A FUNCTIONAL CENTRAL LIMIT THEOREM FOR TRIANGULAR ARRAYS VIA VARIANCE REGULARIZATION WITH CLOSE TO OPTIMAL MIXING RATES

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Abstract. We obtain a functional central limit theorem for sums of triangular arrays satisfying some mixing conditions. The main innovation is that the results are obtained without the standard assumption that the sum of the individual variances has the same order as the variance of the underlying sum. The price for omitting this assumption is that we assume slightly stronger mixing conditions and some growth assumptions (like non-uniform boundedness). The (essentially) optimal mixing condition is \( \sum_n \rho(2^n) < \infty \) (see [11, Theorem 4.1]) and, instead, we assume that \( \sum_n \rho(e^{G(n)}) < \infty \) where \( G(n) \) grows sub-linearly (e.g. \( G(n) = n/\ln(\ln n) \)) and \( \rho(\cdot) \) are the maximal correlation (mixing) coefficients. The proof is based on a reduction to the case that the standard aforementioned condition holds true, which is carried out by a new type of decomposition of the underlying sums into blocks which are “governed” by an essentially arbitrary sub-exponential sequence \( a_j = e^{G(j)} \). We will also discuss results for sufficiently fast strongly (i.e. \( \alpha \)) mixing arrays, whose main purpose is to avoid the assumption in [12, Corollary 2.2], that the sum of the squares of the individual \( \| \cdot \|_{2+\delta} \)-norms has the same order as the variance of the sum. These results will be proven using a reduction to the conditions of [12 Corollary 2.2].

1. Introduction

Limit theorems for partial sums \( S_n = \sum_{i=1}^n \xi_i \) of real-valued weakly stationary mixing sequences \( \{\xi_i\} \) are well studied. The central limit theorem (CLT) states that if \( V_n = \text{Var}(S_n) \to \infty \) then \( \hat{S}_n = (S_n - E[S_n]) / \sigma_n, \sigma_n = \sqrt{V_n} \) converges in distribution to the standard normal law. Under quite general mixing conditions, the CLT goes back to Rosenblatt [18], whose proof was based on the “big block-small block” technique due to Bernstein. Since then, variations of the CLT for weakly stationary sequences were studied in different setups under various authors, using other methods like martingale approximation.

For nonstationary mixing sequences, or, more generally, triangular arrays \( \{\xi_{1,n}, ..., \xi_{n,n}\} \) the CLT itself was studied under various conditions in [19] [13] [6] [21] [10] [15] [16] [17]. In comparison with the stationary case, the main difficulty in the non-stationary case is that the variance \( V_n \) can diverge at an arbitrary, non-linear, rate. For sufficiently well contracting triangular arrays of Markov chains the CLT for \( \hat{S}_n \) was established in the 50’s by R. Dobrushin [5], where it was shown that \( S_n \) can actually be represented as a reverse martingale. We refer to [20] for a more modern presentation of Dobrushin’s CLT and to [14] for the CLT under weaker contraction assumptions.

A more refined result is the functional central limit theorem. In the stationary case it asserts that the random function \( W_n(t) = \hat{S}_{nt} \) converges in distribution towards a Brownian motion. Probably the most common method of proof is the martingale approximation technique due to

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Gordin [8]. While the functional CLT for non-stationary martingales is well understood (see, for instance [11]), the are far less results about martingale approximation for nonstationary mixing sequences (or triangular arrays). The first thing to observe is the nonstationary setup is that the covariance function \( b_n(t,s) = \text{Cov}(\hat{S}_{nt}, \hat{S}_{ns}) \) of the random function \( \hat{S}_{nt} \), where \( \hat{S}_n = (S_n - \mathbb{E}[S_n]) / \sigma_n \), does not converge towards the covariance function of a Brownian motion. Hence, in the nonstationary setup the random functions on \([0,1]\) under consideration are given by \( W_n(t) = \hat{S}_{v_n(t)} \), where \( v_n(t) = \min\{1 \leq k \leq n : \text{Var}(S_{k,n}) \geq tV_n\} \) and \( S_{k,n} = \sum_{j=1}^{k} \xi_{j,n} \).

Very recently there was major progress on the functional CLT for triangular arrays. In [11] Merlevède, Peligrad and Utev answered a question raised by Ibragimov proving that the functional CLT holds if the array satisfies the classical Lindeberg condition and, in addition,

\[ \sum_{j=1}^{n} \text{Var}(\xi_{j,n}) = O(\text{Var}(S_n)) \]

and the \( \rho \)-mixing coefficients (defined by (2.1)) satisfy

\[ \sum_{n} \rho(2^n) < \infty. \]

These result where obtained by a new martingale approximation technique.

In [12] Merlevède and Peligrad applied the martingale approximation techniques developed in [11], and showed (in particular) that the functional CLT holds if the \( \alpha \)-mixing coefficients (defined by (2.5)) satisfy \( \sum_{n} n^{2/\delta} \alpha(n) \) for some \( \delta > 0 \) so that

\[ \sum_{j=1}^{n} \|\xi_{j,n} - \mathbb{E}[\xi_{j,n}]\|_{2+\delta}^2 = O(\text{Var}(S_n)). \]

The goal of this paper is to obtain the functional CLT without condition (1.1) for \( \rho \)-mixing arrays and without condition (1.3) for \( \alpha \)-mixing arrays. In the \( \rho \)-mixing case we will show that there is a “trade-off” between conditions (1.1) and (1.2) and certain growth assumptions and sub-exponential versions of (1.2). For instance, the following theorem will follow from our general results.

1.1. Theorem. Let \( \{\xi_{j,n} : 1 \leq j \leq n\} \) be a triangular arrays which is uniformly bounded in \( L^2 \). Suppose that \( \sum_{n} \rho(\epsilon^{n/n}) < \infty \). Let \( \phi_n \) be the \( \phi \)-mixing coefficients (defined in (2.4)) of the finite sequence \( \{\xi_{j,n} : 1 \leq j \leq n\} \). Suppose that

1. \( \|\xi_{j,n}\|_{L^\infty} \leq K_n; \)
2. \( \phi_n(m_n) < \frac{1}{2} - \epsilon, \)

for some \( \epsilon > 0 \) and two sequences \( K_n \) and \( m_n \) such that \( K_n m_n = O((\ln \ln \sigma_n)^{1/2-\delta}) \) for some \( \delta \in (0, \frac{1}{2}) \). Then \( W_n(\cdot) \) converges in distribution to a standard Brownian motion.

We also obtain similar results when \( \sum_{n} \rho(\epsilon^{n}) < \infty \) for some \( \alpha \in (0,1) \) and \( \sum_{n} \rho(n^q) < \infty \) for some \( q > 0 \), where in these cases we can consider faster growth rates for \( K_n m_n \). We note that formally we assume that the array is uniformly bounded in \( L^2 \), but by dividing by \( s_n = 1 + \max\{\|\xi_{j,n}\|_2 : 1 \leq j \leq n\} \) we can always reduce to that case. By approximating \( \xi_{j,n} \) by bounded variables we also obtain a functional CLT for unbounded random variables (under certain growth assumptions). We also obtain functional CLT’s for \( \alpha \)-mixing arrays.

1.1. Outline of the proof: variance regularization. We will describe the proof in the \( \rho \)-mixing case, and the description of the proof for \( \alpha \)-mixing arrays is similar. Let \( a_j = e^{G(j)} = e^{J(j)} \) be a sequence of sub-exponential growth so that \( \sum_{j} \rho(a_j) < \infty \). The strategy of the proof of Theorem 1.1 and its more general versions is to decompose the sum \( S_n = \sum_{k=1}^{n} \xi_{k,n} \) into
blocks \(X_{j,n} = \sum_{k \in I_{j,n}} \xi_{k,n}\) so that \(S_n = \sum_{j=1}^{u_n} X_{j,n}, I_{j,n} = [x_{j,n}, y_{j,n}]\) is to the left of \(I_{j+1,n}\), and the blocks are “regular” (w.r.t. \((a_j)\)) in the sense that

\[
A_1 a_j \leq \|X_{j,n}\|_{L^2} \leq \max_{k \in I_{j,n}} \left| \sum_{\ell = x_{j,n}}^{x_{j,n}} \xi_{\ell,n} \right|_{L^2} \leq A_2 a_j,
\]

and

\[
\sigma_n^2 = \text{Var}(S_n) \asymp \sum_{j=1}^{u_n} a_j^2.
\]

Once we show that such decomposition exists, we would like to apply [11, Theorem 4.1] (discussed above). However, this requires \(X_{j,n}\) to satisfy the classical Lindeberg condition. Furthermore, once the functional CLT is established for the new array \(\{X_{j,n} : 1 \leq j \leq u_n\}\), we need to show that it implies the functional CLT for the original array \(\{\xi_{j,n} : 1 \leq j \leq n\}\).

It turns out that this task requires a certain “(weak) Lindeberg condition of maximal type”: set \(X_{j,n} = \max_{k \in I_{j,n}} \left| \sum_{\ell = x_{j,n}}^{x_{j,n}} \xi_{\ell,n} \right|\). Then the functional CLT for \(\{\xi_{j,n}\}\) will follow from the functional CLT for \(\{X_{j,n}\}\) if for every \(\varepsilon > 0\),

\[
\lim_{n \to \infty} \sigma_n^{-1} \sum_{j=1}^{u_n} \mathbb{E}[|X_{j,n}| I(|X_{j,n}| \geq \sigma_n \varepsilon)] = 0.
\]

The last step of the proof would be to verify both Lindeberg conditions. This is done by a recent maximal moment inequality due to Merlevéde, Peligrad, and S. Utev ([10, Theorem 6.17]) which holds true in quite a general non-stationary setting, and it yields appropriate upper bounds on \(\|X_{j,n}\|_p/a_j\).

2. Preliminaries and main results

Let \(\{\xi_{j,n}, 1 \leq j \leq n\}\) be a triangular array of real-valued, square integrable centered random variables defined on a common probability space. Let \(S_{k,n} = \sum_{j=1}^{k} \xi_{j,n}, S_n = S_{n,n}\) and set \(\sigma_n = \|S_n\|_{L^2} = \sqrt{\text{Var}(S_n)}\). We assume here that

\[
\lim_{n \to \infty} \sigma_n = \infty
\]

and that\(^1\)

\[
\gamma_n := \max_{j} \|\xi_{j,n}\|_{L^2} = o(\sigma_n).
\]

For each \(t \in [0,1]\), set

\[
v_n(t) = \min\{1 \leq k \leq n : \sigma_{k,n}^2 \geq t \sigma_n^2\},
\]

where \(\sigma_{k,n}^2\) is the variance of \(\sum_{j=1}^{k} \xi_{j,n}\). Consider the random function

\[
W_n(t) = \sigma_n^{-1} \sum_{j=1}^{v_n(t)} \xi_{j,n} = \sigma_n^{-1} S_{v_n(t),n}
\]

on \([0,1]\). Then \(W_n(\cdot)\) is a random element of the Skorokhod space \(D[0,1]\). In this paper we will prove that \(W_n(\cdot)\) converges in \(D[0,1]\) towards a Brownian motion, under a variety of mixing

\(^1\)This is a classical assumption which means that the individual summands are of smaller \(L^2\)-magnitude than the sum itself.
assumptions. The first thing to notice is that $W_n(\cdot)$ is left invariant after replacing $\xi_{j,n}$ by $b_n \xi_{j,n}$, where $b_n > 0$ is a constant depending on $n$. Therefore, we assume here that

$$K := \sup_n \max_{j \leq n} \| \xi_{j,n} \|_{L^2} \leq 1.$$ 

This assumption means that in order to translate the results presented in this paper to the more general case one needs to replace $\xi_{j,n}$ with $\xi_{j,n}/\gamma_n$ and $\sigma_n$ with $s_n := \sigma_n/\gamma_n \to \infty$.

2.1. $\rho$-mixing arrays. Since we will have no restrictions on the joint distribution of $\xi_{j,n}$ and $\xi_{j',n'}$ for $n \neq n'$ we assume here that $\xi_{j,n}$ are defined on a joint probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For any two sub-$\sigma$-algebras $\mathcal{G}$ and $\mathcal{H}$ of $\mathcal{F}$ let

$$\rho(\mathcal{G}, \mathcal{H}) = \sup \{ |E[fg]| : f \in B_{0,2}(\mathcal{G}), g \in B_{0,2}(\mathcal{H}) \}$$

where for any sub-$\sigma$-algebra $\mathcal{G}$, $B_{0,2}(\mathcal{G})$ denotes the space of all square integrable functions $g$ so that $E[g] = 0$ and $E[g^2] = 1$. Then the $k$-th $\rho$-mixing coefficient of the array $\xi$ is defined by

$$(2.1) \quad \rho(k) = \sup_n \sup \{ \rho(\mathcal{F}_n(s), \mathcal{F}_n(s + k, n)) : s \leq n - k \}$$

where $\mathcal{F}_n(s)$ is the $\sigma$-algebra generated by $\xi_{j,n}$ for $j \leq s$ and $\mathcal{F}_n(s + k, n)$ is the $\sigma$-algebra generated by $\xi_{j,n}$ for $s + k \leq j \leq n$. Let us define $\rho(x) = \rho([x])$ for all $x > 0$.

Next, for any two sub-$\sigma$-algebras $\mathcal{G}$ and $\mathcal{H}$ of $\mathcal{F}$ we set

$$\phi(\mathcal{G}, \mathcal{H}) = \sup \{ |\mathbb{P}(B\mid A) - \mathbb{P}(B) | : A \in \mathcal{G}, B \in \mathcal{H}, \mathbb{P}(A) > 0 \}.$$ 

Then for a fixed $n$, the $\phi$-mixing coefficients of the finite sequence $\{\xi_{j,n} : 1 \leq j \leq n\}$ are given by

$$(2.2) \quad \phi_n(k) = \sup \{ \phi_n(\mathcal{F}_n(s), \mathcal{F}_n(s + k, n)) : s \leq n - k \}.$$ 

We assume here that for all $n$ large enough we have

$$(2.3) \quad \phi_n(m_n) < \frac{1}{2} - \varepsilon \text{ for some } m_n < n \text{ and a constant } \varepsilon > 0.$$ 

We will formulate soon a general result, whose main application is the following result:

2.1. Theorem. Assume that $\sigma_n \to \infty$ and that $\frac{\max_{1 \leq k \leq n} \| \xi_{k,n} \|_{\infty}}{K_n}$ for some sequence $K_n$. Then $W_n(\cdot)$ converges in distribution to a standard Brownian motion in the following cases:

1. $\sum_n \rho(qn^s) < \infty$ for some $q > 0$ and $m_n K_n = \sigma_{\frac{2}{1+s} - \varepsilon}$.
2. $\sum_n \rho(e^{(\ln n)^{1/\delta}}) < \infty$ for some $s > 1$ and $m_n K_n = \sigma_{\frac{1}{1+s} - \varepsilon}$ for some $\delta > 0$, where $w_n = (\ln \sigma_n)^{(1-s)/2}$.
3. $\sum_n \rho(e^{\alpha \ln n}) < \infty$ for some $\alpha \in (0, 1)$ and $m_n K_n = \sigma_{\frac{1}{1+s} - \varepsilon}$ for some $\delta > 0$.
4. $\sum_n \rho(e^{\alpha \ln n}) < \infty$ and $m_n K_n = \sigma_{\frac{1}{1+s} - \varepsilon}$ for some $\delta > 0$.

2.2. Remark.

1. the mixing conditions in (i) hold true when $\rho(n) \leq n^{-\varepsilon}$ for some arbitrary small $\varepsilon > 0$.
2. the mixing conditions in (ii) hold true if $\rho(n) \leq e^{-1+\varepsilon \ln n^{1/\delta}}$ for some $s > 1$ and $\varepsilon > 0$.
3. the mixing conditions in (iii) hold true when $\rho(n) \leq (\ln n)^{-s}$ for some $s > 1$.
4. the mixing conditions in (iv) hold true when $\rho(n) \leq \frac{1}{\ln n \ln \ln n}$ for some $s > 2$.

The assumption about $m_n K_n$ holds when both $m_n$ and $K_n$ are bounded, but when $\sigma_n$ satisfies some growth assumptions then one can consider more concrete growth rates for $K_n m_n$. 
2.3. Remark. Let $\phi(k) = \sup_n \phi_n(k)$ be the $\phi$-mixing coefficients of the array $\xi$. Then by [9, Lemma 1.17],
\begin{equation}
\rho(j) \leq 2\sqrt{\phi(j)}.
\end{equation}
Hence we can obtain the functional CLT when the mixing rates in Theorem 2.1 hold true for $\phi(k)$ in place of $\rho(\cdot)$ (which, in particular, implies (2.3)).

We would also like to remark that by approximating $\xi_{j,n}$ by bounded random variables we also obtain a functional CLT under certain type of growth assumptions (see Propositions 5.1 and 5.0).

2.1.1. The general results. Theorem 2.1 follows from the more technical results presented in this section. Let us first consider the following two properties, which for the sake of convenience are presented as definitions.

2.4. Definition. We say that a sequence $(a_j)$ diverges sub-exponentially fast in a good way if $a_j \asymp e^{G(j)}$ for some positive differentiable function $G$ on $[1, \infty)$ so that:

1. $\lim_{x \to \infty} G(x) = \infty$;
2. there are constants $C_1, C_2, c > 0$ and a positive function $H : [1, \infty) \to \mathbb{R}$ so that $\lim_{u \to \infty} H(u) = 0$ and for every $u \geq c$ we have $C_1 H(u) \leq G(x) \leq C_2 H(u)$;
3. there are constants $\varepsilon, \delta \in (0, 1)$ so that for all $u$ large enough we have $e^{G(u)} > (1 - \varepsilon)e^{G(u(1 - \delta))}$.

2.5. Example (Main examples).

1. $a_j = e^{j^\varepsilon}$, with $H(u) \asymp \frac{1}{u}$.
2. $a_j = e^{(\ln j)^s}$ for some $s > 1$, with $H(u) \asymp \frac{(\ln u)^{s-1}}{u}$.
3. $a_j = e^{c j^\alpha}$ where $c > 0$ and $\alpha \in (0, 1)$, with $H(u) \asymp u^{\alpha - 1}$.
4. $a_j = e^{g(j)}$, where $g(j) = \frac{1}{\ln^{d+\delta}(j)}$ for some $d \geq 1$, where $f^{d,\varepsilon} = f \circ f \circ f \circ \ldots \circ f$ ($d$ times).

In this case $H(u) \asymp \frac{1}{\ln^{d,\varepsilon}(u)}$.

Definition 2.4 is included here in order not to treat each one of the cases described in Theorem 2.1 separately (later on we will have certain assumptions which depend on the decay rate of $H(u)$).

2.6. Definition (Regular blocks). Given a sequence $(a_j)$ of positive numbers and two positive constants $C_1 < C_2$, we say that a family of partitions of $\{1, 2, \ldots, n\}$ into intervals (blocks) in the integers

$I_{j,n} = [x_{j,n}, y_{j,n}] \cap \mathbb{N}, j \leq u_n$

is regular (w.r.t $\{\xi_{j,n}\}, (a_j)$, $C_1$ and $C_2$) if

1. $I_{j,n}$ is to the left of $I_{j+1,n}$;
2. for every $1 \leq s_1 < s_2 \leq u_n$, with $X_{j,n} = \sum_{k \in I_{j,n}} \xi_{k,n}$, we have $C_1 a_j \leq \|X_{j,n}\|_{L^2} \leq \max_{k \in I_{j,n}} \left\| \sum_{\ell = x_{j,n}}^k \xi_{\ell,n} \right\|_{L^2} \leq C_2 a_j$;
Lindeberg condition of maximal type if every $\varepsilon > 0$.

Remark. Since $\sigma^2_n \asymp \sum_{j=1}^{u_n} a_j^2$, when $a_j \asymp j^q$ we get $u_n \asymp \frac{\ln n}{q}$. In all the other three special cases, we have

$$\sigma^2_n \asymp \int_{(1-\delta)u_n}^{u_n} e^{2G(x)} = \int_{(1-\delta)u_n}^{u_n} e^{2G(x)}G'(x)/G'(x) \asymp H(u_n)^{-1}e^{2G(u_n)}.$$}

Thus:

1. when $a_j \asymp e^{(\ln j)^{\varepsilon}}$ then $u_n \asymp e^{(\ln \sigma_n)^{1/\varepsilon}}$;
2. when $a_j \asymp e^{j^{\varepsilon}}$ then $u_n \asymp (\ln \sigma_n)^{1/\varepsilon}$;
3. when $a_j \asymp e^{j/\ln^{1-\varepsilon}(j)}$ then $u_n$ is at least of order $\ln(\sigma_n)$ and at most of order $\ln(\sigma_n)^{\ln^{1/\varepsilon}(\sigma_n)}$.

Our main (general) result for $\rho$-mixing arrays is as follows.

2.8. Theorem. Suppose that $\sigma_n \to \infty$. Let $a_j = e^{Q(j)}$ diverge sub-exponentially fast in a good way. Suppose that $\sum_j \rho(a_j) < \infty$. Then are absolute constants $C_1, C_2 > 0$ for which one can construct regular blocks $X_{j,n}, j \leq u_n$ w.r.t. $(\xi_{j,n}, (a_j), C_1$ and $C_2$. Moreover, if the maximal Lindeberg condition holds along the blocks $X_{j,n}$, then $W_n(t)$ converges in distribution towards a standard Brownian motion.

Once Theorem 2.8 is proven, to get the functional CLT we need to verify the maximal Lindeberg condition along the blocks $X_{j,n}$. In Section 5.1 we will discuss several conditions which insure that this Lindeberg condition holds. Then Theorem 2.1 will follow from these general results together with an application of the recent maximal inequality [11, Theorem 6.17].

2.2. $\alpha$ mixing arrays. For any two sub-$\sigma$-algebras $G$ and $H$ of $F$ let

$$\alpha(G, H) = \sup\{||P(A \cap B) - P(A)P(B)|| : A \in G, B \in H\}.$$}

Then the $k$-th $\alpha$-mixing coefficient of the array $\xi$ is defined by

$$\alpha(k) = \sup_n \sup_s \{\alpha(\mathcal{F}_n(s), \mathcal{F}_n(s+k, n)) : s \leq n-k\}$$}

where $\mathcal{F}_n(s)$ is the $\sigma$-algebra generated by $\xi_{j,n}, j \leq s$ and $\mathcal{F}_n(s+k, n)$ is the $\sigma$-algebra generated by $\xi_{j,n}, s+k \leq j \leq n$.

Our main result for $\alpha$-mixing arrays is as follows:

2.9. Theorem. Let

$$\delta_n(m) = \sup_k \sum_{s=m}^{n-k} ||E[\xi_{k+s, n}\xi_{k,n}, ..., \xi_{k,n}]||_{L^2}$$
and suppose that $\delta_n(r) < \frac{1}{4}$ for some $r$ and all $n$ large enough. Then there are absolute constants $C_1, C_2$ for which it is possible to construct regular blocks $X_{j,n}$ corresponding to any constant sequence $a_j = a$ with $a > 0$ large enough. Moreover, the functional CLT holds true under the following two additional conditions:

1. the maximal Lindeberg condition holds along the blocks $X_{j,n}$;
2. for some $\delta > 0$ we have

$$\sup_{j,n} \|X_{j,n}\|_{2+\delta} < \infty$$

and $\sum_n n^{2/\delta} \alpha(n) < \infty$

2.10. Remark. In Lemma 3.2 we show that the condition about $\delta_n(r)$ holds when $\xi_{j,n}$ are uniformly bounded in $L^q$ for some $q > 2$ so that $\sum_{j=1}^{\infty} (\alpha(j))^{1/2 - 1/q} < \infty$.

To complete Theorem 2.9 we have the following proposition, which is a consequence of [11, Theorem 6.17].

2.11. Proposition. The maximal Lindeberg condition and (2.6) hold true when (2.3) holds with a bounded $m_n$ and $p = 2 + \delta$, and the array is uniformly bounded.

A short comparison with the results in [12]. In [12] F. Merlevède and M. Peligrad proved a functional CLT for $\alpha$-mixing arrays, with weaker $\alpha$-mixing coefficients than the strong mixing coefficients considered in the paper. In [12, Corollary 2.2] they assumed that

$$\sum_{j=1}^{n} \|\xi_{j,n}\|_{2+\delta}^2 = O(\sigma_n^2)$$

and $\sum_n n^{2/\delta} \tilde{\alpha}(n) < \infty$, where $\tilde{\alpha}$ are the weaker $\alpha$-mixing coefficients. As demonstrated in the applications of [12, Corollary 2.2], the condition (2.7) entails a certain “structural” assumption on $\xi_{j,n}$. For uniformly bounded sequences, we show that under close mixing rates (but for the uniform mixing coefficients), the functional CLT holds if we replace the assumption (2.7) with the assumption that $\phi_n(m_0) < \frac{1}{4} - \varepsilon$ for some $m_0 \in \mathbb{N}$ and $\varepsilon > 0$ (so there is a certain “trade off” between condition (2.7) and using strong $\alpha$-mixing coefficients instead of weak ones).

3. Regular blocks

3.1. Construction of regular blocks using $\rho$-mixing coefficients. In this section we fix $n$ and omit the subscript $n$, and just write $\xi_{j,n} = \xi_j$. We also set

$$S_k = \sum_{j=1}^{k} \xi_j$$

and for each $B \subset \{1, 2, ..., n\}$ we write

$$S(B) = \sum_{j \in B} \xi_j.$$
The blocks: Let us fix some $A > 1$. Let $b_1$ be the first index $b$ so that $\|S_{b_1}\|_{L^2} \geq Aa_1$. Then
$$A\|S_{b_1}\|_{L^2} \leq Aa_1 + 1$$
where we have used our assumption that $\|\xi\|_{L^2} \leq 1$. Next, let $b_2$ be the first index $b$ so that $b \geq b_1 + a_j$ and $\|S_{b_2}\|_{L^2} \geq Aa_2$. Then
$$\|S_{b_2}\|_{L^2} \leq Aa_2 + 1.$$ Continuing this way we get blocks of the form $B_j = \{b_j, b_j + 1, \ldots, b_j + d_j - 1\}$, $j \leq u_n$ so that
1. $Aa_j \leq \|S(B_j)\|_{L^2} \leq Aa_j + 1$;
2. $B_j$ to the left of $B_{j+1}$;
3. the distance between $B_j$ and $B_{j+1}$ is $a_j$.

Let us also add $\{b_{u_n} + d_{u_n}, \ldots, n\}$ to the last block, and let us denote by $D_j$ the gap (interval) between $B_j$ and $B_{j+1}$. Set
$$Y_j = S(B_j), Z_j = S(D_j), X_j = Y_j + Z_j.$$ Then $\|Z_j\|_{L^2} \leq a_j$ and
$$S_n = \sum_{j=1}^{n} \xi_j = \sum_{j=1}^{u_n} X_j.$$ 

3.1. Remark. $B_j, D_j, Y_j, Z_j$ and $X_j$ depend on $n$.

Our main result here is the following proposition.

3.2. Proposition. Suppose that $\sum_j \rho(a_j) \leq \frac{1}{4}$ and that $A$ is large enough so that
$$\left(\frac{3}{A^2} + 2\sqrt{\frac{3}{A^2}}\right) \leq \frac{1}{2}.$$ Then for every $1 \leq s_1 < s_2 \leq n$ we have
$$\frac{5}{24} \sum_{j=s_1}^{s_2} \text{Var}(X_j) \leq \text{Var}(\sum_{j=s_1}^{s_2} X_j) \leq \frac{7}{2} \sum_{j=s_1}^{s_2} \text{Var}(X_j).$$

The first part of the proof is the following lemma.

3.3. Lemma. Assume that $\sum_j \rho(a_j) \leq \frac{1}{4}$. Then, for all $1 \leq s_1 < s_2 \leq u_n$ we have
$$\frac{1}{2} \sum_{j=s_1}^{s_2} \text{Var}(Y_j) \leq \text{Var}(\sum_{j=s_1}^{s_2} Y_j) \leq \frac{3}{2} \sum_{j=s_1}^{s_2} \text{Var}(Y_j)$$
and
$$\frac{1}{2} \sum_{j=s_1}^{s_2} \text{Var}(Z_j) \leq \text{Var}(\sum_{j=s_1}^{s_2} Z_j) \leq \frac{3}{2} \sum_{j=s_1}^{s_2} \text{Var}(Z_j).$$

Proof. Let us prove the first estimate. First,
$$\text{Var}(\sum_{j=s_1}^{s_2} Y_j) = \sum_{i=s_1}^{s_2} \|Y_i\|^2_{L^2} + 2 \sum_{s_1 \leq i < j \leq s_2} \text{Cov}(Y_i, Y_j).$$
Since the size of the gap between $B_i$ and $B_{i+1}$ is $a_i$, with $A_{i,j} = a_i + \ldots + a_j \geq \max(a_i, a_j)$ we have
$$(3.1) \quad 2 \sum_{s_1 \leq i < j \leq s_2} |\text{Cov}(Y_i, Y_j)| \leq 2 \sum_{s_1 \leq i < j \leq s_2} \rho(A_{i,j})\|Y_i\|_{L^2}\|Y_j\|_{L^2}$$
Thus, using also the Cauchy-Schwartz inequality we have
\[
\sum_{s_1 \leq i < j \leq s_2} \rho(A_{i,j})(\|Y_i\|_{L^2}^2 + \|Y_j\|_{L^2}^2) = \sum_{j=s_1}^{s_2} \|Y_j\|_{L^2}^2 \sum_{i=s_1}^{j-1} \rho(A_{i,j}) + \sum_{i=s_1}^{s_2-1} \|Y_i\|_{L^2}^2 \sum_{j=i+1}^{s_2} \rho(A_{i,j}) \leq \frac{1}{2} \sum_{j=s_1}^{s_2} \|Y_j\|_{L^2}^2.
\]

To prove the corresponding estimate with \(Z_j\) instead of \(Y_j\), we observe that the cardinality \(|B_j|\) satisfies \(|B_j| \geq \|Y_j\|_{L^2} \geq Aa_j\), and so \(|B_j| \geq Aa_j + 1 \geq a_j\). Hence the size of the gap between \(Z_j\) and \(Z_{j+1}\) is at least \(a_j\), and the proof proceeds similarly to the above. \(\square\)

Next, we also need:

3.4. **Lemma.** Assume that \(\sum_j \rho(a_j) \leq \frac{1}{4}\). Then for all \(1 \leq s_1 < s_2 \leq u_n\) we have

\[
\left| \frac{\text{Var}(\sum_{j=s_1}^{s_2} X_j)}{\text{Var}(\sum_{j=s_1}^{s_2} Y_j)} - 1 \right| \leq E(A) := \left( \frac{3}{A^2} + 2\sqrt{\frac{3}{A^2}} \right).
\]

**Proof.** We have

\[
\text{Var} \left( \sum_{j=s_1}^{s_2} X_j \right) = \text{Var} \left( \sum_{j=s_1}^{s_2} Y_j \right) + \text{Var} \left( \sum_{j=s_1}^{s_2} Z_j \right) + 2\text{Cov} \left( \sum_{j=s_1}^{s_2} Y_j, \sum_{j=s_1}^{s_2} Z_j \right).
\]

Observe that \(\|Z_j\|_2 \leq a_j \leq \|Y_j\|_{L^2}/A\), and so by Lemma 3.3 we have

\[
\text{Var} \left( \sum_{j=s_1}^{s_2} Z_j \right) \leq \frac{3}{2} \sum_{j=s_1}^{s_2} \text{Var}(Z_j) \leq \frac{3}{2A} \sum_{j=s_1}^{s_2} \text{Var}(Y_j) \leq \frac{3}{A^2} \text{Var} \left( \sum_{j=s_1}^{s_2} Y_j \right).
\]

Thus, using also the Cauchy-Schwartz inequality we have

\[
\left| \text{Var} \left( \sum_{j=s_1}^{s_2} X_j \right) - \text{Var} \left( \sum_{j=s_1}^{s_2} Y_j \right) \right| \leq \frac{3}{A^2} \text{Var} \left( \sum_{j=s_1}^{s_2} Y_j \right) + 2 \left\| \sum_{j=s_1}^{s_2} Z_j \right\|_{L^2} \left\| \sum_{j=s_1}^{s_2} Y_j \right\|_{L^2}
\leq \left( \frac{3}{A^2} + 2\sqrt{\frac{3}{A^2}} \right) \text{Var} \left( \sum_{j=s_1}^{s_2} Y_j \right).
\]

**Proof of Proposition 3.2** We need to show that if \(E(A) \leq \frac{1}{2}\) then

\[
\frac{5}{24} \sum_{j=s_1}^{s_2} \text{Var}(X_j) \leq \text{Var} \left( \sum_{j=s_1}^{s_2} X_j \right) \leq \frac{21}{6} \sum_{j=s_1}^{s_2} \text{Var}(X_j).
\]

We first recall that

\[
\|Z_j\|_{L^2} \leq a_j \leq \|Y_j\|_{L^2}/A
\]

and therefore

\[
(1 - A^{-1}) \|Y_j\|_{L^2} \leq \|X_j\|_{L^2} = \|Y_j + Z_j\|_{L^2} \leq (1 + A^{-1}) \|Y_j\|_{L^2}.
\]

We also note that \(A^{-1} \leq \frac{1}{6}\) since \(E(A) \leq \frac{1}{2}\). Now Proposition 3.2 follows by Lemma 3.4 and the above estimates. \(\square\)
Proof. For each \( q, p \geq 1 \) and two sub-\( \sigma \)-algebras \( G, H \) let
\[
\varpi_{q,p}(G, H) = \sup \{ \| E[h|G] - E[h]\|_{L^p} : h \in L^\infty(\Omega, H, \mathbb{P}), \| h\|_{L^q} \leq 1 \}.
\]
Then (see \cite[Ch. 4]{2})
\[
\alpha(G, H) = \frac{1}{4} \varpi_{\infty,1}(G, H).
\]
Thus, by applying the Riesz–Thorin interpolation theorem (\cite[Ch.6]{2}) with the operator \( h \to E[h|G] - E[h] \) we see that there is a constant \( C_q > 0 \) so that
\[
\| E[\xi_{k+s,n}|\xi_{k,n}, \ldots, \xi_{1,n}]\|_{L^2} \leq A_q \varpi_{q,2}(G, H) \leq A_q C_q (\alpha(G, H))^{1/2-1/q} \leq (\alpha(s))^{1/2-1/q}
\]
where \( H \) is the \( \sigma \)-algebra generated by \( \xi_{k+s,n} \) and \( G \) is the \( \sigma \)-algebra generated by \( \{\xi_{k,n}, \ldots, \xi_{1,n}\} \).
\hfill \square

The blocks. The construction of the blocks using the coefficient \( \delta_n(r) \) is similar to the \( \rho \)-mixing case, with a small difference. We will need also to control (from below) the size gap between two consecutive “big” and small blocks.

Let us fix some \( A > 1 \) and \( \varepsilon > 0 \) be so that \( A\varepsilon > r \). Let us take \( b_1 \) to be the first time that
\[
\| S_{b_1} \|_2 \geq A.
\]
Set \( Y_1 = S_{b_1} \). Now we take \( \beta_1 \) to be the smallest positive integer so that
\[
\| S_{b_1+\beta_1} - S_{b_1} \|_2 \geq A\varepsilon.
\]
Set \( Z_1 = S_{b_1+\beta_1} - S_{b_1} \). Continuing that way we get blocks \( Y_1, Z_1, Y_2, Z_2, \ldots \) of the form \( \sum_{j \in I} \xi_{j,n} \) for an interval \( I \) so that
\begin{enumerate}
  \item the size of the gap between two consecutive \( Y_j \)'s is at least \( A \);
  \item the size of the gap between two consecutive \( Z_j \)'s is at least \( A\varepsilon \);
  \item \( A \leq \| Y_j \|_2 \leq (A+1), \ A\varepsilon \leq \| Z_j \|_2 \leq A\varepsilon + 1 \).
\end{enumerate}

We prove here the following version of Proposition \cite[3.2]{5}.

3.6. Proposition. If
\[
\left( 6\varepsilon^2 + 2\sqrt{6}\varepsilon \right) \leq \frac{1}{2},
\]
then for every \( 1 \leq s_1 < s_2 \leq u_n \) we have
\[
\frac{5}{24} \sum_{j=s_1}^{s_2} \text{Var}(X_j) \leq \text{Var}\left( \sum_{j=s_1}^{s_2} X_j \right) \leq \frac{7}{2} \sum_{j=s_1}^{s_2} \text{Var}(X_j).
\]
The first part of the proof is the following version of Lemma 3.6.

3.7. **Lemma.** We have

\[
\frac{1}{2} \sum_{j=s_1}^{s_2} \Var(Y_j) \leq \Var\left( \sum_{j=s_1}^{s_2} Y_j \right) \leq \frac{3}{2} \sum_{j=s_1}^{s_2} \Var(Y_j)
\]

and

\[
\frac{1}{2} \sum_{j=s_1}^{s_2} \Var(Z_j) \leq \Var\left( \sum_{j=s_1}^{s_2} Z_j \right) \leq \frac{3}{2} \sum_{j=s_1}^{s_2} \Var(Z_j).
\]

**Proof.** Let us prove the first estimate. First,

\[
\Var\left( \sum_{j=s_1}^{s_2} Y_j \right) = \sum_{i=s_1}^{s_2} \|Y_i\|_2^2 + 2 \sum_{s_1 \leq i < j \leq s_2} \Cov(Y_i, Y_j).
\]

Since the size of the gap between \(Y_i\) and \(Y_{i+1}\) is at least \(r\) and \(\|Y_i\|_2 \geq 1\), we have

\[
(3.2) \quad 2 \left| \sum_{s_1 \leq i < j \leq s_2} \Cov(Y_i, Y_j) \right| \leq 2 \sum_{s_1 \leq i < s_2} \|Y_i\|_2 \left| \sum_{j > i} \sum_{j > i} \E[Y_i \sum_{j > i} Y_j] \right| \leq 2 \delta_n(r) \sum_{s_1 \leq i < s_2} \|Y_i\|_2 \leq \frac{1}{2} \sum_{s_1 \leq i < s_2} \|Y_i\|_2^2.
\]

The proof for the \(Z_j\)'s is similar. \(\square\)

We next need the following version of Lemma 3.4.

3.8. **Lemma.** For all \(1 \leq s_1 < s_2 \leq u_n\) we have

\[
\left| \frac{\Var(\sum_{j=s_1}^{s_2} X_j)}{\Var(\sum_{j=s_1}^{s_2} Y_j)} - 1 \right| \leq D(\varepsilon) := \left( 6\varepsilon^2 + 2\sqrt{6}\varepsilon \right).
\]

**Proof.** We have

\[
\Var\left( \sum_{j=s_1}^{s_2} X_j \right) = \Var\left( \sum_{j=s_1}^{s_2} Y_j \right) + \Var\left( \sum_{j=s_1}^{s_2} Z_j \right) + 2\Cov\left( \sum_{j=s_1}^{s_2} Y_j, \sum_{j=s_1}^{s_2} Z_j \right).
\]

Observe that \(\|Z_j\|_2 \leq (A\varepsilon + 1) \leq 2A\varepsilon \leq 2\varepsilon \|Y_j\|_2\), and so by Lemma 3.4 we have

\[
\Var\left( \sum_{j=s_1}^{s_2} Z_j \right) \leq \frac{3}{2} \sum_{j=s_1}^{s_2} \Var(Z_j) \leq \frac{3 \cdot 4\varepsilon^2}{2} \sum_{j=s_1}^{s_2} \Var(Y_j) \leq (6\varepsilon^2)\Var\left( \sum_{j=s_1}^{s_2} Y_j \right).
\]

Thus, using also the Cauchy-Schwartz inequality we have

\[
\left| \Var(\sum_{j=s_1}^{s_2} X_j) - \Var(\sum_{j=s_1}^{s_2} Y_j) \right| \leq (6\varepsilon^2)\Var(\sum_{j=s_1}^{s_2} Y_j) + 2 \left\| \sum_{j=s_1}^{s_2} Z_j \right\|_{L^2} \left\| \sum_{j=s_1}^{s_2} Y_j \right\|_{L^2} \leq (6\varepsilon^2 + 2\sqrt{6}\varepsilon) \Var(\sum_{j=s_1}^{s_2} Y_j).
\]
The proof of Proposition 3.6 is now completed similarly to the proof of Proposition 3.2.

4. The reduction of the functional CLT to arrays generated by regular blocks

Let \(\sigma_{k,n}^2\) denote the variance of \(\sum_{j=1}^k \xi_{j,n}\). Let also denote by \(s_{k,n}^2\) the variance of \(\sum_{j=1}^k X_{j,n}\). Then \(s_{k,n}^2 = \sigma_{k,n}^2 = \text{Var}(S_n) = \sigma_n^2\). Set

\[
v_n(t) = \inf\{1 \leq k \leq n : \sigma_{k,n}^2 \geq \sigma_n^2 t\}, \quad \tilde{v}_n(t) = \inf\{1 \leq k \leq n : s_{k,n}^2 \geq \sigma_n^2 t\}.
\]

Set also

\[
W_n(t) = \sigma_n^{-1} \sum_{j=1}^{v_n(t)} \xi_{j,n}, \quad W_n(t) = \sigma_n^{-1} \sum_{j=1}^{\tilde{v}_n(t)} X_{j,n}
\]

Next, let us consider \(D[0,1]\) as a metric space with the uniform metric \(d(f,g) = \sup_{t \in [0,1]} |f(t) - g(t)|\). Recall that the Prokhorov (or the Levi-Prokhorov) distance between two probability distributions \(\mu, \nu\) on \(D[0,1]\) is given by

\[
d_P(\mu, \nu) = \inf \{ \varepsilon > 0 : \mu(B) \leq \nu(B^\varepsilon) + \varepsilon \text{ and } \nu(B) \leq \mu(B^\varepsilon) + \varepsilon \text{ for all Borel sets } B\}
\]

where \(B^\varepsilon\) is the \(\varepsilon\)-neighborhood of \(B\). When \(X, Y\) are \(D[0,1]\)-valued random variables with laws \(\mu_X\) and \(\mu_Y\), respectively, we abuse the notations and write \(d_P(X, Y) := d_P(\mu_X, \mu_Y)\). Then a sequence of \(D[0,1]\)-valued random variables \(A_n\) converges in distribution to a \(D[0,1]\)-valued random variable \(A\) if and only if \(d_P(A_n, A)\) converges to 0. We first recall the following result, whose proof is included for the sake of completeness.

4.1. Lemma. Let \(Q_1(t), Q_2(t)\) be two random functions so that \(Q_1(\cdot), Q_2(\cdot) \in D[0,1]\). Let \(L_i\) be the law of \(Q_i, i = 1, 2\). Then for every \(q \geq 1\),

\[
d_P(Q_1, Q_2) = d_P(L_1, L_2) \leq \left\| \sup_{t \in [0,1]} |Q_1(t) - Q_2(t)| \right\|_q
\]

where \(d_P\) is the Prokhorov metric on \(D[0,1]\).

Proof. First, it follows from the definition of \(d_P\) that \(d_P(Q_1, Q_2) \leq \varepsilon_0\) if

\[
P( \sup_{t \in [0,1]} |Q_1(t) - Q_2(t)| \geq \varepsilon_0) \leq \varepsilon_0.
\]

By the Markov inequality we have

\[
P( \sup_{t \in [0,1]} |Q_1(t) - Q_2(t)| \geq \varepsilon_0) \leq \left\| \sup_{t \in [0,1]} |Q_1(t) - Q_2(t)| \right\|_q \varepsilon_0^{-q}
\]

and the lemma follows by taking \(\varepsilon_0 = \|\sup_{t \in [0,1]} |Q_1(t) - Q_2(t)|\|^q_q\).

The approximation of \(W_n(\cdot)\) by \(W_n(\cdot)\) relies on the following result.

4.2. Lemma. Suppose that the maximal Lindeberg condition holds true along the blocks \(X_{j,n}\). Then

\[
\lim_{n \to \infty} \left\| \sup_{t \in [0,1]} |W_n(t) - W_n(t)| \right\|_{L^1} = 0
\]
Proof. Let $I_{j,n} = [x_{j,n}, y_{j,n}]$ be so that $X_{j,n} = \sum_{k \in I_{j,n}} \xi_{k,n}$. Let 

$$X_{j,n} = \max \left\{ \left\| \sum_{k=x_{j,n}}^{m} \xi_{k,n} \right\| : x_{j,n} \leq m \leq y_{j,n} \right\}.$$ 

By considering the block $I_{j,n}$ so that $v_{n}(t) \in I_{j,n}$ we see that 

$$\sup_{t \in [0,1]} |W_{n}(t) - W_{n}(t)| \leq \max\{X_{j,n}/\sigma_{n} : 1 \leq j \leq u_{n}\}.$$ 

Now, for each $\varepsilon > 0$, 

$$\mathbb{E}[\max\{X_{j,n}/\sigma_{n} : 1 \leq j \leq u_{n}\}] \leq \varepsilon + \sigma_{n}^{-1} \sum_{j=1}^{u_{n}} \mathbb{E}[X_{j,n}\|X_{j,n} \geq \varepsilon\sigma_{n}\]].$$ 

Because of the maximal Lindeberg condition, for each fixed $\varepsilon$ the second summand in the above right hand side converges to 0 as $n \to \infty$. \hfill \Box

Proof of Theorem 2.8. The construction of the regular block was already done in Section 3.1 To prove the functional CLT, notice that under the assumptions of Theorem 2.8 the array \{X_{j,n} : 1 \leq j \leq u_{n}\} satisfies the usual Lindeberg conditions. Thus, by applying [11] Theorem 4.1 we see that $W_{n}(\cdot)$ converges in distribution towards a standard Brownian motion. Now Theorem 2.8 follows from Lemma 4.1 and Lemma 4.2. \hfill \Box

5. PROOF OF THEOREM 2.7

All the results that will be formulated in this section hold for any partition into blocks so that $\max_{k \in I_{j,n}} \left\| \sum_{\ell=x_{j,n}}^{k} \xi_{\ell,n} \right\|_{2} = O(a_{j}).$

5.1. On the verification of the maximal Lindeberg condition via maximal moment assumptions. Let us first formulate an abstract condition for the maximal Lindeberg condition to hold true.

5.1. Proposition (A maximal moment assumption). The maximal Lindeberg condition along the blocks $X_{j,n}$ holds true when for some $p > 2$, for all $j \leq u_{n}$ we have 

$$(5.1) \quad \|X_{j,n}\|_{L^{p}} \leq C_{n} a_{j} r(a_{j})$$

for some increasing function $r(\cdot) \geq 1$ so that 

$$(5.2) \quad \lim_{n \to \infty} C_{n}^{p} \min \left\{ r^{p}(e^{G(u_{n})}), e^{-pG(u_{n})-G((1-\delta)u_{n})} R(e^{G(u_{n})}) \right\} \left( H(u_{n}) \right)^{p/2-1} = 0$$

where $R(x) = \int x^{p}r^{p}(x)dx$. In particular, the maximal Lindeberg condition holds when (5.1) holds with a bounded function $r$ and a sequence $C_{n}$ so that $C_{n}(H(u_{n}))^{1/2-1/p} \to 0$.

The first step in the proof of Proposition 5.1 is quite elementary, and it is summarized in the following result.

5.2. Lemma. If there is a number $p > 2$ so that 

$$(5.3) \quad \lim_{n \to \infty} \sum_{j=1}^{u_{n}} \frac{\|X_{j,n}\|_{p}^{p}}{\left( \sum_{j=1}^{u_{n}} a_{j}^{2} \right)^{p/2}} = 0$$

then the maximal Lindeberg condition along the blocks $X_{j,n}$ holds true.
Proof. Let us fix some $\varepsilon > 0$. Then by the Hölder and the Markov inequalities we have
\[ \mathbb{E}[X_j^2 1(|X_j| \geq \varepsilon n)] \leq ||X_j||_p^2 p(||X_j||^{p-2})^{-1} \leq ||X_j||_p^2 e^{-p \sigma_n^2 (p-2)} \]
where $X_{j,n} = X_j$. Next, since
\[ \sigma_n^2 \approx \sum_{j=1}^{n} a_j^2 \]
we have
\[ \sigma_n^{-2} \sum_{j=1}^{n} \mathbb{E}[X_j^2 1(|X_j| \geq \varepsilon n)] \leq C_{\varepsilon, p} \sum_{j=1}^{n} ||X_j||_p^2 \left( \sum_{j=1}^{n} a_j^2 \right)^{p/2} \to 0. \]
\[ \square \]

5.3. Corollary. The maximal Lindeberg conditions along the blocks $\{X_{j,n}\}$ holds true if for some $p > 2$ we have
\[ ||X_{j,n}||_p^p \leq C_n a_j r(a_j) \]
for some increasing strictly positive continuous function $r$ and a sequence $C_n$ so that
\[ \lim_{n \to \infty} C_n \frac{\sum_{j=1}^{n} a_j^p r(a_j)}{\left( \sum_{j=1}^{n} a_j^2 \right)^{p/2}} = 0. \]

Proposition 5.1 follows from Corollary 5.3 and the following lemma.

5.4. Lemma. Let $a_j = e^{G(j)}$ be a sub-exponential sequence as described in Definition 2.4. We suppose that either
\[ \lim_{n \to \infty} C_n r(e^{G(n)})P(H(u_n))^{p/2-1} = 0. \]
where $R$ is the antiderivative of $s(x) = x^p r(x)$, or
\[ \lim_{n \to \infty} C_n r(e^{G(n)})P(H(u_n))^{p/2-1} = 0. \]
Then condition (5.5) holds true for every $p > 2$.

The proof is elementary, and it is included for readers' convenience.

Proof. Suppose first that (5.6) holds true. Since $G(x)$ is eventually monotone, it is enough to prove that
\[ \lim_{n \to \infty} C_n \frac{\int_{1}^{u_n} s(e^{G(x)})dx}{\left( \int_{1}^{u_n} e^{2G(x)}dx \right)^{p/2}} = 0. \]
Moreover, since the integrals over $[(1-\delta)u_n, u_n]$ dominate the integral over $[1, u_n]$, it is sufficient to show that
\[ \lim_{n \to \infty} C_n \frac{\int_{(1-\delta)u_n}^{u_n} s(e^{G(x)})dx}{\left( \int_{(1-\delta)u_n}^{u_n} e^{2G(x)}dx \right)^{p/2}} = 0. \]
Now, with $u = u_n$ and $Z(x) = e^{G(x)}$ we have
\[ \int_{(1-\delta)u_n}^{u} s(e^{G(x)})dx = \int_{(1-\delta)u_n}^{u} s(Z(x)) Z'(x)dx \leq C_2 e^{-G(u(1-\delta))} (H(u))^{-1} \int_{(1-\delta)u_n}^{u} s(Z(x)) Z'(x) \]
\[ \leq C_2 e^{-G(u(1-\delta))} (H(u))^{-1} \int_{(1-\delta)u_n}^{u} s(Z(x)) Z'(x) dx. \]
and, similarly,
\[ \int_{(1-\delta)u}^{u} e^{2G(x)} \, dx = \int_{(1-\delta)u}^{u} (Z(x))^2 Z'(x)/Z'(x) \, dx \geq \frac{1}{3} C_1^{-1}(H(u))^{-1} e^{-G(u)} e^{3G(u)}. \]

Thus, there is a constant \( C = C(p, \varepsilon, C_1, C_2) > 0 \) so that
\[ \frac{\int_{(1-\delta)u}^{u} s(e^{G(x)}) \, dx}{\left( \int_{(1-\delta)u}^{u} e^{2G(x)} \, dx \right)^{p/2}} \leq C G^{-pG(u_n)} - G(u_n(1-\delta))(H(u_n))^{p/2-1} R(e^{G(u_n)}). \]

Now (5.8) follows from (5.6).

Now, suppose that (5.7) holds true. As in the previous case, it is enough to prove that
\[ \lim_{n \to \infty} C_n \frac{\int_{(1-\delta)u_n}^{u_n} s(e^{G(x)}) \, dx}{\left( \int_{(1-\delta)u_n}^{u_n} e^{2G(x)} \, dx \right)^{p/2}} = 0. \]

Let us write \( u_n = u \) and
\[ \int_{(1-\delta)u}^{u} s(e^{G(x)}) \, dx = \int_{(1-\delta)u}^{u} G'(x)s(e^{G(x)})/G'(x) \, dx \]
and
\[ \int_{(1-\delta)u}^{u} e^{2G(x)} \, dx = \int_{(1-\delta)u}^{u} G'(x)e^{2G(x)}/G'(x) \, dx. \]

Then
\[ \int_{(1-\delta)u}^{u} s(e^{G(x)}) \, dx \leq (r(e^{G(u)}))^{p} C_1^{-1}(H(u))^{-1} \int_{(1-\delta)u}^{u} G'(x)e^{pG(x)} \, dx \]
\[ = (r(e^{G(u)}))^{p} C_1^{-1}(H(u))^{-1} \int_{G(u(1-\delta))}^{G(u)} e^{py} \, dy \leq C_1^{-1}(H(u))^{-1} s(e^{G(u)}) \]
and
\[ \int_{(1-\delta)u}^{u} e^{2G(x)} \, dx \geq C_2^{-1}(H(u))^{-1} \int_{(1-\delta)u}^{u} G'(x)e^{2G(x)} \, dx \]
\[ = C_2^{-1}(H(u))^{-1} \int_{G((1-\delta)u)}^{G(u)} e^{2y} \, dy \geq C_2^{-1}(H(u))^{-1} e^{G(u)}(1-\varepsilon). \]

Thus, there is a constant \( C = C(p, \varepsilon, C_1, C_2) > 0 \) so that
\[ \frac{\int_{(1-\delta)u}^{u} s(e^{G(x)}) \, dx}{\left( \int_{(1-\delta)u}^{u} e^{2G(x)} \, dx \right)^{p/2}} \leq C(r(e^{G(u)}))^{p}(H(u))^{p/2-1} \]
and (5.9) follows from (5.7). \( \square \)

5.2. **Proof of Theorem 2.1** Recall next the following maximal moment inequality. If \( \phi_n(m_n) < \frac{1}{2} - \varepsilon \) then by [10, Theorem 6.17], for every \( p > 2 \) and \( k, m \) such that \( k + m \leq n \),
\[ (5.10) \]
\[ \left\| \max \left\{ \sum_{k \leq j < k + l} \xi_{j,n} : 0 \leq l \leq m \right\} \right\|_p \leq C_{\varepsilon,p} \left( m_n \left\| \max_{k \leq j < k + m} \xi_{j,n} \right\|_p + \max_{0 \leq l \leq m} \left\| \sum_{k \leq j < k + l} \xi_{j,n} \right\|_2 \right). \]

Next, using (5.10) and applying Theorem 5.1 with the function \( r(x) = 1 \) yields the following result.
5.5. **Proposition** (Maximal Lindeberg condition for bounded variables). **Under (2.3),** the maximal Lindeberg condition holds true along the blocks $X_{j,n}$ when the random variables $ξ_{j,n}$ are bounded, and with $K_n := \max_{1 \leq j \leq n} \|ξ_{j,n}\|_{L^\infty}$ for some $p > 2$ we have

$$\lim_{n \to \infty} m_n^p K_n^p(\rho(u_n))^{p/2-1} = 0.$$ 

Theorem 2.7 follows from Theorem 2.8, Example 2.5, Remark 2.7 and Proposition 5.5 (when $a_j = j^q$ we use (5.6), and for the other three cases we use (5.7)).

5.3. **Unbounded variables.** The following result follows from Proposition 5.1 by approximating $ξ_{j,n}$ by bounded random variables, and together with Theorem 2.1 they yield the functional CLT under some type of moment growth assumptions.

5.6. **Proposition.** **Under (2.3),** the maximal Lindeberg condition holds true when there is a strictly increasing function $h : [0, \infty) \to [0, \infty)$, a sequence $K_n$ and constants $p_1 > p > 2$, and $C > 0$ so that

$$\lim_{n \to \infty} m_n^p K_n^p(\rho(u_n))^{p/2-1} = 0$$

and for each $1 \leq j \leq u_n$

$$\sum_\ell \|ξ_{\ell,n}\|_{p_1}(\|h(ξ_{\ell,n})\|_{1/p_1-1/p_1}) \leq C a_j K_n m_n.$$ 

5.7. **Example.** When $a_j = j^q$ then $H(u) = \frac{1}{u^q}$ and $u_n \approx \frac{C_1}{\sqrt{n}}$. Let $p > 2$ and $\delta > 0$ be so that $1/2 - 1/p - \delta > 0$. Let us take $m_n$ and $K_n$ of order $n^{1/2-1/p-\delta}$. Then condition (5.11) holds. Let us take $h(x) = x^a$ for some $a > 1$. Then (5.12) will hold if

$$\sum_{\ell=1}^n \|ξ_{\ell,n}\|_{p_1} \|ξ_{\ell,n}\|_{a(1/p_1-1/p_1)} \leq C a_j$$

for some $s > 1$ so that $a/s$ is large enough.

The following result shows that the functional CLT holds under certain assumptions on the sizes of the blocks.

5.8. **Proposition (Blocks with controllable sizes).** The maximal Lindeberg condition holds true when there is a constant $p > 2$ so that

$$\lim_{n \to \infty} m_n^p (H(u_n))^{p/2-1} = 0$$

and there is a strictly increasing function $h : [0, \infty) \to [0, \infty)$ and a constant $C > 0$ so that

$$\sum_\ell \|ξ_{\ell,n}\|_{p_1}(\|h(ξ_{\ell,n})\|_{1/p_1-1/p_1})^{p_1} \leq C a_j.$$ 

In particular, the maximal Lindeberg condition holds true when $|I_{j,n}| \leq a_j k(a_j)$, both $\|ξ_{\ell,n}\|_{p}$ and $\|h(ξ_{\ell,n})\|$ are bounded in $\ell$ and $n$ and (5.13) and (5.14) holds with $h(a_j) \geq 1/(a_j)^{p_1}.

The proof of the proposition is based on approximating $X_{j,n}$ by $\sum_{k \in I_{j,n}} ξ_{j,n} I(|ξ_{j,n}| \leq a_j m_n)$. Taking $h(x) = x^s$ for some $s$ we can consider $i(a_j)$ with polynomial growth in $a_j$, and taking $h(x) = e^{cx^\beta}$ for some $0 < \beta \leq 1$ we can consider $i(a_j)$ with stretched exponential growth in $a_j$.

5.9. **Example.** Let $ξ_{j,n}$ be a uniformly elliptic array of Markov chains (see [4]), and let

$$ξ_{j,n} = f_{j,n}(ξ_{j,n}, ξ_{j+1,n}, ..., ξ_{j+m,n}),$$

where $m$ is some positive integer which does not depend on $j$ and $n$, and $f_{j,n}$ is a uniformly bounded array of functions.
Then (see [4]), there are structural constants $u_{j,n}$ which depend only on $f_{j,n}, \ldots, f_{j+m+2,n}$ (and have an explicit form) so that for all $1 \leq s_1 < s_2 \leq n$,

$$C_3 \sum_{k=s_1}^{s_2} u_{k,n}^2 - C_4 \leq \text{Var} \left( \sum_{k=s_1}^{s_2} \xi_{k,n} \right) \leq C_1 \sum_{k=s_1}^{s_2} u_{k,n}^2 + C_2$$

where $C_i$ are positive constants which do not depend on $n$. Thus, assumptions of the form $|I_{j,n}| \leq a_{j,i}(a_j)$ can be verified under appropriate assumptions on the magnitudes of $u_{j,n}$.

5.4. $\alpha$-mixing arrays. The proof of Theorem 2.9 proceeds similarly to the proof of Theorem 2.8 by applying [12, Corollary 2.2] and the results in Section 3.2. Since all the sufficient conditions (described above) for the maximal Lindeberg condition to hold do not require any assumptions on the $\rho$-mixing coefficients, they also yield explicit conditions for the functional CLT for $\alpha$-mixing triangular arrays. In particular, Proposition 2.11 holds true.

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$u_{j,n}^2$ is the variance of a certain sum related to the functions $f_{j,n}, \ldots, f_{j+m+2,n}$ with respect to an explicit distribution which is described in [4].
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