Turbulent Particle Pair Diffusion, Locality Versus Non-locality: Numerical Simulations

NADEEM A. MALIK †

Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, P.O. Box 5046, Dhahran 31261, Kingdom of Saudi Arabia

Particle pair (relative) diffusion in a field of homogeneous turbulence with generalised power-law energy spectra, $E(k) \sim k^{-p}$ for $1 < p \leq 3$ and $k_1 \leq k \leq k_\eta$ with $k_\eta/k_1 = 10^6$, is investigated numerically using Kinematic Simulation (Kraichnan 1970, Fung et al 1992). If $\Delta = |x_2(t) - x_1(t)|$ is the pair separation distance at time $t$, and $\sigma_\Delta = \sqrt{\langle \Delta^2 \rangle}$ (the angled brackets is the ensemble average over particle pairs), then the results show that:

1. The pair diffusivity scales like $D_p \sim \sigma_\Delta^{\gamma_p}$ with $\gamma_p$ obtained from the simulations such that $\gamma'_p < \gamma_p < 2$, where $\gamma'_p = (1 + p)/2$ is the Richardson locality scaling. The range of $\Delta$ over which these scalings are observed diminishes from above and from below as $p$ increases towards 3.

2. $M(p) = \gamma_p/\gamma'_p > 1$ in the range $1 < p < 3$, and $M$ has a peak at $p_0 \approx 1.8$. This suggests that for spectra close to this in the range $1.5 < p < 2$, which includes Kolmogorov turbulence, local and non-local correlations play comparable roles in the pair diffusion process.

3. The mean square separation scales like $\langle \Delta^2 \rangle_p \sim \tau_p^{\chi_p}$ where $\chi_p = 1/(1 - \gamma_p/2)$ and $\tau_p$ is an adjusted travel time.

4. For Kolmogorov turbulence $p = 5/3$, we observe $D_{5/3} \sim \sigma_{5/3}^{53}$, and $\langle \Delta^2 \rangle_{5/3} \sim \tau^{1.2}$.

5. At $p_* \approx 1.4$ ($E(k) \sim k^{-1.4}$) we observe $D_{p_*} \sim \sigma_{4/3}^{p_*}$, and $\langle \Delta^2 \rangle_{p_*} \sim \tau^3$; these are different to the Richardson 4/3-law and Richardson-Obukov $t^{3}$-regime which occur for $p = 5/3$.

The action of random large scale 'sweeping' by energy containing eddies has little effect on the pair diffusion process.

These results are consistent with Malik’s (2014) theory, and supports the principle upon which the theory is based that both local and non-local correlations are effective in the pair diffusion process inside the inertial subrange. We find no evidence in support of Richardson’s (1926) locality in space theory which is based upon the idea that local correlations alone are effective.

Key words:
1. Introduction

Richardson’s (1926) locality in space hypothesis leads to scalings for the pair diffusivity in Kolmogorov turbulence of the form $D \sim \langle \Delta^{4/3} \rangle$; a generalising of this to turbulence spectra like $E(k) \sim k^{-p}$, $1 < p \leq 3$, implies that $D_p \sim \sigma^\gamma_p \Delta$ where $\gamma_p = (1 + p)/2$. Although there has never been a set of experiments or simulations that has unequivocally proven this theory, Richardson’s locality and the ensuing 4/3-law has become widely accepted in the turbulence community to date. See Salazar and Collins (2009).

However, Malik (2014) has proposed a new theory based upon the idea that both local and non-local correlations are effective in the pair diffusion process inside the inertial subrange, which leads to pair diffusion scalings of the form, $D_p \sim \sigma^\gamma_p \Delta$, where $\gamma_p < \gamma < 2$.

With the tremendous advance in computing power since Fung et al (1992) reported their findings on turbulent pair diffusion using Kinematic Simulations (KS), we are now in a position to re-examine this problem, this time with bigger inertial subranges and much larger ensembles of flow field realisations and particle pairs than in any previous study.

Kinematic Simulation (Kraichnan 1970, Fung et al 1992) is a Lagrangian method for particle diffusion in which the velocity field is prescribed as a sum of energy-weighted Fourier modes. It was used to examine single particle diffusion, Turfus and Hunt (1987), and pair diffusion, Malik (1991), Fung et al (1992), where an approximate $t^3$-regime was observed in short inertial subranges. KS has been validated against Direct Numerical Simulation (Malik and Vassilicos 1999) where it has reproduce particle pair diffusion statistics to good accuracy, including the Lagrangian flatness factor which is a fourth order statistic.

KS has also been used in studies in turbulence with generalised power-law energy spectra of the form $E(k) \sim k^{-p}$ for $p > 1$; for passive particles by Fung and Vassilicos (1996), Malik (1996), Nicolleau and Nokowaski (2011); for studies of turbulent diffusion of inertial particles, by Maxey (1987), Meneguz and Reeks (2011); and as a sub-grid scale model for diffusion of contaminants in estuarine and coastal waters, by Malik et al (1991, 1992).

However, KS is not without controversy because some studies have shown that KS does not reproduce Richardson locality scaling, which is problematic if one assumes locality to be true. Thomson and Devenish (2005) and Nicolleau and Nokowaski (2011) argue that this is because KS does not accurately model the dynamical sweeping of the small inertial scales by the large energy scales which they believe is important in the pair diffusion process.

The main purpose of this study is to investigate numerically turbulent pair diffusion with a view of addressing directly the question of what governs the pair diffusion process in the inertial subrange; is Richardson’s locality hypothesis true, or is Malik’s locality/non-locality hypothesis more consistent with the obtained results?

In section 3 the results from KS for the pair relative diffusivity $D_p$ and related quantities are presented. In section 4 we discuss the significance and implications of the results. However, first it is important to re-examine the KS method in order to establish its suitability for this study. This we do in section 2 below.
2. Kinematic Simulation Method

2.1. Energy spectra and velocity fields

KS generates turbulent-like non-Markovian particle trajectories by releasing particles in flow fields that are incompressible by construction and which satisfy Eulerian statistics up to second order. A turbulent flow field realization is produced as a truncated Fourier series,

\[ \mathbf{u}(x, t) = \sum_{n=1}^{N} \left( (a_n \times \hat{k}_n) \cos (k_n \cdot x + \omega_n t) + (b_n \times \hat{k}_n) \sin (k_n \cdot x + \omega_n t) \right) \tag{2.1} \]

where \( N \) is a suitable number of representative wavemodes, typically hundreds for very long spectral ranges, \( k_\eta/k_1 \gg 1 \). \( k_n \) is a random unit vector \((k_n = \hat{k}_n, k_n = |k_n|)\). The coefficients \( A_n \) and \( B_n \) are chosen such that their orientations are randomly distributed and uncorrelated with any other Fourier coefficient or wavenumber, and their amplitudes are determined by \( (A_n^2) = (B_n^2) = \frac{2}{3} E(k_n) \Delta k_n \), where \( E(k) \). \( k_1 \leq k \leq k_\eta \), is the turbulent energy spectrum. The angled brackets \( \langle \cdot \rangle \) denotes the ensemble average over many flow fields. This construction ensures that the Fourier coefficients are normal to their wavevector \( k_n \) which automatically ensures incompressibility of each flow realization, \( \nabla \cdot \mathbf{u} = 0 \). The flow field ensemble generated in this manner is statistically homogeneous, isotropic and stationary.

The energy spectrum \( E(k) \) can be chosen freely within a finite range of scales. It is common practice in turbulent particle pair studies to prescribe a Kolmogorov-like power law spectrum,

\[ E(k) = C_p \varepsilon^{2/3} L^{5/3} k^{-p} \quad k_1 \leq k \leq k_\eta \quad (= 2\pi/\eta) \tag{2.2} \]

and \( E(k) = 0 \) outside of this range; \( C_p \) is a constant for every \( p \). This form of the energy spectrum is the main interest in pair diffusion studies when the pair separation \( \Delta = \sqrt{\Delta_1^2 + \Delta_2^2 + \Delta_3^2} \) is inside the inertial subrange, \( \eta \ll \Delta \ll L \), far from the smallest and the largest turbulent scales. The largest represented scale of turbulence is \( 2\pi/k_1 \) and smallest is the Kolmogorov scale \( \eta = 2\pi/k_\eta \).

We normalise the constants such that the total energy contained in the range \( k_1 \leq k \leq k_\eta \) is \( 3(u')^2/2 \), where \( u' \) is the rms turbulent velocity fluctuations in each direction. \( \varepsilon \) is determined by integrating the spectrum, \( \int_{k_1}^{k_\eta} E(k) dk = 3(u')^2/2 \), which yields,

\[ (\varepsilon L)^{2/3} = \frac{3(u')^2 (p-1)(k_1 L)^{p-1}}{2C_p \left( 1 - \left( \frac{k_1}{k_\eta} \right)^{p-1} \right)^{p-1}}. \tag{2.3} \]

In our simulations we set, \( K_1 = 1, L = 1 \), and \( C_p = 1.5 \) (Kolmogorov constant) and we also set \( u' = 1 \). Then this yields,

\[ \varepsilon^{2/3} = (p-1) \left( 1 - \left( \frac{k_1}{k_\eta} \right)^{p-1} \right)^{-1}. \tag{2.4} \]

\( p = 1 \) is a singular limit which we do not consider.) The size of the inertial subrange is \( k_\eta/k_1 = 10^6 \). From (2.4), \( v_\eta = (\varepsilon \eta)^{1/3} \) is the velocity microscale, and \( t_\eta = \varepsilon^{-1/3} \eta^{2/3} \) is the time microscale.

The distribution of the wavemodes is geometric, \( k_n = k_1 r^{n-1} \), and the common ratio
is determined from \( r = (k_\eta/k_1)^{1/(N-1)} \). The grid size in wavenode-space of the \( n^{th} \) wavenode is \( \Delta k_n = k_n (\sqrt{r} - 1/\sqrt{r}) \). The frequencies are chosen to be proportional to the eddy-turnover frequencies, i.e. \( \omega_n = 0.5 \sqrt{E_n} E(k_n) \).

A particle trajectory is obtained by integrating the Lagrangian velocity \( u_L(t) \),

\[
\frac{dx}{dt} = u_L(t) = u(x, t). \tag{2.5}
\]

Pairs of trajectories are harvested over a large ensemble of flow realisations and pair statistics are then obtained from it for analysis.

Henceforth, we will attach the subscript 'p' to denote dependence on the energy spectrum; absence of a subscript will denote the Kolmogorov case \( p = 5/3 \).

### 2.2. Large scale sweeping of small scales

Pair diffusion proceeds in response to the strain-rate field induced by eddies comparable to or larger in size than the separation \( \Delta(t) \) at time \( t \). This process occurs in surges as the separation enters into hyperbolic, strain dominated, flow regions where the instantaneous stream lines diverge and the separation vector is aligned in the direction of positive strain. Such a process was called 'structural diffusion' (Fung et al 1992, Malik 1996) and is supported by observations of balloons in the atmosphere, Wilkins (1958), and in particle tracking velocimetry PTV experiments, Malik et al. (1993), Virant et al (1997).

There is evidence quantitatively too that KS produces realistic particle distributions; Meneguz and Reeks (2012) found that the statistics of the segregation in KS generated flow fields for statistically homogeneous isotropic flow fields are similar to those generated by DNS. Furthermore, Malik and Vasilicos (1999) found that particle pair statistics from KS compared well from those from low Reynolds number DNS, as high as the fourth order flatness factor which were in remarkably good agreement.

But recently Thomson and Davenish (2005) and Nicolleau and Nowakowski (2011) obtained results showing that KS does not reproduce Richardson’s scaling law; they found that \( \langle \Delta^2 \rangle \sim t^\alpha \), where \( \alpha \gg 3 \), as high as 6. The latter authors also say that \( \alpha \) depends upon the initial separation, see Fig. 4 in their reference.

These authors believe that Richardson’s locality hypothesis is true in real turbulence, but not in KS where there are non-local effects which alter the effectiveness with which the local eddies disperse the particle pair. Specifically, as there are no dynamical interactions between different scales in KS, the large scales will cut particle pairs right through smaller structures. They thus interpret the KS results as unphysical on the assumption that Richardson locality is true.

On the other hand, as one of the main effects of dynamical sweeping is for the large energy scale eddies to 'carry' the smaller scales, and assuming that other dynamical interactions between large and small scales are statistically small, then we may interpret KS simulations in the absence of explicit large scale sweeping to be in a frame of reference that is carried along by the large scales – which is the appropriate frame in which to investigate pair diffusion studies at the small scales.

A quantitative assessment of its impact in KS can be made in the following manner. We can readily include a large scale energy spectrum in KS simply by extending the energy spectrum to smaller wavenumbers in order to obtain a von Karman energy spectrum
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which is essentially a smooth interpolation between the inertial subrange and large scale spectra which is assumed to scale like $\sim k^4$.

Theoretically, for our analysis, it is sufficient to consider all the large scale energy $E_0$ concentrated at a single wavemode, $k_0 \ll k_1$, and adding it to the inertial subrange spectrum,

$$E(k) \rightarrow E(k) + E_0\delta(k - k_0).$$  \hspace{1cm} (2.6)

A peak in the von Karman spectrum occurs around $k_0 \approx 0.1k_1 \sim 1/10L$. It is readily shown, following a formal derivation similar to Malik (2014), that the contribution of this mode to the pair diffusivity is

$$D^0 \sim \sigma^2 \Delta k_0 \sqrt{E_0} = \frac{\sigma^2 \Delta \sqrt{E_0}}{10L}. \hspace{1cm} (2.7)$$

The question is, when does $D^0$ become dominant over all other contributions to the relative diffusivity, if at all? For this, we require that $D^0 > \text{Min}(D_{nl}^p, D_d^p, D_p)$, which leads to the following condition,

$$E_0 > 100\varepsilon^{2/3}L^{2/3} \text{Min} \left(1, \left(\frac{L}{\sigma_\Delta}\right)^{3-p}, \left(\frac{L}{\sigma_\Delta}\right)^{4-2\gamma_p}\right). \hspace{1cm} (2.8)$$

Using $\varepsilon \sim (u')^3/L$, and normalising $E_0$ by the total energy in the natural von Karman spectrum which is $E_{vk} \approx 25(u')^2$, we obtain the general condition for $1 < p \leq 3$ (ignoring a factor of 4 in view of the various scaling approximations made above),

$$\frac{E_0}{E_{vk}} > \text{Min} \left(1, \left(\frac{L}{\sigma_\Delta}\right)^{3-p}, \left(\frac{L}{\sigma_\Delta}\right)^{4-2\gamma_p}\right) = 1. \hspace{1cm} (2.9)$$

That is,

$$\frac{E_0}{E_{vk}} > 1, \hspace{1cm} (2.10)$$

so $E_0$ must be bigger than the total energy contained in 'natural' turbulence before sweeping can start to affect the inertial subrange scalings. Thus, the random sweeping by the large energy containing wavemodes is unlikely to have much of an effect on the pair diffusion process inside the inertial subrange using KS. We can test for this by comparing cases with $E_0/E_{vk} > 1$, with the base case $E_0 = 0$. This we do in section 3.

2.3. The effect of adjusted travel time

Equation (2.1) in Malik (2014) shows the Richardson pair diffusion law for the pair separation $\langle \Delta^2 \rangle$, including an adjusted travel time $\tau = t - t_\Delta$, which is the time from when the particle pair has decorrelated with, or 'forgotten', its initial condition $\Delta_0$, as proposed by Batchelor (1952) and Fung et al (1992).

Plots of the pair separation directly against the raw time $t$ are extremely sensitive to the initial separation $\Delta_0$. The question is whether such a sensitivity to the initial separation is real or an artifact of presentation on scales with inappropriate coordinates? If the dependency on $\Delta_0$ is a real physical effect, then a dependency on initial separation should persist no matter what time coordinates is chosen.

But if the plots against an appropriate adjusted travel time makes all the different cases
Table 1. The scaling exponents $\gamma_p$ and $\chi_p$ obtained from the simulations. The Richardson locality exponents $\gamma_l$ and $\chi_l$ are also shown.

for a given $p$ collapse on to a single graph, then there is no real dependency of the pair separation process on initial conditions. This is tested in section 3.4.

On the other hand, the pair diffusivity is a function of $\sigma_\Delta$ and independent of initial conditions, and there is no ambiguity is interpreting the plots. It is for this reason that pair diffusion is best studied in terms of the pair diffusivity. In the following results section, we place emphasis on the relative diffusivity, although we will present results for other quantities as well.

3. Results using KS

3.1. Simulation parameters

The spectra $E(k) = C_p|k|^{2/3}L^{5/3-p}k^{-p}$, $k_1 \leq k \leq k_\eta$ were used for $1 < p \leq 3$. Values of $p$ were selected across the whole range; the case $p = 1$ is singular, but we can take $p$ close to this limit; the smallest value of $p$ chosen was 1.01. Table 1 shows all the different cases simulated.

The particle trajectories were obtained by integrating (2.5) using the 4th order Adams-Bashforth predictor-corrector method (4th order Runga-Kutta gives indentical results). Thomson and Devenish (2005) used a variable timestep $\Delta t$ that was small compared to the turnover time of the eddy of the size of the instantaneous pair separation $\Delta(t)$, but larger than the time microscale $t_\eta$. While this speeds up the turnaround time of the calculations, here we want to avoid any extra assumptions so that we can draw unambiguous conclusions from our results. Therefore, in all of our simulations we have kept to a very small but fixed timestep $\Delta t \ll t_\eta$. This has the further advantage of avoiding any smoothening of the particle trajectories that is necessary when using variable timesteps. However, this increases the turnaround time, so a parllel version of the KS code was produced and all simulations were run on the KFUPM High Performance Computing Facility.
We release 8 pairs of particles in each flow realisation placed at the corners of a cube of side 3 units, beginning at (0, 0, 0). This is far enough away for each pair to be independent.

It is crucial to run over a large number of different flow realisations, otherwise the statistics will not converge. Typically the ensemble was around 5000 flow realisations, yielding a harvest of 40,000 particle pair trajectories per simulation.

We ran the simulations for about one large timescale $T = 2\pi/k_1$, which requires around $10^6$ timesteps.

3.2. The effect of initial separation and adjusted time origin

Figure 1(a) shows the plots of the pair separation $\langle \Delta^2 \rangle/\eta^2$ directly against the raw time $t/\eta$, for the case $p = 5/3$ for different initial separations $\Delta_0/\eta$ as indicated on the plot. The plots are sensitive to the initial separation at early times, although for very long times the graphs do converge. The graphs display slopes greatly in excess of the Richardson-Obukov $t^3$ especially at early times. Lines of slope 3 and 5 are shown for comparison.

Plotting the same data as $((\Delta^2/\eta^2)/\eta^2)$ against the adjusted travel time, $\tau = t - t(\Delta_0)$, Fig. 1(b), makes all the different cases collapse onto the same graph over the entire period of time; see also figures 12 and 13 in Fung et al 1992 for a short inertial range of $k_\eta/k_1 = 400$. The travel times $t(\Delta_0)$ are obtained from the data in Fig. 1(a) from when $\langle \Delta^2(t(\Delta_0)) \rangle = \eta^2$. Other cases (not shown) for different powers $p$ yield similar results.

Clearly the pair diffusion is independent of the initial conditions.

3.3. The pair diffusivity $D_p$ and mean square separation $\langle \Delta^2 \rangle_p$

Fig. 2(a) shows the log-log plots of the pair diffusivity $D_p/\eta \nu_\eta$ against $\sigma_\Delta/\eta$ for different energy spectra $p$ (for clarity not all the cases simulated are shown). Lines of slopes 1 and 2 are shown for comparison.

We observe that as $p \to 1$ the slopes of the graphs approach $\gamma_p \to 1$; and as $p \to 3$, the slopes of the graphs approach $\gamma_p \to 2$. The slopes of the graphs for each of the different $p$ are nearly constant over long ranges of the abscissa. Furthermore, the slopes display a smooth transition between the two asymptotic cases at $p = 1$ and 3.

The results in Fig. 2(a) imply an approximate fit of the form,

$$D_p \sim \sigma_\Delta^{\gamma_p}$$

for scaling exponents $\gamma_p$ that are equal to the slopes of the graphs in Fig. 2(a). To see this more clearly, in Fig. 2(b) we have plotted the same data in the form of plots of the compensated relative diffusivity $D_p/\sigma_\Delta^{\gamma_p}$ against $\sigma_\Delta/\eta$, where $\gamma_p$ is the power that gives a near-horizontal line over the longest range of $\Delta$, for each $p$. Table 1 summaries the results obtained.

From Fig. 2(b), the range of $\Delta$ over which the scaling $D_p \sim \sigma_\Delta^{\gamma_p}$ is observed progressively reduces from both ends of the scale. From below because particle separation penetrates further up the inertial subrange before they 'forget' their initial separation due to the diminishing energies contained in the very smallest scales as $p \to 3$. And from above because the long range correlations have increasingly stronger impact as $p \to 3$.

Fig. 3(a) shows log-log plots of the (mean square) pair separation $\langle \Delta^2 \rangle_p/\eta^2$ against the
Figure 1. Pair separation $\langle \Delta^2 \rangle/\eta^2$ plotted against (a) (top) time $t/t_\eta$, (b) (bottom) adjusted travel time $\tau/t_\eta$. 
Figure 2. Pair diffusivity $D_p$ for $p = 1.01, 1.1, 1.3, 1.5, 1.67, 1.9, 2.2, 2.5, 2.9, 3$. (a) (Top) $D_p/\eta \nu_\eta$ against $\sigma_\Delta/\eta$; (b) (bottom) compensated $D_p/\sigma_\Delta^\gamma_p$, against $\sigma_\Delta/\eta$. $\gamma_p$ are shown in Table 1.
Figure 3. Pair separation, from red to grey: $p = 1.01, 1.3, 1.5, 1.7, 1.9, 2.2, 2.5, 3.3$.
(a) (top) $\langle \Delta^2 \rangle_p / \eta^2$ against $t/t_\eta$. (b) (centre) $((\Delta^2)_p - \eta^2)/\eta^2$ against adjusted travel time $\tau_p/t_\eta$. (c) (bottom) $\log((\Delta^2)_p/\delta_p^2)$ against $\tau_p/t_\eta$. 
raw time $t/t_p$ for selected values of $p$. Fig. 3(b) shows log-log plots of $(\langle \Delta^2 \rangle - \eta^2)/\eta^2$ against the adjusted travel times $\tau_p/t_\eta$, for $1 < p < 2$. Figure 3(c) shows log-linear plots of $(\langle \Delta^2 \rangle_p/\delta_p^2)$ against $\tau_p/t_\eta$, for $2 \leq p \leq 3$.

$\delta_p$ is the separation at time $t_p(\Delta_0)$ when the pair are deemed to have forgotten their initial separation $\Delta_0$. For $p < 2$ we have $\Delta_0 < \eta$, and we show the cases with $\delta_p = \eta$. But as $p \to 3$, the energy density in the smallest scales is very small and the ‘memory’ of the initial separation $\Delta_0$ persists for long times even when the separation is inside the inertial subrange. The pair growth is very slow and in order to speed up the simulations, for the cases close to $p = 3$, the initial separation was taken larger than the Kolmogorov scale, $\Delta_0 > \eta$ and $\delta_p > \Delta_0$. (The actual value of $\delta_p$ chosen is obtained from inspection of Fig. 3(a) for every $p$.)

For $p < 2$ the slopes are nearly constant over long periods of the time, indicating that $\langle \Delta^2 \rangle_p \sim \tau^{\chi_p}$, with exponents $\chi_p$ obtained from the simulations. ($\chi_p$ is also related directly to $\gamma_p$ by $\chi_p = 1/(1 - \gamma_p/2)$.)

But as $p \to 3$, then $\gamma_p \to 2$ and $\chi_p \to \infty$ which indicates exponential growth due to the increasing domination of long-range correlations. Therefore, in Fig. 3(c) we plot the cases for $2 \leq p \leq 3$ as log-linear graphs of $(\langle \Delta^2 \rangle_3/\delta_3^2)$ against $\tau_p$. As $p \to 3$, the plots approach a straight line, indicating pure strain dominated exponential growth. The obtained results for $\chi_p$ are shown in Table 1.

Fig. 4 shows plots of the obtained values of $\gamma_p$ against $p$, and the ratio $M(p) = \gamma_p/\gamma_p^l$ against $p$. $M(p)$ shows a peak at $p_m \approx 1.8$.

The fact that $M = 1$ at $p = 3$ does not imply a ‘return’ to locality. The locality scaling happens to be the same as the non-locality scaling in this limit because $\gamma_3^l = \gamma_3^f = 2$.

But the magnitude of the non-local contribution dominates over the local one, such that $D_p^\text{nl}/D_p^\text{l} \gg 1$, as shown in Malik (2014).

The peak at $p_m \approx 1.8$ may be an indicator of the energy spectrum ($\sim k^{-1.8}$) when local and non-local correlations are equally influential in the pair diffusion process. This picture is approximately true in a small range of $p$ centred around the peak value, i.e. for $p$ in the range $1.6 < p < 2$. Interestingly, this range includes Kolmogorov turbulence $p = 5/3$.

From Fig. 4 (and Table 1), for Kolmogorov turbulence $p = 5/3$ we observe that $\gamma_{5/3} \approx 1.53$, and $\chi_{5/3} \approx 4.2$. From Malik’s (2014) scaling, this gives

$$D_{5/3} \sim \varepsilon^{1/3} L^{-0.2} \sigma_\Delta^{1.53}, \quad (\Delta^2)_{5/3} \sim \varepsilon^{1.4} L^{-0.8} \tau^{4.2}, \quad \text{for} \quad p = 5/3. \quad (3.2)$$

Finally, again from Fig. 4 (Table 1), we observe that $\gamma_{p_*} = 4/3$ at $p_* \approx 1.4$. With this turbulence spectrum, $E \sim k^{-1.4}$, the pair diffusion process is independent of the large scale $L$, as noted in Malik (2014), and we obtain,

$$D_{p_*} \sim \varepsilon^{1/3} \sigma_\Delta^{4/3}, \quad (\Delta^2)_{p_*} \sim \varepsilon \tau^{4}. \quad (3.3)$$

This is quite different to the Richardson 4/3-power law and the R-O $t^4$-regime which occurs under the assumption of locality hypothesis for Kolmogorov turbulence $p = 5/3$. 

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3.4. The effect of random large scale sweeping modes

The effect of random large scale modes was tested by placing all the energy in the large (non-inertial) scales as additional random modes at \(k_0 = 0.1k_1\) where there is a peak in the von Karman spectrum, viz \(E_0(k) = E_0\delta(k - k_0)\). See section 2.2.

Fig. 5 shows the pair diffusivities for (a) \(p = 1.01\), (b) \(p = 5/3\), and (c) \(p = 3\), which covers the whole range of interest. On each plot we compare the inertial range case, \(E_0 = 0\) (red), with \(E_0/E_{vk} = 4\) (green) which is the energy content in the von Karman turbulence spectrum, see equation (2.10).

In all three cases, Fig. 5(a)–(c), we observe the same scalings for the different levels of large scale energy \(E_0\). Only the range of validity of the scaling laws diminish slightly. In Fig. 5(b), for \(p = 5/3\), there is an increase in the scatter in the plot for \(E_0/E_{vk} = 4\) presumably due to the large scales causing high frequency noise as it sweeps particles through the smaller eddies. However, the overall scaling remains unchanged.

(In Fig. 5(c) the initial separations in the two plots are different which is why they drop off at different lower values. There is also considerable statistical convergence problems even for \(E_0 = 0\) as \(p \to 3\), which shows up sometimes as wild fluctuations on the plot. Fortunately, this happens mostly is in the range outside of the inertial scaling law range and does not affect our conclusions.)

These results show that large scale random sweeping does not have much effect on the pair diffusion process in the inertial subrange using KS.
Figure 5. Pair diffusivity $D_p/\eta v_\eta$ against the pair separation $\sigma_\Delta/\eta$. From top to bottom: $p =$ (a) 1.01, (b) 5/3, (c) 3. In each plot, the cases for $E_0 = 0$ (red) and $E_0/E_{\nu k} = 4$ (green) are shown.
4. Discussion and Conclusions

Particle pair (relative) diffusion in 3D high Reynolds number homogeneous turbulence with generalised energy spectra of the form $E(k) \sim \varepsilon^{1/3}L^{(5/3-p)}k^{-p}$ for $1 < p \leq 3$ has been investigated with a view of addressing directly the century old problem of what governs the pair diffusion process in the inertial subrange of turbulent motions. Richardson’s (1926) theory is based upon the idea of locality in which only short range correlations of the same scale as the pair separation itself governs this process. Malik’s (2014) theory is based upon the idea that both local and non-local correlations govern this process.

These two theories lead to different scaling laws for the pair diffusivity $D_p$ and related quantities, but they can be tested by a suitable numerical method. Here, we have used the Langrangian method Kinematic Simulation, which produces results that are indicative of the true pair diffusion scalings. In KS, ensembles of realistic particle trajectories are harvested from kinematically prescribed incompressible flow fields. The results described here were obtained from the most comprehensive KS data set produced to date, with ensembles of around 40,000 pairs of particle trajectories from 5,000 flow realisations per run. The accuracy of these trajectories are ensured by using small constant integration timesteps. The size of the inertial subrange was $k_\eta/k_1 = 10^6$.

If $\Delta = |\Delta(t)|$ is the pair separation at time $t$, and $\sigma_\Delta = \sqrt{\langle \Delta^2 \rangle}$, our results show that in the range $1 < p \leq 3$ the pair diffusivity scales like,

$$D_p \sim \sigma_\Delta^{\gamma_p}$$

for scaling exponents $\gamma_p$ that are shown in Table 1, and plotted in Fig. 4 as a function of $p$. We find $\gamma_p$ to be greater than that predicted on the basis of Richardson locality, $\gamma_l = (1 + p)/2$.

In the asymptotic limit $p \to 1$, we find that the pair diffusivity approaches the locality limit $D_p \to D_1 \sim \sigma_\Delta^1$; and in the limit $p \to 3$, it approaches the non-locality limit $D_p \to D_3 \sim \sigma_\Delta^2$.

Furthermore, the range of $\Delta$ over which the $\gamma_p$ scaling is valid diminished from both ends as $p \to 3$; from below due to the diminishing energies in the smallest scales, and from above because non-local (long range) correlations become stronger inside the inertial subrange in this limit.

By using an adjusted travel time scale $\tau_p = t - t_{\Delta_0}$, the pair separation was found to be independent of the initial conditions, as suggested by Batchelor 1952 and Fung et al 1992.

Our results also show that the mean square separation scales like,

$$\langle \Delta^2 \rangle_p \sim \delta_p^2 + \tau_p^{\chi_p}$$

inside the inertial subrange. Table 1 confirms that $\chi_p = 1/(1 - \gamma_p/2)$.

In particular, for Kolmogorov turbulence $p = 5/3$, we find that $\gamma_{5/3} = 1.53$, and $\chi_{5/3} = 4.2$. Thus,

$$D_{5/3} \sim \sigma_\Delta^{1.53}, \quad \langle \Delta^2 \rangle \sim \tau^{4.2}.$$
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1.8, namely for $p$ in the range $1.6 < p < 2$. This range includes Kolmogorov turbulence $p = 5/3$, and we may thus expect non-local correlations to play a significant role in real turbulence.

Malik (2014), suggested that there is a value $p_*$ where the pair diffusion becomes independent of the large scales, which occurs when $\gamma_{p_*} = 4/3$. Our results show that this occurs at $p_* \approx 1.4$. With this spectrum, $E \sim k^{-1.4}$, we obtain a new non-Richardson 4/3-law for the diffusivity, and a new non-R-O $t^3$-regime for the mean square separation, viz

$$D_{p_*} \sim \varepsilon^{1/3} \sigma_{\Delta}^{4/3}, \quad \langle \Delta^2 \rangle_{p_*} \sim \varepsilon_{p_*}^3.$$

Finally, we have found that the 'sweeping' action of random large scale energy containing wavemodes has little effect effect on the pair diffusion process – the scaling laws for the pair diffusivity are remarkably well preserved. However, there is increase in statistical scatter at the small scales, and the range of separation $\Delta$ over which the scalings are observed diminish somewhat.

These results show that the KS is reliable, especially for the case with no sweeping, $E_0 = 0$. We can interpret this case as the reference frame in which the large scales are 'carrying' the smaller scales which is what we expect in the real dynamics of turbulence. Real dynamics will have additional effects, but if we assume that such effects are statistically negligible, then the frame in which $E_0 = 0$ is indeed the appropriate frame in which to investigate pair diffusion, as we have done.

The results presented here are consistent with the predictions from Malik’s (2014) theory, and supports the main hypothesis that the turbulent pair diffusion process is governed by both local and non-local correlations. We have found no results in support of Richardson’s locality hypothesis.

If this picture is true, then non-local correlations must be accounted for as much as local correlations in turbulent diffusion studies and modelling, and it could also have important implications for the general theory of turbulence.

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