Three Dimensional Reductions of Four-Dimensional Quasilinear Systems

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Abstract

In this paper we show that integrable four-dimensional linearly degenerate equations of second order possess infinitely many three-dimensional hydrodynamic reductions. Furthermore, they are equipped with infinitely many conservation laws and higher commuting flows. We show that the dispersionless limits of nonlocal KdV and nonlocal NLS equations (the so-called Breaking Soliton equations introduced by O.I. Bogoyavlenski) are one and two component reductions (respectively) of one of these four-dimensional linearly degenerate equations.

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1 Introduction

This work is inspired by a study of Bogoyavlenskii’s Breaking Soliton equations and especially their dispersionless limit. These equations arise as a simple two-dimensional generalisation of well-known equations, by allowing the Lax pair to depend on an additional independent variable. The analogue of KdV, often called the Breaking Soliton equation (see [2]), written in its nonlocal form is

\[ v_t - \frac{1}{2} v_y \partial_y^{-1} v_x - v v_x + \frac{\epsilon^2}{2} v_{xyy} = 0. \]

(1)

This equation is integrable, possesses a Lax pair and infinitely many commuting flows. What is remarkable is that its dispersionless limit

\[ v_t + v v_x + u v_y = 0, \quad u_y = \frac{1}{2} v_x, \]

(2)

cannot be treated with the standard integrability test for multidimensional quasilinear systems, based on the method of hydrodynamic reductions, since the dispersion relation is degenerate - it reduces to two lines rather than being a conic (for details see Section 2 and [3, 4]). Furthermore, the (2+1)-dimensional non-linear Schrödinger equation, which also appears in [2] as a breaking soliton generalisation of NLS

\[ i \epsilon \psi_t + \epsilon^2 \psi_{xy} \pm 2 \psi \partial_y^{-1}(|\psi|^2)_x = 0, \]

(3)

after an appropriate transformation (the so called Madelung transformation, see Section 5), gives rise in a dispersionless limit to

\[ R^1_t + R^1_x R^1_y + u R^1_y = 0, \quad R^2_t + R^2 R^2_x + u R^2_y = 0, \quad u_y = \frac{1}{2} (R^1_t + R^2_t), \]

(4)

Since both nonlocal systems (1) and (3) are integrable, their dispersionless limits (2) and (4) are also integrable (because they preserve infinitely many conservation laws and higher commuting flows). So the question of how to understand their integrability, and more generally the integrability of the generalisation to \( M \) components (\( \kappa_i \) are constants)

\[ R^i_t + R^i_x R^i_y + u R^i_y = 0, \quad u_y = \sum_{i=1}^{M} \kappa_i R^i_x, \quad i = 1, \ldots, M, \]

arises naturally\(^1\).

\(^1\)The first system (2) is linearisable by a point transformation of the dependent and independent variable \( x = x(v, t) \). Such an approach, however, does not generalise to the multicomponent case.
As already mentioned, the method of hydrodynamic reductions provides a standard, constructive test for the integrability of multidimensional quasilinear systems of first order. A key point is that this method is based on the existence of sufficiently many two-dimensional hydrodynamic reductions (see again [3], [4]).

In this paper we show that integrable linearly degenerate four-dimensional equations of second order also possess infinitely many three-dimensional hydrodynamic reductions.

Among the simplest examples of linearly degenerate four-dimensional integrable equations are (see, for instance, [4], [9], [1]):

\[
U_y\tau = U_{xy}U_z - U_yU_{xz}, \\
U_{x\tau} = U_{t\tau} + U_{xx}U_z - U_xU_{xz}, \\
U_{\sigma\tau} = U_{zz} + U_zU_{x\sigma} - U_{\sigma}U_{xz},
\]

(5)

In all these examples \(U = U(x, t, y, z, \tau, \sigma)\). These four-dimensional quasilinear equations are determined by the following dispersionless Lax pairs (where \(\lambda\) is an arbitrary parameter)

\[
\psi_y = -\frac{1}{\lambda}U_y\psi_x, \quad \psi_{\tau} = \lambda\psi_z + U_z\psi_x, \\
\psi_{t\tau} = (\lambda + U_x)\psi_x, \quad \psi_\tau = \lambda\psi_z + U_z\psi_x, \\
\psi_z = \lambda\psi_\sigma + U_\sigma\psi_x, \quad \psi_{\tau} = \lambda\psi_z + U_z\psi_x,
\]

(6)

respectively, where \(\psi = \psi(x, t, y, z, \tau, \sigma; \lambda)\) in all cases.

The paper is organised as follows: in Section 2 we consider the method of hydrodynamic reductions and its applicability to the aforementioned systems. In Section 3 we introduce three-dimensional hydrodynamic chains arising from these systems. In Section 4 we introduce multidimensional hydrodynamic reductions. We return to Breaking Soliton equations in Section 5.

## 2 The Method of Hydrodynamic Reductions

Without loss of generality we consider the third quasilinear equation (5) from the previous Section (here we only changed the independent variables)

\[
U_xU_{ty} + U_{xz} = U_tU_{xy} + U_{tt},
\]

(7)

Introducing new variables such that \(u = U_x\) and \(a = U_t\) we obtain the four-dimensional two component quasilinear system

\[
u_t = a_x, \quad ua_y + u_z = au_y + a_t
\]

(8)

determined by the dispersionless Lax pair (see (6))

\[
\psi_t = \lambda\psi_x + u\psi_y, \quad \psi_z = \lambda\psi_t + a\psi_y.
\]

(9)

This dispersionless Lax pair is a reduction of the more general dispersionless Lax pair

\[
\psi_t = (\lambda + v)\psi_x + w\psi_y, \quad \psi_z = (\lambda + p)\psi_t + a\psi_y,
\]
which belongs to the class of hyper-Kähler hierarchies (see detail, for instance, in [9]). Moreover, the dispersionless Lax pair (9) together with the quasilinear system (8) is a four-dimensional reduction (such that \( s = -t \) and \( \partial_r = 0 \)) of the six-dimensional two-component quasilinear system (see [4])

\[
    u_t = u a_r - a u_r - a_x, \quad a_s = a u_y - u a_y + u_z,
\]
determined by the dispersionless Lax pair

\[
    \psi_s + \lambda \psi_x + u \psi_y - \lambda u \psi_r = 0, \quad \psi_z - \lambda \psi_t + a \psi_y - \lambda a \psi_r = 0.
\]

In order to apply the method of hydrodynamic reductions (see detail in [4]) we are looking for two-dimensional reductions in the form

\[
    r^i_t = \mu^i r^i_x, \quad r^i_y = \zeta^i r^i_x, \quad r^i_z = \eta^i r^i_x, \quad i = 1, 2, ... N,
\]

where \( N \) is an arbitrary natural number. This means that \( u(x, t, y, z) = \tilde{u}(r(x, t, y, z)) \), \( a(x, t, y, z) = \tilde{a}(r(x, t, y, z)) \), where the \( N \) Riemann invariants \( r^i(x, t, y, z) \) simultaneously solve the three commuting systems (10).

We obtain two consequences, the so called dispersion relation

\[
    \eta^i = \tilde{a} \zeta^i - \tilde{u} \zeta^i \mu^i + (\mu^i)^2 \tag{11}
\]

(as usual here: \( \partial_i \equiv \partial / \partial r^i \)) and the relation between two conservation law densities \( \tilde{u}, \tilde{a} \) and the characteristic velocity \( \mu^k \)

\[
    \mu^i \partial_i u = \partial_i \tilde{a}. \tag{12}
\]

Taking into account the Tsarev conditions\(^2\) (see detail in [10])

\[
    \frac{\partial_k \mu^i}{\mu^k - \mu^i} = \frac{\partial_k \zeta^i}{\zeta^k - \zeta^i} = \frac{\partial_k \eta^i}{\eta^k - \eta^i},
\]

and verifying the compatibility conditions \( \partial_k (\partial_i \tilde{a}) = \partial_i (\partial_k \tilde{a}) \), we obtain the Gibbons-Tsarev type system (cf. [6], [4])

\[
    \partial_k \zeta^i = \frac{\zeta^i (\zeta^k - \zeta^i)}{\mu^k - \mu^i - \tilde{u} (\zeta^k - \zeta^i)} \partial_k \tilde{u}, \quad \partial_k \mu^i = \frac{\zeta^i (\mu^k - \mu^i)}{\mu^k - \mu^i - \tilde{u} (\zeta^k - \zeta^i)} \partial_k \tilde{u}, \quad \partial_k \tilde{u} = \frac{\zeta^i - \zeta^k}{\mu^k - \mu^i - \tilde{u} (\zeta^k - \zeta^i)} \partial_i \tilde{u} \partial_k \tilde{u}. \tag{13}
\]

This system is in involution. Any particular solution (a general solution depends on \( 2N \) arbitrary functions of a single variable) determines three commuting hydrodynamic type systems (8), which can be integrated by the Tsarev generalised hodograph method (see detail in [10]). Each of these hydrodynamic type systems possesses a general solution parameterised by \( N \) arbitrary functions of a single variable. Thus, the method of

\[\text{\footnote{The Tsarev conditions follow from the compatibility conditions } (r^i_t)_y = (r^i_y)_t, (r^i)_z = (r^i_z)_t, (r^i)_y = (r^i_y)_z, \text{ where } N \text{ Riemann invariants } r^i(x, t, y, z) \text{ are common unknown functions for all three commuting flows (10).}}\]
two dimensional hydrodynamic reductions yields solutions parameterised by \(3N\) arbitrary functions of a single variable.

Now we introduce the auxiliary function \(b\) such that \(b = U_y\). This means that \(b_t = a_y\) and \(b_x = u_y\). Then the corresponding function \(\tilde{b}(r(x, t, y, z)) = b(x, t, y, z)\) satisfies the relationship between two conservation law densities \(\tilde{b}, \tilde{u}\) and the characteristic velocity \(\zeta^k\)

\[
\partial_t \tilde{u} = \frac{1}{\zeta^i} \partial_i \tilde{b}. \tag{15}
\]

Then equations (14) reduce to the form

\[
\partial_k \tilde{b} = 0.
\]

Hence, up to reparametrisations \(r^i \to \varphi_i(r^i)\), one has

\[
\tilde{b} = \sum_{m=1}^{N} r^m.
\]

Then other equations (11), (13) become (here \(f_i(r^i)\) are arbitrary functions)

\[
\zeta^i = \frac{1}{v^i}, \quad \mu^i = f_i(r^i) + \frac{\tilde{u}}{v^i}, \quad \eta^i = f_i^2(r^i) + f_i(r^i)\frac{\tilde{u}}{v^i} + \frac{\tilde{a}}{v^i},
\]

where

\[
\frac{\partial_{k} v^i}{v^k - v^i} = \frac{1}{f_k(r^k) - f_i(r^i)}. \tag{16}
\]

Thus the Gibbons-Tsarev type system (13)-(14) determines the commuting triple of two dimensional hydrodynamic type systems (10), where the functions \(\tilde{u}\) and \(\tilde{a}\) can be found by quadratures (see (15) and (12), respectively):

\[
d\tilde{u} = \sum_{m=1}^{N} v^m dr^m, \quad d\tilde{a} = \sum_{m=1}^{N} (f_m(r^m)v^m + \tilde{u})dr^m.
\]

The integrability of these hydrodynamic type systems (10) was investigated in \([7]\). Thus, the integrability (by the method of two dimensional hydrodynamic reductions) of the four-dimensional quasilinear equation of second order (8) is reduced to a construction of the general solution for the hydrodynamic type systems (10).

In the next Section we present a three-dimensional hydrodynamic chain associated with the quasilinear system (8) and its dispersionless Lax pair (9). Such three-dimensional hydrodynamic chains are a convenient tool for the construction of three dimensional hydrodynamic reductions.

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3See similar computations in [4], last formulas on page 2371.  
4See again similar computations in [4], the first formula on page 2372. The integrability of system (16) is presented in [7].
3 Three-Dimensional Hydrodynamic Chains

Under the potential substitution $h = \psi_y$ the dispersionless Lax pair (9) takes the form

$$h_t = \lambda h_x + (uh)_y, \quad h_z = \lambda^2 h_x + [(\lambda u + a)h]_y.$$ 

The asymptotic expansion at ($\lambda \rightarrow \infty$)

$$h = \exp \left( -\sum_{k=0}^{\infty} \frac{A^k}{\lambda^{k+1}} \right) = 1 - \frac{h_0}{\lambda} - \frac{h_1}{\lambda^2} - \frac{h_2}{\lambda^3} - ...$$

leads to a pair of commuting three-dimensional hydrodynamic chains,

$$A^k_t = A^{k+1}_x + uA^k_y, \quad k = 0, 1, ...,$$

$$A^k_z = A^{k+2}_x + uA^{k+1}_y + aA^k_y, \quad k = 0, 1, ...,$$

with two constraints:

$$u_y = A^0_x, \quad a_y = A^0_t.$$ 

These hydrodynamic chains possess infinitely many conservation laws:

$$(h_k)_t = (h_{k+1})_x + (uh_k)_y, \quad k = 0, 1, ...,$$ 

$$(h_k)_z = (h_{k+2})_x + (uh_{k+1} + ah_k)_y,$$

where the first three conservation law densities are

$$h_0 = A^0, \quad h_1 = A^1 - \frac{1}{2} (A^0)^2, \quad h_2 = A^2 - A^0 A^1 + \frac{1}{6} (A^0)^3.$$ 

The two constraints (19) reduce to two additional conservation laws

$$u_y = (h_0)_x, \quad (a - uh_0)_y = (h_1)_x.$$ 

**Remark:** The quasilinear equation (7) can be derived from the hydrodynamic chains (17) and (18), extracting the first two equations from (17), zeroth equation from (18) and both constraints (19), i.e.

$$A^0_t = A^1_x + uA^0_y, \quad A^0_z = A^1_x + aA^0_y, \quad u_y = A^0_x, \quad a_y = A^0_t.$$ 

Eliminating $A^2_x$,

$$A^0_t = A^1_x + uA^0_y, \quad A^0_z = A^1_x + aA^0_y, \quad u_y = A^0_x, \quad a_y = A^0_t,$$

where

$$A^2_x = A^1_t - uA^1_y.$$ 

Introducing a potential function $U$ such that $u = U_x, a = U_t$ and $A^0 = U_y$, (21) reduces to the following pair of equations:

$$A^1_x = U_{yt} - U_x U_{yy}, \quad A^1_t = U_{yz} - U_t U_{yy},$$

with compatibility condition $(A^1)_t = (A^1)_x$ leading to

$$U_{ytt} - U_x U_{yy} = U_{xyz} - U_t U_{xyy},$$

which is nothing but the derivative (with respect to independent variable $y$) of (7).


4 Three-Dimensional Hydrodynamic Reductions

In this Section we extract the most natural three-dimensional hydrodynamic reductions, i.e. $M$ component three-dimensional quasilinear systems, where $M$ is an arbitrary natural number.

I. The first reduction is given by the constraint $A^M = \text{const}$. Then (17) reduces to the following multi-component three dimensional hydrodynamic type systems

$$u_y = A^0_x, \quad A^{M-1}_t = u A^{M-1}_y, \quad A^k_t = A^{k+1}_x + u A^k_y, \quad k = 0, 1, ..., M - 2,$$

where $A^{M-1}$ can be recognised as a Riemann invariant. For instance, if $M = 1$, then

$$u_y = A^0_x, \quad A^0_t = u A^0_y;$$

if $M = 2$, then

$$u_y = A^0_x, \quad A^1_t = A^1_x + u A^0_y, \quad A^1_t = u A^1_y;$$

if $M = 3$, then

$$u_y = A^0_x, \quad A^1_t = A^1_x + u A^0_y, \quad A^2_t = A^2_x + u A^1_y, \quad A^2_t = u A^2_y,$$

etc.

II. The second reduction is given by the constraint $h_M = \text{const}$. Then (20) reduces to a set of multi-component three-dimensional hydrodynamic type systems

$$u_y = (h_0)_x, \quad (h_{N-1})_t = (u h_{N-1})_y, \quad (h_k)_t = (h_{k+1})_x + (u h_k)_y, \quad k = 0, 1, ..., M - 2.$$

For instance, if $M = 1$, then

$$u_y = (h_0)_x, \quad (h_0)_t = (u h_0)_y;$$

if $M = 2$, then

$$u_y = (h_0)_x, \quad (h_0)_t = (h_1)_x + (u h_0)_y, \quad (h_1)_t = (u h_1)_y;$$

if $M = 3$, then

$$u_y = (h_0)_x, \quad (h_0)_t = (h_1)_x + (u h_0)_y, \quad (h_1)_t = (h_2)_x + (u h_1)_y, \quad (h_2)_t = (u h_2)_y,$$

and so on.

III. The ansatz (5) ($\kappa_i$ are arbitrary constants)

$$A^k = \frac{1}{k + 1} \sum_{i=1}^{M} \kappa_i (R^i)^{k+1}, \quad k = 0, 1, ...$$

reduces (17) and (18) to two commuting $M$-component three-dimensional hydrodynamic type systems

$$R^i_t = R^i R^i_x + u R^i_y, \quad R^i_z = (R^i)^2 R^i_x + (u R^i + a) R^i_y, \quad (22)$$

---

5 This is the so-called “waterbag” reduction; see, for instance, [8]
where
\[ u_y = \left( \sum_{m=1}^{M} \kappa_m R^m \right)_x, \quad a_y = \left( \sum_{m=1}^{M} \kappa_m R^m \right)_t. \] (23)

**Lemma:** The two \( M \) component three-dimensional hydrodynamic type systems (22) and (23) commute with each other if and only if the functions \( u(x,t,y,z), a(x,t,y,z) \) satisfy (8).

**Proof:** Given by a straightforward computation.

This \( M \) parametric family of \( M \) component three-dimensional hydrodynamic reductions can be generalised to a family of \( M \) component three-dimensional hydrodynamic reductions parameterised by \( M \) arbitrary functions of a single variable.

**Theorem:** The four-dimensional quasilinear system (8) possesses \( M \) component hydrodynamic reductions (22), (23) where the functions \( u(x,t,y,z), a(x,t,y,z) \) are determined by
\[ u_y = A_0^x, \quad a_y = A_0^t, \] and all moments \( A^k(R) \) are parameterised by \( M \) arbitrary functions \( f_{0k}(R^k) \) of a single variable
\[ A^k = \sum_{m=1}^{M} f_{km}(R^m), \] (24)
where \( f_{k+1,i}^0(R^i) = R^i f_{k,i}^0(R^i), k = 0, 1, \ldots \)

**Proof:** Substituting \( A^k(R) \) into (17) and (18) leads to a sole relationship (here \( \partial_i \equiv \partial / \partial R^i \))
\[ \partial_t A^{k+1} = R^i \partial_i A^k. \] (25)

Compatibility conditions \( \partial_j(\partial_t A^{k+1}) = \partial_t(\partial_i A^{k+1}) \) yield (17). Substituting (24) into (25) implies \( f_{k+1,i}^0(R^i) = R^i f_{k,i}^0(R^i) \). The Theorem is proved.

All three-dimensional hydrodynamic reductions discussed above should be considered as integrable quasilinear systems, because they possess infinitely many conservation laws (see (20)). Since we deal with hyper-Kähler hierarchies, all higher commuting flows are known (see detail, for instance, in [1] and [9]). For instance, the next commuting three-dimensional hydrodynamic chain in the hierarchy is
\[ A_{k+1} = A^x_{k+3} + u A^y_{k+2} + a A^x_{k+1} + c A^y, \]
where
\[ u_y = A_0^x, \quad a_y = A_0^t, \quad c_y = A_0^z. \]

In our opinion, the existence of infinitely many higher commuting flows is sufficient for aforementioned reductions to be integrable. The problem of obtaining solutions is open and will be discussed elsewhere.

5 Breaking Soliton Equations

In the one-component case equations (22) and (23) with \( \kappa = 1/2 \), read:
\[ R_t = RR_x + uR_y, \quad R_z = (R)^2 R_x + (uR + a)R_y, \]
where

\[ u_y = \frac{1}{2} R_x, \quad a_y = \frac{1}{2} R_t. \]

Then the \( z \) equation above is nothing but the dispersionless limit of the second member of Bogojavlenetskii’s breaking soliton hierarchy

\[
v_z = v^2 v_x + \frac{3}{2} (\partial_y^{-1} v_x v_y) + \frac{1}{4} v_y \partial_y^{-1} (v^2)_x
\]

\[
+ \epsilon^2 \left[ -2 (v v_{yy})_x + \frac{3}{4} ((\partial_y^{-1} v_x) v_{xy})_{xx} - (\partial_y^{-1} v_x) v_{yyyy} - \frac{1}{2} (v_x v_y)_y - \frac{1}{6} v_{xxy} - \frac{1}{12} v_y v_{xy} \right] + \epsilon^4 \frac{1}{8} v_{xyyyyy}.
\]

The dispersionless limit is

\[
v_z = v^2 v_x + \frac{1}{4} (3v \partial_y^{-1} v_x + \partial_y^{-1} (v^2)_x)v_y.
\]

Taking into account that \( \partial_y^{-1} v v_x = 4a - 2vu \), we arrive exactly at \( v_z = (v)^2 v_x + (uv + a)v_y \). So we have shown that first two commuting flows of the nonlocal KdV hierarchy in the dispersionless limit are three-dimensional one-component reductions of the four-dimensional linearly degenerate equation of second order (8).

The two-component case is, in a similar way, related to the higher flows of the (2+1)-dimensional NLS. The second flow is

\[
\psi_z = \epsilon^2 \psi_{xyy} - 4 \psi \psi^* \psi_x - 2 \psi_y \partial_y^{-1} (\psi \psi^*)_x + 2 \psi \partial_y^{-1} (\psi_y \psi^*_x - \psi_x \psi^*_y) \quad (26)
\]

In order to obtain the dispersionless limit and derive equation (4) in Section 1 we use a Madelung transformation \( \psi = \sqrt{\rho} e^{iS/\epsilon} \) and introduce the Riemann invariants \( R^1 \) and \( R^2 \) by

\[
R^1 = S_y - 2 \sqrt{\rho}, \quad R^2 = S_y + 2 \sqrt{\rho}, \quad S_x = u.
\]

We do the same here. Then the hydrodynamic limit of (26) becomes

\[
R^1_z = R^2 R_x + (u R^1 + a) R_y, \quad u_y = \frac{1}{2} (R^1 + R^2)_x, \quad a_y = \frac{1}{2} (R^1 + R^2)_t
\]

As in the normal case for multidimensional hydrodynamic systems, each higher flow will introduce a “higher” nested non-locality, for instance the next one is \( c_y = \sum_{i=1}^N \kappa^i R^i_z \).

6 Conclusion

In this paper we considered the four-dimensional quasilinear system (8) determined by its dispersionless Lax pair (9). We extracted \( M \) component three-dimensional hydrodynamic reductions. Although we could not present a method for obtaining general solutions for these reductions, we believe that they are integrable because they possess infinitely many conservation laws and commuting flows. Moreover, our approach is universal, meaning

\[ \text{The entire hierarchy can be constructed by consecutively applying the KdV recursion operator } \mathcal{R} = -\frac{1}{2} \partial_y^2 + v + \frac{1}{2} v_y \partial_y^{-1} \text{ to } u_x \]
that any $D$-dimensional quasilinear system from any hyper-Kähler hierarchy possesses $(D-1)$-dimensional hydrodynamic reductions.

Any multi-dimensional linearly degenerate equation of second order, which necessarily belongs to hyper-Kähler hierarchy possesses infinitely many global solutions (see, for instance, [5]). However, the Breaking Soliton equations have no global solutions. Thus the extraction of corresponding solutions from multi-dimensional linearly degenerate equations of second order is a fascinating and open problem.

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