ON THE DIRICHLET PROBLEM IN THE PLANE WITH POLYNOMIAL DATA

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Abstract. Let $\Omega \subset \mathbb{C}$ be a bounded domain such that there exists an algebraic harmonic function of degree two vanishing on the boundary of $\Omega$. Then we show that the Khavinson-Shapiro conjecture holds for $\Omega$: if the Dirichlet problem on $\Omega$ with all polynomial boundary data have polynomial solutions, then $\Omega$ must be an ellipse. We also prove that if there exists a rational function with a singularity in $\Omega$, such that the Dirichlet problem for its restriction on $\partial \Omega$ along with all polynomial functions have rational solutions, then $\Omega$ must be a disc. This generalizes a well-known result by Bell, Ebenfelt, Khavinson, and Shapiro. Our proofs are purely algebraic.

1. Introduction

Let $\Omega$ be a domain in $\mathbb{C}$. Let $v \in C(\partial \Omega)$. Recall that solving the Dirichlet problem on $\Omega$ with the boundary data $v$ amounts to finding $u \in C^2(\Omega) \cap C(\overline{\Omega})$, such that $u$ is harmonic on $\Omega$ and $u|_{\partial \Omega} = v$. Domains for which the Dirichlet problem with algebraic data admit algebraic solutions have been of considerable interest for some time now. It is of particular interest to consider domains for which the Dirichlet problem with any polynomial data has a polynomial solution. This is known to be the case when $\Omega$ is an ellipse by a short and elegant argument of Fischer, which led Khavinson and Shapiro [KS] to make the following conjecture (they made the conjecture about bounded domains in $\mathbb{R}^n$, we restrict ourselves to $n = 2$ case).

Conjecture 1.1 (Khavinson-Shapiro conjecture). Let $\Omega \subset \mathbb{C}$ be a bounded domain whose boundary consists of finitely many non-intersecting Jordan curves. Suppose that the solution of the Dirichlet problem on $\Omega$ with every polynomial data is again polynomial. Then $\Omega$ must be an ellipse.

At present, the Khavinson-Shapiro conjecture is still wide open. See [CS], [R1], [R2], [LR] for some partial results.

In this paper we show the following.

Theorem 1.1. Let $\Omega \subset \mathbb{C}$ be a bounded domain so that $\partial \Omega$ has no isolated points. Assume that there exist rational holomorphic functions $f, g \in \mathbb{C}(z)$ and a rational harmonic function $r$, not all 0, such that

$$r^4 - 4(f + \bar{g})r^2 + (f - \bar{g})^2 = 0|_{\partial \Omega}.$$ 

Then the Khavinson-Shapiro conjecture holds for $\Omega$. 


The existence of $r, f, g$ as above is (essentially) equivalent to the existence of a harmonic algebraic function of degree 2 that vanishes on $\partial \Omega$.

We also consider the following algebraic generalization of the Khavinson-Shapiro conjecture for arbitrary fields.

**Conjecture 1.2.** Let $F$ be a field. Let $f \in F[z, w]$ be a polynomial such that for any $\phi \in F[z, w]$ there exists $\phi' \in F[z, w]$, so that $\phi - f \phi' \in F[z] + F[w]$. Then either the total degree of $f$ is at most 2, or $f$ is linear in $z$ or $w$.

This conjecture over $\mathbb{C}$ can be seen as a complexified version of [CS, Conjecture 4]. As proved by Render [R2], [CS, Conjecture 4] implies the Khavinson-Shapiro conjecture (assuming that the boundary of the domain has no isolated points).

We show that it is enough to prove the above conjecture for number fields to conclude that it holds for any characteristic 0 field. We also show that proving this conjecture over prime finite fields $\mathbb{F}_p$ for $p \gg 0$ implies its validity for all fields.

We prove the following

**Theorem 1.2.** Let $h_1, h_2 \in F[t]$ be irreducible polynomials over a field $F$ and $\deg(h_1) \leq \deg(h_2)$. Assume that either $\deg(h_1)$ does not divide $\deg(h_2)$, or $\deg(h_1) = \deg(h_2)$ and one of $F[t]/(h_1), F[t]/(h_2)$ is not a splitting field over $F$. Then for any $\phi, \psi \in F[z, w]$, Conjecture 1.2 holds for $h_1(z) \phi + h_2(w) \psi$. In particular, let $a, b$ be square-free odd integers $|a|, |b| > 1$, and assume that $n$ is not a power of 2. Then Conjecture 1.2 holds for $(z^n - a) \phi + (w^n - b) \psi$ for any $\phi, \psi \in \mathbb{Q}(i)[z, w]$.

After polynomials, the next natural choice to consider for the solvability of the Dirichlet problem is the class of rational functions. It is well-known and easy to check that if $\Omega$ is a disk, then for any $v \in C(\partial \Omega)$ such that $v$ is a restriction of a rational function, the Dirichlet problem with the data $v$ has a rational solution. Thus, in the spirit of the Khavinson-Shapiro conjecture, it is natural to ask whether this property characterizes disks. This question was settled affirmatively by the following result of Bell, Ebenfelt, Khavinson, and Shapiro.

**Theorem 1.3** ([BEKS, Theorem 2]). Let $\Omega \subset \mathbb{C}$ be a bounded domain whose boundary consists of finitely many non-intersecting Jordan curves and let $a \in \Omega$. Suppose that the solution of the Dirichlet problem on $\Omega$ with every polynomial data and $\frac{1}{z - a}|_{\partial \Omega}$ is rational. Then $\Omega$ is a disk.

The proof given in [BEKS] relies on some rather nontrivial complex analysis. Under the assumption that $\Omega$ is simply connected, then [BEKS] shows a stronger result: if the Dirichlet problem on $\Omega$ for $z \bar{z}|_{\partial \Omega}, z^2 \bar{z}|_{\partial \Omega}, z^3|_{\partial \Omega}$, and $\frac{1}{z - a}|_{\partial \Omega}$ (for some $a \in \Omega$) admits rational solutions, then $\Omega$ is a disk.

We prove the following generalization of the above theorem to arbitrary (possibly unbounded) domains, and what is more important replacing the rational
function $\frac{1}{z-a}$ with an arbitrary rational function that has a singularity in the domain.

Recall that the upper half plane $\Omega = \{ z | \text{Im}(z) > 0 \}$ also has the property that the Dirichlet problem with any rational data admits a rational solution. Our result shows that a disk (or its complement) and a right half plane are the only such domains (up to removing finitely many points).

**Theorem 1.4.** Let $\Omega \subset \mathbb{C}$ be a domain, such that the Dirichlet problem on $\Omega$ with any polynomial data has a rational solution. Assume moreover that there exists a rational harmonic function $\phi$ with a singularity in $\Omega$, such that the Dirichlet problem on $\Omega$ with the data $\phi|_{\partial \Omega}$ admits a rational solution. Then $\partial \Omega$ except for finitely many points is either a subset of a circle or a line.

In fact, we prove much more general algebraic statement. To state it, recall that given a holomorphic function $u$ (defined on an open subset in $\mathbb{C}$), we say that $u$ is an algebraic function of degree (at most) $n$ if there exist rational functions $r_i(z), 0 \leq i \leq n$ (not all of them zero) such that $\sum_{i=0}^{n} r_i(z)u(z)^i = 0$.

**Theorem 1.5.** Let $\Gamma \subset \mathbb{C}$ be an infinite set. Assume that for any polynomial $u \in \mathbb{C}[z, \bar{z}]$, there exists a rational harmonic function $v$, such that $u = v|_{\Gamma}$. Suppose moreover that there exist algebraic functions $f, g$ of degree at most $n$, so that $f - g$ vanishes on $\Gamma$. Then there exists a nonzero polynomial $\phi \in \mathbb{C}[z, \bar{z}]$ such that the degree of $\phi$ in $z$ and in $\bar{z}$ is at most $n$ and $\phi$ vanishes on $\Gamma$ except for finitely many points.

Our proof relies solely on some elementary algebraic considerations.

**Proposition 2.1.** Let $A$ be a finite dimensional algebra over a field $F$. Let $F_1, F_2$ be subfields of $A$ containing $F$. If $F_1 + F_2 = A$ then $A = F_1$ or $A = F_2$.

**Proof.** Since $\dim_F(F_1 \cap F_2) \geq 1$, then $\dim_F(A) \leq \dim_F F_1 + \dim_F F_2 - 1$. 

On the other hand, as $A$ can be viewed as a vector space over $F_1$ and $F_2$, we conclude that $\dim_F F_1, \dim_F F_2$ must divide $\dim_F(A)$. Therefore
\[ \dim_F(A) = \max(\dim_F F_1, \dim_F F_2). \]
Hence, $A = F_1$ or $A = F_2$.

We also recall the following very simple fact. We include the proof for the reader’s convenience

**Lemma 2.1.** Let $u$ be an algebraic function of degree $n$. Then $z$ is algebraic over $\mathbb{C}(u)$, and any element in $\mathbb{C}(u, z)$ is an algebraic function of degree at most $n$.

**Proof.** As $u$ is algebraic over $\mathbb{C}(z)$, then $u, z$ are algebraically dependent over $\mathbb{C}$, hence $z$ must be algebraic over $\mathbb{C}(u)$. Since $u$ is a root of a degree $n$ polynomial over $\mathbb{C}(z)$, it follows that the degree of the field extension $\mathbb{C}(u, z)/\mathbb{C}(z)$ is at most $n$. Thus, every element in $\mathbb{C}(u, z)$ is an algebraic function of degree at most $n$. 

**Proof of Theorem 1.3** By the assumption on $\Gamma$, there exist nonconstant holomorphic functions $f, g$ defined on a neighborhood of $\Gamma$, such that $f(z) = g(z)|_\Gamma$, and $f, g$ are algebraic of degree at most $n$. Let $\mathcal{O}$ denote the $\mathbb{C}$-algebra of functions defined on $\Gamma$ except possibly for finitely many points. Note that the field of algebraic functions defined on a neighborhood of $\Gamma$ maps into $\mathcal{O}$ by the restriction homomorphism.

Denote by $h$ the image of $f$ (same as the image of $g$) in $\mathcal{O}$. Let $A$ denote the subalgebra of $\mathcal{O}$ generated by images of $z, \bar{z}$ over $\mathbb{C}(h) = F$. Thus, $F$ is a field and $A$ is an algebra over $F$. We identify $z, \bar{z}$ with their images in $\mathcal{O}$. Next we claim that
\[ A = F[z, \bar{z}] = F[z] + F[\bar{z}]. \]
Indeed, since $z, \bar{z}$ are algebraic over $F$ by Lemma 2.1 we have
\[ F(z) = F[z], \quad F(\bar{z}) = F[\bar{z}]. \]
So, $A = F[z, \bar{z}]$. In particular, $\dim_F A < \infty$. We need to show that
\[ z^n \bar{z}^m \in F[z] + F[\bar{z}], \quad n, m \geq 0. \]
This follows from the assumption that there exists a rational harmonic function $u \in \mathbb{C}(z) + \mathbb{C}(\bar{z})$, so that $u = z^n \bar{z}^m|_\Gamma$. Hence
\[ z^n \bar{z}^m \in \mathbb{C}(z) + \mathbb{C}(\bar{z}) \subseteq F[z] + F[\bar{z}]. \]
Put $F_1 = F[z], F_2 = F[\bar{z}]$. Thus $F_1, F_2$ are subfields of $A$ containing $F$ and $A = F_1 + F_2$. Therefore, we can apply Proposition 2.1 to conclude that $A = F_1$ or $A = F_2$. Assume without loss of generality that $\bar{z} \in \mathbb{C}(h)[z]$.

To summarize, we have proved that there exists $u \in \mathbb{C}(f)[z]$ such that $\bar{z} = u|_\Gamma$. It follows from Lemma 2.1 that the degree of $u$ over $\mathbb{C}(z)$ is at most $n$. Therefore,
there exist polynomials $p_i \in \mathbb{C}[z]$, such that for any $z \in \Gamma$ (except for finitely many points)

$$\sum_{i=0}^{n} p_i(z) \bar{z}^i = 0.$$ 

Therefore,

$$\sum_{i=0}^{n} p_i(z) z^i = 0|_{\Gamma}.$$ 

Let $h$ be the greatest common divisor of $\sum_{i=0}^{n} p_i(z) \bar{z}^i$ and $\sum_{i=0}^{n} p_i(z) z^i$ in $\mathbb{C}[z, \bar{z}]$. Then the degree of $h$ in $z, \bar{z}$ is at most $n$ and $h$ vanishes on $\Gamma$ (except for finitely many points), as desired.

3. Results on the Khavinson-Shapiro conjecture

It will be convenient to introduce the following terminology

**Definition 3.1.** Let $F$ be a field. Let $f \in F[z, w]$. We say that $f$ is a Khavinson-Shapiro polynomial (KS-polynomial for short) if for any $\phi \in F[z, w]$ there exist $g_1(z) \in F[z], g_2(w) \in F[w]$, such that $\phi - g_1(z) - g_2(w)$ is a multiple of $\phi$.

Throughout polynomials of the form $f(z) + g(w) \in F[z, w]$ are referred to as harmonic polynomials in $F[z, w]$.

We recall the following classical result of Fischer. We include the usual proof for the reader’s convenience as it is stated over any field (including ones with positive characteristic).

**Lemma 3.1** (Fischer). Let $f \in F[z, w]$ be of degree 2. If $f$ does not divide any nonzero harmonic polynomial, then $f$ is a KS-polynomial.

**Proof.** Denote by $D : F[z, w] \to F[z, w]$ a linear map defined by $D(z^n w^m) = z^{n-1} w^{m-1}$ if $n, m > 0$ and 0 otherwise. Then ker($D$) consists of harmonic polynomials and $D$ is onto. So, by the assumption on $f$ the linear map

$$G : F[z, w] \to F[z, w], G(\phi) = D(f \phi)$$

is injective. On the other hand, for any $m \geq 0$, $G$ preserves the finite dimensional subspace of all polynomials of degree at most $m$. Therefore, $G$ is bijective. Given $\phi \in F[z, w]$, let $\phi_1$ be such that $D(\phi) = D(f \phi_1)$. So, $\phi - f \phi_1$ is harmonic and we are done.

**Lemma 3.2.** Let $\phi \in F[z, w]$ be linear in $z$. Then $\phi$ is a KS-polynomial if and only if $\phi = f(w)z - g(w)$ with linear $f$.

**Proof.** By the assumption $f(w)z - g(w)$ divides $zw + h(z) + h_1(w)$ for some $h, h_1$. Comparing the leading terms in $z$, $f(w)$ must divide a linear polynomial in $w$, hence the result.

□
The proof of the next result is essentially identical to that of Theorem 1.3 so it is omitted.

**Theorem 3.1.** Let \( f \in F[z, w] \) be an irreducible polynomial nonlinear in both \( z, w \) and be a divisor of a nonzero polynomial of the form \( \phi(z)\psi(w) + \phi_1(z)\psi_1(w) \). Then \( f \) is not a KS-polynomial.

The following result shows that in order to resolve Conjecture [1.2] it is enough to treat the case of \( F \) being a finite extension of \( \mathbb{Q} \) or a finite field. Given a polynomial \( f \in F[z, w] \), denote by \( \text{Supp}(f) \) the set of pairs \((n, m)\) for which \( z^n w^m \) has a nonzero coefficient in \( f \).

**Proposition 3.1.** Let \( f \in F[z, w] \) be a KS-polynomial. Then there exists infinitely many primes \( p \) and polynomials \( g \in F_p[z, w] \) so that \( \text{Supp}(g) = \text{Supp}(f) \) and \( g \) is a KS-polynomial. If in addition \( F \) has characteristic 0, then \( F \) can be replaced by a number field.

**Proof.** For \( n, m \geq 1 \), let \( g_{n,m} \in F[z, w] \) be such that \( z^n w^m + fg_{n,m} \in F[z] + F[w] \). Let \( \deg(g_{n,m}) = a_{n,m} \). Put \( b_{nm} = a_{n,m} + \deg(f) + n + m \). Given a natural number \( l \), we denote by \( F[z, w]^l \) the \( F \)-space of polynomials of the total degree \( \leq l \). Let \( \phi_{n,m} : F[z, w]_{a_{nm}} \to zwF[z, w]_{b_{nm}} \) be the projection of the \( F \)-linear map \( h : z^n w^m + fh \) (the projection discarding all monomials not containing both \( z, w \)). Thus \( \phi_{n,m}(g_{n,m}) = 0 \), so \( \phi_{n,m} \) is not injective. Viewing \( \phi_{n,m} \) as a matrix with respect to the monomial basis, its coefficients clearly belong to integer span of coefficients of \( f \) and 1. Therefore, the fact that this matrix has rank less than \( \dim F[z, w]_{a_{nm}} = d(n, m) \) is equivalent to the vanishing of determinants of all \( d(n, m) \) by \( d(n, m) \) minors. Each such minor is a polynomial in \( \mathbb{Z}[f_{ij}] \), where \( f_{ij} \) is the coefficient of \( z^i w^j \) in \( f \).

To summarize, for each triple \( n, m, k \), we have constructed a family of polynomials \( S_{n,m,k} \subset \mathbb{Z}[f_{ij}] \) with the following property. There exists \( g \in F[z, w] \) of degree at most \( k \) so that \( gf + z^n w^m \in F[z] + F[w] \) if and only if all polynomials in \( S_{n,m,k} \) vanish on coefficients of \( f \). Let \( S \) denote the ring generated over \( \mathbb{Z} \) by all nonzero coefficients of \( f \) and their inverses. It is known that for infinitely many primes \( p \), there exists ring homomorphisms \( S \to F_p \). Denoting the image of \( f \) by \( \bar{f} \in F_p[z, w] \), we obtain that \( \bar{f} \) is a KS-polynomial, moreover \( \text{Supp}(\bar{f}) = \text{Supp}(f) \). Similarly, we may replace \( F \) by a number field.

The next observation is crucial.

**Lemma 3.3.** Let \( f \in F[z, w] \) be a KS-polynomial. Let \((a, b) \in F^2 \) be a zero of \( f \). Then either \( a \in F[b] \) or \( b \in F[a] \).

**Proof.** Let \( A \) denote \( F[a, b] \). Thus, \( A \) is a finite extension of \( F \). Since \( f \) is a KS-polynomial, we may conclude that \( A = F[a] + F[b] \). Since both \( F[a], F[b] \) are finite field extensions of \( F \), we are done by Proposition 2.1.

□
Now we can easily prove Theorem 1.2 as follows. In view of Lemma 3.3 it suffices to find $a, b \in F$ roots of $h_1, h_2$ respectively so that $F[a], F[b]$ do not contain each other. If $\deg(h_1)$ does not divide $\deg(h_2)$, since $\dim_F F[a] = \deg(h_1), \dim_F F[b] = \deg(h_2)$, we conclude that $F[a]$ is not a subfield of $F[b]$. Suppose that $\deg(h_1) = \deg(h_2)$ and $F[t]/(h_1)$ is not a splitting field over $F$. Then there exist $a, a' \in F$ roots of $h_1$, such that $a' \notin F(a)$. Then for any $b \in F$ that is a root of $h_2$, we have that $F(b) \neq F(a)$ or $F(b) \neq F(a')$. Let $F(a) \neq F(b)$. Since $[F(a) : F] = \deg(h_1) = \deg(h_2) = [F(b) : F]$, we conclude that $F(a), F(b)$ do not contain each other, as desired. Finally, let $a, b \in \mathbb{Z}$ be square-free odd integers, $|a|, |b| > 1$. Then it follows from the Eisenstein’s irreducibility criterion that $z^n - a, w^n - b$ are irreducible over $\mathbb{Q}(i)$. The splitting field of $z^n - a$ contains $\mathbb{Q}(i)(\xi_n)$, where $\xi_n$ is a primitive $n$-th root of unity. So, $\mathbb{Q}(i)(\xi_n) = \mathbb{Q}(\xi_m)$, where $m = \text{lcm}(4, n)$. So $[\mathbb{Q}(i)(\xi_n) : \mathbb{Q}(i)] = \phi(m)/2$.

It is clear that $\phi(m)/2$ cannot divide $n$, so $\mathbb{Q}(i)(\xi_n)$ is not contained in $\mathbb{Q}(i)(a^{1/n})$ and we are done.

We can easily derive Theorem 1.1 from Theorem 1.5. Since

\[ r^4 - 4(f+\bar{g})r^2 + (f-\bar{g})^2 = (r + \sqrt{f} + \sqrt{g})(r + \sqrt{f} - \sqrt{g})(r - \sqrt{f} + \sqrt{g})(r - \sqrt{f} - \sqrt{g}) \]

it follows that $\partial \Omega$ can be written as a finite union $\bigcup_{i=1}^m \Gamma_i$, so that for each $\Gamma_i$ there exists a harmonic algebraic function of degree 2 vanishing on it. So, by Theorem 1.5 there exists a polynomial $f_i \in \mathbb{C}[z, \bar{z}]$ of order at most 2 in both $z, \bar{z}$ vanishing on $\Gamma$. In fact, such a polynomial must have total degree at most 2. Indeed, since $z\bar{z} + h(z) + \bar{h}(z)|_{\partial \Gamma} = 0$ for some $h \in \mathbb{C}[z]$, we can replace $f_i$ by $\gcd$ of $f_i$ and $z\bar{z} + h(z) + \bar{h}(z)$ which will have the total degree at most 2. Thus, $\partial \Omega$ is a union of finitely many curves each of them being either line, ellipse, parabola or hyperbola. If an infinite subset of $\partial \Omega$ is a zero of an irreducible quadratic function $\phi$, then $\phi|z|^2 + h$ for some real harmonic polynomial $h$. Therefore $h$ is at most quadratic, implying that $\partial \Omega$ is contained in the zero set of an irreducible quadratic polynomial. So, $\Omega$ must be an ellipse. Thus, the only case left to consider is when $\Omega$ a polygon. Then we are done by [R2].

**Acknowledgements.** The paper owes its existence to Steven Bell’s Shoemaker lecture series given at University of Toledo in 2015. I am very grateful to Dima Khavinson for many useful comments.
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