Negativity Fonts, multiqubit invariants and Four qubit Maximally Entangled States

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Recently, we introduced negativity fonts as the basic units of multipartite entanglement in pure states. We show that the relation between global negativity of partial transpose of \(N\)-qubit state and linear entropy of reduced single qubit state yields an expression for global negativity in terms of determinants of negativity fonts. Transformation equations for determinants of negativity fonts under local unitaries (LU’s) are useful to construct LU invariants such as degree four and degree six invariants for four qubit states. The difference of squared negativity and \(N\)-tangle is an \(N\) qubit invariant which contains information on entanglement of the state caused by quantum coherences that are not annihilated by removing a single qubit. Four qubit invariants that detect the entanglement of specific parts in a four qubit state are expressed in terms of three qubit subsystem invariants. Numerical values of invariants bring out distinct features of several four qubit states which have been proposed to be the maximally entangled four qubit states.

I. INTRODUCTION

Entanglement is an intriguing property of quantum systems and its detection, characterization and quantification are important questions in quantum mechanics. For a pure state of bipartite quantum system consisting of two distinguishable subsystems \(A\) and \(B\), each of arbitrary dimension, negativity \([1, 2]\), and linear entropy calculated from reduced density operator of either element, may be chosen as entanglement measures. For tripartite case, besides the quantity of entanglement we must also know whether the entanglement is GHZ-like or W-like \([3]\) and states are grouped into distinct entanglement classes for four qubits \([4, 5]\). An entanglement measure must have value in the range zero for the product state to a maximum value for a maximally entangled state and satisfy the minimal requirement of local unitary invariance \([10]\). Generally accepted measures of entanglement, such as concurrence \([11]\) for two qubits, and three tangle \([12]\) for three qubits, turn out to be such invariants \([8, 13]\). In the case of four qubits, the standard approach from invariant theory, employing the well established W-process by Cayley, has lead to the construction of a complete set of SL-invariants \([14]\). In ref. \([15]\) the invariants up to degree 6 have been determined together with 5 invariants of degree 8. Local unitary invariants have been reported for even number of qubits in ref. \([16]\) and for even and odd number of qubits in \([17, 19]\). Independent of these approaches, a method based on expectation values of antilinear operators with emphasis on permutation invariance of the global entanglement measure \([20, 21]\), has been suggested. Permutation invariance has been highlighted as a demand on global entanglement measures already in Ref. \([12]\) and later in Ref. \([22]\).

Negativity of global partial transpose is a widely used computable measure of free bipartite entanglement. Negativity is based on Peres-Horodecki NPT criterion \([23, 24]\) and is known to be an entanglement monotone \([2]\). A global partial transpose with respect to a sub system \(p\) is obtained by transposing the state of subsystem \(p\) in state operator. In refs. \([28, 29]\), we introduced negativity fonts defined as two by two matrices of probability amplitudes that determine the negative eigen values of four by four submatrices of partially transposed state operators. It was shown that relevant \(N\)-qubit local unitary invariants can be obtained, directly, from transformation properties of determinants of negativity fonts under local unitary transformations. From expression of an invariant in terms of determinants of negativity fonts, one can easily read how subsystems invariants contribute to the composite system invariant. In this article, we obtain an expression for global negativity in terms of determinants of negativity fonts. The squared negativity of \(N\)-qubit partially transposed operator, is found to be the sum of squares of moduli of determinants of all possible negativity fonts. For the sake of completeness, we briefly outline the procedure for constructing two-qubit local unitary (LU) invariants for an \(N\)-qubit state by examining the intrinsic sources of negativity present in global and \(K\)-way partially transposed matrices. In a four qubit state, the entanglement of a three qubit subsystem may arise due to four-way or three-way correlations. We show that four qubit invariants that detect the entanglement

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of two qubits in a four qubit state \( \overline{29} \) are combinations of three qubit invariants. A form of degree six invariant for four qubit states constructed in terms of negativity font determinants demonstrates the ease with which complex invariants can be written down from basic principles and calculated numerically. Numerical values of invariants are found to bring out distinct features of several known four qubit states which have been proposed to be the maximally entangled states.

Definition of negativity fonts and the notation to represent determinants of \( N \)–way and \( K \)–way negativity fonts is given in section II. Transformation equations for determinants of negativity fonts are used to obtain an expression for square of global negativity in terms of determinants of negativity fonts in section III. Section IV details degree two, four and six invariants for a generic four qubit state. Numerical values of invariants and entanglement monotones for states known or conjectured to be maximally entangled four qu bit states are reported and nature of quantum correlations in these states analyzed in section V followed by a summary of results in section VI.

II. DEFINITION OF A \( K \)–WAY NEGATIVITY FONT

Consider a bipartite system consisting of two distinguishable subsystems \( A \) and \( B \), each of arbitrary dimension, in pure state \( \hat{\rho} \). The global negativity \( \overline{1} \) of partial transpose \( \hat{\rho}^T_A \) (partial transpose with respect to \( A \)) is defined as

\[
N^A_G = \frac{1}{d_A - 1} \left( \frac{\| \hat{\rho}^T_A \|}{\| \hat{\rho} \|} - 1 \right),
\]

where \( \| \hat{\rho} \|_1 \) is the trace norm of \( \hat{\rho} \). A general \( N \)–qubit pure state reads as

\[
|\Psi^{A_1A_2...A_N}\rangle = \sum_{i_1i_2...i_N} a_{i_1i_2...i_N} |i_1i_2...i_N\rangle \hat{\rho} = |\Psi^{A_1A_2...A_N}\rangle \langle \Psi^{A_1A_2...A_N}|,
\]

where \( |i_1i_2...i_N\rangle \) are the basis spanning \( 2^N \) dimensional Hilbert space and \( A_p \) is the location of qubit \( p (p = 1 \) to \( N) \). The coefficients \( a_{i_1i_2...i_N} \) are complex numbers. The basis states of a single qubit are labelled by \( i_m = 0 \) and \( 1 \), where \( m = 1, ..., N \). The matrix elements of global partial transpose \( \hat{\rho}^{T_p}_G \) with respect to qubit \( p \) are obtained from \( \hat{\rho} \) through

\[
\langle i_{1}i_{2}...i_{N}\rangle^{T_{p}}|j_{1}j_{2}...j_{N}\rangle = \langle i_{1}i_{2}...i_{p-1}i_{p+1}...i_{N}\rangle^{\hat{\rho}}|j_{1}j_{2}...i_{p-1}i_{p+1}...i_{N}\rangle.
\]

Peres PPT separability criterion \( \overline{23} \) states that the partial transpose \( \hat{\rho}^{T_p}_G \) of a separable state is positive. Rewrite \( N \)–qubit pure state as \( |\Psi^{A_1A_2...A_N}\rangle = \sum_{i_1i_2...i_N} |F\rangle_{00i_3i_4...i_N} \) where

\[
|F\rangle_{00i_3i_4...i_N} = a_{00i_3i_4...i_N} |00i_3i_4...i_N\rangle + a_{10i_3i_4...i_N} |10i_3i_4...i_N\rangle + a_{ii_3i_4+1...i_N+1} |01i_3 + i_4 + 1...i_N + 1\rangle + a_{i1i_3i_4+1...i_N+1} |11i_3 + i_4 + 1...i_N + 1\rangle.
\]

Here \( i_m + 1 = 0 \) for \( i_m = 1 \) and \( i_m + 1 = 1 \) for \( i_m = 0 \). The entanglement of \( \chi^{00i_3i_4...i_N} = |F\rangle_{00i_3i_4...i_N} \langle F| \) is quantified by

\[
(N^A_G(\chi^{00i_3i_4...i_N}))^2 = \det \begin{bmatrix}
a_{00i_3i_4...i_N} & a_{01i_3i_4+1...i_N+1} \\
a_{10i_3i_4...i_N} & a_{11i_3i_4+1...i_N+1}
\end{bmatrix}^2 = 4 |D^{00i_3i_4...i_N}|^2.
\]

Since determinant \( D^{00i_3i_4...i_N} = \det \nu^{00i_3i_4...i_N} \) determines \( N^A_G(\chi^{00i_3i_4...i_N}) \), we refer to \( 2 \times 2 \) matrix of probability amplitudes

\[
\nu_{N}^{00i_3i_4...i_N} = \begin{bmatrix}
a_{00i_3i_4...i_N} & a_{01i_3i_4+1...i_N+1} \\
a_{10i_3i_4...i_N} & a_{11i_3i_4+1...i_N+1}
\end{bmatrix},
\]

as a negativity font of \( N \)–way entanglement in \( |\Psi^{A_1A_2...A_N}\rangle \).

In general, if \( \hat{\rho} \) is a pure state, then the negative eigenvalue of \( 4 \times 4 \) sub-matrix of global partial transpose \( \hat{\rho}^{T_p}_G \) or a \( K \)–way partial transpose \( \hat{\rho}^{T_K}_G \) \( \overline{27} \) in the space spanned by distinct basis vectors \( |i_1i_2...i_3...i_p...i_N\rangle, |j_1j_2...j_p...j_N\rangle \), \( |i_1i_2...j_p...i_N\rangle \), and \( |j_1j_2...i_p...j_N\rangle \) is \( \lambda^- = -\det \begin{bmatrix}
a_{i_1i_2...i_p...i_N} & a_{i_1i_2...j_p...i_N+1} \\
a_{j_1j_2...i_p...j_N} & a_{j_1j_2...j_p...i_N+1}
\end{bmatrix} \) with \( i^+_K^{i_1i_2...i_p...i_N} \) defined as

\[
\nu^{i_1i_2...i_p...i_N}_{K} = \begin{bmatrix}
a_{i_1i_2...i_p...i_N} & a_{j_1j_2...i_p...j_N} \\
a_{i_1i_2...j_p...i_N+1} & a_{j_1j_2...j_p...i_N+1}
\end{bmatrix}.
\]
where \( K = \sum_{m=1}^{N} (1 - \delta_{i_m,j_m}) \) (\( 2 \leq K \leq N \)). Here \( \delta_{i_m,j_m} = 1 \) for \( i_m = j_m \), and \( \delta_{i_m,j_m} = 0 \) for \( i_m \neq j_m \). The \( 2 \times 2 \) matrix \( \nu_k^{i_1j_2...i_pmN} \) defines a \( K \)-way negativity font. To distinguish between different \( K \)-way negativity fonts we shall replace subscript \( K \) in Eq. (11) by a list of qubits for which \( \delta_{i_m,j_m} = 1 \). In other words a \( K \)-way font involving qubits \( A_q \) to \( A_q+K \) such that \( \sum_{m=1}^{N} (1 - \delta_{i_m,j_m}) = K \), reads as

\[
\nu^{i_1j_2...i_pmN}_{(A_1)_{i_1}(A_2)_{i_2}...(A_q)_{i_q}(A_{q+K+1})_{i_{q+K+1}}...(A_N)_{i_N}} = \left[ \begin{array}{ccc} a_{i_1i_2...i_pmN} & a_{i_1i_2...i_pmN1} & a_{i_1i_2...i_pmN2} \\ a_{i_1i_2...i_pmN1} & a_{i_1i_2...i_pmN2} & a_{i_1i_2...i_pmN3} \end{array} \right],
\]

and its determinant is represented by

\[
D^{i_1j_2...i_pmN}_{(A_1)_{i_1}(A_2)_{i_2}...(A_q)_{i_q}(A_{q+K+1})_{i_{q+K+1}}...(A_N)_{i_N}} = \det \nu^{i_1j_2...i_pmN}_{(A_1)_{i_1}(A_2)_{i_2}...(A_q)_{i_q}(A_{q+K+1})_{i_{q+K+1}}...(A_N)_{i_N}}.
\]

Thus the determinant of a \( K \)-way font in an \( N \) qubit state has \( N-K \) subscripts and \( K \) superscripts. In this notation no subscript is needed for determinant of an \( N \)-way negativity font. The general rule to represent the determinants of negativity fonts is that the qubit states are ordered according to the location of the qubits with the states that appear in the subscript not being present in the superscript. One can identify the determinants of negativity fonts with Plücker coordinates in ref. [30], where Plücker coordinate equations of Grassmann variety have been used to construct entanglement monotones for multi-qubit states.

### A. Negativity fonts in \( K \)-way partial transpose

To construct a \( K \)-way partially transposed matrix \( \rho^T_k \) from the state operator \( \hat{\rho} \), every matrix element \( \langle i_1i_2...i_N | \rho^T_k | j_1j_2...j_N \rangle \) is labelled by a number \( K = \sum_{m=1}^{N} (1 - \delta_{i_m,j_m}) \). The \( K \)-way partial transpose \( (K > 2) \) of \( \rho \) with respect to subsystem \( p \) is obtained by selective transposition such that

\[
\langle i_1i_2...i_N | \rho^T_k | j_1j_2...j_N \rangle = \langle i_1i_2...i_{p-1}j_p i_{p+1}...i_N | \hat{\rho} | j_1j_2...j_p-1i_p j_p+1...j_N \rangle,
\]

if \( \sum_{m=1}^{N} (1 - \delta_{i_m,j_m}) = K \), and \( \delta_{i_p,j_p} = 0 \)

(10)

and

\[
\langle i_1i_2...i_N | \rho^T_k | j_1j_2...j_N \rangle = \langle i_1i_2...i_N | \hat{\rho} | j_1j_2...j_N \rangle,
\]

if \( \sum_{m=1}^{N} (1 - \delta_{i_m,j_m}) \neq K \).

(11)

while

\[
\langle i_1i_2...i_N | \rho^T_k | j_1j_2...j_N \rangle = \langle i_1i_2...i_{p-1}j_p i_{p+1}...i_N | \hat{\rho} | j_1j_2...j_p-1i_p j_p+1...j_N \rangle,
\]

if \( \sum_{m=1}^{N} (1 - \delta_{i_m,j_m}) = 1 \) or \( 2 \), and \( \delta_{i_p,j_p} = 0 \)

(12)

and

\[
\langle i_1i_2...i_N | \rho^T_k | j_1j_2...j_N \rangle = \langle i_1i_2...i_N | \hat{\rho} | j_1j_2...j_N \rangle,
\]

if \( \sum_{m=1}^{N} (1 - \delta_{i_m,j_m}) \neq 1 \) or \( 2 \).

(13)
The $K$-way negativity calculated from $K$-way partial transpose of matrix $\rho$ with respect to subsystem $p$, is defined as $N_{Ap}^K = \left( \| \rho_{K}^{T_p} \|_1 - 1 \right)$. Using the definition of trace norm and the fact that $tr(\rho_{K}^{T_p}) = 1$, we get $N_{Ap}^K = 2 \sum_i |\lambda_i^K|$, $\lambda_i^K$ being the negative eigenvalues of matrix $\rho_{K}^{T_p}$. The $K$-way negativity ($2 \leq K \leq N$), defined as the negativity of $K$-way partial transpose, is determined by the presence or absence of $K$-way quantum coherences in the composite system. By $K$-way coherences we mean the type of coherences present in a $K$-qubit GHZ-like state. The negativity $N_{Ap}^K$ is a measure of all possible types of entanglement attributed to $K$-way coherences. It was shown in refs. [25-27] that the global partial transpose of an $N$-qubit state may be written as a sum of $K$-way partial transposes

$$(N^{T_p}_K) = \sum_{K=2}^{N} \rho_{K}^{T_p} - (N - 2)\rho.$$ (14)

By rewriting the global partial transpose as a sum of $K$-way partial transposes, the negativity fonts are distributed amongst $N - 1$ partial transposes. Contributions of partial transposes to global negativity, referred to as partial $K$-way negativities are not unitary invariants, but their values coincide with those of three tangle and concurrences for three qubit canonical state [25].

III. TRANSFORMATION EQUATIONS FOR DETERMINANTS OF NEGATIVITY FONTS, GLOBAL NEGATIVITY AND TWO-QUBIT INVARIANTS

To derive expressions for LU invariants which measure genuine $N$-body quantum correlations present in the state, the transformation equations under LU are written, for negativity fonts characterizing the $N$-way partial transpose and $(N - 1)$ way partial transpose. Two qubit invariants obtained from transformation equations pave the way to construction of $N$-qubit LU invariants to be used to write the entanglement monotones. In the following, an invariant named $I$ represented by $(I_{K})(A_{1...A_{x}})$, is understood to be invariant under the action of local unitaries on qubits $A_1$, $A_2$, ..., $A_x$ of the N qubit system. In general, the superscript outside the bracket will list the qubits in the subsystem of which $I_K$ is an invariant, while subscript lists the remaining qubits and their states. In case no state specification is needed, subscript is redundant as such will not be written. When $(I_{K})$ is an N-qubit invariant both sub and superscripts are redundant and will not be posted. Subscript $K$ in $I_K$ indicates that by suitable choice of local unitaries the invariant can be expressed in terms of determinants of $K$-way negativity fonts. Determinant of an $N$-way negativity font

$$D_{i_1i_2...i_p=0...i_N} = \det \begin{bmatrix} a_{i_1i_2...i_p=0...i_N} & a_{i_1i_2...i_p=0...i_N+1} \\ a_{i_1i_2...i_p=1...i_N} & a_{i_1i_2...i_p=1...i_N+1} \end{bmatrix},$$ (15)

is an invariant of $U_{A_p}$. Local unitary $U_{A_q}$ yields four transformation equations

$$\left(D^{i_1i_2...i_p=0,i_q=0...i_N}\right)^n = \frac{1}{1+|x|^2} \left[D^{i_1i_2...i_p=0,i_q=0...i_N} - |x|^2 D^{i_1i_2...i_p=0,i_q=1...i_N} \right]$$

$$+ x D^{i_1i_2...i_p=0,i_q-1,i_q+1...i_N} - x^* D^{i_1i_2...i_p=0,i_q-1,i_q+1...i_N}$$ (16)

$$\left(D^{i_1i_2...i_p=0,i_q=1...i_N}\right)^n = \frac{1}{1+|x|^2} \left[D^{i_1i_2...i_p=0,i_q=1...i_N} - |x|^2 D^{i_1i_2...i_p=0,i_q=0...i_N} \right]$$

$$+ x D^{i_1i_2...i_p=0,i_q-1,i_q+1...i_N} - x^* D^{i_1i_2...i_p=0,i_q-1,i_q+1...i_N}$$ (17)

$$\left(D^{0,i_p=0...i_q-1,i_q+1...i_N}_{(A_q)}\right)^n = \frac{-x^*}{1+|x|^2} \left[ D^{i_1i_2...i_p=0...i_q-1,i_q+1...i_N} + (x^*)^2 D^{i_1i_2...i_p=0...i_q-1,i_q+1...i_N} \right]$$

$$+ x \left[ D^{i_1i_2...i_p=0...i_q-1,i_q+1...i_N} + D^{i_1i_2...i_p=0...i_q-1,i_q+1...i_N} \right]$$ (18)

$$\left(D^{i_1i_2...i_p=0...i_q-1,i_q+1...i_N}_{(A_q)}\right)^n = \frac{1}{1+|x|^2} \left[ D^{i_1i_2...i_p=0...i_q-1,i_q+1...i_N} + x^2 D^{i_1i_2...i_p=0...i_q-1,i_q+1...i_N} \right]$$

$$+ x \left[ D^{i_1i_2...i_p=0...i_q-1,i_q+1...i_N} + D^{i_1i_2...i_p=0...i_q-1,i_q+1...i_N} \right]$$ (19)
relating \(N\)-way and \((N - 1)\)-way negativity fonts. Eliminating variable \(x\), invariants of \(U^A \rho U^A\) are found to be

\[
(M_N)^{A_p A_q} = \left| \left( D_{i_1 i_2 \ldots i_p = 0, i_q = 0 \ldots i_N} \right)'' + \left| \left( D_{i_1 i_2 \ldots i_p = 0, i_q = 1 \ldots i_N} \right)'' \right|^2 \\
+ \left| \left( D_{i_1 i_2 \ldots i_p = 1, i_q = 0 \ldots i_N} \right)'' + \left| \left( D_{i_1 i_2 \ldots i_p = 1, i_q = 1 \ldots i_N} \right)'' \right|^2 \\
= \left| \left( D_{i_1 i_2 \ldots i_p = 0, i_q = 0 \ldots i_N} \right)'' + \left| \left( D_{i_1 i_2 \ldots i_p = 0, i_q = 1 \ldots i_N} \right)'' \right|^2 \\
+ \left| \left( D_{i_1 i_2 \ldots i_p = 1, i_q = 0 \ldots i_N} \right)'' + \left| \left( D_{i_1 i_2 \ldots i_p = 1, i_q = 1 \ldots i_N} \right)'' \right|^2 \\
\right|,
\]

(20)

which is real, a degree two invariant

\[
(T_N)^{A_p A_q} = \left( D_{i_1 i_2 \ldots i_p = 0, i_q = 0 \ldots i_N} \right)'' - \left( D_{i_1 i_2 \ldots i_p = 0, i_q = 1 \ldots i_N} \right)'' \\
= D_{i_1 i_2 \ldots i_p = 0} = 0 \ldots i_N - D_{i_1 i_2 \ldots i_p = 0} = 1 \ldots i_N,
\]

(21)

a degree four invariant

\[
(I_N)^{A_p A_q} = \left( D_{i_1 i_2 \ldots i_p = 0, i_q = 0 \ldots i_N} \right)'' + \left( D_{i_1 i_2 \ldots i_p = 0, i_q = 1 \ldots i_N} \right)'' \\
- 4 \left( D_{i_1 i_2 \ldots i_p = 0, i_q = 0 \ldots i_N} \right)'' \left( D_{i_1 i_2 \ldots i_p = 0, i_q = 1 \ldots i_N} \right)'' \\
- 4 \left( D_{i_1 i_2 \ldots i_p = 0, i_q = 0 \ldots i_N} \right)'' \left( D_{i_1 i_2 \ldots i_p = 1, i_q = 0 \ldots i_N} \right)'' \\
- 4 \left( D_{i_1 i_2 \ldots i_p = 0, i_q = 1 \ldots i_N} \right)'' \left( D_{i_1 i_2 \ldots i_p = 1, i_q = 1 \ldots i_N} \right)'' \\
\right|,
\]

(22)

and combining Eqs. (21) and (22), we obtain

\[
(P_N)^{A_p A_q} = D_{i_1 i_2 \ldots i_p = 0} = 0 \ldots i_N \left( D_{i_1 i_2 \ldots i_p = 0} = 1 \ldots i_N \right) \\
- D_{i_1 i_2 \ldots i_p = 0} = 0 \ldots i_N \left( D_{i_1 i_2 \ldots i_p = 0} = 1 \ldots i_N \right) \\
= \left( D_{i_1 i_2 \ldots i_p = 0} = 0 \ldots i_N \right)'' \left( D_{i_1 i_2 \ldots i_p = 0} = 1 \ldots i_N \right)'' \\
- \left( D_{i_1 i_2 \ldots i_p = 0} = 0 \ldots i_N \right)'' \left( D_{i_1 i_2 \ldots i_p = 1} = 1 \ldots i_N \right)'' \\
- \left( D_{i_1 i_2 \ldots i_p = 0} = 0 \ldots i_N \right)'' \left( D_{i_1 i_2 \ldots i_p = 1} = 1 \ldots i_N \right)'' \\
\right|.
\]

(23)

Similarly the differences \((M_N)^{A_p A_q} - (I_N)^{A_p A_q}\) and \((M_N)^{A_p A_q} - (T_N)^{A_p A_q}\) are useful to write down different \(N\)-qubit invariants in alternate forms.

Transformation equations under LU for determinants of negativity fonts characterizing \(K\)-way partial transpose and \((K - 1)\) way partial transpose with \(K < N\), yield two qubit invariants \((M_K)^{A_p A_q}\), \((I_K)^{A_p A_q}\), \((T_K)^{A_p A_q}\), and \((P_K)^{A_p A_q}\) analogous to \(N\)-way case.

### A. Global negativity and negativity fonts

It follows from Eq. (20) that by summing up the squared moduli of determinants of all negativity fonts in a partial transpose we obtain an \(N\)-qubit invariant. Recalling that the maximum value that modulus of determinant of a single negativity font may have is \(\frac{1}{\sqrt{2}}\), multiplying the invariant by four leads to an invariant with maximum value equal to one. Next, the relation between global negativity and linear entropy of reduced single qubit state is used to demonstrate that the invariant obtained is nothing but the global negativity defined as in Eq. (1).

Linear entropy, defined as

\[
S = \frac{d_A}{d_A - 1} \left( 1 - Tr (\rho^A)^2 \right)
\]

measures the purity of state \(\rho^A = Tr_B (\hat{\rho})\) and also detects bipartite entanglement of subsystems \(A\) with \(B\). If \(A = A_p\), the \((p^{th})\) qubit of an \(N\)-qubit quantum system, then squared negativity \(\left( N_G^{A_p} \right)^2\) is known to be equal to linear entropy of single qubit reduced state \(\hat{\rho}^{A_p} = tr_{A_1 \ldots A_{p-1} A_{p+1} \ldots A_N} (\hat{\rho})\) that is

\[
\left( N_G^{A_p} \right)^2 = 2 \left( 1 - tr \left( [\hat{\rho}^{A_p}]^2 \right) \right).
\]

(25)
Choosing \( p = 1 \), we write the pure state as \( \hat{\rho} = \sum_{i,j} \rho_{i,j} |i_j\rangle \langle j_i| \), where \( I = \sum_{m=2}^{N} i_m 2^{m-1} \) labels the \((N-1)\) qubit state sans qubit \( A_1 \). Using Eq. (25) and \( \text{tr} (\hat{\rho}^A_i) = 1 \), we obtain

\[
\left( N^A_G \right)^2 = 4 \sum_{i,j} \left( \rho_{1101} \rho_{0011} - \rho_{0011} \rho_{1101} \right).
\]

Next defining \( L = \sum_{m=3}^{N} i_m 2^{m-1} \) and \( M = \sum_{m=3}^{N} j_m 2^{m-1} \), expansion of \( \left( N^A_G \right)^2 \) reads as

\[
\left( N^A_G \right)^2 = 4 \sum_{L,M} \left( \rho_{10L00L} \rho_{00M10M} - \rho_{00L00L} \rho_{10M10M} \right)
+ 4 \sum_{L,M} \left( \rho_{10L00L} \rho_{01M11M} - \rho_{00L00L} \rho_{11M11M} \right)
+ 4 \sum_{L,M} \left( \rho_{11L01L} \rho_{00M10M} - \rho_{01L01L} \rho_{10M10M} \right)
+ 4 \sum_{L,M} \left( \rho_{11L01L} \rho_{01M11M} - \rho_{01L01L} \rho_{11M11M} \right)
\]

which in terms of probability amplitudes has the form

\[
\left( N^A_G \right)^2 = 4 \sum_{L,M} \left| (a_{00L}a_{11M} - a_{10L}a_{01M}) \right|^2
\]

After identifying the determinant \( (a_{00L}a_{11M} - a_{10L}a_{01M}) \) with

\[
\det \nu^L_0 \equiv \det \begin{pmatrix} a_{00i_1...i_N} & a_{01j_3...j_N} \\ a_{10i_3...i_N} & a_{11j_3...j_N} \end{pmatrix},
\]

that is the determinant of a \( K \)-way negativity font, the squared negativity is expressed in terms of determinants of all negativity fonts in \( \hat{\rho}^T_G \) as

\[
\left( N^A_G \right)^2 = 4 \sum_{L,K=2}^{N} \left| \det \nu^L_0 \right|^2.
\]

Global negativity arising due to all the negativity fonts present in \( \hat{\rho}^T_G \) measures the entanglement of qubit \( p \) with it’s complement and is known to be an entanglement monotone [2].

**IV. FOUR QUBIT INVARIANTS**

For \( N = 4 \), with determinants of four-way negativity fonts defined as

\[
D^{00i_3i_4} = \det \begin{pmatrix} a_{00i_3i_4} & a_{01i_3+1,i_4+1} \\ a_{10i_3i_4} & a_{11i_3+1,i_4+1} \end{pmatrix},
\]

four qubit pure state invariant with negativity fonts lying solely in four-way partial transpose is given by

\[
T_4 = D^{0000} + D^{0011} - D^{0010} - D^{0001}.
\]

Invariant \( T_4 \) is identified with degree two invariant \( H \) of ref. [14] which is also one of the hyperdeterminants of Cayley. A four qubit state having quantum correlations of the type present in a four qubit GHZ state, is distinguished from other states by a non zero \( T_4 \). These quantum correlations are lost without leaving any residue, on the loss of a single qubit and are a collective property of four qubit state. It is known [14] that four tangle defined as

\[
\tau_4 = 4 \left| (D^{0000} + D^{0011} - D^{0010} - D^{0001})^2 \right|,
\]
by itself is not enough to detect four qubit genuine entanglement, being non-zero for the product of entangled two qubit states in which case invariants of higher degree are needed to detect GHZ like entanglement.

Local unitary transformations may be used to concentrate the negativity fonts on a selected $\rho_K^{T_p}$ in the expansion of $\rho_K^{T_p}$ given by Eq. (13). When $\rho_K^{T_p} = \rho_4^{T_p}$ and $\tau_4 \neq 0$, we have a GHZ like four qubit state. Four qubit states with each qubit entangled to at least one qubit and $\rho_2 = 0$, allows for two equivalent canonical state descriptions that is

$$\rho_4^{T_p} = \rho_4^{T_p} + \rho_2^{T_p} - \rho, \quad \rho_3^{T_p} = \rho_3^{T_p} - \rho,$$

The class with $\tau_4 = 0$, allows for two equivalent canonical state descriptions that is

$$\rho_4^{T_p} = \rho_4^{T_p} + \rho_2^{T_p} - \rho, \quad \rho_3^{T_p} = \rho_3^{T_p} + \rho_2^{T_p} - \rho.$$

Therefore the difference

$$\Delta_4 = \sum_{p=1}^{4} \left(N_{G_p}^{A_p}\right)^2 - \tau_4,$$

for four qubit pure state may be taken as a measure of three-way plus two-way coherences.

### A. Entanglement of two and three qubits in Four qubit states

As mentioned before, to distinguish between the product of two qubit entangled states with $\tau_4 \neq 0$ and states with all four qubits entangled to each other we need additional invariants. In ref. [28], along with the degree two invariant of Eq. (22), we reported three degree four invariants that detect quantum correlations in a four qubit state. In this section, we list those invariants and identify two distinct types of three qubit invariants that constitute a four qubit invariant. Three-way and two-way negativity font determinants for four qubits are defined as

$$D_{(A_2)_{12}}^{0001} = \det \left( \begin{array}{cc} a_{0021} & a_{0121} \\ a_{1021} & a_{1121} \end{array} \right), \quad D_{(A_3)_{12}}^{0001} = \det \left( \begin{array}{cc} a_{0001} & a_{0101} \\ a_{1001} & a_{1101} \end{array} \right),$$

$$D_{(A_4)_{12}}^{0001} = \det \left( \begin{array}{cc} a_{0011} & a_{0111} \\ a_{1011} & a_{1111} \end{array} \right), \quad D_{(A_4)_{12}}^{0001} = \det \left( \begin{array}{cc} a_{0001} & a_{0101} \\ a_{1001} & a_{1101} \end{array} \right).$$

Using Eq. (21) for four qubits and identifying the terms

$$D_{(A_3)_{0}}^{0001} - D_{(A_3)_{0}}^{0011},$$

$$D_{(A_2)_{0}}^{0001} - D_{(A_2)_{0}}^{0011},$$

as invariants of $U^{A_1} U^{A_4}$, application of Eq. (22) leads to four qubit invariant

$$\left(J_{4}^{A_1 A_4}\right)^{A_1 A_2 A_3 A_4} = \left(D_{(A_3)_{0}}^{0001} - D_{(A_3)_{0}}^{0011} + D_{(A_2)_{0}}^{0001} - D_{(A_2)_{0}}^{0011}\right)^2$$

$$+ 8 \left(D_{(A_2)_{0}}^{0001} D_{(A_2)_{1}}^{0011} + D_{(A_2)_{0}}^{0001} D_{(A_2)_{1}}^{0011}\right)$$

$$- 4 \left(D_{(A_3)_{0}}^{0001} - D_{(A_3)_{1}}^{0011}\right) \left(D_{(A_3)_{0}}^{0001} - D_{(A_3)_{1}}^{0011}\right)$$

$$- 4 \left(D_{(A_2)_{0}}^{0001} - D_{(A_2)_{1}}^{0011}\right) \left(D_{(A_2)_{0}}^{0001} - D_{(A_2)_{1}}^{0011}\right).$$

From the structure of $\left(J_{4}^{A_1 A_4}\right)^{A_1 A_2 A_3 A_4}$ we deduce that four qubit

$$|GHZ\rangle = \frac{1}{\sqrt{2}} \left(|0000\rangle + |1111\rangle\right).$$
state with \((J_{4}^{A_{1}A_{4}})^{A_{1}A_{2}A_{3}A_{4}} = (D^{0000})^2 = \frac{1}{4}\) is unitary equivalent to the state

\[
|1\rangle = \frac{1}{\sqrt{8}} \left( |0000\rangle + |1111\rangle + |0100\rangle - |1011\rangle + |0010\rangle - |1101\rangle + |0110\rangle + |1001\rangle \right),
\]

with four-way coherences transformed to three and two way coherences such that

\[
(D^{0000} - D^{0001} + D^{0010} - D^{0011})^2 = 0,
\]

and

\[
(J_{4}^{A_{1}A_{4}})^{A_{1}A_{2}A_{3}A_{4}} = -4 \left( D_{(A_3)_{0}}^{000} - D_{(A_3)_{0}}^{001} \right) \left( D_{(A_3)_{1}}^{000} - D_{(A_3)_{1}}^{001} \right) = \frac{1}{4}.
\]

In the present context, \(J_{4}^{A_{P}A_{Q}}\) are always four qubit invariants, therefore, the superscript \(A_{1}A_{2}A_{3}A_{4}\) will be understood, from this point on.

To understand the role of three qubit correlations, we rewrite a four qubit state as

\[
|\Psi\rangle = |\Phi_0\rangle |0\rangle_{A_3} + |\Phi_1\rangle |1\rangle_{A_3},
\]

where

\[
|\Phi_0\rangle = \sum_{i_{1}i_{2}i_{4}} a_{i_{1}i_{2}0i_{4}} |i_{1}i_{2}i_{4}\rangle,
|\Phi_1\rangle = \sum_{i_{1}i_{2}i_{4}} a_{i_{1}i_{2}1i_{4}} |i_{1}i_{2}i_{4}\rangle,
\]

are three qubit states characterized by three qubit invariants \((I_{3})^{A_{1}A_{2}A_{4}}_{(A_3)_{0}}\) and \((I_{3})^{A_{1}A_{2}A_{4}}_{(A_3)_{1}}\) with three tangles given, respectively, by

\[
(I_{3})^{A_{1}A_{2}A_{4}}_{(A_3)_{0}} = 4 \left( |(I_{3})^{A_{1}A_{2}A_{4}}_{(A_3)_{0}} |0\rangle_{A_3} \right) = 4 \left( D^{000}_{(A_3)_{0}} - D^{001}_{(A_3)_{0}} \right)^2 - 4D^{000}_{(A_2)_{0}(A_3)_{0}} D^{000}_{(A_2)_{1}(A_3)_{0}},
\]

and

\[
(I_{3})^{A_{1}A_{2}A_{4}}_{(A_3)_{1}} = 4 \left( |(I_{3})^{A_{1}A_{2}A_{4}}_{(A_3)_{1}} |1\rangle_{A_3} \right) = 4 \left( D^{000}_{(A_3)_{1}} - D^{001}_{(A_3)_{1}} \right)^2 - 4D^{000}_{(A_2)_{0}(A_3)_{1}} D^{000}_{(A_2)_{1}(A_3)_{1}}.
\]

A polynomial classification scheme in which families of four qubit are identified through tangle patterns has been suggested recently in [22]. We notice that in the context of four qubits, using Eqs. [16][19] overall three qubit invariant for qubits \(A_1A_2A_4\) may be written as

\[
(I_{3})^{A_{1}A_{2}A_{4}}_{(A_3)_{0}} = \left( D^{000}_{(A_3)_{0}} - D^{001}_{(A_3)_{0}} \right)^2 - 4 \left( D^{000}_{(A_2)_{0}(A_3)_{0}} + D^{000}_{(A_2)_{1}(A_3)_{0}} \right) \left( D^{000}_{(A_2)_{0}(A_3)_{0}} + D^{000}_{(A_2)_{1}(A_3)_{0}} \right);
\]

\[
(I_{3})^{A_{1}A_{2}A_{4}}_{(A_3)_{1}} = \left( D^{000}_{(A_3)_{1}} - D^{001}_{(A_3)_{1}} \right)^2 - 4 \left( D^{000}_{(A_2)_{1}(A_3)_{0}} D^{000}_{(A_2)_{1}(A_3)_{0}} + D^{000}_{(A_2)_{0}(A_3)_{1}} D^{000}_{(A_2)_{0}(A_3)_{1}} \right).
\]

Therefore the term

\[
(P_{3})^{A_{1}A_{2}A_{4}}_{A_3} = 8 \left( D^{000}_{(A_2)_{0}(A_3)_{0}} D^{000}_{(A_2)_{1}(A_3)_{1}} + D^{000}_{(A_2)_{0}(A_3)_{1}} D^{000}_{(A_2)_{1}(A_3)_{0}} \right) - 4 \left( D^{000}_{(A_3)_{0}} - D^{001}_{(A_3)_{0}} \right) \left( D^{000}_{(A_3)_{1}} - D^{001}_{(A_3)_{1}} \right);
\]

\[
= 2 \left( I_{3}^{A_{1}A_{2}A_{4}}_{(A_3)_{0}} + 2 \left( I_{3}^{A_{1}A_{2}A_{4}}_{(A_3)_{1}} - 2 \left( I_{3}^{A_{1}A_{2}A_{4}}_{(A_3)_{0}} \right) \right)
\]

is a three qubit invariant. Since

\[
(I_{4})^{A_{1}A_{2}A_{4}}_{A_3} = \left( D^{0000} - D^{0001} + D^{0010} - D^{0011} \right)^2 - 4 \left( D^{0000}_{(A_2)_{0}} - D^{0011}_{(A_2)_{0}} \right) \left( D^{0000}_{(A_2)_{1}} - D^{0011}_{(A_2)_{1}} \right),
\]
These invariants satisfy the condition
\[ ad = \text{A} \text{ad} = A \text{ad} \]
and are used to define entanglement monotone \( A \text{ad} = A \text{ad} \) in terms of three qubit invariants of sub-system \( A_p A_q A_r \), or \( A_p A_q A_s \). These qubit invariants can be manipulated by unitary transformation on the fourth qubit.

Four qubit invariant obtained by combining the invariants of \( U^{A_1} U^{A_3} \) is
\[ J_4^{(A_1;A_3)} = (D^{0000} - D^{0010} + D^{0001} - D^{0011})^2 + 8 \left( D^{00} (A_2)_0 D^{00} (A_2)_1 (A_4)_1 + D^{00} (A_2)_1 (A_4)_0 D^{00} (A_2)_0 (A_4)_1 \right) - 4 \left( \frac{T}{A} \right)_p \frac{T}{A} \left( D^{00} (A_2)_0 - D^{010} (A_2)_1 \right) \left( D^{00} (A_2)_1 - D^{010} (A_2)_0 \right) - 4 \left( \frac{T}{A} \right)_p \frac{T}{A} \left( D^{00} (A_4)_0 - D^{001} (A_4)_1 \right) \left( D^{00} (A_4)_1 - D^{001} (A_4)_0 \right), \] (46)

and starting with \( U^{A_1} U^{A_2} \) invariants we get
\[ J_4^{(A_1;A_2)} = (D^{0000} - D^{0100} + D^{0010} - D^{0110})^2 + 8 \left( D^{00} (A_2)_0 D^{00} (A_2)_1 (A_4)_1 + D^{00} (A_2)_1 (A_4)_0 D^{00} (A_2)_0 (A_4)_1 \right) - 4 \left( \frac{T}{A} \right)_p \frac{T}{A} \left( D^{00} (A_2)_0 - D^{010} (A_2)_1 \right) \left( D^{00} (A_2)_1 - D^{010} (A_2)_0 \right) - 4 \left( \frac{T}{A} \right)_p \frac{T}{A} \left( D^{00} (A_4)_0 - D^{001} (A_4)_1 \right) \left( D^{00} (A_4)_1 - D^{001} (A_4)_0 \right), \] (47)

These invariants satisfy the condition
\[ \left( \frac{T}{A} \right)_p \frac{T}{A} \left( A^{A_2 A_3 A_4} \right)^2 = \frac{1}{3} \left( J_4^{(A_1;A_2)} + J_4^{(A_1;A_3)} + J_4^{(A_1;A_4)} \right), \] (48)

and are used to define entanglement monotone
\[ \beta_4 = \frac{1}{6} \sum_{m<n} \beta_4^{(A_m A_n)}; \quad \beta_4^{(A_m A_n)} = \frac{4}{3} \left| J_4^{(A_m A_n)} \right|. \] (49)

Consider the entangled states
\[ |B \rangle = a |0000 \rangle + b |1100 \rangle + c |0011 \rangle + d |1111 \rangle, \]
characterized by \( \tau_4 = 4 |ab + bc|^2 \), \( J_4^{(A_1;A_2)} = J_4^{(A_1;A_3)} = (ad + bc)^2 + 8abcd \), and \( J_4^{(A_1;A_4)} = J_4^{(A_1;A_3)} = (ad - bc)^2 \). If \( ad = bc \) then \( \tau_4 = 16 |ad|^2 \), but \( J_4^{(A_1;A_1)} = J_4^{(A_1;A_3)} = 0 \) and the state
\[ |B \rangle_{ad=bc} = (a |00 \rangle + b |11 \rangle) \left( |00 \rangle + \frac{c}{a} |11 \rangle \right), \]
is a product of two qubit entangled states.

B. Sextic Invariant

Set of transformation equations for negativity fonts can be used to obtain additional invariants to discriminate between different types of quantum correlations in four qubit states. In this section an expression for degree six invariant, obtained from set of transformation equations for negativity fonts is given. A sextic invariants \( J_6^{(A_p A_q)} A_p A_q A_r A_s \) may be constructed by starting with a product of three invariants of \( U^{A_p} U^{A_q} \) containing determinants of negativity fonts in \( \rho_G^{T_{A_p}} \). For instance, transformation Eqs. (16), when used to construct an invariant by starting from a
product of three invariants of $U^{A_2}U^{A_3}$ containing determinants of negativity fonts in $\mathcal{T}^{A_2}_G$, yield the invariant

$$
\left(I^6_{A_2A_3}\right)^A_{A_2A_3A_4} = D^{00}_{(A_1)_{0}(A_3)_{0}}D^{00}_{(A_1)_{1}(A_3)_{1}} \left( D^{0000} + D^{0001} - D^{0010} - D^{0100} \right) \\
- D^{00}_{(A_1)_{0}(A_3)_{0}}D^{00}_{(A_1)_{1}(A_3)_{0}} \left( D^{0000} + D^{0001} - D^{0010} - D^{0100} \right) + D^{00}_{(A_1)_{0}(A_3)_{1}} \left( D^{0000} - D^{1000} - D^{0100} \right) \\
-D^{00}_{(A_1)_{1}(A_3)_{0}}D^{00}_{(A_1)_{0}(A_3)_{1}} \left( D^{0000} - D^{0001} + D^{0010} \right) + D^{00}_{(A_1)_{1}(A_3)_{1}} \left( D^{0000} - D^{0001} + D^{0010} \right),
$$

which is the same as invariant $D_{xt}$ of ref. [13]. However, when expressed in terms of negativity fonts, each term gives a clear picture of how negativity fonts may be distributed in the state to generate a non-zero $\left(I^6_{A_2A_3}\right)^A_{A_2A_3A_4}$. Additional degree six invariants can be obtained similarly. The power of sextic invariant lies in distinguishing between states for which degree four invariants have the same value.

V. MAXIMALLY ENTANGLLED FOUR QUBIT STATES

The maximally entangled four qubit GHZ [33] state

$$
|\Psi_{GHZ}\rangle = \frac{1}{\sqrt{2}} \left( |0000\rangle + |1111\rangle \right), \tag{50}
$$

is characterized by a single 4-way negativity font with determinant $D^{0000} = a_{0000}a_{1111} = \frac{1}{2}$, which corresponds to $\tau_4 = 1, \beta_4^{A_4A_4} = \frac{1}{4}$. The state has only four-way correlations therefore $\mathcal{T}^{T}_G = \mathcal{T}^{T}_G$, and $(N^{A_4}_G)^2 = \tau_4$ for ($p = 1 - 4$). The value of degree six invariant $I^6_{A_2A_3}$ is 0 for this state.

To characterize the entanglement of state

$$
|x\rangle = \frac{1}{\sqrt{8}} \left( |0000\rangle - |0011\rangle - |0110\rangle - |0101\rangle \right),
$$

$$
+ \frac{1}{\sqrt{8}} \left( |1100\rangle + |1111\rangle + |1010\rangle + |1001\rangle \right), \tag{51}
$$

expectation values of third, fourth and sixth order filter operators [20, 21] have been used in ref. [34] and the equivalence of the state to some graph states demonstrated [35]. We verify that the state $|x\rangle$ is characterized by $\tau_4 = 0, J^{A_1A_2} = J^{A(1)A(2)} = J^{(A_1)A_2} = J^{(A_2)A_1} = -\frac{1}{4}$, and $J^{(A_3)A_4} = J^{(A_4)A_3} = -\frac{1}{4}$. Therefore, the state has $\beta_4^{A_1A_2} = \beta_4^{A_1A_3} = \beta_4^{A_1A_4} = \beta_4^{A_2A_3} = \beta_4^{A_2A_4} = \beta_4^{A_3A_4} = \frac{1}{4}$, while $\beta_4^{A_1A_2} = \beta_4^{A_1A_3} = \beta_4^{A_1A_4} = \frac{1}{4}$, indicating that the entanglement of state $|x\rangle$ is distinct from that of $|\Psi_{GHZ}\rangle$ (\(\tau_4 = 1, \beta_4^{A_1A_2} = \beta_4^{A_1A_3} = \beta_4^{A_1A_4} = \frac{1}{4} \)). Negativity font formalism provides an easy way to determine the local unitary transformations that transform the state $|x\rangle$ to canonical form that is a state written in terms of minimum number of local basis product states $\mathcal{O}$. In general, by examining the determinants of negativity fonts that contribute to a given invariant, it is possible to use transformation equations to determine local unitaries connecting two unitary equivalent states.

We look at the invariant $J^{A_1A_2}$ for the state $|x\rangle$. Manifestly, the state has four-way and two way fonts, however the only nonzero contribution to this invariant is $J^{A_1A_2} = 8D^{00}_{(A_3)_{0}(A_4)_{0}}D^{00}_{(A_3)_{1}(A_4)_{1}} + 8D^{00}_{(A_3)_{1}(A_4)_{1}}D^{00}_{(A_3)_{0}(A_4)_{1}} = -\frac{1}{4}$.

Local unitary $U^{A_3} = \frac{1}{\sqrt{1 + |x|^2}} \begin{bmatrix} 1 & -x^* \\ x & 1 \end{bmatrix}$, transforms the negativity fonts such that

$$
\left(D^{00}_{(A_3)_{0}(A_4)_{0}}\right)_{ij}'' = \frac{1}{1 + |x|^2} \left(D^{00}_{(A_3)_{0}(A_4)_{0}} + (x^*)^2 D^{00}_{(A_3)_{1}(A_4)_{1}}\right), \tag{52}
$$

$$
\left(D^{00}_{(A_3)_{1}(A_4)_{1}}\right)'' = \frac{1}{1 + |x|^2} \left(D^{00}_{(A_3)_{1}(A_4)_{1}} + x^2 D^{00}_{(A_3)_{0}(A_4)_{0}}\right). \tag{53}
$$
The choice
\[(x^*)^2 = -\frac{D^{00}_{(\chi_3)\ominus(A_4)\ominus}}{D^{00}_{(\chi_3)\ominus(A_4)\ominus}} = 1,\]

makes \(\left(D^{00}_{(\chi_3)\ominus(A_4)\ominus}\right) = 0, (i_3, i_4 = 0, 1)\) and generates 3–way negativity fonts. Next, unitaries \(U^{A_1} = U^{A_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\) on qubits \(A_1\) and \(A_2\) transform the state to canonical form

\[|\chi\rangle_c = \frac{1}{2} (|0000\rangle - |0111\rangle + |1110\rangle + |1001\rangle),\]

with only three and two-way negativity fonts and \(J^{A_1A_2} = -\frac{1}{2}\). Obviously, no entangled pairs \(A_1A_2\) or \(A_1A_3\) can be obtained from \(|\chi\rangle\) on state reduction. Total number of distinct negativity fonts in \(|\chi\rangle_c\) is six that is four 3–way fonts and two 2–way fonts. An interesting feature of \(|\chi\rangle_c\) is that 4–way three qubit invariants are zero for two of the qubits on this state.

Another four qubit state, conjectured to have maximal entanglement in ref. \[36\], is

\[|HS\rangle = \frac{1}{\sqrt{6}} \left(|0011\rangle + |1100\rangle + \exp\left(\frac{i2\pi}{3}\right) (|1010\rangle + |0101\rangle)\right) + \frac{1}{\sqrt{6}} \exp\left(\frac{i4\pi}{3}\right) (|1001\rangle + 0110).\] (54)

Two way negativity fonts \(D^{00}_{(\chi_3)\ominus(A_4)\ominus} = D^{00}_{(\chi_3)\ominus(A_4)\ominus} = \frac{1}{7},\) and 4–way negativity fonts \(D^{0011} = \frac{1}{6}, D^{9001} = \frac{1}{6} (1 - i\sqrt{3}),\) and \(D^{0010} = \frac{1}{6} (1 + i\sqrt{3})\) transform under the action of \(U^{A_3}, U^{A_1}\) generating three-way negativity fonts, however, unlike the state \(|\chi\rangle\), this state cannot be written in a form with only 3–way and 2–way coherences. It is found that in this case three qubit invariants \((P_3)^{A_1A_2A_3}\) as well as \((I_4)^{A_1A_2A_3}\) contribute to \(J^{A_1A_2}\). Similar observations hold for other \(J\) invariants.

Recently, Gilad and Wallach \[39\] have pointed out that three cluster states \[37, 38\]

\[|C_1\rangle = \frac{1}{2} (|0000\rangle + |1100\rangle + |0011\rangle - |1111\rangle)\] (55)

\[|C_2\rangle = \frac{1}{2} (|0000\rangle + |0110\rangle + |1001\rangle - |1111\rangle),\] (56)

\[|C_3\rangle = \frac{1}{2} (|0000\rangle + |1010\rangle + |0101\rangle - |1111\rangle),\] (57)

are the only states that maximize the Renyi \(\alpha\)–entropy of entanglement for all \(\alpha \geq 2\). The state \(|C_1\rangle\) with \(\rho^{T_A}_C = \rho^{T_A}_4 + \rho^{T_A}_2 - \rho, (\tau_4 = 0)\) can be transformed by local unitaries on qubits \(A_1\) and \(A_2\) to the form

\[|C_1\rangle' = |0000\rangle + \left(1100\rangle + |0111\rangle + |1011\rangle,\right]\]

with \(\rho^{T_A}_C = \rho^{T_A}_3 + \rho^{T_A}_2 - \rho.\) A similar observation holds for the states \(|C_2\rangle\), and \(|C_3\rangle\). Calculation of three qubit invariants shows that the distinguishing feature of the states \(|C_1\rangle, |C_2\rangle,\) and \(|C_3\rangle\) is null invariant \((P_3)^{A_4A_3A_2}\) for two of the qubits, while \((I_4)^{A_4A_3A_2}\) is non zero.

Another candidate for maximally entangled state, found through a numerical search in ref. \[40\], is

\[|\Phi\rangle = \frac{1}{2} (|0000\rangle + |1101\rangle) + \frac{1}{\sqrt{2}} (|1011\rangle + |0011\rangle + |0110\rangle - |1110\rangle),\]

This state, just like \(|\chi\rangle_c\), has only three and two way negativity fonts. Unlike \(|\chi\rangle_c\), however, \(J^{A_1A_3} = J^{A_2A_3} = 0,\) because \((I_4)^{A_1A_2A_3} = (P_3)^{A_1A_2A_3}\).
TABLE I: Numerical values of four qubit invariants for $|GHZ\rangle$, $|\chi\rangle$, $|HS\rangle$, state $|\Phi\rangle$, state $|C_1\rangle$, $|C_2\rangle$, $|C_3\rangle$, and state $|\Phi\rangle$.

| State | $(T_4)^2$ | $I^{A_1A_2}_{\delta}$ | $I^{A_1A_3}_{\delta}$ | $I^{A_1A_4}_{\delta}$ | $J_4^{A_1A_3}$ | $J_4^{A_1A_4}$ | $r_{\delta}$ | $\beta_4 = \frac{1}{2} \sum_{j=2}^{4} \beta_{4j}^{A_jA_j}$ | $\frac{1}{2} \sum_{p=1}^{4} \left(N_{p\delta}^{A_p}\right)^2$ | $\Delta_4$ |
|-------|------------|----------------|----------------|----------------|-------------|-------------|-------------|----------------|----------------|-------------|
| $|GHZ\rangle$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $|\chi\rangle$ | $0$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $1$ | $1$ | $1$ |
| $|HS\rangle$ | $0$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $1$ | $1$ |
| $|C_1\rangle$ | $0$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $1$ | $1$ |
| $|C_2\rangle$ | $0$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $1$ | $1$ | $1$ |
| $|C_3\rangle$ | $0$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $1$ | $1$ | $1$ |
| $|\Phi\rangle$ | $0$ | $\frac{1}{2}$ | $0$ | $-\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $1$ | $1$ | $1$ | $1$ |

TABLE II: Numerical values of $(T_4)^2$, sextic invariant $I^{A_2A_3}_{\delta}$, and three qubit invariants for $|GHZ\rangle$, $|\chi\rangle$, $|HS\rangle$, and $|\Phi\rangle$. States.

| State | $(T_4)^2$ | $I^{A_2A_3}_{\delta}$ | $(I^{A_2A_4}_{\delta})^2$ | $(P_2)^{A_2A_3}_{\delta}$ | $(P_3)^{A_2A_4}_{\delta}$ | $(I^{A_2A_3}_{\delta})^2$ | $(P_3)^{A_2A_4}_{\delta}$ |
|-------|------------|----------------|----------------|---------------|---------------|----------------|---------------|
| $|GHZ\rangle$ | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $|\chi\rangle$ | $0$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $|\Phi\rangle$ | $0$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $|C_1\rangle$ | $0$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $|C_2\rangle$ | $0$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $|C_3\rangle$ | $0$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |

In Table II, the numerical values of four qubit invariants $(T_4)^2$, $J_4^{A_1A_2} = J_4^{A_3A_4}$, $J_4^{A_1A_3} = J_4^{A_2A_4}$, and $J_4^{A_1A_4} = J_4^{A_3A_3}$, are listed for $|GHZ\rangle$ state, $|\chi\rangle$ state, $|HS\rangle$ state, cluster states $|C_1\rangle$, $|C_2\rangle$, $|C_3\rangle$, and state $|\Phi\rangle$. Degree six invariant $I_6^{A_2A_3}$ as well as three qubit invariants $(I_3)^{A_pA_qA_r}$ and $(P_3)^{A_pA_qA_r}$ are displayed in Table II.

The state $|\Phi\rangle$ is not different from $|HS\rangle$ state, as far as $4$-way correlations are concerned. However, the degree six invariant $I_6^{A_2A_3}$ is zero for the state $|\Phi\rangle$. The values of degree four invariants are the same for cluster states and state $|\chi\rangle$, but these are not unitary equivalent states. The difference between $|\Phi\rangle$, $|\chi\rangle$, and cluster states lies in the entanglement of three qubit subsystems as is manifest in the values of three qubit invariants in Table II.

We notice that $|GHZ\rangle$ state, $|HS\rangle$ state, $|\chi\rangle$ state, group of states $|C_1\rangle$, $|C_2\rangle$, $|C_3\rangle$ and the state $|\Phi\rangle$ belong to five distinct four qubit entanglement classes. Each state is maximally entangled in its own class with $\frac{1}{4} \sum_{p=1}^{4} \left(N_{p\delta}^{A_p}\right)^2 = 1$ for each qubit, however with different capability for performing information processing tasks.

VI. CONCLUSIONS

To summarize, the transformation equations for negativity fonts under unitary transformations yield relevant $N$-qubit invariants and determine local unitaries relating unitary equivalent states. An expression for global negativity in terms of determinants of negativity fonts has been found. The squared negativity of $N$-qubit partially transposed operator is four times the sum of squared moduli of determinants of all possible negativity fonts. The structure of four qubit invariants of degree four that detect entanglement between pairs of qubits indicates why some of the unitary equivalent states may have different sets of $K$-way coherences. It is shown that a four qubit invariant $J_4^{A_3A_4}$ can be expressed in terms of three qubit invariants for qubits $A_pA_qA_r$, or $A_pA_qA_r$. Three qubit invariants can be manipulated by unitary transformation on the fourth qubit but their value for the canonical state is unique. In the context of four qubit states studied in the article, the two types of three qubit entanglement invariants, each
corresponding to a different type of quantum correlations present in the canonical state, play an important role in distinguishing between states with inequivalent entanglement types. Degree six invariants can also be constructed easily from Eqs. (16-19), as shown by writing the invariant \( \left( I_6^{(A_2 A_3)} \right)^{A_1 A_2 A_3 A_4} \). Decomposition of partially transposed matrix into \( K \)–way partial transposes is a tool to identify the type of quantum correlations which entangle the qubits. We have used the expressions of polynomial invariants in terms of negativity fonts to elucidate the difference in microstructure of some well known four qubit pure states. We conclude that the entanglement in four qubit \(|GHZ\rangle\) state, \(|\chi\rangle\) state, \(|HS\rangle\) state, cluster states \(|C_1\rangle, |C_2\rangle, |C_3\rangle\), and state \(|\Phi\rangle\) is qualitatively different since the states belong to different classes of four qubit entangled states. Cluster states \(|C_1\rangle, |C_2\rangle, |C_3\rangle\), differ from the \(|\chi\rangle\) state, in having different type of three qubit correlations in canonical form. These results indicate that along with composite system invariants, one needs subsystem invariants in canonical form to characterize the entanglement of a state. The four qubit entangled states investigated here do not represent all four qubit entanglement types represented by nine families of four qubit states [4]. However, the results provide insight to formulate efficient criterion for classification of four qubit entangled states. In ref. [29], the general method for writing N-tangle was given. In general, for n-even square of degree two invariant, having only N-way fonts, can be written as a sum of invariants that detect the entanglement of parts of the composite system. As such, the ideas developed for four qubits may be extended to multi qubit systems.

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