ABSTRACT

We perform a linear perturbation analysis of expanding shells driven by expansions of H II regions. The ambient gas is assumed to be uniform. As an unperturbed state, we develop a semi-analytic method for deriving the time evolution of the density profile across the thickness. It is found that the time evolution of the density profile can be divided into three evolutionary phases: deceleration-dominated, intermediate, and self-gravity-dominated phases. The density peak moves relatively from the shock front to the contact discontinuity as the shell expands. We perform a linear analysis taking into account the asymmetric density profile obtained by the semi-analytic method, and imposing the boundary conditions for the shock front and the contact discontinuity while the evolutionary effect of the shell is neglected. It is found that the growth rate is enhanced compared with previous studies based on the thin-shell approximation. This is due to the boundary effect of the contact discontinuity and asymmetric density profile that were not taken into account in previous works.

Key words: H II regions – hydrodynamics – instabilities – shock waves – stars: formation

1. INTRODUCTION

Expanding shells are ubiquitous in the interstellar medium. They are driven by energetic phenomena of massive stars, such as emission of ionizing photons, stellar winds, and supernova explosions. Recently, using 102 samples identified as shells, Deharveng et al. (2010) found evidences of the star formation in more than a quarter of the shells, suggesting that the triggered star formation by H II regions may be an efficient process of the massive star formation. Theoretically, Elmegreen & Lada (1977) presented a sequential star formation scenario where massive star formation takes place through gravitational fragmentation of the expanding shell that is driven by H II regions surrounding massive stars, and the newly formed massive star also triggers the formation of next generation.

To understand the triggered star formation, it is important to investigate how and when the expanding shell fragments through the gravitational instability (GI). Earliest studies were done by using linear analyses of the static dense gas layer confined by the same thermal pressures of hot rarefied gases on both sides (Goldreich & Lynden-Bell 1965; Elmegreen & Elmegreen 1978; Lubow & Pringle 1993). They showed that the GI begins to develop with a scale comparable to the layer thickness and with a growing timescale comparable to the free-fall time of the layer. However, their linear analyses are oversimplified because the actual shells are confined by the shock front (SF) on the leading surface and the contact discontinuity (CD), or the ionization front (IF) on the trailing surface. Moreover, an unbalance between the ram pressure and the thermal pressure causes a decelerating or an accelerating expansion. Many authors have tackled the stability analyses with these effects by mainly using the thin-shell approximation where the perturbed variables are averaged across the thickness. The stability analysis of expanding shells has been investigated by Vishniac (1983), Elmegreen (1994), and Whitworth et al. (1994b). They took into account dilution effects of perturbations owing to the expansion and the mass accretion. Their linear analyses of expanding shells neglected the structure across the thickness and the boundary effect of the CD. Thus, how these effects that they neglected influence the GI are yet unknown. Voit (1988) investigated the stability of asymmetric layers, and found that the asymmetry of the density profile of the shell greatly influences the development of the GI. Moreover, by using shock-like boundary conditions, he found that the different choice of the boundary condition greatly modifies the dispersion relation. However, their analysis is limited to incompressible fluids.

In this paper, we perform a linear analysis taking into account the structure across the thickness and the effects of boundaries, i.e., the SF on the leading surface and the CD on the trailing surface. In order to determine the density profile at all times, we develop a semi-analytic method that well describes the one-dimensional (1D) evolution. This paper extends the study of Voit (1988) to include the compressible effect and the more realistic density profile by taking into account the radial self-gravitational force (Whitworth & Francis 2002). We neglect the effects of expansion and mass accretion through the SF.

In this paper, since we focus on investigating how the boundary effects and asymmetric density profiles influence the GI, we do not apply our result to estimate fragmentation time and scale. We will perform three-dimensional simulation of expanding shells to compare with the results of the linear analysis and present detailed quantitative aspects of the fragmentation process of expanding shells in a companion paper (Iwasaki et al. 2011).

The outline of the paper is as follows: in Section 2, we present a thin-shell model of the expanding shell driven by the H II region. In Section 3, we develop a semi-analytic method of deriving the time evolution of the density profile. We investigate influences of the asymmetric density profile on the dispersion relation of the GI by considering a pressure-confined layer in Section 4. In Section 5, we perform linear analysis of expanding shells by using the density profile obtained in Section 3 and by imposing the approximate SF and the CD boundary conditions. In Section 6, we compare our results with previous works. Our summary is presented in Section 7.

2. THIN-SHELL MODEL DRIVEN BY H II REGION

Massive stars emit ultraviolet photons (hν > 13.6 eV) and produce H II regions around them. Here, we consider a massive star that emits ionizing photons with the photon number luminosity Q_{UV} (s^{-1}), into the ambient gas with uniform density
of $\rho_e = mn_e$, where $n_e$ and $m$ are the number density and the mean mass of the ambient gas particle, respectively. In the standard picture (e.g., Spitzer 1978), the IF initially expands with supersonic speed with respect to the sound speed of ionized gas, $c_{\text{II}}$. The H II region begins to expand due to the pressure difference between the H II region and the ambient gas when the IF reaches the Strömgren radius, $R_{\text{ST}}$, given by

$$R_{\text{ST}} = \left(\frac{3 Q_{\text{UV}}}{4\pi\alpha_B n_{E}^3}\right)^{1/3},$$

(1)

where $\alpha_B$ indicates the case-B recombination coefficient. In this phase, the SF emerges in front of the IF and sweeps up the ambient gas into a dense shell. This paper focuses on the evolution of the shell after the shock emerges. The equation of motion of the shell is given by

$$\frac{d}{dt} \left( M_s \frac{dR_s}{dt} \right) = 4\pi R_s^2 p_{\text{II}},$$

(2)

where $M_s = 4\pi G\rho_{E} R_s^3 / 3$ is the total mass of the shell, i.e., the mass of the ambient gas that initially occupied the volume of the H II region, $R_s$ is the mean radius of the shell, and $p_{\text{II}}$ is the thermal pressure of the H II region. Here, we neglect the pressure of the ambient gas and the thickness of the shell. In the H II region, the detailed balance between the recombination and the ionization is approximately established at all times. Therefore, $p_{\text{II}}$ can be expressed using $R_s$ as follows:

$$p_{\text{II}} = \rho_e c_{\text{II}}^2 \left( \frac{R_{\text{ST}}}{R_s} \right)^{3/2}.$$  

(3)

Using Equation (3), we obtain the solution of Equation (2),

$$R_s(t) = R_{\text{ST}} \left( 1 + \frac{7}{\sqrt{12}} \frac{c_{\text{II}} t}{R_{\text{ST}}} \right)^{4/7}.$$  

(4)

(Hosokawa & Inutsuka 2006). Equations (2) and (4) are valid only in the early phase. As the shell sweeps up the ambient gas and increases its mass, self-gravity influences the expansion. The equation of motion including self-gravity becomes

$$\frac{d}{dt} \left( M_s \frac{dR_s}{dt} \right) = 4\pi R_s^2 p_{\text{II}} - GM_s^2 / 2R_s^2,$$  

(5)

where the second term on the right-hand side represents the self-gravitational force (Whitworth & Francis 2002). The factor of 1/2 in the self-gravity term arises because the gravitational acceleration vanishes at the inner surface; it is $GM_s / R_s^2$ at the outer surface, and the mass-weighted average across the thickness is $GM_s / 2R_s^2$. One can see that self-gravity slows the expansion in Equation (5).

In this paper, for convenience, the units of the time, length, and mass scales are taken to be

$$t_0 = \left( \frac{3\pi}{32G\rho_{E}} \right)^{1/2} = 1.6 n_{E,3}^{-1/2} \text{Myr},$$  

(6)

$$R_0 = \left( \frac{7c_{\text{II}} t_0}{\sqrt{12}} \right)^{4/7} R_{\text{ST}}^{3/7} = 5.9 Q_{\text{UV},49}^{1/7} n_{E,3}^{-4/7} \text{pc},$$  

(7)

and

$$M_0 = \rho_{E} R_0^3 = 5.0 \times 10^3 Q_{\text{UV},49}^{3/7} n_{E,3}^{-5/7} M_\odot,$$  

(8)

respectively, where $Q_{\text{UV},49} = Q_{\text{UV}} / 10^{49} \text{s}^{-1}$ and $n_{E,3} = n_e / 10^3 \text{cm}^{-3}$.

Non-dimensional quantities normalized by $t_0$, $R_0$, and $M_0$ are expressed by using tilde, e.g., $\tilde{R}_s = R_s / R_0$. Using non-dimensional quantities, we can rewrite Equations (3) and (5) as

$$\frac{d}{dt} \left( \tilde{R}_s \frac{d\tilde{R}_s}{dt} \right) = \left( \frac{6}{7} \right)^2 \tilde{R}_s^{1/2} - \frac{\pi^2}{16} \tilde{R}_s^4,$$  

(9)

respectively. We integrate Equation (10) with respect to time with the initial condition $\tilde{R} = \tilde{R}_{\text{ST}}$ at $\tilde{t} = 0$. The initial velocity $d\tilde{R}_{\text{ST}} / d\tilde{t}$ is obtained from Equation (4) with $\tilde{t} = 0$. Figure 1 shows the obtained expansion law with various parameters, $(n_e / \text{cm}^{-3}, Q_{\text{UV}} / \text{s}^{-1}) = (10^3, 10^{49}), (10^2, 10^{49}), (10^3, 10^{48}), (10^4, 10^{48}), (10^3, 10^{48}),$ and $(10^4, 10^{45})$. The difference of these parameter values gives the different values of $\tilde{R}_s$ at $\tilde{t} = 0$ as shown in Figure 1. In Figure 1, it is seen that as the shell expands, the lines quickly approach an asymptotic line that is independent of the parameters. Therefore, the dependence of the expansion law on the parameters is approximately eliminated by using the non-dimensional quantities.

Whitworth & Francis (2002) derived similar thin-shell equations for the shells driven by steady stellar winds. They found a change in the power-law index (from $3/5$ to $1/5$) at the time when self-gravity starts to become important. In the gravity-dominated phase, the shell expands, keeping the force balance between the thermal pressure of the hot bubble and the self-gravity. In the stellar wind case, the steady energy input allows the outward expansion of the shell ($\propto R_s^{4/5}$) even when self-gravity becomes important. On the other hand, in the case with the H II regions, from Equation (10), the gravitational force ($\propto R_s^4$) increases more rapidly than the pressure force by the H II region ($\propto R_s^{4/5}$), suggesting that the shell begins to collapse toward the center at a certain radius. In the numerical calculation, it occurs at $R_s \sim 2.3$ in all parameters. The last term of Equation (10) is valid until only the expansion phase. In reality, besides ionizing photons, the massive star emits strong stellar wind continuously over several tens of million years and...
we obtain through a supernova explosion (Weaver et al. 1977). They may influence the dynamics of the shell in the self-gravity-dominated phase. In this paper, for simplicity, we focus on the expansion phase by the ionizing photons.

3. TIME EVOLUTION OF DENSITY PROFILES: UNPERTURBED STATE

In this section, we derive the time evolution of the density profile of the shell in a semi-analytic way. We assume that the shell is in instantaneous hydrostatic equilibrium at each instant of time. This is a reasonable assumption because the shell is very thin and the sound-crossing time across the thickness is very short compared with the expansion timescale. The equation of the hydrostatic equilibrium in the frame of the shell is given by

\[ \frac{c_s^2}{\rho} \frac{d\rho}{dr} - \frac{d\Phi}{dr} + g_{\text{dec}} = 0, \]

where \( g_{\text{dec}} = -d^2R_c/dt^2 \) is the inertial force owing to the deceleration of the shell, and is assumed to be spatially constant within the shell. In the decelerating shell, the inertia force is parallel to the radial direction. The Poisson equation is given by

\[ \frac{d^2\Phi}{dr^2} + \frac{2}{r} \frac{d\Phi}{dr} \simeq \frac{d^2\phi}{dr^2} = 4\pi G \rho, \]

where the curvature effect is neglected because \( R_c \) is much larger than the thickness. We confirmed that the curvature effect is negligible by comparing density profiles with and without the curvature effect. Substituting Equation (11) into Equation (12), one obtains

\[ \frac{d}{dr} \left( \frac{c_s^2}{\rho} \frac{d\rho}{dr} \right) = -4\pi G \rho. \]

Equation (13) can be solved analytically as follows:

\[ \rho(r) = \rho_0 \left\{ \cosh \left( \frac{r - R_c}{H_0} \right) \right\}^{-2}, \]

where \( R_c \) and \( \rho_0 \) are the radius and the density and where \( dp/dr = 0 \) (cf. Spitzer 1942), and \( H_0 \equiv c_s/\sqrt{2\pi G \rho_0} \) is the scale height.

From Equation (14), if we determine \( \rho_0 \) and \( R_c \), the density profile is completely specified except for the boundaries that are discussed later. Here, the value of \( R_c \) itself loses its physical meaning since the curvature is neglected. Therefore, only \( \rho_0 \) specifies the density profile. The peak density \( \rho_0 \) is determined by the condition of the force balance at \( r = R_{\text{CD}} \).

The gravitational force must vanish at \( r = R_{\text{CD}} \) because the total mass of the hot bubble is negligible. Therefore, from Equation (11), the inner boundary conditions are given by

\[ \frac{c_s^2}{\rho} \frac{d\rho}{dr} \bigg|_{r = R_{\text{CD}}} = g_{\text{dec}}. \]

The column density from \( R_{\text{CD}} \) to \( R_c \) is

\[ \Sigma_{\text{dec}} = \int_{R_{\text{CD}}}^{R_c} \rho dr = \rho_0 H_0 \tanh \left( \frac{R_c - R_{\text{CD}}}{H_0} \right) = \frac{g_{\text{dec}}}{4\pi G}, \]

where we use Equation (15) in the last equality. The characteristic column density \( \Sigma_{\text{dec}} \) represents the amount of the deceleration. The ratio of the column density \( \Sigma_\ast \) to \( \Sigma_{\text{dec}} \) determines the importance of self-gravity relative to deceleration.

From Equation (16) and the pressure equilibrium at the CD, \( \rho(R_{\text{CD}}) \Sigma_{\text{dec}}^2 = P_\text{II}, \) the peak density can be expressed by \( \Sigma_{\text{dec}} \) and \( P_\text{II} \) as follows:

\[ \rho_0 c_s^2 = P_\text{II} + 2\pi G \Sigma_{\text{dec}}. \]

Substituting Equation (16) into Equation (17), one obtains

\[ \rho_0 c_s^2 = P_\text{II} + \frac{g_{\text{dec}}^2}{8\pi G}. \]

The peak density \( \rho_0 \) is determined in the following way. We use the thin-shell model shown in Section 2 to get \( d^2 R_c/dt^2 = -g_{\text{dec}} \) and \( R_c \) at any given times. The pressure of the H II region \( P_\text{II} \) is given by Equation (3). Substituting the obtained \( g_{\text{dec}} \) and \( P_\text{II} \) into Equation (18), we can get \( \rho_0 \) and can specify the functional form of the density profile.

Next, we determine the positions of boundaries, both the CD and the SF. As mentioned above, only the distance relative to \( R_c \) has a physical meaning. The position of the CD and the SF is determined from the pressure balances on both sides which are given by

\[ c_s^2 \rho(R_{\text{CD}}) = P_\text{II} \quad \text{and} \quad c_s^2 \rho(R_{\text{SF}}) = p_\text{II} \left( \frac{dR_c}{dt} \right)^2, \]

respectively, where \( dR_c/dt \) is obtained from the thin-shell model.

3.1. Three Evolutionary Phases of Density Profiles

The time evolution of the density profile is characterized by the ratio of the column density \( \Sigma_\ast \) to the characteristic column density \( \Sigma_{\text{dec}} \). Here, the column density \( \Sigma_\ast \) is given by the thin-shell approximation (see Section 2). The column density \( \Sigma_{\ast} = \rho_0 R_c/3 \) can be derived from the mass conservation, since \( M_\ast = 4\pi G R_c^2 \Sigma_{\ast} \) (see Section 2). The time evolution of the density profile is roughly divided into the following three phases: deceleration-dominated phase (\( \Sigma_\ast < \Sigma_{\text{dec}} \)), intermediate phase (\( \Sigma_{\text{dec}} < \Sigma_\ast < 2\Sigma_{\text{dec}} \)), and self-gravity-dominated phase \( (2\Sigma_{\text{dec}} < \Sigma_\ast) \), depending on the value of \( \Sigma_\ast/\Sigma_{\text{dec}} \). The schematic pictures of the density profiles in these three phases are shown in Figure 2. Figure 3 shows the time evolution of \( \Sigma_\ast/\Sigma_{\text{dec}} \). In the early deceleration-dominated phase, \( R_c \) is outside of the shell and it is in front of the SF, or \( R_{\text{SF}} < R_c \) (see Figure 2(a)). This means that the actual density peak exists at \( R_{\text{SF}} \). As the shell expands, \( \Sigma_\ast \) increases by accretion while \( \Sigma_{\text{dec}} \) decreases by deceleration. In Figure 3, it is seen that \( \Sigma_\ast \) becomes larger than \( \Sigma_{\text{dec}} \) at \( t/t_0 \sim 0.44 \). When \( \Sigma_\ast > \Sigma_{\text{dec}} \), \( R_c \) inside the shell. In the intermediate phase \( (\Sigma_{\text{dec}} < \Sigma_\ast < 2\Sigma_{\text{dec}}) \), \( R_c \) is closer to \( R_{\text{SF}} \) than \( R_{\text{CD}} \) as shown in Figure 2(b). When \( \Sigma_\ast \) becomes larger than \( 2\Sigma_{\text{dec}} \) \( (t/t_0 > 0.57) \), \( R_c \) becomes closer to \( R_{\text{CD}} \) than \( R_{\text{SF}} \) (see Figure 2(c)). Since the period of the intermediate phase is relatively short, roughly speaking, the density profile transforms from the deceleration- to the self-gravity-dominated profiles around \( t/t_0 \sim 0.5 \).

3.2. Comparison with One-dimensional Simulation

The density profile obtained by the above semi-analytic method is compared with the results of 1D simulation. We use the 1D spherically symmetric Lagrangian Godunov method (van Leer 1997). We do not calculate the radiative transfer of ionizing photons and ionized gas, but the cold gas is pushed by an interior pressure whose value is given by Equation (3). The equation of state is assumed to be isothermal. We calculate the expanding...
istic column density $\Sigma$ is proportional to the reciprocal of the typical Mach number $M$.

Figure 2. Schematic pictures of the density profiles of the shell in the (a) deceleration-dominated phase ($\Sigma_0 < \Sigma_{dec}$), (b) intermediate phase ($\Sigma_{dec} < \Sigma < 2\Sigma_{dec}$), and (c) self-gravity-dominated phase ($2\Sigma_{dec} < \Sigma$).

Figure 3. Time evolution of the ratio of the column density $\Sigma$ to the characteristic column density $\Sigma_{dec}$ (Equation (16)). Evolutionary phases corresponding to Figure 2 are labeled.

Figure 4. Snapshots of density profiles for $t/t_0 = 0.5, 0.7, 1.0,\text{ and } 1.26$. The abscissae are the distances from the CD. The thick gray lines represent the results of the 1D simulation. The dashed lines in the upper panel indicate the results of the hydrostatic density profiles. The black solid lines in the upper panel indicate the instantaneous hydrostatic density profiles.

shell around the 41 $M_\odot$ star that is embedded by the uniform ambient gas of $n_E = 10^3$ cm$^{-3}$.

Figure 4 shows the snapshots of density profiles for $t/t_0 = 0.5, 0.7, 1.0,\text{ and } 1.26$. The thick gray lines represent the results of the 1D calculation. The dashed lines show the density profiles obtained from the semi-analytic method. It is seen that the semi-analytic method describes the density profile of the 1D simulation reasonably well. The density profile in the semi-analytic method is slightly lower than the results of the 1D calculation because the actual CD expands a little slower than the mean radius of the shell $R_0$ in the thin-shell approximation. As shown in Section 3.1, one can see that the density peak moves the CD from the SF owing to self-gravity.

3.3. Scaling Law of Unperturbed Density Profiles

As shown in Section 2, the non-dimensional position of the shell, $R_\ast$, is approximately independent of the model parameters $(n_E, Q_{UV})$. Similarly, it is useful to investigate how the density profile depends on the above parameters. The non-dimensional pressures at the CD and the SF are given by Equation (9) and $\bar{P}_s^2$, respectively, that is, they are independent of the parameters. Moreover, the pressure at the density peak $\bar{P}_00 = \bar{\rho}_00 \bar{c}_s^2$ is also independent of the parameters as seen in Equation (18). Thus, noting that the non-dimensional sound speed $\bar{c}_s = c_s t_0 / R_0$ is proportional to the reciprocal of the typical Mach number $\mathcal{M}_0 = 4R_0/(7t_0c_s)$, where the factor of 4/7 arises from Equation (4), we have the scaling laws of $\bar{H}_0$, $\bar{\rho}_00$, and $\bar{t}_f$ given by

$$\bar{H}_0 \propto \bar{c}_s^2 \bar{P}_00^{-1/2} \propto \mathcal{M}_0^{-2},$$

$$\bar{\rho}_00 \propto \bar{c}_s^{-2} \bar{P}_00 \propto \mathcal{M}_0^3,$$

and

$$\bar{t}_f \propto \bar{P}_00^{-1/2} \propto \bar{c}_s \bar{P}_00^{-1/2} \propto \mathcal{M}_0^{-1},$$

respectively, where $t_0 \equiv 1/\sqrt{2\pi G\bar{\rho}_00}$ is the free-fall timescale of the shell. As a result, it is found that the density profiles for various sets of $(n_E, Q_{UV})$ are characterized by a single parameter, that is, the typical Mach number

$$\mathcal{M}_0 = \frac{4 R_0}{7 c_s t_0} \approx 7 \frac{Q_{UV,49}^{1/7}}{T_{c,10}^{1/14}} n_{E,3}^{-1/4},$$

where $T_{c,10} = T_c / 10$ K.

4. INFLUENCE OF ASYMMETRIC DENSITY PROFILE ON GRAVITATIONAL INSTABILITY

As shown in Section 3, the expanding shell has a highly asymmetric density profile and it is expected to influence the GI. In this section, we investigate the influences of the asymmetric density profile on the dispersion relation of the GI. What we discuss here is the extension of the classical stability analysis of the GI in the symmetric layer with respect to the mid-plane (Goldreich & Lynden-Bell 1965; Elmegreen & Elmegreen 1978; Lubow & Pringle 1993) to the GI in the asymmetric layer. The
linear analysis in the incompressible limit has been investigated by Voit (1988).

We take \( z \)-axis parallel to the thickness of the layer, and take the \( x \)-axis as the transverse direction. The density is assumed to peak at \( z = 0 \), and the positions of the boundaries are \( z_1 \) and \( z_2 \) (\( z_1 < z_2 \)). We consider a layer that is subject to a constant deceleration. The deceleration arises from the difference of pressures at the boundaries (\( z = z_1 \) and \( z_2 \)). In this case, the position of the density peak is not in the mid-plane of the layer, the density profile is asymmetric, or \( -z_1 \neq z_2 \). The amount of deceleration directly enhances the degree of asymmetry of the density profile.

4.1. Perturbation Equations

We consider the following perturbations:

\[
\rho(z, x, t) = \rho_0(z) + \delta \rho(z) e^{i(kx-\omega t)},
\]

\[
v_z(z, x, t) = v_z(z) e^{i(kx-\omega t)},
\]

\[
v_x(z, x, t) = v_x(z) e^{i(kx-\omega t)},
\]

\[
\phi(z, x, t) = \phi_0(z) + \delta \phi(z) e^{i(kx-\omega t)}.
\]

The perturbation equations are

\[
-i\omega \delta \rho + \frac{d(\rho_0 v_z)}{dz} + \rho_0 i k v_x = 0, \quad (25)
\]

\[
i\omega v_z = \frac{d}{dz} \left( c_s^2 \frac{\delta \rho}{\rho_0} + \delta \phi \right), \quad (26)
\]

\[
\omega v_x = k \left( c_s^2 \frac{\delta \rho}{\rho_0} + \delta \phi \right), \quad (27)
\]

and

\[
\frac{d^2 \delta \phi}{dz^2} - k^2 \delta \phi = 4\pi G \rho_0, \quad (28)
\]

where the sound speed is assumed to be constant.

4.1.1. Boundary Conditions

To concentrate on the effect of asymmetry of the unperturbed state, we impose the CD boundary conditions at both \( z_1 \) and \( z_2 \) (Goldreich & Lynden-Bell 1965; Elmegreen & Elmegreen 1978) as follows:

\[
\delta \rho(z_1) = \frac{d\rho_0}{dz} \bigg|_{z=z_1}, \quad v_z(z_1) = -i\omega \delta z_1, \quad (29)
\]
the distribution of the Lagrangian density perturbation $\Delta \rho \equiv \delta \rho / \rho_0 \equiv \delta \rho / \rho_0 + v_z d \ln \rho_0 / dz$ for $k H_0 \approx 3.5$ in the even mode. Figures 6(a) and (b) correspond to the P and the SG modes, respectively. The normalization is determined by $|\delta z_2| / z_2 = 0.1$. One can see that the $\Delta \rho$ profiles of the P and SG modes are quite different. In the P mode, $\Delta \rho / \rho_0$ peaks at the mid-plane. The displacement $|\delta z_2| / z_2$ is negligible compared with $\Delta \rho / \rho_0$. The P mode propagates as a longitudinal variation of pressure. Since the layer is no longer symmetric with respect to $z = 0$, perturbations cannot be divided into the even and the odd modes. Figure 7(a) shows the dispersion relation for $k H_0 \approx 2.4$ through the mode exchange. The case with stronger asymmetry with $z_1 = -H_0$ is shown in Figure 7(b). In this case, the difference of the angular frequencies between the two SG modes is larger because the surface gravity at $z_1$ is lower. The frequency of the SG mode associated with $z_1$ becomes smaller than that with $z_2$. Comparing with Figure 7(a), the wavenumber of the mode exchange, as well as the frequency, is smaller. As a result, the frequency range of the P mode is narrower and the range of the SG mode spreads. The P mode $\omega^2 \propto k^2$ is expected to disappear when the wavenumber of the mode exchange is smaller than a critical wavenumber that separates the unstable mode from the stable mode.

Since the layer is no longer symmetric with respect to $z = 0$, perturbations cannot be divided into the even and the odd modes. Figure 7(a) shows the dispersion relation for $z_1 = -H_0$. One can see more complex structure of the mode exchanges around $k H_0 \approx 2.4$ than in Figure 5. For large wavenumbers, the angular frequencies of the two SG modes split because $\omega_{\text{SG},z_1} \approx \omega_{\text{SG},z_2}$. Figure 8(a) shows the cross section of the layer ($z_1 = -H_0$) in the fastest growing mode. The contour indicates the density perturbation normalized by $\rho_0$. Here, we take $\delta \rho_{\text{max}} / \rho_0 = 0.2$ to specify the normalization of the perturbations. The arrows represent the velocity vectors. The boundary surfaces hardly deform, and the gas collapses from all directions to the center ($z = 0, x = 0$). This behavior corresponds to the compressible mode. In Figure 7(a), the unstable branch transforms the P mode around $k H_0 \approx 1$ and it is connected with the SG mode around $k H_0 \approx 2.4$ through the mode exchange. The case with stronger asymmetry with $z_1 = -H_0$ is shown in Figure 7(b). In this case, the difference of the angular frequencies between the two SG modes is larger because the surface gravity at $z_1$ is lower. The frequency of the SG mode associated with $z_1$ becomes smaller than that with $z_2$. Comparing with Figure 7(a), the wavenumber of the mode exchange, as well as the frequency, is smaller. As a result, the frequency range of the P mode is narrower and the range of the SG mode spreads. The P mode $\omega^2 \propto k^2$ is expected to disappear when the wavenumber of the mode exchange is smaller than a critical wavenumber that separates the unstable mode from the stable mode.

Since the layer is no longer symmetric with respect to $z = 0$, perturbations cannot be divided into the even and the odd modes. Figure 7(a) shows the dispersion relation for $z_1 = -H_0$. One can see more complex structure of the mode exchanges around $k H_0 \approx 2.4$ than that in Figure 5. For large wavenumber, the angular frequencies of the two SG modes split because $\omega_{\text{SG},z_1} \approx \omega_{\text{SG},z_2}$. Figure 8(a) shows the cross section of the layer ($z_1 = -H_0$) in the fastest growing mode. The contour indicates the density perturbation normalized by $\rho_0$. Here, we take $\delta \rho_{\text{max}} / \rho_0 = 0.2$ to specify the normalization of the perturbations. The arrows represent the velocity vectors. The boundary surfaces hardly deform, and the gas collapses from all directions to the center ($z = 0, x = 0$). This behavior corresponds to the compressible mode. In Figure 7(a), the unstable branch transforms the P mode around $k H_0 \approx 1$ and it is connected with the SG mode around $k H_0 \approx 2.4$ through the mode exchange. The case with stronger asymmetry with $z_1 = -H_0$ is shown in Figure 7(b). In this case, the difference of the angular frequencies between the two SG modes is larger because the surface gravity at $z_1$ is lower. The frequency of the SG mode associated with $z_1$ becomes smaller than that with $z_2$. Comparing with Figure 7(a), the wavenumber of the mode exchange, as well as the frequency, is smaller. As a result, the frequency range of the P mode is narrower and the range of the SG mode spreads. The P mode $\omega^2 \propto k^2$ is expected to disappear when the wavenumber of the mode exchange is smaller than a critical wavenumber that separates the unstable mode from the stable mode.

The case with $z_1 < 0$ is quite different from the case with $z_1 \geq H_0$. Figure 7(a) shows the dispersion relation for $z_1 = 0.3 H_0$. In Figure 7(c), one can see that the angular frequency of the SG mode $\omega_{\text{SG},z_1}$ is significantly lower than $\omega_{\text{SG},z_2}$ for large wavenumbers. Unlike the case with $z_1 > H_0$, the unstable mode appears to directly connect with the SG mode around $k H_0 \approx 1.2$ as mentioned above. Figure 8(b) shows the cross section of the layer ($z_1 = -0.3 H_0$) in the fastest growing mode. The gas tends to collect toward the density peak $z = 0$ because the unperturbed gravitational potential has the minimum value there. One can see that eigenfunctions in $z > 0$ are similar to those in Figure 8(a). The gas collapses toward the center ($z = 0, x = 0$), leaving behind the gas around $z_2$. However, in the region where $z < 0$, eigenfunctions are quite different. The sound wave can travel between $z_1$ and the density peak many times during the development of the GI. Therefore, collapse toward $z = 0$ is suppressed by the pressure gradient. However, the GI can proceed even in $z < 0$ through

4.3. Asymmetric Layer

In this section, we investigate the dependence of the dispersion relation on the degree of the asymmetry by changing $z_1$. Since the layer is no longer symmetric with respect to $z = 0$, perturbations cannot be divided into the even and the odd modes. Figure 7(a) shows the dispersion relation for $z_1 = -H_0$. One can see more complex structure of the mode exchanges around $k H_0 \approx 2.4$ than that in Figure 5. For large wavenumber, the angular frequencies of the two SG modes split because $\omega_{\text{SG},z_1} \approx \omega_{\text{SG},z_2}$. Figure 8(a) shows the cross section of the layer ($z_1 = -H_0$) in the fastest growing mode. The contour indicates the density perturbation normalized by $\rho_0$. Here, we take $\delta \rho_{\text{max}} / \rho_0 = 0.2$ to specify the normalization of the perturbations. The arrows represent the velocity vectors. The boundary surfaces hardly deform, and the gas collapses from all directions to the center ($z = 0, x = 0$). This behavior corresponds to the compressible mode. In Figure 7(a), the unstable branch transforms the P mode around $k H_0 \approx 1$ and it is connected with the SG mode around $k H_0 \approx 2.4$ through the mode exchange. The case with stronger asymmetry with $z_1 = -H_0$ is shown in Figure 7(b). In this case, the difference of the angular frequencies between the two SG modes is larger because the surface gravity at $z_1$ is lower. The frequency of the SG mode associated with $z_1$ becomes smaller than that with $z_2$. Comparing with Figure 7(a), the wavenumber of the mode exchange, as well as the frequency, is smaller. As a result, the frequency range of the P mode is narrower and the range of the SG mode spreads. The P mode $\omega^2 \propto k^2$ is expected to disappear when the wavenumber of the mode exchange is smaller than a critical wavenumber that separates the unstable mode from the stable mode.

The case with $z_1 < 0$ is quite different from the case with $z_1 \geq H_0$. Figure 7(a) shows the dispersion relation for $z_1 = 0.3 H_0$. In Figure 7(c), one can see that the angular frequency of the SG mode $\omega_{\text{SG},z_1}$ is significantly lower than $\omega_{\text{SG},z_2}$ for large wavenumbers. Unlike the case with $z_1 > H_0$, the unstable mode appears to directly connect with the SG mode around $k H_0 \approx 1.2$ as mentioned above. Figure 8(b) shows the cross section of the layer ($z_1 = -0.3 H_0$) in the fastest growing mode. The gas tends to collect toward the density peak $z = 0$ because the unperturbed gravitational potential has the minimum value there. One can see that eigenfunctions in $z > 0$ are similar to those in Figure 8(a). The gas collapses toward the center ($z = 0, x = 0$), leaving behind the gas around $z_2$. However, in the region where $z < 0$, eigenfunctions are quite different. The sound wave can travel between $z_1$ and the density peak many times during the development of the GI. Therefore, collapse toward $z = 0$ is suppressed by the pressure gradient. However, the GI can proceed even in $z < 0$ through
Figure 7. Dispersion relations for $-z_1/H_0 = (a) 2.0, (b) 1.0, (c) 0.3,$ and (d) 0.0. The ordinate and the abscissa are the same as Figure 5.

Figure 8. Cross sections of the layers with the fastest growing mode for (a) $-z_1 = 2.0H_0$ and (b) $-z_1 = 0.3H_0$. The contour indicates the density perturbation normalized by $\rho_{00}$. The contour levels of the density perturbation take values of 0.04, 0.08, and 0.16.

Figure 9. Dispersion relation of the asymmetric layer for $-z_1/H_0 = 2$ (the thick solid line), 1 (the thick dashed line), 0.3 (the thick dotted line), 0.1 (the thin solid line), 0 (the thin dashed line), and $-0.1$ (the thin dotted line), where $z_2/H_0$ is assumed to be 3.0.

The deformation of the $z_1$ that makes the gravitational potential deeper. From Figure 8(b), one can see that the velocity field is not headed for the density peak ($z = 0$), but arises so that the surface at $z_1$ deforms. Therefore, the features of GI in the region $z > 0$ and $z < 0$ have properties of the “compressible mode” and “incompressible mode,” respectively. If the distance of $z_1$ from the density peak is zero, the layer is unstable for all wavenumbers (see Figure 7(d)).

Figure 9 shows the dispersion relation of the unstable mode for a variety of $z_1$ with $z_2 = 3H_0$. The thick solid and the thick dashed lines correspond to $-z_1/H_0 = 2$ and 1, respectively. For $-z_1/H_0 \geq 1$, the growth rate $-\omega^2/2\pi G \rho_{00}$ decreases as $-z_1/H_0$ decreases. This property is the same as that in symmetric layers (e.g., see Figure 1 in Nagai et al. 1998). On the other hand, for the cases with $-z_1/H_0 = 0.3$ and 0.1, Figure 9 shows that the maximum growth rate increases as $-z_1/H_0$ decreases, indicating the opposite tendency to the case with $-z_1/H_0 > 1$. On the other hand, the wavenumber of the most unstable mode is not much different. One can see that the square growth rate in large wavenumbers is proportional to $\propto k$, while the square growth rate for $-z_1/H_0 > 1$ is proportional to $\propto k^2$. The unstable mode appears to directly
connect with the surface-gravity mode at $z_1$ whose frequency is given by Equation (33). This is also seen in Figure 10 for $k > k_{\text{max}}$. For $-z_1/H_0 = 0$, the surface-gravity mode at $z_1$ becomes unstable for all wavenumbers. In this case, the destabilized surface-gravity wave has the growth rate $\omega_{SG,z_1} = -2\pi G\Sigma(z_1) / c_s^2 < 0$ independent of $k$ for a large wavenumber limit (see Equation (33)). This is the case with a static shell with $\beta_{\text{dec}} = 0$ (Equation (11)). Tomisaka & Ikeuchi (1983) investigated this situation including shell curvature and found that the shell is unstable for all wavenumbers (also see Welter & Schmid-Burgk 1981). For $-z_1/H_0 = -0.1$, the square growth rate increases as $\propto k$ with wavenumbers because $\omega_{SG,z_1} \sim 1/\sqrt{\pi G\Sigma(z_1)} k < 0$ (Equation (11)). This is a well-known scaling law of the Rayleigh–Taylor instability. The enhancement of the growth rate for $-z_1/H_0 < 1$ arises from the combination of the GI and the Rayleigh–Taylor instability.

5. GRAVITATIONAL INSTABILITY OF EXPANDING SHELLS

In the previous section, we focus on the effect of asymmetry of the density profile by imposing the same boundary conditions in both boundaries. In this section, in a more realistic situation, we investigate the stability of expanding shells driven by the expansion of the H II region. The unperturbed density profile at each instant of time is given by the semi-analytic method presented in Section 3. We neglect the curvature effect and solve the perturbation Equations (25)–(28), but $z \to r$. In this case, as well as in the asymmetric density profile, the difference of boundary properties between leading (the SF) and trailing (the CD) surfaces plays important roles in the GI.

5.1. Influences of Boundaries on the Gravitational Instability

Before presenting a linear analysis, we review how the SF and the CD influence the GI through the boundary effect. This point is important in understanding the results of the linear analysis. In the early phase, the shell is highly confined by the ram pressure on the leading surface and by the thermal pressure on the trailing surface. In this phase, the pressure at the boundaries is as large as that at the density maximum, and the thickness of the shell is much smaller than the scale height, $H_0 = c_s/\sqrt{2\pi G\rho_0}$, where $\rho_0$ is the maximum density. Thus, in this phase, the boundary effect can strongly influence the GI. In the later phase, the boundary effect of the CD is expected to be also important because the density peak is close to the CD as shown in Section 3.1. In this section, we summarize how the growth rate of GI is controlled by the different boundary conditions. For simplicity, in Section 5.1, the layer is assumed to be symmetric with respect to the mid-plane, and physical variables are averaged across the thickness.

5.1.1. Shock-confined Layer

Many authors have investigated influences of the SF on the GI (Vishniac 1983, 1994; Elmegreen 1989, 1994; Nishi 1992; Whitworth et al. 1994a; Iwasaki & Tsuribe 2008). The dispersion relation of the shock-confined layer is given by

$$\omega^2 \approx \frac{c_s^2 k^2}{L_s} - 2\pi G\Sigma,$$

(35)

where $\Sigma$ is the column density and $k$ is the transverse wavenumber of the perturbation. This dispersion relation is the same as that for the infinitesimally thin layer. In the highly confined layer, it is well known that the perturbation behaves like the incompressible mode because the sound-crossing time over the thickness is much smaller than the free-fall timescale, $\sim 1/\sqrt{G\rho_0}$ (Elmegreen & Elmegreen 1978; Lubow & Pringle 1993). Therefore, density fluctuation is small. The layer becomes unstable mainly due to the deformation of the surfaces that makes the perturbation of the gravitational potential deeper. Hereafter, we call this mode the “incompressible mode.” The deformation of the SF generates the tangential flow carrying the gas from the convex to the concave regions (seen from the downstream). Therefore, the tangential flow tends to make the SF flat, suggesting that it suppresses the growth of the GI. In the shock-confined layer, the restoring term $c_s^2 k^2$ arises from the tangential flow behind the oblique SF. On the other hand, in the case of the infinitesimally thin layer, this term comes from the pressure gradient. Therefore, the origin of the restoring force is quite different. From Equation (35), the maximum growth rate is given by $\pi G\Sigma/c_s$, and the corresponding wavenumber is given by $\pi G\Sigma/c_s^2$. When $\Sigma$ is small ($\lesssim \rho_0 H_0$), the maximum growth rate is smaller than the inverse of the free-fall timescale, $\sqrt{G\rho_0}$, and the corresponding scale is larger than the scale height $H_0$ that is comparable to the Jeans scale.

5.1.2. Pressure-confined Layer

Next, we review the influence of the CD on the GI. We consider the layer confined by thermal pressure of hot rarefied gases (CD boundary condition) on both sides. The dispersion relation becomes

$$\omega^2 \approx 2\pi G\Sigma L_s k^2 - 2\pi Gk\Sigma,$$

(36)

where $L_s$ is the thickness of the layer, and we consider the large-scale limit where $k \ll 1/L_s$. A detailed derivation of
Equation (36) is shown in Appendix 3 of Iwasaki & Tsuribe (2008). In the pressure-confined layer, the stabilization effect of the tangential flow does not exist. Therefore, the restoring term in Equation (36) is quite different from that in Equation (35). Using the gravitational acceleration at the surfaces as |g| = 2πGΣs, we can express the restoring term in Equation (36) by |g|Ls/k2. Therefore, one can see that the restoring force arises from the surface-gravity wave. The layer with the CD boundary condition is less stabilized compared with the shock boundary condition because the phase velocity of the gravity wave  is much smaller than  when 0. From the dispersion relation (36), the maximum growth rate is comparable to the inverse of the free-fall time of the layer . The corresponding wavelength is about the thickness of the layer .

5.1.3. Expanding Shells

Equations (35) and (36) cannot be applied directly to the GI of the expanding shells because the GI is expected to be stabilized by evolutionary effects, such as the expansion of the shell and the accretion of fresh gas through the SF. Elmegreen (1994) derived the following approximate dispersion relation:

\[ i\omega = \frac{3V_s}{R_s} + \sqrt{\left(\frac{V_s}{R_s}\right)^2 + 2\pi G k \Sigma_s - c_s^2 k^2}. \]  

(37)

The terms with \( V_s/R_s \) come from evolutionary effects that stabilize the GI.

One can see that Equation (37) for the limit of \( V_s/R_s \to 0 \) is the same as Equation (35). Therefore, Elmegreen (1994) and Whitworth et al. (1994b) essentially applied Equation (35) in the context of the GI of the expanding shell. However, they did not take into account the boundary effect of the CD on the trailing surface. Comparing Equations (35) and (36), we suggest that the stability of the thin shell neglecting the effect of the CD suffers due to large stabilizing effects, and it will underestimate the growth rate of GI in expanding shells.

5.2. Boundary Condition

First, we assume that a constant pressure is exerted on the CD all the time (the CD boundary condition; Goldreich & Lynden-Bell 1965; Elmegreen & Elmegreen 1978). The boundary conditions are

\[ \delta \rho(R_{CD}) = -\left. \frac{d\rho}{dr} \right|_{r=R_{CD}} \delta R_{CD}, \quad v_s(R_{CD}) = i\omega \delta R_{CD} \]  

(38)

and

\[ \frac{d\delta \phi}{dr} - k \delta \phi + 4\pi G \rho(R_{CD}) \delta R_{CD} = 0, \]  

(39)

where \( \delta R_{CD} \) is the displacement of the CD.

Next, let us consider the boundary conditions at \( r = R_{SF} \). Since the unperturbed state is assumed to be the hydrostatic configuration, it is impossible to impose the shock boundary conditions self-consistently. In order to treat it self-consistently, a time-dependent initial value problem needs to be solved (Welter 1982; Iwasaki & Tsuribe 2008). Therefore, in this paper, we mimic the shock boundary conditions by introducing the stabilization effect. We consider the following two approximate boundary conditions.

Rigid surface boundary condition (RSBC). Voit (1988) and Usami et al. (1995) assumed that no ripples arise on the surface, or \( \delta R_{SF} = 0 \), where \( \delta R_{SF} \) is the displacement of the SF. The reason why we adopt \( \delta R_{SF} = 0 \) is that the thin-shell linear analysis of the layer confined by rigid surfaces gives the same dispersion relation as that of the shock-confined layer (Equation (35)). More precisely, in the shock-confined layer, the tangential flow boosts the suppression effect against the self-gravity as mentioned in Section 5.1.1. Instead, the RSBC weakens the self-gravity.

Tangential flow boundary condition (TSBC). If the SF is rippled, tangential flow behind the SF is generated. Therefore, we set the tangential velocity \( v_s \) at \( r = R_{SF} \). Linearizing the Rankine–Hugoniot relation, we have

\[ v_s(R_{SF}) = -\left( R_{SF} - \frac{c_s^2}{R_{SF}} \right) i k \delta R_{SF}. \]  

(40)

The detailed derivation of Equation (40) is found in Iwasaki & Tsuribe (2008).

With both of the above boundary conditions (RSBC and TFBC), we also impose the following commonly used boundary conditions:

\[ v_s(R_{SF}) = i\omega \delta R_{SF} \]  

(41)

and

\[ \frac{d\delta \phi}{dr} + k \delta \phi + 4\pi G \rho(R_{SF}) \delta R_{SF} = 0. \]  

(42)

It is well known that the SF of the deceleration shell is subject to hydrodynamical overstability (Vishniac 1983). The linear analysis in this paper cannot capture the Vishniac instability (VI) correctly since the approximate shock boundary conditions are imposed. The effect of the VI is discussed in Section 6.

The numerical method is the same as that in Section 4.1.2.

5.3. Scaling Law of Dispersion Relations

As shown in Section 3.3, it is found that the density profiles are characterized by a single parameter \( M_0 \). This is because the scale height, the peak density, and the free-fall time have the scaling laws with respect to \( M_0 \) as shown in Equations (20)–(22). The same is the case with the perturbation equations and the dispersion relation. The non-dimensional maximum growth rate is and the corresponding wavenumber scale as and respectively. Therefore, in the present model, the evolution of the shell for various sets of \( (n_G, Q_{UV}) \) can be described by a single unperturbed profile and a single time-dependent dispersion relation that are normalized by \( H_0, \rho_0, \) and \( t_f \). The result can be applicable to a wide range of parameters simply by using the scaling relation on \( M_0 \).

5.4. Results

At any time, the unperturbed state is given by the procedure in Section 3. Perturbation Equations (25)–(28) are solved as the eigenvalue and boundary-value problem. As a result, the growth rate, \( \omega(k, t) \), can be obtained as a function of the wavenumber and time.

First, we present the results of the linear analysis in Figure 10 at various epochs. The ordinate and the abscissa axes represent the non-dimensional growth rate and wavenumber \( k H_0/2\pi \). The solid and the dotted lines indicate the results of the linear
analysis using RSBC and TFBC, respectively. We refer the growth rates obtained by using RSBC and TFBC to \( \omega_{\text{RSBC}} \) and \( \omega_{\text{TFBC}} \), respectively. The dependence of the dispersion relation on the parameters (\( n_0 \), \( Q_{\text{UV}} \)) can be eliminated by using non-dimensional growth rate \( \omega t_0 \) and wavenumber \( k H_0 \) as shown in Section 5.3. We have confirmed that the dispersion relation is identical to that of other parameter sets of \((n_0, Q_{\text{UV}})\) by using non-dimensional quantities. Figure 10 shows that the difference between \( \omega_{\text{RSBC}} \) and \( \omega_{\text{TFBC}} \) is negligible although RSBC and TFBC are physically quite different.

In this analysis, we do not take into account evolutionary effects, such as the expansion and accretion of the gas. Therefore, we compare the results of the linear analysis with the dispersion relation of the shock-confined layer (Equation (35)) rather than that of the expanding shell (Equation (37)). One can see that the growth rate is larger than the prediction from the shock-confined layer. As shown in Section 5.1, this difference comes from the gravity-dominated phase (\( t/t_0 > 0.5 \)). This is because \( \omega_{\text{mod}} \) connects with the P mode at \( k^2 \), while \( \omega_{\text{RSBC}} \) connects with the SG mode at \( k \) as shown in Section 4.

The predicted cross section of the shell from the linear analysis for \((Q_{\text{UV}} = 10^{48.78} \text{ s}^{-1}, n_0 = 10^3 \text{ cm}^{-3})\) is shown in Figure 12 using the eigenfunctions. The corresponding time is \( t/t_0 = 1.3 \) and the angular wavenumber is \( l = 52 \). In Figure 12, the gas tends to accumulate onto the peak only through the upper half region \( r > R_c \). This property of the flow can be seen from the direction of arrows in Figure 12. Actually, in the upper-half region, we find \( R_{\text{SF}} - R_c = 1.05 H_0 > H_0 \), which represents how the gas can collapse to the peak because the sound wave cannot travel from \( R_c \) to \( R_{\text{SF}} \) within the free-fall time. On the other hand, in the bottom-half region \( r < R_c \), we find that \( R_c - R_{\text{CD}} = 0.285 H_0 < H_0 \). This indicates that the gas in \( r < R_c \) cannot collapse to the peak because the sound wave can travel from \( R_c \) to \( R_{\text{CD}} \) many times within the free-fall time. Thus, the pressure gradient prevents the compression of gas in the region \( r > R_c \). However, the GI can proceed even in \( r < R_c \) through the deformation of the CD that makes the gravitational potential deeper. Therefore, the features of GI in the region \( r > R_c \) and \( r < R_c \) have the properties of the “compressible mode” and “incompressible mode,” respectively.

6. DISCUSSION

The gravitational fragmentation of expanding shells confined on both sides by the CD was investigated by Dale et al. (2009) numerically and by Wünsch et al. (2010) using analytical approximations. They assumed that the thermal pressure on both sides is the same and temporally constant. Therefore, the density peak is always around the mid-plane of the shell, and the density profile is almost symmetric. In their calculation, the column density decreases with time because the shell expands keeping the mass fixed. Therefore, the pressures at the boundaries approach to the peak pressure. They found that the confining pressure accelerates fragmentation in the later phase and described this effect as “pressure-assisted” gravitational fragmentation. This mode is the same as the incompressible mode in this paper. Wünsch et al. (2010) established a semi-analytic linear analysis that explains the results of Dale et al. (2009).

The linear analysis in this paper cannot describe the VI correctly since the approximate shock boundary conditions are imposed in Section 5. The original analysis by Vishniac (1983) did not find the most unstable mode in the finite scale because the thickness of the shell is neglected. Vishniac & Ryu
\( \text{Figure 13. Growth rate of the VI when the shell expands as } \propto t^{3/7}. \text{ Each line corresponds to } M = 4.7, 10, 7, \text{ and } 4.7. \)

\( \text{Figure 14. Mach number of the shell for } Q_{\text{UV}} = 10^{49} \text{ s}^{-1} \text{ and } n_E = 10^3 \text{ cm}^{-3}. \text{ The solid and the dashed lines indicate the case with } T_c = 10 \text{ K and } 30 \text{ K, respectively.} \)

(1989) derived a simple analytic dispersion relation of the VI for a decelerating isothermal spherical shock wave taking into account the effect of the thickness (also see Ryu & Vishniac 1987). Although their analysis did not include self-gravity, here, we use their dispersion relation (see Equations 19(a) and (b) in their paper) to estimate the effect of the VI. Their dispersion relation depends on the Mach number \( M \) of the shell and the expansion law. For the case with the expanding \( \text{H} \upiota \) regions, the shell expands as \( \propto t^{6/7} \) if the self-gravity is neglected. In this case, the perturbation does not grow exponentially but in a power law \( \propto t^s \), where \( s \) characterizes the growth rate. Figure 13 shows the real part of \( s \) as a function of the angular wavenumber \( l \). One can see that the maximum growth rate \( \text{Re}(s) \) increases with \( M \). The angular scale of the most unstable mode is smaller for larger \( M \). We find that the unstable mode exists only for \( M \gtrsim 4.7 \).

To see the typical value of the Mach number, we consider the expanding shell around the 41 \( M_\odot \) star that is embedded by the uniform ambient gas of \( n_E = 10^5 \text{ cm}^{-3}. \) Figure 14 shows the Mach number of the shell for \( T_c = 10 \text{ K} \) (the solid line) and 30 K (the dashed line). In the early phase when self-gravity is not important (\( t/t_0 < 0.5 \)), since the Mach number is as large as several tens, \( \text{Re}(s) \) is large. The small-scale perturbation with \( l = 10^2 \sim 10^3 \) quickly grows and saturates in the nonlinear stage (Mac Low & Norman 1993). On the other hand, in the later phase (the self-gravity-dominated phase, \( t/t_0 > 0.5 \)), the Mach number is as low as 5–10 as shown in Figure 14. In this phase, \( \text{Re}(s) \sim 1 \) from Figure 13. This means that the growth rate of the perturbations is comparable to the expansion rate \( \propto t^{4/7} \). Therefore, in the self-gravity-dominated phase, the VI is not expected to be important. The VI produces fluctuations in the shell before the GI becomes important.

7. SUMMARY

In this paper, we have performed linear perturbation analysis of decelerating shells created by the expansion of \( \text{H} \upiota \) regions. We summarize our results as follows.

1. We develop a semi-analytic method for describing the density profile in the shell. The time evolution of the density profile of the expanding shell can be divided into three phases: deceleration-dominated, intermediate, and self-gravity-dominated phase. In the deceleration-dominated phase, the density peak is in SF by the inertia force owing to the deceleration. As the shell mass increases and self-gravity becomes important, the density peak is inside the shell, but it is closer to the SF than the CD in the intermediate phase. In the self-gravity-dominated phase, the shell becomes massive and the density peak is closer to the CD than the SF. The evolution is confirmed by 1D hydrodynamical simulation.

2. We show detailed structures of the dispersion relation in the asymmetric layer subjected to a constant deceleration of both the unstable and stable modes by imposing the CD boundary condition from/at both sides.

(a) We discover the mode exchange between the compressible and surface-gravity modes in the stable regime.

(b) In a situation where the distance from one surface \( z_1 \) to the density peak \( z = 0 \) is smaller than the scale height of the self-gravity \( H_0 \) and the distance from the other surface \( z_2 \) to \( z = 0 \) is larger than \( H_0 \), the nature of the GI is quite different from the symmetric case with the same column density and the peak density. The eigenfunction in the region \( 0 < z < z_2 \) is approximately the compressible mode. On the other hand, the eigenfunction in the region \( z_1 < z < 0 \) is approximately the incompressible mode. Moreover, the growth rate is enhanced compared with the symmetric cases through cooperation with the Rayleigh–Taylor instability.

3. We investigate the linear stability of expanding shells driven by \( \text{H} \upiota \) regions taking into account the shock-like boundary condition on the leading surface, the CD boundary condition on the trailing surface, and the asymmetric density profile obtained by the semi-analytic method.

(a) The shell is expected to grow earlier than the prediction of previous studies (Elmegreen 1994; Whitworth et al. 1994b) that are based on the dispersion relation of the shock-confined layer.

(b) In the self-gravity-dominated phase, since the density peak is closer to the CD than to the SF, the CD is expected to deform significantly.

These results provide useful knowledge for the analysis of more detailed nonlinear numerical simulations that is the scope of our companion paper (Iwasaki et al. 2011).

We thank the referee for many constructive comments that improved our paper significantly. This work was supported by Grants-in-Aid for Scientific Research from the MEXT of Japan (K.I.:22864006; S.I.:18540238 and 16077202), and Research Fellowship from JSPS (K.I.:21-1979). This work was based on the results of the companion paper (Iwasaki et al. 2011) where
numerical computations were carried out on PC cluster at Osaka University Cybermedia Center, and Cray XT4 at the CfCA of Numerical Astronomical Observatory of Japan. The page charge of this paper is supported by CfCA.

REFERENCES

Dale, J. E., Wünsch, R., Whitworth, A. P., & Palouš, J. 2009, MNRAS, 398, 1537

Deharveng, L., et al. 2010, A&A, 523, A6

Elmegreen, B. G. 1989, ApJ, 340, 786

Elmegreen, B. G. 1994, ApJ, 427, 384

Elmegreen, B. G., & Elmegreen, D. M. 1978, ApJ, 220, 1051

Elmegreen, B. G., & Lada, C. J. 1977, ApJ, 214, 725

Goldreich, P., & Lynden-Bell, D. 1965, MNRAS, 130, 97

Hosokawa, T., & Inutsuka, S. 2006, ApJ, 646, 240

Iwasaki, K., Inutsuka, S., & Tsuribe, T. 2011, ApJ, 733, 17

Iwasaki, K., & Tsuribe, T. 2008, PASJ, 60, 125

Lubow, S. H., & Pringle, J. E. 1993, MNRAS, 263, 701

Mac Low, M., & Norman, M. L. 1993, ApJ, 407, 207

Nagai, T., Inutsuka, S., & Miyama, S. M. 1998, ApJ, 506, 306

Nishi, R. 1992, Prog. Theor. Phys., 87, 347

Ryu, D., & Vishniac, E. T. 1987, ApJ, 313, 820

Spitzer, L. 1942, ApJ, 95, 329

Spitzer, L. 1978, Physical Processes in the Interstellar Medium (New York: Wiley)

Tomisaka, K., & Ikeuchi, S. 1983, PASJ, 35, 187

Usami, M., Hanawa, T., & Fujimoto, M. 1995, PASJ, 47, 271

van Leer, B. 1997, J. Comput. Phys., 135, 229

Vishniac, E. T. 1983, ApJ, 274, 152

Vishniac, E. T. 1994, ApJ, 428, 186

Vishniac, E. T., & Ryu, D. 1989, ApJ, 337, 917

Voit, G. M. 1988, ApJ, 331, 343

Weaver, R., McCray, R., Castor, J., Shapiro, P., & Moore, R. 1977, ApJ, 218, 377

Weiler, G. L. 1982, A&A, 105, 237

Weiler, G. L., & Schmid-Burgk, J. 1981, AJ, 245, 927

Whitworth, A. P., Bhattal, A. S., Chapman, S. J., Disney, M. J., & Turner, J. A. 1994a, A&A, 290, 421

Whitworth, A. P., Bhattal, A. S., Chapman, S. J., Disney, M. J., & Turner, J. A. 1994b, MNRAS, 268, 291

Whitworth, A. P., & Francis, N. 2002, MNRAS, 329, 641

Wünsch, R., Dale, J. E., Palouš, J., & Whitworth, A. P. 2010, MNRAS, 407, 1963