Random phase approximation in a self-consistent covariant approach: recent applications

Haozhao Liang$^{1,2}$, Yifei Niu$^{1,3}$, Jie Meng$^{4,1,5}$ and Nguyen Van Giai$^2$

$^1$State Key Laboratory of Nuclear Physics and Technology, School of Physics, Peking University, Beijing 100871, China
$^2$Institut de Physique Nucléaire, IN2P3-CNRS and Univ. Paris-Sud, 91406 Orsay, France
$^3$Physics Department, Faculty of Science, University of Zagreb, Croatia
$^4$School of Physics and Nuclear Energy Engineering, Beihang University, Beijing 100191, China
$^5$Department of Physics, University of Stellenbosch, Stellenbosch, South Africa

E-mail: nguyen@ipno.in2p3.fr

Abstract. The relativistic Hartree-Fock (RHF) and Random Phase Approximation (RPA) methods are self-consistently applied to two issues of current interest. The first application is related to the isospin mixing corrections in the problem of super-allowed $0^+ \to 0^+ \beta$-transitions and the unitarity of the CKM matrix. The second application concerns the prediction of inclusive neutrino-nucleus cross-sections, where the results of the present model are compared with other approaches.

1. Introduction

The self-consistent covariant mean field method is well known for its numerous applications to the description of nuclear ground and excited states. The most widely known versions of the method are within the context of the relativistic Hartree approximation: the Relativistic Mean Field (RMF) method [1, 2] and the Random Phase Approximation (RPA) derived from it [3]. In the recent years there have been significant advances in the covariant approaches dealing with the effects of exchange interactions and leading to the Relativistic Hartree-Fock (RHF) and Relativistic Hartree-Fock-Bogoliubov (RHF-B) theories [4, 5] as well as the RHF-RPA description of nuclear excitations [6]. One interesting finding is that, the effects of the tensor $\rho-N$ coupling which contributes mostly in the exchange (Fock) terms can improve somewhat the spurious shell gaps at $N, Z = 56$ and 92 found in RMF parametrizations [7]. Another effect of exchange terms in the particle-hole (p-h) channel is that the isoscalar $\sigma$- and $\omega$-couplings play an important role in the description of isovector excitations like Gamow-Teller states whereas they are absent by definition in the RMF-RPA approach.

In this presentation we shall focus on two recent studies [8, 9] where the RHF-RPA method is used with a degree of self-consistency which cannot be met by the RMF-RPA approach. The first subject concerns the theoretical evaluation of isospin impurity effects in the super-allowed $0^+ \to 0^+ \beta$-transitions. It turns out that the exchange Coulomb effects cannot be disregarded in that problem. The second subject is the prediction of neutrino-nucleus cross sections using RPA wave functions. The important spin-isospin modes (Gamow-Teller, spin dipole, etc...) which determine the cross-sections can be calculated here in a fully self-consistent manner [6] whereas the RMF-RPA must introduce an additional adjusted parameter in the $\pi-N$ couplings.
2. Isospin corrections for superallowed $\beta$ decays

The key issue of the validity of the Standard Model of electro-weak interactions is linked to the unitarity of the Cabbibo-Kobayashi-Maskawa (CKM) matrix. One can have an accurate experimental determination of the leading matrix element $V_{ud}$ by studying the $0^+ \rightarrow 0^+$ superallowed Fermi $\beta$ decays which lead to the value of the vector coupling constant $G_V$ of semi-leptonic weak interactions. One has:

$$V_{ud} = \frac{G_V}{G_F},$$

where $G_F$ is the Fermi coupling constant for purely leptonic decays.

We concentrate on the $\beta$-transitions between a parent state $(N, Z)$ and daughter states $(N \pm 1, Z \mp 1)$, both parent and daughter states having $J^T = 0^+$ and $T = 1$. One has the general relation for the transition strength:

$$|\langle \text{daughter}|T_+|\text{parent}\rangle|^2 = 2(1 - \delta_c),$$

where $\delta_c$ represents the effect of isospin mixing caused by Coulomb and other isospin-breaking forces. Here, we restrict the discussion to the Coulomb force. For each nucleus, one can deduce from the measured $ft$ value a nucleus-independent $Ft$ value:

$$Ft = ft(1 + \delta_R')(1 + \delta_{NS} - \delta_c),$$

where $\delta_R'$ and $\delta_{NS}$ are radiative corrections which can be evaluated [10]. Then, the value of $G_V$ is obtained as:

$$G_V^2 = \frac{K}{2(1 + \Delta_R)Ft},$$

$K$ being a known numerical constant and $\Delta_R$ comes from radiative corrections.

We have calculated the values of $\delta_c$ for a number of $T = 1$ nuclei by charge-exchange RHF-RPA. In all cases the spherical symmetry is assumed and the filling approximation is applied to the last partially occupied orbitals. It must be noted that all meson-induced interactions (isoscalar as well as isovector) participate to the RPA p-h interaction whereas the Coulomb interaction contributes only to the single-particle energies but not to the p-h interaction which is of neutron-proton nature. The $\delta_c$ values calculated in RHF-RPA and RMF-RPA are compared in Table 1. The parameter set DD-ME2 [11] was determined for RMF calculations while PKO1 [4] was constructed for RHF calculations, and they are consequently used here for this comparison. It is seen that a systematic difference is predicted by the two models, the deviations being more pronounced in lighter nuclei. It is easy to understand where the deviations come from by repeating the RHF-RPA calculations without the Coulomb exchange contributions in the mean field (PKO1* column of Table 1). The corresponding results for $\delta_c$ coincide closely with those predicted by DD-ME2. We have concentrated the discussion here to a comparison of PKO1 for the RHF side and DD-ME2 for the RMF side, but similar conclusions are reached [8] if one compares with other RMF models. Thus, one clearly sees that Coulomb exchange contributions are important for this kind of study while the values of $\delta_c$ are rather insensitive to the particular effective Lagrangians as long as they predict nuclear ground states reasonably well.

In Fig.1 the nucleus-independent $Ft$ values are plotted as a function of the charge $Z$ of the daughter nucleus. For comparison the uncorrected experimental $ft$ values [12] and the partially corrected values (including only radiative corrections) are also shown. This figure shows the respective importance of radiative and isospin mixing corrections. The latter corrections become more important as the charge $Z$ increases. We find an average value $\langle Ft\rangle = 3081.4(7)$ leading to $|V_{ud}| = 0.97273(27)$. Combining with the values of $|V_{us}|$ and $|V_{ub}|$ [13] the sum of squared matrix elements of the CKM matrix top row is displayed in Fig.2 for the different cases. Our prediction deviates significantly from the unitarity condition while it agrees with those obtained by other experimental methods, especially with the most recent mirror transition results [14].
Table 1. Isospin symmetry-breaking corrections $\delta_c$ for the $0^+ \rightarrow 0^+$ superallowed transitions obtained by RHF-RPA with PKO1 [4] and RMF-RPA with DD-ME2 [11]. The column PKO1* presents the results obtained with PKO1 without the Coulomb exchange (Fock) term. All values are expressed in %

|        | PKO1 | PKO1* | DD-ME2 |
|--------|------|-------|--------|
| $^{10}$C $\rightarrow$ $^{10}$B | 0.082 | 0.148 | 0.150 |
| $^{14}$O $\rightarrow$ $^{14}$N | 0.114 | 0.178 | 0.197 |
| $^{18}$Ne $\rightarrow$ $^{18}$F | 0.270 | 0.357 | 0.430 |
| $^{26}$Si $\rightarrow$ $^{26}$Al | 0.176 | 0.246 | 0.252 |
| $^{30}$S $\rightarrow$ $^{30}$P | 0.497 | 0.625 | 0.633 |
| $^{34}$Ar $\rightarrow$ $^{34}$Cl | 0.268 | 0.359 | 0.376 |
| $^{38}$Ca $\rightarrow$ $^{38}$K | 0.313 | 0.406 | 0.441 |
| $^{42}$Ti $\rightarrow$ $^{42}$Sc | 0.384 | 0.460 | 0.523 |
| $^{26}$Al $\rightarrow$ $^{26}$Mg | 0.139 | 0.193 | 0.198 |
| $^{34}$Cl $\rightarrow$ $^{34}$S | 0.234 | 0.298 | 0.307 |
| $^{38}$K $\rightarrow$ $^{38}$Ar | 0.278 | 0.344 | 0.371 |
| $^{42}$Sc $\rightarrow$ $^{42}$Ca | 0.333 | 0.395 | 0.448 |
| $^{54}$Co $\rightarrow$ $^{54}$Fe | 0.319 | 0.392 | 0.393 |
| $^{66}$As $\rightarrow$ $^{66}$Ge | 0.475 | 0.571 | 0.572 |
| $^{70}$Br $\rightarrow$ $^{70}$Se | 1.140 | 1.234 | 1.268 |
| $^{74}$Rb $\rightarrow$ $^{74}$Kr | 1.088 | 1.230 | 1.258 |

Figure 1. Corrected $\mathcal{F}t$ values by RHF-RPA with PKO1 (full circles) as a function of the charge $Z$ for the daughter nucleus. The shaded horizontal band gives one standard deviation around the average $\mathcal{F}t$ value. The uncorrected experimental $ft$ values [12] (open squares) and partially corrected ($\delta_c = 0$) $\mathcal{F}t$ values (open triangles) are shown for comparison.

3. Inclusive neutrino-nucleus cross sections

As a second example of RHF-RPA application we consider the charged-current neutrino-nucleus reactions

$$\nu_l + Z \ X_N \rightarrow Z+1 \ X^*_{N-1} + l^-,$$  \hspace{1cm} (5)
where $l$ denotes the charged lepton, e.g., electron or muon. The charged-current neutrino-nucleus cross section reads \[ d\sigma_{\nu} = \frac{V^2}{(2\pi)^2} \sum_{l \text{ spins}} \frac{1}{2J_i + 1} \sum_{M_i M_f} |\langle f | \hat{\mathcal{H}}_W | i \rangle|^2, \] (6)

where $p_l$ and $E_l$ are the momentum and energy of the outgoing lepton, respectively. For charged current reactions this cross section has to be corrected for the distortion of the outgoing lepton wave function by the Coulomb field of the daughter nucleus. In the low energy region this can be done by multiplying the cross section by a Fermi function whereas in the high energy region this effect is handled by the effective momentum approximation (EMA) \[19\].

The Hamiltonian $\hat{\mathcal{H}}_W$ of the weak interaction is expressed in the standard current-current form, i.e., in terms of the nucleon $J^\lambda(x)$ and lepton $j^\lambda(x)$ currents \[17, 18\]

\[ \hat{\mathcal{H}}_W = -\frac{G}{\sqrt{2}} \int dx J^\lambda(x) j^\lambda(x), \] (7)

Denoting the leptonic matrix current as $l^\lambda e^{-iq \cdot x}$, the transition matrix elements read

\[ \langle f | \hat{\mathcal{H}}_W | i \rangle = -\frac{G}{\sqrt{2}} l^\lambda \int dx e^{-iq \cdot x} \langle f | J^\lambda(x) | i \rangle, \] (8)

with the four-momentum transfer $(q^0, \mathbf{q}) = (E_i, p_i) - (E_\nu, p_\nu)$, which must be space-like, i.e., $q^2 = q_0^2 - \mathbf{q}^2 \leq 0$.

Using the set of excited states of the final nucleus calculated with our charge-exchange RHF-RPA model we can obtain the differential neutrino-nucleus cross sections in the extreme relativistic limit (ERL) where the energy of the outgoing lepton is much larger than its rest mass. This leads to the total cross sections if one integrates the differential cross sections over all directions.

There are two issues that we wish to discuss in this short presentation. The first one has to do with which states one should include in the summation over final states. In the daughter

![Figure 2](https://example.com/figure2.png)

**Figure 2.** The sum of squared top row elements of the CKM matrix from super-allowed transitions with corrections calculated by RHF-RPA (PKO1), RMF-RPA (DD-ME2), and shell model (H&T) \[12\]. Values deduced from measurements of neutron decay \[13\], pion $\beta$ decay \[15\] and nuclear mirror transitions \[14\] are also shown. Taken from Ref. \[16\].
nucleus spectrum the excitation energy of the final state is:

\[ E^* = E_{\text{RPA}} - \left( (m_f - m_i) + (m_n - m_p) \right) \, , \quad (9) \]

where \( E_{\text{RPA}} \) is the excitation energy with respect to the ground state of the parent nucleus, \( m_i \) \((m_f)\) is the calculated mass of the parent (daughter) nucleus and \((m_n - m_p)\) is the neutron-proton mass difference. Some authors choose to include in their summation the RPA states whose RPA energies are larger than the threshold value

\[ Q_{\text{th}}^{\exp} = \left[ (m_f^{\exp} - m_i^{\exp}) + (m_n - m_p) \right] \] \quad (10)

However, in microscopic models the predicted values \(m_f - m_i\) may differ somehow from \(m_f^{\exp} - m_i^{\exp}\) and we prefer not to use this empirical threshold value in the RPA summations to keep the consistency of the model. Furthermore, it would be impossible to apply this recipe in exotic regions of the nuclear chart where no experimental data on masses are available.

### Table 2.

Inclusive cross sections of the reactions \(^{16}\text{O}(\nu_e,e^-)^{16}\text{F}\) averaged over Michel flux from the DAR of \(\mu^+\) [20]. The RHF-RPA results are compared with those from Refs. [21, 22, 23, 24, 25, 26]. All units are in \(10^{-42} \text{ cm}^2\).

|                  | all \(E_{\text{RPA}}\) \(E_{\text{RPA}} \geq Q_{\text{th}}^{\exp}\) | all \(E_{\text{RPA}}\) \(E_{\text{RPA}} \geq Q_{\text{th}}^{\exp}\) |
|------------------|------------------------------------------------|------------------------------------------------|
| PKO1             | 12.19                                           | 8.67                                            |
| PKO2             | 12.12                                           | 8.77                                            |
| DD-ME2 [21, 22]  | 16.9                                            | 10.8                                            |
| shell model [23] | 13.1                                            |                                                 |
| SIII [24]        |                                                 | 9.43                                            |
| HFSk [25]        |                                                 | 6.67                                            |
| WSSk [25]        |                                                 |                                                 |
| RPA [26]         |                                                 | 14.55                                           |
|                  |                                                 |                                                 |

The second issue concerns the value of \(g_A\) adopted in different calculations. While the empirical value is \(g_A = 1.262\) some authors use the effective value \(g_A^{\text{eff}} = 1\) introduced by Bohr and Mottelson. There are mainly two reasons for adopting an effective value but we feel that they are not appropriate in the present circumstances. An effective coupling constant might be necessary in a shell model description due to the limited model space but in our RPA calculations the space is large enough to exhaust the total transition strength. The other reason has to do with the long-lasting quenching factor (of about 2/3) observed in the Gamow-Teller strengths. However, the multipole decomposition analysis of Yako et al. [27] has shown that about 88% of Gamow-Teller strength is detected in \((p,n)\) and \((n,p)\) reactions on \(^{90}\text{Zr}\). Furthermore, it is well known that the RPA can reproduce satisfactorily the total \(\mu^-\) capture rate in \(^{16}\text{O}\) without any quenching of \(g_A\) [28, 29]. Thus, we favor the use of \(g_A\) over \(g_A^{\text{eff}}\) in applying our RPA model to predicting neutrino-nucleus cross-sections. We nevertheless mention in Table 2 the results we obtain when applying the other prescriptions in order to make meaningful comparisons with other predictions.

In Table 2 we present our results of inclusive cross sections of the reaction \(^{16}\text{O}(\nu_e,e^-)^{16}\text{F}\). They correspond to neutrinos from the decay at rest of \(\mu^+\) and averaged over the Michel flux [20]. We compare our predictions with those of other models. The strongest disagreements occur when comparing with those of Refs. [23, 26]. The RHF-RPA results are not very sensitive to the particular parametrization - PKO1 or PKO2 - used.
4. Conclusion
The RHF-RPA is a new tool for exploring in a consistent way many different aspects of nuclear excitations. Its main merit is that, once the density-dependent couplings are fixed by the overall ground state properties there is no additional degree of freedom in the effective Lagrangians and the description of the excited states remains consistent with that of the ground states within the random phase approximation.

We have presented here two examples of studies performed with RHF-RPA, choosing to concentrate on the charge-exchange channels and applying the method to problems of current interest. On the question of isospin mixing in super-allowed transitions there should be little room left for model dependence in RPA predictions if the Coulomb force provides the main effects. Our results still do not contain contributions from charge-symmetry and charge-independence breaking forces which are present and are known to affect the isospin mixing [30]. This should be further studied in the RHF-RPA context. The second example shows that it is possible to apply the present approach to evaluate the neutrino-nucleus cross-sections, and this should be quite useful for future experimental studies. In both examples the RHF-RPA method allows one to describe the isospin and spin-isospin excited states more consistently than in the RMF-RPA approach.

Acknowledgements
The authors are grateful to Nils Paar for helpful discussions. This work is partly supported by State 973 Program 2007CB815000, the NSF of China under Grant Nos. 10775004, 10947013 and 10975008.

References
[1] Ring P 1996 Prog. Part. Nucl. Phys. 37 193
[2] Meng J, Toki H, Zhou S G, Zhang S Q, Long W H and Geng L S 2006 Prog. Part. Nucl. Phys. 57 470
[3] Paar N, Vretenar D, Khan E and Colo G 2007 Rep. Prog. Phys. 70 691
[4] Long W H, Van Giai N and Meng J 2006 Phys. Lett. B 640 150
[5] Long W H, Ring P, Van Giai N and Meng J 2010 Phys. Rev. C 81 024308
[6] Liang H, Van Giai N and Meng J 2008 Phys. Rev. Lett. 101 122502
[7] Long W H, Sagawa H, Van Giai N and Meng J 2007 Phys. Rev. C 76 034314
[8] Liang H, Van Giai N and Meng J 2009 Phys. Rev. C 79 064316
[9] Liang H 2010 Ph.D. thesis Université Paris-Sud (unpublished)
[10] Towner I S and Hardy J C 2008 Phys. Rev. C 77 025501
[11] Lalazissis G A, Nikšić T, Vretenar D and Ring P 2005 Phys. Rev. C 71 024312
[12] Hardy J C and Towner I S 2009 Phys. Rev. C 79 055502
[13] Amsler C et al. 2008 Phys. Lett. B 667 1
[14] Naviliat-Cuncic O and Severijns N 2009 Phys. Rev. Lett. 102 142302
[15] Počanić D et al. 2004 Phys. Rev. Lett. 93 181803
[16] Liang H, Van Giai N and Meng J 2010 J. Phys. Conf. Ser. 205 012028
[17] O’Connell J S, Donnelly T W and Walecka J D 1972 Phys. Rev. C 6 719
[18] Walecka J D 1975 Muon Physics (Academic Press, New York)
[19] Engel J 1998 Phys. Rev. C 57 2004
[20] Krakauer D A et al. 1992 Phys. Rev. C 45 2450
[21] Paar N, Vretenar D, Marketin T and Ring P 2008 Phys. Rev. C 77 024608
[22] Paar N private communication
[23] Auerbach N and Brown B A 2002 Phys. Rev. C 65 024322
[24] Lazauskas R and Volpe C 2007 Nucl. Phys. A 792 210
[25] Jachowicz N, Heyde K, Ryckebusch J and Rombouts S 2002 Phys. Rev. C 65 025501
[26] Sajjad Athar M, Ahmad S and Singh S 2006 Nucl. Phys. A 764 551
[27] Yako K et al. 2005 Phys. Lett. B 615 193
[28] Van Giai N, Auerbach N and Mekjian A Z 1981 Phys. Rev. Lett. 46 1444
[29] Marketin T, Paar N, Nikšić T and Vretenar D 2009 Phys. Rev. C 79 054323
[30] Sagawa H, Van Giai N and Suzuki T 1996 Phys. Rev. C 53 2163