Agnostic Detector Error, Wigner Functions, and the Classical Limit of the High-Spin Einstein-Podolsky-Rosen Experiment

Anupam Garg
Department of Physics and Astronomy, Northwestern University, Evanston, Illinois 60208
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The spin-\(j\) Einstein-Podolsky-Rosen experiment is studied with a view to understanding how classical behaviour emerges as \(j \rightarrow \infty\). It is proposed that it is necessary to include detector error, which if it is be to viewed as an essential aspect of the emergence of classicality, should be both minimal, i.e., no more than necessary to wash out quantum mechanical behaviour, and agnostic, by which is meant that one should be able to ascribe it to error in the preparation of the state just as well as to the detector. Errors in the state preparation are discussed via the spin Wigner function. An agnostic error protocol is described which appears to be minimal.

INTRODUCTION

The suggestion that imperfections in the measurement process are necessary in order to understand the connection between quantum and classical mechanics is surely a very old one, and it is hard to pinpoint its exact genesis. It has been made by many authors with varying degrees of emphasis and nuance, and with varying motivations, and we could not possibly know of them all. Some references of which we are aware and which appear relevant to this paper are[1–4]. Our point of view is closest to Kofler and Brukner[3], but goes well beyond it.

That some imperfection is required is suggested even by the elementary example of the quantum mechanical probability distribution for position in a high-energy eigenstate of the simple harmonic oscillator[3]. If we compare this to the classical distribution, it is evident that no matter how large the quantum number becomes, the quantal distribution continues to oscillate ever more rapidly (although their vertical scale stays finite). A perfect position detector would measure these oscillations, so they must be smeared out in some way for the classical distribution to emerge. A general analysis of the smearing process seems prohibitively difficult, and it is not entirely clear on what principles it should be based.

In this paper we investigate this question based on Bohm’s version of the spin-\(j\) Einstein-Podolsky-Rosen experiment. This is an ideal system to study since Bell inequalities continue to be violated with undiminished range as \(j \rightarrow \infty\)[6]. Thus, the oft-made statement that the \(j \rightarrow \infty\) limit corresponds to classical mechanics needs to be examined more closely. The uncertainty principle limits the precision with which pairs of noncommuting variables can be measured (or even assigned values in an ontological sense) simultaneously. The nonclassicality revealed by the violation of Bell inequalities suggests a stronger position, namely, that one must limit the absolute precision with which an individual physical quantity can be measured. In other words, finite precision of measurement should be regarded not just as an unavoidable fact of life, but as an intrinsic ingredient of the classical limit. This is an interesting shift in perspective, for one of the long-standing beliefs of the classical mechanical world view has been that physical quantities can be measured to arbitrary precision.

We propose in particular that any detector error protocol that is to be regarded as irreducible or intrinsic to the classical limit should satisfy two principles: agnosticism and minimalism. By agnosticism we mean that one should not be able to say whether the errors arise in the detection process or in the state preparation process. And by minimalism we mean that they should be no more than is needed to wash out the nonclassical features. Indeed, from this point of view one should speak rather in terms of coarse-graining or smoothing-out ideal quantum mechanical distributions than of error, although for brevity it is convenient to keep doing so. We shall display a protocol that obeys both criteria, and we have also found protocols that disobey one of the two.

In the spin-\(j\) Einstein-Podolsky-Rosen-Bohm experiment, two particles of spin \(j\) in the singlet state, |\(\phi\rangle\), fly toward two far apart detectors. The spin of one particle is measured along a direction \(\hat{a}_1\) and of the other along \(\hat{a}_2\), with outcomes denoted \(m_1\) and \(m_2\). The probability distribution for these outcomes is \(p_{\hat{a}_1,\hat{a}_2}(m_1, m_2) = |\langle \phi | m_1 m_2 \rangle_{\hat{a}_1,\hat{a}_2}|^2\), where \(|m_1 m_2\rangle_{\hat{a}_1,\hat{a}_2} = |j, m_1\rangle_{\hat{a}_1} \otimes |j, m_2\rangle_{\hat{a}_2}\) is the simultaneous eigenstate of \(\hat{J}_1 \cdot \hat{a}_1\) and \(\hat{J}_2 \cdot \hat{a}_2\) with eigenvalues \(m_1\) and \(m_2\). Mermin and Schwarz (MS)[2] discovered the pseudo-factorizable form,

\[
p_{\hat{a}_1,\hat{a}_2}(m_1, m_2) = \int \frac{d^2\hat{n}}{4\pi} p_{\hat{a}_1}(m_1|\hat{n}) p_{\hat{a}_2}(m_2|\hat{n}) - \hat{n}.
\]  

The one-axis functions \(p_{\hat{a}_1}(m_1|\hat{n})\) resemble conditional distributions for outcomes \(m_1\) and \(m_2\) given a particular value for the hidden variable \(\hat{n}\), which is a unit vector that can point in any direction equiprobably. Since \(p_{\hat{a}_1,\hat{a}_2}(m_1, m_2)\)
violates Bell inequalities, the one-axis functions cannot be nonnegative, and they are not. MS showed that

\[ p_{\hat{a}_i}(m|\hat{n}) = \frac{1}{d_j} \sum_{\ell=0}^{2j} \sqrt{d_{\ell} + 1} f_\ell^J(m) P_\ell(\hat{a}_i \cdot \hat{n}), \]  

where \( d_j = 2j + 1 \), and \( f_\ell^J(m) \) is an eigenvector of the real symmetric matrix

\[ F_{mm'}(\theta) = |\hat{z}(j, m|e^{i\theta}j, |j, m')\hat{z}|^2, \]

the eigenvalue being the Legendre polynomial \( P_\ell(\cos \theta) \). The functions \( f_\ell^J(x) \) turn out to be shifted and scaled discrete Chebyshev polynomials [see 7, or Ref. 8, Sec. 22.17.], that are orthogonal with respect to the weight function \( \sum_{m=-j}^{j} \delta(x - m) \). Thus, there are only \( 2j+1 \) of them with degree \( 0 \leq \ell \leq 2j \). Either as polynomials, or as eigenvectors of \( F_{mm'} \), they obey orthonormality and completeness relations, \( d_j \sum_{m=-j}^{j} f_\ell^J(m)f_\ell^J(m') = \delta_{\ell\ell'} \), \( d_j^{-1} \sum_{\ell=0}^{j} f_\ell^J(m)f_\ell^J(m') = \delta_{mm'} \), which fix them uniquely with the convention \( f_\ell^J(j) > 0 \). As special cases, we have \( f_0^J(0) = 1 \), and \( f_1^J(0) = 0 \).

A quantum state can also be characterized by its Wigner function, which is part of the Weyl-Wigner-Moyal formalism [9–11], and which for spin may be implemented as follows [12–17]. Any spin operator may be expanded in terms of the complete set of spherical harmonic tensor operators, \( Y_{\ell m} \), whose Q, P, and Weyl symbols, denoted \( \Phi_{Q,P,W} \), are given by coefficients \( a_{Q,P,W} \) times \( Y_{\ell m}(\hat{n}) \), where \( a_{Q,P,W} = \prod_{k=0}^{\ell} (j + \frac{1}{2} + \frac{1}{2}k) \), and \( a_{W} = \sqrt{a_{Q}a_{P}} \). Further, \( \Phi_{Q,P,W}(\hat{n}) = (d_j/4\pi)(a_{W})^2 \delta_{\ell\ell'} \delta_{mm'} \). These results enable us to construct and go between Q, P, and Weyl representations [18].

The Wigner function for any system is nothing but the normalized Weyl symbol of the density matrix. It purports, but often fails to be, a joint probability density for noncommuting phase space variables, as it is not nonnegative. For our two spin system, the Wigner function, \( W_\rho(\hat{n}_1, \hat{n}_2) \), is nominally the probability that the spins point along \( \hat{n}_1 \) and \( \hat{n}_2 \). For any operator \( G \), \( \langle G \rangle = \text{Tr}(\rho G) = \int d^2\hat{n}_1 d^2\hat{n}_2 W_\rho(\hat{n}_1, \hat{n}_2) Y_G^W(\hat{n}_1, \hat{n}_2) d^2\hat{n}_1 d^2\hat{n}_2 \), where \( Y_G^W \) is the Weyl symbol for \( G \). (Putting \( G = \Phi_{Q}^G = 1 \) gives the normalization of \( W_\rho \).) Hence, for the singlet state,

\[ p_{\hat{a}_1\hat{a}_2}(m_1, m_2) = \int \int d^2\hat{n}_1 d^2\hat{n}_2 W_\rho(\hat{n}_1, \hat{n}_2)s^W(\hat{n}_1|m_1, \hat{a}_1)s^W(\hat{n}_2|m_2, \hat{a}_2), \]

where \( s^W(\hat{n}|m, \hat{a}) \) is the Weyl symbol of the projector \( |m\rangle_{\hat{a}} \). We shall show below that

\[ W_\rho(\hat{n}_1, \hat{n}_2) = \frac{1}{4\pi} \sum_{\ell=0}^{2j} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\hat{n}_1)Y^*_{\ell m}(-\hat{n}_2). \]

This is appealing in that as \( j \to \infty \), the completeness sum of the spherical harmonics yields \( \delta(\hat{n}_1 + \hat{n}_2)/4\pi \), exactly as one would expect for an isotropic state of two classical spinning gyroscopes with net angular momentum zero. The approach to this limit is very singular, however. Using the addition theorem for the \( Y_{\ell m} \)'s and the Christoffel-Darboux theorem [see Ref. 7, Sec. 3.2, or Ref. 8, Eq. 22.12.1], we find (with \( x = -\hat{n}_1 \cdot \hat{n}_2 \))

\[ W_\rho = \frac{1}{(4\pi)^2} \frac{d_j}{1-x}[P_{2j}(x) - P_{2j+1}(x)]. \]

We plot \( W_\rho \) in Fig. 1. That \( W_\rho < 0 \) is of course the standard deficiency of the Wigner function, but as the plots show, as \( j \) gets large, the oscillations in \( W_\rho \) get deeper and rapider [19]. In this way too we see that the singlet does not get more classical as \( j \to \infty \). We show enlarged views of \( W_\rho \) near \( x = 1 \) for \( j = 50 \) in Fig. 1.

To show Eq. (5), and for other technical parts of our analysis, we shall rely repeatedly on two facts. First is the spectral representation of \( f_{mm'}^J \),

\[ F_{mm'}(\theta) = \frac{1}{d_j} \sum_{\ell=0}^{2j} f_\ell^J(m)f_\ell^J(m') P_\ell(\cos \theta). \]

Second is a connection between the polynomials \( f_\ell^J \) and the Weyl-to-Q map,

\[ f_\ell^J(j) = \sqrt{2\ell + 1} a_{\ell \ell}^Q/a_{\ell \ell}^W, \]
FIG. 1: Wigner functions for \( j = 5, 19/2, \) and \( 50 \), as a function of \( x = -\hat{n}_1 \cdot \hat{n}_2 \). The scales on the \( y \) axes should be noted. For \( j = 19/2 \) and \( 50 \), large portions of the vertical range are not shown.

FIG. 2: Enlarged view near \( x = 1 \) of the Wigner function for \( j = 50 \). Once again, the scales on the \( y \) axes should be noted.

which can be proved by induction.

To find \( W_p \) for the singlet, we first find the Q-symbol, \( \Phi^Q_\rho \). With \( |\hat{n}\rangle \) being a spin coherent state \([20, 21]\), we have

\[
\Phi^Q_\rho(\hat{n}_1, \hat{n}_2) = |\langle \phi | (|\hat{n}_1 \rangle \otimes |\hat{n}_2 \rangle) |^2 = d_j^{-1} |\langle -\hat{n}_1 | \hat{n}_2 \rangle |^2 = d_j^{-1} F_j \gamma, \]

with \( \cos \gamma = -\hat{n}_1 \cdot \hat{n}_2 \). Next, we use the spectral representation \([7]\), and multiply each term in the sum over \( \ell \) by \( (a^W_{j\ell} / a^Q_{j\ell})^2 \) to convert to the Weyl symbol. Last, we multiply by \( (d_j/4\pi)^2 \) for normalization, and employ the the connection \([8]\). This leads directly to Eq. \( (5) \).

We include detector error via a matrix \( R \), such that \( R_{mm'} \) is the probability that a particle which has spin \( m \) is detected in the bin for spin \( m' \). Since particles can neither be lost nor appear from nowhere, we must have \( \sum_{m} R_{mm'} = \sum_{m'} R_{mm'} = 1 \), and to be probabilities, we must have \( R_{mm'} \geq 0 \) for all \( m, m' \). Using the same error matrix at both detectors, the measured distribution \( \tilde{p}_{\hat{a}, \hat{a}_s}(m, m_2) \) is given by the same form as Eq. \( (1) \) with the one-axis functions replaced by

\[
\tilde{p}_{\hat{a}, \hat{a}_i}(m|\hat{n}) = \sum_{m'} R_{mm'} p_{\hat{a}, \hat{a}_s}(m'|\hat{n}).
\]  

We will say that the error matrix is sufficient if \( \tilde{p}_{\hat{a}, \hat{a}}(m|\hat{n}) \geq 0 \) for all \( m, \hat{n} \), and \( \hat{a}_s \), and minimally sufficient if the value 0 is attained for some set of parameters. The smoothed one-axis functions then have meaning as conditional probability distributions, and \( \tilde{p}_{\hat{a}, \hat{a}_s}(m_1, m_2) \) is rendered classical in that it cannot violate any Bell inequalities.

In terms of the Wigner function approach, the error matrices effect transformations on the Weyl symbols \( s^W \rightarrow s^W' \)

\[
[s^W(\hat{n}|m, \hat{a}) = \sum_{m'} R_{mm'} s^W(\hat{n}|m', \hat{a})],
\]

which will generally not be equivalent to a transformation of \( W_p \). When it is, the error protocol is agnostic. To see when this is possible, we must examine \( s^W \).

To find \( s^W \), we proceed as we did for \( W_p \): find the Q symbol of \( |m\rangle \hat{a}_a(\hat{a}) \), expand in the \( Y_{m' \ell} \)'s, and multiply each term by \( a^W_{j\ell} / a^Q_{j\ell} \). Now, \( s^Q(\hat{n}|m, \hat{a}) = |\langle \hat{n}|m, \hat{a} \rangle |^2 = F_{jm}(\alpha) \), where \( \cos \alpha = \hat{a} \cdot \hat{n} \). Using the spectral representation \([7]\) and the connection \([8]\), we obtain

\[
s^W(\hat{n}|m, \hat{a}) = \frac{1}{d_j} \sum_{\ell} \sqrt{2\ell + 1} f_{jm}(\hat{n} \cdot \hat{a}).
\]  

That is to say, \( s^W(\hat{n}|m, \hat{a}) \equiv p_{\hat{a}}(m|\hat{n}) \). The MS one-axis functions are the Weyl symbols for projection operators onto the states \( |j, m\rangle \).

The equivalence just noted leads us to consider error matrices that are eigenoperators for the \( f^j_{\ell} (m) \), i.e.,

\[
\sum_{m'} R_{mm'} f^j_{\ell} (m') = c_{\ell} f^j_{\ell} (m),
\]  

which can be proved by induction.
where the $c_\ell$ are arbitrary except that $c_0 = 1$. Then, $\tilde{p}_\hat{a}(m|\hat{n}) = s^W(\hat{n}|m, \hat{a}) = d_j^{-1} \sum_{\ell} \sqrt{2\ell + 1} c_\ell f_\ell^j P_\ell(\hat{n} \cdot \hat{a})$. Because of the orthogonality of the Legendre polynomials, the factors $c_\ell$ can be transferred onto the Wigner function. That is, $\tilde{p}_{\hat{a},\hat{a}}$ can be put in the form $\tilde{\tilde{p}}$, where we leave the single spin Weyl symbols, the $s^W$’s, untouched, and replace $W_\rho$ by

$$W_\rho(\hat{n}_1, \hat{n}_2) = \frac{1}{4\pi} \sum_{\ell=0}^{2j} \sum_{m=-\ell}^{\ell} c_\ell^2 Y_{\ell m}(\hat{n}_1)Y_{\ell m}^*(\hat{n}_2).$$

But this says that the error or coarse-graining protocol affects the state preparation, and not the detection, i.e., is agnostic. Hence, we have shown that error matrices obeying Eq. (11) are agnostic. (They also preserve the isotropy of the singlet state.)

An agnostic but trivial protocol is obtained by taking $c_\ell = \delta_{\ell 0}$, i.e., $R_{mm'} = d_j^{-1}$ for all $m, m'$. Now $\tilde{p}_{\hat{a},\hat{a}}(m_1m_2) = d_j^{-2}$, and the two spins are totally uncorrelated. Hence agnosticism by itself is not a compelling principle, and we must consider sufficiency also.

The special error protocol that we have found that is both agnostic and minimal arises from choosing

$$c_\ell = \frac{a^Q_{j\ell}}{a^W_{j\ell}} = \frac{f_\ell^j(j)}{\sqrt{2\ell + 1}} = \prod_{k=0}^{\ell} \left(\frac{2j + 1 - k}{2j + 1 + k}\right)^{1/2}.$$ \hfill (13)

Then, with $\cos \alpha = \hat{n} \cdot \hat{a}$ and $\cos \gamma = -\hat{n}_1 \cdot \hat{n}_2$,

$$\tilde{p}_\hat{a}(m|\hat{n}) = F_{j m}(\alpha) = \left(\frac{2j}{j - m}\right) \left[\frac{2}{j + 1} (1 + \cos \alpha)\right]^{j+m} \left[\frac{2}{j + 1} (1 - \cos \alpha)\right]^{j-m},$$ \hfill (14)

and

$$W_\rho(\hat{n}_1, \hat{n}_2) = d_j \frac{\sqrt{2}}{(4\pi)^2} F_{j j}(\gamma) = \frac{d_j}{(4\pi)^2} \left(\frac{1 + \cos \gamma}{2}\right)^{2j},$$ \hfill (15)

both of which are nonnegative. This protocol is minimal because the $\tilde{p}_{\hat{a},\hat{a}}(m|\hat{n})$ do in fact vanish when $\hat{n} = \hat{a}$ for any $m_j < j$, and when $\hat{n} = -\hat{a}$ for any $m_j > -j$. These one-axis functions provide an explicit locally realistic model for the resulting coarse-grained distribution $\tilde{p}_{\hat{a},\hat{a}}(m_1m_2)$. It is also minimal from the point of view of state preparation, because $W_\rho(\hat{n}_1, \hat{n}_2)$ does vanish when $\hat{n}_1 = \hat{n}_2$.

Completeness of the $f_\ell^j(m)$ allows any error matrix that obeys Eq. (11) to be written as

$$R_{mm'} = \frac{1}{d_j} \sum_{\ell} c_\ell f_\ell^j(m) f_\ell^j(m').$$ \hfill (16)

(The constraint $c_0 = 1$ follows from $f_0^j(m) = 1$ and the demands $\sum_m R_{mm'} = \sum_{m'} R_{mm'} = 1$.) In addition, we must have $R_{mm'} \geq 0$ for all $m, m'$. We have not been able to establish rigorously that this is so for our special protocol (13), but we have verified it by hand for $d_j \leq 4$, and numerically for $d_j \leq 20$, beyond which our numerical precision is not sufficient. A crude argument is as follows. Eq. (13) implies $R_{mm'} = \frac{1}{d_j} \int_{-1}^{1} dx h_m^j(x) F_{j m'}(x)$, with

$$h_m^j(x) = d_j^{-1} \sum_{\ell=0}^{2j} \sqrt{2\ell + 1} f_\ell^j(m) P_\ell(x).$$

As $j \to \infty$, $F_{j m'}$ and $h_m^j(x)$ are highly peaked functions of $x$ with maxima at $x_{m'} = m'/j$ and $x_m = m/j$, and both of them integrate to $2/d_j$. $F_{j m'}$ is a binomial distribution, which is like a Gaussian of width $\sigma_x = (1 - x_{m'}^2/2j)^{1/2}$ (for $m'$ sufficiently far away from $\pm j$). The sum for $h_m^j(x)$ resembles a Christoffel-Darboux sum, which suggests that for large $j$, $h_m^j(x)$ is much more narrowly peaked with a width of order $1/j^2$. Away from its maximum, it oscillates on a scale $j^{1/2}$ with an approximate frequency $1/2j$. This frequency and the width are both much smaller than the width of the Gaussian, so $h_m^j(x)$ effectively behaves as $d_j^{-1} \delta(x-x_m)$ in the integral. Hence, for $m, m'$ not too close to $\pm j$,

$$R_{mm'} \approx \frac{1}{j \sqrt{2\pi \sigma_x^2}} e^{-(x_m-x_{m'})^2/2\sigma_x^2}.$$ \hfill (17)

This says that as $j \to \infty$, the interval $|m-m'|/j$ over which coarse graining is necessary becomes of order $j^{-1/2}$.

It is not easy to come up with a simple figure of merit for the sufficiency of error matrices for $j > 1/2$, although one choice is to see by how much the spin correlation is reduced. For protocols obeying Eq. (11), this factor is easily shown
to be $c_j^2$, so that for our special protocol [13], \( \langle (J_1 \cdot \hat{a})(J_2 \cdot \hat{b}) \rangle = j^2 \hat{a} \cdot \hat{b}/3 \). However, for \( j = 1/2 \), where the detector error reduces to a single number, it is known not to be the least conservative smoothing procedure. It corresponds to an error rate of 21.1\% [2]. In Ref. [22] we found a truly positive factorizable representation for $p_{\hat{a},\hat{a}}(m_1m_2)$ based on an error rate of 14.6\%. Whether this number can be further reduced for $j = 1/2$, or whether there are analogous constructions for higher $j$ that are less conservative than the protocol [13] remain open questions.

We conclude by mentioning two other error protocols for comparison. In Ref. [23], the author considered

$$R_{mm'} = \frac{1}{2} d_j \int_{x_{m-}}^{x_{m+}} d(\cos \theta) F_{jm'}(\theta),$$

(18)

where $x_{m\pm} = (2m \pm 1)/d_j$. Now $R_{mm'} > 0$, but this protocol is not agnostic. For small $j$ we can explicitly show it is insufficient. The insufficiency decreases for larger $j$, so it is very likely that this defect could be repaired by admixing in the trivial protocol $R_{mm'} = d_j^{-1}$ in an amount that decreases steadily as $j \to \infty$. This yields a gnostic but sufficient protocol. An agnostic but oversufficient protocol is obtained if we choose $c_j = (a_{j\ell}^Q/a_{j\ell}^W)^2$. In this case, we can prove that $R_{mm'} > 0$.

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[1] H. Reichenbach, *The Philosopher Foundations of Quantum Mechanics*, UCLA Press, Berkeley and Los Angeles, 1944 (Reprinted by Dover Publications, 2011).
[2] N. D. Mermin and Gina M. Schwarz, Found. Phys. 12, 101 (1982).
[3] J. Kofler and C. Brukner, Phys. Rev. Lett. 99, 180403 (2007).
[4] J. S. Lundeen et al., Nature Physics 5, 27 (2009).
[5] See, e.g., Fig. 7.2 of R. Shankar, *Principles of Quantum Mechanics*, 2nd ed., Springer (1994); or, Fig. 2.7(b) of D. J. Griffiths, *Introduction to Quantum Mechanics*, 2nd ed., Cambridge University Press (2017).
[6] A. Garg and N. D. Mermin, Phys. Rev. Lett. 49, 901 (1982).
[7] G. Szego, *Orthogonal Polynomials*, Amer. Math. Soc. Colloquium Publications, 23, (1939); Sec. 2.8.
[8] Milton Abramowitz and Irene A. Stegun, *Handbook of Mathematical Functions*, Nat’l Bureau of Standards Applied Math. Series 55 (1964).
[9] H. Weyl, *The Theory of Groups and Quantum Mechanics*, Dover Publications, New York (1950) [translation of *Gruppentheorie und Quantenmechanik*, Hirzel Verlag, Leipzig (1928)]. See Chap. II, Sec. 11 and Chap. IV, Sec. 14.
[10] E. P. Wigner, Phys. Rev. 40, 749 (1932).
[11] J. E. Moyal, Proc. Cambridge Philos. Soc. 45, 99 (1949).
[12] R. L. Stratonovich, Sov. Phys. JETP 31, 1012 (1956).
[13] J. Kutzner, Z. Phys. 259, 177 (1973).
[14] G. S. Agarwal, Phys. Rev. A 24, 2889 (1981).
[15] B. S. Shastry, G. S. Agarwal, and I. Rama Rao, Pramana 11, 85 (1978).
[16] J. C. Varilly and J. M. Gracia-Bondia, Annals of Phys. (NY) 190, 101 (1989).
[17] F. Li, C. Braun, and A. Garg, Europhys. Lett. 102, 60006 (2013).
[18] Very recently, Jens Koch and the author (to be published), have found an exact asymptotic form for this map in terms of exponentials of Faulhaber polynomials of the Laplace-Beltrami operator on the unit sphere.
[19] $(4\pi)^j W_\rho$ equals $(2j + 1)^j$ at $x = 1$, $(-1)^j(2j + 1)$ at $x = -1$, the scale of the oscillations is $\sim j^{1/2}$, and the first zero of $W_\rho$ is at a distance of $7.34/(2j + 1)^2$ from the main peak at $x = 1$. These results, and asymptotic forms for $W_\rho$ will be published elsewhere.
[20] J. M. Radcliffe, J. Phys. A: Gen. Phys. 4, 313 (1971).
[21] F. T. Arecchi, E. Courtens, R. Gilmore, and H. Thomas, Phys. Rev. A 6, 2211 (1972).
[22] A. Garg, Phys. Rev. D 28, 785 (1983).
[23] A. Garg, Ph.D. thesis, Cornell University, 1983.