LIMITS OF THE TRISTRAM–LEVINE SIGNATURE FUNCTION

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Abstract. We show that under a precise condition on the single variable Alexander polynomial, the limit at 1 of the Tristram–Levine signature of a link is determined by the linking matrix.

1. Introduction

Let \( L = L_1 \cup \cdots \cup L_r \) be a link. Recall that the Tristram–Levine signature of \( L \) is defined as:

\[
\sigma_L(z) = \text{sign}\left( (1 - z)S + (1 - \overline{z})S^T \right),
\]

where \( S \) is a Seifert matrix for \( L \), \( z \) is a complex number with modulus 1, and \( \text{sign} \) denotes the signature of a hermitian matrix. Levine–Tristram signature was introduced by Tristram [16] and Levine [9] as a generalization of the Murasugi signature [12]. Levine–Tristram signature is an important link invariant, used e.g. to obstruct concordance of links.

By (1.1), the value of \( \sigma_L(1) \) is zero. Nevertheless, if \( r > 1 \), the limit

\[
\sigma^1 := \lim_{t \to 0, t \neq 0} \sigma_L(e^{it})
\]

can be non-zero. Surprisingly, not much is known about the topological interpretation of \( \sigma^1 \). In [6, Theorem 2.1] it is proved that \( |\sigma^1| \leq r - 1 \); see also [4, Remark 2.2]. This statement generalizes the well-known fact that \( \sigma^1 = 0 \) if \( L \) is a knot.

Before we state the main theorem, recall that the linking matrix of an \( r \)-component link \( L \) is an \( r \times r \) matrix \( A_L = \{a_{ij}\} \), where

\[
a_{ij} = \begin{cases} 
\text{lk}(L_i, L_j) & \text{if } i \neq j \\
-\text{lk}(L_i, L \setminus L_i) & \text{otherwise}.
\end{cases}
\]

The following is the main theorem of this short article.

**Theorem 1.1.** Let \( L \) be an \( r \)-component link. Let \( \Delta_L(t) \) be the single variable Alexander polynomial. Suppose that \( \Delta_L(t) \) is non-zero. If \( (t - 1)^r \) does not divide \( \Delta_L(t) \), then \( \sigma^1 \) is the signature of the linking matrix of \( L \).

**Remark 1.2.** Our definition of the linking matrix is different than the one in LinkInfo [11], where the linking matrix is supposed to have zeros on the diagonal. Our definition is maybe less natural, but has a more geometric meaning; see Lemma 2.3 below. Our convention is used e.g. in connection with Hosokawa theorem [7, 17].

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Theorem 1.1 is proved in Section 4 after we give preparatory results in Sections 2 – 3. Section 5 contains several examples.

Remark 1.3. The methods of the present paper can be used to generalize Theorem 1.1 to the case when $\Delta_L \equiv 0$ or if $(t-1)^r$ divides $\Delta_L(t)$, leading to inequalities for $|\sigma^1 - \text{sign} A_L|$. These statements are more technical than Theorem 1.1 and the proof is therefore, they are not discussed in the present paper.

Convention 1.4. Throughout the paper, we use the notation $\lim_{z \to 1} f(z)$ for $\lim_{z=1, z\neq 1} f(z)$.

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2. Linking matrix

We introduce a useful terminology.

Definition 2.1. A small linking matrix $H_L$ for $L$ is an $(r-1) \times (r-1)$ principal minor of $A_L$.

Lemma 2.2. The congruence class of $H_L$ is well-defined, that is, does not depend on the particular choice of a minor. The signature of $H_L$ is equal to the signature of $A_L$. The dimension of the kernel of $H_L$ is one less than the dimension of $A_L$.

Proof. With the notation of (1.2), we have $\sum_i a_{ij} = 0$ and $\sum_j a_{ij} = 0$. Therefore, the linking matrix is of the form

$$A_L = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,r-1} & -\sum_j a_{1,j} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,r-1} & -\sum_j a_{2,j} \\ \vdots \\ -\sum_i a_{i,1} & -\sum_i a_{i,2} & \cdots & -\sum_i a_{i,r-1} & \sum_i a_{i,j} \end{pmatrix}.$$

With this formula, any row is a linear combination of other rows. Removing a linearly dependent row and the same column from a symmetric matrix does not change the signature and the rank. □

A small linking matrix has a geometric interpretation.

Lemma 2.3. Let $\Sigma$ be a Seifert surface for $L$ and let $H$ be the image of the inclusion induced map $H_1(\partial \Sigma; \mathbb{C}) \to H_1(\Sigma; \mathbb{C})$. Then, the Seifert form restricted to $H$ is congruent to a small matrix $H_L$. In particular the Seifert form restricted to $H$ is symmetric.

Proof. Recall that $L_1, \ldots, L_r$ denote the components of the link. Denote by $[L_1], \ldots, [L_r]$ the classes they represent in $H_1(\Sigma; \mathbb{C})$. These classes span the space $H$, and are subject to the relation $[L_1] + \cdots + [L_r] = 0$.

As $L_i$ and $L_j$ for $i \neq j$ are disjoint, the Seifert form $S([L_i], [L_j])$ is equal to the linking number of $L_i$ and $L_j$. Due to the relation $[L_1] + \cdots + [L_r] = 0$, we infer that $S([L_i], [L_i]) = -\text{lk}(L_i, L_1 \cup \cdots \cup L_{i-1} \cup \cdots \cup L_r)$.

To conclude the proof, we choose a basis $[L_1], \ldots, [L_{r-1}]$ of $H$. In this basis $S$ is represented by the $(r-1) \times (r-1)$ minor of the linking matrix $A_L$. □
Example 2.4. Suppose \( L \) is a two component link. If \( \ell \) denotes the linking number of the two components, then the linking matrix is \( \begin{pmatrix} -\ell & \ell \\ \ell & -\ell \end{pmatrix} \). The small linking matrix is equal to \((-\ell)\). In particular, the signature of the small linking matrix is minus the sign of the linking number.

3. Review of Hodge numbers

Hodge numbers are objects related to so called hermitian variation structures. These were introduced by Némethi \cite{13} as an algebraic framework for describing homological structures associated with the Milnor fibration of an isolated hypersurface singularity. We quickly review the terminology following \cite{13}. We restrict our attention to hermitian structures associated with the Milnor fibration of an isolated hypersurface singularity. For future use, we note that all the structures \( (U, b, h, V) \) and \( (U', b', h', V') \) are isomorphic if there is a map \( \phi: U \to U' \) intertwining \( b, h, V \) and \( b', h', V' \) respectively. This means that \( b = \overline{\phi} b' \phi, h = \phi^{-1} h' \phi \) and \( V = \phi^{-1} V' (\overline{\phi})^{-1} \); see \cite{13} Definition 2.5]. It is worth to stress that in matrix notation, \( h \) transforms by a conjugation, and \( b \) and \( V \) transform by a congruence. We note that there is rather obvious notion of a direct sum of two HVS.

Basic examples of HVS were given by Némethi in \cite{13} Section 2. For any integer \( k \), a unit complex number \( \lambda \) and a sign choice \( u \in \{1, -1\} \), there exists an HVS \( \mathcal{W}_k^\lambda(u) \), such that \( \text{dim} \ U = k \) and \( h \) is a single Jordan block with eigenvalue \( \lambda \). The structures \( \mathcal{W}_k^{(1)} \) and \( \mathcal{W}_k^{(-1)} \) are distinguished by the signature of the pairing \( b \). In this paper we need an explicit form of the structure \( \mathcal{W}_1^1(u) \):

\[
\mathcal{W}_1^1(u) = (\mathbb{C}, 0, 1, u);
\]

compare \cite{13} Example 2.7, item 7].

On the other hand, for any integer \( \ell \) and a complex number \( \mu \) with \( |\mu| \in (0, 1) \), there exists an HVS \( \mathcal{V}_\mu^{2\ell} \). For this structure, \( h \) is a sum of Jordan blocks of size \( k \): one with eigenvalue \( \mu \), another one with eigenvalue \( \overline{\mu}^{-1} \).

Remark 3.1. For future use, we note that all the structures \( \mathcal{V}_\mu^{2\ell} \) and \( \mathcal{W}_\lambda^k(u) \) for \( \lambda \neq 1 \) have non-degenerate pairing \( b \). The pairing \( b \) associated to each of the structures \( \mathcal{W}_\lambda^k(u) \), has one-dimensional kernel.

The classification theorem tells us that each simple HVS can be presented as a direct sum of basic structures. More precisely, we have the following result.
Theorem 3.2 (see [13, Theorem 2.9]). Suppose \( \mathcal{V} \) is a simple HVS. There exists uniquely defined non-negative integers \( p^k_{\lambda}(u) \) and \( q^\ell_{\mu} \) (with \( k, \ell, \lambda, \mu, u \) as above) such that \( \mathcal{V} \) is isomorphic to the sum

\[
\mathcal{V} = \bigoplus_{k, \lambda, u} p^k_{\lambda}(u) \mathcal{W}^k_{\lambda}(u) \oplus \bigoplus_{\ell, \mu} q^\ell_{\mu} \mathcal{V}^{2\ell}_{\mu},
\]

where the symbol \( s\mathcal{V} \) for a non-negative integer \( s \) should be interpreted as the direct sum of \( s \) copies of the structure \( \mathcal{V} \).

We call the integers \( p^k_{\lambda}(u) \) and \( q^\ell_{\mu} \) the Hodge numbers associated with the HVS \( \mathcal{V} \). We will refer to the structures \( \mathcal{W}^k_{\lambda}(u) \) and \( \mathcal{V}^{2\ell}_{\mu} \) as basic structures.

In [1, Section 3.1] there was defined an HVS for a link in \( S^3 \). The following construction is a special case of that construction.

Proposition 3.3. Suppose \( L \subset S^3 \) is a link whose Seifert matrix is \( S \)-equivalent over \( \mathbb{R} \) to a non-degenerate matrix \( S \). Then there exists a well-defined HVS for the link, such that \( \mathcal{V} = (S^T)^{-1} \). The pairing \( b \) of this structure is adjoint to the form \( S - S^T \) and \( h = (S^T)^{-1}S \).

The Hodge numbers associated with the link determine its Alexander polynomial and the Tristram–Levine signature, see [1, Section 4]. Hodge numbers for links satisfy the relations \( p^k_{\lambda}(u) = p^k_{\lambda}(u), q^\ell_{\mu} = q^\ell_{\mu} \).

4. The part of HVS with eigenvalue 1

The next result is due to Keef [8]. We refer to [4, Section 4] for a more detailed exposition of Keef’s results in the present context.

Lemma 4.1. Let \( L \) be a link. The one-variable Alexander polynomial of \( L \) is non-zero if and only if any Seifert matrix of \( L \) is \( S \)-equivalent (over \( \mathbb{Q} \)) to an invertible matrix.

Proof. Let \( S \) be a Seifert matrix of \( L \). By Keef [8, Proposition 3.1] we know that \( S \) is \( S \)-equivalent over \( \mathbb{Q} \) to a block matrix \( S_{\text{deg}} \oplus S_0 \), where \( S_{\text{deg}} \) is invertible and \( S_0 \) is a matrix with all entries zero. The one-variable Alexander polynomial of \( L \) is given by \( \det(tS - S^T) \), which is zero if \( S_0 \) has positive dimension.

Note that the rank of \( S_0 \) is the rank of the torsion-free part of the Alexander module of \( L \) over \( \mathbb{Q}[t, t^{-1}] \), hence it does not depend on the particular choice of the Seifert matrix within its \( S \)-equivalence class. \( \square \)

From now on, we will assume that \( L \) is an \( r \)-component link whose one-variable Alexander polynomial is non-zero. Let \( S \) be an invertible matrix which is \( S \)-equivalent to a Seifert matrix of \( L \). We let \( \mathcal{V}_L = (U_L, b_L, h_L, V_L) \) be the HVS associated with \( L \). In the decomposition (3.3) we group terms with \( \lambda = 1 \) and separately all other terms so as to obtain a decomposition \( \mathcal{V}_L = \mathcal{V}_{z=1} \oplus \mathcal{V}_{z>1} \); compare [2, Section 3.3]. Accordingly, we write:

\[
U_L = U_{z=1} \oplus U_{z>1}, \quad b_L = b_{z=1} \oplus b_{z>1}, \quad h_L = h_{z=1} \oplus h_{z>1}, \quad V_L = V_{z=1} \oplus V_{z>1}.
\]

Note that the key property is that all the eigenvalues of \( h_{z=1} \) are different than 1 and \( h_{z=1} \) is a sum of Jordan blocks with eigenvalue 1.
Given \( \{1\} \), as \( V_L = (S^T)^{-1} \), changing \( S \) by a congruence if needed, we can decompose

the matrix \( S \) as a block sum

\[
S = S_{\pm 1} \oplus S_{\pm 1}; \\
\]

see \([2\) Proposition 3.3.10(d)]\). The following observation follows immediately from the definition of \( U_{\pm 1} \).

**Lemma 4.2.** Let \( p^k_\lambda(u), q^k_\mu \) be the Hodge numbers associated with \( V_L \). Then

\[
\dim U_{\pm 1} = \sum_{k,u} kp^k_\lambda(u). \\
\]

Our next aim is to relate the structure \( V_{\pm 1} \) with the linking matrix of the link \( L \). For links of singularities the result we are going to prove can be found in \([13\) Section 3\], even though it is phrased in a different language.

The case of links with possibly degenerate Seifert matrix requires a technical result. Before we state it, we recall that a **row extension** replaces a matrix \( S_1 \) by a matrix

\[
S_2 = \begin{pmatrix} S_1 & 0 & 0 \\ \xi & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},
\]

where \( \xi \) is a row vector. A **row contraction** is the inverse operation. Column extensions and contractions are defined analogously. There are several conventions regarding these definitions, leading to equivalent formula with different shape; we adopt the convention that is used e.g. by Levine and Murasugi; see \([10, 12]\). We also recall that two square matrices are S-equivalent if one can pass from one to another by a sequence of congruences, row and column extensions, and row and column contractions.

**Lemma 4.3.** Let \( U_1, U_2 \) be two vector spaces over \( \mathbb{R} \), with an embedding \( \iota : U_1 \to U_2 \). Let \( S_i : U_i \to U^*_i, i = 1, 2 \) two maps such that in some choice of basis the matrix representing \( S_2 \) is a row or column extension of a matrix representing \( S_1 \). Then \( \iota \) induces an isomorphism \( \iota' : \ker(S_1 - S_1^T) \to \ker(S_2 - S_2^T) \). The isomorphism \( \iota' \) induces an isometry of forms \((S_1 + S_1^T) \text{ and } (S_2 + S_2^T)\) restricted to \( \ker(S_1 - S_1^T) \), respectively \( \ker(S_2 - S_2^T) \).

**Proof.** Let \( n = \dim U_1 \). We restrict to the case when \( S_2 \) is a row extension of \( S_1 \), the case of the column extension is analogous. In that case, by \([11]\) we have

\[
S_2 - S_1^T = \begin{pmatrix} S_1 - S_1^T & -\xi^T & 0 \\ \xi & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.
\]

For the \((n + 2)\)-dimensional vector \((u_1, u_2, u_3)\) (with \( u_1 \) being an \( n \)-dimensional vector, \( u_2 \) and \( u_3 \) being numbers)

\[
(S_2 - S_1^T) \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} (S_1 - S_1^T)u_1 - \xi^T u_2 \\ \xi \cdot u_1 - u_3 \\ u_2 \end{pmatrix}.
\]

Here \( \xi \cdot u_1 \) is a scalar product. Note that, on the contrary, \( \xi^T u_2 \) (without dot) is a column vector. The map \( \iota' \) is defined as \( \iota'(u) = (u, 0, \xi \cdot u) \). A straightforward calculation involving \([11]\) shows that \( \iota' \) takes \( \ker(S_1 - S_1^T) \) onto \( \ker(S_2 - S_2^T) \).
Our aim is to show that for any \( u, u' \in U_1 \), \( \iota(u)^T(S_2 + S_2^T)\iota(u') = u^T(S_1 + S_1^T)u' \). To this end, using (4.3), we write

\[
S_2 + S_2^T = \begin{pmatrix} S_1 + S_1^T & \xi & 0 \\ \xi & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.
\]

Hence,

\[
\iota(u)^T(S_2 + S_2^T)\iota(u') = \begin{pmatrix} u & 0 & 0 \\ 0 & \xi & 0 \\ \xi \cdot u' & 0 & 1 \end{pmatrix} \begin{pmatrix} S_1 + S_1^T & \xi & 0 \\ \xi & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = u^T(S_1 + S_1^T)u'.
\]

\[\square\]

**Corollary 4.4.** Let \( \tilde{S} \) be a Seifert matrix for an \( r \)-component link \( L \). Suppose \( \tilde{S} \) is \( S \)-equivalent to a non-degenerate matrix \( S \). Then:

- The spaces \( H = \ker(S - S^T) \) and \( \tilde{H} = \ker(\tilde{S} - \tilde{S}^T) \) are isomorphic.
- The forms \( S + S^T \) and \( \tilde{S} + \tilde{S}^T \) restricted to \( H \) and \( \tilde{H} \) respectively have the same rank and signature.

**Proof.** The matrices \( S \) and \( \tilde{S} \) are related by a sequence of congruences and row/column extensions and row/column contractions. By Lemma 4.3, row/column extensions preserve \( \ker(S - S^T) \) and the form \( S + S^T \) restricted to the kernel. Congruences clearly preserves \( \ker(S - S^T) \) and the form \( S + S^T \). \[\square\]

Suppose now \( L \) is a link whose Alexander polynomial is not identically zero. Let \( S \) be an invertible matrix \( S \)-equivalent to a Seifert matrix \( \tilde{S} \) of \( L \) and define \( \mathcal{V}_L = (U_L, b_L, h_L, V_L) \) to be the HVS associated with \( L \). Recall the decomposition \( \mathcal{V}_L = \mathcal{V}_{z=1} \oplus \mathcal{V}_{z=1} \).

**Lemma 4.5.** Suppose the Alexander polynomial of \( L \) is non-zero. The following conditions are equivalent.

(i) The Hodge numbers \( p^k_\ell(u) \) are zero for \( k > 1 \);

(ii) \( U_{z=1} \) is equal to \( \ker b \);

(iii) \( (t-1)^r \) does not divide \( \Delta(t) \).

**Proof.** Each of the structures \( \mathcal{W}_L^k(u) \) has the property that \( \ker b \) is one dimensional. All other basic structures, that is, \( \mathcal{W}_L^k(u) \) for \( \lambda \neq 1 \) or \( \mathcal{V}_{2\ell}^2 \), have non-degenerate \( b \); see Remark 3.1 for an explicit form of these structures in [13, Section 2]. This shows that

\[
\dim \ker b_L = \sum p^k_\ell(u), \quad \ker b_L \subset U_{z=1}.
\]

By (4.3), we immediately show that (i) is equivalent to (ii).

The Alexander polynomial \( \Delta(t) \) is equal to \( \det(h_L - tI) \) up to multiplication by a unit in \( \mathbb{C}[t, t^{-1}] \). In particular, the maximal integer \( \ell \) such that \( (t-1)^\ell \) divides \( \Delta(t) \) is equal to the dimension of the eigenspace of \( h_L \) corresponding to eigenvalue 1. That is, \( \ell = \dim U_{z=1} \). To prove equivalence of (ii) and (iii), it is enough to show that \( \dim \ker b_L = r - 1 \).

By Proposition 3.3, \( \ker b_L = \ker(S^T - S) \). By the first part of Corollary 4.4, we have \( \dim \ker S^T - S = \dim \ker \tilde{S}^T - \tilde{S} \). The form \( \tilde{S}^T - \tilde{S} \) is the intersection form on \( H_1(\Sigma, \mathbb{C}) \). As \( \Sigma \) is an oriented surface with \( r \) boundary components \( \dim \ker(\tilde{S}^T - \tilde{S}) = r - 1 \). \[\square\]
5. Proof of Theorem 1.1

Given the translation between the assumptions of Theorem 1.1 and Hodge numbers we can give the proof of Theorem 1.1.

Lemma 5.1. Let \( \mathcal{V} = (U, b, h, V) \) be a HVS. Suppose \( p_k^1(u) = 0 \) for \( k > 1 \). Then, with \( S = (V^{-1})^T \), the form \( S + S^T \) on \( \ker b \) has signature \( p_1^1(+1) - p_1^1(-1) \).

Proof. Consider the decomposition \( \mathcal{V} = \mathcal{V}_{z=1} \oplus \mathcal{V}_{z=1} \). By Lemma 4.5 we have \( \ker b = U_{z=1} \). Therefore, \( S + S^T \) on \( \ker b \) is precisely \( S_{z=1} + S_{z=1}^T \).

Note that \( p_k^1(u) = 0 \) for \( k > 1 \) implies that \( \mathcal{V}_{z=1} \) is a sum of \( p_1^1(+1) \) copies of \( \mathcal{W}_1^1(+1) \) and \( p_1^1(-1) \) copies of \( \mathcal{W}_1^1(-1) \). Thus, \( \mathcal{V}_{z=1} \) is diagonal with signature \( p_1^1(+1) - p_1^1(-1) \). Therefore, \( S_{z=1} + S_{z=1}^T \) is diagonal with the same signature. \( \square \)

Lemma 5.2. Let \( L \) be link, \( \mathcal{V}_L \) be the HVS associated to it and suppose \( p_k^1(u) = 0 \) for \( k > 1 \). The limit of the signature function \( \lim_{z \to 1} \sigma_L(z) \) is equal to \( p_1^1(+1) - p_1^1(-1) \).

Sketch of proof. The statement follows immediately from Proposition 4.14. For the sake of completeness we recall the main elements of the proof.

The signature function \( z \mapsto \sigma(z) \) can be associated with any simple HVS via the formula resembling (1.1): \( \sigma(z) = \text{sign}((1-z)(V^{-1})^T + (1-z)V^{-1}) \). With this approach the signature is additive with respect direct sums of HVS. In particular, to prove Lemma 5.2 it is enough to understand the behavior of \( \sigma(z) \) for structures \( \mathcal{W}_1^1(u) \) and \( \mathcal{V}_z^H(u) \). Now, for \( \mathcal{V}_z^H(u) \) we have vanishing \( \sigma(z) \) by explicit calculations and for \( \mathcal{W}_1^1(u) \) the limit at 1 of \( \sigma(z) \) can be shown to be zero. From these computations, and using symmetry \( p_k^1(u) = \overline{p_k^1(u)} \), we show that the only contribution to \( \lim_{z \to 1} \sigma(z) \) can come from \( \mathcal{W}_1^1(u) \). In our situation, only \( \mathcal{W}_1^1(u) \) is a summand of \( \mathcal{V}_L \). For this structure, by (3.1), we conclude that \( \sigma(z) = u \) for all \( z \in S^1 \setminus \{1\} \), in particular \( \lim_{z \to 1} \sigma(z) = u \). \( \square \)

Now we are in position to give a proof of the main theorem.

Proof of Theorem 1.1 Let \( \Sigma \) be a Seifert surface for \( L \) and let \( \tilde{S} \) be the Seifert matrix associated to it. Let \( H \Sigma \subset H_1(\Sigma; \mathbb{C}) \) be the kernel of \( \tilde{S} - S^T \). Then \( H \Sigma \) can be identified with the image \( H_1(\partial \Sigma; \mathbb{C}) \rightarrow H_1(\Sigma; \mathbb{C}) \). Next, let \( S \) be an invertible matrix \( S \)-equivalent to \( \tilde{S} \). Write \( H_S = \ker(S - S^T) \). We claim that the following numbers are equal:

(a) The signature of the linking matrix of \( L \);
(b) The signature of the small linking matrix of \( L \);
(c) The signature of \( \tilde{S} + S^T \) restricted to \( \ker(\tilde{S} - S^T) \);
(d) The signature of \( S + S^T \) restricted to \( \ker(S - S^T) \);
(e) The difference \( p_1^1(+1) - p_1^1(-1) \) for the Hodge numbers associated with \( \mathcal{V}_L \);
(f) The limit signature \( \sigma^1 \).

Equality of (a) and (b) is Lemma 2.2. Equality of (b) and (c) is precisely Lemma 2.3. Equality of (c) and (d) follows from Corollary 4.4. Note that until this moment we do not use the assumption of Theorem 1.1. Now, the equality of (d) and (e) follows from Lemma 5.1 which uses Theorem 1.1 via the equivalence of items (i) and (iii) of Lemma 2.5. Finally, equality of (e) and (f) follows from Lemma 5.2 (using again Condition (iii) of Lemma 4.5). \( \square \)
Figure 1. Link L7a2. The orientation is chosen in such a way that the two components have linking number \(-2\). Picture drawn using [5].

6. Examples

6.1. The Hopf link. For the positive Hopf link we have \(\text{lk}(L_1, L_2) = 1\) so the linking matrix becomes

\[
A = \begin{pmatrix}
-1 & 1 \\
1 & -1
\end{pmatrix}
\]

The signature of this matrix is \(-1\) and so is the signature \(\sigma^1\).

6.2. Link L7a2. The link L7a2, see Figure 1, is a two-component link whose Seifert matrix is degenerate. Namely, according to LinkInfo [11] its Seifert matrix is

\[
S = \begin{pmatrix}
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

We have \(\det(tS - S^T) = (3t^6 - 4t^5 + 3t^4)(t - 1)\), and \(3t^2 - 4t + 3\) has two roots on \(S^1 \setminus \{1\}\), namely \(\frac{2}{3} \pm \frac{i\sqrt{5}}{3}\). The assumptions of Theorem 1.1 are satisfied.

The signature function is constant away from \(z = \frac{2}{3} \pm \frac{i\sqrt{5}}{3}\). Therefore, the limit signature at 1 is equal to the signature at any point \(z_0\) on \(S^1 \setminus \{1\}\), such that \(\text{Re } z_0 > \text{Re } z\). With \(z = \frac{4}{3} + i\frac{\sqrt{5}}{3}\), the signature of the matrix \((1 - z)S + (1 - \tau)S^T\) can be computed using Sage [15]. It is equal to 1.
The components of $L$ has linking number $-2$. Therefore, the linking matrix is \[
\begin{pmatrix}
2 & -2 \\
-2 & 2
\end{pmatrix}
\]
and the small linking matrix is $(2)$. The signature of the small linking matrix is $1$; this confirms Theorem 1.1.

6.3. L5a1 link. Consider the Whitehead link (L5a1 on the linkinfo list) depicted in Figure 2. This is a two-component link and the linking number of the two components is $0$. Thus, the linking matrix is the zero $2 \times 2$ matrix and the small linking matrix is $(0)$. In particular, the signature of the small linking matrix is zero.

According to LinkInfo [11], the Seifert matrix of L5a1 is
\[
S = \begin{pmatrix}
1 & 0 & -1 \\
-1 & 1 & 1 \\
0 & 0 & -1
\end{pmatrix}.
\]

The Alexander polynomial $\Delta(t) = \det(tS - S^T) = (t - 1)^3$, therefore the assumptions of Theorem 1.1 are not satisfied.

To calculate $\sigma^1$, we note that the signature function is constant away from the set of roots of the Alexander polynomial $\Delta$. As $\Delta$ has roots only at $t = 1$, we infer that $\sigma^1 = \sigma(-1)$. The latter signature is equal to
\[
\sigma(-1) = \text{sign} \begin{pmatrix}
2 & -1 & -1 \\
-1 & 2 & 1 \\
-1 & 1 & -2
\end{pmatrix}.
\]
The above matrix has $2 > 0$ in the top left corner. The determinant of the top left $2 \times 2$ minor is $3 > 0$ and the determinant of the whole $3 \times 3$ matrix is $-8 < 0$. Hence, $\sigma(-1) = 1$ and so $\sigma^1 = 1$ is not equal to the signature of the small linking matrix. This example shows that the assumption of Theorem 1.1 that $(t - 1)^r$ does not divide the Alexander polynomial is necessary.

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