Left-Right Asymmetric Holographic RG Flow with Gravitational Chern-Simons Term

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Abstract

We consider the holographic renormalization group (RG) flow in three dimensional gravity with the gravitational Chern-Simons term coupled to some scalar fields. We apply the canonical approach to this higher derivative case and employ the Hamilton-Jacobi formalism to analyze the flow equations of two dimensional field theory. Especially we obtain flow equations of Weyl and gravitational anomalies, and derive \( c \)-functions for left and right moving modes. Both of them are monotonically non-increasing along the flow, and the difference between them is determined by the coefficient of the gravitational Chern-Simons term. This is completely consistent with the Zamolodchikov’s \( c \)-theorem for parity-violating two-dimensional quantum field theories.

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1 Introduction

The three dimensional Einstein gravity has been an intriguing theoretical laboratory to study classical as well as quantum gravity. When the negative cosmological constant is added, the theory exhibits even more interesting properties. It has the three dimensional anti-de Sitter space (AdS$_3$) as a vacuum [1], and a black hole solution can be constructed out of the AdS$_3$ geometry by an appropriate identification [2]. Brown and Henneaux [3] succeeded to uncover the conformal symmetry on the boundary of the AdS$_3$ space and derived the left and right Virasoro algebras which share the same central charge. The Cardy’s asymptotic formula of the state counting with the help of the central charge reproduces the correct thermodynamical entropy formula [4].

In [5], we examined the canonical formalism of gravity when the gravitational Chern-Simons term is added to the Einstein-Hilbert action with the negative cosmological constant. Such a theory is often referred to as topologically massive gravity (TMG) [6] and received much attention in recent years [7] - [13]. We investigated the conformal symmetry living on the boundary of the AdS$_3$ space and showed, by the canonical method, that the left and right Virasoro central charges are shifted by an amount equal in magnitude but of opposite sign (see also [14] - [18]).

In our previous paper [19], we studied the three dimensional Einstein gravity without the Chern-Simons term, while a scalar field was added, paying attention to the gauge/gravity correspondence [20] - [22]. Given a certain type of scalar potential, we presented a black hole solution, whose metric exhibits a peculiar property that the spacetime geometry is AdS$_3$ both at spatial infinity and at the horizon. The Virasoro algebras and their central charges were obtained by applying the Brown-Henneaux’s method both at the spatial infinity and at the horizon [1]. Regarding the radial direction in the bulk as the renormalization scale of the dual boundary theory, we discussed renormalization group (RG) flow [26] - [34] and have defined the $c$-function [35] that is monotonically non-increasing toward the infrared direction.

In the present paper we discuss in further detail the RG flow, by including the gravitational Chern-Simons term into the three dimensional Einstein gravity coupled to scalar fields. In the presence of the gravitational Chern-Simons term, the action is not strictly invariant under the diffeomorphism, but receives variations from the boundary term. According to the AdS/CFT correspondence, the non-invariance under the diffeomorphism in the bulk manifests itself in the non-conservation of the boundary energy-momentum tensor, i.e., the gravitational anomaly [14]. Such an anomaly effect was closely connected to the asymmetric central charges of the left and right moving Virasoro algebras.

We apply the canonical formalism adapted for the case of higher derivative theories [36, 37, 38, 5] and derive the Hamilton-Jacobi equation. Thereby we discuss the RG flow of the boundary field theory regarding the radial coordinate as the energy scale. By looking at the Weyl anomaly of energy momentum tensor of the boundary theory, we read off the sum of the left and right $c$-functions. We notice that the energy momentum tensor itself is not defined in a covariant way. However, the Bardeen-Zumino polynomial naturally arises in the Hamilton-Jacobi equation, and makes a modified energy momentum tensor transforming covariantly.

We will go one step further to analyze the momentum constraint to obtain the gravitational anomaly relation on the boundary, and identify the difference of the two $c$-functions.

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6At the horizon we employed covariant formulation of charges [23] [24], which is paid much attention recently to study the Kerr/CFT correspondence [25].
from its coefficient. The left and right $c$-functions thus obtained are monotonically non-increasing toward the infrared region and settle down to the central charges of the boundary CFT. As it turns out, the difference between the left and right $c$-functions is independent of the renormalization scale. This is shown to be consistent with the derivation of the left-right asymmetric $c$-functions for parity-violating two-dimensional quantum field theories $[39]$.  

The organization of this paper is as follows. In section 2, we briefly review $c$-functions of the parity-violating two dimensional quantum field theory. In section 3, we analyze holographic RG flow of the three dimensional gravity with gravitational Chern-Simons term, and derive the left-right asymmetric $c$-functions. Section 4 is devoted to conclusion and discussion. An explicit form of the solution in the gravity theory is given in appendix A.

2 The left-right asymmetric $c$-functions and anomalies

In this section, we review the $c$-theorem $[35]$ for parity-violating two-dimensional quantum field theory and define $c$-functions for left- and right-movers, $c_L(t)$ and $c_R(t)$, which monotonically decrease along the renormalization group flow $[39]$. We emphasize the fact that the difference of the two functions is constant along the RG flow, namely

$$t \frac{d}{dt}(c_L(t) - c_R(t)) = 0, \quad (1)$$

where $t$ is the scaling parameter to be defined below. At the fixed point, if we consider the curved two-dimensional background, the difference of two central charges can be read off from the gravitational anomaly, while the sum of two central charges can be related to the Weyl anomaly. We will briefly discuss these points later in this section. For now, however, we consider flat two-dimensional spacetime to keep close contact with the original proof of Zamolodchikov’s $c$-theorem.

By using complex coordinates $z$ and $\bar{z}$, the conservation law and the symmetrical property of the energy momentum tensor can be written as

$$\partial T + \partial \Theta = 0, \quad \bar{\partial} T + \partial \Theta = 0, \quad (2)$$

where we define $\partial = \partial/\partial z$, $\bar{\partial} = \partial/\partial \bar{z}$, $T = T_{zz}$, $\bar{T} = T_{\bar{z}\bar{z}}$ and $\Theta = T_{z\bar{z}} = T_{\bar{z}z}$. We use these equations to study the scaling properties of two point functions of the energy-momentum tensor. For example, note that by using the right equation we can exchange $\bar{\partial} T$ for $-\partial \Theta$. Then we find

$$\bar{\partial} \langle T(z, \bar{z})T(0, 0) \rangle = -\partial \langle \Theta(z, \bar{z})T(0, 0) \rangle, \quad \bar{\partial} \langle T(z, \bar{z})\Theta(0, 0) \rangle = -\partial \langle \Theta(z, \bar{z})\Theta(0, 0) \rangle. \quad (3)$$

We note that we can generally write these correlators as

$$\langle T(z, \bar{z})T(0, 0) \rangle = \frac{F(z\bar{z})}{z^4}, \quad \langle T(z, \bar{z})\Theta(0, 0) \rangle = \frac{G(z\bar{z})}{z^3\bar{z}}, \quad \langle \Theta(z, \bar{z})\Theta(0, 0) \rangle = \frac{H(z\bar{z})}{z^2\bar{z}^2}. \quad (4)$$

Substituting these into $[35]$, we find $t \frac{d}{dt} F(t) = 3G(t) - t \frac{d}{dt} G(t)$ and $t \frac{d}{dt} G(t) = G(t) + 2H(t) - t \frac{d}{dt} H(t)$, where we define a Lorentz invariant scale parameter $t = z\bar{z}$. Combining these two equations, we can prove a function defined as

$$c_L(t) = 2F(t) - 4G(t) - 6H(t) \quad (5)$$
is monotonically non-increasing along the RG flow, i.e.,

$$t \frac{d}{dt} c_L(t) = -12H(t) \leq 0. \quad (6)$$

Note that this $c$-function has its extremum at the fixed points because the trace of the stress tensor $\Theta = T_{z \bar{z}}$ vanishes when the theory has conformal invariance. The value of the $c$-function at the fixed points is equal to the central charge of the left-moving Virasoro algebra.

In the similar way, by using (2) we can show that another function

$$c_R(t) = 2\bar{F}(t) - 4\bar{G}(t) - 6H(t) \quad (7)$$

is also monotonically non-increasing,

$$t \frac{d}{dt} c_R(t) = -12H(t) \leq 0. \quad (8)$$

Here $\bar{F}(t)$ and $\bar{G}(t)$ are the right-moving counterparts of $F(t)$ and $G(t)$, respectively. From (6) and (8), we can confirm (1). In a parity symmetric theory, $c_L(t)$ is equal to $c_R(t)$ since in terms of the complex coordinate the parity symmetry means the symmetry that exchanges $z$ and $\bar{z}$. In a parity violating case, however, the two $c$-functions, $c_L(t)$ and $c_R(t)$, differ from each other in general, but (1) claims that the difference is a constant.

We now briefly discuss the relation of the $c$-functions to the Weyl and gravitational anomalies. In order to analyse these, here we consider the two dimensional field theory coupled to some curved background.

It is well-known that, at the fixed point of the RG flow, the Weyl anomaly is expressed as

$$\langle T^{ii} \rangle = \frac{1}{24\pi} \frac{c_L + c_R}{2} R, \quad (9)$$

in terms of the sum of the two central charges. It is related to the number of dynamical degrees of freedom of fluctuating fields. Away from the fixed point, the Weyl anomaly (9) receives corrections proportional to beta functions. But nevertheless, the coefficient of the scalar curvature must be still related to the effective degrees of freedom. Hence, it is natural to regard the coefficient as the sum of the two $c$-functions, $c_L(t) + c_R(t)$, along the RG flow.

Now let us discuss the gravitational anomaly. At the fixed point, it is expressed as a violation of momentum conservation

$$\nabla_i \langle T^{ij} \rangle = -\frac{c_L - c_R}{96\pi} c^{jk} \partial_k R. \quad (10)$$

The gravitational anomaly occurs due to the lack of a regularization which preserves general covariance, and the general covariance of the quantum action is broken by parity-violating one-loop diagrams [40]. This is proportional to the difference between two central charges, and thus we obtained the above expression. Away from the fixed point, in contrast to the Weyl anomaly, Eq. (10) does not receive any corrections. For the same reason as before, it is natural to identify the coefficient of the r.h.s. of (10) with the difference of the two $c$-functions, $c_L(t) - c_R(t)$, along the RG flow.

We finally note that, in the presence of the gravitational anomaly, the vacuum expectation value of general energy-momentum tensors is not covariant. The energy-momentum tensor in Eqs. (9) and (10) is therefore understood as modified so as to be covariant by adding the so-called Bardeen-Zumino polynomial [41] (for useful review, see [42]).

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7 From here, we move from Euclidean to Minkowski signature.
3 Holography with gravitational Chern-Simons term

In this section, we consider holographic RG flow of the three dimensional gravity which includes the gravitational Chern-Simons term. We define two different $c$-functions, one for the left-mover and another for the right-mover, the difference of which comes from the existence of the gravitational anomaly.

3.1 ADM decomposition and canonical formalism

We consider the three dimensional gravity with gravitational Chern-Simons term coupled to some scalar fields $\phi^I$. The action is given by

$$\frac{1}{16\pi G_N} \int d^3 x L = \frac{1}{16\pi G_N} \int d^3 x \sqrt{-g} \left\{ \hat{R} - \frac{1}{2} G(\phi) \partial_\mu \phi^I \partial^{\mu} \phi^I - V(\phi) + \frac{\beta}{2} \varepsilon^{\mu \nu \rho} \left( \hat{\Gamma}^\alpha_{\mu \beta} \partial_\nu \hat{\Gamma}_\beta^{\rho \alpha} + \frac{2}{3} \hat{\Gamma}^\alpha_{\mu \beta} \hat{\Gamma}^\beta_{\nu \gamma} \hat{\Gamma}^{\rho \alpha}_{\gamma} \right) \right\},$$

where $G_N$ is the three dimensional Newton constant and the coefficient $\beta$ is some constant. The hat is used for the quantities constructed out of the metric $\hat{g}_{\mu \nu}$, and Greek indices run over the three dimensional coordinates $(r, x^0, x^1)$. Let us parametrize the metric as

$$ds^2 = \hat{g}_{\mu \nu} dx^\mu dx^\nu = N^2 dr^2 + g_{ij} \left( dx^i + N^i dr \right) \left( dx^j + N^j dr \right),$$

where $g_{ij}$ denotes the two dimensional boundary metric and Latin indices run over the two dimensional coordinates $(x^0, x^1)$. Then we perform the Euclidian ADM-decomposition of the Lagrangian as follows:

$$\mathcal{L} = \sqrt{-g} N \left( R - V(\phi) - K_{ij} K^{ij} + K^2 \right) - \frac{\sqrt{-g}}{2N} G_{IJ} \left( \dot{\phi}^J - N^i \partial_i \phi^J \right) \left( \dot{\phi}^I - N^i \partial_i \phi^I \right) - \frac{\sqrt{-g} N}{2} G_{IJ} \partial_i \phi^I \partial^i \phi^J$$

$$+ \beta \sqrt{-g} e^{mn} K_{mk} K_n^k + \beta \sqrt{-g} N \left( 2 \epsilon^{mn} \nabla_n K_m^k - A_{mn} K_{mn} \right) + \beta \sqrt{-g} N^i \left[ 2 \epsilon^{mn} K_i^k \nabla_n K_{mk} + \epsilon^{mn} \nabla_k \left( K_{ni} K_m^k \right) + \frac{1}{2} \epsilon_{ij} \partial^j R + \nabla_k A_i^k \right],$$

where total derivatives are omitted. Here the boundary metric $g_{ij}$ is used in order to raise or lower Latin indices. We use a dot as $\partial / \partial r$ and $\epsilon^{ij}$ denotes the covariantly constant anti-symmetric tensor. In terms of $N, N^i$ and $g_{ij}$, the extrinsic curvature can be written as

$$K_{ij} = \frac{1}{2N} \left( \dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i \right),$$

and $K$ is the trace part of $K_{ij}$. $A_{ij}$ is explicitly written as

$$A^{ij} = \epsilon^{mp} g^{l ij k} T_{m n o}^{l p} \nabla_k \Gamma_{mp}^{o}, \quad T_{mn o}^{l p} := \frac{1}{2} \left( \delta_k^m \delta_o^l \delta_n^i \delta_{i j}^j + \delta_k^m \delta_n^l \delta_o^i \delta_{i j}^j - \delta_k^n \delta_l^m \delta_o^i \delta_{i j}^j \right),$$

and does not behave as a tensor under diffeomorphisms on the boundary because it depends on the affine connection in an explicit way.

In order to construct the Hamilton-Jacobi equation of the system, we consider the canonical formalism with the gravitational Chern-Simons term [5]. Since $K_{ij}$ has an $r$-derivative
term as in (14), the action (13) contains third derivatives with respect to $r$. It is known that the canonical formalism of such a system is formulated by using (modified) Ostrogradsky method [36, 37, 38] in which a Lagrange multiplier is introduced and the extrinsic curvature $K_{ij}$ is treated as an independent variable. A close look at (14) shows that $L$ contains third derivatives of only the traceless part of $K_{ij}$ and there is no $K$ term. We therefore divide $K_{ij}$ into the trace part $K$ and traceless part $H_{ij} = K_{ij} - g_{ij}K/2$. The Lagrangian (13) can be rewritten as

$$L = \sqrt{-g}N \left( R - V(\phi) - H_{ij}H^{ij} + \frac{1}{2}K^2 \right)$$

$$- \frac{\sqrt{-g}}{2N} G_{IJ} \left( \dot{\phi}^I - N^i \partial_i \phi^I \right) \left( \dot{\phi}^J - N^i \partial_i \phi^J \right) - \frac{\sqrt{-g}}{2} G_{IJ} \partial_i \phi^I \partial^i \phi^J$$

$$+ \beta \sqrt{-g} \dot{H}_{mk} \epsilon^{n(m} H^{k)}_n + \frac{1}{2} \beta \sqrt{-g} K \dot{g}_{mk} \epsilon^{n(m} H^{k)}_n + \beta \sqrt{-g} N \left( 2 \epsilon_{mn} \nabla_k \nabla_n H^k_m - A^{mn} K_{mn} \right)$$

$$+ \sqrt{-g} N I \left\{ 2 \epsilon_{mn} K_i^k \nabla_n K_{mk} + \epsilon_{mn} \nabla_k \left( K_m K^k_m \right) + \frac{1}{2} \epsilon_{ij} \dot{\phi}^i R + \nabla_k A^k_i \right\}$$

$$+ v^{ij} (\dot{g}_{ij} - 2 NK_{ij} - 2 \nabla_i N_j), \quad (16)$$

Here $v^{ij}$ is a Lagrange multiplier. We treat $g_{ij}, H_{ij}$ and $K$ as independent variables and $K_{ij}$ in (16) is understood as $K_{ij} = H_{ij} + g_{ij}K/2$. Note that $v^{ij}$ is symmetric but not a tensor.

From this expression, we read off the momenta conjugate to $g_{ij}, H_{ij}$ and $\phi^I$ as

$$\pi^{ij} := \frac{\delta L}{\delta \dot{g}_{ij}} = v^{ij} + \frac{K}{2} \beta \sqrt{-g} \epsilon^{k(i} H^{j)}_k,$$  

$$\Pi^{ij} := \frac{\delta L}{\delta \dot{H}_{ij}} = \beta \sqrt{-g} \epsilon^{k(i} H^{j)}_k,$$  

$$\pi^I := \frac{\delta L}{\delta \dot{\phi}^I} = - \sqrt{-g} N G_{IJ} \left( \dot{\phi}^J - N^i \partial^i \phi^J \right), \quad (19)$$

respectively. Now we find that $\Pi^{ij}$ is not independent of $H_{ij}$ and the system is constrained. Again, such a kind of the constraint should be taken into account by introducing a Lagrange multiplier $f_{ij}$, when we treat $H_{ij}$ and $\Pi^{ij}$ as independent variables. Hence, we add the following term

$$f_{ij} \left( \beta \sqrt{-g} \epsilon^{k(i} H^{j)}_k - \Pi^{ij} \right). \quad (20)$$

Then the total action can be written as

$$S \left[ g_{ij}, H_{ij}, K, \phi^I, \pi^{ij}, \Pi^{ij}, \pi^I, N, N^i, f_{ij}; r_0 \right]$$

$$= \int d^2x \int_r^{r_0} dr \left[ \pi^{ij} \dot{g}_{ij} + \Pi^{ij} \dot{H}_{ij} + \pi^I \dot{\phi}^I - \left( N \mathcal{H} + N^i \mathcal{P}^i \right) + f_{ij} \left( \beta \sqrt{-g} \epsilon^{k(i} H^{j)}_k - \Pi^{ij} \right) \right], \quad (21)$$

where $r$-integration has been cut off at $r = r_0$. Here we introduced Hamiltonian and mo-
\[ \frac{1}{\sqrt{-g}} \mathcal{H} := -R + V(\phi) + H^{kl} H_{kl} - \frac{1}{2} K^2 - \frac{1}{2} \epsilon^{mn} \nabla_k \nabla_m H^{k} + \left( \frac{2}{\sqrt{-g}} \epsilon^{kl} + \beta A^{kl} \right) \left( H_{kl} + \frac{1}{2} g_{kl} K \right), \tag{22} \]

\[ \frac{1}{\sqrt{-g}} \mathcal{P}^i := -2 \beta \epsilon^{mn} K_{ik} \nabla_m K_{nk} - \beta \epsilon^{mn} \nabla_k \left( K_{ik} K^k_{i} \right) + \beta \nabla_j \left( K \epsilon^{k(i} H^{j)} \right) - \frac{1}{2} \beta \epsilon^{ij} \partial_j R - \nabla_j \left( \frac{2}{\sqrt{-g}} \pi_{ij} + \beta A^{ij} \right) + \frac{1}{\sqrt{-g}} \pi_{ij} \partial^j \phi^I. \tag{23} \]

We find \( N, N_i, f_{ij} \) and \( K \) are Lagrange multipliers in (21). Path integrations over them lead to the following constraints:

\[ \mathcal{H} = 0, \quad \mathcal{P}^i = 0, \tag{24} \]
\[ \Pi^{ij} = \beta \sqrt{-g} \epsilon^{k(i} H^{j)}_k, \tag{25} \]
\[ K = \left( \frac{1}{\sqrt{-g}} \pi^{ij} + \frac{\beta}{2} A^{ij} \right) g_{ij} \tag{26} \]

### 3.2 Hamilton-Jacobi equation

Now we consider the Hamilton-Jacobi equation in the bulk. We have six dynamical fields, \( g_{ij}, H_{ij}, \phi^I, \pi^{ij}, \Pi^{ij} \) and \( \pi_I \), together with four auxiliary fields, \( N, N_i, K \) and \( f_{ij} \). In order to obtain a classical solution, \( \tilde{g}_{ij}(r, x), \) e.t.c., we have to give the boundary conditions for dynamical fields which are consistent with the constraints (24), (25) and (26). Since we are interested in the variations of fields along the radial direction, we give the following boundary conditions:

\[ \tilde{g}_{ij}(r_0, x) = g_{ij}(x), \quad \tilde{H}_{ij}(r_0, x) = H_{ij}(x), \quad \tilde{\phi}^I(r_0, x) = \phi^I(x), \tag{27} \]

where \( r = r_0 \) denotes the two-dimensional boundary surface.

We denote as \( \tilde{S} \) the action in which the classical solution is substituted. Clearly this is a functional of boundary conditions \( g_{ij}(x), K_{ij}(x) \) and \( \phi^I(x) \), and should be written as \( \tilde{S}[g_{ij}, H_{ij}, \phi^I; r_0] \). Since the classical solution satisfies the constraints (24), the classical action \( \tilde{S} \) can be written explicitly as

\[ \tilde{S}[g_{ij}, H_{ij}, \phi^I; r_0] = \int d^2 x \int_{r_0} \left( \pi^{ij} \dot{g}_{ij} + \Pi^{ij} \dot{H}_{ij} + \pi^I \dot{\phi}^I \right). \tag{28} \]

Then the variation of the action can be written as

\[ \delta \tilde{S}[g_{ij}, H_{ij}, \phi^I; r_0] = \frac{\partial \tilde{S}}{\partial r_0} \delta r_0 + \int d^2 x \left[ \pi^{ij} \delta g_{ij} + \Pi^{ij} \delta H_{ij} + \pi^I \delta \phi^I \right], \tag{29} \]

where only surface terms contribute to \( \delta \tilde{S} \) due to the classical equations of motion. Here \( \pi_{ij} \) denotes \( \pi_{ij}(r_0, x) \) and similarly for \( \Pi_{ij} \) and \( \pi_I \). We find from this expression that

\[ \frac{\delta \tilde{S}}{\delta g_{ij}} = \pi^{ij}, \quad \frac{\delta \tilde{S}}{\delta H_{ij}} = \Pi^{ij}, \quad \frac{\delta \tilde{S}}{\delta \phi^I} = \pi_I, \quad \frac{\partial \tilde{S}}{\partial r_0} = 0. \tag{30} \]

\(^8\)To deal with second-order differential equations, we of course need to give one more boundary condition for each field, which is assumed to be fixed so that the classical solution is regular inside the bulk \[23, 21, 22\] (see also Ref. \[13\]).
The last equation is obtained by the total differentiation of Eq. (28) with respect to \( r_0 \), and shows that \( \bar{S} \) is independent of the position of the boundary surface. The classical action is therefore determined by the boundary conditions \( g_{ij}, H_{ij} \) and \( \phi^I \), which are constrained by the Hamiltonian one, \( H = 0 \), the momentum one, \( P^i = 0 \), and the others, Eqs. (25) and (26).

### 3.3 Holographic RG equation

Let us now discuss the physical meaning of the Hamiltonian constraint \( H = 0 \). By using the relations of (30), we can rewrite it as

\[
\frac{1}{2} G^{IJ} \left( \frac{1}{\sqrt{-g}} \frac{\delta \bar{S}}{\delta \phi^I} \right) \left( \frac{1}{\sqrt{-g}} \frac{\delta \bar{S}}{\delta \phi^J} \right) - \left( \frac{2}{\sqrt{-g}} \frac{\delta \bar{S}}{\delta g_{kl}} + \beta A^{kl} \right) \left( H_{kl} + \frac{1}{2} g_{kl} K \right) = -R + V(\phi) + \frac{1}{2} g^{ij} G_{IJ} \partial_i \phi^I \partial_j \phi^J.
\]

This is the extension of the flow equation of de Boer, Verlinde and Verlinde \cite{26} by the inclusion of the gravitational Chern-Simons term. From this flow equation, we can determine the classical action \( \bar{S} \) as a functional of boundary variables, \( g_{ij}, H_{ij} \) and \( \phi^I \). To see this, let us assign the weight \( w \) as

\[
w = 0 : g_{ij}, H_{ij}, \phi^I, \Gamma[g, H, \phi^I],
\]

\[
w = 1 : \partial_i,
\]

\[
w = 2 : R, \partial_i \phi^I \partial_j \phi^J, \frac{\delta \Gamma}{\delta g_{ij}}, \frac{\delta \Gamma}{\delta H_{ij}}, \frac{\delta \Gamma}{\delta \phi^I}, A^{ij},
\]

and expand \( \bar{S} \) as

\[
\bar{S} = 16 \pi G_N \Gamma + \sum_{w=0}^{\infty} S^{(w)}_{\text{loc}},
\]

where \( S^{(w)}_{\text{loc}} \) contains local terms with weight \( w \) and only \( \Gamma \) has non-local terms. In the context of the holographic RG, the non-local part \( \Gamma[g, H, \phi] \) can be regarded as the generating functional with respect to the source fields.

Hereafter through the end of this subsection, we assume \( H_{ij} = 0 \) for simplicity. Let us recall in this connection that ordinary solutions such as global AdS and BTZ black hole can satisfy this assumption by performing suitable coordinate transformations, although it is an interesting task to relax this assumption. Under the assumption, we find that the Hamiltonian constraint reduces to

\[
-\frac{1}{2} G^{IJ} \frac{\delta \bar{S}}{\delta \phi^I} \frac{\delta \bar{S}}{\delta \phi^J} - \frac{1}{2} \left( \frac{1}{\sqrt{-g}} \frac{\delta \bar{S}}{\delta g_{kl}} + \frac{\beta}{2} A^{kl} \right) g_{kl} \right)^2 = -R + V(\phi) + \frac{1}{2} g^{ij} G_{IJ} \partial_i \phi^I \partial_j \phi^J.
\]

Here use has been made of (26). We now solve the flow equation (34) order by order with respect to weight \( w \). For \( w = 0 \), defining “superpotential” \( W(\phi) \) by \( S^{(0)}_{\text{loc}} = \int d^2 x \sqrt{-g} W(\phi) \), we find

\[
V = \frac{1}{2} G^{IJ} \frac{\partial W}{\partial \phi^I} \frac{\partial W}{\partial \phi^J} - \frac{1}{2} W^2.
\]

\footnote{Inclusion of higher derivative terms with even parity is investigated in refs. \cite{27}.}
This is an equation to determine $W$, given the potential $V$.

Next, we consider the flow equation for $w = 2$. Since the scalar fields are regarded as coupling constants of the dual field theory, we set $\phi$ to be constant on the two-dimensional surface. Then the $w = 2$ flow equation turns out to be

$$
\delta W = \frac{g_{ij} \left( \frac{2}{\sqrt{-g}} \frac{\delta \Gamma}{\delta g_{ij}} + \frac{\beta}{16\pi G_N} A_{ij} \right) - G^{IJ} \frac{2}{W} \frac{\partial W}{\partial \phi^J} \left( \frac{1}{\sqrt{-g}} \frac{\delta \Gamma}{\delta \phi^I} \right)}{16\pi G_N W R}.
$$

(36)

If the two-dimensional metric is flat, this equation leads to the holographic RG equation of the dual field theory, and the holographic beta function is defined as

$$
\beta^I(\phi) = G^{IJ} \frac{2}{W} \frac{\partial W}{\partial \phi^J}.
$$

(37)

Putting (37) back into (36), we can regard (36) as the Weyl anomaly equation of the dual field theory on the curved backgrounds:

$$
\langle T^i \rangle = \frac{3}{24\pi G_N W R} + \frac{\beta^I(\phi)}{\sqrt{-g}} \frac{1}{\sqrt{-g}} \frac{\delta \Gamma}{\delta \phi^I}.
$$

(38)

Here the covariant energy-momentum tensor $T^{ij}$ is understood as modified by adding the Bardeen-Zumino term, i.e., $\beta A_{ij}/16\pi G_N$ to the ordinary one $(2/\sqrt{-g})\delta \Gamma/\delta g_{ij}$. At the fixed points where $\beta^I(\phi) = 0$ in the second term of the r.h.s of (38), we can read off the sum of the central charges of the left- and right-movers from the coefficients of the scalar curvature. Away from the fixed points, it is legitimate to define the sum of \(c\)-functions for left- and right-movers in the dual field theory as follows:

$$
c_L(\phi) + c_R(\phi) = \frac{6}{G_N W(\phi)}.
$$

(39)

### 3.4 Gravitational Anomaly

We then discuss the physical meaning of the momentum constraint $\mathcal{P}^i = 0$. By using the relations (40), the momentum constraint can be written as

$$
\frac{1}{2} \beta \sqrt{-g} \xi^i \epsilon_{ij} \partial_j R = \xi_i \left\{ -2\beta \sqrt{-g} e^{mn} K^i_m \nabla_n K_m - \beta \sqrt{-g} e^{mn} \nabla_k \left( K_n^i K_m^k \right) + \beta \sqrt{-g} \nabla_j \left( K e^{k(i} H^{j)} \right) \right\}.
$$

(40)

Up to total derivatives, this is equivalent to

$$
-\frac{\beta}{4} \sqrt{-g} \xi^i \epsilon_{ij} \partial_j R = \delta H_{ij} \frac{\delta \mathcal{S}}{\delta H_{ij}} + \delta g_{ij} \left( \frac{\delta \mathcal{S}}{\delta g_{ij}} \frac{1}{2} \beta \sqrt{-g} A^{ij} \right) + \delta \phi^I \frac{\delta \mathcal{S}}{\delta \phi^I},
$$

(41)

where $\delta$ means the infinitesimal field variation with respect to the general coordinate transformation $x^i \rightarrow x^i + \xi^i$. In order to derive this expression, we use

$$
\xi_i \left\{ -2\beta \sqrt{-g} e^{mn} K^i_m \nabla_n K_m - \beta \sqrt{-g} e^{mn} \nabla_k \left( K_n^i K_m^k \right) + \beta \sqrt{-g} \nabla_j \left( K e^{k(i} H^{j)} \right) \right\}
$$

$$
= \sqrt{-g} \xi_i \left\{ -\beta e^{mn} K^i_m \nabla^i K_m + \beta e^{mn} \nabla_m \left( K_i^i K_m^l \right) + \beta \nabla_j \left( K e^{k(i} H^{j)} \right) \right\}
$$

$$
= -\xi_i \Pi^{mn} \nabla^i K_m + 2\xi_i \nabla_m \left( K_i^i \Pi^{mn} \right) + \xi_j \nabla_j \left( K \Pi^{ij} \right)
$$

$$
\simeq -\left( \xi^i \nabla_i K_m + 2K_{im} \nabla_m \xi^i - K \nabla_m \xi_n \right) \Pi^{mn},
$$

(42)
and the constraint of (25).

Eq. (41) means that the classical bulk action $\bar{S}$ is not invariant under two-dimensional diffeomorphisms. More explicitly, by extracting $w = 3$ terms of both sides and set $\phi$ to be constant on the two-dimensional surface, we obtain

$$\nabla_i \langle T^{ij} \rangle = -\frac{3\beta}{G_N} \frac{1}{96\pi} \epsilon^{ij} \partial_j R,$$

(43)

which implies that, in the two-dimensional dual field theory, there is a gravitational anomaly proportional to the Chern-Simons coupling $\beta$. Hence, we can define the difference between two $c$-functions for left- and right-mover as

$$c_L(\phi) - c_R(\phi) = \frac{3\beta}{G_N}.$$

(44)

Combining (39) and (44), we finally obtain the holographic expression of the left-right asymmetric $c$-functions:

$$c_L(\phi) = \frac{3}{G_N} \left( \frac{1}{W(\phi)} + \frac{\beta}{2} \right), \quad c_R(\phi) = \frac{3}{G_N} \left( \frac{1}{W(\phi)} - \frac{\beta}{2} \right).$$

(45)

As we confirm in Appendix A by examples, these $c$-functions are both monotonically non-increasing toward the infrared direction. If the scalar potential $V$ is just a negative constant, i.e.,

$$V = -\frac{2}{\ell^2},$$

(46)

then we find $W(\phi) = 2/\ell$ and (45) turns out to be the central charges

$$c_L = \frac{3}{G_N} \left( \frac{\ell}{2} + \frac{\beta}{2} \right), \quad c_R = \frac{3}{G_N} \left( \frac{\ell}{2} - \frac{\beta}{2} \right).$$

(47)

which we obtained previously by the Brown-Henneaux’s method [5].

4 Conclusion and discussion

In this paper, we investigated the three dimensional gravity with gravitational Chern-Simons term coupled to some scalar fields. This theory admits a solution which interpolates two AdS$_3$ vacua due to the nontrivial profile of the scalar fields in the radial direction. The solution can be interpreted as the holographic RG flow of the two dimensional boundary quantum field theory. We constructed $c$-functions for left and right movers, which are consistent with Zamolodchikov’s $c$-theorem. The presence of the gravitational Chern-Simons term is crucial to obtain the left-right asymmetric ones.

First we decomposed the radial direction of the three dimensional theory in an ADM manner and applied the canonical method. Since the action contains the third derivative term with respect to the radial coordinate, we introduced the Lagrange multiplier and identified the extrinsic curvature as an independent variable. From this prescription we obtained the Hamiltonian and momentum constraints.

By inserting the classical solution with arbitrary boundary variables to the action, we obtained the Hamilton-Jacobi equation from the Hamiltonian constraint. This equation is solved order by order with respect to the weight $w$, and the holographic RG flow equation is
obtained for the case of \( w = 2 \). From this, the beta functions for the scalar fields are derived. Furthermore, by making a comparison with the Weyl anomaly on the boundary field theory, the sum of the left and right \( c \)-functions, \( c_L(\phi) + c_R(\phi) \), is obtained.

On the other hand, the momentum constraint is compared with the gravitational anomaly of the two dimensional quantum field theory. From this, it is possible to read the difference of the two \( c \)-functions, \( c_L(\phi) - c_R(\phi) \), which is constant along the RG flow. That the difference is constant is in perfect agreement with the \( c \)-theorem for parity violating two-dimensional quantum field theories.

In conclusion, by analyzing the three dimensional gravity with gravitational Chern-Simons term coupled to some scalar fields, we obtained the left-right asymmetric \( c \)-functions. These are monotonically non-increasing along the RG flow toward the IR region and precisely agree with the central charges at the fixed points. These results confirm the gauge/gravity correspondence in the presence of the gravitational Chern-Simons term.

In Ref. [7], it was shown in TMG that the AdS\(_3\) vacuum is unstable against the perturbation of the gravitational fields except for the chiral case. As a future work, it is important to study the instability of our solution. It has been known that there exist stable solutions in TMG called warped AdS\(_3\) [12, 13]. It is also an interesting task to study RG flow connecting AdS\(_3\) and warped AdS\(_3\) or two warped AdS\(_3\) vacua. We hope we could come to these problems in our future work.

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**A Explicit form of the solution**

In this appendix, we discuss the holographic RG flow by solving the equations of motion for the action (11). The equations of motion are written as

\[
0 = \dot{R}_{\mu\nu} - \frac{1}{2} G_{IJ}(\phi) \frac{\partial \mu}{\partial \phi^I} \frac{\partial \nu}{\partial \phi^J} - \ddot{g}_{\mu\nu} V(\phi) + \beta \dot{\epsilon}_{\rho\sigma}(\phi) \ddot{\nabla}_\rho \ddot{\nabla}_\sigma,
\]

\[
0 = \ddot{\nabla}_\mu (G_{IJ} \frac{\partial \mu}{\partial \phi^J}) - \frac{1}{2} \frac{\partial G_{IK}}{\partial \phi^J} \frac{\partial \mu}{\partial \phi^J} \frac{\partial \mu}{\partial \phi^K} - \frac{\partial V}{\partial \phi^I}.
\]

(48)

By substituting the ansatz

\[
\dot{s}^2 = dr^2 + e^{-2f(r)} \eta_{ij} dx^i dx^j, \quad \phi^I = \phi^I(r),
\]

(49)

into (48), we obtain a set of differential equations of the form,

\[
0 = -\dddot{f} + \frac{1}{2} G_{IJ}(\phi) \dot{\phi}^I \dot{\phi}^J,
\]

\[
0 = -\dddot{f} + 2f^2 + V(\phi),
\]

\[
0 = -\frac{\partial V}{\partial \phi^I} - 2G_{IJ}(\phi) \ddot{\phi}^I \dot{\phi}^J + \frac{\partial G_{IJ}}{\partial \phi^K} \dot{\phi}^J \dot{\phi}^K - \frac{1}{2} \frac{\partial G_{IK}}{\partial \phi^J} \dot{\phi}^I \dot{\phi}^J + G_{IJ}(\phi) \ddot{\phi}^I.
\]

(50)

(51)

(52)
Note that these equations do not depend on the coefficient $\beta$, and are the same as those for the Einstein gravity with the scalar field. The third equation (52) can be derived by combining (50) and (51), and hence neglected below. By employing the superpotential and the $\beta$-function defined in Eqs. (35) and (37), we can rewrite Eqs. (50) and (51) as follows:

$$\dot{f} = \frac{1}{2} W(\phi),$$

(53)

$$\dot{\phi}^I = G^{IJ}(\phi) \frac{\partial W(\phi)}{\partial \phi^J}.$$  

(54)

In order to solve these equations, we consider the region $r_{\text{IR}} \leq r \leq r_{\text{UV}}$ and the solution where $\dot{\phi}^I = 0$ only at $r = r_{\text{IR}}$ and $r_{\text{UV}}$. As discussed in Eq. (39), $W(\phi)$ is related to the $c$-function and hence should be positive. Furthermore, if we regard $e^{f(r)}$ as the scale of boundary field theory, $t$, and use Eqs. (53) and (54), we can identify the beta function

$$t \frac{d}{dt} c_L(\phi) = t \frac{d}{dt} c_R(\phi) = -\frac{3}{2G_N W(\phi)} \beta^I(\phi) G_{IJ}(\phi) \beta^J(\phi) \leq 0.$$  

(55)

Therefore each $c$-function defined in Eq. (45) is monotonically non-increasing along the RG flow.

Since Eqs. (53) and (54) are just the first order differential equations, we can always find a solution for any superpotential $W(\phi)$ which satisfies the above conditions. As an example, we consider only one scalar field $\phi \equiv \phi^1$ with the metric $G_{11}(\phi) = 1$, and choose the superpotential like

$$W(\phi) = \frac{1}{\ell} (\sin \phi + \alpha),$$

(56)

where $\ell$ is some positive constant and $\alpha > 1$. This is monotonically non-decreasing in the region $-\pi/2 \leq \phi \leq \pi/2$. Now the solution is given by

$$f = \frac{1}{2} \log(\cos \phi) - \alpha \tanh^{-1} \left( \tan \frac{\phi}{2} \right) + b,$$

$$\phi = 2 \arctan \left( \tanh \left( \frac{r-a}{2\ell} \right) \right),$$

(57)

in the region $-\infty < r < \infty$. Namely, the scalar field is represented by a kink solution. The beta function is evaluated as

$$\beta(\phi) = \frac{2 \cos \phi}{\sin \phi + \alpha}.$$  

(58)

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