Reduced-bias estimation of the residual dependence index with unnamed marginals

Jennifer Israelsson
UK Health Security Agency

Emily Black
University of Reading

Cláudia Neves
King's College London

David Walshaw
Newcastle University

Abstract

This paper addresses important weaknesses in current methodology for the estimation of multivariate extreme event distributions. The estimation of the residual dependence index $\eta \in (0, 1]$ is notoriously problematic. We introduce a flexible class of reduced-bias estimators for this parameter, designed to ameliorate the usual problems of threshold selection through a unified approach to the familiar margins standardisation. We derive the associated asymptotic properties. The efficacy of the proposed semi-parametric inference on $\eta$ stems from a hitherto neglected exponentially decaying term in the hidden regular variation characterisation. Simulation studies to assess the performance for finite samples over a range of standard copulas indicate an improved performance, relative to the existing standard methods such as the Hill estimator. Our leading application illustrates how asymptotic independence can be discerned from monsoon-related rainfall occurrences at different locations in Ghana. The considerations involved in extending this framework to feasible inference on the extreme value index attached to domains of attraction are briefly discussed.

KEY WORDS AND PHRASES: Asymptotic independence, Brownian motion, empirical processes, extreme value theory, monsoon rainfall, regular variation theory, tail dependence coefficient

1 Introduction

In recent years, statistical methodology focusing on multivariate extremes has garnered ever greater interest brought by the many challenges faced when attempting to model compound to nearly simultaneous extreme events that occur with only little warning. This paper is motivated by an applied study...
addressing the need for an improved understanding of the extent to which spatial dependence in tropical and El Niño influenced rainfall occurrences can manifest itself at most extreme levels. For example, according to Sang and Gelfand (2009), the asymptotic independence regime must be conjectured when modelling extreme rainfall, especially given the convective nature of extreme tropical precipitation. Although this example focuses on extremal behaviour, having a good grasp of the strength of dependence at play amongst the whole of the data is crucially important in applied science at large. Indeed, quantifying dependence is commonplace in statistics: wherever one sits in the sliding scale of data complexity, from bivariate settings to the infinite-dimensional case in connection to stochastic processes, to multivariate stochastic processes, one often encounters the need of feasibly estimating an estimator’s variance for obtaining statistically meaningful confidence intervals that ward off overly optimistic inference (see e.g. Einmahl et al., 2022, p.12).

The importance of accounting for the strength of the non-asymptotic dependence between two random variables $X$ and $Y$, in the case that it decays in the limit to asymptotic independence

$$\lim_{t \to \infty} \frac{P(X > tx, Y > ty)}{1 - P(X \leq tx, Y \leq ty)} = 0,$$

has since long been recognised (cf. Ledford and Tawn, 1996, and references therein), but it was not until the compelling case made by Sibuya (1960) that the concept of residual dependence (arising with asymptotic independence) really became central to extreme value statistics. The example presented in Sibuya (1960) – that the componentwise maxima of a bivariate normal distribution with correlation coefficient satisfying $|\rho| < 1$ are asymptotically independent – spurred on many important developments in the statistical modelling of extremes events. It provided a compelling argument for devoting efforts to discriminating between full independence and asymptotic independence (Draisma et al., 2004; Coles, 2001), and it has seen greater resurgence in the development of modern statistical methods which focus on smooth transitions from dependence to asymptotic independence when handling multiple variables at a time, with scope to extend to (infinite dimensional) max-stable processes.

The overarching goal of the present paper is to capture penultimate dependence among extreme values, within which context we tackle the problem of reduced bias estimation for the residual dependence index. This index, sometimes termed the coefficient of tail dependence, was introduced by Ledford and Tawn (1996) and further investigated by Ramos and Ledford (2009), de Haan and Zhou (2011) and Eastoe and Tawn (2012). To fix the idea, let $(X_i, Y_i), i = 1, ..., n$, be independent copies of the random vector $(X, Y)$ with joint distribution function $F$, whose marginal distributions we denote by $F_i$.
1.1 Background

and $F_2$, i.e. $F_1(x) := F(x, \infty)$ and $F_2(y) := F(\infty, y)$, assumed continuous. The interest is in evaluating the probability of the two components being large at the same time, more formally that $P(X > u_1, Y > u_2)$, for large $u_1, u_2$, usually much larger than their respective sample componentwise maxima, $M_{X,n} := \max(X_1, \ldots, X_n)$ and $M_{Y,n} := \max(Y_1, \ldots, Y_n)$. From the applied illustrative perspective, this paper will focus on random phenomena that exhibit positive quadrant residual dependence (see e.g. Subramanyam, 1990), entailing that the probability that two random variables are simultaneously large is greater than that if they were independent. In environmental science, this can apply for example to the situation of strong wind-speeds and extreme wave heights; or to two large events of the same physical process occurring at nearby locations, for example heavy rainfall battering two neighbouring cities. In the financial context, different variables (e.g. different financial instruments, such as stocks in different companies) may face common shocks tending to correlate them positively, this leading to overly optimistic inference when ignored, as advocated by Gong and Huser (2022). Since we are restricting focus on the largest values, extreme value theory for threshold exceedances is the right theory to rely upon.

1.1 Background

Unlike univariate extreme value theory, multivariate extreme value distributions cannot be specified through a finite-dimensional parameter family of distributions. Instead, the many facets of multivariate extremes are mirrored in the inherent dependence structure of component-wise maxima which must be dissociated from the limiting extreme behaviour of its marginal distribution functions before a proper characterisation of extreme domains of attractions can be determined.

We assume that the identically distributed random pairs $(X_1, Y_1), \ldots, (X_n, Y_n)$ have common joint distribution function $F$ belonging to the domain of attraction of a bivariate extreme value distribution. Formally, there exist constants $a_n, c_n > 0$ and $b_n, d_n \in \mathbb{R}$ such that, for all continuity points $(x, y)$ of $G$ the following limit relation involving the component-wise partial maxima, $M_{X,n}$ and $M_{Y,n}$, is satisfied:

$$
\lim_{n \to \infty} P \left( \frac{M_{X,n} - b_n}{a_n} \leq x, \frac{M_{Y,n} - d_n}{c_n} \leq y \right) = \lim_{n \to \infty} F^n(a_n + b_n x, c_n + d_n y) = G(x, y).
$$

The domain of attraction condition (1.1) implies convergence of the marginal distributions to the corresponding $G(x, \infty) =: G_1(x)$ and $G(\infty, y) =: G_2(y)$ limits, whereby it is possible to redefine the constants $a_n, c_n, b_n, d_n$ in such a way that both $G_1(x) = \exp\left\{-(1 + \gamma_1 x)^{-1/\gamma_1}\right\}$ and $G_2(y) = \exp\left\{-(1 + \gamma_2 y)^{-1/\gamma_2}\right\}$ are attained, thus laying out the extreme value indices $\gamma_1$ and $\gamma_2$ as the design features for the gener-
alised extreme value (GEV) marginal distributions $G_1$ and $G_2$, respectively. Additionally, in order to formulate a more practical condition for tackling extremal dependence, we define the marginal tail quantile functions $U_i := (1/(1 - F_i))^{-}, i = 1, 2$, where $^{-}$ indicates the generalised inverse function. By virtue of Corollary 6.1.4 in de Haan and Ferreira (2006), it is possible to replace $n$ with $t$ running over the real line, thus enabling a relatively straightforward extension of condition (1.1): with $G_0(x, y) := G((x^{\gamma_1} - 1)/\gamma_1, (y^{\gamma_2} - 1)/\gamma_2)$, for any $(x, y)$ such that $0 < G_0(x, y) < 1$,}

$$\lim_{t \to \infty} \frac{t \left(1 - F(U_1(tx), U_2(ty))\right)}{t \left(1 - F(U_1(t), U_2(t))\right)} = -\log G_0(x, y) - \log G_0(1, 1) =: L(x, y).$$ (1.2)

In addition to noting that $-\log G_1(x) = -\log G_2(x) = 1/x, x > 0$, we also observe that since $L$ is homogeneous of order $-1$, the limiting condition (1.2) implies that $1 - F(U_1(t), U_2(t))$ is of regular variation at infinity with index $-1$ (i.e. $\lim_{t \to \infty} \{1 - F(U_1(tx), U_2(tx))\}/\{1 - F(U_1(t), U_2(t))\} = L(x, x) = x^{-1}L(1, 1) = x^{-1}$, for all $x > 0$). Hence, the above extreme value condition (1.2) suggests that a transformation of the marginal distributions to standard Pareto, with tail distribution function $1 - F(t) = (t^{-1} \wedge 1)$, makes it possible to eschew the influence of the margins and promptly go on to drawing extreme behaviour from the dependence structure alone. Other marginal standardisation transforms amenable to different definitions of dependence structures for extremes, set against tractable dependence measures, are of course possible. A good catalogue can be found in (Beirlant et al., 2004, Sections 8.2-8.3). For the moment, we continue with the characterisation for the domain of attraction of a bivariate extreme value distribution in the form $-\log G_0$ with arbitrary GEV marginals.

It is worth highlighting at this point that (1.2) (and hence (1.1) with the marginals of $G$ standardised as $G_1$ and $G_2$) implies a key domain of attraction condition on the basis of the tail copula:

$$R(x, y) = \lim_{t \to 0} t^{-1} P\left(1 - F_1(X) < t/x, 1 - F_2(Y) < t/y\right)$$

$$= \frac{1}{x} + \frac{1}{y} - \lim_{t \to \infty} t P(X > U_1(tx) \text{ or } Y > U_2(ty)) = \frac{1}{x} + \frac{1}{y} - L(x, y),$$ (1.3) (1.4)

for all $x, y > 0$ such that $0 < G_0(x, y) < 1$, whereby the extreme value dependence structure readily emerges from the combined tail regions $U_j(t.), j = 1, 2$, as in (1.4), for sufficiently large $t$. In particular, if the underlying distribution function $F$ is in the domain of attraction of a bivariate extreme value distribution with independent components, i.e., if $F$ is such that (1.2) holds with $\log G_0(x, y) = \log G_1(x) + \log G_2(y)$ (equivalently if $L$ the limit (1.4) resolves to $x^{-1} + y^{-1}$), then the random pair $(X, Y)$ is
1.2 Our contribution

said to be asymptotically independent as the limit (1.3) becomes zero. This paper is set within the context of asymptotic independence that will eventually render the tail copula condition (1.4), as a standalone, inadequate for the purposes of inference about the strength of association that remains in the initial joint d.f. \( F \).

There are numerous distributions possessing the asymptotic independence property, most often disconnected from the dependence structure prevalent at moderate levels, of which a salient example is the Gaussian copula that is generated by the bivariate normal distribution with correlation coefficient \( |\rho| < 1 \). The latter is asymptotically independent and no sliding window over the tail region on the marginals \( U_j(tx), 0 < x \leq 1 \), for sufficiently large \( t \), can halt this property (see e.g. Ledford and Tawn, 1996; Reiss and Thomas, 2007).

1.2 Our contribution

Momentous works by Ledford and Tawn (1996, 1997) rely on the asymptotically independent framework for extremes through the sub-model characterisation that is induced by (1.2) predicated on the assumption that, if \( F \) belongs to the max-domain of an extreme value distribution with independent components, then on a \( \varepsilon \)-neighbourhood of zero,

\[
\alpha(t) := P(1 - F_1(X) < t, 1 - F_2(Y) < t)
\]

is of lower order than \( P(1 - F_1(X) < t \text{ or } 1 - F_2(Y) < t) \). Since we have already established that the latter is a regularly varying function of index 1, the function \( \alpha \) is henceforth assumed to be regularly varying at zero with index \( 1/\eta \), for some \( \eta \in [0, 1) \), with the aim of exposing and subsequently capturing any remaining dependence that might still be present on the basis of statements (1.2)–(1.4) and similar, which provide the framework for the asymptotic structure for multivariate extreme value theory. It is through getting to grips with the pre-asymptotic behaviour of these structural conditions that the residual dependence index terminology for the otherwise hidden parameter \( \eta \) has arisen (cf. de Haan and Ferreira, 2006, Section 7.6).

Despite the wealth of literature published to date on the estimation of the residual dependence index \( \eta \), this paper adds substantively to the current body of knowledge mainly in two ways:

(i) we introduce a class of smooth estimators for the residual dependence index \( \eta \in (0, 1] \) that ameliorates the problem of threshold selection. Because this class is defined through a gradient of estimators,
1.2 Our contribution

no actual selection of a threshold will be required. The asymptotic normality for the proposed class of estimators is obtained under a new set of regularity conditions at the confluence of those typically used in the characterisation of the max-domains and of the pot-domains of attraction ascribed to threshold exceedances (thus in a reconciliatory stance), in contrast to having to adjudicate between the two approaches (see Bücher et al., 2019; Bücher and Zhou, 2021, for both, max and pot, in comparison). We devise an accompanying Hall-Welsh-type class of models for block maxima (cf. Hall and Welsh, 1985) of which the prototypical distribution is the unit Fréchet that encompasses Pareto-type tails. This class embodies the novel thread of development that runs through this paper. Not only will it serve as a gateway to settle perennial differences between standard Pareto (associated with threshold exceedances) and unit Fréchet marginal transforms (the lore of block maxima) that have been reported in the literature, it will also render a general framework for attaining significant bias reduction even with familiar estimators of $\eta$ (cf. Di Bernardino et al., 2013; Draisma et al., 2004) such as the Hill estimator (Hill, 1975).

(ii) The asymptotic normality of the proposed reduced-bias estimators is proved on the basis of a bespoke tail empirical process whose asymptotic representation is of Donsker type (Kosorok, 2008). Upon inversion, the resulting quantile empirical process uniform approximation established in this paper conforms to analogous results highlighted, e.g. in Draisma et al. (2004); Goegebeur and Guillou (2012); Beirlant et al. (2004). Making use of the developments in (i), reduced-bias estimators are devised in such a way as to harness the improved convergence rate offered by the new Hall-Welsh class of models, thus enabling to narrow the wedge between the actual residual dependence in the data and the level of asymptotic independence that can be effectively handled through the hidden regular variation characterisation.

Statistical procedures for eliciting extremal dependence have typically relied on standardisation of the unknown marginal distribution functions from which the usual pseudo-observations for either Pareto or Fréchet marginals result. Our theoretical results are mirrored in the numerical experiments implemented and demonstrate, for example, that it is not always the case that estimators for the residual index upon Fréchet standardisation exhibit larger bias than those stemming from standardisation to Pareto marginals. This is consistent with (Coles and Walshaw, 1994, p.152), where it is found that the Fréchet transform is preferable since it leads to robust estimation in what would have otherwise produced inadmissible estimates for the extremal dependence coefficient. We prove that, with a minimal but precise adaptation, estimation methods operating on each marginal transform can be made asymptotically equivalent, especially in relation to the Hill estimator, one of the foremost estimators for the tail index (Hill, 1975). Indeed, through a mere shift of unit Fréchet pseudo-observables by $1/2$, the proposed class
of residual index estimators has proved successful in averting common pitfalls such as those alluded in (Goegebeur and Guillou, 2012, point(i), Section 4.3), “The Hill estimator is generally biased, though the bias seems to be a more severe problem for unit Fréchet marginal distributions than for unit Pareto marginal distributions”. In a similar stance, (Draisma et al., 2004, page 254) justify their preference for the standard Pareto marginal transform on the smaller bias of the Hill estimator relative to the alternative based on unit Fréchet. While this could very well be the case with the estimation of the extreme value index for heavy-tailed marginal distributions, in the transposition to the estimation of an extremal dependence coefficient of a joint distribution, the approximation bias is just one facet of the challenge. We demonstrate how such a hindrance can be stymied with a precise shift in the marginal transform and, perhaps more importantly how it can be capitalised upon towards a reduced-bias proposal for estimating the residual dependence index \( \eta \). This is the key challenge that this work seeks to address: to unify these common approaches (for block-maxima and peaks-over-threshold) in their asymptotics and into estimation procedures of enhanced efficacy and broader applicability.

### 1.3 Organisation of the paper

The remainder of this paper is organised as follows. In Section 2 we introduce the proposed class of estimators for \( \eta \in (0, 1] \) central to this paper, including statements of its asymptotic properties of consistency and asymptotic normality that culminate in the reduced-bias estimation in Section 3. The foundational asymptotic results to this endeavour, and in particular an invariance result in the sense of empirical processes, are contained in Section 4. All the necessary proofs of the main results presented in Sections 2 and 3, as well as those of the supporting theorems in Section 4, are collected in Section 5. A great majority of these proofs are non-trivial, especially in the extreme values context. Section 6 contains simulation results for evaluating finite sample performance of representative estimators of the proposed class of estimators in Section 2, with emphasis on their reduced-bias variants encompassing Section 3. Finally, in Section 7, these new estimators for the residual dependence index are applied to monsoon rainfall data, forming a data-set collected across a highly dense network of gauging stations in Ghana.

### 2 Estimation of the residual dependence index: Pareto meets Fréchet

We assume that \( F \) is in the domain of attraction of an extreme value distribution, that is, \( F \) satisfies the extreme value condition (1.1), given in terms of the tail copula with standard Pareto marginals featuring
in (1.4). The starting point to our results is the that, for \( x, y > 0 \),

\[
\lim_{t \to \infty} \frac{P(X > U_1(tx), Y > U_2(ty))}{\tilde{\alpha}(t)} =: S(x, y)
\]

(2.1)

exists and is positive. In line with (1.5), we define

\[
\tilde{\alpha}(t) := P(X > U_1(t), Y > U_2(t)),
\]

which we assume regularly varying (at infinity) with index \(-1/\eta\), for some \( \eta > 0 \). Given the standard Pareto common marginals, we have \( \tilde{\alpha}(t)/t^{-1} \leq 1 \), upon which fact we impose the constraint \( \eta < 1 \) for ensuring the asymptotic independence regime. In addition, the fact that the minimum of independent Pareto random variables is again a Pareto random variable, hence exhibiting an exact power-law for the tail distribution, makes it the canonical choice for the marginal’s transformation. Indeed, in the case of the maximal depth that is exact independence, we find that \( t\tilde{\alpha}(t) = t^{-1} \) and \( \eta = 1/2 \).

Next, we move closer to the simple max-stability mirrored by unit Fréchet margins. Because we are aiming to capture the case of asymptotic (in)dependence, it is useful and possible (without loss of generality) to consider \( x = y \) in (2.1) thus giving rise to the simpler condition that, for all \( x > 0 \),

\[
\lim_{t \to \infty} \frac{tP(-1/\left(\log F_1(X) \vee \log F_2(Y)\right) > tx)}{tP(-1/\left(\log F_2(Y)\right) > t)} = \frac{R_0(x, x)}{R_0(1, 1)}.
\]

(2.2)

Indeed, \( R_0 \) is a transform of the tail copula \( R \) prescribed in (1.3) and therefore is also a distribution function endowed with a certain homogeneity property infused by the domain of attraction condition (1.2) (cf. Section 6.1.5 of de Haan and Ferreira (2006)), and hence bounded. Defining \( \tau(t) := P(-1/\left(\log F_1(X) \vee \log F_2(Y)\right) > t) \), relation (2.2) tells us that \( \lim \sup t \tau(t) \leq 1 \) is the best we can hope for and we must have \( \eta < 1 \), with the former a somewhat lax statement compared to the exact inequality (not limiting) taking place with the standard Pareto marginal transform in (2.1). In the case of near independence, we have that \( \tilde{\alpha}(t) = \mathcal{L}(t)t^{-2} \), with \( \mathcal{L} \) a slowly varying function at infinity (i.e. near independence with \( \eta = 1/2 \)), and additionally maximum depth if \( \mathcal{L}(t) \to 1 \), which again contrasts with the exact independence statement \( \tilde{\alpha}(t) = t^{-2} \) issued with standard Pareto.

We note that \( R_0(1, 1) \) coincides with the dependence coefficient of Sibuya (1960); de Haan and Ferreira (2006).
Estimation of the residual dependence index: Pareto meets Fréchet

On the whole, advantages of Pareto over Fréchet seem self-explanatory: the possibility of replacing limiting statements liable to various degrees of approximation to an extreme value distribution, with exact ones lends legitimate appeal to the canonical Pareto transform. Furthermore, $S$ in homogeneous of order $1/\eta$, thus rendering the familiar dual formulation of (2.2) (cf. Draisma et al. (2004); de Haan and Ferreira (2006); Goegebeur and Guillou (2012)): 

$$
\lim_{t \to \infty} \frac{P\left( \frac{1}{1-F_1(X)} \wedge \frac{1}{1-F_2(Y)} > tx \right)}{P\left( \frac{1}{1-F_1(X)} \wedge \frac{1}{1-F_2(Y)} > t \right)} = S\left( \frac{1}{x}, \frac{1}{x} \right) = x^{-1/\eta}S(1, 1) = x^{-1/\eta},
$$

(2.3)

for all $x > 0$. Specifically, the tail distribution function $F_T := 1 - F_T$ of the random variable

$$
T := \frac{1}{(1 - F_1(X)) \vee (1 - F_2(Y))}
$$

(2.4)

is regularly varying with index $-1/\eta$, for $\eta \in (0, 1]$.

If there is one aspect the limiting condition (2.3) impacts very clearly, it is that the problem of estimating the residual dependence coefficient $\eta$ is inexorably attached to the problem of estimating the tail index within the context of heavy-tailed, univariate extremes. Not for nothing, (de Haan and Ferreira, 2006, p. 265) endorse the Hill estimator as especially purpose-fit when it comes to the problem of estimating the residual dependence index $\eta \in (0, 1]$: consistent estimation is achieved through the corresponding empirical distribution functions rendering the $n$ pseudo-observables of $T$ defined by

$$
T_i^{(n)} := \frac{n + 1}{n + 1 - R(X_i)} \wedge \frac{n + 1}{n + 1 - R(Y_i)},
$$

(2.5)

$i = 1, 2, \ldots, n$, where $R(X_i)$ stands for the rank of $X_i$ among $(X_1, X_2, \ldots, X_n)$, that is, $R(X_i) := \sum_{j=1}^{n} \mathbf{1}\{X_j \leq X_i\}$, and $R(Y_i)$ is the corresponding rank of $Y_i$ among $(Y_1, Y_2, \ldots, Y_n)$. Notably, the Hill estimator (2.6) admits the functional representation:

$$
\hat{\eta}^{(H)} = \frac{n}{m} \int_{T_{n,n-m}}^{\infty} \left( \log x - \log T_{n,n-m} \right) dF_T^{(n)}(x),
$$

(2.6)

where, for each $x$, $F_T^{(n)}(x)$ is the empirical distribution for the sequence of non-independent and identically distributed copies of $T$, whose $(m + 1)$-th descending order statistic is denoted by $T_{n,n-m}$. The class
of estimators introduced in this paper follows from the far more general functional

\[ \hat{n}_{a,b} = \frac{1}{b} \left( \frac{n}{m} \int_{T_{n,m}}^{\infty} \left( \frac{x}{T_{n,m}} \right)^{a} dF_T(x) \right)^{b/a} - 1, \]  

(2.7)

for some \( a, b \in [-\infty, \infty], a \neq 0, \) of which the Hill estimator is a member (suffices to take \( a/b \to -1 \)). However, rather than summoning the pseudo-observables (2.5) as customary with the Hill estimator, all estimators churned out of the \((a, b)\)-functionals (2.7) and alike are purposefully based on the sequence \( \{ E_i^{(n)} \}_{i=1}^{n} \) defined as

\[ E_i^{(n)} := \left\{ -\log \frac{R(X_i)}{n+1} \right\}^{1/2} \left\{ -\log \frac{R(Y_i)}{n+1} \right\}^{-1}. \]  

(2.8)

There is (at a minimum residual) dependence across these identically distributed unit Fréchet random variables with distribution function \( \exp\{-x^{-1}\}, x \geq 0. \)

**Example 1.** The Gaussian copula generated by \( C_{\Sigma}(u, v) = \Phi_{\Sigma}(\Phi^{-1}(u), \Phi^{-1}(v)), 0 < u, v < 1, \) where \( \Phi_{\Sigma} \) denotes the bivariate standard normal distribution function with both marginal means equal to zero, variances one and correlation coefficient \( \theta \in [-1, 1], \) and with \( \Phi \) standing for standard normal distribution function, satisfies (2.1) with \( S(x, y) = (xy)^{-1/(1+\theta)} \) and \( \eta = (1 + \theta)/2, \) because we have that

\[ \hat{\alpha}(t) = c(\theta) (\log t)^{-\theta/(1+\theta)} t^{-2/(1+\theta)} \left( 1 + \frac{\theta}{1 + \theta} \frac{\log(\log t)}{2 \log t} \right), \]

where \( c(\theta) = (1 + \theta)^{3/2} / (1 - \theta)^{1/2} (4\pi)^{-\theta/1+\theta}. \) Details on the derivation of \( \hat{\alpha} \) can be found in Ledford and Tawn (1996); Reiss and Thomas (2007). Of interest is that \( t\hat{\alpha}(t) = L_U t^{-(1-\theta)/(1+\theta)}, \) whose limit as \( t \to \infty \) is \( 0 = R_0(1, 1) \) if \( \theta < 0, \) implying asymptotic independence. Next, we define \( V_i(t) = (-1/ \log F_i)^{\alpha}(t) \) and note that, more generally,

\[ t\bar{\alpha}(t) = tP(X > V_1(t), Y > V_2(t) > t) = t\hat{\alpha}(t) + \frac{1}{2} P(X > U_1(1/(1-e^{-1/t})), Y > U_2(1/(1-e^{-1/t})) \),

whence \( R_0(1, 1) = \lim_{t \to \infty} t\bar{\alpha}(t) = \lim_{t \to \infty} (t + 1/2) \hat{\alpha}(t) = 0, \) but the 1/2-term that is associated with \( L_V(t)t^{-1/\theta} \) is the more important the closer \( \theta \) is to one. Figure 1 displays how this affects the gradient of estimates emanating from (2.7), centred about the Hill estimator (blue dash line), including the case of \( T \) replaced with \( E \) in terms of (2.8) for gaining enhanced performance. The contrasting patterns are a testament both to the importance of a wise specification of marginal transformations, and to the disrup-
Estimation of the residual dependence index: Pareto meets Fréchet

tive nature of the attached slowly varying function $L$. Looking at the estimator's yields pertaining to the values of $q$, simultaneously, it is clear that opting for the Fréchet marginal standardisation allows to take a larger top sample fraction than that mapped out with the Pareto marginal transform. This chimes with the highlight by (Drees and Kaufmann, 1998, see p.150) that in the presence of composite exponential heavy-tailed behavior, estimation of the index of regular variation tends to require a larger number of order statistics than that necessary with Pareto-type behavior having an ultimately constant $L$.

Figure 1: Estimation of $\eta$ with $\hat{\eta}_{a,b}$ for various $a = -b$ and as a function of $m/n$, both by relying on transforms to common standard Pareto marginals (left) and to unit Fréchet marginals (right).

More than building directly on (2.2) for the estimation of the residual dependence index, we seek to exploit intrinsic links between (2.1) and (2.2). As Example 1 suggests, a balanced compromise between standard Pareto and unit Fréchet may reveal a fertile middle ground for harnessing a substantive improvement in the estimation of $\eta$, especially one that leverages on a wider plateau of stability in the estimates’ sample paths (cf. Figure 1). Crucially, the asymptotic equivalence in distribution we introduce next enables us to transition between (2.2) and (2.3). This presently identified and yet untapped link constitutes a significant step forward to the desired estimation for the residual tail index $\eta < 1$ that does not require marginal specifications for filtering out the dependence: employing Taylor’s expansion (twice), as $t \to \infty$, we find that

$$P\left( \frac{1}{\log F_1(X)} \wedge \frac{1}{\log F_2(Y)} > t \right) = P\left( \frac{1}{1 - F_1(X)} \wedge \frac{1}{1 - F_2(Y)} > t + \frac{1}{2} + O_p(t^{-1}) \right) = \tilde{\alpha}(t + o(1)), \tag{2.9}$$

meaning that estimators expressed in (2.7), including Hill’s as in (2.6), all coalesce onto the unit Fréchet marginals shifted by 1/2. This is the key, actionable insight to the results developed in this paper, with potential to be swiftly extended to dimensions greater than 2 (see Eastoe and Tawn, 2012, p.47). More
2.1 Assumptions

details putting this framework into an extreme values context are provided later, as part of Remark 5.

By addressing deviations between the joint (unknown) exceedance probability determined through the unit Fréchet marginal transform (as in (2.2)) and the tail dependence structure $S(1/x, 1/x)$ arising in the limit (2.3) (which is completely specified up to one tail parameter: the residual dependence index $\eta \in [0, 1)$), our proposed framework for sharpening the estimation of $\eta$ configures a semi-parametric inference problem, within which context we introduce a new class of estimators for the index $\eta$ which, while removing the canonical pseudo-observables $T$, sees them replaced with $E + 1/2$ in (2.7), through the device outlined in (2.9).

Let $E_{n,1} \leq \ldots \leq E_{n,n}$ denote the order statistics associated with (2.8) and let $p \in \mathbb{R}, q > 0$ be conjugate constants in the sense that $1/p + 1/q = 1$ (or $1/p + 1/q \to 1$). These constants do not signify a parameter that will be fine-tuned to the estimator’s optimal performance. Instead, $q$ and $p$ must be allowed to vary within a certain gradient of values that will eventually funnel through the desired stable regime for surrendering extremal (residual) dependence in a data-driven way. To this end, we settled on the gradient estimators arising from the dual representation (2.7) with common unit Fréchet margins, i.e.

$$\hat{\eta}^{(S)}_q := \frac{1}{1 - 1/q} \left\{ \frac{1}{m} \sum_{i=0}^{m-1} \left( \frac{1/2 + E_{n,n-i}}{1/2 + E_{n,n-m}} \right)^{1 - \frac{1}{q}} \right\}^{-1}. \tag{2.10}$$

Given the already proved consistency of $\hat{\eta}^{(S)}_q$ for every $q$ satisfying the above class-constraints (cf. Section 2.2), the expectation is that, for a suitable range of extremal-related observations, the gradient pathway of the estimator’s yields will narrow to a tight beam hovering about the true value of the index $\eta$ we wish to determine.

2.1 Assumptions

In order to derive the asymptotic distribution of the class of estimators (2.10) in the present two-dimensional setting for extremes, a second order refinement of (2.2) is required (see Draisma et al., 2004; Goegebeur and Guillou, 2012). Hence, assume that $S$ is continuously differentiable and that there are functions $\tilde{\alpha} > 0$ and $\tilde{\beta}$ such that

$$\lim_{t \to \infty} \frac{\tilde{\alpha}(t) P\left( X > U_1(tx), Y > U_2(ty) \right) - S(x, y)}{\tilde{\beta}(t)} =: D(x, y) \tag{2.11}$$
2.1 Assumptions

exists for all \( x, y \geq 0, x + y > 0 \), with \( \tilde{\alpha}(t) \to \infty \) and \( \tilde{\beta} \) of ultimately constant sign and tending to 0, as \( t \to \infty \), and a function \( D \) which is neither constant nor a multiple of \( S \). In keeping with prevailing theory for bivariate extremes (see e.g. Draisma et al. (2004); Goegebeur and Guillou (2012)), we take \( \tilde{\alpha}(t) = 1/\alpha(t^{-1}) \) in the above, making it possible to assume that the convergence is uniform on \( \{(x, y) \in [0, \infty)^2 \mid x^2 + y^2 = 1\} \) and that

\[
D(x, x) = x^{-1/\eta} x^{-(\eta/\eta\tau)} - 1 \eta \tau.
\]

In addition to \( \tilde{\alpha} > 0 \) being a regularly varying function with index \( 1/\eta \), condition (2.11) implies that \( |\tilde{\beta}| \) is regularly varying with index \( \tau/\eta \geq 0 \). Moreover, we assume that \( l := \lim_{t \to \infty} \tilde{\alpha}(t)/t \) exists and \( l \geq 0 \), which is satisfied so long as \( \tau > 0 \).

Not only do these assumptions imply that (2.11) holds uniformly on \((0, \infty)^2\) (we refer to Draisma et al. (2004) in this respect), these also entail our primary condition upon \( V^* := 1/2 + \left( -1/\log F \right)^{\tau_*} \), in connection with \( \eta < 1 \) (whereby \( l = 0 \)), that there exists a function \( \beta_*(t) \to 0 \), as \( t \to \infty \), not changing sign eventually such that, for all \( x > 0 \),

\[
\lim_{t \to \infty} \frac{V^*(tx)}{\beta_*(t)} = x^\eta x^{-(\eta/\eta\tau)} - 1 \eta \tau =: D_{\eta, \tau_*}(x), \tag{2.12}
\]

with \( \tau_* := \eta \wedge \tau \). It is worth noting at this point that the assumptions outlined in this section are by any measure more stringent than contemporary and largely accepted requirements for ensuring asymptotic normality of other existing estimators of \( 0 < \eta < 1 \) with proven efficacy (de Haan and Ferreira, 2006; Neves, 2009; Alves et al., 2009). A good variety of bivariate distributions satisfying assumption (2.11) can be found in Goegebeur and Guillou (2012). Condition (2.12) and connected limit results are trivially satisfied in many situations of interest (Beirlant et al., 2004; Draisma et al., 2004; Goegebeur and Guillou, 2012). Its implications are discussed at length in Section 4, particularly in relation to Theorem 4.2.

Remark 1. Heeding the development as part of (2.11), we find that the left hand-side of (2.3) is, in a certain sense, proportional to \( l = \lim_{t \to \infty} t \tilde{\alpha}(t) \), with finite non-negative \( l \) in order for the domain of attraction condition to hold. If \( \eta < 1 \) then \( l = 0 \) and there is asymptotic independence. In case of asymptotic dependence, \( l > 0 \) and \( \eta \) must be equal to one. However, \( \eta = 1 \) alone does not guarantee asymptotic dependence.
2.2 Consistency and asymptotic normality

The procedure for bridging the gap between standard Pareto and Fréchet transforms that is predicted on (2.9) realizes the idea that a functional representation for the estimators of the residual tail index relying on the tail quantile process \( \{ 1/2 + E_{n,n-[ms]} \} \), for \( 0 < s < n/m \), transfers seamlessly to the analogous formulation in terms of \( F_{\hat{Y}}^{(n)}(x) \) (i.e. for common standard Pareto marginals). Indeed, Lemma 5.1 ascertains that by merely replacing the latter with the empirical distribution function on the basis of \( E_{n,n-[ms]} - 1/2 \), the asymptotic properties of such estimators, which are usually approximately linear functions of the observable process, remain unchanged.

**Theorem 2.1.** Suppose conditions (2.11) and (2.12) hold. Let \( m := \left[ n/\bar{\alpha} \left( n/k \right) \right] \) be a sequence of positive integers, \( m(n) \to \infty \), with \( k = k(n) \to \infty \) and \( k/n \to 0 \), as \( n \to \infty \). Assume \( \sqrt{m} \beta_*(n/m) \to 0 \). Then, for each \( q > 0 \), estimators \( \hat{\eta}_q^{(S)} \) belonging to the class (2.10) are asymptotically equivalent to their standard Pareto analogue \( \hat{\eta}_q \) (i.e. stemming from (2.7) with \( (a, b) = \left( 1/p, 1/q - 1 \right) \) and conjugated constants \( p, q \)). Concisely,

\[
\sqrt{m} |\hat{\eta}_q^{(S)} - \hat{\eta}_q| \xrightarrow{P} 0,
\]

as \( n \to \infty \).

Since condition \( \sqrt{m} \beta_*(n/m) = o(1) \) imposes an upper bound on the sequence \( m(n) \) that governs the convergence of the joint quantile process, Theorem 2.1 ascertains that, so long as the intermediate number \( m \) of larger pseudo-observables used in the estimation of \( \eta \) is not too large, both gradient estimators \( \hat{\eta}_q^{(S)} \) and \( \hat{\eta}_q \) are consistent and eventually coincide in their asymptotic distributions with probability tending to one. Any remnant differences between these two variants of estimators are culled by an appropriate choice of \( m \) that fulfills the conditions of Theorem 2.1. The second order deterministic bias, concerning approximation bias, of order \( \sqrt{m} \beta_*(n/m) = O(1) \) can nonetheless be made explicit as a function of \( 0 < \eta \leq 1 \) and \( \tau_* > 0 \). This is the case in point for the next theorem featuring the proposed class of estimators of the residual dependence index \( \eta \).

**Theorem 2.2.** Under the conditions of Theorem 2.1, albeit with \( \sqrt{m} \beta_*(n/m) \to \lambda \in \mathbb{R} \), as \( n \to \infty \), for every \( a < 1/(2\eta) \) and \( b/a \to -1 \) in

\[
\hat{\eta}_{a,b}^{(S)} = \frac{1}{b} \left[ \left\{ \int_0^{1} \left( \frac{1}{2} + E_{n,n-[ms]} \right)^{a} ds \right\} \frac{b}{a} - 1 \right],
\]

(2.13)
2.2 Consistency and asymptotic normality

the following convergence in distribution holds:

\[ \sqrt{m} (\hat{\eta}_{a,b}^{(S)} - \eta) \xrightarrow{d} N(\lambda \mu_a, \sigma_a^2), \]

with \( \mu_a = \mu_a(\eta, \tau_*) = (1 - a\eta)/(1 - a\eta + \tau_*) \) and \( \sigma_a^2 = \sigma_a^2(\eta) = \eta^2(1 - a\eta)^2/(1 - 2a\eta). \)

Remark 2. By allowing \( a \to 0 \) in Theorem 2.2, we can retrieve the asymptotic normal distribution of the Hill estimator in a straightforward way. In its quality as a member of the class \((2.7)\) attained for \( a = 1/p \) and \( 1/p + 1/q = 1 \), the Hill estimator is equally considered in Theorem 2.1 by letting \( q \to 1 \). The dual representation in terms of the familiar Pareto pseudo-observables finds its way to Theorem 2.2 by using \( \tau \) in place of \( \tau_* \) everywhere in the theorem (see Theorem 4.2 for assurances).

Theorem 2.2 brings to the fore that the greater the second parameter \( \tau > 0 \), the smaller the second order (dominant) component of the asymptotic bias. This is due to the heightened rate of convergence of the actual underlying bivariate distribution \( F \) to its specific max-stable limit in connection with large \( \tau \). The asymptotic variance is not affected by this but rather it stays subject to \( a < 1/(2\eta) \), \( \eta \in (0,1] \). Through simple algebra one can show that \( \sigma_a^2(\eta) \geq \eta^2 \), for all \( a, \eta \), where equality holds for the Hill estimator (i.e., \( \sigma_a^2(0) = \eta^2 \)). Moreover, the asymptotic variance increases with increasing \(|a|\) but by much less than it does in \( \eta \). Figure 2 aims to highlight the above described properties for the asymptotic variance with varying \(|a| \leq 0.4\).

![Figure 2: Case of \( b = -a \): asymptotic variance of \( \hat{\eta}_{a,b}^{(S)} \) for true \( \eta \in (0, 0.8] \), with \( a \in [-0.4, 0.4] \).](image-url)
3 Reduced bias estimation

The most straightforward, and potentially the most effective way of removing the second order bias inherent to the estimator \( \hat{\eta}^{(S)}_{a,b} \) without increasing the asymptotic variance, is to capitalise on a Hall-Welsh-type representation (in the spirit of the seminal representation by Hall and Welsh (1985)) for effectively pinning down this type of systemic misspecification that comes with approaching the initial distribution \( F \) from a limiting distributional angle. The Hall-Welsh-type model here developed to meet our aim of extending the estimation the residual dependence index \( \eta \), arises as a far-reaching class of regularly varying functions satisfying condition (2.12). It refers to the catch-all for distribution functions whose extremal quantile function \( V^* \) admits the following expansion, as \( t \to \infty \),

\[
V^*(t) = C^n t^\eta \left( 1 + \eta D_1 C^{-\tau} t^{-\tau} + \eta D_2 C^{-\eta} t^{-\eta} \right),
\]

for \( C > 0, D_1, D_2 \neq 0 \), whereby \( \beta_*(t) = \eta \tau_* D_1(Ct)^{-\tau_*}, \) with \( j = 1 \) if \( \tau_* = \tau, j = 2 \) if \( \tau_* = \eta > 0 \). Importantly, such a model for \( V^* \) implies that the speed of convergence to the desired power function in the limit can be enhanced through the location-shift by \( 1/2 \) (cf. Neves, 2009, p.218). This will be the focal point of the reduced bias estimation aspect of this section.

Let \( E_{n,1} \leq E_{n,2} \leq \ldots \leq E_{n,n} \) be the ascending order statistics associated with pseudo-observations \( E_i^{(n)}, i = 1, 2, \ldots, n \), defined in (2.8). The reduced-bias version of estimator (2.10) thus arises from (2.13) with conjugated constants \( p, q \) featuring in Theorem 2.1:

\[
\tilde{\eta}_q(m, m^*) := \tilde{\eta}_q^{(S)} \left\{ 1 - \left( \frac{\tilde{\beta}(n/m)}{1 + 2E_{n,n-m^*}} \right) \left( \frac{1 - \tilde{\eta}_q^{(S)}/p}{1 - \tilde{\eta}_q^{(S)}/p + \hat{\tau}} \right) \right\},
\]

with \( m, m^* \), both intermediate sequences of positive integers, i.e., \( m = m(n) \to \infty, m^* = m^*(n) \to \infty, \)
\( m/n \to 0, \) as \( n \to \infty, \) and \( m^* \leq m \) (implying that \( m^*/n \to 0 \)), and where \( \tilde{\beta}(n/m) \) and \( \hat{\tau} \) denote consistent estimators for \( \beta_*(n/m)/\eta \) and \( \tau_* > 0 \), respectively.

**Theorem 3.1.** Assume conditions of Theorem 2.2 hold. Let \( m^* \) be another intermediate sequence of positive integers such that \( m^* \leq m \). Then,

(i) the estimator \( \tilde{\eta}_q := \tilde{\eta}_q(m, m^*) \) defined in (3.2) is a consistent reduced-bias variant of both estimators
\( \tilde{\eta}^{(S)}_{a,b} \) in (2.13) and \( \tilde{\eta}_{a,b} \) in (2.7) for this choice of conjugate constants.
Basic asymptotic results

(ii) Moreover, if \( m = m(n) \) satisfies \( \sqrt{m} \beta_\star(n/m) = O(1) \), then analogous versions \( \tilde{\eta}_a \) of (2.13) satisfy

\[
\sqrt{m} \left( \tilde{\eta}_a - \eta \right) \xrightarrow{d} Z_a, \tag{3.3}
\]

as \( n \to \infty \), where \( Z_a \) is a normal variable with mean zero and \( \sigma_a^2 > 0 \), the same asymptotic variance provided in Theorem 2.2.

Remark 3. Confidence bounds for the reduced bias estimator (3.2) of the residual dependence index \( \eta \) follow readily from Theorem 3.1(ii).

4 Basic asymptotic results

Semi-parametric inference can be viewed as the statistically meaningful distribution-free methodology that results from piecing together two key components: the study of a statistical functional, which typically serves as an estimator of the relevant parameter (broadly understood), and the study of the underlying empirical process. For example, the Hill estimator relies on a functional of the extreme order statistics that also works inwards to intermediate order statistics for greater efficiency in the estimation of the (positive) index of regular variation. Thus far, and in particular in Section 2, we have examined the former against the backdrop of asymptotic independence. We now lay the groundwork for the latter. For every \( x \), we define tail empirical distribution function as

\[
F_E^{(n)}(x) := \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\left\{ \frac{k}{n+1}(1/2+E_{n,n-i+1}) \geq x \right\}
\]

and its pertaining tail quantile process by

\[
Q_n(s) := \frac{1}{2} + E_{n,n-[ms]}^{(n)}, \quad 0 < s < \frac{n}{m}. \tag{4.1}
\]

Theorem 4.1. Assume condition (2.11) and let \( m := \left\lceil n/\hat{\alpha}(n/k) \right\rceil \) be a sequence of positive integers such that \( m(n) \to \infty \), with \( k = k(n) \to \infty \) and \( k/n \to 0 \), as \( n \to \infty \). Suppose the second order condition (2.12) holds and that \( \sqrt{m} \beta_\star(n/m) = O(1) \). Then, there exits a probability space \((\Omega, \mathcal{A}, P)\) with the random variables in (4.1) and a sequence of standard Brownian motions \( \{W_n(s)\}_{s \geq 0} \) such that, for all \( s_0 > 0 \), with
sufficiently small $\varepsilon > 0$, 
\[
\sup_{0 < s \leq s_0} s^{\eta + \frac{1}{2} + \varepsilon} \left| \sqrt{m} \left( \frac{Q_n(s)}{V^*(n/m)} - s^{-\eta} \right) - \eta s^{-(\eta + 1)} W_n(s) - \sqrt{m} \beta_*(\frac{n}{m}) s^{-\eta} s^{\tau_*} - 1 \right| = o_p(1).
\]

The next theorem is a step anchoring theorem rooted in the theory of regular variation simultaneously aimed at providing a formal justification for the significance of (2.9) to the formulation of Theorem 2.1 and of how it links to the Hall-Welsh-type condition underpinning the bias reduction technique devised in Section 3. Let $V$ be the quantile function associated with standardisation to unit Fréchet, given by:
\[
V(t) := \left( -\frac{1}{\log F} \right)^{-}(t) = U\left( \frac{1}{1 - e^{-1/t}} \right),
\]
in the way of making its relationship with the tail quantile function $U := (1/(1 - F))^{-}$ obvious. By writing concisely $V^*(t) := V(t) + 1/2$, we obtain the formulation for the relevant condition of second order regular variation captured in the next theorem. The proof for Theorem 4.2, presented in Section 5.2, emphasises that $x^{-\eta}V^*(tx)/V^*(t)$ has a similar representation to $x^{-\eta}U^*(tx)/U^*(t)$ (here $U^* := U + 1/2$) but clearly differs from that of $x^{-\eta}U(tx)/U(t)$ by a term of the third order kind.

**Theorem 4.2.** Let $U$ be a tail quantile function of regular variation at infinity with index $\eta \in (0, 1]$, which is denoted by $U \in RV_\eta$, and assume the following second order strengthening of the regular variation of $U$, i.e., for all $x > 0$,
\[
\lim_{t \to \infty} \frac{U(tx)}{U(t)} - x^\eta \beta(t) = x^\eta \frac{x^{-\tau - 1}}{\tau},
\]
with second order parameter $\tau > 0$. Then, $V^*$ is also regularly varying with the same index $\eta$ and is such that, for $x > 0$,
\[
\lim_{t \to \infty} \frac{V^*(tx)}{V^*(t)} - x^\eta \beta_*(t) = \begin{cases} x^\eta \frac{z^{-\tau - 1}}{\tau}, & \tau < \eta; \\ x^\eta \frac{z^{-\eta - 1}}{\eta}, & \tau \geq \eta; \end{cases}
\]
where
\[
\beta_*(t) = \begin{cases} \beta(t), & \tau < \eta; \\ \beta(t) + \frac{\eta}{2V^*(t)}, & \tau = \eta; \\ \frac{\eta}{2V^*(t)}, & \tau > \eta. \end{cases}
\]
Moreover, $|\beta| \in RV_{-\tau}$ with second order parameter governing the speed of convergence given by

$$\tau_* = \begin{cases} 
\tau, & 0 < \tau < \eta, \\
\eta, & 0 < \eta \leq \tau.
\end{cases}$$

**Remark 4.** The reciprocal of the above-stated result holds more generally, so long as $\eta \in (0, 1]$ and $\tau > 0$.

We contend that Theorem 4.2 will form the fulcrum of every view between the Fréchet max-domain of attraction and Pareto-borne domain of attraction in the sense that it offers a significant first step to unifying prior results in the estimation of the extreme value index for heavy tails (Sec. 4.1 of Jacob et al., 2020; Bücher and Zhou, 2021) as well as extending related work beyond the univariate case (e.g. Bücher et al., 2019; Eastoe and Tawn, 2012), including the infinite-dimensional case of max-stable processes where typically random vectors transformed to unit Fréchet margins satisfy a simple max-stability property. We invite the reader to revisit the brief discussion next to Example 1.

## 5 Proofs

This section is comprised of all the proofs and auxiliary lemmas to the asymptotic results encompassing Sections 2 and 3, spliced with the proofs for their respective foundational results presented in Section 4.

Throughout this section we will follow all the notation and conditions from the earlier sections in their relevant instances. In the interest of practicality, we begin with the underpinning results to those in Section 4 and then proceed incrementally to covering those in Sections 2 and 3. The next lemma is of independent interest in the sense that it offers an account of a weak invariance principle for the sequence of normalised tail empirical processes generated by $F_{E}$. No extreme value theory is invoked at this point although this lemma has an integral part in establishing the basic representation in Theorem 4.1 for the tail quantile process $\{Q_n(s)\}$ defined in (4.1). The latter will engage the domain of attraction condition (2.11) by calling upon the second order refinement (2.12) towards a fully-fledged asymptotic independence characterisation and its framework for inference.

**Lemma 5.1.** *Weak invariance principle* Let $r(n)$ be an intermediate sequence defined as $r(n) := (n + 1)\alpha(k/n)$ (entailing $k(n) \to \infty$), and let $k(k(n))$ be such that $k/n \to 0$, $n \to \infty$. Suppose there exists a
positive function $\alpha$ such that
\[\lim_{t \to \infty} \frac{P\left(1 / (\log F(X) \vee \log F(Y)) > tx\right)}{\alpha(t)} = x^{-1/\eta},\] (5.1)
for all $x > 0$. Then,
\[
\sqrt{r(n)} \left\{ \frac{1}{r(n)} \sum_{i=1}^{n} \mathbb{I}\left\{ \frac{k}{n} \left( X_i^{(n)} + \frac{1}{2} \right) \geq x \right\} - \frac{P\left( X > U_1\left(\frac{2}{n} x\right), Y > U_2\left(\frac{2}{n} x\right)\right)}{\alpha(n/k)} \right\} \to W(x^{1/\eta}),
\]
weakly in $D(0, \infty)$, where $W$ denotes standard Brownian motion. 

**Claim 1.** The Gaussian limit established in this lemma curates that of (Draisma et al., 2004, p.270) and therefore supersedes the latter’s corresponding statement appearing in the proof for Lemma 1 in (Goegebeur and Guillou, 2012, Supporting information document).

**Proof.** Consider the random pairs $(X_i, Y_i), i = 1, \ldots, n$, representing i.i.d. copies of $(X, Y)$. The primary focus is on the direct empirical analogue to the copula survival function $C(1 - x, 1 - y)$ such that $C(x, y) = P(1 - F_1(X) < x, 1 - F_2(Y) < y)$. With $F_j^{(n)}, j = 1, 2$ denoting the marginal empirical distribution functions associated with $X$ and $Y$ respectively, and $[a]$ standing for the largest integer less than or equal to $a \in \mathbb{R}$ and , we firstly aim at the development for the joint tail empirical process for $x = y > 0$, in the sense of
\[
\sum_{i=1}^{n} \mathbb{I}\{ X_i \geq X_{n-[kx]+1,n}, Y_i \geq Y_{n-[kx]+1,n} \} = \sum_{i=1}^{n} \mathbb{I}\left\{ \left( -\log F_1^{(n)}(X_i) \right)^{-1} \geq -\frac{1}{F_1^{(n)}(X_{n-[kx]+1,n})}, \left( -\log F_2^{(n)}(Y_i) \right)^{-1} \geq -\frac{1}{F_2^{(n)}(Y_{n-[kx]+1,n})} \right\} = \sum_{i=1}^{n} \mathbb{I}\left\{ \left( -\log F_1^{(n)}(X_i) \right)^{-1} \wedge \left( -\log F_2^{(n)}(Y_i) \right)^{-1} \geq -\log(1 - [kx]/n)^{-1} \right\} = \sum_{i=1}^{n} \mathbb{I}\{ \frac{k}{n} E_i^{(n)} \geq -\frac{k/n}{\log(1 - [kx]/n)} \} =: \sum_{i=1}^{n} I_i^{(n)}(x),
\] (5.2)
where the identically distributed random variables $E_i^{(n)}$ are those defined in (2.8). Owing to the power-series $\sum_{n \geq 0} t^n / (n+1) = -\log(1 - t)/t$, for $|t| < 1$, we can write the following stochastic inequalities for the
Proofs

partial sequence of indicator random variables in (5.2): there exists $\varepsilon > 0$ such that

$$
1 \left\{ \frac{k}{n + 1} E_i^{(n)} \geq \frac{1}{x} \left( 1 - \frac{k x}{n^{1/2}} (1 - \varepsilon \left( \frac{k}{n} \right)^{\varepsilon}) \right) \right\} \leq I_i^{(n)}(x) \leq 1 \left\{ \frac{k}{n + 1} E_i^{(n)} \geq \frac{1}{x} \left( 1 - \frac{k x}{n^{1/2}} (1 + \varepsilon \left( \frac{k}{n} \right)^{\varepsilon}) \right) \right\},
$$

uniformly in $x$ on a compact set bounded away from zero. This gives information about total boundedness in the desired approximation to $F^{(n)}_E$. Specifically, for every $\delta > 0$, we have that

$$
\lim_{\varepsilon' \downarrow 0} \limsup_{n \to \infty} P\left( \max_{1 \leq i \leq n} |I_i^{(n)}(x) - I_{i,\varepsilon'}^{(n)}(x)| > 1 - \delta, \text{ for } 0 \leq x \leq T \right) = 0,
$$

with intervening

$$
I_{i,\varepsilon'}^{(n)}(x) := 1 \left\{ \frac{k}{n} E_i^{(n)} \geq \frac{1}{x} \left( 1 - \frac{k x}{n^{1/2}} (1 \pm \varepsilon' \left( \frac{k}{n} \right)^{\varepsilon'}) \right) \right\}.
$$

This implies in turn that, for each $n \in \mathbb{N}$ and for every $\delta > 0$, there exists an arbitrarily small $\varepsilon' > 0$ as before, such that

$$
P\left( |I_i^{(n)}(x) - I_{i,\varepsilon'}^{(n)}(x)| \leq 1 - \delta, \text{ for all } i = 1, \ldots, m; \text{ for } 0 \leq x \leq T \right) > 1 - \delta,
$$

i.e.,

$$
P\left( \max_{1 \leq i \leq n} |I_i^{(n)}(x) - I_{i,\varepsilon'}^{(n)}(x)| = 0 \text{ for } 0 \leq x \leq T \right) > 1 - \delta.
$$

This configures an equicontinuity property. In relation to (5.2), we note that

$$
\frac{1}{n} \sum_{i=1}^{n} \left| I_i^{(n)}(x) - I_{i,\varepsilon'}^{(n)}(x) \right| \leq \frac{1}{n} \sum_{i=1}^{n} \left| I_i^{(n)}(x) - I_{i,\varepsilon'}^{(n)}(x) \right| \leq \max_{1 \leq i \leq n} |I_i^{(n)}(x) - I_{i,\varepsilon'}^{(n)}(x)|,
$$

whereupon a Skorokhod construction is invoked in order to ascertain that, as $n \to \infty$

$$
\sup_{x \in [0, s_0]} \left| \frac{1}{n} \sum_{i=1}^{n} I_i^{(n)}(x) - \frac{1}{n} \sum_{i=1}^{n} 1 \left\{ \frac{k}{n} E_i^{(n)} + \frac{1}{2} \geq 1/x \right\} \right| = o(1), \text{ a.s. (5.3)}
$$

Next, we deal with the relevant joint tail empirical process on the basis of the intermediate sequence $r(n)$ associated with the desired transform to common shifted-Fréchet margins. To this end, we define the tail empirical function:

$$
1 - F_{E^*}^{(n)}(x) := \frac{1}{n} \sum_{i=1}^{n} 1 \left\{ \alpha^{-1} \left( \frac{r(n)}{n + 1} \right) \left( 1/2 + E_{n,n-i+1} \right) \geq x \right\}.
$$

21
Proofs

We proceed to evaluating its asymptotic mean and variance in two parts. Firstly, we employ the series expansion \( t/(−\log(1 − t)) = 1 − \sum_{n \geq 1} \frac{t^n}{n+1} \), with \( t := k/(nx) = o(1) \), as \( n \to \infty \) (cf. (5.1)), yielding:

\[
\alpha_n = P\left\{ -\log F_1(X) < -\log\left(1 - \frac{k}{nx}\right), -\log F_2(Y) < -\log\left(1 - \frac{k}{nx}\right) \right\} = P\left\{ \frac{1}{-\log F_1(X)} \wedge \frac{1}{-\log F_2(Y)} \geq \frac{n}{k}x\left(1 - \frac{k}{n}x^{-1} \frac{2}{3} (1 + o(1)) \right) \right\}.
\]

Due to the absolute continuity of the marginal distributions \( F_i, i = 1, 2 \), the above implies with \( \mathbb{E}^* := 1/2 + \left(−\log F_1(X) \vee (−\log F_2(Y))\right)^{-1} \) (by analogy with (2.4) and in the spirit of (2.8)) that, for \( \varepsilon > 0 \),

\[
\lim_{n \to \infty} \sum_{i=1}^{\lfloor n/k \rfloor} P\left\{ E_i^* > \frac{n}{k}x\left(1 - \left(\frac{k}{n}\right)^2 (1 + \varepsilon) x^{-2} / 12\right) \right\} < \infty
\]

because by taking \( \delta = x(1 - \vartheta x^{-2+\vartheta}) > 0 \), for some \( \vartheta \in (0, 1) \), we find that the corresponding truncated expected value of the identically distributed random variables \( E_i^* \) is obtained from the approximation, as \( t \to \infty \),

\[
\int_\delta^\infty \mathbb{F}_{E^*}(t) \, dt = \int_\delta^\infty L(t)\left(1 - \exp\left(\frac{1}{t^{1/2}}\right)\right)^{1/n} dt = \int_\delta^\infty L(t)^{t^{-1/n}(1 + t^{-2}/12 + o(t^{-2}))}^{-1/n} dt.
\]

The latter integral exists and is finite for \( 0 < \eta < 1 \) as a result of Karamata’s theorem which states that the slowly varying function \( L \) (i.e. \( \lim_{x \to \infty} L(tx)/L(t) = 1 \), for all \( x > 0 \)) eventually dwindles to the role of a constant in the integral (de Haan and Ferreira, 2006, Appendix B). We notice that the expansion of the exponential skips the term of order \( t^{-1} \) on the right hand-side. Hence,

\[
\lim_{n \to \infty} \sup_{x > 0} \left| \alpha(x) - \mathbb{F}_{E^*}(x) \right| = 0. \tag{5.4}
\]

The remainder of the proof is concerned with the regular variation of \( \alpha(x) \) embedded in (5.1), equivalently the regular variation at zero of \( \alpha \) defined in (1.5), as we draw near the concrete goal of attributing the limiting Gaussian process \( W^* \) to the properly normalised \( 1 - \mathcal{F}^{(n)}_{E^*}(x) \). From (5.4), we have for each \( x > 0 \) that

\[
\mathbb{E}[1 - \mathcal{F}^{(n)}_{E^*}(x)] = \frac{1}{n} \sum_{i=1}^{n} P\left( \frac{1}{2} + E_i^{(n)} \geq \frac{n}{k}x \right) \approx P\left( 1 - F_1(X) \leq \frac{k}{nx}, 1 - F_2(Y) \leq \frac{k}{nx} \right) = C\left( \frac{k}{nx^{-1}}, \frac{k}{nx^{-1}} \right)
\]
and, owing to asymptotic independence of $X$ and $Y$, we have as $n \to \infty$,

$$\text{Var}\left(1 - F_E^*(x)\right) \approx \frac{1}{n^2} \sum_{i=1}^{n} P\left(\frac{1}{2} + E_i^{(n)} \geq \frac{n}{k} x\right) \{1 - P\left(\frac{1}{2} + E_i^{(n)} \geq \frac{n}{k} x\right)\} \approx \frac{1}{n} C\left(\frac{k}{n} x^{-1}, \frac{k}{n} x^{-1}\right) = x^{1/n} \text{O}(\frac{\alpha(k/n)}{n+1}).$$

(5.5)

Secondly, we write concisely the sum of the two building blocks for the limit process of (5.2) suitably normalised in view of the convergence rate identified above:

$$\sqrt{r(n)} \left\{ \frac{1}{r(n)} \sum_{i=1}^{[r(n)]} I_i^{(n)} (x^{-1}) - \frac{P\left(1 - F_1(X) \leq \frac{k}{n^x}, 1 - F_2(Y) \leq \frac{k}{n^x}\right)}{\alpha(k/n)} \right\} \quad \rightarrow_{d} \quad W^*(x)_{0 < x < \infty} \quad (5.6)$$

uniformly for every $s_0 > 0$, where $W^*$ is a Gaussian process with mean zero and covariance given by $E[W^*(x)W^*(y)] = l_1(x \wedge y)$, with $l_1(x) = \lim_{t \to \infty} \pi(t/x) / \pi(t)$ (recall that $\pi(t) = \alpha(1/t)$). From this, and applying the approximation (5.5), we are able to determine that $W^*$ is Brownian motion, specifically $W^* \equiv \{W(x^{1/n})\}_{x > 0}$.

Therefore,

$$\sqrt{r(n)} \left\{ \frac{1 - F_E^*(x)}{\alpha(n/k)} - \frac{P\left(1 - F_1(X) < \frac{k}{n^x}, 1 - F_2(Y) < \frac{k}{n^x}\right)}{\pi(n/k)} \right\} \quad \rightarrow_{d} \quad W^*(x)_{x > 0}, \quad (5.7)$$

weakly in $D(0, \infty)$. This concludes the proof. \hfill \Box

\textbf{Remark 5.} The invariance principle in Lemma 5.1 canvasses flexibility for wider application since it is not justified by extreme value theory (cf. Beirlant et al., 2004, p.346). We have not yet really employed a domain-of-attraction condition at this point because the regular variation of $\alpha$ (asymptotic equivalent of $\alpha$ in (1.5)) is not sufficient nor is it a necessary condition for verifying (1.2). It takes care of the rate

\phantom{23}
Proofs

des of convergence in the common distribution for the marginals, but it has still to be combined with the rate of convergence of the joint distribution to the limiting extremal dependence structure. The suite of examples studied by Schlather (2001) is a good testament to the importance of conjugating the two, often drawing on the auxiliary function which reflects the rate of convergence in \( \lim_{t \to \infty} x^{1/\eta} \pi(t)/\pi(t) = 1 \) dictated by the adopted marginal transform (inter alia, the unit Fréchet). Keeping track of the marginal distributions is an aspect that is accentuated in Remark 6.2.2 of de Haan and Ferreira (2006), emphasised in Coles and Walshaw (1994) and alluded to in Eastoe and Tawn (2012), but suppressed in (Poon et al., 2003, p.937). In the latter, the slowly varying function \( x^{-\eta} \pi(t) \) is suppressed to a constant \( d > 0 \) since no second order refinement is considered. We argue that addressing a second order refinement of this can create greater advantages in tackling estimation bias, as outlined in Example 1.

Before getting underway with the proof of Theorem 4.1, that endows Lemma 5.1 with an extreme value theory capacity, we need yet another approximation result tackling the possibility of different marginal (univariate) conditions at play. Subsequently, rather than focusing tail distribution functions as in (2.9), we shift to their inverses by examining interdependencies of the tail-type quantile functions \( V := (−1/ \log F)^− \) and \( U := (1/(1 − t))^{−} \) pertaining to the unit Fréchet and standard Pareto distributions, respectively.

**Lemma 5.2.** Define \( V^∗(t) := V(t) + 1/2 \). Suppose (4.2) holds for some \( \eta \in (0, 1] \) and \( \tau > 0 \). Then,

\[
\lim_{t \to \infty} \frac{U(t)}{V^∗(t)} = 1.
\]

**Proof.** With the defined \( V \) and \( U \) it holds that \( V(t) = U(1/(1−e^{−1/t})) \), whence

\[
\frac{U(t)}{V^∗(t)} = \frac{U^∗(t)}{U^∗\left(\frac{1}{1-e^{-1/t}}\right)} - \frac{1/2}{U^∗\left(\frac{1}{1-e^{-1/t}}\right)}, \quad U^∗ := U + \frac{1}{2}.
\] (5.8)

with the latter term vanishing as \( t \to \infty \). This proof is hinged on the above representation.

With \( \beta_∗(t) = t^{−\eta} \beta(t)U(t) \), whereby \( |\beta_∗| \in RV_{−\tau} \), the second order regular variation of \( U \) (as in (4.2)) is written as

\[
\lim_{t \to \infty} \frac{(tx)^−\eta U(tx) − t^{−\eta}U(t)}{\beta_∗(t)} = \frac{x^{−\tau} − 1}{\tau},
\] (5.9)
Proofs

for all \( x > 0 \). With \( U^* := U + 1/2 \), this implies in turn that

\[
\frac{(tx)^{-\eta}U^*(tx) - t^{-\eta}U^*(t)}{\beta_*(t)} = \frac{x^{-\tau} - 1}{\tau} (1 + o(1)) + \frac{x^{-\eta} - 1}{2U(t)\beta(t)}, \quad [U^*\beta] \in RV_{-\tau+\eta},
\]

whence,

\[
x^{-\eta} \frac{U^*(tx)}{U^*(t)} - 1 = \left\{ \frac{x^{-\tau} - 1}{\tau} \beta(t) \frac{U(t)}{U^*(t)} + \frac{x^{-\eta} - 1}{2U^*(t)} \right\} (1 + o(1)). \tag{5.10}
\]

Additionally, we note that because \( U \in RV_\eta, \eta > 0 \), we have for any constant \( c \in \mathbb{R} \),

\[
\frac{U(tx) - c}{U(t) - c} = \frac{U(tx)}{U(t)} \left( 1 - \frac{c}{U(t)} \right)^{-1} \left( 1 + o(1) \right) = \frac{U(tx)}{U(t)} (1 + o(1)).
\]

With \( c = 1/2 \) in particular, we find that \( U^*(t) \sim U(t) \), as \( t \to \infty \). Taking \( y = U^*(tx)/U^*(t) \) in the equality \( 1/(1 + y) = 1 - y + y^2/(1 + y), y \neq -1 \), the limiting relation (5.10) entails that

\[
x^y \frac{U^*(t)}{U^*(tx)} - 1 = 1 - \frac{x^{-\eta} U^*(tx)}{U^*(t)} (1 + o(1)) = \frac{x^{-\tau} - 1}{\tau} \beta(t) (1 + o(1)) - \frac{x^{-\eta} - 1}{2U^*(t)} (1 + o(1)).
\]

Finally, Taylor's expansion of \( y/(1 - e^{-y}) \) around zero ascertains the result:

\[
\frac{U^*(t)}{U^*(\frac{1}{1 - e^{-1/t}})} = \left( \frac{1/t}{1 - e^{-1/t}} \right)^{-\eta} = 1 - \frac{\eta}{2} + \frac{\eta(1 + 3\eta)}{24} \frac{1}{t^2} + o(\max(\beta(t), 1/U^*(t))),
\]

which, constrained by the equality in (5.8), enables us to conclude that \( U(t) \sim V^*(t) \), as \( t \to \infty \).

\[\square\]

Proof of Theorem 4.1: For every \( x > 0 \), we define

\[
\Gamma_{n,r(n)}(x) = \frac{1}{r(n)} \sum_{i=0}^{n-1} \mathbf{1}_{\left\{ \alpha^{-\left( \frac{r(n)}{n+1} \right)} (E_{n,n-i}^{(n)} + \frac{1}{n}) > x \right\}}.
\]

With a relatively straightforward rearrangement of Lemma 5.1 aimed at establishing the weak convergence of the above process in \( D[1, \infty) \), and through application of the \( \delta \)-method, we find that

\[
\sqrt{r(n)} \left\{ \frac{\alpha(k/n)}{\Gamma_{n,r(n)}(x)} - \frac{\alpha(k/n)}{\alpha(k/(nx))} \right\}_{0 < x < \infty} \rightarrow \left\{ -x^{2/\eta} W\left( x^{1/\eta} \right) \right\}_{0 < x < \infty}.
\]

By virtue of Vervaat's Lemma (cf. de Haan and Ferreira (2006), Appendix A), the next asymptotic statement in terms of generalised inverses – including \( U_{E^*}(s) = (1/\alpha(s^{-1}))^{-\tau} = \bar{\alpha}^{-\tau}(s) \), with \( E^* := 1/2 + E_1 \wedge \ldots \wedge E_n \geq 0 \))

\[
\} 
\]
Proofs

$E_2$ (cf. (2.11)) – is proved to hold on a compact interval bounded away from zero: for every $\theta \leq s \leq s_0$ and all $\theta, s_0 > 0$, as $n \to \infty$,

$$\left| \sqrt{r(n)} \left\{ \Gamma_{n,r(n)}^{-} \left( \alpha \left( \frac{k}{n} \right) s \right) - \frac{U_{E_r^*} \left( \frac{n}{r(n)} \right)}{U_{E_r^*} \left( \frac{n+1}{r(n)} \right)} \right\} - \eta s^{-\eta-1} W(s) \right| = o_p(1). \quad (5.11)$$

Next, we give the quantile process $\Gamma_{n,r(n)}^{-}$ a closer inspection: it develops as

$$\Gamma_{n,r(n)}^{-} \left( \frac{r(n)}{n} s \right) = \frac{k}{n+1} Q_n \left( \frac{r(n)}{m} s \right) = \frac{k}{n+1} \left( \frac{1}{2} + E_{n,n-\lfloor r(n)s \rfloor}^{(n)} \right).$$

It is fruitful to work with $U$ and $V^*$ in tandem. In particular, through (5.10) in the proof for Lemma 5.2, we find that that $V^* \left( n/r(n) \right) = U_{E_r^*} \left( (n+1)/r(n) \right) \left( 1 + o(\beta_*(n/r(n)) \right)$, whereby

$$s^\eta \Gamma_{n,r(n)}^{-} \left( \frac{r(n)}{n} s \right) = \frac{k}{n+1} s^\eta \left( \frac{1}{2} + E_{n,n-\lfloor r(n)s \rfloor}^{(n)} \right) = \frac{V^* \left( \frac{n}{r(n)} \right)}{U_{E_r^*} \left( \frac{n+1}{r(n)} \right)} \frac{\left( \frac{1}{2} + E_{n,n-\lfloor r(n)s \rfloor}^{(n)} \right)}{V^* \left( \frac{n+1}{r(n)} \right)} \frac{V^* \left( \frac{n}{r(n)} \right)}{s^{-\eta} V^* \left( \frac{n}{r(n)} \right)}.$$

It follows immediately from Theorem 2.1 in Csörgő (1983) that

$$s^\eta \Gamma_{n,r(n)}^{-} \left( \frac{r(n)}{n} s \right) = 1 + O_p \left( \frac{1}{\sqrt{r(n)}} \right). \quad (5.12)$$

as $n \to \infty$. We now work towards strengthening (5.11) through the strong approximation for a weighted quantile process provided in Csörgő and Horváth (1993), p.381 (see also Theorem 5.1.1 of Csörgő (1983)). To this end, we note the following straightforward variant of Theorem B.3.10 of de Haan and Ferreira (2006): under condition (2.12), there exists a function $\beta_0(t) \sim \beta_*(t)$ (and so without loss of generality $\beta_0 \equiv \beta_*$) such that for any $\delta > 0$, there exits $t_0 = t_0(\delta)$ such that for all $tx > t_0$ and $x > 0$,

$$\left| \frac{V^*(tx)/V^*(t) - x^\eta}{\beta_*(t)} - x^\eta \frac{x^{-\tau_*} - 1}{\tau_*} \right| < \delta x^{-\tau_*+\delta}. \quad (5.13)$$

The above is here employed by putting $t = n/r(n)s^{-1}$ and $x = s$. Additionally, as part of Lemma 5.2 we showed that $U_{E_r^*}(t)/V^*(t) - 1 = o(1)$. Because $s^{-1/2+\varepsilon}W^*(s) < \infty$, a.s., application of the triangle inequality upon (5.11) leads to the Gaussian limit of the relevant tail quantile process for all $0 < \varepsilon < 1/2$.
and arbitrarily small $\vartheta > 0$,
\[
\sup_{0 < s \leq s_0} s^{1/2+\varepsilon} \left| \sqrt{r(n)} \left\{ s^n \Gamma_{n,r(n)}^{-} \left( \alpha \left( \frac{k}{n} \right) s \right) - 1 \right\} + O \left( \sqrt{r(n)} \beta_s \left( \frac{n}{r(n)} \right) \right) - \eta s^{-1} W^* (s^n) \right| = o_p(1),
\]
as $n \to \infty$. Finally, in order to obtain a total boundeness result that out-spans the natural support
$[1/(n + 1), 1 - 1/(n + 1)]$ of the quantile process, thus making it practical for extremes, we borrow the arguments as per in the proof of Lemma 6.2 by (Draisma et al., 2004, pp.271-272) in order to rein in stochastic variation in the tails through the proper weight function $q(s) = s^{1/2+\varepsilon}$ which is essentially Chibisov-O’Reilly’s function specialised into the tail empirical processes (see Csörgő and Horváth, 1993; Einmahl, 1997). Then, the fact that $\lim_{\vartheta \downarrow 0} \sup_{0 < s \leq \vartheta} s^{-1/2+\varepsilon} W^* (s^n) = 0$, a.s. ascertains boundedness of the supremum of the relevant weighted process around zero, thus affording $\vartheta \downarrow 0$ in the above. Defining $m := \lfloor r(n) \rfloor$, under the assumptions of theorem, we have that $V^* (n/m)/V^* (n/r(n)) = 1 + o(\beta^*(n/r(n)))$, in such a way that $s_0$ can run on subsequences $n/r(n)$ by a contiguity argument due to the continuous paths of Brownian motion.

\[\square\]

5.1 Proofs for Section 2

Proof of Theorem 2.1: The proof is made at slightly greater generality than the result in the theorem warrants. The proof will evolve around the functional representation of the relevant estimators in terms of the tail empirical process and deploys from their basic representation for $a/b \to -1$ (not necessarily equal to $-1$): with $E^*_{n,n-m} = Q_n(1)$,
\[
\sqrt{m} \left| \tilde{Q}_{a,b}^{(S)} - \tilde{Q}_{a,b} \right| = \sqrt{m} \left\{ \frac{n}{m} \int_{T_{n,n-m}} \left( \frac{x}{T_{n,n-m}^{(n)}} \right)^a dF_t^{(n)}(x) \right\}^{-1} - \left\{ \frac{n}{m} \int_{E^*_T(n,n-m)} \left( \frac{x}{E^*_n(n,n-m)} \right)^a dF_E^{(n)}(x) \right\}^{-1}.\]
This was found to simplify matters. Without loss of generality, we note that either of these terms can be written in the following form for \( \alpha < 1/\eta \):

\[
\sqrt{m} \left\{ \frac{n}{m} \int_{E_{n,n-m}^*}^{\infty} \left( \frac{x}{E_{n,n-m}^*} \right)^{a} dF_E^{(n)}(x) - \left( 1 + \frac{1}{a - 1/\eta} \right) \right\}
\]

(5.14)

Applying Theorem 4.1 with \( s = 1 \), the latter results in

\[
\sqrt{m} \left\{ \frac{V^*(\frac{n}{m})}{E_{n,n-m}^*} \right\}^{1/\eta} - 1 \right\} \frac{1}{a - 1/\eta} + \sqrt{m} \{ I + II \}(1 + O_p(1/\sqrt{m})
\]

where \( I \) and \( II \) are defined as

\[
I := \int_{1}^{\infty} \frac{n}{m} \left( 1 - F_E^{(n)}(xV^*(\frac{n}{m})) \right) a \frac{dx}{x^{1-a}}
\]

\[
II := \int_{E_{n,n-m}^*/V^*(m/n)}^{1} \frac{1}{m} \left( 1 - F_E^{(n)}(xV^*(\frac{n}{m})) \right) a \frac{dx}{x^{1-a}}
\]

The next step is show that \( \sqrt{m} I = O_p(1) \). Indeed, given condition (2.12) holds locally uniformly, then (5.7) coupled with Theorem B.3.14 of de Haan and Ferreira (2006) (see also Theorem 2.3.8 therein) entails that \( \sqrt{m} \beta_s(\alpha^{-}(m/n)) = O(1) \) whereby

\[
|\sqrt{m} I| \leq \sup_{x>0} \sqrt{m} \left| \frac{n}{m} F_E^{(n)}(xV^*(\frac{n}{m})) - x^{-1/\eta} \right| \times \left( \int_{E_{n,n-m}^*/V^*(m/n)}^{1} at^{a-1} dt \right),
\]

with the first terms on the right hand-side bounded with probability one and the second vanishing to zero, also ensured by Theorem 4.1. Theorem 4.1 in conjunction with the dominated convergence theorem ensures that \( \sqrt{m} II = O_p(1) \). We now turn to (5.14) where we put \( g(\eta) = 1 + (a - 1/\eta)^{-1} \). A direct
application of the δ-method enables, as \( n \to \infty \),

\[
\sqrt{m} \left\{ \left( \frac{n}{m} \int_{E_{n,n-m}^{*}}^{\infty} \left( \frac{x}{E_{n,n-m}^{*}} \right)^{a} dF_{E}^{(n)}(x) \right)^{-1} - \frac{1}{g(\eta)} \right\} = \frac{a\eta - 1}{(a\eta + 1 - \eta)^{2}} \sqrt{m} \left\{ 1 - \left( \frac{V_{E}^{*}(\frac{n}{m})}{E_{n,n-m}^{*}} \right)^{1/\eta} \right\} - \frac{(a\eta - 1)}{(a\eta + 1 - \eta)} \sqrt{m} II + o_{p}(1).
\]

It follows from Lemma 5.1 in conjunction with the dual strong approximation provided by Theorem 2.4.8 of de Haan and Ferreira (2006) for the tail quantile process ascribed to standard Pareto marginals that, for every \( \varepsilon > 0 \),

\[
P \left( \lim_{n \to \infty} \left| \left( \frac{V_{E}^{*}(\frac{n}{m})}{E_{n,n-m}^{*}} \right)^{1/\eta} - \left( \frac{U(\frac{n}{m})}{T_{n,n-m}} \right)^{1/\eta} \right| > \varepsilon \right) = 0.
\]

Finally, for some function \( g_{0} \) in a one-to-one correspondence with \( g \),

\[
\lim_{n \to \infty} E \left[ \sqrt{m} \left| \hat{\eta}^{(S)}(\eta) - \hat{\eta}^{(\eta)} \right| \right] \leq \lim_{n \to \infty} E \left[ \sqrt{m} \left| \hat{\eta}^{(S)}(\eta) - g_{0}(\eta) \right| \right] + \lim_{n \to \infty} E \left[ \sqrt{m} \left| \hat{\eta}^{(S)}(\eta) - g_{0}(\eta) \right| \right] = 0,
\]

from which the result in the theorem follows via Markov's inequality. □

The proof of Theorem 2.2 which we lay out next, provides a glimpse of what the above-mentioned function \( g_{0} \) must look like in the general case described by the functional (2.7). The result encompassing this theorem is solely reliant on the weighted approximation to the quantile process established in Theorem 4.1.

**Proof of Theorem 2.2:** The starting point is the distributional representation for the underpinning quantile process that features in Theorem 4.1, both with general \( s \) and specialised on \( s = 1 \). Defining

\[
\{ Z_{n}(s) \}_{0 < s \leq 1} := \left\{ \left( \frac{Q_{n}(s)}{Q_{n}(1)} \right)^{a} - s^{-a\eta} \right\}_{0 < s \leq 1},
\]

we obtain by means of an appropriate Taylor's expansion that

\[
Z_{n}(s) = a\eta s^{-(a\eta+1)} \left( \frac{1}{\sqrt{m}} (W_{n}(s) - sW_{n}(1)) + \beta_{s} \left( \frac{n}{\eta} \right) s^{s-s_{*}} - 1 \right) + O_{p} \left( \frac{1}{\sqrt{m}} \right).
\]

where the \( o_{p} \)-term is uniform on a compact interval bounded away from zero. Hence, for every \( 0 < \varepsilon < \)
Proof of Theorem 4.2:

5.2 Proofs for Section 3

The proof essentially hinges on translating second order regular variation into extended regular variation of an appropriate function related to the former. With the already defined quantile function \( V \), such that \( V(t) = U(1/(1-e^{-t})) \), and \( \beta_*(t) = t^{-\eta} \beta(t) U(t) \) whereby \( |\beta_*| \in RV_{-\tau} \), we have that

\[
\frac{(tx)^{-\eta} V(tx) - t^{-\eta} V(t)}{\beta_*(t)} = \frac{(tx)^{-\eta} V(tx) - t^{-\eta} U(t)}{\beta_*(t)} - \frac{t^{-\eta} V(t) - t^{-\eta} U(t)}{\beta_*(t)}
\]

\[
= \frac{(tx)^{-\eta} U(1/(1-e^{-tx})) - t^{-\eta} U(t)}{\beta_*(t)} - \frac{t^{-\eta} U(1/(1-e^{-t})) - t^{-\eta} U(t)}{\beta_*(t)}.
\]

\[\square\]

5.2 Proofs for Section 3

The first part of the theorem thus follows in a straightforward way via Cramér’s \( \delta \)-method upon

\[\eta_{a,b}(S) = \frac{1}{a} \left\{ \int_0^1 \left( \frac{Q_n(s)}{Q_n(1)} \right)^a ds \right\}^{b/a} - 1\]

with \(-b/a \sim 1\), not depending on \(n\). Finally, given that increments of a Gaussian process are independent normal random variables, then for every \(n\) the limiting integral \( \int_0^1 a^{-\eta} s^{-\eta \tau_1} ds \) resolves to a sum of normals eventually and hence a normal random variable in itself. Finally, in order to derive the variance of the limiting normal random variable, it suffices to consider the process \( Z(s) := \eta s^{-\alpha-1} B(s), 0 \leq s \leq 1 \), for which \( \text{Var}(\int_0^1 Z(s) ds) = E(\int_0^1 \int_0^1 Z(s)Z(t) ds dt) = \eta^2(1 - a\eta)^2(1 - 2a\eta)^{-1} \).
Owing to the second order regular variation for $U$ with index $\eta > 0$ encapsulated in (5.9), which holds locally uniformly for $x > 0$, and by noting that $x(t) = t^{-1/(1-e^{-1/t})} \to 1$, as $t \to \infty$, we find the representation:

$$
\frac{(tx)^{-\eta}V(tx) - t^{-\eta}V(t)}{\beta_*(t)} = \left( \frac{1}{1-e^{-1/(tx)}} \right)^{-\tau} - 1 \frac{1}{\tau} (1 + o(1)) - t^{-\eta} \left( \frac{1}{1-e^{-1/t}} \right)^{-\tau} - 1 \left( 1 + o(1) \right).
$$

Now it is only a matter of applying Taylor’s expansion followed by judicious manipulation of the terms on the right hand-side so that, for all $x > 0$, a suitable representation of subsequent order is available:

$$
\frac{x^{-\eta}V(tx) - t^{-\eta}V(t)}{\beta(t) U(t)/V(t)} = \frac{x^{-\tau} - 1}{\tau} - \frac{1}{2t} x^{-\tau - 1} + o\left( \frac{1}{t} \right).
$$

(5.16)

For tackling $V^*(t) = V(t) + 1/2$, we plug in the above into the extended regular variation development. Hence, as $t \to \infty$,

$$
\frac{(tx)^{-\eta}V^*(tx) - t^{-\eta}V^*(t)}{\beta_*(t)} = \frac{(tx)^{-\eta}V(tx) - t^{-\eta}V(t)}{\beta_*(t)} + \frac{t^{-\eta} x^{-\eta} - 1}{2 \beta_*(t)} = \frac{x^{-\tau} - 1}{\tau} + \frac{t^{-\eta} x^{-\eta} - 1}{2 \beta_*(t)} - \frac{1}{t} x^{-\tau - 1} + o(t^{-1}) + o(1),
$$

for all $x > 0$. Given that in the present setting of asymptotic independence the range $\eta < 1$ is required, the third order term in (5.16) becomes negligible (note that $|t^\eta \beta_*(t)| \in RV_{-\tau+\eta}$ and $\eta < 1 + \tau$, $\tau > 0$), thus resulting in the following expansion for $V^*$:

$$
\frac{x^{-\eta}V^*(tx) - t^{-\eta}V^*(t)}{\beta(t) U(t)/V^*(t)} = \frac{x^{-\tau} - 1}{\tau} + \frac{1}{\beta(t) U(t)} \frac{x^{-\eta} - 1}{2} + o\left( \frac{1}{\beta(t) U(t)} \right).
$$

Under the conditions of the theorem, Lemma 5.2 above now enables replacement of $U$ with $V^*$ everywhere and the desired result detailing second order regular variation for $V^*$ in full thus arises. Specifically,

$$
\frac{V^*(tx)}{V^*(t)} = \beta(t) x^\eta x^{-\tau} - 1 + \frac{\eta}{2 V^*(t)} x^\eta x^{-\eta} - 1 + o(\beta(t)) + o\left( \frac{1}{V^*(t)} \right) \quad (t \to \infty).
$$

□

**Proof of Theorem 3.1**: The first part of the theorem follows from Theorem 2.1 in conjunction with Theorem 4.2. In particular, we note that (4.2) implies (4.3) through suitable adaptation of the auxiliary function.
Simulation results

of second order. In relation to the second part of the theorem, we write with \( \tilde{\eta}_a \equiv \tilde{\eta}_a(m, m^*) \):

\[
\sqrt{m} (\tilde{\eta}_a - \eta) = \sqrt{m} \left\{ \tilde{\eta}_a(S) \left( 1 - \hat{\beta} \left( \frac{n}{m} \right) \frac{1 - a_\eta(S)}{1 - a_\eta(S) + \tau} \right) - \eta \right\} + \sqrt{m} \frac{1}{E_{n,n-m^*}} \tilde{\eta}_a(S) \frac{1 - a_\eta(S)}{2} \frac{1}{1 - a_\eta(S) + \tau}. \tag{5.17}
\]

Subsequently, the methodology devised in Caeiro et al. (2005) ascertains that the residual bias in the first \( \sqrt{m} \)-term is of lower order than that associated with the relevant assumption \( \sqrt{m} \beta_*(n/m) = O(1) \), as \( n \to \infty \), with the resulting asymptotic expansion

\[
\sqrt{m} \left\{ \tilde{\eta}_a(S) \left( 1 - \hat{\beta} \left( \frac{n}{m} \right) \frac{1 - a_\eta(S)}{1 - a_\eta(S) + \tau} \right) - \eta \right\} = Z_a + o_p(\sqrt{m} \beta_*(n/m)), \tag{5.18}
\]

where \( Z_a \) is a Normal random variable with mean zero and variance \( \sigma_a^2 > 0 \), the same variance as in Corollary 2. For concluding the proof, it only remains to show that the last term in (5.17) becomes negligible with probability tending to one. Thus, we apply the equality \( 1/(x + 1) = 1 - x + x^2/(1 + x) \), valid for all \( x \neq -1 \), through the identification

\[
x \equiv x(k, m) := \left( \frac{k}{n + 1} Q_n(1) - 1 \right) = O_p(1/\sqrt{m}),
\]

(cf. (5.12) with \( s = 1 \) and \( m = \lceil r(n) \rceil \)), into (5.18), specifically in that

\[
0 < \sqrt{m} \frac{1}{E_{n,n-m^*}} = \frac{k}{n + 1} \left\{ \sqrt{m} \left( \frac{k}{n + 1} Q_n(1) - 1 \right) + \sqrt{m} \left( \frac{k}{n + 1} Q_n(1) - 1 \right)^2 \right\} = \frac{k}{n + 1} \left\{ O_p(1) + o_p(1) \right\},
\]

thus surrendering the anticipated \( o_p(1) \), albeit this convergence can well be taken to a quicker rate by using \( m^* = \sqrt{m} \) in place of \( m \) that determines the threshold \( Q_n(1) \) so that the resulting \( k^* \) is brought down to ensure faster convergence \( k^*/(n + 1) \to 0 \). This accounts for the precise result in the theorem. \( \square \)

6 Simulation results

The simulation results this section encompasses draw on \( N = 1000 \) replicates of a random sample consisting of \( n \) i.i.d. random pairs \((X_i, Y_i), i = 1, \ldots, n\), from a continuous bivariate distribution function \( F \)
Simulation results

on $\mathbb{R}^2$ with univariate marginal distributions $F_j$, $j = 1, 2$, and whose dependence structure is uniquely determined by the copula function $C: [0, 1]^2 \to [0, 1]$, such that $C(u, v) = F(F_1^{-1}(u), F_2^{-1}(v))$, $0 < u, v < 1$ (Sklar, 1959). The paper by Goegebeur and Guillou (2012) offers a good catalogue of copulas belonging to the domain of attraction of a bivariate extreme value distribution, also evidencing that (residual) dependence is tractable for many bivariate distributions even before the actual ultimate extreme value limit is reached (this refers to a form of pre-asymptotic behaviour).

We choose to focus on the three copula models below, not only for their flexibility in terms of the association being induced into the random pair $(X, Y)$ and covering the three independence regimes around the exact independence case of $\eta = 1/2$, but also because they have been found to emulate the type of asymptotic independence found in the rainfall data analysed as part of the illustrative application of the proposed estimators. We have conducted a much wider simulation study, with many more distributions under scrutiny, simulating with several sample sizes.

This section is devoted to the performance evaluation of the new class of estimators for the residual dependence index by drawing on $N = 1000$ samples, each consisting of $n = 500$ realisations of pairs $(X_i, Y_i)$, $i = 1, 2, \ldots, n$ from the following copulas:

(i) **Frank distribution** with copula function

$$
C_{\theta}(u, v) = -\frac{1}{\theta} \log \left( 1 - \frac{(1 - e^{-\theta u})(1 - e^{-\theta v})}{1 - e^{-\theta}} \right), \quad (u, v) \in [0, 1]^2, \theta > 0,
$$

satisfying the second order condition (4.2) with $\eta = \tau = 1/2$. The value $\eta = 1/2$ indicates exact independence.

(ii) **Ali-Mikhail-Had distribution**, whose copula function is given by

$$
C_{\theta}(u, v) = \frac{uv}{1 - \theta(1-u)(1-v)}, \quad (u, v) \in [0, 1]^2, \theta \in [-1, 1].
$$

For $\theta = -1$ the second order condition (4.2) holds with $\eta = 1/3$ and $\tau = 2\eta = 2/3$. A value $\eta < 1/2$ indicates that joint exceedances of the same probability-threshold occur less frequently than they would have if $X$ and $Y$ were independent random variables.

(iii) **Bivariate Normal**, with Gaussian copula $C_{\theta}(u, v) = \Phi_{\theta}(\Phi^{-1}(u), \Phi^{-1}(v))$ presented in Example 1, where it is shown that, for some correlation coefficient $\theta < 1$, condition (2.1) is satisfied with $\eta = (1 + \theta)/2$. A value $\eta > 1/2$ indicates that $(X, Y)$ exceed the same probability-threshold more
Simulation results

frequently than if $X$ and $Y$ were independent random variables. Although its pertaining second order parameter $\tau = 0$ deems the Gaussian copula beyond the scope of the theoretical results in this paper, as it sits outside the set of assumptions in Section 2.1, it is nonetheless included in this simulation study solely for the purpose of assessing robustness of the new class of estimators.

In order to better evaluate how the proposed gradient-estimators fare in the landscape of estimation of a tail index $\alpha$, say, we call on the general estimator of $\alpha \geq 0$ introduced in Alves et al. (2009). On the basis of the shifted Fréchet pseudo-observables treated in the context of this paper, this estimator which includes the case of super-heavy tails is yet another functional of the tail empirical process defined in (4.1), specifically:

$$
\hat{\eta}^{SH} := \frac{\sum_{i=0}^{m-1} \left( \frac{1}{2} + E_{n,n-m} \right) / \left( \frac{1}{2} + E_{n,n-i} \right) - \sum_{i=0}^{m-1} \left( \frac{1}{2} + E_{n,n-m} \right) / \left( \frac{1}{2} + E_{n,n-i} \right)^2}{2 \sum_{i=0}^{m-1} \left( \frac{1}{2} + E_{n,n-m} \right) / \left( \frac{1}{2} + E_{n,n-i} \right)^2 - \sum_{i=0}^{m-1} \left( \frac{1}{2} + E_{n,n-m} \right) / \left( \frac{1}{2} + E_{n,n-i} \right)^2}.
$$

(6.1)

The result established in Theorem 2.1, in spite of its asymptotic quality, calls to the finite-sample results displayed in the plots (a) to (d) of both Figures 3 and 4: with a suitable and yet flexible choice of threshold rendered by the proportion $m/n$ of joint exceedances used in the estimation, the difference between estimators based on Pareto marginals and the shifted unit Fréchet becomes identical. The plots as part of Example 1 as well as those exhibited later on, in Figure 5 of Section 7 add to this finding.

Regarding reduced-bias estimation, it is worth noting at this point that the shift by $\frac{1}{2}$ entails that the rate of convergence of second order is altered for model (ii), i.e. $\tau = 1/3 < \tau$ (hence translated into a larger approximation bias albeit preserving the systematic or estimation bias), whereas it remains unchanged for the other two copulas (i) and (iii). In all cases, the long stable plateau of stability where the estimation gradient seems to wane demonstrates that the dominant component of the bias has been successfully removed inasmuch as possible for the adopted anchor estimator, namely the Hill estimator.

Most noticeably, the place at which all the estimates paths come most closely together, flatten out, and more often than not are on the cusp of crossing over, seems to pinpoint where we find the empirical mean squared error at its lowest ebb. This highlights the improvement wielded by considering the $q$-gradient lines simultaneously for an accurate identification of the optimal number $m$ of joint exceedances that should be used in the estimation of the residual dependence index $\eta$. For reference, we point out that the asymptotic variance of the Hill estimator (estimates depicted in dash-dotted blue line across Figures 3–5) has asymptotic variance equal to $\eta^2$ and the asymptotic variance of the estimator
Residual dependence in tropical extreme rainfall

defined in (6.1) (sample paths in blue dashed line) is given by

\[ \sigma_{SH}^2 = \frac{(\eta + 1)^2(4\eta^2 + 3\eta + 1)}{(2\eta + 1)^3\eta^3(4\eta + 1)(3\eta + 1)}. \]

This expression is seen as an immediate consequence of Lemma 5.1, by tracing its development thread to Theorem 2.1 and in keeping with its mirroring result in Theorem 2.4 of Alves et al. (2009), followed by application of Cramér’s delta-method.

The robustness aspect of the now introduced reduced-bias estimators in the special Gaussian case comes out as rather striking because, while the bias is constant and at the lowest level we can hope for, the range of intersection and nearly crossover of the gradient lines (spanning \( m/n = 0.25 \) to \( 0.30 \), approx.) is still able to pinpoint the optimal threshold in the sense of minimising the mean squared error, including for the Hill estimator.

For the optimal choice of \( m^* \), specific to the reduced bias estimation, a selection in the order of \( 1/\sqrt{m} \) or \( m^* = [n^{0.3}] \), borne by the proof of Theorem 3.1, is often advantageous. Either of these can be treated as feasible and as well as regarded as generic choices to fall back on even within the wider context of estimating the extreme value index from block maxima with underlying distribution function belonging to the Fréchet domain of attraction. This topic is currently under study from a theoretical perspective.

7 Residual dependence in tropical extreme rainfall

The proposed class of \( q \)-gradient estimators for the residual dependence index \( \eta > 0 \) is here applied to aid the study of extreme values arising in tropical rainfall. Our data consists of daily rainfall measurements collected over 68 years, between the years 1950 and 2017, at 591 irregularly spaced stations across Ghana. The data were collected, processed and quality controlled by the Ghana Meteorological Agency (Israelsson et al., 2020).

Ghana has a strong seasonal rainfall cycle regulated by the West African monsoon. For the purposes of this illustrative analysis, we have singled out a pair of nearby gauging stations (within the range 5-15\( km \)) and a pair of stations distancing 190-200\( km \) from each other. The focus of our analysis is on daily rainfall measurements collected every June in the 68 years worth of available data because we wish to include the main rainy season, with the obvious advantage that it contains the largest proportion of rainy days and likely the highest frequency of extreme rainfall occurrences as well.

In order to ensure that the data are identically distributed, we will only consider the stations in the
Figure 3: Frank copula with $\theta = 0.5$ ($\eta = \tau = 1/2$): (a) and (b) result from $\tilde{\eta}^{(S)}_q$ and $\tilde{\eta}^{(SH)}_q$ with standard Pareto marginals; (c) and (d) analogously for shifted unit Fréchet marginals; (e) and (f) correspond to the reduced bias version $\tilde{\eta}^{(S)}_q (m, \sqrt{m})$. Dashed lines identify $q < 1$ with orange line for the lower bound $q = 0.1$; solid lines pertain to $q > 1$ with red highlighting the upper bound $q = 1.9$. The blue dash-dotted line depicts the Hill estimator and the dashed line pertains to $\tilde{\eta}^{(SH)}_q$. 

Residual dependence in tropical extreme rainfall
Figure 4: Ali-Mikhail-Haq copula with $\theta = -1$ ($\eta = 1/3, \tau = 2/3$): (a) and (b) result from $\hat{\eta}^{(S)}_{\eta}$ and $\hat{\eta}^{(SH)}_{\eta}$ with standard Pareto marginals; (c) and (d) analogously for shifted unit Fréchet marginals; (e) and (f) correspond to the reduced bias version $\tilde{\eta}^{(S)}_{\eta}(m, [\sqrt{m}])$. Dashed lines identify $q < 1$ with orange line for the lower bound $q = 0.1$; solid lines pertain to $q > 1$ with red highlighting the upper bound $q = 1.9$. The blue dash-dotted line depicts the Hill estimator and the dashed line pertains to $\tilde{\eta}^{(SH)}_{\eta}$. 
Figure 5: **Gaussian copula with** $\theta = (\eta = 1/3, \tau = 2/3)$: (a) and (b) result from $\hat{\eta}^{(S)}_q$ and $\hat{\eta}^{(SH)}_q$ with shifted unit Fréchet marginals; (c) and (d) correspond to the reduced bias version $\hat{\eta}^{(S)}_q(m, [\sqrt{m}])$. Dashed lines identify $q < 1$ with orange line for the lower bound $q = 0.1$; solid lines pertain to $q > 1$ with red highlighting the upper bound $q = 1.9$. The blue dash-dotted line depicts the Hill estimator and the dashed line pertains to $\hat{\eta}^{(SH)}_q$. 
Residual dependence in tropical extreme rainfall

southern part of the country (south of $8^\circ$ lat) since there are significant contrasts in rainfall regimes across the country. Namely, the north of Ghana is semi-arid, exhibiting a uni-modal seasonal cycle, peaking in July/August; the south is more humid, with a bi-modal seasonal cycle, with peaks in June and October, and a break in August. None of the individual time series are without missing values, and the proportion of these was found to range between 5% to 95%. As an initial screening, we removed all stations with more than 3000 missing values, roughly equating to 12% of the full time series, which left us with 40 stations from which to select pairs. To minimise distributional variations due to systematic features in the spatial domain, one of the three chosen stations (ASU) is common for the two pairs and the two pairs are roughly aligned on the same bearing. Figure 4 highlights this with BRI located 5-10 km away from ASU, and MAM 190-200 km away. After pre-processing the data recorded at each station, in order to remove inconsistencies, we were left with 1830 bivariate observations validated for our analysis of whether asymptotic independence is present for extreme values in the data. In Israelsson et al. (2020), it is concluded that for moderate values of rainfall, stations located more than 150 km apart appear to no longer exhibit dependence in terms of simultaneous rainfall occurrences. However, some evidence was found of tenuous dependence in very heavy rainfall (i.e. of magnitude $> 50$ mm), during the month of June, even for pairs of stations at such long distance as 150 km apart. Building on these findings, we have settled with 190-200 km as a benchmark for the distance at which we expect to find some evidence of asymptotic independence in extreme tropical rainfall. Importantly, since even during the monsoon season there are many dry days (i.e., daily rainfall amounts $< 1$ mm), we have chosen to conduct estimation of the residual dependence index by drawing only on rainy days with rainfall measurements above the 90%-empirical quantile.

Figure 7 displays the estimates’ paths obtained through the parameterisation $(a, b) = (1/p, 1/q - 1)$, $1/p + 1/q = 1$ for the reduced-bias estimators $\tilde{\eta}_q(m, m^*)$ introduced in Section 3 attached to the values $q = 0.5, 1, 1.5$. We recall that $q = 1$ delivers the Hill estimator, which in the interest of enabling comparison with already published work was not subjected to any bias reduction procedure.

The sample trajectories emanating from the estimators $\tilde{\eta}_q(m, m^*)$ employed (details in the caption) seems to stabilise at the proportion $m/n = 8\%$, a remark that holds for both pairs of locations considered in this analysis. For the stations that lie further apart, the $\eta$-estimates displayed in plot (a) show an appreciable stability region that stretches from $m/n = 6\%$ to $m/n = 10\%$, approximately, with the resulting $\eta$-estimates obtained with the boundary $q = 0.5$ and $q = 1.5$ clasping a value in the order of 0.5 rather tightly. On the other hand, the Hill estimator exhibits a steadily increasing bias which makes it
Figure 6: Map of southern Ghana highlighting the three selected stations.

Figure 7: Reduced-bias estimation of the residual dependence index $\eta \in [0, 1)$: plot (a) corresponds to the long distance residual dependence analysis in the study. The blue thick line gives the sample path for $q = 0.5$, whereas the red solid line corresponds to the $q = 1.5$. Shaded areas indicate their respective 95% confidence bands. The yellow line depicts the Hill estimator; plot (b) is the analogous plot for the nearby stations (see Fig. 6).
Figure 8: Scatter plots of positive daily rainfall transformed to the unit uniform scale for the two pairs of stations: (a) (ASU, MAM); (b) (ASU, BRI).

It is hard to discern a good estimate for $\eta$ in the absence of a formal procedure for threshold selection. This, again, is in stark contrast with the consistently smooth trajectories around 0.5 that emanate from the reduced-bias estimation thus suggesting exact independence as the plausible regime between extreme rainfall occurring at the well separated stations ASU and MAM. The estimates-plot in panel (b) of Figure 7 strikes some consensus amongst the estimators employed as it shows that those reduced-bias estimates stemming from the two estimators $\tilde{\eta}_q(k)$ for enhanced consistency, tend to settle together around the value $\hat{\eta} = 0.77$, with the Hill’s estimates laying in relative proximity. Therefore, the estimation of $\eta$ indicates positive association, in that concomitantly high rainfall values in stations BRI and ASU tend to occur rather more frequently than under the exact independence regime determined for stations ASU and MAM. These estimates provide a measure of the asymptotic independence that is suggested in the two scatter-plots presented in Figure 8. To not observe maximal dependence associated with an estimated value $\eta = 1$, even at such a short distance, is to be expected for tropical rainfall, where sporadic but intense storms tend to affect only a small area of a few square-kilometres. Additionally, in the situation where a convective storm is initiated over one station and travels in the opposite direction to the paired station, only one of them will record rainfall, thus reducing the dependence between them.

As a final note, we clarify that the second order parameters embedded in the definition of $\beta_s$, for equation (3.1) as well as the second order index $\tau > 0$, have been estimated externally using the adaptation of the methodology in Caeiro et al. (2005). In particular, we have followed the guidelines in Caeiro...
et al. (2005) which advise that, for a sample of size $n$, all second order parameters must be estimated at the considerably lower (random) threshold. The general recommendation issued in these studies is to employ $m = \lceil n^{0.999} \rceil$ in the estimation of $\tau$.

Acknowledgements

Cláudia Neves gratefully acknowledges support from UKRI-EPSRC Innovation Fellowship grant EP/S001263/1 and EP/S001263/2. Her work is also partly supported by CEAUL, Faculty of Science, University of Lisbon, DOI: 10.54499/UIDB/00006/2020, https://doi.org/10.54499/UIDB/00006/2020. This work formed part of the PhD research of Jennifer Israelsson, who received funding from the UKRI-EPSRC Centre for Doctoral Training in Mathematics of Planet Earth, grant EP/L016613/1.

We are grateful to the Ghana Meteorological Authority for providing the daily precipitation dataset used in this study.

References

Alves, I. F., de Haan, L., and Neves, C. (2009). A test procedure for detecting super-heavy tails. *Journal of Statistical Planning and Inference*, 139(2):213–227.

Beirlant, J., Goegebeur, Y., Segers, J., and Teugels, J. (2004). *Statistics of Extremes: Theory and Application*. Wiley Series in Probability and Statistics.

Di Bernardino, E., Maume-Deschamps, V., and Prieur, C. (2013). Estimating a bivariate tail: A copula based approach. *Journal of Multivariate Analysis*, 119:81–100.

Bücher, A. and Zhou, C. (2021). A horse race between the block maxima method and the peak–over–threshold approach. *Statistical Science*, 36(3):360 – 378.

Bücher, A., Volgushev, S., and Zou, N. (2019). On second order conditions in the multivariate block maxima and peak over threshold method. *Journal of Multivariate Analysis*, 173:604–619.

Caeiro, F., Gomes, M. I., and Pestana, D. D. (2005). Direct reduction of bias of the classical Hill estimator. *Revstat*, 3:111–136.

Coles, S. (2001). *An Introduction to Statistical Modeling of Extreme Values*. Springer London, London.
Coles, S. G. and Walshaw, D. (1994). Directional modelling of extreme wind speeds. *Journal of the Royal Statistical Society. Series C (Applied Statistics)*, 43(1):139–157.

Csörgő, M. (1983). *Quantile Processes with Statistical Applications*. Society for Industrial and Applied Mathematics.

Csörgő, M. and Horváth, L. (1993). *Weighted Approximations in Probability and Statistics*. Wiley Series in probability and mathematical statistics.

Draisma, G., Drees, H., Ferreira, A., and de Haan, L. (2004). Bivariate tail estimation: dependence in asymptotic independence. *Bernoulli*, 10(2):251–280.

Drees, H. and Kaufmann, E. (1998). Selecting the optimal sample fraction in univariate extreme value estimation. *Stochastic Processes and their Applications*, 75(2):149–172.

Eastoe, E. F. and Tawn, J. A. (2012). Modelling the distribution of the cluster maxima of exceedances of subasymptotic thresholds. *Biometrika*, 99(1):43–55.

Einmahl, J. H. (1997). Poisson and Gaussian approximation of weighted local empirical processes. *Stochastic Processes and their Applications*, 70(1):31–58.

Einmahl, J. H. J., Ferreira, A., de Haan, L., Neves, C., and Zhou, C. (2022). Spatial dependence and space–time trend in extreme events. *The Annals of Statistics*, 50(1):30 – 52.

Goegebeur, Y. and Guillou, A. (2012). Asymptotically unbiased estimation of the coefficient of tail dependence. *Scandinavian Journal of Statistics*, 40(1):174–189.

Gong, Y. and Huser, R. (2022). Asymmetric tail dependence modeling, with application to cryptocurrency market data. *The Annals of Applied Statistics*, 16(3):1822 – 1847.

de Haan, L. and Ferreira, A. (2006). *Extreme Value Theory: An Introduction*. Springer.

de Haan, L. and Zhou, C. (2011). Extreme residual dependence for random vectors and processes. *Advances in Applied Probability*, 43(1):217 – 242.

Hall, P. and Welsh, A. H. (1985). Adaptive estimates of parameters of regular variation. *The Annals of Statistics*, 13(1):331 – 341.
Hill, B. M. (1975). A simple general approach to inference about the tail of a distribution. *Ann. Statist.*, 3(5):1163–1174.

Israelsson, J., Black, E., Neves, C., Torgbor, F. F., Greatrex, H., Tanu, M., and Lamptey, P. N. L. (2020). The spatial correlation structure of rainfall at the local scale over southern Ghana. *Journal of Hydrology: Regional Studies*, 31.

Jacob, M., Vukadinovic-Greetham, D., and Neves, C. (2020). *Forecasting and Assessing Risk of Individual Electricity Peaks*. Springer Briefs in Mathematics of Planet Earth (Open Access).

Kosorok, M. R. (2008). *Introduction to Empirical Processes and Semiparametric Inference*. Springer.

Ledford, A. W. and Tawn, J. A. (1996). Statistics for near independence in multivariate extreme values. *Biometrika*, 83(1):169–187.

Ledford, A. W. and Tawn, J. A. (1997). Modelling dependence within joint tail regions. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 59(2):475–499.

Neves, C. (2009). From extended regular variation to regular variation with application in extreme value statistics. *Journal of Mathematical Analysis and Applications*, 355(1):216–230.

Poon, S.-H., Rockinger, M., and Tawn, J. (2003). Modelling extreme-value dependence in international stock markets. *Statistica Sinica*, 13(4):929–953.

Ramos, A. and Ledford, A. (2009). A new class of models for bivariate joint tails. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 71(1):219–241.

Reiss, R.-D. and Thomas, M. (2007). *Statistical Analysis of Extreme Values*. Birkhäuser Basel.

Sang, H. and Gelfand, A. E. (2009). Hierarchical modeling for extreme values observed over space and time. *Environmental and Ecological Statistics*, 16(3):407–426.

Schlather, M. (2001). Examples for the coefficient of tail dependence and the domain of attraction of a bivariate extreme value distribution. *Statistics & Probability Letters*, 53(3):325–329.

Sibuya, M. (1960). Bivariate extreme statistics, I. *Annals of the Institute of Statistical Mathematics*, 11(3):195–210.
Sklar, A. (1959). Fonctions de répartition à $n$ dimensions et leurs marges. *Publications de l'Institut Statistique de l'Université de Paris*, 8:229–231.

Subramanyam, K. (1990). Some comments on positive quadrant dependence in three dimensions. *Lecture Notes-Monograph Series*, 16:443–449.