A BETTER BOUND ON THE JENSEN’S OPERATOR INEQUALITY

HAMID REZA MORADI, SHIGERU FURUICHI, AND MOHAMMED SABABHEH

Abstract. The primary goal of this paper is to improve the operator version of Jensen inequality. As an application, we provide an improvement for the celebrated Ando’s inequality. Additionally, we give a tight bound for the operator Hölder inequality.

1. Introduction

We begin with fixing some common notations. Let $\mathcal{H}$ be a complex Hilbert space, and $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. An operator $A$ is said to be positive (resp. strictly positive) if and only if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$ (resp. $\langle Ax, x \rangle > 0$ for all non-zero $x \in \mathcal{H}$). For strictly positive operators $A$ and $B$, the $v$-geometric mean is defined as

$$A_\sharp v B = A^{\frac{1}{2}} \left( A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^v A^{\frac{1}{2}} \quad (v \in [0, 1]).$$

A real-valued function $f$ defined on an interval $I$ satisfying

$$(1.1) \quad f \left( (1 - v) A + v B \right) \leq (1 - v) f (A) + v f (B) \quad (v \in [0, 1])$$

for all self-adjoint operators $A, B \in \mathcal{B}(\mathcal{H})$ such that $\sigma (A), \sigma (B) \subset I$ is called an operator convex function, where $\sigma (X)$ means the spectrum of $X \in \mathcal{B}(\mathcal{H})$. The function $f$ is operator concave on $I$, if the inequality (1.1) is reversed. It is an essential fact that $f (t) = t^r$, $r \in [0, 1]$ is operator concave on $(0, \infty)$ and so is operator convex for $r \in [-1, 0] \cup [1, 2]$ on $(0, \infty)$.

A linear map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is called positive (resp. strictly positive) if $\Phi (A) \geq 0$ (resp. $\Phi (A) > 0$) whenever $A \geq 0$ (resp. $A > 0$), and $\Phi$ is said to be normalized if $\Phi (1_\mathcal{H}) = 1_\mathcal{K}$, where $1$ is the identity operator.

Let $f : I \to \mathbb{R}$ be a convex function and $x_1, \ldots, x_n \in I$ and $w_1, \ldots, w_n$ positive numbers with $W_n = \sum_{i=1}^n w_i$. The famous Jensen inequality asserts that

$$(1.2) \quad f \left( \frac{1}{W_n} \sum_{i=1}^n w_i x_i \right) \leq \frac{1}{W_n} \sum_{i=1}^n w_i f (x_i).$$

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In [17], one can find an operator form of (1.2) which says that if $f : I \to \mathbb{R}$ is an operator convex function, then

\begin{equation}
(1.3) \quad f \left( \frac{1}{W_n} \sum_{i=1}^{n} w_i A_i \right) \leq \frac{1}{W_n} \sum_{i=1}^{n} w_i f (A_i)
\end{equation}

whenever $A_1, \ldots, A_n$ are self-adjoint operators with spectra contained in $I$.

The celebrated Choi–Davis–Jensen inequality [3, 4] asserts that if $f : I \to \mathbb{R}$ is an operator convex and, $\Phi : \mathcal{B} (\mathcal{H}) \to \mathcal{B} (\mathcal{K})$ is a normalized positive linear mapping, and $A$ is a self-adjoint operator with spectra contained in $I$, then

\begin{equation}
(1.4) \quad f (\Phi (A)) \leq \Phi (f (A)).
\end{equation}

In the past few years, considerable attentions have been put towards refining or reversing inequalities (1.2), (1.3), and (1.4) and some related inequalities. We refer the interested reader to [11, 14, 15, 18].

The main result of this paper is included in the next section, where we present an improvement of the operator Jensen inequality inspired by the observation of Dragomir in [5]. This refinement enables us to improve the celebrated Ando’s inequality. Additionally, we will refine a known result by Hansen which is related to the perspective of operator convex functions and positive linear maps.

2. Main Results

As it is mentioned in [1, Corollary 1], if $f : I \to \mathbb{R}$ is a convex function, $A_1, \ldots, A_n$ self-adjoint operators with spectra contained in $I$, and $w_1, \ldots, w_n$ positive numbers such that $\sum_{i=1}^{n} w_i = 1$, then

\begin{equation}
(2.1) \quad f \left( \sum_{i=1}^{n} w_i \langle A_i x, x \rangle \right) \leq \sum_{i=1}^{n} w_i \langle f (A_i) x, x \rangle
\end{equation}

where $x \in \mathcal{H}$ with $\|x\| = 1$.

In the following theorem, we make a refinement of the inequality (2.1).

**Theorem 2.1.** Let $f : I \to \mathbb{R}$ be a convex function, $A_1, \ldots, A_n$ self-adjoint operators with spectra contained in $I$, and $w_1, \ldots, w_n$ positive numbers such that $\sum_{i=1}^{n} w_i = 1$. Assume $J \subset \{1, 2, \ldots, n\}$ and $J^c = \{1, 2, \ldots, n\} \setminus J$, $\omega_J \equiv \sum_{i \in J} w_i$, $\omega_{J^c} = 1 - \sum_{i \in J} w_i$. Then for any $x \in \mathcal{H}$ with $\|x\| = 1$,

\begin{equation}
(2.2) \quad f \left( \sum_{i=1}^{n} w_i \langle A_i x, x \rangle \right) \leq \Psi (f, \mathcal{A}, J, J^c) \leq \sum_{i=1}^{n} w_i \langle f (A_i) x, x \rangle
\end{equation}
where

$$\Psi (f, A, J, J^c) \equiv \omega_J f \left( \frac{1}{\omega_J} \sum_{i \in J} w_i \langle A_i x, x \rangle \right) + \omega_{J^c} f \left( \frac{1}{\omega_{J^c}} \sum_{i \in J^c} w_i \langle A_i x, x \rangle \right).$$

The inequality (2.2) reverses if the function $f$ is concave on $I$.

**Proof.** We can replace $x_i$ by $\langle A_i x, x \rangle$ where $x \in H$ and $\|x\| = 1$, in (1.2). Hence, by using [7, Theorem 1.2], we can immediately infer that

$$f \left( \frac{1}{W_n} \sum_{i=1}^n w_i \langle A_i x, x \rangle \right) \leq \frac{1}{W_n} \sum_{i=1}^n w_i f \left( \langle A_i x, x \rangle \right),$$

(2.3)

$$\leq \frac{1}{W_n} \sum_{i=1}^n w_i \langle f (A_i) x, x \rangle$$

where $W_n = \sum_{i=1}^n w_i$. Now a simple calculation shows that

$$\sum_{i=1}^n w_i \langle f (A_i) x, x \rangle = \sum_{i \in J} w_i \langle f (A_i) x, x \rangle + \sum_{i \in J^c} w_i \langle f (A_i) x, x \rangle$$

(2.4)

$$= \omega_J \left( \frac{1}{\omega_J} \sum_{i \in J} w_i \langle f (A_i) x, x \rangle \right) + \omega_{J^c} \left( \frac{1}{\omega_{J^c}} \sum_{i \in J^c} w_i \langle f (A_i) x, x \rangle \right)$$

$$\geq \omega_J f \left( \frac{1}{\omega_J} \sum_{i \in J} w_i \langle A_i x, x \rangle \right) + \omega_{J^c} f \left( \frac{1}{\omega_{J^c}} \sum_{i \in J^c} w_i \langle A_i x, x \rangle \right)$$

$$= \Psi (f, A, J, J^c)$$

where we used the inequality (2.3). On the other hand,

$$\Psi (f, A, J, J^c) = \omega_J f \left( \frac{1}{\omega_J} \sum_{i \in J} w_i \langle A_i x, x \rangle \right) + \omega_{J^c} f \left( \frac{1}{\omega_{J^c}} \sum_{i \in J^c} w_i \langle A_i x, x \rangle \right)$$

(2.5)

$$\geq f \left( \omega_J \left( \frac{1}{\omega_J} \sum_{i \in J} w_i \langle A_i x, x \rangle \right) + \omega_{J^c} \left( \frac{1}{\omega_{J^c}} \sum_{i \in J^c} w_i \langle A_i x, x \rangle \right) \right)$$

$$= f \left( \sum_{i=1}^n w_i \langle A_i x, x \rangle \right).$$

In the above computation we have used the assumption that $f$ is a convex function.

Thus, relation (2.4), together with inequality (2.5), yields the inequality (2.2). □

The following refinements of the arithmetic–geometric–harmonic mean inequality are of interest. Apparently, they have not been stated before either in [5] or the research papers cited therein.
Corollary 2.1. Let $a_1, \ldots, a_n$ be positive numbers and let $\{w_i\}, J, J^c$ be as in Theorem 2.1. Then

\[
\left( \sum_{i=1}^{n} w_i a_i^{-1} \right)^{-1} \leq \left( \frac{1}{\omega J} \sum_{i \in J} w_i a_i^{-1} \right)^{-\omega J} \left( \frac{1}{\omega J^c} \sum_{i \in J^c} w_i a_i^{-1} \right)^{-\omega J^c} \\
\leq \prod_{i=1}^{n} a_i^{w_i} \\
\leq \left( \frac{1}{\omega J} \sum_{i \in J} w_i a_i \right)^{\omega J} \left( \frac{1}{\omega J^c} \sum_{i \in J^c} w_i a_i \right)^{\omega J^c} \\
\leq \sum_{i=1}^{n} w_i a_i
\]

and

\[
\left( \sum_{i=1}^{n} w_i a_i^{-1} \right)^{-1} \leq \left( \omega J \prod_{i \in J} a_i^{\frac{w_i}{\omega J}} + \omega J^c \prod_{i \in J^c} a_i^{\frac{w_i}{\omega J^c}} \right)^{-1} \\
\leq \prod_{i=1}^{n} a_i^{w_i} \\
\leq \omega J \prod_{i \in J} a_i^{\frac{w_i}{\omega J}} + \omega J^c \prod_{i \in J^c} a_i^{\frac{w_i}{\omega J^c}} \\
\leq \sum_{i=1}^{n} w_i a_i.
\]

By virtue of Theorem 2.1, we have the following result:

Corollary 2.2. Let $f : I \rightarrow \mathbb{R}$ be a non-negative increasing convex function, $A_1, \ldots, A_n$ positive operators with spectra contained in $I$, and let $\{w_i\}, J, J^c$ be as in Theorem 2.1. Then

\[
(2.6) \quad f \left( \left\| \sum_{i=1}^{n} w_i A_i \right\| \right) \leq \omega_J f \left( \frac{1}{\omega J} \left\| \sum_{i=1}^{n} w_i A_i \right\| \right) + \omega_{J^c} f \left( \frac{1}{\omega J^c} \left\| \sum_{i=1}^{n} w_i A_i \right\| \right) \leq \left\| \sum_{i=1}^{n} w_i f(A_i) \right\|.
\]

The inequality (2.6) reverses if the function $f$ is non-negative increasing concave on $I$. 
Proof. On account of assumptions, we can write

\[
\sup_{\|x\|=1} f \left( \sum_{i=1}^{n} w_i \langle A_i x, x \rangle \right) = f \left( \sup_{\|x\|=1} \left\langle \sum_{i=1}^{n} w_i A_i x, x \right\rangle \right) \\
= f \left( \left\| \sum_{i=1}^{n} w_i A_i \right\| \right) \\
\leq \omega_J f \left( \frac{1}{\omega_J} \left\| \sum_{i \in J} w_i A_i \right\| \right) + \omega_{J^c} f \left( \frac{1}{\omega_{J^c}} \left\| \sum_{i \in J^c} w_i A_i \right\| \right) \\
\leq \sup_{\|x\|=1} \left\langle \sum_{i=1}^{n} w_i f(A_i) x, x \right\rangle \\
= \left\| \sum_{i=1}^{n} w_i f(A_i) \right\|.
\]

This completes the proof. \qed

A remark on Corollary 2.2 may be added:

Remark 2.1. Let \( A_1, \ldots, A_n \) be positive operators and let \( \{w_i\}, J, J^c \) be as in Theorem 2.1. Then for any \( r \geq 1 \),

\[
(2.7) \quad \left\| \sum_{i=1}^{n} w_i A_i \right\|^r \leq \omega_J \left( \frac{1}{\omega_J} \left\| \sum_{i \in J} w_i A_i \right\| \right)^r + \omega_{J^c} \left( \frac{1}{\omega_{J^c}} \left\| \sum_{i \in J^c} w_i A_i \right\| \right)^r \leq \left\| \sum_{i=1}^{n} w_i A_i^r \right\|.
\]

For \( 0 < r \leq 1 \), the reverse inequalities hold. If the operators are strictly positive, \((2.7)\) is also true for \( r < 0 \).

The multiple version of the inequality (1.4) is proved in [16, Theorem 1] as follows: Let \( f : I \to \mathbb{R} \) be an operator convex, \( \Phi_1, \ldots, \Phi_n \) normalized positive linear mappings from \( \mathcal{B}(H) \) to \( \mathcal{B}(K) \), \( A_1, \ldots, A_n \) self-adjoint operators with spectra contained in \( I \), and \( w_1, \ldots, w_n \) positive numbers such that \( \sum_{i=1}^{n} w_i = 1 \), then

\[
(2.8) \quad f \left( \sum_{i=1}^{n} w_i \Phi_i(A_i) \right) \leq \sum_{i=1}^{n} w_i f(\Phi_i(A_i)).
\]

The following is a refinement of (2.8). This result was found by Moslehian and Kian [19, Corollary 3.2], with a different expression. However, we mimic some ideas of Dragomir [5, Theorem 1] for obtaining it.

Theorem 2.2. Let \( f : I \to \mathbb{R} \) be an operator convex, \( \Phi_1, \ldots, \Phi_n \) normalized positive linear mappings from \( \mathcal{B}(H) \) to \( \mathcal{B}(K) \), \( A_1, \ldots, A_n \) self-adjoint operators with spectra contained in \( I \),
and let \( \{w_i\}, J, J^c \) be as in Theorem 2.1. Then

\[
(2.9) \quad f \left( \sum_{i=1}^{n} w_i \Phi_i (A_i) \right) \leq \Delta (f, \mathbb{A}, J, J^c) \leq \sum_{i=1}^{n} w_i \Phi_i (f (A_i))
\]

where

\[
\Delta (f, \mathbb{A}, J, J^c) \equiv \omega_J f \left( \frac{1}{\omega_J} \sum_{i \in J} w_i \Phi_i (A_i) \right) + \omega_{J^c} f \left( \frac{1}{\omega_{J^c}} \sum_{i \in J^c} w_i \Phi_i (A_i) \right).
\]

The inequality (2.9) reverses if the function \( f \) is operator concave on \( I \).

**Proof.** It is a small exercise to prove that

\[
(2.10) \quad f \left( \frac{1}{W_n} \sum_{i=1}^{n} w_i \Phi_i (A_i) \right) \leq \frac{1}{W_n} \sum_{i=1}^{n} w_i \Phi_i (f (A_i))
\]

where \( W_n = \sum_{i=1}^{n} w_i \). By employing the inequality (2.10) we have

\[
\sum_{i=1}^{n} w_i \Phi_i (f (A_i)) = \sum_{i \in J} w_i \Phi_i (f (A_i)) + \sum_{i \in J^c} w_i \Phi_i (f (A_i))
\]

\[
(2.11) \quad = \omega_J \left( \frac{1}{\omega_J} \sum_{i \in J} w_i \Phi_i (f (A_i)) \right) + \omega_{J^c} \left( \frac{1}{\omega_{J^c}} \sum_{i \in J^c} w_i \Phi_i (f (A_i)) \right)
\]

\[
\geq \omega_J f \left( \frac{1}{\omega_J} \sum_{i \in J} w_i \Phi_i (A_i) \right) + \omega_{J^c} f \left( \frac{1}{\omega_{J^c}} \sum_{i \in J^c} w_i \Phi_i (A_i) \right)
\]

\[
= \Delta (f, \mathbb{A}, J, J^c).
\]

On the other hand, since \( f \) is an operator convex we get

\[
\Delta (f, \mathbb{A}, J, J^c) = \omega_J f \left( \frac{1}{\omega_J} \sum_{i \in J} w_i \Phi_i (A_i) \right) + \omega_{J^c} f \left( \frac{1}{\omega_{J^c}} \sum_{i \in J^c} w_i \Phi_i (A_i) \right)
\]

\[
(2.12) \quad \geq f \left( \omega_J \left( \frac{1}{\omega_J} \sum_{i \in J} w_i \Phi_i (A_i) \right) + \omega_{J^c} \left( \frac{1}{\omega_{J^c}} \sum_{i \in J^c} w_i \Phi_i (A_i) \right) \right)
\]

\[
= f \left( \sum_{i=1}^{n} w_i \Phi_i (A_i) \right).
\]

Combining the two inequalities (2.11) and (2.12), we have the desired inequality.

A special case of (2.9) is the following statement:

**Remark 2.2.** Let \( \Phi_1, \ldots, \Phi_n \) be normalized positive linear mappings from \( \mathcal{B} (\mathcal{H}) \) to \( \mathcal{B} (\mathcal{K}) \), \( A_1, \ldots, A_n \) self-adjoint operators with spectra contained in \( I \), and let \( \{w_i\}, J, J^c \) be as in Theorem 2.1. Then for any \( r \in [-1, 0] \cup [1, 2] \),

\[
\left( \sum_{i=1}^{n} w_i \Phi_i (A_i) \right)^r \leq \omega_J \left( \frac{1}{\omega_J} \sum_{i \in J} w_i \Phi_i (A_i) \right)^r + \omega_{J^c} \left( \frac{1}{\omega_{J^c}} \sum_{i \in J^c} w_i \Phi_i (A_i) \right)^r \leq \sum_{i=1}^{n} w_i \Phi_i (A_i^r).
\]
For $r \in [0, 1]$, the reverse inequalities hold.

The next corollary can be compared to [10, Theorem 1].

**Corollary 2.3.** Let $\Phi_1, \ldots, \Phi_n$ be normalized positive linear mappings from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{K})$, $A_1, \ldots, A_n$ self-adjoint operators with spectra contained in $I$, and let $\{w_i\}, J, J^c$ be as in Theorem 2.1. Then for any $r \geq 1$ and every unitarily invariant norm,

$$
\left\| \left( \sum_{i=1}^{n} w_i \Phi_i (A_i) \right)^r \right\| \leq \left\| \left( \frac{1}{\omega_J} \sum_{i \in J} w_i \Phi_i (A_i^r) \right)^{\frac{1}{r}} + \frac{1}{\omega_{J^c}} \sum_{i \in J^c} w_i \Phi_i (A_i^r) \right\|^{\frac{1}{r}} \leq \sum_{i=1}^{n} w_i \Phi_i (A_i^r).
$$

(2.13)

In particular,

$$
\left\| \left( \sum_{i=1}^{n} w_i X_i^* A_i X_i \right)^r \right\| \leq \left\| \left( \frac{1}{\omega_J} \sum_{i \in J} w_i X_i^* A_i^r X_i \right)^{\frac{1}{r}} + \frac{1}{\omega_{J^c}} \sum_{i \in J^c} w_i X_i^* A_i^r X_i \right\|^{\frac{1}{r}} \leq \sum_{i=1}^{n} w_i X_i^* A_i^r X_i
$$

(2.14)

where $X_1, \ldots, X_n$ are contractions with $\sum_{i=1}^{n} X_i^* X_i = I$.

**Proof.** Of course, the inequality (2.14) is a direct consequence of inequality (2.13), so we prove (2.13). It follows from Remark 2.2 that

$$
\left\| \sum_{i=1}^{n} w_i \Phi_i (A_i) \right\| \leq \left\| \frac{1}{\omega_J} \sum_{i \in J} w_i \Phi_i (A_i) \right\|^\frac{1}{r} + \frac{1}{\omega_{J^c}} \sum_{i \in J^c} w_i \Phi_i (A_i) \right\|^{\frac{1}{r}} \leq \left\| \left( \sum_{i=1}^{n} w_i \Phi_i (A_i) \right)^r \right\|
$$

for any $r \geq 1$. Replacing $A_i$ by $A_i^r$, we get

$$
\left\| \sum_{i=1}^{n} w_i \Phi_i (A_i) \right\| \leq \left\| \frac{1}{\omega_J} \sum_{i \in J} w_i \Phi_i (A_i^r) \right\|^\frac{1}{r} + \frac{1}{\omega_{J^c}} \sum_{i \in J^c} w_i \Phi_i (A_i^r) \right\|^{\frac{1}{r}} \leq \left\| \left( \sum_{i=1}^{n} w_i \Phi_i (A_i^r) \right)^r \right\|
$$

(2.15)
It is well-known that \( \|X\|_p = \|X^r\|^{\frac{1}{r}} \) defines a unitarily invariant norm. So (2.15) implies
\[
\left\| \left( \sum_{i=1}^{n} w_i \Phi_i(A_i) \right)^r \right\| \leq \left\| \left( \omega_J \left( \frac{1}{\omega_J} \sum_{i \in J} w_i \Phi_i(A_i^r) \right)^{\frac{1}{r}} + \omega_{J^c} \left( \frac{1}{\omega_{J^c}} \sum_{i \in J^c} w_i \Phi_i(A_i^r) \right)^{\frac{1}{r}} \right)^r \right\|
\leq \sum_{i=1}^{n} |w_i \Phi_i(A_i^r)|.
\]

The proof is complete. \( \Box \)

Kubo and Ando [13] showed that for every operator mean \( \sigma \) there exists an operator monotone function \( f : (0, \infty) \to (0, \infty) \) such that
\[
(2.16) \quad A \sigma B = A^{\frac{1}{2}} f \left( A^{\frac{1}{2}} B^{-1} A^{\frac{1}{2}} \right) A^{\frac{1}{2}}
\]
for all \( A, B > 0 \). They also proved that if \( f : (0, \infty) \to (0, \infty) \) is operator monotone, the binary operation defined by (2.16) is an operator mean.

We know that (see the estimate (16) in [12]) if \( \sigma \) is an operator mean (in the Kubo-Ando sense) and \( A_i, B_i > 0 \), then
\[
(2.17) \quad \sum_{i=1}^{n} w_i (A_i \sigma B_i) \leq \left( \sum_{i=1}^{n} w_i A_i \right) \sigma \left( \sum_{i=1}^{n} w_i B_i \right).
\]
The following corollary can be regarded as a refinement and generalization of the inequality (2.17).

**Corollary 2.4.** Let \( \sigma \) be an operator mean, \( \Phi_1, \ldots, \Phi_n \) normalized positive linear mappings from \( \mathcal{B}(\mathcal{H}) \) to \( \mathcal{B}(\mathcal{K}) \), \( A_1, \ldots, A_n \), \( B_1, \ldots, B_n \) strictly positive operators with spectra contained in \( I \), and let \( \{w_i\}, J, J^c \) be as in Theorem 2.1. Then
\[
\sum_{i=1}^{n} w_i \Phi_i(A_i \sigma B_i)
\leq \left( \sum_{i \in J} w_i \Phi_i(A_i) \right) \sigma \left( \sum_{i \in J} w_i \Phi_i(B_i) \right) + \left( \sum_{i \in J^c} w_i \Phi_i(A_i) \right) \sigma \left( \sum_{i \in J^c} w_i \Phi_i(B_i) \right)
\leq \left( \sum_{i=1}^{n} w_i \Phi_i(A_i) \right) \sigma \left( \sum_{i=1}^{n} w_i \Phi_i(B_i) \right).
\]
Proof. If $F(\cdot, \cdot)$ is a jointly operator concave, then Theorem 2.2 implies

$$
\sum_{i=1}^{n} w_i \Phi_i(F(A_i, B_i)) \\
\leq \omega_J F\left(\frac{1}{\omega_J} \sum_{i \in J} w_i \Phi_i(A_i), \frac{1}{\omega_J} \sum_{i \in J} w_i \Phi_i(B_i)\right) \\
+ \omega_Jc F\left(\frac{1}{\omega_Jc} \sum_{i \in J^c} w_i \Phi_i(A_i), \frac{1}{\omega_Jc} \sum_{i \in J^c} w_i \Phi_i(B_i)\right) \\
\leq F\left(\sum_{i=1}^{n} w_i \Phi_i(A_i), \sum_{i=1}^{n} w_i \Phi_i(B_i)\right).
$$

(2.18)

It is well-known that $F(A, B) = A\sigma B$ is jointly concave [2], so it follows from (2.18) that

$$
\sum_{i=1}^{n} w_i \Phi_i(A_i \sigma B_i) \\
\leq \omega_J \left(\frac{1}{\omega_J} \sum_{i \in J} w_i \Phi_i(A_i)\right) \sigma \left(\frac{1}{\omega_J} \sum_{i \in J} w_i \Phi_i(B_i)\right) \\
+ \omega_Jc \left(\frac{1}{\omega_Jc} \sum_{i \in J^c} w_i \Phi_i(A_i)\right) \sigma \left(\frac{1}{\omega_Jc} \sum_{i \in J^c} w_i \Phi_i(B_i)\right) \\
= \left(\sum_{i \in J} w_i \Phi_i(A_i)\right) \sigma \left(\sum_{i \in J} w_i \Phi_i(B_i)\right) + \left(\sum_{i \in J^c} w_i \Phi_i(A_i)\right) \sigma \left(\sum_{i \in J^c} w_i \Phi_i(B_i)\right) \\
\leq \left(\sum_{i=1}^{n} w_i \Phi_i(A_i)\right) \sigma \left(\sum_{i=1}^{n} w_i \Phi_i(B_i)\right),
$$

thanks to the homogeneity property of operator means. Hence the proof is completed. \square

By setting $\sigma = \sharp_v (v \in [0, 1])$ and $\Phi_i(X_i) = X_i (i = 1, \ldots, n)$ in Corollary 2.4, we improve the weighted operator Hölder and Cauchy inequalities in the following way:

**Corollary 2.5.** Let $\Phi_1, \ldots, \Phi_n$ be normalized positive linear mappings from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{K})$, $A_1, \ldots, A_n, B_1, \ldots, B_n$ strictly positive operators with spectra contained in $I$, and let $\{w_i\}, J, J^c$ be as in Theorem 2.1. Then for any $v \in [0, 1]$,

$$
\sum_{i=1}^{n} w_i (A_i \sharp_v B_i) \leq \left(\sum_{i \in J} w_i A_i\right) \sharp_v \left(\sum_{i \in J} w_i B_i\right) + \left(\sum_{i \in J^c} w_i A_i\right) \sharp_v \left(\sum_{i \in J^c} w_i B_i\right) \\
\leq \left(\sum_{i=1}^{n} w_i A_i\right) \sharp_v \left(\sum_{i=1}^{n} w_i B_i\right).
$$
In particular,
\[
\sum_{i=1}^{n} w_i (A_i \# B_i) \leq \left( \sum_{i \in J} w_i A_i \right) \# \left( \sum_{i \in J^c} w_i B_i \right) + \left( \sum_{i \in J^c} w_i A_i \right) \# \left( \sum_{i \in J} w_i B_i \right).
\]

Recall that if \( f \) is operator convex, then (2.16) defines the perspective of \( f \) denoted by \( P_f (A \mid B) \), i.e.,
\[
P_f (A \mid B) = A^{\frac{1}{2}} f \left( A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{\frac{1}{2}}.
\]
The operator perspective enjoys the following property:
\[
P_f (\Phi (A) \mid \Phi (B)) \leq \Phi \left( P_f (A \mid B) \right).
\]
This nice inequality has been proved by Hansen [8, 9]. Let us note that the perspective of an operator convex function is operator convex as a function of two variables (see [6, Theorem 2.2]).

So, taking into account above and applying Theorem 2.2, we get the following result:

**Corollary 2.6.** Let \( f : I \to \mathbb{R} \) be an operator convex, \( \Phi_1, \ldots, \Phi_n \) normalized positive linear mappings from \( \mathcal{B} (\mathcal{H}) \) to \( \mathcal{B} (\mathcal{K}) \), \( A_1, \ldots, A_n \) self-adjoint operators with spectra contained in \( I \), and let \( \{w_i\} \), \( J, J^c \) be as in Theorem 2.1. Then
\[
P_f \left( \sum_{i=1}^{n} w_i \Phi_i (A_i) \mid \sum_{i=1}^{n} w_i \Phi_i (B_i) \right)
\leq \omega_J P_f \left( \frac{1}{\omega_J} \sum_{i \in J} w_i \Phi_i (A_i) \mid \frac{1}{\omega_J} \sum_{i \in J} w_i \Phi_i (B_i) \right) + \omega_{J^c} P_f \left( \frac{1}{\omega_{J^c}} \sum_{i \in J^c} w_i \Phi_i (A_i) \mid \frac{1}{\omega_{J^c}} \sum_{i \in J^c} w_i \Phi_i (A_i) \right)
\leq \sum_{i=1}^{n} w_i \Phi_i \left( P_f (A_i \mid B_i) \right).
\]

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(H.R. Moradi) Young Researchers and Elite Club, Mashhad Branch, Islamic Azad University, Mashhad, Iran
E-mail address: hrmoradi@mshdiau.ac.ir

(S. Furuichi) Department of Information Science, College of Humanities and Sciences, Nihon University, 3-25-40, Sakurajyousui, Setagaya-ku, Tokyo, 156-8550, Japan.
E-mail address: furuichi@chs.nihon-u.ac.jp

(M. Sababheh) Department of Basic Sciences, Princess Sumaya University For Technology, Al Jubaiha, Amman 11941, Jordan.
E-mail address: sababheh@psut.edu.jo, sababheh@yahoo.com