SPECTRAL RIGIDITY OF RANDOM SCHRODINGER OPERATORS
VIA FEYNMAN-KAC FORMULAS

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Abstract. We develop a technique for proving number rigidity (in the sense of Ghosh-Peres [35]) of the spectrum of general random Schrödinger operators (RSOs). Our method makes use of Feynman-Kac formulas to estimate the variance of exponential linear statistics of the spectrum in terms of self-intersection local times.

Inspired by recent results concerning Feynman-Kac formulas for RSOs with multiplicative white noise [29, 30, 36] by Gorin, Shkolnikov and the first-named author, we use this method to prove number rigidity for a class of one-dimensional continuous RSOs of the form $-\frac{1}{2}\Delta + V + \xi$, where $V$ is a deterministic potential and $\xi$ is a stationary Gaussian noise. Our results require only very mild assumptions on the domain on which the operator is defined, the boundary conditions on that domain, the regularity of the potential $V$, and the singularity of the noise $\xi$.

1. Introduction

Let $I \subset \mathbb{R}$ be an open interval (possibly unbounded), and let $V : I \to \mathbb{R}$ be a deterministic potential. Let $\xi : I \to \mathbb{R}$ be a centered stationary Gaussian process with a covariance of the form $\mathbb{E}[\xi(x)\xi(y)] = \gamma(x - y)$, where $\gamma$ is an even function or Schwartz distribution. (We refer to Section 2.2 for a formal definition.) In this paper, we investigate the number rigidity of the eigenvalue point processes of random Schrödinger operators (RSOs) of the form

\begin{equation}
\hat{H}_I := -\frac{1}{2}\Delta + V + \xi,
\end{equation}

where $\hat{H}_I$ acts on a subset of functions $f : I \to \mathbb{R}$ that satisfy some fixed boundary conditions (if $I$ has a boundary).

1.1. Random Schrödinger Operators. The spectral theory of RSOs arises naturally in multiple problems in mathematical physics; we refer to [13] for a general introduction to the subject. Looking more specifically at one-dimensional continuous operators, RSOs of the form (1.1) have found applications in the study of random matrices and interacting particle systems, as well as stochastic partial differential equations (SPDEs).

Indeed, if $\xi = \xi_\beta$ is a white noise with variance $1/\beta$ for some $\beta > 0$ (see Example 2.7) and $V(x) = x/2$, then the spectrum of $\hat{H}_{(0,\infty)}$ captures the asymptotic edge fluctuations of a large class of $\beta$-Ensembles [6, 41, 46] due to its connection with the Airy-$\beta$ process (see Section 5). In another direction, the study of the solutions of SPDEs of the form

\begin{equation}
\partial_t u = \frac{1}{2}\Delta u - V u - \xi u = -\hat{H}_I u
\end{equation}

is intimately connected to the spectral theory of $\hat{H}_I$. More specifically, the localization of $\hat{H}_I$’s eigenfunctions is expected to shed light on the geometry of intermittent peaks in (1.2) (e.g., [40, Sections 2.2.3–2.2.4] and references therein). We refer to [14, 17, 27, 28] for a few examples of papers where such ideas have been implemented when $\xi$ is a smooth, white, fractional, or otherwise singular noise (see Examples 2.7–2.10 for definitions of such noises).

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1.2. Spatial Conditioning and Number Rigidity. Point processes are well-studied objects in probability [20, 39], due to their applications in many disciplines (e.g., [3]). One of the simplest point processes is the Poisson process, which is such that the number of points in disjoint sets are independent. In contrast, for point processes with strong correlations, the notion of spatial conditioning (i.e., the distribution of points inside a bounded set conditional on the point configuration outside the set) is of interest. Pioneering work on this subject includes the Dobrushin-Lanford-Ruelle (DLR) formalism (e.g., [21, Sections 1.4-2.4]).

In this paper, we are interested in a form of spatial conditioning known as number rigidity [35]. A point process is said to be number rigid if for every bounded set $A$, the configuration of points outside of $A$ determines the number of points inside of $A$. We refer to [2, 38] for examples of early work on this kind of property. In their seminal paper [35] (see also [31]), Ghosh and Peres introduced (among other things) the notion of number rigidity, and studied its occurrence in two classical point processes. Since then, number rigidity has been shown to have many interesting applications in the theory of point processes (e.g., [7, 8, 9, 31, 32, 45]), and has developed into an active field of research. We refer to [4, 5, 11, 15, 26, 33, 35] for other notions related to number rigidity, such as higher order rigidity, hyperuniformity, linear rigidity, sub-extensivity, and tolerance.

A significant portion of the number rigidity literature is concerned with uncovering sufficient conditions for certain point processes to be number rigid. Though several methods have been found to prove number rigidity (such as DLR equations [22] or deletion tolerance [45]), in most cases one proceeds by controlling the variance of carefully chosen sequences of linear statistics (e.g., [10, 12, 31, 34, 35, 47]). We recall that, given a point process $\mathcal{X}$ on a domain $\Sigma$, the linear statistic associated with a test function $f$ on $\Sigma$ is defined as $\sum_{x \in \mathcal{X}} f(x)$. Following [35, Theorem 6.1], if one concocts a sequence $(f_n)_{n \in \mathbb{N}}$ of test functions such that $f_n \to 1$ uniformly on some set $A \subset \Sigma$ and the variance of $\sum_{x \in \mathcal{X}} f_n(x)$ vanishes, then it follows that the number of points in $A$ is determined by $\mathcal{X}$’s configuration outside of $A$ (see Proposition 2.2). Thus, there is a strong motivation to develop tools that provide upper bounds for the variance of linear statistics of point processes of interest.

1.3. Outline of Results and Method of Proof. To the best of our knowledge, the only RSO whose spectrum is known to be number rigid is the operator

$$\hat{H}_{(0,\infty)} = -\frac{d}{2} \Delta + \frac{c}{2} + \xi_2$$

with a Dirichlet boundary condition at zero, where $\xi_2$ is a white noise with variance 1/2. The proof of this [10] relies on the fact that the eigenvalues of this operator generate the Airy-2 process, which is a determinantal point process (see (5.1)). In this context, our main motivation in this paper is to provide a unified framework to study the number rigidity of the eigenvalues of general RSOs. As a first step in this direction, we develop a new method of proving number rigidity for RSOs by controlling the variance of exponential linear statistics using Feynman-Kac formulas. Informally, our main result is as follows (we point to Theorems 2.19 and 2.20 for precise statements).

**Theorem 1.1** (Informal Statement). Suppose that $\hat{H}_I$ acts on either the full space $I = \mathbb{R}$, the half-line $I = (0, \infty)$ with Dirichlet or Robin boundary condition at zero, or a bounded interval $I = (0, b)$ with Dirichlet, Robin, or mixed boundary conditions. Suppose that $V$ is locally integrable, and that $\xi$ is a smooth, white, fractional, or otherwise singular Gaussian noise.

On the one hand, when $I$ is unbounded, there is an explicit constant $c_\gamma > 0$ (see (2.16)–(2.19)) that only depends on the noise such that if

$$\lim_{|x| \to \infty} V(x)/|x|^{\gamma} = \infty,$$

then $\hat{H}_I$’s spectrum is number rigid. On the other hand, if $I$ is a bounded interval, then $\hat{H}_I$’s spectrum is always number rigid.
Thus, the method developed in this paper applies under very mild assumptions on the noise \( \xi \), the domain \( I \), the boundary conditions on \( I \), and the regularity of the deterministic potential \( V \). However, in cases where the domain \( I \) is unbounded, our method comes at the cost of growth assumptions on \( V \).

**Remark 1.2.** It is worth noting that our main result does not imply rigidity of the Airy-\( \beta \) process for \( \beta \neq 2 \), since the growth condition (1.3) in the case of white noise requires \( V \) to be superlinear (see (2.16)). In fact, we prove that it is not possible to establish the rigidity of the Airy-2 process by using exponential linear functionals (see Proposition 2.22). This suggests (at least for white noise) that (1.3) is the optimal sufficient condition for number rigidity of general RSOs that can be obtained with our semigroup method; see Section 2.5 for more details.

The key steps in the proof of our main result are as follows.

(i) We provide a general condition (see Proposition 2.15) on \( \xi \)'s covariance (i.e., \( \gamma \)) under which exponential functionals \( e^{-t \xi} \) (\( t > 0 \)) of the spectrum of \( \hat{\mathcal{H}}_I \) admit a random Feynman-Kac representation. Apart from classical semigroup theory, the main inspiration for this result is the work on Feynman-Kac formulas for RSOs with white noise [29, 30, 36] pioneered by Gorin and Shkolnikov.

(ii) The Feynman-Kac formulas in (i) give an explicit representation of \( \hat{\mathcal{H}}_I \)'s semigroup in terms of elementary stochastic processes. This allows to reformulate the vanishing of the variance of exponential linear statistics in terms of a corresponding limit for the self-intersection local time of Brownian bridges on \( \mathbb{R} \), or reflected Brownian bridges on the half-line or bounded intervals (see Theorem 4.1).

(iii) The main tool we use to control the Brownian bridge self-intersection local time consists of large deviations results for the self-intersection local time of unconditioned Brownian motion on \( \mathbb{R} \). The latter has been studied extensively; we refer to [16, Chapter 4] and references therein for details. To bridge the gap between the results on the self-intersection local time of Brownian bridges and the unconditioned Brownian motion, we make use of couplings between reflected Brownian motions on different domains, and the absolute continuity of the midpoint of bridge processes with respect to their unconditioned versions.

(iv) By combining (i)–(iii), we obtain our main result (Theorem 2.19), which consists of general sufficient conditions (see (2.13) and (2.14)) for the number rigidity of \( \hat{\mathcal{H}}_I \)'s spectrum in terms of Brownian self-intersection times and the growth rate of \( V \). Then, in Theorem 2.20 we apply this result to white, fractional, singular, and smooth noises.

**1.4. Comparison with Previous Results and Future Work.** Several techniques have been used thus far to control the variance of linear statistics for the purpose of proving number rigidity. Prominent examples include determinantal/Pfaffian or other integrable structure [10, 12, 31, 34, 35], translation invariance and hyperuniformity [34], and finite-dimensional approximations [47]. By using such methods, number rigidity has been established for the zeroes of the planar Gaussian Analytic Function, the Ginibre ensemble, the Sine-\( \beta \) process (for all \( \beta > 0 \)), the Airy-2 process, some Bessel and Gamma point processes, and more. While some of the properties used in those papers are present in some examples of the RSOs in (1.1), none of these results provide sufficient conditions that can be applied to general RSOs.

In closing, we note that the work in this paper raises a number of interesting questions for future research. Most notably, the number rigidity for the Airy-\( \beta \) process with \( \beta \neq 2 \) is still open. As mentioned following the statement of Theorem 1.1, we expect that proving the rigidity of this point process (along with RSOs whose deterministic potentials do not satisfy the growth condition (1.3)) will require new insights; see Section 2.5 for more details. We leave such questions to future papers.

In another direction, we remark that, in principle, none of the elements of the proof outlined in steps (i)–(iv) of Section 1.3 are unique to one-dimensional continuous RSOs. Thus, the method developed in this paper has the potential for substantial generalization. For
example, we expect that the semigroup approach could be used to prove eigenvalue rigidity for RSOs acting on a variety of discrete lattices or continuous space in higher dimensions, making interesting connections with random walk self-intersections and the renormalization theory of singular operators/SPDEs. We will address these questions in future works.

**Organization of Paper.** The rest of this paper is organized as follows. In Section 2, we introduce the setup of the paper, state our main results, and discuss their optimality. Section 3 contains the proof of the Feynman-Kac formulas that form the basis of our method (Propositions 2.15 and 2.16). In Section 4, we use these Feynman-Kac formulas to control the variance of exponential linear statistics, thus proving our main results, Theorems 2.19 and 2.20. Section 5 demonstrates that the variance of exponential linear statistics cannot be used to prove rigidity of the Airy-2 process. Finally, Appendix A and B provide some elementary results on number rigidity and stochastic analysis.

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### 2. Setup and Main Results

This section is organized as follows. In Section 2.1, we give reminders for basic notions regarding number rigidity. In Section 2.2, we introduce the noises considered in this paper. In Section 2.3, we state the Feynman-Kac formulas that form the basis of our proof, and discuss rigorous definitions of the operators $\hat{H}_I$ when the noise cannot be defined pointwise. In Section 2.4, we state our main results. Finally, we discuss the optimality of our results and related open problems in Section 2.5.

#### 2.1. Number Rigidity

Let $\Lambda$ be a point process on $\mathbb{R}$ (i.e., a random locally finite counting measure on $\mathbb{R}$). Given a Borel set $A \subset \mathbb{R}$, we let $\Lambda(A)$ denote the number of points of $\Lambda$ that are inside of $A$, that is,

$$\Lambda(A) := \sum_{\lambda \in \Lambda} 1_{\{\lambda \in A\}}.$$

More generally, for every function $f : \mathbb{R} \to \mathbb{R}$, we use

$$\Lambda(f) := \sum_{\lambda \in \Lambda} f(\lambda)$$

to denote the linear statistic associated with $f$. For any Borel set $A \subset \mathbb{R}$, we let $\mathcal{F}_\Lambda(A) := \sigma\{\Lambda(\bar{A}) : \bar{A} \subset A\}$ denote the $\sigma$-algebra generated by the configuration of points inside of $A$.

**Definition 2.1 ([35]).** We say that $\Lambda$ is **number rigid** if $\Lambda(A)$ is $\mathcal{F}_\Lambda(\mathbb{R} \setminus A)$-measurable for every bounded Borel set $A \subset \mathbb{R}$.

We have the following criterion for number rigidity:

**Proposition 2.2 ([35]).** Let $A \subset \mathbb{R}$ be a bounded Borel set. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of functions satisfying the following conditions.

1. Almost surely, $\Lambda(f_n 1_A) < \infty$ for every $n \in \mathbb{N}$ and $\bar{A} \subset \mathbb{R}$.
2. $|f_n - 1| \to 0$ as $n \to \infty$ uniformly on $A$.
3. $\text{Var}[\Lambda(f_n)] \to 0$ as $n \to \infty$.

Then, $\Lambda(A)$ is $\mathcal{F}_\Lambda(\mathbb{R} \setminus A)$-measurable.

Though this result is by now standard in the rigidity literature, we provide a proof in Appendix A for the reader’s convenience.
2.2. Noise. In this section, we describe the noises \( \xi \) considered in this paper.

**Definition 2.3.** We consider three types of domains \( I \subset \mathbb{R} \) on which \( \xi \) is defined and \( \mathcal{H}_I \) acts: the full space \( I = \mathbb{R} \) (*Case 1*), the half-line \( I = (0, \infty) \) (*Case 2*), and the bounded interval \( I = (0, b) \) for some \( b > 0 \) (*Case 3*).

Let \( C_c = C_c(I) \) denote the set of functions \( f : I \mapsto \mathbb{R} \) that are continuous and compactly supported on \( I \)'s closure. We now introduce the covariance functions that characterize \( \xi \).

**Definition 2.4.** Let \( \gamma \) be an even function or Schwartz distribution (that is, \( \langle f, \gamma \rangle = \langle \tilde{f}, \gamma \rangle \) for every \( f \in C_c \), where \( \tilde{f}(x) := f(-x) \)) such that

\[
\tag{2.1}
(f, g)_\gamma := \int_{\mathbb{R}^2} f(x)\gamma(x - y)g(y) \, dx \, dy, \quad f, g \in C_c
\]

is a semi-inner-product on \( C_c \), that is,

1. \((2.1)\) is finite and well defined for every \( f, g \in C_c \);
2. \((f, g) \mapsto (f, g)_\gamma \) is sesquilinear and symmetric; and
3. \((f, f)_\gamma \geq 0 \) for all \( f \in C_c \).

We denote the seminorm induced by \( \langle \cdot, \cdot \rangle_\gamma \) as

\[
\|f\|_\gamma := \sqrt{(f, f)_\gamma}, \quad f \in C_c.
\]

We say that \( \gamma \) is compactly supported if there exists a compact set \( A \subset \mathbb{R} \) such that \( \langle f, \gamma \rangle = 0 \) whenever \( f(x) = 0 \) for every \( x \in A \).

**Remark 2.5.** In cases where \( \gamma \) is not an almost-everywhere-defined function, the integral over \( \gamma(x - y) \) in (2.1) may not well defined. In such cases, we interpret

\[
\langle f, g \rangle_\gamma := \langle f, g * \gamma \rangle = \langle f * \gamma, g \rangle = \langle f, g, \gamma \rangle = \langle f * \gamma, g \rangle,
\]

where \( \langle \cdot, \cdot \rangle \) denotes the \( L^2 \) inner product.

Throughout this paper, we assume that \( \xi \) satisfies the following assumption.

**Assumption 2.6.** \( \xi \) is a linear functional from \( C_c \) to a space of real-valued random variables such that \( f \mapsto \xi(f) \) is a Gaussian process with mean zero and covariance

\[
\tag{2.3}
\mathbb{E}[\xi(f)\xi(g)] = (f, g)_\gamma, \quad f, g \in C_c
\]

(e.g., [23, Section 6]).

We now present several examples of noises covered by Assumption 2.6.

**Example 2.7 (White).** Let \( \sigma > 0 \) be fixed. We say that \( \xi \) is a white noise with variance \( \sigma^2 \) if \( \gamma = \sigma^2 \delta_0 \), where \( \delta_0 \) denotes the delta Dirac distribution. In this case, the covariance is simply the \( L^2 \) inner product

\[
\tag{2.4}
\mathbb{E}[\xi(f)\xi(g)] = \sigma^2 (f, g),
\]

and \( \xi \) can be constructed as the stochastic integral

\[
\xi(f) := \sigma \int_{\mathbb{R}} f(x) \, dW(x)
\]

with respect to a two-sided Brownian motion \( W \).

**Example 2.8 (Fractional).** Let \( H \in \left( \frac{1}{2}, 1 \right) \) and \( \sigma > 0 \) be fixed. We say that \( \xi \) is a fractional noise with Hurst parameter \( H \) and variance \( \sigma^2 \) if

\[
\gamma(x) := \sigma^2 H(2H - 1)|x|^{2H - 2},
\]

in which case

\[
\tag{2.5}
\mathbb{E}[\xi(f)\xi(g)] = \sigma^2 H(2H - 1) \int_{\mathbb{R}^2} \frac{f(x)g(y)}{|x - y|^{2 - 2H}} \, dx \, dy.
\]
This noise can be constructed as the stochastic integral
\[ \xi(f) := \sigma \int f(x) \, dW^H(x), \]
where \( W^H \) is a two-sided fractional Brownian motion with Hurst parameter \( H \).

**Example 2.9 (L\textsuperscript{p}-Singular).** Let \( 1 \leq p < \infty \). We say that \( \xi \) is an \( L^p \)-singular noise if \( \gamma \) can be decomposed as
\[ \gamma = \gamma_1 + \gamma_2, \]
where \( \gamma_1 \in L^p(\mathbb{R}) \), and \( \gamma_2 \) is uniformly bounded. We can view \( L^p \)-singular noise as a generalization of fractional noise, as \( \gamma_1(x) \sim |x|^{-\epsilon} \) as \( x \to 0 \) for some \( \epsilon \in (0, 1) \), or \( \gamma_1(x) \sim (- \log |x|)^{\epsilon} \) as \( x \to 0 \) for some \( \epsilon > 0 \).

**Example 2.10 (Bounded).** We say that \( \xi \) is a bounded noise if \( \gamma \) is uniformly bounded. In many such cases \( \xi \) gives rise to a pointwise-defined Gaussian process on \( \mathbb{R} \) with covariance function \( E[\xi(x)\xi(y)] = \gamma(x - y) \), whence we can simply define
\[ \xi(f) := \int \mathbb{R} f(x) \xi(x) \, dx. \]

### 2.3. Feynman-Kac Functionals.

We now introduce the random Feynman-Kac semigroups studied in this paper. Much of the notation introduced in this section is directly inspired from [29].

We make the following assumption on the deterministic potential \( V \).

**Assumption 2.11.** \( V : I \to \mathbb{R} \) is bounded below and locally integrable on \( I \)'s closure. If \( I \) is unbounded (i.e., Cases 1 & 2), then we also assume that
\[ \lim_{|x| \to \infty} \frac{V(x)}{|x| \log |x|} = \infty. \]

Next, we introduce some stochastic processes that form the basis of the random Feynman-Kac semigroups that we consider.

**Definition 2.12.** We use \( B \) to denote a standard Brownian motion taking values in \( \mathbb{R} \), \( X \) to denote a reflected standard Brownian motion taking values in \( (0, \infty) \), and \( Y \) to denote a reflected standard Brownian motion taking values in \( (0, b) \). Throughout this paper, we use \( Z \) to denote one of these three processes, depending on which case in Definition 2.3 is being considered, that is
\[ Z = \begin{cases} B & \text{(Case 1)} \\ X & \text{(Case 2)} \\ Y & \text{(Case 3)} \end{cases}. \]

For every \( t > 0 \) and \( x, y \in I \), we denote by
\[ Z^x := (Z|Z(0) = x) \]
the process started at \( x \), and we denote the bridge process from \( x \) to \( y \) in time \( t \) by
\[ Z^x_{t,y} := (Z|Z(0) = x \text{ and } Z(t) = y). \]

We sometimes use \( E^x \) and \( E^x_{t,y} \) to denote the expected value with respect to the law of \( Z^x \) and \( Z^x_{t,y} \), respectively.

We denote the Gaussian kernel by
\[ \mathcal{G}_t(x) := \frac{e^{-x^2/2t}}{\sqrt{2\pi t}}, \quad t > 0, \ x \in \mathbb{R}. \]
We denote the transition kernel of $Z$ by $\Pi_Z$, that is, for every $t > 0$ and $x, y \in I$

$$\Pi_Z(t; x, y) := \begin{cases} G_t(x - y) & \text{(Case 1)} \\ G_t(x - y) + G_t(x + y) & \text{(Case 2)} \\ \sum_{z \in 2\mathbb{Z} \cup \{0\}} G_t(x - z) & \text{(Case 3)} \end{cases}$$

For any $0 \leq s \leq t$, we let $a \mapsto L_{[s,t]}^a(Z) \ (a \in I)$ denote the continuous version of the local time of $Z$ (or its conditioned versions) collected on $[s, t]$, i.e.,

$$\int_s^t f(Z(u)) \, du = \int_s^t L_{[s,t]}^a(Z) \, da = \langle L_{[s,t]}^a(Z), f \rangle$$

for any measurable function $f : I \to \mathbb{R}$. We use the shorthand $L_t(Z) := L_{[0,t]}(Z)$.

As a matter of convention, if $Z = X$ or $Y$, then we distinguish the boundary local time from the above, which we denote as

$$\mathcal{L}_{[s,t]}^c(Z) := \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_s^t 1_{\{c - \epsilon < Z(u) < c + \epsilon\}} \, du$$

for $c \in \partial I$ (i.e., $c = 0$ if $Z = X$ or $c \in \{0, b\}$ if $Z = Y$), also with the shorthand $\mathcal{L}_t^c(Z) := \mathcal{L}_{[0,t]}^c(Z)$.

We are now finally in a position to state our Feynman-Kac formulas.

**Definition 2.13.** Let $\alpha, \beta \in [-\infty, \infty]$. For every $t > 0$ and $x, y \in I$, we define the random kernel

$$\hat{K}(t; x, y) := \begin{cases} \Pi_B(t; x, y) E_t^{x,y} \left[ e^{-\langle L_t(B), V \rangle - \xi(L_t(B))} \right] & \text{(Case 1)} \\ \Pi_X(t; x, y) E_t^{x,y} \left[ e^{-\langle L_t(X), V \rangle - \xi(L_t(X)) + \alpha \mathcal{L}_t^c(X)} \right] & \text{(Case 2)} \\ \Pi_Y(t; x, y) E_t^{x,y} \left[ e^{-\langle L_t(Y), V \rangle - \xi(L_t(Y)) + \alpha \mathcal{L}_t^c(Y) + \beta \mathcal{L}_t^b(Y)} \right] & \text{(Case 3)} \end{cases}$$

where we assume that the noise $\xi$ is independent of $B, X, Y$; hence the expected value $E_t^{x,y}$ is with respect to $B_t^{x,y}, X_t^{x,y}, Y_t^{x,y}$, conditional on $\xi$. We denote by $\hat{K}(t)$ the random integral operator on $L^2(I)$ with the above kernel.

**Remark 2.14.** If $\xi$ can be realized as a pointwise-defined measurable map on $\mathbb{R}$, then it follows from (2.6) and (2.10) that

$$\langle L_t(Z), V \rangle + \xi(L_t(Z)) = \int_0^t V(Z(s)) + \xi(Z(s)) \, ds.$$

Thus, at least formally, the family of operators $(\hat{K}(t))_{t \geq 0}$ corresponds to the semigroup of the RSO (1.1) (see Remarks 2.17 and 2.18 for more details). In this context, in Cases 2 & 3, the parameters $\alpha$ and $\beta$ in Definition 2.13 encode the boundary conditions on $\mathcal{H}_I$’s domain. More specifically, by using the conventions $e^{-\infty} := 0$ and

$$-\infty \cdot \mathcal{L}_t^c(Z) := \begin{cases} 0 & \text{if } \mathcal{L}_t^c(Z) = 0 \\ -\infty & \text{if } \mathcal{L}_t^c(Z) > 0 \end{cases}$$

for any $c \in \partial I$, if we let $\tau_c(Z) := \inf\{t \geq 0 : Z(t) = c\}$ denote the first hitting time of $c$, then we can interpret $e^{-\infty} \mathcal{L}_t^c(Z) := 1_{\{\tau_c(Z) > t\}}$. Thus, taking $\alpha = -\infty$ or $\beta = -\infty$ corresponds to a Dirichlet boundary condition at the respective endpoint. Taking a finite $\alpha$ or $\beta$ corresponds to the Robin boundary conditions $f'(0) + \alpha f(0) = 0$ or $-f'(b) + \beta f(b) = 0$, respectively. Consequently, the setup of Definition 2.13 enables us to treat Dirichlet and Robin boundary conditions at the origin of the half-line, or Diriclet, Robin, and mixed boundary conditions on the bounded interval.

We end this section with the following statements.
Proposition 2.15. Suppose that Assumptions 2.6 and 2.11 hold. Suppose further that
\begin{equation}
\limsup_{t \to 0} \sup_{x \in I} E^x \left[ e^{\theta |L_t(Z)|^2} \right] = 1
\end{equation}
for every \( \theta > 0 \). There exists a point process \( \Lambda = (\Lambda_k)_{k \in \mathbb{N}} \) and a random orthonormal basis \((\Psi_k)_{k \in \mathbb{N}} \) of \( L^2(I) \) satisfying the following conditions.

1. \( -\infty < \Lambda_1 \leq \Lambda_2 \leq \Lambda_3 \leq \cdots \).
2. For every \( t > 0 \), it holds that
\[ \hat{K}(t; x, y) = \sum_{k=1}^{\infty} e^{-t\Lambda_k} \Psi_k(x) \Psi_k(y) \quad \text{almost surely for every } x, y \in I. \]
3. For every \( t > 0 \),
\[ \text{Tr}[\hat{K}(t)] = \int_I \hat{K}(t; x, x) \, dx = \sum_{k=1}^{\infty} e^{-t\Lambda_k} < \infty \quad \text{almost surely.} \]

Proposition 2.16. Condition (2.11) holds for all four types of noises considered in Examples 2.7–2.10.

Propositions 2.15 and 2.16 are proved in Section 3.

Remark 2.17. If the noise \( \xi \) is defined pointwise, then \( \hat{H}_I \) is classically well defined. It then follows from the traditional Feynman-Kac formula (e.g. [19, 44, 49, 50], or [29, Theorem 5.3]) that the point process \( \Lambda \) in Proposition 2.15 is made of \( \hat{H}_I \)'s eigenvalues.

If \( \xi \) is a white noise, then \( \hat{H}_I \) and its eigenvalues can be defined using quadratic forms [29, Proposition 2.8] (see also [6, 25, 42, 46]). That \( \Lambda \) corresponds to this particular eigenvalue point process in this case follows from the Feynman-Kac formula proved in [29, Theorem 2.18] (see also [30, 36]).

Remark 2.18. For other types of noise, to the best of our knowledge there is no rigorous definition of \( \hat{H}_I \) available in the literature. However, we note that the random integral operator with kernel
\[ \hat{K}(t; x, y) := \sum_{k=1}^{\infty} e^{-t\Lambda_k} \Psi_k(x) \Psi_k(y) \]
is strongly continuous, and that its infinitesimal generator is given by the operator
\begin{equation}
\hat{H}_I f := \sum_{k=1}^{\infty} \Lambda_k (\Psi_k, f) \Psi_k
\end{equation}
(whose domain is \( \{ f \in L^2(I) : \sum_{k=1}^{\infty} \Lambda_k^2 (\Psi_k, f)^2 < \infty \} \)). According to Proposition 2.15-(2), \( \mathbb{P}[\hat{K}(t) = \hat{K}(t)] = 1 \) for every \( t > 0 \); hence we propose to define \( \hat{H}_I \) in those cases as (2.12).

2.4. Result. Our main result is as follows.

Theorem 2.19. Suppose that Assumptions 2.6 and 2.11 are satisfied. Suppose that (2.11) holds, and that there exists \( \delta > 0 \) such that
\begin{equation}
\limsup_{t \to 0} t^{-\delta} \left( \sup_{x \in I} E^x \left[ \| L_t(Z) \|_2^2 \right]^{1/\theta} \right) < \infty
\end{equation}
for every \( \theta > 0 \). Let \( \Lambda \) be the point process defined in Proposition 2.15. In Cases 1 & 2 (i.e., \( \hat{H}_R \) or \( \hat{H}_{(0, \infty)} \)), \( \Lambda \) is number rigid if the following growth condition holds.
\[ \lim_{|x| \to \infty} \frac{V(x)}{|x|^{2/(2\alpha-1)}} = \infty \quad \text{if } \gamma \text{ is compactly supported} \]
\[ \lim_{|x| \to \infty} \frac{V(x)}{|x|^{2/(2\beta-1)}} = \infty \quad \text{otherwise}. \]

In Case 3 (i.e., \( \hat{H}_{(0,\delta)} \)), \( \Lambda \) is number rigid if \( \delta > 1 \).
From this theorem, we obtain the following corollary, which specializes (2.14) to the four examples of noises considered earlier.

**Theorem 2.20.** Let \( \xi \) be one of the four types of noises considered in Examples 2.7–2.10. Then, (2.13) holds with

\[
\nu := \begin{cases} 
3/2 & \text{(white noise)} \\
1 + H & \text{(fractional noise with index } H \in (\frac{1}{2}, 1)) \\
2 - 1/2p & \text{(\( L^p \)-singular noise with } p \geq 1) \\
2 & \text{(bounded noise).}
\end{cases}
\]

In particular, if we let \( \Lambda \) be the point process of Proposition 2.15 (which we know exists for these examples thanks to Proposition 2.16), then we obtain the following sufficient conditions for number rigidity: In Cases 1 & 2, \( \Lambda \) is number rigid if the following growth conditions on \( V \) are satisfied.

1. **(White)** If \( \xi \) is a white noise, then
   \[
   \lim_{|x| \to \infty} \frac{V(x)}{|x|} = \infty.
   \]
2. **(Fractional)** If \( \xi \) is a fractional noise with Hurst index \( H \in (\frac{1}{2}, 1) \), then
   \[
   \lim_{|x| \to \infty} \frac{V(x)}{|x|^{2H}} = \infty.
   \]
3. **(\( L^p \)-Singular)** If \( \xi \) is an \( L^p \)-singular noise, then
   \[
   \begin{align*}
   &\lim_{|x| \to \infty} \frac{V(x)}{|x|^{2p/(3p-1)}} = \infty \quad \text{(if } \gamma \text{ is compactly supported)} \\
   &\lim_{|x| \to \infty} \frac{V(x)}{|x|^{4p/(2p-1)}} = \infty \quad \text{(otherwise)}.
   \end{align*}
   \]
4. **(Bounded)** If \( \xi \) is a bounded noise, then
   \[
   \begin{align*}
   &\lim_{|x| \to \infty} \frac{V(x)}{|x|^{2/3}} = \infty \quad \text{(if } \gamma \text{ is compactly supported)} \\
   &\lim_{|x| \to \infty} \frac{V(x)}{|x|^2} = \infty \quad \text{(otherwise)}.
   \end{align*}
   \]

In Case 3, \( \Lambda \) is always number rigid.

Theorem 2.19 is proved in Section 4. The main technical ingredient in this proof is Theorem 4.1, which provides quantitative upper bounds on the variance of the linear statistic \( \sum_k e^{-tA_k} \) as \( t \to 0 \). The result then follows from an application of Proposition 2.2 with test functions of the form \( f_n(x) = e^{-t_n x} \) with \( t_n \to 0 \), by proving that

\[
\lim_{n \to \infty} \text{Var}[\Lambda(f_n)] = \lim_{n \to \infty} \text{Var}[\text{Tr}(\hat{K}(t_n))] = 0
\]

under the conditions stated in Theorem 2.19. Theorem 2.20 is proved in Section 4.6.

2.5. **Questions of Optimality.** The growth assumptions (2.14) raise natural questions concerning the optimality of Theorem 2.19. For instance, when \( \xi \) is a white noise, it is known that the super-linear condition \( V(x)/|x| \to \infty \) in Theorem 2.20 is not necessary for the number rigidity of \( \Lambda \).

**Proposition 2.21 ([10]).** Let \( \xi_2 \) be a white noise with variance 1/2. Let us denote the operator

\[
\hat{H}^{(2)}_{(0, \infty)} := -\frac{1}{2} \Delta + \frac{x}{2} + \xi_2,
\]

with Dirichlet boundary condition at zero. \( \hat{H}^{(2)}_{(0, \infty)} \)'s eigenvalues are number rigid.
Indeed, one may recognize $\hat{H}_{(0,\infty)}^{(2)}$ as the stochastic Airy operator with parameter $\beta = 2$ (up to a multiple of $1/2$), whose eigenvalues form a determinantal point process (e.g., [46, 51]) known as the Airy-2 process. By using this integrable structure, Bufetov showed in [10, Section 3.2] that $\hat{H}_{(0,\infty)}^{(2)}$’s eigenvalues are number rigid. In the following proposition (proved in Section 5), we demonstrate how the linear statistics used in this paper fail to show the rigidity of the Airy-2 process.

Proposition 2.22. With $\hat{H}_{(0,\infty)}^{(2)}$ as in (2.20), it holds that

$$\lim_{t \to 0} \text{Var}\left[ \text{Tr}[e^{-t\hat{H}_{(0,\infty)}^{(2)}}]\right] = (4\pi)^{-1}.$$

In light of Propositions 2.21 and 2.22, it would be interesting to better understand the conditions under which the spectrum of one-dimensional continuous RSOs are number rigid, leading to the following open problem.

Problem 2.23. Suppose that Assumptions 2.6 and 2.11 hold. Given a fixed noise $\xi$, characterize the potentials $V$ such that $\hat{H}_I$’s eigenvalue point process is rigid in Cases 1 & 2.

A second problem of interest would be to uncover optimal conditions under which the variance of $\text{Tr}[\hat{K}(t)]$ vanishes as $t \to 0$.

Problem 2.24. Suppose that Assumptions 2.6 and 2.11 hold. Given a fixed noise $\xi$, characterize the potentials $V$ such that

$$\lim_{t \to 0} \text{Var}\left[ \text{Tr}[\hat{K}(t)]\right] = 0$$

in Cases 1 & 2.

Owing to Proposition 2.22, the following conjecture concerning Problem 2.24 in the case of white noise seems natural.

Conjecture 2.25. Let $\xi$ be a white noise. Suppose that Assumption 2.11 holds. In Cases 1 & 2, if there exists $\kappa, \nu > 0$ such that $V(x) \leq \kappa|x| + \nu$ for all $x \in I$, then

$$\liminf_{t \to 0} \text{Var}\left[ \text{Tr}[\hat{K}(t)]\right] > 0.$$

3. Point Process and Eigenfunctions

In this section, our purpose is to provide sufficient conditions for the Feynman-Kac semigroups (2.13) to have an infinitesimal generator of the form (2.12). Toward this end, by using the argument in [29, Sections 3.2.1 and 3.2.3] verbatim, we obtain the following condition.

Proposition 3.1 ([29]). Suppose that the following conditions hold.

1. For every $t, t > 0$, almost surely, we have

$$\hat{K}(t; x, y) = \hat{K}(t; y, x) \quad \text{and} \quad \int_I \hat{K}(t; x, z)\hat{K}(\bar{t}; z, y) \, dz = \hat{K}(t + \bar{t}; x, y)$$

for every $x, y \in I$.

2. There exists $0 < T \leq \infty$ such that

$$E[\|\hat{K}(t)\|_2^2] < \infty \quad \text{for every } 0 < t < T.$$

3. For every $f \in L^2$, it holds that

$$\lim_{t \to 0} E[\|\hat{K}(t)f - f\|_2^2] = 0.$$

Then, the conclusion of Proposition 2.15 holds.

Remark 3.2. In the case of pointwise-defined noise, conditions (3.1)–(3.3) are characteristic of classical semigroup theory (in fact, stronger results can be proved). For white noise, (3.1)–(3.3) are proved in [29, Sections 5.6–5.8]; see also [30, 36] for previous results along this direction in the special case $I = (0, \infty)$ and $V(x) = x/2$. 
Our main contribution in this section is to provide a generalization of the statements in [29, Sections 5.6–5.8] that can be applied to more general noises, including Examples 2.8 and 2.9. To this effect, we prove the following restatement of Proposition 2.15.

Proposition 3.3. If (2.11) holds, then (3.1)–(3.3) hold.

The remainder of this section is organized as follows. In Section 3.1, we prove Proposition 3.3. Then, in Section 3.2, we apply the condition (2.11) to Examples 2.7–2.10 to prove Proposition 2.16.

3.1. Proof of Proposition 2.15/3.3. To alleviate notation, we introduce the following shorthands.

Notation 3.4. For every $t > 0$, define the path functionals

\[ \mathfrak{A}_t(Z) := -(L_t(Z), V), \]

\[ \mathfrak{B}_t(Z) := \begin{cases} 
0 & \text{(Case 1)} \\
\hat{\alpha}_t^{0}(X) & \text{(Case 2)} \\
\hat{\alpha}_t^{0}(Y) + \hat{\beta}_t^{0}(Y) & \text{(Case 3).} 
\end{cases} \]

3.1.1. Proof of (3.1). Given that the map

\[ f \mapsto -(f, V) - \xi(f), \quad f \in C_c \]

is linear, the proof of (3.1) follows directly from classical semigroup theory, such as [50, (3.6) and (3.7)] (see also [30, (4.23)] and [36, (5.4) and (5.6)]). Indeed, the symmetry $\hat{K}(t; x, y) = \hat{K}(t; y, x)$ follows from time reversal of Brownian paths and the fact that local time is invariant under this transformation, and the semigroup property $\int_t^s \hat{K}(t; x, y) \hat{K}(s; y, z) \, dz = \hat{K}(t + s; x, y)$ follows from the fact that the local time processes $L_t$ and $\mathcal{L}_t$ are additive.

3.1.2. Proof of (3.2). Owing to (3.1), it follows from Fubini’s theorem that

\[
\mathbb{E} \left[ \int_{\mathbb{R}^2} \hat{K}(t; x, y)^2 \, dx \, dy \right] = \mathbb{E} \left[ \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \hat{K}(t; x, y) \hat{K}(t; y, x) \, dy \right) \, dx \right] = \int_{\mathbb{R}^2} \mathbb{E} \left[ \hat{K}(2t; x, x) \right] \, dx.
\]

By (2.3), we know that $\mathbb{E}[e^{-\xi(f)}] = \xi^2 \|\|f\|\|_2^2$ for all $f \in C_c$. Thus, another application of Fubini’s theorem followed by Hölder’s inequality yields

\[
(3.4) \quad \mathbb{E} \left[ \hat{K}(2t; x, x) \right] = \Pi_Z(2t; x, x) \mathbb{E}_{2t}^{x,x} \left[ e^{3\text{ar}(Z) + 3\text{ar}(Z)} \mathbb{E}_t \left[ e^{-\xi(L_{2t}(Z))} \right] \right]

\leq \Pi_Z(2t; x, x) \mathbb{E}_{2t}^{x,x} \left[ e^{3\text{ar}(Z) + 3\text{ar}(Z)} + \frac{1}{t} \|L_{2t}(Z)\|_2^2 \right] \mathbb{E}_t \left[ e^{\|L_{2t}(Z)\|_2^2} \right]^{1/2},
\]

where $\mathbb{E}_z$ denotes the expectation with respect to $\xi$, conditional on $Z_{2t}^{x,x}$. By [29, Lemma 5.9] with $p = 2$, we have that

\[
\int_{\mathbb{R}^2} \Pi_Z(2t; x, x) \mathbb{E}_{2t}^{x,x} \left[ e^{3\text{ar}(Z) + 3\text{ar}(Z)} \right]^{1/2} \, dx < \infty.
\]

Thus, it suffices to prove that, for small enough $t > 0$,

\[
\sup_{x \in \mathbb{R}^2} \mathbb{E}_{2t}^{x,x} \left[ e^{\|L_{2t}(Z)\|_2^2} \right] < \infty.
\]

(3.5)

As it turns out, (3.5) follows from (2.11). The trick that we use to prove this makes several other appearances in this paper: Since the exponential function is nonnegative, for every $\theta > 0$, it follows from the tower property and the Doob $h$-transform that

\[
(3.6) \quad \mathbb{E}_{2t}^{x,x} \left[ e^{\theta\|L_{2t}(Z)\|_2^2} \right] = \mathbb{E} \left[ \mathbb{E}_{2t}^{x,x} \left[ e^{\theta\|L_{2t}(Z)\|_2^2} \right] Z_{2t}^{x,x}(t) \right] = \int_{\mathbb{R}^2} \mathbb{E}_{2t}^{x,x} \left[ e^{\theta\|L_{2t}(Z)\|_2^2} \right] Z_{2t}^{x,x}(t) \, dy.
\]
If we condition on \( Z^x_{2t}(t) = y \), then the paths \((Z^x_{2t}(s) : t \leq s \leq 2t)\) are independent and have respective distributions \( Z^x_{2t} \) and \( Z^y_{2t} \). Since \( \Pi_Z \) is a symmetric kernel, the time-reversed process \( s \mapsto Z^y_{2t}(t-s) \) (for \( 0 \leq s \leq t \)) is equal in distribution to \( Z^x_{t} \). Since local time is additive, \( E_{2t}^{x,x} \left[ e^{\theta \| L_t(Z) \|^2} | Z^x_{2t}(t) = y \right] \) is equal to 
\[
E_{2t}^{x,x} \left[ e^{\theta \| L_t(Z) \|^2} | Z^x_{2t}(t) = y \right] = E_{t}^{y,x} \left[ e^{\gamma \| L_t(Z) \|^2} \right],
\]
for every \( y \in \mathbb{R} \), which we know is finite thanks to (3.8) and references therein for the precise statement we need. We know that the integral operator \( \mathcal{E} \) is strongly continuous. Thus, to prove (3.10), we argue (see [29, (5.19)]) that, given two independent random variables \( u \) and \( v \), an independent copy \( \bar{u} \) of \( u \), and two functionals \( \Phi \) and \( \Theta \), the following holds
\[
E_v \left[ \Phi(u) \Theta(u, v) \right] = E_v \left[ u \Phi(\bar{u}) \Theta(u, v) \Phi(\bar{u}) \Theta(\bar{u}, v) \right] = E_u \left[ \Phi(u) \Phi(\bar{u}) E_v \left[ \Theta(u, v) \Theta(\bar{u}, v) \right] \right] \leq E_u \left[ \Phi(u) \Theta(u, v)^2 \right]^{1/2},
\]
where the last line is obtained by using Fubini’s theorem followed by the Hölder’s inequality. Applying the inequality in the above display to the expression in (3.10) yields
\[
E \left[ \| \bar{K}(t) f - K(t) f \|_2^2 \right] \leq \int f \bigg[ e^{\gamma \| L_t(Z) \|^2} f(\bar{Z}(t)) \bigg] \bigg( e^{-\eta \| L_t(Z) \|^2} - 1 \bigg) \bigg]^{1/2} dx.
\]
(3.11)
Now, we apply the Hölder’s inequality twice more in the r.h.s. of (3.11) to get
\[
\text{r.h.s. of (3.11)} \leq \int f \bigg[ e^{4\gamma \| L_t(Z) \|^2} \bigg] \bigg[ e^{-\eta \| L_t(Z) \|^2} \bigg]^{1/2} \bigg[ f(\bar{Z}(t)) \bigg]^{1/2} dx.
\]
Letting $t$ to 0 in both sides of the above inequality yields

\begin{equation}
\lim_{t \to 0} \text{r.h.s. of } (3.11) \leq \lim_{t \to 0} \int I \sup_{x \in I} E^x \left[ e^{4 \xi(L_t(Z)) + 4 \xi(L_t(Z))} \right] \times \sup_{x \in I} E^x \left[ e^{\xi(L_t(Z))} \right] dx.
\end{equation}

Since $V$ is bounded below, we can find some $c > 0$ such that $\mathfrak{A}(Z) \leq tc$. Combining this with [29, Lemma 5.6] shows

\begin{equation}
\lim_{t \to 0} \sup_{x \in I} E^x \left[ e^{4 \xi(L_t(Z)) + 4 \xi(L_t(Z))} \right] < \infty.
\end{equation}

By expanding

\begin{equation}
E^x \left[ e^{\xi(L_t(Z))} \right] = e^{\xi(L_t(Z))} - 2e^{\xi(L_t(Z))} + 1,
\end{equation}

it follows from (2.11) that

\begin{equation}
\lim_{t \to 0} \sup_{x \in I} E^x \left[ e^{\xi(L_t(Z))} \right] = 0.
\end{equation}

Finally, note that

\begin{equation}
\lim_{t \to 0} \int I \sup_{x \in I} E^x \left[ f(Z(t)) \right] dx = \| f \|_2^2 < \infty.
\end{equation}

Applying (3.13), (3.15) and (3.16), we get (by the dominated convergence theorem) that the right-hand side of (3.12) vanishes, which concludes the proof of (3.3) and therefore Proposition 3.3.

3.2. Proof of Proposition 2.16. We now prove that (2.11) holds for Examples 2.7–2.10. Our proof of this has two steps.

3.2.1. Step 1. Reduction to $L^p$ Norms. The $L^p$ norms of Brownian local time are well-studied objects (e.g., [16, Section 4.2]). Thus, our first step in the proof of Proposition 2.15 is to show that, in Examples 2.7–2.10, $\| \cdot \|_{\gamma}$ is dominated by a combination of $L^p$ norms.

**Lemma 3.5.** There exists a constant $c > 0$ (which only depends on $\gamma$) such that for every $f \in C_c$,

\begin{equation}
\| f \|_2^2 \leq \begin{cases}
\| f \|_2^2 & (\text{white noise}) \\
ct^H (t^{1/2} \| f \|_2^2 + t^{-1} \| f \|_2^2) & (\text{fractional noise with } H \in (\frac{1}{2}, 1)) \\
ct^p(1/1-1/2p) + \| f \|_2^2 & (\text{bounded noise}).
\end{cases}
\end{equation}

**Proof.** We provide a case-by-case argument. If $\xi$ is a white noise, then up to a constant $\| \cdot \|_{\gamma} = \| \cdot \|_2$, so the result is immediate.

For fractional noise, up to a constant, we have that

\begin{equation}
\| f \|_{\gamma}^2 \leq \int GR \| f(\alpha)\gamma(\alpha - b) f(b) \| db = \int GR \| f(\alpha)\gamma(\alpha - b) f(b) \| db.
\end{equation}

By applying the change of variables $(a, b) \mapsto t^{1/2}(a, b)$ to the right-hand side of this equation, we obtain

\begin{equation}
t \int Guters \| f(t^{1/2}a) f(t^{1/2}b) \| db = t^H \int GR \| f(t^{1/2}a) f(t^{1/2}b) \| db.
\end{equation}

Next, we write

\begin{equation}
\int GR \| f(t^{1/2}a) f(t^{1/2}b) \| db = 2 \int \{1_{\| b-a \| < 1} + \int_{\{1_{\| b-a \| > 1}\}}\} \| f(t^{1/2}a) f(t^{1/2}b) \| db.
\end{equation}
On the one hand, by Young’s convolution inequality (e.g., [52]), the first integral (integral over \{|b - a| < 1\}) in the r.h.s. of (3.18) is bounded above by
\[
\left( \int_{-1}^{1} \frac{1}{|z|^{2q}} \, dz \right) \left( \int_{\mathbb{R}} |f(t \hat{a})|^2 \, da \right) = \left( \int_{-1}^{1} \frac{1}{|z|^{2q}} \, dz \right) t^{1/q} \|f\|_{q}^2,
\]
where the right-hand side comes from the change of variables \(a \mapsto t^{-\frac{1}{q}}a\). On the other hand, by the same change of variables, the second integral (integral over \{|b - a| \geq 1\}) is bounded by
\[
\left( \int_{\mathbb{R}} |f(t \hat{a})| \, da \right)^2 = t^{-1} \|f\|_{q}^2.
\]
Substituting these two bounds in the r.h.s. of (3.18) yields the desired bound on \(\|f\|_{\gamma}^2\) in the case of fractional noise with Hurst parameter \(H \in (\frac{1}{2}, 1)\).

Let \(\xi\) be an \(L^p\)-singular noise with decomposition \(\gamma = \gamma_1 + \gamma_2\). Then, the bound on \(\|f\|_{\gamma}^2\) follows from the following use Young’s inequality,
\[
\int_{\mathbb{R}^2} |f(a) \gamma(a - b) f(b)| \, db \, da = \int_{\mathbb{R}^2} |f(a) \gamma_1(a - b) f(b)| \, db \, da + \int_{\mathbb{R}^2} |f(a) \gamma_2(a - b) f(b)| \, db \, da \leq \|\gamma_1\|_p \|f\|_{q}^2 + \|\gamma_2\|_\infty \|f\|_{q}^2
\]
where \(\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2\), or equivalently, \(q = 1/(1 - \frac{1}{2p})\).

Finally, if \(\gamma\) is bounded, then
\[
\int_{\mathbb{R}^2} |f(a) \gamma(a - b) f(b)| \, db \, da \leq \|\gamma\|_\infty \|f\|_{q}^2\]
concluding the proof.

3.2.2. Step 2. \(L^p\) Norm of Local Time. We now conclude the proof of (2.11). Note that, by (3.17), \(\|L_t(Z)\|_{q}^2\) is always bounded by a combination of norms \(\|L_t(Z)\|_{q}^2\) with \(1 \leq q \leq 2\) (indeed, \((1/(1 - 1/2p)) \in (1, 2)\) for \(p \geq 1\). Recall that, thanks to (2.10), \(\|L_t(Z)\|_1 = t\). Thus, by dominated convergence, it suffices to prove that for every \(q \in (1, 2)\), there exists a nonnegative random variable \(R_q \geq 0\) with finite exponential moments in a neighbourhood of zero such that
\[
\sup_{x \in I} \|L_t(Z^x)\|_{q}^2 \leq t^{1+1/q} R_q \quad \text{for small enough } t > 0.
\]

Let us begin with Case 1 which corresponds to \(I = \mathbb{R}\). If we couple \(B^x = x + B^0\) for all \(x\), then straightforward changes of variables with a Brownian scaling imply that
\[
\|L_t(B^x)\|_{q}^2 = \|L_t(B^0)\|_{q}^2 \overset{d}{=} t \left( \int_{\mathbb{R}} L_{t}^{-1/2-q} L_{t}^{q} \, da \right)^{2/q} = t^{1+1/q} \|L_t(B^0)\|_{q}^2
\]
for every \(q > 1\). According to [16, Theorem 4.2.1], for every \(q > 1\) there exists some \(c > 0\) such that
\[
P\left[\|L_1(B^0)\|_{q}^2 > u\right] = e^{-cu^{1/(1-\alpha(1))}}, \quad u \to \infty.
\]
Thus, in Case 1 we have (3.19) with \(R_q \overset{d}{=} \|L_1(B^0)\|_{q}^2\).

Consider now Case 2 where \(I\) is taken to be \((0, \infty)\). By coupling \(X^x(t) = |B^x(t)|\) for all \(t > 0\), we note that for every \(a > 0\), one has \(L_t^a(X^a) = L_t^a(B^a) = L_t^a(B^x) + L_t^{-a}(B^x)\). Therefore,
\[
\|L_t(X^a)\|_{q}^2 \leq 2^{2(q-1)/q} \left( \int_{0}^{\infty} L_{t}^{a} \, da \right)^{2/q} \leq 2^{2(q-1)/q} \|L_t(B^a)\|_{q}^2 \|L_t^{-a}(B^a)\|_{q}^2.
\]
Thus, the proof in Case 2 follows from the same argument used for Case 1.
Finally, consider Case 3 where \( I \) is an interval \((0, b)\) for some \( b > 0 \). We note that we can couple the processes \( Y^x \) and \( B^z \) in such a way that \( Y^x \) is obtained by reflecting the path of \( B^z \) on the boundary of \((0, b)\), namely,

\[
Y^x(t) = \begin{cases} 
B^z(t) - 2kb & \text{if } B^z(t) \in [2kb, (2k + 1)b], \; k \in \mathbb{Z}, \\
|B^z(t) - 2kb| & \text{if } B^z(t) \in [(2k - 1)b, 2kb], \; k \in \mathbb{Z}.
\end{cases}
\]

Under this coupling, we observe that for any \( z \in (0, b) \), one has

\[
L_t^z(Y^x) = \sum_{a \in \mathbb{Z} \pm b} L_t^a(B^z).
\]

The argument that follows is a minor extension of [29, Lemma 5.7], which itself is inspired from the proof of [18, Lemma 2.1]: Under (3.22),

\[
\left( \int_0^b L_t^z(Y^x)^q \, dz \right)^{1/q} = \left( \int_0^b \left( \sum_{k \in \mathbb{Z}} L_t^{k+z}(B^z) + L_t^{k-z}(B^z) \right)^q \, dz \right)^{1/q} 
\]

\[
\leq 2^{(q-1)/q} \sum_{k \in \mathbb{Z}} \left( \int_{-b}^b L_t^{k+z}(B^z)^q \, dz \right)^{1/q}.
\]

Let us denote the maximum and minimum of \( B^z \) as

\[
M^z(t) := \sup_{s \in [0, t]} B^z(s) \quad \text{and} \quad m^z(t) := \inf_{s \in [0, t]} B^z(s).
\]

In order for \( \int_{-b}^b L_t^{k+z}(B^z)^2 \, dz \) to be nonzero, it must be the case that \( M^z(t) \geq k + b \) and \( m^z(t) \leq k - b \), or, equivalently, \( M^z(t) + b \geq k \geq m^z(t) - b \). Thus, for any \( q > 1 \),

\[
\sum_{k \in \mathbb{Z}} \left( \int_{-b}^b L_t^{k+z}(B^z)^q \, dz \right)^{1/q} 
\]

\[
= \sum_{k \in \mathbb{Z}} \left( \int_{-b}^b L_t^{k+z}(B^z)^q \, dz \right)^{1/q} 1_{\{M^z(t) + b \geq k \geq m^z(t) - b\}} 
\]

\[
\leq \left( \sum_{k \in \mathbb{Z}} \int_{-b}^b L_t^{k+z}(B^z)^q \, dz \right)^{1/q} \left( \sum_{k \in \mathbb{Z}} 1_{\{M^z(t) + b \geq k \geq m^z(t) - b\}} \right)^{1-1/q} 
\]

\[
= \left( \int_{\mathbb{R}} L_t^q(B^z)^q \, da \right)^{1/q} \left( \sum_{k \in \mathbb{Z}} 1_{\{M^z(t) + b \geq k \geq m^z(t) - b\}} \right)^{1-1/q} 
\]

\[
\leq c_1 t^{1/q} \left( \sup_{a \in \mathbb{R}} L_t^q(B^a) \right)^{1-1/q} \left( M^z(t) - m^z(t) + c_2 \right)^{1-1/q} 
\]

\[
\leq c_1 t^{1/q} \left( c_2^{-1/q} \left( \sup_{a \in \mathbb{R}} L_t^q(B^a) \right)^{1-1/q} + \left( \sup_{a \in \mathbb{R}} L_t^q(B^a) \cdot (M^z(t) - m^z(t)) \right)^{1-1/q} \right)
\]

where \( c_1, c_2 > 0 \) only depend on \( b \) and \( q \). Indeed, the inequality in the third line follows by Hölder’s inequality, the equality in the fourth line is obtained by noting that \( \sum_{k \in \mathbb{Z}} \int_{-b}^b L_t^{k+z}(B^z)^q \, da \) is equal to \( \int_{\mathbb{R}} L_t^q(B^z)^q \, da \), we get the inequality in the fifth line by noting that \( \int_{\mathbb{R}} L_t^q(B^z)^q \, da \) is bounded by \( \sup_{a \in \mathbb{R}} L_t^q(B^a) \| L_t(B^a) \|_1 \) where \( \| L_t(B^a) \|_1 = t \), and the inequality in the last line follows by bounding \( (M^z(t) - m^z(t) + c_2)^{1-1/q} \) by \( (M^z(t) - m^z(t))^{1-1/q} + c_2^{-1/q} \).

Given that the distributions of the supremum of local time of \( B^z \) and the range \( M^z(t) - m^z(t) \) are independent of the starting point \( x \), by Brownian scaling, we have that

\[
t^{1/q} \left( \sup_{a \in \mathbb{R}} L_t^q(B^a) \right)^{1-1/q} \leq t^{1/2+1/2q} \left( \sup_{a \in \mathbb{R}} L_t^q(B^0) \right)^{1-1/q}
\]
and
\[ t^{1/q} \left( \sup_{a \in \mathbb{R}} L_t^a(B^z) \cdot (M^2(t) - m^2(t)) \right)^{\frac{d}{2}} \leq t \left( \sup_{a \in \mathbb{R}} L_t^a(B^0) \cdot (M^0(1) - m^0(1)) \right)^{\frac{d}{2}}. \]

Note that there exists \( \theta_0 > 0 \) small enough so that
\[ (3.25) \quad \mathsf{E} \left[ \exp \left( \theta_0 \sup_{a \in \mathbb{R}} L_t^a(B^0)^2 \right) \right], \mathsf{E} \left[ e^{\theta_0 \Lambda^0(1) - m^0(1)} \right] < \infty \]
(e.g., the proof of [18, Lemma 2.1] and references therein). Given that \( 4(1 - 1/q) \leq 2 \), for \( q \in (1, 2] \), we then conclude that \( (3.19) \) holds with
\[ R_q \equiv \left( \sup_{a \in \mathbb{R}} L_t^a(B^0) \right)^{2(1-1/q)} + \left( 2 \sup_{a \in \mathbb{R}} L_t^a(B^0)^2 + 2(M^0(1) - m^0(1))^2 \right)^{2(1-1/q)}. \]

4. Variance Estimates

In this section, we provide the main technical contributions of this paper. The chief result in this direction consists of the following variance upper bounds for the trace of \( \hat K(t) \) as \( t \to 0 \).

**Theorem 4.1.** Suppose that \( (2.11) \) and \( (2.13) \) hold. In Cases 1 & 2, assume that there exists \( \kappa, \nu, a > 0 \) such that
\[ (4.1) \quad V(x) \geq |\kappa x|^a - \nu \quad \text{for every } x \in I. \]

In Cases 1 & 2, there exists a constant \( C_0 > 0 \) that only depends on \( a \) such that
\[ (4.2) \quad \text{Var}[\text{Tr}\{\hat K(t)\}] \leq \begin{cases} C_0 e^{2\nu t} & (\text{if } \gamma \text{ is compactly supported}) \\ C_0 e^{2\nu t} & (\text{otherwise}) \end{cases} \quad t^{\beta - 1} \]
for every small enough \( t > 0 \). In Case 3, there exists \( C > 0 \) such that
\[ (4.3) \quad \text{Var}[\text{Tr}\{\hat K(t)\}] \leq C t^{\beta - 1} \]
for small enough \( t > 0 \).

The remainder of this section is organized as follows. The proof of Theorem 4.1 is separated between Sections 4.1 to 4.4. In Section 4.5, we prove Theorem 2.19. Then, in Section 4.6, we prove Theorem 2.20. Before we go on with these proofs, we introduce some notational shortcuts used throughout this section to improve readability.

**Notation 4.2.** For the remainder of Section 4, we use \( C, c > 0 \) to denote constants independent of \( \kappa, \nu, \) and \( a \) whose precise values may change from one equation to the next, and we use \( C_0 > 0 \) to denote such constants that depend on \( a \).

**Notation 4.3.** Let \( Z \) be as in \( (2.8) \), and let \( \tilde Z \) be an independent copy of \( Z \). For every \( t > 0 \), we define the following random functions: for \( (x, y) \in I^2 \),
\[ A_t(x, y) := -\langle L_t(Z_t^{x,x}) + L_t(Z_t^{y,y}), V \rangle, \]
\[ B_t(x, y) := \begin{cases} 0 & (\text{Case 1}) \\ \alpha L_t^0(X_t^{x,x}) + \alpha L_t^0(\tilde Y_t^{y,y}) & (\text{Case 2}) \\ \beta L_t^0(Y_t^{x,x}) + \beta L_t^0(\tilde Y_t^{y,y}) + \alpha L_t^0(\tilde Y_t^{y,y}) + \beta L_t^0(Y_t^{y,y}) & (\text{Case 3}) \end{cases}, \]
\[ C_t(x, y) := \frac{1}{2} \left[ \|L_t^t(Z_t^{x,x})\|^2 + \|L_t^t(Z_t^{y,y})\|^2 \right], \]
\[ D_t(x, y) := \langle L_t(Z_t^{x,x}), L_t(Z_t^{y,y}) \rangle, \]
\[ P_t(x, y) := \Pi_Z(t; x, x) \Pi_Z(t, y, y). \]
4.1. **Step 1. Variance Formula.** The first step in the proof of Theorem 4.1 is the following variance formula, which follows more or less directly from the definition of the Feynman-Kac semigroups in Definition 2.13.

**Lemma 4.4.** Following Notation 4.3, it holds that

\[
\text{Var}[\text{Tr}[\hat{K}(t)]] = \int_{\mathbb{R}^2} \mathcal{P}_t(x,y) \mathbb{E} \left[e^{-(L_t(B^{x,y} + L_t(B^{y,x})))} \mathbb{E}_{\xi} \left[e^{-\xi(L_t(B^{x,y}))} \right]ight] \, dx \, dy.
\]

**Proof.** We only prove the result in **Case 1**, since the other cases follow from exactly the same argument. By Proposition 2.15-(3), and a similar computation to (3.4),

\[
\mathbb{E} \left[\text{Tr}[\hat{K}(t)]\right] = \int_{\mathbb{R}} \Pi_B(t;x,x) \mathbb{E}_t^{x,x} \left[e^{-\langle L_t(B^{x,y}) \rangle} \mathbb{E}_{\xi_t} \left[e^{-\xi(L_t(B^{x,y}))} \right]ight] \, dx,
\]

Via another application of Tonelli, we get

\[
\left(\mathbb{E} \left[\text{Tr}[\hat{K}(t)]\right]\right)^2 = \int_{\mathbb{R}^2} \mathcal{P}_t(x,y) \mathbb{E} \left[e^{-(L_t(B^{x,y} + L_t(B^{y,x})))} \mathbb{E}_{\xi} \left[e^{-\xi(L_t(B^{x,y}))} \right]ight] \, dx \, dy.
\]

4.2. **Step 2. Uniformly Bounded Terms.** By applying Hölder’s inequality to (4.4), we obtain the upper bound

\[
\text{Var}[\text{Tr}[\hat{K}(t)]] \leq \int_{\mathbb{R}^2} \mathcal{P}_t(x,y) \mathbb{E} \left[e^{4A_t(x,y)} \right]^{1/4} \mathbb{E} \left[e^{4B_t(x,y)} \right]^{1/4} \mathbb{E} \left[e^{4C_t(x,y)} \right]^{1/4} \mathbb{E} \left[e^{4D_t(x,y)} - 1 \right]^{1/4} \, dx \, dy.
\]

The second step in the proof of Theorem 4.1 is to show that the terms involving \(B_t(x,y)\) and \(C_t(x,y)\) in (4.6) are uniformly bounded for small \(t\).

**Lemma 4.5.** For any \(\theta > 0\),

\[
\lim_{t \to 0} \sup_{(x,y) \in \mathbb{R}^2} \mathbb{E} \left[e^{\theta B_t(x,y)} \right] < \infty,
\]

\[
\lim_{t \to 0} \sup_{(x,y) \in \mathbb{R}^2} \mathbb{E} \left[e^{\theta C_t(x,y)} \right] < \infty.
\]

**Proof.** We begin with (4.8). By Independence,

\[
\mathbb{E} \left[e^{\theta C_t(x,y)} \right] = \mathbb{E}_t^{x,x} \left[e^{\frac{\theta}{2} \|L_t(Z)\|_{L_2}^2} \right] \mathbb{E}_t^{y,y} \left[e^{\frac{\theta}{2} \|L_t(Z)\|_{L_2}^2} \right]
\]

Thus, by using the argument leading up to (3.9), it suffices to prove that

\[
\lim_{t \to 0} \sup_{x \in I} \mathbb{E}^{x} \left[e^{\theta \|L_t(Z)\|_{L_2}^2} \right] < \infty
\]
for all \( \theta > 0 \), which is given by assumption (2.11).

We now prove (4.7). In Case 1 the result is trivial. In Case 2, by using essentially the same argument leading up to (3.9), we have that

\[
E^x \left[ e^{\theta \mathcal{L}_t^Y(X)} \right] \leq C \cdot E^x \left[ e^{2\theta^2 \mathcal{L}_t^Y(X)} \right].
\]

By coupling \( X^a(s) = |B^a(s)| \) for all \( s \geq 0 \), this yields

\[
E^a \left[ e^{\theta \mathcal{L}_t^Y(X)} \right] \leq C \cdot E^a \left[ e^{2\theta^2 \mathcal{L}_t^Y(B)} \right],
\]

where we define

\[
\mathcal{L}_t^a(B) := \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^t 1_{\{a_\epsilon < B(s) < a + \epsilon\}} \, ds = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^t 1_{\{a_\epsilon < |B(s)| < a + \epsilon\}} \, ds.
\]

for any \( a \in \mathbb{R} \). Thus, by a straightforward application of Hölder’s inequality, it suffices to prove that

\[
\limsup_{t \to 0} \sup_{x \in \mathbb{R}} E^x \left[ e^{\theta \mathcal{L}_t^B(B)} \right] < \infty.
\]

By Brownian scaling, \( \mathcal{L}_t^a(B^x) \overset{d}{=} t^{1/2} \mathcal{L}_t^a(B^{x-1/2}) \). Using the same argument as in the proof of [29, Lemma 5.6], we have that

\[
\sup_{x \in \mathbb{R}} E^x \left[ e^{\theta t^{1/2} \mathcal{L}_t^a(B)} \right] = E^0 \left[ e^{\theta t^{1/2} \mathcal{L}_t^a(B)} \right] < \infty,
\]

and thus the result follows from dominated convergence.

Consider now Case 3. Once again arguing as in (3.9), it suffices to prove that

\[
\limsup_{t \to 0} \sup_{x \in (0, b)} E^x \left[ e^{\theta \mathcal{L}_t^Y(Y)} \right] < \infty, \quad c \in \{0, b\}.
\]

Recall the coupling of \( Y \) and \( B \) in (3.21). Under this coupling, we observe that

\[
\mathcal{L}_t^c(Y^Z) = \left\{ \sum_{a \in 2\mathbb{Z}} \mathcal{L}_t^a(B^Z) \quad (c = 0) \right\} \quad \left\{ \sum_{a \in (2\mathbb{Z} + 1)} \mathcal{L}_t^a(B^Z) \quad (c = b) \right\}
\]

Consider the case \( c = 0 \). According to (4.10), we see that

\[
\mathcal{L}_t^0(Y^Z) \leq \sup_{a \in \mathbb{R}} \mathcal{L}_t^a(B^Z) \cdot n_t,
\]

where \( n_t \) counts the number of intervals of the form \([kb, (k + 1)b] \; (k \in \mathbb{Z}) \) such that

\[
\inf_{kb \leq a \leq (k + 1)b} \mathcal{L}_t^a(B^Z) > 0.
\]

It is easy to see that there exists constants \( c_1, c_2 > 0 \) that only depend on \( b \) such that for every \( t > 0 \), one has \( n_t \leq c_1 \left( M^Z(t) - m^Z(t) + c_2 \right) \), where we denote \( M^Z \) and \( m^Z \) as in (3.24). By Brownian scaling,

\[
\sup_{a \in \mathbb{R}} \mathcal{L}_t^a(B^Z) \overset{d}{=} t^{1/2} \sup_{a \in \mathbb{R}} \mathcal{L}_t^a(B^0),
\]

and

\[
\left( \sup_{a \in \mathbb{R}} \mathcal{L}_t^a(B^Z) \right) \left( M^Z(t) - m^Z(t) \right) \overset{d}{=} t \left( \sup_{a \in \mathbb{R}} \mathcal{L}_t^a(B^0) \right) \left( M^0(1) - m^0(1) \right).
\]

By combining the fact that these terms are independent of \( x \) with (3.25), we obtain (4.9) for \( c = 0 \). The proof for \( c = b \) is nearly identical. \( \square \)
4.3. **Step 3. Vanishing Term.** By combining (4.6) with Lemma 4.5 and (B.1), we obtain the following bound for small enough $t > 0$.

\[(4.12) \quad \text{Var}[\text{Tr}[K(t)]] \leq Ct^{-1} \int_{\mathbb{R}^2} E \left[ e^{4A_t(x,y)} \right]^{1/4} E \left[ \left( e^{D_t(x,y)} - 1 \right)^4 \right]^{1/4} \, dx\,dy.\]

The third step in our proof of Theorem 4.1 is to study the decay rate of $e^{D_t(x,y)} - 1$ as $t \to 0$. To this effect, we have the following lemma.

**Lemma 4.6.** Let $\theta$ be as in (2.13). For any $\theta > 0$,

\[
\sup_{(x,y) \in \mathbb{R}^2} \left( E \left[ e^{D_t(x,y)} - 1 \right] \right)^{1/\theta} \leq Ct^3 \quad \text{for small enough } t > 0.
\]

**Proof.** By combining the inequality $|e^z - 1| \leq e^{|z|} - 1 \leq |z|e^{|z|} (z \in \mathbb{R})$ with

\[|D_t(x, y)| \leq \frac{1}{2} \left( \|L_t(Z_t^{x, x})\|_2^2 + \|L_t(Z_t^{y, y})\|_2^2 \right),\]

and applying the triangle inequality, we see that

\[
\left( E \left[ e^{D_t(x,y)} - 1 \right] \right)^{1/\theta} \leq C \left( E \left[ \|L_t(Z_t^{x, x})\|_2^{2\theta} e^{\theta/2(\|L_t(Z_t^{x, x})\|_2^2 + \|L_t(Z_t^{y, y})\|_2^2)} \right]^{1/\theta}
\]

\[+ E \left[ \|L_t(Z_t^{y, y})\|_2^{2\theta} e^{\theta/2(\|L_t(Z_t^{x, x})\|_2^2 + \|L_t(Z_t^{y, y})\|_2^2)} \right]^{1/\theta} \right).
\]

By using independence of $Z$ and $\bar{Z}$ and applying Hölder’s inequality, the right hand side of the above inequality is bounded by

\[
C \left( E_t^{x,x} \left[ \|L_t(Z_t)\|_2 \right] \right)^{1/20} E_t^{y,y} \left[ e^{\theta/2\|L_t(Z_t)\|_2^2} \right]^{1/20} \left( E_t^{y,y} \left[ \|L_t(Z_t)\|_2 \right] \right)^{1/20}
\]

\[+ E_t^{x,x} \left[ \|L_t(Z_t)\|_2 \right]^{1/20} E_t^{y,y} \left[ e^{\theta/2\|L_t(Z_t)\|_2^2} \right]^{1/20} \left( E_t^{x,x} \left[ \|L_t(Z_t)\|_2 \right] \right)^{1/20} \right).
\]

Thanks to (3.9), it follows from (2.11) that for every $\theta > 0$,

\[\sup_{x \in \mathbb{R}} E_t^{x} \left[ e^{\theta\|L_t(Z_t)\|_2^2} \right] \leq C
\]

for small enough $t > 0$. At this point, to complete the proof of Lemma 4.6, it suffices to show that for every $\theta > 0$,

\[(4.13) \quad \limsup_{t \to 0} t^{-3} \left( \sup_{x \in \mathbb{R}} E_t^{x} \left[ \|L_t(Z_t)\|_2^{2\theta} \right]^{1/\theta} \right) < \infty.
\]

We claim that (4.13) is a consequence of (2.13). To see this, we once again condition on the midpoint of the process $Z_t^{x,x}$: with $g(Z) < \infty$ as in (3.8), we obtain that for any $t \in (0, 1]$,

\[
E_t^{x,x} \left[ \|L_t(Z_t)\|_2^{2\theta} \right] = \int E_t^{x,x} \left[ \|L_t(Z_t)\|_2^{2\theta} Z_t^{x,x}(t/2) = z \right] \Pi_Z(t/2; x, z) \Pi_Z(t/2; z, x) \, dz
\]

\[\leq g(Z) \int E_t^{x,x} \left[ \|L_t/2(Z_t) + L_{t/2,t}(Z_t)\|_2^{2\theta} Z_t^{x,x}(t/2) = z \right] \Pi_Z(t/2; x, z) \, dz
\]

\[\leq 2\max(20,1) g(Z) \int E_t^{x,x} \left[ \|L_t/2(Z_t)\|_2^{2\theta} \Pi_Z(t/2; t, x) \right] \, dz
\]

\[= C E_t^{x} \left[ \|L_t/2(Z_t)\|_2^{2\theta} \right].
\]

where the equality in the second line follows from the Doob h-transform (see (3.6)), the inequality in the second line follows from first applying Minkowski’s inequality to bound $\|L_t/2(Z_t) + L_{t/2,t}(Z_t)\|_2^{2\theta}$ by $2\max(20,1) (\|L_t/2(Z_t)\|_2^{2\theta} + \|L_{t/2,t}(Z_t)\|_2^{2\theta})$, and then using the fact that, under the conditioning $Z_t^{x,x}(t/2) = z$, the local time processes $L_t/2(Z_t^{x,x})$ and
\( L_{|t/2|,t}(Z_{t/2}^{x,z}) \) are i.i.d. copies of \( L_{t/2}'(Z_{t/2}^{x,z}) \). (We refer back to the passage following (3.6) for more details.)

### 4.4. Step 4. Final Estimates

We now wrap up the proof of Theorem 4.1. We argue the result case by case.

We begin with Case 3, as it is the simplest. By Assumption 2.11, \( V \) is bounded below; hence there exists some \( c \geq 0 \) such that \( V(x) \geq -c \) for every \( x \). Therefore, \( e^{A_t(x,y)} \leq e^{2ct} \).

By (4.12), we then have

\[
\text{Var} [ \Tr[K(t)] ] \leq C e^{2ct} t^{-1} \int_{(0,b)^2} \mathbb{E} \left[ \left( e^{D_t(x,y)} - 1 \right)^4 \right]^{1/4} \, dz \, dy,
\]

which proves (4.3) by Lemma 4.6.

Now, we analyze Case 1. If we couple \( B_t^{x,z} := x + B_t^{0,0} \) and similarly for \( B_t^{y,u} \), then it follows from (4.1) that

\[
(4.14) \quad A_t(x,y) \leq 2\nu t - \kappa a \int_0^t \left( |x + B_t^{0,0}(s)|^4 + |y + B_t^{0,0}(s)|^4 \right) \, ds.
\]

Then, by the change of variables \( s \to st \) and a Brownian scaling, we obtain

\[
\text{r.h.s. of (4.14)} = 2\nu t - \kappa a \int_0^t \left( |t^{1/4}x + t^{1/4}B_t^{0,0}(st)|^4 + |t^{1/4}y + t^{1/4}B_t^{0,0}(st)|^4 \right) \, ds
\]

\[= 2\nu t - \kappa a \int_0^t \left( |t^{1/4}x + t^{1/4}B_t^{0,0}(s)|^4 + |t^{1/4}y + t^{1/4}B_t^{0,0}(s)|^4 \right) \, ds.
\]

Let us introduce the following shorthands

\[
\mathcal{B}_{t,x}(s) := |t^{1/4}x + t^{1/4}B_t^{0,0}(s)|^4, \quad \mathcal{B}_{t,y}(s) := |t^{1/4}y + t^{1/4}B_t^{0,0}(s)|^4.
\]

Consider first the case of general \( \gamma \) (i.e., not necessarily compactly supported). If we combine Lemma 4.6 and (4.12) with the bound on \( A_t(x,y) \) from the above display, then we obtain that, for small \( t > 0 \),

\[
(4.15) \quad \text{Var} [ \Tr[K(t)] ] \leq C e^{2ct} t^{3 - 2/a} \int_{\mathbb{R}^2} \mathbb{E} \left[ e^{-\kappa a} f_0^{1/4} \left( \mathcal{B}_{t,x}(s) + \mathcal{B}_{t,y}(s) \right) \, ds \right] \, dx \, dy
\]

\[= C e^{2ct} t^{3 - 2/a} \int_{\mathbb{R}^2} \mathbb{E} \left[ e^{-\kappa a} f_0^{1/4} \left( \mathcal{B}_{t,1/2,s}(s) + \mathcal{B}_{t,1/2,s}(s) \right) \, ds \right] \, dx \, dy
\]

where in the second line we applied the change of variables \( (x,y) \to t^{-1/a}(x,y) \) to get the right-hand side of (4.15). To alleviate notation, let us henceforth write

\[
(4.16) \quad \mathcal{F}_t(x,y) := e^{-\kappa a} f_0^{1/4} \left( \mathcal{B}_{t,1/2,s}(s) + \mathcal{B}_{t,1/2,s}(s) \right) \, ds
\]

noting that the dependence of \( a \) and \( \kappa \) are implicit in this notation. For every fixed \( x,y \in \mathbb{R} \),

\[
(4.17) \quad \lim_{t \to \infty} \mathcal{F}_t(x,y) = e^{-|\kappa x|^2 - |\kappa y|^2}
\]

almost surely. Moreover, for every \( z, \bar{z} \in \mathbb{R} \),

\[
|z + \bar{z}|^4 \geq |z + \bar{z}|_{\min{\{a,1\}}} - 1 \geq |z|_{\min{\{a,1\}}} - |\bar{z}|_{\min{\{a,1\}}} - 1,
\]

and therefore

\[
(4.18) \quad \sup_{t \in (0,1]} \mathcal{F}_t(x,y) \leq \exp \left( -|\kappa x|_{\min{\{a,1\}}} - |\kappa y|_{\min{\{a,1\}}} \right)
\]

\[\times \exp \left( \kappa_{\min{\{a,1\}}} (2 + \sup_{s \in [0,1]} |B_t^{0,0}(s)|_{\min{\{a,1\}}} + \sup_{s \in [0,1]} |\mathcal{B}_t^{0,0}(s)|_{\min{\{a,1\}}} \right).
\]

We recall that the process \( s \to |B_t^{0,0}(s)| \) is a Bessel bridge of dimension one (e.g., [48, Chapter XI]). Thanks to the tail asymptotic in [37, Remark 3.1] (the Bessel bridge is denoted by \( \varphi \) in that paper), we know that Bessel bridge maxima have finite exponential moments of all
orders. Therefore, since the function \(\exp(-|\kappa x|_{\min[a,1]} - |\kappa y|_{\min[a,1]})\) is integrable on \(\mathbb{R}^2\), it follows from the dominated convergence theorem that

\[
\lim_{t \to 0} \int_{\mathbb{R}^2} E[\mathcal{F}_t(x, y)] \, dx \, dy = \int_{\mathbb{R}^2} e^{-|\kappa x|^a - |\kappa y|^a} \, dx \, dy = \left( \frac{2\Gamma(1 + 1/a)}{\kappa} \right)^2.
\]

Combining this limit with (4.15) yields the upper bound

\[
\text{Var}[\text{Tr}[\mathcal{K}(t)]] \leq C_0 e^{2\rho t} \kappa^{2} t^{3-2/a} \quad \text{for small } t > 0
\]

for general \(\gamma\).

Consider now Case 1 when \(\gamma\) is compactly supported. Then, there exists some \(K > 0\) such that \((f, \gamma) = 0\) whenever \(f(z) = 0\) for every \(z \in [-K, K]\). In this situation, in order for the quantity \(D_t(x, y) = (L_t(B^x_t), L_t(B^y_t))_\gamma\) to be nonzero, it must be the case that

\[
\begin{align*}
&\max_{0 \leq s \leq t} B^x_{t,s}(s) + K \geq \min_{0 \leq s \leq t} B^y_{t,s}(s) \quad \text{if } x \leq y, \\
&\max_{0 \leq s \leq t} B^y_{t,s}(s) + K \geq \min_{0 \leq s \leq t} B^x_{t,s}(s) \quad \text{if } x \geq y.
\end{align*}
\]

Looking at the case where \(x \leq y\), this means that

\[
E \left[ \left( e^{D_t(x,y)} - 1 \right)^4 \right]^{1/4} \leq \mathbb{P} \left[ \max_{0 \leq s \leq t} B^x_{t,s}(s) + K \geq \min_{0 \leq s \leq t} B^y_{t,s}(s) \right]^{1/8} \leq C t^B \mathbb{P} \left[ \max_{0 \leq s \leq t} B^x_{t,s}(s) + K \geq \min_{0 \leq s \leq t} B^y_{t,s}(s) \right]^{1/8},
\]

where the inequality in the third line follows by applying Hölder’s inequality and the last inequality is from Lemma 4.6. If we apply a Brownian scaling and use the fact that the maxima of brownian bridges have sub-Gaussian tails, then

\[
\mathbb{P} \left[ \max_{0 \leq s \leq t} B^x_{t,s}(s) + K \geq \min_{0 \leq s \leq t} B^y_{t,s}(s) \right]^{1/8} = \mathbb{P} \left[ \max_{0 \leq s \leq 1} B^x_{1,s}(s) + \max_{0 \leq s \leq 1} B^y_{1,s}(s) \geq (y - x - K)/t^{1/2} \right]^{1/8} \leq C e^{-(y - x - K)^2/2ct}.
\]

A similar bound is obtained when \(x \geq y\). Therefore, by a minor modification of the argument leading up to (4.15), we obtain that, for small \(t > 0\),

\[
\text{Var}[\text{Tr}[\mathcal{K}(t)]] \leq C e^{2\rho t} t^{3-2/a} \int_{\mathbb{R}^2} E \left[ e^{-\kappa^a \int_{\mathbb{R}^2} (\mathcal{R}_t(s) + \mathcal{R}_g(s)) \, ds} \right] e^{-\frac{(x-y)^2}{2ct^{1+2/a}}} \, dx \, dy.
\]

By the change of variables \((x, y) \mapsto t^{-1/a}(x, y)\), the integral in the above display is equal to

\[
\int_{\mathbb{R}^2} E[\mathcal{F}_t(x, y)] e^{-\frac{(x-y)^2}{2ct^{1+2/a}}} \, dx \, dy = \frac{1}{\sqrt{2\pi c}} \cdot t^{1/2-1/a} \int_{\mathbb{R}^2} E[\mathcal{F}_t(x, y)] e^{-\frac{(|x-y| - t^{1/a}K)^2}{2ct^{1+2/a}}} \, dx \, dy,
\]

where we recall that \(\mathcal{F}_t\) is defined as in (4.16). Owing to the inequality \((|x-y| - t^{1/a}K)^2 \geq \min\{|x-y - t^{1/a}K|^2, (x-y + t^{1/a}K)^2\}\), we have

\[
e^{-\frac{(|x-y| - t^{1/a}K)^2}{2ct^{1+2/a}}} \leq e^{-\frac{(x-y - t^{1/a}K)^2}{2ct^{1+2/a}}} + e^{-\frac{(x-y + t^{1/a}K)^2}{2ct^{1+2/a}}}
\]
which yields
\[
\frac{e^{-((x-y)-|t^{1/4}K|)^2/2\sigma t^{1+2/\alpha}}}{\sqrt{2\pi t^{1+2/\alpha}}} \leq \mathcal{G}_{t^{1+2/\alpha}}(x-y-t^{1/4}K) + \mathcal{G}_{t^{1+2/\alpha}}(x-y+t^{1/4}K),
\]
where we recall that $\mathcal{G}_t$ denotes the Gaussian kernel (2.9). Combining this with (4.18) and substituting into (4.20) shows that $\text{Var} [\text{Tr} [\hat{K}(t)]]$ is bounded above by
\[
C_a e^{2\sigma t} t^{-1/2-1/\alpha} \left( \int_{\mathbb{R}^2} e^{-|x|^{\min\{a,1\}} - |y|^{\min\{a,1\}}} \mathcal{G}_{t^{1+2/\alpha}}(x-y-t^{1/4}K) \, dx \, dy 
+ \int_{\mathbb{R}^2} e^{-|x|^{\min\{a,1\}} - |y|^{\min\{a,1\}}} \mathcal{G}_{t^{1+2/\alpha}}(x-y+t^{1/4}K) \, dx \, dy \right).
\]
Owing to a change of variables and the fact that the Gaussian kernel is an approximate identity, the integrals above have the following limits by dominated convergence
\[
\lim_{t \to 0} \int_{\mathbb{R}^2} e^{-|x|^{\min\{a,1\}}} \left( \int_{\mathbb{R}} e^{-|y|^{\min\{a,1\}}} \mathcal{G}_{t^{1+2/\alpha}}(x-y) \, dy \right) \, dx = \int_{\mathbb{R}} e^{-2|\kappa x|^{\min\{a,1\}}} \, dx = \frac{2^{1-1/\min\{a,1\}} \Gamma(1+1/\min\{a,1\})}{\kappa}.
\]
Combining this result with (4.20) yields
\[
\text{Var} [\text{Tr} [\hat{K}(t)]] \leq \frac{C_a e^{2\sigma t}}{\kappa} t^{-1/2-1/\alpha} \quad \text{for small } t > 0
\]
whenever $\gamma$ is compactly supported. This completes the proof of (4.2) for Case 1.

We now finish the proof of Theorem 4.1 by dealing with Case 2. Once again we begin with general $\gamma$. Since $V(x) \geq |x|^a - \nu$,
\[
E \left[ e^{\lambda_k(x,y)} \right]^{1/4} \leq e^{2\sigma t} E \left[ e^{-4\kappa \int_0^t \left( |X^{x,y}(s)|^a + |\dot{X}^{x,y}(s)|^a \right) \, ds} \right]^{1/4}.
\]
Note that we can couple $X$ and $B$ so that $X^x(t) = |B^x(t)|$ for all $t \geq 0$. Then, conditioning on the endpoint corresponds to
\[
X^x_t = \{ |B^x| \, |B^x(t) \in \{ x, -x \} \}.
\]
Using this coupling, by arguing as in [29, Section 5.2.1], it can be shown that for any nonnegative path functional $F$,
\[
E[F(X^x_t)] \leq 2E[F(|B^x_t|)].
\]
Thus, we have that
\[
E \left[ e^{4\lambda_k(x,y)} \right]^{1/4} \leq e^{2\sigma t} E \left[ e^{-4\kappa \int_0^t \left( |x+B^0_0(s)|^a + |x+B^0_0(s)|^a \right) \, ds} \right]^{1/4}.
\]
Combining this with Lemma 4.6, we obtain (4.19) in Case 2 for general $\gamma$ by using the same argument as in Case 1.

Finally, consider Case 2 when $\gamma$ is compactly supported. Similarly to Case 1, by Hölder’s inequality we have the upper bound
\[
E \left[ e^{4\lambda_k(x,y)} \right]^{1/4} \leq P \left( \langle L_t(X^x_t), L_t(X^{x,y}_t) \rangle \gamma \neq 0 \right)^{1/8} E \left[ e^{8\lambda_k(x,y)} \right]^{1/8}.
\]
By (4.22), we get the further upper bound
\[
P \left( \langle L_t(X^x_t), L_t(X^{x,y}_t) \rangle \gamma \neq 0 \right)^{1/8} \leq 2^{1/8} P \left( \langle L_t(|B^x_t|), L_t(|B^{x,y}_t|) \rangle \gamma \neq 0 \right)^{1/8}.
\]
Given that \( L^2_t(|B^{t,x}_t|) = L^2_t(B^{t,x}_t) + L^{-a}_t(B^{t,x}_t) \) for all \( a > 0 \) and similarly for \( \tilde{B}^{y,y}_t \), we can expand \( (L_t(|B^{t,x}_t|), L_t(|\tilde{B}^{y,y}_t|))\), as the sum

\[
\int_{(0,\infty)^2} L^a_t(B^{t,x}_t) \gamma(a-b) L^a_t(B^{y,y}_t) \, db + \int_{(0,\infty)^2} L^{-a}_t(B^{t,x}_t) \gamma(a-b) L^{-b}_t(B^{y,y}_t) \, db
\]

\[
+ \int_{(0,\infty)^2} L^{-a}_t(B^{t,x}_t) \gamma(a-b) L^b_t(B^{y,y}_t) \, db + \int_{(0,\infty)^2} L^a_t(B^{t,x}_t) \gamma(a-b) L^{-b}_t(B^{y,y}_t) \, db.
\]

Let us define the set \( S := (-\infty, 0)^2 \cup (0, \infty)^2 \). Since \( \gamma \) is assumed to be even, by a simple change of variables, the first two terms in the above sum add up to

\[(4.23) \quad \int_S L^a_t(B^{t,x}_t) \gamma(a-b) L^b_t(B^{y,y}_t) \, db, \]

and the last two terms add up to

\[(4.24) \quad \int_S L^a_t(B^{t,x}_t) \gamma(a-b) L^{-b}_t(B^{y,y}_t) \, db. \]

Suppose that \( 0 < x \leq y \). In order for \((4.23)\) to be nonzero, it must be the case that

\[\max_{0 \leq s \leq t} B^{t,x}_t(s) + K \geq \min_{0 \leq s \leq t} \tilde{B}^{y,y}_t(s),\]

and for \((4.24)\) to be nonzero, it must be the case that

\[-\min_{0 \leq s \leq t} \tilde{B}^{y,y}_t(s) + K \geq \min_{0 \leq s \leq t} B^{t,x}_t(s).\]

Thus, by a union bound, followed by Brownian scaling and the fact that Brownian bridge maxima have sub-Gaussian tails, we see that

\[
P\left[ |(L_t(|B^{t,x}_t|), L_t(|\tilde{B}^{y,y}_t|))\gamma| > 0 \right]^{1/8}
\leq P\left[ \max_{0 \leq s \leq 1} B^{0,0}_1(s) + \max_{0 \leq s \leq 1} \tilde{B}^{0,0}_1(s) \geq \frac{y-x-K}{t^{1/2}} \right]^{1/8}
\leq C \left( e^{-\frac{(|x|-|y|-K)^2}{2t}} + e^{-\frac{(|x|+|y|-K)^2}{2t}} \right).
\]

The same bound holds for \( y \leq x \). At this point, given that for any function \( F \),

\[
\int_0^\infty F(x,y) \left( e^{-\frac{(x-y-K)^2}{2t}} + e^{-\frac{(x+y-K)^2}{2t}} \right) \, dy = \int_\mathbb{R} F(x,|y|) e^{-\frac{|y|^2}{2t}} \, dy,
\]

we obtain \((4.21)\) for compactly supported \( \gamma \) by using the same argument as Case 1.

4.5. **Proof of Theorem 2.19.** Let \((t_n)_{n \in \mathbb{N}}\) be a sequence of positive numbers such that \( t_n \to 0 \) as \( n \to 0 \). For every \( n \in \mathbb{N} \), let us define the test function \( f_n(x) := e^{-t_n x} \). This sequence of functions converges to 1 uniformly on compact sets. Moreover, by Proposition 2.15-(3),

\[
\Lambda(f_n) = \sum_{k=1}^\infty e^{-t_n \Lambda_k} = \text{Tr}[\hat{K}(t_n)] < \infty.
\]

Hence, by Proposition 2.2, to prove that \( \Lambda \) is number rigid, it suffices to show that

\[(4.25) \quad \lim_{n \to \infty} \text{Var}[\text{Tr}[\hat{K}(t_n)]] = 0.\]

We now prove that \((4.25)\) holds under the conditions stated in Theorem 2.19.

In Case 3, where we assume that \( 0 > 1 \), \((4.25)\) is immediate from \((4.3)\).
Consider now Cases 1 & 2. If we know that \( V(x)/|x|^a \to \infty \), then for every \( \kappa > 0 \), we can choose \( \nu_\kappa > 0 \) large enough so that \( V(x) \geq |\kappa x|^a - \nu_\kappa \) for every \( x \in I \). As per (2.14), we choose \( a \) such that
\[
\begin{cases} 
0 - 1/2 - 1/a = 0 & \text{(if } \gamma \text{ is compactly supported)} \\
0 - 1 - 2/a = 0 & \text{(otherwise)},
\end{cases}
\]
and thus (4.2) yields
\[
\limsup_{n \to \infty} \text{Var} \left[ \text{Tr}[\hat{K}(t_n)] \right] \leq \begin{cases} 
C_n/\kappa & \text{(if } \gamma \text{ is compactly supported)} \\
C_n/\kappa^2 & \text{(otherwise)}.
\end{cases}
\]
Since \( \kappa > 0 \) was arbitrary, we then obtain (4.25) by taking \( \kappa \to \infty \).

4.6. Proof of Theorem 2.20. We want to prove (2.15). This follows directly from a combination of (3.17), (3.19), and the dominated convergence theorem.

5. Airy-2 Process Counterexample

In this section, we prove Proposition 2.22. For every \( \beta > 0 \), let \( \xi_\beta \) be a Gaussian white noise with variance \( 1/\beta \), and define the operator
\[
\mathcal{H}^{(\beta)}_{t,0,\infty} := -\frac{x^2}{2} + \frac{\xi_\beta}{\beta},
\]
with a Dirichlet boundary condition at zero. The RSO \( 2\mathcal{H}^{(\beta)}_{t,0,\infty} \) is widely known in the literature as the Stochastic Airy Operator (e.g., [24, 46]), and we recall that for every \( \beta > 0 \), the Airy-\( \beta \) point process, which we denote by \( \Phi_\beta \), is defined as the eigenvalue point process of \( -2\mathcal{H}^{(\beta)}_{t,0,\infty} \).

When \( \beta = 2 \), the Airy-\( \beta \) process has an alternative integrable interpretation, namely, \( \Phi_2 \) is the determinantal point process induced by the Airy kernel
\[
\mathcal{K}(x,y) := \begin{cases} 
\frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}(y)\text{Ai}'(x)}{x^2 - y^2} & \text{if } x \neq y \\
\text{Ai}(x)^2 - x \text{Ai}(x)^2 & \text{if } x = y,
\end{cases}
\]
where \( \text{Ai} \) denotes the Airy function
\[
\text{Ai}(x) := \frac{1}{\pi} \int_0^\infty \cos \left( \frac{u^3}{3} + xu \right) \, du, \quad x \in \mathbb{R}.
\]

Let us denote \( f_t(x) := e^{tx} \) for every \( t > 0 \). By standard formulas for the variance of linear statistics of Determinantal point processes (e.g., [31, Equation (8)]), we have that
\[
\text{Var} \left[ \text{Tr}[e^{-2t\mathcal{H}^{(2)}_{t,0,\infty}}] \right] = \text{Var}[\Phi_2(f_t)] = \frac{1}{2} \int_{\mathbb{R}^2} (e^{tx} - e^{ty})^2 \mathcal{K}(x,y)^2 \, dx \, dy.
\]

By expanding the square and using the identity \( \mathcal{K}(x,x) = \int_{\mathbb{R}^2} \mathcal{K}(x,y)^2 \, dy \) (since \( \mathcal{K} \) is a symmetric projection kernel [51, Lemma 2]), we can reformulate this to
\[
\text{Var}[\Phi_2(f_t)] = \int_{\mathbb{R}} e^{2tx} \mathcal{K}(x,x) \, dx - \int_{\mathbb{R}^2} e^{t(x+y)} \mathcal{K}(x,y)^2 \, dx \, dy.
\]

The computation that follows is essentially taken from [43]. We provide the full details for the reader’s convenience. Rewrite the Airy kernel as
\[
\mathcal{K}(x,y) = \int_0^\infty \text{Ai}(u + x)\text{Ai}(u + y) \, du
\]
\[1\text{We note that the variance formula in question is typically only stated for compactly supported functions. The result can easily be improved to (5.2) by using dominated convergence with standard asymptotics for the Airy function such as [1, 10.4.59–10.4.62].}
Then, using Fubini’s theorem, we can write (5.2) as the difference $E_1(t) - E_2(t)$, where

$$E_1(t) := \int_{\mathbb{R}} e^{tx} \left( \int_0^\infty Ai(u + x^2) \, du \right) \, dx = \int_0^\infty \left( \int_{\mathbb{R}} e^{tx} Ai(u + x^2) \, dx \right) \, du,$$

and

$$E_2(t) := \int_{\mathbb{R}^2} e^{tx+y} \left( \int_0^\infty \int_0^\infty Ai(u + x)Ai(v + y)Ai(v + x) \, du \, dv \right) \, dx \, dy$$

$$= \int_0^\infty \int_0^\infty \left( \int_{\mathbb{R}} e^{tx} Ai(u + x)Ai(v + x) \, dx \right)^2 \, du \, dv.$$

We note that the application of Fubini in $E_1(t)$ is justified since the integrand is nonnegative, and in $E_2(t)$ it suffices to check

$$\int_0^\infty \int_0^\infty \left( \int_{\mathbb{R}} e^{tx} |Ai(u + x)Ai(v + x)| \, dx \right)^2 \, du \, dv < \infty.$$

For this, we recall the formula

$$(5.3) \quad \int_{\mathbb{R}} e^{tx} Ai(x + u)Ai(x + v) \, dx = \frac{1}{2\sqrt{\pi t}} \exp \left( \frac{t^3}{12} - \frac{2t}{u + v} - \frac{t(u - v)^2}{4\pi} \right)$$

from [43, Lemma 2.6], and note that by Cauchy-Schwarz, we have

$$\int_{\mathbb{R}} e^{tx} |Ai(u + x)Ai(v + x)| \, dx \leq \left( \int_{\mathbb{R}} e^{tx} Ai(u + x)^2 \, dx \right)^{1/2} \left( \int_{\mathbb{R}} e^{tx} Ai(v + x)^2 \, dx \right)^{1/2}$$

$$= \frac{1}{2\sqrt{\pi t}} \exp \left( \frac{t^3}{12} - \frac{u + v}{2} \right)$$

as desired.

With $E_1(t)$ and $E_2(t)$ established, an application of (5.3) yields

$$E_1(t) = \int_0^\infty \frac{\exp \left( \frac{t^3}{12} - \frac{2tu}{2\sqrt{2\pi}} \right)}{2\sqrt{2\pi t}} \, du = \frac{e^{(2/\sqrt{3})t}}{4\sqrt{2\pi}t^{3/2}}$$

and

$$E_2(t) = \int_0^\infty \int_0^\infty \frac{\exp \left( \frac{t^3}{12} - \frac{(u + v)t}{4\pi} - \frac{(u - v)^2}{2\sqrt{2\pi}} \right)}{4\sqrt{2\pi}t^{3/2}} \, du \, dv = \frac{e^{(2/\sqrt{3})t}}{4\sqrt{2\pi}t^{3/2}} \left( 1 - \text{erf} \left( \frac{\sqrt{2}}{\sqrt{2}} \frac{t^{3/2}}{\sqrt{3}} \right) \right),$$

where $\text{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-w^2} \, dw$ denotes the error function. Thus

$$\lim_{t \to 0} E_1(t) - E_2(t) = \lim_{t \to 0} \frac{e^{(2/\sqrt{3})t}}{4\sqrt{2\pi}t^{3/2}} \text{erf} \left( \frac{\sqrt{2}}{\sqrt{3}} \frac{t^{3/2}}{\sqrt{2}} \right) = \frac{1}{4\pi},$$

concluding the proof of Proposition 2.22.

**Appendix A. Rrigidity**

The following proof is taken from [35, Theorem 6.1].

**Proof of Proposition 2.22.** For every $n$, we can write

$$\Lambda(A) = \Lambda(f_n) - \mathbf{E}[\Lambda(f_n)] + \Lambda((1 - f_n)1_A) - \left( \Lambda(f_n1_{\mathbb{R}^n}) - \mathbf{E}[\Lambda(f_n)] \right).$$

Since the variance of $\Lambda(f_n)$ vanishes, we can choose a sparse enough subsequence $(n_k)_{k \in \mathbb{N}}$ along which $E_{E_1}^{(n_k)} \to 0$ almost surely as $k \to \infty$. Next, we note that

$$|E_2^{(n_k)}| \leq \Lambda(A) \left( \sup_{x \in A} |f_{n_k}(x) - 1| \right),$$
which vanishes almost surely as $k \to \infty$ because $\Lambda$ is locally finite and $A$ is bounded. In particular, $E_A^{(n)}(k) \to \Lambda(A)$ as $k \to \infty$, which completes the proof since $E_A^{(n)}$ is $\mathcal{F}_\Lambda(\mathbb{R} \setminus A)$-measurable for every $n$. □

Appendix B. Some Stochastic Analysis

Proposition B.1. There exists constants $0 < c < C$ such that for every $t \in (0,1]$,
\begin{equation}
ct^{-1/2} \leq \inf_{x \in I} \Pi_Z(t; x, x) \quad \text{and} \quad \sup_{(x, y) \in I^2} \Pi_Z(t; x, y) \leq C t^{-1/2}.
\end{equation}

Proof. In Case 1, the result follows directly from the fact that $\Pi_B(t; x, y) \leq 1/\sqrt{2\pi t}$ and $\Pi_B(t; x, x) = 1/\sqrt{2\pi t}$ for all $x$, $y$ and $t$. A similar argument holds for Case 2. Consider now Case 3. We recall that, by definition,
$$
\Pi_Y(t; x, y) := \sum_{z \in \mathbb{Z}^2} G_t(x - z) = \frac{1}{\sqrt{2\pi t}} \left( \sum_{k \in \mathbb{Z}} e^{-((x-2bk+y)/2t^2} + e^{-(x-2bk-y)^2/2t^2} \right).
$$

On the one hand, note that $t \mapsto e^{-z/t}$ is increasing in $t > 0$ for every $z \geq 0$; hence for every $t \in (0,1]$, one has
$$
\sup_{(x, y) \in (0, b)^2} \left( \sum_{k \in \mathbb{Z}} e^{-((x-2bk+y)/2t^2} + e^{-(x-2bk-y)^2/2t^2} \right)
\leq \sup_{(x, y) \in (0, b)^2} \left( \sum_{k \in \mathbb{Z}} e^{-(x-2bk+y)/2t^2} + e^{-(x-2bk-y)^2/2t^2} \right) < \infty.
$$

On the other hand, by isolating the $k = 0$ term in $\sum_{k \in \mathbb{Z}} e^{-(2bk)^2/2t}$,
$$
\inf_{x \in (0, b)} \left( \sum_{k \in \mathbb{Z}} e^{-(2x-2bk)^2/2t^2} + e^{-(2bk)^2/2t^2} \right) \geq \inf_{x \in (0, b)} \left( \sum_{k \in \mathbb{Z}} e^{-(2x-2bk)^2/2t^2} \right) + 1 \geq 1,
$$
concluding the proof. □

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