CYCILITY AND INDECOMPOSABILITY IN THE BRAUER GROUP OF A $p$-ADIC CURVE

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Abstract. For a $p$-adic curve $X$, we study conditions under which all classes in the $n$-torsion of $\text{Br}(X)$ are $\mathbb{Z}/n$-cyclic. We show that in general not all classes are $\mathbb{Z}/n$-cyclic classes. On the other hand, if $X$ has good reduction and $n$ is prime to $p$, or if $X$ is an elliptic curve over $\mathbb{Q}_p$ with split multiplicative reduction and $n$ is a power of $p$, then we prove that all order $n$ elements of $\text{Br}(X)$ are $\mathbb{Z}/n$-cyclic. Finally, if $X$ has good reduction and its function field $K(X)$ contains all $p^2$-th roots of 1, we show the existence of indecomposable division algebras over $K(X)$ with period $p^2$ and index $p^3$.

1. Introduction

Let $p$ be a prime. By a $p$-adic curve $X$ we mean a smooth geometrically connected projective curve over some finite extension $K$ of $\mathbb{Q}_p$. Let $K(X)$ be the function field of $X$. Division algebras over $K(X)$ have been extensively studied by many different authors (see for instance [Sal97], [Sal98], [HHK09], [BMT11], [PS14], [BMT16], among others). In this paper, we focus on two different aspects about $K(X)$-division algebras: cyclicity and indecomposability. We will be particularly interested in classes of Azumaya algebras over $X$, classified by the Brauer group of $X$. As much as possible, we will try to tackle the “thorny” part of $\text{Br}(K(X))$, namely its $p$-primary component, for which fewer techniques are available, and much less is known about.

Let $F$ be any field, let $n \in \mathbb{N}$ be prime to $\text{char} F$, and write $\text{Br}(F)[n]$ for the $n$-torsion of the Brauer group $\text{Br}(F)$ of $F$. A central simple $F$-algebra is called $\mathbb{Z}/n$-cyclic if it contains a maximal $\mathbb{Z}/n$-cyclic Galois étale subalgebra. Then its class in $\text{Br}(F)[n]$ can be written as the cup product of an element in $H^1(F, \mu_n) = F^\times/n$ and a character in $H^1(F, \mathbb{Z}/n)$; we call such classes $\mathbb{Z}/n$-cyclic as well. Cyclic algebras are the simplest amongst all central simple algebras, and it is known that for a local or global field $F$ all classes in $\text{Br}(F)[n]$ are $\mathbb{Z}/n$-cyclic.

For the function field $K(X)$ of a $p$-adic curve $X$, it is known that not all elements in $\text{Br}(K(X))[n]$ are $\mathbb{Z}/n$-cyclic (see the appendix of [Sal97], by W. Jacob and J.-P. Tignol, for a counter-example when $n = 2$ and $p \neq 2$). In [BMT16] it was shown that, for $n$ prime to $p$, any class in $\text{Br}(K(X))[n]$ is the sum of at most two $\mathbb{Z}/n$-cyclic classes. That result led the author to consider the question of whether all classes in the subgroup $\text{Br}(X)[n]$ of $\text{Br}(K(X))[n]$ are $\mathbb{Z}/n$-cyclic or not. It is not difficult to show that if $X$ has good reduction (see subsection 3.3 for the definition) and $n$ is prime to $p$ then all classes in $\text{Br}(X)[n]$ are $\mathbb{Z}/n$-cyclic; this is done in theorem 13. However, the answer to the above question is negative in general: in

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section [5] for almost all \( p \equiv 1 \pmod{3} \) we show the existence of a \( p \)-adic curve \( X \) of genus 10 having bad reduction and a class in \( \text{Br}(X)[3] \) that is not \( \mathbb{Z}/3 \)-cyclic. Since we believe this construction to be of interest on its own, we made an extra effort to identify and isolate the main points that make it work, proving results in a bit more generality than actually needed.

In the last two sections, we obtain results that also work for the \( p \)-primary part of \( \text{Br}(K(X)) \). In section [6], we study cyclicity for the Brauer group of an elliptic curve \( E \) with split multiplicative reduction. Using Tate’s \( p \)-adic uniformization and Lichtenbaum’s duality, we show in theorem [25] that all classes in \( \text{Br}(E)[n] \) are \( \mathbb{Z}/n \)-cyclic if \( n \) is either prime or satisfies \( \gcd(n, |\mu(K)|) = 1 \), where \( \mu(K) \) is the group of roots of unity in \( K^\times \). As an interesting corollary, we get that if \( E \) is defined over \( \mathbb{Q}_p \) then all classes in \( \text{Br}(E)[p^r] \) are \( \mathbb{Z}/p^r \)-cyclic \( (r \geq 1) \).

For the last result, recall that an \( F \)-division algebra is indecomposable if it cannot be expressed as the tensor product of two nontrivial \( F \)-division algebras. All division algebras of equal period and index are indecomposable, while division algebras of composite period are decomposable, so the problem of producing an indecomposable division algebra is only interesting when the period and index are unequal prime powers. Albert constructed decomposable division algebras of unequal \( (2 \text{-power}) \) period and index in the 1930’s, but indecomposable division algebras of unequal period and index appeared for the first time only in the late 1970’s, in the papers [Sal79] and [ART79]. Since then there have been several constructions, including [Tig87], [JW90], [Jac91], [SVG92], [Kar98], [Bru96], and [McK08]. For a \( p \)-adic curve \( X \) with good reduction, indecomposable algebras over \( K(X) \) were constructed in [BMT11] for several combinations of (prime power) period/indices that are not multiples of \( p \). In the last section, assuming \( K \) contains all \( p^2 \)-th roots of unity, we construct indecomposable algebras over \( K(X) \) of period \( p^2 \) and index \( p^3 \) for any \( p \)-adic curve \( X \) with good reduction (theorem [36]). Here, in contrast with the tame case treated in [BMT11], we do not have general lifting theorems at our disposal, which unfortunately constrains our methods to the above period \( p^2 \) and index \( p^3 \). However, given the paucity of results addressing the \( p \)-primary part of \( \text{Br}(K(X)) \), we still believe this particular construction to be of interest, as the methods here employed may help future investigations of related questions.

2. Notation and setup

Throughout this paper, \( p \) will be a fixed prime, and

- \( K \) will be a \( p \)-adic field, i.e., a finite extension of the field of \( p \)-adic numbers \( \mathbb{Q}_p \);
- \( R \) will be the ring of integers of \( K \), \( \pi \in R \) will be a uniformizer, and \( k = R/(\pi) \) will be its residue field (a finite field of characteristic \( p > 0 \));
- \( \overline{K} \) will be an algebraic closure of \( K \), and \( G_K \stackrel{\text{def}}{=} \text{Gal}(\overline{K}/K) \) will be its absolute Galois group;
- \( s = \text{Spec} \ k \) and \( \eta = \text{Spec} \ K \) will be the closed and generic points of \( \text{Spec} \ R \);
- \( X \) will be a smooth geometrically connected projective curve over \( \eta = \text{Spec} \ K \). We write \( K(X) \) for its function field, and \( X_0 \) for the set of its closed points.

For each closed point \( x \in X_0 \), \( \mathcal{O}_{X,x} \) is a discrete valuation ring, and we denote by

- \( v_x : K(X) \to \mathbb{Z} \cup \{\infty\} \) the associated discrete valuation;
• \( \kappa(x) \) its residue field (a \( p \)-adic field);
• \( \hat{K}(X)_x \) the completion of \( K(X) \) with respect to \( \nu_x \).

We will still write \( \nu_x : \hat{K}(X)_x \to \mathbb{Z} \cup \{ \infty \} \) for the discrete valuation induced by the original valuation \( \nu_x \) on \( K(X) \).

If \( Z \) is a scheme over \( \text{Spec} \ A \) and \( B \) is an \( A \)-algebra, we write \( Z(B) \) for the set of \((\text{Spec} \ B)\)-valued points of \( Z \), and \( Z \otimes_A B \) for the fibered product \( Z \times_{\text{Spec} \ A} \text{Spec} \ B \).

If \( Z \) is defined over \( \text{Spec} \ R \), we denote by
\[
Z_s \overset{\text{def}}{=} Z \times_{\text{Spec} \ R} \mathcal{O} = Z \otimes_R \mathcal{O} \quad \text{and} \quad Z_\eta \overset{\text{def}}{=} Z \times_{\text{Spec} \ R} \mathcal{O} = Z \otimes_R \mathcal{O}
\]
its special and generic fibers, respectively.

For any abelian group \( G \), we respectively write \( G[n] \) and \( G/n \) for the kernel and cokernel of the multiplication-by-\( n \) map \( G \to G \). Moreover, if \( G \) is a topological abelian group, we denote by \( G'' \overset{\text{def}}{=} \text{Hom}_\mathbb{C}(G, \mathbb{Q}/\mathbb{Z}) \) its Pontryagin dual, i.e., the group of all continuous morphisms \( \chi : G \to \mathbb{Q}/\mathbb{Z} \) (here \( \mathbb{Q}/\mathbb{Z} \) is endowed with the discrete topology).

Unless otherwise stated, all cohomology groups will either be Galois or étale cohomology groups. For any field \( F \) and \( n \in \mathbb{N} \) not divisible by \( \text{char} \ F \), we write \( \mu_n \) for the group (or Galois module or étale sheaf) of all \( n \)-th roots in the separable closure \( F^{\text{sep}} \) of \( F \). We also write \( \delta_n : F^\times \to H^1(F, \mu_n) \) for the connecting map relative to the Kummer sequence \( 1 \to \mu_n \to F^{\text{sep}} \to F^{\text{sep}} \to 1 \). If \( n \) is clear from the context, we drop it and simply write \( \delta \) instead.

3. Some auxiliary facts

In this section, we recall some facts that will be used in the proofs of the main results. We advise the reader to skip this section, coming back as needed.

3.1. Ramification. Let \( \hat{K} \) be a discretely valued complete field. Let \( v : \hat{K} \to \mathbb{Z} \cup \{ \infty \} \) be the associated (normalized) valuation, and \( F \) be its residue field. There is a split exact sequence ([Ser79], XII.§3, corollary to proposition 4, p.186)
\[
0 \to \text{Br}(F) \overset{\text{inf}}{\to} \text{Br}(\hat{K})' \overset{\delta_v}{\to} H^1(F, \mathbb{Q}/\mathbb{Z}) \to 0
\]
where \( \text{Br}(\hat{K})' = \ker(\text{Br}(\hat{K}) \to \text{Br}(\hat{K}_{\text{nr}})) \) (here \( \hat{K}_{\text{nr}} \) denotes the maximal unramified extension of \( \hat{K} \)). In case \( F \) is perfect \( \text{Br}(\hat{K})' = \text{Br}(\hat{K}) \), and if \( \text{char} F \nmid n \) then \( \text{Br}(\hat{K})'[n] = \text{Br}(\hat{K})[n] \). The map \( \delta_v \) is called ramification or residue map with respect to \( v \). Intuitively, \( \delta_v \) can be thought of as “valuation for cohomology”, so that the inflation map identifies \( \text{Br}(F) \) with the subgroup of “unramified classes” in \( \text{Br}(\hat{K})' \). By choosing a uniformizer \( \pi_v \in \hat{K} \), the map \( \chi \mapsto \delta_v(\pi_v) \cup \inf(\chi) \) yields a section of \( \delta_v \) for the \( n \)-torsion of the above sequence, and hence a (non-canonical) isomorphism
\[
\text{Br}(F)[n] \oplus H^1(F, \mathbb{Z}/n\mathbb{Z}) \overset{\cong}{\to} \text{Br}(\hat{K})'[n]
\]
\[
(\alpha, \chi) \mapsto \inf(\alpha) + \delta_v(\pi_v) \cup \inf(\chi)
\]
which will be referred to as Witt decomposition of \( \text{Br}(\hat{K})'[n] \). By abuse of notation, we will usually omit the inflation maps, writing \( \alpha + \delta_v(\pi_v) \cup \chi \) instead.

It is fairly easy to compute the index of elements in \( \text{Br}(\hat{K})'[n] \), thanks to ([JW90], theorem 5.15, p.161 for a proof)
Lemma 2 (Nakayama’s index formula). In the above notation, let \( n \in \mathbb{N} \) and \( \beta \in \text{Br}(\hat{K})[n] \) with Witt decomposition \( \beta = \alpha + \delta \pi_n \cup \chi \). Write:

- \(|\chi|\) for the order of the character \( \chi \in H^1(F, \mathbb{Z}/n\mathbb{Z}) \);
- \( F(\chi) \ni F \) for the degree \(|\chi|\) field extension defined by \( \chi \);
- \( \alpha|_{F(\chi)} \in \text{Br}(F(\chi)) \) for the restriction of \( \alpha \) to \( F(\chi) \).

Then the index of \( \beta \) is given by \( \text{ind} \beta = |\chi| \cdot \text{ind} \alpha|_{F(\chi)} \).

Next we define ramification/residue maps in more general contexts. For \( r \in \mathbb{Z} \) and \( n \in \mathbb{N} \) prime to \( \text{char} F \), denote the \( r \)-th Tate twist of the Galois module (or étale sheaf) \( \mathbb{Z}/n\mathbb{Z} \) by

\[
\mathbb{Z}/n\mathbb{Z}(r) = \begin{cases} 
\mu_n^{\otimes r} & \text{if } r \geq 1 \\
\mathbb{Z}/n\mathbb{Z} & \text{if } r = 0 \\
\text{Hom}(\mu_n^{\otimes (-r)}, \mathbb{Z}/n\mathbb{Z}) & \text{if } r \leq -1
\end{cases}
\]

For \( n \in \mathbb{N} \) prime to \( \text{char} F \) and \( i \geq 1 \), there are (noncanonical) split exact sequences (see [GMS03] II.7.9, p.18)

\[
0 \to H^i(F, \mathbb{Z}/n\mathbb{Z}(r)) \xrightarrow{\text{ind}} H^i(\hat{K}, \mathbb{Z}/n\mathbb{Z}(r)) \xrightarrow{\partial_r} H^{i-1}(F, \mathbb{Z}/n\mathbb{Z}(r-1)) \to 0
\]

The map \( \partial_r \) is the ramification or residue map with finite coefficients. When \( i = 2 \) and \( r = 1 \) it is compatible with the previously defined ramification map via the isomorphism \( \text{Br}(\hat{K})[n] = H^2(\hat{K}, \mu_n) \).

Let \( Z \) be an integral regular scheme with function field \( K(Z) \). Let \( D \subset Z \) be an irreducible Weil divisor with generic point \( \nu \in Z \). Then \( O_{Z,\nu} \) is a discrete valuation ring; write \( v_D \colon K(Z) \to \mathbb{Z} \cup \{\infty\} \) and \( \kappa(D) \) for the corresponding valuation and residue field, respectively, and let \( \widehat{K(Z)}_D \) be the completion of \( K(Z) \) with respect to \( v_D \). If \( n \in \mathbb{N} \) is prime to \( \text{char} \kappa(D) \), we define the ramification or residue map \( \partial_D \colon H^i(K(Z), \mathbb{Z}/n\mathbb{Z}(r)) \to H^{i-1}(\kappa(D), \mathbb{Z}/n\mathbb{Z}(r-1)) \) with respect to \( D \) as the composition

\[
H^i(K(Z), \mathbb{Z}/n\mathbb{Z}(r)) \xrightarrow{\text{res}} H^i(\widehat{K(Z)}_D, \mathbb{Z}/n\mathbb{Z}(r)) \xrightarrow{\partial_D} H^{i-1}(\kappa(D), \mathbb{Z}/n\mathbb{Z}(r-1))
\]

Lemma 3. Let \( Z \) be an integral regular scheme of dimension at most 2, and let \( K(Z) \) be its function field.

(i) For any \( n \in \mathbb{N} \) invertible in \( Z \), there is an exact sequence

\[
0 \to \text{Br}(Z)[n] \to \text{Br}(K(Z))[n] \oplus H^1(\kappa(D), \mathbb{Q}/\mathbb{Z})[n]
\]

where \( D \) runs over all prime divisors of \( Z \).

(ii) Suppose that \( Z = \text{Spec} A \) for some excellent 2-dimensional local ring \( A \). Let \( P \in Z \) be its closed point, and \( \kappa(P) \) be the residue field of \( P \). Suppose that \( n \in \mathbb{N} \) is prime to \( \text{char} \kappa(P) \). Then the following sequence is a complex:

\[
\text{Br}(K(Z))[n] \oplus H^1(\kappa(D), \mathbb{Z}/n) \xrightarrow{\partial_P} H^0(\kappa(P), \mu_n^{-1})
\]

where \( D \) runs over all prime divisors of \( Z \), and \( \partial_P = \sum_Q \text{cor}_{\kappa(Q)/\kappa(P)} \cdot \partial_Q \) where \( Q \) runs over all closed points in the normalization of \( D \) in \( \kappa(D) \), and \( \partial_Q \) is the ramification map with respect to the discrete valuation of \( \kappa(D) \) defined by \( Q \).
Proof. The injectivity of $\text{Br}(\mathbb{Z}) \to \text{Br}(K(\mathbb{Z}))$ in (i) is proven in [Mil80] IV.2.6, p.145, while the exactness in the middle term follows from the purity of the Brauer group (see [AG60] 7.4 or [Mil80] IV.2.18 (b), p.153, and also [Sal08], Lemma 6.6). Item (ii) is proven in [Kat86], §1, p.148. □

In particular, for a $p$-adic curve $X$ (as in section 2), the residue field $\kappa(x)$ of any closed point $x \in X_0$ is a $p$-adic field (of characteristic 0), thus we get an exact sequence

$$0 \to \text{Br}(X) \to \text{Br}(K(X)) \bigoplus_{x \in X_0} H^1(\kappa(x), \mathbb{Q}/\mathbb{Z})$$

which allows us to interpret $\text{Br}(X)$ as the subgroup of $\text{Br}(K(X))$ consisting of unramified classes with respect to all $x \in X_0$.

3.2. Lichtenbaum’s duality. Here we collect some facts about Lichtenbaum’s duality (see [Lic69] and also [Sai86], appendix).

**Theorem 5** (Lichtenbaum). Let $X$ be a $p$-adic curve (as in section 2). Then there is a non-degenerate pairing $\langle -, - \rangle : \text{Pic}(X) \times \text{Br}(X) \to \mathbb{Q}/\mathbb{Z}$ inducing an isomorphism between $\text{Br}(X)$ and $\text{Pic}(X)\vee$.

This pairing can explicitly be described as follows. For each $x \in X_0$ write

- $\text{inv}_{\kappa(x)} : \text{Br}(\kappa(x)) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$ for the invariant map with respect to the $p$-adic field $\kappa(x)$;
- $\beta(x) \in \text{Br}(\kappa(x))$ for the image of $\beta \in \text{Br}(X)$ under the natural map $\text{Br}(X) \to \text{Br}(\kappa(x))$ (induced by the inclusion $\text{Spec} \kappa(x) \hookrightarrow X$).

Let $\alpha \in \text{Pic}(X)$, $\beta \in \text{Br}(X)$, and let $\sum_{x \in X_0} n_x x \in \text{Div}(X)$ be a divisor representing $\alpha$. Lichtenbaum’s pairing is given by

$$\langle \alpha, \beta \rangle = \sum_{x \in X_0} n_x \text{inv}_{\kappa(x)} \beta(x)$$

The topology on $\text{Pic}(X)$ is defined as follows. Let $\text{Pic}^0(X)$ be the kernel of the degree map $\deg : \text{Pic}(X) \to \mathbb{Z}$, and $J_X$ be the Jacobian variety of $X$. There is an exact sequence ([CSS86], remark 1.6, p. 169)

$$0 \to \text{Pic}^0(X) \to J_X(K) \xrightarrow{\delta} \text{Br}(K)^{\text{inv}_K} \mathbb{Q}/\mathbb{Z}$$

where $\delta$ has finite image

$$\text{im} \delta = \frac{1}{P} \mathbb{Z}$$

with $P =$ period of $X \overset{\text{def}}{=} \gcd \{ \deg \alpha \mid \alpha \in H^0(K, \text{Pic}(X \otimes_K \mathbb{R})) \}$

(in particular, observe that $\text{Pic}^0(X) = J_X(K)$ whenever $X(K) \neq \emptyset$). We view $J_X(K)$ as a topological group with the usual $p$-adic topology, and $\text{Pic}^0(X)$ as a topological subgroup with the induced topology. From [Mat55], we have

**Lemma 6** (Mattuck). Let $A$ be an abelian variety of dimension $g$ over a $p$-adic field $K$. Then $A(K)$ contains an open subgroup of finite index isomorphic to $\mathbb{R}^g$.
Thus the group $J_X(K)$ is compact, and all its finite index subgroups are open (thus also closed and compact) since any such subgroup contains a finite index subgroup of $R^q$, which is open. In particular, $Pic^0(X)$ is an open compact subgroup of $J_X(K)$. We endow $Pic(X)$ with the unique topology compatible with its group structure, and for which $Pic^0(X)$ is open in $Pic(X)$.

With this topology, all subgroups of $Pic(X)$ of finite index and open. Therefore the $n$-torsion of $Br(X)$ is given by

$$Br(X)[n] \cong Pic(X)^n = \text{Hom}_c(Pic(X), \mathbb{Q}/\mathbb{Z})[n] = \text{Hom}(Pic(X)/n, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$$

Finally, we mention an important consequence of Lichtenbaum’s duality, which is implicit in the proof of theorem 5, p.132 of [Lic69]. See also [Sai86], §9 appendix, p.408 and [Pop88], theorem 4.5, p.172.

**Theorem 7** (Haufe principle). Let $X$ be a $p$-adic curve. Then the following diagonal map is injective:

$$Br(K(X)) \hookrightarrow \prod_{x \in X_0} Br(K(X)_x)$$

(here $K(X)_x$ is the $v_x$-completion of $K(X)$, see notation in section 3).

### 3.3. Integral models.

From now on, we choose once and for all an integral model $\mathcal{X}$ of $X$, i.e., a 2-dimensional regular $R$-scheme such that

- $f: \mathcal{X} \to \text{Spec } R$ is flat and proper;
- there is an isomorphism of $\eta$-schemes $X \cong \mathcal{X}_\eta$;
- the reduced scheme $(\mathcal{X}_s)_{\text{red}}$ is a 1-dimensional (proper) scheme over $s$ whose irreducible components are all regular and have normal crossings (i.e., $\mathcal{X}_s$ only has ordinary double points as singularities).

The existence of such an integral model follows from the resolution of singularities of excellent 2-dimensional schemes ([Lip78]), coupled with the embedded resolution of the special fiber (see [Liu02], corollary IX.2.30 p.404). If $X$ admits a smooth integral model $\mathcal{X}$ over $R$, then we say $X$ has **good reduction** over $R$. If that is the case, then the special fiber $\mathcal{X}_s$ will have a single irreducible component, a proper smooth curve over $s = \text{Spec } k$, and its function field $k(\mathcal{X}_s)$ will be a global field of characteristic $p$.

Observe that since $X$ is geometrically integral by assumption, $H^0(X, \mathcal{O}_X) = K$ by [Sta18], Lemma 0BUG. Then by Stein’s factorization [Sta18], Lemma 0AY8 the map $f: \mathcal{X} \to \text{Spec } R$ has geometrically connected fibers and satisfies $f_* \mathcal{O}_\mathcal{X} = \mathcal{O}_R$.

Recall that a **horizontal prime divisor** $D \subset \mathcal{X}$ is a prime divisor which is not contained in the special fiber $\mathcal{X}_s$. For the next lemma, see [Liu02], proposition VIII.3.4, p.349, and lemma VIII.3.35, p.360.

**Lemma 8.** Let $X$ be a $p$-adic curve (as in section 3), and $\mathcal{X} \to \text{Spec } R$ be an integral model. Then

1. **A horizontal divisor $D \subset \mathcal{X}$ is finite and flat over $\text{Spec } R$. In particular, it is affine:** $D = \text{Spec } S$ for some finite extension of domains $S \supset R$, and the generic point $D_\eta$ of $D$ is the spectrum of a $p$-adic field $\text{Frac } S$.
2. **There is a bijection between the set of horizontal divisors of $\mathcal{X}$ and $X_0$.** Namely, this bijection maps a horizontal divisor $D \subset \mathcal{X}$ to its generic point $x = D_\eta \in X_0$, and conversely the closure $D = \overline{\{x\}}$ of $x \in X_0$ inside $\mathcal{X}$ defines the corresponding horizontal divisor.
(iii) For any closed point \( P \in \mathcal{X}_s \), there is a horizontal divisor \( D \) on \( \mathcal{X} \) intersecting the special fiber \( \mathcal{X}_s \) transversally at \( P \).

The next lemma summarizes the facts we will need concerning the relation between the Brauer and Picard groups of the special and generic fibers of \( \mathcal{X} \).

Lemma 9. In the above notation, let \( C_1, \ldots, C_r \) be the irreducible components of the special fiber \( \mathcal{X}_s \).

(i) (Artin) \( \text{Br}(\mathcal{X}) = \text{Br}(\mathcal{X}_s) = 0 \).

(ii) Assume that \( H^0(\mathcal{X}_s, \mathbb{G}_m) = k^\times \), and let \( n \in \mathbb{N} \) not divisible by \( p \). Then there are canonical isomorphisms
\[
\text{Pic}(\mathcal{X})[n] = \text{Pic}(\mathcal{X}_s)[n]
\]
\[
\text{Pic}(\mathcal{X})/n = \text{Pic}(\mathcal{X}_s)/n = H^2(\mathcal{X}, \mu_n) = H^2(\mathcal{X}_s, \mu_n)
\]

(iii) There is an exact sequence
\[
0 \rightarrow \mathbb{Z} \xrightarrow{i} \mathbb{Z}^r \rightarrow \text{Pic}(\mathcal{X}) \rightarrow \text{Pic}(X) \rightarrow 0
\]

Here \( i(1) = (a_1, \ldots, a_r) \) is the multiplicity vector of the special fiber viewed as an element of \( \text{Div}(\mathcal{X}) : \mathcal{X}_s = \text{div}(\pi) = \sum_{1 \leq i \leq r} a_i C_i \). In particular, if \( X \) has good reduction (i.e., \( \mathcal{X} \) is smooth over \( R \)) then restriction \( \text{Pic}(\mathcal{X}) \cong \text{Pic}(X) \) is an isomorphism.

Proof. Item (i) follows from the canonical isomorphism \( \text{Br}(\mathcal{X}) = \text{Br}(\mathcal{X}_s) \) ([Gro68, théorème 3.1, p. 98]), and the fact that, by class field theory, \( \text{Br}(C_i) = 0 \) for all \( i \), hence \( \text{Br}(\mathcal{X}_s) = 0 \) as well by [Sal97, lemma 3.2, p. 40].

To show (ii), note that for \( i \geq 1 \) the Kummer sequence \( 1 \rightarrow \mu_n \rightarrow \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \rightarrow 1 \) gives a commutative diagram with exact rows (the vertical arrows are restrictions)
\[
\begin{array}{ccccccccc}
0 & \rightarrow & H^{i-1}(\mathcal{X}, \mathbb{G}_m)/n & \rightarrow & H^i(\mathcal{X}, \mu_n) & \rightarrow & H^i(\mathcal{X}_s, \mathbb{G}_m)[n] & \rightarrow & 0 \\
& & \downarrow & \cong & & & \downarrow & \cong & \\
0 & \rightarrow & H^{i-1}(\mathcal{X}_s, \mathbb{G}_m)/n & \rightarrow & H^i(\mathcal{X}_s, \mu_n) & \rightarrow & H^i(\mathcal{X}_s, \mathbb{G}_m)[n] & \rightarrow & 0
\end{array}
\]

The middle vertical arrow is an isomorphism by proper base change ([Mil80, corollary VI.2.7, p. 224]). For \( i = 1 \), since the subgroup \( U^{(1)} \) of \( R^\times / n \) is \( n \)-divisible by Hensel’s lemma, from the exact sequence \( 1 \rightarrow U^{(1)} \rightarrow R^\times \rightarrow k^\times \rightarrow 1 \) the leftmost vertical arrow is also an isomorphism
\[
H^0(\mathcal{X}, \mathbb{G}_m)/n = R^\times / n \xrightarrow{\sim} k^\times / n = H^0(\mathcal{X}_s, \mathbb{G}_m)/n
\]
(recall that \( f : \mathcal{X} \rightarrow \text{Spec } R \) satisfies \( f_* \mathcal{O}_\mathcal{X} = \mathcal{O}_R \) and \( H^0(\mathcal{X}_s, \mathbb{G}_m) = k^\times \) by hypothesis). Hence the rightmost vertical arrow is an isomorphism \( \text{Pic}(\mathcal{X})[n] \xrightarrow{\sim} \text{Pic}(\mathcal{X}_s)[n] \).

For \( i = 2 \), by (i) both groups \( \text{Br}(\mathcal{X}) = H^2(\mathcal{X}, \mathbb{G}_m) \) and \( \text{Br}(\mathcal{X}_s) = H^2(\mathcal{X}_s, \mathbb{G}_m) \) vanish, and we get the remaining isomorphisms to be shown.

To show (iii), observe that exactness at the two rightmost terms follows from [Fu98, proposition 1.8, p. 21] applied to the closed subscheme \( \mathcal{X}_s \) and its open complement \( X = \mathcal{X} \setminus \mathcal{X}_s \), while exactness at the two leftmost terms follows from the intersection theory for arithmetic surfaces (see [Liu02, §9.1.2, p. 381, in particular theorem IX.1.23 on p. 385]).

\[\square\]
3.4. Kummer-Artin-Schreier-Witt theory. Let $p$ be a prime number, $ζ_2 ∈ \mu_{p^2}$ be a primitive $p^2$-th root of unity, and $A = \mathbb{Z}⟨p⟩[ζ_2]$, a discrete valuation ring with uniformizer $λ_2 = ζ_2 - 1$ and residue field $\mathbb{F}_p$. In [SS01], Sekiguchi and Suwa constructed an exact sequence of group schemes over $A$

$$0 \rightarrow \mathbb{Z}/p^2 \rightarrow \mathcal{W}_2 \xrightarrow{Ψ} \mathcal{V}_2 \rightarrow 0$$

whose special fiber is isomorphic to the Artin-Schreier-Witt sequence in characteristic $p > 0$, while its generic fiber is isomorphic to a Kummer type sequence, thus giving an explicit interpolation between the Artin-Schreier-Witt and Kummer theories. In order to describe Sekiguchi and Suwa’s construction, set

$$ζ = ζ_2^p, \quad λ = ζ - 1, \quad λ_2 = ζ_2 - 1, \quad η = \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} λ_2^k, \quad \tilde{η} = \frac{λ^{p-1}}{p} (pη - λ)$$

and consider the polynomials in $A[T]$ ($p$-truncated exponential series)

$$F(T) = \sum_{k=0}^{p-1} \frac{(ηT)^k}{k!}, \quad G(T) = \sum_{k=0}^{p-1} \frac{(λT)^k}{k!}$$

The group schemes $\mathcal{W}_2$ and $\mathcal{V}_2$ are given by

$$\mathcal{W}_2 = \text{Spec } A\left[ T_0, T_1, \frac{1}{λT_0 + 1}, \frac{1}{λT_1 + F(T_0)} \right]$$

$$\mathcal{V}_2 = \text{Spec } A\left[ T_0, T_1, \frac{1}{λ^pT_0 + 1}, \frac{1}{λ^pT_1 + G(T_0)} \right]$$

The comultiplication in $\mathcal{W}_2$ is given by

$$(T_0, T_1) \mapsto (λ_0^F (T_0 \otimes 1, 1 \otimes T_0), λ_1^F (T_0 \otimes 1, T_1 \otimes 1, 1 \otimes T_0, 1 \otimes T_1))$$

and the one in $\mathcal{V}_2$ by

$$(T_0, T_1) \mapsto (λ_0^G (T_0 \otimes 1, 1 \otimes T_0), λ_1^G (T_0 \otimes 1, T_1 \otimes 1, 1 \otimes T_0, 1 \otimes T_1))$$

where $λ_0^F$ and $λ_1^G$ are the polynomials with coefficients in $A$ given by

$$λ_0^F (X_0, Y_0) = λX_0Y_0 + X_0 + Y_0, \quad λ_1^G (X_0, Y_0) = λ^pX_0Y_0 + X_0 + Y_0$$

$$λ_1^F (X_0, X_1, Y_0, Y_1) = λX_1Y_1 + X_1F(Y_0) + F(X_0)Y_0 + \frac{1}{λ}[F(X_0)F(Y_0) - F(λX_0Y_0 + X_0 + Y_0)]$$

$$λ_1^G (X_0, X_1, Y_0, Y_1) = λ^pX_1Y_1 + X_1G(Y_0) + G(X_0)Y_0 + \frac{1}{λ^p}[G(X_0)G(Y_0) - G(λ^pX_0Y_0 + X_0 + Y_0)]$$

Finally the isogeny $Ψ: \mathcal{W}_2 \rightarrow \mathcal{V}_2$ is given by $$(T_0, T_1) \mapsto (Ψ_0(T_0), Ψ_1(T_0, T_1))$$ where $Ψ_t ∈ A[\mathcal{T}_0, \mathcal{T}_1, 1/(λT_0 + 1), 1/(λT_1 + F(T_0))]$ are given by

$$Ψ_0(T_0) = \frac{1}{λ} \left[ \frac{(λT_0 + 1)^p - 1}{λ^p} \right]$$

$$Ψ_1(T_0, T_1) = \frac{1}{λ^p} \left[ \frac{(λT_1 + F(T_0))^p}{λT_0 + 1} - G\left( \frac{(λT_0 + 1)^p - 1}{λ^p} \right) \right]$$
As shown in [SS01], the special fiber of (10) is isomorphic to the Artin-Schreier-Witt sequence (here $W_2$ denotes the group of Witt vectors of length 2 and $F$, the Frobenius map)

$$0 \rightarrow \mathbb{Z}/p^2 \rightarrow W_2 \xrightarrow{F-\text{id}} W_2 \rightarrow 0$$

Moreover, there is a commutative diagram of group schemes over $A$

$$
\begin{array}{ccc}
\mathbb{G}_m^2 & \xrightarrow{\phi} & \mathbb{G}_m^2 \\
\alpha^{(F)} \downarrow & & \downarrow \alpha^{(G)} \\
\mathbb{G}_m^2_{m,A} & \xrightarrow{\Theta} & \mathbb{G}_m^2_{m,A}
\end{array}
$$

whose vertical arrows become isomorphisms over Spec(Frac $A$). Here, if $G_{m,A}^2 = \text{Spec} A[U_0, U_1, 1/U_0, 1/U_1]$, the maps $\alpha^{(F)}$ and $\alpha^{(G)}$ are respectively given by

$$(U_0, U_1) \mapsto (\lambda T_0 + 1, \lambda T_1 + F(T_0)) \quad \text{and} \quad (U_0, U_1) \mapsto (\lambda^p T_0 + 1, \lambda^p T_1 + G(T_0))$$

and $\Theta$, by $(U_0, U_1) \mapsto (U_0^p, U_0^{-1} U_1^p)$.

Let us give the local description of how to lift Artin-Schreier-Witt extensions to Kummer ones. Consider the polynomial

$$(12) \quad c(X, Y) = \frac{X^p + Y^p - (X + Y)^p}{p} \in \mathbb{Z}[X, Y]$$

Let $B$ be an $A$-algebra, and write $\overline{B} \overset{\text{def}}{=} B \otimes_A \mathbb{F}_p = B/(\lambda_2)$ for its reduction modulo $\lambda_2$. To give an Artin-Schreier-Witt extension of $\overline{B}$ is the same as to give a morphism $\overline{\mathfrak{f}}: \text{Spec} \overline{B} \rightarrow \mathbb{V}_2 \otimes_A \mathbb{F}_p = \text{Spec} \mathbb{F}_p[T_0, T_1]$, i.e., to give a pair $(\overline{b}_0, \overline{b}_1) \in \overline{B}^2$ so that $\overline{\mathfrak{f}}$ is defined by $(T_0, T_1) \mapsto (\overline{b}_0, \overline{b}_1)$; then the corresponding Artin-Schreier-Witt extension is given by taking the pullback Spec$(\overline{B}) \times_{\mathbb{V}_2} \mathbb{W}_2 \rightarrow \text{Spec} \overline{B}$ of $\Psi$, i.e., it is the extension $\overline{B}[x_0, x_1] \supset \overline{B}$ defined by equations

$$x_0^p - x_0 = \overline{b}_0, \quad x_1^p - x_1 - c(x_0^p, -x_0) = \overline{b}_1$$

Any lift $f: \text{Spec} B \rightarrow \mathbb{V}_2$ of $\overline{\mathfrak{f}}$ now defines a Kummer extension lifting the above Artin-Schreier-Witt one. Namely, let $(b_0, b_1) \in B^2$ be any lift $(\overline{b}_0, \overline{b}_1)$ with $\lambda^p b_0 + 1$ and $\lambda^p b_1 + G(b_0)$ units in $B$. Then $(T_0, T_1) \mapsto (b_0, b_1)$ defines a morphism $f: \text{Spec} B \rightarrow \mathbb{V}_2$, and taking the pullback Spec$(B) \times_{\mathbb{V}_2} \mathbb{W}_2 \rightarrow \text{Spec} B$ of $\Psi$ with respect to $f$ we get a Kummer extension $B[t_0, t_1] \supset B$ via equations

$$\frac{(\lambda t_0 + 1)^p - 1}{\lambda^p} = b_0, \quad \frac{1}{\lambda^p} \left[ \frac{(\lambda t_1 + F(t_0))^p}{\lambda t_0 + 1} - G(b_0) \right] = b_1$$

or, written in a more Kummerish style,

$$(\lambda t_0 + 1)^p = 1 + b_0 \lambda^p, \quad (\lambda t_1 + F(t_0))^p = (\lambda t_0 + 1)(b_1 \lambda^p + G(b_0))$$

so that $B[t_0, t_1] \supset B$ is built by successively taking $p$-th roots of certain elements.

4. Cyclicity for curves with good reduction

In this section, our goal is to show

**Theorem 13.** Assume that $X$ has good reduction, and let $\mathcal{X} \rightarrow \text{Spec} R$ be a smooth integral model. Let $n \in \mathbb{N}$ with $p \mid n$. The ramification map with respect to the prime divisor $\mathcal{X}_s$ induces an isomorphism

$$\text{Br}(X)[n] = H^1(\mathcal{X}_s, \mathbb{Z}/n)$$
Hence all elements of \( \text{Br}(X)[n] \) are cyclic, namely given any uniformizer \( \pi \in K \), they have the form \( \delta_{\pi} \pi \cup \hat{\chi} \) where \( \hat{\chi} \in H^1(\mathcal{X}, \mathbb{Z}/n) \) is the unique lift of the ramification \( \chi \in H^1(\mathcal{X}_s, \mathbb{Z}/n) \) of \( \beta \) with respect to \( \mathcal{X}_s \).

**Proof.** Let \( k(\mathcal{X}_s) \) be the function field of the special fiber \( \mathcal{X}_s \), and denote by \( \partial_\beta : \text{Br}(K(X))[n] \to H^1(k(\mathcal{X}_s), \mathbb{Z}/n) \) the corresponding ramification map. First we show that \( \partial_\beta \) restricts to a map \( \partial_{t_\lambda} : \text{Br}(X)[n] \to H^1(\mathcal{X}_s, \mathbb{Z}/n) \), i.e., that for \( \beta \in \text{Br}(X)[n] \) the character \( \chi = \partial_{t_\lambda}(\beta) \) is unramified with respect to any closed point \( P \in \mathcal{X}_s \). From sequence (6) we get \( \partial_{x}(\beta) = 0 \) for all \( x \in X_0 \), thus also \( \partial_P(\beta) = 0 \) for all horizontal prime divisors \( D \) of \( \mathcal{X} \) by lemma 8(ii). Applying lemma 8(ii) with \( A = \mathcal{O}_{\mathcal{X}, P} \) we get that \( \partial_P(\chi) = 0 \), i.e., \( \chi \) is unramified at \( P \), as desired. 

Next, given \( \chi \in H^1(\mathcal{X}_s, \mathbb{Z}/n) \) let \( \hat{\chi} \in H^1(\mathcal{X}, \mathbb{Z}/n) \) be its unique lift under the proper base change isomorphism \( H^1(\mathcal{X}, \mathbb{Z}/n) = H^1(\mathcal{X}_s, \mathbb{Z}/n) \) (Mil86, corollary VI.2.7, p. 224). Since \( \pi \) is a uniformizer for the valuation defined by \( \mathcal{X}_s \) and \( \hat{\chi} \) is \( \pi \)-unramified, we have \( \partial_\beta(\delta_{\pi} \pi \cup \hat{\chi}) = \chi \), showing that \( \partial_\beta : \text{Br}(X)[n] \to H^1(\mathcal{X}_s, \mathbb{Z}/n) \) is surjective. Finally, if \( \beta \in \text{Br}(X)[n] \) is such that \( \partial_{t_\lambda}(\beta) = 0 \), then \( \beta \) is unramified on the whole scheme \( \mathcal{X} \), thus \( \beta \in \text{Br}(\mathcal{X}) = 0 \) by lemmas 8(i) and 9(i). This proves \( \partial_\beta : \text{Br}(X)[n] \to H^1(\mathcal{X}_s, \mathbb{Z}/n) \) is also injective. \( \square \)

**Remark 14.** The previous theorem can also be obtained using Lichtenbaum’s duality. We briefly sketch the proof. First recall that from Class Field Theory of global fields of positive characteristic (see Mil86, chapter I, appendix A), for any smooth projective curve \( Z \) over a finite field \( k \), there is a non-degenerate pairing of finite groups

\[
(15) \quad H^1(Z, \mathbb{Z}/n) \times \text{Pic}(Z)/n \to \mathbb{Z}/n
\]

yielding an isomorphism \( H^1(Z, \mathbb{Z}/n) = (\text{Pic}(Z)/n)^\vee \). This pairing is induced by the local perfect pairings

\[
H^1(k(\mathcal{Z}), \mathbb{Z}/n) \times H^1(k(\mathcal{Z}), \mu_n) \to \text{Br}(k(\mathcal{Z})[n] = \mathbb{Z}/n
\]

(here \( k(\mathcal{Z}) \) stands for the completion of the function field \( k(Z) \) of \( Z \) with respect to the valuation given by a closed point \( z \in Z_0 \)).

Explicitly, given \( \chi \in H^1(Z, \mathbb{Z}/n) \) and a divisor \( \sum z \cdot n_z z \in \text{Div}(Z) \) representing a given \( \alpha \in \text{Pic}(Z)/n \), the pairing (15) maps \( (\chi, \alpha) \) to \( \sum z \cdot n_z \cdot \chi|_{\kappa(z)}(\text{Frob}_z) \) where \( \text{Frob}_z \in G_{\kappa(z)} \) is the Frobenius automorphism (here \( \kappa(z) \) denotes the residue field of \( z \), a finite field).

Now to get the theorem we just splice together Lichtenbaum’s duality (theorem 3), lemma 4, the isomorphism \( H^1(\mathcal{X}_s, \mathbb{Z}/n) = (\text{Pic}(\mathcal{X}_s)/n)^\vee \) given by (15) and the proper base change theorem:

\[
\text{Br}(X)[n] \cong \text{Hom}(\text{Pic}(X)/n, \mathbb{Z}/n) \cong \text{Hom}(\text{Pic}(\mathcal{X}_s)/n, \mathbb{Z}/n) \cong H^1(\mathcal{X}_s, \mathbb{Z}/n) ^{\text{PBC}} \cong H^1(\mathcal{X}, \mathbb{Z}/n)
\]

Given the explicit descriptions of (15) and Lichtenbaum’s pairing, it is not difficult to check that this chain of isomorphisms is just the ramification map \( \partial_{x} \) above.
5. Non-cyclic division algebras in the singular case

In this section, we show that for a general $p$-adic curve $X$ with bad reduction, one cannot expect $\text{Br}(X)[n]$ to be comprised solely of $\mathbb{Z}/n$-cyclic algebras. In example 17 below, for infinitely many primes $p$, we exhibit a non $\mathbb{Z}/3$-cyclic class of period 3 in the Brauer group of a certain $p$-adic curve of genus 10. Both for conceptual clarity and in order to facilitate the construction of other similar examples, we work in a more general situation, as described in the following

**Setup 16.** Let $d \in \mathbb{N}$ be prime to $p$, and suppose that $\mu_d \subset K$. Choose homogenous polynomials $F_1, F_2 \in R[x,y,z]$ with $\deg F_i = d$ such that

(i) in $\mathbb{P}_R^2$ the closed subschemes $V_+(F_1)$ and $V_+(F_2)$ intersect the line $V_+(z)$ at two $R$-points $\infty_1$, $\infty_2$, each with multiplicity $d$, and such that $\infty_1 \neq \infty_2$. Here $\infty_i$ denotes the special fiber of $\infty_i$;

(ii) the reductions $F_1, F_2 \in k[x,y,z]$ modulo $\pi$ define geometrically connected smooth curves over $k$

$$C_1 \overset{\text{def}}{=} \text{Proj } k[x,y,z]/(F_1) \quad C_2 \overset{\text{def}}{=} \text{Proj } k[x,y,z]/(F_2)$$

having normal crossings in $\mathbb{P}_k^2$, with at least one $k$-rational common point $P_0 \in C_1 \cap C_2$;

(iii) $\text{Pic}(C_1)[d] \neq 0$.

Define the integral projective $R$-scheme

$$\mathcal{X} \overset{\text{def}}{=} \text{Proj } \frac{R[x,y,z]}{(F_1 \cdot F_2 - \pi z^{2d})} \to \text{Spec } R$$

and the elements $f_1 \overset{\text{def}}{=} F_1/z^d, f_2 \overset{\text{def}}{=} F_2/z^d \in K(\mathcal{X})$ in its function field $K(\mathcal{X})$.

Finally, let $u \in R^\times$ be any unit which is not a $d$-th power.

**Example 17.** As a concrete example, we may choose $d = 3$ and a prime $p \equiv 1 \pmod{3}$, so that there is a primitive third root of unity $\omega \in \mathbb{Z}_p$ by Hensel’s lemma. We will show that for almost all $p$ the polynomials in $\mathbb{Z}_p[x,y,z]$

$$F_1 = y^2 z + y z^2 - x^3 - x^2 z + 7 x z^2 - 5 z^3$$

$$F_2 = x^2 z - y^3 + 26 z^3$$

will satisfy the conditions of the setup (the curve $V_+(F_1) \subset \mathbb{P}_R^2$ is the elliptic curve labeled 91.b2 in the LMFDB database, see http://www.lmfdb.org). Let $\mathcal{X} = V_+(F_1 F_2 - p z^6) \subset \mathbb{P}_R^2$ and $f_i = F_i/z^3 \in K(\mathcal{X})$ as above. Then, in $\mathbb{P}_p^2$, the subschemes $V_+(F_1)$ and $V_+(F_2)$ intersect the line $V_+(z)$ at the $R$-points $\infty_1 = V_+(x,z)$ and $\infty_2 = V_+(y,z)$, both with multiplicity 3.

If $p \neq 2, 3, 7, 13$ then both reductions $\overline{F}_1$ will be elliptic curves over $\mathbb{F}_p$ sharing a common $\mathbb{F}_p$-rational point $P_0 = (-1 : 3 : 1) \in C_1 \cap C_2$. Moreover the resultant of $F_1(x,y,1)$ and $F_2(x,y,1)$ with respect to the variable $x$ is

$$(18) \quad (y - 3)(y^8 + 3 y^7 + 9 y^6 - 66 y^5 - 196 y^4 + 587 y^3 + 1084 y^2 + 3209 y + 9585)$$

which is separable over $\mathbb{F}_p$ as long as $p \neq 2, 3, 7, 97, 29723, 447692787897013$. Then by Bézout’s theorem each of the $3^2 = 9$ roots of (18) will correspond to a different multiplicity 1 intersection point of $C_1$ and $C_2$, thus for almost all $p$ the curves $C_1$ and $C_2$ will be regular and have normal crossings. Finally, $\text{Pic}(C_1)[3] \neq 0$; in fact, if $P_1 = (1 : 0 : 1)$ then $[P_1] - [\infty_1] \in \text{Pic}^0(C_1)[3]$ is a nontrivial element.
We will need the following result concerning the Picard group of reducible curves such as $\mathcal{X}_s$ (see [Hid12], theorem 4.1.5, chapter 4, p.296):

**Lemma 19.** Let $k$ be a field, and $C = C_1 \cup C_2$ be the union of two proper smooth irreducible curves over $k$ such that its components intersect transversally at $r$ points over a finite extension of fields $l \supset k$. Then there is a split exact sequence

$$0 \to T(k) \to \text{Pic}^0(C) \to \text{Pic}^0(C_1) \oplus \text{Pic}^0(C_2) \to 0$$

where $T$ is an algebraic torus that becomes isomorphic to $G_m^{r-1}$ over $l$.

**Lemma 20.** Assume setup [16]. Then

(a) $\mathcal{X}$ is a regular scheme;

(b) The generic fiber $X \overset{\text{def}}{=} \mathcal{X}_\eta = \text{Proj} \ K[x, y, z]/(F_1F_2 - \pi z^{2d})$ of $\mathcal{X}$ is a smooth projective curve over $K$;

(c) The special fiber $\mathcal{X}_s = \text{Proj} \ k[x, y, z]/(F_1F_2)$ of $\mathcal{X}$ is a reduced $k$-scheme whose irreducible components are $C_1$ and $C_2$;

(d) The divisors $[C_1]$ and $[C_2]$ in $\text{Div}(\mathcal{X})$ are principal divisors modulo $d$, more precisely

$$\text{div}(f_1) = [C_1] + d^2 \cdot [\infty_1] - d^2 \cdot [\infty_2]$$

$$\text{div}(f_2) = [C_2] + d^2 \cdot [\infty_2] - d^2 \cdot [\infty_1]$$

In particular, $\text{Pic}(\mathcal{X})$ can be generated by horizontal divisors only (see definition before lemma [3]);

(e) There exists $h \in K(X)^\times$ such that

- $\text{div}(h) = dE$ for some $E \in \text{Div}(\mathcal{X})$;
- the restriction $\overline{h} \in k(C_1)^\times$ to the function field of $C_1$ is well-defined;
- $\overline{h}$ is not a $d$-th power in $k(C_1)^\times$;
- $P_0$ is not a pole of $\overline{h}$, and $\overline{h}(P_0) = \overline{T} \in k(P_0)^\times$ (here $k(P_0)$ denotes the residue field of $P_0$).

**Proof.** To prove (a) we have to show that for all closed points $P \in \mathcal{X}$ the maximal ideal $m_P$ of $O_{\mathcal{X}, P}$ (a 2-dimensional local ring) can be generated by 2 elements. Since $P$ necessarily lies on the special fiber $\mathcal{X}_s$, there are two cases to consider:

- $P$ is a regular point of the special fiber $\mathcal{X}_s$. In that case, if $\overline{T}$ is a uniformizer of the discrete valuation ring $O_{\mathcal{X}_s, P} = O_{\mathcal{X}, P}/(\pi)$ then $m_P = (t, \pi)$ for any lift $t \in O_{\mathcal{X}, P}$ of $\overline{T}$;
- $P$ is one of the nodal points in $C_1 \cap C_2$. Then $m_P = (f_1, f_2, \pi)$, which is clearly equal to $(f_1, f_2)$ since $f_1f_2 = \pi$ in $O_{\mathcal{X}, P}$.

Since $\mathcal{X}$ is regular, so is its open subset $X$, therefore (b) holds. Item (c) is clear.

To show (d), we work with $f_1$, the case for $f_2$ being similar. First notice that, as an element of the ring $O_{\mathcal{X}}(D_+(z))$, $f_1$ is a prime defining the generic point of $C_1$ since (here $\overline{T}_1$ stands for $f_1$ mod $\pi$)

$$O_{\mathcal{X}}(D_+(z))/\langle f_1 \rangle = \frac{R[\underline{z}, \underline{z}]}{(f_1f_2 - \pi, f_1)} = \frac{k[\underline{z}, \underline{z}]}{(\overline{T}_1)} = O_{C_1}(D_+(z))$$

The complement $V_+(z) = \mathcal{X} \setminus D_+(z)$ of the open subset $D_+(z)$ consists of two $R$-points $\infty_1$ and $\infty_2$, hence the support of $\text{div}(f_1)$ is contained in $\{C_1, \infty_1, \infty_2\}$, and all that is left is to compute the valuations $v_{\infty_1}(f_1) = d^2$; then by symmetry $v_{\infty_2}(f_2) = d^2$ and from $f_1f_2 = \pi$ we will automatically get $v_{\infty_2}(f_1) = -d^2$. 

To show $v_{\infty}(f_1) = d^2$, it is enough to work over the 2-dimensional local ring $O_{\mathcal{X}, \infty}$. Without loss of generality, say that $\infty_1 \in D_{+}(y)$, and write $x' = x/y$, $z' = z/y$ and $\phi_1 = F_1/y^d \in R[x', z']$. Let $\xi \in \text{Spec } R[x', z'] \subset \mathbb{P}_R^2$ be the prime corresponding to the generic point of $\infty_1$ (so that $\infty_1 = (\pi, \xi)$ and $O_{\mathcal{X}, \infty_1} = R[x', z']((\pi, \xi)/(\phi_1 \phi_2 - \pi(z')^{2d}))$). Since $\infty_1 \neq \infty_2$ by hypothesis, $\infty_1 \notin V_+(F_2) \subset \mathbb{P}_k^2$ and thus $\phi_2$ is a unit in $O_{\mathcal{X}, \infty_1}$. Therefore

$$\phi_1 \phi_2 = \pi(z')^{2d} \implies f_1 = \frac{\phi_1}{(z')^d} = \frac{\pi}{\phi_2} \cdot (z')^d \quad \text{in } O_{\mathcal{X}, \infty_1},$$

showing that $v_{\infty_1}(f_1) = d \cdot v_{\infty_1}(z')$. Finally, to show that $v_{\infty_1}(z') = d$, we have to compute the length of the $O_{\mathcal{X}, \xi}$-module

$$\frac{O_{\mathcal{X}, \xi}}{(z')} = \frac{R[x', z']_{\xi}}{(\phi_1 \phi_2 - \pi(z')^{2d}, z')} = \frac{R[x', z']_{\xi}}{(\phi_1, z')},$$

(since $\phi_2 \in O_{\mathcal{X}, \xi}^\times$, $\phi_2 \notin (\pi, \xi)$ in $R[x', z']$ and thus $\phi_2 \in R[x', z']^\times$ as well). But the length of the last module is exactly the intersection multiplicity of $V_+(z)$ and $V_+(F_1)$ in $\mathbb{P}_R^2$ at $\infty_1$, which is $d$ by hypothesis.

Finally, to show (e) let $\mathcal{T}_1 \in \text{Pic}(C_1)[d]$ be a nontrivial element, whose existence is guaranteed by setup 16(iii). Since $\mathcal{X}$ is geometrically connected, $H^0(\mathcal{X}, O_{\mathcal{X}}) = k$ and by lemmas 17(ii) and 19 the natural map

$$\text{Pic}(\mathcal{X})[d] = \text{Pic}(\mathcal{X})[d] \to \text{Pic}^0(C_1)[d] \oplus \text{Pic}^0(C_2)[d]$$

is surjective, hence there is $L \in \text{Pic}(\mathcal{X})[d]$ restricting to $\mathcal{T}_1$. We may write $L = O_{\mathcal{X}}(E)$ for some divisor $E \in \text{Div}(\mathcal{X})$ whose support does not contain $C_1$, $C_2$ or any divisor going through any of the (at most $d^2$) intersection points of $C_1$ and $C_2$. In fact, Pic($\mathcal{X}$) is generated by horizontal divisors; moreover, as in the proof of the lemma in [Sal98], we may modify $E$ by a principal divisor using the fact that there is an affine open set Spec $A$ of $\mathcal{X}$ containing all these intersection points, and that the semi-localization of $A$ with respect to the corresponding maximal ideals is a UFD (by Auslander-Buchsbaum’s theorem and [Mat89], exercise 20.5, p.169).

Since $L = O_{\mathcal{X}}(E)$ has order $d$ in Pic(\mathcal{X}), there exists $h \in K(X)^\times$ such that div($h$) = $dE$, and by the choice of $E$ the restriction $\overline{h} \in k(C_1)^\times$ is well-defined, and $P_0$ is neither a zero nor a pole of $\overline{h}$. Thus multiplying $h$ by a unit in $R^\times$, we may further assume that $\overline{h}(P_0) = 1$. Now all that is left is to show that $\overline{h}$ is not a $d$-th power in $k(C_1)^\times$. Write $E_1 \in \text{Div}(C_1)$ for the restriction of $E$ to $C_1$. Then $\mathcal{T}_1 = O_{\mathcal{X}}(E_1)$ in Pic($C_1$) and $\text{div}(\overline{h}) = dE_1$. Hence if $\overline{h} = \overline{\eta}^d$ with $\overline{\eta} \in k(C_1)^\times$ we would have div($\overline{\eta}$) = $dE_1$, which contradicts the fact that $\mathcal{T}_1 = O_{\mathcal{X}}(E_1)$ is not trivial in Pic($C_1$). This finishes the proof of item (e).

\[\square\]

Now we are ready to show

**Claim 21.** Assume setup 17(a) and let $h$ be as in the previous lemma. Choose a primitive root of unity $\zeta_d \in \mu_d$, and define the period $d$ sum of symbol algebras

$$\beta \overset{\text{def}}{=} (f_1, h)_{\zeta_d} + (f_2, u)_{\zeta_d} \in \text{Br}(K(X))[d]$$

Then $\beta \in \text{Br}(X)[d]$. If $m$ and $n$ are respectively the orders of $\overline{h}$ in $k(C_1)^\times/d$ and $u$ in $k^\times/d$ then $mn \mid \text{ind}(\beta)$. In particular, when $d$ is prime $\text{ind}(\beta) > d$, hence $\beta$ is not $\mathbb{Z}/d$-cyclic.
Proof. Let $P \in X_0$ be an arbitrary closed point of the generic fiber of $X$. Then $P$ is the generic point of a unique horizontal divisor on $X$ (lemma $\text{[8]}$); write $v_P : K(X) \to \mathbb{Z} \cup \{\infty\}$ and $\partial_P : \text{Br}(K(X))[d] \to H^1(k(P), \mathbb{Q}/\mathbb{Z})[d]$ for the corresponding discrete valuation and ramification maps. By sequence (4), to prove that $\beta \in \text{Br}(X)[d]$, we have to show that $\partial_P(\beta) = 0$, but this is clear since all $v_P(f_1)$, $v_P(f_2)$, $v_P(h)$, $v_P(u) = 0$ are multiples of $d$ by items (d) and (e) of the previous lemma.

Next let $\hat{K}(X)$ be the completion of $K(X) = K(X)$ with respect to the valuation defined by the prime divisor $C_1$. Observe that $f_1$ is a uniformizer of $\hat{K}(X)$, and that its residue field is $k(C_1)$, the function field of $C_1$. To show that $mn | \text{ind}(\beta)$ it is enough to show that the index of the restriction $\beta|_{\hat{K}(X)}$ is a multiple of $mn$. We just have to apply Nakayma's index formula (lemma 2). Denoting by $\bar{1}$ the same as $\text{inv}(\beta)$, we have to show that one of these invariants equals $m$.

Since $k(C_1)(\bar{1})$ is a global field of characteristic $p$, by Class Field Theory the index of $\beta|_{k(C_1)}(\bar{1})$ is the least common multiple of its local invariants, hence it suffices to show that one of these invariants equals $n$. Since $\bar{1}(P_0) = \bar{T}$ by construction, $P_0$ splits completely in the Kummer extension $k(C_1)(\bar{1})$, hence the local invariant of $\beta|_{k(C_1)}(\bar{1})$ with respect to any point lying over $P_0$ is the same as $\text{inv}_{P_0}(\bar{T})$.

But the latter is equal to the order $n$ of $\bar{1}$ in $k^*/d$ since $\bar{T}$ is a uniformizer of $P_0 \in C_1$ (recall that $C_1$ and $C_2$ have normal crossings by assumption), and we are done. \hfill $\square$

6. Tate curves

In this section, we use Lichtenbaum’s duality in order to study the Brauer group of elliptic curves with split multiplicative reduction, i.e., Tate curves. The nice fact about Tate curves is that they enjoy a $p$-adic uniformization, which will allow us to obtain cycleity results even for the $p$-primary case.

Let $q \in K^\times$ with $|q| < 1$. Recall that the **Tate elliptic curve** $E_q$ is the elliptic curve defined by the affine equation

$$y^2 + xy = x^3 + a_4(q)x + a_6(q)$$

where

$$s_r(q) = \sum_{n \geq 1} \frac{n^r q^n}{1 - q^n} \quad a_4(q) = -5s_3(q) \quad a_6(q) = -\frac{5s_3(q) + 7s_5(q)}{12}$$

For the next result, see [Sil94], theorems V.3.1 on p. 423, V.5.3 on p.442.

**Theorem 22** (Tate). Let $q \in K^\times$ with $|q| < 1$, and $E_q$ be the Tate elliptic curve as defined above.

(i) There is an exact sequence of $G_K$-modules

$$0 \longrightarrow \mathbb{Z} \overset{n \mapsto q^n}{\longrightarrow} \hat{K}^\times \overset{\varphi}{\longrightarrow} E_q(\hat{K}) \longrightarrow 0$$
with the usual trivial $G_K$-action on $\mathbb{Z}$, and Galois actions on $\overline{K}^\times$ and $E_q(\overline{K})$.

In particular, for every algebraic extension $L \supseteq K$, $\varphi$ induces an isomorphism

$$\varphi: E_q(L) \xrightarrow{\sim} L^\times/q\mathbb{Z} \quad (\text{since } H^1(G_L, \mathbb{Z}) = \text{Hom}(G_L, \mathbb{Z}) = 0).$$

(ii) Let $E$ be an elliptic curve over $K$ with split multiplicative reduction. Then $E$ is $K$-isomorphic to $E_q$ for some $q \in K^\times$, $|q| < 1$.

The main technical result in this section is given by the following

**Lemma 23.** Let $q \in K^\times$ with $|q| < 1$, and $E_q$ be the corresponding Tate elliptic curve. For any finite field extension $L \supseteq K$, let $\theta_L: L^\times \hookrightarrow G_L^{ab}$ be the local reciprocity map (see [NS13], chapter V, p.317 and [Ser79], chapter XIII, p.188) and write $H^1_q(G_L, \mathbb{Q}/\mathbb{Z})$ for the kernel of

$$\epsilon_q: H^1(G_L, \mathbb{Q}/\mathbb{Z}) = \text{Hom}_c(G_L^{ab}, \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

$$\chi \mapsto \chi(\theta_L(q))$$

Then there is an isomorphism

$$\xi_q: \text{Br}(E_q \otimes L) \xrightarrow{\sim} H^1_q(G_L, \mathbb{Q}/\mathbb{Z}) \oplus \mathbb{Q}/\mathbb{Z}$$

sitting in a commutative diagram

$$\begin{array}{c}
\text{Br}(E_q \otimes L) \\
\xrightarrow{\text{res}}
\end{array}
\xrightarrow{\xi_q}
\xrightarrow{\text{res} \oplus [L:K]}
\begin{array}{c}
H^1_q(G_K, \mathbb{Q}/\mathbb{Z}) \\
\oplus \mathbb{Q}/\mathbb{Z}
\end{array}$$

**Proof.** First, observe that the continuous map $\theta_L: L^\times \hookrightarrow G_L^{ab}$ induces an isomorphism

$$H^1(G_L, \mathbb{Q}/\mathbb{Z}) = \text{Hom}_c(G_L^{ab}, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\text{def}} \text{Hom}_c(L^\times, \mathbb{Q}/\mathbb{Z})$$

In fact, $\theta_L^\prime$ is injective since $\theta_L$ has dense image. To show that $\theta_L^\prime$ is surjective, let $f \in \text{Hom}_c(L^\times, \mathbb{Q}/\mathbb{Z})$, and let $\pi_L$ be a uniformizer of the ring of integers $R_L$ of $L$, so that $L^\times \cong \mathbb{Z}_L \supseteq U_L$ where $U_L$ denotes the group of units of $R_L$. Since $\mathbb{Q}/\mathbb{Z}$ has discrete topology, $\ker f$ must be open in $L^\times$, i.e., $\ker f \supseteq U_L^{(i)}$ for some $i \geq 1$. Since $L^\times \cong \mathbb{Z}_L \supseteq U_L$, and $f(\pi_L)$ and $U_L/U_L^{(i)}$ have finite orders, $\ker f$ is finite and, thus $\ker f$ has finite index in $L^\times$, therefore $\ker f = N_{M/L}(M^\times)$ is a norm group for some finite abelian extension $M \supseteq L$ by the existence theorem ([Ser79], XIV.§6, theorem 1, p.218). Therefore $f$ will be the image under $\theta_L^\prime$ of the composition

$$G_L^{ab} \xrightarrow{\text{Gal}(M/L)} \frac{L^\times}{N_{M/L}(M^\times)} \xrightarrow{\theta_{M/L}^{-1}} \frac{L^\times}{\ker f} \xrightarrow{f} \mathbb{Q}/\mathbb{Z}$$

where $\theta_{M/L}$ denotes the isomorphism induced on finite quotients by $\theta_L$.

Next, by theorem [22](i), there is an exact sequence of topological groups

$$0 \to \mathbb{Z} \to L^\times \to E_q(L) \to 0$$

which, together with the isomorphism $\theta_L^\prime: H^1(G_L, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \text{Hom}_c(L^\times, \mathbb{Q}/\mathbb{Z})$, gives rise to an exact sequence

$$0 \to \text{Hom}_c(E_q(L), \mathbb{Q}/\mathbb{Z}) \to H^1(G_L, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\epsilon_q} \text{Hom}_c(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}$$
Hence $E_q(L)^{\vee} = \ker \epsilon_q = H^1_q(G_L, \mathbb{Q}/\mathbb{Z})$. Therefore, by Lichtenbaum’s duality (theorem\footnote{Lich69}), we get isomorphisms

$$\Br(E_q \otimes L) \cong \Pic(E_q \otimes L)^{\vee} = (E_q(L) \oplus \mathbb{Z})^{\vee} = H^1_q(G_L, \mathbb{Q}/\mathbb{Z}) \oplus \mathbb{Q}/\mathbb{Z}$$

Here we have used the identifications

$$E_q(L) \overset{\sim}{\to} \Pic^0(E_q \otimes_K L) \quad \Pic(E_q \otimes_K L) \overset{\sim}{\to} \Pic^0(E_q \otimes_K L) \oplus \mathbb{Z} \quad P \mapsto [P] - [O_L] \quad \alpha \mapsto (\alpha - (\deg \alpha) \cdot [O_L], \deg \alpha)$$

where $[P] \in \Pic(E_q \otimes_K L)$ denotes the class of the degree 1 divisor given by $P \in E_q(L)$, and $O_L \in E_q(L)$ is the identity element of the elliptic curve.

Finally, to prove that the diagram in the statement of the lemma commutes, write $f: E_q \otimes_K L \to E_q$ for the natural map, let $n = [L : K]$, and $\sigma_1, \ldots, \sigma_n: L \to \overline{L}$ be the $K$-imersions of $L$ into its separable closure $\overline{L} = \overline{K}$. Then $f_*: \Pic(E_q \otimes_K L) \to \Pic(E_q)$ is given on closed points $P \in E_q(L)$ by

$$f_*[P] = \sum_{1 \leq i \leq n} [\sigma_i(P)] \in \Pic(E_q \otimes_K \overline{K})^{G_K} = \Pic(E_q)$$

where $\Pic(E_q \otimes_K \overline{K})^{G_K} = \Pic(E_q)$ follows from the Hochschild-Serre spectral sequence, and the fact that $E_q$ has a $K$-rational point, so that $\Br(K) \to \Br(E_q)$ is injective (see for example \cite[Lich69], §2, p.122). Hence $f_*[O_L] = n[O_K]$, and there is a commutative diagram

$$\begin{align*}
\Pic(E_q \otimes_K L) & \overset{\sim}{\to} \Pic^0(E_q \otimes_K L) \oplus \mathbb{Z} \\
& \quad \downarrow f_* \downarrow f_* \downarrow f_* \\
\Pic(E_q) & \overset{\sim}{\to} \Pic^0(E_q) \oplus \mathbb{Z} \\
& \quad \downarrow f_* \downarrow f_* \downarrow f_* \\
E_q(K) \oplus \mathbb{Z} & \overset{\sim}{\to} K^\times / q^\mathbb{Z} \oplus \mathbb{Z}
\end{align*}$$

where the rightmost vertical arrow is induced by the norm on the first component, and by multiplication by $n = [L : K]$ on the second. Combining the above diagram with the next one, expressing one of the functorial properties of the local reciprocity map,

$$\begin{align*}
L^\times & \overset{\theta_L}{\to} G_L^{ab} \\
N_{L/K} & \downarrow \\
K^\times & \overset{\theta_K}{\to} G_K^{ab}
\end{align*}$$

taking Pontryagin duals, and applying Lichtenbaum’s duality finishes the proof. $\square$

The following lemma is well-known, but we include its short proof for the convenience of the reader.

**Lemma 24.** Let $m \in \mathbb{N}$ be such that $\gcd(m, |\mu(K)|) = 1$ where $\mu(K)$ denotes the (finite) group of all roots of unity in $K$. Then the character group $H^1(G_K, \mathbb{Q}/\mathbb{Z})$ is $m$-divisible.

**Proof.** Taking $G_K$-invariants of $0 \to (\frac{1}{m}\mathbb{Z})/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \overset{m}{\to} \mathbb{Q}/\mathbb{Z} \to 0$, we obtain an exact sequence

$$H^1(G_K, \mathbb{Q}/\mathbb{Z}) \overset{m}{\to} H^1(G_K, \mathbb{Q}/\mathbb{Z}) \to H^2(G_K, (\frac{1}{m}\mathbb{Z})/\mathbb{Z}) = 0$$
whose last term vanishes by local duality ([NSW00], theorem 7.2.6, p.327):

\[ H^2(G_K, (\frac{1}{m}\mathbb{Z})/\mathbb{Z}) = H^0(G_K, \mu_m)^\vee = \mu_m(K)^\vee = 0 \]

Now we can show the main result in this section.

**Theorem 25.** Let \( m \in \mathbb{N} \), and \( E_q \) be a Tate elliptic curve over \( K \). Suppose that \((p \mid m \text{ is allowed})\n
(i) either \( m \) is prime; or
(ii) \( \gcd(m, |\mu(K)|) = 1. \n
Then all elements of \( \text{Br}(E_q)[m] \) are \( \mathbb{Z}/m \)-cyclic.

**Proof.** By looking at \( \ell \)-primary parts, it suffices to consider the case when \( m = \ell^r \) is a power of some prime \( \ell \). Let \( \beta \in \text{Br}(E_q)[m] \) be an element of order \( m \), and \( \xi_q(\beta) = (\chi, a) \in H^1_q(G_K, \mathbb{Q}/\mathbb{Z}) \oplus \mathbb{Q}/\mathbb{Z} \). Then either \( \chi \) or \( a \) has order \( m = \ell^r \). If \( \chi \in H^1_q(G_K, \mathbb{Q}/\mathbb{Z}) \) has order \( m \), then it corresponds to a surjective morphism \( \chi: G_K \to \mathbb{Z}/m\mathbb{Z} \), hence to a degree \( m \) cyclic extension \( L \supseteq K \). By the commutative diagram in lemma [24] \( \beta|_L = 0 \), i.e., \( \beta \) is \( \mathbb{Z}/m \)-cyclic. If \( \chi \) has order \( \ell^s \) with \( s < r \), then there exists a character \( \tilde{\chi} \in H^1_q(G_K, \mathbb{Q}/\mathbb{Z}) \) of order \( m = \ell^r \) such that \( \ell^{r-s} \tilde{\chi} = \chi \); this is clear in case (i), while in case (ii) it follows from the previous lemma. Then \( \tilde{\chi} \) defines a degree \( \ell^r \) cyclic extension \( L \supseteq K \) such that \( \beta|_L = 0 \), again by the diagram in lemma [24].

Noting that \( |\mu(\mathbb{Q}_p)| = p - 1 \) we obtain the following interesting

**Corollary 26.** Let \( E_q \) be a Tate elliptic curve over \( \mathbb{Q}_p \), and \( r \in \mathbb{N} \). Then all classes in \( \text{Br}(E_q)[p^r] \) are \( \mathbb{Z}/p^r \)-cyclic.

7. **Indecomposable algebras**

In this section, we assume once again that \( X \) is the generic fiber of a smooth projective \( \mathbb{R} \)-curve \( \mathcal{X} \). Our goal is to show that there are indecomposable algebras in the function field \( K(X) \) of \( X \) with period \( p^2 \) and index \( p^3 \), assuming enough roots of unity (theorem [50]). The construction basically follows the strategy in [BMT11], but we now have to struggle with the fact that for the \( p \)-primary case we do not have at our disposal general lifting theorems, as was the case in the prime-to-\( p \) situation treated in that paper. That unfortunately restricts our construction to the above period \( p^2 \) and index \( p^3 \). As in [BMT11], we rely on the following well-known criterion for indecomposability:

**Lemma 27.** Let \( n = p^r \) and \( \beta \in \text{Br}(K(X))[n] \) be such that \( \text{ind}(p\beta) = \frac{1}{p} \text{ind } \beta \). Then \( \beta \) is indecomposable.

We begin with a

**Lemma 28.** Let \( C \) be a smooth projective curve over the finite field \( k = \mathbb{F}_q \). For all sufficiently large integers \( m \gg 0 \), there exist distinct closed points \( P_{\infty}, P_1, P_2 \in C \), all of degree \( m \), such that the divisors \( [P_1] - [P_{\infty}] \) and \( [P_2] - [P_{\infty}] \) are principal.

**Proof.** Let \( g \) be the genus of \( C \), and for each \( m \in \mathbb{N}_{>0} \) denote by \( S(m) \) the set of all closed points \( P \in C \) of degree \( [\kappa(P) : k] = m \). By [HKT08], theorem 9.25, p.346,

\[ \left| S(m) \right| - \frac{q^m}{m} < (2 + 7g) \cdot \sqrt{\frac{q^m}{m}} \text{ for all } m \geq 2 \implies \lim_{m \to \infty} \frac{|S(m)|}{q^m/m} = 1 \]
Hence \(|S(m)|\) asymptotically grows as \(q^m/m\), and if \(m\) is large enough then \(|S(m)| > 2|\text{Pic}^0(C)|\). Fix such an \(m\), and pick an arbitrary point \(Q \in S(m)\). Then in the set \(\{[P] - [Q] \in \text{Div}^0(C) \mid P \in S(m)\}\) there will be three distinct divisors \([P_\infty] - [Q], [P_1] - [Q], [P_2] - [Q]\) having the same image in \(\text{Pic}^0(C)\), hence their differences \([P_1] - [P_\infty]\) and \([P_2] - [P_\infty]\) will have trivial image in \(\text{Pic}^0(C)\), i.e., they will be principal divisors. □

Let \(X \to \text{Spec} R\) be a smooth projective curve as above. Fix \(m \in \mathbb{N}\) such that

- \(m\) is not divisible by \(p\);
- \(m > 2g - 2\), where \(g\) is the genus of \(X_s\);
- \(m \gg 0\) satisfies the conditions of the lemma for \(C = X_s\), so that there are three distinct degree \(m\) points \(P_\infty, P_1, P_2 \in X_s\) such that

\[
\text{div}(\tilde{f}_i) = [P_i] - [P_\infty] \in \text{Div}(X_s) \quad (i = 1, 2)
\]

for some elements \(\tilde{f}_i \in k(X_s)^\times\) in the function field of \(X_s\).

By lemma 8(iii), there is a horizontal divisor \(D_\infty\) on \(X\) intersecting the special fiber \(X_s\) transversally at \(P_\infty \in X_s\). And since \(m > 2g - 2\), \([1111]\) proposition 4.1, p.71, shows that the map \(H^0(X, \mathcal{O}_X(D_\infty)) \to H^0(X_s, \mathcal{O}_{X_s}(P_\infty))\) is surjective, hence we may choose \(f_i \in H^0(X, \mathcal{O}_X(D_\infty)) \subseteq K(X)\) lifting \(\tilde{f}_i\). Then all components of \(\text{div}(f_i)\) are horizontal (since \(\tilde{f}_i \neq 0\), and \(D_\infty\) is the only pole of \(f_i\). Hence

\[
\text{div}(f_i) = [D_i] - [D_\infty] \in \text{Div}(X) \quad (i = 1, 2)
\]

where \(D_i\) is a horizontal divisor on \(X\) restricting (transversally) to \(P_i \in X_s\) for \(i = 1, 2\). Moreover, since \(\mathcal{O}_{X_s, P_i}/(f_i) \supset k\) is a (separable) degree \(m\) field extension, \(\mathcal{O}_{X_s, P_i}/(f_i) \supset R\) is a finite unramified extension of degree \(m\), hence \(\mathcal{O}_{X_s, P_i}/(f_i)\) is a complete \(p\)-adic ring with uniformizer \(\pi\). Now set \(f = f_1f_2\) so that it restricts to \(\mathcal{O} = \mathcal{O}_1\mathcal{O}_2\) and

\[
\text{div}(f) = [D_1] + [D_2] - 2[D_\infty] \in \text{Div}(X)
\]

\[
\text{div}(\mathcal{O}) = [P_1] + [P_2] - 2[P_\infty] \in \text{Div}(X_s)
\]

From now on, suppose that \(p \neq 2\) and \(\mu_{2^p} \subseteq K\). We keep the notation of subsection 3.3 writing \(\zeta_2 \in \mu_{2^p}\) for a primitive \(p^2\)-th root of unity, \(\lambda_2 = \zeta_2 - 1\), and so on. Observe that \(\lambda_2\) is a uniformizer of \(\mathbb{Q}_p(\zeta_2)\), a subfield of \(K\), hence \(\pi \mid \lambda_2\) in \(R\). Consider the degree \(p^2\) Artin-Schreier-Witt extension of \(k(X_s)\) given by

\[
x^p - x = \mathcal{O}
\]

\[
y^p - y - c(x^p, -x) = \mathcal{O}^{-1}
\]

where \(c(X, Y) \in \mathbb{Z}[X, Y]\) is the polynomial \([12]\) defined in subsection 3.3. Since \(p \neq 2\), it is easy to check that both \(P_1\) and \(P_2\) split completely in \(k(X_s)(x) \supset k(X_s)\), and each place above \(P_1\) or \(P_2\) is totally ramified in \(k(X_s)(x, y) \supset k(X_s)(x)\), while \(P_\infty\) is totally ramified in \(k(X_s)(x, y) \supset k(X_s)\). In particular, this shows that the latter extension indeed has degree \(p^2\).

Now we lift \(k(X_s)(x, y) \supset k(X_s)\) to the Kummer extension of \(K(X)\) defined by equations

\[
\frac{(\lambda x + 1)^p - 1}{\lambda^p} = f
\]

\[
\frac{1}{\lambda^p} \left( \frac{(\lambda y + F(x))^p}{\lambda x + 1} - G(f) \right) = \frac{1}{\mathcal{O}}
\]
where $F, G \in R[x]$ are the $p$-truncated exponentials (11). Let $\chi \in H^1(K(X), \mathbb{Z}/p^2)$ be a Kummer character of order $p^2$ associated to this extension, and $W$ (respectively $Y$) be the normalization of $X$ in the Kummer extension of $K(X)$ defined by (31) (respectively (31) and (32)), so that the composition

$$Y \to W \to X$$

defines a degree $p^2$ cyclic cover of $X$. Observe that the function fields of $W$ and $Y$ are given by $K(W) = K(X)(x)$ and $K(Y) = K(X)(x, y)$, respectively.

**Lemma 33.** In the above notation, for $i = 1, 2$ let $A_i$ and $B_i$ be the normalizations of $O_{X, P_i}$ in $K(W)$ and $K(Y)$ respectively. Then

(i) $A_i$ and $B_i$ are regular semilocal rings (hence UFDs), each with exactly $p$ maximal ideals.

(ii) The prime divisor $D_i$ of $X$ splits completely into $p$ disjoint horizontal divisors $D_{i1}, \ldots, D_{ip}$ in Spec $A_i$, i.e.,

$$D_i \times_{X} \text{Spec } A_i = D_{i1}' \sqcup \cdots \sqcup D_{ip}'$$

with each $D_{ij}' \cong D_i$, corresponding bijectively to the maximal ideals of $A_i$. In particular, $f$ is a local parameter for each $D_{ij}'$.

(iii) Each $D_{ij}'$ is totally ramified in Spec $B_i$.

**Proof.** We will prove the lemma for $i = 1$. Observe that $P_1$ is the closed point of $D_1$, $f_2 \in O_{X, P_1}$ (since $f_2(P_1) \neq 0$), both $f_1$ and $f = f_1f_2$ are local parameters for $D_1$ in Spec $O_{X, P_1}$, and $(\pi, f) = (\pi, f_1)$ is the maximal ideal of the 2-dimensional regular local ring $O_{X, P_1}$.

Expanding (31), since $\lambda^{p-1} | p$ in $R \subset O_{X, P_1}$, we get the following monic equation for $x$ with coefficients in $O_{X, P_1}$

$$x^p + \frac{p}{\lambda} x^{p-1} + \frac{p(p-1)}{2\lambda} x^{p-2} + \cdots + \frac{p}{\lambda^{p-1}} x - f = 0$$

whose reduction modulo $\pi$ is (25). Since $O_{X, P_1}[x] \supset O_{X, P_1}$ is an integral extension, if we show that $O_{X, P_1}[x]$ is a regular semilocal ring then $O_{X, P_1}[x]$ will be a UFD (by Auslander-Buchsbaum’s theorem and [Mat89], exercise 20.5, p.169), hence normal, showing also that $A_1 = O_{X, P_1}[x]$. Since (34) reduces to $x^p - x = 0$ modulo the maximal ideal $(\pi, f)$ of $O_{X, P_1}$, the fiber of $(\pi, f)$ under Spec $O_{X, P_1}[x] \to$ Spec $O_{X, P_1}$ is given by Spec $O_{X, P_1}[x]/(\pi, f) = \text{Spec } \kappa(P_1)[x]/(x^p - x)$, a disjoint union of $p$ copies of Spec $\kappa(P_1)$. This shows that $O_{X, P_1}[x]$ is semilocal with $p$ maximal ideals $(x - a, \pi, f)$ for $a = 0, 1, \ldots, p - 1$. Equation (34) shows that $(x - a, \pi, f) = (x - a, \pi)$ in the localization $O_{X, P_1}[x]_{(x - a, \pi, f)}$, therefore $O_{X, P_1}[x]$ is regular.

Since $D_1 \times_{X} \text{Spec } A_1 = \text{Spec } O_{X, P_1}[x]/(f)$, in order to show (ii) we have to show that $O_{X, P_1}[x]/(f)$ is isomorphic to a product of $p$ copies of $O_{X, P_1}$. But $O_{X, P_1}/(f)$ is a complete $p$-adic ring with uniformizer $\pi$, and (34) reduces to the separable equation $x^p - x = 0$ modulo $(\pi, f)$. Therefore the result follows by Hensel’s lemma.

Expanding (32) and multiplying by $(1 + \lambda x)f/y^p$ we get

$$\left(\frac{F(x)^p - (1 + \lambda x)G(f)}{\lambda^p} \cdot f - (1 + \lambda x)\right) \cdot (1/y)^p + \frac{pf}{\lambda^{p-1}} F(x)^p - (1/y)^{p-1}$$

$$+ \cdots + \frac{pf}{\lambda} F(x) \cdot (1/y) + f = 0$$

(35)
The last equation implies that $A_1[1/y] \ni A_1$ is an integral extension. In fact, by (SS01), 5.15, p.236, one has that

$$F(T)^p \equiv (\lambda T + 1) \cdot G \left( \frac{(\lambda T + 1)^p - 1}{\lambda^p} \right) \pmod{\lambda^p}$$

Moreover, $1 + \lambda x$ is a unit in the semilocal ring $A_1 = \mathcal{O}_{x^*P_1}[x]$ since $\lambda$ belongs to the maximal ideal $(\pi, f)$ of $\mathcal{O}_{x^*P_1}$. Therefore the coefficient of $(1/y)^p$ belongs to $A_1^\times$. Equation (35) also shows that the fiber under Spec $A_1[1/y]$ Spec $A_1$ of any maximal ideal $(x - a, \pi, f)$ with $a = 0, 1, \ldots, p - 1$ is Spec $\frac{\mathcal{O}_{x^*P_1}[1/y]}{(1/y)^p}$, i.e., consists of a single maximal ideal $(x - a, \pi, f, 1/y)$. Hence $A_1[1/y]$ is semilocal with $p$ maximal ideals.

Next we show that the 2-dimensional semilocal ring $A_1[1/y]$ is regular, hence normal, thus proving that $B_1 = A_1[1/y]$. In the localization $A_1(x-a, \pi, f)$, we have $(x-a, \pi, f) = (\pi, f)$ by (34), hence in $A_1[1/y]$ Spec $A_1$ we have $(x-a, \pi, f, 1/y) = (1/y, \pi)$ by (35), showing that $A_1[1/y]$ is regular.

Finally, if $D_{1i} \subset \text{Spec } A_1$ is the prime divisor corresponding to the maximal ideal $(x-a, \pi, f)$ of $A_1$ in (ii), since $f$ is a local parameter for $D_{1i} = \text{Spec } A_1(x-a, \pi, f)/(f)$, from (35) we get

$$D_{1i} \times_{\text{Spec } A_1} \text{Spec } B_1 = \text{Spec } A_1(x-a, \pi, f)[1/y] \cap (f, (1/y)^p)$$

Hence the pre-image of $D_{1i}$ in Spec $B_1$ is the divisor locally given by $1/y$, with multiplicity $p$, i.e., $D_{1i}$ is totally ramified in Spec $B_1$. 

We now can show

**Theorem 36.** Suppose $p \neq 2$, and let $X$ be the generic fiber of a smooth projective geometrically connected $R$-curve $\mathcal{X}$. Assume $\mu_p^2 \subset K$, and let $\chi \in H^1(K(X), \mathbb{Z}/p^2)$ be the degree $p^2$ character constructed above, corresponding to the field extension $K(Y) \ni K(X)$. Let $h = f_1/f_2$ and $\psi \in H^1(K, \mathbb{Z}/p^2)$ be the $\pi$-unramified character of order $p^2$. Define

$$\beta \overset{\text{def}}{=} \psi \cup \delta_{\rho^2}(h) + \chi \cup \delta_{\rho^2}(\pi) \in \text{Br}(K(X))[p^2]$$

Then $\beta$ is the class of an indecomposable algebra with $\text{ind } \beta = p^3$.

The proof of the theorem will immediately follow from lemma 27 once we show the next two lemmas.

**Lemma 37.** $p^3 | \text{ind } \beta$ and $p^2 | \text{ind}(p\beta)$.

*Proof.* Let $\hat{K}(X)$ be the completion of $K(X)$ with respect to the discrete valuation defined by the special fiber $\mathcal{X}_s$. Notice that $\hat{K}(X)$ has uniformizer $\pi$ and residue field $k(\mathcal{X}_s)$ (the function field of $\mathcal{X}_s$), a global field of characteristic $p$. Our goal is to show that the restriction of $\beta$ to $\hat{K}(X)$ has index $p^3$. Since the compositum of $\hat{K}(X)(\psi)$ and $\hat{K}(X)(\chi)$ splits $\beta$, and both define $\pi$-unramified extensions of $\hat{K}(X)$, we may apply lemma 2 in order to compute

$$\text{ind } \beta|_{\hat{K}(X)} = |\chi| \cdot \text{ind}(\psi \cup \delta_{\rho^2})(\mathcal{X}_s)(\chi) = p^2 \cdot \text{ind}(\psi \cup \delta_{\rho^2})(\mathcal{X}_s)(\chi)$$

Let $\alpha = (\psi \cup \delta_{\rho^2})(\mathcal{X}_s) \in \text{Br}(k(\mathcal{X}_s))$. We must show that $\text{ind } \alpha|_{\mathcal{X}_s}(\chi) = p$, and for that we need to compute the local invariants of $\alpha|_{\mathcal{X}_s}(\chi)$. Since $\psi$ defines an unramified cover of $\mathcal{X}_s$, and $\text{div}(\tilde{h}) = [P_1] - [P_2] \in \text{Div}(\mathcal{X}_s)$ with $m = [\kappa(P_1) : k]$ prime
to $p$ (where $\kappa = \mathcal{J}_1/\mathcal{J}_2$), $\alpha$ has nontrivial invariants $\pm 1/p^2 \in \mathbb{Q}/\mathbb{Z}$ only at $P_1$ and $P_2$. On the other hand, by construction $k(\mathcal{X}_s)(\chi) \supset k(\mathcal{X}_t)$ is the Artin-Schreier-Witt extension given by equations (29) and (30), which defines local extensions of degree $p$ at the completions of $k(\mathcal{X}_s)$ at $P_1$ and $P_2$. Therefore $\alpha|_{k(\mathcal{X}_s)(\chi)}$ will have local invariants $\pm 1/p \in \mathbb{Q}/\mathbb{Z}$ at those places, and thus its index is $p$, as desired. A similar argument shows that $\text{ind}(p\beta)|_{K(\mathcal{X}_t)} = p^2$. \hfill $\square$

For the next and final lemma, we make some preliminary remarks. First, for any $x \in X_0$ denote by $R_x$ the ring of integers of the $p$-adic field $\kappa(x)$. Then by the valuative criterion of properness the inclusion $x = \text{Spec} \kappa(x) \hookrightarrow X \subset \mathcal{X}$ extends uniquely to an $R_x$-valued point $\text{Spec} R_x \hookrightarrow \mathcal{X}$ (the normalization of the horizontal divisor $\{x\} \subset \mathcal{X}$, see lemma 3(ii)).

Second, let $M \supset K(X)$ be a field extension, let $Z$ be the $p$-adic curve obtained by normalizing $X$ in $M$, and $Z \to X$ be the natural map induced by the inclusion of function fields $K(X) \hookrightarrow M = K(Z)$. By corollary 7 to show that $M$ splits some $\gamma \in \text{Br}(K(X))$, we can proceed locally, showing that for every closed point $z \in Z_0$ with image $x \in X_0$ the completion $\widehat{K(Z)}_z$ of $K(Z)$ at $z$ splits $\gamma|_{\widehat{K(Z)}_z}$.

Third, let $x \in X_0$ and $z \in Z_0$ be a closed point lying over $x$ (with respect to the map $Z \to X$). Let $D_x \overset{\text{def}}{=} \{x\} \subset \mathcal{X}$ be horizontal divisor associated to $x$ (see lemma 3(ii)), write $P_x \in \mathcal{X}$ for the closed point of $D_x$, let $A = \mathcal{O}_{\mathcal{X},P_x}$ and $p \in \text{Spec} A$ be the height 1 prime ideal which is the generic point of $D_x = \text{Spec} A/p$. Let $C \subset K(Z)$ be the normalization of $A$ in $K(Z)$. We claim that $\mathcal{O}_{Z,z} = C_q$ for some height 1 prime ideal $q \in \text{Spec} C$, which is contained in a single maximal ideal of $C$. In fact, since $\mathcal{O}_{Z,z}$ is a localization of the normalization of $\mathcal{O}_{X,x}$ in $K(Z)$, we have the following inclusions of rings, and corresponding maps of spectra

$$
\begin{array}{ccc}
\mathcal{O}_{Z,z} & \hookrightarrow & \mathcal{O}_{X,x} \\
\downarrow & & \downarrow \\
A & \hookrightarrow & \mathcal{O}_{X,x}
\end{array}
\quad \quad \quad 
\begin{array}{ccc}
\text{Spec} C & \hookrightarrow & \text{Spec} \mathcal{O}_{Z,z} \\
\downarrow & & \downarrow \\
\text{Spec} A & \hookrightarrow & \text{Spec} \mathcal{O}_{X,x}
\end{array}
$$

Let $q \in \text{Spec} C$ be the image of the closed point of $\text{Spec} \mathcal{O}_{Z,z}$, so that $q$ lies over $p$. Since $p$ has height 1 and $A \hookrightarrow C$ is integral with $A$ normal, by the going-up and going-down theorems $q$ also has height 1, thus $C_q$ is a discrete valuation ring (being a local noetherian normal domain of dim $C_q = 1$). Then the inclusion $C \hookrightarrow \mathcal{O}_{Z,z}$ induces a local injective map $C_q \hookrightarrow \mathcal{O}_{Z,z}$, which must be an equality since discrete valuation rings are maximal with respect to domination of valuation rings ([Mat89], exercise 10.5, p.77). Finally, we have to check that $q$ is contained in a single maximal ideal of $C$, i.e., that $C/q$ is a local domain. Consider the inclusion $A/p \hookrightarrow C/q$, a finite extension of domains. Since $D_x = \text{Spec} A/p$ is a horizontal divisor, $\text{Spec} A/p$ is finite (and flat) over $\text{Spec} R$ (lemma 3(i)), hence $\text{Spec} C/q \to \text{Spec} R$ is finite with $R$ henselian. Now the result follows from [MB89], theorem I.4.2(b), p.32.

**Lemma 38.** Let $L \supset K$ be the $\pi$-unramified extension of degree $p$. Let $M = L(Y)$ be the compositum of $L$ and $K(Y)$. Then $[M : K(X)] = p^3$ and $M$ splits $\beta$, thus $\text{ind} \beta$ is at most $p^3$. Similarly $\text{ind}(p\beta)$ is at most $p^2$.

**Proof.** First observe that $L$ and $K(Y)$ are linearly disjoint over $K(X)$ (for instance, by lemma 33 the maximal ideals of $\text{Spec} B_1$ lying over $P_1$ define trivial field extensions in $\text{Spec} B_1 \to \text{Spec} \mathcal{O}_{\mathcal{X},P_1}$). Hence $M = L(Y) \supset K(X)$ is a degree
Let $Z$ be the normalization of $X$ in $M$. The inclusions $M \supseteq K(Y) \supseteq K(X)$ define maps of $p$-adic curves $Z \rightarrow Y \rightarrow X$. Fix a closed point $z \in Z_0$, and let $y \in Y_0$ and $x \in X_0$ be its images. Our goal is to show that $K(Z)_z$ splits $\gamma|_{K(X)_x}$.

Let $P_x \in \mathcal{P}_x$ be the closed point of the horizontal divisor $\overline{x}$.

There are a few cases to consider:

(i) $P_x \neq P_1, P_2$. Then $h$ is a unit in $\mathcal{O}_{X,P_x}$, thus also in $\mathcal{O}_{X,x}$. Since $ψ$ is also unramified with respect to $x$, $γ|_{K(X)_x} \in \text{Br}(κ(x))$. But $γ|_{K(X)_x} = (ψ \cup δ_1)|_{κ(x)}$ is already trivial in $\text{Br}(κ(x))$, since $ψ$ is also $π$-unramified and $h \in \mathcal{O}_{X,P_x}$ implies that the image of $h$ in $R_x$ (the valuation ring of $κ(x)$, a normalization of a quotient of $\mathcal{O}_{X,P_x}$) is also a unit.

(ii) $P_x = P_1$ (or the analogous case $P_x = P_2$). Write $B \subseteq K(Y)$ and $C \subseteq M$ for the normalizations of $A = \mathcal{O}_{X,P_1}$ in $K(Y)$ and $M = K(Z)$, respectively. By lemma 33, $B$ is a 2-dimensional regular semilocal ring with $p$ maximal ideals $m_1, \ldots, m_p$. Moreover, since both $f$ and $h$ are local parameters of $D_1$ in $\text{Spec} A$, by the same lemma we may write $h = u t_1^p \ldots t_p^p$ where $u \in B^\times$ and $t_i \in B$ are prime elements defining the irreducible components of $D_1 \times_X \text{Spec} B$, with $t_i \notin m_i$ and $t_i \notin m_j$ for $i \neq j$. By the above remarks, $\mathcal{O}_{Y,y} = B_q$ for some height 1 prime ideal $q \in \text{Spec} B$, say contained in $m_1$ (and not in any other $m_i$ with $i \neq 1$). In $B_{m_1}$, $h = u t_i^p$ with $u' \in B_{m_1}^\times$. Therefore we may write

$$γ|_{M} = (ψ \cup δh)|_{M} = (pψ)|_M \cup δt_1 + (ψ \cup δu')|_M = (ψ \cup δu')|_M$$

since $ψ|_L$ has order $p$ and thus $(pψ)|_M = 0$. But $u' \in B_{m_1}^\times \implies u' \in \mathcal{O}_{Y,y}$, hence the same argument of (i) shows that $(ψ \cup δu')|_{K(Y)} = (ψ \cup δu')|_{κ(y)} = 0$.

Therefore $γ|_{K(Z)_z} = (ψ \cup δu')|_{K(Z)} = 0$ as well.

\[ \square \]

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