An optimal FPT algorithm parametrized by treewidth for \textsc{Weighted-Max-Bisection} given a tree decomposition as advice assuming SETH and the hardness of \textsc{MinConv}

\textbf{Abstract}

The weighted maximal bisection problem is, given an edge weighted graph, to find a bipartition of the vertex set into two sets such that their cardinality differs by at most one and the sum of the weight of the edges between vertices that are not in the same set is maximized. This problem is known to be NP-hard, even when a tree decomposition of width \( t \) and \( O(n) \) nodes is given as an advice as part of the input, where \( n \) is the number of vertices of the input graph. But, given such an advice, the problem is decidable in FPT time in \( n \) parametrized by \( t \). In particular Jansen et al. presented an algorithm with running time \( O(2^t n^3) \). Hanaka, Kobayashi, and Sone enhanced the analysis of the complexity to \( O(2^t (nt)^2) \). By slightly modifying the approach, we improve the running time to \( O(2^n t^2) \) in the RAM model, which is asymptotically optimal in \( n \) under the hardness of \textsc{MinConv}

We prove that this is also asymptotically optimal in its dependence on \( t \) assuming SETH by showing for a slightly easier problem (maximal cut) that there is no \( O(2^t n^{2-\varepsilon}) \) algorithm for any \( \varepsilon < 1 \) under SETH. This was already claimed by Hanaka, Kobayashi, and Sone but without a correct proof.

We also present a hardness result (no \( O(2^t n^{2-\varepsilon}) \) algorithm for any \( \varepsilon > 0 \)) for a broad family of subclasses of the weighted maximal bisection problem that are characterized only by the dependence of \( t \) from \( n \), more precisely, all instances with \( t = f(n) \) for an arbitrary but fixed \( f(n) \in o(\log n) \). This holds even when only considering planar graphs.

Moreover we present a detailed description of the implementation details and assumptions that are necessary to achieve the optimal running time.
1 Introduction

One well known combinatorial graph problem is the weighted maximal cut problem that is to decide whether a given non-negative-weighted undirected graph has a cut greater or equal a given value. The formal definition of a cut is as follows:

**Definition 1** (Cut, Bisection). Let $G = (V, E)$ be an undirected graph. A tuple $(V_1, V_2)$ is called a cut of $G$ iff $V_1 \cap V_2 = \emptyset$ and $V = V_1 \cup V_2$. If additionally $-1 \leq |V_1| - |V_2| \leq 1$, $(V_1, V_2)$ is called a bisection of $G$.

**Definition 2** (Size of a cut). Let $G = (V, E, w)$ be an undirected, non-negative-weighted graph. The size of a cut $(V_1, V_2)$ is given by

$$\text{size}_w(V_1, V_2) = \sum_{v_1 \in V_1, v_2 \in V_2, v_1 v_2 \in E} w(v_1 v_2)$$

The problem maximal bisection is then defined analogously to maximal cut by simply substituting “cut” with “bisection”. We will later provide a detailed encoding of the problem together with the underlying machine model. From here we assume that weights are always non-negative if not stated otherwise. Moreover, we assume that problems are weighted unless stated otherwise.

**Hardness assumptions** Before discussing the complexity of variants of the just defined problems, we introduce the complexity assumptions the hardness results are based on. The first one is a classical assumption:

**Hypothesis 3** (Strong Exponential Time Hypothesis (SETH)). Let $k$-$\text{sat}$ be the problem to decide if a formula, given in CNF with $n$ variables, where each clause has at most $k$ literals, has a satisfying variable assignment. Then it holds that for any $\varepsilon < 1$ there is a $k$ such that $k$-$\text{sat}$ cannot be solved in time $O(2^{\varepsilon n})$.

The other assumption is standard in another area: fine grained complexity. It bounds the running time of $\text{MinConv}$, which is defined as follows (where $[n-1]_0 = \{ i \in \mathbb{N}_0 : 0 \leq i \leq n-1 \}$):

**Problem 4** ($\text{MinConv}$).

**Input**: Sequences $(a_i)_{i \in [n-1]_0} \in \mathbb{Q}^{[n-1]_0}$, $(b_i)_{i \in [n-1]_0} \in \mathbb{Q}^{[n-1]_0}$

**Output**: A sequence $(c_i)_{i \in [n-1]_0}$ where for all $k \in [n-1]_0$ it holds that

$$c_i = \min \{ a_i + b_j : i \in [n-1]_0, \ j \in [n-1]_0, \ k = i + j \}$$

It is notable that $\text{MinConv}$ is a search problem, while $\text{sat}$ is a decision problem. There is a maximization variant called $\text{MaxConv}$ that is equivalent to $\text{MinConv}$ by simply multiplying all values of the sequences by $-1$. The mentioned conjecture now it that the trivial approach in time $O(n^2)$ for this problem is basically optimal:

**Hypothesis 5** ($\text{MinConv}$ hardness hypothesis (MCH) [4, Conj. 1, p. 1]). $\text{MinConv}$ cannot be solved in $O(n^{2-\varepsilon})$ for any $\varepsilon > 0$.

Note that both hypotheses lack the information of their underlying machine model – but we do not see any reason, why they should not be assumed to hold on a register machine with random access and cells of infinite size. In fact, this is often implicitly implied and the machine model is not even mentioned.
Related work Simple-Max-Cut is one of Karp’s 21 NP-complete problems [12] and thus Weighted-Max-Cut is also NP-hard. While the minimization variant of Weighted-Max-Cut can be solved in polynomial time in general [16] and Weighted-Max-Cut can be solved in polynomial time on planar graphs [9, 17], this does not apply to the bisection variants. In particular, Weighted-Max-Cut is NP-hard even on planar graphs [11] and the minimization version is NP-hard even on $d$-regular graphs for fixed $d$ [2]. Moreover both are NP-hard on unit disk graphs [5, 6]. Hanaka, Kobayashi, and Sone showed that Weighted-Max-Bisection is NP-hard for split graphs, comparability graphs, AT-free graphs, bipartite, co-bipartite, and claw-free graphs and the minimization variant is NP-hard for split graphs, and co-comparability graphs [10]. The complexity of Weighted-Max-Bisection on planar graphs is still an open question.

Jansen et al. showed that both bisection problems can be solved in polynomial time on graphs with fixed treewidth [11], given a treewidth-minimal tree decomposition with the number of nodes being linear in the number of vertices as an advice. The running time in [11] is $O(2^t n^3)$ for treewidth $t$ and $n$ vertices. Improving the analysis but without changing the algorithm, [10] achieved a running time of $O(2^t (tn)^2)$. They claim that this running time is optimal in its dependence on $t$ under SETH, but the corresponding proof is significantly incorrect at least two deductions.

If all weights are integers between 1 and some value $W$, Weighted-Max-Bisection can be solved in $O(8^t (tW)^{O(1)} n^{1.864} \log n)$ time [1] by formulating parts of the problem as MAXCONV and using a truly subquadratic algorithm for a special case that follows from the restriction on the weights [8]. Note that this implies that in particular the maximal bisection problem with unweighted edges, that is, every edge has weight 1, is solvable in $O(8^t n^{1.864} \log n)$. In the same paper it was also shown that if Weighted-Max-Bisection can be solved in time $O(f)$ on weighted trees, so can MINCONV, which is considered unlikely to be computable in $O(n^{2-\varepsilon})$ for any $\varepsilon > 0$ [3]. This statement was strengthened in [10]: if Weighted-Max-Bisection on weighted paths can be solved in $O(n^{2-\varepsilon})$ for some $\varepsilon > 0$, MINCONV can be solved in $O(n^{2-\delta})$ for some $\delta > 0$.

Lokshtanov, Marx, and Saurabh showed that under SETH for any $\varepsilon < 1$ there is no algorithm to decide Weighted-Max-Cut (and thus Weighted-Max-Bisection) in time $O(2^{t_{\varepsilon} \cdot \text{poly} \cdot |I|})$ for an instance $I$ that has treewidth $t$ where $|I|$ denotes the size of the encoding of the instance [15].

Our contribution We show that the algorithm in [10] to decide Weighted-Max-Bisection with advice with running time $O(2^t (tn)^2)$ is not optimal under SETH in terms of the dependence on $t$ thus contradicting the claim of Hanaka, Kobayashi, and Sone. We show this by presenting an algorithm with running time $O(2^t n^2)$. Moreover, we present the details that an implementation needs to be aware of in order to achieve the desired running time. This includes the discussion of the machine model, encoding of the problem, encoding and limitation of the advice, proper preprocessing, set operations, and handling arrays indexed by sets.

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1Since 1.864 is a rounded value of the root of a polynomial, one could also use the rounding error to compensate the log factor yielding a running time of $O(8^t (tW)^{O(1)} n^{1.864})$.

2We assume that is a consequence of the following erroneous statement that is implicitly used in [10] Theorem 4, p.7: 

$$\forall f. \left( \exists \varepsilon > 0. f(n, t) \in O((2-\varepsilon)^t \cdot n^2) \right) \iff \left( \exists \varepsilon > 0. f(n, t) \in O(2^{t-\varepsilon} \cdot n^2) \right)$$
Theorem 6. There is an algorithm that decides in time $O(2^t n^2)$ if a weighted graph with $n$ vertices has a bisection greater equal than the input parameter $\gamma$, when a tree decomposition with $\leq 4(n + 1)$ nodes and treewidth $t$ is given as an advice.

Note that we assume that the tree decomposition has $O(n)$ nodes, which is also implicitly made in [10, 11]. Such a decomposition does always exist [14, Lemma 13.1.2, p. 149]. We are neither aware of the complexity of the conversion of an arbitrary tree decomposition to one with $O(n)$ nodes nor of a way to ensure this property while the construction.

Assuming SETH we show that the running time of our algorithm is optimal in its dependence on $t$ when the dependences on $n$ is not allowed to be superpolynomial. We do so by showing that this already holds for the maximal cut problem with advice. This was already claimed in [10] but without a correct proof. 3

Theorem 7. There is no $O(2^t \text{poly } n)$ algorithm to decide Weighted-Max-Cut for any $\varepsilon < 1$, where $n$ is the number of vertices of the graph, even if an advice in form of a tree decomposition with width $t$ and $4(n + 1)$ nodes is given as part of the input.

We also present a hardness result that is somewhat orthogonal to the previously known results. While Hanaka, Kobayashi, and Sone showed the hardness (no $O(n^2)$ algorithm assuming MIN-CONV) of specific graph classes that are mostly containing the class of trees in [10], we show that even subclasses of planar graphs that do not contain the class of trees are hard. Note that we assume that the advice, a tree decomposition of width $t$ with $O(n)$ nodes, is part of the instance description itself.

Theorem 8. Let $f : \mathbb{N}_0 \rightarrow \mathbb{N}$, $f(n) \in o(\log n)$. Let $\mathcal{L}$ be the set of all “yes” instances to Problem 9 restricted to instances where the corresponding graph is planar and has treewidth $t = f(n)$ where $n$ is the number of vertices of the instance. Then for all $\varepsilon > 0$ there is no $O(2^{f(n)} n^{2-\varepsilon})$ algorithm to decide $\mathcal{L}$ unless MCH fails, even if a tree decomposition of width $t$ with $O(n)$ nodes is given as advice.

Organization of this paper. Section 2 introduces necessary formalisms and notation, also including some special notations. In Section 3 we present our algorithm and show the claimed running time. Then in Section 4 we show a hardness result on weighted bisection on the family of sets that are described by the dependency of $t$ on $n$, that is $t = f(n)$ by some sensible function $f(n) \in o(\log n)$.

2 Preliminaries

An undirected (unweighted) graph $G = (V, E)$ is a set $V$ of vertices and a set $E \subseteq \binom{V}{2}$, that is a subset of the set of all cardinality 2 subsets of $V$. An (undirected) weighted graph $G = (V, E, w)$ is an undirected graph $(V, E)$ and a weight function $w : E \rightarrow \mathbb{Q}^+$. For the directed variants simply substitute $\binom{V}{2}$ by $V \times V \setminus \{(v, v) : v \in V\}$. If not stated otherwise, we assume graphs to be undirected and unweighted.

For a graph $G = (V, E)$ (and analogously for the weighted variants), we write $G - M$ for some subset $M \subseteq V$ to denote the graph $(V \setminus M, E \cap \binom{V \setminus M}{2})$, that is, the graph $G$ after deleting all the edges between $M$ and $V \setminus M$.

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3They restated the statement of [15], more precisely Theorem 3: Under SETH the problem Weighted-Max-Cut cannot be solved in time $O(2^t n^2)$ for any $t > 0$. They did not take into account that this hardness cannot be directly transferred to Weighted-Max-Cut with advice.
nodes from $V$ that are in $M$. Moreover we write $G|_M := G - (V \setminus M)$ for the restriction of $G$ to $M$. To simplify notation for $v_1 \in V$, $v_2 \in V$ we write $v_1 v_2 := \{v_1, v_2\}$ to describe a (possible) edge.

A graph that is connected and has no cycles is called a tree. $T = (V, E, r)$, $r \in V$ is a rooted tree with root $r$. For any $v \in V$ we call the subtree that is obtained by removing all nodes from $(V, E)$ that can reach $r$ without reaching $v$ on every way first, the subtree induced by $v$. If such a subtree is mentioned to be rooted, the root is $v$. For a rooted tree $(V, E, r)$, we define the predecessor relation $(\preceq) \in V \times V$ such that $v_1 \preceq v_2$ iff $v_1$ is in the subtree induced by $v_2$. Note that $(\preceq)$ is a (partial) ordering on $V$.

We assume that the natural numbers $\mathbb{N}$ do not include 0 and write $\mathbb{N}_0 = \mathbb{N} \uplus \{0\}$ for the natural numbers unioned with $\{0\}$, where $\uplus$ is the disjunctive union operation on sets. Let $\mathbb{Q}_n^+$ be the non-negative numbers within $\mathbb{Q}$, that is $\mathbb{Q}_n^+ = \{x \in \mathbb{Q} : x \geq 0\}$. The set of numbers from 1 to $n$ for an integer $n \in \mathbb{N}$ is given by $[n] \subseteq \mathbb{N}$ and $[n]_0 := [n] \uplus \{0\}$. For a set $M$ we write $\mathcal{P}(M)$ for its power set.

While $A \to B$ denotes a total function from sets $A$ to $B$, $\to$ denotes a partial function from $A \to B$. We write $\bot$ for the image of a partial function that is undefined.

In context of formal languages, when considering an instance $I$, let $|I|$ be the size of the encoding of the instance.

The underlying machine model is the RAM model, that is a random access register machine. In this model there is an infinite number of memory cells that can hold unbounded integers each. Address resolution is possible in $\mathcal{O}(1)$ as well as all operations of the x86 instruction set where one does not make a distinction between “registers” and “memory cells”\footnote{Some instructions might need some smaller modification on where to store the result, as they originally store the result in a hardcoded register that is not part of its “argument”. Also some details regarding the representation of negative numbers might have to be clarified or changed, but this does not matter for this paper as we only use positive values.}.

In contrast to MINCONV, in the context of bisection, we start with considering the decision problem:

**Problem 9** (Weighted maximal bisection).

**Input:** $(V, E \subseteq V^2, w: E \to \mathbb{Q}_n^+, \gamma \in \mathbb{Q}_n^+)$.

**Question:** Does there exist a bisection $(V_1, V_2)$ of $G = (V, E)$ such that $\text{size}_w(V_1, V_2) \geq \gamma$?

Defining the formal language for a maximal bisection decision problem contains a bunch of details one has to be aware of. First of all, we need our language to be defined in a way such that the advice in form of a tree decomposition is part of the input. Thus we first have to formally define tree decompositions:

**Definition 10** (Tree decomposition). Let $G = (V, E)$ be an undirected graph, $I$ be an arbitrary set and $X = (X_i)_{i \in I}$ be a family of sets such that for any $i \in I$ one has $X_i \subseteq V$. Moreover let $T = (I, H)$ be a tree. Then $(I, X, H)$ is called a tree decomposition of $G$ if and only if $T$ has the following properties:

1. **Node coverage:** $\bigcup X = V$
2. **Edge coverage:** $\forall v_1 v_2 \in E. \exists i \in I. \{v_1, v_2\} \subseteq X_i$
3. **Coherence:** $\forall v \in V. T - \{i \in I : v \notin X_i\}$ is connected
By convention the nodes of the graph $G$ are called \textit{vertices} while the nodes of the tree are just called \textit{nodes}. For a node $i \in I$, the set $X_i$ is called a \textit{bag}. If the node set of a decomposition is sufficiently small, more precisely if $|I| \leq 4 \cdot (|V| + 1)$, we call the decomposition \textit{small}.

A tree decomposition has a specific width, called the \textit{treewidth} of the decomposition.

\textbf{Definition 11 (Treewidth).} Let $G = (V, E)$ be an undirected graph. The \textit{treewidth} of a tree decomposition $(I, X, F)$ of $G$ is given by $\max_{i \in I} |X_i| - 1$. The \textit{treewidth} of the graph $G$, often simply denoted by \textit{treewidth}, is the minimal treewidth of a tree decomposition along all tree decompositions of $G$.

Not only computing a tree decomposition that is minimal in terms of treewidth is NP-hard but also computing the value of the (minimal) treewidth of a graph.\[1\]

We will assume that vertex sets are always given by the unary encoding of their cardinality, that is, $V = ||V||$. This assumption is reasonable as again one can transform any input in this form by some simple preprocessing step\[5\] but will mostly be able to do the transformation while constructing the input graph. The unary encoding is reasonable as a set is usually encoded by listing all its members, thus the intuitive encoding of $V$ would also require $\Theta(n)$ space.

We now need to carefully define the encoding of the instances which also includes the encoding of the advice. Note that the exact form is crucial for the running times one can achieve and is also strongly correlated the the underlying machine model.

\textbf{Definition 12 (Adv$_{STD}$-Max-Bisection).} The language Adv$_{STD}$-Max-Bisection consists of all instances to Problem 9 where the answer to the problems is \textit{“yes”}, given an additional hint in form of a small tree decomposition. The encoding is as follows: $V = ||V||$ is represented by an unary encoded integer of size $|V|$, $E$ and $w$ are encoded as a set of triples $(v_1, v_2, w(v_1v2))$ where $v_1 < v_2$. Therefore every triple is encoded in 3 memory cells, one for each entry, and the set of triples is “encoded” by simply writing triples after each other in any order. The treewidth $t$ is stored in one cell of memory. The tree decomposition $D = (I, X, H)$ is encoded as follows: Again, $I = ||I||$ is represented by an unary encoded integer of size $|I|$. For any $i \in I$, the bag $X_i$ is encoded by a increasing sequence of the vertices $v \in X_i$. The edge set $H$ is encoded analogously to the input graph.

The language Adv$_{STD}$-Max-Cut can be defined analogously. The languages Max-Bisection and Max-Cut are defined the same way but without the treewidth and tree decomposition as input, that is without advice.

It is reasonable to assume that tree decompositions are restricted in the number of nodes by $O(n)$, as otherwise some preprocessing can reduce the tree decomposition into such a form. Note that this assumption is important, as otherwise the size of the encoding of a tree decomposition might exceed the overall running time we want to achieve, implying that one cannot completely read the decomposition in our algorithm.

Formally one would also need to state how to recognize where the edge set ends etc., but since this can be done using a standard approach, we omit those details.

Instead of working with the advice we are given directly, we first convert it into another form that is easier to handle:

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5for example, use an AVL-tree for renaming and rename every element of the input in time $O(|I| \log n + n \log n)$
Definition 13 (Nice Tree Decomposition). Let \( G = (V, E) \) be an undirected graph, \( (I, X, H) \) a tree decomposition of \( G \) and \( T = (I, H, r) \) a rooted tree. Then we call \( (I, X, H, r) \) a nice tree decomposition if and only if for any \( i \in I \) the node \( i \) is of one of the following forms:

1. **Leaf node**: \( i \) has no child node in \( T \) that is, \( i \) is a leaf of the tree \( T \)
2. **Introduce node**: \( i \) has exactly one child \( j \in I \) in \( T \) and \( X_i = X_j \cup \{ v \} \) for some \( v \in V \setminus X_j \) that is, \( i \) introduces a new vertex \( v \in V \setminus X_j \)
3. **Forget node**: \( i \) has exactly one child node \( j \in I \) in \( T \) and \( X_i \cup \{ v \} = X_j \) for some \( v \in X_j \) that is, \( i \) forgets a vertex from \( X_j \)
4. **Join node**: \( i \) has exactly two child nodes \( j \in I, k \in I, j \neq k \), in \( T \) such that \( X_i = X_j = X_k \), joining two branches of the tree \( T \)

Note that this implies that \((I, H)\) is a binary tree.

The conversion of a tree decomposition into a nice tree decomposition can be done in time \( O(n t^2) \) as long as the number of nodes of the decomposition is at most linear in the number of vertices.

Lemma 14 ([13, Lemma 13.1.3, p. 150]). Given a small tree decomposition of a graph \( G \) with width \( t \), one can find a nice tree decomposition of \( G \) with width \( t \) and with at most \( 4n \) nodes in \( O(nt^2) \) time, where \( n \) is the number of vertices of \( G \).

3 The Algorithm

In this section we present the algorithm and analyze its running time. Implementation details regarding set operations and preprocessing can be found in Subsection 3.2. Until then we assume that we are able to put an element in a new set, intersect and union sets in time \( O(1) \) and are able to choose an element of a set in \( O(t) \). Moreover, we assume that the power sets are always in non-decreasing order in terms of cardinality and that constructing and ordering them does not need any relevant extra time. When iterating over a subset \( S \) of vertices within a bag, we assume that we have \( t \) candidates to check, so the running time is \( O(|S| \cdot \text{time-for-function-body} + O(t)) \).

Let \( G = (V, E) \) be an undirected graph and \( (I, X, F, r) \) be a nice tree decomposition with treewidth \( t \) of \( G \). Moreover let

\[
F_i := \bigcup_{j \prec i} X_j \setminus X_i
\]

be the set of vertices that have been forgotten when reaching \( i \) from below. Furthermore let

\[
Y_i := \bigcup_{j \preceq i} X_j
\]

be the set of vertices induced by the subgraph induced by the rooted subtree induced by \( i \). Note that \( F_i = Y_i \setminus X_i \).

The objective of our algorithm is now for every \( i \in I \) to compute the function

\[
B_i: \|F_i\|_0 \times \mathcal{P}(X_i) \to \mathbb{Q}_0^+
\]

where

\[
B_i(l, S) := \max_{S' \in \mathcal{P}(F_i) \mid |S'| = l} \text{size}_{w}(S' \cup S, (Y_i \setminus (S' \cup S)))
\] (1)

...
Join node $O(2^tn)$ per node $O(2^t|F_j \times F_k|)$ per node
Leaf node $O(2^tn)$ per node $O(2^t)$ per node
Introduce node $O(2^tn)$ per node $O(2^t|F_i|)$ per node
Forget node $O(1)$ per node $O(2^t|F_i|)$ per node

Figure 1: Running times of the computations of $B_i$ and $W_i$ for a node $i \in I$ (if there is one child $i$, let $j$ be that child, and if there are two children, let $j,k$ be those children of $i$) depending on the node’s type

that is the size of a maximal $([S] + l, |Y_i \setminus S| - l)$-partition of $Y_i$.

Once these values are computed we can use the following equation to determine the size of the cut of a weighted maximal bisection:

$$\text{MaxBisection}(G,T,X) = \max_{S \in P(X_i)} B_i(l,S)$$

In order to be able to compute the values $B_i(l,S)$ somewhat efficiently, we specialize the computation for every node type. Also – as the number of entries is roughly of the same order of magnitude as $2^tn$, we have to ensure that the needed values of size $w$ do not take too much time.

In order to ensure this, for every node $i \in I$ we compute the partial function $W_i: 2^{X_i} \rightarrow \mathbb{Q}_0^+$ that is $\neq \bot$ for at least all necessary values of the domain needed within the computation of $B_i$.

At some point we will still need the trivial approach to compute size $w$, more precisely for the cases $\text{size}_w(\{v\}, S)$ for some $S \in P(X_i)$ and some $v \in X_i \setminus S$. This can be done in $O(n)$ using the following routine:

Algorithm 1 Computing $\text{size}_w(\{v\}, S)$

1: function VERTEXWEIGHT($v, S$) $\triangleright$ total $O(n)$
2: $\nu \leftarrow 0$ $\triangleright O(1)$
3: for all $z \in \text{neigh}(v) \cap S$ do $\triangleright O(n + |S| \cdot \text{BODY})$
4: if $vz \in E$ then $\triangleright O(1), \text{adjacency matrix}$
5: $\nu \leftarrow \nu + w(v, z)$ $\triangleright O(1), \text{adjacency matrix}$
6: end if
7: end for
8: return $\nu$ $\triangleright O(1)$
9: end function

While we will compute the values for $B_i$ bottom up along the tree, the values of $W_i$ do not depend on each other. In the following for every node type we describe on how to compute $W_i$ and then $B_i$ efficiently, assuming that $B_j$ has already been computed for all $j \in \{ k \in I : k \prec i \}$.

Figure 1 shows the running times $N$ for the computation of $W_i$ and $B_i$ we are aiming for.

Leaf node Let $i \in I$ be a leaf node of $T$. Since $F_i = \emptyset$ the only well-defined case is $l = 0$. For all $S \subseteq X_i$ we have

$$B_i(0, S) = W_i(S, X_i \setminus S)$$
The corresponding code is

**Algorithm 2** Computing entries $B_i$ for a leaf node $i \in I$

1: for all $S \in \mathcal{P}(X_i)$ do  \hspace{1cm} $\triangleright \mathcal{O}(2^t \cdot \text{BODY})$
2: \hspace{0.3cm} $B_i(0, S) \leftarrow W_i(S, X_i \setminus S)$  \hspace{1cm} $\triangleright \mathcal{O}(1)$
3: end for

with the obvious running time of $\mathcal{O}(2^t)$. While the computation of $B_i$ is quite easy for leaf nodes, the computation of the entries of $W_i$ is a little more complex. Note that we need to compute $W_i(S, X_i \setminus S)$ for all $S \in \mathcal{P}(X_i)$. We use the following recursion to do so

$$W_i(\emptyset, X_i) = 0$$
$$W_i(S \uplus \{v\}, X_i \setminus \{v\}) = W_i(S, X_i) - \text{size}_w(\{v\}, S) + \text{size}_w(\{v\}, X_i \setminus (S \uplus \{v\}))$$ (4)

**Algorithm 3** Computing entries of $W_i$ for a leaf node $i \in I$

1: $W_i(\emptyset, X_i) \leftarrow 0$  \hspace{1cm} $\triangleright \mathcal{O}(1)$
2: for all $S \in \mathcal{P}(X_i) \setminus \emptyset$ do  \hspace{1cm} $\triangleright \mathcal{O}(2^t \cdot \text{BODY})$
3: \hspace{0.3cm} $v \leftarrow \text{choose}(X_i \setminus S)$  \hspace{1cm} $\triangleright \mathcal{O}(n)$
4: \hspace{0.6cm} $a_1 \leftarrow \text{vertextweight}(v, S)$  \hspace{1cm} $\triangleright \mathcal{O}(n)$
5: \hspace{0.6cm} $a_2 \leftarrow \text{vertextweight}(v, X_i \setminus (S \uplus \{v\}))$  \hspace{1cm} $\triangleright \mathcal{O}(n)$
6: \hspace{0.6cm} for all $z \in \text{neigh}(v) \cap (X_i \setminus (S \uplus \{v\}))$ do  \hspace{1cm} $\triangleright \mathcal{O}(n)$, adjacency matrix
7: \hspace{0.9cm} if $vz \in E$ then  \hspace{1cm} $\triangleright \mathcal{O}(1)$, adjacency matrix
8: \hspace{1.2cm} \hspace{0.5cm} $a_2 \leftarrow a_2 + w(vz)$
9: \hspace{0.6cm} end if
10: \hspace{0.3cm} end for
11: \hspace{0.3cm} $W_i(S \uplus \{v\}, X_i \setminus \{v\}) \leftarrow W_i(S, X_i) - a_1 + a_2$  \hspace{1cm} $\triangleright \mathcal{O}(1)$
12: end for

The running time for one iteration of the loop lines 2 to 14 is $\mathcal{O}(n + |\text{neigh}(v) \cap S|) \subseteq \mathcal{O}(n)$ and thus the total running of this computation is $\mathcal{O}(2^t n)$.

**Introduce node** Let $i \in I$ be an introduce node which introduces $v \in X_i$ and has a single child node $j \in I$. Because of the edge coverage and coherence properties the neighborhood of $v$ in $G - (V \setminus Y_i)$ is entirely contained in $X_i$. Thus for each $S \in \mathcal{P}(X_i)$ and $l \in ||F_i||_0$ we have

$$B_i(l, S) = \begin{cases} B_j(l, S \setminus \{v\}) + W_i(\{v\}, X_i \setminus S) & v \in S \\ B_j(l, S) + W_i(\{v\}, S) & \text{otherwise} \end{cases}$$ (5)
Algorithm 4 Computing entries of $B_i$ for a introduce node $i \in I$

1: for all $S \in \mathcal{P}(X_i)$ do \hfill $\triangleright \mathcal{O}(2^t \cdot \text{BODY})$
2: for all $l \in [|F_i|]_0$ do \hfill $\triangleright \mathcal{O}(|F_i| \cdot \text{BODY})$
3: if $v \in S$ then \hfill $\triangleright \mathcal{O}(1)$
4: $B_i(l, S) \leftarrow B_j(l, S \setminus \{v\}) + W_i(\{v\}, X_i \setminus S)$ \hfill $\triangleright \mathcal{O}(1)$
5: else
6: $B_i(l, S) \leftarrow B_j(l, S) + W_i(\{v\}, S)$ \hfill $\triangleright \mathcal{O}(1)$
7: end if
8: end for
9: end for

The running time is dominated by the number of iterations of the inner loop (lines 2 to 8) as all lines take $\mathcal{O}(1)$ each execution. Thus we get a running time of $\mathcal{O}(2^t|F_i|)$.

The computation of $W_i$ is quite easy in this case as we only have to determine the values for a fixed $v$:

$$W_i(\{v\}, \emptyset) = 0$$
$$W_i(\{v\}, S) = \text{size}_w(\{v\}, S)$$ \hfill (6)

Note that one would be able to inline this computation for the needed sets between line 1 and line 2 without increasing the asymptotic running time. We did not do this in order to have the computation for $B_i$ of the same form for all nodes that is precomputing $W_i$ and then combining those values and some of $B_i$, $B_j$ and $B_k$ (if there are children $j$ and maybe $k$) via a dynamic program.

Algorithm 5 Computing entries of $W_i$ for a introduce node $i \in I$

1: $W_i(\{v\}, \emptyset) \leftarrow 0$ \hfill $\triangleright \mathcal{O}(1)$
2: for all $S \in \mathcal{P}(X_i)$ do \hfill $\triangleright \mathcal{O}(2^t \cdot \text{BODY})$
3: if $v \notin S$ then \hfill $\triangleright \mathcal{O}(1)$, adjacency matrix
4: $W_i(\{v\}, S) \leftarrow \text{vertextweight}(v, S)$ \hfill $\triangleright \mathcal{O}(n)$
5: else
6: $W_i(\{v\}, S) \leftarrow 0$ \hfill $\triangleright \mathcal{O}(1)$
7: end if
8: end for

The running time for computing $W_i$ is dominated by $\mathcal{O}(2^t n)$ as desired.

Forget node Let $i \in I$ be a forget node with child node $j \in I$ and $X_j \setminus X_i = \{v\}$. For each $S \in \mathcal{P}(X_i)$ and $l \in [|F_i|]_0$ we have

$$B_i(l, S) = \begin{cases} 0 & l = 0 \\ B_j(|F_i| - 1, S \cup \{v\}) & l = |F_i| \\ \max\{B_j(l, S), B_j(l - 1, S \cup \{v\})\} & \text{otherwise} \end{cases}$$ \hfill (7)

Note that in the equation there is no occurrence of a weight, thus we do not need to compute any entry of $W_i$. 

10
Join node  

It remains to show that the running time for computing $B$ claimed. 

The running time is dominated by how often lines 4 to 9 are executed. Each execution takes $O(1)$ time.

Algorithm 6  Computing $B_i$ for a forget node $i \in I$

```
for all $S \in \mathcal{P}(X_i)$ do  \(\mathcal{O}(2^t \cdot \text{BODY})\)
\begin{align*}
&\quad B_i(0, S) \leftarrow B_j(0, S) \quad \mathcal{O}(1) \\
&\quad B_i([F_i], S) \leftarrow B_j([F_i] - 1, S \cup \{ v \}) \quad \mathcal{O}(1) \\
&\quad \text{for all } l \in \{ [F_i] - 1 \} \text{ do} \quad \mathcal{O}(|F_i|) \\
&\quad \quad B_i(l, S) \leftarrow \max \{ B_j(l, S), B_j(l - 1, S \cup \{ v \}) \} \quad \mathcal{O}(1) \\
&\quad \text{end for} \\
&\quad \text{end for}
\end{align*}
```

The running time is dominated by line 5 which is executed $2^t$ times, yielding a running time of $\mathcal{O}(2^t|F_i|)$.

Join node  

Let $i \in I$ be a join node with child nodes $j, k \in I$ and $X_i = X_j = X_k$. For each $S \in \mathcal{P}(X_i)$ and $l \in |F_i|_0$ we have

$$B_i(l, S) = \max_{l_1 + l_2 = l \atop l_1 \in |F_i|_0, l_2 \in |F_k|_0} \left( B_j(l_1, S) + B_k(l_2, S) - \text{size}_w(S, X_i \setminus S) \right)$$  \(8\)

Note that we can reuse Equation 4 and Algorithm 3 to compute the necessary entries of $W_i$ the same way we do in leaf nodes because we are interested in the exact same entries. This can be done in running time $\mathcal{O}(2^t n)$. The computation of $B_i$ does – in contrast to the other cases – not precisely reflect the structure of the corresponding equation. Translation from the corresponding equation. This is only due to our code being easier to analyze: instead of fixing a $l$ and then running through all $l_1$ and $l_2$ such that $l = l_1 + l_2$, we run through all $l_1$ and $l_2$ and set $l := l_1 + l_2$ for this iteration.

Algorithm 7  Calculating entries for a join node

```
for all $S \in \mathcal{P}(X_i)$ do  \(\mathcal{O}(2^t \cdot \text{BODY})\)
\begin{align*}
&\quad \text{for all } l_1 \in |F_j|_0 \text{ do} \quad \mathcal{O}(|F_j| \cdot \text{BODY}) \\
&\quad \quad \text{for all } l_2 \in |F_k|_0 \text{ do} \quad \mathcal{O}(|F_k| \cdot \text{BODY}) \\
&\quad \quad \text{if } l \in |F_i|_0 \text{ then} \quad \mathcal{O}(1) \\
&\quad \quad \quad \nu \leftarrow B_j(l_1, S) + B_k(l_2, S) - W_i(S, X_i \setminus S) \quad \mathcal{O}(1) \\
&\quad \quad \quad \text{if } \nu \geq B_i(l, S) \text{ then} \quad \mathcal{O}(1) \\
&\quad \quad \quad \quad B_i(l, S) \leftarrow \nu \quad \mathcal{O}(1) \\
&\quad \quad \quad \text{end if} \\
&\quad \quad \text{end if} \\
&\quad \quad \text{end for} \\
&\quad \text{end for} \\
&\quad \text{end for}
\end{align*}
```

The running time is dominated by how often lines 4 to 9 are executed. Each execution takes time $O(1)$, the number of executions is $\mathcal{O}(2^t|F_j||F_k|) = \mathcal{O}(2^t|F_j \times F_k|)$ for a single join node as claimed.

It remains to show that the running time for computing $B_i$ for all join nodes $i \in I$ in total is $\mathcal{O}(2^t n^2)$. Therefore we use an approach inspired by 10: Due to the coherence property of tree
decompositions it holds that \( F_j \cap F_k = \emptyset \). Now note that \(|F_j||F_k| = |F_j \times F_k|\). For the sake of the analysis let us annotate the set \( F_j \times F_k \) to the join nodes \( i \) and repeat this step for all join node \( i \in I \) with children \( j, k \). Note that any pair \((v_1, v_2)\) of different vertices \( v_1 \in V, v_2 \in V \setminus \{v_1\} \) can be unambiguously associated with a join node \( i \) with children \( j \) and \( k \) because without loss of generality \( v_1 \in F_j \) and \( v_2 \in F_k \) and – as observed above – \( F_j \cap F_k = \emptyset \). This means, if we add up the running time \( O(2^t|F_j \times F_k|) \) for every join node \( i \) with children \( j, k \), we can never exceed \( O(2^t|V \times V|) = O(2^t n^2) \).

### 3.1 Total running time

As the bounds listed in Figure 1 hold, we know that

1. \( \mathcal{W}_i \) can be computed in time \( O(2^t n) \) per node \( i \in I \) no matter the node type (note that \( |F_i| \leq n \))
2. \( \mathcal{B}_i \) can be computed in time \( O(2^t n) \) per node \( i \in I \) if the node is not a join node
3. \( \mathcal{B}_i \) can be computed in total time \( O(2^t n^2) \) for all join nodes \( i \in I \)

This implies that we can compute the whole table \( \mathcal{B} \) in time \( O(2^t n^2) \) if the table \( \mathcal{W} \) is given. The table \( \mathcal{W} \) can be computed in \( O(2^t n^2) \) also, yielding an overall running time of \( O(2^t n^2) \) to build up \( \mathcal{B} \) including all preprocessing steps. All that is left to do now is to use this table to compute the value of a maximal bisection\(^6\) following Equation 2. Thus we end up with the following code:

---

**Algorithm 8** Computing the value of a weighted maximal bisection

**Input:** A graph \( G = (V,E) \) with weight function \( w: E \rightarrow \mathbb{Q}^+ \) and a tree decomposition \((I,X,F)\)

1. Convert the tree decomposition to a nice tree decomposition \((I,X,F,r)\) \(\triangleright O(n t^2)\)
2. for all \( i \in I \) do \(\triangleright O(n \cdot \text{BODY})\)
3. \( \mathcal{W}_i \) \(\triangleright O(2^t n)\)
4. end for
5. Compute a topological ordering \( \sigma \) of the nodes \( I \) of the tree \( T \) where the edges are in inverted direction in comparison to the direction induced by the root \( r \) \(\triangleright O(n^2)\)
6. for all \( i \in \left\lfloor |I| \right\rfloor \) do \(\triangleright \text{overall } O(2^t n^2) \text{ for this loop}\)
7. \( \mathcal{B}_{\sigma(i)} \)
8. end for
9. \( \nu \leftarrow \bot \)
10. for all \( S \in \mathcal{P}(X_r) \) do \(\triangleright O(2^t \cdot \text{BODY})\)
11. for all \( l \in \left\lfloor |F_r| \right\rfloor \) do \(\triangleright O(|F_r| \cdot \text{BODY}) \subseteq O(n \cdot \text{BODY})\)
12. if \( |S| + l = \lfloor n/2 \rfloor \) then \(\triangleright O(l)\)
13. if \( \mathcal{B}_r(l,S) > \nu \) then \(\triangleright O(1)\)
14. \( \nu \leftarrow \mathcal{B}_r(l,S) \) \(\triangleright O(1)\)
15. end if
16. end if
17. end for
18. end for
19. return \( \nu \) \(\triangleright O(1)\)

---

\(^6\)Formally, we have to decide whether a bisection of size \( \geq \gamma \) exists, but since we compute the biggest value over all bisections, answering this question is then trivial.
It is easy to see that the first loop (lines 2 to 4) takes time $O(2^t n^2)$ as $|I| \in O(n)$. The second loop (lines 6 to 8) takes time $O(2^t n^2)$ as we showed before. The third loop (lines 10 to 18) has a nested loop which itself has a nested statement that is $O(n)$. As all other lines except line 12 within this loop only take $O(1)$, the running time of the third loop is dominated by the number of times line 12 is executed times $O(t)$. This yields $O(2^t nt) \subseteq O(2^t n^2)$. Combining all those running times yield an overall running time for $O(2^t n^2)$ to compute the value of a weighted maximal bisection and thus also to solve the corresponding decision problem. This proofs:

**Theorem 6.** There is an algorithm that decides in time $O(2^t n^2)$ if a weighted graph with $n$ vertices has a bisection greater equal than the input parameter $\gamma$, when a tree decomposition with $\leq 4(n + 1)$ nodes and treewidth $t$ is given as an advice.

### 3.2 Implementation details

In the previous section we showed how one can compute the value of a weighted maximal bisection under the assumption that all the graph operations, set operations as well as cardinality of sets, enumerating all power sets, and storage/access of the computed values is possible in appropriate time. In this section we provide a detailed overview on the underlying data structures, encodings, and some necessary preprocessing of the input.

#### 3.2.1 Computation and Memoization of some values

Our algorithm makes use of the sets $F_i$ and $X_i$ for all $i \in I$ as well as their cardinalities. Moreover for introduce and forget nodes we need to know the newly introduced/just forgotten node $v$. It is easy to see that we can precompute those values in an appropriate time and store them in a lookup table. Due to the triviality of this task we do not provide any further details.

#### 3.2.2 Preprocessing the input graph/tree decomposition

In order to be able to quickly check adjacency of different nodes, evaluate the weight of different edges but also beeing able to efficiently iterate over the neighbourhood within $X_i$ for a $v \in X_i$ and a $i \in I$, we built up some lookup structure before running our algorithm. Those are standard structures: For the whole graph, but also for the induced subgraph of every $X_i$ for every $i$, we built up an adjacency matrix $M_i$ as follows:

**Algorithm 9** Preprocessing the representation of the input graph/tree decomposition

```plaintext
1: Create an adjacency matrix $A$ for $G$ \hspace{1cm} \triangleright O(n^2)
2: for all $i \in I$ do \hspace{1cm} \triangleright O(|I| \cdot \text{BODY}) \subseteq O(n \cdot \text{BODY})
3: Allocate an 2-dimensional array $M_i$ with size $|X_i| \times |X_i|$ \hspace{1cm} \triangleright O(|X_i|^2) \subseteq O(t^2)
4: Initialize $M_i$ with $\perp$ in every entry \hspace{1cm} \triangleright O(|X_i|^2) \subseteq O(t^2)
5: for all $v \in X_i$ do \hspace{1cm} \triangleright O(|X_i| \cdot \text{BODY}) \subseteq O(t \cdot \text{BODY})
6: \hspace{1cm} for all $z \in X_i$ do \hspace{1cm} \triangleright O(|X_i| \cdot \text{BODY}) \subseteq O(t \cdot \text{BODY})
7: \hspace{2.5cm} if $v \neq z$ then \hspace{1cm} \triangleright O(1)
8: \hspace{3cm} $M_i(v, z) \leftarrow A(v, z)$ \hspace{1cm} \triangleright O(1)
9: \hspace{2.5cm} $M_i(z, v) \leftarrow M_i(v, z)$ \hspace{1cm} \triangleright O(1)
10: \hspace{2.5cm} end if
11: \hspace{1cm} end for
12: \hspace{1cm} end for
13: end for
```
This takes at most $O(nt^2 + n^2) \subseteq O(2^it^2)$ time.

### 3.2.3 Set operations

In order to get the desired running time we need to be able to execute specific set operations in sufficiently small running time. In this section we specify how one can do this in detail, mostly by just selecting an appropriate representation and some preprocessing with a running time that does not increase the overall asymptotic running time $O(2^it^2)$.

For $i \in I$, we represent a set $S \in \mathcal{P}(X_i)$ by a bit vector with the $j$-th bit being one if and only if $X_{i}^{(j)} \in S$ where $X_{i}^{(j)}$ is the $j$-th smallest vertex in $X_i$ (which is the $j$-th vertex in the encoding of $X_i$). Other subsets do not occur within our algorithm. We can then intersect two sets $S_1 \in \mathcal{P}(X_i)$, $S_2 \in \mathcal{P}(X_i)$ by combining the corresponding vectors using the bitwise AND ($\&$) operation. On the RAM model this operation takes time $O(1)$. Analogously we can define the union of two sets by using a bitwise OR ($|$) instead. Membership of $v \in X_i$ for $S \in \mathcal{P}(V)$ can be computed as follows ($\ll$ denotes a left shift):

**Algorithm 10** Computing the element function

1: function $\text{elem}(v, S)$
2: \hspace{1em} return $S \cap (1 \ll (v - 1))$
3: end function

The idea is that $1 \ll (v - 1)$ creates a bit vector that is zero except in the component $v$ thus representing $\{v\}$, and then using the intersection routine that returns a true value (a value $\neq 0$) iff the intersection is non-empty.

For the computation of $B_i$ for a join node $i \in I$ we also need a operation $\text{choose}$ that gives us any element $v \in S$ for a non-empty set $S \in \mathcal{P}(X_i) \setminus \{\emptyset\}$. Instead of preprocessing, which would allow us to perform this operation in $O(1)$ time, we use a simpler approach here that is sufficient for our case: We just keep right shifting the bit vector until we find a non-zero bit.

**Algorithm 11** Computing the selection of any element of a set

1: function $\text{choose}(S)$
2: \hspace{1em} $j \leftarrow 1$
3: \hspace{1em} while $S \& 1 = 0$ do
4: \hspace{2em} $S \leftarrow S \gg 1$
5: \hspace{2em} $j \leftarrow j + 1$
6: \hspace{1em} end while
7: \hspace{1em} return $X_{i}^{(j)}$
8: end function

The running time is – obviously – $O(t)$ as the length of a bit vector is upper bounded by the size of the universe $|X_i| < n$.

### 3.2.4 Conversion from sets to array indices

Until now we assumed that we can build up an array that has the power set of some set as its set of indices. Naturally such “arrays” do not exist but can be represented by some dictionary
data structure, that is a structure that allows efficient access to element indexed by some key type. For general data structures of this type, we cannot perform every operation in $O(1)$. Thus we need to be able to somehow have an injective map from the elements of a given power set to a space polynomial in the cardinality of the power set that allows $O(1)$ access. We can do this by using the encoding of a set as bitvector as the key itself, that is, we interpret the bit vector as integer. Doing so leaves one problem: In our algorithms, we use a set $S$ to access the entries of different nodes $i, j \in I$ that have (potentially) different $X_i \neq X_j$, which implies that the set encoding of one node $i$ is not necessarily compatible to that of $j$. Hence we need to provide some conversion between those representations.

We can do this in $O(1)$ even though one can observe that $O(n)$ would also be sufficient. Let $i \in I$ and $j, k \in I$ be the child node(s) if they exist. We again consider the different node types separately, that is, we make a case distinction over the type of $i$:

- **Leaf node**: Leaf nodes do not have any children thus no conversion is needed at all.
- **Introduce node**: Let $\{ v \} = X_i \setminus X_j$, that is, $v$ is the newly introduced vertex in $i$. Let $m \in \mathbb{N}$ such that $X_i^{(m)} = v$. Then the conversion of any set $S \in \mathcal{P}(X_i)$ can be done by simply removing the $m$-th bit:

```
\begin{algorithm}
  \textbf{Algorithm 12} Bit vector conversion at an introduce node $i \in I$ with child $j \in I$
  \textbf{Require:} $v \notin S$
  1: $\triangleright$ Mask that has every bit from 1 to $m - 1$ set to 1
  2: $\mu_1 \leftarrow (1 \ll (m - 1)) - 1$
  3: $\triangleright$ Mask that has every bit from $m$ to $|X_j|$ set to 1
  4: $\mu_2 \leftarrow (1 \ll |X_j|) - 1 - \mu_1$
  5: $\triangleright$ Get the first part of the old bit vector
  6: $\nu_1 \leftarrow S \& \mu_1$
  7: $\triangleright$ Get the second part of the old bit vector
  8: $\nu_2 \leftarrow S \& \mu_2$
  9: $\triangleright$ Remove the bit for $v$
  10: $\nu_2 \gg 1$
  11: $\triangleright$ Glue them together
  12: $S \leftarrow \nu_1 | \nu_2$
\end{algorithm}
```

- **Forget node**: Let $\{ v \} = X_j \setminus X_i$, that is, $v$ is the vertex that just has been forgotten at $i$. Let $m \in \mathbb{N}$ such that $X_i^{(m)} = v$. Then the conversion of any set $S \in \mathcal{P}(X_i)$ can be done by simply inserting a bit at the $m$-th position:
Algorithm 13 Bit vector conversion at an forget node $i \in I$ with child $j \in I$

Require: $v \notin S$

1: $\triangleright$ Mask that has every bit from 1 to $m-1$ set to 1
2: $\mu_1 \leftarrow (1 \ll (m-1)) - 1$
3: $\triangleright$ Mask that has every bit from $m$ to $|X_j|$ set to 1
4: $\mu_2 \leftarrow (1 \ll |X_j|) - 1 - \mu_1$
5: $\triangleright$ Get the first part of the old bit vector
6: $\nu_1 \leftarrow S \& \mu_1$
7: $\triangleright$ Get the second part of the old bit vector
8: $\nu_2 \leftarrow S \& \mu_2$
9: $\triangleright$ Add the bit for $v$
10: $\nu_2 \ll 1$
11: $\triangleright$ Glue them together
12: $S \leftarrow \nu_1 | \nu_2$

• Join node: As $X_i = X_j = X_k$, the identity does the job.

3.2.5 Enumerating elements of the power set

It is obvious that one can enumerate all sets of the power set of $X_i$ for $i \in I$ by simply listing all integers in $[0, |\mathcal{P}(X_i)|]$. This is not sufficient for our cause as we assumed that any enumeration of sets is done such that the sets are non-decreasing regarding their cardinality. Instead of providing a constructive method to enumerate the power set in that way, we simply observe that we can enumerate all sets in $O(2^t)$ and then sort them using some standard algorithm, which yields a running time of $O(2^t t) \subseteq O(2^t n^2)$. Doing so for all $i \in X_i$ yields an overall running time of $O(2^t n^2)$.

3.2.6 A note on the (non-asymptotic) efficiency

All the details we provided are kept as simple as possible, that is, we tried to find the simplest approach that does not increase the overall running time. For the sake of simplicity, many details are less efficient than possible. For example in line Algorithm 12 in Algorithm 8 we take time $O(t)$ to compute the cardinality of the set $S$, which could easily be decreased to $O(1)$ by simply annotating the cardinality during the construction process of the power set.

3.2.7 Computing the bisection

Note that using the same algorithm but annotating for every entry in the table $B$ which values of the children has been used we can also compute the maximal weighted bisection itself, that is the partition of the vertex sets with maximal value. It is easy to see that the time needed for such annotations is always dominated by the computation of the entry we add the annotation to.

4 Hardness and Optimality

To show the optimality in the dependence on $t$ we show that even $\text{ADV}_{\text{STD}}$-$\text{MAX-CUT}$ (which is at most as hard as $\text{ADV}_{\text{STD}}$-$\text{MAX-BISECTION}$) cannot be solved in a better running time in $t$ assuming SETH unless its running time gets superpolynomial in $n$:
Theorem 7. There is no $O(2^t \text{poly } n)$ algorithm to decide $\text{Adv}_{\text{STD-Max-Cut}}$ for any $\varepsilon < 1$.

Using this theorem it is easy to show that the same holds for $\text{Adv}_{\text{STD-Max-Bisection}}$:

Corollary 15. There is no $O(2^t \text{poly } n)$ algorithm to decide $\text{Adv}_{\text{STD-Max-Bisection}}$ for any $\varepsilon < 1$.

Proof. Consider any graph. For every input vertex, add one isolated vertex. The treewidth does not increase. If there is a small tree decomposition given as advice, modifying it to contain the new vertices is not hard (and obviously possible in time polynomial in $n$). We did not change the edge set, thus the possible values of cuts do not change. It is easy to see that the resulting instance is in $\text{Adv}_{\text{STD-Max-Bisection}}$ iff the original instance is in $\text{Adv}_{\text{STD-Max-Cut}}$.

To show Theorem 7 we might be tempted to reuse the result of Lokshtanov, Marx, and Saurabh in [15]:

Theorem 16 ([15, Thm. 3, p. 9]). There is no $O((2 - \varepsilon)^t \text{poly } I)$ algorithm for any $\varepsilon > 0$ deciding $\text{Max-Cut}$ unless SETH fails.

It is worth noting that it is equivalent to state that there is no $O(2^t \text{poly } |I|)$ algorithm for any $\varepsilon > 0$ deciding $\text{Max-Cut}$ unless SETH fails.

The detail that was missed in [10] is that $\text{Adv}_{\text{STD-Max-Cut}}$ is a problem at most as hard as $\text{Max-Cut}$, thus we cannot transfer the statement directly. There is also little hope to finde a reduction from $\text{Max-Cut}$ to $\text{Adv}_{\text{STD-Max-Cut}}$, as arbitrary graphs occur in the instances and computing the treewidth is already NP-hard, thus a polynomial reduction is possible iff P $\neq$ NP. As this is considered very unlikely, we work around this issue: Instead of adding a second reduction, we adjust the existing reduction in order to achieve the desired result. Therefore we need the following statement;

Lemma 17. Let $G = (V, E)$ and $S \subseteq V$. If $G - S$ is a forest, then (given $S$) one can compute a small tree decomposition of $G$ that has width $\leq |S| + 1$ in time $O(\text{poly}(|E| + |V|))$.

Proof. Fix $G = (V, E)$ and $S$. Note that it is trivially possible to compute a small tree decomposition of a tree in time linear in its nodes and edges. Compute the tree decompositions for all connected components in $G' := G - S$ (that are trees by assumption). Rename the node sets (and all the occurrences accordingly) of the decompositions such that the union of all node sets is of the form $[k] \setminus \{1\}$ for some $k \in \mathbb{N}$ and the node sets are disjunct. Add a new node 1 to the forest of tree decompositions and set the associated bag to be $X_1 := \emptyset$. Connect the new node 1 to an arbitrary node of each tree decomposition. It is easy to check that we just constructed a small tree decomposition for $G'$ of width 1. Now modify this decomposition to a small tree decomposition of $G$ as follows: Add the set $S$ to all bags. Again, it is easy to check that we obtain a small tree decomposition of $G$. The same holds for the running time. The width of the constructed decomposition is bounded from above by the term $1 + |S|$, as every bag of the decomposition for $G'$ had at most 2 vertices before we added the vertices within $S$.

We use this statement to modify the reduction in [15, Theorem 3, p. 7] from $\text{sat}$ to $\text{Max-Cut}$ to a reduction from $\text{sat}$ to $\text{Adv}_{\text{STD-Max-Cut}}$. Note that – additionally to the polynomial running time in the input encoding of the reduction – we need to ensure that a $\text{sat}$ instance with
n variables is reduced to an \( \text{Adv}_{\text{STD}} - \text{Max-Cut} \) instance that has treewidth \( n + \mathcal{O}(1) \) (otherwise we do not get the result we need).

**Proof of Theorem 7.** First, let us literally restate the reduction of Lokshtanov, Marx, and Saurabh in [15]:

Given an instance \( \phi \) of SAT we [...] construct an instance \( G_w \) of Weighted Max Cut as follows. [...] We start with making a vertex \( x_0 \). [...] We make a vertex \( \hat{v}_i \) for each variable \( v_i \). For every clause \( C_j \) we make a gadget as follows. We make a path \( \hat{P}_j \) having \( 4 |C_j| \) vertices. All the edges on \( \hat{P}_j \) have weight \( 3n \) [where \( n \) denotes the number of variables in \( \phi \)]. Now, we make the first and last vertex of \( \hat{P}_j \) adjacent to \( x_0 \) with an edge of weight \( 3n \). Thus the path \( \hat{P}_j \) plus the edges from the first and last vertex of \( \hat{P}_j \) to \( x_0 \) form an odd cycle of \( \hat{C}_j \). We will say that the first, third, fifth, etc, vertices are on odd positions on \( \hat{P}_j \) while the remaining vertices are on even positions. For every variable \( v_i \) that appears negatively in \( C_j \) we select a vertex \( p \) at an even position (but not the last vertex) on \( \hat{P}_j \) and make \( \hat{v}_i \) adjacent to \( p \) and \( p \)'s successor on \( \hat{P}_j \) with edges of weight 1. For every variable \( v_i \) that appears negatively in \( C_j \) we select a vertex \( p \) at an odd position on \( \hat{P}_j \) and make \( \hat{v}_i \) adjacent to \( p \) and \( p \)'s successor on \( \hat{P}_j \) with edges of weight 1. We make sure that each vertex on \( \hat{P}_j \) receives an edge at most once in this process. There are more than enough vertices on \( \hat{P}_j \) to accommodate all the edges incident to vertices corresponding to variables in the clause \( C_j \). We create such a gadget for each clause and set the target value \( \gamma = 1 + (12n + 1) \sum_{j=1}^m |C_j| \) [...] This concludes the construction.

Observe that following statements:

1. The number of vertices after the reduction is \( \mathcal{O}(|\phi| + n) = \mathcal{O}(|\phi|) \).
2. For every clause there is exactly one cycle created.
3. Any pair of such cycles does share exactly one vertex: \( x_0 \).
4. Apart from those partially overlapping cycles, there are only \( n \) additional vertices.
5. Every vertex on those cycles that is not \( x_0 \) has degree at most 3 as it is connected to at most one \( \hat{v}_i \).

From **Property 4** and **Property 5** we can now deduce that the number of edges after the reduction is \( \mathcal{O}(|\phi|) \). Thus the encoding of the graph has asymptotically the same size as the encoding of \( \phi \). The correctness of the reduction was shown in [15] Lemma 7 – Lemma 9, p. 9 – 10.

Now observe that **Property 3** and **Property 4** imply that removing all variable vertices \( \hat{v}_i \) and \( x_0 \) does result in a set of paths. This is exactly the statement we need in order to be able to apply **Lemma 17** which completes the proof.

The optimality of our algorithm in its dependence on \( n \) follows directly from the following bound:

**Proposition 18** ([10], Thm. 6, p. 6], [8], Thm. 16, p. 8]). For all \( \varepsilon > 0 \) there is no \( \mathcal{O}(n^{2-\varepsilon}) \) time algorithm for \( \text{Adv}_{\text{STD}} - \text{Max-Bisection} \), even if the instances are restricted to be paths, unless MCH fails.
As we will show, this bound does not only apply in general but for a broad family of quite specific subsets of $\text{ADV}_{\text{STD}}$-$\text{MAX-BISECTION}$ that are not even required to include instances of treewidth 1, which is the treewidth of the instances that are created in the reduction for Proposition 18. In order to show this, we want to transform those treewidth 1 instances to instances with greater treewidth in adequate time without changing the number of vertices or significantly increasing the instance’s encoding size. We are able to do so and additionally ensure that the resulting graph is planar. In order to show this we need the following type of graphs:

**Definition 19 (Grid).** For two integers $l \in \mathbb{N}, k \in \mathbb{N}$, the corresponding grid graph is given by

$$G_{l,k} := ([l] \times [k], \{(i,j)(i',j') : (i,i') \in [l]^2, (j,j') \in [k]^2, |i - i'| + |j - j'| = 1\})$$

Any graph that is isomorphic to $G_{l,k}$ is called a $l \times k$ grid.

Even though the definition appears quite complicated, the structure of those graphs is quite intuitive, see for example Figure 2.

These graphs are important for us as they are obviously planar but also have a quite large treewidth in relation to the number of vertices.

**Lemma 20** ([7, p. 359]). Let $k \in \mathbb{N}_{>1}$ and $G = (V,E)$ be a $k \times k$ grid. Then $G$ has treewidth $k$.

It is quite easy to see that any graph that has a $k \times k$ grid as subgraph has treewidth lower bounded by $k$:

**Observation 21.** Let $k \in \mathbb{N}_{>1}$ and $G$ be a graph that has a subgraph that is a $k \times k$ grid. Then $G$ has treewidth at least $k$.

In fact one can show that any planar graph has a specific treewidth if and only if it has such a grid as minor. Yet we do not need this stronger statement in this paper.

When proving the hardness of $\text{ADV}_{\text{STD}}$-$\text{MAX-CUT}$, we need not only to ensure that the generated instances have the desired treewidth and are planar but also have to provide a tree decomposition with minimal treewidth. As this computation is NP-hard in general, it is not a priori clear that this can be done in truly subquadratic time in the number of vertices for the generated instances. We show that this is in fact possible in time $\mathcal{O}(|V| \log|V|)$ without increasing the instances too much.
Proposition 22. Let $J = (V, E, w, \gamma, t, D)$ be an instance for $\text{Adv}_{\text{STD}}\text{-Max-Bisection}$ where $(V, E)$ is a path (implying $t = 1$). Then for any $\hat{t} \in \left\lfloor \sqrt{|V|} \right\rfloor$ there is an instance $\hat{J} = (V, \hat{E}, \hat{w}, \gamma, \hat{t}, \hat{D})$ such that

1. $(V, \hat{E})$ is planar
2. $\hat{J}$ can be computed in $O(|V| \log |V|)$
3. $|\hat{J}| \in O(|J|)$
4. $J \in \text{Adv}_{\text{STD}}\text{-Max-Bisection} \iff \hat{J} \in \text{Adv}_{\text{STD}}\text{-Max-Bisection}$

Proof. Let $I = (V, E, w, \gamma, t, D)$ be an instance to $\text{Adv}_{\text{STD}}\text{-Max-Bisection}$. Assume without loss of generality that $1 < \hat{t}$ as otherwise $\hat{J} = J$ would be sufficient.

Without loss of generality assume that for any $e \in (V^2)$ it holds that $e \in E \iff \exists i \in V. e = i(i + 1)$. If that would not be the case we could rename the instance in time $O(|V| \log |V|)$ using for example AVL-trees, or even in $O(|V|)$ when choosing a slightly more sophisticated approach. As this is a reduction that does not have to be efficient but only truly subquadratic, we stick with the simpler approach as it sufficient for our cause.

Recall that $V = |V|$ (Definition 12). Now observe that $G|_{[\hat{t}]}$ is a path with $\hat{t}^2$ vertices. We can then transform $E$ into $\hat{E}$ such that for $\hat{G} = (V, \hat{E})$ we have that $\hat{G}|_{[\hat{t}]}$ is a $\hat{t} \times \hat{t}$ grid. Therefor we assume that whenever we add a new edge, the weight function is extended implicitly such that every new edge has weight 0.

Algorithm 14 Extending the path $G|_{[\hat{t}]}$ to a $\hat{t} \times \hat{t}$ grid

1: $\hat{E} \leftarrow E$
2: for all $\nu_1 \in [\hat{t} - 2]_0$ do
3: \hspace{1em} for all $\nu_2 \in [\hat{t} - 2]_0$ do
4: \hspace{2em} $\hat{E} \leftarrow \hat{E} \cup \{ \nu_1 \hat{t} + 1 + \nu_2, (\nu_1 + 2)\hat{t} - \nu_2 \}$
5: \hspace{1em} end for
6: end for

Figure 3: An illustration for the situation in Algorithm 14 when $\hat{t} = 4$. The dashed edges are those that are to be added.

Now using Observation 21 we know that $\hat{G}$ has treewidth at least $\hat{t}$. On the other hand one can easily check that it does not have a greater treewidth as $\hat{G}$ is just a $\hat{t} \times \hat{t}$ grid with a single path
attached. Following the same observation regarding the structure of \( \hat{G} \) we directly get that \( \hat{G} \) is planar.

The running time for the steps described until now is \( O(|V| \log|V|) \) for renaming and \( O(\hat{t}^2) \subseteq O(|V|) \) for adding the edges. Thus \( O(|V| \log|V|) \) upper bounds the overall running time.

We still need to compute a small tree decomposition of \( \hat{G} \). It is easy to see that the following gives a tree-decomposition of a \( \hat{t} \times \hat{t} \) grid with width \( \hat{t} \) and \( \leq \hat{t}^2 \) nodes:

**Algorithm 15** Compute a small tree decomposition of the \( \hat{t} \times \hat{t} \) grid

1: Start with the first row
2: while The current row is not the last row do
3: Let \( M \) be the set of vertices in the current row
4: for all \( i \in [\hat{t}] \) do
5: Create a new node in the decomposition
6: Connect it to the one that was created directly before the new one, if there is one
7: Let \( v \) be the \( i \)-th vertex in the current row
8: Let \( w \) be the \( i \)-th vertex in the next row
9: \( M \leftarrow M \cup \{w\} \)
10: Put \( M \) into the corresponding bag to the new node
11: \( M \leftarrow M \setminus \{v\} \)
12: end for
13: Go to next row and proceed
14: end while

The running time for this algorithm is bounded by the number of executions of the loops: \( O(\hat{t}^2) \). Thus this computation does not dominate the running time of the reduction, as \( O(\hat{t}^2) \subseteq O(|V|) \) by choice of \( \hat{t} \). It is obvious that we can add the remaining part of the input graph, that is the edges that are not part of the just constructed grid, by simply creating one node per edge, in time \( O(|V|) \). The number of nodes in this decomposition is bounded by \( \hat{t}^2 + (|V| + 1) \), thus the decomposition is small.

We added at most \( \hat{t}^2 \) edges, which is at most linear in the number of vertices. As the vertex set is encoded in space linear in the number of vertices this means that by adding edges the size of the encoding did at most grow in a linear fashion as desired. The same holds for the encoding size of the tree decomposition as its size is roughly \( O(\hat{t}^2 + |V|) \subseteq O(|V|) \) as there are \( \hat{t} \) bags with \( \hat{t} + 1 \) elements and additional \( O(|V|) \) bags for the rest of the input path.

Now observe that the size of any cut/bisection is exactly the same in \( \hat{G} \) and \( G \) as they share the same vertex set and all new edges have weight 0. This shows the last claim and finishes the proof.

Additionally to the reduction above we also need a proposition that basically states: if one has a class of instances where \( t \in o(\log n) \), then \textbf{Algorithm 8} solves the problem in truly subquadratic time, that is \( O(n^{2-\varepsilon}) \) for some \( \varepsilon > 0 \).

**Proposition 23.** Let \( g(n) \in o(\log n) \). Then for any \( \varepsilon > 0 \) it holds that there exists \( \delta > 0 \) such that

\[
O(2^{g(n)} \cdot n^{2-\varepsilon}) \subseteq o(n^{2-\delta})
\]
Proof. Fix \( g \). Fix \( \varepsilon \). Note that

\[
2^{g(n)} = \left( n^\log_{2/\varepsilon}(2) \right)^{g(n)} = n^{g(n) \cdot \log_{2/\varepsilon}(2)} = n^{g(n) \cdot \frac{\log 2}{\log n}} = n^{\frac{g(n)}{\log n}}
\]

Now let \( \delta := \varepsilon / 2 \) and note that

\[
2^{g(n)} \cdot 2^{-\varepsilon} = n^{\frac{g(n)}{\log n} \cdot \frac{\log 2}{\log n} \cdot n^{-2\delta} / n^2} = \frac{n^{\frac{g(n)}{\log n} \cdot \frac{\log 2}{\log n}}} {n^{2-\delta}}
\]

Note that for any \( f \) it holds that

\[
f(n) \in \mathcal{O}\left(2^{g(n)} \cdot 2^{-\varepsilon}\right) \iff f(n) \in \mathcal{O}\left(\frac{n^{\frac{g(n)}{\log n} \cdot \frac{\log 2}{\log n}}}{n^{2-\delta}}\right)
\]

\[
\iff \limsup_{n \to \infty} \frac{f(n)}{n^{\frac{g(n)}{\log n} \cdot \frac{\log 2}{\log n}}} < \infty \iff \limsup_{n \to \infty} \frac{n^\delta \cdot f(n)}{n^{\frac{g(n)}{\log n} \cdot \frac{\log 2}{\log n}}} < \infty
\]

Note that \( \lim_{n \to \infty} \frac{g(n)}{\log n} = 0 \) by assumption as \( \log n \in \Theta(\log n) \) and thus \( \lim_{n \to \infty} \frac{n^\delta}{n^{\frac{g(n)}{\log n} \cdot \frac{\log 2}{\log n}}} = \lim_{n \to \infty} n^\delta = \infty \). We can hence deduce that the limit superior above can only be finite if \( \limsup_{n \to \infty} \frac{f(n)}{n^\delta} = 0 \).

Also note that we only consider non-negative functions and hence the limit inferior is always lower bounded by zero, thus the limit inferior and superior are equal and we have that the whole limit converges to 0:

\[
\lim_{n \to \infty} \frac{f(n)}{n^2} = 0 \iff \lim_{n \to \infty} \frac{f(n)}{n^{2-\varepsilon}} = 0 \iff f(n) \in o\left(n^{2-\varepsilon}\right)
\]

With the two propositions we just proved we can now focus on the hardness theorem itself:

**Theorem 8.** Let \( f : \mathbb{N}_0 \to \mathbb{N} \) be a function such that \( f(n) \in o(\log n) \). Let \( L \subseteq \text{ADVSTD-MAX-BISECTION} \) be a set that is build by restricting \( \text{ADVSTD-MAX-BISECTION} \) to instances where the corresponding graph is planar and has treewidth \( t = f(n) \) where \( n \) is the number of vertices of the instance. Then for all \( \varepsilon > 0 \) there is no \( \mathcal{O}(2^{f(n)} \cdot n^{2-\varepsilon}) \) algorithm to decide \( L \) unless MCH fails.

**Proof.** Fix \( f \). Note that

\[
f(n) \in o(\log n) \iff \forall c > 0. \ \exists n_0. \ \forall n > n_0. \ f(n) \leq c \log n
\]

Now let \( c := 1 \) and obtain \( n_0 \). Apply [Proposition 23](#) with \( g := f \). Obtain \( \delta \). Note that

\[
\mathcal{O}(2^{f(n)} \cdot n^{2-\varepsilon}) \subseteq o(n^{2-\delta}) \subseteq \mathcal{O}(n^{2-\delta})
\]

This means that if we are able to decide \( L \) in the stated running time, we decide \( L \) in time subquadratic in \( n \).

If we are now able to do a subquadratic reduction from \( \text{MAXCONV} \) to \( L \), we eventually show the claim as by MCH this problem cannot be solved in subquadratic time.

Consider an instance to \( \text{MAXCONV} \). Use the reduction of [Proposition 18](#) to transform this into an instance for weighted maximal bisection on paths. Let \( n \) be the number of vertices of this
instance. If \( n < n_0 \), we extend the path to a path with \( n_0 + 1 \) nodes by adding new nodes and new edges with weight 0. Since \( n_0 \) is constant, we know that the new number of nodes, \( n' \), is asymptotically equal to \( n \), that is \( n' \in \Theta(n) \). Now observe that

\[
    t = f(n') \leq \log n' \leq \sqrt{n'}
\]

This allows us to apply Proposition 22. Eventually we get a truly subquadratic algorithm for MaxConv which does only exist if MCH fails.

\[\square\]

5 Conclusion

In this paper we show that there is an algorithm to decide AdvSTD-Max-Bisection in time \( O(2^t n^2) \), which is optimal assuming MCH and SETH. We proofed the latter. Moreover, this algorithm is (obviously) also usable to compute the value of a maximal bisection. We gave useful details for implementing the algorithm but also understanding the requirements of it on the underlying machine model. The strong dependency on the machine model raises the question if an algorithm with the same running time is also possible on weaker machine models, especially those, where the size of a memory cell is not unbounded. We showed the hardness of a large family of graphs that are build by their dependency between the number of nodes and the tree width. This raises the question whether better algorithms are possible if \( t \in \omega(\log n) \) or if even instances with this property remain hard.
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