We discuss basic features and new developments in recently proposed induced gauge theory [1] solvable in any number of dimensions in the limit of infinite number of colours. Its geometrical (string) picture is clarified, using planar graph expansion of the corresponding matrix model. New analytical approach is proposed for this theory which is based on its equivalence to an effective two-matrix model. It is shown on some particular examples how the approach works. This approach may be applicable to a wide class of matrix models with tree-like quadratic couplings of matrices.

(This talk was presented on the International Symposium on Lattice Field Theory "LATTICE-92" in Amsterdam, the Netherlands, 15-19 September 1992)

1. Introduction

The recently proposed D-dimensional induced gauge theory solvable in the large N limit [1] has virtually produced some ambitious hopes to provide the long searched Master Field (MF) solution for QCD. This means that we would find a set of field variables that become classical in this limit. In principle, we know such a set: Wilson loop functionals, which satisfy classical loop equations (Makeenko-Migdal equations). But owing to their complicated functional character and essential non-locality they do not advance us very much on the way to an analytical solution of multicolour QCD.

Another idea which so far was developed separately in field theory, was the so called induced gauge theory. Briefly the idea of induction is the following: let us start from a Lagrangian of a field theory containing only massive gauged matter of some kind, with no addition of any kinetic terms for the corresponding gauge field. If we integrate out the matter we obtain a (generally) non-local effective action for only gauge fields. If one then increases the mass of matter field one obtains, in principle, the 1/(mass) expansion of the effective action in terms of local operators.

Of course, things are not as simple as they seem at first sight. We have to satisfy a few conditions to induce a needed gauge theory: the sign of the coupling of an induced interaction should be correct, the power-like divergences should be absent (to be left with only a logarithmic dependence of the induced gauge coupling on the cut-off), non-perturbative (e.g. in all orders of 1/(mass) expansion) locality of the induced action, e.c.

Surprisingly enough, one can formulate a gauge theory of this type: on the one hand, this induces (at least naively, in the logarithmic approximation) the Yang-Mills gauge coupling, and on the other hand, it is solvable by means of the large N matrix model technique. The MF appears to be local in this case, which immediately raises some doubts on the possibility for this theory to serve as a right QCD realization in 4 dimensions. And indeed, as will be clear from the following, the theory obeys symmetry too much (with respect to the centre of the gauge group) to describe a real
QCD \[2\], \[3\]. It is doubtful that this symmetry can be broken spontaneously, unless we make the model more complicated (and thus unsolvable).

Nevertheless, it seems to be interesting in itself that a solvable non-abelian gauge theory of any kind exists in physical dimension.

First of all, the search for an MF for real QCD along these lines should not be necessarily a hopeless enterprise. Some attempts of this kind are on the way \[3\].

Secondly, the corresponding D-dimensional matrix model might be a meaningful string theory with an infinite mass spectrum of physical states. In particular, the analytical methods proposed for the investigation of this model can be used for the solution of any multi-matrix model with the tree-like quadratic form of couplings between the matrices (which includes, say, Q-component Potts models on random graphs) \[6\]. Moreover, the induced gauge model under consideration is itself the model embedded into a tree, rather than into a real D-dimensional space, at least in the large-N limit \[7\]. For large N the space almost falls out from the model and its dimensionality appears only through the number of nearest neighbours of the D-dimensional lattice. In the next orders of 1/N, the space restores little by little, but the trivial space structure of the lowest order also signals that it is not a QCD realization.

On the other hand, as a matrix model, this theory obeys all the features of some string theory, which means that it is a theory of an extended object from the point of view of the internal (transversal) modes of the world sheet given by the corresponding planar graphs. Therefore, amid the tree nature of the model in the embedding space, it can produce an infinite and appropriately scaled spectrum of physical states due to these transversal modes. So, it is worth trying it.

An analytic approach of the model was suggested in \[1\], and a considerable progress was achieved in the papers \[8\], where the critical behaviour of the model was established. In the papers \[10\] (see also \[11\]), the case of Gaussian "induction matter" was solved exactly and the 1D string theory solution was demonstrated. In the paper \[1\], the new critical behaviour of this model was claimed, with a log-of-log singularity of the string susceptibility. This result contradicts in a way the results of \[8\], so that the whole question of critical behaviour in the model needs to be clarified, in our opinion.

In sect.2 we will formulate the model in the continuum and on the lattice, describe the induction idea and discuss the planar graphs picture.

In section 3, the issue of the additional symmetry of the centre of the gauge group is reviewed.

In sect.4 a general master field equation (MFE) will be formulated and the model will be shown to be equivalent to a random surface embedded in a tree in the large-N approximation.

In sect.5 the quantitative approach to the model will be formulated, based on its equivalence to the 2-matrix model (2MM), with the self-consistent effective potential for each of matrices. Using the orthogonal polynomials approach one can then write down a simple functional (and, in principle, solvable) equation. As an application the induced gauge model with a Gaussian scalar potential for any dimension will be solved in section 6 by this method and the result will coincide with the known one.

Sect.7 will review existing results and proposals on the induced gauge theory.

2. Formulation of the model and its physical interpretation

In the continuum version the model under consideration is just a gauged scalar field theory in the adjoint representation of the group SU(N) with the Lagrangian:

$$L = \frac{N}{g_0^2} \text{tr} \left( \left( \partial_\mu \Phi + i (A_\mu, \Phi) \right)^2 + V(\Phi) \right)$$

(1)

Here $\Phi(x)$ is an NxN matrix Hermitian field (a scalar field in the adjoint representation of SU(N)), and $A_\mu(x)$ is a gauge field.

Note that one does not add the Yang-Mills interaction explicitly.

The action for the lattice version of the model looks as:

$$S = \sum_x N \text{tr} \left[ V(\Phi(x)) - \sum_{\mu=1,2,...D} (\Phi(x) - U_\mu(x)\Phi(x + \mu)U_\mu^\dagger(x)) \right]$$

(2)
The partition function is given by:
\[ \int \prod_x d^N \Phi_x \int \prod_{<xy>} (dU)_{SU(N)} \exp -NS \quad (3) \]

The scalar potential can be an arbitrary polynomial
\[ V(\Phi) = m^2 \Phi^2 + \lambda_0 \Phi^4 + ... \quad (4) \]
or even a more complicated function.

Note that no pure Wilson plaquette term was added in (3).

Formally this model can induce the Yang-Mills interaction, if one would believe that the parameters of the scalar potential \( V(\Phi) \) could be adjusted so as to get into the regime of asymptotic freedom. Of course this is not guaranteed at all within this model. Considering this possibility we note that already the simplest one-loop logarithmic contribution induces the Yang-Mills interaction, if one would believe that the parameters of the scalar potential
\[ \mu^2 \rightarrow (m_0^2 - m_{\text{crit}}^2) \quad (5) \]
where the first terms of the \( 1/m \) expansion \( (m^2 = m_0^2 - m_{\text{crit}}^2) \) look as
\[ L_{\text{ind}}(A) = \frac{1}{g_{\text{ind}}^2} \text{tr} F_{\mu\nu}^2 + \text{const} \frac{\text{tr}}{m^2} (\partial_{\mu} F_{\mu\nu})^2 + ... \quad (6) \]
with the induced gauge coupling given by
\[ \frac{1}{g_{\text{ind}}^2} = \frac{N}{96\pi^2} \frac{\text{ln} \Lambda^2}{m^2} \quad (7) \]

From the last formula, we see that the renormalized mass \( m^2 \) must be chosen in such a way that it would be much smaller than the original cut-off scale \( \Lambda \), and much larger than the physical mass scale \( \mu \) of (would be) glueballs. The latter is given by the asymptotic freedom relation:
\[ \ln \frac{m^2}{\mu^2} \rightarrow \frac{48\pi^2}{11N g_0^2} \quad (8) \]

Comparing these two relations we find the scaling law:
\[ \mu^2 \sim (m_0^2 - m_{\text{crit}}^2)^b \quad (9) \]
where
\[ b = \frac{23}{22} \quad (10) \]

One should not take this number too seriously. This naive scaling law was obtained without taking into account the corrections from the scalar field self-interactions and from the effects of hard gluons (note that higher derivative terms in the induced action \( \bar{L} \) are suppressed by \( 1/m^2 \), and not by \( 1/\Lambda \), and hence will contribute to the renormalization of the scaling exponent).

The correct calculation of this scaling law, as well as the whole issue of the existence of the correct physical QCD phase in our theory is so far beyond our technical possibilities. May be, computer simulations, such as those started in [2], can shed some light on it. In the opinion of the author, it is unlikely that this phase could correspond to any local large-N master field approach described here, but the possibility for this phase to exist for finite N cannot be completely excluded.

3. Planar graph representation and extra \( Z_N \) symmetry

Let us try to give some geometrical interpretation of our theory in terms of planar graph expansion, since we will try later to give it the meaning of some improved bosonic string, and the planar graphs usually play the role of a regularization of a world sheet of a string theory.

Let us integrate first over the scalar field in (1). In the large N limit we will get a standard planar Feynman diagram technique, as for scalar field theory, with \( \Phi^3, \Phi^4 \ldots \)-vertices, but the propagators \( G_{kl}^{ij}(x,y,A) \) will be modified owing to the external gauge field: they will be given by sums over paths of P-ordered Wilson factors in adjoint representation:

\[ G_{kl}^{ij}(x,y,A) = \int D\Gamma(x,y) \exp -m^2 \text{Length}(\Gamma) \left\{ P \exp \left[ i \int_{\Gamma(x,y)} dz_{\mu} A_{\mu}(z) \right] \right\}_{ij} \quad (11) \]

The whole planar graph expansion can be represented in this way as standard planar graphs for a scalar field "embedded" into the "external gauge field media". The integration that is left, over the gauge field,
would produce some non-local interactions of different pieces of this discretized world sheet. In principle, we could try to integrate over them in the lattice version of the same representation, using the technique proposed in [13,14], in order to get some final representation of this model in terms of random surfaces, but it would give any promising calculational ideas.

Let us note only that on the lattice one can immediately see how the term like the Wilson plaquette action arises in the strong coupling expansion over the kinetic term in the induced gauge theory. Four terms corresponding to the links around a plaquette give after a Gaussian scalar field integration:

\[ |\text{tr} U(\text{plaquette})|^2 \rightarrow N^2 + \frac{1}{2} a^4 \text{tr} F_{\mu\nu}^2 \]  

(12)

As a consequence of the adjoint representation used in our model, we will always get the induced action for the gauge field with every matrix element of gauge \( U \)-matrix multiplied by some matrix element of its conjugate \( U^* \). As a result we have an extra gauge \( Z_N \) symmetry in our model, the symmetry with respect to the centre of the \( SU(N) \) gauge group. Namely, one can rotate every \( U \) matrix by the Abelian \( Z_N \) factor:

\[ U_{xy} \rightarrow U_{xy} \exp i \frac{2\pi}{N} k_{xy} \]  

(13)

where \( k_{xy} \) is an integer, with the consequence that any Wilson average \( W(C) \) in the fundamental representation will be non-zero (and equal to 1) iff the loop \( C \) forms a tree in the x-space (with zero minimal area of the surfaces spanned on it):

\[ W(C) = \langle 1/N \text{tr} U(C) > = \delta_{0,\text{Area}(C)} \]  

(14)

The same will be true for the adjoint Wilson loop in the large \( N \) limit, since it factorizes in this limit into two fundamental loop averages.

Of course, this kind of superconfinement has nothing in common with the physical confinement and expected area law for fundamental loops.

A way out could be some mechanism of spontaneous breaking of this symmetry. Even though this is a gauge symmetry, one may hope that it may be broken at least in the large \( N \) limit. To analyse this possibility we can use the arguments of [14]. For example, for one plaquette action in an adjoint representation we can use the large \( N \) factorization property:

\[ S_{\text{adj}} = \sum_{\text{plaquettes}} |U(\text{plaquette})|^2 =_{N \rightarrow \infty} N \beta < \frac{1}{\sum_{\text{plaquettes}} \text{tr} U(\text{plaquette})} \]  

(15)

which gives a non-linear equation for the effective coupling \( \bar{\beta} \)

\[ \bar{\beta} = \beta W(\text{plaquette},\bar{\beta}) \]  

(16)

The trivial solution

\[ \bar{\beta} = 0, W(\text{plaquette},\bar{\beta}) = 0 \]  

(17)

corresponds to the non-broken \( Z_N \) symmetry, whereas as a non-trivial solution

\[ \bar{\beta} \neq 0, W(\text{plaquette},\bar{\beta}) \neq 0 \]  

(18)

corresponds to the broken \( Z_N \) symmetry with a possibility of the correct physical behaviour.

It is easy to generalize this picture to the whole induced action of our model, which will include the sums over all loops, and not only over plaquettes.

One should stress that the corresponding large \( N \) phase transition with this symmetry breaking should take place before we approach the continuous limit of the theory.

This scenario seems to be unlikely for the large \( N \) master field approach presented below. It seems that this centre of the group symmetry remains unbroken for infinite \( N \).

4. Master field equation for the eigenvalues of scalar field

In order to see that our model is exactly solvable in the large \( N \) limit let us choose an opposite order of integration in the functional integral: first we will integrate over gauge fields, and then over the (eigenvalues of the) scalar field.

First we demonstrate the idea in the continuum version, but a rigorous treatment will be possible only on the lattice.

We introduce, as usual, the "angular" parametrization of scalar field in terms of eigen-
values $\phi = \text{diag}(\phi_1, \phi_2, \ldots, \phi_N)$ and eigenfunctions $\Omega_{ij} \in SU(N)$:

$$\Phi(x) = \Omega^+(x)\phi(x)\Omega(x)$$  \hspace{1cm} (19)

Putting it into the Lagrangian we obtain:

$$L = N \sum_{k=1}^{N} \left( (\partial_\mu \phi_k)^2 + V(\phi_k) \right) + \sum_{i,j=1}^{N} (\phi_i - \phi_j)^2 |B^i_\mu|^2$$  \hspace{1cm} (20)

where

$$B_\mu = \Omega^+ A_\mu \Omega(x) + i \Omega^+ \partial_\mu \Omega(x)$$  \hspace{1cm} (21)

The field $B_\mu$ is just a gauge rotated $A_\mu$, therefore, naively speaking, the integral over $B_\mu$ is Gaussian. Integrating over it we would obtain a simple effective action for only the eigenvalues, ready for the application of the large N master field approach. But the situation is slightly more complicated: any change of variables: $\Omega \rightarrow P\Omega$, $\phi \rightarrow P^+ \phi P$, where $P$ is a permutation matrix, does not change $\Phi = \Omega^+ \phi \Omega$. Therefore, integrating independently over $B_\mu$, we would overcount the possible configurations of scalar fields.

To avoid overcounting we can, say, impose the condition $\phi_1 < \phi_2 < \ldots < \phi_N$, or we have to subtract one by one the overcounted configurations. The second possibility can be naturally realized in the lattice version.

Returning to the lattice let us note, that after a change of variables at every vertex of the hypercubic lattice, we can again choose the gauge such that the angular degrees of freedom will be absorbed into the gauge fields: $\Omega_{xy} \Omega_{yx}^+ \rightarrow U_{xy}$, because of the group invariance of the Haar measure (on every link of the lattice separately). Then the integral over every link gauge variable can be performed by means of the so-called Itzykson-Zuber-Kharish-Chandra formula [13]:

$$I(\phi, \chi) = \int DU \exp \left( N \text{tr} (\phi - U\chi U^+) \right) \frac{\text{det}_\chi \exp(N(\phi - \chi)^2)}{\Delta(\phi)\Delta(\chi)}$$  \hspace{1cm} (22)

where

$$\Delta(\phi) = \prod_{i<j}(\phi_i - \phi_j)$$  \hspace{1cm} (23)

Taking into account extra $\Delta^2(\phi(x))$ of the Dyson measure for the "angular" parametrization of the scalar field, we arrive at the partition function:

$$Z = \int x,k \prod d\phi_k^i \exp(-S_{eff})$$  \hspace{1cm} (24)

where

$$S_{eff} = \sum_{xy} \log I(\phi_x, \phi_y) + \sum_x \left( \log \Delta^2(\phi_x) - N \sum_{k=1}^{N} V(\phi_k^i) \right)$$  \hspace{1cm} (25)

The above mentioned naive continuum action [20] after integration over $B_\mu$, will correspond only to the diagonal term in the determinant in [22]. All other $(N! - 1)$ terms can be considered as a consequent subtraction of the overcounted permutations of eigenvalues. It would be interesting to find some continuous field theoretical description of this object as a generalization of an ordinary spatial derivative.

Let us remark that in spite the Van-der-Monde determinants in the denominator of [22], the action [21] is not singular at all with respect to the coinciding eigenvalues: the determinant in the numerator has zeros cancelling these singularities, since the whole integral evidently has no such singularities.

Now our effective action [23] depends only on N eigenvalues and we are in position to apply the saddle-point method in the large N limit and derive the corresponding saddle-point equation. Namely, we look for the classical configuration for the eigenvalues obeying the stationarity condition (MFE):

$$\frac{\partial S_{eff}}{\partial \phi^i_x} = 0$$  \hspace{1cm} (26)

From the physical point of view it is natural to expect the spatially homogeneous vacuum for this system, so, identifying $\phi^i_x = \phi_i = \text{const}(x)$

we can write the effective action as

$$S_{eff} = (\text{volume}) (2D)[(\log I(\phi, \phi) + \frac{1}{12D} \log \Delta^2(\phi) - N \sum_{k=1}^{N} V(\phi_k)]$$  \hspace{1cm} (28)

Now the MFE [23] reads as

$$2D \frac{\partial \log I(\phi, \phi)}{\partial \phi_k} + 2 \sum_i \frac{1}{\phi_k - \phi_i} = V'(\phi_k)$$  \hspace{1cm} (29)
or, in terms of density of eigenvalues,
\[
\rho(\phi) = \frac{1}{N} \frac{dk(\phi)}{d\phi}
\]
the basic quantity to be calculated in this approximation:
\[
2D \frac{\partial \log I(\phi, \phi)}{\partial \phi} + 2P \int d\mu \rho(\mu) \frac{1}{\phi - \mu} = V'(\phi)
\]

But we have not yet closed the system of equations, since we still have to find some effective approach for the calculation of \( I(\phi, \phi) \). The original formula of Itzykson and Zuber \([22]\) is valid for any \( N \), but it is not very suitable in the limit of large \( N \), since we have to deal with the sum of \( N! \) sign-changing terms.

An approach to this problem was sketched in \([8]\), where the integral equation for the calculation of the IZ-integral in the large \( N \) limit was derived. Later we will discuss this approach.

In the next section we will propose a new approach based on the effective 2MM.

To conclude this section, let us recall a comment, made in \([6]\), on the equivalence of our induced gauge model and a multi-matrix model embedded into an infinite Bethe tree with the coordination number 2D, in the large \( N \) limit. This Q-matrix model obeys the following action:
\[
S = N \text{tr} \left( \sum_{\langle ij \rangle} \Phi_i \Phi_j + \sum_{i=1}^{Q} V(\Phi_i) \right)
\]
where the first sum runs over the bonds of an infinite tree with the coordination number 2D, and the second over its vertices. Note that if we would gauge the model in the spirit of our induced gauge theory, it would not change it since, on any tree, the gauge variables can be completely absorbed into the angular degrees of freedom of matrices.

On the other hand, integrating by means of the IZ-integral over the relative "angles" on every bond, and looking for the homogeneous saddle point for the eigenvalues (the infinite Bethe tree is certainly translational invariant), we obtain the same effective action \([23]\) and the same MFE \([23]\) or \([31]\) as for the D-dimensional induced gauge model.

Hence, in the large \( N \) limit the models coincide. Of course this coincidence is very worrisome from the point of view of a possible equivalence with QCD. As we see the dependence on the space structure almost disappears from the problem in the main order in \( 1/N \). Its dimensionality \( D \) enters the MFE only through the number of the nearest neighbours on the lattice. So, if we would formulate our theory on the 2-dimensional triangular lattice, we would obtain an effective dimensionality equal to 3, and for the hexagonal lattice it would be \( 3/2 \). Of course, in the next \( 1/N \) correction the dimension of space will enter in a more serious way, but we expect even the large \( N \) vacuum of QCD to be less trivial.

On the other hand, none of these features prevent us from hoping for some meaningful string theory for \( D > 1 \) following from the induced gauge theory. For example, the 1D strings \([22, 23]\) have an infinite mass spectrum with an appropriate scaling, and they are a particular case of our model. So, we can hope for the same properties of it for \( D > 1 \).

Another hope (although a weak one) to attack QCD starting from our model is the possibility to reach the physical QCD behaviour by adjusting also \( N \) to some critical value which is not separated from \( N = \infty \) by a phase transition point, even though the starting point is not an asymptotically free theory. It resembles the hopes related to the strong coupling expansion in Wilson lattice QCD which died when confronted with the computer simulations. The advantage of the induced gauge theory is that it gives, at least, most probably starting from the zero approximation the infinite mass spectrum.

5. The effective two-matrix model and the MFE based on it

In this section we propose a new approach to the induced gauge theory based on its equivalence with an effective 2MM. It is completely different from the one presented in our first paper \([1]\), which used some 2MM representation for the IZ integral. The latter was valid for any \( 1/n \) order, but the resulting integral equations were difficult to analyse.
This time we will work directly in the large N limit and look from the very beginning for the homogeneous MF. In this case, as we know, our effective action is given by (32). Note that the overall factor (volume)\(2D\) does not influence the MFE (29). This fact can be used to reduce the problem to an equivalent 2MM with some modified effective potential.

The 2MM action is a particular case of the action (22) for two matrices. In terms of two systems of eigenvalues \(\{\phi_k\}\) and \(\{\psi_k\}\), the partition function reads:

\[
Z_{2MM} = \int d^N \phi d^N \psi \exp[-N \sum_k (V(\phi_k) + V(\psi_k))] \Delta^2(\phi) \Delta^2(\psi) I(\phi, \psi)
\]

(33)

According to the \(Z_2\) symmetry of this model, there should exist a symmetric saddle point solution \(\phi_k = \psi_k = x_k\).

Comparing the effective action (33) with the effective action (22) at a homogeneous saddle-point, one finds that the induced gauge theory is equivalent to the 2MM with the derivative of the effective potential for the latter, which reads:

\[
V_{eff}'(x_k) = \frac{1}{2D} U'(x_k) + \frac{2D-1}{DN} \sum_{i \neq k} \frac{1}{x_k - x_i}
\]

(34)

It is useful to introduce the density of states, the basic quantity to be calculated in the large N limit:

\[
\rho(x) = \frac{1}{N} \frac{\partial k(x)}{\partial x}
\]

(35)

and the analytical function corresponding to the loop amplitude in 2MM:

\[
W(x) = < \text{tr} \frac{1}{x - \Phi} > = \int \frac{d\rho(y)}{x - y}
\]

(36)

The effective potential then reads:

\[
V_{eff}'(x) = \frac{1}{2D} U'(x) + \frac{2D-1}{D} \text{Re} W(x)
\]

(37)

Hence to find W(x) we have to solve a self-consistent problem: solving the 2MM with the effective potential depending on W(x) we find a self-consistency equation on W(x). This trick allows us to avoid the problem of calculation of the IZ integral, since we will now use a well-elaborated orthogonal polynomial formalism for the 2MM.

Let us review this formalism in the case of an arbitrary potential, using an elegant approach of the paper [16].

We start from the partition function of 2MM in the form [17]:

\[
Z_{2MM} = \int d^N x d^N y \Delta(x) \Delta(y) \exp(N \sum_i (x_i y_i - V(x_i) - V(y_i)))
\]

(38)

One introduces, according to [17], the orthogonal polynomials:

\[
P_n(x) = x^n + a_n-1 x^{n-1} + \ldots + a_0
\]

(39)

obeying the orthogonality condition:

\[
< m|n > = \int dx dy \exp\left(\frac{N}{\hbar} (x y - V(x) - V(y))\right) P_m(x) P_n(y) = \hbar_n \delta_{m,n}
\]

(40)

By means of the Jacobi relations

\[
x|n >= \sum_{l \geq 1} x^n x^{n+l} |n-l >= x_n |n>
\]

\[
\frac{\partial}{\partial x} |n >= \sum_{l \geq 1} p_{n+l} |n-l >= p_n |n>
\]

(41)

These relation, can be effectively used in matrix models of 2D gravity as a KdV-type approach [13,15].

In the large N limit we introduce the rescaled variables

\[
\lambda = \lambda_0 \frac{\Phi}{\hbar} \quad \omega = -\frac{N}{\hbar} \ln(\hat{z}) = \frac{\partial}{\partial \lambda}
\]

(43)

and the KdV-type relation takes the form of the Poisson brackets:

\[
{p(\lambda, \omega), x(\lambda, \omega)} = 1
\]

(44)

where \(p(\lambda, \omega), x(\lambda, \omega)\) are already ordinary functions. We could have started from these KdV-type relations, but we will follow the method of [1].

Considering the matrix element of \(< m | \frac{\partial}{\partial z} | n >\) and integrating by parts inside it, we obtain the equation:

\[
p(z) = V'(x(z)) - x \left( \frac{1}{fz} \right)
\]

(45)
We dropped here the explicit dependence on the cosmological constant $\lambda$ and used the transposition operation for $x(z)$

$$x^T(z) = x\left(\frac{1}{fz}\right)$$  \hspace{1cm} (46)

where $f_n = \frac{h_{n+1}}{h_n} = f(\lambda)$

Eq. (45) is in principle sufficient to define both $x$ and $p$. Since we know that $p$ contains only positive powers of $z$, we can compare the coefficients of non-positive powers. This gives sufficient number of relations to define everything. Particularly simple all this looks for the polynomial potentials $V(x)$, owing to the finite sums in (41). One can, say, easily recover all the results for the Ising model on random graphs [17,19,20].

In a forthcoming paper [21], we will show how effective equations like (44) and (45) are for the complete analysis of the critical regimes of 2MM.

We can now use eq. (45) for our $D$-dimensional induced gauge theory, where we have now to substitute $V(x)$ by $V_{eff}(x)$ from (34). We get

$$p(z) = \frac{1}{2D}U'(x(z)) + \frac{2D-1}{2D}ReW(x(z)) - x\left(\frac{1}{fz}\right)$$  \hspace{1cm} (47)

To close the equation one has to calculate $W(x)$ in terms of $x(z)$. One can obtain this relation from an orthogonal polynomial representation of the eq. (36). It reads:

$$\frac{\partial W(\lambda, x)}{\partial \lambda} = \frac{\partial \log z(\lambda, x)}{\partial x}$$  \hspace{1cm} (48)

where $z(\lambda, x)$ is the function inverse to $x(z)$, to be found from (17).

In the next section we will demonstrate how this equation works for some particular cases.

6. Particular examples

Let us demonstrate how the approach worked out in a previous section works for some particular examples: for usual 2MM, for $D = 0$ (pure 2d-gravity) and for arbitrary $D$ and quadratic potential $U(x)$.

a. Two-Matrix Model: $D = 1/2$

In this case $V_{eff}(x) = U(x)$, and we return to the original 2MM equation (34). A rather complete analysis of it will be done in [21], so we do not continue on this subject here.

b. Pure Gravity: $D = 0$

Now we can retain in (47) only those terms that are singular in $1/D$, which immediately gives the well-known equation for the 1-Matrix Model:

$$2ReW(x) = U'(x)$$  \hspace{1cm} (49)

with all familiar consequences for the pure 2D-gravity following from it.

c. Quadratic Bare Potential for any $D$

In this case we have:

$$U'(x) = m^2 x$$  \hspace{1cm} (50)

It is natural to expect the semi-circle law for the distribution of the eigenvalues here, so we will try the ansatz:

$$W(x) = \frac{1}{2R} \left( x - \sqrt{x^2 - 4R} \right)$$  \hspace{1cm} (51)

where $R(\lambda)$ has to be found. According to (48), this corresponds to

$$x(z) = \frac{1}{z} + Rz$$  \hspace{1cm} (52)

and

$$ReW(x) = \frac{1}{2R} x$$  \hspace{1cm} (53)

If we insert all these expressions into (47), we will find that $p(\lambda, z)$ should be linear in $z$ and in the normalization (43) it looks like

$$p(\lambda, z) = \lambda z$$  \hspace{1cm} (54)

Comparing the coefficients of $\frac{1}{z}$ and $z$ in (47), we find:

$$\frac{1}{R} = \frac{m^2(D - 1) \pm D\sqrt{m^4 - 2(2D - 1)}}{4(2D - 1)}$$  \hspace{1cm} (55)

For $D \geq 1$ and $m^4 \geq 2(2D - 1)$ we choose, of course, the positive root. This result coincides with that one obtained in [3] by Migdal’s method (see also [13]).

This solution is perfectly valid only in the case (which can be called a strong coupling regime) of real $R$ or for the effective mass $M$:

$$M^4 = m^4 - 2(2D - 1) \geq 0$$  \hspace{1cm} (56)
It is clear that the point $M = 0$ corresponds to the critical behaviour of Bethe tree, when its volume becomes infinite. It is hard to believe that some interesting (weak coupling) regime exists beyond this point.

d) Upside-Down Quadratic Potential for D=1

In this case, as we know (see [25] for review) the critical regime corresponds to "upside-down" oscillator potential, so we probably have to take the negative root in (55) and change $x$ into $ix$ in (51), which would give the right distribution of the eigenvalues for the 1D-matrix model in the critical regime describing the 1D-bosonic string theory [22,24].

7. Recent developments

Here we will discuss some alternative approaches to the induced gauge theory.

a. Migdal’s Approach

It is based on an integral equation found in [8] defining the large N limit of the Itzykson-Zuber integral. It allowed some new critical behaviour to be found for $D > 1$ in the model, as well as the first attempts in the spectrum calculations to be made.

We will look at this equation from a point of view that is a bit different of [8], considering it as a loop equation for the pure 2MM. Then one can easily restore the whole equation for the D-dimensional theory, by substituting the bare potential with the effective one, according to formula (34).

Consider an obvious identity for the resolvent in 2MM:

$$W(x) = \int d^{N^2} \Phi d^{N^2} \Psi \exp \left( -N \text{tr} V(\Phi) - N \text{tr} V(\Psi) \right) \frac{\text{tr}}{N x - \sigma_\Phi} \exp(N \text{tr} \Phi \Psi)$$

Integrating by parts in $\Psi$, we get

$$W(x) = \int d^{N^2} \Phi d^{N^2} \Psi \exp N \text{tr} \left( \Phi \Psi - V(\Phi) - V(\Psi) \right) \frac{\text{tr}}{N x + \sigma_\Phi} \frac{1}{V'(\Psi)}$$

The r.h.s. of (58) was calculated in [8] in the large N limit by means of the Riemann-Cauchi method.

The result can be nicely presented as follows:

$$W(x) = -\oint \frac{d\lambda}{2\pi i} \log (x - V'(y) + W(y))$$

where the contour integral encircles the cuts of $W(y)$ but leaves aside the singularity of the logarithm.

It is the loop equation for the 2MM, which was not known before (see [20] for a similar approach).

It is sufficient to substitute the bare potential with the effective one (34) in order to recover the full D-dimensional equation of [8].

Using this approach it was found also in [8] that the singularity of $W(x)$ near the end of the cut should be $x^\alpha$ with $\alpha = 1 + \frac{1}{\pi} \arccos(\frac{D}{2^D-1})$. But the a discrepancy with the case $D=1/2$ (Ising model on random graphs) where $\alpha = 4/3$, is a bit confusing to the author. May be this solution is applicable only to $D \geq 1$.

b. Log(log) Critical Behaviour

Another possible solution was advocated in [7]. It was found that the system for $D \geq 1$ has a critical behaviour of effective 1D-matrix model with the upside-down harmonic potential. The self-consistency equation on the "frequency" leads to a singularity of the free energy $\lambda^2 \log \log(\lambda)$ with respect to the cosmological constant. This is an interesting possibility since it would presumably give a well-scaled infinite mass spectrum, as for the conventional $D=1$ model. Unfortunately it can be well justified only in the limit of large $\lambda$.

The discrepancy between the two approaches is worrisome. It seems that the model still needs deeper understanding before we can view it as a model of an extended object in physical dimensions.

Some related problems and models can be found also in [29,27].

Let us note also that a similar model in two dimensions, but with an $F_{\mu\nu}^2$ term, was investigated in [30].

ACKNOWLEDGEMENTS

I would like to thank D.V.Boulatov, J.-M.Daul, and I.K.Kostov for interesting discussions, and the organizers of "LATTICE-92" for their kind hospitality.
REFERENCES

1. V.A.Kazakov and A.A.Migdal, *Induced QCD at large N*, preprint LPTENS-92/15, PUPT-1322 (June 1992); Nucl.Phys.B, to be published.

2. I.I.Kogan, G.W.Semenoff and N.Weiss, *Induced QCD and hidden local $Z_N$ symmetry*, UBC preprint UBCTP-92-022 (June 1992); I.I.Kogan, G.W.Semenoff, A.Morozov and N.Weis, *Area law and continuum limit in induced QCD*, preprint UBCTP-92-22 (July 1992); I.I.Kogan, G.W.Semenoff, A.Morozov and N.Weis, *Continuum Limits of limits of "induced QCD"*, preprint UBCTP-92/27 (August 1992);

S.Khokhlov and Yu.Makeenko, *The problem of large-N phase transition in Kazakov-Migdal model of induced QCD*, ITEP preprint ITEP-YM-5-92 (July 1992);

3. D.V.Boulatov, *Local symmetry in the Kazakov-Migdal gauge model*, preprint NBI-HE-92-62 (September 1992);

4. S.Khokhlov and Yu.Makeenko, ITEP preprint ITEP-YM-7-92 (July 1992);

5. A.A.Migdal, *Mixed models of induced QCD* preprint LPTENS-92/23 (August 1992) Princeton preprint PUPT-1343 (September 1992);

6. V.A.Kazakov, Nucl.Phys.B (Suppl.) (1987).

7. D.V.Boulatov, preprint NBI-HE-92-78 (November 1992);

8. A.A.Migdal, *Exact solution of induced lattice gauge theory at large N*, Princeton preprint PUPT-1323 (June 1992); preprint LPTENS-92/22 (August 1992); 1/$N$ expansion and particle spectrum in induced QCD, Princeton preprint PUPT-1332 (July 1992); Phase transitions in induced QCD, ENS preprint LPTENS-92/22 (August 1992);

9. D.Gross, *Some remarks about induced QCD*, Princeton preprint PUPT-1335 (August 1992);

10. Yu.Makeenko, *Large N reduction, Master Field and Loop Equations in Kazakov-Migdal model*, preprint ITEP-YM-6-92 (August 1992);

11. M.Caselle, A.D’Adda and S.Panzeri, preprint DFTT 38/92 (July 1992);

12. A.Glocksch and Yue Shen, preprint BNL, (July 1992);

S.Aoki, A.Gocksch and Y.Shen, preprint UTHEP-242, August (1992).

13. V.A.Kazakov, Phys.Lett.128B, 316 (1986).

14. I.K.Kostov, Nucl.Phys.B265 [FS15], 223 (1986).

15. C.Itzykson and J.-B.Zuber, *J.Math.Phys. 21* (1980) 411;

16. M.R.Douglas, proceedings of Cargese Workshop 1990, in "Random Surfaces and Quantum Gravity", ed. by O.Alvarez, et al. (1991).

17. M.L.Mehta, *Comm.Math.Phys. 79*, 327 (1981).

18. M.R.Douglas, Phys.Lett.B238, 176 (1990).

19. V.A.Kazakov, Phys.Lett. 119B, 140 (1985).

20. D.V.Boulatov and V.A.Kazakov, Phys.Lett. 186B, 379 (1987).

21. J.-M.Daul, V.A.Kazakov and I.K.Kostov, to appear.

22. V.A.Kazakov and A.A.Migdal, Nucl. Phys. B311 (1988) 171;

23. I.K.Kostov, Phys.Lett.B215, 499 (1988).

24. E.Brezin, V.A.Kazakov, Al.B.Zamolodchikov, Nucl. Phys. B338 (1990) 673;

D.Gross and N.Milkovic, Phys. Lett. B238 (1990) 217;

P.Ginsparg and J.Zinn-Zustin, Phys. Lett. B240 (1990) 333; G.Parisi, Phys.Lett.

25. V.A.Kazakov, proceedings of Cargese Workshop-1990, in ”Random Surfaces and Quantum Gravity”, ed. by O.Alvarez, et al.(1991).

26. J.Alfaro, preprint CERN-TH-6531/92, (July, 1992).

27. B.Rusakov, *From hermitean matrix model to lattice gauge theory*, preprint TAUP 1996-92 (September, 1992)

28. S.Dalley, preprint PUPT-1310, (March 1992).

29. S.Dalley and I.Klebanov, preprint PUPT-
1333 (July 1992).
30. S. Dalley and I. Klebanov, preprint PUPT-1342 (September 1992).