LOEWNER EQUATIONS ON COMPLETE HYPERBOLIC DOMAINS

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ABSTRACT. We prove that, on a complete hyperbolic domain $D \subset \mathbb{C}^q$, any Loewner PDE associated with a Herglotz vector field of the form $H(z, t) = \Lambda(z) + O(|z|^2)$, where the eigenvalues of $\Lambda$ have strictly negative real part, admits a solution given by a family of univalent mappings $(f_t : D \to \mathbb{C}^q)$ which satisfies $\bigcup_{t \geq 0} f_t(D) = \mathbb{C}^q$. If no real resonance occurs among the eigenvalues of $\Lambda$, then the family $(e^{\Lambda t} \circ f_t)$ is uniformly bounded in a neighborhood of the origin. We also give a generalization of Pommerenke’s univalence criterion on complete hyperbolic domains.

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1. INTRODUCTION

We begin recalling the Loewner equations on the unit disc $\mathbb{D} \subset \mathbb{C}$. The Loewner PDE is the following:

$$\frac{\partial f_t(z)}{\partial t} = -\frac{\partial f_t(z)}{\partial z} H(z, t), \quad \text{a.e. } t \geq 0, z \in \mathbb{D},$$

(1.1)

where $H(z, t) = zp(z, t)$ and $p(z, t) : \mathbb{D} \times \mathbb{R}^+ \to \mathbb{C}$ is measurable in $t \geq 0$, holomorphic in $z \in \mathbb{D}$ and satisfies $\text{Re } p(z, t) < 0$ and $p(0, t) = -1$ for all $t \geq 0$. The second equation is

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the Loewner ODE:

$$\frac{\partial}{\partial t} \varphi_{s,t}(z) = H(\varphi_{s,t}(z), t), \quad \text{a.e. } t \in [s, \infty), z \in \mathbb{D},$$

$$\varphi_{s,s}(z) = z, \quad s \geq 0, z \in \mathbb{D}. \quad (1.2)$$

Both equations were introduced by Loewner in 1923 [17] and used to prove the case \( n = 3 \) of the Bieberbach conjecture. Loewner theory was developed by Pommerenke [20] and Kufarev [16] as a powerful tool in geometric function theory. In fact it is one of the main ingredients of the proof of the Bieberbach conjecture given by de Branges [8] in 1985. Among the extensions of the theory we recall the celebrated theory of Schramm-Loewner evolution [23] introduced in 1999.

Loewner theory was extended to several complex variables by Duran, Graham, Hamada, G. Kohr, M. Kohr, Pfaltzgraff and others [9][12][19]. Recently Bracci, Contreras and Diaz-Madrigal [5][6] (see also [3]) proposed a generalization of the Loewner ODE which has its natural setting in complete hyperbolic manifolds. In the following we denote by \( D \) a complete hyperbolic (in the sense of Kobayashi) domain of \( \mathbb{C}^q \). Recall that a holomorphic vector field \( H : D \rightarrow \mathbb{C}^q \) is said an infinitesimal generator provided the Cauchy problem

$$\begin{cases}
\dot{z}(t) = H(z(t)), \\
z(0) = z_0
\end{cases} \quad (1.4)$$

has a solution \( z : [0, +\infty) \rightarrow D \) for all \( z_0 \in D \).

A Herglotz vector field on a complete hyperbolic domain \( D \subset \mathbb{C}^q \) is a non-autonomous holomorphic vector field \( H(z, t) : D \times \mathbb{R}^+ \rightarrow \mathbb{C}^q \) which is measurable in \( t \geq 0 \), which is an infinitesimal generator for a.e. \( t \geq 0 \) fixed, and such that for any compact set \( K \subset D \) there exists a function \( c_K \in L^d_{\text{loc}}(\mathbb{R}^+, \mathbb{R}^+) \), with \( d \in [1, \infty] \), such that

$$|H(z, t)| \leq c_K(t), \quad z \in K, t \geq 0.$$ 

These vector fields are the natural generalizations of the function \( H(z, t) = -zp(z, t) \) in (1.1). The Loewner ODE studied in [3][6]

$$\begin{cases}
\frac{\partial}{\partial t} \varphi_{s,t}(z) = H(\varphi_{s,t}(z), t), \quad \text{a.e. } t \in [s, \infty), z \in D, \\
\varphi_{s,s}(z) = z, \quad s \geq 0, z \in D,
\end{cases} \quad (1.3)$$

has a locally absolutely continuous (in the variable \( t \)) solution defined for all \( 0 \leq s \leq t \) given by a family \( \varphi_{s,t} : D \rightarrow D \) of univalent mappings which is a \( \mathbb{R}^+ \)-evolution family, that is which satisfies \( \varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u} \) for all \( 0 \leq s \leq u \leq t \) and \( \varphi_{s,s}(z) = z \) for all \( s \geq 0 \).

Let \( H(z, t) \) be a Herglotz vector field on \( D \). In [4] we proved that a family of univalent mappings \( (f_t : D \rightarrow \mathbb{C}^q) \) is locally absolutely continuous (in the variable \( t \)) and solves the Loewner PDE

$$\frac{\partial f_t(z)}{\partial t} = -df_t(z)H(z, t), \quad \text{a.e. } t \geq 0, z \in D, \quad (1.4)$$
if and only if it solves the functional equation
\[ f_t \circ \varphi_{s,t}(z) = f_s(z), \quad 0 \leq s \leq t, \ z \in D, \]
where \((\varphi_{s,t})\) is the solution of (1.3).

The solution \((f_t : D \to \mathbb{C}^q)\) satisfies \(f_s(D) \subset f_t(D)\) for all \(0 \leq s \leq t\). A family of univalent mappings with this property is called a \(\mathbb{R}^+\)-Loewner chain. A \(\mathbb{R}^+\)-evolution family \((\varphi_{s,t})\) and a \(\mathbb{R}^+\)-Loewner chain \((f_t)\) are associated if (1.5) holds.

We now introduce the special Herglotz vector fields that we are going to study in this paper. A Herglotz vector field \(H(z, t)\) is of dilation type if
\[ H(z, t) = \Lambda(z) + O(|z|^2), \quad t \geq 0, \]
where the eigenvalues of \(\Lambda \in \mathcal{L}(\mathbb{C}^q)\) have strictly negative real part, and the term \(O(|z|^2)\) may depend on \(t\).

We recall the following recent result by Graham, Hamada, G. Kohr and M. Kohr [12].

**Theorem 1.1** ([12]). Let \(H(z, t) = \Lambda(z) + O(|z|^2)\) be a dilation Herglotz vector field on the unit ball \(B \subset \mathbb{C}^q\), and assume that
\[ 2 \max \{\text{Re} \langle \Lambda(z), z \rangle : |z| = 1\} < \min \{\text{Re} \lambda : \lambda \in \text{sp}(\Lambda)\}. \]
Then the Loewner PDE (1.4) admits a locally absolutely continuous univalent solution \((f_t : B \to \mathbb{C}^q)\) such that \(\cup_{t \geq 0} f_t(B) = \mathbb{C}^q\). The family \((e^{\Lambda t} \circ f_t)\) is uniformly bounded in a neighborhood of the origin.

In [2] we introduced \(\mathbb{N}\)-evolution families and \(\mathbb{N}\)-Loewner chains, that is the discrete-time analogues of \(\mathbb{R}^+\)-evolution families and \(\mathbb{R}^+\)-Loewner chains. Solving equation (1.5) for discrete times we proved the following result.

**Theorem 1.2** ([2]). Let \(H(z, t) = \Lambda(z) + O(|z|^2)\) be a dilation Herglotz vector field on the unit ball \(B \subset \mathbb{C}^q\). Assume that \(\Lambda\) is diagonal. Then the Loewner PDE (1.4) admits a locally absolutely continuous univalent solution \((f_t : B \to \mathbb{C}^q)\) such that \(\cup_{t \geq 0} f_t(B) = \mathbb{C}^q\). The family \((e^{\Lambda t} \circ f_t)\) is uniformly bounded in a neighborhood of the origin if no real resonance of the form
\[ \text{Re} \left( \sum_{j=1}^{q} k_j \alpha_j \right) = \text{Re} \alpha_l, \quad k_j \geq 0, \quad \sum_{j=1}^{q} k_j \geq 2 \]
occurs among the eigenvalues \((\alpha_j)\) of \(\Lambda\).

The same result was obtained independently with different methods by Voda [24], assuming \(\max \{\text{Re} \langle \Lambda(z), z \rangle : |z| = 1\} < 0\) instead of assuming \(\Lambda\) diagonal.

Notice that condition (1.6) avoids real resonances. Recall that by [2] Counterexample 2) the Loewner PDE associated with the autonomous dilation Herglotz vector field on \(B \subset \mathbb{C}^2\)
\[ H(z, t) = (\alpha z_1, 2\alpha z_2 + cz_1^2), \]
where \(|\alpha| < 1/2\) and \(c \in \mathbb{C}^\ast\) is small enough, does not admit any solution \((f_t: \mathbb{B} \to \mathbb{C}^q)\) such that the family \((e^{\Lambda t} \circ f_t)\) is uniformly bounded in a neighborhood of the origin. In this case a real resonance occurs.

In this paper we generalize Theorem 1.2 to any dilation Herglotz vector field on a complete hyperbolic domain \(D \subset \mathbb{C}^q\). We should mention that, to our knowledge, this is the first existence result for the Loewner PDE (1.4) on such domains. We start by solving equation (1.5) for discrete times. Let \((\varphi_{n,m})\) be a \(\mathbb{N}\)-evolution family. We show that a family of tangent to identity univalent mappings \((h_n: D \to \mathbb{C}^q)\) which is uniformly bounded near the origin solves the non-autonomous Schröder equation

\[ h_m \circ \varphi_{n,m} = e^{\Lambda(m-n)} \circ h_n. \]  

(1.7)

if and only if \((\varphi_{n,m})\) is associated with the \(\mathbb{N}\)-Loewner chain \((f_n) \doteq (e^{-\Lambda n} \circ h_n)\).

Equation (1.7) shows a strong connection between Loewner theory and the theory of basins of attraction of discrete non-autonomous complex dynamical systems grown around Bedford’s conjecture: see [1][11][14][18][25][22]. To solve equation (1.7) we use techniques from this theory, in particular from [18]. Indeed we need a non-autonomous version of the Poincaré-Dulac method, whose homological equation is replaced by a difference equation in the space \(\mathcal{H}_i\) of homogeneous polynomial mappings of degree \(i\),

\[ H_{n+1} = e^\Lambda \circ H_n \circ e^{-\Lambda} + B_n, \]

(1.8)

where \((H_n)\) is an unknown bounded sequence in \(\mathcal{H}_i\) and \((B_n)\) is a bounded sequence in \(\mathcal{H}_i\). In order to find a bounded solution of (1.8) we study the spectral and dynamical properties of the linear operator \(H \mapsto e^\Lambda \circ H \circ e^{-\Lambda}\) acting on \(\mathcal{H}_i\) and we show that the obstruction to the existence of solutions is given by real resonances.

This method provides a family of univalent mappings \((f_n: r\mathbb{B} \subset D \to \mathbb{C}^q)_{n \in \mathbb{N}}\) satisfying (1.5) but defined only for integer times and in a little neighborhood of the origin. Then we extend this family to all \(t \in \mathbb{R}^+\) and \(z \in D\).

The main result of this paper is thus the following.

**Theorem 1.3.** Let \(D \subset \mathbb{C}^q\) be a complete hyperbolic domain and let \(H(z,t) = \Lambda(z) + O(|z|^2)\) be a dilation Herglotz vector field on \(D\). Then the Loewner PDE (1.4) admits a locally absolutely continuous univalent solution \((f_t: \mathbb{B} \to \mathbb{C}^q)\) such that \(\cup_{t \geq 0} f_t(D) = \mathbb{C}^q\). The family \((e^{\Lambda t} \circ f_t)\) is uniformly bounded in a neighborhood of the origin if no real resonance occurs among the eigenvalues of \(\Lambda\).

We also generalize to complete hyperbolic domains the classical univalence criterion in the unit disk due to Pommerenke [20, Folgerung 6].

**Theorem 1.4.** Let \(D \subset \mathbb{C}^q\) be a complete hyperbolic domain and let \(H(z,t) = \Lambda(z) + O(|z|^2)\) be a dilation Herglotz vector field on \(D\). Let \((f_t: D \to \mathbb{C}^q)\) be a family of holomorphic mappings which solves the Loewner PDE (1.4) and assume that the family \((e^{\Lambda t} \circ f_t)\) is uniformly bounded in a neighborhood of the origin is an univalent family. Then for all \(t \geq 0\) the mapping \(f_t\) is univalent.
2. Local conjugacy

We start recalling some basic definitions.

**Definition 2.1.** Let $\mathbb{T} = \mathbb{N}$ or $\mathbb{R}^+$. Let $D$ be a domain of $\mathbb{C}^q$. A $\mathbb{T}$-**evolution family** is a family of univalent mappings $(\varphi_{\alpha, \beta} : D \to D)_{\alpha \leq \beta \in \mathbb{T}}$ such that

i) $\varphi_{\alpha, \alpha} = \text{id}$ for all $\alpha \geq 0$,

ii) $\varphi_{\alpha, \beta} = \varphi_{\gamma, \beta} \circ \varphi_{\alpha, \gamma}$ for all $0 \leq \alpha \leq \gamma \leq \beta$.

A family of univalent mappings $(f_\alpha : D \to \mathbb{C}^q)$ is a $\mathbb{T}$-**Loewner chain** if $f_\alpha(D) \subset f_\beta(D)$ for all $0 \leq \alpha \leq \beta$.

**Remark 2.2.** Let $(f_\alpha : D \to \mathbb{C}^q)$ be a $\mathbb{T}$-Loewner chain. Then there exists a unique associated $\mathbb{T}$-evolution family $(\varphi_{\alpha, \beta} = f_\beta^{-1} \circ f_\alpha)$.

One has the following uniqueness result for $\mathbb{T}$-Loewner chains.

**Theorem 2.3** ([4]). Let $(\varphi_{\alpha, \beta})$ be a $\mathbb{T}$-evolution family on $D$ and let $(f_\alpha : D \to \mathbb{C}^q)$ be an associated $\mathbb{T}$-Loewner chain. If $(g_\alpha : D \to \mathbb{C}^q)$ is a subordination chain associated with $(\varphi_{\alpha, \beta})$ then there exists a holomorphic mapping $\Psi : \bigcup_{\alpha \in \mathbb{T}} f_\alpha(D) \to \mathbb{C}^q$ such that $(g_\alpha = \Psi \circ f_\alpha)$.

In what follows we focus on special types of $\mathbb{N}$-evolution families and $\mathbb{N}$-Loewner chains.

**Definition 2.4.** We denote $\mathcal{L}(\mathbb{C}^q)$ and $\mathcal{A}(\mathbb{C}^q)$ the sets of $\mathbb{C}$-linear endomorphisms and $\mathbb{C}$-linear automorphisms of $\mathbb{C}^q$. Let $A \in \mathcal{A}(\mathbb{C}^q)$. The **spectrum** $\sigma(A)$ of $A$ is the set of its eigenvalues. The **spectral radius** $\rho(A)$ is defined as $\max_{\lambda \in \sigma(A)} |\lambda|$.

Let $D$ be a domain in $\mathbb{C}^q$ containing 0. A $\mathbb{N}$-**evolution family** $(\varphi_{n,m})$ on $D$ is a **dilation** $\mathbb{N}$-evolution family if for all $n \geq 0$,

$$\varphi_{n,n+1}(z) = A(z) + O(|z|^2),$$

(2.1)

with $A \in \mathcal{A}(\mathbb{C}^q)$ such that $\rho(A) < 1$. A $\mathbb{N}$-**Loewner chain** $(f_n : D \to \mathbb{C}^q)$ is a **locally bounded** $\mathbb{N}$-Loewner chain if for all $n \geq 0$,

$$f_n(z) = A^{-n}(z) + O(|z|^2),$$

where $A \in \mathcal{A}(\mathbb{C}^q)$ is such that $\rho(A) < 1$ and the family $(A^n \circ f_n)$ is uniformly bounded in a neighborhood of the origin.

On complete hyperbolic domains, the dynamics of dilation $\mathbb{N}$-evolution families is uniformly contractive, as the following lemma shows. A reference for complete hyperbolic manifolds and the Kobayshi distance is [15].

**Lemma 2.5.** Let $D \subset \mathbb{C}^q$ be a complete hyperbolic domain and let $(\varphi_{n,m})$ be dilation $\mathbb{N}$-evolution family on $D$. Then the basin of attraction of the origin at time $n \geq 0$

$$\mathcal{B}(n) = \{ z \in D : \lim_{m \to \infty} \varphi_{n,m}(z) = 0 \}$$

is the whole $D$, and for all $n \geq 0$ the convergence $\lim_{m \to \infty} \varphi_{n,m}(z) = 0$ is uniform on compact subsets.
Proof. Up to a linear change of coordinates, we may assume that \( \max_{z \in \mathbb{C}^q} \left| \frac{|A(z)|}{|z|} \right| < 1 \).

Lemma A.2 yields then that there exists \( \varepsilon > 0 \) such that the Kobayashi ball \( \Omega(0, \varepsilon) \) centered in the origin of radius \( \varepsilon \) is contained in the set \( \bigcap_{m \geq 0} \mathfrak{A}(m) \). For all \( n \geq 0 \), the set \( \mathfrak{A}(n) \) is an open subset of \( D \). Indeed, if \( z \in \mathfrak{A}(n) \), there exists \( m > 0 \) such that \( \varphi_{n,m}(z) \subset \Omega(0, \varepsilon/2) \). Since holomorphic mappings decrease the Kobayashi distance, one has

\[
\varphi_{n,m}(\Omega(z, \varepsilon/2)) \subset \Omega(0, \varepsilon) \subset \mathfrak{A}(m),
\]

thus \( \Omega(z, \varepsilon/2) \subset \mathfrak{A}(n) \).

The set \( \mathfrak{A}(n) \) is also a closed subset of \( D \). Indeed let \( z \) be a point in the closure of \( \mathfrak{A}(n) \). Then there exist a point \( w \in \mathfrak{A}(n) \) such that \( k_D(z, w) < \varepsilon/2 \). Let \( u > 0 \) be such that \( \varphi_{n,u}(w) \in \Omega(0, \varepsilon/2) \). Since holomorphic mappings decrease the Kobayashi distance one has

\[
\varphi_{n,u}(z) \subset \Omega(0, \varepsilon) \subset \mathfrak{A}(u),
\]

thus \( z \in \mathfrak{A}(n) \).

Since \( D \) is connected one has \( \mathfrak{A}(n) = D \). The convergence is local uniform and hence uniform on compact subsets.

\( \square \)

A triangular mapping is a mapping \( T: \mathbb{C}^q \to \mathbb{C}^q \) whose components \( T^{(i)}(z) \) satisfy

\[
T^{(1)}(z) = \lambda_1 z_1, \quad T^{(i)}(z) = \lambda_i z_i + t^{(i)}(z_1, z_2, \ldots, z_{i-1}), \quad 2 \leq i \leq q,
\]

where \( \lambda_i \in \mathbb{C} \) and \( t^{(i)} \) is a polynomial in \( i-1 \) variables fixing the origin. Its degree is the maximum of the degree of its components. If \( \lambda_i \neq 0 \) for all \( 1 \leq i \leq q \), the mapping \( T \) is called a triangular automorphism. This is indeed an automorphism of \( \mathbb{C}^q \), since we can iteratively write its inverse, which is still a triangular automorphism. Since the composition of two triangular automorphisms is still a triangular automorphism, they form a subgroup of \( \text{aut}(\mathbb{C}^q) \). A triangular dilation \( \mathbb{N} \)-evolution family is a dilation \( \mathbb{N} \)-evolution family \( (T_{n,m}, \mathbb{C}^q) \) such that each \( T_{n,m+1} \), and hence every \( T_{n,m} \), is a triangular automorphism of \( \mathbb{C}^q \). A triangular dilation \( \mathbb{N} \)-evolution family \( (T_{n,m}) \) has uniformly bounded coefficients if the family \( (T_{n,n+1}) \) has uniformly bounded coefficients. A triangular dilation \( \mathbb{N} \)-evolution family \( (T_{n,m}) \) has uniformly bounded degree if the family \( (T_{n,n+1}) \) has uniformly bounded degree.

Definition 2.6. Let \( D \subset \mathbb{C}^q \) be a complete hyperbolic domain. A dilation \( \mathbb{N} \)-evolution family \( (\varphi_{n,m}: D \to D) \) and a triangular dilation \( \mathbb{N} \)-evolution family \( (T_{n,m}) \) with uniformly bounded degree and uniformly bounded coefficients are locally conjugate if there exists, on a ball \( rB \subset D \) satisfying

\[
\varphi_{n,m}(rB) \subset (rB), \quad 0 \leq n \leq m,
\]
a uniformly bounded family of holomorphic mappings \((h_n : \mathbb{R} \to \mathbb{C}^q)\) such that \(h_n(z) = z + O(|z|^2)\) for all \(n \geq 0\), and such that
\[
h_m \circ \varphi_{n,m}(z) = T_{n,m} \circ h_n(z), \quad z \in \mathbb{R}, \quad 0 \leq n \leq m.
\] (2.2)

**Proposition 2.7.** Let \(D \subset \mathbb{C}^q\) be a complete hyperbolic domain. Assume that a dilation \(N\)-evolution family \((\varphi_{n,m} : D \to D)\) and a triangular dilation \(N\)-evolution family \((T_{n,m})\) with uniformly bounded degree and uniformly bounded coefficients are locally conjugate. Then for each fixed \(n \geq 0\) the sequence \((T_{n,m}^{-1} \circ h_m \circ \varphi_{n,m})_{m \geq n}\) is eventually defined on each compact subset \(K \subset D\), its limit
\[
h^n_e \doteq \lim_{m \to \infty} T_{n,m}^{-1} \circ h_m \circ \varphi_{n,m}
\]
exists uniformly on compacta on \(D\), and satisfies \(h^n_e|_{\mathbb{R}} = h_n\). The family \((h^n_e : D \to \mathbb{C}^q)\) satisfies
\[
h^n_m \circ \varphi_{n,m}(z) = T_{n,m} \circ h^n_e(z), \quad z \in D, \quad 0 \leq n \leq m.
\]

**Proof.** Let \(K \subset D\) be a compact subset. By Lemma 2.5 for all \(n \geq 0\) there exists \(u \geq n\) such that \(\varphi_{n,u}(K) \subset \mathbb{R}\). Then for \(m \geq u\),
\[
T_{n,m}^{-1} \circ h_m \circ \varphi_{n,m}|_K = T_{n,u}^{-1} \circ (T_{n,m}^{-1} \circ h_m \circ \varphi_{u,m}) \circ \varphi_{n,u}|_K = T_{n,u}^{-1} \circ h_u \circ \varphi_{n,u}|_K
\]
by (2.2), thus the sequence \((T_{n,m}^{-1} \circ h_m \circ \varphi_{n,m})_{m \geq n}\) converges uniformly on compacta. By (2.2) we have,
\[
T_{n,m}^{-1} \circ h_m \circ \varphi_{n,m}(z) = h_n(z), \quad z \in \mathbb{R}, \quad n \leq m,
\]
thus \(h^n_e|_{\mathbb{R}} = h_n\).
Finally
\[
h^n_m \circ \varphi_{n,m} = \lim_{j \to \infty} T_{m,j}^{-1} \circ h_j \circ \varphi_{m,j} \circ \varphi_{n,m} = T_{n,m} \circ \lim_{j \to \infty} T_{n,j}^{-1} \circ h_j \circ \varphi_{n,j} = T_{n,m} \circ h^n_e.
\]

**Definition 2.8.** We call the mappings \(h^n_e\) intertwining mappings. Notice that since \(h^n_e|_{\mathbb{R}} = h_n\), the family \((h^n_e : D \to \mathbb{C}^q)\) is uniformly bounded in a neighborhood of the origin. From now on we will denote \(h^n_e\) simply by \(h_n\).

**Proposition 2.9.** Let \(D \subset \mathbb{C}^q\) be a complete hyperbolic domain. Assume that a dilation \(N\)-evolution family \((\varphi_{n,m} : D \to D)\) and a triangular dilation \(N\)-evolution family \((T_{n,m})\) with uniformly bounded degree and uniformly bounded coefficients are locally conjugate. Then each intertwining mapping \(h_n : D \to \mathbb{C}^q\) is univalent.

**Proof.** Assume that there exist \(z \neq w\) in \(D\) and \(n \geq 0\) such that \(h_n(z) = h_n(w)\). Then by (2.2),
\[
h_m(\varphi_{n,m}(z)) = h_m(\varphi_{n,m}(w)), \quad 0 \leq n \leq m.
\] (2.3)
By Lemma A.3 there exists a ball \(s \mathbb{B}\) such that for all \(m \geq 0\) the mapping \(h_m|_{s \mathbb{B}}\) is univalent. By Lemma 2.5 there exists \(m \geq n\) such that \(\varphi_{n,m}(z) \subset \varphi_{n,m}(w) \subset s \mathbb{B}\). But
\( \varphi_{n,m}(z) \neq \varphi_{n,m}(w) \) since \( \varphi_{n,m} \) is a univalent mapping, hence (2.3) contradicts the univalence of \( h_m|_{\mathbb{B}} \).

\[ \square \]

### 3. Non-autonomous Poincaré-Dulac method

For a detailed exposition of the classical Poincaré-Dulac method, see [21, Appendix]. We will need the non-autonomous version of the Poincaré-Dulac method developed in [18] in the case of \( \mathbb{N} \)-evolution families of holomorphic automorphisms of \( \mathbb{C}^q \). We will give alternative proofs of this method in Propositions 3.4 and 3.6 in which we show also that, in absence of real resonances (defined below), it is possible to find a local conjugacy between a dilation \( \mathbb{N} \)-evolution family and its linear part.

In what follows we identify a linear automorphism \( A \in \mathcal{A}(\mathbb{C}^q) \) with its associated matrix with respect to the canonical basis.

**Definition 3.1.** A real multiplicative resonance for \( A \in \mathcal{A}(\mathbb{C}^q) \) with eigenvalues \( \lambda_i \) is an identity

\[
|\lambda_j| = |\lambda_1^{i_1} \cdots \lambda_q^{i_q}|,
\]

where \( i_j \geq 0 \), and \( \sum_j i_j \geq 2 \). If for every \( 1 \leq j \leq q \) we have \( |\lambda_j| < 1 \), real multiplicative resonances can occur only in a finite number. Moreover, if \( 0 < |\lambda_q| \leq \cdots \leq |\lambda_1| < 1 \), then

\[
|\lambda_j| = |\lambda_1^{i_1} \cdots \lambda_q^{i_q}| \Rightarrow i_j = i_{j+1} = \cdots = i_q = 0.
\]

**Definition 3.2.** An automorphism \( A \in \mathcal{A}(\mathbb{C}^q) \) is in optimal form if

i) \( A \) is in lower-triangular \( \varepsilon \)-Jordan normal form for some \( \varepsilon > 0 \), that is in lower triangular Jordan normal form with the underdiagonal multiplied by \( \varepsilon \),

ii) if the diagonal of \( A \) is \( (\lambda_1, \ldots, \lambda_q) \) then \( 1 > |\lambda_1| \geq \cdots \geq |\lambda_q| > 0 \),

iii) one has \( \max_{z \in \mathbb{C}^q} \frac{|A(z)|}{|z|^i} < 1 \).

Note that any linear automorphism can be put in optimal form by a linear change of coordinates.

Let \( A \in \mathcal{A}(\mathbb{C}^q) \) be in optimal form. For \( 1 \leq j \leq q \) let \( \pi_j : \mathbb{C}^q \to \mathbb{C} \) be the projection to the \( j \)-th coordinate. Let \( i \geq 2 \) and let \( \mathcal{H}_i \) be the vector space of all holomorphic maps \( H : \mathbb{C}^q \to \mathbb{C}^q \) whose components \( \pi_j \circ H \) are homogeneous polynomials of degree \( i \). A basis for this vector space is easily described: let \( 1 \leq j \leq q \), let \( I \in \mathbb{N}^q \) be a multi-index of absolute value \( |I| = i \), and define \( X_I^j \) such that

\[
\pi_l \circ X_I^j \equiv \delta_{l,j} z^I, \quad 1 \leq l \leq q.
\]

The set \( \mathcal{B} \ni \{ X_I^j : 1 \leq j \leq q, |I| = q \} \) is a basis of \( \mathcal{H}_i \). Next we define a splitting of \( \mathcal{H}_i \) by specifying a partition of the basis \( \mathcal{B} \).

We set \( X_I^j \in \mathcal{B}_r \) if \( |\lambda_j \lambda^{-I}| = 1 \). The real resonant subspace \( \mathcal{R}_i \) is the vector subspace spanned by the vectors in \( \mathcal{B}_r \).

We set \( X_I^j \in \mathcal{B}_s \) if \( |\lambda_j \lambda^{-I}| < 1 \). The stable subspace \( \mathcal{S}_i \) is the vector subspace spanned by the vectors in \( \mathcal{B}_s \).
We set $X_i^j \in \mathcal{B}_s$ if $|\lambda_j \lambda^{-I}| > 1$. The **unstable subspace** $\mathcal{U}_i$ is the vector subspace spanned by the vectors in $\mathcal{B}_u$.

This defines the splitting $\mathcal{H}_i = \mathcal{R}_i \oplus \mathcal{S}_i \oplus \mathcal{U}_i$, with projections $\pi_r$, $\pi_s$, and $\pi_u$.

If $F \in \mathcal{L}(\mathbb{C}^q)$, then $H \mapsto H \circ F$ and $H \mapsto F \circ H$ are endomorphisms of $\mathcal{H}_i$. We define the linear operator $\Gamma : \mathcal{H}_i \to \mathcal{H}_i$ as $H \mapsto A \circ H \circ A^{-1}$.

The next lemma justifies the terms “stable” and “unstable”.

**Lemma 3.3.** The stable subspace $\mathcal{S}_i$ is $\Gamma$-totally invariant and $\rho(\Gamma|_{\mathcal{S}_i}) < 1$. Indeed

$$\text{sp}(\Gamma|_{\mathcal{S}_i}) = \{\lambda_j \lambda^{-I} : X_i^j \in \mathcal{B}_s\}.$$ 

The unstable subspace $\mathcal{U}_i$ is $\Gamma$-totally invariant and $\rho(\Gamma^{-1}|_{\mathcal{U}_i}) < 1$. Indeed

$$\text{sp}(\Gamma|_{\mathcal{U}_i}) = \{\lambda_j \lambda^{-I} : X_i^j \in \mathcal{B}_u\}.$$ 

**Proof.** Since the $\Gamma$-invariance is an straightforward calculation, we prove the statement concerning the spectrum of $\Gamma|_{\mathcal{S}_i}$. The automorphism $A$ is conjugate to any automorphism obtained multiplying the underdiagonal by a positive constant. Thus there exists a continuous path $\gamma : [0,1] \to \mathcal{A}(\mathbb{C}^q)$ such that $\gamma(0) = A$ and $\gamma(1) = (\lambda_1 z_1, \ldots, \lambda_q z_q)$, with $\gamma(0)$ conjugated to $\gamma(t)$ for all $t \in [0,1]$.

Let $M \in \mathcal{A}(\mathbb{C}^q)$. Define $\Xi(M) \in \mathcal{A}(\mathcal{S}_i)$ as $H \mapsto M \circ H \circ M^{-1}$. If $B = M \circ A \circ M^{-1}$, the linear operator $\Gamma|_{\mathcal{S}_i} = \Xi(A)$ is conjugate to the linear operator $\Xi(B)$. Indeed

$$B \circ H \circ B^{-1} = M \circ A \circ M^{-1} \circ H \circ M \circ A^{-1} \circ M^{-1},$$

thus $\Xi(B) = \Xi(M) \circ \Xi(A) \circ \Xi(M)^{-1}$.

We have $\lim_{t \to 1} \Xi(\gamma(t)) = \Xi(\lambda_1 z_1, \ldots, \lambda_q z_q)$ and $\Gamma|_{\mathcal{S}_i} = \Xi(A) = \Xi(\gamma(0))$ is conjugate to $\Xi(\gamma(t))$ for all $t \in [0,1)$. Thus

$$\text{sp}(\Gamma|_{\mathcal{S}_i}) = \text{sp}(\Xi(\lambda_1 z_1, \ldots, \lambda_q z_q)).$$

It is easy to see that the linear operator $\Xi(\lambda_1 z_1, \ldots, \lambda_q z_q)$ is diagonalizable and that the basis $\mathcal{B}_s$ is a basis of eigenvectors such that

$$[\Xi(\lambda_1 z_1, \ldots, \lambda_q z_q)](X_i^j) = \lambda_j \lambda^{-I} X_i^j.$$ 

Thus

$$\text{sp}(\Gamma|_{\mathcal{S}_i}) = \text{sp}(\Xi(\lambda_1 z_1, \ldots, \lambda_q z_q)) = \{\lambda_j \lambda^{-I} : X_i^j \in \mathcal{B}_s\}.$$ 

The same argument works for the spectrum of $\Gamma|_{\mathcal{U}_i}$. 

**Proposition 3.4.** Let $D \subset \mathbb{C}^q$ be a complete hyperbolic domain. Let $(\varphi_{n,m} : D \to D)$ be a dilation $\mathbb{N}$-evolution family such that $\varphi_{n,m+1}(z) = A(z) + O(|z|^2)$ with $A$ in optimal form. Then for each $i \geq 2$ there exist

1. a family $(k_n^i)$ of polynomial maps $k_n^i(z) = z + O(|z|^2)$ with uniformly bounded degree and uniformly bounded coefficients, and
ii) a triangular dilation evolution family \((T_{n,m}^i)\) with \(T_{n,n+1}^i(z) = A(z) + O(|z|^2),\)

\[ \deg T_{n,n+1}^i \leq i - 1, \]

and uniformly bounded coefficients such that for all \(n \geq 0,\)

\[ k_{n+1}^i \circ \varphi_{n+1}^i - T_{n,n+1}^i \circ k_n^i = O(|z|^i). \]  

(3.2)

If no multiplicative real resonance occurs among the eigenvalues of \(A,\) then the family \((T_{n,m}^i)\) is the linear family \((A^{m-n}).\)

**Proof.** For \(i = 2\) set \(T_{n+1,n}^2 = A, k_2^i = \text{id},\) and we are done since \(A\) is a triangular mapping. Now assume that (3.2) holds for \(i \geq 2.\) We can rewrite (3.2) as

\[ k_{n+1}^i \circ \varphi_{n+1}^i - T_{n,n+1}^i \circ k_n^i = P_{n,n+1} + O(|z|^{i+1}), \]

(3.3)

where \((P_{n,n+1})\) is a bounded sequence in \(H_i.\) Define \(R_{n,n+1} = \pi_r(P_{n,n+1})\) which is in the real resonant subspace \(\mathcal{R}_i,\) and \(N_{n,n+1} = P_{n,n+1} - R_{n,n+1} \in \mathcal{S}_i \oplus \mathcal{U}_i.\) Set

\[ T_{n,n+1}^i = T_{n,n+1}^i + R_{n,n+1}, \]

which is still a triangular dilation \(\mathbb{N}\)-evolution family with uniformly bounded degree and uniformly bounded coefficients since \(R_{n,n+1}\) is a triangular mapping thanks to (3.1), and set

\[ k_{n+1}^i = k_n^i + H_n \circ k_n^i, \]

where \((H_n)\) is an unknown bounded sequence in \(H_i.\)

\[ k_{n+1}^i \circ \varphi_{n+1}^i - T_{n,n+1}^i \circ k_n^i = \]

\[ = (k_{n+1}^i + H_{n+1} \circ k_{n+1}^i) \circ \varphi_{n+1}^i - (T_{n,n+1}^i + R_{n,n+1}) \circ (k_n^i + H_n \circ k_n^i) \]

\[ = P_{n,n+1} - R_{n,n+1} + H_{n+1} \circ A - A \circ H_n + O(|z|^{i+1}) \]

\[ = N_{n,n+1} + H_{n+1} \circ A - A \circ H_n + O(|z|^{i+1}). \]

Thus to end the proof we need to prove the existence of a bounded sequence \((H_n)\) of elements of \(H_i\) which satisfies

\[ N_{n,n+1} = A \circ H_n - H_{n+1} \circ A, \]

(3.4)

that is a bounded solution \((H_n)\) of the homological difference equation

\[ H_{n+1} = A \circ H_n - H_{n+1} \circ A^{-1}. \]

Define \(B_n = -N_{n,n+1} \circ A^{-1}.\) In the proof of Lemma 3.3 we proved that \(\mathcal{S}_i\) and \(\mathcal{U}_i\) are invariant by the linear operator \(H \mapsto H \circ A^{-1},\) thus \(B_n \in \mathcal{S}_i \oplus \mathcal{U}_i.\) Define \(B_n^s = \pi_s(B_n),\) \(B_n^u = \pi_u(B_n).\) If \(n \geq 1\) it is easy to prove by induction that

\[ H_n = \Gamma^n(H_0) + \sum_{j=0}^{n-1} \Gamma^j(B_{n-1-j}) = \Gamma^n(H_0) + \sum_{j=0}^{n-1} \Gamma^{n-1-j}(B_j). \]

(3.5)
We have

\[ H_n = \Gamma^n(H_0) + \sum_{j=0}^{n-1} \Gamma^j(B_{n-1-j}^s) + \sum_{j=0}^{n-1} \Gamma^j(B_{n-1-j}^u) \]

\[ = \sum_{j=0}^{n-1} \Gamma^j(B_{n-1-j}^s) + \Gamma^{n-1}(\Gamma(H_0) + \sum_{j=0}^{n-1} \Gamma^{-j}(B_j^u)). \]

Recall that if \( V \) is a complex vector space, and \( L \in \mathcal{L}(V) \), then the spectral radius of \( L \) satisfies \( \rho(L) = \inf_{\|\cdot\| \in \mathcal{I}} \{ \|L\| \} \), where \( \mathcal{I} \) is the set of all operator norms induced by a norm on \( V \). Hence by Lemma 3.3 there exist a norm \( \| \cdot \|_s \) on \( S_i \) and a norm \( \| \cdot \|_u \) on \( U_i \) such that \( \| \Gamma \|_s < 1 \), \( \| \Gamma^{-1} \|_u < 1 \). Define a norm on \( S_i \oplus U_i \) by

\[ \| H \| = \| \pi_s(H) \|_s + \| \pi_u(H) \|_u. \]

Since \( (B_n^s) \) is bounded, there exists \( C > 0 \) such that

\[ \| \sum_{j=0}^{n-1} \Gamma^j(B_{n-1-j}^s) \| \leq \sum_{j=0}^{\infty} \| \Gamma^j(B_{n-1-j}^s) \|_s \leq C, \quad n \geq 0. \]

Since \( (B_n^u) \) is bounded, \( \sum_{j=0}^{\infty} \| \Gamma^{-j}(B_j^u) \|_u < +\infty \), thus we can define

\[ H_0 = -\Gamma^{-1}(\sum_{j=1}^{\infty} \Gamma^{-j}(B_j^u)) \in U_i. \]

With this definition,

\[ \| H_n \| \leq C + \| \Gamma^{n-1}(\sum_{j=n}^{\infty} \Gamma^{-j}(B_j^u)) \|_u = C + \sum_{j=1}^{\infty} \| \Gamma^{-j}(B_{n-1+j}^u) \|_u, \]

and since

\[ \| \sum_{j=1}^{\infty} \Gamma^{-j}(B_{n-1+j}^u) \|_u \leq \sum_{j=1}^{\infty} \| \Gamma^{-j}(B_{n-1+j}^u) \|_u \leq C', \]

we have \( \| H_n \| \leq C + C' \).

If no multiplicative real resonance occurs among the eigenvalues of \( A \), then

\[ \mathcal{R}_i = \emptyset, \quad \text{for all } i \geq 2, \]

and thus \( T_{n,m}^{i+1} = T_{n,m}^i \) for all \( i \geq 2 \), which gives

\[ T_{n,m}^i = T_{n,m}^2 = A^{m-n}, \quad \text{for all } i \geq 2. \]

\[ \square \]

Remark 3.5. Let \( p \geq 0 \) be the smallest integer such that \( |\lambda_p| < |\lambda_q| \). Then if \( i \geq p \) we have \( \pi_r(P_{n,n+1}) = 0 \) in \( \mathcal{H}_i \). Hence \( T_{n,n+1}^i = T_{n,n+1}^p \) for any \( i \geq p \).
Proposition 3.6. Let $D \subset \mathbb{C}^q$ be a complete hyperbolic domain. Let $(\varphi_{n,m}, : D \to D)$ be a dilation $\mathbb{N}$-evolution family such that $\varphi_{n,n+1}(z) = A(z) + O(|z|^2)$ with $A$ in optimal form. Then there exists a triangular dilation $\mathbb{N}$-evolution family $(T_{n,m})$ with bounded degree and bounded coefficients locally conjugate to $(\varphi_{n,m})$. If no multiplicative real resonance occurs among the eigenvalues of $A$, then $(\varphi_{n,m})$ is locally conjugate to its linear part $(A^{m-n})$.

Proof. Let $\alpha$ be such that $\max_{z \in \mathbb{C}^q} \frac{|A(z)|}{|z|} < \alpha < 1$. Let $(T_{n,m}^i)$ and $(k_n^i)$ be the families given by Proposition 3.4. Let $p \geq 0$ be as in previous remark. Define $(T_{n,m}) = (T_{n,m}^p)$. Let $\beta > 0$ be the constant given by Lemma A.4 for $(T_{n,m})$. Let $\ell \geq 0$ be an integer such that $\alpha^\ell < 1/\beta$, and define $(k_{n}) = (k_{n}^\ell)$. By Proposition 3.4

$$k_{n+1} \circ \varphi_{n,n+1} - T_{n,n+1} \circ k_{n} = O(|z|^\ell),$$

thus

$$T_{n,n+1}^{-1} \circ k_{n+1} \circ \varphi_{n,n+1} - k_{n} = O(|z|^\ell).$$

By Lemma A.2 there exists $r > 0$ (we can assume $0 < r < 1/2$) such that on $rB$ we have $|\varphi_{n,n+1}(z)| \leq \alpha |z|$ and $|T_{n,n+1}(z)| \leq \alpha |z|$ for all $n \geq 0$. Thus for $\zeta \in rB$ we have

$$|\varphi_{0,m}(\zeta)| < r\alpha^m.$$

Thanks to Lemma A.1 there exists $C > 0$ such that on $rB$,

$$|T_{m,m+1}^{-1} \circ k_{m+1} \circ \varphi_{m,m+1}(\zeta) - k_{m}(\zeta)| \leq C|\zeta|^\ell, \quad m \geq 0.$$

Hence

$$|T_{m,m+1}^{-1} \circ k_{m+1} \circ \varphi_{0,m+1}(\zeta) - k_{m} \circ \varphi_{0,m}(\zeta)| \leq C|\varphi_{0,m}(\zeta)|^\ell \leq C r^\ell \alpha^{\ell m}.$$

Let $\Delta$ be the unit polydisc. There exists $sB \subset rB$ such that

$$T_{m,m+1}^{-1} \circ k_{m+1} \circ \varphi_{0,m+1}(sB) \subset \frac{1}{2}\Delta$$

and

$$k_{m} \circ \varphi_{0,m}(sB) \subset \frac{1}{2}\Delta.$$

Indeed the families $(k_{m})$ and $(T_{m,m+1}^{-1})$ are uniformly bounded on $rB$ and thus equicontinuous in $0$.

Hence Lemma A.4 applies to get on $sB$,

$$|T_{0,m+1}^{-1} \circ k_{m+1} \circ \varphi_{0,m+1}(\zeta) - T_{0,m}^{-1} \circ k_{m} \circ \varphi_{0,m}(\zeta)| \leq C r^\ell (\beta \alpha^\ell)^m.$$

Likewise it is easy to see that for all $m \geq n \geq 0$, $n \geq 0$,

$$|T_{n,m+1}^{-1} \circ k_{m+1} \circ \varphi_{n,m+1}(\zeta) - T_{n,m}^{-1} \circ k_{m} \circ \varphi_{n,m}(\zeta)| \leq C r^\ell (\beta \alpha^\ell)^{m-n}.$$

Since $\alpha^\ell < 1/\beta$ for all $n \geq 0$ there exists a holomorphic mapping $h_n$ on $sB$ such that

$$h_n = \lim_{m \to \infty} T_{n,m}^{-1} \circ k_{m} \circ \varphi_{n,m}$$
uniformly on compacta. Each $h_n$ is bounded by $|k_n| + \sum_{j=0}^{\infty} Cr^j (\beta \alpha^j)^j$, hence they are uniformly bounded. Moreover
\[
h_m \circ \varphi_{n,m} = \lim_{j \to \infty} T_{m,j}^{-1} \circ k_j \circ \varphi_{m,j} \circ \varphi_{n,m} = \lim_{j \to \infty} T_{n,m} \circ T_{j,n} \circ k_j \circ \varphi_{n,j} = T_{n,m} \circ h_n.
\]

If no multiplicative real resonance occurs among the eigenvalues of $A$, then $(T_{n,m}) = (A^{m-n})$.

We can now prove an existence result for $\mathbb{N}$-Loewner chains.

**Proposition 3.7.** Let $D \subset \mathbb{C}^q$ be a complete hyperbolic domain. Let $(\varphi_{n,m} : D \to D)$ be a dilation $\mathbb{N}$-evolution family, $\varphi_{n,n+1}(z) = A(z) + O(|z|^2)$. Then there exists a $\mathbb{N}$-Loewner chain $(f_n : D \to \mathbb{C}^q)$ with $\bigcup_{n \geq 0} f_n(D) = \mathbb{C}^q$ associated with $(\varphi_{n,m})$, which is locally bounded if no multiplicative real resonance occurs among the eigenvalues of $A$.

**Proof.** Up to a linear change of coordinates, we can assume that $A$ is in optimal form. By Proposition 3.6 there exists a triangular dilation $\mathbb{N}$-evolution family $(T_{n,m})$ with bounded degree and bounded coefficients locally conjugate to $(\varphi_{n,m})$. Let $(h_n)$ be the family of intertwining mappings given by Proposition 2.7. Then $(T_{0,n}^{-1} \circ h_n)$ is a $\mathbb{N}$-Loewner chain $(f_n)$ associated with $(\varphi_{n,m})$. Indeed
\[
T_{0,m}^{-1} \circ h_m \circ \varphi_{n,m}(z) = T_{0,m}^{-1} \circ T_{n,m} \circ h_n(z) = T_{0,n}^{-1} \circ h_n(z), \quad z \in D, \ 0 \leq n \leq m.
\]

By Lemma A.3 there exists a ball $s\mathbb{B} \subset \bigcap_{n \geq 0} h_n(D)$. Hence
\[
\bigcup_{n \geq 0} T_{0,n}^{-1}(h_n(D)) \supset \bigcup_{n \geq 0} T_{0,n}^{-1}(s\mathbb{B}) = \mathbb{C}^q
\]
by Lemma A.3.

If no multiplicative real resonance occurs among the eigenvalues of $A$, then $(T_{n,m}) = (A^{m-n})$, and the chain $(A^{-n} \circ h_n)$ is locally bounded. 

Now we can go back to the continuous-time setting.

**Definition 3.8.** A $\mathbb{R}^+$-evolution family $(\varphi_{s,t})$ on $D$ is a dilation $\mathbb{R}^+$-evolution family if for all $0 \leq s \leq t$,
\[
\varphi_{s,t}(z) = e^{\Lambda(t-s)}(z) + O(|z|^2), \quad (3.6)
\]
where the eigenvalues of $\Lambda \in \mathcal{L}(\mathbb{C}^q)$ have strictly negative real part.

A $\mathbb{R}^+$-Loewner chain $(f_t : D \to \mathbb{C}^q)$ is a locally bounded $\mathbb{R}^+$-Loewner chain if for all $t \geq 0$,
\[
f_t(z) = e^{-\Lambda t}(z) + O(|z|^2),
\]
where the eigenvalues of $\Lambda \in \mathcal{L}(\mathbb{C}^q)$ have strictly negative real part and the family $(e^{\Lambda t} \circ f_t)$ is uniformly bounded in a neighborhood of the origin.

If we restrict time to integer values in a dilation $\mathbb{R}^+$-evolution family $(\varphi_{s,t})$ we obtain its discretized dilation $\mathbb{N}$-evolution family $(\varphi_{n,m})$. We have
\[
\varphi_{n,n+1}(z) = e^{\Lambda}(z) + O(|z|^2).
\]
An additive real resonance is an identity
\[
\text{Re} \left( \sum_{j=1}^{N} k_j \alpha_j \right) = \text{Re} \alpha_l,
\]
where \( k_j \geq 0 \) and \( \sum_j k_j \geq 2 \). Recall that \( \alpha \) is an eigenvalue of \( \Lambda \) with algebraic multiplicity \( m \) if and only if \( e^\alpha \) is an eigenvalue of \( e^A \) with algebraic multiplicity \( m \). Hence additive real resonances of \( \Lambda \) correspond to multiplicative real resonances of \( e^A \).

**Lemma 3.9.** Let \( D \) be a complete hyperbolic domain. Let \( (\varphi_{s,t} : D \to D) \) be a dilation \( \mathbb{R}^+ \)-evolution family, and let \( (\varphi_{n,m} : D \to D) \) be its discretized evolution family. Assume there exists a \( \mathbb{N} \)-Loewner chain \( (f_n) \) associated with \( (\varphi_{n,m}) \). Then we can extend it in a unique way to a \( \mathbb{R}^+ \)-Loewner chain associated with \( (\varphi_{s,t}) \). If \( (f_n) \) is a locally bounded \( \mathbb{N} \)-Loewner chain, then also \( (f_s) \) is locally bounded.

**Proof.** For all \( s \in \mathbb{R}^+ \) define \( f_s = f_j \circ \varphi_{s,j} \), where \( j \) is an integer such that \( s \leq j \). The family \( (f_s) \) is a \( \mathbb{R}^+ \)-Loewner chain associated with \( (\varphi_{s,t}) \) (cf. [2, Lemma 8.5]). If \( (f_n) \) is a locally bounded \( \mathbb{N} \)-Loewner chain, then there exists \( r > 0 \) such that the family \( (e^{An} \circ f_n) \) is uniformly bounded on the Kobayashi ball \( \Omega(0, r) \) centered in the origin of radius \( r > 0 \). For each \( s \geq 0 \) define \( m_s \) as the smallest integer greater than \( s \). One has
\[
e^{As} \circ f_s = e^{As} \circ f_{m_s} \circ \varphi_{s,m_s} = e^{A(s-m_s)} \circ e^{Am_s} \circ f_{m_s} \circ \varphi_{s,m_s},
\]
and since \( \varphi_{s,m_s}(\Omega(0, r)) \subset \Omega(0, r) \) and \( m_s - s \leq 1 \), the family \( (e^{As} \circ f_s) \) is uniformly bounded on \( \Omega(0, r) \).

**Proposition 3.10.** Let \( D \subset \mathbb{C}^q \) be a complete hyperbolic domain. Let \( (\varphi_{s,t} : D \to D) \) be a dilation \( \mathbb{R}^+ \)-evolution family, \( \varphi_{s,t}(z) = e^{A(t-s)}(z) = O(\abs{z}^2) \). Then there exists a \( \mathbb{R}^+ \)-Loewner chain \( (f_s : D \to \mathbb{C}^q) \) with \( \bigcup_{t \geq 0} f_s(D) = \mathbb{C}^q \) associated with \( (\varphi_{s,t}) \), which is locally bounded if no additive real resonance occurs among the eigenvalues of \( \Lambda \).

**Proof.** Let \( (\varphi_{n,m} : D \to D) \) be the discretized evolution family of \( (\varphi_{s,t} : D \to D) \). Since no additive real resonance occurs in \( \Lambda \), no multiplicative real resonance occurs in \( A = e^A \). The result follows from Proposition 3.7 and Lemma 3.9.

**Remark 3.11.** If no additive real resonance occurs among the eigenvalues of \( \Lambda \) then by the proof of Proposition 3.10 there exists a family \( (h_n) \) of tangent to identity polynomial mappings of uniformly bounded degree and uniformly bounded coefficients such that
\[
f_s = \lim_{m \to \infty} e^{-Am} \circ h_m \circ \varphi_{s,m}.
\]

4. THE LOEWNER PDE

**Definition 4.1.** Let \( D \) be a domain in \( \mathbb{C}^q \) containing \( 0 \) and let \( d \in [1, +\infty] \). A dilation Herglotz vector field of order \( d \geq 1 \) on \( D \) is a mapping
\[
G : D \times \mathbb{R}^+ \to \mathbb{C}^q
\]
satisfying
a) for all \( z \in D \) the map \( t \mapsto H(z, t) \) is measurable,
b) for a.e. \( t \geq 0 \) the map \( z \mapsto H(z, t) \) is an infinitesimal generator on \( D \) of the form
\[
H(z, t) = \Lambda(z) + O(|z|^2)
\]
where the eigenvalues of \( \Lambda \in \mathcal{L}(\mathbb{C}^q) \) have strictly negative real part.
c) for any compact set \( K \subset D \) and there exists a function \( c_K \in L^d_{\text{loc}}(\mathbb{R}^+, \mathbb{R}^+) \) such that
\[
|H(z, t)| \leq c_K(t), \quad z \in K, t \geq 0.
\]
The partial differential equation
\[
\frac{\partial f_t(z)}{\partial t} = -d f_t(z) H(z, t) \quad \text{a.e.} \quad t \geq 0, \quad z \in D,
\] (4.1)
where \( H(z, t) \) is a dilation Herglotz vector field, is called the Loewner PDE.

The following is our main result.

**Theorem 4.2.** Let \( D \subset \mathbb{C}^q \) be a complete hyperbolic domain and let \( H(z, t) = \Lambda(z) + O(|z|^2) \) be a dilation Herglotz vector field on \( D \) of order \( d \geq 1 \). Then the Loewner PDE (4.1) admits a solution given by a family of univalent mappings \((f_t: D \to \mathbb{C}^q)\), such that \( \cup_{t \geq 0} f_t(D) = \mathbb{C}^q \), and which is locally absolutely continuous of order \( d \) in the following sense: for any compact set \( K \subset D \) there exists a function \( k_K \in L^d_{\text{loc}}(\mathbb{R}^+, \mathbb{R}^+) \) such that
\[
|f_s(z) - f_t(z)| \leq \int_s^t k_K(\xi)d\xi, \quad z \in K, \ 0 \leq s \leq t.
\] (4.2)

If no additive real resonance occurs among the eigenvalues of \( \Lambda \) then the family \((e^{\Lambda t} \circ f_t)\) is uniformly bounded in a neighborhood of the origin. Any locally absolutely continuous solution given by a family of holomorphic mappings \((g_t: D \to \mathbb{C}^q)\), such that \( \cup_{t \geq 0} g_t(D) = \mathbb{C}^q \), where \( \Psi: \mathbb{C}^q \to \mathbb{C}^q \) is holomorphic.

**Proof.** Since by assumption \( D \) is complete hyperbolic, \([3]\) yields that the solution of the Loewner ODE
\[
\begin{aligned}
\frac{\partial}{\partial t} \varphi_{s,t}(z) &= H(\varphi_{s,t}(z), t), \quad \text{a.e.} \quad t \in [s, \infty), \\
\varphi_{s,s}(z) &= z, \quad s \geq 0.
\end{aligned}
\] (4.3)
is a \( \mathbb{R}^+ \)-evolution family \((\varphi_{s,t}(z) = e^{\Lambda(t-s)}(z) + O(|z|^2))\) which is locally absolutely continuous of order \( d \) in the following sense: for any compact set \( K \subset D \) there exists a function \( C_K \in L^d_{\text{loc}}(\mathbb{R}^+, \mathbb{R}^+) \) such that
\[
|\varphi_{s,t}(z) - \varphi_{s,u}(z)| \leq \int_u^t C_K(\xi)d\xi, \quad z \in K, \ 0 \leq s \leq u \leq t.
\] (4.4)
By Proposition \([3.10]\) there exists a \( \mathbb{R}^+ \)-Loewner chain \((f_t: D \to \mathbb{C}^q)\) with \( \cup_{t \geq 0} f_t(D) = \mathbb{C}^q \) associated with \((\varphi_{s,t})\). By \([4\ ]\ \text{Theorem 4.10}\) the chain \((f_t)\) is of locally absolutely
continuous of order $d$, and by [4, Theorem 5.2] it solves the Loewner PDE (4.1). If no additive real resonance occurs among the eigenvalues of $\Lambda$ then by Proposition 3.10 the chain $(f_t)$ is locally bounded. Any locally continuous solution of (4.1) is by [4, Theorem 5.2] a $\mathbb{R}^+$-Loewner chain associated with $(\varphi_{s,t})$, thus by Theorem 2.3 it is of the form $(\Psi \circ f_t)$, where $\Psi: \mathbb{C}^q \to \mathbb{C}^q$ is holomorphic.

□

5. An univalence criterion

In this section we generalize a classical univalence criterion in the unit disk due to Pommerenke. We first need to generalize the notion of locally bounded $\mathbb{R}^+$-Loewner chains in order to include families of non-necessarily univalent mappings.

**Definition 5.1.** Let $D$ be a domain in $\mathbb{C}^q$ containing 0. A family $(f_t: D \to \mathbb{C}^q)$ is a *locally bounded $\mathbb{R}^+$-subordination chain* if

i) for all $0 \leq s \leq t$ there exists a holomorphic mapping $\varphi_{s,t} \in \text{Hol}(D,D)$ fixing 0 and satisfying $f_s = f_t \circ \varphi_{s,t}$, called *transition mapping*;

ii) $f_t(z) = e^{-\Lambda t}(z) + O(|z|^2)$ for all $t \geq 0$, the eigenvalues of $\Lambda \in \mathcal{L}(\mathbb{C}^q)$ have strictly negative real part, and the family $(e^{\Lambda t} \circ f_t)$ is uniformly bounded in a neighborhood of the origin.

Now we can state Pommerenke’s criterion.

**Theorem 5.2** ([20, Folgerung 6]). If $(f_t: \mathbb{D} \to \mathbb{C})$ is a locally bounded $\mathbb{R}^+$-subordination chain, then for all $t \geq 0$ the mapping $f_t$ is univalent.

This criterion has been generalized to the unit ball $\mathbb{B} \subset \mathbb{C}^q$, with different hypotheses, by Pfaltzgraff [19, Theorem 2.3], and by Graham and Kohr [13, Theorem 8.1.6]. To generalize Pommerenke’s criterion we do not assume the subordination chain to solve a Loewner PDE: we only assume continuity. A locally bounded $\mathbb{R}^+$-subordination chain $(f_t: D \to \mathbb{C}^q)$ is *continuous* if the mapping $t \mapsto f_t$ is continuous with respect to the topology of uniform convergence on compacta on $\text{Hol}(D,\mathbb{C}^q)$.

**Proposition 5.3.** Let $D \subset \mathbb{C}^q$ be a complete hyperbolic domain. If $(f_t: D \to \mathbb{C}^q)$ is a continuous locally bounded $\mathbb{R}^+$-subordination chain, then for all $t \geq 0$ the mapping $f_t$ is univalent.

**Proof.** We have to show that $f_t$ is univalent for all $t \geq 0$. The identity principle yields that for all $0 \leq s \leq t$ there exists a unique transition mapping $\varphi_{s,t} \in \text{Hol}(D,D)$ satisfying $f_s = f_t \circ \varphi_{s,t}$ and fixing 0, since $f_t$ is locally invertible at 0. Thus the family $(\varphi_{s,t})_{0 \leq s \leq t}$ satisfies $\varphi_{s,s} = \text{id}$ for all $s \geq 0$. Moreover one has

$$\varphi_{u,t} \circ \varphi_{s,u} = \varphi_{s,t}, \quad 0 \leq s \leq u \leq t.$$ 

Indeed for all $0 \leq s \leq u \leq t$,

$$f_t \circ \varphi_{u,t} \circ \varphi_{s,u} = f_u \circ \varphi_{s,u} = f_s = f_t \circ \varphi_{s,t},$$
and the assertion follows since the transition mapping is unique. Moreover by the chain rule \( df_s(0) = df_t(0) \circ d\varphi_{s,t}(0) \), hence \( d\varphi_{s,t}(0) = e^{\Lambda(t-s)} \).

We claim that \( \lim_{t \to s^+} \varphi_{s,t} = \text{id} \) for all \( s \geq 0 \). Indeed since \( (\varphi_{s,t} : D \to D)_{0 \leq s \leq t} \) is a normal family and \( \varphi_{s,t}(0) = 0 \) for all \( 0 \leq s \leq t \), any sequence \( (\varphi_{s,t_n}) \) with \( t_n \to s \) admits a subsequence \( (\varphi_{s,t_{n_k}}) \) converging on compacta to a mapping \( \varphi \in \text{Hol}(D,D) \), and by \( f_s = f_{t_{n_k}} \circ \varphi_{s,t_{n_k}} \) we obtain \( f_s = f_s \circ \varphi \), thus \( \varphi = \text{id} \). This proves that \( \lim_{t \to s^+} \varphi_{s,t} = \text{id} \). In the same way, \( \lim_{s \to t^-} \varphi_{s,t} = \text{id} \).

This implies that \( \varphi_{s,t} \) is univalent for all \( 0 \leq s \leq t \). Indeed suppose there exists \( 0 < s < t \) and \( z \neq w \) contained in a Kobayashi ball \( \Omega(0, \ell) \) such that \( \varphi_{s,t}(z) = \varphi_{s,t}(w) \). Set \( r = \inf\{ u \in [s, t] : \varphi_{s,u}(z) = \varphi_{s,u}(w) \} \). Since \( \lim_{u \to s^+} \varphi_{s,u} = \text{id} \) uniformly on compacta, we have \( r > s \). If \( u \in (s, r) \),

\[
\varphi_{u,r}(\varphi_{s,u}(z)) = \varphi_{u,r}(\varphi_{s,u}(w)),
\]

and since \( \varphi_{s,u}(z) \neq \varphi_{s,u}(w) \), the mapping \( \varphi_{u,r} \) is not univalent on

\[
\bigcup_{u \in (s, r)} \varphi_{s,u}(z) \cup \varphi_{s,u}(w) \subset \Omega(0, \ell).
\]

Since \( D \) is complete hyperbolic, by [15, Proposition 1.1.9] one has \( \Omega(0, \ell) \subset \subset D \). But \( \lim_{u \to r^-} \varphi_{u,r} = \text{id} \) uniformly on compacta which is a contradiction since the identity mapping is univalent.

Define \( h_s = e^{\Lambda s} \circ f_s \), for all \( s \geq 0 \). By hypothesis the family \((h_s : D \to \mathbb{C}^q)\) is uniformly bounded in a neighborhood of the origin. From \( f_t \circ \varphi_{s,t} = f_s \) we obtain

\[
h_t \circ \varphi_{s,t} = e^{\Lambda(t-s)} \circ h_s.
\]

Fix \( s \geq 0 \). The dilation \( N \)-evolution family \((\varphi_{s+n, s+m})_{0 \leq n \leq m} \) is locally conjugate to \((e^{\Lambda(m-n+s)})_{0 \leq n \leq m} \) by means of the intertwining mappings \((h_{s+n})_{n \geq 0} \). By Proposition 2.9 the mapping \( h_s \) is univalent, thus \( f_s = e^{-\Lambda s} \circ h_s \) is univalent. \( \square \)

As a corollary one easily obtains the following.

**Corollary 5.4.** Let \( D \subset \mathbb{C}^q \) be a complete hyperbolic domain and let \( H(z, t) = \Lambda(z) + O(|z|^2) \) be a dilation Herglotz vector field on \( D \). Let \((f_t : D \to \mathbb{C}^q)\) be a locally absolutely continuous family of holomorphic mappings which solves the Loewner PDE \((4.1)\) and assume that the family \((e^{\Lambda t} \circ f_t)\) is uniformly bounded in a neighborhood of the origin. Then for all \( t \geq 0 \) the mapping \( f_t \) is univalent.

**Appendix A. Auxiliary Lemmas**

For the convenience of the reader, we recall here some auxiliary Lemmas, in the form used in the proofs.

**Lemma A.1.** [2 Lemma 2.2] Let \( A \in \mathcal{L}(\mathbb{C}^q) \). Let \( F \) be a family of holomorphic mappings \((f : rB \to \mathbb{C}^q)\), bounded by \( M > 0 \), and let \( k \geq 2 \) such that \(|f(z) - A(z)| = O(|z|^k)\) for all \( f \in F \). Then there exists \( C_k > 0 \) such that \(|f(z) - A(z)| \leq C_k |z|^k\) for all \( z \in rB \).
Lemma A.2. [2, Lemma 2.3] Let $A \in \mathcal{L}(\mathbb{C}^q)$, and let $D$ be a domain containing $0$. Let $\mathcal{F}$ be a family of holomorphic mappings $(f : D \to \mathbb{C}^q)$, bounded by $M > 0$, and satisfying $f(z) = A(z) + O(|z|^2)$. Let $\alpha > 0$ be such that $\max_{z \in \mathbb{C}^q} \frac{|A(z)|}{|z|} < \alpha$. Then there exists $s > 0$ such that if $f \in \mathcal{F}$ then $|f(z)| \leq \alpha|z|$ for all $|z| \leq s$.

Lemma A.3. [2, Lemma 2.5] Let $A \in \mathcal{A}(\mathbb{C}^q)$, and let $D$ be a domain containing $0$. Let $\mathcal{F}$ be a family of holomorphic mappings $(f : D \to \mathbb{C}^q)$, bounded by $M > 0$, and satisfying $f(z) = A(z) + O(|z|^2)$. There exist $r > 0$ and $s > 0$ such that if $f \in \mathcal{F}$ then $f$ is univalent on $rB$, and $sB \subset f(rB)$.

The following Lemma is stated in [18, Lemma 11] as a simple generalization of [21, Lemma 1, Appendix]. A proof can be found in [2, Corollary 4.4, Lemma 4.5].

Lemma A.4. [2, Corollary 4.4] Let $\Delta \subset \mathbb{C}^q$ be the unit polydisc.
Let $(T_n,m)$ be a triangular dilation $\mathbb{N}$-evolution family of uniformly bounded degree and uniformly bounded coefficients. Then
\begin{enumerate}
  \item there exists $\beta \geq 0$ such that for all $k \geq 0$,
    \[ |T_{0,k}^{-1}(z) - T_{0,k}^{-1}(z')| \leq \beta^k |z - z'|, \quad z, z' \in \frac{1}{2}\Delta. \]
  \item $T_{0,n}(z) \to 0$ uniformly on compacta and for each neighborhood $V$ of $0$ we have
    \[ \bigcup_{n=1}^{\infty} T_{0,n}^{-1}(V) = \mathbb{C}^q. \]
\end{enumerate}

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