Turbulent magnetohydrodynamic dynamo action in a spherically bounded von Kármán flow at small magnetic Prandtl numbers

Klaus Reuter\textsuperscript{1,2,4}, Frank Jenko\textsuperscript{2} and Cary B Forest\textsuperscript{3}

\textsuperscript{1} Computing Center of the Max Planck Society and the IPP, 85748 Garching, Germany
\textsuperscript{2} Max-Planck-Institut für Plasmaphysik (IPP), EURATOM Association, 85748 Garching, Germany
\textsuperscript{3} Department of Physics, University of Wisconsin, Madison, 1150 University Avenue, Madison, WI 53706, USA
E-mail: khr@ipp.mpg.de

Abstract. Turbulent magnetohydrodynamic (MHD) dynamo action in a spherically bounded electrically conducting flow is investigated numerically. A large-scale two-vortex flow driven by a constant body force is simulated. The numerical setup models the spherical Madison Dynamo Experiment, which uses an impeller-driven flow of liquid sodium. The study focuses on small magnetic Prandtl numbers (\(Pm\)), the regime relevant to liquid sodium experimental flows. The critical magnetic Reynolds number (\(Rm_c\)) of the dynamo model is determined. It initially rises steeply quasi-linearly as a function of the Reynolds number (\(Re\)) by about a factor of 10. Finally, it starts to flatten for \(Pm \gtrsim 0.1\). Further investigations yield that the initial rise of the stability curve is caused in concert with large- and small-scale fluctuations of the velocity field. As an inertial range of turbulence develops with increasing \(Re\), small-scale dynamo modes become unstable, indicating a transition from large-scale (dipolar) to small-scale dynamo action. It is argued that the flattening of the stability curve is related to a saturation of detrimental large-scale velocity fluctuations, the activation of small-scale dynamo action, and the separation of resistive and viscous cutoff scales for \(Pm < 1\). Moreover, it is shown that only the turbulent fluctuations obtained by subtracting the precomputed mean flow from the dynamically evolving flow can act as a small-scale dynamo.

\textsuperscript{4} Author to whom any correspondence should be addressed.
1. Introduction

It is the generally accepted theory that magnetic fields in geo- and astrophysical objects are generated by the magnetohydrodynamic (MHD) dynamo effect, a process that converts the kinetic energy of an electrically conducting flow into magnetic energy [1]. Dynamo action is described by dynamo theory, which is extensively studied in analytical and numerical models [2, 3]. At the turn of the millennium, two experimental setups using liquid sodium as a working fluid were independently successful in demonstrating a self-excited fluid dynamo in the laboratory [4, 5]. These experiments used optimized flows guided by pipes that intentionally limited the influence that turbulence could have on the dynamo process. Natural dynamos, on the other hand, are highly turbulent. For example, the fluid Reynolds number of the liquid outer core of the Earth is of the order of \( Re \sim 10^8 \)–\( 10^9 \) [3, 6].

Motivated by the numerical study of a simple two-vortex flow by Dudley and James (the ‘\( s2r2 \)’ flow) [7], several experimental groups have been working on realizing a turbulent dynamo in an \( s2r2 \) sodium flow. Experimental setups in spherical geometry were built at Maryland [8, 9] and Madison [10–15], whereas at Cadarache, a cylindrical containment was chosen [16–20]. Until now, self-excited turbulent dynamo action has only been demonstrated in the VKS dynamo experiment at Cadarache. Surprisingly, the VKS dynamo generated an axial dipole [16], contrary to earlier kinematic calculations that had predicted a magnetic eigenmode transverse to the mean flow’s axis of symmetry [21]. Depending on the boundary conditions, the rotation rate and direction, and the material of the impellers, the dynamo exhibits complex behavior with bifurcations, oscillations and reversals [20].

For the Madison Dynamo Experiment, linear calculations based on an experimentally measured mean flow from a water model had predicted the onset of a dynamo [10]. It has, however, turned out that turbulent fluctuations in the sodium flow prevent the system from self-excitation. The dynamo threshold of a flow is expressed by its critical magnetic Reynolds number, \( Rm_c \). The magnetic Reynolds number \( Rm \) quantifies induction effects in relation to
Ohmic dissipation, and in general depends on the flow geometry. In the present experimental setup, turbulence increases the critical magnetic Reynolds number above which magnetic fields are amplified to some unknown value that is larger than $R_m_c$ of the mean flow. The numerical determination of the dynamo threshold and the role of turbulence on the spherical $s2t2$ dynamo are the main goals of this work.

With liquid sodium as a working fluid, turbulence is inevitable in dynamo experiments. The magnetic Prandtl number $Pm$, which is given by the ratio of the kinematic viscosity to the magnetic diffusivity of a fluid and, hence, is identical to the ratio of the magnetic Reynolds number $Rm$ to the fluid Reynolds number $Re$, is for liquid sodium at laboratory conditions a constant of the order of $10^{-5}$. Therefore, if an experimental flow is driven sufficiently strongly to overcome a dynamo threshold of typically about $Rm_c \approx 100$, the flow is highly turbulent with $Re \sim 10^7$. Somewhat more flexibility in the control parameters is possible with plasma as a working fluid [22]. Simulating an experimental sodium flow at realistic values of $Re$ in order to determine $Rm_c$ is far beyond the capabilities of any supercomputer. Instead, results from numerically just tractable parameter values of $Re = O(10^3)$ need to be extrapolated to predict an asymptotic behavior. On the other hand, numerical simulations easily capture $Rm = O(10^2)$.

In this paper, we report on numerical investigations of magnetic field generation in a turbulent $s2t2$ flow within a spherical container comparable to the setup of the Madison Dynamo Experiment. First, the governing equations and their numerical treatment are presented. Next, an exploration of the two-dimensional (2D) parameter space in the experimentally relevant regime $Pm < 1$ is reported. Scalings of the growth rates of the magnetic energy as functions of $Re$ and $Rm$ are presented. The curve of marginal stability (or critical magnetic Reynolds number, $Rm_c$) is determined within the numerically tractable limits. It rises steeply initially and levels off for $Pm \lesssim 0.1$. Additional investigations into the role of large- and small-scale turbulent fluctuations with respect to dynamo action in the two regimes of the stability curve are discussed. Finally, the paper closes with a summary and a discussion of the results.

2. Numerical setup

The magnetic ($B$) and velocity ($v$) fields that describe an incompressible electrically conducting fluid are governed by the induction equation,

$$\frac{\partial B}{\partial t} = \nabla \times (v \times B) + \lambda \nabla^2 B,$$

(1)

and the Navier–Stokes equation,

$$\rho \left( \frac{\partial v}{\partial t} + (v \cdot \nabla) v \right) = -\nabla p + \rho v \nabla^2 v + j \times B + F,$$

(2)

along with the constraints $\nabla \cdot B = \nabla \cdot v = 0$. Here, $\lambda$ is the magnetic diffusivity, $\rho$ is the mass density, $p$ is the pressure, $\nu$ is the viscosity, $j = \mu_0^{-1} \nabla \times B$ is the current density, $F$ is a volume force term and $\mu_0$ is the permeability of free space. For the numerical treatment, it is convenient to write the equations in the well-known non-dimensional form, where the problem is characterized by two control parameters, the Reynolds number $Re = LV\nu^{-1}$ and the magnetic Reynolds number $Rm = LV\lambda^{-1}$. The parameters $L$ and $V$ denote length and velocity scales that are characteristic of the physical system under consideration. In this paper, the non-dimensional characteristic velocity used to calculate $Re$ and $Rm$ for a particular simulation is $\overline{v}_{rms}$, where $v_{rms} = \sqrt{\langle v^2 \rangle}$ is the spatial rms velocity. Here, the angle brackets denote averaging in space, and
the overline denotes averaging in time, which is performed during the quasi-stationary phase of the flow after all initial bifurcations. The characteristic time scale of the flow is given by the eddy turnover time \( \tau_\nu = L/V \), whereas the magnetic diffusion time \( \tau_\sigma = Rm \tau_\nu \) is the time scale relevant to the magnetic field.

We solve numerically the non-dimensional form of the MHD equations (1) and (2) in a sphere, using the parallel version of the \textsc{Dynamo} code [23, 24]. This employs the standard pseudo-spectral method based on a poloidal–toroidal decomposition of the vector fields in combination with spherical harmonic expansions, the modes being labeled with azimuthal \( \ell \) and zonal \( m \) mode numbers. Radial derivatives are approximated by finite differences of the fourth order. Outer boundary conditions are the potential field solution for the magnetic field (i.e. assuming no currents outside the fluid conductor) and the zero-slip condition for the velocity field. The forcing is designed to produce an \( s^2t^2 \) type of flow [7]. In cylindrical coordinates \( (r, \phi, z) \), the forcing function \( \mathbf{F} \) reads

\[
F_r = 0,
F_\phi = \mathrm{sgn}(z)m(z) \left\{ \begin{array}{ll}
\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(r - 0.26)^2}{0.0072}\right), & \text{if } r \leq 0.26, \\
\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(r - 0.26)^2}{0.00245}\right), & \text{if } r > 0.26,
\end{array} \right.
\]

\[
F_z = \mathrm{sgn}(z)m(z) \left\{ \begin{array}{ll}
0.33 \sqrt{2\pi} \exp\left(-\frac{(r - 0.15)^2}{0.02}\right), & \text{if } r \leq 0.15, \\
0.33 \sqrt{2\pi} \exp\left(-\frac{(r - 0.15)^2}{0.0128}\right), & \text{if } r > 0.15,
\end{array} \right.
\]

(3)

where \( m(z) \) is a mask function used to level off the driving in the \( z \)-direction. The mask function reads

\[
m(z) = \left\{ \begin{array}{ll}
\exp\left(-\frac{(|z| - 0.3)^2}{0.005}\right), & \text{if } |z| \leq 0.3, \\
1, & \text{if } 0.3 < |z| < 0.5, \\
\exp\left(-\frac{(|z| - 0.5)^2}{0.005}\right), & \text{if } |z| \geq 0.5.
\end{array} \right.
\]

(4)

The particular form of \( \mathbf{F} \) was designed to obtain laminar flows similar to those driven by the body force (labeled \( \mathbf{F}' \) in the following) that was used in earlier work [25, 26]. For numerical reasons, the spatially smooth forcing of the form (3) proved to be of advantage at high fluid Reynolds numbers. Independent of these details of the forcing, the overall flow geometry that develops in the simulations is of the \( s^2t^2 \) type: in each hemisphere, the fluid streams out towards the poles of the sphere, along the outer boundary to the equator, where it rolls back in to the center of the sphere. Separated by an equatorial shear layer, the two flow cells rotate in the opposite direction. Figure 1(a) shows streamlines of an \( s^2t^2 \) flow.

For sufficiently large \( Rm \), this flow is a dynamo, the transverse dipolar magnetic eigenmode growing fastest for laminar and weakly turbulent cases. The generation of this dipole can be understood from a simple frozen-flux picture based on stretching, twisting and folding of magnetic field lines [12]. This basic transverse dipolar magnetic field structure that breaks the rotational symmetry of the flow is shown in figure 1(b).
Figure 1. Streamlines (a) and magnetic field lines (b) of an $s2\ell2$ dynamo. Since the numerical simulations presented in this work are related to the Madison Dynamo Experiment, the forcing and the axis of symmetry are indicated by impellers.

The numerical resolution was chosen such that magnetic and kinetic power spectra in terms of $\ell$ show a drop-off by typically at least two orders of magnitude between the peak in the spectrum and its tail; for the simulations at the largest $Re$, a cutoff mode number of $\ell = 146$ was used, in concert with 768 radial points. Moreover, the power balance derived from the MHD equations was continuously monitored during the simulations, and the left- and right-hand sides of the equations were allowed to differ by no more than about 1.

3. A numerical investigation of turbulent dynamo action at small $Pm$

3.1. Problem setup

The stability curve as a function of Reynolds number, $Rm_c(Re)$, is of central interest to the Madison Dynamo Experiment and the focus of this paper. Its determination requires a scan in the 2D parameter space spanned by $Re$ and $Rm$. In the following, the methodology of the investigation is outlined, before we turn to a discussion of the actual results.

3.1.1. Hydrodynamic and subsequent magnetohydrodynamic (MHD) simulations. As a first step, hydrodynamic runs are performed independently at various values of the viscosity. At the beginning of each run, the velocity field is initialized with pseudo-random noise of small amplitude. In the following, the Navier–Stokes equation (2) is integrated forward in time with $\mathbf{B} = 0$. Momentum is injected via the body force $\mathbf{F}$ as defined by equation (3), which is kept constant in time and at identical amplitude for all runs. A hydrodynamic run is performed until the flow reaches a statistically stationary state, i.e. after potential initial transitions have occurred. The quantities used to monitor the hydrodynamic system are the characteristic velocity $v_{rms}$ and the spectrally decomposed kinetic energy as functions of time. Depending on the Reynolds number $Re$, a hydrodynamic run is performed for times of the order of 10–100
turnover times. The latter comparatively long runs are necessary in the regime of weak turbulence, which is governed by \( m = 2 \) wave motion [26]. Between different runs, \( Re \) is increased by decreasing the viscosity while keeping the driving constant. The characteristic velocity increases weakly with decreasing viscosity and, hence, makes only a minor contribution to the numerical value of \( Re \).

The second step is to introduce pseudo-random seed magnetic fields and continue the hydrodynamic runs as full MHD simulations. Between different runs, the magnetic Reynolds number \( Rm \) is varied by changing the resistivity. At each value of \( Re \) under consideration, the system’s evolution is computed for various values of \( Rm \) in order to obtain cases with growing and decaying magnetic fields. All the runs are performed in the kinematic regime, i.e. the magnetic field is negligibly small and does not affect the flow, the ratio of the magnetic to the kinetic energy in the system being initially of the order of \( 10^{-12} \)–\( 10^{-15} \). The simulation time is about one magnetic diffusion time for most of the runs. From the simulations, the fields and a number of derived quantities, such as spectral kinetic and magnetic energies, are obtained as functions of time. Figure 2 shows, as an example, spectra of the kinetic energy for a selection of runs. This figure will be discussed in more detail in section 4.

3.1.2. Determination of the growth rate of the magnetic energy. Time traces of the magnetic energy obtained from turbulent runs show irregular temporal fluctuations. As an example, figure 3 displays time series of the magnetic energy for three magnetic Reynolds numbers, all computed from an identical flow at \( Re \approx 1760 \). The growth rates are computed by a least-squares fit of a straight line to the evolution of \( \ln(B^2) \) over time. In order to compensate for the dependency of the magnetic diffusion time on the magnetic Reynolds number, the growth rates are normalized to the eddy-turnover time scale \( \tau_e \).

Difficulties and inaccuracies occur due to the relatively long time correlation of the system, especially close to criticality. For example, the blue (dotted) curve in figure 3 clearly decreases.
Figure 3. Magnetic energy as a function of the eddy-turnover time for three different magnetic Reynolds numbers. The flow is turbulent with $Re \approx 1760$.

on average over $140\tau_\nu$. On shorter time scales (say $25\tau_\nu$), however, phases of slow growth alternate with phases of rapid decay. Sufficiently long simulation times are therefore crucial, which are at the same time difficult to achieve given the computational costs of the runs, especially at large Reynolds numbers. Moreover, it is desirable to evaluate the error in the determined growth rate. A plausible definition of an error estimator reads

$$\Delta \gamma = \frac{2\sigma_r}{\Delta t},$$

where $\sigma_r$ is the rms of the residuals of the least squares fit and $\Delta t$ is the length of the time trace under consideration. In the following, the data obtained from the parameter scan are analyzed and discussed.

3.2. Results: influence of turbulence on the growth rate of the magnetic energy

3.2.1. Growth rates as functions of $Re$ at fixed $Rm$. The growth/decay rates of the magnetic energy are shown as functions of the fluid Reynolds number in figure 4(a). For each curve, the magnetic diffusivity is kept constant whereas the viscosity is varied. Because the characteristic velocity used to normalize $Re$ and $Rm$ changes weakly under variation of $Re$, each curve represents a range in $Rm$ as indicated.

Considering the cases at the two lowest magnetic Reynolds numbers for which the broadest range in $Re$ is available, the effects induced by the developing turbulence are clearly seen. Starting from positive values at small $Re$, the curves first decrease steeply and flatten in the following, starting from values of $Re \approx 900$. Finally, a slight increase in the rates is seen. Figure 4(b) displays the same data as a function of the magnetic Prandtl number $Pm$. In particular, the red curve ($Rm \approx 100$) indicates a decoupling of the magnetic decay rate from $Re$ for magnetic Prandtl numbers smaller than $Pm \approx 0.1$. For comparison, the Madison Dynamo Experiment does achieve $Rm \sim 100$ but runs at a much smaller magnetic Prandtl number of $Pm \sim 10^{-5}$ [12].
**Figure 4.** Growth rates of the magnetic energy on the eddy-turnover time scale as functions of $Re$ (a) and $Pm$ (b). The error bars were computed using equation (5).

In the range $Re \approx 1000–1500$ (and in the respective range in $Pm$), the scalings do not appear smooth. The reason is not known and could be related to a transition towards small-scale dynamo action that takes place in that range (see below). Moreover, equation (5) is likely to underestimate the error, at least in that range.

In the following, we turn to an investigation of the growth/decay rates under a variation of the magnetic Reynolds number.

### 3.2.2. Growth rates as functions of $Rm$ at fixed $Re$

Figure 5(a) shows the growth/decay rates of the magnetic energy as functions of $Rm$, thereby comparing all the runs performed for the purpose of this study. The zero crossing of each curve, $\gamma (Re, Rm_c) = 0$, defines the dynamo threshold $Rm_c$ at the respective value of $Re$. The global overview shown in figure 5(a) yields qualitative and quantitative changes occurring to $\gamma (Re)$ as $Re$ is increased. In the laminar and weakly turbulent limit, the curves cross the marginal rate the steepest. As $Re$ is increased, $\gamma (Rm)$ flattens, leading to a decrease in the absolute values of the growth/decay rates. At $Re \approx 92$ and $Re \approx 329$, the growth rates were computed for $Rm$ far beyond the threshold which yields a saturation to a flat-top, symptomatic of a large-scale dynamo mechanism (see section 4.1). To accurately determine the critical magnetic Reynolds numbers, the range close to criticality is shown separately for low and high $Re$ in figures 5(b) and (c).

Figure 5(b) focuses on the zero crossing in the low $Re$ regime. Error bars that were omitted in (a) for reasons of clarity are included in (b). The error, as defined by equation (5), is in most cases of the order of 1–10%, but larger relative errors occur close to criticality. The relatively large density of available data points legitimates the use of Akima splines to smoothly interpolate the curves.

Figure 5(c) displays the scaling of the growth rates for simulations at higher Reynolds numbers. Due to the high costs of these runs, fewer data points are available. For the same reason, the simulations are shorter and, hence, the relative error is larger than in the previous cases. A linear interpolation between the data points is used.

### 3.3. Results: stability curve of the turbulent $s2t2$ dynamo

From the zeros of the growth rates, the critical magnetic Reynolds number of the $s2t2$ dynamo is determined as a function of $Re$. At each $Re$, the error in the critical magnetic Reynolds number
Figure 5. Growth rates of the magnetic energy on the eddy-turnover time scale \( \tau_v \) as functions of the magnetic Reynolds number \( Rm \) for various fluid Reynolds numbers \( Re \). The curves for \( Re \lesssim 680 \) were interpolated using Akima splines. Error bars were computed according to formula (5). (a) Global overview of the scalings. (b) Low \( Re \) regime close to criticality. (c) High \( Re \) regime close to criticality.

is determined from the error in the growth rates, according to the formula

\[
\Delta Rm_c = \left( \frac{\gamma_1 + \Delta \gamma_1}{(\gamma_1 + \Delta \gamma_1) - (\gamma_0 + \Delta \gamma_0)} - \frac{\gamma_1}{\gamma_1 - \gamma_0} \right) (Rm_1 - Rm_0).
\]

Here, subscript 1 refers to the dynamo case found at the lowest \( Rm = Rm_1 \), whereas the subscript 0 refers to the non-dynamo case found at the highest \( Rm = Rm_0 \), with \( Rm_1 > Rm_0 \).

The stability curve is shown by the red (solid) line in figure 6. It answers the long-standing and experimentally relevant question of whether the dynamo threshold of the \( s2\tau2 \) flow becomes independent of the fluid Reynolds number for sufficiently large \( Re \).

Let us discuss the shape of \( Rm_c(Re) \) in detail. As \( Re \) is increased starting from the laminar regime, the stability curve \( Rm_c(Re) \) exhibits four characteristic features. Firstly, the dynamo threshold starts out plateau-like at a small value of \( Rm \approx 21 \) where the two data points shown in the plot correspond to a laminar case and to a time-periodic flow with \( m = 2 \).
structures [26]. Details of this wave motion and its influence on the dynamo process depend on the forcing and are discussed below. Secondly, beyond the wave-dominated regime, we observe a more or less linear increase in $Rm_c(Re)$ by approximately a factor of 10, which starts for $Re \gtrsim 100$ and extends up to $Re \lesssim 1000$. Thirdly, the stability curve shows a steep increase after this quasi-linear phase and reaches an absolute maximum. To exclude the possibility of a runaway value causing this distinct maximum, the error bar of the respective data point was reduced by extending the simulation times of the respective runs up to affordable limits. Fourthly, beyond the maximum, the dynamo threshold decreases again and shows saturation to a level of about $Rm_c \approx 250$ for $Pm \lesssim 0.15$. The question of whether the stability curve finally levels off for large $Re$ is very likely to be confirmed, although a downward trend is seen between the rightmost two points. This central result will be discussed in more detail in section 4.2.

In addition to the critical magnetic Reynolds number of the turbulent flow, the dynamo threshold of the axisymmetric mean flow (determined from the turbulent flow at the respective $Re$) is shown in figure 6. This curve rises moderately, starting from the laminar level before it saturates for $Re \gtrsim 250$ to a plateau at $Rm_c \approx 50$, which is in line with linear calculations based on ad hoc or experimental mean flows [7, 10].

To relate the findings presented in previous publications [25, 26] to the current investigation, the inset in figure 6 shows results from an analogous study that, however, uses a body force $F'$ slightly different from $F$ to drive the flow. Differences exist around $Re \approx 100$, where $Rm_c(Re)$ is significantly smaller than the dynamo threshold of the respective mean flows.
This phenomenon is caused by wave motion [26]. Obviously, the mean flows arising from $F'$ are less efficient dynamos than the mean flows resulting from $F$. Nevertheless, the global picture is consistent: as $Re$ is increased, $Rm_c(Re)$ rises steeply and separates from the threshold of the mean flow.

The shape of the stability curve is strikingly similar to previous numerical simulations in very different configurations [27–31]. We will discuss this aspect in detail in sections 4.3 and 4.4. Before we turn to investigations of the physics leading to particular details of the stability curve, we discuss some properties of the magnetic field solutions first.

3.4. Results: properties of the magnetic field solutions

The detrimental influence of turbulence on the efficiency of the $s2r2$ dynamo was illustrated and quantified in the preceding sections. Turbulence has implications not only for the magnetic field’s growth rate but also for the structure of the magnetic fields generated by the dynamo. The present section is concerned with the properties of the magnetic solutions obtained during the kinematic phase of the MHD simulations under variation of the control parameters $Re$ and $Rm$.

For $Re \lesssim 1000$ and $Pm < 1$, the dynamos produce large-scale dipolar magnetic fields. In these cases, the solutions are dominated by the transverse dipole mode, which clearly results from the stretch–twist–fold mechanism driven by the $s2r2$ mean flow [12]. In compliance with Cowling’s theorem, non-axisymmetric turbulent fluctuations in the velocity field may nevertheless drive axisymmetric ($m = 0$) magnetic field components.

Energy spectra of the magnetic fields in terms of the spherical harmonic degree $\ell$ are shown for various combinations of $Re$ and $Rm$ in figure 7. All the curves were computed from a series of magnetic field snapshots during the kinematic phases of the MHD runs, covering about one magnetic diffusion time (if possible). For each snapshot, the spectrum was individually calculated and normalized to unity before averaging was performed, in order to compensate for the fact that the magnetic fields in general grow or decay.

Figure 7(a) displays magnetic spectra from a parameter scan in $Rm$ with $Pm$ held fixed at low values between 0.2 and 0.33. All the cases shown are dynamos. As the Reynolds number is increased, the spectra flatten more and more. There are three cases (at the three largest $Re$), which peak at $\ell \approx 6, \ldots, 8$, in contrast to the other cases, which peak at $\ell = 1$. The latter is indicative of the conventional transverse dipole solution generated by the large-scale background flow. The former systems produce magnetic fields on spatial scales smaller than the system scale, in addition to the dipole mode, which is not negligible. The capability of the system to generate small-scale fields depends not only on $Re$ but also on $Pm$, as a comparison of the blue (dotted) curve (1759, 0.20) with the green (dashed) curve (1759, 0.33) shows. Evidently, there is a critical $Rm$ that governs a transition between large-scale and small-scale dynamo action.

To address the possibility of the generation of small-scale magnetic fields in the moderately turbulent regime, figure 7(b) shows spectra from a parameter scan in $Pm$ at a constant $Re \approx 329$. For comparison, the spectrum of the kinetic energy is also shown. As $Pm$ is increased, the dominance of the $\ell = 1$ dipole mode weakens until at $Pm = 1$ the spectrum finally peaks at $\ell = 3$, i.e. the system prefers an octupole solution.

In the turbulent regime, the production of magnetic fields on scales smaller than the outer scale occurs at smaller $Pm$ compared with the moderately turbulent case, as figure 7(c) implies.
Figure 7. Time-averaged normalized spectra of the magnetic energy (M) inside the sphere computed from field snapshots during the kinematic phase of the MHD runs. The asterisk marks a case that is not a dynamo. For direct comparison, spectra of the kinetic energy are shown (K). (a) Cases with $Pm \approx 0.2, \ldots, 0.33$. (b) Scan in $Rm$ at $Re \approx 329$. (c) Scan in $Rm$ at $Re \approx 1759$.

It shows a scan in $Pm$ at constant $Re \approx 1759$. The dynamo at $Pm = 0.2$ is slightly dominated by the transverse dipole mode. For $Pm = 0.33$, the peak is found at $\ell \approx 5$, whereas it moves to $\ell \approx 10$ for $Pm = 0.5$.

The shift towards larger degrees peaking around $\ell = 7–10$ potentially indicates a qualitative change in the mechanism of magnetic field generation from the large-scale to a small-scale dynamo mechanism. Whether this small-scale dynamo mechanism is acting independently as a fluctuation dynamo or in concert with the large-scale dynamo maintained by the mean flow is investigated in more detail below. A further possible explanation for the origin of magnetic energy at small scales could be turbulent induction due to the existence of the large-scale dipole mode in the form of a non-local transfer of energy in $k$ space. Such effects are difficult to untangle; see the following section for a discussion.

4. Discussion of the results

The numerical results presented in the previous section yield a rather complex picture, which requires further investigation. As the turbulence develops, dynamo action is initially hindered, which is indicated by growth rates decreasing with $Re$ at fixed $Rm$, and by the critical magnetic Reynolds number $Rm_c(Re)$, which increases monotonously with $Re$. Above a certain value of $Re$, however, the critical magnetic Reynolds number $Rm_c(Re)$ suddenly drops and flattens.
A small downward trend is present between the two rightmost available points. Hence, above a critical value of the control parameter $Re$, the dynamo threshold appears to decouple from the dynamics at the smallest spatial scales. Simultaneously, magnetic energy starts to grow on scales smaller than the outer scale for $Pm < 1$, indicative of a small-scale dynamo mechanism acting on top of the mean flow. In the present section, the observed phenomena are investigated and interpreted in detail.

In section 4.1, we first discuss the quasi-linear range of the stability curve, i.e. $Re \lesssim 1000$, during which a significant increase in $Rm_c$ by about a factor of 10 occurs. Section 4.2 focuses on the upper part of the curve, which displays a rollover followed by a flattening that potentially indicates an asymptotic regime with constant $Rm_c$. The results from a numerical experiment considering magnetic field generation merely due to the fluctuating part of the velocity field are discussed in section 4.3. A comparison to related work is drawn in section 4.4, before we proceed to the conclusions.

4.1. The quasi-linear regime of $Rm_c(Re)$

The laminar time-independent flow is clearly a very efficient dynamo, as the comparably low critical magnetic Reynolds number and the steep scaling of the growth rate as a function of $Rm$ indicate (see the black (solid) line in figure 5(a)). As the fluid Reynolds number is increased, hydrodynamic instabilities occur and the previously laminar flow becomes time dependent. Just above the threshold, this time dependence is periodic wave motion.

Weak turbulence characterized by periodic large-scale $m = 2$ wave motion was shown in previous work to support dynamo action in the sense that such flows are sometimes better dynamos than the axisymmetric mean flow, which is similar to the laminar flow [26]. For these previous studies, a forcing $F'$ slightly different from $F$ was used. Due to subtle changes in the resulting flows, the wave effects are of minor importance in the current study. The MHD runs performed at $Re \approx 92$ give potential evidence for their presence, where $Rm_c \approx 20.9$ and the threshold of the mean flow is approximately 22. Nevertheless, similar to the flows studied in [26], the flows driven by the body force $F$ show a distinct peak at $m = 2$ in the energy spectra (see the inset of figure 2). Compared with the mean flows resulting from $F'$, the mean flows studied in this paper are clearly more efficient dynamos, as figure 6 and its inset show. This fact and the possibility that the $m = 2$ waves in the flows driven by $F$ could well have an inefficient time dependence far from an optimum frequency explain the minor importance of wave-driven dynamo action in the present context. On the other hand, larger growth rates are obtained in the case of the wave-influenced dynamo ($Re \approx 92$) compared with the laminar dynamo at identical $Rm$ before the curve rolls over (see again the two leftmost curves in figure 5(a)), which can be interpreted as a beneficial effect due to wave motion. It is likely that the wave motion in the simulations is linked to the $m = 2$ shear layer instability observed by Ravelet et al in hydrodynamic experiments with a cylindrical $s2r2$ flow [32].

Increasing the Reynolds number beyond the wave-dominated regime triggers further instabilities in the flow. The $m = 2$ wave structures break up and spawn irregularly fluctuating large-scale structures, mainly localized at the polar regions of the sphere. Starting in the upper part of the quasi-linear regime of the stability curve, an inertial range of turbulence develops, indicated by a Kolmogorov-type power law in the kinetic energy spectrum (see figure 2). This leads us to a question central to the understanding of the stability curve: the specific role that turbulent fluctuations at large and small spatial scales play on the dynamo process. Before we
discuss these aspects arising from time dependence, we first turn to an investigation of the influence of the mean flow.

4.1.1. Role of the axisymmetric background flow. In the quasi-linear regime of $Rm_c(Re)$ and for $Pm < 1$, the turbulent $s2\ell2$ flow produces large-scale magnetic fields as the magnetic spectra in figure 7 indicate. Given these phenomena, we expect that the magnetic field is produced by the large-scale background flow. Certainty that the mean flow produces the large-scale magnetic field is obtained from investigating the scaling of $\gamma$ as a function of $Rm$.

If the outer scale motions amplify the magnetic field, the growth rate cannot be larger than the characteristic rate of stretching of these motions $[31]$. In the limit of $Rm \gg Rm_c$, this argument yields the scaling

$$\gamma \sim \frac{\gamma L}{\sqrt{Rm}}. \quad (7)$$

If, on the other hand, the inertial range motions amplify the magnetic field, the growth rate of the magnetic energy should scale with $Rm$ as

$$\gamma \sim \frac{v_{rms}}{L} \sqrt{Rm}, \quad (8)$$

which holds in the limit of large $Rm$ and under the assumption of a local transfer of energy in $k$ space $[31]$.

The saturation of $\gamma(Rm)$ for $Rm \gg Rm_c$, which is clearly observed from the scans in $Rm$ at $Re \approx 92$, $Re \approx 329$ and $Re \approx 858$ (see figure 5(a)), is indicative of dynamo action caused by the outer-scale motions of the background $s2\ell2$ flow. The decrease in the growth rates seen with increasing $Re$ is consequently caused by fluctuations perturbing this large-scale dynamo. In the following, we investigate the specific role of large- and small-scale velocity fluctuations. Here, large scale denotes the first few mode numbers $\ell$ that contribute to the background flow. The term small scale refers to higher modes.

4.1.2. Fluctuations at large spatial scales. In the flows under investigation, turbulence develops on top of a strong $s2\ell2$ background flow. To untangle the influence of the mean flow versus that of the fluctuations, it is helpful to measure the intensity of the fluctuations in relation to the strength of the mean flow.

Following Dubrulle et al who investigated the turbulent Taylor–Green (TG) flow in 3D-periodic box simulations $[33, 34]$, we introduce two measures $\delta$ and $\delta_2$ to globally quantify the level of large-scale turbulent fluctuations of a time-dependent velocity field $v$. These measures are defined as

$$\delta = \frac{\langle v^2 \rangle}{\langle \mathbf{v}^2 \rangle}, \quad \delta_2 = \frac{\sqrt{\langle \mathbf{v}^2 \rangle^2 - \langle v^2 \rangle^2}}{\langle \mathbf{v}^2 \rangle}. \quad (9)$$

Similar to the ‘turbulence intensity’, which is calculated from data measured at a single point in space, the ‘noise intensity’ $\delta$ quantifies the intensity of the turbulent fluctuations globally in the volume under consideration, with strong emphasis on fluctuations at large scales. In the definitions (9), the overline denotes averaging in time and the angle brackets averaging in space. Note that the quantity $(\delta - 1)$ is just the ratio of the kinetic energy in the fluctuations to the kinetic energy in the mean field. The variable $\delta_2$ quantifies the fluctuations of the total

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kinetic energy about the mean total kinetic energy. Since the spectrum of the turbulent $s2t2$ flow, which is driven at the large scale $\ell = 2$, drops off quite rapidly, the fluctuations in the low wavenumbers mainly contribute to the amplitude of $\delta$, whereas high wavenumbers play only a minor role. Experimental work based on a cylindrical $s2t2$ flow by Cortet et al [35] arrived at the same conclusion from measured data. An on-the-fly computation of these statistical measures is implemented in the DYNAMO code, which produces as a spin off an increasingly converging mean flow.

In addition, the same group of authors investigated the role of large-scale and small-scale noise in so-called stochastic-kinematic simulations by feeding synthetic noise at small/large wavenumbers and at different correlation times (in relation to $\tau_\nu$) into the velocity field [33, 34]. It was found that large-scale noise is detrimental to the dynamo process in all cases, especially when the correlation times are long. On the other hand, small-scale noise proved to have little effect in any case. For the problem of the turbulent TG dynamo, it was discovered that the critical magnetic Reynolds number $Rm_c(Re)$ increases in a correlated way with the large-scale hydrodynamic fluctuation level $\delta(Re)$. Therefore, it was concluded that the large-scale fluctuations in the velocity field are detrimental to self-excitation, but once developed their effect ceases to contribute further above a certain $Re$. This reasoning was also supported by the results of the stochastic-kinematic simulations.

The values of $\delta$ and $\delta_2$ obtained from hydrodynamic and kinematic MHD simulations of the $s2t2$ flow are plotted in figure 8. The quantity $\delta(Re)$ rises up to $Re \approx 1000$, and then converges to a value of about 2 for larger Reynolds numbers, i.e. $\delta$ becomes independent of the increasingly smaller-scale motions of turbulence. The values of $\delta_2$ are below 0.1.

The increase in $\delta(Re)$ strikingly coincides with the quasi-linear regime of the stability curve $Rm_c(Re)$. The adjacent hump in the stability curve is likely to be of transient nature and potentially linked to a transition to small-scale dynamo action. In any case, the detrimental effect of the large-scale turbulent fluctuations levels off for $Re \gtrsim 1000$, as indicated by the saturation of $\delta(Re)$. The fact that the value of $\delta(Re) \approx 2$ in saturation is significantly smaller in the $s2t2$ flow than in the TG flow is likely to be attributed to the presence of outer physical boundary

Figure 8. Hydrodynamic fluctuation measures $\delta$ and $\delta_2$ as functions of the Reynolds number. Note that $\delta_2$ is plotted by a factor of 10 amplified.
conditions in the $s2t2$ model that limit the extent to which large-scale fluctuations may grow. Nevertheless, the same correlation as in [33] is clearly observed.

Interestingly, hydrodynamic experiments with $s2t2$ flows in cylindrical containers yielded a similar evolution of $\delta(Re)$ (or related variables) [32, 35]. In particular, Cortet et al [35] found a value of $\delta \approx 2$ for a configuration without equatorial annulus.

In summary, the saturation of $\delta(Re)$ is indicative of a saturation of the energy in the large-scale turbulent fluctuations for $Re \gtrsim 1000$. We therefore conclude that the development of such fluctuations contributes significantly to the initial quasi-linear increase in the stability curve (which ends at about the same $Re$) but ceases to contribute further larger $Re$.

4.1.3. Fluctuations at small scales. To systematically investigate the role of the fluctuations on spatial scales smaller than the scale of the outer fluid motions, it is desirable to have a controlled influence on the amplitudes of such small-scale modes in relation to the amplitudes of the modes on the outer scale. Damping of modes with predominantly large wavenumbers is possible within certain limitations by introducing a wavenumber-dependent viscous dissipation.

In the following, a numerical experiment is reported that utilizes a fourth-order hyperdiffusion operator of the form $(\nabla^2 - \epsilon \nabla^4)$. The term proportional to the square of the Laplacian emphasizes viscous dissipation at large wavenumbers, leading to stronger viscous dissipation of energy from these modes compared to the conventional Laplacian diffusion operator. The multiplicative parameter $\epsilon$ allows one to adjust the relative importance of hyperviscous ($\propto k^4$) to viscous dissipation ($\propto k^2$).

The numerical experiment discussed in the following operates with a flow at $Re \approx 300$, $Rm \approx 150$. This flow is a large-scale dynamo with a transverse dipole field and is located in the quasi-linear regime of $Rm_c(Re)$. Initially, a hydrodynamic simulation is performed using the conventional Laplacian diffusion operator. This run is then continued with viscous hyperdiffusion in a series of MHD simulations, each of them at a different value of $\epsilon$. The goal of the study is to determine the growth rate of the magnetic energy as a function of the parameter $\epsilon$.

The influence of hyperdiffusion on the hydrodynamic spectrum is as follows. As the hyperdiffusion parameter $\epsilon$ is increased, the spectra displayed in figure 9(a) steepen due to increased viscous dissipation at large wavenumbers. At low wavenumbers/large scales that represent the background flow, the changes are marginal.

Figure 9(b) shows the growth rate of the magnetic energy as a function of $\epsilon$ computed from MHD runs based on the flows simulated with hyperdiffusion. For comparison with the hyperdiffusion cases that use finite differences on a seven-point stencil to compute radial derivatives, the growth rates obtained from simulations using Laplacian diffusion are shown, computed using a five-point stencil and a seven-point stencil. Both rates agree well. The values of the growth rates from the hyperdiffusion cases initially scatter around the conventional Laplacian value. As $\epsilon$ is increased above a critical value of about $10^{-3}$, the growth rate steeply increases.

The numerical experiment provides evidence that the dynamo under investigation is comparably inefficient as long as there is a certain amount of energy in the small-scale fluid motions. As soon as sufficient energy is drawn from large mode numbers, the efficiency of the dynamo increases. By a converse argument, we may conclude that fluctuations at spatial scales smaller than the large-scale flow are detrimental to the process of magnetic field generation, as are the fluctuations at large scales (see the previous section). The question remains as to which of these two effects wins. Given the fact that the derivatives of $\delta(Re)$ satisfy $\delta' > 0$, $\delta'' < 0$ whereas
the stability curve rises roughly linearly, one may conjecture that the influence of the large-scale fluctuations decreases whereas effects on smaller scales become increasingly important (which is also consistent with the picture of the development of turbulence). Moreover, one has to keep in mind that both effects are not clearly separable since the large-scale and small-scale dynamics of a turbulent flow are not independent. Nevertheless, the values of $\delta(v)$ are close to 1.5 in all the runs shown in figure 9, indicating consistency in the large-scale properties of the simulated flows.

It is important to point out that this result is valid in the initial more or less linear regime of the stability curve. In this case, the dynamo at $Pm < 1$ is driven by the outer-scale motions and produces a large-scale magnetic dipole. In the saturated regime of $Rm_c(Re)$ with $Pm < 1$, magnetic fields are produced on scales smaller than the system scale. A small-scale or fluctuation dynamo would rely on the small-scale motions and therefore show a contrary effect. Due to limitations in computing time and the fact that using hyperdiffusion restricts the maximum allowed length of the time-step, an analogous scan in $\epsilon$ could not yet be performed at large $Re$.

4.2. The stability curve in the limit of large $Re$

For $Re \gtrsim 1000$, the stability curve of the $s2r2$ dynamo displays a rollover, followed by a flattening starting from $Re \gtrsim 1800$ (see figure 6). In the following, we discuss this result, which is of great importance for the Madison Dynamo Experiment.

4.2.1. Kolmogorov phenomenology for $Pm < 1$ in the limit of weak magnetic fields. The finding that the stability curve flattens in the limit of large $Re$ and small $Pm$ is consistent with the following argument. Let us consider a weak magnetic field in a turbulent flow within the framework of the Kolmogorov–Richardson phenomenology. The characteristic rate

Figure 9. (a) Time-averaged spectra of the kinetic energy for a flow at $Re \approx 300$. Cases with the standard Laplacian diffusion operator computed on a five- and a seven-point radial stencil are compared with cases using a hyperdiffusion scheme (which also uses a seven-point stencil). (b) Magnetic growth rate as a function of the hyperdiffusion parameter $\epsilon$, for the flows introduced in (a) at $Rm \approx 150$. 

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of stretching of magnetic field lines by the turbulent eddies at scale \( l \) is of the order of
\[
u l / l = \epsilon^{1/3} l^{-2/3},
\]
where \( \epsilon \) is the turbulent energy flux. The scale at which the stretching rate
becomes comparable to the rate of resistive diffusion \( \lambda / l^2 \) defines the resistive cutoff below
which Joule dissipation overcomes induction \([36]\). This resistive scale is given by
\[
l_{\lambda} \sim \left( \frac{\lambda^3}{\epsilon} \right)^{1/4} \sim \frac{L}{Rm^{3/4}},
\]
in perfect analogy to the definition of the viscous scale \( l_v = L \text{Re}^{-3/4} \). Let us assume the
situation \( \text{Re} \gg Rm \gg 1 \) and compare both scales. It follows that \( L \gg l_{\lambda} \gg l_v \), i.e. the resistive
scale lies inside the inertial range. As \( \text{Re} \) is increased while \( Rm \) is kept fixed, the dissipation
scales separate more and more. Below the resistive scale, the magnetic field lacks structure
due to magnetic diffusion, which dominates \( \propto l^{-2} \) over induction \( \propto l^{-2/3} \). Therefore, it is
unaffected by the turbulent velocity field, which extends down to the viscous scale.

4.2.2. Role of large-scale turbulent fluctuations. As was pointed out previously, the intensity
\( \delta(\text{Re}) \) of large-scale turbulent fluctuations becomes independent of \( \text{Re} \) for \( \text{Re} \gtrsim 1000 \)
(see figure 8). Above threshold, such fluctuations can consequently not contribute further to the
increase in \( Rm_c \), which is compatible with the flattening of \( Rm_c(\text{Re}) \) for large \( \text{Re} \). This
reasoning does, however, not consider the role of fluid motions at small scales, which leads
us to discuss the phenomenon of small-scale dynamo action.

4.2.3. Small-scale dynamo action at \( Pm < 1 \). As the stability curve levels off, the turbulent
\( s^2 \alpha \) flow has developed an inertial range (see figure 2). In this regime, the system is capable
of producing magnetic fields on a broad range of spatial scales, which indicates a qualitative
change in the dynamo mechanism. This transition not only depends on \( \text{Re} \) but also on \( Rm \), as is
demonstrated by means of a flow with \( \text{Re} \approx 1759, \ Rm \approx 352 \) (see figure 7(c)). Its spectral peak
is located at \( \ell = 1 \), indicative of the large-scale dynamo mechanism caused by the mean flow.
Nevertheless, the spectrum is relatively flat at low mode numbers compared with dynamos at
smaller \( \text{Re} \). Increasing \( Rm \) further to \( Rm \approx 587 \) causes the spectral peak of the magnetic energy
to move to \( \ell \approx 7 \). A well-resolving scan in \( Rm \) would unveil information on the transition from
the large-scale to the small-scale dynamo, especially on how far above \( Rm_c \) the small-scale
regime begins and if this transition is smooth. Whether magnetic field generation at small scales
contributes—together with other effects—to the flattening of the stability curve is likely but
cannot be proved currently. We will pick up this question again in section 4.4, which presents a
comparison of this study with previous work.

Certainty that the inertial range motions of turbulence amplify small-scale magnetic fields
as a fluctuation dynamo could in principle be obtained by confirming a scaling of the form \( \gamma \propto Rm^{1/2} \)
(see equation (8) and the work by Schekochihin et al [31] for further details). However, a
numerical verification of this scaling has not been achieved yet, even in computationally efficient
Cartesian box simulations due to the extreme demands in resolution and CPU time that become
necessary to satisfy limit \( \text{Re} \gg Rm \gg 1 \), in which the aforementioned scaling is expected to
hold \([37, 38]\).

As the decoupling of \( Rm_c \) from \( \text{Re} \) has now been supported by numerical and theoretical
arguments, it is important to view this result in relation to the Madison Dynamo Experiment.
An estimate is given in the concluding section 5.
4.3. Dynamo action from fluctuations without mean flow

An interesting question related to the evidence of magnetic fields growing at scales smaller than the system scale is whether and above which $Rm'_c$, the fluctuations of the velocity field are capable of amplifying magnetic fields in the absence of mean flow. To shed light on this issue, a set of MHD simulations was carried out solving a modified induction equation of the form

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v}' \times \mathbf{B}) + \lambda \nabla^2 \mathbf{B},$$

where the fluctuating part of the velocity field is given by

$$\mathbf{v}'(t) = \mathbf{v}(t) - \bar{\mathbf{v}}.$$ (12)

The mean flow $\bar{\mathbf{v}}$ is computed from a previous kinematic MHD simulation that is identical to the one used for the computation of the mean flow’s dynamo threshold. The velocity field $\mathbf{v}(t)$ is the solution of the Navier–Stokes equation, which is simultaneously integrated, however, without the Lorentz force term.

A characteristic velocity $U'$ of the fluctuations is, as an estimate, defined a posteriori as the difference of the characteristic velocity of the turbulent flow (as used to normalize Re) and that of the mean flow. The value of $U'$ is then used to normalize the critical magnetic Reynolds number $Rm'_c$ of the fluctuations.

We first investigate a flow located in the quasi-linear regime of $Rm_c(Re)$. For this study, time traces of the magnetic energy from simulations at $Re \approx 561$ are investigated, comparing regular MHD simulations with runs in which the mean flow is removed before computation of the induction term. The conventional dynamo threshold is $Rm_c \approx 111$. A run using $\mathbf{v}'$ shows a decaying magnetic field at $Pm = 1$. Purely fluctuation-driven magnetic field growth is indeed observed for $Pm = 1.2$ and $Pm = 1.5$. The first of these cases grows marginally; hence the dynamo threshold for purely fluctuation-driven growth is approximately $Rm'_c \approx 167 \approx 1.5 Rm_c$. With $Rm = Rm'_c$, however, a conventional MHD run is a large-scale dynamo. Nevertheless, for the purely fluctuation-driven dynamo, the value of $Rm'_c$ is plausible since the fluctuation dynamo at $Pm > 1$ is known to excite for comparably small $Rm$ (see [31] and references therein). For comparison, both the values of $Rm'_c$ that we have computed are indicated in figure 6.

An identical numerical experiment was carried out for a flow at $Re \approx 1760$. The fluctuations cause marginal magnetic field growth for $Pm = 0.67$, whereas a run at $Pm = 0.6$ is subcritical. The dynamo threshold of the fluctuations is $Rm'_c \approx 382 \approx 1.36 Rm_c$. Interestingly, this value is in between a case that is still a large-scale dynamo ($Rm \approx 352$) and a case that is a small-scale dynamo ($Rm \approx 587$, see figure 7). Moreover, $Rm'_c$ is consistent with the large dynamo threshold reported for the fluctuation dynamo at $Pm < 1$ [31].

At both values of $Re$ under investigation, the magnetic field spectra turn out to be very similar when comparing the regular and the fluctuation-driven dynamos. The peak in the spectrum is located at $\ell \approx 5$ for $Re \approx 561$, $Pm = 1.2$, and at $\ell \approx 10$ for $Re \approx 1760$, $Pm = 0.67$. In all cases, the magnetic fields grow significantly faster at identical $Pm$ in the regular MHD simulations.

4.4. Comparison to related work in periodic-box geometry

The findings discussed in this paper are in line with the results from studies of large-scale MHD flows carried out by Mininni et al [27, 29] and Ponty et al [28, 30], which motivates a
comparison in detail. The important difference from this work is that the aforementioned studies solved the MHD equations in a cubic domain under periodic-boundary conditions, which allows for the use of an efficient Fourier pseudo-spectral method, but without including solid walls by using e.g. a zero-slip condition. In view of the dynamo problem at $Pm < 1$, the emphasis of the investigations was to monitor the dynamo’s behavior as $Pm$ is lowered, similar to the aim of this study. The MHD flows were driven by large-scale body forces, such as the helical Roberts forcing [27, 29] and the non-helical TG forcing [28–30].

4.4.1. Shape of the critical magnetic Reynolds number $Rm_c(Re)$. In both models, the stability curve $Rm_c(Re)$ shows an initial increase by about a factor of 10 compared to the dynamo threshold of the laminar flow. For $Re \gtrsim 1000$, the stability curves settle to values between 150 and 250. These results are in line with the findings for the s2t2 dynamo. A rollover as seen in figure 6 is not observed, and the flat-top behavior starts rather abruptly. In addition, the saturation of $Rm_c(Re)$ was shown to coincide with the development of an inertial range of turbulence indicated by a $k^{-5/3}$ energy spectrum. In this paper, a similar finding is reported for the s2t2 dynamo (see figure 2). Moreover, in reference [27], turbulent eddies smaller than the scale of the forcing were shown to amplify magnetic field, thereby contributing to the saturation of $Rm_c(Re)$ (see below).

4.4.2. Influence of large-scale fluctuations on $Rm_c(Re)$. The saturation of the large-scale fluctuation level $\delta(Re)$ and of the stability curve $Rm_c(Re)$ were shown to coincide in the TG model, and it was argued—supported by the results of stochastic-kinematic simulations—that this coincidence is causal [33, 39]. For the turbulent s2t2 flow, both curves were demonstrated to rise jointly, but as $\delta(Re)$ saturates immediately in the following, the stability curve increases steeply to a maximum and rolls over before it flattens. While the nature of the hump in the stability curve is potentially related to the transition to small-scale dynamo behavior, the effects due to the saturation of $\delta(Re)$ discovered in the TG model can be confirmed for the s2t2 flow.

4.4.3. Small-scale dynamo action at $Pm < 1$. A transition to small-scale dynamo action was discovered for the TG model in the regime $Pm < 1$ as well. Similar to the finding presented in figure 7(a), the peak in the magnetic spectrum moves from small to larger wavenumbers as $Re$ is increased [28]. To address the question of which scales of the turbulent velocity field actually contribute to magnetic field generation, Mininni et al [27] calculated spectral energy transfer functions, which could be done by 3D Fourier decomposition implemented for the purpose of the pseudo-spectral scheme. It was discovered that the magnetic field draws energy from wavenumbers larger than the injection wavenumber of the forcing by up to a factor of 10, i.e. the small scales and the large scales of the flow amplify the magnetic field in concert. This scenario is likely to apply to the s2t2 system as well; however, spectral transfer functions cannot be computed currently due to the lack of a radial spectral decomposition in the Dynamo code.

Moreover, it is interesting to relate the results from the s2t2 dynamo model to investigations of the fluctuation dynamo at $Pm < 1$ by Schekochihin et al [31, 40]. These studies considered magnetic field generation in a turbulent flow randomly driven by a non-helical homogeneous body force injecting energy into all Fourier modes in the form of $\delta$-correlated (white) noise. Dynamo action was confirmed to take place in the system that is completely lacking any mean flow. The stability curve of this fluctuation dynamo looks qualitatively similar to the curves of
the aforementioned large-scale dynamos; however, lying at about a factor of 3 higher in $Rm$ as the comparison between the fluctuation dynamo and the TG dynamo studied by Ponty et al in figure 1(b) of [31] indicates. A common feature between the fluctuation dynamo’s stability curve and the stability curve of the s2t2 flow (figure 6) is the pronounced maximum before saturation occurs. Interestingly, the stability curves of the TG or Roberts model show a flat top in contrast. A second, related common feature is the rise in magnetic growth rates towards the largest (smallest) simulated $Re$ ($Pm$) (see figure 4).

The comparably large critical magnetic Reynolds number of the dynamo based on the fluctuating part of the turbulent s2t2 flow is in line with the results found by Schekochihin et al [31] for the non-helical white-noise-driven fluctuation dynamo (see section 4.3).

5. Summary and conclusions

This paper reports on the results of an extensive 2D parameter scan in the fluid Reynolds number $Re$ and the magnetic Reynolds number $Rm$, which are the control parameters of the dynamo problem posed by a mechanically driven incompressible MHD flow. A program of MHD simulations of a spherically bounded s2t2 flow driven by an impeller model was carried out in order to determine the growth/decay rates of the magnetic energy $\gamma$ as functions of the control parameters, focusing on magnetic Prandtl numbers $Pm = Rm/Re < 1$. From the zeros of the rates $\gamma$ as functions of $Rm$, the critical magnetic Reynolds number $Rm_c$ was determined as a function of $Re$. This stability curve $Rm_c(Re)$ separates the parameter space into a lower part ($Rm < Rm_c$), in which the flow is stable under infinitesimal magnetic perturbations, and an upper part ($Rm > Rm_c$), in which infinitesimal seed magnetic fields are amplified and grow exponentially in a statistical sense. The study was motivated by the liquid sodium Madison Dynamo Experiment, which is currently restricted to the lower stable region of the parameter space, i.e. it does not produce a self-excited dynamo. The experimental difficulties are caused by vigorous turbulence in the MHD flow, which has Reynolds numbers of $Re \approx 10^7$. Given the fixed magnetic Prandtl number of liquid sodium of $Pm \approx 10^{-5}$, the experimental flow attains magnetic Reynolds numbers of $Rm \approx 100$. Simulating a flow at such large $Re$ in order to determine $Rm_c$ is far beyond the capabilities of any supercomputer. Instead, results from numerically just tractable parameter values of $Re = O(10^3)$ need to be extrapolated to predict an asymptotic behavior.

As turbulence develops in MHD simulations of the s2t2 flow, the growth rate $\gamma$ of the magnetic energy is immediately affected. Keeping $Rm$ fixed at e.g. the experimentally relevant value of $Rm \approx 100$, the initially positive rate steeply decreases, passes through zero and finally tends to saturate at a finite negative value. Moreover, the growth rate $\gamma$ as a function of $Rm$ flattens strongly with increasing $Re$. The shape of the critical magnetic Reynolds number $Rm_c$ defined by the zeros of the function $\gamma(Re, Rm_c)$ takes the following form: starting from low values of $Rm_c \approx 20$ in the laminar regime, the stability curve shows a quasi-linear increase up to values of $Rm_c \approx 200$, which ends at $Re \approx 1000$. In the following, a steep increase is observed to a maximum of $Rm_c \approx 400$, followed by a roll over. Finally, a flattening at a level of $Rm_c \approx 250$ is indicated by the numerical data, starting from $Re \gtrsim 1800$, i.e. the stability curve potentially starts to decouple from $Re$ for $Pm \lesssim 0.15$. A slight downward trend does exist between the two rightmost data points; however, the large computational costs of the simulations currently impede further exploration towards larger $Re$. Actual decoupling of $Rm_c$ from $Re$ to a constant
value is plausible as the viscous dissipation scale decreases with decreasing $Pm$, whereas the resistive dissipation scale remains constant and larger than the viscous scale. These features of the stability curve are shown for the first time in a bounded flow with realistic physical boundary conditions.

In this study, the mere numerical determination of the stability curve was complemented by additional investigations of the underlying physics. It was found that the initial quasi-linear increase of $Rm_c(Re)$ is correlated with the development of large-scale fluctuations. For $Re \gtrsim 1000$, this fluctuation level saturates and, in conclusion, also its negative impact on the dynamo process. At the same time, results from numerical experiments using hyperdiffusion indicate that small-scale fluctuations are also detrimental to the dynamo efficiency in the linear regime of $Rm_c(Re)$. As an inertial range of turbulence emerges for $Re \gtrsim 1000$, the MHD flow develops the capability to produce magnetic fields on scales smaller than the system size for $Pm < 1$. Such magnetic field amplification on small scales is likely to contribute to the observed saturation of the stability curve. Moreover, results from numerical experiments provide evidence that a small-scale dynamo is possible purely from the velocity field fluctuations, i.e. when the mean flow is subtracted from the actual flow. The threshold to this type of fluctuation dynamo is, however, larger than the saturated level of $Rm_c$.

Extrapolating a saturation level of $Rm_c \approx 200$ in the limit of large $Re$, the magnetic Reynolds number of the Madison Dynamo Experiment lies below the threshold to self-excitation by probably up to a factor of 2. The error from the procedure employed to determine the stability curve is of the order of 10%. Moreover, the impeller model used in the simulations is a simple body force kept constant in time, which is not quite an accurate representation of the experimental reality. As predicted by linear calculations based on a measured experimental flow [10], the experiment would self-excite from the axisymmetric mean flow in the absence of turbulence: the mean flow has a flat dynamo threshold of about $Rm_c(\overline{v}) \approx 50$, which becomes independent of the fluid Reynolds number of the underlying turbulent flow for $Re \gtrsim 250$. Recent experimental modifications, such as the installation of an equatorial ring baffle, improve the flow by damping large-scale fluctuations, e.g. between the hemispheres [41]. In the light of the finding of this paper that large-scale fluctuations quantified by $\delta(Re)$ lead to a drastic increase in $Rm_c$, these upgrades may prove essential to reduce the critical magnetic Reynolds number of the experiment. It might, however, in addition be necessary to make use of boundary conditions that are known to reduce $Rm_c$, such as soft-iron impellers and outer stagnant fluid layers. Similar modifications helped the VKS dynamo experiment reach a state of self-excitation [16]; albeit more recent investigations show that the VKS flow only requires some of the aforementioned boundary conditions to work as a dynamo [20]. Finally, we turn towards the question about the implication of the observation of small-scale dynamo action in simulations for the Madison Dynamo Experiment. In this paper, we have studied turbulent dynamo action during the kinematic phase, i.e. when the growing magnetic field does not affect turbulent flow. Preliminary results from ongoing simulations suggest that a growing small-scale dynamo becomes a large-scale dynamo with the basic transverse dipole mode during saturation. A detailed report will be published elsewhere. Measurements of the magnetic field during the growth and saturation phase of the spherical dynamo experiment are required to verify these numerical results. In saturation, the cylindrical VKS dynamo experiment shows a large-scale magnetic field that is, surprisingly and contradictory to earlier numerical predictions, aligned with the axis of symmetry of the device [16].
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References

[1] Moffatt H K 1978 Magnetic Field Generation in Electrically Conducting Fluids (Cambridge: Cambridge University Press)
[2] Rüdiger G and Hollerbach R 2004 The Magnetic Universe (Weinheim: Wiley)
[3] Dormy E and Soward A M (ed) 2007 Mathematical Aspects of Natural Dynamos (London: Chapman and Hall)
[4] Gailitis A et al 2000 Detection of a flow induced magnetic field eigenmode in the Riga dynamo facility Phys. Rev. Lett. 84 4365
[5] Stieglitz R and Müller U 2001 Experimental demonstration of a homogeneous two-scale dynamo Phys. Fluids 13 561
[6] Buffett B and Matsui H 2007 Core turbulence ed D Gubbins and E Herrero-Bervera Encyclopedia of Geomagnetism and Paleomagnetism (Berlin: Springer)
[7] Dudley M L and James R W 1989 Time-dependent kinematic dynamos with stationary flows Proc. R. Soc. A 425 407
[8] Peffley N L, Cawthorne A B and Lathrop D P 2000 Toward a self-generating magnetic dynamo: the role of turbulence Phys. Rev. E 61 5287
[9] Lathrop D P, Shew W L and Sisan D R 2001 Laboratory experiments on the transition to MHD dynamos Plasma Phys. Control. Fusion 43 A151
[10] Forest C B, Bayliss R A, Kendrick R D, Nornberg M D, O’Connell R and Spence E J 2002 Hydrodynamic and numerical modeling of a spherical, homogeneous dynamo experiment Magnetohydrodynamics 38 107–20
[11] Nornberg M D, Spence E J, Kendrick R D, Jacobson C M and Forest C B 2006 Intermittent magnetic field excitation by a turbulent flow of liquid sodium Phys. Rev. Lett. 97 044503
[12] Nornberg M D, Spence E J, Kendrick R D, Jacobson C M and Forest C B 2006 Measurements of the magnetic field induced by a turbulent flow of liquid metal Phys. Plasmas 13 055901
[13] Spence E J, Nornberg M D, Jacobson C M, Kendrick R D and Forest C B 2006 Observation of a turbulence-induced large scale magnetic field Phys. Rev. Lett. 96 055002
[14] Spence E J, Nornberg M D, Jacobson C M, Parada C A, Taylor N Z, Kendrick R D and Forest C B 2007 Turbulent diamagnetism in flowing liquid sodium Phys. Rev. Lett. 98 164503
[15] Spence E J, Nornberg M D, Bayliss R A, Kendrick R D and Forest C B 2008 Fluctuation-driven magnetic fields in the madison dynamo experiment Phys. Plasmas 15 055910
[16] Monchaux R et al 2007 Generation of a magnetic field by dynamo action in a turbulent flow of liquid sodium Phys. Rev. Lett. 98 044502
[17] Ravelet F et al 2008 Chaotic dynamos generated by a turbulent flow of liquid sodium Phys. Rev. Lett. 101 074502
[18] Monchaux R et al 2009 The von Kármán sodium experiment: turbulent dynamical dynamos Phys. Fluids 21 035108
[19] Berhanu M et al 2009 Bistability between a stationary and an oscillatory dynamo in a turbulent flow of liquid sodium J. Fluid Mech. 641 217
[20] Berhanu M et al 2010 Dynamo regimes and transitions in the VKS experiment Eur. Phys. J. B 77 459

New Journal of Physics 13 (2011) 073019 (http://www.njp.org/)
[21] Ravelet F, Chiffaudel A, Daviaud F and Léorat J 2005 Toward an experimental von kármán dynamo: numerical studies for an optimized design Phys. Fluids 17 7104

[22] Spence E J, Reuter K and Forest C B 2009 A spherical plasma dynamo experiment Astrophys. J. 700 470

[23] Bayliss R A, Forest C B, Nornberg M D, Spence E J and Terry P W 2007 Numerical simulations of current generation and dynamo excitation in a mechanically forced turbulent flow Phys. Rev. E 75 026303

[24] Reuter K, Jenko F, Forest C B and Bayliss A R 2008 A parallel implementation of an MHD code for the simulation of mechanically driven, turbulent dynamos in spherical geometry Comput. Phys. Commun. 179 245

[25] Reuter K, Jenko F and Forest C B 2009 Hysteresis cycle in a turbulent, spherically bounded MHD dynamo model New J. Phys. 11 013027

[26] Reuter K, Jenko F, Tilgner A and Forest C B 2009 Wave-driven dynamo action in spherical magnetohydrodynamic systems Phys. Rev. E 80 056304

[27] Mininni P D, Ponty Y, Montgomery D C, Pinton J-F, Politano H and Pouquet A 2005 Dynamo regimes with a nonhelical forcing Astrophys. J. 626 853

[28] Ponty Y, Mininni P D, Montgomery D C, Pinton J-F, Politano H and Pouquet A 2005 Numerical study of dynamo action at low magnetic Prandtl numbers. Phys. Rev. Lett. 94 164502

[29] Mininni P D 2006 Turbulent magnetic dynamo excitation at low magnetic Prandtl number Phys. Plasmas 13 056502

[30] Ponty Y, Laval J-P, Dubrulle B, Daviaud F and Pinton J-F 2007 Subcritical dynamo bifurcation in the Taylor–Green flow Phys. Rev. Lett. 99 224501

[31] Schekochihin A A, Iskakov A B, Cowley S C, McWilliams J C, Proctor M R E and Yousef T A 2007 Fluctuation dynamo and turbulent induction at low magnetic Prandtl numbers New J. Phys. 9 300

[32] Ravelet F, Chiffaudel A and Daviaud F 2008 Supercritical transition to turbulence in an inertially-driven von Karman closed flow J. Fluid Mech. 601 339

[33] Laval J-P, Blaineau P, Leprovost N, Dubrulle B and Daviaud F 2006 Influence of turbulence on the dynamo threshold Phys. Rev. Lett. 96 204503

[34] Dubrulle B, Blaineau P, Mafra Lopes O, Daviaud F, Laval J-P and Dolganov R 2007 Bifurcations and dynamo action in a Taylor–Green flow New J. Phys. 9 308

[35] Cortet P-P, Diribarne P, Monchaux R, Chiffaudel A, Daviaud F and Dubrulle B 2009 Normalized kinetic energy as a hydrodynamical global quantity for inhomogeneous anisotropic turbulence Phys. Fluids 21 025104

[36] Moffatt K 1961 The amplification of a weak applied magnetic field by turbulence in fluids of moderate conductivity J. Fluid Mech. 11 625

[37] Boldyrev S and Cattaneo F 2004 Magnetic-field generation in Kolmogorov turbulence Phys. Rev. Lett. 92 144501

[38] Malyshevkin L M and Boldyrev S 2010 Magnetic dynamo action at low magnetic Prandtl numbers Phys. Rev. Lett. 105 215002

[39] Ponty Y, Mininni P D, Laval J-P, Alexakis A, Baerenzung J, Daviaud F, Dubrulle B, Pinton J-F, Politano H and Pouquet A 2008 Linear and nonlinear features of the Taylor–Green dynamo C. R. Phys. 9 749

[40] Iskakov A B, Schekochihin A A, Cowley S C, McWilliams J C and Proctor M R E 2007 Numerical demonstration of fluctuation dynamo at low magnetic Prandtl numbers Phys. Rev. Lett. 98 208501

[41] Kaplan E J, Clark M M, Nornberg M D, Rahbarnia K, Rasmus A M, Taylor N Z and Forest C B 2011 Reducing global turbulent resistivity by eliminating large eddies in a spherical liquid-sodium experiment Phys. Rev. Lett. 106 254502

New Journal of Physics 13 (2011) 073019 (http://www.njp.org/)