On the Penalty term for the Mixed Discontinuous Galerkin Finite Element Method for the Biharmonic Equation

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Abstract
In this paper, we present an analysis of the effect of penalty term in the mixed discontinuous Galerkin finite element method for the biharmonic equation. We split the biharmonic problem $\Delta^2 u = f$ into two second order problems by introducing an auxiliary variable $v = -\Delta u$. We prove that choosing the penalty term $\alpha_k = \sigma_0 |e_k|^{-1}p^2 < \sigma_0 |e_k|^{-3}p^2$ for a sufficiently large $\sigma_0$, ensures optimal rate of convergence in the $L^2$ and the energy norm for the approximations $u_h$ and $v_h$. Finally, we present numerical experiments to validate our theoretical results.

Keywords: Finite elements, Discontinuous Galerkin finite element method, Biharmonic problem, Optimal error estimates.

1. Introduction

Discontinuous Galerkin methods are popular finite element techniques which use discontinuous polynomials to construct approximate solutions. The local nature of approximation offers flexibility in using higher order polynomials with non–uniformity in the degree of approximation and in adaptive methods. For a review of the various discontinuous Galerkin methods we refer the reader to \cite{1, 5, 10}. For fourth order problems, conforming methods require imposition of $C^1$–continuity across the inter element boundaries and are computationally expensive. Several finite element methods have been proposed including the mixed finite element method \cite{4, 6, 9}, $C^0$–interior penalty methods \cite{3, 7} and so on to relax the continuity requirements. The idea behind the mixed finite element method is to split the fourth order problem into two second order problems by introducing an auxiliary variable $v = -\Delta u$. The $C^1$–continuity requirement is relaxed and the system is then approximated using a $C^0$–finite element method.

An $hp$ mixed discontinuous Galerkin finite element method was proposed by Gudi et al. \cite{8}. Using a primal formulation leads to integrals involving higher order derivatives which can be avoided by using a mixed formulation. However, the method yielded sub-optimal convergence rates for piecewise quadratic elements and no significant convergence rates for piecewise linear elements.

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This is due to the choice of the penalty parameter in the weak formulation which was taken proportional to the inverse cube of edge/face diameter $e_k$, a common choice in super-penalization in non-symmetric interior penalty Galerkin methods [10]. However, modifying the penalty parameter by taking it proportional to the inverse of the edge $e_k$, essentially reducing the size of the penalty term, yields optimal convergence.

In this paper, we consider the mixed formulation of the biharmonic equation
\[ \Delta^2 u = f \quad \text{in } \Omega, \] (1)
subject to the clamped boundary conditions
\[ u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial \Omega, \]
where $\Omega \subset \mathbb{R}^n$, $n = 2, 3$ is a bounded and convex domain with smooth boundary $\partial \Omega$ and $n$ is the unit outward normal to $\partial \Omega$. We assume that the data $f$ is sufficiently smooth so that there exists a unique solution to the problem in $H^4(\Omega)$. We introduce a new variable $v = -\Delta u$ and split the biharmonic equation (1) into two equations as
\[ -\Delta v = f \quad \text{in } \Omega, \]
\[ -\Delta u = v \quad \text{in } \Omega, \]
\[ u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial \Omega. \] (2)
We then discretize the problem using the mixed discontinuous Galerkin method and prove some error estimates for the finite element solution to study the convergence of the method with respect to the mesh size $h$. We then perform some numerical experiments to validate the theoretical results.

The paper is organized as follows. In Section 2 we derive the weak formulation of the biharmonic problem. The proof of well-posedness of the problem is identical to [8] and hence we omit the same. In Section 3 we discuss the error estimates for $h-$refinement and we derive an optimal estimate for the discrete solution of the primal variable $u_h$ in the $L^2$ and energy norm and sub-optimal estimate for the auxiliary variable in the $L^2$-norm. Finally, in Section 4 we perform numerical experiments to validate our theoretical results established in Section 3.

2. Weak Formulation

Let $\hat{\Omega} = \bigcup_{K \in T_h} \hat{K}$ be a uniform partition of $\Omega$. Let us denote the edges of $T_h$ by $e_k$ where $k = 1, \cdots, M_h$. Let $\Gamma_I = \{e_1, e_2, \cdots e_{p_h}\}$ denote the set of all interior edges, $\Gamma_D = \{e_{p_h+1}, \cdots, e_{M_h}\}$, the boundary edges and $\Gamma = \Gamma_I \cup \Gamma_D$. For each edge $e_k = K_i \cap K_j$, associate a unit normal $n_k$ outward from $K_i$, so that $n|_{K_i} = -n|_{K_j}$. Let $h_K$ denote the diameter of each $K \in T_h$ and $h = \max_{K \in T_h} h_K$. We define the broken Sobolev space
\[ H^s(T_h) = \{ v \in L^2(\Omega) : v_{|K_i} \in H^s(K) \forall K \in T_h \}, \] (3)
where $H^s(K)$ denotes the standard Sobolev space of order $s$. The associated broken norm and semi-norm are defined respectively, by

$$
\|v\|_{s,T_h} = \left( \sum_{K \in T_h} \|v\|_{s,K}^2 \right)^{1/2} \quad \text{and} \quad |v|_{s,T_h} = \left( \sum_{K \in T_h} |v|_{s,K}^2 \right)^{1/2},
$$

where $\|v\|_{s,K}$ and $|v|_{s,K}$ denotes the standard Sobolev norm and seminorm on $K$, respectively. We denote the $L^2$ norm by $\| \cdot \|$ and its innerproduct by $(\cdot, \cdot)$. We set $V = H^2(T_h)$.

Let $u, v$ be sufficiently smooth functions. We consider the first equation in (2), multiply by some $\varphi \in V$, and integrate over the domain to obtain

$$
- \int_{\Omega} (\Delta v) \varphi \, dx = \int_{\Omega} f \varphi \, dx.
$$

(4)

Applying Green’s formula and using the fact that $[v] = 0$ in $\Gamma_I$, we obtain

$$
\sum_{K \in T_h} \int_K \nabla v \cdot \nabla \varphi \, dx - \sum_{e_k \in \Gamma} \int_{e_k} \{v \cdot n_k\} [\varphi] \, ds - \sum_{e_k \in \Gamma_1} \int_{e_k} \{\nabla \varphi \cdot n_k\} [v] \, ds
$$

$$
= \sum_{K \in T_h} \int_K f \varphi \, dx.
$$

(5)

Since $[u] = 0$ in $\Gamma$, we see that

$$
J(u, \varphi) = \sum_{e_k \in \Gamma} \int_{e_k} \alpha_k [u][\varphi] \, ds = 0,
$$

(6)

where $\alpha_k$ is a positive real constant. Adding $J(u, \varphi)$ to (5) we obtain

$$
\sum_{K \in T_h} \int_K \nabla u \cdot \nabla \varphi \, dx - \sum_{e_k \in \Gamma} \int_{e_k} \{v \cdot n_k\} [\varphi] \, ds - \sum_{e_k \in \Gamma_1} \int_{e_k} \{\nabla \varphi \cdot n_k\} [v] \, ds
$$

$$
+ \sum_{e_k \in \Gamma_1} \int_{e_k} \alpha_k [u][\varphi] \, ds = \sum_{K \in T_h} \int_K f \varphi \, dx,
$$

(7)

which yields the weak formulation of the first equation. Similarly for the second equation, multiplying some $\chi \in V$, integrating over $\Omega$ and using the fact that $\nabla u \cdot n = 0$ on $\Gamma_D$, $[u] = 0$ in $\Gamma$ and $[v] = 0$ in $\Gamma_I$, we obtain

$$
\sum_{K \in T_h} \int_K \nabla u \cdot \nabla \chi \, dx - \sum_{e_k \in \Gamma_1} \int_{e_k} \{v \cdot n_k\} [\varphi] \, ds - \sum_{e_k \in \Gamma_1} \int_{e_k} \{\nabla \varphi \cdot n_k\} [v] \, ds
$$

$$
+ \sum_{e_k \in \Gamma} \int_{e_k} \alpha_k [u][\varphi] \, ds = \sum_{K \in T_h} \int_K f \varphi \, dx.
$$

(8)
We define the bilinear forms
\[ B(w, z) = \sum_{K \in T_h} \int_K \nabla w \cdot \nabla z \, dx - \sum_{e_k \in \Gamma} \int_{e_k} \{ \nabla w \cdot n_k \} [z] \, ds - \sum_{e_k \in \Gamma} \int_{e_k} \{ \nabla z \cdot n_k \} [w] \, ds, \]
\[ B_E(w, z) = \sum_{K \in T_h} \int_K \nabla w \cdot \nabla z \, dx - \sum_{e_k \in \Gamma} \int_{e_k} \{ \nabla w \cdot n_k \} [z] \, ds - \sum_{e_k \in \Gamma} \int_{e_k} \{ \nabla z \cdot n_k \} [w] \, ds. \]

Now the weak formulation of the problem is as follows: Find \((u, v) \in V \times V\), such that
\[ B(v, \varphi) + J(u, \varphi) = (f, \varphi), \]
\[ B(\chi, u) = (v, \chi), \] (9)
for all \((\varphi, \chi) \in V \times V\). Define a finite dimensional subspace \(V_h\) of \(V\) as
\[ V_h = D^k(T_h) = \{ v \in H^s(T_h) : v|_K \in P_k(K) \text{ for all } K \in T_h \}, \]
where \(P_k(K)\) denotes the space of polynomials of degree \(\leq k\). Now the discontinuous Galerkin finite element weak formulation reads: Find \((u_h, v_h) \in V_h \times V_h\) such that
\[ B(v_h, \varphi) + J(u_h, \varphi) = (f, \varphi), \]
\[ B(\chi, u_h) = (v_h, \chi), \] (10)
for all \((\varphi, \chi) \in V_h \times V_h\). For the purpose of error analysis, we define the following mesh dependent energy norm
\[ |||w|||^2 = \left( \sum_{K \in T_h} \int_K |\nabla w|^2 \, dx + \sum_{e_k \in \Gamma} \int_{e_k} \frac{\sigma_1}{|e_k|} [w]^2 \, ds \right). \]

Below, we state some properties of the bilinear form \(B(\cdot, \cdot)\) without proof.

Lemma 2.1. For sufficiently large constant \(\sigma_1\), it can be shown that for any \(w_h \in V_h\),
\[ C|||w_h|||^2 \leq B_E(w_h, w_h) + \sum_{e_k \in \Gamma} \int_{e_k} \frac{\sigma_1}{|e_k|} [w_h]^2 \, ds, \] (11)
where \(C\) is a constant independent of \(h\) and \(p\). It can also be shown that for all \(w, q \in V\) there exists a constant independent of \(h\) such that
\[ |B(w, q)| \leq C|||w||| \, |||q|||. \] (12)

The inequalities (11) and (12) refer to the coercivity and the boundedness property of the bilinear form \(B(\cdot, \cdot)\). For proofs, we refer the reader to \[8\] and the references therein. We recall the following trace inequality on the finite element space for further use in the error analysis.

Lemma 2.2. Let \(v_h \in V_h\). There exists a constant \(C > 0\) such that
\[ \|\nabla^l v_h\|_{L^2(e_k)} \leq C h^{-l/2}_K \|\nabla^l v_h\|_{L^2(K)}, \quad l = 0, 1. \] (13)

We state the inverse inequality without proof.
Lemma 2.3. Let $v_h \in V_h$. There exists a constant $C > 0$ such that

$$|v_h|_{H^1(K)} \leq C h_K^{\frac{1}{2}} \|v_h\|_{L^2(K)}. \quad (14)$$

For proofs of the trace inequality and the inverse inequality, we refer the reader to [2] and [10].

We are interested to study the effect of the penalty parameter on the convergence of the discrete solution. Let us set

$$a_k = \sigma_0 |e_k|^{-1} p_k^2. \quad (15)$$

where $\sigma_0$ is a constant which will be defined later and $i$ is an integer which describes the degree of penalization. It is well known in literature that for the case of the Non–symmetric interior penalty Galerkin (NIPG) method for second order elliptic problems, the choice of the penalty parameter plays a crucial role on the convergence of the solution. If $i = 1$, under normal penalization, the convergence in the $L^2$–norm for the piecewise quadratic case is suboptimal. However, for the case when $i = 3$, under super penalization, the convergence is optimal. Here, we observe that the penalization term with $i = 1$ produces optimal convergence in all cases, as opposed to $i = 3$ where optimal convergence rates are observed only for piecewise cubic elements. The latter case was well studied by Gudi et al. in [8]. In the next section, we derive optimal error estimates with $i = 1$.

3. Error Analysis

Define the auxiliary projection $\Pi_h : H^s(T_h) \rightarrow D^k(T_h)$ by

$$B_E(\phi - \Pi_h \phi, \chi) + \sum_{e_k \in \Gamma} \int_{e_k} \sigma_0 |\phi - \Pi_h \phi| |\chi| ds = 0. \quad (16)$$

The auxiliary projection can be shown to satisfy optimal error estimates in the $L^2$ and energy norms. We state and prove the following Lemma.

Lemma 3.1. For a given $\phi \in V$, there exists a unique $\Pi_h \phi \in V_h$ satisfying (16). Moreover, there exists positive constants independent of $h$ such that

$$|||\phi - \Pi_h \phi||| \leq C h^{\mu-1} \|\phi\|_{s,T_h} \quad (17)$$

$$\|\phi - \Pi_h \phi\| \leq C h^{\mu} \|\phi\|_{s,T_h}. \quad (18)$$

Proof. The projection $\Pi_h$ can be viewed as a discrete solution of the Poisson equation using the Symmetric Interior Penalty Galerkin (SIPG) Method with homogeneous Dirichlet boundary condition. Using standard arguments the desired results can be obtained, see for example [10].

Subtracting (10) from (9), we obtain the error equations

$$B(v - v_h, \varphi) + J(u - u_h, \varphi) = 0, \quad (19)$$

$$B(\chi, u - u_h) - (v - v_h, \chi) = 0, \quad (20)$$

for $(\varphi, \chi) \in V_h \times V_h$. Set $e_u := u - u_h$ and $e_v := v - v_h$. Using the auxiliary projection, we split the error as:

$$e_u = u - u_h = (u - \Pi_h u) - (u_h - \Pi_h u) := \eta_u - \theta_u,$$
\[ e_v = u - u_h := (v - \Pi_h v) - (v_h - \Pi_h v) := \eta_v - \theta_v, \]

and rewrite the error equations as

\begin{align*}
B(\theta_v, \varphi) + J(\theta_u, \varphi) &= B(\eta_v, \varphi) + J(\eta_u, \varphi), \\
B(\chi, \theta_h) - (\theta_v, \chi) &= B(\chi, \eta_u) - (\eta_v, \chi).
\end{align*}

(21)

(22)

Now we state and prove the following theorem.

**Theorem 3.1.** Let \( u_h, v_h \) satisfy (10). Then there exists some positive constant \( C \) independent of \( h \) such that

\[
\| u - u_h \| \leq Ch^{\mu-1}\| u \|_{s, T_h} + Ch^{\bar{\mu}-1}\| u \|_{s+2, T_h},
\]

\[
\| u - u_h \| \leq Ch^{\mu}\| u \|_{s, T_h} + Ch^{\bar{\mu}}\| u \|_{s+2, T_h},
\]

\[
\| v - v_h \| \leq Ch^{\mu-1}\| u \|_{s, T_h} + Ch^{\bar{\mu}-1}\| u \|_{s+2, T_h},
\]

where \( \mu = \min\{k + 1, s\} \) and \( \bar{\mu} = \min\{k + 1, \bar{s}\} \).

**Proof.** We split the proof into four parts where we estimate three intermediate terms and combine them into a single estimate.

(I) We set \( \varphi = \theta_u \) in (21), \( \chi = \theta_v \) in (22) and subtract the resulting equations to obtain

\[
\| \theta_v \|^2 + J(\theta_u, \theta_u) = B(\eta_v, \theta_u) + J(\eta_u, \theta_u) - B(\theta_v, \eta_u) + (\eta_v, \theta_v).
\]

Using the definition of the auxiliary projection, we obtain

\[
B(\eta_v, \theta_u) = B_E(\eta_v, \theta_u) + \sum_{e_k \in \Gamma_D} \int_{e_k} \frac{\partial \theta_u}{\partial n_k} \eta_v \, ds,
\]

\[
= - \sum_{e_k \in \Gamma_D} \int_{e_k} \frac{\sigma_1}{|e_k|} \eta_v \theta_u \, ds + \sum_{e_k \in \Gamma_D} \int_{e_k} \frac{\partial \theta_u}{\partial n_k} \eta_v \, ds.
\]

Using the boundedness property in Lemma 2.1, the trace inequality in Lemma 2.2, the Young’s inequality, we obtain the estimate

\[
B(\eta_v, \theta_u) \leq Ch^{2\bar{\mu}-2}\| u \|^2_{s+2, T_h} + \frac{1}{4} J(\theta_u, \theta_u) + \frac{1}{2} \| \theta_u \|^2,
\]

where we have used the approximation property of \( \Pi_h \) in Lemma 3.1 and \( v = -\Delta u \). Similarly we observe that

\[
J(\eta_u, \theta_u) \leq Ch^{2\mu-2}\| u \|^2_{s, T_h} + \frac{1}{4} J(\theta_u, \theta_u),
\]

\[
B(\theta_v, \eta_u) \leq Ch^{2\mu-2}\| u \|^2_{s, T_h} + \frac{1}{2} \| \theta_v \|^2,
\]

\[
(\eta_v, \theta_v) \leq Ch^{2\mu}\| u \|^2_{s+2, T_h} + \frac{1}{2} \| \theta_v \|^2.
\]

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Similarly, from Lemma 2.1 and the error equation (22), we have

\[ \| \theta_c \|^2 + J(\theta_c, \theta_a) \leq C h^{2\mu - 2} \| u \|^2_{x, \mathcal{T}_h} + C h^{2\mu - 2} \| u \|^2_{x + 2, \mathcal{T}_h} + 2(\| \theta_c \|^2 + \| \theta_a \|^2). \]  

(II) From the coercivity property in Lemma 2.1 and the error equation (21), we have

\[ \| \theta_c \|^2 \leq C \left( \sum_{K \in \mathcal{T}_h} \int_K | \nabla \theta_c |^2 + \sum_{e_k \in e_c} \int_{e_k} \frac{\sigma_1}{|e_k|} | \theta_c |^2 \, ds \right) \]

\[ \leq C \left( B(\eta_c, \theta_c) + J(e_u, \theta_v) + \sum_{e_k \in e_c} \int_{e_k} \left\{ \frac{\partial \theta_v}{\partial n_k} \right\} [\theta_c] \, ds + \sum_{e_k \in e_{c1}} \int_{e_k} \left\{ \frac{\partial \theta_c}{\partial n_k} \right\} [\theta_c] \, ds \right. \]

\[ + \sum_{e_k \in e_{c2}} \int_{e_k} \frac{\sigma_1}{|e_k|} | \theta_c |^2 \, ds \right). \]

Similarly, using Lemma 2.1 to 2.3, Lemma 3.1 and the Young’s inequality, we obtain the bounds

\[ B(\eta_c, \theta_c) \leq C h^{2\mu - 2} \| u \|^2_{x + 2, \mathcal{T}_h} + \frac{1}{4} \| \theta_c \|^2, \]

\[ J(e_u, \theta_v) \leq (J(e_u, \theta_v) + (J(\theta_v, \theta_v)))^{1/2} \leq C \| e_u \|^2_{x + 2, \mathcal{T}_h} + \frac{1}{4} \| \theta_v \|^2, \]

\[ \sum_{e_k \in e_{c1}} \int_{e_k} \left\{ \frac{\partial \theta_v}{\partial n_k} \right\} [\theta_c] \, ds \leq \frac{C}{2\sigma_0} \| \theta_c \|^2 + \frac{1}{8} \| \theta_v \|^2, \]

\[ \sum_{e_k \in e_{c2}} \int_{e_k} \left\{ \frac{\partial \theta_c}{\partial n_k} \right\} [\theta_c] \, ds \leq \frac{C}{2\sigma_0} \| \theta_c \|^2 + \frac{1}{8} \| \theta_v \|^2. \]

Assuming that \( k_1 \sigma_0 \leq \sigma_1 \leq k_2 \sigma_0 \), we obtain the following bound for the last term

\[ \sum_{e_k \in e_{c1}} \int_{e_k} \frac{\sigma_1}{|e_k|} | \theta_c |^2 \, ds \leq k_2 \| \theta_c \|^2. \]

Combining all the bounds and using the triangle inequality on \( \| e_u \| \), we obtain

\[ \| \theta_v \|^2 \leq C h^{2\mu - 2} \| u \|^2_{x, \mathcal{T}_h} + C h^{2\mu - 2} \| u \|^2_{x + 2, \mathcal{T}_h} + \frac{C}{4 - C (k_2 - \frac{1}{2\sigma_0})} \| \theta_u \|^2. \]  

(III) Similarly, from Lemma 2.1 and the error equation (22), we obtain

\[ \| \theta_u \|^2 \leq C \left( \sum_{K \in \mathcal{T}_h} \int_K | \nabla \theta_u |^2 \, dx + \sum_{e_k \in e_c} \int_{e_k} \frac{\sigma_1}{|e_k|} [\theta_u]^2 \, ds \right), \]

\[ \leq C \left( B(\eta_u, \theta_u) + (e_v, \theta_u) + \sum_{e_k \in e_c} \int_{e_k} \left\{ \frac{\partial \theta_u}{\partial n_k} \right\} [\theta_u] \, ds \right). \]
\[
+ \sum_{e_k \in \Gamma_1} \int_{e_k} \left( \frac{\partial \theta_u}{\partial n_k} \right) [\theta_u] \, ds.
\]

Constructing similar bounds and using the triangle inequality on \( \|e_v\| \), we obtain the estimate
\[
\||| \theta_u ||| \leq C h^{2\mu - 2} \|u\|_{\sigma, \mathcal{T}_h} + C h^{2\bar{\mu} - 2} \|u\|_{\bar{\sigma} + 2, \mathcal{T}_h} + C \left( \||| \theta_v ||| \right)^2 + J(\theta_u, \theta_u) + C \||| \theta_u \|||, \tag{25}
\]
where \( C_1 \) is a constant different from \( C \). Applying the bound derived in \( 23 \), we obtain
\[
\||| \theta_u ||| \leq C h^{2\mu - 2} \|u\|_{\sigma, \mathcal{T}_h} + C h^{2\bar{\mu} - 2} \|u\|_{\bar{\sigma} + 2, \mathcal{T}_h} + C_1 \left( \||| \theta_v ||| \right)^2 + \||| \theta_v \|||^2 + C \||| \theta_u \|||^2. \tag{26}
\]
Again, using the upper bound on \( \||| \theta_v \||| \) derived in \( 24 \), we obtain
\[
\||| \theta_u \||| \leq C h^{2\mu - 2} \|u\|_{\sigma, \mathcal{T}_h} + C h^{2\bar{\mu} - 2} \|u\|_{\bar{\sigma} + 2, \mathcal{T}_h} + C \||| \theta_u \|||^2 + C \||| \theta_u \|||^2. \tag{27}
\]
By setting \( \sigma_0 \geq \sigma^*_0 \geq 0 \) i.e., a constant sufficiently big, we can ensure the following estimate for \( \||| \theta_v \||| \):
\[
\||| \theta_v \||| \leq C h^{2\mu - 2} \|u\|_{\sigma, \mathcal{T}_h} + C h^{2\bar{\mu} - 2} \|u\|_{\bar{\sigma} + 2, \mathcal{T}_h} + C \||| \theta_u \|||^2. \tag{28}
\]
Using \( 27 \), we observe that
\[
\||| \theta_v \|||^2 + J(\theta_u, \theta_u) \leq C h^{2\mu - 2} \|u\|_{\sigma, \mathcal{T}_h} + C h^{2\bar{\mu} - 2} \|u\|_{\bar{\sigma} + 2, \mathcal{T}_h} + C \||| \theta_v \|||^2. \tag{29}
\]
(IV) Finally, we use the Aubin–Nitsche duality argument to derive the estimate for the error \( e_u \) in the \( L^2 \) norm. Consider the dual problem
\[
-\Delta \phi = z \quad \text{in} \quad \Omega,
-\Delta \psi = \phi \quad \text{in} \quad \Omega,
\psi = \frac{\partial \psi}{\partial n} = 0 \quad \text{on} \quad \partial \Omega,
\]
where the functions \( \phi \) and \( \psi \) satisfy the regularity result
\[
\|\phi\|_{H^1} + \|\psi\|_{H^1} \leq C \|z\|.
\]
Take \( z = e_u \), multiply by \( e_u \) and integrate over the domain to obtain
\[
B(\phi, e_u) + J(\theta_u, \theta_u) = \|e_u\|^2. \tag{30}
\]
Next multiply by \( e_v \) and integrate to obtain
\[
B(e_v, \psi) - (e_v, \phi) = 0. \tag{31}
\]
Adding the resulting equations, we obtain
\[ \|e_u\|^2 = B(\phi, e_u) + J(e_u, \psi) + B(e_v, \psi) - (e_v, \phi). \]

Denote \( \eta_\phi = \phi - I_h \phi \) and \( \eta_\psi = \psi - I_h \psi \) where \( I_h \) is an interpolation operator satisfying optimal estimates in the energy norm (see for example [8, p.143]). Using the orthogonality result (19) and (20), we obtain the following equation,
\[ \|e_u\|^2 = B(\eta_\phi, e_u) + J(e_u, \eta_\psi) + B(e_v, \eta_\psi) - (e_v, \eta_\phi). \]

Using the regularity estimate for the functions \( \phi \) and \( \psi \) and constructing similar error bounds, we obtain
\[ \|e_u\| \leq C \left( h \|e_u\| + Ch^{\min\{p+1,2\}} \left( \|e_v\| + J(e_u, e_u)^{1/2} \right) + Ch^{\min\{\mu + \theta - 2\}} \|u\|_{s+2,T_h} \right), \]
where \( \theta = \min \{k + 1, 4\} \). Using the estimates (27), (28) and (29) and the triangle inequality, we obtain
\[ \|e_u\| \leq C \left( h^{\min\{\mu + \theta - 2, 2\}} \|u\|_{s,T_h} + h^{\min\{\mu + \theta - 2, 2\}} \|u\|_{s+2,T_h} \right), \]
\[ = C \left( h^p \|u\|_{s,T_h} + h^p \|u\|_{s+2,T_h} \right), \]
which yields the desired estimate for \( e_u \). The other estimates follow from (28), (29) and the triangle inequality. Hence the proof is complete.

4. Numerical Experiments

In this section, we present some numerical experiments to validate the theoretical results. To perform the grid refinement analysis, we choose the exact solution \( u(x, y) = 1000 x^4 y^4 (1-x)^4 (1-y)^4 \) and calculate the right hand side. We then calculate the DGFEM solution \( u_h(x, y) \) and compute the error in the \( L^2 \) and the energy norms. The numerical experiments were conducted using FreeFem++. We consider two cases with different choices of penalty parameters and compare the results of the numerical scheme. In all the numerical results, we set the parameter \( \sigma_0 = 1 \).

First, we compute the rates of convergence for the DGFEM considered by Gudi et al. [8]. From Table 1 we observe that the rates of convergence for the linear DGFEM is not significant and a sub-optimal convergence in the \( L^2 \)-norm was observed for piecewise quadratic elements. The penalty parameter was chosen as \( \alpha_k = \sigma_0 |e_k|^{-1} p^2 \) for the mixed DGFEM. However, the choice of the penalty parameter is crucial for the mixed method as choosing a higher value may result in sub-optimal convergence.

In Table 2 we present the convergence results by choosing a lower value of the penalty parameter with \( \alpha_k = \sigma_0 |e_k|^{-1} p^2 \). The constant \( \sigma_0 \) in all the cases was set to be equal to 1. We observe a significant improvement in the convergence rates for piecewise linear elements and for piecewise quadratic elements. We observe that on choosing a lesser value of penalty, the solution becomes more accurate with refinement. This is especially strong in the linear case, where the solution converges rapidly (rate close to \( \approx 2.5 \)) to the exact solution and the magnitude of the \( L^2 \)-error \( \|u - u_h\| \) is significantly lower in the last iteration. This is illustrated in Figures 1 and 2 where convergence is
Figure 1: Convergence of the DGFEM solution using $\alpha_k =\sigma_0|e_k|^{-3}$ for piecewise linear elements. We observe that the approximate solution does not converge to the exact solution. The error values and the rates of convergence are summarized in Table 1.
Table 1: Error values and rates of convergence of the Mixed DG FEM discussed by Gudi et al. for the biharmonic equation. The penalty parameter is chosen to be $\alpha_k = \sigma_0 |e_k|^{-3} p^2$ with $\sigma_0 = 1$. The rate of convergence for the linear DGFEM is not significant and a sub-optimal convergence for the piecewise quadratic elements is observed which is an observation made in [8].

Table 2: Error values and rates of convergence for the mixed DGFEM for the biharmonic equation. The rate of convergence is close to $p+1$ in the $L^2$-norm and close to $p$ in the energy norm indicating optimal rates of convergence. The penalty parameter is chosen to be $\alpha_k = \sigma_0 |e_k|^{-1} p^2$ with $\sigma_0 = 1$.

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observed only in Figure 2 when $\alpha_k = \sigma_0 |e_k|^{-1}$. We observe a higher convergence rate for piecewise quadratic elements in the $L^2$ and energy norms while the error magnitude is comparable to the previous case in Table 1. We observe that optimal convergence rates are preserved for the piecewise cubic case, although the error magnitudes are higher than that observed from Table 1.
Figure 2: Convergence of the DGFEM solution using $\alpha_k = \sigma_0|e_k|^{-1}$ for piecewise linear elements. We observe that the DGFEM solution converges to the exact solution. The error values and the rates of convergence are summarized in Table 2.

Similar observations were made for the convergence in the auxiliary variable $v$ which is summarized in Table 3. Gudi et al. [8] observed a sub-optimal convergence rate ($\approx p - 1$) for the auxiliary variable in the $L^2$ norm. This was not observed in the case when the penalty parameter was chosen as $\alpha_k = \sigma_0|e_k|^{-1}p^2$ and the convergence rates are optimal in the $L^2$ norm. The observed rate of convergence is better than the theoretically established result in the previous section which predicts a sub-optimal convergence in $v$. We observe that the current choice of penalty term works well to approximate the auxiliary variable.
\[ \alpha_k = \sigma_0 |e_k|^{-1} p^2 \]

\[ \alpha_k = \sigma_0 |e_k|^{-3} p^2 \]

| \( p \) | \( N \) | \( \| v - v_h \| \) | \( O(\| v - v_h \|) \) | \( \| v - v_h \| \) | \( O(\| v - v_h \|) \) |
|---|---|---|---|---|---|
| 1 | 10 | 0.062298 | - | 0.030978 | - |
|   | 20 | 0.020921 | 1.574214 | 0.024028 | 0.366553 |
|   | 30 | 0.010340 | 1.738122 | 0.026058 | -0.200086 |
|   | 40 | 0.006243 | 1.753678 | 0.026978 | -0.120520 |
|   | 50 | 0.004227 | 1.747744 | 0.027434 | -0.075169 |
| 2 | 8 | 0.013329 | - | 0.009947 | - |
|   | 16 | 0.001495 | 3.155931 | 0.001876 | 2.406888 |
|   | 24 | 0.000413 | 3.172947 | 0.001082 | 1.356340 |
|   | 32 | 0.000167 | 3.146058 | 0.000802 | 1.043306 |
|   | 40 | 0.000083 | 3.129984 | 0.000643 | 0.988573 |
| 3 | 6 | 0.007547 | - | 0.002982 | - |
|   | 12 | 0.000615 | 3.618003 | 0.00382 | 2.966170 |
|   | 18 | 0.000118 | 4.075293 | 0.002222 | 1.333628 |
|   | 24 | 0.000035 | 4.206286 | 0.00138 | 1.658336 |
|   | 30 | 0.000014 | 4.272284 | 0.00091 | 1.856309 |

Table 3: The rates of convergence for the auxiliary variable for the DGFEM with respect to the mesh size and penalty parameter. We observe optimal convergence rate for the auxiliary variable in the \( L^2 \) and the energy norm for the current choice of penalty parameter and suboptimal rate of \( p - 1 \) for the choice in [8]. We set \( \sigma_0 = 1 \).

5. Conclusion

In this work, we have considered a mixed Discontinuous Galerkin Finite Element Method to solve the biharmonic equation subject to clamped boundary conditions. We have derived the weak formulation of the problem and established error estimates for \( h \)-refinement. We performed a series of numerical experiments using FreeFem++, and verified the theoretical results. We observed that the choice of the penalty term is crucial and must be chosen to be of the form \( \alpha_k = \sigma_0 |e_k|^{-1} p^2 \) to obtain optimal error estimates in \( h \)-refinement. Significant improvements in convergence rates for the piecewise linear and quadratic elements were observed as a result.

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