Finite and infinite hypergeometric sums involving the digamma function

Juan L. González-Santander
Department of Mathematics, Universidad de Oviedo, 33007 Oviedo, Spain.
E-mail: gonzalezmarjuan@uniovi.es
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Abstract. We calculate some finite and infinite sums containing the digamma function in closed-form. For this purpose, we differentiate selected reduction formulas of the hypergeometric function with respect to the parameters applying some derivative formulas of the Pochhammer symbol. Also, we compare two different differentiation formulas of the generalized hypergeometric function with respect to the parameters. For some particular cases, we recover some results found in the literature. Finally, all the results have been numerically checked.

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1. Introduction

A large number of finite sums and series involving the digamma function have been compiled by Hansen [1] and more recently by Brychkov [2]. Some authors have contributed to enhance this compilation, such as Miller [3], who used reduction formulas of the Kampé de Fériet function; and Cvijović [4], who used the derivative of the Pochhammer symbol. Sums involving the digamma function occur in the expressions of the derivatives of the Mittag-Leffler function and the Wright function with respect to parameters [5, 6]. Also, they occur in the derivation of asymptotic expansions for Mellin-Barnes integrals [7].

The aim of this paper is the derivation of several apparently new results by using also the derivative of the Pochhammer symbol to known reduction formulas of the hypergeometric function. Nevertheless, for the last result given in this paper, we use other approach. For this purpose, we compare the expression of the first derivative of the generalized hypergeometric function with respect to the parameters given in [8] to the one given in [9]. As a consistency test, for many particular values of the results obtained, we recover expressions given in the literature. In addition, we have checked
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all the derived expressions with the aid of MATHEMATICA since sometimes we found some erratums in the literature.

This paper is organized as follows. In Section 2, we present some basic properties of the Pochhammer symbol, the beta and the digamma functions. In addition, we set the notation we use throughout the paper. In Sections 3 and 4 we derive some results for finite and infinite sums respectively involving the digamma function. Finally, we collect our conclusions in Section 5.

2. Preliminaries

The Pochhammer symbol is defined as [10, Eqn. 18:12:1]

\[(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)},\]  

(1)

where \(\Gamma(x)\) denotes the gamma function. Also, the beta function, defined as [11, Eqn. 1.5.3]

\[B(x,y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt,\]

\[\text{Re} \ x > 0, \ \text{Re} \ y > 0,\]

satisfies the property [11, Eqn. 1.5.5]

\[B(x,y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)}.\]  

(2)

For \(0 \leq z \leq 1\), the incomplete beta function is defined as [10, Eqn. 58:3:1]

\[B_z(x,y) = \int_0^z t^{x-1} (1-t)^{y-1} dt.\]

Next, we state some properties of the Pochhammer symbol, i.e. the reflection formula [10, Eqn. 18:5:1],

\[(-x)_n = (-1)^n (x-n+1)_n,\]  

(3)

the properties [10] Eqn. 18:5:7&2:12:3],

\[(x)_{n+1} = x (x+1)_n,\]  

(4)

\[\left(\frac{1}{2}\right)_n = \frac{(2n)!}{4^n n!},\]  

(5)

and the differentiation of the Pochhammer symbol [10] Eqn. 18:10:1]

\[\frac{d}{dx} (x)_n = (x)_n [\psi(x+n) - \psi(x)],\]  

(6)

thus

\[\frac{d}{dx} \left[\frac{1}{(x)_n}\right] = \frac{1}{(x)_n} [\psi(x) - \psi(x+n)],\]  

(7)

where \(\psi(x)\) denotes the digamma function [10, Ch. 44]

\[\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)},\]
with the following properties [11, Eqns. 1.3.3-9]

\[ \psi(z + 1) = \frac{1}{z} + \psi(z), \quad (8) \]

\[ \psi(1 - z) - \psi(z) = \pi \cot(\pi z), \quad (9) \]

\[ \psi(z) + \psi\left(z + \frac{1}{2}\right) + 2 \log 2 = 2\psi(2z), \quad (10) \]

\[ \psi(1) = -\gamma, \quad (11) \]

\[ \psi\left(\frac{1}{2}\right) = -\gamma - \log 4, \quad (12) \]

\[ \psi(n + 1) = -\gamma + H_n, \quad (13) \]

\[ \psi\left(n + \frac{1}{2}\right) = -\gamma - \log 4 + 2H_{2n} - H_n, \quad (14) \]

where

\[ H_n = \sum_{k=1}^{n} \frac{1}{k} \]

is the \(n\)-th harmonic number.

Throughout the paper, we adopt the notation [12, p. 797]

\[ \beta(z) = \frac{1}{2} \left[ \psi\left(\frac{z + 1}{2}\right) - \psi\left(\frac{z}{2}\right) \right]. \quad (15) \]

Also, \( _pF_q (z) \) denotes the generalized hypergeometric function, usually defined by means of the hypergeometric series [13, Sect. 16.2]:

\[ _pF_q \left( \begin{array}{c} (a_p) \\ \left( \begin{array}{c} (b_q) \\ \end{array} \right) \end{array} \bigg| z \right) = _pF_q \left( \begin{array}{c} (a_1, \ldots, a_p) \\ \left( \begin{array}{c} (b_1, \ldots, b_q) \\ \end{array} \right) \end{array} \bigg| z \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}, \]

whenever this series converge and elsewhere by analytic continuation. Finally, we use the notation:

\[ ((a_p))_k = (a_1)_k \cdots (a_p)_k. \]

### 3. Finite sums involving digamma function

**Theorem 1** The following summation formula holds true:

\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(a)_k}{(c)_k} \psi(a + k) \]

\[ = \frac{(c - a)_n}{(c)_n} [\psi(a) - \psi(c - a + n) + \psi(c - a)]. \quad (16) \]

**Proof.** Chu-Vandermonde summation formula [14, Corollary 2.2.3] is given by

\[ _2F_1 \left( \begin{array}{c} -n, a \\ \frac{c}{c} \end{array} \bigg| 1 \right) = \frac{(c - a)_n}{(c)_n}, \quad n \in \mathbb{N}. \quad (17) \]
According to (3) and (1), we have
\[ 2F_1 \left( \begin{array}{c} -n, a \\ c \end{array} \right| 1 \right) = \sum_{k=0}^{\infty} \frac{(-n)_k (a)_k}{k! (c)_k} \]
\[ = \sum_{k=0}^{\infty} \frac{(-1)^k (n-k+1)_k (a)_k}{k! (c)_k} \]
\[ = \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(n+1) (a)_k}{k! \Gamma(n-k+1) (c)_k} \]
\[ = \sum_{k=0}^{\infty} \frac{(-1)^k (a)_k}{k! (c)_k}. \]

Apply (6) to differentiate (17) with respect to the parameter \( a \). On the one hand, we have
\[ \frac{\partial}{\partial a} \left[ \frac{(c-a)_n}{(c)_n} \right] = -\frac{(c-a)_n}{(c)_n} \left[ \psi(c+a+n) - \psi(c-a) \right], \tag{18} \]
and, on the other hand,
\[ \frac{\partial}{\partial a} \left[ 2F_1 \left( \begin{array}{c} -n, a \\ c \end{array} \right| 1 \right) \right]
\[ = \sum_{k=0}^{n} \frac{n}{k} \frac{(-1)^k d(a)_k}{(c)_k} \]
\[ = \sum_{k=0}^{n} (-1)^k \frac{n}{k} \frac{(a)_k}{(c)_k} \psi(a+k) - \psi(a) \sum_{k=0}^{n} \frac{n}{k} \frac{(-1)^k (a)_k}{(c)_k} \]
\[ = \sum_{k=0}^{n} (-1)^k \frac{n}{k} \frac{(a)_k}{(c)_k} \psi(a+k) - \psi(a) \frac{(c-a)_n}{(c)_n}. \tag{19} \]

Equating (18) to (19), we obtain (16), as we wanted to prove. \( \blacksquare \)

**Corollary 2** For \( a = 1 \), taking into account (17), we get
\[ \sum_{k=0}^{n} \frac{(-1)^k \psi(k+1)}{(n-k)! (c)_k} \]
\[ = \frac{(c-1)_n}{n! (c)_n} \left[ -\gamma - \psi(c-1+n) + \psi(c-1) \right]. \]

**Theorem 3** Similarly to (16), if we perform the derivative with respect to the \( c \) parameter and apply (7), we will obtain
\[ \sum_{k=0}^{n} (-1)^k \frac{n}{k} \frac{(a)_k}{(c)_k} \psi(c+k) \]
\[ = \frac{(c-a)_n}{(c)_n} [\psi(c+n) + \psi(c-a) - \psi(c-a+n)]. \]
Corollary 4 For $a = 1$, we get
\[
\sum_{k=0}^{n} (-1)^k \psi(c + k) \frac{(c - k)!}{(n - k)!} (c)_k
\]
\[= \frac{c - 1}{n! (c - 1 + n)} \left[ \frac{1}{c - 1 + n} + \psi(c - 1) \right].
\]

Corollary 5 Taking the limit $c \to 1$ in (20), and applying (3), we obtain
\[
\sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} \psi(k + 1) = \frac{1}{n}.
\]

Theorem 6 For $n \in \mathbb{N}$, the following finite sum holds true:
\[
\sum_{k=0}^{n} 2^k \binom{a}{k} \frac{(2n - k - 1)!}{k! (n - k)!} \psi(a + k)
\]
\[= \frac{2^{2(n-1)}}{n} \left\{ \left( \frac{1 + a}{2} \right)^n \left[ \psi\left( \frac{1 + a}{2} + n \right) + \psi\left( \frac{a}{2} \right) + \log 4 \right] + \left( \frac{a}{2} \right)_n \left[ \psi\left( \frac{a}{2} + n \right) + \psi\left( \frac{1 + a}{2} \right) + \log 4 \right] \right\}.
\]

Proof. Apply the reflection formula (3) and the property (5) to the reduction formula [15]
\[2F_1\left( \begin{array}{c} -n, a \\ -2n + 1 \end{array} \right| 2 \right) = \frac{1}{\left( \frac{1}{2} \right)_n} \left[ \left( \frac{1 + a}{2} \right)_n + \left( \frac{a}{2} \right)_n \right], \quad n \in \mathbb{N},
\]
to arrive at
\[
\sum_{k=0}^{n} 2^k \binom{a}{k} \frac{(2n - k - 1)!}{k! (n - k)!} = \frac{2^{2n-1}}{n} \left[ \left( \frac{1 + a}{2} \right)_n + \left( \frac{a}{2} \right)_n \right].
\]
Differentiate (22) with respect to parameter $a$, taking into account (10) to obtain (21), as we wanted to prove.

Corollary 7 For $a = 1$, taking into account (13), (14) and (5), Eqn. (21) is reduced to
\[
\sum_{k=0}^{n} \frac{2^k (2n - k - 1)!}{(n - k)!} \psi(k + 1)
\]
\[= \frac{(n - 1)!}{4} \left[ 4^n (H_n - 2\gamma) + \left( \frac{2n}{n} \right) (2 H_{2n} - H_n - 2\gamma) \right],
\]
or equivalently, reversing the sum order,
\[
\sum_{k=0}^{n} \frac{(n + k - 1)!}{2^k k!} \psi(n + 1 - k)
\]
\[= \frac{(n - 1)!}{2^{n+2}} \left[ 4^n (H_n - 2\gamma) + \left( \frac{2n}{n} \right) (2 H_{2n} - H_n - 2\gamma) \right].
\]
4. Infinite sums involving digamma function

Theorem 8 For \( \Re(c - a - b) > 0 \), the following infinite series holds true:

\[
\sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k! (c)_k} \psi(a + k) = \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)} [\psi(c - a) - \psi(c - a - b) + \psi(a)].
\] (23)

**Proof.** Differentiate Gauss summation formula [14, Theorem 2.2.2]:

\[
_{2}F_{1} \left( a, b \mid c \right| 1 \right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k! (c)_k} = \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)},
\] (24)

with respect to the parameter \( a \).

**Remark 9** In [1, Addendum. Eqn. 55.4.5.2], we found an equivalent form, but with an erratum:

\[
\sum_{k=0}^{\infty} \frac{(b)_k (c)_k}{k! (a)_k} [\psi(c - k) - \psi(c)] \neq \frac{\Gamma(a) \Gamma(a - b - c)}{\Gamma(a - b) \Gamma(a - c)} [\psi(a - c) - \psi(a - b - c)],
\]

where we have to change in the sum \( \psi(c - k) \) by \( \psi(c + k) \). Also, the condition seems to be wrong.

**Corollary 10** For the particular case \( c = 2 \) and \( b = 1/2 \), taking into account (8), (9), we recover the formula given in [2, Eqn. 6.2.1(67)],

\[
\sum_{k=0}^{\infty} \frac{(a)_k (\frac{1}{2})_k}{k! (k+1)!} \psi(a + k) = \frac{2 \Gamma\left(\frac{3}{2} - a\right)}{\sqrt{\pi} \Gamma(2 - a)} \left[ \frac{1}{1 - a} + \pi \cot(\pi a) + 2 \psi(a) - \psi\left(\frac{3}{2} - a\right) \right],
\] (25)

where we have to change the validity of (25) to \( \Re a < 1 \).

**Corollary 11** For \( b > 0 \), the following expansion of the beta function holds true:

\[
\Beta(a, b) = - \sum_{k=0}^{\infty} \frac{(-b)_k}{k!} \psi(a + k).
\]

**Proof.** Calculate the following limit, taking into account (11) and (8):

\[
\lim_{x \to 0} \frac{\psi(x)}{\Gamma(x)} = \lim_{x \to 0} \frac{1}{\Gamma(x)} \left[ \psi(x + 1) - \frac{1}{x} \right] = - \lim_{x \to 0} \frac{1}{\Gamma(x + 1)} = -1.
\] (26)
Take $c = a$ in (23), and apply (26) and (2), to obtain
\[
\sum_{k=0}^{\infty} \frac{(-b)^k}{k!} \psi(a + k) = \lim_{c \to a} \frac{\Gamma(c) \Gamma(c - a + b)}{\Gamma(c - a) \Gamma(c + b)} \left[ \psi(c - a) - \psi(c - a + b) + \psi(a) \right] = -\frac{\Gamma(a) \Gamma(b)}{\Gamma(a + b)} = -B(a, b),
\]

\[\text{Remark 12} \quad \text{If we differentiate Gauss summation formula (24) with respect to parameter } c \text{ and we apply (7), we will obtain for } \Re(c - a - b) > 0,
\]
\[
\sum_{k=0}^{\infty} \frac{(a)_k (1-a)_k}{k!} \frac{(b)_k}{(c)_k} \psi(c + k) = \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)} \left[ \psi(c - a) + \psi(c - b) - \psi(c - a - b) \right],
\]
which is equivalent to [1, Addendum. Eqn. 55.4.5.1], but the condition $\Re(a + b - c) < 1$ seems to be wrong.

\[\text{Theorem 13} \quad \text{The following series holds true:}
\]
\[
\sum_{k=0}^{\infty} \frac{(a)_k (1-a)_k}{2^k k!} \frac{(b)_k}{(c)_k} \psi(b + k) = \frac{\sqrt{\pi} \Gamma(b)}{2^b \Gamma \left( \frac{a+b}{2} \right) \Gamma \left( \frac{b-a+1}{2} \right)} \left[ \psi \left( \frac{a+b}{2} \right) + \psi \left( \frac{b-a+1}{2} \right) + \log 4 \right].
\]

\[\text{Proof. Differentiate the summation formula [12 Eqn. 7.3.7(8)]}
\]
\[
_{2}F_{1} \left( a, 1-a \left| b \right| \right) = \sum_{k=0}^{\infty} \frac{(a)_k (1-a)_k}{2^k k!} \frac{(b)_k}{(c)_k} = \frac{2^{1-b} \sqrt{\pi} \Gamma(b)}{\Gamma \left( \frac{a+b}{2} \right) \Gamma \left( \frac{b-a+1}{2} \right)},
\]
with respect to parameter $b$. \[\square\]

\[\text{Corollary 14} \quad \text{Take } a = c \text{ and apply (12) to obtain}
\]
\[
\sum_{k=0}^{\infty} \frac{(1-a)_k}{2^k k!} \psi(a + k) = \psi(a) - \gamma.
\]

\[\text{Theorem 15} \quad \text{For } |z| < 1, \text{ the following series holds true:}
\]
\[
\sum_{k=0}^{\infty} \frac{(b)_k z^k}{(k+1)!} \psi(k + b) = \frac{(1-z)^{1-b}}{z(1-b)^2} \left[ (1-b) \log (1-z) - \left[ 1 - (1-z)^{b-1} \right] [1 + (1-b) \psi(b)] \right].
\]
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Proof. Differentiate the reduction formula [12, Eqn. 7.3.1(125)],

\[ \binom{a + 1, b}{a} \frac{d}{db} \binom{b}{k + 1} \psi (k + b) \]

with respect to the parameter \(b\). \(\blacksquare\)

Remark 16 Taking the limit \(b \rightarrow 1\), we arrive at

\[ \sum_{k=0}^{\infty} z^k \frac{\psi (k + 1)}{k + 1} = \log (1 - z) \frac{2 \gamma + \log (1 - z)}{2z}, \]

\[ |z| < 1, \]

which is equivalent to [2, Eqn. 6.2.1(2)].

Remark 17 For \(b = 2\), and taking into account (13) for \(n = 1\), i.e. \(\psi (2) = 1 - \gamma\), we arrive at

\[ \sum_{k=1}^{\infty} z^k \frac{\psi (k + 1)}{k + 1} = \frac{\gamma z + \log (1 - z)}{z - 1}, \]

\[ |z| < 1, \]

which is equivalent to [2, Eqn. 6.2.1(1)].

Theorem 18 For \(|z| < 1\), the following infinite sum holds true:

\[ \sum_{k=0}^{\infty} \frac{z^k (a + 1)_k (b)_k}{k! (a)_k} \psi (k + b) = \frac{[\psi (b) - \log (1 - z)] [1 - (1 - \frac{b}{a}) z] + z/a}{(1 - z)^{1+b}}. \]

Proof. Differentiate the following reduction formula [13, Eqn. 15.4.19]

\[ \binom{a + 1, b}{a} \frac{d}{db} \binom{b}{k} \psi (k + b) = \binom{a + 1, b}{a} \frac{d}{db} \binom{b}{k} \psi (k + b) \]

with respect to parameter \(b\). \(\blacksquare\)

Corollary 19 For the particular case \(b = a\), we obtain

\[ \sum_{k=0}^{\infty} \frac{z^k (a + 1)_k}{k!} \psi (k + a) \]

\[ = \frac{\psi (a) - \log (1 - z) + z/a}{(1 - z)^{1+a}}. \]

Corollary 20 For the particular case \(b = 1\), we obtain

\[ \sum_{k=0}^{\infty} z^k (a + k) \psi (k + 1) \]

\[ = \frac{z - [\gamma + \log (1 - z)] [a + (1 - a) z]}{(1 - z)^2}. \]
Theorem 21 The following series holds true:
\[
\sum_{k=0}^{\infty} \frac{(k+1)!}{(b)_k} 2^{-k} \psi (k + b) = 2 \left[ (b - 1) \psi (b) - 1 \right] + 4 \left[ 2b - 3 - (b - 1) (b - 2) \psi (b) \right] \beta (b - 1)
+ 4 (b - 1) (b - 2) \beta' (b - 1).
\] (28)

Proof. Differentiate the reduction formula [12, Eqn. 7.3.7(18)]
\[
_2F_1 \left( 1, \frac{1}{2} \left| \frac{1}{2} \right. \right) = \sum_{k=0}^{\infty} \frac{(k+1)!}{(b)_k} 2^{-k} = 2 \left( b - 1 \right) \beta (b - 1),
\]
with respect to parameter \( b \). \( \square \)

Remark 22 If we differentiate the reduction formula [12, Eqn. 7.3.7(17)]:
\[
_2F_1 \left( 1, 1 \left| \frac{1}{2} \right. \right) = \sum_{k=0}^{\infty} \frac{k!}{(b)_k} 2^{-k} = 2 \left( b - 1 \right) \beta (b - 1),
\]
with respect to parameter \( b \), we will obtain [2, Eqn. 6.2.1(64)]
\[
\sum_{k=0}^{\infty} \frac{2^{-k} k!}{(b)_k} \psi (k + b) = 2 \left[ (b - 1) \psi (b) - 1 \right] \beta (b - 1) - 2 (b - 1) \beta' (b - 1).
\] (29)

Corollary 23 Subtracting (29) from (28), we arrive at
\[
\sum_{k=0}^{\infty} \frac{k!}{(b)_k} 2^{-k-1} \psi (k + b) = (b - 1) \left[ \psi (b) + (2b - 3) \beta' (b - 1) \right] + [4b - 5 - (b - 1) (2b - 3) \psi (b)] \beta (b - 1) - 1.
\]

Theorem 24 For \( |z| < 1 \), the following infinite sum holds true:
\[
\sum_{k=0}^{\infty} \frac{z^k (a + 1)_k (b)_k}{(a)_k (c)_k} \psi (c + k) = \psi (c - 1) \ _3F_2 \left( 1, a + 1, b \left| \frac{1}{a}, c \right. \right) \left( \frac{b, c - 1, c - 1}{b, c - 1} \right)
+ \frac{1}{a (1 - z)^{1+b}} \left\{ \frac{a + (b - a)}{c - 1} \ _3F_2 \left( \frac{b, c - 1, c - 1}{b, c - 1} \left| \frac{z}{z - 1} \right. \right) \right\}
+ \frac{b (c - 1) z}{c^2 (z - 1)} \ _3F_2 \left( \frac{b + 1, c, c}{c + 1, c + 1} \left| \frac{z}{z - 1} \right. \right) \right\}
\] (30)

Proof. On the one hand, consider the reduction formula [12, Eqn. 7.4.4(94)]
\[
_3F_2 \left( \frac{-k, a, b}{a + \ell, b + n} \left| \frac{1}{1} \right. \right)
\]
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for the particular case \( \ell = 1 \), \( n = 1 \), to obtain

\[
3F_2 \left( \begin{array}{c}
-k, a, b \\
a + 1, b + 1
\end{array} \left| 1 \right. \right) = \frac{k!a b}{(a)_{k+1} (b)_{k+1}} \left[ \frac{(a)_{k+1} - (b)_{k+1}}{a - b} \right].
\]

Take the limit \( a \to b \) and apply (30) as well as the property (4), to obtain

\[
3F_2 \left( \begin{array}{c}
-k, b, b \\
b + 1, b + 1
\end{array} \left| 1 \right. \right) = \frac{k!b^2}{[b]_{k+1}^2} \lim_{a \to b} \left[ \frac{(a)_{k+1} - (b)_{k+1}}{a - b} \right]
\]

\[
= \frac{k!b^2}{[b]_{k+1}^2} \frac{d}{dx} [(x)_{k+1}]_{x=b}
\]

\[
= \frac{k!b}{(b + 1)_{k}} [\psi (b + 1 + k) - \psi (b)]. \tag{31}
\]

On the other hand, from (27), we have

\[
2F_1 \left( \begin{array}{c}
\alpha + k + 2, \beta + k + 1 \\
\alpha + k + 1
\end{array} \left| z \right. \right) = \left( 1 + \frac{\beta - \alpha}{\alpha + k + 1} z \right) (1 - z)^{-2 - \beta - k}. \tag{32}
\]

Now, equate the results given in [9] and [8],

\[
D^m_\beta \left[ \begin{array}{c}
\cdots (a_i) \\
\beta
\end{array} \left| b, (b_q) \right. \left| z \right. \right]
\]

\[
= \frac{m! (-1)^m ((a_p))_1}{b_{m+1} ((b_q))_1}
\]

\[
\sum_{k=0}^{\infty} \frac{z^{k+1} ((a_p + 1)_k)}{k! (k + 1)! ((b_q + 1)_k)} m_{m+1}F_{m+1} \left( \begin{array}{c}
-k, b, \ldots, b \\
b + 1, \ldots, b + 1
\end{array} \left| 1 \right. \right)
\]

\[
= m! (-1)^m z \sum_{k=0}^{\infty} \frac{(-z)^k ((a_p + 1)_{k+1})}{k! (k + 1)! ((b_q + 1)_{k+1})} m_{m+1}F_{m+1} \left( \begin{array}{c}
(a_p) + k + 1 \\
(b_q) + k + 1, k + 2
\end{array} \left| z \right. \right),
\]

for the particular case \( m = 1 \), \( (a_p) = (\alpha + 1, \beta, 1) \) and \( (b_q) = (\alpha) \), to obtain

\[
= \frac{(\alpha + 1) \beta}{b^2 \alpha} \sum_{k=0}^{\infty} \frac{z^k (\alpha + 2)_k (\beta + 1)_k}{k! (\alpha + 1)_k} 3F_2 \left( \begin{array}{c}
-k, b, b \\
b + 1, b + 1
\end{array} \left| 1 \right. \right)
\]

\[
= \sum_{k=0}^{\infty} \frac{(-z)^k (\alpha + 1)_{k+1} (\beta)_{k+1}}{k! (\alpha)_{k+1} (k + b)^2} 2F_1 \left( \begin{array}{c}
\alpha + k + 2, \beta + k + 1 \\
\alpha + k + 1
\end{array} \left| z \right. \right).
\]

Next, insert (31) and (32), and simplify the result using (4), to arrive at

\[
\sum_{k=0}^{\infty} \frac{z^k (\alpha + 2)_k (\beta + 1)_k}{(\alpha + 1)_k (b + 1)_k} [\psi (b + 1 + k) - \psi (b)]
\]
\[
\frac{1}{b(\alpha+1)(1-z)^{2+\beta}} \sum_{k=0}^{\infty} \frac{(\beta+1)_k [(b)_k]^2}{k! [(b+1)_k]^2} \left[ \alpha + k + 1 + (\beta - \alpha) z \right] \left( \frac{z}{z-1} \right)^k.
\]

Grouping terms,
\[
\sum_{k=0}^{\infty} \frac{z^k (\alpha+2)_k (\beta+1)_k}{(\alpha+1)_k (b+1)_k} \psi(b + 1 + k)
\]
\[= \psi(b) \sum_{k=0}^{\infty} \frac{z^k (\alpha+2)_k (\beta+1)_k (1)_k}{k! (\alpha+1)_k (b+1)_k}
\]
\[+ \frac{1}{b(\alpha+1)(1-z)^{2+\beta}} \left\{ (\alpha + 1 + (\beta - \alpha) z) \sum_{k=0}^{\infty} \frac{(\beta+1)_k [(b)_k]^2}{k! [(b+1)_k]^2} \left( \frac{z}{z-1} \right)^k
\]
\[+ \frac{(\beta+1)b^2}{(b+1)^2} \left( \frac{z}{z-1} \right) \sum_{k=0}^{\infty} \frac{(\beta+2)_k [(b+1)_k]^2}{k! [(b+2)_k]^2} \left( \frac{z}{z-1} \right)^k \right\},
\]

and recasting the sums with hypergeometric functions (renaming the parameters), we finally arrive at (30), as we wanted to prove. \[\Box\]

**Remark 25** For the particular case \(a = b\), and taking into account the reduction formula [12, Eqn. 7.3.1(119)]
\[
\left. \begin{array}{l}
\binom{1}{a} \binom{c}{z} = z^{1-c} (1-z)^{c-a-1} (c-1) B_z (c-1, a-c+1),
\end{array} \right\}
\]
we obtain for \(|z| < 1\)
\[
\sum_{k=0}^{\infty} \frac{z^k (a)_k}{(c)_k} \psi(c + k)
\]
\[= \psi(c-1) z^{1-c} (1-z)^{c-a-1} (c-1) B_z (c-1, a-c+1)
\]
\[+ \frac{1}{(1-z)^a} \left\{ \frac{1}{c-1} 3\!F_2 \left( \begin{array}{c} a-1, c-1, c-1 \end{array} \left| \begin{array}{c} z \\ z-1 \end{array} \right. \right) + \frac{(c-1) z}{c^2 (z-1)} 3\!F_2 \left( \begin{array}{c} a, c, c \end{array} \left| \begin{array}{c} z \\ z-1 \end{array} \right. \right) \right\},
\]
which is a non-trivial alternative form of the result given in [4]:
\[
\sum_{k=0}^{\infty} \frac{z^k (a)_k}{(c)_k} [\psi(c + k) - \psi(c)]
\]
\[= \frac{az}{c^2 (1-z)^{a+1}} 3\!F_2 \left( \begin{array}{c} a+1, c, c \end{array} \left| \begin{array}{c} z \\ z-1 \end{array} \right. \right),
\]
a \in \mathbb{C}, |z| < 1.
5. Conclusions

We have calculated some finite and infinite sums involving the digamma function differentiating some reduction formulas of the hypergeometric function with respect to the parameters and applying the differentiation formulas of the Pochhammer symbol given in (6) and (7). It is worth noting that this method can be applied to many other reduction formulas of hypergeometric and generalized hypergeometric functions. Here we have only selected some interesting new cases, some of which have allowed us to detect errors in the literature. Also, as a consistency test, we have recovered some formulas found in the literature from some particular cases of the results obtained.

Nevertheless, in (30), we have applied another approach, wherein we have compared the differentiation formulas given in (33) for a particular case of the parameters. This approach is not as straightforward as the other one. However, note that the particular case given in (34) applying this method provides a non-trivial alternative form of the result (35) found in the literature.

Finally, we point out that all the sums presented in this paper have been numerically checked with MATHEMATICA and they are available at https://shorturl.at/CFG24.

References

[1] Hansen E R 1975 A Table of Series and Products (Englewood Cliffs, NJ (USA) Prentice-Hall)
[2] Handbook of (S
[3] Miller A R 2006 Journal of Physics A: Mathematical and General 39 3011
[4] Cvijović D 2008 Journal of Physics A: Mathematical and Theoretical 41 455205
[5] Apelblat A 2020 Mathematics 8 657
[6] Apelblat A and González-Santander J L 2021 Mathematics 9 3255
[7] Paris R B and Kaminski D 2001 Asymptotics and Mellin-Barnes integrals vol 85 (Cambridge University Press)
[8] Fejzullahu B X 2017 Integral Transforms and Special Functions 28 781–788
[9] Sofotasios P and Brychkov Y A 2018 Integral Transforms and Special Functions 29 852–865
[10] Oldham K B, Myland J and Spanier J 2009 An Atlas of functions: with equator, the atlas function calculator (Springer)
[11] Lebedev N N 1965 Special Functions and their applications (Prentice-Hall Inc.)
[12] Prudnikov A P, Brychkov Y A and Marichev O I 1986 Integrals and Series: More special functions vol 3 (CRC press)
[13] Olver F W, Lozier D W, Boisvert R F and Clark C W 2010 NIST Handbook of mathematical functions (Cambridge University Press)
[14] Andrews G E, Askey R, Roy R, Roy R and Askey R 1999 Special Functions vol 71 (Cambridge university press Cambridge)
[15] Qureshi M, Jabee S and Ahamad D 2022 TWMS Journal of Applied and Engineering Mathematics 12 52