Large deviation asymptotics for the left tail of the sum of dependent positive random variables

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Abstract

We study the left tail behavior of the logarithm of the distribution function of a sum of dependent positive random variables. Asymptotics are computed under the assumption that the marginal distribution functions decay slowly at zero, meaning that their logarithms are slowly varying functions. This includes parametric families such as log-normal, gamma, Weibull and many distributions from the financial mathematics literature. We show that the logarithmic asymptotics of the sum in question depend on a characteristic of the copula of the random variables which we term weak lower tail dependence function, and which is computed explicitly for several families of copulas in this paper. In applications, our results may be used to quantify the diversification of long-only portfolios of financial assets with respect to extreme losses. As an illustration, we compute the left tail asymptotics for a portfolio of options in the multi-dimensional Black-Scholes model.

Key words: tail behavior, large deviations, regular variation, tail dependence, portfolio diversification, Gaussian copula

1 Introduction

We consider the tail behavior of the sum of \( n \) dependent positive random variables:

\[
X = \sum_{i=1}^{n} X_i
\]

This problem has received considerable attention in the literature, but mainly in the insurance context, where the random variables \( X_1, \ldots, X_n \) represent losses from individual claims, and one is interested in the right tail asymptotics of \( X \), so as to estimate the probability of having a very large aggregate loss. In this setting, provided the variables \( X_1, \ldots, X_n \) are sufficiently fat-tailed (subexponential)

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under various assumptions on the dependence structure, it can be shown that
the right tail behavior of \( X \) is determined by the single variable with the fattest
tail. We refer to [1, 2, 8, 10, 16, 17, 25, 5] and the references therein for precise
statements and proofs in various contexts of this result, known as the “principle
of single big jump”.

In this paper, we focus on the finance context, where the random variables
\( X_1, \ldots, X_n \) represent the prices of individual assets and \( X \) represents a long-
one-only portfolio of an investor. In this context, to estimate the probability of a
very large loss, one needs to focus on the left tail asymptotics of \( X \). Owing to
the positivity of the variables \( X_1, \ldots, X_n \), the behavior of the left tail of \( X \) turns
out to be very different from that of the right tail. Indeed, for \( \{X \geq x\} \) it is
enough that at least one of \( X_i \) satisfies \( X_i \geq x \), while for \( X \leq x \), it is necessary
that all \( X_i \) satisfy \( X_i \leq x \). It is then intuitively clear that the dependence
among \( X_1, \ldots, X_n \) plays a more important role in the left-tail asymptotics than
in the right-tail one.

When the variables \( X_1, \ldots, X_n \) are independent, the tail behavior of \( X \) can
be studied with characteristic function / Laplace transform methods. For ex-
ample, the following result, which covers, e.g., the gamma distribution, is a
straightforward consequence of the Tauberian theorem (see [3]).

**Proposition 1.** Assume that \( X_1, \ldots, X_n \) are independent and that for each \( i \),
the distribution function \( F_i(x) \) satisfies

\[
F_i(x) \sim \frac{x^{\rho_i} l_i(x)}{\Gamma(1 + \rho_i)}, \quad x \to 0,
\]

where \( \rho_i \geq 0 \) and \( l_i \) is slowly varying at zero. Then, the distribution function \( F \)
of \( X \) satisfies

\[
F(x) \sim \frac{\prod_{i=1}^n \Gamma(1 + \rho_i)}{\Gamma(1 + \rho_1 + \cdots + \rho_n)} \prod_{i=1}^n F_i(x), \quad x \to 0.
\]

However, for distribution functions which are not regularly varying, the prod-
uct of marginal probabilities \( P[X_i \leq x] \) does not provide a good approximation
for the tail of \( X \). For instance, when \( X_i \) follows the inverse Gaussian law with
density

\[
f_i(x) = \frac{\mu_i}{x^{\frac{3}{2}} \sqrt{2\pi}} e^{-\frac{(\lambda x - \mu_i)^2}{2x}},
\]

the sum \( X \) has density

\[
f_i(x) = \frac{\sum_{i=1}^n \mu_i}{x^{\frac{3}{2}} \sqrt{2\pi}} e^{-\frac{(\lambda x - \sum_{i=1}^n \mu_i)^2}{2x}}.
\]

As \( x \to 0 \), the distribution functions can be shown to satisfy

\[
F_i(x) \sim \frac{2x}{\mu_i \sqrt{2\pi}} e^{-\frac{\mu_i^2}{2x} + \lambda \mu} \quad \text{and} \quad F(x) \sim \frac{2x}{\sqrt{2\pi} \sum_{i=1}^n \mu_i} e^{-\frac{(\sum_{i=1}^n \mu_i)^2}{2x} + \lambda \sum_{i=1}^n \mu_i},
\]
which means that $F(x)$ decays much faster than $\prod_{i=1}^n F_i(x)$ as $x \to 0$.

When the variables $X_1, \ldots, X_n$ are dependent, the law of $X$ is more difficult to analyze, and very few results are available in the literature. Wüthrich \[24\] considers the left-tail asymptotics for a sum of identically distributed random variables in the domain of attraction of Weibull and Gumbel distributions (for the minimum), with dependence given by an Archimedean copula with a regularly varying generator. He finds that in these cases

$$P[X \leq nx] \sim C P[X_1 \leq x]$$

for some constant $C$, as $u$ tends to the lower bound of the support of distribution of $X_1$. In other words, the tail dependence in the Archimedean copula is so strong that asymptotically all variables become perfectly correlated and no diversification effects come into play.

However, for weaker tail dependence patterns, the situation may be very different. For instance, when $X_i$, $i = 1, \ldots, n$ are exponentials of components of a Gaussian vector (in other words, log-normal random variables with a Gaussian copula), the tail behavior of $X$ may depend on the entire covariance matrix of the Gaussian vector, and the left tail of $X$ may be much thinner than the tails of $X_1, \ldots, X_n$. This has been shown in \[9\] for $n = 2$ and more recently in \[11\] in the general case. For example, when $X_1, \ldots, X_n$ are identically distributed such that $\log X_i \sim N(\mu, \sigma^2)$, and the correlation between $X_i$ and $X_j$ is equal to $\rho$ for all $i \neq j$ with $|\rho| < 1$,

$$P[X \leq x] \sim C \left( \log \frac{1}{x} \right) \exp \left( -\frac{n}{2\sigma^2(1 + \rho(n-1))} \left\{ \log \frac{x - \mu}{n} \right\}^2 \right),$$

for some constant $C$. We see that for any value of $\rho$ the tail of $X$ is thinner than the tail of $X_1$ and for $\rho = 0$, $F(x)$ decays much faster than $\prod_{i=1}^n F_i(x)$ as $x \to 0$.

These motivating examples show that it does not seem possible, under sufficiently general assumptions, to express the asymptotics of $F(x)$ in terms of the asymptotics of $F_i(x)$ for $i = 1, \ldots, n$. For this reason, in this paper we consider a weaker logarithmic formulation, and study the limiting behavior of

$$\frac{\log P[X_1 + \cdots + X_n \leq x]}{\min_i \log P[X_i \leq x]} \quad (1)$$

as $x \to 0$.

From the mathematical point of view, the limit of this expression, which we compute explicitly in many cases, provides a large deviations estimate for the left tail of the sum of positive dependent random variables in terms of their marginal distributions.

From the applied point of view, it can be seen as a measure of asymptotic diversification of a portfolio of dependent risks. A value close to 1 indicates that the portfolio is poorly diversified, since its behavior under extreme scenarios is similar to that of the component with the thinnest tail. By contrast,
a large value corresponds to good diversification. Portfolio diversification with respect to extreme risks has recently been studied in the context of fat-tailed distributions satisfying the property of multivariate regular variation \cite{21,20,6}. The present paper complements these references by studying the left tail of a portfolio of positive assets, to which the multivariate regular variation theory does not readily apply.

We compute the limit of (1) under the following assumptions on the marginal laws.

- The logarithms of distribution functions of $X_i$ are slowly varying at 0. This assumption includes all distributions with regularly varying left tail as well as parametric families such as log-normal, gamma, Weibull and many distributions from the financial mathematics literature.

- The logarithms of the distribution functions of $X_i$ are equivalent, up to a constant, to a common function:

  $$\log F_i(x) \sim \lambda_i \log F(x).$$

  This assumption ensures that the laws of components have similar asymptotic behavior, but nevertheless is not very restrictive: for example, $X_i$ with different $i$-s can follow log-normal distributions with different parameters, or have regularly varying tails with different indices.

Under the above assumptions, we show that the limit of (1) can be expressed in terms of the coefficients $\lambda_i$ and of a characteristic of the copula of $X_1, \ldots, X_n$, which we term weak lower tail dependence function. This function is related to the weak lower tail dependence coefficient introduced in the literature in the two-dimensional case (see remark \ref{remark1}) and is defined by

$$\chi(\alpha_1, \ldots, \alpha_n) = \lim_{u \to 0} \frac{\min_i \log u^{\alpha_i}}{\log C(u^{\alpha_1}, \ldots, u^{\alpha_n})}, \quad \alpha_1, \ldots, \alpha_n \geq 0.$$  

In the particular case when the logarithmic tails of $X_1, \ldots, X_n$ are all equivalent to each other (e.g., when $\lambda_1 = \cdots = \lambda_n$), it follows that the limit of (1) does not depend on the marginal distribution of $X_1, \ldots, X_n$ and is determined exclusively by the copula-dependent quantity

$$\chi = \lim_{u \to 0} \frac{\log u}{\log C(u, \ldots, u)}.$$  

Note that for each fixed $x$, the value (1) depends both on the copula and the marginal distributions. This result also provides a new interpretation of the weak tail dependence coefficient and shows that for analyzing the tail behavior of sums of dependent random variables (portfolios of dependent risks), this measure of tail dependence is more relevant than the strong lower tail dependence coefficient defined by

$$\lambda = \lim_{u \to 0} \frac{C(u, \ldots, u)}{u}.$$
Our second main contribution is to compute the weak tail dependence function for commonly used families of copulas. Of particular interest is the result for the Gaussian copula since in applications one often has to deal with random variables $X_i$ which are not Gaussian or log-normal themselves, but whose dependence is nevertheless given by the Gaussian copula. For example, this happens when $X_1, \ldots, X_d$ are non-linear functions of risk factors which form a Gaussian random vector (see section 4 for a concrete example). More generally, the Gaussian copula is by far the most popular way of introducing dependence between non-Gaussian risk factors in finance and other domains [15, 19]. It is therefore important to understand the tail behavior of $\sum_{i=1}^{d} X_i$ when the dependence of $X_1, \ldots, X_d$ is given by a Gaussian copula, for various marginal distributions. We show that even though the strong tail dependence coefficient is zero for the Gaussian copula (see e.g. [7]), the weak tail dependence function has a nontrivial form and allows to quantify in a precise fashion the diversification effect of Gaussian dependence.

**Remarks on notation** Throughout this paper, we write $f \sim g$ as $x \to a$ whenever $\lim_{x \to a} \frac{f(x)}{g(x)} = 1$ and $f \lesssim g$ whenever $\limsup_{x \to a} \frac{f(x)}{g(x)} \leq 1$. We recall that a function $f$ is called slowly varying as $x \to 0$ whenever $\lim_{x \to 0} \frac{f(\alpha x)}{f(x)} = 1$ for all $\alpha > 0$.

We also recall that the copula of a random vector $(Y_1, \ldots, Y_d)$ is a function $C : [0, 1]^d \to [0, 1]$, satisfying the assumptions

- $dC$ is a positive measure in the sense of Lebesgue-Stieltjes integration,
- $C(u_1, \ldots, u_d) = 0$ whenever $u_k = 0$ for at least one $k$,
- $C(u_1, \ldots, u_d) = u_k$ whenever $u_i = 1$ for all $i \neq k$,

and such that

$$P[Y_1 \leq y_1, \ldots, Y_d \leq y_d] = C(P[Y_1 \leq y_1], \ldots, P[Y_d \leq y_d]), \quad (y_1, \ldots, y_d) \in \mathbb{R}^d.$$

A copula exists by Sklar’s theorem and is uniquely defined whenever the marginal distributions of $Y_1, \ldots, Y_d$ are continuous. We refer to [22] for details on copulas.

## 2 Tail asymptotics

**Definition 1.** The weak lower tail dependence function $\chi(\alpha_1, \ldots, \alpha_n)$ of a copula $C$ is defined by

$$\chi(\alpha_1, \ldots, \alpha_n) = \lim_{u \to 0} \frac{\min_i \log u^{\alpha_i}}{\log C(u^{\alpha_1}, \ldots, u^{\alpha_n})},$$

whenever the limit exists and is finite for all $\alpha_1, \ldots, \alpha_n \geq 0$ such that $\alpha_k > 0$ for at least one $k$. The weak lower tail dependence coefficient of a copula $C$ is defined by

$$\chi = \chi(1, \ldots, 1) = \lim_{u \to 0} \frac{\log u}{\log C(u, \ldots, u)}.$$
whenever the limit exists.

**Remark 1.** The weak lower tail dependence coefficient defined above is closely related to the “coefficient of tail dependence” introduced in [18] in the 2-dimensional setting. It was further studied in [4] under the name “dependence measure $\bar{\chi}$” and in a number of other papers including [23]. In particular, [13] gives the values of this index (in the two-dimensional case) for various families of copulas. The copula appears in the denominator to obtain an expression which is increasing with respect to the concordance order of copulas, meaning that stronger dependence corresponds to larger values of $\chi$.

The weak lower tail dependence function $\chi(\alpha_1, \ldots, \alpha_n)$ of a copula is order 0 homogeneous: for all $r > 0$,

$$\chi(r\alpha_1, \ldots, r\alpha_n) = \chi(\alpha_1, \ldots, \alpha_n).$$

It is increasing with respect to the concordance order of copulas and admits the following bounds (the upper bound is due to the Frechet-Hoeffding upper bound on the copula):

$$0 \leq \chi(\alpha_1, \ldots, \alpha_n) \leq 1.$$ 

The upper bound is attained for the complete dependence copula $C_\parallel(u_1, \ldots, u_n) = \min(u_1, \ldots, u_n)$. On the other hand, for the independence copula $C_\perp(u_1, \ldots, u_n) = u_1 \ldots u_n$, we get

$$\chi(\alpha_1, \ldots, \alpha_n) = \frac{\max_i \alpha_i}{\sum_i \alpha_i}.$$ 

Weak lower tail dependence functions for various copula families will be computed in section 3.

The following theorem is the main result of this paper.

**Theorem 1.** Let $X_1, \ldots, X_n$ be random variables with values in $(0, \infty)$ with marginal distribution functions $F_1, \ldots, F_n$ and copula $C$ satisfying the following assumptions.

- For each $k = 1, \ldots, n$, $F_k$ is slowly varying at zero and satisfies
  $$\log F_k(x) \sim \lambda_k \log F(x)$$
  for some constant $\lambda_k > 0$ and some function $F$.
- The copula $C$ admits a weak lower tail dependence function $\chi$.

Then,

$$\lim_{x \downarrow 0} \frac{\log \mathbb{P}[X_1 + \cdots + X_n \leq x]}{\min_i \log \mathbb{P}[X_i \leq x]} = \frac{1}{\chi(\lambda_1, \ldots, \lambda_n)}.$$ 

**Proof.** We first establish an upper bound on $\mathbb{P}[X_1 + \cdots + X_n \leq x]$.

$$\mathbb{P}[X_1 + \cdots + X_n \leq x] \leq \mathbb{P}[X_1 \leq x, \ldots, X_n \leq x] = C(F_1(x), \ldots, F_n(x)).$$

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By assumption of the theorem, for any $\varepsilon > 0$ and $x$ small enough,

$$ F_k(x) \leq F(x)^{\lambda_k (1-\varepsilon)}, \; k = 1, \ldots, n. $$

Therefore,

$$ P[X_1 + \cdots + X_n \leq x] \leq C(F(x)^{\lambda_1 (1-\varepsilon)}, \ldots, F(x)^{\lambda_n (1-\varepsilon)}) $$

and by definition of the weak lower tail dependence function, for $x$ small enough, we then have

$$ P[X_1 + \cdots + X_n \leq x] \leq F(x)^{\chi^{-1}(\lambda_1, \ldots, \lambda_n)(1-\varepsilon)^2 \max_i \lambda_i}. $$

On the other hand,

$$ P[X_1 + \cdots + X_n \leq x] \geq P[X_1 \leq \frac{x}{n}, \ldots, X_n \leq \frac{x}{n}], $$

which, by a computation similar to the above one leads to the lower bound

$$ P[X_1 + \cdots + X_n \leq x] \geq F(x/n)^{\chi^{-1}(\lambda_1, \ldots, \lambda_n)(1+\varepsilon)^2 \max_i \lambda_i}. $$

Taking the logarithms and using the fact that $\varepsilon$ is arbitrary and $\log F$ is slowly varying shows that

$$ \lim_{x \downarrow 0} \frac{\log P[X_1 + \cdots + X_n \leq x]}{\max_i \lambda_i \log F(x)} = \chi^{-1}(\lambda_1, \ldots, \lambda_n) $$

and therefore

$$ \lim_{x \downarrow 0} \frac{\log P[X_1 + \cdots + X_n \leq x]}{\log \min_i P[X_i \leq x]} = \chi^{-1}(\lambda_1, \ldots, \lambda_n). $$

\[
\square
\]

**Corollary 1.** Let $X_1, \ldots, X_n$ be random variables with values in $(0, \infty)$ with marginal distribution functions $F_1, \ldots, F_n$ and copula $C$ satisfying the following assumptions.

- For each $k = 1, \ldots, n$, $F_k$ is slowly varying at zero and satisfies

  $$ \log F_k(x) \sim \log F(x) $$

  for some function $F$.

- The copula $C$ admits a weak lower tail dependence coefficient $\chi$.

Then,

$$ \lim_{x \downarrow 0} \frac{\log P[X_1 + \cdots + X_n \leq x]}{\min_i \log P[X_i \leq x]} = \frac{1}{\chi}. $$
3 Weak lower tail dependence function for common copula families

The degree of tail dependence in a given copula function may be quantified by the strong tail dependence coefficient, which is defined (for the left tail) by

$$\lambda_L = \lim_{u \downarrow 0} \frac{C(u, \ldots, u)}{u},$$

whenever the limit exists. When $\lambda > 0$, the copula is said to have strong tail dependence in the left tail. Strong tail dependence coefficients for different copula families are listed, for instance, in [22, 13]. In particular, it is known that the Gaussian copula does not have strong tail dependence. The following simple result shows that if a copula has strong tail dependence in the left tail, its weak tail dependence function is equal to the upper bound. Weak and strong tail dependence are thus “orthogonal” notions, which are relevant for different dependence regimes.

**Proposition 2.** Assume that a copula function $C$ has strong tail dependence in the left tail with coefficient $\lambda_L > 0$. Then, the weak lower tail dependence function of $C$ is equal to the upper bound:

$$\chi(\alpha_1, \ldots, \alpha_n) = 1.$$

**Proof.** From the definition of $\lambda_L$, for any $\varepsilon > 0$ and $u$ sufficiently small,

$$C(u, \ldots, u) \geq (\lambda_L - \varepsilon)u.$$

Using the fact that the copula is increasing in each argument, we have, for $u$ sufficiently small,

$$\frac{\log C(u^{\alpha_1}, \ldots, u^{\alpha_n})}{\log u} \leq \frac{\log(\lambda_L - \varepsilon) + \max(\alpha_1, \ldots, \alpha_n) \log u}{\log u},$$

which shows that

$$\limsup_{u \downarrow 0} \frac{\log C(u^{\alpha_1}, \ldots, u^{\alpha_n})}{\log u} = \max(\alpha_1, \ldots, \alpha_n).$$

Combining this with the Frechet-Hoeffding upper bound on the copula, the proof is complete.

The above proposition implies in particular that for all copulas of elliptical distributions with regularly varying tails, including, in particular, the $t$-copula, which are known to have strong tail dependence [14], the weak tail dependence function is equal to 1.

The Gaussian copula with correlation matrix $R$ is the unique copula of any Gaussian vector with correlation matrix $R$ and nonconstant components (it does not depend on the mean vector and on the variances of the components). The
following proposition characterizes the weak lower tail dependence function of
the Gaussian copula. In this proposition, we define

\[ \Delta_n := \{ w \in \mathbb{R}^n : w_i \geq 0, i = 1, \ldots, n, \sum_{i=1}^n w_i = 1 \} . \]

**Proposition 3.** Let \( C \) be a Gaussian copula with correlation matrix \( R \) with \( \det R \neq 0 \). Then,

\[ \chi(\alpha_1, \ldots, \alpha_n) = \max_i \alpha_i \min_{w \in \Delta_n} w^T \Sigma w, \quad \text{for all } \alpha_1, \ldots, \alpha_n > 0, \]

where the matrix \( \Sigma \) has coefficients \( \Sigma_{ij} = \frac{R_{ij}}{\sqrt{\alpha_i \alpha_j}}, 1 \leq i, j \leq n \).

**Proof.** Let \((X_1, \ldots, X_n)\) be a centered Gaussian vector with covariance matrix \( \Sigma \) defined above. From results in [12], one can deduce that there exist positive constants \( c \) and \( C \) such that, for all \( z \) sufficiently small,

\[ c \frac{e^{-\frac{2}{2 \inf_{w \in \Delta_n} w^T \Sigma w} \leq \mathbb{P}[X_1 \leq z, \ldots, X_n \leq z] \leq C \frac{e^{-\frac{2}{2 \inf_{w \in \Delta_n} w^T \Sigma w}}}{\bar{n}^n e^{-\frac{1}{2} \sum_{i=1}^n w_i^2}} \]

where \( \bar{n} = \# \{ i = 1, \ldots, n : \tilde{w}_i > 0 \} \) and \( \tilde{w} = \arg \inf_{w \in \Delta_n} w^T \Sigma w \). This means that

\[ \log \mathbb{P}[X_1 \leq z, \ldots, X_n \leq z] \sim -\frac{z^2}{2 \inf_{w \in \Delta_n} w^T \Sigma w} \]

as \( z \to -\infty \). Applying this to a single Gaussian variable yields \( \mathbb{P}[X_i \leq z] \sim -\frac{z^2 \alpha_i}{2} \) as \( z \to -\infty \). Now combine these estimates to get, for \( \varepsilon \) and \( z \) small enough,

\[ -\frac{z^2(1 + \varepsilon)}{2 \inf_{w \in \Delta_n} w^T \Sigma w} \leq \log \mathbb{P}[X_1 \leq z, \ldots, X_n \leq z] = \log C(\mathbb{P}[X_1 \leq z], \ldots, \mathbb{P}[X_n \leq z]) \]

\[ \leq \log C(e^{-\frac{z^2 \alpha_1(1-\varepsilon)}{2}}, \ldots, e^{-\frac{z^2 \alpha_n(1-\varepsilon)}{2}}). \]

Letting \( u = e^{-\frac{z^2(1-\varepsilon)}{2}} \), this leads to

\[ \frac{1 + \varepsilon}{(1 - \varepsilon) \inf_{w \in \Delta_n} w^T \Sigma w} \log u \leq \log C(u^{\alpha_1}, \ldots, u^{\alpha_n}). \]

Dividing by \( \min_i \log u^{\alpha_i} \), and using the fact that \( \varepsilon \) is arbitrary, we finally get

\[ \max_i \alpha_i \inf_{w \in \Delta_n} w^T \Sigma w \geq \limsup_{u \to 0} \frac{\min_i \log u^{\alpha_i}}{\log C(u^{\alpha_1}, \ldots, u^{\alpha_n})}. \]

The lower bound may be obtained in a similar fashion.

**Finally we recall that given a function \( \phi : [0, 1] \to [0, \infty] \) which is continuous, strictly decreasing and such that its inverse \( \phi^{-1} \) is completely monotonic, the Archimedean copula with generator \( \phi \) is defined by

\[ C(u_1, \ldots, u_n) = \phi^{-1}(\phi(u_1) + \cdots + \phi(u_n)). \]
The following simple result gives the weak lower tail dependence function for an Archimedean copula. The case when \( \log \phi^{-1} \) is regularly varying includes for example the Gumbel copula with \( \phi^{-1}(t) = \exp(-t^{1/\theta}) \) and several other families.

**Proposition 4.** Let \( C \) be an Archimedean copula with generator function \( \phi \).

(i) If \( \log \phi^{-1} \) is regularly varying at \( +\infty \) with index \( \lambda > 0 \), then,

\[
\chi(\alpha_1, \ldots, \alpha_n) = \frac{\max(\alpha_1, \ldots, \alpha_n)}{(\alpha_1^{1/\lambda} + \cdots + \alpha_n^{1/\lambda})^\lambda}
\]

(ii) If \( \log \phi^{-1} \) is slowly varying at \( +\infty \), then

\[
\chi(\alpha_1, \ldots, \alpha_n) = 1
\]

**Proof.** Assume first that \( \log \phi^{-1} \) is regularly varying with index \( \lambda > 0 \). By definition of \( \chi \),

\[
\chi(\alpha_1, \ldots, \alpha_n) = \lim_{u \to 0} \frac{\max(\alpha_1, \ldots, \alpha_n) \log u}{\log \phi^{-1}(\phi(u^{\alpha_1}) + \cdots + \phi(u^{\alpha_n}))}
\]

\[
= \lim_{u \to 0} \frac{\max(\alpha_1, \ldots, \alpha_n) \log u}{\log \phi^{-1}(\phi(e^{\alpha_1 \log u}) + \cdots + \phi(e^{\alpha_n \log u}))}
\]

By the inversion theorem for regularly varying functions \([3]\), the function \( u \to \phi(e^u) \) is regularly varying at \(-\infty \) with index \( \frac{1}{\lambda} \). Therefore, for any \( \varepsilon > 0 \) and \( u \) sufficiently small,

\[
(1 - \varepsilon)(\alpha_1^{1/\lambda} + \cdots + \alpha_n^{1/\lambda})\phi(u) \leq \phi(e^{\alpha_1 \log u}) + \cdots + \phi(e^{\alpha_n \log u}) \leq (1 + \varepsilon)(\alpha_1^{1/\lambda} + \cdots + \alpha_n^{1/\lambda})\phi(u),
\]

and we conclude using the regular variation of \( \log \phi^{-1} \) and the fact that \( \varepsilon \) is arbitrary. The proof for the case when \( \log \phi^{-1} \) is slowly varying is similar.

\Box

### 4 An application to finance

In this section we show how the asymptotic results obtained in this note may be used to analyze the tail behavior of a portfolio of options. Fix a time horizon \( T \) and let \((X_1, \ldots, X_n)\) denote the vector of logarithmic returns of \( n \) risky assets under real-world measure over this time horizon. The asset prices at date \( T \) are then given by \( S_t = e^{\lambda t} \) for \( i = 1, \ldots, n \) where we have assumed without loss of generality that the initial values of all assets are normalized to 1. We suppose that the \( n \) risky assets follow the multidimensional Black-Scholes model. This
means that the distribution of the vector \((X_1, \ldots, X_n)\) is Gaussian, and we denote by \(\Sigma T\) its covariance matrix and by \(\mu T\) its mean vector.

We are interested in the tail behavior a long-only portfolio of European call options written on \(n\) risky assets. To simplify the discussion we assume that the portfolio contains exactly one option on each of the risky assets, but the setting can obviously be extended to an arbitrary number of options. The log-strikes of the options will be denoted by \((k_1, \ldots, k_n)\) and the maturity dates by \((T_1, \ldots, T_n)\), where \(T_i > T\) for \(i = 1, \ldots, n\). Assuming that the interest rate is zero, the price of \(i\)-th option at date \(T\) is given by the Black-Scholes formula:

\[
P_i = e^{X_i} N(d_1) - e^{k_i} N(d_2), \quad d_{1,2} = \frac{X_i - k_i}{\sigma_i \sqrt{T_i - T}} \pm \frac{\sigma_i \sqrt{T_i - T}}{2}, \quad \sigma_i = \sqrt{\Sigma_{ii}},
\]

where \(N\) is the standard normal distribution function.

The following proposition clarifies the asymptotic behavior of the probability \(\mathbb{P}[P_1 + \cdots + P_n \leq z]\) as \(z \to 0\).

**Proposition 5.** As \(z \to 0\),

\[
\log \mathbb{P}[P_1 + \cdots + P_n \leq z] \sim -\frac{\log \frac{1}{\inf_{w \in \Delta_n} w^t Rw}}{\inf_{w \in \Delta_n} w^t Rw},
\]

where \(R\) is a \(n \times n\) matrix with elements given by \(R_{ij} = \frac{\Sigma_{ij} T}{\sigma_i \sigma_j \sqrt{(T_i - T)(T_j - T)}}\).

**Proof.** \(P_1, \ldots, P_n\) are obviously increasing and continuous functions of the Gaussian random variables \((X_1, \ldots, X_n)\). Therefore, the copula of \((P_1, \ldots, P_n)\) is the Gaussian copula with correlation matrix with elements \(\rho_{ij} = \frac{\Sigma_{ij}}{\sigma_i \sigma_j}\). It remains to characterize the asymptotic behavior of the distribution functions of \(P_1, \ldots, P_n\).

Let \(\tilde{X}_i = \frac{X_i - \mu_i T}{\sigma_i \sqrt{T_i - T}}\) for \(i = 1, \ldots, n\) and define

\[
f_i(x) = e^{\mu_i T + x \sigma_i \sqrt{T_i - T}} N(d_1(x)) - e^{k_i} N(d_2(x)), \quad d_{1,2}(x) = x \sqrt{\frac{T - T_i}{T - T}} - \frac{\mu_i T + k_i}{\sigma_i \sqrt{T_i - T}} \pm \frac{\sigma_i \sqrt{T_i - T}}{2}.
\]

Then, \(\tilde{X}_i\) is a standard normal random variable. From the well-known equivalence

\[
N(x) \sim \frac{e^{-x^2/2}}{|x| \sqrt{2\pi}}, \quad x \to -\infty,
\]

one easily deduces that

\[
f_i(x) \sim \frac{\sigma_i (T_i - T)^{3/2}}{x^2 T \sqrt{2\pi}} e^{k_i \frac{d_2^2(x)}{2}}, \quad x \to -\infty.
\]

Taking the logarithm, we obtain

\[
\log f_i(x) \sim -\frac{x^2 T}{2(T_i - T)}, \quad x \to -\infty
\]
and
\[ f_i^{-1}(u) \sim \sqrt{\frac{2}{T} \frac{T_i - T}{T_i}} \log \frac{1}{u}, \quad u \to 0. \]

Therefore, the distribution function of \( P_i \) satisfies
\[
\log \mathbb{P}[P_i \leq x] = \log N(f_i^{-1}(x)) \sim -\frac{f_i^{-1}(x)^2}{2} \sim \frac{T_i - T}{T} \log \frac{1}{x}, \quad x \downarrow 0,
\]
so that the assumptions of Theorem 1 are satisfied with \( \lambda_i = \frac{T_i - T}{T} \) and \( F(x) = \frac{1}{x} \), and the result follows by applying Proposition 3 and Theorem 1.

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