ON THE DERIVED DG FUNCTORS

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Abstract. Assume that abelian categories $A$, $B$ over a field admit countable direct limits and that these limits are exact. Let $F : D^+_dg(A) \to D^+_dg(B)$ be a DG quasi-functor such that the functor $Ho(F) : D^+(A) \to D^+(B)$ carries $D^{>0}(A)$ to $D^{>0}(B)$ and such that, for every $i > 0$, the functor $H^i F : A \to B$ is effaceable. We prove that $F$ is canonically isomorphic to the right derived DG functor $RH^0(F)$. We also prove a similar result for bounded derived DG categories in a more general setting. We give an example showing that the corresponding statements for triangulated functors are false. We prove a formula that expresses Hochschild cohomology of the categories $D^{b}_{dg}(A), D^{b}_{dg}(A)$ as the Ext groups in the abelian category of left exact functors $A \to IndB$.

1. Main results

Let $A$ and $B$ be abelian categories, and let

$RF_{tri} : D^+(A) \to D^+(B)$

be the right derived functor of some left exact functor $F : A \to B$. Then, the corresponding cohomological $\delta$-functor $R^* F = H^* RF_{tri} : A \to B$ has the following property: $H^i RF_{tri} = 0$ for $i < 0$, effaceable for $i > 0$ and $H^0 RF_{tri} \simeq F$. Conversely, according to a result of Grothendieck ([G]) every cohomological $\delta$-functor $T^* : A \to B$ satisfying the above property is canonically isomorphic to the right derived functor $R^* F$. The purpose of this paper is to extend this extremely useful characterization of $R^* F$ to the derived category level. Unfortunately, Verdier’s notion of triangulated functor is too poor to allow such a simple characterization of the derived functors (see Remark 1.1). In order to get a meaningful statement one has to consider triangulated functors with some kind of enrichment. Arguably the most useful notion here is the one of DG quasi-functor (or essentially equivalent notion of $A_{\infty}$-functor). Indeed, works of Keller and Drinfeld ([K2], [Dri]) provide a canonical DG enhancement $D^+_{dg}(A)$ of Verdier’s triangulated derived category. Roughly, a DG quasi-functor $F : D^+_{dg}(A) \to D^+_{dg}(B)$ is a diagram of the form

$D^+_{dg}(A) \leftarrow C \rightarrow D^+_{dg}(B)$

(1.1)

where $C$ is a DG category, $S$, $G$ DG functors, and $S$ is a homotopy equivalence. Every quasi-functor ([1]) yields a triangulated functor $Ho(F) : D^+(A) \to D^+(B)$, but the converse is not true in general. Nevertheless, many of the natural triangulated functors come together with a DG enhancement. For example, the triangulated derived functor $RF$ can be canonically promoted to a DG quasi-functor ([Dri] §5). The main result of this paper states that under certain mild assumptions on abelian categories $A$ and $B$ the DG quasi-functors isomorphic to the DG derived ones are precisely the DG quasi-functors satisfying Grothendieck’s condition above. To state the result we need to introduce some notations.
Let $k$ be a commutative ring. Denote by $\text{Mod}(k)$ the category of $k$-modules. We shall say that $k$-linear category\footnote{i.e., a category enriched over $\text{Mod}(k)$.} is $k$-flat if, for every two objects $X, Y$, the $k$-module $\text{Hom}(X, Y)$ is flat. Given a $k$-linear exact category $\mathcal{A}$ we denote by $D^b_{dg}(\mathcal{A})$ the corresponding bounded derived DG category over $k$. This the DG quotient (\cite{Dri}) of the DG category $C^b_{dg}(\mathcal{A})$ of bounded complexes by the subcategory of acyclic ones (\cite{N}, §1). The homotopy category of $D^b_{dg}(\mathcal{A})$ is the triangulated derived category $D^b(\mathcal{A})$ as defined in (\cite{N}). Let $\mathcal{B}$ be another $k$-linear abelian category, $D^b_{dg}(\mathcal{B})$ the corresponding bounded derived DG category, and let $\mathcal{T}(D^b_{dg}(\mathcal{A}), D^b_{dg}(\mathcal{B}))$ be the triangulated category of DG quasi-functors $\mathcal{F} : D^b_{dg}(\mathcal{A}) \to D^b_{dg}(\mathcal{B})$ (\cite{Dri}, §16.1). Given such $\mathcal{F}$ and an integer $i$ we denote by $H^i\mathcal{F} : \mathcal{A} \to \mathcal{B}$ the composition

$\mathcal{A} \to D^b_{dg}(\mathcal{A}) \xrightarrow{\mathcal{F}} D^b_{dg}(\mathcal{B}) \xrightarrow{H^i} \mathcal{B}.$

**Theorem 1.** Let $\mathcal{A}$ be a small $k$-flat exact category and $\mathcal{B}$ a small abelian $k$-linear category.

1. Assume that a DG quasi-functor

$$\mathcal{F} : D^b_{dg}(\mathcal{A}) \to D^b_{dg}(\mathcal{B})$$

has the following property:

(P) The functors $H^i\mathcal{F} : \mathcal{A} \to \mathcal{B}$ are 0 for every $i < 0$ and effaceable (i.e., for every object $X \in \mathcal{A}$, there is an admissible monomorphism $X \to Y$ such that the induced morphism $H^i\mathcal{F}(X) \to H^i\mathcal{F}(Y)$ is 0) for every $i > 0$.

Then the functor $\mathcal{F} = H^0\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is left exact, has a right derived DG quasi-functor (\cite{Dri}, §5)

$$RF : D^b_{dg}(\mathcal{A}) \to D^b_{dg}(\mathcal{B}),$$

and there is a unique isomorphism $\mathcal{F} \simeq RF$ such that the induced automorphism $F = H^0(\mathcal{F}) \simeq H^0(RF) = F$ equals $\text{Id}$. Conversely, the right derived DG quasi-functor of any left exact functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ satisfies property (P).

2. For every two DG quasi-functors $\mathcal{F}, \mathcal{G} \in \mathcal{T}(D^b_{dg}(\mathcal{A}), D^b_{dg}(\mathcal{B}))$ satisfying property (P) and every $i < 0$

$$\text{Hom}_{\mathcal{T}(D^b_{dg}(\mathcal{A}), D^b_{dg}(\mathcal{B}))}(\mathcal{F}, \mathcal{G}[i]) = 0,$$

$$\text{Hom}_{\mathcal{T}(D^b_{dg}(\mathcal{A}), D^b_{dg}(\mathcal{B}))}(\mathcal{F}, \mathcal{G}) = \text{Hom}_{\text{Fct}(\mathcal{A}, \mathcal{B})}(H^0\mathcal{F}, H^0\mathcal{G}).$$

Here $\text{Fct}(\mathcal{A}, \mathcal{B})$ denotes the category of all $k$-linear functors $\mathcal{A} \to \mathcal{B}$.

**Remark 1.1.** The analogous statement for triangulated functors is false (in contrast to DG enhanced functors). Here is an example of a triangulated functor $\mathcal{F}_{\text{tri}} : D^b(\text{Mod}(\mathbb{C}[x])) \to D^b(\text{Mod}(\mathbb{C}[x]))$ such that $H^i\mathcal{F}_{\text{tri}}$ is 0 for $i < 0$ and effaceable for $i > 0$ but $\mathcal{F}_{\text{tri}}$ is not isomorphic (as a triangulated functor) to a right derived functor. Recall (\cite{KS}, §10.1.9) that a triangulated functor $\mathcal{F}_{\text{tri}}$ is a pair $(\mathcal{F}_{\text{add}}, \tau)$, where $\mathcal{F}_{\text{add}}$ is an additive functor and $\tau$ is an isomorphism of functors $\mathcal{F}_{\text{add}} \circ T \simeq T \circ \mathcal{F}_{\text{add}}$ (here $T$ is the translation functor: $T(X) = X[1]$), preserving the class of distinguished triangles. Consider $\mathcal{F}_{\text{tri}} = (\text{Id}, \tau) : D^b(\text{Mod}(\mathbb{C}[x])) \to D^b(\text{Mod}(\mathbb{C}[x]))$, where $\tau : \text{Id} \circ T = T \xrightarrow{-1} T = T \circ \text{Id}$ is the multiplication by $-1$. We claim that $\mathcal{F}_{\text{tri}}$ is...
are exact. Indeed, such isomorphism would be given by an automorphism $S$ of the identity functor $Id : D^b(\text{Mod}(\mathbb{C}[x])) \to D^b(\text{Mod}(\mathbb{C}[x]))$ with the following property: for every $M \in D^b(\text{Mod}(\mathbb{C}[x]))$

$$S(M)[1] = -S(M[1]) : M[1] \to M[1].$$

Taking $M = \mathbb{C}$ and observing that $\text{Hom}(\mathbb{C}, \mathbb{C}[1]) = \mathbb{C}$ we see that no such $S$ may exist.

**Remark 1.2.** It is likely that the $k$-flatness assumption on $\mathcal{A}$ is abundant. However, I cannot prove this.

We have a similar result for bounded from below derived DG categories. If $\mathcal{A}$ is a $k$-linear abelian category we will write $D^+_{dg}(\mathcal{A})$ for the bounded from below derived DG category of $\mathcal{A}$ and $D^+_{ad}(\mathcal{A})$ for the corresponding triangulated category. Let $D^{\geq n}(\mathcal{A})$ be the full subcategory of $D^+(\mathcal{A})$ that consists of complexes with trivial cohomology in degrees less than $n$. We say that a DG quasi-functor

$$F : D^+_{dg}(\mathcal{A}) \to D^+_{dg}(\mathcal{B})$$

has property ($P'$) if

$$(P') \text{ Ho}(F)(D^{\geq 0}(\mathcal{A})) \subset D^{\geq 0}(\mathcal{B}) \text{ and, for every } i > 0, \text{ the functor } H^iF : A \to B \text{ is effaceable.}$$

**Theorem 2.** Let $k$ be a field and let $\mathcal{A}, \mathcal{B}$ be small abelian $k$-linear categories. Assume that the both categories are closed under countable direct limits and that these limits are exact.

1. Let $F \in \mathcal{T}(D^+_{dg}(\mathcal{A}), D^+_{dg}(\mathcal{B}))$ be a DG quasi-functor satisfying property ($P'$) and $F := H^0F : A \to B$. The functor $F$ admits a right derived DG quasi-functor $RF : D^+_{dg}(\mathcal{A}) \to D^+_{dg}(\mathcal{B})$ and there is a unique isomorphism $F \simeq RF$ such that the induced automorphism $F = H^0(F) \simeq H^0(RF) = F$ equals $Id$. Conversely, a right derived DG quasi-functor of any left exact functor $F : A \to B$ satisfies property ($P'$).

2. For every two DG quasi-functors $F, G \in \mathcal{T}(D^+_{dg}(\mathcal{A}), D^+_{dg}(\mathcal{B}))$ satisfying property ($P'$) and every $i < 0$

$$\text{Hom}_{\mathcal{T}(D^+_{dg}(\mathcal{A}), D^+_{dg}(\mathcal{B}))}(F, G[i]) = 0,$$

$$\text{Hom}_{\mathcal{T}(D^+_{dg}(\mathcal{A}), D^+_{dg}(\mathcal{B}))}(F, G) = \text{Hom}_{\text{Map}(\mathcal{A}, \mathcal{B})}(H^0F, H^0G).$$

The main ingredient of the proof of Theorem 2 is the following construction. Let $\text{Sh}(\mathcal{A}^o \otimes_k \mathcal{B})$ be the category of $k$-linear contravariant functors $\mathcal{A}^o \otimes_k \mathcal{B} \to \text{Mod}(k)$ that are left exact with respect to the both arguments. Every $k$-linear left exact functor $F : A \to B$ yields $s(F) \in \text{Sh}(\mathcal{A}^o \otimes_k \mathcal{B})$:

$$s(F)(X \otimes X') = \text{Hom}_B(X', F(X)).$$

Let $\mathcal{T}^+ \subset \mathcal{T}(D^+_{dg}(\mathcal{A}), D^+_{dg}(\mathcal{B}))$ be the full triangulated subcategory whose objects are quasi-functors $F$ such that $\text{Ho}(F)(D^{\geq 0}(\mathcal{A})) \subset D^{\geq n}(\mathcal{B})$ for some $n$. Using key Lemma 2.1 we construct a fully faithful embedding

$$(1.2) \mathcal{T}^+ \hookrightarrow D(\text{Sh}(\mathcal{A}^o \otimes_k \mathcal{B}))$$

that carries every DG quasi-functor $F$ satisfying property ($P'$) to $s(F) \in \text{Sh}(\mathcal{A}^o \otimes_k \mathcal{B}) \subset D(\text{Sh}(\mathcal{A}^o \otimes_k \mathcal{B})).$
As another application of (1.2) we compute the Hochschild cohomology of the derived DG category. Recall (see, e.g. [K1], §5.4) that the Hochschild cohomology of a DG category \( \mathcal{C} \) can be interpreted as

\[
HH^*(\mathcal{C}, \mathcal{C}) = \text{Hom}_{\mathcal{T}(\mathcal{C}, \mathcal{C})}(\text{Id}_\mathcal{C}, \text{Id}_\mathcal{C}[i]).
\]

The composition in \( \mathcal{C} \) makes \( HH^*(\mathcal{C}, \mathcal{C}) \) a graded commutative algebra over \( k \).

**Theorem 3.** Let \( k \) be a field and let \( A \) be a small abelian \( k \)-linear category. There is an isomorphism of algebras

\[
HH^*(D^+_dg(A), D^+_dg(A)) \simeq Ext^*_\mathcal{Sh}(\mathcal{A}^o \otimes_k \mathcal{A})(s(Id_A), s(Id_A)).
\]

If, in addition, \( A \) is closed under countable direct limits and that these limits are exact

\[
HH^*(D^+_dg(A), D^+_dg(A)) \simeq Ext^*_\mathcal{Sh}(\mathcal{A}^o \otimes_k \mathcal{A})(s(Id_A), s(Id_A)).
\]

**Remark 1.3.** The category \( \mathcal{Sh}(\mathcal{A}^o \otimes_k \mathcal{A}) \) has a tensor structure that extends the tensor structure on the category of left exact endofunctors \( \mathcal{A} \to \mathcal{A} \) given by the composition. This can be used to promote (1.4), (1.5) to isomorphisms of Gerstenhaber algebras (see, e.g. [K1], §5.4).

**Notation.** Given a category \( \mathcal{C} \) we denote by \( \mathcal{C}^o \) the opposite category. If \( \mathcal{C} \) is a DG category we will write \( Ho \mathcal{C} \) for the corresponding homotopy category ([Dri], §2.7). For example, \( Ho \mathcal{C}(\text{Mod}(k)) \) denotes the homotopy category of complexes of \( k \)-modules. The derived category of right DG modules over a DG category \( \mathcal{C} \) will be denoted by \( D(\mathcal{C}) \) ([Dri], §2.3)\(^2\). We will write \( \mathcal{C}_o \) for the DG category of semi-free DG \( \mathcal{C}^o \)-modules ([BV], 1.6.1). We have a canonical equivalence of triangulated categories \( Ho \mathcal{C}_o \to D(\mathcal{C}) \) ([BV], 1.6.4). For DG categories \( \mathcal{C}, \mathcal{C}' \) we denote by \( \mathcal{T}(\mathcal{C}, \mathcal{C}') \) the category of DG quasi-functors ([Dri], §16.1). If \( \mathcal{C}' \) is a pretriangulated ([Dri], §2.4) \( \mathcal{T}(\mathcal{C}, \mathcal{C}') \) has a canonical structure of triangulated category. If \( F \in \mathcal{T}(\mathcal{C}, \mathcal{C}') \) we will write \( Ho(F) \) for the corresponding functor between the homotopy categories.

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2. Proofs

**Proof of theorem 1.** Let \( \mathcal{T}^+ \subset \mathcal{T} := \mathcal{T}(D^+_dg(A), D^+_dg(B)) \) be the full triangulated subcategory whose objects are quasi-functors \( F \) such that \( H^i F = 0 \) for sufficiently small \( i \). To prove Theorem we shall construct (in Lemma 2A below) a fully faithful embedding of \( \mathcal{T}^+ \) into derived category of a certain abelian category \( \mathcal{Sh}(\mathcal{A}^o \otimes_k \mathcal{B}) \) that takes every functor \( F \in \mathcal{T}^+ \) satisfying property (P) to an object of the heart \( \mathcal{Sh}(\mathcal{A}^o \otimes_k \mathcal{B}) \subset \mathcal{Sh}(\mathcal{A}^o \otimes_k \mathcal{B}) \).

Under our flatness assumption on \( A \), the category \( \mathcal{T} \) is a full subcategory of the derived category \( D(D^+_dg(A), D^+_dg(B)) \) of right DG modules over \( D^+_dg(A) \otimes_k D^+_dg(B) \) that consists of all \( M \in D(D^+_dg(A), D^+_dg(B)) \) such that, for every \( X \in D^+_dg(A) \), the module \( M(X) \in D(D^+_dg(B)) \) belongs to the essential image of the Yoneda embedding \( D^+_dg(B) \to D(D^+_dg(B)) \) ([Dri], §16.1).

\(^2\)Drinfeld’s notation for this category is \( D(\mathcal{C}) \). We use \( D(\mathcal{C}) \) to avoid confusion with Verdier’s derived category of an abelian category \( \mathcal{C} \) that is denoted by \( D(\mathcal{C}) \).
Consider the restriction functor
\[ \mathbb{D}(D^b_{dg}(A)^o \otimes_k D^b_{dg}(B)) \xrightarrow{\beta} \mathbb{D}(A^o \otimes_k B) \]
induced by the DG quasi-functor \( A^o \otimes_k B \to D^b_{dg}(A)^o \otimes_k D^b_{dg}(B) \). By definition, the triangulated category \( \mathbb{D}(A^o \otimes_k B) \) is the derived category of the abelian category \( PSh := PSh(A^o \otimes_k B) \) of \( k \)-linear presheaves i.e., the category of \( k \)-linear contravariant functors \( A^o \otimes_k B \to \text{Mod}(k) \). Consider a Grothendieck topology on \( A^o \otimes_k B \) whose covers are maps of the form \( f \otimes g : Y \otimes Y' \to X \otimes X' \), where \( X, Y \in A^o \), \( X', Y' \in B \), and \( f : Y \to X, g : Y' \to X' \) are admissible epimorphisms \(^3\) i.e., a sieve \( C \) over \( X \otimes X' \) is a covering sieve if there exist \( f : Y \to X, g : Y' \to X' \) as above such that \( Y \otimes Y' \xrightarrow{f \otimes g} X \otimes X' \in C \). The axioms of Grothendieck topology (see, e.g. [KS], §16.1) are immediate except for the one which is the following statement: for every cover \( Y \otimes Y' \xrightarrow{f \otimes q} X \otimes X' \) and every morphism \( Z \otimes Z' \xrightarrow{o} X \otimes X' \) there exists a cover \( T \otimes T' \xrightarrow{\phi \otimes q} Z \otimes Z' \) and a morphism such that \( T \otimes T' \xrightarrow{\psi} Y \otimes Y' \) such that \( (f \otimes q) \circ \psi = \phi \circ (p \otimes q) \), which is a consequence of the base change axiom of exact category ([Q], §2). Let \( Sh := Sh(A^o \otimes_k B) \) be the subcategory of \( PSh \) that consists of objects satisfying the sheaf property. Explicitly, objects of the category \( Sh(A^o \otimes_k B) \) are contravariant functors \( A^o \otimes_k B \to \text{Mod}(k) \) that are left exact with respect to the both arguments. The embedding \( Sh \to PSh \) has a left adjoint functor (sheafification)
\[ ^\gamma : PSh \to Sh, \]
which is exact ([KS], §17.4). We denote by \( \gamma : D(PSh) \to D(Sh) \) the induced functor between the derived categories. The composition
\[ \mathbb{D}(D^b_{dg}(A)^o \otimes_k D^b_{dg}(B)) \xrightarrow{\beta} D(PSh) \xrightarrow{\gamma} D(Sh) \]
is not fully faithful in general, however, we have the following result.

**Lemma 2.1.** Let \( \mathbb{D}^+ \subset \mathbb{D}(D^b_{dg}(A)^o \otimes_k D^b_{dg}(B)) \) be the full subcategory whose objects are DG modules \( M \) such that \( \beta(M) \in \mathbb{D}^+(PSh) \). Then the functor
\[ \mathbb{S} : \mathbb{D}^+ \xrightarrow{\beta} \mathbb{D}^+(PSh) \xrightarrow{\gamma} \mathbb{D}^+(Sh) \]
is an equivalence of categories.

**Proof.** The category \( \mathbb{D}^b_{dg}(A)^o \otimes_k \mathbb{D}^b_{dg}(B) \) is a DG quotient of the category \( C^b_{dg}(A)^o \otimes_k C^b_{dg}(B) \) by the full subcategory whose objects are of the form \( X^* \otimes X'^* \), where either \( X^* \) or \( X'^* \) is acyclic. It then follows from ([Dri], Theorem 1.6.2) that the functor
\[ \beta : \mathbb{D}(D^b_{dg}(A)^o \otimes_k D^b_{dg}(B)) \to \mathbb{D}(C^b_{dg}(A)^o \otimes_k C^b_{dg}(B)) = D(PSh) \]
is fully faithful and that its essential image consists of all DG-modules \( M \in \mathbb{D}(C^b_{dg}(A)^o \otimes_k C^b_{dg}(B)) \) that carry every \( X \otimes X' \) with the above property to an acyclic complex. Identifying the category \( \mathbb{D}(C^b_{dg}(A)^o \otimes_k C^b_{dg}(B)) \) with \( D(PSh) \) and observing that the subcategories of acyclic complexes in the homotopy categories \( HoC^b_{dg}(A), HoC^b_{dg}(B) \) are generated by short exact sequences ([N], §1) we exhibit \( \mathbb{D}(D^b_{dg}(A)^o \otimes_k D^b_{dg}(B)) \) as a full subcategory \( \mathfrak{R} \subset D(PSh) \) whose objects are complexes \( F^* \) of presheaves satisfying the following two conditions:

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\(^3\)By definition, admissible epimorphisms \( Y \to X \) in \( A^o \) are admissible monomorphisms \( X \to Y \) in \( A \).
For any exact sequence $0 \to Z \to Y \to X \to 0$ in $\mathcal{A}^o$ and any $X' \in \mathcal{B}$ the total complex of
\[(2.1) \quad F^i(X \otimes X') \to F^i(Y \otimes X') \to F^i(Z \otimes X')
\]
is acyclic.

For any $X \in \mathcal{A}^o$ and any exact sequence $0 \to Z' \to Y' \to X' \to 0$ in $\mathcal{B}$ the total complex of
\[
F^i(X \otimes X') \to F^i(X \otimes Y') \to F^i(X \otimes Z')
\]
is acyclic.

Observe that, for every $F^i \in \mathcal{R}$ and an exact sequence $0 \to Z \to Y \to X \to 0$ in $\mathcal{A}^o$, we have a long exact sequence of $k$-modules
\[(2.2) \quad \cdots H^{m-1}(F(Z \otimes X')) \to H^m(F(X \otimes X')) \to H^m(F(Y \otimes X')) \to H^m(F(Z \otimes X')) \to \cdots
\]

The equivalence of categories $\beta : \mathcal{D}(\mathcal{D}_{dg}(\mathcal{A})^o \otimes_k \mathcal{D}_{dg}(\mathcal{B})) \xrightarrow{\sim} \mathcal{R} \subset \mathcal{D}(\mathcal{PSh})$
carries $\mathbb{D}^+$ to the subcategory $\mathcal{R}^+$ of $\mathcal{R}$ that consists of bounded from below complexes.

The derived category of sheaves $\mathcal{D}(\mathcal{Sh})$ is the quotient of the derived category of presheaves by the subcategory $\mathcal{I}_{lac} \subset \mathcal{D}(\mathcal{PSh})$ of locally (for our Grothendieck topology on $\mathcal{A}^o \otimes_k \mathcal{B}$) acyclic complexes ([BV], §1.1). We shall prove that
\[(2.3) \quad \mathcal{R}^+ \subset \mathcal{I}_{lac},
\]
where $\mathcal{I}_{lac}$ denotes the right orthogonal complement to $\mathcal{I}_{lac}$ in $\mathcal{D}(\mathcal{PSh})$ ([BV], §1.1);
\[(2.4) \quad \text{Hom}_{\mathcal{D}(\mathcal{PSh})}(G^\cdot, F^\cdot) = 0.
\]

for every $G^\cdot \in \mathcal{I}_{lac}$ and $F^\cdot \in \mathcal{R}^+$. Without loss of generality we may assume that $F^\cdot$ has trivial cohomology in negative degrees: $F^i = F^0 \to F^1 \to \cdots$. Let $F^\cdot = \tilde{F}^0 \to \tilde{F}^1 \to \cdots$ be the corresponding complex of sheaves. Since the category of sheaves has enough injective objects (see, e.g. [KS], Theorems 9.6.2, 18.1.6) there exists a complex $I^\cdot = I^0 \to I^1 \to \cdots$ of injective sheaves together with a morphism $\tilde{F}^\cdot \to I^\cdot$ which is an isomorphism in the derived category of sheaves. Let us show that the composition
\[
\delta : F^\cdot \to \tilde{F}^\cdot \to I^\cdot
\]
is an isomorphism in the category derived category of presheaves. Indeed, every injective sheaf, viewed as a presheaf, is an object of $\mathcal{R}$. Thus $I^\cdot$ and $\text{cone}(\delta)$ are in $\mathcal{R}^+$. Assuming that $\text{cone}(\delta) \neq 0$ choose the smallest integer $m$ such that
\[
0 \neq H^m(\text{cone}(\delta)) \in \mathcal{PSh}.
\]
Then there exist an object $X \otimes X' \in \mathcal{A}^o \otimes_k \mathcal{B}$ and a nonzero element $a \in H^m(\text{cone}(\delta))(X \otimes X')$. Since the sheafification of $H^m(\text{cone}(\delta))$ is 0 there exists a cover $p : Y \otimes Y' \to X \otimes X'$ such that
\[
0 = p^*a \in H^m(\text{cone}(\delta))(Y \otimes Y').
\]
Writing $p$ as a composition
\[
X \otimes X' \xrightarrow{1 \otimes q} Y \otimes X' \xrightarrow{\cdot 1} X \otimes X'
\]
we may assume $(f \otimes 1)^* a = 0$ (otherwise, we replace $X \otimes X'$ by $Y \otimes X'$). Let us look at a fragment of the long exact sequence \( (2.2) \) applied to $F = cone(\delta)$ and the exact sequence $0 \to Z \to Y \to X \to 0$:

$$H^{m-1}(cone(\delta))(Z \otimes X') \to H^{m}(cone(\delta))(X \otimes X') \to H^{m}(cone(\delta))(Y \otimes X').$$

Since by our assumption $H^{m-1}(cone(\delta)) = 0$, it follows that $(f \otimes 1)^*$ is injective and, hence, $a = 0$. This contradiction proves that $cone(\delta) = 0$ i.e., $\delta$ is a quasi-isomorphism. Thus, to complete the proof of \( (2.3) \) it suffices to show that

$$Hom_{D(PSh)}(G', I) = 0,$$

for every $G' \in I_{\text{Iac}}$ and every bounded from below complex of injective sheaves $I$. Indeed, every morphism $h : G' \to I$ in the derived category is represented by a diagram in $C(PSh(A^0 \otimes k B))$

$$G' \leftarrow G'' \xrightarrow{h'} I,$$

where the first arrow is a quasi-isomorphism (and, in particular, $G'' \in I_{\text{Iac}}$). If $h'$ is homotopic to 0 then $h = 0$ in the derived category. Thus, it is enough to show that

$$Hom_{K(PSh)}(G'', I) = 0$$

where $K(PSh)$ denotes the homotopy category of complexes. We have

$$Hom_{K(PSh)}(G'', I) \xrightarrow{\sim} Hom_{K(Sh)}(G'', I) \xrightarrow{\sim} Hom_{D(Sh)}(G'', I).$$

The first arrow is an isomorphism because all terms of the complex $I$ are sheaves; the second arrow is an isomorphism by \( (KS, \text{Lemma 13.2.4}) \). Finally, $Hom_{D(Sh)}(G'', I) = 0$ because the sheafification $G''$ is 0 in $D(Sh)$.

To finish the proof of the lemma, we observe that, for every triangulated category $C$ and its full triangulated subcategory $I$, the composition

$$I^+ \to C \to C/I$$

is a fully faithful embedding: for every $X, Y \in C$

$$Hom_{C/I}(X, Y) := \colim_{f : X' \to X} Hom_C(X', Y),$$

where the colimit is taken over the filtrant category of pairs $(X' \in C, f : X' \to X)$ such that $cone f \in I$. If $Y \in I^+$, then

$$Hom_{C}(X, Y) \xrightarrow{\sim} Hom_{C}(X', Y),$$

and, hence,

$$Hom_{C/I}(X, Y) = Hom_C(X, Y).$$

Applying this remark to $C = D(PSh)$, $I = I_{\text{Iac}}$ and using \( (2.3) \) we conclude that the functor $R^+ \xrightarrow{\gamma} D(Sh)$ is fully faithful and, hence, so is the composition $D^+ \xrightarrow{\sim} R^+ \xrightarrow{\gamma} D(Sh)$. The essentially image the functor $R^+ \xrightarrow{\gamma} D(Sh)$ is $D^+(Sh)$ because every complex of injective sheaves viewed as a complex of presheaves is on object of $\mathcal{R}^+$. \hfill \Box

**Remark 2.2.** Applying Lemma \( (2.1) \) to $k = \mathbb{Z}$ and $\mathcal{A}$ = the category of free abelian groups of finite rank we obtain the following statement: for every small abelian category $\mathcal{B}$

$$\mathbb{D}^+(D_{dg}^b(\mathcal{B})) \xrightarrow{\sim} D^+(PSh(\mathcal{B})) = D^+(Ind(\mathcal{B})).$$
where \( \mathbb{D}^+(D^b_{dg}(\mathcal{B})) \) is a full subcategory of \( \mathbb{D}(D^b_{dg}(\mathcal{B})) \) that maps to \( D^+(PSh(\mathcal{B})) \) under the restriction functor (and the ind-completion \( Ind(\mathcal{B}) \) is just another name for \( PSh(\mathcal{B}) \) [KS, §8.6]). Note the functor

\[
\mathbb{D}(D^b_{dg}(\mathcal{B})) \to D(Ind(\mathcal{B}))
\]

is not an equivalence of categories in general. In fact, the functor \( \mathbb{D} \) factors as

\[
\mathbb{D}(D^b_{dg}(\mathcal{B})) \xrightarrow{\phi} HoC(Ind(\mathcal{B}))/HoC^b_{ac}(\mathcal{B}) \xrightarrow{p} D(Ind(\mathcal{B})),
\]

where \( HoC^b_{ac}(\mathcal{B}) \) is the smallest triangulated subcategory of the homotopy category of acyclic complexes \( HoC_{ac}(Ind(\mathcal{B})) \) that contains finite acyclic complexes \( HoC^b_{ac}(\mathcal{B}) \) and closed under arbitrary direct sums; the functor \( p \) is the projection

\[
HoC(Ind(\mathcal{B}))/HoC^b_{ac}(\mathcal{B}) \to HoC(Ind(\mathcal{B}))/HoC_{ac}(Ind(\mathcal{B})).
\]

The equivalence \( \phi \) can be constructed as follows. Let \( C^b_{ac}(\mathcal{B}) \) be the full subcategory of the DG category \( C(Ind(\mathcal{B})) \) whose objects are those of \( HoC^b_{ac}(\mathcal{B}) \). The DG quasi-functor \( D^b_{dg}(\mathcal{B}) \to C(Ind(\mathcal{B}))/C^b_{ac}(\mathcal{B}) \) extends uniquely to a quasi-functor

\[
\phi_{dg}: D^b_{dg}(\mathcal{B}) \to C(Ind(\mathcal{B}))/C^b_{ac}(\mathcal{B})
\]

that commutes with arbitrary direct sums ([BV], §1.6.1). Define

\[
\phi := Ho\phi_{dg}.
\]

Let us show that \( \phi \) is an equivalence of categories. The subcategory \( HoC^b_{ac}(\mathcal{B}) \subset HoC(Ind(\mathcal{B})) \) is generated by compact objects (e.g., objects of \( HoC^b_{ac}(\mathcal{B}) \)); it follows that the projection \( HoC(Ind(\mathcal{B}))/HoC^b_{ac}(\mathcal{B}) \) carries compact objects of \( HoC(Ind(\mathcal{B})) \) to compact objects of the quotient category ([BV], §1.4.2). In particular, in the commutative diagram

\[
\begin{array}{ccc}
D^b_{dg}(\mathcal{B}) & = & D^b_{dg}(\mathcal{B}) \\
i & \downarrow & \downarrow j \\
\mathbb{D}(D^b_{dg}(\mathcal{B})) & \xrightarrow{\phi} & HoC(Ind(\mathcal{B}))/HoC^b_{ac}(\mathcal{B})
\end{array}
\]

the image of \( j \) consists of compact objects. The same is true for the image of \( i \) ([BV], §1.7). The functors \( i, j \) are fully faithful and their images generate the categories \( \mathbb{D}(D^b_{dg}(\mathcal{B})) \), \( HoC(Ind(\mathcal{B}))/HoC^b_{ac}(\mathcal{B}) \) respectfully. It follows that \( \phi \) is an equivalence of categories.

In general, (e.g., if \( \mathcal{B} \) is the category of finitely generated modules over a finite group) the projection \( p \) is not conservative. However, if the category \( \mathcal{B} \) has finite homological dimension the objects of \( D^b_{dg}(\mathcal{B}) \) are compact in \( D^b_{dg}(Ind(\mathcal{B})) \) and the above argument proves that \( \mathbb{D} \) is an equivalence of categories.

**Corollary 2.3.** The composition

\[
S : \mathcal{T}^+ \xrightarrow{\alpha} \mathbb{D}(D^b_{dg}(\mathcal{A})) \otimes_k D^b_{dg}(\mathcal{B})) \xrightarrow{\beta} D(PSh) \xrightarrow{\gamma} D(Sh)
\]

is a fully faithful embedding.

---

4 Indeed, under our finiteness assumption every complex in \( D^b_{dg}(\mathcal{B}) \) is quasi-isomorphic to a finite complex of projective objects. Thus it is enough to show that every projective object of \( \mathcal{B} \) is compact in \( D(Ind(\mathcal{B})) \). This is clear because every such object is projective and compact in \( Ind(\mathcal{B}) \).
Consider the Yoneda embedding
\[ s : Fun(A, B) \to PSh \]
that takes a functor \( F \in Fun(A, B) \) to the presheaf
\[ s(F)(X \times X') = Hom_B(X', F(X)). \]
If \( F \) is left exact then \( s(F) \) is actually a sheaf.

Let \( \mathcal{F} \in T \) be a DG quasi-functor satisfying property \((P)\). It follows from the definition of \( \mathcal{T}^+ \) given at the beginning of this section that \( \mathcal{F} \in \mathcal{T}^+ \). We shall prove that \( S(\mathcal{F}) \sim \to s(H^0\mathcal{F}) \). Having in mind applications to Theorem 2, we will actually show a slightly more general statement. Namely, let us extend the functor \((2.7)\) to a larger category:
\[ S' : \mathcal{T}(D^b_{dg}(A), D^+_d(B)) \xrightarrow{\alpha_i} \mathcal{D}(D^b_{dg}(A) \otimes_k D^+_d(B)) \xrightarrow{\beta_i} D(PSh) \xrightarrow{\gamma} D(Sh). \]

**Lemma 2.4.** Let \( \mathcal{F} \in \mathcal{T}(D^b_{dg}(A), D^+_d(B)) \) be a DG quasi-functor such that \( H^i\mathcal{F} \) is zero for \( i < 0 \) and effaceable for \( i > 0 \). Set \( s(F) = s(H^0\mathcal{F}) \subset Sh \subset D(Sh) \) \( ^5 \). Then \( S'(\mathcal{F}) \in D(Sh) \) is canonically isomorphic to \( s(F) \).

**Proof.** By definition, the cohomology presheaves of the complex \( \beta'\alpha'(\mathcal{F}) \in D(PSh) \) are given by the formula
\[ H^i(\beta'\alpha'(\mathcal{F}))(Y) = \text{Hom}_{D^+(B)}(X', Ho(\mathcal{F})(X)[i]). \]
Since the negative cohomology of the complex \( Ho(\mathcal{F})(X) \in D^+(B) \) vanish the same is true for \( \beta'\alpha' \mathcal{F} \) and
\[ H^0(\beta'\alpha'(\mathcal{F}))(Y) = \text{Hom}_{D^+(B)}(X', H^0\mathcal{F}(X)) = s(F). \]
It remains to prove that for \( i > 0 \) the sheafification of the presheaf \( H^i(\beta'\alpha'(\mathcal{F})) \) equals zero. Given an integer \( j \) define presheaves \( G^{i,j} \) to be
\[ G^{i,j}(X \otimes X') = \text{Hom}_{D^+(B)}(X', \tau_{\leq j}(Ho(\mathcal{F})(X))[i]). \]
We shall show by induction on \( j \) that for every \( i > 0 \) and every \( j \) the sheafification of \( G^{i,j} \) is zero. This would complete the proof since \( G^{i,j} \sim \to H^i(\beta'\alpha'(\mathcal{F}))(X \otimes X') \) for \( j \geq i \).

For every \( i > 0 \) and every element \( v \) of
\[ G^{i,0}(X \otimes X') = Ext^i_B(X', H^0\mathcal{F}(X)) \]
there exists an epimorphism \( Y' \to X' \) such that \( v \) is annihilated by the map
\[ Ext^i_B(Y', H^0\mathcal{F}(X)) \to Ext^i_B(Y', H^0\mathcal{F}(X)) \]
\((\text{KS, Exercise } 13.17)\). This proves that the sheafification of \( G^{i,0} \) is 0. For the induction step, consider the distinguished triangle
\[ \tau_{\leq j}(Ho(\mathcal{F})(X)) \to \tau_{\leq j+1}(Ho(\mathcal{F})(X)) \to H^{j+1}(\mathcal{F}(X)[-j - 1]) \]
and the corresponding long exact sequence
\[ \to G^{i,j}(X \otimes X') \to G^{i,j+1}(X \otimes X') \to \text{Hom}_{D^+(B)}(X', H^{j+1}\mathcal{F}(X)[-j - 1 + i]) \to . \]
It follows that \( G^{i,j+1} \) fits in a long exact sequence
\[ \to G^{i,j} \to G^{i,j+1} \to Ext^{-j-1}_B(\cdot, H^{j+1}\mathcal{F}(\cdot)) \to . \]
\(^5\)The vanishing of \( H^i\mathcal{F} \) implies that \( F \) is left exact and, hence, \( s(F) \) is a sheaf.
The sheafification of $G^{i,j}$ is 0 by the induction assumption, the sheafification of $\Ext_B^{i,j-1}(-, H^{i+1}F(-))$ is 0 because the functor $H^{i+1}F$ is effaceable. Hence, the sheafification of $G^{i,j+1}$ is 0 as well. \hfill \Box

Now we are ready to prove the second part of the theorem. Given quasi-functors $F, G \in \mathcal{T}$ satisfying property (P) we have by Lemmas 2.4.14.12

\begin{equation}
\Hom_{\mathcal{T}}(F, G[i]) \overset{\sim}{\to} \Hom_{\text{Di}(Sh)}(S(F), S(G)[i]) \overset{\sim}{\to} \Ext_{Sh}^i(s(H^0F), s(H^0G)).
\end{equation}

In particular, $\Hom_{\mathcal{T}}(F, G[i])$ is isomorphic to $\Hom_{\text{Fun}(A,B)}(H^0F, H^0G)$ if $i = 0$ (because $s : \text{Fun}(A, B) \to \text{PSh}$ is fully faithful) and to 0 for $i < 0$.

To prove the first part of the theorem we need to recall some facts about DG categories and derived functors. Let $f : C_1 \to C_2$ be a DG functor between small DG categories. Then the restriction functor $f_* : \mathcal{D}(C_2) \to \mathcal{D}(C_1)$ admits a left and a right adjoint functors (the derived induction and the co-induction functors)

\begin{equation}
f^*, f^i : \mathcal{D}(C_1) \to \mathcal{D}(C_2)
\end{equation}

([Dri], §14.12). In particular, we have the canonical morphisms

\begin{equation}
Id \to f_* f^*, \quad f_* f^i \to Id
\end{equation}

\begin{equation}
Id \to f^* f_*, \quad f^* f_* \to Id.
\end{equation}

It also follows from the adjunction property that $f^*$ commutes with arbitrary direct sums and that $f^*$ commutes with arbitrary direct products. If the the functor $Ho(f) : Ho(C_1) \to Ho(C_2)$ is fully faithful so is $f_*$ and the first two morphisms in (2.10) are isomorphisms.

Recall the definition of the derived DG quasi-functor $RF$ of a left exact functor $F : A \to B$ from ([Dri], §16). Consider the functor

\begin{equation}
\mathcal{T}(A, D^b_{dg}(B)) \hookrightarrow \mathcal{D}(C^b_{dg}(A)^{\text{op}} \otimes_k D^b_{dg}(B)) \xrightarrow{f^*} \mathcal{D}(D^b_{dg}(A)^{\text{op}} \otimes_k D^b_{dg}(B))
\end{equation}

induced by the projection

\[ f : C^b_{dg}(A)^{\text{op}} \otimes_k D^b_{dg}(B) \to D^b_{dg}(A)^{\text{op}} \otimes_k D^b_{dg}(B). \]

Given a $k$-linear functor $F \in \text{Fun}(A, B) \to \mathcal{T}(A, D^b_{dg}(B))$ define the “derived functor”

\begin{equation}
"RF" = f^*(F) \in \mathcal{D}(D^b_{dg}(A)^{\text{op}} \otimes_k D^b_{dg}(B)).
\end{equation}

The right derived DG quasi-functor $RF : D^b_{dg}(A) \to D^b_{dg}(B)$ if it exists is an object of $\mathcal{T}(D^b_{dg}(A), D^b_{dg}(B))$ whose image in $\mathcal{D}(D^b_{dg}(A)^{\text{op}} \otimes_k D^b_{dg}(B)) \supset \mathcal{T}(D^b_{dg}(A), D^b_{dg}(B))$ is “RF”.

**Lemma 2.5.** Assume that $F$ is left exact. Then “RF” $\in \mathcal{D}^+ \subset \mathcal{D}(D^b_{dg}(A)^{\text{op}} \otimes_k D^b_{dg}(B))$ the functor $S : \mathcal{D}^+ \to \mathcal{D}(Sh)$ takes “RF” to $s(F)$.

**Proof.** Let $\beta : \mathcal{D}(D^b_{dg}(A)^{\text{op}} \otimes_k D^b_{dg}(B)) \to \mathcal{D}(PSh)$ be the restriction functor, and let $\gamma : \mathcal{D}(PSh) \to \mathcal{D}(Sh)$ be the sheafification functor. As explained in ([Dri], §5) the presheaves $H^i(\beta(\text{"RF"}))$ can be computed as follows:

\begin{equation}
H^i(\beta(\text{"RF"}))(X \otimes X') = \text{colim}_Q \Hom_{D^b(B)}(X', F(Y)[i]),
\end{equation}

where the colimit is taken over the filtrant category $Q$ of pairs $(Y \in HoC_{dg}(A), f \in \Hom_{\text{Fun}(C_{dg}(A))}(X, Y))$ such that $cone(f)$ is acyclic. As the subcategory $Q' \subset Q$ consisting of pairs $(Y, f)$ with $Y^j = 0$ for $j < 0$ is cofinal in $Q$ the category $Q$ in
equation (2.12) can be replaced by \( Q' \). This proves that \( \text{"RF"} \in \mathbb{D}^+ \). Let us show that \( \gamma \circ \beta (\text{"RF"}) \simeq s(F) \). We have
\[
H^0(\text{"RF"})(X \otimes X') = \text{colim}_{Q'} \text{Hom}_{D^+(B)}(X', F(Y')) \simeq \text{colim}_{Q'} \text{Hom}_{D^+(B)}(X', F(Y)) = s(F)(X \otimes X').
\]
It remains to prove that, for every \( i > 0 \) the sheafification of \( H^i(\text{"RF"}) \) is 0. Let \( s \) be a section of \( H^i(\beta(\text{"RF"}))(X \otimes X') \) represented by an element
\[
\tilde{s} \in \text{Hom}_{D^+(B)}(X', F(Y))[i]
\]
, where \( X \to Y^0 \to Y^1 \to \cdots \) is an object of \( Q' \). Looking at the diagram
\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y^0 \\
\downarrow f & & \downarrow \text{Id} \\
Y^0 & \xrightarrow{\text{Id}} & Y^0 \to 0 \to \cdots
\end{array}
\]
we see that the pullback \((f \otimes \text{Id})^* s \in H^i(\beta(\text{"RF"}))(Y^0 \otimes X') \) is represented by an element of the group \( \text{Hom}_{D^+(B)}(X', F(Y^0)[i]) = \text{Ext}_B(X', F(Y^0)) \). If \( i > 0 \) every element of this group is annihilated by the map \( \text{Ext}_B(X', F(Y^0)) \to \text{Ext}_B(Y', F(Y^0)) \) for some epimorphism \( Y' \to X' \).

□

Let us prove the first part of the theorem. Let \( F \in \mathcal{T} \subset \mathbb{D}(D^b_{dg}(A)^o \otimes_k D^b_{dg}(B)) \) be a DF quasi-functor satisfying property (P) and \( F = H^0 F \). We need to construct an isomorphism \( F \simeq \text{"RF"} \). By Lemmas 2.4, 2.5 \( F \simeq \text{"RF"} \in \mathbb{D}^+ \). By Lemma 2.4 the functor \( S : \mathbb{D}^+ \to D(Sh) \) is fully faithful. Thus, constructing an isomorphism \( F \simeq \text{"RF"} \) is equivalent to producing an isomorphism \( S(F) \simeq S(\text{"RF"}) \) in \( D(Sh) \) which is done in Lemmas 2.4, 2.5. Theorem 1 is proved.

**Proof of theorem 2.** Let \( \mathcal{T}^+ \subset \mathcal{T} := \mathcal{T}(D^b_{dg}(A), D^b_{dg}(B)) \) be the full triangulated subcategory whose objects are quasi-functors \( F \) such that \( \text{Ho}(\mathcal{F})(D^+_{\geq 0}(A)) \subset D^+_{\geq n}(B) \) for some \( n \). We shall prove that the composition
\[
\mathcal{T}^+ \to \mathbb{D}(D^b_{dg}(A)^o \otimes_k D^b_{dg}(B)) \xrightarrow{\text{Res}_S} \mathbb{D}(D^b_{dg}(A)^o \otimes_k D^b_{dg}(B)) \to D(Sh)
\]
is a fully faithful embedding. Here \( \text{Res} \) denotes the restriction functor induced by the embedding
\[
(2.13) \quad D^b_{dg}(A)^o \otimes_k D^b_{dg}(B) \to D^b_{dg}(A)^o \otimes_k D^b_{dg}(B).
\]

To show this we need to introduce a bit of notation. If \( \mathcal{C} \) is an abelian category closed under countable direct sums and
\[
X^0 \xrightarrow{\phi_0} X^1 \xrightarrow{\phi_1} X^2 \xrightarrow{\phi_2} \cdots
\]
is a diagram of complexes \( X^i \in C(\mathcal{C}) \) set
\[
\text{hocolim} X^i = \text{cone}(\bigoplus_i X^i \xrightarrow{v} \bigoplus_i X^i) \in C(\mathcal{C}),
\]
where \( v_{X^i} := \text{Id}_{X^i} - \phi_i : X^i \to \bigoplus_i X^i \). There is a canonical morphism
\[
\text{hocolim} X^i \to \text{colim} X^i,
\]
which is a quasi-isomorphism if countable direct limits in \( \mathcal{C} \) are exact. If this is the case, every morphism \( X^i \to X^{i'} \) of the diagrams that is a term-wise quasi-isomorphism
induces a quasi-isomorphism of the homotopy colimits. Dually, for a category $\mathcal{C}$ closed under countable products and a diagram

$$\cdots \to X_2 \xrightarrow{\phi_1} X_1 \xrightarrow{\phi_0} X_0$$

set

$$\text{holim } X_i = \text{cone}(\prod_i X_i \xrightarrow{v_i} \prod_i X_i)[-1],$$

where $v_i := p_i - \phi_i p_{i+1} : \prod X_i \to X_i$ and $p_i : \prod X_i \to X_i$ are the projections.

Let $\mathbb{D}' \subset \mathbb{D}(D_{dg}^+(A) \otimes_k D_{dg}^+(B))$ be the full subcategory whose objects covariant DG functors $M : D_{dg}^+(A) \otimes_k D_{dg}^+(B)^{op} \to C(\text{Mod}(k))$ such that, for every $X \in D^+_{dg}(A)$ and $X' \in D_{dg}^+(B)$, the canonical morphism

$$(2.14) \quad M(X \otimes X') \to \text{holim } M(X \otimes \tau_{<i} X'),$$

is a quasi-isomorphism and for every $X \in D^+_{dg}(A)$ and every bounded $X' \in D^b_{dg}(B)$ the canonical morphism

$$(2.15) \quad \text{hocolim } M(\tau_{<i} X \otimes X') \to M(X \otimes X'),$$

is a quasi-isomorphism.

**Remark 2.6.** Since countable direct limits are exact in $\mathcal{B}$, the morphism $\text{hocolim } \tau_{<i} X' \to X'$ is a quasi-isomorphism. Thus, property (2.14) is implied by the following: for every integer $n$ and a countable collection $X^n \in D_{dg}^{>n}(\mathcal{B})$, the morphism

$$M(X \otimes \oplus_i X^n) \to \prod_i M(X \otimes X^n)$$

is a quasi-isomorphism.

**Remark 2.7.** Since directed limits are exact in $\text{Mod}(k)$ property (2.15) is equivalent to the following: for every $X \in D_{dg}^+(A)$ and $X' \in B$,

$$(2.16) \quad \text{colim } H^0(M(\tau_{<i} X \otimes X')) \xrightarrow{\sim} H^0(M(X \otimes X')).$$

**Lemma 2.8.** Then the restriction functor

$$\mathbb{D}' \xrightarrow{\text{Res}} \mathbb{D}(D_{dg}^b(A)^{op} \otimes_k D_{dg}^b(B))$$

is an equivalence of categories.

**Proof.** We shall first consider the restriction

$$f_* : \mathbb{D}(D_{dg}^+(A)^{op} \otimes_k D_{dg}^+(B)) \to \mathbb{D}(D_{dg}^+(A)^{op} \otimes_k D_{dg}^b(B))$$

and prove that $f'$ and $f_*$ define mutually inverse equivalences of categories

$$(2.17) \quad \mathbb{D}(D_{dg}^{+}(A)^{op} \otimes_k D_{dg}^b(B)) \simeq \mathbb{D'},$$

where $\mathbb{D}'$ is a full subcategory of $\mathbb{D}(D_{dg}^+(A)^{op} \otimes_k D_{dg}^+(B))$ whose objects are DG functors $M$ satisfying property (2.14). Let us check that

$$(2.18) \quad f'(\mathbb{D}(D_{dg}^+(A)^{op} \otimes_k D_{dg}^b(B))) \subset \mathbb{D}.$$
For every DG functor $f : \mathcal{C}_1 \to \mathcal{C}_2$ between DG categories over a field the functor $f^! : \mathcal{D}((\mathcal{C}_1) \to \mathcal{D}((\mathcal{C}_2)$ admits the following concrete description: if $M : \mathcal{C}_1 \to C(\text{Mod}(k))$ is a contravariant DG functor and $X \in \mathcal{C}_2$

\[
(2.19) \quad f^!(M)(X) = \text{Hom}_{\mathcal{D}_d}(\mathcal{C}_1)(f^*g \text{Hom}_{\mathcal{C}_2}(\cdot, X), M).
\]

Here $\mathcal{D}_d((\mathcal{C}_1)$ denotes the DG derived category of right $\mathcal{C}_1$-modules, $f^*g$ the derived restriction functor, and $\text{Hom}_{\mathcal{C}_2}(\cdot, X)$ is the image of $X$ under the Yoneda embedding $\mathcal{C}_2 \to \mathcal{D}_{d}(\mathcal{C}_2)$.

We shall prove that

\[
\text{hocolim} \text{Hom}(\cdot, X \otimes \tau_{<i}X') \to f_*\text{Hom}(\cdot, X \otimes X')
\]

is an isomorphism in $\mathcal{D}(D^+_d(A)^o \otimes_k D^+_d(B))$. Together with (2.19) it will imply (2.18). By definition of the tensor product of DG categories, for every $Y \otimes Y' \in D^+_d(A)^o \otimes_k D^+_d(B)$,

\[
\text{Hom}(Y \otimes Y', X \otimes X') = \text{Hom}(Y, X) \otimes_k \text{Hom}(Y', X').
\]

Hence, it is enough to check that

\[
\text{hocolim} \text{Hom}_{D^+_d(B)}(Y', \tau_{<i}Y) \to \text{Hom}_{D^+_d(B)}(Y', Y)
\]

is a quasi-isomorphism, for every $Y' \in D^+_d(B)$. Using the exactness of direct limits in $\text{Mod}(k)$ the last assertion is reduced to the formula

\[
\text{colim} \text{Hom}_{D^+_d(B)}(Y', \tau_{<i}Y) \simeq \text{Hom}_{D^+_d(B)}(Y', Y),
\]

which holds because $\text{Hom}_{D^+_d(B)}(Y', \tau_{>i}Y) = 0$ for large $i$. This proves (2.18).

Since $\text{Ho}(f)$ is fully faithful we have

\[
f_*f^! \sim \text{Id}.
\]

Let us check that every $M \in \mathbb{D}'$ the canonical morphism $M \to f^!f_*M$ is an isomorphism. Set $G = \text{cone}(M \to f^!f_*M)$. As we have just proved $G \in \mathbb{D}(D^+_d(A)^o \otimes_k D^+_d(B))$. On the other hand, the isomorphism $f_*f^!f_* \simeq f_*$ shows that $f_*G = 0$. Hence $G = 0$ by (2.14).

Next, consider the DG functor

\[
g : D^+_d(A)^o \otimes_k D^+_d(B) \to D^+_d(A)^o \otimes_k D^+_d(B)
\]

and show that $g^*$ and $g_*$ define mutually inverse equivalences of categories

\[
(2.21) \quad \mathbb{D}(D^+_d(A)^o \otimes_k D^+_d(B)) \simeq \mathbb{D}',
\]

where $\mathbb{D}'$ is a full subcategory of $\mathbb{D}(D^+_d(A)^o \otimes_k D^+_d(B))$ whose objects are DG functors $F$ satisfying property (2.18). Let us check that

\[
(2.21) \quad g^*(\mathbb{D}(D^+_d(A)^o \otimes_k D^+_d(B))) \subset \mathbb{D}'.
\]

If $M \in \mathbb{D}(D^+_d(A)^o \otimes_k D^+_d(B))$ is a functor representable by

\[
Y \otimes Y' \in D^+_d(A)^o \otimes_k D^+_d(B)
\]

then $g^*M$ is represented by the same object $Y \otimes Y'$ (viewed as an object of $D^+_d(A)^o \otimes_k D^+_d(B)$). Hence (2.18) is implied by the formula

\[
\text{hocolim} \text{Hom}_{D^+_d(A)}(Y, \tau_{<i}X) \simeq \text{Hom}_{D^+_d(A)}(Y, X), \quad Y \in D^+_d(A)
\]
proved above (with \( \mathcal{A} \) replaced by \( \mathcal{B} \)). Since \( g^* \) commutes with arbitrary direct sums and since \( \mathbb{D}(D^b_{\mathcal{A}}(A))^\circ \otimes_k D^b_{\mathcal{A}}(B)) \) is the smallest triangulated subcategory that contains representable functors and closed under direct sums \( g^*(M) \in \mathbb{D}(D^+_{\mathcal{B}}(A))^\circ \otimes_k D^b_{\mathcal{B}}(B)) \) for all \( M \). By (2.15) the functor \( g_* \) is conservative when restricted to \( \mathbb{D}'' \) and the adjoint functor \( g^* \) is fully faithful (because \( Ho(g) \) is fully faithful). It follows that \( Id \xrightarrow{\sim} g_*g^* \) and \((g^*g_*)|_{\mathbb{D}''} \xrightarrow{\sim} Id.\)

Combining (2.17) and (2.21) we see that the functors \( Res \) and \( f^!g^* \) define mutually inverse equivalences between the category \( \mathbb{D}^f \) and \( \mathbb{D}(D^b_{\mathcal{A}}(A))^\circ \otimes_k D^b_{\mathcal{B}}(B)).\)

\[
\square
\]

Consider the composition

\[
(2.22) \quad \mathbb{D}^f \xrightarrow{Res} \mathbb{D}(D^b_{\mathcal{A}}(A))^\circ \otimes_k D^b_{\mathcal{B}}(B)) \xrightarrow{\beta} D(PSh) \rightarrow D(Sh).
\]

Combining Lemmas 2.1 and 2.8 we get the following.

**Corollary 2.9.** Let \( \mathbb{D}^f \subset \mathbb{D}^f \) be the full subcategory whose objects are DG modules \( M \) such that \( \beta \circ Res(M) \in D^+(PSh). \) Then (2.22) induces an equivalence of categories \( S : \mathbb{D}^f \xrightarrow{\sim} D^+(Sh)\).

**Lemma 2.10.** The functor \( \mathcal{T} \mapsto \mathbb{D}(D^+_{\mathcal{A}}(A))^\circ \otimes_k D^+_{\mathcal{B}}(B)) \) carries \( \mathcal{T}^+ \) into \( \mathbb{D}^f \).

**Proof.** Let us show that every \( \mathcal{F} \in \mathcal{T} \) satisfies property (2.6). By definition of \( \mathcal{T} \), for every \( X \in D^+_{\mathcal{A}}(A) \) there exists \( Y \in D^+_{\mathcal{B}}(B) \) and an isomorphism

\[
\mathcal{F}(X \times ?) \cong Hom_{D^+_{\mathcal{B}}(B)}(?, Y)
\]

in the derived category of right \( D^+_{\mathcal{B}}(B) \)-modules. Property (2.6) follows because

\[
Hom_{D^+_{\mathcal{B}}(B)}(?, X^i, Y) \rightarrow \prod_i Hom_{D^+_{\mathcal{B}}(B)}(X^i, Y).
\]

is a quasi-isomorphism.

Let us show that every \( \mathcal{F} \in \mathcal{T}^+ \) satisfies property (2.6). Denote by \( Ho(\mathcal{F}) : D^+_{\mathcal{A}}(A) \rightarrow D^+(B) \) the triangulated functor associated with \( \mathcal{F} \). By definition of \( Ho(\mathcal{F}) \) there is a functorial isomorphism

\[
(2.23) \quad Ho^i(\mathcal{F}(X \otimes X')) \cong Hom_{D^+(B)}(X', Ho(\mathcal{F})(X))
\]

In order to check (2.23) we will prove a stronger statement: for every \( X' \in \mathcal{B} \) the morphism

\[
(2.24) \quad Hom_{D^+(B)}(X', Ho(\mathcal{F})(\tau_{<n} X)) \rightarrow Hom_{D^+(B)}(X', Ho(\mathcal{F})(X))
\]

is an isomorphism for sufficiently large \( n \). By definition of \( \mathcal{T}^+ \)

\[
Ho(\mathcal{T}^+ \circ N(A)) \subset D^{>0}(B),
\]

for some \( N \). Then, for every \( n > N \),

\[
Ho(\mathcal{T}^+ \circ cone(\tau_{<n} X \rightarrow X)) \in D^{>0}(B)
\]

and, hence,

\[
Hom_{D^+(B)}(X', Ho(\mathcal{T}^+ \circ cone(\tau_{<n} X \rightarrow X))) = 0.
\]

\[
\square
\]
By Lemma 2.4, \( S : T^+ \to D(Sh) \).

By Lemma 2.4, \( S \) carries every quasi-functor \( F \) satisfying property \( (P') \) to \( s(H^0F) \in Sh \). This proves the second part of Theorem 2. For the first part, let \( F \in Fun(A,B) \) be a \( k \)-linear functor, and let

\[
(RF \in \mathbb{D}(D_{dg}^+(A)^o \otimes_k D_{dg}^+(B)))
\]

be the “derived functor” (see (2.11)). To complete the proof of Theorem it suffices to show the following.

**Lemma 2.11.** Assume that \( F \) is left exact. Then \( RF \in \mathbb{D}^{f+} \) and \( S(RF) \sim \to s(F) \).

**Proof.** Let us show that \( RF \) satisfies property (2.11). According Remark 2.6 it will suffice to show that, for every integer \( n \), \( Y_i \in D_{dg}^{\geq n}(B) \) and \( X \in HoC^+(A) \)

\[
H^0(RF)(X \otimes \oplus_i X^n)) \sim \to \prod_i H^0(RF)(X \otimes X^n)).
\]

We have (Dr., \S 5)

\[
H^0(RF)(X \otimes X')) \simeq \text{colim}_{Q_X} \text{Hom}_{D^+(B)}(X', F(Y)),
\]

where \( Q_X \) is the filtrant category of pairs

\[
(Y \in HoC^+_+(A), f \in \text{Hom}_{HoC^+_d(A)}(X, Y))
\]

such that \( cone(f) \) is acyclic. If \( X \in HoC^{\geq n}(A) \) the subcategory \( Q'_X \subset Q_X \) formed by pairs \((Y, f)\) with \( Y \in HoC^{\geq n}(A) \) is cofinal in \( Q_X \) and, hence, \( Q_X \) in equation (2.27) can be replaced by \( Q'_X \). Thus, it is enough to prove that the category \( Q_X \) has the following property: for every countable collection \( w_i = (Y_i, f_i) \in Q_X, (i = 1, 2, \cdots) \), there exists \( v \in Q_X \) such that, for every \( i \), the set \( Mor_{Q_X}(w_i, v) \) is not empty. In fact, the object

\[
v = (cone(\bigoplus_i X \xrightarrow{\phi_i} \bigoplus_i Y_i), g),
\]

where \( \phi_j : X \to \bigoplus_i Y_i \) equals \( f_j - f_{j-1} \) and \( g \) is induced by the morphisms \( X \xrightarrow{f_j} Y_1 \). Thus, \( Q_X \) does the job.

Let us show that \( RF \) satisfies property (2.15). As we explained in Remark 2.7, it suffices to show that

\[
\text{colim} H^0(RF(\tau_{<\cdot}X \otimes X')) \sim \to H^0(RF)(X \otimes X'))
\]

for every \( X' \in B \). In fact, formula (2.27) with \( Q_{\tau_{\leq}, X} \) replaced by \( Q'_{\tau_{\leq}, X} \) shows that

\[
H^0(RF(\tau_{>\cdot}X \otimes X')) = 0 \text{ for } i > 0. \quad \text{Hence, } H^0(RF(\tau_{<\cdot}X \otimes X')) \sim \to H^0(RF)(X \otimes X')) \text{ is an isomorphism for } i > 1. \quad \text{This proves that } RF \in \mathbb{D}^{f+}.
\]

For the second claim, observe that the restriction \( \text{Res}_d(RF) \in \mathbb{D}(D_{dg}^o(A)^o \otimes_k D_{dg}^+(B)) \) is the bounded ”derived functor” \( (2.11) \). Thus, we are done by Lemma 2.5.

**Proof of theorem** Apply Corollary 2.3 and equation (2.26).
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ON THE DERIVED DG FUNCTORS

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Abstract. Assume that abelian categories \( A, B \) over a field admit countable direct limits and that these limits are exact. Let \( F : D^+_{dg}(A) \to D^+_{dg}(B) \) be a DG quasi-functor such that the functor \( Ho(F) : D^+(A) \to D^+(B) \) carries \( D^{>0}(A) \) to \( D^{>0}(B) \) and such that, for every \( i > 0 \), the functor \( H^iF : A \to B \) is effaceable. We prove that \( F \) is canonically isomorphic to the right derived DG functor \( RH^0(F) \).

We also prove a similar result for bounded derived DG categories and a formula that expresses Hochschild cohomology of the categories \( D_{dg}^b(A), D_{dg}^b(A) \) as the \( Ext \) groups in the abelian category of left exact functors \( A \to Ind A \). The proofs are based on a description of Drinfeld’s category of quasi-functors as the derived category of a category of sheaves.

1. Main results

Let \( A \) and \( B \) be abelian categories, and let

\[ RF_{tri} : D^+(A) \to D^+(B) \]

be the right derived functor of some left exact functor \( F : A \to B \). Then, the corresponding cohomological \( \delta \)-functor \( R^iF = H^iRF_{tri} : A \to B \) has the following property: the functor \( H^iRF_{tri} \) is 0 for \( i < 0 \), effaceable for \( i > 0 \), and \( H^0RF_{tri} \) is isomorphic to \( F \). Conversely, according to a result of Grothendieck (\[G\]), every cohomological \( \delta \)-functor \( T^* : A \to B \) satisfying the above property is canonically isomorphic to the right derived functor \( R^*F \). The purpose of this paper is to extend this extremely useful characterization of \( R^*F \) to the derived category level. Unfortunately, Verdier’s notion of triangulated functor seems too poor to allow such a simple characterization of the derived functors. In order to get a meaningful statement one has to consider triangulated functors with some kind of enrichment. Arguably the most useful notion here is the one of DG quasi-functor (or essentially equivalent notion of \( A_\infty \)-functor). Indeed, works of Keller and Drinfeld (\[K2\], \[Dri\]) provide a canonical DG enhancement \( D_{dg}^+(A) \) of Verdier’s triangulated derived category. Roughly, a DG quasi-functor \( F : D_{dg}^+(A) \to D_{dg}^+(B) \) is a diagram of the form

\[ D_{dg}^+(A) \leftarrow S C \xrightarrow{G} D_{dg}^+(B), \]

where \( C \) is a DG category, \( S \) and \( G \) are DG functors, and, in addition, \( S \) is a homotopy equivalence. Every quasi-functor (\[L1\]) yields a triangulated functor \( Ho(F) : D^+(A) \to D^+(B) \), but the converse is not true in general. Nevertheless, many of the natural triangulated functors come together with a DG enhancement. For example, the triangulated derived functor \( RF \) can be canonically promoted to a DG quasi-functor (\[Dri\] \S5). The main result of this paper states that under certain mild assumptions on abelian categories \( A \) and \( B \) the DG quasi-functors isomorphic
to the DG derived ones are precisely the DG quasi-functors satisfying Grothendieck’s condition above. To state the result we need to introduce a bit of notation.

Let $k$ be a commutative ring. Denote by $\text{Mod}(k)$ the category of $k$-modules. We shall say that a $k$-linear category is flat if, for every two objects $X, Y$, the $k$-module $\text{Hom}(X,Y)$ is flat. Given a $k$-linear exact category $\mathcal{A}$ we denote by $D^b_\text{dg}(\mathcal{A})$ the corresponding bounded derived DG category over $k$. This is the DG quotient $\left(D\mathcal{A}\right)$ of the DG category $C^b_{\text{dg}}(\mathcal{A})$ of bounded complexes by the subcategory of acyclic ones $\left[N\right]$, §1. The homotopy category of $D^b_\text{dg}(\mathcal{A})$ is the triangulated derived category $D^b(\mathcal{A})$ as defined in $\left[N\right]$. Let $\mathcal{B}$ be another $k$-linear abelian category, $D^b_\text{dg}(\mathcal{B})$ the corresponding bounded derived DG category, and let $T(D^b_\text{dg}(\mathcal{A}), D^b_\text{dg}(\mathcal{B}))$ be the triangulated category of DG quasi-functors $\mathcal{F}: D^b_\text{dg}(\mathcal{A}) \to D^b_\text{dg}(\mathcal{B})$ ([$D\mathcal{A}$], §16.1). Given such $\mathcal{F}$ and an integer $i$ we denote by $H^i\mathcal{F}: \mathcal{A} \to \mathcal{B}$ the composition

$$\mathcal{A} \to D^b_\text{dg}(\mathcal{A}) \xrightarrow{\mathcal{F}} D^b_\text{dg}(\mathcal{B}) \xrightarrow{H^i} \mathcal{B}.$$ 

**Theorem 1.** Let $\mathcal{A}$ be a small $k$-flat exact idempotent complete category$^1$ and $\mathcal{B}$ a small abelian $k$-linear category.

1. Assume that a DG quasi-functor

$$\mathcal{F}: D^b_\text{dg}(\mathcal{A}) \to D^b_\text{dg}(\mathcal{B})$$

has the following property:

(P) The functor $H^i\mathcal{F}: \mathcal{A} \to \mathcal{B}$ is $0$ for every $i < 0$ and effaceable $^2$ for every $i > 0$.

Then the functor $F := H^0\mathcal{F}: \mathcal{A} \to \mathcal{B}$ is left exact, has a right derived DG quasi-functor ([$D\mathcal{A}$], §5)

$$RF: D^b_\text{dg}(\mathcal{A}) \to D^b_\text{dg}(\mathcal{B}),$$

and there is a unique isomorphism $\mathcal{F} \simeq RF$ such that the induced automorphism $F = H^0(F) \simeq H^0(RF) = F$ equals $\text{Id}$. Conversely, the right derived DG quasi-functor of any left exact functor $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ satisfies property (P).

2. For every two DG quasi-functors $\mathcal{F}, \mathcal{G} \in T(D^b_\text{dg}(\mathcal{A}), D^b_\text{dg}(\mathcal{B}))$ satisfying property (P) and every $i < 0$, we have

$$\text{Hom}(T(D^b_\text{dg}(\mathcal{A}), D^b_\text{dg}(\mathcal{B}))(\mathcal{F}, \mathcal{G}[i])) = 0,$$

$$\text{Hom}(T(D^b_\text{dg}(\mathcal{A}), D^b_\text{dg}(\mathcal{B}))(\mathcal{F}, \mathcal{G})) = \text{Hom}_{\text{Fct}}(\mathcal{A}, \mathcal{B})(H^0\mathcal{F}, H^0\mathcal{G}).$$

Here $\text{Fct}(\mathcal{A}, \mathcal{B})$ denotes the category of all $k$-linear functors $\mathcal{A} \to \mathcal{B}$.

**Remark 1.1.** I do not know if the analogous statement holds for merely triangulated functors.

**Remark 1.2.** It is likely that the $k$-flatness assumption on $\mathcal{A}$ is unnecessary. However, I can not prove this.

---

$^1$ i.e., a category enriched over $\text{Mod}(k)$.

$^2$ An additive category is called idempotent complete if any its morphism $p: X \to X$ such that $p \circ p = p$ is the projection on a direct summand of a decomposition $X \simeq Y \oplus Z$.

$^3$ That is, for every object $X \in \mathcal{A}$, there exists an admissible monomorphism $X \hookrightarrow Y$ such that the induced morphism $H^0\mathcal{F}(X) \to H^0\mathcal{F}(Y)$ is $0$. 
We have a similar result for bounded from below derived DG categories. If \( \mathcal{A} \) is a \( k \)-linear abelian category we will write \( D^+_d(A) \) for the bounded from below derived DG category of \( \mathcal{A} \) and \( D^+(\mathcal{A}) \) for the corresponding triangulated category. Let \( D^\geq n(\mathcal{A}) \) be the full subcategory of \( D^+(\mathcal{A}) \) that consists of complexes with trivial cohomology in degrees less then \( n \). We say that a DG quasi-functor
\[
\mathcal{F}: D^+_d(A) \to D^+_d(B)
\]
has property \((P')\) if
\((P')\) The functor \( Ho(\mathcal{F}) \) takes every object of the category \( D^\geq 0(\mathcal{A}) \) to an object of \( D^\geq 0(B) \) and, for every \( i > 0 \), the functor \( H^i \mathcal{F}: \mathcal{A} \to \mathcal{B} \) is effaceable.

**Theorem 2.** Let \( k \) be a field and let \( \mathcal{A}, \mathcal{B} \) be small abelian \( k \)-linear categories. Assume that both categories are closed under countable direct limits and that these limits are exact.

1. Let \( \mathcal{F} \in T(D^+_d(A), D^+_d(B)) \) be a DG quasi-functor satisfying property \((P')\) and \( F := H^0 \mathcal{F}: \mathcal{A} \to \mathcal{B} \). The functor \( F \) admits a right derived DG quasi-functor \( RF: D^+_d(A) \to D^+_d(B) \) and there is a unique isomorphism \( \mathcal{F} \cong RF \) such that the induced automorphism \( F = H^0(\mathcal{F}) \cong H^0(RF) = F \) equals \( Id \). Conversely, a right derived DG quasi-functor of any left exact functor \( F: \mathcal{A} \to \mathcal{B} \) satisfies property \((P')\).

2. For every two DG quasi-functors \( \mathcal{F}, \mathcal{G} \in T(D^+_d(A), D^+_d(B)) \) satisfying property \((P')\) and every \( i \), we have
\[
\text{Hom}_{T(D^+_d(A), D^+_d(B))}(\mathcal{F}, \mathcal{G}[i]) = 0,
\]
\[
\text{Hom}_{T(D^+_d(A), D^+_d(B))}(\mathcal{F}, \mathcal{G}) = \text{Hom}_{\text{Fct}(A, B)}(H^0(\mathcal{F}), H^0(\mathcal{G})).
\]

The main ingredient of the proof of Theorem 2 is the following construction. Let \( \text{Sh}(A^\circ \otimes_k B) \) be the category of \( k \)-linear contravariant functors \( A^\circ \otimes_k B \to \text{Mod}(k) \) that are left exact with respect to both arguments. Every \( k \)-linear left exact functor \( F: \mathcal{A} \to \mathcal{B} \) yields \( s(F) \in \text{Sh}(A^\circ \otimes_k B) \):
\[
s(F)(X \otimes X') = \text{Hom}_B(X', F(X)).
\]

Let \( T^+ \subset T(D^+_d(A), D^+_d(B)) \) be the full triangulated subcategory whose objects are quasi-functors \( \mathcal{F} \) such that \( Ho(\mathcal{F})(D^\geq 0(\mathcal{A})) \subset D^\geq n(\mathcal{B}) \) for some \( n \). Using key Lemma 2.1 we construct a fully faithful embedding
\[
T^+ \hookrightarrow D(\text{Sh}(A^\circ \otimes_k B))
\]
that carries every DG quasi-functor \( \mathcal{F} \) satisfying property \((P')\) to \( s(F) \in \text{Sh}(A^\circ \otimes_k B) \subset D(\text{Sh}(A^\circ \otimes_k B)) \).

**Remark 1.3.** In ([I], Th. 8.9), Toën gave an analogous description of the category of quasi-functors between the derived DG categories of (quasi-)coherent sheaves.

As another application of (1.2) we compute the Hochschild cohomology of a derived DG category. Recall (see, e.g. [K2], §5.4, [I], §8.1) that the Hochschild cohomology of a DG category \( \mathcal{C} \) can be interpreted as
\[
HH^i(\mathcal{C}, \mathcal{C}) = \text{Hom}_{T(\mathcal{C}, \mathcal{C})}(\text{Id}_C, \text{Id}_C[i]).
\]
The composition in \( \mathcal{C} \) makes \( HH^*(\mathcal{C}, \mathcal{C}) \) a graded commutative algebra over \( k \).
Theorem 3. Let \( k \) be a field, and let \( \mathcal{A} \) be a small abelian \( k \)-linear category. There is an isomorphism of algebras
\[
HH^s(D^b_{dg}(\mathcal{A}), D^{b*}_{dg}(\mathcal{A})) \simeq Ext^s_{\mathcal{A}^o \otimes_k \mathcal{A}}(s(Id_{\mathcal{A}}), s(Id_{\mathcal{A}})).
\]
If, in addition, \( \mathcal{A} \) is closed under countable direct limits and that these limits are exact, we have
\[
HH^s(D^+_{dg}(\mathcal{A}), D^{+*}_{dg}(\mathcal{A})) \simeq Ext^s_{\mathcal{A}^o \otimes_k \mathcal{A}}(s(Id_{\mathcal{A}}), s(Id_{\mathcal{A}})).
\]

Remark 1.4. This is a remarkable phenomenon the Hochschild cohomology does not change we “enlarge” the DG category. A similar result, that the Hochschild cohomology of a small DG category coincides with the Hochschild cohomology of its DG ind-completion, is due to Toën (\cite{T}, \S8). An analogous statement for Grothendieck abelian categories was proved by Lowen and Van den Bergh (\cite{LV}).

Remark 1.5. The category \( Sh(\mathcal{A}^o \otimes_k \mathcal{A}) \) has a tensor structure that extends the tensor structure on the category of left exact endofunctors \( \mathcal{A} \to \mathcal{A} \) given by the composition. This can be used to promote \( (1.4), (1.5) \) to isomorphisms of Gerstenhaber algebras (see, e.g. \cite{K1}, \S5.4).

Notation. Given a category \( \mathcal{C} \) we denote by \( \mathcal{C}^o \) the opposite category. If \( \mathcal{C} \) is a DG category we will write \( Ho\mathcal{C} \) for the corresponding homotopy category (\cite{Dri}, \S2.7). For example, \( Ho\mathcal{C}(Mod(k)) \) denotes the homotopy category of complexes of \( k \)-modules. The derived category of right DG modules over a DG category \( \mathcal{C} \) will be denoted by \( \mathbb{D}(\mathcal{C}) \) (\cite{Dri}, \S2.3)\footnote{Drinfeld’s notation for this category is \( D(\mathcal{C}) \). We use a different notation to avoid a possible confusion with Verdier’s derived category of an abelian category \( \mathcal{C} \) that is denoted by \( D(\mathcal{C}) \).}. We will write \( \mathbb{D}(\mathcal{C})^s \) for the DG category of semi-free right DG modules over \( \mathcal{C} \) (\cite{BV}, 1.6.1). We have a canonical equivalence of triangulated categories \( Ho\mathbb{C} \xrightarrow{\sim} \mathbb{D}(\mathcal{C}) \) (\cite{BV}, 1.6.4). For DG categories \( \mathcal{C}, \mathcal{C}' \) we denote by \( T(\mathcal{C}, \mathcal{C}') \) the category of DG quasi-functors (\cite{Dri}, \S16.1). If \( \mathcal{C}' \) is a pretriangulated (\cite{Dri}, \S2.4) \( T(\mathcal{C}, \mathcal{C}') \) has a canonical structure of triangulated category. If \( \mathcal{F} \in T(\mathcal{C}, \mathcal{C}') \) we will write \( Ho(\mathcal{F}) \) for the corresponding functor between the homotopy categories. The expression “direct limit” always means “filtrant direct limit” (\cite{KS}, \S3).

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2. Proofs

Proof of theorem\footnote{Drinfeld’s notation for this category is \( D(\mathcal{C}) \). We use a different notation to avoid a possible confusion with Verdier’s derived category of an abelian category \( \mathcal{C} \) that is denoted by \( D(\mathcal{C}) \).} Let \( T^+ \subset T := T(D^b_{dg}(\mathcal{A}), D^{b*}_{dg}(\mathcal{B})) \) be the full triangulated subcategory whose objects are quasi-functors \( \mathcal{F} \) such that \( H^i\mathcal{F} = 0 \) for sufficiently small \( i \). To prove the Theorem, we shall construct (in Lemma 2.1 below) a fully faithful embedding of \( T^+ \) into the derived category of a certain abelian category \( Sh(\mathcal{A}^o \otimes_k \mathcal{B}) \) that takes every functor \( \mathcal{F} \in T^+ \) satisfying property (P) to an object of the heart \( Sh(\mathcal{A}^o \otimes_k \mathcal{B}) \subset D(Sh(\mathcal{A}^o \otimes_k \mathcal{B})) \).

Under our flatness assumption on \( \mathcal{A} \), the category \( T \) is a full subcategory of the derived category \( \mathbb{D}(D^b_{dg}(\mathcal{A})^o \otimes_k D^{b*}_{dg}(\mathcal{B})) \) of right DG modules over \( D^b_{dg}(\mathcal{A})^o \otimes_k D^{b*}_{dg}(\mathcal{B}) \)

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that consists of all \( M \in \mathbb{D}(D^+_{dg}(A)^{op \otimes k} D_{dg}(B)) \) such that, for every \( X \) in \( D^+_{dg}(A)^{op} \), the module \( M(X) \in \mathbb{D}(D^+_{dg}(B)) \) belongs to the essential image of the Yoneda embedding \( D^+_{dg}(B) \to \mathbb{D}(D^+_{dg}(B)) \) ([Dri], §16.1).

Consider the restriction functor

\[
\mathbb{D}(D^+_{dg}(A)^{op \otimes k} D_{dg}(B)) \xrightarrow{\beta} \mathbb{D}(A^{op \otimes k} B)
\]

induced by the DG quasi-functor \( A^{op \otimes k} B \to D^+_{dg}(A)^{op \otimes k} D_{dg}(B) \). By definition, the triangulated category \( \mathbb{D}(A^{op \otimes k} B) \) is the derived category of the abelian category \( PSh := PSh(A^{op \otimes k} B) \) of \( k \)-linear presheaves i.e., the category of \( k \)-linear contravariant functors \( A^{op \otimes k} B \to Mod(k) \). Consider a Grothendieck topology on \( A^{op \otimes k} B \) whose covers are maps of the form \( f \otimes g : Y \otimes Y' \to X \otimes X' \), where \( X, Y \in A \), \( X', Y' \in B \), and \( f : Y \to X \), \( g : Y' \to X' \) are admissible epimorphisms \( \text{i.e.} \), a sieve \( C \) over \( X \otimes X' \) is a covering sieve if there exist \( f : Y \to X \), \( g : Y' \to X' \) as above such that \( Y \otimes Y' \xrightarrow{f \otimes g} X \otimes X' \in C \). The axioms of Grothendieck topology (see, e.g. [KS], §16.1) are immediate except for the one which is the following statement: for every cover \( Y \otimes Y' \xrightarrow{f \otimes g} X \otimes X' \) and every morphism \( Z \otimes Z' \xrightarrow{\phi} X \otimes X' \) there exists a cover \( T \otimes T' \xrightarrow{\psi} Z \otimes Z' \) and a morphism \( T \otimes T' \xrightarrow{\nu} Y \otimes Y' \) such that \( (f \otimes g) \circ \psi = \phi \circ (p \otimes q) \), which is a consequence of the base change axiom of exact category ([Q], §2). Let \( Sh := Sh(A^{op \otimes k} B) \) be the subcategory of \( PSh \) that consists of objects satisfying the sheaf property. Explicitly, objects of the category \( Sh(A^{op \otimes k} B) \) are contravariant functors \( A^{op \otimes k} B \to Mod(k) \) that are left exact with respect to both arguments. The embedding \( Sh \to PSh \) has a left adjoint functor (sheafification)

\[
\gamma : PSh \to Sh,
\]

which is exact ([KS], §17.4). We denote by \( \gamma : D(PSh) \to D(Sh) \) the induced functor between the derived categories. The composition

\[
\mathbb{D}(D^+_{dg}(A)^{op \otimes k} D_{dg}(B)) \xrightarrow{\beta} D(PSh) \xrightarrow{\gamma} D(Sh)
\]

is not fully faithful in general, however, we have the following result.

**Lemma 2.1.** (cf. [T], Th. 8.9) Let \( D^+ \subset \mathbb{D}(D^+_{dg}(A)^{op \otimes k} D_{dg}(B)) \) be the full subcategory whose objects are DG modules \( M \) such that \( \beta(M) \) is bounded from below. Then the functor

\[
S : D^+ \xrightarrow{\beta} D^+(PSh) \xrightarrow{\gamma} D^+(Sh)
\]

is an equivalence of categories.

**Proof.** The category \( D^+_{dg}(A)^{op \otimes k} D_{dg}(B) \) is the DG quotient of the category \( C^b_{dg}(A)^{op \otimes k} C_{dg}(B) \) by the full subcategory whose objects are of the form \( X^- \times X' \), where either \( X^- \) or \( X' \) is acyclic. It then follows from ([Dri], Theorem 1.6.2) that the functor

\[
\beta : \mathbb{D}(D^+_{dg}(A)^{op \otimes k} D_{dg}(B)) \to \mathbb{D}(C^b_{dg}(A)^{op \otimes k} C_{dg}(B)) = D(PSh)
\]

is fully faithful and that its essential image consists of all DG-modules \( M \in \mathbb{D}(C^b_{dg}(A)^{op \otimes k} C_{dg}(B)) \) that carry every \( X \otimes X' \) with the above property to an acyclic complex. Identifying the category \( \mathbb{D}(C^b_{dg}(A)^{op \otimes k} C_{dg}(B)) \) with \( D(PSh) \) and observing that the subcategories of acyclic complexes in the homotopy categories \( HoC^b_{dg}(A), HoC^b_{dg}(B) \)

\[\text{By definition, admissible epimorphisms } Y \to X \text{ in } A \text{ are admissible monomorphisms } X \to Y \text{ in } A.\]
are generated by short exact sequences \([\mathbb{N}, \S 1]\) we exhibit \(\mathbb{D}(\mathcal{D}^b_\mathcal{A}(\mathcal{A})^p \otimes_k \mathcal{D}^b_\mathcal{B}(\mathcal{B}))\) as a full subcategory \(\mathcal{R} \subset \mathcal{D}(\mathcal{PSh})\) whose objects are complexes \(F^\cdot\) of presheaves satisfying the following two conditions:

- For any exact sequence \(0 \rightarrow Z \rightarrow Y \rightarrow X \rightarrow 0\) in \(\mathcal{A}^p\) and any \(X' \in \mathcal{B}\) the total complex of

\[
F^\cdot(X \otimes X') \rightarrow F^\cdot(Y \otimes X') \rightarrow F^\cdot(Z \otimes X')
\]

is acyclic.

- For any \(X \in \mathcal{A}^p\) and any exact sequence \(0 \rightarrow Z' \rightarrow Y' \rightarrow X' \rightarrow 0\) in \(\mathcal{B}\) the total complex of

\[
F^\cdot(X \otimes X') \rightarrow F^\cdot(X \otimes Y') \rightarrow F^\cdot(X \otimes Z')
\]

is acyclic.

Observe that, for every \(F^\cdot \in \mathcal{R}\) and an exact sequence \(0 \rightarrow Z \rightarrow Y \rightarrow X \rightarrow 0\) in \(\mathcal{A}^p\), we have a long exact sequence of \(k\)-modules

\[
\cdots H^m(Z \otimes X') \rightarrow H^m(F^\cdot(X \otimes X')) \rightarrow H^m(F^\cdot(Y \otimes X')) \rightarrow H^m(F^\cdot(Z \otimes X')) \rightarrow \cdots
\]

The equivalence of categories

\[
\beta : \mathbb{D}(\mathcal{D}^b_\mathcal{A}(\mathcal{A})^p \otimes_k \mathcal{D}^b_\mathcal{B}(\mathcal{B})) \xrightarrow{\sim} \mathcal{R} \subset \mathcal{D}(\mathcal{PSh})
\]

carries \(\mathbb{D}^+\) to the subcategory \(\mathcal{R}^+\) of \(\mathcal{R}\) that consists of bounded from below complexes.

The derived category of sheaves \(\mathcal{D}(\mathcal{Sh})\) is the quotient of the derived category of presheaves by the subcategory \(\mathcal{I}_{\mathcal{Sh}} \subset \mathcal{D}(\mathcal{PSh})\) of locally (for our Grothendieck topology on \(\mathcal{A}^p \otimes_k \mathcal{B}\)) acyclic complexes \((\mathcal{BV}, \S 1.11)\). We shall prove that

\[
\mathcal{R}^+ \subset \mathcal{I}_{\mathcal{Sh}},
\]

where \(\mathcal{I}_{\mathcal{Sh}}\) denotes the right orthogonal complement to \(\mathcal{I}_{\mathcal{Sh}}\) in \(\mathcal{D}(\mathcal{PSh})\) \((\mathcal{BV}, \S 1.1)\); i.e.

\[
\hom_{\mathcal{D}(\mathcal{PSh})}(G^\cdot, F^\cdot) = 0.
\]

for every \(G^\cdot \in \mathcal{I}_{\mathcal{Sh}}\) and \(F^\cdot \in \mathcal{R}^+\). Without loss of generality we may assume that \(F^\cdot\) has trivial cohomology in negative degrees: \(F^\cdot = F^0 \rightarrow F^1 \rightarrow \cdots\). Let \(\tilde{F}^\cdot = F^0 \rightarrow \tilde{F}^1 \rightarrow \cdots\) be the corresponding complex of sheaves. Since the category of sheaves has enough injective objects (see, e.g. \([\mathcal{K}, \mathcal{S}], \text{Th. 9.6.2, 18.1.6}\) there exists a complex \(I = I^0 \rightarrow I^1 \rightarrow \cdots\) of injective sheaves together with a morphism \(\tilde{F}^\cdot \rightarrow I^\cdot\) which is an isomorphism in the derived category of sheaves. Let us show that the composition

\[
\delta : F^\cdot \rightarrow \tilde{F}^\cdot \rightarrow I^\cdot
\]

is an isomorphism in the derived category of presheaves. Indeed, every injective sheaf, viewed as a presheaf, is an object of \(\mathcal{R}\). Thus \(I^\cdot\) and \(\cone\delta\) are in \(\mathcal{R}^+\). Assuming that \(\cone\delta \neq 0\) choose the smallest integer \(m\) such that

\[
0 \neq H^m(\cone\delta) \in \mathcal{PSh}.
\]

Then, there exist an object \(X \otimes X' \in \mathcal{A}^p \otimes_k \mathcal{B}\) and a nonzero element \(a \in H^m(\cone\delta)(X \otimes X')\). Since the sheafification of \(H^m(\cone\delta)\) is 0 there exists a cover \(p : Y \otimes Y' \rightarrow X \otimes X'\) such that

\[
0 = p^*a \in H^m(\cone\delta)(Y \otimes Y').
\]

Writing \(p\) as a composition

\[
Y \otimes Y' \xrightarrow{1 \otimes g} Y \otimes X' \xrightarrow{f \otimes 1} X \otimes X'
\]
we may assume \((f \otimes 1)^* a = 0\) (otherwise, we replace \(X \otimes X'\) by \(Y \otimes X'\)). Let us look at the following fragment of the long exact sequence \((2.2)\) applied to \(F = \text{cone}(\delta)\) and the exact sequence \(0 \to Z \to Y \overset{f}{\to} X \to 0:\)

\[H^{m-1}(\text{cone}(\delta))(Z \otimes X') \to H^m(\text{cone}(\delta))(X \otimes X') \to H^m(\text{cone}(\delta))(Y \otimes X').\]

Since, by our assumption, \(H^{m-1}(\text{cone}(\delta)) = 0\), it follows that \((f \otimes 1)^*\) is injective and, hence, \(a = 0\). This contradiction proves that \(\text{cone}(\delta) = 0\) i.e., \(\delta\) is a quasi-isomorphism. Thus, to complete the proof of \((2.4)\) it suffices to show that

\[\text{Hom}_{D(PSh)}(G', I) = 0,\]

for every \(G' \in \mathcal{I}_{\text{lac}}\) and every bounded from below complex of injective sheaves \(I\).

Indeed, every morphism \(h : G' \to I\) in the derived category is represented by a diagram in \(C(PSh(\mathcal{A}^o \otimes_k \mathcal{B}))\)

\[G' \leftarrow G'' \overset{h'}{\to} I,\]

where the first arrow is a quasi-isomorphism (and, in particular, \(G'' \in \mathcal{I}_{\text{lac}}\)). If \(h'\) is homotopic to 0 then \(h\) is 0 in the derived category. Thus, it is enough to show that

\[\text{Hom}_{K(PSh)}(G'', I) = 0,\]

where \(K(PSh)\) denotes the homotopy category of complexes. We have

\[\text{Hom}_{K(PSh)}(G'', I) \xrightarrow{\sim} \text{Hom}_{K(Sh)}(G'', I) \xrightarrow{\sim} \text{Hom}_{D(Sh)}(G'', I).\]

The first arrow is an isomorphism because all terms of the complex \(I\) are sheaves; the second arrow is an isomorphism by \((\text{KS}, \text{Lemma 13.2.4})\). Finally, the group \(\text{Hom}_{D(Sh)}(G'', I)\) is trivial because the sheafification \(G''\) is 0 in \(D(Sh)\).

To finish the proof of the lemma, we observe that, for every triangulated category \(\mathcal{C}\) and its full triangulated subcategory \(\mathcal{I}\), the composition

\[\mathcal{I}^+ \to \mathcal{C} \to \mathcal{C}/\mathcal{I}\]

is a fully faithful embedding: for every \(X, Y \in \mathcal{C}\)

\[\text{Hom}_{\mathcal{C}/\mathcal{I}}(X, Y) := \text{colim} \text{Hom}_{\mathcal{C}}(X', Y),\]

where the colimit is taken over the filtrant category of pairs \((X' \in \mathcal{C}, f : X' \to X)\) such that \(\text{cone} f \in \mathcal{I}\). If \(Y \in \mathcal{I}^+\), then

\[\text{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{\sim} \text{Hom}_{\mathcal{C}}(X', Y),\]

and, hence,

\[\text{Hom}_{\mathcal{C}/\mathcal{I}}(X, Y) = \text{Hom}_{\mathcal{C}}(X, Y).\]

Applying this remark to \(\mathcal{C} = D(PSh), \mathcal{I} = \mathcal{I}_{\text{lac}}\) and using \((2.4)\) we conclude that the functor \(\mathcal{R}^+ \xrightarrow{\gamma} D(Sh)\) is fully faithful and, hence, so is the composition \(\mathbb{D}^+ \xrightarrow{\gamma} \mathcal{R}^+ \xrightarrow{\gamma} D(Sh)\). The essential image the functor \(\mathcal{R}^+ \xrightarrow{\gamma} D(Sh)\) coincides with \(D^+(Sh)\) because every complex of injective sheaves viewed as a complex of presheaves is an object of \(\mathcal{R}^+\).

\[\text{Remark 2.2.}\] Applying Lemma \((2.1)\) to \(k = \mathbb{Z}\) and \(\mathcal{A}\) being the category of free abelian groups of finite rank we obtain the following statement: for every small abelian category \(\mathcal{B}\)

\[\mathbb{D}^+(D^h_{dg}(\mathcal{B})) \xrightarrow{\sim} D^+(PSh(\mathcal{B})) = D^+(Ind(\mathcal{B})).\]
where $\mathbb{D}^+(\mathcal{D}^b_{dg}(\mathcal{B}))$ is the full subcategory of $\mathbb{D}(\mathcal{D}^b_{dg}(\mathcal{B}))$ that maps to $D^+(PSh(\mathcal{B}))$ under the restriction functor (and the ind-completion $Ind(\mathcal{B})$ is just another name for $PSh(\mathcal{B})$ ([KS], §8.6)). Note the functor

$$\mathbb{D}(\mathcal{D}^b_{dg}(\mathcal{B})) \rightarrow D(Ind(\mathcal{B}))$$

is not an equivalence of categories in general. In fact, the functor (2.5) factors as

$$\mathbb{D}(\mathcal{D}^b_{dg}(\mathcal{B})) \xrightarrow{\phi} HoC(Ind(\mathcal{B}))/\text{HoC}_{ac}^b(\mathcal{B}) \xrightarrow{p} D(Ind(\mathcal{B})),$$

where $\text{HoC}_{ac}^b(\mathcal{B})$ is the smallest triangulated subcategory of the homotopy category of acyclic complexes $\text{HoC}_{ac}(Ind(\mathcal{B}))$ that contains finite acyclic complexes $\text{HoC}_{ac}^b(\mathcal{B})$ and closed under arbitrary direct sums; the functor $p$ is the projection

$$\text{HoC}(Ind(\mathcal{B}))/\text{HoC}_{ac}^b(\mathcal{B}) \rightarrow \text{HoC}(Ind(\mathcal{B}))/\text{HoC}_{ac}(Ind(\mathcal{B})).$$

The equivalence $\phi$ can be constructed as follows. Let $\text{C}_{ac}^b(\mathcal{B})$ be the full subcategory of the DG category $C(\mathcal{B})$ whose objects are those of $\text{HoC}_{ac}^b(\mathcal{B})$. The DG quasi-functor $\mathcal{D}^b_{dg}(\mathcal{B}) \rightarrow C(\mathcal{B})/\text{HoC}_{ac}^b(\mathcal{B})$ extends uniquely to a quasi-functor

$$\phi_{dg} : \mathcal{D}^b_{dg}(\mathcal{B}) \rightarrow C(\mathcal{B})/\text{HoC}_{ac}^b(\mathcal{B})$$

that commutes with arbitrary direct sums ([BV], §1.6.1). Define

$$\phi := \text{HoC}_{dg}.$$

Let us show that $\phi$ is an equivalence of categories. The subcategory $\text{HoC}_{ac}^b(\mathcal{B}) \subset \text{HoC}(Ind(\mathcal{B}))$ is generated by compact objects (e.g., objects of $\text{HoC}_{ac}^b(\mathcal{B})$); it follows that the projection $\text{HoC}(Ind(\mathcal{B}))/\text{HoC}_{ac}^b(\mathcal{B})$ carries compact objects of $\text{HoC}(Ind(\mathcal{B}))$ to compact objects of the quotient category ([BV], §1.4.2). In particular, in the following commutative diagram

$$\begin{array}{ccc}
\mathcal{D}(\mathcal{D}^b_{dg}(\mathcal{B})) & \xrightarrow{i} & \mathcal{D}^b_{dg}(\mathcal{B}) \\
\downarrow & & \downarrow j \\
\mathcal{D}(\mathcal{D}^b_{dg}(\mathcal{B})) & \xrightarrow{\phi} & \text{HoC}(Ind(\mathcal{B}))/\text{HoC}_{ac}^b(\mathcal{B})
\end{array}$$

the image of $j$ consists of compact objects. The same is true for the image of $i$ ([BV], §1.7). The functors $i,j$ are fully faithful and their images generate the categories $\mathcal{D}(\mathcal{D}^b_{dg}(\mathcal{B}))$, $\text{HoC}(Ind(\mathcal{B}))/\text{HoC}_{ac}^b(\mathcal{B})$ respectfully. It follows that $\phi$ is an equivalence of categories.

In general, (e.g., if $\mathcal{B}$ is the category of finitely generated modules over a finite group) the projection $p$ is not conservative. However, if the category $\mathcal{B}$ has finite homological dimension the objects of $\mathcal{D}^b_{dg}(\mathcal{B})$ are compact in $\mathcal{D}^b_{dg}(Ind(\mathcal{B}))$ and the above argument proves that (2.5) is an equivalence of categories.

**Corollary 2.3.** The composition

$$\begin{array}{ccc}
\mathcal{T}^+ & \xrightarrow{\alpha} & \mathcal{D}(\mathcal{D}^b_{dg}(\mathcal{A})^p \otimes_k \mathcal{D}^b_{dg}(\mathcal{B})) \\
\xrightarrow{\beta} & & \mathcal{D}(PSh) \\
\xrightarrow{\gamma} & & \mathcal{D}(Sh)
\end{array}$$

is a fully faithful embedding.

\footnote{Indeed, under our finiteness assumption every complex in $\mathcal{D}^b_{dg}(\mathcal{B})$ is quasi-isomorphic to a finite complex of projective objects. Thus it is enough to show that every projective object of $\mathcal{B}$ is compact in $\mathcal{D}(Ind(\mathcal{B}))$. This is clear because every such object is projective and compact in $Ind(\mathcal{B})$.}
Consider the Yoneda embedding

$$s : \text{Fun}(A, B) \rightarrow \text{PSh}$$

that takes a functor $F \in \text{Fun}(A, B)$ to the presheaf

$$s(F)(X \times X') = \text{Hom}_{\text{B}}(X', F(X)).$$

If $F$ is left exact then $s(F)$ is actually a sheaf.

Let $F \in T$ be a DG quasi-functor satisfying property $(P)$. It follows from the definition of $T^+$ given at the beginning of this section that $F \in T^+$. We shall prove that $S(F) \sim s(H^0F)$. Having in mind applications to Theorem 2, we will actually show a slightly more general statement. Namely, let us extend the functor (2.7) to a larger category:

$$S' : T(D^b_{dg}(A), D^+_d(B)) \rightarrow D(PSh) \rightarrow D(Sh).$$

**Lemma 2.4.** Let $F \in T(D^b_{dg}(A), D^+_d(B))$ be a DG quasi-functor such that $H^iF$ is zero for $i < 0$ and effaceable for $i > 0$. Set $s(F) = s(H^0F) \subset \text{Sh} \subset D(Sh)$. Then the complex $S'(F) \in D(Sh)$ is canonically quasi-isomorphic to $s(F)$.

**Proof.** By definition, the cohomology presheaves of the complex $\beta'\alpha'(F) \in D(Sh)$ are given by the formula

$$H^i(\beta'\alpha'(F))(X \otimes X') = \text{Hom}_{D^+(B)}(X', H_0(F)(X)[i]).$$

Since the negative cohomology of the complex $Ho(F)(X) \in D^+(B)$ vanishes the same is true for $\beta'\alpha'$ and, thus, we have

$$H^0(\beta'\alpha'(F))(X \otimes X') = \text{Hom}_{D^+(B)}(X', H^0(F)(X)) = s(F).$$

It remains to prove that for every $i > 0$ the sheafification of the presheaf $H^i(\beta'\alpha'(F))$ equals zero. Given an integer $j$ define presheaves $G^{i,j}$ to be

$$G^{i,j}(X \otimes X') = \text{Hom}_{D^+(B)}(X', \tau_{\leq j}(Ho(F)(X))[i]).$$

We shall show by induction on $j$ that for every $i > 0$ and every $j$ the sheafification of $G^{i,j}$ is 0. This would complete the proof since $G^{i,j}$ is isomorphic to $H^i(\beta'\alpha')(X \otimes X')$ for $j \geq i$. For every $i > 0$ and every element $v$ of the group

$$G^{i,0}(X \otimes X') = \text{Ext}^i_B(X', H^0(F)(X))$$

there exists an epimorphism $Y' \rightarrow X'$ such that $v$ is annihilated by the map

$$\text{Ext}^i_B(Y', H^0(F)(X)) \rightarrow \text{Ext}^i_B(Y', H^0(F)(X))$$

([KS], Exercise 13.17). This proves that the sheafification of $G^{i,0}$ is 0. For the induction step, consider the distinguished triangle

$$\tau_{\leq j}(Ho(F)(X)) \rightarrow \tau_{\leq j+1}(Ho(F)(X)) \rightarrow H^{j+1}(F)(X)[-j - 1]$$

and the corresponding long exact sequence

$$\rightarrow G^{i,j}(X \otimes X') \rightarrow G^{i,j+1}(X \otimes X') \rightarrow \text{Hom}_{D^+(B)}(X', H^{j+1}(F)(X)[-j - 1 + i]) \rightarrow .$$

It follows that $G^{i,j+1}$ fits in a long exact sequence

$$\rightarrow G^{i,j} \rightarrow G^{i,j+1} \rightarrow \text{Ext}^{j-1}_B(\cdot, H^{j+1}(F)(\cdot)) \rightarrow .$$

The vanishing of $H^iF$ implies that $F$ is left exact and, hence, $s(F)$ is a sheaf.
The sheafification of $G^{i,j}$ is 0 by the induction assumption, the sheafification of $\text{Ext}^{i,j+1}_{\mathcal{B}}(\cdot, H^{j+1}F(\cdot))$ is 0 because the functor $H^{j+1}F$ is effaceable. Hence, the sheafification of $G^{i,j+1}$ is 0 as well. □

Now we are ready to prove the second part of the theorem. Given quasi-functors $\mathcal{F}, \mathcal{G} \in \mathcal{T}$ satisfying property (P) we have by Lemmas 2.13, 2.14

(2.8) $\text{Hom}_\mathcal{T}(\mathcal{F}, \mathcal{G}[i]) \xrightarrow{\sim} \text{Hom}_{\mathcal{D}(\text{Sh})}(S(\mathcal{F}), S(\mathcal{G})[i]) \xrightarrow{\sim} \text{Ext}^i_{\mathcal{S}}(s(H^0\mathcal{F}), s(H^0\mathcal{G})).$

In particular, $\text{Hom}_\mathcal{T}(\mathcal{F}, \mathcal{G}[i])$ is isomorphic to $\text{Hom}_{\mathcal{D}(\mathcal{A}, \mathcal{B})}(H^0\mathcal{F}, H^0\mathcal{G})$ for $i = 0$ (since the functor $s: \text{Fun}(\mathcal{A}, \mathcal{B}) \to \mathcal{PSh}$ is fully faithful) and to 0 for $i < 0$.

To prove the first part of the theorem we need to recall some facts about DG categories and derived functors. Let $f: \mathcal{C}_1 \to \mathcal{C}_2$ be a DG functor between small DG categories. Then the restriction functor $f_*: \mathcal{D}(\mathcal{C}_2) \to \mathcal{D}(\mathcal{C}_1)$ admits a left and a right adjoint functors (the derived induction and co-induction functors)

(2.9) $f^*, f^! : \mathcal{D}(\mathcal{C}_1) \to \mathcal{D}(\mathcal{C}_2)$

([Dri], §14.12). In particular, we have the canonical morphisms

(2.10) $\text{Id} \to f_* f^*, \quad f_* f^! \to \text{Id}$

$\text{Id} \to f^! f_* \to \text{Id}.$

It also follows from the adjunction property that $f^*$ commutes with arbitrary direct sums and that $f^!$ commutes with arbitrary direct products. If the the functor $H_0(f): H_0(\mathcal{C}_1) \to H_0(\mathcal{C}_2)$ is fully faithful so is $f_*$ and the first two morphisms in (2.10) are isomorphisms.

Recall the definition of the derived DG quasi-functor $RF$ of a left exact functor $F: \mathcal{A} \to \mathcal{B}$ from ([Dri], §16). Consider the functor

$\mathcal{T}(\mathcal{A}, D^b_{dg}(\mathcal{B})) \hookrightarrow \mathcal{D}(C^b_{dg}(\mathcal{A}) \otimes_k D^b_{dg}(\mathcal{B})) \xrightarrow{f^*} \mathcal{D}(D^b_{dg}(\mathcal{A}) \otimes_k D^b_{dg}(\mathcal{B}))$

induced by the projection

$f: C^b_{dg}(\mathcal{A}) \otimes_k D^b_{dg}(\mathcal{B}) \to D^b_{dg}(\mathcal{A}) \otimes_k D^b_{dg}(\mathcal{B}).$

Given a $k$-linear functor $F \in \text{Fun}(\mathcal{A}, \mathcal{B}) \to \mathcal{T}(\mathcal{A}, D^b_{dg}(\mathcal{B}))$ we define the “derived functor”

(2.11) $\text{RF} = f^*(F) \in \mathcal{D}(D^b_{dg}(\mathcal{A})^{ op} \otimes_k D^b_{dg}(\mathcal{B})).$

The right derived DG quasi-functor $RF: D^b_{dg}(\mathcal{A}) \to D^b_{dg}(\mathcal{B})$, if it exists, is an object of $\mathcal{T}(D^b_{dg}(\mathcal{A}), D^b_{dg}(\mathcal{B}))$ whose image in $\mathcal{D}(D^b_{dg}(\mathcal{A})^{ op} \otimes_k D^b_{dg}(\mathcal{B})) \supset \mathcal{T}(D^b_{dg}(\mathcal{A}), D^b_{dg}(\mathcal{B}))$ is “RF”.

**Lemma 2.5.** Assume that $F$ is left exact. Then “RF” $\in \mathcal{D}^+ \subset \mathcal{D}(D^b_{dg}(\mathcal{A})^{ op} \otimes_k D^b_{dg}(\mathcal{B}))$ and the functor $S: \mathcal{D}^+ \hookrightarrow D(\text{Sh})$ takes “RF” to $s(F)$.

**Proof.** Let $\beta: \mathcal{D}(D^b_{dg}(\mathcal{A})^{ op} \otimes_k D^b_{dg}(\mathcal{B})) \to D(\mathcal{PSh})$ be the restriction functor, and let $\gamma: D(\mathcal{PSh}) \to D(\text{Sh})$ be the sheafification functor. As explained in ([Dri], §5) the presheaves $H^n(\beta(\text{RF}))$ can be computed as follows:

(2.12) $H^n(\beta(\text{RF}))(X \otimes X') = \text{colim}_Q \text{Hom}_{\mathcal{D}(\mathcal{B})}(X', F(Y)[i]),$

where the colimit is taken over the filtrant category $Q$ of pairs $(Y^\circ \in H^0\mathcal{C}_{dg}(\mathcal{A}), f \in \text{Hom}_{\mathcal{PSh}}(\mathcal{C}_{dg}(\mathcal{A}), (X, Y'))$ such that cone$(f)$ is acyclic. As the subcategory $Q' \subset Q$ consisting of pairs $(Y^\circ, f)$ with $Y^j = 0$ for $j < 0$ is cofinal in $Q$, the category $Q$ in the
that \( \gamma_i > 0 \) it remains to prove that, for every equation (2.12) can be replaced by \( Q \)

be the section of \( X \) where \( t \) is a fully faithful embedding. Here \( \text{Res}_{D}^{X} \) objects of \( D \) are closed under countable direct sums and we need to construct an isomorphism \( \mathcal{F} \cong H^0F \) together with an isomorphism \( F \cong \mathcal{F} \). By Lemma 2.1 the functor \( \text{Sh}(X,F) \) is represented by an element of the group \( \text{Ext}^1(X,F) \). For any positive element of this group is annihilated by the map \( \text{Ext}^1(X,F) \rightarrow \text{Ext}^1(X,F) \) for some epimorphism \( Y' \rightarrow X' \).

Let us prove the first part of the theorem. Let \( \mathcal{F} \in T \subset D(\mathcal{D}_g(A)^o \otimes_k \mathcal{D}_g(B)) \) be a DG quasi-functor satisfying property (P) together with an isomorphism \( \mathcal{F} \cong H^0F \). We need to construct an isomorphism \( \mathcal{F} \cong \text{Sh}(X,F) \). By Lemma 2.1, the functor \( S : D \rightarrow D(\text{Sh}) \) is fully faithful. Thus, constructing an isomorphism \( \mathcal{F} \cong \text{Sh}(X,F) \) is equivalent to producing an isomorphism \( S(\mathcal{F}) \cong \text{Sh}(X,F) \) in \( D(\text{Sh}) \) which was done in Lemmas 2.3, 2.4. Theorem 1 is proved.

Proof of theorem 2. Let \( T^+ \subset T := (\mathcal{D}_g(A), \mathcal{D}_g(B)) \) be the full triangulated subcategory whose objects are quasi-functors \( \mathcal{F} \) such that, for some integer \( n \), we have

\[ H\alpha(F)(D^2(A)) \subset D^n(B). \]

We shall prove that the composition

\[ T^+ \hookrightarrow D(\mathcal{D}_g(A)^o \otimes_k \mathcal{D}_g(B)) \xrightarrow{Res} D(\mathcal{D}_g(A)^o \otimes_k \mathcal{D}_g(B)) \rightarrow D(\text{Sh}) \]

is a fully faithful embedding. Here \( \text{Res} \) denotes the restriction functor induced by the embedding

\[ (2.13) \quad D^b_{dg}(A)^o \otimes_k D^b_{dg}(B) \rightarrow D^b_{dg}(A)^o \otimes_k D^b_{dg}(B). \]

To show this we need to introduce a bit of notation. If \( C \) is an abelian category closed under countable direct sums and

\[ X^0 \xrightarrow{\phi_0} X^1 \xrightarrow{\phi_1} X^2 \xrightarrow{\phi_2} \ldots \]

is a diagram of complexes \( X^i \in C(C) \), we set

\[ \text{hocolim } X^i = \text{cone}(\bigoplus_i X^i \xrightarrow{\phi_i} \bigoplus_i X^i) \in C(C), \]

where \( \phi_{|X^i} := \text{Id}_{X^i} - \phi_i : X^i \rightarrow \oplus_i X^i \). There is a canonical morphism

\[ \text{hocolim } X^i \rightarrow \text{colim } X^i, \]
Remark 2.6. is a quasi-isomorphism if countable direct limits in \( C \) are exact. If this is the case, every morphism \( X^i \rightarrow X^0 \) of diagrams that is a term-wise quasi-isomorphism induces a quasi-isomorphism of the homotopy colimits. Dually, for a category \( C \) closed under countable products and a diagram

\[
\cdots \rightarrow X_2 \xrightarrow{\varphi_1} X_1 \xrightarrow{\varphi_0} X_0,
\]

we set

\[
\text{holim } X_i = \text{cone}(\prod_i X_i) \rightarrow \prod_i X_i)[-1],
\]

where \( v_i := p_i - \phi_i p_{i+1} : \prod_i X_i \rightarrow X_i \) and \( p_i : \prod_i X_i \rightarrow X_i \) are the projections.

Let \( \mathbb{D}^f \subseteq \mathbb{D}(D_{dg}^+(A) \otimes_k D_{dg}^+(B)) \) be the full subcategory whose objects are the covariant DG functors \( M : D_{dg}^+(A) \otimes_k D_{dg}^+(B) \rightarrow C(\text{Mod}(k)) \) such that, for every \( X \in D_{dg}^+(A) \) and \( X' \in D_{dg}^+(B) \), the canonical morphism

\[
M(X \otimes X') \rightarrow \text{holim } M(X \otimes \tau_{<i} X'),
\]

is a quasi-isomorphism, and, for every \( X \in D_{dg}^+(A) \) and every bounded \( X' \in D_{dg}^+(B) \), the canonical morphism

\[
\text{hocolim } M(\tau_{<i} X \otimes X') \rightarrow M(X \otimes X'),
\]

is a quasi-isomorphism.

**Remark 2.6.** Since countable direct limits are exact in \( B \), the morphism \( \text{hocolim } \tau_{<i} X' \rightarrow X' \) is a quasi-isomorphism. Thus, property \((2.14)\) is implied by the following: for every integer \( n \) and a countable collection \( X^{n_i} \in D_{dg}^+(B) \), the morphism

\[
M(X \otimes \bigoplus_i X^{n_i}) \rightarrow \prod_i M(X \otimes X^{n_i})
\]

is a quasi-isomorphism.

**Remark 2.7.** Since directed limits are exact in \( \text{Mod}(k) \) property \((2.15)\) is equivalent to the following: for every \( X \in D_{dg}^+(A) \) and \( X' \in B \), we have

\[
\text{colim } H^0(M(\tau_{<i} X \otimes X')) \xrightarrow{\sim} H^0(M(X \otimes X')).
\]

**Lemma 2.8.** The restriction functor

\[
\mathbb{D}^f \xrightarrow{\text{Res}} \mathbb{D}(D_{dg}^+(A) \otimes_k D_{dg}^+(B))
\]

is an equivalence of categories.

**Proof.** We shall first consider the restriction

\[
f_\ast : \mathbb{D}(D_{dg}^+(A) \otimes_k D_{dg}^+(B)) \rightarrow \mathbb{D}(D_{dg}^+(A) \otimes_k D_{dg}^+(B))
\]

and prove that \( f^! \) and \( f_\ast \) define mutually inverse equivalences of categories

\[
\mathbb{D}(D_{dg}^+(A) \otimes_k D_{dg}^+(B)) \simeq \mathbb{D}',
\]

where \( \mathbb{D}' \) is the full subcategory of \( \mathbb{D}(D_{dg}^+(A) \otimes_k D_{dg}^+(B)) \) whose objects are DG functors \( M \) satisfying the property \((2.14)\). Let us check that

\[
f^!(\mathbb{D}(D_{dg}^+(A) \otimes_k D_{dg}^+(B))) \subseteq \mathbb{D}'.
\]

\footnote{For the last property, it suffices to assume that countable direct sums are exact in \( C \).}
For every DG functor $f : C_1 \to C_2$ between DG categories over a field, the functor $f^\star : \mathcal{D}(C_1) \to \mathcal{D}(C_2)$ admits the following concrete description: if $M : C_1 \to C(\text{Mod}(k))$ is a contravariant DG functor and $X$ is an object of $C_2$, we have

\[(2.19)\quad f^\star(M)(X) = \text{Hom}_{\mathcal{D}_d}(c_1)(f^d_*\text{Hom}_{C_2}(\cdot, X), M) .\]

Here $\mathcal{D}_d(C_1)$ denotes the DG derived category of right $C_1$-modules, $f^d_*$ the derived restriction functor, and $\text{Hom}_{C_2}(\cdot, X)$ is the image of $X$ under the Yoneda embedding $C_2 \to \mathcal{D}_d(C_2)$.

We shall prove that

\[\text{hocolim} \text{Hom}(\cdot, X \otimes_{\tau_i} X') \to f_*\text{Hom}(\cdot, X \otimes X')\]

is an isomorphism in $\mathcal{D}(D^+_d(A)^o \otimes_k D^b_d(B))$. Together with (2.19) it will imply (2.18). By definition of the tensor product of DG categories, for every $Y \otimes Y' \in D^+_d(A)^o \otimes_k D^b_d(B)$,

\[\text{Hom}(Y \otimes Y', X \otimes X') = \text{Hom}(Y, X) \otimes_k \text{Hom}(Y', X').\]

Hence, it is enough to check that the morphism

\[\text{hocolim} \text{Hom}_{D^+_d(B)}(Y', \tau_{<i} Y) \to \text{Hom}_{D^+_d(B)}(Y', Y)\]

is a quasi-isomorphism, for every $Y' \in D^b_d(B)$. Using the exactness of direct limits in $\text{Mod}(k)$ the last assertion is reduced to the formula

\[\text{colim} \text{Hom}_{D^+(B)^o}(Y', \tau_{<i} Y) \simeq \text{Hom}_{D^+(B)^o}(Y', Y),\]

which holds because the group $\text{Hom}_{D^+(B)^o}(Y', \tau_{>i} Y)$ is trivial for large $i$. This proves the assertion (2.18).

Since the functor $\text{Ho}(f)$ is fully faithful, we have

\[f_*f^\star \sim \text{Id}.\]

Let us check that for every $M \in \mathcal{D}'$ the canonical morphism $M \to f^\star f_* M$ is an isomorphism. Set $G = \text{cone}(M \to f^\star f_* M)$. As we have just proved $G$ belongs to $\mathcal{D}'(D^b_d(A)^o \otimes_k D^+_d(B))$. On the other hand, the isomorphism $f_* f^\star f_* \simeq f_*$ shows that $f_* G$ is 0. Hence, $G$ is 0 by (2.19).

Next, consider the DG functor

\[g : D^b_d(A)^o \otimes_k D^b_d(B) \to D^+_d(A)^o \otimes_k D^b_d(B)\]

and show that $g^*$ and $g_*$ define mutually inverse equivalences of categories

\[(2.20)\quad \mathcal{D}(D^b_d(A)^o \otimes_k D^b_d(B)) \simeq \mathcal{D}'^o,\]

where $\mathcal{D}'$ is a full subcategory of $\mathcal{D}(D^+_d(A)^o \otimes_k D^b_d(B))$ whose objects are DG functors $F$ satisfying property (2.15). Let us check that

\[(2.21)\quad g^*(\mathcal{D}(D^b_d(A)^o \otimes_k D^b_d(B))) \subset \mathcal{D}'^o.\]

If $M \in \mathcal{D}(D^b_d(A)^o \otimes_k D^b_d(B))$ is a functor representable by

\[Y \otimes Y' \in D^b_d(A)^o \otimes_k D^b_d(B)\]

then $g^* M$ is represented by the same object $Y \otimes Y'$ (viewed as an object of $D^+_d(A)^o \otimes_k D^b_d(B)$). Hence (2.16) is implied by the formula

\[\text{hocolim} \text{Hom}_{D^+_d(A)}(Y, \tau_{<i} X) \simeq \text{Hom}_{D^+_d(A)}(Y, X), \quad Y \in D^b_d(A).\]
proved above (with $\mathcal{A}$ replaced by $\mathcal{B}$). Since $g^*$ commutes with arbitrary direct sums and since $\mathbb{D}(D_{dg}^+(\mathcal{A}))$ is the smallest triangulated subcategory that contains representable functors and closed under direct sums, $g^*(M)$ is an object of $\mathbb{D}(D_{dg}^+(\mathcal{A}) \otimes_k D_{dg}^b(\mathcal{B}))$ for every $M$. By (2.15), the functor $g^*$ is conservative when restricted to $\mathbb{D}$ and the adjoint functor $g^*$ is fully faithful (because $\text{Ho}(g)$ is fully faithful). Hence, we have

$$Id \sim \to g_* g^*, \quad (g^* g_*)_{|\mathbb{D}''} \sim \to Id.$$ Combining equations (2.14) and (2.20) we see that the functors $\text{Res}$ and $f^! g^*$ define mutually inverse equivalences between the category $\mathbb{D}^f$ and the category $\mathbb{D}(D_{dg}^+(\mathcal{A}) \otimes_k D_{dg}^b(\mathcal{B})).$

Consider the composition

(2.22) $$\mathbb{D}^f \xrightarrow{\text{Res}} \mathbb{D}(D_{dg}^+(\mathcal{A}) \otimes_k D_{dg}^b(\mathcal{B})) \xrightarrow{\beta} D(PSh) \to D(Sh).$$ Combining Lemmas 2.1 and 2.8 we get the following.

**Corollary 2.9.** Let $\mathbb{D}^{f+} \subset \mathbb{D}^f$ be the full subcategory whose objects are DG modules $M$ such that $\beta \circ \text{Res}(M)$ is bounded from below. Then (2.22) induces an equivalence of categories

$$S : \mathbb{D}^{f+} \sim \to D^+(Sh).$$

**Lemma 2.10.** The functor $T \mapsto \mathbb{D}(D_{dg}^+(\mathcal{A}) \otimes_k D_{dg}^b(\mathcal{B}))$ carries $T^+$ into $\mathbb{D}^{f+}$.

**Proof.** Let us show that every $\mathcal{F} \in T$ satisfies property (2.6). By definition of $T$, for every $X \in D_{dg}^+(\mathcal{A})$, there exists $Y \in D_{dg}^+(\mathcal{B})$ and an isomorphism

$$\mathcal{F}(X \times ?) \simeq \text{Hom}_{D_{dg}^+(\mathcal{B})}(?, Y)$$

in the derived category of right $D_{dg}^+(\mathcal{B})$-modules. Property (2.6) follows because the morphism

$$\text{Hom}_{D_{dg}^+(\mathcal{B})}(? \otimes X^n, Y) \to \prod_i \text{Hom}_{D_{dg}^+(\mathcal{B})}(X^i, Y).$$

is a quasi-isomorphism.

Let us show that every $\mathcal{F} \in T^+$ satisfies the property (2.7). Denote by $\text{Ho}(\mathcal{F}) : D_{dg}^+(\mathcal{A}) \to D^+(\mathcal{B})$ the triangulated functor associated with $\mathcal{F}$. By definition of $\text{Ho}(\mathcal{F})$ there is a functorial isomorphism

(2.23) $$\text{H}^0(\mathcal{F}(X \otimes X')) \simeq \text{Hom}_{D_{dg}^+(\mathcal{B})}(X', \text{Ho}(\mathcal{F})(X))$$

In order to check (2.7) we will prove a stronger statement: for every $X' \in \mathcal{B}$ the morphism

(2.24) $$\text{Hom}_{D_{dg}^+(\mathcal{B})}(X', \text{Ho}(\mathcal{F}(\tau_{<n} X))) \to \text{Hom}_{D_{dg}^+(\mathcal{B})}(X', \text{Ho}(\mathcal{F})(X))$$

is an isomorphism for sufficiently large $n$. By definition of $T^+$ we can find an integer $N$ such that the functor $\text{Ho}(\mathcal{F})$ carries every object of $D^{>N}(\mathcal{A})$ to an object $D^{>0}(\mathcal{B})$. In particular, for every $n > N$, the complex $\text{Ho}(\text{cone}(\tau_{<n} X \to X))$ has trivial cohomology in non-positive degrees. Hence, we have

$$\text{Hom}_{D_{dg}^+(\mathcal{B})}(X', \text{Ho}(\text{cone}(\tau_{<n} X \to X))) = 0.$$ 

□
Combining Lemma 2.10 and Corollary 2.9 we get a fully faithful embedding

\[(2.25)\]

\[S : T^+ \hookrightarrow D(Sh).\]

By Lemma 2.4, \(S\) carries every quasi-functor \(F\) satisfying property \((P')\) to \(s(H^0F) \in Sh\). This proves the second part of Theorem 2. For the first part, let \(F \in Fun(\mathcal{A}, \mathcal{B})\) be a \(k\)-linear functor, and let

\[(2.26)\]

\[\text{"RF"} \in D(D^+_d(A) \otimes_k D^+_d(B))\]

be the "derived functor" (see (2.11)). To complete the proof of Theorem it suffices to show the following.

**Lemma 2.11.** Assume that \(F\) is left exact. Then \(\text{"RF"}\) is an object of \(\mathbb{D}^f+\) and \(S(\text{"RF"})\) is isomorphic to \(s(F)\).

**Proof.** Let us show that \(\text{"RF"}\) satisfies property (2.14). According Remark 2.6 it will suffice to show that, for every integer \(n\), \(Y^i \in D^+_d(B)\) and \(X \in HoC^+(A)\)

\[H^0(\text{"RF"}(X \otimes \oplus_i X^n)) \xrightarrow{\sim} \prod_i H^0(\text{"RF"}(X \otimes X^n)).\]

We have (Dr5, §5)

\[(2.27)\]

\[H^0(\text{"RF"}(X \otimes X')) \simeq \text{colim}_{Q_X} \text{Hom}_{D^+(B)}(X', F(Y)),\]

where \(Q_X\) is the filtrant category of pairs

\[(Y \in HoC^+_d(A), f \in \text{Hom}_{HoC^+_d(A)}(X,Y))\]

such that \(\text{cone}(f)\) is acyclic. If \(X\) is in \(HoC^\geq n(A)\) the subcategory \(Q_X' \subset Q_X\) formed by pairs \((Y, f)\) with \(Y \in HoC^\geq n(A)\) is cofinal in \(Q_X\) and, hence, \(Q_X\) in equation (2.27) can be replaced by \(Q_X'\). Thus, it is enough to prove that the category \(Q_X\) has the following property: for every countable collection \(w_i = (Y_i,f_i) \in Q_X', (i = 1, 2, \cdots)\), there exists \(v \in Q_X\) such that, for every \(i\), the set \(\text{Mor}_{Q_X}(w_i, v)\) is not empty. In fact, the object

\[v = (\text{cone}(\bigoplus_i X \xrightarrow{\phi} \bigoplus_i Y_i), g),\]

where \(\phi_j : X \to \bigoplus_i Y_i\) equals \(f_j - f_{j-1}\) and \(g\) is induced by the morphisms \(X \xrightarrow{f_i} Y_i \hookrightarrow \bigoplus_i Y_i\), does the job.

Let us show that \(\text{"RF"}\) satisfies property (2.15). As we explained in Remark 2.7 it suffices to show that

\[\text{colim} H^0(\text{"RF"}(\tau_{<i} X \otimes X')) \xrightarrow{\sim} H^0(\text{"RF"}(X \otimes X')),\]

for every \(X' \in B\). In fact, formula (2.27) with \(Q_{\tau_{>i} X}\) replaced by \(Q_{\tau_{>i} X}'\) shows that \(H^0(\text{"RF"}(\tau_{>i} X \otimes X'))\) is trivial for \(i > 0\). Hence, the morphism \(H^0(\text{"RF"}(\tau_{<i} X \otimes X')) \to H^0(\text{"RF"}(X \otimes X'))\) is an isomorphism for \(i > 1\). This proves that \(\text{"RF"}\) belongs to \(\mathbb{D}^f+\).

For the second claim, observe that the restriction \(\text{Res}(\text{"RF"}) \in D(D^+_d(A) \otimes_k D^+_d(B))\) is the bounded "derived functor" (2.11). Thus, we are done by Lemma 2.5. \(\square\)

**Proof of theorem 3.** Apply Corollary 2.3 and equation (2.25).
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