Von Neumann Regular Semiring

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Abstract. The aim of this action is a study and investigate "Von Neumann regular" semirings, some related concepts, e.g. reduced semirings; duo semiring, quasi-duo, and weakly duo semirings; regular, weakly regular and strongly regular semirings, also investigated. Some known results related to those concepts in rings were converted to semirings. Another aim of this paper is characterization Von Neumann Regular condition by the principal right ideal generated by an idempotent element.

Key words. Semirings, reduced semiring; duo, quasi-duo semiring, weakly duo Semiring; regular, weakly regular, strongly regular; Boolean semiring; semifield; Nilpotent

1. Introduction

The concept of Von Neumann Regular introduced in (ring theory) in 1936 by J. Von Neumann [1], also was studied in semirings, through much research [2], [3], [4], [5]. A semiring \( \mathcal{R} \) is referred to as ‘simply regular’ or ‘Von Neumann regular’ if \( \forall a \in \mathcal{R} \exists x \in \mathcal{R} \exists axa = a \ [3] \)." A non-empty set \( \mathcal{R} \) with two bilateral operations (\( + \)) and(\( \cdot \) is referred to as a semiring if:

(1) \(( \mathcal{R}, +)\) is a commutative monoid with identity element 0;
(2) \(( \mathcal{R}, \cdot)\) is a monoid with identity element \( 1 \neq 0 \);
(3) Both the distributive laws hold in \( \mathcal{R} \);
(4) \( a \cdot 0 = 0 \cdot a = 0 \) for all \( a \in \mathcal{R} \).[6]

A nonempty subset \( I \) of a semiring \( \mathcal{R} \) is called a (left, right) ideal if \( a, b \in I \) and \( r \in \mathcal{R} \) implies \( a + b \in I \) and \( (ra \in I, ar \in I \) respectively [6]. An ideal \( I \) from a semiring \( \mathcal{R} \) is called subtractive if \( a, a + b \in I, b \in \mathcal{R} \) implies \( b \in I \). A semiring \( \mathcal{R} \) is called yoked if for each \( x \) and \( y \) in the semiring \( \mathcal{R} \), \( x + h = y \) or \( x = y + h \) for some \( h \) in the semiring \( \mathcal{R} \).[8] A semiring \( \mathcal{R} \) is called cancellative if for every \( a, b, c \in \mathcal{R} \) such that \( a + c = b + c \) then \( a = b \). This paper consisting of three sections. In section one, we study semirings which that contain no non-

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zero nilpotent elements; such semirings are called reduced semiring. We give some of their basic properties and provide some examples. **Section two** is devoted to exhibiting several preliminary results on duo semiring, quasi-duo semirings and weakly duo semiring. In **section three**, the properties and definitions of strongly regular, regular and weakly regular semiring were studied.

2. Reduced semiring

In this section, we study semirings that contain no non-zero nilpotent elements; such semirings are called reduced semirings. We give some of their basic properties and provide some examples.

**Definition 2.1.** (see [6], p. 43)

An element \(x\) at a semiring \(R\) is referred to as "nilpotent" if there exists a positive integer \(n\) satisfying \(x^n = 0\). We will denote the set for every nilpotent elements from \(R\) by \(N\).

**Lemma 2.2.** (see [6], p. 43, 44)

Let \(N\) be the set of all nilpotent elements of \(R\), then \(N\) is an ideal of \(R\).

**Definition 2.3.** [9]

A semiring \(R\) is referred to as "reduced" if \(R\) contains no non-zero nilpotent elements.

**Example 1**

The semiring of integers modulo 6, \(\mathbb{Z}_6\), is reduced while \(\mathbb{Z}_8\) is not reduced, since 2, 4, 6 are nilpotent elements of \(\mathbb{Z}_8\).

**Definition 2.4.** [10]

"A right annihilator" of a non-zero element \(a\) in a semiring \(R\) is defined by

\[ r(a) = \{ b \in R : ab = 0 \}. \]

A left annihilator \(l(a)\) is similarly defined.

**Proposition 2.5.** [9]

Let \(R\) be "a reduced semiring". Then, for every \(a \in R\)

1. \(r(a) = l(a)\)
2. \(r(a) = r(a^2)\)
3. \(R/r(a)\) is reduced

**Definition 2.6.** [1]

An ideal \(I\) from a semiring \(R\) is referred to as essential if and only if \(I \cap H \neq 0\) for every nonzero ideal of \(R\).

**Example 2**

1. Let \(\mathbb{Z}_8\) be the semiring of (integers modulo 8) and \(I=\langle 2 \rangle, J=\langle 4 \rangle\), then \(I\) and \(J\) are essential ideals in \(\mathbb{Z}_8\).

2. Let \(\mathbb{Z}_6\) be the semiring of (integers modulo 6), then \(I=\langle 2 \rangle\) is not essential in \(\mathbb{Z}_6\).

3. Let \(R = ( \mathbb{N} \cup \{ \infty \}, \min, +)\) be the semiring wherever \(\mathbb{N}\) is the natural numbers, thus the ideals from \(R\) are the form \(I=\{n, n+1, \ldots \}\) or \(\{ \infty \}\)

Since :

(i) \(I = \{ n, n+1, n+2, \ldots \} \cup \{ \infty \}\) closed under addition ( min ).

(ii) Let \(r\) be any element belongs to \(R\) and \(a\) be any element belongs to \(I, r+a \geq n\), then \(r+a \in I\) this implies closed under multiplication by elements of \(R\) (+).

Now, suppose \(J\) is a non-zero ideal contained in \(R\). indeed :

(i) \(\infty \in R, a \in J\) then \(\infty + a = \infty \in J\)

(ii) Let \(n\) be the smallest element of \(J\). Then \(J\) is an ideal, \(1 \in R\) implies \(1+n \in J\).

Then \(\{ n, n+1, n+2, \ldots \} \subseteq J\) → \(J=\{ n, n+1, \ldots \} \cup \{ \infty \}\).

On the another hand every non-zero ideal from \(R\) is essential from \(R\) (the zero element from \(R\) is \(\infty\)), if \(J \cap K = \{ \infty \}\) then \(J = \{ \infty \}\) or \(K = \{ \infty \}\).

\(J = \{ n, n+1, \ldots \} \cup \{ \infty \}, K = \{ m, m+1, \ldots \} \cup \{ \infty \}\).

\(J \cap K = J\) if \(n > m\) or \(J \cap K = K\) if \(n < m\) – if \(J \neq \{ \infty \}\) and \(K \neq \{ \infty \}\). Then \(J \cap K \neq \{ \infty \}\).
Definition 2.7. [1]
Let \( x \) an element in a semiring \( \mathcal{R} \). Then \( x \) is referred to as "a right singular" iff \( r(x) \) is essential ideal in \( \mathcal{R} \). The set of all "right singular elements" in \( \mathcal{R} \) is denoted by "\( r(\mathcal{R}) \)".

A left singular ideal, denoted by \( l(\mathcal{R}) \), is similarly defined.

Example 3
1. Let \( Z_{12} \) be the semiring of integers modulo 12. Then \( r(6) \) and \( r(0) \) are the only essential ideals in \( Z_{12} \). Therefore, \( r(\mathcal{R}) = l(\mathcal{R}) = \{0, 6\} \).

2. By (Example 2(3)), if \( m \neq \infty \), then \( r(m) = \{ k \in \mathbb{N} \cup \{\infty\} \mid m+k = \infty \} = \{\infty\} \) not essential in \( \mathcal{R} \).

By a similar argument in [12], the following result can be proved.

Proposition 2.8.
If \( l(\mathcal{R}) \) contains no non-zero nilpotent elements, then \( l(\mathcal{R}) = 0 \).

Proof:
Since \( l(\mathcal{R}) \neq 0 \), then there exists an \( x \in l(\mathcal{R}) \) essential in \( \mathcal{R} \).

Thus \( l(\mathcal{R}) \cap \mathcal{R}x \neq 0 \) for each \( x \in \mathcal{R} \). In particular when \( x=1 \), then there exists \( rz \in l(z) \cap \mathcal{R}z \) with \( rz \neq 0 \). So, \( (rz)z = 0 \). Since \( (rz)^2 = (rz)(rz) = rz(rz) = 0 \), then \( rz \in Z(\mathcal{R}) \) and \( rz \) is nilpotent in \( \mathcal{R} \).

Now, \( (rz)^3 = (rz)(rz) = r(zr)z = 0 \). This implies that \( l(\mathcal{R}) = 0 \) and \( \mathcal{R} \) is a reduced semiring.

3. Duo and quasi-duo semiring

The present section is devoted to exhibiting several preliminary results on "duo semirings", "quasi-duo semirings", and "weakly duo semirings". We shall begin this section with the following definition.

Definition 3.1. [7]

The semiring \( \mathcal{R} \) is referred to as right (left) duo if every right (left) ideal of \( \mathcal{R} \) is a two-sided ideal.

The following definition is analogous to a similar one in ring theory (see [13]).

Definition 3.2.

A semirings \( \mathcal{R} \) is referred to as "left (right) quasi-duo" if each maximal (left) right ideal of \( \mathcal{R} \) is a two-sided ideal.

A right (quasi-duo) semiring form a non-trivial generalization of right duo semiring.

Definition 3.3. [14]

An element \( x \) of a semiring \( \mathcal{R} \) is (a unit) if and only if there exists (a necessarily unique) element \( x^{-1} \) of \( \mathcal{R} \) satisfying \( xx^{-1} = 1 = x^{-1}x \).

The following definition is analogous to a similar one in ring theory (see [13]).

Definition 3.4.

A semiring \( \mathcal{R} \) is referred to as (weakly right (left) duo), if for every \( x \in \mathcal{R} \), there exists a positive integer \( m \) such that \( x^m \mathcal{R} (\mathcal{R}x^m) \) is a two-sided ideal of \( \mathcal{R} \).

Note that, every "weakly right (left) duo semiring" is "right (left) quasi-duo".
Definition 3.5. [10]

"The Jacobson radical" of a semiring \( \mathbb{R} \), denoted by \( J(\mathbb{R}) \), is the set
\[
J(\mathbb{R}) = \cap \{ M : M \text{ is a maximal ideal of } \mathbb{R} \}.
\]

Definition 3.6. [7]

A semiring \( \mathbb{R} \) is called semi-simple if \( J(\mathbb{R}) = 0 \).

Corollary 3.7. [15]

Any proper ideal of a semiring \( \mathbb{R} \) is a subset of a maximal ideal of \( \mathbb{R} \).

The following result is analogous to a similar one in ring theory (see [16], p. 109).

Lemma 3.8.

\( J(\mathbb{R}) = \{ a \in \mathbb{R} : \mathbb{R}a \preceq \mathbb{R} \} \).

Proof :

(\( \Rightarrow \)) \( \mathbb{R}a \preceq \mathbb{R} \), \( C \) is a maximal ideal of \( \mathbb{R} \), such that \( a \notin C \rightarrow \mathbb{R}a + C = \mathbb{R} \rightarrow a\mathbb{R} \) is not small in \( \mathbb{R} \), a contradiction. This implies \( a \in \cap C \), where \( C \) is a maximal ideal of \( \mathbb{R} \).

(\( \Leftarrow \)) Let \( a \in \cap C \), where \( C \) is a maximal ideal of \( \mathbb{R} \). Assume \( a\mathbb{R} + U = \mathbb{R} \), for some proper ideal \( U \) of \( \mathbb{R} \). We can assume that \( U \) is a maximal ideal of \( \mathbb{R} \) by corollary (3.7.). But \( a \in U \rightarrow \mathbb{R}a \subseteq U \rightarrow \mathbb{R}a + U = U \neq \mathbb{R} \), a contradiction. Therefore \( \mathbb{R}a \preceq \mathbb{R} \). □

The following result is analogous to a similar one in ring theory (see [17]).

Proposition 3.9.

Let \( \mathbb{R} \) be "a right quasi-duo semiring". Then \( \mathbb{R}/J(\mathbb{R}) \) is "a reduced semiring".

Proof :

It is enough to prove that any nilpotent element belongs to \( J(\mathbb{R}) \). That is, to prove if \( x \in \mathbb{R} \) and \( x^m = 0 \) for some \( m \in \mathbb{Z}^+ \), then \( x \in J(\mathbb{R}) = \{ x \in \mathbb{R} : \mathbb{R}x \preceq \mathbb{R} \} \) by lemma (3.8.). Suppose that \( \mathbb{R}a + K = \mathbb{R} \) where \( K \) is a left ideal from \( \mathbb{R} \), we want to show that \( K=\mathbb{R} \) which implies \( x \in J(\mathbb{R}) \), by corollary (3.7.), we can assume that \( K \) is a maximal ideal of \( \mathbb{R} \), multiplying both side by \( x \) from right we get \( \mathbb{R}x^2 + Kx = \mathbb{R}x \rightarrow \mathbb{R}x^2 + Kx + K = \mathbb{R} \), continuing in this way, we end up with \( Kx^{n-1} + \cdots + Kx^2 + Kx + K = \mathbb{R} \), (\( \mathbb{R}x^n = 0 \)). Since \( \mathbb{R} \) is "left quasi-duo", and \( K \) is maximal, then hence \( K=\mathbb{R} \). □

4. Regular, Strongly Regular, Weakly Regular

In this section, the definitions, and properties of regular, weakly regular and strongly regular semirings are given.

Definition 4.1. [2]

A semiring \( \mathbb{R} \) is said to be "Von Neumann Regular" if, for any \( x \in \mathbb{R} \), there exists \( y \in \mathbb{R} \) such that \( x = yxy \).

The following definition is analogous to a similar one in ring theory (see [18]).

Definition 4.2.

A semiring \( \mathbb{R} \) is said to be unit regular if, for every \( a \in \mathbb{R} \), there exists a unit \( u \) in \( \mathbb{R} \) such that \( a = uau \).

Definition 4.3. [19]

"A commutative semiring" \( \mathbb{R} \) is referred to as (a semifield) if each non-zero element in \( \mathbb{R} \) has a (multiplicative) inverse in \( \mathbb{R} \).

Definition 4.4. (see [6], p. 7)

The Boolean semiring is the commutative semiring \( B = \{ 0,1 \} \), formed by the two-elements, and defined by \( 1 + 1 = 1 \).

Example 1

1- Every semifield is regular.

2- Every Boolean semiring is regular.
3- Let $\mathbb{R}^+ = \{ r \geq 0 | r \in \mathbb{R} \}$ be the semiring and $\mathbb{R}_i = \begin{bmatrix} \mathbb{R}^+ & \mathbb{R}^+ \\ 0 & \mathbb{R}^+ \end{bmatrix}$, it's clear that $\mathbb{R}_i$ is "a non-commutative semiring" with identity $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, but not "regular" because

$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ for all $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in \mathbb{R}_i$.

The following result is analogous to a similar one in ring theory (see [20]).

**Lemma 4.5.**

Let $v \in \mathbb{R}$, if $v$ is "unit regular", then $v = eu$ for some idempotent element $e$ and some unit element $u$.

**Proof:**

Suppose that $x$ is a unit regular. Then there exists a unit $v \in \mathbb{R}$ such that $xvx = x$. Let $e = xv$. Then $e^2 = xv = e$, so $e$ is an idempotent element of $\mathbb{R}$. Let $u = v^j$, then we have $x = eu$.

The following definition is analogous to a similar one in ring theory (see [21]).

**Definition 4.6.**

A semiring $\mathbb{R}$ is referred to as CI-semiring if each idempotent element from $\mathbb{R}$ is central, $(a \in \mathbb{R}$ is central if $ab = ba \forall b \in \mathbb{R}$).

The following definition is analogous to a similar one ring theory (see [22]).

**Definition 4.7.**

A semiring $\mathbb{R}$ is called "strongly regular" if, for each $r \in \mathbb{R}$, there exists $s, t \in \mathbb{R}$ such that $r = rsx = trx$.

**Remark 2**

Every "strongly regular semiring" is "regular", (clear)

We call $\mathbb{R}$ $\pi$-regular (unit $\pi$-regular) semiring if for any $x \in \mathbb{R}$, there exists a positive integer $m$ and an element $y$ (a unit $u$) of $\mathbb{R}$ such that $x^m = x^m y x^m (x^m y x^m)$.

**Lemma 4.8.**

Let $\mathbb{R}$ a semiring. Then the following statements equivalent conditions:

1- $\mathbb{R}$ is $\pi$-regular
2- $\mathbb{R}$ is $\pi$-regular
3- The chain $z \mathbb{R} \supset z^2 \mathbb{R} \supset \ldots$ terminates.

**Proof:**

1 $\Rightarrow$ 2: $z^m \in z^{m+1} \mathbb{R} \rightarrow z^m = z^{m+1} r$ for some $r \in \mathbb{R}$, $z^m s = z^{m+1} r s \in z^{m+1} \mathbb{R} \rightarrow z^m \mathbb{R} \subseteq z^{m+1} \mathbb{R}$, $z^{m+1} r = z^m (x r) \in z^m \mathbb{R} \rightarrow z^{m+1} \mathbb{R} \subseteq z^m \mathbb{R}$.

2 $\Rightarrow$ 3: trivial.

3 $\Rightarrow$ 1: trivial. \(\square\)

**Definition 4.9.**

An element $z$ in a semiring $\mathbb{R}$ is called right $\pi$-regular if, it satisfies the equivalent conditions in lemma(4.8).

**Definition 4.10.**

An element $k \in \mathbb{R}$ is referred to as (strongly $\pi$-regular) if it is both left and right "$\pi$-regular", and $\mathbb{R}$ is referred to as "a strongly $\pi$-regular semiring" if each element is "strongly $\pi$-regular".

**Remark 3**

Every strongly $\pi$-regular semiring is $\pi$-regular. (clear)

**Definition 4.11.** [3]

A semiring $\mathbb{R}$ is referred to as right (left) "weakly regular" if $H^2 = H$ for each right (left) ideal $H$ of $\mathbb{R}$, equivalently "if $w \in \mathbb{R} w \mathbb{R}$ (w $\in \mathbb{R} w \mathbb{R}$) for every $w \in \mathbb{R}$. $\mathbb{R}$ is referred as to "weakly regular" if it is both right and left "weakly regular".
Remark 4

Every “regular semiring” is “weakly regular”.
In case $\mathbb{R}$ is commutative semiring then $\mathbb{R}$ is regular if and only if $\mathbb{R}$ is weakly regular.[3]

The following result is analogous to a similar one ring theory (see [23])

Proposition 4.12.

Let $\mathbb{R}$ be a right weakly regular, cancellative and yoked semiring. Then $\mathbb{R} = \mathbb{R} \oplus \mathbb{R}$ for any right non-zero divisor element $a$ of $\mathbb{R}$.

Proof:

Let $a$ be a right non-zero divisor element of $\mathbb{R}$. Then $\mathbb{R} = (a\mathbb{R})^2$ (since $a \in (a\mathbb{R})^2$). Assume that $rat \in \mathbb{R} = (a\mathbb{R})^2$, then by yoked property either $1 + h = rat$ or $1 = rat + h$ ... \(1\) $\rightarrow a + ah = arat$ or $a = arat + ah$, since $a$ and $arat \in (a\mathbb{R})^2$, then by subtractive, we get $ah \in (a\mathbb{R})^2$ $\rightarrow ah = auav$ for some $u, v \in \mathbb{R}$. Again, by yoked property either $h = s + uav$ or $h + s = uav$ for some $s \in \mathbb{R} \rightarrow as + uav = auav$ or $ah + as = ah$.

By cancellative property, we have $as = 0 \rightarrow s = 0 (a$ is non - zero divisor $) \rightarrow h = uav \in \mathbb{R}ah$, then by \(1\) $\in \mathbb{R} \mathbb{R} \rightarrow \mathbb{R} \mathbb{R} = \mathbb{R}$. □

The following definition is analogous to a similar one in ring theory (see [24])

Definition 4.13.

A semiring $\mathbb{R}$ is called right (left) "weakly $\pi$-regular" if $\forall x \in \mathbb{R}$ there exists a natural number $n$ such that $x^n = x^{n+1} \mathbb{R}$ \(x^\mathbb{R} = \mathbb{R}x^n\), $\mathbb{R}$ is "weakly $\pi$-regular" if it is both right and left "weakly $\pi$-regular".

Remark 5

Every "$\pi$-regular semiring" is "weakly $\pi$-regular".

The following result is analogous to a similar one in ring theory (see [23])

Proposition 4.14.

For a semiring $\mathbb{R}$, the following are equivalent:

(a) $\mathbb{R}$ is "Von Neumann regular".
(b) For each $a$ in $\mathbb{R}$, there exists an "idempotent" $e$ in $\mathbb{R}$ such that $a\mathbb{R} = e\mathbb{R}$.

Proof:

(a)$\Rightarrow$ (b) Since $\mathbb{R}$ is a "Von Neumann regular semiring", then for every element $a$ in $\mathbb{R}$ there exists an element $b$ in $\mathbb{R}$ such that $a = aba$. Now we put $e = ab$ yields $e = e^2$ for some $e$ in $\mathbb{R}$, $a\mathbb{R} = e\mathbb{R}$ (since $; a\mathbb{R} = (aba)\mathbb{R} = ea\mathbb{R} \subseteq e\mathbb{R}, e\mathbb{R} = (ab)\mathbb{R} \subseteq a\mathbb{R}$).

(b)$\Rightarrow$ (a) Assume $a\mathbb{R} = e\mathbb{R}$ where $e$ is an idempotent element. Then $a = ex$ for some $x$ in $\mathbb{R}$.

Now, $a = ex = e^2x = ea$. Let $e = ab$ (since $e \in a\mathbb{R} \rightarrow e = ab$) we get $a = aba$. So $\mathbb{R}$ is Von Neumann regular semiring. □

Definition 4.15. [10]

An ideal $I$ from the semiring is referred to as "direct summand" of $\mathbb{R}$ if there exists an ideal $J$ of $\mathbb{R}$ such that $\mathbb{R} = I + J$ and $I \cap J = 0$. We usually write $\mathbb{R} = I \oplus J$.

The following result is analogous to similar one in ring theory (see [23])

Proposition 4.16.

A cancellative and yoked semiring $\mathbb{R}$ is " Von Neumann regular" if and only if every principal right ideal of $\mathbb{R}$ is a direct summand.

Proof:

Let $\mathbb{R}$ be a von Neumann regular semiring, if $\emptyset \neq a \in \mathbb{R}$, then by proposition (4.14) $a\mathbb{R} = e\mathbb{R}$ for some idempotent element $e$ of $\mathbb{R}$. To prove $e\mathbb{R}$ is a direct summand of $\mathbb{R}$. Assume that $e$ is an idempotent element of $\mathbb{R}$ and $J = e\mathbb{R}$. If $e$ is a not zero-divisor, then $f: \mathbb{R} \rightarrow e\mathbb{R}$ defined by $r \rightarrow er$ is an isomorphism, so, $e\mathbb{R}$ is "a direct summand" of $\mathbb{R}$. If $e$ is a zero-divisor, and $eu = 0$ (for some $u \in \mathbb{R}$).

Claim: $\mathbb{R} = e\mathbb{R} + u\mathbb{R}$ for some $u$ such that $eu = 0$. We need to consider that $\mathbb{R}$ is yoked. In this case either $e + u = 1$ or $e = 1 + u$ for some $u \in \mathbb{R}$. If $e + u = 1$,
then $\mathcal{R} = e\mathcal{R} + u\mathcal{R}$ and since $e(e^+u) = e \to e^+eu = e \to eu = e \to eu = 0$. 
$x \in e\mathcal{R} \cap u\mathcal{R} \to x = er = us$ for some $r, s \in \mathcal{R}$. $x = er = er^{-1}x$ and $ex = eu = 0$, so $x = 0$. Then $\mathcal{R} = e\mathcal{R} \oplus u\mathcal{R}$. In case $e = 1 + u$, also we get $eu = 0$, too, and 
$e\mathcal{R} \cap u\mathcal{R} = 0$. On the other hand $e = 1 + u \to 0 = eu = u^+u^2 \to e^+u^2 = 1 + u$, by cancellative 
property then $1 = e + u^2 \to r = er + u^2r \in e\mathcal{R} + u\mathcal{R}$; $\forall r \in \mathcal{R} \to \mathcal{R} = e\mathcal{R} \oplus u\mathcal{R}$. Therefore $l = Re$ is "a 
direct summand" of $\mathcal{R}$.

Conversely, let $\mathcal{R} = a\mathcal{R} \oplus K$, for some ideal $K$ of $\mathcal{R}$. Now $1 = ar + k$ for some $r$ in $\mathcal{R}$ and $k$ in $K$, and 
$a = ara + ka$, but $ka \in a\mathcal{R} \cap K = 0$ implies that $a = ara$ and $\mathcal{R}$ is "Von Neumann regular" 
semiring. □

The following result is analogous to similar one in ring theory (see [17])

Proposition 4.17.

Let $\mathcal{R}$ be "a right duo semiring". The following statements are equivalent:

1. $\mathcal{R}$ is a right "weakly regular semiring";
2. $\mathcal{R}$ is "a strongly regular semiring".
3. $\mathcal{R}$ is "Von Neumann regular";

Proof:

$(1) \to (2)$. By proposition(4.12.) $\mathcal{R} = a\mathcal{R} \to 1 = rat \to a = ar at \to a = a(atas)t$, for some $s \in \mathcal{R}$, then 
$a = a'st \to a = a'b$, where $b = st$.

$(2) \to (3)$. $\mathcal{R}$ is strongly regular, then for each $a \in \mathcal{R}$ \exists $b, c$ such that $a = a'b = ca^2 
\to a = ab = aba (ab = ba$, since $\mathcal{R}$ is a right duo semiring).

$(3) \to (1)$. \forall $a \in \mathcal{R}$ \exists $b \exists a = ab \to ar = abar \to a \in a\mathcal{R}a\mathcal{R}$

This implies $\mathcal{R}$ is a right "weakly regular semiring". □

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