THETA SUMS OF HIGHER INDEX

JAE-HYUN YANG

Abstract. In this paper, we obtain some behaviours of theta sums of higher index for the Schrödinger-Weil representation of the Jacobi group associated with a positive definite symmetric real matrix of degree \( m \).

1. Introduction

For a given fixed positive integer \( n \), we let

\[
\mathbb{H}_n = \{ \Omega \in \mathbb{C}^{(n,n)} \mid \Omega = \Omega^t, \quad \text{Im} \Omega > 0 \}
\]

be the Siegel upper half plane of degree \( n \) and let

\[
\text{Sp}(n, \mathbb{R}) = \{ g \in \mathbb{R}^{(2n,2n)} \mid g J_n g^t = J_n \}
\]

be the symplectic group of degree \( n \), where \( F^{(k,l)} \) denotes the set of all \( k \times l \) matrices with entries in a commutative ring \( F \) for two positive integers \( k \) and \( l \), \( t M \) denotes the transpose of a matrix \( M \), \( \text{Im} \Omega \) denotes the imaginary part of \( \Omega \) and

\[
J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.
\]

Here \( I_n \) denotes the identity matrix of degree \( n \). We see that \( \text{Sp}(n, \mathbb{R}) \) acts on \( \mathbb{H}_n \) transitively by

\[
g \cdot \Omega = (A \Omega + B)(C \Omega + D)^{-1},
\]

where \( g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(n, \mathbb{R}) \) and \( \Omega \in \mathbb{H}_n \).

For two positive integers \( n \) and \( m \), we consider the Heisenberg group

\[
H_{\mathbb{R}}^{(n,m)} = \{ (\lambda, \mu; \kappa) \mid \lambda, \mu \in \mathbb{R}^{(m,n)}, \kappa \in \mathbb{R}^{(m,m)}, \kappa + \mu^t \lambda \text{ symmetric} \}
\]

endowed with the following multiplication law

\[
(\lambda, \mu; \kappa) \circ (\lambda', \mu'; \kappa') = (\lambda + \lambda', \mu + \mu'; \kappa + \kappa' + \lambda'^t \mu' - \mu'^t \lambda').
\]

We let

\[
G^J = \text{Sp}(n, \mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)} \quad \text{(semi-direct product)}
\]

be the Jacobi group endowed with the following multiplication law

\[
\left( g, (\lambda, \mu; \kappa) \right) \cdot \left( g', (\lambda', \mu'; \kappa') \right) = \left( g g', (\lambda + \lambda', \mu + \mu'; \kappa + \kappa' + \lambda'^t \mu' - \mu'^t \lambda') \right)
\]

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with \(g, g' \in \text{Sp}(n, \mathbb{R}), (\lambda, \mu; \kappa), (\lambda', \mu'; \kappa') \in \text{H}_\mathbb{R}^{(n,m)}\) and \((\tilde{\lambda}, \tilde{\mu}) = (\lambda, \mu)g'\). Then we have the natural transitive action of \(G^J\) on the Siegel-Jacobi space \(\mathbb{H}_{n,m} := \mathbb{H}_n \times \mathbb{C}^{(m,n)}\) defined by

\[
(g, (\lambda, \mu; \kappa)) \cdot (\Omega, Z) = \left( (A\Omega + B)(C\Omega + D)^{-1}, (Z + \lambda \Omega + \mu)(C\Omega + D)^{-1} \right),
\]

where \(g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(n, \mathbb{R}), (\lambda, \mu; \kappa) \in \text{H}_\mathbb{R}^{(n,m)}\) and \((\Omega, Z) \in \mathbb{H}_{n,m}\). Thus \(\mathbb{H}_{n,m}\) is a homogeneous Kähler space which is not symmetric. In fact, \(\mathbb{H}_{n,m}\) is biholomorphic to the homogeneous space \(G^J/K^J\), where \(K^J \cong U(n) \times S(m, \mathbb{R})\). Here \(U(n)\) denotes the unitary group of degree \(n\) and \(S(m, \mathbb{R})\) denote the abelian additive group consisting of all \(m \times m\) symmetric matrices. We refer to [1, 2, 4], [17]-[29] for more details on materials related to the Siegel-Jacobi space, e.g., Jacobi forms, invariant metrics, invariant differential operators and Maass-Jacobi forms.

The Weil representation for a symplectic group was first introduced by A. Weil in [11] to reformulate Siegel’s analytic theory of quadratic forms (cf. [9]) in terms of the group theoretical theory. It is well known that the Weil representation plays a central role in the study of the transformation behaviors of theta series. In [28], Yang constructed the Schrödinger-Weil representation \(\omega_M\) of the Jacobi group \(G^J\) associated with a positive definite symmetric real matrix \(M\) of degree \(n\) explicitly.

This paper is organized as follows. In Section 2, we review the Schrödinger-Weil representation \(\omega_M\) of the Jacobi group \(G^J\) associated with a symmetric positive definite matrix \(M\) and recall the basic actions of \(\omega_M\) on the representation space \(L^2(\mathbb{R}^{(m,n)})\) which were expressed explicitly in [28]. In Section 3, we define the theta sum \(\Theta^{[M]}_f(\tau, \phi; \lambda, \mu, \kappa)\) of higher index and obtain some properties of the theta sum. The theta sum \(\Theta^{[M]}_f(\tau, \phi; \lambda, \mu, \kappa)\) is a generalization of the theta sum defined by J. Marklof [6].

**Notations:** We denote by \(\mathbb{Z}, \mathbb{R}\) and \(\mathbb{C}\) the ring of integers, the field of real numbers and the field of complex numbers respectively. \(\mathbb{C}^\times\) denotes the multiplicative group of nonzero complex numbers and \(\mathbb{Z}^\times\) denotes the set of all nonzero integers. \(T\) denotes the multiplicative group of complex numbers of modulus one. The symbol “:=” means that the expression on the right is the definition of that on the left. For two positive integers \(k\) and \(l\), \(F^{(k,l)}\) denotes the set of all \(k \times l\) matrices with entries in a commutative ring \(F\). For a square matrix \(A \in F^{(k,k)}\) of degree \(k\), \(\sigma(A)\) denotes the trace of \(A\). For any \(M \in F^{(k,l)}\), \(^tM\) denotes the transpose of a matrix \(M\). \(I_n\) denotes the identity matrix of degree \(n\). We put \(i = \sqrt{-1}\). For a positive integer \(m\) we denote by \(S(m, F)\) the additive group consisting of all \(m \times m\) symmetric matrices with coefficients in a commutative ring \(F\).

### 2. The Schrödinger-Weil Representation

In this section we review the Schrödinger-Weil representation of the Jacobi group \(G^J\) (cf. [28], Section 3).

Throughout this section we assume that \(M\) is a positive definite symmetric real \(m \times m\) matrix. We let

\[
L = \left\{ (0, \mu; \kappa) \in \text{H}_\mathbb{R}^{(n,m)} \mid \mu \in \mathbb{R}^{(m,n)}, \kappa = \mu^t \in \mathbb{R}^{(m,m)} \right\}.
\]
be a commutative normal subgroup of $H_R^{(n,m)}$ and $\chi_M : L \to \mathbb{C}$ be the unitary character of $L$ defined by

$$\chi_M((0, \mu; \kappa)) := e^{\pi i \sigma(M\kappa)}, \quad (0, \mu; \kappa) \in L.$$ 

The representation $\mathcal{W}_M = \text{Ind}_{H_R^{(n,m)}} H^L \chi_M$ induced by $\chi_M$ from $L$ is realized on the Hilbert space $H(\chi_M) \cong L^2(\mathbb{R}^{m,n}, d\xi)$. $\mathcal{W}_M$ is irreducible (cf. [11], Theorem 3) and is called the Schrödinger representation $\mathcal{W}_M$ of the Heisenberg group $H_R^{(n,m)}$ with the central character $\chi_M$. We refer to [11, 12, 13, 14, 15, 16] for more details on representations of the Heisenberg group $H_R^{(n,m)}$ and their related topics. Then $\mathcal{W}_M$ is expressed explicitly as

$$\mathcal{W}_M(h_0)f(\lambda) = e^{\pi i \sigma(M\kappa)} f(\lambda + \lambda_0),$$

where $h_0 = (\lambda_0, \mu_0; \kappa_0) \in H$ and $\lambda \in \mathbb{R}^{m,n}$. See Formula (2.4) in [28] for more detail on $\mathcal{W}_M$. We note that the symplectic group $Sp(n, \mathbb{R})$ acts on $H^{(n,m)}_R$ by conjugation inside $G^J$. For a fixed element $g \in Sp(n, \mathbb{R})$, the irreducible unitary representation $\mathcal{W}_M^g$ of $H^{(n,m)}_R$ defined by

$$\mathcal{W}_M^g(h) = \mathcal{W}_M(ghg^{-1}), \quad h \in H^{(n,m)}_R$$

has the property that

$$\mathcal{W}_M^g((0, 0; \kappa)) = \mathcal{W}_M((0, 0; \kappa)) = e^{\pi i \sigma(M\kappa)} \text{Id}_{H(\chi_M)}, \quad \kappa \in S(m, \mathbb{R}).$$

Here $\text{Id}_{H(\chi_M)}$ denotes the identity operator on the Hilbert space $H(\chi_M)$. According to Stone-von Neumann theorem, there exists a unitary operator $R_M(g)$ on $H(\chi_M)$ with $R_M(I_{2n}) = \text{Id}_{H(\chi_M)}$ such that

$$R_M(g)\mathcal{W}_M(h) = \mathcal{W}_M^g(h)R_M(g) \quad \text{for all } h \in H^{(n,m)}_R.$$ 

We observe that $R_M(g)$ is determined uniquely up to a scalar of modulus one.

From now on, for brevity, we put $G = Sp(n, \mathbb{R})$. According to Schur’s lemma, we have a map $c_M : G \times G \to T$ satisfying the relation

$$R_M(g_1g_2) = c_M(g_1, g_2)R_M(g_1)R_M(g_2) \quad \text{for all } g_1, g_2 \in G.$$

We recall that $T$ denotes the multiplicative group of complex numbers of modulus one. Therefore $R_M$ is a projective representation of $G$ on $H(\chi_M)$ and $c_M$ defines the cocycle class in $H^2(G, T)$. The cocycle $c_M$ yields the central extension $G_M$ of $G$ by $T$. The group $G_M$ is a set $G \times T$ equipped with the following multiplication

$$(g_1, t_1) \cdot (g_2, t_2) = (g_1g_2, t_1t_2 c_M(g_1, g_2)^{-1}), \quad g_1, g_2 \in G, \ t_1, t_2 \in T.$$ 

We see immediately that the map $R_M : G_M \to GL(H(\chi_M))$ defined by

$$R_M(g, t) = t R_M(g) \quad \text{for all } (g, t) \in G_M$$

is a true representation of $G_M$. As in Section 1.7 in [5], we can define the map $s_M : G \to T$ satisfying the relation

$$c_M(g_1, g_2)^2 = s_M(g_1)^{-1}s_M(g_2)^{-1}s_M(g_1g_2) \quad \text{for all } g_1, g_2 \in G.$$


Thus we see that
\begin{equation}
G_{2, M} = \{ (g, t) \in G, M \mid t^2 = s_M(g)^{-1} \}
\end{equation}
is the metaplectic group associated with $M$ that is a two-fold covering group of $G$. The restriction $R_{2, M}$ of $\bar{R}_M$ to $G_{2, M}$ is the Weil representation of $G$ associated with $M$.

If we identify $h = (\lambda, \mu; \kappa) \in H^{(n, m)}_R$ (resp. $g \in Sp(n, \mathbb{R})$) with $(I_{2n}, (\lambda, \mu; \kappa)) \in G^J$ (resp. $(g, (0, 0); 0)) \in G^J$, every element $\tilde{g}$ of $G^J$ can be written as $\tilde{g} = hg$ with $h \in H^{(n, m)}_R$ and $g \in Sp(n, \mathbb{R})$. In fact,
\begin{equation}
(g, (\lambda, \mu; \kappa)) = (I_{2n}, ((\lambda, \mu)g^{-1}; \kappa)) (g, (0, 0; 0)) = ((\lambda, \mu)g^{-1}; \kappa) \cdot g.
\end{equation}
Therefore we define the projective representation $\pi_M$ of the Jacobi group $G^J$ with cocycle $c_M(g_1, g_2)$ by
\begin{equation}
\pi_M(hg) = \varpi_M(h) R_M(g), \quad h \in H^{(n, m)}_R, \quad g \in G.
\end{equation}

We let
\begin{equation}
G^J_M = G_M \ltimes H^{(n, m)}_R
\end{equation}
be the semidirect product of $G_M$ and $H^{(n, m)}_R$ with the multiplication law
\begin{equation}
((g_1, t_1), (\lambda_1, \mu_1; \kappa_1)) \cdot ((g_2, t_2), (\lambda_2, \mu_2; \kappa_2)) = ((g_1, t_1)(g_2, t_2), (\tilde{\lambda} + \lambda_2, \bar{\mu} + \mu_2; \kappa_1 + \kappa_2 + \tilde{\mu} \mu_2 - \mu \tilde{\lambda}),
\end{equation}
where $(g_1, t_1), (g_2, t_2) \in G_{2, M}$, $(\lambda_1, \mu_1; \kappa_1), (\lambda_2, \mu_2; \kappa_2) \in H^{(n, m)}_R$ and $(\tilde{\lambda}, \bar{\mu}) = (\lambda, \mu)g_2$. If we identify $h = (\lambda, \mu; \kappa) \in H^{(n, m)}_R$ (resp. $(g, t) \in G_M$) with $(I_{2n}, 1, (\lambda, \mu; \kappa)) \in G^J_M$ (resp. $(g, (0, 0); 0)) \in G^J_M$, we see easily that every element $((g, t), (\lambda, \mu; \kappa))$ of $G^J_M$ can be expressed as
\begin{equation}
((g, t), (\lambda, \mu; \kappa)) = ((I_{2n}, 1), ((\lambda, \mu)g^{-1}; \kappa)) ((g, t), (0, 0; 0)) = ((\lambda, \mu)g^{-1}; \kappa) (g, t).
\end{equation}
Now we can define the true representation $\overline{\varpi}_M$ of $G^J_M$ by
\begin{equation}
\overline{\varpi}_M(h \cdot (g, t)) = t \pi_M(hg) = t \varpi_M(h) R_M(g), \quad h \in H^{(n, m)}_R, \quad (g, t) \in G_M.
\end{equation}

We recall that the following matrices
\begin{align*}
t(b) &= \begin{pmatrix} I_n & b \\ 0 & I_n \end{pmatrix} \quad \text{with any } b = t^t b \in \mathbb{R}^{(n, n)}, \\
g(\alpha) &= \begin{pmatrix} t^t \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \quad \text{with any } \alpha \in GL(n, \mathbb{R}), \\
\sigma_n &= \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}
\end{align*}
generate the symplectic group $G = Sp(n, \mathbb{R})$ (cf. [3] p. 326], [7] p. 210]). Therefore the following elements $h_t(\lambda, \mu; \kappa)$, $t(b; t)$, $g(\alpha; t)$ and $\sigma_n; t$ of $G_M \ltimes H^{(n, m)}_R$ defined by
\begin{align*}
h_t(\lambda, \mu; \kappa) &= ((I_{2n}, t), (\lambda, \mu; \kappa)) \quad \text{with } t \in T, \quad \lambda, \mu \in \mathbb{R}^{(m, n)} \text{ and } \kappa \in \mathbb{R}^{(m, m)}, \\
t(b; t) &= ((t(b), t), (0, 0; 0)) \quad \text{with any } b = t^t b \in \mathbb{R}^{(n, n)}, \quad t \in T, \\
g(\alpha; t) &= ((g(\alpha), t), (0, 0; 0)) \quad \text{with any } \alpha \in GL(n, \mathbb{R}) \text{ and } t \in T, \\
\sigma_n; t &= ((\sigma_n, t), (0, 0; 0)) \quad \text{with } t \in T.
\end{align*}
generate the group $G_{\mathcal{M}} \rtimes H^{(n,m)}_{\mathbb{R}}$. We can show that the representation $\widetilde{\omega}_{\mathcal{M}}$ is realized on the representation $H(\chi_{\mathcal{M}}) = L^2(\mathbb{R}^{(m,n)})$ as follows: for each $f \in L^2(\mathbb{R}^{(m,n)})$ and $x \in \mathbb{R}^{(m,n)}$, the actions of $\widetilde{\omega}_{\mathcal{M}}$ on the generators are given by

\begin{align}
\widetilde{\omega}_{\mathcal{M}}(h(\lambda, \mu; \kappa)) f(x) &= t e^{\pi i \sigma(\mathcal{M}(\kappa + \mu \lambda + 2 x^t \mu))} f(x + \lambda), \\
\widetilde{\omega}_{\mathcal{M}}(t(b; t)) f(x) &= t e^{\pi i \sigma(\mathcal{M}(x b^t x))} f(x), \\
\widetilde{\omega}_{\mathcal{M}}(g(\alpha; t)) f(x) &= t \det \alpha |f(x^t \alpha)|,
\end{align}

(2.10) \hspace{1cm} (2.11) \hspace{1cm} (2.12)

(2.13) \quad \widetilde{\omega}_{\mathcal{M}}(\sigma_n; t) f(x) = t (\det \mathcal{M})^{\frac{n}{2}} \int_{\mathbb{R}^{(m,n)}} f(y) e^{-2 \pi i \sigma(\mathcal{M} y^t x)} dy.

Let

$$G_{2,\mathcal{M}}^J = G_{2,\mathcal{M}} \rtimes H^{(n,m)}_{\mathbb{R}}$$

be the semidirect product of $G_{2,\mathcal{M}}$ and $H^{(n,m)}_{\mathbb{R}}$. Then $G_{2,\mathcal{M}}^J$ is a subgroup of $G_{\mathcal{M}}^J$ which is a two-fold covering group of the Jacobi group $G^J$. The restriction $\omega_{\mathcal{M}}$ of $\widetilde{\omega}_{\mathcal{M}}$ to $G_{2,\mathcal{M}}^J$ is called the Schrödinger-Weil representation of $G_{\mathcal{M}}^J$ associated with $\mathcal{M}$.

**Remark 2.1.** In the case $n = m = 1$, $\omega_{\mathcal{M}}$ is dealt in [1] and [6].

**Remark 2.2.** The Schrödinger-Weil representation is applied usefully to the theory of Maass-Jacobi forms [5].

### 3. Theta Sums of Higher Index

Let $\mathcal{M}$ be a positive definite symmetric real matrix of degree $m$. We recall the Schrödinger representation $\mathcal{M}$ of the Heisenberg group $H^{(n,m)}_{\mathbb{R}}$ associated with $\mathcal{M}$ that is given by Formula (2.1) in Section 2. We note that for an element $(\lambda, \mu; \kappa)$ of $H^{(n,m)}_{\mathbb{R}}$, we have the decomposition

$$(\lambda, \mu; \kappa) = (\lambda, 0; 0) \circ (0, \mu; 0) \circ (0, \kappa - \lambda^t \mu).$$

We consider the embedding $\Phi_n : SL(2, \mathbb{R}) \rightarrow Sp(n, \mathbb{R})$ defined by

$$\Phi_n \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) := \begin{pmatrix} aI_n & bI_n \\ cI_n & dI_n \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}).$$

(3.1)

For $x, y \in \mathbb{R}^{(m,n)}$, we put

$$(x, y)_{\mathcal{M}} := (x_{\mathcal{M}})^t y_{\mathcal{M}} \quad \text{and} \quad \|x\|_{\mathcal{M}} := \sqrt{(x, x)_{\mathcal{M}}}.$$

According to Formulas (2.11)-(2.13), for any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}) \leftrightarrow Sp(n, \mathbb{R})$ and $f \in L^2(\mathbb{R}^{(m,n)})$, we have the following explicit representation

$$[R_M(M)f](x) = \begin{cases} |a|^{\frac{m}{2}} e^{ab\|x\|_{\mathcal{M}}^2} \pi^i f(ax) & \text{if } c = 0, \\
(\det \mathcal{M})^{\frac{n}{2}} |c|^{-\frac{m}{2}} \int_{\mathbb{R}^{(m,n)}} e^{\frac{a(M x \cdot y, M)}{c}} \pi^i f(y) dy & \text{if } c \neq 0,
\end{cases}$$

(3.2)
where
\[ \alpha(M, x, y, M) = a \|x\|^2_M + d \|y\|^2_M - 2(x, y)_M. \]

Indeed, if \( a = 0 \) and \( c \neq 0 \), using the decomposition
\[
M = \begin{pmatrix} 0 & -c^{-1} \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}
\]
and if \( a \neq 0 \) and \( c \neq 0 \), using the decomposition
\[
M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \ c^{-1} \\ 0 \ a^{-1} \end{pmatrix} \begin{pmatrix} ac & ad \\ 0 & (ac)^{-1} \end{pmatrix},
\]
we obtain Formula (3.2).

If \( M_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, M_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \) and \( M_3 = \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} \in SL(2, \mathbb{R}) \) with \( M_3 = M_1 M_2 \), the corresponding cocycle is given by
\[ c_M(M_1, M_2) = e^{-i \pi mn \text{sign}(c_1c_2c_3)/4}, \]
where
\[
\text{sign}(x) = \begin{cases} -1 & (x < 0) \\ 0 & (x = 0) \\ 1 & (x > 0). \end{cases}
\]

In the special case when
\[
M_1 = \begin{pmatrix} \cos \phi_1 & -\sin \phi_1 \\ \sin \phi_1 & \cos \phi_1 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} \cos \phi_2 & -\sin \phi_2 \\ \sin \phi_2 & \cos \phi_2 \end{pmatrix},
\]
we find
\[ c_M(M_1, M_2) = e^{-i \pi mn (\sigma_{\phi_1} + \sigma_{\phi_2} - \sigma_{\phi_1} + \phi_2)/4}, \]
where
\[ \sigma_{\phi} = \begin{cases} 2\nu & \text{if } \phi = \nu \pi \\ 2\nu + 1 & \text{if } \nu \pi < \phi < (\nu + 1)\pi. \end{cases} \]

It is well known that every \( M \in SL(2, \mathbb{R}) \) admits the unique Iwasawa decomposition
\[ M = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^{1/2} & 0 \\ 0 & v^{-1/2} \end{pmatrix} \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}, \]
where \( \tau = u + iv \in \mathbb{H}_1 \) and \( \phi \in [0, 2\pi) \). This parametrization \( M = (\tau, \phi) \) in \( SL(2, \mathbb{R}) \) leads to the natural action of \( SL(2, \mathbb{R}) \) on \( \mathbb{H}_1 \times [0, 2\pi) \) defined by
\[ (a \ b \ c \ d) (\tau, \phi) := \left( \frac{a \tau + b}{c \tau + d}, \phi + \text{arg}(c \tau + d) \mod 2\pi \right). \]

**Lemma 3.1.** For two elements \( g_1 \) and \( g_2 \) in \( SL(2, \mathbb{R}) \), we let
\[
g_1 = \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1^{1/2} & 0 \\ 0 & v_1^{-1/2} \end{pmatrix} \begin{pmatrix} \cos \phi_1 & -\sin \phi_1 \\ \sin \phi_1 & \cos \phi_1 \end{pmatrix}
\]
and
\[
g_2 = \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_2^{1/2} & 0 \\ 0 & v_2^{-1/2} \end{pmatrix} \begin{pmatrix} \cos \phi_2 & -\sin \phi_2 \\ \sin \phi_2 & \cos \phi_2 \end{pmatrix}
\]
be the Iwasawa decompositions of \( g_1 \) and \( g_2 \) respectively, where \( u_1, u_2 \in \mathbb{R}, \, v_1 > 0, v_2 > 0 \) and \( 0 \leq \phi_1, \phi_2 < 2\pi \). Let

\[
g_3 = g_1 g_2 = \begin{pmatrix}1 & u_3 \\ 0 & 1\end{pmatrix} \begin{pmatrix}v_3^{1/2} & 0 \\ 0 & v_3^{-1/2}\end{pmatrix} \begin{pmatrix}\cos \phi_3 & -\sin \phi_3 \\ \sin \phi_3 & \cos \phi_3\end{pmatrix}
\]

be the Iwasawa decomposition of \( g_3 = g_1 g_2 \). Then we have

\[
u_3 = \frac{A}{(u_2 \sin \phi_1 + \cos \phi_1)^2 + (v_2 \sin \phi_1)^2},
\]

\[
v_3 = \frac{v_1 v_2}{(u_2 \sin \phi_1 + \cos \phi_1)^2 + (v_2 \sin \phi_1)^2}
\]

and

\[
\phi_3 = \tan^{-1} \left[ \frac{(v_2 \cos \phi_2 + u_2 \sin \phi_2) \tan \phi_1 + \sin \phi_2}{(-v_2 \sin \phi_2 + u_2 \cos \phi_2) \tan \phi_1 + \cos \phi_2} \right],
\]

where

\[
A = u_1 (u_2 \sin \phi_1 + \cos \phi_1)^2 + (u_1 v_2 - v_1 u_2) \sin^2 \phi_1 + v_1 u_2 \cos^2 \phi_1 + v_1 (u_2^2 + v_2^2 - 1) \sin \phi_1 \cos \phi_1.
\]

Proof. If \( g \in SL(2, \mathbb{R}) \) has the unique Iwasawa decomposition (3.4), then we get the following

\[
a = v^{1/2} \cos \phi + u v^{-1/2} \sin \phi,
\]

\[
b = -v^{1/2} \sin \phi + u v^{-1/2} \cos \phi,
\]

\[
c = v^{-1/2} \sin \phi, \quad d = v^{-1/2} \cos \phi,
\]

\[
u = (ac + bd) (c^2 + d^2)^{-1}, \quad v = (c^2 + d^2)^{-1}, \quad \tan \phi = \frac{c}{d}.
\]

We set

\[
g_3 = g_1 g_2 = \begin{pmatrix}a_3 & b_3 \\ c_3 & d_3\end{pmatrix}.
\]

Since

\[
u_3 = (a_3 c_3 + b_3 d_3) (c_3^2 + d_3^2)^{-1}, \quad v = (c_3^2 + d_3^2)^{-1}, \quad \tan \phi_3 = \frac{c_3}{d_3},
\]

by an easy computation, we obtain the desired results. \( \square \)

Now we use the new coordinates \((\tau = u + iv, \phi)\) with \(\tau \in \mathbb{H}_1\) and \(\phi \in [0, 2\pi)\) in \(SL(2, \mathbb{R})\). According to Formulas (2.11)-(2.13), the projective representation \(R_M\) of \(SL(2, \mathbb{R}) \hookrightarrow Sp(n, \mathbb{R})\) reads in these coordinates \((\tau = u + iv, \phi)\) as follows:

\[
[R_M(\tau, \phi) f](x) = v \frac{\max}{\max} e^{u|x|} e^{\pi i} [R_M(i, \phi) f] (v^{1/2} x),
\]

where \(f \in L^2(\mathbb{R}^{(m,n)}), \, x \in \mathbb{R}^{(m,n)}\) and

\[
[R_M(i, \phi) f](x)
\]

\[
\begin{cases} f(x) & \text{if } \phi \equiv 0 \mod 2\pi, \\
(f(-x)) & \text{if } \phi \equiv \pi \mod 2\pi, \\
\det(M)^{\frac{m}{2}} \sin \phi^{-\frac{m}{2}} \int_{\mathbb{R}^{(m,n)}} e^{B(x,y,\phi,M)\pi i} f(y)\,dy & \text{if } \phi \neq 0 \mod \pi.
\end{cases}
\]

(3.7)
Here
\[ B(x, y, \phi, \mathcal{M}) = \frac{\|x\|_{\mathcal{M}}^2 + \|y\|_{\mathcal{M}}^2 \cos \phi - 2(x, y)_{\mathcal{M}}}{\sin \phi}. \]

Now we set
\[ S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \]

We note that
\[ R_{\mathcal{M}}(i, \frac{\pi}{2}) f \] for \( f \in L^2(\mathbb{R}^{(m,n)}) \).

**Remark 3.1.** For Schwartz functions \( f \in \mathcal{S}(\mathbb{R}^{(m,n)}) \), we have
\[ \lim_{\phi \to 0^\pm} |\sin \phi|^{-\frac{mn}{2}} \int_{\mathbb{R}^{(m,n)}} e^{B(x,y,\phi,\mathcal{M})}\pi i f(y)dy = e^{\pm i \pi mn/4} f(x) \neq f(x). \]

Therefore the projective representation \( R_{\mathcal{M}} \) is not continuous at \( \phi = \nu \pi (\nu \in \mathbb{Z}) \) in general.

If we set
\[ \tilde{R}_{\mathcal{M}}(\tau, \phi) = e^{-i \pi mn \sigma_\phi/4} R_{\mathcal{M}}(\tau, \phi), \]
\( \tilde{R}_{\mathcal{M}} \) corresponds to a unitary representation of the double cover of \( \text{SL}(2, \mathbb{R}) \) (cf. Formula (2.6) and [5]). This means in particular that
\[ \tilde{R}_{\mathcal{M}}(i, \phi) \tilde{R}_{\mathcal{M}}(i, \phi') = \tilde{R}_{\mathcal{M}}(i, \phi + \phi'), \]
where \( \phi \in [0, 4\pi) \) parametrises the double cover of \( \text{SO}(2) \subset \text{SL}(2, \mathbb{R}) \).

We observe that for any element \((g, (\lambda, \mu; \kappa)) \in G^I \) with \( g \in \text{Sp}(n, \mathbb{R}) \) and \((\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)} \), we have the following decomposition
\[ (g, (\lambda, \mu; \kappa)) = (I_{2n}, ((\lambda, \mu)g^{-1}; \kappa)) (g, (0, 0; 0)) = ((\lambda, \mu)g^{-1}; \kappa) \cdot g. \]

Thus \( \text{Sp}(n, \mathbb{R}) \) acts on \( H_{\mathbb{R}}^{(n,m)} \) naturally by
\[ g \cdot (\lambda, \mu; \kappa) = ((\lambda, \mu)g^{-1}; \kappa), \quad g \in \text{Sp}(n, \mathbb{R}), \ (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)}. \]

**Definition 3.1.** For any Schwartz function \( f \in \mathcal{S}(\mathbb{R}^{(m,n)}) \), we define the function \( \Theta_f^{[\mathcal{M}]} \) on the Jacobi group \( \text{SL}(2, \mathbb{R}) \times H_{\mathbb{R}}^{(n,m)} \hookrightarrow G^I \) by
\[ \Theta_f^{[\mathcal{M}]}(\tau, \phi; \lambda, \mu, \kappa) := \sum_{\omega \in \mathbb{Z}^{(m,n)}} [\pi_{\mathcal{M}} ((\lambda, \mu; \kappa)(\tau, \phi)) f(j)], \]
where \((\tau, \phi) \in \text{SL}(2, \mathbb{R}) \) and \((\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)} \). The function \( \Theta_f^{[\mathcal{M}]} \) is called the theta sum of index \( \mathcal{M} \) associated to a Schwartz function \( f \). The projective representation \( \pi_{\mathcal{M}} \) of the Jacobi group \( G^I \) was already defined by Formula (2.8). More precisely, for \( \tau = u + iv \in \mathbb{H}_1 \) and \((\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)} \), we have
\[ \Theta_f^{[\mathcal{M}]}(\tau, \phi; \lambda, \mu, \kappa) = e^{\frac{\pi}{2} i \sigma_\phi (\mathcal{M} (\kappa + \mu^T \lambda))} \sum_{\omega \in \mathbb{Z}^{(m,n)}} e^{\pi i \left\{ u \| \omega + \lambda \|_{\mathcal{M}}^2 + 2(\omega, \mu)_{\mathcal{M}} \right\}} [R_{\mathcal{M}}(i, \phi) f] \left( \nu^{1/2} (\omega + \lambda) \right). \]
Lemma 3.2. We set \( f_\phi := \tilde{R}_M(i, \phi)f \) for \( f \in \mathcal{S}(\mathbb{R}^{m,n}) \). Then for any \( R > 1 \), there exists a constant \( C_R \) such that for all \( x \in \mathbb{R}^{m,n} \) and \( \phi \in \mathbb{R} \),
\[
|f_\phi(x)| \leq C_R (1 + \|x\|_M)^{-R}.
\]

Proof. Following the arguments in the proof of Lemma 4.3 in [6], pp. 428-429, we get the desired result. \( \Box \)

Theorem 3.1 (Jacobi 1). Let \( M \) be a positive definite symmetric integral matrix of degree \( m \) such that \( M\mathbb{Z}^{(m,n)} = \mathbb{Z}^{(m,n)} \). Then for any Schwartz function \( f \in \mathcal{S}(\mathbb{R}^{m,n}) \), we have
\[
\Theta^{[M]}_F \left( -\frac{1}{\tau}, \phi + \arg \tau; -\mu, \lambda, \kappa \right) = \left( \det M \right)^{-\frac{n}{2}} c_M(S, (\tau, \phi)) \Theta^{[M]}_F (\tau, \phi; \lambda, \mu, \kappa),
\]
where
\[
c_M(S, (\tau, \phi)) := e^{i \pi mn \text{sign}_{\phi} \text{sign}(\sin \phi \sin(\phi + \arg \tau))}.
\]

Proof. First we recall that for any Schwartz function \( \varphi \in \mathcal{S}(\mathbb{R}^{m,n}) \), the Fourier transform \( \mathcal{F}_\varphi \) of \( \varphi \) is given by
\[
(\mathcal{F}_\varphi)(x) = \int_{\mathbb{R}^{m,n}} \varphi(y) e^{-2\pi i \sigma(y^t x)} dy.
\]
Now we put
\[
S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL(2, \mathbb{Z}) \hookrightarrow Sp(n, \mathbb{R})
\]
and for any \( F \in \mathcal{S}(\mathbb{R}^{m,n}) \), we put
\[
F_M(x) := F(M^{-1}x), \quad x \in \mathbb{R}^{m,n}.
\]
According to Formula (2.13), for any \( F \in \mathcal{S}(\mathbb{R}^{m,n}) \),
\[
[R_M(S)F](x) = \left( \det M \right)^{\frac{n}{2}} \int_{\mathbb{R}^{m,n}} F(y) e^{-2\pi i \sigma(My^t x)} dy = \left( \det M \right)^{-\frac{n}{2}} \int_{\mathbb{R}^{m,n}} F(M^{-1}y) e^{-2\pi i \sigma(y^t x)} dy = \left( \det M \right)^{-\frac{n}{2}} \int_{\mathbb{R}^{m,n}} F_M(y) e^{-2\pi i \sigma(y^t x)} dy = \left( \det M \right)^{-\frac{n}{2}} [\mathcal{F}_F M](x).
\]
Thus we have
\[
(3.10) \quad \mathcal{F}_M = \left( \det M \right)^{\frac{n}{2}} R_M(S)F \quad \text{for} \quad F \in \mathcal{S}(\mathbb{R}^{m,n}).
\]

By Lemma 3.1, we get easily
\[
S \cdot (\tau, \phi) = \left( -\frac{1}{\tau}, \phi + \arg \tau \right).
\]

If we take \( F = \pi_M((\lambda, \mu ; \kappa)(\tau, \phi))f \) for \( f \in \mathcal{S}(\mathbb{R}^{m,n}) \), a fixed element \( (\lambda, \mu ; \kappa) \in H_{\mathbb{R}}^{(m,n)} \) and an fixed element \( (\tau, \phi) \in SL(2, \mathbb{R}) \), then it is easily seen that \( F \in \mathcal{S}(\mathbb{R}^{m,n}) \).
Hence from (3.13) we obtain the desired formula
\[ [R_M(S)F](x) = [R_M(S)\pi_M((\lambda, \mu; \kappa)(\tau, \phi))f] (x), \quad x \in \mathbb{R}^{(m,n)} \]
According to Poisson summation formula, we have
\[ [\mathcal{W}_M((\lambda, \mu)S^{-1}; \kappa)R_M(S)R_M(\tau, \phi)f] (x) \]
On the other hand,
\[ \sum_{\omega \in \mathbb{Z}^{(m,n)}} [\mathcal{F}_M](\omega) = \sum_{\omega \in \mathbb{Z}^{(m,n)}} F_M(\omega). \]
According to Poisson summation formula, we have
\[ (\det \mathcal{M})^{\frac{n}{2}} c_M(S, (\tau, \phi))^{-1} \pi_M\left((-\mu, \lambda; \kappa) - \frac{1}{\tau}, \phi + \arg \tau \right) f(x). \]
Thus we obtain
\[ (3.12) \quad [R_M(S)F](x) = c_M(S, (\tau, \phi))^{-1} \left[ \pi_M\left((-\mu, \lambda; \kappa) - \frac{1}{\tau}, \phi + \arg \tau \right) f \right](x). \]
It follows from (3.10) and (3.12) that
\[ \sum_{\omega \in \mathbb{Z}^{(m,n)}} [\mathcal{F}_M](\omega) = (\det \mathcal{M})^{\frac{n}{2}} c_M(S, (\tau, \phi))^{-1} \]
\[ \times \sum_{\omega \in \mathbb{Z}^{(m,n)}} \left[ \pi_M\left((-\mu, \lambda; \kappa) - \frac{1}{\tau}, \phi + \arg \tau \right) f \right](x) \]
\[ = (\det \mathcal{M})^{\frac{n}{2}} c_M(S, (\tau, \phi))^{-1} \Theta_f^{[\mathcal{M}]}\left(-\frac{1}{\tau}, \phi + \arg \tau; -\mu, \lambda, \kappa \right). \]
On the other hand,
\[ \sum_{\omega \in \mathbb{Z}^{(m,n)}} F_M(\omega) = \sum_{\omega \in \mathbb{Z}^{(m,n)}} F(\mathcal{M}^{-1}\omega) \]
\[ = \sum_{\omega \in \mathbb{Z}^{(m,n)}} [\pi_M((\lambda, \mu; \kappa)(\tau, \phi))f] (\mathcal{M}^{-1}\omega) \]
\[ = \sum_{\omega \in \mathbb{Z}^{(m,n)}} [\pi_M((\lambda, \mu; \kappa)(\tau, \phi))f] (\omega) \quad (\therefore \mathcal{M}^{-1}\mathbb{Z}^{(m,n)} = \mathbb{Z}^{(m,n)}) \]
\[ = \Theta_f^{[\mathcal{M}]}(\tau, \phi; \lambda, \mu, \kappa). \]
Hence from (3.13) we obtain the desired formula
\[ \Theta_f^{[\mathcal{M}]}\left(-\frac{1}{\tau}, \phi + \arg \tau; -\mu, \lambda, \kappa \right) = (\det \mathcal{M})^{-\frac{n}{2}} c_M(S, (\tau, \phi)) \Theta_f^{[\mathcal{M}]}(\tau, \phi; \lambda, \mu, \kappa). \]
If
\[ S = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \quad (\tau, \phi) = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \quad \text{and} \quad S \cdot (\tau, \phi) = \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} \in SL(2, \mathbb{R}), \]
according to Lemma 3.1, we get easily
\[ c_1c_2c_3 = (u^2 + v^2)^{1/2} \sin \phi \sin(\phi + \arg \tau), \]
where
\[ (\tau, \phi) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{u^2 + v^2} & 0 \\ 0 & \sqrt{\frac{1}{u^2 + v^2}} \end{pmatrix} \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \]
is the Iwasawa decomposition of \((\tau, \phi) \in SL(2, \mathbb{R})\). Thus we obtain
\[ c_M(S, (\tau, \phi)) = e^{i \pi mn \text{sign}(c_1c_2c_3)} = e^{i \pi mn \text{sign}(\sin(\phi + \arg \tau))}. \]

This completes the proof. \(\square\)

**Theorem 3.2** (Jacobi 2). Let \(M = (M_{kl})\) be a positive definite symmetric integral \(m \times m\) matrix and let \(s = (s_{kl}) \in \mathbb{Z}^{(m,n)}\) be integral. Then we have
\[ \Theta_f^M(\tau + 2, \phi; \lambda, s - 2\lambda + \mu, \kappa - s^i \lambda) = \Theta_f^M(\tau, \phi; \lambda, \mu, \kappa) \]
for all \((\tau, \phi) \in SL(2, \mathbb{R})\) and \((\lambda, \mu; \kappa) \in H^{(n,m)}_R\).

**Proof.** For brevity, we put \(T_* = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\). According to Lemma 3.1, for any \((\tau, \phi) \in SL(2, \mathbb{R})\), the multiplication of \(T_*\) and \((\tau, \phi)\) is given by
\[ (3.14) \quad T_*(\tau, \phi) = (\tau + 2, \phi). \]

For \(s \in \mathbb{R}^{(m,n)}\), \((\lambda, \mu; \kappa) \in H^{(n,m)}_R\) and \((\tau, \phi) \in SL(2, \mathbb{R})\), according to (3.14),
\[ \pi_M((0, s; 0)T_*) \pi_M(\lambda, \mu; \kappa)(\tau, \phi) \]
\[ = \pi_M(0, s; 0) R_M(T_*) \pi_M(\lambda, \mu; \kappa) R_M(\tau, \phi) \]
\[ = \pi_M(0, s; 0) \pi_M((\lambda, \mu) T_*^{-1}; \kappa) R_M(T_*) R_M(\tau, \phi) \]
\[ = c_M(T_*(\tau, \phi))^{-1} \pi_M(\lambda, s - 2\lambda + \mu; \kappa - s^i \lambda) R_M(T_*(\tau, \phi)) \]
\[ = \pi_M(\lambda, s - 2\lambda + \mu; \kappa - s^i \lambda) R_M(\tau + 2, \phi) \]
\[ = \pi_M((\lambda, s - 2\lambda + \mu; \kappa - s^i \lambda)(\tau + 2, \phi)). \]

Here we used the fact that \(c_M(T_*(\tau, \phi)) = 1\) because \(T_*\) is upper triangular.

On the other hand, according to the assumptions on \(M\) and \(s\), for \(f \in \mathcal{S}(\mathbb{R}^{(m,n)})\) and \(\omega \in \mathbb{Z}^{(m,n)}\), using Formulas (2.1), (2.11) or (3.6), we have
\[ \left[ \pi_M((0, s; 0)T_*) \pi_M((\lambda, \mu; \kappa)(\tau, \phi)) f \right](\omega) \]
\[ = \left[ \pi_M(0, s; 0) R_M(T_*) \pi_M((\lambda, \mu; \kappa)(\tau, \phi)) f \right](\omega) \]
\[ = e^{2\pi i \sigma(M\omega^i)} e^{2\|\omega\|_M^2 \pi i} \left[ R_M(i, 0) \pi_M((\lambda, \mu; \kappa)(\tau, \phi)) f \right](\omega) \]
\[ = \left[ \pi_M((\lambda, \mu; \kappa)(\tau, \phi)) f \right](\omega). \]

Here we used the facts that
\[ e^{2\pi i \sigma(M\omega^i)} = 1, \quad e^{2\|\omega\|_M^2 \pi i} = 1 \quad \text{and} \quad R_M(i, 0)f = f \quad (\text{cf. (3.7)}). \]
Therefore for \( f \in \mathcal{S}(\mathbb{R}^{(m,n)}) \),

\[
\Theta^M_f(\tau + 2, \phi; \lambda, s - 2 \lambda + \mu, \kappa - s^t \lambda)
= \sum_{\omega \in \mathbb{Z}^{(m,n)}} \left[ \pi_M((\lambda, s - 2 \lambda + \mu, \kappa - s^t \lambda)(\tau + 2, \phi)) f\right](\omega)
= \sum_{\omega \in \mathbb{Z}^{(m,n)}} \left[ \pi_M((0, s; 0) T_\ast \pi_M(\lambda, \mu; \kappa)(\tau, \phi)) f\right](\omega)
= \sum_{\omega \in \mathbb{Z}^{(m,n)}} \left[ \pi_M((\lambda, \mu; \kappa)(\tau, \phi)) f\right](\omega)
= \Theta^M_f(\tau, \phi; \lambda, \mu, \kappa).
\]

This completes the proof. \( \square \)

**Theorem 3.3 (Jacobi 3).** Let \( \mathcal{M} = (\mathcal{M}_{kl}) \) be a positive definite symmetric integral \( m \times m \) matrix and let \((\lambda_0, \mu_0; \kappa_0) \in \mathcal{H}_z^{(m,n)} \) be an integral element of \( \mathcal{H}_r^{(n,m)} \). Then we have

\[
\Theta^M_f(\tau, \phi; \lambda + \lambda_0, \mu + \mu_0, \kappa + \kappa_0 + \lambda_0^t \mu - \mu_0^t \lambda)
= e^{\pi i \sigma(\mathcal{M}(\kappa_0 + \mu_0^t \lambda_0))} \Theta^M_f(\tau, \phi; \lambda, \mu, \kappa)
\]

for all \((\tau, \phi) \in SL(2, \mathbb{R})\) and \((\lambda, \mu; \kappa) \in \mathcal{H}_r^{(n,m)}\).

*Proof.* For any \( f \in \mathcal{S}(\mathbb{R}^{(m,n)}) \), we have

\[
\sum_{\omega \in \mathbb{Z}^{(m,n)}} \left[ \mathcal{W}_M(\lambda_0, \mu_0; \kappa_0) \pi_M((\lambda, \mu; \kappa)(\tau, \phi)) f\right](\omega)
= \sum_{\omega \in \mathbb{Z}^{(m,n)}} \left[ \mathcal{W}_M(\lambda_0 + \lambda, \mu_0 + \mu; \kappa_0 + \kappa + \lambda_0^t \mu - \mu_0^t \lambda) \pi_M(\tau, \phi)\right](\omega)
= \sum_{\omega \in \mathbb{Z}^{(m,n)}} \left[ \pi_M((\lambda_0 + \lambda, \mu_0 + \mu; \kappa_0 + \kappa + \lambda_0^t \mu - \mu_0^t \lambda)(\tau, \phi)) f\right](\omega)
= \Theta^M_f(\tau, \phi; \lambda + \lambda_0, \mu + \mu_0, \kappa + \kappa_0 + \lambda_0^t \mu - \mu_0^t \lambda).
\]
On the other hand, for any \( f \in \mathcal{S}(\mathbb{R}^{(m,n)}) \), we have
\[
\sum_{\omega \in \mathbb{Z}^{(m,n)}} \left[ \mathcal{W}_M(\lambda_0, \mu_0; \kappa_0) \pi_M((\lambda, \mu; \kappa)(\tau, \phi)) f \right](\omega)
\]
\[
= \sum_{\omega \in \mathbb{Z}^{(m,n)}} e^{\pi i \sigma(M(\kappa_0 + \mu_0^2 \lambda_0 + 2 \omega^2 \mu_0))} \left[ \pi_M((\tau, \phi; \lambda, \mu, \kappa)f \right](\omega + \lambda_0)
\]
\[
= e^{\pi i \sigma(M(\kappa_0 + \mu_0^2 \lambda_0))} \sum_{\omega \in \mathbb{Z}^{(m,n)}} [\pi_M((\tau, \phi; \lambda, \mu, \kappa)f](\omega + \lambda_0) \quad (\because \mu_0 \text{ is integral})
\]
\[
= e^{\pi i \sigma(M(\kappa_0 + \mu_0^2 \lambda_0))} \sum_{\omega \in \mathbb{Z}^{(m,n)}} [\pi_M((\tau, \phi; \lambda, \mu, \kappa)f](\omega) \quad (\because \lambda_0 \text{ is integral})
\]
\[
= e^{\pi i \sigma(M(\kappa_0 + \mu_0^2 \lambda_0))} \Theta_f^{[M]}(\tau, \phi; \lambda, \mu, \kappa).
\]

Finally we obtain the desired result. \( \Box \)

We put \( V(m, n) = \mathbb{R}^{(m,n)} \times \mathbb{R}^{(m,n)} \). Let
\[
G^{(m,n)} := SL(2, \mathbb{R}) \ltimes V(m, n)
\]
be the group with the following multiplication law
\[
(3.15) \quad (g_1, (\lambda_1, \mu_1)) \cdot (g_2, (\lambda_2, \mu_2)) = (g_1 g_2, (\lambda_1, \mu_1) g_2 + (\lambda_2, \mu_2)),
\]
where \( g_1, g_2 \in SL(2, \mathbb{R}) \) and \( \lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}^{(m,n)} \).

We define
\[
\Gamma^{(m,n)} := SL(2, \mathbb{Z}) \ltimes H^Z_n.
\]

Then \( \Gamma^{(m,n)} \) acts on \( G^{(m,n)} \) naturally through the multiplication law (3.15).

**Lemma 3.3.** \( \Gamma^{(m,n)} \) is generated by the elements
\[
(S, (0, 0)), \quad (T_b, (0, s)) \quad \text{and} \quad (I_2, (\lambda_0, \mu_0)),
\]
where
\[
S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T_b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \ s, \lambda_0, \mu_0 \in \mathbb{Z}^{(m,n)}.
\]

**Proof.** Since \( SL(2, \mathbb{Z}) \) is generated by \( S \) and \( T_b \), we get the desired result. \( \Box \)

We define
\[
\Theta_f^{[M]}(\tau, \phi; \lambda, \mu) = \sum_{\omega \in \mathbb{Z}^{(m,n)}} e^{\pi i \left\{ u\|\omega + \lambda\|_M^2 + 2(\omega, \mu)\chi \right\}} [R_M(i, \phi)f \left( v^{1/2}(\omega + \lambda) \right).}
\]

**Theorem 3.4.** Let \( \Gamma^{(m,n)}_{[2]} \) be the subgroup of \( \Gamma^{(m,n)} \) generated by the elements
\[
(S, (0, 0)), \quad (T_s, (0, s)) \quad \text{and} \quad (I_2, (\lambda_0, \mu_0)),
\]
where
\[
T_s = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \ s, \lambda_0, \mu_0 \in \mathbb{Z}^{(m,n)}.
\]
Let $M = (M_{kl})$ be a positive definite symmetric unimodular integral $m \times m$ matrix such that $MZ^{(m,n)} = Z^{(m,n)}$. Then for $f, g \in \mathcal{S}(R^{(m,n)})$, the function

$$\Theta_f^{[M]}(\tau, \phi; \lambda, \mu) \Theta_g^{[M]}(\tau, \phi; \lambda, \mu)$$

is invariant under the action of $\Gamma^{(m,n)}_{[2]}$ on $G^{(m,n)}$.

**Proof.** The proof follows directly from Theorem 3.1 (Jacobi 1), Theorem 3.2 (Jacobi 2) and Theorem 3.3 (Jacobi 3) because the left actions of the generators of $\Gamma^{(m,n)}_{[2]}$ are given by

$$(((\tau, \phi), (\lambda, \mu)) \mapsto \left(\left(-\frac{1}{\tau}, \phi + \arg \tau\right), (-\mu, \lambda)\right),$$

$$(((\tau, \phi), (\lambda, \mu)) \mapsto ((\tau + 2, \phi), (\lambda, s - 2\lambda + \mu))$$

and

$$(((\tau, \phi), (\lambda, \mu)) \mapsto ((\tau, \phi), (\lambda + \lambda_0, \mu + \mu_0)).$$

$\square$

**References**

[1] R. Berndt and R. Schmidt, *Elements of the Representation Theory of the Jacobi Group*, Birkhäuser, 1998.
[2] M. Eichler and D. Zagier, *The Theory of Jacobi Forms*, Progress in Math., 55, Birkhäuser, Boston, Basel and Stuttgart, 1985.
[3] E. Freitag, *Siegsche Modulfunktionen*, Grundlehren de mathematischen Wissenschaften 55, Springer-Verlag, Berlin-Heidelberg-New York (1983).
[4] M. Itoh, H. Ochiai and J.-H. Yang, *Invariant Differential Operators on the Siegel-Jacobi Space*, submitted (2015).
[5] G. Lion and M. Vergne, *The Weil representation, Maslov index and Theta series*, Progress in Math., 6, Birkhäuser, Boston, Basel and Stuttgart, 1980.
[6] J. Marklof, *Pair correlation densities of inhomogeneous quadratic forms*, Ann. of Math., 158 (2003), 419-471.
[7] D. Mumford, *Tata Lectures on Theta I*, Progress in Math. 28, Boston-Basel-Stuttgart (1983).
[8] A. Pitale, *Jacobi Maass forms*, Abh. Math. Sem. Hamburg 79 (2009), 87–111.
[9] C. L. Siegel, *Indefinite quadratische Formen und Funktionentheorie I und II*, Math. Ann. 124 (1951), 17–54 and Math. Ann. 124 (1952), 364–387; Gesammelte Abhandlungen, Band III, Springer-Verlag (1966), 105–142 and 154–177.
[10] A. Weil, *Sur certains groupes d’operateurs unitaires*, Acta Math., 111 (1964), 143–211; Collected Papers (1964-1978), Vol. III, Springer-Verlag (1979), 1–69.
[11] J.-H. Yang, *Harmonic Analysis on the Quotient Spaces of Heisenberg Groups*, Nagoya Math. J., 123 (1991), 103–117.
[12] J.-H. Yang, *Harmonic Analysis on the Quotient Spaces of Heisenberg Groups II*, J. Number Theory, 49 (1) (1994), 63–72.
[13] J.-H. Yang, *A decomposition theorem on differential polynomials of theta functions of high level*, Japanese J. of Mathematics, the Mathematical Society of Japan, New Series, 22 (1) (1996), 37–49.
[14] J.-H. Yang, *Fock Representations of the Heisenberg Group $H^{(g,h)}_\mathbb{R}$*, J. Korean Math. Soc., 34, no. 2 (1997), 345–370.
[15] J.-H. Yang, *Lattice Representations of the Heisenberg Group $H^{(g,h)}_\mathbb{R}$*, Math. Annalen, 317 (2000), 309–323.
[16] J.-H. Yang, *Heisenberg Group, Theta Functions and the Weil Representation*, Kyung Moon Sa, Seoul (2012).
[17] J.-H. Yang, *The Siegel-Jacobi Operator*, Abh. Math. Sem. Univ. Hamburg 63 (1993), 135–146.
[18] J.-H. Yang, Remarks on Jacobi forms of higher degree, Proc. of the 1993 Workshop on Automorphic Forms and Related Topics, the Pyungsan Institute for Mathematical Sciences, Seoul (1993), 33–58.
[19] J.-H. Yang, Singular Jacobi forms, Trans. of American Math. Soc. 347, No. 6 (1995), 2041-2049.
[20] J.-H. Yang, Construction of vector valued modular forms from Jacobi forms, Canadian J. of Math. 47 (6) (1995), 1329-1339.
[21] J.-H. Yang, A note on a fundamental domain for Siegel-Jacobi space, Houston Journal of Mathematics, Vol. 32, No. 3 (2006), 701–712.
[22] J.-H. Yang, Invariant metrics and Laplacians on Siegel-Jacobi space, Journal of Number Theory, 127 (2007), 83–102.
[23] J.-H. Yang, A partial Cayley transform of Siegel-Jacobi disk, J. Korean Math. Soc. 45, No. 3 (2008), 781-794.
[24] J.-H. Yang, Invariant metrics and Laplacians on Siegel-Jacobi disk, Chinese Annals of Mathematics, Vol. 31 B(1) (2010), 85-100.
[25] J.-H. Yang, A Note on Maass-Jacobi Forms II, Kyungpook Math. J. 53 (2013), 49-86.
[26] J.-H. Yang, Y.-H. Yong, S.-N. Huh, J.-H. Shin and G.-H. Min, Sectional Curvatures of the Siegel-Jacobi Space, Bull. Korean Math. Soc. 50 (2013), No. 3, pp. 787-799.
[27] J.-H. Yang, Geometry and Arithmetic on the Siegel-Jacobi Space, Geometry and Analysis on Manifolds, In Memory of Professor Shoshichi Kobayashi (edited by T. Ochiai, A. Weinstein et al), Progress in Mathematics, Volume 308, Birkhäuser, Springer International Publishing AG Switzerland (2015), 275-325.
[28] J.-H. Yang, Covariant maps for the Schroedinger-Weil representation, Bull. Korean Math. Soc. 52 (2015), No. 2, pp. 627-647.
[29] C. Ziegler, Jacobi Forms of Higher Degree, Abh. Math. Sem. Hamburg 59 (1989), 191–224.

Department of Mathematics, Inha University, Incheon 22212, Korea
E-mail address: jhyang@inha.ac.kr