Geodesic models generated by Lie symmetries

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Abstract We study the junction condition relating the pressure to the heat flux at the boundary of a shearing and expanding spherically symmetric radiating star when the fluid particles are travelling in geodesic motion. The Lie symmetry generators that leave the junction condition invariant are identified and the optimal system is generated. We use each element of the optimal system to transform the partial differential equation to an ordinary differential equation. New exact solutions, which are group invariant under the action of Lie point infinitesimal symmetries, are found. We obtain families of traveling wave solutions and self-similar solutions, amongst others. The gravitational potentials are given in terms of elementary functions, and the line elements can be given explicitly in all cases. We show that the Friedmann dust model is regained as a special case, and we can connect our results to earlier investigations.

Keywords radiating stars · junction conditions · Lie symmetries

1 Introduction

Exact models of relativistic radiating stars are required to investigate physical phenomena such as particle production, temperature profiles, the cosmic censorship hypothesis, and gravitational collapse of stars. The interior spacetime of a relativistic radiating star matches with the exterior Vaidya \textsuperscript{1} solution. The junction condition relating the pressure with the heat flux at the boundary of the star, which was first developed by Santos \textsuperscript{2}, must be satisfied. Kolassis et al. \textsuperscript{3} discussed the dissipative effects of fluid particles travelling in geodesic motion; the

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Friedmann dust solution is regained in the absence of heat flow. The exact solution obtained in this model has been widely used to investigate physical features of stars. The physical investigations include modelling radiating gravitational collapse in spherical geometry with neutrino flux by Grammenos and Kolassis [1]. Tomimura and Nunes [2] describe realistic astrophysical processes in the presence of heat flow. The temperature in casual thermodynamics for particles travelling in geodesic motion produces higher central values than the Eckart theory as shown by Govender et al. [3]. Thirukkanesh and Maharaj [4] generated new classes of solutions for geodesic fluid trajectories. Later Govender and Thirukkanesh [5] extended the model to include a nonvanishing cosmological constant. Recently, Ivanov [6] performed a general analysis of perfect fluid spheres with heat flow which contains the geodesic model as a special case. Note that these treatments that have been mentioned do not include shear.

The effects of shear and anisotropic pressure were studied by Chan [7], Herrera and Santos [8], Herrera et al. [9] and Thirukkanesh et al. [10]. Euclidean stars in general relativity may be modelled with nonvanishing shear; in Euclidean stars both the areal and proper radii are equal. Particular shearing solutions were found by Herrera and Santos [11], Govender et al. [12] and Govinder and Govender [13]. Naidu et al. [14] obtained the first exact solution with shear by considering the geodesic motion of fluid particles. Rajah and Maharaj [15] extended this result and obtained new classes of solutions by transforming the junction condition to a Riccati equation and solving it. Thirukkanesh and Maharaj [16] obtained new classes of exact solutions in terms of elementary functions without assuming a separable form for the gravitational potentials. The presence of shear changes the nature of the boundary condition and it is more difficult to integrate in general.

Generating exact solutions to the boundary condition of a shearing radiating star in geodesic motion, using the symmetry approach, is the main objective of this paper. In the past the Lie theory of differential equations has been used to great effect in solving the Einstein field equations in cosmology [20, 21, 22, 23, 24, 25, 26] and in higher dimensions [27]. With the help of the Lie symmetry theory of differential equations Govinder and Govender [16] obtained an exact solution for the boundary condition for an Euclidean star. We believe that their treatment has been the first group theoretic approach to solve the boundary condition. Later Abebe et al. [28] generated two classes of exact solutions for a conformally flat model by solving the junction condition exactly in the presence of anisotropic pressures using Lie symmetries. We expect that the application of the Lie symmetry analysis to a shearing and expanding radiating star when the fluid particles are in geodesic motion is likely to provide new insights.

We briefly introduce the shearing and expanding model, when the fluid particles are in geodesic motion, and present the junction condition in Sect. 2. We use a geometric approach to generate exact solutions. In Sect. 3 we find the Lie point symmetries that are admitted by the boundary condition. The optimal system for the symmetries is found to which all group invariant solutions can be transformed. The junction condition is transformed into an ordinary differential equation for each symmetry in Sect. 4, Sect. 5 and Sect. 6. By analysing the relevant ordinary differential equations, solutions are found to the boundary condition. We show the connection to known results and obtain limiting metrics. We make some concluding remarks in Sect. 7.
2 The model

The assumption that the fluid particles of radiating stars are in geodesic motion is reasonable in relativistic astrophysics modelling. The line element for the interior spacetime of a radiating star in geodesic motion with shear and expansion can be written as

$$ds^2 = -dt^2 + B^2 dr^2 + Y^2 d\Omega^2$$

(1)

where $Y = Y(r,t)$ and $B = B(r,t)$ are the metric functions and $d\Omega^2 \equiv d\theta^2 + \sin^2 \theta d\phi^2$. The fluid four-velocity $u^a$ is comoving and is given by

$$u^a = \delta^a_0$$

and the energy momentum has the form

$$T_{ab} = (\mu + p) u_a u_b + p g_{ab} + q_a u_b + q_b u_a + \pi_{ab}$$

(2)

where $\mu$ is the energy density of the fluid, $p$ is the isotropic pressure, $q_a$ is the heat flux and $\pi_{ab}$ is the stress tensor. The heat flow vector $q$ takes the form

$$q^a = (0, q, 0, 0)$$

since $q^a u_a = 0$ and the heat is assumed to flow in the radial direction. The stress tensor has the form

$$\pi_{ab} = \left( p_\parallel - p_\perp \right) \left( n_a n_b - \frac{1}{2} h_{ab} \right)$$

where $p_\parallel$ is radial pressure, $p_\perp$ is tangential pressure, $n$ is a unit radial vector given by $n^a = \frac{1}{B} \delta^a_1$, and $h_{ab}$ is the projection tensor. The isotropic pressure is given by

$$p = \frac{1}{3} \left( p_\parallel + 2 p_\perp \right)$$

in terms of the radial pressure, $p_\parallel$, and the tangential pressure, $p_\perp$.

The kinematical and dynamical quantities can be generated from the general treatment [13] by setting $A = 1$. The four-acceleration $\dot{u}^a$, the expansion scalar $\Theta$, and the magnitude of the shear scalar $\sigma$ are given by

$$\dot{u}^a = 0$$

(3a)

$$\Theta = \frac{2}{B} \left( \frac{Y_t}{Y} \right) + \frac{B_t}{B}$$

(3b)

$$\sigma = \frac{1}{3} \left( \frac{Y_t}{Y} - \frac{B_t}{B} \right)$$

(3c)

respectively for the line element (1). The Einstein field equations for the interior matter distribution become

$$\mu = 2 \left( \frac{B_t Y_t}{B Y} \right) + \frac{1}{Y^2} + \frac{Y^2}{Y^2} - \frac{1}{B^2} \left( \frac{2 Y_{tt}}{Y} + \frac{Y_{rr}}{Y^2} - \frac{2 B_r}{B} \frac{Y}{Y} \right)$$

(4a)

$$p_\parallel = -2 \frac{Y_{tt}}{Y} - \frac{Y_t}{Y^2} + \frac{1}{B^2} \frac{Y^2}{Y^2} - \frac{1}{Y^2}$$

(4b)

$$p_\perp = \frac{1}{B^2} \left( \frac{B_t Y_r}{B Y} + \frac{Y_{rt}}{Y} \right) - \left( \frac{B_{tt}}{B} + \frac{B_t Y_t}{B Y} + \frac{Y_{tt}}{Y} \right)$$

(4c)

$$q = -2 \frac{B_t}{B} \left( \frac{Y_{rt}}{Y} + \frac{B_t Y_r}{B Y} \right)$$

(4d)
for the line element (1) and matter distribution (2). The subscripts stand for partial derivatives with respect to the independent variables $t$ and $r$. The equations (4) describe the gravitational interactions in the interior of a geodesic shearing and expanding spherically symmetric star with heat flux and anisotropic pressure.

The exterior spacetime, describing the region outside the stellar boundary, is described by the Vaidya metric

$$ds^2 = -\left(1 - \frac{2m(v)}{R}\right)dv^2 - 2dvdR + R^2d\Omega^2$$ (5)

The matching of the exterior spacetime (5) with the interior spacetime (1) leads to the following set of junction conditions for the radiating star with shear in geodesic motion

\begin{align}
&dt = \left(1 - \frac{2m(v)}{R}\right)\left(\frac{dR_S}{dv}\right)dv\\
&(Y)_\Sigma = R_S(v)\\
&m(v) = \left[\frac{Y}{2} \left(1 + Y^2 - \frac{Y^2}{B}\right)\right]_\Sigma\\
&(p_\parallel) = (Bq)_\Sigma
\end{align} (6)

where $\Sigma$ is the hypersurface that defines the boundary of the radiating sphere.

The junction condition (6d) was established by Santos [2] for the first time in the case of shear-free spacetimes. Later it was extended by Glass [29] for spacetimes with nonzero shear. From (4b), (4d) and (6d) we have

$$2B^2YY_{tt} + B^2Y_t^2 - Y_t^2 + 2BY_{rt} - 2BtYY_t + B^2 = 0$$ (7)

at the boundary. Equation (4) is the fundamental differential equation governing the evolution of a shearing relativistic radiating star in geodesic motion. It is a complicated nonlinear partial differential equation and difficult to solve in general. We will attempt to integrate (4) to complete the model using the Lie theory of extended groups applied to differential equations.

### 3 Lie symmetry analysis

As mentioned earlier, the Lie symmetry method has been successfully used to generate exact solutions in General Relativity. In this paper we attempt to find solutions for the boundary condition of an expanding and shearing radiating star when the fluid particles are travelling in geodesic motion using the Lie symmetry approach.

An $n$th order differential equation

$$F(r, t, B, Y, B_r, Y_r, B_t, Y_t, B_{rt}, Y_{rt}, B_{tt}, Y_{tt}, \ldots) = 0$$ (8)

where $B = B(r, t)$ and $Y = Y(r, t)$, admits a Lie point symmetry of the form

$$G = \xi_t (r, t, B, Y) \frac{\partial}{\partial t} + \xi_r (r, t, B, Y) \frac{\partial}{\partial r} + \eta_t (r, t, B, Y) \frac{\partial}{\partial Y} + \eta_r (r, t, B, Y) \frac{\partial}{\partial B}$$ (9)
provided that

$$G^{[n]} F|_{F=0} = 0 \quad (10)$$

where $G^{[n]}$ is the $n$th prolongation of the symmetry $G$ [30,31]. The process is algorithmic and so can be implemented by computer algebraic packages. Using PROGRAM LIE [32], we find that the junction condition (7) admits the infinite-dimensional set of symmetries

$$G_1 = \frac{\partial}{\partial t} \quad (11a)$$

$$G_2 = -B f'(r) \frac{\partial}{\partial B} + f(r) \frac{\partial}{\partial r} \quad (11b)$$

$$G_3 = B \frac{\partial}{\partial B} + Y \frac{\partial}{\partial Y} + t \frac{\partial}{\partial t} \quad (11c)$$

where $f(r)$ is an arbitrary function of $r$. These symmetries tell us that (7) admits translational invariance (in $t$) as well as scaling invariance in $B$ and $r$ (together) as well as $B$, $Y$ and $t$ (together). When $f(r)$ is constant we have translational invariance in $r$ as well. The appearance of translational invariance in both independent variables indicates that traveling wave solutions may exist.

Each symmetry and any linear combination of the symmetries may be helpful in solving (7). Any group invariant solution obtained by using these symmetries can be transformed to the group invariant solution obtained by the symmetries in the optimal system which is a subalgebra of the symmetries in (11). To obtain the subalgebra of the symmetries we begin with the nonzero vector

$$G = a_1 G_1 + a_2 G_2 + a_3 G_3 \quad (12)$$

removing the coefficients $a_i$ of $G$ in a systematic way using applications of adjoint maps to $G$ to obtain

$$G_1 = \frac{\partial}{\partial t} \quad (13a)$$

$$a G_1 + G_2 = -B \frac{\partial}{\partial B} + f(r) \frac{\partial}{\partial r} \quad (13b)$$

$$a G_2 + G_3 = (1 - a f'(r)) B \frac{\partial}{\partial B} + Y \frac{\partial}{\partial Y} + t \frac{\partial}{\partial t} \quad (13c)$$

which is the optimal set of one-dimensional subalgebras of the symmetries in (11).

### 4 Invariance under $G_1$

Using the generator

$$G_1 = \frac{\partial}{\partial t} \quad (14)$$

we determine the invariants from the surface condition

$$\frac{dt}{1} = \frac{dr}{0} = \frac{dB}{0} = \frac{dY}{0} \quad (15)$$

We obtain the invariants $r$ and

$$B = h(r), \quad Y = g(r) \quad (16)$$

for the generator $G_1$. Thus the gravitational potentials are static for the generator $G_1$ and the heat flux [41] must vanish. We do not pursue this case further as the star is not radiating.
5 Invariance under $aG_1 + G_2$

The generator

$$aG_1 + G_2 = a \frac{\partial}{\partial t} - B f'(r) \frac{\partial}{\partial B} + f(r) \frac{\partial}{\partial r}$$  \hspace{1cm} (17)$$

yields the surface condition

$$\frac{dt}{a} = \frac{dr}{f(r)} = \frac{dY}{0} = - \frac{dB}{B f'(r)}$$  \hspace{1cm} (18)$$

from which we determine the invariants as

$$x = \int \frac{dr}{f(r)} - \frac{t}{a} \equiv f(r) - \frac{t}{a} \hspace{1cm} (19a)$$

$$B = \frac{h(x)}{f(r)} \hspace{1cm} (19b)$$

$$Y = g(x) \hspace{1cm} (19c)$$

for the generator $aG_1 + G_2$. When $f(r) = 1$, our independent variable becomes

$$x = r - \frac{1}{a} t$$  \hspace{1cm} (20)$$

and so we are in the realm of traveling waves solutions, with wave speed equal to $1/a$.

Using the transformation (19) equation (7) becomes

$$2ag'gh' - 2ag''h + \left( a^2 + (2gg'' + g'^2)\right) h^2 = a^2 g'^2$$  \hspace{1cm} (21)$$

which is a Riccati equation in $h$. It is difficult to integrate (21) in general. We can find particular solutions for $h$ by specifying the functional form of $g$.

We note solutions to the boundary condition with dependence on the variable $x$ given in (19a) have not been found previously for particles traveling in geodesic motion in the stellar interior. The geodesic model of Thirukkanesh and Maharaj [19], containing earlier models, has the potential

$$Y = [R_1(r)t + R_2(r)]^\alpha$$  \hspace{1cm} (22)$$

where $\alpha = 2/3, 1$. The functional dependence in (19c) is different from that in (22) since $g(x)$ is arbitrary. It is only in special cases, for particular forms of $R_1(r)$ and $R_2(r)$, that (19c) can be brought into the form (22). Therefore the classes of solution corresponding to (19) are new. This is not surprising since the Thirukkanesh and Maharaj [19] models were generated using a method that transforms (7) to a first order separable equation. In our case we are seeking group invariant solutions to the second order differential equation (7) using the Lie theory of differential equations. We achieve this by restricting the coefficients in (21) which generate forms for the function $g$. 
5.1 Case I: 

We set 

\[ 2gg'' + g'^2 = 0 \]  

Then \[ 2gg'' + g'^2 = 0 \]  

(23) can be integrated to give 

\[ g(x) = (b + cx)^{2/3} \]  

(24) 

where \( b \) and \( c \) are arbitrary constants of integration. On substituting (24) into (21), we have 

\[ 12c(b + cx)h' + 4c^2h + 9a(b + cx)^{2/3}h^2 = 4ac^2 \]  

(25) 

which is a simpler Riccati equation in \( h \). On integration we obtain 

\[ h(x) = \frac{2c}{3(b + cx)^{1/3}} \left( \frac{\exp \left[ \frac{3a}{c} (b + cx)^{1/3} + d \right] - 1}{\exp \left[ \frac{3a}{c} (b + cx)^{1/3} + d \right] + 1} \right) \]  

(26) 

where \( d \) is an arbitrary constant of integration. 

The gravitational functions are given by 

\[ B = \frac{2}{3} \int f(r) \left( b + c \left( f(r) - \frac{d}{a} \right) \right)^{1/3} \]  

\[ \times \left( \frac{\exp \left[ \frac{3a}{c} (b + c \left( f(r) - \frac{d}{a} \right) \right]^{1/3} + d \right] - 1}{\exp \left[ \frac{3a}{c} (b + c \left( f(r) - \frac{d}{a} \right) \right]^{1/3} + d \right] + 1} \right) \]  

(27a) 

\[ Y = \left( b + c \left( f(r) - \frac{d}{a} \right) \right)^{2/3} \]  

(27b) 

which satisfy the boundary condition (7). The line element becomes 

\[ ds^2 = -dt^2 + \frac{4}{9} \left[ \frac{c}{f(r) \left( b + c \left( f(r) - \frac{d}{a} \right) \right)^{1/3}} \right. \]  

\[ \times \left( \frac{\exp \left[ \frac{3a}{c} (b + c \left( f(r) - \frac{d}{a} \right) \right]^{1/3} + d \right] - 1}{\exp \left[ \frac{3a}{c} (b + c \left( f(r) - \frac{d}{a} \right) \right]^{1/3} + d \right] + 1} \right)^2 dr^2 \]  

\[ + \left( b + c \left( f(r) - \frac{d}{a} \right) \right)^{4/3} d\Omega^2 \]  

(28) 

We believe that this is a new solution for geodesic motion. 

The functional dependence on the spacetime variables \( t \) and \( r \) in the metric functions is that of a travelling wave. This becomes clearer if we select a particular form of the function \( f(r) \). We consider the special case \( f(r) = 1, a = -1, b = 0, c = 1, d = 0 \). Then we obtain the line element 

\[ ds^2 = -dt^2 + \frac{4}{9} \left[ \frac{1}{(r + t)^{1/3}} \left( \frac{\exp \left( 3(r + t)^{1/3} \right) - 1}{\exp \left( 3(r + t)^{1/3} \right) + 1} \right) \right] \]  

\[ dr^2 \]  

\[ + [r + t]^{4/3} d\Omega^2 \]  

(29)
which has an explicit travelling wave solution form. (Observe that the special case \(29\) arises essentially since the generator \(17\) has the reduced form \(aG_1 + G_2 = -\frac{\partial}{\partial r} + \frac{\partial}{\partial t}\).) The spacetime is well-behaved as translational invariance under \(t\) removes the singularity that would occur at \(t = r = 0\).

5.2 Case II: \(2ag'' = 0\)

If we set
\[
2ag'' = 0
\]
then a simple integration gives
\[
g(x) = bx + c
\]
Equation \(21\) becomes
\[
2ab(c + bx)\theta' + \left(a^2 + b^2\right) \theta^2 = a^2b^2
\]
which is also a Riccati equation in \(\theta\). This can be integrated to give
\[
\theta(x) = \frac{ab}{\sqrt{a^2 + b^2}} \left(\frac{d(c + bx)^{\sqrt{a^2 + b^2} + 1}}{d(c + bx)^{\sqrt{a^2 + b^2} - 1}} - 1\right)
\]
where \(d\) is a nonzero constant.

The gravitational potentials have the form
\[
B = \frac{ab}{f(r)\sqrt{a^2 + b^2}} \left(\frac{d(c + b(\tilde{f}(r) - \frac{t}{a}))^{\sqrt{a^2 + b^2} + 1}}{d(c + b(\tilde{f}(r) - \frac{t}{a}))^{\sqrt{a^2 + b^2} - 1}} - 1\right)
\]
\[
Y = c + b \left(\tilde{f}(r) - \frac{t}{a}\right)
\]
which is a solution for the boundary condition \(14\). The line element becomes
\[
ds^2 = -\frac{a}{f(r)\sqrt{a^2 + b^2}} \left(\frac{d(c + b(\tilde{f}(r) - \frac{t}{a}))^{\sqrt{a^2 + b^2} + 1}}{d(c + b(\tilde{f}(r) - \frac{t}{a}))^{\sqrt{a^2 + b^2} - 1}} - 1\right)^2 dr^2
\]
\[
+ \left(c + b \left(\tilde{f}(r) - \frac{t}{a}\right)\right)^2 d\Omega^2
\]
We believe that this solution is new.

A simple form can be regained from \(35\). We set \(a = -1, b = 1, d = 1, c = 0\) and \(f(r) = 1\) to obtain
\[
ds^2 = -dt^2 + \frac{1}{2} \left[\frac{[(r+t)]^{\sqrt{2}} - 1}{[(r+t)]^{\sqrt{2} + 1}}\right]^2 dr^2 + [(r+t)]^2 d\Omega^2
\]
in terms of elementary functions. The time translational symmetry ensures that this spacetime is also well-behaved. Again this particular metric has the corresponding reduced Lie generator \(aG_1 + G_2 = -\frac{\partial}{\partial r} + \frac{\partial}{\partial t}\) as in Sect. 5.1.
5.3 Case III: $a^2 + 2gg'' + g'^2 = 0$

We have also considered the case
\[ a^2 + 2gg'' + g'^2 = 0 \]  
(37)

The condition (37) can be integrated to give
\[ \pm \frac{b}{a^3} \arctan \left( a \sqrt{\frac{g(x)}{b - a^2 g(x)}} \right) = x + c \]  
(38)

This an implicit solution involving the function $g(x)$. For this case (39) becomes
\[ 2g'gh' - 2gg''h = ag'^2 \]  
(39)

which is a linear equation in $h$. This equation may be integrated to obtain
\[ h(x) = g' \left( \frac{a}{2} \int \frac{dx}{g} + d \right) \]  
(40)

where $d$ is a constant of integration, and (40) gives $h$ in terms of $g$ explicitly. The gravitational potentials therefore are

\[ B = g' \left( \frac{a}{2} \int \frac{dx}{g} + d \right) \]  
(41a)

\[ Y = g \]  
(41b)

which are functions of $x = f(r) - \frac{4}{a}$, and $g$ is given by (38). The line element is
\[ ds^2 = -dt^2 + \left[ g' \left( \frac{a}{2} \int \frac{dx}{g} + d \right) \right]^2 dr^2 + g^2 d\Omega^2 \]  
(42)

which is written in terms of the function $g$ only.

It is possible to rewrite the solution given above parametrically. We introduce a new parameter $u$ for convenience. Then (38) can be represented as
\[ x = \frac{b}{a^3} \left( u - \frac{1}{2} \sin(2u) \right) - c \]  
(43a)

\[ g = \frac{b}{2a^2} (1 - \cos(2u)) \]  
(43b)

in a parametric representation. In this case the line element (12) becomes
\[ ds^2 = -dt^2 + b \left[ \frac{1}{2a^2} (1 - \cos(2u)) \right]^2 \times \left\{ \left[ \frac{b}{a^2} \left( d - \frac{a^2}{b} \cot(x) \right) \sin(2x) \right]^2 dr^2 + d\Omega^2 \right\} \]  
(44)

where the relationship between the $u$ and $x$ is given by (43a).
6 Invariance under $aG_2 + G_3$

By using the generator

$$aG_2 + G_3 = (1 - af'(r))B \frac{\partial}{\partial B} + Y \frac{\partial}{\partial Y} + t \frac{\partial}{\partial t} + af(r) \frac{\partial}{\partial r}$$

we find the invariants from the invariant surface condition

$$\frac{dt}{t} = \frac{dr}{af(r)} = \frac{dB}{(1 - af'(r))B} = \frac{dY}{Y}$$

which are given by

$$x = \frac{t}{\exp \left( \int \frac{dt}{af(r)} \right)} \equiv \frac{t}{\exp \left( \int f(r) \right)}$$

$$B = h(x) \exp \left( \int \frac{dr}{af(r)} \right) = h(x) \exp \left( \int f(r) \right)$$

$$Y = g(x)t$$

for the symmetry $aG_2 + G_3$. We observe that the new independent variable $x$ has a self-similar form. This is particularly evident when $f(r) = r/a$ as then we have

$$x = \frac{t}{r}$$

With transformation (47) equation (7) becomes

$$2ax^3 g'gh' - 2ax^2 g (2g' + xg'') h + a^2 \left( 1 + g^2 + x^2 g^2 + 2xg (3g' + xg'') \right) h^2 = x^4 g'$$

Equation (49) is a different Riccati equation in $h$ from those considered previously. As in Sect. 5 we can find group invariant solutions to (49) by restricting the coefficients which produce functional forms for the function $g$.

6.1 Case I: $g^2 + x^2 g'^2 + 2xg (3g' + xg'') = 0$

In this case we have

$$g^2 + x^2 g'^2 + 2xg (3g' + xg'') = 0$$

To integrate we set

$$p(x) = (xg(x))^{3/2}$$

Then (51) becomes

$$p''(x) = 0$$

with solution

$$p(x) = b + cx$$

and so (53) has the general solution

$$g(x) = \frac{(b + cx)^{2/3}}{x}$$
where $b$ and $c$ are arbitrary constants of integration. The transformation (51) was picked out due to the fact that (50) possesses eight Lie point symmetries and so is linearisable to the free particle equation.

On substituting equation (54) into (49) we have

$$6a \left(3 b^2 + 4 b c x + c^2 x^2 \right) h' - 4 a c^2 x h - 9 a^2 (b + c x)^2/3 h^2 = - \left(9 b^2 + 6 b c x + c^2 x^2 \right)$$

which is also a Riccati equation in $h$. This can be integrated to give

$$h(x) = \frac{(3b + cx)}{3a(b + cx)^{1/3}} \left(1 - k + (k + 1) \exp \left(\frac{3}{a} \left(b + \frac{ct}{\exp(f(r))}\right)^{1/3}\right)\right)$$

where $k$ is an arbitrary constant of integration.

The gravitational potentials become

$$B = \frac{\exp \left(\tilde{f}(r) \right)}{3a f(r) \left(b + \frac{ct}{\exp(f(r))}\right)^{1/3}} \times \left(\frac{1 - k + (k + 1) \exp \left(\frac{3}{a} \left(b + \frac{ct}{\exp(f(r))}\right)^{1/3}\right)}{1 - k - (k + 1) \exp \left(\frac{3}{a} \left(b + \frac{ct}{\exp(f(r))}\right)^{1/3}\right)}\right)$$

$$Y = \exp \left(\tilde{f}(r) \right) \left(b + \frac{ct}{\exp(f(r))}\right)^{2/3}$$

which is a particular solution for the master equation. The line element is

$$ds^2 = -dt^2 + \left[\frac{\exp \left(\tilde{f}(r) \right)}{3a f(r) \left(b + \frac{ct}{\exp(f(r))}\right)^{1/3}} \times \left(\frac{1 - k + (k + 1) \exp \left(\frac{3}{a} \left(b + \frac{ct}{\exp(f(r))}\right)^{1/3}\right)}{1 - k - (k + 1) \exp \left(\frac{3}{a} \left(b + \frac{ct}{\exp(f(r))}\right)^{1/3}\right)}\right)\right]^2 dr^2$$

$$+ \left[\exp \left(\tilde{f}(r) \right) \left(b + \frac{ct}{\exp(f(r))}\right)^{2/3}\right]^2 d\Omega^2$$

We believe that the metric (58) has not been found before. Note that the gravitational potentials are given in terms of elementary functions. There is simplification in the form of the line elements for particular values of the parameter $k$. We consider these cases below.
6.1.1 $k = \pm 1$

If we set the arbitrary constant of integration $k = \pm 1$ in equation (58) then we obtain the metric

$$ds^2 = -dt^2 + \left[ \frac{\exp \left( \tilde{f}(r) \right) \left( 3b + \frac{ct}{\exp(\tilde{f}(r))} \right)}{3a f(r) \left( b + \frac{ct}{\exp(\tilde{f}(r))} \right)^{1/3}} \right]^2 dr^2 + \exp \left( 2 \tilde{f}(r) \right) \left( b + \frac{ct}{\exp(\tilde{f}(r))} \right)^{4/3} d\Omega$$

which is a simple form. If we set $\exp \left( \frac{1}{3} \tilde{f}(r) \right) = \beta(r)$, $b = 0$, $c = 1$ and utilize a translation of time ($t \to t + \alpha$) then (58) becomes

$$ds^2 = -dt^2 + t^{4/3} \left[ dr^2 + r^2 d\Omega \right]$$

which is the Friedmann dust model in the absence of heat flux. This is a desirable feature as the Friedmann dust model also arises as a special case in the analyses of Kolassis et al. [3], Thirukkanesh and Maharaj [7,19], Naidu et al. [17] and Rajah and Maharaj [18].

6.1.2 $k = 0$

The case $k = 0$ in equation (58) is also of physical interest. This case contains earlier investigations of geodesic configurations with shear experiencing gravitational collapse. Hence the group invariant approach followed in this paper does regain other physically viable models. If we set $\exp \left( \frac{1}{3} \tilde{f}(r) \right) = \beta(r)$, $b = 0$, $c = 1$ and utilize a translation of time ($t \to t + \alpha$) then (58) becomes

$$ds^2 = -dt^2 + (t + \alpha)^{4/3} \left[ \beta'(r)^2 \left( \frac{1 + \exp \left( \frac{3(t+\alpha)^{1/3}}{\beta(r)} \right)}{1 - \exp \left( \frac{3(t+\alpha)^{1/3}}{\beta(r)} \right)} \right)^2 dr^2 + \beta(r)^2 d\Omega \right]$$

This solution is related to the first category of the Rajah and Maharaj [18] models. In the Rajah and Maharaj [18] model there is an additional function of integration which is absent in (61). When this quantity is set to be unity then (61) is exactly the same as the Rajah-Maharaj metric. Observe that if we further set $\beta(r) = r$ and $\alpha = 0$ then the line element (61) has the form

$$ds^2 = -dt^2 + t^{4/3} \left[ \left( \frac{1 + \exp \left( \frac{3^{1/3}}{r} r^{1/3} \right)}{1 - \exp \left( \frac{3^{1/3}}{r} r^{1/3} \right)} \right)^2 dr^2 + r^2 d\Omega \right]$$

This solution was first obtained by Naidu et al. [17] for a shearing radiating star in geodesic motion. (Note that the arbitrary function of integration in the Rajah-Maharaj metric has to be set to be unity in [17] to regain (62). Thus, in the
solution space of (7), their set of solutions overlap with our set of solutions.) They analysed heat dissipation and pressure anisotropy and showed that this was a realistic description of matter configuration undergoing gravitational collapse.

For sufficiently large values of $t$ the expression

$$ \left( \frac{1 + \exp \left( \frac{2t}{r} \right)}{1 - \exp \left( \frac{2t}{r} \right)} \right)^2 $$

approaches unity and the line element (62) is approximately

$$ ds^2 \approx -dt^2 + t^{4/3} \left[ dr^2 + r^2 d\Omega^2 \right] $$

which is the limiting Friedmann dust model in the absence of heat flux. In the Rajah and Maharaj [18] model the Friedmann dust model is regained exactly when a function of integration is set to be zero. In the case of the metric (62) the dust model only arises approximately.

If we set $\exp \left( \tilde{f}(r) \right) = r$, $b = 0$, $k = 0$ and $c = 1$ in (58) then we have

$$ ds^2 = -dt^2 + \left( \frac{t}{r} \right)^{4/3} \left[ \frac{1}{9} \left( \frac{1 + \exp \left[ \frac{3 (\frac{t}{r})^{1/3}}{1 - \exp \left[ \frac{3 (\frac{t}{r})^{1/3}}{1} \right]} \right]}{1 - \exp \left[ \frac{3 (\frac{t}{r})^{1/3}}{1} \right]} \right)^2 \left[ dr^2 + r^2 d\Omega^2 \right] $$

in terms of the self-similar variable $t/r$. We observe that the self-similar variable indicates the existence of a homothetic Killing vector. Wagh and Govinder [33] found a self-similar vector in shearing spherically symmetric spacetimes. Recently, Abebe et al. [28] obtained new models for a conformally flat radiating star in which a particular class contains the self-similar variable.

6.2 Case II: $2g' + xg'' = 0$

If we set

$$ 2g' + xg'' = 0 $$

then we obtain the function

$$ g(x) = \frac{b}{x} + c $$

where $b$ and $c$ are arbitrary constants of integration. On substituting equation (66) into (49) we have

$$ 2ab(b + cx)h' - a^2 \left( 1 + c^2 \right) h^2 = -b^2 $$

which is a Riccati equation in $h$. This can be integrated to give

$$ h(x) = \frac{b}{a\sqrt{1 + c^2}} \left( \frac{1 - k(b + cx)^{\sqrt{1 + c^2}}}{1 + k(b + cx)^{\sqrt{1 + c^2}}} \right) $$

where $k$ is a nonzero constant.
The gravitational potentials have the form

\[
B = \frac{b \exp \left( \tilde{f}(r) \right)}{a \sqrt{1 + c^2 f(r)}} \left( \frac{1 - k \left( b + \frac{ct}{\exp(f(r))} \right) \sqrt{1 + c^2 f(r)}}{1 + k \left( b + \frac{ct}{\exp(f(r))} \right) \sqrt{1 + c^2 f(r)}} \right) \tag{69a}
\]

\[
Y = b \exp \left( \tilde{f}(r) \right) + ct \tag{69b}
\]

which is a particular solution for the master equation (7). The metric is given by

\[
ds^2 = -dt^2 + \left[ \frac{b \exp \left( \tilde{f}(r) \right)}{a \sqrt{1 + c^2 f(r)}} \left( \frac{1 - k \left( b + \frac{ct}{\exp(f(r))} \right) \sqrt{1 + c^2 f(r)}}{1 + k \left( b + \frac{ct}{\exp(f(r))} \right) \sqrt{1 + c^2 f(r)}} \right) \right]^2 dr^2
\]

\[
+ \left[ b \exp \left( \tilde{f}(r) \right) + ct \right]^2 d\Omega^2 \tag{70}
\]

which is another group invariant model.

If we set \( \exp \left( \tilde{f}(r) \right) = r, a = 1/\sqrt{2}, b = 1, c = 1 \) and \( k = 1 \) then the line element (70) becomes

\[
ds^2 = -dt^2 + \left( 1 - \frac{1 + t}{1 + (1 + \frac{t}{r}) \sqrt{2}} \right)^2 dr^2 + r^2 \left( 1 + \frac{t}{r} \right)^2 d\Omega^2 \tag{71}
\]

which has a simple form.

7 Conclusion

We considered a shearing and expanding relativistic radiating star when the fluid particles are in geodesic motion. This model was analysed with the Lie infinitesimal generators applicable to differential equations. We studied in particular the junction condition which relates the radial pressure to the heat flux. Three Lie point symmetries admitted by this equation were found and an optimal system was obtained. The symmetries were used to reduce the governing highly nonlinear partial differential equation to ordinary differential equations. By solving the reduced ordinary differential equations, and transforming to the original variables, we obtained exact solutions for the master equation. It was particularly pleasing to observe that we were able to provide families of traveling wave solutions as well as families of self-similar solutions. Both types of solutions have been found to have great application in a variety of areas of mathematical physics [34,35].

Our classes of solutions contain new and previously obtained solutions. (We utilised the computer software package Mathematica [36] for some of the integrations and to verify the correctness of all solutions.) We regained the Friedmann dust model as a special case of one family of solutions. In addition, the connection to the previous models of Naidu et al. [17] and Rajah and Maharaj [18] was shown.
Therefore we have demonstrated that the Lie method is a useful tool in modelling gravitational behaviour in collapse.

This approach has allowed us to solve the rather complicated equation (7). In its original form, this equation is very difficult to solve. We used appropriate group invariants to reduce the equation to Riccati ODEs. While this allowed us to make some progress, the resulting equations did not yield to the standard approaches for solving Riccati equations. However, by making simplifying assumptions, we were able to solve the equations. It is remarkable that the simplifications also ensured that the standard techniques could be applied. In fact, the simplified Riccati equations could then be transformed into second order linear equations with constant coefficients! This is a elegant happenstance and completely unexpected. None of this would have been revealed if not for the Lie symmetry approach.

The physical features of the models generated here will be studied in greater detail in the future. In particular, the traveling wave collapse and self-similar collapse should be of great interest.

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