Abstract. A permutation $\pi$ of a multiset is said to be a \textit{quasi-Stirling} permutation if there does not exist four indices $i < j < k < \ell$ such that $\pi_i = \pi_k$ and $\pi_j = \pi_\ell$. For a multiset $M$, denote by $Q_M$ the set of quasi-Stirling permutations of $M$. The \textit{quasi-Stirling polynomial} on the multiset $M$ is defined by $Q_M(t) = \sum_{\pi \in Q_M} t^{\text{des}(\pi)}$, where $\text{des}(\pi)$ denotes the number of descents of $\pi$. By employing generating function arguments, Elizalde derived an elegant identity involving quasi-Stirling polynomials on the multiset $\{1^2, 2^2, \ldots, n^2\}$, in analogy to the identity on Stirling polynomials. In this paper, we derive an identity involving quasi-Stirling polynomials $Q_M(t)$ for any multiset $M$, which is a generalization of the identity on Eulerian polynomial and Elizalde’s identity on quasi-Stirling polynomials on the multiset $\{1^2, 2^2, \ldots, n^2\}$. We provide a combinatorial proof the identity in terms of certain ordered labeled trees. Specializing $M = \{1^2, 2^2, \ldots, n^2\}$ implies a combinatorial proof of Elizalde’s identity in answer to the problem posed by Elizalde. As an application, our identity enables us to show that the quasi-Stirling polynomial $Q_M(t)$ has only real roots and the coefficients of $Q_M(t)$ are unimodal and log-concave for any multiset $M$, in analogy to Brenti’s result for Stirling polynomials on multisets.

Keywords: quasi-Stirling permutation, quasi-Stirling polynomial, ordered labeled tree, real root.

AMS Subject Classifications: 05A05, 05C30

1 Introduction

Let $M = \{1^{k_1}, 2^{k_2}, \ldots, n^{k_n}\}$ be a multiset where $k_i$ is the number of occurrences of $i$ in $M$ and $k_i \geq 1$. A permutation $\pi$ of a multiset $M$ is said to be a \textit{Stirling} permutation if $i < j < k$ and $\pi_i = \pi_k$, then $\pi_j > \pi_i$. Stirling permutations were originally introduced by Gessel and Stanley [9] in the case of the multiset $M = \{1^2, 2^2, \ldots, n^2\}$. Denote by $Q_M$ the set of Stirling permutations of $M$.

For a permutation $\pi = \pi_1\pi_2\ldots\pi_n$, the number of descents of $\pi$, denoted by
des(\pi), is defined by

\[ des(\pi) = |\{i \mid \pi_i > \pi_{i+1}, 1 \leq i \leq n-1\}| + 1. \]

The polynomial

\[ Q_M(t) = \sum_{\pi \in Q_M} t^{des(\pi)} \]

is called the \textit{Stirling polynomial} on the multiset \( M \). When \( M = \{1,2,\ldots,n\} \), the Stirling polynomial reduces to Eulerian polynomial \( A_n(t) \). Let

\[ \sum_{m=0}^{\infty} B_M(m)t^m = \frac{Q_M(t)}{(1-t)^{K+1}}, \]

where \( M = \{1^{k_1},2^{k_2},\ldots,n^{k_n}\} \) and \( K = k_1 + k_2 + \ldots + k_n \). Gessel and Stanley \[9\] proved that \( B_M(m) = S(n+m,m) \) when \( M = \{1^2,2^2,\ldots,n^2\} \), where \( S(n+m,m) \) is the number of partitions of \([m+n]\) into \( m \) blocks. Brenti \[2\], \[3\] investigated Stirling permutations of an arbitrary multiset \( M \) and has obtained algebraic properties of Stirling polynomials. He also proved that \( B_M(m+1) \) is a Hilbert polynomial for any multiset \( M \). Analogous problems have been studied for Legendre-Stirling polynomials \[6\] and Jacobi-Stirling polynomials \[10\]. Dzhumadil’davaev and Yeliussizov \[5\] provided combinatorial interpretations of \( B_M(m) \) in terms of \( P \)-partitions, set partitions and permutations.

In analogy to Stirling permutations, Archer, Gregory, Pennington and Slayden \[1\] introduced \textit{quasi-Stirling} permutations. A permutation \( \pi \) of a multiset is said to be a \textit{quasi-Stirling} permutation if there does not exist four indices \( i < j < k < \ell \) such that \( \pi_i = \pi_k \) and \( \pi_j = \pi_\ell \). For a multiset \( M \), denote by \( \overline{Q}_M \) the set of quasi-Stirling permutations of \( M \). For example, if \( M = \{1,2^2\} \), we have \( \overline{Q}_M = \{122,221,212\} \). Archer, Gregory, Pennington and Slayden \[1\] noticed that when \( M = \{1^2,2^2,\ldots,n^2\} \),

\[ |\overline{Q}_M| = n!C_n, \]

where \( C_n = \frac{1}{n+1}\binom{2n}{n} \) is the \( n \)th Catalan number. They also posed the following intriguing conjecture.

\textbf{Conjecture 1.1 (1)} Let \( M = \{1^2,2^2,\ldots,n^2\} \). The number of \( \pi \in \overline{Q}_M \) with \( des(\pi) = n \) is equal to \( (n+1)^{n-1} \).

Elizalde \[7\] confirmed Conjecture 1.1 by establishing a bijection between such quasi-Stirling permutations and edge-labeled plane (ordered) rooted trees. He further proved that the number of \( \pi \in \overline{Q}_M \) with \( des(\pi) = n \) is equal to \( ((k-1)n+1)^{n-1} \) when \( M = \{1^k,2^k,\ldots,n^k\} \) and \( k \geq 2 \).
The polynomial
\[ \sum_{\pi \in Q_M} t^{\text{des}(\pi)} \]
is called the quasi-Stirling polynomial on the multiset \( M \). For instance, if \( M = \{1, 2^2, 3\} \), then we have \( Q_M(t) = t + 7t^2 + 4t^3 \). By employing generating function arguments, Elizalde \cite{elizalde} derived that
\[ \sum_{m=0}^{\infty} \frac{m^n}{n+1} \binom{n+m}{m} t^m = \frac{Q_M(t)}{(1-t)^{2n+1}}, \tag{1.1} \]
when \( M = \{1^2, 2^2, \ldots, n^2\} \) and asked for a combinatorial proof.

Motivated by previous results for Stirling polynomials on multisets, we investigate quasi-Stirling polynomials on multisets and derive the following generalization of Elizalde’s formula (1.1).

**Theorem 1.2** Let \( M = \{1^{k_1}, 2^{k_2}, \ldots, n^{k_n}\} \) and \( K = k_1 + k_2 + \ldots + k_n \) with \( k_i \geq 1 \). We have
\[ \sum_{m=0}^{\infty} \frac{m^n}{K-n+1} \binom{K-n+m}{m} t^m = \frac{Q_M(t)}{(1-t)^{K+1}}, \tag{1.2} \]

In this paper, we shall provide a combinatorial proof of (1.2) in terms of certain ordered labeled trees. Specializing \( M = \{1^2, 2^2, \ldots, n^2\} \) in (1.2) implies a combinatorial proof of (1.1) in answer to the problem posed by Elizalde \cite{elizalde}. Notice that by letting \( M = \{1, 2, \ldots, n\} \), we are led to the well-known identity \cite{euler} involving Eulerian polynomials
\[ \sum_{m=0}^{\infty} m^n t^m = \frac{A_n(t)}{(1-t)^{n+1}}. \]

Notice that for any multiset \( M = \{1^k, 2, 3, \ldots, n\} \) with \( k \geq 1 \), the quasi-Stirling permutations of \( M \) are equivalent to Stirling permutations of \( M \). The following corollary follows immediately from Theorem 1.2.

**Corollary 1.3** Let \( M = \{1^{k_1}, 2^{k_2}, \ldots, n^{k_n}\} \) and \( K = k_1 + k_2 + \ldots + k_n \) with \( k_i \geq 1 \). Then we have \( Q_M(t) = Q_{M'}(t) \), where \( M' = \{1^{K-n+1}, 2, 3, \ldots, n\} \).

It is well known result of Frobenius that the roots of the Eulerian polynomials are real, distinct, and nonpositive. Bóna \cite{bona} proved the real realrootedness of Stirling polynomial \( Q_M(t) \) in the case of \( M = \{1^2, 2^2, \ldots, n^2\} \). Brenti \cite{brenti} showed that the Stirling polynomial \( Q_M(t) \) has only real roots for any multiset \( M \). Combining Corollary 1.3 and Brenti’s result, it is clear that the quasi-Stirling polynomial \( Q_M(t) \) has the same property as the Stirling polynomial \( Q_M(t) \), which is an extension of Elizalde’s result \cite{elizalde} for \( M = \{1^2, 2^2, \ldots, n^2\} \).
Corollary 1.4 For any multiset $\mathcal{M}$, the quasi-Stirling polynomial $\overline{Q}_{\mathcal{M}}(t)$ has only real roots.

Unimodal and log-concave sequences arise frequently in combinatorics. It is well known that the coefficients of a polynomial with nonnegative coefficients and with only real roots are log-concave and that log-concavity implies unimodality. As an application of this result, Corollary 1.4 implies the following property of the coefficients of $\overline{Q}_{\mathcal{M}}(t)$.

Corollary 1.5 For any multiset $\mathcal{M}$, the coefficients of $\overline{Q}_{\mathcal{M}}(t)$ are unimodal and log-concave.

Notice that specializing $\mathcal{M} = \{1^2, 2^2, \ldots, n^2\}$ in Corollary 1.5 recovers Elizalde’s result [7].

2 Proof of Theorem 1.2

The objective of this section is to provide a combinatorial proof of Theorem 1.2. Recall that an ordered tree is a tree with one designated vertex, which is called the root, and the subtrees of each vertex are linearly ordered. In an ordered tree $T$, the level of a vertex $v$ in $T$ is defined to be the length of the unique path from the root to $v$. A vertex $v$ is said to be at odd level (resp. at even level) if the level of $v$ is odd (resp. even). In order to prove 1.2, we need to consider a new class of ordered labeled trees $T$ which verify the following properties:

(i) the vertices are labeled by the elements of the multiset $\{0\} \cup \mathcal{M}$, where $\mathcal{M} = \{1^{k_1}, 2^{k_2}, \ldots, n^{k_n}\}$ with $k_i \geq 1$;

(ii) the root is labeled by 0;

(iii) for a vertex $v$ at odd level, if $v$ is labeled by $i$, then $v$ has exactly $k_i - 1$ children and the children of $v$ have the same label as that of $v$ in $T$.

Let $\mathcal{T}_\mathcal{M}$ denote the set of such ordered labeled trees. For example, a tree $T \in \mathcal{T}_\mathcal{M}$ with $\mathcal{M} = \{1, 2, 3^2, 4, 5^3, 6, 7^2\}$ is illustrated in Figure 1.
Figure 1: A tree $T \in \mathcal{T}_M$ with $M = \{1, 2, 3^2, 4, 5^3, 6, 7^2\}$.

Define the number of cyclic descents of a sequence $\pi = \pi_1\pi_2\ldots\pi_n$ to be

$$cdes(\pi) = |\{i \mid \pi_i > \pi_{i+1}\}|$$

with the convention $\pi_{n+1} = \pi_1$.

In an ordered labeled tree $T$, let $u$ be a vertex of $T$. Suppose that $u$ has $\ell$ children $v_1, v_2, \ldots, v_\ell$ listed from left to right. The number of cyclic descents of $u$, denoted by $cdes(u)$, is defined to be the number $cdes(uv_1v_2\ldots v_\ell)$. The number of cyclic descents of $T$ is defined to be

$$cdes(T) = \sum_{u \in V(T)} cdes(u),$$

where $V(T)$ denotes the vertex set of $T$. If $T$ has only one vertex, let $cdes(T) = 1$. For example, if let $T$ be a tree as shown in Figure 1, we have $cdes(T) = 5$.

In order to prove (1.2), we establish a bijection between $\mathcal{T}_M$ and $\mathcal{Q}_M$. Let $\pi$ be a permutation, the leftmost (resp. rightmost) entry of $\pi$ is denoted by $\text{first}(\pi)$ (resp. $\text{last}(\pi)$). Similarly, for an ordered labeled tree $T$, we denote by $\text{first}(T)$ (resp. $\text{last}(T)$) the leftmost (resp. rightmost) child of the root. If a tree $T$ has only one vertex, let $\text{first}(T) = +\infty$ and $\text{last}(T) = -\infty$. Similarly, for an empty permutation $\epsilon$, we let $\text{first}(\epsilon) = +\infty$ and $\text{last}(\epsilon) = -\infty$.

**Theorem 2.1** There exists a bijection $\phi$ between $\mathcal{T}_M$ and $\mathcal{Q}_M$ such that

$$(cdes, \text{first}, \text{last})T = (\text{des}, \text{first}, \text{last})\phi(T)$$

for any $T \in \mathcal{T}_M$.

**Proof.** First we give a recursive description of the map $\phi$ from $\mathcal{T}_M$ to $\mathcal{Q}_M$. Let $T \in \mathcal{T}_M$. If $T$ has only one vertex, let $\phi(T) = \epsilon$. Otherwise, suppose that $\text{first}(T) = r$.
Case 1. The leftmost child of the root is a leaf. Let $T_0$ be the tree obtained from $T$ by removing the leftmost child of the root together with the edge incident to it. Define $\phi(T) = r\phi(T_0)$.

Case 2. The leftmost child of the root has $k$ children. For $1 \leq i \leq k$, let $T_i$ be the subtree rooted at the $i$-th child of the leftmost child of the root. Denote by $T'_i$ the tree obtained from $T_i$ by relabeling its root by 0. Let $T_0$ be the tree obtained from $T$ by removing all the subtrees $T_1, T_2, \ldots, T_k$ and all the vertices labeled by $r$ together with the edges incident to them. Define $\phi(T) = r\phi(T'_1)r\phi(T'_2)\ldots r\phi(T'_k)r\phi(T_0)$.

It is routine to check that we have $\phi(T) \in \overline{\mathcal{Q}}_M$. Now we proceed to show that $(cdes, first, last)T = (des, first, last)\phi(T)$ by induction on the cardinality of the multiset $M$. Obviously, the statement holds when $|M| = 0$. Assume that $(cdes, first, last)T = (des, first, last)\phi(T)$ for any $T \in \mathcal{T}_{M'}$ with $|M'| < |M|$. From the construction of $\phi$, it is easily seen that $first(T) = first(\phi(T))$. Moreover, if $T_0$ has only one vertex, one can easily check that $last(T) = r = last(\phi(T))$. Otherwise, we have $last(T_0) = last(T)$ and $last(\phi(T)) = last(\phi(T_0))$. By induction hypothesis, we have $last(T) = last(T_0) = last(\phi(T_0)) = last(\phi(T))$ as desired. It remains to show that $cdes(T) = des(\phi(T))$. We have two cases.

Case 1. The leftmost child of the root is a leaf. In this case, we have $\phi(T) = r\phi(T_0)$. It is routine to check that

$$cdes(T) = cdes(T_0) + \chi(r > first(T_0)).$$

Here $\chi(S) = 1$ if the statement $S$ is true, and $\chi(S) = 0$ otherwise. Then by induction hypothesis, we have

$$cdes(T) = cdes(T_0) + \chi(r > first(T_0)) = des(\phi(T_0)) + \chi(r > first(\phi(T_0))) = des(\phi(T)).$$

Case 2. The leftmost child of the root has $k$ children. Then we have $\phi(T) = r\phi(T'_1)r\phi(T'_2)\ldots r\phi(T'_k)r\phi(T_0)$. It is easily seen that

$$cdes(T) = cdes(T_0) + \chi(r > first(T_0)) + \sum_{i=1}^k (cdes(T'_i) - 1 + \chi(r > first(T'_i))) + \chi(r < last(T'_i)),$$

and $des(\phi(T))$ is given by

$$des(\phi(T_0)) + \chi(r > first(\phi(T_0))) + \sum_{i=1}^k (des(\phi(T'_i)) - 1 + \chi(r > first(\phi(T'_i)))) + \chi(r < last(\phi(T'_i))).$$

By induction hypothesis, it is easy to verify that $cdes(T) = des(\phi(T))$. 


So far, we have concluded that \( \phi \) is map from \( T_M \) to \( Q_M \) satisfying that

\[
(c_{\text{des}}, \text{first}, \text{last})T = (\text{des}, \text{first}, \text{last})\phi(T)
\]

for any \( T \in T_M \). It is clear that the construction of \( \phi \) is reversible, and hence the map \( \phi \) is the desired bijection. This completes the proof.

For example, let \( T \) be the tree as shown in Figure 1. By applying the map \( \phi \), we get a permutation \( \phi(T) = 27175633545 \).

Relying on Theorem 2.1 quasi-Stirling polynomials can be reformulated as follows:

\[
Q_M(t) = \sum_{\pi \in Q_M} t^{\text{des}(\pi)} = \sum_{T \in T_M} t^{c_{\text{des}}(T)}.
\]

(2.1)

Let \( e \) be an edge of an ordered labeled tree \( T \). If \( e \) connects vertices \( u \) and \( v \) and \( u \) is the parent of \( v \), then we call \( v \) the south endpoint of \( e \). An edge whose south endpoint is an unlabeled leaf is called a half-edge.

Let \( T \in T_M \) and let \( u \) be a vertex of \( T \) at even level. Now we may attach some half-edges to the vertex \( u \) by obeying the following rules:

- the inserted half-edges serve as walls to separate the children of \( u \) into (possibly empty) compartments such that the vertices in the same compartment are increasing from left to right;
- if \( v \) is the last child of \( u \) and \( v > u \), then there is some half-edge inserted to the right of \( v \);
- if \( v \) is the first child of \( u \) and \( u > v \), then there is some half-edge inserted to the left of \( v \).

Repeat the above procedure for all the vertices at even level of \( T \), we will get a tree with some half-edges inserted. Denote by \( T_{M,m}^* \) the set of such trees with \( m \) half-edges inserted. For example, a tree \( T \in T_{M,9}^* \) with \( M = \{1,2,3^2,4,5^3,6,7^2\} \) is illustrated on the right of Figure 2.

Relying on (2.1), the coefficient of \( t^m \) of the right-hand side of (1.2) can be interpreted as the number of ordered labeled trees in \( T_{M,m}^* \). In order to provide a combinatorial proof of (1.2), it suffices to show that

\[
|T_{M,m}^*| = \frac{m^n}{K - n + 1} \binom{K - n + m}{m}
\]

(2.2)

for \( M = \{1^{k_1},2^{k_2},\ldots,n^{k_n}\} \) and \( k_1 + k_2 + \ldots + k_n = K \) with \( k_i \geq 1 \). To this end, we need to consider a class of ordered labeled trees satisfying that
(i) the vertices are labeled by the elements of the multiset \( \{0\} \cup \mathcal{M} \) and a leaf at odd level may be unlabeled, where \( \mathcal{M} = \{1^{k_1}, 2^{k_2}, \ldots, n^{k_n}\} \);

(ii) the root is labeled by 0;

(iii) for each vertex \( v \) at odd level, if \( v \) is labeled by \( i \), then \( v \) has exactly \( k_i - 1 \) children and the children of \( v \) have the same label as that of \( v \);

(iv) for each internal vertex \( v \) at even level, the children of \( v \) are separated into blocks satisfying that each block is either an unlabeled leaf or consists of children which are increasing from left to right in \( T \).

Denote by \( \mathcal{B}^*_T \mathcal{M},m \) the set of such ordered labeled trees with \( m \) blocks. For example, a tree \( T \in \mathcal{B}^*_T \mathcal{M},9 \) with \( \mathcal{M} = \{1, 2, 3^2, 4, 5^3, 6, 7^2\} \) is illustrated on the left of Figure 2, where the vertices in the same block are grouped together by a circle.

The bijective proof of (2.2) relies on the following two results.

**Theorem 2.2** There is a bijection \( \psi : \mathcal{B}^*_T \mathcal{M},m \to \mathcal{T}^{*}_m \mathcal{M} \).

**Theorem 2.3** Let \( \mathcal{M} = \{1^{k_1}, 2^{k_2}, \ldots, n^{k_n}\} \) and \( k_1 + k_2 + \ldots + k_n = K \) with \( k_i \geq 1 \). We have

\[
|\mathcal{B}^*_T \mathcal{M},m| = \frac{m^n}{K-n+1} \left( K - n + m \right) .
\] (2.3)

**Proof of Theorem 2.2.** Let \( T \in \mathcal{B}^*_T \mathcal{M},m \) and let \( u \) be an internal vertex at even level of \( T \). A block of \( u \) is said to be *trivial* if the block consists of an unlabeled leaf. The map \( \psi : \mathcal{B}^*_T \mathcal{M},m \to \mathcal{T}^{*}_m \mathcal{M} \) can be described as follows. For an internal vertex \( u \) at even level of \( T \), attach a half-edge to \( u \) to the right of the rightmost vertex of each non-trivial block of \( u \). Assume that the children of \( u \) to the right of the last half-edge incident to \( u \) are \( v_1, v_2, \ldots, v_t \) with \( v_1 < v_2 < \ldots < v_t < u \). Then rearrange \( v_1, v_2, \ldots, v_t \) as the last \( t \) children of \( u \), which are arranged in increasing order from left to right. Repeat the above procedure for each internal vertex at even level of \( T \). Define \( \psi(T) \) to be the resulting tree. It is routine to check that \( \psi(T) \in \mathcal{T}^{*}_m \mathcal{M} \) and thus \( \psi \) is well defined.

In order to show that \( \psi \) is bijection, we proceed to describe a map \( \psi' : \mathcal{T}^{*}_m \mathcal{M} \to \mathcal{B}^*_T \mathcal{M},m \). Let \( T \in \mathcal{T}^{*}_m \mathcal{M} \) and let \( u \) be an internal vertex at even level of \( T \). Assume that the children of \( u \) to the right of the last half-edge incident to \( u \) are \( v_1, v_2, \ldots, v_t \). Clearly, we have \( v_1 < v_2 < \ldots < v_t < u \). Rearrange the vertices \( v_1, v_2, \ldots, v_t \) as the leftmost \( t \) children of \( u \), arranged in increasing order from left to right. Then the half-edges serve as walls to separate the children of \( u \) into (possibly empty) compartments such that the vertices in the same compartment are increasing from
left to right. Remove each half-edge e incident to u if the compartment to the left of e is non-empty. Then let each non-empty compartment (resp. each unlabeled leaf) be a block of u. Repeat the above procedure for each internal vertex at even level of T. Define \( \psi'(T) \) to be the resulting tree. It is easy to check that \( \psi'(T) \in \mathcal{BT}_{\mathcal{M},m} \). Moreover, the maps \( \psi \) and \( \psi' \) are inverses of each other, and thus \( \psi \) is a bijection as desired. This completes the proof.

For example, let \( T \) be a tree in \( \mathcal{BT}_{\mathcal{M},9} \) with \( \mathcal{M} = \{1, 2, 3^2, 4, 5^3, 6, 7^2\} \) as illustrated on the left of Figure 2. By applying the map \( \psi \), we obtain a tree \( \psi(T) \in \mathcal{T}_{\mathcal{M},9}^* \) as shown on the right of Figure 2.

![Figure 2: An example of the bijection \( \psi \).](image)

In the following, we shall present a bijective enumeration of trees in \( \mathcal{BT}_{\mathcal{M},m} \) by exhibiting a Pr"{u}fer-like code.

Let \( \mathcal{M} = \{1^{k_1}, 2^{k_2}, \ldots, n^{k_n}\} \) and \( I = \{i \mid k_i > 1\} \). For \( k_i > 1 \), let \( \mathcal{M}_i = \{i_1, i_2, \ldots, i_{k_i-1}\} \). Denote by \( \mathcal{P}_{\mathcal{M},m} \) the set of pairs \((P,S)\) where

- \( P \) is an \( m \)-multiset on the set \( \{0\} \cup (\bigcup_{i \in I} \mathcal{M}_i) \);
- \( S \) is a sequence \((a_1,b_1)(a_2,b_2)\ldots(a_n,b_n)\) satisfying that \( a_i \in P \) and \( b_i \) is a positive integer not larger than the number of occurrences of \( a_i \) in \( P \);
- \( a_n = 0 \)

We need a combinatorial identity before we get the enumeration of the pairs in \( \mathcal{P}_{\mathcal{M},m} \).

**Lemma 2.4** For \( m \geq 1 \), we have

\[
\sum_{\ell=1}^{m} \ell\binom{n+m-\ell-1}{n-1} = \binom{n+m}{n+1}.
\]
Proof. It is apparent that the statement holds for $m = 1$. Now we assume $m \geq 2$. The right-hand side of (2.4) can be interpreted as the number of $(n + 1)$-subsets $P$ of $[n + m] = \{1, 2, \ldots, n + m\}$. Suppose that the second largest number appear in $P$ is equal to $\ell + 1$ for some $1 \leq \ell \leq m$. Then we have $\ell$ ways to choose the minimal element of $P$ from $[\ell]$, and we can choose the remaining $n - 1$ elements of $P$ from $[n + m] \setminus [\ell + 1]$ in $\binom{n + m - \ell - 1}{n - 1}$ ways. Therefore, the total number of ways to choose such a subset $P$ is given by $\ell^n(n + m - \ell - 1)$. Then the result follows, completing the proof. 

Lemma 2.5 Let $\mathcal{M} = \{1^{k_1}, 2^{k_2}, \ldots, n^{k_n}\}$ and $K \leq k_1 + k_2 + \ldots + k_n$ with $k_i \geq 1$. We have

$$|\mathcal{P}_{\mathcal{M}, m}| = \frac{m^n}{K - n + 1} \binom{K - n + m}{m}.$$ 

Proof. Given a positive integer $\ell$ with $\ell \leq m$, we can generate a pair $(P, S) \in \mathcal{P}_{\mathcal{M}, m}$ in which the number of occurrences of 0 in $P$ equals $\ell$. It is apparent that we have $\binom{K - n + m - \ell - 1}{m - \ell}$ ways to choose such a multiset $P$. Once $P$ is determined, we can choose $b_n$ in $\ell$ ways, and each $(a_i, b_i)$ in $m$ ways for $1 \leq i \leq n - 1$. Then the total number of different ways to choose a pair $(P, S) \in \mathcal{P}_{\mathcal{M}, m}$ in which the number of occurrences of 0 in $P$ equals $\ell$ is given by $m^{n-1} \ell \binom{K - n + m - \ell}{m - \ell}$. This yields that

$$|\mathcal{P}_{\mathcal{M}, m}| = m^{n-1} \sum_{\ell=1}^{m} \ell \binom{K - n + m - \ell - 1}{m - \ell}.$$ 

By (2.4), we have

$$|\mathcal{P}_{\mathcal{M}, m}| = \frac{m^n}{K - n + 1} \sum_{\ell=1}^{m} \ell \binom{K - n + m - \ell - 1}{m - \ell} = \frac{m^n}{K - n + 1} \binom{K - n + 1}{m}$$

as desired, completing the proof. 

For a tree $T \in \mathcal{B}T_{\mathcal{M}, m}$, a vertex $v$ is said to be even (resp. odd) if $v$ is at even (resp. odd) level.

Lemma 2.6 There exists a bijection $\theta : \mathcal{B}T_{\mathcal{M}, m} \rightarrow \mathcal{P}_{\mathcal{M}, m}$.

Proof. Let $T \in \mathcal{B}T_{\mathcal{M}, m}$. First we obtain a tree $T'$ on the vertex set $\mathcal{M} \cup \{0\}$ by removing all the half-edges from $T$. Starting from $T_0 = T'$, for $i = 1, 2, \ldots, n$, let $T_i$ be the tree obtained from $T_{i-1}$ by deleting the largest odd vertex $l_i$ together with its children if all the children of the odd vertex $l_i$ are leaves or the odd vertex $l_i$ is a leaf. Set $\theta(T) = (P, S)$ where

- $P$ is a multiset on the set $\{0\} \cup \bigcup_{i \in I} \mathcal{M}_i$ such that the
– for all \( x \in [n] \), the number of occurrences of \( x_j \) in \( P \) equals \( c \) if and only if the children of the \( j \)-th child of the odd vertex \( x \) are separated into \( c \) blocks (both children and blocks are ordered from left to right);

– the number of occurrences of \( 0 \) in \( P \) equals \( c \) if and only if the children of the root \( 0 \) are separated into \( c \) blocks.

• \( S \) is a sequence \((a_1, b_1)(a_2, b_2)\ldots(a_n, b_n)\) satisfying that

  – \( a_i = x_j \) for some \( x \in [n] \) if and only if the odd vertex \( l_i \) lies in the \( b_i \)-th block of the \( j \)-th child of the odd vertex \( x \);

  – \( a_i = 0 \) if and only if the vertex \( 0 \) is the parent of the odd vertex \( l_i \) and the odd vertex \( l_i \) lies in the \( b_i \)-th block of the vertex \( 0 \) in \( T \).

It is apparent that we have \( \theta(T) \in \mathcal{P}_M \).

Conversely, given a pair \((P, S) \in \mathcal{P}_M, m\), we can recover a tree \( T \in \mathcal{BT}_M, m \) by the following procedure.

(a) For \( 1 \leq i \leq n \), let \( l_i \) be the largest element of \([n] \cup \{0\} \setminus \{l_1, l_2, \ldots, l_{i-1}\} \) such that \((l_i)_j \) does not appear in the sequence \((a_i, a_{i+1}, \ldots, a_n)\) for all \( 1 \leq j \leq k_{l_i} - 1 \);

(b) If \( a_i = x_j \) for some \( x \in [n] \), then let \( l_i \) be an odd vertex lying in the \( b_i \)-th block of the \( j \)-th child of the odd vertex \( x \). Let the odd vertex \( l_i \) have \( k_{l_i} - 1 \) children and let all the children of the odd vertex \( l_i \) be labeled by \( l_i \).

(c) If \( a_i = 0 \), then let \( l_i \) be an odd vertex lying in the \( b_i \)-th block of the vertex \( 0 \). Let the odd vertex \( l_i \) have \( k_{l_i} - 1 \) children and let all the children of the odd vertex \( l_i \) be labeled by \( l_i \).

(d) Let \((x, y) \neq (a_i, b_i)\) for any \( i \in [n] \), where \( x \in P \) and \( y \) is a positive integer not larger than the number of the occurrences of \( x \) in \( P \).

  – If \( x = i_j \) for some \( 1 \leq i \leq n \) and \( 1 \leq j \leq k_i - 1 \), we attach a half-edge to the \( j \)-th child of the odd vertex \( i \) such that the new inserted unlabeled leaf lies in its \( y \)-th block;

  – If \( x = 0 \), we attach a half-edge to the vertex \( 0 \) such that the new inserted unlabeled leaf lies in its \( y \)-th block.

It is clear that the map \( \theta \) is reversible, and therefore the map \( \theta \) is a bijection. This completes the proof.
For example, let $T$ be a tree in $\mathcal{BT}_{M,9}$ with $M = \{1, 2, 3^2, 4, 5^3, 6, 7^2\}$ as shown in Figure 3. By applying the map $\theta$, we get a pair $\theta(T) = (P, S) \in \mathcal{P}_{M,9}$, where

$$P = \{0^2, 3_1, 5_1^2, 5_2^3, 7_1\}$$

and

$$S = (5_1, 1)(5_2, 3)(5_1, 1)(0, 2)(0, 1)(7_1, 1)(0, 1).$$

Combining Lemmas 2.5 and 2.6, we are led to a combinatorial proof of Theorem 2.3. Combining Theorems 2.2 and 2.3, we obtain a combinatorial proof of (2.2), thereby providing a combinatorial proof of Theorem 1.2.

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