A generalization of the 1935 Einstein-Podolsky-Rosen (EPR) argument for measurements with continuous variable outcomes is presented to establish criteria for the demonstration of the EPR paradox, for situations where the correlation between spatially separated subsystems is not perfect. Two types of criteria for EPR correlations are determined. The first type are based on measurements of the variances of conditional probability distributions and are necessary to reflect directly the situation of the original EPR paradox. The second weaker set of EPR criteria are based on the proven failure of (Bell-type) local realistic theories which could be consistent with a local quantum description for each subsystem. The relationship with criteria sufficient to prove entanglement is established, to show that any demonstration of EPR correlations will also signify entanglement. It is also shown how a demonstration of entanglement between two spatially separated subsystems, if not able to interpreted as a violation of a Bell-type inequality, may be interpreted as a demonstration of the EPR correlations. In particular it is explained how the experimental observation of two-mode squeezing using spatially separated detectors will signify not only entanglement but EPR correlations defined in a general sense.

I. INTRODUCTION

In 1935 Einstein, Podolsky and Rosen [1] (EPR) presented their famous argument in an attempt to show that quantum mechanics is incomplete. EPR defined a premise called local realism, which is assumed in all classical theories. The premise of realism implies that if one can predict with certainty the result of a measurement of a physical quantity at a location \( A \), without disturbing the system at \( A \), then the results of the measurement are predetermined. In these circumstances there is an “element of reality” corresponding to this physical quantity, the element of reality being a variable that assumes one of the set of values that are the predicted results of the measurement. The locality (no action-at-a-distance) assumption implies that a measurement performed at a spatially separated location \( B \) cannot induce any immediate change to the subsystem at \( A \). Local realism is defined as the dual premise, where EPR’s realism and locality are both assumed.

For certain correlated spatially separated particles EPR showed that, if quantum mechanics is to be consistent with local realism, the position and momentum of a single localised particle must be simultaneously predetermined with absolute definiteness. Such simultaneous determinacy for both position and momentum is not predicted for any quantum state. While EPR took the view that local realism must be valid and therefore argued that quantum mechanics was incomplete, the argument is perhaps best viewed as a demonstration of the inconsistency between quantum mechanics as we know it (that is without “completion”) and local realism.

Observation of an EPR correlation, while not as conclusive as observing a violation of a Bell inequality [2,3], can be carried out at high detector efficiency. It is possible to measure continuous variable EPR correlations using fields, where the conjugate “position” and “momentum” observables are replaced by the two orthogonal non-commuting quadrature phase amplitudes of the field [4]. These field quadrature amplitudes are measurable using homodyne detection schemes. A specific criterion, a violation of an inferred Heisenberg Uncertainty Principle (H. U. P.), to demonstrate the continuous variable EPR paradox for real experiments was put forward in 1989 [5]. This proposal employed a two-mode squeezed state [6] as the source.

The first experimental achievement, using the parametric oscillator [7], of the 1989 EPR criterion, for efficient measurements with continuous variable outcomes, was reported by Ou et al. [8] in 1992. Recently Zhang et al. [9] detected EPR correlations between the intense output fields of the parametric oscillator above threshold. Silberhorn et al. [10] have detected such correlations for pulsed fields, and there have been further theoretical proposals [11,12]. The EPR fields have proven significant in enabling the experimental realization of a continuous variable quantum teleportation [13], and may have application also to continuous variable quantum cryptography [14].

It is known that for the two-mode squeezed state which exhibits the EPR paradox for continuous variable quadrature amplitude measurements, a violation of a Bell inequality [2,3] is not possible, for the same sorts of measurements. This is transparent [2] on recognizing that the Wigner function is positive for such a state and can act as a local hidden variable theory for the predictions of such measurements. It is for this reason that the Bell inequality must be replaced with an EPR or entanglement criterion in certain quantum cryptographic or teleportation schemes involving continuous variable measurements.

In view of this, in order to understand the application
II. GENERALIZATION OF THE EPR ARGUMENT

In order to demonstrate the existence of continuous variable EPR correlations for real experiments, we need to extend the EPR argument to situations where the results of measurements between the spatially separated subsystems need not be maximally correlated. This is also necessary if we are to link EPR criteria with entanglement criteria, since it is certainly true that not all entangled states are perfectly correlated.

EPR originally argued as follows. Consider two spatially separated subsystems at $A$ and $B$. EPR considered two observables $\hat{x}$ (the “position”) and $\hat{p}$ (“momentum”) for subsystem $A$, where $\hat{x}$ and $\hat{p}$ do not commute, so that ($C$ is nonzero)

of the continuous variable EPR correlations to areas of research such as quantum cryptography, and to assist in required proofs of security, our first objective in this first paper is to fully define EPR correlations through a generalization of the EPR paradox (rather than through violations of Bell inequalities) for spatially separated systems where the correlation need not be maximum. In Sections 2 and 3 we review and further develop the generalization of the EPR argument to provide criteria for demonstrating EPR correlations for situations where the correlation between spatially separated subsystems is less than optimal. To enable experimental proof of EPR correlations, the criteria are to be expressed in terms of measurable probabilities and correlations.

Two types of EPR criteria are established. The first, a stronger set (Sections 3a,b and c), are based on reduced variances of conditional probability distributions, and represent most directly the application of EPR’s premises of realism and no action-at-a-distance and are therefore in the spirit of the original EPR paradox. The second are a weaker set, established indirectly: through the failure of any local realistic theory which is also consistent with a local quantum description for each subsystem, to adequately predict the quantum statistics.

Such EPR criteria are of a fundamental significance in providing the condition for the experimental demonstration of the EPR paradox. The point is made, as has been made previously [3], that the demonstration of EPR is not itself be a demonstration of the failure of all local hidden variable (or local realistic) theories. It is however a failure of all such theories that could be consistent with each of the spatially separated subsystems being depicted by a quantum state, and as such is a confirmation of quantum inseparability (entanglement [7]) for the striking situation of spatially separated subsystems. The EPR criteria define conditions for the onset of a fundamental philosophical conflict between the viewpoint of a believer in local realism, and the believer that the quantum state represents the ultimate (most complete) description of a physical system.

It may be the case that further different types of measurements will show outright the failure of all local hidden variable theories, by way of violation of a Bell inequality. It is still relevant however to recall the significance of the EPR demonstrations alone, for the following reasons. First, such demonstrations have actually been performed with reasonable spatial separations (though not with causally separated events) for high detection efficiencies. This means that the conclusions are loop-hole free in that they do not require the auxiliary assumptions associated with the category of Bell inequality tests, that have been performed using as subsystems fields with significant spatial separations, but inefficient detectors [14] or vice versa [17].

Second, the EPR correlations have been proven experimentally for macroscopic systems, and also where measurements outcomes are continuous and can be translated with some minor experimental modification to a macroscopic outcome domain [18]. This is not the case for Bell tests which are difficult in such regimes [19]. For the sake of the completeness of this paper then the philosophy and significance of the EPR argument and the significance of its demonstration is revised in Section 4.

In Section 5 the link between a demonstration of entanglement and a demonstration of the EPR paradox is established. Here we define a demonstration of entanglement as an experimentally measured violation of a necessary criterion for separability, where the criterion must be expressed in terms of measurable probabilities or correlations in the fashion of a Bell-inequality. Such entanglement criteria have been derived for measurements with continuous variable outcomes by Duan et al [20] and Simon [21]. We prove explicitly an entanglement criterion which is based on the achievement of two-mode squeezing between certain spatially separated quadrature amplitudes. It is discussed then how such a suitably performed demonstration of entanglement will also demonstrate the failure of a weaker EPR criterion. It is also proved explicitly that the demonstration of an EPR paradox must always imply a demonstration of entanglement: the EPR criteria are signatures of entanglement. The results of this paper have been presented in brief in a previous preprint [21] (quant-ph 0103142), but are derived here in full detail.

The continuous variable measurements that demonstrate the EPR paradox for the two-mode squeezed state are not predicted to directly show a violation of a Bell inequality. However for certain quantum cryptographic protocols one may replace the Bell-inequality by an EPR-criterion to test for security. A proof of security for the scheme similar to that proposed by Ekert [22], but based on the use of EPR criteria, rather than Bell inequalities, is presented in a second paper [23].

II. GENERALIZATION OF THE EPR ARGUMENT
Suppose one may predict with certainty the result of measurement \( \hat{x} \), based on the result of a measurement performed at \( B \). Also, for a different choice of measurement at \( B \), suppose one may predict the result of measurement \( \hat{p} \) at \( A \). Such correlated systems are predicted by quantum theory. Assuming “local realism” (discussed in Section 1) EPR deduce the existence of an “element of reality”, \( \hat{x} \), for the physical quantity \( \hat{x} \); and also an element of reality, \( \hat{p} \), for \( \hat{p} \). Local realism implies the existence of two hidden variables \( \hat{x} \) and \( \hat{p} \) that simultaneously predetermine, with no uncertainty, the values for the result of an \( \hat{x} \) or \( \hat{p} \) measurement on subsystem \( A \), should it be performed. This hidden variable state for the subsystem \( A \) alone is not describable within quantum mechanics, since simultaneous eigenstates of \( \hat{x} \) and \( \hat{p} \) do not exist. Hence, EPR argued, if quantum mechanics is to be compatible with local realism, we must regard quantum mechanics to be incomplete.

We now need to extend the EPR argument to situations where the result of measurement \( \hat{x} \) at \( A \) cannot be predicted with absolute certainty \(^{[1][2]} \). The assumption of local realism allows us to deduce the existence of an “element of reality” of some type for \( \hat{x} \) at \( A \), since we can make a prediction of the result at \( A \), without disturbing the subsystem at \( A \), under the locality assumption. This prediction is subject to the result \( x^B_i \) of a measurement, \( \hat{x}^B \) say, performed at \( B \). (Throughout this paper \( i \) is used to label the possible results, discrete or otherwise, of the measurement \( \hat{x}^B \)). The predicted results for the measurement at \( A \), based on the measurement at \( B \), are however no longer a set of definite numbers with zero uncertainty, but become fuzzy, being described by a set of distributions \( P(x|\hat{x}^B) \) giving the probability of a result for the measurement at \( A \), conditional on a result \( x^B_i \) for measurement at \( B \). We define \( \Delta^2 \hat{x} \) to be the variance of the conditional distribution \( P(x|\hat{x}^B) \).

Similarly we may infer the result of measurement \( \hat{p} \) at \( A \), based on a (different) measurement, \( \hat{p}^B \) say, at \( B \). Denoting the results of the measurement \( \hat{p}^B \) at \( B \) by \( p^B_j \), we then define the probability distribution, \( P(p|\hat{p}^B) \) which is the predicted result of the measurement for \( \hat{p} \) at \( A \) conditional on the result \( p^B_j \) for the measurement \( \hat{p}^B \) at \( B \). The variance of the conditional distribution \( P(p|\hat{p}^B) \) is denoted by \( \Delta^2 \hat{p} \).

### III. Signatures of the EPR Paradox

For a given experiment one could in principle measure the individual variances \( \Delta^2 \hat{x} \) of the conditional distributions \( P(x|\hat{x}^B) \) (and also \( \Delta^2 \hat{p} \) for the \( P(p|\hat{p}^B) \)). If each of the variances satisfy

\[
\Delta^2 \hat{x} = 0
\]

(1)

(2)

\[
\Delta^2 \hat{p} = 0
\]

(for all \( i, j \)) of course there is no difficulty in establishing that this would imply the demonstration of the original EPR paradox.

This situation however is not practical for continuous variable measurements. Instead of considering the problem of simultaneous eigenstates as originally proposed by EPR, we suggest instead a different and experimentally realizable criterion based on the Heisenberg Uncertainty Principle (H. U. P.)

\[
\Delta \hat{x} \Delta \hat{p} \geq C
\]

(3)

For the sake of notational convenience we now consider in the remainder of the paper that appropriate scaling enables \( \hat{x} \) and \( \hat{p} \) to be dimensionless and \( C = 1 \).

#### A. Strong EPR correlations using bounded conditional distributions

EPR correlations however would be demonstrated in a convincing manner if the experimentalist could measure each of the conditional distributions \( P(x|\hat{x}^B) \) and establish that each of the distributions is very narrow, in fact constrained so that

\[
P(x|\hat{x}^B) = 0 \quad \text{if} \quad |x - \mu_i| > \delta
\]

\[
P(p|\hat{p}^B) = 0 \quad \text{if} \quad |p - \nu_j| > \delta
\]

(4)

Here \( \mu_i \) is the mean of the conditional distribution \( P(x|\hat{x}^B) \) and \( \nu_j \) is the mean of the conditional distribution \( P(p|\hat{p}^B) \).

In this case the assumption of local realism would imply, since the measurement \( \hat{x}^B \) at \( B \) will always imply the result of \( \hat{x} \) at \( A \) to be within the range \( \mu_i \pm \delta \), that the result of the measurement at \( A \) is predetermined to be within a bounded range of width \( 2\delta \). In a straightforward extension of EPR’s argument, we replace the words “predict with certainty” with “predict with certainty that the result is constrained to be within the range \( \mu_i \pm \delta \)”, and then define an “element of reality” with this intrinsic bounded blurring or fuzziness \( \delta \).

After considering the \( \hat{p} \) and \( \hat{p}^B \) correlations, and where \( \delta < 1 \), the conclusions of the paradox follow. This is because the predetermined precision associated with the “elements of reality” describing the subsystem at \( A \) could not be given by any quantum state, in view of the uncertainty relation \(^{[4]} \). This situation represents the spirit of the original EPR gedanken experiment in its truest form. Such narrow distributions over the entire outcome domain \( x^B_i \) (and \( p^B_j \) ) have however to my knowledge not been experimentally established. In the case of the two-mode squeezed state \(^{[3]} \) to be discussed later, this situation is never strictly mathematically satisfied for finite values of the squeezing parameter \( r \) as defined in equation

\[\ldots\]
B. 1989 EPR criterion: Violation of an Inferred Heisenberg Uncertainty Principle

We next consider the situation where an experimenter has demonstrated that for every outcome \( x_i^B \) (and \( p_j^B \)) for the measurement \( \hat{x}^B \) (and \( \hat{p}^B \)) performed at \( B \), the variance \( \Delta_i x \) (and \( \Delta_j p \)) of the appropriate conditional distribution satisfies

\[
\begin{align*}
\Delta_i x &< 1 \\
\Delta_j p &< 1
\end{align*}
\]

(5)

for all \( i, j \). The measurement at \( B \) always allows an inference of the result at \( A \) to a precision better than given by the uncertainty bound \( 1 \).

In this case we do not predict a result at \( A \) “with certainty”, as in EPR’s original paradox. The measurement \( x_i^B \) at \( B \) however does predict with a certain probability constraints on the result for \( \hat{x} \) at \( A \). Following the EPR argument, which assumes no action-at-a-distance, so that the measurement at \( B \) does not cause any instantaneous influence to the system at \( A \), one can attribute a probabilistic predetermined “element of reality” to the system at \( A \). There is a similar predicted result for the measurement \( \hat{p} \) at \( A \) based on a result of measurement at \( B \), and a corresponding predetermined description based on the no-action-at-a-distance assumption.

The important point in establishing the EPR paradox for this more general yet practical situation is that under the EPR premises the predetermined statistics (or generalised “elements of reality”) for the physical quantities \( \hat{x} \) and \( \hat{p} \) are attributed simultaneously to the subsystem at \( A \). Assuming no action-at-a-distance, the choice of the experimenter (Bob) at \( B \) to infer information about either \( \hat{x} \) or \( \hat{p} \) cannot actually induce the result of the measurement at \( A \). As there is no disturbance created by Bob’s measurement, the (appropriately extended) EPR definition of realism is that the prediction for \( x \) is something (a probabilistic “element of reality”) that can be attributed to the subsystem at \( A \), whether or not Bob makes his measurement. This is also true of the prediction for \( \hat{p} \), and therefore the two “elements of reality” representing the physical quantities \( \hat{x} \) and \( \hat{p} \) exist to describe the predictions for \( \hat{x} \) and \( \hat{p} \) simultaneously.

The paradox can then be established by proving the impossibility of such a simultaneous level of prediction for both \( \hat{x} \) and \( \hat{p} \) for any quantum description of the subsystem \( A \) alone. By this we mean explicitly that there can be no procedure allowed, within the predictions of quantum mechanics, to make simultaneous inferences by measurements performed at \( B \) or any other location, of both the result \( \hat{x} \) and \( \hat{p} \) at \( A \), to the precision indicated by \( \Delta_i x < 1 \), \( \Delta_j p < 1 \). (Recall that the inference of the result at \( A \) by measurement at \( B \) is actually a measurement of \( \hat{x} \) performed with the accuracy determined by the \( \Delta_i x \). Simultaneous measurements of \( \hat{x} \) and \( \hat{p} \) to the accuracy \( \Delta_i x \) are not possible (predicted by quantum mechanics). The reduced density matrix describing the state at \( A \) after such measurements would violate the H. U. P.)

More generally in an experiment we could obtain a mixed situation, where for example, the conditional variance \( \Delta_i^2 x \) for some of the \( i \) might be greater than \( 1 \). The exact boundary at which we can claim an EPR paradox needs defining.

A simpler quantitative, experimentally testable criterion for EPR was proposed in 1989 [3]. The 1989 inferred H.U.P. criterion is based on the average variance of the conditional distributions for inferring the result of measurement \( \hat{x} \) (and also for \( \hat{p} \)). The EPR paradox is demonstrated when the product of the average errors in the inferred results for \( \hat{x} \) and \( \hat{p} \) violate the corresponding Heisenberg Uncertainty Principle. The spirit of the original EPR paradox is present, in that one can perform a measurement on \( B \) to enable an estimate of the result \( x \) at \( A \) (and similarly for \( \hat{p} \)).

We define

\[
\begin{align*}
\Delta_{inf}^2 x &= \sum_i P(x_i^B)\Delta_i^2 x \\
\Delta_{inf}^2 p &= \sum_j P(p_j^B)\Delta_j^2 p
\end{align*}
\]

Here \( \Delta_{inf}^2 \hat{x} \) is the average variance for the prediction (inference) of the result \( x \) for \( \hat{x} \) at \( A \), conditional on a measurement \( \hat{x}^B \) at \( B \). Here \( i \) labels all outcomes of the measurement \( \hat{x} \) at \( A \), and \( \mu_i \) and \( \Delta_i x \) are the mean and standard deviation, respectively, of the conditional distribution \( P(x|x_i^B) \), where \( x_i^B \) is the result of the measurement \( \hat{x}^B \) at \( B \). We define a \( \Delta_{inf}^2 \hat{p} \) similarly to represent the weighted variance for the prediction (inference) of the result \( \hat{p} \) at \( A \), based on the result of the measurement at \( B \). Here \( P(p_j^B) \) is the probability for a result \( x_i^B \) upon measurement of \( \hat{x}^B \), and \( P(p_j) \) is defined similarly.

The criterion to demonstrate the EPR paradox, the signature of the EPR paradox, is

\[
\Delta_{inf} x \Delta_{inf} p < 1.
\] (7)

This criterion is a clear criterion for the demonstration of the EPR paradox, by way of the argument presented above. Such a prediction for \( \hat{x} \) and \( \hat{p} \) with the average inference variances given, cannot be achieved by any quantum description of the subsystem alone. This EPR criterion has been achieved experimentally. It is most useful in the proof of security for the quantum cryptographic proposal discussed in the companion paper [2].
C. Best estimates criterion

It is not always convenient to measure each conditional distribution \( P(x|a_i^B) \) and \( P(p|a_i^B) \). It is possible to construct other measurements based on the measurement of a sufficiently reduced noise in the appropriate sum or difference \( \hat{x} - g \hat{x}^B \) and \( \hat{p} + y \hat{p}^B \) (where here \( g \) is a number). This sort of measurement was proposed in [5] in 1989 as a means to demonstrate EPR correlations, and is closely linked to squeezing and quantum-nondemolition measurements.

We therefore consider a more general situation where an estimate or prediction \( x_{est} \) is given of the result for \( \hat{x} \) at \( A \), based on a result \( x_i^B \) at \( B \) for the measurement \( \hat{x}^B \). For each result \( x_i^B \) we define the rms error

\[
\delta_i^2 = \sum_x P(x|x_i^B)(x - x_{est})^2.
\]

(8)

The average error in the inference based on the particular estimate is given by

\[
\Delta_{inf,est} \hat{x} = \sum_i P(x_i^B) \delta_i^2
\]

\[
= \sum_{x,i} P(x,x_i^B)(x - x_{est})^2.
\]

(9)

The best estimate (meaning that which will minimize the root mean square error \([24]\)) of the outcome of \( \hat{x} \) at \( A \), based on a result \( x_i^B \) for the measurement at \( B \), is given when \( x_{est} = \mu_i \), since this minimizes each \( \delta_i^2 \). The quantity

\[
\Delta_{inf,est}^2 \hat{x} = \sum_{x,i} P(x_i^B) \Delta_i^2
\]

\[
= \sum_{x,i} P(x,x_i^B)(x - \mu_i)^2
\]

(10)

then defines the minimum average variance for the inference of the result of a measurement \( \hat{x} \) at \( A \), based on the result of the measurement \( x^B \) at \( B \). Generally

\[
\Delta_{inf,est} \hat{x} \geq \Delta_{inf} \hat{x}
\]

and also \( \Delta_{inf,est} \hat{p} \geq \Delta_{inf} \hat{p} \).

The observation of a violation of the “inferred Heisenberg Uncertainty principle”

\[
\Delta_{inf,est} \hat{x} \Delta_{inf,est} \hat{p} < 1
\]

(12)

is a demonstration of the EPR paradox. Here “elements of reality” \( \hat{x}, \hat{p} \) simultaneously attributed to system \( A \) by local realism would have a degree of definiteness not compatible, in the EPR sense, with the uncertainty principle. The criterion \([13]\) is a violation of the inferred Heisenberg Uncertainty Principle, a signature of an EPR paradox experiment defined in 1989.

1. Linear estimation

As an example of the 1989 criterion, we propose upon a result \( x_i^B \) for the measurement at \( B \) that the predicted value for the result \( x \) at \( A \) is given linearly by the estimate \( x_{est} = g x_i^B + d \). For example suppose our two systems at \( A \) and \( B \) are harmonic oscillators, with boson operators \( \hat{a} \) and \( \hat{b} \) respectively, and become correlated as a result of a coupling described by the interaction Hamiltonian \( H_I = i \hbar \kappa (\hat{a} \hat{b}^\dagger - \hat{a}^\dagger \hat{b}) \), which acts for a finite time \( t \). For vacuum initial states this interaction generates two-mode squeezed light \([3]\)

\[
|\psi > = \sum_{n=0}^{\infty} c_n |n > _{a} \ |n > _{b}
\]

(13)

where \( c_n = \tanh^n r / \cos \kappa t \) and \( \kappa = \kappa t \). This simple quantum state was shown to be EPR-correlated in reference \([3]\), and has been used to model continuous-variable EPR-correlated fields to date. Here we define the quadrature phase amplitudes

\[
\hat{x} = \hat{X}_a = (\hat{a} + \hat{a}^\dagger)
\]

\[
\hat{p} = \hat{P}_a = (\hat{a} - \hat{a}^\dagger) / i
\]

\[
x^B = \hat{X}_b = (\hat{b} + \hat{b}^\dagger) / i.
\]

\[
\hat{p}^B = \hat{P}_b = (\hat{b} - \hat{b}^\dagger) / i.
\]

(14)

The Heisenberg uncertainty relation for the orthogonal amplitudes of mode \( \hat{a} \) is \( \Delta^2 X_a \Delta^2 P_a \geq 1 \).

The size of the deviation \( \delta_i = x - (g x_i^B + d) \) in the linear estimate \( x_{est} \) can then be measured (or calculated). We simultaneously measure \( \hat{x} \) at \( A \) and \( \hat{x}^B \) at \( B \), to determine \( x \) and \( x_i^B \) and then to calculate initially for given \( x_i^B \)

\[
\langle \delta_i^2 \rangle = \sum_x P(x,x_i^B)(x - (g x_i^B + d))^2 / P(x_i^B).
\]

Averaging over the different values of \( x_i^B \) we obtain as a measure of error in our inference, based on the linear estimate:

\[
\Delta_{inf,L}^2 \hat{x} = \sum_{x,i} P(x_i^B) \langle \delta_i^2 \rangle
\]

\[
= \sum_{x,x_i^B} P(x,x_i^B) \{x - (g x_i^B + d)^2 \}
\]

\[
= \langle (\hat{x} - (g \hat{x}^B + d))^2 \rangle.
\]

(15)

The best linear estimate \( x_{est} \) is the one that will minimize \( \Delta_{inf,L}^2 \hat{x} \). This corresponds to the choice \([24]\) \( d = - (\hat{x} - g \hat{x}^B) \). Denoting \( \delta_0 = \hat{x} - g \hat{x}^B \), our choice of estimate optimized with respect to \( d \) gives a minimum error

\[
\Delta_{inf,L}^2 \hat{x} = \langle \delta_0^2 \rangle = \langle \delta_0^2 \rangle.
\]

(16)

The best choice for \( g \) is discussed in \([3]\). The two-mode squeezed state will predict \([3]\) \( g = \tanh 2\kappa t \) the correlations \( X_a = X_b \), and \( P_a = - P_b \) to give

\[
\Delta_{inf,L}^2 \hat{x} = \Delta_{inf,L}^2 \hat{p} = 1 / \cosh 2\kappa t
\]

(17)
The EPR correlations are predicted possible for all nonzero values of the two-mode squeeze parameter \( r \).

If the estimate \( x_{est} \) corresponds to the mean of the conditional distribution \( P(x|x^B) \) then the variance \( \Delta^2_{inf,L} \hat{x} \) will correspond to the average conditional variance \( \Delta^2_{inf} x = \sum x P(x^B) \Delta x^2 \) specified in Section 3b above. This is the case, with a certain choice of \( g \), for the two-mode squeezed state used to model continuous variable EPR states generated to date.

In general the variances of type \( \Delta^2_{inf,L} \hat{x} \) based on estimates will be greater than or equal to the optimal evaluated from the conditionals (this was shown in Section 3b): we have \( \Delta_{inf,L} \hat{x} \geq \Delta_{inf} x \) and \( \Delta_{inf,L} \hat{p} \geq \Delta_{inf} p \). The observation of

\[
\Delta_{inf,L} \hat{x} \Delta_{inf,L} \hat{p} < 1
\]

by way of the 1989 EPR criterion \([12]\), will then also imply the situation of the EPR paradox.

### D. Generalised hidden variable EPR criteria

Our objective was to determine the boundaries at which one could sensibly claim EPR correlations. To determine a whole set of more general EPR criteria, we first need to consider what is meant by the assumption of “local realism” as applied to less strongly correlated subsystems, and to then define situations where this assumption would imply an “incompleteness” of quantum mechanics. EPR’s “realism” was defined originally only in the context of perfect correlation, and therefore it must be pointed out that any generalization actually goes further than EPR.

We propose then that the most general application of the EPR premise of local realism to correlated systems is to simply postulate the existence of a some local realistic theory to describe the correlations. If we can show that in order to predict the quantum correlations correctly, we need to use local realistic states (“elements of reality”) that predetermine with sufficient definiteness the results of \( \hat{x} \) and \( \hat{p} \) measurements, so that the level of definiteness cannot be represented by a quantum local state for the subsystem, then we have the situation of the EPR argument in its most general sense. Namely, that if the premise of local realism is valid, then we need to “complete” quantum mechanics (to introduce hidden variables) to correctly predict the quantum statistics.

EPR’s “realism” implied, for the perfect predictions they considered, the existence of variables that predetermine with an absolute definiteness the predictions for the results of measurements \( \hat{x} \) and \( \hat{p} \). These variables are ‘hidden variables” since they cannot be represented by any quantum state, and therefore exist to supplement (complete) quantum mechanics. While EPR themselves did not use the term “hidden variable”, it is generally understood that this is the consequence of their argument \([3]\), the hidden variable being the mathematical symbol of the “element of reality”.

EPR’s argument is based on the assumption of the validity of a local realism defined only with respect to perfectly correlated systems. We propose the following, that the most general meaning of the assumption of local realism, as applied to any more weakly correlated system, is the ability to correctly describe the predictions of measurements through some actual local realistic theory, and that such theories are symbolised mathematically by a general “local hidden variable theory” of the type considered by Bell \([2,3]\).

The EPR criteria derived from this proposal are quite different to the constraints (Bell inequalities) derived by Bell, which are derived from the assumption that a local hidden variable theory will correctly predict the experimental outcomes. The distinction between EPR and Bell tests is discussed in Section 4.

However it must be stressed that it was EPR’s intention (presumably) to discuss the incompatibility of the “completeness of quantum mechanics” with “local realism” (to imply the need for hidden variables) in an intuitive, physical way, without introducing any particular mathematical formulation for the meaning of local realism. This is an important philosophical and historical point. In this regard the criteria discussed in Sections 3a,b, and c above must be regarded as stronger EPR criteria, necessary for the demonstration of the EPR paradox itself. It is mentioned also at this point that the previous EPR criteria are not only stronger in this philosophical sense, but are mathematically stronger. By this we mean that the EPR criteria given in Sections 3a,b and c are a subset of the more general EPR criteria we derive here, as will be shown in Section 5 \([25]\).

To show however that the criteria we consider in this section still fit into the category of general EPR criteria, it is simply argued that if local realism as applied to systems of arbitrary correlation (the starting point of a generalised EPR argument) is to be valid, then there is a need to propose a working mathematical theory satisfying local realism in order to predict the results of experimental measurements. The question then is whether any kind of local realism can be constructed using something different to the mathematical basis of the local realistic theories studied by Bell. To my knowledge there is no such mathematical representation \([22]\).

In order to derive mathematical criteria, we formulate the mathematical description of local realism along the lines of Bell \([22]\). Such general local realistic theories do not “a priori” put any restrictions \([24]\) on the degree of definiteness of the prediction for the results of measurements, for a system described by a given local realistic (hidden variable) state. For more weakly correlated systems, since the result at \( A \) is not completely determined by the result at \( B \), we introduce local realistic (local hid-
den?) variables $\{\lambda\}$ to symbolize possible states of the subsystem $A$, but where now for each such local variable specification of the subsystem, there can be an unspecified fuzziness, symbolized by the variances $\Delta^2_{\lambda}x$ and $\Delta^2_{\lambda}p$ that give the predicted values for the result of the $\hat{x}$ or $\hat{p}$ measurement respectively [27]. It is interesting that the concept of a “blurred” reality was discussed qualitatively by Schrodinger [14] in his reply to EPR in 1935. This approach is illustrated in Figure 1.

At $A$ there is the choice to measure either $\hat{x}$ or $\hat{p}$, a choice denoted by different variables, $0$ and $\pi/2$ respectively, of the parameter $\theta$. Similarly at $B$ there is the choice, denoted by $\phi$, to measure $\hat{x}^B$ or $\hat{p}^B$. The (hidden) variable values $\{\lambda\}$ determine the results, or probabilities for results, of measurements if performed. There will be a probability $p^A_{\theta}(\theta, \lambda)$ for the result $x$ of a measurement $\theta$ at $A$, given the hidden variable state $\{\lambda\}$; similarly a $p^B_{\phi}(\phi, \lambda)$ is defined. The $\Delta^2_{\lambda}x$, $\Delta^2_{\lambda}p$ are the variances of $p^A_{\theta}(\theta = 0, \lambda)$, $p^A_{\theta}(\theta = \pi/2, \lambda)$ respectively. In accordance with EPR’s locality (no action-at-a-distance) assumption, this probability distribution is independent of the experimenter’s choice $\phi$ of simultaneous measurement at $B$. Assuming a general local hidden variable theory the joint probability $P_{\theta,\phi}(x, x^B)$ of obtaining an outcome $x$ at $A$ and $x^B$ at $B$ is of the form ($\rho(\lambda)$ is the probability distribution for the $\{\lambda\}$)

$$P_{\theta,\phi}(x, x^B) = \int_{\lambda} \rho(\lambda) \, p^A_{\theta}(\theta, \lambda)p^B_{\theta,\phi}(\phi, \lambda) \, d\lambda \quad (19)$$

These general local hidden variable theories were considered by Bell [9].

![Diagram 1](image)

**FIG. 1.** The local realist will describe the correlated statistics through classical variables $\lambda$ (that can be shared by the two subsystems at $A$ and $B$). These classical variables depict possible states of the subsystems. The $\Delta^2_{\lambda}x$ and $\Delta^2_{\lambda}p$ are the variances of the distributions $p^A_{\theta}(\theta = 0, \lambda)$ and $p^A_{\theta}(\theta = \pi/2, \lambda)$, respectively, that denote the probability of a result $x$ upon measurement of $\hat{x}$ ($\theta = 0$) or $\hat{p}$ ($\theta = \pi/2$), given that the system is in a particular state $\lambda$. Similar definitions exist for the subsystem at $B$. The system is EPR-correlated in a general sense when it can be demonstrated that in order to predict the observed statistics, $\Delta_{\lambda}x \Delta_{\lambda}p < C$ (where $\Delta_{\lambda}x \Delta_{\lambda}p \geq 1$ is the uncertainty relation) for at least one of the possible states $\lambda$.

In order to demonstrate EPR correlations, or the EPR paradox in the most general way, we assume the validity of a local realistic description and then must prove the necessity of using hidden variable states with a simultaneously well-defined prediction for “position” and “momentum”, so that $\Delta_{\lambda}x \Delta_{\lambda}p < 1$. To test for EPR correlations, the objective then is to demonstrate the failure of any local realistic theory [13] satisfying the auxiliary assumption

$$\Delta_{\lambda}x \Delta_{\lambda}p \geq 1 \quad (20)$$

(and a similar restriction is placed on the variances of the $p^A_{\theta,\phi}(\phi, \lambda)$) to describe the statistics of the correlated, spatially-separated fields. This would indicate that at least part of the time, if local realism, as manifested through any actual local realistic theory, is to correctly describe the statistics, the local subsystem at $A$ must be in a truly hidden variable state satisfying $\Delta_{\lambda}x \Delta_{\lambda}p < 1$, a description that cannot be given by a local quantum wavefunction for the subsystem alone. The “incompleteness of quantum mechanics” from the point of view of the local realist then follows: he/she must introduce “hidden” variables to “complete” quantum mechanics so as to enable a local realistic description. This is the situation of EPR correlations in their most generalized (weakest) sense.

To derive criteria sufficient to demonstrate EPR correlations, we assume the general local realistic description [13] with the proviso [21] and consider the whole set of inequalities or other constraints following necessarily from these assumptions. Such constraints are derived explicitly in Section 5, where it is also shown that these are constraints sufficient to quantum inseparability.

**E. Summary of EPR criteria**

We classify two types of EPR correlations. The first type are based on variances of conditional distributions reflecting the ability to infer a result for measurement at $A$ based on a measurements at spatially separated location $B$. These strong EPR correlations are evidenced through EPR criteria such as that of 1989, the inferred H. U. P. EPR criterion ([12]), and are necessary for the demonstration of EPR correlations defined in the spirit of the original EPR paradox (where the premise of local realism is defined and employed with no assumptions made regarding the mathematical form a local realistic theory would take). In terms of quadrature phase amplitude measurements this strong EPR criterion is satisfied when

$$\Delta(X_a - gX_b)\Delta(P_a + hP_b) < 1 \quad (21)$$

where the quadrature amplitudes have been defined in Section 3c, and $g, h$ are parameters chosen to minimize the variances.
The second generalised and weaker type of EPR correlations are demonstrated through the failure of local realistic (Bell-type local hidden variable) descriptions to predict the measured statistics of the spatially separated fields, unless we allow for a degree of definiteness in their prediction for results of an \( \hat{x} \) or \( \hat{p} \) measurement. These correlations demonstrate that if we do assume the correctness of local realism through the existence of an actual (Bell-type) local realistic theory, then the hidden variables describing each local subsystem must give such definite predictions for position and momentum that they cannot also be represented by a local quantum state for that subsystem. In this way an inconsistency between the assumption of local realistic theories and the completeness of quantum mechanics is established. Although not based on conditional measurements, these weaker correlations therefore still reflect the essence of the EPR argument.

The explicit forms for some of these weaker constraints are derived in Section 5. It is also shown in Section 5 that this set of constraints includes all of the stronger 1989 inferred H. U. P. EPR criteria discussed above. An example of a weaker EPR criterion, which is not a stronger EPR criterion, is derived in Section 5c and is written in terms of quadrature phase amplitude measurements as

\[
\Delta(X_a - X_b)\Delta(P_a + P_b) < 2 \tag{22}
\]

This criterion is closely related to the criterion for the observation of a two-mode squeezing. Two-mode squeezing is said to be observed when \( \Delta^2(X_a - X_b) < 2 \), or \( \Delta^2(P_a + P_b)^2 < 2 \). In the limit of perfect correlation, the optimal \( g, h \) values for (21) are 1, and here the weaker nature of the last criterion is apparent. The constraints sufficient to prove quantum inseparability derived recently by Duan et al and Simon are also examples of generalised (weaker) EPR criteria.

IV. SIGNIFICANCE OF THE DEMONSTRATION OF THE EPR CORRELATIONS AND RELATIONSHIP TO BELL INEQUALITIES

In order to appreciate the significance of an experimen- tational demonstration of EPR, it is crucial to realize that local realistic descriptions may be entirely compatible, depending on the statistics, with a local quantum description for each of the subsystems \( A \) and \( B \). Where EPR correlations cannot be proven, the observed statistics could be explained using local variables or parameters to represent possible states of the subsystem that have an intrinsic fuzziness associated with their predictions for measurements, so that this fuzziness could be entirely the result of a quantum state for each localized subsystem.

A local realist takes the viewpoint that (hidden) variables exist to predetermine (but perhaps with some un- specified indeterminacy) the result of measurements for subsystem \( A \) in a way which is compatible with EPR’s no action-at-a-distance (locality). But at the onset of the experimental proof of EPR correlations, the local realist’s view is one that “completes” quantum mechanics, since the (hidden) variables representing the local subsystems define the result for measurement so precisely that they cannot also be represented by a quantum state for the subsystem \( A \) or \( B \), itself. In this manner, at the onset of EPR correlations, a dispute is established between the local realist and the believer in the ultimate completeness of the quantum state; though not between the local realist and the believer in quantum mechanics (presumably EPR were in this category) who is perfectly prepared to supplement quantum mechanics with hidden variables.

From the hidden variable form (14) various constraints (called Bell-type inequalities) may be derived, that do not require the assumption of any auxiliary assumptions such as (20). Where one shows a violation of such a constraint (which is also an example of a separability criterion), then we have a violation of a Bell-type inequality. From such a violation one may draw the stronger conclusion than can be drawn from a demonstration of EPR paradox itself. In the case where one violates a Bell inequality, it is proved that local realism (or all local hidden variable theories) itself is invalid, and that the predictions of the particular quantum states in this case cannot be equivalently represented by any more “complete” theory (in which hidden variables are introduced) still satisfying local realism.

The two-mode squeezed state (13) predicts EPR correlations satisfying the EPR condition \( \Delta_{m,f} x \Delta_{m,f} \hat{x} < 1 \). However it is well-known that the joint probabilities \( P_{\theta,\phi}(x, y) \) for the results of measurements \( \hat{x}, \hat{p} \) can be predicted from a local hidden variable theory (19), derived from the Wigner function, which is positive for the two-mode squeezed state. In such a local hidden theory the Wigner function c-numbers \( x \) and \( p \) take on the role of position and momentum hidden variables; the Wigner function, being positive, gives the probability distribution for the hidden variables \( x, p \). This implies that the Bell inequalities will not be violated in this case for direct \( \hat{x}, \hat{p} \) measurements. (Of course Bell inequalities may be violated for other measurements (not \( \hat{x}, \hat{p} \)) on this quantum state; this is a different issue, however).

The existence of a local hidden variable theory that would predict the EPR \( \hat{x}, \hat{p} \) correlations might lead to the interpretation that the EPR experiment, demonstrating \( \Delta_{m,f} \hat{x} \Delta_{m,f} \hat{x} < 1 \), reflected a situation in which quantum and local realistic (classical) domains are not distinguishable, and that “there is no real paradox”. This is not the case.

The local realistic hidden variable theory used to give the quantum predictions is, necessarily, not actually quantum theory, since it must incorporate a description \( \{ \lambda_n \} \) for a state of the subsystem at \( A \) or \( B \) in which the
x and $p$ are prespecified to a variance better than the uncertainty principle. At the proof of demonstration of EPR correlations, the quantum mechanics theorist cannot generate a separable description in which each subsystem can be represented locally by a quantum state; his/her quantum state is necessarily entangled. (This is shown in Section 5). The separable local hidden variable theory based on the Wigner function is not a separable (local) theory in quantum mechanics since these simultaneously well-defined $x$ and $p$ are not quantum states.

The philosophical viewpoint of the local realist can be compatible with the quantum description only if quantum mechanics “completed” to incorporate the local hidden variables (for example, so that the c-numbers of the quantum Wigner function are the positions and momenta of the particle) and to therefore provide a local (separable) description. The conflict between the local realist and the believer in the completeness of quantum mechanics still exists, and there is the paradox of EPR, in spite of the fact that there is a local realistic description for the quantum predictions in this case.

V. LINK WITH ENTANGLEMENT CRITERIA

The EPR criteria are closely linked to criteria for entanglement. The demonstration of entanglement may be defined as the measured experimental violation of any one of the set of criteria following necessarily from the assumption of separability, where a separable quantum state is defined as being expressible by a density matrix of the form

$$\rho = \sum_\lambda P_\lambda \rho^A_\lambda \rho^B_\lambda$$

(23)

where $\sum_\lambda P_\lambda = 1$. Here $\lambda$ is simply a discrete or continuous label for quantum states, and no longer refers to hidden variables. Necessary conditions for separability for finite-dimensional systems were explored by Peres and Horodecki et al [28]. Such necessary conditions using continuous variable measurements have been derived by Duan et al [29] and Simon [30]. Here we are concerned with criteria expressed, as are Bell inequalities, in terms of measurable expectation values or probabilities for results of experimental measurements that may be performed on the system.

The prediction $P_{\theta,\phi}(x,y)$ given a separable quantum state is of the form

$$P_{\theta,\phi}(x,y) = \sum_\lambda P_\lambda \ p^A_\lambda(\theta,\lambda)p^B_\lambda(\phi,\lambda)$$

(24)

where $p^A_\lambda(\theta,\lambda) = \langle x | \rho^A_\lambda | x \rangle$ and $p^B_\lambda(\phi,\lambda) = \langle y | \rho^B_\lambda | y \rangle$ and here $|x\rangle$ and $|y\rangle$ are eigenstates of the operators representing measurements at $A$ and $B$ respectively. In the Sections 5b and c below, we prove certain constraints to follow necessarily from the assumption of quantum separability (23). The violation of these constraints is then sufficient to demonstrate entanglement.

The predictions based on quantum separability must allow for all possible quantum density operators and therefore we make no further assumptions except to assume the separable nature of the decomposition, and the restriction put on the statistics due to general quantum bounds such as the uncertainty relation satisfied by each local quantum state $\rho^A_\lambda$ and $\rho^B_\lambda$.

In the proofs given in Section 5b and c, we assume the separable form (23) and apply the constraint that the local quantum density operator must allow the Heisenberg uncertainty principle, to derive the necessary separability criteria. Using the same algebra, the same constraints are derivable from the local hidden variable form (18) with the auxiliary constraint (20), meaning that violation of these constraints will also imply EPR correlations in the most general sense, as defined in Section 3d. In this way it is seen that the generalised EPR constraints are identical to those derived as necessary conditions of separability where the additional quantum restriction in connection with the uncertainty relation is assumed.

Other separability criteria, using the uncertainty product bound, have been derived by Duan et al and Simon [20]. These criteria are important because they have been shown to be necessary and sufficient to demonstrate entanglement for certain quantum states (Gaussian states) relevant to many experimental situations. These constraints also follow from the assumptions (19) and (24), and are thus examples of generalized EPR criteria, as defined in Section 3d.

In deriving a particular constraint from the assumption of quantum separability (23), one either derives a result based on (23) (or (18)) alone, or else makes further assumptions regarding a general property of a quantum state $\rho^A_\lambda$ such as satisfaction of the uncertainty relation (or (20)). The constraints derived based only on (23) (or (18)) are called Bell-type inequalities and their violation is therefore proof of failure of all local hidden variable theories (13), as well as being demonstrations of inseparability. The violation of the remaining constraints (that are based on an additional assumption), cannot imply failure of all local hidden variables, but nonetheless may be classified as a violation of a generalised EPR criterion.

To summarize, provided measurements and spatial separations between subsystems $A$ and $B$ allow justification of the locality assumption, the measured violations of one of the necessary criteria for separability (where an extra constraint relating to general quantum bound such as the uncertainty relation is assumed) are not only a demonstration of entanglement, but are then none other than a demonstration of EPR correlations in the generalized sense.
A. EPR-criteria as signatures of entanglement

If we demonstrate the EPR paradox in its generalized form, then it follows through the very meaning of the EPR paradox that we must use an entangled source and that demonstration of EPR correlations must imply entanglement. A separable source as given by (23) has the interpretation that it is always in one of the factorizable states $\rho_A \rho_B$ with probability $P_A$; in each case the subsystem at $A$ being describable by the quantum state $\rho_A$ and the subsystem $B$ being described by the quantum state $\rho_B$. These states represent local descriptions where for each such description the predictions (“elements of reality”) for the position and momentum measurements are sufficiently indefinite so that the uncertainty bound (20) is satisfied. The predicted statistics must be compatible with this local fuzzy description, and in being so cannot satisfy the general EPR criterion discussed in relation to equation (20) in Section 3d.

B. 1989 Inferred H. U. P. EPR criterion as a signature of entanglement

We will first prove that separability will always imply $\Delta_{inf,\lambda}^2 \Delta_{inf,\lambda}^2 \hat{x} \Delta_{inf,\lambda}^2 \hat{p} \geq 1$, meaning that a satisfaction of the 1989 criterion as given by (12) for EPR correlations will always imply entanglement. (The same algebra is proof that the strong 1989 EPR criterion follows necessarily from the assumptions (13) and (23), and therefore that the 1989 EPR criterion is also a weaker EPR criteria of the type discussed in Section 3d.)

The conditional probability of result $x$ for measurement $\hat{x}$ at $A$ given a simultaneous measurement of $\hat{B}$ at $B$ with result $x_B$ is $P(x|x_B) = P(x, x_B)/P(x_B)$ where, assuming separability (23),

$$P(x, x_B) = \sum_{\lambda} P_A \rho_A(x_B) P_A(x)$$  

(25)

Here $|x\rangle, |x_B\rangle$ are the eigenstates of $\hat{x}, \hat{B}$ respectively, and $P_A(x) = |x\rangle \rho_A^A |x\rangle, P_A(x_B) = |x_B\rangle \rho_B^B |x_B\rangle$. The mean $\mu_i$ of this conditional distribution is

$$\mu_i = \sum_x x P(x|x_B)$$

$$= \sum_{\lambda} P_A \rho_A(x_B) \langle x \rangle_{\lambda}/P(x_B)$$

(26)

where $\langle x \rangle_{\lambda} = \sum_x x P_A(x)$. The variance $\Delta_i^2 x$ of the distribution $P(x|x_B)$ is

$$\Delta_i^2 x = \sum_{\lambda} P_A \rho_A(x_B) \sum_x P_A(x) (x - \mu_i)^2 /P(x_B)$$

(27)

For each state $\lambda$, the mean square deviation $\sum P_A(x) (x - d)^2$ is minimized with the choice $d = \langle x \rangle_{\lambda}$ [24]. Therefore for the choice $d = \mu_i$,

$$\Delta_i^2 x \geq \left\{ \sum_{\lambda} P_A \rho_A(x_B) \sum_x P_A(x)(x - \langle x \rangle_{\lambda})^2 /P(x_B) \right\} = \left\{ \sum_{\lambda} P_A \rho_A(x_i) \sigma^2(x) /P(x_B) \right\}$$

(28)

where $\sigma^2(x)$ is the variance of $P_A(x)$. Taking the average variance over the $x_B$ we get (recalling that for any set of statistics (11) holds)

$$\Delta_{inf,\lambda}^2 \Delta_{inf,\lambda}^2 \hat{x} \Delta_{inf,\lambda}^2 \hat{p} \geq \sum_{x_B} P(x_B) \left\{ \sum_{\lambda} P_A \rho_A(x_B) \sigma^2(x) /P(x_B) \right\}$$

$$= \sum_{\lambda} P_A \sigma^2(x_B) \sum_{x_B} P_A(x_B)$$

(29)

Also $\Delta_{inf,\lambda}^2 \hat{p} \geq \sum_{\lambda} P_A \sigma^2(p)$, where $\sigma^2(p)$ is the variance of $P_A(p) = \langle |p\rangle \rho_A^A |p\rangle$, $|p\rangle$ being the eigenstate of $\hat{p}$. This implies (from the Cauchy-Schwarz inequality)

$$\Delta_{inf,\lambda}^2 \Delta_{inf,\lambda}^2 \hat{x} \Delta_{inf,\lambda}^2 \hat{p} \geq \sum_{\lambda} P_A \sigma^2(x_B) \sum_{\lambda} P_A \sigma^2(p)$$

$$| \sum_{\lambda} P_A \sigma^2(x_{\lambda}) \sigma^2(p_{\lambda}) |^2.$$  

(30)

For any $\rho_A^A$ it is constrained, by the uncertainty relation, that $\sigma^2(x_{\lambda}) \sigma^2(p_{\lambda}) \geq 1$. We therefore conclude that for a separable quantum state

$$\Delta_{inf,\lambda}^2 \Delta_{inf,\lambda}^2 \hat{x} \Delta_{inf,\lambda}^2 \hat{p} \geq \Delta_{inf,\lambda}^2 \hat{x} \Delta_{inf,\lambda}^2 \hat{p} \geq 1.$$  

(31)

The experimental observation then of the EPR criterion $\Delta_{inf,\lambda}^2 \Delta_{inf,\lambda}^2 \hat{x} \Delta_{inf,\lambda}^2 \hat{p} < 1$ (or $\Delta_{inf,\lambda}^2 \Delta_{inf,\lambda}^2 \hat{x} \Delta_{inf,\lambda}^2 \hat{p} < 1$), as given by (12), will imply inseparability (that is entanglement).

1. Linear estimates

In general the variances of type $\Delta_{inf,\lambda}^2 \hat{x}$ defined in Section 3c based on linear estimates will be greater than or equal to the optimal (minimal) variances evaluated from the conditionals (this was shown in Sections 3b and c): we have $\Delta_{inf,\lambda}^2 \hat{x} \geq \Delta_{inf,\lambda}^2 \hat{x}$ and $\Delta_{inf,\lambda}^2 \hat{p} \geq \Delta_{inf,\lambda}^2 \hat{p}$. The separable state must then always predict $\Delta_{inf,\lambda}^2 \hat{x} \Delta_{inf,\lambda}^2 \hat{p} \geq 1$ and the observation of

$$\Delta_{inf,\lambda}^2 \hat{x} \Delta_{inf,\lambda}^2 \hat{p} < 1$$

(32)

implies quantum inseparability, for any $g$ and $d$.

Given that the linear inference method of the previous section is that so far actually used in the two-mode squeezing EPR experiments, it is worthwhile to demonstrate that $\Delta_{inf,\lambda}^2 \hat{x} \Delta_{inf,\lambda}^2 \hat{p} < 1$ implies inseparability explicitly. Such a proof is useful in that we will also prove...
that the observation of two-mode squeezing with respect to certain observables is sufficient to demonstrate entanglement (or a generalized EPR correlation). Our proof, first presented in [24], is from first principles, although it has also been previously discussed by Braunstein et al. [24] how the observation of $\Delta_{inf,L}\hat{x}\Delta_{inf,L}\hat{p} < 1$ may imply inseparability, based on criterion for inseparability derived by Duan et al.

Separability will imply, upon optimizing $d$ but keeping $g$ general (use [24])

$$\Delta_{inf,L}\hat{x}^2 = (\langle \hat{x}^2 \rangle - \langle g\hat{x}B - \langle \hat{x}B \rangle \rangle^2) = \sum_{x,x'} \sum_{\lambda} P_\lambda \langle x|\hat{x}_B^2|\rho_\lambda^A\rho_\lambda^B\{\hat{x} - g\hat{x}B - \langle \hat{x}B \rangle\}^2|x|\rangle \langle x|$$

$$= \sum_{\lambda} P_\lambda (\langle \hat{\delta}_0 - \langle \hat{\delta}_0 \rangle \rangle^2 \rangle \geq \sum_{\lambda} P_\lambda (\langle \hat{\delta}_0 - \langle \hat{\delta}_0 \rangle \rangle^2 \rangle \lambda) \geq 1$$

Here $\hat{\delta}_0 = g\hat{x}B$ and $\langle \hat{\delta}_0 \rangle$ denotes the average for state $\lambda$ given by density operator $\rho_\lambda = \rho_\lambda^A\rho_\lambda^B$. Since $\rho_\lambda$ factorizes, $\langle \hat{x}\hat{x}B \rangle_\lambda = \langle \hat{x}\rangle_\lambda\langle \hat{x}B \rangle_\lambda$. We have

$$\Delta_{inf,L}\hat{x}^2 = \sum_{r} P_\lambda (\langle \hat{x}_r^2 \rangle - \langle \hat{x}_r \rangle^2) = \sum_{\lambda} P_\lambda (\Delta_{x_B}\hat{x} + g^2\Delta_{x_B}^2\hat{x}B)$$

(34)

where $\Delta_{x_B}\hat{x} = \sigma_{x_B}(x)$ and $\Delta_{x_B}^2\hat{x}B = \langle \hat{x}B \rangle \lambda - \langle \hat{x}B \rangle \lambda$. Also

$$\Delta_{inf,L}\hat{p}^2 = \sum_{\lambda} P_\lambda (\Delta_{x_B}^2\hat{p} + h^2\Delta_{x_B}^2\hat{p}B)$$

(35)

where $\Delta_{x_B}^2\hat{p} = \sigma_{x_B}(p)$ and $\Delta_{x_B}^2\hat{p}B$ is the measurement at $B$ used to infer the result at $\hat{p}$ at $A$. It follows (take $\Delta_{x_B}\hat{p}^2 \geq 1$)

$$\Delta_{inf,L}\hat{x}\Delta_{inf,L}\hat{p}^2 \geq \sum_{\lambda} P_\lambda (\sigma_{x_B}(x) + g^2\Delta_{x_B}\hat{x}B)$$

$$= \sum_{\lambda} P_\lambda (\sigma_{x_B}(p) + h^2\Delta_{x_B}^2\hat{p}B).$$

(36)

Separability implies

$$\Delta_{inf,L}\hat{x}\Delta_{inf,L}\hat{p}^2 \geq \sum_{\lambda} P_\lambda (\sigma_{x_B}(x) + g^2\Delta_{x_B}\hat{x}B) \geq \sum_{\lambda} P_\lambda (\sigma_{x_B}(p) + h^2\Delta_{x_B}^2\hat{p}B)$$

(37)

We notice immediately that

$$\Delta_{inf,L}\hat{x}\Delta_{inf,L}\hat{p}^2 \geq 1 + g^2h^2$$

(38)

meaning that the experimental observation of the EPR criterion

$$\Delta_{inf,L}\hat{x}\Delta_{inf,L}\hat{p} < 1$$

(39)

implies not only EPR correlations but entanglement (inseparability). In terms of the quadrature amplitudes defined in Section 3c, we may use this criterion to write the following criterion sufficient to demonstrate not only (strong) EPR correlations in the fashion of the 1935 EPR paradox, but to demonstrate entanglement.

$$\Delta(X_a - gX_b)\Delta(P_a + gP_b) < 1$$

(40)

C. An EPR-entanglement criterion based on the observation of two-mode squeezing

In fact inseparability (and the generalized weaker EPR correlations of the type discussed in Section 3d) may be deduced through a weaker (more easily achieved) criterion based on the observation of a two-mode squeezing. Upon taking $g = 1$, we see from [15] that another constraint following necessarily from the assumption of separability follows. Separability implies

$$\Delta_{inf,L}\hat{x}\Delta_{inf,L}\hat{p}^2$$

$$= \langle \hat{x} - \langle \hat{x} \rangle - \langle \hat{x}B \rangle \rangle^2 \langle \hat{p} - \langle \hat{p} \rangle \rangle^2 \langle \hat{p}B - \langle \hat{p}B \rangle \rangle^2 \geq 1$$

(33)

where we have used the inequality $x + 1/x \geq 2$ and where $\hat{x}, \hat{p}$ are the observables for the measurements made for system $B$, to allow inference of the result $\hat{x}$ and $\hat{p}$ respectively at $A$. The criterion [11], following necessarily from the assumption of quantum separability, then implies the following criterion sufficient to demonstrate entanglement (inseparability)

$$\Delta_{inf,L}\hat{x}\Delta_{inf,L}\hat{p}^2$$

$$= \langle \hat{x} - \langle \hat{x} \rangle - \langle \hat{x}B \rangle \rangle^2 \langle \hat{p} - \langle \hat{p} \rangle \rangle^2 \langle \hat{p}B - \langle \hat{p}B \rangle \rangle^2 < 4$$

(36)

which may be rewritten in terms of the quadrature amplitudes as

$$\Delta^2(X_a - X_b)\Delta^2(P_a + P_b) < 4$$

(43)

This observation of this entanglement criterion may be identified as a “two-mode squeezing” criterion for entanglement, since the individual criterion

$$\langle \hat{x} - \langle \hat{x} \rangle - \langle \hat{x}B \rangle \rangle^2 < 2$$

(44)

($\Delta^2(X_a - X_b) < 2$) is the criterion for the observation of a two-mode squeezing. In this way we see that fields that are two-mode squeezed with respect to both $X_a - X_b$ and $P_a + P_b$, so as to violate [11], are necessarily entangled, and EPR correlated in the general sense discussed Sections 3d. This criteria for a generalized demonstration of
EPR correlations is generally more easily achieved than that given by (40) which is based on conditional probabilities. However it is pointed out that for the ideal two-mode squeezed state (13) both the strong and the weak EPR criterion are met for any nonzero value of the squeeze parameter $r$.

VI. CONCLUSION

In conclusion, we have established certain criteria sufficient to demonstrate EPR correlations. These EPR correlations are proven, not by violations of Bell inequalities, but by demonstrating a sufficient correlation between results of measurements performed at two spatially separated locations, where it is necessary to consider both “position” and “momentum” measurements.

The EPR criteria are also sufficient to prove entanglement. Such criteria are particularly useful to situations where measurements have continuous variable outcomes, where violations of Bell-type inequalities are not so readily constructed. In this case entanglement and EPR correlations are demonstrable using highly efficient quadrature phase amplitude measurements on two-mode squeezed light. In a second paper (23) the application of strong EPR criteria to give proof of security in continuous variable cryptography is presented.

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VII. APPENDIX

In Section 3 the demonstration of strong EPR correlations most in spirit with the original paradox was discussed. Here we require to measure $\Delta \lambda x = 0, \Delta \lambda p = 0$. We aim to prove as a matter of completeness that any local realistic theory in order to predict these perfect EPR correlations will involve hidden variables where the results of measurement for $\hat{x}$ and $\hat{p}$ are predetermined with zero uncertainty. We argue as follows. The assumption of a local realistic theory as discussed in Sections 3d implies the local hidden variable form (13). The predictions for the conditional distributions based on (13) are then

$$P(x|x_i^B) = \sum_\lambda P_\lambda P_\lambda(x,x_i^B)/(\sum_\lambda P_\lambda P_\lambda(x_i^B))$$

$$= \sum_\lambda f_\lambda(x_i^B)P_\lambda(x|x_i^B)$$

where we use a discrete summation over the possible hidden variable states $\lambda$ for convenience of notation only. Here $P_\lambda$ (the discrete from of $\rho(\lambda)$) is the probability of the system being in the hidden variable state denoted by $\lambda$; $P_\lambda(x,x_i^B)$ is the probability for results $x$ and $x_i^B$ respectively upon joint measurement $\hat{x}$ and $\hat{x}_i^B$ for the state $\lambda$; $P_\lambda(x_i^B)$ is the probability for result $x_i^B$ upon measurement of $\hat{x}_i^B$, given the system is in $\lambda$, and we define the fraction $f_\lambda(x_i^B) = P_\lambda(x_i^B)/(\sum_\lambda P_\lambda P_\lambda(x_i^B))$. The (45) would always imply

$$\Delta^2_i x \geq \sum_\lambda f_\lambda(x_i^B)\Delta^2_{\lambda,i} x$$

where $\Delta^2_{\lambda,i} x$ is the variance of the conditional distribution $P_\lambda(x|x_i^B)$. (See equations (28) for a more complete explanation of a similar result). Recalling from (13) that $P_\lambda(x,x_i^B) = P_\lambda(x)P_\lambda(x_i^B)$ we see that $\Delta_\lambda x = \Delta_\lambda i x$ where $\Delta_\lambda x$ is the variance of the distribution $P_\lambda(x)$ ($P_\lambda(x)$ being the probability that the result of $\hat{x}$ is $x$, given that the system is in the hidden variable state $\lambda$). Immediately then we see that where $\Delta_\lambda x = 0$, each $\Delta_\lambda x = 0$. This means that the system, which according to the local realistic assumption is describable as being in one of the states depicted by $\lambda$ with a probability $P_\lambda$, must for every $\lambda$ have a zero uncertainty $\Delta_\lambda x$ in the prediction for result of measurement $\hat{x}$. The result of $\hat{x}$ is, under the local realism assumption, predetermined with zero uncertainty. A similar conclusion is drawn for the predetermined nature of $\hat{p}$, and the EPR argument follows.

Generally the measurement of zero variances is difficult. However suppose the conditional distributions $P(x|x_i^B)$ are for each $i$ measured to be sufficiently narrow, so that there is a zero probability of obtaining a result $x$ for $\hat{x}$ at $A$ which deviates from the mean $\mu_i$ of $P(x|x_i^B)$ by an amount greater than $\delta$. It follows from (13) that this must also be true for each of the $P_\lambda(x|x_i^B)$, and this in turn implies $\Delta^2_{\lambda,i} x \leq \delta^2, \Delta^2_{\lambda,i} x$ being the variance of $P_\lambda(x/x_i^B)$. This is true for all $\lambda$ (and for all $i$). Recalling again that $P_\lambda(x,x_i^B) = P_\lambda(x)P_\lambda(x_i^B)$ so that $\Delta_\lambda x = \Delta_\lambda i x$, we can then conclude that the system, if compatible with local realism, must always be in a state where the result for $\hat{x}$ is predetermined to an uncertainty $\Delta_\lambda x \leq \delta$ where $\delta^2 < 1$. Applying the same logic to measurements $\hat{p}$, we obtain the EPR paradox.

While this case of very strong EPR correlation is most similar to the original EPR argument, the weaker constraints described in Sections 3d and 5 still imply EPR-type correlations, in the sense that it is proved that any theory compatible with local realism can only predict the measured correlations, if at least one of the $\lambda$ (a $\lambda_0$ say) has a sufficiently definite simultaneous prediction for its result of $\hat{x}$ and $\hat{p}$ ($\Delta_{\lambda_0} x \Delta_{\lambda_0} x \leq 1$).

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[24] This choice $d = \mu_i$ will minimize the rms error $\sqrt{\langle (x|\Delta x - d)^2 \rangle}$ where $x|\Delta x$ refers to results $x$ at A, conditional on a result $x|\Delta x$ at B.

[25] The proven failure of one of the more general EPR criteria (based on [19] and [24]) will not necessarily imply that the 1989 EPR inferred H. U. P. criterion [12] has been met. How this difference comes about may be understood by the following example. This stronger 1989 EPR criterion requires certain restrictions on the average of the variances of the conditional distributions. Consider a system with the following local hidden variable description: suppose the system is in a hidden variable state $\{1, 1, 1, 2\}$ with probability $P_1$; or in an alternative state $\{3, 2, 1, 4\}$ with probability $P_2$, where the values of the hidden variables give the precise results of measurements $\hat{x}, \hat{p}, \hat{x}_B, \hat{p}_B$ respectively if measured. In this case then the predicted results for measurement given a particular hidden variable state are definite: $\Delta x = 0$. The result 1 for measurement $\hat{x}_B$ at B is correlated with both 1 and 3 for measurement $\hat{x}$ at A. The measured variance in the conditional distribution $P(x|\Delta x = 1)$, the probability of result $x$ for $\hat{x}$ given result $\hat{x}_B = 1$ for $\hat{x}_B$, can be substantial despite the definiteness of the local hidden variable description. The failure to demonstrate a direct EPR situation through conditional measurements does not however mean that the statistics can be explained through an entirely local realistic description (as given practically by (19)) where local subsystems could be represented by quantum states.

[26] It is certainly true however that for the case of perfect EPR correlations where all $\Delta x, \Delta p = 0$, it is required of any local realistic theory that all $\Delta x = \Delta p = 0$ to give hidden variables with definite predictions for $\hat{x}$ and $\hat{p}$. This is proved explicitly in the Appendix (as is the case of near-perfect correlation), and gives the situation of the original EPR paradox discussed in Section 3a.

[27] The degree of fuzziness however cannot be determined directly from the values $\Delta^2 x, \Delta^2 p$ for the variances of the conditional distributions. This is because poor correlation between results for A and B can be described, in a way perfectly consistent with local realism, using local realistic (hidden) variables with varying definiteness $\Delta_{A,B} \neq 0$, depending on the degree of correlation between the underlying local realistic variables describing the subsystems A and B. For example we may have poorly-correlated variables with definite predictions ($\Delta A, \Delta B$ are zero), or well-correlated variables for A and B which have with fuzzy predictions ($\Delta_{A,B}$ are large).

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