New relations in the algebra of the Baxter Q-operators.

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Abstract

We consider irreducible cyclic representations of the algebra of mon-odromy matrices corresponding to the $R$-matrix of the six-vertex model. In roots of unity the Baxter Q-operator can be represented as a trace of a tensor product of $L$-operators corresponding to one of these cyclic representations and satisfies the $TQ$-equation. We find a new algebraic structure generated by these $L$-operators, and as a consequence, by the $Q$-operators.

Introduction.

In his papers [1] Baxter introduced the $Q$-operator and used it for solving of the six-vertex model. The $Q$-operators form a family that commutes with a family of the transfer-matrices $T(u)$, with the $TQ$-equation satisfying. The latter equation relates two families with each other and is a key for solving the model.

In [2] (see also [3], [4]) the expressions for the Boltzmann weights of the chiral Potts model were found, that are solutions of the star-triangle relation. The $R$-matrix $S$ of the model can be represented as a product of four such Boltzmann weights.

Light on the algebraic structure of the $Q$-operators in the particular case of the six-vertex model and its relation to the $R$-matrix of the chiral Potts model shed the paper by Bazhanov and Stroganov [5]. In the $N$-th roots of unity ($N$ is a prime number) they found the $N$-dimensional cyclic representation $\mathcal{L}$ of the Yang–Baxter algebra related to the $R$-matrix of the usual six-vertex model.
The trace over the $N$-dimensional quantum space of a tensor product of $L$-operators possesses the properties of the $Q$-operator. In particular, it satisfies the $TQ$-equation.

In the paper by Tarasov [6] irreducible cyclic representations of the algebra of monodromy matrices corresponding to the $R$-matrix of the six-vertex model in roots of unity were described.

Recently the $Q$-operator came to the centre of attention again. For some models of statistical physics it was shown [8], [7], [9], [10] that the $Q$-operator is a quantum analogue of the Bäcklund transformation.

In the paper [5] mentioned above the $R$-matrix of the chiral Potts model $S$ was derived as an operator intertwining tensor products of two cyclic representations of the algebra of monodromy matrices, with they multiplying at first in one order and then in the inverse order.

As it was shown by A.Odesskii (unpublished), the four factors generating the $R$-matrix of the chiral Potts model $S$ are actually intertwiners that provide some elementary isomorphisms of cyclic representations of the $L$-operators algebra.

In this work we clarify the conditions under which two cyclic representations are equivalent and find the corresponding intertwiner. We also solve the same problem for two tensor products of a pair of cyclic representations. The obtained intertwiners are a generalization of the well-known vertex weights of the chiral Potts model and satisfy some modification of the star-triangle equations.

The plan of the paper is the following. In Section 1 we introduce the notion of cyclic representations of the algebra of $L$-operators. In Section 2 we discuss different definitions of cyclic representations. In Section 3 we derive the $TQ$-equation. In Section 4 we discuss some special cases of elementary isomorphisms acting on cyclic representations of the algebra of monodromy matrices. In Section 5 we find them in the general case. We show that the intertwiners of these elementary isomorphisms satisfy a generalized star-triangle relation of the chiral Potts model. In Section 6 we write some relations in the algebra of $Q$-operators. In Section 7 we discuss some questions related to possible future investigations. In Appendices we prove some formulae used in the paper.

1 Cyclic representations of the Yang–Baxter algebra.

Following to [6], we introduce the definition of the $R$-matrix:

$$R(u) = \begin{bmatrix}
1 - u\omega & \omega(1 - u) & u(1 - \omega) \\
\omega(1 - u) & u(1 - \omega) & 1 - u \\
1 - \omega & 1 - u & 1 - u\omega
\end{bmatrix}. \quad (1)$$
It is connected with the algebra $U_q(sl_2)$ \cite{11}, \cite{12} and can be obtained from the $R$-matrix of the usual six-vertex model by means of a simple transformation (see Section 2). For short we denote $\cal M = End$ $C^2$ so that $R(u) = \cal M \otimes \cal M$.

The algebra of monodromy matrices $A$ is the algebra with generators $A(u)$, $B(u)$, $C(u)$, $D(u)$, $H$, $H^{-1}$ and relations

\begin{align*}
R(u) \begin{bmatrix} \hat{\omega} & H \\ L(u) & 1 \end{bmatrix} &= \begin{bmatrix} \hat{\omega} & H \\ L(u) & 1 \end{bmatrix} R(u), \\
[H \otimes H, L(u)] &= 0, \\
H H^{-1} &= H^{-1} H = 1,
\end{align*}

(2)

The indices '1' and '2' over $L$ denote a two-dimensional space in which the corresponding $L$-operator multiplies by the $R$-matrix. At that both $L$-operators act in the same auxiliary space.

As indicated in \cite{6}, in the algebra $A$ one can introduce the coproduct $\Delta$:

\begin{align*}
\Delta(L(u)) &= L_1(u) L_2(u) \in \cal M \otimes A \otimes A, \\
\Delta(H) &= H \otimes H.
\end{align*}

By lower indices '1' and '2' are denoted the quantum spaces, in which the corresponding $L$-operators act. At that the $L$-operators considering as two-dimensional matrices (each matrix element is an operator in one of the two quantum spaces), multiply with each other according to the usual rule of multiplication of matrices.

Thus a tensor product of some representations of the algebra $A$ is a representation of $A$ too.

Let us now define the quantum determinant:

$$
\det_q L(u) = D(u)A(u\omega^{-1}) - C(u)B(u\omega^{-1}).
$$

(3)

One can verify that $H^{-1}\det_q L(u)$ is a central element of the algebra $A$.

Hereafter we set $\omega^N = 1$. As shown in \cite{3} in this case the center of the algebra $A$ increases, namely, the following operators become central:

$$
\langle O \rangle(u) = \prod_{k=0}^{N-1} O(u\omega^k), \quad O = A, B, C, D,
$$

and thus we can define the matrix of central elements:

$$
\langle L \rangle = \begin{bmatrix} \langle A \rangle & \langle B \rangle \\ \langle C \rangle & \langle D \rangle \end{bmatrix}.
$$

One can show \cite{3} that $L = L_1 L_2$ satisfies the equation:

$$
\Delta(\langle L \rangle) = \langle L_1 \rangle \langle L_2 \rangle, \quad \langle \det_q L \rangle = \det \langle L \rangle.
$$
The $N$-dimensional cyclic representation $\pi$ of the algebra $\mathcal{A}$ has the form:

$$L(u, p_1, p_2) = \begin{pmatrix} c_1 c_2 Z - b_1 b_2 u & -u(b_1 d_2 - c_1 a_2 Z)X \\ X^{-1}(d_1 b_2 - a_1 c_2 Z) & d_1 d_2 - a_1 a_2 u Z \end{pmatrix}$$

$$H_\pi = hZ, \quad p_i = (a_i, b_i, c_i, d_i), i = 1, 2.$$ (4)

The action of the operators $X, Z$ on the standard basis in $\mathbb{C}^N$ reads as follows:

$$Z|k\rangle = \omega^k|k\rangle, \quad X|k\rangle = |k + 1\rangle, \quad (k = 0, \ldots, N - 1, |N\rangle \equiv |0\rangle).$$

We also have

$$\langle L(p_1, p_2)(v) = \begin{pmatrix} c_1^N c_2^N - b_1^N b_2^N v & -v(b_1^N d_2^N - c_1^N a_2^N) \\ d_1^N b_2^N - a_1^N c_2^N & d_1^N d_2^N - a_1^N a_2^N v \end{pmatrix} \right).$$

Though in formulae (4) there are eight (in addition to $v$) parameters, the $N$-dimensional representation depends only on six of them: the substitution

$$a_1 \to \lambda a_1, \quad a_2 \to \lambda^{-1} a_2, \quad c_1 \to \lambda c_1, \quad c_2 \to \lambda^{-1} c_2, \quad b_1 \to b_1, \quad b_2 \to b_2, \quad d_1 \to d_1, \quad d_2 \to d_2,$$

where $\lambda$ is an arbitrary number, does not change the operator $L(u, p_1, p_2)$. The same is true for the substitution

$$b_1 \to \lambda b_1, \quad b_2 \to \lambda^{-1} b_2, \quad d_1 \to \lambda d_1, \quad d_2 \to \lambda^{-1} d_2, \quad a_1 \to a_1, \quad a_2 \to a_2, \quad c_1 \to c_1, \quad c_2 \to c_2.$$

Apart from, the projective equivalence class of the $L$-operators depends only on four additional parameters, since

$$L(\lambda p_1, p_2) = \lambda L(p_1, p_2), \quad L(p_1, \mu p_2) = \mu L(p_1, p_2),$$

where $\lambda, \mu$ are arbitrary numbers.

Two representations $L_1(u, p_1, p_2)L_2(u, p_3, p_4)$ and $L_1(u, p_3, p_4)L_2(u, p_1, p_2)$ are equivalent if and only if one can choose $p_i, i = 1, 2, 3, 4$, satisfying the conditions

$$a_1^N b_1^N, \quad c_1^N d_1^N = \lambda_{\pm},$$ (5)
where $\lambda_\pm$ independent of $i$ (Appendix 3). The intertwiner given by the equation
\[
S(p_1, p_2, p_3, p_4) L_1(u, p_1, p_2) L_2(u, p_3, p_4) = L_1(u, p_3, p_4) L_2(u, p_1, p_2) S(p_1, p_2, p_3, p_4),
\]
has explicit expression through Boltzmann weights of the chiral Potts model $W_{pq}$:
\[
S(p_1, p_2, p_3, p_4) = F(p_2, p_3; X_1 X_2^{-1}) G(p_1, p_3; Z_1 X_2^{-1}),
\]
\[
G(p, q; \omega^k) = \sum_{i=1}^N \omega^k W_{pq}(i),
\]
\[
F(p_2, p_3; X_1 X_2^{-1}) = \sum_{i=1}^N \omega^k \prod_{j=1}^N \omega^k W_{pq}(j).
\]

2 Cyclic representations of the Yang–Baxter algebra in the form of Bazhanov–Stroganov.

Parallel with the $L$-operators introduced in the previous section one can consider their version related to another choice of the $R$-matrix.

Let us consider the usual $R$-matrix of the ice model,
\[
R_{ice}(x) = \begin{bmatrix}
  x \omega_1 - x^{-1} \omega_1^{-1} & x^{-1} - x & \omega_1 - \omega_1^{-1} & x - x^{-1} \\
  \omega_1 - \omega_1^{-1} & x \omega_1 - x^{-1} \omega_1^{-1} & \omega_1 - \omega_1^{-1} & x - x^{-1}
\end{bmatrix},
\]
and corresponding relations in the Yang–Baxter algebra,
\[
R_{ice}(x) L_{ice} (xy) L_{ice} (y) = L_{ice} (y) L_{ice} (xy) R_{ice}(x).
\]

The $N$-dimensional representation of the algebra [3] one can write in the following form:
\[
L_{ice}(y, p_1, p_2) = \begin{bmatrix}
  y^{-1} c_1 c_2 Z_1 - b_1 b_2 y Z_1^{-1} & -(b_1 b_2 Z_1^{-1} - c_1 a_2 Z_1) X \\
  \omega_1 X^{-1} (d_1 b_2 Z_1^{-1} - a_1 c_2 Z_1) & y^{-1} d_1 d_2 Z_1^{-1} - a_1 a_2 \omega_1^2 y Z_1
\end{bmatrix},
\]
where the operators $Z_1, X$ satisfy the relations
\[
Z_1^N = 1, \quad X^N = 1, \quad Z_1 X = \omega_1 X Z_1.
\]

Let us make the following substitution:
\[
R_{12} \rightarrow C_1^{-1}(xy) C_2^{-1}(y) R_{12} C_2(y) C_1(xy), \quad L_1(xy) \rightarrow C_1(xy)^{-1} L_1(xy) C_1(xy),
\]
}\[
R_{12} \rightarrow C_1^{-1}(xy) C_2^{-1}(y) R_{12} C_2(y) C_1(xy), \quad L_1(xy) \rightarrow C_1(xy)^{-1} L_1(xy) C_1(xy),
\]
\[
R_{12} \rightarrow C_1^{-1}(xy) C_2^{-1}(y) R_{12} C_2(y) C_1(xy), \quad L_1(xy) \rightarrow C_1(xy)^{-1} L_1(xy) C_1(xy),
\]
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R_{12} \rightarrow C_1^{-1}(xy) C_2^{-1}(y) R_{12} C_2(y) C_1(xy), \quad L_1(xy) \rightarrow C_1(xy)^{-1} L_1(xy) C_1(xy),
\]
\[
R_{12} \rightarrow C_1^{-1}(xy) C_2^{-1}(y) R_{12} C_2(y) C_1(xy), \quad L_1(xy) \rightarrow C_1(xy)^{-1} L_1(xy) C_1(xy),
\]
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R_{12} \rightarrow C_1^{-1}(xy) C_2^{-1}(y) R_{12} C_2(y) C_1(xy), \quad L_1(xy) \rightarrow C_1(xy)^{-1} L_1(xy) C_1(xy),
\]
\[
R_{12} \rightarrow C_1^{-1}(xy) C_2^{-1}(y) R_{12} C_2(y) C_1(xy), \quad L_1(xy) \rightarrow C_1(xy)^{-1} L_1(xy) C_1(xy),
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R_{12} \rightarrow C_1^{-1}(xy) C_2^{-1}(y) R_{12} C_2(y) C_1(xy), \quad L_1(xy) \rightarrow C_1(xy)^{-1} L_1(xy) C_1(xy),
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R_{12} \rightarrow C_1^{-1}(xy) C_2^{-1}(y) R_{12} C_2(y) C_1(xy), \quad L_1(xy) \rightarrow C_1(xy)^{-1} L_1(xy) C_1(xy),
\]
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R_{12} \rightarrow C_1^{-1}(xy) C_2^{-1}(y) R_{12} C_2(y) C_1(xy), \quad L_1(xy) \rightarrow C_1(xy)^{-1} L_1(xy) C_1(xy),
\]
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R_{12} \rightarrow C_1^{-1}(xy) C_2^{-1}(y) R_{12} C_2(y) C_1(xy), \quad L_1(xy) \rightarrow C_1(xy)^{-1} L_1(xy) C_1(xy),
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\]
\[
R_{12} \rightarrow C_1^{-1}(xy) C_2^{-1}(y) R_{12} C_2(y) C_1(xy), \quad L_1(xy) \rightarrow C_1(xy)^{-1} L_1(xy) C_1(xy),
\]
\[
R_{12} \rightarrow C_1^{-1}(xy) C_2^{-1}(y) R_{12} C_2(y) C_1(xy), \quad L_1(xy) \rightarrow C_1(xy)^{-1} L_1(xy) C_1(x
\[ L_2(y) \to C_2^{-1}(y)L_2(y)C_2(y), \]

where
\[ C(y) = \begin{bmatrix} 1 & 0 \\ 0 & y \end{bmatrix}, \]

and the index of the matrix \( C \) indicates the space this matrix acts in.

As a result of this substitution, the equation \( RLL = LLR \) is valid, as before.

At that
\[
R(x) = \begin{bmatrix}
x \omega_1 - x^{-1} \omega_1^{-1} & x(x_1 - x^{-1}) \\
\omega_1 y^{-1}X^{-1}(d_1 b_2 Z_1^{-1} - c_1 a_2 Z_1) & x \omega_1 - x^{-1} \omega_1^{-1}
\end{bmatrix},
\]

and the \( L \)-operator is given by the formula:

\[
L(y, p_1, p_2) = \begin{bmatrix}
y^{-1} c_1 c_2 Z_1 - b_1 b_2 y Z_1^{-1} & -y(b_1 d_2 Z_1^{-1} - c_1 a_2 Z_1)X \\
\omega_1 y^{-1}X^{-1}(d_1 b_2 Z_1^{-1} - a_1 c_2 Z_1) & y^{-1} d_1 d_2 Z_1^{-1} - a_1 a_2 \omega_1^2 y Z_1
\end{bmatrix}.
\]

It is the last \( L \)-operator that was found in the paper [5]. We call it the cyclic representation of the algebra of monodromy matrices in the form of Bazhanov–Stroganov.

Let us now multiply \( L(y) \) by \( y Z_1 \) and introduce the following notations:

\[ v = y^2, \quad Z = Z_1^2, \quad \omega = \omega_1^2. \]

Then we obtain the operator [6]. At that the \( R \)-matrix becomes (in order the equation \( RLL = LLR \) being valid as before):

\[ R(u) = -\omega_1 x K_1 R_{BS}(x) K_2^{-1}, \]

where \( u = x^2, \)

\[ K = \omega_1^{(\sigma^z - 1)/2} = \begin{bmatrix} 1 & 0 \\ 0 & \omega_1^{-1} \end{bmatrix}. \]

The lower index of the matrix \( K \) is denoted the space it acts in. As it can be seen without difficulty the matrix \( R \) coincides with [6] and the operator \( L \) coincides with [6].

We emphasize that hereinafter we denote the cyclic representations in the form of Tarasov by \( L \) and the cyclic representations in the form of Bazhanov–Stroganov by \( L \) in order to avoid confusion.

Matrix elements of the \( L \)-operators we denote by \( L^{j\beta}_i \) and \( L^{j\beta}_i \), \( i, j = 0, 1, \alpha, \beta = 0, ..., N - 1, \) correspondingly.
3 The Q-operator and the $TQ$-equation.

The transfer matrix built with the aim of $\mathcal{L}(u)$,

$$Q(u) = \text{tr}_0 \mathcal{L}_{10}(u) \mathcal{L}_{20}(u) ... \mathcal{L}_{n0}(u),$$

where the trace is calculated in the $N$-dimensional space, possesses a very important property which allows to say about it as about the Baxter $Q$-operator \[5\]. Namely, it satisfies the $TQ$-equation. Let us prove this statement.

We consider the equation

$$R^{ij}_{12}(u) L^{k_1 \beta}_{i j \alpha} (uv) L^{k_2 \gamma}_{j i \beta} (v) = L^{j \beta}_{1 \alpha} (u) L^{j \gamma}_{1 \beta} (uv) R^{k_1 k_2}_{j j} (u).$$

(12)

It can be shown graphically as in Fig.1.

If the indices $i_1$, $k_1$ are fixed, the equation (12) is an operator in the tensor product $C^2 \times C^N$. Let us act by this operator on the vector $\psi_{k_2 \gamma}$, that belongs to the kernel of the operator $\mathcal{L}_{23}(v)$, that is, on the vector satisfying the equation

$$L^{j \beta}_{j \alpha} (v) \psi_{k_2 \gamma} = 0.$$

A nontrivial kernel of $\mathcal{L}_{23}(v)$ there exists only if the spectral parameter $v$ equals some special values $v = v_\ast$:

$$v_\ast^2 = \frac{c_1 d_1}{a_1 b_1}$$
or

\[ v^2 = \frac{c_2d_2}{a_2b_2}. \]

One can see from Fig.1 that the kernel of the operator \( L_{23}(v_\ast) \) is a subspace which is invariant with regard to the tensor product \( L_{13}(uv_\ast)R_{12}(u) \). In this case the supplement of the kernel considering as a coset space is an invariant space too. Therefore the matrix of the operator \( L_{13}(uv_\ast)R_{12}(u) \) has a block-diagonal form:

\[
L_{13}(uv_\ast)R_{12}(u) = \begin{bmatrix} P_1 & * \\ 0 & P_2 \end{bmatrix},
\]

where all blocks are \( N \)-dimensional matrices, and we denote the matrix elements inessential for us by the star. Here we introduce the following way of ordering of the basis vectors: the first \( N \) vectors generate the kernel and the others \( N \) vectors generate its supplement.

Let the following equation be valid under some values of the parameters of \( L(v_\ast) \):

\[
P_1 = \phi_1 L(uv_\ast \lambda), \quad P_2 = \phi_2 L(uv_\ast \lambda^{-1}).
\]

Then after multiplying \( n \) copies of the operator \( L_{13}(uv_\ast)R_{12}(u) \) in the spaces ‘2’ and ‘3’ (see Fig.2) and taking the trace we obtain the equation

\[
\tilde{Q}(uv_\ast)T(u) = \phi_1^n \tilde{Q}(uv_\ast \lambda) + \phi_2^n \tilde{Q}(uv_\ast \lambda^{-1}),
\]

where

\[
\tilde{Q}(uv_\ast) = tr L_{13}(uv_\ast)L_{13}(uv_\ast)\ldots L_{13}(uv_\ast),
\]

and \( T(u) \) is the usual transfer matrix of the ice model. Making the substitution

\[
Q(u) = \tilde{Q}(uv_\ast),
\]

we obtain the \( TQ \)-equation.

It turns out that if we use the definition (11) for the operator \( L(v) \), to make the calculations to the end is impossible. This is the case because there is not any set of the parameters such as the conditions (13) satisfy.

However, one can redefine the operator \( L(v) \) in order the condition (13) and, consequently, the equation (14), to be valid [5]. For that \( p_1 \) has to depend on the spectral parameter \( v \) and \( p_2 \) has not to change:

\[
p_1(v) = \{a_1v^{-1}, b_1, c_1, d_1v\}, \quad p_2(v) = \{a_2, b_2, c_2, d_2\}.
\]

It is shown in Appendix A that \( Q(u) = Q(u, p_1(u), p_2) \) actually satisfies the \( TQ \)-equation:

\[
Q(u)T(u) = (u - u^{-1})^n Q(u\omega) + (u\omega - u^{-1}\omega^{-1})^n Q(u\omega^{-1}).
\]

Here \( T(u) \) is the usual transfer matrix of the ice model.

7
4 The elementary isomorphisms in the special case.

We consider two representations of the algebra of $L$-operators: $L(u, p_1, p_2)$ and $L(u, p_2, p_1)$. Let the parameters $p_1$ and $p_2$ be such that these representations are equivalent. We introduce the operator $G$ satisfying the equation

$$G(Z)L(u, p_1, p_2) = L(u, p_2, p_1)G(Z).$$  \hspace{1cm} (15)

Here $G(Z)$ acts in $N$-dimensional space.

We now consider two tensor products of a pair of cyclic representations: $L_1(u, p_1, p_2)L_2(u, p_3, p_4)$ and $L_1(u, p_1, p_3)L_2(u, p_2, p_4)$ (pay attention to the permutation $p_2 \leftrightarrow p_3$). Let the parameters $p_1$, $p_2$, $p_3$, $p_4$ be such that the two representations are equivalent. We introduce the operator $F$ intertwining them:

$$F(X_1X_2^{-1})L_1(u, p_1, p_2)L_2(u, p_3, p_4) =$$

$$= L_1(u, p_1, p_3)L_2(u, p_2, p_4)F(X_1X_2^{-1}).$$  \hspace{1cm} (16)

Here $F(X_1X_2^{-1})$ acts in the $N$-dimensional space too.

It turns out that the conditions (15) are certainly sufficient $G$ and $F$ to exist. If they obey, the two operators are given by the following formulae:

$$G(p_1, p_2; \omega^k)G(p_1, p_2; 1) = \prod_{j=1}^{k} \frac{d_1b_2 - a_1c_2\omega^j}{b_1d_2 - c_1a_2\omega^j},$$  \hspace{1cm} (17)

$$F(p_1, p_2; \omega^k)F(p_1, p_2; 1) = \prod_{j=1}^{k} \frac{\omega a_1d_2 - d_1a_2\omega^j}{c_1b_2 - b_1c_2\omega^j}.  \hspace{1cm} (18)$$
Here we denote by $G(p_1, p_2; \omega^k)$ and $F(p_1, p_2; \omega^k)$ the diagonal matrix elements of the $N$-dimensional matrices $G(p_1, p_2; Z)$ and $F(p_1, p_2; X_1X_2^{-1})$ in the eigenbasis corresponding to each of them ($G$ and $F$ cannot be brought to the diagonal form at the same time).

In the following section we calculate the two operators in more general case, and now we note that the formulae (17), (18) coincide with the formulae (7).

The existence of elementary isomorphisms $G$ and $F$ explains the factorisation of the $R$-matrix of the chiral Potts model. Permuting the pairs, we obtain the chain:

$$(p_1, p_2)(p_3, p_4) \xrightarrow{F_2} (p_1, p_3)(p_2, p_4) \xrightarrow{G_1, G_2} (p_3, p_1)(p_4, p_2) \xrightarrow{F_1} (p_3, p_4)(p_1, p_2).$$

From this the factorisation of the $R$-matrix becomes evident.

## 5 The general case.

### 5.1 The $G$-operator.

Let us consider two representations of the algebra of $L$-operators: $L(u, p_1, \bar{p}_1)$ and $L(u, p_2, \bar{p}_2)$. We want to clarify when they are equivalent and find the corresponding intertwiner which is a generalization of the operator $G$ introduced in the previous part. For simplicity we denote this generalized intertwiner by the same symbol $G$.

The two representations are equivalent if the following equations are valid (Appendix C):

\[
\begin{cases}
    a_1^N \bar{a}_1^N = a_2^N \bar{a}_2^N, & b_1^N \bar{b}_1^N = b_2^N \bar{b}_2^N, \\
    \frac{c_1^N \bar{d}_1^N}{a_1^N \bar{b}_1^N} = \frac{c_2^N \bar{d}_2^N}{a_2^N \bar{b}_2^N}, & \frac{\bar{c}_1^N \bar{d}_2^N}{\bar{a}_1^N \bar{b}_1^N} = \frac{\bar{c}_2^N \bar{d}_2^N}{\bar{a}_2^N \bar{b}_2^N}, \\
    \frac{d_1^N \bar{d}_1^N}{a_1^N \bar{a}_1^N} = \frac{d_2^N \bar{d}_2^N}{a_2^N \bar{a}_2^N}, & d_1^N \bar{b}_1^N - a_1^N \bar{c}_1^N = d_2^N \bar{b}_2^N - a_2^N \bar{c}_2^N. 
\end{cases}
\]

(19)

We consider the simplest case when we extract the $N$-th roots by the simple striking out the letter $N$. As the result we obtain the system:

\[
\begin{cases}
    a_1 \bar{a}_1 = a_2 \bar{a}_2, & b_1 \bar{b}_1 = b_2 \bar{b}_2, \\
    \frac{c_1 \bar{d}_1}{a_1 \bar{b}_1} = \frac{c_2 \bar{d}_2}{a_2 \bar{b}_2}, & \frac{\bar{c}_2 \bar{d}_2}{\bar{a}_2 \bar{b}_2} = \frac{\bar{c}_1 \bar{d}_1}{\bar{a}_1 \bar{b}_1}, \\
    d_1 \bar{d}_1 = d_2 \bar{d}_2, & d_1 \bar{b}_1^N - a_1 \bar{c}_1^N = d_2 \bar{b}_2^N - a_2 \bar{c}_2^N. 
\end{cases}
\]

(20)
We find the operator $G$ satisfying the equation

$$GL(u,p_1,\bar{p}_1) = L(u,p_2,\bar{p}_2)G.$$ 

If the conditions (20) obey then the operator $G$ exists. We prove this now.

Let us make use of the following ansaz:

$$G = G(Z).$$

We obtain the system:

$$
\begin{align*}
G(Z)A_1 &= A_2 G(Z), \\
G(Z)B_1 &= B_2 G(Z), \\
G(Z)C_1 &= C_2 G(Z), \\
G(Z)D_1 &= D_2 G(Z).
\end{align*}
$$

We choose a basis $|k\rangle$, $k = 0, \ldots, N - 1$ (mod $N$):

$$Z|k\rangle = \omega^k |k\rangle, \quad X|k\rangle = |k + 1\rangle.$$ 

It is clear that in this basis the matrix $G(Z)$ is diagonal. Let us find its nonzero matrix elements.

The first equation is

$$G(Z) \left[ c_1 \tilde{c}_1 Z - b_1 \bar{b}_1 u \right] = \left[ c_2 \tilde{c}_2 Z - b_2 \bar{b}_2 u \right] G(Z).$$

We act on the vector $|k\rangle$ by the left-hand and the right-hand sides and compare the coefficients at different powers of $u$. As the result we obtain the following restrictions on the parameters:

$$c_1 \tilde{c}_1 = c_2 \tilde{c}_2, \quad b_1 \bar{b}_1 = b_2 \bar{b}_2. \tag{21}$$

In the similar way, from the fourth equation,

$$G(Z) \left[ d_1 \tilde{d}_1 - a_1 \bar{a}_1 \omega u Z \right] = \left[ d_2 \tilde{d}_2 - a_2 \bar{a}_2 \omega u Z \right],$$

we obtain that

$$d_1 \tilde{d}_1 = d_2 \tilde{d}_2, \quad a_1 \bar{a}_1 = a_2 \bar{a}_2. \tag{22}$$

The second equation is

$$G(Z)X^{-1} \left[ d_1 \tilde{b}_1 - a_1 \bar{c}_1 Z \right] = \left[ d_2 \tilde{b}_2 - a_2 \bar{c}_2 Z \right] G(Z).$$

From this it follows that

$$G(\omega^{k+1}) = \frac{d_1 \tilde{b}_1 - a_1 \bar{c}_1 \omega^{k+1}}{d_2 b_2 - a_2 \bar{c}_2 \omega^{k+1}} G(\omega^k), \tag{23}$$

10
where by \( G(\omega^k) \), \( k = 0, ..., N - 1 \), are denoted the diagonal matrix elements of the matrix \( G(Z) \).

In the same way from the third equation,

\[
G(Z) \left[ b_1 \bar{d}_1 - c_1 \bar{a}_1 Z \right] X = \left[ b_2 \bar{d}_2 - c_2 \bar{a}_2 Z \right] XG(Z),
\]

one can easily derive that

\[
G(\omega^{k+1}) = \frac{b_2 \bar{d}_2 - c_2 \bar{a}_2 \omega^{k+1}}{b_1 \bar{d}_1 - c_1 \bar{a}_1 \omega^{k+1}} G(\omega^k). \tag{24}
\]

Since \( G(\omega^k) \) has a single meaning it must be

\[
(d_1 b_1 - a_1 \bar{c}_1 \omega^{k+1})(b_1 \bar{d}_1 - c_1 \bar{a}_1 \omega^{k+1}) = (d_2 b_2 - a_2 \bar{c}_2 \omega^{k+1})(b_2 \bar{d}_2 - c_2 \bar{a}_2 \omega^{k+1}).
\]

Comparing coefficients at different powers of \( \omega \) and taking into account (21), (22), we obtain an additional condition:

\[
\bar{c}_1 \bar{d}_1 \bar{a}_1 + c_1 d_1 = \frac{\bar{c}_1 \bar{d}_2}{a_2 b_2} - \frac{c_2 d_2}{a_2 b_2} \tag{25}
\]

Apart from, we obtain from the periodicity condition \( G(\omega^{N+1}) = G(\omega) \):

\[
d_1^N b_1^N - a_1^N \bar{c}_1^N = d_2^N b_2^N - a_2^N \bar{c}_2^N. \tag{26}
\]

Using the gauge symmetries of the \( L \)-operators, one can set

\[
a_1 = \bar{a}_2, \quad b_1 = \bar{b}_2. \tag{27}
\]

Then from (26) it follows that (26), (21), (23), (25) are valid.

Thus \( G(\omega^k) \) exists and is given by the recurrence relation (23).

One can rewrite \( G(\omega^k) \) in terms of \( p_1, \bar{p}_1 \) without difficult. We substitute

\[
\bar{c}_2 = \frac{c_1 \bar{c}_1}{c_2}
\]

into (26) and express \( c_2 \) in terms of \( p_1, \bar{p}_1 \):

\[
c_2 = \bar{c}_1 \sqrt{\frac{b_1^{N+1} d_1^{N+1} - c_1^{N+1} \bar{a}_1^{N+1}}{d_1^{N-1} b_1^{N-1} - a_1^{N-1} \bar{c}_1^{N-1}}} = \Lambda(p_1, \bar{p}_1) \bar{c}_1,
\]

where we introduce a new function

\[
\Lambda(p_1, p_2) = \sqrt{\frac{b_1^{N} d_1^{N} - c_1^{N} \bar{a}_1^{N}}{d_1^{N} b_1^{N} - a_1^{N} \bar{c}_1^{N}}}.\]
From this one can obtain

\[
\frac{G(p_1, \bar{p}_1; \omega^k)}{G(p_1, \bar{p}_1; 1)} = \Lambda(p_1, \bar{p}_1)^k \prod_{j=1}^{k} \frac{d_1 \bar{b}_1 - a_1 \bar{c}_1 \omega^j}{d_1 b_1 - \bar{a}_1 c_1 \omega^j}.
\]

We emphasize that \( G \) depends only on \( p_1, \bar{p}_1 \).

So, the found operator \( G \) generates an isomorphism of the two representations of the algebra of monodromy matrices, \( L(u, p_1, \bar{p}_1) \), \( L(u, p_2, \bar{p}_2) \), with the parameters \( p_2, \bar{p}_2 \) expressing in terms of \( p_1, \bar{p}_1 \) in the following way:

\[
a_2 = \bar{a}_1, \quad \bar{a}_2 = a_1, \\
b_2 = b_1, \quad \bar{b}_2 = b_1, \\
c_2 = \Lambda(p_1, \bar{p}_1) \bar{c}_1, \quad \bar{c}_2 = \Lambda(p_1, \bar{p}_1)^{-1} c_1, \\
d_2 = \Lambda(p_1, \bar{p}_1)^{-1} \bar{d}_1, \quad \bar{d}_2 = \Lambda(p_1, \bar{p}_1) d_1.
\]

The operator \( G \) found here is a generalization of the operator \( G \) in \([1]\). In order to come to such special case we must set

\[
p_2 = \bar{p}_1, \quad \bar{p}_2 = p_1.
\]

At that there is an additional constraint on the parameters \( p_1, \bar{p}_1 \):

\[
\Lambda(p_1, \bar{p}_1) = 1.
\]

**Note.** We extract the N-th roots by simple striking out the letter \( N \). But in all probability the general case comes to this one. A complete investigation can be transacted by the following way. In order two representations to be equivalent it is necessary their centres coincide. However, when we derive our conditions, we does not compare all central elements. We must add the condition of equality of the corresponding quantum determinants to our system of equations. We do not study completely the question about what comes the additional condition to. It seems that it can be used for investigating how one must extract the N-th root, and it strongly restricts a number of variants.

### 5.2 The \( F \)-operator.

Let us consider two representations of the algebra of \( L \)-operators: \( L_1(u, p_1, \bar{p}_1) \), \( L_2(u, p_2, \bar{p}_2) \) and

\[
L_1(u, p_3, \bar{p}_3) \), \( L_2(u, p_4, \bar{p}_4) \). We want to find the conditions under which these two representations are equivalent and calculate the corresponding intertwiner. The latter is a generalization of the operator \( F \) introduced in the previous section.

The matrix of the central elements:

\[
\langle L(u, p, \bar{p}) \rangle = \left[
\begin{array}{cc}
  c^N \bar{c}^N - \bar{b}^N \bar{b}^N u & -u(\bar{b}^N \bar{d}^N - c^N \bar{a}^N) \\
  d^N \bar{b}^N - a^N \bar{c}^N & d^N \bar{d}^N - a^N \bar{a}^N u
\end{array}
\right].
\]

(29)
The necessary condition of the equivalence of the two representations is the coincidence of their centres. Therefore

\[ \langle L_1(u,p_1,\bar{p}_1) \rangle \langle L_2(u,p_2,\bar{p}_2) \rangle = \langle L_1(u,p_3,\bar{p}_3) \rangle \langle L_2(u,p_4,\bar{p}_4) \rangle. \]

From this it follows that

\[
\det(L_1(u,p_1,\bar{p}_1)) \cdot \det(L_2(u,p_2,\bar{p}_2)) = \det(L_1(u,p_3,\bar{p}_3)) \cdot \det(L_2(u,p_4,\bar{p}_4)).
\]

(30)

In Appendix C it is shown that from these conditions one can derive the following relations between the parameters (we do not consider the trivial case \( p_3 = p_1, \bar{p}_3 = \bar{p}_1, p_4 = p_2, \bar{p}_4 = \bar{p}_2 \)):

\[
\begin{align*}
 b_1^N b_1^N &= b_3^N b_3^N, & b_2^N b_2^N &= b_4^N b_4^N, \\
 a_1^N a_1^N &= a_3^N a_3^N, & a_2^N a_2^N &= a_4^N a_4^N, \\
 d_1^N b_1^N &= d_3^N b_3^N, & a_2^N c_2^N &= a_4^N c_4^N, \\
 c_2^N d_2^N = \frac{c_3^N d_3^N}{a_3^N b_3^N}, & c_3^N d_3^N = \frac{c_4^N d_4^N}{a_4^N b_4^N}, \\
 c_2^N d_2^N = \frac{c_4^N d_4^N}{a_4^N b_4^N}, & c_3^N d_3^N = \frac{c_4^N d_4^N}{a_4^N b_4^N}, \\
 b_2^N c_2^N &= \frac{b_2^N c_2^N}{a_2^N b_2^N} - \frac{c_2^N d_2^N}{a_2^N d_2^N}, \\
 a_1^N d_1^N &= \frac{a_1^N d_1^N}{a_2^N b_2^N} - \frac{a_1^N d_1^N}{a_2^N d_1^N}.
\end{align*}
\]

(31)

We strike out the letter \( N \) again and obtain:

\[
\begin{align*}
 b_1 b_1 &= b_3 b_3, & b_2 b_2 &= b_4 b_4, \\
 a_1 a_1 &= a_3 a_3, & a_2 a_2 &= a_4 a_4, \\
 d_1 b_1 &= d_3 b_3, & a_2 c_2 &= a_4 c_4, \\
 c_2 d_2 = \frac{c_3 d_3}{a_3 b_3}, & c_3 d_1 = \frac{c_3 d_3}{a_3 b_3}, \\
 c_2 d_2 = \frac{c_4 d_4}{a_4 b_4}, & c_4 d_4 = \frac{c_4 d_4}{a_4 b_4}.
\end{align*}
\]

(32)

Let us recall about the gauge symmetries of the \( L \)-operator. The substitution

\[
\begin{align*}
 a &\rightarrow \lambda a, & \bar{a} &\rightarrow \lambda^{-1} \bar{a}, \\
 c &\rightarrow \lambda c, & \bar{c} &\rightarrow \lambda^{-1} \bar{c},
\end{align*}
\]

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\[ b \rightarrow b, \quad \bar{b} \rightarrow \bar{b}, \quad d \rightarrow d, \quad \bar{d} \rightarrow \bar{d} \]
does not change the operator \( L(u, p, \bar{p}) \). The same is valid for the substitution
\[ b \rightarrow \lambda b, \quad \bar{b} \rightarrow \lambda^{-1} \bar{b}, \]
\[ d \rightarrow \lambda d, \quad \bar{d} \rightarrow \lambda^{-1} \bar{d}, \]
\[ a \rightarrow a, \quad \bar{a} \rightarrow \bar{a}, \quad c \rightarrow c, \quad \bar{c} \rightarrow \bar{c}. \]
Using the gauge degrees of freedom, one can set
\[ b_1 = b_3, \quad b_2 = b_4, \quad a_1 = a_3, \quad a_2 = a_4. \tag{33} \]
From this and from \((32)\) it follows that
\[ \bar{b}_1 = \bar{b}_3, \quad \bar{b}_2 = \bar{b}_4, \quad \bar{a}_1 = \bar{a}_3, \quad \bar{a}_2 = \bar{a}_4, \quad \bar{c}_1 = c_3, \quad \bar{c}_2 = c_4. \tag{34} \]
Apart from the equations satisfy:
\[ \bar{c}_3 \bar{d}_3 = c_2 d_2 \frac{a_3 \bar{b}_3}{a_2 b_2}, \quad c_4 d_4 = \bar{c}_1 \bar{d}_1 \frac{a_2 b_2}{a_1 b_1}. \tag{35} \]
Recalling the formulae \((31)\), we obtain:
\[ \bar{c}_3^N = c_2^N \bar{b}_3^N \frac{b_2^N b_1^N c_3^N}{c_2^N \bar{b}_1^N} - \bar{a}_1^N d_2^N, \tag{36} \]
\[ d_4^N = \frac{\bar{a}_1^N a_2^N b_2^N c_1^N}{\bar{a}_1^N b_1^N} - a_1^N \bar{d}_1^N. \tag{37} \]
Now we want to find the matrix \( F \) intertwining the two representations in question. In particular, we want to prove that the conditions \((33) - (37)\) are not only necessary but sufficient for the existence of \( F \).
So, we write
\[ F L_1(p_1, \bar{p}_1) L_2(p_2, \bar{p}_2) = L_1(p_3, \bar{p}_3) L_2(p_4, \bar{p}_4) F. \tag{38} \]
We find the operator \( F \) in the form \( F(X_1 X_2^{-1}) \), where \( X_1, X_2 \) are the matrix of shift acting in the first and in the second \( N \)-dimensional spaces correspondingly.
As shown in Appendix \[\text{[ ]}\], \( F \) actually exists and is expressed in terms of \( p_1, \bar{p}_1, p_2, \bar{p}_2 \) by the formula:
\[
\frac{F(\bar{p}_1, p_2; \omega^k)}{F(\bar{p}_1, p_2; 1)} = \Omega(\bar{p}_1, p_2)^{-k} \prod_{j=1}^{k} \frac{\bar{c}_1 b_2 - \bar{a}_1 d_2 \omega^j}{b_1 c_2 - d_1 a_2 \omega^j},
\]
where for the further convenience we define a new function
\[
\Omega(p_1, p_2) = \sqrt{\frac{b_2^N c_2^N - d_2^N \bar{a}_1^N}{c_2^N b_1^N - a_2^N \bar{a}_1^N}}
\]

We emphasize that \( F \) depends only on the parameters \( \bar{p}_1 \) and \( p_2 \). At that the action of \( F \) is given by the formulae
\[
\begin{align*}
a_3 &= a_1, & \bar{a}_3 &= \bar{a}_1, & a_4 &= a_2, & \bar{a}_4 &= \bar{a}_2, \\
b_3 &= b_1, & \bar{b}_3 &= \bar{b}_1, & b_4 &= b_2, & \bar{b}_4 &= \bar{b}_2, \\
c_3 &= c_1, & \bar{c}_3 &= \bar{c}_1, & c_4 &= \bar{c}_2, & \bar{c}_4 &= \bar{c}_2, \\
d_3 &= d_1, & \bar{d}_3 &= \bar{d}_1, & d_4 &= \bar{d}_2, & \bar{d}_4 &= \bar{d}_2,
\end{align*}
\]

(39)

It is clear that the obtained expression for \( F \) is gauge invariant.

Recall that the tensor product of two \( L \)-operators possesses another symmetry. Namely, one can insert the unity between the two factors:
\[
L_1(p_3, \bar{p}_3)L_2(p_4, \bar{p}_4) = L_1(p_3, \bar{p}_3)M^{-1}ML_2(p_4, \bar{p}_4),
\]

where an arbitrary matrix \( 2 \times 2 \) is denoted by \( M \).

We set
\[
M = \begin{bmatrix} \bar{b}_1/b_2 & 0 \\ 0 & \bar{a}_1/a_2 \end{bmatrix}.
\]

We apply the additional symmetry to our \( L \)-operators. As the result, we obtain the product of new \( L \)-operators, whose parameters are expressed in terms of \( p_1, p_1, p_2, p_2 \) in the following way:
\[
\begin{align*}
a_3 &= a_1, & \bar{a}_3 &= \bar{a}_1, & a_4 &= a_2, & \bar{a}_4 &= \bar{a}_2, \\
b_3 &= b_1, & \bar{b}_3 &= b_1, & b_4 &= b_2, & \bar{b}_4 &= b_2, \\
c_3 &= c_1, & \bar{c}_3 &= \bar{c}_1, & c_4 &= \bar{c}_2, & \bar{c}_4 &= \bar{c}_2, \\
d_3 &= d_1, & \bar{d}_3 &= \bar{d}_1, & d_4 &= \bar{d}_2, & \bar{d}_4 &= \bar{d}_2,
\end{align*}
\]

(40)

At that the expression for \( F(\omega^{k+1}) \), of course, is invariable.

The case considered in the paper [6], is the special case of the approach in question and can be obtained if we set in all formulæ
\[
p_3 = p_1, & \bar{p}_3 = p_2, & p_4 = \bar{p}_1, & \bar{p}_4 = \bar{p}_2.
\]

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In particular, it is not difficult to derive the $F$-operator in (7). At that there is an additional constraint on the parameters $\bar{p}_1, \bar{p}_2$:

$$\Omega(\bar{p}_1, \bar{p}_2) = 1.$$

The found operators $G$ and $F$ satisfy the relation that generalizes the star-triangle relation of the chiral Potts model [2]. Namely, the equation is valid:

$$G(p, q; Z_1)F(\bar{p}, r; X_1X_2^{-1})G(\bar{q}, \bar{r}; Z_1) = \mu F(q, r; X_1X_2^{-1})G(p, r'; Z_1)F(p', q'; X_1X_2^{-1}),$$

where $\mu$ is a constant, and the intertwiners depend on the parameters that are expressed in terms of $p$, $q$, $r$ by the following formulae:

$$
\begin{align*}
\begin{cases}
a'_{ij} = a_{ij}, \\
b'_{ij} = b_{ij}, \\
c'_{ij} = \Omega(q, r)c_{ij}, \\
d'_{ij} = \Omega(q, r)^{-1}d_{ij},
\end{cases}
\quad
\begin{cases}
a'_{i} = a_{i}, \\
b'_{i} = b_{i}, \\
c'_{i} = \Omega(q, r)c_{i}, \\
d'_{i} = \Omega(q, r)^{-1}d_{i},
\end{cases}
\quad
\begin{cases}
a'_{p} = a_{p}, \\
b'_{p} = b_{p}, \\
c'_{p} = \Lambda(p, r)\Lambda^{-1}(p', q) \Lambda^{-1}(p'q'r')c_{p}, \\
d'_{p} = \Lambda(p, r)\Lambda^{-1}(p'q'r') \Lambda^{-1}(p'q'r')d_{p},
\end{cases}
\quad
\begin{cases}
a'_{p} = a_{p}, \\
b'_{p} = b_{p}, \\
c'_{p} = \Lambda(p, q)c_{p}, \\
d'_{p} = \Lambda(p, q)^{-1}d_{p},
\end{cases}
\quad
\begin{cases}
a'_{p} = a_{p}, \\
b'_{p} = b_{p}, \\
c'_{p} = \Lambda(p, r)c_{p}, \\
d'_{p} = \Lambda(p, r)^{-1}d_{p}.
\end{cases}
\end{align*}
$$

The proof of these statement one can find in Appendix E.

Note. The stated gives a foundation to say about the existence of a new algebraic structure related to the cyclic representations of the monodromy matrices algebra.

Let us consider a Hopf algebra with generators $L^i_j(p_1, p_2)$, $i, j = 0, 1, p_1, p_2 \in C^4$.

The coproduct is given by

$$\Delta \left( L^i_j \right) = \left( L^k_l \right)_1 \left( L^i_j \right)_2.$$
The relations in the Hopf algebra:

\[ G(p_1, p_2)L^1_k(p_1, p_2) = L^1_k(\tilde{p}_1, \tilde{p}_2)G(p_1, p_2), \]

\[ F(\tilde{p}_1, p_2)\left( L^k_k(p_1, \tilde{p}_1)\right)_1 \left( L^k_k(p_2, \tilde{p}_2)\right)_2 = \left( L^k_k(\tilde{p}_2, \tilde{p}_1)\right)_1 \left( L^k_k(p_1, \tilde{p}_2)\right)_2 F(\tilde{p}_1, p_2), \]

\[ \left( L^k_k(p_1, \tilde{p}_1)\right)_1 \left( L^k_k(p_2, \tilde{p}_2)\right)_2 = \left( L^k_k(p_1, \tilde{p}_2)\right)_1 \left( L^k_k(p_2, \tilde{p}_1)\right)_2, \]

where an arbitrary two-dimensional diagonal matrix is denoted by \( M \), and the parameters in the right-hand sides of the relations are expressed in terms of the parameters in the left-hand sides by the formulae (28), (40). We mean the sum over repeating indices.

6 The algebra of the \( Q \)-operators.

Besides the operator \( Q(u) \) introduced in the Section 3 and related to the cyclic representations of the algebra of monodromy matrices in the form by Bazhanov–Stroganov, one can consider the operator \( Q(u) \),

\[ Q(u) = \text{tr}_0L_{10}(u)L_{20}(u)\ldots L_{k_0}(u), \]

where the trace is calculated in the \( N \)-dimensional space and the cyclic representations of the monodromy matrices algebra in the form by Tarasov are denoted by \( L_{10}(u) \).

The operators \( Q(u) \) generate the algebra with the relations following from the properties of the operators \( L(u) \):

\[ Q(\lambda p_1, \bar{p}_1) = \lambda^k Q(p_1, \bar{p}_1), \] (42)

\[ Q(p_1, \mu \bar{p}_1) = \mu^k Q(p_1, \bar{p}_1), \] (43)

\[ Q(a_1, b_1, c_1, d_1, \bar{a}_1, \bar{b}_1, \bar{c}_1, \bar{d}_1) = Q(\lambda a_1, b_1, \lambda c_1, d_1, \lambda^{-1} \bar{a}_1, \bar{b}_1, \lambda^{-1} \bar{c}_1, \bar{d}_1), \] (44)

\[ Q(a_1, b_1, c_1, d_1, \bar{a}_1, \bar{b}_1, \bar{c}_1, \bar{d}_1) = Q(a_1, \mu b_1, c_1, \mu d_1, \bar{a}_1, \mu^{-1} \bar{b}_1, \bar{c}_1, \mu^{-1} \bar{d}_1), \] (45)

\[ Q(a_1, b_1, c_1, d_1, \alpha a_1, \beta b_1, \gamma c_1, \delta d_1) = \]

\[ = Q(\alpha \bar{a}_1, \beta \bar{b}_1, \gamma \bar{c}_1, \delta \bar{d}_1, a_1, b_1, \gamma^{-1} c_1, \delta^{-1} d_1), \]

\[ Q(a_1, b_1, c_1, d_1, \bar{a}_1, \bar{b}_1, \bar{c}_1, \bar{d}_1)Q(a_2, b_2, c_2, d_2, \bar{a}_2, \bar{b}_2, \bar{c}_2, \bar{d}_2) = \]

\[ = Q(a_1, b_1, c_1, d_1, \beta \bar{a}_1, \alpha \bar{b}_1, \alpha \bar{c}_1, \beta \bar{d}_1) \]

\[ \times Q(\beta^{-1} a_2, \alpha^{-1} b_2, \alpha^{-1} c_2, \beta^{-1} d_2, \bar{a}_2, \bar{b}_2, \bar{c}_2, \bar{d}_2), \]

\[ Q(a_1, b_1, c_1, d_1, \bar{a}_1, \bar{b}_1, \bar{c}_1, \bar{d}_1)Q(a_2, b_2, c_2, d_2, \bar{a}_2, \bar{b}_2, \bar{c}_2, \bar{d}_2) = \]

\[ Q(a_1, b_1, c_1, d_1, a_2, b_2, \Omega c_2, \Omega^{-1} d_2) \]

\[ \times Q(\bar{a}_1, \bar{b}_1, \Omega^{-1} \bar{c}_1, \Omega \bar{d}_1, a_2, \bar{b}_2, \bar{c}_2, \bar{d}_2), \] (48)
where $\alpha, \beta, \lambda, \mu$ are arbitrary numbers,

$$\Lambda = \Lambda(p_1, \bar{p}_1), \quad \Omega = \Omega(\bar{p}_1, p_2).$$

One can find the proof of all these relations in Appendix F.

7 Discussion.

We note some questions which can become the topics of the future investigations.

The approach used in this paper can be applied in more general cases, namely it would be interesting to study the elementary isomorphisms intertwining the cyclic representations of the monodromy matrices algebra related to the elliptical $R$-matrix and also the $R$-matrix corresponding to the quantum algebra $U_q(sl_n)$.

The spectrum of the six-vertex model transfer matrix in roots of unity is degenerate [5], [14]. Some finite dimensional representations of the $Q$-operators algebra correspond to non-one-dimensional eigensubspaces of the transfer matrix. The $Q$-operators act on these spaces in nontrivial way, since they do not generally commute with each other.

Therefore an investigation of finite dimensional representations of the $Q$-operators algebra given by [12] -- [18] can shed light on the properties of the transfer matrix spectrum.

It is also interesting to clarify the relationship between the algebra of $Q$-operators and the $U(A_1^{(1)})$ symmetry found in the paper [14].

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Appendices.

A $TQ$-equation.

So, we want to find the kernel of the operator, $L_{23}(v; p_1(v), p_2) = \tilde{L}_{23}(v)$, where

$$p_1(v) = \{a_1 v^{-1}, b_1, c_1, d_1 v\}, \quad p_2 = \{a_2, b_2, c_2, d_2\}.$$

We have

$$\tilde{L}(v) = \begin{bmatrix}
    v^{-1}c_1c_2Z - b_1b_2vZ^{-1} & -v(b_1d_2Z^{-1} - c_1a_2Z)X \\
    \omega v^{-1}X^{-1}(d_1b_2vZ^{-1} - a_1c_2v^{-1}Z) & d_1d_2Z^{-1} - a_1a_2\omega^2Z
\end{bmatrix}.$$

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Hereinafter we use the following basis $|\alpha\rangle$, $\alpha = 0, ..., N - 1 (mod N)$:

$Z|\alpha\rangle = \omega^{\alpha}|\alpha\rangle$, $X|\alpha\rangle = |\alpha + 1\rangle$.

Let us consider $2N$-dimensional vector $\Psi$:

$$\Psi = \left( \begin{array}{c} \Phi_1 \\ \Phi_2 \end{array} \right),$$

where $\Phi_1, \Phi_2$ are $N$-dimensional vectors.

We act on $\Psi$ by the operator $\hat{L}_{23}(v)$ and compare the result with zero. We obtain:

$$\begin{align*}
(c_1c_2Z - b_1b_2v^2Z^{-1})\Phi_1 - v^2(b_1d_2Z^{-1} - c_1a_2Z)X\Phi_2 &= 0, \\
\omega X^{-1}v^{-1}(d_1b_2vZ^{-1} - a_1c_2v^{-1}Z)\Phi_1 + (d_1d_2Z^{-1} - a_1a_2\omega^2)\Phi_2 &= 0.
\end{align*}$$

(49)

It is not difficult to see that this system has a solution if and only if

$$v^2 = v_2^2 = \frac{c_2d_2}{a_2b_2}.$$

The vectors generating the kernel are

$$\Psi_\alpha = \left( \begin{array}{c} -d_2|\alpha\rangle \\ b_2|\alpha - 1\rangle \end{array} \right) = -d_2|0, \alpha\rangle + b_2|1, \alpha - 1\rangle.$$

Now we must act on the vectors $\Psi_\alpha$ by the operator $\hat{L}_{13}(uv_*)\mathcal{R}_{12}(u)$ (it is especially worth to note that here we can employ the usual matrix multiplication in the two-dimensional space).

We write

$$\hat{L}_{13}(uv_*) = \begin{bmatrix} A(uv_*) & B(uv_*) \\ C(uv_*) & D(uv_*) \end{bmatrix}, \quad \mathcal{R}_{12}(u) = \begin{bmatrix} a(u) & b(u) \\ c(u) & d(u) \end{bmatrix}.$$

Then

$$\hat{L}_{13}(uv_*)\mathcal{R}_{12}(u) = \begin{bmatrix} a(u)A(uv_*) + c(u)B(uv_*) & b(u)A(uv_*) + d(u)B(uv_*) \\ a(u)C(uv_*) + c(u)D(uv_*) & b(u)C(uv_*) + d(u)D(uv_*) \end{bmatrix}.$$

Acting on the vector $\Psi_\alpha$ by each of the four matrix elements, we obtain, for example,

$$(\hat{L}\mathcal{R})^0_\alpha \Psi_\alpha = [a(u)A(uv_*) + c(u)B(uv_*)][-d_2|0, \alpha\rangle + b_2|1, \alpha - 1\rangle] =
\begin{align*}
&= -d_2(u\omega - u^{-1}\omega^{-1})(v_2^*-v^{-1}c_1c_2\omega^\alpha - b_1b_2uv_2\omega^{-\alpha})|0, \alpha\rangle +
\end{align*}$$

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+\omega_2 \{(u - u^{-1})(v_s^{-1}u^{-1}c_1c_2\omega^{-\alpha} - b_1b_2uv_s\omega^{1-\alpha})|1, \alpha - 1\} + \\
u(\omega - \omega^{-1})(-uv_s)(b_1d_2\omega^{-\alpha} - c_1a_2\omega^\alpha)|0, \alpha) = \\
= (u - u^{-1})(v_s^{-1}u^{-1}c_1c_2\omega^{-\alpha} - b_1b_2uv_s\omega^{1-\alpha})\Psi_\alpha.

Similarly,

\begin{equation}
(\hat{\mathcal{R}})_1^1\Psi_\alpha = (u - u^{-1})(d_1d_2\omega^{-\alpha} - a_1a_2\omega^{2+\alpha})\Psi_\alpha,
\end{equation}

\begin{equation}
(\hat{\mathcal{R}})_0^0\Psi_\alpha = (u - u^{-1})(-v_s)(b_1d_2\omega^{-\alpha} - c_1a_2\omega^{1+\alpha})\Psi_{\alpha+1},
\end{equation}

\begin{equation}
(\hat{\mathcal{R}})_{10}^0\Psi_\alpha = (u - u^{-1})\omega v_s^{-1}u^{-1}(d_1d_2uv_s\omega^{1-\alpha} - a_1c_2u^{-1}v_s^{-1}\omega^{1-\alpha})\Psi_{\alpha-1}.
\end{equation}

It is clear that the kernel \(\hat{\mathcal{L}}_{13}(v_s)\) is actually an invariant subspace with regard to the operator \(\hat{\mathcal{L}}_{13}(uv_s)\mathcal{R}_{12}(u)\).

It is not difficult to see that the obtained \(N\)-dimensional matrix is proportional to \(\hat{\mathcal{L}}(uv_s)\):

\begin{equation}
\hat{\mathcal{L}}_{13}(uv_s)\mathcal{R}_{12}(u)\big|_{\ker} = (u - u^{-1})\hat{\mathcal{L}}(uv_s, \omega).
\end{equation}

Let us now consider the vectors that belong to the supplement of the kernel. If we make a factorization by the vectors of the kernel then this supplement must be an invariant space with regard to the operator \(\hat{\mathcal{L}}_{13}(uv_s)\mathcal{R}_{12}(u)\). We choose the basis

\[\Psi'_\alpha = (0, \alpha).\]

Making all computations we find (mod \(\Psi_\alpha\)):

\begin{equation}
(\hat{\mathcal{R}})_0^0\Psi'_\alpha = (u\omega - u^{-1}\omega^{-1})(v_s^{-1}u^{-1}c_1c_2\omega^\alpha - b_1b_2uv_s\omega^{-\alpha})\Psi'_\alpha.
\end{equation}

\begin{equation}
(\hat{\mathcal{R}})_1^1\Psi'_\alpha = (u\omega - u^{-1}\omega^{-1})(d_1d_2\omega^1\alpha - a_1a_2\omega^{1+\alpha})\Psi'_\alpha,
\end{equation}

\begin{equation}
(\hat{\mathcal{R}})_0^0\Psi'_\alpha = (u\omega - u^{-1}\omega^{-1})(-v_s)(b_1d_2\omega^{-\alpha} - c_1a_2\omega^\alpha)\Psi'_{\alpha+1},
\end{equation}

\begin{equation}
(\hat{\mathcal{R}})_{10}^0\Psi'_\alpha = (u\omega - u^{-1}\omega^{-1})\omega v_s^{-1}u^{-1}(d_1d_2uv_s\omega^{-\alpha} - a_1c_2u^{-1}v_s^{-1}\omega^\alpha)\Psi'_{\alpha-1}.
\end{equation}

As in the previous case, it is not difficult to see that the found \(N\)-dimensional matrix is proportional to \(\hat{\mathcal{L}}(uv_s, \omega^{-1})\):

\begin{equation}
\hat{\mathcal{L}}_{13}(uv_s)\mathcal{R}_{12}(u)\big|_{\ker^{-1}} = (u\omega - u^{-1}\omega^{-1})\hat{\mathcal{L}}(uv_s, \omega^{-1}).
\end{equation}

Thus we obtain that the conditions \([13]\) actually satisfy. At that

\[\lambda = \omega, \quad \phi_1 = u - u^{-1}, \quad \phi_2 = u\omega - u^{-1}\omega^{-1}.
\]

As the result, the \(TQ\)-equation is valid:

\[Q(u)T(u) = (u - u^{-1})^nQ(u\omega) + (u\omega - u^{-1}\omega^{-1})^nQ(u\omega^{-1}),\]

where

\[Q(u) = tr_3\hat{\mathcal{L}}_{13}(u)\hat{\mathcal{L}}_{13}(u)...\hat{\mathcal{L}}_{13}(u).\]
B  The conditions of the equivalence of representations. The Ferma curve.

Let $N$ be odd.

We now prove that the representations $L_1(u,p_1,p_2)L_2(u,p_3,p_4)$ and $L_1(u,p_3,p_4)L_2(u,p_1,p_2)$ are equivalent in the general case if and only if we can choose $p_i, i = 1, 2, 3, 4,$ satisfying

$$a_i^N + b_i^N = \lambda \pm c_i^N + d_i^N = \lambda \pm.$$ (50)

For the convenience we make the following substitution

$$a_i^N \to a_i, \quad b_i^N \to b_i, \quad c_i^N \to c_i, \quad d_i^N \to d_i.$$

We have

$$\langle L(u,p_1,p_2) \rangle = \begin{bmatrix} c_1c_2 - b_1b_2u & -u(b_1d_2 - c_1a_2) \\ d_1b_2 - a_1c_2 & d_1d_2 - a_1a_2u \end{bmatrix},$$ (51)

$$\langle L(u,p_3,p_4) \rangle = \begin{bmatrix} c_3c_4 - b_3b_4u & -u(b_3d_4 - c_3a_4) \\ d_3b_4 - a_3c_4 & d_3d_4 - a_3a_4u \end{bmatrix}.$$ (52)

Two representations $L_\pi, L_\pi'$ are equivalent if there exists an isomorphism $P:

$$L_\pi' = PL_\pi P^{-1},$$

that is, elements of $L_\pi$ and $L_\pi'$ are in one-to-one correspondence.

The central elements $\langle L_\pi \rangle$ and $\langle L_\pi' \rangle$ must coincide. If $\pi = \pi_1 \times \pi_2$, $\pi' = \pi_2 \times \pi_1$, then

$$\langle L_{\pi_1} \rangle \langle L_{\pi_2} \rangle = \langle L_{\pi_2} \rangle \langle L_{\pi_1} \rangle.$$

From this it follows that

$$\langle L(u,p_1,p_2) \rangle \langle L(u,p_3,p_4) \rangle = \langle L(u,p_3,p_4) \rangle \langle L(u,p_1,p_2) \rangle.$$

Multiplying the matrices we obtain five equations. Only three of them are independent:

$$\frac{b_1d_2 - c_1a_2}{d_1b_2 - a_1c_2} = \frac{b_3d_4 - c_3a_4}{d_3b_4 - a_3c_4} = s, \quad (53)$$

$$\frac{a_1a_2 - b_1b_2}{b_1d_2 - c_1a_2} = \frac{a_3a_4 - b_3b_4}{b_3d_4 - c_3a_4} = q, \quad (54)$$

$$\frac{c_1c_2 - d_1d_2}{b_1d_2 - c_1a_2} = \frac{c_3c_4 - d_3d_4}{b_3d_4 - c_3a_4} = r, \quad (55)$$

where $s, q, r$ are arbitrary constants.
We want to find the constraints on \( p_i \) under which this system has solutions. We have:

\[
\begin{align*}
    b_1 d_2 - c_1 a_2 &= s (d_1 b_2 - a_1 c_2), \\
    a_1 a_2 - b_1 b_2 &= q (b_1 d_2 - c_1 a_2), \\
    c_1 c_2 - d_1 d_2 &= r (b_1 d_2 - c_1 a_2).
\end{align*}
\]  

(56)

It turns out that (56) can satisfy if \( p_1 \) and \( p_2 \) are points of the curve obtained by crossing of two planes (the projective symmetry of the operator \( L \)):

\[
\begin{align*}
    \alpha_1 a_i + \beta_1 b_i + \gamma_1 c_i + \delta_1 d_i &= 0, \\
    \alpha_2 a_i + \beta_2 b_i + \gamma_2 c_i + \delta_2 d_i &= 0, \quad i = 1, 2.
\end{align*}
\]

Let us find these planes. We have from the last system:

\[
\begin{align*}
    a_i &= \lambda_i c_i + \mu_i d_i, \\
    b_i &= \nu_i c_i + \eta_i d_i, \quad i = 1, 2.
\end{align*}
\]  

(57)

We substitute (57) into (56) and obtain the equations for coefficients:

\[
\begin{align*}
    \eta_1 &= s \eta_2, \\
    \lambda_2 &= s \lambda_1, \\
    \nu_1 &= \mu_2, \\
    \nu_2 &= \mu_1, \\
    1 &= -r \lambda_2, \\
    -1 &= r \eta_1, \\
    \nu_1 &= \mu_2, \\
    \lambda_1 \lambda_2 - \nu_1 \nu_2 &= -q \lambda_2, \\
    \mu_1 \mu_2 - \eta_1 \eta_2 &= q \eta_1, \\
    \lambda_1 \mu_2 - \nu_1 \eta_2 &= q \nu_1 - q \mu_2, \\
    \mu_1 \lambda_2 - \nu_2 \eta_1 &= 0.
\end{align*}
\]
Solving the system, we find:

\[
\begin{aligned}
\nu_1 &= \mu_2 = \mu, \\
\mu_1 &= \nu_2 = \nu, \\
\eta_1 &= \lambda_2 = \lambda, \\
\lambda_1 &= \eta_2 = \eta, \\
\lambda &= -1/r, \\
\nu \mu &= (q + \eta) \lambda, \\
s &= 1.
\end{aligned}
\]

Apart from, since the points \(p_1\) and \(p_2\) lie on the same curve, it must be

\[
\begin{aligned}
\lambda_1 &= \lambda_2 = \lambda, \\
\mu_1 &= \mu_2 = \mu, \\
\nu_1 &= \nu_2 = \nu, \\
\eta_1 &= \eta_2 = \eta.
\end{aligned}
\]

Consequently,

\[
\begin{aligned}
\lambda &= \eta, \\
\nu &= \mu.
\end{aligned}
\]

As the result,

\[
\begin{aligned}
a_1 + b_1 &= a_2 + b_2, & a_1 - b_1 &= a_2 - b_2, \\
c_1 + d_1 &= c_2 + d_2, & c_1 - d_1 &= c_2 - d_2.
\end{aligned}
\]

Similarly, it must be

\[
\begin{aligned}
a_3 + b_3 &= a_4 + b_4, & a_3 - b_3 &= a_4 - b_4, \\
c_3 + d_3 &= c_4 + d_4, & c_3 - d_3 &= c_4 - d_4.
\end{aligned}
\]

We return to the old notations:

\[
p_i \rightarrow p_i^N, \quad i = 1, 2, 3, 4.
\]

In principle, the points \(p_1, p_2, p_3, p_4\) can be related in two different ways:

1) \[
\begin{aligned}
a_1^N + b_1^N &= a_2^N + b_2^N, & a_1^N - b_1^N &= a_2^N - b_2^N, \\
c_1^N + d_1^N &= c_2^N + d_2^N, & c_1^N - d_1^N &= c_2^N - d_2^N.
\end{aligned}
\]

2) \[
\begin{aligned}
a_3^N + b_3^N &= a_4^N + b_4^N, & a_3^N - b_3^N &= a_4^N - b_4^N, \\
c_3^N + d_3^N &= c_4^N + d_4^N, & c_3^N - d_3^N &= c_4^N - d_4^N.
\end{aligned}
\]

This can be explained by the existence of two roots of the equation

\[
\nu^2 = \mu^2 = -q/r + 1/r^2.
\]

However, one should recall about symmetries of the operator \(L(u, p_1, p_2)\):

one can substitute

\[
\begin{aligned}
b_1 \rightarrow \lambda b_1, \quad b_2 \rightarrow \lambda^{-1} b_2, \quad d_1 \rightarrow \lambda d_1, \quad d_2 \rightarrow \lambda^{-1} d_2,
\end{aligned}
\]

at that \(L(u, p_1, p_2)\) does not change. With the aim of such substitution, where \(\lambda = -1\), the curve 2) reduces to the curve 1) (recall that \(N\) is odd).
The conditions of equivalence of representations in the general case.

We introduce the notations:

\[ \phi = d^N \bar{b}^N, \quad \psi = a^N \bar{c}^N, \quad \beta = b^N \bar{b}^N, \]
\[ \delta = a^N \bar{a}^N, \quad \mu = \frac{c^N d^N}{a^N b^N}, \quad \lambda = \frac{\bar{c}^N \bar{d}^N}{\bar{a}^N \bar{b}^N}. \]  

Then \( \langle L(u, p, \bar{p}) \rangle \) can be written in the form

\[
\langle L(u, p, \bar{p}) \rangle = \begin{bmatrix}
\frac{\mu \beta \psi}{\phi} - \beta u & -u \left( \frac{\lambda \beta \delta}{\psi} - \frac{\mu \beta \delta}{\phi} \right) \\
\phi - \psi & \frac{\lambda \delta \phi}{\psi} - \delta u
\end{bmatrix}.
\]  

We consider two representations of the algebra of \( L \)-operators, \( L(u, p_1, \bar{p}_1) \) \( L(u, p_2, \bar{p}_2) \), and want to find the conditions under which they are equivalent.

The necessary condition of the equivalence of the two representations is a coincidence of centres of these representations. We have:

\[
\langle L(u, p_1, \bar{p}_1) \rangle = \langle L(u, p_2, \bar{p}_2) \rangle.
\]  

We obtain from this by comparing coefficients of the different powers of \( u \):

\[
\delta_1 = \delta_2, \quad \beta_1 = \beta_2, \quad \phi_1 - \psi_1 = \phi_2 - \psi_2,
\]  

\[
\frac{\lambda_1 \phi_1}{\psi_1} = \frac{\lambda_2 \phi_2}{\psi_2}.
\]  

In addition to this, from (60) it follows that

\[
\det \langle L(u, p_1, \bar{p}_1) \rangle = \det \langle L(u, p_2, \bar{p}_2) \rangle.
\]  

In this equation the left-hand and the right-hand sides are polynomials of the second powers of \( u \). The roots of the left polynomial:

\[
u_1 = \lambda_1, \quad \bar{u}_1 = \mu_1.
\]

The roots of the right polynomial:

\[
u_2 = \lambda_2, \quad \bar{u}_2 = \mu_2.
\]

It is clear that the roots of the left-hand and the right-hand sides must coincide. We consider the following case:

\[
\lambda_1 = \mu_2, \quad \lambda_2 = \mu_1.
\]  

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Rewriting \((61)\rightarrow(63)\) in terms of \(a_i, b_i, c_i, d_i\), we obtain
\[
\begin{align*}
\frac{c_1^N d_2^N}{a_1^N b_2^N} &= \frac{c_2^N d_2^N}{a_2^N b_2^N}, \\
\frac{d_1^N d_2^N}{a_1^N a_2^N} &= \frac{d_2^N d_2^N}{a_2^N a_2^N}, \\
\frac{d_1^N b_1^N - a_1^N c_1^N}{a_1^N} &= \frac{d_2^N b_2^N - a_2^N c_2^N}{a_2^N}.
\end{align*}
\]

We now consider two representations of the \(L\)-operators algebra: \(L_1(u, p_1, \bar{p}_1)L_2(u, p_2, \bar{p}_2)\) and \(L_1(u, p_3, \bar{p}_3)L_2(u, p_4, \bar{p}_4)\). We want to find the conditions under which these two representations are equivalent.

We have:
\[
\det(L_1(u, p_1, \bar{p}_1)) \cdot \det(L_2(u, p_2, \bar{p}_2)) = \det(L_1(u, p_3, \bar{p}_3)) \cdot \det(L_2(u, p_4, \bar{p}_4)).
\]

Each determinant is a square polynomial with regard to \(u\). It is not difficult to find its roots. They are
\[
u = \lambda, \quad \bar{u} = \mu.
\]

As the result, we see that the left-hand side of \((64)\) vanishes if \(u_1 = \lambda_1, \bar{u}_1 = \mu_1, u_2 = \lambda_2, \bar{u}_2 = \mu_2\), and the right-hand side vanishes if \(u_3 = \lambda_3, \bar{u}_3 = \mu_3, u_4 = \lambda_4, \bar{u}_4 = \mu_4\).

It is clear that the left and the right roots coincide. We consider the following case:
\[
\mu_2 = \lambda_3, \quad \mu_1 = \mu_3, \quad \lambda_2 = \lambda_4, \quad \lambda_1 = \lambda_4.
\]

We have:
\[
\langle L_1(u, p_1, \bar{p}_1) \rangle \langle L_2(u, p_2, \bar{p}_2) \rangle = \langle L_1(u, p_3, \bar{p}_3) \rangle \langle L_2(u, p_4, \bar{p}_4) \rangle,
\]
\[
\mu_1, \lambda_1 \quad \mu_2, \lambda_2 \quad \mu_1, \mu_2 \quad \lambda_1, \lambda_2
\]
that is two roots \(\lambda_1\) and \(\mu_2\) change trade.

Let \(u = \lambda_2\). We act from the right on the vector \(\Psi_1\) (the right zero vector of the operator \(\langle L_2(\lambda_2, p_2, \bar{p}_2) \rangle\)) by the both sides of the equation \((65)\):
\[
\Psi_1 = \left( \begin{array}{c} -\lambda_2 \delta_2 \\
\psi_2 \end{array} \right), \quad \langle L_2(\lambda_2, p_2, \bar{p}_2) \rangle \Psi_1 = 0.
\]

This vector is the right zero vector for \(\langle L_2(\lambda_2, p_4, \bar{p}_4) \rangle\) too. Consequently, the following equation is valid:
\[
\frac{\delta_2}{\psi_2} = \frac{\delta_4}{\psi_4}.
\]

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Let now \( u = \mu_1 \). We act from the left on the left zero vector \( \Psi_2 \) of the operator \( \langle L_1(\mu_1, p_1, \bar{p}_1) \rangle \) by the both sides of the equation:

\[
\Psi_2 = \begin{pmatrix} \phi_1 \\ \mu_1 \beta_1 \end{pmatrix}, \quad \Psi_2 \langle L_1(\mu_1, p_1, \bar{p}_1) \rangle = 0.
\]

Since this vector is the left zero vector of the operator \( \langle L_1(\mu, p_3, \bar{p}_3) \rangle \), we have

\[
\frac{\phi_1}{\beta_1} = \frac{\phi_3}{\beta_3}.
\]

It is clear that (65) is valid if we insert between the two factors in the right-hand side the unity \( 1 = MM^{-1} \), where \( M \) is a two-dimensional matrix:

\[
M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}.
\]

Using this gauge symmetry, one can set

\[
\beta_3 = \beta_1, \quad \delta_4 = \delta_2,
\]

from which it follows immediately that

\[
\phi_3 = \phi_1, \quad \psi_4 = \psi_2.
\]

Multiplying the matrices in (65) and comparing coefficients of the powers of \( u \), it is not difficult to see that

\[
\begin{cases}
\beta_4 = \beta_2, & \delta_3 = \delta_1, \\
\psi_3 = \phi_4 \frac{\mu_2 \psi_1}{\lambda_1 \phi_2}, \\
\psi_3 \beta_2 - \delta_1 \phi_4 = \psi_1 \beta_2 - \delta_1 \phi_2.
\end{cases}
\]

From the two last equations we obtain

\[
\phi_4 = \frac{\lambda_1 \phi_2 (\beta_2 \psi_1 - \delta_1 \phi_2)}{\beta_2 \mu_2 \psi_1 - \lambda_1 \delta_1 \phi_2}, \quad \psi_3 = \frac{\mu_2 \psi_1 (\beta_2 \psi_1 - \delta_1 \phi_2)}{\beta_2 \mu_2 \psi_1 - \lambda_1 \delta_1 \phi_2}.
\]

Collecting all obtained equations we have in addition to (66)

\[
\begin{aligned}
\beta_1 &= \beta_3, & \beta_2 &= \beta_4, & \delta_1 &= \delta_3, & \delta_2 &= \delta_4, \\
\phi_1 &= \phi_3, & \psi_2 &= \psi_4, & \mu_2 &= \lambda_3, \\
\mu &= \mu_3, & \lambda_2 &= \lambda_4, & \lambda_1 &= \mu_4.
\end{aligned}
\]
D The operator $F$.

The matrix equation (38) can be written in the form of a system of equations corresponding to the four matrix elements. We have:

$$L_1(p_1, \bar{p}_1)L_2(p_2, \bar{p}_2) = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \cdot \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} = \begin{bmatrix} A_1A_2 + B_1C_2 & A_1B_2 + B_1D_2 \\ C_1A_2 + D_1C_2 & C_1B_2 + D_1D_2 \end{bmatrix}.$$  

As the result, we have the system:

$$\begin{cases}
F(X_1X_2^{-1})|A_1A_2 + B_1C_2| = [A_3a_4 + B_3C_4]F(X_1X_2^{-1}), \\
F(X_1X_2^{-1})|A_1B_2 + B_1D_2| = [A_3B_4 + B_3D_4]F(X_1X_2^{-1}), \\
F(X_1X_2^{-1})|C_1A_2 + D_1C_2| = [C_3a_4 + D_3C_4]F(X_1X_2^{-1}), \\
F(X_1X_2^{-1})|C_1B_2 + D_1D_2| = [C_3B_4 + D_3D_4]F(X_1X_2^{-1}).
\end{cases}$$

We choose a basis $|k_1, k_2\rangle$, $k_1, k_2 = 0, ..., N - 1$ (mod $N$):

$$X_1|k_1, k_2\rangle = \omega^{k_1}|k_1, k_2\rangle, \quad X_2|k_1, k_2\rangle = \omega^{k_2}|k_1, k_2\rangle,$$

$$Z_1|k_1, k_2\rangle = |k_1 - 1, k_2\rangle, \quad Z_2|k_1, k_2\rangle = |k_1, k_2 - 1\rangle.$$

In this basis the matrix $F(X_1X_2^{-1})$ is diagonal. Let us compute its nonzero matrix elements.

Substituting the expressions for $A_i$, $B_i$, $C_i$, $D_i$, $i = 1, 2$, we have, for example, for the first equation:

$$F(X_1X_2^{-1}) \left[ (c_1\bar{c}_1Z_1 - b_1\bar{b}_1u)(c_2\bar{c}_2Z_2 - b_2\bar{b}_2u) - u(b_3\bar{d}_3 - c_3\bar{a}_3)|X_1X_2^{-1}(d_2\bar{b}_2 - a_2\bar{c}_2Z_2) \right] =$$

$$= \left[ (c_3\bar{a}_3Z_1 - b_3\bar{b}_3u)(c_4\bar{c}_4Z_2 - b_4\bar{d}_4u) - u(b_3\bar{d}_3 - c_3\bar{a}_3Z_1)X_1X_2^{-1}(d_3\bar{b}_3 - a_3\bar{c}_3Z_2) \right] F(X_1X_2^{-1}).$$

Opening the parenthesis and acting by the left-hand and the right-hand sides on the vector $|k_1, k_2\rangle$ and comparing coefficients of the linearly independent
vectors and different powers of \(u\), we obtain:

\[
\begin{align*}
&c_1 \tilde{c}_1 c_2 \tilde{c}_2 = c_3 \tilde{c}_3 c_4 \tilde{c}_4, \\
&b_1 \tilde{b}_1 b_2 \tilde{b}_2 = b_3 \tilde{b}_3 b_4 \tilde{b}_4, \\
&b_1 \tilde{d}_1 d_2 b_2 = b_3 \tilde{d}_3 d_4 b_4, \\
&c_1 \tilde{a}_1 a_2 \tilde{c}_2 = c_3 \tilde{a}_3 a_4 \tilde{c}_4, \\
&F(\omega^{k+1}) = \frac{b_3 \tilde{c}_3 (d_3 a_4 \omega^{k+1} - \tilde{b}_3 c_4)}{b_1 \tilde{c}_2 (d_1 a_2 \omega^{k+1} - b_1 c_2)} F(\omega^k) = \frac{c_1 \tilde{b}_2 (\tilde{a}_1 d_2 \omega^{k+1} - \tilde{c}_1 b_2)}{c_3 \tilde{b}_4 (\tilde{a}_3 d_4 \omega^{k+1} - \tilde{c}_3 b_4)} F(\omega^k),
\end{align*}
\]

where \(F(\omega^k), k = 0, ..., N - 1\), are the diagonal matrix elements of the matrix \(F\).

Using the obtained restrictions on \(a_i, b_i, c_i, d_i\), we can reduce this system to the smaller one:

\[
\begin{align*}
&\tilde{c}_1 c_2 = \tilde{c}_3 c_4, \\
&\tilde{d}_1 d_2 = \tilde{d}_3 d_4, \\
&F(\omega^{k+1}) = \frac{\tilde{d}_3 a_4 \omega^{k+1} - \tilde{b}_3 c_4}{\tilde{d}_1 a_2 \omega^{k+1} - b_1 c_2} F(\omega^k) = \frac{\tilde{a}_1 d_2 \omega^{k+1} - \tilde{c}_1 b_2}{\tilde{a}_3 d_4 \omega^{k+1} - \tilde{c}_3 b_4} F(\omega^k)
\end{align*}
\]

One can easily see that the second equation can be derived from the first if we use the constraints

\[
\tilde{c}_3 \tilde{d}_3 = c_2 d_2 \frac{\tilde{a}_3 b_3}{a_2 b_2}, \quad c_4 d_4 = \tilde{c}_1 d_1 \frac{a_2 b_2}{\tilde{a}_1 b_1}
\]

Since \(F(\omega^k)\) has a single meaning, we must to equal the two fractions in terms of which \(F(\omega^k)\) is expressed. We obtain:

\[
(\tilde{d}_3 a_4 \omega^{k+1} - \tilde{b}_3 c_4)(\tilde{a}_3 d_4 \omega^{k+1} - \tilde{c}_3 b_4) = (\tilde{d}_1 a_2 \omega^{k+1} - \tilde{b}_1 c_2)(\tilde{a}_1 d_2 \omega^{k+1} - \tilde{c}_1 b_2).
\]

Comparing coefficients of different powers of \(\omega\), we obtain:

\[
\begin{align*}
&\tilde{a}_1 \tilde{d}_1 a_2 d_2 = \tilde{a}_3 \tilde{d}_3 a_4 d_4, \\
&\tilde{b}_1 \tilde{c}_1 b_2 c_2 = \tilde{b}_3 \tilde{c}_3 b_4 c_4, \\
&\tilde{\alpha}_1 \tilde{b}_1 a_2 b_2 \left( \frac{\tilde{c}_1 d_1}{a_1 b_1} + \frac{c_2 d_2}{a_2 b_2} \right) = \tilde{a}_4 b_4 \tilde{a}_3 \tilde{b}_3 \left( \frac{\tilde{c}_3 d_3}{a_3 b_3} + \frac{c_4 d_4}{a_4 b_4} \right).
\end{align*}
\]

If we take into account the restrictions on \(a_i, b_i, c_i, d_i\) again, then we have from this:

\[
\tilde{c}_1 c_2 = \tilde{c}_3 c_4,
\]

that is, the same that earlier.
We also have from the evident equation
\[ F(\omega^N + k) = F(\omega^k) \]
that
\[ c_1^N b_2^N - a_1^N d_2^N = c_1^N b_4^N - a_3^N d_4^N. \]  \quad (68)
We now prove that (67), (68) follow from (33)−(37). We have:
\[ \psi_3 = a_3^N c_3^N = \frac{c_2^N b_1^N}{b_2^N} \frac{a_1^N}{a_1^N} \frac{d_2^N}{d_2^N}, \]
\[ \phi_4 = d_4^N c_4^N = \frac{d_1^N a_2^N}{a_1^N} \frac{b_2^N}{b_2^N} \frac{e_1^N}{e_1^N} \frac{d_2^N}{d_2^N}. \]
Expressing from this \( c_3^N \) and \( d_4^N \) and substituting them into (68), we obtain the identity. Apart from, we have:
\[ \frac{c_1}{d_4} = \frac{c_2 a_1 b_1}{a_1 a_2 b_2}. \]
Multiplying the last equation by
\[ c_4 d_4 = c_1 d_1 \frac{a_3 b_2}{a_1 b_1}, \]
we obtain (68).
Thus,
\[ F(\omega^{k+1}) = \frac{d_4^N a_2^N - b_1^N c_2^N}{d_2^N} \frac{c_1 b_2 - a_1 d_2 \omega^{k+1}}{d_2^N} F(\omega^k). \]

E  The star-triangle equation.

Here we prove the star-triangle equation,
\[ G(Z_1)F(X_1X_2^{-1})G(Z_1) = \mu F(X_1X_2^{-1})G(Z_1)F(X_1X_2^{-1}), \]  \quad (69)
and find \( \mu \).
So, we have:
\[ (p, q)(r, s) \xrightarrow{F} (p, r')(q', s) \xrightarrow{G} (r'', p')(q', s) \]
\[ \downarrow G \]
\[ (q, \bar{r})(p, s) \xrightarrow{F} (q, \bar{r})(p'', s) \xrightarrow{G} (r'', q'')(p'', s) \]
Now if \( r'' , q'' , \tilde{p}' \), obtained in the two different ways, coincide exactly within the gauge transformations, then the equation (69) is actually valid.

We recall the definition of the functions \( \Lambda(p_1,p_2) \), \( \Omega(p_1,p_2) \):

\[
\Lambda(p_1,p_2) = \sqrt{\frac{b_1^N d_2^N - c_1^N a_2^N}{d_1^N b_2^N - a_1^N c_2^N}}, \tag{70}
\]

\[
\Omega(p_1,p_2) = \sqrt{\frac{c_1^N b_2^N - a_1^N d_2^N}{b_1^N c_2^N - d_1^N a_2^N}}. \tag{71}
\]

Let us make all computations for the first chain:

1. \( (p,q)(r,s) \stackrel{F}{\rightarrow} (p',q')(q'',s) \).
   
   We have:
   \[
a'_p = a_r, \quad a'_q = a_q,
b'_p = b_r, \quad b'_q = b_q,
c'_r = \Omega(q,r) c_r, \quad c'_q = \Omega(q,r)^{-1} c_q,
d'_r = \Omega(q,r)^{-1} d_r, \quad d'_q = \Omega(q,r) d_q.
\]

2. \( (p,q') (q'',s) \stackrel{G}{\rightarrow} (r'',p')(q'',s) \).
   
   We have:
   \[
a''_r = a'_r, \quad a''_p = a_p,
b''_r = b'_r, \quad b''_p = b_p,
c''_r = \Lambda(p',q') c'_r, \quad c''_q = \Lambda(p',q')^{-1} c_q,
d''_r = \Lambda(p',q')^{-1} d'_r, \quad d''_p = \Lambda(p',q') d_p.
\]

3. \( (r'',p')(q'',s) \stackrel{F}{\rightarrow} (r'',q'')(p'',s) \).
   
   We have:
   \[
a''''_r = a''_r, \quad a''''_p = a''_p,
b''''_r = b''_r, \quad b''''_p = b''_p,
c''''_q = \Omega(p'',q') c'''_q, \quad c''''_p = \Omega(p'',q')^{-1} c'''_p,
d''''_q = \Omega(p'',q')^{-1} d'''_q, \quad d''''_p = \Omega(p'',q') d'''_p.
\]

Now we make all computations for the second chain:

1. \( (p,q)(r,s) \stackrel{G}{\rightarrow} (\tilde{q},\tilde{p})(r,s) \).
We have:
\[ \tilde{a}_q = a_q, \quad \tilde{a}_p = a_p, \]
\[ \tilde{b}_q = b_q, \quad \tilde{b}_p = b_p, \]
\[ \tilde{c}_q = \Lambda(p,q)c_q, \quad \tilde{c}_p = \Lambda(p,q)^{-1}c_p, \]
\[ \tilde{d}_q = \Lambda(p,q)^{-1}d_q, \quad \tilde{d}_p = \Lambda(p,q)d_p. \]

2. \((\tilde{q}, \tilde{p})(r, s) \xrightarrow{F} (\tilde{q}, \tilde{r})(p'', s)\).

We have:
\[ \tilde{a}_r = a_r, \quad a''_p = \tilde{a}_p, \]
\[ \tilde{b}_r = b_r, \quad b''_p = \tilde{b}_p, \]
\[ \tilde{c}_r = \Omega(\tilde{p}, r)c_r, \quad c''_p = \Omega(\tilde{p}, r)^{-1}\tilde{c}_p, \]
\[ \tilde{d}_r = \Omega(\tilde{p}, r)^{-1}\tilde{d}_r, \quad d''_p = \Omega(\tilde{p}, r)d_p. \]

3. \((\tilde{q}, \tilde{r})(p'', s) \xrightarrow{G} (r'', q'')(p'', s)\).

We have:
\[ a''_r = \tilde{a}_r, \quad a''_q = \tilde{a}_q, \]
\[ b''_r = \tilde{b}_r, \quad b''_q = \tilde{b}_q, \]
\[ c''_r = \Lambda(\tilde{q}, \tilde{r})c_r, \quad c''_q = \Lambda(\tilde{q}, \tilde{r})^{-1}\tilde{c}_q, \]
\[ d''_r = \Lambda(\tilde{q}, \tilde{r})^{-1}\tilde{d}_r, \quad d''_q = \Lambda(\tilde{q}, \tilde{r})\tilde{d}_q. \]

It remains to verify that the obtained \(L\)-operators actually coincide. Comparing the parameters \(r'', q'', p''\), obtained in the two different ways, we conclude that the equation (41) satisfies if
\[ \Lambda(p, r')\Omega(p', q') = \Lambda(p, q)\Omega(\tilde{p}, r), \]
\[ \Omega(q, r)\Lambda(p, r') = \Omega(\tilde{p}, r)\Lambda(\tilde{q}, \tilde{r}). \]  

(72)

It is not difficult to show that (72) satisfies identically. Thus, we prove (41) for some unknown \(\mu\). We now find \(\mu^N\).

It is clear that
\[ \mu^N = \frac{\det G(p, q) \cdot \det F(\tilde{p}, r) \cdot \det G(\tilde{q}, \tilde{r})}{\det F(q, r) \cdot \det G(p, r') \cdot \det F(p', q')} \]  

(73)

One can derive the determinants \(G(p_1, p_2)\) and \(F(p_1, p_2)\) without difficult. Each of these matrices can be reduced to the diagonal form (not at the same
or, in more details,
where \( \lambda \) represent the \( F \) the relations in the algebra of the following formulae:

\[
\frac{G(p_1, p_2; \omega^k)}{G(p_1, p_2; 1)} = \left( \frac{b^N_1 d^N_2 - c^N_1 a^N_2}{d^N_1 b^N_2 - a^N_1 c^N_2} \right)^{k/N} \prod_{j=1}^{k} \frac{d_1 b_2 - a_1 c_2 \omega^j}{b_1 d_2 - c_1 a_2 \omega^j},
\]

\[
\frac{F(p_1, p_2; \omega^k)}{F(p_1, p_2; 1)} = \left( \frac{b^N_1 c^N_2 - d^N_1 a^N_2}{c^N_1 b^N_2 - a^N_1 d^N_2} \right)^{k/N} \prod_{j=1}^{k} \frac{c_1 b_2 - a_1 d_2 \omega^j}{b_1 c_2 - d_1 a_2 \omega^j}.
\]

We set
\[ G(p_1, p_2; 1) = F(p_1, p_2; 1) = 1. \]

Then
\[
\det G(p_1, p_2) = \left( \frac{b^N_1 d^N_2 - c^N_1 a^N_2}{d^N_1 b^N_2 - a^N_1 c^N_2} \right)^{N-1} \prod_{k=1}^{N-1} \prod_{j=1}^{k} \frac{d_1 b_2 - a_1 c_2 \omega^j}{b_1 d_2 - c_1 a_2 \omega^j},
\]

\[
\det F(p_1, p_2) = \left( \frac{b^N_1 c^N_2 - d^N_1 a^N_2}{c^N_1 b^N_2 - a^N_1 d^N_2} \right)^{N-1} \prod_{k=1}^{N-1} \prod_{j=1}^{k} \frac{c_1 b_2 - a_1 d_2 \omega^j}{b_1 c_2 - d_1 a_2 \omega^j}.
\]

**F The relations in the algebra of the \( Q \)-operators.**

The relations in the \( Q \)-operators algebra follow from the properties of the cyclic representations of the \( L \)-operators algebra. We now prove this.

The relations (12), (13), (14), (15) become evident if we recall that
\[ Q(u) = \text{tr}_0 L_1(u) L_2(u) \ldots L_n(u), \]
and the following symmetries are the case:

\[ L(\lambda p_1, \bar{p}_1) = \lambda L(p_1, \bar{p}_1), \]
\[ L(p_1, \mu \bar{p}_1) = \mu L(p_1, \bar{p}_1), \]
\[ L(a_1, b_1, c_1, d_1, \bar{a}_1, \bar{b}_1, \bar{c}_1, \bar{d}_1) = L(\lambda a_1, b_1, \lambda c_1, d_1, \lambda^{-1} \bar{a}_1, \bar{b}_1, \lambda^{-1} \bar{c}_1, \bar{d}_1), \]
\[ L(a_1, b_1, c_1, d_1, \bar{a}_1, \bar{b}_1, \bar{c}_1, \bar{d}_1) = L(a_1, \mu b_1, c_1, \mu d_1, \bar{a}_1, \mu^{-1} \bar{b}_1, \bar{c}_1, \mu^{-1} \bar{d}_1), \]
where \( \lambda, \mu \) are arbitrary numbers.

The relation (20) follows from another symmetry:
\[ L_1(p_1, \bar{p}_1) L_2(p_2, s) = L_1(p_1, \bar{p}_1) M M^{-1} L_2(p_2, s), \]
or, in more details,
\[ L_1(a_1, b_1, c_1, d_1, \bar{a}_1, \bar{b}_1, \bar{c}_1, \bar{d}_1) L_2(a_2, b_2, c_2, d_2, \bar{a}_2, \bar{b}_2, \bar{c}_2, \bar{d}_2) = \]
\[ = L_1(a_1, b_1, c_1, d_1, \beta \bar{a}_1, \alpha \bar{b}_1, \alpha \bar{c}_1, \beta \bar{d}_1) \]
\[ \times L_2(\beta^{-1} a_2, \alpha^{-1} b_2, \alpha^{-1} c_2, \beta^{-1} d_2, \bar{a}_2, \bar{b}_2, \bar{c}_2, \bar{d}_2), \]

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where $\alpha$, $\beta$ are arbitrary numbers. Here
\[
M = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.
\]

The relation (58) is obtained from the equation
\[
GL(a_1, b_1, c_1, d_1, \bar{a}_1, \bar{b}_1, \bar{c}_1, \bar{d}_1)G^{-1} = L(a_1, b_1, \Lambda c_1, \Lambda^{-1}d_1, \bar{a}_1, \bar{b}_1, \Lambda^{-1}\bar{c}_1, \Lambda\bar{d}_1),
\]
where
\[
\Lambda = \Lambda(p_1, \bar{p}_1) = \sqrt{\frac{b_1^N d_1^N - c_1^N \bar{a}_1^N}{d_1^N \bar{a}_1^N - a_1^N c_1^N}},
\]
and the relation (59) is obtained from
\[
FL_1(a_1, b_1, c_1, d_1, \bar{a}_1, \bar{b}_1, \bar{c}_1, \bar{d}_1)L_2(a_2, b_2, c_2, d_2, \bar{a}_2, \bar{b}_2, \bar{c}_2, \bar{d}_2)F^{-1} =
\]
\[
= L_1(a_1, b_1, c_1, d_1, a_2, b_2, \Omega c_2, \Omega^{-1}d_2)
\]
\[
\times L_2(\bar{a}_1, \bar{b}_1, \Omega^{-1}\bar{c}_1, \Omega\bar{d}_1, \bar{a}_2, \bar{b}_2, \bar{c}_2, \bar{d}_2),
\]
where
\[
\Omega = \Omega(p_1, p_2) = \sqrt{\frac{c_1^N b_2^N - \bar{a}_1^N d_2^N}{b_1^N c_2^N - \bar{d}_1^N a_2^N}}.
\]

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