ON THE REPRESENTATION THEORY OF A QUANTUM
GROUP ATTACHED TO THE FOMIN-KIRILLOV ALGEBRA $\mathcal{FK}_3$

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Abstract. Let $\mathcal{D}$ be the Drinfeld double of $\mathcal{FK}_3 \# kS_3$. The simple $\mathcal{D}$-modules were described in [18]. In the present work, we describe the indecomposable summands of the tensor product between them. We classify the extensions of the simple modules and show that $\mathcal{D}$ is of wild representation type. We also investigate the projective modules and their tensor products.

1. Introduction

An important property of the category of modules over a Hopf algebra is that it is a tensor category. Several works address the study of the tensor structure for various families of Hopf algebras: the small quantum group $u_q(sl_2)$ [13,19], the (generalized) Taft algebras [6,7,12,15] and their Drinfeld doubles [4,5,8,23], the Drinfeld doubles of finite groups [22], the non-semisimple Hopf algebras of low dimension [21] and the pointed Hopf algebras over $kS_3$ [10] (these are liftings of $\mathcal{FK}_3$).

In particular, the small quantum group $u_q(sl_2)$ is a quotient of the Drinfeld double of a Taft algebra by central group-like elements. Thus, their representation theory have in common significant features: (1) the simple modules are parametrized by the simple modules over the corresponding corradical and (2) the tensor product of two simple modules decomposes into the direct sum of simple and projective modules, see loc. cit.

These features are generalization of well-known results in Lie theory. Indeed, the simple modules over a semisimple Lie algebra are parametrized by the weights of the Cartan subalgebra, while the tensor products of simple modules are described by the Clebsch-Gordon formula. Moreover, (1) holds for a more general kind of Hopf algebras which for short we call finite quantum groups, see for instance [3,14,18]. These are the Drinfeld doubles of the bosonization of a finite-dimensional Nichols algebra by a finite-dimensional semisimple Hopf algebra. Notice that a Taft algebra can be presented as a bosonization of the quantum line $k(x \mid x^n = 0)$ over the cyclic group of order $n$, the first example of a finite-dimensional Nichols algebra.

A valuable consequence of (2) is, roughly speaking, that the simple modules generate a fusion subcategory in a quotient category. The motivating question for our work was: will (2) also hold for other finite quantum groups?

The first example of a finite-dimensional Nichols algebra over a non-abelian group is the Fomin-Kirillov algebra $\mathcal{FK}_3$ [9]. It is a Nichols algebra over $S_3$ [17]. The main result of our work is the following.

2000 Mathematics Subject Classification. 16W30.

C. V. was partially supported by CONICET, Secyt (UNC), FONCyT PICT 2016-3957, Programa de Cooperación MINCyT-FWO, MathAmSud project GR2HOPF and ESCALA Docente AUGM.
Theorem. Let $\mathcal{D}$ be the Drinfeld double of $\mathcal{F}K_3 \# kS_3$. Given two simple $\mathcal{D}$-modules $L_1$ and $L_2$, the indecomposable summands of the tensor product $L_1 \otimes L_2$ are described in Propositions 4.1, 4.3, 4.7, 4.9, 4.10, 5.5 and 5.6.

We find out that some of these summands are not either simple or projective (we schematize them in Figures 1, 2 and 3). That is, (2) does not hold in this example. Instead, all the non-simple non-projective summands have the following in common.

- They have simple head and simple socle. Moreover, these are isomorphic.
- Being graded, the socle and the head are concentrated in the same homogeneous components.
- They are not either highest-weight modules or lowest-weight modules.

In this context, the weights are the simple modules over the Drinfeld double of $S_3$. Therefore some weights have dimension greater than one and the tensor product of two weights is not necessarily a weight, but it is the direct sum of various weights. These facts complicate the computations. However, the use of the following properties helps to simplify things. These properties hold for any finite quantum group and are not present in the above references.

First of all, as $\mathcal{D}$ is graded and finite-dimensional, we can restrict our attention to the category of graded modules. Let $\mathcal{N} = \bigoplus_{i \in \mathbb{Z}} \mathcal{N}(i)$ be such a module and $\text{ch}^* \mathcal{N}$ its graded character, i.e. its representative in the Grothendieck ring of the category of graded $\mathcal{D}(S_3)$-modules. Then

- The graded composition factors of $\mathcal{N}$ are given by $\text{ch}^* \mathcal{N}$.

In fact, the graded characters of the simple modules form a $\mathbb{Z}[t, t^{-1}]$-basis of the Grothendieck ring of the category of graded $\mathcal{D}$-modules [20, Theorem 3.3].

In order to know the indecomposable summands of $\mathcal{N}$, we need to know how its composition factors are connected. For this purpose, we need to compute the action of the space of generators $V$ of $\mathcal{F}K_3$ on a homogeneous weight $S$ of $\mathcal{N}$, i.e. a simple $\mathcal{D}(S_3)$-submodule of $\mathcal{N}(i)$. Here, we shall use that

- The action $V \otimes S \rightarrow \mathcal{N}(i - 1)$ is a morphism of $\mathcal{D}(S_3)$-modules.

The last fact is also useful to classify the extensions of simple $\mathcal{D}$-modules, see Lemmas 3.1 and 3.3. As a consequence we show that $\mathcal{D}$ is of wild representation type.

In Section 2 we recall the structure of $\mathcal{D}$ and summarize all the notation and conventions. We study the extensions of the simple modules in Section 3 and their tensor products in Section 4. Finally, we describe the projective modules and their tensor products in Section 5. In the Appendix we give the action of the generators of $\mathcal{D}$ on the simple modules.

Acknowledgments. We thank Nicolás Andruskiewitsch for suggesting us this project and for stimulating discussions. Part of this work was carried out during a visit of B. P. to the Universidad Nacional de Córdoba in the framework of “MathAmSud project GR2HOPF”, a visit of C. V. to the Universidade Federal do Rio Grande do Sul in the framework of “Programa ESCALA Docente de la Asociación de Universidades del Grupo Montevideo”, and also during a research stay of C. V. in the University of Clermont Ferrand (France) supported by CONICET.
2. Preliminaries

We summarize all the information needed for our work. We follow the notation and conventions of [18,20]. The reader can find there more details about the following.

2.1. The quantum group attached to the Fomin-Kirillov algebra \( \mathcal{FK}_3 \). We begin by fixing the notation related to the finite quantum group \( \mathcal{D} \).

2.1.1. The quantum group attached to the Fomin-Kirillov algebra \( \mathcal{FK}_3 \).

\( \mathcal{D}(\mathbb{S}_3) \) denotes the Drinfeld double of \( \mathbb{S}_3 \). It is a subalgebra of \( \mathcal{D} \) generated by the group-like elements of \( \mathbb{S}_3 \) and the dual elements \( \delta_g, g \in \mathbb{S}_3 \).

We identify every \( \mathbb{S}_3 \)-module with \( \mathbb{S}_3 \)-modules. They are isomorphic via \( \gamma(x) \mapsto x \).

The Grothendieck ring of the category of \( \mathbb{S}_3 \)-modules. It is an involution and it is easy to check that \( S^2(h) = sgn \cdot h \cdot sgn \forall h \in \mathcal{D} \).

2.2. Graded Modules.

Our objects of study are the finite-dimensional \( \mathbb{Z} \)-graded left modules over \( \mathcal{D} \), graded modules for short. We denote \( R^* \) the Grothendieck ring of the category of graded modules. It is a \( \mathbb{Z}[t, t^{-1}] \)-module via the shift functor \( t \cdot N = N[1] \) where \( N[1](i) = N(i-1) \), with \( N = \bigoplus_{i \in \mathbb{Z}} N(i) \) a graded module.

We will consider the \( \mathcal{D} \)-modules as modules over the distinguished subalgebras \( \mathcal{D}^{\leq 0} \), \( \mathcal{D}^{\geq 0} \) and \( \mathcal{D}(\mathbb{S}_3) \) by restricting the action. As these are graded subalgebras of \( \mathcal{D} \), a graded module also is graded over them. Moreover, \( N \) is simultaneously a graded \( \mathcal{B}(V) \)-module and \( \mathcal{B}(V) \)-module in the category of \( \mathcal{D}(\mathbb{S}_3) \)-modules since \( \mathcal{D}^{\leq 0} \) and \( \mathcal{D}^{\geq 0} \) are bosonizations. In particular, we deduce the following.

2.3. Weights.

The simple \( \mathcal{D}(\mathbb{S}_3) \)-modules are parametrized by the set

\[ \Lambda = \{ \varepsilon = (\varepsilon, +), (\varepsilon, -), (\varepsilon, \rho), (\sigma, +), (\sigma, -), (\tau, 0), (\tau, 1), (\tau, 2) \}. \]

We identify every weight \( \lambda \in \Lambda \) with the simple \( \mathcal{D}(\mathbb{S}_3) \)-module \( M(\lambda) \) in [18, §5.2].

The Grothendieck ring of the category of \( \mathcal{D}(\mathbb{S}_3) \)-modules is the abelian group \( K = \mathbb{Z}\Lambda \) endowed with the product \( \lambda \cdot \mu = M(\lambda) \otimes M(\mu) \) and unit \( \varepsilon \). These tensor products were explicitly given in [18, §5.2.4]. We will often use these fusion rules.
in the coming section. The Grothendieck ring of $D(G)$, for any finite group $G$, was described in [22].

Highest and lowest weights (modules) are defined as usual, keeping in mind the grading on $D$, cf. [20] §2.2. For instance, a highest-weight is a weight $\lambda$ which is a simple $D^{\geq 0}$-module with $V \cdot \lambda = 0$.

2.4. Characters. Let $N$ be a $D(S_3)$-module. The character $\text{ch} N$ is the representative of $N$ in the Grothendieck ring $K$. If $N$ is a graded $D(S_3)$-module, we denote $N(i)$ its homogeneous component of degree $i$. In that case, its graded character is $\text{ch} N = \sum_{i \in \mathbb{Z}} \text{ch} N(i) t^i \in K[t, t^{-1}]$. For instance,

$$\text{ch} \mathfrak{B}(V) = \text{ch} \mathfrak{B}(V) = \varepsilon + (\sigma, \cdot) t^{-1} + (\tau, 1) + (\tau, 2) \cdot t^{-2} + (\sigma, \cdot) t^{-3} + \varepsilon t^{-4}.$$

(1)

Then $\text{ch}^\bullet : R^\bullet \rightarrow K[t, t^{-1}]$ is a ring homomorphism and $\text{ch}^\bullet N^* = \overline{\text{ch}^\bullet N}$, where $\overline{P(t, t^{-1})} = p(t^{-1}, t)$ for any $p \in K[t, t^{-1}]$. Notice that there exist polynomials $p_{N, \lambda} \in \mathbb{Z}[t, t^{-1}]$ such that

$$\text{ch}^\bullet N = \sum_{\lambda \in \Lambda} p_{N, \lambda} \lambda \iff N \simeq \bigoplus_{\lambda \in \Lambda} p_{N, \lambda} \cdot \lambda.$$

2.5. The simple modules. Given $\lambda \in \Lambda$, $L(\lambda)$ denotes the simple module of highest-weight $\lambda$. This is graded and every simple module is isomorphic to some $L(\lambda)$ [18] Theorem 3. In [18] we also show that

$$\text{ch}^\bullet L(\varepsilon) = \varepsilon,$$

$$\text{ch}^\bullet L(e, \rho) = (e, \rho) + (\sigma, +) t^{-1} + (\tau, 0) t^{-2},$$

$$\text{ch}^\bullet L(\sigma, -) = (\sigma, -) + ((\tau, 1) + (\tau, 2)) t^{-1} + (\sigma, -) t^{-2},$$

$$\text{ch}^\bullet L(\tau, 0) = (\tau, 0) + (\sigma, +) t^{-1} + (e, \rho) t^{-2},$$

$$\text{ch}^\bullet L(\lambda) = \lambda \cdot \text{ch}^\bullet \mathfrak{B}(V), \ \forall \lambda \in \Lambda_{sp} := \{(e, -), (\sigma, +), (\tau, 1), (\tau, 2)\}.$$

These simple modules are self-dual except for

$$L(e, \rho)^* \simeq L(\tau, 0) \quad \text{and} \quad L(\tau, 1)^* \simeq L(\tau, 2).$$

We denote by $\overline{\lambda}$ the lowest-weight of $L(\lambda)$. Explicitly

$$\overline{(e, \rho)} = (\tau, 0), \ \overline{(\tau, 0)} = (e, \rho) \quad \text{and} \quad \overline{\lambda} = \lambda \ \forall \lambda \in \Lambda \setminus \{(e, \rho), (\tau, 0)\}.$$

(2)

The simple modules also are distinguished by their lowest-weights [18] Theorem 4.

The set $\{\text{ch}^\bullet L(\lambda) \mid \lambda \in \Lambda\}$ is a $\mathbb{Z}[t, t^{-1}]$-basis of $R^\bullet$ by [20] Theorem 3.2. Then, for every graded module $N$ there are unique Laurent polynomials $p_{N, L(\lambda)}$ such that

$$\text{ch}^\bullet N = \sum_{\lambda \in \Lambda} p_{N, L(\lambda)} \text{ch}^\bullet L(\lambda).$$

We can deduce the following information from these polynomials.

Remark 2.3. Assume that $p_{N, L(\lambda)} = \sum a_{N, L(\lambda), i} t^i$ with $a_{N, L(\lambda), i} \neq 0$.

(i) $N$ has $a_{N, L(\lambda), i}$ composition factors isomorphic to $L(\lambda)[i]$.

(ii) If $L(\lambda)$ is projective, then $a_{N, L(\lambda), i} L(\lambda)[i]$ is a direct summand of $N$. 

(iii) There exists a weight \( S \subset \mathbb{N}(i) \) isomorphic to \( \lambda \) such that \( DS/X \simeq L(\lambda)[i] \) for some maximal submodule of \( DS \).

(iv) Let \( S \subset \mathbb{N}(i) \) be a weight isomorphic to \( \lambda \) and \( X \) a maximal submodule of \( DS \). Then \( DS/X \simeq L(\mu)[j] \) such that \( a_{\mu, L(\mu)[j]} \neq 0 \) and \( \lambda[i] \) is a weight of \( L(\mu)[j] \). In particular, \( DS/X \simeq L(\lambda)[i] \) if \( \lambda[i] \) is not a weight of any composition factor \( L(\mu)[j] \) of \( N \) with \( \mu \neq \lambda \) or \( j \neq i \).

In fact, (i), (ii) and (iii) are clear, cf. [20, §3.2]. Since every composition factor of \( DS \) is a composition factor of \( \mathcal{N} \), (iv) holds.

2.6. Verma modules. Given a highest-weight \( \lambda \in \Lambda \), the induced module

\[
M(\lambda) = D \otimes_{D_{\geq 0}} \lambda \simeq \mathfrak{B}(V)(V) \otimes \lambda \tag{3}
\]

is called Verma module; where the isomorphism is of \( \mathbb{Z} \)-graded \( D_{\leq 0} \)-modules. This is the universal highest-weight module of weight \( \lambda \). Its head is isomorphic to \( L(\lambda) \) and its socle is \( L(\mu) \) with \( \overline{\mu} = \mathfrak{B}(\mathfrak{m}_{\mu}(V)) \otimes \lambda \) [18] Theorems 3 and 4.

In this case, the Verma modules are self-dual except \( M(\tau, 1)^* \simeq M(\tau, 2) \) by [20, (11)] since \( \text{ch} \mathfrak{B}(\mathfrak{m}_{\mu}(V)) = 0 \). By [18, Theorem 6] \( M(e, -), M(\sigma, +), M(\tau, 1) \) and \( M(\tau, 2) \) are simple and hence they are projective by [20, Corollary 4.6]. Their graded characters are

\[
\text{ch}^* M(\varepsilon) = (1 + t^4) \text{ch}^* L(\varepsilon) + t^{-1} \text{ch}^* L(\sigma, -),
\]

\[
\text{ch}^* M(e, \rho) = \text{ch}^* L(e, \rho) + t^{-1} \text{ch}^* L(\sigma, -) + t^{-2} \text{ch}^* L(\tau, 0),
\]

\[
\text{ch}^* M(\sigma, -) = (1 + t^{-2}) \text{ch}^* L(\sigma, -) + t^{-1} \text{ch}^* L(\tau, 0) + (t^{-1} + t^{-3}) \text{ch}^* L(\varepsilon),
\]

\[
\text{ch}^* M(\tau, 0) = \text{ch}^* L(\tau, 0) + t^{-1} \text{ch}^* L(\sigma, -) + t^{-2} \text{ch}^* L(e, \rho),
\]

\[
\text{ch}^* M(\lambda) = \text{ch}^* L(\lambda), \quad \forall \lambda \in \Lambda_{sp}.
\]

In fact, we can calculate explicitly the polynomials \( p_{M(\mu), L(\lambda)} \) or use [18, Theorems 7, 8, 9 and 10] where we have computed the lattice of submodules of the Verma modules.

2.7. co-Verma modules. Given a lowest-weight \( \mu \), the induced module

\[
W(\mu) = D \otimes_{D_{\geq 0}} \mu \simeq \mathfrak{B}(V) \otimes \mu \tag{4}
\]

is called co-Verma module; the isomorphism is of \( \mathbb{Z} \)-graded \( D_{\geq 0} \)-modules. By [20, Theorem 3.4] we know that \( \text{ch}^* W(\lambda^*) = \text{ch}^* M(\lambda)^* \). Hence

\[
\text{ch}^* W(\varepsilon) = (1 + t^4) \text{ch}^* L(\varepsilon) + t^3 \text{ch}^* L(\sigma, -),
\]

\[
\text{ch}^* W(e, \rho) = t^2 \text{ch}^* L(\tau, 0) + t^3 \text{ch}^* L(\sigma, -) + t^4 \text{ch}^* L(e, \rho),
\]

\[
\text{ch}^* W(\sigma, -) = (t^2 + t^4) \text{ch}^* L(\sigma, -) + t^3 \text{ch}^* L(e, \rho) + (t + t^3) \text{ch}^* L(\varepsilon),
\]

\[
\text{ch}^* W(\tau, 0) = t^2 \text{ch}^* L(e, \rho) + t^3 \text{ch}^* L(\sigma, -) + t^4 \text{ch}^* L(\tau, 0),
\]

\[
\text{ch}^* W(\lambda) = t^4 \text{ch}^* L(\lambda^*), \quad \forall \lambda \in \Lambda_{sp}.
\]

We give more information about the structure of the co-Verma modules.

Lemma 2.4. (i) The socle of \( W(\lambda) \) is isomorphic to \( L(\lambda) \) for all \( \lambda \notin \Lambda_{sp} \).
(i) The head of $W(\lambda)$ is isomorphic to $L(\lambda)$ for all $\lambda \notin \Lambda_{sp}$.

(ii) The socle of $W(\lambda)/soc W(\lambda)$ is isomorphic to $L(\sigma,-)$ if $(\sigma,-) \neq \lambda \notin \Lambda_{sp}$.

(iii) The socle of $W(\sigma,-)/soc W(\sigma,-)$ is isomorphic to $2L(\varepsilon) \oplus L(\epsilon,\rho) \oplus L(\tau,0)$.

(iv) The unique maximal submodule of $W(\lambda)$ is the preimage of the socle of $W(\lambda)/soc W(\lambda)$ for all $\lambda \notin \Lambda_{sp}$.

(vi) $W(\lambda) \simeq M(\lambda)$ for all $\lambda \in \Lambda_{sp}$. In particular, they are simple and projective.

Proof. Arguing as in [18, §3.1], we can see that a co-Verma module has simple head and simple socle. As a simple module is characterized by its highest-weight, (i) and (ii) follow from (2) and $\mathcal{B}^{n_{top}}(V) = \varepsilon$.

(iii) If $(\sigma,-) \neq \lambda \notin \Lambda_{sp}$, then $W(\lambda)$ has three composition factors because of the graded character. As the head of $W(\lambda)$ is simple we deduce that the socle of $W(\lambda)/soc W(\lambda)$ also has to be simple and isomorphic to $L(\sigma,-)$.

Notice that (v) follows from (iii) for $\lambda \neq (\sigma,-)$.

(iv) Notice that any weight $\lambda \in \{\varepsilon,(\epsilon,\rho),(\tau,0)\}$ is contained in the maximal submodule of $W(\sigma,-)$ because of $ch^* L(\sigma,-)$. In particular, $\lambda$ is a lowest-weight for $\lambda$ of degree 1, see $ch^* W(\sigma,-)$. Hence the submodule $D\lambda$ of $W(\sigma,-)$ is a quotient of the co-Verma module $W(\lambda)$. Therefore the socle of $W(\sigma,-)/soc W(\sigma,-)$ contains a submodule isomorphic to $L(\varepsilon) \oplus L(\epsilon,\rho) \oplus L(\tau,0)$ by (i)–(iii).

The other copy of $L(\varepsilon)$ corresponds to the weight $\varepsilon$ of degree 3, see $ch^* W(\sigma,-)$. In fact, it is a highest-weight in $W(\sigma,-)/soc W(\sigma,-)$, since this quotient has no homogeneous component of degree 4. Then the submodule $D\varepsilon$ of $W(\sigma,-)/soc W(\sigma,-)$ is a quotient of the Verma module $M(\varepsilon)$. We claim that $D\varepsilon \simeq L(\varepsilon)$ and (iv) follows. Otherwise, $D\varepsilon$ has a composition factor isomorphic to $L(\sigma,-)$ by [18, Theorem 8]. From $ch^* W(\sigma,-)$ we deduce that this composition factors corresponds to the head of $W(\sigma,-)$. But this is not possible because $\varepsilon$ is contained in the maximal submodule of $W(\sigma,-)$. This finishes the proof of (iv) which implies (v).

(vi) As $\mathcal{B}^{n_{top}}(V) \otimes \mu \simeq \mu$ is a highest-weight, we see that $W(\lambda) \simeq M(\lambda)$ for all $\lambda \in \Lambda_{sp}$ because these are simple.

□

3. Extensions of simple modules

In this section we classify the extensions of the form

$$0 \longrightarrow L(\mu) \overset{i}{\longrightarrow} E \overset{\pi}{\longrightarrow} L(\lambda) \longrightarrow 0.$$  

(5)

Lemma 3.1. If $\lambda, \mu \in \{\varepsilon,(\epsilon,\rho),(\tau,0)\}$ or $\lambda = \mu = (\sigma,-)$, then $E \simeq L(\mu) \oplus L(\lambda)$.

Proof. Since $D$ is finite-dimensional, the space of extensions and the space of graded extensions are isomorphic. Thus, we can assume that $E$ and the maps $i$ and $\pi$ are graded. Let $i$ be a section of $\pi$ as graded $D(\mathbb{S}_3)$-modules. Up to a shift, $E \simeq L(\mu)[\ell] \oplus L(\lambda)$ as graded $D(\mathbb{S}_3)$-modules.

By the form of the graded quotients of the Verma and co-Verma modules, cf. [18 §4] and Lemma 2.4, the lemma follows if we show that either $\iota(\lambda)$ is a highest-weight of $E$ or $\iota(\lambda)$ is a lowest-weight of $E$; recall that $\lambda$ denotes the lowest-weight of $L(\lambda)$. Recall also that the restrictions of the action maps $V \otimes \iota(\lambda) \longrightarrow L(\mu)(1- \ell)$ and $V \otimes \iota(\lambda) \longrightarrow L(\mu)(-1- \ell)$ are morphisms of graded $D(\mathbb{S}_3)$-modules, Remark 2.2.

Next, we proceed case-by-case keeping in mind the character of $L(\mu)$ and the fusion rules for the simple $D(\mathbb{S}_3)$-modules given in [18 §2.5.4].

If $\lambda = \varepsilon$, then $V \cdot \iota(\lambda) = 0 = V \cdot \iota(\lambda)$ because $(\sigma,-)$ is not a weight of $L(\mu)$.
If \( \lambda = (\varepsilon, \rho) \) and \( \nabla \cdot \iota(\lambda) \neq 0 \), then \( \mu \neq \varepsilon \) and \( \nabla \cdot \iota(\lambda) \simeq (\sigma, +) \simeq L(\mu)(-1) \). This forces \( \ell = 2 \). Hence \( V \cdot \iota(\lambda) = 0 \) because \( L(\mu)[-2](\sigma) = L(\mu)(-3) = 0 \). The case \( \lambda = (\tau, 0) \) is analogous.

If \( \lambda = (\tau, -) \) and \( \nabla \cdot \iota(\lambda) \neq 0 \), then \( \nabla \cdot \iota(\lambda) \subseteq L(\sigma, -)(1) \) and we can conclude that \( V \cdot \iota(\lambda) = 0 \) as above.

For \( \mu = (\sigma, -) \) and \( \lambda \in \{ \varepsilon, (\rho, (\sigma, 0)) \} \), there are two nonequivalent distinguished graded extensions. We set \( E(\lambda)_{1,0} = M(\lambda)/\operatorname{soc} M(\lambda) \) and \( E(\lambda)_{0,1} = W(\lambda)/\operatorname{soc} W(\lambda) \). Then, by [18] Theorems 8, 9 and 10 and Lemma 2.4 it follows that

\[ 0 \rightarrow L(\sigma, -)[1] \rightarrow E(\lambda)_{1,0} \rightarrow L(\lambda) \rightarrow 0 \quad \text{and} \quad 0 \rightarrow L(\sigma, -)[3] \rightarrow E(\lambda)_{0,1} \rightarrow L(\lambda)[2] \rightarrow 0. \]

**Definition 3.2.** Let \( s, t \) be scalars and \( \lambda \in \{ \varepsilon, (\rho, (\tau, 0)) \} \). We denote \( E(\lambda)_{s,t} \) the Baer sum \( sE(\lambda)_{1,0} + tE(\lambda)_{0,1} \), if \( s \neq 0 \) or \( t \neq 0 \), and \( E(\lambda)_{0,0} = L(\sigma, -) \oplus L(\lambda) \).

Therefore \( E(\lambda)_{s,t} \) fits in an extension as in [5] and its dual \( E(\lambda)^*_{s,t} \) is an extension of \( L(\sigma, -) \) by \( L(\lambda) \).

**Lemma 3.3.** Let \( E, \lambda \) and \( \mu \) as in [5].

(i) If \( \lambda \in \{ \varepsilon, (\rho, (\tau, 0)) \} \) and \( \mu = (\sigma, -) \), then \( E \simeq E(\lambda)_{s,t} \) for some \( s, t \in k. \)

Moreover, it is a graded extension if and only if it is isomorphic (up to shifts) to either \( E(\lambda)_{1,0}, E(\lambda)_{0,1} \) or \( E(\lambda)_{0,0} \).

(ii) If \( \lambda = (\sigma, -) \) and \( \mu \in \{ \varepsilon, (\rho, (\tau, 0)) \} \), then \( E \simeq E(\lambda)^*_{s,t} \) for some \( s, t \in k. \)

Moreover, it is a graded extension if and only if it is isomorphic (up to shifts) to either \( E(\lambda)^*_{1,0}, E(\lambda)^*_{0,1} \) or \( E(\lambda)^*_{0,0} \).

**Proof.** (i) As in the above lemma, it is enough to prove that if \( E \) is a nontrivial graded extensions, then \( E \simeq E(\lambda)_{1,0} \) or \( E \simeq E(\lambda)_{0,1} \). We prove only the case \( \lambda = (\varepsilon, \rho) \); the others are similar.

Let \( \iota \) be a section of \( \pi \) as graded \( D(S_3) \)-modules. Up to a shift, \( E \simeq L(\sigma, -)[\ell] \oplus L(\varepsilon, \rho)[2] \) as graded \( D(S_3) \)-modules and the restriction \( \nabla \otimes \iota(\varepsilon, \rho) \rightarrow L(\sigma, -)[3-\ell] \) is a morphism of graded \( D(S_3) \)-modules. If \( \iota(\rho) \) is a highest-weight in \( E \), then \( E \simeq E(\rho, (\sigma, 0)) \). Otherwise, \( \nabla \cdot \iota(\varepsilon, \rho) \simeq (\sigma, -) \) is homogeneous. Then \( \ell = 3 \) or \( \ell = 5 \) by \( \chi^* L(\sigma, -) \). On the other hand, \( V \cdot \iota(\tau, 0) \subseteq L(\sigma, -)[-1-\ell] \). Thus, \( \iota(\tau, 0) \) is a lowest-weight of \( E \) and hence \( E \simeq E(\rho, (\sigma, 0)) \). Moreover, this forces that \( \ell = 3 \) because \( \chi^* E(\rho, (\sigma, 0)) - t^3 \chi^* L(\sigma, -) \).

(ii) is equivalent to (i) because \( (\sigma, -)^* = (\sigma, -) \) and \( \mu^* \in \{ \varepsilon, (\rho, (\tau, 0)) \} \).

By the above lemmas the separated quiver of \( D \) is

\[
\begin{array}{ccc}
L(\varepsilon) & \overset{}{\rightarrow} & L(\varepsilon)' \\
\downarrow & & \downarrow \\
L(\sigma, -)' & \overset{}{\rightarrow} & L(\sigma, -)'
\end{array}
\]

Then, we deduce the following proposition, see for instance [1] §4.2 for details.
Proposition 3.4. \( \mathcal{D} \) is of wild representation type.

4. The tensor products of non-projective simple modules

In this section we describe the tensor products of the simple modules. We will use the bases of the simple modules and the action over these basis elements given in the Appendix. Also we will often use the fusion rules of the simple \( \mathcal{D}(S_3) \)-modules given in [18 §2.5.4].

4.1. How to compute the indecomposable submodules. We explain the general strategy which we shall follow to compute the indecomposable summands. These ideas apply to any graded module \( N = \oplus_{i \in \mathbb{Z}} N(i) \) and finite quantum group. See also [18 §3.2].

Assume that \( \text{ch}^* N = \sum_{\lambda \in \Lambda} \text{ch}^* L(\lambda) \) and \( \sigma_{N,L(\lambda),i} \neq 0 \). In view of Remark 2.3, we shall start by computing the submodules \( \mathcal{D}\lambda \) generated by the weights \( \lambda \subset N(i) \). Among these, we will first consider the weights \( \lambda \) such that \( i \) is either maximal or minimal because this implies that \( \mathcal{D}\lambda \) is a quotient of either the Verma module \( M(\lambda) \) or the co-Verma module \( W(\lambda) \). In fact, \( \lambda \) will be either a highest or lowest weight. We know these quotients from [18 §4] and Lemma 2.4, respectively.

For the remaining weights, we will repeatedly use that the action maps \( V \otimes \lambda \rightarrow N(i-1) \) and \( V \otimes \lambda \rightarrow N(i+1) \) are morphisms of \( \mathcal{D}(S_3) \)-modules; this is Remark 2.2 with \( \lambda \) instead of \( N(i) \). Therefore \( \mathcal{D}\lambda \) will be generated by the successive images of the former maps. Thus, we shall decompose \( V \otimes \lambda \) (respectively \( V \otimes \lambda \)) into a direct sum of weights and apply the action on each summand. This restriction morphism will be zero or an injection by Schur Lemma. Hence it is enough to compute the action in a single element of each weight. The knowledge of \( \text{ch} N(i-1) \) (respectively \( \text{ch} N(i+1) \)) will help to make less computations.

Finally, we shall analyze the intersections of the submodules \( \mathcal{D}\lambda \).

4.2. The tensor product \( L(\tau, 0) \otimes L(e, \rho) \).

Proposition 4.1. It holds that

\[
\text{ch}^* (L(\tau, 0) \otimes L(e, \rho)) = \text{ch}^* L(\tau, 1) + \text{ch}^* L(\tau, 2) + t^{-2} \text{ch}^* L(\varepsilon).
\]

Therefore \( L(\tau, 0) \otimes L(e, \rho) \simeq L(\tau, 1) \oplus L(\tau, 2) \oplus L(\varepsilon)[-2] \) as graded modules.

Proof. As \( \text{ch}^* \) is a ring homomorphism and using the formulae of [18 §5.2], we have that

\[
\text{ch}^* (L(\tau, 0) \otimes L(e, \rho)) = (\tau, 0)(e, \rho) + t^{-1}(\sigma, +)((\tau, 0) + (e, \rho)) + t^{-2}(\sigma, 0)(\tau, 0) + (e, \rho) + t^{-3}(\sigma, +)(\tau, 0) + (e, \rho) + t^{-4}(\tau, 0)(e, \rho) + \\
= (\tau, 1) + (\tau, 2) + t^{-1}(\sigma, -)((\tau, 1) + (\tau, 2)) + t^{-2}(\tau, 1) + (\tau, 2)((\tau, 1) + (\tau, 2)) + \\
+ t^{-2} \varepsilon + t^{-3}(\sigma, -)((\tau, 1) + (\tau, 2)) + t^{-4}(\tau, 1) + (\tau, 2).
\]

Then, (6) is a straightforward computation.

By (6) and Remark 2.3, the simple modules \( L(\tau, 1) \) and \( L(\tau, 2) \) are direct summands of \( L(\tau, 0) \otimes L(e, \rho) \). Thus, the isomorphism holds because \( L(\tau, 0) \otimes L(e, \rho) \) has only three composition factors. \( \Box \)
Remark 4.2. The weights \((\tau, 1)\) and \((\tau, 2)\) of the degree zero component are obviously highest-weights generating the simple submodules \(L(\tau, 1)\) and \(L(\tau, 2)\). The element generating the submodule \(L(\varepsilon)\) is
\[
d = a_6 \otimes b_2 + a_7 \otimes b_1 + a_3 \otimes b_4 + a_5 \otimes b_3 + a_4 \otimes b_5 + a_1 \otimes b_6 + a_2 \otimes b_7,
\]
where the elements \(a_i, b_j\) are presented in the Appendix. In fact, using the Appendix, we see that \(y_{(12)} \cdot d = 0\) and \(x_{(12)} \cdot d = 0\).

4.3. The tensor product \(L(\sigma, -) \otimes L(\sigma, -)\). As in (6) we can see that
\[
\text{ch}^* (L(\sigma, -) \otimes L(\sigma, -)) = \text{ch}^* L(\tau, 1) + \text{ch}^* L(\tau, 2) + 2t^{-1} \text{ch}^* L(\sigma, -) + (1 + 2t^{-2} + t^{-4}) \text{ch}^* L(\varepsilon) + (1 + t^{-2} + t^{-4}) \text{ch}^* L(e, \rho) + \text{ch}^* L(\tau, 0).
\]
Therefore \(L(\tau, 1)\) and \(L(\tau, 2)\) are graded direct summands of \(L(\sigma, -) \otimes L(\sigma, -)\) by Remark 2.3. The aim of this subsection is to show the next proposition. We give the proof after some preparatory lemmas. Recall the socle filtration \(\{\text{soc}^i A\}_{i \geq 1}\) is given by the preimages of \(\text{soc}(A/\text{soc}^{i-1} A)\) for \(i > 1\).

Proposition 4.3. There exists a graded indecomposable module \(A\) with \(\text{ch}^* A = 2t^{-1} \text{ch}^* L(\sigma, -) + (1 + 2t^{-2} + t^{-4}) \text{ch}^* L(\varepsilon) + (1 + t^{-2} + t^{-4}) (\text{ch}^* L(e, \rho) + \text{ch}^* L(\tau, 0))\) such that \(A^* \simeq A\) and
\[
L(\sigma, -) \otimes L(\sigma, -) \simeq L(\tau, 1) \oplus L(\tau, 2) \oplus L(\varepsilon) \oplus A.
\]
Moreover
\[
\text{soc} A = t^{-1} L(\sigma, -),
\]
\[
\text{soc}^2 A/\text{soc} A \simeq (1 + 2t^{-2} + t^{-4}) L(\varepsilon) \oplus (1 + t^{-2}) L(e, \rho) \oplus (1 + t^{-2}) L(\tau, 0),
\]
\[
\text{soc}^3 A/\text{soc}^2 A \simeq t^{-1} L(\sigma, -),
\]
\[
\text{soc}^3 A = A.
\]

Figure 1 helps the reader to visualize the module \(A\) and to follow the proof of the following lemmas.

**Figure 1.** The dots represent the weights of \(A\). Each shadow region correspond to a composition factor whose highest-weight is in the top. The actions of \(V\) and \(\overline{V}\) are illustrated by the arrows.
By the fusion rules \[18\] §2.5.4, \( L(\sigma, -) \otimes L(\sigma, -) \) has four copies of the weight \( \varepsilon \) in degree \(-2\). In fact, these are
\[
\begin{align*}
\varepsilon_{-2,0} &= c_1 \otimes c_8 + c_2 \otimes c_9 + c_3 \otimes c_{10}, \\
\varepsilon_{-1,-1,1} &= c_4 \otimes c_5 + c_5 \otimes c_4, \\
\varepsilon_{0,-2} &= c_6 \otimes c_1 + c_9 \otimes c_2 + c_{10} \otimes c_3, \\
\varepsilon_{-1,-1,2} &= c_6 \otimes c_7 + c_7 \otimes c_6;
\end{align*}
\]
the subindices refer to the degree of \( c_i \), see the Appendix. We will see that the direct summand \( L(\varepsilon) \) in the proposition is the following submodule.

\[\text{Lemma 4.4.} \text{ Let } \varepsilon_{-2} = -\zeta^2 \varepsilon_{-1,-1,1} + \varepsilon_{-1,-1,2} + (1 - \zeta^2) \varepsilon_{0,-2} - (1 - \zeta^2) \varepsilon_{-2,0}. \text{ Then the submodule generated by } \varepsilon_{-2} \text{ is isomorphic to } L(\varepsilon).\]

\[\text{Proof.} \text{ Using the Appendix, we see that } x_{(12)} \varepsilon_{-2} = 0 = y_{(12)} \varepsilon_{-2}. \]

On the other hand, the weight \( \varepsilon \) of \( A \) in degree \(-2\) will be
\[
\varepsilon_{-2}' = 18 \varepsilon_{-1,-1,1} - 6 \varepsilon_{-1,-1,2} + 6 \varepsilon_{-2,0} + 6 \varepsilon_{0,-2}.
\]
The socle of \( A \) will be generated by
- \( s = (\zeta c_7 - c_3) \otimes c_8 - c_{10} \otimes (\zeta c_7 - c_3) + \zeta^2 (c_6 - c_4) \otimes c_{10} - \zeta^2 c_8 \otimes (c_6 - c_4). \)

Let \( S \) be the \( D(S_3) \)-module generated by \( s \).

\[\text{Lemma 4.5.} \text{ Let } \lambda \text{ be an homogeneous weight of } (L(\sigma, -) \otimes L(\sigma, -))(\ell) \text{ and } D\lambda \text{ denote the submodule generated by } \lambda. \text{ Hence}
\]
(i) \( DS \simeq L(\sigma, -) \) with highest-weight \( S \simeq (\sigma, -) \).
(ii) \( \text{If } \lambda \in \{\varepsilon, (e, \rho), (\tau, 0)\} \text{ and } \ell = 0, \text{ then } \lambda \text{ is a highest-weight and } D\lambda \text{ is an extension of } L(\lambda) \text{ by } DS. \)
(iii) \( \text{If } \lambda \in \{\varepsilon, (e, \rho), (\tau, 0)\} \text{ and } \ell = -4, \text{ then } \lambda \text{ is a lowest-weight and } D\lambda \text{ is an extension of } L(\lambda) \text{ by } DS. \)
(iv) \( \text{If } \lambda = \zeta \varepsilon'_2, \text{ then } D\lambda \text{ is an extension of } L(\lambda) \text{ by } DS. \)
(v) \( \text{Let } A' \text{ be the sum of all above submodules. Then } A' \text{ is indecomposable with simple socle } DS. \)

\[\text{Proof.} \text{ By the fusion rules } [18] \text{ §2.5.4, the homogeneous weight } \varepsilon \text{ of degree zero is spanned by } \varepsilon_0 = c_6 \otimes c_8 + c_9 \otimes c_3 + c_{10} \otimes c_{10}. \text{ Clearly, this is a highest-weight. Then } D\varepsilon_0 \text{ is a quotient of the Verma module } M(\varepsilon) \text{ via the morphism } \pi : M(\varepsilon) \to D\varepsilon_0, \pi(x \otimes 1) = x \cdot \varepsilon_0 \text{ for all } x \in \mathcal{B}(V). \text{ Using the Appendix, we see that } (1 - \zeta) x_{(23)} \cdot \varepsilon_0 = s \text{ and } x_{(23)} \cdot \varepsilon_0 = 0. \text{ By inspecting the quotients of } M(\varepsilon) \text{ in } [18] \text{ Theorem 8], we deduce (i) and (ii) for } \lambda = \varepsilon. \)

The elements \( t = c_6 \otimes c_8 + \zeta^2 c_9 \otimes c_3 + c_{10} \otimes c_{10} \) and \( u = c_6 \otimes c_8 + c_{10} \otimes c_8 + c_9 \otimes c_{10} \) generate the highest-weights \( (e, \rho) \) and \( (\tau, 0) \) in degree zero, respectively; again, this holds by the fusion rules \[18\] §2.5.4. Then \( Dt \) and \( Du \) are quotient of the Verma modules \( M(e, \rho) \) and \( M(\tau, 0) \), respectively. We finish the proof of (ii) by noting that \( V \cdot t \) and \( V \cdot u \) are contained in \( S \). In fact,
\[
s = \zeta^{-1} (1 - (23)) x_{(23)} \cdot t = (\zeta - 1)(1 - (23)) x_{(12)} \cdot u.
\]

(iii) The homogeneous weights \( \varepsilon, (e, \rho) \) and \( (\tau, 0) \) of degree \(-4\) are generated by
\[
\varepsilon_{-4} = c_1 \otimes c_1 + c_2 \otimes c_2 + c_3 \otimes c_3,
\]
\[
v = c_1 \otimes c_1 + \zeta^2 c_2 \otimes c_2 + \zeta c_3 \otimes c_3 \text{ and } u = c_1 \otimes c_2 + c_3 \otimes c_1 + c_2 \otimes c_3.
\]
respectively, cf. [18, §2.5.4]. Clearly, these are lowest-weights and we have that
\[(1 - (12))y_{(12)} \cdot \varepsilon_{-4} = (1 - (12))y_{(12)} \cdot v = (1 - (12))y_{(13)} \cdot w.\]
Moreover, this element is \(x_{(13)}x_{(12)}x_{(23)} \cdot \varepsilon_0\) which generates the lowest-weight of \(DS\) thanks to [18, Theorem 8]. This means that \(\nabla \cdot \varepsilon_{-4}, \nabla \cdot v\) and \(\nabla \cdot w\) are contained in \(DS\). Hence (iii) follows from Lemma 2.4.

(iv) We have that
\[x_{(12)} \cdot \varepsilon'_{-2} = (1 - \zeta)x_{(13)}x_{(12)}x_{(23)} \cdot \varepsilon_0 \quad \text{and} \quad y_{(12)} \cdot \varepsilon'_{-2} = (13)s\]
belong in \(DS\). Therefore \(D\varepsilon'_{-2} = k\varepsilon_{-2} \oplus DS\) as \(D(S_3)\)-modules and (iv) follows. (v) is a direct consequence of the above. \(\square\)

By (7) and Remark 2.3, there is a graded submodule \(N\) such that
\[L(\sigma, -) \otimes L(\sigma, -) \simeq L(\tau, 1) \oplus L(\tau, 2) \oplus N.\]
Notice that \(k\varepsilon_{-2}\) and \(A'\) are submodules of \(N\) such that \(k\varepsilon_{-2} \cap A' = 0\) and \(ch^\star N = ch^\star A' + t^{-2}ch^\star L(\varepsilon_2) + t^{-1}ch^\star L(\sigma, -)\).

**Lemma 4.6.** Let \(\lambda = (\sigma, -)\) be an homogeneous weight of degree \(-1\) or \(-3\) which is not contained in \(DS\). Hence \(D\lambda \supset A'\) and \(D\lambda / A' \simeq L(\sigma, -)\).

**Proof.** Since \(L(\sigma, -), L(\tau, 1)\) and \(L(\tau, 2)\) are self-dual, so is \(N\). Moreover, as graded modules \(N \simeq N^*[-4]\).

If \(\lambda\) is of degree \(-1\), then the space of weights \((\sigma, -)\) in \(N(-1)\) is \(N = \lambda \oplus S\). We claim that \(\nabla \cdot N = N(0) = A'(0)\). In fact, let \(\mu\) be a weight of \(N(0)\) and \(\mu^* \subset (N(0))^*\) the dual space of \(\mu\). We see that
\[\langle \mu^*, \nabla \cdot N \rangle = \langle \mu^*, \nabla D(S_3) \cdot N \rangle = \langle S(\nabla D(S_3)) \cdot \mu^*, N \rangle = \langle \nabla \cdot \mu^*, N \rangle \neq 0,\]
and it is non-zero because \(N \simeq N^*[-4]\) and Lemma 4.5(iii).

In a similar way, we can show that \(V \cdot \tilde{N} = N(-4) = A'(-4)\) where \(\tilde{N}\) is the space of weights \((\sigma, -)\) in \(N(-3)\). Also, we can show that \(V \cdot N\) has a weight \(\mu_1 \simeq \varepsilon\) and \(\nabla \cdot N\) has a weight \(\mu_2 \simeq \varepsilon\), both weights are of degree \(-2\).

We claim that \(\mu_1 = \mu_2\). Indeed, the space of weights \(\varepsilon\) of \(D\lambda / DA'(0)\) is \(\mu_1 + \mu_2 + k\varepsilon_{-4}\) where \(k\varepsilon_{-4}\) is the trivial weight of \(A'(-4)\). On the other hand, \((\sigma, -)\) is a highest-weight generating \(D\lambda / DA'(0)\) and hence \(D\lambda / DA'(0)\) is a quotient of \(M(\sigma, -)\). As \(M(\sigma, -)\) has only two copies of \(\varepsilon\) we deduce that \(\mu_1 = \mu_2\).

Finally, the element
\[z = 3(c_4 \circ c_2 + c_5 \circ c_3) + 2(\zeta - 1)(c_3 \circ c_6 + c_2 \circ c_7) + (4\zeta^2 - \zeta)(c_2 \circ c_5 + c_3 \circ c_4)\]
belongs in a weight \((\sigma, -)\) in \(A'(-3)\) by the fusion rules. Moreover, we have that
\[\varepsilon'_{-2} = (1 + (13) + (23))y_{(12)} \cdot z.\]
Therefore \(k\varepsilon'_{-2} = \mu_1 = \mu_2\). This finishes the proof. \(\square\)

**Proof of Proposition 4.3.** Let \(\lambda\) be as in Lemma 4.6. Then \(A = D\lambda\) satisfies the properties of the statement by Lemmas 4.4, 4.6 and 4.6. \(\square\)
4.4. The case \( L(e, \rho) \otimes L(e, \rho) \). As in \([6]\) we can see that

\[
\text{ch}^\bullet (L(e, \rho) \otimes L(e, \rho)) = (8)
\]

\[
= \text{ch}^\bullet L(e, -) + (1 + t^{-2} + t^{-4}) \text{ch}^\bullet L(\varepsilon) + (1 + t^{-2}) \text{ch}^\bullet L(e, \rho) + 2t^{-1}L(\sigma, -).
\]

Therefore \( L(e, -) \) is a graded direct summand of \( L(e, \rho) \otimes L(e, \rho) \).

**Proposition 4.7.** Let \( B \) be a graded complement of \( L(e, -) \). Then \( B \) is indecomposable and

\[
L(e, \rho) \otimes L(e, \rho) \simeq L(e, -) \oplus B
\]

as graded modules. Moreover,

\[
\text{soc } B = H \simeq t^{-1}L(\sigma, -),
\]

\[
\text{soc}^2 B / \text{soc } B \simeq (1 + t^{-2} + t^{-4})L(\varepsilon) \oplus (1 + t^{-2})L(e, \rho),
\]

\[
\text{soc}^3 B / \text{soc}^2 B \simeq t^{-1}L(\sigma, -),
\]

\[
\text{soc}^3 B = B.
\]

**Proof.** By Remark 2.3, there exists \( B \) such that \( L(e, \rho) \otimes L(e, \rho) \simeq L(e, -) \oplus B \). We will show in Lemma 4.8 that such a \( B \) satisfies the required properties. \( \square \)

Figure 2 helps the reader to visualize the module \( B \) and to follow the proof of the next lemma.

**Figure 2.** The dots represent the weights of \( B \). Each shadow region correspond to a composition factor whose highest-weight is in the top. The actions of \( V \) and \( \overline{V} \) are illustrated by the arrows.

We define the elements \( h, h' \in B(-1) \) by

\[
h = b_4 \otimes (b_7 - b_6) - (b_7 - b_6) \otimes b_4 \quad \text{and} \quad h' = b_4 \otimes b_7 - b_4 \otimes b_6.
\]

Using the fusion rule \([18, (15)]\) we obtain that \( \mathcal{D}(S_3)h \simeq \mathcal{D}(S_3)h' \simeq (\sigma, -) \). Moreover, the space of weights \((\sigma, -) \) of \( B(-1) \) is \( \mathcal{D}(S_3)h \oplus \mathcal{D}(S_3)h' \) by \([8]\).

Let \( H \) be the submodule generated by \( h \). It is a highest-weight module since \( y_{(12)}h = 0 \), which is a straightforward computation using the Appendix.

**Lemma 4.8.** Let \( \lambda \) be a homogeneous weight of \( B(\ell) \) and \( \mathcal{D}\lambda \) denote the submodule generated by \( \lambda \).
Assume that
\[ \text{Proof. } y \]
Recall the quotients of
\[ \lambda \]
Clearly,
\[ 2.3. \text{ Since } \]
\[ \ker \]
Hence,
\[ D \]
weight
\[ L \]
of
\[ \zeta \]
Hence
\[ (12) \]
·
\[ b \]
vi) If
\[ S \]
(vii) If
\[ \lambda \]
λ
\[ \text{ of } \zeta \]
\[ 3 \]
(ii)
\[ \lambda \]
(iv)
\[ \text{ If } \lambda = (e, \rho) \text{ and } \ell = -2, \text{ then } D \lambda \text{ is is an extension of } L(e, \rho) \text{ by } H. \]
(vi) In this case
\[ \lambda \]
\[ \text{ is a quotient of the Verma module } \lambda \]
\[ \text{ is a highest-weight. Then } \]
\[ \text{ and } \]
\[ \pi \]
\[ \text{ by } \]
\[ \mathcal{B}(0) \]
\[ \lambda \]
\[ \text{ of } L(\sigma, -) \rightarrow D \lambda \rightarrow L(\varepsilon) \]
\[ \text{ by } \]
\[ \text{ Hence } H \text{ is a submodule of } D \lambda \text{ we deduce that } H \simeq L(\sigma, -) \text{ and } (ii) \text{ and } (iii) \text{ follow. } \]

In case [iii] \( m = b_1 \otimes b_2 + b_4 \otimes b_4 + b_5 \otimes b_5 \) is a basis of \( \lambda \), cf. \[ 18 \] §2.5.4]. Then we see that \( y_{(12)} \cdot m = h \) and \( x_{(12)} \cdot m = (13)x_{(13)}x_{(23)} \cdot h. \]

For (iv) \( n = b_1 \otimes b_2 + b_2 \otimes b_1 \) is a basis of \( \lambda \) and we have that \( x_{(12)} \cdot n = 0 \) and \( y_{(12)} \cdot n = (13)x_{(13)}x_{(23)} \cdot h. \]

A basis of \( \lambda \) is formed by \( b_0 \otimes b_0 \) and \( b_7 \otimes b_7 \). Clearly \( \lambda \) is a highest-weight. Then \( D \lambda \) is a quotient of the Verma module \( M(e, \rho) \). Let \( \pi : M(e, \rho) \rightarrow D \lambda \) be the induced morphism, which is analogous to that in the case (i). Since \( B \) has no composition factors isomorphic to \( L(\tau, 0) \) by \[ 8 \], we deduce that the socle of \( M(e, \rho) \) is contained in \( \ker \pi \), see \[ 18 \] Theorem 10]. Finally, we have that \( \pi(\epsilon_0) = -h \), cf. \[ 18 \] §4.6.

In this case \( D \lambda \) has a composition factor isomorphic to \( L(e, \rho)[-2] \) by Remark 2.3. Since \( \text{ch}^* L(e, \rho) \simeq (e, \rho) + t^{-1}(\sigma, -) + t^{-2}(\tau, 0) \), \( \mathcal{B}^2(V) \lambda \) contains the unique weight \( \mu = (\tau, 0) \) of \( \mathcal{B}(-4) \) and \( D \mu = D \lambda \). Notice that \( \mu \) is a lowest-weight. Hence, \( D \lambda \) is a quotient of the co-Verma module \( W(\tau, 0) \). We have that \( p = \zeta^2 b_0 \otimes b_0 + b_2 \otimes b_4 + \zeta b_5 \otimes b_5 \) belongs in \( \lambda \) and \( y_{(12)} \cdot p = h. \) Then \( D \lambda \) is an extension of \( L(\lambda) \) by \( H \) thanks to Lemma 2.4.

Let \( u = sh + th' \) be a generator of \( \lambda \) for some \( s, t \in k \) with \( t \neq 0 \), that is \( D(\mathbb{S}_3)u = \lambda \). Then, the next elements are linearly independent:
\[ y_{(12)} \cdot u = -t(b_5 - b_0) \otimes (b_7 - b_0), \]
\[ (13)y_{(12)} \cdot u = -t(\zeta b_7 - \zeta^2 b_6) \otimes (\zeta b_7 - \zeta^2 b_6), \]
\[ (23)y_{(12)} \cdot u = -t(\zeta^2 b_7 - \zeta b_6) \otimes (\zeta^2 b_7 - \zeta b_6). \]

Hence \( \nabla \cdot \lambda \) coincides with the \( D(\mathbb{S}_3) \)-submodule \( (e, \rho) \oplus \varepsilon \) contained in \( \mathcal{B}(0) \) by the fusion rules. Therefore the submodules in \[ (i) \] \[ (ii) \] and \[ (v) \] are contained in \( D \lambda \).
On the other hand, $x_{(23)}x_{(12)}x_{(13)} \cdot u = tn$ and $x_{(13)}x_{(12)}x_{(13)} \cdot u = -2tb_1 \otimes b_1$, where the second element belongs in the weight $(\tau, 0)$ of $B(-4)$ by the fusion rules. Hence $D\lambda$ contains the submodules in (iv) and (vi).

Finally, $(1 + (13) + (23))x_{(12)} \cdot u = -tm$. Then $\text{ch}^* D\lambda = \text{ch}^* B$ and (vii) follows. □

4.5. The case $L(\sigma, -) \otimes L(e, \rho)$. In $K[t, t^{-1}]$ it holds that

$$\text{ch}^* (L(\sigma, -) \otimes L(e, \rho)) = 2t^{-1} \text{ch}^* L(\tau, 0) + (1 + t^{-2}) \text{ch}^* L(\sigma, -).$$

Therefore $L(\sigma, +)$ is a graded direct summand of $L(\sigma, -) \otimes L(e, \rho)$.

**Proposition 4.9.** Let $C$ be a graded complement of $L(\sigma, +)$. Then $C$ is indecomposable, $\text{ch}^* C = 2t^{-1} \text{ch}^* L(\tau, 0) + (1 + t^{-2}) \text{ch}^* L(\sigma, -)$ and

$L(\sigma, -) \otimes L(e, \rho) \simeq L(\sigma, +) \oplus C$

as graded modules. Moreover, the socle filtration of $C$ satisfies

- $\text{soc} C \simeq t^{-1}L(\tau, 0)$,
- $\text{soc}^2 C / \text{soc} C \simeq (1 + t^{-2})L(\sigma, -)$,
- $\text{soc}^3 C / \text{soc}^2 C \simeq t^{-1}L(\tau, 0)$,
- $\text{soc}^4 C = C$.

**Figure 3.** The dots represent the weights of $C$. Each shaded area corresponds to a composition factor whose highest-weight is in the top. The actions of $V$ and $\overline{V}$ are illustrated by the arrows.

**Proof.** The weight $\lambda = (\sigma, -)$ of $C(0)$ is a highest-weight. Then we have a projection $\pi : M(\sigma, -) \to D\lambda$ and hence $D\lambda$ is a quotient of $M(\sigma, -)$ by one of the submodules given in [18, Theorem 7]. By $\text{ch}^* C$, we see that either $D\lambda \simeq L(\sigma, -)$ or $D\lambda$ is an extension of $L(\sigma, -)$ by $L(\tau, 0)$.

By the fusion rules [18, §2.5.4], $t = c_8 \otimes (b_6 + b_7)$ generates $\lambda$. Let $o_0 \in M(\sigma, -)$ be as in [18, Lemma 23]. Then

$$0 \neq \pi(o_0) = x_{(13)} \cdot t + x_{(12)} \cdot (c_9 \otimes (\zeta^2 b_6 + \zeta b_7)) + x_{(23)} \cdot (c_{10} \otimes (\zeta b_6 + \zeta^2 b_7))$$
Therefore \( D\lambda \) is an extension of \( L(\sigma, -) \) by \( L(\tau, 0) \). In particular, this shows that \( t^{-1}L(\tau, 0) \subseteq \text{soc} C \).

If \( u \in M(\sigma, -) \) as in [18] Lemma 23, we have

\[
\pi(u) = \zeta^2 x_{(12)}x_{(13)}x_{(12)}(13) \cdot t - \zeta x_{(12)}x_{(13)}x_{(23)}(23) \cdot t + x_{(13)}x_{(12)}x_{(23)} \cdot t
\]

\[
= 3\left(\zeta^2 c_2 \otimes b_3 + \zeta c_3 \otimes b_5 - c_4 \otimes b_2 + c_1 \otimes b_4 + c_7 \otimes b_1\right).
\]

On the other hand, lowest-weight \( \mu = (\sigma, -) \subset C(-4) \) is generated by \( t' = c_3 \otimes b_1 + c_2 \otimes b_2 \), cf. [18] §2.5.4. We have that

\[
y_{(12)} \cdot t' - \zeta^2 y_{(23)}(13) \cdot t' - \zeta y_{(13)}(23) \cdot t' =
\]

\[
= (1 - \zeta)c_2 \otimes b_3 + (\zeta^2 - 1)c_3 \otimes b_5 - \frac{3\zeta^2}{\zeta - 1} c_4 \otimes b_2 + (\zeta - \zeta^2)c_1 \otimes b_4 + \frac{3\zeta^2}{\zeta - 1} c_7 \otimes b_1
\]

\[
= \frac{\zeta^2}{\zeta - 1} \pi(u).
\]

Therefore \( D\mu \) is an extension of \( L(\sigma, -) \) by \( L(\tau, 0) \) thanks to Lemma 2.4 and \( D\mu \cap D\lambda \simeq L(\tau, 0) \).

Let \( N \) denote the space of weights \( (\tau, 0) \) contained in \( (L(\sigma, -) \otimes L(e, \rho))(−1) \). By the fusion rules [18] §2.5.4, \( N \) is generated by

\[
\{c_4 \otimes b_6, \ c_6 \otimes b_7, \ c_8 \otimes b_3 + c_9 \otimes b_5 + c_{10} \otimes b_4\}.
\]

Hence \( \nabla \cdot N \subset \lambda \). In fact,

\[
\zeta t = (1-(12))y_{(13)} \cdot (c_4 \otimes b_6),
\]

\[
\zeta^2 t = (1-(12))y_{(13)} \cdot (c_6 \otimes b_7)
\]

and

\[
(\zeta - \zeta^2)t = (1-(12))y_{(13)} \cdot (c_8 \otimes b_3 + c_9 \otimes b_5 + c_{10} \otimes b_4).
\]

In particular, if \( \nu \simeq (\tau, 0) \) is a weight of \( C(-1) \) which is different from \( D(S_3) \cdot \pi(\omega_0) \), then \( \nabla \cdot \nu = \lambda \). Hence \( D\nu \) contains \( D\lambda \).

Let \( \eta \simeq (e, \rho) \) be a weight of \( D\nu \) which is different from \( D(S_3) \cdot \pi(u) \). We claim that \( \nabla \cdot \eta = \mu \). Otherwise, \( \eta \) should be a lowest-weight because of \( \text{ch}^* C \). Hence \( D\nu \) is a quotient of \( W(e, \rho) \) with two composition factors isomorphic to \( \text{ch}^* L(\tau, 0) \). However, this can not happen by Lemma 2.4 and our claim follows.

Therefore \( D\nu = C \) because \( \text{ch}^* D\nu = \text{ch}^* C \).

4.6. The remainder cases. The functor \( L(e) \otimes - \) is the identity and \( M \otimes N \simeq N \otimes M \) because \( D \) is quasitriangular. Thus, we finish the description of the tensor product between non-projective simple modules with the next proposition.

**Proposition 4.10.** We have that

\[
L(\tau, 0) \otimes L(\tau, 0) \simeq L(e, -) \oplus B^*,
\]

\[
L(\sigma, -) \otimes L(\tau, 0) \simeq L(\sigma, +) \oplus C^*.
\]

**Proof.** It follows from dualizing the isomorphisms of Propositions 4.7 and 4.9.
5. The projective modules

We denote by \( P(\lambda) \) the projective cover of \( L(\lambda) \), \( \lambda \in \Lambda \). Since \( \mathcal{D} \) is symmetric \([16]\), \( P(\lambda) \) also is the injective hull of \( L(\lambda) \).

Up to shifts, \( P(\lambda) \) admits a unique \( \mathbb{Z} \)-grading \([11]\). We fix one such that \( \lambda \) is a homogeneous weight of degree 0 generating \( P(\lambda) \). Thus, \( P(\lambda) \) also is the projective cover and the injective hull of \( L(\lambda) \) as a graded module, cf. \([20]\) Lemma 3.1.

Let \( R^\bullet_{proj} \) denote the Grothendieck ring of the subcategory of projective modules. The sets \( \{ \text{ch}^* P(\lambda) \mid \lambda \in \Lambda \} \), \( \{ \text{ch}^* M(\lambda) \mid \lambda \in \Lambda \} \) and \( \{ \text{ch}^* W(\lambda) \mid \lambda \in \Lambda \} \) are \( \mathbb{Z}[t, t^{-1}] \)-bases of \( R^\bullet_{proj} \) \([20]\) Remark 3.2. Then, for every graded projective module \( P \), there are polynomials \( p_{P,P(\lambda)}, p_{P,M(\lambda)} \) and \( p_{P,W(\lambda)} \) in \( \mathbb{Z}[t, t^{-1}] \) satisfying

\[
\text{ch}^* P = \sum_{\lambda \in \Lambda} p_{P,P(\lambda)} \text{ch}^* P(\lambda) \iff P \simeq \oplus_{\lambda \in \Lambda} p_{P,P(\lambda)} P(\lambda) \quad \text{as graded modules.}
\]

(10)

\[
\text{ch}^* P = \sum_{\lambda \in \Lambda} p_{P,M(\lambda)} \text{ch}^* M(\lambda) \iff P \simeq \oplus_{\lambda \in \Lambda} p_{P,M(\lambda)} M(\lambda) \quad \text{as graded } \mathcal{D}^{\leq 0}\text{-modules.}
\]

(11)

\[
\text{ch}^* P = \sum_{\lambda \in \Lambda} p_{P,W(\lambda)} \text{ch}^* W(\lambda) \iff P \simeq \oplus_{\lambda \in \Lambda} p_{P,W(\lambda)} W(\lambda) \quad \text{as graded } \mathcal{D}^{\geq 0}\text{-modules.}
\]

(12)

The graded BGG Reciprocity \([20]\) Corollary 3.6 and Theorem 4.9] states that

\[
p_{P,M(\lambda)} = \frac{1}{t^{\text{ht}(\lambda)}} p_{M(\lambda),L(\mu)} = t^4 p_{P(\mu),W(\lambda)}.
\]

(13)

for all \( \mu, \lambda \in \Lambda \). Therefore,

\[
\text{ch}^* P(\epsilon) = (1 + t^4) \text{ch}^* M(\epsilon) + (t + t^3) \text{ch}^* M(\sigma, -),
\]

\[
\text{ch}^* P(\epsilon, \rho) = \text{ch}^* M(\epsilon, \rho) + t \text{ch}^* M(\sigma, -) + t^2 \text{ch}^* M(\tau, 0),
\]

\[
\text{ch}^* P(\sigma, -) = (1 + t^2) \text{ch}^* L(\sigma, -) + t \text{ch}^* M(\epsilon) + t \text{ch}^* M(\epsilon, \rho) + t \text{ch}^* M(\tau, 0),
\]

\[
\text{ch}^* P(\tau, 0) = \text{ch}^* M(\tau, 0) + t \text{ch}^* M(\sigma, -) + t^2 \text{ch}^* M(\epsilon, \rho),
\]

\[
\text{ch}^* P(\lambda) = \text{ch}^* M(\lambda), \quad \forall \lambda \in \Lambda_{sp}.
\]

We give more information on the structure of the indecomposable projective modules, cf. \([20]\) Remark 4.4. In the following, if \( M(\lambda)[\ell] \) is a graded shift of a Verma module, we shall denote its highest-weight by \( 1 \otimes \lambda[\ell] \). We will omit \( \ell \) if it is zero.

**Proposition 5.1.** As graded \( \mathcal{D}^{\leq 0} \)-modules,

\[
P(\sigma, -) = M(\sigma, -)[2] \oplus M(\epsilon)[1] \oplus M(\epsilon, \rho)[1] \oplus M(\tau, 0)[1] \oplus M(\sigma, -).
\]

The action of \( \nabla \) satisfies:

\[
\nabla \cdot (1 \otimes (\sigma, -)[2]) = 0,
\]

\[
\nabla \cdot (1 \otimes \epsilon[1]) = 1 \otimes (\sigma, -)[2],
\]

\[
\nabla \cdot (1 \otimes (\epsilon, \rho)[1]) = 1 \otimes (\sigma, -)[2],
\]

\[
\nabla \cdot (1 \otimes (\tau, 0)[1]) = 1 \otimes (\sigma, -)[2].
\]
Moreover, the projection of $\nabla \cdot (1 \otimes (\sigma, -))$ over $M(\lambda)[1]$ is equal to $(1 \otimes \lambda)[1]$ for all $\lambda \in \{\varepsilon, (\varepsilon, \rho), (\tau, 0)\}$.

Therefore

(i) The submodule generated by $1 \otimes (\sigma, -)[2]$ is isomorphic to $M(\sigma, -)[2]$.

(ii) The submodule generated by $1 \otimes \lambda[1]$ is equal to $M(\sigma, -)[2] \oplus M(\lambda)[1]$ as graded $D^{\leq 0}$-module for all $\lambda \in \{\varepsilon, (\varepsilon, \rho), (\tau, 0)\}$.

(iii) $P(\sigma, -)$ is generated by the homogeneous weight $1 \otimes (\sigma, -)$ of degree 0.

(iv) The following are standard filtrations of $P(\sigma, -)$

$M(\sigma, -)[2] \subset \mathcal{D} \cdot (1 \otimes \lambda_1[1]) \subset \mathcal{D} \cdot (1 \otimes \lambda_1[1]) + \mathcal{D} \cdot (1 \otimes \lambda_2[1])$

$\subset \mathcal{D} \cdot (1 \otimes \lambda_1[1]) + \mathcal{D} \cdot (1 \otimes \lambda_2[1]) + \mathcal{D} \cdot (1 \otimes \lambda_3[1]) \subset P(\sigma, -)$

where $\{\lambda_1, \lambda_2, \lambda_3\} = \{\varepsilon, (\varepsilon, \rho), (\tau, 0)\}$.

Proof. The structure of $D^{\leq 0}$-module of $P(\sigma, -)$ follows by (11). A direct consequence of this isomorphism is that $M(\lambda)[\ell]$ is a graded submodule of $P(\sigma, -)$ if $1 \otimes \lambda[\ell]$ is a highest-weight. But $P(\sigma, -)$ has only one Verma submodule because its socle is simple. Then we see that such a Verma module is $M(\sigma, -)[2]$.

To calculate the $\nabla$-actions, we shall use the grading on $P(\sigma, -)$ which ensures that $\nabla \cdot (1 \otimes \lambda[\ell]) \subseteq P(\sigma, -)[\ell \lambda + 1]$. Then the action of $\nabla$ on $(1 \otimes (\sigma, -))[2]$ is zero because $P(\sigma, -)[3] = 0$. This shows (i).

By the graded character, $P(\sigma, -)[2] = 1 \otimes (\sigma, -)[2]$ and then $0 \neq \nabla \cdot (1 \otimes \lambda[1]) \subset 1 \otimes (\sigma, -)[2]$. Hence the equality holds, because $1 \otimes (\sigma, -)$ is a weight, and (ii) follows.

We now analyze the action on $1 \otimes (\sigma, -)$. We have that

$\nabla \cdot (1 \otimes (\sigma, -)) \subset (1 \otimes (1 \otimes \sigma, -)[1]) \oplus (1 \otimes (\sigma, -)[1]) \oplus M(\sigma, -)[2](1)$. 

If the projection of $\nabla \cdot (1 \otimes (\sigma, -))$ over $1 \otimes \lambda[1]$ is zero for some $\lambda \in \{\varepsilon, (\varepsilon, \rho), (\tau, 0)\}$, then the submodule $N$ generated by $1 \otimes (\sigma, -)$ satisfies $P(\sigma, -)/N \simeq M(\lambda)[1]$ by (ii)

But this is not possible since $P(\sigma, -)$ has simple head. Hence the projection is equal to $1 \otimes \lambda[1]$ because it is a weight. In particular, we see that (iii) holds.

The filtrations in (iv) are standard by (ii) and (iii).

The demonstrations of the next results are analogous to Proposition 5.1

**Proposition 5.2.** As graded $D^{\leq 0}$-modules,

$P(\varepsilon) = M(\varepsilon)[4] \oplus M(\sigma, -)[3] \oplus M(\sigma, -)[1] \oplus M(\varepsilon)$.

The action of $\nabla$ satisfies:

$\nabla \cdot (1 \otimes \varepsilon[4]) = 0$,

$\nabla \cdot (1 \otimes (\sigma, -)[3]) = 1 \otimes \varepsilon[4]$.

Moreover, the projection of $\nabla \cdot (1 \otimes \varepsilon)$ over $M(\sigma, -)[1]$ is equal to $1 \otimes (\sigma, -)[1]$.

Therefore

(i) $\mathcal{D} \cdot (1 \otimes \varepsilon[4]) \simeq M(\varepsilon)[4]$.

(ii) $\mathcal{D} \cdot (1 \otimes (\sigma, -)[3]) = M(\varepsilon)[4] \oplus M(\sigma, -)[3]$ as graded $D^{\leq 0}$-modules.

(iii) $\mathcal{D} \cdot (1 \otimes (\sigma, -)[1]) + \mathcal{D} \cdot (1 \otimes (\sigma, -)[3]) = M(\varepsilon)[4] \oplus M(\sigma, -)[3] \oplus M(\sigma, -)[1]$ as graded $D^{\leq 0}$-modules.

(iv) $P(\varepsilon) = \mathcal{D} \cdot (1 \otimes \varepsilon)$. 

The following is a standard filtration of $P(\varepsilon)$

$$M(\varepsilon)[4] \subset \mathcal{D} \cdot (1 \otimes (\sigma, -))[3] \subset \mathcal{D} \cdot (1 \otimes (\sigma, -))[1] + \mathcal{D} \cdot (1 \otimes (\sigma, -))[3] \subset P(\varepsilon).$$

Proof. The equality for the action of $V$ over $1 \otimes \varepsilon[4]$ and $1 \otimes (\sigma, -)[3]$ is a direct consequence of the grading. Hence, we can deduce that the projection of $V \cdot (1 \otimes \varepsilon)$ over $M(\sigma, -)[1]$ is equal to $1 \otimes (\sigma, -)[1]$ arguing as in the above proposition. For [iii] note that $V \cdot (1 \otimes (\sigma, -)[1]) \subset P(\varepsilon)[2] \subset M(\varepsilon)[4] \oplus M(\sigma, -)[3].$ \hfill \Box

**Proposition 5.3.** As graded $\mathcal{D}^{\leq 0}$-modules,

$$P(e, \rho) = M(\tau, 0)[2] \oplus M(\sigma, -)[1] \oplus M(e, \rho).$$

The action of $V$ satisfies:

$$V \cdot (1 \otimes (\tau, 0)[2]) = 0,$$

$$V \cdot (1 \otimes (\sigma, -)[1]) = 1 \otimes (\tau, 0)[2].$$

Moreover, the projection of $V \cdot (1 \otimes (e, \rho))$ over $M(\sigma, -)[1]$ is equal to $1 \otimes (\sigma, -)[1].$

Therefore

(i) $\mathcal{D} \cdot (1 \otimes (\tau, 0)[2]) \simeq M(\tau, 0)[2].$

(ii) $\mathcal{D} \cdot (1 \otimes (\sigma, -)[1]) = M(\tau, 0)[2] \oplus M(\sigma, -)[1]$ as graded $\mathcal{D}^{\leq 0}$-modules.

(iii) $P(e, \rho) = \mathcal{D} \cdot 1 \otimes (e, \rho).$

(iv) The following is a standard filtration of $P(e, \rho)$

$$M(\tau, 0)[2] \subset \mathcal{D} \cdot (1 \otimes (\sigma, -)[1]) \subset P(e, \rho).$$ \hfill \Box

**Proposition 5.4.** As graded $\mathcal{D}^{\leq 0}$-modules,

$$P(\tau, 0) = M(e, \rho)[2] \oplus M(\sigma, -)[1] \oplus M(\tau, 0).$$

The action of $V$ satisfies:

$$V \cdot (1 \otimes (e, \rho)[2]) = 0,$$

$$V \cdot (1 \otimes (\sigma, -)[1]) = (1 \otimes (e, \rho)[2]).$$

Moreover, the projection of $V \cdot (1 \otimes (\tau, 0))$ over $M(\sigma, -)[1]$ is equal to $1 \otimes (\sigma, -)[1].$

Therefore

(i) $\mathcal{D} \cdot (1 \otimes (e, \rho)[2]) \simeq M(e, \rho)[2].$

(ii) $\mathcal{D} \cdot (1 \otimes (\sigma, -)[1]) = M(e, \rho)[2] \oplus M(\sigma, -)[1]$ as graded $\mathcal{D}^{\leq 0}$-modules.

(iii) $P(\tau, 0) = \mathcal{D} \cdot 1 \otimes (\tau, 0).$

(iv) The following is a standard filtration of $P(\tau, 0)$

$$M(e, \rho)[2] \subset \mathcal{D} \cdot (1 \otimes (\sigma, -)[1]) \subset P(\tau, 0).$$ \hfill \Box
5.1. The induced modules. Given $\lambda \in \Lambda$, we set
\[
\text{Ind}(\lambda) = \mathcal{D} \otimes_{\mathcal{D}(G)} \mathbf{1} \cong \mathfrak{B}(V) \otimes \mathfrak{B}(\overline{V}) \otimes \lambda \cong M(\text{ch} \mathfrak{B}(\overline{V}) \cdot \lambda),
\] 
(14)
where the isomorphisms are of $\mathbb{Z}$-graded $\mathcal{D} \otimes \mathcal{D}^\mathbb{Z}$-modules \cite{20} Definition 2.7. Thanks to \cite{20} Theorem 4.9 the induced modules help to describe the product in $R_{\text{proj}}^\bullet$. By \cite{20} (34), $\text{Ind}(\mu) \cong \oplus_{\lambda \in \Lambda} \text{Ind}(\lambda, \mu) \cdot \text{Ind}(\lambda)$.

Therefore
\[
\text{Ind}(\varepsilon) \cong P(\varepsilon) \oplus P(\tau, 2)[2] \oplus P(\tau, 2)[2],
\]
\[
\text{Ind}(e, -) \cong (1 + t^4) \cdot P(e, -) \oplus (t + t^2) \cdot P(\varepsilon, \varepsilon) \oplus P(\tau, 2)[2] \oplus P(\tau, 2)[2],
\]
\[
\text{Ind}(e, \rho) \cong P(e, \rho) \oplus (t + t^2) \cdot P(\varepsilon, \varepsilon) \oplus P(\tau, 0)[2] \oplus P(\tau, 1)[2] \oplus P(\tau, 1)[2] \oplus P(\tau, 2)[2],
\]
\[
\text{Ind}(\sigma, -) \cong (1 + t^2) \cdot P(\sigma, -) \oplus P(\sigma, +) \oplus P(\tau, 1) \oplus (t + t^2) \cdot P(\tau, 2),
\]
\[
\text{Ind}(\sigma, +) \cong (t + t^3) \cdot P(e, -) \oplus P(e, \rho)[1] \oplus (1 + 2t^2 + t^4) \cdot P(\sigma, +) \oplus P(\tau, 0)[1] \oplus (t + t^2) \cdot P(\tau, 1) \oplus (t + t^2) \cdot P(\tau, 2),
\]
\[
\text{Ind}(\tau, 0) \cong P(e, \rho)[2] \oplus (t + t^3) \cdot P(\sigma, +) \oplus P(\tau, 0) \oplus P(\tau, 1)[2] \oplus P(\tau, 2)[2],
\]
\[
\text{Ind}(\tau, i) \cong P(e, -)[2] \oplus P(\sigma, -)[1] \oplus (t + t^3) \cdot P(\sigma, +) \oplus P(\tau, j)[2]
\]
for $\{i, j\} = \{1, 2\}$.

5.2. The tensor products of projective modules. For $\lambda_1, \lambda_2 \in \Lambda$, it holds that
\[
P(\lambda_1) \otimes P(\lambda_2) \cong \oplus_{\lambda, \mu \in \Lambda} P_{\text{proj}}(\lambda, \mu) \cdot \text{Ind}(\lambda \cdot \mu),
\]
by \cite{20} Theorem 4.9. The polynomials $p_{\text{proj}}(\lambda, \mu)$ were given at the beginning of this section and $p_{\text{proj}}(\lambda, \mu) = t^{-4}p_{\text{proj}}(\lambda, \mu)$, recall \cite{13}. The products of weights are in \cite{18} §2.5.4. Thus, the tensor products of the projective modules follow by long and tedious computations. For instance,
\[
P(\varepsilon) \otimes P(\varepsilon) \cong t^{-4}(t^6 + t^4 + 4t^2 + 1)P(\varepsilon) \oplus 2t^{-4}(1 + t^2)^2 P(e, -) \oplus t^{-2}(1 + t^2)^3 P(e, \rho)
\]
\[
\oplus 2t^{-2}(1 + t^2 + t^4)(1 + t^2)^2 P(\sigma, -) \oplus 8t^{-1}(1 + t^2 + t^4)(1 + t^2) P(\sigma, +)
\]
\[
\oplus t^{-2}(1 + t^2)^3 P(\tau, 0) \oplus t^{-2}((1 + t^4)^2 + 2(1 + t^2)^4) P(\tau, 1) \oplus P(\tau, 2).
\]

In the case of the simple projective modules, their fusion rules follow directly since $L(\lambda) \cong P(\lambda) \cong M(\lambda) \cong W(\lambda)$.

Proposition 5.5. Let $\lambda, \mu \in \Lambda_{sp}$. Hence $L(\lambda) \otimes L(\mu) \cong \text{Ind}(\lambda \cdot \mu)$.

5.3. Simple tensoring by projective modules. To conclude our work, we need to analyze the products $L(\lambda) \otimes L(\mu)$ with $\lambda \in \Lambda_{sp}$ and $\mu \notin \Lambda_{sp}$. In this case $L(\lambda)$ is projective and hence so are these tensor products. Thus, we can use the graded character to obtain the following isomorphisms thanks to \cite{10}.

Proposition 5.6. Let $\{i, j\} = \{1, 2\}$. The next isomorphisms hold in the category of graded modules.
\[
L(e, -) \otimes L(e, \rho) \cong t^{-2}L(\tau, 0)
\]
\[
L(e, -) \otimes L(\tau, 0) \cong t^{-2}P(e, \rho)
\]
\[
L(e, -) \otimes L(\sigma, -) \cong t^{-1}(L(\tau, 1) \otimes L(\tau, 2)) \oplus (1 + t^{-2}) L(\sigma, +)
\]
\(L(\tau, i) \otimes L(e, \rho) \simeq (1 + t^{-2})L(\tau, j) \oplus t^{-1}L(\sigma, +) \oplus t^{-2}P(e, \rho)\)
\(L(\tau, i) \otimes L(\tau, 0) \simeq (1 + t^{-2})L(\tau, j) \oplus t^{-1}L(\sigma, +) \oplus t^{-2}P(\tau, 0)\)
\(L(\tau, i) \otimes L(\sigma, -) \simeq t^{-1}(L(e, -) \oplus L(\tau, j)) \oplus (1 + t^{-2})L(\sigma, +) \oplus t^{-2}L(\sigma, -)\)
\(L(\sigma, +) \otimes L(\tau, 0) \simeq L(\sigma, +) \otimes L(e, \rho) \simeq t^{-1}(L(\tau, 1) \oplus L(\tau, 2)) \oplus (1 + t^{-2})L(\sigma, +) \oplus t^{-2}P(\sigma, -)\)

\(\simeq (1 + t^{-2})(L(e, -) \oplus L(\tau, 1) \oplus L(\tau, 2)) \oplus 2t^{-1}L(\sigma, +) \oplus t^{-2}(P(e, \rho) \oplus P(\tau, 0))\)

\(\square\)

**Appendix**

We give here the action of the generators of \(\mathcal{D}\) on the simple modules \(L(\lambda)\) for \(\lambda \notin \Lambda_{\mathrm{sp}}\). We have computed them identifying \(L(\lambda)\) with the socle of a Verma module. Then, we use [18, Appendix A] to calculate the action of \(y_{(12)}\) and the action of \(x_{(12)}\) is just the multiplication in \(\mathfrak{B}(V)\). The actions of the remainder \(y_{(ij)}\) and \(x_{(ij)}\) were deduced from the above using that the action is a morphism of \(\mathcal{D}(S_3)\)-modules. For instance, \(y_{(23)}c_2 = (13)(y_{(12)}c_1)\).

**Bases for the simple modules.** The isomorphisms listed below are of graded \(\mathcal{D}(S_3)\)-modules. These are obtained by identifying the elements of the respective ordered bases. We keep the notation of [18 §2.5.2].

- \(L(\tau, 0)\) has a homogeneous basis \(\{a_i \mid 1 \leq i \leq 7\}\) such that
  \(\mathbb{k}\langle a_1, a_2 \rangle \simeq \mathbb{k}\{[123], [132]\} \simeq (e, \rho), \quad \deg a_1 = \deg a_2 = -2,\)
  \(\mathbb{k}\langle a_3, a_4, a_5 \rangle \simeq \mathbb{k}\{[12], [13], [23]\} \simeq (\sigma, +), \quad \deg a_3 = \deg a_4 = \deg a_5 = -1,\)
  \(\mathbb{k}\langle a_6, a_7 \rangle \simeq \mathbb{k}\{[123], [132]\} \simeq (\tau, 0), \quad \deg a_6 = \deg a_7 = 0,\)

The first weight corresponds to \(C\) in [18 §4.5] and the last one to \(\mathfrak{B}^{\text{top}}(V) \otimes (e, \rho)\).

- \(L(e, \rho)\) has a homogeneous basis \(\{b_i \mid 1 \leq i \leq 7\}\) such that
  \(\mathbb{k}\langle b_1, b_2 \rangle \simeq \mathbb{k}\{[123], [132]\} \simeq (\tau, 0), \quad \deg b_1 = \deg b_2 = -2,\)
  \(\mathbb{k}\langle b_3, b_4, b_5 \rangle \simeq \mathbb{k}\{[23], [12], [13]\} \simeq (\sigma, +), \quad \deg b_3 = \deg b_4 = \deg b_5 = -1,\)
  \(\mathbb{k}\langle b_6, b_7 \rangle \simeq \mathbb{k}\{[132], [123]\} \simeq (e, \rho), \quad \deg b_6 = \deg b_7 = 0,\)

The first weight corresponds to \(G\) in [18 §4.6] and the last one to \(\mathfrak{B}^{\text{top}}(V) \otimes (\tau, 0)\).

- \(L(\sigma, -)\) has a homogeneous basis \(\{c_i \mid 1 \leq i \leq 10\}\) such that
  \(\mathbb{k}\langle c_1, c_2, c_3 \rangle \simeq \mathbb{k}\{[12], [23], [13]\} \simeq (\sigma, -), \quad \deg c_1 = \deg c_2 = \deg c_3 = -2,\)
  \(\mathbb{k}\langle c_4, c_5 \rangle \simeq \mathbb{k}\{[123], [132]\} \simeq (\tau, 1), \quad \deg c_4 = \deg c_5 = -1,\)
  \(\mathbb{k}\langle c_6, c_7 \rangle \simeq \mathbb{k}\{[123], [132]\} \simeq (\tau, 2), \quad \deg c_6 = \deg c_7 = -1,\)
  \(\mathbb{k}\langle c_8, c_9, c_{10} \rangle \simeq \mathbb{k}\{[12], [23], [13]\} \simeq (\sigma, -), \quad \deg c_8 = \deg c_9 = \deg c_{10} = 0,\)

The listed weights correspond to \(R, N_1, N_2\) and \(\mathfrak{B}^{\text{top}}(V) \otimes (\sigma, -)\) of [18 §4.3], respectively.
• $L(\varepsilon) = k(d_1)$ is one-dimensional of degree 0.

**Action on the bases.** In this section we give the necessary calculations with the basis elements in order to simplify the proofs given in the article.

| $a_i$ | $a_j$ | $a_k$ |
|-------|-------|-------|
| $a_1$ | $a_2$ | $a_3$ |
| $a_4$ | $a_5$ | $a_6$ |
| $a_7$ | | |

| $x_{12}$ | $x_{13}$ | $x_{23}$ |
|----------|----------|----------|
| $x_{12}a_1 = 0$ | $x_{13}a_1 = 0$ | $x_{23}a_1 = 0$ |
| $x_{12}a_2 = 0$ | $x_{13}a_2 = 0$ | $x_{23}a_2 = 0$ |
| $x_{12}a_3 = a_1 - a_2$ | $x_{13}a_3 = \zeta^2 a_1 - \zeta a_2$ | $x_{23}a_3 = 0$ |
| $x_{12}a_4 = 0$ | $x_{13}a_4 = 0$ | $x_{23}a_4 = \zeta a_1 - \zeta^2 a_2$ |
| $x_{12}a_5 = 0$ | $x_{13}a_5 = 0$ | $x_{23}a_5 = 0$ |
| $x_{12}a_6 = a_5$ | $x_{13}a_6 = a_5$ | $x_{23}a_6 = a_3$ |
| $x_{12}a_7 = -a_4$ | $x_{13}a_7 = -a_4$ | $x_{23}a_7 = -a_5$ |

| $y_{12}$ | $y_{13}$ | $y_{23}$ |
|----------|----------|----------|
| $y_{12}a_1 = a_3$ | $y_{13}a_1 = \zeta a_3$ | $y_{23}a_1 = \zeta^2 a_4$ |
| $y_{12}a_2 = -a_3$ | $y_{13}a_2 = -\zeta^2 a_3$ | $y_{23}a_2 = -\zeta a_4$ |
| $y_{12}a_3 = 0$ | $y_{13}a_3 = 0$ | $y_{23}a_3 = a_6$ |
| $y_{12}a_4 = -a_7$ | $y_{13}a_4 = -a_7$ | $y_{23}a_4 = 0$ |
| $y_{12}a_5 = a_6$ | $y_{13}a_5 = a_6$ | $y_{23}a_5 = -a_7$ |
| $y_{12}a_6 = 0$ | $y_{13}a_6 = 0$ | $y_{23}a_6 = 0$ |
| $y_{12}a_7 = 0$ | $y_{13}a_7 = 0$ | $y_{23}a_7 = 0$ |

| $b_{12}$ | $b_{13}$ | $b_{23}$ |
|----------|----------|----------|
| $b_{12} = b_2$ | $b_{13} = b_2$ | $b_{23} = b_2$ |
| $b_{12} = b_1$ | $b_{13} = b_1$ | $b_{23} = b_1$ |
| $b_{12} = b_3$ | $b_{13} = b_3$ | $b_{23} = b_3$ |
| $b_{12} = b_4$ | $b_{13} = b_4$ | $b_{23} = b_4$ |
| $b_{12} = b_5$ | $b_{13} = b_5$ | $b_{23} = b_5$ |
| $b_{12} = b_7$ | $b_{13} = b_7$ | $b_{23} = b_7$ |
| $b_{12} = b_6$ | $b_{13} = b_6$ | $b_{23} = b_6$ |

| $x_{12}$ | $x_{13}$ | $x_{23}$ |
|----------|----------|----------|
| $x_{12}b_1 = 0$ | $x_{13}b_1 = 0$ | $x_{23}b_1 = 0$ |
| $x_{12}b_2 = 0$ | $x_{13}b_2 = 0$ | $x_{23}b_2 = 0$ |
| $x_{12}b_3 = b_1$ | $x_{13}b_3 = -b_2$ | $x_{23}b_3 = 0$ |
| $x_{12}b_4 = 0$ | $x_{13}b_4 = b_1$ | $x_{23}b_4 = -b_2$ |
| $x_{12}b_5 = -b_2$ | $x_{13}b_5 = 0$ | $x_{23}b_5 = b_1$ |
| $x_{12}b_6 = b_4$ | $x_{13}b_6 = \zeta^2 b_5$ | $x_{23}b_6 = \zeta b_3$ |
| $x_{12}b_7 = -b_4$ | $x_{13}b_7 = -\zeta b_5$ | $x_{23}b_7 = -\zeta^2 b_3$ |
\[
\begin{align*}
 y_{(12)}b_1 &= b_3 \\
y_{(12)}b_2 &= -b_5 \\
y_{(12)}b_3 &= 0 \\
y_{(12)}b_4 &= b_6 - b_7 \\
y_{(12)}b_5 &= 0 \\
y_{(12)}b_6 &= 0 \\
y_{(12)}b_7 &= 0 \\
(12)c_1 &= -c_1 \\
(12)c_2 &= -c_3 \\
(12)c_3 &= -c_2 \\
(12)c_4 &= c_5 \\
(12)c_5 &= c_4 \\
(12)c_6 &= c_7 \\
(12)c_7 &= c_6 \\
(12)c_8 &= -c_8 \\
(12)c_9 &= -c_10 \\
(12)c_{10} &= -c_9 \\
x_{(12)}c_1 &= 0 \\
x_{(12)}c_2 &= 0 \\
x_{(12)}c_3 &= 0 \\
x_{(12)}c_4 &= \zeta c_2 \\
x_{(12)}c_5 &= \zeta c_3 \\
x_{(12)}c_6 &= \zeta c_2^2 \\
x_{(12)}c_7 &= \zeta^2 c_3 \\
x_{(12)}c_8 &= 0 \\
x_{(12)}c_9 &= \frac{1}{\zeta^2}(c_6 - \zeta c_4) \\
x_{(12)}c_{10} &= \frac{1}{\zeta^2}(c_7 - \zeta c_5) \\
y_{(12)}c_1 &= 0 \\
y_{(12)}c_2 &= \frac{1}{\zeta^2}(c_4 - \zeta c_6) \\
y_{(12)}c_3 &= \frac{1}{\zeta^2}(c_5 - \zeta c_7) \\
y_{(12)}c_4 &= \zeta c_9 \\
y_{(12)}c_5 &= \zeta^2 c_9 \\
y_{(12)}c_6 &= \zeta^2 c_8 \\
y_{(12)}c_7 &= \zeta c_9 \\
y_{(12)}c_8 &= 0 \\
y_{(12)}c_9 &= 0 \\
y_{(12)}c_{10} &= 0 \\
y_{(13)}b_1 &= b_4 \\
y_{(13)}b_2 &= -b_3 \\
y_{(13)}b_3 &= 0 \\
y_{(13)}b_4 &= 0 \\
y_{(13)}b_5 &= \zeta b_6 - \zeta^2 b_7 \\
y_{(13)}b_6 &= 0 \\
y_{(13)}b_7 &= 0 \\
(13)c_1 &= -c_2 \\
(13)c_2 &= -c_1 \\
(13)c_3 &= -c_3 \\
(13)c_4 &= \zeta^2 c_5 \\
(13)c_5 &= \zeta c_4 \\
(13)c_6 &= \zeta c_7 \\
(13)c_7 &= \zeta^2 c_6 \\
(13)c_8 &= -c_9 \\
(13)c_9 &= -c_{10} \\
(13)c_{10} &= -c_9 \\
x_{(13)}c_1 &= 0 \\
x_{(13)}c_2 &= 0 \\
x_{(13)}c_3 &= 0 \\
x_{(13)}c_4 &= \zeta^2 c_1 \\
x_{(13)}c_5 &= c_2 \\
x_{(13)}c_6 &= \zeta c_1 \\
x_{(13)}c_7 &= c_2 \\
x_{(13)}c_8 &= \frac{1}{\zeta}(c_6 - c_4) \\
x_{(13)}c_9 &= \frac{1}{\zeta^2}(c_7 - c_5) \\
x_{(13)}c_{10} &= 0 \\
y_{(13)}c_1 &= \frac{1}{\zeta^2}(c_4 - \zeta c_6) \\
y_{(13)}c_2 &= \frac{1}{\zeta^2}(c_5 - \zeta c_7) \\
y_{(13)}c_3 &= 0 \\
y_{(13)}c_4 &= \zeta c_8 \\
y_{(13)}c_5 &= \zeta^2 c_9 \\
y_{(13)}c_6 &= \zeta^2 c_8 \\
y_{(13)}c_7 &= \zeta c_9 \\
y_{(13)}c_8 &= 0 \\
y_{(13)}c_9 &= 0 \\
y_{(13)}c_{10} &= 0 \\
(23)c_1 &= -c_4 \\
(23)c_2 &= -c_2 \\
(23)c_3 &= -c_1 \\
(23)c_4 &= \zeta c_5 \\
(23)c_5 &= \zeta^2 c_4 \\
(23)c_6 &= \zeta^2 c_7 \\
(23)c_7 &= \zeta c_6 \\
(23)c_8 &= -c_{10} \\
(23)c_9 &= -c_9 \\
(23)c_{10} &= -c_8 \\
x_{(23)}c_1 &= 0 \\
x_{(23)}c_2 &= 0 \\
x_{(23)}c_3 &= 0 \\
x_{(23)}c_4 &= \zeta c_1 \\
x_{(23)}c_5 &= \zeta c_2 \\
x_{(23)}c_6 &= \zeta c_3 \\
x_{(23)}c_7 &= \zeta^2 c_1 \\
x_{(23)}c_8 &= \frac{1}{\zeta}(c_7 - c_5) \\
x_{(23)}c_9 &= 0 \\
x_{(23)}c_{10} &= \frac{1}{\zeta^2}(c_6 - c_4) \\
y_{(23)}c_1 &= \frac{1}{\zeta^2}(c_4 - \zeta c_6) \\
y_{(23)}c_2 &= \frac{1}{\zeta^2}(c_5 - \zeta c_7) \\
y_{(23)}c_3 &= 0 \\
y_{(23)}c_4 &= \zeta^2 c_{10} \\
y_{(23)}c_5 &= \zeta c_8 \\
y_{(23)}c_6 &= \zeta^2 c_{10} \\
y_{(23)}c_7 &= \zeta^2 c_8 \\
y_{(23)}c_8 &= 0 \\
y_{(23)}c_9 &= 0 \\
y_{(23)}c_{10} &= 0
\end{align*}
\]

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