Comments On Hamiltonian Formalism
Of \textit{AdS/CFT} Correspondence

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\textbf{Abstract}

As a toy model to search for Hamiltonian formalism of the \textit{AdS/CFT} correspondence, we examine a Hamiltonian formulation of the \textit{AdS}_2/\textit{CFT}_1 correspondence emphasizing unitary representation theory of the symmetry. In the course of a canonical quantization of the bulk scalars, a particular isomorphism between the unitary irreducible representations in the bulk and boundary theories is found. This isomorphism defines the correspondence of field operators. It states that field operators of the bulk theory are field operators of the boundary theory by taking their boundary values in a due way. The Euclidean continuation provides an operator formulation on the hyperbolic coordinates system. The associated Fock vacuum of the bulk theory is located at the boundary, thereby identified with the boundary CFT vacuum. The correspondence is interpreted as a simple mapping of the field operators acting on this unique vacuum. Generalization to higher dimensions is speculated.
It has recently been conjectured [1] that supergravity (and string theory) on the universal cover of $AdS_{d+1}$ (often called $CAdS_{d+1}$) times a compact manifold is equivalent to a conformal field theory (CFT) living on the boundary of $CAdS_{d+1}$. The agreement of the spectrum of supergravity fluctuations with the spectrum of operators in the conformal field theory provides some evidence for this conjecture [2, 3, 4], [5, 6]. In particular correlation functions in the boundary CFT are derived [4, 7] from the dependence of the supergravity action near the boundary.

In spite of these phenomenological evidence a rationale of this $AdS/CFT$ correspondence seems to be little revealed. Frameworks for the Hamiltonian formulation are proposed [4], [8] to obtain an insight on it, but, presumably due to a complication of representation theory of the symmetry of $AdS_{d+1}$ for general $d$ and our little knowledge of higher dimensional CFT, still make it mysterious.

In this article, to obtain the clue, avoiding such difficulties, we investigate the Hamiltonian or operator formalism of the $AdS_2/CFT_1$ correspondence. Since the dimensionality itself seems [4] irrelevant for the correspondence, one can expect that study of this toy model provides some insight on our understanding of the $AdS/CFT$ correspondence.

First we examine the canonical quantization of massive scalar fields on $CAdS_2$ emphasizing the following two perspectives; i) unitary representation theory of symmetry group, and ii) normalizability of wave functions. It turns out that these two perspectives are not equivalent in the process of quantization although they could become consistent with each other. This is because at some region of mass$^2$ the normalizable wave functions constitute two unitary irreducible representations different from each other. Therefore classification of the scalar particles by the symmetry requires another means more than their mass. This phenomenon turns out to be a manifestation of the $AdS/CFT$ correspondence. As its explanation, an isomorphism between the unitary irreducible representations in the bulk and boundary theories is constructed in a specific manner. This isomorphism also shows that field operators of the bulk theory are field operators of the boundary theory by taking their boundary values in a due way. This picture of the $AdS/CFT$ correspondence is supplemented by an argument on the Euclidean theory.

The Euclidean continuation provides an operator formulation on the two-dimensional hyperbolic space. In particular it is defined on the hyperbolic coordinate system. The associated
Fock vacuum is located at the boundary of the hyperbolic space. Therefore it is identified with the vacuum of the boundary theory. With this identification both field operators of the bulk and boundary theories can act simultaneously on the unique vacuum. The correspondence now becomes the aforementioned simple mapping of the field operators.

Finally, we address two observations. One is related with the path-integral argument of the Euclidean theory based on the parabolic or elliptic coordinate system. The other is a speculation on the generalization to higher dimensions.

Note added: The $AdS_2/CFT_1$ correspondence is considered also in [9] from the different viewpoint and mapping of some bulk operators to the boundary theory, which are not equivalent to ours, is introduced there.

**Preliminary on $AdS_2$**

1 + 1 dimensional anti-de Sitter spacetime $AdS_2$ is topologically $S^1 \times \mathbb{R}$ and described by the metric

$$ds^2 = \frac{-(dt)^2 + (d\rho)^2}{\cos^2 \rho},$$

(1)

The ranges of the coordinates $t$ and $\rho$ are $0 \leq t < 2\pi$ and $-\pi/2 < \rho < \pi/2$. The Minkowski time $t$ is periodic. The universal cover $CAdS_2$ is topologically $\mathbb{R} \times \mathbb{R}$. The range of $t$ becomes $-\infty < t < +\infty$. The isometry group is $SU(1,1)$ [1]. The group action may be conveniently described in the lightcone coordinates $(u,v)$. They are introduced by $u = e^{i(t+\rho)}$ and $v = e^{i(t-\rho)}$. $SU(1,1)$ acts on $(u, v)$ in the following manner

$$\left(\begin{array}{c} \alpha \\ \beta \end{array}\right) \circ (u, v) = \left(\begin{array}{c} \alpha u + \beta \\ \beta u + \alpha \end{array}\right),$$

(2)

One can easily see that this group action preserves the $AdS_2$ metric $dudv/(u + v)^2$ in the lightcone coordinates. $AdS_2$ is a homogeneous space of $SU(1,1)$. We want to describe it

---

\[ SU(1,1) = \left\{ \left(\begin{array}{c} \alpha \\ \beta \end{array}\right) \mid \|\alpha\|^2 - \|\beta\|^2 = 1 \right\}. \]
explicitly for a later convenience. Let us introduce the following group elements

\[
\begin{align*}
k_t & \equiv \begin{pmatrix} e^{it/2} \\ e^{-it/2} \end{pmatrix}, \\
g_x & \equiv \begin{pmatrix} \cosh x/2 & i \sinh x/2 \\ -i \sinh x/2 & \cosh x/2 \end{pmatrix}, \\
a_\sigma & \equiv \begin{pmatrix} \cosh \sigma/2 & \sinh \sigma/2 \\ \sinh \sigma/2 & \cosh \sigma/2 \end{pmatrix}.
\end{align*}
\]

The lightcone coordinates are related with the coordinates \((t, \rho)\) through the group action, \((u, v) = k_t \, g_x(\rho) \circ (1, 1)\). Here \(x(\rho)\) is determined by \(\cosh x(\rho) = 1/\cos \rho\). Notice that the little group at \((u, v) = (1, 1)\) consists of \(a_\sigma\). Therefore AdS\(_2\) is the homogeneous space \(SU(1, 1)/\mathbb{R}\).

As the bases of the Lie algebra \(su(1, 1)\) we take the ones which generate \(a_\sigma, g_x\) and \(k_t\):

\[
\begin{align*}
X & \equiv \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}, \\
Y & \equiv \begin{pmatrix} i/2 \\ -i/2 \end{pmatrix}, \\
Z & \equiv \begin{pmatrix} i/2 \\ -i/2 \end{pmatrix}.
\end{align*}
\]

They satisfy the algebra, \([Z, X] = Y, [Z, Y] = -X, [X, Y] = -Z\). The Killing vectors corresponding to these generators are denoted by \(x, y, z\). They turn out to have the forms

\[
\begin{align*}
x & = \cos t \sin \rho \partial_t + \sin t \cos \rho \partial_\rho, \\
y & = \sin t \sin \rho \partial_t - \cos t \cos \rho \partial_\rho, \\
z & = -\partial_t.
\end{align*}
\]

We also introduce the complexified bases, \(E \equiv X - iY, F \equiv -X - iY\) and \(H \equiv -2iZ\). These satisfy the algebra, \([H, E] = 2E, [H, F] = -2F, [E, F] = -H\). Denote the corresponding complexified Killing vectors by \(e, f\) and \(h\). They have the forms

\[
\begin{align*}
e & = ie^{-it}(\cos \rho \partial_\rho - i \sin \rho \partial_t), \\
f & = ie^{it}(\cos \rho \partial_\rho + i \sin \rho \partial_t), \\
h & = 2i \partial_t.
\end{align*}
\]

The Casimir element \(C\) of \(su(1, 1)\) is given by \(C = (X^2 + Y^2 - Z^2)/2\). Insertion of the Killing vectors into \(2C\) leads to the scalar laplacian \(\Box_{AdS_2} = \cos^2 \rho (-\partial_t^2 + \partial_\rho^2)\) on AdS\(_2\).

The boundary of AdS\(_2\) consists of two circles located at \(\rho = \pm \pi/2\). In the lightcone coordinates it is described by \(u = -v\). SU(1, 1) acts as one-dimensional conformal symmetry

\[
a_\sigma = e^{\sigma X}, g_x = e^{x Y}, k_t = e^{t Z}.
\]
on the boundary. Consider the boundary circle at \( \rho = \pi/2 \). The Minkowski time \( t \) provides the coordinate. The conformal transformation, \( t \to t^g \), can be read from (2) as

\[
e^{i(t^g + \frac{\pi}{2})} = \frac{\alpha e^{i(t + \frac{\pi}{2})} + \beta}{\beta e^{i(t + \frac{\pi}{2})} + \bar{\alpha}},
\]

where \( g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \).

**Massive scalar field on \( CAdS_2 \)**

Consider a free scalar field theory on \( CAdS_2 \). Let us parametrize its mass by \( m^2 = s(s-1) \). The field equation for this scalar field \( \Phi(t, \rho) \) is simply given by

\[
\{-\Box_{AdS_2} + s(s-1)\} \Phi(t, \rho) = 0.
\]

(8)

Formal solutions of this equation can be described using the Gegenbauer function

\[
\phi_{\lambda}^{\pm} = e^{\pm i\omega t} (\cos \rho)^{\lambda} C_{\omega-\lambda}^{\lambda}(\sin \rho)
\]

where the parameter \( \lambda \) is equal to \( s \) or \( 1 - s \), and \( \omega \) is an arbitrary non-negative real number. \( C_{\alpha}^{\lambda} \) denotes the Gegenbauer function defined by the hypergeometric function

\[
C_{\alpha}^{\lambda}(z) = \frac{\Gamma(\alpha + 2\lambda)}{\Gamma(\alpha + 1)\Gamma(2\lambda)} F(\alpha + 2\lambda, -\alpha, \lambda + \frac{1}{2}; \frac{1}{2}; 1 - z^2),
\]

(10)

which satisfies

\[
(1 - z^2) \frac{d^2w}{dz^2} - (2\lambda + 1) \frac{dw}{dz} + \alpha(\alpha + 2\lambda)w = 0.
\]

(11)

Notice that the formal solution \( \phi_{\omega}^{\lambda} \) has the equivalent expression by the Legendre function of the first kind, \( \phi_{\omega}^{\lambda} \sim e^{\pm i\omega t} (\cos \rho)^{1/2} P_{\omega-\lambda}^{\frac{1}{2}}(\sin \rho) \).

We start with the discussion on the normalizability of the formal solutions. For this purpose we introduce the field theoretical scalar product \( (\ , \ ) \) on the solutions of (8) by

\[
(\phi, \psi) \equiv -i \int_{-\pi/2}^{\pi/2} d\rho \{\phi(t, \rho) \partial_t \bar{\psi}(t, \rho) - \partial_t \phi(t, \rho) \bar{\psi}(t, \rho)\},
\]

(12)

where \( \bar{\psi} \) is the complex conjugation of \( \psi \). This scalar product is formally independent of \( t \).
As one may easily observe it, the formal solutions \( \phi_{\omega}^{\lambda,\pm} \) with general values of \( \lambda \) and \( \omega \) are not normalizable with this scalar product. If one requires the normalizability on the wave function, the allowed values of the parameters are restricted. In fact, the \( \cos \rho \)-factor in \( \phi_{\omega}^{\lambda,\pm} \) imposes the condition: \( \lambda \) is real and greater than \(-1/2\). At the same time the property of the Gegenbauer function requires the quantization of the energy \( \omega : \omega = n + \lambda \) where \( n = 0, 1, 2, \cdots \). All the formal solutions \( \phi_{\omega}^{\lambda,\pm} \) which satisfy these two conditions are normalizable. Since \( \lambda \) is equal to \( s \) or \( 1 - s \), the above condition on \( \lambda \) implies that \( s \) must be real, which means \( m^2 \geq -1/4 \).

Normalizable solutions \( \phi_{n,\lambda,\pm} \) provides representations of \( su(1, 1) \). Denote the spaces spanned by these wave functions by \( \mathcal{H}_{\pm}^{\lambda} \equiv \bigoplus_{n \geq 0} C\phi_{n,\lambda,\pm} \). The unitarity of the representation may be examined by looking at the group action on the orthonormal bases of the representation space. They can be given by the following functions and their complex conjugations

\[
\phi_{n}^{\lambda}(t, \rho) \equiv c(\lambda) \left[ \frac{n!}{\Gamma(n+2\lambda)} \right] e^{-i(n+\lambda)(t+\pi/2)} (\cos \rho)^{\lambda} C_{n}^{\lambda}(\sin \rho),
\]

(13)

where \( c(\lambda) \equiv \frac{\Gamma(\lambda)}{2^{\lambda-1/2}} \) is the normalization constant independent of \( n \). Due to the orthogonality of the Gegenbauer polynomials, \( \phi_{n}^{\lambda} \) and \( \phi_{n}^{\lambda,\lambda} \) satisfy the relations

\[
(\phi_{m}^{\lambda}, \phi_{n}^{\lambda}) = \delta_{m,n}, \quad (\phi_{m}^{\lambda,\lambda}, \phi_{n}^{\lambda}) = -\delta_{m,n},
\]

(14)

The orthonormal bases of \( \mathcal{H}_{\pm}^{\lambda} \) are respectively given by \( \{\phi_{n}^{\lambda}\}_{n \geq 0} \) and \( \{\phi_{n}^{\lambda,\lambda}\}_{n \geq 0} \). The representations can be read from the action of the complex Killing vectors on these orthonormal wave functions

\[
e \phi_{n}^{\lambda} = \sqrt{(n+1)(n+2\lambda)} \phi_{n+1}^{\lambda}, \quad f \phi_{n}^{\lambda} = \sqrt{n(n-1+2\lambda)} \phi_{n-1}^{\lambda},
\]

\[
h \phi_{n}^{\lambda} = 2(n+\lambda) \phi_{n}^{\lambda}.
\]

\[
e \phi_{n}^{\lambda,\lambda} = -\sqrt{n(n-1+2\lambda)} \phi_{n-1}^{\lambda,\lambda}, \quad f \phi_{n}^{\lambda,\lambda} = -\sqrt{(n+1)(n+2\lambda)} \phi_{n+1}^{\lambda,\lambda},
\]

\[
h \phi_{n}^{\lambda,\lambda} = -2(n+\lambda) \phi_{n}^{\lambda,\lambda}.
\]

(15)

These explicit forms show that when \( \lambda > 0 \) the representations on \( \mathcal{H}_{\pm}^{\lambda} \) are irreducible and unitary. They can be thought as slight generalizations of the discrete series of the unitary

\[
3 \int_{-1}^{1} dx (1-x^2)^{\lambda-1/2} C_{m}^{\lambda}(x) C_{n}^{\lambda}(x) = \frac{\pi \Gamma(n+2\lambda)}{2^{n+1} \Gamma(\lambda+1) \Gamma(n+\lambda)} \delta_{m,n}.
\]
irreducible representations of $SU(1,1)$ \cite{1}. (Actually, if one restricts $\lambda$ to positive integers, they constitute the discrete series.) On the other hand, when $-1/2 < \lambda \leq 0$, the representations on $\mathcal{H}^\pm$ are not unitary. This can be read from eqs.(13) related with $\phi^\lambda_n$ and $\bar{\phi}^\lambda_n$. Since $SU(1,1)$ is the symmetry group of $CAdS_2$ we require that stable scalar particles on it should belong to the unitary representations of $su(1,1)$. This puts a further physical restriction on the allowed value of $\lambda$. It must satisfy $\lambda > 0$.

From the perspective of the above representation theory the allowed space of the normalizable solutions of (8) can be summarized as follows\footnote{Due to the symmetry of $s$ in $m^2$ we can assume $s \geq 1/2$}: In the case of $s \geq 1$, it is given by $\mathcal{H}^s \equiv \mathcal{H}^s_+ \oplus \mathcal{H}^s_-$. Here $\mathcal{H}^s_+$ respectively consist of positive and negative energy modes. Therefore the scalar field $\Phi$ belongs to one unitary irreducible representation of $su(1,1)$. In the case of $1/2 \leq s < 1$, the allowed space of the normalizable solutions is $\mathcal{H}^s \oplus \mathcal{H}^{1-s}$. The scalar field $\Phi$ belongs to two different unitary irreducible representations. If one classifies the scalar particles on $CAdS_2$ by its symmetry these two unitary irreducible representations should be distinguished. But, as we have seen it, the consideration only on the field equation is insufficient to do this.

This incompleteness of the discrimination of particles is a manifestation of the AdS/CFT correspondence. To explain why it is, we first observe the boundary behaviors of the orthonormal wave functions $\phi^\lambda_n, \bar{\phi}^\lambda_n (\lambda > 0)$. Of course they behave as $\sim (\cos \rho)^\lambda$ and approach to zero near the boundary of $CAdS_2$. But what we want to know is how they approach to zero. At the boundary $\rho = \pi/2$ it may be captured as follows\footnote{$C^\lambda_n(1) = \frac{\Gamma(n+2\lambda)}{\Gamma(2\lambda)n!}$}

$$
\begin{align*}
\lim_{\rho \to \pi/2} \frac{\Gamma(2\lambda)}{c(\lambda)} (\cos \rho)^{-\lambda} \phi^\lambda_n(t, \rho) &= \beta^\lambda_n(t), \\
\lim_{\rho \to \pi/2} \frac{\Gamma(2\lambda)}{c(\lambda)} (\cos \rho)^{-\lambda} \bar{\phi}^\lambda_n(t, \rho) &= \alpha^\lambda_n(t),
\end{align*}
$$

\text{(16)}

where

$$\beta^\lambda_n(t) \equiv \sqrt{\frac{\Gamma(n+2\lambda)}{n!}} e^{-i(n+\lambda)(t+\frac{\pi}{2})}, \quad \alpha^\lambda_n(t) \equiv \bar{\beta}^\lambda_n(t).$$

\text{(17)}

Due to the parity of the Gegenbauer function, $C^\lambda_n(-x) = (-)^n C^\lambda_n(x)$, the description at the other boundary reduces to the above ones. In the next section we will see that $\{\beta^\lambda_n\}_{n \geq 0}$ and
\{\alpha_n^\lambda\}_{n \geq 0} \text{ constitute the orthonormal bases of the unitary irreducible representations of } su(1, 1) \text{ living on the boundary. We assume this result for a while. Denote the spaces spanned by these boundary wave functions by } W^\lambda_+ \equiv \bigoplus_{n \geq 0} C\beta_n^\lambda \text{ and } W^\lambda_- \equiv \bigoplus_{n \geq 0} C\alpha_n^\lambda. \text{ The L.H.S. of eqs.(16) define the isomorphism between the unitary irreducible representations living on } CAdS_2 \text{ and its boundary}

\mathcal{H}^\lambda_\pm \xrightarrow{\sim} W^\lambda_\pm. \tag{18}

For the scalar field \(\Phi^\lambda\) which belongs to \(\mathcal{H}^\lambda\), the image \(\lim_{\rho \to \pm \pi/2} (\cos \rho)^{-\lambda} \Phi^\lambda(t, \rho)\) describes a wave function on the boundary which belongs to \(W^\lambda \equiv W^\lambda_+ \oplus W^\lambda_-\).

**Boundary CFT revisited**

Here we simply describe the unitary irreducible representations of \(su(1, 1)\) on the boundary of \(CAdS_2\). Related conformal quantum mechanical models are investigated in [11], and recently in [12].

We begin with the case of \(AdS_2\). Let us consider the boundary circle located at \(\rho = \pi/2\). Let \(\varphi(t)\) be a tensor field of degree \(h\) on this circle. It will transform covariantly under the conformal group action

\[\varphi(t)(dt)^h = \varphi(t^g)(dt^g)^h \quad g \in SU(1, 1).\] \tag{19}

The (complexified) generators \(E, F\) and \(H\) act on \(\varphi\) as the following differential operators \(\hat{e}, \hat{f}\) and \(\hat{h}\)

\[
\hat{e} = e^{-i(t + \frac{\pi}{2})}(h + \partial_t), \quad \hat{f} = e^{i(t + \frac{\pi}{2})(-h + i\partial_t)}, \\
\hat{h} = 2i\partial_t. \tag{20}
\]

The corresponding Casimir element becomes \(C = h(h - 1)/2\). We can expand \(\varphi\) by the modes \(e^{-in(t + \frac{\pi}{2})}(\equiv \varphi_n)\ (n \in \mathbb{Z})\). The representation can be written as

\[
\hat{e} \varphi_n = (n + h) \varphi_{n+1}, \quad \hat{f} \varphi_n = (n - h) \varphi_{n-1}, \\
\hat{h} \varphi_n = 2n \varphi_n. \tag{21}
\]

The cases of \(h\) being positive integers are physically reasonable. In these cases there appear the invariant subspaces which provide the highest or lowest weight representations of the conformal
Let \( h = l \in \mathbb{Z}_{\geq 0} \). The subspaces, \( \bigoplus_{n \geq 0} C \varphi_{n+l} \) and \( \bigoplus_{n \geq 0} C \varphi_{-n-l} \) are invariant under the action \((21)\). Actually they are respectively \( \mathcal{W}^l_+ \) and \( \mathcal{W}^l_- \). Taking the bases \( \beta^l_n \) and \( \alpha^l_n \) the representations can be read as

\[
\begin{align*}
\hat{e} \beta^l_n &= \sqrt{(n+1)(n+2l)} \beta^l_{n+1}, & \hat{f} \beta^l_n &= \sqrt{n(n-1+2l)} \beta^l_{n-1}, \\
\hat{h} \beta^l_n &= 2(n+l) \beta^l_n.
\end{align*}
\]

\[
\begin{align*}
\hat{e} \alpha^l_n &= -\sqrt{n(n-1+2l)} \alpha^l_{n-1}, & \hat{f} \alpha^l_n &= -\sqrt{(n+1)(n+2l)} \alpha^l_{n+1}, \\
\hat{h} \alpha^l_n &= -2(n+l) \alpha^l_n.
\end{align*}
\]

These coincide with the representation \((13)\) of the scalar field \( \Phi^{\lambda=l} \). In fact, introducing the hermitian norms such that \( \beta^l_n \) and \( \alpha^l_n \) are respectively orthonormal in \( \mathcal{W}^l_+ \) and \( \mathcal{W}^l_- \), we obtain the unitary irreducible integrable representations of \( su(1,1) \) which belong to the discrete series. Therefore the scalar field \( \Phi^{\lambda=l} \) living on \( AdS_2 \) can appear on the boundary as the tensor field \( \varphi^l \) which belongs to \( \mathcal{W}^l \) by

\[
\varphi^l(t) = \lim_{\rho \to \pi/2} \frac{\Gamma(2l)}{c(l)} (\cos \rho)^{-l} \Phi^l(t,\rho).
\]

(23)

Now we discuss the case of \( CAdS_2 \). Consider the boundary \( \mathbb{R} \) located at \( \rho = \pi/2 \). Let \( \varphi(t) \) be a tensor field of degree \( s \) with the quasi-periodicity

\[
\varphi(t + 2\pi) = e^{-2\pi is} \varphi(t).
\]

(24)

(We can also regard \( \varphi \) as a twisted tensor field on the boundary circle of \( AdS_2 \).) The generators \( E, F \) and \( H \) act on it as the differential operators \( \hat{e}, \hat{f} \) and \( \hat{h} \) replacing \( h \) by \( s \) in \((20)\). \( \varphi \) and its complex conjugate \( \bar{\varphi} \) are expanded respectively by \( e^{-i(n+s)(t+\pi/2)} \) and \( e^{i(n+s)(t+\pi/2)} \). Taking the same route as in the previous case we can obtain \( \mathcal{W}^s_+ \) and \( \mathcal{W}^s_- \) as the invariant subspaces. They are spanned by \( \beta^s_n \) and \( \alpha^s_n \). On these invariant subspaces the differential operators act as

\[
\begin{align*}
\hat{e} \beta^s_n &= \sqrt{(n+1)(n+2s)} \beta^s_{n+1}, & \hat{f} \beta^s_n &= \sqrt{n(n-1+2s)} \beta^s_{n-1}, \\
\hat{h} \beta^s_n &= 2(n+s) \beta^s_n, \\
\hat{e} \alpha^s_n &= -\sqrt{n(n-1+2s)} \alpha^s_{n-1}, & \hat{f} \alpha^s_n &= -\sqrt{(n+1)(n+2s)} \alpha^s_{n+1}, \\
\hat{h} \alpha^s_n &= -2(n+s) \alpha^s_n.
\end{align*}
\]

(25)
which coincide with the unitary irreducible non-integrable representations \([15]\). This means that the scalar field \(\Phi^s\) on \(CAdS_2\) can appear on the boundary as the tensor field \(\varphi^s\) which belongs to \(\mathcal{W}^s\) by

\[
\varphi^s(t) = \lim_{\rho \to \pi/2} \frac{\Gamma(2s)}{c(s)} (\cos \rho)^{-s} \Phi^s(t, \rho).
\]  

(26)

**Canonical quantization of \(\Phi^s\)**

The canonical quantization of the scalar field \(\Phi^s(t, \rho)\) will be performed following the standard recipe \([13]\). Using the orthonormal bases \(\phi_n^s\) and \(\bar{\phi}_n^s\), the field can be expanded as

\[
\Phi^s(t, \rho) = \sum_{n \geq 0} a_n \phi_n^s(t, \rho) + \sum_{n \geq 0} a_n^\dagger \bar{\phi}_n^s(t, \rho).
\]  

(27)

The canonical quantization will be implemented by adopting the commutation relations

\[
[a_m, a_n^\dagger] = \delta_{mn}, \quad [a_m, a_n] = [a_m^\dagger, a_n^\dagger] = 0.
\]  

(28)

The Fock vacuum and its dual are introduced by the conditions

\[
\forall a_n |\text{vac}\rangle = 0, \quad <\text{vac}| \forall a_n^\dagger = 0.
\]  

(29)

The generators of \(su(1,1)\) are constructed from the Noether currents and realized on the Fock space \(\mathcal{F}^s = \bigoplus_{n_1, \ldots, n_k} \mathbb{C} a_{n_1}^\dagger \cdots a_{n_k}^\dagger |\text{vac}\rangle\) by the following operators

\[
E = \sum_{n \geq 0} \sqrt{(n + 1)(n + 2s)} \ a_{n+1}^\dagger a_n, \quad F = \sum_{n \geq 0} \sqrt{n(n - 1 + 2s)} \ a_{n-1}^\dagger a_n,
\]  

\[
H = \sum_{n \geq 0} 2(n + s) \ a_n^\dagger a_n.
\]  

(30)

We notice that the Fock vacuum is \(su(1,1)\)-invariant. This invariance might be thought as a trivial consequence of the canonical quantization procedure. But, its implication seems to be non-trivial. The isomorphism \([18]\) of the representations through the wave functions on the \(CAdS_2\) and its boundary naturally leads to the identification of the Fock vacuum of the bulk theory and the vacuum of the boundary CFT. This means the \(su(1,1)\)-invariance of the Fock vacuum is nothing but the conformal invariance of the boundary CFT vacuum \(1\).

\[6\] We will investigate this observation again in the discussion on the Euclidean operator formalism.
To check the consistency of our formulation of the \( AdS_2/CFT_1 \) correspondence let us examine the boundary behavior of the Wightman function, \( \langle \text{vac} | \Phi^s(t_1, \rho_1) \Phi^s(t_2, \rho_2) | \text{vac} \rangle \). Using the mapping (16) of the wave functions we can derive

\[
\lim_{\rho_1, \rho_2 \to \pi/2} \left( \frac{\Gamma(2s)}{c(s)} \right)^2 (\cos \rho)^{-2s} \langle \text{vac} | \Phi^s(t_1, \rho_1) \Phi^s(t_2, \rho_2) | \text{vac} \rangle = \sum_{n \geq 0} \beta_n^s(t_1) \alpha_n^s(t_2)
\]

\[
= \Gamma(2s) \frac{e^{is(t_1+t_2)}}{(e^{i\rho_1} - e^{i\rho_2})^{2s}}.
\]

This is nothing but the two-point function \( \langle \varphi^s(t_1) \varphi^s(t_2) \rangle \) of the boundary conformal field \( \varphi^s \). To present it in a familiar form, it is convenient to change the coordinate from \( t \) to \( x \) by \( x = \tan t/2 \). The conformal field \( \varphi^s(t) \) transforms to \( \hat{\varphi}^s(x) \) according to the relation

\[
\varphi^s(t) dt = \hat{\varphi}^s(x) dx.
\]

In particular it holds

\[
\langle \hat{\varphi}^s(x_1) \hat{\varphi}^s(x_2) \rangle = \langle \varphi^s(t_1) \varphi^s(t_2) \rangle \frac{(dt_1)}{(dx_1)} \frac{(dt_2)}{(dx_2)}. \tag{31}
\]

This is the two-point function of the corresponding conformal quantum mechanics [11].

**Operator formulation for Euclidean theory**

We want to examine the operator formulation of the Euclidean theory by simply replacing the Minkowski time \( t \) by \(-i\tau\). The scalar field \( \Phi^s \) now lives in the Euclidean \( CAdS_2 \). The Euclideanization of \( CAdS_2 \) is two-dimensional hyperbolic space \( H_2 \), that is, the hyperbolic plane or the Poincare disk. Notice that the Euclideanization of \( AdS_2 \) is not \( H_2 \), but the quotient space \( H_2/\mathbb{Z} \), where the \( \mathbb{Z} \)-action can be derived from the periodicity of the time. To be explicit, we consider the hyperbolic plane, that is, the upper half-plane \( C_+ = \{ w = x + iy, y > 0 \} \) endowed with the Poincare metric, \( \frac{(dx)^2 + (dy)^2}{y^2} \). The coordinate \((\tau, \rho)\) of the Euclidean \( CAdS_2 \) are related with \((x, y)\) by \( x = e^{-\tau} \sin \rho \) and \( y = e^{-\tau} \cos \rho \). (With this transform the Poincare metric becomes \( \frac{(dr)^2 + (d\rho)^2}{\cos^2 \rho} \).) We assume that the mode expansion (27) of \( \Phi^s \) on \( CAdS_2 \) also provides the mode expansion on \( H_2 \) after the Euclidean continuation. Notice that the following behavior of \( \phi^s_n \) and \( \bar{\phi}^s_n \) as \( \tau \to \pm \infty \)

\[
\lim_{\tau \to +\infty} \phi^s_n(\tau, \rho) = \infty, \quad \lim_{\tau \to -\infty} \phi^s_n(\tau, \rho) = 0,
\]

\[
\lim_{\tau \to +\infty} \bar{\phi}^s_n(\tau, \rho) = 0, \quad \lim_{\tau \to -\infty} \bar{\phi}^s_n(\tau, \rho) = \infty. \tag{33}
\]
Since \( \lim_{\tau \to +\infty} (x, y) = (0, 0) \) and \( \lim_{\tau \to -\infty} (x, y) = \infty \) we can see that the Euclidean positive (negative) energy modes \( \phi_n^+ (\bar{\phi}_n^-) \) diverge at \((x, y) = (0, 0)\) \((x, y) = \infty\) and converge to zero at \((x, y) = \infty\) \((x, y) = (0, 0)\). From the perspective of the Euclidean operator formalism this means that the Fock vacuum and its dual associated with the mode expansion are located at \(\infty\) and \((0, 0)\). They are on the boundary of \(H_2\). This also implies the vacua of the bulk and boundary theories actually coincide.

In the Euclidean theory the operators \(E, F\) and \(H\) given in (30) will generate respectively the special conformal transformation, translation and dilation of one-dimensional conformal symmetry. Let us check this. The isometry group of the hyperbolic plane is \(SL(2; \mathbb{R})\). The group action is given by \(g \circ w = \frac{aw + b}{cw + d}\) where \(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2; \mathbb{R})\). Fixing the isomorphism \(SL(2 : \mathbb{R}) \cong SU(1, 1)\) by \(g \to \gamma g \gamma^{-1}\), where \(\gamma = \frac{1}{\sqrt{i+1}} \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}\), let us take \(\gamma^{-1}X\gamma\), \(\gamma^{-1}Y\gamma\) and \(\gamma^{-1}Z\gamma\) as the bases of \(sl(2 : \mathbb{R})\). Denote the corresponding Killing vectors on the hyperbolic plane by \(x_{H_2}, y_{H_2}\) and \(z_{H_2}\). They have the forms

\[
\begin{align*}
x_{H_2} &= -\partial_\tau, \\
y_{H_2} &= -\cosh \tau \cos \rho \partial_\rho + \sinh \tau \sin \rho \partial_\rho, \\
z_{H_2} &= -\sinh \tau \cos \rho \partial_\rho + \cosh \tau \sin \rho \partial_\rho.
\end{align*}
\]

After the Euclidean continuation the complex Killing vectors on \(CA_{dS_2}\) can be written in terms of \(x_{H_2}, y_{H_2}\) and \(z_{H_2}\) as

\[
\begin{align*}
e &= -i(y_{H_2} - z_{H_2}), \\
y &= -i(y_{H_2} + z_{H_2}), \\
h &= 2x_{H_2}.
\end{align*}
\]

Therefore the operators \(E, F\) and \(H\) represent \(\gamma^{-1}(Y - Z)\gamma\), \(\gamma^{-1}(Y + Z)\gamma\) and \(2\gamma^{-1}X\gamma\) of \(sl(2 : \mathbb{R})\), which respectively generate the special conformal transform, translation and dilation on the boundary of the hyperbolic plane.

These identifications of the vacua besides the field operators suggest that the Fock space of the bulk theory itself has an interpretation in terms of the boundary physics. Notice that the one particle states of \(\Phi^s\) are nothing but \(\mathcal{W}_s^s\). It can be thought as the Hilbert space of one-body conformal quantum mechanics. The identification of the Fock space \(\mathcal{F}^s\) with the infinite
sum of the symmetric product of $W_s^+$, namely $F^s = \bigoplus_{l \geq 0} \text{sym}(W_{s+}^s \times \cdots \times W_{s+}^s)$, implies that the boundary theory is $\infty$-body conformal quantum mechanics rather than one-body.

**Outlook**

Two observations are addressed.

**(i) Comparison with path-integral formulation**

Our operator formulation of Euclidean theory depends on the coordinate $(\tau, \rho)$ of the hyperbolic space $H_2$. This coordinate system is hyperbolic in the following sense: It can be determined by the $SL(2 : \mathbb{R})$ parametrization, $w = \gamma^{-1}a_{\tau}g_{x(\rho)}\gamma \circ i$, of the upper half-plane and there $a_{\tau}$ is a hyperbolic element of $SU(1,1)$. While this, the path-integral arguments \[4, 7\] are based on the coordinates $(x,y)$ of the upper half-plane or the radial coordinates $(\theta, r)$ of the Poincare disk. These coordinate systems are respectively parabolic and elliptic in the sense that they are realized by the $SL(2 : \mathbb{R})$ parametrizations,

\[
\begin{pmatrix}
1 & x \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
y^{1/2} & 0 \\
0 & y^{-1/2}
\end{pmatrix}
\circ i
\]

and $w = \gamma^{-1}k_{\theta}a_{\sigma(r)}\gamma \circ i$. In the $CAdS_2$ case the path integral argument of the AdS/CFT correspondence may be summarized as follows: Take the hyperbolic plane with the parabolic coordinates $(x,y)$ for simplicity. Consider a tensor field of degree $s$ on the boundary. Let us denote it by $\varphi_0^s(x)$. It transforms covariantly under the $SL(2 : \mathbb{R})$

\[
\varphi_0^s(x)(dx)^s = \varphi_0^s(g \circ x)(d(g \circ x))^s \quad g \in SL(2 : \mathbb{R}).
\]

This defines a representation of $SL(2 : \mathbb{R})$ on the space of $\varphi_0^s$. Denote it $V^s$. There exists a natural pairing between $\varphi_0^s$ and $\varphi_0^{1-s}$

\[
< \varphi_0^{1-s}, \varphi_0^s > = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dx \varphi_0^{1-s}(x)\varphi_0^s(x),
\]

which is interpreted \[4\] as the interaction between the bulk and boundary theories. Consider the generalized Poisson integral \[14\],

\[
\int_{-\infty}^{+\infty} dx' \left( \frac{y}{y^2 + (x-x')^2} \right)^s \varphi_0^{1-s}(x'),
\]

\[\sigma(r) = \ln \frac{1+r}{1-r}\]
which can be rewritten in terms of the representation theory as

\[ \langle \varphi_0^{1-s}, V^s \left( \begin{array}{cc} 1 & x \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{array} \right) \varphi_0^s \rangle, \]  

where \( \varphi_0(u) \equiv \frac{1}{(1+u^2)^{1/4}} \). This provides a formal solution of the bulk scalar field equation with \( m^2 = s(s-1) \). It is divergent on the boundary. By the semi-classical argument using this solution the correct two-point function of the boundary theory is derived \([4, 7]\) from the bulk theory action. Those problems of the unitarity of representations and the normalizability of wave functions, which play the crucial role in the operator formulation on the hyperbolic coordinate system, do not appear, at least, superficially. The physical picture of the correspondence becomes completely different depending on the coordinate system which one takes. It is interesting to construct the Bogoliubov transforms between the vacua constructed on the coordinate systems of different types.

(ii) Generalization to higher dimensions

As a simple application, we speculate on the case of three dimensions. \( CAdS_3 \) is topologically \( \mathbb{R} \times \mathbb{R}^2 \) and endowed with the \( AdS_3 \) metric \( -\frac{(dt)^2+(dr)^2+\sin^2 \rho (d\theta)^2}{\cos^2 \rho} \). The ranges of the coordinates are \(-\infty < t < +\infty, 0 \leq \rho < \pi/2 \) and \( 0 \leq \theta < 2\pi \). The identification, \( t \sim t + 2\pi \), yields \( AdS_3 \). The Euclideanization of \( CAdS_3 \) is the three-dimensional hyperbolic space \( H_3 \). Consider the three-dimensional upper half-space, \( \{(x, y, z) \mid z > 0\} \) endowed with the hyperbolic metric \( \frac{(dx)^2+(dy)^2+(dz)^2}{z^2} \). With the change of coordinates, \( x = e^{-\tau} \sin \rho \cos \theta, \ y = e^{-\tau} \sin \rho \sin \theta \) and \( z = e^{-\tau} \cos \rho \), the metric acquires the form, \( \frac{(d\tau)^2+(dp)^2+\sin^2 \rho (d\theta)^2}{\cos^2 \rho} \). So, it is actually the Euclideanization of \( CAdS_3 \).

In the hyperbolic coordinate system \( (\tau, \rho, \theta) \), the operator formulation of the bulk scalars tells the Fock vacuum of the bulk theory is located at the boundary of \( H_3 \). The \( so(2,2) \)-invariance of the Fock vacuum of the bulk theory automatically implies the \( sl(2 : \mathbb{C}) \)-invariance of the boundary 2d CFT vacuum by the identification of these two. Field operators of the bulk and boundary theories can act on this unique vacuum. We can expect that the \( AdS/CFT \) correspondence is realized along the same line as in the case of \( CAdS_2 \). Namely, the bulk theory field operators become the boundary theory field operators by taking their boundary values in such a way to define the isomorphism between the unitary irreducible representations living on
the bulk and the boundary.

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