Physical measures for the geodesic flow tangent to a transversally conformal foliation

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Abstract

We consider a transversally conformal foliation $F$ of a closed manifold $M$ endowed with a smooth Riemannian metric whose restriction to each leaf is negatively curved. We prove that it satisfies the following dichotomy. Either there is a transverse holonomy-invariant measure for $F$, or the foliated geodesic flow admits a finite number of physical measures, which have negative transverse Lyapunov exponents and whose basin cover a set full for the Lebesgue measure. We also give necessary and sufficient conditions for the foliated geodesic flow to be partially hyperbolic in the case where the foliation is transverse to a projective circle bundle over a closed hyperbolic surface.

Introduction

The existence of a transverse holonomy-invariant measure for a foliation whose leaves have dimension 2 or more is a rare phenomenon. The ergodic study of a foliation classically refers to the statistical description of Brownian paths tangent to its leaves: see [24]. In this paper we develop a different viewpoint and study the ergodic properties of geodesics tangent to the leaves of foliations.

All along this work $(M, F)$ stands for a smooth (i.e. of class $C^\infty$) closed foliated manifold of codimension $q$ endowed with a smooth Riemannian metric $g$. Up to passing to a double cover, we will always assume that our foliations are oriented. We will make the two following hypotheses

1. every leaf $L$ has negative sectional curvature for the induced metric $g_{|L}$;
2. the foliation $F$ is transversally conformal.

The first hypothesis is satisfied for example by every foliation transverse to a fiber bundle over a closed negatively curved manifold (see §4.1). Moreover for every foliation by surfaces without transverse holonomy-invariant measure there exists a Riemannian metric on the ambient space such that the first hypothesis is satisfied (see Theorem C). The second hypothesis is satisfied by every codimension 1 foliation and by (singular) holomorphic foliations on complex surfaces. It means that the holonomy pseudogroup of $F$ consists of conformal local diffeomorphisms of $\mathbb{R}^q$ (i.e. their derivatives at every point are similitudes of the Euclidean space).

We shall denote by $\tilde{M}$ the unit tangent bundle of the foliation $F$ i.e. the set of unit vectors tangent to $F$. Unit tangent bundles of leaves of $F$ form a foliation of $\tilde{M}$ denoted by $\tilde{F}$. The foliated geodesic flow is the smooth and leaf-preserving flow of $\tilde{F}$ denoted by $G_t$ which induces on

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each leaf its geodesic flow. Since the leaves are negatively curved $G_t$ exhibits a weak form of hyperbolicity called \textit{foliated hyperbolicity} which is defined and analyzed by Bonatti, Gómez-Mont and Martínez in \cite{12}. It means that there exist two continuous and $G_t$-invariant subfoliations of $\tilde{\mathcal{F}}$, called \textit{stable and unstable foliations} and denoted by $W^s$ and $W^u$, whose leaves are respectively uniformly exponentially contracted and expanded by $G_t$. This notion resembles the classical definition of \textit{partial hyperbolicity} for flows, which will be defined in §4.3.1, the transverse direction of the foliation playing the role of the central direction. But there is a main difference: the contraction, or expansion, in the transverse direction does not need to be dominated by the hyperbolicity inside the leaves. We will later discuss this matter.

The goal of that paper is twofold. Firstly we wish to describe the ergodic properties of the flow. Secondly we wish to discuss the relations between partial hyperbolicity and foliated hyperbolicity through the study of special examples: the foliations transverse to a projective circle bundle over a surface.

\textbf{Finiteness of SRB measures.} Recall that an \textit{SRB measure} or \textit{physical measure} for $G_t$ is a $G_t$-invariant probability measure $\mu$ whose basin (the set of $v \in \tilde{M}$ such that the averages of Dirac masses along the orbit of $v$ converges to $\mu$ in the weak$^*$ sense) has positive Lebesgue measure.

They are named after by Sinai, Ruelle and Bowen who introduced them for uniformly hyperbolic dynamics in \cite{45, 43, 15}. The question of the existence and finiteness of SRB measures for partially hyperbolic dynamics was studied by Bonatti, Viana in \cite{14} and together with Alves in \cite{5}. It is proven in \cite{14} that a partially hyperbolic diffeomorphism which is “mostly contracting” in the center direction has a finite number of SRB measures and that the union of their basins is full for the Lebesgue measure. Our situation is similar and we propose to prove the following dichotomy.

\textbf{Theorem A.} Let $\mathcal{F}$ be a smooth transversally conformal foliation of a closed manifold $M$. Assume that $M$ is endowed with a smooth Riemannian metric such that every leaf is negatively curved for the restricted metric. Then we have the following dichotomy

- either there exists a transverse measure invariant by holonomy;
- or $\tilde{\mathcal{F}}$ has a finite number of minimal sets each of which supports a unique SRB measure for $G_t$. These measures have negative transverse Lyapunov exponents and the union of their basins is full for the Lebesgue measure.

In the latter case it follows in particular that these SRB measures for $G_t$ are the unique ones and that $\mathcal{F}$ has finitely many minimal sets as well.

We define the \textit{transverse Lyapunov exponent} of an ergodic $G_t$-invariant measure $\mu$ as the following limit independent of the choice of a $\mu$-typical $v \in \tilde{M}$

$$\lambda_h(\mu) = \lim_{t \to \infty} \frac{1}{t} \log \|D_{h_{G_t}} v\omega\|,$$

where $\omega \in \mathcal{N}_F = TM/T\mathcal{F}$, the normal bundle of $\mathcal{F}$, and $h_{G_t,v}(\omega)$ denotes the holonomy map along the orbit segment $G_{[0,t]}(v)$. The fact that the number is independent of $\omega$ (i.e. that there is a unique transverse exponent) follows from the condition that $\mathcal{F}$ is transversally conformal (see §1.4.1). The question of knowing what happens when there is more than one transverse Lyapunov exponent (when the foliation is not transversally conformal) seems a difficult one. Bonatti, Eskin and Wilkinson announce that they can treat the case of foliations transverse to a projective $\mathbb{C}P^q$-fiber bundle over a closed hyperbolic surface.\footnote{Private conversation with Christian Bonatti}
This dichotomy for foliations is reminiscent of a whole series of works initiated by Furstenberg (see for example [23, 32, 31]) and culminating with Avila-Viana’s invariance principle [7] and Malicet’s study of random walks in the group of circle homeomorphisms [35], which develop the following general principle. When composing randomly homeomorphisms of certain manifolds (for example the circle) either a probability measure is globally preserved, or there is some contraction in the dynamics.

Deroin and Kleptsyn were the first to include foliations inside the family of dynamical systems exhibiting this feature. In a wonderful paper [18], which motivates the present paper, they proved Theorem A with Garnett’s harmonic measures for \( F \) (see [24]) instead of SRB measures and they do not need the assumption on the sectional curvatures of the leaves. We also mention Fornæss-Sibony’s work on harmonic currents of laminations [22]. In [13] Bonatti, Gómez-Mont and Martínez showed how, in the case of foliations by hyperbolic manifolds, to deduce Theorem A from Deroin-Kleptsyn’s result and from the bijective correspondence between harmonic measures and a special class of invariant measures called Gibbs \( u \)-states that we shall define below (see [3, 8, 38]). Our proof of Theorem A is independent of the study of the foliated Brownian motion and uses Pesin’s theory of nonuniformly hyperbolic dynamical systems.

As in [12], we follow the line of reasoning of [14] and the candidates to be SRB measures are the so called Gibbs \( u \)-states, which are those \( G_t \)-invariant measures whose conditional measures in local unstable leaves (the leaves of \( W^u \)) are equivalent to the Lebesgue measure. Our strategy to prove our main result is to establish that when the foliation has no transverse invariant measure, it is transversally mostly contracting, in the sense that the transverse Lyapunov exponent is negative for every Gibbs \( u \)-state. We will then follow the “mostly contracting scenario” developed by [14] for partially hyperbolic systems, and Theorem A will follow.

**Theorem B.** Let \( F \) be a smooth transversally conformal foliation of a closed manifold \( M \). Assume that \( M \) is endowed with a smooth Riemannian metric such that every leaf \( L \) is negatively curved for the restriction \( g|_L \). Assume that there is no transverse holonomy invariant measure for \( F \). Then the transverse Lyapunov exponent of every Gibbs \( u \)-state is negative.

For the proof of this theorem we use Pesin theory as well as a criterion for the existence of a transverse invariant measure given in [1] (see Theorem 2.2). In the latter the first author proved that if \( G_t \) admits a measure which is a Gibbs \( s\)-state, i.e. an invariant measure whose conditional measures in both stable and unstable leaves are equivalent to the Lebesgue measure, then it has to be totally invariant i.e. has to be locally the product of the Liouville measure of the leaves of \( \hat{F} \) by a transverse invariant measure for \( \hat{F} \). In particular there exists a transverse invariant measure for \( \hat{F} \), and thus there exists one for \( F \) as well.

Our strategy is the following. Assuming the existence of a Gibbs \( u \)-state \( \mu \) with a nonnegative transverse Lyapunov exponent, which by ergodic decomposition we may assume to be ergodic, we want to prove that it is a Gibbs \( s \)-state which, by our criterion, implies the existence of a transverse invariant measure. In order to do this, we will prove that Pesin’s entropy formula (see [41]) holds along the stable foliation: see §2.2. Ledrappier-Young’s work about entropy [34] will then imply that \( \mu \) must be a Gibbs \( s \)-state, and the result will follow.

**Foliations by surfaces.** It is not clear that for a given foliation there must exist an ambient Riemannian metric inducing negatively curved metrics in the leaves.

However, Ghys showed us a nice argument in order to prove that such a phenomenon is quite common in the world of two-dimensional foliations. Since it does not appear in the literature we propose to give in Appendix the proof the following result that we attribute to Ghys.
Theorem C (Ghys). Let $(M, \mathcal{F})$ be a closed manifold foliated by hyperbolic Riemann surfaces and $g$ be a smooth Riemannian metric on $M$. Then there exists a Riemannian metric $g'$ conformally equivalent to $g$ such that the Gaussian curvature of every leaf $L$ is negative for the restricted metric $g'|_L$.

We shall define in Appendix what is a foliation by hyperbolic surfaces, and note that this is a purely topological condition on the foliation, which does not depend on the structure of Riemann surface foliation we chose. Let us emphasize that every two-dimensional foliation without transverse invariant measure must be a foliation by hyperbolic surfaces: see Proposition 6.3. As a consequence we deduce that for every transversally conformal two-dimensional foliation the following dichotomy holds true.

- Either it has a transverse holonomy-invariant measure.
- Or there exists a smooth Riemannian metric of the ambient space such that every leaf is negatively curved for the restricted metric. For such a metric the second alternative of Theorem A holds true.

Let us describe a foliation for which the second alternative holds true. It was originally constructed by Hirsch. Let us consider a solid torus $T$ and drill out an inner solid torus, which is the neighbourhood of a $(2, 1)$ braid. We obtain a 3-dimensional manifold with two boundary components, which are tori, and is naturally foliated by pairs of pants (discs with two holes) transverse to the boundary. This foliation induces two circle foliations of the boundary components. Now glue the two components by a diffeomorphism which preserves the circle foliations. We obtain a boundaryless 3-dimensional manifold endowed with a minimal codimension 1-foliation. The leaves, which are obtained by glueing pairs of pants, are easy to describe: they are all diffeomorphic to a Cantor tree, i.e. a sphere minus a Cantor set. Except of course a countable number of leaves corresponding to the “periodic points” of the glueing which have genus one and are tori minus a Cantor set. We refer to [4] for more details about the construction. Theorem A applies to this beautiful foliation.

Partially hyperbolic examples. We now raise the problem of the relation between partial and foliated hyperbolicities. We illustrate it by a detailed study of special examples, namely foliations transverse to a circle bundle over a hyperbolic surface with projective holonomy. We propose a link between partial hyperbolicity of the foliated geodesic flow and a purely topological condition on the bundle: the value of its Euler number.

Recall that circle bundles over a closed surface $\Sigma$ of genus $g$ are classified by an integer, their Euler number, and that those admitting a transverse foliation are precisely those whose Euler number is, in absolute value, less than $2g - 2$ (this is Milnor-Wood’s inequality for which we refer to [39, 49]). The Euler number of a circle bundle $\Pi : M \to \Sigma$ is denoted by $\text{Eu}(\Pi)$.

A smooth foliation $\mathcal{F}$ transverse to a circle bundle $\Pi : M \to \Sigma$ is obtained from its holonomy representation $\text{hol} : \pi_1(\Sigma) \to \text{Diff}^\infty(S^1)$ by a process called suspension and described in §4.1. When the foliated bundle has a projective holonomy group (i.e. the fibers are identified with the real projective line $\mathbb{R}P^1$ and the holonomy representation takes its values in the group $\text{PSL}_2(\mathbb{R})$ of projective transformations of $\mathbb{R}P^1$) we say that the data $(\Pi, M, \Sigma, \mathcal{F})$ is a foliated $\mathbb{R}P^1$-bundle with projective holonomy.

Let $(\Pi, M, \Sigma, \mathcal{F})$ be a foliated $\mathbb{R}P^1$-bundle with projective holonomy and suppose $\Sigma$ is endowed with a hyperbolic metric $m$. Say a smooth Riemannian metric $g$ on $M$ is admissible if for every leaf $L$ the restriction $\Pi|_L : (L, g|_L) \to (\Sigma, m)$ is a Riemannian cover. In that case $G_f$ preserves the fibers
of the bundle $\Pi_* : \tilde{M} \to T^1 \Sigma$ induced by the differential of $\Pi$, and the fiber direction, which as we shall prove is invariant by the flow, is a good candidate for being the central direction.

Our next result provides a topological condition on a circle foliated bundle with projective holonomy to possess a partially hyperbolic foliated geodesic flow (we refer to §4.3.1 for the definition of partially hyperbolic flows). This provides new geometric examples of partially hyperbolic dynamical systems.

**Theorem D.** Let $\Sigma$ be a closed surface of genus $g \geq 2$.

1. If $(\Pi, M, \Sigma, \mathcal{F})$ is a foliated $\mathbb{R}P^1$-bundle with projective holonomy satisfying $|\text{Eu}(\Pi)| < 2g - 2$ then there exists a hyperbolic metric on $\Sigma$ such that for every admissible Riemannian metric on $M$ the foliated geodesic flow of $\mathcal{F}$ is partially hyperbolic.

2. For every hyperbolic metric on $\Sigma$ there exists a foliated $\mathbb{R}P^1$-bundle with projective holonomy such that for every admissible Riemannian metric on $M$ the foliated geodesic flow $G_t$ is partially hyperbolic. Moreover $\text{Eu}(\Pi)$ can be made arbitrary in $[3 - 2g, ..., 0, ..., 2g - 3]$.

This result is a consequence of a result coming from the field of 3-dimensional Anti-de Sitter geometry proven recently and independently by Guéritaud-Kassel-Wolff in [29] and Deroin-Tholozan in [19].

Recall that a hyperbolic metric $m$ on a closed surface $\Sigma$ gives rise to a Fuchsian representation $\rho : \pi_1(\Sigma) \to \text{PSL}_2(\mathbb{R})$, i.e. a discrete and faithful projective representation, which is well defined up to conjugacy by an element of $\text{PSL}_2(\mathbb{R})$. Recall also that a foliated $\mathbb{R}P^1$-bundle with projective holonomy gives rise to a holonomy representation $\text{hol} : \pi_1(\Sigma) \to \text{PSL}_2(\mathbb{R})$ (see §4.1). In the works cited above one can find a very interesting notion of domination of projective representations of a surface group. We say that the representation $\rho$ dominates the representation $\text{hol}$ if the translation lengths for $\rho$ are uniformly larger than for $\text{hol}$ (we refer to §4.2 for more details). Theorem D is then a consequence of the main results of these papers (see Theorem 4.3) and of the following result.

**Theorem E.** Let $(\Pi, M, \Sigma, \mathcal{F})$ be a foliated $\mathbb{R}P^1$-bundle with projective holonomy over a closed surface $\Sigma$ endowed with a hyperbolic metric $m$. Endow $M$ with an admissible Riemannian metric. Let $\rho : \pi_1(\Sigma) \to \text{PSL}_2(\mathbb{R})$ denote a Fuchsian representation associated to $m$ and $\text{hol} : \pi_1(\Sigma) \to \text{PSL}_2(\mathbb{R})$ denote the holonomy representation of $\mathcal{F}$. Then the foliated geodesic flow $G_t$ is partially hyperbolic if and only if $\rho$ dominates $\text{hol}$.

**Fuchsian foliations.** To complete our study of foliated circle bundles with projective holonomy, it remains to treat the case of maximal Euler number. By Goldman’s thesis [28] such a projective foliated bundle has maximal Euler number if and only if it is Fuchsian, i.e. its holonomy representation is Fuchsian. Teichmüller’s theory implies that a Fuchsian representation can’t dominate another one so that the foliated geodesic flow of such a foliation can’t be partially hyperbolic. We go further and compute the transverse Lyapunov exponent of the unique SRB measure (the uniqueness of the SRB measure, which also follows from our work, was first obtained in [13]).

Assume that $\rho, \text{hol} : \pi_1(\Sigma) \to \text{PSL}_2(\mathbb{R})$ are both Fuchsian. Denote respectively by $\Gamma_1$ and $\Gamma_2$ their images, and by $m_1$, $m_2$ the hyperbolic metrics they are associated to. A classical construction provides a biholöder orbit equivalence $H : T^1 \Sigma \to T^1 \Sigma$ between their geodesic flows (see §5.1 for more details about the construction). The reparametrization of the flow-lines provides an additive cocycle $a : \mathbb{R} \times T^1 \Sigma \to \mathbb{R}$ which is Liouville-integrable (for $m_1$). As a consequence of Birkhoff’s ergodic theorem, there exists a number $\chi > 0$ that we call the average reparametrization of the geodesic flow such that for Liouville-almost every $v \in T^1 \Sigma$, $a(t, v) \sim \chi t$ as $t$ tends to infinity. Thurston
proved that we always have $\chi \geq 1$ and that equality holds if and only if $m_1$ and $m_2$ represent the same Teichmüller class (see [48]).

The following result shows that not only the foliated geodesic flow of a Fuchsian foliation is not partially hyperbolic, but the dynamics in the fiber direction is typically (for the SRB measure) strictly more contracting than in the stable direction inside the leaves.

**Theorem F.** Assume that the hypothesis of Theorem E hold and that both $\rho, \text{hol} : \pi_1(\Sigma) \to \text{PSL}_2(\mathbb{R})$ are Fuchsian. Let $\chi$ be the associated average reparametrization of the geodesic flow. Then the transverse Lyapunov exponent $\lambda^0$ of the unique SRB measure equals $-\chi$.

*In particular when $\rho$ and $\text{hol}$ are not conjugated we find $|\lambda^0| > 1$.***

**Outline of the work.** In Section 1 we give the necessary material which will be used throughout the text. Section 3 is where we prove Theorems A and B. Section 4 is devoted to the presentation of the notion of domination of projective representation and to proving Theorems D and E. Fuchsian foliations are treated in Section 5 and Theorem E is proven there. In Appendix, we give Ghys’ argument for proving Theorem C. The final Section contains a certain number of question and research perspectives that we find interesting.

1 Preliminaries

1.1 Transversally conformal foliations

**Foliations.** A smooth foliation $\mathcal{F}$ of codimension $q$ of a smooth closed manifold $M$ (here smooth means $C^\infty$) is determined by a family $(U_i, \gamma_i)_{i \in I}$ where:

1. $(U_i)_{i \in I}$ is a finite open cover of $M$;

2. for every $i \in I$, $\gamma_i : U_i \to \mathbb{R}^q$ is a submersion such that for every $x \in U_i \cap U_j$ there exists a local diffeomorphism $\tau_{ij}$ of $\mathbb{R}^q$ defined in neighborhood of $\gamma_i(x)$ such that the equality

$$\gamma_j = \tau_{ij} \circ \gamma_i,$$

holds in a neighborhood of $x$ in $U_i$ (see [27]).

The $\gamma_i$ are called distinguished maps and the $\tau_{ij}$, the holonomy maps. Two points $x, y$ are said to be on the same leaf if there exists a sequence $x = x_0, x_1, \ldots, x_k, x_{k+1} = y$ such that $x_i$ and $x_{i+1}$ belong to the same chart $U_i$ and $\gamma_j(x_i) = \gamma_j(x_{i+1})$. The leaves of $\mathcal{F}$ are smooth immersed manifolds of the same dimension $p$.

The set of vectors tangent to $\mathcal{F}$ is a subbundle $T\mathcal{F} \subset TM$ called the tangent bundle of $\mathcal{F}$. The normal bundle of $\mathcal{F}$ is by definition $\mathcal{N}_\mathcal{F} = TM/T\mathcal{F}$. The choice of a smooth Riemannian metric of $M$ identifies $\mathcal{N}_\mathcal{F}$ with $T\mathcal{F}^\perp$.

**Holonomy.** The maps $\tau_{ij}$ defined above generate a pseudogroup of local smooth diffeomorphisms of $\mathbb{R}^q$ (see [27]) called the holonomy pseudogroup $\mathcal{P}$.

Let $c$ be a path tangent to $\mathcal{F}$ and $U_{i_0}, \ldots, U_{i_k}$ be a chain of distinguished charts covering $c$. Choosing a point of $U_{i_j} \cap U_{i_{j+1}}$ $\tau_{ij}$ gives a holonomy map $\tau_{ij}$. The holonomy map along $c$ is the composition $h_c = \tau_{i_k, i_{k-1}} \circ \ldots \circ \tau_{i_1}$. The germ of $h_c$ at $c(0)$ only depends on the homotopy class of $c$.

A more geometric, and equivalent, approach to holonomy is to consider $T_1$ and $T_2$, two local transversals to $\mathcal{F}$, two points $x \in T_1, y \in T_2$ and a path $c$ tangent to that leaf joining $x$ and $y$. Then
we will sometimes write abusively $h_c = \Phi(1,.) : S_1 \to S_2$ where $S_1, S_2$ are relatively compact open sets of $T_1, T_2$ and $\Phi : [0,1] \times S_1 \to M$ is a smooth map that has the following properties. Firstly $\Phi(.,x) = c$. Secondly for every $z \in S_1$ $\Phi(.,z)$ is a path tangent to the leaf of $z$ joining $z$ to some $z' \in S_2$.

The derivatives of holonomy maps forms naturally a linear cocycle of the normal bundle $N_F$ called the linear holonomy.

**Transversally conformal foliations.** Say the foliation $F$ is transversally conformal if all maps $\tau_{i,j}$ are conformal transformations of $\mathbb{R}^q$, meaning that their derivatives are everywhere similitudes of the Euclidian space.

If one prefers that means that there exists $|.|$, a transverse metric for $F$, such that for every path $c$ tangent to $F$, every $x$ inside the domain of $h_c$ and every $v \in N_F(x)$, one has

$$|D_x h_c(v)| = \lambda |v|,$$

for some $\lambda > 0$ independent of $v$.

**Remark 1.1.** Every codimension 1 foliation is transversally conformal.

### 1.2 Invariant measures

**Transverse invariant measures.** Let $(T_i)_{i \in I}$ be a complete system of transversals to the foliation, i.e. a finite family of transversals to $F$ whose union meets every leaf. A transverse invariant measure is a family of finite nonnegative measures $(\nu_i)_{i \in I}$ satisfying

1. $\nu_i(T_i) > 0$ for some $i \in I$;
2. if for $i, j \in I$ there is a holonomy map $h : S_i \to S_j$ between two open sets $S_i \subset T_i$ and $S_j \subset T_j$,
   then for any Borel set $A_i \subset S_i$ we have $\nu_i(A_i) = \nu_j(h(A_i))$.

**Totally invariant measures.** The Riemannian metric induces a Riemannian structure on each leaf. Hence every leaf is endowed with a natural volume form. Assume that the foliation $F$ possesses a transverse invariant measure $(\nu_i)_{i \in I}$. Then, if $\nu_i(T_i) > 0$, it is possible inside a corresponding foliated chart $U_i$ to integrate the volume of the plaques (i.e. the connected components of the intersections of leaves of $F$ with $U_i$) against $\nu_i$. We obtain this way a measure $m_i$ in the chart $U_i$.

Since the family of measures $(\nu_i)_{i \in I}$ is holonomy-invariant, these local measures glue together and provide a finite measure $m$ on $M$. Such a measure will be from now one called totally invariant.

### 1.3 The foliated geodesic flow and foliated hyperbolicity

In what follows we assume the existence of a smooth Riemannian metric $g$ on $M$ such that the sectional curvature of every leaf $L$ for the restricted metric $g_L$ is negative. By compactness of $M$ this implies that the sectional curvatures of every leaf $L$ are uniformly pinched between two negative constants $-b^2 < -a^2 < 0$. 

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1.3.1 The foliated geodesic flow

**Unit tangent bundle.** Let \( \hat{M} \) denote the unit tangent bundle of \( F \), i.e. the subbundle of \( T^1M \) consisting of those unit vectors tangent to \( F \). It is a closed manifold endowed with a smooth foliation denoted by \( \hat{F} \) whose leaves are the unit tangent bundles of leaves of \( F \). We will denote by \( pr : \hat{M} \to M \) the canonical basepoint projection associating its basepoint to each vector \( v \in \hat{M} \).

We will denote by \( \hat{L}_v \) the leaf of \( v \in \hat{M} \), i.e. \( T^1L_x \) where \( L_x \) is the leaf of the basepoint \( x \) of \( v \).

Note that the foliations \( F \) and \( \hat{F} \) have the same holonomy pseudogroups. In order to establish this fact one can lift via \( pr \) a foliated atlas for \( F \): this is explained in details for example in [1, §3.1]. In particular we have the

**Proposition 1.2.** There exists a transverse invariant measure for \( F \) if and only if there exists one for \( \hat{F} \).

The foliation \( F \) is transversally conformal if and only if \( \hat{F} \) is transversally conformal.

On \( T^1M \) there is a special smooth Riemannian metric called the *Sasaki metric* (see [40, §1.3.1]). It induces a smooth Riemannian metric on \( \hat{M} \subset T^1M \) denoted by \( \hat{g} \).

**Foliated geodesic flow.** We call foliated geodesic flow the leaf-preserving flow of \( (\hat{M}, \hat{F}) \) which induces on each leaf \( \hat{L} \) its geodesic flow. This flow will be denoted by \( (t, v) \in \mathbb{R} \times \hat{M} \to G_t(v) \).

The foliated geodesic flow is smooth on \( \hat{M} \). Indeed geodesics of leaves \( L \) are the solutions of the geodesic equation which is a second order ODE whose local coefficients are given by the Christoffel symbols of \( g_L \). These symbols depend only on the 1-jet of \( g_L \) (see [20, Chapter 2, Section 3]). We deduce that the equation defining \( G_t \) has smooth coefficients and the smoothness of the flow follows.

1.3.2 Foliated hyperbolicity

Even if leaves are not compact they are all immersed in a compact manifold which implies some sort of recurrence in their geometry (in particular the sectional curvatures are uniformly pinched between two negative constants). This enables one to recover some results which classically hold for uniform hyperbolicity. In what follows of this article the norm \( ||.|| \) is induced by the Riemannian metric \( \hat{g} \).

**Invariant bundles.** When all sectional curvatures are negative the foliated geodesic flow exhibit a weak form of hyperbolicity called *foliated hyperbolicity* in [12]. We explain below what it means.

Following [9, Chapter IV], one proves that there are two continuous and \( DG_t \)-invariant sub-bundles of \( T\hat{F} \) of the same dimension \( p - 1 \) (\( p \) being the dimension of \( F \)) denoted by \( E^s \) and \( E^u \), and respectively called stable and unstable bundles, such that

\[
\begin{align*}
\frac{a}{b} e^{-bt} ||v_s|| &\leq ||DG_t(v_s)|| \leq \frac{b}{a} e^{-at} ||v_s|| & \text{if } v_s \in E^s \text{ and } t > 0 \\
\frac{a}{b} e^{-bt} ||v_u|| &\leq ||DG_{-t}(v_u)|| \leq \frac{b}{a} e^{-at} ||v_u|| & \text{if } v_u \in E^u \text{ and } t > 0.
\end{align*}
\]  

(1.1)

We then have the following continuous and \( DG_t \)-invariant splitting of the tangent bundle of the foliation

\[
T\hat{F} = E^s \oplus \mathbb{R}X \oplus E^u,
\]  

(1.2)
$X$ being the generator of the foliated geodesic flow. We will also set $E^{cs} = E^s \oplus \mathbb{R} X$ and $E^{cu} = E^u \oplus \mathbb{R} X$. These continuous and $DG_t$-invariant subbundles are respectively called center-stable and center-unstable bundles.

**Invariant foliations.** Consider the (foliated) stable and unstable manifolds of $v \in \hat{M}$

$$W^s(v) = \{ y \in \hat{L}_v; \text{dist}(G_t(v), G_t(w)) \to 0 \}$$

$$W^u(v) = \{ y \in \hat{L}_v; \text{dist}(G_{-t}(v), G_{-t}(w)) \to 0 \}.$$  

As explained in [1, 12] the usual Stable Manifold Theorem applies in that case and these sets are smooth immersed submanifolds of $\hat{M}$ included in the leaf of $v$.

They form two continuous subfoliations of $\hat{F}$, denoted by $W^s$ and $W^u$ tangent to the distributions $E^s$ and $E^u$. We call them the stable and unstable foliations. The saturations of $W^s$ and $W^u$ by $G_t$ form two continuous subfoliations of $\hat{F}$ tangent to $E^{cs}$ and $E^{cu}$ called center-stable and center-unstable foliations and denoted by $W^{cs}$ and $W^{cu}$. The leaves of $W^{cs}$ and $W^{cu}$ passing through $v \in \hat{M}$ are denoted by $W^{cs}(v)$ and $W^{cu}(v)$ and called respectively the center-stable and center-unstable manifolds of $v$.

### 1.4 Lyapunov exponents

Two kinds of Lyapunov exponents will be considered. In §1.4.1 we will define the transverse Lyapunov exponents of a $G_t$-invariant measure, which measures how the leaves separate along the orbit segments. In §1.4.2 we will consider the “classical” Lyapunov exponents of the flow, which describe the asymptotic behavior of the orbits of $G_t$: we define there the Oseledets’ splitting. Finally, in §1.4.3 we describe the Lyapunov spectrum of $G_t$-invariant measures, proving in particular that the transverse Lyapunov exponent is realized as a “classical” Lyapunov exponent.

#### 1.4.1 Transverse Lyapunov exponent

The linear holonomy over the foliated geodesic flow defines a linear cocycle on $\mathcal{N}\hat{F}$ over $G_t$, so we can apply Oseledets’ theorem to that cocycle. Using that the foliation $\hat{F}$ is transversally conformal (see Proposition 1.2) we obtain the following

**Proposition 1.3.** There exists a Borel set $\mathcal{X}_0 \subset \hat{M}$ which is $G_t$-invariant and full for every $G_t$-invariant measure such that for every $v \in \mathcal{X}_0$ and $\omega \in \mathcal{N}\hat{F}(v)$ the following number is well defined and independent of $\omega$ and of the transverse metric $|.|$

$$\lambda^h(v) = \lim_{t \to \infty} \frac{1}{t} \log |D_h G_{[0,t]}(v) \omega|$$

where $h_{G_{[0,t]}(v)}$ denotes the holonomy map along the orbit segment $G_{[0,t]}(v)$.

#### 1.4.2 Oseledets’ splitting

**Oseledets’ theorem.** Considering the linear cocycle given by the derivative of $G_1 : \hat{M} \to \hat{M}$ we find a Borel set $\mathcal{X}_1 \subset \hat{M}$ full for every $G_t$-invariant measure such that for every $v \in \mathcal{X}_1$ there exists a splitting

$$T_v \hat{M} = E_1(v) \oplus ... \oplus E_k(v)$$
which is $DG_t$-invariant and such that for every $i \in \{1, \ldots, k\}$ and every $\omega \in E_i(v)$ the Lyapunov exponent

$$\lambda_i(v) = \lim_{t \to -\infty} \frac{1}{t} \log \|D_v G_t \omega\|$$

is well defined.

**Stable and unstable Lyapunov exponents.** The bundles $E^s$ and $E^u$ are $DG_t$-invariant. Oseleeds’ theorem applied to the restriction of $DG_1$ to these bundles provides a Borel set $\mathcal{X}_2 \subset \mathcal{X}_1$ full for every $G_t$-invariant measure such that for every $v \in \mathcal{X}_2$ there exist two splittings

$$E^s(v) = E^s_1(v) \oplus \ldots \oplus E^s_{k_s}(v)$$

$$E^u(v) = E^u_1(v) \oplus \ldots \oplus E^u_{k_u}(v)$$

which are $DG_t$-invariant and such that, when $\star = u$ or $s$, for every $i \in \{1, \ldots, k_\star\}$ and $\omega \in E^\star_i(v)$ the Lyapunov exponent

$$\lambda^\star_i(v) = \lim_{t \to -\infty} \frac{1}{t} \log \|D_v G_t \omega\|$$

is well defined.

Note that these numbers belong to the family of Lyapunov exponents $(\lambda_1(v), \ldots, \lambda_k(v))$. Moreover the “tangential” Lyapunov exponents at $v \in \mathcal{X}_2$ of $G_t$ (i.e. those of the linear cocycle $(DG_t)|_{T_v \mathcal{X}}$) are precisely $(\lambda^s_1(v), \ldots, \lambda^s_k(v), 0, \lambda^u_1(v), \ldots, \lambda^u_k(v))$, where $0$ corresponds to the flow direction.

Moreover one has $\lambda^s_i(v) < 0$ and $\lambda^u_j(v) > 0$ for every $v \in \mathcal{X}_2$, $i \in \{1, \ldots, k_s\}$ and $j \in \{1, \ldots, k_u\}$. Denote for $v \in \mathcal{X}_2$

$$\Lambda^s(v) = \sum_{i=1}^{k_s} \dim E^s_i(v) \lambda^s_i(v) < 0$$

(1.4)

and

$$\Lambda^u(v) = \sum_{j=1}^{k_u} \dim E^u_j(v) \lambda^u_j(v) > 0.$$  

(1.5)

Since the foliated geodesic flow preserves the Liouville measure inside the leaves it comes that

$$\Lambda^s(v) + \Lambda^u(v) = 0.$$  

(1.6)

**1.4.3 Lyapunov spectrum**

**Linear Poincaré flow.** Recall that $X$ denotes the generator of the foliated geodesic flow. This vector field is everywhere nonzero so we can define the normal bundle $\mathcal{N}_X = X^\perp$. The linear Poincaré flow $\Psi_t$ defines a linear cocycle of $\mathcal{N}_X$ over $G_t$ (see [6, §2.6]). Recall that for every $v \in \overline{M}$, we have

$$\Psi_t(v) = \pi_{G_t(v)} \circ D_v G_t : \mathcal{N}_X(v) \to \mathcal{N}_X(G_t(v)),$$

where $\pi_\nu : T_v \overline{M} \to \mathcal{N}_X(v)$ denotes the orthogonal projection.

Oseleeds’ theorem applies for that cocycle and provides a Borel set $\mathcal{X}_3 \subset \mathcal{X}_2$ full for every $G_t$-invariant measure such that for every $v \in \mathcal{X}_3$ there is a splitting

$$\mathcal{N}_X(v) = F_1(v) \oplus \ldots \oplus F_l(v)$$

which is $\Psi_t$-invariant and such that for every $i \in \{1, \ldots, l\}$ and every $\omega \in \mathcal{N}_X(v)$ the Lyapunov exponent

$$\chi_i(v) = \lim_{t \to -\infty} \frac{1}{t} \log \|\Psi_t(v)\omega\|$$

(10)
is well defined. Moreover it is noted in [6, §2.7.2.1] that since the variation of the angles between subspaces defining Oseledets splitting (1.3) is subexponential along the orbits of $G_t$, the Lyapunov exponents $\chi_i$ coincide with the Lyapunov exponents $\lambda_j$ off the flow direction and the space $F_i$ coincide with the orthogonal projection to $\mathcal{N}_X$ of some space $E_j$.

**Lyapunov spectrum.** Now we are ready to conclude our discussion and to identify the Lyapunov spectrum of the foliated geodesic flow.

The metric $\hat{g}$ identifies $\mathcal{N}_\hat{g}$ and $T\hat{F}_u^{-1}$. Note that since $X$ is tangent to $\hat{\mathcal{F}}$ we have $\mathcal{N}_\hat{g} \subset \mathcal{N}_X$ and we can decompose $\mathcal{N}_X = \mathcal{N}_\hat{g} \oplus \mathcal{N}'$ where $\mathcal{N}' = \mathcal{N}_X \cap T\hat{\mathcal{F}}$ is a $\Psi_t$-invariant bundle. Lyapunov exponents of the cocycle $(\Psi_t)_{\mathcal{N}'}$ are precisely the stable and unstable Lyapunov exponents defined above. Define

$$\Psi_t^{\hat{\mathcal{F}}}(v) = \pi_{\hat{g}^{-1}(v)} \circ \Psi_t(v) : \mathcal{N}_\hat{g}(v) \to \mathcal{N}_\hat{g}(G_t(v))$$

and argue as in the previous paragraph in order to get that the Lyapunov exponents of $\Psi_t$ off the $\mathcal{N}'$ direction.

Finally remark that the latter cocycle coincides with the linear holonomy over the foliated geodesic flow, i.e. for every $v \in \hat{M}$

$$\Psi_t^{\hat{\mathcal{F}}}(v) = D_v h_{G_0}(v)$$

and hence that cocycle has only one Lyapunov exponent which is the *transverse Lyapunov exponent* defined in §1.4.1. Hence the following proposition follows from our discussion.

**Proposition 1.4.** There exists a Borel set $\mathcal{F} \subset \hat{M}$ full for every $G_t$-invariant measure such that for every $v \in \mathcal{F}$ the Lyapunov exponents of $G_t$ at $v$ exist and are precisely given by the list

$$(\lambda^s_1(v), ..., \lambda^s_{k_s}(v), 0, \lambda^u_1(v), ..., \lambda^u_{k_u}(v), \lambda^{(\hat{\mathcal{F}})}(v)),$$

where a priori $\lambda^{(\hat{\mathcal{F}})}$ can be equal to $\lambda^s_i, 0$ or $\lambda^u_j$.

**Definition 1.5.** We define the transverse Lyapunov exponent of a $G_t$-invariant measure $\mu$ as the integral

$$\lambda^{(\hat{\mathcal{F}})}(\mu) = \int_{\hat{M}} \lambda^{(\hat{\mathcal{F}})}(v) d\mu(v).$$

The average sums $\Lambda^s(\mu)$ and $\Lambda^u(\mu)$ are defined similarly.

### 1.5 Gibbs $u$-states

**Definition.** A Gibbs $u$-state for $G_t$ is a $G_t$-invariant probability measure on $\hat{M}$ whose conditional measures in local unstable manifolds are equivalent to the Lebesgue measure.

A *Gibbs s-state* for $G_t$ is a Gibbs $u$-state for $G_{-t}$: its conditional measures in local stable manifolds are equivalent to the Lebesgue measure.

A *Gibbs su-state* is a probability measure on $\hat{M}$ which is both a Gibbs $s$-state and a Gibbs $u$-state.

Gibbs $u$-states were first introduced by Pesin-Sinaï in [42] and then by Bonatti-Viana in [14] in the context of partially hyperbolic dynamics.

**Example.** Suppose $\mathcal{F}$, and therefore $\hat{\mathcal{F}}$ as well, has a family of transverse invariant measures $(\nu_i)_{i \in I}$. As explained in §1.2 this measure can be combined with the Liouville measure in the plaques of $\hat{\mathcal{F}}$ so as to construct a totally invariant measure $\mu$ on $\hat{M}$.

Since $G_t$ preserves the Liouville measure of the leaves it must preserve $\mu$. Moreover by absolute continuity of $\mathcal{W}^s$ and $\mathcal{W}^u$ it must have Lebesgue disintegration in both stable and unstable plaques. Such a measure is a Gibbs $su$-state.
**Existence.** Define the unstable Jacobian at \( v \in \hat{M} \) by the formula

\[
\text{Jac}^u G_t(v) = \det(DvG_t|E^u(v)).
\]

The following theorem shows the existence of Gibbs \( u \)-states. It is a consequence of the usual distortion control (see Lemma 1.7). Its proof follows the line of reasoning of [11, Section 11.2.2]

**Theorem 1.6.** Let \((M, \mathcal{F})\) be a closed manifold endowed with a smooth Riemannian metric \( g \) such that every leaf \( L \) is negatively curved for the induced metric \( g|_L \). Then

1. for every \( v \in \hat{M} \) and every Borel set \( D^u \subset W^u_\text{loc}(v) \) with positive Lebesgue measure, any accumulation point of the following family of measures, indexed by \( T \in (0, \infty) \), is a Gibbs \( u \)-state

\[
\mu_T = \frac{1}{T} \int_0^T G_t^* \left( \frac{\text{Leb}^u_{D^u}}{\text{Leb}^u(D^u)} \right) dt.
\]

Moreover its densities along unstable plaques, denoted by \( \psi^u_w \), are uniformly log-bounded and satisfy the following for \( z_1, z_2 \in W^u_\text{loc}(w) \)

\[
\frac{\psi^u_w(z_2)}{\psi^u_w(z_1)} = \lim_{t \to \infty} \text{Jac}^u G_{-t}(z_2) \text{Jac}^u G_{-t}(z_1)^{-1}.
\]

2. all ergodic components of a Gibbs \( u \)-state are Gibbs \( u \)-states with local densities in the unstable plaques that are uniformly log-bounded and satisfy (1.7);

3. every Gibbs \( u \)-state for \( G_t \) is a measure whose local densities in the unstable plaques are uniformly log-bounded and satisfy (1.7).

**Distortion control.** The metric \( \hat{g} \) induces a Riemannian distance on \( W^u(v) \) denoted by \( \text{dist}_u \).

The following distortion control may be proven exactly as in the classical case for which we refer to [37, Lemma 3.2 Chapter III]. It relies on the fact that the flow contracts uniformly exponentially fast the unstable manifolds in the past, as well as on the fact that \( v \mapsto \text{Jac}^u G_t(v) \) is uniformly \( C^1 \) along unstable manifolds.

**Lemma 1.7.** There exist constants \( C_0 > 0 \) such that for every \( v, w \in \hat{M} \) lying on the same unstable leaf and for every \( t > 0 \)

\[
\frac{\text{Jac}^u G_{-t}(v)}{\text{Jac}^u G_{-t}(w)} \leq C_0 \exp(\text{dist}_u(v, w)).
\]

In particular, the left hand side of (1.8) has a limit as \( T \) goes to \( \infty \).

### 2 Transversally mostly contracting foliated geodesic flows

We wish to prove the following dichotomy for transverse Lyapunov exponents of Gibbs \( u \)-states of \( G_t \). Either there exists a totally invariant measure (see §1.2), and such a measure is a Gibbs \( u \)-state that has zero transverse Lyapunov exponent. Or all Gibbs \( u \)-states have negative Lyapunov exponents: this is the transversally mostly contracting case.
2.1 Transverse Lyapunov exponents for totally invariant measures

We consider here totally invariant measures on \( \hat{M} \). As already mentioned in §1.5 such a measure is \( G_t \)-invariant. We prove that their transverse Lyapunov exponent always vanishes. This requires no curvature assumption.

The companion theorem for totally invariant harmonic measures has been proven in [18, Proposition 3.1]. The proof is very similar Deroin and Kleptsyn use the symmetry of the heat kernel, whereas we use a well known flip function on the unit tangent bundle of a leaf which preserves the Liouville measure.

**Proposition 2.1.** Let \( M \) be a closed manifold endowed with a transversally conformal foliation \( F \) and with a smooth Riemannian metric \( g \). Then the transverse Lyapunov of the geodesic flow of every transverse holonomy-invariant measure vanishes.

**Proof.** Define the flip function as the involution \( \iota : v \in \hat{M} \mapsto -v \). This map preserves each leaf \( \hat{L} \) of \( \hat{F} \) and preserves also the Liouville measure: see [40, Lemma 1.34]. We deduce that the flip function \( \iota \) preserves every totally invariant measure.

Assume that there exists a totally invariant measure \( \mu \). There exists a set full for \( \mu \), denoted by \( X \), which we can assume to be symmetric (i.e. invariant by \( \iota \): recall that \( \iota \) preserves \( \mu \)), such that for every \( v \in X \), the number \( \lambda^\iota(v) \) exists. By symmetry, the transverse Lyapunov exponent at \( \iota(v) \) is also well defined. Note that \( h_{G_{0,t}(v)} = h_{G_{-t,0}(v)} \) so that we have

\[
\lambda^\iota(\iota(v)) = -\lambda^\iota(v).
\]

Integrating against \( \mu \) and using that \( \iota \) preserves \( \mu \), we find that \( \lambda^\iota(\mu) = -\lambda^\iota(\mu) \): the transverse Lyapunov exponent for \( \mu \) vanishes. \( \square \)

2.2 Existence of transverse invariant measures: proof of Theorem B

We intend to prove here Theorem B. We assume the existence of a Gibbs \( u \)-state for \( G_t \) with a non-negative transverse Lyapunov exponent: the goal is to prove the existence of a transverse measure invariant for the holonomy of \( F \).

**Existence of a transverse invariant measure.** The first author proved in [1, Theorem A] the following criterion for the existence invariant measure.

**Theorem 2.2.** Let \( (M, F) \) be a closed foliated manifold. Suppose each leaf \( L \) is endowed with a Riemannian metric \( g_L \) such that the assignment \( L \mapsto g_L \) varies continuously in local charts for the smooth topology. Finally suppose all the leaves have negative sectional curvatures.

Let \( \mu \) be a probability measure on \( \hat{M} \) which is a Gibbs \( su \)-state for the foliated geodesic flow \( G_t \) as defined in §1.5. Then \( \mu \) is totally invariant. In particular \( \hat{F} \) and \( F \) both possess a transverse invariant measure.

**Idea of the proof.** It is noted in [1, Theorem 3.7] that the foliations \( W^u \) and \( W^s \) are absolutely continuous in the leaves of \( F \): using the usual distortion controls we see that the usual proof for Anosov dynamics applies (see [37, Theorem 3.1]).

Consider a foliated atlas for \( \hat{F} \) whose charts have the local product structure (see [1, Proposition 3.4] and disintegrate the Gibbs \( su \)-state in these charts.

Using that \( \mu \) is \( G_t \)-invariant and has Lebesgue disintegration both along stable and unstable leaves it is proven, using the absolutely continuity of \( W^u \) and \( W^s \), that the disintegration in local
charts of $\mu$ has to be equivalent to the Liouville measure (see [1, Lemmas 4.3 and 4.4] for the precise line of reasoning).

The densities of the conditional measures in local unstable manifolds of a Gibbs $u$-state are prescribed dynamically: see Theorem 1.6. A careful analysis of the densities with respect to the Liouville measure of $\mu$ in local charts show that they have to be constant: this is [1, Proposition 4.5].

Finally we are able to deduce that $\mu$ has to be locally the product of the Liouville measure of the leaves by some transverse measure which consequently has to be holonomy invariant. The theorem follows.

Entropic Gibbs $u$-states. Starting with a Gibbs $u$-state with nonnegative transverse Lyapunov exponent we want to use the above criterion and to prove that it has Lebesgue disintegration along the stable foliation. In order to do so, our strategy is to prove that the inverse flow satisfies Pesin's entropy formula 2.9 and finally to conclude using Ledrappier-Young's work [34].

The first step is to prove the following general proposition about metric entropy of Gibbs $u$-states. Such a statement may be found in [33] in a much more general context. In ours the proof is much shorter and shall be postponed until the end of the section.

**Proposition 2.3.** Let $\mu$ be an ergodic Gibbs $u$-state for $G_t$. Then

$$h_\mu \geq \Lambda^u(\mu),$$

where $h_\mu$ denotes the metric entropy of $G_t$ for the measure $\mu$.

**Proof of Theorem B.** We shall now assume that $\mu$ is a Gibbs $u$-state for $G_t$ whose transverse Lyapunov exponent is nonnegative. Note first that we can assume that $\mu$ is ergodic.

**Lemma 2.4.** The foliated geodesic flow possesses an ergodic Gibbs $u$-state with nonnegative transverse Lyapunov exponent.

**Proof.** The transverse Lyapunov exponent of a Gibbs $u$-state is the integral of those of its ergodic components, which are also Gibbs $u$-states (see Theorem 1.6). As a consequence, since there exists a Gibbs $u$-state with nonnegative Lyapunov exponent, there must exist an ergodic one with the same property. 

From now on $\mu$ is supposed to be ergodic. Since it has a nonnegative transverse Lyapunov exponent it follows from Proposition 1.4 that the only negative Lyapunov exponents of $\mu$ are the Lyapunov exponents in the stable direction $\lambda_1^s(\mu)$. Then Ruelle's inequality (see [44]) applied to $G_{-t}$ (which has the same measure entropy as $G_t$) implies that

$$h_\mu \leq -\Lambda^s(\mu).$$

Remember that since $G_t$ preserves the Liouville measure of the leaves, we have $\Lambda^u(\mu) = -\Lambda^s(\mu)$. Hence using Proposition 2.3, we see that the following Pesin's formula holds true

$$h_\mu = -\Lambda^s(\mu).$$

(2.9)

By Ledrappier-Young (see [34]), this equality holds if and only if $\mu$ is a Gibbs $u$-state for $G_{-t}$, or if one prefers, a Gibbs $s$-state for $G_t$.

As a consequence we find that if $\mu$ has nonnegative transverse Lyapunov exponent, then it is a Gibbs $su$-state and, by Theorem 2.2, has to be totally invariant, i.e. locally the product of the Liouville measure by a transverse invariant measure, thus concluding the proof of Theorem B.
2.3 Proof of Proposition 2.3

In what follows, μ is an ergodic Gibbs u-state for the foliated geodesic flow.

Metric entropy according to Katok. We will use Katok’s characterization of the metric entropy of a flow. If μ is an ergodic $G_t$-invariant measure on $\widetilde{M}$, define the Bowen’s balls $B_{t,r}(v)$ as follows: for $v \in \widetilde{M}, t \geq 0$ and $r > 0$, we say that $w \in B_{t,r}(v)$ if for every $s \in [0,t]$, \( \text{dist}(G_s(v), G_s(w)) \leq r \).

Given $\delta \in (0,1)$, denote by $N(t, r, \delta)$ the minimal number of Bowen’s ball $B_{t,r}(v)$ needed to cover a set of measure greater than $1 - \delta$. Katok proved in [30] that the following relation holds for every $\delta \in (0,1)$

$$ h_\mu = \lim_{t \to 0} \lim_{t \to -\infty} \frac{\log N(t, r, \delta)}{t}. \quad (2.10) $$

Approaching the Lyapunov exponents. Oseledets’ theorem applied to the cocycle $DG_t^{E_u} : E^u \to E^u$ provides a set $\mathcal{X}^u$ full for $\mu$ such that for every $v \in \mathcal{X}_u$, the following equality holds

$$ \lim_{t \to -\infty} \frac{1}{t} \log \text{Jac}^u G_t(v) = \Lambda^u(\mu) = \Lambda^u. $$

Fixing a small $\varepsilon > 0$ as well as a time $t > 0$, we shall define the following measurable set

$$ \mathcal{X}_{t,\varepsilon}^u = \left\{ v \in \mathcal{X}^u, \left| \frac{1}{t} \log \text{Jac}^u G_t(v) - \Lambda^u \right| < \varepsilon \right\}, \quad (2.11) $$

and notice that $\lim_{t \to -\infty} \mu(\mathcal{X}_{t,\varepsilon}^u) = 1$ for every parameter $\varepsilon > 0$.

From now on, we fix a small $\varepsilon > 0$, which we will let tend to zero at the end of the proof.

Atlas for the unstable foliation. Consider a finite atlas $(U_i, \phi_i)_{i \in I}$ for the unstable foliation such that $U_i$ reads as a union of local unstable manifolds $D^u(v_i) = \mathcal{W}^u_{loc}(v_i)$.

Since the foliation $\mathcal{W}^u$ is continuous in the $C^\infty$-topology, we find universal bounds for diameters and volumes of the unstable plaques.

For every chart $U_i$ with $\mu(U_i) > 0$, we can disintegrate $\mu$ in the local unstable manifolds of $U_i$. The conditional measures have local densities with respect to the volume which are uniformly log-bounded by a constant independent of $i$: see Theorem 1.6.

Unstable volume of Bowen’s balls. The following lemma is a useful consequence of the distortion controls.

Lemma 2.5. 1. There exists a constant $r_0 > 0$ such that for every $r < r_0$, $w \in \widetilde{M}, t \geq 0$ and every measurable set $V \subset \widetilde{M}$ contained in an unstable plaque containing $w$ we have

$$ B_{t,r}(w) \cap V = G_{-t}(\mathcal{W}^u_r(w)) \cap V. $$

2. There exists a positive constant $C_0 > 0$ such that for every $r < r_0$ and every unstable plaque $D^u = D^u(v_i)$ and every $w \in \widetilde{M}$ we have

$$ \text{Leb}^u(B_{t,r}(w) \cap D^u) \leq C_0 \frac{\text{Leb}^u(\mathcal{W}^u_r(w))}{\text{Jac}^u G_t(w)}. $$

Proof. The first part of the lemma follows classically from the fact that $G_t$ expands uniformly the unstable foliation $\mathcal{W}^u$.

The second part follows from the first one, from the classical distortion control, as well as from the fact that unstable plaques are uniformly bounded. □
Covering the local unstable manifolds. Fixing a small $\delta > 0$, we get $T_0$ such that for every $t \geq T_0$, and every $i \in I$ with $\mu(U_i) > 0$, the relative mass $\mu(\mathcal{X}^u_{t,\epsilon} \cap U_i) / \mu(U_i)$ is less that $\delta$ (we adopt the notation $\epsilon \mathcal{X}$ for the complement of a set $X$). In particular, the mass of $\epsilon \mathcal{X}^u_{t,\epsilon}$ is less that some constant times $\delta$.

Now choose $r$ smaller than the $r_0$ given by Lemma 2.5 and cover $\mathcal{X}^u_{t,\epsilon}$ by $p$ Bowen’s balls $B_{t,\epsilon}(w_j)$, $w_1, \ldots, w_p \in \hat{M}$. Of course we ask that all these Bowen’s balls intersect $\mathcal{X}^u_{t,\epsilon}$. Moreover, up to consider Bowen balls associated to $2r$ instead to $r$, which don’t affect our argument, we can ask that all $w_j$ belong to $\mathcal{X}^u_{t,\epsilon}$. We now have two facts

1. $\mu((\bigcup_{j=1}^p B_{t,\epsilon}(w_j) \cap U_i) \leq \delta \mu(U_i)$.
2. $\mu_{U_i}$ has Lebesgue disintegration in the unstable plaques $D^u(x_i)$ with local densities which are uniformly $\log$-bounded (see Theorem 1.6).

From this and Lemma 2.5, we find a plaque $D^u$ and uniform constants $C_1, \delta_1 > 0$ such that

$$\text{Leb}^u\left(\bigcup_{j=1}^p B_{t,\epsilon}(w_j) \cap D^u\right) \geq (1 - \delta_1)\text{Leb}^u(D^u).$$

(2.12)

$$\text{Leb}^u(B_{t,\epsilon}(w_j) \cap D^u) \leq C_1 \frac{\text{Leb}^u(W^u_{t,\epsilon}(w_j))}{\text{Jac}^n G_t(w_j)}.$$  

(2.13)

Putting (2.12) and (2.13) together and using a uniform upper bound for $\text{Leb}^u(W^u_{t,\epsilon}(v))$, we get a constant $C_r > 0$ depending only on $r$ such that

$$\text{Leb}^u(D^u) \leq \sum_{j=1}^p \text{Leb}^u(B_{t,\epsilon}(w_j) \cap D^u)$$

$$\leq C_r \sum_{j=1}^p \frac{1}{\text{Jac}^n G_t(w_j)}.$$

Since $w_j \in \mathcal{X}^u_{t,\epsilon}$ for every $j$, we see that by definition of $\mathcal{X}^u_{t,\epsilon}$ (see (2.11)) $\frac{1}{\text{Jac}^n G_t(w_j)} \geq \Lambda - \epsilon$. Finally there exists a constant $C_r' > 0$ depending only on $r$ such that the following lower bound for $p$ holds

$$C_r' \exp \left[ r(\Lambda - \epsilon) \right] \leq p.$$  

(2.14)

Lower bound (2.14) holds for every open cover by Bowen’s balls of $\mathcal{X}^u_{t,\epsilon}$, which is of measure $\geq 1 - \delta$: it also provides a lower bound for $N(t, r, \delta)$. Finally, we find $\lim_{t \to -\infty} t^{-1} \log N(t, r, \delta) \geq \Lambda - \epsilon$. Since $\epsilon$ is arbitrary, we deduce that the metric entropy is greater than the sum of the unstable Lyapunov exponents as desired. \qed

### 3 Finiteness of SRB measures and of minimal sets

We intend here to prove Theorem A, i.e. that when $\mathcal{F}$ has no transverse invariant measure $G_t$ has finitely many SRB measures, that their transverse Lyapunov exponents are negative and that $\mathcal{F}$ has finitely many minimal sets. We assume here that $\mathcal{F}$ does not have transverse invariant measure. Theorem B, that we just proved, implies that every Gibbs $u$-state for $G_t$ has negative transverse Lyapunov exponent.
3.1 Finiteness of SRB measures

**Proposition 3.1.** Let $\mathcal{F}$ be a transversally conformal foliation of a closed manifold $M$. Assume that $M$ is endowed with a smooth Riemannian metric such that every leaf is negatively curved. Assume that all Gibbs $u$-states have negative transverse Lyapunov exponent.

1. Every ergodic Gibbs $u$-state is a SRB measure.

2. There is a finite number of Gibbs $u$-states.

3. The supports of the SRB measures are disjoint minimal sets of $W^{cu}$.

**Proof.** The proofs of Items 1. and 2. are given by Bonatti, Gómez-Mont and Martínez in [12, p.16-17]. The idea is to use Pesin’s stable manifold theory (which we will introduce later on) as well as the strong similarity between our context and that of “mostly contracting” partially hyperbolic dynamical diffeomorphisms and to reproduce the line of reasoning of [14].

The fact that the supports of two different SRB measures are disjoint can be shown by copying verbatim the proof of [14, Lemma 2.9].

Now let us show that the support of every SRB measure $\mu$ is a minimal set of $W^{cu}$. Since $\mu$ is a Gibbs $u$-state, its support $\text{Supp}(\mu)$ is a closed and $W^{cu}$-saturated set. Let $K^{cu} \subset \text{Supp}(\mu)$ be a $W^{cu}$-minimal set (which is in particular a $G_t$-invariant set). Using Theorem 1.6 we construct an ergodic Gibbs $u$-state $\mu'$ with $\text{Supp}(\mu') \subset K^{cu}$. Since we proved that the supports of two different ergodic Gibbs $u$-states are disjoint, it must be the case that $\mu = \mu'$ from which we deduce that $K^{cu} = \text{Supp}(\mu)$. The proposition follows.

**Remark 3.2.** The argument shows that every $W^{cu}$-minimal set is the support of a unique SRB measure.

3.2 A bijective correspondence between minimal sets and ergodic Gibbs $u$-states

**Proposition 3.3.** Let $\mathcal{F}$ be a transversally conformal foliation of a closed manifold $M$. Assume that $M$ is endowed with a smooth Riemannian metric such that every leaf is negatively curved.

If all Gibbs $u$-states have negative transverse Lyapunov exponent, there is a natural bijective correspondence between minimal sets of $\mathcal{F}$ and ergodic Gibbs $u$-states for $G_t$. More precisely every minimal set of $\mathcal{F}$ supports a unique Gibbs $u$-state of $G_t$ and every ergodic Gibbs $u$-state for $G_t$ is supported inside a minimal set of $\mathcal{F}$.

**A geometric property.** For a set $K \subset \bar{M}$ we denote $W^s(K)$ its stable manifold i.e. the union of the stable manifolds $W^s(\nu)$ of elements $\nu \in K$. The sets $W^u(K), W^c(K), W^s(K)$ and $W^{cu}(K)$ are defined analogously.

**Lemma 3.4.** The following properties hold true for every $\nu \in \bar{M}$

1. $W^u(W^c(\nu)) = W^{cu}(\nu)$;

2. $W^s(W^{cu}(\nu))$ has full volume in $\bar{L}_\nu$.

**Proof.** The first property is clear because $W^{cu}(\nu)$ has been defined as the saturation of $W^u(\nu)$ in the flow direction and because unstable manifolds are invariant by the flow (i.e. $G_t(W^u(\nu)) = W^u(G_t(\nu))$). In order to prove the second one, let us work in the universal cover.

Let $L$ be a leaf of $\mathcal{F}$ and $\bar{L}$ be its universal cover. The sectional curvature of $\bar{L}$ is everywhere pinched between two negative constants. Consequently $\bar{L}$ is compactified by adding the sphere
at infinity $\tilde{L}(\infty)$ defined as the set of equivalence classes of geodesic rays for the relation “stay at bounded distance”.

Lifts to $T^1\tilde{L}$ of stable manifolds of $G_t$ are denoted by $\tilde{W}^s(\cdot)$. Manifolds $\tilde{W}^u(\cdot)$, $\tilde{W}^{cs}(\cdot)$ and $\tilde{W}^{cu}(\cdot)$ are defined analogously.

In order to prove the second property, it is enough to prove the following equality for every $v \in T^1\tilde{L}$

$$\tilde{W}^s(\tilde{W}^{cu}(v)) = T^1\tilde{L} \setminus \tilde{W}^{cs}(-v), \quad (3.15)$$

Indeed $\tilde{W}^{cs}(-v)$ is a strict submanifold of $T^1\tilde{L}$ and therefore has volume zero.

For $v \in T^1\tilde{L}$ denote by $v(\infty) = \lim_{t \to -\infty} c_v(t) \in \tilde{L}(\infty)$ and $v(-\infty) = \lim_{t \to -\infty} c_v(-t) \in \tilde{L}(\infty)$, where $c_v$ is the geodesic directed by $v$. It is well known (see for example [9]) that for every $v \in T^1\tilde{L}$, $v(-\infty) \neq v(\infty)$ and that $w \in \tilde{W}^{cs}(v)$ (resp. $w \in \tilde{W}^{cu}(v)$) if and only if $v(\infty) = w(\infty)$ (resp. $v(-\infty) = w(-\infty)$). This implies that $\tilde{W}^s(\tilde{W}^{cu}(v)) \cap \tilde{W}^{cs}(-v) = \emptyset$ for every $v \in T^1\tilde{L}$.

Now let $\xi = v(-\infty)$. Let $w \in T^1\tilde{L}$ and $\xi' = w(\infty)$. If $w \notin \tilde{W}^{cs}(-v)$ then $\xi' \neq \xi$ and there exists a directed geodesic starting at $\xi$ and ending at $\xi'$. This geodesic is precisely the intersection of $\tilde{W}^{cu}(v)$ with $\tilde{W}^{cs}(w)$. In particular it intersects $\tilde{W}^s(w)$. This implies that $w \in \tilde{W}^s(\tilde{W}^{cu}(v))$.

One deduces that (3.15) holds true, and the lemma follows.

**Basins of Gibbs $u$-states.** We now prove that the intersection between the basin of an ergodic Gibbs $u$-state and a typical leaf is large. Recall that the basin of $\mu$ is defined as the set

$$\mathcal{B}(\mu) = \left\{ v \in \tilde{M}; \frac{1}{T} \int_0^T \delta_{G_t[v]} d t \to \mu \right\}.$$

**Lemma 3.5.** Let $\mu$ be an ergodic Gibbs $u$-state for $G_t$. Then there is a Borel set $\mathcal{X} \subset \tilde{M}$ full for $\mu$ such that for every $v \in \mathcal{X}$, $\mathcal{B}(\mu) \cap L_v$ has full volume in $L_v$.

**Proof.** First notice that $\mathcal{B}(\mu)$ is $\tilde{W}^s$-saturated. Using the second item of Lemma 3.4 as well as the absolute continuity of $\tilde{W}^s$ inside leaves of $\tilde{F}$ (see for example [1, Theorem 3.7]) it is enough to prove the existence of a Borel set $\mathcal{X}$ full for $\mu$ such that for every $v \in \mathcal{X}$, $\mathcal{B}(\mu) \cap \tilde{W}^{cu}(v)$ has full volume in $\tilde{W}^{cu}(v)$.

Denote by $\mathcal{X}_1 \subset \tilde{M}$ the set of points $v \in \tilde{M}$ such that $\text{Leb}^{cu}$-almost every point of $W_1^{cu}(v)$ belongs to $\mathcal{B}(\mu)$. Since $\mu$ is an ergodic Gibbs $u$-state, $\mathcal{X}_1$ is full for $\mu$.

We define $\mathcal{X} = \bigcap_{n \in \mathbb{Z}} G_n(\mathcal{X}_1)$. Fix $v \in \mathcal{X}$ and denote $v_m = G_m(v)$ for $m \in \mathbb{Z}$.

**Claim.** For every $m \in \mathbb{Z}$ the basin $\mathcal{B}(\mu)$ contains a full volume subset of $W^u(G_{[0,1]}(v_m))$.

Establishing the claim suffices to prove the lemma. Indeed by the first item of 3.4 we have $W^{cu}(v) = \bigcup_{m \in \mathbb{Z}} W^u(G_{[0,1]}(v_m))$.

Now in order to prove this claim note that, if we set $\chi = \operatorname{Min} \{ \| (DG_1)_t [v] \| > 1 \}$, we have for $n \geq -m$

$$W_{\chi^{n+m}}^u(G_{[0,1]}(v_m)) \subset G_{n+m} \left( W_1^u(G_{[0,1]}(v_m)) \right),$$

in such a way that

$$W^u(G_{[0,1]}(v_m)) = \bigcup_{n \in \mathbb{Z}} G_{n+m} \left( W_1^u(G_{[0,1]}(v_m)) \right).$$

Since $v \in \mathcal{X}$ we have $v_{-n} \in \mathcal{X}_1$ for every $n$ and $\text{Leb}^{cu}$-almost every point of $W_1^{cu}(G_{[0,1]}(v_m))$ belongs to $\mathcal{B}(\mu)$. Using that $\mathcal{B}(\mu)$ is $G_t$-invariant this property also holds for $G_{n+m}(W_1^{cu}(G_{[0,1]}(v_{-n})))$. Finally we deduce that $\text{Leb}^{cu}$-almost every $v \in W^u(G_{[0,1]}(v_m))$ belongs to $\mathcal{B}(\mu)$ and the claim, as well as the lemma, follows.
**Pesin manifolds.** Assume here that \( \mu \) is an ergodic Gibbs \( u \)-state whose transverse Lyapunov exponent is negative. By Proposition 1.4, \( \mu \) is hyperbolic in the sense of Pesin: its Lyapunov exponents do not vanish.

*Pesin’s stable manifold theory* (see for example [14]) implies that \( \mu \)-almost every \( v \in \hat{M} \) admits a Pesin stable manifold \( W^s_{\text{Pes}}(v) \) (of dimension \( \dim W^s + \dim \mathcal{F} \)) and that the “foliation” \( (W^s_{\text{Pes}}(v))_{v \in \hat{M}} \) is absolutely continuous (see [14, p.164] for the definition).

Pesin center-stable foliation \( W^{cs}_{\text{Pes}}(v) \) is the saturation of \( W^s_{\text{Pes}}(v) \) in the flow direction. Note that these manifolds are transverse to \( \mathcal{W}^u \). We denote respectively by \( W^s_{\text{Pes}} \) and \( W^{cs}_{\text{Pes}} \) the stable and center-stable Pesin foliations.

**Lemma 3.6.** Let \( \mu \) be an ergodic Gibbs \( u \)-state for \( G_1 \) with negative transverse Lyapunov exponent. Then there exists a Borel set \( \mathcal{Y} \subset \hat{M} \) full for \( \mu \) such that for every \( v \in \mathcal{Y} \) there exists a Borel set \( \Gamma_v \subset W^s_{\text{loc}}(v) \) of full volume such that:

1. \( \Gamma_v \) is included in the basin of \( \mu \);
2. every point of \( \Gamma_v \) admits a Pesin stable manifold.

Moreover there exists \( \varepsilon = \varepsilon(v) > 0 \) as well as a Borel set \( \Gamma_{v, \varepsilon} \subset \Gamma_v \) of positive volume such that for every \( w \in \hat{M} \) with \( \text{dist}(v, w) \leq \varepsilon \), \( W^s_{\text{Pes}}(\Gamma_{v, \varepsilon}) \) induces a holonomy map between \( W^s_{\text{loc}}(v) \) and \( W^s_{\text{loc}}(w) \).

In particular the basin of \( \mu \) intersects \( W^s_{\text{loc}}(w) \) in a set of positive volume whenever \( \text{dist}(v, w) \leq \varepsilon \).

**Proof.** Pesin’s theory and Birkhoff’s theorem imply the existence of Borel subset \( \mathcal{Y}_1 \subset \hat{M} \) included in the basin of \( \mu \) such that every point of \( \mathcal{Y}_1 \) possesses a Pesin stable manifold.

Since \( \mu \) is a Gibbs \( u \)-state, inside a foliated chart for \( \mathcal{W}^u \) the conditional measure of \( \mu \) in local unstable manifolds are equivalent to the Lebesgue measure. This implies in particular that there exists a Borel subset \( \mathcal{Y}_2 \subset \hat{M} \) full for \( \mu \) such that for every \( v \in \mathcal{Y}_2 \), \( \mathcal{Y}_1 \cap W^s_{\text{loc}}(v) \) has full volume in \( W^s_{\text{loc}}(v) \).

The remaining part of the lemma follows directly from the absolute continuity of \( W^{cs}_{\text{Pes}} \). \( \square \)

**The key lemma.** We are now ready to state the main ingredient of the proof of Proposition 3.3. In the sequel we adopt the following notation: if \( X \subset \hat{M} \) then \( \text{Cl}(X) \) denotes its closure.

**Lemma 3.7.** Assume that every Gibbs \( u \)-state of \( G_1 \) has a negative transverse Lyapunov exponent. Let \( \mu \) be an ergodic Gibbs \( u \)-state. Then for every \( v \in \text{Supp}(\mu) \), \( \mu \) is the only Gibbs \( u \)-state for \( G_1 \) supported inside \( \text{Cl}(\hat{L}_v) \).

**Proof.** First note that, by Proposition 3.1, \( \text{Supp}(\mu) \) is a minimal set for \( \mathcal{W}^{cs} \) which implies that for every \( v \in \text{Supp}(\mu) \) we have \( \text{Supp}(\mu) \subset \text{Cl}(\hat{L}_v) \). We deduce that \( \text{Cl}(\hat{L}_v) \) does not depend on \( v \in \text{Supp}(\mu) \) and in particular we can assume that \( v \in \mathcal{X} \), the set constructed in Lemma 3.5.

For such a \( v \in \mathcal{X} \) set \( K = \text{Cl}(\hat{L}_v) \) and let \( \mu' \) be an ergodic Gibbs \( u \)-state satisfying \( \text{Supp}(\mu') \subset K \) (see Theorem 1.6). We will prove that \( \mu = \mu' \), and the lemma will follow.

First note that by hypothesis \( \mu' \) has a negative transverse Lyapunov exponent so we can apply Lemma 3.6 to \( \mu' \). It provides an element \( v' \in K \) such that \( \mathcal{B}(\mu') \) contains a full volume subset of \( W^s_{\text{loc}}(v') \), each point of which admitting a Pesin stable manifold.

Because \( v \in \mathcal{X} \), Lemma 3.5 provides \( v'' \in \hat{L}_v \cap \mathcal{B}(\mu) \) arbitrarily close to \( v' \).

Using one more time Lemma 3.6 we find that \( \Gamma = \mathcal{B}(\mu') \cap W^s_{\text{loc}}(v'') \) has positive volume in \( W^s_{\text{loc}}(v'') \).

The basin of any ergodic \( G_1 \)-invariant measure is \( \mathcal{W}^{cs} \)-saturated so \( \mathcal{W}^{cs}(\Gamma) \subset \mathcal{B}(\mu') \). Finally use the absolute continuity of \( \mathcal{W}^{cs} \) inside \( \hat{L}_{v''} = \hat{L}_v \) to prove that \( \mathcal{W}^{cs}(\Gamma) \) has positive volume in \( \hat{L}_v \).

Since by 3.5 \( \mathcal{B}(\mu) \) has full volume in \( \hat{L}_v \) we conclude that \( \mathcal{B}(\mu') = \mathcal{B}(\mu) \) and hence that \( \mu = \mu' \). \( \square \)
Proof of Proposition 3.3. We assume in all the following that all Gibbs \( u \)-states for \( G_t \) have negative transverse Lyapunov exponents. We divide the proof of Proposition 3.3 into two halves.

Lemma 3.8. Let \( K \) be a minimal set of \( \hat{\mathcal{F}} \). There exists a unique Gibbs \( u \)-state supported inside \( K \).

Proof. A minimal set \( K \) is closed and \( \hat{\mathcal{F}} \)-saturated. Therefore applying in \( K \) the proof of Theorem 1.6 provides that Gibbs \( u \)-states supported in \( K \) exist and that ergodic components of such a measure are also supported in \( K \). Lemma 3.7 immediately implies the uniqueness of the Gibbs \( u \)-state.

Lemma 3.9. Every ergodic Gibbs \( u \)-state is supported inside a minimal set of \( \hat{\mathcal{F}} \).

Proof. Let \( \mu \) be an ergodic Gibbs \( u \)-state for \( G_t \). We will prove that for every \( v \in \text{Supp}(\mu) \), the set \( \text{Cl}(\hat{L}_v) \) is minimal for \( \hat{\mathcal{F}} \).

Let \( v \in \text{Supp}(\mu) \). The set \( \text{Cl}(\hat{L}_v) \) is closed and \( \hat{\mathcal{F}} \)-saturated so it contains \( K \), a minimal set for \( \hat{\mathcal{F}} \). By Lemma 3.8 there is a Gibbs \( u \)-state \( \mu' \) supported inside \( K \subset \text{Cl}(\hat{L}_v) \).

But by Lemma 3.7 \( \mu \) is the unique Gibbs \( u \)-state supported inside \( \text{Cl}(\hat{L}_v) \) so it must be the case that \( \mu = \mu' \). In particular \( \mu \) is supported inside a minimal set of \( \hat{\mathcal{F}} \).

4 Some partially hyperbolic foliated geodesic flows

4.1 Suspensions and cocycles

We now wish to discuss a natural question: when is the transverse dynamics of the flow dominated by the hyperbolicity inside the leaves? In other words, when is the foliated geodesic flow partially hyperbolic?

In the sequel, we shall focus on foliations transverse to a \( \mathbb{RP}^1 \)-fiber bundle over a hyperbolic surface and with projective holonomy. We will show that whether the foliated geodesic flow of such a foliation is partially hyperbolic or not depends on a natural topological condition on the fiber bundle: the value of its Euler class. In particular this provides new geometrical examples of partially hyperbolic flows.

Suspension. Let \( \Sigma \) be a closed Riemann surface of genus higher than 2. Its universal cover is conformally equivalent to the upper half plane \( \mathbb{H} \). Let \( \text{hol} : \pi_1(\Sigma) \to \text{PSL}_2(\mathbb{R}) \) be a projective representation of its fundamental group. Note that \( \text{PSL}_2(\mathbb{R}) \) acts on \( \mathbb{RP}^1 \) by homographies and thus identifies with a subgroup of circle diffeomorphisms.

The group \( \pi_1(\Sigma) \) acts diagonally on \( \mathbb{H} \times \mathbb{RP}^1 \), the action on the first factor being by deck transformations and on second one being by hol. The quotient is a closed manifold \( M \), called the suspended manifold. It is endowed with a structure of circle bundle \( \Pi : M \to \Sigma \) and with a foliation \( \mathcal{F} \) called the suspended foliation transverse to the fibers. The representation \( \text{hol} : \pi_1(\Sigma) \to \text{PSL}_2(\mathbb{R}) \) is called the holonomy representation.

Note that holonomy has the following topological interpretation. If \( c \) is a closed path on \( \Sigma \) and \( \gamma \in \pi_1(\Sigma) \) denotes its homotopy class, then there is, for every \( x \) belonging to the fiber of \( p = c(0) \) a unique lift of \( c \) starting at \( x \). Then the ending point of this lift depends only on \( \gamma \) and is equal to \( \text{hol}(\gamma)^{-1}(x) \).

Closed hyperbolic surfaces. The upper half plane is naturally endowed with its Poincaré metric \( ds^2 = (dx^2 + dy^2)/y^2 \). This metric is hyperbolic, i.e. of curvature constant equal to \( -1 \). The action
on $\mathbb{RP}^1$ of $\text{PSL}_2(\mathbb{R})$ extends to an action on $\mathbb{H}$. This group then identifies with the group of direct isometries of $\mathbb{H}$.

By uniformization, a smooth hyperbolic metric $m$ on $\Sigma$ gives rise to a Fuchsian representation, i.e. a faithful and discrete representation $\rho : \pi_1(\Sigma) \to \text{PSL}_2(\mathbb{R})$ which is well defined up to conjugacy by an element of $\text{PSL}_2(\mathbb{R})$. Say two metrics represent the same point in the Teichmüller space if one is the image of the other by a diffeomorphism isotopic to the identity. It is equivalent to having their Fuchsian representations conjugated in $\text{PSL}_2(\mathbb{R})$.

**Admissible metrics on the suspended manifold.** When $\text{hol} : \pi_1(\Sigma) \to \text{PSL}_2(\mathbb{R})$ is given it is possible to lift via $\Pi$ the Riemannian metric $m$ to the leaves so that in restriction to each leaf $\Pi$ is a Riemannian cover. Using a partition of the unity it is possible to combine this metric with a smooth family of Riemannian metrics of the fibers.

This gives a smooth Riemannian metric on $M$ whose restriction in each leaf is locally isometric to $m$, in restriction to each fiber is conformally equivalent to the usual angle metric (because the codimension of $\mathcal{F}$ equals 1), and which makes fibers and leaves orthogonal. Such a metric will be called an admissible metric associated to $m$.

**Foliated geodesic flow as a cocycle.** The differential of $\Pi$ induces an $\mathbb{RP}^1$-bundle $\Pi^* : \hat{M} \to T^1\Sigma$ which is transverse to the foliation $\hat{\mathcal{F}}$. For $w \in T^1\Sigma$ we set $F_{*,w} = \Pi^{-1}_{*}(w)$.

Since $\Pi$ is a local isometry when restricted to the leaves, it sends geodesics of the leaves on geodesics of the base. As a consequence the foliated geodesic flow $G_t : \hat{M} \to \hat{M}$ projects down via $\Pi_{*}$ to the geodesic flow of $T^1\Sigma$ which we denote by $g_{t} : T^1\Sigma \to T^1\Sigma$. In particular it preserves the fibers of $\Pi_{*}$ and for every $w \in T^1\Sigma$ and $t \in \mathbb{R}$ the map

$$A_t(w) = (G_t)_{F_{*,w}} : F_{*,w} \to F_{*,g_{t}(w)}$$

identifies with the holonomy along the orbit segment $g_{[0,t]}(w)$ and in particular belongs to $\text{PSL}_2(\mathbb{R})$. Moreover it satisfies the cocycle relation

$$A_{t_1 + t_2}(w) = A_{t_1}(g_{t_2}(w))A_{t_2}(w).$$

All this implies that the foliated geodesic flow $G_t$ is a projective cocycle over the geodesic flow $g_t$.

### 4.2 Domination of representations

Recently a notion of domination of representations appeared in the theory of 3-dimensional Anti-de Sitter geometry [19, 29]. We wish to show here that it gives rise to new examples of partially hyperbolic dynamical systems.

**Domination.** The translation length of an element $P \in \text{PSL}_2(\mathbb{R})$ is by definition

$$l_P = \inf_{z \in \mathbb{H}} \text{dist}(Pz, z) \geq 0.$$

**Remark 4.1.** If $P$ is an elliptic element (i.e. conjugated to a rotation) it has a fixed point in $\mathbb{H}$ and $l_P = 0$.

If $P$ is a parabolic or hyperbolic element of $\text{PSL}_2(\mathbb{R})$ (i.e. respectively conjugated to a translation or a homothety) then $l_P$ coincide with the modulus of the logarithm of the derivative at any of its fixed points. In particular it vanishes in the case $P$ is parabolic.
Lemma 4.2. Let \( P, Q \in \text{PSL}_2(\mathbb{R}) \) such that \( Q \) is hyperbolic. Then for every \( k \in \mathbb{Z} \setminus \{0\} \)
\[
\frac{l_{P^k}}{l_{Q^k}} = \frac{l_P}{l_Q}.
\]

Proof. It is a fairly direct application of Remark 4.1 that for every \( P \in \text{PSL}_2(\mathbb{R}) \) and \( k \in \mathbb{Z} \), \( l_{P^k} = k l_P \).
The lemma follows. \( \square \)

The marked length spectrum of a projective representation \( \phi : \pi_1(\Sigma) \to \text{PSL}_2(\mathbb{R}) \) is by definition the collection \( \ell_\phi = (l_{\phi(\gamma)})_{\gamma \in \pi_1(\Sigma)} \). Say \( \phi_1 \) dominates \( \phi_2 \) if there exists \( \kappa < 1 \) such that \( \ell_{\phi_2} < \kappa \ell_{\phi_1} \).

Domination by Fuchsian representation. We will make use of the following theorem proven independently and with different methods by Guéritaud-Kassel-Wolff in [29] (to which we refer for details about the Euler class) and by Deroin-Tholozan in [19]. Recall that, as we mentioned in the introduction, by Goldman’s work [28] non-Fuchsian representations of the fundamental group of a closed surface with genus \( g \) are precisely those who have Euler number strictly smaller, in absolute value, than \( 2g - 2 \).

Theorem 4.3. Let \( \Sigma \) be a closed Riemann surface of genus \( g \geq 2 \).

1. Let \( \text{hol} : \pi_1(\Sigma) \to \text{PSL}_2(\mathbb{R}) \) be a non-Fuchsian projective representation. Then it is dominated by a Fuchsian representation \( \rho : \pi_1(\Sigma) \to \text{PSL}_2(\mathbb{R}) \).

2. Reciprocally any Fuchsian representation \( \rho : \pi_1(\Sigma) \to \text{PSL}_2(\mathbb{R}) \) dominates some non-Fuchsian representation whose Euler class can be prescribed in \( \{3 - 2g, ..., 0, ..., 2g - 3\} \).

4.3 Partially hyperbolic foliated geodesic flows

The goal of that paragraph is to give a proof of Theorem E. Together with Theorem 4.3 it will give a proof of Theorem D. First we need to recall some facts about partial hyperbolicity.

4.3.1 Partially hyperbolic flows

Definition. A non-singular flow \( \Phi_t : N \to N \) on a Riemannian manifold \( N \) generated by a vector field \( X \) is said to be partially hyperbolic if there exists a decomposition of the normal bundle of \( X \) of the form
\[
\mathcal{N}_X = E^s_N \oplus E^c_N \oplus E^u_N
\]
and two constants \( C, \lambda > 0 \) such that

1. the bundles \( E^s_N, E^c_N, E^u_N \) are invariant by the linear Poincaré flow \( \Psi_t : \mathcal{N}_X \to \mathcal{N}_X \) of \( \Phi_t \);

2. for \( x \in N \) and every \( v^s \in E^s_N(x) \) and \( v^u \in E^u_N(x) \),
\[
||\Psi_t(x) v^s|| \leq C \exp(-\lambda t) ||v^s||
\]
\[
||\Psi_{-t}(x) v^u|| \leq C \exp(-\lambda t) ||v^u||.
\]

3. the decomposition is dominated in the sense that for every \( t > 0 \), \( x \in N \) and \( v^s \in E^s_N(x) \), \( v^c \in E^c_N(x) \) and \( v^u \in E^u_N(x) \)
\[
\frac{||\Psi_t(x) v^c||}{||\Psi_t(x) v^u||} \leq C \exp(-\lambda t) \quad \text{and} \quad \frac{||\Psi_{-t}(x) v^c||}{||\Psi_{-t}(x) v^u||} \leq C \exp(-\lambda t).
\]

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Criterion for domination. We will give a criterion for partial hyperbolicity due to Mañé [36].

**Lemma 4.4.** Suppose the linear Poincaré flow $\Psi_t : \mathcal{N}_X \to \mathcal{N}_X$ preserves a decomposition $\mathcal{N}_X = E \oplus F$ such that for every $\Phi_t$-invariant probability measure $\mu$

$$\lambda^+_E(\mu) < \lambda^-_F(\mu),$$

where $\lambda^+_E(\mu)$ and $\lambda^-_F(\mu)$ stand respectively for the greatest Lyapunov exponent of $\mu$ along $E$ and the lowest Lyapunov exponent of $\mu$ along $F$.

Then the decomposition $\mathcal{N}_X = E \oplus F$ is dominated.

**Proof.** This argument is quite classical so we only give a glimpse of the proof. Using the invariance and the continuity of the decomposition it is enough to prove that it is dominated for $\Psi_1$.

**Claim.** For every $x \in N$ there exists a integer $n_x > 0$ such that for every $v \in E(x)$ and $w \in F(x)$

$$\frac{||\Psi_{n_x}(x)v||}{||\Psi_{n_x}(x)w||} < \frac{1}{2}.$$  

(4.16)

Proving the previous claim clearly suffices to prove the domination: use the continuity of the Poincaré flow to prove that $n_x$ is locally constant and the compactness of $N$ to give a uniform upper bound for $n_x$. The domination then follows easily.

Suppose the claim does not hold for some $x \in N$. Then for every integer $n > 0$ we have

$$\frac{1}{n} \log |||\Psi_n(x)\|_{E(x)}||| - \frac{1}{n} \log m(\Psi_n(x)\|_{F(x)}) \geq \frac{\log 2}{n},$$

(4.18)

where $|||\cdot|||$ and $m(.)$ stand respectively for the operator norm and conorm associated to the norm $||\cdot||$.

There exists a strictly increasing sequence of integers $(n_k)_{k \geq 0}$ and a $\Phi_1$-invariant measure $\eta$ such that

$$\frac{1}{n_k} \sum_{i=0}^{n_k} \delta_{\Phi_i(x)} \to \eta.$$  

(4.17)

Setting $h_1(x) = \log |||\Psi_1(x)\|_{E(x)}|||$ and $h_2(x) = \log m(\Psi_1(x)\|_{F(x)})$ which are continuous functions of $x \in N$ we see that

$$\frac{1}{n_k} \log |||\Psi_{n_k}(x)\|_{E(x)}||| \leq \frac{1}{n_k} \sum_{i=0}^{n_k-1} h_1 \circ \Phi_i(x),$$

(4.18)

$$\frac{1}{n_k} \log m(\Psi_{n_k}(x)\|_{F(x)}) \geq \frac{1}{n_k} \sum_{i=0}^{n_k-1} h_2 \circ \Phi_i(x).$$

(4.19)

Putting together (4.16), (4.17), (4.18) and (4.19) it follows that

$$\lambda^+_E(\eta) - \lambda^-_F(\eta) = \int_N h_1(x)d\eta(x) - \int_N h_2(x)d\eta(x) \geq 0,$$

where $\lambda^+_E(\eta)$ and $\lambda^-_F(\eta)$ represent respectively the greatest and lowest Lyapunov exponents along $E$ and $F$ of the diffeomorphism $\Phi_1$ for $\eta$.

This does not contradict yet our hypothesis. But if $\mu$ denotes the average of the measures $\Phi_t \eta$, for $t \in [0,1]$ one easily proves using the commutation formula $\Phi_s \circ \Phi_t = \Phi_{t} \circ \Phi_s$ that $\mu$ is $\Phi_t$-invariant for every $t$ and has the same Lyapunov exponents as $\eta$. Hence $\mu$ is an invariant measure which does not satisfy the hypothesis of the lemma. This proves the claim by contraposition. □
4.4 Domination of representations implies partial hyperbolicity

We intend here to prove the first half of Theorem E by showing that if hol is dominated by $\rho$ then the corresponding foliated geodesic flow $G_t : \tilde{M} \to \tilde{M}$ is partially hyperbolic. So let us assume that $\rho, \text{hol} : \pi_1(\Sigma) \to \text{PSL}_2(\mathbb{R})$ are projective representations such that $\rho$ is Fuchsian and dominates hol.

**Domination along periodic orbits.** Let $\nu \in \tilde{M}$ be a periodic point for the foliated geodesic flow and $\mu_\nu$ be the $G_t$-invariant measure supported by the corresponding periodic orbit.

**Lemma 4.5.** If $\rho$ dominates hol and if $\kappa \in (0, 1)$ denotes the domination constant, we have

$$|\lambda^h(\mu_\nu)| \leq \kappa.$$ 

*Proof.* Let $T_0 > 0$ be the period of $\nu$. The projection of the orbit $\Theta(\nu)$ of $\nu$ is a periodic orbit $\Theta(w)$ for $g$, where $w = \Pi_\nu(\nu)$.

The free homotopy class of $\Theta(w)$ is the conjugacy class of some element $\gamma \in \pi_1(\Sigma)$ and the holonomy map $\tau_\gamma$ over $\Theta(w)$ is conjugated to $\text{hol}(\gamma)^{-1}$. In particular it lies in $\text{PSL}_2(\mathbb{R})$ and, since $\Theta(\nu)$ is closed, has a periodic point in $\mathbb{R}^1$: it has to be conjugated to a rational rotation, a parabolic or hyperbolic element. In the first case the holonomy over $\Theta(w)$ is conjugated to an isometry and we clearly have $\lambda^h(\mu_\nu) = 0$. In the remaining cases, this implies that $\nu$, which is periodic for $\tau_\gamma$, has to be a fixed point of $\tau_\gamma$.

We deduce two things. Firstly the orbits $\Theta(\nu)$ and $\Theta(w)$ have the same length which is equal to $T_0$. Secondly the holonomy map $h_{G_0, T_0}(\nu)$ of $\tilde{\mathcal{F}}$ along the closed orbit $G_{[0, T_0]}(\nu)$ is conjugated to $\text{hol}(\gamma)^{-1}$.

By definition of $\rho$ the length of the closed geodesic $\Theta(w)$ is $l_\rho(\gamma) = l_{\rho(\gamma)^{-1}}$. By Remark 4.1 the logarithm of the derivative at the fixed point $\nu$ of $\text{hol}(\gamma)^{-1}$ is $\pm l_{\text{hol}(\gamma)^{-1}}$ (note that these quantities are constant on the conjugacy class of $\gamma$).

Using the domination of hol by $\rho$ one sees that there exists $\kappa \in (0, 1)$ independent of $\gamma \in \pi_1(\Sigma)$ such that we have $l_{\text{hol}(\gamma)^{-1}} < \kappa l_{\rho(\gamma)^{-1}}$.

This implies that for every positive integer $k$ we have by Lemma 4.2

$$\left| \frac{1}{kT_0} \log D_{\nu} h_{G_{[0, T_0]}(\nu)} \right| = \frac{l_{\text{hol}(\gamma)^{-1}}}{l_{\rho(\gamma)^{-1}}} \leq \kappa.$$  \(4.20\)

Since the left hand side of (4.20) tends to $\lambda^h(\mu_\nu)$ as $k$ tends to infinity, the lemma follows. \qed

**Partial hyperbolicity.** Note that the Poincaré linear flow $\Psi_t$ of $G_t$ preserves a decomposition

$$\mathcal{N}_X = E^s_{\mathcal{N}} \oplus E^c_{\mathcal{N}} \oplus E^u_{\mathcal{N}},$$

where $E^c_{\mathcal{N}}$ denotes the tangent space of the fibers, and $E^s_{\mathcal{N}}, E^u_{\mathcal{N}}$ represent respectively the orthogonal projections on $\mathcal{N}_X$ of the stable and unstable directions of the flow. These bundles are $1$-dimensional.

As we saw in §1.4.1 the Lyapunov exponent at $\nu$ along $E^c_{\mathcal{N}}$ is precisely $\lambda^h(\nu)$.

Moreover, since the metric on the leaves is of constant curvature $-1$, it is clear from the commutation relations between horocyclic and geodesic flows that for every $\nu \in \tilde{M}$ we have

$$\lambda^u(\nu) = -\lambda^s(\nu) = 1.$$ 

In order to prove the partial hyperbolicity of the flow $G_t$ we are going to use the criterion stated in Lemma 4.4. It is enough to prove the
Lemma 4.6. There exists $\kappa \in (0, 1)$ such that for every ergodic $G_t$-invariant probability measure $\mu$ we have

$$|\lambda^0(\mu)| \leq \kappa.$$  

Proof. It is clear that it suffices to treat the case where $\mu$ is an ergodic $G_t$-invariant measure with $\lambda^0(\mu) \neq 0$. In that case the measure $\mu$ is an ergodic hyperbolic measure in the sense of Pesin: all its Lyapunov exponents are non-zero.

Since the flow $G_t$ is $C^\infty$ we can use Katok’s closing lemma (see [30]). In that case there exists a sequence $(v_k)_{k \geq 0}$ of periodic points for $G_t$ such that

$$\mu_{v_k} \xrightarrow{k \to \infty} \mu,$$

in the weak$^*$-sense, where we recall that $\mu_{v_k}$ is the $G_t$-invariant measure supported by the periodic orbit $O(v_k)$.

The transverse Lyapunov exponent of a $G_t$-invariant measure $\nu$ is in our case given by an integral

$$\lambda^0(\nu) = \int_M \log \|\Psi'(v)_{|E^c(v)}\| \, d\nu(v),$$

in particular it varies continuously with $\nu$ and we have by Lemma 4.5

$$|\lambda^0(\mu)| = \lim_{k \to \infty} |\lambda^0(\mu_k)| \leq \kappa.$$  

\[\square\]

4.5 Partial hyperbolicity implies domination of representations

End of the proof of Theorem E. We assume here that the foliated geodesic flow $G_t$ is partially hyperbolic. We want to find $\kappa < 1$ so that for every $\gamma \in \pi_1(\Sigma)$, $l_{\text{hol}(\gamma)} \leq \kappa l_{\rho(\gamma)}$. It is obvious from Remark 4.1 that it is enough to treat the case where $\text{hol}(\gamma)$ is hyperbolic because otherwise $l_{\text{hol}(\gamma)} = 0$.

Let $\gamma \in \pi_1(\Sigma)$ be such that $\text{hol}(\gamma)$ is hyperbolic. There exists a unique periodic orbit of the geodesic flow of $T^1\Sigma$, denoted by $O(w)$, whose free homotopy class is the conjugacy class of $\gamma$. As we have already noticed, the length of the orbit equals $l_{\rho(\gamma)}$.

The holonomy $\tau_w$ over $\Theta(w)$ is conjugated to $\text{hol}(\gamma)^{-1}$ and in particular it is a hyperbolic element of $\text{PSL}_2(\mathbb{R})$ with the same translation length as $\text{hol}(\gamma)^{-1}$. It implies that $\tau_w$ has a repelling fixed point $v$, which is also fixed by all of its powers. By Remark 4.1 the logarithm of the derivative of $\tau^k_w$ at $v$ equals $l_{\text{hol}(\gamma)^{-1}}$. Now the partial hyperbolicity at $v$ implies the existence of uniform $C, \lambda > 0$ such that if $T_0 = l_{\rho(\gamma)} = l_{\rho(\gamma)^{-1}}$ denotes the period of $w$ we have for every $k > 0$

$$\frac{D\tau^k_w(v)}{e^{kT_0}} \leq Ce^{-k\lambda T_0}.$$  

Note that $\log D\tau^k_w(v) = l_{\text{hol}(\gamma)^{-1}} = k l_{\text{hol}(\gamma)^{-1}}$ and $k T_0 = k l_{\rho(\gamma)^{-1}}$. Taking the logarithm, dividing by $k$ and denoting $\kappa = 1 - \lambda < 1$ provides

$$l_{\text{hol}(\gamma)^{-1}} \leq \frac{\log C}{k} + \kappa l_{\rho(\gamma)^{-1}},$$

for every $\gamma \in \pi_1(\Sigma)$ and $k > 0$. We deduce that $\text{hol}$ is dominated by $\gamma$. Theorem E then follows.  

\[\square\]
5 Foliations associated to Fuchsian representations

5.1 Reparametrization of the geodesic flow of a hyperbolic surface

We now turn to the case where both representations $\rho, \text{hol} : \pi_1(\Sigma) \to \text{PSL}_2(\mathbb{R})$ are Fuchsian and propose to prove Theorem F. In that case, as we shall see, the foliated geodesic flow is never partially hyperbolic. In fact we will prove that in that case, the transverse Lyapunov exponent of the foliated geodesic flow of the suspended foliation for the SRB measure is, in absolute value, no less than the stable and unstable Lyapunov exponents. Moreover, in the case where the two representations are not conjugate by an element of $\text{PSL}_2(\mathbb{R})$ we prove that the modulus of this exponent is strictly greater. The two representations correspond to two hyperbolic metrics on $\Sigma$ that we shall denote respectively by $m_1$ and $m_2$. Associated to these metrics there are two uniformizing maps $\Pi_1 : \mathbb{D} \to \Sigma$ and $\Pi_2 : \mathbb{D} \to \Sigma$ and two Fuchsian groups $\Gamma_1 = \rho(\pi_1(\Sigma))$ and $\Gamma_2 = \text{hol}(\pi_1(\Sigma))$

**Boundary correspondence.** Lifting any diffeomorphism of $\Sigma$ via $\Pi_1$ and $\Pi_2$, we get a diffeomorphism $h : \mathbb{H} \to \mathbb{H}$ which is bilipschitz and conjugates the actions of $\rho$ and $\text{hol}$ i.e. for every $\gamma \in \pi_1(\Sigma)$

$$h \circ \rho(\gamma) = \text{hol}(\gamma) \circ h.$$ 

This bilipschitz diffeomorphism of $\mathbb{H}$ extends continuously as a bihölder homeomorphism still denoted by $h : \mathbb{RP}^1 \to \mathbb{RP}^1$ which conjugates the actions of $\rho$ and $\text{hol}$ (see Section 5.9 of Thurston’s notes [47] for all these facts).

We call $h : \mathbb{RP}^1 \to \mathbb{RP}^1$ the **boundary correspondence** associated to $\rho$ and $\text{hol}$. This correspondence does not depend on the diffeomorphism we lifted and is the unique homemorphism of $\mathbb{RP}^1$ with that equivariance property.

**Orbit equivalence of the geodesic flows.** The geodesic flow of $T^1\mathbb{H}$ shall be denoted by $\tilde{G}_t$. There is an identification between $T^1\mathbb{H}$ and the set of oriented triples of $\mathbb{RP}^1$, denoted by $S^{(3)}$, that associates to every vector $v$ the triple $(pr_+(v), pr_0(v), pr_-(v))$ where

- $pr_+(v) \in \mathbb{RP}^1$ is the extremity of the geodesic ray determined by $-v$;
- $pr_-(v) \in \mathbb{RP}^1$ is the extremity of the geodesic ray determined by $v$;
- $pr_0(v) \in \mathbb{RP}^1$ is the extremity of the geodesic orthogonal to $v$ which satisfies $pr_+(v) < pr_0(v) < pr_-(v)$ for the orientation.

This identification is an equivariance for the actions of $\pi_1(\Sigma)$ given on $T^1\mathbb{H}$ by differentials of hyperbolic isometries, and on $S^{(3)}$ by the diagonal action.

Moreover, the geodesic starting at $pr_+(v)$ and ending at $pr_-(v)$ is parametrized by the point $pr_0(v)$. More precisely, as $v$ evolves according to the geodesic flow, the point $pr_0(v)$ evolves along the differential equation given by the vector field $Y_v$ obtained by pulling back the vector field $x\partial_x$ of $\mathbb{RP}^1$ by the unique Möbius transformation sending respectively $pr_+(v), pr_0(v), pr_-(v)$ on $0, 1, \infty$.

By evaluating the orientation preserving equivariance $h : \mathbb{RP}^1 \to \mathbb{RP}^1$ on triples of points, we get an equivariant and bihölder homeomorphism $H : S^{(3)} \to S^{(3)}$ which descends to the quotient and provides an orbit equivalence between the geodesic flows on the unit tangent bundles corresponding respectively to the metrics $m_1$ and $m_2$. We will conveniently identify $H$ with a homeomorphism of $T^1\mathbb{H}$.
Average reparametrization of the geodesic flow. Define the function \(a : \mathbb{R} \times T^1 \mathcal{H} \rightarrow \mathbb{R}\) defined for \((t, v) \in \mathbb{R} \times T^1 \mathcal{H}\):

\[
H \circ \tilde{G}_t(v) = G_{a(t, v)} \circ H(v).
\]

It can be proven that \(a\) descends to a Liouville-integrable (for \(m_1\)) additive cocycle of \(T^1 \Sigma\). Birkhoff’s additive ergodic theorem then ensures the existence of the following number, that will be referred to as the average reparametrization of the geodesic flow, for Liouville almost every \(v \in T^1 \Sigma\)

\[
\chi = \lim_{t \to -\infty} \frac{a(t, v)}{t} > 0. \tag{5.21}
\]

This number has already been considered by Thurston and Wolpert with a slightly different definition in terms of intersection number of the Liouville geodesic currents corresponding to the two metrics. In [48], Wolpert called this number the average \(m_2\)-length of a \(m_1\)-geodesic and gave a proof of the following theorem.

Theorem 5.1 (Thurston). Let \(m_1, m_2\) be two hyperbolic metrics on the same closed surface \(\Sigma\). Then the average reparametrization of the geodesic flow is \(\geq 1\) with equality if and only if the two hyperbolic metrics represent the same Teichmüller class.

5.2 Non-partially hyperbolic foliated geodesic flows

Canonical foliated geodesic flow. Assume for the moment that \(\rho = \text{hol}\). By suspension of hol we obtain the so-called canonical foliation \((\mathcal{M}^{\text{can}}, \mathcal{F}^{\text{can}})\) that we endowed with an admissible Riemannian metric. We can look at the foliated geodesic flow denoted by \(G^{\text{can}}_t\) acting on \(\mathcal{M}^{\text{can}}\). It can be lifted as a flow of \(T^1 \mathcal{H} \times \mathbb{RP}^1\) still denoted by \(\tilde{G}_t\). It is possible to consider three sections \(\sigma^{*, \text{can}} : T^1 \mathcal{H} \rightarrow T^1 \mathcal{H} \times \mathbb{RP}^1, \star = +, 0, –\) defined by

\[
\sigma^{*, \text{can}} = (1d, pr_\star).
\]

One can prove that these sections descend to the quotient and provide three sections \(\sigma^{*, \text{can}} : T^1 \Sigma \rightarrow \mathcal{M}^{\text{can}}, \star = +, 0, –\). The sections \(\sigma^{+, \text{can}}\) and \(\sigma^{-, \text{can}}\) commute with the geodesic flows and are respectively the sections of largest expansion and contraction defined in [13].

Trivialization. As in [13] (see also Section VIII.1.3 of the first author’s thesis [2] for the precise construction in this particular context), we can find an equivariant and fiber preserving analytic map \(\Phi : T^1 \mathcal{H} \times \mathbb{RP}^1 \rightarrow T^1 \mathcal{H} \times \mathbb{RP}^1\) which:

- sends respectively the sections \(\widetilde{\sigma}^{+, \text{can}}, \widetilde{\sigma}^{0, \text{can}}, \widetilde{\sigma}^{-, \text{can}}\) on the sections corresponding to the constant functions respectively equal to 0, 1 and \(\infty\).
- sends the vector field \(Y_\nu\) defined above on the vector field \(x \partial_x\).

For this consider \(\Phi(v, x) = (v, P_\nu(x))\) for \((v, x) \in T^1 \mathcal{H} \times \mathbb{RP}^1\), where \(P_\nu\) denotes the unique Möbius transform sending the triple \((pr_+(v), pr_0(v), pr_-(v))\) on \((0, 1, \infty)\)

Horizontal and vertical components. Define on \(T^1 \mathcal{H} \times \mathbb{RP}^1\) the vector field \(\tilde{X}\) which generates the lift to \(T^1 \mathcal{H} \times \mathbb{RP}^1\) of the canonical geodesic flow, which is still denoted by \(\tilde{G}_t\).

The vertical component of \(\tilde{X}\) is by definition the vector field \(\tilde{Y}\) on \(T^1 \mathcal{H} \times \mathbb{RP}^1\) which is tangent to the fibers \(|v| \times \mathbb{RP}^1\) and induces on any such fiber the vector field \(Y_\nu = P_\nu^*(x \partial_x)\) where \(P_\nu\) has been defined above. The following lemma is essentially due to Bonatti, Gómez-Mont and Vila (see [13, §8.2]), we give below a glimpse of its proof in our context.
Lemma 5.2. 1. The vector field \( \tilde{Y} \), as well as the sum \( \tilde{Z} = \tilde{X} + \tilde{Y} \), are invariant by the diagonal action of \( \pi_1(\Sigma) \) on \( T^1\mathbb{H} \times \mathbb{R}^1 \) and commute with \( \tilde{X} \).

2. The vector field \( \tilde{\Phi}_* \tilde{Y} \) is tangent to the fibers \( \{v\} \times \mathbb{R}^1 \) and induces the vector field \( x\partial_x \) in these fibers;

3. The flow of \( \tilde{\Phi}_* \tilde{Z} \) preserves the fibers and and sends fiber to fiber as the identity.

Proof. In order to prove the first item it is enough to see that \( \tilde{Y} \) commutes with the foliated geodesic flow, which is a direct consequence of the definition of the three sections: we only have to check that \( Y_v = Y_{\tilde{G}_t(v)} \) for every \( v \in T^1\mathbb{H} \) (this is left to the reader).

The second item is a trivial consequence of the definition of \( \tilde{Y} \).

In order to prove the third item it is enough to prove that the flow \( \tilde{Z}_t \) of \( \tilde{Z} \) is fiber preserving and commutes with the three sections \( \tilde{\sigma}^{\pm,can} \), \( \star = +,0,- \). Indeed under these conditions the flow of \( \tilde{\Phi}_* \tilde{Z} \) preserves the fibers (since \( \tilde{\Phi} \) is fiber preserving) and sends fiber to fiber as a Möbius transformation fixing 0, 1 and \( \infty \). As a consequence it has to be transversally the identity.

Since \( \tilde{X} \) and \( \tilde{Y} \) commute, and since their flows both send fibers on fibers we get that \( \tilde{Z}_t \) does as well.

Since \( \tilde{\sigma}^{\pm,can} \) are zeros of \( Y_v \) for every \( v \) and since \( \tilde{G}_t \) commutes with these sections, one easily gets that \( \tilde{Z}_t \) commutes as well with these sections.

By definition \( \tilde{Y}_t(\tilde{\sigma}^{0,can}(v)) = (v, pr_0(\tilde{G}_t(v))) \) and \( \tilde{G}_t(\tilde{\sigma}^{0,can}(v)) = (\tilde{G}_t(v), pr_0(v)) \). Since \( \tilde{Y} \) and \( \tilde{X} \) commute we have

\[
\tilde{Z}_t(\tilde{\sigma}^{0,can}(v)) = \tilde{Y}_t \circ \tilde{G}_t(\tilde{\sigma}^{0,can}(v)) = (\tilde{G}_t(v), pr_0(\tilde{G}_t(v))) = \tilde{\sigma}^{0,can}(\tilde{G}_t(v)),
\]

and the lemma follows.

All the objects above are equivariant for the diagonal action of \( \pi_1(\Sigma) \): they all descend to the quotient (the notation of these objects is doing just by omission of the “tilde”-character). The foliated geodesic flow of \( \tilde{M}^{can} \) then satisfies for every time \( t \in \mathbb{R} \) \( G_t^{can} = Y_{-t} \circ Z_t \).

The SRB measure. The unique SRB measure for \( G_t^{can} \) is precisely \( \mu^{+,can} = \sigma^{+,can} \), Liou (see [13, Theorems 1.1 and 1.6]). With the description made above, we can prove the following

Proposition 5.5. The transverse Lyapunov exponent of the canonical foliated geodesic flow for its unique SRB measure is equal to \(-1\).

Proof. First the value of the transverse Lyapunov of the SRB measure is independent of the choice of a transverse metric. We are going to use the pullback by \( \Phi \) of the usual metric of the fibers (identified with \( \mathbb{R}^1 \)) in order to compute it.

It follows from the discussion above that we can write \( G_t^{can} = Y_{-t} \circ Z_t \) where, after the smooth fiber-preserving change of coordinates \( \Phi \), \( Y_t \) coincide in each fiber with the flow \( (x,t) \mapsto e^t x \) and \( Z_t \) induces the identity map between fibers. The metric of the fibers is by definition sent onto the usual metric of \( \mathbb{R}^1 \).

A typical point for the SRB measure is given by \( \sigma^{+,can}(v) \). In order to compute the transverse Lyapunov at this point, it is enough to compute the Lyapunov exponent at 0 of \( (x,t) \mapsto e^{-t} x \), which is clearly \(-1\).
General Fuchsian foliation. We now turn to the case where the two Fuchsian representations \( \rho, \text{hol} \) are a priori disjoint and perform the suspension as indicated before.

Let \( h \) be the boundary correspondence between \( \Gamma_1 \) and \( \Gamma_2 \); it provides a biholder orbit equivalence of the geodesic flows corresponding to the two Fuchsian representations \( \rho_1, \text{hol} \), which gives an equivariant biholder orbit equivalence \( (H, Id) : T^1\mathbb{H} \times \mathbb{RP}^1 \to T^1\mathbb{H} \times \mathbb{RP}^1 \) between foliated geodesic flow corresponding to hol and the canonical one.

This in turn provides a biholder orbit equivalence \( \hat{H} : \hat{M} \to \hat{M} \) such that for every \( v \in \hat{M} \) and \( t \in \mathbb{R} \):

\[
G_t(v) = \hat{H}^{-1} \circ G_{\text{can}}(t, v) \circ \hat{H}(v).
\]

Using now that \( \hat{H} \), although being only Hölder continuous in the horizontal direction, is smooth in the fiber direction, and the easy fact that the SRB measure of \( G_t \) is precisely given by \( \mu^+ = \hat{H}^* \mu^+ \text{can} \), we conclude the proof of Theorem F:

The transverse Lyapunov exponent of \( G_t \) for its unique SRB measure is equal to \(-\chi\).

6 Appendix. Negatively curved metrics in the leaves of a foliation by surfaces

In this paragraph \( M \) is a closed manifold endowed with a smooth foliation by surfaces \( \mathcal{F} \) and with a smooth Riemannian metric \( g \). Our goal is to prove Ghys’ theorem C: we want to find in the conformal class of \( g \) a metric whose restriction to each leaf has negative Gaussian curvature.

6.1 Harmonic measures and Gauss-Bonnet theorem

Harmonic measures. There is well defined foliated Laplace-Beltrami operator defined on the space \( C^{0,2}(M) \) of continuous functions \( \phi : M \to \mathbb{R} \) which are of class \( C^2 \) inside the leaves that we denote by \( \Delta^{\mathcal{F}} \). By definition for every \( x \in M \) and \( \phi \in C^{0,2}(M) \) we have \( \Delta^{\mathcal{F}} \phi(x) = \Delta_{L_x} \phi(x) \) where \( L_x \) is the leaf of \( x \) and \( \Delta_{L_x} \) denotes the Laplace operator for the restricted metric \( g_{L_x} \).

Definition 6.1 (Harmonic measures). A harmonic measure for \( \mathcal{F} \) is a probability measure \( m \) on \( M \) such that for every \( \phi \in C^{0,2}(M) \)

\[
\int_M \Delta^{\mathcal{F}} \phi \, dm = 0. \tag{6.22}
\]

Garnett proved in [24] the existence of harmonic measures.

Remark 6.2. Note that since \( M \) is compact, by using a partition of unity and convolution, we can find for every \( \phi \in C^{0,2}(M) \) a sequence \( (\phi_n)_{n \in \mathbb{N}} \) of smooth functions on \( M \) converging uniformly to \( \phi \) and whose derivatives of first and second orders inside the leaves converge uniformly to those of \( \phi \). This means in particular that \( \Delta^{\mathcal{F}} \phi_n \to \Delta^{\mathcal{F}} \phi \) uniformly. Hence to prove that \( m \) is harmonic it is enough that it vanishes on the Laplacians of smooth functions, i.e. that (6.22) holds for every \( \phi \in C^\infty(M) \).

Foliations by hyperbolic surfaces. Using isothermal coordinates the metric \( g \) gives \( \mathcal{F} \) a structure of Riemann surface foliation (see [17, Theorem 3.2.] for more details).

Say \( \mathcal{F} \) is a foliation by hyperbolic surfaces if the universal cover of every leaf is conformally equivalent to the unit disc \( \mathbb{D} \). As noted in [17] the property of being a foliation by hyperbolic surfaces is topological and is independent of the choice of a metric \( g \).

It comes from [26, Lemma 2.1] that
Proposition 6.3. Let \((M, \mathcal{F})\) be a closed manifold foliated by surfaces and \(g\) be a Riemannian metric on \(M\). If \(M\) does not possess a transverse invariant measure, then all of its leaves are hyperbolic.

Gauss-Bonnet theorem. Hereafter we let \(\kappa(x)\) denote the Gaussian curvature at \(x\) of the leaf \(L_x\). This is a continuous function of \(x \in M\).

Even if a priori we don’t have \(\kappa < 0\) everywhere, Ghys proved in [25] the following foliated analogue of Gauss-Bonnet theorem.

Theorem 6.4 (Ghys). Let \((M, \mathcal{F})\) be a foliated manifold foliated by hyperbolic surfaces and \(g\) be a Riemannian metric on \(M\). Then for every harmonic measure \(m\) we have

\[
\int_M \kappa \, dm < 0.
\]

6.2 Proof of Theorem C

Conformal change of metric. Let \((M, \mathcal{F})\) be a closed manifold foliated by surfaces and \(g\) be a Riemannian metric. Let \(\phi : M \to \mathbb{R}\) be a smooth function and \(g' = e^{2\phi} g\). The Gaussian curvature of \(g'\) at \(x\) of \(L_x\), denoted by \(\kappa'(x)\), is related to \(\kappa(x)\) by the following formula (see [25])

\[
\kappa'(x) = e^{-2\phi(x)} \left( \kappa(x) - \Delta^\mathcal{F} \phi(x) \right).
\]

Recall that we want to prove Theorem C which states the existence of a metric \(g'\) conformally equivalent to \(g\) such that \(\kappa' < 0\) everywhere.

The key lemma. Theorem C is then a consequence of the following key lemma.

Lemma 6.5 (Ghys). Assume that all leaves of \(\mathcal{F}\) are hyperbolic. Then there exists a smooth function \(\phi : M \to \mathbb{R}\) such that for every \(x \in M\)

\[
\kappa(x) - \Delta^\mathcal{F} \phi(x) < 0.
\]

Proof. This argument is due to Ghys [25, 26] and uses an idea based on Hahn-Banach’s theorem that goes back to Sullivan’s work on foliated cycles [46]. A similar argument is used in Deroin-Kleptsyn’s work: see [18, Lemma 3.5].

Let \(\mathcal{C} = C^0(M)\) denote the Banach space of continuous functions of \(M\) endowed with the supremum norm \(||\cdot||_\infty\). Let \(\mathcal{H}\) be the closure in \(\mathcal{C}\) of the space \(\{\Delta^\mathcal{F} \phi; \phi \in C^\infty(M)\}\). This is a closed subspace of a Banach space so the quotient \(\mathcal{C} / \mathcal{H}\) is naturally a Banach space and \(\Pi : \mathcal{C} \to \mathcal{C} / \mathcal{H}\) is continuous and open.

Claim. The space \(\mathcal{H}\) of harmonic measures is identified with the space of positive and continuous linear forms of \(\mathcal{C} / \mathcal{H}\).

Proof of the claim. Define the orthogonal complement of \(\mathcal{H}\) as the closed space of continuous linear forms \(m\) defined in \(\mathcal{C}\) such that \(m(h) = 0\) for every \(h \in \mathcal{H}\). This space identifies isometrically with the topological dual of \(\mathcal{C} / \mathcal{H}\).

A harmonic measure is a Radon measure vanishing on every element \(\Delta^\mathcal{F} \phi, \phi \in C^\infty(M)\) so it must vanish on every element of \(\mathcal{H}\), which is by definition a uniform limit of such functions. Now by Riesz representation theorem, positive elements of the orthogonal complement of \(\mathcal{H}\) are Radon measures vanishing in particular on every laplacian \(\Delta^\mathcal{F} \phi, \phi \in C^\infty(M)\): they are harmonic measures by Remark 6.2.

Consider now the open cone \(\Lambda^- \subset \mathcal{C}\) of negative continuous functions and its projection \(\hat{\Lambda}^- = \Pi(\Lambda^-) \subset \mathcal{C} / \mathcal{H}\). Let \(\hat{\kappa} = \Pi(\kappa) \in \mathcal{C} / \mathcal{H}\).
Claim. We have $\hat{\kappa} \in \hat{\Lambda}^-$. 

Proof of the claim. Suppose the contrary. By continuity and openness of $\Pi$, $\hat{\Lambda}^-$ is a nonempty open convex subset of the normed vector space $\mathcal{C}/\mathcal{H}$ and $\hat{\kappa} \notin \hat{\Lambda}^-$. Hahn-Banach’s theorem (see [16, Lemme I.3]) states that there exists $m \in (\mathcal{C}/\mathcal{H})'$ and $a \in \mathbb{R}$ such that for every $u \in \hat{\Lambda}^-$, $m(u) < a = m(\hat{\kappa})$.

Let us evaluate $m$ on elements of the form $\lambda u$, $u \in \hat{\Lambda}^-$, $\lambda > 0$. Letting $\lambda$ tend to infinity we see that $m \leq 0$ on $\hat{\Lambda}^-$. Letting $\lambda$ tend to zero we see that $a \geq 0$.

Hence the linear form $m$, correspond to an element of the orthogonal complement of $\mathcal{H}$ which is nonpositive on nonpositive functions.

Using the first claim we see that we found a harmonic measure $m$ such that $\int_\mathcal{U} \kappa dm = a \geq 0$: this contradicts Ghys’ Gauss-Bonnet Theorem 6.4.

Finally to conclude the proof of Lemma 6.5 note that the previous claim implies that there exists $h \in \mathcal{H}$ such that $\kappa - h \in \Lambda^-$ i.e. $\kappa - h < 0$ on $M$. Now by definition of $\mathcal{H}$ there must exist a smooth function $\phi$ such that $\kappa - \Delta^\phi \phi < 0$ on $M$. □

7 Final questions

As a conclusion we would like to address some questions related to our work, which we find natural and interesting.

7.1 Prescribing the curvature

Let $\mathcal{F}$ be a smooth foliation by hyperbolic surfaces of a closed manifold $M$. Ghys’ theorem implies the existence of a smooth Riemannian metric on $M$ whose restriction to any leaf is negatively curved. But we don’t know nothing about the curvature. The following problem seems interesting.

Question 1. Let $(M, \mathcal{F})$ be a smooth foliation of a smooth compact manifold by hyperbolic surfaces and $f : M \to (-\infty, 0)$ be a smooth negative function. Does there exist a Riemannian metric on $M$ such that $f$ coincides with the Gaussian curvature of every leaf?

The following laminated version seems simpler and is very much in the spirit of Candel’s simultaneous uniformization theorem. It concern the prescription of the curvature for a leafwise metric, i.e. an assignment $L \to g_L$ where $g_L$ is a Riemannian metric on the leaf $L$ which varies transversally continuously in local charts in the smooth topology (but which does not come a priori from a smooth ambient Riemannian metric). It might be a combination of Candel’s work with Berger’s PDE argument to uniformize surfaces [10].

Question 2. Let $(M, \mathcal{F})$ be a lamination of a compact metric space by hyperbolic surfaces. Let $f : M \to (-\infty, 0)$ be a continuous function which is smooth in restriction to the leaves and varies transversally continuously in local charts in the smooth topology. Does there exists a leafwise metric (as defined above) such that $f$ coincide with the Gaussian curvature of every leaf?

7.2 About partial hyperbolicity

Perturb our examples. One natural question in the spirit of the study of partially hyperbolic diffeomorphisms with mostly contracting center (see [21]) is to study how SRB measures (and hence minimal sets of foliations) can collapse when we perturb foliations (in the Epstein topology for example).
We provided new examples of partially hyperbolic dynamis tangent to the leaves of a foliation \( \mathcal{F} \). It would be interesting to perturb them and study the dynamical and ergodic properties of these perturbations.

In particular a perturbation of the tangent bundle to \( \mathcal{F} \) should give rise to new partially hyperbolic flows tangent to some non-integrable plane field: it could be interesting to study the dynamical properties of these examples.

**Non projective circle bundles.** Our examples of partially hyperbolic foliated geodesic flows are related to *projective* representations \( \text{hol}: \pi_1(\Sigma) \to \text{PSL}_2(\mathbb{R}) \) and rely heavily on a geometrical result by Deroin-Tholozan and Guéritaud-Kassel-Wolff (see Theorem 4.3). It would be nice to be able to treat the more general case of representations \( \text{hol}: \pi_1(\Sigma) \to \text{Diff}_+(S^1) \).

**Question 3.** *Is there a natural notion of domination of representations of surface groups in the groups of orientation preserving diffeomorphisms of the circle? If so what are the representations that can be dominated by Fuchsian ones?*

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