Improved Quantum LDPC Decoding Strategies for the Misidentified Quantum Depolarization Channel

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Abstract—The importance of the channel mismatch effect in determining the performance of quantum low-density parity-check codes has very recently been pointed out. It was found that an order of magnitude degradation in the qubit error performance was found even if optimal information on the channel identification was assumed. However, although such previous studies indicated the level of degradation in performance, no alternate decoding strategies had been proposed in order to reduce the degradation. In this work we fill this gap by proposing new decoding strategies that can significantly reduce the qubit error performance degradation by as much as 50%. Our new strategies for the quantum LDPC decoder are based on previous insights from classical LDPC decoders in mismatched wireless channels, where an asymmetry in performance is known as a function of the estimated bit-flip probability. We show how similar asymmetries carry over to the quantum depolarizing channel, and show with that when a weighted estimate of the depolarization coherence parameter to larger values is assumed we find significant performance improvement.

I. INTRODUCTION

Low density parity check (LDPC) codes have become a mainstay of wireless communications. Originally proposed by Gallager in 1963 [1], LDPC codes lay largely unnoticed (although see [2], [9], [10]) until their re-discovery in the mid-90’s [11, 12]. Since then hundreds of papers have been published outlining the near optimal performance of LDPC codes over a wide range of noisy wireless communication channels. In almost all of these previous works it was assumed that the characteristics of the noisy wireless channel was known. However, the reality is that in many cases an exact determination of the wireless channel is unavailable. Indeed, several works have in fact investigated the case where a channel mismatch (or channel misidentification) occurs, which in turn impacts on the performance of the LDPC decoder (e.g. [13], [14]).

From the perspective of the work reported on here, the most interesting aspect of such channel mismatch studies is the asymmetry in the LDPC code performance as a function the channel crossover probability for the binary symmetric channel (BSC). In fact, the main focus of the work described here is an investigation of whether such asymmetric LDPC code performance carries over from the classical BSC to quantum LDPC codes operating over the quantum depolarizing channel.

Since the discovery of CSS (Calderbank, Shor and Steane) codes [9] [10] and stabilizer codes [11], it has been known how quantum error-correction codes can be developed in a similar manner to classical codes. Quantum LDPC codes based on finite geometry were first proposed in [12], followed by the bicycle codes proposed in [13]. Their research explored the conjecture that the best quantum error-correcting codes will be closely related to the best classical codes, and in [14] Poulin et al. proposed serial turbo codes for quantum error correction.

A more detailed history on the development of QECC can be found elsewhere e.g. [15], [16]. More recently, many works attempting to improve quantum LDPC code performance have been published [17], [18], and [19] based on quasi-cyclic structure since it reduces the complexity of encoding and decoding.

Recently in [20] the impact of channel mismatch effects on the performance of quantum low-density parity-check codes was highlighted. In investigations of the performance of quantum LDPC codes it has been assumed that perfect knowledge of the quantum channel exists. Of course in practice this is not the case. In this work we utilize optimal estimates of the channel derived from quantum Fisher information about the channel parameters. Even in this optimal situation, the use of the unbiased estimator to estimate the level of channel noise produces an approximately order of magnitude degradation to the performance. We note, however, that the use of quantum entangled states will aid an estimating the noise level of the channel. However, there is a practical trade-off in hardware complexity between entanglement consumption and code performance.

In this paper, we further investigate the behavior and the robustness of the sum-product decoding algorithm when simulating over the quantum depolarizing channel. Interestingly, an asymmetry behavior in performance is observed as a function of the estimated channel flip probability, showing that the performance of a quantum LDPC code would experience a reduced degradation when the channel is overestimated instead of underestimated, given the overestimated channel knowledge still within the threshold limit of the code. Based on these observations, a new decoding strategy is proposed that can improve a quantum codes performance by as much as 50%.

In section II we discuss the behavior of the classical sum-product decoder under channel mismatch conditions. In
section III we briefly review quantum communications and the stabilizer formalism for describing QECCs, and discuss their relationship to classical codes. In section IV we explore the behavior of a quantum decoder when simulating over a quantum depolarizing channel and show how the decoding strategy we outline here leads to a significant improvement in performance relative to decoders that simply utilize the estimated channel parameter. Lastly, in section V we draw some conclusions and discuss future works.

II. Behavior of Classical Sum-Product Decoder

It is well known in classical coding that low-density parity-check codes are good rate achievable codes [1] [21], given an optimal decoder. The best algorithm known to decode them is the sum-product algorithm, also known as iterative probabilistic decoding or belief propagation (BP). The performance of sparse-graph codes can be improved if knowledge about the channel is known at the decoder side. However, in practical situations the decoder unlikely to know the channel’s characteristics exactly, thus, the robustness of the decoder to channel mismatches is also an important issue when designing practical codes.

In [7], MacKay et al investigated the sensitivity of Gallager’s codes [1] to the assumed noise level (classical bit-flip probability) when decoded by belief propagation. A useful result therein is that the belief propagation decoder for LDPC codes appears to be robust to channel mismatches because the block error probability is not a very sensitive function of the assumed noise level. In addition, an underestimation of channel characteristics deteriorates the performance more compare to an overestimation of channel characteristics. This behavior is shown in Fig. 1.

Our results shown in Fig. 1 are for a rate half code of block length $N = 2040$ and a binary symmetric channel. The code is a $(3, 6)$ regular LDPC codes which is constructed with the length of the cycle maximized.

By inspection, the plotted result shows a similar behavior to that found by MacKay in [7]. The vertical straight line indicates the true value of the noise level, and the minimum point of the plot is approximately at the intersection between two lines. This infers that an optimal performance of a practical sum-product decoder can be achieved when the input of decoder is the true noise level. The slope towards the left of the graph is more steeper than the slope towards the right, indicating that underestimation of the noise level degrades the performance more so than overestimation does. However, when the estimated noise level is far too large, there is a significant increase in the error probability. Such higher bit flip probabilities can be thought of as the classical Shannon’s limit (in this case, the Shannon’s limit for rate 1/2 code is 0.11), which theoretically represents the maximum allowable noise level under reliable transmission at a certain rate.

III. Quantum LDPC Codes

A. Preliminary on Quantum Stabilizer Codes

A stabilizer generator $S$ that encodes $K$ qubits in $N$ qubits consists of a set of Pauli operators on the $N$ qubits closed under multiplication, with the property that any two operators in the set commute, so that every stabilizer can be measured simultaneously. An example of a stabilizer generator $S$ is shown below for $K = 1, N = 5$ representing a rate $\frac{1}{5}$ quantum stabilizer code,

$$
S = \begin{pmatrix}
Z & Z & X & I & X \\
X & Z & Z & I & \\
I & X & Z & Z & X \\
X & I & X & Z & Z
\end{pmatrix}.
$$

Consider now a set of error operators $\{E_{\alpha}\}$ taking a state $|\psi\rangle$ to the corrupted state $E_{\alpha}|\psi\rangle$. A given error operator either commutes or anti-commutes with each stabilizer $S_i$ (row of the generator $S$) where $i = 1 \ldots N - K$. If the error operator commutes with $S_i$ then

$$
S_i E_{\alpha} |\psi\rangle = E_{\alpha}S_i |\psi\rangle = E_{\alpha}|\psi\rangle
$$

and therefore $E_{\alpha} |\psi\rangle$ is a $+1$ eigenstate of $S_i$. Similarly, if it anti-commutes with $S_i$, the eigenstate is $-1$. The measurement outcome of $E_{\alpha} |\psi\rangle$ is known as the syndrome.

B. Conversion between Quantum and Classical Codes

To connect quantum stabilizer codes with classical LDPC codes it is useful to describe any given Pauli operator on $N$ qubits as a product of an $X$-containing operator, a $Z$-containing operator and a phase factor $(+1, -1, i, -i)$. For example, the first row of matrix (1) can be expressed as

$$
ZZXIX = (IIXIX) \times (ZZIII).
$$

Thus, we can directly express the $X$-containing operator and $Z$-containing operator as separate binary strings of length $N$. In the $X$-containing operator a $1$ represents the $X$ operator
In the context of LDPC quantum error correction codes, it is
A. Quantum Channel Estimation
investigate is the widely adopted polarization channel.
the decoding performance under asymmetrical estimates of
not satisfy the constraint (5).
C. CSS Codes
Example 1: For example, the set of stabilizers in (1) appears
A = (A1 | A2) =

(0 0 1 0 1 1 1 0 0)
(1 0 0 1 0 0 1 1 0)
(0 1 0 0 1 0 0 1 0)
(1 0 1 0 0 0 0 1 1)

Due to the requirement that stabilizers must commute, a
constraint on a general matrix A can be written as e.g. [13].
A1A T 2 + A2A T 1 = 0. (5)
Note that the quantum syndrome can be conceptually con-
sidered as an equivalent to the classical syndrome Ae, where
A is a binary parity-check matrix and e is a binary error vector.
To summarize, the property of stabilizer codes can be
directly inferred from classical codes. Any binary parity-check
matrix of size M × 2N that satisfies the constraint in (5) defines
a quantum stabilizer code with rate R = K N that encodes K
qubits into N qubits.
C. CSS Codes
As mentioned earlier, an important class of codes are the
CSS Codes [9] [10]. These have the form
A = 

H 0
0 G

where H and G are M H × N and M G × N matrices,
respectively. (M H does not necessary equal to M G). Requiring
HGT = 0 ensures that constraint (5) is satisfied.
If G = H, the resulting CSS code structure is called a
dual-containing code. Most classical (good) LDPC codes do
not satisfy the constraint (5).
IV. Simulations for Improved Channel Decoding
Motivated by the decoding asymmetry discussed above for
classical LDPC codes, we now wish to explore whether a
similar asymmetry in decoding performance is achieved for
quantum LDPC codes. As stated above several well known
classes of quantum codes, such as quantum stabilizer codes
and CSS codes can be designed from existing classical codes.
Upon construction of such codes we will then investigate the
decoding performance under asymmetrical estimates of
the quantum channel parameters. The quantum channel we
investigate is the widely adopted polarization channel.
A. Quantum Channel Estimation
The issue of quantum channel identification (quantum pro-
cess tomography) is of fundamental importance for a range of
practical quantum information processing problems (e.g. [15]).
In the context of LDPC quantum error correction codes, it is
normally assumed that the quantum channel is known perfectly
in order for the code design to proceed. In reality of course,
perfect knowledge of the quantum channel is not available -
only some estimate of the channel is available. To make
progress we will assume a depolarization channel with some
parameter f d .
Given some initial system state |Ψ s ⟩, a decoherence model
1Note although same symbol use in section II, its new usage here should
clear from the context.
where $L_f$ is the symmetric logarithmic derivative defined implicitly by
\[ 2\partial_f \rho_f = L_f \rho_f + \rho_f L_f, \]
and where $\partial_f$ signifies partial differential w.r.t. $f$. With the quantum Fisher information in hand, the quantum Cramer-Rao bound can then be written as
\[ \text{mse} \left[ \hat{f} \right] \geq (N_m J(f))^{-1} \]
where $\text{mse} \left[ \hat{f} \right]$ is the mean square error of the unbiased estimator $\hat{f}$, and $N_m$ is the number of independent quantum measurements.

The performance results in [20] are obtained by randomly estimating a flip probability from a truncated normal distribution at the decoder side, given the mean square error of the unbiased estimation $\hat{f}$. In return, the performance is degraded approximately an order of magnitude.

B. Quantum Decoding algorithm

The appropriate decoding algorithm to decode quantum LDPC codes is based on the classical sum-product algorithm since the most common quantum channel model, namely depolarizing channel, is analogous to the classical 4-ary symmetric channel. The received values at the decoder side can be mapped to measurement outcomes $s \in \{1, -1\}^M$ (syndrome) of the received qubit sequence, and this syndrome is then used in error estimation and recovery. Assuming an initial quantum state representing a codeword, the initial probabilities $p_i$ for the $i$th qubit of the state undergoing an $X$, $Y$, or $Z$ error are
\[ p_i = \left\{ \begin{array}{ll}
  f & \text{for } X, Y, \text{ or } Z \\
  1-f & \text{for } I
\end{array} \right., \quad (7) \]
where $f$ is the flip probability known at the decoder.

The standard BP algorithm operates by sending messages along the edges of the Tanner graph. Let $u_{b_i \rightarrow c_j}$ and $u_{c_j \rightarrow b_i}$ denote the messages sent from bit node $i$ to check node $j$ and messages sent from check node $j$ to bit node $i$, respectively. Also denote $N(b_i)$ as the number of neighbors of bit node $i$, and define $N(c_j)$ as the number of neighbors of check node $j$.

To initialize our algorithm, each qubit node sends out a message to its neighboring qubit node given by
\[ u_{c_j \rightarrow b_i} = \sum_{t_1 \ldots , t \in \{1 \oplus c_j \cdot g = s_j\}} \prod_{b' \in N(c_j) \backslash b_i} u_{b' \rightarrow c_j} \quad (8) \]
where $N(c_j) \backslash b_i$ denotes all neighbors of check node $j$ except qubit node $i$, and the summation is over all possible error sequences $t_1 \ldots , N$. Each bit node then sends out a message to its neighboring checks given by
\[ u_{b_i \rightarrow c_j} = f \prod_{c_j \in N(b_i) \backslash c_j} u_{c_j \rightarrow b_i} \quad (9) \]
where $N(b_i) \backslash c_j$ denotes all neighbors of qubit node $i$ except check node $j$. Equations (8) and (9) operate iteratively until the message is correctly decoded or the maximum pre-determined iteration number is reached.

C. Behavior of Quantum Sum-Product Decoder over depolarizing channel

In this section, we investigate the dependence of the performance of a Quantum LDPC code on the estimated flip probability $\hat{f}$ of a depolarizing channel using the same quantum LDPC code simulated in [20]. In each decoding process, the decoder performed an iterative message passing algorithm (Sum-product decoding algorithm) until it either found a valid codeword or timed out after a maximum number of 200 iterations. If the maximum number of iterations was reached then the decoding process was considered a failure. Conversely, whenever a valid codeword was found, it was the correct one regardless whether it was the actual transmitted codeword. Thus, the simulation plots herein is the block/qubit error probability (BLER/QBER) as a function of noise level, where the block error probability is defined
\[ \text{BLER} = \frac{\text{Number of failures}}{\text{Total number of runs}}. \]

The noise vectors were generated to have weight exactly $fN$, where $N$ was the block length of the code $N = 1034$ and $f$ was the true flip probability for the depolarizing channel. The decoder assumed an estimated flip probability $\hat{f}$. We varied the value of $\hat{f}$ while the the true flip probability $f$ is fixed. The result of our simulation is shown in Figure 2.

![Fig. 2. Block error probability as a function of assumed noise level when the actual noise level is fixed.](image-url)
knowledge. The trend of the curve in Figure 2 also shows the impact on block error probability caused when overestimation of the flip probability is less than that arising from an overestimation of the channel flip probability. When the assumed flip probability reaches a limit (beyond the limit of 0.0417 for a classical rate 3/4 code), there is a catastrophic increase in the error probability. This result indicates that in the quantum case (just as found in the classical case) a limit to the degradation in performance may occur if in any estimate of the channel flip probability, an overestimate of the flip probability is utilized. We investigate this possibility further in what follows.

**D. Improved Decoding of Channel**

Consider now the case where a decoder can only attain partial channel information by probing the quantum channel using un-entangled or entangled quantum states (only one measurement each, i.e. $N_m = 1$). Given such partial information we will then weight our estimate of the channel parameter (at the decoder side) to large values (rather than smaller values) of the estimated flip probability. A schematic of our new decoding strategy is shown in Fig. 3.

![Diagram of quantum decoder strategy for misidentified depolarizing channels](image)

Fig. 3. An outline of the quantum decoder strategy for misidentified depolarizing channels.

Re-simulated results for the case A (unentangled state) and $B$ (entangled state) discussed in [20] section III are plotted in Fig. 4 and Fig. 5. Instead of estimating a noise level randomly from a truncated normal distribution (in a range of 0 to 1) characterized by mean $\hat{f}$ and variance $(N_f(f))^{-1}$. Our new simulation results, shown in Fig. 4 and Fig. 5, attempt to mimic the situation where to any estimate of the total flip probability $\hat{f}$ an additional $\Delta f$ is added at the decoder side (with an upper limit on the flip probability related to Shannon’s limit for the classical code rate). In the simulations, shown $\Delta f/\hat{f}$ was set at 0.5. It is this new estimate that is referred to as “improved decoder” in the figures.

The quantum LDPC code used here has block length of $N = 1034$ qubits with its quantum code rate $R_q^f = 1/2$. The decoding process terminates if 100 block errors are collected or the maximum iteration number (200) is reached. We can clearly see that by weighting the estimated channel information $\hat{f}$ higher (overestimating), an approximately 50% improvement in performance for both using entangled quantum states and without entangled states.

Various other simulations were run with $\Delta f/\hat{f}$ was set at other ratios. However, we find that setting $\Delta f/\hat{f}$ at 0.5 resulted in close to optimal performance for the specific code studied. It is straightforward to carry out a numerical fit to the curves shown Fig. 4 and Fig. 5, producing “cost” functions for the codes as a function of $\Delta f/\hat{f}$. Differentiating such curves with respect to $\Delta f/\hat{f}$ will lead to the value $\Delta f/\hat{f} \sim 0.5$. We conjecture that all quantum codes applied in misidentified depolarizing channels will show similar performance than that showed here. The specific value of $\Delta f/\hat{f}$, however, will likely vary from code to code. Such studies form part of our ongoing work in this area.

In practical situations, a trade-off between entanglement consumption and quantum LDPC code performance is one of important aspect when partial channel information is available to the decoder. Theoretically, when the number of entanglement pairs used is very large, the performance should approach the case where perfect channel knowledge is known at the decoder side. However, a reduction in the number of required entanglement pairs could also yield a near optimal performance if a constant overestimation of channel is utilized when estimating the quantum channel.

![Graph showing QBER/BLER vs True Flip Probability N = 1034 for Case A](image)

Fig. 4. The performance of the quantum stabilizer code with length $N = 1034$ for Case A. The dash lines plots the BLER of the code and the solid line plots the QBER of the code.
V. Conclusion

In this work we have investigated possible improvements in the decoding strategies of quantum LDPC decoders in the quantum depolarization channel. The importance of the channel mismatch effect in determining the performance of quantum low-density parity-check codes has very recently been shown to lead to an order of magnitude degradation in the qubit error performance. In this work we have illustrated how such a performance gap in the qubit error performance can be substantially reduced. The new strategies for quantum LDPC decoding we provided here are based on previous insights from classical LDPC decoders in mismatched channels, where an asymmetry in performance is known as a function of the estimated bit-flip probability. We first showed how similar asymmetries carry over to the quantum depolarizing channel. We then showed that when a weighted estimate of the depolarization coherence parameter to larger values is assumed, significant performance improvement by as much as 50% was found. For the specific quantum code studied here we found that a specific estimate of the channel flip probability, increasing that estimate by half as much again provided the most improved decoding performance.

Future work will further investigate asymmetric decoder performance in other quantum channels for which multiple parameters must be estimated in order to define the channel. We conjecture that all quantum channels which are misidentified, or for which only partial channel information is available, will benefit from similar decoding strategies to those outlined here. The use of pre-existing quantum entanglement between sender and receiver in the presence of asymmetric decoder performance will also be investigated in future work. The use of such pre-existing quantum entanglement is known to greatly expand the range of classical LDPC codes that can be “reused” in the quantum setting.

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