NEGATIVE DEFINITE FUNCTIONS ON GROUPS
WITH POLYNOMIAL GROWTH

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Abstract. The aim of this work is to show that on a locally compact, second countable, compactly generated group \( G \) with polynomial growth and homogeneous dimension \( d_h \), there exist a continuous, proper, negative definite function \( \ell \) with polynomial growth dimension \( d_\ell \) arbitrary close to \( d_h \).

1. Introduction and statement of the results

The length function \( \ell \) associated to a finite set of generators of a countable, discrete, finitely generated group \( \Gamma \), may be used to reveal interesting aspects of the group itself. This is the spirit of metric group theory (see [12]). In a famous result of U. Haagerup [11], for example, the length functions on the free groups \( \mathbb{F}_n \) with \( n \geq 2 \) generators, has been used to prove that, even if these groups are not amenable, they still have an approximation property: there exists a sequence \( \varphi_n \in c_0(\mathbb{F}_n) \) of normalized, positive definite functions, converging pointwise to the constant function. Correspondingly, the trivial representation is weakly contained in a \( C_0 \)-representation (one whose coefficients vanish at infinity) (see [10]). These properties of \( \mathbb{F}_n \) only depend to the fact that the length function is negative definite and proper. The class of groups where a length function has the latter properties has been characterized in several independent ways and form the object of intensive investigations (see [2]). On another, somewhat opposite, side, a deeply studied class of groups are those possessing the Kazhdan property (T), i.e. those where every negative definite function is bounded (see for example [1]). Finer asymptotic properties of length functions are connected to other fundamental properties of groups. For example, if \( \ell \) has polynomial growth (see below for the definition) then \( \Gamma \) is amenable and its von Neumann algebra \( vN(\Gamma) \) is hyperfinite (weak closure on an increasing sequence of matrix algebras).

However, length functions are not necessarily negative definite and one may wonder how far is a given length function from being negative definite. Reversing the point of view, in the present work we prove that on finitely generated, discrete groups \( \Gamma \) or, more generally, on compactly generated, locally compact, second countable groups \( G \), with polynomial growth, there always exists a continuous, negative definite function with a (polynomial) growth arbitrarily close to the one of \( \Gamma \) or \( G \).

The consideration of negative definite functions and their growth properties are also meaningful in Noncommutative Potential Theory (see [3], [6]) and Quantum Probability (see [4]). As an application of the main results, we provide a natural construction of a noncommutative Dirichlet form on the group von Neumann algebra \( \lambda(G)'' \) with an upper spectral growth dimension \( d_S \) bounded above by the polynomial growth homogeneous dimension \( d_h \) of \( G \).

Date: December 8, 2015.

1991 Mathematics Subject Classification. Primary 20F65; Secondary 20F69, 57M07.

Key words and phrases. Group, polynomial growth, negative definite function, homogeneous dimension.

This work was supported by Italy I.N.D.A.M. France C.N.R.S. G.D.R.E.-G.R.E.F.I. Geometrie Noncommutative and by Italy M.I.U.R.-P.R.I.N. project N. 2012TC7588-003.
Let $G$ be a locally compact, second countable, compactly generated group with identity $e \in G$ and let $\mu$ be one of its left Haar measures.

The main results of the work are the following.

**Theorem 1.1.** Suppose that $G$ has polynomial growth and homogeneous dimension $d_h$. Then, for all $d > d_h$, there exists a continuous, proper, negative type function $\ell$ with polynomial growth such that
\[
\mu\{s \in G \mid \ell(s) \leq x\} = O(x^d), \quad x \to +\infty.
\]

**Theorem 1.2.** Suppose that $G$ has polynomial growth and homogeneous dimension $d_h$. Then, there exists a continuous, proper, negative type function $\ell$ with polynomial growth such that
\[
\forall d > d_h, \quad \mu\{s \in G \mid \ell(s) \leq x\} = O(x^d) \quad x \to +\infty.
\]

As a straightforward application of these results to Noncommutative Potential Theory we have the following corollary. Further investigations will be discussed in [7].

**Corollary 1.3.** Let $\Gamma$ be a discrete, finitely generated group with polynomial growth and homogeneous dimension $d_h$. Let $\lambda(\Gamma)^n$ be the von Neumann algebra generated by the left regular representation and let $\tau$ be its trace. Then there exists on $L^2(\lambda(\Gamma)^n, \tau)$ a noncommutative Dirichlet form $(\mathcal{E}, \mathcal{F})$ with discrete spectrum and upper spectral dimension $d_s \leq d_h$.

Here the upper spectral dimension $d_s$ of the Dirichlet form is defined as
\[
d_s := \inf\{d > 0 : \limsup_{x \to +\infty} \frac{\#\{\lambda \in \text{sp}(L) : \lambda \leq x\}}{x^d} < +\infty\}
\]
where $\text{sp}(L)$ denotes the spectrum of the self-adjoint, nonnegative operator $(L, \text{dom}(L))$ associated to $(\mathcal{E}, \mathcal{F})$
\[
\mathcal{E}[a] = \|\sqrt{\nu}a\|_2^2, \quad a \in \mathcal{F} = \text{dom}(L).
\]

2. **Groups with polynomial growth**

Let $K \subset G$ be a compact, symmetric, generator set with non empty interior
\[
G = \bigcup_{n=1}^{\infty} K^n, \quad K^{-1} = K, \quad K^\circ \neq \emptyset,
\]
where
\[
K^{-1} := \{s^{-1} \in G : s \in K\}, \quad K^n := \{s_1 \ldots s_n \in G : s_k \in K, k = 1, \ldots, n\}.
\]
We may always assume that $K$ is a neighborhood of the identity $e \in K^\circ \subset G$.

Let us recall that $G$ has polynomial growth if
\[
\exists c, d > 0 \text{ such that } \mu(K^n) \leq c(n+1)^d, \quad n \geq 1
\]
or, equivalently, if
\[
\frac{1}{\log(n+1)} \log(\mu(K^n)) \leq d + \frac{1}{\log(n+1)} \log(c), \quad n \geq 1
\]
and in that case its homogeneous dimension is defined as
\[
d_h = \limsup_{n \to \infty} \frac{1}{\log n} \log(\mu(K^n)).
\]
The class of locally compact, second countable, compactly generated groups with polynomial growth includes nilpotent, connected, real Lie groups and finitely generated, nilpotent, countable discrete groups (see [8] and [12] Chapter VII S 26). For example, the homogeneous dimension of the discrete Heisenberg group is 4 (see [12] Chapter VII S 21).

Recall also the M. Gromov’s characterization [9] of finitely generated, countable discrete groups with polynomial growth as those which have a nilpotent subgroup of finite index (see also [12] Chapter VII S 29 and [13] for a generalization to locally compact groups).

**Remark 2.1.** Let us observe that since $K$ is a neighborhood of the identity $e \in G$, we have $K^{n-1} \subset K^{n-1}K^o \subset (K^n)^o$ so that
\[
\bigcup_{k=0}^{\infty} (K^n)^o = G
\]
and we have an open cover of $G$. As a consequence

**every compact set in $G$ is contained in $K^n$ for some $n \geq 1$.**

### 3. Proof of the Theorems

In the following, we will consider the sequence $\{\alpha_n\}_{n=1}^{\infty} \subset [0, +\infty)$ defined by
\[
\alpha_1 := 0, \quad \frac{\mu(K^n)}{\mu(K^{n-1})} := 1 + \alpha_n, \quad n \geq 2.
\]
Equation (2.2) can thus be written as
\[
\frac{1}{\log(n+1)} \sum_{k=1}^{n} \log(1 + \alpha_k) \leq d + \frac{1}{\log(n+1)} \log(c) \leq d', \quad n \geq 1,
\]
with $d' := d + \log_2 c$.

#### 3.1. Some preliminary estimates

The proof of the theorems relies on the following lemmas.

**Lemma 3.1.** For any $\beta \in (0, 1)$ the set
\[
E_\beta := \{ n \in \mathbb{N}^* : \alpha_n > n^{-\beta} \}
\]
has vanishing density.

**Proof.** Let us fix $K > 0$ such that $\log(1 + n^{-\beta}) \geq Kn^{-\beta}$ for all $n \geq 1$. By equation (3.1), we have
\[
d' \log(n+1) \geq \sum_{k\in [1,n] \cap E_\beta} \log(1 + \alpha_k) \geq Kn^{-\beta} \sharp \{[1,n] \cap E_\beta\}
\]
so that
\[
\frac{1}{n} \sharp \{[1,n] \cap E_\beta\} \leq \frac{d' \log(n+1)}{K n^{1-\beta}} \to 0 \quad \text{as} \quad n \to \infty.
\]

**Lemma 3.2.** For any $\beta \in (0, 1)$ and $\gamma > 1/\beta$, the set
\[
F_{\beta, \gamma} := \{ n \in \mathbb{N} : \lfloor n^\gamma, (n+1)^\gamma \rfloor \cap \mathbb{N} \subset E_\beta \}
\]
has vanishing density.
Proof. For any $\varepsilon \in (0, 1/2)$ and any integer $N \geq 2$, we have
\[
d' \geq \frac{1}{\log(N+1)^\gamma} \sum_{(\varepsilon N)^\gamma \leq k \leq N^\gamma} \log(1 + \alpha_k) \geq \frac{1}{\log(N+1)^\gamma} \sum_{n \in [\varepsilon N, N-1] \cap F_{\beta, \gamma}} \sum_{k \in [n^\gamma, (n+1)^\gamma]} \log(1 + \alpha_k) \geq \frac{K}{\gamma \log(N+1)} \sum_{n \in [\varepsilon N, N-1] \cap F_{\beta, \gamma}} \sum_{k \in [n^\gamma, (n+1)^\gamma]} k^{-\beta} N \geq 1.
\]
For any fixed $n \in [\varepsilon N, N-1] \cap F_{\beta, \gamma}$, there exists $t \in (n, n+1)$ such that $(n+1)^\gamma - n^\gamma = \gamma t^{\gamma-1}$. As $n \geq \varepsilon N$, we have $t/N > n/N \geq \varepsilon$ so that for $n \in [\varepsilon N, N-1] \cap F_{\beta, \gamma}$ the following bound holds true
\[
\sum_{n^\gamma \leq k < (n+1)^\gamma} k^{-\beta} \geq N^{-\beta \gamma} ((n+1)^\gamma - n^\gamma - 2) = N^{-\beta \gamma} (N t^{\gamma-1} - 2) > N^{-\beta \gamma} (\varepsilon N)^{\gamma-1} - 2).
\]
Thus we have
\[
d' \geq \frac{K}{\gamma \log(N+1)} \sum_{n \in [\varepsilon N, N-1] \cap F_{\beta, \gamma}} \left( \gamma^{\gamma-1} N^{(1-\beta)\gamma-1} + o(N^{(1-\beta)\gamma-1}) \right) \geq \frac{K}{\log(N+1)} \left( \gamma^{\gamma-1} N^{(1-\beta)\gamma-1} + o(N^{(1-\beta)\gamma-1}) \right) \#([\varepsilon N, N-1] \cap F_{\beta, \gamma})
\]
and we may deduce
\[
d' \geq \frac{K}{\gamma^{\gamma-1} N^{(1-\beta)\gamma}} \left( \sum_{n \in [\varepsilon N, N-1] \cap F_{\beta, \gamma}} 1 + o(1) \right) \geq \frac{\#([\varepsilon N, N-1] \cap F_{\beta, \gamma})}{N}.
\]
For $N$ big enough, we have
\[
\frac{\#([\varepsilon N, N-1] \cap F_{\beta, \gamma})}{N} \leq \varepsilon
\]
and finally
\[
\frac{\#([1, N-1] \cap F_{\beta, \gamma})}{N} \leq \frac{\#([1, \varepsilon N] \cap F_{\beta, \gamma})}{N} + \frac{\#([\varepsilon N, N-1] \cap F_{\beta, \gamma})}{N} \leq 2\varepsilon.
\]
\[\square\]

3.2. Proofs of the theorems. To build up a negative type function having the properties required in Theorem 1.2, we begin to construct a suitable sequence of positive definite functions.

Fix $\beta \in (0, 1)$ and $\gamma > 1/\beta$. For any fixed $n \not\in F_{\beta, \gamma}$, we can chose $k(n) \in [n^\gamma, (n+1)^\gamma] \setminus E_{\beta}$ in such a way that
\[
n^\gamma \leq k(n) \leq (n+1)^\gamma, \quad k(n)^{-\beta} \geq \alpha_{k(n)} = \frac{\mu(K^{k(n)+1}) - \mu(K^{k(n)})}{\mu(K^{k(n)})}.
\]
Denote by $\mathcal{B}(L^2(G, \mu))$ the Banach algebra of all bounded operators on the Hilbert space $L^2(G, \mu)$ and by $\lambda : G \to \mathcal{B}(L^2(G, \mu))$ the left regular unitary representation of $G$, defined as $\lambda(s)a(t) := a(s^{-1}t)$ for $a \in L^2(G, \mu)$ and $s, t \in G$.  

Consider now the function $\xi_n := \mu(K^{k(n)})^{-1/2} \chi_{K^{k(n)}} \in L^2(G, \mu)$, with unit norm, and set

$$\omega_n : G \to \mathbb{R} \quad \omega_n(s) := \langle \xi_n, \lambda(s)\xi_n \rangle_{L^2(G)} = \frac{\mu(sK^{k(n)} \cap K^{k(n)})}{\mu(K^{k(n)})}.$$  

By construction we have that

- $\omega_n \in C_c(G)$ is a continuous, normalized, positive definite function with compact support.
- Moreover
- $\omega_n(s) \geq 1 - n^{-\beta \gamma}$ for all $s \in K$.

In fact, by the translation invariance of the Haar measure and since $K^{k(n)} \supset sK^{k(n)}$ for $s \in K$, we have

$$\omega_n(s) = \frac{\mu(sK^{k(n)} \cap K^{k(n)})}{\mu(K^{k(n)})} \geq \frac{\mu(sK^{k(n)} \cap sK^{k(n)} - 1)}{\mu(K^{k(n)})} = \frac{\mu(sK^{k(n)} \cap K^{k(n)} - 1)}{\mu(K^{k(n)})}$$

$$= \frac{\mu(K^{k(n)} \cap K^{k(n)} - 1)}{\mu(K^{k(n)})} \leq \frac{\mu(K^{k(n)} - 1)}{\mu(K^{k(n)})} = (1 + \alpha_k(n))^{-1} \geq 1 - \alpha_k(n)$$

$$\geq 1 - k(n)^{-\beta} \geq 1 - n^{-\beta \gamma}.$$

**Lemma 3.3.** For $n \notin F_{\beta, \gamma}$ and any integer $p \geq 1$, we have

$$\omega_n(s) \geq 1 - pn^{-\beta \gamma} \quad s \in K^p.$$  

**Proof.** Consider $s = s_1 \cdots s_p \in K^p$ for some $s_1, \ldots, s_p \in K$ and notice that

$$2(1 - \omega(s)) = ||\xi_n - s\xi_n||^2 = ||\xi_n - s_1 \cdots s_p \xi_n||^2$$

$$= ||\xi_n - s_1 \xi_n + s_1 \xi_n - s_1 s_2 \xi_n + \cdots + s_1 \cdots s_{p-1} \xi_n - s_1 \cdots s_p \xi_n||^2$$

$$\leq ((||\xi_n - s_1 \xi_n|| + ||\xi_n - s_2 \xi_n|| + \cdots + ||\xi_n - s_p \xi_n||)^2$$

$$\leq p^2 \sup_{\sigma \in K} ||\xi_n - \sigma\xi_n||^2$$

$$= 2p^2 \sup_{\sigma \in K} (1 - \omega_n(\sigma)) \leq 2p^2 n^{-2\beta \gamma}.$$  

**Proof of Theorem 1.1.** By previous lemma, the series $\sum_{n \notin F_{\beta, \gamma}} (1 - \omega_n(s))$ converges uniformly on $K^p$ for any integer $p \geq 1$. Hence, it converges uniformly on each compact subset of $G$ and

$$\ell : G \to [0, +\infty) \quad \ell(s) = \sum_{n \notin F_{\beta, \gamma}} (1 - \omega_n(s))$$

is a continuous, negative definite function. The function $\ell$ is proper: in fact, if $m \in \mathbb{N}$ is greater or equal to the integer part of $(N + 1)^\gamma$, then, for $s \notin K^{2m}$, we have $\omega_n(s) = 0$ for all $n \in [1, N] \setminus F_{\beta, \gamma}$ so that

$$\ell(s) \geq \sharp\{[1, N] \setminus F_{\beta, \gamma}\} \quad s \notin K^{2m}.$$  

According to Lemma 3.2 we can write $\sharp\{[1, N] \setminus F_{\beta, \gamma}\} = N(1 - \varepsilon_N)$ for $\varepsilon_N \to 0$. The set $\{s \mid \ell(s) \leq N(1 - \varepsilon_N)\}$ is contained in $K^{2m}$, where $m$ is the integer part of $(N + 1)^\gamma + 1$, whose volume $\leq c \cdot 2^d (N + 1)^\gamma + 1)^d = O(N^{\gamma d})$. For fixed $\varepsilon > 0$ and $N$ large enough, we have, for $x \in [N - 1, N]$,

$$\mu\{\ell \leq x(1 - \varepsilon)^{-1}\} \leq c' N^{\gamma d} \leq c'' x^{\gamma d}$$

so that $\mu\{\ell \leq x\} \leq c'' x^{\gamma d}$.  

$\square$
Proof of Theorem 1.2. Let us choose a decreasing sequence $d_m$ converging to $d_h$, and for all $m$, choose a function $\ell_m$ such that $\ell_m(x) = O(x^{d_m})$, $x \to +\infty$ and $\sup_K \ell_d(x) = 1$ (by normalization). Hence $\ell \leq k^2$ on $K^k$. It is enough to set $\ell = \sum_m 2^{-m} \ell_m : \{\ell \leq x\} \subset \{\ell_m \leq 2^m x\}$, whose measure is $O(x^{d_m})$.

□

Proof of Corollary 1.3. Recall first that the GNS space $L^2(\lambda(\Gamma)'', \tau)$ can be identified, at the Hilbert space level, with $l^2(\Gamma)$. Consider now the continuous, proper, negative type function $\ell$ constructed in Theorem 1.2. By [5 S 10.2], the quadratic form $\mathcal{E} : \mathcal{F} \to [0, +\infty)$

$$\mathcal{E}[a] = \sum_{s \in \Gamma} \ell(s) |a(s)|^2,$$

defined on the subspace $\mathcal{F} \subset l^2(\Gamma)$ where the sum is finite, is a noncommutative Dirichlet form whose associated nonnegative, self-adjoint operator on $l^2(\Gamma)$ is the multiplication operator by $\ell$ given by

$$\text{dom}(L) := \{a \in l^2(\Gamma) : \ell \cdot a \in L^2(G, \mu)\} \quad (La)(s) := \ell(s) \cdot a(s) \quad s \in G.$$

The stated result thus follows from Theorem 1.2 and the straightforward identification $\text{sp}(L) = \{\ell(s) \in [0, +\infty) : s \in \Gamma\}$.

□
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