We present here some concepts and notation introduced in \([17]\). Let \( M(\infty) \) denote the family of measures \( \{m(\cdot)\} \) on the Borel subsets of \([0, \infty) \times U\) satisfying \( m([0, t] \times U) = t \) for all \( t \geq 0 \). By the weak convergence \( m_n(\cdot) \to m(\cdot) \) in \( M(\infty) \) we mean \( \lim_{n \to \infty} \int f(s, \alpha)m_n(ds \times d\alpha) = \int f(s, \alpha)m(ds \times d\alpha) \) for any continuous function \( f(\cdot) : [0, \infty) \times U \mapsto \mathbb{R} \) with compact support.

A random measure \( m(\cdot) \) with values in \( M(\infty) \) is said to be an admissible relaxed control for \((1.1)\) if \( \int_U \int_0^1 f(s, \alpha)m(ds \times d\alpha) \) is progressively measurable with respect to \( F^\varepsilon_t := \mathcal{F}^1_t \) for each bounded continuous function \( f(\cdot) \). With a relaxed control \( m(\cdot) \), let \( m_t^\varepsilon = \lim_{s \to t} \frac{1}{s-t} \int_s^t m(ds \times d\alpha) \), the model \((1.1)\) becomes

\[
\begin{align*}
   dX^\varepsilon(t) &= X^\varepsilon(t)[a_1 - b_1X^\varepsilon(t) - c_1Y^\varepsilon(t)]dt + \frac{1}{\varepsilon}X^\varepsilon(t)r_1(\xi^\varepsilon(t))dt \\
   dY^\varepsilon(t) &= Y^\varepsilon(t)[a_2 - h(Y^\varepsilon(t))m_t - b_2Y^\varepsilon(t) + c_2X^\varepsilon(t)]dt + \frac{1}{\varepsilon}Y^\varepsilon(t)r_2(\xi^\varepsilon(t))dt
\end{align*}
\]

Let \( \mathcal{P}(\mathcal{M}) \) be the space of invariant probability measures with Prohorov’s topology. A relaxed control is said to be Markov if there exists a measurable function \( \tilde{m} \) such that \( m_t = \tilde{m}(X^\varepsilon(t)) \), \( t \geq 0 \). For \( z \in \mathbb{R}^2_+ \), \( w \in \mathcal{S} \) and \( u \in \mathcal{M} \), define

\[
F(z, w) = \left( xr_1(w), yr_2(w) \right)^\top
\]

and

\[
G(z, u) = \left( x[a_1 - b_1x - c_1y], y[a_2 - h(y)u - b_2y + c_2x] \right)^\top.
\]

By an ergodicity argument (see, for example, \([11, 12, 14, 24]\)), it can be shown that if \( -a_2 + c_2 \frac{a_1}{b_1} < 0 \) then for any admissible control \( u(t) \), \( Y^\varepsilon(t) \) tends to 0 with probability 1, which implies

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \Phi \left( h(Y^\varepsilon(t))Y^\varepsilon(t)u(t) \right) dt = 0 \ a.s.
\]

Thus, to avoid the trivial limit, we assume throughout this paper that

\[
-a_2 + c_2 \frac{a_1}{b_1} > 0.
\]

Define the operator

\[
\mathcal{L}^\varepsilon \phi(z, w) = \frac{1}{\varepsilon}Q\phi(z, w) + \frac{1}{\varepsilon} \frac{\partial \phi(z, w)}{\partial z} F(z, w) + \frac{\partial \phi(z, w)}{\partial z} G(z, u),
\]
where \( \phi : \mathbb{R}_+^2 \times \mathcal{S} \mapsto \mathbb{R} \) is continuous and have continuous derivative with respect to the first variable, \( \frac{\partial \phi(z, w)}{\partial z} \). Denote by \( \mathbb{P}_{z,w} \) and \( \mathbb{E}_{z,w} \) the probability measure and the corresponding expectation of the process \( (Z^\varepsilon(\cdot), \xi^\varepsilon(\cdot)) \) with initial condition \((z, w)\). Note that \( \mathbb{P}_{z,w} \) and \( \mathbb{E}_{z,w} \) depends implicitly on the control \( m(t) \). For any bounded stopping times \( \tau_1 \leq \tau_2 \), we have
\[
\mathbb{E}_{z,w} \phi(Z^\varepsilon(\tau_2), \xi^\varepsilon(\tau_2)) = \mathbb{E}_{z,w} \phi(Z^\varepsilon(\tau_1), \xi^\varepsilon(\tau_1)) + \mathbb{E}_{z,w} \int_{\tau_1}^{\tau_2} \mathcal{L}^\varepsilon_{m_s} \phi(Z^\varepsilon(s), \xi^\varepsilon(s)) ds
\]
given that the expectations involved exist.

A random measure \( m(\cdot) \) with values in \( \mathcal{M}(\infty) \) is said to be an admissible relaxed control for (2.4) if \( \int_{\mathbb{V}} \int_0^t f(s, \alpha)m(ds \times d\alpha) \) is independent of \( \{W_i(t + s) - W_i(t) , s > 0 , i = 1, 2\} \) for each bounded continuous function \( f(\cdot) \). Under a relaxed control \( m(\cdot) \), the controlled diffusion (2.4) becomes
\[
\begin{align*}
\frac{dX(t)}{dt} &= X(t)[\alpha_1 - b_1 X(t) - c_1 Y(t)] dt + X(t)(\sigma_{11}dW_1(t) + \sigma_{12}dW_2(t)) \\
\frac{dY(t)}{dt} &= Y(t)[\alpha_2 - h(Y(t))m_t - b_2 Y(t) + c_2 X(t)] dt + Y(t)(\sigma_{12}dW_1(t) + \sigma_{22}dW_2(t)).
\end{align*}
\]
(2.7)
The generator for the controlled diffusion process (2.7) is
\[
\mathcal{L}_u \phi(z) = \frac{\partial \phi(z)}{\partial x} x[\alpha_1 - b_1 x - c_1 y] + \frac{\partial \phi(z)}{\partial y} y[\alpha_2 - h(y)u - b_2 y + c_2 x] \\
+ \frac{1}{2} \left( a_{11} \frac{\partial^2 \phi(z)}{\partial x^2} x^2 + 2a_{12} \frac{\partial^2 \phi(z)}{\partial x \partial y} xy + a_{22} \frac{\partial^2 \phi(z)}{\partial y^2} y^2 \right).
\]

**Definition 2.1.** A relaxed control \( m(\cdot) \) for (2.7) is said to be Markov if there exists a measurable function \( v : \mathbb{R}_+^2 \mapsto \mathcal{P}(\mathcal{M}) \) such that \( m_t = v(Z(t)), t \geq 0 \). A Markov control \( v \) is a relaxed control satisfying that \( v(z) \) is a Dirac measure on \( \mathcal{M} \) for each \( z \in \mathbb{R}_+^2 \). Denote the set of Markov controls and relaxed Markov controls by \( \Pi_M \) and \( \Pi_{RM} \), respectively. With a relaxed Markov control, \( Z(t) \) is a Markov process that has the strong Feller property in \( \mathbb{R}_+^{2,\circ} \); see [1] Theorem 2.2.12. Since the diffusion is nondegenerate in \( \mathbb{R}_+^{2,\circ} \), if the process \( Z(t) \) has an invariant probability measure in \( \mathbb{R}_+^{2,\circ} \), the invariant measure is unique, denoted by \( \eta_v \). In this case, the control \( v \) is said to be stable.

3 Main Results

First, we need the existence and uniqueness of positive solutions to (2.7) for any admissible relaxed control.

**Lemma 3.1.** If \( m(\cdot) \) is an admissible relaxed control for (2.4) (or (2.7)), then there exists a unique nonanticipative solution to (2.7) with initial value \( z = (x, y) \in \mathbb{R}_+^2 \) satisfying
\[
1. \mathbb{P}_z\{X(t) > 0 , t \geq 0\} = 1 \text{ (resp. } \mathbb{P}_z\{X(t) > 0 , t \geq 0\} = 1) \text{ if } x > 0 \text{ (resp. } y > 0) \text{, and } \mathbb{P}_z\{X(t) = 0, t \geq 0\} = 1 \text{ (resp. } \mathbb{P}_z\{Y(t) = 0, t \geq 0\} = 1) \text{ if } x = 0 \text{ (resp. } y = 0).
\]
2. \[ \mathbb{E}_2 \sup_{t \leq T} |Z(t)|^2 \leq K(1 + |z|^2) \]

where \( K \) depends only on \( T \).

**Proof.** This lemma can be proved by arguments in [17, Theorem 1] or [4, Theorem 2.2.2]. Note that the coefficients in (2.7) do not satisfy the linear growth condition. However, using a truncation argument and a Khaminskii-type method in [21], we can easily prove the existence of a unique solution to (2.7) satisfying claim 1. Moreover, we can estimate

\[
\begin{align*}
    d[c_2X(t) + c_1Y(t)] &= c_2X(t)\left(\bar{a}_1 - b_1X(t)\right)dt + c_1Y(t)\left(\bar{a}_2 - h(Y(t))\bar{m}_t - b_2Y(t)\right)dt \\
    &\quad + c_2X(t)(\sigma_{11}dW_1(t) + \sigma_{12}dW_2(t)) + c_1Y(t)(\sigma_{12}dW_1(t) + \sigma_{22}dW_2(t)) \\
    &\leq [c_2\bar{a}_1X(t) + c_1\bar{a}_2X(t)]dt \\
    &\quad + c_2X(t)(\sigma_{11}dW_1(t) + \sigma_{12}dW_2(t)) + c_1Y(t)(\sigma_{12}dW_1(t) + \sigma_{22}dW_2(t)).
\end{align*}
\]

In this estimate, the right-hand side is linear in \( X(t) \) and \( Y(t) \). Using standard arguments, (e.g., [15, Theorem 3.5] or [28, Proposition 3.5]), we can obtain the moment estimate, the second claim of this lemma. \( \square \)

With this lemma, in each finite interval, we can approximate \( Z^\varepsilon(t) \) by \( Z(t) \), which is proved in [17, Theorem 5].

**Lemma 3.2.** For any compact set \( K \in \mathbb{R}^2_+ \), \( \{(Z^\varepsilon(\cdot),m(\cdot)), t \geq 0\} \) with \( Z^\varepsilon(0) \in K \) is tight in \( D[0,\infty) \times M(\infty) \). If \( (Z^{\varepsilon_k}(\cdot),m^{(k)}(\cdot)) \) converges weakly to \( (\tilde{Z}(\cdot),\tilde{m}(\cdot)) \) as \( k \to \infty \) with \( \varepsilon_k \to 0 \) as \( k \to \infty \), then there exists independent Brownian motions \( W_1(t) \) and \( W_2(t) \) such that \( \tilde{m}(\cdot) \) is progressively measurable with respect to the filtration generated by \( W_1(t), W_2(t) \), and \( \tilde{Z} \) satisfying (2.7) with \( (Z(\cdot),m(\cdot)) \) replaced by \( (\tilde{Z}(\cdot),\tilde{m}(\cdot)) \).

We need the following lemma, whose proof is postponed to the appendix.

**Lemma 3.3.** The following claims hold.

- For any admissible relaxed control \( m(\cdot) \), we have that
  \[
  \limsup_{t \to \infty} \frac{1}{T} \int_0^T \Phi \left( Y(t)h(Y(t))\bar{m}_t \right)dt \leq \tilde{C} \quad \text{a.s.}
  \] (3.1)
  for some constant \( \tilde{C} \).

- Every relaxed Markov control is stable and there exists \( \tilde{C} > 0 \) such that
  \[
  \int_{\mathbb{R}^2_+ \times U} [1 + \Phi(yh(y)u)]^2 \pi_v(dz \times du) \leq \tilde{C}
  \] (3.2)
  for any relaxed Markov control \( v \), where \( \pi \) is a measure in \( \mathbb{R}^2_+ \times \mathcal{M} \) defined by
  \[
  \pi_v(dz \times du) = [v(z)(du)] 	imes \eta_v(du).
  \]
The family $\{\eta_\nu : \nu \in \Pi_{RM}\}$ is tight in $\mathbb{R}^{2,\circ}_+$. [Recall that $\eta_\nu$ is the invariant measure.]

With this lemma, letting

$$
\rho^* = \sup_{\nu \in \Pi_{RM}} \left\{ \int_{\mathbb{R}^{2,\circ} \times U} \Phi(gh(y)u)\pi_\nu(dz \times du) \right\},
$$

we have the following result from [4, Theorems 3.7.11 and 3.7.14].

**Theorem 3.1.** The Hamilton-Jacobi-Bellman (HJB) equation

$$
\max_{u \in U} \left[ L_u V(x) + c(x, u) \right] = \rho
$$

admits a solution $V^* \in C^2(\mathbb{R}^{2,\circ}_+)$ satisfying $V^*(0) = 0$ and $\rho = \rho^*$. A relaxed Markov control is optimal if and only if it satisfies

$$
\frac{\partial V^*}{\partial y} \left[ y(-a_2 - h(y)\overline{v}(z) - b_2y + c_2x) + \Phi(gh(y)v(z)) \right] = \max_{u \in U} \frac{\partial V^*}{\partial y} \left[ y(-a_2 - h(y)u - b_2y + c_2x) + \Phi(gh(y)u) \right],
$$

where $\overline{v}(z) = \int_M u[v(z)(du)]$.

The existence of an optimal Markov control can be derived from a well-known selection theorem; see e.g., [13, pp. 199-200]. Let $v^*$ be an optimal Markov control. There exists a sequence of $v_n : \mathbb{R}^{2,\circ}_+ \mapsto U$ such that $v_n(z)$ is locally Lipschitz in $z$ and $\lim_{n \to \infty} v_n = v$ almost everywhere in $\mathbb{R}^{2,\circ}_+$. Since every Markov control is stable, and the family $\{\nu_\nu, \nu \in \Pi_{RM}\}$ is tight on $\mathbb{R}^{2,\circ}_+$, we have from [4, Lemma 3.2.6] that

$$
\lim_{n \to \infty} \rho_{v_n} = \rho_{v^*} = \rho^*.
$$

This indicates that we can always find a $\delta$-optimal Markov control that is locally Lipschitz. We state here the main result of this paper.

**Theorem 3.2.** For any $\delta > 0$, there exists a locally Lipschitz Markov control $u^\delta$ such that

$$
J^\varepsilon(u^\delta) := \liminf_{T \to \infty} \frac{1}{T} \int_0^T \Phi(h(Y^\varepsilon(t)Y^\varepsilon(t))u^\delta(t))dt \leq \rho^* + \delta
$$

and that for sufficiently small $\varepsilon > 0$, we have

$$
J^\varepsilon(u^\delta) \geq \gamma^\varepsilon - 3\delta.
$$

The result above is known as chattering-type theorem. It connects relaxed controls and that of ordinary controls, and indicates that for any relaxed control, we can find a locally Lipschitz control to approximate the relaxed control. This is important because even though relaxed controls facilitate the establishment of the desired asymptotic results. Such control sets are much larger than the usual ordinary controls and cannot be used in the real applications. Thus viable approximation will be much appreciated. In view of [17, Theorem 8], we proceed to verify the following conditions to prove the desired result.
(C1) There is an \( \varepsilon_0 > 0 \) such that \( \{ Z^\varepsilon(u, t), u \in PM^\varepsilon, 0 \leq t < \infty, \varepsilon \leq \varepsilon_0 \} \) is \( P_{z, w} \)-tight in \( \mathbb{R}_+^{2, \circ} \) for each \( (z, w) \in \mathbb{R}_+^{2, \circ} \times \mathcal{S} \).

(C2) There is a \( \delta \)-optimal Markov control \( u(z) \) that is locally Lipschitz in \( z \) for any \( \delta > 0 \).

Condition [C2] has been verified in our manuscript; see (3.3). Since the dynamics of \( Z^\varepsilon(t) \) is dominated by negative quadratic terms when \( Z^\varepsilon(t) \) is large, it is easy to prove the tightness of \( \{ Z^\varepsilon(u, t), u \in PM^\varepsilon, 0 \leq t < \infty, \varepsilon \leq \varepsilon_0 \} \) in \( \mathbb{R}_+^{2, \circ} \). However, we need the tightness in \( \mathbb{R}_+^{2, \circ} \) to achieve the near optimality. To do that we need to analyze the behavior of \( Z^\varepsilon(u, t) \) near the boundary. Inspired by [7], we utilize the ergodicity of the system on the boundary and a property of the Laplace transform to construct a function \( V^\varepsilon(z, w) \) satisfying the inf-compact condition in \( \mathbb{R}_+^{2, \circ} \), i.e.,

\[
\lim_{R \to \infty} \inf \left\{ V^\varepsilon(z, w) : z + \frac{1}{x} + \frac{1}{y} > R \right\} = \infty
\]

and that

\[
\mathbb{E}_{z, w} V^\varepsilon(Z^\varepsilon(t), \xi^\varepsilon(t)) \leq C(1 + V(z, w))
\]

for any control \( u \in PM^\varepsilon \) and \( t \geq 0 \). Clearly, (C1) is proved if such a function is constructed.

In contrast to the technique used in [7], which is applied to a process in a compact space, the verification in our case is more difficult because the space \( \mathbb{R}_+^{2, \circ} \) is not compact and we have to treat a family of singularly perturbed processes rather than a single process.

4 Proofs of Results

First, when \( p_0, p_1, p_2 > 0 \) are sufficiently small, we have

\[
2p_0 + p_1b_1 + p_2c_2 < b_1, \quad \text{and} \quad 2p_0 + p_1c_1 + p_2b_2 < c_1.
\]

(4.1)

We can also choose \( p_1 \) and \( p_2 \) such that

\[
p_1a_1 - p_2a_2 < 0.
\]

(4.2)

By (2.6) and (4.2), we have

\[
\lambda = \frac{1}{11} \min \left\{ p_1a_1 - p_2a_2, p_2 \left( -a_2 + \frac{a_1c_2}{b_1} \right) \right\} > 0
\]

(4.3)

In view of (2.2) and (2.3), there exist bounded functions \( r_3(w) \) and \( r_4(w) \) such that \( Qr_3(w) = r_1(w) \) and \( Qr_4(w) = r_2(w) \). Let \( V(x, y) = \frac{1 + c_2x + c_1y}{x^{p_1}y^{p_2}} \). Define

\[
V_1(z, w) := xr_3(w)\frac{\partial V(z)}{\partial x} + yr_4(w)\frac{\partial V(z)}{\partial y}.
\]
We have
\[ \frac{\partial V(z)}{\partial z} \cdot G(z, u) = V(x, y) \left[ -p_1(a_1 - b_1 x - c_1 y) - p_2(-a_2 - h(y)u - b_2 y + c_2 x) + \frac{c_2(a_1 - b_1 x) + c_1(-a_2 - h(y)u - b_2 y)}{1 + c_2 x + c_1 y} \right] \leq (H_1 1_{\{z < H\}} - 3 - K_2 - p_0(1 + |z|)) V(z). \] (4.8)

Let \( V^\varepsilon(z, w) = V(z) + \varepsilon V_1(z, w) \), we have from (4.5) that
\[ (1 - \varepsilon K_2) V(z) \leq V^\varepsilon(z, w) \leq (1 + \varepsilon K_2) V(z), \quad z \in \mathbb{R}^2_+, s \in S. \] (4.9)

If \( \varepsilon > 0 \) is sufficiently small such that
\[ \varepsilon K_2 \leq p_0; \quad (H_1 + 3)\varepsilon K_2 < 1; \] (4.10)
using (4.6), (4.7), (4.4), and (4.8), we can estimate
\[
\mathcal{L}_u^\varepsilon V(z, w) = \frac{\partial V(z)}{\partial z} \left[ \frac{1}{\varepsilon} F(z, w) + G(z, u) \right] \\
+ \varepsilon \frac{\partial V_1(z, w)}{\partial z} \left[ \frac{1}{\varepsilon} F(z, w) + G(z, m) \right] + \frac{1}{\varepsilon} Q V_1(z, w)
\]
\[
\leq \left( H_1 1_{\{z < H\}} - 3 - K_2 - p_0(1 + |z|) \right) V(z) + K_2 V(z) + \varepsilon K_2(1 + |z|) V(z)
\]
\[
\leq \left( (H_1 + 1) 1_{\{z < H\}} - 2 \right) V(z)
\]
\[
\leq \left( (H_1 + 2) 1_{\{z < H\}} - 1 \right) V^\varepsilon(z, w),
\]
where the last two lines follow from (4.9) and (4.10). By virtue of (4.11), standard arguments show that
\[
\mathbb{E}_{z,w} V^\varepsilon(Z(t)) \leq e^{(H_1 + 2)t} V^\varepsilon(z), \ t \geq 0, \ z \in \mathbb{R}^{2^o}_+, \ w \in \mathcal{S}
\]
Let \( \tau^\varepsilon = \inf \{ s \geq 0 : Z^\varepsilon(s) \leq H \} \). Since \( \mathcal{L}_u^\varepsilon V^\varepsilon(z, w) \leq -V^\varepsilon(z, w) \) if \( z \geq H \), we have that
\[
\mathbb{E}_{z,w} e^{t \wedge \tau^\varepsilon} V^\varepsilon(Z(t \wedge \tau^\varepsilon), \xi^\varepsilon(t \wedge \tau^\varepsilon)) = V^\varepsilon(z) + \mathbb{E}_{z,w} \int_0^{t \wedge \tau^\varepsilon} e^s \left[ V^\varepsilon(s) + \mathcal{L}_u^\varepsilon V^\varepsilon(Z^\varepsilon(s), \xi^\varepsilon(s)) \right] ds
\]
\[
\leq V^\varepsilon(z), \ \text{for } t \geq 0, \ z \in \mathbb{R}^{2^o}_+, \ w \in \mathcal{S}.
\]

**Lemma 4.1.** There exist \( L > 0 \) and \( \varepsilon_1 > 0 \) such that for all \( \varepsilon < \varepsilon_1 \),
\[
\frac{1}{V_1^\varepsilon(Z^\varepsilon(t), \xi^\varepsilon(t))} \leq L e^{(H_1 + 2)t} \frac{1 + |z|^2}{V_1^\varepsilon(z, w)}, \ \text{for } (z, w) \in \mathbb{R}^{2^o}_+ \times \mathcal{S}, \ t \geq 0.
\]

**Proof.** Let \( \tilde{V}(z) = (1 + c_2 x + c_1 y)x^{p_1}y^{p_2} \). Construct a perturbed Lyapunov function
\[
\tilde{V}^\varepsilon(z, w) = \tilde{V}(z) + \varepsilon \left( x r_3(w) \frac{\partial \tilde{V}(z)}{\partial x} + y r_4(w) \frac{\partial \tilde{V}(z)}{\partial y} \right)
\]
Similar to estimates in (4.11), we can find \( K_3 > 0 \) such that
\[
(1 - \varepsilon K_3) \tilde{V}(z) \leq \tilde{V}^\varepsilon(z, w) \leq (1 + \varepsilon K_3) \tilde{V}(z)
\]
and
\[
\mathbb{E}_{z,w} \tilde{V}^\varepsilon(Z^\varepsilon(t), \xi^\varepsilon(t)) \leq e^{(H_1 + 2)t} \tilde{V}^\varepsilon(z, w)
\]
when \( \varepsilon \) is sufficiently small. On the other hand, for any \((z, w) \in \mathbb{R}^{2^o}_+ \times \mathcal{S}, \) we have
\[
\frac{1}{V(z)} \leq \tilde{V}(z) \leq (1 + c_2 x + c_1 y)^2 \frac{1}{V(z)}.
\]
which combined with (4.9) and (4.15) implies that
\[
\frac{1}{V_1^\varepsilon(z, w)} \leq \frac{1}{(1 - \varepsilon K_2)V(z)} \leq \frac{1}{(1 - \varepsilon K_2)(1 - \varepsilon K_3)^2} \tilde{V}^\varepsilon(z, w)
\]

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and
\[ \tilde{V}^\varepsilon(z, w) \leq (1 + \varepsilon K_3)\tilde{V}(z) \leq (1 + \varepsilon K_3) \frac{(1 + c_2 x + c_1 y)^2}{\tilde{V}(z)} \leq (1 + \varepsilon K_2)^2 \frac{(1 + c_2 x + c_1 y)^2}{\tilde{V}(z, w)}. \quad (4.19) \]

Applying (4.18) and (4.19) to (4.16), we can easily obtain (4.14) for suitable \( L > 0 \) when \( \varepsilon \) is sufficiently small.

**Lemma 4.2.** There are \( \hat{K} > 0 \) and \( \varepsilon_2 > 0 \) such that for any \( \varepsilon < \varepsilon_2 \), and any admissible control \( m(\cdot) \) for (1.1), we have
\[
E_{z,w} \int_0^t |Z^\varepsilon(s)|^2 ds \leq \hat{K}(1 + |z| + t),
\]
and
\[
E_z \int_0^t |Z(s)|^2 ds \leq \hat{K}(1 + |z| + t).
\]

**Proof.** Let \( V_2(z) = 1 + c_2 x + c_1 y \) and
\[
V_3(z, w) := x r_3(w) \frac{\partial V_2(z)}{\partial x} + y r_4(w) \frac{\partial V_2(z)}{\partial y}.
\]
We can find a \( K_4 > 0 \) satisfying
\[
|V_3(z, w)| \leq K_4 V_2(z), \ (z, w) \in \mathbb{R}_+^2 \times \mathcal{S}. \quad (4.20)
\]
and
\[
\left| \frac{\partial V_3(z, w)}{\partial z} \cdot F(z, w) \right| \leq K_4 V_2(z), \ (z, w) \in \mathbb{R}_+^2 \times \mathcal{S}. \quad (4.21)
\]
\[
\left| \frac{\partial V_3(z, w)}{\partial z} \cdot G(z, u) \right| \leq K_4(1 + |z|) V_2(z), \ (z, w) \in \mathbb{R}_+^2 \times \mathcal{S}, u \in \mathcal{M}. \quad (4.22)
\]
We have
\[
\frac{\partial V_2(z)}{\partial z} \cdot F(z, u) = c_2 x [a_1 - b_1 x] + c_1 y [a_2 - h(y)u - b_2 y].
\]
Let \( \beta \in (0, (c_2 b_1) \land (c_1 b_2)) \). Clearly, we can choose a \( K_5 > 0 \) such that
\[
\frac{\partial V_2(z)}{\partial z} \cdot F(z, u) \leq K_5 - V_2(z) - \beta(x^2 + y^2) \forall (x, y) \in \mathbb{R}_+^2, u \in [0, M]. \quad (4.23)
\]
Let
\[
V_2^\varepsilon(z, w) = V_2(z) + \varepsilon V_3(z, w)
\]
Similar to (4.11), from (4.20), (4.21), and (4.22), we have
\[
\mathcal{L}_u V_3^\varepsilon(z, w) \leq 2 K_5 - \frac{\beta |z|^2}{2}
\]
for sufficiently small $\varepsilon$. As a result,

$$
\mathbb{E}_{z,w} V_{3}^{\varepsilon}(Z^{\varepsilon}(t), \xi^{\varepsilon}(t)) = V_{3}^{\varepsilon}(z, w) + \mathbb{E}_{z,w} \int_{0}^{t} \mathcal{L}^{\varepsilon}_{m_{t}} V_{3}^{\varepsilon}(Z^{\varepsilon}(ds), \xi^{\varepsilon}(s))ds \\
\leq V_{3}^{\varepsilon}(z, w) + 2K_{5}t - \frac{\beta}{2} \int_{0}^{t} \mathbb{E}_{z,w} |Z^{\varepsilon}(t)|^{2},
$$

(4.24)

which leads to

$$
\frac{\beta}{2} \int_{0}^{t} \mathbb{E}_{z,w} |Z^{\varepsilon}(t)|^{2} \leq V_{3}^{\varepsilon}(z, w) + 2K_{5}t
$$

The first claim of the lemma follows directly from the above estimate. The second claim can be derived by applying Itô’s formula for $V_{2}(z)$ to (2.7) and then proceeding like (4.24).

**Lemma 4.3.** There is a $\tilde{K} > 0$ such that

$$
\left| \mathbb{E}_{z,w} \left[ \ln V(Z^{\varepsilon}(T)) \right] - \ln V(z) - \mathbb{E}_{z,w} \int_{0}^{T} \mathcal{L}_{m_{t}} \ln V(Z^{\varepsilon}(t), \xi^{\varepsilon}(t))dt \right| \leq \tilde{K}(1 + T)\varepsilon,
$$

for any admissible relaxed control $m(\cdot)$.

**Proof.** Let

$$
g_{1}(z, w) = \int_{S} \chi(w, d\tilde{w}) \frac{\partial(\ln V(z))}{\partial z} \cdot F(z, \tilde{w}),
$$

and

$$
g_{2}(z, w) = \int_{S} \chi(w, d\tilde{w}) \left[ \frac{\partial g_{1}(z, w)}{\partial z} F(x, \tilde{w}) + \frac{\partial(\ln V(z))}{\partial z} \cdot G(x, u) - \mathcal{L}_{u} \ln V(z) \right].
$$

Note that $g_{2}$ does not depend on $u$ since there is no $u$ dependence in

$$
\frac{\partial(\ln V(z))}{\partial z} \cdot G(x, u) - \mathcal{L}_{u} \ln V(z) = \frac{1}{2} \frac{a_{11}c_{2}^{2}x^{2} + a_{22}c_{1}^{2}y^{2} + 2a_{12}c_{1}c_{2}xy}{(1 + c_{2}x + c_{1}y)^{2}} - \frac{c_{3}x_{a_{11}} + c_{1}ya_{22}}{1 + c_{2}x + c_{1}y}.
$$

Moreover, direct calculations show that $\frac{\partial(\ln V(z))}{\partial z} \cdot F(z, w)$ and $\frac{\partial g_{1}(z, w)}{\partial z} \cdot F(x, y)$ are bounded along with $\frac{\partial(\ln V(z))}{\partial z} \cdot G(x, u) - \mathcal{L}_{u} \ln V(z)$. Consequently, $g_{i}(z, w), i = 1, 2$ are also bounded in $\mathbb{R}_{+}^{2} \times \mathcal{S}$. As a result, we have from [3 Formula (4.21)] that

$$
\left| \mathcal{L}_{u}^{\varepsilon}[\ln V(z) + \varepsilon g_{1}(z, w) + \varepsilon^{2}g_{2}(z, w)] - \mathcal{L}_{u} \ln V(z) \right| \leq K_{6}\varepsilon \text{ for all } (z, w) \in \mathbb{R}_{+}^{2} \times \mathcal{S}
$$

for some constant $K_{6} > 0$ independent of $m$. Combining this and the equality

$$
\begin{align*}
\mathbb{E}_{z,w} \left[ \ln V(Z^{\varepsilon}(T)) + \varepsilon g_{1}(Z^{\varepsilon}(T), \xi^{\varepsilon}(T)) + \varepsilon^{2}g_{2}(Z^{\varepsilon}(T), \xi^{\varepsilon}(T)) \right] \\
= \ln V(z) + \varepsilon g_{1}(z, w) + \varepsilon^{2}g_{2}(z, w) \\
+ \mathbb{E}_{z,w} \int_{0}^{T} \mathcal{L}^{\varepsilon}_{m_{t}}[\ln V(Z^{\varepsilon}(t)) + \varepsilon g_{1}(Z^{\varepsilon}(t), \xi^{\varepsilon}(t)) + \varepsilon^{2}g_{2}(Z^{\varepsilon}(t), \xi^{\varepsilon}(t))]dt,
\end{align*}
$$

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we obtain

\[
\begin{align*}
\left| \mathbb{E}_{z,w}[\ln V(Z^\varepsilon(T)) + \varepsilon g_1(Z^\varepsilon(T), \xi^\varepsilon(T)) + \varepsilon^2 g_2(Z^\varepsilon(T), \xi^\varepsilon(t))] \right. \\
\left. - \ln V(z) - \varepsilon g_1(z, w) - \varepsilon^2 g_2(z, w) - \mathbb{E}_{z,w} \int_0^T \mathcal{L}_{m_i}[\ln V(Z^\varepsilon(t), \xi^\varepsilon(t))] dt \right| \leq K_6 T \varepsilon.
\end{align*}
\]

By the boundedness of \(g_i(z, w), i = 1, 2\), we deduce that

\[
\begin{align*}
\left| \mathbb{E}_{z,w}[\ln V(Z^\varepsilon(T))] - \ln V(z) - \mathbb{E}_{z,w} \int_0^T \mathcal{L}_{m_i}[\ln V(Z^\varepsilon(t), \xi^\varepsilon(t))] dt \right| \leq (K_6 T + K_7) \varepsilon.
\end{align*}
\]

for some \(K_7 > 0\). The lemma is therefore proved. \(\square\)

Define \(f, g : \mathbb{R}^2_+ \mapsto \mathbb{R}\) by

\[
f(x, y) = p_1(a_1 - b_1 x - c_1 y) + p_2(-a_2 - b_2 y + c_2 x)
\]

and

\[
g(x, y) = \frac{c_2 x (\overline{a}_1 - b_1) + c_1 y (\overline{a}_2 - b_2)}{1 + c_2 x + c_1 y} - \frac{1}{2} \frac{a_{11} c_2^2 x^2 + a_{22} c_1^2 y^2 + 2 a_{12} c_1 c_2 x y}{(1 + c_2 x + c_1 y)^2}.
\]

**Lemma 4.4.** For any \(H > 0\) and \(k_0 > 1\), there exist \(T_1 = T_1(H, \varepsilon_0, k_0) > 0\) and \(\delta = \delta(H, \varepsilon_0, k_0) > 0\) such that for any admissible control \(m(\cdot)\), and \(z \in D_{\delta,H} := ([0, H] \times [0, \delta]) \cup ([0, \delta] \times [0, H])\), we have

\[
\frac{1}{t} \int_0^t \mathbb{E}_z f(Z(s)) ds > 9 \lambda, \text{ and } \frac{1}{t} \int_0^t \mathbb{E}_z g(Z(s)) ds \leq \lambda, \forall t \in [T_1, T_2],
\]

and

\[
\frac{1}{t} \int_0^t \mathbb{E}_z h(Y(s)) ds \leq \frac{\lambda}{p_2 M}, \forall t \in [T_1, T_2]
\]

where \(T_2 = (k_0 + 1)T_1\) and \(\lambda\) is defined in (4.3).

The results in this lemma are obtained by analyzing the behavior of \(Z(t)\) near the boundary. The proof is postponed to the appendix.

**Lemma 4.5.** With \(H, k_0, T_1, T_2, \delta\) as given in Lemma 4.4, there is an \(\varepsilon_3 > 0, \theta \in (0, 1)\) such that for any \(\varepsilon \in (0, \varepsilon_3)\). Let \(D_{\delta,H}^0 = ([0, H] \times (0, \delta]) \cup ((0, \delta) \times (0, H])\). For any admissible control \(m(\cdot), (z, w) \in D_{\delta,H} \times S\), we have

\[
\mathbb{E}_{z,w}[V^\varepsilon(Z^\varepsilon(t), \xi^\varepsilon(t))]^\theta \leq e^{-\lambda t}[V^\varepsilon(z, w)]^\theta, t \in [T_1, T_2].
\]

**Proof.** Since \(D_{\delta,H}\) is a compact set, by virtue of Lemma 4.4 and [17, Theorem 5], (which tell us we can approximate solutions to (2.3) by the corresponding solutions to (2.7)), there is an
\( \varepsilon_2 > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_2) \), and for any admissible control \( m(\cdot), (z, w) \in D_{\delta,H} \times \mathcal{S} \), we have
\[
\frac{1}{t} \int_0^t \mathbb{E}_{z,w} f(Z^\varepsilon(s)) ds > 8\lambda, \quad t \in [T_1, T_2], \tag{4.27}
\]
and
\[
\frac{1}{t} \int_0^t \mathbb{E}_{z,w} g(Z^\varepsilon(s)) ds < 2\lambda, \quad t \in [T_1, T_2], \tag{4.28}
\]
for any admissible control. In view of (4.31) and Lemma 4.3, when combining (4.32) and (4.9), we have that
\[
(4.27) \quad \text{and} \quad (4.28)
\]
have linear growth rates. Thus, (4.27) and (4.28) can still be obtained from the uniform integrability in Lemma 4.2 combined with the weak convergence.

On the other hand, \( \mathcal{L}_u \ln V(z, w) = -f(z) + g(z) - \frac{c_1 y h(y) u}{1 + c_2 x + c_1 y} + p_2 h(y) u \)
\[
\leq -f(z) + g(z) + M p_2 h(y). \tag{4.30}
\]
It follows from (4.27), (4.28), (4.29), and (4.30) that
\[
\frac{1}{t} \int_0^t \mathbb{E}_{z,w} \mathcal{L}_m \ln V(Z^\varepsilon(s), \xi^\varepsilon(s)) ds \leq -4\lambda, \quad t \in [T_1, T_2], \tag{4.31}
\]
for any admissible control. In view of (4.31) and Lemma 4.3 when \( \varepsilon \) is sufficiently small, we have
\[
\mathbb{E}_{z,w} \left[ \ln V(Z^\varepsilon(t)) \right] - \ln V(z) \leq -3\lambda t, \quad t \in [T, k_0 T], \tag{4.32}
\]
for any admissible control. By (4.11) and Lemma 4.4, there is a \( \tilde{K} \) depending only on \( T_1, T_2 \) and \( H \) such that
\[
\max \left\{ \mathbb{E}_{z,w} \exp(-Y^\varepsilon(t), \mathbb{E}_{z,w} \exp(Y^\varepsilon(t)) \right\} < \tilde{K}, \quad \varepsilon \in (0, \varepsilon_2), z \in D_{\delta,H}^0, w \in \mathcal{S}, t \in [T_1, T_2]
\]
for any admissible control. By Lemma 4.1 there is a \( \tilde{K}_2 > 0 \) such that
\[
\ln \left( \mathbb{E}_{z,w} \left[ \frac{V^\varepsilon(Z^\varepsilon(t), \xi^\varepsilon(t))}{V(z, w)} \right] \right) \leq \theta \mathbb{E}_{z,w} \left[ Y^\varepsilon(t) + \theta^2 \tilde{K}_2 \right]
\]
\[
\leq -2\lambda \theta t + \theta^2 \tilde{K}_2, \quad (z, w) \in D_{\delta,H}^0 \times \mathcal{S}, t \in [T_1, T_2], \quad \theta \in [0, 0.5].
\]
Letting $\theta = \lambda T_1 [\hat{K}_2]^{-1} \wedge 0.5$, we have

$$
E_{z,w} [V^\varepsilon(Z^\varepsilon(t), \xi^\varepsilon(t))]^\theta \leq e^{-\lambda \theta t}[V^\varepsilon(z, w)]^\theta, \ (z, w) \in D_{\delta,H}^0 \times \mathcal{S}, \ t \in [T_1, T_2].
$$

\[ \square \]

**Lemma 4.6.** Let $\theta$ satisfy the conclusion of Lemma 4.5. There are $q \in (0,1)$ and $C > 0$ such that

$$
E_{z,w} [V^\varepsilon(Z^\varepsilon(T_2), \xi^\varepsilon(T_2))]^\theta \leq q[V^\varepsilon(z, w)]^\theta + C,
$$

for any relaxed Markov control $u^\varepsilon \in PM^\varepsilon$ when $\varepsilon$ is sufficiently small.

**Proof.** Applying Jensen’s inequality to (4.12) and (4.13), we have that for $t \geq 0$,

$$
E_{z,w} e^{\theta(t \wedge \tau^\varepsilon)} [V^\varepsilon(Z^\varepsilon(t \wedge \tau^\varepsilon), \xi^\varepsilon(t \wedge \tau^\varepsilon))]^\theta \leq [V^\varepsilon(z, w)]^\theta
$$

(4.33)

and

$$
E_{z,w} [V^\varepsilon(Z^\varepsilon(t), \xi^\varepsilon(t))]^\theta \leq e^{(H_1+2)\theta t}[V^\varepsilon(z, w)]^\theta.
$$

(4.34)

Since $\tilde{D}_{\delta,H} := (0, H]^2 \setminus D_{\delta,H}$ is a compact subset of $\mathbb{R}^2_+$,

$$
C := e^{(H_1+2)\theta T_2} \sup_{z \in \tilde{D}_{\delta,H}, w \in \mathcal{S}} [V^\varepsilon(z, w)]^\theta < \infty.
$$

By virtue of (4.34) and Lemma 4.4, we have

$$
E_{z,w} [V^\varepsilon(Z^\varepsilon(t), \xi^\varepsilon(t))]^\theta \leq C + e^{-\theta \lambda}[V^\varepsilon(z, w)]^\theta, \ \forall (z, w) \in (0, H]^2 \times \mathcal{S}, \ t \in [T_1, T_2].
$$

(4.35)

We have the following estimate.

$$
E_{z,w} e^{\theta(T_2 \wedge \tau^\varepsilon)} [V^\varepsilon(Z^\varepsilon(T_2 \wedge \tau^\varepsilon), \xi^\varepsilon(T_2 \wedge \tau^\varepsilon))]^\theta
$$

\begin{align*}
= & E_{z,w} 1_{\{\tau^\varepsilon < k_0 T_1\}} e^{\lambda(T_2 \wedge \tau^\varepsilon)} [V^\varepsilon(Z^\varepsilon(T_2 \wedge \tau^\varepsilon), \xi^\varepsilon(T_2 \wedge \tau^\varepsilon))]^\theta \\
+ & E_{z,w} 1_{\{k_0 T_1 \leq \tau^\varepsilon < T_2\}} e^{\beta_1 \lambda(T_2 \wedge \tau^\varepsilon)} [V^\varepsilon(Z^\varepsilon(T_2 \wedge \tau^\varepsilon), \xi^\varepsilon(T_2 \wedge \tau^\varepsilon))]^\theta \\
+ & E_{z,w} 1_{\{\tau^\varepsilon \geq T_2\}} e^{\beta_2 \lambda\tau^\varepsilon} [V^\varepsilon(Z^\varepsilon(T_2 \wedge \tau^\varepsilon), \xi^\varepsilon(T_2 \wedge \tau^\varepsilon))]^\theta
\end{align*}

(4.36)

$$
\geq E_{z,w} 1_{\{\tau^\varepsilon \leq k_0 T_1\}} [V^\varepsilon(Z^\varepsilon(\tau^\varepsilon), \xi^\varepsilon(\tau^\varepsilon))]^\theta \\
+ e^{\lambda k_0 T} E_{z,w} 1_{\{k_0 T \leq \tau^\varepsilon < T_2\}} [V^\varepsilon(Z^\varepsilon(\tau^\varepsilon), \xi^\varepsilon(\tau^\varepsilon))]^\theta \\
+ e^{\lambda T_2} E_{z,w} 1_{\{\tau^\varepsilon \geq T_2\}} [V^\varepsilon(Z^\varepsilon(T_2), \xi^\varepsilon(T_2))]^\theta.
$$

With a relaxed Markov control $u^\varepsilon \in PM^\varepsilon$, the process $(Z^\varepsilon(t), \xi^\varepsilon(t))$ is a Markov-Feller process. Thus, we have from (4.35) that

$$
E_{z,w} 1_{\{\tau^\varepsilon \leq k_0 T_1\}} [V^\varepsilon(Z^\varepsilon(T_2), \xi^\varepsilon(T_2))]^\theta
$$

\begin{align*}
\leq & E_{z,w} 1_{\{\tau^\varepsilon \leq k_0 T_1\}} [C + e^{-\theta \lambda(T_2 \wedge \tau^\varepsilon)} [V^\varepsilon(Z^\varepsilon(\tau^\varepsilon), \xi^\varepsilon(\tau^\varepsilon))]^\theta] \\
\leq & C + e^{-\theta \lambda T_1} E_{z,w} 1_{\{\tau^\varepsilon \leq k_0 T_2\}} [V^\varepsilon(Z^\varepsilon(\tau^\varepsilon), \xi^\varepsilon(\tau^\varepsilon))]^\theta.
\end{align*}

(4.37)
Similarly, it follows from (4.33) and the inequality \((H_1 + 2)T \leq \lambda(k_0 - 1)\) that
\[
\mathbb{E}_{z,w} 1_{\{k_0T_1 \leq \tau \leq T_2\}} \left[ V^\varepsilon(Z^\varepsilon(T_2), \xi^\varepsilon(T_2)) \right]^\theta \\
\leq \mathbb{E}_{z,w} 1_{\{k_0T_1 \leq \tau \leq T_2\}} e^{\theta(H_1+2)\varepsilon(T_2 - \tau^\varepsilon)} \left[ V^\varepsilon(Z^\varepsilon(\tau^\varepsilon), \xi^\varepsilon(\tau^\varepsilon)) \right]^\theta \\
\leq e^{\theta H_1 + 2}\theta T_1 \mathbb{E}_{z,w} 1_{\{k_0T_1 \leq \tau \leq T_2\}} \left[ V^\varepsilon(Z^\varepsilon(\tau^\varepsilon), \xi^\varepsilon(\tau^\varepsilon)) \right]^\theta \\
\leq e^{-\theta \lambda T_1} e^{\theta \lambda k_0 T_1} \mathbb{E}_{z,w} 1_{\{k_0T_1 \leq \tau \leq T_2\}} \left[ V^\varepsilon(Z^\varepsilon(\tau^\varepsilon), \xi^\varepsilon(\tau^\varepsilon)) \right]^\theta.
\] (4.38)

Moreover,
\[
\mathbb{E}_{z,w} 1_{\{\tau \geq T_2\}} \left[ V^\varepsilon(Z^\varepsilon(T_2), \xi^\varepsilon(T_2)) \right]^\theta = e^{-\theta \lambda T_1} e^{\theta \lambda T_2} \mathbb{E}_{z,w} 1_{\{\tau \geq T_2\}} \left[ V^\varepsilon(Z^\varepsilon(T_2), \xi^\varepsilon(T_2)) \right]^\theta. \quad (4.39)
\]

Owing to (4.37), (4.38), and (4.39), we have
\[
\mathbb{E}_{z,w}[V^\varepsilon(Z^\varepsilon(T_2), \xi^\varepsilon(T_2))]^\theta \leq C + e^{-\theta \lambda T_1} \mathbb{E}_{z,w} e^{\theta (T_2 - \tau^\varepsilon)} \left[ V^\varepsilon(Z^\varepsilon(T_2 \land \tau^\varepsilon), \xi^\varepsilon(T_2 \land \tau^\varepsilon)) \right]^\theta.
\]

This together with (4.33) concludes the proof with \(q = e^{-\theta \lambda T_1}\).

\[\square\]

**Theorem 4.1.** With \(q\) and \(C\) given in Lemma 4.6, for sufficiently small \(\varepsilon\), we have
\[
\mathbb{E}_{z,w}[V^\varepsilon(Z^\varepsilon(t), \xi^\varepsilon(t))]^\theta \leq e^{(H_1+2)T_2 q^{1/(2T_2)}}[V^\varepsilon(z, w)]^\theta + \frac{C}{1 - q}, \quad \text{for any relaxed Markov control } u \in PM^\varepsilon.
\]

**Proof.** By the Markov property, we have
\[
\mathbb{E}_{z,w}[V^\varepsilon(Z^\varepsilon((k+1)T_2), \xi^\varepsilon((k+1)T_2))]^\theta \leq q \mathbb{E}_{z,w}[V^\varepsilon(Z^\varepsilon(kT_2), \xi^\varepsilon(kT_2))]^\theta + C, \quad k \in \mathbb{N}.
\]

Using this inequality recursively, we obtain
\[
\mathbb{E}_{z,w}[V^\varepsilon(Z^\varepsilon(kT_2), \xi^\varepsilon(kT_2))]^\theta \leq q^n[V^\varepsilon(z, w)]^\theta + \frac{C(1 - q^n)}{1 - q}.
\] (4.41)

The assertion of this theorem follows from (4.41) and (4.33) \(\square\)

**Proof of Theorem 3.2.** Since
\[
\lim_{r \to \infty} \left( \inf_{\{z|v_{x-1}v_{y-1} > r, w \in S\}} [V^\varepsilon(z, w)]^\theta \right) = \infty, \quad \text{and } q < 1,
\]
the conclusion of Theorem 42 clearly implies Condition (C1). Theorem 3.2 is therefore proved. \(\square\)
5 Concluding Remarks

Our main effort in this paper is to demonstrate that we can obtain near-optimal policies for average-cost per unit time yield for a predator-prey model under fast-varying jump noise by using a near optimal strategy of a controlled diffusion model. Due to the technical complexity of the proofs, we made some simplifications in the model in order to facilitate the presentation but still preserve important properties of the model. The main result, Theorem 3.2 still hold true if the following generalizations are made.

(a) The coefficients $a_i, b_i, c_i, i = 1, 2$ depend on the state of $\xi^\varepsilon(t)$.

(b) The wideband noise in (1.1), which is linear in the current setup, can be replaced by nonlinear terms.

(c) The assumption on $\xi(t)$ in Section 2 can be reduced to the condition that $\xi(t)$ a stationary zero mean process which is either (i) strongly mixing, right continuous and bounded, with the mixing rate function $\phi(\cdot)$ satisfying $\int_0^\infty \phi^{1/2}(s)ds < \infty$, or (ii) stationary Gauss-Markov with an integrable correlation function as in [17].

With the generalization specified in (a) above, the proofs carry over, although the notations are more complicated. With (b), we need some additional conditions imposed on the wideband noise parts to obtain certain boundedness of the solutions to the limit diffusion equation.

Throughout the paper, we assume that $\xi(t)$ is an ergodic Markov process, under which we can utilize the Fredholm alternative to construct Lyapunov functions for the wideband noise model (1.1) based on those for the controlled diffusion (2.4). If that assumption is replaced by (c), it is slightly more complicated to construct Lyapunov functions for the wideband noise model (1.1). However, it is doable using the perturbed Lyapunov method in [17]. In such a setup, however, we need to work mainly with convergence of probability measures.

In this paper, we consider the situation that only the predator is harvested. It is also interesting to deal with the optimization problem of harvesting both species under the constraint that the extinction of each species is avoided. Moreover, time-average optimal harvesting problems for different ecological models also deserve careful study. Our methods can be generalized to treat harvested ecological models of higher dimensions.

A Appendix

This appendix provides several technical results. These results are collected in a number of lemmas.

Lemma A.1. Let $Y$ be a random variable, $\theta_0 > 0$ a constant, and suppose

$$\mathbb{E} \exp(\theta_0 Y) + \mathbb{E} \exp(-\theta_0 Y) \leq K_1.$$
Then the log-Laplace transform \( \phi(\theta) = \ln \mathbb{E} \exp(\theta Y) \) is twice differentiable on \( [0, \frac{\theta_0}{2}] \) and
\[
\frac{d\phi}{d\theta}(0) = \mathbb{E} Y, \quad \text{and} \quad 0 \leq \frac{d^2\phi}{d\theta^2}(\theta) \leq K_2, \theta \in \left[ 0, \frac{\theta_0}{2} \right)
\]
for some \( K_2 > 0 \) depending only on \( K_1 \) and \( \theta_0 \). Moreover,
\[
\phi(\theta) \leq \theta \mathbb{E} Y + \theta^2 K_2, \quad \text{for} \quad \theta \in [0, 0.5\theta_0).
\]

**Proof.** The lemma is proved in [23]. \( \square \)

**Lemma A.2.** For any \( p > 0 \) and \( T > 0 \), \( H > 0 \), there is a \( \kappa_{p,T} > 0 \) such that for any admissible control \( m(\cdot) \), we have
\[
\mathbb{E}_z(1 + |Z_t|)^p \leq \kappa_{p,T}, t \in [0, T], z \in [0, H]^2.
\] (A.1)

Moreover,
\[
\mathbb{E}_z|X_t|^2 \leq x^2 \kappa_T^2, \mathbb{E}_z|Y_t|^2 \leq y^2 \kappa_T^2, t \in [0, T], z \in [0, H]^2.
\] (A.2)

**Proof.** By a straightforward computation, we can show that
\[
\mathcal{L} u(1 + c_2 x + c_1 y)^p \leq C_p (1 + c_2 x + c_1 y)^p, \quad z \in \mathbb{R}^2_+.
\]

This implies that
\[
\mathbb{E}_z(1 + c_2 X(t) + c_1 Y(t))^p \leq e^{C_p t} (1 + c_2 x + c_1 y)^p.
\]

Choosing a suitable \( \kappa_{p,T} \), we obtain (A.1). Now, using the function \( U(z) = (1 + c_2 x + c_1 y)^p x^2 \),
\[
\mathcal{L}_u U(z) = U(z) \left[ \frac{c_2(a_1 - b_1 x) + c_1(-a_2 - h(y) u - b_2 y)}{1 + c_2 x + c_1 y} + 2(a_1 - b_1 x) \right]
\]
\[
+ U(z) \left[ \frac{p - 1}{2} \frac{a_{11} c_2 x^2 + a_{22} c_1 y^2 + 2a_{12} c_1 c_2 x y}{(1 + c_2 x + c_1 y)^2} + a_{11} + 2 \frac{p(a_{11} + a_{12}) c_2 x + (a_{12} + a_{22})}{(1 + c_2 x + c_1 y)} \right]
\]

When \( p > 0 \) is sufficiently large, it can be seen that there is a \( \tilde{C}_p > 0 \) satisfying
\[
\mathcal{L}_u U(z) \leq \tilde{C}_p U(z), \quad \text{for} \quad z \in \mathbb{R}^2_+.
\] (A.3)

Thus,
\[
\mathbb{E}_z|X(t)|^2 \leq \mathbb{E}_z U(Z(t)) \leq U(z) e^{\tilde{C}_p t} \leq x^2 (1 + c_2 x + c_1 y)^p e^{\tilde{C}_p t}.
\]

The above estimate and a similar estimate for \( \mathbb{E}_z|Y(t)|^2 \) lead to (A.2). \( \square \)
Let $f(\cdot)$ and $g(\cdot)$ be defined as in (4.25) and (4.26). Since $f(z), g(z), h(y)$ are Lipschitz, there is a $\ell > 0$ such that

$$|f(z) - f(\tilde{z})| \vee |g(z) - g(\tilde{z})| \leq \ell(|z - \tilde{z}|), \quad z, \tilde{z} \in \mathbb{R}^2_+ \quad \text{(A.4)}$$

and

$$|h(y) - h(\tilde{y})| \leq \ell(|y - \tilde{y}|), \quad y, \tilde{y} \geq 0. \quad \text{(A.5)}$$

**Lemma A.3.** Let $\tilde{X}(t) > 0$ and $\tilde{Y}(t) > 0$ satisfy

$$\begin{cases}
    d\tilde{X}(t) = \tilde{X}(t)\left[\bar{a}_1 - b_1 X(t)\right] dt + \tilde{X}(t)(\sigma_{11}dW_1(t) + \sigma_{12}dW_2(t)) \\
    d\tilde{Y}(t) = \tilde{Y}(t)\left[-\frac{a_2}{2} - a_{22} - b_2 Y(t)\right] dt + \tilde{Y}(t)(\sigma_{12}dW_1(t) + \sigma_{22}dW_2(t)).
\end{cases} \quad \text{(A.6)}$$

Then there exists a $T_0 > 0$ such that

$$\frac{1}{t} \int_0^t \mathbb{E}_x f(\tilde{X}(s), 0) ds \geq 10\lambda; \quad \frac{1}{t} \int_0^t \mathbb{E}_x g(\tilde{X}(s), 0) ds \leq \frac{\lambda}{2}; \quad \text{(A.7)}$$

and

$$\frac{1}{t} \int_0^t \mathbb{E}_y \tilde{Y}(s) ds \leq \frac{\lambda}{2(1 + M)} \ell \quad \text{(A.8)}$$

for any $t > T_0$ and $x, y \in [0, H]$.

**Proof.** Define the occupation measure

$$\pi^x_t(\cdot) = \int_0^t \mathbb{P}_x\{\tilde{X}(s) \in \cdot\} ds$$

Let $d_1$ and $d_2 > 0$ be such that

$$x(\bar{a}_1 - b_1 x) \leq d_1 - x - d_2 x^2 \quad \text{for any } x \in \mathbb{R}_+. \quad \text{(A.9)}$$

Then we have

$$\mathbb{E}_x \tilde{X}(t) = x + \mathbb{E}_x \int_0^t \tilde{X}(s)[\bar{a}_1 - b_1 X(s)] ds$$

$$= x + d_1 t - d_2 \mathbb{E}_x \int_0^t \tilde{X}^2(s) ds,$$

which leads to

$$\mathbb{E}_x \int_0^t \tilde{X}^2(s) ds \leq \frac{x}{d_2 t} + d_1 \leq \frac{2H}{d_2 t} + d_1, \quad x \in [0, H]. \quad \text{(A.10)}$$

On the other hand, it follows from Itô's formula that

$$\mathbb{E}_x e^{t} \tilde{X}(t) = x + \mathbb{E}_x \int_0^t e^s (X(s) + X(s)(\bar{a}_1 - b_1 X(s))) ds$$

$$\leq x + d_1 \mathbb{E}_x \int_0^t e^s ds$$

$$\leq x + d_1 e^t.$$
which implies
\[ \mathbb{E}_x \tilde{X}(t) \leq xe^{-t} + d_1. \] (A.11)

We have from Itô’s formula that
\[ \mathbb{E}_x \ln(1 + c_2 \tilde{X}(t)) = \ln(1 + c_2x) + \mathbb{E}_x \int_0^t g(\tilde{X}(s), 0)ds. \] (A.12)

In view of (A.11), we have that
\[ \lim_{t \to \infty} \frac{\mathbb{E}_x \ln(1 + c_2 \tilde{X}(t))}{t} = 0 \text{ uniformly for } x \in [0, H]. \] (A.13)

Owing to (A.11) and (A.12), we have
\[ \lim_{t \to \infty} \frac{1}{t} \mathbb{E}_x \int_0^t g(\tilde{X}(s), 0)ds = 0 \text{ uniformly for } x \in [0, H]. \] (A.14)

To proceed, note that the process \( \{\tilde{X}(t)\} \) has exactly two ergodic invariant probability measures on \( \mathbb{R}_+ \): \( \delta_0 \), the Dirac measure concentrated on 0 and \( \mu^* \) on \( (0, \infty) \) (see [12] for the density of \( \mu^* \)), while \( \{\tilde{Y}(t)\} \) admits \( \delta_0 \) as its unique invariant probability measure on \( \mathbb{R}_+ \).

Thus, every invariant probability measures \( \nu \) of \( \{\tilde{X}(t)\} \) has the form \( \nu = \delta \delta_0 + (1 - \delta) \mu^* \) for some \( \delta \in [0, 1] \). Direct calculations show that
\[ \int_{\mathbb{R}_+} f(x, 0) \delta(dx) = p_1a_1 - p_2a_2 > 10\lambda, \]
and
\[ \int_{\mathbb{R}_+} f(x, 0) \mu^*(dx) = p_2 \left( -a_2 + \frac{a_1c_2}{b_1} \right) > 10\lambda. \]

Thus, for any invariant probability measures \( \nu \) of \( \{\tilde{X}(t)\} \), we have
\[ \int_{\mathbb{R}_+} f(x, 0)\nu(dx) > 10\lambda. \] (A.15)

We now prove that there is a \( \tilde{T}_1 > 0 \) such that
\[ \frac{1}{t} \int_0^t \mathbb{E}_x f(\tilde{X}(s), 0)ds \geq 10\lambda \text{ for } t \geq \tilde{T}_1, x \in [0, H]. \] (A.16)

Suppose this claim is false, then we can find \( x_n \in [0, H], t_n > 0 \) such that \( \lim_{n \to \infty} t_n = \infty \) and
\[ \int_0^\infty f(\bar{x}, 0)\pi_{t_n}(d\bar{x}) < 10\lambda. \]

In view of (A.10), the family \( \{\pi_{t_n}, n \in \mathbb{N}\} \) is tight in \( \mathbb{R}_+ \). We can extract a subsequence, still denoted by \( \{\pi_{t_n}\} \), that converges weakly to a probability measure \( \nu \). Since \( \pi_t \) is the empirical measure of the process \( \tilde{X}(t) \), it is well-known that \( \nu \) is an invariant probability
measure on \( \mathbb{R}_+ \) of the process \( \{\tilde{X}(t)\} \). By (A.15) and the uniform integrability (A.10), we must have \( \lim_{n \to \infty} \int_{\mathbb{R}_+} f(\tilde{x},0)n_t^n(x)dx = \int_{\mathbb{R}_+} f(\tilde{x},0)\tilde{\nu}(d\tilde{x}) > 10\lambda \), which contradicts the assumption. On the other hand, since \( \delta_0 \) is the unique invariant probability measure of \( \tilde{Y}(t) \) in \( \mathbb{R}_+ \), similar arguments show that there exists \( \tilde{T}_2 > 0 \) such that (A.8) holds for \( t \geq \tilde{T}_2, y \in [0, H] \). Combining this, (A.14), and (A.16), we obtain the desired results.

\[ \textbf{Lemma A.4.} \] For any \( H > 0 \) and \( T > 0 \), there exists a \( \tilde{\kappa}_T \) depending on \( H \) and \( T \) such that

\[
\mathbb{E}_z |X(t) - \tilde{X}(t)| \leq \tilde{\kappa}_T \sqrt{y}, \tag{A.17}
\]

and

\[
\mathbb{E}_z Y(t) \leq \mathbb{E}_z \tilde{Y}(t) + \tilde{\kappa}_T x \tag{A.18}
\]

for any \( z \in [0, H]^2 \) and for any admissible relaxed control.

\[ \text{Proof.} \]

\[
d[X(t) - \tilde{X}(t)] = [X(t) - \tilde{X}(t)][a_1 - b_1(\tilde{X}(t) + X(t))]dt - c_3X(t)Y(t)dt + [X(t) - \tilde{X}(t)](\sigma_{12}dW_1(t) + \sigma_{22}dW_2(t)).
\]

By Hölder’s inequality and (A.1) and (A.2), we have for any \( s \in [0, T], z \in [0, H]^2 \)

\[
\mathbb{E}_z |(X(s) - \tilde{X}(s))X(s)Y(s)| \leq \left[\mathbb{E}_z (X(s) - \tilde{X}(s))^2\right]^{\frac{1}{2}} \left[\mathbb{E}_z X^4(s)\right]^{\frac{1}{2}} \left[\mathbb{E}_z Y^4(s)\right]^{\frac{1}{2}} \leq \tilde{\kappa}_1 y
\]

for some \( \tilde{\kappa}_1 \) depending only on \( H \) and \( T \). Applying Itô’s formula yields

\[
\mathbb{E}_z [X(s) - \tilde{X}(s)]^2 = \mathbb{E}_z \int_0^s [X(s) - \tilde{X}(s)]^2 \left[2\tilde{\sigma}_1 + a_{11} - 2b_1(\tilde{X}(s) + X(s))\right]ds
\]

\[
- 2c_3 \mathbb{E}_z \int_0^s (X(t) - \tilde{X}(s))X(s)Y(s)ds
\]

\[
\leq (2\tilde{\sigma}_1 + a_{11}) \mathbb{E}_z \int_0^s [X(s) - \tilde{X}(s)]^2 + 2c_3\tilde{\kappa}_1 Ty.
\]

An application of Gronwall’s inequality leads to

\[
\mathbb{E}_z [X(t) - \tilde{X}(t)]^2 \leq 2c_3\tilde{\kappa}_1 Tye^{(2\tilde{\sigma}_1 + a_{11})t}, t \in [0, T].
\]

Subsequently, (A.17) is obtained by applying Hölder’s inequality to the estimate above.

Let \( U(z) \) and \( \tilde{C}_p \) be as in Lemma (A.2) and \( \zeta = \inf \left\{ t \geq 0 : U(Z(t)) \geq \frac{a_{21}}{4} \right\} \). If \( t < \zeta \) then

\[
X(t) \leq \frac{a_{21}}{2}.
\]

Thus, by a comparison theorem, \( \mathbb{P}_z \{ Y(t) \leq \tilde{Y}(t), t \leq \zeta \} = 1 \). By virtue of Itô’s formula,

\[
\mathbb{E}_z U(Z(t \wedge \zeta)) \leq U(z) + \tilde{C}_p \mathbb{E}_z \int_0^{t \wedge \zeta} U(Z(s))ds
\]

\[
\leq U(z) + \tilde{C}_p \mathbb{E}_z \int_0^t U(Z(s \wedge \zeta))ds.
\]
In view of Gronwall’s inequality, we have
\[ \mathbb{E}_z U(Z(t \wedge \zeta)) \leq U(z) e^{\tilde{C}_p t}. \]

Thus, for \( z \in [0, H]^2 \), we have
\[ \mathbb{P}_z \{ \zeta < T \} \leq (1 + c_2 x + c_1 y) x^2 \frac{4 e^{\tilde{C}_p t}}{a_2^2} \leq \tilde{\kappa}_2 x^2 \]
for some \( \tilde{\kappa}_2 \) depending on \( H \) and \( T \). As a result,
\[ \mathbb{E}_z Y(s) ds \leq \mathbb{E}_z \mathbf{1}_{\{ \zeta < t \}} \tilde{Y}(t) + \mathbb{E}_z \mathbf{1}_{\{ \zeta < t \}} Y(t) \]
\[ \leq \mathbb{E}_z \tilde{Y}(t) + (\mathbb{P}_z \{ \zeta < t \}) \mathbb{E}_z [Y(t)]^2 \]

for some \( \tilde{\kappa}_4 \) depending only on \( H \) and \( T \). As a result, \( \mathbb{E}_z f(Z(s)) ds \geq 9 \lambda, t \in [T_1, T_2] \)
when \( z \in [0, H]^2 \) and either \( x \) or \( y \) is sufficiently small. The other claims can be proved similarly. As a result of (A.4), (A.2), (A.7) and (A.17), we have for \( t \in [T_1, T_2] \) that
\[ \frac{1}{t} \int_0^t \mathbb{E}_z f(Z(s)) ds \geq 10 \lambda - \ell (\tilde{\kappa}_T x + \tilde{\kappa}_T y) \]
\[ \geq 9 \lambda \]
when \( z \in [0, H]^2 \) and \( y \) is sufficiently small. Similarly, by (A.4), (A.2), and (A.18), we have
\[ \frac{1}{t} \int_0^t \mathbb{E}_z f(Z(s)) ds \geq f(0, 0) - \frac{\ell}{t} \int_0^t \mathbb{E}_z \| f(0, 0) - f(Z(s)) \| ds \]
\[ \geq f(0, 0) - \frac{\ell}{t} \int_0^t \mathbb{E}_z (X(s) + Y(s)) ds \]
\[ \geq f(0, 0) - \lambda - \ell (\tilde{\kappa}_T x + \tilde{\kappa}_T x) \]
\[ \geq 9 \lambda \]
when \( z \in [0, H]^2 \) and \( x \) is sufficiently small. Similarly, using (A.7), (A.8), and (A.5), we can prove the remaining results of Lemma 4.4.

Proof of Lemma 4.4. We shall show that
Proof of Lemma 3.3. Similar to (1.23), we have
\[
\mathcal{L}uV_2(z) \leq K_5 - \beta |z|^2, \quad \text{for any } z \in \mathbb{R}^2_+.
\]
Thus,
\[
\frac{V_2(Z(t))}{t} \leq \frac{V_2(z)}{t} + K_5 t - \beta \frac{1}{t} \int_0^t |Z(s)|^2 ds + \frac{\widetilde{M}(t)}{t} \tag{A.19}
\]
where
\[
\widetilde{M}(t) = \int_0^t \left[ c_2 X(s)(\sigma_{11}dW_1(s) + \sigma_{12}dW_2(s)) + c_1 Y(t)(\sigma_{12}dW_1(t) + \sigma_{22}dW_2(t)) \right].
\]
By the strong law of large number for martingales,
\[
limit_{t \to \infty} \sup \left( M(t) - \frac{\beta}{2} \int_0^t |Z(s)|^2 ds \right) \leq 0 \quad \text{a.s.} \tag{A.20}
\]
Since \( \liminf_{t \to \infty} \frac{V_2(Z(t))}{t} \geq 0 \), it follows from (A.19) and (A.20) we obtain that
\[
limsup \frac{1}{t} \int_0^t |Z(s)|^2 ds \leq \frac{2K_5}{\beta} \quad \text{a.s.}
\]
Since \( \Phi(yh(y)u) \leq K_6(1+y) \) for some \( K_6 > 0 \), we can easily obtain the desired results: (3.1) and (3.2).

To show the tightness of the family \( \{\eta_\nu : \nu \in \Pi_{RM}\} \) in \( \mathbb{R}^{2,\circ}_+ \), we can derive
\[
\int_{\mathbb{R}^{2,\circ}_+} V(z)\eta_\nu(dz) \leq \frac{C}{1-q}
\]
from the estimate
\[
E_z[V(Z(t))]^\theta \leq e^{(H_1+2)T_2q^{1/(2T_2)}}[V(z)]^\theta + \frac{C}{1-q}
\]
where \( \theta \) and \( q \) are constants in Theorem 4.1. The above estimate can be proved in the same manner as in the proof of Theorem 4.1 with the perturbed Lyapunov function \( V^\varepsilon(z,w) \) replaced with \( V(z) \). Alternatively, it can be shown simply by letting \( \varepsilon \to 0 \) in (4.40). \( \square \)

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