UNVEILING THE UNIVERSALITY OF I-LOVE-Q RELATIONS

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ABSTRACT

The recent discovery of the universal I-Love-Q relations connecting the moment of inertia, tidal deformability, and the spin-induced quadrupole moment of compact stars is intriguing and totally unexpected. In this paper, we provide numerical evidence showing that the universality can be attributed to the incompressible limit of the I-Love-Q relations. The fact that modern equations of state are stiff, with an effective adiabatic index larger than about two, above the nuclear density range is the key to establishing the universality for neutron stars and quark stars with typical compactness from about 0.1 to 0.3. On the other hand, the I-Love-Q relations of low-mass neutron stars near the minimum mass limit depend more sensitively on the underlying equation of state because these stars are composed mainly of softer matter at low densities. However, the I-Love-Q relations for low-mass quark stars can still be represented accurately by the incompressible limit. We also study the I-Love relation connecting the moment of inertia and tidal deformability analytically in Newtonian gravity and show why the I-Love-Q relation is weakly dependent on the underlying equation of state and can be attributed to its incompressible limit.

Key words: equation of state – gravitation – stars: neutron

1. INTRODUCTION

Neutron stars (NSs) are prime examples of physical systems that require an understanding of both strongly interacting many-body nuclear physics and general relativity (GR). With the density in their cores reaching a few times the normal nuclear density, that is, the regime where theoretical calculations for dense nuclear matter are poorly constrained, the equation of state (EOS) in the cores of NSs is not well understood. Different EOS models would generally predict quite different physical quantities related to NSs. From a nuclear physicist’s point of view, it would thus be interesting to use the observed physical quantities of NSs to place constraints on EOS models. For example, the discovery of NSs with masses of about ~2\(M_{\odot}\) (Demorest et al. 2010; Antoniadis et al. 2013) would place a strong constraint on the underlying EOS model (see, e.g., Lattimer 2012 for discussions). A relativist, however, might be more interested in looking for EOS-insensitive relationships connecting the different physical quantities, since one could then use these relations to test the underlying gravitational theory in the strong-field limit despite our ignorance of the supranuclear density EOS.

It is well established that different EOS models would generally lead to quite different stellar structure and global quantities of NSs, such as their masses and radii. It is thus quite surprising that approximately EOS-insensitive relations concerning various different physical quantities, such as the moments of inertia, compactness, and oscillation modes of nonrotating NSs, can indeed be found (Bejger & Haensel 2002; Lattimer & Schutz 2005; Andersson & Kokkotas 1996, 1998; Benhar et al. 1999, 2004; Tsui & Leung 2005a, 2005b; Lau et al. 2010). More recently, the discovery of the so-called I-Love-Q relations in GR (Yagi & Yunes 2013a, 2013b) has gained a lot of interest. The I-Love-Q relations connect the moment of inertia, the spin-induced quadrupole moment, and the tidal deformability of NSs with EOS-insensitive relations.

Since the original discovery by Yagi & Yunes (2013a, 2013b), the investigation of the I-Love-Q relations has been extended to include rapid rotation. Initially, Doneva et al. (2014b) used the rotation frequency to characterize rotation and found that the I-Q universal relation is broken. However, more recent numerical works by Pappas & Apostolatos (2014) and Chakrabarti et al. (2014), which are supported by the analytical study of Stein et al. (2014), found that the I-Q relation remains approximately EOS-independent for rapidly rotating stars when rotation is characterized by a suitable dimensionless parameter. Besides the extension to rapid rotation, the effects of magnetic field (Haskell et al. 2014) and tidal deformation in NS–NS inspirals (Maselli et al. 2013) on the universal relations have also been studied. More recently, universal relations connecting higher multipole moments have also been investigated (Pappas & Apostolatos 2014; Stein et al. 2014; Yagi et al. 2014a; Chatziioannou et al. 2014). As proposed in Yagi & Yunes (2013a, 2013b), one interesting astrophysical application of the I-Love-Q relations is to use them to break the degeneracy between the NS quadrupole moment and the NS’s individual spins in the gravitational-wave analysis for an inspiraling NS binary.

From a relativist’s point of view, as discussed above, the I-Love-Q relations are interesting because they are insensitive to EOS models to within 1%, and hence they could in principle be used to test GR by comparing the relations with the relevant observed quantities. Furthermore, it would be interesting to study whether similar universal relations exist in alternative theories of gravity. In the original work of Yagi & Yunes (2013a, 2013b), besides discovering the I-Love-Q relations in GR, they also found similar universal relations in dynamical Chern–Simons gravity, although the relations differ from the GR ones. They further suggested that if the moment of inertia and the tidal deformability of the double-binary pulsar J0737-3039 can be measured to 10% and 60% accuracy, respectively, then the Chern–Simons theory can be constrained much better than

\(^2\) Note that approximately EOS-independent relations connecting the f-mode oscillation frequency to the mass and moment of inertia of NSs, accurate to within the 1% level, have also been found (Lau et al. 2010).
current tests by six orders of magnitude. Recently, the I-Love-Q relations were studied in Eddington-inspired Born–Infield gravity (Sham et al. 2014) and scalar-tensor theories (Pani & Berti 2014; Doneva et al. 2014a). It was found that the I-Love-Q relations in these theories agree with those in GR to within a few percent level.

The universal I-Love-Q relations are totally unexpected and the reason why these relations exist is not yet understood. In the original work, Yagi & Yunes (2013a, 2013b) suggested two possible reasons. The first reason is that the physical quantities considered depend mostly on the NS outer layer, where all EOS models are similar. The second reason is that the relations approach the black hole limit as the NS compactness increases, and hence the internal structure of NS becomes unimportant. More recently, Yagi et al. (2014b) suggested that the isodensity contours of realistic NSs can be approximated by elliptical isodensity contours, and the assumption of the self-similarity of these surfaces plays a crucial role in the universality of I-Love-Q relations (see Section 5 for a discussion).

In this paper, we propose that although realistic EOS models can differ from one another quite significantly, they are nevertheless already stiff enough in the nuclear density range to lead to the universality of the I-Love-Q relations. The stiffness of an EOS is measured by its effective adiabatic index $\Gamma$. While it is well known that $\Gamma$ generally depends quite sensitively on the density and varies from one EOS to another, the fact that modern realistic EOS models are typically stiff enough and have $\Gamma \gtrsim 2$ above the nuclear density range is enough to explain the universality of the I-Love-Q relations. We will study analytically how the I-Love relation in Newtonian gravity depends on the stellar density profile, and hence the underlying EOS model, which is parameterized by a perturbation parameter $\delta$. We demonstrate that the prescribed density profile with the value $\delta = 1$ can yield the NS structure constructed by realistic EOS models very well. Furthermore, we show that the perturbation expansion point $\delta = 0$, which corresponds to an incompressible stellar model (with $\Gamma = \infty$ formally), is a stationary point for the I-Love relation in the sense that the dependence of the relation on the stellar density profile is second order in $\delta$, and hence is weakly dependent on the EOS. Our analysis suggests that in view of the I-Love relation, different realistic EOS models are generally stiff enough to be modeled well by a single incompressible EOS, and hence lead to an approximate universality.

The plan of this paper is as follows. In Section 2, we show numerically that the I-Love-Q relations obtained from realistic EOSs can be accurately represented by those of the incompressible EOS. We then provide a Newtonian analysis in Section 3 to show why this is the case for the I-Love relation. In Section 4, we study when the universality of I-Love-Q relations would break down. Section 5 compares our finding with the recent work of Yagi et al. (2014b) in which the elliptical isodensity approximation is suggested to play a crucial role in the universality. Finally, our conclusions are summarized in Section 6. Unless otherwise noted, we use geometric units where $G = c = 1$.

2. I-Love-Q Relations

Here, we will study the I-Love-Q relations numerically in the slow-rotation approximation. In GR, the moment of inertia $I$ of a star is obtained by expanding the expression $I = J/\Omega$ to first order in $\Omega$, where $J$ and $\Omega$ are the angular momentum and angular velocity of the star, respectively. The spin-induced quadrupole moment $Q$ characterizes the deformation of the star due to rotation and must be calculated at second order in $\Omega$. The tidal deformability $\lambda$ measures the deformation of a star due to the tidal effect created by a companion star and is defined by $\lambda_{ij} = -\lambda \varepsilon_{ij}$, where $Q_{ij}$ is the traceless quadrupole moment tensor of the star and $\varepsilon_{ij}$ is the tidal tensor that induces the deformation (Flanagan & Hinderer 2008). The methodologies to calculate $L$, $Q$, and $\lambda$ in GR can be found in (Hartle 1967; Hartle & Thorne 1968; Flanagan & Hinderer 2008; Hinderer 2008; Damour & Nagar 2009; Binnington & Poisson 2009; Yagi & Yunes 2013a; Urbanec et al. 2013).

The universal I-Love-Q relations (Yagi & Yunes 2013a, 2013b) concern the dimensionless quantities $I \equiv I/M^3$, $Q \equiv -Q/(M^3\chi^2)$, and $\lambda \equiv \lambda/M^4$ (with $M$ and $\chi \equiv J/M^2$ being the mass and dimensionless spin parameter, respectively). The I-Love-Q relations are EOS-insensitive in the sense that if one plots $I$ against $\lambda$ (and similarly $Q$ against $\lambda$), then one would find that the results of the different EOS models generally agree to a percent level (see, e.g., Figure 1 of Yagi & Yunes 2013a). In this work, we will consider the following nuclear matter EOS models: model APR (Akmal et al. 1998), model BB2 (Baldo et al. 1997), model FPS (Lorenz et al. 1993), model SLy4 (Douchin & Haensel 2000), model UU (Wiringa et al. 1988), and model WS (Lorenz et al. 1993; Wiringa et al. 1988). For comparison, we also consider quark stars (QSs) governed by the MIT bag model EOS with a bag constant of $B = 70.2$ MeV fm$^{-3}$ (see, e.g., Witten 1984; Alcock et al. 1986).

To motivate our analytic investigation of the I-Love relation in Section 3, which uses an incompressible stellar configuration as a background for perturbative expansion, here we first compare the I-Love-Q relations of realistic EOS models with those of an incompressible stellar model. In particular, we will use the results of an incompressible model as a benchmark to compare with those of realistic EOS models. To achieve the comparison, for a fixed $\lambda$ we define the fractional difference

$$\Delta \lambda \equiv \frac{\tilde{I} - I_{\text{incom}}}{I_{\text{incom}}} = \left( \frac{\lambda - \lambda_{\text{incom}}}{\lambda_{\text{incom}}} \right),$$

where $\tilde{I}$ and $I_{\text{incom}}$ are the scaled moments of inertia of a realistic EOS model and the incompressible model, respectively. Similarly, we define the fractional difference $\Delta Q$ for the scaled spin induced quadrupole moment. In Figure 1, we plot $\Delta Q$ (upper panel) and $\Delta \lambda$ (lower panel) against $\ln \lambda$ using our chosen EOS models. Note that each curve terminates at the maximum mass limit for the corresponding EOS in the figure. It should be pointed out that $\lambda$ decreases with increasing compactness. That is, general relativistic effects become important as $\lambda$ decreases. The compactness of NSs increases from about 0.07 to 0.3 as $\ln \lambda$ decreases from 10 to about 1. Figure 1 shows that the results of realistic EOS models generally agree very well with those of the incompressible model. In particular, even in the ultrarelativistic regime near the maximum mass limits (i.e., $\ln \lambda$ in the range between one and two), the incompressible results still agree with realistic EOS results to about 3% and 0.1% for $\lambda$ and $Q$, respectively.

Figure 1 shows that the difference between the incompressible and realistic-EOS results generally varies with the EOS models according to their stiffness, as expected. For NS models, the APR and UU EOSs have maximum mass limits at about 2.2 $M_{\odot}$ and are stiff among the chosen EOS models. These two models have $\Delta \lambda$ at about 0.3% for $\ln \lambda = 2$. On the other hand, the FPS EOS is relatively soft (with a maximum mass 1.8 $M_{\odot}$) and the corresponding $\Delta \lambda$ increases to about 0.6% for the
same value $\ln \bar{\lambda} = 2$. We also see from Figure 1 that the fractional differences for QSs governed by the MIT bag model are generally smaller than those of NSs for $\ln \bar{\lambda}$ larger than about five. This is expected as the density profile of QSs is rather uniform and can be modeled well by the incompressible stellar model. The deviations become comparable to those of NSs only near the maximum mass limit.

The stiffness of an EOS can be represented by the effective adiabatic index $\Gamma$ that is defined by $\Gamma \equiv (\rho/P)\partial P/\partial \rho$, where $\rho$ is the rest mass density and $P$ is the pressure. In Figure 2, we plot $\Gamma$ against the energy density $\epsilon$ for the APR and FPS EOSs for comparison. The fact that the effective adiabatic index for the APR EOS is higher than that of the FPS EOS in the high-density regime above a few times the nuclear density $2.8 \times 10^{14}$ g cm$^{-3}$ is the main reason why the APR EOS has a larger maximum-mass limit. To further illustrate the effect of the stiffness of the underlying EOS on the universal I-Love-Q relations, we repeat the analysis using a polytropic EOS model given by $P = K\rho^\Gamma$, where $K$ and $\Gamma$ are constants. We choose five different cases for the adiabatic index: $\Gamma = 1.6, 1.8, 2.0, 2.2, \text{ and } 2.5$. We plot $\Delta \bar{Q}$ and $\Delta \bar{I}$ against $\ln \bar{\lambda}$ for these cases in Figure 3. It can clearly be seen from the figure that there is a monotonic trend as we vary the value of $\Gamma$. In particular, similar to the results for realistic EOSs as shown in Figure 1, the fractional differences $\Delta \bar{Q}$ and $\Delta \bar{I}$ of the polytropic EOS decrease as $\Gamma$ is increased.

3. Analytical Study of the I-Love Relation

In the previous section, we provided numerical evidence showing that the I-Love-Q relations of realistic EOSs and the incompressible model agree very well, with fractional differences of the order of 1% or less for NSs spanning compactness from about 0.1 to 0.3. Here, we will study the I-Love relation analytically in the Newtonian limit and show that the incompressible limit is a crucial link to the observed universality of the I-Love relation.

In the Newtonian limit, the main equation for calculating the (quadrupolar $l = 2$) tidal deformability $\lambda$ is (see, e.g., Equation (60) of Yagi & Yunes 2013a)

$$\frac{d^2 h}{dr^2} + \frac{2}{r}\frac{dh}{dr} - \left(\frac{6}{r^2} - 4\pi \rho \frac{dP}{d\rho}\right) h = 0, \quad (2)$$

where $\rho$ and $P$ are the mass density and pressure, respectively. The metric function $h$ (in the Newtonian limit) is related to the dimensionless tidal deformability $\bar{\lambda} = \lambda/M^3$ by

$$\bar{\lambda} = \frac{2 - y(R)}{3[3 + y(R)]} C^{-5}, \quad (3)$$

where $R$ is the radius, $C = M/R$ is the compactness, and $y(R) = Rh'(R)/h(R)$. On the other hand, the dimensionless moment of inertia $\bar{I} = I/M^3$ is

$$\bar{I} = \frac{\int_0^R \rho r^4 dr}{24\pi^2 \left(\int_0^R \rho r^2 dr\right)^3}. \quad (4)$$

The starting point of our analysis is to assume that the density inside a compact star can be modeled well by

$$\rho = \rho_0(1 - \delta x^2), \quad (5)$$
where $\rho_0$ is the central density and $x = r/R$. The parameter $\delta$ is defined in the range $[0, 1]$ and is used to mimic different density profiles of different EOS models. The case $\delta = 0$ corresponds to the incompressible limit, which we will expand the I-Love relation about perturbatively. For the case $\delta = 1$, if we replace $\rho$ by the energy density $\varepsilon$ (and similarly for $\rho_0$), then the profile is known as the Tolman VII model in GR and has been shown to give reasonably good approximations for NSs constructed with different realistic EOS models (Lattimer & Prakash 2001; Postnikov et al. 2010).

In Figure 4, we plot the energy density (normalized by the central value) against $r/R$ for NSs constructed from our chosen EOS models. All the stellar models in Figure 4 have the same compactness $C = 0.1$. The profile given by the Tolman VII model is also plotted in the figure for comparison. It is seen that the Tolman VII model can mimic the density profiles of realistic EOSs quite well. In particular, inside a large part of the stars, the density profiles of realistic EOSs are generally slightly higher than that of the Tolman VII model. Note also that the profile for QS is well above the Tolman VII profile.

In our Newtonian analysis, Equation (5) with the parameter $\delta \ll 1$ would represent approximately the density profiles of NS models constructed from different realistic EOSs. On the other hand, for $\delta \approx 0$, the resulting density profile would be used to mimic the structure of a QS. Using Equation (5), we can rewrite Equation (2) as

$$x^2 \left(1 - \frac{3}{5} \delta x^2 \right) \frac{d^2 \tilde{h}}{dx^2} - \left(6 - \frac{48}{5} \delta x^2 \right) \tilde{h} = 0,$$

where $\tilde{h} = rh$. Equation (6) has a formal solution given by the hypergeometric function $\frac{\gamma}{\delta} F_1$:

$$\tilde{h} = x^3 \frac{\gamma}{\delta} F_1 \left(\frac{5 - \sqrt{65}}{4}, \frac{5 + \sqrt{65}}{4}, \frac{3}{2}, \frac{3}{5} \delta x^2 \right).$$

The dimensionless tidal deformability $\tilde{\lambda}$ can then be calculated by Equation (3), where the value $y(R)$ is given by

$$y(R) = 2 - \frac{65}{7} \frac{\gamma}{\delta} F_1 \left(\frac{9 - \sqrt{65}}{4}, \frac{9 + \sqrt{65}}{4}, \frac{3}{2}, \frac{3}{5} \delta x^2 \right) - \frac{15 (1 - \delta)}{5 - 3\delta}.$$

where $\delta \ll 1$. The second-order term in $\delta$ vanishes in the expansion. The dependence of the I-Love relation on the expansion parameter $\delta$, and hence implicitly the underlying EOS model, can be seen in Figure 5. The figure shows that the function $f(\delta)$ changes at most by only 0.1% as $\delta$ varies from 0 to 1. This weak dependence on $\delta$ can also be shown analytically by expanding Equation (10) perturbatively about the incompressible limit $\delta = 0$, which gives (up to the second order of $\delta$)

$$f(\delta) \approx \tilde{\lambda}^{-5/2} = \frac{2 - y(R)}{3(3 + y(R))} \left[\frac{2(7 - 5\delta)}{7(5 - 3\delta)}\right]^{-5/2},$$

where $y(R)$ is given by Equation (8). The dependence of the I-Love relation on the parameter $\delta$, and hence implicitly the underlying EOS model, can be seen in Figure 5. The figure shows that the function $f(\delta)$ changes at most by only 0.1% as $\delta$ varies from 0 to 1. This weak dependence on $\delta$ can also be shown analytically by expanding Equation (10) perturbatively about the incompressible limit $\delta = 0$, which gives (up to the second order of $\delta$)

$$\tilde{\lambda}^{-5/2} = \frac{2 - y(R)}{3(3 + y(R))} \left[\sqrt{\frac{5}{8} - \frac{1}{588} \delta^2 + ...}\right].$$
that the approximate similarity of EOSs in this region cannot explain the universality of the I-Love-Q relations. However, it should be emphasized that although \( I \) and \( \bar{Q} \) depend sensitively on the underlying EOS, it is their specific combination that leads to the cancelation of the strong dependence on EOS, and hence the existence of the universality. This point is reflected in our analysis by noting that although \( I \) and \( \bar{Q} \) depend non-trivially on the parameter \( \delta \), their combination \( \bar{I}_{\delta}^{5/2} \) turns out to depend weakly on \( \delta \), hence leading to the universal I-Love relation.

4. BREAKDOWN OF THE UNIVERSALITY

In Yagi & Yunes (2013a, 2013b), it was shown analytically in the Newtonian limit that the I-Love-Q relations are very similar for the incompressible \( (\Gamma = \infty) \) and \( \Gamma = 2 \) polytropic stellar models, although the reason that this was the case was not understood. In the above, we have provided support for the analysis carried out in Yagi & Yunes (2013a, 2013b).

Furthermore, we have shown that the incompressible limit is a stationary point and the I-Love relation depends weakly on the parameter \( \delta \), and hence on the underlying stellar structure and EOS. Our conclusion is valid for stellar density profiles that can be modeled well by Equation (5), which essentially covers the region above the Tolman VII profile in Figure 4. A similar for the incompressible (\( \Gamma = \infty \)) and polytropic (\( \Gamma = 2 \)) stellar models with density profiles that are far from and below the Tolman VII profile would show a much stronger dependence of the I-Love-Q relations on the underlying EOS models.

When would our analysis and the universality break down? Figure 1 already shows that the dependence of the I-Love-Q relations on the underlying EOS model becomes more apparent in the ultra-relativistic regime near the maximum mass limits (i.e., \( \ln \bar{\lambda} \approx 0.2 \)). This may not be that surprising because it is well known that gravity in GR is stronger than that in Newtonian theory, and hence GR tends to destabilize a star. This destabilizing effect can be interpreted as an effective softening of the underlying EOS in such a way that the resulting density profile cannot be modeled well by Equation (5). This can be seen from Figure 6 where we plot the energy density profiles of NSs with different compactness using the SLy4 EOS. The Tolman VII model is also plotted for comparison (solid line). The profile corresponding to \( \ln \bar{\lambda} = 8.24 \) and \( C = 0.12 \) is a typical NS model (dashed line) that can be modeled well by the Tolman VII model. This star lies in the domain where the universality of the I-Love-Q relations has the highest accuracy (<1% level). On the other hand, the maximum-mass NS model (dashed-dotted line) has the values \( \ln \bar{\lambda} = 1.44 \) and \( C = 0.30 \). It can be seen that while the density profile of this configuration is still reasonably close to the Tolman VII model, it is nevertheless located below the Tolman VII profile. This model shows a larger deviation (e.g., a few percent for \( \Delta \bar{Q} \)) from the incompressible limit as shown in Figure 1.

In Figure 6, we also plot the profile for a 0.1 \( M_\odot \) low-mass NS (dotted line) with \( \ln \bar{\lambda} = 19.3 \) and \( C = 4.3 \times 10^{-3} \). This model is near the minimum mass of stable NSs that can be supported by the SLy4 EOS. We see that the density profile of this star cannot be modeled well by Equation (5). The density profile of this model is also far away from the region of validity of our analysis, namely, the region above the Tolman VII profile in the figure. We thus expect that the universality of the I-Love-Q relations would break down. In order to check our expectation, we extend Figure 1 to the low-mass region using the FPS, SLy4, and MIT bag models and plot the results in Figure 7. It can be seen that the I-Love-Q relations for low-mass NSs become more sensitive to the EOS as \( \bar{\lambda} \) increases, and they also deviate significantly from the incompressible limit. Note, however, that the sensitivity on the EOS can only be seen indirectly by subtracting the I-Love-Q relations by their incompressible limits. Near the minimum-mass limit at about \( \ln \bar{\lambda} = 20 \), the fractional differences \( \Delta \bar{Q} \) and \( \Delta \bar{I} \) increase to about 20% and 7%, respectively. However, contrary to the case for NSs, the I-Love-Q relations for low-mass QSs can still be accurately represented by the incompressible limit. On physical grounds, the large deviation of the I-Love-Q relations from the incompressible limit for low-mass NSs is due to the fact that these stars have low densities and their interiors are on average dominated by relatively softer matter with a smaller \( \Gamma \). For instance, a low-mass NS with \( \ln \bar{\lambda} \approx 20 \) and \( C \approx 3 \times 10^{-3} \) constructed from the SLy4 EOS would have...
about 70% of its total mass attributed to matter with $\Gamma < 2$. On the other hand, QSs can be modeled by the incompressible model very well, regardless of their masses.

The different behaviors of the I-Love-Q relations for low-mass NSs and QSs might point to a method that allows us, at least in principle, to distinguish between these two types of objects observationally. However, can low-mass compact stars be formed in the first place? It is known that hot proto-NSs formed in supernova explosions generally have a large minimum mass, hence rendering the existence of low-mass NSs unlikely. Nevertheless, it is still not ruled out that low-mass NSs can be formed via fragmentation during the formation of proto-NSs (Popov et al. 2007; Horowitz 2010). This is still an open question for further investigation.

5. COMPARISON WITH ELLIPTICAL ISODENSITY APPROXIMATION

Approximate universal relations among multipole moments have also been found recently (Stein et al. 2014; Yagi et al. 2014a; Chatziioannou et al. 2014; Yagi et al. 2014b). In particular, Yagi et al. (2014b) found that the isodensity contours of realistic EOSs can be approximated by elliptical isodensity contours and that the relaxation of the self-similarity assumption of these surfaces can destroy the universal relations among multipole moments. They further suggested that the universal I-Q relation between $I$ and $Q$ would be affected by the elliptical isodensity approximation in roughly the same way. In this paper, based on our numerical evidence and Newtonian perturbative analysis, we propose that the universality of I-Love-Q relations can be attributed to the incompressible limit of these relations. The physical reason is due to the fact that modern realistic EOSs are stiff in the nuclear density range.

Is there any connection between our study and the elliptical isodensity approximation as discussed by Yagi et al. (2014b)? A first hint can be seen in Figure 17 of Yagi et al. (2014b) in which the variation of the eccentricity inside a slowly rotating Newtonian star (modeled by the polytropic EOS) is shown to decrease as the stiffness of the EOS increases. In particular, the elliptical isodensity approximation becomes exact in the incompressible limit. Yagi et al. (2014b) also found that the eccentricity profile is almost constant for QSs with large $I$ (corresponding to small compactness), and hence the elliptical isodensity approximation becomes highly accurate (see Figure 15 of Yagi et al. 2014b). We can understand this result by noting that the average effective adiabatic index of quark matter inside QSs increases with decreasing compactness. On the other hand, Yagi et al. (2014b) found that there are no universal relations among multipole moments and between $I$ and $Q$ for noncompact stars. They also found that the variation of eccentricity inside noncompact stars is large and suggested that the loss of universality for these stars could be a consequence of the breakdown of the elliptical isodensity approximation. From our point of view, however, the loss of universality for noncompact stars is due to the fact that the underlying EOS is much softer than the EOS of nuclear matter.

Generally speaking, for a given star, the validity of the elliptical isodensity approximation, which is a criterion for the I-Love-Q universality, as suggested in Stein et al. (2014), Chatziioannou et al. (2014), Yagi et al. (2014b), needs to be verified by numerical methods on a case by case basis. However, from the above discussion, it is evident that the high stiffness of the EOS can directly lead to the validity of the elliptical isodensity approximation. Therefore, we believe that the stiffness of the EOS plays a more fundamental role in the observed universality. The requirement that nuclear-matter EOSs only need to be stiff is a physically appealing and simple explanation for the universality.

On the other hand, Yagi et al. (2014b) further suggested that the validity of the elliptical isodensity approximation, and hence the observed universality, are due to an approximate symmetry emerging in relativistic stars, which are composed of nuclear matter with high stiffness and which have large compactness. It is crucial to pinpoint the exact cause of the universality. Does either high stiffness or large compactness in itself lead to the universality? Does the universality arise from the concerted effects of both high stiffness and large compactness? As we have already seen in Figures 1 and 3 of the present paper, the accuracy of the I-Love-Q relations slightly deteriorates towards the large compactness end and, as mentioned above, is better for high-stiffness EOSs. Similarly, as shown in Figures 2, 15, and 16 of Yagi et al. (2014b), the accuracy of the elliptical isodensity approximation also worsens with increasing stellar compactness. Therefore, we conclude that the high stiffness of nuclear matter is the cause of the universality. The effect of GR in fact softens the stiffness of matter and adversely affects the universal relations.

6. CONCLUSION

In this paper, we have provided numerical evidence showing that the universality of I-Love-Q relations can be attributed to the incompressible limits of these relations. For typical NSs with compactness in the range from about 0.05 to 0.2, the incompressible limit can model the I-Love-Q relations of realistic EOSs to much better than 1%. Furthermore, using a generalized Tolman VII model density profile, we have carried out a perturbative analysis of the I-Love relation in Newtonian gravity and have shown that the relation can be represented by the incompressible limit accurately and is weakly dependent on the EOS because the incompressible limit is a stationary point. The leading dependence of the I-Love relation on the density profile, and hence the underlying EOS model, turns out to be second order in the expansion parameter $\delta$ that is used to mimic realistic NSs (when $\delta \approx 1$) and QSs (when $\delta \approx 0$). We also demonstrate numerically that the I-Love-Q relations for low-mass NSs, which are composed mainly of softer nuclear matter, deviate significantly from the incompressible limit and become more sensitive on the underlying EOS. On the other hand, the I-Love-Q relations for low-mass QSs can still be represented accurately by the incompressible limit because QSs can be modeled well by the incompressible stellar model regardless of their masses.

One of the possible reasons for the universality of I-Love-Q relations, as suggested in Yagi & Yunes (2013a, 2013b), is that these relations approach the black hole limit as the NS compactness increases. In this work, we show that there exists a “softer” limit, the incompressible limit, to which the I-Love-Q relations for different realistic EOSs converge, and hence a universality is established. Besides the I-Love-Q relations, it is also known that the $f$-mode oscillation frequency and moment of inertia of compact stars also display a universality (Lau et al. 2010). It would be interesting to study whether the two different sets of $f$-mode and I-Love-Q universal relations have a common

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3 Since the EOS for noncompact stars such as the sun is well understood, Yagi et al. (2014b) used different opacity laws to mimic the effects of different EOS models in their work.
origin or not. A first hint can be seen in Figure 8, in which we plot the fractional difference for the real part of the \( f \)-mode oscillation frequency \( \Delta \bar{\omega}_r \) (defined similarly as \( \Delta \bar{I} \) in Equation (1)) against \( \ln \bar{I} \) for three different polytropic EOS models, where \( \bar{\omega}_r \) is the scaled frequency \( M\omega_r \). We see that the trend of the results is essentially the same as those in Figure 3 for the I-Love-Q relations. The incompressible limit can generally model the \( f \)-mode universal relation to a percent level. This strongly suggests that the incompressible limit also plays an important role in the \( f \)-mode universal relations. We conjecture that in view of the \( f \)-mode and I-Love-Q relations, which is supported by our numerical evidence, their universalities originate from the fact that modern realistic EOSs turn out to be stiff enough to be modeled well by the incompressible limit of these relations.

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Figure 8. \( \Delta \bar{\omega}_r \) is plotted against \( \ln \bar{I} \) for polytropic models.