CURRENTS, PRIMITIVE COHOMOLOGY CLASSES AND SYMPLECTIC HODGE THEORY

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ABSTRACT. Tseng and Yau developed primitive cohomology theories on symplectic manifolds. In addition, they proposed a definition of primitive homology, and proved that there is a natural homomorphism from the primitive homology to the primitive cohomology. Inspired by the work of Tseng and Yau, we develop a new approach to the symplectic Hodge theory, and prove in this paper that there is a Poincaré duality between the primitive homology and cohomology for any compact symplectic manifold with the Hard Lefschetz property. Among other things, we introduce a De Rham complex of real flat chains on symplectic manifolds, and use it to give a dual chain description of the symplectic Hodge adjoint operator.

For projective Kähler manifolds, the Poincaré duality between the primitive cohomology and homology provides a new geometric interpretation of primitive cohomology classes from the viewpoint of symplectic Hodge theory, which is very different from the ones what algebraic geometers had before. As an application, we use the primitive version of the Poincaré duality theorem to investigate the support of symplectic Harmonic representatives of Thom classes, and provide an answer to a question asked by Guillemin.

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1. Introduction

Symplectic Hodge theory was introduced by Ehresmann and Libermann \cite{EL49, L55}, and was rediscovered by Brylinski \cite{Bry88}. A symplectic form induces a non-degenerate bi-linear pairing on the space of differential forms. By mimicking the construction in Riemannian Hodge theory one can define the symplectic Hodge star operator $\star$. In this context, a differential form $\alpha$ is said to be symplectic Harmonic if and only if $d\alpha = d^\Lambda \alpha = 0$, where $d^\Lambda = \pm \star d \star$ is the symplectic Hodge adjoint operator.

A $2n$ dimensional symplectic manifold is said to satisfy the Hard Lefschetz property if and only if for any $0 \leq k \leq n$, the Lefschetz map

$$L^k : H^{n-k}(M) \to H^{n+k}(M) \quad [\alpha] \mapsto [\omega^k \land \alpha]$$

is an isomorphism. A remarkable theorem of Mathieu \cite{Ma95} asserts that on a compact symplectic manifold every De Rham cohomology class admits a symplectic Harmonic representative if and only if the manifold satisfies the Hard Lefschetz property.

Harmonicity is a much flabbier property in symplectic Hodge theory than in Riemannian Hodge theory. One does not expect the Harmonic representative of a Thom class to exhibit any interesting global features. However, Bahramgiri \cite{Ba06} proved a beautiful result that a compact oriented submanifold $N$ of a symplectic manifold $M$ is co-isotropic if and only if its canonical Thom current is symplectic Harmonic. He then constructed a symplectic smoothing operator for currents on symplectic manifolds, and used it to prove that for any tubular neighborhood $U$ of a co-isotropic submanifold $N$, there exists a Harmonic representative $\tau_N$ of the Thom class of $N$ which is supported entirely inside $U$. This stands in contrast with Riemannian Hodge theory, where any Harmonic form which has a zero of infinite order is identically zero, cf. \cite{AKS62}.

In addition, Bahramgiri also proved that if $\tau_N$ is a symplectic Harmonic representative of the Thom class of a symplectic submanifold $N$, then the support of $\tau_N$ is the entire manifold $M$. This motivated Victor Guillemin to ask the following fundamental question in symplectic Hodge theory.
Question: What can we say about the support of symplectic Harmonic representatives of the Thom classes of isotropic submanifolds? More generally, can we give a characterization of the submanifolds of a symplectic manifold whose Thom class admits a symplectic Harmonic representative that is not supported on the entire manifold?

In a different direction, L. Tseng and S. T. Yau [TY09], [TY10] developed primitive cohomology theory on symplectic manifolds. On a 2n-dimensional symplectic manifold \((M, \omega)\), for any \(0 \leq k \leq n\), a k-form \(\alpha\) is said to be primitive if and only if

\[
\omega^{n-k+1} \wedge \alpha = 0.
\]

(1.2)

Let \(P^r(M)\) be the space of primitive r-forms which are closed under the symplectic adjoint operator \(d^\Lambda\). One version of the primitive cohomology group \(PH_d(M)\) introduced in [TY09] is as follows

\[
PH_d^r(M) = \ker \frac{d \cap P^r(M)}{dP^{r-1}(M)}, \quad 1 \leq r \leq n.
\]

(1.3)

If \(n < l \leq 2n - 1\), let \(C_l(M)\) be the space of smooth co-isotropic l-chains which have co-isotropic boundaries; if \(l = n\), let \(C_n(M)\) be the space of smooth co-isotropic n-chains. Denote by \(\partial\) the usual boundary operator. [TY09] introduced the following definition of primitive homology.

\[
PH_l(M) = \ker \frac{\partial \cap C_l(M)}{\partial C_{l+1}(M)}, \quad n \leq l \leq 2n - 1.
\]

(1.4)

It is shown in [TY09] that the canonical current of any closed oriented cycle in \(C_l(M)\) induces a cohomology class in \(PH_d^{2n-l}(M)\). In the literature, Tseng and Yau are probably the first mathematicians who see that there may be a Poincaré duality between the primitive cohomology and homology.

Let us pause for a moment and explain why such a duality theorem is a very non-trivial result in symplectic Hodge theory. For simplicity, let us assume that the symplectic manifold \((M, \omega)\) is compact and satisfies the Hard Lefschetz property. As we explain in Proposition 2.7 under such assumptions the above definition of primitive cohomology agrees with the usual one used by algebraic geometers. In particular, a cohomology class \([\alpha] \in H^k(M)\) is primitive if and only if \([\omega^{n-k+1}] \wedge [\alpha] = 0\), \(0 \leq k \leq n\). In this situation, the usual Poincaré duality theorem asserts that the Poincaré dual of \([\alpha]\) is represented by a \(2n - k\) dimensional homology cycle \(C\). It follows that for any closed \((k - 2)\)-form \(\beta\) we have that

\[
\int_C \omega^{n-k+1} \wedge \beta = 0.
\]

(1.5)

However, under the same assumption the primitive version of the duality theorem would assert that there exists a \(2n - k\) dimensional cycle \(C\)
Poincaré dual to $[\alpha]$ such that Equation 1.5 holds for any differential form $\beta$ of degree $k - 2$. This is by no means obvious.

To continue this discussion, let us give a more precise definition of co-isotropic chains. Equip $\mathbb{R}^{2n}$ with the standard symplectic structure. A $p$-chain element on $M$ is a simplicial $p$-simplex $\sigma$ in $\mathbb{R}^{2n}$, together with a symplectomorphism from an open neighborhood of $\sigma$ onto an open set in $M$. A $p$-chain element is called co-isotropic if the interior of the simplex is a co-isotropic submanifold of $\mathbb{R}^{2n}$. Finally, a finite $p$-chain on $M$ is a finite linear combination of $p$-chain elements on $M$. It is called co-isotropic if it is a finite linear combination of co-isotropic chain elements.

In the usual treatment of the Poincaré duality between cohomology and homology on a compact orientable manifold, one first proves that the Poincaré duality holds on $\mathbb{R}^m$, and then extends it to the manifold using a Mayer-Vietoris sequence type argument. However, there are some extraordinary technical difficulties when one tries to prove the duality between the primitive cohomology and homology by following these traditional lines.

First, it is a rather rigid condition that finite chains are embedded into the manifold by local symplectomorphisms. It is hard to establish the local Poincaré lemma for the primitive homology. Indeed, the usual cone construction in smooth singular homology fails. And one can not overcome the difficulty by choosing a polyhedral subdivision of the manifold, since in the symplectic setting we do not have a PL structure which is compatible with the symplectic structure. Second, the requirement that $\partial T$ is co-isotropic for any $T \in C_*(M)$ makes it difficult to establish a Mayer-Vietoris sequence for the primitive homology. In addition, the differential complex that Tseng and Tau used to define the primitive cohomology consists of co-closed primitive forms. The co-closedness condition makes it impossible to establish a Mayer-Vietoris exact sequence for the primitive cohomology.

Inspired by the above-mentioned pioneering work of Tseng and Yau [TY09], and by Guillemin’s question concerning symplectic Harmonic representatives of Thom classes, we develop in the present paper a new approach to the symplectic Hodge theory, and prove that there is a Poincaré duality between the primitive cohomology and homology for any compact symplectic manifold with the Hard Lefschetz property. Among other things, we introduce a De Rham complex of real flat chains on symplectic manifolds, and use it to give a dual chain description of the symplectic Hodge adjoint operator.

More precisely, we define a real flat chain on a symplectic manifold to be a compactly supported current which is the limit of a sequence of finite chains. Here the limit is used in the sense as defined in [DR84 Sec. 10], which is much stronger than the usual weak limit in the functional analysis. This notion of flat chains is a natural extension of flat chains on an Euclidean space first introduced in [WH57]. We note that the usual definition of a flat chain extends to a convex open coordinate neighborhood on a manifold without any difficulty. On such a coordinate neighborhood, using...
the fundamental duality between flat chains and flat forms, we prove that our definition of a flat chain agrees with the usual one used in the geometric measure theory. On a symplectic manifold, we show that a flat chain introduced in our paper is a finite sum of flat chains in the usual sense which each sits inside a convex open coordinate neighborhood. (See Definition 4.4, Lemma 4.6, and Theorem 4.12 for more details.)

For any compact symplectic manifold, we prove that the De Rham cohomology of real flat chains computes the usual De Rham cohomology with real coefficients. More importantly, we show that the complex of flat chains is a $\mathfrak{sl}_2$ sub-module of the distributional De Rham complex. One can therefore use this complex to do symplectic Hodge theory. If the symplectic manifold satisfies the Hard Lefschetz property, the Poincaré duality between the primitive cohomology and homology would follow from the symplectic $d^\Lambda$-lemma for real flat chains on the symplectic manifold. (See Theorem 7.11 for more details.)

The Poincaré duality between the primitive cohomology and homology provides us a much deeper understanding of primitive cohomology classes. It asserts that on a symplectic manifold with the Hard Lefschetz property every primitive cohomology class with positive degree is Poincaré dual to a primitive real flat chain, which is shown to be the limit of a sequence of finite co-isotropic chains on the symplectic manifold. On a compact projective Kähler manifold, this offers us a new geometric interpretation of primitive cohomology classes, which is very different from what algebraic geometers had before, cf. [N93], [Sch10].

Combining this result with the symplectic smoothing operator construction in [Ba06], one sees immediately that every primitive cohomology class of positive degree is represented by a smooth symplectic Harmonic differential form which is not supported on the entire manifold. As an immediate application, we prove the following theorem, which provides an answer to the question asked by Victor Guillemin.

**Theorem 1.1.** Assume that $(M, \omega)$ is a $2n$ dimensional compact symplectic manifold with the Hard Lefschetz property, and that $N$ is a compact oriented submanifold of $M$ with Thom class $[\tau_N]$. If $\text{codim} \ (N)$ is odd, or if $\text{codim} \ (N) = 2p$ is even and $[\omega]^{n-p} \wedge [\tau_N] = 0$, then $N$ must admit a symplectic Harmonic representative which is not supported everywhere on $M$. As a special case, the Thom class of a compact oriented isotropic submanifold always admits a symplectic Harmonic representative which is not supported on the whole manifold.

In the other direction, suppose that $N$ is a compact oriented submanifold of a compact symplectic manifold $M$ with even codimension $2p$, and that $T_N$ is the Thom class of $N$. Bahramgiri [Ba06, Thm.1] had a beautiful simple proof that if $[\omega^{n-p}] \wedge T_N \neq 0$, then any symplectic Harmonic representative of $T_N$ must be nowhere vanishing on $M$. (See [TY09, Lemma 4.1] for an account available on arxiv.) Combining this result with Theorem 1.1, we get a rather complete understanding on when a compact oriented submanifold
of $M$ would admit a symplectic Harmonic form that is not supported on the entire manifold.

We had no previous background in the geometric measure theory. We are led to this very beautiful theory naturally by the questions arising from symplectic Harmonic theory. After we completed the first draft of the current paper, we noticed that a homology theory of integral flat chains ([F69 4.4]) has been treated for the the pairs of subsets in $\mathbb{R}^m$. This is certainly closely related to the cohomology theory we develop in the present paper. But as we are mainly interested in applications in symplectic Harmonic theory, our viewpoint is somewhat different. In addition, we also noticed that a dual description for the Riemannian Hodge star operator has been worked out by J. Harrison very elegantly in the general framework called chainlet geometry, cf. [Ha06]. The dual description for the symplectic Hodge operator given in the present paper can be regarded as a symplectic counterpart of the main result established in [Ha06].

The methods developed in the present paper may have a wider area of application. It may be applied to study the other versions of primitive cohomology theory introduced in [TY09] and [TY10]. As we do not need the underlying space to be smooth to have a well developed geometric integration theory, it may also be adapted to study the intersection homology theory of singular symplectic spaces such as Poisson symplectic stratified spaces introduced in [LSV11].

In the past, the symplectic Hodge theory has been studied mostly using algebraic tools such as Lie algebra representations and spectral sequences. The dual description of the symplectic Hodge adjoint operator given in the present paper reveals an unexpected close connection between the symplectic Hodge theory and the geometric measure theory. In a follow-up paper [L11], we will further explore this connection, and discuss possible non-trivial applications of the duality theorem established in the present paper to symplectic topology.

2. Preliminaries

2.1. Review of Symplectic Hodge theory.

In this section we present a brief review of background materials in symplectic Hodge theory. For more details, we refer to [Bry88], [Yan96], [Ba06], [TY09] and [TY10]. Throughout this section, we assume that $(M, \omega)$ is a $2n$ dimensional symplectic manifold.

On the symplectic manifold $(M, \omega)$, the Lefschetz map $L$, the dual Lefschetz map $\Lambda$, and the degree counting map $H$ are defined as follows.
\[ \begin{align*}
L : \Omega^s(M) &\to \Omega^{s+2}(M), \quad \alpha \mapsto \alpha \wedge \omega, \\
\Lambda : \Omega^s(M) &\to \Omega^{s-2}(M), \quad \alpha \mapsto \pi_\ast \alpha, \\
H : \Omega(M) &\to \Omega(M), \quad H(\alpha) = \sum_{k=0}^{n} (n-k) \Pi^k(\alpha),
\end{align*} \]

where \( \pi = \omega^{-1} \) is the canonical Poisson bi-vector associated to \( \omega \), and

\[ \Pi^k : \Omega(M) = \bigoplus_{i=0}^{2n} \Omega^i(M) \to \Omega^k(M) \]

is the projection map.

The actions of \( L, \Lambda \) and \( H \) on \( \Omega(M) \) satisfy the following commutator relations.

\[ \begin{align*}
[\Lambda, L] &= H, \\
[H, \Lambda] &= 2\Lambda, \\
[H, L] &= -2L.
\end{align*} \]

Therefore they define a representation of the Lie algebra \( sl(2) \) on \( \Omega(M) \). Although the \( sl_2 \)-module \( \Omega(M) \) is infinite dimensional, there are only finitely many eigenvalues of \( H \). \( sl_2 \)-modules of this type are studied in great detail in [Ma95] and [Yan96]. Among other things, the following result is proved in [Yan96].

**Lemma 2.1.** Assume that \((M, \omega)\) is a \( 2n \) dimensional symplectic manifold.

1) For any \( 0 \leq r \leq n \), the Lefschetz map

\[ L^{n-r} : \Omega^r(M) \to \Omega^{2n-r}(M), \quad \alpha \mapsto \omega^{n-r} \wedge \alpha \]

is an isomorphism;

2) Let \( \alpha \in \Omega^k(M) \) with \( 0 \leq k \leq n \). Then \( \alpha \) is primitive if and only if

\[ \Lambda \alpha = 0. \]

3) any differential form \( \alpha_k \in \Omega^k(M) \) admits a unique Lefschetz decomposition

\[ \begin{align*}
\alpha_k &= \sum_{r \geq \max\left(\frac{k-n}{2}, 0\right)} \frac{L^r}{r!} \beta_{k-2r},
\end{align*} \]

where \( \beta_{k-2r} \) is a primitive form of degree \( k - 2r \).

Since the symplectic structure \( \omega \) is a non-degenerate two form, using it to identify one forms with one vectors we obtain a non-degenerate bi-linear pairing on the space of one forms. This pairing further extends to a non-degenerate bi-linear pairing \( (\cdot, \cdot) \) on the space of differential \( k \)-forms. In this context, we define the symplectic Hodge star operator \( \ast \) as follows.

\[ \ast \alpha_k \wedge \beta_k = (\alpha_k, \beta_k) \frac{\omega^n}{n!}, \]

\[ (\alpha_k, \beta_k) \quad \text{is the bi-linear pairing on one forms.} \]
where both $\alpha_k$ and $\beta_k$ are differential k-forms. We collect here some useful facts concerning the symplectic Hodge star operator.

**Lemma 2.2.**

1) $\star^2 = \text{id}$.

2) (Weil’s identity) For any $0 \leq k \leq n$, if $\alpha$ is a primitive k-form, then

$$\frac{L^r}{r!} \alpha = (-1)^{k+1} \frac{L^{n-k-r}}{(n-k-r)!} \alpha.$$ 

On the space of differential k-forms, the symplectic Hodge adjoint operator of the exterior differential $d$, is given by

$$d \Lambda \alpha_k = (-1)^{k+1} \star d \star \alpha_k.$$ 

It is straightforward to check that $d$ anti-commutes with $d \Lambda$. In this context, a differential form $\alpha$ is said to be symplectic Harmonic if and only if $d \alpha = d \Lambda \alpha = 0$.

The following commutator relations are important.

$$[d, L] = 0, \quad [d \Lambda, \Lambda] = 0, \quad [d, \Lambda] = d \Lambda,$$

$$[d \Lambda, L] = d, \quad [d^2 \Lambda, L] = 0, \quad [d^2 \Lambda, \Lambda] = 0.$$ 

For any $0 \leq k \leq n$, denote by $P^k(M)$ the space of primitive k-forms on $M$. The following lemma is an easy consequence of the commutator relations given in Equation 2.5.

**Lemma 2.3.** ([TY09, Lemma 2.4]) Let $\alpha \in P^k(M)$ with $0 \leq k \leq n$. The action of the differential operators $(d, d \Lambda, d^2 \Lambda)$ on $\alpha$ has the following form:

1) if $k < n$, then $d \alpha = A_k + LA_k$;

2) $d \Lambda \alpha = -HA_k = -(n-i+1)A_k$;

3) if $k < n$, then $d^2 \Lambda \alpha = -\frac{n-k}{n-k+1} d^2 \Lambda A_k$.

Here $A_k, A_{k+1} \in P^*(M)$ are primitive forms.

**Proof.** The first two assertions are proved in [TY09, Lemma 2.4]. The last assertion follows from the first two, and the commutator relation $[d \Lambda, L] = d$. $\square$

We give a definition of primitive cohomology on a symplectic manifold as follows.

**Definition 2.4.** Let $(M, \omega)$ be a 2n dimensional symplectic manifold. For any $0 \leq r \leq n$, the r-th primitive cohomology group, $PH^r(M)$, is defined as follows.

$$PH^r(M) = \ker(L^{n-r+1} : H^r(M) \to H^{2n-r+2}(M)).$$

On a projective Kähler manifold, the notion of primitive cohomology given in Definition 2.4 is exactly the one used by algebraic geometers, cf. [V07, p. 4]. In fact, when the manifold $M$ satisfies the Hard Lefschetz property, it is not hard to see that the primitive cohomology given in Definition
is naturally isomorphic to the one given in Equation 1.3. For completeness, we provide a proof here using the following symplectic $\dd^\Lambda$-lemma independently established by Merkulov [Mer98] and Guillemin [Gui01].

**Theorem 2.5.** Suppose that $M$ is a compact symplectic manifold with the Hard Lefschetz property. Then we have

\[ \ker d \cap \text{im} \dd^\Lambda = \text{im} d \cap \ker d^\Lambda = \text{im} \dd^\Lambda. \]

The following result is quite useful when one applies the symplectic $\dd^\Lambda$-lemma to primitive differential forms.

**Lemma 2.6.** ([TY10]) Let $\alpha \in P^k(M)$ with $0 \leq k \leq n$. Suppose that there exists a $k$-form $\gamma$ such that $\alpha = \dd^\Lambda \gamma$. Then there exists a primitive $k$-form $\beta$ such that $\alpha = \dd^\Lambda \beta$.

**Proof.** Lefschetz decompose $\gamma$ as follows

\[ \gamma = \beta_k + L\beta_{k-2} + L^2\beta_{k-4} + \cdots. \]

Here $\beta_{k-2i}$ is a primitive differential form of degree $k - 2i$, $i = 0, 1, 2, \cdots$.

Since $[\dd^\Lambda, L] = 0$, we get that

\[ \alpha = \dd^\Lambda \beta_k + L\dd^\Lambda \beta_{k-2} + L^2\dd^\Lambda \beta_{k-4} + \cdots. \]

Since $[\dd^\Lambda, \Lambda] = 0$, the differential operator $\dd^\Lambda$ maps primitive forms to primitive forms. Therefore the right hand side of Equation 2.6 is the Lefschetz decomposition of $\alpha$. Since $\alpha$ itself is a primitive form, it follows from the uniqueness of the Lefschetz decomposition that

\[ \alpha = \dd^\Lambda \beta_k. \]

This completes the proof. \(\square\)

**Proposition 2.7.** Suppose that $M$ is a compact $2n$ dimensional symplectic manifold with the Hard Lefschetz property. Then for any $0 \leq r \leq n$,

\[ \text{PH}^r_d(M) \cong \text{PH}^r(M). \]

**Proof.** First we observe that on any symplectic manifold $M$, for any $0 \leq r \leq n$, there is a natural homomorphism

\[ \text{PH}^r_d(M) \to \text{PH}^r(M), [\alpha]_{\text{PH}_d} \mapsto [\alpha]_{\text{PH}}. \]

Assume that $M$ is compact and satisfies the Hard Lefschetz property. We need to prove that this homomorphism is an isomorphism. When $r = 0$, this is trivially true. We may assume that $r > 0$. Suppose that $\alpha$ is a closed form in $P^r(M)$ such that $[\alpha]_{\text{PH}} = 0$. Since by definition $\alpha$ is $d^\Lambda$-closed, $\alpha$ is both $d$-exact and $d^\Lambda$-closed. It follows from Theorem 2.5 that $\alpha = \dd^\Lambda \gamma$ for some $r$-form $\gamma$. Since $\alpha$ is primitive, by Lemma 2.6 we can assume that $\gamma$ is a primitive $k$-form. Since $[d^\Lambda, \Lambda] = 0$ and since $\gamma$ is a primitive form, $d^\Lambda \gamma$ is also a primitive form. It follows that $[\alpha]_{\text{PH}_d} = 0$. This proves that the homomorphism (2.7) is injective.
If \([\alpha]_{PH} \in PH^r(M)\), then by definition \(L^{n-r+1}\alpha\) represents a trivial cohomology class in \(H^{2n-r+2}(M)\). Thus \(L^{n-r+1}\alpha = d\beta_{2n-r+1}\) for a \((2n-r+1)\)-form \(\beta_{2n-r+1}\). It then follows from Lemma 2.1 that \(\beta_{2n-r+1} = L^{n-r+1}\eta\) for a \((r-1)\)-form \(\eta\). Thus \(L^{n-r+1}(\alpha - d\eta) = 0\). Note that \(\alpha - d\eta\) is both \(d\)-closed and primitive. So it must be \(d^A\)-closed as well. Thus \(\alpha - d\eta\) represents a cohomology class in \(PH^r_d(M)\) whose image under the homomorphism (2.7) is \([\alpha]_{PH}\). This proves that the homomorphism (2.7) is also surjective.

□

It is important to note that if a compact symplectic manifold satisfies the Hard Lefschetz property, then its De Rham cohomology admits a unique Lefschetz decomposition.

**Theorem 2.8.** ([Yan96]) On a compact symplectic manifold \(M\) with the Hard Lefschetz property, we have the following Lefschetz decomposition

\[
H^k(M) = \bigoplus_r L^r PH^{k-2r}(M).
\]

It is well known that on a differential manifold one can apply a smoothing operator to any current and get a smooth differential form. In the proof of [Ba06, Theorem 2], Bahramgiri constructed a symplectic smoothing operator for currents on symplectic manifolds. This is an important construction in symplectic Harmonic theory. We summarize the properties of the symplectic smoothing operator in the following theorem.

**Theorem 2.9.** ([Ba06]) On a symplectic manifold \((M, \omega)\), for any open set \(W \subset M\), there is a symplectic smoothing operator \(S\) satisfies the following properties.

a) If a current \(T\) is of degree \(k\), \(S(T)\) is a smooth differential \(k\)-form;

b) if a current \(T\) is supported inside \(W\), then \(S(T)\) is supported inside \(W\);

c) if a current \(T\) is symplectic Harmonic, then \(S(T)\) is a symplectic Harmonic differential form;

d) if a current \(T\) is primitive, then \(S(T)\) is a primitive differential form.

2.2. Review of the theory of compactly supported currents.

In this section, we present a quick review of the standard theory of compactly supported currents in the context of locally convex topological vector spaces [R73]. We follow closely the exposition given in the classic textbook [DR84].

Throughout this section, we assume that \(M\) is an \(m\) dimensional differential manifold. We denote by \(B_p(M)\) the space of \(C^p\) forms on \(M\), where \(p\) is a finite integer \(\geq 0\) or \(p = \infty\). To simplify the notation, when it is clear from the context what manifold we are referring to, we would write \(B_p\) instead of \(B_p(M)\); and when \(p = \infty\), we simply write \(B\) instead of \(B_\infty\).

For any non-negative integer \(0 \leq i \leq p\), and any compact set \(K\) which lies in an open coordinate neighborhood \(U\), we define a semi-norm \(\| \cdot \|_k\)
on $\mathcal{B}_p$ as follows: using the coordinate system $\{x_1, x_2, \cdots, x_m\}$ on $U$, if the restriction of a $k$-form $\phi$ to $U$ has an expression as

$$\sum_{i_1 < i_2 < \cdots < i_k} f_{i_1 i_2 \cdots i_k} dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k},$$

then

$$\|\phi\|_K^i = \sup \{|D^j f_{i_1 i_2 \cdots i_k}|, 0 \leq |j| \leq i, x \in K\},$$

where $j = (j_1, j_2, \cdots, j_m)$ is a multi-index, $|j| = j_1 + j_2 + \cdots + j_m$, $D^l = D_1^{l_1} \cdots D_m^{l_m}$, and $D_t = \frac{\partial}{\partial x_t}$. The family of all such semi-norms induces a translation invariant Hausdorff topology on $\mathcal{B}_p$, and turns $\mathcal{B}_p$ into a locally convex topological vector space.

Let $\mathcal{D}_p$ be the space of forms in $\mathcal{B}_p$ with compact support. For each compact set $K \subset M$ we define

$$\mathcal{D}_{p,K} = \mathcal{B}_p \cap \{ \phi \in \mathcal{B}_p, \text{supp } \phi \subset K \}$$

and observe that $\mathcal{D}_{p,K}$ is closed in $\mathcal{B}_p$. Note that

$$\mathcal{D}_p = \bigcup_K \{ \mathcal{D}_{p,K}, K \text{ is a compact subset of } M \}.$$ 

We endow $\mathcal{D}_p$ with the largest topology such that the inclusion maps from each subset $\mathcal{D}_K$ is continuous.

From now on, when we refer to $\mathcal{D}_p$ or $\mathcal{B}_p$, we mean the topological vector space $\mathcal{D}_p$ or $\mathcal{B}_p$ with the topology we just described in the previous paragraphs. Without considering the topologies on them, we will use $\Omega(M)$ and $\Omega_c(M)$ to denote the vector spaces of differential forms and differential forms with compact support on $M$ respectively.

We will denote by $\mathcal{D}'_p(M)$ the topological dual of $\mathcal{D}_p(M)$. A continuous $\mathbb{R}$-linear functional in $\mathcal{D}'_p(M)$ is called a current on $M$. We say that a current $T \in \mathcal{D}'_p$ is zero on an open set $V \subset M$ if $T(\phi) = 0$ for any $\phi \in \mathcal{D}_p$ supported inside $V$. It is an elementary fact that there exists a maximal open set in $M$ such that $T$ is zero. The complement of this maximal open set is called the support of $T$.

In this paper, we are mainly interested in compactly supported currents. As explained in [DR84], the space of compactly supported currents in $\mathcal{D}'_p$ can be naturally identified with $\mathcal{B}'_p$, the topological dual of $\mathcal{B}_p$. A set of forms $S \subset \mathcal{B}_p$ is bounded in $\mathcal{B}_p$ if and only if there exists a covering of $M$ by a collection of compact sets $K$ such that each $K$ is contained in a coordinate neighborhood $U$ and such that there exists a positive constant $M^i_K > 0$ that satisfies

$$\|\phi\|_K^i \leq M^i_K, \quad \forall K, \quad \forall 0 \leq i \leq p.$$ 

A set of forms $S \subset \mathcal{D}_p$ is bounded in $\mathcal{D}_p$ if it is bounded in $\mathcal{B}_p$ and if the supports of all the forms in $S$ is contained in a single compact set $K$.

Let $V$ be the topological vector space $\mathcal{D}_p$ or $\mathcal{B}_p$. De Rham introduced the following topology $\mathcal{T}$ on $V'$, the topological dual of $V$. 


Definition 2.10. ([DR84, Sec.10]) For any collection of finitely many bounded subsets \( S_1, S_2, \ldots, S_q \) in \( V \), and any \( r > 0 \), define

\[
W^r_{S_1 S_2 \cdots S_q} = \{ T \in V', \ |T(\phi)| < r, \ \forall \phi \in \cap_{i=1}^q S_i \}. 
\]

The topology \( \mathcal{I} \) on \( V' \) is the collection of all unions of sets of the form

\[
f + W^r_{S_1 S_2 \cdots S_q}, \ f \in V'.
\]

It is straightforward to check that with the above topology both \( \mathcal{B}'_p \) and \( \mathcal{D}'_p \) are locally convex topological vector spaces. The following result is an immediate consequence of Definition 2.10.

Proposition 2.11. ([DR84, Sec. 10]) Let \( \{T_k\}_{k=1}^{\infty} \) be a sequence in \( \mathcal{B}'_p \), and \( T \) a compactly supported current in \( \mathcal{B}'_p \). Then the sequence \( \{T_k\} \) converges to \( T \) in \( \mathcal{B}'_p \) if and only if it converges to \( T \) uniformly on any bounded subset in \( \mathcal{B}_p \).

Observe that for any non-negative integers \( p \), the natural inclusion map \( i : \mathcal{B} \hookrightarrow \mathcal{B}_p \) is continuous; moreover, its image is dense in \( \mathcal{B}_p \). A linear functional in \( \mathcal{B}' \) is said to be continuous to order \( p \) if it is continuous with respect to the topology on \( \mathcal{B} \) inherited from \( \mathcal{B}_p \) as a subspace. Since the image of \( i \) is dense in \( \mathcal{B}_p \), the dual map \( i^* : \mathcal{B}_p' \to \mathcal{B}' \) is also injective. It is proved in [DR84, Sec. 10] the image of \( i^* \) in \( \mathcal{B}' \) coincides with the space of all functionals in \( \mathcal{B}' \) continuous to order \( p \). In other words, \( \mathcal{B}_p' \) can be naturally identified with linear functionals in \( \mathcal{B}' \) continuous to order \( p \).

We say that a compactly supported current \( T \) in \( \mathcal{B}_p' \) is of degree \( k \) if \( T(\phi) = 0 \) for any \( \phi \in \Omega(M) \) whose degree is not \( k \). For a \( k \) dimensional compactly supported current \( T_i \), its degree is defined to be \( m - k \). We denote by \( \mathcal{B}_p^{r k} \) the space of all compactly supported currents in \( \mathcal{B}_p' \) of degree \( k \).

Given a \( q \) dimensional current \( T \) in \( \mathcal{B}_p' \), its boundary is by definition a \( q - 1 \) dimensional current given by

\[
\partial T(\phi) = T(d\phi), \ \forall \phi \in \mathcal{B}^{q-1},
\]

and its differential is defined by

\[
dT = (-1)^{q+1} \partial T.
\]

Note that the exterior differential \( d : \mathcal{B} \to \mathcal{B} \) is a continuous linear mapping. Therefore both \( \partial T \) and \( dT \) are well defined compactly supported currents in \( \mathcal{B}' \). Since \( d^2 = \partial^2 = 0 \), we have a differential complex

\[
0 \to \mathcal{B}^0 \xrightarrow{d} \mathcal{B}^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{B}^m \to 0.
\]

The \( i \)-th compactly supported distributional De Rham cohomology \( H_{\mathcal{B}}^{i-\infty}(M) \) is defined to be the \( i \)-th cohomology of the differential complex (2.13).

Let \( f \) be a smooth map from a manifold \( X \) into a manifold \( Y \). Then the pullback map \( f^* : \mathcal{B}_p(Y) \to \mathcal{B}_p(X) \) is a continuous linear mapping. Thus it induces a pushforward map from \( \mathcal{B}'_p(X) \) to \( \mathcal{B}'_p(Y) \) in the following way.

\[
f_* : \mathcal{B}'_p(X) \to \mathcal{B}'_p(Y), \ f_*(T)(\phi) = T(f^* \phi), \ \forall \phi \in \mathcal{B}_p(Y).
\]
It is straightforward to check that this pushforward map is a continuous map; moreover, it commutes with the boundary map, and sign commutes with the differential $d$. More precisely, let $\dim X = m$ and $\dim Y = n$, then we have that

$$
\partial f_\ast(T) = f_\ast(\partial T), \quad df_\ast(T) = (-1)^{m+n}f_\ast dT, \quad \forall T \in \mathcal{B}_p'(X).
$$

In this paper, we are particularly interested in the following situation. Let $U$ be an open set of a manifold $M$. Then the inclusion map $i : U \hookrightarrow M$ induces a pushforward map

$$
i_\ast : \mathcal{B}_p'(U) \to \mathcal{B}_p'(M).
$$

Using Lemma 2.11 and a cutoff function, it is easy to show the following result.

**Lemma 2.12.** The map (2.14) is an injective map. Moreover, suppose that $\{T_k\}$ is a sequence in $\mathcal{B}_p'(U)$, and that $T$ is a compactly supported current in $\mathcal{B}_p'(U)$. Then $\{T_k\}$ converges to $T$ in $\mathcal{B}_p'(U)$ if and only if $\{i_\ast(T_k)\}$ converges to $i_\ast(T)$ in $\mathcal{B}_p'(M)$.

**Remark 2.13.** By abuse of notations, in this paper we usually do not distinguish a compact supported current $T \in \mathcal{B}_p'(U)$ from its image under the pushforward map $i_\ast(T) \in \mathcal{B}_p'(M)$.

Now suppose that $M$ is an $m$ dimensional manifold covered by two open sets $U$ and $V$. Then sequences of inclusions

$$
M \leftarrow U \sqcup V \leftarrow U \cap V
$$

give rise to sequences of pushforward maps

$$
\mathcal{B}^i(M) \leftarrow \sum \mathcal{B}^i(U) \oplus \mathcal{B}^i(V) \leftarrow \mathcal{B}^i(U \cap V) \leftarrow (-i_\ast(T), i_\ast(T)) \leftarrow T
$$

This gives us a Mayer-Vietoris sequence

$$
0 \leftarrow \mathcal{B}^i(M) \leftarrow \mathcal{B}^i(U) \oplus \mathcal{B}^i(V) \leftarrow \mathcal{B}^i(U \cap V) \leftarrow 0.
$$

**Proposition 2.14.** The Mayer-Vietoris sequence (2.15) of compactly supported currents is exact.

**Proof.** The argument given in [BT82, Prop. 2.7]for the exactness of the Mayer-Vietoris sequence of forms with compactly support extends to the present situation.

Therefore we have a long exact sequence

$$
\cdots \leftarrow H_c^{-\infty, i}(U \cup V) \leftarrow H_c^{-\infty, i}(U) \oplus H_c^{-\infty, i}(V) \leftarrow H_c^{-\infty, i}(U \cap V) \leftarrow H_c^{-\infty, i-1}(U \cup V) \leftarrow \cdots
$$
Let $\Omega_c(M)$ be the space of compactly supported forms on $M$. Any compactly supported form $\alpha \in \Omega_c(M)$ defines a compactly supported current in $\mathcal{B}'(M)$ as follows.

$$\mathcal{B}(M) \to \mathbb{R}, \quad \phi \mapsto \int_M \alpha \wedge \phi.$$ 

Therefore there is a natural chain homomorphism

(2.17) \hspace{1cm} \Omega_c(M) \to \mathcal{B}'(M).

The following results are standard and we refer to [DR84] and [Mel01] for detailed proofs.

**Theorem 2.15.** The homomorphism (2.17) induces an isomorphism.

$$H^*_c(M) \cong H^{-\infty,*}_c(M).$$

**Theorem 2.16.**

$$H^{\infty,i}_{c}(\mathbb{R}^m) = \begin{cases} 0, & \text{if } 0 \leq i < m \\ \mathbb{R}, & \text{if } i = m. \end{cases}$$

**Theorem 2.17.** On a connected manifold every closed current of degree zero is equal to a constant.

**Lemma 2.18.** ([Mel01, Lemma 3]) In $\mathbb{R}^m$ the complex of currents with support at the origin has homology which is one-dimensional and is in dimension $m$.

### 3. Polyhedral Chains and Flat Chains in $\mathbb{R}^m$

In this section, we collect some basic facts on polyhedral and flat chains in $\mathbb{R}^m$, and refer to [WH57] and [F69] for more background materials. We also prove a few technical results that we need later in the paper. In particular, for a sequence of polyhedral chains, there are two notions of convergence which are important to us: the convergence as a sequence of currents introduced by De Rham [DR84] as we explained in Section 2.2 and the convergence with respect to the flat norm introduced by Whitney [WH57]. We prove that in an Euclidean space these two notions are essentially equivalent to each other.

An oriented $p$ simplex $\sigma = [v_0, v_1, \ldots, v_p]$ in $\mathbb{R}^m$ is the convex hull of $p + 1$ geometrically independent points $\{v_0, v_1, \ldots, v_p\}$, i.e.,

$$\sigma = \{x_0v_0 + x_1v_1 + \cdots + x_pv_p \in \mathbb{R}^m, \quad x_0 + x_1 + \cdots + x_p = 1, \quad x_i \geq 0, \quad i = 0, 1, \ldots, p\},$$

together with an orientation induced by the multi-vector

$$(v_1 - v_0) \wedge (v_2 - v_0) \wedge \cdots \wedge (v_p - v_0).$$

In this paper, we use the standard Euclidean metric structure on $\mathbb{R}^m$ implicitly. When we refer to distance in $\mathbb{R}^m$, we mean distance with respect to the standard Euclidean metric. For any set $A$ in $\mathbb{R}^m$, the diameter of $A$, denoted by $\text{diam}(A)$, is defined to be the supermum of the distance
between any two points in $A$. Let $D^n(\sigma)$ be the collection of all oriented $p$-simplices in the $n$-th barycentric subdivision of a $p$-simplex $\sigma$. Then for any $\tau \in D^n(\sigma)$, we have
\[
diam(\tau) \leq \left(\frac{p}{p+1}\right)^n diam(\sigma).
\]

More importantly, the Euclidean metric structure determines a Hausdorff measure $\mathcal{H}^k$ on any $k$ dimensional subset in $\mathbb{R}^m$. Let $\sigma$ be a $p$-simplex in $\mathbb{R}^m$. For any continuous function $f$ on $\mathbb{R}^m$, we define the integration of $f$ on $\sigma$ to be
\[
\int_{\sigma} f := \int_{\sigma} f d\mathcal{H}^p.
\]

In particular, we define the volume of the $p$-simplex $\sigma$ to be
\[
(3.1) \quad \text{vol}(\sigma) = \mathcal{H}^p(\sigma).
\]

Note that for any $0 \leq p \leq m$, a $p$-simplex $\sigma$ determines a unique $p$ dimensional affine plane $E_\sigma$ which can be naturally identified with $\mathbb{R}^p$ by a translation. Under this identification, the Hausdorff measure gets identified with the Lebesgue measure $L^p$ on $\mathbb{R}^p$. In calculations made in this paper, we will sometimes use this identification.

Note that the Euclidean inner product structure determines two Euclidean volume form on a $p$-simplex $\sigma$ which differ from each other only by a sign. Now suppose that $\sigma$ is an oriented $p$-simplex. Then the orientation of $\sigma$ determines on it a unique Euclidean volume form $\gamma_\sigma$. For a Borel measurable differential $p$-form in $\mathbb{R}^m$, its restriction to $E_\sigma$ admits a representation as $\alpha = f\gamma_\sigma$, where $f$ is a Borel measurable function on $E_\sigma$. We define the integration of $\alpha$ over $\sigma$ by
\[
\int_{\sigma} \alpha := \int_{\sigma} f.
\]

Let $\sigma$ be an oriented $k$-simplex in $\mathbb{R}^m$. It defines a canonical $k$-dimensional compactly supported current in $\mathcal{B}_1$ as follows.
\[
\mathcal{B}_1 \to \mathbb{R}, \quad \phi \mapsto \int_{\sigma} \phi.
\]

**Definition 3.1.** A compactly supported current $T$ in $\mathcal{B}_1(\mathbb{R}^m)$ is called a $k$ dimensional real polyhedral chain in $\mathbb{R}^m$ if it admits a representation as a finite linear combination of $k$-simplices in $\mathbb{R}^m$. That is to say, there exist real scalars $a_1, \cdots, a_l$ and $k$-simplices $\sigma_1, \cdots, \sigma_l$ such that
\[
(3.2) \quad T(\phi) = \sum_{i=1}^{l} a_i \int_{\sigma_i} \phi, \quad \forall \phi \in \mathcal{B}_1(\mathbb{R}^m).
\]
It is important to note that by definition a polyhedral chain may have infinitely many different representations. For instance, let $D(\sigma)$ be the collection of all $k$ dimensional simplices in a simplicial subdivision of a $k$-simplex $\sigma$, and let any simplex $\tau$ in $D(\sigma)$ be equipped with an orientation induced by that of $\sigma$. Then we have that

$$\int_{\sigma} \phi = \sum_{\tau \in D(\sigma)} \int_{\tau} \phi, \; \forall \phi \in \mathcal{B}_1.$$ 

On the space of polyhedral chains, there are two important norms: the mass norm and the Whitney’s flat norm. We first observe that in Equation 3.2 we can always assume that any two distinct simplices $\sigma_i$ and $\sigma_j$ intersect each other along a common proper subface. This is due to the following elementary fact on simplicial complexes in $\mathbb{R}^m$.

**Lemma 3.2.** ([M66, Theorem 7.10]) Let $L_1$ and $L_2$ be two finite simplicial complexes in $\mathbb{R}^m$. There are simplicial subdivisions $L_1'$ and $L_2'$ of $L_1$ and $L_2$ respectively, such that $L_1' \cup L_2'$ is a simplicial complex.

We define the mass norm and the flat norm as follows.

**Definition 3.3.** The mass norm of a polyhedral $p$-chain $T = \sum_i a_i \sigma_i$ is defined as

$$M(T) = \sum_i |a_i| \text{vol}(\sigma_i).$$

Here we assume that any two distinct simplices $\sigma_i$ and $\sigma_j$ intersect each other along a common proper subface.

**Definition 3.4.** The flat norm of a polyhedral $p$-chain $T = \sum_i a_i \sigma_i$ in $\mathbb{R}^m$ is defined as

$$(3.3) \quad \|T\|_b = \inf \{M(T - \partial \tau) + M(\tau)\},$$

where the infimum is taken over all polyhedral $(p+1)$-chains $\tau$ in $\mathbb{R}^m$.

More generally, given any open set $U \subset \mathbb{R}^m$ that contains the support of the polyhedral chain $T$, if we require the infimum in the equation (3.3) to be taken over all polyhedral $(p+1)$-chains $\tau$ in $U$, then we get the flat norm $\|T\|_{b,U}$ of $T$ in $U$. In general, this flat norm depends on the region $U$. However, when $U$ is a convex open set in $\mathbb{R}^m$, it is shown in [WH57, Ch. VIII] that

$$\|T\|_{b,U} = \|T\|_b.$$ 

In this paper, we are going to use this fact in the following case. Given two balls $B_1 \subset B$, then $\|T\|_{b,B_1} = \|T\|_{b,B}$. To simplify the notation, for any open ball $B$ in $\mathbb{R}^m$, we will simply denote by $\|T\|_b$ the flat norm of $T$ in $B$.

**Definition 3.5.** Let $B$ be an open ball in $\mathbb{R}^m$. The vector space of polyhedral $p$-chains equipped with the flat norm is called the space of polyhedral $p$-chains in $B$. 
and is denoted by $\mathcal{P}_k(B)$. The completion of this normed space is the Banach space of real flat $p$-chains in $B$, denoted by $\mathcal{F}_k(B)$.

The following result is an immediate consequence of Definition 3.4.

**Lemma 3.6.** ([WH57]) For any polyhedral chain $T$, we have that $\|T\|_b \leq M(T)$ and that $\|\partial T\|_b \leq \|T\|_b$.

Let $\sigma = [v_0, v_1, \ldots, v_p]$ be an oriented $p$-simplex in $\mathbb{R}^m$, and $b$ a point in $\mathbb{R}^m$. We say that $b$ and $\sigma$ are geometrically independent if $\{b, v_0, \ldots, v_p\}$ is a set of $p + 2$ many geometrically independent points. We denote by $b \ast \sigma$ the oriented $(p + 1)$-simplex $[b, v_0, v_1, \ldots, v_p]$. More generally, let $T = \sum_{i=1}^{k} a_i \sigma_i$ be a polyhedral $p$-chain in $\mathbb{R}^m$, where for each $i$, $\sigma_i$ is an oriented $p$-simplex. We say that $b$ and $T$ are geometrically independent if $b$ and $\sigma_i$ are geometrically independent for any $1 \leq i \leq k$. If $b$ and $T$ are geometrically independent, we define the polyhedral $(p + 1)$-chain $b \ast T$ to be $b \ast T = \sum_{i} a_i (b \ast \sigma_i)$.

The following result is elementary. We refer to [Ad04, Lemma 6.2] for a proof.

**Lemma 3.7.** Let $B$ be an open ball in $\mathbb{R}^m$, and $T = \sum_{i=1}^{k} a_i \sigma_i$ a polyhedral $p$-chain in $B$ with $0 \leq p < m$. Suppose that $b$ is a point in $B$ such that $b$ and $\sigma_i$ are geometrically independent for any $1 \leq i \leq k$. Then there exists a constant $L > 0$ that depends only on the diameter of $B$, such that $\|b \ast T\|_b \leq L \cdot \|T\|_b$.

For any $0 \leq k \leq m$, The Euclidean inner product on $\mathbb{R}^m$ induces a norm on $\wedge_k \mathbb{R}^m$ which we denote by $\| \cdot \|$. A $k$-vector $w \in \wedge_k \mathbb{R}^m$ is called simple if $w = v_1 \wedge \cdots \wedge v_k$ for some collection of vectors $\{v_1, \cdots, v_k\} \subset \mathbb{R}^m$. The comass of a $k$-form $\varphi \in \wedge^k \mathbb{R}^m$ is defined as

$$\|\varphi\| := \sup\{\langle \varphi, w \rangle, \ w \in \wedge_k \mathbb{R}^m \text{ is simple and } |w| \leq 1\}.$$

We now describe a fundamental duality theorem [WH57, P. VIII] in geometric integration theory which was first proved by J. H. Wolfe in his 1948 Harvard thesis. (See also [Hei05] for a modern treatment.) It turns out to be particularly useful for our paper.

Throughout the rest of this section, we assume that $B$ is an open ball in $\mathbb{R}^m$, and $W$ is an arbitrary open set in $\mathbb{R}^m$. A $p$-form $\alpha$ on $W$ is called flat if both $\alpha$ and $d\alpha$ lie in $L^\infty(W)$, where $d\alpha$ is in the sense of currents.
The vector space of flat p-forms on $W$ is denoted by $\tilde{F}^k(W)$. It is a Banach space under the flat norm

\begin{equation}
\|\alpha\|_b = \max(\|\alpha\|_\infty, \|d\alpha\|_\infty),
\end{equation}

where the $L^\infty$-norm $\|\alpha\|_\infty$ for a form $\alpha$ stands for the $L^\infty$-norm of the pointwise comass as we described in Equation \ref{eq:comass}.

\[\|\alpha\|_\infty = \text{ess supp}(\|\alpha(x)\|, \alpha(x) \in \wedge^n \mathbb{R}^m, x \in W).\]

Given a polyhedral chain $T$ in $W$, we will also use the following notation in this paper.

\begin{equation}
\|\phi\|_\infty := \text{ess supp}(\|\alpha(x)\|, \alpha(x) \in \wedge^n \mathbb{R}^m, x \in \text{supp } T).
\end{equation}

A simple calculation shows that the following result is true. It is elementary but quite useful for our paper.

**Lemma 3.8.** Let $W$ be an open set in $\mathbb{R}^m$. Let $T = \sum_i a_i \sigma_i$ be a real polyhedral $k$-chain in $W$, and $\phi$ a flat $k$-form in $\tilde{F}^k(W)$. Then we have that

\[|T(\phi)| \leq M(T) \cdot \|\phi\|_\infty^T.\]

Let $\Omega^k_c(W)$ be the space of smooth compactly supported $k$-forms in $W \subset \mathbb{R}^m$. We equip $\Omega^k_c(W)$ with the flat norm \ref{eq:flat_norm}, and denote the resulting normed space by $F^k(W)$. Note that the topology on $F^k(W)$ induced by the norm is weaker than the topology on $\Omega^k(W)$ that we discussed in Section \ref{sec:smooth_forms}. The dual space of this normed space is a Banach space when normed by the dual norm. We denote the dual space by

\[F_k(W) = \left(\Omega^k_c(W)\right)'.\]

It is clear that there is a canonical embedding

\[P_k(W) \subset F_k(W),\]

which extends to an embedding

\begin{equation}
\mathcal{F}_k(W) \hookrightarrow F_k(W).
\end{equation}

We state the fundamental duality theorem in the following form.

**Theorem 3.9.** (\cite{Hei05}) If $W = B$ is an open ball in $\mathbb{R}^m$, then the embedding \ref{eq:embedding} is isometric. As a result, the space $\tilde{F}^k(B)$ of flat $k$-forms is the Banach space dual of the space $\mathcal{F}_k(B)$ of flat $k$-chains.

We next discuss a canonical embedding of $\mathcal{B}'_1(W)$ into $F(W)$.

**Lemma 3.10.** There is a canonical embedding

\begin{equation}
\mathcal{B}'_1(W) \hookrightarrow F(W).
\end{equation}
Proof. Suppose that $T$ is a compactly supported current in $\mathcal{B}'_1$. Set

$$S = \{\phi \in \mathcal{D}, \|\phi\|_b \leq 1\}.$$ 

It is easy to see that $S$ is a bounded subset in $\mathcal{B}_1(W)$. Let $K$ be any compact subset of $W$. Then there exists a constant $C$ such that

$$|T(\phi)| \leq C\|\phi\|_K^1 \leq C\|\phi\|_b \leq C, \forall \phi \in S.$$ 

Here $\|\cdot\|^1_K$ is the semi-norm we defined in Equation 2.9. This proves that $T$ is a continuous linear functional in $F(W)$.

$\square$

We emphasize that the following result does not depend on the fundamental duality theorem.

**Proposition 3.11.** Let $\{T_k\}$ be a sequence in $\mathcal{B}'_1(W)$, and $T$ a compactly supported current in $\mathcal{B}'_1(W)$. Identify $\mathcal{B}'_1(W)$ with a subspace in $F(W)$ using the canonical embedding (3.11). Assume that all the supports of $T_k$ are contained in a single compact set. Then $\{T_k\}$ converges to $T$ in $\mathcal{B}'_1(W)$ if and only if $\{T_k\}$ converges to $T$ with respect to the flat norm on $F(W)$.

**Proof.** Since the topology of compactly supported forms is induced by the flat norm, the topology given in Definition 2.10, the ”only if” direction is trivial. We shall only prove the ”if” direction.

Assume that $\{T_k\}$ converges to $T$ with respect to the flat norm. We show that it converges to $T$ in $\mathcal{B}'_1(W)$. Fix a compact set $K$ in $W$ such that all the supports of $T_k$ and $T$ are contained in $K$. Choose two open sets $U$ and $V$ in $W$ with $K \subset U \subset \overline{U} \subset V$, and $K_1 := \overline{V} \subset W$ being compact.

Let $\rho$ be a smooth cutoff function such that $\rho = 1$ on $U$ and $\rho = 0$ outside $V$. Let $S$ be a bounded subset in $\mathcal{B}_1(W)$. Set

$$S_\rho = \{\rho \phi, \phi \in S\}.$$ 

Observe that $\forall \phi \in S$, $\|\rho \phi\|_K^1 = \|\rho \phi\|_b$. It follows that $S_\rho$ is a bounded subset in $F(W)$ with respect to the flat norm. In particular, there exists a constant $R > 0$ such that

$$\|\rho \phi\|_b < R, \forall \phi \in S.$$ 

Since $\{T_k\}$ converges to $T$ with respect to the flat norm, $\forall \epsilon > 0$, there exists $N > 0$, such that

$$\|T_k - T\|_b < \frac{\epsilon}{R}, \forall k \geq N.$$ 

As a result, we have that

$$|T_k(\rho \phi) - T(\rho \phi)| \leq \|T_k - T\|_b \cdot \|\rho \phi\|_b < \epsilon, \forall k \geq N, \forall \phi \in S.$$ 

Note that for any $\phi \in S$, $(1 - \rho)\phi$ is supported outside $K$. It follows that

$$T_k(\rho \phi) = T_k(\phi), \forall k \geq 1, \text{ and } T(\rho \phi) = T(\phi).$$
Therefore we have that
\[ |T_k(\phi) - T(\phi)| < \varepsilon, \forall k \geq N, \forall \phi \in S. \]
This completes the proof. \(\square\)

An argument similar to the proof of Proposition 3.11 gives us Lemma 3.12.

**Lemma 3.12.** The image of the embedding map (3.8) consists of compactly supported continuous linear functionals in \(F(W)\).

The following result is an immediate consequence of Theorem 3.9 and Proposition 3.11.

**Theorem 3.13.** Let \(B\) be an open ball in \(\mathbb{R}^m\), and \(\{T_k\}_{k=1}^{\infty}\) a sequence of polyhedral chains in \(B\). Assume that the supports of all polyhedral chains \(\{T_k\}\) are contained in a single compact set. Then the sequence \(\{T_k\}\) converges to a compactly supported current \(T\) in \(\mathcal{B}'_1(B)\) if and only if it converges to \(T\) with respect to the flat norm (3.4).

By Lemma 3.12 and Theorem 3.9, the space of compactly supported flat chains in \(B\) embeds into \(\mathcal{B}'_1(B)\). The next result asserts that its image is a closed subspace in \(\mathcal{B}'_1(B)\). It is an easy consequence of Theorem 3.13.

**Corollary 3.14.** Suppose that \(B\) is an open ball in \(\mathbb{R}^m\), that \(\{T_k\}\) is a sequence of flat chains in \(F(B)\), and that \(T\) is a compactly supported current in \(\mathcal{B}'_1(B)\). If \(\{T_k\}\) converges to \(T\) in \(\mathcal{B}'_1(B)\), then \(T \in F(B)\).

The following technical lemma will play an important role in Section 4.3. It can be derived from the general discussions in [WH57, Ch. VII] on the product between a sharp function and a sharp chain. However, since the case we need is rather elementary, we present a direct proof below.

**Lemma 3.15.** Let \(W\) be an open set in \(\mathbb{R}^m\), \(\sigma\) a \(p\)-simplex in \(W\), and \(\rho\) a \(C^1\) function on \(W\). Then there exists a sequence of polyhedral chains \(\{T_k\}\) in \(W\) such that \(\lim_{k \to \infty} T_k = \rho \cdot \sigma\) in \(\mathcal{B}'_1\) and such that
\[
\text{supp } T_k \subset (\text{supp } \sigma) \cap (\text{supp } \rho).
\]

**Proof.** Let \(D^k(\sigma)\) be the collection of all \(p\)-simplices in the \(k\)-th barycentric subdivision of \(\sigma\). For any \(\tau \in D^k(\sigma)\), denote by \(b_\tau\) the barycenter of \(\tau\), and equip \(\tau\) with an orientation induced by that of \(\sigma\). Define
\[
T_k = \sum_{\tau \in D^k(\sigma)} \rho(b_\tau) \cdot \tau.
\]
By construction, all the supports of \(T_k\) are contained in the single compact set \(\text{supp } \sigma\). To prove Lemma 3.15, by Proposition 3.11 it suffices to show that that \(\{T_k\}\) converges to \(\rho \cdot \sigma\) with respect to the flat norm.
Now observe that for any $k$-form $\phi \in F^k(W)$, we have that
\[
|T_k(\phi) - \rho \cdot \sigma(\phi)| = \left| \sum_{\tau \in D^k(\sigma)} \rho(\beta_\tau) \tau(\phi) - \sum_{\tau \in D^k(\sigma)} \tau(\phi) \right|
\leq \sum_{\tau \in D^k(\sigma)} \left| \int_{\tau} (\rho(\beta_\tau) \phi - \rho \phi) \right|
\leq \sum_{\tau \in D^k(\sigma)} \M(\tau) \| (\rho - \rho(\beta_\tau)) \phi \|_\sigma^\infty \quad (\text{By Lemma } 3.8)
\leq \sum_{\tau \in D^k(\sigma)} \vol(\tau) \diam(\tau) \|\rho\|_b \|\phi\|_b
\leq \vol(\sigma) \left( \frac{p}{p+1} \right)^k \diam(\sigma) \|\rho\|_b \|\phi\|_b.
\]
It follows immediately
\[
\|T_k - \rho \cdot \sigma\|_b \leq \vol(\sigma) \left( \frac{p}{p+1} \right)^k \diam(\sigma) \|\rho\|_b \|\phi\|_b.
\]
Letting $k \to \infty$, we get that $\lim_{k \to \infty} \|T_k - \rho \cdot \sigma\|_b = 0$.

Let $T$ be a compactly supported flat chain in an open ball $B$. By definition, there exists a sequence of polyhedral chains $\{T_k\}$ that converges to $T$ with respect to the flat norm in $B$. The following result asserts that this sequence of polyhedral chains can be chosen such that all the supports of $\{T_k\}$ are contained in a single compact set in $B$.

**Lemma 3.16.** Let $B$ be an open ball in $\R^m$, and $T$ a compactly supported real flat chain in $F(B)$. Then there exists a sequence of polyhedral chains $\{T_k\}$ that converges to $T$ in $W^1_1(B)$.

**Proof.** Let $K$ be the compact support of $T$. Choose two balls $B_1$ and $B_2$ concentric with $B$ such that $K \subset B_1 \subset B_2 \subset B$. Choose a smooth cutoff function $\rho$ such that $\rho = 1$ on $B_1$ and $\rho = 0$ outside $B_2$. By construction, we have that $\rho \cdot T = T$. By definition of flat chains, there is a sequence of polyhedral chains $\{T_k\}$ that converges to $T$ with respect to the flat norm. By Lemma 3.15, each $\rho \cdot T_k$ is a flat chain. Using Theorem 3.9, it is easy to see that $\{\rho \cdot T_k\}$ converges to $T = \rho \cdot T$ with respect to the flat norm.

For any $k \geq 1$, by Lemma 3.15 there exists a polyhedral chain $Q_k$ such that
\[
\|Q_k - \rho \cdot T_k\|_b < \frac{1}{k^2}, \text{ and such that } \supp Q_k \subset (\supp \sigma) \cap (\supp \rho).
\]
It is clear that $\{Q_k\}$ converges to $T$ with respect to the flat norm; moreover, all the supports of $Q_k$ are contained in a single compact set $B_2$ in $B$. Lemma 3.16 now follows easily from Theorem 3.13.

\[\square\]
It is important to note that the invariance of flat chains under Lipschitz maps. Let \( U \) be an open set in \( \mathbb{R}^n \), and \( X \) a subset of \( X \). A function \( f : X \rightarrow \mathbb{R}^m, X \subset \mathbb{R}^n \), is said to be a \( L \)-Lipschitz from \( X \) to \( \mathbb{R}^m \) if there exists a constant \( L \geq 0 \) such that

\[
|f(x) - f(y)| \leq L|x - y|, \quad \forall x, y \in A.
\]

In particular, if \( f \) is a smooth map from an open set in \( \mathbb{R}^n \) to an open set \( \mathbb{R}^m \), and if \( X \) is a compact subset of \( U \), then \( f : X \rightarrow V \) is a Lipschitz map. We refer to \([WH57, \text{Ch. X}]\) for a proof of the following result.

**Theorem 3.17.** Let \( f \) be an \( L \)-Lipschitz map from an open set \( U \subset \mathbb{R}^n \) into an open set \( W \subset \mathbb{R}^m \). Then for any polyhedral \( k \)-chain \( A \in \mathcal{F}(U) \), and any open set \( V \subset \mathbb{R}^m \) with \( f(\text{supp } A) \subset V \subset W \), and \( V \) being compact, there exists a sequence of polyhedral chains \( \{A_n\} \) in \( V \) that converges to the Lipschitz chain \( f(A) \) with respect to the flat norm. Moreover, if \( n = m \), and if \( f \) is a bi-Lipschitz map from \( U \) to \( V \), then \( f(A) \in \mathcal{F}_k(\mathbb{R}^n) \).

We conclude this section with the following result, which will be used in the proof of Theorem 4.9.

**Proposition 3.18.** Let \( B \) be an open ball in \( \mathbb{R}^m \), \( \{T_n\}_{n=1}^\infty \) a sequence of polyhedral \( p \)-chains with \( 0 \leq p < m \) that converges in \( \mathcal{B}_1 \). Suppose that \( b \) is a point in \( B \) such that \( b \) and \( T_n \) are geometrically independent for any \( n \geq 1 \). Then \( \{b \ast T_n\}_{n=1}^\infty \) converges to a \( p + 1 \) dimensional real flat chain \( Q \) in \( \mathcal{B}_1 \). Moreover, if \( \{T_n\} \) converges to zero in \( \mathcal{B}_1 \), then \( Q = 0 \).

**Proof.** Since \( \{T_n\} \) converges to \( T \) in \( \mathcal{B}_1 \), the supports of all polyhedral chains \( T_n \) are contained in a single compact set. It follows that the supports of all the polyhedral chains \( b \ast T_n \) are contained in a single compact set. By Theorem 3.13 to prove the first assertion it suffices to show that \( \forall \epsilon > 0 \), there exists an integer \( N > 0 \) such that

\[
||b \ast T_i - b \ast T_j||_b < \epsilon, \quad \forall i, j \geq N.
\]

This follows immediately from Lemma 3.7. The second assertion can be proved similarly.

\[\square\]

### 4. Real Flat Chains on Symplectic Manifolds

#### 4.1. The De Rham Complex of Flat Chains on Symplectic Manifolds

Throughout this section, we assume that \((M, \omega)\) is a \(2n\) dimensional symplectic manifold. A symplectically embedded \( p \)-chain element, or simply a \( p \)-chain element \( T \) in \( M \), is by definition a \( p \) dimensional compactly supported current in \( \mathcal{B}_1(M) \) which admits a representation

\[
T(\phi) = \int_\sigma (h^{-1})^* \phi, \quad \forall \phi \in \mathcal{B}_1,
\]
where $\sigma$ is a $p$-simplex in $\mathbb{R}^{2n}$, and $h$ is a symplectomorphism from an open set $U$ in $M$ onto an open neighborhood $O$ of $\sigma$ in $\mathbb{R}^{2n}$ with the standard symplectic structure. We will denote a $p$-chain element $T$ by $[\sigma, h]$.

**Definition 4.1.** A finite $p$-chain $T$ in a symplectic manifold $M$ is a $p$-dimensional compactly supported current in $\mathcal{B}_1^p(M)$ which admits a representation as a finite linear combination of $p$-chain elements in $M$,

$$T = \sum_{i=1}^{k} a_i[(\sigma_i, h_i)],$$

where for any $1 \leq i \leq k$, $a_i$ is a real number, $\sigma_i$ is a $p$-simplex in $\mathbb{R}^{2n}$, and $h_i$ is a symplectomorphism from an open set $U_i$ in $M$ onto an open neighborhood $O_i$ of $\sigma_i$ in $\mathbb{R}^{2n}$ with the standard symplectic structure. We will denote by $A_p(M)$ the space of all finite $p$-chains in $M$.

When a compactly supported current $T$ is a finite $p$-chain in $M$, for any $p$-form $\phi \in \mathcal{B}_1$, we would also write $T(\phi)$ as $\int_T \phi$.

Let $M$ be a $2n$ dimensional symplectic manifold, $[\sigma, h]$ a $p$-chain element. We say that $[\sigma, h]$ is a co-isotropic chain element if the interior of $\sigma$ is a co-isotropic submanifold of $\mathbb{R}^{2n}$ with the standard symplectic structure. A finite $p$-chain $T$ is called a co-isotropic finite $p$-chain if it admits a representation as a finite linear combination of co-isotropic $p$-chain elements. The notion of a finite isotropic chain can be defined in a similar fashion.

**Example 4.2.** Let $B$ be an open symplectic ball in $\mathbb{R}^{2n}$ with the standard symplectic structure $\omega_0$, and $T = \sum_{i=1}^{l} a_i \sigma_i$ a $k$-dimensional polyhedral chain in $B$. Then $T$ must be a finite chain in $B$. Moreover, $T$ is co-isotropic if and only if for any $0 \leq i \leq l$, the interior of $\sigma_i$ is a co-isotropic submanifold of $\mathbb{R}^{2n}$.

Let $[[v_0, v_1, \ldots, v_p], h]$ be a $p$-chain element in $M$. Its boundary is given as follows.

$$\partial[[v_0, v_1, \ldots, v_p], h]] = \sum_{i} (-1)^{i-1}[[v_0, \ldots, \hat{v}_i, \ldots, v_p], h]].$$

Let $T = \sum_{i=1}^{k} a_i[[\sigma_i, h_i]]$ be a finite chain in $M$. Then its boundary $\partial T$ is given by

$$\partial T = \sum_{i} a_i[\partial \sigma_i, h_i].$$

It is easy to see that in this context, we have the following Stokes’ Formula.
Lemma 4.3. ([DR84, Sec. 6]) If $T$ is a finite $p$-chain in $M$, then for any $(p-1)$-form $\alpha \in \mathcal{B}^\prime (M)$, we have that

\[
\int_{\partial T} \alpha = \int_T d\alpha.
\]

(4.3)

It follows that for any finite $p$-chain in $M$, its boundary $\partial T$ given in Equation 4.2 agrees with the boundary $\partial T$ given in Equation 2.11. As a result, the definition of $\partial T$ does not depend on a particular representation of $T$ as a finite linear combination of chain elements in $M$. Moreover, it is easy to check that the boundary operator is not going to destroy the finiteness condition. If $T$ is a finite chain, then $\partial T$ is also a finite chain.

Definition 4.4. Let $(M, \omega)$ be a $2n$ dimensional symplectic manifold. For any $0 \leq i \leq 2n$, the space of real flat chains in $M$ with degree $i$ is defined to be

\[
\mathcal{F}^i(M) = \{ T \in \mathcal{B}^\prime_1, T = \lim_{k \to \infty} T_k \text{ in } \mathcal{B}^\prime_1, \text{ where } \{ T_k \} \subset A_{2n-i}(M) \}.
\]

(4.4)

To simplify the notation, we simply write $\mathcal{F}^i$ instead of $\mathcal{F}^i(M)$ when it is clear from the context to which manifold we are referring. An element $T$ in $\mathcal{F}^i$ will be called a real flat chain of degree $i$.

Remark 4.5. a) The notion of a flat chain given in Definition 4.4 is a natural extension of a flat chain on $\mathbb{R}^m$ introduced in [WH57]. For a symplectic ball $B$ in $\mathbb{R}^{2n}$ with the standard symplectic structure, we prove in Lemma 4.6 that $T$ is a flat chain in $B$ as given in Definition 4.4 if and only if $T$ is a flat chain in $B$ as given in Definition 3.5 and $T$ is compactly supported. For a general symplectic manifold, we prove in Theorem 4.12 that a flat chain given in Definition 4.4 is a finite sum of flat chains given in Definition 3.5. (See Theorem 4.12 for the precise statement.)

b) By definition, $\mathcal{F}^i(M)$ is the sequential closure of $A_{2n-i}(M)$ in $\mathcal{B}^\prime_1(M)$. We prove in Corollary 4.13 that $\mathcal{F}^i(M)$ itself is sequentially closed in $\mathcal{B}^\prime_1(M)$.

c) We emphasize here that we have followed the the lead of [F69, 4.12] by requiring that a flat chain is compactly supported. Throughout the rest of this paper, when we are using the notion of a flat chain, we use it in the sense of Definition 4.4.

Lemma 4.6. Let $B$ be a symplectic ball in $\mathbb{R}^{2n}$ with the standard symplectic structure. Then the condition that $T$ is a flat chain in $B$ as given in definition 4.4 is equivalent to that $T$ is a compactly supported flat chain in $B$ as given in Definition 3.5

Proof. Assume that $T$ is a flat chain in $B$ as given in Definition 4.4. Then there is a sequence of finite chains $(T_k)$ in $B$ that converges to $T$ in $\mathcal{B}^\prime_1$. By Theorem 3.17 for any $k \geq 1$, $T_k$ is a flat chain in $B$. It follows immediately from Corollary 3.14 that $T$ is a flat chain as given in Definition 3.5.
Assume that $T$ is a compactly supported flat chain as given in Definition 3.5. By Lemma 3.16, there exists a sequence of polyhedral chains $\{T_k\}$ that converges to $T$ in $\mathcal{B}'_1$. Thus $T$ must be a flat chain as given in Definition 4.4.

We postpone the proof of the following lemma to Section 4.3.

**Lemma 4.7.** Assume that $(M, \omega)$ is a $2n$ dimensional symplectic manifold. If a sequence of flat chains $\{T_k\}$ converges to $T$ in $\mathcal{B}'_1$, then

$$dT = \lim_{k \to \infty} T_k, \quad \partial T = \lim_{k \to \infty} \partial T_k,$$

in $\mathcal{B}'_1$.

As a result, if $T \in F^i$, then $dT \in F^{i+1}$.

By Lemma 4.7, we get the following cohomological differential complex.

\[\begin{align*}
0 \to F^0 \xrightarrow{d} F^1 \xrightarrow{d} \cdots \xrightarrow{d} F^{2n} \to 0,
\end{align*}\]

and we define

\[H^i(F, d) = \frac{\ker (F^i \xrightarrow{d} F^{i+1})}{\text{im} (F^{i-1} \xrightarrow{d} F^i)}, \quad \forall 0 \leq i \leq 2n.\]

4.2. Poincaré Lemma.

We establish in this section the Poincaré lemma for $H^*(F, d)$. Given a symplectic ball $B$ in $\mathbb{R}^{2n}$, we first prove a simple lemma which provides us an useful necessary condition for a sequence of zero dimensional polyhedral chains to converge in $\mathcal{B}'_1(B)$.

**Lemma 4.8.** Let $B$ be a symplectic ball in $\mathbb{R}^{2n}$ with the standard symplectic structure $\omega_0$. Let $\{T_k\}$ be a sequence of zero dimensional polyhedral chains in $B$ that converges in $\mathcal{B}'_1$. For any $k \geq 1$, write

\[T_k = \sum_i a_{ki} \delta(x_{ki}),\]

where $a_{ki}$ is a real scalar, and $\delta(x_{ki})$ is the Dirac delta function at point $x_{ki}$ in $B$. Set $a_k = \sum_i a_{ki}$. Then the sequence $\{a_k\}$ must converge.

**Proof.** Let $f(x) \equiv 1$ be the identify function on $B$. It is clear that $f \in \mathcal{B}'_1$. Lemma 4.8 follows easily from the fact that the sequence $\{T_k(f)\}$ must converge.

**Theorem 4.9.** (Poincaré Lemma) Let $B$ be a symplectic ball in $\mathbb{R}^{2n}$ equipped with the standard symplectic structure $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$, and let $\{F(B), d\}$ be
the differential complex defined in (4.5). Then

$$H^i(F(B), d) = \begin{cases} 0 & \text{if } 0 \leq i < 2n \\ \mathbb{R} & \text{if } i = 2n. \end{cases}$$

Proof. If $i = 0$, assume that $T \in F^0$ such that $dT = 0$. Then by Theorem 2.17, we must have $T = C$ for some constant $C$. However, since $T$ is by definition a compactly supported current, we must have $C = 0$ on $B$.

We next prove the case that $0 < i \leq 2n$. Suppose that $T \in F^i$ such that $dT = 0$. It follows from Lemma 4.6 that $T$ is a compactly supported flat chain as given in Definition 3.5. By Lemma 3.16 there exists a sequence of $2n - i$ dimensional polyhedral chains $(T_k)$ in $B$ that converges to $T$ in $\mathcal{B}'$.

For any $0 < i \leq 2n$, the support of each $T_k$ is contained in the union of finitely many $2n - i$ dimensional planes $E_{k1}, \ldots, E_{kl}$. Note that for any fixed $k$, the complement of $\bigcup_{i=1}^{k} E_{ki}$ in $B$ is an open and dense subset. By the Baire’s category theorem, the intersection of all such complements is an open and dense subset in $B$. Choose a point $b$ in this open and dense set. Then $b$ and $T_k$ are geometrically independent for any $k \geq 1$.

By Proposition 3.18 $(b \ast T_k)_{k=1}^{\infty}$ converges to an element $Q \in F^{i+1}$ in $\mathcal{B}'$. It follows from Lemma 4.7 that

$$\partial(Q) = \partial \left( \lim_{k \to \infty} b \ast T_k \right) = \lim_{k \to \infty} \partial(b \ast T_k).$$

If $0 < i < 2n$, we note that by Lemma 4.7

$$\partial T = \lim_{k \to \infty} \partial T_k = 0 \text{ in } \mathcal{B}'_1.$$

It follows that

$$\partial(Q) = \lim_{k \to \infty} -b \ast (\partial T_k) + \lim_{k \to \infty} T_k = T.$$

This proves the case $0 < i < 2n$.

If $i = 2n$, consider the representation of $T_k$ in Equation 4.7. Using the same notation as in Lemma 4.8 we have that

$$\partial Q = (- \lim_{k \to \infty} a_k) \delta(b) + T.$$

Here $\delta(b)$ is the Dirac delta function at point $b$. Note that by Lemma 4.8 $\lim_{k \to \infty} a_k$ does exist. The case $i = 2n$ now follows easily from Lemma 2.18.

□

4.3. Mayer-Vietoris exact sequence.

In this section we establish the Mayer-Vietoris sequence for the De Rham complex of real flat chains on a symplectic manifold. We first prove that the product of a finite chain and a $C^1$ function is a flat chain.
Lemma 4.10. Let $(M, \omega)$ be a $2n$ dimensional symplectic manifold, and let $T = \sum_{i=1}^{l} a_i[(\sigma_i, h_i)]$ be a finite $p$-chain on $M$. Here for each $i$, $\sigma_i$ is a $p$-simplex in $\mathbb{R}^{2n}$, and $h_i$ is a symplectomorphism from an open set $U_i$ in $M$ onto an open neighborhood $O_i$ of $\sigma_i$ in $\mathbb{R}^{2n}$. Suppose that $\rho$ is a $C^1$ function on $M$, then we have that $\rho \cdot T \in \mathcal{F}(M)$. Moreover, if $\rho$ is compactly supported inside an open set $U \subset M$, then $\rho \cdot T \in \mathcal{F}(U)$.

Proof. By Lemma 3.15, for any $1 \leq i \leq l$, there exists a sequence of polyhedral chains $Q_{ij}$ that converges to $(\rho \circ h_i^{-1}) \cdot \sigma_i$ in $\mathcal{B}'_1(O_i)$. It follows that for any fixed $1 \leq i \leq l$ we get a sequence of finite chains $[(Q_{ij}, h_i)]$ that converges to the compactly supported current $\rho \cdot [(\sigma_i, h_i)]$ in $\mathcal{B}'_1(U_i)$. Moreover, all the supports of $Q_{ij}$ are contained in a single compact set $K := \cup_i (\text{supp} \sigma_i \cap \text{supp} \rho)$. It follows from Lemma 2.12 that

$$T_j = \sum_{i=1}^{l} a_i[(Q_{ij}, h_i)]$$

converges to $\rho \cdot T$ in $\mathcal{B}'_1(M)$.

If $\text{supp} \rho \subset U$, then all the supports of the finite chains $T_i$ are supported inside a single compact subset in $U$. It is easy to see that $\{T_i\}$ converges to $\rho \cdot T$ in $\mathcal{B}'_1(U)$. Therefore $\rho \cdot T \in \mathcal{F}(U)$.

□

Proposition 4.11. Suppose that $M$ is a $2n$ dimensional symplectic manifold, that $T$ is a flat chain in $\mathcal{F}(M)$, and that $\rho$ is a compactly supported smooth function supported inside an open set $U \subset M$. Then $\rho \cdot T \in \mathcal{F}(U)$.

Proof. Let $K$ be the compact support of $\rho$. Have it covered with finitely many Darboux coordinate chars $(U_i, h_i)$ in $U$ such that for each $i$, $h_i$ is a symplectomorphism from $U_i$ onto a symplectic ball $B_i$ in $\mathbb{R}^{2n}$ with the standard symplectic structure. Choose a partition of unit $\{\lambda_i\}$ subordinate to $\{U_i\}$. Set $\rho_i = \lambda_i \cdot \rho$. By construction, each $\rho_i$ is compactly supported, and $\rho = \sum_i \rho_i$.

Since $T$ is a flat chain in $\mathcal{F}(M)$, by definition, there is a sequence of flat chains $\{T_k\}$ that converges to $T$ in $\mathcal{B}'_1(M)$. By Lemma 4.10 for each fixed $i$, $\{\rho_i \cdot T_k\}$ is a sequence of flat chains in $\mathcal{F}(U_i)$. It is clear that $\{\rho_i \cdot T_k\}$ converges to $\rho_i \cdot T$ in $\mathcal{B}'_1(U_i)$, and that all the supports of $\rho \cdot T_k$ are contained in a single compact set. Since for each fixed $i$, $h_i$ is a symplectomorphism from $U_i$ onto a symplectic ball $B_i$ in $\mathbb{R}^{2n}$, it follows from Lemma 4.6 and Corollary 3.14 that $\rho_i \cdot T$ is a compactly supported flat chain in $\mathcal{F}(U_i)$. A straightforward check of definitions shows that

$$\rho \cdot T = \sum_i \rho_i \cdot T \in \mathcal{F}(U).$$

□
A close examination of the proof of Proposition 4.11 gives us the following result.

**Theorem 4.12.** Let \((M, \omega)\) be a 2n dimensional symplectic manifold, and \(T\) a flat chain in \(\mathcal{F}(M)\). Then there exists a finite sequence of compactly supported currents \(\{T_i\}_{i=1}^l \subset \mathcal{F}(M)\) which satisfies the following conditions.

1) For any \(i\), there exists a Darboux coordinate chart \((U_i, h_i)\) such that \(h_i(U_i)\) is a symplectic ball in \(\mathbb{R}^{2n}\) with the standard symplectic structure, and such that \(T_i\) is supported inside \(U_i\).

2) For any \(i\), if we use \(h_i\) to identify \(U_i\) with its image in \(\mathbb{R}^{2n}\), then \(T_i\) is a compactly supported flat chain in \(\mathcal{F}(U_i)\) in the sense of [F69, 4.12].

3) \(T = \sum_{i=1}^l T_i\).

The following result asserts that the space of flat chains is sequentially closed in \(\mathcal{B}'^1\).

**Corollary 4.13.** Let \((M, \omega)\) be a 2n dimensional symplectic manifold, \(\{T_k\}\) a sequence of flat chains in \(M\), and \(T\) a compactly supported current in \(\mathcal{B}'^1\). If \(\{T_k\}\) converges to \(T\) in \(\mathcal{B}'^1\), then \(T\) is also a flat chain in \(M\).

**Proof.** Since \(\{T_k\}\) converges to \(T\) in \(\mathcal{B}'^1\), all the supports of \(T_k\) and \(T\) are contained in a single compact set \(K\). Cover \(K\) with finitely many Darboux coordinate charts \((U_i, h_i)\) such that \(B_i := h_i(U_i)\) is a symplectic ball in \(\mathbb{R}^{2n}\) with the standard symplectic structure. Choose a partition of unit \(\{\rho_i\}\) subordinate to \(\{U_i\}\). Then for any fixed \(i\), \(\rho_i \cdot T_k\) converges to \(\rho_i \cdot T\) in \(\mathcal{B}'^1(U_i)\). It follows from Lemma 4.6 and Proposition 3.14 that each \(\rho_i \cdot T_k\) is a flat chain in \(U_i\). A check of definitions shows that \(T = \sum \rho_i \cdot T\) is a flat chain in \(M\). 

We present a proof of Lemma 4.7 below.

**Proof.** By Theorem 4.12, it suffices to give a proof in the case that \(M\) is a symplectic ball in \(\mathbb{R}^{2n}\). By Lemma 3.16, there is a sequence of polyhedral chains \(\{T_k\}\) in \(B\) that converges to \(T\) with respect to the flat norm (3.4); moreover, all the supports of \(T_k\) are contained in a single compact set in \(B\). By Lemma 3.6 we have that that 

\[
\|\partial T_k - \partial T\|_b \leq \|T_k - T\|_b.
\]

It follows immediately that \(\{dT_k\}\) converges to \(dT\) with respect to the flat norm (3.4) in \(B\). Moreover, since for any \(k\),

\[
\text{supp}(dT_k) \subset \text{supp}T_k,
\]

we conclude that all the supports of \(dT_k\) are contained in a single compact set in \(B\). Lemma 4.7 now follows immediately from Theorem 3.13.
The following lemma is an important step towards establishing the exactness of the Mayer-Vietoris sequence for the De Rham complex of real flat chains.

**Lemma 4.14.** Suppose that $M$ is a $2n$ dimensional symplectic manifold covered by two open sets $U, V$. If $T$ is a flat chain in $\mathcal{F}(M)$, then there exist flat chains $T_U \in \mathcal{F}(U)$ and $T_V \in \mathcal{F}(V)$ such that $T = T_U + T_V$.

**Proof.** Let $\{\rho_U, \rho_V\}$ be a partition of unit subordinate to the open cover $\{U, V\}$ of $M$ such that $\text{supp } \rho_U \subset U$ and $\text{supp } \rho_V \subset V$. By construction, we have that

$$T = \rho_U \cdot T + \rho_V \cdot T.$$ 

Moreover, it follows from Proposition 4.11 that $\rho_U \cdot T \in \mathcal{F}(U)$, and that $\rho_V \cdot T \in \mathcal{F}(V)$. This completes the proof. □

Let $M$ be a $2n$ dimensional symplectic manifold covered by two open sets $U, V$. We define two natural maps

(4.8) $\mathcal{F}^*(U \cap V) \to \mathcal{F}^*(U) \oplus \mathcal{F}^*(V)$, $T \mapsto (-i_*(T), i_*(T))$,

(4.9) $\mathcal{F}^*(U) \oplus \mathcal{F}^*(V) \to \mathcal{F}^*(M)$, $(T_1, T_2) \mapsto i_*(T_1) + i_*(T_2)$,

where $i_*$ denote the pushforward maps induced by the obvious inclusion maps. These two maps gives rise to the following Mayer-Vietoris sequence.

(4.10) $0 \leftarrow \mathcal{F}^*(M) \leftarrow \mathcal{F}^*(U) \oplus \mathcal{F}^*(V) \leftarrow \mathcal{F}^*(U \cap V) \leftarrow 0$

**Proposition 4.15.** The Mayer-Vietoris sequence (4.10) is exact.

**Proof.** The only step requires a proof is that Map 4.9 is surjective. This follows immediately from Lemma 4.14. □

Then standard facts in homological algebra gives us the following long exact sequence

(4.11)

$$\cdots \leftarrow H^i(\mathcal{F}(U \cup V)) \leftarrow H^i(\mathcal{F}(U)) \oplus H^i(\mathcal{F}(V)) \leftarrow H^i(\mathcal{F}(U \cap V)) \leftarrow H^{i-1}(\mathcal{F}(U \cup V)) \leftarrow \cdots$$

Note that the chain map given by the inclusion

$$\mathcal{F}(M) \hookrightarrow B^*(M)$$

induces a natural homomorphism of cohomologies

(4.12) $H^*(\mathcal{F}(M), d) \to H_{\infty,*}^*(M)$.

In view of the long exact sequence (2.16), the standard Mayer-Vietoris argument as explained in [BT82, Sec. 5] provides us the following result.

**Theorem 4.16.** For any compact symplectic manifold $(M, \omega)$, the natural homomorphism (4.12) is an isomorphism.
5. \( \mathfrak{sl}_2 \) MODULE STRUCTURE

5.1. \( \mathfrak{sl}_2 \) module structure on distributional De Rham complex.

In this section, we discuss \( \mathfrak{sl}_2 \)-module structure on the space of compactly supported currents on a \( 2n \) dimensional symplectic manifold. To set up the stage, we first review the wedge product construction between a form and a current, and the inner product construction between a multivector and a current.

Let \( \alpha \) be a smooth differential \( r \)-form, and \( Z \) a smooth \( s \)-vector field on a differential manifold \( M \). We have the following two maps

\[
\mathcal{B}_p \to \mathcal{B}_p, \phi \mapsto \alpha \wedge \phi, \quad \mathcal{B}_p \to \mathcal{B}_p, \phi \mapsto \iota_Z \phi.
\]

It is easy to see that they are both continuous linear mappings. Now for a compactly supported current \( T \) of degree \( k \), define \( \alpha \wedge T \) and \( \iota_Z T \) as follows.

\[
\alpha \wedge T(\phi) = (-1)^k T(\alpha \wedge \phi), \quad \iota_Z T(\phi) = T(\iota_Z \phi), \quad \forall \phi \in \mathcal{B}_p.
\]

Then both \( \alpha \wedge T \) and \( \iota_Z T \) are compactly supported currents in \( \mathcal{B}'_p \).

**Definition 5.1.** Let \((M, \omega)\) be a \( 2n \) dimensional symplectic manifold, \( \pi = \omega^{-1} \) the canonical poisson bi-vector, and \( \Pi^i : \mathcal{B} \to \mathcal{B}^i \) the projection operator. Define the Lefschetz map \( L \), the dual Lefschetz map \( \Lambda \), and the degree counting map \( H \) on \( \mathcal{B}' \) as follows.

\[
(LT)(\alpha) = T(\omega \wedge \alpha), \quad (\Lambda T)(\alpha) = T(\iota_\pi \alpha), \quad (HT)(\alpha) = T(-\sum_i (n-i) \Pi^i \alpha),
\]

\( \forall T \in \mathcal{B}', \forall \alpha \in \mathcal{B} \).

In this context, a compactly supported current \( T \) of degree \( i \) is said to be primitive if \( L^{n-i+1} T = 0 \), where \( 0 \leq i \leq n \).

**Lemma 5.2.** Consider the maps given in Definition 5.1. We have the following commutator relations.

\[
[\Lambda, L] = H, \quad [H, \Lambda] = 2\Lambda, \quad [H, L] = -2L.
\]

Therefore they define a natural \( \mathfrak{sl}_2 \) module structure on \( \mathcal{B}' \).

**Proof.** It is an immediate consequence of Definition 5.1 and the usual commutator relations on forms given in Equation 2.2. \( \square \)

Observe that although \( \mathcal{B}' \) is an infinite dimensional \( \mathfrak{sl}_2 \)-module, it has the property that \( H \) has only finitely many eigenvalues. The following result is a direct consequence of [Yan96, Corollary, 2.5, 2.6].

**Proposition 5.3.** Let \((M, \omega)\) be a \( 2n \) dimensional symplectic manifold. We have the following results.
1) \[ L^k : \mathcal{B}^{n-k} \to \mathcal{B}^{n+k}, \ T \mapsto \omega^{n-k} \wedge T \]

is an isomorphism for any \(0 \leq k \leq n\).

2) A current \(T\) in \(\mathcal{B}^k\) is primitive if and only if \(\Lambda T = 0\), where \(0 \leq k \leq n\).

3) Any \(T \in \mathcal{B}^k\) admits a unique Lefschetz decomposition as follows.

\[
T = \sum_{r \geq \max\left(\frac{k-n}{2}, 0\right)} \frac{L^r}{r!} T_{k-2r},
\]

where \(T_{k-2r}\) is a primitive compactly supported current in \(\mathcal{B}^{k-2r}\).

**Definition 5.4.** For any \(0 \leq i \leq 2n\), define the symplectic Hodge star operator by

\[
(\star T)(\alpha) = T(\star \alpha), \ T \in \mathcal{B}^i, \ \alpha \in \mathcal{B}^i.
\]

And define the symplectic Hodge adjoint operator \(d^\Lambda\) by

\[
(d^\Lambda T)(\alpha) = T(d^\Lambda \alpha), \ T \in \mathcal{B}^i, \ \alpha \in \mathcal{B}^{2n-i+1}.
\]

A compactly supported current \(T\) is called (symplectic) Harmonic if and only if \(dT = d^\Lambda T = 0\).

It is easy to see that by definition

\[
d^\Lambda T = (-1)^{i+1} \star T \star, \ \forall T \in \mathcal{B}^i.
\]

Moreover, using the commutator relations on forms given in Equation 2.5 it is straightforward to check that we have the following commutator relations on currents.

**Lemma 5.5.**

\[
[d, L] = 0, \ [d^\Lambda, \Lambda] = 0, \ [d, \Lambda] = d^\Lambda, \ [d^\Lambda, L] = d, \ [d, d^\Lambda] = 0.
\]

The following result is an easy consequence of the commutator relations given in Lemma 5.5.

**Lemma 5.6.** Consider the Lefschetz decomposition of a compactly supported current \(T\) of degree \(k\) as given in Equation 5.1. Then

1) there are non-commutative polynomials \(\Phi_{k,r}(L, \Lambda)\) such that

\[
T_{k-2r} = \Phi_{k,r}(L, \Lambda) T;
\]

2) each \(T_{k-2r}\) is \(d\)-closed and primitive if \(T\) is Harmonic.

We conclude this section by deducing the Weil’s identity for currents from the usual Weil’s identity for differential forms.

**Lemma 5.7.** (Weil’s identity)

For any \(0 \leq r \leq n\), and any primitive current \(T \in \mathcal{B}^n\), we have the following Weil’s identity

\[
(\star \frac{L^r}{r!} T) = (-1)^{\frac{p(p+1)}{2}} \frac{L^{n-p-r} T}{(n-p-r)!}.
\]
Proof. Given a test form $\alpha \in \mathcal{B}^{2r+p}$, Lefschetz decompose $\alpha$ into

$$\alpha = \sum_{k \geq \max \left( \frac{2r+p-n}{2}, 0 \right)} L^k \alpha_k \frac{k!}{k!},$$

where $\alpha_k$ is a primitive $(2r+p-2k)$-form. Set $i_k = 2r+p-2k$. Then we have

$$< \frac{LT}{r!}, \alpha > = < \frac{LT}{r!}, *\alpha >$$

$$= < \frac{LT}{r!}, \sum_{k \geq \max \left( \frac{2r+p-n}{2}, 0 \right)} (-1)^{\frac{i_k(i_k+1)}{2}} \frac{L^{n-2r-p+k} \alpha_k}{(n-i_k-k)!} >$$

$$= \sum_{k \geq \max \left( \frac{2r+p-n}{2}, 0 \right)} < \frac{LT}{r!}, (-1)^{\frac{i_k(i_k+1)}{2}} \frac{L^{n-2r-p+k} \alpha_k}{(n-i_k-k)!} >$$

By primitivity, $L^{n-2r-p+2k+1} \alpha_k = 0$, and $L^{n-p+1T} = 0$. It follows that

$$(5.3) \quad < L^T L^{n-2r-p+k} \alpha_k > = 0, \quad \forall k \neq r.$$ 

Consequently we have that

$$< * \frac{LT}{r!}, \alpha > = < \frac{LT}{r!}, (-1)^{\frac{p(p+1)}{2}} \frac{L^{n-p-r} \alpha_r}{(n-p-r)!} >$$

$$= < (-1)^{\frac{p(p+1)}{2}} \frac{L^{n-p-r}T}{(n-p-r)!}, \frac{L^r \alpha_r}{r!} >$$

$$= < (-1)^{\frac{p(p+1)}{2}} \frac{L^{n-p-r}T}{(n-p-r)!}, \sum_{k \geq \max \left( \frac{2r+p-n}{2}, 0 \right)} L^k \alpha_k \frac{k!}{k!} >$$

$$= < (-1)^{\frac{p(p+1)}{2}} \frac{L^{n-p-r}T}{(n-p-r)!}, \alpha > .$$

This completes the proof of Lemma 5.7. \qed

5.2. $\mathfrak{sl}_2$ module structure on the space of real flat chains.

In this section, we prove that on a symplectic manifold the space of real flat chains inherits a $\mathfrak{sl}_2$ submodule structure from the canonical $\mathfrak{sl}_2$-module structure on $\mathcal{B}'$ that we discussed in Section 5.1. To set up the stage, we first clarify the notion of an oriented parallelepiped in an Euclidean space $\mathbb{R}^m$. In the symplectic setting, it will be much easier to see how the Lefschetz map $L$ and the dual Lefschetz map $\Lambda$ act on an oriented parallelepiped. In this section, we deal almost exclusively with oriented parallelepipeds in $\mathbb{R}^{2n}$ with the standard symplectic structure, except for our treatment of the elementary result stated in Lemma 5.8.
Let $V$ be a $k$-dimensional subspace of $\mathbb{R}^m$, and $u$ a point in $\mathbb{R}^m$. A $k$-dimensional oriented parallelepiped $P$ in the affine space $u + V$ consists of an ordered basis $\{e_1, \ldots, e_k\}$ in $V$, a set of the form

$$(5.4) \quad u + \{x_1 e_1 + \cdots + x_k e_k, \quad a_1 \leq x_1 \leq b_1, \ldots, a_k \leq x_k \leq b_k\},$$

and an orientation induced by either $e_1 \land \cdots \land e_k$ or $-e_1 \land \cdots \land e_k$. Here we assume that $a_i < b_i$, $i = 1, \ldots, k$.

It is clear that any oriented parallelepiped $P$ generated a compactly supported current in $\mathcal{B}'_1(\mathbb{R}^m)$ in a canonical way. If $P$ is supported inside an open set $W$ in $\mathbb{R}^m$, its canonical current can also be regarded as an element in $\mathcal{B}'_1(W)$. Throughout the rest of this paper, when we refer to an oriented parallelepiped, we mean the canonical current generated by it. It will be clear from the context in which topological vector space this canonical current lives.

Observe that an oriented parallelepiped is uniquely determined by its support and orientation. To simplify the notation, we use the following convention in this paper. If the support of $P$ is given as in Equation 5.4 and if $P$ has the same orientation as $e_1 \land \cdots \land e_k$, then we simply write

$$(5.5) \quad P = u + [a_1, b_1] \times \cdots \times [a_k, b_k] \times \mathbb{O}_{m-k},$$

if $P$ has the opposite orientation, we write

$$(5.6) \quad P = u - [a_1, b_1] \times \cdots \times [a_k, b_k] \times \mathbb{O}_{m-k}.$$

Here $\mathbb{O}_{m-k} = (0, \ldots, 0)$; moreover, we implicitly extend $\{e_1, \ldots, e_k\}$ to a full ordered basis $\{e_1, \ldots, e_m\}$ in $\mathbb{R}^m$, and use it to identify $\mathbb{R}^m$ with $\mathbb{R} \times \cdots \times \mathbb{R}^m$.

When we are using notations in Equation 5.5 and Equation 5.6 in this paper, it will be clear from the context if we are referring to an oriented parallelepiped or its support.

Let $P$ be an oriented parallelepiped that is supported on a set given in Equation 5.4. We make two simple observations. First, the mass $M(P)$ coincides with the Hausdorff measure $\mathcal{H}^n(P)$. Therefore $M(P)$ is a non-zero constant multiple of $(b_1 - a_1) \cdots (b_k - a_k)$ with the constant depending only on the Euclidean norm of the basis vector $\{e_1, \ldots, e_k\}$ in $V \subset \mathbb{R}^m$. Second, if we denote by $||e_i||$ the Euclidean norm of $e_i$ with respect to the standard inner product in $\mathbb{R}^m$, then the diameter of the support of $P$ is at most $\max_{1 \leq i \leq k} (k \cdot ||a_i - b_i|| \cdot ||e_i||)$.

The support of an oriented parallelepiped along with all its faces form a rectilinear cell complex as defined in [M66, Definition 7.6]. Hence it admits a simplicial subdivision, cf. [M66, Lemma 7.6]. Orient each simplex in the simplicial subdivision with the orientation inherited from the oriented parallelepiped. Then the oriented parallelepiped can be naturally identified with a polyhedral chain in $\mathbb{R}^m$. More generally, a polyhedral chain
in $\mathbb{R}^m$ is called a polyhedral chain of oriented parallelepipeds if it admits a representation as a finite linear combination of oriented parallelepipeds.

**Lemma 5.8.** Let $\sigma = [v_0, v_1, \ldots, v_p]$ be a $p$-simplex in $\mathbb{R}^m$. Then there exists a sequence of polyhedral chains of oriented parallelepipeds $\{T_k\}$ that converges to $\sigma$ in $B^1_{\flat}$; moreover, all the supports of $\{T_k\}$ are contained in that of $\sigma$.

**Proof.** Without the loss of generality, we assume that $\sigma$ lies on a $p$-dimensional subspace $V$ of $\mathbb{R}^m$, and use the edge vectors $\{e_1 := v_1 - v_0, \ldots, e_p := v_p - v_0\}$ to identify $V$ with $\mathbb{R}^p \cong \mathbb{R}^p \times \{(0, \ldots, 0)\}$ $\subset \mathbb{R}^m$. Note that for any fixed positive integer $k$, we can decompose $\mathbb{R}^p$ into the union of the collection of the supports of all oriented parallelepipeds of the following form

$$[a_1 + \frac{i_1}{2^k}, a_1 + \frac{i_1 + 1}{2^k}] \times \cdots \times [a_p + \frac{i_p}{2^k}, a_p + \frac{i_p + 1}{2^k}], \quad (a_1, \ldots, a_p) \in Z^p = \mathbb{Z} \times \cdots \times \mathbb{Z}.$$  

Here for any $1 \leq j \leq p$, $i_j$ runs over all integer points between $1$ and $2^k - 1$. Let $D_k$ be the collection of all such oriented parallelepipeds given in Equation 5.7 whose supports are contained in that of $\sigma$. By our convention, each oriented parallelepiped in $D_k$ is equipped with an orientation induced by the multi-vector $e_1 \wedge \cdots \wedge e_p$. Let $T_k$ be the sum of all oriented parallelepipeds in $D_k$. By construction, all the supports of $T_k$ are contained in that of $\sigma$. It suffices to show that $\{T_k\}$ converges to $\sigma$ with respect to the flat norm.

Let $\text{Bd}(\sigma)$ be the boundary of $\sigma$. For any small positive number $r > 0$, set

$$A_r = \{x \in \sigma, \text{ the distance from } x \text{ to } \text{Bd}(\sigma) \leq r\}.$$  

Denote by $\mu$ the Lebesgue measure in $\mathbb{R}^p$. Since $\text{Bd}(\sigma)$ is of Lebesgue measure zero in $\mathbb{R}^p$, it is easy to see that for any $\varepsilon > 0$, there exists $r > 0$ such that $\mu(A_r) < \varepsilon$. We further claim that there exists an integer $N > 0$, such that

$$\text{supp } \sigma \setminus \left( \bigcup_{P \in D_k} \text{supp } P \right) \subset A_r, \quad \forall k \geq N.$$  

Note that if a point $x \in \sigma$ lies in an oriented parallelepiped $P$ of the form given in Equation 5.7 and if $P$ is not supported inside $\text{supp } \sigma$, then $P$ will intersect the boundary of $\sigma$. However, the distance between any two points in $P$ is at most $\max_{1 \leq i \leq p} \left( \frac{P}{2^k} ||e_i|| \right)$. Therefore there must exist an integer $N > 0$ such that for any $k \geq N$, the distance from $x$ to $\text{Bd}(\sigma)$ is less than $r$. This proves the claim.

Therefore, for any $k \geq N$, we have that

$$||\sigma - T_k||_b \leq M(\sigma - T_k) \leq \mu(A_r) < \varepsilon$$  

This completes the proof. □
Throughout the rest of this section, we assume that $\mathbb{R}^{2n}$ is equipped with the standard symplectic structure

$$\omega_0 = \sum_{i=1}^{2n} dx_i \wedge dx_{n+i},$$

where $\{x_1, \cdots, x_{2n}\}$ are the standard Darboux coordinates. For any $1 \leq i \leq 2n$, we will denote by $\partial_i$ the vector field $\frac{\partial}{\partial x_i}$ on $\mathbb{R}^{2n}$.

Without the loss of generality, we may assume that an arbitrarily given $p$-dimensional oriented parallelepiped $P$ is supported on a set of the form

$$u + \{a_1 \leq x_1 \leq b_1, \cdots, a_p \leq x_p \leq b_p, x_{p+1} = x_{p+2} = \cdots = x_{2n} = 0\},$$

where $u \in \mathbb{R}^m, a_i < b_i, i = 1, \cdots, p$. Given such an oriented parallelepiped, using the convention we explained preceding the proof of Lemma 5.8, we simply write

$$P = u + [a_1, b_1] \times \cdots \times [a_p, b_p] \times O_{2n-p}$$

if it has an orientation given by the multi-vector $\partial_1 \wedge \cdots \wedge \partial_p$; and we write

$$P = u - [a_1, b_1] \times \cdots \times [a_p, b_p] \times O_{2n-p}$$

if it has an opposite orientation.

**Lemma 5.9.** Let $B$ be a symplectic ball in $\mathbb{R}^{2n}$ with the standard symplectic structure $\omega_0 = \sum_{i=1}^{n} dx_{2i-1} \wedge dx_{2i}$, and let $P$ be a $p$-dimensional oriented parallelepiped.

Then for any $1 \leq i \leq n$ there exists a sequence of $(p-2)$-dimensional polyhedral chains of oriented parallelepipeds $\{T_k\}$ in $B$ such that

$$(dx_{2i-1} \wedge dx_{2i}) \wedge P = \lim_{k \to \infty} T_k \text{ in } \mathcal{B}_1';$$

moreover, we have that

$$\text{supp } T_k \subset \text{supp } P, \forall k \geq 1.$$ 

**Proof.** To simplify notations, we assume $P$ is an oriented parallelepiped supported on a set as given in Equation 5.8 with $u$ being the origin of $\mathbb{R}^{2n}$. Without the loss of generality, we assume that $P$ has an orientation induced by that of the multi-vector $\partial_1 \wedge \cdots \wedge \partial_p$. We write the current $P$ as follows.

(5.9)  
$$\chi(P) \delta(x_{p+1}) \cdots \delta(x_{2n}) dx_{p+1} \wedge \cdots \wedge dx_{2n},$$

where $\chi(P)$ is the characteristic function of $P$ in the the variable $x_j$, $1 \leq j \leq p$, $\delta(x_j)$ is the Dirac delta function in variables $x_j$, $j > p$, at $x_j = 0$.

Let $r = [\frac{p}{2}]$ be the greatest integer less than or equal to $\frac{p}{2}$. We see immediately that to prove Lemma 5.9 it suffices to show that for $1 \leq i \leq r$, there exists a sequence of $(p-2)$-dimensional polyhedral chains of oriented parallelepipeds $\{T_k\}$ that converges to $(dx_{2i-1} \wedge dx_{2i}) \wedge P$ in $\mathcal{B}_1'$. As the proof for each $i$ is identical, we shall only prove the case $i = 1$. 

For any positive integer $k \geq 1$, and any $1 \leq i, j \leq k$, set $a_1^0 = a_1$, $a_2^0 = a_2$, and define

\[
a_i^1 = a_1 + \frac{i}{k}(b_1 - a_1), \quad a_i^2 = a_2 + \frac{j}{k}(b_2 - a_2),
\]

\[
P_{ij} = [a_i^{i-1}, a_1^1] \times [a_j^{j-1}, a_2^1] \times [a_3, b_3] \times \cdots \times [a_p, b_p] \times O_{2n-k},
\]

\[
T_k = \sum_{i,j=1}^{k} \frac{1}{k^2} ((a_i^1, a_j^1)) \times [a_3, b_3] \times \cdots \times [a_p, b_p] \times O_{2n-p}.
\]

Note that all the supports of $T_k$ are contained in the support of $P$ by construction. To prove Lemma 5.9, by Theorem 3.13 it suffices to show that \{T_k\} converges to $dx_1 \wedge dx_2 \wedge P$ with respect to the flat norm.

It is easy to see that in this context we have that

\[
M(P) = H^p(P) = \sum_{i,j=1}^{k} H^p(P_{ij}) = \sum_{i,j=1}^{k} M(P_{ij}).
\]

For any $1 \leq i, j \leq k$, define the map $\pi_{ij}$ as follows.

\[
\pi_{ij} : \mathbb{R}^{2n} = \mathbb{R}^2 \times \mathbb{R}^{2n-2} \to \mathbb{R}^{2n} = \mathbb{R}^2 \times \mathbb{R}^{2n-2}
\]

\[
(x_1, x_2, x_3, \cdots x_{2n}) \mapsto (a_1^1, a_2^1, x_3, \cdots, x_{2n}).
\]

Then for any $(p-2)$-form $\phi \in F^{p-2}(B)$, a direct calculation shows that

\[
T_k(\phi) = \sum_{i,j=1}^{k} P_{ij} (dx_1 \wedge dx_2 \wedge \pi_{ij}^* \phi).
\]

Consequently, we have that

\[
|| (dy_1 \wedge dy_2 \wedge P)(\phi) - T_k(\phi) ||
\]

\[
= | \sum_{i,j=1}^{k} (P_{ij}(dx_1 \wedge dx_2 \wedge \phi) - P_{ij}(dx_1 \wedge dx_2 \wedge \pi_{ij}^* \phi)) |
\]

\[
\leq \sum_{i,j=1}^{k} |P_{ij}(dx_1 \wedge dx_2 \wedge (\phi - \pi_{ij}^* \phi))|
\]

\[
\leq \sum_{i,j=1}^{k} M(P_{ij}) \cdot ||dx_1 \wedge dx_2 \wedge (\phi - \pi_{ij}^* \phi)||_{\infty} \quad \text{(By Lemma 3.8)}
\]

\[
\leq \sum_{i,j=1}^{k} C \cdot M(P_{ij}) ||\phi||_b \leq \frac{C}{k} M(P) ||\phi||_b.
\]

Here $C$ is a constant that depends only on the Euclidean norm of the standard symplectic basis in $\mathbb{R}^{2n}$. It follows that

\[
||(dx_1 \wedge dx_2 \wedge P) - T_k||_b \leq \frac{C}{k} M(P).
\]
This completes the proof of Lemma 5.9.

Lemma 5.10. Let $B$ be a symplectic ball in $\mathbb{R}^{2n}$ with the standard symplectic structure $\omega_0 = \sum_{i=1}^{n} dx_{2i-1} \wedge dx_{2i}$. Let $P$ be a $p$ dimensional oriented parallelepiped. Then for any $1 \leq i \leq n$, and any open set $U \supseteq \text{supp}P$, there exists a sequence of $(p+2)$-dimensional polyhedral chains of oriented parallelepipeds $\{T_k\}$ in $B$ such that

$$\iota_{\alpha_{2i}}\iota_{\alpha_{2i-1}}P = \lim_{k \to \infty} T_k \text{ in } \mathcal{B}_1,$$

and such that

$$\text{supp } T_k \subset U, \forall k \geq 1.$$

Proof. We assume that the oriented parallelepiped $P$ has the same form as in the proof of Lemma 5.10. In view of Equation 5.9, to prove the first assertion in Lemma 5.10, it suffices to prove that for any $r$ with $2r-1 > p$, there is a sequence of $(p+2)$-dimensional polyhedral chains of oriented parallelepipeds $\{T_k\}$ in $B$ that converges to $\iota_{\alpha_{2r}}\iota_{\alpha_{2r-1}}P$ in $\mathcal{B}_1$. Now define

$$T_k = k^2 P \times [0, \frac{1}{k}] \times [0, \frac{1}{k}] \times \mathbb{O}_{2n-p-2}.$$

Here the first copy of $[0, \frac{1}{k}]$ sits inside the $(2r-1)$-th component of $\mathbb{R}^{2n} = \mathbb{R} \times \cdots \mathbb{R}$, and the second copy sits inside the $(2r)$-th component. It is clear that for any open subset $U$ of $B$ that contains $\text{supp } P$, $\text{supp } T_k \subset U$ except for finitely many $k$. So after reindexing the first few initial terms, we have that $\text{supp } T_k \subset U$ for any $k$.

Consider the map

$$\pi_k : \mathbb{R}^p \times ([0, \frac{1}{k}] \times [0, \frac{1}{k}]) \to \mathbb{R}^{2n},$$

$$(x_1, \cdots, x_p, x_{2r-1}, x_{2r}) \mapsto (x_1, \cdots, x_p, 0, \cdots, 0).$$

For any $(p+2)$-form $\phi \in \mathbb{F}^{p+2}(B)$, a direct calculation shows that

$$(\iota_{\alpha_{2r}}\iota_{\alpha_{2r-1}}P)(\phi) = T_k(\pi_k^* \phi).$$

Therefore we have that

$$|(\iota_{\alpha_{2r}}\iota_{\alpha_{2r-1}}P)(\phi) - T_k(\phi)| = |T_k(\pi_k^* \phi - \phi)|$$

$$\leq M(T_k) \cdot \|\pi_k^* \phi - \phi\|_{\infty}$$

$$\leq \frac{1}{k} M(T_k) \cdot \|\phi\|_b.$$
Here \( C \) is a constant that may depend on the choice of a complement to \( R^p \) in \( R^{2n} \), and the choice of a basis in the complement. But it does not depend on \( k \). Let \( k \to \infty \) we get that \( \|t_{0_{2r}, t_{0_{2r-1}} P - T_k}\| \to 0 \). This finishes the proof of Lemma 5.10.

Lemma 5.11. Let \( B \) be a symplectic ball in \( R^{2n} \) with the standard symplectic structure, and \( \sigma \) a \( p \)-simplex in \( B \). Let \( L \) and \( \Lambda \) be the Lefschetz map and the dual Lefschetz map as given in Definition 5.7 respectively. Then

a) there exists a sequence of polyhedral \((p-2)\)-chains \( \{T_k\} \) in \( B \) that converges to \( L\sigma \) in \( B'_1 \) and that satisfies
\[
\text{supp} \ T_k \subset \text{supp} \ \sigma, \ \forall \ k;
\]

b) for any open set \( U \supset \text{supp} \ \sigma \), there exists a sequence of polyhedral \((p+2)\)-chains \( \{Q_k\} \) in \( B \) that converges to \( \Lambda \sigma \) in \( B'_1 \) and that satisfies
\[
\text{supp} \ Q_k \subset U, \ \forall \ k.
\]

Proof. Lemma 5.11 is an easy consequence of Lemma 5.8, Lemma 5.9, Lemma 5.10 and Corollary 3.14.

Theorem 5.12. Let \((M, \omega)\) be a \( 2n \) dimensional symplectic manifold. Let \( L \) and \( \Lambda \) be the Lefschetz map and the dual Lefschetz map as given in Definition 5.1 respectively, and let \( T \) be a flat chain in \( F^i(M) \). Then we have that
\[
HT = (n - i)T, \ LT \in F^{i+2}, \ \Lambda T \in F^{i+2}.
\]
As a result, \( F \) is a \( \mathfrak{sl}_2 \) sub-module of the space of compactly supported currents with the canonical \( \mathfrak{sl}_2 \)-module structure that we discussed in Section 5.7.

Proof. The first assertion \( HT = (n - i)T \) is trivially true. Since \( T \) is a flat chain on \( M \), there is a sequence of finite chains \( \{T_k\} \) that converges to \( T \) in \( B'_1 \). By Lemma 5.11 it is easy to see that for any \( k \), both \( LT_k \) and \( \Lambda T_k \) are flat chains on \( M \). It follows immediately from Corollary 4.13 that \( LT \in F^{i+2} \) and that \( \Lambda T \in F^{i-2} \).

On a \( 2n \) dimensional symplectic manifold, a real flat chain \( T \) of degree \( i \) is said to be primitive if it is a primitive current as given in Definition 5.1, i.e., \( L^{n-i+1} T = 0 \). In view of Proposition 5.3, \( T \) is primitive if and only if \( \Lambda T = 0 \). We say that \( T \) is symplectic Harmonic if and only if \( dT = d^\Lambda T = 0 \). Similar to Proposition 5.1, we have the following result.

Corollary 5.13. Let \((M, \omega)\) be a \( 2n \) dimensional symplectic manifold. We have the following results.

1) \[
L^k : F^{n-k} \to F^{n+k}, \ T \mapsto \omega^{n-k} \wedge T
\]
is an isomorphism for any \( 0 \leq k \leq n \).
2) Any \( T \in F^k \) admits a unique Lefschetz decomposition as follows.

\[
T = \sum_{r \geq \max(\frac{k-n}{2}, 0) \text{ and } 0} L^r T_{k-2r},
\]

where \( T_{k-2r} \) is a primitive flat chain of degree \( k - 2r \).

For any \( 0 \leq k \leq n \), let \( P^k_F \) be the space of primitive flat chains of degree \( k \). The argument used in the proof of Lemma 2.3 extends to the present situation and gives us the next result, cf. [TY09, Lemma 2.4].

**Lemma 5.14.** Let \( T \in P^k_F \) with \( 0 \leq k \leq n \). The action of the differential operators \( (d, d^\Lambda, dd^\Lambda) \) on \( T \) has the following form:

1) if \( k < n \), then \( d\alpha = A_{k+1} \Lambda_{k-1} \);
2) \( d^\Lambda \alpha = -HA_{k-1} = -(n-i+1)A_{k-1} \);
3) if \( k < n \), then \( dd^\Lambda \alpha = -n \Lambda_{k+1} \).

Here \( A_{k-1}, A_{k+1} \in P^*_F \) are primitive flat chains.

6. The symplectic adjoint operator and primitive flat chains

6.1. Dual description of the symplectic adjoint operator.

In this section, we prove that if \( T \) is a real flat chain in a symplectic manifold \( M \), then both \( \ast T \) and \( d^\Lambda T \) are real flat chains in \( M \).

**Theorem 6.1.** Let \( (M, \omega) \) be a \( 2n \) dimensional symplectic manifold, and \( T \in F^p \). Then we have that \( \ast T \in F^{2n-p} \).

**Proof.** Lefschetz decompose \( T \) as follows.

\[
T = \sum_{r \geq \max(\frac{p-n}{2}, 0) \text{ and } 0} L^r \frac{T_{p-2r}}{(p-2r)!},
\]

where \( T_{p-2r} \) is a primitive flat chain of degree \( p - 2r \). By Lemma 5.7 we have that

\[
\ast T = \sum_{r \geq \max(\frac{p-n}{2}, 0) \text{ and } 0} L^{n-p+r} \frac{T_{p-2r}}{(n-p+r)!}.
\]

It follows immediately from Theorem 5.12 that \( \ast T \in F^{2n-p} \). \( \square \)

The following result is an immediate consequence of Theorem 6.1.

**Theorem 6.2.** Let \( (M, \omega) \) be a \( 2n \) dimensional symplectic manifold, and \( T \in F^p \). Then we have that \( d^\Lambda T \in F^{p-1} \).

It may be instructive to see how the Hodge star act on an oriented parallelepiped directly. We present a concrete example below, in which we compute the Hodge star of a co-isotropic oriented parallelepiped.
Example 6.3. Consider $\mathbb{R}^4$ with the standard symplectic structure $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$. Here $\{x_1, x_2, x_3, x_4\}$ are the standard Darboux coordinates on $\mathbb{R}^4 \cong \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Let $P$ be an oriented isotropic parallelepiped in $\mathbb{R}^4$ given as follows.

$$P = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \times \{0\}.$$ 

By the convention we explained in Section 5.2, $P$ is equipped with an orientation induced by the 3-vector

$$\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3}.$$ 

We explain that $*P$ must be the limit of a sequence of isotropic polyhedral chains in $\mathcal{B}'_1(\mathbb{R}^4)$. We first write the current $P$ as follows.

$$P = \chi(P) \delta(x_4) \, dx_4.$$ 

Here $\chi(P)$ is the characteristic function of $P$ in the variables $x_1, x_2, x_3$, $\delta(x_4)$ is the Dirac delta function in the variable $x_4$ at $x_4 = 0$. We have that

$$*P = \chi(P) \delta(x_4) \, dx_1 \wedge dx_2 \wedge dx_4.$$ 

For any $k \geq 1$, $1 \leq i, j \leq k$, let

$$a_i^j = a_1 + \frac{i}{k}(b_1 - a_1), \quad a_j^j = a_2 + \frac{j}{k}(b_2 - a_2),$$

and let $T_{ij}$ be the isotropic oriented parallelepiped given by

$$\{(a_i^j, a_j^j)\} \times [a_3, b_3] \times \{0\}.$$ 

Define

$$T_k = \frac{1}{k^2} \sum_{i,j=1}^k T_{ij}.$$ 

Then for any $k$, $T_k$ is an isotropic polyhedral chain. A calculation similar to the one given in the proof of Lemma 5.9 shows that $\{T_k\}$ converges to $T$ in $\mathcal{B}'_1$.

Remark 6.4. A similar construction applies to any arbitrarily given co-isotropic oriented parallelepiped in $\mathbb{R}^{2n}$. Indeed, it can be shown that the Hodge star of any co-isotropic oriented parallelepiped in $\mathbb{R}^{2n}$ is the limit of a sequence of isotropic polyhedral chains. The difficulties are only notational. In Section 6.7 we are going to prove that a primitive flat chain of positive degree must be the limit of a sequence of co-isotropic chains in $\mathcal{B}'_1$. Combining this result with the method we explained in Example 6.3, it is not hard to show that on a symplectic manifold $M$, the Hodge star of a primitive flat chain with positive degree is the limit of a sequence of isotropic finite chains in $\mathcal{B}'_1$. 

6.2. Geometric interpretation of primitive flat chains.

In this section, we prove that on a symplectic manifold \( M \), for any primitive real flat chain \( T \) of positive degree in \( M \), there exists a sequence of co-isotropic finite chains that converges to \( T \) in \( \mathcal{B}_1^{\prime}(M) \). We begin with a simple technical lemma that we need in this section.

**Lemma 6.5.** Suppose that \( U \) and \( V \) are two non-empty open sets in \( \mathbb{R}^m \) with \( V \subset U \) and \( V \) being compact. Then there exist constants \( m_r, r = 1, 2, 3, \ldots \), depending only on the diameter of \( U \), and a function \( \lambda \) such that

1) \( 0 \leq \lambda \leq 1 \);
2) \( \lambda(x) = \begin{cases} 1 & \text{if } x \in V, \\ 0 & \text{if } x \in \mathbb{R}^m \setminus U; \end{cases} \);
3) \( \forall \) multi-index \( j \) with \( ||j|| = r \), \( \forall x \in \mathbb{R}^m \), we have that \( |D^j \lambda(x)| \leq m_r \), where the notation \( D^j \) is used in the sense explained in the paragraph below Equation (2.9).

**Proof.** Only the last condition imposed on \( \lambda \) requires some explanation. It is explained in [R73, ch. 1.42] how to construct such a cut-off function on the real line. Using the argument in [CCL94] it is easy to extend it to the case for \( \mathbb{R}^m \). \[ \square \]

**Lemma 6.6.** Let \( B \) be a symplectic ball in \( \mathbb{R}^{2n} \) with the standard symplectic structure \( \omega_0 = \sum_{i=1}^{n} dx_{2i-1} \wedge dx_{2i} \) and let \( T \) be a primitive flat chain in \( \mathcal{F}^p(B) \) with \( 1 \leq p \leq n \). Then there exists a sequence of co-isotropic polyhedral chains \( \{T_k\} \) that converges to \( T \) in \( \mathcal{B}_1^{\prime}(B) \).

**Proof.** By Lemma 4.6, there exists a sequence of \((2n - k)\) dimensional polyhedral chains \( \{Q_k\} \) in \( B \) that converges to \( T \); moreover, all the supports of \( Q_k \) are contained in a single compact set in \( B \). For each \( k \), we write

\[
Q_k = \sum_i a_{ki} \sigma_{ki},
\]

where \( \sigma_{ki} \)'s are \((2n - p)\)-simplices in \( B \), and \( a_{ki} \)'s are non-zero real scalars. Using Lemma 3.2, we assume here that if \( \sigma_{ki} \cap \sigma_{kj} \neq \emptyset \) and if \( i \neq j \), then \( \sigma_{ki} \cap \sigma_{kj} \) is a common proper surface of both \( \sigma_{ki} \) and \( \sigma_{kj} \).

If \( p = 1 \), then for dimensional reasons \( \{Q_k\} \) itself is a sequence of co-isotropic polyhedral chains that converges to \( T \). So we may assume that \( p \geq 2 \).

For each \( k \geq 1 \), let \( \sigma_{k1}, \ldots, \sigma_{kk} \) be all the co-isotropic \((2n - p)\)-simplices appeared on the right hand side of Equation 6.1 and let \( \sigma_{k(i+1)} \), \ldots, \( \sigma_{kq_k} \) be all the \((2n - p)\)-simplices appeared on the right hand side of Equation
which are not co-isotropic. Set

\[ T_k = \sum_{i=1}^{i_k} a_{ki} \sigma_{ki}, \quad \tilde{T}_k = \sum_{i=i_k+1}^{i_k+q_k} a_{ki} \sigma_{ki}. \]

To prove Lemma 6.6 by Theorem 3.13 it suffices to prove that \( \|\tilde{T}_k\|_b \) converges to 0.

We first note that each \( T_k \) is primitive, i.e., \( \omega_0^{n-p+1} \wedge T_k = 0 \). Moreover, since for any fixed \( k \) and any \( i_k + 1 \leq i \leq i_k + q_k, \sigma_{ki} \) is not co-isotropic, the restriction of \( \omega_0^{n-p+1} \) to the interior of \( \sigma_{ki} \) is nowhere vanishing. Therefore there exists a \( (p-2) \)-form \( \alpha_{ki} \) on \( B \) of the form \( cdx_{l_1} \wedge \cdots \wedge dx_{l_{p-2}} \), where \( 1 \leq l_1 < l_2 \cdots < l_{p-2} \leq 2n \) and \( c \) is a constant which depends only on the Euclidean norm of the standard symplectic basis in \( \mathbb{R}^{2n} \), such that

\[
(6.2) \quad a_{ki} \int_{\sigma_{ki}} \omega_0^{n-p+1} \wedge \alpha_{ki} = |a_{ki}| \mathcal{H}^p(\sigma_{ki}),
\]

where \( \mathcal{H}^p \) is the \( p \)-dimensional Hausdorff measure on \( \mathbb{R}^{2n} \).

For any \( k \) and \( i_k + 1 \leq i \leq i_k + q_k \), choose two open sets \( V_{ki} \) and \( U_{ki} \) in \( B \) with \( \overline{V_{ki}} \subset U_{ki} \) and \( \overline{V_{ki}} \) being compact, such that

1) \( \mathcal{H}^p(\text{supp} \sigma_{ki} \setminus V_{ki}) \leq \frac{1}{q_k 2^k (\sum_{i>i_j} |a_{ij}|)} \)

2) \( U_{ki} \cap U_{ks} = \emptyset \) if \( i_k + 1 \leq s \neq l \leq q_k \).

For each \( i \) and \( k \), choose a cut-off function \( 0 \leq \rho_{ki} \leq 1 \) such that \( \rho_{ki} = 1 \) on \( V_{ki} \) and \( \rho_{ki} = 0 \) outside \( U_{ki} \), and such that it also satisfies the third condition imposed in Lemma 6.5. Set

\[
\beta_k = \sum_{i=i_k+1}^{i_k+q_k} \rho_{ki} \omega_0^{n-p+1} \wedge \alpha_{ki}.
\]

Then it follows from Lemma 6.5 that \( \{\beta_k\} \) is a bounded subset in \( \mathcal{B}_1(B) \). Since \( \{Q_j\} \) converges to \( T, \forall \epsilon > 0 \), there exists \( N_1 \) such that

\[
|Q_j(\beta_k) - T(\beta_k)| = |\tilde{T}_j(\sum_{i=i_k+1}^{i_k+q_k} \rho_{ki} \omega_0^{n-p+1} \wedge \alpha_{ki})| < \frac{\epsilon}{2}, \forall j \geq N_1, \forall k \geq 1.
\]

Here we use the fact that \( T \) is primitive and each \( T_j \) is also primitive. In particular, we have that

\[
(6.3) \quad |\sum_{i=i_j+1}^{i_j+q_j} a_{ji} \sigma_{ji} (\rho_{ji} \omega_0^{n-p+1} \wedge \alpha_{ji})| < \frac{\epsilon}{2}, \forall j \geq N_1.
\]
However, by Equation\[\ref{eq:6.2}\] for any \(i > i_j\), \(a_{ji}\sigma_{ji}(\rho_{ji}\omega_0^{n-p+1} \wedge \alpha_{ji}) > 0\). It then follows from Equation\[\ref{eq:6.3}\] that

\[
\sum_{i=i_j+1}^{i_j+q_j} |a_{ji}\sigma_{ji}(\rho_{ji}\omega_0^{n-p+1} \wedge \alpha_{ji})| < \frac{\epsilon}{2}
\]

Note that for each fixed \(i > i_j\), a simple calculation shows that

\[
|a_{ji}\sigma_{ji}(\rho_{ji}\omega_0^{n-p+1} \wedge \alpha_{ji}) - a_{ji}\sigma_{ji}(\omega_0^{n-p+1} \wedge \alpha_{ji})| \\
\leq |a_{ji}| \cdot Hp(\text{supp } \sigma_{ji} \setminus V_{ji}) \leq \frac{1}{q_j2^j}.
\]

It follows that

\[
\sum_{i=i_j+1}^{i_j+q_j} |a_{ji}\sigma_{ji}(\rho_{ji}\omega_0^{n-p+1} \wedge \alpha_{ji}) - a_{ji}\sigma_{ji}(\omega_0^{n-p+1} \wedge \alpha_{ji})| < \frac{1}{2^j}.
\]

Choose an integer \(N \geq N_1\) such that \(\forall j \geq N, \frac{1}{2^j} < \frac{\epsilon}{2}\). Then \(\forall j \geq N,\)

\[
M(\tilde{T}_j) = \sum_{i=i_j+1}^{i_j+q_j} a_{ji}\sigma_{ji}(\omega_0^{n-p+1} \wedge \alpha_{ji}) \\
\leq \sum_{i=i_j+1}^{i_j+q_j} |a_{ji}\sigma_{ji}(\rho_{ji}\omega_0^{n-p+1} \wedge \alpha_{ji}) - a_{ji}\sigma_{ji}(\omega_0^{n-p+1} \wedge \alpha_{ji})| \\
+ \sum_{i=i_j+1}^{i_j+q_j} |a_{ji}\sigma_{ji}(\rho_{ji}\omega_0^{n-p+1} \wedge \alpha_{ji})| < \frac{1}{2^j} + \frac{\epsilon}{2} < \epsilon
\]

This finishes the proof of our claim. Since by Lemma\[\ref{lem:3.6}\]|\(\tilde{T}_j|_b \leq M(\tilde{T}_j)\), Lemma\[\ref{lem:6.6}\] follows immediately.

\[\square\]

**Theorem 6.7.** Let \((M, \omega)\) be a \(2n\) dimensional symplectic manifold, and \(T\) a primitive real flat chain in \(\mathcal{F}(M)\), \(1 \leq i \leq n\). Then there exists a sequence of finite co-isotropic chains \(\{T_k\}\) that converges to \(T\) in \(\mathcal{B}_i(M)\).

**Proof.** Theorem\[\ref{thm:6.7}\] follows immediately from Theorem\[\ref{thm:4.12}\] and Lemma\[\ref{lem:6.6}\].

\[\square\]

7. **The symplectic \(dd^A\)-lemma and the Poincaré duality**

7.1. **The symplectic \(dd^A\)-lemma.**
In this section, assuming that the symplectic manifold $(M, \omega)$ is compact and satisfies the Hard Lefschetz property, we establish the symplectic $\dd^A$-lemma for the De Rham complex of real flat chains. We follow the argument used in [Gui01] to prove the symplectic $\dd^A$-lemma for differential forms, but have it simplified somehow using a result in [Ca05].

**Lemma 7.1.** Assume that $(M, \omega)$ is a compact $2n$ dimensional symplectic manifold. Then there is a natural $\mathfrak{sl}_2$-module isomorphism

$$H^*(M) \cong H^*(\mathcal{F}, d).$$

Moreover, if $(M, \omega)$ satisfies the Hard Lefschetz property, then for any $0 \leq k \leq n$, the Lefschetz map

$$L^{n-k} : H^k(\mathcal{F}, d) \to H^{2n-k}(\mathcal{F}, d), \quad [T] \mapsto \omega^{n-k} \wedge [T],$$

is an isomorphism.

**Proof.** Since $M$ is compact, by Theorem 2.15 there is a natural isomorphism

$$H(M) \cong H^{-\infty}(M).$$

By Theorem 4.16 there is another natural isomorphism

$$H^{-\infty}(M) \cong H(\mathcal{F}, d).$$

It is easy to see that these two isomorphisms are isomorphisms of $\mathfrak{sl}_2$-modules. Lemma 7.1 follows immediately. \qed

**Lemma 7.2.** Assume that $(M, \omega)$ is a compact symplectic manifold with the Hard Lefschetz property, and $T \in \mathcal{F}^p(M)$. Consider the Lefschetz decomposition of $T$ in Equation 5.1. If $T$ is Harmonic and $d$ exact, then each $T_r$ is $d$-exact.

**Proof.** Write

$$T = \sum_{r \geq \max\left(\frac{p+2n}{2}, 0\right)} L^r T_r,$$

where $T_r$ is a primitive flat chain in $\mathcal{F}^{p-2r}$. By Lemma 5.6, each $T_r$ is a closed current and so represents a cohomology class in $H^{p-2r}(\mathcal{F}, d)$. Note that $T_{q+1} = 0$ is $d$-exact. Let us assume by induction that $T_r$ is $d$-exact for $r > k$ and conclude that $T_k$ is $d$-exact. By the induction hypothesis,

$$T' = T - \sum_{r > k} L^r T_r = \sum_{r \leq k} L^r T_r$$

is $d$-exact. Applying $L^{n-p+k}$ we get from the above identity that

$$L^{n-p+k} T' = L^{n-p+2k} T_k + \sum_{r < k} L^{n-(p-2r)+(k-r)} T_r.$$

Since $T_r$ is a primitive flat chain with degree $p-2r$, we have that

$$L^{n-(p-2r)+(k-r)} T_r = 0, \quad \forall \ r < k.$$
It follows that
\begin{equation}
L^{n-p+k} = L^{n-p+2k}.
\end{equation}
The left hand side of Equation 7.4 is d-exact. It follows from Lemma 7.1 that \( T_k \) is d-exact as well.

If \( T \in \mathcal{F}^p \) is Harmonic and d-exact, it then follows easily from Lemma 7.2 and the Weil’s identity that \( \ast T \) must be d-exact as well. This proves the next result.

**Lemma 7.3.** If \( T \in \mathcal{F}^p \) is Harmonic and d-exact, then it is d^\wedge-exact.

In order to prove Theorem 7.5 we also need the following result.

**Lemma 7.4.** If \( T \in \mathcal{F}^p \) is Harmonic and d^\wedge-exact, then it must be d-exact.

**Proof.** If \( T \in \mathcal{F}^p \) is Harmonic and d^\wedge-exact, then \( \ast T \) is Harmonic and d exact. It follows that \( \ast T \) must be d^\wedge-exact. Therefore \( T \) must be d-exact. \qed

**Theorem 7.5.** Assume that \((M, \omega)\) is a compact symplectic manifold with the Hard Lefschetz property. The following symplectic d^\wedge-lemma holds in the space of real flat chains.

\[
\ker d \cap \operatorname{im} d^\wedge = \ker d^\wedge \cap \operatorname{im} d = \operatorname{im} d d^\wedge.
\]

**Proof.** The algebraic proof used in [Ca05, Theorem 4.3] carries over to the present situation verbatim. \qed

### 7.2. Duality between primitive homology and cohomology.

In this section, we establish the Poincaré duality between the primitive homology and cohomology. First, we use primitive flat chains to modify the definition of primitive homology as given in Equation 1.4.

**Definition 7.6.** Let \((M, \omega)\) be a symplectic manifold of dimension \(2n\). For any \(n \leq i \leq 2n\), define
\begin{equation}
C_i(M) = \{ T \in \mathcal{F}^{2n-i}, \ T \text{ is primitive, and } d^\wedge T = 0 \}.
\end{equation}
For any \(n \leq i \leq 2n-1\), we define the i-th primitive homology \(PH_i(M)\) as follows.

\[
PH_i(M) = \frac{\ker d \cap C_i(M)}{\operatorname{im} d \cap C_{i+1}(M)}.
\]

We recall that a real flat chain \( T \) in \( \mathcal{F}^{2n-i} \) is primitive if and only if \( L^{i-n+1} T = 0 \) or equivalently, \( \Lambda T = 0 \).

**Lemma 7.7.** The primitive homology \(PH_i(M)\) is well defined. That is to say, if \( T \in C_i(M) \) for some \( n+1 \leq i \leq 2n \), then \( dT \in C_{i-1}(M) \).
Proof. By definition we have that $\Lambda T = d^\Lambda T = 0$. It then follows from the commutator relation $[d, \Lambda] = d^\Lambda$ that $\Lambda d T = 0$. This proves that $d T$ is primitive. Hence $\partial T$ must also be primitive. Since $d$ anti-commutes with $d^\Lambda$, $d T$ is also $d^\Lambda$-closed. Therefore $\partial T$ is also $d^\Lambda$-closed. □

Remark 7.8. 1) For a polyhedral chain $T$, the conditions used in Equation 7.5 is equivalent to that both $T$ and $\partial T$ are co-isotropic. We refer to [TY09, Sec. 4.1] for an explanation.

2) Assume that $T$ is a flat chain in $C_i(M)$. By Theorem 6.7, if $n \leq i \leq 2n - 1$, then there exists a sequence of finite co-isotropic $i$-chains that converges to $T$ in $B'(M)$; moreover, if $n < i \leq 2n - 1$, then there also exists a sequence of finite co-isotropic $(i - 1)$-chains that converges to $\partial T$ in $B'(M)$.

3) In view of the definition of primitive homology given in Equation 1.4, it may be interesting to ask if any primitive flat chains in $C_*(M)$ can be approximated by a sequence of finite co-isotropic chains which have co-isotropic boundaries, or equivalently, by a sequence of finite co-isotropic chains which are $d^\Lambda$-closed. We do not have an answer to this question, and we think that it may be too good to be true.

Definition 7.9. Let $(M, \omega)$ be a symplectic manifold of dimension $2n$. For any $0 \leq i \leq n$, define
\[
C_i(M) = \{T \in F^i, T \text{ is primitive, and } d^\Lambda T = 0\}.
\]
For any $1 \leq i \leq n$, define the $i$-th primitive cohomology $PH^i_{d^\Lambda}(C^*, d)$ to be
\[
PH^i_{d^\Lambda}(C^*, d) = \frac{\ker d \cap C_i(M)}{\im d(C_{i-1}(M))}.
\]

Theorem 7.10. Let $(M, \omega)$ be a $2n$ dimensional compact symplectic manifold with the Hard Lefschetz property. Then for any $0 \leq i \leq n$, there is a natural isomorphism $PH^i(M) \cong PH^i_{d^\Lambda}(C^*, d)$.

In particular, any primitive cohomology class in $PH^i(M)$ is represented by a symplectic Harmonic primitive flat chain in $F^i(M)$.

Proof. For any $0 \leq i \leq n$, set $PH^i(F, d) = \ker (L^{n-i+1} : H^i(F, d) \to H^{2n-i+2}(F, d))$.

It follows easily from Lemma 7.1 that there is a natural isomorphism $PH^i(M) \cong PH^i(F, d)$.

Using the symplectic $dd^\Lambda$-lemma established in Theorem 7.5, the same argument as we gave in the proof of Proposition 2.7 shows that there is a natural isomorphism $PH^i_{d^\Lambda}(C^*, d) \cong PH^i(F, d)$.
Combining the isomorphism (7.6) and the isomorphism (7.7), we get a natural isomorphism
\[ \text{PH}^i_{d\Lambda}(C^*, d) \cong \text{PH}^i(M). \]

We are ready to prove the Poincaré duality between the primitive cohomology and the primitive homology.

**Theorem 7.11. (Poincaré Duality)** Assume that \((M, \omega)\) is a compact 2n dimensional symplectic manifold with the Hard Lefschetz property. Then for any \(0 \leq i \leq n\), there is a natural isomorphism
\[ \text{PH}_{2n-i}(M) \cong \text{PH}^i(M). \]

**Proof.** It is clear by definition that there is a natural isomorphism
\[ \text{PH}_{2n-i}(M) \cong \text{PH}^i_{d\Lambda}(C^*, d). \]

Theorem 7.11 follows immediately from Theorem 7.10.

The following result is important in the proof of Theorem 1.1.

**Corollary 7.12.** Assume that \((M, \omega)\) is a compact symplectic manifold with the Hard Lefschetz property. Then every primitive cohomology class with degree greater than zero is represented by a primitive symplectic Harmonic form which is not supported everywhere on the entire manifold \(M\).

**Proof.** By Theorem 7.10, every primitive cohomology class of degree \(k\) is represented by a closed primitive current \(T\) of degree \(k\). By Theorem 6.7, there exists a sequence of finite chains \(\{T_k\}\) of dimension \(2n - k\) that converges to \(T\) in \(B^i\). When \(k > 0\), by the Baire category theorem, the complement of \(\bigcup_k \text{supp } T_k\) forms an open and dense subset of \(M\). Since we have that \(\text{supp } T \subset \bigcup_k \text{supp } T_k\), it follows that \(\text{supp } T\) can not be supported on the entire manifold \(M\). Now the assertion of Corollary 7.12 follows from Theorem 2.9.

We now use the main results established in this section to investigate the following complex of primitive differential forms on a 2n dimensional symplectic manifold \((M, \omega)\) introduced in [TY10].

\[
0 \to P' n \to P n \xrightarrow{d\Lambda} P_{n-1} \xrightarrow{d\Lambda} \cdots \xrightarrow{d\Lambda} P 1 \xrightarrow{d\Lambda} P 0 \xrightarrow{d\Lambda} 0.
\]

Here \(P'k\) is the space of \(d\Lambda\)-closed primitive \(k\)-forms, and \(Pk\) is the space of primitive \(k\)-forms. For any \(0 \leq k < n\), denote by \(\text{PH}^k_{d\Lambda}(M)\) the \(k\)-th cohomology of the complex (7.8). Due to the commutator relation \([d\Lambda, \Lambda] = 0\), \(d\Lambda\) must map primitive forms to primitive forms. So the differential complex (7.8) is well-defined.
Now for any $0 \leq k < n$, consider the natural inclusion map
\[(7.9) \quad p^k \rightarrow p^\prime k.\]

**Proposition 7.13.** Let $(M, \omega)$ be a compact $2n$ dimensional symplectic manifold with the Hard Lefschetz property. Then for any $0 \leq k < n$, the inclusion map (7.9) induces the following isomorphism of cohomologies.
\[(7.10) \quad \text{PH}^k_{\Lambda d}(M) \rightarrow \text{PH}^k_{\Lambda d}(M).\]

*Proof.* Suppose that $\alpha = d\beta \in P^k$ for some $(k - 1)$-form $\beta \in P^{k-1}$. Then $\alpha$ is both $d$-exact and $d^\Lambda$-closed. It follows form Theorem 2.5 that there exists a $k$-form $\varphi$ such that $\alpha = d\Lambda \varphi$. Since $\alpha$ is primitive, by Lemma 2.6 we can assume that $\varphi$ is a primitive k-form. By Lemma 2.3 $d\Lambda \varphi = d^\Lambda \eta$ for some primitive form $\eta$. This proves that the homomorphism (7.10) induced by the inclusion map (7.9) is well-defined.

Let $\alpha \in P^k$ be a representative of a cohomology class in $\text{PH}^k_{\Lambda d}(M)$ such that $\alpha$ is $d^\Lambda$-exact. So $\alpha$ is both $d^\Lambda$-exact and $d$-closed. By Theorem 2.5 there exists a $k$-form $\gamma$ such that $\alpha = d\Lambda \gamma$. By Lemma 2.6 we can assume that $\gamma$ is a primitive k-form. Since $d\Lambda \gamma$ is also primitive, $\alpha$ must represent a trivial cohomology class in $\text{PH}^k_{\Lambda d}(M)$. This proves that the homomorphism (7.10) is injective.

Now let $\beta$ be a representative of a cohomology class in $\text{PH}^k_{\Lambda d}(M)$. By definition, $d^\Lambda \beta = 0$. Since $d$ anti-commutes with $d^\Lambda$, $d\beta$ is both $d^\Lambda$-closed and $d$-exact. By Theorem 2.5 there exists a $(k + 1)$-form $\eta$ such that $d\beta = d\Lambda \eta$. Since $\beta$ is $d^\Lambda$-closed and since $d^\Lambda = [d, \Lambda]$, we have that $\Lambda d\beta = 0$. It follows that $d\beta$ is also a primitive form. By Lemma 2.6 we can assume that $\eta$ is a primitive $(k + 1)$-form. As a result, $d^\Lambda \eta$ must also be primitive. Note that $d(\beta - d^\Lambda \gamma) = 0$. We conclude that $\beta - d^\Lambda \gamma$ is a symplectic Harmonic primitive k-form which represents the same cohomology class as $\beta$ in $\text{PH}^k_{\Lambda d}(M)$. This proves that the homomorphism (7.10) is surjective. 

\[\square\]

For any $0 \leq k \leq n$, let $P^k_F$ be the space of primitive flat chains of degree $k$, and let $C^k$ be the space of $d^\Lambda$-closed primitive flat chains of degree $k$ as given in Definition 7.9. Consider the following differential complex.
\[(7.11) \quad 0 \rightarrow C^n \stackrel{i}{\rightarrow} P^n_F \stackrel{d^\Lambda}{\rightarrow} P^{n-1}_F \rightarrow \cdots \rightarrow P^1_F \stackrel{d^\Lambda}{\rightarrow} P^0_F \stackrel{d^\Lambda}{\rightarrow} 0.\]

For any $0 \leq k < n$, denote by $\text{PH}^k_{d^\Lambda}(\mathcal{F}(M))$ the $k$-th cohomology of the complex (7.11). Now consider the inclusion map
\[(7.12) \quad C^k \rightarrow P^k_F,\]
where $0 \leq k < n$.

In view of the symplectic $dd^\Lambda$-lemma established in Theorem 7.5, an argument similar to the one used in the proof of Proposition 7.13 gives us the following result.
Proposition 7.14. Let \((M, \omega)\) be a compact symplectic manifold with the Hard Lefschetz property. Then for any \(0 \leq k < n\), the inclusion map (7.12) induces the following isomorphism of cohomologies.

\[(7.13) \quad \text{PH}_{d+}(C^*, d) \to \text{PH}_d^k(F(M))\]

The following result asserts that every cohomology class in \(\text{PH}_d^*(M)\) can be represented by a \(d^\wedge\)-closed primitive real flat chain.

Theorem 7.15. Let \((M, \omega)\) be a compact symplectic manifold with the Hard Lefschetz property. Then for any \(0 \leq k < n\), there is a natural isomorphism
\[
\text{PH}_d^k(M) \cong \text{PH}_d^k(F(M)).
\]

Proof. Theorem 7.15 is an immediate consequence of Proposition 2.7, Proposition 7.13, Proposition 7.14, and Theorem 7.10.

Remark 7.16. 1) In [TY10], \(\text{PH}_d^*(M)\) is defined using the so-called \(\partial_-\) differential operator on a symplectic manifold \(M\). However, when acting on primitive forms of a given degree, \(\partial_-\) differs from \(d^\wedge\) only by a multiple of a non-zero constant.

2) Let \((M, \omega)\) be a compact symplectic manifold with the Hard Lefschetz property. A result similar to Theorem 7.15 also holds for the primitive cohomology \(\text{PH}_d^*(M)\) introduced in [TY10].

8. SYMPLECTIC HARMONIC REPRESENTATIVES OF THOM CLASSES

In this section, we establish Theorem 1.1 and provide an answer to the question asked by Victor Guillemin.

Lemma 8.1. Let \((M, \omega)\) be a compact \(2n\) dimensional symplectic manifold, \(N\) an oriented compact isotropic submanifold of \(M\), and \([\tau_N]\) the Thom class of \(N\). Then \([\omega \wedge \tau_N] = 0\).

Proof. Without the loss of generality, we may assume that \(\text{codim } N \leq 2\). Since \(N\) is an isotropic submanifold of \(M, \omega|_N = 0\). Now for any close form \(\beta\) of degree \(\dim N - 2\), we have
\[
\int_M \beta \wedge \omega \wedge \tau_N = \int_N \beta \wedge \omega = 0.
\]

It then follows from the Poincaré duality that \([\omega \wedge \tau_N] = 0\).

Lemma 8.2. Let \((M, \omega)\) be a \(2n\)-dimensional symplectic manifold with the Hard Lefschetz property. Let \(N\) be a compact oriented isotropic submanifold of \(M\) with codimension \(k\), and \([\tau_N]\) the Thom class of \(N\). Then there exist a closed primitive \(k\)-form \(\alpha_k\), such that
\[
[\tau_N] = \Lambda^{n-k}[\alpha_k].
\]
Proof. Due to the Lefschetz decomposition (2.8), $\tau_N$ is cohomologous to the following form

$$L^{n-k} \alpha_k + L^{n-k+1} \land \alpha_{k-2} + L^{n-k+2} \land \alpha_{k-4} + \cdots.$$ 

where $\alpha_{k-2i}$ is a closed primitive form of degree $k-2i$ for each $i \geq 0$. However, by Lemma 8.1, $[\omega \land \tau_N] = 0$. Now using the definition of primitive forms, an easy calculation shows that $[\omega^{n-k+1} \land \alpha_k] = 0$ and that

$$0 = [\omega \land \tau_N] = L^{n-k} [\alpha_{k-2}] + L^{n-k+1} [\alpha_{k-4}] + \cdots.$$ 

By the uniqueness of the Lefschetz decomposition (2.8) we have that $[\alpha_{k-2}] = [\alpha_{k-4}] = \cdots = 0$. □

Lemma 8.3. Let $(M, \omega)$ be a $2n$-dimensional symplectic manifold with the Hard Lefschetz property. Let $N$ be a compact oriented submanifold of $M$ with codimension $k$, and $[\tau_N]$ the Thom class of the submanifold $N$. Consider the following Lefschetz decomposition of $[\tau_N]$.

$$[\tau_N] = \sum_{r \geq \max\left(\frac{k-n}{2}, 0\right)} L^r [\alpha_{k-2r}],$$

where $\alpha_{k-2r}$ is a closed primitive differential form of degree $k-2r$. If $k$ is odd, or if $k = 2p$ is even and $L^{n-p} [\tau_N] = 0$, then all the non-zero primitive cohomology classes $[\alpha_{k-2r}]$ appeared on the righthand side of Equation (8.1) are of degree greater than zero.

Proof. When $k$ is odd, Lemma 8.3 follows from dimension consideration. Now assume that $k = 2p$ is even and write

$$[\tau_N] = \sum_{r=1}^{p} L^r [\alpha_{2p-2r}],$$

where $\alpha_{2p-2r}$ is a closed primitive differential form of degree $2p-2r$. It suffices to show that $[\alpha_0] = 0$. Applying $L^{n-p}$ to the both sides of Equation (8.2) we get that

$$0 = \sum_{r=1}^{p} L^{n-p+r} [\alpha_r] = \sum_{r=1}^{p-1} L^{n-p+r} [\alpha_{2p-2r}] + L^n [\alpha_0].$$

Note that for any $1 \leq r \leq p-1$,

$$L^{n-p+r} [\alpha_{2p-2r}] = L^{(p-r)-1} L^{n-2(p-r)+1} [\alpha_{2p-2r}] = 0,$$

since $[\alpha_{2p-2r}]$ represents a primitive cohomology class of degree $2p-2r$. It follows that $L^n [\alpha_0] = 0$. Since the symplectic manifold $M$ satisfies the Hard Lefschetz property, we must have that $[\alpha_0] = 0$. □
Now we are ready to prove Theorem 1.1.

Proof. It is a direct consequence of Lemma 8.2, Lemma 8.3 and Corollary 7.12. □

Remark 8.4. Without the assumption that the symplectic manifold satisfies the Hard Lefschetz property, it is easy to use [L04, Prop. 2.3] to show that there are examples of compact oriented isotropic submanifolds whose Thom classes do not have any symplectic Harmonic representatives.

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