Fractal-like frontier structure of avalanches in a sandpile model

A G Buzykin¹, I A Kuznetsov², A N Ipatov² and D A Parshin²

¹ Peter the Great St. Petersburg Polytechnic University, St. Petersburg, Russia
² Alferov University, Russian Academy of Sciences, St. Petersburg, Russia

e-mail: Buzykin7@gmail.com

Abstract. The self-organized criticality (SOC) phenomena for dynamical systems with spatial degrees of freedom is commonly observed in application to different aspects of natural sciences. In our study we also consider our sandpile model as such a system. Since the first mention in 1987, a sandpile model, which is a common example of spatiotemporal evolution, became widely applied for avalanche-like processes. With a simple computer simulation we show that an avalanche as a result of natural system evolution has a fractal border structure, and approves SOC phenomena of a sandpile model. We also demonstrate Gutenberg-Richter dependence for avalanche process in a sandpile model.

1. Introduction

In 1977 Benoît Mandelbrot put forward his fundamental idea of many natural structures having fractal structure with their inherent properties as self-similarity or scale invariance [1]. Due to that, the widespread nature of power-law patterns was explained, describing time-dependent and spatial correlations between various parameters of fractal structures. However, the observation of Mandelbrot did not explain the very reason for the appearance of numerous fractal forms in the world around us.

It was cleared out in 1987 with the birth of the work by Bak, Tang and Wiesenfeld [2], that a self-similarity lays in a wide-spread phenomenon of self-organization. According to that, in many open dissipative (complex) systems, the formation of a self-organized critical state is possible in a wide range of external impacts. This effect does not require precise tuning, as, for example, a second type phase transition. In comparison to the phase transition, we do not need the setting parameter to be critical, while the temperature for a phase shift ought to be so. In the presence of a slow flux of energy and (or) particles, the complex system drifts to the critical state on its own and sometimes remains for a rather long time. But since this critical state is on the threshold of stability, it (with a continuous impact on the system) loses the stability, following with an “avalanche” formed. After this evolution, the system returns back to the stable state. The avalanche repeats again and again, as the system slowly drifts towards an unstable critical state.

The simplest model of such a self-organized system is a pile of sand. Grains of sand are slowly added to the sandpile, causing avalanches (the Sandpile model[2]). At a critical angle of inclination, adding a single grain of sand to a pile may cause a small avalanche of sand grains, or a giant avalanche. The size distribution of avalanches is intended to behave according to Gutenberg-Richter law, which demonstrates dependence of earthquake frequency distribution by magnitude, comparable to our
sandpile model by SOC phenomena[3]. The spatial frontiers of the avalanches are not smooth and have a sharp fractal structure. For that reason, we find it interesting to calculate the fractal dimension of the avalanche frontiers, as well as demonstrate the Gutenberg-Richter law, assuming avalanche surface area logarithm as a magnitude of the effect.

2. Methods of research

The sandpile model described as a system of square cells. For that purpose, a computer simulation is obtained with a square field of 400×400 cells. The state of up to 3 grains of sand in a single cell is supposed as stable. The state in which a single cell contains 4 or more sand grains is supposed to be unstable, and results into an avalanche. To simulate an avalanche, we assume that an unstable cell gives one sand grain to each neighbor cell. Every neighbor cell is a square, placed on each side of a square cell. If a cell’s border occurs to be a field border, in case of avalanche we assume that as many grains leave the cell and the whole system, as many borders of the field a single cell contains.

Let us now discuss an avalanche process. While it starts from a single grain dropped, it may switch neighbor cells into critical state. The cells turned into critical state during an avalanche process are called “A” type. Number of such cells is considered as the avalanche square. A cell which contains less than 3 grains is not critical, and called “B” type. The avalanche process ends when there is not a single cell in critical state left, and all existing cells are considered as “B” type. Some of “A” type cells may border with “B” type cells. The set of such borders is considered as perimeter of the avalanche. The example of a small avalanche is shown on the figure 1.

![Figure 1. An example of a small avalanche for Bak, Tang and Wiesenfeld model from [4].](image)

The simulation is performed as a computer program, which drops a single grain iteratively into a random cell, and starts an avalanche if a cell in critical state is observed during the same iteration.

3. Results and discussion

According to the Mandelbrot, a planar figure considered as fractal if its perimeter and square act according to the following dependence (1):
\[ S \sim \frac{P^2}{D} \]  

where D is a fractal dimension [1,3-4]. In order to calculate the fractal dimension of the figure, shaped by the avalanche, we need to plot the dependence of square logarithm \( \text{lg}(S) \) from perimeter logarithm \( \text{lg}(P) \).

![Figure 2. A plot, summarizing the statistics of avalanches as the dependence of \( \text{lg}(S) \) from \( \text{lg}(P) \).](image)

The blue line on the figure 2 shows the linear approximation, from which \( \tan(a)=1.61 \), and so, the fractal dimension \( D=1.24 \), what, apparently, leads to an assumption that the figure is fractal[1,3]. This result is also in good accordance with [5,6]

The Gutenberg-Richter law shows the earthquake distribution from its magnitude in general case. Here we assume the square of the avalanche as a magnitude. It is useful to take into consideration that square also shows a number of iterations of critical cells stabilization taken in a single avalanche process, what makes avalanche process familiar to an earthquake[8]. The Gutenberg-Richter law in general case may be represented by (2) [4], where b, a parameter of our interest, is \( b=1 \) in case of natural earthquake. N is a number of earthquakes of certain magnitude (M).

\[ N(M) = 10^{-bM} \]  

On the figure 3 we plot such a dependence carried out from statistics of avalanche in a sandpile model. Parameter b occurs to be \( b=1.09 \) in our case, what is truly close to real earthquake model.
4. Conclusions
In this work we have demonstrated the SOC behavior of the BTW sandpile model. Our results are in good agreement with the ideas and results of the pioneers [1,2] as well as up to date research [5, 7]. We have to conclude, that, however the existing Abelian general model with its developments [see 8 for good review, 9 for three dimensional extension] may be more certain in modelling the real sand, it may be interesting to study the performance from the SOC point of view for a more simple Bak, Tang and Wiesenfeld model. The fractal-like frontier structure, approved by computer simulation is a good example of such a SOC behavior. At the same time, good correlation between earthquake and sandpile model demonstrates good statistical description and study possibilities of SOC models[10,11]. The idea of self-organized criticality made possible to explain the reason for the existence of fractals and power laws in the world around us, as well as many other previously unexplored natural phenomena [3,4], various effects in solid state physics, particularly in high-temperature superconductivity [12], electromagnetic radiation [11], and many others. Authors hope that the idea of fractal frontier structure in the self-organized system with avalanche way of stabilization will also find its application in some avalanche-like process one day.

References
[1] Mandelbrot B Fractals 1977 (WH Freeman: Form, chance and dimension)
[2] Bak P, Tang C, Wiesenfeld K 1987 Phys. Rev. Lett 59 381
[3] Bunde A, Havlin S Fractals in Science 1994 (Springer-Verlag) pp 29-45
[4] Bak P How Nature Works: the science of self-organized criticality 2013 (Springer Science & Business Media) p 53
[5] Najafi M N, Cheraghalizadeh J, Luković M and Herrmann H J 2014 Phys. Rev. E 101 032116
[6] Majumdar S M 1992 Phys. Rev. Lett. 68 2329
[7] Saberi A, Moghimi-Araghi S, Dashti-Naserabadi H, and Rouhani S 2009 Phys. Rev. E 79 031121

[8] Dhar D 2006 Physica A. 369 (1) 29-70

[9] Dashti-Naserabadi H and Najafi M N 2015 Phys. Rev. E 91 052145

[10] Olami Z, Feder H J S and Christensen K 1992 Phys. Rev. Lett. 68 1244

[11] Rabinovitch A et al. 2001 Phys. Rev. E 65 011401

[12] Field S et al. 1995 Phys. Rev. Lett. 74 1206-09