C\(^1\),\(^1\) regularity for solutions to the degenerate \(L_p\) Dual Minkowski problem

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Abstract
In this paper, we study C\(^1\),\(^1\) regularity for solutions to the degenerate \(L_p\) Dual Minkowski problem. Our proof is motivated by the idea of Guan and Li’s work on C\(^1\),\(^1\) estimates for solutions to the Aleksandrov problem.

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1 Introduction

The classic Minkowski problem is one of the cornerstones in the Brunn–Minkowski theory of convex bodies. Its solution has many applications in various fields of geometry and analysis, see [39] for an overview. An important counterpart of the classic Minkowski problem is the famous Aleksandrov problem characterizing the integral Gauss curvature, which is introduced and completely solved by Aleksandrov [1].

Oliker [37] has shown that there is a PDE associated with the Aleksandrov problem:

\[
\frac{h}{(|\nabla h|^2 + h^2)^{\frac{n}{2}}} \det(\nabla^2 h + hI) = f \text{ on } S^{n-1},
\]

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where \( f \) is a given positive function defined on the unit sphere \( S^{n-1} \), \( h \) is an unknown function on \( S^{n-1} \). Here \( \nabla \) is the covariant derivative with respect to an orthonormal frame on \( S^{n-1} \), \( I \) is the unit matrix of order \( n - 1 \). When \( f \) is a smooth positive function, the solution to the Aleksandrov problem (1) is smooth, see \([38]\) and \([37]\). When \( f \) is a smooth, but only nonnegative, Guan and Li \([12]\) have shown solutions to the Aleksandrov problem (1) are at least \( C^{1,1} \) in dimension \( n = 3, 4 \). For higher dimensions, they have obtained the same conclusion under some further hypothesis on \( f \).

Recently, the \( L_p \) dual Minkowski problem was introduced in \([36]\). For a given positive function \( f \) defined on the unit sphere \( S^{n-1} \), the \( L_p \) dual Minkowski problem is concerned with the solvability of the Monge-Ampère type equation

\[
\frac{h^{1-p}}{(|\nabla h|^2 + h^2)^{\frac{n-q}{2}}} \det(\nabla^2 h + hI) = f \quad \text{on} \quad S^{n-1},
\]

for some support function \( h \) of a hypersurface \( M \) in the Euclidean space \( \mathbb{R}^n \) enclosing the origin. The \( L_p \) dual Minkowski problem unifies the Aleksandrov problem \((p = 0, q = 0)\), the dual Minkowski problem \((p = 0)\) and the \( L_p \)-Minkowski problem \((q = n)\). The dual Minkowski problem was first proposed by Huang, Lutwak, Yang and Zhang in their recent groundbreaking work \([18]\) and then followed by \([4,15,17,29,46,47]\). The \( L_p \)-Minkowski problem was introduced by Lutwak \([34]\) in 1993 and has been extensively studied since then; see e.g. \([5,10,48]\) for the logarithmic Minkowski problem \((p = 0)\), \([22,23,31–33,49]\) for the centroaffine Minkowski problem \((p = -n)\), and \([9,21,35,40]\) for other cases of the \( L_p \)-Minkowski problem. For the general \( L_p \) dual Minkowski problem, much progress has already been made in \([3,7,8,19,20,28]\).

When \( f \) is a smooth positive function, the solution to the \( L_p \) dual Minkowski problem (2) is smooth provided either \( p > q \) or \( pq \geq 0 \) and \( f \) is even (see \([20]\) and \([7]\)). It is natural to ask that when \( f \) is smooth, but only nonnegative, are the solutions to the \( L_p \) dual Minkowski problem (2) smooth? In this case, we encounter with certain degenerate Monge-Ampère type equation. Regularity of solutions to degenerate Monge-Ampère type equations has been investigated in \([2,6,11,13,16,24–27,30,41–43]\) and the references therein. The global \( C^{1,1} \) regularity of degenerate Monge-Ampère type equations has been obtained in \([13]\) and one cannot expect regularity higher than \( C^{1,1} \) in general \([45]\).

In this paper, we study \( C^{1,1} \) regularity for solutions to the \( L_p \) dual Minkowski problem (2) when \( f \) is smooth enough, but only nonnegative. For low dimensions \( n = 3, 4 \), we can obtain the following result.

**Theorem 1**  Suppose \( p > q > 0 \) and \( n = 3 \) or \( n = 4 \). Let \( f \) be a smooth, nonnegative, nonzero and even function on \( S^{n-1} \). Then, there exists a generalized solution \( h \in C^{1,1}(S^{n-1}) \) satisfying the Equ. (2).

For higher dimensions, we need some additional hypothesis. To statement our result we recall the following Condition 1 which was introduced by Guan-Li in \([12]\).

**Condition 1**  \( f \in C^2(S^{n-1}) \) is nonnegative and there exist a constant \( A \) such that

\[
|\nabla(f^{\frac{1}{n-2}})| \leq A \quad \text{on} \quad S^{n-1},
\]

and

\[
\Delta(f^{\frac{1}{n-2}}) \geq -A \quad \text{on} \quad S^{n-1}.
\]
It is clearly for nonnegative function $f \in C^2(S^{n-1})$, the condition (3) is equivalent to

$$|\nabla f(x)| \leq (n - 2)Af^{1-\frac{1}{p+1}}(x),$$

and the condition (4) is equivalent to

$$f(x)\Delta f(x) - \frac{n-2}{n-3}|\nabla f(x)|^2 \geq -(n - 2)Af^{2-\frac{1}{p+1}}(x).$$

We also introduce the following Condition 2.

**Condition 2** $f \in C^2(S^{n-1})$ is nonnegative, $q < 2$ and there exist some constants $A$ such that

$$f \Delta f - \frac{3-q}{2-q} |\nabla f|^2 \geq -Af^{2-\frac{1}{p+2}} \text{ on } S^{n-1}. \quad (5)$$

When $q = 0$, Condition 2 was introduced by Guan-Li in [12]. Clearly, if $f$ satisfies Condition 2, then $f$ satisfies the condition (4).

Using the idea in [12], we can prove the following theorem.

**Theorem 2** Suppose $p > q > 0$, $f$ is a nonnegative, nonzero and even function on $S^{n-1}$. Then, there exists a generalized solution $h \in C^{1,1}(S^{n-1})$ satisfying the Eq. (2), provided $f \in C^{1,\alpha}(S^{n-1})$ ($0 < \alpha < 1$) satisfies either Condition 1 or Condition 2.

**Remark 1** In [12], Guan-Li showed that for $n = 3$, all nonnegative functions $f \in C^{1,1}(S^2)$ satisfied Condition 1 and all nonnegative function $f \in C^{3,1}(S^3)$ satisfied Condition 1 for $n = 4$. Thus, Theorem 1 is just a direct corollary of Theorem 2.

**Remark 2** The condition that $f$ is nonzero in Theorem 2 is necessary, otherwise we can see that $h \equiv +\infty$ by Lemma 1. When $f$ is a smooth positive function, the existence of smooth solutions to the $L_p$ dual Minkowski problem (2) is attained in [20] provided $p > q$, thus it is a natural question to ask whether we can drop the assumptions that $q > 0$ and $f$ is an even function in Theorem 2.

**Remark 3** It is worth pointing out that Chou-Wang [9, Theorem E] also treated the degenerate case of $L_p$ Minkowski problem for $1 < p < n$ and $f \geq c_0 > 0$. In their case, “degenerate” means that the generalized solution of $L_p$ Minkowski problem may contain zero points. Our results imply $C^{1,1}$ regularity for solutions to another degenerate $L_p$ Minkowski problem when $p > n$ and $f \geq 0$, where the “degenerate” means that the function $f$ may contain zero points.

The organization of the paper is as follows. $C^0$, $C^1$ and $C^2$ estimates are given in Sect. 2. In Sect. 3 we prove Theorem 2.

## 2 A priori estimates

In this section, we will establish $C^0$, $C^1$ and $C^2$ estimates for solutions to (2). The key is that those estimates must be independent of $\min_{S^{n-1}} f$. 
2.1 Basic properties of convex hypersurfaces

We first recall some basic properties of convex hypersurfaces in $\mathbb{R}^n$; see [44] for details. Let $M$ be a smooth, closed, uniformly convex hypersurface in $\mathbb{R}^n$ enclosing the origin. The support function $h$ of $M$ is defined as

$$h(x) := \max_{y \in M} \langle y, x \rangle, \quad \forall x \in S^{n-1},$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^n$.

The convex hypersurface $M$ can be recovered by its support function $h$. In fact, writing the Gauss map of $M$ as $\nu_M$, we parametrize $M$ by $X: S^{n-1} \to M$ which is given as

$$X(x) = \nu_M^{-1}(x), \quad \forall x \in S^{n-1}.$$ 

Note that $x$ is the unit outer normal vector of $M$ at $X(x)$. On the other hand, one can easily check that the maximum in the definition (6) is attained at $y = \nu_M^{-1}(x)$, namely

$$h(x) = \langle x, X(x) \rangle, \quad \forall x \in S^{n-1}. \tag{7}$$

Let $\nabla$ be the corresponding connection on $S^{n-1}$. Differentiating the both sides of (7), we have

$$\nabla_i h = \langle \nabla_i x, X(x) \rangle + \langle x, \nabla_i X(x) \rangle.$$ 

Since $\nabla_i X(x)$ is tangent to $M$ at $X(x)$, there is

$$\nabla_i h = \langle \nabla_i x, X(x) \rangle,$$

which together with (7) implies that

$$X(x) = \nabla h(x) + h(x)x, \quad \forall x \in S^{n-1}.$$ 

The radial function $\rho$ of the convex hypersurface $M$ is defined as

$$\rho(u) := \max \{ \lambda > 0 : \lambda u \in M \}, \quad \forall u \in S^{n-1}. \tag{8}$$

Note that $\rho(u)u \in M$. If we connect $u$ and $x$ through the following equality:

$$\rho(u)u = X(x) = \nabla h(x) + h(x)x. \tag{8}$$

By virtue of (8), there is

$$\rho^2 = |\nabla h|^2 + h^2,$$

which implies that

$$|\nabla h| \leq \rho. \tag{9}$$

By (7) and (8), we have

$$\max_{S^{n-1}} h = \max_{S^{n-1}} \rho. \tag{10}$$

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2.2 $C^0$ estimate and the gradient estimate

Now, we begin to prove $C^0$ estimate.

**Lemma 1** Assume $p > q$ and $f$ is a nonnegative and $C^2$ function. Let $h \in C^1(S^{n-1})$ be a solution to (2), then there exists a positive constant $C$ depending on $p$, $q$ and $\max_{S^{n-1}} f$ such that

$$\min_{S^{n-1}} h \geq C.$$  

**Proof** Assume $h$ attains its minimum at $x_0$, using the Eq. (2), it is straightforward to see that

$$h^{q-p}(x_0) \leq \max_{S^{n-1}} f.$$  

Note that $p > q$, thus

$$h(x_0) \geq \frac{1}{[\max_{S^{n-1}} f]^{p-q}}.$$  

So, we complete the proof. \square

**Lemma 2** Assume $p > q > 0$ and $f$ is a $C^2$, even and nonzero function. Let $h \in C^1(S^{n-1})$ be a solution to (2), then there exists a positive constant $C$ depending on $p$, $q$, $n$ and $\max_{S^{n-1}} f$ such that

$$\max_{S^{n-1}} h \leq C.$$  

Thus,

$$\max_{S^{n-1}} |\nabla h| \leq C,$$

where $C$ is a positive constant depending on $p$, $q$, $n$ and $\max_{S^{n-1}} f$.

**Proof** Write

$$h_{\max} = \max_{x \in S^{n-1}} h(x) = h(x_0)$$

for some $x_0 \in S^{n-1}$. Since $f$ is even, according to the uniqueness in Lemma 4.2 in [20], $h$ is also even. Thus, we have by the definition of support function that

$$h(x) \geq h_{\max}|\langle x, x_0 \rangle|, \quad \forall x \in S^{n-1}.$$  

Thus,

$$\int_{S^{n-1}} h^p(x)f(x)dx \geq h_{\max}^p \int_{S^{n-1}} |\langle x, x_0 \rangle|^p f(x)dx.$$  

(11)

Since $f$ is nonzero, $\max_{S^{n-1}} f > 0$. Assume $f$ attains its maximum at $y$, there exist a ball $B(y, \delta) \subset S^{n-1}$ such that

$$f(x) \geq \frac{1}{2} \max_{S^{n-1}} f, \quad \forall x \in B(y, \delta).$$  

(12)
Substituting (12) into (11), we have

\[
\int_{\mathbb{S}^{n-1}} h^p(x) f(x) dx \geq \frac{1}{2} h^{p \max}_{\mathbb{S}^{n-1}} f \int_{B(y, \delta)} |\langle x, x_0 \rangle|^p dx
\]

\[
\geq \frac{1}{2} h^{p \max}_{\mathbb{S}^{n-1}} \min_{z \in \mathbb{S}^{n-1}} \int_{B(y, \delta)} |\langle x, z \rangle|^p dx
\]

\[
\geq C h^{p \max}_{\mathbb{S}^{n-1}}.
\]  

(13)

Using the Eq. (2), it is straightforward to see that

\[
\int_{\mathbb{S}^{n-1}} h^p(x) f(x) dx = \int_{\mathbb{S}^{n-1}} \rho^q(u) du
\]

\[
\leq |\mathbb{S}^{n-1}|(\max_{\mathbb{S}^{n-1}} \rho)^q
\]

\[
= |\mathbb{S}^{n-1}| h^{q \max}_{\mathbb{S}^{n-1}}.
\]  

(14)

where we use (10) to get the last equality and $|\mathbb{S}^{n-1}|$ is the volume of $\mathbb{S}^{n-1}$. Thus, combining (13), (14) and the fact that $p > q$, we obtain

\[
h^{\max} \leq C.
\]

Thus, we get the upper bound of $h$. The gradient estimate follows from the upper bound of $h$, (9) and (10) consequently. \qed

2.3 $C^2$ estimate

First, we give some notations. For a $(0, 2)$ tensor field $b = \{b_{ij}\}$ on $\mathbb{S}^{n-1}$, the coordinate expression of its covariant derivative $\nabla b$ and the second covariant derivative $\nabla^2 b$ are denoted by

\[
\nabla b = (b_{ij;k}), \quad \nabla^2 b = (b_{ij;kl}).
\]

However, the coordinate expression of the covariant differentiation will be denoted by indices without semicolons, e.g.

\[
h_i, \quad h_{ij} \quad \text{or} \quad h_{ijk}
\]

for a function $h : \mathbb{S}^{n-1} \to \mathbb{R}$. To prove the $C^2$ estimate, we first recall a simple algebraic inequality.

**Lemma 3** For any $n - 1$ real number $a_1, a_2, \ldots, a_{n-1}$ satisfying

\[
\min\{a_1, a_2, \ldots, a_{n-1}\} \leq 0 \quad \text{and} \quad \max\{a_1, a_2, \ldots, a_{n-1}\} \geq 0,
\]

then we have the following inequality

\[
\sum_{i=1}^{n-1} a_i^2 \geq \frac{1}{n-2} \left( \sum_{i=1}^{n-1} a_i \right)^2.
\]  

(15)
Proof Without loss of generality, we assume
\[ a_1 \leq a_2 \leq \cdots \leq a_s \leq 0 \leq a_{s+1} \leq \cdots \leq a_{n-1}, \quad 2 \leq s \leq n-2, \]
and
\[ |a_1 + a_2 + \cdots + a_s| \geq a_{s+1} + \cdots + a_{n-1}. \]
Thus, using Cauchy-Schwartz inequality, we have
\[ \sum_{i=1}^{n-1} a_i^2 \geq \frac{1}{s} (\sum_{i=1}^{s} a_i)^2 \geq \frac{1}{n-2} (\sum_{i=1}^{n-1} a_i)^2. \]
So, we complete the proof.

Lemma 4 Let \( h \in C^4(S^{n-1}) \) be a solution to (2). Then there exists a positive constant \( C \) depending on \( n, \max_{S^{n-1}} h, \min_{S^{n-1}} h, \max_{S^{n-1}} |\nabla h|, A \) and \( \max_{S^{n-1}} f \) such that
\[ |h|_{C^2(S^{n-1})} \leq C \]
provided \( f \in C^2(S^{n-1}) \) satisfies either Condition 1 or Condition 2.

Proof Set \( b_{ij} = h_{ij} + h \delta_{ij} \) and \( b = \{b_{ij}\} \), let
\[ H(x) = \text{tr} b = (n-1)h(x) + \Delta h, \]
which is clearly nonnegative since the matrix \( b = \nabla^2 h + hI \) is nonnegative definite. Then, there exists a point \( x_0 \in S^{n-1} \) such that
\[ H(x_0) = \max_{x \in S^{n-1}} H(x). \]
If \( H(x_0) \leq 1 \), then our result holds. So, we assume \( H(x_0) \geq 1 \). By choosing a suitable orthonormal frame, we may assume \( \{h_{ij}(x_0)\} \) is diagonal. Then,
\[ 0 = \nabla_i H(x_0) = \sum_k b_{kk;i} \]
and
\[ \nabla_i \nabla_i H(x_0) = \sum_k b_{kk;ii} \leq 0. \]
Since \( \{b_{ij}\} \) is nonnegative definite, its inverse matrix \( \{b^{ij}\} \) is also nonnegative definite. Thus, we have at \( x_0 \)
\[ 0 \geq b^{ii} H_{ii} \]
\[ = b^{ii} \sum_k b_{kk;ii} \]
\[ = b^{ii} \left( \sum_k b_{ii;kk} - (n-1)b_{ii} + \sum_k b_{kk} \right) \]
\[ = \sum_k b^{ii} b_{ii;kk} - (n-1)^2 + \sum_i b^{ii} H, \]
where we use the Ricci identity
\[ b_{ii;kk} = b_{kk;ii} - b_{kk} + b_{ii}. \]
We rewrite the Eq. (2) as
\[
\det \frac{1}{n-2} (\nabla h + h I) = \varphi \frac{1}{n-2}, \tag{20}
\]
where
\[
\varphi = \frac{(|\nabla h|^2 + h^2)^{\frac{n-q}{2}}}{h^{1-p}} f.
\]
Differentiating (20) twice, we can obtain at \(x_0\)
\[
\sum_k b^{ij} b_{ij;kk} = b^{ik} b^{lj} \sum_p b_{ij;p} b_{kl;p} - \frac{1}{n-2} b^{ij} b^{kl} \sum_p b_{ij;p} b_{kl;p}
\]
\[
+ \varphi^{-2} \left[ \varphi \Delta \varphi - \frac{n-3}{n-2} |\nabla \varphi|^2 \right]
\]
\[
= b^{ij} b^{ij} \sum_p (b_{ij;p})^2 - \frac{1}{n-2} b^{ij} b^{kl} \sum_p b_{ij;p} b_{kl;p}
\]
\[
+ \varphi^{-2} \left[ \varphi \Delta \varphi - \frac{n-3}{n-2} |\nabla \varphi|^2 \right]
\]
\[
\geq \sum_i \sum_p (b^{ii} b_{ii;p})^2 - \frac{1}{n-2} \sum_p (\sum_i b^{ii} b_{ii;p})^2
\]
\[
+ \varphi^{-2} \left[ \varphi \Delta \varphi - \frac{n-3}{n-2} |\nabla \varphi|^2 \right]
\]
\[
\geq \varphi^{-2} \left[ \varphi \Delta \varphi - \frac{n-3}{n-2} |\nabla \varphi|^2 \right], \tag{21}
\]
where we use the inequality (15) to get the last inequality by noticing that \(\sum_i b_{ii;k}(x_0) = 0\).
Substituting (21) into (19), we arrive at \(x_0\)
\[
0 \geq \varphi^{-2} \left[ \varphi \Delta \varphi - \frac{n-3}{n-2} |\nabla \varphi|^2 \right] - (n-1)^2 + H \sum_i b^{ii}. \tag{22}
\]
Set \(\psi = \frac{(|\nabla h|^2 + h^2)^{\frac{n-q}{2}}}{h^{1-p}} f\), thus \(\varphi = \psi f\). It follows consequently
\[
\varphi^{-2} \left[ \varphi \Delta \varphi - \frac{n-3}{n-2} |\nabla \varphi|^2 \right]
\]
\[
= f^{-2} \left[ f \Delta f - \frac{n-3}{n-2} |\nabla f|^2 \right] + \frac{2}{n-2} f^{-1} \psi^{-1} \nabla f \nabla \psi
\]
\[
+ \psi^{-2} \left[ \psi \Delta \psi - \frac{n-3}{n-2} |\nabla \psi|^2 \right]. \tag{23}
\]
Differentiating $\psi$ twice, we have
\[
\nabla_i \psi = (n - q) \frac{(|\nabla h|^2 + h^2)^{\frac{n-q-2}{2}}}{h^{1-p}} \sum_k h_k h_k
\]
\[
+ (n - q) \frac{(|\nabla h|^2 + h^2)^{\frac{n-q-2}{2}}}{h^{1-p}} \nabla h_i + (p - 1) \frac{(|\nabla h|^2 + h^2)^{\frac{n-q}{2}}}{h^{2-p}} h_i
\]
\[
= (n - q) \frac{(|\nabla h|^2 + h^2)^{\frac{n-q-2}{2}}}{h^{1-p}} h_i h_i + C(h, h^{-1}, \nabla h),
\]
(24)

\[
|\nabla \psi|^2 = (n - q)^2 \frac{(|\nabla h|^2 + h^2)^{n-q-2}}{h^{2(1-p)}} \sum_i h_i^2 h_i^2
\]
\[
+ C(h, h^{-1}, \nabla h) * b,
\]
(25)

and
\[
\psi \Delta \psi = (n - q) \frac{(|\nabla h|^2 + h^2)^{n-q-2}}{h^{2(1-p)}} \left[ (|\nabla h|^2 + h^2) h_{ii}^2 + (n - q - 2) \sum_i h_i^2 h_i^2 \right]
\]
\[
+ C(h, h^{-1}, \nabla h) * b
\]
\[
\geq (n - q)(n - q - 1) \frac{(|\nabla h|^2 + h^2)^{n-q-2}}{h^{2(1-p)}} \sum_i h_i^2 h_i^2 - C(h, h^{-1}, \nabla h) H,
\]
(26)

where $C(h, h^{-1}, \nabla h)$ denotes some quantity depending on $h, h^{-1}, \nabla h$ and may change from line to line.

Combining (25) with (26), we can arrive
\[
\psi^{-2} \left[ \psi \Delta \psi - \frac{n - 3}{n - 2} |\nabla \psi|^2 \right] \geq \psi^{-2} \frac{2 - q}{(n - q)(n - 2)} |\nabla \psi|^2
\]
\[
- C(h, h^{-1}, \nabla h) H.
\]
(27)

Using Cauchy-Schwartz inequality, we obtain
\[
\frac{2}{n - 2} \int \psi^{-1} \nabla f \nabla \psi \leq \frac{n - q}{(n - 2)(2 - q)} \int |\nabla f|^2
\]
\[
+ \frac{2 - q}{(n - q)(n - 2)} \psi^{-2} |\nabla \psi|^2.
\]
(28)

Substituting (27) and (28) into (23), it yields
\[
\psi^{-2} \left[ \psi \Delta \psi - \frac{n - 3}{n - 2} |\nabla \psi|^2 \right]
\]
\[
\geq f^{-2} \left[ f \Delta f - \frac{3 - q}{2 - q} |\nabla f|^2 \right] - C(h, h^{-1}, \nabla h) H.
\]
(29)

Putting (29) into (22), we arrive at $x_0$ due to Condition 2
\[
0 \geq -A f^{-1} - (n - 1)^2 + H \sum_i b_{ii} - C(h, h^{-1}, \nabla h) H.
\]
(30)

Now we need to estimate $\sum_i b_{ii}$. Without loss of generality, we assume
\[
b_{11} \leq b_{22} \leq \cdots \leq b_{n-1, n-1}.
\]
It follows that $b_{n-1} \geq \frac{H}{n-1}$. Thus,
\[
\sum_i b^{ii} \geq \sum_{i=1}^{n-2} b^{ii} \geq (n-2) \left( \prod_{i=1}^{n-2} b^{ii} \right)^{\frac{1}{n-2}} \\
= (n-2) \left( \frac{b_{n-1} n-1}{\det b} \right)^{\frac{1}{n-2}} \\
\geq C(n) H^{\frac{1}{n-2}} (\det b)^{-\frac{1}{n-2}}. 
\] (31)

Plugging the above inequality into (30), we have
\[
(1-1)^2 f^{\frac{1}{n-2}} A + C(n) H^{1+\frac{1}{n-2}} \left[ \frac{f}{\det b} \right]^{\frac{1}{n-2}} - f^{\frac{1}{n-2}} C(h, h^{-1}, \nabla h) H \\
\geq C(n, h, h^{-1}, \nabla h) H^{1+\frac{1}{n-2}} - f^{\frac{1}{n-2}} C(h, h^{-1}, \nabla h) H. 
\]

Thus, we conclude from above
\[
H \leq C(n, h, h^{-1}, \nabla h). 
\]

This gives an upper bound of $\max_{\mathcal{B}^n} H$.

Next, we will prove an upper bound of $\max_{\mathcal{B}^n} H$ when $f$ satisfies Condition 1, it follows by (24) and (3)
\[
\frac{2}{n-2} f^{-1} \left( \frac{\psi}{n-2} \frac{\nabla \psi}{\nabla \psi} \right) \leq 2(n-q) A f^{-\frac{1}{n-2}} H \\
+ 2A f^{-\frac{1}{n-2}} C(h, h^{-1}, \nabla h). 
\] (32)

Substituting (32) and (4) into (23), we have
\[
\varphi^{-2} \left[ \varphi \Delta \varphi - \frac{n-3}{n-2} |\nabla \varphi|^2 \right] \\
\geq -\frac{A}{n-2} f^{-\frac{1}{n-2}} - 2(n-q) A f^{-\frac{1}{n-2}} C(h, h^{-1}, \nabla h) H \\
-2A f^{-\frac{1}{n-2}} C(h, h^{-1}, \nabla h) - C(h, h^{-1}, \nabla h) H. 
\]

Substituting the above inequality into (22) and using $H(x_0) \geq 1$, we obtain
\[
0 \geq -C(h, h^{-1}, \nabla h) A H - f^{\frac{1}{n-2}} C(h, h^{-1}, \nabla h) H - C(h, h^{-1}, \nabla h) \\
+ f^{\frac{1}{n-2}} \sum_i b^{ii} H, 
\]

which implies
\[
f^{\frac{1}{n-2}} \sum_i b^{ii} \leq C(h, h^{-1}, \nabla h) A + C(h, h^{-1}, \nabla h) (\max f) \frac{1}{n-2}. 
\]

Then, it follows from (31)
\[
H^{\frac{1}{n-2}} \leq C(h, h^{-1}, \nabla h) A + C(h, h^{-1}, \nabla h) (\max f) \frac{1}{n-2}. 
\]

We have thus complete the proof. \(\square\)
3 The proof of Theorem

We first recall Lemma 2.2 in [12].

Lemma 5 Let $f_1, f_2 \in C^{1,1}(S^{n-1})$ be two nonnegative functions satisfying, for some positive constants $a, b, A$, that

$$af_i \Delta f_i - b|\nabla f_i|^2 \geq -Af_i^{2 - \frac{1}{p-n}}, \quad \forall x \in S^{n-1}.$$ 

Then $f = f_1 + f_2$ satisfies

$$af \Delta f - b|\nabla f|^2 \geq -2Af^{2 - \frac{1}{p-n}}, \quad \forall x \in S^{n-1}.$$ 

Now we begin to prove Theorem 2.

Proof Set $f_\epsilon = f + \epsilon$ for positive small $\epsilon$. It follows from [20] that we can find an even function $h_\epsilon \in C^{4,\alpha}$ satisfying (2). Note that $f$ is even and nonzero, so is $f_\epsilon$. We know from Lemma 5 that $f_\epsilon$ satisfies either Condition 1 or Condition 2. Using Lemma 1, Lemma 2 and Lemma 4, we have $\{|h_\epsilon|_{C^{2}(S^{n-1})}\}$ is uniformly bounded by some universal constant independent of $\epsilon$. Let $h = \lim_{\epsilon \to 0} h_\epsilon$. Thus, $h$ is a generalized solution to (2) and $h \in C^{1,1}(S^{n-1})$. Therefore Theorem 2 is proved. \hfill \qed

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