ALGEBRAIC GEOMETRY OVER ALGEBRAIC STRUCTURES. II.
FOUNDATIONS

E. Yu. Daniyarova, A. G. Myasnikov, and V. N. Remeslennikov

UDC 512.57+512.7

Abstract. In this paper, we introduce elements of algebraic geometry over an arbitrary algebraic structure. We prove so-called unification theorems that describe coordinate algebras of algebraic sets in several different ways.

1. Introduction

Quite often relations between sets of elements of a fixed algebraic structure $\mathcal{A}$ can be described in terms of equations over $\mathcal{A}$. In the classical case where $\mathcal{A}$ is a field, the area of mathematics that studies such relations is algebraic geometry. Therefore, it is reasonable to use the same name in the general case.

Algebraic geometry over algebraic structures is a new area of research in modern algebra. Nevertheless, there are already several breakthrough results for particular algebraic structures as well as an interesting development of a general theory.

To date, the most developed branch of algebraic geometry over algebraic structures is algebraic geometry over groups. Most notable is the solution of the main problem of algebraic geometry: the classification of algebraic sets and coordinate groups in the case of free groups. The classification of coordinate groups is given in the language of free constructions and is a result of the joint effort of many mathematicians. The most important papers in this direction are by R. C. Lyndon [43], K. I. Appel [1], R. Bryant [5], G. Makanin [44], A. Razborov [58,59], R. I. Grigorchuk and P. F. Kurchanov [27], Z. Sela [71–73], A. Myasnikov, V. Remeslennikov, and D. Serbin [51, 53, 60]. The final results were obtained in a breakthrough series of papers by O. Kharlampovich and A. Myasnikov [38–41].

Significant progress was made in algebraic geometry over free metabelian groups [10,61–65,67,68]. The case of solvable groups was considered in [34,54,69]. In the last few years, considerable progress has been made towards understanding algebraic geometry over partially commutative groups. Here we would like to mention the following papers: [6–8,35,48,78].

Algebraic geometry over algebraic structures is also being developed for algebraic structures other than groups. Nice results were obtained in algebraic geometry over commutative monoids with cancellation [50,75,76]. Progress has been achieved in algebraic geometry over nonassociative algebras, namely over Lie algebras [11,13,15–17,22,66,70], and over anti-commutative algebras [21].

Note that there are lots of papers on solving particular equations over particular algebras. In this short introduction, we do not pretend to account for all these papers; rather, we only mention those papers that demonstrate the importance and necessity of algebraic geometry over algebraic structures.

The accumulated analysis of the structure of algebraic sets and coordinate algebras over particular algebraic structures (groups, monoids, rings, algebras, etc.) creates a need for a general framework. From this perspective, there are general results that hold when one studies algebraic geometry over an arbitrary algebraic structure; we refer to such results, and, more generally, to such a viewpoint, as the universal algebraic geometry. Research in this area has been initiated in a series of papers by B. I. Plotkin [55–57], G. Baumslag, O. G. Kharlampovich, A. G. Myasnikov, and V. N. Remeslennikov [3,38,39,52].

Universal algebraic geometry, firstly, is a transfer of general notions and ideas from algebraic geometry over particular algebraic structures to the case of an arbitrary algebraic structure; secondly, it is the

Translated from Fundamentalnaya i Prikladnaya Matematika, Vol. 17, No. 1, pp. 65–106, 2011/12.

1072-3374/12/1853-0389 © 2012 Springer Science+Business Media, Inc.
formulation and proof of general results without the use of properties of a concrete algebraic structure; thirdly, it is the development of a general theory with its own naturally arising problems and goals. One can point out several papers with general results for particular algebraic structures. Most of the results in these papers are proven using techniques and properties specific for the structures considered. Universal algebraic geometry presents standard and universal means of proving those results using the framework of universal algebra and model theory.

This paper is the second in our series of papers on universal algebraic geometry. In the first paper of this series, [18], we present the background material from universal algebra and model theory as needed for universal algebraic geometry and discuss how model-theoretic notions and ideas work in universal algebraic geometry. As this paper is a continuation of the authors’ previous paper [18], we suggest that the reader consult that paper prior to reading this one. For the sake of convenience, and in an attempt to make the paper more self-contained, we present some of the more essential notations and definitions from [18] (see Sec. 2).

The main aim of our previous paper [18] is to prove the so-called unification theorems (Theorem A and Theorem B), which give a description of coordinate algebras of irreducible algebraic sets from several different viewpoints. Let us note that, following R. Hartshorne [36], in our papers all irreducible algebraic sets are nonempty.

**Theorem A.** Let \( A \) be an equationally Noetherian algebraic structure in language \( L \). Then for a finitely generated algebraic structure \( C \) of \( L \) the following conditions are equivalent:

1. \( \forall A \subseteq \forall C \), i.e., \( C \in \text{Ucl}(A) \);
2. \( \exists A \supseteq \exists C \);
3. \( C \) embeds into an ultrapower of \( A \);
4. \( C \) is discriminated by \( A \);
5. \( C \) is a limit algebraic structure over \( A \);
6. \( C \) is an algebraic structure defined by a complete atomic type in the theory \( \forall A \) in \( L \);
7. \( C \) is the coordinate algebra of an irreducible algebraic set over \( A \) defined by a system of equations in the language \( L \).

We begin the current paper with a detailed exposition of the foundations of universal algebraic geometry. In Sec. 3, we introduce the basic notions of algebraic geometry over an arbitrary algebraic structure \( A \): equation over \( A \), algebraic set over \( A \), radical, coordinate algebra, the Zariski topology, and the notions of irreducible sets and equationally Noetherian algebras.

The main results of this paper are the following theorems.

**Theorem 5.6.** The category \( \text{AS}(A) \) of algebraic sets over an algebraic structure \( A \) and the category \( \text{CA}(A) \) of coordinate algebras of algebraic sets over \( A \) are dually equivalent.

**Theorem C.** Let \( A \) be an equationally Noetherian algebraic structure in a language \( L \). Then for a finitely generated algebraic structure \( C \) of \( L \) the following conditions are equivalent:

1. \( C \in \text{Qvar}(A) \), i.e., \( \forall q(A) \subseteq \forall q(C) \);
2. \( C \in \text{Pvar}(A) \);
3. \( C \) embeds into a direct power of \( A \);
4. \( C \) is separated by \( A \);
5. \( C \) is a subdirect product of finitely many limit algebraic structures over \( A \);
6. \( C \) is an algebraic structure defined by a complete atomic type in the theory \( \forall q(A) \) in \( L \);
7. \( C \) is the coordinate algebra of an algebraic set over \( A \) defined by a system of equations in the language \( L \).

Theorem C continues a series of unification theorems in algebraic geometry that we have begun in [18]. Theorem A gives a description of coordinate algebras of irreducible algebraic sets. In classical algebraic geometry over a field, irreducible algebraic sets determine the whole picture. Unlike the classical case, in
algebraic geometry over an arbitrary algebraic structure $\mathcal{A}$ it is not so and here we need a description of all algebraic sets and all coordinate algebras. Here Theorem C is helpful.

Let us note that items (5) in Theorems A and C give a description of coordinate algebraic structures via limit algebraic structures. Limit algebraic structures (for the most part, groups) have become the object of an intense study in modern algebra [9,25,28–31,33]. The definitions of a limit algebraic structure and of an algebraic structure defined by a complete atomic type require a lot of preliminary material and are omitted in this paper (see [18, Subsections 4.2 and 5.1]).

In the previous [18] paper and this one, we suppose that a language $L$ is functional, i.e., it has no predicates. This restriction is not a fundamental matter: all the results proved here remain true in the case of an arbitrary signature $L$. However, if $L$ has predicates, then the definitions of all notions that we introduce will become more complicated, the volume of the paper will become bigger, and the reader will need more grounding to understand the paper. We will describe the case of an arbitrary signature in an addition to this paper.

Summarising, in our work we set up the foundation of universal algebraic geometry. The presented material can be considered as a guide for studying algebraic geometry over particular algebraic structures. In Sec. 7 of the paper, we present several open problems in algebraic geometry over free monoids, free Lie algebras, and free associative algebras. Before applying universal algebraic geometry to a particular group, ring, monoid, etc., we draw attention to the following remark.

There are three different segments of algebraic geometry over a particular algebraic structure:

1. coefficient-free algebraic geometry;
2. Diophantine algebraic geometry;
3. algebraic geometry with coefficients in some algebraic structure $\mathcal{A}$ and solutions in some extension $\mathcal{A} < \mathcal{B}$ (usually, in some saturated model).

While laying the foundations of algebraic geometry over groups in [3,52], the authors choose the universal way for explaining the material: they talk about algebraic geometry over a group $H$ with coefficients in a given group $G$, $G \leq H$. Along these lines, the notions of a $G$-group, $G$-homomorphism, $G$-formula, etc. arise naturally. Obviously, this approach is useful for all the three above-mentioned segments of algebraic geometry over groups: for coefficient-free algebraic geometry (set $G = 1$), for Diophantine algebraic geometry (set $G = H$). The situation is the same in algebraic geometry over Lie algebras [14], monoids, rings, and so on. Note that for semigroups and any other algebra without the trivial subalgebra such “universal” approach does not work. However, universal algebraic geometry provides an instrument for analyzing three of the above segments in a uniform way, using one technique: it just suffices to choose the ground language $L$ appropriately. For instance, when studying coefficient-free algebraic geometry over a semigroup $G$, one should choose the language $L = \{\cdot\}$. For Diophantine algebraic geometry over $G$ it is only natural to take the extended language $L_G$ as the ground language (see Sec. 2 for the definition of $L_G$). Similarly, for algebraic geometry over a semigroup $H$ with coefficients in $G$, $G \leq H$, the signature $L_G$ also works well.

Mathematical logic, model theory, and universal algebra are the background of universal algebraic geometry. Hence, it is only natural that the choice of the ground language $L$ plays a crucial role in universal algebraic geometry, as all definitions that we give depend on the ground language $L$. When we talk about algebraic structures, formulas, and theories, we always assume that a certain language is fixed, hence no confusion arises. Therefore, when one considers our definitions and results in the context of particular algebraic structures (e.g., groups, monoids, algebras, etc.), it is necessary to point out the language in which this group (monoid, algebra, etc.) is considered.

2. Preliminaries

In this section, we present the basic notation from model theory that we use in this paper. For more detailed information we refer to [18,26,46].
Let $L$ be a first-order functional language, $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set of variables, $T_L(X)$ be the set of all terms of $L$ with variables in $X$, $T_L(X)$ be the absolutely free $L$-algebra with basis $X$, and $A_{tL}(X)$ be the set of all atomic formulas of $L$ with variables in $X$.

Typically we denote algebraic structures in $L$ by capital calligraphic letters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ and their universes (the underlying sets) by the corresponding capital Latin letters $A, B, C, \ldots$. Algebraic structures in a functional language are termed \emph{algebras}.

In this paper, we use some operators that map a class $K$ of $L$-algebras into another one. For the sake of convenience we collect here the list of all these operators:

- $S(K)$ — the class of subalgebras of algebras from $K$;
- $P(K)$ — the class of direct products of algebras from $K$;
- $P_\omega(K)$ — the class of finite direct products of algebras from $K$;
- $P_s(K)$ — the class of subdirect products of algebras from $K$;
- $P_f(K)$ — the class of filterproducts of algebras from $K$;
- $P_u(K)$ — the class of ultraproducts of algebras from $K$;
- $L(K)$ — the class of direct limits of algebras from $K$;
- $\overline{L}_s(K)$ — the class of epimorphic direct limits of algebras from $K$;
- $L_{\omega}(K)$ — the class of algebras in which all finitely generated subalgebras belong to $K$;
- $P\text{var}(K)$ — the least prevariety including $K$;
- $Q\text{var}(K)$ — the least quasi-variety including $K$, i.e., $Q\text{var}(K) = \text{Mod}(\text{Th}_{\eta}(K))$;
- $U\text{cl}(K)$ — the universal class of algebras generated by $K$, i.e., $U\text{cl}(K) = \text{Mod}(\text{Th}_v(K))$;
- $\text{Res}(K)$ — the class of algebras that are separated by $K$;
- $\text{Dis}(K)$ — the class of algebras that are discriminated by $K$;
- $K_\mathcal{E}$ — the addition of the trivial algebra $\mathcal{E}$ to $K$, i.e., $K_\mathcal{E} = K \cup \{\mathcal{E}\}$;
- $K_u$ — the class of finitely generated algebras from $K$.

Here we denote by $\text{Th}_{\eta}(K)$ ($\text{Th}_v(K)$, $\text{Th}_\mathcal{E}(K)$) the set of all quasi-identities (respectively, universal sentences, existential sentences) that are true in all structures from $K$.

For an arbitrary class $K$ of $L$-algebras one has

\[
\text{Dis}(K) \subseteq \text{Ucl}(K) = \text{SP}_u(K) \subseteq \text{Qvar}(K),
\]

\[
\text{Dis}(K) \subseteq \text{Res}(K) = \text{SP}(K) = \text{Pvar}(K) \subseteq \text{Qvar}(K).
\]

According to Gorbunov [26] and in contrast to [18], we assume that the direct product for the empty set of indices coincides with the trivial $L$-algebra $\mathcal{E}$. In particular, $\mathcal{E} \in P(K)$, $\mathcal{E} \in P_\omega(K)$, $\mathcal{E} \in P_s(K)$, for an arbitrary class of $L$-algebras $K$. However, while defining a filter product we assume that the set of indices is nonempty.

Let us recall the definitions of separation and discrimination.

**Definition 2.1.** An $L$-algebra $C$ is \emph{separated} by a class of $L$-algebras $K$ if for any pair of nonequal elements $c_1, c_2 \in C$ there is a homomorphism $h: C \to B$ for some $B \in K$ such that $h(c_1) \neq h(c_2)$.

**Definition 2.2.** An $L$-algebra $C$ is \emph{discriminated} by $K$ if for any finite set $W$ of elements from $C$ there is a homomorphism $h: C \to B$ for some $B \in K$ whose restriction onto $W$ is injective.

If $C$ is separated (discriminated) by a class $K = \{B\}$, then we say that $C$ is separated (discriminated) by algebra $B$. Let us note that in the definitions above we do not claim that a homomorphism $h$ is an epimorphism. As follows from the definitions of separation and discrimination, the trivial algebra $\mathcal{E}$ is separated by a class $K$ anyway, and $\mathcal{E}$ is discriminated by $K$ if and only if there exists a homomorphism $h: \mathcal{E} \to B$ for some $B \in K$, i.e., $B$ has a trivial subalgebra. This circumstance allows one to reserve inclusion $\text{Dis}(K) \subseteq \text{Ucl}(K)$ and identity $\text{Res}(K) = \text{SP}(K)$ in the customary form.

Unification Theorem A has been proved in [18] for equationally Noetherian algebras. However, the following result holds in a more general case too.
Proposition 2.3 ([18]). Let \( \mathcal{A} \) be an algebra in a language \( \mathcal{L} \). Then for a finitely generated \( \mathcal{L} \)-algebra \( \mathcal{C} \) the following conditions are equivalent:

- \( \text{Th}_v(\mathcal{A}) \subseteq \text{Th}_v(\mathcal{C}) \), i.e., \( \mathcal{C} \in \text{Ucl}(\mathcal{A}) \);
- \( \text{Th}_e(\mathcal{A}) \supseteq \text{Th}_e(\mathcal{C}) \);
- \( \mathcal{C} \) embeds into an ultrapower of \( \mathcal{A} \);
- \( \mathcal{C} \) is a limit algebra over \( \mathcal{A} \);
- \( \mathcal{C} \) is an algebra defined by a complete atomic type in the theory \( \text{Th}_v(\mathcal{A}) \) in \( \mathcal{L} \).

For an \( \mathcal{L} \)-algebra \( \mathcal{A} \), we denote by \( \mathcal{L}_\mathcal{A} = \mathcal{L} \cup \{ c_a \mid a \in \mathcal{A} \} \) the language \( \mathcal{L} \) extended by elements from \( \mathcal{A} \) as new constant symbols.

An algebra \( \mathcal{B} \) in the language \( \mathcal{L}_\mathcal{A} \) is called an \( \mathcal{A} \)-algebra if \( \mathcal{B} \models \text{Diag}(\mathcal{A}) \). This means that \( \mathcal{A} \) embeds into \( \mathcal{B} \) and the corresponding embedding \( \lambda : \mathcal{A} \to \mathcal{B} \) is fixed. Recall that the diagram \( \text{Diag}(\mathcal{A}) \) of \( \mathcal{A} \) is the set of all atomic sentences from \( \text{At}_\mathcal{L}(\mathcal{A}) \) or their negations that are true in \( \mathcal{A} \).

Let \( \mathcal{B} \) and \( \mathcal{C} \) be \( \mathcal{A} \)-algebras, and \( h : \mathcal{B} \to \mathcal{C} \) a \( \mathcal{L}_\mathcal{A} \)-homomorphism. We usually refer to \( h \) as an \( \mathcal{A} \)-homomorphism. Similarly, we define \( \mathcal{A} \)-separation and \( \mathcal{A} \)-discrimination. The prevariety of \( \mathcal{B} \) in the language \( \mathcal{L}_\mathcal{A} \) we denote by \( \text{Pvar}_{\mathcal{A}}(\mathcal{B}) \), the quasivariety — by \( \text{Qvar}_{\mathcal{A}}(\mathcal{B}) \), and the universal closure — by \( \text{Ucl}_{\mathcal{L}}(\mathcal{B}) \). Such a notation is especially convenient when \( \mathcal{B} = \mathcal{A} \). In this case, we must distinguish \( \mathcal{A} \) as \( \mathcal{L} \)-algebra and \( \mathcal{A} \) as \( \mathcal{L}_\mathcal{A} \)-algebra. Correspondingly, for instance, we must point out what class we are concerned with: \( \text{Ucl}(\mathcal{A}) \) or \( \text{Ucl}_{\mathcal{A}}(\mathcal{A}) \).

### 3. Elements of Algebraic Geometry

Let \( \mathcal{L} \) be a functional language and \( \mathcal{A} \) be an \( \mathcal{L} \)-algebra.

In this section, we introduce the basic notions of universal algebraic geometry: equation in the language \( \mathcal{L} \) algebraic set over the algebra \( \mathcal{A} \), radical, coordinate algebra, the Zariski topology, irreducible set, and equationally Noetherian algebra.

#### 3.1. Equations and Algebraic Sets

**Definition 3.1.** Atomic formulas from \( \text{At}_\mathcal{L}(\mathcal{X}) \) are called *equations* in \( \mathcal{L} \) with variables in \( \mathcal{X} \). Any subset \( S \subseteq \text{At}_\mathcal{L}(\mathcal{X}) \) is called a *system of equations* in \( \mathcal{L} \).

Sometimes, to emphasize that formulas are from \( \mathcal{L} \), we call such equations (and systems of equations) *coefficient-free equations*, meanwhile, in the case where \( \mathcal{L} = \mathcal{L}_\mathcal{A} \), we refer to such equations as *equations with coefficients in the algebra \( \mathcal{A} \) or \( \mathcal{A} \)-equations*.

When someone looks for solutions of equations and systems of equations in an algebra \( \mathcal{A} \), it is said to be *algebraic geometry over the algebra \( \mathcal{A} \)*. Algebraic geometry over an algebra \( \mathcal{A} \) in the language \( \mathcal{L}_\mathcal{A} \) is called *Diophantine*. If \( \mathcal{B} \) is an \( \mathcal{A} \)-algebra, then investigation into algebraic geometry over \( \mathcal{B} \) as over \( \mathcal{L}_\mathcal{A} \)-algebra is called *algebraic geometry over \( \mathcal{B} \) with coefficients in \( \mathcal{A} \).*

We term the set

\[
\mathcal{A}^n = \{ (a_1, \ldots, a_n) \mid a_i \in \mathcal{A} \}
\]

the *affine n-space* over algebra \( \mathcal{A} \), and we sometimes refer to its elements as *points*. A point \( p = (a_1, \ldots, a_n) \in \mathcal{A}^n \) is called a *root* of an equation \( (t_1 = t_2) \), \( t_1, t_2 \in \text{Th}_{\mathcal{L}}(\mathcal{X}) \), if \( \mathcal{A} \models (t_1 = t_2) \) via interpretation \( x_i \mapsto a_i, i = 1, n \). Further, a point \( p \) is a root of a system of equations \( S \subseteq \text{At}_\mathcal{L}(\mathcal{X}) \) if it is a root of every equation from \( S \).

**Definition 3.2.** Let \( S \) be a system of equations in the language \( \mathcal{L} \) in variables \( \mathcal{X} \). The set of all roots of the system \( S \) in the affine \( n \)-space \( \mathcal{A}^n \) will be denoted by \( \mathcal{V}_\mathcal{A}(S) \) (or briefly \( \mathcal{V}(S) \)):

\[
\mathcal{V}_\mathcal{A}(S) = \{ (a_1, \ldots, a_n) \in \mathcal{A}^n \mid t_1^A(a_1, \ldots, a_n) = t_2^A(a_1, \ldots, a_n) \ \forall \ (t_1 = t_2) \in S \}.
\]

The set \( \mathcal{V}_\mathcal{A}(S) \) is called the *algebraic set* over the algebra \( \mathcal{A} \) defined by the system \( S \).

A system \( S \) is called *inconsistent* over \( \mathcal{A} \) if \( \mathcal{V}_\mathcal{A}(S) = \emptyset \); otherwise it is called *consistent*. We say two systems of equations \( S_1 \) and \( S_2 \) are *equivalent* over \( \mathcal{A} \) and write \( S_1 \sim_\mathcal{A} S_2 \) if \( \mathcal{V}_\mathcal{A}(S_1) = \mathcal{V}_\mathcal{A}(S_2) \).
Example 3.3. Any points, affine $n$-spaces, and direct products of algebraic sets give standard examples of algebraic sets.

1. In Diophantine algebraic geometry over an algebra $A$, every point from the affine $n$-space $A^n$ is an algebraic set. For a point $(a_1, \ldots, a_n) \in A^n$ one has
   \[ S = \{ x_1 = c_{a_1}, \ldots, x_n = c_{a_n} \}, \quad V(S) = \{ (a_1, \ldots, a_n) \}. \]

2. The affine $n$-space $A^n$ is an algebraic set for the degenerate system $S = \{ x = x \}$.

3. Let $Y \subseteq A^n$ and $Z \subseteq A^m$ be algebraic sets over $A$. Then $Y \times Z \subseteq A^{n+m}$ is an algebraic over $A$ too. Indeed, if $Y = V(S)$, $S \subseteq \text{Aut}(x_1, \ldots, x_n)$, and $Z = V(S')$, $S' \subseteq \text{Aut}(x'_1, \ldots, x'_m)$, then
   \[ Y \times Z = V(S \cup S'), \quad S \cup S' \subseteq \text{Aut}(x_1, \ldots, x_n, x'_1, \ldots, x'_m). \]

Now we show some examples of algebraic sets over specific algebras: free group, free Lie algebra, min-max structure.

Example 3.4. Let us consider a free algebra $F$ with a free base $a_1, \ldots, a_n$ in some variety $\Theta$ and the following equation over $F$:
   \[ w(x_1, \ldots, x_n) = w(a_1, \ldots, a_n), \]
where $w(a_1, \ldots, a_n)$ is an element in $F$. Then algebraic set for equation (1) is
   \[ Y = \{ (\varphi(a_1), \ldots, \varphi(a_n)) \mid \varphi \in \text{St}(w) \}, \]
where
   \[ \text{St}(w) = \{ \varphi \in \text{End}(F) \mid \varphi(w) = w \}. \]
So, the algebraic set $Y$ is parametrized by endomorphisms from the stabilizer $\text{St}(w)$. Recall that an element $w(a_1, \ldots, a_n) \in F$ is termed testing if $\text{St}(w) \subseteq \text{Aut}(F)$. Thus, if $w$ is a testing element, then the algebraic set $Y$ has a parametrization by means of automorphisms.

If $F$ is a finitely generated free group, then there exists an effective algorithm to determine $\text{St}(w)$ for any (cyclic) testing element $w \in F$ [47]. Therefore, there exists an effective algorithm for finding all solutions of equation (1).

In particular, the stabilizer $\text{St}(w)$ of the commutator $w = [a_1, a_2]$ in $F = \langle a_1, a_2 \rangle$ is isomorphic to the free group of rank 2. Indeed, $\text{St}(w) = \langle \text{tr}_1, \text{tr}_2 \rangle$, where $\text{tr}_1$ is the automorphism of $F$ defined by $a_1 \rightarrow a_2 a_1, a_2 \rightarrow a_2$, and $\text{tr}_2$ is defined by $a_1 \rightarrow a_1, a_2 \rightarrow a_1 a_2$.

Example 3.5. Let $L$ be a free Lie algebra of finite rank over a field $k$. An algebraic set $Y$ is called bounded if it enters into some finite-dimensional subspace of $L^n$ as a $k$-linear space. Bounded algebraic sets over $L$ have been classified in [22]. Any finite-dimensional subspace in $L$ supplies an elementary example of bounded algebraic set. For linear subspace $W$ in $L$ with basis $v_1, \ldots, v_m$ we have
   \[ s_1(x) = [x, v_1], s_2(x) = [[x, v_1], v_2, v_1]], \ldots, s_m(x) = [s_{m-1}(x), s_{m-1}(v_m)], \quad V(s_m) = W. \]
While $W$ is an algebraic set in one variable $x$, the similar algebraic sets in $n$ variables are called $n$-parallelepipeds. By an $n$-parallelepiped $W$ we mean a Cartesian product of an $n$-tuple of finite-dimensional subspaces $W_1, \ldots, W_n$ in $L$:
   \[ W = W_1 \times \cdots \times W_n. \]
The dimension of the $n$-parallelepiped $W$ is defined by
   \[ \dim(W) = \dim(W_1) + \cdots + \dim(W_n). \]
An algebraic set $Y \subseteq L^n$ is bounded by a parallelepiped $W$ if $Y \subseteq W$.

Theorem ([22]). Let $W$ be an $n$-parallelepiped over the free Lie algebra $L$ over a field $k$. Algebraic sets over the algebra $L$ bounded by the parallelepiped $W$ are in one-to-one correspondence with algebraic sets over the field $k$ defined by systems of equations in $\dim(W)$ variables.

394
In [66], it has been shown that the equation \([x, a] + [y, b] = 0\) \((a, b \in L, [a, b] \neq 0)\) has a complicated solution over \(L\) and its algebraic set is not bounded. However, the same equation is easy to solve over a free anti-commutative algebra \(A\); and its algebraic set over \(A\) is bounded [21].

**Example 3.6.** The algebraic structure \(M_\mathbb{R} = (\mathbb{R}; \max, \min, +, -, 0, 1)\) with the obvious interpretation of the symbols from the signature on \(\mathbb{R}\) is an example of the so-called min-max structure.

**Theorem (23).** A set \(Y \subseteq \mathbb{R}^n\) is algebraic over \(M_\mathbb{R}\) if and only if it is closed in the topology induced by the Euclidean metric on \(\mathbb{R}^n\).

The following two lemmas will be helpful further; they show how equivalence between systems of equations is preserved when we move from algebra \(A\) to its direct powers and filterpowers.

**Lemma 3.7.** Let \(A\) be an \(L\)-algebra and \(C\) be a subalgebra of some direct power of \(A\). Then for two system of equations \(S_1, S_2 \subseteq \text{Att}_L(X)\) the condition \(S_1 \sim A S_2\) implies \(S_1 \sim C S_2\).

*Proof.* Let \(C\) be a subalgebra of \(\prod_{i \in I} A^{(i)}\). For a point \(p = (c_1, \ldots, c_n) \in C^n\) let us write \(c_j = (a_j^{(i)})_{i \in I}\), \(a_j^{(i)} \in A\), \(j = 1, \ldots, n\). We have \(p \in V_C(S_1)\) if and only if \((a_1^{(i)}, \ldots, a_n^{(i)}) \in V_A(S_1)\) for every \(i \in I\). Since \(S_1 \sim_A S_2\), the latter is equal to \((a_1^{(i)}, \ldots, a_n^{(i)}) \in V_A(S_2)\) for every \(i \in I\). Therefore, \(p \in V_A(S_1)\) if and only if \(p \in V_C(S_2)\), i.e., \(S_1 \sim C S_2\).

**Lemma 3.8.** Let \(A\) be an \(L\)-algebra and \(C\) be a subalgebra of some filterpower of \(A\). Then for a system of equations \(S \subseteq \text{Att}_L(X)\) and a finite subsystem \(S_0 \subseteq S\) the condition \(S \sim A S_0\) implies \(S \sim C S_0\).

*Proof.* Suppose that \(S \sim A S_0\) and \(C\) is a subalgebra of \(\prod_{i \in I} A^{(i)}/D\), where \(D\) is a filter on \(I\). Since the inclusion \(V_C(S_0) \supseteq V_C(S)\) is obvious, we need to prove only the converse inclusion \(V_C(S_0) \subseteq V_C(S)\).

Assume that a point \(p = (c_1, \ldots, c_n) \in C^n\) is a root of \(S_0\). We write \(c_j = (a_j^{(i)})_{i \in I}/D, j = 1, \ldots, n\), and denote \(p^{(i)} = (a_1^{(i)}, \ldots, a_n^{(i)}) \in A^n\). For each equation \((t = s) \in S_0\) there exists an element \(J\) in \(D\) such that \(t(p^{(i)}) = s(p^{(i)})\) for all \(i \in J\). Since \(S_0\) is finite, there is an element \(J_0 \in D\) such that \(t(p^{(i)}) = s(p^{(i)})\) for all \(i \in J_0\) and each index \(i \in J_0\). As \(V_A(S_0) = V_A(S)\), we have \(t(p^{(i)}) = s(p^{(i)})\) for all \(i \in J_0\) and every equation \((t = s) \in S\). Hence, the point \(p\) is a solution of \(S\). Thus, the inclusion \(V_C(S_0) \subseteq V_C(S)\) holds.

### 3.2. Radicals.

With every algebraic set \(Y\) we associate two important objects: its radical \(\text{Rad}(Y)\) and the coordinate algebra \(\Gamma(Y)\). In this subsection, we will discuss radicals, and in the next one we will discuss coordinate algebras.

**Definition 3.9.** For a subset \(Y \subseteq A^n\), we term the following set of atomic formulas from \(\text{At}_L(x_1, \ldots, x_n)\) the radical of the set \(Y\):

\[
\text{Rad}(Y) = \{(t_1 = t_2) \mid t_1^A(a_1, \ldots, a_n) = t_2^A(a_1, \ldots, a_n) \forall (a_1, \ldots, a_n) \in Y\}.
\]

The radical \(\text{Rad}(Y)\) of an algebraic set \(Y\) uniquely defines it, i.e., an algebraic set \(Y_1\) coincides with an algebraic set \(Y_2\) if and only if \(\text{Rad}(Y_1) = \text{Rad}(Y_2)\) (see Lemma 3.11).

The radical of a system of equations \(S \subseteq \text{Att}_L(X)\) over an \(L\)-algebra \(A\) is the set \(\text{Rad}(V_A(S))\). We denote it \(\text{Rad}_A(S)\) (or briefly \(\text{Rad}(S)\)). Atomic formulas from \(\text{Rad}_A(S)\) are called consequences of the system \(S\) over \(A\). So, atomic formula \((t_1 = t_2)\) is a consequence of \(S\) over \(A\) if and only if \(V(S) \subseteq V\{t_1 = t_2\}\), i.e., \(S \cup \{t_1 = t_2\} \sim_A S\). In other words, \(\text{Rad}(S)\) is the maximal system of equations that is equivalent to \(S\). The radical of an inconsistent system \(S\) coincides with \(\text{Att}_L(X)\).

By \([S]\) we denote the congruent closure of \(S\), i.e., the least congruent subset of \(\text{Att}_L(X)\), containing \(S\) [18]. It is obvious that \([S] \subseteq \text{Rad}(S)\).

**Definition 3.9.** We call a subset \(S \subseteq \text{At}_L(X)\) the radical ideal over \(A\) if \(S = \text{Rad}_A(Y)\) for some \(Y \subseteq A^n\).
One can consider Rad and V as operators. Thus, Rad is the operator of the calculation of the radical for sets \( Y \subseteq A^n \), and V is the operator of the calculation of algebraic sets for systems \( S \subseteq A^L(X) \). In the next lemma, we gather elementary properties of these operators.

**Lemma 3.11.** The following holds.

1. A subset \( Y \subseteq A^n \) is algebraic over \( A \) if and only if \( Y = V_A(\text{Rad}(Y)) \).
2. A subset \( S \subseteq A^L(X) \) is a radical ideal over \( A \) if and only if \( S = \text{Rad}_A(S) \).
3. For any sets \( Y_1, Y_2 \subseteq A^n \) one has \( Y_1 \subseteq Y_2 \implies \text{Rad}(Y_1) \supseteq \text{Rad}(Y_2) \).
4. For any systems of equations \( S_1, S_2 \subseteq A^L(X) \) one has \( S_1 \subseteq S_2 \implies V(S_1) \supseteq V(S_2) \implies \text{Rad}(S_1) \subseteq \text{Rad}(S_2) \).
5. For any algebraic sets \( Y_1, Y_2 \subseteq A^n \) one has \( Y_1 = Y_2 \iff \text{Rad}(Y_1) = \text{Rad}(Y_2) \).
6. For any family of subsets \( \{Y_i \subseteq A^n, i \in I\} \) one has \( \text{Rad}\left( \bigcup_{i \in I} Y_i \right) = \bigcap_{i \in I} \text{Rad}(Y_i) \).
7. For any family of systems of equations \( \{S_i \subseteq A^L(X), i \in I\} \) one has \( V\left( \bigcup_{i \in I} S_i \right) = \bigcap_{i \in I} V(S_i) \).

In particular, the intersection of any family of algebraic sets in \( A^n \) is an algebraic set.

**Proof.** Straightforward. \( \square \)

Lemma 3.11 gives the following method for calculating the radical \( \text{Rad}(Y) \) of an arbitrary nonempty set \( Y \subseteq A^n \). With a point \( p = (a_1, \ldots, a_n) \in A^n \) we associate the homomorphism \( h_p : T^L(X) \to A \) defined by \( h_p(t) = t^A(a_1, \ldots, a_n) \). Clearly,

\( t_1 \sim_{\text{ker } h_p} t_2 \iff (t_1 = t_2) \in \text{Rad}('\{p\}') \).

**Lemma 3.12.** Let \( Y \) be a nonempty algebraic set over an algebra \( A \). Then

\[
\theta_{\text{Rad}(Y)} = \bigcap_{p \in Y} \text{ker } h_p.
\]  

**Proof.** Indeed, by Lemma 3.11, \( \text{Rad}(Y) = \bigcap_{p \in Y} \text{Rad}(\{p\}) \). \( \square \)

### 3.3. Coordinate Algebras

Let \( S \subseteq A^L(X) \) be a system of equations and \( Y = V_A(S) \). It is not hard to see that the radical \( \text{Rad}(Y) \) is a congruent set of atomic formulas. Hence, it defines a congruence on the absolutely free \( L \)-algebra \( T^L(X) \) that we denote by \( \theta_{\text{Rad}(Y)} \) [18]:

\( t_1 \sim_{\theta_{\text{Rad}(Y)}} t_2 \iff (t_1 = t_2) \in \text{Rad}(Y), \quad t_1, t_2 \in T^L(X) \).

**Definition 3.13.** The \( L \)-structure

\( \Gamma(Y) = T^L(X)/\theta_{\text{Rad}(Y)} \)

is called the coordinate algebra of the algebraic set \( Y \).
When \( Y = V_A(S) \), we also refer to \( \Gamma(Y) \) as the coordinate algebra of the system \( S \) over \( A \) and write \( \Gamma_A(S) \) (or \( \Gamma(S) \)).

If \( S \sim_A A_{\text{tL}}(X) \), then \( \Gamma(S) \) is the trivial algebra \( \mathcal{E} \). For instance, one has \( S \sim_A A_{\text{tL}}(X) \) if \( S \) is inconsistent over \( A \).

**Definition 3.14.** We say that an \( L \)-algebra \( C \) is a coordinate algebra over an \( L \)-algebra \( A \) if \( C \cong \Gamma(Y) \) for some algebraic set \( Y \over A \).

One of the principal goals of algebraic geometry over an algebra \( A \) is to describe algebraic sets over \( A \) up to isomorphism (the definition of isomorphism of algebraic sets see in Sec. 5.1). We will show that this problem has two equivalent forms: the problem of classifying radicals and the problem of classifying coordinate algebras over \( A \).

While every algebraic set may be restored in a unique manner from its radical, it may be restored from its coordinate algebra just up to isomorphism. The following result gives a specification of algebraic sets by means of sets of homomorphisms. It shows how one can restore an algebraic set from its coordinate algebra.

**Lemma 3.15.** Let \( Y \) be a nonempty algebraic set over an \( L \)-algebra \( A \). Then points of \( Y \) are in one-to-one correspondence with \( L \)-homomorphisms from \( \text{Hom}(\Gamma(Y),A) \).

**Proof.** Indeed, every homomorphism \( h: T_L(X)/\theta_{\text{Rad}(Y)} \rightarrow A \) is uniquely defined by the images of elements \( x_i/\theta_{\text{Rad}(Y)}, i = 1, \ldots, n \), i.e., by a point \((a_1, \ldots, a_n) \in A^n \) with \( t_1^A(a_1, \ldots, a_n) = t_2^A(a_1, \ldots, a_n) \) for all \((t_1, t_2) \in \text{Rad}(Y) \). Clearly, the set of all appropriate points \((a_1, \ldots, a_n) \) coincides with \( Y \). \( \square \)

**Corollary 3.16.** Points of a nonempty algebraic set \( Y \) over an \( A \)-algebra \( B \) are in one-to-one correspondence with \( A \)-homomorphisms from \( \text{Hom}_A(\Gamma(Y),B) \).

In the classical algebraic geometry over a field, one can consider the coordinate ring as the ring of polynomial functions. Let us discuss a similar idea in universal algebraic geometry.

**Definition 3.17.** For a nonempty set \( Y \subseteq A^n \) and a term \( t \in T_L(X) \) we refer to the map \( t^Y: Y \rightarrow A \) defined by

\[ t^Y(p) = t^A(a_1, \ldots, a_n), \quad p = (a_1, \ldots, a_n), \quad p \in Y, \]

as a term function on \( Y \). We call the set \( T(Y) \) of all term functions on \( Y \) with the obvious interpretation of signature symbols from \( L \) the algebra of term functions on \( Y \).

**Lemma 3.18.** For a nonempty algebraic set \( Y \) over an \( L \)-algebra \( A \) one has \( \Gamma(Y) \cong T(Y) \).

**Proof.** Let \( h: T_L(X) \rightarrow T(Y) \) be the epimorphism defined by \( h(t) = t^Y, \ t \in T_L(X) \). One has \( T_L(X)/\ker h \simeq T(Y) \). On the other hand, \( t_1 \sim_{\ker h} t_2 \) if and only if \((t_1, t_2) \in \text{Rad}(Y), t_1, t_2 \in T_L(X) \). Therefore, \( T_L(X)/\ker h \simeq \Gamma(Y) \). \( \square \)

**Example 3.19.** Let \( Y = \{(a_1, \ldots, a_n)\} \) be a singleton algebraic set from Example 3.3. Then the coordinate algebra \( \Gamma(Y) \) is \( A \)-isomorphic to the algebra \( A \). Indeed, it is easy to see that \( T(Y) \cong_A A \).

The empty set \( \emptyset \) is not necessarily an algebraic set over an algebra \( A \).

**Example 3.20.** Let \( L = \{\cdot, -1, e\} \) be the language of groups and \( G \) be a group. Every equation \( t(x_1, \ldots, x_n) = s(x_1, \ldots, x_n) \) in \( L \) has at least one root in \( G \), namely, \( x_1 = \cdots = x_n = e \). Thus, the empty set is not algebraic over \( (G; L) \).

On the other hand, if \( L \) is a language containing at least two constant symbols \( c_1, c_2 \), and \( A \) is an \( L \)-algebra with \( c_1^A \neq c_2^A \), then the empty set is algebraic over \( A \), since \( V_A(\{c_1 = c_2\}) = \emptyset \).

**Lemma 3.21** (on the empty set and the trivial algebra). For an \( L \)-algebra \( A \) the following hold.

1. The empty set is an algebraic over \( A \) if and only if \( A \) has not a trivial subalgebra.
2. If the empty set is algebraic over \( A \) then \( V_A(A_{\text{tL}}(X)) = \emptyset \) for every finite set \( X \).
(3) The trivial algebra $E$ is a coordinate algebra over $A$ anyway. Moreover, if $Y$ is an algebraic set over $A$ such that $E = \Gamma(Y)$, then $Y$ is either irreducible or $Y = \emptyset$ (the definition of irreducible algebraic set see in Sec. 3.4).

(4) The trivial algebra $E$ is the coordinate algebra of an irreducible algebraic set over $A$ if and only if $A$ has a trivial subalgebra.

Proof. Suppose that $A$ has a trivial subalgebra $E = \langle \{e\}; L \rangle$. Then for every term $t \in T_L(X)$ one has $t(e, \ldots, e) = e$. Thus, every system of equations $S$ has a root $p = (e, \ldots, e)$, and the empty set is not algebraic over $A$. Now assume that $\emptyset$ is not algebraic over $A$. Then there exists an element $e \in A$ such that $t(e) = e$ for all terms $t \in T_L(\{x\})$. It is clear that the element $e$ forms a trivial subalgebra of $A$. We have just proven item (1). To show (2), assume that $\emptyset$ is an algebraic set over $A$. Then there exists a natural number $n$ and an inconsistent over $A$ system of equations $S(x_1, \ldots, x_n)$. Thus, $S'(x) = S(x, \ldots, x)$ is inconsistent too. Hence, for every finite set $X$ one has the inconsistent system $S' \subseteq A(x) = A_{L,X}$, thus $V_A(A_{L,X}) = \emptyset$.

The first statement in item (3) is obvious as far as $E = \Gamma(A_{L,X})$. The second one will be proven in Corollary 3.36. To prove item (4), let us assume that $A$ has a trivial subalgebra; then, according to item (1), the empty set is not an algebraic set over $A$. By item (3), it means that the trivial algebra is the coordinate algebra of an irreducible algebraic set over $A$. Conversely, suppose that $A$ has no a trivial subalgebra, then the empty set is algebraic over $A$. If even so $E$ is the coordinate algebra of an irreducible algebraic set $Y$ over $A$, $Y = V(S)$, $S \subseteq A_{L,X}$, then $\text{Rad}(Y) = A_{L,X}$. However, according to item (2), in this case $V_A(A_{L,X}) = \emptyset$, which means that $Y = \emptyset$. The latter contradicts the definition of irreducible set.

Remark 3.22. If an $L$-algebra $A$ has a trivial subalgebra, then there exists an element $e \in A$ such that $c^A = e$ for all constant symbols $c \in L$. Suppose that we are studying Diophantine algebraic geometry over a nontrivial group $G$. Then the trivial subgroup 1 of $G$ is not a trivial subalgebra of $G$ in terms of model theory. As the ground language here is $L$, the trivial subgroup 1 is not $L$-substructure of $G$ at all.

The following proposition and its corollaries are helpful for the problem of classifying of coordinate algebras over $A$.

Proposition 3.23. Let $A$ be an algebra in $L$. Then for a finitely generated algebra $C$ of $L$ the following conditions are equivalent:

1. $C \in \text{Pvar}(A)$;
2. $C$ embeds into a direct power of $A$;
3. $C$ is separated by $A$;
4. $C$ is the coordinate algebra of an algebraic set over $A$ defined by a system of equations in $L$.

Proof. The equivalence (1) $\iff$ (2) $\iff$ (3) has been proved in [18, Lemma 3.5] in the form

$\text{Pvar}(A) = \text{SP}(A) = \text{Res}(A)$.

Suppose that $C$ is the coordinate algebra of an algebraic set $Y$ over $A$. If $Y = \emptyset$, then $\Gamma(Y) = E$, and inclusion $C \in \text{Pvar}(A)$ is evident. So, we assume that $Y$ is nonempty. Equality (2) induces the monomorphism

$h: T_L(X)/\theta_{\text{Rad}(Y)} \to \prod_{p \in Y} T_L(X)/\ker h_p$

[18, Lemma 3.1]. Since $T_L(X)/\ker h_p \cong \text{Im} h_p$ is a subalgebra of $A$, we have the embedding

$h: T_L(X)/\theta_{\text{Rad}(Y)} \to A[Y]$.

Thus, we have proved the implication (4) $\implies$ (2).

Let us show the implication (3) $\implies$ (4). Suppose that $C$ is a finitely generated $L$-algebra from $\text{Res}(A)$ with a finite generating set $X = \{x_1, \ldots, x_n\}$. If $C$ is the trivial algebra, then there is nothing to prove.
Assume that $C$ is nontrivial, and $C = \langle X \mid S \rangle$ is a presentation of $C$ in the generators $X$, where $S \subseteq \text{At}_L(X)$. The latter means that $C \cong \mathcal{T}_L(X)/\theta_S$. It is sufficient to show that $C = \Gamma(S)$, i.e., $\text{Rad}_A(S) = [S]$. Since $C$ is separated by $A$, for any atomic formula $(t = s) \notin [S]$ there exists a homomorphism $h: C \to A$ with $t^A(h(x_1), \ldots, h(x_n)) \neq s^A(h(x_1), \ldots, h(x_n))$. Obviously, $(h(x_1), \ldots, h(x_n)) \in V_A(S)$, whence $(t = s) \notin \text{Rad}_A(S)$. This proves that $\text{Rad}_A(S) = [S]$. \hfill \Box

**Corollary 3.24.** Let $C$ be the coordinate algebra of an algebraic set over an algebra $A$ and $\langle X \mid S \rangle$ be a presentation of $C$ in the generators $X$ with $S \subseteq \text{At}_L(X)$. Then $[S]$ is a radical ideal over $A$.

*Proof.* Since $C$ is separated by $A$, we may repeat the arguments above. \hfill \Box

**Corollary 3.25.** The class of all coordinate algebras of algebraic sets over an algebra $A$ coincides with $\text{Pvar}(A)_\omega$.

**Corollary 3.26.** Let a finitely generated $L$-algebra $C$ imbed into a direct product of coordinate algebras of some algebraic sets over $A$. Then $C$ is the coordinate algebra of an algebraic set over $A$.

**Corollary 3.27.** For any algebraic set $Y$ over algebra $A$ one has $\Gamma(Y) \in \text{Qvar}(A)$. In particular, $\Gamma(Y)$ satisfies all identities and quasi-identities in $L$ that hold in $A$.

*Proof.* This follows from the inclusion $\text{Pvar}(A) \subseteq \text{Qvar}(A)$. \hfill \Box

**Corollary 3.28.** Let $S$ be a consistent system of equations over $A$-algebra $B$. Then the coordinate algebra $\Gamma_B(S)$ is an $A$-algebra too.

*Proof.* All algebras from $\text{Pvar}_A(B)$ except the trivial algebra $E$ are $A$-algebras [18, Corollary 3.16]. If $\Gamma_B(S) = E$ and $S$ is consistent, then the empty set $\emptyset$ is not algebraic over $B$. Hence, by Lemma 3.21, $B$ has a trivial $L_A$-subalgebra. This is possible if and only if $A \cong E$. \hfill \Box

The asserted connection between the classification of algebraic sets up to isomorphism and the classification of them coordinate algebras will be discussed in Sec. 5, and now let us prove the first two results in this direction.

**Lemma 3.29.** Let $Y$ and $Z$ be algebraic sets in $A^n$ such that $Y \subseteq Z$. Then there exists an epimorphism $h: \Gamma(Z) \to \Gamma(Y)$. Moreover, if the inclusion $Y \subseteq Z$ is strict, then the epimorphism $h$ is proper.

*Proof.* As $Y \subseteq Z$, we see that $\text{Rad}(Y) \supseteq \text{Rad}(Z)$, i.e., $\theta_{\text{Rad}(Y)} \geq \theta_{\text{Rad}(Z)}$. Hence, there exists the natural epimorphism $h: \Gamma(Z) \to \Gamma(Y)$. If $Y \neq Z$, then $\text{Rad}(Y) \neq \text{Rad}(Z)$, whence the epimorphism $h$ is not a monomorphism. \hfill \Box

**Lemma 3.30.** Let $Y \subseteq A^n$ and $Z \subseteq A^m$ be algebraic sets over $A$. Suppose there exists an epimorphism $h: \Gamma(Z) \to \Gamma(Y)$. Then there exists an algebraic subset $Y' \subseteq Z$ with $\Gamma(Y) \cong \Gamma(Y')$. Moreover, if $h$ is proper, then the inclusion $Y' \subseteq Z$ is strict.

*Proof.* Let us introduce the notation

$$\Gamma(Z) = \mathcal{T}_L(\{x'_1, \ldots, x'_m\})/\theta_{\text{Rad}(Z)}.$$  

Since $h$ is an epimorphism, the coordinate algebra $\Gamma(Y)$ is generated by the set

$$X' = \{h(x'_i/\theta_{\text{Rad}(Z)}), \ i = 1, m\}.$$  

Thus, there exists $S' \subseteq \text{At}_L(X')$ such that $\Gamma(Y) \cong \langle X' \mid S' \rangle$. It is obvious that $[S'] \supseteq \text{Rad}(Z)$. By Corollary 3.24, $[S']$ is a radical ideal over $A$. Hence, for the algebraic set $Y' = V_A(S')$ we have $\text{Rad}(Y') = [S']$. It is clear that $\Gamma(Y) \cong \Gamma(Y')$. Since $[S'] \supseteq \text{Rad}(Z)$, we have the inclusion $Y' \subseteq Z$. Furthermore, if $h$ is proper, then the inclusion $[S'] \supseteq \text{Rad}(Z)$ is strict, and, therefore, the inclusion $Y' \subseteq Z$ is also strict. \hfill \Box
3.4. The Zariski Topology and Irreducible Sets. There are three perspectives for investigation in the algebraic geometry over a given algebra $A$: algebraic, geometric, and logic. The geometric approach is connected with the examination of the affine space $A^n$ as a topological space.

Following [3], we define the Zariski topology on $A^n$, where algebraic sets over $A$ form a pre-basis of closed sets, i.e., closed sets in this topology are obtained from the algebraic sets by finite unions and arbitrary intersections.

Remark 3.31. Suppose that $A_1$ and $A_2$ are algebras with the same universe set $A$. Let $T_i$ be the family of algebraic sets $Y \subseteq A^n$ over $A_i$, $i = 1, 2$. In general, $T_1$ and $T_2$ are different families. Then the affine space $A^n$ possesses two Zariski topologies. For the sake of good order we assume everywhere below that the language $L$ and $L$-algebra $A$ with universe $A$ are fixed.

For a subset $Y \subseteq A^n$, we denote by $\bar{Y}$ its closure in the Zariski topology on $A^n$ and by $Y^{ac}$ the least algebraic set over $A$ that contains $Y$. It is clear that $Y^{ac} = V_A(\text{Rad}(Y)) = \bigcap_{Y \subseteq Z} \{Z, \text{Z is algebraic set over } A\}$.

In the classic algebraic geometry, when $A$ is a field, the notions of $\bar{Y}$ and $Y^{ac}$ coincide, because in this case all sets closed in the Zariski topology are algebraic. Algebraic structures with such property are called equational domains. We discuss equational domains in one of the next articles on the universal algebraic geometry.

In the general case, we have only the inclusion $\bar{Y} \subseteq Y^{ac}$ for a subset $Y \subseteq A^n$. It is clear that $\bar{Y} = Y^{ac}$ if and only if $\bar{Y}$ is an algebraic set. Lemma 3.33 below shows that the identity $\bar{Y} = Y^{ac}$ holds for every irreducible set $Y$.

Definition 3.32. A subset $\emptyset \neq Y \subseteq A^n$ is called irreducible if for all closed subsets $Y_1, Y_2 \subseteq A^n$ the inclusion $Y \subseteq Y_1 \cup Y_2$ implies $Y \subseteq Y_1$ or $Y \subseteq Y_2$; otherwise, it is called reducible. The empty set is not considered to be irreducible.

For example, any singleton set $\{p\}, p \in A^n$, is irreducible.

A nonempty closed set is irreducible if and only if it is not a union of two proper closed subsets. In an arbitrary topological space $(W, \mathcal{T})$, a subset $Y \subset W$ is irreducible if and only if its closure $\bar{Y}$ is irreducible. In our case, this fact has the following specification.

Lemma 3.33. Let $Y \subseteq A^n$ be an irreducible set. Then $\bar{Y} = Y^{ac}$, i.e., $\bar{Y}$ is an irreducible algebraic set over $A$. In particular, every closed irreducible subset $Y \subseteq A^n$ is algebraic.

Proof. Let $\bar{Y} = \bigcap_{i \in I} \{Y_1^i \cup \cdots \cup Y_{m_i}^i\}$, where $Y_j^i$ are algebraic sets. For each $i \in I$ we have $Y \subseteq Y_1^i \cup \cdots \cup Y_{m_i}^i$, whence there exists $j(i) \in \{1, \ldots, m_i\}$ such that $Y \subseteq Y_{j(i)}^i$. So, $\bar{Y} = \bigcap_{i \in I} Y_{j(i)}^i$ is an algebraic set, and, therefore, $\bar{Y} = Y^{ac}$.

In an arbitrary topological space $(W, \mathcal{T})$, every irreducible subset $Y \subset W$ is contained in a maximal irreducible subset $Z$ (this follows from the Zorn lemma, since the union of sets from an ascending chain of irreducible sets is an irreducible set), which is, of course, closed (since the closure of irreducible set is irreducible). The maximal irreducible subsets $Z \subset W$ are called irreducible components of $W$. The irreducible components cover $W$ (since every point $p \in W$ forms an irreducible set, which is contained in a maximal irreducible set). In our case, this topological fact turns into the following statement.

Lemma 3.34. Every nonempty subset $Y \subseteq A^n$ closed in the Zariski topology is a union of maximal irreducible algebraic over $A$ subsets $Y_i \subseteq Y$ that are irreducible components of $Y$. 400
Proof. Let us cover $Y$ with the induced Zariski topology by its irreducible components $\{Y_i, i \in I\}$. Each of them is a maximal irreducible closed subset in $Y$. Then $Y_i$ is closed and irreducible in $A^n$, and by Lemma 3.33, $Y_i$ is an algebraic set over $A$. 

A stronger result than Lemma 3.34 holds in the classical algebraic geometry over a field: here every nonempty closed set is a finite union of irreducible components. In the general case, such a result holds if an algebra $A$ is equationally Noetherian (we will discuss such algebras in Section 4). Anyway, Lemma 3.34 shows the importance of studying irreducible algebraic sets and, correspondingly, their coordinate algebras.

**Lemma 3.35.** Let $Y \subseteq A^n$ be a nonempty algebraic set over $A$. Then the following conditions are equivalent:

- $Y$ is irreducible;
- $Y$ is not a finite union of proper algebraic subsets.

**Proof.** It follows from the definition that if $Y$ is a finite union of proper algebraic subsets, then it is reducible. Conversely, let us assume that an algebraic set $Y$ is reducible, i.e., $Y$ is a union of two closed proper subsets: $Y = Y_1 \cup Y_2$. We can write $Y_1 = \bigcap_{i \in I} Z_i$ and $Y_2 = \bigcap_{j \in J} W_j$, where $Z_i, W_j$ are finite unions of algebraic sets. Thus, $Y = \bigcap_{i \in I, j \in J} Z_i \cup W_j$. Since $Y \neq Y_1$ and $Y \neq Y_2$, there exist $i \in I$ and $j \in J$ such $Y \notin Z_i$ and $Y \notin W_j$. Therefore, $Y = (Y \cap Z_i) \cup (Y \cap W_j)$ is a decomposition of algebraic set $Y$ into a finite union of proper algebraic subsets. 

**Corollary 3.36.** The algebraic set $V_A(\text{At}_L(X))$ is either irreducible or the empty set. Correspondingly, the trivial algebra $E$ is either the coordinate algebra of an irreducible algebraic set or $E = \Gamma(\emptyset)$.

**Proposition 3.37.** Let $A$ be an algebra in $L$. Then for a finitely generated algebra $C$ of $L$ the following conditions are equivalent:

1. $C$ is discriminated by $A$;
2. $C$ is the coordinate algebra of an irreducible algebraic set over $A$ defined by a system of equations in $L$.

**Proof.** First of all, we consider the case where $C$ is the trivial algebra $E$. By definition, $E$ is discriminated by $A$ if and only if $A$ has a trivial subalgebra. At once, by Lemma 3.21, the trivial algebra $E$ is the irreducible coordinate algebra of an irreducible algebraic set over $A$ if and only if $A$ has a trivial subalgebra.

Assume that $C \neq E$. Let us first prove the implication (2) $\implies$ (1). Suppose, to the contrary, that $C$ is the coordinate algebra of an irreducible algebraic set $Y = V(S), C \simeq T_L(X)/\theta_{\text{Rad}(Y)}$, and $C$ is not discriminated by $A$. Thus, there exist atomic formulas $(t_i = s_i) \in \text{At}_L(X), (t_i = s_i) \notin \text{Rad}(Y), i = 1, \ldots, m$, such that for any homomorphism $h: C \rightarrow A$ there exists an index $i \in \{1, \ldots, m\}$ with $h(t_i/\theta_{\text{Rad}(Y)}) = h(s_i/\theta_{\text{Rad}(Y)})$. Hence, for any $p \in Y$ there exists an index $i \in \{1, \ldots, m\}$ with $t_i^A(p) = s_i^A(p)$. Put $Y_i = V(S \cup \{t_i = s_i\}), i = 1, \ldots, m$. We have $Y = Y_1 \cup \cdots \cup Y_m$, moreover, the sets $Y_1, \ldots, Y_m$ are proper closed subsets of $Y$. This contradicts the irreducibility of $Y$.

Let us prove (1) $\implies$ (2). Since $\text{Dis}(A) \subseteq \text{Res}(A)$, we have, by Proposition 3.23, that $C = \Gamma(Y)$ for some algebraic set $Y$ over $A$ ($Y \neq \emptyset$, because $C \neq E$). To prove that $Y$ is irreducible, it suffices to reverse the argument above. Indeed, suppose that $Y = Y_1 \cup \cdots \cup Y_m$ for some proper algebraic subsets $Y_i$. From $Y_i \subseteq Y$, by Lemma 3.11, it follows that $\text{Rad}(Y) \subseteq \text{Rad}(Y_i)$. So, there exists an atomic formula $(t_i = s_i) \in \text{Rad}(Y_i) \setminus \text{Rad}(Y), i = 1, \ldots, m$. This implies that there is no homomorphism $h: C \rightarrow A$ with $h(t_i/\theta_{\text{Rad}(Y)}) \neq h(s_i/\theta_{\text{Rad}(Y)})$ for all $i = 1, \ldots, m$, which contradicts $C \in \text{Dis}(A)$.

**Corollary 3.38.** The class of all coordinate algebras of irreducible algebraic sets over an algebra $A$ coincides with $\text{Dis}(A)_\omega$. 

401
4. Equationally Noetherian Algebras

Let $\mathcal{A}$ be an algebra in a functional language $L$, and $\mathcal{B}$ be an $\mathcal{A}$-algebra.

**Definition 4.1.** An algebra $\mathcal{A}$ is called *equationally Noetherian* (with respect to $L$-equations) if for any positive integer $n$ and any system of equations $S \subseteq \text{At}_{L}(x_1, \ldots, x_n)$ there exists a finite subsystem $S_0 \subseteq S$ such that $V_{\mathcal{A}}(S) = V_{\mathcal{A}}(S_0)$.

If an $\mathcal{A}$-algebra $\mathcal{B}$ is equationally Noetherian with respect to $\mathcal{A}$-equations, then we also say that it is $\mathcal{A}$-*equationally Noetherian*.

For a given algebra $\mathcal{A}$: How can we establish if $\mathcal{A}$ is equationally Noetherian or not? The natural way to answer this question is to examine all systems of equations $S$ in order to check whether $S$ is equivalent to some of its finite subsystems or not. N. S. Romanovskii has called our attention to the question: Should we check inconsistent systems as well as consistent systems? As usual, in concrete algebraic structures the examination of inconsistent systems is trivial. However, the following problem is natural.

**Problem 4.2.** Find an algebra $\mathcal{A}$ such that every system consistent over $\mathcal{A}$ is equivalent to its finite subsystem and there exists an inconsistent system that is not equivalent to some finite subsystem over $\mathcal{A}$.

In [77], A. N. Shevlyakov has constructed an example of a commutative idempotent semigroup $\mathcal{A}$ in the language with a countable set of constants that is “equationally Noetherian with respect to consistent systems,” but is not equationally Noetherian.

The following statement gives alternative ways to examine whether $\mathcal{A}$ is equationally Noetherian.

**Statement 4.3.** For an $L$-algebra $\mathcal{A}$ the following conditions are equivalent:

1. $\mathcal{A}$ is equationally Noetherian;
2. for any finite set $X$ and any system $S \subseteq \text{At}_{L}(X)$ there exists finite system $S_0 \subseteq [S]$ such that $V_{\mathcal{A}}(S) = V_{\mathcal{A}}(S_0)$;
3. for any positive integer $n$ the Zariski topology on $\mathcal{A}^n$ is Noetherian, i.e., it satisfies the descending chain condition on closed subsets;
4. for any positive integer $n$ every chain $Y_1 \supset Y_2 \supset Y_3 \supset \ldots$ of distinct algebraic sets in $\mathcal{A}^n$ is finite;
5. every chain $\Gamma(Y_1) \rightarrow \Gamma(Y_2) \rightarrow \Gamma(Y_3) \rightarrow \ldots$ of proper epimorphisms of coordinate algebras of algebraic sets $Y_1, Y_2, Y_3, \ldots$ over $\mathcal{A}$ is finite;
6. for any finite set $X$ the set of atomic formulas $\text{At}_{L}(X)$ satisfies the ascending chain condition on radical ideals over $\mathcal{A}$.

**Proof.** The implication (1) $\implies$ (2) is trivial. To show (2) $\implies$ (1), note that for every atomic formula $c = (t = s) \in [S]$ there exists a finite subsystem $S_c \subseteq S$ such that $S_c \vdash (t = s)$. Therefore, if $V_\mathcal{A}(S) = V_\mathcal{A}(S_0)$ for a finite system $S_0 \subseteq [S]$, then one has

$$V_\mathcal{A}(S) = V_\mathcal{A}\left( \bigcup_{c \in S_0} S_c \right).$$

The equivalences (1) $\iff$ (3) and (3) $\iff$ (4) have been proved in [18, Lemma 4.11, Remark 4.8]. The implication (5) $\implies$ (4) follows from Lemma 3.29, and the converse implication (4) $\implies$ (5) from Lemma 3.30. The equivalence (4) $\iff$ (6) follows from Lemma 3.11.

As has been announced, for equationally Noetherian algebras Lemma 3.34 attains the form of a well-know theorem from classical algebraic geometry.
**Theorem 4.4** ([18]). *Let $A$ be an equationally Noetherian algebra. Then any nonempty subset $Y \subseteq A^n$ closed in the Zariski topology (in particular, any nonempty algebraic set) is a finite union of irreducible algebraic sets (irreducible components): $Y = Y_1 \cup \cdots \cup Y_m$. Moreover, if $Y_i \not\subseteq Y_j$ for $i \neq j$, then this decomposition is unique up to a permutation of components.*

Denote by $N$ the class of all equationally Noetherian $L$-algebras. Section 2 presented a list of operators. What operators from that list map $N$ to $N$?

**Statement 4.5.** *Let $A$ be an equationally Noetherian $L$-algebra. Then the following $L$-algebras are equationally Noetherian too:

1. every subalgebra of $A$;
2. every filterpower, direct power, ultrapower of $A$;
3. the coordinate algebra $\Gamma(Y)$ of an algebraic set $Y$ over $A$;
4. every algebra that is separated or discriminated by $A$;
5. every algebra from $\text{Qvar}(A)$, $\text{Ucl}(A)$;
6. every limit algebra over $A$;
7. every finitely generated algebra defined by a complete atomic type in the theory $\text{Th}_{qi}(A)$ or $\text{Th}_{v}(A)$.*

**Proof.** Item (1) is obvious. Item (2) follows from Lemma 3.8. Item (3) follows from items (1), (2) and Proposition 3.23. Item (4) is true because of $\text{Dis}(A) \subseteq \text{Res}(A) = \text{SP}(A)$ [18]. Item (5) follows from $\text{Ucl}(A) \subseteq \text{Qvar}(A) = \text{SP}_{l}(A)$ [45]. Every limit algebra over $A$ embeds into an ultrapower of $A$ [18, Corollary 5.7], which proves item (6). Every finitely generated algebra defined by a complete atomic type in the theory $\text{Th}_{qi}(A)$ (or $\text{Th}_{v}(A)$) belongs to $\text{Qvar}(A)$ (or $\text{Ucl}(A)$) [18, Lemma 4.7]. Thus, we have item (7).

So, the class $N$ is closed under ultrapowers.

**Problem 4.6.** Is the class $N$ closed under ultraproducts?

As $N$ is closed under subalgebras, the problem above is equivalent to the following problem [45].

**Problem 4.7.** Is the class $N$ axiomatizable?

The negative solution of this problem has been presented in [77] for the class of equationally Noetherian commutative idempotent semigroups in the language with countable set of constants.

**Example 4.8** (positive examples). Every algebra $A$ in the list below is $A$-equationally Noetherian:

- a Noetherian commutative ring;
- a linear group over a Noetherian ring (in particular, a free group, a polycyclic group, a finitely generated metabelian group) [3,5,32];
- a torsion-free hyperbolic group [73];
- a free solvable group [34];
- a finitely generated metabelian (or nilpotent) Lie algebra [14].

About equationally Noetherian property for the universal enveloping algebras of wreath products of Abelian Lie algebras see [70].

**Example 4.9** (negative examples). The following algebras are not equationally Noetherian:

- infinitely generated nilpotent groups [52];
- the wreath product $A \wr B$ of a non-Abelian group $A$ and an infinite group $B$ [4];
- the min-max structures $\mathcal{M}_{\mathbb{R}} = (\mathbb{R}; \max, \min, \cdot, +, -, 0, 1)$ and $\mathcal{M}_{\mathbb{N}} = (\mathbb{N}; \max, \min, +, 0, 1)$ [23].

**Lemma 4.10.** *Let $A$ be an equationally Noetherian $L$-algebra. The universal closure $\text{Ucl}(A)$ contains the trivial algebra $E$ if and only if $A$ has a trivial subalgebra.*
Proof. It is clear that $\mathcal{E} \subseteq \mathcal{A}$ implies $\mathcal{E} \in \text{Ucl}(\mathcal{A})$. Suppose that $\mathcal{A}$ does not have a trivial subalgebra. By Lemma 3.21, $V(At_L(x)) = \emptyset$. Hence, there exists a finite system $S_0 \subset At_L(x)$ such that $V(S_0) = \emptyset$, i.e., the universal formula

$$\forall x \left( \bigvee_{(t=s) \in S_0} t(x) \neq s(x) \right)$$

(3)

holds in $\mathcal{A}$. However, (3) is false in $\mathcal{E}$, so $\mathcal{E} \notin \text{Ucl}(\mathcal{A})$. □

5. The Theorem on Duality of the Category of Algebraic Sets and the Category of Coordinate Algebras

Let $L$ be a functional language and $A$ be an algebra in $L$.

In Sec. 5.1, we introduce two categories: the category $\mathbf{AS}(\mathcal{A})$ of algebraic sets over $\mathcal{A}$ and the category $\mathbf{CA}(\mathcal{A})$ of coordinate algebras of algebraic sets over $\mathcal{A}$. In Sec. 5.2, we prove that these categories are dually equivalent (Theorem 5.6). In Sec. 5.3, we discuss how Theorem 5.6 is useful when classifying algebraic sets.

5.1. The Category of Algebraic Sets and the Category of Coordinate Algebras. Objects of $\mathbf{CA}(\mathcal{A})$ are all coordinate algebras of algebraic sets over $\mathcal{A}$. For two coordinate algebras $C_1$ and $C_2$ in $\mathbf{CA}(\mathcal{A})$ the set of morphisms $\text{Mor}(C_1, C_2)$ coincides with the set $\text{Hom}(C_1, C_2)$ of all $L$-homomorphisms from $C_1$ into $C_2$. Note that the trivial algebra $\mathcal{E}$ is the terminal object in $\mathbf{CA}(\mathcal{A})$. This means that for every object $C$ in $\mathbf{CA}(\mathcal{A})$ there is exactly one morphism from $C$ to $\mathcal{E}$.

Objects of $\mathbf{AS}(\mathcal{A})$ are all algebraic sets over $\mathcal{A}$. To define morphisms in $\mathbf{AS}(\mathcal{A})$, we need the notion of a term map.

Definition 5.1. A map $\varphi: A^n \to A^m$ is called a term map if there exist terms $t_1, \ldots, t_m \in T_L(x_1, \ldots, x_n)$ such that

$$\varphi(a_1, \ldots, a_n) = (t_1^A(a_1, \ldots, a_n), \ldots, t_m^A(a_1, \ldots, a_n))$$

(4)

for all $(a_1, \ldots, a_n) \in A^n$. For two nonempty algebraic sets $Y \subseteq A^n$ and $Z \subseteq A^m$, a map $\varphi: Y \to Z$ is called a term map if it is a restriction on $Y$ of some term map $\varphi: A^n \to A^m$ such that $\varphi(Y) \subseteq Z$.

Remark 5.2. Note that a term map $\psi: A^n \to A^m$ defined by terms $s_1, \ldots, s_m \in T_L(x_1, \ldots, x_n)$ may induce the same term map $\psi: Y \to Z$ as $\varphi: Y \to Z$ above. This happens if and only if $(t_i = s_i) \in \text{Rad}(Y)$ for all $i = 1, \ldots, m$. Thereby, any term map $\varphi: Y \to Z$ is uniquely defined by an ordering set of term functions $(t'_1, \ldots, t'_m) \in T(Y)$ such that $(t'_1(p), \ldots, t'_m(p)) \in Z$ for all $p \in Y$.

As the family $\text{Mor}(Y, Z)$ of morphisms from an object $Y$ to an object $Z$ in $\mathbf{AS}(\mathcal{A})$ we take the set of all term maps $\varphi: Y \to Z$. Furthermore, $\text{id}_Y$ is the identical map on $Y$. If the empty set $\emptyset$ is algebraic over $\mathcal{A}$, then we place it into $\mathbf{AS}(\mathcal{A})$ as the initial object. This means that for every object $Y$ in $\mathbf{AS}(\mathcal{A})$ there is exactly one arrow (morphism) from $\emptyset$ to $Y$.

As usual, one can define the notion of an isomorphism in the categories $\mathbf{CA}(\mathcal{A})$ and $\mathbf{AS}(\mathcal{A})$. Thus, algebraic sets $Y \subseteq A^n$ and $Z \subseteq A^m$ are isomorphic if and only if there exist term maps $\varphi: Y \to Z$ and $\psi: Z \to Y$ such that $\psi \circ \varphi = \text{id}_Y$ and $\varphi \circ \psi = \text{id}_Z$.

Example 5.3. Let $L = \{\cdot, ^{-1}, e\}$ be the language of groups, $G$ be a group, and $Y \subseteq G^n$ be an algebraic set over $G$ for a system of equations in the extended language $L_G$ (with coefficients in $G$). Then for every element $h \in G$ the shift

$$Yh = \{ (g_1h, \ldots, g_nh) \mid (g_1, \ldots, g_n) \in Y \}$$

of $Y$ is an algebraic set over $G$ that is $L_G$-isomorphic to $Y$. Indeed, if $Y = V(S(x_1, \ldots, x_n))$, then $Yh = V(S(x_1h^{-1}, \ldots, x_nh^{-1}))$. An isomorphism between $Y$ and $Yh$ is established by term maps $\varphi, \psi: G^n \to G^n$:

$$\varphi(g_1, \ldots, g_n) = (g_1h, \ldots, g_nh), \quad \psi(g_1, \ldots, g_n) = (g_1h^{-1}, \ldots, g_nh^{-1}).$$

It is evident that $\varphi \circ \psi = \text{id}_{G^n}$ and $\psi \circ \varphi = \text{id}_{G^n}$.
Lemma 5.4. Let $\varphi: A^n \to A^m$ be a term map. Then the following holds.

1. If $Z$ is an algebraic set in $A^m$, then $\varphi^{-1}(Z)$ is an algebraic set in $A^n$.
2. The map $\varphi$ is continuous in the Zariski topology.
3. If $Y$ is an irreducible subset in $A^n$, then $\varphi(Y)$ is an irreducible subset in $A^m$.
4. Isomorphic algebraic sets are irreducible and reducible simultaneously.

Proof. Let $t_1, \ldots, t_m$ be the terms from (4). Suppose that $Z = V(S')$, where $S' \subseteq \mathbb{A}_d(x'_1, \ldots, x'_m)$. Taking $S = S'(t_1(x_1, \ldots, x_n), \ldots, t_m(x_1, \ldots, x_n))$, we have $\varphi^{-1}(Z) = V(S)$. This proves item (1). Item (2) follows from item (1), since algebraic sets form a closed pre-base of the Zariski topology. Every continuous map between topological spaces maps irreducible sets into irreducible ones, whence we have item (3). Item (4) easily follows from item (3). \qed

The following result holds in Diophantine algebraic geometry. Its proof is similar to the proof of the corresponding result in the classical algebraic geometry over a field [74].

Lemma 5.5. Let $\mathcal{A}$ be an $L$-algebra and $Y \subseteq A^n$ and $Z \subseteq A^m$ be algebraic sets over $\mathcal{A}$ defined by systems of equations with coefficients in $\mathcal{A}$. The algebraic set $Y \times Z$ is irreducible if and only if $Y$ and $Z$ are irreducible (irreducibility is considered with respect to the Zariski topology for $\mathcal{A}$ as $L_\mathcal{A}$-algebra).

Proof. Suppose that $Y$ is a reducible algebraic set, i.e., $Y$ is a finite union of proper algebraic subsets: $Y = Y_1 \cup \cdots \cup Y_d$. It follows that $Y \times Z = (Y_1 \times Z) \cup \cdots \cup (Y_d \times Z)$ is a decomposition of $Y \times Z$ into the finite union of proper algebraic subsets, whence $Y \times Z$ is reducible.

Now assume that $Z$ is an irreducible algebraic set and $Y \times Z = W_1 \cup \cdots \cup W_d$ is a decomposition of $Y \times Z$ into a finite union of proper algebraic subsets. We show that $Y$ is reducible in this case.

Every point $p \in A^n$ forms an algebraic set $\{p\}$ over $\mathcal{A}$. Moreover, algebraic sets $Z$ and $\{p\} \times Z$ are isomorphic. In particular, $\{p\} \times Z$ is an irreducible algebraic set. If $p \in Y$, then $\{p\} \times Z \subseteq W_1 \cup \cdots \cup W_d$. This implies that $\{p\} \times Z \subseteq W_i$ for some $i \in \{1, \ldots, d\}$. Denote by $Y_i$ the set $\{p \in Y \mid \{p\} \times Z \subseteq W_i\}$, $i = 1, \ldots, d$. One has $Y = Y_1 \cup \cdots \cup Y_d$ and $Y \neq Y_i$ for all $i = 1, \ldots, d$.

Let us check that $Y_i$ is an algebraic set for each $i = 1, \ldots, d$. For a point $p' \in Z$ denote by $Y_{i,p'}$ the set $\{p \in Y \mid \{p\} \times \{p'\} \subseteq W_i\}$. As the set $(Y \times \{p'\}) \cap W_i$ is algebraic and $Y_{i,p'} \times \{p'\} = (Y \times \{p'\}) \cap W_i$, we see that $Y_{i,p'}$ is an algebraic set. Finally, note that $Y_i = \bigcap_{p' \in Z} Y_{i,p'}$. Hence, $Y_i$ is an algebraic set. \qed

5.2. The Theorem on Dual Equivalence. This subsection requires basic notions and ideas of category theory. We refer to [2] everywhere.

Theorem 5.6. The category $\text{AS}(\mathcal{A})$ of algebraic sets over an algebra $\mathcal{A}$ and the category $\text{CA}(\mathcal{A})$ of coordinate algebras of algebraic sets over $\mathcal{A}$ are dually equivalent.

Proof. To prove the theorem, we need to construct a contravariant functor $F: \text{AS}(\mathcal{A}) \to \text{CA}(\mathcal{A})$, i.e., a map such that

(F1) if $\varphi: Y \to Z$ is a morphism of $\text{AS}(\mathcal{A})$, then $F(\varphi): F(Z) \to F(Y)$ is a morphism of $\text{CA}(\mathcal{A})$;
(F2) $F(\text{id}_Y) = \text{id}_{F(Y)}$ for every object $Y$ of $\text{AS}(\mathcal{A})$;
(F3) if $\psi: Z \to W$ is a morphism of $\text{AS}(\mathcal{A})$, then $F(\psi \circ \varphi) = F(\varphi) \circ F(\psi)$.

After that we need to show that $F$ is a dual equivalence. There are several equivalent definitions of dual equivalence in [2]. We take the one most convenient for our needs. A functor $F$ is a dual equivalence if

(E1) $F$ is fully faithful, i.e., for every objects $Y$ and $Z$ of $\text{AS}(\mathcal{A})$ and every morphism $h \in \text{Hom}(F(Z), F(Y))$ there is one and only one morphism $\varphi \in \text{Hom}(Y, Z)$ such that $h = F(\varphi)$;
(E2) $F$ is representative, i.e., for any object $\mathcal{C}$ of $\text{CA}(\mathcal{A})$ there is an object $Y$ of $\text{AS}(\mathcal{A})$ for which $F(Y)$ is isomorphic to $\mathcal{C}$.

To define the functor $F$, we put $F(Y) = \Gamma(Y)$ for an algebraic set $Y$ of $\text{AS}(\mathcal{A})$. Also we must define $F$ on morphisms. Let $Y$ and $Z$ be objects of $\text{AS}(\mathcal{A})$. If $Y = \varnothing$, then $F(Y) = E$. Furthermore, $\text{Mor}(Y, Z)$
has a unique arrow \( \varphi \) and \( \text{Hom}(\Gamma(Z), \mathcal{E}) \) has a unique morphism \( h \). So we put \( F(\varphi) = h \). If \( Y \neq \emptyset \) and \( Z = \emptyset \), then \( \text{Mor}(Y, Z) = \emptyset \).

Now suppose that \( Y \subseteq A^n \) and \( Z \subseteq A^m \) are nonempty algebraic sets in \( \text{AS}(A) \) and \( \varphi \in \text{Mor}(Y, Z) \) is a morphism defined by term functions \( t_1^Y, \ldots, t_m^Y \in T(Y) \). For defining the morphism \( F(\varphi) \in \text{Hom}(\Gamma(Z), \Gamma(Y)) \) it will be convenient to think of coordinate algebras as term functions algebras, which is possible due to Lemma 3.18. The algebra \( \Gamma(Z) \cong T(Z) \) generates by coordinate term functions \( x_1^Z, \ldots, x_m^Z \) on \( Z \), so it is sufficient to define a morphism \( h = F(\varphi) \) on these generators. Let us put

\[
h(x_1^Z) = t_1^Y, \ldots, h(x_m^Z) = t_m^Y.
\]

(5)

It is necessary to show that the morphism \( h \) is well defined, i.e., for each atomic formula \( t = s \) in \( m \) variables \( t(x_1^Z, \ldots, x_m^Z) = s(x_1^Z, \ldots, x_m^Z) \) implies \( t(t_1^Y, \ldots, t_m^Y) = s(t_1^Y, \ldots, t_m^Y) \). Identity \( t(x_1^Z, \ldots, x_m^Z) = s(x_1^Z, \ldots, x_m^Z) \) means that \( (t = s) \in \text{Rad}(Z) \), and since \( \varphi(Y) \subseteq Z \), then

\[
t(t_1^Y(p), \ldots, t_m^Y(p)) = s(t_1^Y(p), \ldots, t_m^Y(p))
\]

(6)

for all \( p \in Y \). This is what was required.

It is not hard to see that (F1), (F2), (F3), and (E2) hold. Let us check (E1). Suppose that \( \varphi: Y \to Z \) defined by term functions \( t_1^Y, \ldots, t_m^Y \in T(Y) \) and \( \psi: Y \to Z \) defined by term functions \( s_1^Y, \ldots, s_m^Y \in T(Y) \) are distinct morphisms of \( \text{AS}(A) \). Hence, there exists \( i \in \{1, \ldots, m\} \) such that \( t_i^Y \neq s_i^Y \). So, \( h = F(\varphi) \) and \( g = F(\psi) \) are distinct homomorphisms, because \( t_i^Y = h(x_i^Z) \neq g(x_i^Z) = s_i^Y \). Hence, the functor \( F \) is faithful.

To establish that \( F \) is full, consider an arbitrary homomorphism \( h: T(Z) \to T(Y) \) defined by (5). Term functions \( t_1^Y, \ldots, t_m^Y \) define a term map \( \varphi: Y \to A^m \). Since \( h \) is well defined, for every atomic formula \( (t = s) \in \text{Rad}(Z) \) and each point \( p \in Y \) we have identity (6), therefore, \( \varphi(Y) \subseteq Z \). Hence, \( F(\varphi) = h \), and \( F \) is full.

**Corollary 5.7.** Two algebraic sets \( Y \) and \( Z \) over algebra \( A \) are isomorphic if and only if \( \Gamma(Y) \cong \Gamma(Z) \).

**Proof.** Indeed, every fully faithful functor preserves and reflects isomorphisms [2].

**Corollary 5.8.** Let \( Y \) and \( Z \) be nonempty algebraic sets over algebra \( A \). There exists a one-to-one correspondence \( F \) between term maps from \( \text{Mor}(Y, Z) \) and \( L \)-homomorphisms from \( \text{Hom}(\Gamma(Z), \Gamma(Y)) \).

**Definition 5.9.** We say that an \( L \)-algebra \( C \) is an irreducible coordinate algebra over an \( L \)-algebra \( A \) if \( C \cong \Gamma(Y) \) for some irreducible algebraic set \( Y \) over \( A \).

If \( C \cong \Gamma(Y) \) and \( C \cong \Gamma(Z) \), then the algebraic sets \( Y \) and \( Z \) are isomorphic, by Corollary 5.7. By Lemma 5.4, isomorphic algebraic sets are irreducible and reducible simultaneously. Thus, irreducible coordinate algebras are well defined.

**Lemma 5.10.** Let \( Y \subseteq A^n \) and \( Z \subseteq A^m \) be algebraic sets over an algebra \( A \). Let \( \varphi \in \text{Mor}(Y, Z) \) and \( h \in \text{Hom}(\Gamma(Z), \Gamma(Y)) \) be morphisms such that \( F(\varphi) = h \):

\[
\begin{array}{ccc}
Y & \xrightarrow{\varphi} & Z \\
F \downarrow & & \downarrow F \\
\Gamma(Y) & \leftarrow h & \Gamma(Z)
\end{array}
\]

Then the following holds.

(1) If \( h \) is an epimorphism, then \( \varphi \) is a monomorphism.

(2) If \( \varphi \) is an epimorphism, then \( h \) is a monomorphism.

(3) Furthermore, \( h \) is a monomorphism if and only if \( \varphi(Y)^{ac} = Z \) (see Sec. 3.4).

(4) Suppose that \( Y \) is irreducible. Then \( h \) is a monomorphism if and only if \( \overline{\varphi(Y)} = Z \).

406
Proof. To rationalize the first two statements, one may refer to the appropriate results from the category theory; however, we prefer to give direct proofs. By Lemma 3.18, we may think of \( \Gamma(Y) \) and \( \Gamma(Z) \) as algebras of term functions \( T(Y) \) and \( T(Z) \).

First, suppose that \( \phi \) is not a monomorphism. Then there exist distinct points \( p_1, p_2 \in Y \) such that \( \phi(p_1) = \phi(p_2) \). Let \( p_1 = (a_1^1, \ldots, a_n^1) \) and \( p_2 = (a_1^2, \ldots, a_n^2) \). We may assume that \( a_1^1 \neq a_1^2 \). Denote by \( x^Y_1 \) the term function \( x^Y_1 : Y \to A \) defined by term \( x_1 \in T \). Then for an arbitrary term function \( t^Z \in T(Z) \) we have \( h(t^Z)(p_1) = h(t^Z)(p_2) \), so \( h(t^Z) \neq x^Y_1 \). Hence, \( x^Y_1 \notin h(T(Z)) \), and \( h \) is not an epimorphism. This proves item (1).

Item (2) follows from item (3). Let us prove (3). By definition \( h \) is injective if

\[
t^Z = s^Z \iff h(t^Z) = h(s^Z) \quad \text{for all } t^Z, s^Z \in T(Z).
\]

The identity \( \phi(Y)^{ac} = Z \) is equivalent to

\[
t^Z = s^Z \iff \phi(Y)^{ac} = s^{\phi(Y)^{ac}} \quad \text{for all } t, s \in T_L(x_1, \ldots, x_m).
\]

Furthermore, for arbitrary \( t, s \in T_L(x_1, \ldots, x_m) \) one has

\[
\begin{align*}
h(t^Z) = h(s^Z) & \iff t(y^Y_1(p), \ldots, y^Y_m(p)) = s(y^Y_1(p), \ldots, y^Y_m(p)) \quad \text{for all } p \in Y \\
& \iff t(\phi(p)) = s(\phi(p)) \quad \text{for all } p \in Y \iff (t = s) \in \text{Rad}(\phi(Y)) \\
& \iff (t = s) \in \text{Rad}(\phi(Y)^{ac}) \iff \phi(Y)^{ac} = s^{\phi(Y)^{ac}},
\end{align*}
\]

where \( y^Y_1, \ldots, y^Y_m \) are term functions that define the morphism \( \phi \). This implies item (3).

Finally suppose that \( Y \) is irreducible. Then \( \phi(Y) \) is irreducible, by Lemma 5.4 too, and by Lemma 3.33 \( \phi(Y)^{ac} = \phi(Y) \). \( \square \)

5.3. Classification of Algebraic Sets and Coordinate Algebras. It is important to remember that one of the major problems of algebraic geometry over a given algebra \( A \) lies in classifying algebraic sets over the algebra \( A \) up to isomorphism. According to Theorem 5.6, this problem is equivalent to the problem of classifying coordinate algebras over \( A \).

Suppose that we have attained a classification of coordinate algebras over \( A \). Then algebraic sets over \( A \) may be found as \( \text{Hom} \)'s. The corresponding idea is explained in Lemma 3.15. Sometimes the expression of algebraic sets in terms of \( \text{Hom} \)'s is reasonable, as in Example 5.13, sometimes not. For instance, there is a simple description of coordinate groups for equations in one variable over free metabelian group, while corresponding algebraic sets have no clear representation [63].

Besides the description of all algebraic sets over \( A \), it is very important to find a classification of irreducible algebraic sets over \( A \) and their coordinate algebras. Lemma 3.34 shows that every algebraic set may be decomposed in a union of maximal irreducible algebraic sets (irreducible components). Moreover, if the case where \( A \) is an equationally Noetherian algebra, such decomposition is finite and unique by Theorem 4.4.

Proposition 3.23 is effective for the description of coordinate algebras over \( A \), and Proposition 3.37 is helpful for the description of irreducible coordinate algebras over \( A \). In the case where \( A \) is an equationally Noetherian algebra, it is possible to take more informative results: unification theorems A and C (see Sec. 6).

The following lemma shows a way for the description of all coordinate algebras when having a classification of irreducible coordinate algebras.

Lemma 5.11. A finitely generated \( L \)-algebra \( C \) is the coordinate algebra of an algebraic set over \( L \)-algebra \( A \) if and only if it is a subdirect product of coordinate algebras of irreducible algebraic sets over \( A \).

Proof. First, suppose that \( Y \) is an algebraic set over \( A \). By Lemma 3.34, there exist irreducible algebraic sets \( Y_i, i \in I \), over \( A \) such that \( Y = \bigcup_{i \in I} Y_i \). Hence, by Lemma 3.11, we have \( \text{Rad}(Y) = \bigcap_{i \in I} \text{Rad}(Y_i) \). This
implies that there exists subdirect embedding \( \Gamma(Y) \rightarrow \prod_{i \in I} \Gamma(Y_i) \) [18, Lemma 3.1]. The converse statement is true by Corollary 3.26.

**Corollary 5.12.** Let \( \mathcal{A} \) be an equationally Noetherian \( \mathbf{L} \)-algebra. A finitely generated \( \mathbf{L} \)-algebra \( \mathcal{C} \) is the coordinate algebra of an algebraic set over \( \mathcal{A} \) if and only if it is a subdirect product of finitely many coordinate algebras of irreducible algebraic sets over \( \mathcal{A} \).

**Proof.** Theorem 4.4 implies what is required.

The following example is taken from [52]. A classification of coordinate groups over an Abelian group \( A \) has been found in that paper. Furthermore, this classification allows one to describe algebraic sets over \( A \). Also in [52], coordinate groups of irreducible algebraic sets over \( A \) have been classified.

**Example 5.13.** Let \( A \) be a fixed Abelian group and \( L_A \) be the language of Abelian groups with constants from \( A \), i.e., \( L_A = \{+, -, 0, c_a, a \in A\} \). We consider \( A \) as \( L_A \)-structure.

Recall that the period of an Abelian group \( A \) is the minimal positive integer \( m \), if it exists, such that \( mA = 0 \); and \( \infty \) otherwise. Let \( T(A) \) be the torsion part of \( A \) and \( T(A) \cong \bigoplus_p T_p(A) \) be the primary decomposition of \( T(A) \). Here and below in this example, \( p \) is a prime number. Denote by \( e(A) \) the period of \( A \), and by \( e_p(A) \) the period of \( T_p(A) \).

**Theorem** ([52]). Let \( C \) be a finitely generated \( A \)-group. Then \( C \) is the coordinate group of an algebraic set over \( A \) if and only if the following conditions holds:

1. \( C \cong A \oplus B \), where \( B \) is a finitely generated Abelian group;
2. \( e(A) = e(C) \) and \( e_p(A) = e_p(C) \) for every prime number \( p \).

Now it is easy to describe an algebraic set \( Y \) corresponding to the coordinate group \( C = A \oplus B \). Fix a primary cyclic decomposition of the group \( B \):

\[
B \cong \langle a_1 \rangle \oplus \cdots \oplus \langle a_r \rangle \oplus \langle b_1 \rangle \oplus \cdots \oplus \langle b_t \rangle,
\]

here \( a_i \)'s are generators of infinite cyclic groups and \( b_j \)'s are generators of finite cyclic groups of orders \( p_j^{m_j} \). For positive integer \( n \) denote by \( A[n] \) the set \( \{a \in A \mid na = 0\} \). By Lemma 3.15, points form algebraic set \( Y \) are in one-to-one correspondence with \( A \)-homomorphisms from \( \text{Hom}_A(A \oplus B, A) \); therefore,

\[
Y = A \oplus \cdots \oplus A \oplus A[p_1^{m_1}] \oplus \cdots \oplus A[p_t^{m_t}]^r.
\]

For a positive integer \( k \) and a prime number \( p \), we denote by \( \alpha_{p^k}(A) \) the dimension, if it exists, of the factor-group \( A[p^k]/A[p^{k-1}] \) as a vector space over the finite field with \( p \) elements and \( \infty \) otherwise.

**Theorem** ([52]). Let \( C \) be a finitely generated \( A \)-group. Then \( C \) is the coordinate group of an irreducible algebraic set over \( A \) if and only if the following conditions holds:

1. \( C \cong A \oplus B \), where \( B \) is a finitely generated Abelian group;
2. \( e(A) = e(C) \) and \( e_p(A) = e_p(C) \) for every prime number \( p \);
3. \( \alpha_{p^k}(A) = \alpha_{p^k}(C) \) for each prime number \( p \) and positive integer \( k \).

### 6. Unification Theorems for Equationally Noetherian Algebras

The following unification theorems help to describe coordinate algebras of algebraic sets. We first formulate the theorems and then prove them.

Fix a functional language \( \mathbf{L} \).

**Theorem A.** Let \( \mathcal{A} \) be an equationally Noetherian algebra in \( \mathbf{L} \). Then for a finitely generated algebra \( \mathcal{C} \) of \( \mathbf{L} \) the following conditions are equivalent:

1. \( \text{Th}_\mathbf{L}(\mathcal{A}) \subseteq \text{Th}_\mathbf{L}(\mathcal{C}) \), i.e., \( \mathcal{C} \in \text{Ucl}(\mathcal{A}) \);
2. \( \text{Th}_\mathbf{L}(\mathcal{A}) \supseteq \text{Th}_\mathbf{L}(\mathcal{C}) \);

408
(3) $C$ embeds into an ultrapower of $A$;
(4) $C$ is discriminated by $A$;
(5) $C$ is a limit algebra over $A$;
(6) $C$ is an algebra defined by a complete atomic type in the theory $\text{Th}_{\forall}(A)$ in $L$;
(7) $C$ is the coordinate algebra of an irreducible algebraic set over $A$ defined by a system of equations in the language $L$.

The following Theorem B is a particular case of Theorem A for $L = L_A$. We present it especially for the needs of Diophantine algebraic geometry and algebraic geometry with coefficients in algebra $A$.

**Theorem B** (with coefficients in $A$). Let $A$ be an algebra in a functional language $L$ and $B$ be an $A$-equationally Noetherian $A$-algebra. Then for a finitely generated $A$-algebra $C$ the following conditions are equivalent:

1. $\text{Th}_{\forall, A}(B) \subseteq \text{Th}_{\forall, A}(C)$, i.e., $C \in \text{Ucl}_A(B)$;
2. $\text{Th}_{\exists, A}(B) \supseteq \text{Th}_{\exists, A}(C)$;
3. $C$ $A$-embeds into an ultrapower of $B$;
4. $C$ is $A$-discriminated by $B$;
5. $C$ is an $A$-limit algebra over $B$;
6. $C$ is an algebra defined by a complete atomic type in the theory $\text{Th}_{\forall, A}(B)$ in the language $L_A$;
7. $C$ is the coordinate algebra of an irreducible algebraic set over $B$ defined by a system of equations with coefficients in $A$.

**Remark 6.1.** In Diophantine algebraic geometry, when $A = B$, the first two items in Theorem B can be formulated in a more precise form: $C \equiv_{\forall, A} A$ and $C \equiv_{\exists, A} A$, respectively. The notation $C \equiv_{\forall, A} A$ implies that any universal sentence in the language $L_A$ holds in $C$ if and only if it holds in $A$.

Theorem A gives the description of irreducible coordinate algebras. Theorem C below gives a description of all coordinate algebras.

**Theorem C.** Let $A$ be an equationally Noetherian algebra in $L$. Then for a finitely generated algebra $C$ of $L$ the following conditions are equivalent:

1. $C \in \text{Qvar}(A)$, i.e., $\text{Th}_{\forall}(A) \subseteq \text{Th}_{\forall}(C)$;
2. $C \in \text{Pvar}(A)$;
3. $C$ embeds into a direct power of $A$;
4. $C$ is separated by $A$;
5. $C$ is a subdirect product of finitely many limit algebras over $A$;
6. $C$ is an algebra defined by a complete atomic type in the theory $\text{Th}_{\forall}(A)$ in $L$;
7. $C$ is the coordinate algebra of an algebraic set over $A$ defined by a system of equations in the language $L$.

The following Theorem D is a particular case of Theorem C, as well as Theorem B for Theorem A.

**Theorem D** (with coefficients in $A$). Let $A$ be an algebra in a functional language $L$ and $B$ be an $A$-equationally Noetherian $A$-algebra. Then for a finitely generated $A$-algebra $C$ the following conditions are equivalent:

1. $C \in \text{Qvar}_A(B)$, i.e., $\text{Th}_{\forall_A}(B) \subseteq \text{Th}_{\forall_A}(C)$;
2. $C \in \text{Pvar}_A(B)$;
3. $C$ $A$-embeds into a direct power of $B$;
4. $C$ is $A$-separated by $B$;
5. $C$ is a subdirect product of finitely many $A$-limit algebras over $B$;
6. $C$ is an algebra defined by a complete atomic type in the theory $\text{Th}_{\forall_A}(B)$ in the language $L_A$;
7. $C$ is the coordinate algebra of an algebraic set over $B$ defined by a system of equations with coefficients in $A$. 

409
Remark 6.2. In Diophantine algebraic geometry, when \( A = \mathcal{B} \), the first two items in Theorem D can be formulated in the form \( \text{Qvar}_A(A) = \text{Qvar}_A(\mathcal{C}) \) and \( \text{Pvar}_A(A) = \text{Pvar}_A(\mathcal{C}) \), respectively.

Corollary 6.3. If an algebra \( \mathcal{A} \) is equationally Noetherian, then the following identities hold:

\[
\text{Ucl}(A)_\omega = \text{Dis}(A)_\omega, \quad \text{Qvar}(A)_\omega = \text{Pvar}(A)_\omega, \quad \text{Qvar}(A)_\omega = \mathcal{P}_\omega(\text{Ucl}(A)_\omega).
\]

Proof. The first identity follows from the equivalence (1) \( \iff \) (4) in Theorem A, the second identity — from the equivalence (1) \( \iff \) (2) in Theorem C. The third identity follows from the equivalence (1) \( \iff \) (5) in Theorem C and equivalence (1) \( \iff \) (5) in Theorem A.

Proof of Theorem A. Theorem A has been proved in [18]. Now we just give a more precise review for the case where \( \mathcal{C} \) is the trivial algebra \( \mathcal{E} \). The special case \( \mathcal{C} = \mathcal{E} \) has been omitted in the proof in [18]. By Proposition 3.37, the trivial algebra \( \mathcal{E} \) is the coordinate algebra of an irreducible algebraic set over \( \mathcal{A} \) if and only if \( \mathcal{E} \) is discriminated by \( \mathcal{A} \). By definition, \( \mathcal{E} \) is discriminated by \( \mathcal{A} \) if and only if \( \mathcal{A} \) has a trivial subalgebra. By Lemma 4.10, \( \mathcal{A} \) has a trivial subalgebra if and only if \( \mathcal{E} \in \text{Ucl}(\mathcal{A}) \). Thus, items (1), (4), and (7) are equivalent. Finally, by Proposition 2.3, items (1), (2), (3), (5), and (6) are equivalent anyway.

Proof of Theorem C. The equivalence (2) \( \iff \) (3) \( \iff \) (4) \( \iff \) (7) has been proved in Proposition 3.23. Corollary 3.27 shows that implication (7) \( \implies \) (1) holds. The equivalence (1) \( \iff \) (6) has been proved in [18, Lemma 4.7]. The implication (5) \( \implies \) (1) is easy. Indeed, every limit algebra over \( \mathcal{A} \) lies in \( \text{Ucl}(\mathcal{A}) \) [18, Corollary 5.7]. Moreover, \( \text{Ucl}(\mathcal{A}) \subseteq \text{Qvar}(\mathcal{A}) \), and the quasivariety \( \text{Qvar}(\mathcal{A}) \) is closed under operators \( \mathcal{P} \) and \( \mathcal{S} \). Thus, if an algebra \( \mathcal{C} \) is a subdirect product of limit algebras over \( \mathcal{A} \), then \( \mathcal{C} \in \text{Qvar}(\mathcal{A}) \).

By Corollary 5.12, the coordinate algebra of an algebraic set over equationally Noetherian algebra \( \mathcal{A} \) is a subdirect product of finitely many coordinate algebras of irreducible algebraic sets over \( \mathcal{A} \). According to Theorem A, coordinate algebras of irreducible algebraic sets over \( \mathcal{A} \) are limit algebras over \( \mathcal{A} \), whence we have the implication (7) \( \implies \) (5).

Now we prove the last implication (1) \( \implies \) (4). Suppose that \( \mathcal{C} \in \text{Res}(\mathcal{A}) \). It suffices to show that \( \mathcal{C} \notin \text{Qvar}(\mathcal{A}) \). Let \( X = \{x_1, \ldots, x_n\} \) be a finite set of generators of \( \mathcal{C} \) and \( \langle X \mid S \rangle \) be a presentation of \( \mathcal{C} \) in the generators \( X \), where \( S \subseteq \text{At}_L(X) \). The latter means that \( \mathcal{C} \cong \mathcal{T}_L(X)/\theta_S \).

Since \( \mathcal{A} \) does not separate \( \mathcal{C} \), there is an atomic formula \( (t = s) \in \text{At}_L(X) \), \( (t = s) \notin [S] \), such that \( h(t/\theta_S) = h(s/\theta_S) \) for any homomorphism \( h: \mathcal{C} \to \mathcal{A} \). This means that \( t^A(p) = s^A(p) \) for any point \( p \in V_\mathcal{A}(S) \), i.e., \( (t = s) \in \text{Rad}_\mathcal{A}(S) \). Since \( \mathcal{A} \) is equationally Noetherian, there exists a finite subsystem \( S_0 \subseteq S \) such that \( \text{Rad}_\mathcal{A}(S_0) = \text{Rad}_\mathcal{A}(S) \). Therefore, the following quasi-identity holds in \( \mathcal{A} \):

\[
\forall y_1 \ldots \forall y_n \left( \bigwedge_{(t_0 = s_0) \in S_0} t_0(y) = s_0(y) \longrightarrow t(y) = s(y) \right). \tag{7}
\]

On the other hand, the formula

\[
\bigwedge_{(t_0 = s_0) \in S_0} t_0(y) = s_0(y) \longrightarrow t(y) = s(y)
\]

is false in \( \mathcal{C} \) under the interpretation \( y_i \mapsto x_i \), \( i = 1, \ldots, n \), whence \( \mathcal{C} \notin \text{Qvar}(\mathcal{A}) \).

Note that if \( V_\mathcal{A}(S) = \emptyset \), then our reasoning is still true. In this case, the premise in quasi-identity (7) is identically false in \( \mathcal{A} \).

Unification theorems A and C are formulated for an equationally Noetherian algebra \( \mathcal{A} \). However, in the reasoning used for some implications in their proofs, the equationally Noetherian property is not required, namely, one has the following remark.

410
Remark 6.4. The following implications and equivalences from Theorems A and C hold for an arbitrary algebra \( A \):

\[
\text{Theorem A: } \{(4) \iff (7)\} \implies \{(1) \iff (2) \iff (3) \iff (5) \iff (6)\};
\]

\[
\text{Theorem C: } \{(5)\} \implies \{(1) \iff (6)\} \iff \{(2) \iff (3) \iff (4) \iff (7)\}.
\]

Theorem C gives a classification of coordinate algebras over an equationally Noetherian algebra \( A \) as finitely generated algebras in the quasivariety \( \text{Qvar}(A) \). Therefore, the following characterizations of quasivariety \( \text{Qvar}(K) \) of a class \( K \) of \( L \)-algebras are helpful here:

\[
\text{Qvar}(K) = \text{SP}_f(K)_e = \text{SP}_u(K) = \text{SP}_u\omega(K) = \text{SL}_s\omega(K) = \text{SP}s(K).
\]

The first of these identities is due to Malcev [45, Sec. 11, Theorem 4], and the others are due to Gorbunov [26, Corollary 2.3.4, Theorem 2.3.6].

We demonstrate an application of unification theorems on the following example.

Example 6.5. Algebraic geometry over the additive monoid of natural numbers has been studied by P. Morar and A. Shevlyakov [50,75,76]. The authors consider \( \mathbb{N} \) in several signatures \( L \). We discuss here the simplest case from these papers.

Let \( L = \{+, 0\} \) be the basic signature with binary function + and constant 0. Let \( N = (\mathbb{N}; +, 0) \) be the additive monoid of natural numbers in the language \( L \) with the obvious interpretation of symbols from \( L \).

A commutative \( L \)-monoid \( M \) is called positive if \( x + y = 0 \) implies \( x = y = 0 \) for all \( x, y \in M \). The monoid \( M \) is named monoid with cancellation if \( x + z = y + z \) implies \( x = y \) for all \( x, y, z \in M \).

Theorem ([50]). For any finitely generated \( L \)-monoid \( M \) the following condition are equivalent:

1. \( M \) is the coordinate monoid of an algebraic set over \( N \);
2. \( M \) is separated by \( N \);
3. \( M \) is a commutative positive monoid with cancellation;
4. quasi-identities
   \[
   \forall x, y \ (x + y = y + x), \quad \forall x, y \ (x + y = 0 \rightarrow x = 0), \quad \forall x, y, z \ (x + z = y + z \rightarrow x = z)
   \]
   hold in \( M \);
5. \( M \) is in the \( \text{Ucl}(N) \);
6. \( M \) is discriminated by \( N \);
7. \( M \) is the coordinate monoid of an irreducible algebraic set over \( N \).

Corollary 1. Every algebraic set over \( N \) is irreducible.

Corollary 2. \( \text{Ucl}(N) = \text{Qvar}(N) \).

7. Open Problems and Questions

We hope that this series of papers on universal algebraic geometry will be a helpful guide for creating algebraic geometry over classical algebraic structures. In the nearest future, we plan to publish the papers “Equationally Noetherian property and compactness” [19] and “Equational domains and co-domains” [20] along this series.

As has been mentioned in the Introduction, nowadays we know well the structure of algebraic sets and coordinate groups over a free group \( F \) of finite rank: irreducible coordinate groups over \( F \) are finitely generated limit groups over \( F \). In our view, there is sufficient background for studying algebraic geometry for other classical algebraic structures.
7.1. Free Semigroup (Monoid). Let \( S \) be a free non-Abelian semigroup or a free monoid.

There is a well-known result due to G. Makanin that the problem of solvability for a system of equations over a free semigroup is algorithmically solvable. Also there exist works on evaluation of the complexity for such algorithm. Nevertheless, we still do not have a good description of sets of all solutions for systems of equations over \( S \) (even for quadratic equations!) and their coordinate semigroups.

Since, by unification theorem A, the notion of coordinate algebra is equivalent to the notion of limit algebra for equationally Noetherian algebraic structures, we formulate the following problem.

**Problem 7.1.** It is interesting to find a description of limit semigroups over free non-Abelian semigroup \( S \).

7.2. Free Lie Algebra. Let \( L \) be the free Lie algebra of finite rank \( r \geq 2 \) over a field \( k \).

**Problem 7.2.** It is interesting to develop general techniques for solving equations over \( L \) and develop the algebraic geometry over \( L \).

The guidelines for solving this problem have been set up in [3, 18, 52]. In [22], E. Daniyarova and V. Remeslennikov produced results that are specific for the free Lie algebra \( L \) (see Example 3.5 of current paper). The so-called bounded algebraic geometry over a free Lie algebra \( L \) has been completely examined in [22]. It turns out that the algebraic geometry over \( L \) contains the totally Diophantine algebraic geometry of the ground field \( k \).

It is well known that quadratic equations, their algebraic sets, and coordinate groups have played a significant role in the creation of the algebraic geometry over the free group \( F \). In our view, in the case of a free Lie algebra, linear equations may play a similar role.

By \( U(L) \) we denote the universal enveloping algebra of \( L \). The algebra \( L \) possesses the natural structure of a \( U(L) \)-module.

**Definition 7.3.** An equation of the form
\[
x_1 \varphi_1 + x_2 \varphi_2 + \cdots + x_n \varphi_n = w,
\]
where \( w \in L \) and \( \varphi_i \in U(L), i = 1, n, \) is called a *linear* equation over \( L \).

Let us note that every expression \( x \varphi \) \((\varphi \in U(L))\) may be written as a sum of terms in the form
\[
\ldots \left[ [x, v_1], v_2 \right], \ldots , v_m \right] , \quad v_1, \ldots , v_n \in L.
\]

V. Remeslennikov and R. Stöhr in [66] have demonstrated that the structure of the solution of so simple an equation as \([x, a] + [y, b] = 0\), \(a, b \in L, a \neq b\), is complicated. However, the coordinate algebra for the equation \([x, a] + [y, b] = 0\) may be calculated quite easily.

**Problem 7.4.** It is interesting to develop specific techniques for solving linear equations over \( L \), and find the corresponding algebraic sets and coordinate algebras.

7.3. Free Associative Algebra. Let \( A \) be a free associative algebra of finite rank \( r \geq 2 \) over a field \( k \).

We know almost nothing about solutions of systems of equations over \( A \). Thus, we present the following “testing problem” for realizing the algebraic geometry over \( A \).

**Problem 7.5.** It is interesting to develop the bounded algebraic geometry over \( A \) in such a manner as has been done over the free Lie algebra \( L \).

7.4. Equationally Noetherian Property. Unification theorems show that the most perspective algebras for investigation into algebraic geometry are equationally Noetherian algebras. Thus, we present the following open problems on the equationally Noetherian property for some classical algebras.

**Problem 7.6.** Is the free non-Abelian Lie algebra of finite rank over a field equationally Noetherian or not?

**Problem 7.7.** Is the free non-Abelian associative algebra of finite rank over a field equationally Noetherian or not?

**Problem 7.8.** When is the free product of equationally Noetherian groups equationally Noetherian?
REFERENCES

1. K. I. Appel, “One-variable equations in free groups,” Proc. Am. Math. Soc., 19, 912–918 (1968).
2. M. Barr and C. Wells, “Toposes, triples and theories,” Theory Appl. Categ., 1, 1–289 (2005).
3. G. Baumslag, A. Myasnikov, and V. Remeslennikov, “Algebraic geometry over groups. I. Algebraic sets and ideal theory,” J. Algebra, 219, 16–79 (1999), http://ofim.okno.ru/~remesl/articles/algeom1.pdf.
4. G. Baumslag, A. Myasnikov, and V. Roman’kov, “Two theorems about equationally Noetherian groups,” J. Algebra, 194, 654–664 (1997).
5. R. Bryant, “The verbal topology of a group,” J. Algebra, 48, 340–346 (1977).
6. M. Casals-Ruiz and I. Kazachkov, “Elements of algebraic geometry and the positive theory of partially commutative groups,” Can. J. Math., 62, No. 3, 481–519 (2010), arXiv:math.GR/0710.4077.
7. M. Casals-Ruiz and I. Kazachkov, On Systems of Equations over Free Partially Commutative Groups, Preprint, arXiv:math.GR/0810.4867.
8. M. Casals-Ruiz and I. Kazachkov, On Systems of Equations over Free Products of Groups, Preprint, arXiv:math.GR/0903.2096.
9. C. Champetier and V. Guirardel, “Limit groups as limits of free groups: Compactifying the set of free groups,” Israel J. Math., 146, 1–76 (2005), arXiv:math.GR/0401042.
10. O. Chapuis, “∀-free metabelian groups,” J. Symb. Logic, 62, 159–174 (1997).
11. I. V. Chirkov and M. A. Shevelin, “Zero divisors in amalgamated free products of Lie algebras,” Sib. Math. J., 45, No. 1, 188–195 (2004).
12. I. M. Chiswell and V. N. Remeslennikov, “Equations in free groups with one variable,” J. Group Theory, 3, No. 4, 445–466 (2000).
13. E. Daniyarova, Algebraic Geometry over Free Metabelian Lie Algebras. III. Q-Algebras and the Coordinate Algebras of Algebraic Sets, Preprint, OmGU, Omsk (2005).
14. E. Daniyarova, “Foundations of algebraic geometry over Lie algebras,” Vestn. Omsk. Univ. Kombin. Metody v Algebre i Logike, 8–39 (2007); Preprint No. 131, Inst. Math. SB RAS (2004).
15. E. Daniyarova, I. Kazachkov, and V. Remeslennikov, “Algebraic geometry over free metabelian Lie algebras. I. U-algebras and universal classes,” J. Math. Sci., 135, No. 5, 3292–3310 (2006), arXiv:math.AG/07103871.
16. E. Daniyarova, I. Kazachkov, and V. Remeslennikov, “Algebraic geometry over free metabelian Lie algebras. II. Finite fields case,” J. Math. Sci., 135, No. 5, 3311–3326 (2006), arXiv:math.AG/07103872.
17. E. Yu. Daniyarova, I. V. Kazachkov, and V. N. Remeslennikov, “Semidomains and metabelian product of metabelian Lie algebras,” J. Math. Sci., 131, No. 6, 6015–6022 (2005), arXiv:math.AG/07103873.
18. E. Daniyarova, A. Miasnikov, and V. Remeslennikov, “Unification theorems in algebraic geometry,” Algebra Discrete Math., 1, 80–112 (2008), arXiv:math.AG/08082522.
19. E. Daniyarova, A. Miasnikov, and V. Remeslennikov, “Algebraic geometry over algebraic structures. III. Equationally Noetherian property and compactness,” Southeast Asian Bull. Math., 35, No. 1, 35–68 (2011), arXiv:math.AG/10024243.
20. E. Daniyarova, A. Miasnikov, and V. Remeslennikov, “Algebraic geometry over algebraic structures. IV. Equational domains and co-domains,” Algebra Logika, 49, No. 6, 715–756 (2010).
21. E. Yu. Daniyarova and I. V. Oskul, “Linear and bilinear equations over a free anticommutative algebra,” Vestn. Omsk. Univ. Kombin. Metody v Algebre i Logike, 38–49 (2008).
22. E. Daniyarova and V. Remeslennikov, “Bounded algebraic geometry over free Lie algebras,” Algebra Logic, 44, No. 3, 148–167 (2005), http://ofim.okno.ru/~remesl/articles/bounded_eng.pdf.
23. Yu. S. Dvorzhetsky and M. V. Kotov, “Min-max algebraic structures,” Vestn. Omsk. Univ. Kombin. Metody v Algebre i Logike, 130–136 (2008).
24. D. Eizenbud, Commutative Algebra with a View towards Algebraic Geometry, Graduate Texts Math., Vol. 150, Springer, Berlin (1995).
25. A. Gaglione and D. Spellman, “Some model theory of free groups and free algebras,” *Houston J. Math.*, **19**, 327–356 (1993).

26. V. A. Gorbunov, *Algebraic Theory of Quasivarieties*, Plenum (1998).

27. R. I. Grigorchuk and P. F. Kurchanov, “On quadratic equations in free groups,” *Contemp. Math.*, **131**, No. 1, 159–171 (1992).

28. D. Groves, “Limits of (certain) CAT(0) groups. I. Compactification,” *Algebraic Geometric Topology*, **5**, 1325–1364 (2005), [arXiv:math.GR/0404440](http://arxiv.org/abs/math.GR/0404440).

29. D. Groves, *Limits of (Certain) CAT(0) Groups. II. The Hopf Property and the Shortening Argument*, Preprint (2004), [arXiv:math.GR/0408080](http://arxiv.org/abs/math.GR/0408080).

30. D. Groves, *Limit Groups for Relatively Hyperbolic Groups. I. The Basic Tools*, Preprint (2004), [arXiv:math.GR/0412492](http://arxiv.org/abs/math.GR/0412492).

31. D. Groves, “Limit groups for relatively hyperbolic groups. II. Makanin–Razborov diagrams,” *Geom. Topol.*, **9**, 2319–2358 (2005), [arXiv:math.GR/0503045](http://arxiv.org/abs/math.GR/0503045).

32. V. Guba, “Equivalence of infinite systems of equations in free groups and semigroups to finite subsystems,” *Mat. Zametki*, **40**, No. 3, 321–324 (1986).

33. V. Guirardel, “Limit groups and group acting freely on $\mathbb{R}^n$-trees,” *Geom. Topol.*, **8**, 1427–1470 (2004), [arXiv:math.GR/0306306](http://arxiv.org/abs/math.GR/0306306).

34. C. K. Gupta and N. S. Romanovskii, “The property of being equationally Noetherian for some soluble groups,” *Algebra Logic*, **46**, No. 1, 28–36 (2007).

35. C. K. Gupta and E. I. Timoshenko, “Partially commutative metabelian groups: Centralizers and elementary equivalence,” *Algebra Logic*, **48**, No. 3, 173–192 (2009).

36. R. Hartshorne, *Algebraic Geometry*, Graduate Texts Math., Vol. 52, Springer, Berlin (1977).

37. E. Hrushovski, “The Mordell–Lang conjecture for function fields,” *J. Amer. Math. Soc.*, **9**, 667–690 (1996).

38. O. Kharlampovich and A. Myasnikov, “Irreducible affine varieties over free group I: Irreducibility of quadratic equations and Nullstellensatz,” *J. Algebra*, **200**, No. 2, 472–516 (1998).

39. O. Kharlampovich and A. Myasnikov, “Irreducible affine varieties over free group II: Systems in triangular quasi-quadratic form and description of residually free groups,” *J. Algebra*, **200**, No. 2, 517–570 (1998).

40. O. Kharlampovich and A. Myasnikov, “Algebraic geometry over free groups: Lifting solutions into generic points,” *Contemp. Math.*, **378**, 213–318 (2005), [arXiv:math.GR/0407110](http://arxiv.org/abs/math.GR/0407110).

41. O. Kharlampovich and A. Myasnikov, “Elementary theory of free nonabelian groups,” *J. Algebra*, **302**, No. 2, 451–552 (2006).

42. M. V. Kotov, “Equationally Noetherian property and close properties,” *Southeast Asian Bull. Math.*, **35**, No. 3, 419–429 (2011).

43. R. C. Lyndon, “Groups with parametric exponents,” *Trans. Am. Math. Soc.*, **96**, 518–533 (1960).

44. G. Makanin, “Equations in free groups,” *Izv. Akad. Nauk SSSR, Ser. Mat.*, **46**, No. 6, 1199–1273 (1982).

45. A. I. Malcev, *Algebraic Structures* [in Russian], Nauka, Moscow (1970).

46. D. Marker, *Model Theory: An Introduction*, Springer, New York (2002).

47. J. McCool and A. Pietrowski, “Some finitely presented subgroups for the automorphism group of a free group,” *J. Algebra*, **35**, 205–213 (1975).

48. A. A. Mishchenko, “Universal equivalence of partially commutative nilpotent $\mathbb{Q}$-groups of class 2,” *Vestn. Omsk. Univ. Kombin. Metody v Algebre i Logike*, 61–68 (2008).

49. A. A. Mishchenko and A. V. Treyer, “Commuting graphs for partially commutative nilpotent $\mathbb{Q}$-groups of class 2,” *Sib. Electron. Math. Rep.*, **4**, 460–481 (2007).

50. P. Morar and A. Shevlyakov, “Algebraic geometry over the additive monoid of natural numbers: systems of coefficient free equations,” in: O. Bogopolski, ed., et al., *Combinatorial and Geometric Group Theory. Dortmund and Ottawa–Montreal Conferences. Selected papers of the conferences on*
“Combinatorial and geometric group theory with applications” (GAGTA), Dortmund, Germany, August 27–31, 2007, “Fields workshop in asymptotic group theory and cryptography,” Ottawa, Canada, December 14–16, 2007, and the workshop on “Action on trees, non-Archimedian words, and asymptotic cones,” Montreal, Canada, December 17–21, 2007, Trends Math., Birkhäuser, Basel (2010), pp. 261–278 (2010).

51. A. Myasnikov and V. Remeslennikov, “Exponential groups 2: Extension of centralizers and tensor completion of CSA-groups,” Int. J. Algebra Comput., 6, No. 6, 687–711 (1996), arXiv:math.GR/9507203.

52. A. Myasnikov and V. Remeslennikov, “Algebraic geometry over groups II: Logical foundations,” J. Algebra, 234, 225–276 (2000), http://ofim.okno.ru/~remesl/articles/algeom2.pdf.

53. A. Myasnikov, V. Remeslennikov, and D. Serbin, “Regular free length functions on Lyndon’s free \(Z(t)\)-group \(F_{Z(t)}\),” Contemp. Math., 378, 37–77 (2005), http://ofim.okno.ru/~remesl/articles/lyndon.pdf.

54. A. Myasnikov and N. Romanovskii, Krull Dimension of Solvable Groups, Preprint (2008), arXiv:math.GR/0808.2932.

55. B. Plotkin, “Varieties of algebras and algebraic varieties,” Israel J. Math., 96, No. 2, 511–522 (1996).

56. B. Plotkin, “Varieties of algebras and algebraic varieties. Categories of algebraic varieties,” Sib. Adv. Math., 7, No. 2, 64–97 (1997).

57. B. Plotkin, “Algebras with the same (algebraic) geometry,” Proc. Steklov Inst. Math., 242, 165–196 (2003), arXiv:math.GM/0210194.

58. A. Razborov, “On systems of equations in a free groups,” Izv. Akad. Nauk SSSR, Ser. Mat., 48, No. 4, 779–832 (1982).

59. A. Razborov, “On systems of equations in a free groups,” in: A. J. Duncan, N. D. Gilbert, and J. Howie, eds., Combinatorial and Geometric Group Theory. Edinburgh 1993, London Math. Soc. Lect. Notes Ser., Vol. 204, Cambridge Univ. Press, Cambridge (1995), pp. 269–283.

60. V. Remeslennikov, “∃-free groups,” Sib. Math. J., 30, No. 6, 998–1001 (1989), http://ofim.okno.ru/~remesl/articles/efreegroups1.pdf.

61. V. Remeslennikov, “Dimension of algebraic sets in free metabelian groups,” Fundam. Prikl. Mat., 7, No. 3, 873–885 (2001).

62. V. Remeslennikov and N. Romanovskii, “Metabelian products of groups,” Algebra Logic, 43, No. 3, 190–197 (2004), http://ofim.okno.ru/~remesl/articles/remrom2_eng.pdf.

63. V. Remeslennikov and N. Romanovskii, “Irreducible algebraic sets in metabelian groups,” Algebra Logic, 44, No. 5, 336–347 (2005), http://ofim.okno.ru/~remesl/articles/remrom3_eng.pdf.

64. V. Remeslennikov and R. Stöhr, “On the quasivariety generated by a non-cyclic free metabelian group,” Algebra Colloq., 11, 191–214 (2004), http://ofim.okno.ru/~remesl/articles/remstohr1.pdf.

65. V. Remeslennikov and R. Stöhr, “On algebraic sets over metabelian groups,” J. Group Theory, 8, 491–513 (2005), http://ofim.okno.ru/~remesl/articles/remstohr2.pdf.

66. V. Remeslennikov and R. Stöhr, “The equation \([x, u] + [y, v] = 0\) in free Lie algebras,” Int. J. Algebra Comput., 17, No. 5/6, 1165–1187 (2007), http://ofim.okno.ru/~remesl/articles/remstohr3.pdf.

67. V. Remeslennikov and E. Timoshenko, “On topological dimension of \(u\)-groups,” Sib. Math. J., 47, No. 2, 341–354 (2006), http://ofim.okno.ru/~remesl/articles/topdim_eng.pdf.

68. N. Romanovskii, “Algebraic sets in metabelian groups,” Algebra Logic, 46, No. 4, 274–280 (2007).

69. N. Romanovskii, “Equational Noetherianness of rigid soluble groups,” Algebra and Logic, 48, No. 2, 147–160 (2009).

70. N. S. Romanovskii and I. P. Shestakov, “Equationally Noetherism for universal enveloping algebras of wreath products of Abelian Lie algebras,” Algebra Logic, 47, No. 4, 269–278 (2008).

71. Z. Sela, “Diophantine geometry over groups I: Makanin–Razborov diagrams,” Publ. Math. IHES, 93, 31–105 (2001).
72. Z. Sela, “Diophantine geometry over groups VI: The elementary theory of a free group,” *GAFA*, 16, 707–730 (2006).
73. Z. Sela, *Diophantine Geometry over Groups VII: The Elementary Theory of a Hyperbolic Group*, Preprint.
74. I. R. Shafarevich, *Basic Algebraic Geometry*, Berlin, Springer (1974).
75. A. N. Shevlyakov, “Algebraic geometry over natural numbers. The classification of coordinate monoids,” *Groups, Complexity Cryptology*, 2, No. 1, 91–111 (2010).
76. A. N. Shevlyakov, “Algebraic geometry over the additive monoid of natural numbers: Irreducible algebraic sets,” *Tr. Inst. Mat. Mekh. UrO RAN*, 16, No. 4, 258–269 (2010).
77. A. N. Shevlyakov, “Commutative idempotent semigroups at the service of the universal algebraic geometry,” *Southeast Asian Bull. Math.*, 35, No. 1, 111–136 (2011).
78. E. I. Timoshenko, “Universal equivalence of partially commutative metabelian groups,” *Algebra Logic*, 49, No. 2, 177–196 (2010).

Evelina Daniyarova  
Institute of Mathematics, Siberian Branch of the Russian Academy of Science,  
644099, Pevtsova 13, Omsk, Russia  
E-mail: evelina.omsk@list.ru

Alexei Myasnikov  
Schaefer School of Engineering and Science, Department of Mathematical Sciences,  
Stevens Institute of Technology, Castle Point on Hudson, Hoboken NJ 07030-5991, USA  
E-mail: amiasnikov@gmail.com

Vladimir Remeslennikov  
Institute of Mathematics, Siberian Branch of the Russian Academy of Science,  
644099, Pevtsova 13, Omsk, Russia  
E-mail: remesl@ofim.oscsbras.ru