SOME IDENTITIES OF CHEBYSHEV POLYNOMIALS ARISING FROM NON-LINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we investigate some properties of Chebyshev polynomials arising from non-linear differential equations. From our investigation, we derive some new and interesting identities on Chebyshev polynomials.

1. Introduction

As is well known, the Chebyshev polynomials of the first kind, $T_n(x)$, $(n \geq 0)$, are defined by the generating function

$$
\frac{1 - t^2}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} T_n(x) \frac{t^n}{n!}, \quad \text{ (see [1, 3, 8, 17, 21]).}
$$

The higher-order Chebyshev polynomials are given by the generating function

$$
\left(\frac{1 - t^2}{1 - 2xt + t^2}\right)^\alpha = \sum_{n=0}^{\infty} T_n^{(\alpha)}(x) t^n,
$$

and Chebyshev polynomials of the second kind are denoted by $U_n$ and given by generating function

$$
\frac{1}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} U_n(x) t^n, \quad \text{ (see [1, 3, 12, 17]).}
$$

The higher-order Chebyshev polynomials of the second kind are also defined by

$$
\left(\frac{1}{1 - 2xt + t^2}\right)^\alpha = \sum_{n=0}^{\infty} U_n^{(\alpha)}(x) t^n.
$$

The Chebyshev polynomials of the third kind are defined by the generating function

$$
\frac{1 - t}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} V_n(x) t^n, \quad \text{ (see [1, 7, 8, 17]).}
$$

and the higher-order Chebyshev polynomials of the third kind are also given by the generating function

$$
\left(\frac{1 - t}{1 - 2xt + t^2}\right)^\alpha = \sum_{n=0}^{\infty} V_n^{(\alpha)}(x) t^n.
$$

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Finally, we introduce the Chebyshev polynomials of the fourth kind defined by the generating function

\begin{equation}
\frac{1 + t}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} W_n(x) t^n.
\end{equation}

The higher-order Chebyshev polynomials of the fourth kind are defined by

\begin{equation}
\left( \frac{1 + t}{1 - 2xt + t^2} \right)^\alpha = \sum_{n=0}^{\infty} W_n^{(\alpha)}(x) t^n.
\end{equation}

It is well known that the Legendre polynomials are defined by the generating function

\begin{equation}
\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} p_n(x) t^n, \quad \text{(see [2, 20])}
\end{equation}

Chebyshev polynomials are important in approximation theory because the roots of the Chebyshev polynomials of the first kind, which are also called Chebyshev nodes, are used as nodes in polynomial nodes (see [19]).

The Chebyshev polynomials of the first kind and of the second kind are solutions of the following Chebyshev differential equations

\begin{equation}
(1 - x^2) y'' - xy' + n^2 y = 0,
\end{equation}

and

\begin{equation}
(1 - x^2) y'' - 3xy' + n(n + 2) y = 0.
\end{equation}

These equations are special cases of the Strum-Liouville differential equation (see [1–3]).

The Chebyshev polynomials of the first kind can be defined by the contour integral

\begin{equation}
T_n(z) = \frac{1}{4\pi i} \oint \frac{(1 - t^2)}{1 - 2tz + t^2} t^{-n-1} dt,
\end{equation}

where the contour encloses the origin and is traversed in a counterclockwise direction (see [1, 19, 21]). The formula for \( T_n(x) \) is given by

\begin{equation}
T_n(x) = \sum_{m=0}^{\left[ n/2 \right]} \binom{n}{2m} x^{n-2m} (x^2 - 1)^m.
\end{equation}

From (1.13), we note that

\begin{equation}
2 (x - t) (1 - 2xt + t^2)^{-2} = \sum_{n=0}^{\infty} nU_n(x) t^n - 1.
\end{equation}

Thus, by (1.14), we get

\begin{equation}
(2xt - 2t^2) (1 - 2xt + t^2)^{-2} = \sum_{n=0}^{\infty} nU_n(x) t^n.
\end{equation}

From (1.13) and (1.15), we can derive the following equation:

\begin{equation}
\frac{(2xt - 2t^2) + (1 - 2xt + t^2)}{(1 - 2xt + t^2)^2} = \frac{1 - t^2}{(1 - 2xt + t^2)^2}.
\end{equation}
\[
= \sum_{n=0}^{\infty} (n + 1) U_n(x) t^n.
\]

Note that
\[
\frac{1 - t^2}{(1 - 2xt + t^2)^2} = \left( \frac{1 - t^2}{1 - 2xt + t^2} \right) \left( \frac{1}{1 - 2xt + t^2} \right) = \left( \sum_{l=0}^{\infty} T_l(x) t^l \right) \left( \sum_{m=0}^{\infty} U_m(x) t^m \right) = \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} T_l(x) U_{n-l}(x) \right) t^n.
\]

From (1.16) and (1.17), we have
\[
U_n(x) = \frac{1}{n+1} \sum_{l=0}^{n} T_l(x) U_{n-l}(x).
\]

The Chebyshev polynomials have been studied by many authors in the several areas (see [1–21]).

In [11], Kim-Kim studied non-linear differential equations arising from Changhee polynomials and numbers related to Chebyshev polynomials.

In this paper, we study non-linear differential equations arising from Chebyshev polynomials and give some new and explicit formulas for those polynomials.

2. Differential equations arising from Chebyshev polynomials and their applications

Let
\[
F = F(t,x) = \frac{1}{1 - 2tx + t^2}.
\]

Then, by (1.11), we get
\[
F^{(1)} = \frac{d}{dt} F(t,x) = 2 (x - t) F^2.
\]

From (2.2), we note that
\[
2F^2 = (x - t)^{-1} F^{(1)}.
\]

By using (2.3) and (2.2), we obtain the following equations:
\[
2^2 \cdot 2F^3 = (x - t)^{-3} F^{(1)} + (x - t)^{-2} F^{(2)},
\]
\[
2^3 \cdot 2 \cdot 3F^4 = 3 (x - t)^{-5} F^{(1)} + 3 (x - t)^{-4} F^{(2)} + (x - t)^{-3} F^{(3)}
\]
and
\[
2^4 \cdot 2 \cdot 3 \cdot 4F^5 = 3 \cdot 5 (x - t)^{-6} F^{(1)} + 3 \cdot 5 (x - t)^{-5} F^{(2)} + (3 \cdot 2) (x - t)^{-4} F^{(3)} + (x - t)^{-4} F^{(4)},
\]
where
\[ F^N = F \times \cdots \times F \] (N-times) and \[ F^{(N)} = \left( \frac{d}{dt} \right)^N F(t, x). \]

Continuing this process, we set
\[ 2^N N! F^{N+1} = \sum_{i=1}^{N} a_i (N) (x - t)^{i-2N} F^{(i)}, \] (2.7)

where \( N \in \mathbb{N} \).

From (2.7), we note that
\[
2^N N! F^{N} (N + 1) F^{(1)} = \sum_{i=1}^{N} a_i (N) (2N - i) (x - t)^{i-2N-1} F^{(i)} + \sum_{i=1}^{N} a_i (N) (x - t)^{i-2N} F^{(i+1)}.
\] (2.8)

By (2.2) and (2.8), we get
\[
2^N N! F^{N} (2(x - t) F^2) = \sum_{i=1}^{N} a_i (N) (2N - i) (x - t)^{i-2N-1} F^{(i)} + \sum_{i=1}^{N} a_i (N) (x - t)^{i-2N} F^{(i+1)}.
\] (2.9)

Thus, from (2.9), we have
\[
2^{N+1} (N + 1)! F^{N+2} = \sum_{i=1}^{N} a_i (N) (2N - i) (x - t)^{i-2(N+1)} F^{(i)}
+ \sum_{i=2}^{N+1} a_{i-1} (N) (x - t)^{i-2(N+1)} F^{(i)}.
\] (2.10)

On the other hand, by replacing \( N \) by \( N + 1 \), in (2.7), we get
\[
2^{N+1} (N + 1)! F^{N+2} = \sum_{i=1}^{N+1} a_i (N + 1) (x - t)^{i-2(N+1)} F^{(i)}.
\] (2.11)

Comparing the coefficients on both sides of (2.10) and (2.11), we have
\[
a_1 (N + 1) = (2N - 1) a_1 (N), \quad a_{N+1} (N + 1) = a_N (N),
\] (2.12) (2.13)

and
\[
a_i (N + 1) = a_{i-1} (N) + (2N - i) a_i (N), \quad (2 \leq i \leq N).
\] (2.14)

Moreover, by (2.4) and (2.7), we get
\[
2 F^2 = (x - t)^{-1} F^{(1)} = a_1 (1) (x - t)^{-1} F^{(1)}.
\] (2.15)

By comparing the coefficients on both sides of (2.15), we get
\[
a_1 (1) = 1.
\] (2.16)
Now, by (2.12) and (2.16), we have

\[(2.17) \quad a_1 (N + 1) = (2N - 1) a_1 (N) = (2N - 1) (2N - 3) a_1 (N - 1) = (2N - 1) (2N - 3) (2N - 5) a_1 (N - 2) \]

\[
\vdots
\]

\[
= (2N - 1) (2N - 3) (2N - 5) \cdots 1 \cdot a_1 (1) = (2N - 1)!!,
\]

where \((2N - 1)!!\) is Arfken’s double factorial.

From (2.13), we easily note that

\[(2.18) \quad a_{N+1} (N + 1) = a_N (N) = \cdots = a_1 (1) = 1.
\]

For \(2 \leq i \leq N\), from (2.14), we can derive the following equation:

\[(2.19) \quad a_i (N + 1) = a_{i-1} (N) + (2N - i) a_i (N) = a_{i-1} (N) + (2N - i) a_{i-1} (N - 1) + (2N - 2 - i) a_i (N - 1)
\]

\[
\vdots
\]

\[
= \sum_{k=0}^{N-i} \prod_{l=0}^{k-1} (2 (N - l) - i) a_{i-1} (N - k) \prod_{l=0}^{N-i} (2 (N - l) - i) a_i (i)
\]

\[
= \sum_{k=0}^{N-i} 2^k \binom{N - i}{k} a_{i-1} (N - k) + 2^{N-i+1} \binom{N - i}{\frac{i}{2}} a_{i-1} (N - \frac{i}{2})
\]

\[
= \sum_{k=0}^{N-i+1} 2^k \binom{N - i}{k} a_{i-1} (N - k),
\]

where \((x)_n = x (x - 1) \cdots (x - n + 1)\), \(n \geq 1\) and \((x)_0 = 1\).

As the above is also valid for \(i = N + 1\), by (2.19), we get

\[(2.20) \quad a_i (N + 1) = \sum_{k=0}^{N+1-i} 2^k \binom{N - i}{k} a_{i-1} (N - k),
\]

where \(2 \leq i \leq N + 1\).

Now, we give an explicit expression for \(a_i (N + 1)\).

From (2.17) and (2.20), we can derive the following equations:

\[(2.21) \quad a_2 (N + 1) = \sum_{k_1=0}^{N-1} 2^{k_1} \binom{N - \frac{i}{2}}{k_1} a_1 (N - k_1)
\]

\[
= \sum_{k_1=0}^{N-1} 2^{k_1} \binom{N - \frac{i}{2}}{k_1} (2 (N - k_1 - 1) !!),
\]
Theorem 1. The nonlinear differential equations

\[ \begin{align*}
(2.22) \quad a_3 (N + 1) &= \sum_{k_2=0}^{N-2} 2^{k_2} \left( N - \frac{3}{2} \right)_{k_2} a_2 (N - k_2) \\
&= \sum_{k_2=0}^{N-2} \sum_{k_1=0}^{N-2-k_2} 2^{k_1+k_2} \left( N - \frac{3}{2} \right)_{k_2} \left( N - k_2 - \frac{4}{2} \right)_{k_1} (2 (N - 2 - k_1 - k_2) - 1)!!, \\
\end{align*} \]

and

\[ \begin{align*}
(2.23) \quad a_4 (N + 1) &= \sum_{k_3=0}^{N-3} 2^{k_3} \left( N - \frac{4}{2} \right)_{k_3} a_3 (N - k_3) \\
&= \sum_{k_3=0}^{N-3} \sum_{k_2=0}^{N-3-k_3} \sum_{k_1=0}^{N-3-k_3-k_2} 2^{k_1+k_2+k_3} \left( N - \frac{4}{2} \right)_{k_3} \left( N - k_3 - \frac{5}{2} \right)_{k_2} \left( N - k_3 - k_2 - \frac{6}{2} \right)_{k_1} \\
& \times (2 (N - 3 - k_1 - k_2 - k_3) - 1)!!. \\
\end{align*} \]

Thus, we see that, for \( 2 \leq i \leq N + 1, \)

\[ \begin{align*}
(2.24) \quad a_i (N + 1) &= \sum_{k_{i-1}=0}^{N-i+1} \sum_{k_{i-2}=0}^{N-i+1-k_{i-1}} \cdots \sum_{k_1=0}^{N-i+1-k_{i-1} \cdots - k_2} 2^{\sum_{j=1}^{i-1} k_j} \\
& \times \prod_{j=2}^{i} \left( N - \sum_{l=j}^{i-1} k_l - \frac{2i-j}{2} \right)_{k_{j-1}} \left( 2 \left( N - i + 1 - \sum_{j=1}^{i-1} k_j \right) - 1 \right)!!.
\end{align*} \]

Therefore, we obtain the following theorem.

Theorem 1. The nonlinear differential equations

\[ 2^N N! F^{N+1} = \sum_{i=1}^{N} a_i (N) (x-t)^{i-2N} F^{(i)}, \quad (N \in \mathbb{N}) \]

has a solution \( F = F(t,x) = \frac{1}{1 - 2tx + t^2}, \) where

\[ a_1 (N) = (2N - 3)!!, \]

\[ a_i (N) = \sum_{k_{i-1}=0}^{N-i \cdots - k_{i-1}} \cdots \sum_{k_1=0}^{N-i \cdots - k_{i-1} \cdots - k_2} 2^{\sum_{j=1}^{i-1} k_j} \\
\times \prod_{j=2}^{i} \left( N - \sum_{l=j}^{i-1} k_l - \frac{2i-j}{2} \right)_{k_{j-1}} \left( 2 \left( N - i + 1 - \sum_{j=1}^{i-1} k_j \right) - 1 \right)!! \]

\( (2 \leq i \leq N). \)

From (1.3) and (1.9), we note that

\[ \sum_{n=0}^{\infty} U_n (x) t^n = \frac{1}{1 - 2xt + t^2} \]
\[ \frac{1}{\sqrt{1 - 2xt + t^2}} = \left( \sum_{l=0}^{\infty} p_l(x) t^l \right) \left( \sum_{m=0}^{\infty} p_m(x) t^m \right) = \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} p_l(x) p_{n-l}(x) \right) t^n. \]

Thus, from (2.25), we have
\[ U_n(x) = \sum_{l=0}^{n} p_l(x) p_{n-l}(x). \]

From (1.4), we obtain
\[ (2.26) \quad 2^N N! F_{N+1} = 2^N N! \sum_{n=0}^{\infty} U_n^{(N+1)}(x) t^n. \]

On the other hand, by Theorem 1, we get
\[ (2.27) \quad 2^N N! F_{N+1} = \sum_{i=1}^{N} a_i(N) (x-t)^{i-2N} F(i) \]
\[ = \sum_{i=1}^{N} a_i(N) \left( \sum_{m=0}^{\infty} \left( \frac{2N + m - i - 1}{m} \right) x^{i-2N-m} t^m \right) \left( \sum_{l=0}^{\infty} U_{i+l}(x) (l+i) t^l \right) \]
\[ = \sum_{i=1}^{N} a_i(N) \sum_{n=0}^{\infty} \left\{ \sum_{l=0}^{n} \left( \frac{2N + n - l - i - 1}{n-l} \right) x^{i-2N-n+l} U_{i+l}(x) (l+i) \right\} t^n \]
\[ = \sum_{n=0}^{\infty} \left\{ \sum_{i=1}^{N} a_i(N) \sum_{l=0}^{n} \left( \frac{2N + n - l - i - 1}{n-l} \right) x^{i+l-2N-n} U_{i+l}(x) (l+i) \right\} t^n. \]

Comparing the coefficients on the both sides of (2.26) and (2.27), we obtain the following theorem.

**Theorem 2.** For \( N \in \mathbb{N}, \) and \( n \in \mathbb{N} \cup \{0\}, \) the following identity holds.

\[ U_n^{(N+1)}(x) = \frac{1}{2^N N!} \sum_{i=1}^{N} a_i(N) \sum_{l=0}^{n} \left( \frac{2N + n - l - i - 1}{n-l} \right) U_{i+l}(x) x^{i+l-2N-n} (l+i). \]

The higher-order Legendre polynomials are given by the generating function
\[ (2.28) \quad \left( \frac{1}{\sqrt{1 - 2xt + t^2}} \right)^{\alpha} = \sum_{n=0}^{\infty} p_n^{(\alpha)}(x) t^n. \]

Thus, by (1.4) and (2.27), we get
\[ (2.29) \quad \sum_{n=0}^{\infty} U_n^{(\alpha)}(x) t^n = \left( \frac{1}{1 - 2xt + t^2} \right)^{\alpha}. \]
\[ \frac{1}{\sqrt{1 - 2xt + t^2}}^{2^\alpha} = \left( \sum_{l=0}^{\infty} p_l^{(\alpha)}(x) t^l \right) \left( \sum_{m=0}^{\infty} p_m^{(\alpha)}(x) t^m \right) = \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \frac{\alpha}{n} \sum_{l=0}^{l} \frac{\alpha}{n} \right) t^n. \]

From (2.29), we note that
\[ U_n^{(\alpha)}(x) = \sum_{l=0}^{n} p_l^{(\alpha)}(x) p_n^{(\alpha)}(x). \]

Therefore, we obtian the following corollaries.

**Corollary 3.** For \( N \in \mathbb{N} \) and \( n \in \mathbb{N} \cup \{0\} \), we have
\[ \sum_{l=0}^{n} p_l^{(N+1)}(x) p_{n-l}^{(N+1)}(x) = \frac{1}{2^N N!} \sum_{i=1}^{N} a_i(N) \sum_{l=0}^{n} \left( \begin{array}{c} 2N + n - l - i - 1 \\ n - l \end{array} \right) U_{l+i}(x) (l + i) x^{i+l-2N-n}. \]

**Corollary 4.** For \( N \in \mathbb{N} \) and \( n \in \mathbb{N} \), we have
\[ U_n^{(N+1)}(x) = \frac{1}{2^N N!} \sum_{i=1}^{N} a_i(N) \sum_{l=0}^{n} \left( \begin{array}{c} 2N + n - l - i - 1 \\ n - l \end{array} \right) x^{i+l-2N-n} (l + i) x^{i + l - 2N} p_{l+i-1}(x). \]

By (1.6), we get
\[ 2^N N! F^{(N+1)} \]
\[ = 2^N N! (1 - t)^{-N-1} \left( \frac{1 - t}{1 - 2xt + t^2} \right)^{N+1} = 2^N N! \left( \sum_{m=0}^{\infty} \frac{N + m}{m} t^m \right) \left( \sum_{i=0}^{\infty} V_i^{(N+1)}(x) t^i \right) = 2^N N! \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \frac{N + n - l}{n - l} \right) V_n^{(N+1)}(x) t^n. \]

On the other hand, by Theorem II we have
\[ 2^N N! F^{(N+1)} = \sum_{i=1}^{N} a_i(N) (x - t)^{i-2N} F^{(i)} \]
\[ = \sum_{i=1}^{N} a_i(N) (x - t)^{i-2N} \left( \frac{d}{dt} \right)^i \left( \frac{1 - t}{1 - t \cdot 1 - x t + t^2} \right) \]

From Leibniz formula, we note that
\[ \left( \frac{d}{dt} \right)^i \left( \frac{1 - t}{1 - 2xt + t^2} \cdot \frac{1}{1 - t} \right) \]
For Theorem 5.

Therefore, by (2.31) and (2.34), we obtain the following theorem.

By (2.32) and (2.33), we get

From (1.8), we note that

Now, we observe that

On the other hand, by Theorem 1, we get

\[
\sum \nolimits_l \begin{pmatrix} i \cr l \end{pmatrix} \left( \frac{d}{dt} \right)^{i-l} \frac{1}{1-t} \left( \frac{d}{dt} \right)^l \frac{1-t}{1-2xt+t^2} \]

\[
= \sum \nolimits_l \begin{pmatrix} i \cr l \end{pmatrix} (i-l)! (1-t)^{-i+l-1} \left( \frac{d}{dt} \right)^l \frac{1-t}{1-2xt+t^2} \]

\[
= \sum \nolimits_l \begin{pmatrix} i \cr l \end{pmatrix} (i-l)! \sum \nolimits_{s=0}^{\infty} \binom{i-l+s}{s} t^s \sum \nolimits_{p=0}^{\infty} V_{l+p} (x) (p+l)_t t^p \]

\[
= \sum \nolimits_l \frac{i!}{l!} \sum \nolimits_{s=0}^{\infty} \binom{i-l+s}{s} t^s \sum \nolimits_{p=0}^{\infty} V_{l+p} (x) (p+l)_t t^p.
\]

By (2.32) and (2.33), we get

(2.34) \quad 2^N N! F^{N+1}

\[
= \sum \nolimits_{n=0}^{\infty} \left\{ \sum \nolimits_{i=1}^{N} \sum \nolimits_{l=0}^{i} a_i (N) \frac{i!}{l!} \sum \nolimits_{m+s+p=n} \binom{2N+m-i-1}{m} \binom{i-l+s}{s} (p+l)_t \right\} x^{i-2N-m} V_{l+p} (x) t^n.
\]

Therefore, by (2.31) and (2.34), we obtain the following theorem.

**Theorem 5.** For \( N \in \mathbb{N} \) and \( n \in \mathbb{N} \cup \{0\} \), we have the following identity:

\[
\sum \nolimits_{l=0}^{n} \binom{N+n-l}{n-l} V_l^{(N+1)} (x)
\]

\[
= \frac{1}{2^N N!} \sum \nolimits_{i=1}^{N} \sum \nolimits_{l=0}^{i} a_i (N) \frac{i!}{l!} \sum \nolimits_{m+s+p=n} \binom{2N+m-i-1}{m} \binom{i-l+s}{s} (p+l)_t \cdot x^{i-2N-m} V_{l+p} (x).
\]

From (1.8), we note that

(2.35) \quad 2^N N! F^{N+1}

\[
= 2^N N! (1+t)^{-N-1} \left( \frac{1+t}{1-2xt+t^2} \right)^{N+1}
\]

\[
= 2^N N! \left( \sum \nolimits_{m=0}^{\infty} \binom{N+m}{m} (-1)^m t^m \right) \left( \sum \nolimits_{l=0}^{\infty} W_l^{(N+1)} (x) t^l \right)
\]

\[
= 2^N N! \sum \nolimits_{n=0}^{\infty} \sum \nolimits_{l=0}^{n} (-1)^{n-l} \binom{N+n-l}{n-l} W_l^{(N+1)} (x) t^n.
\]

On the other hand, by Theorem 1, we get

(2.36) \quad 2^N N! F^{N+1} = \sum \nolimits_{i=1}^{N} a_i (N) (x-t)^{i-2N} \left( \frac{d}{dt} \right)^i \left\{ \frac{1}{1+t} \cdot \frac{1+t}{1-2xt+t^2} \right\}.
\]

Now, we observe that

(2.37) \quad \left( \frac{d}{dt} \right)^i \left\{ \left( \frac{1}{1+t} \right) \left( \frac{1+t}{1-2xt+t^2} \right) \right\}
\[ \sum_{l=0}^{i} \binom{i}{l} (-1)^{i-l} (i-l)! \left( \frac{1}{1+t} \right)^{i-l+1} \left( \frac{d}{dt} \right)^l \left( \frac{1+t}{1-2xt+t^2} \right) \]
\[ = \sum_{l=0}^{i} \binom{i}{l} (-1)^{i-l} (i-l)! \sum_{s=0}^{\infty} \binom{i-l+s}{s} (-1)^s t^s \sum_{p=0}^{\infty} W_{p+l}(x) (p+l)_1 t^p. \]

From (2.36) and (2.37), we have
\[ 2^N N! F_1^{N+1} \]
\[ = \sum_{n=0}^{\infty} \left\{ \sum_{i=1}^{N} a_i \binom{N}{i} \sum_{l=0}^{i} (-1)^{i-l} \frac{l!}{l} \sum_{m+s+p=n} (-1)^s \binom{2N+m-i-1}{m} \right. \]
\[ \times \left. \binom{i-l+s}{s} (p+l)_1 x^{i-2N-m} W_{p+l}(x) \right\} t^n. \]

Therefore, by (2.36) and (2.37), we obtain the following theorem.

**Theorem 6.** For \( N \in \mathbb{N} \) and \( n \in \mathbb{N} \cup \{0\} \), the following identity is valid:
\[ \sum_{l=0}^{n} (-1)^{n-l} \binom{N+n-l}{n-l} W_l^{(N+1)}(x) \]
\[ = \frac{1}{2^N N! N_{n-l}} \sum_{i=1}^{N} (-1)^{i-l} a_i \binom{N}{i} \frac{i!}{i} \sum_{m+s+p=n} (-1)^s \binom{2N+m-i-1}{m} \]
\[ \times \binom{i-l+s}{s} (p+l)_1 x^{i-2N-m} W_{p+l}(x). \]

From (1.31), we have
\[ 2^N N! F_1^{N+1} \]
\[ = 2^N N! \left( \frac{1}{1-t^2} \cdot \frac{1-t^2}{1-2xt+t^2} \right)^{N+1} \]
\[ = 2^N N! \left( \frac{1}{1-t} \right)^{N+1} \left( \frac{1}{1+t} \right)^{N+1} \left( \frac{1-t^2}{1-2xt+t^2} \right)^{N+1} \]
\[ = 2^N N! \left( \sum_{l=0}^{\infty} \binom{N+l}{l} t^l \right) \left( \sum_{m=0}^{\infty} \binom{m+N}{m} (-1)^m t^m \right) \left( \sum_{p=0}^{\infty} T_p^{(N+1)}(x) t^p \right) \]
\[ = 2^N N! \sum_{n=0}^{\infty} \left( \sum_{i=m+p=n} \binom{N+l}{l} \binom{m+N}{m} (-1)^m T_p^{(N+1)}(x) \right) t^n. \]

On the other hand, by Theorem 6 we get
\[ 2^N N! F_1^{N+1} \]
\[ = \sum_{i=1}^{N} a_i \binom{N}{i} (x-t)^{i-2N} F^{(i)} \]
Theorem 7. For \( a_i, N \) and \( 2N = 1 \), we obtain the following theorem.

From Leibniz formula, we note that the following equations:

\[
\left( \frac{d}{dt} \right)^i \left\{ \left( \frac{1}{1-t} \right) \cdot \left( \frac{1 - t^2}{1 - 2xt + t^2} \right) \right\} = \sum_{l=0}^{i} \binom{i}{l} (i-l)! \sum_{s=0}^{\infty} \binom{i + s - l}{s} t^s \sum_{p=0}^{\infty} T_{p+l}(x) (p + l)_l t^p,
\]

and

\[
\left( \frac{d}{dt} \right)^i \left\{ \left( \frac{1}{1-t} \right) \left( \frac{1 - t^2}{1 - 2xt + t^2} \right) \right\} = \sum_{l=0}^{i} \binom{i}{l} (i-l)! (-1)^{i-l} \sum_{s=0}^{\infty} \binom{i - l + s}{s} (-1)^s t^s \sum_{p=0}^{\infty} T_{p+l}(x) (p + l)_l t^p.
\]

By \( 2.40, 2.41, \) and \( 2.42 \), we obtain

\[
2^N N! F^{N+1}
\]

\[
= \frac{1}{2} \sum_{i=1}^{N} a_i(N) (x - t)^{i-2N} \sum_{l=0}^{i} \binom{i}{l} (i-l)! \sum_{s=0}^{\infty} \binom{i + s - l}{s} t^s \sum_{p=0}^{\infty} T_{p+l}(x) (p + l)_l t^p
\]

\[
+ \frac{1}{2} \sum_{i=1}^{N} a_i(N) (x - t)^{i-2N} \sum_{l=0}^{i} \binom{i}{l} (i-l)! (-1)^{i-l} \sum_{s=0}^{\infty} \binom{i - l + s}{s} (-1)^s t^s \sum_{p=0}^{\infty} T_{p+l}(x) (p + l)_l t^p
\]

\[
\times \sum_{p=0}^{\infty} T_{p+l}(x) (p + l)_l t^p
\]

\[
= \frac{1}{2} \sum_{n=0}^{\infty} \sum_{i=1}^{N} a_i(N) \frac{i!}{i!} \sum_{m+s+p=n} \left( 2N + m - i - 1 \right) \binom{i + s - l}{s} \binom{m}{m} (p + l)_l
\]

\[
\times \times a_i(N) \frac{i!}{i!} \binom{i}{l} (i-l)! \sum_{p=0}^{\infty} T_{p+l}(x) t^n + \frac{1}{2} \sum_{n=0}^{\infty} \sum_{i=1}^{N} a_i(N) \frac{i!}{i!} (-1)^{i-l}
\]

\[
\times \sum_{m+s+p=n} (-1)^s \binom{2N + m - i - 1}{m} \binom{i + s - l}{s} (p + l)_l x^{i-2N-m} T_{p+l}(x) t^n.
\]

Therefore, by \( 2.40 \) and \( 2.41 \), we obtain the following theorem.

Theorem 7. For \( n \in \mathbb{N} \cup \{0\} \) and \( N \in \mathbb{N} \), we have the following identity

\[
2^{N+1} N! \sum_{s+m+p=n} \binom{N + s}{s} \binom{m + N}{m} (-1)^m T_p^{(N+1)}(x)
\]

\[
= \sum_{i=1}^{N} a_i(N) \frac{i!}{i!} \sum_{m+s+p=n} \left( 2N + m - i - 1 \right) \binom{i + s - l}{s} \binom{m}{m} (p + l)_l
\]
\[ x^{i-2N-m}T_{p+l}(x) + \sum_{i=1}^{N} \sum_{l=0}^{i} a_i(N) \frac{i!}{l!} (-1)^{i-l} \sum_{m+s+p=n} (-1)^s \binom{2N+m-i-1}{m} \]

\[ \times \binom{i+s-l}{s} \binom{p+l}{l} x^{i-2N-m}T_{p+l}(x). \]

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