A Lower Bound for the Optimization of Finite Sums

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Abstract
This paper presents a lower bound for optimizing a finite sum of \( n \) functions, where each function is \( L \)-smooth and the sum is \( \mu \)-strongly convex. We show that no algorithm can reach an error \( \varepsilon \) in minimizing all functions from this class in fewer than \( \Omega(n + \sqrt{n(\kappa - 1)\log(1/\varepsilon)}) \) iterations, where \( \kappa = L/\mu \) is a surrogate condition number. We then compare this lower bound to upper bounds for recently developed methods specializing to this setting. When the functions involved in this sum are not arbitrary, but based on i.i.d. random data, then we further contrast these complexity results with those for optimal first-order methods to directly optimize the sum. The conclusion we draw is that a lot of caution is necessary for an accurate comparison. In interest of completeness, we also provide a self-contained proof of the classical result on optimizing smooth and strongly convex functions under a first-order oracle.

1. Introduction
Many machine learning setups lead to the minimization a convex function of the form

\[
x_f^* = \arg\min_{x \in \Omega} f(x), \quad \text{where} \quad f(x) = \frac{\mu}{2} \|x\|^2 + \frac{1}{n} \sum_{i=1}^{n} g_i(x),
\]

where \( \Omega \) is a convex, compact set. When the functions \( g_i \) are also convex, then the overall optimization problem is convex, and can in principle be solved using any off-the-shelf convex minimization procedure. In the machine learning literature, two primary techniques have typically been used to address such convex optimization problems. The first approach (called the batch approach) uses the ability to evaluate the function \( f \) along with its gradients, Hessian etc. and applies first- and second-order methods to minimize the objective. The second approach (called the stochastic approach) interprets the average in Equation (1) as an expectation and uses stochastic gradient methods, randomly sampling a \( g_i \) and using its gradient and Hessian information as unbiased estimates for those of the function \( f \).\(^1\) Both these classes of algorithms have extensive literature on upper bounds on complexities of specific methods. More fundamentally, there are also lower bound results on the minimum black-box complexity of the best-possible algorithm to solve convex minimization problems. In several broad problem classes, these lower bounds further coincide with the known upper bounds for specific methods, yielding a rather comprehensive general theory for reasoning about convex optimization problems.

However, a recent line of work in the machine learning literature, recognizes that the specific problem (1) of interest has additional structure beyond a general convex minimization problem. For instance, the average in defining the function \( f \) is over a fixed number \( n \) of functions, whereas

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1. There is a body of literature that recognizes the ability of stochastic optimization to minimize testing error rather than training error in machine learning contexts (see e.g. (Bottou and Bousquet, 2008)), but we will focus on training error aspects for this paper.
typical complexity results on stochastic optimization allow for the expectation to be with respect to a continuous random variable, meaning that the finite average turns into an integration over a continuous set. Recent works (Le Roux et al., 2012; Shalev-Shwartz and Zhang, 2013; Johnson and Zhang, 2013) make further assumptions that the functions $g_i$ involved in this sum are smooth, and the function $f$ is of course strongly convex by construction. Under these conditions, the algorithms studied in these works have the following properties: (i) the cost of each iteration is identical to stochastic optimization methods, and (ii) the convergence rate of the method is linear. The results are surprising since the existing lower bounds on stochastic optimization dictate that the error can decrease no faster than $\Omega(1/k)$ after $k$ iterations under such assumptions (Nemirovsky and Yudin, 1983), leaving an exponential gap compared to these new results. It is of course not a contradiction due to the finite sum structure of the problem (1) (following the terminology of Bertsekas (2012), we will call the setup of optimizing a finite sum incremental optimization hereafter).

Given this recent and highly interesting line of work, it is natural to ask just how much better can one do in this model of minimizing finite sums. Put another way, can we specialize the existing lower bounds for stochastic or batch optimization, to yield results for this new family of functions. The aim of such a result would be to understand the fundamental limits on any possible algorithm for this family of problems, and whether better algorithms are possible at all than the existing ones. Answering such questions is the goal of this work. To this end, we define the Incremental First-order Oracle (IFO) complexity model, where an algorithm picks an index $i \in \{1, 2, \ldots, n\}$ and a point $x \in \Omega$ and the oracle returns $g'_i(x)$. We consider the setting where each function $g_i$ is $L$-smooth (that is, it has $L$-Lipschitz continuous gradients). In this setting, we demonstrate that no method can achieve $\|x_K - x^*_f\| \leq \epsilon \|x^*_f\|$ for all functions $f$ of the form (1), without performing $K = \Omega\left(n + \sqrt{n(L/\mu - 1)} \log(1/\epsilon)\right)$ calls to the IFO. As we will discuss following this main result, this lower bound is close to the complexity upper bounds of the recent ASDCA and SPDC methods (Shalev-Shwartz and Zhang, 2014; Zhang and Xiao, 2014). At the same time, there is no method which has a precisely matching upper bound on its complexity, meaning that there is further room for improving either the upper or the lower bounds for this class of problems.

Following the statement of our main result, we will also discuss the implications of these lower bounds, and contrast them with lower bounds for the black-box first-order minimization of smooth and strongly convex functions (Nemirovsky and Yudin, 1983; Nesterov, 2004) for the typical machine learning problems that have inspired this line of work. In particular, we will demonstrate that caution is needed in comparing the results between these two complexity models, and the newly designed methods for the IFO need not be much better (and can sometimes be much worse) than existing optimal algorithms for the first-order black-box setting under typical considerations. We believe that this is an important observation that should be carefully accounted for in designing future algorithms in the IFO model.

As a prerequisite for our result, we need the result on black-box first-order complexity of minimizing smooth and strongly convex functions. The most accessible version of the result is given by Nesterov (2004), who makes additional assumptions on the algorithm. The desired result does appear in the book of Nemirovsky and Yudin (1983), perhaps the most definitive treatment of complexity considerations in convex optimization. Since the book is out of print, we provide a self-contained proof of this result in our paper in Appendix A which might be of independent interest. In fact, we establish a slight variation on the original result, in order to help prove our main result. Our main result will invoke this construction multiple times to design each of the components $g_i$ in the optimization problem (1).
The remainder of this paper is organized as follows. The next section formally describes the complexity model and the structural assumptions. We then state the main result, followed by a discussion of consequences for typical machine learning problems. The proofs are deferred to the subsequent section.

2. Setup and main result

Let us begin by formally describing the class of functions we will study in this paper. For the definition, we first remind the reader of some standard notation. A function \( g \) is called \( L \)-smooth, if it has \( L \)-Lipschitz continuous gradients, that is

\[
\forall x, y \in \Omega \quad \| g'(x) - g'(y) \|_* \leq L \| x - y \|,
\]

where \( \| \cdot \|_* \) is the norm dual to \( \| \cdot \| \). In this paper, we will only concern ourselves with scenarios where \( \Omega \) is a convex subset of a separable Hilbert space, with \( \| \cdot \| \) being the (self-dual) norm associated with the inner product. A function \( g \) is called \( \mu \)-strongly convex if

\[
\forall x, y \in \Omega \quad g(y) \geq g(x) + \langle g'(x), y - x \rangle + \frac{\mu}{2} \| x - y \|^2.
\]

Given these definitions, we now define the family of functions being studied in this paper.

**Definition 1** Let \( \mathcal{F}_{n}^{\mu,L}(\Omega) \) denote the class of all convex functions \( f \) with the form (1), where each \( g_i \) is \( (L - \mu) \)-smooth and convex.

Note that \( f \) is \( \mu \)-strongly convex and \( L \)-smooth by construction, and hence \( \mathcal{F}_{n}^{\mu,L}(\Omega) \subseteq \mathcal{S}_{\mu,L}(\Omega) \) where \( \mathcal{S}_{\mu,L}(\Omega) \) is the set of all \( \mu \)-strongly convex and \( L \)-smooth functions. However, as we will see in the sequel, it can often be a much smaller subset, particularly when the smoothness of the global function is much better than that of the local functions. We now define a natural oracle for optimization of functions with this structure.

**Definition 2 (Incremental First-order Oracle — IFO)** For a function \( f \in \mathcal{F}_{n}^{\mu,L}(\Omega) \), the Incremental First-order Oracle (IFO) takes as input a point \( x \in \Omega \) and index \( i \in \{1, 2, \ldots, n\} \) and returns the pair \((f_i(x), f'_i(x))\).

It is easy to see that there are several algorithms that can be implemented using an IFO. For instance, a standard gradient algorithm would take the current iterate \( x_k \), and invoke the IFO with \((x_k, i)\) in turn with \( i = \{1, 2, \ldots, n\} \), in order to assemble the gradient of \( f \). A stochastic gradient algorithm would take the current iterate \( x_k \) along with a randomly chosen index \( i \) as inputs to IFO. Most interesting to our work, however, are recent algorithms like SAG, SVRG and SDCA (Le Roux et al., 2012; Johnson and Zhang, 2013; Shalev-Shwartz and Zhang, 2013) which can all be implemented with an IFO as well.

Based on an oracle, we now consider optimization algorithms which invoke the oracle \( K \) times, at iterates \( x_0, \ldots, x_{K-1} \) and use the oracle’s responses to finally output an estimate \( x_K \) for the minimizing solution \( x^* \). Our goal is to bound the smallest number of queries \( K \) needed for any method to ensure an error \( \varepsilon \| x^*_i \| \) in \( \| x_K - x^*_i \| \), uniformly for all \( f \in \mathcal{F}_{n}^{\mu,L}(\Omega) \). As is common in the optimization of smooth and strongly convex functions, a condition number like quantity plays the key role in determining the fundamental complexity of this optimization problem. Specifically, for
Consider a black-box optimization algorithm for the problem (1) which accesses \( f \) only using an IFO, and guarantees that \( \| x_f^g \| = \gamma \) and
\[
\| x_f^g - x_k \| \geq \gamma q^2 \quad \text{with} \quad q = \frac{1 + \frac{\kappa - 1}{n} - 1}{1 + \frac{\kappa - 1}{n} + 1}, \quad \kappa = \frac{L}{\mu} \quad \text{and} \quad t = \begin{cases} 0 & \text{if } K < n \\ \frac{K}{n} & \text{otherwise} \end{cases}.
\]

In order to better interpret the result of the theorem, we state the following direct corollary which lower bounds the number of steps need to attain an accuracy of \( \varepsilon \| x_f^g \| \).

**Corollary 1** Consider a black-box optimization algorithm for (1) which accesses \( f \) only using an IFO, and guarantees that \( \| x_f^g - x_k \| \leq \varepsilon \| x^* \| \) for any \( \varepsilon < 1 \). Then there is a function \( f \in \mathcal{F}^\mu_{L}(\ell_2) \) on which the algorithm must perform at least \( K = \Omega(n + \sqrt{n(\kappa - 1)} \log(1/\varepsilon)) \) calls to the IFO.

The first term in the lower bound simply asserts that one needs to make at least one query per \( g_i \), in order to even see each component of \( f \) which is clearly necessary for any optimization method. The second term, which is more important since it depends on the desired accuracy \( \varepsilon \), asserts that the problem becomes harder as the number of elements \( n \) in the sum increases or as the problem conditioning worsens. Again, both these behaviors are qualitatively expected. Indeed as \( n \to \infty \), the finite sum approaches an integral, and the IFO becomes equivalent to a generic stochastic-first order oracle for \( f \), under the constraint that the stochastic gradients are also Lipschitz continuous. Due to \( \Omega(1/\varepsilon) \) complexity of stochastic strongly-convex optimization (with no dependence on \( n \)), we do not expect the linear convergence of Corollary 1 to be valid as \( n \to \infty \). Also, we certainly expect the problem to get harder as the ratio \( L/\mu \) degrades. Indeed if all the functions \( g_i \) were identical, whence the IFO becomes equivalent to a standard first-order oracle, the optimization complexity similarly depends on \( \Omega(\sqrt{\kappa - 1} \log(1/\varepsilon)) \).

Whenever presented with a lower bound, it is natural to ask how it compares with the upper bounds for existing methods. We now compare our lower bound to upper bounds for standard optimization schemes for \( \mathcal{S}^\mu_{L}(\ell_2) \) as well as specialized ones for \( \mathcal{F}^\mu_{L}(\ell_2) \). We specialize to \( \| x_f^g \| = 1 \) for this discussion.

**Comparison with optimal gradient methods:** As mentioned before, \( \mathcal{F}^\mu_{L}(\ell_2) \subseteq \mathcal{S}^\mu_{L}(\ell_2) \), and hence standard methods for optimization of smooth and strongly convex objectives apply. These methods need \( n \) calls to the IFO for getting the gradient of \( f \), followed by an update. Using Nesterov’s optimal gradient method (Nesterov, 2004), one needs at most \( \mathcal{O}(\sqrt{\kappa} \log(1/\varepsilon)) \) gradient evaluations to reach \( \varepsilon \)-optimal solution for \( f \in \mathcal{S}^\mu_{L}(\ell_2) \), resulting in at most \( \mathcal{O}(n \sqrt{\kappa} \log(1/\varepsilon)) \) calls to the IFO. Comparing with our lower bound, there is a suboptimality of at most \( \mathcal{O}(\sqrt{n}) \) in this result.
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Since this is also the best possible complexity for minimizing a general \( f \in S^H_L(\ell_2) \), we conclude that there might indeed be room for improvement by exploiting the special structure here. Note that there is an important caveat in this comparison. For \( f \) of the form (1), the smoothness constant for the overall function \( f \) might be much smaller than \( L \), and the strong convexity term might be much higher than \( \mu \) due to further contribution from the \( g_i \). In such scenarios, the optimal gradient methods will face a much smaller condition number \( \kappa \) in their complexity. This issue will be discussed in more detail in Section 3.

Comparison with the best known algorithms: Two algorithms recently developed for problem setting (1) offer complexity guarantees that are close to our lower bound. The SPDC algorithm (Zhang and Xiao, 2014) reaches an optimization error \( \varepsilon \) after less than \( \mathcal{O}((n + \sqrt{n(\kappa - 1)})) \log(1/\varepsilon) \) calls to the oracle. The ASDCA algorithm (Shalev-Shwartz and Zhang, 2014) reaches the same bound up to some additional logarithmic factors.

Note that the first term of \( n \) crucially multiplies the \( \log(1/\varepsilon) \) term, rather than being additive as in our lower bound. Consequently, these bounds are always larger than the lower bound. As we can see, the most significant gap between the upper and the lower bound is the presence of the factor of \( n \) additively or multiplicatively with the \( \log(1/\varepsilon) \) term, indicating that there is room to either improve the lower bound, or for algorithms with a better complexity.

Room for better lower bounds? One natural question to ask is whether the gap between the lower bounds and existing upper bounds reveals a room for better algorithms, or is it due to the weakness of the lower bound. As will become clear from the proof, a better lower bound is not possible for the hard problem instance which we construct. The structure of our lower bound instance is such that it is a quadratic problem, with each \( g_i \) acting on a disjoint subset of coordinates. The precise structure of these \( g_i \) is identical to the worst case functions in smooth and strongly convex optimization, the construction of which is shown in Appendix A. In this setting, one can use the conjugate gradient method to optimize each \( g_i \) independently, since the IFO boils down to have \( n \) disjoint first-order oracles in this case. Known results on the convergence of the conjugate gradient algorithm applied to quadratic objectives (see e.g. Shewchuk (1994), Equation (52)) yield a nearly matching upper bound (up to a \( \log(n) \) factor) for solving our lower bound instance. As a result, one cannot hope for a better lower bound without modifying the construction.

3. Consequences for optimization in machine learning

With all the relevant results in place now, we will compare the efficiency of the different available methods in the context of solving typical machine learning problems. Recall the definitions of the constants \( L \) and \( \mu \) from before. In general, the full objective \( f \) (1) has its own smoothness and strong convexity constants, which need not be the same as \( L \) and \( \mu \). To that end, we define \( L_f \) to be the smoothness constant of \( f \), and \( \mu_f \) to the strong convexity of \( f \). It is immediately seen that \( L \) provides an upper bound on \( L_f \), while \( \mu \) provides a lower bound on \( \mu_f \).

In order to provide a meaningful comparison for incremental as well as batch methods, we follow Zhang and Xiao (2014) and compare the methods in terms of their batch complexity, that is, how many times one needs to perform \( n \) calls to the IFO in order to ensure that the optimization

2. The authors in ASDCA state the upper bound as \( \tilde{\mathcal{O}}((n + \min(n(\kappa - 1))) \log(1/\varepsilon)) \), with \( \tilde{\mathcal{O}} \) suppressing log factors in \( n, \kappa \) and \( \log(1/\varepsilon) \). However, the minimum under \( \tilde{\mathcal{O}} \) does not matter since \( n + \kappa - 1 \geq 2\sqrt{n(\kappa - 1)} \) by AM-GM inequality, so that the SPDC bound is just as good up to constant factors.
| Algorithm          | Batch complexity                                           | Adaptive?  |
|--------------------|-----------------------------------------------------------|------------|
| ASDCA, SDPC (Shalev-Shwartz and Zhang, 2014) | $\tilde{O}\left(\left(1 + \sqrt{\frac{L - \mu}{\mu n}}\right) \log \frac{1}{\epsilon}\right)$ | no         |
| (Zhang and Xiao, 2014) |                                                            |            |
| SAG (Schmidt et al., 2013) | $\tilde{O}\left(\left(1 + \frac{L}{\mu n}\right) \log \frac{1}{\epsilon}\right)$ | to $\mu_f$ |
| AGM† (Nesterov, 2007) | $\tilde{O}\left(\sqrt{\frac{L}{\mu_f}} \log \frac{1}{\epsilon}\right)$ | to $\mu_f$ and $L_f$ |

Table 1: A comparison of the batch complexities of different methods. A method is adaptive to $\mu_f$ or $L_f$, if it does not need the knowledge of these parameters to run the algorithm and obtain the stated complexity upper bound. †Although the simplest version of AGM does require the specification of $\mu_f$ and $L_f$, Nesterov also discusses an adaptive variant with the same bound up to additional logarithmic factors.

error for the function $f$ is smaller than $\epsilon$. When defining batch complexity, Zhang and Xiao (2014) observed that the incremental and batch methods have dependence on $L$ versus $L_f$, but did not consider the different strong convexities that play a part for different algorithms. Doing so, we see that the batch complexities can be summarized as in Table 1.

Based on the table, we see two main points of difference. First, the incremental methods rely on the smoothness of the individual components. That this is unavoidable is clear, since even the worst case lower bound of Theorem 1 depends on $L$ and not $L_f$. As Zhang and Xiao (2014) observe, $L_f$ can in general be much smaller than $L$. They attempt to address the problem to some extent by using non-uniform sampling, thereby making sure that the best of the $g_i$ and the worst of the $g_i$ have a similar smoothness constant under the reweighting. This does not fully bridge the gap between $L$ and $L_f$ as we will show next. However, more striking is the difference in the lower curvature across methods. To the best of our knowledge, all the existing analyses of coordinate ascent require a clear isolation of strong convexity, as in the function definition (1). These methods then rely on using $\mu$ as an estimate of the curvature of $f$, and cannot adapt to any additional curvature when $\mu_f$ is much larger than $\mu$. Our next example shows this can be a serious concern for many machine learning problems.

In order to simplify the following discussion we restrict ourselves to perhaps the most basic machine learning optimization problem, the regularized least-squares regression, also known as ridge regression:

$$f(x) = \frac{\mu}{2} \|x\|^2 + \frac{1}{n} \sum_{i=1}^{n} g_i(x) \quad \text{with} \quad g_i(x) = (\langle a_i, x \rangle - b_i)^2,$$

(2)

where $a_i$ is a data point and $b_i$ is a scalar target for prediction. It is then easy to see that $g''_i(x) = a_i a_i^\top$ so that $f \in T_{n,L}^{\mu} (\Omega)$ with $L = \max_i (\mu + \|a_i\|^2)$. To simplify the comparisons, assume that $a_i \in \mathbb{R}^d$ are drawn uniformly from the sphere $\|a_i\| = R$. Therefore $\mathbb{E}[a_i] = 0$, $\mathbb{E}[a_i a_i^\top] = \Sigma = (R^2/d) I$ where $I$ is the identity matrix, and $L = \mu + R^2$. Since each function $g_i$ has the same smoothness constant in
our example, the importance sampling techniques (Zhang and Xiao, 2014) cannot hope to do better under these assumptions.

In order to follow the development of Table 1, we need to evaluate the constants \( L_f \) and \( \mu_f \) in this setup. Note that in this special case, the constants \( L_f \) and \( \mu_f \) are given by the upper and lower eigenvalues respectively of the matrix \( \mu I + \hat{\Sigma} \), where \( \hat{\Sigma} = \sum_{i=1}^{n} a_i a_i^\top / n \) represents the empirical covariance matrix. In order to understand the scaling of this covariance matrix, we invoke a standard result on matrix concentration (Vershynin, 2012, Corollary 5.52):

\[
\lambda_{\min}(\Sigma) - c \sqrt{\frac{\|\Sigma\| R^2}{n} \log \frac{1}{\delta}} \leq \lambda(\hat{\Sigma}) \leq \lambda_{\max}(\Sigma) + c \sqrt{\frac{\|\Sigma\| R^2}{n} \log \frac{1}{\delta}},
\]

with probability at least \( 1 - \delta \) for a universal constant \( c \). Recalling our choice of \( \Sigma \),

\[
\frac{R^2}{d} - \frac{cR^2}{\sqrt{dn} \sqrt{\log(1/\delta)}} \leq \lambda(\hat{\Sigma}) \leq \frac{R^2}{d} + \frac{cR^2}{\sqrt{dn} \sqrt{\log(1/\delta)}}.
\]

Consequently, we can now specialize the results of Table 1 to this setup by using

\[
L_f = \mu + \frac{R^2}{d} + \frac{cR^2}{\sqrt{dn} \sqrt{\log(1/\delta)}}, \quad \text{and} \quad \mu_f = \mu + \frac{R^2}{d} - \frac{cR^2}{\sqrt{dn} \sqrt{\log(1/\delta)}}.
\]

In order to succinctly compare algorithms, we define the quantity

\[
\kappa_i \triangleq \frac{R^2}{\mu} = \frac{L - \mu}{\mu} = \kappa - 1,
\]

and we use the notation \( \Gamma_{\text{ALG}} \) to represent the batch complexity of ALG without the \( \log(1/\epsilon) \) term, which is common across all methods. Then we see that the upper bound for \( \Gamma_{\text{ASDCA}} \) is

\[
\Gamma_{\text{ASDCA}} \leq 1 + 2 \kappa_i.
\]

To discuss the batch complexities of SAG and AGM, we further assume that the ratio \( n/d \) satisfies \( n/d \geq 4c^2 \log(1/\delta) \) where \( c \) is the same universal constant as in Equation 3, thereby ensuring that \( \lambda_{\min}(\hat{\Sigma}) > 0 \). We will discuss the high-dimensional settings where this might not hold later in this section. It is readily seen that this condition on \( n/d \) ensures \( \mu_f \geq \mu + R^2/(2d) \), that is, the loss term indeed contributes added strong convexity to the global function. The batch complexity of SAG can now be simplified as

\[
\Gamma_{\text{SAG}} \leq 1 + \frac{L}{\mu_f n} \leq 1 + \frac{1}{n} \frac{L}{\mu + R^2/(2d)} = 1 + \frac{1}{n} \frac{1 + \kappa_i}{1 + \kappa_i/(2d)}.
\]

Finally, we discuss the complexity of AGM, whence the smoothness constant \( L_f \) can also be upper bounded by \( \mu + 3R^2/(2d) \) under our assumption on the ratio \( n/d \). Noting that error reduction in accelerated gradient schemes (AGM) depends on \( \sqrt{\kappa_f} \), we obtain

\[
\Gamma_{\text{AGM}} \leq \sqrt{\kappa_f} \leq \sqrt{\frac{\mu + R^2}{\mu + \frac{R^2}{2d}}} = \sqrt{\frac{1}{1 + \frac{2\kappa_i}{2d + \kappa_i}}},
\]

Table 2 compares these methods depending on the value of \( \kappa_i \).
Algorithm Moderate $\kappa_i$ ($\kappa_i = \mathcal{O}(n)$) Large $\kappa_i$ ($\kappa_i \gg n$) 

| ASDCA, SPDC (Eq. (4)) | $\mathcal{O}\left(\log \frac{1}{\varepsilon}\right)$ | $\mathcal{O}\left(\sqrt{n} \log \frac{1}{\varepsilon}\right)$ |
|------------------------|---------------------------------|-------------------------------|
| SAG (Eq. (5))          | $\mathcal{O}\left(\log \frac{1}{\varepsilon}\right)$ | $\mathcal{O}\left(\log \frac{1}{\varepsilon}\right)$ |
| AGM (Eq. (6))          | $\mathcal{O}\left(\log \frac{1}{\varepsilon}\right)$ | $\mathcal{O}\left(\log \frac{1}{\varepsilon}\right)$ |

Table 2: A comparison of the batch complexities of different methods for the regularized least squares objective (2) when $n/d \geq 4c^2 \log(1/\varepsilon)$. Observe how the complexity of ASDCA continues to grow with $\kappa_i$, while those for SAG and AGM do not when $\kappa \gg n$. For small and moderate $\kappa_i$, all algorithms show comparable complexities.

**Problems with $\kappa_i = \mathcal{O}(n)$:** This setting is quite interesting for machine learning. Yet the analysis does not show any benefit in using incremental algorithms over a fully adaptive accelerated gradient. To understand this counter-intuitive result, consider how the complexity of incremental algorithms depend on the ratios $R^2/\mu$ (ASDCA, SPDC) or $L/\mu_f$ (SAG), which potentially grow with $n$, thereby negating the gains these methods have in cheaper complexity of each update. On the other hand, the complexity of the adaptive AGM depends on the ratio $L_f/\mu_f$, which converges towards the condition number of $\Sigma$, a quantity independent of $n$ and specifically equal to 1 in our example.\(^3\)

**Problems with large $\kappa_i$:** In this setting, the coordinate ascent methods seem to be at a disadvantage, because the average loss term provides additional strong convexity, which is exploited by both SAG and AGM, but not by ASDCA or SPDC methods. Indeed, we find that the complexity term $\Gamma_{\text{ASDCA}}$ can be made arbitrarily large as $\kappa_i$ grows large. However, the contraction factors for both SAG and AGM are still $\mathcal{O}(1)$ in this setting, leading to a large gap between the complexities.

**High-dimensional settings ($n/d \ll 1$):** In this setting, the global strong convexity can not really be larger than $\mu$ for the function (2), since the Hessian of the averaged loss has a non-trivial null space. It would appear then, that SAG is forced to use the same problem dependent constants as ASDCA/SPDC, while AGM gets no added benefit in strong convexity either. In this case, the comparison just depends on the difference in smoothness constants, which is already discussed in detail in Zhang and Xiao (2014). We will point out an important caveat in this reasoning though. In such high-dimensional problems, one is often enforcing a low-dimensional structure in machine learning settings for generalization. Examples of these low-dimensional structures are sparse vectors and low-rank matrices. In such structures, the global Hessian matrix can still satisfy restricted versions of strong convexity and smoothness conditions, which are often sufficient for batch optimization methods to succeed (Agarwal et al., 2012). In such situations, the comparison might once again resemble that of Table 2, and we leave such development to the reader.

In a nutshell, the superiority of incremental algorithm for the optimization of machine learning training error is far more subtle than suggested by their worst case bounds. Among the incremental algorithms, SAG has favorable complexity results in all regimes despite the fact that both ASDCA

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\(^3\) In general, the condition number of $\Sigma$ might have a dependence on $d$ for particularly ill-conditioned problems even at the population level.
and SPDC offer better worst case bounds. Thanks to its adaptive properties, AGM matches the complexity of the incremental algorithms, despite the fact that the latter methods offer better worst-case bounds on problems from the class $F^\mu_n(\Omega)$. It is worth observing that all our comparisons are ignoring constants, and in some cases logarithmic factors, which of course play a role in the running time of the algorithms in practice. In the regimes where the complexities of different methods is comparable in Table 2, perhaps the most satisfactory comparison can only be done empirically on a large testbed of diverse problems. Note also that the worst case bounds for the incremental methods account for the worst possible choice of the $n$ functions in the sum. Better results might be possible when they are based on i.i.d. random data. Such results would be of great interest within the machine learning community.

The main goal of this discussion is to draw attention to the fact that batch first-order methods and incremental gradient methods face very different problem dependent constants in their complexity results (and this is not uniform even across all incremental methods). An accurate comparison requires a lot of caution, and some efforts in this direction were made in a couple of the previous works (Schmidt et al., 2013; Zhang and Xiao, 2014). But the focus was mostly on the difference in the smoothness constants across methods. As our analysis above highlights, even the strong convexity constants are different for different methods.

4. Proof of main result

In this section, we outline and provide the proof of Theorem 1. Our high-level strategy is the following. We will first construct the function $f : \ell_2 \rightarrow \mathbb{R}$ such that each $g_i$ acts on only the projection of a point $x$ onto a smaller basis, with the bases being disjoint across the $g_i$. Since the $g_i$ are separable, we then demonstrate that optimization of $f$ under an IFO is equivalent to the optimization of each $g_i$ under a standard first-order oracle. The functions $g_i$ will be constructed so that they in turn are smooth and strongly convex with appropriate constants. Hence, we can invoke the known result for the optimization of smooth and strongly convex objectives under a first-order oracle, obtaining a lower bound on the complexity of optimizing $f$. In the following sections, we will formalize this intuitive sketch.

4.1 Construction of a separable objective

We start with a simple definition in order to state things concisely.

**Definition 3** Let $e_1, e_2, \ldots$ denote the canonical basis vectors of $\ell_2$, and let $Q_i$, $i = 1 \ldots n$, denote the orthonormal families $Q_i = [e_1, e_{n+i}, e_{2n+i}, \ldots, e_{kn+i}, \ldots]$.

For ease of presentation, we also extend the transpose notation for matrices over operators in $\ell_2$ in the natural manner (to avoid stating adjoint operators each time).

**Definition 4** Given a finite or countable orthonormal family $S = [s_1, s_2, \ldots] \subset \ell_2$ and $x \in \ell_2$, let

$$Sx = \sum_{i=1}^{\infty} x[i]s_i \quad \text{and} \quad S^Tx = (\langle s_i, x \rangle)_{i=1}^{\infty},$$

where $s_i$ is assumed to be zero when $i$ is greater than the size of the family.
Remark 1 Both $Sx$ and $S^T x$ are square integrable and therefore belong to $\ell_2$.

Using the above notation, we first establish some simple identities for the operators $Q_i$ defined above.

**Lemma 1** Simple calculus yields the following identities:

$$Q_i^T Q_i = I \quad \sum_{i=1}^{n} Q_i Q_i^T = I \quad \|Q_i x\|^2 = \|x\|^2 \quad \sum_{i=1}^{n} \|Q_i^T x\|^2 = \|x\|^2.$$ 

**Proof** We start with the first claim. For any basis vector $e_j$, it is easily checked that $Q_i e_j = e_{(j-1)\bmod n}$. By definition of $Q_i^T$, it further follows that $Q_i^T e_{(j-1)\bmod n} = e_j$. Linearity now yields $Q_i^T Q_i x = x$ for any $x \in \ell_2$, giving the first claim. For the second claim, we observe that $Q_i Q_i^T e_j = 0$ unless $\bmod(j, n) = i$, in which case $Q_i Q_i^T e_j = e_j$. This implies the second claim. The third claim now follows from the first one, since $\langle Q_i x, Q_i x \rangle = \langle x, Q_i^T Q_i x \rangle = \|x\|^2$. Similarly the final claim follows from the second claim.

We now define the family of separable functions that will be used in order to establish our lower bound.

$$f(x) = \frac{\mu}{2} \|x\|^2 + \frac{1}{n} \sum_{i=1}^{n} h_i(Q_i^T x) \quad \text{where } h_i(x) \in S^{0, L-\mu}(\ell_2). \quad (7)$$

The first thing to check is that these functions even belong to $S^{0, L}(\ell_2)$, which we establish next.

**Proposition 1** All functions of the form (7) belong to $S^{0, L}(\ell_2)$.

**Proof** We simply need to prove that the functions $g_i(x) = h_i(Q_i^T x)$ belong to $S^{0, L-\mu}(\ell_2)$. Using $g_i'(x) = Q_i h_i'(Q_i^T x)$ and lemma 1, we can write $$\|g_i'(x) - g_i'(y)\|^2 = \|Q_i (h_i'(Q_i^T x) - h_i'(Q_i^T y))\|^2 = \|h_i'(Q_i^T x) - h_i'(Q_i^T y)\|^2 \leq (L-\mu)^2 \|Q_i^T (x-y)\|^2 \leq (L-\mu)^2 \|x-y\|^2.$$ 

4.2 Decoupling the optimization across components

We will now like to assert that the separable structure of $f$ allows us to reason about optimizing the different $h_i$ separately. This would, however, not be very convenient since the $h_i$ are not strongly convex by themselves. As a result, we first do an equivalent rewriting of $f$ into a more convenient structure. Observe that functions of the form (7) can be rewritten as

$$f(x) = \frac{\mu}{2} \|x\|^2 + \frac{1}{n} \sum_{i=1}^{n} h_i(Q_i^T x) = \frac{\mu}{2} \sum_{i=1}^{n} \|Q_i^T x\|^2 + \frac{1}{n} \sum_{i=1}^{n} h_i(Q_i^T x)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{\mu}{2} \|Q_i^T x\|^2 + h_i(Q_i^T x) \right] \Delta \frac{1}{n} \sum_{i=1}^{n} f_i(Q_i^T x),$$

where we again invoked Lemma 1. By construction, the functions $f_i$ belong to $S^{n \mu, L-\mu + n \mu}$ and are applied to disjoint subsets of the $x$ coordinates. Therefore, when the function is known to have form (7), problem (1) can be written as

$$x^* = \sum_{i=1}^{n} Q_i x_i^* \quad x_i^* = \underset{x \in \ell_2}{\arg \min} f_i(x), \quad \text{where } f_i(x) = \frac{\mu}{2} \|x\|^2 + h_i(x). \quad (8)$$
Any algorithm that solves optimization problem (1) therefore has the side effect of solving all the problems listed in (8).

We are almost done, but for one minor detail. Note that we want to obtain a lower bound where the IFO is invoked for a pair \((i, x)\) and responds with \((h_i(Q_i^T x), \partial h_i(Q_i^T x)/\partial x)\). In order to claim that this suffices to optimize each \(f_i\) separately, we need to argue that a first-order oracle for \(f_i\) can be obtained from this information, knowing solely the structure of \(f\) and not the functions \(h_i\). Since the strong convexity constant \(\mu\) is assumed to be known to the algorithm, the additional \((n\mu/2)||x||^2\) part in defining \(f_i\) from \(h_i\) is also known to the algorithm. As a result, using just the structure of \(f\), a first-order oracle for \(f_i\) is obtained by returning \((h_i(Q_i^T x) + (n\mu/2)||Q_i^T x||^2, \partial h_i(Q_i^T x)/\partial x + n\mu Q_iQ_i^T x)\). Hence, an IFO for \(f\) can be used to simulate the response of a first-order oracle for the \(f_i\) depending on the index \(i\) used to query the IFO. Furthermore, an IFO invoked with the index \(i\) reveals no information about \(f_j\) for any other \(j\) based on the separable nature of our problem. Hence, the IFO for \(f\) offers no additional information beyond having a standard first-order oracle for each \(f_i\). Based on these details, we are now ready to prove Theorem 1.

### 4.3 Proof of Theorem 1

Based on the discussion above, we can now pick any \(i \in \{1 \ldots n\}\) and view our algorithm as a complicated setup whose sole purpose is to optimize function \(f_i \in S^\mu.\ell_2\). Indeed, given the output \(x_K\) of an algorithm using an IFO for the function \(f\), we can declare \(x^*_K = Q_i^T x_K\) as our estimate for \(x^*_i\). It is easy to check based on Lemma 1 that

\[
\|x_K - x^*_i\|^2 = \sum_{i=1}^{n} \|Q_i^T(x_K - x^*_i)\|^2 = \sum_{i=1}^{n} \|x^*_K - x^*_i\|^2.
\]

In order to establish the theorem, we now invoke the classical result on the black-box optimization of functions using a first-order oracle. The specific form of the result stated here is proved in Appendix A.

**Theorem 2 (Nemirovsky-Yudin)** Consider a first order black box optimization algorithm for problem (9) that performs \(K \geq 0\) calls to the oracle and returns an estimate \(x_K\) of the minimum. For any \(\gamma > 0\), there exists a function \(f \in S^{\mu.\ell_2}\) such that \(\|x^*_i\| = \gamma\) and

\[
\|x^*_i - x_K\| \geq \gamma q^{2K} \text{ with } q = \frac{\sqrt{K} - 1}{\sqrt{K} + 1} \text{ and } K = \frac{L}{\mu}.
\]

At a high-level, our oracle will make an independent choice of one of the functions that witness the lower bound in Theorem 2 for each \(f_i\). The details on the construction of these functions can be found in Appendix A.

Suppose the IFO is invoked \(K_i\) times on index \(i\), so that \(K = K_1 + K_2 + \ldots + K_n\). We first establish the theorem for the case when \(K < n\). The basic idea is that unless the algorithm sees queries each function at least once, the oracle can declare a really bad \(f_i\) for the algorithm’s response \(x_K\) after the fact, and hence no non-trivial accuracy is possible in this case.

**Proposition 2** Under conditions of Theorem 1, suppose further that \(K < n\). Then there exists an oracle such that for any algorithm, there is a function \(f \in F^\mu.\ell_2\) such that we have the lower bound \(\|x^*_i - x_K\| \geq \gamma\).
Proof  Since $K < n$, there is at least one function $f_i$ for which $K_i = 0$. Since the oracle knows $K$ in advance, it can answer any queries on $f_j$ for $j \neq i$ with $\gamma_j = 0$ (this is done by taking the function $f$ from Theorem 2 which attains the lower bound with $\gamma = 0$). Finally, the algorithm declares $x_K$. Since information on $f_i$ was never revealed, the oracle can declare $f_i$ after inspecting $x_i^K$ such that $\|x_i^K - x_i^*\| \geq \|x_i^*\| = \gamma_i$. Since $x_i^*$ is the only part of $x^*$ which is non-zero, we also get $\gamma_i = \gamma$, which completes the proof.

For the remainder, we can assume without loss of generality that $K_i > 0$ for each $i$. As a particular consequence, we see that this ensures $K \geq n$. Then we can appeal to Theorem 2 for each $f_i$ in turn to obtain

$$\|x_K - x_i^*\|^2 = \sum_{i=1}^{n} \|x_K^i - x_i^*\|^2 \geq \sum_{i=1}^{n} \gamma_i^2 q^{4K_i} \geq \gamma^2 \sum_{i=1}^{n} \gamma_i^2 q^{4K_i}/n,$$

where the last inequality is a consequence of Jensen’s inequality applied to the convex function $q^{2\alpha}$ for $\alpha \geq 1$. Finally, we note that the oracle has no way to discriminate amongst the $\gamma_i$ values when $K_i > 0$, so that it will end up setting $\gamma_i = \gamma / \sqrt{n}$. With this setting, we now obtain the lower bound

$$\|x_K - x_i^*\|^2 \geq \gamma^2 q^{4K/n},$$

for $K > n$, along with $\|x_K - x_i^*\|^2 \geq \gamma^2/2$ for $K < n$.

This completes the proof of the Theorem. In order to further establish Corollary 1, we need an additional technical lemma.

Lemma 2  $\forall x > 1$, $\log\left(\frac{\sqrt{x} - 1}{\sqrt{x} + 1}\right) > \frac{-2}{\sqrt{x} - 1}$.

Proof  The function $\phi(x) = \log\left(\frac{\sqrt{x} - 1}{\sqrt{x} + 1}\right) + \frac{2}{\sqrt{x} - 1}$ is continuous and decreasing on $(1, +\infty)$ because

$$\phi'(x) = \frac{1}{(\sqrt{x} - 1)(\sqrt{x} + 1) \sqrt{x}} - \frac{1}{(x - 1) \sqrt{x} - 1} = \frac{1}{(x - 1) \sqrt{x}} - \frac{1}{(x - 1) \sqrt{x} - 1} < 0.$$ The result follows because $\lim_{x \to \infty} \phi(x) = 0$. ■

Now we observe that we have at least $n$ queries due to the precondition $\varepsilon < 1$ and Proposition 2, which yields the first term in the lower bound. Based on Theorem 1 and this lemma, the corollary is now immediate.

5. Discussion

The results in this paper were motivated by recent results and optimism on exploiting the structure of minimizing finite sums, a problem which routinely arises in machine learning. Our main result provides a lower bound on the limits of gains that might be possible in this setting, allowing us to do a more careful comparison of this setting with regular first-order black box complexity results.
As discussed in Section 3, the results seem mixed when the sum consists of \( n \) functions based on random data drawn i.i.d. from a distribution. In this setting, we find that in some interesting regimes, the methods specializing to the IFO can do no better than optimal gradient methods. In general, we observe that the problem dependent constants that appear in different methods can be quite different, even though this is not always recognized. We believe that accounting for these opportunities might open door to more interesting algorithms and analysis.

Of course, there is another and a possibly more important aspect of optimization in machine learning which we do not study in this paper. In typical machine learning problems, the goal of optimization is not just to minimize the objective \( f \)—usually called the training error—to a numerical precision. In most problems, we eventually want to reason about test error, that is the accuracy of the predictions we make on unseen data. There are existing results (Bottou and Bousquet, 2008) which highlight the optimality of single-pass stochastic gradient optimization methods, when test error and not training error is taken into consideration. So far, we do not have any clear results comparing the efficacy of methods designed for the problem (1) in minimizing test error directly. We believe this is an important question for future research, and one that will perhaps be most crucial eventually for the adoption of these methods in the machine learning community.

We believe that there are some important open questions for future works in this area, which we will conclude with:

1. Is it possible to obtain better complexity upper bounds when the \( n \) functions involved in the sum (1) are based on random data, rather than being \( n \) arbitrary functions? In particular, can the incremental methods exploit global rather than local smoothness properties in this setting?

2. Can the coordinate ascent methods be adaptive to the global strong convexity, rather than needing its isolation explicitly?

3. What are the test error properties of incremental methods for machine learning problems? Specifically, can one do better than just adding up the optimization and generalization errors, and follow a more direct approach as the stochastic optimization literature?

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Appendix A. Optimization of a strongly convex smooth functions

The most accessible derivation of this classic lower bound Nesterov (2004) relies on the simplifying assumption that the successive points \( x_k \) lie in the span of the gradients previously returned by the oracle. This section provides a derivation of the lower bound that does not rely on this assumption and is critical for Theorem 1 where no such assumptions are made.

This section considers algorithms that produce an approximate solution of the optimization problem

\[
x_f^* = \arg\min_{x \in \ell_2} f(x) = \frac{\mu}{2} \|x\|^2 + g(x) \quad \text{where} \quad f(x) \in S^{\mu,L}(\ell_2)
\]

using, as sole knowledge of function \( f \), an oracle that returns the value \( f(x) \) and the gradient \( f'(x) \) on points successively determined by the algorithm. Note that this writing of \( f \) is without loss of generality, since any \( \mu \)-strongly convex function can be written in the form (9) where \( g \) is convex.

**Remark 2** We could equivalently consider an oracle that reveals \( g(x_k) \) and \( g'(x_k) \) instead of \( f(x_k) \) and \( f'(x_k) \) because these quantities can be computed from each other (since \( \mu \) is known.)

At a high-level, our proof will have the following structure. We will first establish that any algorithm for solving the minimization problem (9) for all \( f \in S^{\mu,L}(\ell_2) \) will be forced to play the point \( x_K \) in the span of the previous iterates and gradients. This essentially shows that the restriction made by Nesterov is not too serious. The second part of the proof constructs a resisting oracle for such algorithms whose final query point falls within the span of the previous responses. Combining these ingredients, we obtain the desired lower bound.

A.1 Restriction of final solution to span

Consider an algorithm that calls the oracle on \( K > 1 \) successive points \( x_0, \ldots, x_{K-1} \). The first part of the proof describes how to pick the best possible \( x_K \) on the basis of the oracle answers and the algorithm’s queries.

**Definition 5** For any \( \gamma \geq 0 \), let \( \mathcal{S}^{\mu,L}_\gamma(\ell_2) \) be the set of all functions \( f \in S^{\mu,L}(\ell_2) \) that reach their minimum in a point \( x_f^\gamma \) such that \( \|x_f^\gamma\| = \gamma \).

**Definition 6** Let \( \mathcal{G}^f_\gamma \subset \mathcal{S}^{\mu,L}_\gamma(\ell_2) \) be the set of the functions of \( \mathcal{S}^{\mu,L}_\gamma(\ell_2) \) whose values and gradients coincide with those of \( f \) on points \( x_0 \ldots x_{K-1} \). Let \( H_f^\gamma \in \ell_2 \) be the set of their minima.

When the function \( f \) is clear from the context, we will drop the superscript for brevity. Since all functions in \( \mathcal{G} \) are compatible with the values returned by the oracle, running our algorithm on any of them would perform the same calls to the oracle and obtain the same answers. Therefore, in order to offer the best guarantee on \( \|x_K - x_f^\gamma\|^2 \) without further knowledge of the function \( f(x) \), our algorithm must choose \( x_K \) to be the center of the smallest ball containing \( H_f^\gamma \).

**Definition 7** Let \( P = \text{Span}\{x_0, \ldots, x_{K-1}, f'(x_0), \ldots, f'(x_{K-1})\} \). Let \( \Pi_P(x) \) be the orthogonal projection of point \( x \) on \( P \) and let \( M_P(x) = 2\Pi_P(x) - x \) be its mirror image with respect to \( P \).

Stated differently, we know that any point \( x \) can be decomposed into \( \Pi_P(x) \) and \( \Pi_{P^\perp}(x) \) such that \( x = \Pi_P(x) + \Pi_{P^\perp}(x) \). Then the above definition yields \( M_P(x) = \Pi_P(x) - \Pi_{P^\perp}(x) \), which is the natural reflection of \( x \) with respect to the subspace \( P \).
Proposition 3  The set $H_f$ is symmetric with respect to $P$.

Proof  Consider an arbitrary point $x^*_h \in H_f$ which minimizes a function $h \in \mathcal{G}_f$. Since function $x \mapsto h(M_P(x))$ also belongs to $\mathcal{G}_f$, its minimum $M_P(x^*_h)$ also belongs to $H_f$. \hfill \blacksquare \\

Corollary 2  The center of the smallest ball enclosing $H_f$ belongs to $P$.

We are now in a position to present the main ingredient of our proof that allows us to state a more general result than Nesterov. In particular, we demonstrate that the assumption made by Nesterov about the iterates lying in the span of previous gradients can be made almost without loss of generality. The key distinction is that we can only make it on the step $K$, where the algorithm is constrained to produce a good answer, while Nesterov assumes it on all iterates, somewhat restricting the class of admissible algorithms.

Lemma 3  For any $\gamma > 0$ and any algorithm $A$ that performs $K \geq 1$ calls of the oracle and produces an approximate solution $x^*_K(f)$ of problem (9), there is an algorithm $B$ that performs $K$ calls of the oracle and produces an approximate solution $x^*_K(f) \in \text{Span}\{x_0, \ldots, x_{K-1}, f'(x_0) \ldots f'(x_{K-1})\}$ for all $f \in S^L_\gamma(\ell_2)$ such that

$$
\sup_{f \in S^L_\gamma(\ell_2)} \|x^B_K - x^*_f\|^2 \leq \sup_{f \in S^L_\gamma(\ell_2)} \|x^A_K - x^*_f\|^2.
$$

Proof  Suppose for contradiction that there is an algorithm $A$ for which the assertion is false. Consider the algorithm $B$ which runs algorithm $A$ for any $f$ and then returns the center of the smallest ball enclosing $H_f$ as $x^B_K$. Corollary 2 ensures that $x^B_K$ belongs to $P$, and hence falls within the span as posited. Let $x^A_K = x^B_K + \delta$, with $\delta \neq 0$. Let $\bar{x} = \arg\max_{x \in H_f} \|x - x^A_K\|$. By definition of $x^B_K$, we know that $\|x^B_K - \bar{x}\| \leq \|x^A_K - \bar{x}\|$. Furthermore, we know that $\bar{x} = \arg\min_f \bar{f}(x)$ for some $\bar{f} \in \mathcal{G}_f$. Hence, the oracle can declare $\bar{f}$ to be the function being optimized, which is consistent with all the information revealed to the algorithm, when $A$ declares its solution $x^A_K$. Stated differently, we have demonstrated that $\sup_{f \in \mathcal{G}_f} \|x^B_K - x^*_f\|^2 \leq \sup_{f \in \mathcal{G}_f} \|x^A_K - x^*_f\|^2$ for any subset $\mathcal{G}_f^\gamma \subset S^L_\gamma(\ell_2)$ of functions, where the oracle responses were based on the function $g$. Since this assertion holds for every $g \in S^L_\gamma(\ell_2)$, this completes the proof. \hfill \blacksquare \\

Lemma 3 means that we can restrict the analysis to algorithms that pick their final estimate $x_K$ in the subspace $P$ that results from the execution of the algorithm. In order to establish a lower bound for such an algorithm, it is sufficient to construct a function $f_K$ whose minimum is located sufficiently far away from this subspace. We construct this function by running the algorithm against a resisting oracle, which is quite standard in these lower bound proofs. Each call to the resisting oracle picks a new objective function $f_k$ among all the $S^L(\ell_2)$ functions that agree with the values and gradients returned by all previous calls to the oracle. This constraint ensures that the optimization algorithm would have reached the same state if it had been run against function $f_k$ instead of the resisting oracle.
A.2 Construction of a resisting oracle

We start by defining the basic structure of the function which will be used by our oracle to construct hard problem instances. This structure is identical to that used by Nesterov.

**Definition 8 (Nesterov)** Fix $\rho > 0$ and let $N_{\mu,L}$ denote the function

$$N_{\mu,L}(x) = \frac{L-\mu}{8} \left( (x[1])^2 + \sum_{i=1}^{\infty} (x[i+1] - x[i])^2 - 2\rho x[1] \right) + \frac{\mu}{2} \|x\|^2.$$ 

**Proposition 4** $N_{\mu,L} \in S_{\mu,L}^{\ell_2}$ and reaches its minimum in $x_N^* = (\rho q_i q_{i-1})_i$ with $q = \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1}$.

**Proof** The assertions $\mu I \preceq N''_{\mu,L} \preceq LI$ and $N'_{\mu,L}(x_N^*) = 0$ follow from direct calculation, as shown in Nesterov (2004, p. 67).

**Remark 3** We can arbitrarily choose the value of $\|x_N^*\|$ by appropriately selecting $\rho$.

We also need some other properties of the function, which are also present in Nesterov’s analysis.

**Proposition 5** Let $[e_1, e_2, \ldots]$ be the canonical basis of $\ell_2$ and let $R_k = \text{Span}(e_1 \ldots e_k)$.

$$x \in R_k \Rightarrow N'_{\mu,L}(x) \in R_{k+1}.$$ 

**Proof** Through a direct calculation, it is easy to verify that

$$\frac{\partial}{\partial x[i]} N_{\mu,L}(x) = \begin{cases} \frac{L-\mu}{4} (x[i] + (x[i] - x[i+1] - 2\rho) + \mu x[i] & \text{for } i = 1, \\ \frac{L-\mu}{4} (2x[i] - x[i+1] - x[i-1]) & \text{for } i > 1. \end{cases}$$

The statement directly follows from this.

We now recall our earlier definition of the matrix notation for orthonormal families in Definition 4. The resisting oracle we construct will apply the function $N_{\mu,L}$ to appropriately rotated versions of the point $x$, that is it constructs functions of the form $N_{\mu,L}(S^\top x)$, where the orthonormal operators $S$ will be constructed appropriately to ensure that the optimal solution is sufficiently far away from the span of algorithm’s queries and the oracle’s responses. Before we define the oracle, we need to define the relevant orthogonalization operations.

**Definition 9 (Gram-Schmidt)** Given a finite orthonormal family $S$ and a vector $v$, the Gram-Schmidt operator $\text{Gram}(S,v)$ augments the orthonormal family, ensuring that $v$ lies in the new span.

$$\text{Gram}(S,v) = \begin{cases} S & \text{if } v \in \text{Span}(S) \\ \left[ S, \frac{v - SS^\top v}{\|v - SS^\top v\|} \right] & \text{otherwise} \end{cases}$$

Our resisting oracle incrementally constructs orthonormal families $S_k$ and defines the functions $f_k(x)$ as the application of function $N_{\mu,L}$ to the coordinates of $x$ expressed an orthonormal basis of $\ell_2$ constructed by completing $S_k$. 

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Definition 10 (Resisting oracle) Let $S_{-1}$ be an empty family of vectors. Each call $k = 0 \ldots K-1$ of the resisting oracle performs the following computations and returns $y_k = f_k(x_k)$ and $g_k = f'_k(x_k)$.

\begin{align}
S_k &= \text{Gram}(\text{Gram}(S_{k-1}, x_k), v_k) \quad \text{for some } v_k \notin \text{Span}(S_{k-1}, x_k). \\
\bar{S}_k &= [S, \ldots] \\
y_k &= f_k(x_k) = N_{\mu, L}(\bar{S}_k^T x_k) \\
g_k &= f'_k(x_k) = \bar{S}_k N'_{\mu, L}((\bar{S}_k^T x_k))
\end{align}

Step (11) augments $S_{k-1}$ to ensure that $\text{Span}(S_k)$ contains both $x_k$ and an arbitrary additional vector. This construction ensures that $\dim(S_k) \leq 2k + 2$. Step (12) nominally constructs an orthonormal basis $\bar{S}_k$ of $\ell_2$ by completing $S_k$. This is mostly for notational convenience because the additional basis vectors have no influence on the results produced by oracle. Step (13) computes the value of $y_k = f_k(x_k)$ by applying the function $N_{\mu, L}$ to the coordinates $\bar{S}_k^T x_k$ of vector $x_k$ in basis $\bar{S}_k$. Since $x_k$ belongs to the span of the first $\dim(S_k) - 1$ basis vectors, $\bar{S}_k^T x_k \in R_{\dim(S_k) - 1}$. Finally, step (14) computes the gradient $g_k = f'_k(x_k)$. Note that $g_k \in S_k$ because proposition 5 ensures that $N'_{\mu, L}((\bar{S}_k^T x_k)) \in R_{\dim(S_k)}$.

**Proposition 6** The resisting oracle satisfies the following properties:

(a) $S_k = \text{Span}\{x_0 \ldots x_{K-1}, f'(x_0) \ldots f'(x_{K-1})\}$ \quad $\dim(S_k) \leq 2k + 2$ .

(b) $\forall i < k \quad y_i = f_k(x_i) \quad g_i = f'_k(x_i)$ .

**Proof** Property (a) holds by construction (see discussion above). Property (b) holds because both $x_i$ and $g_i$ belong to $\text{Span}(S_i)$. Therefore $y_i = f_k(x_i)$ because $S_i^T x_i = \bar{S}_k^T x_i$ and $g_i = f'_k(x_i)$ because $N'_{\mu, L}(S_i^T x_i) = N'_{\mu, L}(\bar{S}_k^T x_k) \in R_{\dim(S_k)}$. \hfill \(\blacksquare\)

### A.3 Proof of Theorem 2

We now have all the ingredients to establish the main result of this appendix on the complexity of optimizing smooth and strongly convex functions. Given our work so far, we know that the solution $X_k$ lives in a $2K + 2$ dimensional subspace of $\ell_2$. We also know that our resisting oracle constructs orthonormal operators $S_k$, so that the optimal solution of the function $f$ being constructed can be as far away as possible from this subspace. The next proposition, which almost establishes the theorem, essentially quantifies just how far the optimum lies from this span.

**Proposition 7** The minimum $x^*$ of function $f_{K-1}$ satisfies

$$\text{dist}[x^*, \text{Span}(S_{K-1})] \geq \|x^*\| q^{2K} \quad \text{with } q = \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \quad \text{and } \kappa = \frac{\mu}{L}.$$ 

**Proof** Any vector $x \in \text{Span}(S_{K-1})$ is such that $(\bar{S}_{K-1})^T x \in R_{\dim(S_{K-1})} \subseteq R_{2K}$. Meanwhile, equation (13) and proposition A.2 imply that $(\bar{S}_{K-1})^T x^* = (\rho q^i)_{i=1}^\infty$. Therefore

$$\|x^* - x\|^2 = \|S_{K-1}^T x^* - (\bar{S}_{K-1})^T x\|^2 \geq \sum_{i=2K+1}^{\infty} (\rho q^i - 0)^2 = q^{4K} \sum_{i=1}^{\infty} (\rho q^i)^2 = q^{4K} \|x^*\|^2.$$ \hfill \(\blacksquare\)
Proposition 7 and lemma 3 then directly yield the theorem. Indeed, the theorem is trivial when \( K = 0 \). Consider otherwise an algorithm \( B \) known to pick its answer \( x_B^K \) in \( \text{Span}(x_0 \ldots x_{K-1}, f'(x_0) \ldots f'(x_K)) \). For an appropriate choice of constant \( \rho \), proposition 7 constructs a function that satisfies the theorem. Finally, for any algorithm \( A \), lemma 3 implies that there is a function \( f \in S_{\mu,L}^{\Omega}(\ell_2) \) such that \( \|x^*_f - x_A^K\| \geq \|x^*_f - x_B^K\| \).

Lemma 2 then yields the corollary.

**Corollary 3** In order to guarantee that \( \|x^* - x_K\| \leq \varepsilon \|x^*\| \) for \( \varepsilon < 1 \), any first order black box algorithm for the optimization of \( f \in S_{\mu,L}(\ell_2) \) must perform at least \( K = \Omega(\sqrt{\kappa-1} \log(1/\varepsilon)) \) calls to the oracle.

Since this lower bound is established in the case where \( \Omega = \ell_2 \), it should be interpreted as the best dimension independent guarantee that can be offered by a first order black box algorithm for the optimization of \( L \)-smooth \( \mu \)-strongly convex functions.