Approximate symmetries, pseudo-Goldstones, and the second law of thermodynamics

Jay Armas,1,2,* Akash Jain,1,2,† and Ruben Lier3,4, ‡

1 Institute for Theoretical Physics, University of Amsterdam, 1090 GL Amsterdam, The Netherlands
2 Dutch Institute for Emergent Phenomena, 1090 GL Amsterdam, The Netherlands
3 Max Planck Institute for the Physics of Complex Systems, 01187 Dresden, Germany
4 Wurzburg-Dresden Cluster of Excellence ct.qmat, 01187 Dresden, Germany

We propose a general hydrodynamic framework for systems with spontaneously broken approximate symmetries. The second law of thermodynamics naturally results in relaxation in the hydrodynamic equations, and enables us to derive a universal relation between damping and diffusion of pseudo-Goldstones. We discover entirely new physical effects sensitive to explicitly broken symmetries. We focus on systems with approximate U(1) and translation symmetries, with direct applications to pinned superfluids and charge density waves. We also comment on the implications for chiral perturbation theory.

Symmetry has proved to be a powerful organisational tool in physics for characterising and classifying phases of matter. Knowledge about the symmetries of a physical system, and whether these are spontaneously broken by the low-energy ground state, is often sufficient to develop an effective theory describing its long-distance late-time behaviour. Symmetries are useful even when they are only approximate. The canonical example of this is the extremely successful effective theory for pions as Goldstones of spontaneously broken SU(2) chiral symmetry. In this context, due to nonzero quark masses, the symmetry is only approximate and the effective theory can be systematically corrected to account for the pion mass.

In this paper, we draw general lessons about effective theories featuring this pseudo-spontaneous pattern of symmetry breaking. We are interested in physical systems where an approximate global symmetry is spontaneously broken by the low-energy ground state, leading to a slightly massive pseudo-Goldstone field $\phi(x)$. The explicitly broken symmetry means that the associated Noether charge conservation is weakly violated, giving rise to physical effects such as relaxation, damping, and pinning. Pseudo-spontaneous symmetry breaking is common across the phase space of matter, due to inherent defects, inhomogeneities, and impurities in materials. Examples include pinned crystals [1, 2], charge density waves [3–6], pinned superfluids [7, 8], electrons in graphene [9], pinned nematics [10, 11], and pions in chiral perturbation theory [12–14], among many others.

In recent years, there have been several efforts towards developing hydrodynamic techniques for systems with spontaneously broken approximate symmetries, aimed at explaining experimental and holographic results; see e.g. [5, 6, 15–25]. A rigorous hydrodynamic framework for these systems, however, is still missing in the literature. This is the primary goal of this paper. We formulate a complete hydrodynamic theory for thermal systems exhibiting pseudo-spontaneous symmetry breaking.

The key accomplishment of our construction is to show that damping of pseudo-Goldstones and charge (or momentum) relaxation follow from the second law of thermodynamics. In particular, we derive the relation $\Omega = D_\phi k_0^2$ among the pseudo-Goldstone damping rate $\Omega$, attenuation $D_\phi$, and correlation length $1/k_0$, first noted in holographic models [16, 18, 20]. We emphasise that our derivation only relies on the second law; see [11] for a derivation using the Schwinger-Keldysh effective field theory or locality of the equations of motion [26].

Surprisingly, we find that dissipative effects also lead to many new transport coefficients in the hydrodynamic theory that have not been identified in past literature. The most notable of these is a coefficient $\lambda$ that enters the Josephson equation for the pseudo-Goldstone at leading order as $\partial_t \phi = \lambda \mu + \ldots$, where $\mu$ is the chemical potential associated with the approximately conserved charge.

In the context of pinned crystals, this equation becomes $\partial_t \delta \phi = \lambda u^i + \ldots$, where $\delta \phi^i$ is the displacement field and $u^i$ the fluid velocity. While $\lambda = 1$ when the symmetry is exact, we can generically have $\lambda \neq 1$. Another such coefficient $\lambda_T$ enters the entropy/heat flux of pinned crystals at leading order as $s^i = (s + \lambda_T) u^i + \ldots$. Physically, the coefficients of this type result in a modification of the speed of sound dependent on the strength of explicit symmetry breaking.

Pinned simple diffusion.—To highlight the striking features of our construction, we start with a simple toy model where the only conserved quantity is a scalar charge density $n$. This could be mass, energy, number of particles, electromagnetic charge, etc. We later generalise this construction to approximately conserved momenta resulting in a theory of pinned crystals.

According to Noether’s theorem, we expect that the system exhibits an associated global symmetry. However, this symmetry could be spontaneously broken in the low-energy ground state, giving rise to a Goldstone field $\phi(x)$. The equilibrium configurations of the system can be described by the free energy $F = \int d^4x (F(\phi) - K_{\text{ext}} \phi)$, where $K_{\text{ext}}$ is an external source coupled to $\phi$. The free energy density $F(\phi)$ obeys a Goldstone shift symmetry $\phi(x) \to \phi(x) - \Lambda$. This forbids a mass term like $F \sim \frac{1}{2} m^2 \phi^2$ in the free energy density, rendering the
Goldstone massless.

The situation is qualitatively different when the said conservation is only approximate, because the shift symmetry need not be respected. In practice, however, we find it convenient to artificially restore the symmetry by coupling the system to a background field \( \Phi(x) \) that also shifts as \( \Phi(x) \rightarrow \Phi(x) - \Lambda \). The background field explicitly breaks the global symmetry by picking out a preferred phase \( \Phi \). We can now write a mass term in \( \mathcal{F} \)

\[
\mathcal{F} = -p + \frac{1}{2} f_s \partial_t \phi \partial^t \phi + \frac{1}{2} \ell^2 m^2 (\phi - \Phi)^2, \tag{1}
\]

where \( \ell \) is a bookkeeping parameter that controls the strength of explicit symmetry breaking. The free energy \( F \) with (1) can be understood as a generalised Ginzburg-Landau model that accounts for explicit symmetry breaking with an arbitrary source \( \Phi \); see e.g. [27]. The mass term can be thought of as an “elastic potential” that tends to align the phase \( \phi \) with the background phase \( \Phi \). Varying (1) results in a configuration equation

\[
f_s (\partial_t \partial^t \phi - \ell_s^2 \phi) + m^2 \ell^2 \Phi + K_{\text{ext}} = 0, \tag{2a}
\]

where \( \ell_s = m/\sqrt{F_s} \) is the finite inverse correlation length for \( \phi \), denoting it to a massive pseudo-Goldstone. Typically, this massive field can be integrated out from the long-wavelength effective theory. However, if the symmetry is still approximately preserved, i.e. \( \ell \) is sufficiently small, the pseudo-Goldstone can still affect the long-wavelength spectrum. Using the Noether procedure, the static Ward identity

\[
\partial_j j^i = K_{\text{ext}} + m^2 \ell^2 (\phi - \Phi), \tag{2b}
\]

follows from (1), where \( j^i = -f_s \partial^i \phi \) is the charge flux. As expected, the mass term results in a violation of charge conservation even in the absence of external sources.

When we leave thermal equilibrium, we can no longer start with a free-energy and must rely on the framework of hydrodynamics to proceed. Firstly, we have a Josephson equation giving dynamics to \( \phi \) which, generalising (2a), we take to have the schematic form

\[
K + K_{\text{ext}} = 0. \tag{3a}
\]

We also have a conservation equation, generalising (2b), describing the dynamics of charge density \( n \)

\[
\partial_t n + \partial_i j^i = -K - \ell L, \tag{3b}
\]

where \( L \) is some operator causing explicit symmetry breaking. We will also need a new energy conservation equation implementing the first law of thermodynamics

\[
\partial_t \epsilon + \partial_i \epsilon^i = -K \partial_t \phi - \ell L \partial_t \Phi, \tag{3c}
\]

where \( \epsilon, \epsilon^i \) are the energy density and flux respectively. To complete these equations, we must give a set of constitutive relations for \( j^i, \epsilon^i, K, L \) in terms of \( n, \epsilon, \phi, \Phi \), arranged order-by-order in gradients.

We implement the gradient counting scheme where \( \phi \sim \mathcal{O}(\partial^{-1}) \), making its gradients \( \mathcal{O}(\partial^0) \); see [28, 29]. We ascribe the scaling \( \mathcal{O}(\partial) \) to the symmetry breaking parameter \( \ell \) and require that all dependence on the background phase \( \Phi \) must appear via \( \psi = \ell (\phi - \Phi) \sim \mathcal{O}(\partial^0) \). This ensures that setting \( \ell = 0 \) restores the symmetry.

The most important ingredient in a hydrodynamic theory is the local second law of thermodynamics. It necessitates the existence of an entropy density \( s^i \) and flux \( s^i \) such that

\[
\partial_t s^i + \partial_i s^i \geq 0, \tag{4}
\]

is satisfied for every solution of the conservation equations; see [30]. Despite being an inequality, this requirement is extremely powerful and is known to give strong constraints on the constitutive relations [31]. At leading order in gradients, we simply have \( s^i = s(\epsilon, \sigma, \phi) \). We can define the temperature \( T \), chemical potential \( \mu \), “superfluid density” \( f_s \), pseudo-Goldstone mass parameter \( m \), and the grand-canonical free-energy density \( \mathcal{F} \) via

\[
T \text{d}s = d\epsilon - \mu \text{d}n - \frac{1}{2} f_s \text{d}(\partial_t \phi \partial^t \phi) - m^2 \psi \text{d}\psi, \quad \mathcal{F} = \epsilon - Ts^t - \mu n. \tag{5}
\]

The entropy density will, in general, admit gradient corrections. We consider these in the appendix.

For clarity, let us assume the dynamics to be isothermal, i.e. \( T = T_0 \), so that energy conservation decouples from the charge conservation and Josephson equations. In this case, the second law constraints result in

\[
j^i = -f_s \partial^i \phi - \sigma_n \partial^i \mu, \quad K = \partial_t (f_s \partial^t \phi) + \ell m^2 \psi - (\sigma_\phi + \ell \sigma_\chi)(\partial_t \phi - \mu) + \sigma_\chi \partial_t \psi, \quad L = m^2 \psi + \sigma_\phi \partial_t \psi - (\sigma_\chi + \ell \sigma_\phi)(\partial_t \phi - \mu). \tag{6}
\]

Setting \( \mu = \mu_0 \), \( \partial_t \phi = \partial_t \Phi = \mu_0 \), at leading order in gradients, we recover the equilibrium version of these equations derived from (1). We also find four dissipative coefficients \( \sigma_n, \sigma_\phi, \sigma_\chi, \sigma_\phi \) that satisfy the inequality relations \( \sigma_n, \sigma_\phi \geq 0 \) and \( \sigma_\phi \geq \sigma_\chi^2/\sigma_\phi \). We provide a detailed derivation in the appendix.

Linearised fluctuations.—To highlight the physical implication of this model, we set \( \Phi = \mu_0 t, K_{\text{ext}} = 0 \), and linearly expand the equations around the solution \( \mu = \mu_0, \phi = \mu_0 t \). We find

\[
j^i = -f_s \partial^i \delta \phi - \sigma_n \partial^i \delta \mu, \quad \partial_t \delta \phi = \lambda \delta \mu - \Omega \delta \phi + D_\phi \partial_t \delta \phi, \quad \ell L = \frac{\chi}{\chi} \omega_0^2 \delta \phi + \Gamma \delta \mu + (1 - \lambda) f_s \partial_t \partial^t \phi, \tag{7a}
\]

where we have defined the susceptibility \( \chi \), pinning frequency \( \omega_0 \), pseudo-Goldstone attenuation constant \( D_\phi \).
damping constant $\Omega$, charge relaxation coefficient $\Gamma$, and a new coefficient $\lambda$ as

$$
\chi = \frac{\partial n}{\partial \mu}, \quad \omega_0^2 = \frac{\lambda^2 \ell^2 m^2}{\chi}, \quad D_\phi = \frac{f_s}{\sigma_\phi}, \quad \Omega = \frac{\ell^2 m^2}{\sigma_\phi}, \quad 
\Gamma = \frac{\ell^2}{\chi} \left( \sigma_\phi - \frac{\sigma_\chi}{\sigma_\phi} \right), \quad \lambda = 1 + \frac{\ell \sigma_\chi}{\sigma_\phi}.
$$

(7b)

Solving the equations and assuming $\ell \sim O(k)$, we find a damped sound mode with dispersion relations

$$
\omega = \pm \sqrt{\omega_0^2 + v_s^2 k^2 - \frac{i}{2} (k^2 (D_n + D_\phi) + \Gamma + \Omega)}, \quad (8)
$$

where $v_s^2 = \lambda^2 f_s/\chi$ and $D_n = \sigma_n/\chi$. The second law inequality constraints imply that $D_n, D_\phi, \Omega, \Gamma \geq 0$, ensuring that the sound pole remains in the lower-half plane.

From (7), it is possible to make a few interesting observations. For instance, we have proved the damping-attenuation relation

$$
\Omega = D_\phi k_0^2,
$$

(9)

where $k_0 = \omega_0/v_s$ [32]. We also see that our model naturally gives rise to charge relaxation $\Gamma$, without needing to introduce it by hand. Interestingly, we also find a new coefficient $\lambda \neq 1$ appearing in front of the $\mu$ term in the Josephson equation. It affects the dispersion relations by modifying the speed of sound in the presence of small explicit symmetry breaking, and has not appeared in the literature before.

It might be tempting to try and absorb the new coefficient $\lambda$ via some rescaling of fields and coefficients. However, it is possible to find a Kubo formula for $\lambda$ using the static response functions

$$
\lambda^2 = -v_s^2 \frac{G_{nn}^R(\omega = 0, k = 0)}{G_{jj}^R(\omega = 0, k = 0)},
$$

(10)

where the speed of sound $v_s$ can be read off using the singularity structure of the dispersion relations (8). This gives $\lambda$ a concrete physical meaning. We have included further details regarding the physical implications of $\lambda$ in the appendix.

The discussion can be extended to account for temperature fluctuations and conserved momenta, leading to a theory of explicitly broken superfluids. This theory was recently considered in the holographic context in [8]. It will be interesting to revisits their results in the view of our new $\lambda$ coefficient, along with other similar coefficients that can appear in the energy flux and stress tensor. We will discuss this in more detail in another publication.

**Pinned viscoelastic crystals.**—The hydrodynamic theory for pinned crystals can be constructed similarly to the U(1) case. In $d$ spatial dimensions, a static configuration of a crystal can be described by the spatial distribution of its lattice sites $\phi^I = 1, \ldots, d(x)$, called the crystal fields. We can define the strain tensor as $u_{1J} = \frac{1}{2}(h_{1J} - \delta_{1J}/\alpha_0^2)$, where $h_{1J}$ is the inverse of $h^{IJ} = \partial^I \phi^J \partial^J \phi^J$ and $\alpha_0$ is a constant parametrising the “inverse lattice spacing” of the crystal. We can always rescale the fields to set $\alpha_0 = 1$.

When the crystal is homogeneous, the free-energy density $\mathcal{F}$ obeys a global spatial shift symmetry $\partial^I (x) \to \phi^I (x) + a^I$, and $\phi^I$ can be understood as Goldstones of spontaneously broken translations. However, when the crystal has slight inhomogeneities, possibly due to defects or impurities, this shift symmetry can be violated, and $\phi^I$ become pseudo-Goldstones of approximate translation symmetry. Analogous to the U(1) case, we artificially restore the symmetry by introducing a set of background fields $\Phi^I (x)$, also shifting as $\Phi^I (x) \to \Phi^I (x) + a^I$. In the present case, $\Phi^I (x)$ can be interpreted as describing the spatial configuration of a fixed background lattice coupled to our physical crystal of interest. This allows us to introduce a mass term in the free-energy density

$$
\mathcal{F} = -p + \frac{1}{2} (B - \frac{2}{3} G)(a^I)^2 + G u_{1J} u_{1J} + m_2^2 \psi^I \psi^I,
$$

(11)

where $\psi^I = \ell (\phi^I - \Phi^I)$ is the misalignment tensor. $B$ and $G$ are bulk and shear moduli respectively. $I, J, \ldots$ indices are raised/lowered using $h^{IJ}, h_{IJ}$.

To describe the dynamical evolution of this system, we need to formulate the theory of pinned viscoelastic hydrodynamics following the construction of [28, 29]. Firstly, analogous to (3a), we have a set of Josephson equations for the crystal fields

$$
K_I + K_I^{\text{ext}} = 0,
$$

(12a)

where $K_I$ is an unknown operator and $K_I^{\text{ext}}$ are sources coupled to $\phi^I$. Assuming the crystal to exhibit Galilean symmetry, we also have momentum conservation and continuity equations

$$
\partial_t \pi^i + \partial_j \pi^{ij} = K_I \partial^i \phi^I + \ell L_I \partial^i \Phi^I,
$$

$$
\partial_t \rho + \partial_i \pi^i = 0,
$$

(12b)

where $\pi^i$ is the momentum density, $\tau^{ij}$ the stress tensor, $\rho$ the mass density, and $L_I$ an operator causing explicitly broken translations. These have to be supplemented with the energy conservation equation arising from the first law of thermodynamics

$$
\partial_t \epsilon + \partial_i \epsilon^i = -K_I \partial_i \psi^I - \ell L_I \partial_i \Phi^I.
$$

(12c)

We can now proceed and derive a set of constitutive relations for $\tau^{ij}, \epsilon^i, K_I, L_I$ in terms of $\pi^i, \epsilon, \rho, \phi^I, \Phi^I$, arranged in a gradient expansion, and obtain constraints due to the second law of thermodynamics. Imposing Galilean invariance, at leading order in gradients, the entropy density is given by $s^i = s(\epsilon, \rho, h^{IJ}, \psi^I)$, where $\epsilon = \epsilon - \frac{1}{2} \rho \bar{u}^2$ is the Galilean-invariant “internal energy density” and $\bar{u} = \pi^i / \rho$ is the fluid velocity. We can define the temperature $T$, chemical potential $\mu$, elastic...
stress tensor $r_{IJ}$, pseudo-Goldstone mass $m$, and free-energy $F$ via

$$T ds = d\varepsilon - \mu d\rho + \frac{1}{2} r_{IJ} dh^{IJ} - m^2 \psi_I d\psi^I,$$

$$F = \varepsilon - Ts^i - \mu \rho.$$  \hspace{1cm} (13)

The entropy density can also admit first order gradient corrections, which we consider in detail in the appendix.

Similarly to the U(1) case, restricting to an isothermal regime, i.e. $T = T_0$, energy conservation decouples from the rest of the system, and we obtain the allowed set of constitutive relations

$$\tau^{ij} = \rho u^i u^j - F \delta^{ij} - r_{IJ} \phi^I \partial^i \phi^J - 2\eta \delta^{(i} u^{j)} - \zeta \partial_k u^k \delta^{ij},$$

$$K_I = -\partial_i (r_{IJ} \partial^j \phi^I) - \ell m^2 \psi_I - \langle \sigma_\phi \rangle h_{IJ} \frac{d\phi^J}{dt} + \langle \sigma_x \rangle h_{IJ} \frac{d\phi^J}{dt},$$

$$L_I = m^2 \psi_I + \langle \sigma_\phi \rangle h_{IJ} \frac{d\phi^J}{dt} - \langle \sigma_x \rangle \chi \langle \phi \rangle h_{IJ} \frac{d\phi^J}{dt}.$$  \hspace{1cm} (14)

where $d/\partial t = \partial_i + u^i \partial_i$ and angular brackets around indices denote a symmetric traceless combination. The five dissipative coefficients $\eta, \zeta, \sigma_\phi, \sigma_x$ follow the inequalities $\eta, \zeta, \sigma_\phi \geq 0$ and $\sigma_x \geq \sigma_\phi \chi \sigma$. A detailed derivation relaxing the isothermal assumption appears in the appendix.

**Linear pinned crystals.**—In the small strain regime, the equation of state of the crystal can be written as (11), except that $\alpha_0$ in $u_{IJ}$ should now be promoted to $\alpha(T, \mu)$. We can still set $\alpha(T_0, \mu_0) = 1$ by rescaling the fields, but its thermodynamic derivatives are generically nontrivial.

Setting $\Phi^I = x^I$ and $K^\text{ext}_I = 0$, and expanding around $\phi^I = \delta_I^I (x^I - \phi^I)$, $\mu = \mu_0$, $u^I = 0$ we can obtain

$$\tau^{ij} = (p + B \alpha_m \delta \mu) \delta^{ij} - 2\eta \delta^{(i} u^{j)} - \zeta \partial_k u^k \delta^{ij} - B \delta_{ij} \delta \phi^k \delta \phi^k - 2G \delta^{(i} \delta \phi^j),$$

$$\partial_i \delta \phi^I = \lambda \delta \phi^I - \Omega \delta \phi^I + \gamma_m \partial^I \mu$$

$$+ 2D_\phi \partial_i \partial^j \delta \phi^j + D_\phi \partial_i \delta \phi^j \partial^j \delta \phi^j,$$

$$\ell L^i = -\frac{\beta}{\lambda} \alpha_m \delta \phi^i - \Gamma \pi^i - (1 - \frac{1}{2} B \alpha_m \delta \mu)$$

$$+ (1 - 1) \left( B \delta^j \partial_k \delta \phi^j + 2G \partial_k \partial^j \delta \phi^j \right).$$  \hspace{1cm} (15a)

We have defined pinning frequency $\omega_0$, mass expansion coefficient $\alpha_m$, pseudo-Goldstone attenuation constants $D_\phi$ and damping constant $\Omega$, momentum relaxation $\Gamma$, and coefficients $\lambda, \gamma_m$ as

$$\omega_0^2 = \frac{\lambda^2 \ell^2 m^2}{\rho}, \hspace{0.5cm} \alpha_m = \frac{d \alpha}{\alpha \partial \mu}, \hspace{0.5cm} \gamma_m = -\frac{B \alpha_m}{\sigma_\phi},$$

$$D_\phi = \frac{G}{\sigma_\phi}, \hspace{0.5cm} D_\phi = \frac{B + 2\ell^2 m^2}{\sigma_\phi}, \hspace{0.5cm} \Omega = \frac{\ell^2 m^2}{\sigma_\phi},$$

$$\Gamma = \frac{\ell^2}{\rho} \left( \sigma_\phi - \frac{\sigma_\phi^2}{\sigma_\phi} \right), \hspace{0.5cm} \lambda = 1 + \frac{\ell \sigma_x}{\sigma_\phi}. \hspace{1cm} (15b)$$

Looking at the mode spectrum, we obtain a damped sound mode in the longitudinal and transverse sectors similar to (8). We also find a crystal diffusion mode in the longitudinal sector. These expressions are given in the appendix. Lifting the isothermal assumption leads to an energy diffusion mode coupled with crystal diffusion; see e.g. [29].

Analogously to the U(1) case, using (15) we recover the damping-attenuation relation from [16, 18, 20] where $k_0 = \omega_0 / v_\perp$ with $v_\perp^2 = \lambda^2 G / \rho$. The momentum relaxation $\Gamma$ also arises naturally in our model, along with the coefficient $\lambda$ affecting the Josephson equation. Upon including thermal fluctuations, we find another damping-attenuation relation similar to (16) in the energy flux. We also find a new pinning-sensitive coefficient $\chi T$ in the energy flux; see the appendix for more details.

**Discussion.**—In this paper we introduced a general hydrodynamic framework for dissipative systems with spontaneously broken approximate symmetries. Our construction builds upon the technology of forced fluid dynamics from [33, 34], by systematically coupling the hydrodynamic equations to pseudo-Goldstone fields $\phi(x)$ and fixed background phase fields $\Phi(x)$, responsible for spontaneously and explicitly breaking the symmetries respectively [35]. We illustrated how the interplay between the two field ingredients gives rise to physical effects such as damping, pinning, and relaxation. In particular, we showed that the elusive relation between the damping $\Omega$ and attenuation $D_\phi$ of pseudo-Goldstones follows simply by imposing the second law of thermodynamics in the presence of background fields $\Phi(x)$. The second law also requires the relaxation coefficient $\Gamma$ to be non-negative.

In addition to providing a rigorous mathematical language for systems with pseudo-spontaneously broken symmetries, we also found entirely new physical effects that have not been discussed in previous literature. Namely, we discovered new transport coefficients sensitive to the explicit nature of symmetry breaking that modify the hydrodynamic and Josephson equations at the thermodynamic level. These coefficients result in a modification of the speed of the damped sound mode and affect the hydrodynamic correlators in a non-trivial way.

We primarily focused on systems with approximate U(1) or approximate spatial translation symmetry. However, the framework developed here is equally relevant for other physical situations exhibiting a pseudo-spontaneous pattern of symmetry breaking. For instance, a hydrodynamic theory for pions recently appeared in [14], featuring a pseudo-spontaneously broken SU(2) chiral symmetry [12, 13, 36]. In particular, [14] noted that the damping-attenuation relation (9) for pions follows from the second law of thermodynamics. However, their analysis does not include additional pinning-
sensitive coefficients such as $\lambda$. It is straight-forward to generalise the U(1) case analysed here to a SU(2) pion field $\phi^a$ coupled to a fixed background SU(2) phase $\Phi^a$, where $a, b, \ldots$ denote SU(2) Lie algebra indices. The linearised Josephson equation will take the schematic form

$$\partial_t \delta \phi^a = \lambda^a \delta \mu^b - \Omega^a \delta \phi^b + D_{\phi^a} \partial_t \delta \phi^b,$$

(17)

with the damping-attenuation relation $\Omega^a b = D_{\phi^a} \delta \phi^b$. The coefficient $\lambda^a b$ is equal to $\delta^a b$ in the absent of explicit symmetry breaking, but can acquire corrections when the symmetry is weakly broken, modifying the mode spectrum of chiral perturbation theory.

Holographic models with pseudo-spontaneous pattern of symmetry breaking have been discussed in multiple works; see e.g. [16–25]. It would be interesting to develop the relativistic versions of the hydrodynamic theories formulated in this paper and revisit their holographic applications in light of the new transport coefficients that we have identified. We leave this direction for future work.

JA and AJ are partly supported by the Netherlands Organization for Scientific Research (NWO) and by the Dutch Institute for Emergent Phenomena (DIEP) cluster at the University of Amsterdam. AJ is funded by the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 101027527.

\* j.armas@uva.nl
\* a.jain@uva.nl
\* rubenl@pks.mpg.de

[1] Michael M. Fogler and David A. Huse, “Dynamical response of a pinned two-dimensional wigner crystal,” Phys. Rev. B 62, 7553–7570 (2000).

[2] R. Chitra, T. Giamarchi, and P. Le Doussal, “Pinned wigner crystals,” Phys. Rev. B 65, 035312 (2001).

[3] H. Fukuyama and P. A. Lee, “Dynamics of the charge-density wave. i. impurity pinning in a single chain,” Phys. Rev. B 17, 535–541 (1978).

[4] G. Grünner, “The dynamics of charge-density waves,” Rev. Mod. Phys. 60, 1129–1181 (1988).

[5] Luca V. Delacrétaz, Blaise Goutéraux, Sean A. Hartnoll, and Anna Karlsson, “Theory of hydrodynamic transport in fluctuating electronic charge density wave states,” Physical Review B 96 (2017), 10.1103/physrevb.96.195128.

[6] Luca V. Delacrétaz, Blaise Goutéraux, Sean A. Hartnoll, and Anna Karlsson, “Theory of collective magnophonon resonance and melting of a field-induced Wigner solid,” Phys. Rev. B 100, 085140 (2019), arXiv:1904.04872 [cond-mat.mes-hall].

[7] Aristomenis Donos, Polydoros Kailidis, and Christina Pantelidou, “Dissipation in holographic superfluids,” JHEP 9, 134 (2021), arXiv:2107.03680 [hep-th].

[8] Martin Ammon, Daniel Areán, Matteo Baggioi, Seán Gray, and Sebastian Grieninger, “Pseudo-spontaneous $U(1)$ Symmetry Breaking in Hydrodynamics and Holography,” (2021), arXiv:2111.10305 [hep-th].

[9] Andrew Lucas and Kin Chung Fong, “Hydrodynamics of electrons in graphene,” J. Phys. Condens. Matter 30, 053001 (2018), arXiv:1710.08425 [cond-mat.str-el].

[10] A.C. Eringen, Microcontinuum Field Theories: I. Foundations and Solids (Springer New York, 2012).

[11] Luca V. Delacrétaz, Blaise Goutéraux, and Vaios Zio- gas, “Damping of Pseudo-Goldstone Fields,” (2021), arXiv:2111.13459 [hep-th].

[12] D. T. Son and M. A. Stephanov, “Pion propagation near the qcd chiral phase transition,” Phys. Rev. Lett. 88, 202302 (2002).

[13] D. T. Son and M. A. Stephanov, “Real-time pion propagation in finite-temperature qcd,” Phys. Rev. D 66, 076011 (2002).

[14] Eduardo Grossi, Alexander Soloviev, Derek Teaney, and Fanglida Yan, “Transport and hydrodynamics in the chiral limit,” Phys. Rev. D 102, 014042 (2020), arXiv:2005.02885 [hep-th].

[15] Sašo Grozdanov, Andrew Lucas, and Napat Poovuttikul, “Holography and hydrodynamics with weakly broken symmetries,” Phys. Rev. D 99, 086012 (2019), arXiv:1810.10016 [hep-th].

[16] Andrea Amoretti, Daniel Areán, Blaise Goutéraux, and Daniele Musso, “Universal relaxation in a holographic metallic density wave phase,” Phys. Rev. Lett. 123, 211602 (2019), arXiv:1812.08118 [hep-th].

[17] Aristomenis Donos, Daniel Martin, Christina Pantelidou, and Vaios Ziogas, “Hydrodynamics of broken global symmetries in the bulk,” JHEP 10, 218 (2019), arXiv:1905.00308 [hep-th].

[18] Aristomenis Donos, Daniel Martin, Christina Pantelidou, and Vaios Ziogas, “Incoherent hydrodynamics and density waves,” Class. Quant. Grav. 37, 045005 (2020), arXiv:1906.03132 [hep-th].

[19] Andrea Amoretti, Daniel Areán, Blaise Goutéraux, and Daniele Musso, “Gapless and gapped holographic phonons,” JHEP 01, 058 (2020), arXiv:1910.11330 [hep-th].

[20] Martin Ammon, Matteo Baggioi, and Amadeo Jiménez-Alba, “A Unified Description of Translational Symmetry Breaking in Holography,” JHEP 09, 124 (2019), arXiv:1904.05785 [hep-th].

[21] Tomas Andrade, Matteo Baggioi, and Alexander Krikun, “Phase relaxation and pattern formation in holographic gapless charge density waves,” JHEP 03, 292 (2021), arXiv:2009.05551 [hep-th].

[22] Matteo Baggioi, Sebastian Grieninger, and Li Li, “Magnophonons & type-B Goldstones from Hydrodynamics to Holography,” JHEP 09, 037 (2020), arXiv:2005.01725 [hep-th].

[23] Aristomenis Donos, Christina Pantelidou, and Vaios Ziogas, “Incoherent hydrodynamics of density waves in magnetic fields,” JHEP 05, 270 (2021), arXiv:2101.06230 [hep-th].

[24] Andrea Amoretti, Daniel Areán, Daniel K. Brattan, and Nicodemo Magnoli, “Hydrodynamic magneto-transport in charge density wave states,” JHEP 05, 027 (2021), arXiv:2101.05343 [hep-th].

[25] Andrea Amoretti, Daniel Areán, Daniel K. Brattan, and Luca Martinova, “Hydrodynamic magneto-transport in holographic charge density wave states,” JHEP 11, 011 (2021), arXiv:2107.00519 [hep-th].
[26] See [11] for critical comments on the derivation of [37, 38].

[27] P.C. Hohenberg and A.P. Krekhov, “An introduction to the ginzburg–landau theory of phase transitions and nonequilibrium patterns,” Physics Reports 572, 1–42 (2015).

[28] Jay Armas and Akash Jain, “Viscoelastic hydrodynamics and holography,” JHEP 01, 126 (2020), arXiv:1908.01175 [hep-th].

[29] Jay Armas and Akash Jain, “Hydrodynamics for charge density waves and their holographic duals,” Phys. Rev. D 101, 121901 (2020), arXiv:2001.07357 [hep-th].

[30] Akash Jain, A universal framework for hydrodynamics, Ph.D. thesis, Durham U., CPT (2018).

[31] L.D. Landau and E.M. Lifshitz, Fluid Mechanics, v. 6 (Elsevier Science, 2013).

[32] This relation was recently derived in [11] by invoking the locality of constitutive relations. However, our derivation merely follows as a consequence of the second law of thermodynamics.

[33] Sayantani Bhattacharyya, R. Loganayagam, Shiraz Minwalla, Suresh Nampuri, Sandip P. Trivedi, and Spenta R. Wadia, “Forced Fluid Dynamics from Gravity,” JHEP 02, 018 (2009), arXiv:0806.0006 [hep-th].

[34] Jay Armas, Jakob Gath, Vasilis Niarchos, Niels A. Obers, and Andreas Vigand Pedersen, “Forced Fluid Dynamics from Blackfolds in General Supergravity Backgrounds,” JHEP 10, 154 (2016), arXiv:1606.09644 [hep-th].

[35] In the context of holography, forced fluid dynamics and explicit symmetry breaking have been studied in multiple works, see e.g. [39–43].

[36] Akash Jain, “Theory of non-Abelian superfluid dynamics,” Phys. Rev. D 95, 121701 (2017), arXiv:1610.05797 [hep-th].

[37] Matteo Baggioli, “Homogeneous holographic viscoelastic models and quasicrystals,” Phys. Rev. Res. 2, 022022 (2020), arXiv:2001.06228 [hep-th].

[38] Matteo Baggioli and Michael Landry, “Effective Field Theory for Quasicrystals and Phasons Dynamics,” SciPost Phys. 9, 062 (2020), arXiv:2008.05339 [hep-th].

[39] Tomás Andrade and Benjamin Withers, “A simple holographic model of momentum relaxation,” JHEP 05, 101 (2014), arXiv:1311.5157 [hep-th].

[40] Tomás Andrade and Simon A. Gentle, “Relaxed superconductors,” JHEP 06, 140 (2015), arXiv:1412.6521 [hep-th].

[41] Tomás Andrade and Alexander Krikun, “Commensurability effects in holographic homogeneous lattices,” JHEP 05, 039 (2016), arXiv:1512.02465 [hep-th].

[42] Mike Blake, “Momentum relaxation from the fluid/gravity correspondence,” JHEP 09, 010 (2015), arXiv:1505.06992 [hep-th].

[43] Tomás Andrade, Simon A. Gentle, and Benjamin Withers, “Drude in D major,” JHEP 06, 134 (2016), arXiv:1512.06263 [hep-th].

[44] Lars Onsager, “Reciprocal relations in irreversible processes. i.” Phys. Rev. 37, 405–426 (1931).

[45] Lars Onsager, “Reciprocal relations in irreversible processes. ii.” Phys. Rev. 38, 2265–2279 (1931).

[46] Akash Jain, “Effective field theory for non-relativistic hydrodynamics,” JHEP 10, 208 (2020), arXiv:2008.03994 [hep-th].
**Supplementary Material**

**Details of pinned simple diffusion**

In this appendix we provide details of the second law analysis in the pinned simple diffusion model. We also turn on a background gauge field $A_i$, $A_i$ throughout the discussion to facilitate the computation of hydrodynamic correlation functions. In addition, we do not assume the system to be isothermal as in the bulk of the paper. 

**Second law constraints.**—We can define the gauge covariant derivatives of $\phi$ and $\Phi$ as

$$
\xi_i = \partial_i \phi + A_i, \quad \xi_i = \partial_i \phi + A_i,
$$

In the presence of background gauge fields, the Josephson and charge conservation eqs. (3a)-(3b) remain unchanged, but the energy conservation (3c) generalises to

$$
\partial_t \epsilon + \partial_t \epsilon^i = E_{ij}^i - K \xi_i - (\ell L \Xi)_i, \quad (A1)
$$

where $E_i = \partial_i A_t - \partial_t A_i$. Let us parametrise the entropy density as

$$
S = S + \epsilon, \quad (A2)
$$

where $S$ are possible gradient corrections. Using the thermodynamic relations (5) and the conservation equations, we find

$$
\partial_t s^i + \partial_i s^i = -\frac{1}{T^2} \xi^i (\partial_i T) - J^i (\partial_i \mu - \frac{E_i}{T})
- \frac{1}{T^2} \mathcal{K} (\xi_i - \mu) - \frac{\ell}{T} \mathcal{L} (\xi_i - \mu) + \partial_i S + \partial_i s^i, \quad (A3)
$$

where we have identified the constitutive relations

$$
\epsilon^i = -f_i \xi^i \xi_t + \mathcal{E}^i, \quad j^i = -f_i \xi^i + J^i, \quad K = \partial_i (f_i \xi^i) - \ell m^2 \psi + K, \quad L = m^2 \psi + L,
$$

$$
\epsilon^i = \frac{1}{T} \xi^i - \mu \frac{\ell}{T} J^i + S^i. \quad (A4)
$$

The right-hand side of (A4) is required to be a positive semi-definite quadratic form. Truncating at first order in gradients, there are two kinds of solutions to (A4), First, we have the “non-hydrostatic sector”, where $\epsilon, S^i$ are identically zero and we simply have

$$
\begin{pmatrix}
\frac{1}{T^2} \mathcal{E}^i \\
\mathcal{J}^i \\
\mathcal{K}^i \\
\mathcal{L}^i
\end{pmatrix} =
\begin{pmatrix}
\frac{1}{T^2} K_{ij}^i & \gamma_{ij}^i & \gamma_{ij}^i & \gamma_{ij}^i \\
\gamma_{ij}^i & \sigma_{ij}^i & \gamma_{ij}^i & \gamma_{ij}^i \\
\gamma_{ij}^i & \gamma_{ij}^i & \sigma_{ij}^i & \sigma_{ij}^i \\
\gamma_{ij}^i & \gamma_{ij}^i & \gamma_{ij}^i & \sigma_{ij}^i
\end{pmatrix}
\begin{pmatrix}
\partial_i T \\
T \partial_i \mu - E_i \\
\xi_t - \mu \\
\ell (\xi_t - \mu)
\end{pmatrix}, \quad (A5)
$$

The objects appearing in the matrix here have to be constructed out of the zero-gradient structures $\delta^{ij}, e^{ij}...$, and $\xi^i$, supplemented with coefficients that are arbitrary functions of $T, \mu, \partial_i \phi \partial_i \phi, \psi$. If we were only interested in the terms that contribute linearly to the constitutive relations, we can ignore any dependence on $\xi^i, \partial_i \phi \partial_i \phi, \psi$. Further imposing parity-symmetry, we have the allowed coefficients

$$
\kappa^{ij} = \kappa \delta^{ij}, \quad \sigma_{ij}^i = \sigma_\phi \delta^{ij}, \quad \gamma_{ij}^i = \gamma \delta^{ij}, \quad \gamma_{ij}^i = \gamma \delta^{ij}, \quad \sigma_\phi, \quad \sigma_\Phi, \quad \sigma_\chi, \quad \sigma_\chi', \quad (A7)
$$

while all vector coefficients vanish. All coefficients are functions of $T$ and $\mu$. Onsager’s reciprocity relations [44, 45] further impose $\gamma' = \gamma$ and $\sigma_\phi' = \sigma_\phi$. The second law results in the inequality relations

$$
\kappa, \sigma_\phi \geq 0, \quad \sigma_n \geq \gamma^2 / \kappa, \quad \sigma_\Phi \geq \sigma_\chi^2 / \sigma_\phi. \quad (A8)
$$

In addition, we have the “hydrostatic sector”, characterised by corrections to the entropy density

$$
S = f_1 \partial_i \xi^i - \frac{\ell}{T} \xi^i \xi_t. \quad (A9)
$$

The factor of $\ell$ in front of $f_1$ is necessary because all dependence on $\Phi$ must be expressible as a combination involving $\psi$. Indeed $\Xi_i = \xi_i - \partial_i \psi$. The coefficient $f_1$ characterises the response of the system due to a background superfluid velocity due to the presence of $\Phi$. If we only focus on linear corrections to the constitutive relations, (A4) means that we only need to consider the entropy density up to quadratic order in fields. Therefore, without loss of generality, we can take the coefficient $f_1$ to be constant, while $f_1$ can be taken to be a linear function of $\epsilon$ and $n$. Plugging (A9) into (A4), we derive the hydrostatic constitutive relations

$$
\mathcal{E}_{\text{hs}}^i = -T \partial_t f_1 - \ell \partial_i \xi^i (\xi_t + \Xi_t^i), \quad (A10)
$$

$$
\mathcal{J}_{\text{hs}}^i = -T \partial_t f_1 - f_s \xi^i (\Xi_t + \Xi_t^i), \quad (A11)
$$

$$
\mathcal{K}_{\text{hs}} = T \partial_t \psi f_1 + T \sigma_\phi \left( \frac{\partial f_1}{\partial \epsilon} + \frac{\partial f_1}{\partial n} \right) \partial_t \xi^i + f_s \partial_t \Xi_t^i, \quad (A12)
$$

along with

$$
S^i = -f_1 \partial_i \xi^i + \left( \partial_i f_1 + \frac{\ell f_s}{T} \Xi_t \right) (\xi_t - \mu) \quad + \frac{\ell f_s}{T} \xi_t (\Xi_t - \mu), \quad (A13)
$$

where we have used the first-order equations of motion

$$
\partial_t \epsilon = \mu \sigma_\phi (\xi_t - \mu), \quad \partial_t n = \sigma_\phi (\xi_t - \mu). \quad (A14)
$$

Demanding $S$ to be invariant under time-reversal symmetry, the coefficient $f_1$ is not allowed. Additionally, it can
be checked that the coefficient $\tilde{f}_s$ does not contribute to the linearised equations of motion, when coupled to a homogeneous background $\Phi = \mu at$. Note also that, linearly, these coefficients can be removed by a redefinition of the pseudo-Goldstone field $\phi \to \phi + (T f_1 + \tilde{f}_s \psi)/f_s$ and are only physical if one has an unambiguous macroscopic notion of the pseudo-Goldstone field. For these reasons, we have not considered these coefficients in the remainder of our discussion.

**Modes.**—Focusing on isothermal fluctuations and employing the definitions in (7), we can read obtain the damped sound modes

$$\omega = \pm \sqrt{\omega_0^2 + v_s^2 k^2 - \frac{1}{4} (\Gamma - \Omega + (D_n - D_\phi) k^2)^2}$$

$$- \frac{i}{2} (k^2 (D_n + D_\phi) + \Gamma + \Omega).$$  \hspace{1cm} (A13)

Expanding this expression for $k^2 \ll 1$, we find

$$\omega = -\frac{i}{2} (\Gamma + \Omega) \pm \sqrt{\omega_0^2 - \frac{1}{4} (\Gamma - \Omega)^2}$$

$$- \frac{i}{2} k^2 \left( D_n + D_\phi \pm \frac{v_s^2}{4} (\Gamma - \Omega) (D_n - D_\phi) \right).$$ \hspace{1cm} (A14)

The sound modes (8) in the main text can be obtained from here by ignoring the $\mathcal{O}(k^4, \ell^2 k^2)$ corrections, i.e. assuming $\omega_0 \gg \Gamma, \Omega$.

**Correlation functions.**—We can also obtain the hydrodynamic predictions for the retarded correlation functions. Specifically, at zero momentum we have

$$G_{nn}(\omega) = \frac{\chi_\omega (\omega + i\Omega)}{(\omega + i\Gamma)(\omega + i\Omega) - \omega_0^2},$$

$$G_{\phi\phi}(\omega) = \frac{\chi_\omega}{\omega_0^2} \frac{\chi_\omega (\omega + i\Gamma)}{(\omega + i\Gamma)(\omega + i\Omega) - \omega_0^2},$$

$$G_{nn}(\omega) = \frac{i\omega \lambda}{(\omega + i\Gamma)(\omega + i\Omega) - \omega_0^2},$$

$$G_{ij}(\omega) = -f_s \delta_{ij} + i\sigma_n \omega \delta_{ij},$$ \hspace{1cm} (A15)

while all other correlators vanish. In general, these can be used to derive Kubo formulas for the various transport coefficients. In particular, it is easy to see that the Kubo formula (10) for $\lambda$ remains intact.

**Goldstone charge and $\lambda$ coefficient.**

We defined the pseudo-Goldstone field $\phi$ to have unit charge under $U(1)$ transformations, i.e. $\phi \to \phi - \Lambda$. Normally, $\phi$ does not have any well-defined macroscopic meaning and we can equally work with a rescaled field $\tilde{\phi} = q \phi$ with charge $q$. As it turns out, by choosing the charge $q = 1/\lambda$, we can entirely remove $\lambda$ from the hydrodynamic equations in the absence of background sources. Nonetheless, $\lambda$ resurfaces upon coupling the system to background electromagnetic fields and thus has non-trivial effects on the correlation functions.

In order to see this explicitly, we consider the equations of motion following from (7), but turning on the gauge field $A_t, A_i$. Note that we can always keep the background phase $\Phi = \mu at$ fixed by means of a gauge transformation. Expressed in terms of $\tilde{\phi}$, we find

$$\partial_t \tilde{\phi} = -\Gamma \tilde{\phi} + D_n (\partial_\tau \tilde{\phi} + \partial_i \partial^i \tilde{\phi} + \partial_\tau \partial^i \tilde{\phi} + q \partial_i A^i)$$

$$+ q \lambda \chi \left( \partial_\tau \partial^i \tilde{\phi} + q \partial_i A^i \right) - \frac{1}{q} \frac{\omega_0^2}{\chi_\omega} \tilde{\phi},$$

$$\partial_\tau \tilde{\phi} = q \lambda \tilde{\phi} - \Omega \tilde{\phi} + D_\phi \left( \partial_\tau \partial^i \tilde{\phi} + q \partial_i A^i \right),$$ \hspace{1cm} (A16)

where $\delta_{\lambda} = \delta_{\mu} - A_t$. Note that $n = n_0 + \chi \delta_{\mu}$. We have rescaled the superfluid density $f_s = f_s/q^2$. As we can clearly see, upon setting $q = 1/\lambda$, the coefficient $\lambda$ only appears coupled to $A_t$. This suggests that $\lambda$ should not appear in the $G_{nn}^R$, $G_{\phi\phi}^R$, and $G_{n\phi}^R$, correlators, which is indeed what we see from (A15). If one were only interested in these correlators, $\lambda$ could be safely ignored.

However, $\lambda$ will still non-trivially affect the flux correlators due to its coupling to $A_i$. In fact, it is easy to see from (A15) that the flux correlator now becomes

$$G_{ij}^F(\omega) = -\frac{\tilde{f}_s}{\sqrt{\lambda}} \delta_{ij} + i\sigma_n \omega \delta_{ij}.$$ \hspace{1cm} (A17)

This ensures that the Kubo formula (10) for $\lambda$ remains intact.

**Second law constraints in pinned viscoelastic hydrodynamics.**

We now give details about the second law analysis for pinned viscoelastic hydrodynamics. We introduce a gauge field $A_t, A_i$, which can be used to compute the correlations of $n, \tau_i$ respectively. For technical simplicity, we will omit introducing sources for $\tau^i, \ell^t, \ell^i$, which would require using Newton-Cartan geometry; see e.g. [46]. The energy and momentum conservation equations take the form

$$\partial_t \epsilon + \partial_i \ell^i = E_i j^i - K_i \partial_\tau \phi^i - \ell L_j \partial_\tau \Phi^i,$$

$$\partial_i \tau^i + \partial_\tau \tau^i = E^t \rho + F^i j^i + K_j \partial_\tau \phi^i + \ell L_j \partial_\tau \Phi^i,$$

$$\partial_t n + \partial_\tau n = 0,$$ \hspace{1cm} (A18)

while the Josephson and continuity equations remains the same as (12a)-(12b). We parametrise the entropy density to be

$$s^i = s + S,$$ \hspace{1cm} (A19)
where all the objects in the coefficient matrix have to be most general first order gradient corrections in $s_{ij}$ and all others zero. Here $s_{ij}$ is required to be a positive semi-definite quadratic form. It was shown in [28, 29] that, assuming the crystal to be isotropic and parity-preserving, and focusing only on the $s_{ij}$, we need to consider the "hydrostatic sector", we need to consider the matrix above. Assuming the crystal to be isotropic and parity-preserving, and focusing only on the terms that contribute to the linearised constitutive relations, we find

\begin{align}
\sigma_{\epsilon}^{ij} &= \sigma_{\epsilon} \delta^{ij}, \\
\eta^{ijkl} &= \eta (\delta^{ik} \delta^{jl} - \delta^{il} \delta^{jk}) + (\zeta - \frac{2}{3} \eta) \delta^{ij} \delta^{kl}, \\
\sigma_{\phi}^{ij} &= \sigma_{\phi} \delta_{ij}, \\
\gamma_{\phi}^{ij} &= \gamma_{\phi} \delta^{ij}, \\
\gamma_{\Phi}^{ijkl} &= \gamma_{\Phi} \delta^{ij} \delta^{kl}.
\end{align}

where $\gamma_{\phi} = \gamma_{\Phi} = 0$. All coefficients are functions of $T$ and $\mu$. The second law results in a set of inequality constraints on these coefficients

\begin{align}
\eta, \zeta, \sigma_{\epsilon}, \sigma_{\phi} \geq 0, \\
\sigma_{\Phi} \geq \sigma_{\phi} / \sigma_{\phi}.
\end{align}

For the "hydrostatic sector", we need to consider the most general first order gradient corrections in $s$. It was already found in [28, 29] that, assuming the crystal to be isotropic, there are no allowed terms in the absence of the background field $\Phi^I$. However, in the presence of $\Phi^I$ we can include the term

\begin{align}
S = \frac{1}{T} \tilde{f}_{IJ} \gamma^{IJ},
\end{align}

where

\begin{align}
\gamma^{IJ} &= \frac{1}{2} \left( -2 \partial_k \phi^I \partial^k \phi^J + T \ell \delta^{IJ} - \ell \delta^{IJK} \right),
\end{align}

is defined so that, linearly, we have $\gamma^{IJ} \approx \ell \partial^I \partial^J \Phi^I$. We can further require that $\tilde{f}_{IJ}$ is at least linear in fluctuations, because the constant contribution can be removed using a total derivative term $\partial_t \Phi^I$. We hence have

\begin{align}
\tilde{f}_{IJ} = \alpha h_{IJ} + \tilde{C}_{IJ KL} u^K L.
\end{align}

We can remove one component of $\tilde{C}_{IJ KL}$ using the redefinition of pseudo-Goldstone fields $\phi^I \rightarrow \phi^I + a \psi^I$.

### Linear pinned viscoelastic crystals

Linearising the equations on a homogeneous background $\Phi^I = x^I$, we can obtain the Josephson equation

\begin{align}
\partial_t \delta_x^I &= \lambda u^I - \Omega \delta_x^I + \eta_\gamma \delta x^I + 2 D^I_\Phi \partial_j \delta x^j + \nu \eta_\gamma \langle \Phi^I \rangle,
\end{align}

where

\begin{align}
\gamma &= \frac{-B \alpha_m}{\sigma_\phi}, \\
\gamma_T &= \frac{\gamma_\phi}{\sigma_\phi} - \frac{B \alpha_T}{\sigma_\phi}, \\
\alpha_m &= -d \frac{\partial \ln \alpha}{\partial \mu}, \\
\alpha_T &= -d \frac{\partial \ln \alpha}{\partial T}.
\end{align}

Here $\alpha_m$ is the mass expansion coefficient, while $\alpha_T$ is the thermal expansion coefficient. For the conservation equations, we find

\begin{align}
\epsilon^I &= (\epsilon + p + T \lambda_T) u^I + T \Omega \delta_x^I - \kappa_m \delta_x^I - \kappa_\gamma \delta x^I, \\
\tau^{ij} &= \rho_\gamma \delta^{ij} - 2 \lambda \delta^{ij} \delta_x^I + \lambda \delta_x^I \delta_x^{ij} + \kappa_\gamma \delta x^I, \\
\ell L^I &= -\rho \omega_0^2 \delta^{Ij} - \Gamma \delta^{ij} + \lambda \delta_x^I \delta_x^{ij} - \ell \chi^{ij},
\end{align}

where $\epsilon^I = (\epsilon + p + T \lambda_T) u^I + T \Omega \delta_x^I - \kappa_m \delta_x^I - \kappa_\gamma \delta x^I$.
where we have further defined the mechanical pressure $p_m$, mass conductivity $\kappa_m$, thermal conductivity $\kappa$, heat damping coefficient $\Omega_s$, and a new coefficient $\lambda_T$ as

$$p_m = p - \lambda B d \delta \ln \alpha,$$

$$\kappa_m = \frac{T_\gamma \sigma \alpha}{\sigma \phi}, \quad \kappa = T \sigma \epsilon + \frac{T_\gamma \sigma \alpha}{\sigma \phi} - \frac{T \gamma \sigma \alpha}{\sigma \phi},$$

$$\Omega_s = \frac{\gamma \ell^2 m^2}{\sigma \phi}, \quad \lambda_T = \ell \left( \gamma \phi - \frac{\sigma \phi}{\sigma \phi} \right), \quad (A33)$$

as well as

$$\chi^{ij} = 2 \left( \tilde{G} - \frac{\sigma \phi}{\sigma \phi} \right) \delta^{ij},$$

$$+ \delta^{ij} \left( \tilde{B} - \frac{\sigma \phi}{\sigma \phi} B \right) \delta \phi^k,$$

$$+ \delta \phi \left( \tilde{D}^{ij} \partial \phi^k \right), \quad (A34)$$

Note that $\chi^{ij}$ identically drops out from the equations of motion, and is the only contribution that contains $f_{ij}$. However, $\chi^{ij}$ still non-trivially affects the stress tensor and respective correlation functions. For the record, let us also note the heat/entropy flux

$$s^i = (s + \lambda_T) u^i + \Omega_s \delta \phi^i - \frac{\kappa_m}{T} \partial \phi^i - \frac{\kappa}{T} \partial \phi T$$

$$- \gamma \left( \tilde{D}^{ij} \partial \phi^k \right), \quad (A35)$$

From here, we derive another damping-attenuation relation in energy/entropy/heat flux

$$\Omega_s = \gamma \phi D^{ij} \kappa_0^2, \quad (A36)$$

where $k_0^2 = \ell^2 m^2/G$. This relation recently appeared in [11], where the authors derived it using the locality of hydrodynamic constitutive relations. We also find a new coefficient $\lambda_T$ that modifies the energy and entropy flux at thermodynamic level, and contributes to sourcing momenta.

### Mode spectrum of pinned viscoelastic crystals

We can use the linearised equations of motion to derive the mode spectrum of pinned viscoelastic hydrodynamics. In the transverse sector, we find a phonon sound mode with the dispersion relation similar to the U(1) case, namely

$$\omega = \pm \sqrt{\omega_0^2 + v_0^2 k^2} - \frac{i}{2} \left( k^2 (D_x^2 + D_y^2) + \Gamma + \Omega \right), \quad (A37)$$

where

$$v_0^2 = \frac{\lambda^2 G}{\rho}, \quad D_x^2 = \frac{\eta}{\rho}, \quad (A38)$$

The longitudinal sector is considerably more involved. Focusing on isothermal configurations, we find a damped sound mode and a crystal diffusion mode

$$\omega = \pm \sqrt{\omega_0^2 + v_0^2 k^2} - \frac{i}{2} \left( D_x^2 - \frac{\rho m^2}{\delta \phi^k} \Omega k^2 + \Gamma + \Omega \right),$$

$$\omega = - \frac{i k^2 \rho}{\delta \phi^k} \left( D_x^2 k^2 + \Omega \right), \quad (A39)$$

where we have defined

$$v_0^2 = \frac{\rho m^2 + \lambda B m}{\delta \phi^k}, \quad D_x^2 = \frac{\rho m^2 + \lambda B m}{\delta \phi^k} + \frac{\zeta + 2 d - 1 \eta}{\rho} \quad (A40)$$

along with mechanical mass density $\rho = \rho + \lambda B \sigma \alpha$, and susceptibility $\chi = \partial \rho/\partial \mu$. We have taken $\ell \sim O(\delta)$ in the expressions above. We again note that $\lambda$ non-trivially affects the various speeds of mode propagation. While solving the linearised equations, it is useful to note that in the isothermal limit, $\chi \delta \mu = \rho - B \lambda \rho \delta \phi^k$.

Turning back on the background fields, we can compute the following correlation functions at zero momentum

$$G^{R}_{\pi \pi}(\omega, k = 0) = \rho \delta \pi - \frac{\rho \omega (\omega + i \Omega \delta \pi)}{\omega + i \Omega}(\omega + i \Omega - \omega_0^2),$$

$$G^{R}_{\phi \phi}(\omega, k = 0) = \delta \phi \lambda^2 \frac{\rho \omega (\omega + i \Omega \delta \phi)}{\omega + i \Omega}(\omega + i \Omega - \omega_0^2),$$

$$G^{R}_{\tau \tau}(\omega, k = 0) = -i \omega \lambda \delta \tau \left( \omega + i \Omega \right) \left( \omega + i \Omega \right) \left( \omega + i \Omega - \omega_0^2 \right). \quad (A41)$$

Computing $\tau^{ij}$ correlators is beyond the scope of this work and would require coupling the system to curved space. Note that the coefficients (A27) do not affect these three correlators above.

### Comparison with previous works

In the previous version of our paper, we pointed out certain discrepancies with the work of [11]. These have now been resolved by the authors of [11] in an updated version of their paper; we present a detailed comparison below.

Let us start with the U(1) model. Our constitutive relations in (7) trivially reduce to those in [11] upon setting $\sigma = f_s = 0$ (i.e. $\lambda = 1$) and matching the conventions $\phi \rightarrow -\phi$, $A_t \rightarrow -A_t$, $A_r \rightarrow -A_r$. Our results also match with [8] in this limit upon matching the conventions $\phi \rightarrow -\phi$. In an updated version of their paper, the authors of [11] verified that their formalism does allow for nonzero $\sigma$ and $f_s$ in the presence of background
We have again used a hat for the coefficients in \( A44 \). These should be compared to the coefficients in \( A34 \) exactly match our \( A44 \), modulo the new coefficients \( \lambda_T \) due to explicit symmetry breaking.

These should be compared to [11] in the Galilean setting, i.e. upon setting \( j^i = \pi^i \). Note that the displacement field \( u^i \) of [11] is identified with our \( \delta \phi^i \), their fluid velocity \( v^i \) is our \( u^i \), their heat current \( j_Q \) is our \( T s^i \). Firstly, the Galilean constraint implies for their transport coefficients

\[
\hat{n} = \chi_{\pi\pi}, \quad \hat{\gamma}_{\lambda c} = -\chi_{n\lambda\xi}(B + G), \quad \hat{\Omega}_n = \hat{\sigma}_0 = \hat{\alpha}_0 = \hat{\gamma}_1 = 0.
\]  

We have again used a hat for the coefficients in [11] to avoid confusion. The mapping between the remaining coefficients follows as

\[
\hat{p} = p_m, \quad \hat{\chi}_{\pi\pi} = \rho, \quad \hat{\chi}_{n\lambda} = \frac{B\alpha}{B + G}, \quad \hat{\chi}_{s\lambda} = \frac{B\alpha T}{B + G}, \quad \hat{\xi} = \frac{1}{\sigma_\phi}, \quad \hat{\gamma}_{3h} = \gamma_T.
\]  

(A46)