THE PARAMETRIX PROBLEM FOR TODA EQUATION WITH STEPLIKE INITIAL DATA

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1. Introduction

The Toda rarefaction problem is related to the analysis of the long-time asymptotic behaviour of the Cauchy problem solution for the doubly infinite Toda lattice

\[
\begin{align*}
\dot{b}(n,t) &= 2(a(n,t)^2 - a(n-1,t)^2), \\
\dot{a}(n,t) &= a(n,t)(b(n+1,t) - b(n,t)), \\
(n,t) &\in \mathbb{Z} \times \mathbb{R}_+,
\end{align*}
\]

with symmetric steplike initial data \(a(-n,0) = a(n,0), b(-n,0) = -b(n,0)\), such that \(a(n,0) \to \frac{1}{2}\) and \(b(n,0) \to \pm \hat{b}\), \(\hat{b} > 0\), as \(n \to \pm \infty\). From a physical point of view it is interesting to study asymptotics of the solution in the regime when \(t \to +\infty, n \to \infty\) with the ratio \(\xi := \frac{n}{t}\) slow varying. The Toda rarefaction wave demonstrates qualitatively different behaviour depending on the value of the background constant \(\hat{b}\), where we distinguish between the cases \(\hat{b} > 1\) and \(0 < \hat{b} \leq 1\).

The behaviour of the rarefaction wave depends also on a value of \(\xi\) varying in some intervals of \(\mathbb{R}\). In particular, for \(\hat{b} > 1\) one can observe four sectors with different asymptotics of the solution on the \((n,t)\) half plane. These sectors are divided by the rays corresponding to the leading and the back wave fronts \((\xi = \pm 1)\), and to the ray \(\xi = 0\). In the regions ahead of the leading wave front and behind the back wave front, an application of the Inverse Scattering Transform (IST) method results straightforward in adjusting soliton asymptotics of the solution on the respective backgrounds, while in two middle sectors the classical IST does not lead to desirable results. On the other hand, in the middle sectors the Nonlinear Steepest Descent (NSD) method, developed in [3] for modified KdV equation, proves to be more efficient. For the Toda lattice this method was pioneered by Deift at all in [2].

In this seminal work the NSD approach was applied for the first time for a vector form of the oscillating Riemann-Hilbert problem (RHP) resulting in establishing asymptotics in a physically important transitional region near the ray \(\xi := \frac{n}{t} \sim 0\), as \(t \to +\infty\). Recall that NSD approach proceeds by a sequence of transformations (conjugations/deformations) that convert the original RHP into an equivalent RHP (RHP - equiv) with a jump matrix \(v^{equiv}\) of the form \(v^{equiv} = v^{mod} + v^{err}\), where \(v^{mod}\) is a jump matrix for an explicitly solvable RHP (RHP-mod), and the entries of an error matrix \(v^{err}\) are exponentially small with respect to \(t\) except of a finite number of small vicinities of critical (parametrix) points. Solving the RHP corresponding to \(v^{mod}\) yields the principal term of asymptotic expansion of the solution with respect to large \(t\). To estimate the error term one has to rescale the RHP - equiv in vicinities of the parametrix points and solve respective local RH problems. One has to mention that for steplike solutions a contribution of the parametrices in asymptotics is perceptible only for the second or even the third term of the expansion. By means
of the same NSD approach for a vector RHP in [6], the long-time asymptotic behaviour of the solution was studied in all main regions of \((n,t)\) half-plane for more general initial data:

\[
a(n,0) \to a, \quad b(n,0) \to b, \quad \text{as } n \to -\infty, \\
(1.2) \quad a(n,0) \to \frac{1}{2}, \quad b(n,0) \to 0, \quad \text{as } n \to +\infty,
\]

where \(a > 0\) and \(b \in \mathbb{R}\) satisfy the condition

\[
(1.3) \quad 1 < b - 2a.
\]

By an analogy the problem (1.1)-(1.3) is also called the Toda rarefaction problem. Note that the initial value problem (1.1)–(1.2) is uniquely solvable for any constants \(a > 0, b \in \mathbb{R}\) and any initial data which approach their limiting constants with a polynomial rate. Moreover, for each \(t \neq 0\) the solution tends as \(n \to \pm \infty\) to the same constants, and with the same rate as the initial data (cf. [4]). However, an application of NSD approach require a faster speed of approximation, namely, in [2] and in [6] it was assumed that

\[
(1.4) \quad \sum_{n=1}^{\infty} e^{\nu n} \left| a(-n,0) - a \right| + \left| b(-n,0) - b \right| + \left| a(n,0) - \frac{1}{2} \right| + \left| b(n,0) \right| < \infty,
\]

for some \(\nu > 0\). Again, one can expect (see [11]) that the long-time asymptotics of the solution for (1.1)–(1.4) are determined qualitatively by the mutual location of the intervals \([b - 2a, b + 2a]\) and \([-1,1]\), and by the discrete spectrum \(\lambda_1, \ldots, \lambda_N\) of the underlying Jacobi operator

\[
(1.5) \quad H(t)y(n) := a(n-1,t)y(n-1) + b(n,t)y(n) + a(n,t)y(n+1), \quad n \in \mathbb{Z}.
\]

In particular, in [6] it was shown that for \(t \to +\infty\) the solution \(\{a(n,t), b(n,t)\}\) of the problem (1.1)-(1.3):

- In the region \(n > t\) is asymptotically close to the right constants \(\{\frac{1}{2}, 0\}\) plus a sum of solitons corresponding to the eigenvalues \(\lambda_j < -1\).
- In the region \(0 < n < t\):

\[
(1.6) \quad a(n,t) = \frac{n}{2t} + O\left(\frac{1}{t}\right), \quad b(n,t) = 1 - \frac{n}{t} + O\left(\frac{1}{t}\right).
\]

- In the region \(-2at < n < 0\):

\[
(1.7) \quad a(n,t) = -\frac{n}{2t} + O\left(\frac{1}{t}\right), \quad b(n,t) = b - 2a - \frac{n}{t} + O\left(\frac{1}{t}\right).
\]

- In the region \(n < -2at\), the solution of (1.1)-(1.3) is close to the left background constants \(\{a, b\}\) plus a sum of solitons corresponding to the eigenvalues \(\lambda_j > b + 2a\).

By an analogy with the KdV rarefaction waves (cf. [1],[9],[10]), in [6] it was conjectured that the error terms \(O(t^{-1})\) are uniformly bounded with respect to \(n\) for \(\varepsilon t \leq n \leq (1 - \varepsilon)t\) in (1.6), and for \((-2a + \varepsilon)t \leq n \leq -\varepsilon t\) in (1.7), where \(\varepsilon > 0\) is an arbitrary small value. Moreover, it was conjectured that the influence of parameters is not perceptible for the first two terms of the asymptotic expansion in the two middle regions. More detailed analysis of transformations from original RHP to RHP-equiv allowed then to derive a precise formula for the second term of asymptotic expansion.
Our paper is a continuation of [6]. We solve rigorously the parametrix problems associated with the region $0 < n < t$ and complete the asymptotic analysis, justifying the asymptotics obtained in [6]. In particular, we show that the parametrix problems solutions do not contribute indeed in the first two terms of asymptotic expansion. Note, that in [6] it was assumed that points $b-2a$ and $b+2a$ are non-resonant. In this paper resonances are admitted there. The presence of a resonance at the edge of the spectrum of operator (1.5) implies a non-$L^2$ singularity of the jump matrix in the original RHP. In turn it requires an additional discussion of the statement of the RHP, of the solution uniqueness, and modifications in the transformation steps which lead to RHP- equiv. We investigate the region $0 < n < t$ only.

The asymptotical analysis in the region $-2at < n < 0$ is sequent if one considers the Toda lattice solution
\[
\hat{a}(n,t) = \frac{1}{2a}a(-n - 1, t), \quad \hat{b}(n,t) = \frac{1}{2a} \left(b - b(-n, t)\right).
\]
This solution corresponds to the initial profile
\[
\hat{a}(n,0) \to \frac{1}{2a}, \quad \hat{b}(n,0) \to 0, \quad n \to -\infty,
\]
and the asymptotics of it in the region $0 < n < t$ immediately implies the asymptotics of solution for (1.1)-(1.2) in the region $-2at < n < 0$. As for the soliton regions, they are studied rigorously in [8] for the decaying case. In the steplike case the analysis is very similar. The paper has the following structure. In section 2 we recall some necessary facts from scattering theory for Jacobi operator with steplike backgrounds and study the unique solvability of the initial vector RHP, taking into account a possible presence of resonances. In section 3 we lists some conjugation/deformation invertible transformations given in [6] which reduce the initial meromorphic RHP to a holomorphic RHP with jumps close as $t \to \infty$ to constant matrices except of vicinities of two points, where we pose and solve the parametrix problem (Section 4). In section 5 we perform a completion of the asymptotic analysis related to the Cauchy type integrals and singular integral equations.

2. Statement of the Riemann-Hilbert problem

Let \( \{a(n,t), b(n,t)\} \) be the solution for the initial value problem (1.1)-(1.4) and let \( H(t) \) be the Jacobi operator (1.5). Consider the underlying spectral problem:
\[
H(t)y(n) = \lambda y(n), \quad \lambda \in \mathbb{C}.
\]
Introduce also the left and the right background operators:
\[
H y(n) := \frac{1}{2}y(n-1) + \frac{1}{2}y(n+1), \quad n \in \mathbb{Z},
\]
\[
H_1 y(n) := ay(n-1) + by(n) + ay(n+1), \quad n \in \mathbb{Z}.
\]
Under condition (1.5) they have disjoint spectra \( \sigma(H) = [-1,1] \) and \( \sigma(H_1) = [b-2a, b+2a] \) with the mutual location as depicted in Fig. 1.

Operator \( H(t) \) has the continuous spectrum of multiplicity 1, consisting of the union of the background spectra plus a finite number of eigenvalues
\[
\{\lambda_j\}_{j=1}^N \subset \mathbb{R} \setminus ([-1,1] \cup [b-2a, b+2a]).
\]
Denote a meromorphic function in \( D \)

Consider now the scattering relations (2.3)

where \( \lambda \) is the spectral parameter.

Instead of the spectral parameter \( \lambda \) we use its Joukowsky transform \( z \):

\[
\lambda = \frac{1}{2} (z + z^{-1}), \quad |z| \leq 1.
\]

Denote

\[
q_1 := z(b - 2a), \quad q_2 := z(b + 2a), \quad I := [q_2, q_1],
\]

where \( z(\lambda) = \lambda - \sqrt{\lambda^2 - 1} \). Note that the map \( z \mapsto \lambda \) is a bijection between the sets \( \mathcal{D} := (D \setminus I) \) and \( \mathbb{C} \setminus ([-1,1] \cup [b - 2a, b + 2a]) \), where \( D := \{z : |z| < 1\} \). At the same time \( z \mapsto \lambda \) is a bijection between the sets \( \overline{\mathcal{D}} := \text{clos} \mathcal{D} \) and \( \text{clos}(\mathbb{C} \setminus ([-1,1] \cup [b - 2a, b + 2a])) \), if we treat the closure as adding to boundaries.

Next, on the boundary \( \partial D \) the Wronskian can vanish only on the set \( \{-1, 1, q_1, q_2\} \).

By definition, a point \( p \in \{-1, 1, q_1, q_2\} \) is called a resonant point if \( W(p,0) = 0 \). In this case

\[
W(p, t) = C(t)\sqrt{z - p}(1 + o(1)), \quad \text{as } z \to p, \quad C(t) \neq 0 \quad \forall t \geq 0.
\]

Consider now the scattering relations

\[
T(z, t)\psi_1(z, n, t) = \psi(z, n, t) + R(z, t)\psi(z, n, t), \quad |z| = 1,
\]

where \( T(z, t) \) and \( R(z, t) \) are the right transmission and reflection coefficients. Since \( T(z, t) = (z - z^{-1})(2W(z, t))^{-1} \), the transmission coefficient can be continued as a meromorphic function in \( D \). Due to (2.5) it has continuous limiting values on \( T \).
and on the sides of the interval \((q_2, q_1)\). Moreover, its modulo does not have a jump on \(I\). Define now the function

\[
\chi(z) := \frac{2a(\zeta(z - i0) - \zeta(z - i0)^{-1})}{z - 1 - z}|T(z, 0)|^2, \quad z \in I.
\]

The function \(\chi(z)\) is continuous on the interval \(I\) except of possibly the endpoints \((2.8)\). If \(q_i\) is a resonant point then

\[
\chi(z) = C(z - q_i)^{-1/2}(1 + o(1)), \quad C \neq 0, \quad z \to q_i,
\]

otherwise \(\chi(q_i) = 0\) (in the non-resonant case).

Denote now \(R(z) := R(z, 0), \quad \gamma_j := \gamma_j(0)\). As is known (cf. [2],[7],[11],[15]), the IST approach allows us to restore uniquely the solution of \((1.1)-(1.3)\) from a minimal set of the initial scattering data

\[
\{R(z), \quad z \in \mathbb{T}, \quad \chi(z), \quad z \in I, \quad \lambda_j, \gamma_j, \quad j = 1, \ldots, N;\}
\]

This is the data which are involved in the right Marchenko equation for the step-like case \((1.2)-(1.3)\). Note that the time-dependent Marchenko equation encloses the values \(R(z, t), \quad |T(z, t)|\) (cf. \((2.6)\)) and \(\gamma_j(t), \quad j = 1, \ldots, N\), whose evolution due to the Toda flow is given by: \(\gamma_j(t) = \gamma_j(0)\exp((z_j - z_j^{-1})t)\),

\[
R(z, t) = R(z)e^{(z-z^{-1})t}, \quad z \in \mathbb{T}; \quad |T(z, t)|^2 = |T(z)|^2e^{(z-z^{-1})t}, \quad z \in I.
\]

In \(\mathcal{D}\) introduce a vector-function \(m(z) = (m_1(z, n, t), \quad m_2(z, n, t)):\)

\[
m(z, n, t) = (T(z, t)\psi_1(z, n, t)\psi_n, \quad \psi(z, n, t)\psi_n).
\]

The space and time variables are treated here as parameters. We omit them in the notation of \(m\) whenever it is possible.

**Lemma 2.1 (5).** The components of the vector function \((2.9)\) satisfy

\[
m_1(z) = \prod_{j=n}^{\infty} 2a(j, t)\left(1 + 2z\sum_{m=n}^{\infty} b(m, t)\right) + O(z^2),
\]

\[
m_1(z) m_2(z) = 1 + O(z^2), \quad \text{as} \quad z \to 0.
\]

The first component \(m_1(z)\) is a meromorphic function in \(\mathcal{D}\) with poles at \(z_j\), the second one is a holomorphic function continuous up to the boundary. We extend \(m\) to the set \(\mathcal{D}^* := \mathbb{C} \setminus (\overline{\mathbb{D}} \cup I^*)\) where \(I^* := \{q_2^{-1}, q_1^{-1}\}\), by the following symmetry condition \(m(z^{-1}) = m(z)\sigma_1\), where \(\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\) is the first Pauli matrix. With this extension, the second component \(m_2(z)\) is a meromorphic function in \(\mathbb{C} \setminus (\overline{\mathbb{D}} \cup I^*)\) with poles at the points \(z_j^{-1}\).

Endow the circle \(\mathbb{T}\) with counterclockwise orientation and intervals \(I, I^*\) with orientation towards the origin of the circle \(\mathbb{T}\). As always a positive side of a contour is the one which lies to the left as one traverses the contour in the direction it is oriented; the negative side lies to the right respectively. We will use a common abbreviation \(m_+\) which denotes the limit of \(m(z)\) from the positive/negative side respectively of the contour \(\Gamma := \mathbb{T} \cup I \cup I^*.\) Using these abbreviations we implicitly assume the existence of respective limits. For \(m(z)\) these limits are continuous on \(\Gamma\) with the only possible singularities at points \(q_i\) and \(q_i^{-1}, \quad i = 1, 2.\)
Proposition 2.2. Suppose that the initial data of the Cauchy problem (1.1)–(1.3), satisfy (1.4). Let the set (2.8) be the right scattering data of the operator $H(0)$ (1.5). Then the vector-valued function $m(z) = m(z,n,t)$ defined by (2.9) is the unique solution of the following vector Riemann–Hilbert problem: Find a meromorphic in $\mathbb{C} \setminus \Gamma$ function $m(z)$ with poles at the points $z_j$, $z_j^{-1}$, which satisfies:

I. The jump condition $m(z) = m(z)v(z)$, where

$$v(z) = \begin{cases} 
\begin{pmatrix} 0 & -Re\Phi(z) \nu \left(z^{-1}\right) e^{-2\Phi(z)} \\
R(z) e^{2\Phi(z)} & 1 \end{pmatrix}, & z \in \mathbb{T}, \\
\begin{pmatrix} 1 & 0 \\
\chi(z) e^{2\Phi(z)} & 1 \end{pmatrix}, & z \in \mathbb{I}, \\
\begin{pmatrix} 1 & -\chi(z^{-1}) e^{-2\Phi(z)} \\
0 & 1 \end{pmatrix}, & z \in \mathbb{I}^*.
\end{cases}$$

Here $\chi(z)$ is given by (2.6) and the phase function $\Phi(z) = \Phi(z,n,t)$ is defined by

$$\Phi(z) = \frac{1}{2} \left( z - z^{-1} \right) + \xi \log z, \quad \xi := \frac{n}{t} \in \mathbb{R}.$$  

II. The symmetry condition

$$m(z^{-1}) = m(z)\sigma_1.$$  

III. The normalization condition

$$m_1(0)m_2(0) = 1, \quad m_1(0) > 0.$$  

IV. The pole condition

$$\text{Res}_{z_j} m(z) = (-z_j \gamma_j m_2(z_j) e^{2\Phi(z_j)}, \ 0);$$

$$\text{Res}_{z_j^{-1}} m(z) = (0, \ z_j^{-1} m_1(z_j^{-1}) \gamma_j e^{2\Phi(z_j)}).$$

V. If a point $q_i$ (2.3) is the resonant point, that is if $\chi(z)$ satisfies (2.7), then

$$m(z) = \left( C_1(z - q_i)^{-1/2}, \ C_2 \left(1 + o(1)\right) \right), \quad C_1 \neq 0, \quad z \to q_i,$$

with an analogous singularity of the second component $m_2(z)$ at the point $q_i^{-1}$. If $\chi(q_i) = 0$ then $m(z)$ has limiting values as $z \to q_i$, $z \in \mathcal{D}$ and $z \to q_i^{-1}, z \in \mathcal{D}^*$.  

This proposition is proved in Appendix.

Note that the symmetry condition II plays a crucial role in ascertaining the solution uniqueness for the vector RHP. That is why we will perform only those transformations which will preserve this symmetry and also the normalization condition IV. To this end we introduce a few constraints on our conjugation/deformation steps. The original RHP I-V is in agreement with these constraints already. Namely, suppose that after some steps we got an equivalent RHP for a vector function $\tilde{m}$ with a jump matrix $\tilde{v}$ on a contour $\tilde{\Gamma}$. Then:

1. The jump contour $\tilde{\Gamma}$ is symmetric with respect to the map $z \mapsto z^{-1}$.

2. The direction on parts of $\tilde{\Gamma}$ are chosen in a way that the jump matrix satisfies the symmetry condition

$$\left( \tilde{v}(z) \right)^{-1} = \sigma_1 \tilde{v}(z^{-1}) \sigma_1.$$  


(3) The vector \( \tilde{m} \) satisfies the symmetry condition \( \tilde{m}(z^{-1}) = \tilde{m}(z)\sigma_1 \) for all \( z \in \mathbb{C} \setminus \tilde{\Gamma} \), moreover, \( \tilde{m}_1(0)\tilde{m}_2(0) = 1 \).

(4) Let \( \Gamma' \subset \tilde{\Gamma} \) be a symmetric sub-contour and let \( d : \mathbb{C} \setminus \Gamma' \to \mathbb{C} \) be a sectionally analytic function. Suppose that \( d(z^{-1}) = d(z)^{-1} \) for \( z \in \mathbb{C} \setminus \Gamma' \) and \( d(0) > 0 \). Then the conjugation

\[
\hat{m}(z) = \tilde{m}(z)[d(z)]^{-\sigma_3}, \quad \text{where} \quad [d(z)]^{-\sigma_3} := \begin{pmatrix} d^{-1}(z) & 0 \\ 0 & d(z) \end{pmatrix},
\]

is in agreement with the constraints above. To simplify further consideration note that transformation \((2.14)\) convert the jump matrix \( \hat{v} \) into a jump matrix

\[
\hat{v} = \begin{cases} 
\begin{pmatrix} \hat{v}_{11} & \hat{v}_{12}d^2 \\
\hat{v}_{21}d^{-2} & \hat{v}_{22} \end{pmatrix}, & z \in \tilde{\Gamma} \setminus \Gamma', \\
\begin{pmatrix} \frac{d_{-}d^{-1}}{d_+}\hat{v}_{11} & \hat{v}_{12}d_+d_-^{-1} \\
\hat{v}_{21}d_+d_-^{-1}d_+^{-1} & \frac{d_{-}d^{-1}}{d_+}\hat{v}_{22} \end{pmatrix}, & z \in \Gamma'.
\end{cases}
\]

Recall now that the asymptotic behavior as \( z \to 0 \) of the solution of the original RHP I–V depends on the "slow" variable \( \xi = \frac{n}{t} \), and is determined in essential by the signature table of the real part of the phase function. As it was mentioned in Introduction, we confine ourselves by studying the solution of RHP I–V for \( \xi \in [\varepsilon, 1 - \varepsilon] \). The signature table of \( \text{Re } \Phi(z, \xi) \) for such \( \xi \) is shown in Fig. 2.

Evidently, \( \text{Re } \Phi(z) = 0 \) as \( z \in T \). The other curve \( \text{Re } \Phi(z) = 0 \) intersects \( T \) at the stationary points \( \zeta_0 \) and \( \overline{\zeta}_0 \) of \( \Phi \), where \( \zeta_0 = -\xi + \sqrt{\xi^2 - 1} \). Our next section consists in a short description of the conjugation/deformation steps performed in [6] which led to a model RHP for \( \xi \in [\varepsilon, 1 - \varepsilon] \). In [6] the resonances at points \( q_i \) were not admitted. That is why in Section 3 we make also some additional transformations for the resonant cases.

3. Reduction to the model problem

Step 1. First of all we get rid of singularities of \( m \) at the eigenvalues. The approach to replace the residue conditions by additional jumps on contours around
the eigenvalues was developed in [2], [7]. Let
\[ P(z) = \prod_{z_j \in (-1,0)} \frac{|z_j| z^{-1}}{z - z_j} \]
be the Blaschke product corresponding to negative \( z_j \) (if any). It satisfies \( P(0) > 0 \), \( P(z^{-1}) = P^{-1}(z) \). Let \( \delta \) be sufficiently small such that the circles \( T_j = \{ z : |z - z_j| = \delta \} \) around the eigenvalues do not intersect and lie away from \( T \cup I \) (the precise value of \( \delta \) will be chosen later). Set
\[ A(z) = \begin{cases} \left( 1 \frac{z - z_j}{z_j e^{i \Phi(z_j)}} \right) [P(z)]^{-\sigma_1}, & |z - z_j| < \delta, \ z_j \in (-1,0), \\ \left( 0 1 \right), & |z - z_j| = \delta, \ z_j \in (0,1), \\ \left( 0 1 \right), & |z^{-1} - z_j| < \delta, \ j = 1, \ldots, N, \\ \sigma_1 A(z^{-1}) \sigma_1, & \text{else.} \end{cases} \]
We consider the circles \( T_j \) as contours with counterclockwise orientation. Denote their images under the map \( z \mapsto z^{-1} \) by \( T_j^* \) and orient them clockwise.

Redefine the solution of RHP I–V by
\[ m^{\text{ini}}(z) = m(z)A(z), \quad z \in \mathbb{C}. \]
Then \( m^{\text{ini}}(z) \) is a holomorphic vector function in \( \mathbb{C} \setminus \{ \Gamma \cup T^\delta \} \), where \( T^\delta := \bigcup_j T_j \cup T_j^* \), and solves the jump problem \( m^{\text{ini}}_+ = m^{\text{ini}}_- \sigma_1 \Phi(z_j) \), for \( z \in \Gamma \cup T^\delta \), where
\[ v^{\text{ini}}(z) = \begin{cases} v(z), & z \in \Gamma, \\ B(z), & z \in \cup_j T_j, \\ \sigma_1 \Phi(z^{-1})^{-1} \sigma_1, & z \in \cup_j T_j^*. \end{cases} \]
\[ (3.1) \]
\[ B(z) = \begin{cases} \left( 1 \frac{z - z_j}{z_j e^{i \Phi(z_j)}} \right)^{-1}, & z \in T_j, \ z_j \in (-1,0), \\ \left( 0 1 \right), & z \in T_j^*, \ z_j \in (0,1). \end{cases} \]
Note that \( \| B(z) - I \| \leq C \exp \left( -t \inf_j |\text{Re} \Phi(z_j)| \right) \) for \( z \in T^\delta \) when \( 0 < \xi < 1 \). Here the matrix norm is to be understood as the maximum of the absolute value of its elements.

Step 2. Set \( z_0 = e^{i \theta_0} \), where \( \cos \theta_0 = 1 - 2\xi, \ \theta_0 \in (0, \pi) \), and introduce a symmetric contour
\[ \Sigma := \{ z \in \mathbb{T} : \text{Re} \ z \leq \text{Re} \ z_0 = \cos \theta_0 \}, \]
oriented in the same way as \( T \), i.e., from \( z_0 \) to \( \overline{z}_0 \). Introduce the function
\[ g(z) = \frac{1}{2} \int_{z_0}^z \sqrt{\left( 1 - \frac{1}{s z_0} \right) \left( 1 - \frac{z_0}{s} \right) (1 + s)} \frac{ds}{s}, \]
where the square root in the integrand is defined in \( \mathbb{C} \setminus \Sigma \) and \( \sqrt{s} > 0 \) for \( s > 0 \).
Lemma 3.1. [6] The function $g(z)$ has the following properties:

(a) Function $g$ has a jump along the arc $\Sigma$ with $g_+(z) = -g_-(z) > 0$ as $z \neq z_0^{\pm 1}$;
(b) $\lim_{z \to 0} \Phi(z) - g(z) = K(\xi) \in \mathbb{R}$, where $\frac{d}{d\xi}K(\xi) = -\log \xi$;
(c) $g(z_0) = 0$;
(d) $g(z^{-1}) = -g(z)$ as $z \in \mathbb{C} \setminus \Sigma$.

The signature table for $g(z)$ is given in Fig. 3. From Lemma 3.1 and from the oddness of the phase function $\Phi(z)$ it follows that the function $d(z) := e^{i(\Phi(z) - g(z))}$ satisfies our symmetry constraints. Set

$$m^{(1)}(z) = m^{ini}(z)[d(z)]^{-\sigma_3},$$

then the vector function $m^{(1)}(z)$ solves jump problem $m^+_1(z) = m^-_1(z)v^{(1)}(z)$ with

$$v^{(1)}(z) = \begin{cases} 
0 & \text{if } z \in T \setminus \Sigma, \\
R(z)e^{2\gamma(z)} & \text{if } z \in \Sigma, \\
E(z), & \text{if } z \in \Xi,
\end{cases}$$

where $R(z) := R(z)P^{-2}(z)$, $\Xi := I \cup I^* \cup T^\delta$, and

$$E(z) := \begin{pmatrix} 
1 & 0 \\
\sigma_1(E(z^{-1}))^{-1}\sigma_1 & 1 \\
[d(z)]^{\sigma_3}B(z)[d(z)]^{-\sigma_3}, & \text{if } z \in T^\delta.
\end{pmatrix}$$

As it is shown in [6], for sufficiently small $\delta > 0$ in the case if the points $q_1$ and $q_2$ are nonresonant, the following estimate is valid:

$$\|E(z) - I\|_{L^\infty(\Xi)} \leq Ce^{-\frac{J(\delta)}{2}}, \quad J(\delta) > 0.$$
Step 3: Here we perform a standard lenses mechanism related to the upper-lower factorization of the jump matrix \((3, 8)\). Till this step we did not use the decaying condition \((1.4)\). To proceed further one has to specify the value of \(\nu\) there. Indeed, the lenses mechanism requires a holomorphic continuation of the right reflection coefficient in a small vicinity of the arc \(\mathbb{T} \setminus \Sigma\). Denote 
\[
\hat{\psi}(z, n) := \psi(z, n, t)|_{t=0}, \\
\hat{\psi}_1(z, n) := \psi(z, n, t)|_{t=0}.
\]
Since
\[
R(z) = -\frac{\langle \hat{\psi}_1(z), \psi(z) \rangle}{\langle \hat{\psi}_1(z), \psi(z) \rangle},
\]
and \(\overline{R}(z) = R(z^{-1}) = R^{-1}(z)\) as \(z \in \mathbb{T}\), we need an analytical continuation \(\tilde{\psi}(z, n)\) of the function \(\hat{\psi}(z, n)\), which is defined initially on \(z \in \mathbb{T}\). Evidently, this continuation can be represented via the transformation operator
\[
\tilde{\psi}(z, n) = \sum_{m=n}^{\infty} K(n, m) z^{-m}, \quad z \in \mathbb{T},
\]
where for \(m > n > 0\) the following estimate holds:
\[
|K(n, m)| \leq C(n) \sum_{l=\lceil\frac{m+n}{2}\rceil}^{\infty} \{|a(n, 0) - \frac{1}{2}| + |b(n, 0)|\}, \quad 0 < C(n) < C.
\]

Thus for arbitrary \(0 < \nu\) in \((1.4)\) the right reflection coefficient can be continued analytically in the domain \(\mathcal{P}_\nu := \{z : e^{-\nu} < |z| < 1, \ z \notin I\}\), with possible poles at points of the discrete spectrum which find themselves inside this ring with a cut. The lenses mechanism which we apply holds for an arbitrary small \(\nu > 0\) if there are no resonances at points \(q_1\) and \(q_2\), for \(\nu > -\log q_2\) if the point \(q_1\) is not resonant, and for \(\nu > -\log q_1\) if there is a resonance at \(q_1\). The case of the nonresonant points \(q_1\) and \(q_2\) is described in \([6]\). Assume that the point \(q_2\) is the resonant one and \(\nu > -\log q_1\). Let \(C\) be a contour close to the arc \(\mathbb{T} \setminus \Sigma\) with endpoints \(z_0\) and \(\overline{z}_0\) and clockwise orientation as depicted on Figure 3. We presume that there are no points of the discrete spectrum between this contour and the arc \(\mathbb{T} \setminus \Sigma\). Choose another contour \(C_I\) close the interval \(I\) such that \(C_I \subset \mathcal{P}_\nu\) and there are no points of the discrete spectrum inside \(C_I\). Orient this contour counterclockwise.

Let now \(C^*\) (resp., \(C^*_I\)) be the image of \(C\) (resp., \(C_I\)) under the map \(z \mapsto z^{-1}\), oriented clockwise (resp., counterclockwise) as well. Let \(\Omega\) consist of two domains:
\( \mathcal{C}_I \) without points of \( I \). Denote the symmetric domains by \( \Omega^* \) as in Fig. 3 and set

\[
V(z) := \begin{cases} 
1 & z \in \Omega, \\
-\mathcal{R}(z)e^{2g(z)} & z \in \Omega^*, \\
\sigma_1(V(z))^{-1}\sigma_1 & z \in \Omega^*.
\end{cases}
\]

Set

\[
m^{(2)}(z) = m^{(1)}(z)V(z), \quad z \in \Omega \cup \Omega^*; \quad m^{(2)}(z) = m^{(1)}(z), \quad z \in \mathbb{C} \setminus (\Omega \cup \Omega^*).
\]

The new vector function has not jump on \( T \setminus \Sigma \). Instead it has additional jumps on the new contours and satisfies \( m^{(2)}_+(z) = m^{(2)}_-(z)v^{(2)}(z) \), where (cf. \( 3.3 \), \( 3.11 \))

\[
v^{(2)}(z) = \begin{cases} 
v^{(1)}(z), & z \in \Sigma \cup T^\delta, \\
V(z), & z \in \mathcal{C} \cup \mathcal{C}_I, \\
V^{-1}(z)E(z)V_+(z), & z \in I, \\
\sigma_1[v^{(2)}(z^{-1})]^{-1}\sigma_1, & z \in \mathcal{C}^* \cup \mathcal{C}_I^* \cup I^*.
\end{cases}
\]

Note that this step again preserves both the normalization and the symmetry conditions. Next, we observe that

\[
V^{-1}(z)E(z)V_+(z) = \left( P^{-2}(z) \left[ R_-(z) + \chi(z) - R_+(z) \right] e^{2g(z)} \right), \quad z \in I.
\]

Recall now that that

\[
R_- = \frac{\langle \psi_1, \bar{\psi} \rangle}{\langle \bar{\psi}_1, \psi \rangle}, \quad \chi = -\frac{\langle \bar{\psi}_1, \psi \rangle}{\langle \bar{\psi}_1, \psi \rangle}, \quad R_+ = -\frac{\langle \psi, \bar{\psi}_1 \rangle}{\langle \psi_1, \bar{\psi} \rangle}.
\]

As the Wronskian of any two Jost solutions is an anti-symmetric operation, by the Plucker identity

\[
\langle a, b \rangle \langle c, d \rangle + \langle a, c \rangle \langle d, b \rangle + \langle a, d \rangle \langle b, c \rangle = 0,
\]

setting \( a = \bar{\psi}, \ b = \psi, \ c = \bar{\psi}_1, \ d = \psi_1 \), we have \( R_-(z) + \chi(z) - R_+(z) \equiv 0 \) and

\[
V^{-1}(z)E(z)V_+(z) = \mathbb{I}, \quad \text{for} \quad z \in I.
\]

Thus, this step allows us to get rid of a possible singularity of the jump matrix in the resonant point under assumption that the initial data tend to the right background constant exponentially fast with an exponent which depends on a location of the resonant point. Since this singularity is not \( L_2 \) this transformation will allow us further to stay in the frameworks of the standard schemes of investigation of the Cauchy-type integrals.

Note that in the case of the absence of resonances in \( q_1 \) and \( q_2 \) we do not need to introduce contours \( \mathcal{C}_I \) and \( \mathcal{C}_I^* \) and redefine \( m^{(1)} \) inside them, because estimate \( 3.3 \) is already suitable for further considerations. To put all cases (resonant and nonresonant) in a single scheme we can either set \( v^{(2)}(z) = \mathbb{I} \) on \( \mathcal{C}_I \cup \mathcal{C}_I^* \). Anyway, if we denote

\[
\tilde{\Xi} := I \cup I^* \cup T^\delta \cup \mathcal{C}_I \cup \mathcal{C}_I^*,
\]

we observe that \( v^{(2)}(z^{-1}) = \sigma_1[v^{(2)}(z)]^{-1}\sigma_1 \) for \( z \in \tilde{\Xi} \) and

\[
\|v^{(2)}(z) - \mathbb{I}\|_{L^\infty(\tilde{\Xi})} \leq Ce^{-\mu}, \quad C = C(\nu, \epsilon) > 0, \quad \mu = \mu(\nu, \epsilon) > 0.
\]
Step 4: For \( z \in \mathbb{C} \setminus \Sigma \) introduce the function
\[
q(z, z_0) = \sqrt{(z_0 - z)(z_0 z - 1)}.
\]
Let \( [q(z, z_0)]_+ \) be its values from the positive side of \( \Sigma \). Set
\[
(3.7) \quad \tilde{d}(z) = \exp \left( \frac{q(z, z_0)}{2\pi i} \int_{z_0}^\sigma \frac{R(s)}{q(z, z_0)[s - z]} ds \right).
\]
As is shown in [6], this function uniquely solves the following scalar conjugation problem: find a holomorphic function \( \tilde{d}(z) \) on \( \mathbb{C} \setminus \Sigma \), such that
\[
(3.8) \quad \tilde{d}_+(z)\tilde{d}_-(z) = R(z)R^{-1}(-1), \quad z \in \Sigma,
\]
\[
(3.9) \quad (i) \quad \tilde{d}(z^{-1}) = \tilde{d}^{-1}(z), \quad z \in \mathbb{C} \setminus \Sigma; \quad (ii) \quad \tilde{d}(\infty) > 0.
\]
Define now \( m^{(3)}(z) = m^{(2)}(z)[\tilde{d}(z)]^{-\sigma_3}, \quad z \in \mathbb{C} \). From the previous considerations we conclude that \( m^{(3)}(z) \) is the unique solution of the following RHP, which is in fact RHP-equivalent: find a holomorphic vector function \( \tilde{m}(z) \) in \( \mathbb{C} \setminus (\Sigma \cup \mathcal{C} \cup \mathcal{C}^* \cup \tilde{\Xi}) \), which is continuous up to the boundary and has the following properties:

- It solves the jump problem \( \tilde{m}_+(z) = \tilde{m}_-(z)\tilde{v}(z) \) with
\[
(3.10) \quad \tilde{v}(z) = \begin{cases} 
\left( \begin{array}{cc} 0 & -R(-1) \\
R(-1) & \frac{dg(s)}{ds} \frac{\alpha_2(s)}{\alpha_1(s)} \end{array} \right), & z \in \Sigma,
\end{cases}
\]
\[
\begin{cases} 
\left( \begin{array}{cc} 1 & 0 \\
-\tilde{d}^{-2}(z)R(z)e^{-2tg(z)} & 1 \\
\tilde{d}^2(z)R(z)e^{2tg(z)} & 0 \\
0 & 1 \\
[\tilde{d}(z)]\sigma_3, v^{(2)}(z)[\tilde{d}(z)]^{-\sigma_3}, & z \in \tilde{\Xi},
\end{array} \right) \quad & z \in \mathcal{C},
\end{cases}
\]
\[
\begin{cases} 
\left( \begin{array}{cc} [\tilde{d}(z)]\sigma_3, v^{(2)}(z)[\tilde{d}(z)]^{-\sigma_3}, & z \in \tilde{\Xi},
\end{array} \right) \quad & z \in \mathcal{C}^*.
\end{cases}
\]

From (3.11) it follows that
\[
(3.11) \quad \|\tilde{v}(z) - I\|_{L^\infty(\tilde{\Xi})} \leq Ce^{-\mu z}, \quad C = C(\nu, \epsilon) > 0, \quad \mu = \mu(\nu, \epsilon) > 0.
\]

- \( \tilde{m}(z^{-1}) = \tilde{m}(z)\sigma_1 \) and \( \tilde{m}_1(0)\tilde{m}_2(0) = 1 \); moreover, \( \tilde{v}(z) \) has the symmetry property (2.13).
- For small \( z \) the original vector function \( m \) in (2.9) and \( \tilde{m}(z) \) are connected by the following transformation
\[
(3.12) \quad \tilde{m}(z) = m(z)[q(z)]^{-\sigma_3}, \quad q(z) = d(z)P(z)e^{t(\Phi(z) - g(z))}.
\]
Denote
\[
(3.13) \quad v^{\text{mod}}(z) = \begin{cases} 
(0 & -R(-1) \\
R(-1) & 0 \
\end{cases}, \quad z \in \Sigma.
\]
Let \( B \) and \( B^* \) be small vicinities of points \( z_0 \) and \( z_0^* \) respectively. We suppose that these vicinities do not have joint points on their boundaries. They are symmetric with respect to the map \( z \mapsto z^{-1} \) and are located on positive distances bigger than \( \frac{\text{Im} z}{2\pi} \) from points \( \pm 1 \). Put \( \bar{\Gamma} := \Sigma \cup \mathcal{C} \cup \mathcal{C}^* \cup \tilde{\Xi} \), where \( \tilde{\Xi} \) is defined by (3.5). Taking into account the signature table for \( \text{Re} g(z) \) for \( \xi \in [\epsilon, 1 - \epsilon, \text{we observe that} \)
\[
(3.14) \quad \|\tilde{v}(z) - v^{\text{mod}}(z)\|_{L^\infty(\bar{\Gamma} \setminus (\bar{\Gamma} \cap (B \cup B^*)))} \leq Ce^{-tU}, \quad U = U(\epsilon) > 0.
\]
Therefore one can assume that the solution of the RHP–equiv (3.10) can be well approximated in \( \mathbb{C} \setminus (B \cup B^* ) \) by a solution of the following model RHP: find a holomorphic vector function in \( \mathbb{C} \setminus \Sigma \) satisfying the jump condition
\[
M^\text{mod}(z) = m^\text{mod}(z)v^\text{mod}(z), \quad z \in \Sigma,
\]
the symmetry condition \( m^\text{mod}(z^{-1}) = m^\text{mod}(z)\sigma_1 \), and the normalization condition
\[
m^\text{mod}_1(0) > 0, m^\text{mod}_1(0)m^\text{mod}_2(0) = 1.
\]

The solution of this problem is unique (see lemma 4.1 in appendix). In \([6]\) it was proved that
\[
m^\text{mod}(z) = (\alpha, \alpha)M^\text{mod}(z), \quad \text{where} \quad (\sin \frac{\theta}{4})^{-1/2};
\]
\[
M^\text{mod}(z) = \begin{pmatrix}
\frac{\beta(z) + \beta^{-1}(z)}{2} & \frac{\beta(z) - \beta^{-1}(z)}{2i} \\
\frac{\beta(z) - \beta^{-1}(z)}{2i} & \frac{\beta(z) + \beta^{-1}(z)}{2}
\end{pmatrix};
\]
\[
\beta(z) = \left(\frac{z_0 z - 1}{z_0 - z}\right)^{\pm 1/4}, \quad \text{for} \quad \mathcal{R}(1) = \mp 1.
\]

Here the branch of the fourth root is chosen with the cut along the negative half axis and \( 1^{1/4} = 1 \). The case \( \mathcal{R}(1) = -1 \) corresponds to the nonresonant case at 1, while the case \( \mathcal{R}(1) = 1 \) is related to the resonant case. Note that \( M^\text{mod}(z) = M^\text{mod}(z,n,t) \) solves the following model matrix RHP: find a holomorphic matrix function \( M^\text{mod}(z) \) on \( \mathbb{C} \setminus \Sigma \) satisfying the following jump and symmetry conditions,
\[
M^\text{mod}_+(z) = M^\text{mod}_-(z)v^\text{mod}(z), \quad z \in \Sigma; \quad M^\text{mod}(z^{-1}) = \sigma_1 M^\text{mod}(z)\sigma_1.
\]
The solution of this matrix RHP is not unique.

4. The parametrix problem

\[
\bar{\Gamma} = \Sigma \cup \mathcal{C} \cup \mathcal{C}^* \cup \tilde{\mathcal{C}} \cup \partial \mathcal{B} \cup \partial \mathcal{B}^*, \quad \Sigma_B = \bar{\Gamma} \cap B,
\]
\[
\Sigma_1 = \Sigma \cup \mathcal{B}, \quad \Sigma_2 = \mathcal{C} \cup \mathcal{B}, \quad \Sigma_3 = \mathcal{C}^* \cup \mathcal{B}.
\]

Properties of the function \( g \) don’t allow us to use the same arguments considering close to \( z_0, \bar{z}_0 \) points of the contour \( \Sigma \), so an additional consideration of this part is required. Technically this problem is similar to the one considered in \([13]\).

Here we give the lemma proved in \([6]\), which allows us to replace the jump matrix \( v^{(3)}(z) \) inside \( B \) by another approximately close matrix.

**Lemma 4.1.** The function \( \tilde{d}(z) \) satisfying (3.3)–(3.9) has the following asymptotic behavior in a vicinity of \( z_0 \),
\[
\tilde{d}^{-2}(z)\mathcal{R}(z) = \mathcal{R}(-1) + O(\sqrt{z - z_0}), \quad z \notin \Sigma, \quad \frac{\tilde{d}_+(z)}{\tilde{d}_-(z)} = 1 + O(\sqrt{z - z_0}), \quad z \in \Sigma.
\]

**Proof.** We will use the representation (3.7). To simplify notation set
\[
r(s) = \log(\mathcal{R}(s)\mathcal{R}^{-1}(-1)), \quad q(s,z_0) = \sqrt{(z_0 - z)(z_0 z - 1)}.
\]

Then
\[
\log \tilde{d}(z) = \frac{q(z, z_0)}{2\pi i} \int_{z_0}^{\bar{z}_0} \frac{r(s)ds}{[q(s, z_0)]_+(s - z)} = J_1(z) + J_2(z),
\]

where

\[
J_1(z) = \int_{z_0}^{\bar{z}_0} \frac{r(s)ds}{[q(s, z_0)]_+(s - z)}, \quad J_2(z) = \int_{z_0}^{\bar{z}_0} \frac{r(s)ds}{[q(s, z_0)]_- (s - z)}.
\]
where

\[
J_1(z) = \frac{q(z, z_0)}{2\pi i} \int_{z_0}^{z} \frac{(r(s) - r(z))ds}{q(s, z_0) + (s - z)} \]

\[
J_2(z) = \frac{q(z, z_0)}{2\pi i} \int_{z_0}^{z} ds \frac{1}{q(s, z_0) + (s - z)}.
\]

Since \(r(s) - r(z) \sim (s - z)\), the integral in \(J_1(z)\) is Hölder continuous in a vicinity of \(z_0\). Therefore,

\[
J_1(z) = I(z_0)\sqrt{z - z_0}(1 + o(1)), \quad \text{(4.3)}
\]

\[
I(z_0) = \frac{1}{2}[q(z, z_0)]^2 = \frac{1}{2}[q(z, z_0)]^2 + \frac{1}{[q(z, z_0)]^2}, \quad z \in \Sigma,
\]

and \((q(z, z_0))^{-1} \to 0\) as \(z \to \infty\), we have

\[
\frac{1}{2}[q(z, z_0)]^2 = \frac{1}{2\pi i} \int_{z_0}^{z} ds \frac{1}{q(s, z_0) + (s - z)},
\]

and \(J_2(z) = \frac{r(z_0)}{2}(1 + O(z - z_0))\). Therefore,

\[
\log \tilde{d}(z) = \frac{1}{2} \log \frac{R(z_0)}{R(-1)} + I(z_0)\sqrt{z - z_0} + o(\sqrt{z - z_0}),
\]

and (4.2) follows from (4.3) in a straightforward manner. \(\square\)

Using this lemma we have

\[
v_{\text{par}}(z) := e^{-tg_{-}(z)\sigma_3} S e^{tg_{+}(z)\sigma_3},
\]

where

\[
S = \begin{cases} 
  S_1 := \begin{pmatrix} 0 & -R(-1) \\ R(-1) & 1 \end{pmatrix}, & z \in \Sigma_1, \\
  S_2 := \begin{pmatrix} 1 & 0 \\ -R(-1) & 1 \end{pmatrix}, & z \in \Sigma_2, \\
  S_3 := \begin{pmatrix} 1 & R(-1) \\ 0 & 1 \end{pmatrix}, & z \in \Sigma_3,
\end{cases}
\]

where \(\Sigma_i\) defined by (4.1) and oriented outwards.

Note, that if \(M_{\text{par}}\) solves (4.17), then the matrix function

\[
M(z) = M_{\text{par}}(z)e^{-tg(z)\sigma_3}
\]

solves the constant jump problem

\[
M_+(z) := M_-(z)S
\]

with the normalization \(M \sim M_{\text{mod}}e^{-tg(z)\sigma_3}\) on \(\partial B\).

To simplify our considerations we will next use a change of coordinates

\[
w(z) = \left( \frac{3tg(z)}{2} \right)^{2/3}.
\]
Lemma 4.2. In a vicinity of $z_0$,

$$g(z) = C(\theta_0)(z - z_0)^{\frac{3}{2}}e^{i(\frac{\pi}{4} - \frac{3\theta_0}{2})}(1 + o(1)),$$

$$C(\theta_0) = \sqrt{\frac{2}{3}}(\sin \theta_0)^{\frac{3}{2}} \cos \frac{\theta_0}{2} > 0.$$ 

Proof. The function $g(z)$ is given by (3.2). Then it’s derivative is

$$\frac{d}{dz}g(z) = \frac{1}{2} \sqrt{\left(1 - \frac{1}{z_0^2}\right) \left(1 - \frac{z_0}{z}\right) \frac{1 + z}{z}} = \sqrt{z - z_0}f(z),$$

where $f(z) = \frac{1}{2} \frac{1 + z}{z} - \frac{1}{z_0} - \text{holomorphic in a vicinity of } z_0$, and

$$f(z_0) = 2\sqrt{2}e^{i(\frac{\pi}{4} - \frac{3\theta_0}{2})} \sqrt{\sin \theta_0} \cos \frac{\theta_0}{2}.$$

So

$$g'(z) = 2\frac{3}{2}(\sin \theta_0)^{\frac{3}{2}} \cos \frac{\theta_0}{2} e^{i(\frac{\pi}{4} - \frac{3\theta_0}{2})} \sqrt{z - z_0} (1 + O((z - z_0))),$$

and

$$g(z) = C(\theta_0)e^{i(\frac{\pi}{4} - \frac{3\theta_0}{2})}(z - z_0)^{\frac{3}{2}}(1 + o(1)), \quad z \to z_0.$$ 

Since $\theta_0 \in (\frac{\pi}{2}, \pi)$, $\sin \theta_0, \cos \frac{\theta_0}{2} > 0$ and $C(\theta_0) > 0$. □

Then we have

$$(4.5) \quad w(z) = t^{\frac{3}{2}} \left(\frac{3}{2}C(\theta_0)\right)^{\frac{3}{4}} e^{i(\frac{\pi}{4} - \theta_0)}(z - z_0)(1 + o(1)), \quad z \to z_0.$$ 

Choose the set $B$ as the preimage under the map $z \mapsto w$ of the circle $D_\rho = \{ w \in \mathbb{C} : |w| < t^{2/3}(\frac{3}{2}C(\theta_0))^{2/3} \rho, \quad \rho < \text{Im } z_0/2 \}$, centered at $w = 0$. Seeing we have some flexibility of choosing the contours $C$ and $C^*$ we can set them such that $\Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ are mapped onto the straight lines $(\Gamma_1 \cup \Gamma_2 \cup \Gamma_3) \cap D_\rho$ (see Fig. 4), where

$$\Gamma_2 = \{ w \in \mathbb{C} : \arg w = \frac{4\pi}{3} \}, \quad \Gamma_3 = \{ w \in \mathbb{C} : \arg w = \frac{2\pi}{3} \}, \quad \Gamma_1 = [0, +\infty).$$

Figure 4. The map $z \to w$

Lemma 4.3. $w(z)$ maps $\Sigma_1$ into $[0, A]$, $A > 0$. 
Proof. Since \( \arg(z - z_0) = \theta_0 - \frac{3\pi}{2} \), then
\[
e^{i(\frac{\pi}{4} - \frac{3\theta_0}{2})}(z - z_0)^{\frac{3}{2}} = e^{-2\pi i|z - z_0|} = |z - z_0|.
\]
\[\square\]

Consider first the generic nonresonant case where
\[
S_1 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},
\]
and the function \( \beta(z) \) is locally given by (3.18)
\[
\beta(z) = \frac{w - 1/4}{\gamma(w)}, \quad w \in \mathbb{D}_\rho,
\]
where \( \gamma \) is holomorphic, and
\[
(4.6) \quad \gamma(w) = \beta(z(w))w^{1/4} = 2^{\frac{1}{4}}(\sin \theta_0)^{1/3}(\cos \frac{\theta_0}{2})^{1/6}e^{i\frac{\pi}{12}}\left(1 + O\left(t^{-2/3}w\right)\right).
\]
Combining the fact that \( \gamma(w) \) holomorphic with (3.18) we can represent matrix \( M^{\text{mod}}(z) \) as
\[
M^{\text{mod}}(z) = M_0[\gamma(w)]^{\sigma_3}w^{-\frac{2\pi}{3}}M_0^{-1}.
\]
Where \( M_0 = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \). Since \( \mathbb{D}_\rho \) grows as \( t \to \infty \), if we find a matrix solution \( A(w) \) of the jump problem
\[
(4.7) \quad A_+ = A_-S_j \quad \text{on} \quad \Gamma_j,
\]
with the normalization condition
\[
(4.8) \quad A(w) = w^{-\frac{2\pi}{3}}(M_0^{-1} + O(w^{-3/2}))e^{-\frac{2\pi}{3}w^{3/2}\sigma_3}, \quad \text{as} \quad w \to \infty,
\]
in any direction with respect to \( w \) (here the term \( O(w^{-3/2}) \) has been written a-posteriori), then the matrix
\[
(4.9) \quad M^{\text{par}}(z) = M_0[\gamma(w)]^{\sigma_3}A(w)e^{\frac{2\pi}{3}w^{3/2}\sigma_3}
\]
will satisfy the condition
\[
(4.10) \quad M^{\text{par}}(z) = M^{\text{mod}}(z)(I + O(\rho^{-3/2}t^{-1})), \quad \text{as} \quad t \to \infty, \quad z \in \partial B.
\]
So the parametrix problem (4.17) was reduced to the problem (4.7), (4.8). The solution of the last one can be given in terms of Airy functions. To this end set
\[
y_1(w) := \text{Ai}(w) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \exp\left(\frac{1}{3}z^3 - wz\right)dz,
\]
\[
y_2(w) := e^{\frac{2\pi i}{3}}\text{Ai}(e^{\frac{2\pi i}{3}}w),
\]
\[
y_3(w) := e^{\frac{2\pi i}{3}}\text{Ai}(e^{\frac{2\pi i}{3}}w).
\]
Functions \( y_i(z) \) are entire functions and connected with each other by the relation
\[
(4.11) \quad y_1(w) + y_2(w) + y_3(w) = 0.
\]
Set
\[(4.12)\]
\[\Omega_1 = \{ w : \arg w \in \left(0, \frac{2\pi}{3}\right) \}, \quad \Omega_2 = \{ \arg w \in \left(\frac{2\pi}{3}, \frac{4\pi}{3}\right) \}, \quad \Omega_3 = \mathbb{C} \setminus \{\Omega_1 \cup \Omega_2\}.\]

The asymptotics of the Airy functions (cf. [12], (9.7.5), (9.7.6)) are
\[(4.13)\]
y_1(w) = \begin{cases} 
\frac{1}{2\sqrt{\pi}w^{3/2}}e^{-\frac{2}{3}w^{3/2}}(1 + O(w^{-3/2})), & w \in \Omega_1, \\
\frac{i}{2\sqrt{\pi}w^{3/2}}e^{\frac{2}{3}w^{3/2}}(1 + O(w^{-3/2})), & w \in \Omega_3,
\end{cases}\]
\[(4.14)\]
y_2(w) = -\frac{1}{2\sqrt{\pi}w^{1/4}}e^{\frac{2}{3}w^{1/2}}(1 + O(w^{-3/2})), & w \in \Omega_1 \cup \Omega_2,
\[(4.15)\]
y_3(w) = -\frac{1}{2\sqrt{\pi}w^{1/4}}e^{-\frac{2}{3}w^{1/2}}(1 + O(w^{-3/2})), & w \in \Omega_2 \cup \Omega_3,

and can be differentiated with respect to \(w\). Set
\[(4.16)\]
\[A_i(w) := \sqrt{\pi} \begin{pmatrix} y_1(w) & y_2(w) \\
y_1'(w) & -y_2'(w) \end{pmatrix}, \quad w \in \Omega_1.\]

Then \(\det A_1(w) = 1\) (cf. [12], (9.2.8)), and by (4.13), (4.14) we have the correct normalization (1.8) in \(\Omega_1\). Furthermore, functions \(A_i(w), \quad i = 2, 3\) can be found as next \(A_2(w) := A_1S_2(w), \quad A_3(w) := A_2S_3(w)\). And we should check two things: the condition (4.8) and the monodromy condition \(A_3(w)S_1 = A_1(w)\). So, by (4.11)
\[\begin{align*}
A_2(w) &= A_1(w)S_2 = \sqrt{\pi} \begin{pmatrix} y_1(w) & y_2(w) \\
y_1'(w) & -y_2'(w) \end{pmatrix} \begin{pmatrix} 1 & -1 \\
0 & 1 \end{pmatrix} \\
&= \sqrt{\pi} \begin{pmatrix} -y_3(w) & y_2(w) \\
y_3'(w) & -y_2'(w) \end{pmatrix}, \\
A_3(w) &= A_2(w)S_3 = \sqrt{\pi} \begin{pmatrix} -y_3(w) & y_2(w) \\
y_3'(w) & -y_2'(w) \end{pmatrix} \begin{pmatrix} 1 & 0 \\
1 & 1 \end{pmatrix} \\
&= \sqrt{\pi} \begin{pmatrix} -y_3(w) & -y_1(w) \\
y_3'(w) & y_1'(w) \end{pmatrix}.
\end{align*}\]

In virtue of (4.13) – (4.15) matrix \(A_i(w), \quad i = 1, 2, 3\) obeys the normalization (1.8) in \(\Omega_1\). For the monodromy condition we have
\[A_3(w)S_1 = \sqrt{\pi} \begin{pmatrix} -y_3(w) & -y_1(w) \\
y_3'(w) & y_1'(w) \end{pmatrix} \begin{pmatrix} 0 & 1 \\
-1 & 1 \end{pmatrix} = \sqrt{\pi} \begin{pmatrix} y_1(w) & y_2(w) \\
y_1'(w) & -y_2'(w) \end{pmatrix} = A_1(w).
\]

So, \(A(w) = A_i(w), \quad w \in \Omega, \quad i = 1, 2, 3\) is the solution we looked for.

So a result of our previous investigations can be formulated as the following

**Theorem 4.4.** The matrix solution of the jump problem
\[(4.17)\]
\[M^\text{par}_+(z) = M^\text{par}_-(z)e^{\text{par}_+(z)}, \quad z \in \mathcal{B} \cap (\Sigma_1 \cup \Sigma_2 \cup \Sigma_3)\]
is given by
\[M^\text{par}(z) = M_0[\gamma(w)]^\text{par}_jA_j(w(z))e^{\frac{2}{3}w^{3/2}(z)^\text{par}_j}, \quad w(z) \in \Omega_j, \quad j = 1, 3,
\]
where \(M_0, \gamma(w), w(z)\) defined by (4.10), (4.5), respectively. And \(\Omega_1, A_1\) are given by (4.12), (4.16) and \(A_j = A_{j-1}S_j, \quad j = 2, 3\).
5. The conclusion of the asymptotic analysis

In this section we are going to give substance to the proposition that the solution $\tilde{m}^{(3)}$ of the RH problem (3.10) is well approximated in some sense by $(\alpha \ a) M^{\par}$.

Set

\begin{equation}
\tilde{m}(z) = m^{(3)}(z)(M^{\as}(z))^{-1}, \quad M^{\as}(z) := \begin{cases} M^{\par}(z), & z \in \mathcal{B}, \\ M^{\mod}(z), & z \in \mathbb{C} \setminus \mathcal{B}. \end{cases}
\end{equation}

The vector function $\tilde{m}$ must solve the jump problem

\begin{equation}
\hat{m}_{\pm}(z) = \hat{m}(z)\hat{v}(z),
\end{equation}

where

\begin{equation}
\hat{v}(z) = \begin{cases} M^{\par}(z)v^{(3)}(z)(M^{\par}(z))^{-1}, & z \in \Sigma_{B}, \\ M^{\par}(z)(M^{\mod}(z))^{-1} - \mathbb{I}, & z \in \partial \mathcal{B}, \\ \mathbb{I}, & z \in \mathbb{C} \setminus (\Sigma_{B} \cup \partial \mathcal{B}). \end{cases}
\end{equation}

and the symmetry and normalization conditions:

\begin{equation}
\hat{m}(z)^{-1}\hat{m}(z)\sigma_{1}, \quad \hat{m}_{1}(0) > 0, \quad \hat{m}_{1}(0)\hat{m}_{2}(0) = 1.
\end{equation}

Sets $\Sigma_{B}, \mathcal{B}$ etc. defined by (4.1) Consider $W(z) = \hat{v}(z) - \mathbb{I}$.

\begin{equation}
W(z) = \begin{cases} M^{\par}(z)v^{(3)}(z)(M^{\par}(z))^{-1} - \mathbb{I}, & z \in \Sigma_{B}, \\ M^{\par}(z)(M^{\mod}(z))^{-1} - \mathbb{I}, & z \in \partial \mathcal{B}, \\ \mathbb{I}, & z \in \mathbb{C} \setminus (\Sigma_{B} \cup \partial \mathcal{B}). \end{cases}
\end{equation}

Where $v^{\mod}(z) = \begin{pmatrix} 0 & -\mathcal{R}(-1) \\ \mathcal{R}(-1) & 0 \end{pmatrix}$ and $v^{\par}(z) = e^{-t_{g_{-}}(z)\sigma_{3}} S e^{t_{g_{+}}(z)\sigma_{3}}$, i.e.

\begin{equation}
v^{\par}(z) = \begin{cases} 0 & \mathcal{R}(-1) \\ -\mathcal{R}(-1) & e^{-2t_{g_{+}}(z)} \end{cases}, \quad z \in \Sigma_{1}, \\
\begin{pmatrix} 1 & 0 \\ -\mathcal{R}(-1)e^{2t_{g_{+}}(z)} & 1 \end{pmatrix}, \quad z \in \Sigma_{2}, \\
\begin{pmatrix} 1 & \mathcal{R}(-1)e^{-2t_{g_{+}}(z)} \\ 0 & 1 \end{pmatrix}, \quad z \in \Sigma_{3}.
\end{equation}

Also recall that

\begin{equation}
v^{(3)}(z) = \begin{cases} 0 & -\mathcal{R}(-1) \\ \mathcal{R}(-1) & d^{(z)}(\mathcal{R}(z)e^{-2t_{g_{+}}(z)}) \end{cases}, \quad z \in \Sigma, \\
\begin{pmatrix} 1 & 0 \\ -d^{(z)}(\mathcal{R}(z)e^{2t_{g_{+}}(z)}) & 1 \end{pmatrix}, \quad z \in \mathcal{C}, \\
\begin{pmatrix} 1 & \mathcal{R}(z)e^{-2t_{g_{+}}(z)} \\ 0 & 1 \end{pmatrix}, \quad z \in \mathcal{C}^{*}.
\end{equation}

Now let's make some observations about matrix $W(z)$.

- In virtue of (3.14) and boundness of $M^{\mod}(z)$ on the part $\Sigma \setminus \mathcal{B}$ we have

\[ \|W\|_{L^\infty(\Sigma \setminus \Sigma_{B})} \leq C e^{-\ell U} \]
Now, consider the function $g(z) = \text{Re}(z) + O(z - z_0)e^{-2t(g(z))}$, $z \in \Sigma_j$.

As $g(z) = \text{Re}(z)$ on $\Sigma_B$ from (4.2) and (4.4) follows

\begin{equation}
(5.5) \quad u(z) = \left(C_jI(z_0)\sqrt{|z - z_0|}\right)e^{-2t(g(z))} + O(z - z_0)e^{-2t(g(z))}, \quad z \in \Sigma_j.
\end{equation}

Now, consider the function $I(z_0)$ defined by (4.3)

\[ I(z_0) = \frac{\sqrt{1 - z_0^2}}{2\pi i} \int_{\Sigma_0} \frac{(\log R(s) - \log R(z_0))}{[\sqrt{(z_0 - s)(z_0 - 1)]_+ (s - z_0)}} ds. \]

Denote $f(z_0, z, s) := \frac{\log R(s) - \log R(z_0)}{(s - z)\sqrt{z_0 - 1}}$.

\[ \int_{\Sigma_0} f(z_0, z, s) ds = 2f(z_0, z, s)\sqrt{z_0 - s} + \int_{\Sigma_0} f'(z_0, z, s)\sqrt{z_0 - s} ds, \]

where the last integral can be computed.

On the arc $(-1, \tau_0)$ the function $\frac{1}{\sqrt{z_0 - s}}$ does not have any singularities and is differentiable. On this arc the function $h(z_0, z, s) := \frac{\log R(s) - \log R(z_0)}{(s - z)\sqrt{z_0 - 1}}$ is differentiable too.

\[ \int_{-1}^{\tau_0} h(z_0, z, s) ds = 2h(z_0, z, s)\sqrt{z_0 s - 1} - \int_{-1}^{\tau_0} h'(z_0, z, s)\sqrt{z_0 s - 1} ds, \]

and the last integral can be computed too. This consideration shows us that the function $I(z_0)$ is differentiable as $\xi \in \mathcal{I}$.

Next, let $(z_0 + C_j^2 \rho_j)$ be the end points of the contours $\Sigma_j$. Recall that $\rho_j \geq \rho \geq \sqrt{2e}$. Then

\[ \int_{\Sigma_j} u(z) dz = |s = z - z_0| = C_jI(z_0) \int_0^{\rho_j} \frac{1}{4} e^{-2t(g(s))} ds + O(t^{-4/3}). \]

By lemma 1.2 in a vicinity of $z_0$ we have

\[ \int \frac{1}{4} e^{-2t(g(s))} ds = \int \frac{1}{4} e^{-2C(\varphi_0)ts^{3/2}} e^{-2t(g(s)-s^{3/2})} ds, \]

and

\[ e^{-2t(g(s)-s^{3/2})} \sim e^{Cs^{5/2}} = \sum_{k=0}^{\infty} (-1) \frac{\mathcal{C}_k k^{5k/2}}{k!}. \]

\[ \int \frac{1}{4} e^{-2t(g(s))} ds \sim \left[ y := ts^{2/3}, \quad s^{5k/2} = \left(\frac{2}{t}\right)^{5k/3} \right] \sim \frac{3}{2} \int \sum_{k=0}^{\infty} \frac{\mathcal{C}_k}{k!} y^{5k} t^{-1-\frac{5k}{3}} e^{-Cy} dy. \]

\[ \int y^{5k} t^{-1-\frac{5k}{3}} e^{-Cy} dy = O(t^{-\alpha}), \quad \alpha > 1, \quad \text{when } k > 0, \]
so we conclude
\[ \int_{\Sigma} u(z)dz = \frac{H_j(z_0)}{t} + O(t^{-\alpha}), \quad \alpha > 1, \]
where the function \( H_j(z_0) \) is differentiable and uniformly bounded with respect to \( \xi \in \mathcal{I} \). We also have the following estimate
\[ \|u\|_{L^1(\Sigma_B)} = O(t^{-1}), \quad z \in \Sigma_B. \]
Moreover, the change of the variable of integration \( y := (2C(\varphi_0) t)^{2/3} s \) implies the similar estimate in \( L^p(\Sigma_B) \)
\[ \|u\|_{L^p(\Sigma_B)} = O(t^{-\frac{2+p}{p}}), \quad z \in \Sigma_B. \]
Using the same arguments and taking into account that the matrix entries \([M^\text{par}]_{mk}(z)\) are bounded for \( z \in \Sigma_B \)
uniformly with respect to \( \xi \in \mathcal{I}, \) and using (4.13) and (5.5), we get:
\[ \int_{\Sigma_B} u(z) [M^\text{par}]_{mk}(z) [(M^\text{par})^{-1}]_{pq}(z)dz = \frac{h_{m,k,p,q}(z_0)}{t} + O(t^{-1/3}). \]
The functions \( h_{m,k,p,q}(z_0) \) are bounded with respect to \( \xi \in \mathcal{I} \) (5.6) implies
\[ \int_{\Sigma_B} W(z)dz = \frac{F_B(z_0)}{t} + O(t^{-1/3}), \]
where the matrix \( F_B(z_0) \) is bounded for \( \xi \in \mathcal{I}. \) From (5.4), (4.9), (3.18) and (5.5) it follows that the function \( W(z) \) does not have a singularity at the point \( z_0. \) Next we note that the function \( g(z) \) (3.2) has a continuous derivative with respect to \( z_0 \) for \( \xi \in \mathcal{I}, \) so the function \( F_B(z_0) \) is differentiable. The boundedness allows us to get the following estimates
\[ \|W\|_{L^1(\Sigma_B)} = O(t^{-1}), \quad \|W\|_{L^\infty(\Sigma_B)} = O(t^{-1/3}). \]
Moreover, from (5.4) it follows that
\[ \int_{\partial B} W(z)dz = \frac{F_{\partial B}(z_0)}{t \rho^{1/2}} + O(t^{-1/3}), \]
where the matrix \( F_{\partial B}(z_0) \) have the same properties as \( F_B(z_0), \) since
\[ W(z) = \frac{1}{t^{2q}g(z)} \begin{pmatrix} -7 & 7 \\ 5 & -5 \end{pmatrix} + O(t^{-2}). \]
The last statement follows from (12), (9.7.5), (9.7.6) and (4.10).
Next, the matrix \( M^\text{mod}(z) \) and its inverse are bounded with an estimate \( O(\rho^{-1/4}) \)
on the remaining part of the contour \( \tilde{\Sigma}. \) Using (5.4), (3.14) and (3.11) we conclude
\[ \int_{\Sigma \setminus (\Sigma_B \cup \partial B)} W(z)dz = \tilde{F}_{\text{mod}}(z_0, \rho, t), \quad \|\tilde{F}_{\text{mod}}(z_0, \rho, t)\| \leq C \rho^{-1/4} e^{-\frac{\xi}{4}}, \]
where the matrix norm of \( F_{\text{mod}}(z_0, \rho, t) \) is uniformly bounded with respect to \( \xi \) and \( \rho \) for \( t \in [T_0, \infty) \) and \( \xi \in \mathcal{I}. \) We can conclude also
\[ \|W(z)\|_{L^1(\tilde{\Sigma} \setminus (\Sigma_B \cup \partial B))} \leq O(e^{-ct}), \quad \|W(z)\|_{L^\infty(\tilde{\Sigma} \setminus (\Sigma_B \cup \partial B))} \leq O(e^{-ct}). \]
The contour $\tilde{\Sigma}$ does not contain the point $z = 0$, so multiplying the error matrix $W(z)$ by $z^p, \ p \in \mathbb{R}$ preserves previous estimates:
\[
\int_{\Sigma_0} |z^p W(z)| \, dz \leq \|z^p\|_{C(\tilde{\Sigma})} \|W(z)\|_{L^1(\tilde{\Gamma})}.
\]

The previous estimates allow us to formulate the following

**Lemma 5.1.** The following estimates holds uniformly for $\xi \in I, 1 \leq p \leq \infty$
\[
\|W\|_{L^p(\Gamma)} = O\left(t^{-\frac{1}{3} - \frac{2}{3}p}\right).
\]

To solve the RH problem (5.2) – (5.3) we need to find out the behavior of the vector $m^{(3)}(z)$ at any “appropriate” point. In our case we choose $z = 1$ as such a point. Observe that at the point $z = 1$ the vector-function $m(z) = (T(z, t)\psi_\ell(z, n, t)z^n \psi(z, n, t)z^{-n})$ has the following structure
\[
m(1) = \begin{cases} 
(0, \tilde{\eta}) & \text{in a non-resonant case,} \\
(2\hat{\eta}, \hat{\eta}) & \text{in a resonant case,}
\end{cases}
\]

where $\tilde{\eta}(n, t)$ and $\hat{\eta}(n, t)$ are bounded functions with respect to all arguments. It can be easily checked that after steps 1 – 4 from the section 3 at the point $z = 1$ the vector-function $m^{(3)}(z)$ has the next structure
\[
m^{(3)}(1) = (\eta, \eta),
\]
where $\eta = \eta(n, t)$ is a bounded function with respect to all arguments.

**Lemma 5.2.** The solution of the RH problem (5.2) – (5.3) has the following form
\[
\hat{m}(z) = (\alpha \alpha) + \frac{f(\xi, \rho)}{t} + z \frac{\hat{f}(\xi, \rho)}{t} + o(t^{-1})o(z),
\]
where $\alpha = \alpha(\xi)$ defined by (3.16).

**Proof.** Let $\mathcal{C}$ denote the Cauchy operator associated with $\tilde{\Gamma}$ and with kernel $\Omega(z, s) := \frac{1}{2\pi i} \left( \frac{s + \zeta}{1 - \zeta} - \frac{s + 1}{\lambda}\right)$:
\[
(\mathcal{C}h)(z) = \frac{1}{2\pi i} \int_\tilde{\Gamma} h(s)\Omega(z, s) \frac{ds}{s}, \quad k \in \mathbb{C} \setminus \tilde{\Gamma},
\]

where $h = (h_1 \ h_2) \in L^2(\tilde{\Gamma}) \cap L^\infty(\tilde{\Gamma})$. Let $\mathcal{C}_+ f$ and $\mathcal{C}_- f$ be its non-tangential limiting values from the left and right sides of $\tilde{\Gamma}$, respectively. These operators will be bounded with bound depending on the contour, that is on $z_0$. However, since we can choose our contour scaling invariant at least locally, scaling invariance of the Cauchy kernel implies that we can get a bound which is uniform on compact sets.

Using the Cauchy operator $\mathcal{C}$ let’s introduce the operator $\mathcal{C}_W : L^2(\tilde{\Gamma}) \cap L^\infty(\tilde{\Gamma}) \rightarrow L^2(\tilde{\Gamma})$ by $\mathcal{C}_Wf = \mathcal{C}_-(fW)$, where $W$ is our error matrix. Then,
\[
\|\mathcal{C}_W\|_{L^2(\tilde{\Gamma}) \rightarrow L^2(\tilde{\Gamma})} \leq C\|W\|_{L^\infty(\tilde{\Gamma})} \leq O(t^{-1/3}).
\]

Utilizing the Neumann series representation
\[
(\mathbb{I} - \mathcal{C}_W)^{-1} = \sum_{j=0}^{\infty} \mathcal{C}_W^j
\]
we obtain
\[
\| (\mathbb{I} - \mathbf{C}_W)^{-1} \|_{L^2(\Gamma) \to L^2(\Gamma)} \leq \sum_{j=0}^{\infty} \| \mathbf{C}_W \|_{L^2(\Gamma) \to L^2(\Gamma)}^j = \frac{1}{1 - \| \mathbf{C}_W \|_{L^2(\Gamma) \to L^2(\Gamma)}},
\]
and by (5.11)
\[
(5.12) \quad \| (\mathbb{I} - \mathbf{C}_W)^{-1} \|_{L^2(\Gamma) \to L^2(\Gamma)} \leq \frac{1}{1 - O(t^{-1/3})}
\]
for sufficiently large \( t \). Now, for \( t \gg 1 \) we can define next vector function
\[
\mu(z) = (\eta \eta) + (\mathbb{I} - \mathbf{C}_W)^{-1} \mathbf{C}_W \left( (\eta \eta) \right)(z),
\]
here the component \( \eta \) is defined by (5.10).
\[
(5.13) \quad \| \mu(z) - (\eta \eta) \|_{L^2(\Gamma)} \leq \| (\mathbb{I} - \mathbf{C}_W)^{-1} \|_{L^2(\Gamma) \to L^2(\Gamma)} \| \mathbf{C}_W \|_{L^2(\Gamma) \to L^2(\Gamma)} \| W \|_{L^2(\Gamma)}.
\]
Then by (5.12) and lemma 5.1
\[
\| \mu(z) - (\eta \eta) \|_{L^2(\Gamma)} \leq O(t^{-2/3}).
\]
The solution of the RH problem (5.2) can be expressed by
\[
\hat{m}(z) = (\eta \eta) + \frac{1}{2\pi i} \int_{\tilde{\Gamma}} \mu(s)W(s)\Omega(z, s) \frac{ds}{s} + G(z),
\]
Rewriting this equation we can obtain
\[
\hat{m}(z) = (\eta \eta) + \frac{1}{2\pi i} \int_{\tilde{\Gamma}} (\eta \eta) W(s)\Omega(z, s) \frac{ds}{s} + G(z),
\]
where
\[
G(z) := \frac{1}{2\pi i} \int_{\tilde{\Gamma}} (\mu(s) - (\eta \eta)) W(s)\Omega(z, s) \frac{ds}{s}.
\]
Further, decomposition of the function \( \Omega(z, s) \) in a vicinity of the point \( z = 0 \) has the following form
\[
\Omega(z, s) = \frac{1}{2} \left( 1 - \frac{s+1}{s-1} + \frac{2}{s} z \right) + O(z^2) = -\frac{1}{s-1} + \frac{1}{s} z + O(z^2)s^{-2}.
\]
Observe that our contour \( \tilde{\Gamma} \) does not contain points \( z = 0 \) and \( z = 1 \). Consequently, on this contour function \( \Omega(z, s) \) is bounded with respect to \( s \) (and even uniformly bounded). Using this fact and (5.13) allows us to estimate the function \( G(z) \):
\[
|G(z)| \leq \| W \|_{L^2(\Gamma)} \| \mu(k) - (\eta \eta) \|_{L^2(\Gamma)} (1 + O(z)) = O(t^{-4/3}) + O(z)O(t^{-4/3}).
\]
\[
\frac{1}{2\pi i} \int_{\tilde{\Gamma}} (\eta \eta) W(s)\Omega(z, s) \frac{ds}{s} = (\eta \eta) \frac{F_1(z, t, \xi)}{t} + (\eta \eta) \frac{F_2(z, t, \xi)}{t} + O(z^2)O(t^{-1}),
\]
here matrices \( F_i(z, t, \xi), \quad i = 1, 2 \) are uniformly bounded with respect to all arguments. So, using (5.7), (5.8) and (5.9) for \( z \to 0 \) we have
\[
\hat{m}(z) = (\eta \eta) + \frac{f(\xi, \rho)}{t} + \frac{f(\xi, \rho)}{t} + o(t^{-1})o(z).
\]
Furthermore, using this equation and (5.1) we get
\[ m^{(3)}(z) = \hat{m}(z)M^{\text{as}}(z) = (\eta \quad \eta) M^{\text{mod}}(0) + M^{\text{mod}}(0)O(t^{-1}), \quad \text{as} \quad z \to 0. \]
So,
\[ \hat{m}(z) \approx (\eta \quad \eta) + O(t^{-1}), \quad z \in \mathbb{C}, \]
and by (5.1), (3.17)
\[ m^{(3)}(z) = \hat{m}(z)M^{\text{mod}}(z) + O(t^{-1}) = (\eta \quad \eta) M^{\text{mod}}(z) + O(t^{-1}). \]
The function \( m^{(3)}(z) \) satisfies the normalization condition (2.11) and the symmetry condition (2.10) and \( M^{\text{mod}}(z) \) satisfies the symmetry condition: \( M^{\text{mod}}(z^{-1}) = \sigma_1 M^{\text{mod}}(z) \sigma_1 \). So for sufficiently large \( t \) at the point \( z = 0 \) we get
\[ m^{(3)}(0) = \left( m^{(3)}_1, m^{(3)}_2 \right) = (\eta \quad \eta) \sigma_1 M^{\text{mod}}(\infty) \sigma_1, \]
and as follows from (3.16)
\[ \eta = \alpha(\xi) + O(t^{-1}). \]

\[ \square \]

Remark 5.3. From the discussion on the differentiability of the functions \( F_\mathcal{B}, F_\partial \mathcal{B} \) and \( \tilde{F}^{\text{mod}} \) it follows that the functions \( f(\xi, \rho) \) and \( \tilde{f}(\xi, \rho) \) are differentiable with respect to \( \xi \in \mathcal{I} \).

Since by (3.12) \( m^{(3)}(z) = m(z) \left[ \tilde{d}(z)P(z)e^{t(\Phi(z) - g(z))} \right]^{-\sigma_3} \), then from (5.1) we have
\[ m(z) = \hat{m}(z)M^{\text{mod}}(z) \left( \tilde{d}(z)P(z)e^{t(\Phi(z) - g(z))} \right)^{\sigma_3}. \]

The last equation with lemma 5.2 and theorem 5.1 \[6\] allow us to formulate the following

**Theorem 5.4.** For arbitrary small number \( \varepsilon > 0 \) the following asymptotics for the solution of the Toda lattice (1.1) – (1.4) as \( t \to \infty \) are valid in the region \( \varepsilon t \leq n \leq (1 - \varepsilon)t \):
\[ a(n,t) = \frac{n}{2t} + O(t^{-1}), \]
\[ b(n,t) = 1 - \frac{n}{t} + O(t^{-1}), \]
and here terms \( O(t^{-1}) \) does not contains terms from the solution of the parametrix problem.

**Appendix A. Solution of the problem I – V**

The existence of the solution of the RH problem I – V was proved in \[2\](2.13 – 2.15, 2.17, 2.18). To prove the uniqueness we will need the following

**Lemma A.1.** Let \( f(z) = (f_1(z) \quad f_2(z)) \) and \( g(z) = (g_1(z) \quad g_2(z)) \) be two solutions of the RH problem I – V. Then \( f(z) = c(z)g(z) \), where \( c(z) \) is a scalar function without jumps on the contour \( \mathcal{T} \cup \mathcal{I} \cup \mathcal{I}^* \) and \( \lim_{z \to \infty} c(z) = 1 \).
Proof. Consider the function $S(z)$ defined by
\[
S(z) := \begin{pmatrix} f_1(z) & f_2(z) \\ g_1(z) & g_2(z) \end{pmatrix}.
\]
Then
(A.1) \hspace{1cm} S^+(z) = S^-(z)v(z).

Furthermore, let $s(z) := \det S(z)$. Since $\det v(z) = 1$ by (A.1) $s^+(z) = s^-(z)$, i.e. the function $s(z)$ does not have a jump along the contour. Denote
(A.2) \hspace{1cm} h(z) = f(z) - g(z).

By the pole condition (2.12)
\[
\text{Res}_{z_j} [f_1(z)g_2(z) - g_1(z)f_2(z)] = -z_j \gamma_j e^{2i\Phi(z_j)} [f_2(z_j)g_2(z_j) - g_2(z_j)f_2(z_j)] \equiv 0,
\]
this means, that the function $s(z)$ does not have poles at eigenvalues in non-resonant case. In a resonant case $s(z) = O((z-p_j)^{-1/2})$, \hspace{1cm} $p_j \in \{q_1, q_2, -1, 1, q_1^*, q_2^*\}$, so $s(z)$ is a bounded function at these points. In virtue of the normalization (2.11) and the symmetry (2.10) conditions the function $s(z)$ is bounded at $\infty$. So, $s(z)$ is bounded holomorphic function, then by Liouville’s theorem $s(z) \equiv \text{const}$. Using the symmetry condition once more, we have
\[
s(z^{-1}) = f_2(z)g_1(z) - f_1(z)g_2(z) \equiv -s(z).
\]

Then $s(1) = -s(1)$ and $s(z) \equiv 0$. Consequently $f(z) = c(z)g(z)$, where $c(z)$ has no jumps on the jump contour. Moreover from the normalization condition (2.11) follows $\lim_{z \to \infty} c(z) = 1$. \hspace{1cm} \square

Proof. In virtue of Lemma [A.1] showing that the associated vanishing problem, where the normalization condition (2.11) is replaced by:
\[
|h_1(\infty)| + |h_2(\infty)| = 0,
\]
here the vector-function $h(z) = (h_1(z) \hspace{1cm} h_2(z))$ defined by (A.2), has only the trivial solution is enough to prove the uniqueness. Consider the scalar bounded (meromorphic) function
\[
F(z) := h_1(z)\overline{h_1(z^{-1})} + h_2(z)\overline{h_2(z^{-1})}
\]
By Cauchy’s residue theorem
\[
\int_{C_\varepsilon} F(z) \frac{idz}{z} = 2\pi i \left( \sum_{z_j} \frac{i}{z_j} \text{Res}_{z_j} F(z) + i \text{Res}_{z=0} F(z) \right).
\]
From the vanishing condition follows $F(0) = 0$, then
\[
\int_{C_\varepsilon} F(z) \frac{idz}{z} = -2\pi \sum_{z_j} \frac{1}{z_j} \text{Res}_{z_j} F(z).
\]
First, we consider the integral of $F(z)$ on the part $C_\varepsilon^{(1)}$ of the contour $C_\varepsilon$ (see Fig. 5), such as $C_\varepsilon^{(1)} \rightarrow \mathbb{T}$, \hspace{1cm} $\varepsilon \rightarrow 0$. When $|z| \rightarrow 1 \pm 0$, \hspace{1cm} $F(z) = F^\pm(z)$.
\[
F^+(z) = h_{1,+}(z)\overline{h_{1,-}(z)} + h_{2,+}(z)\overline{h_{2,-}(z)}
\]
From the statement of the initial RHP we have

\[
\begin{pmatrix} h_1(z) & h_2(z) \end{pmatrix}_+ = \begin{pmatrix} h_1(z) & h_2(z) \end{pmatrix}_- \begin{pmatrix} 0 & -R(z)e^{2\Phi(z)} \\ R(z)e^{2\Phi(z)} & 1 \end{pmatrix},
\]

\[
h_{1,+}(z) = h_{2,-}(z)R(z)e^{2\Phi(z)},
\]

\[
h_{2,+}(z) = -h_{1,-}(z)\overline{R(z)}e^{-2\Phi(z)} + h_{2,-}.
\]

Then

\[
F_+(z) = h_{1,-}(z)h_{2,-}(z)R(z)e^{2\Phi(z)} - h_{1,-}(z)\overline{h_{2,-}(z)R(z)e^{-2\Phi(z)}} + h_{2,-}(z)\overline{h_{2,-}(z)}
\]

\[
= 2i \text{Im} \frac{h_{1,-}(z)h_{2,-}(z)R(z)e^{2\Phi(z)}}{h_{2,-}(z)}|h_{2,-}(z)|^2.
\]

\[
\int_T \frac{\text{Re } F_+(z)}{z} \, \frac{idz}{z} = -\int_0^{2\pi} |h_{2,-}(e^{i\theta})|^2 \, d\theta \in \mathbb{R},
\]

\[
2i \int_T \frac{\text{Im } F_+(z)}{z} \, \frac{idz}{z} = -2\int_0^{2\pi} \text{Im } F_+(e^{i\theta}) \, d\theta \in \mathbb{R}.
\]

Next, we consider the integral on the part \( C^{(2)}_\varepsilon \), where \( C^{(2)}_\varepsilon \to I, \varepsilon \to 0 \). Firstly, we note \( F(\xi) \to F_{\pm}(z) \), as \( \xi \to z \pm 0i, \quad z \in I \).

\[
F_+(z) = h_{1,-}(z)\overline{h_{2,+}(z)} + h_{2,-}(z)\overline{h_{1,+}(z)}
\]

From the definition of the matrix \( v(z) \)

\[
h_{1,+}(z) = h_{1,-}(z) + h_{2,-}(z)\chi(z)e^{2\Phi(z)},
\]

\[
h_{2,+}(z) = h_{2,-}.
\]
So
\[
F_+(z) = h_1, -h_2, -(z) + h_2, -h_1, -(z) + h_2, -h_2, -(z) e^{2i\Phi(z)}
\]
\[
= h_1, -h_2, -(z) + h_1, -h_2, -(z) + |h_2, -(z)|^2 \chi(z) e^{2i\Phi(z)}
\]
\[
= 2 \text{Re} h_1, -h_2, -(z) + i|h_2, -(z)|^2 \chi(z) e^{2i\Phi(z)}.
\]
Analogously, for \( F_- (z) \) we have
\[
F_- (z) = 2 \text{Re} h_2, -(z) h_1, -(z) - |h_2, -(z)|^2 \chi(z) e^{2i\Phi(z)}.
\]

Note, that \( \text{Re} F_- (z) = \text{Re} F_+ (z), \quad \text{Im} F_- (z) = -\text{Im} F_+ (z) \). Then as \( \varepsilon \to 0 \) we obtain
\[
\int_{C_{\varepsilon}^{(2)}} F(\xi) \frac{id\xi}{\xi} = i \int_{q_1}^{q_2} \text{Re} F_+(z) \frac{dz}{z} - \int_{q_1}^{q_2} \text{Im} F_+(z) \frac{dz}{z} + i \int_{q_1}^{q_2} \text{Re} F_-(z) \frac{dz}{z} - \int_{q_1}^{q_2} \text{Im} F_-(z) \frac{dz}{z} = - \int_{q_1}^{q_2} \text{Im} F_+(z) \frac{dz}{z} = - \int_{q_1}^{q_2} |h_2, -(z)|^2 \chi(z) e^{2i\Phi(z)} \frac{dz}{z}.
\]

\[
i \int_{C_\varepsilon} F(z) \frac{dz}{z} = \int_{0}^{2\pi} |h_2, -(e^{i\theta})|^2 d\theta - \int_{q_1}^{q_2} |h_2, -(z)|^2 \chi(z) e^{2i\Phi(z)} \frac{dz}{z} - 2\int_{0}^{2\pi} \text{Im} F_+(e^{i\theta}) d\theta.
\]

From the pole condition (2.12) we have
\[
\text{Res}_{z_j} h_1(z) h_2(\overline{z}) = -z_j \gamma_j e^{2i\Phi(z)} |h_2(z_j)|^2
\]
and
\[
i \int_{C_\varepsilon} F(z) \frac{dz}{z} = 4\pi \sum_{j} \gamma_j e^{2i\Phi(z_j)} |h_2(z_j)|^2.
\]

Note that \( \int_{0}^{2\pi} \text{Im} F_+(e^{i\theta}) d\theta = 0 \) then
\[
\int_{0}^{2\pi} |h_2, -(e^{i\theta})|^2 d\theta + \int_{q_1}^{q_2} |h_2, -(z)|^2 \chi(z) e^{2i\Phi(z)} \frac{dz}{z} + 4\pi \sum_{j} \gamma_j e^{2i\Phi(z_j)} |h_2(z_j)|^2 = 0.
\]

we obtain
- \( h_2(z_j) = 0 \), and consequently \( \text{Res}_{z_j} h_1(z) = 0 \), so the vector function \( h(z) \) does not have any poles.
- \( h_2(z) = 0, \quad z \in \mathbb{T} \cup I, \) so \( h_2(z) \equiv 0, \quad z \in \mathbb{D}, \) as it is analytic function and the set \( I \) has a condensation point. Then, by the symmetry condition (2.10) \( h_1(z) \equiv 0, \quad z \in \mathbb{C} \setminus \mathbb{D} \).
- From (2.10) we obtain \( h_1(z) = h_2(z^{-1}) = 0, \quad z \in \mathbb{D} \).
Combining these statements we conclude
\[ h(z) = (h_1(z) \ h_2(z)) = (0 \ 0). \]

\[ \square \]

**Appendix B. Uniqueness for the model problem**

**Lemma B.1.** The solution of the model RH problem is unique.

**Proof.** As follows from proof of the uniqueness for the problem I–V and A.1 to prove the uniqueness of the model problem, it suffices to prove that the associated RH problem with the normalization condition replaced by
\[ |h_1(\infty)| + |h_2(\infty)| = 0, \]
where vector-function \( h(z) \) is a difference between two not identical solutions of the problem (3.15), has only the trivial solution. To this end we consider the following function
\[ F(z) := h_1(z)h_2(z). \]

Note that
\[ F_\pm(z) = h_{1,\pm}(z)h_{2,\mp}(z), \]
From (3.13) we have
\[ h_{1,+}(z) = h_{2,-}(z)R(-1), \ h_{2,+}(z) = -h_{1,-}(z)R(-1), \]
then
\[ F_+(z) = h_{2,-}(z)h_{2,-}(z)R(-1) = R(-1)|h_{2,-}(z)|^2, \]
\[ F_-(z) = -h_{1,-}(z)h_{1,-}(z)R(-1) = -R(-1)|h_{1,-}(z)|^2. \]

Since the function \( h(z) \) does not have any poles inside the contour \( C_\varepsilon \) and taking into account that \( C_\varepsilon \to \Sigma, \ \varepsilon \to 0 \) and that contours \( C_\varepsilon^{(1)} \) and \( C_\varepsilon^{(2)} \) (see Fig. 6) oriented oppositely we get
\[ \int_{C_\varepsilon} F(z) \frac{dz}{z} = R(-1) \int_{\Sigma} (|h_{2,-}(z)|^2 + |h_{1,-}(z)|^2) \frac{dz}{z} = 0. \]

Consequently we have \( h_{2,-}(z) = h_{1,-}(z) = 0 \) and using the symmetry condition we conclude \( (h_1(z) \ h_2(z)) = (0 \ 0). \)

\[ \square \]
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