Matrix Model for the Transition Expansion of Dispersive Corrections to Nucleon-Nucleus Total Cross Sections

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Abstract

A finite-dimensional matrix model for the nucleon-nucleon cross section operator is used to calculate the dispersive correction to nucleon-nucleus total cross sections, and the leading terms in its expansion in the number of inelastic transitions in the high-energy limit where the longitudinal momentum transfers can be ignored. Results for matrices of different dimensions, but giving the same second and third cross section moments, are compared for scattering from $^{208}$Pb. The leading second order terms in the expansion are accurate to better than 10%, but always larger than the exact result. The 3rd order terms overcorrect, but are nearly cancelled by the 4th order terms. The 5th and 6th order terms are smaller, and also tend to cancel, but not to the degree of the 3rd and 4th order terms. The series converges more rapidly for smaller matrix dimensions, and in each order the magnitudes of the terms increase with the dimension of the matrix.
I Introduction

One consequence of the composite structure of nucleons is a decrease in nucleon-nucleus total cross sections due to transitions between different internal states of the projectile nucleon. This decrease can be calculated \(^1\),\(^2\) using an operator to represent the nucleon-nucleon total cross section, with the matrix elements representing the probability amplitudes for forward scattering transitions between different states of the nucleon.

If the longitudinal momentum transfer due to the different masses of these states cannot be ignored this dispersive correction can only be calculated by expanding in terms of the number of inelastic transitions between states of different mass. Because of the difficulty of this calculation, and uncertainties in the nature of the cross section operator, only the leading term in this expansion has been evaluated \(^3\), and then only approximately. (A closely related effect in quasi-elastic electron scattering can complicate the analysis of color transparency effects in these reactions \(^4\).) In this paper we take a first step toward testing the convergence of this expansion by using finite matrix models for the cross section operator and taking the high energy limit in which the longitudinal momentum transfer vanishes.

Section II reviews the formulas and notation expressing the dispersive corrections to the hadron-nucleus total cross sections in terms of an integral over the generating function for the cross section probability distribution function. The transition expansion is then reviewed in Sect. III, where it is shown that each term in the expansion can be represented as a sum over products of transition amplitudes weighted by a function of the differences among cross sections in the different nucleonic states. In Sect. IV the cross section operator is represented by a finite dimensional matrix depending on two parameters, one of which fixes the second moment of the operator, the other the third moment. With this simple form for the cross section operator one can find relatively simple analytic expressions for the low order terms in the transition expansion. These are evaluated and the results presented in Sect. V. The results are summarized and discussed in the concluding Sect. VI.

II Hadron-Nucleus Cross Sections

In this paper it is assumed that the energy of the incident hadron is high enough that the longitudinal momentum transfers associated with inelastic scattering can be ignored. Nucleon overlap in the target nucleus is also ignored, along with nuclear correlations. For heavy nuclei the total cross section for the scattering of an incident hadron from a nucleus of nucleon number \(A\) is then given by

\[
\sigma(A) = \sigma^G(A) - \sigma^D(A),
\]

where
\[ \sigma^{G,D}(A) = 2 \int d^2b \, G^{G,D}(t(A,b)), \]  
with the Glauber contribution
\[ G^G(t) = 1 - \exp(-t) \]  
and the “dispersive” correction due to diffractive excitation
\[ G^D = \langle 1 | \exp(-\hat{x}t) | 1 \rangle - \exp(-t). \]  
Here the expectation value is in the first, or ground, mass eigenstate of the projectile. The dimensionless thickness function \( t \) at impact parameter \( b \) is given by
\[ t(A,b) = A \sigma T(A,b)/2, \]  
where \( \sigma \) is the projectile-nucleon total cross section and \( T(A,b) \) is the usual thickness function, i.e. the integral of the nuclear density, normalized to unity, along a straight line at constant impact parameter. The dimensionless thickness function decreases from a maximum value of approximately \((3\sigma/4\pi r_0^2)A^{1/3}\) at zero impact parameter to zero for impact parameters well outside the nuclear radius \( R \approx r_0 A^{1/3} \), with \( r_0 \approx 1.14 \text{fm} \). Below the values \( A = 208 \) and \( \sigma = 39.8 \text{mb} \) are used, giving the maximum value of \( t \) as about 4.3. The detailed shape of \( t \) as a function of \( b \) depends on the shape of the nuclear density: below a Woods-Saxon form will be assumed.

In Eqn. 4 the operator
\[ \hat{x} = \hat{\sigma} / \langle 1 | \hat{\sigma} | 1 \rangle \]  
is the dimensionless cross section operator. The matrix elements of the cross section operator \( \hat{\sigma} \) itself give the cross sections for transitions among the various mass eigenstates of the projectile. Thus
\[ \sigma = \langle 1 | \hat{\sigma} | 1 \rangle \]  
is the total cross section for the projectile to interact with a single nucleon, while, assuming all forward amplitudes are pure imaginary, the total forward cross section for diffraction dissociation is
\[ \frac{d\sigma}{dt}|_{t=0} = \pi \sum_{j \neq 1} | j | \hat{\sigma} / 4\pi | 1 \rangle |^2 \]  
\[ = \sigma^2 | 1 | \hat{x}^2 | 1 \rangle - 1 | / (16\pi). \]  
Since there is diffraction dissociation the operators \( \hat{\sigma} \) and \( \hat{x} \) are clearly not diagonal in the space of mass eigenstates. They can, however, be diagonalized using their eigenstates \( | \alpha \rangle \), where
\[ \hat{x}|\alpha\rangle = x_{\alpha}|\alpha\rangle. \quad (10) \]

Then the generating function for the probability distribution, needed to evaluate Eqn. 4, is

\[ <1|\exp(-\hat{x}t)|1> = \sum_{\alpha} P_{\alpha} \exp(-x_{\alpha}t), \quad (11) \]

where

\[ P_{\alpha} = |<1|\alpha>|^2 \quad (12) \]

is the probability of finding the projectile in the scattering eigenstate \(|\alpha\rangle\) when it is in the mass ground state \(|1\rangle\). Of course \(\hat{x}\) will have in general a continuous as well as a discrete spectrum, so that the sum in Eqn. 11 should be interpreted as a sum plus an integral.

In this paper, however, \(\hat{x}\) will be represented by a finite dimensional matrix determined by two parameters, chosen to conform to the following constraints:

1. The off-diagonal elements \(<i|\hat{x}|j>\) should decrease as \(|i-j|\) increases.
2. The diagonal elements \(<j|\hat{x}|j>\) should increase, or at least not decrease, as \(j\) increases.

Both of these requirements are suggested by non-relativistic wave function models: the overlap of the two wavefunctions should decrease with increasing quantum number separation, and the spatial extent of the wavefunctions should increase with increasing quantum numbers. (Related work using finite matrices to describe inelastic scattering can be found in [5, 6, 7].)

Although the longitudinal momentum transfer is ignored in the high energy limit used here, it would be reasonably easy to include it along the lines of [3] if the masses of the excited states were known. This is in fact one of the motivations for developing a finite matrix model.

### III Transition Expansion

The operator \(\hat{x}\) can be separated into two components, one diagonal and the other off-diagonal (inelastic) in the mass eigenstates:

\[ \hat{x} = \hat{x}_d + \hat{x}_I, \quad (13) \]

where

\[ <i|\hat{x}_d|j> \equiv \delta_{ij} <i|\hat{x}|i> . \quad (14) \]

Then, just as in the case of time-dependent perturbation theory,

\[ \hat{U}(t) \equiv \exp(-\hat{x}t) \quad (15) \]
can be expanded in powers of $\hat{x}_I$:

$$\hat{U}(t) = e^{-\hat{x}_I d}[1 + \sum_{n=1}^\infty \hat{U}^{(n)}_I(t)],$$  

(16)

where the $n$th term is given by the $t$-ordered integral

$$\hat{U}^{(n)}_I(t) = (-1)^n \int_0^t dt_n \ldots dt_1 \theta(t_n - t_{n-1}) \ldots \theta(t_2 - t_1) \hat{x}_I(t_n) \ldots \hat{x}_I(t_1),$$  

(17)

with

$$\hat{x}_I(t) \equiv e^{\hat{x}_I d} \hat{x}_I e^{-\hat{x}_I d}.$$  

(18)

The diffractive contribution to $G(t)$ can then be expanded in powers of $\hat{x}_I$, which is equivalent to an expansion in the number of transitions between different mass eigenstates:

$$G^D(t) = \sum_{n=2}^\infty G^{D(n)}(t),$$  

(19)

where

$$G^{D(n)}(t) = e^{-t} < 1|\hat{U}^{(n)}_I(t)|1 >.$$  

(20)

Inserting complete sets of mass eigenstates, this can be written as

$$G^{D(n)}(t) = \frac{((-t)^n e^{-t}/n!)}{\sum_{j_1, \ldots, j_{n-1}} < 1|\hat{x}_I|j_{n-1} > \ldots < j_1|\hat{x}_I|1 >} f^{(n)}(y_{j_{n-1}}, \ldots, y_{j_1}),$$  

(21)

where

$$y_j = (x_{jj} - 1)t$$  

(22)

and the functions $f^{(n)}$ are defined as the ordered integrals

$$f^{(n)}(y_{n-1}, \ldots, y_1) \equiv \frac{n!}{\sum_{j_1, \ldots, j_{n-1}} < 1|\hat{x}_I|j_{n-1} > \ldots < j_1|\hat{x}_I|1 >} e^{-y_n-1(u_n - u_{n-1})} \ldots e^{-y_1(u_2 - u_1)}.$$  

(23)

These symmetric functions are normalized to equal one when all arguments vanish, and are monotonically decreasing functions of each argument.

The leading term in this expansion for $G^D$ is well known and often used to estimate $G^D$, especially when the longitudinal momentum transfers are not negligible:

$$G^{D(2)}(t) = (t^2/2)e^{-t} \sum_{j \neq 1} < 1|\hat{x}|j > < j|\hat{x}|1 > f^{(2)}((x_{jj} - 1)t),$$  

(24)
where
\[
f^{(2)}(y) = 2(e^{-y} - 1 + y)/y^2. \tag{25}
\]
One in principle needs to know all matrix elements of $\hat{x}$ to evaluate $G^{D(2)}$ and higher order terms, although in practice the sum over $j$ may converge rapidly.

If all diagonal matrix elements of $\hat{x}$ are equal then $\hat{x}_d = 1$, all $y_j$s vanish, and the above expressions simplify to
\[
G^{D(n)} = \langle 1|\hat{x}^n|1 \rangle (-(t)^n/e^{-t}/n!), \tag{26}
\]
where $\hat{x}_1 \equiv \hat{x} - 1$ is the shifted operator.

IV Matrix Model

The dimensionless cross section operator $\hat{x}$ determines $G^D(t)$ and its expansion in inelastic transitions (assuming that the mass eigenstates are known). As mentioned above, $\hat{x}$ may in general have a discrete and/or a continuous spectrum. (The parameterizations of $\hat{x}$, for example, are based upon a purely continuous spectrum for $\hat{x}$.) Here it is assumed, however, that this operator can be approximately represented by a two-parameter matrix of finite dimension $N$ in the space of (discrete) mass eigenstates:
\[
\langle i|\hat{x}|j \rangle = \delta_{ij}(j - 1)d + f|^{i-j}l, \tag{27}
\]
so that the diagonal elements $1, 1 + d, 1 + 2d, \ldots$ increase linearly with $i = j$ if $d > 0$ while the off-diagonal elements decrease geometrically with $|i - j|$ if $0 < f < 1$. Since
\[
\langle 1|\hat{x}^2|1 \rangle = 1 + f^2 + f^4 + \ldots + f^{2(N-1)} = (1 - f^{2N})/(1 - f^2), \tag{28}
\]
the parameter $f$ is fixed by $\langle 1|\hat{x}^2|1 \rangle$, which is in turn fixed by forward diffraction dissociation according to Eqn. 9. The second parameter $d$ can then be determined from
\[
\langle 1|\hat{x}^3|1 \rangle = \langle 1|\hat{x}^2|1 \rangle = (2 + d)[f^2 + 2f^4 + \ldots + (n-1)f^{2(N-1)}]. \tag{29}
\]
For any $N$, $f$ and $d$ can therefore be chosen to match the second and third moments of $\hat{x}$ (the only constraint being that $d$ should be non-negative): all higher moments and $G^D(t)$ are then completely determined.

This parameterization of $\hat{x}$ is rather arbitrary and leads to what should be considered as a “toy” model. Its main advantages are that it has enough flexibility to match any desired $\langle x^2 \rangle$ and $\langle x^3 \rangle$ (with the constraint noted above) and that the lower order terms in the transition expansion are relatively easy to evaluate. One can think of many other models with these properties: to choose among the possibilities one would require a rather complete realistic model for the composite structure of the nucleon.
In this model the product of matrix elements in each term in the sum in Eqn. (21) is just the parameter $f$ raised to some integer power. This multiplies a function $f^{(n)}$ which is a simple symmetric analytic function of its $n - 1$ arguments. Starting with $f^{(2)}$, given by Eqn. (25), these functions can be calculated from recursion relations and power series:

$$f^{(n+1)}(y_1, y_2, \ldots, y_n) = -(n + 1)[f^{(n)}(y_2, y_3, \ldots, y_n) - f^{(n)}(y_1, y_3, \ldots, y_n)]/(y_2 - y_1)$$

$$= 1 - (y_1 + y_2 + \ldots + y_n)/(n + 2) + \ldots \quad (30)$$

### V Evaluation for Nucleon-\textsuperscript{208}Pb Total Cross Section

The formulas above can be used to calculate the dispersive corrections to any total cross section and their expansions in terms of the number of inelastic transitions for cross section matrices of any dimension. Here only the example of nucleons on the heavy nucleus \textsuperscript{208}Pb will be considered, taking 39.8 mb for the nucleon-nucleon total cross section $\sigma$ and using a Woods-Saxon density with radius $R = 6.75 fm$ and surface thickness parameter $a_0 = 2.3 fm$ for the \textsuperscript{208}Pb nucleus. (These parameters give an un-corrected Glauber cross section of 3022 mb.) Here the values $< x^2 > = 1.25$, as in Ref. 1, and $< x^3 > = 1.9$, which is close to the smallest value giving non-negative values of the parameter $d$ according to Eqn. 29, are used for the 2nd and 3rd moments. (Ref. 1 takes $< x^3 > = 1.75$ based upon an approximate analysis of nucleon-deuteron inclusive diffractive scattering.)

The exact integrands $4\pi bG_D$ in these models, along with the contributions from terms of order $n = 2, 3, \ldots, 6$ in inelastic scattering, are shown in Figs. 1 and 2 for matrix dimensions $N = 3$ and $N = 6$, respectively. The contributions from different $n's$ alternate in sign, as required by Eqn. 21, with the positive even terms comparable in magnitude to the preceding negative odd terms. This can be understood by considering the lowest order terms in powers of the parameter $f^2$, which is never much larger than 0.2. A simple analysis shows that the leading terms in the $G_D^{(3)}$ and $G_D^{(4)}$ are both of order $f^4$, while the leading terms in $G_D^{(5)}$ and $G_D^{(6)}$ are both of order $f^6$. Furthermore, the larger the dimension of the matrix the more terms are included in the sum in Eqn. 21, and thus the larger the magnitude of $G_D^{(n)}$. The additional terms are in general of higher order in $f^2$, however, so this increase converges for large $N$.

The results for the cross section corrections after integration over impact parameter are shown in Table 1. These results reflect the behavior of the integrands: The $n = 2$ term is accurate to better than 10%, while the 3rd and 4th order terms range from about 20 to 40% of the exact result, but cancel to better than 4%. The 5th and 6th order terms are smaller, and also of opposite sign, but do not cancel to the same degree. The sum of the $n = 2$ through
$n = 6$ terms in the expansion gives results which are accurate to better than 1% for $N=3$, but only to about 5% for $N=6$. (The continuous distribution models of Ref. 1 give somewhat higher values, ranging from 199 to 232 mb, for the dispersive correction to the total nucleon-$^{208}\text{Pb}$ cross sections.)

The calculations above have been repeated for larger values of $\langle x^3 \rangle$ up to 2.5. The exact value for the dispersive correction and the magnitudes of terms in the expansion all increase with the dimension of the matrix and decrease with increasing $\langle x^3 \rangle$ (presumably because the diagonal elements of the matrix become relatively larger as $\langle x^3 \rangle$ and therefore $d$ increase). The near-cancellation of the 3rd and 4th order terms seems to be general, so that including these terms never improves the accuracy significantly.

VI Conclusion

For the particular matrix models used above, the lowest order term in the transition expansion for the diffractive correction to the total cross section gives fairly good accuracy, but is always a bit too large. Including higher order terms does not improve the accuracy significantly, and in fact adding only the next-to-leading 3rd order term always gives poorer accuracy, with a correction less than the exact result.

It would be interesting to repeat these calculations for other matrix models to see how general these results are, and to include the longitudinal momentum transfers which appear for finite energies. (Preliminary calculations indicate that for constant nuclear densities the formulas above can still be used, except that the arguments $y_j$ in Eqn. 21 become complex, with imaginary parts depending on the nucleon's mass spectrum.) Including the longitudinal momenta, for example, might change the phases of the terms in the expansion enough to interfere with the cancellation of the 3rd and 4th order terms.

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FIGURES

FIG. 1 The integrands $4\pi b G^D$ for the dispersive corrections to the nucleon-$^{208}\text{Pb}$ total cross section using a 3-dimensional matrix model as described in the text, with $\langle x^2 \rangle = 1.25$ and $\langle x^3 \rangle = 1.9$. The solid curve is the exact result, the labelled dashed curves the contributions of order 2 through 6 in inelastic transitions.

FIG. 2 The same as FIG. 1, but using a 6-dimensional matrix model.

TABLE

TABLE 1 The dispersive corrections to the nucleon-$^{208}\text{Pb}$ total cross sections for matrix models of different dimensions, including the exact results, the contributions from orders 2 through 6 in inelastic transitions, and the sums of these five contributions.

| Matrix Dimension | Dispersive Corrections to Total Cross Sections (mb) | Order in Inelastic Transitions | Sum |
|------------------|---------------------------------------------------|--------------------------------|------|
|                  | Exact                                             | 2 | 3  | 4  | 5  | 6  |      |
| 3                | 137.64                                            | 143.41 | -30.37 | 29.46 | -8.51 | 4.58 | 138.56 |
| 4                | 143.08                                            | 154.82 | -47.13 | 47.75 | -20.80 | 11.80 | 146.44 |
| 5                | 145.18                                            | 158.53 | -53.21 | 57.02 | -29.60 | 18.37 | 151.12 |
| 6                | 145.82                                            | 159.58 | -54.95 | 60.29 | -33.61 | 22.13 | 153.43 |
Figure 1

Integrand (fm)

$N=3$

$b$ (fm)

Figure 1
Figure 2

Integrands (fm)

N=6

b (fm)

Figure 2