Bounds for Curves in Abelian Varieties

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Abstract

A uniform bound of intersection multiplicities of curves and divisors on abelian varieties is proved by algebraic geometric methods. It extends and improves a result obtained by A. Buium with a different method based on Kolchin’s differential algebra. The problem is modeled after the “abc-Conjecture” of Masser-Oesterlé for abelian varieties over the function field of a curve. As an application a finiteness theorem will be proved for maps from a curve into an abelian variety omitting an ample divisor.

1 Introduction

Buium [Bu98] proved the following:

Theorem A. Let $C$ be a smooth compact curve, let $A$ be an abelian variety, and let $D$ be an effective reduced divisor on $A$ which does not contain any translate of a non-trivial abelian subvariety of $A$. Then there exists a number $N$ depending on $C$, $A$ and $D$ such that for every morphism $f : C \to A$, either $f(C) \subset D$ or the multiplicities $\text{mult}_x f^*D, x \in C$ of the pulled-back divisor $f^*D$ are uniformly bounded by $N$.

This theorem together with his former one [Bu94] was motivated by the so-called abc-Conjecture of Masser-Oesterlé on abelian varieties. There is also such a bound of multiplicities in the Second Main Theorem for (transcendental) holomorphic curves in abelian and semi-abelian varieties by [NWY00] and [NWY99]. In Theorem A, if a finite subset $S$ of $C$ is fixed and if $f^{-1}D \subset S$ is imposed, then the bound of such multiplicities immediately follows from the quasi-projective algebraicity of the corresponding moduli space of mappings ([N88]). Therefore, it is a delicate but important point in Theorem A that $f$ is allowed to take values in $D$ at arbitrary points of $C$.

In this article we provide an algebraic geometric proof by making use of jet bundles, a method different from the one used by Buium [Bu98] who relied on the theory of Kolchin’s differential algebra. By our proof we have the following:

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(i) The condition that the given divisor $D$ contains no translate of non-trivial abelian subvarieties can be removed, as conjectured in [Bu98]:

(ii) The bound $N$ depends only on the numerical data involved; the genus $g$ of the compact curve $C$, the dimension of $A$, and the top self-intersection number $D^{\dim A}$.

Moreover, we show that Theorem A permits a generalization to the semi-abelian case and we give an application to a finiteness theorem for maps from a curve into an abelian variety omitting an ample divisor (Theorem 8.1).

Our main result is the following:

**Main Theorem.** Let $A$ be a semi-abelian variety (i.e., a connected commutative reductive algebraic group), let $A \hookrightarrow \bar{A}$ be a smooth equivariant algebraic compactification, let $\bar{D}$ be an effective reduced ample divisor on $\bar{A}$, let $D = \bar{D} \cap A$, and let $C$ be a smooth algebraic curve with smooth compactification $C \hookrightarrow \bar{C}$. Then there exists a number $N \in \mathbb{N}$ such that for every morphism $f : C \to A$ either $f(C) \subset D$ or $\text{mult}_x f^*D \leq N$ for all $x \in C$. Furthermore, the number $N$ depends only on the numerical data involved as follows:

(i) The genus of $\bar{C}$ and the number $\#(\bar{C} \setminus C)$ of the boundary points of $C$,

(ii) the dimension of $A$,

(iii) the toric variety (or, equivalently, the associated “fan”) which occurs as closure of the orbit in $\bar{A}$ of the maximal connected linear algebraic subgroup $T \cong (\mathbb{C}^*)^t$ of $A$,

(iv) all intersection numbers of the form $D^h \cdot B_{i_1} \cdots B_{i_k}$, where the $B_{i_j}$ are closures of $A$-orbits in $\bar{A}$ of dimension $n_j$ and $h + \sum_j n_j = \dim A$.

In particular, if we let $A$, $\bar{A}$, $C$ and $D$ vary within a flat connected family, then we can find a uniform bound for $N$. For abelian varieties this specializes to the following result:

**Theorem 1.1** There is a function $N : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that the following statement holds: Let $C$ be a smooth compact curve of genus $g$, let $A$ be an abelian variety of dimension $n$, let $D$ be an ample effective divisor on $A$ with intersection number $D^n = d$. Let $f : C \to A$ be a morphism. Then either $f(C) \subset D$ or $\text{mult}_x f^*D \leq N(g, n, d)$ for all $x \in C$.

**Remark.** If $D$ is an effective divisor on a compact complex torus $M$ which is not ample, then there exists a subtorus $\text{St}(D)$ of $A$ stabilizing $D$ such that $D$ is the pull-back of an ample divisor on $A/\text{St}(D)$. Thus one can easily generalize our theorem to the non-ample case.

For toric varieties we obtain
Theorem 1.2 Let $\bar{A}$ be a toric variety with open orbit $A$, let $C$ be an affine curve, let $D$ be an ample divisor on $\bar{A}$. Then there exists a number $N$ depending on the genus of the smooth compactification $\bar{C}$ of $C$, $\#(\bar{C} \setminus C)$ and the numerical data of $(A, \bar{A}, D)$ such that for every morphism $f : C \to A$ we have either $f(C) \subset D$ or $\text{mult}_x f^*D \leq N, x \in C$.

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2 Basic idea

The goal is to achieve a bound on the possible multiplicities for $f^*D$ where $D$ is a divisor on an abelian variety $A$, $C$ is a curve and $f$ runs through all morphisms $f : C \to A$ except those with $f(C) \subset \text{Supp} D$ (the support of $D$).

A rather naive idea would be to define subsets $Z_k$ of the space $\text{Mor}(C, A)$ of all morphisms from $C$ to $A$ by requiring that $f \in Z_k$ if and only if $f^*D$ has multiplicity $\geq k$ somewhere. Then the $Z_k$ form a decreasing sequence of subsets of $\text{Mor}(C, A)$. Now, if $\text{Mor}(C, A)$ were an algebraic variety and $Z_k$ were algebraic subvarieties, this sequence $Z_k$ would eventually have to stabilize, i.e., there would be a number $N$ such that $Z_k = Z_N$ for all $k \geq N$. Unfortunately $\text{Mor}(C, A)$ is not necessarily an algebraic variety, but it may have infinitely many irreducible components. This infinitude make things complicated.

Our approach is to embed $\text{Mor}(C, A)$ into a larger connected space $H$, such that the $Z_k$ can be extended to a sequence of algebraic subsets $\tilde{Z}_k$ which eventually stabilizes. This is done in the following way: Every morphism $f : C \to A$ induces a morphism from the Albanese $\text{Alb}(C)$ to $A$ which in turn induces a holomorphic map between the respective universal coverings. The crucial fact here is that a morphism between compact complex tori necessarily lifts to an affine-linear map between the corresponding universal coverings. Clearly the space $\text{Lin}(C^n, C^m)$ of all affine-linear maps from one $C^n$ to a $C^m$ is connected, and also carries a natural structure as an algebraic variety.

Now let $H = C \times D \times \text{Lin}(C^n, C^m)$. Then we can define subsets $\tilde{Z}_k \subset H$ in the following way: $(p, q, \phi) \in \tilde{Z}_k$ if and only if there is a holomorphic map germ $f : (C, p) \to (A, q)$ such that $f^*D$ has multiplicity $\geq k$ at $p$ and $f$ lifts to an affine-linear map whose linear part is $\phi$. Then $H$ is an algebraic variety, and the subsets $\tilde{Z}_k$ form a descending sequence of closed algebraic subvarieties, which eventually stabilizes.

Note that, up to the choice of a base point, the universal covering of the Albanese torus of a projective variety can be canonically identified with the dual vector space of the space of global holomorphic 1-forms. Correspondingly, in the proof itself we discuss induced maps between
these vector spaces rather than maps between the universal covering of the respective Albanese tori.

3 Basic notions

3.1 Commutative reductive groups

We recall some basic terminologies.

Definition 3.1 (i) A complex Lie group $G$ is said to be reductive if $G$ itself is the only complex Lie subgroup $H \subset G$ containing a maximal compact subgroup of $G$.

(ii) A (complex) semi-torus is a connected commutative reductive complex Lie group.

Every semi-torus $A$ admits a short exact sequence of commutative complex Lie groups

$\begin{align*}
1 \to T \to A \to M \to 1,
\end{align*}$

where $M$ is a compact complex torus and $T \cong (\mathbb{C}^*)^t$ for some $t \in \mathbb{N}$.

A commutative algebraic group $T$ is called a semi-abelian variety if there exists a short exact sequence of algebraic groups as in (3.2) (with $M$ being an abelian variety).

Neither presentation (3.2), nor the compact torus $M$ is unique in the complex-analytic category. However, if $A$ is a semi-abelian variety, then there is one and only one such algebraic presentation. This fact can be deduced in the following way: The theorem of Chevalley implies that there is a linear algebraic subgroup $T$ of $A$ such that $M = A/T$ is an abelian variety ([Bo91]). This linear algebraic subgroup $T$ is unique, because every morphism from a linear algebraic group to an abelian variety is constant. Thus, for a semi-abelian variety $A$ there is exactly one presentation

$\begin{align*}
1 \to T \to A \to M \to 1,
\end{align*}$

such that $T$, $M$ and the morphisms are all algebraic.

A different way to characterize semi-tori is the following: A complex Lie group $A$ is a semi-torus if and only if it is isomorphic to a quotient $(\mathbb{C}^n,+)/\Gamma$, where $\Gamma$ is a discrete subgroup of $\mathbb{C}^n$ spanning $\mathbb{C}^n$ as complex vector space.

Definition 3.3 Let $A$ be a complex semi-torus. An equivariant smooth compactification $A \hookrightarrow \overline{A}$ is said to be quasi-algebraic if every isotropy subgroup of the $A$-action on $\overline{A}$ is reductive.

If $A$ is a semi-abelian variety, then evidently every algebraic compactification $\overline{A}$ is quasi-algebraic (but not conversely).
Lemma 3.4 Let $A \rightarrow \tilde{A}$ be a smooth quasi-algebraic compactification of a complex semi-torus and let $D$ be an (effective, reduced) $A$-invariant divisor on $\tilde{A}$. Then $D$ is a divisor with only simple normal crossings.

Proof. Let $p \in \text{Supp } D$ and $A_p = \{ a \in A : a(p) = p \}$. By assumption, $A_p$ is reductive. This implies that the $A_p$-action can be linearized near $p$, i.e. there is an $A_p$-equivariant biholomorphism between an open neighborhood of $p$ in $\tilde{A}$ and an open neighborhood of 0 in the vector space $V = T_p \tilde{A}$. It follows that a neighborhood of $p$ in $D$ is isomorphic to a union of vector subspaces of codimension one in $V$. From the assumption that the $A_p$-action is effective, one can deduce that this is a transversal union, i.e. $D$ is a simple normal crossings divisor near $p$. Q.E.D.

3.2 Toric varieties

An equivariant compactification of $T = (\mathbb{C}^*)^t$ is called a toric variety. Here we are interested only in toric varieties which are nonsingular, compact and moreover projective. We recall some well-known properties of such toric varieties (see, e.g., [O85]).

Let $\tilde{T}$ be a projective smooth equivariant compactification of $T = (\mathbb{C}^*)^t$. Then $\tilde{T}$ is simply connected and the homology $H_*(\tilde{T}, \mathbb{Z})$ is generated (as $\mathbb{Z}$-module) by the fundamental classes of closures of $T$-orbits. In particular, the Chern classes of the tangent bundle are linear combinations of fundamental classes of orbit closures.

There are only finitely many $T$-orbits, and each $T$-invariant divisor has only simple normal crossings. Every ample line bundle on $\tilde{T}$ is already very ample. Note that $H^1(\tilde{T}, \mathcal{O}_{\tilde{T}}) = 0$ and hence that every topologically trivial line bundle on $\tilde{T}$ is already holomorphically trivial.

There are only countably many non-isomorphic compact toric varieties. Toric varieties can be classified by certain combinatorial data, called fans (cf. [O85]).

Lemma 3.5 Let $\tilde{T}$ be a smooth projective toric variety. Then there exists a number $k \in \mathbb{N}$ such that for every ample line bundle $H$ and every reduced $T$-invariant hypersurface $D$ the line bundle $H^k \otimes L(-D)$ is ample.

Proof. There are only finitely many $T$-invariant reduced hypersurfaces $D_1, \ldots, D_r$ on $\tilde{T}$ and only finitely many $T$-invariant curves $C_1, \ldots, C_s$. We choose $k$ such that $k > D_i \cdot C_j$ for all $i \in \{1, \ldots, r\}$, $j \in \{1, \ldots, s\}$. Then $(kc_1(H) - D_i) \cdot C_j > 0$ for all $i, j$. By the toric Nakai criterion (see [O85], Theorem 2.18) it follows that $H^k \otimes L(-D_i)$ is ample for every $i$. Q.E.D.
3.3 Compactifications of semi-tori

Let $A$ be a semi-torus with presentation

\[(3.6) \quad 1 \to T \to A \to M \to 1,\]

where $M$ is a compact complex torus and $T \cong (\mathbb{C}^*)^t$. An equivariant compactification $\tilde{A} \hookrightarrow A$ is called compatible with the given presentation, if the projection $\pi : A \to M$ extends to a holomorphic map $\tilde{\pi} : \tilde{A} \to M$. In this case, $\tilde{T} = \tilde{\pi}^{-1}(1)$ is a toric variety. Furthermore

$$\tilde{A} = A \times_T \tilde{T}.$$  

(Here $A \times_T \tilde{T}$ denotes the quotient of $A \times \tilde{T}$ under the diagonal action of $T$.) We say that $A \hookrightarrow \tilde{A}$ is a compactification of type $\tilde{T}$.

Let us now discuss the case where $A$ is a semi-abelian variety. We claim that in this case every smooth algebraic equivariant compactification $A \hookrightarrow \tilde{A}$ is compatible with the unique algebraic presentation. Indeed, let $A \hookrightarrow \tilde{A}$ be a smooth equivariant algebraic compactification. Since $A$ acts effectively, for every $x \in \tilde{A}$ and $g \in A_x = \{g \in A; g(x) = x\} \setminus \{1\}$ the automorphism of $\tilde{A}$ given by $g$ induces a non-trivial automorphism of the local ring at $x$ and therefore on the space of $k$-jets (see the next subsection 3.4) at $x$ for some $k \in \mathbb{N}$. It follows that $A_x$ admits a faithful linear representation. Thus $(A_x)^0 \subset T$. Therefore every algebraic equivariant compactification is compatible with the unique algebraic presentation.

**Lemma 3.7** Let $A \hookrightarrow \tilde{A}$ be a smooth equivariant compactification compatible with a presentation as in (3.6). Let $U$ be the maximal compact subgroup of $T$. Then $\tilde{A} \to M$ can be described as a locally trivial fiber bundle whose transition functions are locally constant functions with values in $U$.

**Proof.** Let $K$ be a maximal compact subgroup of $A$, and $V = \text{Lie} K \cap i\text{Lie} K$. For a sufficiently small open neighborhood $U$ of $0$ in $V$ the map $U \ni v \mapsto \exp(v) \cdot e_M$ is a biholomorphism onto an open neighborhood of the unit $e_M \in M$. Every $q \in M$ can be written in the form $q = \pi(k)$ for some $k \in K$. Then $U \ni v \mapsto \exp(v) \cdot \pi(k)$ parameterizes an open neighborhood of $q$ in $M$ and the diagram

$$(U \times \tilde{T}) \ni (v, x) \mapsto \exp(v) \cdot k \cdot x$$

$\downarrow_{\text{pr}_1}$

$U \ni v \mapsto \exp(v) \cdot \pi(k)$

gives a local trivialization for this neighborhood. Everything is canonical except for the choice of $k$. Since $\pi(k)$ must equal $q$, the element $k$ is well-defined up to multiplication by an element in $U = K \cap \pi^{-1}(e_M)$. Hence the local trivializations constructed in this way realize $\tilde{\pi}$ as a fiber bundle whose transition functions are locally constant with values in $U$. Q.E.D.
3.4 Sheaf of logarithmic 1-forms

Let \( N \) be a complex manifold, and let \( D \) be a divisor on \( N \) with only normal crossings. For \( p \in N \) we take a local coordinate neighborhood \( U(x_1, \ldots, x_n) \) such that \( D \cap U = \{ x_1 \cdots x_k = 0 \} \) \((0 \leq k \leq n)\). Then the sheaf \( \Omega^1_N(\log D) \) of logarithmic 1-forms along \( D \) is the sheaf of those meromorphic 1-forms locally generated over the structure sheaf \( \mathcal{O}_N \) by

\[
\frac{dx_1}{x_1}, \ldots, \frac{dx_k}{x_k}, dx_{k+1}, \ldots, dx_n.
\]

The sheaf \( \Omega^1_N(\log D) \) is locally free, and the dual bundle of its associated vector bundle is called the logarithmic tangent bundle, denoted by \( T(N, \log D) \) ([I76]).

Let \( M, N \) be complex manifolds and \( D, E \) divisors with only normal crossings on \( M \) resp. \( N \). Let \( F : M \setminus D \to N \setminus E \) be a holomorphic map which extends to a meromorphic map from \( M \) to \( N \). Then \( F^* \Omega^1_N(\log E) \subset \Omega^1_M(\log D) \).

In general, let \( X \) be a reduced complex space and let \( E \) be a reduced complex subspace of \( X \) (we will actually deal only with algebraic \( X \)). By the well-known Hironaka resolution there is a proper holomorphic mapping \( \lambda : \tilde{X} \to X \) such that \( \tilde{X} \) is smooth, and \( \tilde{E} = \pi^{-1}(E) \) is a divisor with only normal crossings. Then we have \( \Omega^1_{\tilde{X}}(\log \tilde{E}) \) defined as above.

We define the sheaf of logarithmic 1-forms on \( X \) along \( E \) by its direct image sheaf:

\[
(3.8) \quad \Omega^1_X(\log E) =: R^0\lambda_*\Omega^1_{\tilde{X}}(\log \tilde{E}).
\]

Let \( \lambda' : \tilde{X}' \to X \) be another resolution such that \( \tilde{X}' \) is smooth and \( \tilde{E}' = \lambda'^{-1}E \) is a divisor with only normal crossings. Then

\[
R^0\lambda_*\Omega^1_{\tilde{X}}(\log \tilde{E}) \cong R^0\lambda'_*\Omega^1_{\tilde{X}'}(\log \tilde{E}').
\]

(see [I76]). Hence \( \Omega^1_X(\log E) \) is well-defined.

Let \( X \) be a (possibly non-compact) smooth complex algebraic variety. By [I77] we have the quasi-Albanese variety \( \text{Alb}(X) \) and the quasi-Albanese mapping \( \alpha_X : X \to \text{Alb}(X) \). If \( X \) is compact, the quasi-Albanese variety is just the usual Albanese variety. In general, the quasi-Albanese variety \( \text{Alb}(X) \) is a semi-abelian variety. As in the case of Albanese varieties, we have a universal property: For a semi-Abelian variety \( A \) and a morphism \( f : X \to A \), there is a unique morphism \( \tilde{f} : \text{Alb}(X) \to A \) such that \( f = \tilde{f} \circ \alpha_X \).

3.5 Jet bundles

Let \( \Delta = \{ |z| < 1 \} \subset \mathbb{C} \) be the unit disk with center at the origin. The space \( J^k_p(X) \) of \( k \)-jets at a point \( p \) of a complex manifold \( X \) can be defined as the quotient space of all holomorphic
mappings of the space germ \((\Delta, 0)\) into the space germ \((X, p)\) by the equivalence relation given as follows: \(f \sim g\) if their Taylor series expansions agree up to degree \(k\). Set
\[
J^k(X) = \bigcup_{p \in X} J^k_p(X),
\]
which is called the \(k\)-jet bundle over \(X\). For \(k = 1\) there is a natural isomorphism \(J^1(X) \cong T(X)\) with the holomorphic tangent bundle \(T(X)\) over \(X\). For \(k \geq 2\) \(J^k(X)\) carries no natural vector bundle structure. However, the \(\mathbb{C}^*\)-action on \((\Delta, 0)\) induces a natural \(\mathbb{C}^*\)-action on \(\text{Spec} \, \mathbb{C}[t]/(t^{k+1})\) via
\[
c \cdot j(t) = j(ct), \quad c \in \mathbb{C}^*, \; j(t) \in \text{Spec} \, \mathbb{C}[t]/(t^{k+1}),
\]
and hence on \(J^k_p(X)\). This \(\mathbb{C}^*\)-action can be used to define the notion of polynomials of weighted degree: A function \(P\) on \(J^k_p(X)\) is called a polynomial of weighted degree \(d\) if \(P(c \cdot j) = c^d P(j)\).

The points of \(J^k_p(X)\) are separated by polynomials of weighted degree \(\leq k\).

Let \(Y \subset X\) be an analytic subspace with defining ideal sheaf \(\mathcal{I}_Y\). For \(p \in Y\) we define the \(k\)-jet space of \(Y\) by
\[
J^k_p(Y) = \{ j \in J^k_p(X); (j^* \mathcal{I}_Y)_0 \subset (t^{k+1}) \}, \quad J^k(Y) = \bigcup_{p \in Y} J^k_p(Y).
\]
Then \(J^k(Y)\) is a complex subspace of \(J^k(X)\). When \(Y\) is non-singular, \(J^k(Y)\) coincides the one defined as above with \(X = Y\). Because \(J^k(Y)\) is independent of the used embedding \(Y \hookrightarrow X\), the \(k\)-jet space \(J^k(Y)\) over \(Y\) is defined for a (reduced) complex space \(Y\).

Equivalently, for any complex space \(X\) we can define the space \(J^k_p(X)\) of \(k\)-jets at a point \(p \in X\) as the space of holomorphic mappings from \(\text{Spec} \, \mathbb{C}[t]/(t^{k+1})\) to \(X\) which map the geometric point of \(\text{Spec} \, \mathbb{C}[t]/(t^{k+1})\) to \(p\).

The maximal ideal of \(\text{Spec} \, \mathbb{C}[t]/(t^{k+1})\) is denoted by \(m(k)\).

A holomorphic mapping \(\phi : X \to Y\) between complex spaces induces \(\mathbb{C}^*\)-equivariant mappings
\[
d^k \phi : J^k(X) \to J^k(Y), \quad \text{i.e., } c d^k \phi(j) = d^k \phi(cj)
\]
for \(j \in J^k_p(X), \; c \in \mathbb{C}^*\).

Let \(A\) be a semi-abelian variety, and let \(\tilde{A}\) be a projective algebraic equivariant compactification. Then \(A \hookrightarrow \tilde{A}\) is quasi-algebraic in the sense of Definition 3.3. Hence the \(A\)-invariant divisor \(\partial A\) has only simple normal crossings (Lemma 3.4). We have the logarithmic tangent bundle \(T(\tilde{A}, \log \partial A)\) and logarithmic jet bundles \(J^k(\tilde{A}, \log \partial A)\) over \(\tilde{A}\) ([176], [N86]). For \(k = 1\) we have
\[
(3.9) \quad J^1(\tilde{A}, \log \partial A) \cong T(\tilde{A}, \log \partial A).
\]
Moreover, there is a natural bundle morphism,

\[ J^k(\bar{A}, \log \partial A) \to J^k(\tilde{A}). \]  

(3.10)

**Proposition 3.11** Let \( A \hookrightarrow \bar{A} \) be a quasi-algebraic smooth equivariant compactification of a complex semi-torus \( A \). Then the logarithmic jet bundles \( J^k(\bar{A}, \log \partial A) \) are trivialized as

\[ J^k(\bar{A}, \log \partial A) \cong \bar{A} \times (m(k) \otimes \text{Lie} A). \]

**Proof.** We first show that the logarithmic tangent bundle is trivialized by the \( A \)-fundamental vector fields. Since we required the compactification to be quasi-algebraic in the sense of Definition 3.3, all the isotropy groups are reductive. Therefore the action of every isotropy subgroup is linearizable in some open neighborhood. Thus for every point on \( \bar{A} \) we can find a system of local coordinates in which the \( A \)-fundamental vector fields are simply

\[ \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_h}, z_{h+1} \frac{\partial}{\partial z_{h+1}}, \ldots, z_n \frac{\partial}{\partial z_n}, \]

for some \( h, n \in \mathbb{N} \), where \( n - h \) equals the dimension of the isotropy group. Hence we have

\[ \mathbf{T}(\bar{A}, \log \partial A) \cong \bar{A} \times \text{Lie} A. \]  

(3.12)

For \( k \geq 2 \), (3.12) induces a global trivialization of \( J^k(\bar{A}, \log \partial A) \) in the following way: We regard the \( k \)-jets over \( A \) as maps from \( S_k = \text{Spec} \mathbb{C}\{t\}/(t^{k+1}) \) to \( A \). As before, let \( m(k) \) denote the maximal ideal \( (t) \) of \( \mathbb{C}\{t\}/(t^{k+1}) \). For \( p \in A \) and \( \alpha = \sum_i \alpha_i \otimes v_i \in m(k) \otimes \text{Lie} A \) we define a map (germ) from \( \text{Spec} \mathbb{C}\{t\}/(t^{k+1}) \) by

\[ \alpha : t \mapsto \exp \left( \sum_i \alpha_i(t)v_i \right) \cdot p. \]

A calculation in local coordinates shows that this gives a trivialization, \( J^k(\bar{A}, \log \partial A)|_A \cong A \times (m(k) \otimes \text{Lie} A) \). This trivialization holomorphically extends over \( \bar{A} \) (see [N86]). Q.E.D.

Now fix a point \( p \in \bar{A} \) and consider the induced map \( m(k) \otimes \text{Lie} A \to J^k_p(\bar{A}, \log \partial A) \). Let \( V \) be a complex vector space and let \( E \) be a vector space of linear mappings from \( V \) to \( \text{Lie} A \). Then we obtain a map \( m(k) \otimes V \times E \to J^k(\bar{A}, \log \partial A) \) induced by the natural pairing \( V \times E \to \text{Lie} A \). Observe that this map \( m(k) \otimes V \times E \to J^k(\bar{A}, \log \partial A) \) is polynomial of degree \( \leq k \) in \( E \).

**4 Proof of the Main Theorem in absolute case**

Here we deduce the following key lemma.
Lemma 4.1 Let $A$, $B$ be complex semi-abelian varieties with smooth equivariant algebraic compactifications $\tilde{A}$ and $\tilde{B}$. Let $\phi : B \to A$ be a morphism. Then there is a linear map $\lambda_\phi \in \text{Lin}(\text{Lie } B, \text{Lie } A)$ such that in view of Proposition 3.11

\[
d^k \phi : J^k(\tilde{B}, \log \partial B)|_B \to J^k(\tilde{A}, \log \partial A)|_A
\]

for every holomorphic map-germ $\phi : (C, p) \to (A, q)$ with the property that $d\phi(v) = \tilde{\phi}(v)$ near $p$ for every element $v \in \text{Lie } B$, regarded as vector field on $C$.

We claim that the $Z_k$ form a descending sequence of closed algebraic subsets. Indeed, $J^k(C)$ (resp. $J^k(D)$) is a closed algebraic subvariety of $J^k(B)|_C \cong C \times (\mathfrak{m}(k) \otimes \text{Lie } B)$ (resp. $J_k(A)$). Thus $J_p^k(C)$ (resp. $J_q^k(D)$) can be regarded as a closed subvariety of $\mathfrak{m}(k) \otimes \text{Lie } B$ (resp. $\mathfrak{m}(k) \otimes \text{Lie } A$). Then

\[
Z_k = \{ (\lambda, p, q) \in H \times C \times D : (\text{id}_{\mathfrak{m}(k)} \otimes \lambda) \left( J_p^k(C) \right) \subset J_q^k(D) \},
\]

and the algebraicity of the sets $Z_k$ becomes evident.

By Noetherianity of the algebraic Zariski topology we may conclude that there is a number $N$ such that $Z_k = Z_N$ for all $k \geq N$.

For any morphism $\phi : C \to A$ let $\tilde{\phi}$ denote the associated linear map in $H$. If there exists a pair $(p, q) \in C \times D$ such that $\phi(p) = q$ and $(\tilde{\phi}, p, q) \in Z_N$, then $(\tilde{\phi}, p, q) \in Z_k$ for all $k \geq N$ and consequently $\text{mult}_p \phi^*D \geq k$ for all $k$ which in turn implies that $\phi(C) \subset D$ by identity principle.

On the other hand, if there is no such pair, then $\text{mult}_p \phi^*D < N$ for all $p \in C$.

This finished the proof for the Main Theorem except for the dependence of the number $N$. In the rest of this paper, we will prove the said numerical dependence of $N$ by constructing certain parameter spaces of the objects which appeared in the above proof.
5 Line bundles on semi-abelian variety

5.1 Notation

Throughout this section $A$ denotes a semi-abelian variety. There is a short exact sequence of morphisms of algebraic groups

$$1 \to T \to A \xrightarrow{\pi} M \to 1,$$

where $T \cong (\mathbb{C}^*)^t$, $t \in \mathbb{N} \cup \{0\}$ and $M$ is an abelian variety. Let $m = \dim M$ and $n = \dim A$. Let $T \hookrightarrow \bar{T}$ denote a smooth projective equivariant algebraic compactification of $T \cong (\mathbb{C}^*)^t$; i.e. $\bar{T}$ is a toric variety. Let $A \hookrightarrow \bar{A}$ denote a smooth algebraic equivariant compactification of type $\bar{T}$, compatible with the above presentation; i.e. $\pi : A \to M$ extends to an equivariant holomorphic map $\bar{\pi} : \bar{A} \to M$.

5.2 Line bundles

The $A$-orbits in $\bar{A}$ of a given codimension are in one-to-one correspondence with the $T$-orbits of that codimension in $\bar{T}$. In particular, there is a bijective correspondence between $T$-invariant divisors on $\bar{T}$ and $A$-invariant divisors on $\bar{A}$. Since $A$ acts with only finitely many orbits on $\bar{A}$, a divisor on $\bar{A}$ is $A$-invariant if and only if its support is contained in the boundary $\partial A$.

The presentation

$$1 \to T \to A \xrightarrow{\pi} M \to 1$$

realizes $A$ as a $T$-principal bundle over $M$. Choosing a vector subspace $W$ of Lie $A$ transversal to Lie $T$, we obtain a splitting of the holomorphic tangent bundle $T(\bar{A}) = H_{\bar{A}} \oplus V_{\bar{A}}$ into horizontal and vertical tangent bundles, where $H_{\bar{A}}$ is the subbundle spanned by the fundamental vector fields coming from $W$ and $V_{\bar{A}}$ is given as the kernel of $d\bar{\pi} : T_{\bar{A}} \to T_M$. Note that $H_{\bar{A}}$ is a bundle with flat connection, because $W$ is a commutative subalgebra of the Lie algebra Lie $A$.

We choose a Kähler form $\Omega_0$ on $\bar{T}$. By averaging, we may assume that $\Omega_0$ is $U$-invariant where $U$ denotes the maximal compact subgroup of $A$. Recall that $\bar{\pi} : \bar{A} \to M$ can be realized as a fiber bundle with $U$ as structure group (Lemma 3.7). Using these facts it is clear that $\Omega_0$ induces a hermitian metric on the bundle $V_{\bar{A}}$ of vertical tangent vectors. Extending it by zero on $H_{\bar{A}}$ we obtain a closed semi-positive $(1,1)$-form $\Omega$ on $\bar{A}$ which is Kähler on each fiber of $\bar{\pi} : \bar{A} \to M$.

In this way we have

**Lemma 5.1** There is a closed semi-positive $(1,1)$-form $\Omega$ on $\bar{A}$ such that

(i) $\Omega$ is positive definite on every fiber of $\bar{A} \to M$, 

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(ii) if $Z$ is an $A$-invariant divisor, then $\Omega^t|_Z \equiv 0$ (where $t = \dim T$).

For a divisor $E$ on $\bar{A}$ we denote by $L(E)$ its associated line bundle on $\bar{A}$.

**Lemma 5.2** Let $Z$ be a divisor on $\bar{A}$ such that $Z \cap A$ is effective. Assume that there are a line bundle $L_0 \in \text{Pic}(M)$ and a divisor $E$ with $\text{Supp} E \subset \partial A$, satisfying $L(Z) \otimes \bar{\pi}^* L_0^{-1} \cong L(E)$. Then $c_1(L_0) \geq 0$.

**Proof.** Assume the contrary. Recall that $M$ is a compact complex torus with universal covering $\pi_0 : \mathbb{C}^m \to M$ (with $m = \dim M$). We may regard the Chern class $c_1(L_0)$ as bilinear form on the vector space $\mathbb{C}^m$. Since $c_1(L_0)$ is not semi-positive definite, there is a vector $v \in \mathbb{C}^m$ with $c_1(L_0)(v, v) < 0$. Set $W = \{ w \in \mathbb{C}^m : c_1(L_0)(v, w) = 0 \}$. Let $\mu$ be a semi-positive $(1, 1)$-form on $\mathbb{C}^m$ such that $\mu(v, \cdot) \equiv 0$ and $\mu|_{W \times W} > 0$. Let $\Omega$ be as in Lemma 5.1, and consider the $(n-1, n-1)$-form $\omega$ on $\bar{A}$ given by

$$(5.3) \quad \omega = \Omega^t \wedge \bar{\pi}^* \mu^{m-1}, \quad m = \dim M.$$ 

By construction we have $\omega \wedge \bar{\pi}^* c_1(L_0) < 0$.

Let $Z = Z' + Z''$ so that $Z'$ is effective and no component of $Z'$ is contained in $\partial A$, and $\text{Supp} Z'' \subset \partial A$. By the Poincaré duality,

$$\int_{\bar{A}} c_1(L(Z)) \wedge \omega = \int_Z \omega.$$ 

Since $\omega \wedge c_1(L(E)) = 0$, we have

$$\int_{\bar{A}} c_1(L(Z)) \wedge \omega = \int_{\bar{A}} \bar{\pi}^* c_1(L_0) \wedge \omega < 0.$$ 

On the other hand,

$$\int_Z \omega = \int_{Z'} \omega + \int_{Z''} \omega.$$ 

Note that $\int_{Z'} \omega \geq 0$, because $Z'$ is effective and $\omega \geq 0$, and that $\int_{Z''} \omega = 0$, because $\text{Supp} Z'' \subset \partial A$, and $\Omega^t$ vanishes on $\partial A$ by Lemma 5.1 (ii). Thus we deduced a contradiction. **Q.E.D.**

**Proposition 5.4**

(i) For every line bundle $L$ on $\bar{A}$ there exists a line bundle $L_0$ on $M$ and an $A$-invariant divisor $D$ on $\bar{A}$ such that $L = \bar{\pi}^* L_0 \otimes L(D)$.

(ii) Let $L_1, L_2 \in \text{Pic}(M)$ and let $D_1, D_2$ be $A$-invariant divisors on $\bar{A}$. Assume that

$$\bar{\pi}^* L_1 \otimes L(D_1) \cong \bar{\pi}^* L_2 \otimes L(D_2).$$

Then there is a topologically trivial line bundle $H \in \text{Pic}^0(M)$ such that $L_1 = L_2 \otimes H$ and $\pi^* H \cong L(D_2 - D_1)$. 

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**Proposition 5.5** Let \( \bar{\pi} \) be the fundamental classes of \( M \) with \( \bar{\pi} \) an ample divisor on \( M \). Let \( \bar{\pi}^* \mathcal{O}(L) \) be an invertible coherent sheaf on \( M \). Then \( \bar{\pi}^* \mathcal{O}(L) \) is topologically trivial on \( \bar{\pi} \). Hence, \( \bar{\pi}^* \mathcal{O}(L) \) is topologically trivial on \( \bar{\pi} \). By the same arguments as in (i), this implies that this bundle comes from \( M \). Q.E.D.

**Proof.** (i) Recall that for the toric variety \( \bar{T} \) the cohomology group \( H^2(\bar{T}, \mathbb{Z}) \) is generated by the fundamental classes of invariant divisors. Therefore there exists an \( A \)-invariant divisor \( D \) on \( \bar{A} \) such that \( L \otimes L(D) \) is topologically trivial on \( \bar{T} \). For a line bundle on a toric variety, the topological triviality implies the holomorphic triviality. Hence, \( L \otimes L(D) \) is trivial along the \( \bar{\pi} \)-fibers. Therefor \( R^0 \bar{\pi}_* \mathcal{O}(L) \otimes L(D) \) is an invertible coherent sheaf on \( M \) and thus \( L = \bar{\pi}^* L_0 \otimes L(D) \) for some line bundle \( L_0 \in \text{Pic}(M) \).

(ii) Since \( (D_1 - D_2) \cap A = \emptyset \) and \( L(D_1 - D_2) \cong \bar{\pi}^*(L_2 \otimes L_1^{-1}) \), we may apply Lemma 5.2 to \( Z = D_1 - D_2 \) and \( L_0 = L_2 \otimes L_1^{-1} \). Hence \( c_1(L_2 \otimes L_1^{-1}) \geq 0 \). Similarly, \( c_1(L_1 \otimes L_2^{-1}) \geq 0 \). Thus \( c_1(L_2 \otimes L_1^{-1}) = 0 \). Because \( H^2(M, \mathbb{Z}) \) is torsion-free (\( M \) is a torus), \( L_2 \otimes L_1^{-1} \) is topologically trivial. By the same arguments as in (i), this implies that this bundle comes from \( M \). Q.E.D.

**Proposition 5.5** Let \( L_0 \) be a line bundle on \( M \) and let \( D \) be an \( A \)-invariant divisor on \( \bar{A} \). Let \( L = \bar{\pi}^* L_0 \otimes L(D) \). Then \( L \) is ample on \( \bar{A} \) if and only if both \( L_0 \) and \( D \cap \bar{T} \) are ample.

**Proof.** The ampleness of \( L_0 \) follows from that of \( L \) in the same way as in the proof of Lemma 5.2. Furthermore, \( L|_F \cong L(D)|_T \). Hence \( L \) being ample implies that both \( L_0 \) and \( D \cap \bar{T} \) are ample.

For the converse, recall that ampleness is equivalent to being positive. Since \( \bar{A} \to T \) is a topologically trivial bundle, the positivities of \( c_1(L_0) \) and \( c_1(D|_T) \) imply the positivity and therefore the ampleness of \( L \). Q.E.D.

### 5.3 Very ampleness criterion

We keep the notation in the previous section.

For every ample line bundle \( L \) on a projective manifold \( X \) there exists a number \( m \), depending on both \( X \) and the polynomial \( P_L(k) = \chi(X, L^k) \) such that \( L^m \) is very ample (Matsusaka’s theorem, see [Ma72]). It is well-known that \( m \) can be chosen as 3 if \( X \) is an abelian variety (see, e.g., [Mu70]). Here we prove a similar result for the case where \( X \) is a smooth equivariant compactification of a semi-abelian variety.

We will employ the following auxiliary fact on line bundles on toric varieties:

**Lemma 5.6** Let \( L \) be an ample line bundle on a smooth compact toric variety \( \bar{T} = F \) and let \( p \in \bar{T} \). Then there exists an effective \( T \)-invariant divisor \( D \) with \( p \notin \text{Supp} D \) and \( L = L(D) \).

**Proof.** On a toric variety every ample line bundle is already very ample. Thus there are sections \( \tau \in H^0(\bar{T}, L) \) with \( \tau(p) \neq 0 \). Now consider the \( T \)-action on the \( H^0(\bar{T}, L) \). This is a linear action of the commutative reductive group \( T \cong (\mathbb{C}^*)^l \) and thus completely diagonalizable. Then by the general theory of linear algebraic groups (cf., e.g., [Bo91]) there is a “character” \( \chi \),
Lemma 5.9 Let \( L \) be a line bundle vanishing at \( p \) divisor \( D \). Thus \( L \) we obtain a claim is proved.

Theorem 5.7 Let \( F \) be a toric variety and let \( L \) be an ample line bundle on an equivariant compactification \( \bar{A} \) of a semi-abelian variety \( A \) of type \( F \). Then \( L^3 \) is very ample on \( \bar{A} \).

Proof. By Propositions 5.4 and 5.5, \( L = \pi^*L_0 \otimes L(D) \); here \( L_0 \) is ample on \( M \) and \( D \) is an \( A \)-invariant divisor on \( \bar{A} \) such that \( D \cap F \) is ample on \( F \) for every fiber \( F \) of \( \bar{F} \).

By a standard result on line bundles on abelian varieties \( L_0^3 \) is very ample on \( M \).

Claim 5.8 Let \( p \in M \) and \( F = \bar{F}^{-1}(p) \). Then every section \( \sigma \in H^0(F, L^3) \) extends to a section of \( L^3 \) on \( \bar{A} \).

Recall that \( F \cong \bar{T} \) is a toric variety. Hence \( L|_F \) being ample implies that \( L|_F \) is already very ample. Furthermore \( H^1(F, \mathcal{O}) = \{0\} \). As a consequence the connected Lie group \( T \) acts trivially on \( \text{Pic}(F) \). For this reason \( L|_F \) is \( T \)-invariant and we obtain a \( T \)-action on \( H^0(F, L) \). Similarly we obtain a \( T \)-action on \( H^0(F, L^3) \). Let \( \sigma \in H^0(F, L^3) \). Then there are characters \( \chi_i : T \rightarrow \mathbb{C}^* \) and sections \( \sigma_i \in H^0(F, L^3) \) such that \( \sigma = \sum \chi_i \sigma_i \) and \( t^*\sigma_i = \chi_i(t) \sigma_i \) for all \( t \in T \). Now \( \text{div}(\sigma_i) \) is a \( T \)-invariant divisor on \( F \) for every \( i \). These divisors extend to \( A \)-invariant divisors \( D_i = A \cdot \text{div}(\sigma_i) \) on \( \bar{A} \) and \( \sigma_i \) extend to sections \( \bar{\sigma}_i \in H^0(\bar{A}, L(D_i)) \) with \( \text{div}(\bar{\sigma}_i) = D_i \). Now \( L(3D - D_i) \) is topologically trivial for each \( i \). Hence there are topologically trivial line bundles \( H'_i \in \text{Pic}^0(M) \) such that \( L(3D - D_i) \cong \pi^*H'_i \). Since \( \text{Pic}^0(M) \) is a divisible group, there are line bundles \( H_i \in \text{Pic}^0(M) \) with \( H'_i = \pi^*H'_i \). The ampleness of \( L_0 \in \text{Pic}(M) \) implies that \( L_0 \otimes H_i \) is also ample and that consequently \( (L_0 \otimes H_i)^3 \cong L_0^3 \otimes H'_i \) is very ample. It follows that \( L_0^3 \otimes H'_i \) admits a section not vanishing at \( p \). Hence there exists an element \( \zeta_i \in H^0(\bar{A}, \pi^*(L_0^3 \otimes H'_i)) \) such that \( \zeta_i \) has no zero on \( F \). We have

\[
\bar{\sigma}_i \otimes \zeta_i \in H^0(\bar{A}, \pi^*(L_0^3 \otimes H'_i) \otimes L(D_i)) \cong H^0(\bar{A}, L^3).
\]

Thus each \( \sigma_i \) and consequently also their sum \( \sigma = \sum \sigma_i \) extends to a section of \( L^3 \) over \( \bar{A} \). The claim is proved.

Using Lemma 5.6 one can show that for every point \( p \in \bar{A} \) there is an effective \( A \)-invariant divisor \( D \) (depending on \( p \)) and an ample line bundle \( L_0 \) on \( M \) such that \( L \cong \pi^*L_0 \otimes L(D) \). Thus \( L^3 \) can be realized as the tensor product of a line bundle \( L(3D) \) with a section \( \sigma \) not vanishing at \( p \) and a line bundle \( \pi^*L_0^3 \) which is the pull-back of a very ample line bundle on \( M \).

Combining these facts with Claim 5.8, we can easily verify that \( L^3 \) is very ample on \( \bar{A} \). Q.E.D.

Lemma 5.9 Let \( \bar{T} \) be a toric variety. Then there exists a number \( l_0 \) such that for every ample line bundle \( L \) over \( \bar{T} \) the bundle \( L^{l_0} \otimes K_{\bar{T}}^{-1} \) is ample, too.
Proof. There are finitely many $T$-stable curves $C_1, \ldots, C_r$. By the toric Nakai criterion ([O85], Theorem 2.18) a line bundle $L$ on $\bar{T}$ is ample if and only if $\deg(L|_{C_j}) > 0$ for all $j \in \{1, \ldots, r\}$. It follows that every $l_0$ with $\deg(K_{\bar{T}}^{-1}|_{C_j}) > -l_0, 1 \leq j \leq r$, has the desired property. Q.E.D.

Proposition 5.10 Let $A \hookrightarrow \bar{A}$ and $\bar{T}$ be as above. Then there exists a number $l$ depending only on the toric variety $\bar{T}$ such that for every ample line bundle $L$ on $\bar{A}$, the line bundle $L^l \otimes K_{\bar{A}}^{-1}$ is ample.

Proof. By Propositions 5.4 and 5.5 there is an ample line bundle $L_0$ on $M$ and an $A$-invariant divisor $D \in \text{Div}(\bar{A})$ such that $D|_F$ is ample for every fiber $F$ of $\bar{\pi}$. If $v_1, \ldots, v_m$ is a basis for $\text{Lie} A$, then

$$K_{\bar{A}}^{-1} = \text{div}(v_1 \wedge \ldots \wedge v_m).$$

Thus $K_{\bar{A}}^{-1} = \partial A$. In particular the anticanonical bundle $K_{\bar{A}}^{-1}$ is induced by an $A$-invariant divisor. The adjunction formula implies $K_{\bar{A}}^{-1}|_F = K_F^{-1}$.

Let $l_0$ be a number as in Lemma 5.9. Then $l_0 D|_F - K_{\bar{T}}$ is an ample divisor on the fiber $F$ of $\bar{\pi} \to M$. Thus $l_0 D - K_{\bar{A}}$ is an $A$-invariant divisor on $\bar{A}$ whose restriction to every fiber is ample. On the other hand, $(L_0)^{l_0}$ is ample on $M$. Thus

$$(L_0)^{l_0} \otimes L(l_0 D - K_{\bar{A}}) \cong L^{l_0} \otimes K_{\bar{A}}^{-1}$$

is ample on $\bar{A}$ by Proposition 5.5. Q.E.D.

Corollary 5.11 For every toric variety $\bar{T}$ there exists a number $l$ such that for every semi-abelian variety $A$ with smooth equivariant compactification $A \hookrightarrow \bar{A}$ of type $\bar{T}$ and every ample line bundle $L$ on $\bar{A}$ the tensor power $L^l$ is very ample and moreover $h^0(L^k) = \chi(L^k), k \geq l$.

Proof. We may choose $l$ such that $L^l$ is very ample (Proposition 5.7) and that furthermore $L^l \otimes K_{\bar{A}}^{-1}$ is ample (Proposition 5.10). By the Kodaira Vanishing Theorem, the ampleness of $L^k \otimes K_{\bar{A}}^{-1}$ with $k \geq l$ implies $h^0(L^k) = \chi(L^k)$. Q.E.D.

6 Parametrizing spaces

6.1 Characterizing compactifications of semi-abelian varieties

We start with a preparatory lemma.

Lemma 6.1 Let $A$ be a semi-abelian variety. Then $\text{Aut}(A)^0 = A$, where $\text{Aut}(A)^0$ denotes the connected component of the group of all variety automorphisms of $A$. 

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Proof. We have to show that every automorphism \( \phi \in \text{Aut}(A)^0 \) is given as translation by some element \( a \in A \). It suffices to show that \( \phi = \text{id}_A \) if \( \phi \in \text{Aut}(A)^0 \) and \( \phi(e_A) = e_A \).

Let

\[ 1 \to T \to A \xrightarrow{\pi} M \to 1 \]

be a presentation of \( A \), where \( M \) is an abelian variety and \( T \cong (C^*)^t \). Observe that every algebraic morphism from \( C^* \) to an abelian variety is necessarily constant. Therefore: If \( \phi \) is an automorphism of the algebraic variety \( A \), then \( \phi \) must map fibers of \( \pi : A \to M \) into fibers and thereby induce an automorphism \( \phi_0 \) of \( M \). Now \( \phi \in \text{Aut}(A)^0 \) implies that \( \phi \) acts trivially on the fundamental group \( \pi_1(A) \) and therefore \( \phi_0 \) acts trivially on the fundamental group \( \pi_1(M) \).

Together with \( \phi_0(e_M) = e_M \) this implies that \( \phi_0 = \text{id}_M \). Thus \( \phi \) can act only along the fibers of \( \pi \). Moreover, since the fundamental group of the fiber \( T \) injects into the fundamental group of \( A \), it is clear that for each fiber of the projection map \( \pi : A \to M \) the restriction of \( \phi \) is homotopic to the identity. Now the only automorphisms of \( (C^*)^t \) which acts trivially on the fundamental group \( \pi_1((C^*)^t)) \cong Z^t \) are translations. Hence \( \phi \) must be given in the form

\[ \phi : x \mapsto \zeta(\pi(x)) \cdot x, \]

where \( \zeta : M \to T \) is a holomorphic map. But \( M \) is compact, hence every holomorphic map from \( M \) to \( (C^*)^t \) is constant. Thus \( \zeta \) is constant, and in fact \( \zeta \equiv e_A \), because \( \phi(e_A) = a_A \), i.e., \( \phi = \text{id}_A \). Q.E.D.

**Proposition 6.2** Let \( X \) be a smooth projective variety with \( n = \text{dim } X \) and let \( D \) be an effective reduced divisor on \( X \). Let \( V \) denote the vector space of those vector fields on \( X \) which are everywhere tangent to \( D \). Then \( X \) is an equivariant algebraic compactification of a semi-abelian variety \( A \) with \( A = X \setminus D \) if and only if the following conditions are fulfilled:

(i) \( \text{dim } V = n = \text{dim } X \).

(ii) For every \( v, w \in V \) the vector fields \( v \) and \( w \) commute.

(iii) Let \( v_i, 1 \leq i \leq n \), be bases of \( V \). Then the zero divisor of \( \wedge_{i=1}^n v_i \) equals \( D \).

(iv) For \( v \in V \) and \( p \in \{ x \in X : v_x = 0 \} \) let \( \Phi_{v,p} \) denote the induced linear endomorphism of \( T_pX \). Then \( \Phi_{v,p} \) is nilpotent if and only if \( v = 0 \).

Proof. Conditions (ii) and (iii) ensure that \( \text{Aut}(X) \) contains a connected commutative subgroup \( A_0 \) (corresponding to the Lie algebra \( V \)) which has an open orbit \( \Omega \), and that this open orbit \( \Omega \) is precisely \( X \setminus D \). The connected component of \( \text{Aut}(X) \) is an algebraic group. Let \( A \)
denote the Zariski closure of $A_0$ in $\text{Aut}(X)$. Since $A_0$ stabilizes $D$, its Zariski closure $A$ must stabilize $D$, too. But this implies $\text{Lie}(A_0) \subset V$. Hence $A = A_0$.

For $p \in \Omega$ let $A_p = \{a \in A : a(p) = p\}$. Because $A$ is commutative, $A_p = A_q$ for all $p, q \in \Omega$. Since $\Omega$ is dense in $X$, it follows that $A_p = \{\text{id}_X\}$. Hence $\Omega \cong A$.

As a connected commutative algebraic group, either $A$ is a semi-abelian variety, or it contains an algebraic subgroup $U$ isomorphic to the additive group $(\mathbb{C}, +)$. If $A$ contains such a subgroup $U$, then every non-trivial $U$-orbit is affine and one-dimensional, and therefore contains a $U$-fixed point $p \in X$ in its closure. Being unipotent, elements of $U$ induce unipotent endomorphisms of $T_pX$. This however implies that a vector field $v \in \text{Lie}(U)$ induces a nilpotent endomorphism of $T_pX$, contradicting property (iv). Thus $A$ cannot contain an algebraic subgroup isomorphic to the additive group and therefore must be a semi-abelian variety.

Conversely, let $A \hookrightarrow \bar{A}$ be a smooth algebraic equivariant compactification of a semi-abelian variety $A$. Let $G$ denote the group of all automorphisms of $\bar{A}$ stabilizing the boundary $\partial A$. Then every $g \in G$ stabilizes $A$ as well, hence $G \hookrightarrow \text{Aut}(A)$. But $\text{Aut}(A)^0 = A$ (Lemma 6.1). Hence $G^0 = A$, and the fundamental vector fields of the $A$-action on $\bar{A}$ are the only vector fields on $\bar{A}$ which are everywhere tangent to the divisor $D = \partial A$. This implies properties (i)–(iii). Property (iv) follows from the fact that all the isotropy groups of the $A$-action on $\bar{A}$ are algebraic subgroups and therefore reductive. Q.E.D.

Thus, we obtain the following:

**Theorem 6.3** Let $p : U \to S$ be a flat family of smooth projective varieties, let $\Theta$ be an effective divisor on $U$, and let $S_0$ be the set of points $s \in S$ such that $p^{-1}(s) \setminus \Theta$ is a semi-abelian variety with $p^{-1}(s) \setminus \Theta \hookrightarrow p^{-1}(s)$ as a smooth equivariant compactification. Then $S_0$ is a Zariski open subset of its Zariski closure in $S$.

**Proof.** By the preceding Proposition 6.2, the condition $s \in S_0$ translates into four conditions (i), . . . , (iv), each of which is closed, or at least locally closed for a flat family. Hence the assertion follows. Q.E.D.

### 6.2 Curves

Let $C$ be a smooth compact complex algebraic curve of genus $g$, and let $E$ be a divisor on $C$ with $\deg(E) = 2g + 1$. Then $\deg(K_C - E) = -3$, since $\deg(K_C) = 2g - 2$. By Riemann-Roch’s theorem, $E$ is very ample,

$$h^0(C, L(E)) = \dim H^0(C, L(E)) = \deg(E) + 1 - g = g + 2,$$

and

$$\chi(C, L(lE)) = l \deg(E) + 1 - g = l(2g + 1) + (1 - g)$$
for all $l \in \mathbb{N}$. Hence every smooth compact algebraic curve of genus $g$ can be embedded into $\mathbb{P}^{g+1}$ with Hilbert polynomial $P_1(l) = l(2g + 1) + (1 - g)$.

**Proposition 6.4** Let $g, d \in \mathbb{N}$. Then there exist projective algebraic varieties $U_1$ and $Q_1$, a proper surjective morphism $\pi_1 : U_1 \to Q_1$, and a divisor $\Sigma_1 \subset U_1$ such that for every algebraic curve $C'$ of genus $g$ with $d$ punctures, there exists a point $p \in Q_1$ (not necessarily unique) such that $(\pi_1^{-1}(p) \setminus \Sigma_1) \cong C'$.

**Proof.** Let $\pi : U \to Q_0$ be the universal family of the Hilbert scheme of compact algebraic curves of genus $g$ with the Hilbert polynomial $P_1(l)$ as above. Define

$$\pi_1 : U_1 = \{(u_0, u_1, \ldots, u_d) \in U^{d+1} : \pi(u_0) = \ldots = \pi(u_d)\} \to Q_0 \times U^d = Q_1$$

by $\pi_1(u_0, \ldots, u_d) = (\pi(u_0), u_1, \ldots, u_d)$. Moreover we define $\Sigma_1$ as the “diagonal divisor” by

$$\Sigma_1 = \{(u_0, u_1, \ldots, u_d) \in U_1 : u_0 \in \{u_i\}_{i=1}^d\}.$$ 

Q.E.D.

### 6.3 Semi-abelian varieties

**Proposition 6.5** Let $(\mathbb{C}^*)^I \cong T \hookrightarrow \bar{T}$ be a smooth projective toric variety, let $\{\alpha_i\}_{i \in I}$ be a $\mathbb{Z}$-module basis of $H_{2d}(\bar{T}, \mathbb{Z})$ with $\alpha_i \in H_{2d_i}(\bar{T}, \mathbb{Z})$ ($d_i \in \{0, \ldots, \dim T\}$) and let $s_i \in \mathbb{N}$ for $i \in I$. Then there exist projective algebraic varieties $U_2$ and $Q_2$, a proper surjective morphism $\pi_2 : U_2 \to Q_2$, divisors $\Sigma_2, \Theta \in \text{Div}(U_2)$ such that for every semi-abelian variety $A$ of dimension $n \geq t$ with smooth equivariant compactification $A \hookrightarrow \bar{A}$ of type $\bar{T}$ we have the following property: If there exists an ample divisor $D \in \text{Div}(\bar{A})$ with $\alpha_i \cdot [D]^{n-d_i} = s_i$ for all $i \in I$, then there exists a point $q \in Q_2$ and an isomorphism $\zeta : \bar{A} \to \pi_2^{-1}(q) \subset U_2$ such that $A = \bar{A} \setminus \zeta^{-1}(\Sigma_2)$ and $D = \zeta^{-1}(\Theta)$.

**Proof.** First note that topologically $\bar{A} \cong \bar{T} \times M$ with $M = A/T$ and that $H_*(T, \mathbb{Z})$ is generated by the invariant divisors of $\bar{T}$ while $T_M$ is trivial. Therefore the Chern classes of $T_A$ can be expressed in terms of the generators of $H_2(\bar{T}, \mathbb{Z})$. Via Hirzebruch-Riemann-Roch’s theorem it follows that the Hilbert polynomial of $D$ is determined by the conditions $\alpha_i \cdot [D]^{n-d_i} = s_i$ ($i \in I$).

Furthermore $\bar{T}$ determines a number $l$ such that $lD$ is very ample with $h^0(L(lD)) = \chi(L(lD))$ (Corollary 5.11). Hence these conditions also determine a polynomial $P_2$ and a number $k$ such that $\bar{A}$ can be embedded into $\mathbb{P}^k(\mathbb{C})$ with Hilbert polynomial $P_2$.

The boundary $\partial A$ of $A$ in $\bar{A}$ is a divisor and its intersection numbers are determined by the choice of the toric variety $\bar{T}$. Thus the Hilbert polynomial of this divisor $\partial A$ is also determined by the choice of the toric variety.
By the general theory of Hilbert schemes we now obtain a morphism between projective algebraic varieties \( \pi_2 : U_2 \to Q_2 \) and divisor \( \Sigma_2, \Theta \in \text{Div}(U_2) \) satisfying the following property:

For every \((A, \tilde{A}, D)\) with the given numerical data we can find a point \( q \in Q_2 \) such that there is an isomorphism \( \zeta : \tilde{A} \to \pi_2^{-1}(q) \) with \( \zeta^{-1}(\Theta) = D \) and \( \zeta^{-1}(\Sigma_2) = \partial A \).

By Proposition 5.2 we may replace \( Q_2 \) by a closed subspace and therefore assume that there is a Zariski-open subset \( W \subset Q_2 \) such that the fiber \( \pi_2^{-1}(q) \) is a smooth equivariantly compactified semi-abelian variety for every \( q \in W \). Q.E.D.

7 Proof of the Main Theorem in general case

7.1 Compactified total space of a coherent sheaf

If \( E \to X \) is a vector bundle of rank \( n \) over a compact complex space, then \( E \) admits a compactification in the form of an embedding into a \( \mathbb{P}^n \)-bundle \( \tilde{E} \) over \( X \). This construction can be generalized to coherent sheaves.

Following Grauert [G62], we define a linear space over \( X \) as a holomorphic map of complex spaces \( \pi : V \to X \) together with holomorphic maps \( \mu : V \times_X V \to V \) and \( \nu : \mathbb{C} \times V \to V \), satisfying the conditions:

(i) \( \pi(\mu(x, y)) = \pi(x) = \pi(y) \) for all \((x, y) \in V \times_X V\) and \( \pi(\nu(t, v)) = \pi(v) \) for all \( t \in \mathbb{C} \) and \( v \in V \).

(ii) Each fiber of \( \pi \) is a complex vector space, where the vector addition is given by \( \mu \) and the scalar multiplication by \( \nu \).

(iii) For every point \( x \in X \) there is an open neighbourhood \( U \) and a number \( n \in \mathbb{N} \) such that there is a commutative diagram

\[
\begin{array}{ccc}
V|_U & \xrightarrow{\phi} & \mathbb{C}^n \times U \\
\downarrow \pi & & \downarrow \text{pr}_2 \\
U & \xrightarrow{id} & U,
\end{array}
\]

where \( V|_U = \pi^{-1}(U) \) and \( \phi \) is an embedding, linear on every fiber of \( \pi \).

As described in [G62], for every coherent sheaf \( S \) on a complex space \( X \) there exists a "linear space" \( V \to X \) such that \( S \) is isomorphic to the sheaf of those holomorphic functions on \( V \) which are linear on each fiber. Furthermore, if \( X \) is compact, there is a natural compactification \( V \hookrightarrow \tilde{V} \) induced by \( \mathbb{C}^n \hookrightarrow \mathbb{P}^n \) in each fiber. The points in the boundary \( \partial V = \tilde{V} \setminus V \) correspond to complex lines in the fibers of \( \pi \).
7.2 Proof of the Main Theorem

Let \( U_1 \xrightarrow{\pi_1} Q_1 \) and \( U_2 \xrightarrow{\pi_2} Q_2 \), together with \( \Sigma_i \in \text{Div}(Q_i) \) and \( \Theta \in \text{Div}(Q_2) \) be as constructed in Propositions 6.4 and 6.5, respectively. We set \( \pi = \pi_1 \times \pi_2 : U = U_1 \times U_2 \to Q = Q_1 \times Q_2 \), \( p_i : Q \to Q_i \), where \( p_i \) are the natural projections. We define \( Q_i^* \) as the open subset of \( Q_i \) consisting of all those points \( q_i \in Q_i \) such that

(i) the fiber \( \pi_i^{-1}(q_i) \) is irreducible, reduced and smooth,

(ii) \( \pi_i^{-1}(q_i) \not\subset \Sigma_i \), \( i = 1, 2 \), and \( \pi_2^{-1}(q_2) \not\subset \Theta \).

Set \( Q^* = Q_1^* \times Q_2^* \).

The points of \( Q^* \) are the “nice” points in which we are really interested.

For each of the morphisms \( \pi_i : U_i \to Q_i \), the sheaf of horizontal logarithmic differential forms along \( \Sigma_i \) is defined to be the subsheaf of those logarithmic differential forms whose induced differentials on fibers vanish. This is a coherent sheaf as well as the quotient sheaf of all differential forms divided by the horizontal ones. This quotient sheaf we call the “sheaf of vertical logarithmic differential forms” denoted by \( \Omega^1_{\text{vert}}(U_i, \log \Sigma_i)/Q_i \). Define

\[
S_i = p_i^{-1}\pi_{i*}\Omega^1_{\text{vert}}(U_i, \log \Sigma_i)/Q_i.
\]

This is a coherent sheaf over \( Q \). Moreover, it is a locally free sheaf over \( Q^* \). By Grauert’s theory of linear spaces (see §7.1), there are linear spaces \( S_i \to Q \) such that \( S_i \) is the sheaf of functions on \( S_i \) which are linear on fibers.

Set

\[
S_i^* = S_i|_{Q^*}, \quad U_i^* = U_i|_{Q^*}, \quad i = 1, 2.
\]

The fibers of \( S_i^* \to Q^* \) can be identified with the space of logarithmic vector fields on the quasi-Albanese variety of the fibers of \( U_i^* \to Q_i^* \). Let \( J^k_{\text{vert}}(U_i^*) \) denote the relative \( k \)-jet space, i.e., the inverse image of zero jets of \( \pi_{i*} : J^k(U_i^*) \to J^k(Q_i^*) \). Let \( L \to Q \) be the space of all relative linear morphisms from \( S_1 \) into \( S_2 \) over \( Q \). We set \( L^* = L|_{Q^*} \).

By Proposition 3.11 we can identify \( \mathfrak{m}(k) \otimes S_{i_{q_i}}^* \) with \( J^k(U_{i_{q_i}}^*, \log \Sigma_{i_{q_i}}) \) for \( q_i \in Q_i^* \), and

\[
J^k(U_{i_{q_i}}^*, \log \Sigma_{i_{q_i}})|_{U_{i_{q_i}}^* \setminus \Sigma_{i_{q_i}}} \cong J^k_{\text{vert}}(U_i^*)|_{U_{i_{q_i}}^* \setminus \Sigma_{i_{q_i}}}.
\]
As in (4.3), we next define algebraic subsets $Z_k \subset L^* \times Q^* ((U_1^* \setminus \Sigma_1) \times (\Theta^* \setminus \Sigma_2))$, $k = 1, 2, \ldots$

by

$$Z_k = \left\{ (\lambda, p, q) \in L^* \times Q^* ((U_1^* \setminus \Sigma_1) \times (\Theta^* \setminus \Sigma_2)) : \right.$$

$$\left. (\text{id}_{m(k)} \otimes \lambda) \left( J^k_{\text{vert}} (U_1^*)_{p_1(p)} \right) \subset J^k_{\text{vert}} (\Theta^*)_{p_2(q)} \right\}.$$

Since $\{Z_k\}_k$ is a decreasing sequence of algebraic subsets, it terminates. Thus there exists a number $N$ such that $Z_k = Z_N$ for all $k \geq N$.

Let $q \in Q^*$, $C = \pi_1^{-1}(p_1(q)) \setminus \Sigma_1$, $A = \pi_1^{-1}(p_2(q)) \setminus \Sigma_2$ and $D = A \cap \Theta$. Let $f : C \to A$ be a morphism. Then $C$ may be regarded as a closed subvariety of $\text{Alb}(C)$, and there is a morphism $\tilde{f} : \text{Alb}(C) \to A$ such that $\tilde{f}|_C = f$. If $\text{mult}_x f^*D \geq k$ with $x \in C$, then by construction $(d\tilde{f}_x, x, f(x)) \in Z_k$. If $k \geq N$, then $f(C)$ is tangent to $D$ at $f(x)$ with infinite order; that is, $f(C) \subset D$. This finishes the proof of the Main Theorem. Q.E.D.

8 Application

As an application of the Main Theorem we prove

**Theorem 8.1** Let $C$ be an affine algebraic curve, let $A$ be an abelian variety, and let $D$ be an ample effective reduced divisor on $A$. Then either there is a non-constant morphism from $C$ into $D$ or there are only finitely many morphisms from $C$ into $A \setminus D$.

Before we prove the theorem we need the following.

**Lemma 8.2** Let $C$ be a smooth algebraic curve with smooth algebraic compactification $C \hookrightarrow \bar{C}$, let $A$ be an abelian variety, and let $D$ be an ample hypersurface in $A$. Let $f : \bar{C} \to A$ be a morphism with $f(C) \subset A \setminus D$, and let $f_a : C \to A$ be given by $f_a(t) = a + f(t)$. Set

$$P = \{ a \in A : f_a(C) \subset A \setminus D \}.$$

Then either there exists a point $p \in A$ with $f_p(C) \subset D$ or $P$ is finite.

**Proof.** Let $S = \bar{C} \setminus C$. Let $d = \text{deg} f^*D$. Let $(S_i)_{i \in I}$ denote the finite family of all distinct effective divisors on $\bar{C}$ with $\text{deg}(S_i) = d$ and $\text{Supp} S_i \subset S$. Define

$$P_i = \{ a \in A : f_a^*(D) = S_i \}$$

for $i \in I$, and

$$P_\infty = \{ a \in A : f_a(C) \subset D \}.$$
Then $P$ is the disjoint union of $P_\infty$ and $P_i, i \in I$. Assume that $P_\infty = \emptyset$. Then $P_i$ is compact. Now consider

$$F_i : P_i \times C \to A \setminus D$$

given by $F_i(a, t) = f_a(t) = a + f(t)$. Note that $A \setminus D$ is affine, because $D$ is assumed to be ample. Thus the compactness of $P_i$ implies that $F_i$ is locally constant with respect to the first variable. Due to the equality $F_i(a, t) = a + f(t)$ this can be true only if $P_i$ is finite. Hence $P$ is finite. \textit{Q.E.D.}

\textbf{Corollary 8.3} Assume that every morphism from $C$ to $D$ is constant. Then $\text{Mor}(C, A \setminus D)$ is embedded into $\text{Mor}(\bar{C}, A)$ as a discrete subset.

\textit{Proof.} Let $f : C \to A \setminus D$ be a morphism, and let $U$ be a Stein open neighbourhood of the neutral element $e$ in $A$. Then $W = \{g : \bar{C} \to A : f(x) - g(x) \in U, \forall x \in \bar{C}\}$ is an open neighbourhood (for the compact-open topology) of $f$ in $\text{Mor}(\bar{C}, A)$. However, if $f(x) - g(x) \in U$ for all $x \in \bar{C}$, then $x \mapsto f(x) - g(x)$ is constant, because $\bar{C}$ is compact and $U$ is Stein. Thus every $g \in W$ is given as $g : x \mapsto a + f(x)$ for some $a \in A$. Therefore the assertion follows from above Lemma 8.2. \textit{Q.E.D.}

\textit{Proof of Theorem 8.1.} Assume that every morphism from $C$ to $D$ is constant. By the Main Theorem this implies that there exists a number $N$ such that for every morphism $\bar{f} : \bar{C} \to A$ the multiplicities of $\bar{f}^*D$ are uniformly bounded by $N$. This leads to $\deg \bar{f}^*D \leq N \cdot \#(\bar{C} \setminus C)$ for all holomorphic maps $\bar{f} : \bar{C} \to A$ with $\bar{f}(C) \subset A \setminus D$. Thus $\text{Mor}(C, A \setminus D)$ is contained in a union of finitely many irreducible components of $\text{Mor}(\bar{C}, A)$. It follows from Corollary 8.3 that $\dim \text{Mor}(C, A \setminus D) = 0$. Therefore $\text{Mor}(C, A \setminus D)$ must be finite. \textit{Q.E.D.}

\textit{Remark 8.4} (i) By Dethloff-Lu [DL01], every holomorphic mapping $f : C \to A \setminus D$ extends holomorphically to $\bar{f} : \bar{C} \to A$, and so $f$ is an algebraic morphism. Thus, Theorem 8.1 implies that there are only finitely many non-constant holomorphic mappings of $C$ into $A \setminus D$.

(ii) If $D$ contains no translate of a non-trivial Abelian subvariety of $A$, then $A \setminus D$ is complete hyperbolic and hyperbolically embedded into $A$ ([Gr77]). In this case, the uniform bound for $\deg \bar{f}, f \in \text{Mor}(C, A \setminus D)$, is a consequence of [Nog88].

(iii) If $D$ admits a non-constant morphism from $C$, then there may be infinitely many non-constant morphisms $f : C \to A$ with $f^{-1}D \subset S$. In fact, let $E$ be an elliptic curve, and
set \( A = E^2 \). Then \( D = \{0\} \times E + E \times \{0\} \) is ample. Set \( \bar{C} = E \) and \( C = E \setminus \{0\} \). There are infinitely many morphisms

\[
f_a : x \in C \rightarrow (x, a) \in A \setminus D, \quad a \in C.
\]

In this case, \( \bar{f}_0(C) \subset D \).

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