Review of Hadron Structure Calculations on a Lattice

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Thanks for your material!

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Outline

- Lattice QCD Gold-plated Observables
  - nucleon axial charge, e/m radii, magnetic moment, quark momentum fraction and their systematic uncertainties

- Hadron Wave Functions
  - Nucleon and resonance wave functions and distribution amplitudes

- Hadron Form Factors
  - Vector & axial nucleon form factors,
  - Delta axial form factors, Lambda electric form factor
  - timelike vector and scalar pion form factors

- Decomposition of the Proton Spin
  - contributions from light & strange quarks and glue

- Parton Distributions on a Lattice
  - PDFs and TMDs
Lattice QCD Gold-Plated Observables

Isovector (u-d)
• axial charge
• Dirac & Pauli (or electric & magnetic) radii
• magnetic moment
• quark momentum fraction

✦ Best stochastic precision (forward or near-forward kinematics)
✦ No disconnected diagrams
✦ (typically) simple renormalization
✦ Well-known experimentally
Gold-plated observables

Drama of the Axial Charge

\[ \langle N(p) | \bar{q} \gamma^\mu \gamma^5 q | N(p) \rangle = g_A \bar{u}_p \gamma^\mu \gamma^5 u_p , \]

Experiment (W.A.) [PDG'12] \( g_A^{\text{ave}} = 1.2701(25) \)

Many lattice calculations underestimated \( g_A \) by 10-15%
Nucleon Axial Charge: Excited State Effects?

High-statistics study
[S.Dinter et al (ETMC) arXiv:1112.2931]

$$m_\pi \approx 380 \text{ MeV}$$

Variational method
[Ben Owen, et al. (CSSM) Phys.Lett. B 723 (2013) 217]

$$m_\pi \approx 290 \text{ MeV}$$

2-state fits
[H.W.Lin et al (PNDME) arXiv:1306.5435]

“Summation”
[T.D.Rae (CLS-Mainz);
S.Capitani et al, Phys.Rev. D86 (2012) 074502]
Gold-plated observables

Nucleon Axial Charge: Lattice Size Effects?

**Fig. 2**: The pion mass $m_\pi$ as a function of lattice size for two ensembles at $\beta = 5.29$. The solid line shows a fit of eq. (10) to the data. The dashed line shows the NLO result, eq. (7), fit to the smallest mass point.

The pion mass extrapolates indeed to a finite value in the chiral limit, in good agreement with the expected result (8). This also has an effect on $m_\pi$ in the region of small, but nonvanishing, quark masses [25]. We thus expect the finite size correction to be effectively given by

$$m_\pi(L) = m_\pi(\infty) + m_\pi L (1 + \Delta_0)$$ (10)

with the parameter $c(m_\pi)$ rapidly dropping to zero at larger pion masses.

**IV. EXTRAPOLATION TO INFINITE VOLUME**

In the following fits we take $f_0 = 86$ MeV [26]. There is some freedom in the value of the pion mass $m_\pi$ to take in eqs. (5), (6) and (10). We choose $m_\pi = m_\pi(\infty)$ in $\lambda$, $\lambda(y)$ and $c(m_\pi)$, and $m_\pi = m_\pi(L)$ otherwise.

Let us first consider the pion mass. In Fig. 2 we show the fits of eq. (10) to $m_\pi$ for two of our lattice ensembles. The corrections to $m_\pi$ are well described by this equation. Apart from $m_\pi(\infty)$, we have one free parameter, $c(m_\pi)$, only. Equally good fits are obtained for $\beta = 5.40$, $\kappa = 0.13660$ and 0.13640. The parameter $c(m_\pi)$ is found to vanish with a large inverse power of the pion mass.

The finite size corrections predicted by the NLO expression

$$\delta g_A(m_\pi L \approx 2.7) \approx 3(\pm 3)\%$$

**Fit** $g_A(L_s, L_t) = g_A^\infty + B e^{-m_\pi L_s} + C e^{-m_\pi L_t}$

**At** $m_\pi \approx 250$ MeV:

$$g_A(m_\pi L_s = 4) - g_A^\infty = -0.009(54)$$

$$g_A(m_\pi L_t = 4) - g_A^\infty = -0.016(39)$$

**[C.Alexandrou et al, 1303.5979]**

- $N_f=2$ Clover QCDSF $a=0.076, 0.071, 0.06$ fm
- $N_f=2+1$ TMF $a=0.056$ fm

**[R.Horsley et al (QCDSF), 1302.2233]**

- $m_\pi L \approx 2.7$
- $m_\pi \approx 250$ MeV

**Multiple syst.effects?**

$(L_s, L_t)$-dependence with Wilson fermions [J.R.Greene (LHPC), prelim.]
Gold-plated observables

Nucleon Dirac Radius

\[ F^{u-d}(Q^2) \approx F(0) \left[ 1 - \frac{1}{6} Q^2 \langle r_1^2 \rangle^{u-d} + \mathcal{O}(Q^4) \right] \]

ChPT predicts divergence \( \sim \log m_{\pi}^2 \)

Larger \( L_S \), smaller \( Q_{\text{min}}^2 \) are desirable
Gold-plated observables

**Dirac Radius: Excited States**

**2-state fits**

[H.W.Lin et al (PNDME) arXiv:1306.5435]

\[ m_\pi = 220 \text{ MeV} \]

\[ m_\pi = 310 \text{ MeV} \]

**Excited states problem:**
Worse below 200 MeV?

**[T.D.Rae (CLS-Mainz)]**

**[J.R.Green et al (LHPC), 1209.1687]**

**t_{sink} - t_{source} = 0.93, 1.16, 1.39 \text{ fm}**
Radius: Finite Volume Corrections

FVE corrections to nucleon electric radius
[J.M.Hall et al, arXiv:1210.6124 (to appear in PLB)]

\[ (r_1^2)^{u-d}(L_s, L_t) = (r_1^2)^{u-d}(\infty) + Be^{-m_\pi L_s} + Ce^{-m_\pi L_t} \]

\[ \delta (r_1^2)^{u-d} \bigg|_{m_\pi L_s=4} = 0.008(38) \text{ fm}^2 \]
\[ \delta (r_1^2)^{u-d} \bigg|_{m_\pi L_t=4} = 0.003(28) \text{ fm}^2 \]

\( m_\pi \approx 250 \text{ MeV} \)

Fit

\( m_\pi \approx 250 \text{ MeV} \)

\( \text{dependence with Wilson fermions} \) [J.R.Green (LHPC), prelim.]
Anomalous Magnetic Moment

\[ \kappa_v = F_2^{u-d}(Q^2 = 0) \]

Larger \( L_S \), smaller \( Q_{\text{min}}^2 \) are desirable
Gold-plated observables

**Quark Momentum Fraction**

\[
\langle x \rangle_{u-d} = \int dx \ x \ (u(x) + \bar{u}(x) - d(x) - \bar{d}(x))
\]

**Phenomenology:**

\[
\langle x \rangle_{u-d}^{\overline{MS}(2 \ GeV)} = 0.155(5)
\]

\[
\langle N(p) | \bar{q} \gamma_{\{\mu} \bar{D}_{\nu}\} q | p \rangle = \langle x \rangle_{u-d} \bar{u}_p \gamma_{\{\mu} p_{\nu\}} u_p
\]

![Graph showing quark momentum fraction]
Quark Momentum Fraction: Excited States

Gold-plated observables

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[T.D.Rae (CLS-Mainz)]

[J.R.Green et al (LHPC) arXiv:1209.1687]

[S.Dinter et al (ETMC) arXiv:1112.2931]

[S.Collins et al (U.Regensburg), Sec.3B]
(Sub)Summary: Gold-plated observables

- (finally) Exciting developments at the physical pion mass
- Removing excited states is necessary in most cases
- Agreement is reassuring, but much more work is required to ensure quality control.
Hadron Wave Functions

• Wave functions of the Roper state and n=2 radial nucleon excitation

• LC Wave functions (distribution amplitudes) of the nucleon and negative parity excitations
Nucleon & Radial Resonance Wave Functions

Variational method in a basis of 4 nucleon operators

\[ \chi_1^{(S)}(\vec{x}) = \epsilon^{abc} \left[ \left( \bar{u}^{T_a}_{(S)} C \gamma_5 \bar{d}^b_{(S)} \right) \bar{u}^c_{(S)} \right] \vec{x} \]

with varying smearing radius \( S = 0.21, 0.32, 0.54 \) and 0.78 fm and find energy eigenvectors

Calculate w.f. of \( d \)-quark w.r.t. 2 \( u \) quarks:

\[ \chi_1(\vec{x}, \vec{z}) = \epsilon^{abc} \left( u^{T_a}(\vec{x}) C \gamma_5 d^b(\vec{x} + \vec{z}) \right) u^c(\vec{x}) \]

\[ \psi^d_\alpha(\vec{p}, t; \vec{z}) = \text{const} \cdot \sum_{\vec{x}} e^{-i\vec{p} \cdot \vec{x}} \langle \chi_1(\vec{x}, \vec{z}, t) \chi_1^{(S)}(t) \rangle \psi^{(S)}_\alpha \]

Nf=2+1 dynamical \( O(a) \)-improved Wilson fermions, \( m_\pi = 156 \text{ MeV} \)
Nucleon & Radial Resonance Wave Functions

\( m_\pi = 156 \text{ MeV} \)

- \( n = 0 \) Nucleon
- \( n = 1 \) "Roper"
- \( n = 2 \)

[D.Roberts et al (CSSM), arXiv:1304.0325 (to appear in Phys.Lett.B)]
Nucleon and N* Distribution Amplitudes

LC Fock valence state of a Baryon
\[ |N^{(*)}, \uparrow\rangle = \text{const} \int \frac{[dx] \varphi^{(*)}(x_i)}{2\sqrt{24}x_1x_2x_3} \{ |u^\uparrow(x_1)u^\uparrow(x_2)d^\uparrow(x_3)\rangle - |u^\uparrow(x_1)d^\uparrow(x_2)u^\uparrow(x_3)\rangle \} \]

\[ \varphi(x_i; \mu^2) = 120x_1x_2x_3 \left\{ 1 + c_{10}(x_1 - 2x_2 + x_3) \left( \frac{\alpha_S(\mu)}{\alpha_S(\mu_0)} \right)^{\frac{8}{3\beta_0}} + c_{11}(x_1 - x_3) \left( \frac{\alpha_S(\mu)}{\alpha_S(\mu_0)} \right)^{\frac{20}{9\beta_0}} + \ldots \right\} \]

Compute moments of DA on a lattice: \[ \langle O_{\alpha\beta\gamma}(x)|N(0)\rangle \rightarrow \langle \Omega|O_{\alpha\beta\gamma}(x)|N\rangle \]

\{O(x)\} : local 3-quark operators with up to 2 derivatives

\[ \varphi^{lmn} = \int [dx] x_1^l x_2^m x_3^n \varphi(x_1, x_2, x_3) \]

\{c_{1j}, c_{2j}\} \leftrightarrow \{\varphi^{lmn} | l + m + n = 1, 2\}

\[ m_\pi = 290 \text{ MeV}, \ a = 0.072 \text{ fm} \]
Select Hadron Form Factor Results

- Vector form factors of the nucleon
- Axial form factors of the nucleon
- Strange quark contributions to the nucleon form factors
- Axial form factors of Delta(1232)
- Electric form factor of Lambda(1405)
- Timelike vector form factor of the pion
- Scalar form factor and radius of the pion
**Nucleon Vector Form Factors (u-d)**

\[
\langle P + q | \bar{q} \gamma^\mu q | P \rangle = \bar{U}_{P+q} \left[ F_1(Q^2) \gamma^\mu + F_2(Q^2) \frac{i\sigma^{\mu\nu} q_\nu}{2M_N} \right] U_P
\]

- \( m_\pi = 354 \text{ and } 210 \text{ MeV} \)
- \( N_f=2+1+1 \) Twisted mass fermions & earlier works: QCDSF, LHP, RBC
  - [C.Alexandrou et al (ETMC), arXiv:1303.5979]
Nucleon Vector Form Factors (u-d)

\[ \langle P + q | \bar{q} \gamma^\mu q | P \rangle = \bar{U}_{P+q} \left[ F_1(Q^2) \gamma^\mu + \frac{F_2(Q^2)}{2M_N} i\sigma^{\mu\nu} q_\nu \right] U_P \]

\[ m_\pi = 310 \text{ and } 220 \text{ MeV} \]

\[ m_\pi = 354 \text{ and } 210 \text{ MeV} \]

Nf=2+1+1 Twisted mass fermions & earlier works
[C.Alexandrou et al (ETMC), arXiv:1303.5979]

Nf=2+1+1 HISQ + Clover(v) fermions
2-state fits to suppress exc.states
[T.Bhattacharya et al (PNDME)]
Nucleon Vector Form Factors (u-d)

\[ \langle P + q | \bar{q} \gamma^\mu q | P \rangle = \bar{U}_{P+q} \left[ F_1(Q^2) \gamma^\mu + F_2(Q^2) \frac{i \sigma^\mu\nu q_\nu}{2M_N} \right] U_P \]

- \( m_\pi = 310 \) and 220 MeV
- \( m_\pi = 149 \) MeV

\( m_\pi = 354 \) and 210 MeV

Nf=2+1+1 Twisted mass fermions & earlier works
[C.Alexandrou et al (ETMC), arXiv:1303.5979]

Nf=2+1+1 HISQ + Clover(v) fermions
2-state fits to suppress exc.states
[T.Bhattacharya et al (PNDME)]

Nf=2+1 clover-imp.Wilson, “summation” to suppress excited states
[J.R.Green et al (LHPC)]
**Nucleon Axial & Pseudoscalar Form Factors**

\[
\langle P + q | \bar{q} \gamma^\mu \gamma^5 q | P \rangle = \bar{U}_{P+q} \left[ G_A(Q^2) \gamma^\mu \gamma^5 + G_P(Q^2) \frac{\gamma^5 q^\mu}{2M_N} \right] U_P
\]

- **G_A(Q^2)**: Axial form factor
- **G_P(Q^2)**: Pseudoscalar form factor

**Dipole fit**

\[
G_A(Q^2) \sim \frac{A}{(1 + \frac{1}{6}(r_A)^2Q^2)^2}
\]

**Pole fit**

\[
G_P(Q^2) \sim \frac{A}{(m_{pole})^2 + Q^2 + C}
\]

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**Data Points**

- **N_f=2 TMF a=0.089 fm m_\pi=296 MeV**
- **N_f=2+1 TMF a=0.089 fm m_\pi=354 MeV**
- **N_f=2+1 Hybrid a=0.124 fm L=3.5 fm m_\pi=356 MeV**
- **N_f=2+1 Hybrid a=0.124 fm**

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[C.Alexandrou (ETMC), 1303.5979]
Nucleon S-Quark Vector Form factors

\[ G_s^{E,M,A}(Q^2) \]

\[ m_\pi = 416 \text{ MeV} \]

-0.04
-0.02
0
0.02
0.04
0
0.05
0.1
0.15
0.2
0.25
0.3
\[ a_s^2q^2 \]

\[ G_s^{E,M,A}(Q^2) \]

\[ m_\pi = 416 \text{ MeV} \]

-0.04
-0.02
0
0.02
0.04
0
0.05
0.1
0.15
0.2
0.25
0.3
\[ a_s^2q^2 \]

\[ G_s^{E,M,A}(Q^2) \]

\[ m_\pi = 416 \text{ MeV} \]

-0.04
-0.02
0
0.02
0.04
0
0.05
0.1
0.15
0.2
0.25
0.3
\[ a_s^2q^2 \]

Point-split current
Local current

[ R. Babich et al, (DISCO Collab.) Phys.Rev.D85, 054510 ]

\[ |G_s^{E,M,A}| \lesssim 1\% \text{ of } |G_{s u/d}^{E,M,A}| \]

[T. Doi (ChiQCD), 1010.2834]
\[ \langle \Delta^+ | \bar{q} \gamma^\mu \gamma^5 \tau^3 q | \Delta^+ \rangle = -\bar{u}_\sigma \left[ g_{1}(Q^2) \gamma^\mu \gamma^5 + g_{3}(Q^2) \frac{q^\mu \gamma^5}{2M_\Delta} \right] u_\tau + \frac{q^\sigma q^7}{4M_\Delta^2} \left( h_{1}(Q^2) \gamma^\mu \gamma^5 + h_{3}(Q^2) \frac{q^\mu \gamma^5}{2M_\Delta} \right) u_\tau \]
Delta(1232) Axial & Pseudoscalar Form Factors

\[ \langle \Delta^+ | \bar{q} \gamma^\mu \gamma^5 \tau^3 q | \Delta^+ \rangle = -\bar{u}_\sigma \left[ g^{\sigma \tau} \left( g_1(Q^2) \gamma^\mu \gamma^5 + g_3(Q^2) \frac{q^{\mu} \gamma^5}{2M_\Delta} \right) + \frac{q^{\sigma} q^{\tau}}{4M_\Delta^2} \left( h_1(Q^2) \gamma^\mu \gamma^5 + h_3(Q^2) \frac{q^{\mu} \gamma^5}{2M_\Delta} \right) \right] u_\tau \]

\[ \langle \Delta^+ | \bar{q} \gamma^5 \tau^3 q | \Delta^+ \rangle = -\bar{u}_\sigma \left[ g^{\sigma \tau} \tilde{g}(Q^2) \gamma^5 + \frac{q^{\sigma} q^{\tau}}{4M_\Delta^2} \tilde{h}(Q^2) \gamma^5 \right] u_\tau \]
**Λ(1405) Electric Form Factor**

6x6 Variational analysis: 2 octets + 1 singlet \( \otimes \) N=16,100 smearing

\[ G_E(Q^2 = 0.16 \text{ GeV}^2) \]

\[ m_{\pi}^2 [\text{GeV}^2] \]

In \( \Lambda(1405) \leftrightarrow \bar{K}N \), virtual cloud of \( \bar{K} = (s \bar{q}_{\text{light}}) \) enhances \( \langle r^2 \rangle^s \) and shrinks \( \langle r^2 \rangle^{u,d} \)
Timelike Pion Form Factor

$$|\langle \Omega | J_\mu | (\pi^+\pi^-)_{l=1} \rangle|^2 \longrightarrow |F_\pi(t = E_{\pi\pi}^2)|^2$$

[H.B. Meyer, PRL 107:072002(2011); arxiv:1105.1892]

2+1 dyn. Overlap fermions

Experiment

2+1 dyn. Overlap fermions

$$m_\pi = 380 \text{ MeV}$$

$$m_\pi = 290 \text{ MeV}$$

$$m_\pi = 140 \text{ MeV}$$

$$F_\pi(E)$$

$$E [\text{GeV}]$$

Vector-meson dominance fits

[Rom Feng (JLQCD); Poster sessn.]
Scalar Radius of the Pion

\[ F_s(Q^2) = \langle \pi^+(p+q)|m_u\bar{u}u + m_d\bar{d}d|\pi^+(p)\rangle \]

[V.Guelpers, H.Wittig, G.von Hippel]

**Nf=2 O(a)-improved Wilson Fermions**

\[ m_\pi = 300 \ldots 520 \text{ MeV} \]
\[ a = 0.12 \text{ fm} \]

![NLO ChPT](image1)

\[ m_\pi = 280 \ldots 650 \text{ MeV} \]
\[ a = 0.063 \text{ fm} \]

![NNLO ChPT](image2)

Agreement with phenomenology

[Colangelo et al, Nucl.Phys.B603,125] :

\[ \langle r^2 \rangle_s = 0.61(4) \text{ fm}^2 \]

Large disconnected contributions
Origin of the Nucleon Spin

Proton spin puzzle:
1989 EMC experiment finds
\[ \Delta \Sigma = \sum_q (\Delta q + \Delta \bar{q}) = 0.2 \ldots 0.3 \]

Spin sum rule:
\[
J_{\text{glue}} + \sum_q J_q = \frac{1}{2}, \quad J_q = \frac{1}{2} \Delta \Sigma_q + L_q
\]

Quark Spin:
\[
\langle N(p)|\bar{q}\gamma^\mu \gamma^5 q|N(p)\rangle = (\Delta \Sigma_q) [\bar{u}_p \gamma^\mu \gamma^5 u_p]
\]

Angular momentum \((J_q)\):
\[
J_{q,\text{glue}} = \frac{1}{2} \left[A^{q,\text{glue}}_{20}(0) + B^{q,\text{glue}}_{20}(0)\right]
\]

where \(A_{20}, B_{20}\) are E.-M. tensor form factors:
\[
\langle N(p + q)| T_{\mu\nu}^{q,\text{glue}} |N(p)\rangle \rightarrow \left\{A_{20}, B_{20}, C_{20}\right\}(Q^2)
\]

\[
T^{q}_{\mu\nu} = \frac{\bar{q} \gamma\{\mu \vec{D}_\nu\} q}{4} \quad T^{\text{glue}}_{\mu\nu} = G^{a}_{\mu\lambda} G^{a}_{\nu\lambda} - \frac{1}{4} \delta_{\mu\nu} (G_{\mu\nu})^2
\]
Quark Angular Momentum and Spin (Connected)

\[ J_u \approx 40 - 50\% \]
\[ |J_d| \lesssim 10\% \]
\[ |L_{u+d}| \lesssim \frac{1}{2} \Delta \Sigma_{u+d} \]

(*) not including disconnected diagrams!
Disconnected Quark Angular Momentum

[K.F.Liu (ChiQCD), arXiv:1203.6388]

\[ T(q^2), T_2(q^2) \text{ (DI)} \]

\[ \langle x \rangle_{u+d, (DI)} = 0.076(14) \]
\[ 2J_{u+d, (DI)} = 0.072(14) \]

(chiral extrapolation values)

\[ \langle x \rangle_s = 0.024(6) \]
\[ 2J_s = 0.023(7) \]

\[ m_\pi = 478 \ldots 650 \text{ MeV} \]

statistical error is well under control and reliable

Origin of the Nucleon Spin
Gluon Momentum and Angular Momentum

[K.F.Liu (ChiQCD), arXiv:1203.6388]
(Quenched fermions)

Suppress UV fluctuations with the overlap operator:

\[ \hat{G}_{\mu\nu} = \frac{1}{c_T a^2} \text{Tr}_{\text{spin}} [\sigma_{\mu\nu} D_{ov}(x, x)] + O(a) \]

\[ \langle x \rangle_{\text{glue}} = T_1(0) = 0.313(56) \]

\[ 2J_{\text{glue}} = T_1(0) + T_2(0) = 0.254(76) \]

[QCDSF (R. Horsley et al) Phys.Lett.B714:312]
(Quenched fermions)

Background “field”:

\[ S_{\text{gauge}} \rightarrow S_{\text{gauge}} - \lambda a \cdot \left( T_{00} - \frac{1}{3} T_{ii} \right) \]
\[ \frac{1}{2} \left[ (E^a)^2 + (B^a)^2 \right] \]
\[ \frac{1}{2} \left[ - (E^a)^2 + (B^a)^2 \right] \]

\[ \langle x \rangle_{\text{glue}} = - \frac{2}{3m_N} \frac{\partial m_N}{\partial \lambda} \]

\[ m_\pi = 314 \ldots 555 \text{ MeV} \]
Angular momentum: Quenched studies

$2L^q = 2J^q - \Delta \Sigma ^q$

$(\Delta \Sigma )^u_{\text{disc}} = (\Delta \Sigma )^d_{\text{disc}} \approx (\Delta \Sigma )^s_{\text{disc}} \approx -0.12(1)$

$2L_{u+d} \approx 0.49 = 0.0|_{\text{conn}} + 0.49|_{\text{disc}}$
Origin of the Nucleon Spin

**(Disconnected) Light Quarks Spin**

**S. J. Dong et al, '95**
**SESAM '99**
**QCDSF '11**
**ETMC '12**
**S. Meinel '13**
**ETMC '13**

\[
|\Delta q_{(u,d)}^{\text{disc.}}| \lesssim 0.06
\]

**\(m_\pi = 373\) MeV**

[**C. Alexandrou et al (ETMC), 2013**]

**LHPC preliminary**

\[
m_\pi \approx 317\text{ MeV}
\]

**\([S. Meinel '13 (LHPC)]\)**

(Using hierarchical probing)

**K. Orginos 1302.4018**
Strange Quark Spin

Origin of the Nucleon Spin

\begin{align*}
\Delta m^2 &= 0.05(24) \\
\Delta s &= 0.005(24)
\end{align*}

Stochastic estimation of the quark loop

\begin{align*}
m_\pi &= 293 \ldots 495 \text{ MeV} \\
m_\pi &= 285 \text{ MeV} \\
m_\pi &= 416 \text{ MeV}
\end{align*}

\begin{align*}
\text{[M.Engelhardt, Phys.Rev.D86, 114510]} \\
\text{[G.Bali et al (QCDSF) PRL 108, 222001]} \\
\text{[R.Babich et al, (DISCO Collab.) Phys.Rev.D85, 054510]}
\end{align*}

\begin{align*}
\text{Background “field”} \\
S &= S_{\text{SLiNC}} + \lambda \sum_x [\bar{s} \gamma_3 \gamma_5 s]_x \\
\frac{\partial E_H}{\partial \lambda} &= \langle N | \bar{s} \gamma_3 \gamma_5 s | N \rangle \\
\Delta \Sigma^s &= 0.005(24)
\end{align*}

\[\text{[QCDSF, '13]}\]
(Sub)Summary: Nucleon Spin

★ Quark spin from connected contractions agrees with phenomenology

★ Total quark orbital angular momentum is consistent with zero (using only connected data for $J_q$ and $S_q$) although individual $L_u$ and $L_d$ are not zero

★ Older quenched calculations indicate $L_{u+d} \sim 50\%$ (mostly due to disconnected contractions)

★ Newer dynamic fermion calculations yield much smaller values and imply $L_{u+d} \sim 20-30\%$

★ Need update for gluon angular momentum with dynamical fermions
Parton Distribution Functions on a Lattice

1. (TMD) PDFs = Quark-bilinear correlators separated by a light-cone shift

2. Relax the LC condition: slightly spacelike 4-vector $n$

3. Boost the system:
   Spatial separation is suitable for lattice QCD

4. Recover LC physics in $n \cdot P \to \infty$ limit
TMDs from Lattice: Formalism

Transverse momentum-dependent (TMD) parton distributions

\[
\tilde{\Phi}^{[\Gamma]}(x, \vec{b}_\perp; P, S, \ldots) = \int \frac{db^-}{4\pi} e^{ix(b^- P^+)} \frac{\langle P, S| \bar{q}(0) \Gamma U(C_b) q(b) | P, S \rangle}{\{\text{soft factor}\}}
\]

\[
\tilde{\Phi}^{[\Gamma]}(x, \vec{k}_\perp; P, S, \ldots) = \int \frac{d^2 b_\perp}{(2\pi)^2} \tilde{\Phi}^{[\Gamma]}(x, \vec{b}_\perp; P, S, \ldots)
\]

\[C_b \text{ is process-dependent}\]

[M. Engelhardt, B. Mush, A. Shaefe, Ph. Hagler]

Gauge link structure:

In matrix element

\[
\tilde{\Phi}^{[\Gamma]}_{\text{unsbtr.}}(b, P, S, \ldots) \equiv \frac{1}{2} \langle P, S| \bar{q}(0) \Gamma U[0, \ldots, b] q(b) | P, S \rangle
\]

Staple-shaped gauge link \(U[0, \eta v, \eta v + b, b]\)

\[l + N(P) \rightarrow l' + h(P_h) + X\]

LC limit: Collins-Soper parameter

\[\hat{\zeta} = \frac{P \cdot v}{m_N |v|} \rightarrow \infty\]

incorporates SIDIS final state effects
TMDs from Lattice: T-odd momentum shift (1)

x-integrated TMDs (moments) with finite $\overline{b}_T^2 \neq 0$ as an UV-regulator

Sivers Shift: \[
\langle k_y \rangle^{Sivers}(\overline{b}_T^2) \equiv m_N \frac{\overline{f}_1^{[1]}(\overline{b}_T^2)}{\overline{f}_1^{[1]}(0)} \overline{b}_T^2 \rightarrow 0 \int dx \int d^2 k_\perp \cdot k_y \cdot \Phi[\gamma^+](x, k_\perp) \int dx \int d^2 k_\perp \cdot 1 \cdot \Phi[\gamma^+](x, k_\perp)
\]

To compute an x-moment, specify kinematics: $\int dx \rightarrow b \cdot P = 0$

To compute a $k_y$-moment, select Lorentz structure [B.Mush, Phys.Rev.D85, 094510]
TMDs from Lattice: T-odd momentum shift (1)

x-integrated TMDs (moments) with finite $\vec{b}_T^2 \neq 0$ as an UV-regulator

Sivers Shift:

\[
\langle k_y \rangle^{\text{Sivers}} (\vec{b}_T^2) \equiv m_N \frac{\tilde{f}_1^{[2][1]}(\vec{b}_T^2)}{\tilde{f}_1^{[1][0]}(\vec{b}_T^2)} \xrightarrow{\vec{b}_T^2 \to 0} \frac{\int dx \int d^2k_\perp \cdot k_y \cdot \Phi[\gamma^+](x, \vec{k}_\perp)}{\int dx \int d^2k_\perp \cdot 1 \cdot \Phi[\gamma^+](x, \vec{k}_\perp)}
\]

Transverse coordinate dependence

Sivers Shift (SIDIS), $u,d$ – quarks

$\zeta = 0.39$, $m_\pi = 518$ MeV

[M.Engelhardt, B.Mush, A.Shaefer, Ph.Hagler]
TMDs from Lattice: T-odd momentum shift (1)

x-integrated TMDs (moments) with finite $\vec{b}_T^2 \neq 0$ as an UV-regulator

Sivers Shift:

$$\langle k_y \rangle_{\text{Sivers}} (\vec{b}_T^2) \equiv m_N \frac{\tilde{f}_1^{[1](1)} (\vec{b}_T^2)}{\tilde{f}_1^{[1](0)} (\vec{b}_T^2)} \to \int dx \int d^2 k_\perp \cdot k_y \cdot \Phi^{[\gamma^+]}(x, k_\perp)$$

Transverse coordinate dependence

Light-cone limit: $\hat{\zeta} \to \infty$

[M. Engelhardt, B. Mush, A. Shaefer, Ph. Hagler]
TMDs from Lattice: T-odd momentum shift (2)

x-integrated TMDs (moments) with finite $\vec{b}_T^2 \neq 0$ as an UV-regulator

Boer-Mulders Shift:

$\langle k_y \rangle_{BM} (\vec{b}_T^2) \equiv m_N \frac{\tilde{h}_1^{[1][1]}(\vec{b}_T^2)}{\tilde{f}_1^{[1][0]}(\vec{b}_T^2)} \ \vec{b}_T^2 \to 0 \ \int dx \int d^2 \vec{k}_\perp \cdot k_y \cdot \Phi^{[\sigma^{x,0}]}(x, \vec{k}_\perp)$

$\int dx \int d^2 \vec{k}_\perp \cdot 1 \cdot \Phi^{[\gamma]}(x, \vec{k}_\perp)$

**Boer–Mulders Shift**

- total
- contrib. from $\tilde{a}_4$

- up – quarks, $\tilde{\zeta} = 0.39,$ $|b| = 0.36$ fm
- connected only
- partial statistics only

**m$_\pi$$\bar{A}_{4B}/\bar{A}_{2B}$**

- up-quarks
- $\tilde{\zeta} = 1.01$
- $|b| = 0.36$ fm
- $m_\pi = 518$ MeV

[Boer-Mulders Shift: avg. y-momentum of transv. polarized quarks in an unpolarized proton]

**proton**

**pion**

[S. N. Syritsyn]
TMDs from Lattice: T-odd momentum shift (2)

x-integrated TMDs (moments) with finite $\vec{b}_T^2 \neq 0$ as an UV-regulator

Boer-Mulders Shift:

$$\langle k_y \rangle^{BM} (\vec{b}_T^2) \equiv m_N \frac{\tilde{h}_1^{[1][1]} (\vec{b}_T^2)}{\tilde{f}_1^{[1][0]} (\vec{b}_T^2)} \xrightarrow{\vec{b}_T^2 \to 0} \int dx \int d^2 k_\perp \cdot k_y \cdot \Phi^{[\sigma^{x,+}]}(x, \vec{k}_\perp)$$

proton

$$\int dx \int d^2 k_\perp \cdot 1 \cdot \Phi^{[\gamma^+]}(x, \vec{k}_\perp)$$

pion

Transverse coordinate dependence

Boer–Mulders Shift (SIDIS), u–d – quarks

$\zeta = 0.39,$

$m_\pi = 518$ MeV

Results: Boer-Mulders shift

Dependence of SIDIS limit on $|\vec{b}_T|$

$\frac{\tilde{h}_1^{[1][1]} (\vec{b}_T^2)}{\tilde{f}_1^{[1][0]} (\vec{b}_T^2)}$

up-quarks

$\hat{\zeta} = 0$

$\hat{\zeta} = 1.01$

$\hat{\zeta} = 2.03$

Transverse coordinate dependence

$$m_\pi = 518 \text{ MeV}$$

[O.Engelhardt, B.Mush, A.Shaefer, Ph.Hagler]
TMDs from Lattice: T-odd momentum shift (2)

x-integrated TMDs (moments) with finite $\vec{b}_\perp^2 \neq 0$ as an UV-regulator

Boer-Mulders Shift: 

$$
\langle k_y \rangle^{BM} (\vec{b}_T^2) \equiv m_N \frac{\tilde{h}_{1}^{[1](1)} (\vec{b}_T^2)}{\tilde{f}_{1}^{[1](0)} (\vec{b}_T^2)} \xrightarrow{\vec{b}_T^2 \to 0} \int dx \int d^2 k_\perp \cdot k_y \cdot \Phi[^{\sigma^x,+}] (x, \vec{k}_\perp) / \int dx \int d^2 k_\perp \cdot 1 \cdot \Phi[^{\gamma^+}] (x, \vec{k}_\perp)
$$

proton

Boer–Mulders Shift (SIDIS), $u–d$ – quarks

| $m_\pi$ | Value |
|--------|-------|
| 518 MeV | $20^3$ |
| 369 MeV | $20^3$ |
| 369 MeV | $28^3$ |

$|b_T| = 0.36 \text{ fm}$

Light-cone limit: $\hat{\zeta} \to \infty$

pion

$P \sim (1, 0, 0)$

$P \sim (1, 1, 0)$

$P \sim (1, 1, 1)$

Contribution $A_4$ only
PDFs From Lattice: Spatial Quark Correlations

Definition of a parton distribution function:

\[
q(x, \mu) = \int \frac{dx}{4\pi} e^{ix(z-P_+)} \langle P | \bar{q}(z-) \gamma^+ \exp \left[ -ig \int_0^{z-} dt A_+(t) \right] q(0) | P \rangle
\]

Instead, boost the hadron and make gauge link spatial

\[
\tilde{q}(x, \mu, P_z) = \int \frac{dx}{4\pi} e^{ix(zP_z)} \langle P | \bar{q}(z) \gamma^+ \exp \left[ -ig \int_z^z dt A_z(t) \right] q(0) | P \rangle + \mathcal{O}\left( \frac{\Lambda_{\text{QCD}}^2}{P_z^2}, \frac{M^2}{P_z^2} \right)
\]

Equivalent to “static” virtual photon \( q^\mu = (0, \vec{Q}) \) and boosted hadron \( P_z = \frac{Q}{2x} \)

[X.-D. Ji, arXiv:1305.1539]
PDFs From Lattice: Preliminary Results

\[ \int \frac{dx}{4\pi} e^{ix(zP_z)} \langle P | \bar{q}(z) \gamma^\gamma | q(0) | P \rangle \]

\[ m_\pi = 310 \text{ MeV} \quad a = 0.12 \text{ fm} \quad P_z = \frac{2\pi}{L} \{1, 2, 3\} \]

[H.W. Lin, S. Cohen, J.-W. Chen, X. Ji]
PDFs From Lattice: Preliminary Results

\[ \int \frac{dx}{4\pi} e^{ix(zPz)} \langle P|\bar{q}(z)\gamma^\mu \exp \left[ -ig \int_0^z dt A_\mu(t) \right] q(0)|P\rangle \]

[H.W.Lin, S.Cohen, J.-W.Chen, X.Ji]
PDFs From Lattice: Preliminary Results

\[ \int \frac{dx}{4\pi} e^{ix(zP_z)} \langle P|\bar{q}(z)\gamma^{z}\gamma^{5} \exp \left[ -ig \int_{0}^{z} dt A_{z}(t) \right] q(0)|P \rangle \]

\[ \frac{(\Delta u - \Delta d)}{g_{A}} \]

[H.W.Lin, S.Cohen, J.-W.Chen, X.Ji]
PDFs From Lattice: Preliminary Results

\[ \int \frac{dx}{4\pi} e^{i x (z P_z)} \langle P|\bar{q}(z) \sigma^{x,y} \exp \left[ -ig \int_{0}^{z} dt A_z(t) \right] q(0)|P\rangle \]

[H.W.Lin, S.Cohen, J.-W.Chen, X.Ji]
Summary

- Encouraging Hadron Structure results at the physical pion mass: axial charge, radius, vector form factors. Although clearing up systematic effects is still to be done.

- Excited states require close attention: variational methods look most promising.

- Background field methods: potential demonstrated for glue momentum fraction.

- New approach to computing parton distribution functions on a lattice: the first results look promising; theory side needs more work.