DETERMINATION OF THE GENUS OF SURFACES FROM THE SPECTRUM OF SCHRODINGER OPERATORS ATTACHED TO HEIGHT FUNCTIONS

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Abstract. Using results on inverse spectral problems, in particular the so-called new wave invariants attached to a classical equilibrium, we show that it is possible to determine the Morse index of height functions. For compact Riemannian surfaces \( M \subset \mathbb{R}^3 \) this imply that we can retrieve the topology (via the genus).

Our results are independent from the choice of a metric on \( M \) and can be obtained from the choice of a ‘generic’ height-function. For surfaces of genus zero, diffeomorphic to a 2-sphere, the method allows to detect the convexity, or the local convexity of the surface.

keywords: Micro-local analysis; Schrödinger operators; Inverse spectral problems.

1. Introduction.

1.1. Basic definitions and setting. We are here interested in applications of the inverse spectral problem for certain special differential operators in the semi-classical regime. Let \( X \) be compact (boundaryless) Riemannian manifold equipped with a strictly positive density and \( \Delta_X \) the (positive) induced Laplace-Beltrami operator. In particular, we consider the so called \( h \)-quantized Schrödinger operator:

\[
P_h = h^2 \Delta_X + V, \text{ on } L^2(X),
\]

also called semi-classical Schrödinger operator, where the potential \( V \) is measurable and bounded from below on \( X \).

Remark 1. With some mild conditions on \( V \), we could also assume that \( X \) is non-compact. But to simplify we stay in the compact situation.

By a standard result, see \cite{3}, when \( V \) is bounded from below \( P_h \) has a self-adjoint realization on a dense subset of \( L^2(X) \). To this quantum operator \( P_h \) we can associate a classical counterpart with the Hamiltonian, or total energy, function on the phase space:

\[
p(x, \xi) = ||\xi||_x^2 + V(x) \text{ on } T^*X.
\]

Here the notation:

\[
||\xi||_x^2 = \sum_{i,j} g_{ij} \xi_i \xi_j, \quad g_{ij} = G^{-1},
\]
designs the norm (or scalar product at \( x \)) induced by the Riemannian metric \( G = g^{ij}(x) \) at \( x \). We note \( \Phi_t \) the Hamiltonian flow of \( H_p = \partial_\xi p_\cdot \partial_\xi - \partial_x p_\cdot \partial_x \).

Here we are mainly interested in an asymptotic relation between the semi-classical eigenvalues \( \{ \lambda_j(h) : j \in \mathbb{N} \} \) of \( P_h \):

\[
P_h \phi_j(x, h) = \lambda_j(h) \phi_j(x, h), \quad \phi_j \in L^2(X), \quad \text{as } h \to 0^+,
\]

and the set of fixed point \( \mathcal{P} \) (see below) for the map \( \Phi_t \) (viewed as a map on \( \mathbb{R}_t \times T^*X \)). We refer to the introduction of [7] for a general presentation of this kind of relation between quantum and classical mechanics.

In the last section of this article we will work with compact orientable surfaces \( M \) of \( \mathbb{R}^3 \) and Schrödinger operators \( h^2 \Delta_M + z \) attached to height-functions \( z \) on \( M \).

**Spectral statistics.**

Consider an interval \( I = [E_1, E_2] \) with \( E_1 < E_2 \) and \( I(\varepsilon) = [E_1 - \varepsilon, E_2 + \varepsilon] \). For each \( \varepsilon > 0 \) the pullback \( p^{-1}(I(\varepsilon)) \) is a compact subset of \( T^*X \) and by a standard argument, see [3], it follows that the spectrum \( \sigma(P_h) \cap I(\varepsilon) \) is discrete and consists for each \( h \) in a sequence:

\[
\lambda_1(h) \leq \lambda_2(h) \leq \ldots \leq \lambda_j(h),
\]

of eigenvalues of finite multiplicities, if \( \varepsilon \) and \( h \) are positive. In general no formula is known to compute the eigenvalues \( \lambda_j(h) \) and to get pertinent information about the spectrum (and the classical dynamics) it is interesting to study the following spectral distributions:

\[
\Upsilon(E, h, \varphi) = \sum_{\lambda_j(h) \in I(\varepsilon)} \varphi(\frac{\lambda_j(h) - E}{h}),
\]

where \( \varphi \in \mathcal{S}(\mathbb{R}) \) is a test function, conveniently chosen, see below.

This kind of problem leads to a mathematically rigorous version of the so-called Gutzwiller formula (see [13]), a formula intensively used in physics and quantum-chemistry. For example see [14] for various applications in physics and quantum chaos with also many references. The so-called semi-classical approximation consists in studying the asymptotic behavior of Eq.(1) as the parameter \( h \) tends to zero. For a rigorous mathematical study of this problem a non-exhaustive list of references is [5, 8, 17, 18].

**Wave and new wave invariants.**

To study Eq.(1), a classical approach (see [11, 5] and section 3 of [7]) is to study the asymptotic behavior, as \( h \to 0^+ \), of the localized trace:

\[
\omega(E, h, t) = \text{Tr} \left( \Theta(P_h) e^{-\frac{i}{h}(P_h - E)} \right), \quad \Theta \in C^\infty_0.
\]

I will follow now the terminology used in [15]. For \( E \) regular it is known since a while that \( \Omega \) admits an asymptotic expansion of the form:

\[
\omega(E, h, t) \sim \sum_{j=-n}^{\infty} a_j(E, t) h^j, \quad \text{as } h \to 0^+.
\]
Definition 2. The coefficients $a_j(E, t)$ are some distributions on the line $\mathbb{R}_t$ and are called wave invariants of $P_h$.

When $E = E_c$ is critical certain new coefficients appear in the asymptotic expansion of $\omega(E, h, t)$ as $h$ tends to 0. In particular, one can predict a general expansion in the form:

$$\omega(E_c, h, t) \sim \sum_{k=0}^{n-1} \sum_{j=-n_0}^{\infty} a_{j,k}(E_c, t) h^k \log(h)^j, \quad \text{as } h \to 0^+,$$

for some $p \in \mathbb{N}^*$, see [4] for the singularity near $t = 0$ and [7] for many examples of new wave invariants. In this work we only use the top-order coefficients of Eq. (2) near a non-degenerate singularity (see Theorem 6 below).

Definition 3. These extra distributional coefficients appearing in Eq. (2) are called new wave invariants.

In general, the top order coefficient w.r.t. $h$ of the expansion involving the new wave invariants contains many information on the shape of the symbol.

2. Hypotheses and semi-classical results.

We recall first the general result determining the wave invariants at a regular energy level. Consider $X$ a closed smooth Riemannian manifold, $n = \dim(X)$ and $V \in C^\infty(X)$ a positive potential. Let $\{\lambda_j(h) : j \in \mathbb{N}\}$ be the spectrum of the Schrödinger operator $S_h = h^2 \Delta_X + V$ where $\Delta_X$ is the Laplacian on $X$ (here given as a positive operator). This spectrum of $S_h$ is always discrete, and with finite multiplicities, when $X$ is compact or if $V$ is 'confining', i.e. $V(x) \to \infty$ when $d(x, x_0) \to \infty$ for some $x_0 \in X$. The justification is that both conditions insure that the level-sets $\Sigma_E$ defined below are compact and that the resolvent of $S_h$ is a compact operator.

Let $\Phi_t$ be the Hamilton flow of $H(x, \xi) = ||\xi||_2^2 + V(x)$ on $T^*X$ and given $E > 0$ define the energy surface:

$$\Sigma_E = \{(x, \xi) \in T^*X : H(x, \xi) = E\}.$$

Recall that the flow can be viewed as a map $\Phi_t : \Sigma_E \to \Sigma_E$ (conservation of the energy). Also when the surfaces $\Sigma_E$ are compact the general theory of differential equations insures that the flow is complete (property of the maximal solutions of a Cauchy-problem). To simplify notations we write $z = (x, \xi) \in T^*X$. We recall that $E$ is regular (or non-critical) when $dH \neq 0$ everywhere on $\Sigma_E$ and critical otherwise. Below, we use the subscript $E_c$ to distinguish out critical values of $H$. The so-called period manifold of $\Phi_t$ on $\Sigma_E$ is:

$$\mathcal{P} = \{(T, z) \in \mathbb{R} \times \Sigma_E : \Phi_T(z) = z\}.$$

At a non-critical energy level.

When $E$ is non-critical, we have, see [11] or [5], the following general result concerning the wave invariants at the energy $E$: 
Theorem 4 (Semi-classical trace formula at a regular level.).

Assume that $E$ is regular and that the restriction of $\Phi_t$ to $\Sigma_E$ is a clean flow (see section 2 of [11]). Then there exists a sequence of distributions on the real line, $\{\gamma_k\}_k$, such that for every test function $\varphi$ with Fourier-transform $\hat{\varphi} \in C^\infty_0$:

$$\text{Tr} \varphi\left( \frac{S_h - E}{h} \right) = \sum_{j=0}^{\infty} \varphi\left( \frac{\lambda_j(h) - E}{h} \right) \sim \sum_{j=1}^{\infty} \gamma_j(\hat{\varphi}) h^{-n+jc_j(h)}.$$

Moreover, the supports of the distributions $\gamma_j$ are contained in the sets of periods of the closed trajectories of $\Phi_t$ on $\Sigma_E$.

For a better description of the coefficients appearing in Theorem 4 we refer to [11, 5] (we do not need their explicit expressions here), see also [9] for a similar high-energy result concerning elliptic operators on a compact manifold. Also, under our hypotheses the trace in Theorem 4 and the functional $\Upsilon(E,h,\varphi)$ are equal modulo a function of fast decay w.r.t. $h$.

Such a coefficient, of order $O(h^\infty)$, is negligible in semi-classical asymptotic expansions.

The idea we want to use here is that by a clever choice of supp($\hat{\varphi}$) we can eliminate the wave invariants appearing in Theorem 4:

- If $\hat{\varphi}$ is flat at the origin the set:
  $$\{\{0\} \times \Sigma_E\} \subset P,$$
  does not contribute.
- If supp($\hat{\varphi}$) $\subset [-T_0,T_0]$, for $T_0$ small enough then no periodic orbit:
  $$\{(T,z) : \Phi_T(z) = z\} \subset P,$$
  will contribute to the asymptotic expansion.

At a critical energy level. We allow now the presence of critical points for $H$ and we impose the type of singularity:

(A1) The potential $V$ is a Morse function on $X$.

A fortiori, in $I$ there is finitely many critical values $E^1_c,...,E^l_c$ and in $p^{-1}(I)$ finitely many fixed points $z^1_0,...,z^l_0$ of the energy function $p$.

Remark 5. The number of critical points $z^j_0 = (x_0,0)$ is equal to the number of critical energy levels. Otherwise $V$ would not be a Morse function on $X$.

Next, we impose two conditions on our test function $\varphi$:

(A2) $\hat{\varphi}$ is flat at 0, i.e. $\hat{\varphi}^{(j)}(0) = 0$, $\forall j \in \mathbb{N}$.

(A3) For some sufficiently small $T$ we have supp($\hat{\varphi}$) $\subset [-T,T]$.

A fundamental property is that the singularity of $\Upsilon(s,h,\varphi)$ as $s \to E_c$ describes partially the singularity of $V$. In fact with conditions (A2) and (A3) we will only see the new wave invariants attached to the critical point $z^j_0$ in $\Sigma_{E_j}$. We have:
Theorem 6 (New wave invariants at a critical level).
Under the conditions \((A_1), (A_2)\) and \((A_3)\) we have:
\[
\text{Tr} \varphi \left( \frac{S_h - E_j}{h} \right) \sim \sum_{j=0}^{\infty} h^j c_j(\hat{\varphi}).
\]
The leading coefficient is of the form:
\[
c_0(\varphi) = \frac{e^{i\pi m_0/2}}{(2\pi)} \int_{\mathbb{R}} \frac{\hat{\varphi}(t)}{|\det(d\Phi_t(z_0^j) - \text{Id})|^{1/2}},\ m_0 \in \mathbb{Z}.
\]
We refer to \([11, 7, 6, 16]\) for a proof. Observe that \((A_3)\) implicitly insures that \(\det(d\Phi_t(z_0^j) - \text{Id}) \neq 0\). Because of our implicit choice for \(\varphi\), not all the new wave invariants are present in this formula. The other new wave invariants are studied:
- In \([4]\), near \(t = 0\).
- In \([6, 16]\), near a period of \(d\Phi_t(z_0^j)\).

The explicit determination of all wave invariants, near a critical point of arbitrary signature, is a somehow complicated analytic problem involving oscillatory integrals with degenerate phases. For an operator which is not a Schrödinger operator some new terms can generally appear at a period of \(d\Phi_t(z_0^j)\) (see \([6]\)).

**New wave invariants.** In our setting, the top-order coefficient, given by the Duistermaat-Guillemin-Uribe density, is indeed a smooth function as long as we stay away from any period of the linearized flow at the point \(z_0^j\). When \(X = \mathbb{R}^n\) an explicit computation, done in \([16]\) in a suitable system of linear coordinates, shows that:
\[
d\nu_t(z_0) = \frac{1}{| \prod_{j=1}^{r} \sinh(\alpha_j(z_0)t) \prod_{j=r+1}^{n} \sin(\alpha_j(z_0)t)|}.
\]
We must simply retain that the density \(d\nu_t(z_0)\) determines:
- The signature \((n - r, r)\) of the Hessian of \(V\) at \(z_0\).
- Eigenvalues \(\alpha_j(z_0)\).

The last affirmation follows via Taylor-series and evaluation at several times.

**Remark 7.** In general, if the metric and the height function are unknown, the spectral expectation determines only the numbers \(\alpha_j\) and not the respective eigenvalues of \(G(x_0)\) and \(d^2V(x_0)\). A similar indetermination is already valid for linear combinations of harmonic oscillators on \(\mathbb{R}^n\).

It follows that, when the potential is Morse-function on \(X\), we can retrieve the morse index of \(X\) by several successive applications of Theorem 6: we have only to cross finitely many critical energy levels and to collect the index at each energy.
3. Application to Surfaces.

There is a nice application to compact smooth surfaces $M \subset \mathbb{R}^3$ equipped with a Riemannian metric (not necessarily the metric of $\mathbb{R}^3$ restricted to $M$). We assume that $M \subset \mathbb{R}^3$ is smooth, boundaryless, orientable and that $M$ carries a smooth Riemannian metric $G$, fixed once for all. We take $\Delta_M$ as the Laplace-Beltrami operator attached to this metric (following the convention of geometers we may assume that $\Delta_M$ is positive). Let us chose as potential $V$ a height function. We can assume $V$ to be positive, this can always be achieved via a translation, $M$ being compact. If we embed $M$ in $\mathbb{R}^3$, via some coordinates $(x,y,z)$, we can chose $V$ as the projection on the $z$ axis. It is a standard result of topology, see chapter 6 of [2], that for almost embedding $V$ will be a Morse function.

Then, for any choice of a smooth Riemannian metric on $M$, we have:

**Proposition 8.** Under the previous conditions on $V$, the semi-classical spectrum of $P_h = h^2\Delta_M + V(x)$, defined as an unbounded operator on $L^2(M)$, determines the topology of $M$.

**Remark 9.** Observe that the knowledge of the metric is not required. We only need a kinetic energy operator which is micro-locally elliptic and with a principal symbol nowhere degenerated (see below). The knowledge of $V$ is also not required. We only need to recover the number of critical points of $V$ and their signature to conclude.

**Proof of Proposition 8.** We will use a variational argument w.r.t. the energy $E$. Since $M$ is compact our potential has a maximum $E_{\text{max}}$ and it will be sufficient to perform spectral estimates below $E_{\text{max}}$. Let $\lambda_j(h)$ be the spectrum of $P_h$, each eigenvalue being repeated according to it’s multiplicity.

Since our potential is a Morse function, by Sard’s theorem, we obtain that the energy function $p(x,\xi)$ has only finitely many critical values $E_j^c$, $j \in \{1,\ldots,N\}$, attached to single critical points. When $\text{supp}(\hat{\varphi})$ is small enough and does not contains the origin we have:

$$\gamma(E,h,\varphi) \sim \begin{cases} O(h^{\infty}), & \text{for } E \text{ non-critical,} \\ \epsilon_j^c(\varphi) + O(h), & \text{for } E = E_j^c \text{ critical.} \end{cases}$$

Hence, the semi-classical spectrum determines each critical value $E_j^c$ of $V$.

Now for $j$ fixed we can use a simple micro-local argumentation. The only critical point on $\Sigma_{E_j^c}$ is of the form $z_j^0 = (x_j^0,0)$ with $V(x_j^0) = E_j^c$. We pick a function in $\psi \in C_0^\infty(T^*M)$ such that $0 \leq \psi \leq 1$ everywhere and $\psi = 1$ in a neighborhood of $z_j^0$. Always with our conditions ($A_2$) and ($A_3$) on $\text{supp}(\hat{\varphi})$, we have:

$$\Upsilon(E,h,\varphi) = \text{Tr} \left( \psi^{\mu}(x, h D_x) \varphi \left( \frac{P_h - E_j^c}{h} \right) \right) + O(h^{\infty}).$$
The important fact here is that on $\text{supp}(1 - \psi)$ there is no critical point of $p$. Hence with condition $(A_2)$ and $(A_3)$ we have:

$$\text{Tr} \left( (1 - \psi^w(x, hD_x)) \varphi \left( \frac{P_h - E^j_c}{h} \right) \right) = \mathcal{O}(h^{\infty}),$$

which easily follows from a non-stationary phase argument. On $\text{supp}(\psi)$, which can be chosen arbitrary small up to an error of order $\mathcal{O}(h^{\infty})$, we can use local coordinates around $x^0_j$ and the Laplace operator has the form:

$$-h^2 \sum_{i,j} \sqrt{g} \frac{\partial}{\partial x_i} \frac{1}{\sqrt{g}} g_{ij} \frac{\partial}{\partial x_j} + V = -h^2 \sum_{i,j} g_{ij}(x) \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} + V + h^2 \sum_{i,j} \sqrt{g} \frac{\partial}{\partial x_i} \left( \frac{1}{\sqrt{g}} g_{ij}(x) \right) \frac{\partial}{\partial x_j}.$$

Here we write the metric $G = g^{ij}$, $G^{-1} = g_{ij}$ and $g = \det G$. Hence, in the sense of the $h$-calculus, we have $p_h := p_0 + hp_1$ with a principal symbol:

$$p_0(x, \xi) = \sum_{i,j} g_{ij}(x) \xi_j \xi_i + V(x),$$

and a sub-principal symbol:

$$p_1(x, \xi) = \sqrt{g}(x) \sum_{i,j} \left( \frac{\partial}{\partial x_i} \frac{1}{\sqrt{g}} g_{ij} \right) x_j \xi_j.$$

Observe that $p_1 = 0$ at every point where $\xi = 0$ and the sub-principal symbol $p_1$ will play no essential role for the estimates below. You can also use the convention that $h^2\Delta$ is the quantization of $||\xi||^2$. This makes no difference for the spectral estimates below.

Next, since $V$ is independent of the choice of the metric on $M$ we can freely assume that near the origin:

$$g_{ij}(x) = \text{Id} + \mathcal{O}(||x||).$$

In this system of local coordinates we obtain that:

$$\text{Tr} \left( \psi^w(x, hD_x) \varphi \left( \frac{P_h - E^j_c}{h} \right) \right) \sim C^j_0(\hat{\varphi}) + \mathcal{O}(h),$$

where:

$$C^j_0(\hat{\varphi}) = C \int_{\mathbb{R}} \nu_t(z^0_j) \hat{\varphi}(t) dt, \quad C \in \mathbb{C}^*.$$
In our setting we can apply standard results on the linearized flow, see [1], to compute the density \( \nu_t(z_0^j) \). This density, of the form given by Eq. (3), determines the number of positive and negative eigenvalues of \( V \) at \( x_0^j \).

In general position, define the Morse index of a critical point \( x_0 \) as the dimension of the negative eigenspace in \( x_0 \). If \( z \) is Morse function we denote by \( N_j(z) \) the number of critical points of \( z \) with index \( j \). Since for a surface we have only \( j = 0, 1, 2 \) we can retrieve the Euler characteristic:

\[
N_0(z) - N_1(z) + N_2(z) = \chi(M) = 2(1 - g(M)),
\]

where \( g(M) \) is the genus of \( M \).

**Remark 10.** Of course this approach is still valid in dimension \( n > 2 \) but then the genus is no more a sufficient topological invariant. For a nice overview on Morse theory and indexes see [2] or [10] for surfaces. Observe that, for an unknown metric and an unknown height function, we can still retrieve the Morse index of \( V \) but not the Hessian of \( V \). This is because in formula (3) we can only retrieve the ratios \( \alpha_j(z_0) \) of eigenvalues of \( G(x_0^j) \) and \( d^2V(x_0^j) \) in a given system of coordinates.

The notion of Morse-Smale function (see, e.g., [2] p.158) is here central. Morse-Smale functions are moreover dense in every \( C^r \)-spaces (\( r \geq 1 \)) (Kupka-Smale-Theorem, p.159 and remark 6.7 p.160 in [2]). To a Smale-Morse function is attached the Morse-Smale-Floer complex and this complex is isomorphic to the complex giving the singular homology (Theorem on Morse homology, Theorem 7.4 of [2]). For a surface it is known that the topology is given by the genus or the Euler characteristic and this one is also given by the Euler-Poincaré characteristic of the complex of homology. That a certain Morse-function determines the topology of a surface is also contained in the book [10] page 70.

**About the choice of a height function.**

For certain simple surfaces, e.g., convex surfaces diffeomorphic to a 2-sphere, there is no bad embedding since the height-function \( z \) always shows up a strict minimum and a strict maximum. Such a height function is also a perfect Morse function, i.e. a Morse function with exactly 2 critical points attached respectively to a strict minimum and a strict maximum. For a surface of genus 0 we can still have several critical points at the same critical value. This is not generic and unstable under a small perturbation of \( z \).

For surfaces of higher genus some 'bad' embeddings are possible if the height function is chosen transverse to a level set (non-generic choice).

The first embedding (here \( z \) is the axe of symmetry) is not favorable: the set of critical point consists of 2 circles. These circles are manifolds of critical points attached respectively to a maxima and a minima of \( V \) of energies \( E_{\min} < E_{\max} \). In that situation, a compact manifold of critical points of dimension 1, we can here anyhow apply the results of [4] or [16]. If we still assume that conditions \( (A_2) \) and \( (A_3) \) are satisfied, each of these circles
contributes as a 1-dimensional submanifold in the 4-dimensional phase space:

\[ \Upsilon(E_{\text{min}}, \varphi, h) \sim C_1 \int_{\theta \in S^1} \int_{t \in \text{supp} (\hat{\varphi})} \frac{h^{-\frac{1}{2}} \hat{\varphi}(t)}{\sin(\alpha(\theta)t)} \frac{dt}{\sqrt{|t|}} d\theta + O(h^{\frac{1}{2}}), \quad C_1 \in \mathbb{C}^* \]

for the circle of minima and:

\[ \Upsilon(E_{\text{max}}, \varphi, h) \sim C_2 \int_{\theta \in S^1} \int_{t \in \text{supp} (\hat{\varphi})} \frac{h^{-\frac{1}{2}} \hat{\varphi}(t)}{\sinh(\alpha(\theta)t)} \frac{dt}{\sqrt{|t|}} d\theta + O(h^{\frac{1}{2}}), \quad C_2 \in \mathbb{C}^* \]

for the circle of minima. Observe that the order w.r.t. \( h \) is now \( -1/2 \).

Remark 11. Both formulae for \( \Upsilon(E_{\text{max}}, \cdot) \) and \( \Upsilon(E_{\text{min}}, \cdot) \) easily follow from an application of a stationary phase method with a compact manifold of critical point and a non-degenerate transverse Hessian.

Here \( \alpha(\theta) \) is the ratio of the eigenvalues of the linearized operator in the transverse direction to \( \Sigma_{E_{\text{min}}} \simeq \Sigma_{E_{\text{max}}} \simeq S^1 \) evaluated at the point \( \theta \in S^1 \). Observe that \( \alpha(\theta) \) is negative for \( E = E_{\text{max}} \) (hyperbolic flow) and positive for \( E = E_{\text{min}} \) (periodic flow). Observe that, only from the spectral estimates, we can still see:

- Hyperbolic contributions: unstable equilibria at the maximal energy.
- Trigonometric contributions: stable equilibria at the minimal energy.

The previous situation can be generalized for a smooth curve \( \gamma \), necessarily isomorphic to \( S^1 \), of critical points with a non-degenerate transverse Hessian at each point of \( \gamma \).

Remark 12. For a generic choice of the metric \( G \) on \( \mathbb{T}^2 \) the function \( \theta \mapsto \alpha(\theta) \) is not constant along \( S^1 \). Unfortunately, the spectrum of the associated Schrödinger operator \( -h^2 \Delta_\theta + z \) only determines the average of the density along the circles. To get a better description here requires to perform eigenfunction estimates. See, e.g. [4] for this point.
For the second embedding, where $x$ is the axe of symmetry, $z$ is a Morse function and we meet successively the critical points:

- a singularity of type $(0, 2)$: strict minimum,
- a singularity of type $(1, 1)$: first saddle point,
- a singularity of type $(1, 1)$: second saddle point,
- a singularity of type $(2, 0)$: strict maximum.

This gives:

$$\chi(T^2) = 1 - 2 + 1 = 0 \Rightarrow g(T^2) = 1.$$ 

The situation of the second embedding is generic and stable (e.g., w.r.t. a little deformation of the height function). This allows to retrieve $T^2$, up to a smooth deformation.

**Remark 13.** Results concerning Morse-functions are not specific to height-functions and Proposition 8 can be generalized to $h^2 \Delta_M + V$ where $V$ is a Morse-function on $M$. The interest here is the evident physical interpretation: the choice of $V(x) = z(x)$, $x \in M$ is equivalent to put a particle, forced to move freely on $M$ along geodesics, in a constant gravitation field. Also a height-function gives a function independent of the choice of the metric on $M$. For example, this is not the case of a potential $V(x)$ depending (locally) on the geodesic distance $d(x_0, x)$ on $M$. Such a potential depends on $G$ and can be singular at conjugate points.

**Convexity and measure.**

Assume that $g(M) = 0$ then if $M$ is convex every choice of a height-function gives a perfect Morse function. But if $M$ is not convex certain choice of the potential give locally a number of critical points greater than 2, with the same Morse index.

![Figure 2. A non-convex surface: locally we see 3 critical points.](image)
From the point of view of statistical mechanics it could be interesting to put a probability measure \( \mu(z) \) on all choice possible for \( z \) and to average the spectral estimates with respect to \( \mu \). This is here simply equivalent to chose a probability measure on \( \mathbb{S}^2 \) since the full problem is invariant under translation. A similar construction is possible for the choice of the metric \( G \): if \( G \) is in a bounded set of metrics \( G_\alpha \), estimates given by conditions \((A_2)\) and \((A_3)\) are still globally valid and so are our conclusions. One could average the results with respect to some probability measure \( \mu(\alpha) \).

We could also obtain the contributions of a surface \( M \) carrying a flat section in the following sense:

There exist an open subset \( U \subset \mathbb{R}^3 \) and a two-dimensional plane \( P \subset \mathbb{R}^3 \) such that \( U \cap M = U \cap P \).

The flat section can be interpreted a 2-dimensional subset of critical points when the height function \( z \) is chosen transversally to this section. The associated result is simply the Lebesgue measure of the flat section. Observe that, as predicted by the general theory of Morse-functions, this situation is not generic and not stable under a small perturbation of \( z \).

**Final Remark.** At a first look it might seem childish to use semi-classical methods. But the 'high-energy' method (see e.g. [9]) is not working: when the energy \( E \) is larger than the maximum of the potential \( E_{\text{max}} \) we have that the kinetic energy is bounded from below by \( ||\xi||^2 \geq E - E_{\text{max}} > 0 \).

By ellipticity of the Laplacian, it is not possible to produce any new wave invariant in this regime.

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