Negativity and topological order in the toric code

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(Dated: May 7, 2014)

In this manuscript we study the behaviour of the entanglement measure dubbed negativity in the context of the toric code model. Using a replica method introduced recently by Calabrese, Cardy and Tonni [Phys. Rev. Lett. 109, 130502 (2012)], we obtain an exact expression which illustrates how the non-local correlations present in a topologically ordered state reflect in the behaviour of the negativity of the system. We find that the negativity has a leading area-law contribution, if the subsystems are in direct contact with one another (as expected in a zero-range correlated model). We also find a topological contribution directly related to the topological entropy, provided that the partitions are topologically non-trivial in both directions on a torus. We further confirm by explicit calculation that the negativity captures only quantum contributions to the entanglement. Indeed, we show that the negativity vanishes identically for the classical topologically ordered 8-vertex model, which on the contrary exhibits a finite von Neumann entropy, inclusive of topological correction.

INTRODUCTION

In the effort to understand and quantify classical vs quantum correlations in many body systems, a number of different measures of entanglement have been proposed in recent years. The von Neumann entropy, for instance, is obtained from a bi-partition of the system $S = A \cup B$: $S^{(A)}_v = -\text{Tr} \rho_A \ln \rho_A$, where $\rho_A = \text{Tr}_B \rho$. This definition, however, is a measure of quantum correlations between $A$ and $B$ only if $\rho$ is a pure state. In order to apply it to mixed states, one ought to symmetrise it and compute the mutual information $S^{(A)}_v + S^{(B)}_v - S^{(A/B)}_v$, which nonetheless remains sensitive to classical as well as quantum correlations, and it is therefore only an upper bound on the entanglement between $A$ and $B$.

Providing an explicit measure of entanglement that applies to both mixed and pure states, and that is of practical use, has been a tall order. In recent years it was proposed to use of a quantity called negativity, which was first introduced in Ref. [1] and later proven to be an entanglement monotone by several authors [2-5]. The negativity $N$ (or, equivalently, the logarithmic negativity $E$), is defined from the trace norm $\|\rho^{T_B}\|_1$ of the partial transpose over subsystem $B$ of the density matrix $\rho$,

$$N \equiv \frac{\|\rho^{T_B}\|_1 - 1}{2} \quad (1)$$

$$E \equiv \ln \|\rho^{T_B}\|_1, \quad (2)$$

where $\|\rho^{T_B}\|_1$ is the sum of the absolute values of the eigenvalues $\lambda_i$ of $\rho^{T_B}$. If all the eigenvalues are positive then $N = 0$ (recall that $\sum_i \lambda_i = 1$), and $N > 0$ otherwise – hence its name. The existence of negative eigenvalues is directly related to the fact that $A$ and $B$ are not separable, as discussed e.g., in Ref. [5].

Despite the availability of an explicit formulation, the calculation of the negativity of a many body quantum system remains an arduous task which has been carried out in the literature mostly in 1D. Recently, Calabrese, Cardy and Tonni [6] devised a new scheme based on a replica approach, which allows to compute the negativity in conformally invariant field theories (see also Refs. [7] and [8]).

Here we apply the method introduced in Ref. [6] to compute the negativity of the toric code model [9]. We are able to obtain an exact expression which illustrates the microscopic origin of the different contributions to the negativity, depending on the nature of the partition of the system. In addition to the expected area-law contribution if the subsystems are in direct contact with one another, we also find that the non-local correlations present in a topologically ordered state affect the behaviour of the negativity. This topological contribution relates directly to the topological entropy and appears only if the partitions are topologically non-trivial, which is consistent with the fact that topologically trivial disconnected subsystems are separable in a topologically ordered state.

With this calculation we also show that the negativity captures only the off-diagonal (‘quantum’) contribution to the topological entropy [10, 11] and it is insensitive to the diagonal part. Indeed, we find that $N = 0$ for the classical topologically ordered 8-vertex model (which on the contrary has a non-vanishing topological entropy [12]).

THE TORIC CODE MODEL

The toric code is a system of spin-1/2 degrees of freedom $\sigma_i$ living on the bonds $i$ of a square lattice (periodic boundary conditions will be assumed throughout). The Hamiltonian of the system can be written as [9]:

$$H = -\lambda_A \sum_s A_s - \lambda_B \sum_p B_p, \quad (3)$$

where $s$ ($p$) label the sites (plaquettes) of the lattice, and $A_s = \prod_{i \in s} \sigma_i^z, B_p = \prod_{i \in p} \sigma_i^z$.

The ground state (GS) is 4-fold degenerate, according to the 4 topological sectors identified by the expectation value of winding loop operators. Within each sector, the GS is given by the equal amplitude superposition of all tensor product basis states $\otimes_i |\sigma_i^z\rangle$ belonging to that sector. Following the notation in Refs. [13] and [14] we define $|0\rangle \equiv \otimes_i |\sigma_i^z = +1\rangle$ and we introduce the group $G$ generated by products of $A_s$ operators. Notice that one has to define elements $g \in G$ modulo the
identity \( \prod A_s = \mathbb{I} \) in order for the inverse of \( g \) to be uniquely defined (in which case, \( g^{-1} = g \)). The dimension (i.e., the number of elements) of \( G \) is therefore \( |G| = 2^{N(s)} - 1 \), where \( N(s) \) is the number of sites on the lattice. A GS can then be written explicitly as:

\[
|\psi_0\rangle = \frac{1}{|G|^{1/2}} \sum_{g \in G} g|0\rangle. \tag{4}
\]

**CHOICE OF PARTITIONS**

Here we are interested in computing the negativity of the system, in order to understand its relation to the topological correlations between two subsystems \( A_1 \) and \( A_2 \) without knowledge of the rest of the system \( S = A_1 \cup A_2 \cup B \). Since \( N \) is a measure of separability of subsystems \( A_1 \) and \( A_2 \), one expects \( N = 0 \) if \( A_1 \) and \( A_2 \) are topologically trivial. This expectation is confirmed by the calculation below.

In order to understand the behaviour of the negativity, we consider progressively more involved choices of partitions, as illustrated in Fig. 1. This work will form the basis to understand how to identify the topological contribution in \( N \).

**NEGATIVITY OF THE TORIC CODE**

The density matrix of subsystem \( A = A_1 \cup A_2 \) is given by \( \rho_A = \text{Tr}_B \rho \), where \( \rho = |\psi_0\rangle \langle \psi_0| \) is prepared in the pure state (4):

\[
\rho_A = \frac{1}{|G|} \sum_{g \in G, g' \in G} g_A |0_A\rangle \langle 0_A| g'_{A_2} g_2 |0_B\rangle \langle 0_B|, \tag{5}
\]

where we introduced the notation \( |0\rangle = |0_A\rangle \otimes |0_B\rangle \) and \( g = g_A \otimes g_2 \). It is convenient to redefine \( g' \rightarrow gg' \). Note that, for any given \( g \), this mapping for \( g' \) is 1-to-1 in \( G \). The trace over \( B \) imposes then that \( g' \) acts trivially on \( B \) \((|0_B\rangle g'_2 |0_B\rangle = |0_B\rangle) \), i.e., \( g' \in G_A \equiv \{g' \in G | g'_2 = \mathbb{I}_B \} \). We thus arrive at the expression:

\[
\rho_A = \frac{1}{|G|} \sum_{g \in G, g' \in G_A} g_A |0_A\rangle \langle 0_A| g_A g'_A. \tag{6}
\]

Next, we take the partial transpose over \( A_2 \). Given that we can choose all elements of \( \rho_A \) to be real, the transpose is equivalent to the adjoint of the part of \( \rho_A \) acting on \( A_2 \):

\[
\rho_A^{T_2} = \frac{1}{|G|} \sum_{g \in G, g' \in G_A} (g_A |0_A\rangle \langle 0_A| g_A g'_A) \otimes (g_{A_2} g'_{A_2} |0_{A_2}\rangle \langle 0_{A_2}| g_{A_2} g'_{A_2})), \tag{7}
\]

Following Ref. 8, we want to obtain the trace norm of \( \rho_A^{T_2} \) with a replica approach as the analytic continuation for \( n \rightarrow 1/2 \) of \( \text{Tr} \left( \rho_A^{T_2} \right)^n \). Therefore, we need to compute

\[
\text{Tr} \left( \rho_A^{T_2} \right)^n = \frac{1}{|G|^n} \sum_{g_1 \ldots g_n \in G} \sum_{g_1' \ldots g_n' \in G_A} (0_A | g_1 A_1 g_2 A_2 g_1' A_1 | 0_A) \ldots (0_A | g_{n-1} A_1 g_n A_1 g_n' A_1 | 0_A) (0_A | g_n A_1 g_1 A_2 g_n' A_2 | 0_A) \ldots (0_A | g_{n-1} A_2 g_n A_2 g_n' A_2 | 0_A) (0_A | g_{n-1} A_2 g_n A_2 g_n' A_2 | 0_A).
\]

In general, the subgroup \( G_A \subset G \) decomposes into the product \( G_{A_1} \cdot G_{A_2} \cdot G_{A_1 A_2} \), where the quotient group

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**FIG. 1.** (Colour online) – Examples of tri-partitions of the system into \( S = A_1 \cup A_2 \cup B \). The top panel illustrates a topologically trivial choice for \( A_1 \) and \( A_2 \), which corresponds to vanishing negativity. Although subsystems \( A_1 \) and \( A_2 \) are no longer trivial in the middle panel, as they wind around the system in the vertical direction, the negativity remains zero. The bottom panel illustrates a choice of partitions with finite logarithmic negativity between \( A_1 \) and \( A_2 \), where \( \mathcal{E} \) exhibits both a boundary as well as a topological contribution.
$G_{A_1,A_2} \equiv G_A/(G_{A_1},G_{A_2})$ is defined as the set of $g \in G_A$ that are equivalent up to the action of elements of $G_{A_1}$ and $G_{A_2}$. For disjoint, topologically trivial partitions (top panel in Fig. 1), $G_{A_1,A_2} = \{\mathbb{1}\}$ and $G_A = G_{A_1} \cdot G_{A_2}$.

This is no longer the case, for instance, with the choice of partitions in the middle panel in Fig. 1. Here $A$ divides $B$ into two disconnected portions $B = B_1 \cup B_2$. The product of all star operators acting on each component has more than 2 disconnected components, another independent element of $G_{A_1,A_2}$ can be found per additional component (cf. Refs. 13, 14).

If $A_1$ and $A_2$ share a common edge, as is the case in the bottom panel of Fig. 1 then the system allows for single star boundary star operators acting simultaneously (and only) on the two subsystems. In this case, each boundary star operator is an additional generator of $G_{A_1,A_2}$.

In general, for each element $g^\ell \in G_A$ there exists a unique decomposition $g^\ell = \tilde{g}^\ell \tilde{g}^\ell_1 \cdots \tilde{g}^\ell_n$, with $\tilde{g}^\ell \in G_{A_1}$, $\tilde{g}^\ell_1 \in G_{A_2}$, and $\theta^\ell_1 \in G_{\tilde{A}_1,\tilde{A}_2}$. Accordingly, one can write the first $n - 1$ expectation values in each row of Eq. (8) as:

$$\langle 0_A | \tilde{g}^\ell_{\ell - 1} A_1 | 0_A \rangle$$

for $\ell = 2, \ldots, n$. It is then convenient to redefine $g^\ell \rightarrow \tilde{g}^\ell \equiv g^\ell - 1 \tilde{g}^\ell_{\ell - 1} \tilde{g}^\ell_{\ell - 1}$, $\tilde{g}^\ell = \tilde{g}^\ell_1 \cdots \tilde{g}^\ell_n$. Once again, this is a 1-to-1 mapping $g^\ell \rightarrow \tilde{g}^\ell$ in $G$ given $g^\ell_{\ell - 1}, \tilde{g}^\ell_{\ell - 1},$ and $\tilde{g}^\ell$. Upon fixing the first term, say $\tilde{g}_1 = g_1$, the new definition of $\tilde{g}^\ell$ is nothing but a re-labelling of $n - 1$ mute indices in the summation over $g_1 \cdots g_n$. This simplifies the above expectation values to:

$$\langle 0_A | \tilde{g}^\ell_{\ell - 1} A_1 | 0_A \rangle$$

which vanish unless $\tilde{g}^\ell_{\ell - 1}$ acts trivially on $A_1$ and $\tilde{g}^\ell$ acts trivially on $A_2$.

The last expectation value in each of the two rows in Eq. (8) needs to be dealt with separately. Upon combining the chain of mappings $g^\ell \rightarrow \tilde{g}^\ell$ one obtains $g^\ell = \prod_{\ell=1}^{n-1} \tilde{g}^\ell \tilde{g}^\ell_{\ell - 1} g^\ell_{\ell - 1} | g_A |$. For disjoint, topologically trivial partitions (top panel in Fig. 1), one obtains $g^\ell = \prod_{\ell=1}^{n-1} \tilde{g}^\ell_{\ell - 1} \tilde{g}^\ell_{\ell - 1} g^\ell_{\ell - 1} | g_A |$.

Notice that the dependence on $\tilde{g}_1 = g_1$ has disappeared. We can further simplify these expressions using Eqs. (11). Since each $\tilde{g}^\ell$ acts trivially on $A_1$ and $\tilde{g}^\ell$ acts trivially on $A_2$ for $\ell = 2, \ldots, n$, then this is true of their product. Therefore, $\prod_{\ell=2}^{n} \tilde{g}^\ell_{\ell - 1} A_1 = \prod_{\ell=2}^{n} \tilde{g}^\ell_{\ell - 1} A_2 = \prod_{\ell=2}^{n} \tilde{g}^\ell_{\ell - 1}$.

By definition, products of $\theta^\ell_1 \in G_{\tilde{A}_1,\tilde{A}_2}$ other than the identity cannot be decomposed into products of $\tilde{g}^\ell_{\ell - 1} \in G_{A_1}$ and $\tilde{g}^\ell_{\ell - 1} \in G_{A_2}$. Therefore, the above equations separately imply that $\prod_{\ell=1}^{n-1} \theta^\ell_1 = \mathbb{1}$, $\prod_{\ell=1}^{n-1} \tilde{g}^\ell_{\ell - 1} A_1 = \mathbb{1}$, and $\prod_{\ell=1}^{n-1} \tilde{g}^\ell_{\ell - 1} A_2 = \mathbb{1}$. We thus arrive at the expression:

$$\text{Tr} \left( \rho_A^{\ell_1 \cdots \ell_n} \right) = \frac{|G_{A_1}|^{n-1} |G_{A_2}|^{n-1}}{|G|^{n-1}} \sum_{\theta^\ell_1 \cdots \theta^\ell_n \in G_{A_1, A_2}} \prod_{\ell=2}^{n} \left( \sum_{\tilde{g}^\ell \in G} \langle 0_A | \tilde{g}^\ell_{\ell - 1} A_1 | 0_A \rangle \langle 0_A_2 | \tilde{g}^\ell A_2 | 0_A_2 \rangle \right) \langle 0 | \prod_{\ell=1}^{n} \theta^\ell | 0 \rangle.$$  

The term in round brackets acts as a projector: it vanishes identically unless the action of $\theta^\ell_{\ell - 1}, A_1 \theta^\ell A_2$ can be matched.
by \( \hat{g}_t \). In that case, the expectation values equal 1, \( \hat{g}_t \) is uniquely selected over subsystem \( \mathcal{A} \), and the summation over \( \hat{g}_t \in G \) contributes a factor \(|G_{\mathcal{B}}|\).

Let us consider for example the three cases illustrated in Fig. 1. The case in the top panel corresponds to \( G_{\mathcal{A}_1, \mathcal{A}_2} = \{1\} \); the summation over \( \theta_t \) is not present and the result greatly simplifies. As discussed in more detail in the Appendix, this choice of partition leads to vanishing negativity.

In the middle panel of Fig. 1 the partitions \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are no longer topologically trivial (in the sense that they wind around the system in one direction). Here \( G_{\mathcal{A}_1, \mathcal{A}_2} = \{1, k\} \), where \( k \) is given by the product of all star operators acting on \( \mathcal{B}_1 \). Nonetheless, after some consideration one can see that the expectation values between round brackets in Eq. (12) do not impose any limitations on the choice of \( \theta_t \). The only difference to the previous case is that the summation over \( \theta_t \) subject to the constraint \( \{0\} \prod_{\ell=1}^n \theta_{\ell} |0\) results in an additional factor \( 2^{([G_{\mathcal{A}_1, \mathcal{A}_2}] - 1)(n-1)} = 2^n - 1 \), and once again the negativity vanishes.

Let us finally consider the partition in the bottom panel in Fig. 1. Subsystem \( \mathcal{B} \) has only one component; however, subsystems \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) have now two direct boundaries with one another (recall that the figure has periodic boundary conditions). The group \( G_{\mathcal{A}_1, \mathcal{A}_2} \) is generated by all the star operators acting simultaneously (and exclusively) on \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \). One can also define the operator \( k \) given by the product of all the stars acting on at least one spin in \( \mathcal{B} \). However, the product of all \( \mathcal{A}_1 - \mathcal{A}_2 \) boundary star operators times the operator \( k \) is nothing but the product of all stars acting solely on \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \), which is an operator that belongs to \( G_{\mathcal{A}_1} \cdot G_{\mathcal{A}_2} \). Therefore, \( k \) is in fact equivalent to the product of all \( \mathcal{A}_1 - \mathcal{A}_2 \) boundary stars, and \( |G_{\mathcal{A}_1, \mathcal{A}_2}| = 2^{N_{\mathcal{A}_1, \mathcal{A}_2} - 1} \). In the following, it is convenient to think of the group \( G_{\mathcal{A}_1, \mathcal{A}_2} \) as the group generated by \( k \) and by products of \( \mathcal{A}_1 - \mathcal{A}_2 \) boundary star operators defined modulo the product of all \( \mathcal{A}_1 - \mathcal{A}_2 \) boundary star operators.

One can verify that the action of \( k \) on the expectation values in Eq. (12) is immaterial, much as is the case for the partition in the middle panel of Fig. 1 considered earlier. On the contrary, products of \( \mathcal{A}_1 - \mathcal{A}_2 \) boundary star operators (identified modulo the product of all of them) play a crucial role. If \( \theta_{\ell-1} \) and \( \theta_{\ell} \) differ in this respect, it is then not possible to find any \( \hat{g}_t \in G \) such that \( \hat{g}_t \mathcal{A} = \theta_{(\ell-1)} \mathcal{A}_1 \theta_{\ell} \mathcal{A}_2 \). The expectation values in Eq. (12) vanish unless all \( \theta_t \) have the same contribution of products of \( \mathcal{A}_1 - \mathcal{A}_2 \) boundary star operators. This leads to a significant difference in the behaviour of \( \text{Tr} \left( \rho_{\mathcal{A}}^{T_n} \right) \) for even or odd \( n \). Indeed, if \( n \) is odd, the product of \( \theta_t \) can be the identity only if all \( \theta_t = 1 \). Vice versa, if \( n \) is even, the product always equals the identity irrespective of the choice of \( \theta_t \). As a result, Eq. (12) becomes

\[
\text{Tr} \left( \rho_{\mathcal{A}}^{T_n} \right) = f(n) \left[ \frac{2|G_{\mathcal{A}_1}| |G_{\mathcal{A}_2}| |G_{\mathcal{B}}|}{|G|} \right]^{n-1}, \tag{13}
\]

where \( f(n) = 1 \) for \( n \) odd, and \( f(n) = 2^{N_{\mathcal{A}_1, \mathcal{A}_2} - 1} \) if \( n \) is even.

The behaviour of the factor \( f(n) \) leads to a different analytic continuation of \( \text{Tr} \left( \rho_{\mathcal{A}}^{T_n} \right) \) for even and odd \( n \). If we follow the odd sequence, then \( \text{Tr} \left( \rho_{\mathcal{A}}^{T_n} \right) \to 1 \) for \( n \to 1 \), as expected for a quantity that converges to the sum of the eigenvalues of a density matrix operator. If instead we follow the even sequence, we obtain \( \|\rho_{\mathcal{A}}^{T_2}\|_1 = \lim_{n \to 1/2} \text{Tr} \left( \rho_{\mathcal{A}}^{T_2} \right)^{2n} = 2^{N_{\mathcal{A}_1, \mathcal{A}_2} - 1} \) and therefore \( N = (2^{N_{\mathcal{A}_1, \mathcal{A}_2} - 1} - 1)/2, \) \( \mathcal{E} = (N_{\mathcal{A}_1, \mathcal{A}_2} - 1) \ln 2 \).

The leading behaviour is akin to the well-known area law observed in the scaling of the entanglement entropy. The correction of order one is instead universal and it is directly related to the topological entropy \( \gamma \) of the quantum system \( \mathcal{A} \).

**CONCLUSIONS**

We performed an exact calculation of the negativity for the toric code model using different choices of partitions. We find that the negativity has a leading area-law contribution, if the subsystems are in direct contact with one another, as expected in a zero-range correlated model. We also find a topological contribution reflecting the topological nature of the quantum state, provided that subsystem \( \mathcal{B} \) (which is traced out) does not span the system in either direction. This topological contribution is directly related to the topological entropy \( \gamma \). As in the case of the von Neumann entropy, a direct measure of \( \gamma \) likely requires either a subtraction scheme or finite-size extrapolation [10, 11].

It is interesting to recall that other approaches to probe the topological nature of the system, typically based on the von Neumann entropy, yield a non-vanishing value of \( \gamma \) also for classical topologically ordered systems [12] (e.g., in the 8-vertex model). A straightforward calculation (see Appendix) shows that the negativity vanishes identically in the classical 8-vertex model, consistently with the expectation that \( N > 0 \) is a measure of quantum entanglement only.

Comparing the calculations in this paper with the work in Refs. [13] and [14] one expects that the topological contribution to the negativity vanishes in the 2D toric code at any finite temperature in the thermodynamic limit. Contrary to the behaviour of the topological entropy, this ought to be true also for the toric code in 3D, which reduces to a classical \( \mathbb{Z}_2 \) gauge theory at finite temperature. It will be interesting to see whether the finite size behaviour of the negativity at finite temperature is able to discern the low temperature phase of the classical \( \mathbb{Z}_2 \) gauge theory from the trivial paramagnetic phase at high temperature, despite the fact that \( N \) vanishes in both cases in the thermodynamic limit. Finally, only in the 4D toric code one might expect quantum topological correlations to actually survive at finite temperature, and thus \( N > 0 \) for \( T > 0 \).

To some extent the toric code is a rather special example
of topological order with precisely ‘zero-ranged’ local correlations. It will be interesting to see extensions of the calculation of the negativity to other topologically non-trivial states in dimensions larger than one. One could perhaps start from perturbations of the toric code introduced via stochastic matrix form decomposition \cite{14}, where the GS wavefunction is known exactly throughout the phase diagram. These perturbations introduce finite correlations and eventually drive the system across a so-called conformal critical point. It may also be possible to study the behaviour of the negativity at such critical points by means of conformal field theoretic techniques \cite{12, 17, 18}. This work could lead the way to the much more challenging and interesting question of investigating the behaviour of the negativity in quantum Hall states and other topologically ordered phases of matter.

After this work was completed, private communication with G. Vidal revealed that, together with A. Lee, they had independently arrived at similar results \cite{19}. The author is deeply indebted to G. Vidal for spotting an inconsistency in the first version of this manuscript.

ACKNOWLEDGMENTS

This paper owes its existence to P. Calabrese, who introduced the author to the concept of negativity and stimulated interest in an exact calculation for the toric code model. This work was supported by EPSRC Grant EP/K028960/1.

APPENDIX

Classical 8-vertex model

The 8-vertex model is a combinatorial problem of arrows on the bonds of the square lattice, with the hard constraint that the number of incoming arrows at every vertex is even (counting 0 as an even number). Taking advantage of the bipartite nature of the lattice, we can define arrows going from sublattice A to sublattice B as positive spins, and all others are negative. This establishes a 1-to-1 mapping between 8-vertex configurations and $\sigma^x$ tensor product states that minimise the energy of the plaquette term in the toric code Hamiltonian \cite{3}. All 8-vertex configurations can be obtained from a reference configuration, say the spin polarized $|0\rangle$, by acting with elements of $G$.

The partition function of the 8-vertex model can thus be written in ket-bra notation as

$$\rho = \frac{1}{|G|} \sum_{g \in G} g |0\rangle \langle 0| g.$$ \hspace{1cm} (14)

Taking the trace over $B$ is straightforward, since $g_B^2 = 1$, and $\rho_A$ remains diagonal:

$$\rho_A = \frac{1}{|G|} \sum_{g \in G} g_A |0_A\rangle \langle 0_A| g_A = \rho_A^{T_2}. \hspace{1cm} (15)$$

In order to use the replica approach, we need to compute

$$\text{Tr} \left( \rho_A^{T_2} \right)^n = \frac{1}{|G|^n} \sum_{g_1 \ldots g_n \in G} \langle 0_A | g_1 A g_2 A | 0_A \rangle \ldots \langle 0_A | g_{(n-1)} A g_n A | 0_A \rangle \langle 0_A | g_n A g_1 A | 0_A \rangle.$$  \hspace{1cm} (16)

It is convenient to redefine $g_{\ell} \rightarrow \tilde{g}_{\ell} = g_{\ell-1} g_{\ell}, \ell = 2, \ldots, n$, with the choice $\tilde{g}_1 = g_1$. All expectation values simplify to $\langle 0_A | g_{\ell} A | 0_A \rangle$ for $\ell = 2, \ldots, n$ except for the last one, where the chain of mappings leads to $\tilde{g}_n = \prod_{\ell=1}^{n-1} g_{\ell} g_1$ and therefore to the expectation value $\langle 0_A | \prod_{\ell=1}^{n-1} \tilde{g}_{\ell} | 0_A \rangle$. Once again, the dependence on $\tilde{g}_1$ has disappeared, and the expectation values impose that all other $\tilde{g}_\ell$ acts trivially on $A$ ($\tilde{g}_\ell \in G_B$ for $\ell = 2, \ldots, n$). As a results, we obtain:

$$\text{Tr} \left( \rho_A^{T_2} \right)^n = \frac{|G_B|^{n-1}}{|G|^{n-1}}, \hspace{1cm} \left[ 2^{-N(s)} - N(s)^2 \right]^{n-1}. \hspace{1cm} (16)$$

where $|G| = 2^{N(s)} - 1$ and $|G_B| = 2^{N(s)} + 1$ (the latter is due to the fact that the product of all stars acting on at least one spin in $A_1$ is an element of $G_B$ that cannot be written as a product of stars belonging to $G_B$, see e.g., Ref. \cite{14}). Here $N(s) = N_{A_1}^{(s)} + N_{A_2}^{(s)} + N_{A_3}^{(s)}$, where $N_{A_1}^{(s)}$ is the number of star operators acting simultaneously on spins in $A$ and $B$.

The three contributions in the final expression of Eq. \cite{16} correspond, respectively, to the classical entropy (scaling with the volume of subsystem $A$), the area law, and a classical topological contribution.

Due to the diagonal nature of the density matrix, the topologically trivial vs non-trivial character of $A_1$ and $A_2$ does not play a role (whereas the fact that $B$ is non-trivial plays a crucial part in this case). As a result, the even and odd analytic continuations of $\text{Tr} \left( \rho_A^{T_2} \right)^n$ coincide and in the limit $n \rightarrow 1$ we obtain $\|\rho_A^{T_2}\|_1 = 1$, and therefore $N = 0, E = 0$.

Topologically trivial partitions

It is interesting to briefly consider what happens to the calculation of $\text{Tr} \left( \rho_A^{T_2} \right)^n$ when the partitions $A_1$ and $A_2$ are disjoint and topologically trivial. As discussed earlier, $\theta_\ell = \bar{1}$ is the only choice throughout Eq. \cite{12}.

In this case, $G_A = G_{A_1} \cdot G_{A_2}$ and $G_B \equiv G_{A_1} \cap G_{A_2}$. Therefore,

$$\text{Tr} \left( \rho_A^{T_2} \right)^n = \left[ \frac{|G_B| |G_{A_1}| |G_{A_2}|}{|G|} \right]^{n-1} = \left[ 2^{-N(s)} - N(s)^2 \right]^{n-1}, \hspace{1cm} (17)$$

where we used the fact that $N(s) = N_{A_1}^{(s)} - N_{A_2}^{(s)} = N_{A_3}^{(s)}$. Once again we recognise the leading area law and a topological contribution. However, the latter arises from the
topologically non-trivial nature of $\mathcal{B}$ rather than $\mathcal{A}_1$ or $\mathcal{A}_2$, and it does not contribute to the negativity between the latter two subsystems. Indeed, the even and odd analytic continuations of $\text{Tr} \left( \rho_{\mathcal{A}}^{T_2} \right)^n$ coincide, and in the limit $n \to 1$ we obtain $\|\rho_{\mathcal{A}}^{T_2}\|_1 = 1$, and therefore $N = 0$, $\mathcal{E} = 0$.

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