SEARCHING FOR FRACTAL STRUCTURES IN THE
UNIVERSAL STEENROD ALGEBRA AT ODD PRIMES

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Abstract. Unlike the \( p = 2 \) case, the universal Steenrod Algebra \( \mathbb{Q}(p) \) at odd primes does not have a fractal structure that preserves the length of monomials. Nevertheless, when \( p \) is odd we detect inside \( \mathbb{Q}(p) \) two different families of nested subalgebras each isomorphic (as length-graded algebras) to the respective starting element of the sequence.

1. Introduction

Let \( p \) be any prime. The so-called universal Steenrod algebra \( \mathbb{Q}(p) \) is an \( \mathbb{F}_p \)-algebra extensively studied by the authors (see, for instance, \([2] - [12]\)). On its first appearance, it has been described as the algebra of cohomology operations in the category of \( H_\infty \)-ring spectra (see \([16]\)). Invariant-theoretic descriptions of \( \mathbb{Q}(p) \) can be found in \([11]\) and \([15]\). When \( p \) is an odd prime, the augmentation ideal of \( \mathbb{Q}(p) \)

\[
S_p = \{ z_{\epsilon,k} \mid (\epsilon, k) \in \{0, 1\} \times \mathbb{Z} \}
\]

subject to the set of relations

\[
R_p = \{ R(\epsilon, k, n), S(\epsilon, k, n) \mid (\epsilon, k, n) \in \{0, 1\} \times \mathbb{Z} \times \mathbb{N}_0 \},
\]

where

\[
R(\epsilon, k, n) = z_{\epsilon, pk-1-n} z_{0,k} + \sum_j (-1)^j \binom{(p-1)(n-j)-1}{j} z_{\epsilon, pk-1-j} z_{0,k-n+j},
\]

and

\[
S(\epsilon, k, n) = z_{\epsilon, pk-n} z_{1,k} + \sum_j (-1)^j \binom{(p-1)(n-j)-1}{j} z_{\epsilon, pk-j} z_{1,k-n+j}
\]

\[
+ (1-\epsilon) \sum_j (-1)^j \binom{(p-1)(n-j)}{j} z_{1,pk-j} z_{0,k-n+j}.
\]

Such relations are known as generalized Adem relations.

The algebra \( \mathbb{Q}(p) \) is related to many Steenrod-like operations. For instance to those acting on the cohomology of a graded cocommutative Hopf algebra \((6, 13)\), or the Dyer-Lashof operations on the homology of infinite loop spaces \((11\) and \(17)\). Details of such connections, at least for \( p = 2 \), can be found in \([5]\). In particular, the

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ordinary Steenrod algebra \( A(p) \) is a quotient of \( Q(p) \). At odd primes, the algebra epimorphism is determined by

\[
\zeta : z_{\epsilon,k} \mapsto \begin{cases} 
\beta^s P^k & \text{if } k \geq 0, \\
0 & \text{otherwise.}
\end{cases}
\]

The kernel of the map \( \zeta \) turns out to be the principal ideal generated by \( z_{0,0} - 1 \).

All monic monomials in \( Q(p) \), with the exception of \( z_0 = 1 \) have the form

\[
z_I = z_{\epsilon_1,i_1} z_{\epsilon_2,i_2} \cdots z_{\epsilon_m,i_m},
\]

where the string \( I = (\epsilon_1,i_1; \epsilon_2,i_2; \ldots; \epsilon_m,i_m) \) is the label of the monomial \( z_I \). By length of a monomial \( z_I \) of type (1.6) we just mean the integer \( m \), while the length of any \( p \in \mathbb{F}_p \subset Q(p) \) is defined to be 0. Since Relations (1.3) and (1.4) are homogeneous with respect to length, the algebra \( Q(p) \) can be regarded as a graded object.

A monomial and its label are said to be admissible if \( i_j \geq p i_{j+1} + \epsilon_{j+1} \) for any \( j = 1,\ldots,m-1 \). We also consider \( z_0 = 1 \in \mathbb{F}_p \subset Q(p) \) admissible. The set \( \mathcal{B} \) of all monic admissible monomials forms an \( \mathbb{F}_p \)-linear basis for \( Q(p) \) (see (11)).

Through two different approaches, in [8] and [10] it has been shown that \( Q(2) \) has a fractal structure given by a sequence of nested subalgebras \( Q_s \), each isomorphic to \( Q \). The interest in searching for fractal structures inside algebras of (co-)homology operations initially arose in [13], where such structures were used as a tool to establish the nilpotence height of some elements in \( A(p) \). Results in the same vein are in [13].

Recently, in [7] the authors proved that no length-preserving strict monomorphisms turn out to exist in \( Q(p) \) when \( p \) is odd. Hence no descending chain of isomorphic subalgebras starting with \( Q(p) \) exists for \( p > 2 \). Results in [7] did not exclude the existence of fractal structures for proper subalgebras of \( Q(p) \). As a matter of fact, the subalgebras \( Q^0 \) and \( Q^1 \) generated by the \( z_{0,h} \)'s and the \( z_{1,k} \)'s respectively (together with 1) turn out to have self-similar shapes, as stated in our Theorem 1.1.

**Theorem 1.1.** Let \( p \) be any odd prime. For any \( \epsilon \in \{0,1\} \) there is a chain of nested subalgebras of \( Q(p) \)

\[
Q_0^\epsilon \supset Q_1^\epsilon \supset Q_2^\epsilon \supset \cdots \supset Q_s^\epsilon \supset Q_{s+1}^\epsilon \supset \cdots
\]

each isomorphic to \( Q_0^\epsilon = Q^\epsilon \) as length-graded algebras.

Theorem 1.1 relies on the existence of two suitable algebra monomorphisms

\[
\phi : Q^0 \longrightarrow Q^0 \quad \text{and} \quad \psi : Q^1 \longrightarrow Q^1.
\]

Indeed, we just set \( Q^\epsilon = \phi^s(Q^0) \) and \( Q_s^\epsilon = \phi^s(Q^1) \), the restrictions \( \phi |_{Q^\epsilon} \) and \( \psi |_{Q_s^\epsilon} \) being the desired isomorphism between \( Q_s^\epsilon \) and \( Q_{s+1}^\epsilon \) (\( \epsilon \in \{0,1\} \)).

For sake of completeness we point out that the algebra \( Q(p) \) can also be filtered by the internal degree of its elements, defined on monomials as follows:

\[
|z_I| = \sum h(2i_h(p-1) + \epsilon_{i_h}), \quad \text{if } I = (\epsilon_1,i_1; \epsilon_2,i_2; \ldots; \epsilon_m,i_m)
\]

\[
0 \quad \text{if } I = \emptyset.
\]

In spite of its geometric importance, the internal degree will not play any role here.
2. A FIRST DESCENDING CHAIN OF SUBALGEBRAS

We first need to establish some congruential identities. Let \( \mathbb{N}_0 \) denote the set of all non-negative integers. Fixed any prime \( p \), we write
\[
(2.1) \quad \sum_{i \geq 0} \gamma_i(m) p^i \quad (0 \leq \gamma_i(m) < p)
\]
to denote the \( p \)-adic expansion of a fixed \( m \in \mathbb{N}_0 \). The following well-known Lemma is a standard device to compute mod \( p \) binomial coefficients.

**Lemma 2.1 (Lucas’ Theorem).** For any \( (a, b) \in \mathbb{N}_0 \times \mathbb{N}_0 \), the following congruential identity holds.
\[
(2.2) \quad \binom{a}{b} \equiv \prod_{i \geq 0} \binom{\gamma_i(a)}{\gamma_i(b)} \pmod{p}.
\]

**Proof.** See [13, p. 260] or [19, I 2.6]. Equation (2.2) follows the usual conventions: \( \binom{0}{0} = 1 \), and \( \binom{i}{0} = 0 \) if \( 0 \leq l < r \).

Congruence (2.2) immediately yields
\[
(2.3) \quad \binom{p^r a}{p^r b} \equiv \frac{a}{b} \pmod{p} \quad \text{for every } r \geq 0,
\]
since, in both cases, we find on the right side of (2.2) the same products of binomial coefficients, apart from \( r \) extra factors all equal to \( \binom{0}{0} = 1 \).

**Corollary 2.2.** For any \( (\ell, t, h) \in \mathbb{N}_0 \times \mathbb{N}_0 \times \{1, \ldots, p\} \), the following congruential identity holds.
\[
(2.4) \quad \binom{p^t \ell - h}{pt} \equiv \binom{\ell - 1}{t} \pmod{p}.
\]

**Proof.** Since \( p^t \ell - h = (p - h) + p(\ell - 1) \), we have \( \gamma_0(p^t \ell - h) = p - h \). Note also that \( \gamma_0(pt) = 0 \). According to Lemma 2.1, we get
\[
(2.5) \quad \binom{p^t \ell - h}{pt} \equiv \binom{p - h}{0} \frac{\rho(\ell - 1)}{pt} \pmod{p}.
\]
We now use Congruence 2.3 for \( r = 1 \), and the fact that \( \binom{k}{0} = 1 \) for all \( k \in \mathbb{N}_0 \).

In order to make notation less cumbersome, we set
\[
(2.6) \quad A(k, j) = \binom{(p - 1)(k - j) - 1}{j}.
\]

**Corollary 2.3.** Let \((n, j)\) a couple of positive integers. Whenever \( j \not\equiv 0 \pmod{p} \), the binomial coefficient \( A(pn, j) \) is divisible by \( p \).

**Proof.** If a fixed positive integer \( j \) is not divisible by \( p \), then there exists a unique couple \((l, h) \in \mathbb{N} \times \{1, \ldots, p - 1\}\) such that \( j = pl - h \). Hence, setting
\[
T = (p - 1)(n - l) + h - 1,
\]
we get
\[
(2.7) \quad A(pn, j) = \binom{(p - h - 1) + pT}{(p - h) + p(l - 1)} \equiv \binom{p - h - 1}{p - h} \frac{T}{\ell - 1} \pmod{p}
\]
by Lemma 2.1 and Equation 2.3. Since \( p - h - 1 < p - h \), the first factor on the right side of Equation (2.7) is zero, so the result follows. \(\square\)
Lemma 2.4. Let \((s,n,j)\) a triple of positive integers. Whenever \(j \not\equiv 0 \pmod{p^s}\), the binomial coefficient \(A(p^s n,j)\) is divisible by \(p\).

Proof. We proceed by induction on \(s\). The \(s = 1\) case is essentially Corollary 2.3. Suppose now \(s > 1\). The hypothesis on \(j\) is equivalent to the existence of a suitable \((b,i) \in \mathbb{N} \times \{1, \ldots, p^s - 1\}\) such that \(j = p^s b - i\). Likewise, we can write \(i = pl - r\), for a certain \((l,r) \in \{1, \ldots, p^{s-2}\} \times \{0, \ldots, p-1\}\).

We now distinguish two cases. If \(r = 0\), the binomial coefficient \(A(p^s n,j)\) has the form \(\binom{n-l}{p^s p^l r}\) where

\[
\ell = (p-1)(p^{s-1}n - p^{s-1}b) + h, \quad h = 1, \quad \text{and} \quad t = p^{s-1}b - l.
\]

By Corollary 2.2 we get

\[
A(p^s n,j) \equiv A(p^{s-1} n, p^{s-1} b - l) \pmod{p},
\]

and the latter is divisible by \(p\) by the inductive hypothesis.

Assume now \(1 \leq r \leq p-1\). In this case,

\[
A(p^s n,j) = \binom{r-1 + pT'}{r + p(p^{s-1} b - l)}
\]

where \(T' = (p-1)(p^{s-1} n - p^{s-1} b + l) - r\). Therefore, by Lemma 2.4 we get

\[
A(p^s n,j) \equiv \binom{r-1}{r} \cdot \binom{T'}{p^{s-1} b - l} \pmod{p}.
\]

The right side of Equation 2.9 vanishes, since \(r-1 < r\), and the proof is over. \(\square\)

Lemmas and Corollaries proved so far will be helpful to reduce, in some particular cases, the number of potentially non-zero binomial coefficients in (1.3) and in (1.4). For instance, for any \((h,n) \in \mathbb{Z} \times \mathbb{N}_0\), relations of type \(R(\epsilon, p^s h - \alpha_s, p^s n)\), where

\[
\alpha_s = \frac{p^s - 1}{p-1} \quad (s \geq 1),
\]

only involve generators in the set

\[
T_{(\epsilon,s)} = \{z_{\epsilon, p^s m - \alpha_s} | m \in \mathbb{Z}\}
\]

as stated in the following Proposition.

Proposition 2.5. Let \((\epsilon, k, n, s)\) a fixed 4-tuple in \(\{0,1\} \times \mathbb{Z} \times \mathbb{N}_0 \times \mathbb{N}\). The polynomial \(R(\epsilon, p^s k - \alpha_s, p^s n)\) in (1.3) is actually equal to

\[
z_{\epsilon, p^s (pk-1-n) - \alpha_s} \cdot z_{0, p^s k - \alpha_s} \cdot \sum_j (-1)^j A(n,j) z_{\epsilon, p^s (pk-1-j) - \alpha_s} \cdot z_{0, p^s (k-n+j) - \alpha_s}.
\]

Proof. By definition (see (1.3)), \(R(\epsilon, p^s k - \alpha_s, p^s n)\) is equal to

\[
z_{\epsilon, p^s (pk-\alpha_s) - 1 - p^s n} \cdot z_{0, p^s k - \alpha_s} \cdot \sum_l (-1)^l A(p^s n, l) z_{\epsilon, p^s (pk-\alpha_s) - 1 - l} \cdot z_{0, p^s k - \alpha_s} \cdot \cdot p^s n + l}.
\]

According to Lemma 2.4, the only possible non-zero coefficients in the sum above occur when \(l \equiv 0 \pmod{p^s}\). Thus, we set \(l = p^s j\) and write \(R(\epsilon, p^s k - \alpha_s, p^s n)\) as

\[
z_{\epsilon, p^s (pk-\alpha_s) - 1 - p^s n} \cdot z_{0, p^s k - \alpha_s} \cdot \sum_j (-1)^{p^s j} A(p^s n, p^s j) z_{\epsilon, p^s (pk-\alpha_s) - 1 - p^s j} \cdot z_{0, p^s k - \alpha_s} \cdot \cdot p^s n + p^s j}.
\]

In such polynomial we can replace \(z_{\epsilon, p^s (pk-\alpha_s) - 1 - p^s n}\) and \(z_{\epsilon, p^s (pk-\alpha_s) - 1 - p^s j}\) by

\[
z_{\epsilon, p^s (pk-1-n) - \alpha_s} \quad \text{and} \quad z_{\epsilon, p^s (pk-1-j) - \alpha_s}.
\]
respectively, since \( p\alpha_s + 1 = p^s + \alpha_s \). Finally, applying Equation (2.3) as many times as necessary, and recalling that we are supposing \( p \) odd, we get
\[
(2.11) \quad (-1)^{p^s \cdot j} A(p^s n, p^s j) \equiv (-1)^j A(n, j) \pmod{p}.
\]

As a consequence of Proposition 2.6, the admissible expression of any non-admissible monomial with label \((\epsilon, p^s h_1 - \alpha_s; 0, p^s h_2 - \alpha_s; \ldots; 0, p^s h_m - \alpha_s)\) involves only generators in \( T_{(s, \alpha)} \).

That’s the reason why, for any non-negative integer \( s \), there is a well-defined \( \mathbb{F}_p \)-algebra \( Q_0^s \) generated by the set \( \{1\} \cup T_{(0, s)} \) and subject to relations
\[
R(0, p^s h - \alpha_s, p^s n) = 0 \quad \forall n \in \mathbb{N}_0.
\]

Thus \( Q_0^0 \) and \( Q_0^1 \) are the subalgebras of \( Q(p) \) generated by the sets
\[
\{1\} \cup \{z_{0, h} \mid h \in \mathbb{Z}\} \quad \text{and} \quad \{1\} \cup \{z_{0, ph-1} \mid h \in \mathbb{Z}\}
\]
respectively. The former has been simply denoted by \( Q^0 \) in Section 1. The arithmetic identity
\[
(2.12) \quad p^{s+1} h - \alpha_{s+1} = p^s (ph - 1) - \alpha_s,
\]
implies that \( Q_s^0 \subset Q_{s+1}^0 \).

**Lemma 2.6.** A monomial of type
\[
(2.13) \quad z_I = z_{c, p^s h_1 - \alpha_s} z_{0, p^s h_2 - \alpha_s} \cdots z_{0, p^s h_m - \alpha_s}
\]
is admissible if and only if \( h_i \geq ph_i + 1 \) for any \( i = 1, \ldots, m - 1 \).

**Proof.** Admissibility for a monomial of type (2.13) is tantamount to the condition
\[
p^s h_i - \alpha_s \geq p(p^s h_{i+1} - \alpha_s) \quad \forall i \in \{1, \ldots, m - 1\}.
\]
Inequalities above are equivalent to
\[
h_i \geq ph_{i+1} - \frac{p^s - 1}{p^s} \quad \forall i \in \{1, \ldots, m - 1\},
\]
and the ceiling of the real number on the right side is precisely \( ph_{i+1} \). \hfill \Box

**Proposition 2.7.** An \( \mathbb{F}_p \)-linear basis for \( Q_s^0 \) is given by the set \( B_{Q_s^0} \) of its monic admissible monomials.

**Proof.** In [11] it is explained the procedure to express any monomial in \( Q(p) \) as a sum of admissible monomials. As Proposition 2.6 shows, the generalized Adem relations required to complete such procedure starting from a monomial in \( Q_0^s \) only involve generators actually available in the set at hands. \hfill \Box

So far, we have established the existence of the following descending chain of algebra inclusions:
\[
Q_0^0 = Q_0^1 \supset Q_1^0 \supset Q_2^0 \supset \cdots \supset Q_s^0 \supset Q_{s+1}^0 \supset \cdots,
\]

On the free \( \mathbb{F}_p \)-algebra \( \mathbb{F}_p \{1\} \cup T_{(0, 0)} \) we now define a monomorphism \( \Phi \) acting on the generators as follows
\[
(2.14) \quad \Phi(1) = 1 \quad \text{and} \quad \Phi(z_{0, k}) = z_{0, pk-1}.
\]
We set \( \Phi^0 = 1_{\mathbb{F}_p (g_s)} \) and \( \Phi^s = \Phi \circ \Phi^{s-1} \) for \( s \geq 1 \).
Proposition 2.8. For each \( s \geq 0 \),
\[
\Phi^s(z_{0,i_1} \cdots z_{0,i_m}) = z_{0,p^s \alpha_1 \cdots \alpha_s} z_{0,p^s \alpha_m \cdots \alpha_s},
\]
and
\[
\Phi^s(R(0,k,n)) = R(0,p^s k - \alpha s, p^s n).
\]

Proof. Equations (2.15) and (2.16) are trivially true for \( s = 0 \). For \( s \geq 1 \) use an inductive argument taking into account (2.12) and Proposition 2.5. □

Proposition 2.9. Let \( \pi : F_p \langle \{1\} \cup T(0,0) \rangle \to Q^0 \) be the quotient map. There exists an algebra monomorphism \( \phi \) such that the diagram

\[
\begin{array}{ccc}
F_p \langle \{1\} \cup T(0,0) \rangle & \xrightarrow{\Phi} & F_p \langle \{1\} \cup T(0,0) \rangle \\
\pi & \downarrow & \pi \\
Q^0 & \xrightarrow{\phi} & Q^0
\end{array}
\]

commutes.

Proof. By Equation (2.16), it follows in particular that
\[
\Phi(R(0,k,n)) = R(0,pk - 1, pn).
\]

Therefore there exists a well-defined algebra map
\[
\phi : z_{0,i_1} z_{0,i_2} \cdots z_{0,i_m} \in Q^0 \mapsto z_{0,p_{i_1-1}} z_{0,p_{i_2-1}} \cdots z_{0,p_{i_m-1}} \in Q^0.
\]

Such map is injective since the set \( B_{Q^0} \) – an \( F_p \)-linear basis for \( Q^0 \) according to Proposition 2.7 – is mapped onto admissibles by Lemma 2.6. □

Corollary 2.10. The algebra \( Q_s^0 \) is isomorphic to its subalgebra \( Q_{s+1}^0 \).

Proof. By Propositions 2.8 and 2.9 we can argue that \( \phi^s(Q^0) = Q_s^0 \). Hence the map
\[
\phi|_{Q_s^0} : \text{Im } \phi^s \to \text{Im } \phi^{s+1}
\]
gives the desired isomorphism. □

Corollary 2.10 proves Theorem 1.1 for \( \epsilon = 0 \).

3. A SECOND DESCENDING CHAIN OF SUBALGEBRAS

The aim of this Section is to provide a proof for the \( \epsilon = 1 \) case of Theorem 1.1. We choose to follow as close as possible the line of attack put forward in Section 2.

Proposition 3.1. Let \((k,n,s)\) a fixed triple in \( \mathbb{Z} \times \mathbb{N}_0 \times \mathbb{N} \). In (1.4) the polynomial
\[
S(1,p^s k, p^s n)
\]

is actually equal to
\[
z_1 p^s (pk - n) z_1 p^s + \sum_j (-1)^{j+1} A(n,j) z_1 p^s (pk - j) z_1 p^s (k-n+j).
\]

Proof. By definition (see (1.4),
\[
S(1,p^s k, p^s n) = z_1 p^s (pk - n) z_1 p^s + \sum_l (-1)^{l+1} A(p^s n, l) z_1 p^{s+1} k - l z_1 p^s (k-l) p^s n + l.
\]
According to Lemma 2.4, the only possible non-zero coefficients in the sum above are those with \( l \equiv 0 \mod p^s \). Setting \( l = p^s j \), the polynomial (3.1) becomes
\[
z_{1,p^s+1} z_{1,p^s j} + \sum_j (-1)^{p^s j+1} A(p^s n, p^s j) z_{1,p^s+1} z_{1,p^s j} z_{1,p^s n+1}.
\]
The result now follows from Equation (2.11).

Proposition 3.1 implies that relations of type \( S(1,p^s h, p^s n) \) only involve generators of type \( z_{1,p^s m} \), therefore the admissible expression of any non-admissible monomial with label \( (1,p^s h_1; 1,p^s h_2; \ldots; 1,p^s h_m) \) only involves generators in the set
\[
\mathcal{T}_{(1,s)} = \{ z_{1,p^s m} \mid m \in \mathbb{Z} \}.
\]
So it makes sense to define \( \mathcal{Q}_s^1 \) as the \( \mathbb{F}_p \)-algebra generated by the set \( \{ 1 \} \cup \mathcal{T}_{(1,s)} \) and subject to relations
\[
S(1,p^s h, p^s n) = 0 \quad \forall n \in \mathbb{N}_0.
\]
Each \( \mathcal{Q}_s^1 \) is actually a subalgebra of \( \mathcal{Q}(p) \). We have inclusions \( \mathcal{Q}_s^1 \supset \mathcal{Q}_{s+1}^1 \). In Section 1, the algebra \( \mathcal{Q}_0^1 \) has been simply denoted by \( \mathcal{Q}^1 \).

**Lemma 3.2.** A monomial of type
\[
z_{1,p^s h_1} z_{1,p^s h_2} \cdots z_{1,p^s h_m}
\]
in \( \mathcal{Q}_s^1 \subset \mathcal{Q}(p) \) is admissible if and only if \( h_i \geq ph_{i+1} + 1 \quad \forall i \in \{1, \ldots, m-1\} \).

**Proof.** By definition, the monomial (3.3) is admissible if and only if
\[
p^s h_i \geq p(p^s h_{i+1}) + 1 \quad \forall i \in \{1, \ldots, m-1\}.
\]
Inequalities above are equivalent to
\[
h_i \geq ph_{i+1} + \frac{1}{p^s} \quad \forall i \in \{1, \ldots, m-1\},
\]
and the ceiling of the real number on the right side is precisely \( ph_{i+1} + 1 \).

**Proposition 3.3.** An \( \mathbb{F}_p \)-linear basis for \( \mathcal{Q}_s^1 \) is given by the set \( \mathcal{B}_{\mathcal{Q}_s^1} \) of its monic admissible monomials.

**Proof.** Follows verbatim the proof of Proposition 2.7, just replacing “Proposition 2.5” by “Proposition 3.1” and \( \mathcal{Q}_0^1 \) by \( \mathcal{Q}_s^1 \).
Proposition 3.4. Let \( \pi' : \mathbb{F}_p \{ \{1\} \cup T'_{(1,0)} \} \to Q^1 \) be the quotient map. There exists an algebra monomorphism \( \psi \) such that the diagram

\[
\begin{array}{ccc}
\mathbb{F}_p \{ \{1\} \cup T'_{(1,0)} \} & \xrightarrow{\psi} & \mathbb{F}_p \{ \{1\} \cup T'_{(1,0)} \} \\
\pi' \downarrow & & \downarrow \pi \\
Q^1 & \xrightarrow{\psi} & Q^1
\end{array}
\]

(3.5)

commutes.

Proof. Since \( \Psi^*(z_{1,t_1} \cdots z_{1,t_m}) = z_{1,p^s \alpha_1} \cdots z_{1,p^s \alpha_m} \), by Proposition 3.1 we argue that

\( \Psi^*(S(1,k,n)) = S(1,p^s k, p^s n) \).

Therefore there exists a well-defined algebra map

\( \psi : z_{1,1} \cdots z_{1,n} \in Q^1 \mapsto z_{1,pi_1} \cdots z_{1,pi_n} \in Q^1. \)

Such map is injective since the set \( B_{Q^1} \) – an \( \mathbb{F}_p \)-linear basis for \( Q^1 \) according to Proposition 3.3 – is mapped onto admissibles by Lemma 3.2.

\[\square\]

Corollary 3.5. The algebra \( Q^1_s \) is isomorphic to its subalgebra \( Q^1_{s+1} \).

Proof. By Equation (3.6) and Proposition 3.4 we can argue that \( \psi^*(Q^1) = Q^1_s \).

Thus, the desired isomorphism is given by

\( \psi|_{Q^1} : \text{Im} \psi^* \mapsto \text{Im} \psi^{s+1}. \)

\[\square\]

4. Further Substructures

For each \( s \in \mathbb{N}_0 \), we define \( V_s \) to be the \( \mathbb{F}_p \)-vector subspace of \( Q(p) \) generated by the set of monomials

\( U_s = \{ z_{1,p^s \alpha_1} \cdots z_{1,p^s \alpha_m} | m \geq 2, (h_1, \ldots, h_m) \in \mathbb{Z}^m \}. \)

Equation 2.12 implies that \( V_s \supset V_{s+1} \). None of the \( V_s \)'s is a subalgebra of \( Q(p) \), nevertheless, by Proposition 2.5 and the nature of relations (1.3) it follows that \( V_s \) can be endowed with a right \( Q^0_s \)-module structure just by considering multiplication in \( Q(p) \). By using once again Lemma 2.6 and the argument along the proof of Proposition 2.7, we get

Proposition 4.1. An \( \mathbb{F}_p \)-linear basis for \( V_s \) is given by the set \( B_{V_s} \) of its monic admissible monomials.

Proposition 4.2. The map between sets

\[
z_{1,i_1} z_{0,i_2} \cdots z_{0,i_m} \in U_o \quad \mapsto \quad z_{1,pi_1} z_{0,pi_2} \cdots z_{0,pi_m} \in U_o
\]

can be extended to a well-defined injective \( \mathbb{F}_p \)-linear map \( \lambda : \text{Im} \to \text{Im} \).

Moreover

\[
\lambda^s(V_0) = V_s \subset V_0.
\]

Proof. As in the proof of Proposition 2.7, Equation 2.12 and Proposition 2.5 show that the polynomial \( R(c,k,n) \in \mathbb{F}_p(S_p) \) in mapped onto \( R(c,p^s k - \alpha_s, p^s n) \) through the \( s \)-th power of the \( \mathbb{F}_p \)-linear map

\[
\lambda : z_{e_1,i_1} z_{e_2,i_2} \cdots z_{e_m,i_m} \in \mathbb{F}_p(S_p) \mapsto z_{e_1,pi_1} z_{e_2,pi_2} \cdots z_{e_m,pi_m} \in \mathbb{F}_p(S_p).
\]
Hence there are two maps $\tilde{\Lambda}$ and $\lambda$ such that the diagram

$$
\begin{array}{ccc}
\mathbb{F}_p(S_p) & \xrightarrow{\Lambda} & \mathbb{F}_p(S_p) \\
\uparrow & & \uparrow \\
\mathbb{F}_p(U_0) & \xrightarrow{\tilde{\Lambda}} & \mathbb{F}_p(U_0) \\
\downarrow \pi'' & & \downarrow \pi'' \\
V_0 & \xrightarrow{\lambda} & V_0
\end{array}
$$

(4.3)

commutes, where $\pi'' : \mathbb{F}_p(U_0) \to V_0$ is the quotient map. Finally, taking into account Equation 2.12, one checks that

$$
\lambda^s(\tilde{z}_1, i_1, \tilde{z}_0, i_2, \ldots, \tilde{z}_0, i_m) = z_1, p^s, i_1 - \alpha, z_0, p^s, i_2 - \alpha, \ldots, z_0, p^s, i_m - \alpha.
$$

Since Equation (4.4) implies (4.1), the proof is over. \(\square\)

We now introduce a category $\mathcal{K}$ whose objects are couples $(M, R)$, with $R$ being any ring, and $M$ any right $R$-module. A morphism between two objects $(M, R)$ and $(N, S)$ is given by a couple $(f, \omega)$ where $f : M \to N$ is group homomorphism and $\omega : R \to S$ is a ring homomorphism, furthermore

$$
f(mr) = f(m) \omega(r) \quad \forall (m, r) \in (M, R).
$$

The category $\mathcal{K}$ is partially ordered by “inclusions”. More precisely we say that $(M, R) \subseteq (M', R')$ if $M$ is a subgroup of $M'$ and $R$ is a subring of $R'$.

**Theorem 4.3.** The objects in $\mathcal{K}$ of the descending chain

$$(V_0, Q_0^0) \triangleright (V_1, Q_1^0) \triangleright \cdots \triangleright (V_s, Q_s^0) \triangleright (V_{s+1}, Q_{s+1}^0) \triangleright \cdots
$$

are all isomorphic.

**Proof.** By Proposition 1.2 it follows that $\lambda|_{V_s} : V_s \to V_{s+1}$ is an isomorphism between $\mathbb{F}_p$-vector spaces. Thus, recalling Corollary 2.10 the desired isomorphism in $\mathcal{K}$ is given by

$$(\lambda|_{V_s}, \phi|_{Q_s^0}) : (V_s, Q_s^0) \to (V_{s+1}, Q_{s+1}^0).
$$

5. A final remark

Theorem 1.1 in \cite{7} says that no strict algebra monomorphism in $\mathbb{Q}(p)$ exists when $p$ is odd. Hence there is no chance to find algebra endomorphisms over $\mathbb{Q}(p)$ extending the maps $\phi$ and $\psi$ defined in Sections 2 and 3 respectively. Just to give an idea about the obstructions you come up with, consider the $\mathbb{F}_p$-linear map

$$
\Theta : \mathbb{F}_p(S_p) \to \mathbb{F}_p(S_p)
$$

defined on monomials as follows

$$
\Theta(z_{e_1, i_1} z_{e_m, i_m}) = z_{e_1, p i_1} \cdots z_{e_m, p i_m}.
$$

Neither the map $\Theta$ nor the map $\Lambda$ introduced in Section 4 stabilizes the entire set (1.2). Indeed, take for instance

$$
R(0, 0, 0) = z_0_{-1} z_{0,0} \quad \text{and} \quad S(1, 0, 0) = z_{1,0} z_{1,0}.
$$
The polynomial
\[(5.1) \Theta(R(0,0,0)) = z_{0,-p}z_{0,0}\]
does not belong to the set \(\mathcal{R}_p\). In fact, the only polynomial in \(\mathcal{R}_p\) containing \((5.1)\) as a summand is
\[R(0,0,p-1) = z_{0,-1-(p-1)}z_{0,0} + z_{0,0}z_{0,-p+1}.\]
Similarly, the polynomial
\[\Lambda(S(1,0,0)) = z_{1,-1}z_{1,-1}\]
does not belong to the set \(\mathcal{R}_p\), since it consists of a single admissible monomial, whereas each element in \(\mathcal{R}_p\) always contains a non-admissible monomial among its summands.

References

[1] S. Araki, T. Kudo, Topology of \(H_n\)-spaces and \(H\)-squaring operations, Mem. Fac. Sci. Kyusyu Univ. Ser. A 10 (1956), 85–120.
[2] M. Brunetti, A. Ciampella, L. A. Lomonaco, The Cohomology of the Universal Steenrod algebra, Manuscripta Math., 118 (2005), 271–282.
[3] M. Brunetti, A. Ciampella, L. A. Lomonaco, An Embedding for the \(E_2\)-term of the Adams Spectral Sequence at Odd Primes, Acta Mathematica Sinica, English Series 22 (2006), no. 6, 1657–1666.
[4] M. Brunetti, A. Ciampella, A Priddy-type koszulness criterion for non-locally finite algebras, Colloquium Mathematicum 109 (2007), no. 2, 179–192.
[5] M. Brunetti, A. Ciampella, L. A. Lomonaco, Homology and cohomology operations in terms of differential operators, Bull. London Math. Soc. 42 (2010), no. 1, 53–63.
[6] M. Brunetti, A. Ciampella, L. A. Lomonaco, An Example in the Singer Category of Algebras with Coproducts at Odd Primes, Vietnam J. Math. 44 (2016), no. 3, 463–476.
[7] M. Brunetti, A. Ciampella, L. A. Lomonaco, Length-preserving monomorphisms for some algebras of operations, Bol. Soc. Mat. Mex. 23 (2017), no. 1, 487–500.
[8] M. Brunetti, A. Ciampella, The Fractal Structure of the Universal Steenrod Algebra: an invariant-theoretic description, Applied Mathematical Sciences, Vol. 8 no. 133 (2014), 6681–6687.
[9] M. Brunetti, L. A. Lomonaco, Chasing non-diagonal cycles in a certain system of algebras of operations, Ricerche Mat. 63 (2014), no. 1, suppl., S57–S68.
[10] A. Ciampella, On a fractal structure of the universal Steenrod algebra, Rend. Accad. Sci. Fis. Mat. Napoli, vol. 81, (4) (2014), 203–207.
[11] A. Ciampella, L. A. Lomonaco, The Universal Steenrod Algebra at Odd Primes, Communications in Algebra 32 (2004), no. 7, 2589–2607.
[12] A. Ciampella, L. A. Lomonaco, Homological computations in the universal Steenrod algebra, Fund. Math. 183 (2004), no. 3, 245–252.
[13] I. Karaca, Nilpotence relations in the mod \(p\) Steenrod algebra, J. Pure Appl. Algebra 171 (2002), no. 2–3, 257–264.
[14] A. Liulevicius, The factorization of cyclic reduced powers by secondary cohomology operations, Mem. Amer. Math. Soc. 42 (1962).
[15] Lomonaco L. A., Dickson invariants and the universal Steenrod algebra. Topology, Proc. 4th Meet., Sorrento/Italy 1988, Suppl. Rend. Circ. Mat. Palermo, II. Ser. 24 (1990), 429–443.
[16] J. P. May, A General Approach to Steenrod Operations, Lecture Notes in Mathematics. 168, Berlin: Springer, 153–231 (1970).
[17] J. P. May, Homology operations on infinite loop spaces, Algebraic topology (Proc. Sympos. Pure Math., Vol. XXII, Univ. Wisconsin, Madison, Wis., 1970), pp. 171–185. Amer. Math. Soc., Providence, R.I. (1971).
[18] K. G. Monks, Nilpotence in the Steenrod algebra, Bol. Soc. Mat. Mexicana (2) 37 (1992), no. 1-2, 401–416 (Papers in honor of José Adem).
[19] N. E. Steenrod, *Cohomology Operations*, lectures written and revised by D. B. A. Epstein, Ann. of Math. Studies 50, Princeton Univ. Press, Princeton, NJ (1962).

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