Scalable Implementation of Boson Sampling with Trapped Ions

C. Shen, Z. Zhang and L.-M. Duan

Department of physics, University of Michigan, Ann Arbor, USA
and Center for Quantum Information, IIIS, Tsinghua University, China

Boson sampling solves a classically intractable problem by sampling from a probability distribution given by matrix permanents. We propose a scalable implementation of Boson sampling using local transverse phonon modes of trapped ions to encode the Bosons. The proposed scheme allows deterministic preparation and high-efficiency readout of the Bosons in the Fock states and universal mode mixing. With the state-of-the-art trapped ion technology, it is feasible to realize Boson sampling with tens of Bosons by this scheme, which would outperform the most powerful classical computers and constitute an effective disproof of the famous extended Church-Turing thesis.

PACS numbers: 03.67.Ac, 37.10.Ty, 89.70.Eg

What is the ultimate computational power of physical devices? That is a deep question of great importance for both physics and computer science. The famous extended Church-Turing thesis (ECTT) postulates that a (classical) probabilistic Turing machine can efficiently simulate the computational power of any physical devices ("efficiently" here means with a polynomial overhead) [1]. The recent development in quantum computation brings doubt to the ECTT with discovery of superfast quantum algorithms. The most well known example is Shor’s algorithm to factorize a large number in polynomial time with a quantum computer [2]. Classically, whether factoring is hard is not settled (a "hard" problem means its solution requires exponential time). No efficient classical algorithm has been found yet to solve factoring, but it wouldn’t be very surprising if one finds one as this will not induce dramatic change to the computational complexity theory.

Recently, Ref. [1] introduces another problem, called Boson sampling, which is hard for classical computers but can be solved efficiently with a quantum machine. Boson sampling is defined as a problem to predict the probabilities of the measurement outcomes in the Fock basis for M Bosonic modes, which start in definite Fock states and undergo a series of mode mixing defined in general by a unitary matrix. By definition, this problem can be efficiently solved with a quantum machine, but classically its solution requires sampling of a probability distribution given by matrix permanents with an exponentially large number of possible outcomes. Computation of the matrix permanent is known to be #P-hard (much harder than the more well-known class of the NP-hard problems) [3]. Ref. [1] rigorously proved that Boson sampling is classically intractable unless the so-called polynomial hierarchy in the computational complexity theory collapses, which is believed to be extremely unlikely. In this sense, compared with the factoring problem, although Boson sampling has no immediate practical applications, it is a problem much harder for classical computers to solve. A demonstration of Boson sampling with a quantum machine thus constitutes an effective disproof of the famous ECTT. Because of this far-reaching theoretical implication, experimental demonstration of the Boson sampling has raised strong interest recently. Several publications this year have reported proof-of-principle demonstrations of the Boson sampling with up to three photons [4–7]. The key challenge for the next-step experiments is to scale up the number of Bosons. The demonstration using photons based on the spontaneous parametric down conversion source has difficulty in terms of scalability [4–7]. The success probability decreases very rapidly with the number of photons due to the probabilistic nature of the single-photon source and the significant photon loss caused by the detector and the coupling inefficiencies. This, in practice, limits the number of Bosons below 10, which is still within the simulation range of classical computers.

In this paper, we propose a scalable scheme to realize Boson sampling using the transverse phonon modes of trapped ions. Compared with the implementation using photons, this scheme has the following desirable features: First, the Fock states of the phonons can be prepared in a deterministic fashion and there is no limitation to the number of Bosons that one can realize with this system. We encode the Bosons using the local transverse phonon modes [8], and the state initialization can be done through simple Doppler cooling and one step of the sideband cooling that applies to any number of ions. Second, we find a technique to do projective detection of the phonon numbers for all the ions through sequential spin quantum jump measurements. This gives an implementation of number-resolving phonon detectors with near perfect efficiency, much higher than the efficiency of typical single-photon detectors. Finally, we prove that universal coherent mixing of different phonon modes can be achieved through a combination of the inherent Coulomb interaction and simple laser-induced phase shifts of the ions. Through this scheme, it is feasible to realize Boson sampling for tens of phonons with the state-of-the-art trapped ion technology. This scale has gone beyond the simulation capability of any classical computers and corresponds to the most interesting experimental region for
test of the ECTT [1, 9].

The problem of Boson sampling is defined as follows: we have $M$ input Bosonic modes $a_i$ ($i = 1, 2, ..., M$), which undergo coherent mode mixing described in general by a unitary matrix $\Lambda$, with the output modes given by $b_i = \sum_j A_{ij} a_j$. The input modes are prepared in a Fock (number) state $|T\rangle = |t_1, t_2, ..., t_M\rangle$, where $t_i$ is an integer denoting the occupation number of the mode $a_i$. We measure the output modes $b_i$ in the Fock basis and the probability to get the outcome $|S\rangle = |s_1, s_2, ..., s_M\rangle$ is given by [1, 10]

$$P(S|T) = \frac{|P_{\text{err}}(\Lambda(S,T))|^2}{\prod_{i=1}^M s_i! \prod_{i} j_! t_i!}$$

(1)

where $P_{\text{err}}(\cdot)$ denotes the matrix permanent and $\Lambda(S,T)$ is a sub-matrix of $\Lambda$ formed by taking $s_j$ copies of the $j$-th column and $t_i$ copies of the $i$-th row of the matrix $\Lambda$. Since the total number of Bosons is conserved $N = \sum_i a_i^\dagger a_i = \sum_j b_j^\dagger b_j$, the sub-matrix $\Lambda(S,T)$ has dimension $N \times N$. Due to the hardness to calculate the matrix permanent, it becomes impossible to sample the probability distribution $P(S|T)$ with any classical computer when the number of Bosons $N$ increases beyond $20 \sim 30$. An experimental demonstration of a quantum machine that can successfully perform this job therefore provides strong evidence against the ECTT.

To realize Boson sampling with trapped ions, we consider a chain of $N$ harmonic traps with a transverse trapping frequency $\omega_x$ significantly larger than the axial one $\omega_z$. The Bosons are represented by the local transverse phonon modes $a_i$ associated with each ion $i$ ($i = 1, 2, ..., M$), all with the oscillation frequency $\omega_x$. The Coulomb interaction between the ions introduces a small perturbation to the oscillation frequency of the local phonon modes, with the interaction Hamiltonian described by [11]

$$H_c = \sum_{1 \leq i < j \leq M} \hbar t_{i,j} \left( a_i^\dagger a_j + a_i a_j^\dagger \right)$$

(2)

where the hopping rates $t_{i,j} = t_0/|z_{i0} - z_{j0}|^3$ and $t_0 = e^2/(8\pi\epsilon_0 m\omega_x)$. Here, $z_{i0}$ denotes the axial equilibrium position of the $i$th ion with mass $m$ and charge $e$. The Hamiltonian (1) is valid under the condition $t_{i,j} \ll \omega_x$, which is always satisfied for the parameters considered in this paper. To make the scheme more scalable and eliminate the challenging requirement of resolving phonon sidebands for a large ion chain, we use the local transverse phonon modes to represent the target Bosons instead of the conventional normal modes.

To initialize the local phonon modes $a_i$ to the desired Fock states, first we cool them to the ground state by laser cooling. The routine Doppler cooling achieves a temperature $T_D \sim \hbar\Gamma/(2k_B)$ ($\Gamma$ is the natural bandwidth of the excited state and $k_B$ is the Boltzmann constant), with the corresponding thermal phonon number $\bar{n}_x = k_B T_D/\hbar \omega_x \sim \Gamma/(2\omega_x)$, which is about $1 \sim 2$ under typical values of $\omega_x \approx 2\pi \times (5 \sim 10)$ MHz and $\Gamma \approx 2\pi \times 20$ MHz. The sideband cooling can further push the transverse modes to the ground state with $\bar{n}_x \approx 0$ [12]. For the axial modes, we only require their thermal motion to be much less than the ion spacing, which is satisfied already under routine Doppler cooling. As all the local transverse modes have the same frequency (with $t_{i,j} \ll \omega_x$), we only need to apply one step of the sideband cooling independent of the number of ions, with the laser detuning set at $-\omega_x$. The off-resonant process in the sideband cooling limits $\bar{n}_x \sim \gamma/\omega_x$, where $\gamma$ is the rate of the sideband cooling which needs to be comparable with the phonon hopping rate $t_{i,i+1}$. For a harmonic trap, we take $t_0 = [e^2/(4\pi\epsilon_0 m\omega_x^2)]^{1/3}$ as the length unit so that the ion spacings in this unit take universal dimensionless values (of the order of 1) independent of the ion species and the trap frequency [13]. The hopping rate $t_{i,i+1} \sim t_0/\bar{n}_x = \omega_x^2/(2\omega_y)$ and the thermal phonon number after the sideband cooling $\bar{n}_x \sim t_{i,i+1}/\omega_x \sim \omega_x^2/(2\omega_y^2) < 10^{-2}$ with a typical $\omega_y \approx 2\pi \times (0.3 \sim 1)$ MHz. After cooling of all the transverse modes to the ground state, we can then set them to any desired Fock states through a sequence of laser pulses blue detuned at $\omega_x$ [14]. Note that the ion spacing is about or larger than $10 \mu m$ under our choice of the parameters, and under such a spacing it is reasonable to assume individual addressing of different ions with focused laser beams. The focused beam can prepare different local modes $a_i$ to different Fock states $|n_i\rangle$.

For implementation of the Boson sampling, without loss of generality we actually can choose $n_i = 1$, which requires only one pulse for preparation. To make the phonon hopping negligible during the preparation step, the sideband Rabi frequency $\Omega$ needs to be large compared with the hopping rate $t_{i,i+1} \sim \omega_x^2/(2\omega_y) \sim 2\pi \times (10 \sim 100)$ kHz, which is easy to satisfy under typical laser power.

After the state initialization, we need to coherently mix different phonon modes. The inherent Coulomb interaction described by the Hamiltonian (1) serves this purpose, however, it is constantly on without a tuning knob and we need to introduce additional control parameters to realize different unitary transformations between the $M$ modes. To achieve this goal, we introduce a simple operation which induces a controllable phase shift for any local phonon mode at any desired time. A laser pulse with duration $t_p$ and detuning $\delta$ to the sideband induces an additional Hamiltonian $H_L = \hbar (\Omega_i^2/\delta) a_i^\dagger a_i$ ($\Omega_i$ is the sideband Rabi frequency associated to the target ion $i$), which gives a phase shift $U_{\phi_i} = e^{i\phi_i a_i^\dagger a_i}$ to the mode $a_i$ with $\phi_i = \Omega_i^2 t_p/\delta$. We choose $\Omega_i^2/\delta \gg t_{i,i+1}$ so that the pulse can be considered to be instantaneous over the time scale of phonon tunneling.

The operation $U_{\phi_i}$ and the Coulomb interaction $H_c$ together are universal in the sense that a combination
of them can make any unitary transformation on the $M$ phonon modes represented by the $M \times M$ matrix $\Lambda$. Now we prove this statement. It is known that any unitary transformation $\Lambda$ on $M$ Bosonic modes can be decomposed as a sequence of neighboring beam-splitter-type of operations and individual phase shifts [15]. The beam splitter operation for the modes $(j, j + 1)$ is represented by the Hamiltonian $H^{(j)}_{bs} = \hbar t_{j,j+1} \left(a_ja_{j+1}^\dagger + a_{j+1}a_j^\dagger \right)$. To realize $H^{(j)}_{bs}$, we just need to cut off all the other interaction terms in the Coulomb Hamiltonian given by Eq. (1) except for a specific pair $(j, j + 1)$. This can be achieved through the idea of dynamical decoupling using the fast phase shifts $U_{\phi_i}$ with $\phi_i = \pi$ [16]. Note that a Hamiltonian term $H_{ij} = \hbar t_{i,j} \left(a_i a_j^\dagger + a_j a_i^\dagger \right)$ can be effectively turned off for an evolution time $t$ if we apply an instantaneous $\pi$-phase shift $U_{\phi_i=\pi}$ at time $t/2$ to the mode $a_i$ to flip the sign of $H_{ij}$ to $-H_{ij}$ for the second half period of the evolution. The interaction Hamiltonian $H_c$ has long-range tunneling, but it decays fast with distance $d$ through $1/d^3$ scaling. If we take the first order approximation to keep only the nearest neighbor tunneling, the Hamiltonian has the form $H_{NN} = \sum_{i=1}^{M-1} \hbar t_{i,i+1} \left(a_i a_{i+1}^\dagger + a_{i+1} a_i^\dagger \right)$. The Hamiltonian $H_{NN}$ can be used to realize the required coupling $H^{(j)}_{bs}$ for an arbitrary $j$ if we apply $\pi$ phase shifts at time $t/2$ to every other modes in the ion chain except for the pair $(j, j + 1)$ as illustrated in Fig. 1(a). This kind of decoupling can be extended and we can simulate the Hamiltonian $H_{NN}$ (and thus $H^{(j)}_{bs}$) with the original long range Hamiltonian $H_c$ to an arbitrary order of approximation. Suppose we cut the interaction range in $H_c$ to the $k$th order (i.e., we neglect the terms in $H_c$ that scale as $1/d^3_{ij}$ with $|i-j| > k$), we can shrink the interaction range from $k$ to $k-1$ by applying one step of dynamical decoupling with the pattern of $\pi$-phase shifts illustrated in Fig. 1(b). This step can be continued until one reaches $H_{NN}$ through concatenation of the dynamical decoupling [16]. This proves that the Coulomb interaction Hamiltonian $H_c$, together with the phase shifts $U_{\phi_i}$ on single ions, can realize any beam splitter operations and thus be universal for construction of arbitrary unitary operations on the $M$ phononic modes. We should note that the above proof of universality based on the idea of dynamical decoupling is constructive but does not necessarily give the optimal decomposition of a given unitary operation $\Lambda$. Direct optimization of a sequence of control parameters $\phi_i$ in $U_{\phi_i}$ and the evolution time could give much more efficient construction of a given unitary matrix.

The final step of the Boson sampling is detection of all the phononic modes in the Fock basis. The conventional method of measuring the phonon number distribution of a single mode by recording the spin oscillation from red or blue sideband pulses is not applicable here as it cannot measure correlation of different phonon modes in the Fock basis [14]. What we need is a projective measurement of each mode in the Fock basis which gives information of arbitrary high order correlations between different modes. For trapped ions, a projective measurement of its spin (internal) state can be done with a very high efficiency through the quantum jump technique using a cycling transition. However, the spin detection gives only binary measurement outcomes ("dark" or "bright"). We need to figure out a way to perform projective measurements of the Fock states (with multiple possible outcomes) for each phonon mode through the binary spin detection. This is achieved through a consecutive detection scheme with the following steps: (1) First, to illustrate the idea, we consider a single ion with its phonon mode in an arbitrary state $|n\rangle$ and its spin prepared in the dark state $|D\rangle$ (see Fig. 2a). (2) Through the well known adiabatic transition technique [17], we make a complete population transfer from $|n+1\rangle |D\rangle$ to $|n\rangle |B\rangle$ for all the Fock components $|n\rangle$ by chirping the frequency of a laser pulse across the red detuning at $-\omega_x$ (see Fig. 2b for the population distribution after this step). (3) We make a carrier transition $|n\rangle |D\rangle \rightarrow |n\rangle |B\rangle$ with a $\pi$-pulse to flip the dark and the bright states (see Fig. 2c). (4) After this step, we immediately measure the spin state of the ion through the quantum jump detection. With probability $|c_0|^2$, the outcome is "bright". In this case the measurement is finished and we know the phonon is in the $|n=0\rangle$ state. Otherwise, the spin is in the dark state and the phonon is in the $|n \geq 1\rangle$ components (see Fig. 2d for the population distribution in this case). When the spin is in the dark state, the ion does not scatter any photons during the quantum jump measurement. So its phonon state will not be influenced by this measurement. This feature is important for this consecutive measurement scheme. (5) Now with the phonons in the $|n \geq 1\rangle$ components, we just repeat the steps (2)-(3)-(4) until finally we get the outcome "bright" for the spin detection. We conclude that the phonon is in the Fock state $|n=1\rangle$ if the outcome "bright" occurs (with proba-
probability $|c_l|^2$ after $l$ repetitions of the above steps. (6) The above consecutive measurement scheme can be extended straightforwardly to measure $M$ local phonon modes in the Fock basis independently with $M$ ions. The only requirement is that the phonon tunneling between different modes is negligible during the measurement process. The slowest step of the measurement is the quantum jump detection of the ion spin state. Recently, there has been experimental report of high efficiency ($> 99\%$) spin state detection within $10 \mu s$ detection time \cite{18}. The typical phonon hopping rate between the neighboring ions in our scheme is in the range of $t_{i,i+1} \sim 2\pi \times (10 \sim 100) \text{ kHz}$, and this hopping rate can be significantly reduced during the detection through either an expansion of the ion chain along the $z$ direction right before the direction by lowering the axial trap frequency or application of a few dynamical decoupling pulses to turn off the neighboring tunneling during the detection. As the hopping scales as $1/d^3$, a moderate increase of the effective distance $d$ will significantly reduce the tunneling and push it below the kHz level. We should note that for the Boson sampling algorithm, the output phonon number per mode is typically small (the conventional photon detectors actually can only distinguish 0 and 1 photons), and the number of repetitions in our consecutive measurement scheme is either zero or very few in most cases.

In summary, we have proposed a scalable scheme to realize the Boson sampling algorithm by use of the local transverse phonon modes of trapped ions. The scheme allows deterministic preparation and high-efficiency read-out of the phonon Fock states and universal manipulation of the phonon modes through a combination of inherent Coulomb interaction and individual phase shifts. Several dozens of ions have been successfully trapped experimentally to form a linear chain, and in principle there is no limitation to the number of ions that can be manipulated in a linear Paul trap by use of anharmonic axial potentials \cite{19}. This scheme thus opens the perspective to realize Boson sampling for dozens of phonons with the state-of-the-art trapped ion technology, which would beat the capability of any classical computers and give the first serious experimental test of the extended Church-Turing thesis.

We thank Kihwan Kim for discussions. This work was supported by the NBRPC (973 Program) 2011CBA00300 (2011CBA00302), the IARPA MUSIQC program, the ARO and the AFOSR MURI programs, and the DARPA OLE program.

\begin{thebibliography}{99}
\bibitem{1} S. Aaronson, A. Arkhipov, in Proceedings of the ACM Symposium on Theory of Computing (ACM, NY, 2011), pp. 333-342.
\bibitem{2} P. W. Shor, SIAM J. Comput., 26(5):1484–1509 (1997); Earlier version in IEEE FOCS (1994).
\bibitem{3} L. G. Valiant, Theoretical Comput. Sci., 8(2), 189-201 (1979).
\bibitem{4} J. B. Spring et. al., Science 339, 798 (2013).
\bibitem{5} M. A. Broome et. al., Science 339, 794 (2013).
\bibitem{6} M. Tillmann et. al., Nature Photon. 7, 540 (2013).
\bibitem{7} A. Crespi et. al., Nature Photon. 7, 545 (2013).
\bibitem{8} S.-L. Zhu, C. Monroe, L.-M. Duan, Phys. Rev. Lett. 97, 050505 (2006).
\bibitem{9} S. Aaronson and A. Arkhipov, arXiv:1309.7460; C. Gogolin, M. Kliesch, L. Aolita, J. Eisert, arXiv:1306.3995.
\bibitem{10} S. Scheel, arXiv: quant-ph/0406127 (2004).
\bibitem{11} D. Porras and J. I. Cirac, Phys. Rev. Lett. 93, 263602 (2004); X.-L. Deng, D. Porras and J. I. Cirac, Phys. Rev. A 77,033403 (2008).
\bibitem{12} F. Diedrich, et. al., Phys. Rev. Lett. 62, 403 (1989); C. Monroe et. al., Phys. Rev. Lett. 75, 4011 (1995); B. E. King, et. al., Phys. Rev. Lett. 81, 1525 (1998); G. Poulsen, et. al., Phys. Rev. A 86, 051402(R).
\bibitem{13} D. F. V. James, Applied Physics B 66, 181 (1998).
\bibitem{14} D. M. Meekhof et. al., Phys. Rev. Lett. 76, 1796 (1995).
\bibitem{15} M. Reck, A. Zeilinger, H. J. Bernstein, and P. Bertani, Phys. Rev. Lett. 73, 58 (1994).
\bibitem{16} For a review, see H. K. Ng, D. A. Lidar, J. Preskill, Phys. Rev. A 84, 012305 (2011).
\bibitem{17} T. Watanabe, S. Nomura, K. Toyoda, and S. Urabe, Phys. Rev. A 84, 033412 (2011).
\bibitem{18} R. Noek, et. al., arXiv: quant-ph/1304.3511 (2013).
\bibitem{19} G.-D. Lin et. al., Europhys. Lett. 86, 60004(2009).
\end{thebibliography}
Figure 2: A consecutive measurement scheme to perform projective detection of the phonon mode in the Fock basis. (a) The initial state configuration right before the measurement. (b-d) The state configuration after the blue sideband transition, the carrier transition, and the quantum jump detection. These three steps are repeated until one finally registers the "bright" state (see the text for details).