On Weighted Greedy-Type Bases

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Abstract
We study weights for the thresholding greedy algorithm, aiming to extend previous work on sequential weights $\varsigma$ on $\mathbb{N}$ to weights $\omega$ on $\mathcal{P}(\mathbb{N})$. We revisit major results on weighted greedy-type bases in this new setting including characterizations of $\omega$-(almost) greedy bases and the equivalence between $\omega$-semi-greedy bases and $\omega$-almost greedy bases. Some new results are encountered along the way. For example, we show that there exists an $\omega$-greedy unconditional basis that is not $\varsigma$-almost greedy for any weight sequence $\varsigma$. Moreover, a basis is unconditional if and only if it is $\omega$-greedy for some weight $\omega$. Similarly, a basis is quasi-greedy if and only if it is $\omega$-almost greedy for some weight $\omega$.

Keywords Thresholding greedy algorithm · Greedy · Almost greedy · Semi-greedy · Partially greedy · Weight

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1 Introduction

1.1 Background

Let $(X, \| \cdot \|)$ be a Banach space over the field $K = \mathbb{R}$ or $\mathbb{C}$ with a semi-normalized Schauder basis $B = (e_n)_{n=1}^\infty$ satisfying

$$0 < c_1 := \inf_n \|e_n\| \leq \sup_n \|e_n\| =: c_2 < \infty. \quad (1.1)$$

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Let \((e_n^*)_{n=1}^{\infty} \subset \mathbb{X}^*\) be the biorthogonal functionals such that \(e_n^*(e_m) = \delta_{n,m}\). Every \(x \in \mathbb{X}\) can be uniquely written as the series \(\sum_{n=1}^{\infty} e_n^*(x)e_n\). Recall that the partial sum operators, defined as \(S_m(x) = \sum_{n=1}^{m} e_n^*(x)e_n\), are uniformly bounded. We let \(K_b := \sup_m \|S_m\|\). It is easy to verify that for a semi-normalized basis, the corresponding biorthogonal functionals are also semi-normalized, i.e.,

\[
0 < c^*_1 := \inf_n \|e_n^*\| \leq \sup_n \|e_n^*\| =: c^*_2 < \infty. \tag{1.2}
\]

Konyagin and Temlyakov (1999) introduced the **Thresholding Greedy Algorithm (TGA)** to approximate each vector \(x\) using finite linear combinations of basis vectors. In particular, for each \(x \in \mathbb{X}\) and \(m \in \mathbb{N}\), as \(\Lambda_m(x)\) is called a **greedy set** of order \(m\) of \(x\) if \(|\Lambda_m(x)| = m\) and

\[
\min_{n \in \Lambda_m(x)} |e_n^*(x)| \geq \max_{n \notin \Lambda_m(x)} |e_n^*(x)|.
\]

For \(x \in \mathbb{X}\), the TGA produces a sequence of approximating greedy sums \((G_m(x))_{m=1}^{\infty}\), where \(G_m(x) := \sum_{n \in \Lambda_m(x)} e_n^*(x)e_n\). Here \(G_m(x)\) depends on \(\Lambda_m(x)\). A basis is quasi-greedy if there exists \(C \geq 1\) such that

\[
\|x - G_m(x)\| \leq C\|x\|, \quad \forall x \in \mathbb{X}, \ \forall m \in \mathbb{N}, \ \forall \Lambda_m(x). \tag{1.3}
\]

The least constant \(C\) satisfying (1.3) is denoted by \(C_\ell\), and we say \(B\) is \(C_\ell\)-suppression quasi-greedy. A basis is greedy if the TGA gives essentially the best approximation, i.e., there exists a constant \(C \geq 1\) such that

\[
\|x - G_m(x)\| \leq C\sigma_m(x), \quad \forall x \in \mathbb{X}, \ \forall m \in \mathbb{N}, \ \forall \Lambda_m(x),
\]

where

\[
\sigma_m(x) := \inf \left\{ \left\| x - \sum_{n \in A} a_ne_n \right\| : A \subset \mathbb{N}, \ |A| = m, a_n \in \mathbb{K} \right\}.
\]

A basis is almost greedy if the TGA gives essentially the best projection approximation, i.e., there exists a constant \(C \geq 1\) such that

\[
\|x - G_m(x)\| \leq C\tilde{\sigma}_m(x), \quad \forall x \in \mathbb{X}, \ \forall m \in \mathbb{N}, \ \forall \Lambda_m(x),
\]

where

\[
\tilde{\sigma}_m(x) := \inf \left\{ \left\| x - \sum_{n \in A} e_n^*(x)e_n \right\| : A \subset \mathbb{N}, \ |A| = m \right\}.
\]

A beautiful theorem of Konyagin and Temlyakov (1999) characterizes greedy bases as being unconditional and democratic (defined later.) In the same spirit, Dilworth et al. (2003b) characterized almost greedy bases as being quasi-greedy and democratic.

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As a variant of (almost) greedy bases, one can introduce the weighted version, where a weight sequence \( \varsigma = (s_n)_{n=1}^{\infty} \in (0, \infty)^{\mathbb{N}} \) is involved. For early work on weighted greedy bases, see Kerkyacharian et al. (2006, Section 4) and Temlyakov (2011, pp. 19–23). Given a set \( A \subset \mathbb{N} \), the weight of \( A \) is \( s(A) := \sum_{n \in A} s_n \). For \( \alpha \geq 0 \), we define the error \( \sigma_\alpha(x) \) and \( \tilde{\sigma}_\alpha(x) \)

\[
\sigma_\alpha(x) := \inf \left\{ \| x - \sum_{n \in A} a_n e_n \| : |A| < \infty, s(A) \leq \alpha, a_n \in \mathbb{K} \right\}, \quad \text{and}
\tilde{\sigma}_\alpha(x) := \inf \left\{ \| x - \sum_{n \in A} e^*_n(x)e_n \| : |A| < \infty, s(A) \leq \alpha \right\}.
\]

**Definition 1.1** (Kerkyacharian et al. 2006, Definition 4.1) and (Dilworth et al. 2018, Definition 1.1) A basis \( B \) is

1. \( \varsigma \)-greedy if there exists a constant \( C \geq 1 \) such that

\[
\| x - G_m(x) \| \leq C \sigma_\varsigma(s(\Lambda_m(x))) (x), \quad \forall x \in \mathbb{X}, \forall m \in \mathbb{N}, \forall \Lambda_m(x). \quad (1.4)
\]

2. \( \varsigma \)-almost greedy if there exists a constant \( C \geq 1 \) such that

\[
\| x - G_m(x) \| \leq C \tilde{\sigma}_\varsigma(s(\Lambda_m(x))) (x), \quad \forall x \in \mathbb{X}, \forall m \in \mathbb{N}, \forall \Lambda_m(x). \quad (1.5)
\]

Weighted greedy-type bases have received much attention and witnessed progresses in various directions: see Berasategui and Lassalle (2023), Berná et al. (2019), Berná (2020), Dilworth et al. (2018), Kerkyacharian et al. (2006) and Khurana (2020). Specifically, Berná et al. (2019), Dilworth et al. (2018) and Kerkyacharian et al. (2006) characterized weighted greedy and weighted almost greedy bases; Berasategui and Lassalle (2023), Berná (2020) and Dilworth et al. (2018) studied weighted weak semi-greedy bases and weighted semi-greedy bases; Khurana (2020) investigated weighted partially greedy and weighted reverse partially greedy bases. It follows trivially from Khurana (2020, Remark 3.3 and Theorem 3.11) that there exists a basis that is weighted partially greedy but is not partially greedy. Furthermore, Dilworth et al. (2018, Remark 4.10) gives an example of a basis that is weighted greedy but is not almost greedy.

**1.2 Main Goals**

This paper extends sequential weights \( \varsigma \) to weights on sets \( \omega \) and revisit existing results in the new setting. First, we characterize weighted (almost) greedy bases (Sect. 2). As a corollary, we show that a basis is unconditional if and only if it is \( \omega \)-greedy for some \( \omega \). Similarly, a basis is quasi-greedy if and only if it is \( \omega \)-almost greedy for some \( \omega \) (see Corollary 2.11.)
For our next result, it is worth noting that Dilworth et al. (2018) briefly discussed a generalization of sequential weights \( \varsigma \), denoted by \( \nu \). Here \( \nu \) satisfies

\[
\nu(\emptyset) = 0 \quad \text{and} \quad \nu(A) \leq \nu(B) \implies \nu(A \setminus B) \leq \nu(B \setminus A), \quad \text{for all} \ A, B \subset \mathbb{N}.
\]

In Dilworth et al. (2018, Remark 2.7), the authors provided an example of a weight \( \nu \) on subsets of \( \mathbb{N} \) that cannot be obtained by any sequential weight \( \varsigma \). However, it was not known if a \( \nu \)-weighted basis must be \( \varsigma \)-weighted for some sequential weight \( \varsigma \).

Motivated by this, we leverage recent results to obtain an example of an \( \omega \)-greedy basis that is not \( \varsigma \)-greedy for any \( \varsigma \) (Theorem 3.4.)

Last but not least, we study \( \omega \)-semi-greedy and \( \omega \)-partially greedy bases. We establish the equivalence between \( \omega \)-semi-greedy and \( \omega \)-almost greedy bases (Theorem 4.2), thus extending the same equivalence known to hold for sequential weights \( \varsigma \).

To prepare for the next section, we give a formal definition of a weight \( \omega \) on sets and \( \omega \)-(almost) greedy bases.

**Definition 1.2** Let \( \mathcal{P}(\mathbb{N}) \) be the power set of \( \mathbb{N} \). A weight on sets is a nonnegative function \( \omega : \mathcal{P}(\mathbb{N}) \to [0, \infty) \) such that

- \( \omega(\emptyset) = 0 \),
- \( \omega(A) \in (0, \infty] \) for each nonempty \( A \subset \mathbb{N} \).

For a given Schauder basis \( B = (e_n)_{n=1}^{\infty} \) and \( B \subset \mathbb{N} \), define

\[
\sigma_B^\omega(x) := \inf \left\{ \left\| x - \sum_{n \in A} a_n e_n \right\| : A \in \mathbb{N}^{< \infty}, \omega(A \setminus B) \leq \omega(B \setminus A), a_n \in \mathbb{K} \right\},
\]

where \( \mathbb{N}^{< \infty} \) is the set of all finite subsets of \( \mathbb{N} \).

The following definition of \( \omega \)-greedy bases covers the classical weighted bases.

**Definition 1.3** (Extension of Kerkyacharian et al. (2006, Definition 4.1)) A basis \( B \) is \( \omega \)-greedy if there exists a constant \( C \geq 1 \) such that for all \( x \in \mathbb{X} \), \( m \in \mathbb{N} \), and \( \Lambda_m(x) \),

\[
\| x - G_m(x) \| \leq C \sigma_{\Lambda_m(x)}^\omega(x).
\]

The least constant \( C \) is denoted by \( C_\omega^g \).

In a similar manner, we define \( \omega \)-almost greedy. For a basis \( B = (e_n)_{n=1}^{\infty} \) and \( B \subset \mathbb{N} \), let

\[
\tilde{\sigma}_B^\omega(x) := \inf \left\{ \| x - P_A(x) \| : A \in \mathbb{N}^{< \infty}, \omega(A \setminus B) \leq \omega(B \setminus A) \right\},
\]

where \( P_A(x) = \sum_{n \in A} e_n^*(x)e_n \).

**Definition 1.4** (Extension of Dilworth et al. (2018, Definition 1.1)) A basis \( B \) is \( \omega \)-almost greedy if there exists a constant \( C \geq 1 \) such that for all \( x \in \mathbb{X} \), \( m \in \mathbb{N} \), and \( \Lambda_m(x) \),

\[
\| x - G_m(x) \| \leq C \tilde{\sigma}_{\Lambda_m(x)}^\omega(x).
\]
The least constant $C$ is denoted by $C_{ω}^ω$.

2 Characterizations of ω-(Almost) Greedy Bases

In order to characterize $ω$-(almost) greedy bases, we need the notion of unconditionality and $ω$-Property (A).

Definition 2.1 A basis $B$ is unconditional if there exists $C ≥ 1$ such that
\[
\|P_A(x)\| ≤ C\|x\|, \quad \forall x ∈ X, \; \forall A ⊂ N.
\]
In this case, we say that $B$ is $C$-suppression unconditional. The least such $C$ is denoted by $K_s$. For an unconditional basis, there also exists a constant $K_u$ such that
\[
\left\|\sum_{n=1}^{N} a_n e_n\right\| ≤ K_u \left\|\sum_{n=1}^{N} b_n e_n\right\|
\]
for all $N ≥ 1$ and for all scalars $a_n, b_n$ with $|a_n| ≤ |b_n|$.

Let
\[
1_A = \sum_{n ∈ A} e_n \; \text{and} \; 1_ε A = \sum_{n ∈ A} ε_n e_n,
\]
where $ε = (ε_n)_{n=1}^{∞} ⊂ K$ and $|ε_n| = 1$. For $x ∈ X$, supp $(x) := \{n : e^*_n(x) ≠ 0\}$, $\|x\|_{∞} := \sup_n |e^*_n(x)|$, and we write $A ∪ B ∪ x$ to indicate that $A, B$, and supp $(x)$ are pairwise disjoint. Finally, sgn $(e^*_n(x)) = \begin{cases} 0 & \text{if } e^*_n(x) = 0, \\ e^*_n(x)/|e^*_n(x)| & \text{if } e^*_n(x) ≠ 0. \end{cases}$

Definition 2.2 (Extension of Berná et al. (2019, Definition 1.3)) A basis $B$ has $ω$-Property (A) if there exists $C ≥ 1$ such that
\[
\|x + 1_ε A\| ≤ C\|x + 1_δ B\|
\]
for all $x ∈ X$ with $\|x\|_{∞} ≤ 1$, $A, B ∈ \mathbb{N}^{<\infty}$ with $ω(A) ≤ ω(B)$ and $A ∪ B ∪ x$, and signs $(ε), (δ)$. The least constant $C$ is denoted by $C_{ω}^ω$.

Theorem 2.3 (Extension of Berná et al. (Berná et al. 2019, Theorem 4.1)) Let $B$ be a basis and $ω$ be a weight on subsets of $\mathbb{N}$.

1. If $B$ is $C_{ω}^ω$-ω-greedy, then $B$ is $C_{ω}^ω$-suppression unconditional and satisfies $C_{ω}^ω$-ω-Property (A).
2. If $B$ is $K_s$-suppression unconditional and satisfies $C_{ω}^ω$-ω-Property (A), then $B$ is $K_s C_{ω}^ω$-ω-greedy.

First, we need an useful reformulation of $ω$-Property (A).
Lemma 2.4  (Extension of Berná et al. (2019, Theorem 4.1)) A basis $B$ has $C_\omega^\omega$-Property (A) if and only if

$$\|x\| \leq C_\omega^\omega \|x - P_A(x) + 1_{\mathcal{E}B}\|,$$

(2.1)

for all $x \in X$ with $\|x\|_\infty \leq 1$, $A, B \in \mathbb{N}^{<\infty}$ with $\omega(A) \leq \omega(B)$ and $B \cap (A \cup \text{supp}(x)) = \emptyset$, and sign $(\varepsilon)$.

**Proof**  Assume that $B$ has $C_\omega^\omega$-Property (A). Let $x, A, B, (\varepsilon)$ be chosen as in (2.1). We have

$$\|x\| = \left\| x - P_A(x) + \sum_{n \in A} e_n^*(x)e_n \right\| \leq \sup_{(\delta)} \|x - P_A(x) + 1_{\delta A}\| \leq C_\omega^\omega \|x - P_A(x) + 1_{\mathcal{E}B}\|,$$

as desired.

Next, assume that $B$ satisfies (2.1). Let $x, A, B, (\varepsilon), (\delta)$ be chosen as in Definition 2.2. Let $y = x + 1_{\mathcal{E}A}$. By (2.1),

$$\|x + 1_{\mathcal{E}A}\| = \|y\| \leq C_\omega^\omega \|y - P_A(y) + 1_{\delta B}\| = C_\omega^\omega \|x + 1_{\mathcal{E}B}\|.$$

This completes our proof.  

The following result of Albiac and Wojtaszczyk (2006) will be used in due course.

**Proposition 2.5** (Albiac and Wojtaszczyk 2006, Proposition 2.1) Let $B = (e_n)_{n=1}^\infty$ be a $K_s$-suppression unconditional basis. Fix $N \in \mathbb{N}$. For any scalars $a_1, \ldots, a_N, b_1, \ldots, b_N$ so that either $a_n = 0$ or $\text{sgn}(a_n) = \text{sgn}(b_n)$ and $|a_n| \leq |b_n|$ for all $1 \leq n \leq N$, we have

$$\left\| \sum_{n=1}^N a_n e_n \right\| \leq K_s \left\| \sum_{n=1}^N b_n e_n \right\|.$$

**Proof of Theorem 2.3**  Assume that $B$ is $C_\omega^g_\omega$-greedy. Let $x \in X$ and $B \in \mathbb{N}^{<\infty}$. Write

$$y = \sum_{n \in B} (\alpha + e_n^*(x))e_n + P_{B^c}(x),$$

where $\alpha$ is chosen sufficiently large such that $B$ is a greedy set of $y$. Then

$$\|P_{B^c}(x)\| = \|y - G_{|B|}(y)\| \leq C_\omega^g \sigma_B^\omega(y) \leq C_\omega^\omega \|y - \alpha 1_B\| = C_\omega^\omega \|x\|.$$

Hence, $B$ is $C_\omega^\omega$-suppression unconditional. Next, we prove $\omega$-Property (A). We choose $x, A, B, (\varepsilon), (\delta)$ as in Definition 2.2. Set $y = x + 1_{\mathcal{E}A} + 1_{\delta B}$. Then $B$ is a greedy set of $y$. We have

$$\|x + 1_{\mathcal{E}A}\| = \|y - G_{|B|}(y)\| \leq C_\omega^g \sigma_B^\omega(y) \leq C_\omega^\omega \|y - P_A(y)\| = C_\omega^\omega \|x + 1_{\mathcal{E}B}\|.$$
This completes the first part of the proof.

Now we assume that $\mathcal{B}$ is $K_\cdot$-suppression unconditional and satisfies $C^\omega_p$-Property (A). Take $x \in X$ with a greedy set $A \subset \mathbb{N}$. By the standard density argument, we can assume that $x$ is finitely supported. Let $B \subset \mathbb{N}$ satisfy $\omega(B \setminus A) \leq \omega(A \setminus B)$. Also, choose arbitrary $(b_n)_{n \in B} \subset \mathbb{K}$. Let $\alpha := \min_{n \in A} |e_n^p(x)|$. We have

$$
\| x - P_A(x) \|
\leq C^\omega_p \left\| x - P_A(x) - P_{B \setminus A}(x) + \alpha \sum_{n \in A \setminus B} \text{sgn}(e_n^p(x)) e_n \right\| \quad \text{by Lemma 2.4}
$$

$$
= C^\omega_p \left\| P_{(A \cup B)^c}(x) + \alpha \sum_{n \in A \setminus B} \text{sgn}(e_n^p(x)) e_n \right\|
$$

$$
\leq K_\cdot C^\omega_p \left\| P_{(A \cup B)^c}(x) + \sum_{n \in B} (e_n^p(x) - b_n) e_n + P_{A \setminus B}(x) \right\| \quad \text{by Proposition 2.5}
$$

$$
= K_\cdot C^\omega_p \left\| x - \sum_{n \in B} b_n e_n \right\|.
$$

This completes our proof that $\mathcal{B}$ is $K_\cdot C^\omega_p$-greedy. \qed

When we do not need tight estimates, the notion of $\omega$-disjoint (super)democracy can play the role of $\omega$-Property (A), providing other characterizations of $\omega$-greedy bases.

**Definition 2.6** (Extension of Kerkyacharian et al. (2006, Definition 4.2)) A basis $\mathcal{B}$ is $\omega$-disjoint democratic ($\omega$-disjoint superdemocratic, respectively) if there exists $C \geq 1$ such that

$$
\|1_A\| \leq C\|1_B\|, (\|1_\varepsilon A\| \leq C\|1_\delta B\|, \text{respectively},)
$$

for all $A, B \subset \mathbb{N}$ with $\omega(A) \leq \omega(B)$ and $A \cap B = \emptyset$ and signs ($\varepsilon$), ($\delta$). The least constant $C$ is denoted by $C^\omega_{d,\sqcup}$ (and $C^\omega_{sd,\sqcup}$, respectively.)

**Remark 2.7** A basis $\mathcal{B}$ is said to be $\omega$-(super)democratic if in Definition 2.6, we drop the requirement $A \cap B = \emptyset$; $\mathcal{B}$ is said to be (super)democratic if it is $\omega$-(super)democratic for $\omega$ being the cardinality weight, i.e., $\omega(A) = |A|$, $\forall A \subset \mathbb{N}$.

It follows from Proposition 6.1 that Definition 2.6 is truly an extension of definition Kerkyacharian et al. (2006, Definition 4.2).

**Theorem 2.8** (Extension of Berná et al. (2019, Theorem 4.1) and Kerkyacharian et al. (2006, Theorem 4.1)) Let $\mathcal{B}$ be a basis and $\omega$ be a weight on subsets of $\mathbb{N}$. The following are equivalent:

1. $\mathcal{B}$ is $\omega$-greedy,
2. $\mathcal{B}$ is unconditional and satisfies $\omega$-Property (A),
(3) \( B \) is unconditional and \( \omega \)-disjoint superdemocratic,

(4) \( B \) is unconditional and \( \omega \)-disjoint democratic.

**Proof** By Theorem 2.3, we know that (1) \( \iff \) (2). It follows immediately from definitions that \( \omega \)-Property (A) \( \implies \) \( \omega \)-disjoint superdemocratic \( \implies \) \( \omega \)-disjoint democratic. Hence, (2) \( \implies \) (3) \( \implies \) (4). It remains to show that (4) \( \implies \) (2). Let \( x, A, B, (\varepsilon), (\delta) \) be chosen as in Definition 2.2. We have

\[
\| x + 1_{\delta A} \| \leq \| x \| + \| 1_{\delta A} \| \leq \| x \| + K_u \| 1_A \| \\
\leq \| x \| + K_u C^\omega_{d,\omega} \| 1_B \| \\
\leq K_x \| x + 1_{\delta B} \| + K_u^2 C^\omega_{d,\omega} \| x + 1_{\delta B} \| \\
= (K_x + K_u^2 C^\omega_{d,\omega}) \| x + 1_{\delta B} \|. 
\]

This completes our proof. \( \square \)

For \( \omega \)-almost greedy bases, corresponding results hold. We include the proof of the next theorem in the Appendix.

**Theorem 2.9** (Extension of Berná et al. (2019, Theorem 4.3)) Let \( B \) be a basis and \( \omega \) be a weight on subsets of \( \mathbb{N} \).

1. If \( B \) is \( C_{\omega}^{\omega} \)-\( \omega \)-almost greedy, then \( B \) is \( C_{\alpha}^{\omega} \)-suppression quasi-greedy and satisfies \( C_{\omega}^{\omega} \)-\( \omega \)-Property (A).
2. If \( B \) is \( C_{\ell} \)-suppression quasi-greedy and satisfies \( C_{\omega} \)-\( \omega \)-Property (A), then \( B \) is \( C_{\ell} C_{\omega}^{\omega} \)-\( \omega \)-almost greedy.

**Theorem 2.10** (Extension of Berná et al. (2019, Theorem 4.3) and Dilworth et al. (2018, Theorem 2.6)) Let \( B \) be a basis and \( \omega \) be a weight on subsets of \( \mathbb{N} \). The following are equivalent:

1. \( B \) is \( \omega \)-almost greedy,
2. \( B \) is quasi-greedy and satisfies \( \omega \)-Property (A),
3. \( B \) is quasi-greedy and \( \omega \)-disjoint superdemocratic,
4. \( B \) is quasi-greedy and \( \omega \)-disjoint democratic.

**Corollary 2.11** (1) A basis \( B \) is unconditional if and only if \( B \) is \( \omega \)-greedy for some weight \( \omega \).

(2) A basis \( B \) is quasi-greedy if and only if \( B \) is \( \omega \)-almost greedy for some weight \( \omega \).

**Proof** (1) If \( B \) is \( \omega \)-greedy for some weight \( \omega \), then \( B \) is unconditional by Theorem 2.3. Conversely, suppose that \( B \) is unconditional. Define the weight \( \omega \) on a set \( A \)

\[
\omega(A) = \begin{cases} 
\| 1_A \| & \text{if } A \text{ is finite,} \\
\infty & \text{if } A \text{ is infinite.} 
\end{cases}
\]

By Theorem 2.8, it suffices to show that \( B \) is \( \omega \)-disjoint democratic. This is clearly true since for two finite sets \( A, B \) with \( \omega(A) \leq \omega(B) \), we get \( \| 1_A \| \leq \| 1_B \| \) by the definition of \( \omega \).

The proof of (2) is similar to that of (1). \( \square \)
3 A Set-Weighted-Greedy Basis That is Not Sequence-Weighted-Greedy

The following theorem provides a necessary condition for a basis to be \( \zeta \)-greedy for some weight sequence \( \zeta \).

**Theorem 3.1** If a basis \( B = (e_n)_{n=1}^\infty \) is \( \zeta \)-(almost) greedy for some weight sequence \( \zeta \), then either \( B \) is (almost) greedy or there exists a subsequence \( (e_{n_k})_{k=1}^\infty \) equivalent to the canonical basis of \( c_0 \).

Observe that \( \zeta \)-Property (A) is \( \omega \)-Property (A) when the weight \( \omega \) on sets is determined by a weight sequence \( \zeta \). Particularly, Property (A) (first introduced in Albiac and Wojtaszczyk 2006 and later extended in Dilworth et al. 2014) is \( \zeta \)-Property (A) when \( \zeta = (1, 1, \ldots) \).

**Definition 3.2** (Berná et al. 2019, Definition 1.3) A basis \( B \) has \( \zeta \)-Property (A) if there exists \( C \geq 1 \) such that

\[
\|x + 1_{\varepsilon A}\| \leq C\|x + 1_{\delta B}\|,
\]

for all \( x \in X \) with \( \|x\|_\infty \leq 1 \), \( A, B \in \mathbb{N}^{<\infty} \) with \( s(A) \leq s(B) \) and \( A \cup B \cup x \), and signs \( (\varepsilon), (\delta) \). The least constant \( C \) is denoted by \( C^\zeta_B \). As a special case, a basis \( B \) is said to have Property (A) if it has \( \zeta \)-Property (A) for \( \zeta = (1, 1, \ldots) \).

**Proof of Theorem 3.1** We assume that \( B \) is \( \zeta \)-greedy for some weight sequence \( \zeta = (s(n))_{n=1}^\infty \). By Berná et al. (2019, Theorem 4.1), \( B \) is unconditional and has \( \zeta \)-Property (A).

If \( 0 < \inf s(n) \leq \sup s(n) < \infty \), then Berná et al. (2019, Proposition 3.5) implies that \( B \) has Property (A). According to Dilworth et al. (2014, Theorem 2), we know that \( B \) is greedy.

If \( \sup s(n) = \infty \), then Berná et al. (2019, Proposition 3.10) states that \( B \) is equivalent to the canonical basis of \( c_0 \) and thus, is greedy.

If \( \inf s(n) = 0 \), then by Berná et al. (2019, Proposition 3.10), \( (e_n)_{n=1}^\infty \) has a subsequence \( (e_{n_k})_{k=1}^\infty \) that is equivalent to the canonical basis of \( c_0 \).

The proof of the almost greedy case is similar.

We now state the existence of an \( \omega \)-greedy basis that is not \( \zeta \)-almost greedy for any weight sequence \( \zeta \). We can, in particular, require the weight \( \omega \) to have a more rigid structure than in Definition 1.2. For conciseness, we let \( \omega_n := \omega([n]) \).

**Definition 3.3** A structured weight is a nonnegative function \( \omega : \mathcal{P}(\mathbb{N}) \rightarrow [0, \infty] \) such that

(a) \( \omega(\emptyset) = 0 \),
(b) \( \omega(A) < \infty \) if \( |A| < \infty \),
(c) \( \omega(A) \in (0, \infty] \) for each nonempty \( A \subset \mathbb{N} \),
(d) \( \omega(A) \rightarrow 0 \) as \( \sum_{n \in A} \omega_n \rightarrow 0 \) (with respect to (wrt) the Euclidean metric on the real line),
(e) \( \omega(A) \to \infty \) as \( \sum_{n \in A} \omega_n \to \infty \) (wrt the Euclidean metric on the real line),

(f) There exists an arbitrarily large number \( N \in \mathbb{N} \) such that there exists an \( \varepsilon > 0 \)

satisfying \( \omega([N, n)) - \omega_n > \varepsilon \) for all \( n \in \mathbb{N}, n \neq N \).

Conditions (a), (b), and (c) are almost the same as what we have in Definition 1.2, except that we now require the weight to be finite on finite sets. Conditions (d) and (e) are reasonable. Condition (d) states that the weight on a set approaches 0 when the sum of weights of its singletons approaches 0, while (e) states the same condition with 0 replaced by \( \infty \). In what follows, we will specify whether we need structured weights in our results.

**Theorem 3.4** There exists a basis that is \( \omega \)-greedy for some structured weight \( \omega \) on sets but is not \( \varsigma \)-almost greedy for any weight sequence \( \varsigma \) on positive integers.

**Proof** By Theorem 3.1, an unconditional basis that is neither democratic nor has a subsequence equivalent to the canonical basis of \( c_0 \) is not \( \varsigma \)-almost greedy on any \( \varsigma \).

Set \( P := \{2^k : k \geq 1\} \), \( a_n = 1/n^{1/2} \) and \( b_n = 1/n \) for \( n \geq 1 \). Let \( X \) be the completion of \( c_0 \) under the following norm: for \( x = (x_1, x_2, \ldots) \in c_0 \), define

\[
\|x\| := \left( \sup_{\sigma} \sum_{i \in P} a_{\sigma(i)} |x_i| \right) + \left( \sup_{\pi} \sum_{i \notin P} b_{\pi(i)} |x_i| \right),
\]

where \( \sigma : P \to \mathbb{N} \) and \( \pi : \mathbb{N} \setminus P \to \mathbb{N} \) are bijections. Let \( B = (e_n)_{n=1}^\infty \) be the canonical basis. Clearly, \( B \) is unconditional and normalized. However, \( B \) is not democratic. Indeed, fix \( N \in \mathbb{N} \) and set \( A = \{3^1, 3^2, \ldots, 3^N\}, B = \{2^1, 2^2, \ldots, 2^N\} \). We have

\[
\|1_A\| = \sum_{n=1}^N \frac{1}{n} \sim \ln(N) \text{ and } \|1_B\| = \sum_{n=1}^N \frac{1}{\sqrt{n}} \sim \sqrt{N}.
\]

Since \( \|1_B\|/\|1_A\| \sim \sqrt{N}/\ln(N) \to \infty \) as \( N \to \infty \), we know that \( B \) is not democratic and thus, not almost greedy.

By the proof of Corollary 2.11, \( B \) is \( \omega \)-greedy for the following weight \( \omega \)

\[
\omega(A) = \begin{cases} 
\|1_A\| & \text{if } A \text{ is finite,} \\
\infty & \text{if } A \text{ is infinite.}
\end{cases}
\]

It is easy to check that \( \omega \) is a structured weight.

We claim that there is no subsequence of \( B = (e_n)_{n=1}^\infty \) that is equivalent to the canonical basis of \( c_0 \). Indeed, pick any subsequence \( (e_{n_k})_{k=1}^\infty \) of \( B \). For \( N \in \mathbb{N} \), we have

\[
\| \sum_{k=1}^N e_{n_k} \| \geq \sum_{n=1}^N \frac{1}{n} \sim \ln(N).
\]
Hence, \((e_{n_k})_{k=1}^\infty\) is not equivalent to the canonical basis of \(c_0\). Therefore, Theorem 3.1 and the fact that \(B\) is not almost greedy tell us that \(B\) is not \(\varsigma\)-almost greedy for any weight sequence \(\varsigma\).

\[\square\]

4 \(\omega\)-Semi-Greedy Bases

We define the \(\omega\)-version of the classical semi-greedy bases (first introduced in Dilworth et al. 2003a), whose sequentially weighted version was studied first in Dilworth et al. (2018) then in Berná (2020). Corresponding to each greedy set \(\Lambda_m(x)\), there is a so-called Chebyshev greedy sum of order \(m\), denoted by \(CG_m(x)\), such that

1. \(\text{supp}(CG_m(x)) \subset \Lambda_m(x)\) and
2. \(\|x - CG_m(x)\| = \min \left\{ \left\| x - \sum_{n \in \Lambda_m(x)} a_n e_n \right\| : (a_n) \subset K \right\}\).

**Definition 4.1** (Extension of Dilworth et al. (2018, Definition 4.1)) A basis \(B\) is \(\omega\)-semi-greedy if there exists a constant \(C \geq 1\) such that for all \(x \in X\), \(m \in \mathbb{N}\), and \(\Lambda_m(x)\),

\[\|x - CG_m(x)\| \leq C \sigma^{\omega}_{\Lambda_m(x)}(x)\]

The least constant \(C\) is denoted by \(C^{\omega}_s\).

The main goal of this section is to establish the following theorem.

**Theorem 4.2** (Extension of Berná (2020, Theorem 1.10)) Let \(\omega\) be a structured weight. Then \(B\) is \(\omega\)-semi-greedy if and only if it is \(\omega\)-almost greedy.

The next result is the key to case analysis in the proof of Theorem 4.2.

**Proposition 4.3** (Extension of Dilworth et al. (2018, Proposition 4.5) and Berná (2020, Proposition 2.4)) Let \(B\) be a \(C^{\omega}_s\)-\(\omega\)-semi-greedy basis, where \(\omega\) is structured.

1. If \(B \in \mathbb{N}^{\prec \infty}\) and \(\omega(B) \leq \limsup_{n \to \infty} \omega_n\). Then we have
   \[\sup_{(e)} \| 1_{e_B} \| \leq 2 K_b C^{\omega}_s c_2,\]
   where \(c_2\) is in (1.1).
2. If \(\sup_n \omega_n = \infty\) or \(\sum_n \omega_n < \infty\), then \(B\) is equivalent to the canonical basis of \(c_0\).
3. If \(\inf_n \omega_n = 0\), then \(B\) contains a subsequence equivalent to the canonical basis of \(c_0\).

**Remark 4.4** The conclusions in Proposition 4.3 still hold if our basis \(B\) is \(C^{\omega}_{sd,\sqcup}\)-disjoint superdemocratic. The proof is left for interested readers.
Proof (1) Pick \( B \in \mathbb{N}^{<\infty} \) and a sign \((\varepsilon)\). Choose \( N_1 > \max B \) be the number in condition (f) of a structured weight such that there exists \( \varepsilon > 0 \) satisfying \( \omega([N_1, n]) > \omega_n + \varepsilon \) for all \( n \neq N_1 \). It follows that

\[
\limsup_{n \to \infty} \omega([N_1, n]) = \limsup_{n \to \infty} \omega_n + \varepsilon \geq \omega(B) + \varepsilon.
\]

Pick \( N_2 > N_1 \) such that \( \omega([N_1, N_2]) > \omega(B) \). (This is possible due to condition (b).) Set \( x := 1_{\varepsilon B} + e_{N_1} + e_{N_2} \). Then \([N_1, N_2]\) is a greedy set of \( x \). Let \( \|x - C G_2(x)\| = \|1_{\varepsilon B} + \alpha_1 e_{N_1} + \alpha_2 e_{N_2}\| \) for some \( \alpha_1, \alpha_2 \in \mathbb{K} \). We have

\[
\|1_{\varepsilon B}\| \leq K_b \|1_{\varepsilon B} + \alpha_1 e_{N_1} + \alpha_2 e_{N_2}\| \leq K_b C_s^\omega \sigma_{[N_1, N_2]}(x) \\
\leq K_b C_s^\omega \|\varepsilon N_1 + e_{N_2}\| \leq 2K_b C_s^\omega c_2.
\]

(2) If \( \sup_{n} \omega_n = \infty \), then by (1), \( \sup_{(\varepsilon)} \|1_{\varepsilon B}\| \leq 2K_b C_s^\omega c_2, \forall B \in \mathbb{N}^{<\infty} \). Hence, the basis is equivalent to the canonical basis of \( c_0 \). If \( \sum_n \omega_n < \infty \), then choose \( N \in \mathbb{N} \) such that \( \sum_{n=N+1}^{\infty} \omega_n \) is so small that \( \omega(E) \leq \omega_1 \) for all \( E \subseteq \mathbb{N}_{N+1} \). This can be done due to condition (d) of \( \omega \). We claim that for any \( B \in \mathbb{N}^{<\infty} \) and any sign \((\varepsilon)\), we have \( \|1_{\varepsilon B}\| = O(1) \). Let \( B_1 = B \cap [1, N] \) and \( B_2 = B \cap [N + 1, \infty) \). Observe that

\[
\|1_{\varepsilon B}\| \leq \|1_{\varepsilon B_1}\| + \|1_{\varepsilon B_2}\| \leq N c_2 + \|1_{\varepsilon B_2}\|.
\]

Set \( x := e_1 + 1_{\varepsilon B_2} \). Then \([1]\) is a greedy set of \( x \). Let \( \|x - C G_1(x)\| = \|\alpha e_1 + 1_{\varepsilon B_2}\| \) for some \( \alpha \in \mathbb{K} \). Since \( \omega(B_2) < \omega_1 \), we have

\[
\|1_{\varepsilon B_2}\| \leq (K_b + 1) \|\alpha e_1 + 1_{\varepsilon B_2}\| \leq (K_b + 1) C_s^\omega \sigma_{[1]}(x) \\
\leq (K_b + 1) C_s^\omega \|e_1\| \leq (K_b + 1) C_s^\omega c_2.
\]

This completes our proof that \( \|1_{\varepsilon B}\| = O(1) \) and so, \( B \) is equivalent to the canonical basis of \( c_0 \).

(3) Choose a subsequence \( (n_k)_{k=1}^{\infty} \) such that \( \sum_{k=1}^{\infty} \omega_{n_k} < \infty \) and apply (2). \( \square \)

**Theorem 4.5** (Extension of Dilworth et al. (2018, Theorem 4.8) and Berná (2020, Theorem 1.10 (a))) Let \( \omega \) be a structured weight. If a basis \( B \) is \( \omega \)-semi-greedy, then it is quasi-greedy and \( \omega \)-superdemocratic.

**Proof** Suppose that \( \sum_{n=1}^{\infty} \omega_n < \infty \) or \( \sup_n \omega_n = \infty \). By Proposition 4.3, we know that \( B \) is equivalent to the canonical basis of \( c_0 \), and the desired conclusion follows trivially. For the rest of the proof, let us assume that \( \sum_{n=1}^{\infty} \omega_n = \infty \) and \( \sup_n \omega_n < \infty \).

**Quasi-greedy:** Let \( x \in \mathbb{K} \) with \( \|x\|_\infty \leq 1 \), \( \text{sup}(x) < \infty \), and a greedy set \( \Lambda_m(x) \).

**Case 1:** \( \omega(\Lambda_m(x)) \leq \limsup_{n \to \infty} \omega_n \). By Proposition 4.3, we have

\[
\sup_{(\varepsilon)} \|1_{\varepsilon \Lambda_m(x)}\| \leq 2K_b C_s^\omega c_2.
\]
By norm convexity,
\[ \| P_{\Lambda_m(x)}(x) \| \leq \max_n | e^*_n(x) | \sup_{(x)} \| 1 \Lambda_{m}(x) \| \]
\[ \leq \sup_n \| e^*_n \| \| x \| \cdot 2 K_b C_s^{\omega} c_2 \leq 2 K_b C_s^{\omega} c_2 c_2^* \| x \| , \]
where \( c_2 \) and \( c_2^* \) are in (1.1) and (1.2), respectively.

Case 2: \( \omega(\Lambda_m(x)) > \lim \sup_{n \to \infty} \omega_n \). We build a finite set \( E \) as follows: choose \( N > \max \text{supp}(x) \) such that \( \omega_N \leq \omega(\Lambda_m(x)) \). Let \( k \) be the smallest positive integer verifying
\[ \omega([N, N+1, \ldots, N+k]) \leq \omega(\Lambda_m(x)) < \omega([N, N+1, \ldots, N+k, N+k+1]). \]
We know such \( k \) exists due to \( \sum_n \omega_n = \infty \) and condition (e) of a structured weight. Let \( A = [N, N+1, \ldots, N+k] \) and \( B = A \cup [N+k+1] \). Define
\[ y := x - P_{\Lambda_m(x)}(x) + \alpha 1_B, \]
where \( \alpha := \min_{n \in \Lambda_m(x)} | e^*_n(x) | \). Since \( B \) is a greedy set of \( y \), by \( C_s^{\omega} \)-\( \omega \)-semi-greediness, there exist \( (b_n)_{n \in B} \subset \mathbb{K} \) such that
\[ \| x - P_{\Lambda_m(x)}(x) \| \leq K_b \| x - P_{\Lambda_m(x)}(x) + \sum_{n \in B} b_n e_n \| \leq K_b C_s^{\omega} \sigma_B^{\omega}(y) \]
\[ \leq K_b C_s^{\omega} \| x \| + \alpha 1_B \| \leq K_b C_s^{\omega} (\| x \| + \alpha \| 1_B \| + \alpha \| e_{N+k+1} \|). \]
(4.1)

Pick \( j \in \Lambda_m(x) \). We have
\[ \alpha \| e_{N+k+1} \| \leq \alpha c_2 \leq c_2 | e^*_j(x) | \leq c_2 \| e^*_j \| \| x \| \leq c_2 c_2^* \| x \|. \]
(4.2)

It remains to bound \( \alpha \| 1_A \| \). Let \( z := x + \alpha 1_A \). Since \( \Lambda_m(x) \) is a greedy set of \( z \), \( C_s^{\omega} \)-\( \omega \)-semi-greediness gives \( (t_n)_{n \in \Lambda_m(x)} \subset \mathbb{K} \) such that
\[ \| \alpha 1_A \| \leq (K_b + 1) \| \sum_{n \in \Lambda_m(x)} t_n e_n + P_{\Lambda_m(x)}(x) + \alpha 1_A \| \]
\[ \leq (K_b + 1) C_s^{\omega} \sigma_{\Lambda_m(x)}^{\omega}(x) \leq (K_b + 1) C_s^{\omega} \| x \|. \]
(4.3)

From (4.1), (4.2), and (4.3), we have shown that
\[ \| x - P_{\Lambda_m(x)}(x) \| = O(\| x \|). \]

This completes our proof that \( B \) is quasi-greedy.

\( \omega - \text{superdemocratic} \): Let \( A, B \in \mathbb{N}^{<\infty} \) with \( \omega(A) \leq \omega(B) \). Pick signs \( (\varepsilon), (\delta) \).
Case 1: $\omega(A) \leq \omega(B) \leq \limsup_{n \to \infty} \omega_n$. By Proposition 4.3, we know that

$$\|1_{\varepsilon A}\| \leq 2K_b C_s^\omega c_2.$$  

On the other hand, if $j = \min B$, then

$$\|1_{\delta B}\| \geq \|e_j\|/K_b \geq c_1/K_b,$$

where $c_1$ is in (1.1). Therefore,

$$\|1_{\varepsilon A}\| \leq 2K_b^2 C_s^\omega c_2 c_1 \|1_{\delta B}\|.$$  

Case 2: $\omega(B) > \limsup_{n \to \infty} \omega_n$. As when we prove quasi-greediness, choose $E$ and $F = E \cup \{N\}$ such that $A \cup B < E < \{N\}$ and $\omega(E) \leq \omega(B) < \omega(F)$. Set $x := 1_{\varepsilon A} + 1_{F}$. Then $F$ is a greedy set of $x$. By $C_s^\omega$-semi-greediness, there exist $(a_n)_{n \in F} \subset \mathbb{K}$ such that

$$\|1_{\varepsilon A}\| \leq K_b \left\|1_{\varepsilon A} + \sum_{n \in F} a_n e_n\right\| \leq K_b C_s^\omega \sigma_F^\omega(x) \leq K_b C_s^\omega \|1_F\|.$$

(4.4)

Now, let $y = 1_{\delta B} + 1_{E}$. Since $B$ is a greedy set of $y$, by $C_s^\omega$-semi-greediness, we obtain

$$\|1\| \leq (K_b + 1) \left\|\sum_{n \in B} b_n e_n + 1_{E}\right\| \leq C_s^\omega (K_b + 1) \sigma_B^\omega(y) \leq C_s^\omega (K_b + 1) \|1_{\delta B}\|,$$

(4.5)

for some $(b_n)_{n \in B} \subset \mathbb{K}$. Furthermore, if $u = \min E$,

$$\|1_F\| \leq \|1_E\| + \|e_N\| \leq \|1_E\| + c_2 \leq \|1_E\| + \frac{c_2}{c_1} \|e_u\| \leq \left(\frac{c_2}{c_1} K_b + 1\right) \|1_E\|.$$

(4.6)

From (4.4), (4.5), and (4.6), we obtain

$$\|1_{\varepsilon A}\| \leq (C_s^\omega)^2 K_b (K_b + 1) \left(\frac{c_2}{c_1} K_b + 1\right) \|1_{\delta B}\|.$$  

Hence, $B$ is $\omega$-superdemocratic.

The proof of the next theorem is similar to that of Dilworth et al. (2003a, Theorem 3.2) with obvious modifications, so we move the proof to the Appendix.

**Theorem 4.6** (Extension of Dilworth et al. (2018, Theorem 4.3) and Berná (2020, Theorem 1.10 (b))) *If a basis $B$ is quasi-greedy and $\omega$-disjoint superdemocratic, then it is $\omega$-semi-greedy.*
Proof of Theorem 4.2  The theorem follows from Theorems 2.8, 4.5, and 4.6.

5 ω-Partially Greedy Bases

Partially greedy bases were first introduced and characterized in Dilworth et al. (2003b) to compare the performance of the TGA to that of the partial sum operators \((S_m)_{m=1}^\infty\). In this section, we characterize \(\omega\)-partially greedy bases and prove the existence of \(\omega\)-partially greedy bases that are not \(\varsigma\)-partially greedy for any sequence weight \(\varsigma\). For each \(m \geq 0\), let \(L_m := \{1, 2, \ldots, m\}\).

Definition 5.1  (Extension of Berná et al. (2019, Definition 6.1)) A basis is said to be \(\omega\)-partially greedy if there exists \(C \geq 1\) such that for all \(x \in \mathbb{X}, m \in \mathbb{N}, \) and \(\Lambda_m(x)\), we have

\[
\|x - G_m(x)\| \leq C\sigma_{\Lambda_m(x)}(x),
\]

where

\[
\sigma_{\Lambda}(x) := \inf \{\|x - S_k(x)\| : \omega(L_k \setminus A) \leq \omega(A \setminus L_k)\}.
\]

The least such \(C\) is denoted by \(C_{p\omega}\).

We shall characterize \(\omega\)-partial greediness, covering characterizations of \(\varsigma\)-partially greedy bases in the literature. In Berasategui et al. (2021), the authors introduce partial symmetry for largest coefficients (PSLC).

Definition 5.2  (Extension of Berasategui et al. (2021, Definition 1.9)) A basis is \(C\omega\)-PSLC if

\[
\|x + 1_{\varepsilon A}\| \leq C\|x + 1_{\delta B}\|,
\]

for all \(x \in \mathbb{X}\) with \(\|x\|_{\infty} \leq 1\), \(A, B \in \mathbb{N}^{<\infty}\) with \(\omega(A) \leq \omega(B)\) and \(A < \text{supp}(x) \sqcup B\), and signs \((\varepsilon), (\delta)\). The least constant \(C\) is denoted by \(C_{p\omega}\).

Theorem 5.3  Let \(B\) be a basis and \(\omega\) be a weight on subsets of \(\mathbb{N}\).

(1) If \(B\) is \(C_{p\omega}\)-\(\omega\)-partially greedy, then \(B\) is \(C_{\omega}\)-suppression quasi-greedy and is \(C_{p\omega}\)-\(\omega\)-PSLC.

(2) If \(B\) is \(C_{\ell}\)-suppression quasi-greedy and is \(C_{p\omega}\)-\(\omega\)-PSLC, then \(B\) is \(C_{\ell}C_{p\omega}\)-\(\omega\)-partially greedy.

Proof  Similar to the proof of Theorem 2.9. □

Definition 5.4  (Extension of Berná et al. (2019, Definition 6.2)) A basis \(B\) is said to be \(\omega\)-conservative (\(\omega\)-superconservative, respectively) if there exists \(C \geq 1\) such that

\[
\|1_A\| \leq C\|1_B\|, (\|1_{\varepsilon A}\| \leq C\|1_{\delta B}\|, \text{ respectively}),
\]

for all \(A, B \in \mathbb{N}^{<\infty}\) with \(A < B\) and \(\omega(A) \leq \omega(B)\) and signs \((\varepsilon), (\delta)\).
We have the following equivalences, whose proof is similar to that of Theorem 2.8 and is left for interested readers.

**Theorem 5.5** (Extension of Berná et al. (2019, Theorem 6.4); see also Berasategui et al. (2021, Proposition 1.13 and Theorem 1.14) and Berná (2021, Theorem 4.2)) Let \( B \) be a basis and \( \omega \) be a weight on subsets of \( \mathbb{N} \). The following are equivalent:

1. \( B \) is \( \omega \)-partially greedy,
2. \( B \) is quasi-greedy and is \( \omega \)-PSLC,
3. \( B \) is quasi-greedy and \( \omega \)-superconservative,
4. \( B \) is quasi-greedy and \( \omega \)-conservative.

The following is an analog of Theorem 3.1.

**Theorem 5.6** If a basis \( B = (e_n)_{n=1}^\infty \) is \( \varsigma \)-partially greedy for some weight sequence \( \varsigma = (s(n))_{n=1}^\infty \) with \( \inf s(n) > 0 \), then \( B \) is partially greedy.

**Proof** We assume that \( B \) is \( \varsigma \)-partially greedy for some weight sequence \( \varsigma = (s(n))_{n=1}^\infty \). By Berná et al. (2019, Theorem 6.4), \( B \) is quasi-greedy and is \( \varsigma \)-conservative.

If \( 0 < \inf s(n) \leq \sup s(n) < \infty \), then Khurana (2020, Proposition 4.5) implies that \( B \) is conservative. According to Dilworth et al. (2003b, Theorem 3.4), \( B \) is partially greedy.

If \( \sup s(n) = \infty \), then Khurana (2020, Proposition 4.1) states that \( B \) is equivalent to the canonical basis of \( c_0 \) and thus, is greedy. \( \square \)

**Theorem 5.7** There exists a Schauder basis that is \( \omega \)-partially greedy for some structured weight \( \omega \) on sets but is not \( \varsigma \)-partially greedy for any weight sequence \( \varsigma = (s(n))_{n=1}^\infty \) with \( \inf s(n) > 0 \).

**Proof** The basis \( B \) in Sect. 3 is not conservative. To see this, simply pick \( A = \{2, 2^2, \ldots, 2^N\} \) and \( B = \{3^{N+1}, \ldots, 3^{2N}\} \). We have \( \|1_A\|/\|1_B\| \sim \sqrt{N}/\ln(N) \to \infty \) as \( N \to \infty \). Hence, \( B \) is not partially greedy due to Dilworth et al. (2003b, Theorem 3.4). Applying Theorem 5.6, we obtain the desired conclusion. \( \square \)

**Remark 5.8** Theorem 5.7 is sharp in the sense that we cannot drop the requirement \( \inf s(n) > 0 \). Indeed, Khurana (2020) characterized \( \varsigma \)-partially greedy bases by quasi-greediness and the so-called \( \varsigma \)-left-Property (A). By Khurana (2020, Remark 3.3), any basis trivially satisfies \( \varsigma \)-left-Property (A) with \( \varsigma = (s(n))_{n=1}^\infty = (2^{-n})_{n=1}^\infty \). Hence, if we have an \( \omega \)-partially greedy, it is quasi-greedy by Theorem 5.3 and has \( \varsigma \)-left-Property (A) for \( s(n) = 2^{-n} \). Therefore, the basis is automatically \( \varsigma \)-partially greedy.

### 6 Questions and Discussion

We list several open questions for future research.

**Q1** We show that for a structured weight \( \omega \), a basis is \( \omega \)-almost greedy if and only if it is \( \omega \)-semi-greedy. Does the result hold for a larger class of weights?
Q2 For weights in Definition 1.2, is an \(\omega\)-disjoint superdemocratic basis also \(\omega\)-superdemocratic? If not, what minimal condition(s) to put on \(\omega\) so that the two properties are equivalent.

For the second question, we know that for a structured weight, an \(\omega\)-disjoint superdemocratic is \(\omega\)-superdemocratic.

Proposition 6.1 For a structured weight \(\omega\), a basis \(B\) is \(\omega\)-superdemocratic if and only if \(B\) is \(\omega\)-disjoint superdemocratic.

Proof Assume that \(B\) is \(\omega\)-disjoint superdemocratic. Let \(A, B \in \mathbb{N}^{<\infty}\) with \(\omega(A) \leq \omega(B)\). Pick signs \((\varepsilon), (\delta)\). If \(\sum_{n=1}^{\infty} \omega_n < \infty\) or \(\sup_n \omega_n = \infty\), then by Proposition 4.3 and Remark 4.4, \(B\) is equivalent to the canonical basis of \(c_0\), and the desired conclusion follows trivially. For the rest of the proof, we assume that \(\sum_{n=1}^{\infty} \omega_n = \infty\) and \(\sup_n \omega_n < \infty\).

Case 1: \(\omega(A) \leq \limsup_{n \to \infty} \omega_n\). We proceed as in the proof of Theorem 4.5 to show \(\|1_{\varepsilon A}\| \lesssim \|1_{\delta B}\|\).

Case 2: \(\omega(A) > \limsup_{n \to \infty} \omega_n\). Choose \(E = E \cup \{N\}\) such that \(A \cup B < E < \{N\}\) and \(\omega(E) \leq \omega(A) < \omega(F)\). By \(C_{sd,\cup}^\omega\)-\(\omega\)-disjoint superdemocracy and (4.6), we have

\[
\|1_{\varepsilon A}\| \leq C_{sd,\cup}^\omega \|1_F\| \leq C_{sd,\cup}^\omega \left(\frac{c_2}{c_1}K_b + 1\right) \|1_E\| \leq (C_{sd,\cup}^\omega)^2 \left(\frac{c_2}{c_1}K_b + 1\right) \|1_{\delta B}\|.
\]

Therefore, \(B\) is \(\omega\)-superdemocratic.

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Declarations

Conflict of interest The author has no funding or conflicts of interests to declare.

7 Appendix

7.1 Proof of Theorem 2.9

The key input is the uniform boundedness of the truncation function. For each \(\alpha > 0\), we define the truncation function \(T_\alpha\) as follows: for \(b \in \mathbb{K}\),

\[
T_\alpha(b) = \begin{cases} 
\text{sgn}(b)\alpha, & \text{if } |b| > \alpha, \\
b, & \text{if } |b| \leq \alpha.
\end{cases}
\]

We define the truncation operator \(T_\alpha : \mathbb{K} \to \mathbb{K}\) as

\[
T_\alpha(x) = \sum_{n=1}^{\infty} T_\alpha(e_n^*(x))e_n = \alpha 1_{\varepsilon T_\alpha(x)} + P_{Y_\alpha^c(x)}(x),
\]
where $\Gamma_\alpha(x) = \{ n : |e_n^*(x)| > \alpha \}$ and $\epsilon_n = \text{sgn}(e_n^*(x))$ for all $n \in \Gamma_\alpha(x)$. The operator $T_\alpha$ is well-defined as $|\Gamma_\alpha(x)| < \infty$ for all $\alpha > 0$ and $x \in \mathbb{X}$.

**Theorem 7.1** (Berná et al. 2017, Lemma 2.5) Let $B$ be $C_\ell$-suppression quasi-greedy. Then for any $\alpha > 0$, $\| T_\alpha \| \leq C_\ell$.

**Proof of Theorem 2.9** Assume that $B$ is $C_{\omega}^\omega_{\omega}$-almost greedy. Let $x \in \mathbb{X}$ and $A$ be a greedy set of $x$. We have

$$\| x - P_A(x) \| \leq C_{\omega}^\omega_{\omega} \sum_{n \in A \setminus B} \text{sgn}(e_n^*(x))e_n$$

The proof of $C_{\omega}^\omega_{\omega}$-Property (A) uses the exact argument as in the proof of Theorem 2.3, so we skip it.

Now assume that $B$ is $C_\ell$-suppression quasi-greedy and satisfies $C_{\omega}^\omega_{\omega}$-Property (A). Let $x \in \mathbb{X}$ and $A$ be a greedy set of $x$. Let $B \in \mathbb{N}^{<\infty}$ such that $\omega(B \setminus A) \leq \omega(A \setminus B)$. By Lemma 2.4 and Theorem 7.1, we have

$$\| x - P_A(x) \| \leq C_{\omega}^\omega_{\omega} \sum_{n \in A \setminus B} \text{sgn}(e_n^*(x))e_n$$

This completes our proof that $B$ is $C_\ell C_{\omega}^\omega_{\omega}$-almost greedy. \qed

### 7.2 Proof of Theorem 4.6

**Lemma 7.2** (Berná et al. 2017, Lemma 2.3) Let $B$ be a $C_\ell$-suppression quasi-greedy basis and $x \in \mathbb{X}$. If $A$ is a greedy set of $x$, then

$$\min_{n \in A} |e_n^*(x)| \left\| \sum_{n \in A} \epsilon_n e_n \right\| \leq 2C_\ell \| x \|,$$  \hspace{1cm} (7.1)

where $\epsilon_n = \text{sgn}(e_n^*(x))$.

**Proof of Theorem 4.6** Let us assume that $B$ is $C_\ell$-suppression quasi-greedy and $C_{\omega,\omega}^\omega_{\omega,\omega}$-disjoint superdemocratic. Let $x \in \mathbb{X}$ with $|\text{supp}(x)| < \infty$ and $\Lambda_m(x)$ be a greedy set. Fix $\epsilon > 0$. Let $y = \sum_{n \in A} a_n e_n$, where $A \in \mathbb{N}^{<\infty}$, $\omega(A \setminus \Lambda_m(x)) \leq \omega(\Lambda_m(x) \setminus A)$ and $\| x - y \| < \sigma_{\Lambda_m(x)}(\epsilon) + \epsilon$. Write $x - y = \sum_{n=1}^\infty b_n e_n$, where $b_n = e_n^*(x) - a_n$ if $n \in A$ and $b_n = e_n^*(x)$ if $n \notin A$. We shall find a vector $w$ with $\text{supp}(w) \subset \Lambda_m(x)$ such that

$$\| x - w \| \leq C_\ell (1 + 4C_{\omega,\omega}^\omega_{\omega,\omega} C_\ell)(\sigma_{\Lambda_m(x)}(\epsilon) + \epsilon).$$  \hspace{1cm} (7.2)
Set $\alpha := \max_{n \notin \Lambda_m(x)} |e_n^*(x)|$. If $\alpha = 0$, then choose $w = x$ and we are done. Assume that $\alpha > 0$. Consider the following vector:

$$z := \sum_{n \in \Lambda_m(x)} T_\alpha(b_n)e_n + P_{\Lambda_m(x)}(x) \tag{7.3}$$

$$= \sum_{n \in \Lambda_m(x)} T_\alpha(b_n)e_n + \sum_{n \notin A \cup \Lambda_m(x)} T_\alpha(b_n)e_n + \sum_{n \in A \setminus \Lambda_m(x)} e_n^*(x)e_n \tag{7.4}$$

$$= \sum_{n \in \Lambda_m(x)} T_\alpha(b_n)e_n + \sum_{n \in A \setminus \Lambda_m(x)} (e_n^*(x) - T_\alpha(b_n))e_n. \tag{7.5}$$

We claim that $x - z$ is a choice for $w$. Indeed, using (7.3), we know that $\text{supp}(w) = \text{supp}(x - z) \subset \Lambda_m(x)$. By Theorem 7.1, we have

$$\left\| \sum_{n=1}^{\infty} T_\alpha(b_n)e_n \right\| \leq C_\ell \|x - y\|. \tag{7.6}$$

Note that $|e_n^*(x) - T_\alpha(b_n)| \leq 2\alpha$ for all $n \in A \setminus \Lambda_m(x)$. Let $\eta = (\text{sgn}(e_n^*(x) - y))_{n=1}^{\infty}$. We have

$$\left\| \sum_{n \in A \setminus \Lambda_m(x)} (e_n^*(x) - T_\alpha(b_n))e_n \right\| \leq 2\alpha \sup_{(\delta)} \|1_{A \setminus \Lambda_m(x)} \|$$

$$\leq 2C_{sd,\cup}^{\omega} \min_{n \in \Lambda_m(x) \setminus A} |e_n^*(x) - y| \|1_{\Lambda_m(x) \setminus A} \| \leq 2C_{sd,\cup}^{\omega} \min_{n \in \Lambda_m(x) \setminus A} |e_n^*(x) - y| \|1_{\Lambda_m(x) \setminus A} \|. \tag{7.7}$$

Let $B := \{n : |e_n^*(x) - y| \geq \min_{n \in \Lambda_m(x) \setminus A} |e_n^*(x) - y)| \}$. Then $B$ is a greedy set of $x - y$ and $\Lambda_m(x) \setminus A \subset B$. Therefore, we obtain

$$\|1_{\Lambda_m(x) \setminus A} \| \leq C_\ell \|1_{\eta B} \|$$

and so, by (7.1),

$$\left\| \sum_{n \in A \setminus \Lambda_m(x)} (e_n^*(x) - T_\alpha(b_n))e_n \right\| \leq 2C_{sd,\cup}^{\omega} C_\ell \min_{n \in \Lambda_m(x) \setminus A} |e_n^*(x) - y| \|1_{\eta B} \|$$

$$\leq 4C_{sd,\cup}^{\omega} C_\ell^2 \|x - y\|. \tag{7.8}$$

Using (7.4), (7.5), and (7.6), we obtain (7.2). Therefore,

$$\|x - CG_m(x)\| \leq C_\ell (1 + 4C_{sd,\cup}^{\omega} C_\ell^2 (\sigma_{\Lambda_m(x)}(x) + \varepsilon)).$$

Letting $\varepsilon \to 0$ completes the proof. \qed
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