**DENSELY LOCALLY MINIMAL GROUPS**

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**Abstract.** We study locally compact groups having all dense subgroups (locally) minimal. We call such groups densely (locally) minimal. In 1972 Prodanov proved that the infinite compact abelian groups having all subgroups minimal are precisely the groups \( \mathbb{Z}_p \) of \( p \)-adic integers. In [31], we extended Prodanov’s theorem to the non-abelian case at several levels. In this paper, we focus on the densely (locally) minimal abelian groups.

We prove that in case that a topological abelian group \( G \) is either compact or connected locally compact, then \( G \) is densely locally minimal if and only if \( G \) either is a Lie group or has an open subgroup isomorphic to \( \mathbb{Z}_p \) for some prime \( p \). This should be compared with the main result of [9]. Our Theorem C provides another extension of Prodanov’s theorem: an infinite locally compact group is densely minimal if and only if it is isomorphic to \( \mathbb{Z}_p \). In contrast, we show that there exists a densely minimal, compact, two-step nilpotent group that neither is a Lie group nor it has an open subgroup isomorphic to \( \mathbb{Z}_p \).

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1. Introduction

A Hausdorff group \((G, \tau)\) is called *minimal* if there exists no Hausdorff group topology on \( G \) which is strictly coarser than \( \tau \). Equivalently, if every continuous isomorphism \( G \to H \), with a Hausdorff group \( H \), is a topological isomorphism. Introducing independently this notion, Doïtchinov and Stephenson [14, 28] also provided the first examples of minimal groups which are not compact. For more information on minimal groups we refer the reader to [3, 13, 15, 25] (see also the survey [10] and the book [12]). Note that locally compact groups need not be minimal. Actually, Stephenson proved the following.

**Fact 1.1.** [28, Theorem 1] A minimal locally compact abelian group is compact.

Morris and Pestov [23] introduced the following class of groups which contains all minimal groups and all locally compact groups. A Hausdorff group \((G, \tau)\) is called *locally minimal* if there exists a neighborhood \( V \) of the identity of \( G \), such that for every coarser Hausdorff group topology \( \sigma \subseteq \tau \) with \( V \in \sigma \) one has \( \sigma = \tau \).

Neither minimality nor local minimality is inherited by all subgroups. For this reason, we studied in [31] the locally compact groups having all subgroups (locally) minimal, introducing the following terminology.

**Definition 1.2.** [31, Definition 1.2] A topological group \( G \) is said to be *hereditarily (locally) minimal*, if every subgroup of \( G \) is (locally) minimal.

Prodanov proved that the \( p \)-adic integers are the only infinite hereditarily minimal compact groups, namely:

**Fact 1.3.** [25] An infinite compact abelian group \( K \) is isomorphic to \( \mathbb{Z}_p \) for some prime \( p \) if and only if \( K \) is hereditarily minimal.

In [31] we extended this result to non-abelian groups at various levels of non-commutativity (in particular, see the solvable case [31, Theorem D]). As a starting point of our current research consider the following extension of Prodanov’s theorem which unifies Lie theory and \( p \)-adic numbers “under the same umbrella”.

**Fact 1.4.** [9, Corollary 1.11] For a locally compact group \( K \) that is either abelian or connected the following conditions are equivalent:

(a) \( K \) is a hereditarily locally minimal group;

(b) \( K \) either is a Lie group or has an open subgroup isomorphic to \( \mathbb{Z}_p \) for some prime \( p \).

In this paper, we study the topological groups having all dense subgroups (locally) minimal. We mainly focus on locally compact abelian groups.

**Definition 1.5.** A topological group \( G \) is said to be *densely (locally) minimal*, if every dense subgroup of \( G \) is (locally) minimal.

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Here are some examples of compact abelian groups that are not densely locally minimal.

Example 1.6.

(1) The Pontryagin dual $\hat{\mathbb{Q}}$ of the discrete group $\mathbb{Q}$ is monothetic, i.e., $\hat{\mathbb{Q}} = \mathbb{C}$ for some (infinite) cyclic subgroup $C$ of $\hat{\mathbb{Q}}$. Obviously $C$ is (algebraically) isomorphic to $\mathbb{Z}$, but it does not carry the $p$-adic topology for any prime $p$. So $C$ is not minimal, since the $p$-adic topologies are the only minimal topologies on $\mathbb{Z}$ (see for example, [10, page 5]). Moreover, as $\hat{\mathbb{Q}}$ is a compact torsion-free abelian group we deduce that $C$ is not even locally minimal by [2, Corollary 4.7].

(2) By [9, Lemma 3.1], $\mathbb{Z}_p^2$ has a dense subgroup that fails to be locally minimal.

(3) If $p \neq q$ are primes, then $\mathbb{Z}_p \times \mathbb{Z}_q$ is not densely locally minimal by [9, Lemma 3.2].

In fact, Theorem A, which generalizes in part Fact 1.4, characterizes all compact abelian groups that are densely locally minimal. This answers [9, Question 5.3] in the positive for compact groups.

**Theorem A.** For a compact abelian group $K$, the following conditions are equivalent:

(a) $K$ is a hereditarily locally minimal group;
(b) $K$ is a densely locally minimal group;
(c) $K$ either is a Lie group or has an open subgroup isomorphic to $\mathbb{Z}_p$ for some prime $p$.

For compact two-step nilpotent groups conditions (a) and (b) of Theorem A are not equivalent (see Example 3.11).

Theorem A and the next result, which provides another connection to Fact 1.4, are both proved in Section 3 by using some preliminary results from Section 2.

**Theorem B.** For a connected locally compact abelian group $G$, the following conditions are equivalent:

(a) $G$ is a hereditarily locally minimal group;
(b) $G$ is a densely locally minimal group;
(c) $G$ is a Lie group.

In case a topological abelian group is either compact or connected locally compact, then it is densely locally minimal if and only if it is hereditarily locally minimal by Theorem A and Theorem B. Negatively answering [9, Question 5.3], we now show that this equivalence is not true in general.

**Example 1.7.** Obviously, if a locally minimal group has no proper dense subgroups, it is densely locally minimal. In particular, this implication is true for locally compact groups.

Khan [22, Proposition 5.1] proved that a non-discrete locally compact abelian group $G$ has no proper dense subgroups if and only if $G$ is totally disconnected, every compact subgroup of $G$ is torsion, and $pG$ is an open subgroup of $G$ for every prime $p$. In particular, such a group neither is a Lie group nor it contains a copy of $\mathbb{Z}_p$ for any prime $p$. By Fact 1.4, $G$ is not hereditarily locally minimal.

For a concrete example, consider the following construction. Equip $K = \mathbb{Z}(2^\omega)$ with the usual product topology, and consider the group $G = \mathbb{Z}(2^\omega)\times$, equipped with the smallest group topology having $K$ open. Then $G$ is non-discrete, locally compact, divisible, abelian group, and one can check that $G$ has no proper dense subgroups (see for example [4, §7.2]).

Apart from taking subgroups, the class of minimal groups is also not stable under taking Hausdorff quotients, so Dikranjan and Prodanov [11] introduced the following stronger notion: a topological group is **totally minimal** if all of its Hausdorff quotients are minimal. In this paper, we also consider the topological groups having all dense subgroups totally minimal.

**Definition 1.8.** A topological group $G$ is said to be **densely totally minimal**, if every dense subgroup of $G$ is totally minimal.

By the Total Minimality Criterion (see Fact 2.2(3)), a group is densely totally minimal if and only if it is totally minimal and every dense subgroup is totally dense. This implies the following reformulation of T. Soundararajan [27] (see also [3, Theorem 3.1] and [12, Exercise 5.5.6]).

**Fact 1.9.** An infinite compact abelian group that is densely totally minimal is isomorphic to $\mathbb{Z}_p$ for some prime $p$.

As a densely totally minimal group is obviously densely minimal, we extend the above fact as follows.

**Theorem C.** For an infinite locally compact abelian group $K$, the following conditions are equivalent:

(a) $K$ is a hereditarily minimal group;
(b) $K$ is a densely minimal group;
(c) $K$ is isomorphic to $\mathbb{Z}_p$ for some prime $p$.

We prove Theorem C in Section 4 where dense minimality and hereditary minimality are compared in general. In particular, we see that these properties are not equivalent even for compact two-step nilpotent groups (see Example 3.11).

In [31, Proposition 3.9], we proved that a hereditarily minimal locally compact group is totally disconnected, while Example 4.6 shows that there exist densely totally minimal compact groups that are pathwise connected.

Section 5 collects some open questions and final remarks. Furthermore, Proposition 5.5 and Question 5.6 deal with the Hilbert-Smith Conjecture.

The next diagram describes some of the interrelations between the properties considered so far. The concepts, which are introduced in Definition 1.2 and Definition 1.3, are abbreviated here to HM, HLM, DM and DLM. The double arrows denote implications that always hold. The single arrows denote implications valid under some additional assumptions reported in brackets.

(1): This implication holds true for abelian groups that are either compact (Theorem A), or connected locally compact (Theorem B).
(2): This implication holds true for compact torsion-free groups (for a more general result see [31, Theorem B]).
(3): This implication holds true for locally compact abelian groups (Theorem C), but fails even for compact two-step nilpotent groups (Example 3.11).
(4): This implication holds true for compact groups having no finite normal non-trivial subgroups, that are either abelian ([2, Corollary 4.7]) or totally disconnected ([31, Proposition 3.11]).

2. Notation and preliminary results

We denote by $\mathbb{Z}$ the group of integers, and by $\mathbb{N}$ and $\mathbb{N}_+$ its subsets of non-negative integers and positive natural numbers, respectively. The groups of rationals, reals and the unit circle are denoted, respectively, by $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{T}$. For $n \in \mathbb{N}_+$, we denote by $\mathbb{Z}(n)$ the finite cyclic group with $n$ elements. For a prime number $p$, $\mathbb{Q}_p$ stands for the field of $p$-adic numbers, $\mathbb{Z}_p$ is its subring of $p$-adic integers and $\mathbb{Z}(p^\infty)$ is the quasicyclic $p$-group (the Prüfer group).

Let $G$ be a group. The abbreviation $K \leq G$ is used to denote a subgroup $K$ of $G$ and $e$ denotes the identity element. If $A$ is a non-empty subset of $G$, we denote by $\langle A \rangle$ the subgroup of $G$ generated by $A$. In particular, if $x$ is an element of $G$, then $\langle x \rangle$ is a cyclic subgroup. If $F = \langle x \rangle$ is finite, then $x$ is called a torsion element. We denote by $t(G)$ the torsion part of a group $G$, and $G$ is called torsion if $t(G) = G$, while $G$ is called torsion-free if $t(G)$ is trivial. If $G$ is abelian, then $t(G) \leq G$. For an abelian group $G$ and $n \in \mathbb{N}_+$, let $nG = \{ng : g \in G\}$. Then $G$ is bounded if $nG$ is trivial for some $n \in \mathbb{N}_+$, and a bounded group is torsion. If $G = nG$ for every $n \in \mathbb{N}_+$, then $G$ is called divisible.

All topological groups in this paper are assumed to be Hausdorff. The closure of $H$ in $G$ is denoted by $\overline{H}$, and $e(G)$ is the connected component of $G$. The weight and density of $G$ are denoted by $w(G)$ and $d(G)$, respectively. The Pontryagin dual of a locally compact abelian group $G$ is denoted by $\hat{G}$.

All unexplained terms related to general topology can be found in [16]. For background on abelian groups, see [17].

There exist useful criteria for establishing the minimality, local minimality, and total minimality of a dense subgroup of a minimal, locally minimal, and totally minimal group, respectively. These criteria are based on the following definitions.

**Definition 2.1.** Let $H$ be a subgroup of a topological group $G$.

1. $H$ is *essential* in $G$ if $H \cap N \neq \{e\}$ for every non-trivial closed normal subgroup $N$ of $G$.
2. $H$ is *locally essential* in $G$ if there exists a neighborhood $V$ of $e$ in $G$ such that $H \cap N \neq \{e\}$ for every non-trivial closed normal subgroup $N$ of $G$ which is contained in $V$.
3. $H$ is *totally dense* in $G$ if $H \cap N$ is dense in $N$ for every closed normal subgroup $N$ of $G$. 
Obviously, a totally dense subgroup of $G$ is dense and essential in $G$, and an essential subgroup is locally essential.

**Fact 2.2.** Let $H$ be a dense subgroup of a topological group $G$.

(1) **Minimality Criterion** | $H$ is minimal if and only if $G$ is minimal and $H$ is essential in $G$ (for compact $G$ see also [24, 28]).

(2) **Local Minimality Criterion** | $H$ is locally minimal if and only if $G$ is locally minimal and $H$ is locally essential in $G$.

(3) **Total Minimality Criterion** | $H$ is totally minimal if and only if $G$ is totally minimal and $H$ is totally dense in $G$.

By the Minimality Criterion, a topological group is densely minimal if and only if it is minimal, and every dense subgroup is essential. Similarly, by the Local Minimality Criterion, a topological group is densely locally minimal if and only if it is locally minimal, and every dense subgroup is locally essential.

The following easy lemma is used in our main theorems.

**Lemma 2.3.** If a direct product is densely (locally) minimal, then each of its factors is densely (locally) minimal.

**Proof.** Let $K_i$, $i \in I$, be topological groups, and assume $K = \prod_{i \in I} K_i$ is densely (locally) minimal. Fix an index $i \in I$, and let $H_i$ be a dense subgroup of $K_i$. Then $H = H_i \times \prod_{i \in I \setminus \{i\}} K_j$ is a dense subgroup of $K$, so $H$ is (locally) minimal. Thus $H_i$ is (locally) minimal itself. □

Using Example 1.6(2)-(3) and Lemma 2.3 we obtain the following.

**Corollary 2.4.** Let $G = \prod_{p} \mathbb{Z}_p^{\kappa_p}$ be infinite, where $\kappa_p$ is a cardinal for every prime $p$. Then $G$ is densely locally minimal if and only if $G = \mathbb{Z}_p$ for some prime $p$.

The following lemma generalizes [9, Lemma 3.3], in which only the case when $k$ is a prime number and $\alpha = \omega$ is considered. Recall that a topological group is NSS if it has a neighborhood of the identity that does not contain non-trivial subgroups.

**Lemma 2.5.** If $k$ is a positive integer, and $\alpha$ is an infinite cardinal, then the compact group $\mathbb{Z}(k)^{\alpha}$ is not densely locally minimal. In particular, an infinite compact bounded abelian group is not densely locally minimal.

**Proof.** We show that the direct sum $\mathbb{Z}(k)^{\alpha}$ is not locally minimal. Consider any countable partition of $\alpha$ into infinite subsets, $\alpha = \bigcup_{i \in \mathbb{N}} P_i$, with infinite $P_i$’s. For each $i \in \mathbb{N}$, let $D_i$ be the diagonal of $\mathbb{Z}(k)^{P_i}$. Then $\prod_{i \in \mathbb{N}} D_i$ is a closed subgroup of $\prod_{i \in \mathbb{N}} \mathbb{Z}(k)^{P_i} = \mathbb{Z}(k)^{\alpha}$ that is not NSS and that trivially meets $\mathbb{Z}(k)^{\alpha}$. Hence $\mathbb{Z}(k)^{\alpha}$ fails to be locally minimal by [9, Lemma 2.10].

For the second assertion, let $G$ be an infinite bounded abelian group. Then $G$ has the form $G = \prod_{i=1}^n \mathbb{Z}(k_i)^{\alpha_i}$ for positive integers $n$, $k_1, \ldots, k_n$, and cardinal numbers $\alpha_1, \ldots, \alpha_n$, with, say, $\alpha_1$ infinite. Then $\mathbb{Z}(k_1)^{\alpha_1}$ is not densely locally minimal by the first part of the proof, so that $G$ is not densely locally minimal by Lemma 2.3. □

### 3. Proof of Theorem A

In this section we prove Theorem A and Theorem B, while Theorem C is proved in Section 4.

According to [8], a topological abelian group $G$ is $w$-divisible if $w(G) = w(mG) \geq \omega$ for every $m \in \mathbb{N}_+$. One of the key ingredients in the proof of Theorem A and Theorem C is the following splitting theorem for compact abelian groups.

**Fact 3.1.** [8, Theorem 1.3] Let $K$ be a compact abelian group. Then $K$ splits topologically in a direct product $K = K_{tor} \times K_d$, where $K_{tor}$ is a compact bounded abelian group, while the compact abelian group $K_d$ is $w$-divisible.

The dual of the above result, namely a factorization theorem for discrete abelian groups, is proved in [18].

Recall that a topological group is linearly topologized if it has a local base at the identity consisting of open subgroups (the term non-archimedean is also used by some authors). For example, profinite groups are linearly topologized, and indeed in Theorem A we apply the following fact to profinite groups.

**Fact 3.2.** [8, Proposition 3.4(c)] Let $G$ be a topological abelian group and let $H$ be a subgroup of $G$. If $H$ is locally essential in $G$, then for every closed and linearly topologized subgroup $N$ of $G$, there exists an open subgroup $V$ of $N$ such that $H \cap V$ is essential in $V$.

The following results which appear in [7] are frequently used in the sequel.

**Fact 3.3.**

1. If $K$ is a $w$-divisible compact abelian group, then $K$ admits a dense free abelian subgroup $F$ with $|F| = d(K)$. 
(2) Let $N$ be a quotient of $G$ with $h : G \to N$ the canonical projection. If $H$ is a subgroup of $G$ such that $h(H)$ is dense in $N$ and $H$ contains a dense subgroup of $ker h$, then $H$ is dense in $G$.

The following lemma is used in the subsequent Proposition 3.5.

**Lemma 3.4.** Let $G$ be a topological abelian group, and $\kappa$ be a cardinal such that:

1. $d(G) \leq \kappa \cdot \omega$;
2. $G$ has a subgroup $R \cong \mathbb{Z}_p^\kappa$ for some prime number $p$.

If $G$ is densely locally minimal, then $\kappa$ is finite. In particular, $G$ is separable.

**Proof.** Assume by contradiction $\kappa$ to be infinite, and let $H$ be a dense subgroup of $G$ with $|H| = d(G) \leq \kappa \cdot \omega = \kappa$. Then $H$ is locally minimal, so it is locally essential in $G$ by the Local Minimality Criterion. Using Fact 3.2, we find an open subgroup $U$ of $R$ (necessarily isomorphic to $\mathbb{Z}_p^\kappa$), such that $H \cap U$ is essential in $U$ and clearly $|H \cap U| \leq |H| \leq \kappa$.

As $\kappa$ is infinite, the group $\mathbb{Z}_p^\kappa$ has a family of size $2^\kappa$ of closed subgroups isomorphic to $\mathbb{Z}_p$ with pairwise trivial intersection (see [12]). The essentiality of $H \cap U$ in $U$ implies that $|H \cap U| \geq 2^\kappa$, a contradiction. $\square$

In the next proposition, we consider some conditions guaranteeing that a quotient of a densely locally minimal compact abelian group is a finite power of $\mathbb{T}$.

**Proposition 3.5.** Let $G$ be a compact abelian group, $N$ be a closed subgroup of $G$, and $\kappa$ be a cardinal such that $d(N) \leq \kappa \cdot \omega$ and $G/N \cong \mathbb{T}^\kappa$. If $G$ is densely locally minimal, then $\kappa$ is finite.

**Proof.** It suffices to check that $G$ satisfies conditions (1) and (2) of Lemma 3.4 when $\kappa$ is infinite. So let $k = \kappa \cdot \omega$ be infinite, and let $q : G \to G/N$ be the canonical projection.

It is easy to see that $\mathbb{T}^\kappa$ is w-divisible, so there exists a dense free subgroup $B$ of $\mathbb{T}^\kappa$ with $|B| \leq \kappa$ by Fact 3.3(1). As $B$ is free, there exists a free subgroup $B_1$ of $G$ such that $q(B_1) = B$ and $|B_1| = |B| \leq \kappa$. Let $D$ be a dense subgroup of $N$ of minimal cardinality $d(N) \leq \kappa$. Then $H = B_1 + D$ is dense in $G$ by Fact 3.3(2), and $|H| \leq \kappa$. In particular, $d(G) \leq \kappa$.

We now prove that condition (2) is satisfied. First note that $\mathbb{T}^\kappa$ contains $\mathbb{Z}_p^\kappa$ for every prime $p$. For the compact subgroup $A = q^{-1}(L)$ we have $A/N \cong L \cong \mathbb{Z}_p^\kappa$, that is torsion-free. Being divisible, the Pontryagin dual $\hat{A}/N$ is topologically isomorphic to a direct summand of $\widehat{A}$ (see [13] Theorem 21.2). So $\widehat{A}/N \cong A/N \cong \mathbb{Z}_p^\kappa$ is (topologically isomorphic to) a subgroup of $A$, hence of $G$. $\square$

The following result is folklore.

**Fact 3.6.** If $f : G \to H$ is a continuous surjective homomorphism of compact groups, then $f(c(G)) = c(H)$.

Now we apply Proposition 3.5 to a particular instance of Theorem A, where we prove that condition (b) implies condition (c), under some additional assumptions.

**Proposition 3.7.** Let $K$ be a compact abelian group and assume that $K$ has a subgroup $N \cong F \times \mathbb{Z}_p$, where $F$ is finite and $p$ is a prime number, and such that $K/N \cong \mathbb{T}^\kappa$ for some cardinal $\kappa$.

If $K$ is densely locally minimal, then $\kappa = 0$ (i.e., $K = N$), so in particular $K$ has an open subgroup isomorphic to $\mathbb{Z}_p$.

**Proof.** Let $q : K \to K/N \cong \mathbb{T}^\kappa$ be the quotient homomorphism with $ker q = N$. As $d(N) = \omega$, $\kappa$ is finite by Proposition 3.5. Assume by contradiction that $\kappa \neq 0$, so let $\kappa = n \in \mathbb{N}_+$.

By Fact 3.6

$$c(q(K)) = c(\mathbb{T}^\kappa) = \mathbb{T}^n.$$ 

Since both $\mathbb{T}^n$ and $F \times \mathbb{Z}_p$ are metrizable, the group $K$ is also metrizable (see [1], Corollary 3.3.20), and so is $c(K)$. It is known that a compact metrizable connected abelian group is monothetic (see [20] Example 8.75]), i.e., it contains a dense cyclic subgroup (that is necessarily infinite, if the group is non-trivial). Fix a dense (infinite) cyclic subgroup $B$ of $c(K)$ and call $b$ its generator. Moreover, fix also a cyclic subgroup $(z)$ dense in $\mathbb{Z}_p$. Now let $x := z + b$ and $F = F + \{x\}$.

**Claim 1** $H$ is dense in $K$.

**Proof.** Let $A$ be the closure of $H$ in $K$. In order to prove that $A = K$ we first prove that $q(A) = q(K) = \mathbb{T}^n$. Since $q(x) = q(b)$ and $q(F) = 0$, we deduce that $q(H) = q(b)$. As $(b)$ is dense in $c(K)$ and $q(c(K)) = \mathbb{T}^n$, it follows that $q(H)$ is dense in $\mathbb{T}^n$. So, $q(A) = q(H) = \mathbb{T}^n$ and indeed $q(c(A)) = \mathbb{T}^n$ by Fact 3.6.

Next we show that $c(K)$ is contained in $A$. By a dimension theorem given in [24], we obtain

$$dim(c(A)) = dim(c(K)) = dim(\mathbb{T}^n) = n,$$

as $ker q = N$ is zero dimensional. Moreover, since the quotient group $c(K)/c(A)$ is connected and

$$dim(c(K)/c(A)) = dim(c(K) - dim(c(A)) = 0,$$
we conclude that \( c(K) = c(A) \) is contained in \( A \).

To end the proof of the claim, i.e., the proof of the equality \( A = K \), it suffices to see that \( A + c(K) = K \). Consider the quotient map \( s : K \to K/c(K) \cong N/(c(K) \cap N) \), as \( K = c(K) + N \). Since \( b \in c(K) \), \( s(b) = 0 \), so \( s(x) = s(z) \). Then \( s(H) = s(F + (z)) \). Therefore \( s(A) = s(H) = s(F + (z)) = s(F + (z)) = s(N) \), as \( F + (z) = N \). Thus \( s(A) = K/c(K) \), it follows that \( A + c(K) = K \).

**Claim 2.** \( H \) is not locally essential in \( K \).

**Proof.** Assume \( H \) to be locally essential in \( K \) and let \( U \) be an open neighborhood of \( e \) in \( K \) with compact closure, and witnessing the local essentiality of \( H \). As \( \bigcap_{n \in \mathbb{N}} p^nZ_p = \{ e \} \subseteq U \), by the compactness of \( U \) we find \( n \in \mathbb{N} \) such that \( p^nZ_p \cap U \subseteq U \). Then \( p^nZ_p \cap H \) is non-trivial, so it is infinite and \( H \cap N \) is infinite as well. Since \( F \leq N \), by the modular law we obtain \( H \cap N = (F + (x)) \cap N = F + ((x) \cap N) \). In particular, also \( (x) \cap N \) is infinite so has finite index in \( (x) \). Furthermore, \( |H/H \cap N| = |(F + (x))|(F + ((x) \cap N))| \leq |(x) / (x) \cap N| < \infty \). It follows that \( q(H) = (H + N) / N \cong H / H \cap N \) is finite. This contradicts the fact that \( q(H) \) is dense in \( T \cong K / N \).

Claim 1 and Claim 2 complete the proof of Proposition 3.7, contradicting the assumption that \( K \) is densely locally minimal.

**Proof of Theorem A.** We have to prove that for a compact abelian group \( K \), the following conditions are equivalent:

(a) \( K \) is a hereditarily locally minimal group;

(b) \( K \) is a densely locally minimal group;

(c) \( K \) either is a Lie group or \( K \) has an open subgroup isomorphic to \( \mathbb{Z}_p \) for some prime \( p \).

(a) \( \Leftrightarrow \) (c): Follows from Fact 1.4.

(a) \( \Rightarrow \) (b): It is clear by the definition.

(b) \( \Rightarrow \) (c): Write \( K = K_{tor} \times K_d \), where \( K_{tor} \) is a compact bounded abelian group and \( K_d \) is w-divisible, as in Fact 3.1. By Lemma 2.3 both \( K_{tor} \) and \( K_d \) are densely locally minimal, so \( K_{tor} \) is finite by Lemma 2.5. In the rest of the proof, we show that \( G = K_d \) either is a Lie group or has an open subgroup isomorphic to \( \mathbb{Z}_p \) for some prime \( p \), so that \( K \) satisfies condition (c).

As \( G \) is a compact abelian group, there exists a closed profinite subgroup \( N \) of \( G \) such that \( G / N \cong T^\kappa \) for some cardinal \( \kappa \) (see [20, Proposition 8.15]). Let \( q : G \to G / N \) be the canonical projection.

We break the proof into three steps.

**Step 1.** By our assumptions on \( G \) and using Fact 3.3(1), there exists a dense free subgroup \( F \) of \( G \). As \( G \) is densely locally minimal, we deduce that \( F \) is locally essential by the Local Minimality Criterion. Since \( N \) is profinite, Fact 2.4 provides an open subgroup \( O \) of \( N \) (so of finite index), such that \( F \cap O \) is essential in \( O \). As \( F \cap O \) is torsion-free, this yields that \( O \) is torsion-free. Thus \( O \cap t(N) = \{ 0 \} \), and as \( [N : O] \) is finite we deduce that \( t(N) \) is finite. So, by [17, Theorem 27.5], we obtain \( N \cong N_1 \times t(N) \) with \( N_1 \) torsion-free.

**Step 2.** Assume that \( N \) is finite with \( |N| = m \). If \( \kappa \) is infinite, then clearly \( d(N) = m \leq \kappa \), contradicting Proposition 3.6.

As \( \kappa \) is finite, Pontryagin duality implies that \( N \cong \widehat{N} \cong \widehat{G} / \mathbb{T}^\kappa \). Using the three-space property of finitely generated groups, we deduce that \( \widehat{G} \) is finitely generated. By [11, Theorem 15.5], a (discrete) finitely generated abelian group has the form \( A \times \mathbb{Z}^n \), where \( A \) is finite and \( n \in \mathbb{N} \), so \( \widehat{G} \cong \widehat{A} \times \mathbb{Z}^n \) is a Lie group.

**Step 3.** We are left now with the case when \( N = N_1 \times t(N) \) is infinite, i.e., \( N_1 \) is infinite. As \( N_1 \) is compact, totally disconnected and torsion-free, the dual of \( N_1 \) is discrete, torsion and divisible. Thus \( \widehat{N}_1 \cong \bigoplus_p \mathbb{Z}(p^\infty)^{(\omega_p)} \) and \( N_1 \cong \bigoplus_p \mathbb{Z}_p / p^{\aleph_0} \). We show that \( N_1 \) is isomorphic to \( \mathbb{Z}_p \) for some prime \( p \).

Otherwise, \( N_1 \) is not densely locally minimal, by Corollary 2.4. According to Lemma 2.3 also \( N \) has a dense subgroup \( T \) that is not locally essential. Since \( G / N \cong T^\kappa \) is w-divisible, Fact 3.3(1) provides a dense free subgroup \( F_1 \) of \( T^\kappa \) with \( |F_1| \leq \kappa \cdot \omega \), and we can choose a free subgroup \( F_2 \) of \( G \) that trivially meets \( N \) and that is sent onto \( F_1 \) by the quotient map \( q \). Consider the subgroup \( K = T + F_2 \) of \( G \). Since \( T \) is a dense subgroup of \( N \) and \( q(K) \) is dense in \( T^\kappa \), Fact 3.3(2) implies that \( K \) is dense in \( G \). Using the fact that \( T \) is not locally essential in \( N \), and that \( F_2 \cap N \) is trivial, one can see that \( K \) is not locally essential in \( G \), a contradiction. Then, \( N_1 \cong \mathbb{Z}_p \) and \( N \cong \mathbb{Z}_p \times t(N) \) with \( t(N) \) finite. Then \( \kappa = 0 \) by Proposition 3.7, i.e., \( G = N \) has an open subgroup isomorphic to \( \mathbb{Z}_p \).

As a corollary of Theorem A we obtain the following.

**Corollary 3.8.** If a compact torsion-free abelian group \( K \) is densely locally minimal, then \( K \) is isomorphic to \( \mathbb{Z}_p \) for some prime \( p \).
Proof. By our assumption on $K$, we obtain that $K$ is hereditarily locally minimal according to Theorem A. Moreover, since $K$ is also torsion-free, we deduce that it is hereditarily minimal by [31, Theorem B]. Hence $K \cong \mathbb{Z}_p$ for some prime $p$ by Fact 1.3.

By Theorem A, a compact abelian group is densely locally minimal precisely when it is hereditarily locally minimal. In Example 3.11, we show, in contrast, that there exist compact two-step nilpotent groups that are densely minimal, but not hereditarily locally minimal.

The following easy result appears in [31]. We apply it in Corollary 3.10 to two-step nilpotent topological groups i.e., to groups $G$ satisfying $G' \leq Z(G)$, where $Z(G)$ is the center and $G'$ is the derived subgroup of $G$.

**Fact 3.9.** Let $G$ be a group and $G'$ be its derived subgroup. If $H$ is a subgroup of $G$ that is either non-central normal or non-abelian, then $H \cap G'$ is non-trivial.

**Corollary 3.10.** Let $G$ be a two-step nilpotent topological group.

1. The center $Z(G)$ is essential in $G$.
2. If $H$ is a non-abelian subgroup of $G$, then $H \cap Z(G)$ is non-trivial.

**Example 3.11.** Consider the ring multiplication $w: \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Z}_p$, $w(a, b) = ab$ and let $G = H(w) = (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$ be the induced generalized Heisenberg group (see [10]). The group $G$ is isomorphic to the following subgroup of the special linear group $\text{SL}(3, \mathbb{Z}_p)$:

$$\left\{ \left( \begin{array}{ccc} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{array} \right) | a, b, c \in \mathbb{Z}_p \right\}.$$  

It is known that $G$ is a compact two-step nilpotent group, and $Z(G) = (\mathbb{Z}_p \times \{0\} \rtimes \{0\}) \cong \mathbb{Z}_p$. As $G$ contains a copy of $\mathbb{Z}_p \times \mathbb{Z}_p$, we deduce by Fact 1.3 that $G$ is not hereditarily locally minimal.

Now we show that $G$ is densely minimal, so let $H$ be a dense subgroup of $G$. In view of the Minimality Criterion, we have to prove that $H$ is essential in $G$. So let $N$ be a non-trivial closed normal subgroup of $G$, and we show that $H \cap N$ is non-trivial. By Corollary 3.10, the subgroup $N_1 = N \cap Z(G)$ is non-trivial, and it is closed in $Z(G)$. Clearly, the dense subgroup $H$ of $G$ is non-abelian, so $H_1 = H \cap Z(G)$ is non-trivial by Corollary 3.10(2). As every non-trivial subgroup of $\mathbb{Q}_p$ is essential (see [31, Lemma 2.5]), and $Z(G)$ is closed in $\mathbb{Q}_p$, it follows that every non-trivial subgroup of $Z(G)$ is essential in $Z(G)$, and in particular $H_1$ is essential in $Z(G)$. As $N_1$ is closed in $Z(G)$, this implies that $H_1 \cap N_1$ is non-trivial. In particular, $H \cap N$ is non-trivial.

**Proof of Theorem B.** We just need to prove $(b) \Rightarrow (c)$, i.e., if a connected locally compact abelian group $G$ is densely locally minimal, then it is a Lie group.

Assume that $G$ is a connected locally compact abelian group. It is known that $G \cong \mathbb{R}^n \times K$, where $n \in \mathbb{N}$ and $K$ is a connected compact group. Then $K$ is densely locally minimal by Lemma 2.3 moreover, $K$ is connected, so it is a Lie group by Theorem A. Thus $G$ itself is a Lie group. \qed

4. DENSE MINIMALITY VS HEREDITARY MINIMALITY

We start this section proving another sufficient condition for a compact abelian group to be isomorphic to some $\mathbb{Z}_p$.

**Lemma 4.1.** If a $w$-divisible compact abelian group $K$ is densely minimal, then it is isomorphic to $\mathbb{Z}_p$ for some prime $p$.

**Proof.** By Fact 3.3(1), $K$ has a dense free abelian subgroup $H$. Then $H$ is essential in $K$ by the Minimality Criterion, and $H$ is torsion-free, so $K$ is torsion-free as well. Then $K \cong \mathbb{Z}_p$ for some prime $p$ by Corollary 3.8. \qed

The following result is contained in the proof of [1, Theorem 1.3], but we give a proof of this property for the sake of the reader. Together with the above Lemma 4.1 and the splitting theorem Fact 3.1 it will provide the proof of Theorem C.

**Lemma 4.2.** [2] Let $p$ be a prime number, and $F$ be a finite abelian group such that $K = \mathbb{Z}_p \times F$ is densely minimal. Then $F$ is trivial.

**Proof.** By contradiction, let $F = \{c_1, c_2, \ldots, c_n\}$ be non-trivial. Pick elements $\xi_1, \xi_2, \ldots, \xi_n$ in $\mathbb{Z}_p$ independent with 1, and let $H$ be the subgroup of $K$ generated by $\mathbb{Z} \times \{0\}$ and $\{(\xi_i, c_i) : i = 1, 2, \ldots, n\}$.

Considering the projection $p: K \to F$, we deduce by Fact 3.8(2) that $H$ is a dense subgroup of $K$, hence essential by the Minimality Criterion. An element $h \in H$ has the form $h = (k, 0) + \sum_{i=1}^n k_i(\xi_i, c_i)$, so the closed nontrivial subgroup $\{0\} \times F$ of $K$ trivially meets $H$, a contradiction. \qed
Proof of Theorem C. We have to prove that for an infinite locally compact abelian group $K$, the following conditions are equivalent:

(a) $K$ is a hereditarily minimal group;
(b) $K$ is a densely minimal group;
(c) $K$ is isomorphic to $\mathbb{Z}_p$ for some prime $p$.

By Fact 1.1 we may assume that $K$ is compact.

(a) $\iff$ (c) follows from Fact 1.3.

(a) $\implies$ (b): It is clear by the definition.

(b) $\implies$ (c): Write $K = K_{tor} \times K_d$, where $K_{tor}$ is a compact bounded abelian group and $K_d$ is w-divisible, as in Fact 3.1. By Lemma 2.3, both $K_{tor}$ and $K_d$ are densely minimal, so $K_{tor}$ is finite by Lemma 2.5 while $K_d$ is isomorphic to $\mathbb{Z}_p$ by Lemma 1.1. Now apply Lemma 1.2.

By Theorem C, a locally compact abelian group is densely minimal precisely when it is hereditarily minimal, and these properties characterize the $p$-adic integers among the infinite locally compact abelian groups. Example 3.11 shows that the equivalence between those two properties need not hold relaxing the abelianity to two-step nilpotency, even for compact groups.

The next example shows that a compact metabelian group which is simultaneously hereditarily locally minimal and densely minimal need not be hereditarily minimal. In what follows, $A^*$ denotes the multiplicative group of the invertible elements of a ring $A$.

Example 4.3. Let $G = (\mathbb{Q}_p, +) \times \mathbb{Q}_p^*$ and consider its compact subgroup $H = \mathbb{Z}_p \times \mathbb{Z}_p^*$. By [31, Theorem A], the group $G$ (in particular, also its subgroup $H$) is hereditarily locally minimal. By [31, Corollary 2.11], both $G$ and $H$ are also densely minimal. Since $\mathbb{Z}_p^* \cong \mathbb{Z}_p \times F$, where $F$ is a finite non-trivial group, it follows that $\mathbb{Z}_p^*$ is not hereditarily minimal by Fact 1.3. In particular, $H$ and $G$ are not hereditarily minimal.

We do not know if there exists a hereditarily minimal locally compact group which is neither discrete nor compact (see [31, Question 7.3]). On the other hand, the group $G = (\mathbb{Q}_p, +) \times \mathbb{Q}_p^*$ is an example of a densely minimal locally compact group which is neither discrete nor compact.

Recall that a (topologically) simple group is a group whose only normal subgroups (closed normal subgroups) are the trivial group and the group itself. Obviously, a minimal topologically simple group is totally minimal. Moreover, every dense subgroup of a topologically simple group is totally dense, so the Total Minimality Criterion implies:

Lemma 4.4. A minimal topologically simple group is densely totally minimal.

The following example shows that there exists a discrete (densely) totally minimal group which is not hereditarily minimal.

Example 4.5. A discrete minimal group is also called non-topologizable. The first example of a non-topologizable group was provided by Shelah [26] under the assumption of the Continuum Hypothesis. His example is simple and torsion-free, so this group is also (densely) totally minimal, yet not hereditarily minimal, as discrete hereditarily minimal groups are torsion by [31, Lemma 3.5].

A hereditarily minimal locally compact group is totally disconnected (see [31, Proposition 3.9]), while the next example shows that there exist densely totally minimal, compact, pathwise connected groups.

Example 4.6. Let $n \geq 5$ be an odd number and consider the special orthogonal group $G = \text{SO}(n, \mathbb{R})$ of degree $n$ over the reals. The group $G$ is compact, and simple (see for example [10, Theorem 7.5]), so it is densely totally minimal by Lemma 4.4. On the other hand, it is also known that $G$ is pathwise connected (see for example [29]).

5. Open questions and concluding remarks

By Theorem B, a densely locally minimal, connected, locally compact abelian group is Lie. Can we omit the assumption ‘abelian’ in the hypotheses of Theorem B? In other words:

Question 5.1. Let $G$ be a densely locally minimal connected locally compact group. Is $G$ a Lie group?

Dikranjan and Stoyanov classified all hereditarily minimal abelian groups.

Fact 5.2. [14] Let $G$ be a topological abelian group. Then the following conditions are equivalent:

(1) each subgroup of $G$ is totally minimal;
(2) $G$ is hereditarily minimal;
DENSELY LOCALLY MINIMAL GROUPS

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