Identifying codes in line digraphs *

C. Balbuena¹, C. Dalfó², B. Martínez-Barona¹

¹Departament d’Enginyeria Civil i Ambiental
Universitat Politècnica de Catalunya
²Departament de Matemàtica
Universitat de Lleida
e-mails: {m.camino.balbuena,berenice.martinez}@upc.edu, cristina.dalfo@matematica.udl.cat

Abstract

Given an integer $\ell \geq 1$, a $(1, \leq \ell)$-identifying code in a digraph is a dominating subset $C$ of vertices such that all distinct subsets of vertices of cardinality at most $\ell$ have distinct closed in-neighbourhood within $C$. In this paper, we prove that every $k$-iterated line digraph of minimum in-degree at least 2 and $k \geq 2$, or minimum in-degree at least 3 and $k \geq 1$, admits a $(1, \leq \ell)$-identifying code with $\ell \leq 2$, and in any case it does not admit a $(1, \leq \ell)$-identifying code for $\ell \geq 3$. Moreover, we find that the identifying number of a line digraph is lower bounded by the size of the original digraph minus its order. Furthermore, this lower bound is attained for oriented graphs of minimum in-degree at least 2.

Mathematics Subject Classifications: 05C69, 05C20.

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1 Introduction

In this paper we study the concept of $(1, \leq \ell)$-identifying codes in line digraphs, where $\ell \geq 1$ is an integer. In [2], the authors studied the $(1, \leq \ell)$-identifying codes in digraphs, and gave some sufficient conditions for a digraph of minimum in-degree $\delta^- \geq 1$ to admit

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a \((1, \leq \ell)\)-identifying code for \(\ell = \delta^-, \delta^- + 1\). Regarding line graphs, Foucaud, Gravier, Naserasr, Parreau, and Valicov \[9\] studied \((1, \leq 1)\)-identifying codes and Junnila and Laihonen \[11\] studied \((1, \leq \ell)\)-identifying codes for \(\ell \geq 2\).

We consider simple digraphs without loops or multiple edges. Unless otherwise stated, we follow the book by Bang-Jensen and Gutin \[3\] for terminology and definitions.

Let \(D\) be a digraph with vertex set \(V(D)\) and arc set \(A(D)\). A vertex \(u\) is adjacent to a vertex \(v\) if \((u, v) \in A(D)\). If both arcs \((u, v), (v, u) \in A(D)\), then we say that they form a digon. A digraph is symmetric if \((u, v) \in A(D)\) implies \((v, u) \in A(D)\), so it can be studied as a graph. A digon is often referred as a symmetric arc of \(D\).

An oriented graph is a digraph without digons. The out-neighborhood of a vertex \(u\) is \(N^+(u) = \{v \in V : (u, v) \in A(D)\}\) and the in-neighborhood of \(u\) is \(N^-(u) = \{v \in V(D) : (v, u) \in A(D)\}\). The closed in-neighborhood of \(u\) is \(N^-([u] = \{u\} \cup N^-(u)\) Given a vertex subset \(U \subseteq V(D)\), let \(N^-[U] = \bigcup_{v \in U} N^-[v]\) and \(N^+[U] = \bigcup_{v \in U} N^+[v]\). A dominating set is a subset of vertices \(S \subseteq V\) such that \(N^+[S] = V\). The out-degree of \(u\) is \(d^+(u) = |N^+(u)|\), and its in-degree \(d^-(u) = |N^-(u)|\). We denote by \(\delta^+ = \delta^+(D)\) the minimum out-degree of the vertices in \(D\), and by \(\delta^- = \delta^-(D)\) the minimum in-degree. The minimum degree is \(\delta = \delta(D) = \min\{\delta^+, \delta^\}\). A digraph \(D\) is said to be \(d\)-in-regular if \(|N^-(v)| = d\) for all \(v \in V(D)\), and \(d\)-regular if \(|N^+(v)| = |N^-(v)| = d\) for all \(v \in V(D)\). A path from \(u\) to \(v\) is denoted by \((u, \ldots v)\) or also by \(u \rightarrow v\). A path of order \(k\) is called a \(k\)-path. A digraph \(D\) is said to be strongly connected when, for any pair of vertices \(u, v \in V(D)\), there always exists a \(u \rightarrow v\) path. For any pair of vertices \(u, v \in V(D)\), we denote by \(\text{dist}(u, v)\) the distance from \(u\) to \(v\) in \(D\), that is, \(\text{dist}(u, v) = \min\{k \mid \text{ there is a } k\text{-path in } D \text{ from } u \text{ to } v\}\). For each vertex \(v \in V(D)\), we denote by \(\omega^-(v) = \{(u, v) \in A(D)\}\) and \(\omega^+(v) = \{(v, u) \in A(D)\}\). For a natural number \(k\), a \(k\)-cycle is a directed cycle of order \(k\).

For a given integer \(\ell \geq 1\), a vertex subset \(C \subseteq V(D)\) is a \((1, \leq \ell)\)-identifying code in \(D\) if it is a dominating set and for all distinct subsets \(X, Y \subseteq V(D)\), with \(1 \leq |X|, |Y| \leq \ell\), we have

\[
N^-[X] \cap C \neq N^-[Y] \cap C. \tag{1}
\]

The definition of a \((1, \leq \ell)\)-identifying code for graphs was introduced by Karpovsky, Chakrabarty and Levitin \[10\], and its definition can be obtained from \[1\] by omitting the superscript signs minus. Thus, the definition for digraphs is a natural extension of the concept of \((1, \leq \ell)\)-identifying codes in graphs. A \((1, \leq 1)\)-identifying code is known as an identifying code. Thus, an identifying code of a graph is a dominating set, such that any two vertices of the graph have distinct closed neighborhoods within this set. Identifying codes model fault-diagnosis in multiprocessor systems, and these are used in other applications, such as the design of emergency sensor networks. For more information on these applications, see Karpovsky, Chakrabarty, and Levitin \[10\] and Laifenfeld, Trachtenberg, Cohen and Starobinski \[11\].

Note that if \(C\) is a \((1, \leq \ell)\)-identifying code in a digraph \(D\), then the whole set of vertices \(V(D)\) also is. Thus, a digraph \(D\) admits some \((1, \leq \ell)\)-identifying code if and only
if for all distinct subsets \(X, Y \subset V(D)\) with \(|X|, |Y| \leq \ell\), we have

\[
N^- [X] \neq N^- [Y].
\] (2)

We recall that a transitive tournament of 3 vertices is denoted by \(TT_3\), as shown in Figure 1.

**Remark 1.1.** Let \(D\) be a \(TT_3\)-free digraph. Then, for every arc \((x, y)\) of \(D\), we have \(N^-(x) \cap N^-(y) = \emptyset\) and \(N^+(x) \cap N^+(y) = \emptyset\).

## 2 Identifying codes in line digraphs

In the line digraph \(LD\) of a digraph \(D\), each vertex represents an arc of \(D\). Thus, \(V(LD) = \{uv : (u, v) \in A(D)\}\); and a vertex \(uv\) is adjacent to a vertex \(wz\) if and only if \(v = w\), that is, when the arc \((u, v)\) is adjacent to the arc \((w, z)\) in \(D\). For any integer \(k \geq 1\), the \(k\)-iterated line digraph \(L^k D\) is defined recursively by \(L^k D = LL^{k-1} D\), where \(L^0 D = D\). From the definition, it is evident that the order of \(LD\) equals the size of \(D\), that is, \(|V(LD)| = |A(D)|\). Due to the bijection between the set of arcs in the digraph \(D\) and the set of vertices in the digraph \(LD\), when it is clear from the context, we use \(uv\) to denote both the arc in \(A(D)\) and the vertex in \(V(LD)\). Hence, for each vertex \(v \in V(D)\), the set of arcs \(\omega^+(v)\) in \(D\) corresponds to a set of vertices in \(LD\). If \(D\) is a strongly connected digraph different from a directed cycle with minimum degree \(\delta\), then the iterated line digraph \(L^k D\) has minimum degree \(\delta\) and diameter \(\text{diam}(L^k D) = \text{diam}(D) + k\). See Aigner [1], Fiol, Yebra, and Alegre [5], and Reddy, Kuhl, Hosseini, and Lee [12].

A large known family of digraphs obtained with the line digraph technique is the family of Kautz digraphs. The Kautz digraph of degree \(d\) and diameter \(k\) is defined as the \((k - 1)\)-iterated line digraph of the symmetric complete digraph of \(d + 1\) vertices \(K_{d+1}\), that is, \(K(d, k) \cong L^{k-1} K_{d+1}\). For instance, the Kautz digraph \(K(2, 2)\), shown in Figure 2, is the line digraph of the symmetric complete digraph on three vertices.

Line digraphs were characterized by Heuchenne [8] with the following property: A digraph \(D\) is a line digraph if and only if it has no multiple arcs, and for any pair of vertices \(u\) and \(v\), either \(N^-(u) \cap N^-(v) = \emptyset\) or \(N^-(u) = N^-(v)\). A similar characterization is obtained replacing \(N^-\) by \(N^+\).
Figure 2: The Kautz digraph $K(2, 2)$ as the line digraph of the symmetric complete digraph $K_3$.

The semigirth $\gamma$ was defined by Fàbrega and Fiol [4] as follows.

**Definition 2.1.** [4] Let $D$ be a digraph with minimum degree $\delta$. Let $\gamma = \gamma(D)$, for $1 \leq \gamma \leq \text{diam}(D)$, be the greatest integer such that, for any $x, y \in V(G)$:

1. if $\text{dist}(x, y) < \gamma$, the shortest $x \to y$ path is unique and there are no paths of length $\text{dist}(x, y) + 1$;

2. if $\text{dist}(x, y) = \gamma$, there is only one shortest $x \to y$ path.

Note that, as $D$ has no loops, $\gamma \geq 1$. In [4] it was also proved that, if $D$ is a strongly connected digraph without loops and different from a directed cycle, then $\gamma(L^kD) = \gamma + k$.

From now on we are going to consider strongly connected digraphs.

**Remark 2.1.** If $D$ is a digraph with $\gamma(D) \geq 2$, then:

(i) $D$ is $TT_3$-free,

(ii) the paths of length two are unique.

Observe that for any line digraph $LD$ different from a directed cycle, $\gamma(LD) \geq 2$, therefore by Remark 2.1(ii) any line digraph is $TT_3$-free. As a consequence, we can write the following result.

**Proposition 2.1.** The line digraph of a strongly connected digraph of order at least 3 admits a $(1, \leq 1)$-identifying code. □

The following result is a direct consequence of Remark 2.1(ii) and the definition of line digraph.

**Lemma 2.1.** Let $LD$ be a line digraph.

(i) If $u, v \in V(D)$ are two different vertices such that $N^-(u) \cap N^-(v) \neq \emptyset$, then $N^+(u) \cap \ N^+(v) = \emptyset$.
(ii) There are no two digons incident with the same vertex.

In [2], the authors proved that if $D$ is a digraph of minimum in-degree $\delta^-$ admitting a $(1, \leq \delta^- + 1)$-identifying code, then the vertices of minimum in-degree does not lay on a digon. In the following theorem, we give sufficient and necessary conditions for a line digraph to admit a $(1, \leq 2)$-identifying code.

**Theorem 2.1.** Let $LD$ be a line digraph different from a 4-cycle and such that the vertices of in-degree 1 (if any) does not lay on a digon. Then, $LD$ admits a $(1, \leq 2)$-identifying code if and only if $LD$ satisfies the following conditions:

(i) there are no 3-cycles with at least 2 vertices of in-degree 1 (see Figure 3 (a) where the vertices of in-degree one are indicated in black color);

(ii) there do not exist four vertices $x, x', y, y'$ such that $N^-(x) = \{y, y'\}$, $N^-(y') = \{x'\}$ and $x \in N^-(x') \cap N^-(y)$ (see Figure 3 (b) where the vertices of in-degree one are indicated in black color and the vertices of in-degree two in gray color);

(iii) there do not exist two vertices $x, y \in V(LD)$ such that $N^-(x) = \{y, y'\}$, $N^-(y) = \{x, x'\}$ and $N^-(x') \cap N^-(y') \neq \emptyset$ (see Figure 3 (c) where the vertices of in-degree two are indicated in gray color).

**Proof.** Let $LD$ be a line digraph satisfying the hypothesis of the theorem. First, suppose that $LD$ does not satisfy (i). Hence, let $(z, y, x, z)$ be a 3-cycle such that $d^-(x) = 1 = d^-(y)$ (see Figure 3 (a)). Then, $N^-([x, z]) = \{x, y\} \cap N^-[z] = N^-([y, z])$, implying that $LD$ does not admit an identifying code. Second, suppose that $LD$ does not satisfy (ii). Let $X = \{x, x'\}$ and $Y = \{y, y'\}$, where $x, x', y, y'$ are four different vertices of $LD$ such that $N^-(x) = \{y, y'\}$, $N^-(y') = \{x'\}$, and $x \in N^-(x') \cap N^-(y)$ (see Figure 3 (b)). Hence, by the Heuchenne’s condition $N^-(x') = N^-(y)$, it follows that

$$N^-[X] = N^-(x) \cup N^-(x') \cup \{x, x'\} = \{y, y'\} \cup N^-(y) \cup \{x, x'\} = \{y, y'\} \cup N^-(y) \cup N^-(y') = N^-[Y].$$
Therefore, \( LD \) does not admit a \((1, \leq 2)\)-identifying code. Now, suppose that \( LD \) does not satisfy \((iii)\). Let \( X = \{x, x'\} \) and \( Y = \{y, y'\} \), where \( N^-(x) = \{y, y'\}, N^-(y) = \{x, x'\}, \) and \( N^-(x') \cap N^-(y') \neq \emptyset \) (see Figure 3 (c)). Since, by the Heuchenne’s condition \( N^-(x') = N^-(y') \), it follows that

\[
N^-[X] = N^-(x) \cup N^-(x') \cup \{x, x'\} = \{y, y'\} \cup N^-(y') \cup N^-(y) = N^-[Y].
\]

Therefore, \( LD \) does not admit a \((1, \leq 2)\)-identifying code.

For the converse, let \( X, Y \subset V(LD) \) be two different subsets such that \( 1 \leq |X| \leq |Y| \leq 2 \) and \( N^-[X] = N^-[Y] \). By Proposition 2.1 \(|Y| = 2 \). If \(|X| = 1\), say \( X = \{x\} \), then for all \( y \in Y \setminus X \), since \( N^-[Y] = N^-[X] = N^-[x] \), it follows \( N^-[y] \subset N^-[x] \). Hence, \( y \in N^-(x)\).

If \( d^-(y) = 1 \), there is at least one vertex \( z \in N^-(y) \cap N^-(x) \) because there are no vertices of in-degree 1 laying on a digon, and clearly the same happens if \( d^-(y) \geq 2\), reaching a contradiction to Remark 2.1 \((i)\). Hence, \(|X| = 2\), and there are two cases to be considered.

First, let us suppose that \( X \cap Y \neq \emptyset \). Let \( X = \{x, z\} \) and \( Y = \{y, z\} \). If there is an arc between \( x \) and \( y \), say \( xy \in A(LD) \), then by Remark 2.1 \((i)\), \( N^-(x) \cap N^-(y) = \emptyset \). Then, \( N^-(x) \subset N^-[z] \) and \( N^-(x) \cap N^-(z) \subset N^-[z] \cup \{y\} \). Consider \( d^-(x) \geq 2 \) and let \( u \in N^-(x) \setminus \{y\} \).

Hence, \( u \in N^-[z] \). If \( u = z \), then \( N^-(x) \cap N^-(z) = \emptyset \). Hence, by Remark 2.1 \((i)\) and \((ii)\), \( N^-[z] \cap N^-(y) = \emptyset \), implying that \( N^-(y) = \emptyset \), a contradiction since \( \delta^- \geq 1 \). Then, \( u \in N^-(z) \cap N^-(x) \) implying, by the Heuchenne’s condition that \( N^-(z) = N^-(x) \), hence \( y \in N^-(z) \), which is a contradiction since \( N^-(y) \subset N^-[z] \). Now suppose that \( d^-(x) = 1 \), then \( N^-(x) = \{y\} \). Since \( x \in N^-[Y] \) and \( x \) does not lay on a digon, \( x \in N^-(z) \). Since, \( x \notin N^-(y) \), \( N^-(y) \cap N^-(z) = \emptyset \), implying that \( N^-(y) = \{z\} \) because \( N^-(y) \subset N^-[z] \).

Therefore, \( (x, z, y, x) \) is a 3-cycle of \( LD \) with two vertices of in-degree 1, implying that \( LD \) does not satisfy \((i)\). Now, suppose that there are no arcs between \( x \) and \( y \). Since \( x \in N^-[Y] \) and \( y \in N^-[X] \), it follows that \( x, y \in N^-(z) \). Hence, by Remark 2.1 \((i)\) and \((ii)\), \( N^-(x) \cap (N^-(z) \cup N^-(y)) = \emptyset \), implying that \( N^-(x) = \{z\} \), a contradiction since there are no vertices of in-degree 1 laying on a digon.

Now let \( X \cap Y = \emptyset \), with \( X = \{x, x'\} \) and \( Y = \{y, y'\} \). In order to \( y \in N^-(x) \), that is, \( xy \in A(LD) \). Then, by Remark 1.1 \( N^-(x) \cap N^-(y) = \emptyset \) implying that \( N^-(y) \subset N^-(x') \cup \{x, x'\} \). Since \( x \in N^-[Y] \), there are two cases to be considered.

First, suppose that \( x \in N^-(y) \). Then, \( d^-(x), d^-(y) \geq 2 \), since both vertices lay on a digon. If there is \( u \in N^-(y) \setminus (X \cup Y) \), then \( u \in N^-(x') \), implying that \( x \in N^-(x') \) by the Heuchenne’s condition. Hence, since \( x' \in N^-[Y] \) and \( N^-(x') = N^-(y') \), it follows that \( x' \in N^-(y') \). Furthermore, \( y' \in N^-(x') \) or \( y' \in N^-(x) \). If \( y' \in N^-(x') \), then by the Heuchenne’s condition \( N^-(x') \cap N^-(y') = \emptyset \). Moreover, by Remark 2.1, \( x, y \notin N^-(y') \), implying that \( N^-[X] \cap N^-(y') = \{x'\} \). Hence, \( d^-(y') = 1 \), because \( N^-[X] = N^-[Y] \), a contradiction because \( y' \) lay on a digon. If \( y' \in N^-(x) \), then \( N^-(y') \cap (N^-(x) \cup N^-(x')) = \emptyset \), implying that \( N^-(y') = \{x'\} \) and \( N^-(x) = \{y, y'\} \). Therefore, \( LD \) does not satisfy \((ii)\). Reasoning similarly for \( x \) as we did for \( y \), we can assume that \( N^-(y) \subset X \cup Y \) and
Corollary 2.2. Let $N^-(x) \subseteq X \cup Y$. If $x' \in N^-(x)$, then $x' \notin N^-(y)$, implying that $y' \in N^-(y)$. Then, by Lemma 2.1 (ii) and Remark 2.1 (i), $x, y \notin N^-(x') \cup N^-(y')$, implying that there is a vertex $u \in (N^-(x') \cap N^-(y')) \setminus (X \cup Y)$. Therefore, LD does not satisfy (iii). If $x' \in N^-(y)$, then $y' \notin N^-(y)$, implying that $y' \in N^-(x)$. Then, by Lemma 2.1 (ii) and Remark 2.1 (i), $x, y \notin N^-(x') \cup N^-(y')$, implying that there is a vertex $u \in (N^-(x') \cap N^-(y')) \setminus (X \cup Y)$. Therefore, LD does not satisfy (iii).

Now, suppose that $x \in N^-(y') \setminus N^-(y)$. Then, $N^-(x) \cap (N^-(y) \cup N^-(y')) = \emptyset$, implying that $N^-(x) = \{y\}$. Hence, since $y' \in N^-[X]$, it follows that $y' \in N^-(x')$, implying that $N^-(y') \cap (N^-(x') \cup N^-(x)) = \emptyset$, and consequently $N^-(y') \subseteq \{x, x'\}$. Since $x' \in N^-[Y]$, there are two cases to be considered. If $x' \in N^-(y)$, then $N^-([y') \cap (N^-(y) \cup N^-(y'))$, and by Lemma 2.1 (ii) and since LD is TTT$_3$-free, $N^-(x) = \{y'\}$, implying that $d^-(y) = d^-(y') = 1$. Hence, LD would be a 4-cycle, since LD is a strongly connected digraph, a contradiction. Therefore, $x' \in N^-(y')$, implying that $d(x') \geq 2$. Since $x \in N^-(y') \setminus N^-(y)$ and $y' \in N^-(x') \setminus N^-(x)$, it follows that $N^-(y) \cap N^-(y') = \emptyset$ and $N^-(x) \cap N^-(x') = \emptyset$, respectively. Then, $x' \notin N^-(y)$ and $y \notin N^-(x')$. Then, there is $u \in N^-(x') \setminus (X \cup Y)$, implying that $u \in N^-(y)$ and, hence, LD does not satisfy (iii). Thus, this completes the proof.

Notice that, according to the above theorem, if a line digraph with minimum in-degree $\delta^- \geq 2$ does not admit a $(1, \leq 2)$-identifying code, then $\delta^- = 2$. Note that $\gamma(L^kD) = k + 1 \geq 3$ if $k \geq 2$, which implies by Definition 2.1 that $L^kD$ does not contain two vertices satisfying the hypothesis of Theorem 2.1. Therefore, we have the following result.

**Corollary 2.1.** Let $L^kD$ be a line digraph with minimum in-degree $\delta^- \geq 2$.

(i) If $k \geq 2$, then $L^kD$ admits a $(1, \leq 2)$-identifying code.

(ii) If $k = 1$ and $\delta^- \geq 3$, then LD admits a $(1, \leq 2)$-identifying code.

(iii) If $D$ is a 2-in-regular digraph and $k \geq 1$, then $L^kD$ admits a $(1, \leq 2)$-identifying code if and only if $L^kD$ does not contain the subdigraph of Figure 3 (c).

**Corollary 2.2.** For each $n \geq 3$, the Kautz digraph $K(n, 2) = LK_{n+1}$ admits a $(1, \leq 2)$-identifying code.

By Corollary 2.1 (iii), the Kautz digraph $K(2, 2) = LK_3$ (see Figure 2) does not admit a $(1, \leq 2)$-identifying code. Then, the condition $k \geq 2$ in Corollary 2.1 (i) is necessary.

**Remark 2.2.** Let $D$ be a digraph with minimum in-degree $\delta^- \geq 2$. Then, there exists a vertex $u \in V(D)$ such that $d^+(u) \geq 2$. It is enough to observe that if $d^+(u) < 2$ for all $u \in V(D)$, then we would reach the contradiction:

$$2|V(D)| \leq \sum_{v \in V(D)} d^-(v) = \sum_{v \in V(D)} d^+(v) \leq |V(D)|.$$
Consequently, any line digraph \(LD\) with minimum in-degree \(\delta^- \geq 2\) contains at least two vertices with the same in-neighborhood by the Heuchenne’s condition.

**Proposition 2.2.** Let \(LD\) be a line digraph with minimum in-degree \(\delta^- \geq 2\), then \(LD\) does not admit a \((1, \leq 3)\)-identifying code.

**Proof.** By Remark 2.2 there are two different vertices \(u, v \in V(LD)\) such that \(N^-(u) = N^-(v)\). Moreover, \(LD\) has \(\delta^+ \geq 1\) because it is strongly connected. Let \(w \in N^+(u)\), thus by Remark 1.1 \(w \neq v\). Then, \(X = \{u, v, w\}\) and \(Y = \{v, w\}\) are two different sets such that \(N^-[X] = N^-[Y]\), implying that \(LD\) does not admit a \((1, \leq 3)\)-identifying code. \(\square\)

3 The identifying number of a line digraph

Foucaud, Naserasr, et al. [6] characterized the digraphs that only admit as identifying code the whole set of vertices. Let us introduce the terminology used for this characterization.

Given two digraphs \(D_1\) and \(D_2\) on disjoint sets of vertices, we denote \(D_1 \oplus D_2\) the disjoint union of \(D_1\) and \(D_2\), that is, the digraph whose vertex set is \(V(D_1) \cup V(D_2)\) and whose arc set is \(A(D_1) \cup A(D_2)\). Given a digraph \(D\) and a vertex \(x \notin V(D)\), \(x \triangle(D)\) is the digraph with vertex set \(V(D) \cup \{x\}\), and whose arcs are the arcs of \(D\) together with each arc \((x, v)\) for every \(v \in V(D)\).

**Definition 3.1.** We define \((K_1, \oplus, \triangle)\) to be the closure of the one-vertex graph \(K_1\) with respect to the operations \(\oplus\) and \(\triangle\). That is, the class of all graphs that can be built from \(K_1\) by repeated applications of \(\oplus\) and \(\triangle\).

Foucaud, Naserasr, et al. [7] proved that for any digraph \(D\), \(\overrightarrow{\gamma}^{ID}(D) = |V(D)|\) if and only if \(D \in (K_1, \oplus, \triangle)\). Since, as they pointed out, every element \(D \in (K_1, \oplus, \triangle)\) is the transitive closure of a rooted oriented forest, if \(LD\) is a line digraph with minimum in-degree \(\delta^- \geq 2\), then \(LD \notin (K_1, \oplus, \triangle)\). Hence, \(\overrightarrow{\gamma}^{ID}(LD) \leq |V(LD)| - 1\), where \(\overrightarrow{\gamma}^{ID}(D)\) denotes the minimum size of an identifying code of a digraph \(D\). Next, we establish better upper bounds on \(\overrightarrow{\gamma}^{ID}(LD)\).

With this goal, we define the relation \(\sim\) over the set of vertices \(V(LD)\) as follows. For all \(u, v \in V(LD)\), \(u \sim v\) if and only if \(N^-(u) = N^-(v)\). Clearly, \(\sim\) is an equivalence relation. For any \(u \in V(LD)\), \(\{v \in V(LD) : v \sim u\}\).

**Lemma 3.1.** Let \(D\) be digraph and \(C\) an identifying code of \(LD\). Then, for any vertex \(w \in V(LD)\), \(|[w]_\sim \setminus C| \leq 1\).

**Proof.** Let \(w \in V(LD)\) and \(u, v \in [w]_\sim \setminus C\). Then, \(N^-(u) = N^-(v)\) and, since \(u, v \notin C\), it follows that \(N^-[u] \cap C = N^-(u) \cap C = N^-(v) \cap C = N^-[v] \cap C\), which is a contradiction if \(u \neq v\). \(\square\)

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Definition 3.2. Given a digraph $D$, a subset $C$ of $A(D)$ is an arc-identifying code of $D$ if $C$ is both:

- an arc-dominating set of $D$, that is, for each arc $uv \in A(D)$, $(\{uv\} \cup \omega^-[u]) \cap C \neq \emptyset$,
- and

an arc-separating set of $D$, that is, for each pair $uv,wz \in A(D)$ ($uv \neq wz$), $(\{uv\} \cup \omega^-[u]) \cap C \neq (\{wz\} \cup \omega^-[w]) \cap C$.

Hence, a line digraph $LD$ admits a $(1, \leq \ell)$-identifying code if and only if $D$ admits a $(1, \leq \ell)$-arc-identifying code. As a consequence, the minimum size of an identifying code of a digraph $D$, $−→γ_ID(D)$, is equivalent to the minimum size of an arc-identifying code of its line digraph $LD$.

Theorem 3.1. Let $D$ be a strongly connected digraph with minimum in-degree $δ^− \geq 2$. Then,

$$\gamma^{ID}(LD) \geq |A(D)| - |V(D)|.$$ 

Proof. By Remark 2.1, $LD$ admits an identifying code. Let $C$ be an arc-identifying code of $D$. Then, by Lemma 3.1,

$$|C| \geq \sum_{V_{\geq 2}(D)} (d^+_D(u) - 1)$$

$$= \sum_{V_{\geq 2}(D)} d^+_D(u) - |V_{\geq 2}(D)| + \sum_{V_1^+(D)} d^+_D(u) - \sum_{V_1^+(D)} d^+_D(u)$$

$$= \sum_{V(D)} d^+_D(u) - |V_{\geq 2}(D)| - |V_1^+(D)|$$

$$= |A(D)| - |V(D)|.$$ 

Theorem 3.2. Let $D$ be a strongly connected digraph of order at least 3, and let $C \subseteq A(D)$. Then, $C$ is an arc-identifying code of $D$ if and only if $C$ satisfies the following conditions:

(i) for all $v \in V(D)$, $|\omega^+(v) \setminus C| \leq 1$, and if $|\omega^+(v) \setminus C| = 1$, then $\omega^-(v) \cap C \neq \emptyset$;

(ii) for all $uv \in C$, if $vu \in C$ or $|\omega^+(v) \setminus C| = 1$, then $(\omega^-(v) \cup \omega^-(u)) \setminus \{uv, vu\} \cap C \neq \emptyset$. 

\[\square\]
Proof. First suppose that $C$ is an arc-identifying code of $D$. The first part of (i) follows directly from Lemma 3.1. For the second one, let $v \in V(D)$ be such that $|\omega^+(v) \setminus C| = 1$ and let $vx \in \omega^+(v) \setminus C$. Hence, $(\{vx\} \cup \omega^-(v)) \cap C = \omega^+(v) \cap C$. Since $C$ is an arc-identifying code, $(\{vx\} \cup \omega^-(v)) \cap C \neq \emptyset$, hence $C$ satisfies (i). To prove (ii), let $uv \in C$ be such that $((\omega^-(u) \cup \omega^-(v)) \setminus \{vu, uv\}) \cap C = \emptyset$. If $uv \in C$, then $(\{uv\} \cup \omega^-(u)) \cap C = \{uv, vu\} = (\{vu\} \cup \omega^-(v)) \cap C$, contradicting that $C$ is an arc-identifying code. Hence, $vu \notin C$. If $|\omega^+(v) \setminus C| = 1$, let say $\omega^+(v) \setminus C = \{vx\}$, then $(\{uv\} \cup \omega^-(u)) \cap C = \{uv\} = N^-[vx] \cap C$, a contradiction. Therefore, $C$ satisfies (ii).

Now, suppose that $C$ is a set of arcs of $D$ satisfying (i) and (ii), and let us show that $C$ is an arc-identifying code. Let us show that $C$ is an arc-dominating set of $D$. Let $ab \in A(D)$. By (i), $\omega^+(a) \subseteq C$ or $\omega^-(a) \cap C \neq \emptyset$, implying that $(\{ab\} \cup \omega^-(a)) \cap C = \emptyset$. Therefore, $C$ is an arc-dominating set of $D$. Next, let us prove that $C$ is an arc-separating set of $D$. On the contrary, suppose that there are two different arcs $ab$ and $cd$, such that $(\{ab\} \cup \omega^-(a)) \cap C = (\{cd\} \cup \omega^-(c)) \cap C$. First, let us assume that $ab, cd \notin C$. If we take an arc $uv \in (\{ab\} \cup \omega^-(a)) \cap C = (\{cd\} \cup \omega^-(c)) \cap C$, then $v = a = c$, implying that $ab, cd \in \omega^+(v) \setminus C$, contradicting (i). Second, let us assume that $ab \in C$, hence, $c = b$. If $bd \notin C$, by (ii), $((\omega^-(a) \cup \omega^-(b)) \setminus \{ba, ab\}) \cap C \neq \emptyset$, implying that $(\{ab\} \cup \omega^-(a)) \cap C \neq (\{bd\} \cup \omega^-(b)) \cap C$, a contradiction. Therefore, $bd \in C$ implying that $d = a$ because our assumption $(\{ab\} \cup \omega^-(a)) \cap C = (\{cd\} \cup \omega^-(c)) \cap C$. Hence, again by (ii), $((\omega^-(a) \cup \omega^-(b)) \setminus \{ab, cd\}) \cap C \neq \emptyset$, implying that $(\{ab\} \cup \omega^-(a)) \cap C = (\{cd\} \cup \omega^-(c)) \cap C$, a contradiction. Therefore, $C$ is an arc-separating set. This completes the proof.

Now we present an algorithm for constructing an arc-identifying code of a given strongly connected oriented graph with minimum in-degree $\delta^- \geq 2$.

**Algorithm 3.1. Constructing an arc-identifying code $C$ of a given strongly connected digraph $D$ with minimum in-degree $\delta^- \geq 2$ and without digons.**

1: Let $U^- := \{v \in V(D) : N^-(v) \subseteq V^+_1(D)\}$, $U := \emptyset$ and $C := \emptyset$
2: while $U^- \setminus U \neq \emptyset$ do
3: let $v \in U^- \setminus U$ and $f \in N^-(v)$
4: replace $U$ by $U \cup \{v\}$ and $C$ by $C \cup \{fv\}$
5: end while
6: let $X := V^+_1(D)$ and $Y := U^-$
7: let $xy \in A(D)$ such that $x \in V(D) \setminus X$ and $y \in V(D) \setminus Y$
8: replace $Y$ by $Y \cup (N^+(x) \setminus \{y\})$, $X$ by $X \cup \{x\}$ and $C$ by $C \cup (\omega^+(x) \setminus \{xy\})$
9: while $Y \neq V(D)$ do
10: while $N^- (y) \setminus X \neq \emptyset$ do
11: let $t \in N^- (y) \setminus X$ and let $z \in N^+(t) \setminus \{y\}$
12: replace $Y$ by $Y \cup (N^+(t) \setminus \{z\})$, $X$ by $X \cup \{t\}$, $C$ by $C \cup (\omega^+(t) \setminus \{tz\})$, $t$ by $x$ and $z$ by $y$
13: end while
14: if $N^- (y) \setminus X = \emptyset$ then
15: choose an arc $uv$ of $D$ such that $v \notin Y$
16: end if
17: end while

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Corollary 3.1. Let $C$ be an oriented and strongly connected graph with minimum in-degree $\delta \geq 2$. Then, Algorithm 3.1 produces a subset $C \subseteq A(D)$ of

$$|C| = |A(D)| - |V(D)| + |\{v \in V(D) : N^-(v) \subseteq V_1^+(D)\}|,$$

satisfying the requirements of Theorem 3.2.

**Proof.** By construction, given Algorithm 3.1, we can check that for every $x \in V(D)$ we get $|\omega^+(x) \setminus C| \leq 1$. Let $v \in V(D)$. Since the algorithm finishes when the set $Y$ is equal to $V(D)$, it follows that $v \in Y$ at a certain step of the algorithm. Then, $v \in N^+(t) \setminus \{z\}$ for a certain $t$ and $z$ in the algorithm, which implies that $tv \in \omega^+(t) \setminus \{tz\} \subseteq C$. Then, $\omega^-(v) \cap C \neq \emptyset$ and Theorem 3.2 (i) holds. Finally, since $D$ is oriented, for all $uv \in C$, clearly $vu \notin A(D)$, and we have $|\omega^-(u) \cup (\omega^-(v) \setminus \{uv\})\cap C| \geq 1$ because $\omega^-(u) \cap C \neq \emptyset$. Hence, Theorem 3.2 (ii) also holds. Therefore, $C$ is an arc-identifying code of $D$ and $|C| = |A(D)| - |V(D)| + |\{v \in V(D) : N^-(v) \subseteq V_1^+(D)\}|$.

As a consequence of Theorems 3.1 and 3.3, we can conclude the following corollary.

**Corollary 3.1.** Let $D$ be a strongly connected oriented graph with minimum in-degree $\delta \geq 2$. Then, the following assertions hold.

(i) $\gamma^{ID}(LD) = |A(D)| - |V(D)| + |\{v \in V(D) : N^-(v) \subseteq V_1^+(D)\}|$ if $\delta^+ = 1$;

(ii) $\gamma^{ID}(LD) = |A(D)| - |V(D)|$ if $\delta^+ \geq 2$.

Next, we also present a result for all Hamiltonian digraphs of minimum degree at least two, not necessarily oriented.

**Theorem 3.4.** Let $D$ be a Hamiltonian strongly connected digraph with minimum in-degree $\delta \geq 3$ and out-degree $\delta^+ \geq 2$. Then, $\gamma^{ID}(LD) = |A(D)| - |V(D)|$. 

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Proof. Let $H = (x_1, x_2, \ldots, x_n, x_1)$ denote a Hamiltonian cycle of $D$. Let $C = A(D) \setminus A(H)$. Let us show that $C$ satisfies the requirements of Theorem 3.3. By definition of $C$, $|\omega^+(v) \cap C| = 1$ and $|\omega^-(v) \cap C| = 1$. Since $\delta \geq 2$, Theorem 3.2(i) follows directly. To show Theorem 3.2(ii), observe that $|\omega^-(v) \cap C| = 1$ implies that $|\omega^-(v) \cap C| = d^-(v) - 1 \geq 2$ because $\delta \geq 3$. Therefore, for all $uv \in C$, $((\omega^-(v) \cup \omega^-(u)) \setminus \{uv, vu\}) \cap C \neq \emptyset$ by Theorem 3.2(ii) holds. Thus, $C$ is an arc-identifying code of $D$ and $\gamma^{ID}(LD) = |A(D)| - |V(D)|$ by Theorem 3.1.

Corollary 3.2. The identifying number of a Kautz digraph $K(d, k)$ is $\gamma^{ID}(K(d, k)) = d^k - d^{k-2}$ for $d \geq 3$ and $k \geq 2$.

Proof. Note that $K(d, 2) = LK_{d+1}$. Since $K_{d+1}$ is Hamiltonian and $d \geq 3$, by Theorem 3.4 $\gamma^{ID}(K(d, 2)) = \gamma^{ID}(LK_{d+1}) = d(d + 1) - (d + 1) = d^2 - 1$, and the result holds for $k = 2$. For any $k \geq 3$, the Kautz digraph $K(d, k) = L^{k-1}K_{d+1} = LL^{k-2}K_{d+1} = LK(d, k-1)$. Since $K(d, k-1)$ is a Hamiltonian digraph and $d \geq 3$, by Theorem 3.4 $\gamma^{ID}(K(d, k)) = \gamma^{ID}(LK(d, k-1)) = d^k + d^{k-1} - (d^{k-1} + d^{k-2}) = d^k - d^{k-2}$ and the result holds.

To extend the Corollary 3.2 to $K(2, k)$ we need the 1-factorization of Kautz digraphs obtained by Tvrzik [13]. This 1-factorization uses the following operation.

Definition 3.3. [13] If $x = x_1 \ldots x_k \in V(K(d, k))$, then

- $\sigma_1(x) = x_2 \ldots x_{k-1}x_kx_1$ if $x_1 \neq x_k$
- $\sigma_1(x) = x_2 \ldots x_{k-1}x_kx_2$ if $x_1 = x_k$

Let $Inc : V(K(d, k)) \times \mathbb{Z}_d \rightarrow V(K(d, k))$ denote a binary operation such that

$$ Inc(x_1 \ldots x_{k-1}x_k, i) = x_1 \ldots x_{k-1}x'_k $$

where

$$ x'_k = \begin{cases} 
  x_k + i \mod (d+1) & \text{if } x_{k-1} > x_k \text{ and } x_{k-1} > x_k + i \\
  x_{k-1} + d + 1 & \text{if } x_{k-1} < x_k \text{ and } x_{k-1} + d + 1 > x_k + i; \\
  x_k + i + 1 \mod (d+1) & \text{otherwise.}
\end{cases} $$

Then, the generalized $K$-shift operation is defined as follows:

$$ \sigma^{+i}_1(x) = Inc(\sigma_1(x), i), $$

$$ \sigma^{+i}_k = \sigma^{+i}_1 \circ \sigma^{+i}_{k-1}. $$

Theorem 3.5. [13] The arc set of $K(d, k)$ can be partitioned into $d$ 1-factors $\mathcal{F}_0, \ldots, \mathcal{F}_{d-1}$ such that the cycles of $\mathcal{F}_i$ are closed under the operation $\sigma^{+i}_1$.

Theorem 3.6. The identifying number of a Kautz digraph $K(2, k)$ is $\gamma^{ID}(K(2, k)) = 2^k - 2^{k-2}$ for $k \geq 2$. 

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Proof. It is easy to check in Figure 2 that $C = \{uv, vw, wu\}$ is an identifying code of $K(2, 2)$, then $\gamma^{ID}(K(2, 2)) = 3$ and the theorem holds for $k = 2$. Suppose that $k \geq 3$ and let us study the Kautz digraph $K(2, k - 1)$. By Theorem 3.5 we can consider a partition of the arcs of $K(2, k - 1)$ into two 1-factors $F_0$ and $F_1$, such that the cycles of $F_i$ are closed under the operation called $\sigma_i^+$, given in Definition 3.3. It is not difficult to see that the relation $\sigma_i^+$ preserves digons, implying that all the digons of $K(2, k - 1)$ belong to the family $F_0$. Hence, since $F_1$ is a 1-factor of $K(2, k - 1)$, it is clear that the set of arcs in $F_1$, say $A_1$, satisfies the conditions of Theorem 3.2. Therefore, $A_1$ is an arc-identifying code of $K(2, k - 1)$, that is, an identifying code of $K(2, k)$. Therefore, $\gamma^{ID}(K(2, k)) = |A_1| = |V(K(2, k - 1))| = 3 \cdot 2^{k-2} = 2^k - 2^{k-2}$, and the proof is complete.  

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