ROOT SYSTEMS, AFFINE SUBSPACES, AND PROJECTIONS

PAOLA CELLINI AND MARIO MARIETTI

Abstract. We tackle several problems related to a finite irreducible crystallographic root system $\Phi$ in the real vector space $E$. In particular, we study the combinatorial structure of the subsets of $\Phi$ cut by affine subspaces of $E$ and their projections. As byproducts, we obtain easy algebraic combinatorial proofs of refinements of Oshima’s Lemma and of a result by Kostant, a partial result towards the resolution of a problem by Hopkins and Postnikov, and new enumerative results on root systems.

Keywords: Root system; Root polytope; Weyl group

1. Introduction

Let $\Phi$ be a finite irreducible crystallographic root system and let $\Pi$ be a set of simple roots of $\Phi$. Given a subset $S$ of $\Pi$ and $\beta \in \Phi$, with $\beta = \sum c_\alpha(\beta)\alpha$, we set

$$\Phi_{S,\beta} = \left\{ \gamma = \sum_{\alpha \in \Pi} c_\alpha(\gamma)\alpha \in \Phi : c_\alpha(\gamma) = c_\alpha(\beta) \text{ for all } \alpha \in S \right\},$$

$$\Phi_{S,\mathbb{Z}\beta} = \left\{ \gamma = \sum_{\alpha \in \Pi} c_\alpha(\gamma)\alpha \in \Phi : \exists k \in \mathbb{Z} \text{ with } c_\alpha(\gamma) = kc_\alpha(\beta) \text{ for all } \alpha \in S \right\}.$$  

If $\text{Supp}(\beta) \cap S = \emptyset$, the set $\Phi_{S,\beta}$ coincides with $\Phi_{S,\mathbb{Z}\beta}$ and is the standard parabolic subsystem generated by $\Pi \setminus S$. So the nontrivial case happens when $\text{Supp}(\beta) \cap S \neq \emptyset$.

By definition, $\Phi_{S,\beta}$ is the intersection of $\Phi$ with the affine subspace that contains $\beta$ and has $\text{Span}(\Pi \setminus S)$ as directional linear subspace. The subset $\Phi_{S,\mathbb{Z}\beta}$ is, in fact, a root subsystem of $\Phi$ and is a union of strata each obtained by intersecting $\Phi$ with an affine subspace parallel to $\text{Span}(\Pi \setminus S)$.

In this paper, we study several problems concerning the subsets $\Phi_{S,\beta}$ and $\Phi_{S,\mathbb{Z}\beta}$ and their applications.

Section 2 provides the notational conventions used throughout the paper.

In Section 3, we give a uniform case-free description of the root polytope of an arbitrary (possibly reducible) root system. The root polytope of a root system is the convex hull of all roots and, in the irreducible case, it is studied in [2], [3], [4]. The results in this section are needed in Section 7.
In Section 4, we study the root subsystem $\Phi_{S,\beta}$ and find two of its simple systems that provide in a natural way the existence of the minimum and the maximum of $\Phi_{S,\beta}$ viewed as a subposet of the root poset.

As consequences of the results in this section, we obtain easy case-free algebraic combinatorial proofs of a result by Kostant, of a result by Oshima, and some refinements. Kostant’s result, which is presented in [9, Section 2], asserts the irreducibility of a certain representation of the semisimple subalgebra $\mathfrak{g}_{\Pi \setminus S}$ associated with $\Pi \setminus S$ (see Remark 4.3). Oshima’s result [10, Lemma 4.3], which is Corollary 4.7 in this paper, has received attention recently for its applications and is referred to as “Oshima’s Lemma”. The purpose of Dyer and Lehrer in [5] is to provide an elementary self-contained proof of (a generalised version of) Oshima’s Lemma. Our proof here is very elementary, self-contained, and independent of the representation theory of semisimple complex Lie algebras (see [5]). Oshima’s Lemma states that the action of $W(\Pi \setminus S)$ partitions the roots in $\Phi_{S,\beta}$ according to their length, hence in at most two orbits. In the representation theory interpretation, this means that the weights of the irreducible representation studied by Kostant are divided in at most two orbits. Indeed, the fact that the roots of maximal length constitute a single orbit follows directly from the irreducibility of this representation. We provide general necessary and sufficient conditions for establishing whether $\Phi_{S,\beta}$ contains roots of one or two lengths (Proposition 4.12). Moreover, we provide an easy combinatorial criterion on the pair $(S, \beta)$ for $\Phi_{S,\beta}$ to contain long roots (Proposition 4.14). Though this last criterion is uniform, our proof requires some case by case check.

In Sections 5 and 6, we study in more detail the case of codimension 1, when $S$ consists of a single simple root $\alpha$. In this case, we obtain a partition of $\Phi$ into subsets $\Phi_{\alpha,k} = \Phi \cap \{ x \in \mathbb{E} : (x, \tilde{\omega}_\alpha) = k \}$ that are all intervals w.r.t. the usual partial order, except for the 0-level $\Phi_{\alpha,0}$, which is a possibly reducible root subsystem. In particular, we analyse the projections of the subsets $\Phi_{\alpha,k}$ on the hyperplane $\text{Span}(\Pi \setminus \{\alpha\})$. As a consequence of that analysis, we obtain a partial result towards the resolution of a problem by Hopkins and Postnikov. In [8], Hopkins and Postnikov show that the projection of the root polytope of $\Phi$ fall inside the polytope formed by expanding by a factor of 2 the root polytope of the root subsystem generated by $\Pi \setminus \{\alpha\}$ (see Lemma 7.1 ). Their proof uses the classification of root systems and finding a case-free proof is an open problem in [7]. In Section 7, we give a uniform case-free argument that proves that the result of Hopkins and Postnikov is equivalent to the fact that a certain number $r_\alpha$, defined by a uniform formula (7.5.1), is smaller than 2, for each simple root $\alpha$. Unfortunately, this final check is done case-by-case.

In Section 8, we give some enumerative results on irreducible root systems.
Except for the last part of the proof of Proposition 4.14 (which is not needed in the rest of the paper), its Lemma 4.13 and the final check in Section 7 discussed above, all uniform statements are proved independently of the classification of root systems.

2. Notational conventions

Let \( \Phi \) be a finite (reduced, possibly reducible) crystallographic root system in the real vector space \( E = \text{Span}_\mathbb{R} \Phi \) endowed with the positive definite bilinear form \((\cdot,\cdot)\). We identify \( E \) with its dual space via its bilinear form.

We fix our notation on the root system and its Weyl group in the following list:

- \( \Phi^\vee \) the set of coroots of \( \Phi \),
- \( \alpha^\vee \) the coroot associated with the root \( \alpha \in \Phi \),
- \( \Pi \) the set of simple roots of \( \Phi \),
- \( \Pi^\vee \) the set of simple coroots of \( \Phi \),
- \( \Omega = \{ \omega_\alpha : \alpha \in \Pi \} \) the set of fundamental weights (the dual basis of \( \Pi^\vee \)),
- \( \tilde{\Omega} = \{ \tilde{\omega}_\alpha : \alpha \in \Pi \} \) the set of fundamental co-weights (the dual basis of \( \Pi \)),
- \( \Phi^+ \) the set of positive roots w.r.t. \( \Pi \),
- \( C \) the fundamental chamber associated with \( \Pi \),
- \( c_\alpha(\beta) \) the \( \alpha \)-coordinate of \( \beta \) w.r.t. \( \Pi \): \( \beta = \sum_{\alpha \in \Pi} c_\alpha(\beta) \alpha \),
- \( \text{Supp}(\alpha) \) = \( \{ \alpha_i \in \Pi : c_i(\alpha) \neq 0 \} \), the support of \( \alpha \),
- \( W \) the Weyl group of \( \Phi \),
- \( s_\alpha \) the reflection with respect to \( \alpha \),
- \( W(S) \) the subgroup of \( W \) generated by \( \{ s_\alpha : \alpha \in S \} \) (for \( S \subseteq \Phi \)),
- \( \Phi(S) \) the root subsystem generated by \( S \) (for \( S \subseteq \Phi \)),
- \( P_\Phi \) the convex hull of all roots in \( \Phi \), called the root polytope of \( \Phi \).

For any \( x \in E \), \( x \) dominant means \( x \in C \).

The root poset of \( \Phi \) (w.r.t. the basis \( \Pi \)) is \( \Phi^+ \) with the standard partial order, i.e., for all \( \alpha, \beta \in \Phi^+ \), \( \alpha \leq \beta \) if and only if \( \beta - \alpha \) is a nonnegative linear combination of roots in \( \Phi^+ \). The root poset is the transitive closure of the relation \( \alpha \triangleright \beta \) if and only if \( \beta - \alpha \) is a simple root. The root poset hence is ranked by the height function.

In the irreducible case, we denote by \( \theta \) the highest root in \( \Phi \), which is the maximum of the root poset, and by \( m_\alpha \) the coefficient \( c_\alpha(\theta) \), for \( \alpha \in \Pi \). Furthermore, we denote by \( \hat{\Phi} \) the affine root system associated to \( \Phi \) and realized as follows. We extend \( E \) to a vector space \( E \oplus \mathbb{R}\delta \), and extend the bilinear form of \( E \) to a semidefinite positive form on \( E \oplus \mathbb{R}\delta \) by the condition that \( \mathbb{R}\delta \) is the kernel of the new form. Then we set:
$\alpha_0 = -\theta + \delta,$
\[\hat{\Pi} = \{\alpha_0\} \cup \Pi,\]
\[\hat{\Phi} = \{\beta + k\delta : \beta \in \Phi, k \in \mathbb{Z}\}\] the affine root system with simple system $\hat{\Pi},$
\[\hat{\Phi}^+ = \{\beta + k\delta : \beta \in \Phi^+, k \in \mathbb{Z}_{\geq 0}\} \cup \{-\beta + k\delta : \beta \in \Phi^+, k \in \mathbb{Z}_{\geq 0}\}\] the set of (real) positive roots of $\hat{\Phi}$ w.r.t. $\hat{\Pi}.$

For $k \in \mathbb{N},$ we let $[1, k] = \{1, \ldots, k\}.$

3. Root polytopes of possibly reduced root systems

Recall that we denote by $P_\Phi$ the convex hull of all roots in $\Phi,$ and we call it the root polytope of $\Phi.$ For all irreducible $\Phi,$ a uniform explicit description of the root polytope $P_\Phi$ is given in [3]. In this section, we extend some of the results in [3] to the reducible case.

Let us first recall some results from [3]. If $G$ is a graph with set of vertices $V$ and $U$ is a subset of $V,$ we denote by $G \setminus U$ the subgraph of $G$ induced by the subset of vertices $V \setminus U.$

**Definition 3.1.** Let $\Phi$ be irreducible and $\hat{\Gamma}$ be its extended Dynkin diagram. For any subset $I$ of $\Pi,$ we say that $I$ is $\hat{\Pi}$-extremal if $\hat{\Gamma} \setminus I$ is connected.

We say that the root $\alpha$ is $\hat{\Pi}$-extremal if $\{\alpha\}$ is.

**Remark 3.2.**

1. If $\Phi$ is of type $A_n,$ all roots in $\Pi$ are $\hat{\Pi}$-extremal. For all other types, the extremal roots are the leaves of the extendend Dynkin diagram other than the affine vertex.

2. Let $\theta$ be the highest root of $\Phi$ (supposed irreducible). A subset $I$ is $\hat{\Pi}$-extremal if and only if $\{-\theta\} \cup \Pi \setminus I$ is the simple system of an irreducible root subsystem of $\Phi.$

Let $\Phi$ be irreducible with highest root $\theta = \sum_{\alpha \in \Pi} m_\alpha \alpha,$ and set $o_\alpha = \tilde{\omega}_\alpha / m_\alpha$ for all $\alpha \in \Pi.$

Given $I \subseteq \Pi,$ we set
\[F_I = \text{Conv}\{\gamma \in \Phi : (\gamma, o_\alpha) = 1 \text{ for all } \alpha \in I\}\]

**Proposition 3.3, Proposition 3.5, and Theorem 5.5 of [3] imply the following result.**

**Theorem 3.3.** Let $\Phi$ be irreducible.

1. For each $\hat{\Pi}$-extremal subset $I,$ the set $F_I$ is a face of $P_\Phi$ of dimension $rk(\Phi) - |I|.$

2. For each subset $I$ of $\Pi,$ there exists a $\hat{\Pi}$-extremal subset $\bar{I}$ such that $F_I = F_{\bar{I}}.$

3. For each face $F$ of $P_\Phi,$ there exist some $w \in W$ and a unique $\hat{\Pi}$-extremal subset $I$ such that $F = w(F_I).$
Following [3], we call the faces of type $F_I$ the \textit{standard parabolic faces}. Note that the standard parabolic facets are exactly the faces of type $F_{[\alpha]}$ where $\alpha$ is a $\hat{\Pi}$-extremal root.

**Remark 3.4.** When $\Phi$ is irreducible, for each subset $I$ of $\Pi$, there exists a root $\alpha \in I$ such that $\alpha$ is not orthogonal to $\Pi \setminus I$. If $I$ is $\hat{\Pi}$-extremal, for such a root $\alpha$ we have that $I \setminus \{\alpha\}$ is $\hat{\Pi}$-extremal too, if nonempty. This implies that any standard parabolic face either is a facet, or is contained in a standard parabolic face of higher dimension. In particular, each standard parabolic face is contained in a standard parabolic facet.

Now we suppose that $\Phi$ be possibly reducible.

For any face $F$ of $\mathcal{P}_\Phi$, we denote by $\hat{F}$ the relative interior of $F$, i.e. its interior within the affine hull of $F$. The relative interior $\hat{F}$ of $F$ consists of all strictly positive convex linear combinations of the vertices of $F$. If $F$ and $F'$ are different faces, then $\hat{F} \cap \hat{F'} = \emptyset$.

**Definition 3.5.** Let $F$ be a face of $\mathcal{P}_\Phi$. We call $F$ \textit{standard} provided that either $\hat{F} \cap \mathcal{C} \neq \emptyset$, or $F = \emptyset$.

The Weyl group $W$ acts on the faces of $\mathcal{P}_\Phi$. We prove that the standard faces are a set of representatives of the orbits of this action. Given a face $F$ with vertex set $V$, we denote by $\beta_F$ the barycenter of $V$, i.e. $\beta_F = \frac{1}{|V|} \sum_{v \in V} v$.

**Theorem 3.6.** Let $\Phi$ be a (possibly reducible) root system and let $F$ be a face of $\mathcal{P}_\Phi$.

1. $F$ is standard if and only if the barycenter $\beta_F$ is dominant. In this case, the stabilizer $\text{Stab}_W(F)$ of $F$ is the standard parabolic subgroup of $W$ generated by $\Pi \cap \beta_F^\perp$.
2. The orbit $W \cdot F$ of $F$ under the action of $W$ contains exactly one standard face.

**Proof.** (1) Given $\gamma \in \Phi$, we denote by $s_\gamma$ the reflection through $\gamma$ and by $H_\gamma$ the hyperplane fixed by $s_\gamma$.

For each $w \in W$, if $w(F) \neq F$ then $\hat{F} \cap w(\hat{F}) = \emptyset$. In particular, for each $\gamma \in \Phi$, since $H_\gamma \cap \hat{F}$ is fixed by $s_\gamma$, if $s_\gamma(F) \neq F$ then $H_\gamma \cap \hat{F} = \emptyset$, and $\hat{F}$ lies in one of the two open half spaces determined by $H_\gamma$. If, in addition, $\gamma > 0$ and $\hat{F} \cap \mathcal{C} \neq \emptyset$, then $(x, \gamma) > 0$ for all $x \in \hat{F} \cap \mathcal{C}$, and hence $(x, \gamma) \geq 0$ for all $x \in F$.

The stabilizer $\text{Stab}_W(F)$ of $F$ coincides with the stabilizer of the barycenter $\beta_F$, hence $\text{Stab}_W(F)$ is the parabolic subgroup of $W$ generated by $\{s_\gamma : (\gamma, \beta_F) = 0\}$. We have just seen that, if $F$ is standard, then $(\gamma, \beta_F) > 0$ for all $\gamma \notin \text{Stab}_W(F)$, hence $\beta_F \in \mathcal{C}$.

Conversely, if $\beta_F \in \mathcal{C}$, then $F$ is standard by definition, since $\beta_F \in \hat{F}$.

Furthermore, if $\beta_F \in \mathcal{C}$, then $\text{Stab}_W(F)$ is a standard parabolic subgroup, namely, $\text{Stab}_W(F) = W(\Pi \cap \beta_F^\perp)$.

(2) Since $\mathcal{C}$ is a fundamental domain for $W$, there is one and only one face in $W \cdot F$ with dominant barycenter, hence $W \cdot F$ contains exactly one standard face by (1). □
As a direct consequence of Theorem 3.6 we have the following result in the irreducible case.

**Corollary 3.7.** Let $\Phi$ be irreducible. The standard faces of $P_\Phi$ are exactly the standard parabolic faces.

**Proof.** Since the barycenters of the standard parabolic faces are dominant (see [3, Proposition 4.4]), the statement follows by Theorem 3.6. □

As we see below, each dominant point belongs to the cone of some standard face.

**Proposition 3.8.** Let $\Phi$ be a (possibly reducible) root system and let $F$ be a face of $P_\Phi$ such that $F \cap C \neq \emptyset$. Then there is a standard subface $F'$ of $F$ such that $F \cap C = F' \cap C$.

**Proof.** If $F$ is a vertex of $P_\Phi$, then $F$ is standard. If $F$ is not standard, then there is a proper subface $F'$ of $F$ such that $F \cap C = F' \cap C$. The assertion follows by induction on the dimension of $F$. □

Proposition 3.8 directly implies the following result.

**Corollary 3.9.** Let $\Phi$ be a (possibly reducible) root system and let $\partial P_\Phi$ be the boundary of $P_\Phi$, i.e. the union of its faces. Then $C \cap \partial P_\Phi$ is contained in the union of the standard faces. In particular, for each nonzero $x \in C$, there exist a positive real number $r$ and a standard face $F$ such that $rx \in F$.

We now determine explicitly the standard faces in the case $\Phi$ is any (possibly reducible) root system.

Let $\Phi_1, \ldots, \Phi_k$ be the irreducible components of $\Phi$, $E_i = \text{Span}(\Phi_i)$, $\Pi_i = \Pi \cap \Phi_i$, $C_i = C \cap E_i$, and let $P_i = P_\Phi \cap E_i$ be the root polytope of $\Phi_i$. We say that $x \in E$ is $\Phi_i$-dominant provided that $x \in C_i$.

Let $F$ be a face of $P_\Phi$ and

$$F_i = F \cap E_i.$$ 

If $F = P_\Phi$, then $F_i = P_i$ for all $i \in [1,k]$. Conversely, if $F \neq P_\Phi$, then $F_i \neq P_i$ for all $i \in [1,k]$. By a *proper face*, we intend a possibly empty face different from the whole polytope (this definition slightly differs from the usual one but is more convenient for our purposes).

**Proposition 3.10.** Let $\Phi$ be a (possibly reducible) root system.

1. The map sending a face $F$ to $(F_1, \ldots, F_k)$, where $F_i = F \cap E_i$, is a bijection between the proper faces of $P_\Phi$ and the $k$-tuples $F_1, \ldots, F_k$, where $F_i$ is a proper face of $P_i$ for every $i \in [1,k]$.

2. A proper face $F$ has dimension

$$\dim F = k - 1 + \sum_{i=1}^k \dim F_i.$$
(3) A face \( F \) of \( \mathcal{P}_\Phi \) is standard if and only if \( F_i \) is standard for all \( i \in [1, k] \).

**Proof.** (1) Let \( F \) be a proper face of \( \mathcal{P}_\Phi \), and denote by \( H_F \) a supporting hyperplane for \( F \), i.e., a hyperplane such that \( F = \mathcal{P}_\Phi \cap H_F \) and \( \mathcal{P}_\Phi \) is contained in one of the two halfspaces cut by \( H_F \). Since \( \Phi \) is symmetric and spans \( \mathbb{E} \), the hyperplane \( H_F \) does not contain the origin, hence \( H_F \cap \mathbb{E}_i \) is either empty, or a hyperplane in \( \mathbb{E}_i \). Since \( F_i = F \cap \mathbb{E}_i = \mathcal{P}_\Phi \cap H_F \cap \mathbb{E}_i = \mathcal{P}_i \cap (H_F \cap \mathbb{E}_i) \), it follows that \( F_i \) is a proper (possibly empty) face of \( \mathcal{P}_i \). Since \( F = \text{Conv}(F_1 \cup \cdots \cup F_k) \), the map in the statement is injective.

Conversely, for any choice of proper faces \( F_1, \ldots, F_k \) of \( \mathcal{P}_1, \ldots, \mathcal{P}_k \), respectively, consider the convex hull \( F \) of \( F_1 \cup \cdots \cup F_k \). Clearly, \( F \cap \mathbb{E}_i = F_i \), for every \( i \in [1, k] \). We need to show that \( F \) is a proper face of \( \mathcal{P}_\Phi \). Indeed, since the supporting hyperplanes of the \( \mathcal{P}_i \) do not contain the origin, there exist \( y_i \in \mathbb{E}_i \) such that \( (y_i, x) \leq 1 \) for all \( x \in \mathcal{P}_i \), and \( F_i = \{ x \in \mathcal{P}_i : (x, y_i) = 1 \} \), for every \( i \in [1, k] \). If we define \( y = y_1 + \cdots + y_k \), we obtain \( (y, x) \leq 1 \) for all \( x \in \mathcal{P}_\Phi \) and \( (y, x) = 1 \) if and only if \( x \in F \).

(2) Since a face of a polytope is the convex hull of the vertices of the polytope that belong to the face, \( F \) is the convex hull of \( F \cap \Phi \), and \( F_i \) is the convex hull of \( F \cap \Phi_i \). If \( F \) is proper and nonempty, then \( \dim F = \dim(\text{Span}(F \cap \Phi)) - 1 \) and \( \dim F_i = \dim(\text{Span}(F \cap \Phi_i)) - 1 \). The dimension formula follows by the preceding formulas (which hold also for the empty face, since it has dimension \(-1 \) by definition).

(3) The set of vertices of \( F \) is the union of the sets of vertices of the subfaces \( F_i \). Hence, the barycenter \( \beta_F \) of \( F \) is a positive linear combination of the barycenters \( \beta_{F_i} \) of the nonempty faces \( F_i \), and \( \beta_F \) is dominant if and only if all these barycenters \( \beta_{F_i} \) are \( \Phi_i \)-dominant. This implies the claim by Theorem 3.6. \( \square \)

Now let \( F \) be standard. For \( i \in [1, k] \), denote by \( \theta_i \) the highest root of \( \Phi_i \). For \( \alpha \in \Pi_i \), define \( m_\alpha \) by the condition \( \theta_i = \sum_{\alpha \in \Pi_i} m_\alpha \alpha \), and set \( o_\alpha = \hat{\omega}_i/m_\alpha \). By Theorem 3.3, Corollary 3.7, and Proposition 3.10 for every nonempty \( F_i = F \cap \mathbb{E}_i \), there exists a unique \( \hat{\Pi}_i \)-extremal subset \( \Pi_i(F) \) of \( \Pi_i \) such that:

\[
F_i = \{ x \in \mathcal{P}_i : (x, o_\alpha) = 1 \text{ for all } \alpha \in \Pi_i(F) \}.
\]

We let

\[
o_i(F) = \frac{1}{|\Pi_i(F)|} \sum_{\alpha \in \Pi_i(F)} o_\alpha.
\]

For \( F_i = \emptyset \), we set \( o_i(F) = 0 \). Since \( (x, o_\alpha) \leq 1 \) for all \( \alpha \in \Pi_i \) and \( x \in \mathcal{P}_i \), we have

\[
(x, o_i(F)) \leq 1, \text{ for all } x \in \mathcal{P}_i, \text{ and } F_i = \{ x \in \mathcal{P}_i : (x, o_i(F)) = 1 \}.
\]

The following result holds.
Proposition 3.11. Let $\Phi$ be a (possibly reducible) root system. If $F$ is a standard face of $P_{\Phi}$, then \( \left( x, \sum_{i=1}^{k} o_i(F) \right) \leq 1 \) for all $x \in P_{\Phi}$ and

\[
F = \left\{ x \in P_{\Phi} : \left( x, \sum_{i=1}^{k} o_i(F) \right) = 1 \right\}.
\]

Proof. Let $x \in P_{\Phi}$. From any expression of $x$ as a convex linear combination of the roots in $\Phi$, we easily obtain a subset $J$ of $[1, k]$ and nonzero elements $x_i \in P_i$, for all $i \in J$, such that $x$ is a strictly positive convex linear combination of such elements $x_i$. By (3.10.1) above, \( (x_i, o_i(F)) \leq 1 \) for all $i \in J$, hence \( \left( x, \sum_{i=1}^{k} o_i(F) \right) = \left( x, \sum_{i \in J} o_i(F) \right) \leq 1 \). Moreover, \( \left( x, \sum_{i \in J} o_i(F) \right) = 1 \) if and only if \( (x_i, o_i(F)) = 1 \) for all $i \in J$, hence if and only if $x_i \in F_i$ for all $i \in J$: this is equivalent to $x \in F$. \qed

In the following corollary, we isolate the special case of facets. The proof is a direct application of the preceding results.

Corollary 3.12. Let $\Phi$ be a (possibly reducible) root system and let $F$ be a facet of $P_{\Phi}$. Then $F_i \neq \emptyset$ and \( \dim F_i = \dim E_i - 1 \), for all $i \in [1, k]$. In particular, $F$ is standard if and only if $F_i$ is a standard facet of $P_i$. In this case, for all $i \in [1, k]$, the set $\Pi_i(F)$ is a singleton containing a $\hat{\Pi}_i$-extremal root $\alpha_i \in \Pi_i$, hence $o_i(F) = o_{\alpha_i}$ and the supporting hyperplane of $F$ is

\[
\{ x \in E : (x, o_{\alpha_1} + \cdots + o_{\alpha_k}) = 1 \}.
\]

Conversely, for each $k$-tuple $(\alpha_1, \ldots, \alpha_k) \in \Pi_1 \times \cdots \times \Pi_k$ such that $\alpha_i$ is $\hat{\Pi}_i$-extremal for all $i \in [1, k]$, the hyperplane

\[
\{ x \in E : (x, o_{\alpha_1} + \cdots + o_{\alpha_k}) = 1 \}
\]

supports a standard facet of $P_{\Phi}$.

Corollary 3.12 and Remark 3.4 together imply that each standard face is a subface of a standard facet. Hence, by Corollary 3.9, we obtain the following result.

Corollary 3.13. Let $x$ be a dominant element in $E$. Then $\text{Conv}(W \cdot x) \subseteq P_{\Phi}$ if and only if

\[
(x, o_{\alpha_1} + \cdots + o_{\alpha_k}) \leq 1,
\]

for each $k$-tuple $(\alpha_1, \ldots, \alpha_k) \in \Pi_1 \times \cdots \times \Pi_k$ such that $\alpha_i$ is $\hat{\Pi}_i$-extremal for all $i \in [1, k]$. 
4. Root systems and affine subspaces

In this section, we study the intersections of an irreducible root system with certain affine subspaces.

Let $\Phi$ be irreducible.

Given a subset $S$ of $\Pi$ and $\beta \in \Phi$, with $\beta = \sum_{\alpha \in \Pi} c_{\alpha} \alpha$, we set

$$\Phi_{S,\beta} = \left\{ \gamma = \sum_{\alpha \in \Pi} c_{\alpha} \alpha : c_{\alpha}(\gamma) = c_{\alpha}(\beta) \text{ for all } \alpha \in S \right\},$$

$$\Phi_{S,\mathbb{Z},\beta} = \left\{ \gamma = \sum_{\alpha \in \Pi} c_{\alpha} \alpha : \exists k \in \mathbb{Z} \text{ with } c_{\alpha}(\gamma) = kc_{\alpha}(\beta) \text{ for all } \alpha \in S \right\}.$$

**Remark 4.1.** Let $\beta \in \Phi$. If $\Phi$ were reducible and $\Psi$ were the irreducible component of $\Phi$ containing $\beta$, then $\Phi_{S,\beta} = \Psi_{S,\beta}$ and $\Phi_{S,\mathbb{Z},\beta} = \Psi_{S,\mathbb{Z},\beta}$ for all nonempty $S$. Thus, there is no loss of generality in our assumption that $\Phi$ be irreducible.

Recall that $\delta$ denotes the indivisible imaginary root of the affine root system $\widehat{\Phi}$ (see Section 2).

**Proposition 4.2.** Let $S \subseteq \Pi$ and $\beta \in \Phi^+$ with $\text{Supp}(\beta) \cap S \neq \emptyset$.

1. $\Phi_{S,\mathbb{Z},\beta}$ is a root subsystem of $\Phi$.
2. $\Phi_{S,\beta}$ is an interval of the root poset, i.e. it has minimum and maximum and contains every root in between. Moreover, both

$$\{\min \Phi_{S,\beta}\} \cup (\Pi \setminus S) \quad \text{and} \quad \{-\max \Phi_{S,\beta}\} \cup (\Pi \setminus S)$$

are simple systems of $\Phi_{S,\mathbb{Z},\beta}$; in particular, $\min \Phi_{S,\beta}$ and $\max \Phi_{S,\beta}$ have the same length.
3. Let $\Psi = \Phi_{S,\mathbb{Z},\beta}$ and choose $\{\gamma\} \cup (\Pi \setminus S)$ as a set of simple roots of $\Psi$, with either $\gamma = \min \Phi_{S,\beta}$ or $\gamma = -\max \Phi_{S,\beta}$. Then either

$$\Phi_{S,\beta} = \Psi_{\{\gamma\},\gamma} \quad \text{or} \quad \Phi_{S,\beta} = -\Psi_{\{\gamma\},\gamma},$$

respectively.

**Proof.**

1. The assertion is clear.
2. Let $U = \text{Span}(\{(\Pi \setminus S) \cup \{\beta + \delta\})$ and $\widehat{\Phi}_U = \widehat{\Phi} \cap U$. Then $\Phi_{S,\beta} + \delta \subseteq \widehat{\Phi}_U$.

Since $\text{Supp}(\beta) \cap S$ is nonempty, $\mathbb{R}\delta \cap U = \{0\}$, hence the restriction of $(\cdot, \cdot)$ to $U$ is positive definite and $\widehat{\Phi}_U$ is a finite dimensional crystallographic root system, naturally isomorphic to $\Phi_{S,\mathbb{Z},\beta}$ through the natural projection $E + \mathbb{R}\delta \to E$. We can choose $\widehat{\Phi}_U \cap \mathbb{R}^+$ as a positive system for $\widehat{\Phi}_U$. The simple system determined by this choice is $(\Pi \setminus S) \cup \{\gamma + \delta\}$, for a certain $\gamma \in \Phi_{S,\beta}$ with $\gamma \leq \beta$. Indeed, since all roots in $\Phi_{S,\beta} + \delta$ must be non-negative linear combination of the basis, $\gamma = \min \Phi_{S,\beta}$.
If we make the above construction with \(-\beta + \delta\) in place of \(\beta + \delta\), we obtain, as a simple system for \(\hat{\Phi}_U\), \((\Pi \setminus S) \cup \{-\gamma + \delta\}\), with \(\gamma = \max \Phi_{S,\beta}\).

By definition of \(\Phi_{S,\beta}\), every root between \(\min \Phi_{S,\beta}\) and \(\max \Phi_{S,\beta}\) in the root poset belongs to \(\Phi_{S,\beta}\).

Finally, it is clear that, if \((\Pi \setminus S) \cup \{\gamma + \delta\}\) is a basis of \(\hat{\Phi}_U\), then \((\Pi \setminus S) \cup \{\gamma\}\) is a basis of \(\Phi_{S,\beta}\).

\[ \begin{array}{c} 3 \end{array} \] The assertion follows by \(1\) and \(2\). \(\square\)

**Remark 4.3.** Proposition 4.2 implies a representation theoretical result due to Kostant, which is presented in [9, Section 2]. We briefly recall this result. Let \(g\) be a simple Lie algebra with root system \(\Phi\), and \(p_{\Pi \setminus S}\) be the standard parabolic subalgebra of \(g\) corresponding to \(\Pi \setminus S\). Then

\[ p_{\Pi \setminus S} = g_{\Pi \setminus S} \oplus h_{(\Pi \setminus S)^+} \oplus n_{\Phi, (\Pi \setminus S)}, \]

where \(g_{\Pi \setminus S}\) is the semisimple subalgebra generated by the root spaces \(g_\alpha\) with \(\alpha \in \Pi \setminus S\), \(h_{(\Pi \setminus S)^+}\) is defined as \(\{h \in h : \alpha(h) = 0, \forall \alpha \in \Pi \setminus S\}\) and is the center of the reductive subalgebra \(g_{\Pi \setminus S} \oplus h_{(\Pi \setminus S)^+}\), and \(n_{\Phi, (\Pi \setminus S)} = \bigoplus_{\gamma \in \Phi^+ \cap (\Pi \setminus S)} g_\gamma\). The adjoint representation of \(g\) restricts to a representation of \(g_{\Pi \setminus S} \oplus h_{(\Pi \setminus S)^+}\) on the nilpotent summand \(n_{\Phi, (\Pi \setminus S)}\). For any \(\beta \in \Phi^+\), if \(\sum_{\alpha \in \Pi} c_\alpha \alpha\), the action of \(h_{(\Pi \setminus S)^+}\) on \(g_\beta\) depends only on \(\sum_{\alpha \in S} c_\alpha \alpha\). Hence, \(\bigoplus_{\gamma \in \Phi_{S,\beta}} g_\gamma\), which we denote by \(n_{\Phi_{S,\beta}}\), is a \(h_{(\Pi \setminus S)^+}\)-isotypic component. Moreover, \(n_{\Phi_{S,\beta}}\) is stable under the action of \(g_{\Pi \setminus S}\). Kostant’s results is that, whenever \(\text{Supp}(\beta) \cap S \neq \emptyset\), the subrepresentation of \(g_{\Pi \setminus S}\) on \(n_{\Phi_{S,\beta}}\) is irreducible. The whole set of weights of this irreducible representation is \(\Phi_{S,\beta}\), hence its unique highest weight is the maximum of \(\Phi_{S,\beta}\). We show that the set of weights \(\Phi_{S,\beta}\) is partitioned into one or two \(W(\Pi \setminus S)\)-orbits according to Propositions 4.11, 4.12, and 4.14 (see also Remark 4.9).

We also note that, by a general result of representation theory (see [10, p. 204]), the set of weights \(\Phi_{S,\beta}\) is contained in the convex hull of the orbit of the highest weight under the action of the Weyl group of \(g_{\Pi \setminus S}\), hence

\[ \text{Conv}(\Phi_{S,\beta}) = \text{Conv}(W(\Pi \setminus S) \cdot \max \Phi_{S,\beta}). \]

**Remark 4.4.** Let \(\alpha \in \Pi\) and \(S = \{\alpha\}\). Then \(\Phi_{S,\Pi \setminus S} = \Phi\) and \(\{\min \Phi_{S,\alpha}\} \cup (\Pi \setminus S) = \Pi\). By Proposition 4.2 also \(\{- \max \Phi_{S,\alpha}\} \cup (\Pi \setminus S)\) is a simple system of \(\Phi\). When \(m_\alpha = 1\), \(\max \Phi_{S,\alpha} = \theta\) and, since the affine simple root \(\alpha_0\) satisfies \((\alpha_0, x) = (-\theta, x)\) for all \(x \in \mathbb{E}\), we reobtain the following well-known fact:

the coefficient \(m_\alpha\) is equal to 1 if and only if the Dynkin diagram of \(\Pi\) is equal to the subdiagram \(\Pi \cup \{\alpha_0\} \setminus \{\alpha\}\) of the extended Dynkin diagram of \(\Phi\).

Recall that \(\tilde{\omega}_\alpha\), for each \(\alpha \in \Pi\), denotes the fundamental co-weight corresponding to \(\alpha\), i.e. the element of \(\mathbb{E}\) defined by the conditions \((\tilde{\omega}_\alpha, \alpha) = 1\) and \((\tilde{\omega}_\alpha, \alpha') = 0\) for
Let \( \alpha' \in \Pi \setminus \{\alpha\} \). Let

\[
A = \text{Span}(\Pi \setminus S), \quad \text{so that} \quad A^\perp = \text{Span}(\tilde{\alpha}_\alpha : \alpha \in S),
\]

and denote by \( \pi_A \) and \( \pi_A^\perp \) the orthogonal projections from \( E \) onto \( A \) and \( A^\perp \), respectively.

**Lemma 4.5.** For all \( \gamma \in \Phi_{S,\beta} \) and \( w \in W(\Pi \setminus S) \),

1. \( \pi_A(\gamma) = \pi_A^\perp(\beta) \), hence \( \pi_A(\gamma) = \gamma - \pi_A^\perp(\beta) \),
2. \( \pi_A \circ w(\gamma) = w \circ \pi_A(\gamma) \).

**Proof.** The equalities in (1) follow directly from the definitions. Equality (2) follows easily from (1), since \( W(\Pi \setminus S) \) stabilizes \( A \) and fixes \( A^\perp \) pointwise. \( \square \)

Given a subset \( T \) of \( \Pi \), we say that \( v \in E \) is \( T\text{-dominant} \) provided that \( (v, \alpha) \geq 0 \) for all \( \alpha \in T \). We note that, for all \( v \in E \), \( v \) is \( (\Pi \setminus S)\text{-dominant} \) if and only if \( \pi_A(v) \) belongs to the fundamental chamber of the possibly reducible root system \( \Phi(\Pi \setminus S) \).

**Proposition 4.6.** Let \( S \subseteq \Pi \) and \( \beta \in \Phi^+ \), with \( \text{Supp}(\beta) \cap S \neq \emptyset \). Then

1. \( \max \Phi_{S,\beta} \) and \( -\min \Phi_{S,\beta} \) are both \( (\Pi \setminus S)\text{-dominant} \);
2. if \( \Phi_{S,\beta} \) contains some long roots, then \( \max \Phi_{S,\beta} \) is the unique long \( (\Pi \setminus S)\text{-dominant} \) root; otherwise, \( \max \Phi_{S,\beta} \) is the unique (short) \( (\Pi \setminus S)\text{-dominant} \) root.

**Proof.** Assertion (1) is implicit in Proposition 4.2 (2), and may also be seen directly since, for each \( \gamma \in \Phi \), if \( (\gamma, \alpha) < 0 \) for some \( \alpha \in \Pi \setminus S \), then \( \gamma + \alpha \in \Phi \).

Let us prove (2). We first show that, for each \( \gamma \in \Phi_{S,\beta} \), there exists a \( (\Pi \setminus S)\text{-dominant} \) root of the same length as \( \gamma \). The parabolic subgroup \( W(\Pi \setminus S) \) acts on the set of long (short) roots in \( \Phi_{S,\beta} \). By Lemma 4.3, the orthogonal projection \( \pi_A : E \to A \) restricts to a bijection from the \( W(\Pi \setminus S) \)-orbit of \( \gamma \) to the \( W(\Pi \setminus S) \)-orbit of \( \pi_A(\gamma) \). Since \( W(\Pi \setminus S) \cdot \pi_A(\gamma) \) contains a unique element belonging to the fundamental chamber of \( \Phi(\Pi \setminus S) \), the orbit \( W(\Pi \setminus S) \cdot \gamma \) contains a unique \( (\Pi \setminus S)\text{-dominant} \) root.

Next, we prove that, if \( \gamma \) is a \( (\Pi \setminus S)\text{-dominant} \) root in \( \Phi_{S,\beta} \) other than \( \max \Phi_{S,\beta} \), then \( \max \Phi_{S,\beta} \) is a long root and \( \gamma \) is short. Since \( \gamma < \max \Phi_{S,\beta} \), we have \( \max \Phi_{S,\beta} - \gamma \in \mathbb{N}(\Pi \setminus S) \), and hence \( (\max \Phi_{S,\beta}, \gamma^\vee) \geq (\gamma, \gamma^\vee) = 2 \), since \( (\alpha, \gamma^\vee) \geq 0 \) for all \( \alpha \in \Pi \setminus S \). This implies that \( \max \Phi_{S,\beta} \) is long and \( \gamma \) is short.

Hence, if \( \Phi_{S,\beta} \) contains either only long roots or both long and short roots, then \( \max \Phi_{S,\beta} \) is the unique long \( (\Pi \setminus S)\text{-dominant} \) root. Moreover, if \( \max \Phi_{S,\beta} \) is short, then it is the unique short \( (\Pi \setminus S)\text{-dominant} \) root in \( \Phi_{S,\beta} \), and there is no long root in \( \Phi_{S,\beta} \). \( \square \)

As a consequence, we obtain an algebraic combinatorial short proof of Oshima’s Lemma (see [10]), which is case-free and independent of the representation theory of semisimple complex Lie algebras (see [5]).

**Corollary 4.7** (Oshima’s Lemma). Let \( S \subseteq \Pi \) and \( \beta \in \Phi^+ \), with \( \text{Supp}(\beta) \cap S \neq \emptyset \). Then
(1) if \( \Phi_{S,\beta} \) contains a long (short) root, then it contains exactly one \((\Pi \setminus S)\)-dominant long (short) root;

(2) \( W(\Pi \setminus S) \) is transitive on the set of long (short) roots in \( \Phi_{S,\beta} \).

Proof. Let us prove (1). By Proposition 4.6(2), if \( \Phi_{S,\beta} \) contains some long roots, then it contains exactly one \((\Pi \setminus S)\)-dominant long root.

It remains to show that, if \( \Phi_{S,\beta} \) contains some short roots, then it contains exactly one \((\Pi \setminus S)\)-dominant short root. We prove it by duality.

For a subset \( T \) of \( \Phi \), let \( T^\vee = \{ \alpha^\vee : \alpha \in T \} \). Let \( \beta_s \) be any short root in \( \Phi_{S,\beta} \). Then \( \gamma \) is a short root in \( \Phi_{S,\beta} \) if and only if \( \gamma^\vee \) is a long root in \( \Phi_{S^\vee,\beta^\vee} \), and \( \gamma \) is \((\Pi \setminus S)\)-dominant root if and only if \( \gamma^\vee \) is \((\Pi^\vee \setminus S^\vee)\)-dominant. Hence, we are done.

Part (2) is, in fact, equivalent to Part (1) since, as already noticed, Lemma 4.5 implies that the orthogonal projection \( \pi_A : E \rightarrow A \) restricts to a bijection from the \( W(\Pi \setminus S) \)-orbit of each \( \gamma \in \Phi_{S,\beta} \) to the \( W(\Pi \setminus S) \)-orbit of \( \pi_A(\gamma) \).

\( \square \)

Remark 4.8. If \( \Phi_{S,\beta} \) contains some short roots, then the unique short \((\Pi \setminus S)\)-dominant root is also the unique maximal short root in \( \Phi_{S,\beta} \). Indeed, if \( \gamma \in \Phi_{S,\beta} \) is not \((\Pi \setminus S)\)-dominant, for \( \alpha \in \Pi \setminus S \) with \((\gamma, \alpha) < 0\), we have that \( s_\alpha(\gamma) > \gamma \), so \( \gamma \) cannot be maximal among the short roots.

Remark 4.9. In Remark 4.3, we have seen that \( \max \Phi_{S,\beta} \) is the highest weight of an irreducible representation of the semisimple Lie algebra associated to \( \Phi(\Pi \setminus S) \). Corollary 4.7 shows that the action of \( W(\Pi \setminus S) \) on the weights of this representation has one orbit for each length of the roots \( \Phi_{S,\beta} \), hence at most two orbits. In particular, in the simply laced case there is a single orbit action. By Proposition 4.6 there is a single orbit action also when \( \max \Phi_{S,\beta} \) is short.

In the rest of the section, we study when \( \Phi_{S,\beta} \) contains roots of one or two lengths.

Recall that the lacing number of \( \Phi \) is the maximal number of edges connecting two vertices of the Dynkin diagram of \( \Phi \), i.e. \( \max\{ (\alpha, \alpha)/ (\beta, \beta) : \alpha, \beta \in \Phi \} \). As usual, in simply laced root system, we consider all roots as long. The following result is an exercise of Bourbaki’s [1]. We include a proof for completeness.

Lemma 4.10. [1] Exercise 20, Chapter VI, §1 Let \( \gamma = \sum_{\alpha \in \Pi} c_{\alpha} \alpha \in \Phi \). Then, \( \gamma \) is long if and only if, for all short \( \alpha \) in \( \Pi \), the coefficient \( c_{\alpha} \) is a multiple of the lacing number of \( \Phi \).

Proof. Let \( \gamma \) be a long root. The coroot \( \gamma^\vee \) satisfies

\[
\gamma^\vee = \frac{2\gamma}{(\gamma, \gamma)} = \sum_{\alpha \in \Pi_\ell} \frac{2}{(\gamma, \gamma)} c_{\alpha} \alpha + \sum_{\alpha \in \Pi_s} \frac{2}{(\gamma, \gamma)} c_{\alpha} \alpha = \sum_{\alpha \in \Pi_\ell} c_{\alpha} \alpha^\vee + \sum_{\alpha \in \Pi_s} (\alpha, \alpha \gamma, \gamma) c_{\alpha} \alpha^\vee ,
\]

where \( \Pi_\ell \) and \( \Pi_s \) denote, respectively, the set of simple long and short roots. Now, for all short root \( \alpha \), the ratio \((\alpha, \alpha)/(\gamma, \gamma)\) coincides with the inverse of the lacing number of
Φ; since the coefficients of $\gamma^\vee$ must be integers, $c_\alpha$ is a multiple of the lacing number, for each short $\alpha \in \Pi$.

Conversely, let $\gamma = \sum_{\alpha \in \Pi} d_\alpha \alpha$ be a root such that $d_\alpha$ is a multiple of the lacing number of $\Phi$, for all short $\alpha \in \Pi$. We prove that $\gamma$ is long by contradiction. Suppose that it is short. The coroot $\gamma^\vee$ satisfies

$$\gamma^\vee = \frac{2\gamma}{(\gamma, \gamma)} = \sum_{\alpha \in \Pi_t} \frac{2}{(\gamma, \gamma)} d_\alpha \alpha + \sum_{\alpha \in \Pi_s} \frac{2}{(\gamma, \gamma)} d_\alpha \alpha = \sum_{\alpha \in \Pi_t} \frac{(\alpha, \alpha)}{(\gamma, \gamma)} d_\alpha \alpha^\vee + \sum_{\alpha \in \Pi_s} d_\alpha \alpha^\vee.$$ 

Since $(\alpha, \alpha)/(\gamma, \gamma)$ coincides with the lacing number of $\Phi$ for each long root $\alpha$, all coefficients of $\gamma^\vee$ are multiples of the lacing number. This is a contradiction since a (co)root cannot be a multiple of an element in the (co)root lattice, since any (co)root can be a basis element of the (co)root lattice.

A direct consequence of Lemma 4.10 is the following result.

**Proposition 4.11.** If all short simple roots belong to $S$, then all roots in $\Phi_{S,\beta}$ share the same length. □

Lemma 4.10, together with Proposition 4.2(3), yields a further proof of the fact that, if $\min \Phi_{S,\beta}$ is short, then all roots in $\Phi_{S,\beta}$ are short (see Proposition 4.6(2). Indeed, by Proposition 4.2(3) $\Phi_{S,\beta}$ consists of all the roots in $\Phi_{S,\beta}$ that, with respect to the basis $(\Pi \setminus S) \cup \{\min \Phi_{S,\beta}\}$, have the $\min \Phi_{S,\beta}$-coordinate equal to 1.

Similarly, we may obtain a criterion for $\Phi_{S,\beta}$ to contain short roots in case $\min \Phi_{S,\beta}$ is long.

**Proposition 4.12.** Let $S \cap \text{Supp}(\beta) \neq \emptyset$.

1. The set $\Phi_{S,\beta}$ contains only short roots if and only if $\min \Phi_{S,\beta}$ is short.
2. The set $\Phi_{S,\beta}$ contains only long roots if and only if $\min \Phi_{S,\beta}$ is long and the connected component of $\min \Phi_{S,\beta}$ in the Dynkin diagram of $(\Pi \setminus S) \cup \{\min \Phi_{S,\beta}\}$ contains only long simple roots.
3. The set $\Phi_{S,\beta}$ contains both long and short roots if and only if $\min \Phi_{S,\beta}$ is long and the connected component of $\min \Phi_{S,\beta}$ in the Dynkin diagram of $(\Pi \setminus S) \cup \{\min \Phi_{S,\beta}\}$ contains some short simple root.

**Proof.** Part (1) has already been proved.

Parts (2) and (3) follow from the following argument.

The roots in $\Phi_{S,\beta}$ belong to the irreducible component of $\min \Phi_{S,\beta}$ in $\Phi_{S,\beta}$. Hence, if all simple roots in such connected component are long, then all roots in $\Phi_{S,\beta}$ are long. On the contrary, if such connected component contains some simple short root, then the sum of all simple roots in such connected component is a root (it is a general well-known fact that the sum of the simple roots of an irreducible root system is a root), belongs to $\Phi_{S,\beta}$, and moreover, by Lemma 4.10 is a short root. □
Lemma 4.10 provides a necessary condition on $\beta$ in order that $\Phi_{S,\beta}$ contain some long roots, namely, that for any short simple root $\alpha \in S$, the coefficient $c_{\alpha}(\beta)$ should be a multiple of the lacing number. In Proposition 4.14 we see that this is also a sufficient condition, provided $S \cap \text{Supp}(\beta) \neq \emptyset$. The proofs of Proposition 4.14 and its Lemma 4.13 require the classification of the irreducible root systems.

We note that in case $S \cap \text{Supp}(\beta) = \emptyset$, the above condition on $\beta$ is trivially satisfied, but $\Phi_{S,\beta}$ is the root subsystem generated by $\Pi \setminus S$, and hence, if $S$ contains all long simple roots, then $\Phi_{S,\beta}$ only contains short roots.

Lemma 4.13. If $\mu \in \Phi^+$ is a short root and satisfies $\text{Supp}(\mu) = \Pi$, then there exists a short simple root $\gamma$ such that $\mu + \gamma \in \Phi^+$ and $c_{\gamma}(\mu)$ is not a multiple of the lacing number.

Proof. We prove the assertion by a case-by-case analysis. For each type, we list all short roots $\mu$ satisfying $\text{Supp}(\mu) = \Pi$ and, for each such $\mu$, we provide a short simple root $\gamma$ with the required properties. The simple roots are denoted according to the Tables in [1].

If $\Phi$ is of type $B_l$, then $\mu = \sum_{i=1}^{l} \alpha_i$ and $\gamma = \alpha_l$.

If $\Phi$ is of type $C_l$, then $\mu = \sum_{i=1}^{j-1} \alpha_i + 2 \sum_{i=j}^{l-1} \alpha_i + \alpha_l$, with $1 < j \leq l$, and $\gamma = \alpha_{j-1}$.

If $\Phi$ is of type $F_4$, then either $\mu = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ and $\gamma = \alpha_3$, or $\mu = \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4$ and $\gamma = \alpha_4$, or $\mu = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4$ and $\gamma = \alpha_4$, or $\mu = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4$ and $\gamma = \alpha_3$.

If $\Phi$ is of type $G_2$, then either $\mu = \alpha_1 + \alpha_2$, or $\mu = 2\alpha_1 + \alpha_2$, and, in both cases, $\gamma = \alpha_1$.

Proposition 4.14. Let $S \cap \text{Supp}(\beta) \neq \emptyset$. The set $\Phi_{S,\beta}$ contains long roots if and only if, for every short root $\alpha$ in $S$, the coefficient $c_{\alpha}(\beta)$ is a multiple of the lacing number.

Proof. Let $\gamma = \sum_{\alpha \in \Pi} c_{\alpha}(\gamma)\alpha$ be a long root in $\Phi_{S,\beta}$. By Lemma 4.10 the coefficient $c_{\alpha}(\gamma)$ is a multiple of the lacing number for every short root $\alpha$ in $S$ and so, in particular, for every short root $\alpha$ in $S$. By the definition of $\Phi_{S,\beta}$, we have $c_{\alpha}(\beta) = c_{\alpha}(\gamma)$, for all $\alpha \in S$.

Conversely, suppose that $c_{\alpha}(\beta)$ is a multiple of the lacing number for each short root $\alpha$ in $S$. If $\beta$ is long, then we are done. So we suppose that $\Phi$ is not simply laced and $\beta$ is short. By symmetry, we may also suppose $\beta \in \Phi^+$. Let $\mu_*$ be a maximal root in the set $\{\mu \in \Phi_{S,\beta} : \mu$ is short and $\text{Supp}(\mu) \subseteq \text{Supp}(\beta)\}$, which is not empty since it contains $\beta$. It is well known that the support of a root is connected and hence $\Phi(\text{Supp}(\beta))$ is a standard parabolic irreducible subsystem of $\Phi$. Thus, if $\text{Supp}(\beta)$ contains only short roots, then, by the classification of the irreducible root systems, $\text{Supp}(\beta)$ is a simple root system of type $A$. This is impossible since $c_{\alpha}(\beta) > 1$ for $\alpha \in S \cap \text{Supp}(\beta)$. Hence, $\text{Supp}(\beta)$ is a simple system of an irreducible not simply laced root system. By applying Lemma 4.13 with $\mu_*$ and $\text{Supp}(\beta)$ in place of $\mu$ and $\Pi$, we find a short simple root $\gamma \in \text{Supp}(\beta)$ such that $\mu_* + \gamma$ is short and $c_{\gamma}(\mu_*)$ is not a multiple of the lacing number. In particular,
\( \gamma \not\in S \) and hence \( \mu_* + \gamma \) still belongs to \( \Phi_{S, \beta} \). By the maximality of \( \mu_* \), the root \( \mu_* + \gamma \) is long. \( \square \)

5. Root systems and affine hyperplanes

In this section, we analyse in more details the case of codimension 1 of the intersections studied in Section 4.

Let \( \Phi \) be an irreducible root system (see Remark 4.1). Throughout this section, we fix a simple root \( \alpha \in \Pi \). According to its coordinate \( c_\alpha(\beta) \), each root \( \beta = \sum_{\gamma \in \Pi} c_\gamma(\beta) \gamma \) lies in one of the \( 2m_\alpha + 1 \) parallel affine hyperplanes \( \{ x \in E : (x, q_\omega \alpha) = c_\alpha \} \), with \( -m_\alpha \leq c_\alpha \leq m_\alpha \).

To lighten the notation, we set:

- \( \Phi_{\alpha, k} = \Phi \cap \{ x \in E : (x, q_\omega \alpha) = k \} \)
- \( \Phi_{\alpha, Zk} = \Phi \cap \{ x \in E : k \text{ divides } (x, q_\omega \alpha) \} \)
- \( \Phi_\alpha = \Phi \langle \Pi \setminus \{ \alpha \} \rangle \)
- \( W_\alpha = W \langle \Pi \setminus \{ \alpha \} \rangle \)
- \( \mu_{\alpha, k} = \max \Phi_{\alpha, k} \), for \( k \in [1, m_\alpha] \) (see Proposition 4.2)

In the notation of Section 4, \( \Phi_{\alpha, k} = \Phi_{(\alpha), \beta} \) and \( \Phi_{\alpha, Zk} = \Phi_{(\alpha), Z\beta} \), where \( \beta = \sum_{\gamma \in \Pi} c_\gamma(\beta) \gamma \) is any root with \( c_\alpha(\beta) = k \). Moreover, \( \Phi_\alpha = \Phi_{a,0} \), \( \Phi_{a, Zk} \supseteq \Phi_{a, Zk'} \) whenever \( k \) divides \( k' \), and \( \Phi_{a, Zk} \cap \Phi_{a, Zl} = \Phi_{a, Zl} \) where \( l \) is the least common multiple of \( k \) and \( h \). For \( k = 1 \), we have \( \Phi_{a, Zk} = \Phi \).

Notice that, by Proposition 4.2(2), the \( \Phi_{\alpha, k} \), with \( k \neq 0 \), provide a partition of \( \Phi \setminus \Phi_\alpha \) into intervals.

By Proposition 4.6 and Corollary 4.7, we have the two following possibilities:

- if all roots in \( \Phi_{\alpha, k} \) have the same length, then
  \[ \Phi_{\alpha, k} = W_\alpha \cdot \mu_{\alpha, k}, \]

- if \( \Phi_{\alpha, k} \) contains both short and long roots, then \( \mu_{\alpha, k} \) is long, the set \( \{ \beta \in \Phi_{\alpha, k} : \beta \text{ is short} \} \) has a maximum, denoted \( \sigma_{\alpha, k} \), and
  \[ \Phi_{\alpha, k} = W_\alpha \cdot \mu_{\alpha, k} \cup W_\alpha \cdot \sigma_{\alpha, k}. \]

By the last formula in Remark 4.3, however, in both cases

\[ \text{Conv}(\Phi_{\alpha, k}) = \text{Conv}(W_\alpha \cdot \mu_{\alpha, k}). \]

Recall that the subset \( \Phi_{a, Zk} \), with \( k \leq m_\alpha \), is a root subsystem of \( \Phi \), by Proposition 4.2(1). For instance, if \( \Phi \) has type \( G_2 \), then the root subsystem \( \Psi_{a, k} \) has type \( A_1, G_2, A_1 \times A_1 \), or \( A_2 \) depending on whether \( k \) has the value 0, 1, 2, or 3.
Let \( A = \text{Span}(\Pi \setminus \{\alpha\}) \). For \( x \in \mathbb{E} \) and \( X \subseteq \mathbb{E} \), we let \( \bar{x} = \pi_A(x) \) and \( \overline{X} = \pi_A(X) \). For all \( x \in \mathbb{E} \), we have \( \bar{x} = x - \frac{k}{(\omega_\alpha, \omega_\alpha)} \omega_\alpha \). In particular,

\[
\bar{x} = x - \frac{k}{(\omega_\alpha, \omega_\alpha)} \omega_\alpha,
\]

for all \( \gamma \in \Phi_{\alpha,k} \).

Let \( \gamma \in \Phi_{\alpha,k} \) be a root of the same length as \( \mu_{\alpha,k} \). The transitivity of \( W_\alpha \) on the long roots in \( \Phi_{\alpha,k} \) and \( \gamma = \mu_{\alpha,k} \) imply (5.0.1)

\[
\text{Conv}(\Phi_{\alpha,k}) = \text{Conv}(\overline{\Phi_{\alpha,k}}) = \text{Conv}(W_\alpha \cdot \bar{x}).
\]

Thus, since \( \bar{x} \) is a weight of \( \Phi_\alpha \), we have that \( \text{Conv}(\Phi_{\alpha,k}) \) is the weight polytope of \( \bar{x} \), relatively to \( \Phi_\alpha \) in the space \( A \). Clearly, \( \bar{x} = 0 \) if and only if \( \gamma \in A \perp \).

**Proposition 5.1.** Fix \( \alpha \in \Pi \) and \( k \in [1, m_\alpha] \). Let \( \gamma \in \Phi_{\alpha,k} \) be a root of the same length as \( \mu_{\alpha,k} \). Enumerate the irreducible components \( \Psi_1, \ldots, \Psi_t \) of \( \Phi_\alpha \) so that there exists \( h \in [1, t] \) satisfying \( \gamma \not\perp \Psi_i \) for \( 1 \leq i \leq h \) and \( \gamma \perp \Psi_i \) for \( h < i \leq t \). Then \( \overline{\Phi_{\alpha,k}} \subseteq \text{Span}(\Psi_1 \cup \cdots \cup \Psi_h) \) and

\[
\dim(\text{Conv}(\Phi_{\alpha,k})) = \dim(\text{Conv}(\overline{\Phi_{\alpha,k}})) = \text{rk}(\Psi_1 \cup \cdots \cup \Psi_h).
\]

**Proof.** The first equality follows from the fact that \( \text{Conv}(\Phi_{\alpha,k}) \) belongs to the affine hyperplane \( \overline{h} + \frac{k}{(\omega_\alpha, \omega_\alpha)} \omega_\alpha \), and the restriction of \( \pi_A \) to this affine hyperplane is the translation by \(- \frac{k}{(\omega_\alpha, \omega_\alpha)} \omega_\alpha \).

Since \( \gamma \perp \Psi_i \) for \( h < i \leq t \), we have \( W_\alpha \cdot \gamma \subseteq \text{Span}(\Psi_1 \cup \cdots \cup \Psi_h) + \gamma \). By (5.0.2), in order to prove the second equality, it suffices to prove that, for all \( \beta \in \Pi \cap (\Psi_1 \cup \cdots \cup \Psi_h) \), there exist \( \gamma_1, \gamma_2 \in W_\alpha \cdot \gamma \) such that \( \gamma_1 - \gamma_2 = c \beta \), for some \( c \neq 0 \).

Let \( \beta \in \Pi \cap \Psi_i \), for some \( i \in [1, h] \). Since \( \gamma \not\perp \Psi_i \), there exist \( l \) (possibly \( l = 1 \)) and a path \( (\beta_1, \ldots, \beta_l) \) in the Dynkin diagram of \( \Psi_i \) such that \( \beta_1 = \beta \), \( \beta_s \perp \gamma \) for \( s \in [1, l - 1] \), and \( \beta_l \not\perp \gamma \). Hence, for all \( j \in [1, l - 1] \),

\[
s_{\beta_j} \cdots s_{\beta_l}(\gamma) = c_j \beta_j + \cdots + c_l \beta_l + \gamma
\]

for certain nonzero real \( c_j, \ldots, c_l \). We can take the elements \( s_{\beta_1} \cdots s_{\beta_l}(\gamma) \) and \( s_{\beta_2} \cdots s_{\beta_l}(\gamma) \) as \( \gamma_1 \) and \( \gamma_2 \), respectively. \( \square \)

We state the following lemma for later reference.

**Lemma 5.2.** Let \( \alpha \in \Pi \). The roots \( \alpha \) and \( \mu_{\alpha,1} \) have the same length (the maximal length).

**Proof.** The equality of the lengths follows from Proposition 4.2, since \( \alpha = \min \Phi_{\alpha,1} \). The length is maximal by Proposition 4.6. \( \square \)

As a direct consequence of Proposition 5.1 and Lemma 5.2, we obtain the following result.
Corollary 5.3. Let $\alpha \in \Pi$ and let $\Psi$ be any irreducible component of $\Phi_\alpha$. Let $k \in [1, m_\alpha]$ and denote by $\tilde{\Phi}_{\alpha,k}$ the set of roots of $\Phi_{\alpha,k}$ of the same length as $\mu_{\alpha,k}$.

1. For every $\gamma, \gamma' \in \tilde{\Phi}_{\alpha,k}$, we have $\gamma \perp \Psi$ if and only if $\gamma' \perp \Psi$. 
2. For every $\gamma \in \tilde{\Phi}_{\alpha,1}$, we have $\gamma \not\perp \Psi$.

In particular, $\dim(\text{Conv}(\Phi_{\alpha,1})) = \text{rk}(\Phi) - 1$.

Proof. (1) The proof is straightforward from Proposition 5.1.

(2) Being $\Phi$ irreducible, $\alpha \not\perp \Psi$. Since $\alpha$ belongs to $\Phi_{\alpha,1}$ and has the same length as $\mu_{\alpha,1}$ by Lemma 5.2, $\gamma \not\perp \Psi$ follows by (1). The assertion on the dimension follows by Proposition 5.1.

6. Projections

Let $\Phi$ be an irreducible root system. Fix an arbitrary simple root $\alpha \in \Pi$. In this section, we study the inclusion of the projected polytopes $\text{Conv}(\Phi_{\alpha,k}) = \text{Conv}(\Phi_{\alpha,k})$ in the root polytope $P_{\Phi_\alpha} = \text{Conv}(\Phi_\alpha)$ and, more generally, the scale factors $r$ for which $\text{Conv}(\Phi_{\alpha,k}) \subseteq r \cdot P_{\Phi_\alpha}$.

Lemma 6.1. Let $k \in [1, m_\alpha]$ and $r \in \mathbb{R}$.

1. The inclusion $\text{Conv}(\Phi_{\alpha,k}) \subseteq r \cdot P_{\Phi_\alpha}$ holds if and only if $\mu \in r \cdot P_{\Phi_\alpha}$ for some $\mu \in \Phi_{\alpha,k}$ of maximal length in $\Phi_{\alpha,k}$.
2. We have $\Phi_{\alpha,k} \subseteq r \cdot P_{\Phi_\alpha}$ if and only if $\Phi_{\alpha,-k} \subseteq r \cdot P_{\Phi_\alpha}$.

Proof. (1) By Remark 4.3, Proposition 4.6 and Corollary 4.7, the set of vertices of $\text{Conv}(\Phi_{\alpha,k})$ is the set of all elements of maximal length in $\Phi_{\alpha,k}$, and this set is a single $W_\alpha$-orbit. Since $W_\alpha$ and the projection $\pi_k$ commute by Lemma 4.5 and since $\Phi_\alpha$ is $W_\alpha$-stable, if a single vertex of $\text{Conv}(\Phi_{\alpha,k})$ belongs to $r \cdot P_{\Phi_\alpha}$, then all vertices do. The claim follows.

(2) We have $\Phi_{\alpha,-k} = -\Phi_{\alpha,k}$, hence $\Phi_{\alpha,-k} = -\Phi_{\alpha,k}$, so the claim holds since $-\Phi_\alpha = \Phi_\alpha$.

By Lemma 6.1 (1), we obtain a simple criterion for checking the inclusion of $\text{Conv}(\Phi_{\alpha,1})$ in the dilations of $P_{\Phi_\alpha}$.

Proposition 6.2. Let $r \in \mathbb{R}$. We have $\text{Conv}(\Phi_{\alpha,1}) \subseteq r \cdot P_{\Phi_\alpha}$ if and only if $\mu \in r \cdot P_{\Phi_\alpha}$.

Proof. By Lemma 5.2, the root $\alpha$ has maximal length in $\Phi_{a,1}$. The assertion follows by Lemma 6.1 (1).

Remark 6.3. Let $1 < k \leq m_\alpha$, $\gamma = \min \Phi_{\alpha,k}$, and $\Psi = \Phi_{\alpha,\mathbb{Z}k}$. By Proposition 4.2 and (3), the roots $\{\gamma\} \cup \Pi \setminus \{\alpha\}$ form a simple system for the root subsystem $\Psi$ and $\Phi_{\alpha,0} = \Psi_{\gamma,0}$, while $\Phi_{\alpha,k} = \Psi_{\gamma,1}$.
The root system $\Psi$ may be reducible; however
\[ r\Psi_{\gamma,1} \subseteq \text{Conv}(\Psi_{\gamma,0}) = P_{\Psi_{\gamma,0}} \] if and only if
\[ r\Psi_{\gamma,1}^* \subseteq \text{Conv}(\Psi_{\gamma,0}^*) = P_{\Psi_{\gamma,0}^*}, \]
where $\Psi^*$ denotes the component of $\gamma$ in $\Psi$.

Recall that, for each $\epsilon \in \Pi$, we denote by $\omega_{\epsilon}$ the fundamental weight corresponding to $\epsilon$, i.e. the element of $E$ defined by the conditions $(\omega_{\epsilon}, \epsilon^\vee) = 1$ and $(\omega_{\alpha}, \epsilon'^\vee) = 0$ for $\epsilon' \in \Pi \setminus \{\epsilon\}$. Note that, for $\epsilon \in \Pi \setminus \{\alpha\}$, the projection $\overline{\omega}_{\epsilon}$ is the fundamental weight corresponding to $\epsilon$ in the subsystem $\Phi_{\alpha}$. We let $N[\alpha]$ denote the set $\{\epsilon \in \Pi \setminus \{\alpha\} : \epsilon \not\perp \alpha\}$ of the neighbours of $\alpha$ in the Dynkin diagram of $\Phi$. By the following result, $\overline{\alpha}$ is a negative linear combination of the fundamental weights associated with $N[\alpha]$ in the standard parabolic subsystem $\Phi_{\alpha}$.

**Proposition 6.4.** The projection $\overline{\alpha}$ satisfies
\[ \overline{\alpha} = \sum_{\epsilon \in N[\alpha]} (\alpha, \epsilon^\vee) \overline{\omega}_\epsilon. \]

**Proof.** The set $\{\overline{\omega}_\epsilon : \epsilon \in \Pi \setminus \{\alpha\}\}$ is the dual basis of $(\Pi \setminus \{\alpha\})^\vee$ in the hyperplane $A$. Hence, $\sum_{\epsilon \in N[\alpha]} (\alpha, \epsilon^\vee) \overline{\omega}_\epsilon$ has the same scalar products as $\alpha$ with the basis $(\Pi \setminus \{\alpha\})^\vee$ of $A$. This implies $\sum_{\epsilon \in N[\alpha]} (\alpha, \epsilon^\vee) \overline{\omega}_\epsilon = \overline{\alpha}$. \hfill \Box

**Remark 6.5.** If $\alpha$ is a leaf in the Dynkin diagram, the sum in Proposition 6.4 consists of merely one addend, say $(\alpha, \epsilon^\vee) \overline{\omega}_\epsilon$. Moreover, in such cases, the coefficient $(\alpha, \epsilon^\vee)$ is always equal to $-1$ (and hence $\alpha$ projects to $-\overline{\omega}_\epsilon$) except:

- in $C_n$, when $\alpha = \alpha_n$, and in this case $\overline{\alpha}_n = -2\overline{\omega}_{n-1}$,
- in $G_2$, when $\alpha = \alpha_2$, and in this case $\overline{\alpha}_2 = -3\overline{\omega}_1$.

Also notice that $N[\alpha]$ consists of at most three elements, hence $\sum_{\epsilon \in N[\alpha]} (\alpha, \epsilon^\vee) \overline{\omega}_\epsilon$ consists of at most three summands. When $k = 1$, we have $\alpha = \min \Phi_{\alpha,k}$ and $\Phi_{\alpha,2k} = \Phi$. For $k > 1$, by Proposition 4.2, $\Phi_{\alpha,2k}$ is a root system with simple system $\{\min \Phi_{\alpha,k}\} \cup \Pi \setminus \{\alpha\}$, hence also the projection $\min \Phi_{\alpha,k}$ can be written as a linear combination of three or less elements in $\{\overline{\omega}_\epsilon : \epsilon \in \Pi \setminus \{\alpha\}\}$. In fact, by a direct check, we see that, if $k > 1$, then $\min \Phi_{\alpha,k}$ can be written as a linear combination of two or less elements in $\{\overline{\omega}_\epsilon : \epsilon \in \Pi \setminus \{\alpha\}\}$. Otherwise, $\Pi$ and $\{\min \Phi_{\alpha,k}\} \cup \Pi \setminus \{\alpha\}$ would be two non isomorphic simple systems that produce isomorphic root subsystem after removing their trivalent vertex. This is impossible.

7. **On a problem by Hopkins and Postnikov**

In this section, we use the results in the preceding sections to tackle a problem by Hopkins and Postnikov.
In [8], Hopkins and Postnikov find certain applications of the root polytope of an irreducible root system $\Phi$. In particular, they need the following lemma concerning dilations of projections of the root polytope (see [8, Lemma 2.10]).

**Lemma 7.1 (Hopkins-Postnikov).** Let $U$ be a nonzero subspace of $E$ spanned by a subset of $\Phi$. Let $\Phi_U$ be the root subsystem $\Phi \cap U$ of $\Phi$. Then there exists some $r \in \mathbb{R}$, with $1 \leq r < 2$, such that the projection on $U$ of the root polytope $P_\Phi$ of $\Phi$ is contained in the dilation $r \cdot P_{\Phi_U}$ of the root polytope of $\Phi_U$.

Hopkins and Postnikov prove Lemma 7.1 by using Theorem 3.3 in a case-by-case check, and ask for a uniform proof that does not require the classification of root systems. In this section, we give a shorter proof, which unfortunately, in its last part, requires the classification (see Remark 7.6). Towards this goal, consider the following lemma.

**Lemma 7.2.** There exists some positive real number $r$, with $r < 2$, such that $$\alpha \in r \cdot P_{\Phi_{\alpha'}}$$ for all $\alpha \in \Pi$.

Lemma 7.2 is in fact equivalent to Lemma 7.1.

**Proposition 7.3.** Lemma 7.1 holds true for all root systems if and only if Lemma 7.2 holds true for all root systems.

**Proof.** Clearly, Lemma 7.1 implies Lemma 7.2.

Conversely, suppose that Lemma 7.2 holds true for all root systems. Let us prove Lemma 7.1.

As explained in [8, Appendix A], we may reduce to the case when $U$ is a maximal standard parabolic subspace, i.e., $U$ is the hyperplane generated by a codimension 1 standard parabolic subsystem of $\Phi$. Hence the proof of Lemma 7.1 amounts to show that there exists some $r \in \mathbb{R}$, with $1 \leq r < 2$, such that, for all $\alpha \in \Pi$,

$$\Phi_{\alpha,k} \subseteq r \cdot P_{\Phi_{\alpha}}$$

for all $k \in [1, m_\alpha]$. Furthermore, Remark 6.3 implies that it suffices to show that there exists some $r \in \mathbb{R}$, with $1 \leq r < 2$, such that, for all $\alpha \in \Pi$,

$$\Phi_{\alpha,1} \subseteq r \cdot P_{\Phi_{\alpha}}.$$

Hence, the result follows by Proposition 6.2.

For each $\varepsilon \in N[\alpha]$, let $\Phi^\varepsilon$ be the component of $\varepsilon$ in $\Phi_\alpha$ and $\Pi^\varepsilon = \Pi \cap \Phi^\varepsilon$. Note that, since $\Phi$ is irreducible, for $\varepsilon$ varying in $N[\alpha]$, the $\Phi^\varepsilon$ exhaust the components of $\Phi_\alpha$. Moreover, they are distinct and at most 3. Also, let $\theta^\varepsilon$ be the highest root of $\Phi^\varepsilon$ and, for all $\eta \in \Pi^\varepsilon$, let $m_\eta$ be defined by the condition $\theta^\varepsilon = \sum_{\eta \in \Pi^\varepsilon} m_\eta \eta$, and let $a_\eta = \tilde{\omega}_\eta / m_\eta$. 

□
Proposition 7.4. Let \( \Phi \) be irreducible and \( \alpha \in \Pi \).

\[
\min\{r : \Phi_{\alpha,1} \subseteq r \cdot \mathcal{P}_{\Phi_\alpha}\} = \min\{r : \overline{\alpha} \in r \cdot \mathcal{P}_{\Phi_\alpha}\} = \sum_{\varepsilon \in N[\alpha]} -(\alpha, \varepsilon) c_\varepsilon
\]

where

\[
c_\varepsilon = \max\{(\overline{\varepsilon}, o_\eta) : \eta \text{ is a } \hat{\Pi}_\varepsilon\text{-extremal root in } \Pi_\varepsilon\}.
\]

Proof. The first equality follows by Proposition 6.2. Let us prove the second equality.

Since \( \mathcal{P}_{\Phi_\alpha} = -\mathcal{P}_{\Phi_\alpha} \), we have \( \overline{\alpha} \in r \cdot \mathcal{P}_{\Phi_\alpha} \) if and only if \( -\overline{\alpha} \in r \cdot \mathcal{P}_{\Phi_\alpha} \). By Proposition 6.4

\[
-\overline{\alpha} = \sum_{\varepsilon \in N[\alpha]} -(\alpha, \varepsilon) \overline{\varepsilon},
\]

is dominant relatively to \( \Phi_\alpha \) and hence, in order to check its inclusion in \( r \cdot \mathcal{P}_{\Phi_\alpha} \), we may use Corollary 3.13. If we number the elements in \( N[\alpha] \), say \( N[\alpha] = \{\varepsilon_1, \ldots, \varepsilon_k\} \), by Corollary 3.13 we obtain:

\[
\min\{r : \Phi_{\alpha,1} \subseteq r \cdot \mathcal{P}_{\Phi_\alpha}\} = \max\{(-\overline{\alpha}, o_{\eta_1} + \cdots + o_{\eta_k})\}
\]

\[
= \max\{-(\alpha, \varepsilon) (\overline{\varepsilon}_1, o_{\eta_1}) - \cdots - (\alpha, \varepsilon_k) (\overline{\varepsilon}_k, o_{\eta_k})\}
\]

where the maximum is taken over all \( k \)-tuples \((\eta_1, \ldots, \eta_k) \in \Pi^{\varepsilon_1} \times \cdots \times \Pi^{\varepsilon_k}\) such that \( \eta_i \) is \( \hat{\Pi}_{\varepsilon_i}\)-extremal. Hence, the assertion follows. \( \Box \)

Note that \((\alpha, \varepsilon) = -1\) except when \((\alpha, \alpha) > (\varepsilon, \varepsilon)\), in which case \((\alpha, \varepsilon) = -(\alpha, \alpha)/(\varepsilon, \varepsilon)\).

In order to prove Lemma 7.2 we show \( \sum_{\varepsilon \in N[\alpha]} -(\alpha, \varepsilon) c_\varepsilon < 2 \), by using the classification of root systems.

Let \( \Phi \) be of type \( X_n \), with the simple roots \( \{\alpha_1, \ldots, \alpha_n\} \) numbered according to the Tables in [1]. Let

\[
c_i[X_n] = \max\{(\omega_i, o_j) : \alpha_j \text{ is } \hat{\Pi}\text{-extremal}\}.
\]

By definition, \((\omega_i, o_j) = (\omega_i, \overline{\omega}_j)/m_i\). Furthermore, \((\omega_i, \overline{\omega}_j)\), for \( j \in [1, n] \), are the coordinates of \( \omega_i \) with respect to \( \{\alpha_1, \ldots, \alpha_n\} \), and thus are explicit in the tables of [1]. Notice also that \((\omega_i, \overline{\omega}_j))_{i,j} = \text{ the inverse of the Cartan matrix } ((\alpha_i, \alpha_j))_{i,j}\).

For \( X = A \), all simple roots are \( \hat{\Pi}\)-extremal and all \( m_j \) are 1, and hence \( c_i[A_n] = \max_{1 \leq j \leq n} \{(\omega_i, \overline{\omega}_j)\} \). In all other cases, the \( \hat{\Pi}\)-extremal simple roots are very few. In the following lemma, we list the values of \( c_i[X_n] \) needed in the proof of Lemma 7.2 (\( c_i[A_n] \) is actually needed only for \( i = 1, 2, 3 \)).

Lemma 7.5. The following equalities hold:

- \( c_1[A_n] = i(n + 1 - i)/(n + 1) \),
- \( c_1[B_n] = c_1[D_n] = 1 \),
- \( c_1[C_n] = 1/2 \),
- \( c_n[B_n] = c_n[D_n] = n/4 \),
- \( c_n[C_n] = n/2 \).
\[ c_6[E_6] = 4/3, \]
\[ c_7[E_7] = 3/2. \]

Proof of Lemma 7.2. Let \( \alpha \in \Pi \) and let
\[
(7.5.1) \quad r_\alpha = \sum_{\varepsilon \in N[\alpha]} -\langle \alpha, \varepsilon^\vee \rangle c_\varepsilon
\]
By Proposition 7.4, it suffices to prove \( r_\alpha < 2 \). The proof consists in a case-by-case computation.

Henceforward, \( \{\alpha_1, \ldots, \alpha_n\} \) is the simple system of \( \Phi \) with the numbering of \([\Pi]\), and \( \alpha = \alpha_i \) for some \( i \in [1, n] \).

Let \( \Phi \cong A_n \). Then \( \Phi_\alpha \cong A_{i-1} \times A_{n-i} \), where we intend \( A_0 \) as the empty component. For each \( \varepsilon \in N[\alpha] \), we can view \( \varepsilon_\alpha \) as the first fundamental weight for the component of \( \varepsilon \) in \( \Phi_\alpha \), hence \( r_\alpha = c_1[A_{i-1}] + c_1[A_{n-i}] \).

Let \( \Phi \cong X_n \), with \( X = B, C, D \). Reasoning as in case \( A_n \), when either \( X = B, C \) and \( i \in [1, n-2] \), or \( X = D \) and \( i \in [1, n-4] \), we find
\[
r_\alpha = c_1[A_{i-1}] + c_1[X_{n-i}].
\]
For \( X = B, C \) and \( i = n - 1 \), we have
\[
r_\alpha = c_1[A_{n-2}] + t c_1[A_1] \quad \text{where} \quad t = \begin{cases} 2 & \text{if } X = B, \\ 1 & \text{if } X = C. \end{cases}
\]
For \( X = B, C \) and \( i = n \), we have
\[
r_\alpha = t c_1[A_{n-1}] \quad \text{where} \quad t = \begin{cases} 1 & \text{if } X = B, \\ 2 & \text{if } X = C. \end{cases}
\]
For \( X = D \), we have:
\[
r_\alpha = \begin{cases} c_1[A_{n-4}] + c_2[A_3] & \text{if } i = n - 3, \\ c_1[A_{n-3}] + c_1[A_1] + c_1[A_1] & \text{if } i = n - 2, \\ c_2[A_{n-1}] & \text{if } i = n - 1, n. \end{cases}
\]
Substituting the values of \( c_i[X] \) listed in Lemma 7.5, we find \( r_\alpha < 2 \) in all cases.
If $\Phi \cong E_n$ ($n = 6, 7, 8$), then

$$r_\alpha = \begin{cases} 
  c_{n-1} [D_{n-1}] \leq \frac{7}{4} & \text{if } i = 1, \\
  c_3 [A_{n-1}] \leq \frac{15}{8} & \text{if } i = 2, \\
  c_1 [A_1] + c_2 [A_{n-2}] \leq \frac{27}{14} & \text{if } i = 3, \\
  c_1 [A_1] + c_4 [A_{n-4}] \leq \frac{59}{30} & \text{if } i = 4, \\
  c_2 [A_4] + c_4 [A_{n-5}] \leq \frac{39}{15} & \text{if } i = 5, \\
  c_3 [D_5] + c_4 [A_{n-6}] \leq \frac{23}{12} & \text{if } i = 6, \\
  c_6 [E_6] + c_4 [A_{n-7}] \leq \frac{11}{6} & \text{if } i = 7 \leq n, \\
  c_7 [E_7] = \frac{3}{2} & \text{if } i = 8 = n.
\end{cases}$$

If $\Phi \cong F_4$, then

$$r_\alpha = \begin{cases} 
  c_3 [C_3] = \frac{3}{2} & \text{if } i = 1, \\
  c_1 [A_1] + 2 c_1 [A_2] = \frac{11}{6} & \text{if } i = 2, \\
  c_1 [A_1] + c_2 [A_2] = \frac{7}{6} & \text{if } i = 3, \\
  c_3 [B_3] = \frac{3}{4} & \text{if } i = 4
\end{cases}$$

If $\Phi \cong G_2$, then

$$r_\alpha = \begin{cases} 
  c_1 [A_1] = \frac{1}{2} & \text{if } i = 1, \\
  3 c_1 [A_1] = \frac{3}{2} & \text{if } i = 2.
\end{cases}$$

\[\square\]

**Remark 7.6.** In [8], Hopkins and Postnikov show, by a uniform argument, that Lemma 7.1 can be reduced to check that, for each $\alpha \in \Pi$ and each $k \in [1, m_\alpha]$, the projections of the (one or two) $\Phi_\alpha$-dominant roots belonging $\Phi_{\alpha,k}$ fall inside $2 \cdot \mathcal{P}_{\Phi_\alpha}$. This final check is done case-by-case. Here, our case-by-case analysis amounts to just one check for each $\alpha \in \Pi$, i.e. that the uniform expression $\sum_{\varepsilon \in N[\alpha]} - (\alpha, \varepsilon^\vee) c_\varepsilon$ of Proposition 7.4 specialises to a number smaller than 2.

8. Enumerative formulas

In this section, we use the results in the previous sections to give some enumerative results.

For each subset $X$ of $\Pi$, we use the shorthand notation $W_X$ for $W(\Pi \setminus X)$. Let $\alpha \in \Pi$, $W_\alpha = W_{\{\alpha\}}$, $A = \text{Span}(\Pi \setminus \{\alpha\})$, $\pi_A$ be the projection on $A$, and, for $x \in E$, $x = \pi_A(x)$. We recall that $N[\alpha]$ denote the set $\{\varepsilon \in \Pi \setminus \{\alpha\} : \beta \not\perp \alpha\}$ of the neighbours of $\alpha$ in the Dynkin diagram of $\Pi$. We also set $\tilde{N}[\alpha] = N[\alpha] \cup \{\alpha\}$.

Then $\tilde{N}(\alpha)$ is the set of all simple roots that are not orthogonal to $\alpha$. 
Lemma 8.1. Let $\Phi$ be an irreducible root system and $\alpha \in \Pi$. The number of the roots in $\Phi_{\alpha,1}$ having the same length as $\alpha$ is
\[
\frac{|W_\alpha|}{|W_{\tilde{N}[\alpha]}|}.
\]

Proof. By Corollary 4.7, the parabolic subgroup $W_\alpha$ acts transitively both on the set long roots of $\Phi_{\alpha,1}$ and on the set short roots of $\Phi_{\alpha,1}$.

Recall the well known fact that the stabilizer in a Weyl group $W$ of a vector $v \in \mathbb{C}$ is the standard parabolic subgroup generated by the reflections trough the simple roots orthogonal to $v$. By Proposition 6.4, $\alpha$ is a negative linear combination of the weights of $\Phi_\alpha$ corresponding to the simple roots in $N[\alpha]$. Hence, the stabilizer in $W_\alpha$ of $\alpha$, and hence of $\alpha$, is the parabolic subgroup $W_{\tilde{N}[\alpha]}$. The assertion follows.

The proof of the following result uses the classification of root systems.

Proposition 8.2. Let $\Phi$ be a classical irreducible root system and $\alpha \in \Pi$ be a leaf in the Dynkin diagram. Then, the number of the roots in $\Phi^+ \setminus \Phi_\alpha$ having the same length as $\alpha$ is
\[
\frac{|W_\alpha|}{|W_{\tilde{N}[\alpha]}|}.
\]

Proof. If $m_\alpha = 1$, we have $\Phi^+ \setminus \Phi_\alpha = \Phi_{\alpha,1}$ and we conclude by Lemma 8.1. For the classical types, this happens exactly when the leaf $\alpha$ is long. If $\alpha$ is short, we obtain the claim by duality, since $W(S^\vee) = W(S)$, for each $S \subseteq \Phi^+$, and the coroots corresponding to the short roots in $\Phi^+$ whose supports contain $\alpha$ are exactly the long roots in $(\Phi^\vee)^+$ whose supports contain $\alpha^\vee$.

It is well known that the number of roots in the irreducible root system $\Phi$ equals the product of the rank of $\Phi$ times its Coxeter number (Kostant formula). Let $h_\Phi$ be the Coxeter number of $\Phi$. The following result is straightforward by Proposition 8.2.

Corollary 8.3. Let $\Phi$ be a root system of type $A_n$ or $D_n$. If $\alpha \in \Pi$ is a leaf of the Dynkin diagram of $\Phi$, then
\[
h_\Phi n - h_\Phi, n - 1 = 2 \frac{|W_\alpha|}{|W_{\tilde{N}[\alpha]}|}.
\]

Let $\Phi$ be an irreducible root system and let $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r\}$ be the set of all simple roots of a fixed length $t$, ordered in such a way that $\Phi(\Pi \setminus \{\varepsilon_1, \ldots, \varepsilon_i\})$ is irreducible for each $i \in [1, r]$. Such an ordering exists, is unique in the multiple laced case, and is not unique in the simply laced case. For each $i \in [1, r]$, let also
\[
\Pi^{(i)} = \Pi \setminus \{\varepsilon_j : j < i\}, \quad W^{(i)} = W(\Pi^{(i)}),
\]
(so $\Pi^{(i)} = \Pi$). By definition, $\varepsilon_i$ is a leaf of the Dinkyn diagram of $\Pi^{(i)}$, for $i \in [1, r]$; we denote by $\varepsilon'_i$ the unique root in $\Pi^{(i)}$ such that $\mathcal{N}[\varepsilon_i] \cap \Pi^{(i)} = \{\varepsilon_i, \varepsilon'_i\}$.

Adapting the previous notation, let

$$W_{\varepsilon_i}^{(i)} = W(\Pi^{(i)} \setminus \{\varepsilon_i\}), \quad W_{\varepsilon_i, \varepsilon'_i}^{(i)} = W(\Pi^{(i)} \setminus \{\varepsilon_i, \varepsilon'_i\}).$$

**Corollary 8.4.** Let $\Phi$ be a classical root system, $\Phi^+_t$ be the set of all positive roots of length $t$ in $\Phi$, and $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r\}$ be the set of all simple roots of length $t$, ordered in such a way that $\Phi(\Pi \setminus \{\varepsilon_1, \ldots, \varepsilon_i\})$ is irreducible for each $i \in [1, r]$. Then

$$|\Phi^+_t| = \sum_{i=1}^r \frac{|W_{\varepsilon_i}^{(i)}|}{|W_{\varepsilon_i, \varepsilon'_i}^{(i)}|}.$$  

**Proof.** By definition, $\Pi^{(i+1)} = \Pi^{(i)} \setminus \{\varepsilon_i\}$, hence, by Proposition 8.2, $|W_{\varepsilon_i}^{(i)}| / |W_{\varepsilon_i, \varepsilon'_i}^{(i)}|$ is the number of positive roots of length $t$ in $\Phi(\Pi^{(i)}) \setminus \Phi(\Pi^{(i+1)})$. The claim follows directly, since the roots of length $t$ are contained in $\Phi \setminus \Phi(\Pi^{(r)})$. \hfill $\square$

**Remark 8.5.** Corollary 8.4 implies, in particular, that the number $\sum_{i=1}^r \frac{|W_{\varepsilon_i}^{(i)}|}{|W_{\varepsilon_i, \varepsilon'_i}^{(i)}|}$ does not depend on the particular ordering of $\{\varepsilon_1, \ldots, \varepsilon_r\}$, provided that each $\varepsilon_i$ is a leaf of the Dynkin diagram of $\Pi \setminus \{\varepsilon_j : j < i\}$. This is trivial when the vertices of the Dynkin diagram have degree at most two. It is not trivial for type D. For type E, it is false.

Indeed, consider type $E_6$. If the sequence starts with the leaf (denoted $\alpha_2$ in $\Pi$) adjacent to the trivalent root, then

$$\sum_{i=1}^r \frac{|W_{\varepsilon_i}^{(i)}|}{|W_{\varepsilon_i, \varepsilon'_i}^{(i)}|} = \frac{|W[A_5]|}{|W[A_2]| \times |W[A_2]|} + (\text{number of positive roots in } A_5) = 35,$$

where $W[X_n]$ is the Weyl group of the root system of type $X_n$. On the other hand, if the sequence starts with another leaf, then

$$\sum_{i=1}^r \frac{|W_{\varepsilon_i}^{(i)}|}{|W_{\varepsilon_i, \varepsilon'_i}^{(i)}|} = \frac{|W[D_5]|}{|W[A_4]|} + (\text{number of positive roots in } D_5) = 36.$$

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Dipartimento di Ingegneria e Geologia, Università di Chieti – Pescara, Viale Pindaro 42, 65127 Pescara, ITALY

Email address: pcellini@unich.it

Dipartimento di Ingegneria Industriale e Scienze Matematiche, Università Politecnica delle Marche, Via Brecce Bianche, 60131 Ancona, ITALY

Email address: m.marietti@staff.univpm.it