Quantum nonlocality in the sequential correlation scenario

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We study the possibility of the multiple uses of a single copy of a quantum state shared by Alice, Bob and Charlie through the sharing of tripartite genuine nonlocality measured sequentially by several observers at one side. The interesting result is that arbitrarily many independent Charlies can have an quantum violation of the corresponding inequalities given by the definitions about the genuine multipartite nonlocality with the single Alice and Bob.

I. INTRODUCTION

Quantum nonlocality is of the great importance in both understanding the conceptual foundations of quantum theory and quantum information processing such as building quantum protocols to decrease communication complexity \cite{1,2} and providing secure quantum communication \cite{3,4}.

Recently, the problem about whether or not multiple observers can share the nonlocality from an entangled quantum system has gained the extensive attention. In \cite{5}, the authors give a positive answer and show that the nonlocality of a single particle from an entangled pair can be shared among multiple observers who act sequentially and independently of each other. In \cite{6}, they prove that the number of observers can be arbitrarily large based on the sequential measurements at one side. In \cite{2,5,6}, contrary to the bipartite quantum systems, the authors show that in the tripartite systems at most two Charlies could simultaneously demonstrate genuine tripartite nonlocality with single Alice and Bob.

In this work, we study the constraints about the tripartite genuine nonlocality and get that arbitrarily many independent Charlies can have an quantum violation of the corresponding inequalities with the single Alice and Bob.

II. QUANTUM GENUINE NONLOCALITY IN THE SEQUENTIAL CORRELATION SCENARIO

First, let we consider a scenario where a tripartite state $\rho_{ABC}$ is initially shared by three observers, Alice, Bob and Charlie1. A single Alice and Bob want to establish nonlocal correlations with a sequence of Charlies who measure one of three particles (see Fig. 1). Charlies only can send their postmeasurement states to the next and the measurement choices and outcomes of each Charlie are not shared. In particular, every observer chooses randomly one of two dichotomic measurements, which can be denoted by $(X, A)$, $(Y, B)$, and $(Z_k, C_k)$, respectively. And the corresponding conditional probability distribution is denoted by $P(abc|XYZ)$. To begin, after Charlie1 chooses a random uniformly input $Z_1$ and performs the corresponding measurement, recording the outcome $C_1$. Then he passes his particle to Charlie2. The postmeasurement state can be described through the Lüders rule:

$$\rho_{ABC_2} = \frac{1}{2} \sum_{z_1, c_1} (I \otimes I \otimes \sqrt{F^1_{c_1|z_1}}) \rho_{ABC_1} (I \otimes I \otimes \sqrt{F^1_{c_1|z_1}}),$$

where $F^1_{c_1|z_1} \geq 0$, $\sum_{c_1} F^1_{c_1|z_1} = I$.

Second, different from the ones in bipartite systems, states in tripartite systems can be not only entangled or non-locally correlated, but also genuinely entangled or genuinely non-locally correlated. Quantum nonlocality can be revealed via violations of various Bell inequalities. For the tripartite quantum system, except for the well-known Svetlichny inequalities\cite{9}, in \cite{10} other three-qubit genuine nonlocality, three-way nonlocal correlations, have been studied. Here we mainly study the nonsignaling genuine nonlocality.

If the conditional probability correlations $P(abc|XYZ)$ can be written as in the hybrid local-nonlocal form \cite{10},

$$P(abc|XYZ) = \sum_{\lambda} q_{\lambda} P_{\lambda}(a|XY)P_{\lambda}(c_k|Z) + \sum_{\mu} q_{\mu} P_{\mu}(ac_k|XZ_k)P_{\mu}(b|Y) + \sum_{\nu} q_{\nu} P_{\nu}(bc_k|YZ_k)P_{\nu}(a|X),$$

where $\forall a, X, Y, P_{\lambda}(a|XY) = P_{\lambda}(a|XY')$, $\forall b, X', Y, P_{\lambda}(b|XY) = P_{\lambda}(b|X'Y)$,

$\forall a, X, Z, Z', P_{\lambda}(a|XZ_k) = P_{\lambda}(a|XZ'_k)$,

$\forall c_k, X, X', Z_k, P_{\lambda}(c_k|XZ_k) = P_{\lambda}(c_k|X'Z'_k)$,

$\forall b, Y, Z, Z', P_{\lambda}(b|YZ_k) = P_{\lambda}(b|YZ'_k)$,

$\forall c_k, Y, Y', Z_k, P_{\nu}(c_k|YZ_k) = P_{\nu}(c_k|Y'Z'_k)$, $0 \leq q_{\lambda}$, $q_{\mu}, q_{\nu} \leq 1$ and $\sum_{\lambda} q_{\lambda} + \sum_{\mu} q_{\mu} + \sum_{\nu} q_{\nu} = 1$, then the

FIG. 1: Sharing the genuine tripartite nonlocality with multiple Charlies: A quantum state $\rho$ is initially distributed between Alice, Bob and Charlie1. After Charlie1 performed his randomly selected measurement and recorded the outcomes, he passed the post-measurement quantum state to Charlie2 who then repeated the process. In especial, the measurement choices and outcomes of each Charlie are not conveyed.
correlations are called nonsignal-local. Otherwise we call them genuinely nonsignal nonlocal.

III. THE MEASUREMENT STRATEGY AND THE RESULT ABOUT THE QUANTUM GENUINE NONLOCALITY

Here we study the three-qubit quantum systems. Suppose the started quantum state is the generalized GHZ ones $|GGHZ\rangle = \cos(\alpha)|000\rangle + \sin(\alpha)|111\rangle$, where $\{|0\rangle, |1\rangle\}$ is the computational basis in the qubit system. The measurement method which makes Alice, Bob and n Charlies share the genuine nonsignal nonlocality is as follows: Alice’s and Bob’s measurements are

\[
X_{0|0} = \frac{I + \sigma_3}{2}, \quad (2)
\]

\[
X_{0|1} = \frac{I + \sigma_1}{2}, \quad (3)
\]

\[
Y_{0|0} = \frac{I + \sigma_3}{2}, \quad (4)
\]

\[
Y_{0|1} = \frac{I + \sigma_1}{2}, \quad (5)
\]

and for $k = 1, 2, 3, \cdots, n$, the $k$th Charlie’s measurements are

\[
Z_{0|0}^k = \frac{I + \cos(\theta)\sigma_3 - \sin(\theta)\sigma_1}{2}, \quad (6)
\]

\[
Z_{0|1}^k = \frac{I + \gamma_k[\cos(\theta)\sigma_3 + \sin(\theta)\sigma_1]}{2}, \quad (7)
\]

where for $\forall j \in 0, 1, \sum_{i=0}^3 F_{ij}^k = I, F_{ij}^k \geq 0, F \in \{A, B, C\}, \{\sigma_i, i = 1, 2, 3\}$ are the standard Pauli matrices, and the sequence $\{0 \leq \gamma_k \leq 1\}$ is given

\[
\gamma_k(\delta) = \begin{cases} 
\frac{(1+\epsilon)[2^{k-1}-\cos(\delta)]}{2^{k-1}(1+\sqrt{1-\gamma_1^2})}, & \text{if } \gamma_{k-1} \in [0, 1] \\
\infty, & \text{otherwise}
\end{cases}
\]

Here $0 < \delta \leq \frac{\pi}{4}, \epsilon > 0$ and when $k = 1$, $\gamma_1 = (1 + \epsilon)\frac{1-\cos(\delta)}{\sin(\delta)}$.

**Lemma 1.** When $0 < \delta \leq \frac{\pi}{4}, \epsilon > 0$, the above defined sequence $\{\gamma_k(\delta)\}_k$ satisfies: for $1 \leq k \leq n$, $0 \leq \gamma_k(\delta) \leq 1$.

**Theorem 1.** For every $n \in \mathbb{N}$, there exists a sequence $\{\gamma_k(\delta)\}_k$, such that $\rho_{ABC}^{\gamma_k(\delta)}(\{GGHZ\})$ is genuinely nonsignal nonlocal.

**[Proof]**. In order to obtain the violations of genuinely nonsignal nonlocality, we consider the following inequality, called $NS_2^k$ inequalities:

\[
\langle N S_2^k \rangle \equiv \langle X_0 Y_0 \rangle + \langle X_0 Z_0^k \rangle + \langle Y_0 Z_0^k \rangle - \langle X_1 Y_1 Z_0^k \rangle + \langle X_1 Y_1 Y_1 \rangle \leq 3, \quad (8)
\]

where $\langle X_i Y_j Z_m^k \rangle = \sum_{abc} (-1)^{a+b+c} p(abc) X_i Y_j Z_m^k$.

Let $X_i = X_{0|i} - X_{1|i}, Y_j = Y_{0|j} - Y_{1|j}, Z_m^k = Z_{0|mn}^k - Z_{1|mn}^k, |GGHZ\rangle = \cos(\alpha)|000\rangle + \sin(\alpha)|111\rangle$, according to the inequality $[8]$, we can get the values about $\langle N S_2 \rangle$ of $|GGHZ\rangle$, when $k \geq 2$,

\[
\langle N S_2^k \rangle = 1 + (1 + \gamma_k)[\cos(\theta) + \sin(\theta)\sin(2\alpha)] \left(\frac{\Pi_{j=1}^{k-1}(1 + \sqrt{1 - \gamma_j^2})}{2^{k-1}} + \frac{\gamma_j}{2^{k-1}}\right). \quad (9)
\]

When $k = 1$, $N S_2^1 = 1 + (1 + \gamma_1)[\cos(\theta) + \sin(\theta)\sin(2\alpha)]$.

Based on the sequence of $\gamma_k$ of the Lemma 1 and $|GGHZ\rangle$, we can get that for $k \in \mathbb{N}, N S_2^k > 3$ which means $\rho_{ABC}^{\gamma_k(\delta)}(|GGHZ\rangle)$ is genuinely nonsignal nonlocal.

IV. CONCLUSIONS AND DISCUSSIONS

Being different from the results in [3, 8], we prove that in tripartite quantum systems, arbitrarily many independent Charlies can share the genuine quantum nonlocality with the single Alice and Bob when they are distributed in the generalized GHZ states. It also would be interesting to explore the possibility of this sharing of quantum correlations in multipartite high quantum systems.

**Acknowledgments** This project is supported by the National Natural Science Foundation of China (Grants No.11725417, No.11974057), NSAF(Grant No. U1930403), and Science Challenge Project(Grant No.2018005).

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