Bounds for Hochschild cohomology of block algebras

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Abstract

We show that for any block algebra $B$ of a finite group over an algebraically closed field of prime characteristic $p$ the dimension of $HH^n(B)$ is bounded by a function depending only on the nonnegative integer $n$ and the defect of $B$. The proof uses in particular a theorem of Brauer and Feit which implies the result for $n = 0$.

Let $p$ be a prime and $k$ an algebraically closed field of characteristic $p$. Let $G$ be a finite group and $B$ a block algebra of $kG$; that is, $B$ is an indecomposable direct factor of $kG$ as a $k$-algebra. A defect group of $B$ is a minimal subgroup $P$ of $G$ such that $B$ is isomorphic to a direct summand of $B \otimes_k P$ as a $B$-$B$-bimodule. The defect groups of $B$ form a $G$-conjugacy class of $p$-subgroups of $G$, and the defect of $B$ is the integer $d(B)$ such that $p^{d(B)}$ is the order of the defect groups of $B$. The weak Donovan conjecture states that the Cartan invariants of $B$ are bounded by a function depending only on the defect $d(B)$ of $B$. As a consequence of a theorem of Brauer and Feit [3], the number of isomorphism classes of simple $B$-modules is bounded by a function depending only on $d(B)$. Thus the weak Donovan conjecture would imply that the dimension of a basic algebra of $B$ is bounded by a function depending on $d(B)$. This in turn would imply that the dimension of the term in any fixed degree $n$ of the Hochschild complex of a basic algebra of $B$ is bounded by a function depending on $n$ and $d(B)$. Since Hochschild cohomology is invariant under Morita equivalences, we would thus get that the dimension of $HH^n(B)$ is bounded by a function depending on $n$ and $d(B)$. The purpose of this note is to show that this consequence of the weak Donovan conjecture does indeed hold.

**Theorem 1.** There is a function $f : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ such that for any integer $n \geq 0$, any finite group $G$ and any block algebra $B$ of $kG$ with defect $d$ we have

$$\dim_k(HH^n(B)) \leq f(n, d)$$

For $n = 0$ this follows from the aforementioned theorem of Brauer and Feit [3], since $HH^0(B) \cong Z(B)$. Using Tate duality, the theorem above extends to Tate cohomology for negative $n$. A result of Kulshammer and Robinson [7, Theorem 1] implies that it suffices to show theorem 1 for finite groups with a non-trivial normal $p$-subgroup. We follow a slightly different strategy in the proof below, reducing the problem directly to finite groups with a non-trivial central $p$-subgroup.

**Remark 2.** We make no effort to construct a best possible bound; we define the function $f$ in theorem 1 inductively as follows: we set $f(0, 0) = 1$, $f(n, 0) = 0$ for $n > 0$; for all $d > 0$, $f(0, d)$ is the largest integer less or equal to the bound $\frac{1}{2}p^{2d} + 1$ given in the Brauer-Feit theorem (one
could, of course, also take \( f(0,1) = p \) and \( f(0,d) = p^{2d-2} \) for \( d \geq 2 \); cf. [6, Ch. VII, 10.14]), and for \( n > 0, d > 0 \) we set

\[
f(n,d) = p \cdot c(d) \cdot \sum_{i=0}^{n} f(i,d - 1)
\]

where \( c(d) \) is the maximum of the numbers of subgroups in any finite group of order \( p^d \).

Let \( G \) be a finite group and \( U \) a \( kG \)-module. We denote as usual by \( U^G \) the subspace of \( G \)-fixed points in \( U \). If \( H \) is a subgroup of \( G \) then \( U^G \subseteq U^H \), and there is a *trace map* \( \text{tr}^H_U : U^H \to U^G \) sending \( u \in U^H \) to \( \sum_{x \in [G/H]} xu \), where \([G/H] \) is a set of representatives of the \( H \)-cosets in \( G \); one checks that this map is independent of the choice of \([G/H] \) and that its image, denoted \( U^G_H \), is contained in \( U^G \). For \( Q \) a \( p \)-subgroup of \( G \), we denote the Brauer construction of \( U \) with respect to \( Q \) by \( U(Q) = U^Q / \sum_{R:R \leq Q} U^Q_R \) and by \( \text{Br}_Q^U : U^Q \to U(Q) \) the canonical surjection, called *Brauer homomorphism*. A block algebra \( B \) of \( kG \) can be viewed as an indecomposable \( k(G \times G) \)-module, with \((x,y) \in G \times G \) acting by left multiplication with \( x \) and right multiplication with \( y^{-1} \). For \( H \) a subgroup of \( G \), we denote by \( \Delta H \) the ‘diagonal’ subgroup \( \Delta H = \{(h,h) \mid h \in H\} \) in \( G \times G \). In particular, the action of \( \Delta G \) on \( B \) can be identified with the conjugation action of \( G \) on \( B \). The Brauer construction applied to \( B \) with respect to \( \Delta Q \) is canonically isomorphic to \( k\text{C}_{\Delta G}(Q)c \), where \( Q \) is a \( p \)-subgroup of \( G \) and \( c = \text{Br}_\Delta Q(1_B) \). A \( B \)-Brauer pair is a pair \((Q,c)\) consisting of a \( p \)-subgroup \( Q \) of \( G \) and of a block idempotent \( e \) of \( k\text{C}_{\Delta G}(Q) \) satisfying \( e\text{Br}_\Delta Q(1_B) = e \). The set of \( B \)-Brauer pairs is a \( G \)-poset in which the maximal pairs are all conjugate. The maximal \( B \)-Brauer pairs are exactly the \( B \)-Brauer pairs \((Q,e)\) for which \( Q \) is a defect group of \( B \). See [2] and [9, §11, §40] for details. In what follows we use without further comment the canonical graded isomorphism \( HH^*(B) \cong H^*(\Delta G;B) \); see [8, (3.2)]). The following result is certainly well-known but not always stated in exactly the form we need it; we therefore give a proof for the convenience of the reader.

**Proposition 3.** Let \( G \) be a finite group, \( B \) be a block algebra of \( kG \) and \( Q \) a \( p \)-subgroup of \( G \). Set \( b = 1_B \) and \( c = \text{Br}_Q(b) \). Suppose that \( c \neq 0 \) and set \( B_Q = k\text{C}_{\Delta G}(Q)cb \). Then we have a direct sum decomposition of \( k\text{N}_{G \times G}(\Delta Q) \)-modules

\[
\text{Res}_{k\text{N}_{G \times G}(\Delta Q)}^G(B) = B_Q \oplus C_Q
\]

such that multiplication by \( b \) is an isomorphism of \( k\text{N}_{G \times G}(\Delta Q) \)-modules \( k\text{C}_{\Delta G}(Q)c \cong B_Q \) and such that \( C_Q(\Delta Q) = \{0\} \).

The proof we present here uses the following well-known lemma, which is a special case of expressing relative projectivity in terms of the splitting of adjunction maps (the general theme behind this is developed in [4], [5], for instance).

**Lemma 4.** Let \( \alpha : B \to A \) be a homomorphism of \( k \)-algebras. Suppose that \( B \) is isomorphic to a direct summand of \( A \) as a \( B \)-bimodule. Then \( \alpha \) is injective and \( \text{Im}(\alpha) \) is a direct summand of \( A \) as a \( B \)-bimodule.

**Proof.** The left or right action of an element \( b \in B \) on \( A \) is given by left or right multiplication with \( \alpha(b) \). Let \( \iota : B \to A \) and \( \pi : A \to B \) be \( B \)-bimodule homomorphisms satisfying \( \pi \circ \iota = \text{Id}_B \). Then \( \iota(1_B) \) commutes with \( \text{Im}(\alpha) \), the map \( \beta \) sending \( a \in A \) to \( \alpha(\iota(1_B)) \) is an \( A \)-bimodule
endomorphism of $A$, and we have $\beta(\alpha(b)) = \alpha(b)\iota(1_A) = \iota(b)$, hence $\beta \circ \alpha = \iota$. Thus $\pi \circ \beta \circ \alpha = \text{Id}_B$, which shows that as a $B$-$B$-bimodule homomorphism, $\alpha$ is split injective with $\pi \circ \beta$ as a retraction.

Proof of Proposition 3. For any block of $kN_G(Q)$ which appears in $kN_G(Q)c$, the block $B$ of $kG$ is the corresponding ‘induced’ block. By [1, §14, Lemma 1], $kN_G(Q)c$ is isomorphic to a direct summand of $B$ as a $kN_G(Q)$-$kN_G(Q)$-bimodule, and thus of $cBc$, as a $kN_G(Q)c$-$kN_G(Q)c$-bimodule. By lemma 4, multiplication by $b$ induces an algebra homomorphism $kN_G(Q)c \to cBc$ which is split injective as a homomorphism of $kN_G(Q)c$-$kN_G(Q)c$-bimodules. Since $kC_G(Q)c$ is a direct summand of $kN_G(Q)c$ as an $N_G\times_G(\Delta Q)$-module we get that $kC_G(Q)c \cong B_Q$ and that $B_Q$ is a direct summand of $B$ as an $N_G\times_G(\Delta Q)$-module. Moreover, $B(\Delta Q) \cong B_Q$, and hence any complement $C_Q$ of $B_Q$ in $B$, as an $N_G\times_G(\Delta Q)$-module, satisfies $C_Q(\Delta Q) = \{0\}$.

We will make use of the following well-known fact on transfer in cohomology (we include a short proof for the convenience of the reader).

**Lemma 5.** Let $G$ be a finite group, $H$ a subgroup of $G$ and $V$ a $kH$-module. Let $U$ be a direct summand of $\text{Ind}^G_H(V)$. Then $H^*(G;U) = \text{tr}_{H}^G(H^*(H;\text{Res}^G_H(U)))$.

**Proof.** By Higman’s criterion there is a $kH$-endomorphism $\varphi$ of $U$ such that $\text{Id}_U = \text{tr}_{H}^G(\varphi)$. Let $n \geq 0$ and let $\zeta : \Omega^n(k) \to U$ be a $kG$-homomorphism, representing an element in $H^n(G;U)$. Then $\zeta = \text{Id}_U \circ \zeta = \text{tr}_{H}^G(\varphi \circ \zeta)$, whence the result.

This is applied in the following situation:

**Lemma 6.** Let $G$ be a finite group, $B$ a block algebra of $kG$ and $P$ a defect group of $B$. We have $H^*(\Delta G;B) = \text{tr}_{\Delta G}^G(\text{Res}_{\Delta G}^G(B))$.

**Proof.** As a $k(G \times G)$-module, $B$ has vertex $\Delta P$ and trivial source, thus is isomorphic to a direct summand of $\text{Ind}^{G \times G}_{\Delta P}(k)$. Mackey’s formula shows that $\text{Res}^{G \times G}_{\Delta P}(B)$ is still relatively $\Delta P$-projective, hence lemma 5 implies the result. Alternatively, this follows from the fact that $b = 1_B$ can be written as a relative trace of the form $b = \text{Tr}_{\Delta G}^G(y)$ for some $y \in B^{\Delta P}$.

**Proposition 7.** Let $G$ be a finite group and $B$ be a block algebra of $kG$. Set $b = 1_B$ and for every $B$-Brauer pair $(Q, e)$ set $B_{(Q,e)} = kC_G(Q)e b$. Then $B_{(Q,e)}$ is a direct summand of $B$ as a $k(C_G(Q) \times C_G(Q))\Delta Q$-module, isomorphic to $kC_G(Q)e$.

In particular, $H^*(\Delta Q; B_{(Q,e)})$ is a direct summand, as a graded vector space, of $H^*(\Delta Q; B)$, and we have

$$H^*(\Delta G;B) = \sum_{(Q,e)} \text{tr}_{\Delta G}^G(\text{Res}^G_{\Delta Q}(H^*(\Delta Q;B_{(Q,e)})))$$

where in the sum $(Q,e)$ runs over a set of representatives of the $G$-conjugacy classes of $B$-Brauer pairs.

**Proof.** The proof adapts techniques that have been used in the proof of a result of Watanabe [10, Lemma 1]. Clearly $H^*(\Delta G;B)$ contains the right side in the displayed equation. We need to show that $H^*(\Delta G;B)$ is contained in the right side. Since any summand of the right side of the
form $\text{tr}^G_{\Delta Q}(H^*(\Delta Q; B_{(Q,e)}))$ depends only on the $G$-conjugacy class of $(Q,e)$ it suffices to prove the inclusion

$$H^*(\Delta G; B) \subseteq \sum_Q \text{tr}^G_{\Delta Q}(H^*(\Delta Q; B_Q))$$

where $Q$ runs over the $p$-subgroups of $G$ for which $\text{Br}_Q(b) \neq 0$. Note that this makes sense since $B_Q$ is a direct summand of $B$ as a $k\Delta Q$-module, hence $H^*(\Delta Q; B_Q)$ is a subspace of $H^*(\Delta Q; B)$, to which we then apply the transfer map $\text{tr}^G_{\Delta Q}$. Since $H^*(\Delta G; B) = \text{tr}^G_{\Delta P}(H^*(\Delta P; B))$ by lemma 6 it suffices to show that the right side contains $\text{tr}^G_{\Delta R}(H^*(\Delta R; B))$ for any $p$-subgroup $R$ of $G$. This will be shown by induction. For $R = \{1\}$ this holds trivially because $B_{\{1\}} = B$ and $C_{\{1\}} = \{0\}$. For $R \neq \{1\}$ we have a direct sum decomposition $B = B_R \oplus C_R$ of $k\mathcal{N}_{G \times G}(\Delta R)$-modules as in proposition 3, and hence

$$H^*(\Delta R; B) = H^*(\Delta R; B_R) + H^*(\Delta R; C_R)$$

Since $C_R(\Delta R) = \{0\}$ we have

$$H^*(\Delta R; C_R) \subseteq \sum_{S:S < R} \text{tr}_{\Delta S}^R(H^*(\Delta S; B))$$

by lemma 5. Applying the transfer map $\text{tr}^G_{\Delta Q}$ yields

$$\text{tr}^G_{\Delta R}(H^*(\Delta R; C_R)) \subseteq \sum_{S:S < R} \text{tr}^G_{\Delta S}(H^*(\Delta S; B))$$

hence

$$\text{tr}^G_{\Delta R}(H^*(\Delta R; B)) \subseteq \text{tr}^G_{\Delta R}(H^*(\Delta R; B_R)) + \sum_{S:S < R} \text{tr}^G_{\Delta S}(H^*(\Delta S; B))$$

The result follows by induction.

\[\square\]

**Lemma 8.** Let $G$ be a finite group and $B$ be a block algebra of $kG$. Set $b = 1_B$ and for every $B$-Brauer pair $(Q,e)$ set $B_{(Q,e)} = kC_G(Q)e$. For any integer $n \geq 0$ we have

$$\dim_k(\text{tr}^G_{\Delta Q}((H^n(\Delta Q; B_{(Q,e)})))) \leq \dim_k(H^n(\Delta QC_G(Q); kQC_G(Q)e))$$

**Proof.** Clearly $\dim_k(\text{tr}^G_{\Delta Q}((H^n(\Delta Q; B_{(Q,e)})))) \leq \dim_k(\text{tr}^G_{\Delta Q}(H^n(\Delta QC_G(Q); B_{(Q,e)}))))$. Moreover, since $B_{(Q,e)} \cong kC_G(Q)e$ is isomorphic to a direct summand of $kQC_G(Q)e$, the lemma follows. \[\square\]

**Lemma 9.** Let $G$ be a finite group, $B$ a block of $kG$ and $Z$ a subgroup of order $p$ of $Z(G)$. Set $\bar{G} = G/Z$ and denote by $\bar{B}$ the image of $B$ in $k\bar{G}$ under the canonical algebra homomorphism $kG \rightarrow k\bar{G}$. For any integer $n \geq 0$ we have

$$\dim_k(H^n(\Delta \bar{G}; B)) \leq p \cdot \sum_{i=0}^n \dim_k(H^i(\Delta \bar{G}; B))$$
Proof. The Lyndon-Hochschild-Serre spectral sequence associated with $G$, $Z$, $\bar{G}$ and $B$ endowed with the conjugation action of $G$ reads

$$H^i(\Delta \bar{G}; H^3(\Delta Z; B)) \Rightarrow H^{i+j}(\Delta G; B)$$

Since $\Delta Z$ acts trivially on $kG$, hence on $B$, we have $H^i(\Delta Z; B) \cong H^i(\Delta Z; k) \otimes_k B \cong B$, where the last isomorphism uses that we have $H^i(\Delta Z; k) \cong k$ because $Z$ is cyclic. Thus $H^n(\Delta G; B)$ is filtered by subquotients of $H^i(\Delta \bar{G}; B)$, with $0 \leq i \leq n$; in particular, $\dim_k(H^n(\Delta G; B)) \leq \sum_{i=0}^n \dim_k(H^i(\Delta \bar{G}; B))$. Let $z$ be a generator of $Z$. As a $k\Delta \bar{G}$-module, $B$ has a filtration of the form

$$B \supseteq B(1-z) \supseteq B(1-z)^2 \supseteq \cdots \supseteq B(1-z)^{p-1} \supseteq \{0\}$$

and since $B$ is projective as a right $kZ$-module, the quotient of any two consecutive terms in this filtration is isomorphic to $B$. Thus the appropriate long exact sequences in cohomology imply that $\dim_k(H^n(\Delta \bar{G}; B)) \leq p \cdot \dim_k(H^i(\Delta \bar{G}; B))$, whence the result.

Proof of Theorem 1. Let $f$ be the function defined in remark 2. Note that $f(n, d) \geq f(n, d-1)$ for all $n \geq 0$ and all $d > 0$. Denote by $c(d)$ the maximum of the numbers of subgroups in finite groups of order $p^d$. As mentioned before, theorem 1 holds for $n = 0$. Clearly theorem 1 holds for $d = 0$ because a defect zero block is a matrix algebra. Let $n$ and $d$ be a positive integers. Then $tr_{\Delta \bar{G}}^G(H^n(1; B)) = \{0\}$. Thus, by proposition 7 and lemma 8 we have $\dim_k(H^n(B)) \leq \sum_{(Q, e)} \dim_k(H^n(kQC_G(Q)e))$ where in the sum $(Q, e)$ runs over a set of representatives of the $G$-conjugacy classes of non-trivial $B$-Brauer pairs. Any such pair $(Q, e)$ has a conjugate with $Q$ contained in a fixed defect group $P$, and hence the number of summands in this sum is at most $c(d)$. Moreover, $Z(QC_G(Q))$ contains $Z(Q)$, and hence $QC_G(Q)$ has a non-trivial central subgroup $Z_Q$ of order $p$. After replacing $(Q, e)$ by a suitable $G$-conjugate, we may assume that $QC_P(Q)$ is a defect group of $e$ viewed as a block of $kQC_G(Q)$; in particular the defect groups of $e$ have order at most $|P| = p^d$. Thus the defect groups of the image $\bar{e}$ of $e$ in $kQC_G(Q)/Z_Q$ have order at most $|P|/p = p^{d-1}$, hence $\dim_k(H^n(kQC_G(Q)/Z_Q\bar{e})) \leq f(n, d-1)$. It follows from lemma 9 that $\dim_k(H^n(kQC_G(Q)e)) \leq p \cdot \sum_{i=0}^n f(i, d-1)$. Together with the above remarks we get the inequality $\dim_k(H^n(B)) \leq p \cdot c(d) \cdot \sum_{i=0}^n f(i, d-1) = f(n, d)$, as required.

Remark 10. The strong version of Donovan’s conjecture states that for a fixed integer $d \geq 0$ there should be only finitely many Morita equivalence classes of blocks with defect at most $d$. If true, this would imply that there are only finitely many isomorphism classes of Hochschild cohomology algebras of blocks with defect at most $d$; this remains an open problem.

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