A conical approximation of constant scalar curvature Kähler metrics of Poincaré type

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Abstract

Let \((X, L_X)\) be a polarized manifold and \(D\) be a smooth hypersurface such that \(D \in |L_X|\). In this paper, we show that if there is no nontrivial holomorphic vector field on \(D\) and \(\text{Aut}_0((X, L_X); D)\) is trivial, then constant scalar curvature Kähler metrics of Poincaré type on \(X \setminus D\) can be approximated by constant scalar curvature Kähler metrics with cone singularities of sufficiently small angle along \(D\). This result implies log K-semistability of \(((X, L_X); D)\) with angle 0.

1 Introduction

In Kähler geometry, the existence of constant scalar curvature Kähler (cscK) metrics is a fundamental problem. There are many works on the existence of cscK metrics on the complement of a divisor on a compact Kähler manifold. In particular, Auvray have studied cscK (more generally, extremal Kähler) metrics of Poincaré type, i.e., with cusp singularities \([\text{Au}1, \text{Au}2, \text{Au}3]\). On the other hand, Guenancia \([\text{Gu}]\) and Biquard-Guenancia \([\text{BG}]\) showed that some complete Kähler-Einstein metric can be realized as the limit of a sequence of Kähler-Einstein metrics with cone singularities along a divisor. As an analogue of Guenancia’s result \([\text{Gu}]\) for cscK metrics, we prove the following theorem in this paper.

**Theorem 1.1.** Let \((X, L_X)\) be an \(n\)-dimensional polarized manifold with a smooth hypersurface \(D \in |L_X|\). Assume that there is no nontrivial holomorphic vector field on \(D\) and \(\text{Aut}_0((X, L_X); D)\) is trivial. If \(X \setminus D\) admits a cscK metric \(\omega_0^{\text{cscK}} \in c_1(L_X)\) of Poincaré type, then there exists a cscK cone metric \(\omega_\beta^{\text{cscK}} \in c_1(L_X)\) of cone angle \(2\pi \beta\) along \(D\) for any \(0 < \beta \ll 1\). Moreover, \(\omega_\beta^{\text{cscK}}\) converges to \(\omega_0^{\text{cscK}}\) as \(\beta \to 0\), in the sense of topology of \(C^4,\alpha(X \setminus D)\) for some \(-1 \ll \alpha < 0\).

Note that we consider the construction of a cscK cone metric in the fixed cohomology class \(c_1(L_X)\) (cf. \([\text{Gu}]\)). Here, \(C^4,\alpha(X \setminus D)\) denotes the weighted Hölder space (see §2.1 in this paper and \([\text{Au}2]\)) and \(\text{Aut}_0((X, L_X); D)\) denotes the connected component of the identity in the holomorphic automorphism group preserving \(D\). The hypothesis on \(\text{Aut}_0((X, L_X); D)\) is a generic condition since we can find such a divisor generically by replacing \(L_X\) with \(mL_X\) for sufficiently large \(m\) (see Proposition 2.11 in \([\text{AHZ}]\)). In fact,
the standard elliptic theory implies that $\omega_{cscK}^{\beta} \to \omega_0^{cscK}$ in $C^\infty_{\text{loc}}(X \setminus D)$ (see §3.6 in [Aub]).

In particular, the sequence of metric spaces $(X \setminus D, \omega_{cscK}^{\beta}, p)$ converge to $(X \setminus D, \omega_0^{cscK}, p)$ in the sense of the pointed Gromov-Hausdorff topology where $p \in X \setminus D$ is a fixed point.

It is expected that the existence of cscK cone metrics is equivalent to log $K$-stability ([Do2], see also [AHZ]). Zheng [Zhe2] showed that the existence of cscK cone metrics is equivalent to log geodesic stability. By using this result, we can show that the existence of cscK cone metrics implies $\text{Aut}_0((X, L_X); D)$-uniform log $K$-stability for normal test configurations [AHZ]. Thus, we immediately obtain the following corollary.

**Corollary 1.2.** Assume that there is no nontrivial holomorphic vector field on $D$ and $\text{Aut}_0((X, L_X); D)$ is trivial. If $X \setminus D$ admits a cscK metric of Poincaré type, then $((X, L_X); D)$ is uniformly log $K$-stable with sufficiently small angle $2\pi \beta$ for normal test configurations.

In [J.Sun], J. Sun proved that the existence of a cscK metrics of Poincaré type (constructed by Calabi ansatz) on the total space of some line bundle over a cscK manifold with negative scalar curvature implies log K-semistability with angle 0. Since the log Donaldson-Futaki invariant depends linearly on $\beta$ ([Do2], see also [AHZ]), we have the following result which is a generalization of Sun’s result.

**Corollary 1.3.** Assume that there is no nontrivial holomorphic vector field on $D$ and $\text{Aut}_0((X, L_X); D)$ is trivial. If $X \setminus D$ admits a cscK metric of Poincaré type, then $((X, L_X); D)$ is log $K$-semistable with cone angle 0 for normal test configurations.

This result is a partial solution of Székelyhidi’s conjecture when $H^0(D, TD) = 0$ and $\text{Aut}_0((X, L_X); D)$ is trivial (see [Sz1, AAS, Sz]). On the other hand, J. Sun and S. Sun conjectured that if $D$ has a cscK metric with nonpositive scalar curvature, then $((X, L_X); D)$ is log $K$-semistable with cone angle 0 (see Conjecture 1.1 in [SS]). This conjecture has already been solved when $D$ has a scalar-flat Kähler metric in [S.Sun] (see also Introduction of [SS]). Auvray showed that the existence of a cscK metric of Poincaré type on $X \setminus D$ implies the existence of a cscK metric with negative scalar curvature on the divisor $D \in |L_X|$ (see Remark 2.12 in this paper). So, Corollary 1.3 is a solution of Conjecture 1.1 in [SS] under the (stronger) hypothesis that $X \setminus D$ admits a cscK metric of Poincaré type (see the following diagram).

\[
\begin{array}{ccccccccc}
X \setminus D : \text{Poincaré type cscK} & \xrightarrow{\text{Auvray’s result [Au3]}} & D : \text{negative cscK} \\
\downarrow \text{Conjecture [Sz]} & & & & & & \downarrow \text{Conjecture [SS]} \\
((X, L_X); D) : \text{log K-stable for angle 0} & \xrightarrow{\text{Corollary 1.3}} & ((X, L_X); D) : \text{log K-semistable for angle 0}
\end{array}
\]

**Remark 1.4.** Let $\Sigma_D$ be the average of the scalar curvature of a Kähler metric in the class $c_1(L_X|D)$. It is known that $((X, L_X); D)$ is log $K$-unstable with angle $2\pi \beta$ if $\beta < \frac{\Sigma_D}{n(n-1)}$ in [AHZ]. In our setting: $D \in |L_X|$, Auvray’s topological constraint [Au1] tells us that the average of the scalar curvature on $D$ is negative: $\Sigma_D < 0$ (see Remark 2.12 in this paper). Thus, Corollary 1.3 does not contradict the result in [AHZ].
This paper is organized as follows. In Section 2, we recall the definitions of Kähler metrics of Poincaré/cone singularities. In addition, we define a background Kähler metric with cone singularities, which converges to a cscK metric of Poincaré type. Finally, we consider a fixed point formula on the weighted Hölder space, which characterizing a cscK metric with cone singularities. In Section 3, we prove Theorem 1.1.

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2 Preliminaries

In this section, we consider the following. Firstly, we recall the definitions of Kähler metrics of Poincaré type and the weighted Hölder space by following [Au2]. Secondly, we recall the definition of Kähler metrics with cone singularities along a divisor by following [Zhe1]. Finally, we define a Kähler metric with cone singularities, which converges locally to a cscK metric of Poincaré type. (We will show that its scalar curvature is asymptotic to the average value of the scalar curvature in Lemma 3.3 later.) In addition, we consider a fixed point formula which characterizing a cscK cone metric.

2.1 Kähler metrics of Poincaré type and the weighted Hölder space

Let $(X, \theta_X)$ be an $n$-dimensional compact Kähler manifold and $D$ be a smooth hypersurface in $X$. We recall the definition of Kähler metrics of Poincaré type for the simplest case when $D$ is a smooth divisor, by following [Au1, Au2, Au3]. In this paper, $(U; z^1, ..., z^n)$ denotes a holomorphic coordinate chart such that $U \cap D = \{z^1 = 0\}$ and $|z^1| < 1$.

Definition 2.1 ([Au2]). A Kähler metric $\Theta$ on $X \setminus D$ is said to be of Poincaré type if for all holomorphic coordinate chart $(U; z^1, ..., z^n)$, $\Theta$ is quasi-isometric to the standard cusp metric

$$\omega_{cusp} := \frac{\sqrt{-1}dz^1 \wedge d\overline{z}^1}{|z^1|^2 \log^2 |z^1|^2} + \sum_{j=2}^{n} \sqrt{-1}dz^j \wedge d\overline{z}^j,$$

and its derivatives are bounded at any order with respect to this model metric.

We say that $\Theta$ has class $[\theta_X]$ if it can be written as $\Theta = \theta_X + \sqrt{-1}\partial\overline{\partial}\varphi$ for some smooth function $\varphi$ on $X \setminus D$ such that $\varphi = O(1 + \log(-\log |z^1|))$ and its derivatives are bounded at any positive order with respect to the model metric above.

Remark 2.2. Kähler metrics of Poincaré type can be defined for a simple normal crossing divisor $D$ (see [Au1, Au2, Au3]).

For Kähler metrics of Poincaré type, there is a Banach space of functions on $X \setminus D$ defined by Kobayashi [Ko] and Auvray [Au2] by using quasi-coordinates introduced by Cheng-Yau [CY]. We recall quickly the definition of it for the reader’s convenience (for more detail, see [CY, Ko, Au2]). We consider the standard cusp metric $\omega_{cusp}$ on the
punctured disc \( \Delta^* := \{0 < |z| < 1\} \subset \mathbb{C} \). For \( \delta \in (0,1) \), we define the following holomorphic map: \( \varphi_{\delta} : \frac{3}{4}\Delta \to \Delta^*, \zeta \mapsto \exp \left(-\frac{1+\delta}{1-\delta}d\zeta\right) \). Here, \( \Delta := \{|z| < 1\} \subset \mathbb{C} \). It is known that \( \bigcup_{\delta \in (0,1)} \varphi_{\delta}(\frac{3}{4}\Delta) = \Delta^* \) (see Section 2 in [Ko]). Directly, we note that the pull back of this Kähler metric:

\[
\varphi_{\delta}^*\omega_{\text{cusp}} = \frac{\sqrt{-1}d\zeta \wedge d\bar{\zeta}}{(1-|\zeta|^2)^2},
\]

is independent of \( \delta \in (0,1) \) and is \( C^\infty \)-quasi-isometric to the Euclidean metric on \( \frac{3}{4}\Delta \). By using this holomorphic map \( \varphi_{\delta} \), we define a holomorphic map \( \Phi_{\delta} : \mathcal{P} := \frac{3}{4}\Delta \times \Delta^{n-1} \to \Delta^* \times \Delta^{n-1}, (\zeta_1, z_2, ..., z_n) \mapsto (\varphi_{\delta}(\zeta_1), z_2, ..., z_n) \). For \( k \in \mathbb{Z}_{\geq 0}, \alpha \in (0,1) \), we define the \( C^{k,\alpha}(U \setminus D) \)-norm by

\[
\|f\|_{C^{k,\alpha}(U \setminus D)} := \sup_{\delta \in (0,1)} \|\Phi_{\delta}^*f\|_{C^{k,\alpha}(\mathcal{P})}.
\]

Here, we have identified \( U \setminus D \simeq \Delta^* \times \Delta^{n-1} \). We take an open subset \( U_0 \subset X \setminus D \) and a finite covering \( \{U_i\}_{i=1}^N \) of \( D \) such that \( U_0 \cup \bigcup_{i=1}^N U_i = X \).

**Definition 2.3** (The Hölder space on \( X \setminus D \), [Ko], see also §1.1 in [Au2]). For \( k \in \mathbb{Z}_{\geq 0}, \alpha \in (0,1) \), the Hölder space \( C^{k,\alpha}(X \setminus D) \) is defined by the norm

\[
\|f\|_{C^{k,\alpha}(X \setminus D)} := \|f\|_{C^{k,\alpha}(U_0 \setminus D)} + \max_{i=1, ..., N} \|f\|_{C^{k,\alpha}(U_i \setminus D)},
\]

where the norm \( \|f\|_{C^{k,\alpha}(U_0 \setminus D)} \) is defined by (2.3).

For a defining section \( \sigma_D \in H^0(X, L_X) \) of \( D \) and a Hermitian metric \( h_X \) on \( L_X \), we set a function \( t \) on \( X \setminus D \) by

\[
t := \log \|\sigma_D\|_{h_X}^{-2}.
\]

We use \( t \) as the weight function in order to define the weighted Hölder space below.

**Definition 2.4** (The weighted Hölder space on \( X \setminus D \), §3.1 in [Au2]). For \( \eta \in \mathbb{R} \), the weighted Hölder space \( C^{k,\alpha}_{\eta} = C^{k,\alpha}_{\eta}(X \setminus D) \) and its norm are defined by

\[
C^{k,\alpha}_{\eta}(X \setminus D) := \{f \in C^{k,\alpha}_{\text{loc}}(X \setminus D) \mid t^{-\eta}f \in C^{k,\alpha}(X \setminus D)\},
\]

\[
\|f\|_{C^{k,\alpha}_{\eta}(X \setminus D)} := \|t^{-\eta}f\|_{C^{k,\alpha}(X \setminus D)}.
\]

Note that if \( f \in C^{k,\alpha}_{\eta} \) for \( \eta < 0 \), \( f \) and its derivatives decay at infinity.

It is expected that the existence of cscK (more generally, extremal Kähler) metrics of Poincaré type is equivalent to algebro-geometric stability as follows.

**Conjecture 2.5** (§3.1 in [Sz1], see also [AAS, Sc] for more precise statements). \( (X, L_X); D \) has a cscK (extremal) Kähler metric of Poincaré type iff it is (relative) K-polystable for angle \( 0 \).
2.2 Kähler metrics with cone singularities

In this subsection, we recall the definition of Kähler metrics with cone singularities along $D$. In addition, we recall the definition of cscK metric with cone singularities along $D$. We take a real parameter $\beta \in (0, 1]$.

**Definition 2.6 ([Zhe1]).** A Kähler metric $\omega$ on $X \setminus D$ is said to be a Kähler cone metric of cone angle $2\pi \beta$ if $\omega$ is quasi isometric to the flat cone metric:

$$\omega_{\text{cone}} := \frac{\beta^2 \sqrt{-1} dz^1 \wedge d\bar{z}^1}{|z^1|^{2(1-\beta)}} + \sum_{j=2}^{n} \sqrt{-1} dz^j \wedge d\bar{z}^j,$$

(2.8)

on the holomorphic coordinate chart $(U; z^1, ..., z^n)$ such that $D = \{z^1 = 0\}$.

We recall the definition of cscK metrics with cone singularities introduced by Zheng [Zhe1]. Let $\Omega$ be a Kähler class and $\omega_0 \in \Omega$ be a Kähler metric. We set

$$c_1(X, D, \beta) := c_1(X) - (1 - \beta)c_1(L_D),$$

where $L_D$ denotes the line bundle associated with $D$. Fix a smooth form $\theta \in c_1(X, D, \beta)$. Let $s$ be a defining section of $D$ and $h$ be a Hermitian metric on $L_D$. The symbol $\Theta_D$ denotes the curvature form multiplied by $\sqrt{-1}$, i.e., $\Theta_D = -\sqrt{-1}\partial\bar{\partial}\log h$. The $\partial\bar{\partial}$-lemma tells us that there exists a smooth function $f$ satisfying

$$\text{Ric}(\omega_0) = \theta + (1 - \beta)\Theta_D + \sqrt{-1}\partial\bar{\partial}f.$$

We can find the Kähler cone metric $\omega_\theta$ of angle $2\pi \beta$ by solving the following singular complex Monge-Ampère equation

$$\omega_\theta^n = e^f \|s\|_h^{2\beta - 2} \omega_0^n.$$

Note that the Kähler metric $\omega_\theta$ satisfies

$$\text{Ric}(\omega_\theta) = \theta + 2\pi(1 - \beta)[D].$$

**Definition 2.7 (Def 3.1 in [Zhe1]).** A cscK cone metric $\omega_{\text{cscK}}$ in the class $\Omega$ is the solution of the following coupled system in the sense of currents on $X$:

$$\frac{\omega_{\text{cscK}}^n}{\omega_\theta^n} = e^F, \quad \Delta_{\omega_{\text{cscK}}} F = \text{tr}_{\omega_{\text{cscK}}} \theta - S_{\beta}.$$

(2.9)

Here, $S_{\beta}$ denotes the topological constant defined by

$$S_{\beta} := \frac{nc_1(X, D, \beta)\Omega^{n-1}}{\Omega^n}.$$

(2.10)

We can easily check that the scalar curvature of cscK cone metric $\omega_{\text{cscK}}$ is equal to $S_{\beta}$ on $X \setminus D$. Conversely, we have
Lemma 2.8. Let $\omega \in \Omega$ be a Kähler cone metric of angle $2\pi \beta$. Assume that the scalar curvature of $\omega$ is equal to $S_{\beta}$, i.e., $S(\omega) = S_{\beta}$ on $X \setminus D$. Then, $\omega$ is a cscK cone metric, i.e., $\omega$ satisfies the coupled system (2.9) in Definition 2.7.

Proof. We define a bounded function $F$ by

$$F := \log \frac{\omega^n}{\omega^0_n}.$$ 

Note that $F$ is smooth on $X \setminus D$ (in particular, $F \in C^{2,\alpha,\beta}$ (see Lemma 3.3 in [Zhe1])). On local holomorphic coordinates $(U; z^1, ..., z^n)$ such that $D = \{z^1 = 0\}$, we can write $\|s\|^2_n = |z^1|^2 e^{-a}$ for some smooth function $a$. Note that $a$ satisfies $\Theta_D = -\sqrt{-1} \partial \bar{\partial} a$. So, we have

$$\Delta_\omega F = \Delta_\omega \log \frac{\omega^n}{\omega^0_n} = n\sqrt{-1} \partial \bar{\partial} \log (|z^1|^{2(1-\beta)} \omega^n) \wedge \omega^{n-1} - n\sqrt{-1} \partial \bar{\partial} \log (e^{f+(1-\beta)a} \omega^0_n) \wedge \omega^{n-1} = -S_{\beta} - \text{tr}_\omega (\sqrt{-1} \partial \bar{\partial} f + (1-\beta) \Theta_D - \text{Ric}_0) \wedge \omega^{n-1}$$

$$= -S_{\beta} + \text{tr}_\omega \Theta.$$ 

Note that the singularities of the volume form $\omega^n$ are canceled by multiplying the factor $|z^1|^{2(1-\beta)}$. Thus, this equation holds in the sense of currents on $X$. \hfill \Box

The existence of cscK cone metrics is related to algebro-geometric stability which is called log $K$-polystability.

**Conjecture 2.9** (Log Yau-Tian-Donaldson conjecture (see [AHZ])). The pair $((X, L_X); D)$ admits a cscK cone metric iff it is log $K$-polystable.

The definitions of log $K$-polystability is given in [Do2] (see also Section 3.1 of [AHZ]). The necessary condition of the existence of cscK cone metric is shown as follows.

**Theorem 2.10** ([AHZ]). If $((X, L_X); D)$ admits a cscK cone metric of angle $2\pi \beta$, then it is $\text{Aut}_0((X, L_X); D)$-uniformly log $K$-stable with angle $2\pi \beta$.

It is also proved that the existence of cscK cone metric implies log $K$-(semi/poly)stability in [AHZ]. The definition of uniform log $K$-stability is also given in Section 3.1 of [AHZ].

2.3 Kähler cone metrics approximating a cscK metric of Poincaré type and fixed point formula

We assume the existence of cscK metrics of Poincaré type on $X \setminus D$ throughout this paper. In this subsection, we define a Kähler metric with cone singularities along $D$ of cone angle $2\pi \beta$, which converges to a cscK metric of Poincaré type when $\beta \to 0$. In addition, we introduce a fixed point formula which characterizes a cscK cone metric.

We consider a polarized manifold $(X, L_X)$ with a smooth hypersurface $D \in |L_X|$. Let $h_X$ be a Hermitian metric on $L_X$ such that its Chern curvature form multiplied by $\sqrt{-1}$
is a Kähler form $\theta_X$, i.e., $\theta_X = -\sqrt{-1} \partial \overline{\partial} \log h_X$. For a defining section $\sigma_D \in H^0(X, L_X)$ of $D$, we assume that $\|\sigma_D\|_{h_X}^2 < e^{-2}$ by scaling. We define a function $t(>2)$ on $X \setminus D$ by

$$t := \log \|\sigma_D\|_{h_X}^{-2}. \quad (2.11)$$

We set

$$S := \frac{n(c_1(X) - c_1(L_X)) \cup c_1(L_X)^{n-1}}{c_1(L_X)^n}, \quad S_D := \frac{(n-1)(c_1(X) - c_1(L_X))|D| \cup (c_1(L_X)|_D)^{n-2}}{(c_1(L_X)|_D)^{n-1}}.$$

Note that $S$ is the average value of the scalar curvature of Kähler metrics of Poincaré type on $X \setminus D$ in the class $[\theta_X]$ and $S_D$ is the average value of the scalar curvature of Kähler metrics on $D$ in the class $[\theta_X|_D]$. Auvray showed the topological constraint for cscK metrics of Poincaré type in the class $[\theta_X]$.

**Theorem 2.11 ([Au1]).** If $X \setminus D$ admits a cscK metric of Poincaré type, then the following inequality holds:

$$S_D > S. \quad (2.12)$$

**Remark 2.12.** Since we assume that $D \in |L_X|$, we have $S = \frac{n}{n-1} S_D$. So, Theorem 2.11 implies the negativity of $S_D$ under the assumption that there is a cscK metric of Poincaré type, i.e.,

$$0 > S - S_D = \frac{n}{n-1} S_D - S_D = \frac{1}{n-1} S_D.$$

By Theorem 2.11 we set the following well-defined positive number:

$$a_0 := \frac{2}{S_D - S} > 0. \quad (2.13)$$

For sufficiently large $\lambda > 0$, we directly define a Kähler metric of Poincaré type by

$$\theta_X - a_0 \sqrt{-1} \partial \overline{\partial} \log(\lambda + t) = \left(1 - \frac{a_0}{\lambda + t}\right) \theta_X + \frac{a_0 \sqrt{-1} \partial t \wedge \overline{\partial} t}{(\lambda + t)^2}. \quad (2.14)$$

We can compute as follows:

$$\lambda + t = \lambda + \log \|\sigma_D\|_{h_X}^{-2} = \log \|e^{-\lambda/2} \sigma_D\|_{h_X}^{-2}.$$

By replacing $\sigma_D$ with $e^{-\lambda/2} \sigma_D$, we write $\lambda + t$ as $t$ from now on. By using this simple symbol $t$, we define a Kähler metric of Poincaré type $\tilde{\omega}_0$ by

$$\tilde{\omega}_0 := \theta_X - a_0 \sqrt{-1} \partial \overline{\partial} \log t. \quad (2.15)$$

The reason why we put the coefficient $a_0$ is that the scalar curvature $S(\tilde{\omega}_0)$ is asymptotic to the average value of the scalar curvature $S$ (see Lemma 3.3). The properties of cscK (more generally, extremal Kähler) metrics of Poincaré type is well studied by Auvray as follows.
If \( \omega^\text{cscK}_c = \tilde{\omega}_0 + \sqrt{-1} \partial \bar{\partial} \varphi^\text{cscK}_c \) is a cscK metric of Poincaré type, then there exists a cscK metric \( \theta^\text{cscK}_D = \theta_X|_D + \sqrt{-1} \partial \bar{\partial} \psi_D \) on \( D \). Moreover, there is \( \delta > 0 \) such that
\[
\varphi^\text{cscK}_c = p^* \psi_D + O(t^{-\delta})
\]
at any differential order near \( D \), where \( p(z^1, z^2, ..., z^n) = (z^2, ..., z^n) \) on \( (U; z^1, ..., z^n) \) such that \( D = \{ z^1 = 0 \} \).

In this paper, we assume the non-existence of nontrivial holomorphic vector field on \( D \), so a cscK metric on \( D \) is unique in the Kähler class \( c_1(L_X|_D) \) ([Do1]). So, by replacing \( \theta_X \) so that the restricted Kähler metric \( \theta_X|_D \) is a cscK metric, we can consider that the function \( \varphi^\text{cscK}_c \) in Theorem 2.13 decays near \( D \) because we may assume that \( \psi_D = 0 \) from the uniqueness of cscK metrics on \( D \). Through out this paper, we fix the following notations.

**Definition 2.14.** We assume that
\[
\omega^\text{cscK}_0 := \tilde{\omega}_0 + \sqrt{-1} \partial \bar{\partial} \varphi^\text{cscK}_c = \theta_X - a_0 \sqrt{-1} \partial \bar{\partial} \log t + \sqrt{-1} \partial \bar{\partial} \varphi^\text{cscK}_c
\]
is a cscK metric on \( X \setminus D \) of Poincaré type. By Theorem 2.13,

- \( \theta_X|_D \) is a cscK metric on \( D \) and
- the positive number \( \delta > 0 \) satisfies \( \varphi^\text{cscK}_c = O(t^{-\delta}) \) at any differential order near \( D \).

Next, we define a Kähler metric with cone singularities approximating \( \omega^\text{cscK}_0 \). Set
\[
\mathcal{S}_\beta := nc_1(X, D, \beta) \cup c_1(L_X)^{n-1},
\]
\[
\mathcal{S}_{D, \beta} := (n - 1)c_1(X, D, \beta)|_D \cup (c_1(L_X)|_D)^{n-2}.
\]
Recall that \( c_1(X, D, \beta) = c_1(X) - (1 - \beta)c_1(L_X) \) for the smooth divisor \( D \in |L_X| \). For sufficiently small \( \beta > 0 \), we define the following well-defined number :
\[
a_\beta := \frac{2}{\mathcal{S}_{D, \beta} - \mathcal{S}_\beta} > 0.
\]
Define functions \( f_\beta(t) \) on \( X \setminus D \) dy
\[
f_\beta(t) := \frac{\beta a_\beta}{e^{\beta t} - 1}, \quad f_0(t) := \frac{a_0}{t}.
\]
In addition, we set
\[
G_\beta(t) := \int_2^t f_\beta(y)dy = \begin{cases} \frac{a_\beta \log ((1 - e^{-t\beta})/(1 - e^{-2\beta}))}{(\beta > 0)} & \\
\frac{a_0(\log t - \log 2)}{(\beta = 0)}
\end{cases}
\]
Since \((1 - e^{-\beta y})/\beta \to y \ (\beta \to 0)\), we have \(G_\beta(t) \to a_0(\log t - \log 2)\) and \(f_\beta(t) \to f_0(t)\) as \(\beta \to 0\) locally. By the direct computation, we have
\[
\theta_X - \sqrt{-1}\partial\overline{\partial}G_\beta(t) := (1 - f_\beta(t)) \theta_X + \beta f_\beta(t) \sqrt{-1}\partial t \wedge \overline{\partial}t
\]
\[= \left(1 - \frac{\beta a_\beta}{e^{\beta t} - 1}\right) \theta_X + \left(\frac{\beta}{1 - e^{-\beta t}}\right)^2 a_\beta e^{-\beta t} \sqrt{-1}\partial t \wedge \overline{\partial}t. \tag{2.24}\]
Note that \(\theta_X - \sqrt{-1}\partial\overline{\partial}G_\beta(t)\) is a Kähler cone metric of cone angle \(2\pi\beta\) since the quadratic term \(e^{-\beta t} \sqrt{-1}\partial t \wedge \overline{\partial}t\) included in \((2.24)\) is quasi isometric to \(\sqrt{-1}dz^1 \wedge \overline{dz^1}/|z^1|^{2(1-\beta)}\) near \(D = \{z^1 = 0\}\). Here, we write \(\|\sigma_D\|_{h_X}^2 = |z|^2 e^{-a}\) for some smooth function \(a\).

**Definition 2.15** (The background Kähler cone metric). We define a Kähler cone metric by
\[
\omega_\beta := \theta_X - \sqrt{-1}\partial\overline{\partial}G_\beta(t) + \sqrt{-1}\partial\overline{\partial}\varphi_{cscK},
\]
where \(\varphi_{cscK}\) is the function on \(X \setminus D\) in Definition 2.14.

**Remark 2.16.** Note that \(\omega_\beta\) is a Kähler metric with cone singularities of angle \(2\pi\beta\) along \(D\) since the function \(\varphi_{cscK}\) decays near \(D\) (see Definition 2.14) and does not affect the asymptotic behavior of the Kähler cone metric \(\theta_X - \sqrt{-1}\partial\overline{\partial}G_\beta(t)\). Moreover, \(\omega_\beta\) converges locally to \(\omega_0^{cscK}\) as \(\beta \to 0\). (In general, \(\omega_\beta\) is not a cscK cone metric.)

**Remark 2.17.** In [Gu], Guenancia uses the function \(\log((1 - e^{-t})/\beta)\) in order to define a Kähler metric with cone singularities. So, the function \(G_\beta(t)\) is equal to the function in [Gu] up to the coefficient \(a_\beta\) (and additive constants). The reason why we put coefficient \(a_\beta\) is that we consider the construction of a cscK cone metric in the fixed cohomology class \(c_1(L_X)\) and the scalar curvature of the background cone metric \(\omega_\beta\) is asymptotic to \(S_\beta\). In [Gu], the cohomology class which contains a Kähler-Einstein metric with cone singularities, is \(c_1(K_X) + (1 - \beta)c_1(D)\), so it depends on cone angle \(2\pi\beta\). In this case, the average values \(S_\beta\) and \(S_{D,\beta}\) are equal to \(n\) and \(n - 1\) respectively and the corresponding coefficient is 2 (independent of \(\beta\)!). Thus, this problem of the setting of coefficients does not arise in [Gu].

Finally, we consider the fixed point formula on \(C_{q,\alpha}^{4,\alpha}\) which characterizes a cscK cone metric. For a function \(\phi_\beta \in C_{q,\alpha}^{4,\alpha}\), \(\eta < 0\), we consider the following expansion:
\[
S(\omega_\beta + \sqrt{-1}\partial\overline{\partial}\phi_\beta) = S(\omega_\beta) + L_{\omega_\beta}(\phi_\beta) + Q_{\omega_\beta}(\phi_\beta). \tag{2.26}\]
Here, \(L_{\omega_\beta} : C_{q,\alpha}^{4,\alpha} \to C_{q,\alpha}^{0,\alpha}\) is the linearization of the scalar curvature operator and \(Q_{\omega_\beta}\) is the remaining nonlinear term. We characterize a solution of the equation:
\[S(\omega_\beta + \sqrt{-1}\partial\overline{\partial}\phi_\beta) = S_\beta, \quad \phi_\beta \in C_{q,\alpha}^{4,\alpha},\] as a fixed point \(\phi_\beta \in C_{q,\alpha}^{4,\alpha}\) given by
\[
\phi_\beta = -L_{\omega_\beta}^{-1}(S(\omega_\beta) - S_\beta + Q_{\omega_\beta}(\phi_\beta)). \tag{2.28}\]
Note that \(\omega_\beta + \sqrt{-1}\partial\overline{\partial}\phi_\beta\) in \((2.27)\) is a cscK cone metric by Lemma 2.18.

**Remark 2.18.** For a Kähler cone metric of angle \(2\pi\beta\), there are the function spaces denoted by \(C_{k,\alpha}^{k,\alpha}\) for \(k = 0, 1, \ldots, 4\), defined by Donaldson [Do2] and Li-Zheng [LZ]. By considering these function space, we can show the existence of a cscK cone metric on \(X \setminus D\) under the assumption that \(D\) is of sufficiently large degree (Aharonov-Zelditch [AZ]). In our case, we consider the functional space \(C_{q,\alpha}^{4,\alpha}\) since we want to use Sëktnan’s result [Sc] later.
3 cscK cone metrics and convergence

In this section, we prove Theorem 1.1 by constructing a fixed point in $C^{4,\alpha}_n$, which characterizes a cscK cone metric (2.26). From now on, we fix the exponent $\alpha \in (0,1)$. In order to show that, we study the following.

1. The weighted Hölder norm of the scalar curvature of the Kähler cone metric $\omega_\beta$, i.e., $\|S(\omega_\beta) - S_\beta\|_{C^{0,\alpha}_n(X \setminus D)}$ can be made small arbitrarily as $\beta \to 0$. Here, the weight $\eta$ is independent of $\beta$.

2. The linearization of the scalar curvature operator has a bounded inverse, i.e., there is $L_{\omega_\beta}^{-1}$ and its operator norm is independent of cone angle $2\pi \beta$.

3.1 The estimate of the scalar curvature

In this subsection, we prove the following estimate.

**Proposition 3.1.** For a weight $\eta \in (-\delta,0)$, we have

$$
\|S(\omega_\beta) - S_\beta\|_{C^{0,\alpha}_n(X \setminus D)} = O((-\beta \log \beta)^{\eta+\delta}).
$$

(3.1)

Here, the number $\delta$ is given in Definition 2.14.

In order to prove this proposition, we divide $X \setminus D$ into a neighborhood of $D$ and a region away from $D$. Firstly, we study the asymptotic behavior of the scalar curvature of the Kähler metric $\theta_X - \sqrt{-1}\partial\bar{\partial}G_\beta(t)$ on a neighborhood of $D$.

**Lemma 3.2.** If $\theta_D := \theta_X|_D$ is a cscK metric on $D$, we have

$$
S(\theta_X - \sqrt{-1}\partial\bar{\partial}G_\beta(t)) - S_\beta = O(\|\sigma_D\|^{2(1-\beta)}) = O(e^{(\beta-1)t})
$$

as $t \to \infty$ (equivalently, $\sigma_D \to 0$).

**Proof.** From (2.21), the volume form of $\theta_X - \sqrt{-1}\partial\bar{\partial}G_\beta(t)$ can be written as

$$
(\theta_X - \sqrt{-1}\partial\bar{\partial}G_\beta(t))^n = (1 + O(\|\sigma_D\|^2))(\frac{\beta}{1 - e^{-\beta t}})^2 na_\beta e^{-\beta t} \theta_X^{-1} \wedge \partial t \wedge \bar{\partial}t
$$

(3.3)

So, the Ricci form of this Kähler metric is

$$
\text{Ric}(\theta_X - \sqrt{-1}\partial\bar{\partial}G_\beta(t))
$$

$$
= -\sqrt{-1}\partial\bar{\partial} \log (1 + O(\|\sigma_D\|^2)) + 2\sqrt{-1}\partial\bar{\partial} \log (1 - e^{-\beta t}) + \beta \theta_X + \text{Ric}\theta_X
$$

$$
= -\sqrt{-1}\partial\bar{\partial} \log (1 + O(\|\sigma_D\|^2)) + \sqrt{-1}\partial\bar{\partial} \left( (S_\beta - S_{D,\beta}) G_\beta(t) \right) + \beta \theta_X + \text{Ric}\theta_X.
$$

In order to obtain the following three estimates;

$$
\text{tr}_{\theta_X - \sqrt{-1}\partial\bar{\partial}G_\beta(t)} \sqrt{-1}\partial\bar{\partial} \log (1 + O(\|\sigma_D\|^2)) = O(\|\sigma_D\|^{2(1-\beta)}),
$$

(3.4)

$$
\text{tr}_{\theta_X - \sqrt{-1}\partial\bar{\partial}G_\beta(t)} \sqrt{-1}\partial\bar{\partial} \left( (S_\beta - S_{D,\beta}) G_\beta(t) \right) = S_\beta - S_{D,\beta} + O(\|\sigma_D\|^{2(1-\beta)}),
$$

(3.5)

$$
\text{tr}_{\theta_X - \sqrt{-1}\partial\bar{\partial}G_\beta(t)} (\beta \theta_X + \text{Ric}\theta_X) = \beta(n - 1) + S(\theta_D) + O(\|\sigma_D\|^{2(1-\beta)}),
$$

(3.6)
we use the following linear algebraic result (see [Zha, p.24] and [Ao, §3.1 and §3.2]). We consider the following matrix

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

and assume that $D$ and $S := A - BD^{-1}C$ are invertible. Then, $T$ is invertible and the inverse matrix of $T$ is given by

$$T^{-1} = \begin{bmatrix} S^{-1} & -S^{-1}BD^{-1} \\ -D^{-1}CS^{-1} & D^{-1} + D^{-1}CS^{-1}BD^{-1} \end{bmatrix}.$$  

(3.7)

By taking normal holomorphic coordinates in [Ao, Prop 5], we can write

$$\theta_X - \sqrt{-1} \partial \bar{\partial} G_\beta(t) = \begin{bmatrix} \frac{a_\beta}{|z|^{2(1-\beta)}} & 0 \\ 0 & g_{\bar{\beta}j} \end{bmatrix} + O(|z|^2)$$

near $D$, where we write $\theta_X = ig_{\bar{\beta}j}dz^j \wedge d\bar{z}^j$. By applying the formula (3.7), we have

$$\left(\theta_X - \sqrt{-1} \partial \bar{\partial} G_\beta(t)\right)^{-1} = \begin{bmatrix} \frac{|z|^{2(1-\beta)}}{a_\beta} & 0 \\ 0 & g_{\bar{\beta}j} \end{bmatrix} + O(|z|^{2(1-\beta)}).$$

Here, $(g_{\bar{\beta}j}) = (g_{\bar{\beta}j})^{-1}$. Thus, we obtain directly the estimate (3.4) and (3.5).

Secondly, we study the estimate of $S(\theta_X - \sqrt{-1} \partial \bar{\partial} G_\beta(t))$ away from $D$. We show the following.

**Proposition 3.4.** On $V_\beta := \{t \leq (-\beta \log \beta)^{-1}\} \subset X \setminus D$, we have

$$S(\theta_X - \sqrt{-1} \partial \bar{\partial} G_\beta(t)) - S(\tilde{\omega}_0) = O(\beta)$$

(3.9)
Proof. Note that
\[
\theta_X - \sqrt{-1} \partial \bar{\partial} G_\beta (t) = (1 - f_\beta(t)) \theta_X + \dot{f}_\beta(t) \sqrt{-1} dt \wedge \bar{dt}
\]
and
\[
\tilde{\omega}_0 = \theta_X - \sqrt{-1} \partial \bar{\partial} G_0 (t) = (1 - f_0(t)) \theta_X + \dot{f}_0(t) \sqrt{-1} dt \wedge \bar{dt}.
\]
Their volume forms are given by
\[
(\theta_X - \sqrt{-1} \partial \bar{\partial} G_\beta (t))^n = (1 - f_\beta(t))^{n-1} \left( 1 - f_\beta(t) + \dot{f}_\beta(t) \| \partial t \|^2_{\theta_X} \right) \theta^n_X
\]
and
\[
\tilde{\omega}_0^n = (1 - f_0(t))^{n-1} \left( 1 - f_0(t) + \dot{f}_0(t) \| \partial t \|^2_{\theta_X} \right) \theta^n_X
\]
respectively. So, the Ricci forms of these Kähler metrics are given by
\[
\text{Ric}(\theta_X - \sqrt{-1} \partial \bar{\partial} G_\beta (t)) = \text{Ric}(\theta_X) - (n-1) \sqrt{-1} \partial \bar{\partial} \log (1 - f_\beta(t)) - \sqrt{-1} \partial \bar{\partial} \log \left( 1 - f_\beta(t) + \dot{f}_\beta(t) \| \partial t \|^2_{\theta_X} \right).
\]
and
\[
\text{Ric}(\tilde{\omega}_0) = \text{Ric}(\theta_X) - (n-1) \sqrt{-1} \partial \bar{\partial} \log (1 - f_0(t)) - \sqrt{-1} \partial \bar{\partial} \log \left( 1 - f_0(t) + \dot{f}_0(t) \| \partial t \|^2_{\theta_X} \right).
\]
Thus, in order to study the difference of their scalar curvature, it suffices to show that any derivatives of \( f_\beta \) are uniformly continuous and converges uniformly to the derivatives of \( f_0 \) on \( V_\beta \). Since \( t > 2 \), we have
\[
f_\beta(t) < 1/2.
\]
Recall that
\[
f_\beta(t) = \frac{\beta a_\beta}{e^{\beta t} - 1}.
\]
Directly, we have
\[
\dot{f}_\beta(t) = -f_\beta(t)^2 e^{\beta t} / a_\beta.
\]
Thus, the family of functions \( \{ f_\beta \} \) is uniformly continuous. Inductively, we obtain
\[
\ddot{f}_\beta(t) = -\beta f_\beta(t)^2 e^{\beta t} / a_\beta + 2 f_\beta(t)^3 e^{2\beta t} / a_\beta^2,
\]
so any differential of \( f_\beta \) is uniformly continuous.

To show the uniform convergence of the family \( \{ f_\beta \} \), we use the following elementary inequalities:
\[
0 \geq \frac{\beta}{e^{\beta t} - 1} - \frac{1}{t} \geq \frac{e^{-\beta t} - 1}{t} \geq -\beta e^{\beta t}.
\]
On \( V_\beta \), there exists a constant \( C > 0 \) independent of \( \beta \) such that
\[
|f_0(t) - f_\beta(t)| < C \beta.
\]
This estimate implies that for any \( k \), there exists \( C_k > 0 \) independent of \( \beta > 0 \) such that
\[
|f_\beta^{(k)}(t) - f_0^{(k)}(t)| < C_k \beta \quad \text{on} \quad V_\beta.
\]
So, we have finished proving this proposition. \( \square \)
Recall the definitions of $\omega^\text{cscK}_0$ and $\omega_\beta$:

$$\omega^\text{cscK}_0 := \theta_X - \sqrt{-1}\partial\bar{\partial}G_0(t) + \sqrt{-1}\partial\bar{\partial}\varphi_{\text{cscK}}; \quad \omega_\beta := \theta_X - \sqrt{-1}\partial\bar{\partial}G_\beta(t) + \sqrt{-1}\partial\bar{\partial}\varphi_{\text{cscK}}.$$ 

So, the difference of the scalar curvatures $S(\omega^\text{cscK}_0)$ and $S(\omega_\beta)$ comes from the function $G_\beta(t)$ and its derivatives. By applying Proposition 3.3 to these metrics, we have

**Lemma 3.5.** On $V_\beta$, we have

$$\|S(\omega^\text{cscK}_0) - S(\omega_\beta)\|_{C^{0,\alpha}_\eta(V_\beta)} \leq C_\beta.$$ (3.16)

for some $C > 0$ independent of $\beta$.

**Proof of Proposition 3.7**

On $V_\beta$, we can write as follows:

$$S(\omega_\beta) - S_3 = S(\omega_\beta) - S(\omega^\text{cscK}_0) + S(\omega^\text{cscK}_0) - S_3.$$

Note that we have $S(\omega^\text{cscK}_0) - S_3 = -\beta n$. Lemma 3.9 gives the estimate (3.11) on $V_\beta$.

Since $\omega_\beta = \theta_X - \sqrt{-1}\partial\bar{\partial}G_\beta(t) + \sqrt{-1}\partial\bar{\partial}\varphi_{\text{cscK}}$ on the complement of $V_\beta$, Lemma 3.3 implies the estimate (3.11). \qed

### 3.2 The bounded inverse of the linearization

For a Kähler metric $\omega$, the linearization of the scalar curvature operator $L_\omega$ satisfies

$$L_\omega = -D^*_\omega D_\omega + (\nabla^{1,0} S(\omega), \nabla^{1,0})_\omega.$$ 

Here, $D_\omega = \overline{\partial} \circ \nabla^{1,0}$ and $\nabla^{1,0}$ denotes the $(1,0)$-gradient with respect to $\omega$. We call $D^*_\omega D_\omega$ the Lichnerowicz operator. If $D_\omega \phi = 0$, $\nabla^{1,0} \phi$ is a holomorphic vector field.

In this subsection, we show the uniform lower estimate of the Lichnerowicz operator $D^*_\omega D_\omega : C^{4,\alpha}_\eta(X \setminus D) \to C^{0,\alpha}_\eta(X \setminus D)$.

**Theorem 3.6.** Assume that $\text{Aut}_0((X, L_X); D)$ is trivial and $H^0(D, TD) = 0$. Then, there exist $\kappa > 0$ and $\beta_0 > 0$ which satisfy the following property. For $\eta \in (-\kappa, 0)$, there exists $K > 0$ independent of $\beta \in [0, \beta_0)$ such that

$$\|D^*_\omega D_\omega \phi\|_{C^{0,\alpha}_\eta} \geq K \|\phi\|_{C^{4,\alpha}_\eta}, \quad \forall \phi \in C^{4,\alpha}_\eta.$$

**Remark 3.7.** The constant $K > 0$ in Theorem 3.6 depends on $\alpha, \kappa, \eta$ and $\delta$.

In order to study the inverse operator of the Lichnerowicz operator $D^*_\omega D_\omega : C^{4,\alpha}_\eta(X \setminus D) \to C^{0,\alpha}_\eta(X \setminus D)$, we use the following proposition in [Se] for the Lichnerowicz operator $D^*_\omega^\text{cscK} D_\omega^\text{cscK} : C^{4,\alpha}_\eta(X \setminus D) \to C^{0,\alpha}_\eta(X \setminus D)$ when $\beta = 0$.

**Proposition 3.8** (Prop 4.3 in [Se] when $r = h^0(D, TD) = 0$ and $\text{Aut}_0((X, L_X); D)$ is trivial). Assume that $H^0(D, TD) = 0$ and $\text{Aut}_0((X, L_X); D)$ is trivial. There is $\kappa > 0$ with the following properties: For $\eta \in (-\kappa, 0)$, we have
The Lichnerowicz operator can be written locally as
\[ \text{Im}(\mathcal{L}^*_{\omega_0^c} \mathcal{L}_{\omega_0^c}^* : \mathcal{C}^{4,\alpha}_\eta \to \mathcal{C}^{4,\alpha}_\eta) = C^{4,\alpha}_\eta. \]

After this, we fix the symbol \( \kappa \) satisfying Theorem 3.6. Thus if \( \eta \in (-\kappa,0) \) and \( H^0(D,TD) = 0 \) and \( \text{Aut}_0((X,L_X);D) \) is trivial, the Lichnerowicz operator \( \mathcal{L}_{\omega_0^c}^* \mathcal{L}_{\omega_0^c}^* : \mathcal{C}^{4,\alpha}_\eta \to \mathcal{C}^{4,\alpha}_\eta \) is isomorphic.

**Remark 3.9.** Sektnan showed more general result for the case when \( X \) and \( D \) admit nontrivial holomorphic vector fields (see Proposition 4.3 in [Se]). In our case, we assume that \( H^0(D,TD) = 0 \) and \( \text{Aut}_0((X,L_X);D) \) is trivial since we want to deal with the case that the Lichnerowicz operator is isomorphic.

**Lemma 3.10.** The following map is continuous with respect to the operator norm:
\[ [0,1] \ni \beta \mapsto \mathcal{L}_{\omega_\beta}^* \mathcal{L}_{\omega_\beta} \in \text{Map}(\mathcal{C}^{4,\alpha}_\eta(X \setminus D) \to \mathcal{C}^{4,\alpha}_\eta(X \setminus D)). \]

**Proof.** The Lichnerowicz operator can be written locally as \( \mathcal{L}_{\omega_\beta}^* \mathcal{L}_{\omega_\beta} = g_\beta^{-\frac{1}{2}} g_\beta^{-\frac{1}{2}} \nabla_i \nabla_j \nabla_k \nabla_l \), where we write \( \omega_\beta = \sqrt{-1} g_\beta dz^i \wedge d\bar{z}^j \) and \( (g_\beta^{-\frac{1}{2}}) = (g_\beta^{-\frac{1}{2}})^{-1} \) (see §4.1 in [Sz2]). Since the operator norm of \( \mathcal{L}_{\omega_\beta}^* \mathcal{L}_{\omega_\beta} \) is defined by
\[ \|\mathcal{L}_{\omega_\beta}^* \mathcal{L}_{\omega_\beta} \|_{\mathcal{C}^{4,\alpha}_\eta \to \mathcal{C}^{4,\alpha}_\eta} := \sup\{\|\mathcal{L}_{\omega_\beta}^* \mathcal{L}_{\omega_\beta} \phi\|_{\mathcal{C}^{4,\alpha}_\eta} : \|\phi\|_{\mathcal{C}^{4,\alpha}_\eta} \leq 1\}, \]

it suffices to show that
\[ \|g_\beta^{-\frac{1}{2}} - g_\beta^{-\frac{1}{2}}\|_{\mathcal{C}^{4,\alpha}_0 \to \mathcal{C}^{4,\alpha}_0} \to 0 \]
and any derivatives of \( g_\beta^{-\frac{1}{2}} \) converge to \( g_0^{-\frac{1}{2}} \) as \( \beta \to 0 \). This convergence follows from the fact that \( \omega_\beta = (1 - f_\beta(t)) \theta_X + f_\beta(t) \sqrt{-1} \bar{\partial} t \wedge \partial t + \sqrt{-1} \partial \bar{\partial} \psi_{cscK} \)
and any derivatives \( f_\beta^{(k)} \) converges to \( f_0^{(k)} \) uniformly when \( \beta \to 0 \). \( \square \)

**Proof of Theorem 3.1.**

Fix \( \eta \in (-\kappa,0) \). By Proposition 3.8 we know that \( \mathcal{L}_{\omega_\beta}^* \mathcal{L}_{\omega_\beta} \) is isomorphic at \( \beta = 0 \). It follows from the closed mapping theorem (see page 77 of [Yo]) that there is \( K_0 \) satisfying
\[ \|\mathcal{L}_{\omega_0^c}^* \mathcal{L}_{\omega_0^c}^* \|_{\mathcal{C}^{4,\alpha}_\eta \to \mathcal{C}^{4,\alpha}_\eta} \geq K_0 \|\phi\|_{\mathcal{C}^{4,\alpha}_\eta}, \forall \phi \in \mathcal{C}^{4,\alpha}_\eta. \]
Set \( \beta_0 := \sup\{\beta \geq 0 \mid \exists \mathcal{L}_{\omega_{\beta_0}}^* \mathcal{L}_{\omega_{\beta_0}}^{-1} : C^{4,\alpha}_\eta \to C^{4,\alpha}_\eta \} \). From Lemma 3.10, we have \( \beta_0 > 0 \). \( \square \)

### 3.3 Proof of Theorem 1.1

In this subsection, we show the map \( \mathcal{N}_\beta : C^{4,\alpha}_\eta \to C^{4,\alpha}_\eta \) defined by
\[ \mathcal{N}_\beta(\phi_\beta) = -L_{\omega_\beta}^{-1}(S(\omega_\beta) - \sum_{\beta} + Q_{\omega_\beta}(\phi_\beta)) \]
has a fixed point, i.e., there exists a cscK cone metric of angle \( 2\pi \beta \). In order to show this, we prove that the map \( \mathcal{N}_\beta : C^{4,\alpha}_\eta \to C^{4,\alpha}_\eta \) is a contraction map.
Remark 3.11. This proof is an application of the construction of a cscK metric on the blowing-up of a cscK manifold without nontrivial holomorphic vector field by Arezzo-Pacard [AP1, AP2].

Recall that $L_{\omega_\beta} = -D_{\omega_\beta}^* D_{\omega_\beta} + (\nabla^{1,0} S(\omega_\beta), \nabla^{1,0} \phi)_{\omega_\beta}$. Since $S(\omega_\beta)$ converges to the constant $S$ as $\beta \to 0$, Theorem 3.10 implies

Theorem 3.12. Assume that $\text{Auto}_0((X, L_X); D)$ is trivial and $H^0(D, TD) = 0$. Then, there exist $\kappa > 0$ and $\beta_0 > 0$ which satisfy the following property. For $\eta \in (-\kappa, 0)$, there exists $\hat{K} > 0$ independent of $\beta \in [0, \beta_0)$ such that

$$\|L_{\omega_\beta}\phi\|_{C^0_\eta} \geq \hat{K}\|\phi\|_{C^4_\eta}, \forall \phi \in C^4_\eta.$$  

Namely, the operator norm of $L_{\omega_\beta}^{-1} : C^0_\eta \to C^4_\eta$ is bounded by $1/\hat{K}$.

Remark 3.13. The constant $\hat{K} > 0$ in Theorem 3.12 depends on $\alpha, \kappa, \eta$ and $\delta$.

We need the following lemma.

Lemma 3.14. There exists $c_0 > 0$ independent of sufficiently small $\beta > 0$ such that if $\|\phi\|_{C^4_\eta} \leq c_0$, we have

$$\|L_{\omega_\beta + \sqrt{-1}\partial\bar{\partial}\phi} - L_{\omega_\beta}\|_{C^4_\eta \to C^4_\eta} \leq \frac{\hat{K}}{2}$$

and $\omega_\beta + \sqrt{-1}\partial\bar{\partial}\phi$ is positive definite.

Proof. Let us write locally as $\omega_\beta = \sqrt{-1}g_{\phi}dz^i \wedge d\bar{z}^j$. The Lemma follows from the simple relation:

$$g_{\phi}^{-1} - g^{\bar{z}} = -g_{\phi}^{-1} (g_{\phi} - g) g^{-1},$$

because the weight $\eta$ is negative. \qed

By using Theorem 3.12 and Lemma 3.14, we can show that $N_\beta$ is a contraction map.

Lemma 3.15. Let $c_0$ be the constant given in Lemma 3.14. If $\|\phi\|_{C^4_\eta} \leq c_0$, we have

$$\|N_\beta(\phi) - N_\beta(\psi)\|_{C^4_\eta} \leq \frac{1}{2}\|\phi - \psi\|_{C^4_\eta}.$$  

Proof. Since the operator $N_\beta$ is defined by $N_\beta(\phi) := -L_{\omega_\beta}^{-1}(S(\omega_\beta) - \mathbb{S}_\beta + Q_{\omega_\beta}(\phi))$, we have

$$N_\beta(\phi) - N_\beta(\psi) = -L_{\omega_\beta}^{-1}(Q_{\omega_\beta}(\phi) - Q_{\omega_\beta}(\psi)).$$

By the mean value theorem, there exists $\chi = t\phi + (1-t)\psi$ for $t \in [0, 1]$ such that

$$DQ_{\omega_\beta, \chi}(\phi - \psi) = Q_{\omega_\beta}(\phi) - Q_{\omega_\beta}(\psi).$$

Here, $DQ_{\omega_\beta, \chi}$ denotes the derivative of $Q_{\omega_\beta}$ at $\chi$. We write $\omega_\chi = \omega_\beta + \sqrt{-1}\partial\bar{\partial}\chi$. By differentiating the equation $S(\omega_\beta) + L_{\omega_\beta}(\chi + sf) + Q_{\omega_\beta}(\chi + sf) = S(\omega_\chi) + sL_{\omega_\chi}f + Q_{\omega_\chi}(sf)$ at $s = 0$, we obtain

$$DQ_{\omega_\beta, \chi} = L_{\omega_\chi} - L_{\omega_\beta}.$$ 

We can easily show that $\|\chi\|_{C^4_\eta} \leq c_0$. By Theorem 3.12 and Lemma 3.14 we finish the proof. \qed
Proof of Theorem 1.1

We take $\alpha \in (0, 1)$, $\delta$, $\kappa$ and $\eta$ as before (see Definition 2.14 and Proposition 3.8). Fix a small positive number $\epsilon > 0$ so that $\delta + \eta > \epsilon$.

$$U_\beta := \{ \phi \in C^{4, \alpha}_\eta \mid \| \phi \|_{C^{4, \alpha}_\eta} \leq c_0 (\log \beta)^{-\epsilon} \}$$  \hfill (3.22)

Lemma 3.15 implies that the map $N_\beta$ is a contraction map on $U_\beta$. From (3.1), we can find a sufficiently small $\beta > 0$ so that

$$\| S(\omega_\beta) - S_3 \|_{C^{0, \alpha}_\eta} \leq c_0 \hat{K} (\log \beta)^{-\epsilon}.$$  \hfill (3.23)

For $\phi \in U_\beta$, we have

$$\| N_\beta(\phi) \|_{C^{4, \alpha}_\eta} \leq \| N_\beta(\phi) - N_\beta(0) \|_{C^{4, \alpha}_\eta} + \| L_{\omega_\beta}^{-1}(S(\omega_\beta) - S_3) \|_{C^{4, \alpha}_\eta}$$  \hfill (3.24)

$$\leq \frac{1}{2} c_0 (\log \beta)^{-\epsilon} + \hat{K}^{-1} \| S(\omega_\beta) - S_3 \|_{C^{0, \alpha}_\eta}$$  \hfill (3.25)

$$\leq c_0 (\log \beta)^{-\epsilon}.$$  \hfill (3.26)

So, we note that $N_\beta(U_\beta) \subset U_\beta$. Therefore, we can find $\phi_\beta \in U_\beta$ so that $\omega_\beta + \sqrt{-1} \partial \bar{\partial} \phi_\beta$ is a cscK cone metric for sufficiently small cone angle $2\pi \beta$. Moreover, by definition of $U_\beta$, the sequence of cscK cone metrics $\omega_\beta + \sqrt{-1} \partial \bar{\partial} \phi_\beta$ locally converges to $\omega_0^{\text{cscK}}$.

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