The Cauchy problem for metric-affine $f(R)$-gravity in presence of a Klein-Gordon scalar field

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We study the initial value formulation of metric-affine $f(R)$-gravity in presence of a Klein-Gordon scalar field acting as source of the field equations. Sufficient conditions for the well-posedness of the Cauchy problem are formulated. This result completes the analysis of the same problem already considered for other sources.

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I. INTRODUCTION

Among the various attempts to extend General Relativity, $f(R)$-gravity can be considered a useful paradigm capable of preserving the good and well-established results of the Einstein gravity without imposing a priori the form of the action. This approach has recently acquired great interest in cosmology and in quantum field theory in order to cure the shortcomings of standard theory of gravity at ultra-violet and infra-red scales. In particular, $f(R)$-gravity, not excluded by observations [1] could result useful to address cosmological puzzles as dark energy and dark matter, up to now not probed at fundamental scales. On the other hand, observations and experiments could, in principle, help to reconstruct the effective form of the theory by solving inconsistencies and shortcomings present at various scales in General Relativity (for reviews, see [2–5]).

However, because of further gravitational degrees of freedom, emerging in assuming actions not linear in the Ricci scalar $R$, the initial value problem becomes an urgent issue to be correctly formulated and addressed. Furthermore, also if the initial value problem is well-formulated, stability against perturbations has to be studied and the causal structure has to be preserved. If both these requirements are satisfied, the initial value problem of the theory is also well-posed.

The Cauchy problem for General Relativity is well-formulated and well-posed [7–10]. Such a result should be achieved also for $f(R)$-gravity, if one wants to consider it a viable extension of the Einstein theory and compare the two approaches.

The crucial point with respect to General Relativity comes from the fact that the further degrees of freedom could lead to an ill-formulated initial value problem. In fact, their role has to be clearly understood in order to discriminate among ghost modes, standard massless modes or further massive gravitational modes [6]. Such an analysis can be correctly addressed by conformally transforming $f(R)$-gravity from the Jordan frame to the Einstein frame. The extra-degrees of freedom give rise to auxiliary scalar fields, minimally coupled to the standard gravitational Hilbert-Einstein action. In this way, the analysis results simplified.

However, being $f(R)$-gravity a gauge theory, the initial value formulation depends on suitable constraints and gauges that mean a consequent choice of coordinates so that the Cauchy problem can result well-formulated and, possibly, well-posed. The debate on the well-formulation and the well-posedness of the Cauchy problem of $f(R)$ theories, in metric and Palatini approaches, has recently given several interesting and sometime contrasting results [11–14].

It is possible to show that the Cauchy problem of metric-affine $f(R)$-gravity is well-formulated and well-posed in vacuum, while it can be, at least, well-formulated for various form of matter fields [14]. The reason of the apparent contradiction with respect to the results in [11] lies on the above mentioned gauge choice. Following [9], Gaussian normal coordinates can be adopted. This choice, introducing further constraints on the Cauchy data surface, results more suitable to set the initial value problem in such a way that the well-formulation can be easily achieved. As a general remark, we can say that the well-position cannot be achieved for any metric-affine $f(R)$-gravity theory but it has to be formulated specifying, case by case, the source term in the field equations. However, it is straightforward to demonstrate that, in vacuum case, as well as for electromagnetic and generic Yang-Mills fields acting as sources, the Cauchy problem results always well-formulated and well-posed since it is possible to show that $f(R)$-gravity reduces to $R + \Lambda$, that is the General Relativity plus a cosmological constant [14].

In [12], we have addressed the Cauchy problem for metric-affine $f(R)$-gravity, in the Palatini approach, and with torsion, assuming perfect-fluid matter as source. Performing the conformal transformation from the Jordan to the Einstein frame and following the approach by Bruhat, adopted for General Relativity [7, 8, 23], we have formulated sufficient conditions to ensure the well-posedness of the Cauchy problem. Moreover, we have shown that the set of functions $f(R)$ satisfying the stated conditions is actually not empty.

Here, we want to show that analogous results hold also in the case in which the source is a Klein-Gordon scalar field. This is a delicate case due to the fact that second-order partial derivatives in the Klein-Gordon equation could give rise to inconsistencies when coupled with gravitational degrees of freedom (for a detailed discussion see [10]).

The layout of the paper is the following. In Sec. II, we give a summary of $f(R)$-gravity in metric-affine formulation à la Palatini and with torsion. Sec. III is devoted to the discussion of the Cauchy problem in presence of a Klein-Gordon scalar field.
acting as source of the field equations. We use the arguments in [7,8,13,23] to show the consistency with the analogous well-formulation and well-position of General Relativity. In particular, it is possible to show that the well-position conditions are capable of selecting self-consistent \( f(R) \)-models. The relevant example \( f(R) = R + \alpha R^2 \) is discussed in Sec.IV. Conclusions are drawn in Sec.V.

II. PRELIMINARIES ON METRIC-AFFINE \( f(R) \)-GRAVITY

The pairs \((g,\Gamma)\) constitute the gravitational fields in the metric–affine formulation of \( f(R) \)-theories of gravity: \( g \) is a pseudo-Riemannian metric and \( \Gamma \) a linear connection on the space-time manifold \( \mathcal{M} \). The connection \( \Gamma \) is torsionless in the Palatini approach but it is not requested to be metric-compatible. On the other hand, in the approach with torsion, the dynamical connection \( \Gamma \) is forced to be metric in the approach with torsion. The field equations are derived from the action

\[
\mathcal{A}(g,\Gamma) = \int \left( \sqrt{|g|} f(R) + \mathcal{L}_m \right) \, ds
\]

where \( f(R) \) is a real function, \( R(g,\Gamma) = g^{ij} R_{ij} \) (with \( R_{ij} := R^h_{\;i\;h\;j} \)) is the curvature scalar associated with the connection \( \Gamma \) and \( \mathcal{L}_m ds \) is the matter Lagrangian.

Assuming that the matter Lagrangian does not depend on the dynamical connection, the field equations are

\[
f'(R) R_{ij} - \frac{1}{2} f(R) g_{ij} = \Sigma_{ij}, \tag{2a}
\]

\[
T_{ij}^h = - \frac{1}{2 f'(R)} \frac{\partial f'(R)}{\partial x^p} \left( \delta_p^i \delta_j^h - \delta_p^j \delta_i^h \right) \tag{2b}
\]

for \( f(R) \)-gravity with torsion [16], and

\[
f'(R) R_{ij} - \frac{1}{2} f(R) g_{ij} = \Sigma_{ij}, \tag{3a}
\]

\[
\nabla_k (f'(R) g_{ij}) = 0, \tag{3b}
\]

for \( f(R) \)-gravity in the Palatini approach [17,21]. The quantity \( \Sigma_{ij} := - \frac{1}{\sqrt{|g|}} \frac{\delta \mathcal{L}_m}{\delta g^{ij}} \) is the stress-energy tensor. From Eqs. (2a) and (3a), we obtain the relation

\[
f'(R) R - 2 f(R) = \Sigma \tag{4}
\]

where the curvature scalar \( R \) is linked to the trace of the stress-energy tensor \( \Sigma := g^{ij} \Sigma_{ij} \). From now on, we shall suppose that the relation (4) is invertible as well as that \( \Sigma \neq \text{const.} \) (this implies \( f(R) \neq \alpha R^2 \) which is compatible with \( \Sigma = 0 \)). Under these restrictions, the curvature scalar \( R \) can be expressed as a suitable function of \( \Sigma \), namely

\[
R = F(\Sigma). \tag{5}
\]

If \( \Sigma = \text{const.} \), General Relativity plus the cosmological constant is immediately recovered [16]. Defining the scalar field

\[
\varphi := f'(F(\Sigma)), \tag{6}
\]

we can put the Einstein–like field equations of both à la Palatini and with torsion theories in the same form [16,21], that is

\[
\ddot{R}_{ij} - \frac{1}{2} \ddot{g}_{ij} = \frac{1}{\varphi} \Sigma_{ij} + \frac{1}{\varphi^2} \left( - 3 \frac{\partial \varphi}{\partial x^i} \frac{\partial \varphi}{\partial x^j} + \varphi \ddot{\nabla}_j \frac{\partial \varphi}{\partial x^i} + \frac{3}{4} \frac{\partial \varphi}{\partial x^h} \frac{\partial \varphi}{\partial x^k} \dot{g}^{hk} g_{ij} - \varphi \ddot{\nabla}_k \frac{\partial \varphi}{\partial x^h} g_{ij} - V(\varphi) g_{ij} \right), \tag{7}
\]

where the effective potential

\[
V(\varphi) := \frac{1}{4} \left[ \varphi F^{-1}(f')^{-1}(\varphi) + \varphi^2 (f')^{-1}(\varphi) \right], \tag{8}
\]
for the scalar field \( \varphi \) has been introduced. In Eq. (7), \( \tilde{R}_{ij} \), \( \tilde{R} \) and \( \tilde{\nabla} \) denote, respectively, the Ricci tensor, the scalar curvature and the covariant derivative associated with the Levi-Civita connection of the metric \( g_{ij} \). Therefore, if the dynamical connection \( \Gamma \) is not coupled with matter, both the theories (with torsion and \( a \) la Palatini) generate identical Einstein-like field equations. Moreover, it can be shown [16] that the Einstein–like Eqs. (7) (together with Eqs. (6)) are deducible from a scalar-tensor theory with Brans-Dicke parameter \( \omega_0 = -3/2 \). To see this point, we recall that the action functional of a (purely metric) scalar–tensor theory is given by

\[
\mathcal{A}(g, \varphi) = \int \left[ \sqrt{|g|} \left( \varphi \tilde{R} - \frac{\omega_0}{\varphi} \varphi_i \varphi^i - U(\varphi) \right) + \mathcal{L}_m \right] ds
\]

(9)

where \( \varphi \) is the scalar field, \( \varphi_i := \frac{\partial \varphi}{\partial x^i} \) and \( U(\varphi) \) is the potential of \( \varphi \). The matter Lagrangian \( \mathcal{L}_m(g_{ij}, \psi) \) is a function of the metric and some matter fields \( \psi \). \( \omega_0 \) is the so called Brans–Dicke parameter \(^1\).

The field equations, derived by varying with respect to the metric and the scalar field, are

\[
\tilde{R}_{ij} - \frac{1}{2} \tilde{R} g_{ij} = \frac{1}{\varphi} \Sigma_{ij} + \frac{\omega_0}{\varphi^2} \left( \varphi_i \varphi_j - \frac{1}{2} \varphi h \varphi^h g_{ij} \right) + \frac{1}{\varphi} \left( \tilde{\nabla}_j \varphi_i - \tilde{\nabla}_i \varphi^h g_{ij} \right) - \frac{U}{2 \varphi} g_{ij}
\]

(10)

and

\[
\frac{2 \omega_0}{\varphi^2} \tilde{\nabla}_i \varphi^h + \tilde{R} - \frac{\omega_0}{\varphi^2} \varphi h \varphi^h - U' = 0
\]

(11)

where \( \Sigma_{ij} := -\frac{1}{\sqrt{|g|}} \frac{\delta \mathcal{A}}{\delta g^{ij}} \) and \( U' := \frac{dU}{d\varphi} \). Taking the trace of Eq. (10) and using it to replace \( \tilde{R} \) in eq. (11), one obtains the equation

\[
(2 \omega_0 + 3) \tilde{\nabla}_h \varphi^h = \Sigma + \varphi U' - 2U
\]

(12)

By a direct comparison, it is easy to see that for \( \omega_0 = -\frac{3}{2} \) and \( U(\varphi) = \frac{2}{\varphi} V(\varphi) \) (where \( V(\varphi) \) is defined in Eq. (8)) eqs. (10) become formally identical to the Einstein–like equations (7) for a metric–affine \( f(R) \) theory. Moreover Eq. (12) reduces to the algebraic equation

\[
\Sigma + 2 V'(\varphi) - \frac{6}{\varphi} V(\varphi) = 0
\]

(13)

relating the matter trace \( \Sigma \) to the scalar field \( \varphi \). In particular, it is straightforward to verify that (under the condition \( f'' \neq 0 \)) Eq. (13) expresses the inverse relation of Eq. (6), namely

\[
\Sigma + 2 V'(\varphi) - \frac{6}{\varphi} V(\varphi) = 0 \quad \iff \quad \Sigma = F^{-1}((f')^{-1}(\varphi))
\]

(14)

being \( F^{-1}(X) = f'(X)X - 2f(X) \). Metric–affine \( f(R) \) theories (with torsion or \( a \) la Palatini) are then dynamically equivalent to scalar–tensor theories with Brans–Dicke parameter \( \omega_0 = -\frac{3}{2} \).

III. THE CAUCHY PROBLEM OF METRIC-AFFINE \( f(R) \)-GRAVITY IN PRESENCE OF KLEIN-GORDON SCALAR FIELD

Let us consider now the Cauchy problem for metric–affine \( f(R) \)-gravity where a Klein-Gordon scalar field acting as a source. We shall derive sufficient conditions ensuring the well-posedness of the problem. This result is obtained making use of the above stated dynamical equivalence between metric-affine \( f(R) \)-theories and \( \omega_0 = -\frac{3}{2} \) scalar–tensor theories (see also [12–14]).

Let us start by defining a Klein-Gordon scalar field \( \psi \) whose dynamics is given by the self-interacting potential \( U(\psi) = \frac{1}{2} m^2 \psi^2 \). The associated stress-energy tensor is

\[
\Sigma_{ij} \equiv \frac{\partial \psi}{\partial x^i} \frac{\partial \psi}{\partial x^j} - \frac{1}{2} g_{ij} \left( \frac{\partial \psi}{\partial x^p} \frac{\partial \psi}{\partial x^q} g^{pq} + m^2 \psi^2 \right)
\]

(15)

\(^1\) However, it is worth noticing that this is a Brans–Dicke–like theory, not a proper Brans-Dicke theory, due to the presence of the self-interacting potential. This fact, in particular the presence of the potential, have to be stressed since misleading conclusions on the analogy between \( f(R) \)-gravity and Brans-Dicke theory could be drawn. For a recent discussion on this topic see [23].
The corresponding Klein-Gordon equation is given by
\[ \tilde{\nabla}_j \frac{\partial \psi}{\partial x^j} g^{ij} = m^2 \psi \] (16)

where \( \tilde{\nabla} \) denotes the Levi-Civita covariant derivative induced by the metric \( g_{ij} \). The trace of the tensor (15) is
\[ \Sigma := \Sigma_{ij} g^{ij} = -\frac{\partial \psi}{\partial x^p} \frac{\partial \psi}{\partial x^q} g^{pq} - 2m^2 \psi^2. \] (17)

Furthermore, let us consider a scalar–tensor theory, with Brans-Dicke parameter \( \omega_0 = -\frac{3}{2} \) and potential \( U(\phi) = \frac{2}{\phi^2} V(\phi) \), coupled with this Klein-Gordon field. The corresponding field equations are the Einstein-like Eqs. (7), Eq. (13) relating the scalar field \( \phi \) to the trace \( \Sigma \), and the Klein-Gordon Eq. (16). In order to discuss the Cauchy problem of such a theory, following [12, 14], we begin by performing the conformal transformation \( \bar{g}_{ij} = \phi g_{ij} \). The Einstein-like Eqs. (7) assume then the simpler form (see for example [16, 21])
\[ \bar{\tilde{\nabla}}_i \Sigma_{ij} - \frac{1}{2} \bar{\tilde{\nabla}}_i \bar{\tilde{\nabla}}_j \bar{g}_{ij} = \frac{1}{\phi^2} V(\phi) \bar{g}_{ij} \] (18)

where \( \bar{\tilde{\nabla}}_i \) denotes the covariant derivative associated with the conformal metric \( \bar{g}_{ij} \). Similarly, a direct calculation shows that the Klein-Gordon equation, expressed in terms of the conformal metric \( \bar{g}_{ij} \), becomes
\[ -\frac{\partial \psi}{\partial x^i} \bar{g}^{ij} + \phi \bar{\tilde{\nabla}}_j \frac{\partial \psi}{\partial x^i} \bar{g}^{ij} = m^2 \psi \] (19)

where \( \bar{\tilde{\nabla}}_j \) denotes the covariant derivative associated with the conformal metric \( \bar{g}_{ij} \). Also the trace \( \Sigma \) can be expressed in function of \( \bar{g}_{ij} \), that is
\[ \Sigma = -\frac{\partial \psi}{\partial x^p} \frac{\partial \psi}{\partial x^q} \phi \bar{g}^{pq} - 2m^2 \psi^2. \] (20)

Now, the relation (13) links the scalar field \( \phi \) to the Klein–Gordon field \( \psi \), its partial derivatives \( \frac{\partial \psi}{\partial x^i} \) and the conformal metric \( \bar{g}_{ij} \). In Eqs. (18), the quantity
\[ T_{ij} := \frac{1}{\phi} \Sigma_{ij} - \frac{1}{\phi^2} V(\phi) \bar{g}_{ij} \] (21)

plays the role of the effective stress-energy tensor. Furthermore, it is worth noticing that conservation laws can be related in the Jordan and in the Einstein frame [12, 14], that is

**Proposition 1.** Eqs. (7), (8) and (13) imply the usual conservation laws \( \tilde{\nabla}^i \Sigma_{ij} = 0 \).

**Proposition 2.** The condition \( \tilde{\nabla}^i \Sigma_{ij} = 0 \) is equivalent to the condition \( \tilde{\nabla}^i T_{ij} = 0 \).

In the following discussion, Proposition 2 plays a crucial role. The Klein-Gordon Eq. (16) implies the conservation laws \( \tilde{\nabla}^i \Sigma_{ij} = 0 \) and then the identity \( \tilde{\nabla}^i T_{ij} = 0 \). It turns out that the latter allows to use harmonic coordinates to deal with the Cauchy problem, according to the lines considered in [7, 8, 10].

We start by rewriting the Einstein–like Eqs. (18) in the equivalent form
\[ \bar{R}_{ij} = T_{ij} - \frac{1}{2} \bar{g}_{ij} \] (22)

Assuming harmonic coordinates, i.e. local coordinates satisfying the conditions
\[ \nabla_p \nabla_{x^i} = -g^{pq} \Gamma^i_{pq} = 0, \] (23)
we can express Eqs. (22) as (see, for example, [3, 10])
\[ \bar{g}^{pq} \frac{\partial^2 \bar{g}_{ij}}{\partial x^p \partial x^q} = f_{ij}(\bar{g}, \partial \bar{g}, \psi, \partial \psi), \] (24)

where \( f_{ij} \) indicate suitable functions depending only on the metric \( \bar{g} \), the scalar field \( \psi \) and their first-order derivatives.
Moreover, from now on, we assume that Eq. \( (13) \) is solvable with respect to the variable \( \varphi \). In other words, we suppose to be able to derive from Eq. \( (13) \) a function of the form

\[
\varphi = \varphi \left( \bar{g}, \psi, \frac{\partial \psi}{\partial x^p}, \frac{\partial \psi}{\partial x^s}, \bar{g}^{pq} \right)
\]

expressing the scalar field \( \varphi \) in terms of the metric \( \bar{g} \), the Klein–Gordon field \( \psi \) and its first order derivatives. In particular, from Eq. \( (20) \), it is easily seen that the dependence of \( \varphi \) on the derivatives of \( \psi \) has to be necessarily of the above form. The requirement on Eq. \( (13) \) depends on the explicit form of the potential \( V(\varphi) \). The latter is determined by the function \( f(R) \) through the relation \( \delta \). Therefore, the above assumption becomes a rule to select viable \( f(R) \)-models. In addition, from Eq. \( (25) \), we derive the identity

\[
\frac{\partial \varphi}{\partial x^t} = \frac{\partial \varphi}{\partial \left( \frac{\partial \psi}{\partial x^p}, \frac{\partial \psi}{\partial x^q}, \bar{g}^{pq} \right)} 2 \frac{\partial \psi}{\partial x^t} \frac{\partial^2 \psi}{\partial x^p \partial x^q} + f_1(\bar{g}, \bar{g}, \psi, \psi). \tag{26}
\]

Inserting Eq. \( (26) \) in Eq. \( (19) \) and taking Eqs. \( (23) \) into account, we obtain the final form of the Klein–Gordon equation given by

\[
\left( \bar{g}^{ip} - \frac{2}{\varphi} \frac{\partial \varphi}{\partial \left( \frac{\partial \psi}{\partial x^p}, \frac{\partial \psi}{\partial x^q}, \bar{g}^{pq} \right)} \frac{\partial \psi}{\partial x^j} \bar{g}^{jq} \frac{\partial \psi}{\partial x^q} \bar{g}^{pq} \right) \frac{\partial^2 \psi}{\partial x^t \partial x^p} = f(\bar{g}, \bar{g}, \psi, \psi) \quad \tag{27}
\]

In Eqs. \( (26) \) and \( (27) \), \( f_1 \) and \( f \) indicate suitable functions of \( \bar{g}_{ij}, \psi \) and their first order derivatives only. Eqs. \( (24) \) and \( (27) \) describe a second order quasi-diagonal system of partial differential equations for the unknowns \( \bar{g}_{ij} \) and \( \psi \). The matrix of the principal parts of such a system is diagonal and its elements are the differential operators

\[
\bar{g}^{pq} \frac{\partial^2}{\partial x^p \partial x^q} \tag{28a}
\]

and

\[
\left( \bar{g}^{ip} - \frac{2}{\varphi} \frac{\partial \varphi}{\partial \left( \frac{\partial \psi}{\partial x^p}, \frac{\partial \psi}{\partial x^q}, \bar{g}^{pq} \right)} \frac{\partial \psi}{\partial x^j} \bar{g}^{jq} \frac{\partial \psi}{\partial x^q} \bar{g}^{pq} \right) \frac{\partial}{\partial x^t \partial x^p} \tag{28b}
\]

The operator \( (28a) \) is noting else but the wave-operator associated with the metric \( \bar{g}_{ij} \), while the operator \( (28b) \) is very similar to the sound-wave-operator arising from the analysis of the Cauchy problem for General Relativity coupled with an irrotational perfect fluid [7, 23]. To discuss the Cauchy problem for the system \( (24) \) and \( (27) \), we can then follow the same arguments developed in [7, 23]. In particular, if the quadratic form associated with \( (28b) \) is of Lorentzian signature and if the characteristic cone of the operator \( (28b) \) is exterior to the metric cone, the system \( (24) \) and \( (27) \) is causal and hyperbolic in Leray sense [15, 24]. In such a circumstance, the corresponding Cauchy problem is well-posed in suitable Sobolev spaces. Still following [7, 23], if the signature of \( \bar{g}_{ij} \) is \( (+ - - -) \), the required conditions are satisfied when the vector \( \frac{\partial \psi}{\partial x^p} \bar{g}^{ij} \) is timelike and the inequality

\[
- \frac{2}{\varphi} \frac{\partial \varphi}{\partial \left( \frac{\partial \psi}{\partial x^p}, \frac{\partial \psi}{\partial x^q}, \bar{g}^{pq} \right)} < 0 \quad \tag{29}
\]

holds. On the contrary, if the signature of \( \bar{g}_{ij} \) is \( (- + +) \), the inequality \( (29) \) has to be inverted. As already mentioned, the explicit expression of the function \( (25) \) depends on that of the potential \( \delta \) which is determined by the function \( f(R) \). Therefore, the requirement \( (29) \) (or, equivalently, its opposite) can be a criterion to single out viable \( f(R) \)-models as shown in [12]. An example to illustrate the result is given in the next section.

**IV. THE \( f(R) = R + \alpha R^2 \) CASE**

Let us consider the model \( f(R) = R + \alpha R^2 \). Taking into account that \( F^{-1}(X) = f'(X) - 2f(X) = -X \) and \( (f')^{-1}(\varphi) = \frac{\varphi - 1}{2\alpha} \), from the definition \( \delta \), we get the identity

\[
V(\varphi) = \frac{1}{8\alpha}(\varphi - 1)^2 \varphi \quad \tag{30}
\]
representing the effective potential for the considered model. Eq. 30 together with eqs. 13 and 20 yield

$$\varphi = \frac{\left(\frac{1}{2\alpha} + 2m^2\psi^2\right)}{\left(\frac{1}{2\alpha} - \frac{\partial \psi}{\partial x^\alpha} \frac{\partial \psi}{\partial t} \sqrt{\bar{g}^{st}}\right)}$$  \hspace{1cm} (31)$$

that expresses the scalar field $\varphi$ as a function of the metric $\bar{g}_{ij}$, the Klein–Gordon field $\psi$ and its first order derivatives. Directly from (31) it follows

$$-\frac{2}{\varphi} \frac{\partial \varphi}{\partial \psi} \frac{\partial \psi}{\partial x^\alpha} \frac{\partial \psi}{\partial t} \sqrt{\bar{g}^{st}} = -\frac{2}{\left(\frac{1}{2\alpha} - \frac{\partial \psi}{\partial x^\alpha} \frac{\partial \psi}{\partial t} \sqrt{\bar{g}^{st}}\right)}$$  \hspace{1cm} (32)$$

In the signature $(+ - - -)$ for the metric $\bar{g}_{ij}$, it is immediately seen that the requirement (29) is automatically satisfied if $\alpha < 0$ and if $\bar{g}^{pq} \frac{\partial \psi}{\partial x^p}$ is a timelike vector field. If the signature is $(- + + +)$, the condition becomes $\alpha > 0$. It is worth noticing that the condition $\alpha < 0$ ensures the well–posedness of the Cauchy problem for the model $f(R) = R + \alpha R^2$ also when coupled with a perfect fluid [12].

V. CONCLUSIONS

According to the prescriptions given in [7, 8, 23], we have formulated sufficient conditions to ensure the well–posedness of the Cauchy problem for metric-affine $f(R)$-theories à la Palatini and with torsion, in presence of Klein-Gordon scalar field acting as source. As in the case of perfect fluid [12], the procedure allows to select physically viable models. The presented results refute the criticisms advanced by some authors [11] about the viability of metric-affine $f(R)$-theories. The key points of the demonstration are that the conservation laws are preserved under the conformal transformation and the Bruhat arguments can be applied if suitable differential operators are defined for the Klein-Gordon field. The $f(R) = R + \alpha R^2$ case is paradigmatic. The well-posedness strictly depends on the sign of the parameter $\alpha$ in connection with the kind of signature of the metric tensor. As shown also in cosmological context, this can discriminate between physical and unphysical models [25].

In conclusion, the Cauchy problem can be, in general, well-formulated for $f(R)$-gravity in metric-affine as well as in metric formalism [11, 14]. However, the well-posedness strictly depends on the source and the parameters of the theory. (e.g. $\alpha$ in the above example). In the case of Klein-Gordon scalar field and perfect-fluid matter, it works, essentially, because the problem can be reduced to the Einstein frame by a conformal transformation and the Bruhat arguments can be applied.

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