Grothendieck–Teichmüller and Batalin–Vilkovisky

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Abstract. It is proven that, for any affine supermanifold $M$ equipped with a constant odd symplectic structure, there is a universal action (up to homotopy) of the Grothendieck–Teichmüller Lie algebra $\mathfrak{grt}_1$ on the set of quantum BV structures (i.e. solutions of the quantum master equation) on $M$.

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1. Introduction

Let $M$ be a finite dimensional affine $\mathbb{Z}$-graded manifold $M$ over a field $\mathbb{K}$ equipped with a constant degree 1 symplectic structure $\omega$. In particular, the ring of functions $\mathcal{O}_M$ is a Batalin–Vilkovisky algebra, with Batalin–Vilkovisky operator $\Delta$ and bracket $\{\ ,\ \}$. A degree 2 function $S \in \mathcal{O}_M[[u]]$ is a solution of the quantum master equation on $M$ if

$$u\Delta S + \frac{1}{2}\{S, S\} = 0,$$

where $u$ is a formal variable of degree 2. In other words $S$ is a Maurer–Cartan element in the differential graded (dg) Lie algebra $(\mathcal{O}_M[[u]][1], u\Delta, \{\ ,\ \})$.

The Grothendieck–Teichmüller group $GRT_1$ is a pro-unipotent group introduced by Drinfeld in [3]; we denote its Lie algebra by $\mathfrak{grt}_1$. In this paper we show the following result.

**THEOREM 1.1** There is an $L_\infty$ action of the Lie algebra $\mathfrak{grt}_1$ on the differential graded Lie algebra $(\mathcal{O}_M[[u]][1], u\Delta, \{\ ,\ \})$ by $L_\infty$ derivations. In particular, it follows that there is an action of $GRT_1$ on the set of gauge equivalence classes of formal solutions of the quantum master equation, i.e. on gauge

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$^1$See [11] for an introduction into the geometry of the BV formalism.
equivalence classes of Maurer–Cartan elements in the differential graded Lie algebra \((h\mathcal{O}_M[[u]][[[h]]][1], u\Delta, \{ , , \})\), where \(h\) is a formal deformation parameter of degree 0.

Our main technical tool is a version of the Kontsevich graph complex, \((\text{GC}_2[[u]], d_u)\) which controls universal deformations of \((\mathcal{O}_M[[u]][1], u\Delta, \{ , , \})\) in the category of \(L_\infty\) algebras. Using the main result of [13] we show in Section 2 that

\[ H^0(\text{GC}_2[[u]], d_u) \simeq \mathfrak{g}\mathfrak{t}_1 \]

and then use this isomorphism in Section 3 to prove the Main Theorem.

1.1. SOME NOTATION

In this paper \(K\) denotes a field of characteristic 0. If \(V = \bigoplus_{i \in \mathbb{Z}} V^i\) is a graded vector space over \(K\), then \(V[k]\) stands for the graded vector space with \(V[k]^i := V^{i+k}\). For \(v \in V^i\), we set \(|v| := i\). The phrase differential graded is abbreviated by dg. The \(n\)-fold symmetric product of a (dg) vector space \(V\) is denoted by \(\otimes^n V\), and the full symmetric product space by \(\otimes^\bullet V\). For a finite group \(G\) acting on a vector space \(V\), we denote via \(V^G\) the space of invariants with respect to the action of \(G\), and by \(V_G\) the space of coinvariants \(V_G = V/\{gv - v | v \in V, g \in G\}\). As we always work over a field \(K\) of characteristic zero, we have a canonical isomorphism \(V_G \cong V^G\).

We use freely the language of operads. For a background on operads we refer to the textbook [10]. For an operad \(\mathcal{P}\) we denote by \(\mathcal{P}\{k\}\) the unique operad which has the following property: for any graded vector space \(V\) there is a one-to-one correspondence between representations of \(\mathcal{P}\{k\}\) in \(V\) and representations of \(\mathcal{P}\) in \(V[-k]\); in particular, \(\mathcal{E}nd_V\{k\} = \mathcal{E}nd_V[-k]\).

2. A Variant of the Kontsevich Graph Complex

2.1. FROM OPERADS TO LIE ALGEBRAS

Let \(\mathcal{P} = (\mathcal{P}(n))_{n \geq 1}\) be an operad in the category of dg vector spaces with the partial compositions \(\circ_i : \mathcal{P}(n) \otimes \mathcal{P}(m) \to \mathcal{P}(m+n-1), 1 \leq i \leq n\). Then the map

\[
[a, b] := \sum_{i=1}^n a \circ_i b - (-1)^{|a||b|} \sum_{i=1}^m b \circ_i a
\]

makes the vector space \(\mathcal{P} := \prod_{n \geq 1} \mathcal{P}(n)\) into a dg Lie algebra [4, 5]. Moreover, the Lie algebra structure descends to the subspace of coinvariants \(\mathcal{P}_S := \prod_{n \geq 1} \mathcal{P}(n)_S\). Via the identification of invariants and coinvariants \(\mathcal{P}_S \cong \mathcal{P}^S\), we furthermore obtain a Lie algebra structure on the space of invariants \(\mathcal{P}^S := \prod_{n \geq 1} \mathcal{P}(n)^S\) as well.
2.2. AN OPERAD OF GRAPHS AND THE KONTSEVICH GRAPH COMPLEX

For any integers \( n \geq 1 \) and \( \geq 0 \) we denote by \( G_{n,l} \) a set of graphs,\(^2\) \( \{ \Gamma \} \), with \( n \) vertices and \( l \) edges such that (i) the vertices of \( \Gamma \) are labelled by elements of \( [n] := \{1, \ldots, n\} \), (ii) the set of edges, \( E(\Gamma) \), is totally ordered up to an even permutation. For example, 1→ 2 ∈ \( G_{2,1} \). The group \( \mathbb{Z}_2 \) acts freely on \( G_{n,l} \) for \( l \geq 2 \) by changes of the total ordering; its orbit is denoted by \( \{ \Gamma, \Gamma_{opp} \} \). Let \( \mathbb{K}\langle G_{n,l} \rangle \) be the vector space over a field \( \mathbb{K} \) spanned by isomorphism classes, \( [\Gamma] \), of elements of \( G_{n,l} \) modulo the relation \(^3\) \( \Gamma_{opp} = -\Gamma \), and consider a \( \mathbb{Z} \)-graded \( S \)-module,

\[
\text{Gra}(n) := \bigoplus_{l=0}^{\infty} \mathbb{K}\langle G_{n,l} \rangle[l].
\]

Note that graphs with two or more edges between any fixed pair of vertices do not contribute to \( \text{Gra}(n) \), so that we could have assumed right from the beginning that the sets \( G_{n,l} \) do not contain graphs with multiple edges. The \( S \)-module, \( \text{Gra} := \{ \text{Gra}(n) \}_{n \geq 1} \), is naturally an operad with the operadic compositions given by

\[
o_1 : \text{Gra}(n) \otimes \text{Gra}(m) \longrightarrow \text{Gra}(m+n-1) \qquad \Gamma_1 \otimes \Gamma_2 \longrightarrow \sum_{\Gamma \in G_{1,0}^{i_1, i_2}} (-1)^{\sigma_2} \Gamma
\]

where \( G_{i_1, i_2} \) is the subset of \( G_{n+m-1} \# E(\Gamma_1) + \# E(\Gamma_2) \) consisting of graphs, \( \Gamma \), satisfying the condition: the full subgraph of \( \Gamma \) spanned by the vertices labeled by the set \( \{i, i+1, \ldots, i+m-1\} \) is isomorphic to \( \Gamma_2 \), and the quotient graph, \( \Gamma/\Gamma_2 \), obtained by contracting that subgraph to a single vertex, is isomorphic to \( \Gamma_1 \). The sign \( (-1)^{\sigma_2} \) is determined by the equality

\[
\bigwedge_{e \in E(\Gamma)} e = (-1)^{\sigma_2} \bigwedge_{e' \in E(\Gamma_1)} e' \wedge \bigwedge_{e'' \in E(\Gamma_2)} e''.
\]

The unique element in \( G_{1,0} \) serves as the unit element in the operad \( \text{Gra} \). The associated Lie algebra of \( S \)-invariants, \( (\text{Gra}(-2))^S, [\xi, \eta] \) is denoted, following notations of [13], by \( fGC_2 \). Its elements can be understood as graphs from \( G_{n,l} \) but with labeling of vertices forgotten, e.g.

\[\bullet \bullet = \frac{1}{2} \left( \bullet \rightarrow 2 + \bullet \rightarrow 1 \right) \in fGC_2.\]

The cohomological degree of a graph with \( n \) vertices and \( l \) edges is \( 2(n-1) - l \). It is easy to check that \( \bullet \rightarrow \bullet \) is a Maurer–Cartan element in the Lie algebra \( fGC_2 \). Hence, we obtain a dg Lie algebra

\[\langle fGC_2, [\xi, \eta], d := [\bullet \rightarrow \bullet, \eta] \rangle.\]

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\(^2\)A graph \( \Gamma \) is, by definition, a 1-dimensional \( CW \)-complex whose 0-cells are called vertices and 1-dimensional cells are called edges. The set of vertices of \( \Gamma \) is denoted by \( V(\Gamma) \) and the set of edges by \( E(\Gamma) \).

\(^3\)Abusing notations we identify from now an equivalence class \( [\Gamma] \) with any of its representative \( \Gamma \).
One may define a dg Lie subalgebra, $\mathcal{G}_C^2$, spanned by connected graphs with at least trivalent vertices and no edges beginning and ending at the same vertex. It is called the Kontsevich graph complex \cite{7}. We leave it to the reader to verify that the subspace $\mathcal{G}_C^2$ is indeed closed under both the differential and the Lie bracket. We refer to \cite{13} for a detailed explanation of why studying the dg Lie subalgebra $\mathcal{G}_C^2$ rather than full Lie algebra $\mathfrak{g}_C^2$ should be enough for most purposes. The cohomologies of $\mathcal{G}_C^2$ and $\mathfrak{g}_C^2$ were partially computed in \cite{13}.

**THEOREM 2.1** (\cite{13}).

(i) $H^0(\mathcal{G}_C^2, d) \simeq \mathfrak{grt}_1$.

(ii) For any negative integer $i$, $H^i(\mathcal{G}_C^2, d) = 0$.

We shall introduce next a new graph complex which is responsible for the action of $GRT_1$ on the set of quantum master functions on an odd symplectic supermanifold.

### 2.3. A VARIATION OF THE KONTSEVICH GRAPH COMPLEX

The graph $\bullet \vDash \bullet \in \mathfrak{g}_C^2$ has degree $-1$ and satisfies

\[
\begin{bmatrix}
\bullet \vDash \bullet
\end{bmatrix} = \begin{bmatrix}
\bullet \vDash \bullet
\end{bmatrix} = 0.
\]

Let $u$ be a formal variable of degree 2 and consider the graph complex $\mathfrak{g}_C^2[[u]]$ with the differential

\[d_u := d + u\Delta, \quad \text{where} \quad \Delta := \begin{bmatrix}
\bullet \vDash \bullet
\end{bmatrix} .\]

The subspace $\mathcal{G}_C^2[[u]] \subset \mathfrak{g}_C^2[[u]]$ is a subcomplex of $(\mathfrak{g}_C^2[[u]], d_u)$.

**PROPOSITION 2.1.** $H^0(\mathcal{G}_C^2[[u]], d_u) \simeq \mathfrak{grt}_1$ and $H^i(\mathcal{G}_C^2[[u]], d_u) = 0$.

**Proof.** Consider a decreasing filtration of $\mathcal{G}_C^2[[u]]$ by the powers in $u$. The first term of the associated spectral sequence is

\[\mathcal{E}_1 = \bigoplus_{i \in \mathbb{Z}} \mathcal{E}_i, \quad \mathcal{E}_i^j = \prod_{p \geq 0} H^{i-2p}(\mathcal{G}_C^2, d) u^p\]

with the differential equal to $u\Delta$. As $H^0(\mathcal{G}_C^2, d) \simeq \mathfrak{grt}_1$ and $H^{\leq -1}(\mathcal{G}_C^2, d) = 0$, one gets the desired result. $H^0(\mathfrak{g}_C^2[[u]], d_u) \simeq \mathfrak{grt}_1$.

The projections $(\mathcal{G}_C^2[[u]], d_u) \to (\mathcal{G}_C^2, d)$ and $(\mathfrak{g}_C^2[[u]], d_u) \to (\mathfrak{g}_C^2, d)$ sending $u$ to 0 are maps of Lie algebras and induce isomorphisms in degree 0 cohomology. Since the isomorphisms of Theorem 2.1 (i) are maps of Lie algebras as shown in \cite{13}, so are the maps in the above Proposition. \qed
Remark 2.2. Let $\sigma$ be an element of $\mathfrak{grt}_1$ and let $\Gamma^{(0)}_\sigma$ be any cycle representing the cohomology class $\sigma$ in the graph complex $(\mathbb{G}C_2, d)$. Then one can construct a cocycle,

$$\Gamma^u = \Gamma^{(0)}_\sigma + \Gamma^{(1)}_\sigma u + \Gamma^{(2)}_\sigma u^2 + \Gamma^{(3)}_\sigma u^3 + \ldots,$$

representing the cohomology class $\sigma \in \mathfrak{grt}_1$ in the complex $(\mathbb{G}C_2[[u]], d_u)$ by the following induction:

1st step: As $d\Gamma^{(0)}_\sigma = 0$, we have $d(\Delta_1/\Gamma^{(0)}_\sigma) = 0$. As $H^{-1}(\mathbb{G}C_2, d) = 0$, there exists $\Gamma^{(1)}_\sigma$ of degree $-2$ such that $\Delta_1/\Gamma^{(0)}_\sigma = -d\Gamma^{(1)}_\sigma$ and hence

$$(d + u\Delta)(\Gamma^{(0)}_\sigma + \Gamma^{(1)}_\sigma u) = 0 \mod O(u^2).$$

n-th step: Assume we have constructed a polynomial $\sum_{i=1}^n \Gamma^{(i)}_\sigma u^i$ such that

$$(d + u\Delta) \sum_{i=1}^n \Gamma^{(i)}_\sigma u^i = 0 \mod O(u^{n+1}).$$

Then $d(\Delta_1/\Gamma^{(n)}_\sigma) = 0$, and, as $H^{-2n-1}(\mathbb{G}C_2, d) = 0$, there exists a graph $\Gamma^{(n+1)}_\sigma$ in $\mathbb{G}C_2$ of degree $-2n - 2$ such that $\Delta_1/\Gamma^{(n)}_\sigma = -d\Gamma^{(n+1)}_\sigma$. Hence, $(d + u\Delta) \sum_{i=1}^{n+1} \Gamma^{(i)}_\sigma u^i = 0 \mod O(u^{n+2}).$

3. Quantum BV Structures on Odd Symplectic Manifolds

3.1. MAURER–CARTAN ELEMENTS AND GAUGE TRANSFORMATIONS

Let $(g = \bigoplus_{i \in \mathbb{Z}} g^i, [\ , \ ], d)$ be a dg Lie algebra and consider the dg Lie algebra $g_h := h g[[h]] =: \bigoplus_{i \in \mathbb{Z}} g^i_h$, where $h$ is a formal deformation parameter. The group $G := \exp(g^0_h)$ (which is, as a set, $g^0_h$ equipped with the standard Baker–Campbell–Hausdorff multiplication) acts on $g^1_h$,

$$\gamma \to \exp(h) \cdot \gamma := e^{ad_h} \gamma = e^{ad_h} - 1 \frac{ad_h}{ad_h} dh,$$

preserving its subset of Maurer–Cartan elements

$$\mathcal{MC}(g_h) = \left\{ \gamma \in g^1_h | d\gamma + \frac{1}{2} [\gamma, \gamma] = 0 \right\}.$$

We call the $G$-orbits in $\mathcal{MC}(g_h)$ the gauge equivalence classes of Maurer–Cartan elements.

The group of $L_\infty$ automorphism of $g$ acts on $\mathcal{MC}(g_h)$ by the formula

$$F \cdot \gamma := \sum_{n \geq 1} \frac{1}{n!} F_n(\gamma, \ldots, \gamma).$$
where $F_n$ is the $n$-th component of the $L_\infty$ morphism. In particular, let $f$ be an $L_\infty$ derivation of $g$ without linear term. It exponentiates to an $L_\infty$ automorphism $\exp(f)$ of $g$, which acts on $MC(g_{\hbar})$, and in particular on the set of gauge equivalence classes. By a small calculation one may check that if we change $f$ by homotopy, i.e. by adding $dh$ for some degree 0 element $h$ of the Chevalley–Eilenberg complex of $g$, then the induced actions of $\exp(f)$ and $\exp(f + dh)$ on the set of gauge equivalence classes agree.

### 3.2. QUANTUM BV MANIFOLDS

Let $M$ be a $\mathbb{Z}$-graded manifold equipped with an odd symplectic structure $\omega$ (of degree 1). There always exist so-called Darboux coordinates, $(x^a, \psi_a)_{1 \leq a \leq n}$, on $M$ such that $|\psi_a| = -|x^a| + 1$ and $\omega = \sum_a dx^a \wedge d\psi_a$. The odd symplectic structure makes, in the obvious way, the structure sheaf into a Lie algebra with brackets, $\{ , \}$, of degree $-1$. A less obvious fact is that $\omega$ induces a degree $-1$ differential operator, $\Delta_{\omega}$, on the invertible sheaf of semidensities, $Ber(M)^\frac{1}{2}$ [6]. Any choice of a Darboux coordinate system on $M$ defines an associated trivialization of the sheaf $Ber(M)^\frac{1}{2}$; if one denotes the associated basis section of $Ber(M)^\frac{1}{2}$ by $D_{x,\psi}$, then any semidensity $D$ is of the form $f(x, \psi)D_{x,\psi}$ for some smooth function $f(x, \psi)$, and the operator $\Delta_{\omega}$ is given by

$$\Delta_{\omega}(f(x, \psi)D_{x,\psi}) = \sum_{a=1}^{n} \frac{\partial^2 f}{\partial x^a \partial \psi_a} D_{x,\psi}.$$ 

Let $u$ be a formal parameter of degree 2. A quantum master function on $M$ is a $u$-dependent semidensity $D$ which satisfies the equation

$$\Delta_{\omega} D = 0$$

and which admits, in some Darboux coordinate system, a form

$$D = e^{\frac{S}{2}} D_{x,\psi},$$

for some $S \in \mathcal{O}_M[[u]]$ of total degree 2, where $\mathcal{O}_M$ is the algebra of functions on $M$. In the literature it is this formal power series in $u$ which is often called a quantum master function. Let us denote the set of all quantum master functions on $M$ by $QM(M)$. It is easy to check that the equation $\Delta_{\omega} D = 0$ is equivalent to the following one,

$$u\Delta S + \frac{1}{2}\{S, S\} = 0,$$  \hspace{1cm} (2)

where $\Delta := \sum_{a=1}^{n} \frac{\partial^2}{\partial x^a \partial \psi_a}$. This equation is often called the quantum master equation, while a triple $(M, \omega, S \in QM(M))$ a quantum BV manifold.
Let us assume from now on that \( M \) is affine or formal (i.e., we work with \( \infty \)-jets of functions at some point) and that a particular Darboux coordinate system is fixed on \( M \) up to affine transformations\(^4\) so that the algebra of function on \( M \) is \( \mathcal{O}_M \cong K[x^a, \psi_a] \) or \( \mathcal{O}_M \cong K[[x^a, \psi_a]] \).

For later reference we will also consider solutions of (2) that depend on a formal deformation parameter \( \hbar \) of degree 0, \( S \in \hbar \mathcal{O}_M[[u]][[\hbar]] \). We will call the set of such \( S \) the set of formal solutions of the quantum master equation and denote it by \( \mathcal{Q}\mathcal{M}_\hbar(M) \).

### 3.3. AN ACTION OF \( GRT_1 \) ON QUANTUM MASTER FUNCTIONS

The constant odd symplectic structure on \( M \) makes \( \mathcal{O}_M \) into a representation

\[
\rho : \text{Gra}(n) \longrightarrow \text{End}_V(n) = \text{Hom}_{\text{cont}}(\mathcal{O}_M^\otimes n, \mathcal{O}_M)
\]

\[
\Gamma \longrightarrow \Phi_\Gamma
\]

of the operad \( \text{Gra} \) as follows:

\[
\Phi_\Gamma(S_1, \ldots, S_n) := \pi( \prod_{e \in E(\Gamma)} \Delta_e(S_1(x_{(1)}, \psi_{(1)}, u) \otimes S_2(x_{(2)}, \psi_{(2)}, u) \otimes \ldots \otimes S_n(x_{(n)}, \psi_{(n)}, u)))
\]

where, for an edge \( e \) connecting vertices labeled by integers \( i \) and \( j \),

\[
\Delta_e = \sum_{a=1}^n \frac{\partial}{\partial x_{(i)}^a} \frac{\partial}{\partial \psi_{a(j)}} + \frac{\partial}{\partial \psi_{a(i)}} \frac{\partial}{\partial x_{(j)}^a}
\]

with the subscript \((i)\) or \((j)\) indicating that the derivative operator is to be applied to the \( i \)-th of \( j \)-th factor in the tensor product. The symbol \( \pi \) in (4) denotes the multiplication map,

\[
\pi : V^\otimes n \longrightarrow V
\]

\[
S_1 \otimes S_2 \otimes \ldots \otimes S_n \longrightarrow S_1 S_2 \cdots S_n.
\]

Let \( V := \mathcal{O}_M[[u]] \). Then by \( u \)-linear extension, we obtain a continuous representation (in the category of topological \( K[[u]] \)-modules)

\[
\text{Gra}[[u]] \longrightarrow \text{End}_V = \text{Hom}_{\text{cont}}(V^\otimes, V).
\]

The space \( V[1] \) is a topological dg Lie algebra with differential \( u \Delta \) and Lie bracket \( \{ , \} \). These data define a Maurer–Cartan element, \( \gamma_{QM} := u \Delta \oplus \{ , \} \) in the

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\(^4\)This is not a serious loss of generality as any quantum master equation can be represented in the form (2). Our action of \( GRT_1 \) on \( \mathcal{Q}\mathcal{M}_\hbar(M) \) depends on the choice of an affine structure on \( M \) in exactly the same way as the classical Kontsevich's formula for a universal formality map [8] depends on such a choice. A choice of an appropriate affine connection on \( M \) and methods of the paper [2] can make our formulae for the \( GRT_1 \) action invariant under the group of symplectomorphisms of \((M, \omega)\); we do not address this \textit{globalization} issue in the present note.
Lie algebra \((\text{End}_V(-2))^S \subset CE^*(V, V)\), where \(CE^*(V, V)\) is the Lie algebra of coderivations

\[
CE^*(V, V) = (\text{Coder}(\odot^{* \geq 1}(V[2])), [ \ , \ ] \text{ with } CE^*(V, V)_{(m)} := \text{Hom}(\odot^{* \geq m+1}(V[2]), V[2]),
\]
of the standard graded co-commutative coalgebra, \(\odot^{* \geq 1}(V[2])\), co-generated by a vector space \(V\). The set \(MC(CE^*(V, V))\) can be identified with the set of \(L_\infty\) structures on the space \(V[1]\).

The map sending an operad \(P\) to the Lie algebra of invariants \(\prod_n P\{-2\}(n)^{S_n}\) is functorial. Hence, from the representation (4) we obtain a map of graded Lie algebras

\[
fGC_2[\{u\}] \cong (\text{Gra}\{-2\}[[u]])^S \rightarrow (\text{End}_V(-2))^S \subset CE^*(V, V)
\]

One checks that the Maurer–Cartan element \(\gamma_{Q, M} \in CE^*(V, V)\). Hence, we obtain a morphism of dg Lie algebras

\[
(fGC_2[\{u\}], [ \ , \ ], dh) \rightarrow (CE^*(V, V), [ \ , \ ], \delta := [\gamma_{Q, M}, \ ]),
\]
and by restriction a morphism

\[
\Phi: (GC_2[\{u\}], [ \ , \ ], dh) \rightarrow (CE^*(V, V), [ \ , \ ], \delta := [\gamma_{Q, M}, \ ]),
\]
Hence, we also obtain a morphism of their cohomology groups,

\[
\text{grt}_1 \simeq H^0(GC_2[\{u\}], d_u) \rightarrow H^0(CE^*(V, V), \delta).
\]

Let \(\sigma\) be an arbitrary element in \(\text{grt}_1\) and let \(\Gamma^{u}_\sigma\) be a cocycle representing \(\sigma\) in the graph complex \((GC_2[\{u\}], d_u)\). We may assume that \(\Gamma^{u}_\sigma\) consists of graphs with at least 4 vertices; see [13]. Then the element \(\Phi(\Gamma^{u}_\sigma)\) describes an \(L_\infty\) derivation of the Lie algebra \(V[1]\) without the linear term. By exponentiation we obtain an \(L_\infty\) automorphism,

\[
F^\sigma = \{F^\sigma_n : \odot^n V \rightarrow V[2-2n]\}_{n \geq 1},
\]
of the dg Lie algebra \((V[1], u\Delta, \{ \ , \ \})\) with \(F^\sigma_1 = \text{Id}\). Hence, for any formal quantum master function \(S \in QM_h(M)\) the series

\[
S^\sigma := S + \sum_{n \geq 2} \frac{1}{n!} F^\sigma_n (S, \ldots, S)
\]
gives again a formal quantum master function. The induced action on gauge equivalence classes of such functions is well defined, i.e. it does not depend on the representative $\Gamma^\mu_\sigma$ chosen. This is the acclaimed homotopy action of $GRT_1$ on $QM_h(M)$ for any affine odd symplectic manifold $M$.

**Remark 3.1.** As pointed out by one of the referees, there is also a stronger notion of “homotopy action” that holds in our setting. We will only consider the infinitesimal version. Then, we do not only have a Lie algebra morphism $grt_1 \rightarrow H^0(CE^\bullet(V, V))$, but an $L_\infty$ morphism $grt_1 \rightarrow CE^\bullet(V, V)$ as follows. First, consider the truncated version $(GC_2[[u]])^{tr}$ of the dg Lie algebra $GC_2[[u]]$, which is by definition the same as $GC_2[[u]]$ in negative degrees, zero in positive degrees, and consists of the degree zero cocycles in degree zero. By Proposition 2.1 the canonical projection $(GC_2[[u]])^{tr} \rightarrow grt_1$ is a quasi-isomorphism. Hence we can obtain the desired $L_\infty$ morphism $grt_1 \rightarrow CE^\bullet(V, V)$ by lifting the zig-zag

$$grt_1 \xrightarrow{\sim} (GC_2[[u]])^{tr} \xrightarrow{\approx} CE^\bullet(V, V).$$

This proves the first claim of the main Theorem.

**Remark 3.2.** It is a well-known result due to Tamarkin [12] that the Grothendieck Teichmüller group $GRT_1$ acts on the operad of chains of the little disks operad. In fact, one can show that this $GRT_1$ action extends to an action on the operad of chains of the framed little disks operad, which is quasi-isomorphic to the Batalin–Vilkovisky operad. Hence, one obtains in particular an action of $GRT_1$ on the set of Batalin–Vilkovisky algebra structures on any vector space, and on their deformations, up to homotopy. In our setting the algebra $\mathcal{O}_M$ is an algebra over the framed little disks operad. Any solution $S = S_0 + uS_1 + u^2S_2 + \cdots$ of the master equation (2) yields a deformation of the Batalin–Vilkovisky structure on $\mathcal{O}_M$, up to homotopy. Concretely, to $S$ one may associate a $BV_{\infty}^{com}$-structure (see [9] or [1, section 5.3]), whose $n$-th order “BV” operator is defined as $\Delta_n := [S_n, \cdot]$ (notation as in [1, section 5.3]). The $GRT_1$ action on solutions of the master equation described above can hence be seen as a shadow of this more general action of $GRT_1$ on the framed little disks operad. However, we leave the details to elsewhere.

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5The series trivially converges since we work in the formal setting, i.e. $S = h(\cdots)$. Ideally, of course, one hopes to have a nonzero convergence radius in $h$, but we cannot guarantee this.
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