Fisher information of orthogonal polynomials I

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Dedicated to Jesús Dehesa on the occasion of his 60th birthday

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Abstract
Following the lead of J. Dehesa and his collaborators, we compute the Fisher information of the Meixner-Pollaczek, Meixner, Krawtchouk and Charlier polynomials.

1 Introduction

The Fisher information $I_\theta (\mu)$ of a random variable $X$ with distribution $\mu(x; \theta)$, where $\theta$ is a continuous parameter, is defined by

$$I_\theta (\mu) = \mathbb{E} \left\{ \left[ \frac{\partial}{\partial \theta} \ln (\mu) \right]^2 \right\} . \quad (1)$$

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It is named after R. A. Fisher (17 February 1890 – 29 July 1962), who invented the concept of maximum likelihood estimator and discovered many of its properties. Among other results, he proved that if $\hat{\theta}$ is the maximum likelihood estimator of $\theta$, we have the following asymptotic normality of $\hat{\theta}$

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{D} \mathcal{N}
\begin{pmatrix} 0, \frac{1}{I_\theta(\mu)} \end{pmatrix},$$

where $\mathcal{N}(\cdot, \cdot)$ denotes the normal distribution and $n$ is the sample size. Over the years, the concept of Fisher information has found many applications in physics [4], biology [9], engineering [7], etc.

**Example 1** The negative binomial distribution.

Let $0 \leq p \leq 1$, $r > 0$ and

$$\mu(k; p, r) = \binom{r + k - 1}{k} p^k (1 - p)^r, \quad k = 0, 1, \ldots.$$

Then, we have

$$I_p(\mu) = \sum_{k=0}^{\infty} \left[ \frac{r + k - r (1 - p)}{p} \right]^2 \binom{r + k - 1}{k} p^k (1 - p)^r = \frac{r}{p (1 - p)^2}. \quad (2)$$

**Example 2** The binomial distribution.

Let $0 \leq p \leq 1$, $n \in \mathbb{N}$ and

$$\mu(k; p, n) = \binom{n}{k} p^k (1 - p)^{n-k}, \quad k = 0, 1, \ldots, n.$$

Then, we have

$$I_p(\mu) = \sum_{k=0}^{n} \left[ \frac{k - pn}{p (1 - p)} \right]^2 \binom{n}{k} p^k (1 - p)^{n-k} = \frac{n}{p (1 - p)}. \quad (3)$$

**Example 3** The Poisson distribution.

Let $\lambda > 0$ and

$$\mu(k; \lambda) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 0, 1, \ldots.$$

Then, we have

$$I_\lambda(\mu) = \sum_{k=0}^{\infty} \left( \frac{k - \lambda}{\lambda} \right)^2 \frac{\lambda^k}{k!} e^{-\lambda} = \frac{1}{\lambda}. \quad (4)$$

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In [8], J. Sánchez-Ruiz and J. Dehesa introduced the concept of Fisher information of orthogonal polynomials. They considered a sequence of real polynomials orthogonal with respect to the weight function \( \rho(x) \) on the interval \([a, b] \)

\[
\int_{a}^{b} P_n(x)P_m(x)\rho(x)dx = h_n\delta_{n,m}, \quad n, m = 0, 1, \ldots,
\]

with \( \text{deg}(P_n) = n \). Introducing the normalized density functions

\[
\rho_n(x) = \frac{[P_n(x)]^2 \rho(x)}{h_n},
\]

they defined the Fisher information corresponding to the densities \( \rho_n(x) \) by

\[
I(n) = \int_{a}^{b} \left( \frac{\rho_n'(x)}{\rho_n(x)} \right)^2 dx,
\]

which they referred to as the Fisher information of the polynomial \( P_n(x) \). Applying (6) to the classical hypergeometric polynomials, they calculated \( I(n) \) for the Jacobi, Laguerre and Hermite polynomials.

In this work, we extend their ideas to some families of orthogonal polynomials. We use a concept of Fisher information closer to (1), i.e., information content with respect to a parameter.

The paper is organized as follows: Section 2 contains some general results on hypergeometric polynomials which we have been unable to find explicitly in the literature, although they may be known. In Section 3 we compute the Fisher information of the Meixner-Pollaczek, Meixner, Krawtchouk and Charlier polynomials.

2 Preliminaries

Let \( P_n(x) \) be a family of orthogonal polynomials satisfying

\[
\sum_{x=0}^{\infty} P_n(x)P_m(x)\rho(x) = h_n\delta_{n,m}, \quad n, m = 0, 1, \ldots
\]
We define
\[ \rho_n(x) = \frac{[P_n(x)]^2 \rho(x)}{h_n}, \quad n = 0, 1, \ldots \] (8)
and
\[ I_\theta(P_n) = \sum_{x=0}^\infty \left[ \frac{\partial}{\partial \theta} \rho_n(x) \right]^2 \frac{1}{\rho_n(x)}, \quad n = 0, 1, \ldots \] (9)
Note that
\[ \sum_{x=0}^\infty \rho_n(x) = 1, \quad n = 0, 1, \ldots \] (10)

**Theorem 4** Let \( P_n(x) \) be a family of polynomials defined by
\[ P_n(x) = \; \qquad \left[ \frac{-n,-x}{c} \right] z(\theta), \quad n = 0, 1, \ldots, \]
where \( \; \qquad \left[ \frac{-n,-x}{c} \right] \) is the hypergeometric function [6]
\[ \; \qquad \left( \frac{a,b}{c} \right) \; \qquad \left[ \frac{a,b}{c} \right] z^k = \sum_{k=0}^\infty \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} \]
and \((\cdot)_k\) denotes the Pochhammer symbol. Then, \( P_n(x) \) satisfies the following:
1. \((n + c) P_{n+1}(x) + [(n - x) z - 2n - c] P_n(x) - n (z - 1) P_{n-1}(x) = 0.\) (11)
2. \[ \frac{\partial P_n}{\partial \theta} = n \frac{z'}{z} [P_n(x) - P_{n-1}(x)] , \quad n = 0, 1, \ldots \] (12)
3. \[ \sum_{x=0}^\infty P_n(x) P_m(x) \rho(x) = h_n \delta_{n,m}, \quad n, m = 0, 1, \ldots , \]
with \[ \rho(x) = \frac{(c)_x}{(1-z)^x x!} \] (13)
and \[ h_n = \left( 1 - \frac{1}{z} \right)^c \frac{(1-z)^n}{(c)_n} n!, \quad n = 0, 1, \ldots \] (14)
Proof. The three term recurrence equation (11) is a direct consequence of the contiguous relation [2, 2.8 (31)]

\[ [2 - 2a - (b - a)z] F + a(1 - z)F(a + 1) - (c - a)F(a - 1) = 0, \]

where

\[ F = \binom{a, b}{c} z. \]

The differentiation formula (12) follows from the identity [1, (2.5.5)].

\[ z \frac{dF}{dz} = a \left[ F(a + 1) - F(a) \right]. \]

To prove (13), (14), we use the formula [2, 2.5.2 (12)]

\[ \sum_{k=0}^{\infty} \binom{\lambda}{k} s^k \binom{-k, b}{-\lambda} z \binom{-k, \beta}{-\lambda} \rho(x) = \left(1 + s \right)^{b + \beta} \binom{b, \beta}{-\lambda} \left[ 1 + s(1 - z) \right] \left[ 1 + s(1 - \zeta) \right], \]

with \( \lambda = -c, b = -n, \beta = -m \) and \( s = (z - 1)^{-1} \). Taking into account that

\[ \rho(x) = \frac{(c) x}{(1 - z)^{x}} = \left(\frac{-c}{x}\right)(z - 1)^{-x}, \]

we have

\[ \sum_{x=0}^{\infty} \binom{-n, -x}{c} \binom{-m, -x}{c} \rho(x) = \left(1 - \frac{1}{z}\right)^{c} \left(\frac{z - \zeta}{z}\right)^{n+m} \binom{-n, -m}{c} \binom{1 - z}{(z - \zeta)^2}. \]

Assuming that \( n \leq m \), we get

\[ \sum_{x=0}^{\infty} \binom{-n, -x}{c} \binom{-m, -x}{c} \rho(x) \]

\[ = \left(\frac{z - 1}{z}\right)^{c} \left(\frac{1}{z}\right)^{n+m} \sum_{k=0}^{n} \binom{-n}{k} \binom{-m}{k} \binom{1 - z}{k} \binom{k}{(c)_k} \binom{2k}{(z - \zeta)^{n+m-2k}}. \]

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Replacing \( \zeta \) by \( z \) in (15), we obtain
\[
\sum_{x=0}^{\infty} 2F_1 \left[ \begin{array}{c} -n, -x \\ c \end{array} \right] z 2F_1 \left[ \begin{array}{c} -m, -x \\ c \end{array} \right] \rho (x) \\
= \left( 1 - \frac{1}{z} \right)^c \frac{\frac{(1 - z)^n}{n!}}{(c)_n} \delta_{n,m}
\]
and the result follows. ■

**Corollary 5** Let
\[
P_n(x) = 2F_0 \left[ \begin{array}{c} -n, -x \\ - \end{array} \right] z (\theta), \quad n = 0, 1, \ldots
\]

Then,

1. \[
P_{n+1} (x) + [(x - n) z - 1] P_n (x) - nz P_{n-1} (x) = 0. \quad (16)
\]
2. \[
\frac{\partial P_n}{\partial \theta} = n z' \left[ P_n (x) - P_{n-1} (x) \right], \quad n = 0, 1, \ldots \quad (17)
\]
3. We have
\[
\rho (x) = \frac{\left[ -z (\theta) \right]^{-x}}{x!} \\
\]
and
\[
h_n = \left[ -z (\theta) \right]^n n! \exp \left[ -\frac{1}{z (\theta)} \right], \quad n = 0, 1, \ldots, \quad (19)
\]
where \( \rho (x) \) and \( h_n \) were defined in (7).

**Proof.** The results follow immediately using the limit relation [5] (0.4.5)]
\[
2F_0 \left[ \begin{array}{c} -n, -x \\ - \end{array} \right] z (\theta) = \lim_{\lambda \to \infty} 2F_1 \left[ \begin{array}{c} -n, -x \\ \lambda c \end{array} \right] \lambda cz (\theta). \quad (20)
\]
■
3 Main results

We shall now use that results of the previous section and compute the Fisher information of the Meixner-Pollaczek, Meixner, Krawtchouk and Charlier polynomials.

**Theorem 6** Let

\[ P_n(x) = \binom{-n,-x}{c} z(\theta) , \quad n = 0, 1, \ldots. \]

Then,

\[ I_\theta (P_n) = \left( \frac{z'}{z} \right)^2 (1 - z)^{-1} \left[ 2n^2 + (2n + 1) c \right] , \quad n = 0, 1, \ldots. \] (21)

**Proof.** From (8), (13) and (14), we have

\[ \rho_n(x) = \left( 1 - \frac{1}{z} \right)^{-c} [P_n(x)]^2 \frac{(c)_x (c)_n}{(1 - z)^{x+n} x! n!} . \] (22)

Hence,

\[ \frac{\partial}{\partial \theta} \rho_n(x) = - \left( \frac{z}{z - 1} \right)^{c-1} \frac{(c)_x (c)_n}{(1 - z)^{x+n} x! n!} P_n(x) \left[ (c + nz + xz) z'P_n(x) + 2z(1 - z) \frac{\partial P_n}{\partial \theta} \right] . \] (23)

Using (12) and (11) in (23), we obtain

\[ \frac{\partial}{\partial \theta} \rho_n(x) = - \left( \frac{z}{z - 1} \right)^{c-1} \frac{(c)_x (c)_n}{(1 - z)^{x+n} x! n!} z'P_n(x) \left[ (c - nz + xz + 2n) P_n(x) + 2n(z - 1)P_{n-1}(x) \right] \] (24)

\[ = - \left( \frac{z}{z - 1} \right)^{c-1} \frac{(c)_x (c)_n}{(1 - z)^{x+n} x! n!} z'P_n(x) \left[ (n + c) P_{n+1}(x) + n(z - 1)P_{n-1}(x) \right] . \]

Therefore,

\[ \left[ \frac{\partial}{\partial \theta} \rho_n(x) \right]^2 \frac{1}{\rho_n(x)} = \left( \frac{z}{z - 1} \right)^{c-2} \frac{(c)_x (c)_n}{(1 - z)^{x+n} x! n!} (z')^2 [(n + c) P_{n+1}(x) + n(z - 1)P_{n-1}(x)]^2 \]

\[ = (1 - z)^{-1} \left( \frac{z'}{z} \right)^2 [(n + 1)(c + n) \rho_{n+1}(x) + n(c + n - 1) \rho_{n-1}(x)] \] (25)

\[ + 2 \left( \frac{z}{z - 1} \right)^{c-2} \frac{(c)_n}{(1 - z)^{n} n!} (z')^2 n(n + c)(z - 1)P_{n+1}(x)P_{n-1}(x) \rho(x) , \]
where we have used (13) and (22).

Summing (25), while taking (7), (9) and (10) into account, the result follows.

**Corollary 7** Let

\[ P_n(x) = {}_2F_0 \left[ \begin{array}{c} -n, -x \\ - \end{array} \right| z(\theta) \right], \quad n = 0, 1, \ldots. \]

Then,

\[ I_\theta(P_n) = - \left( \frac{z'}{z} \right)^2 \frac{(2n + 1)}{z}, \quad n = 0, 1, \ldots. \]

**Proof.** The result follows from (16)-(19) and the same steps used in the proof of Theorem 6. It can also be proved directly by using (20) in (21), since

\[
\lim_{\lambda \to \infty} \left( \frac{\lambda cz'}{\lambda cz} \right)^2 (1 - \lambda cz)^{-1} \left[ 2n^2 + (2n + 1) \lambda c \right] = - \left( \frac{z'}{z} \right)^2 \frac{(2n + 1)}{z}.
\]

We have now all the elements to state our main result.

**Theorem 8** The Fisher information of the Meixner, Krawtchouk and Charlier polynomials is given by:

1. **Meixner**

\[ I_c(M_n) = \frac{2n^2 + (2n + 1)\beta}{c(c-1)^2}, \quad n = 0, 1, \ldots. \]

2. **Krawtchouk**

\[ I_p(K_n) = \frac{2n^2 - (2n + 1)N}{p(p-1)}, \quad n = 0, 1, \ldots, N. \]

3. **Charlier**

\[ I_a(C_n) = \frac{2n + 1}{a}, \quad n = 0, 1, \ldots. \]
Proof. The result follows from Theorem 6, Corollary 7 and the hypergeometric representations \[5\]

\[M_n(x; \beta, c) = \binom{-n}{\beta} \frac{1}{c}, \quad \beta > 0, \quad 0 < c < 1,\]

\[K_n(x; p, N) = \binom{-n}{-N} \frac{1}{p}, \quad 0 < p < 1, \quad N = 0, 1, \ldots,\]

\[C_n(x; a) = \binom{-n}{-1} \frac{1}{a}, \quad a > 0.\]

Remark 9 When \(n = 0\), we recover the Fisher information of the negative binomial (2), binomial (3) and Poisson (4) distributions.

We now compute the Fisher information of the Meixner-Pollaczek polynomials using some of the results obtained in the discrete case.

Theorem 10 The Fisher information of the Meixner-Pollaczek polynomials is given by:

\[I_\phi \left( P_n^{(\lambda)} \right) = \int_{-\infty}^{\infty} \left[ \frac{\partial}{\partial \theta} \rho_n(x) \right]^2 \frac{1}{\rho_n(x)} dx = \frac{2 \left[ n^2 + (2n + 1) \lambda \right]}{\sin^2(\phi)}, \quad n = 0, 1, \ldots,\]

with \(\rho_n(x)\) defined as in \[8\].

Proof. The Meixner-Pollaczek polynomials have the hypergeometric representation \[5\]

\[P_n^{(\lambda)}(x; \phi) = \frac{(2\lambda)_n}{n!} e^{i\phi} \binom{-n, \lambda + ix}{2\lambda} \left[ 1 - e^{-2i\phi} \right], \quad \lambda > 0, \quad 0 < \phi < \pi.\]

They satisfy the orthogonality relation

\[\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(2\phi - \pi)x} |\Gamma(\lambda + ix)|^2 P_m^{(\lambda)}(x; \phi) P_n^{(\lambda)}(x; \phi) dx = \frac{\Gamma(n + 2\lambda)}{[2 \sin(\phi)]^{2\lambda} n!} \delta_{n,m}, \quad n, m = 0, 1, \ldots\]

(27)
and the recurrence relation
\[ (n + 1) P_{n+1}^{(\lambda)} - 2 [x \sin (\phi) + (n + \lambda) \cos (\phi)] P_n^{(\lambda)} + (n + 2\lambda - 1) P_{n-1}^{(\lambda)} = 0, \]  
\[ (28) \]

From (12) and (26), we have
\[ \frac{\partial P_n^{(\lambda)}}{\partial \phi} = n \cot (\phi) P_n^{(\lambda)} - \frac{(n + 2\lambda - 1)}{\sin (\phi)} P_{n-1}^{(\lambda)}, \]

while (8) and (27) give
\[ \rho_n(x) = \frac{e^{(2\phi-\pi)x} |\Gamma (\lambda + ix)|^2 [2 \sin (\phi)]^{2\lambda} n! \left[ P_n^{(\lambda)}(x; \phi) \right]^2}{2\pi \Gamma (n + 2\lambda)}. \]  
\[ (29) \]

Note that
\[ \int_{-\infty}^{\infty} \rho_n(x) dx = 1, \quad n = 0, 1, \ldots. \]  
\[ (30) \]

Differentiating (29) with respect to \( \phi \), we obtain
\[ \frac{\partial \rho_n}{\partial \phi} = 2\rho_n(x) \left\{ [x + (n + \lambda) \cot (\phi)] P_n^{(\lambda)} - \frac{(n + 2\lambda - 1)}{\sin (\phi)} P_{n-1}^{(\lambda)} \right\} \]
or, using (28),
\[ \frac{\partial \rho_n}{\partial \phi} = \frac{\rho_n(x)}{\sin (\phi) P_n^{(\lambda)}} [(n + 1) P_{n+1}^{(\lambda)} - (n + 2\lambda - 1) P_{n-1}^{(\lambda)}]. \]  
\[ (31) \]

Therefore,
\[ \left[ \frac{\partial}{\partial \theta} \rho_n(x) \right]^2 = \frac{1}{\rho_n(x)} = \frac{\rho_n(x)}{\sin^2 (\phi) P_n^{(\lambda)}} \left[ (n + 1)^2 \left( P_{n+1}^{(\lambda)} \right)^2 \right. \]
\[ \left. - 2(n + 1)(n + 2\lambda - 1) P_{n+1}^{(\lambda)} P_{n-1}^{(\lambda)} + (n + 2\lambda - 1)^2 \left( P_{n-1}^{(\lambda)} \right)^2 \right] \]
\[ = \frac{1}{\sin^2 (\phi)} \left[ (n + 1)(n + 2\lambda) \rho_{n+1}(x) + n(n + 2\lambda - 1) \rho_{n-1}(x) \right] \]
\[ - 2(n + 1)(n + 2\lambda - 1) \frac{[2 \sin (\phi)]^{2\lambda} n!}{\Gamma (n + 2\lambda)} \rho(x) P_{n+1}^{(\lambda)} P_{n-1}^{(\lambda)} \],
where
\[
\rho(x) = \frac{e^{(2\phi-\pi)x} |\Gamma(\lambda + ix)|^2}{2\pi}.
\]

Integrating (32) and using the orthogonality relation (27) and (30), we get
\[
\int_{-\infty}^{\infty} \left[ \frac{\partial}{\partial \theta} \rho_n(x) \right]^2 \frac{1}{\rho_n(x)} dx = \frac{1}{\sin^2(\phi)} [(n + 1)(n + 2\lambda) + n(n + 2\lambda - 1)]
\]
and the result follows. ■

4 Conclusions and further directions

We have computed the Fisher information of the Meixner, Krawtchouk and Charlier polynomials, which can be viewed in a sense as extensions of the Fisher information of the negative binomial, binomial and Poisson distributions, respectively. We are working on trying to extend the same framework to include other discrete orthogonal polynomials, namely the Racah and Hahn families.

We have also obtained the Fisher information of the Meixner-Pollaczek polynomials. It would be very interesting to calculate the Fisher information of the Wilson and the rest of the Hahn families (continuous, dual and continuous dual). Finally, the Fisher information of q-orthogonal polynomials doesn’t seem to have been considered yet.

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References

[1] G. E. Andrews, R. Askey, and R. Roy. Special functions, volume 71 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1999.
[2] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi. *Higher transcendental functions. Vol. I*. Robert E. Krieger Publishing Co. Inc., Melbourne, Fla., 1981.

[3] R. A. Fisher. *Statistical methods and scientific inference*. Hafner Press [A Division of Macmillan Publishing Co., Inc.], New York, 1973.

[4] B. R. Frieden. *Science from Fisher information*. Cambridge University Press, Cambridge, 2004.

[5] R. Koekoek and R. F. Swarttouw. The Askey-scheme of hypergeometric orthogonal polynomials and its $q$-analogue. Technical Report 98-17, Delft University of Technology, 1998. [http://aw.twi.tudelft.nl/~koekoek/askey/](http://aw.twi.tudelft.nl/~koekoek/askey/)

[6] N. N. Lebedev. *Special functions and their applications*. Dover Publications Inc., New York, 1972.

[7] G. R. Mohtashami Borzadaran. Relationship between entropies, variance and Fisher information. In *Bayesian inference and maximum entropy methods in science and engineering (Gif-sur-Yvette, 2000)*, volume 568 of *AIP Conf. Proc.*, pages 139–144. Amer. Inst. Phys., Melville, NY, 2001.

[8] J. Sánchez-Ruiz and J. S. Dehesa. Fisher information of orthogonal hypergeometric polynomials. *J. Comput. Appl. Math.*, 182(1):150–164, 2005.

[9] G. Zheng and J. L. Gastwirth. Fisher information in randomly sampled sib pairs and extremely discordant sib pairs in genetic analysis for a quantitative trait locus. *J. Statist. Plann. Inference*, 130(1-2):299–315, 2005.