The translated Whitney–Lah numbers: 
generalizations and $q$-analogues

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Abstract: In this paper, we derive some combinatorial formulas for the translated Whitney–Lah numbers which are found to be generalizations of already-existing identities of the classical Lah numbers, including the well-known Qi’s formula. Moreover, we obtain $q$-analogues of the said formulas and identities by establishing similar properties for the translated $q$-Whitney numbers.

Keywords: Lah numbers, translated Whitney–Lah numbers, Qi’s formula, $q$-analogues.

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1 Introduction

The (unsigned) Lah numbers, denoted by $L(n, k)$, count the number of partitions of a set $X$ with $n$ elements into $k$ nonempty linearly ordered subsets. These numbers are known to satisfy the following basic combinatorial properties:

- explicit formula

$$L(n, k) = \frac{n!}{k!} \binom{n - 1}{k - 1};$$

- recurrence relation

$$L(n + 1, k) = L(n, k - 1) + (n + k)L(n, k);$$

- exponential generating function

$$\sum_{n=0}^{\infty} L(n, k) \frac{t^n}{n!} = \frac{1}{k!} \left( \frac{t}{1 - t} \right)^{k}.$$
The numbers $L(n,k)$ are often defined as coefficients of rising factorials in terms of falling factorials. That is
\[ \langle t \rangle_n = \sum_{k=0}^{n} L(n,k) (t)_k, \] (4)
where
\[ \langle t \rangle_n = t(t+1)(t+2) \cdots (t+n-1) \]
is the rising factorial of $t$ of order $n$ and
\[ (t)_k = t(t-1)(t-2) \cdots (t-k+1) \]
is the falling factorial of $t$ of order $k$ with $\langle t \rangle_0 = (t)_0 = 1$ and $(-t)_n = (-1)^n \langle t \rangle_n$. The Lah numbers are actually closely-related with the well-known Stirling numbers. To illustrate this, we first recall that the Stirling numbers of the first and second kinds, denoted by $\left[ \begin{array}{c} n \\ j \end{array} \right]$ and $\left\{ \begin{array}{c} n \\ j \end{array} \right\}$, respectively, are defined as coefficients in the expansions of the relations
\[ (t)_n = \sum_{j=0}^{n} (-1)^{n-j} \left[ \begin{array}{c} n \\ j \end{array} \right] t^j \] (5)
and
\[ t^n = \sum_{j=0}^{n} \left\{ \begin{array}{c} n \\ j \end{array} \right\} (t)_j. \] (6)
Notice that putting $-t$ in place of $t$ in (5) yields
\[ \langle t \rangle_n = \sum_{j=0}^{n} \left[ \begin{array}{c} n \\ j \end{array} \right] t^j. \] (7)
By substituting (6) in the right-hand side of (7), we get
\[ \langle t \rangle_n = \sum_{j=0}^{n} \left[ \begin{array}{c} n \\ j \end{array} \right] \sum_{k=0}^{j} \left\{ \begin{array}{c} j \\ k \end{array} \right\} (t)_k = \sum_{k=0}^{n} \left( \sum_{j=k}^{n} \left[ \begin{array}{c} n \\ j \end{array} \right] \left\{ \begin{array}{c} j \\ k \end{array} \right\} \right) (t)_k. \]

By combining this with (4) and comparing the coefficients of $(t)_k$, we are able to write
\[ L(n,k) = \sum_{j=k}^{n} \left[ \begin{array}{c} n \\ j \end{array} \right] \left\{ \begin{array}{c} j \\ k \end{array} \right\}. \] (8)

It is important to note that here, the numbers $\left[ \begin{array}{c} n \\ j \end{array} \right]$ particularly refer to the “unsigned” Stirling numbers of the first kind which count the number of permutations of the $n$-element set $X$ into $j$ disjoint cycles. Similarly, the Stirling numbers of the second kind $\left\{ \begin{array}{c} n \\ j \end{array} \right\}$ can be combinatorially interpreted as the number of partitions of $X$ into $j$ nonempty blocks. With this, the Bell numbers $B_n$ are defined as the total number of partitions of the $n$-element set $X$. That is,
\[ B_n = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\}. \] (9)
The paper of Petkovšek and Pisanski [20], and the books of Comtet [4] and Chen and Kho [2] contain detailed discussions on the Lah, Stirling and Bell numbers, including their respective combinatorial properties and interpretations. In addition to these, Qi [21] recently obtained an explicit formula for the Bell numbers expressed in terms of both the Lah numbers and the Stirling numbers of the second kind, viz.

\[ B_n = \sum_{k=1}^{n} (-1)^{n-k} \left( \sum_{\ell=1}^{k} L(k, \ell) \right) \binom{n}{k}. \]  

(10)

The results of this paper are organized as follows. In Section 2, we present the translated Whitney numbers and derive some formulas which generalize already-existing identities for the classical Lah numbers, including one that will generalize (10). In Section 3, we establish the \( q \)-analogues of some of the results in Section 2 using as framework the translated \( q \)-Whitney numbers.

## 2 Translated Whitney numbers

In 2013, Belbachir and Bousbaa [1] introduced the translated Whitney numbers using a combinatorial approach which involves “mutations” of some elements of a given finite set. To be more precise, the translated Whitney numbers of first kind, denoted by \( \tilde{w}_\alpha(n, k) \), were defined as the number of permutations of \( n \) elements with \( k \) cycles such that the elements of each cycle can mutate in \( \alpha \) ways, except the dominant one while the translated Whitney numbers of the second kind, denoted by \( \tilde{W}_\alpha(n, k) \), were defined as the number of partitions of the \( n \)-element set into \( k \) subsets such that the elements of each subset can mutate in \( \alpha \) ways, except the dominant one. These numbers were shown to satisfy the recurrence relations [1, Theorems 2 and 8]

\[ \tilde{w}_\alpha(n, k) = \tilde{w}_\alpha(n-1, k-1) + \alpha(n-1)\tilde{w}_\alpha(n-1, k) \]  

(11)

and

\[ \tilde{W}_\alpha(n, k) = \tilde{W}_\alpha(n-1, k-1) + \alpha k \tilde{W}_\alpha(n-1, k), \]  

(12)

and the horizontal generating functions [1, Theorems 4 and 10]

\[ (t|\alpha)_n = \sum_{k=0}^{n} \tilde{w}_\alpha(n, k)x^k \]  

(13)

and

\[ x^n = \sum_{k=0}^{n} \tilde{W}_\alpha(n, k)(t|\alpha)_k, \]  

(14)

where \((t|\alpha)_n\) denotes the generalized factorial of \( t \) of increment \( \alpha \) given by

\[ (t|\alpha)_n = \prod_{i=0}^{n-1} (t - i\alpha), \quad (t|\alpha)_0 = 1. \]

In the same paper, Belbachir and Bousbaa [1] also defined translated Whitney–Lah numbers, denoted by \( \hat{w}_\alpha(n, k) \), as the number of ways to distribute the set \( \{1, 2, \ldots, n\} \) into \( k \) ordered lists such that the elements of each list can mutate with \( \alpha \) ways, except the dominant one. The values of the numbers \( \hat{w}_\alpha(n, k) \) can be computed using the recurrence relation [1, Theorem 13]
\( \hat{w}(\alpha)(n, k) = \hat{w}(\alpha)(n - 1, k - 1) + \alpha(n + k - 1)\hat{w}(\alpha)(n - 1, k) \)  
\( (t - \alpha)_n = \sum_{k=0}^{n} \hat{w}(\alpha)(n, k) (t|\alpha)_k. \)

Similar to what is observed in equation (8), the translated Whitney–Lah numbers may also be expressed as sum of products of \( \hat{w}(\alpha)(n, k) \) and \( \tilde{W}(\alpha)(n, k) \) as follows [1, Corollary 14]

\[ \hat{w}(\alpha)(n, k) = \sum_{j=k}^{n} \tilde{w}(\alpha)(n, j)\tilde{W}(\alpha)(j, k). \]

It is evident that the translated Whitney and Whitney–Lah numbers are generalizations of the Stirling and Lah numbers, respectively. This may be verified by simply setting \( \alpha = 1 \) in the defining relations of the former. Recently, Mansour et al. [16] defined the recurrence relation

\[ u(n, k) = u(n - 1, k - 1) + (a_{n-1} + b_k)u(n - 1, k) \]

for two sequences \((a_i)_{i \geq 0}\) and \((b_i)_{i \geq 0}\) with boundary conditions given by

\[ u(n, 0) = \prod_{i=0}^{n-1} (a_i + b_0), \quad u(0, k) = \delta_{0,k}, \]

where

\[ \delta_{i,j} = \begin{cases} 
0, & \text{if } i \neq j \\
1, & \text{if } i = j 
\end{cases} \]

is the Kronecker delta. Notice that if \( a_{n-1} = \alpha(n - 1) \) and \( b_k = \alpha k \), the above recurrence relation coincides with equation (15). Moreover, the following useful formula was first established in the same paper:

\[ u(n, k) = \sum_{j=0}^{k} \left( \prod_{i=0}^{n-1} (b_j + a_i) \right) / \prod_{i=0}^{n-1, i \neq j} (b_j - b_i). \]

In a later paper, Mansour et al. [17] used the identity in (19) to derive an explicit formula for a certain generalization of the translated Whitney numbers (see [17, Equation 19]). We also note of another related paper by Mansour and Shattuck [19] which provide additional insights on Lah numbers.

Now, for \( a_i = \alpha i \) and \( b_j = \alpha j \), we utilize equation (19) to obtain an explicit formula for \( \hat{w}(\alpha)(n, k) \) given in the next theorem.

**Theorem 2.1.** The translated Whitney–Lah numbers satisfy the following explicit formula:

\[ \hat{w}(\alpha)(n, k) = \frac{\alpha^{n-k}}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \langle j \rangle_n. \]

This theorem allows us to write the numbers \( \hat{w}(\alpha)(n, k) \) in a closed form similar to (1). It is implied in the proof of the succeeding corollary.
Corollary 2.1.1. The translated Whitney–Lah numbers satisfy the following relation:
\[ \hat{w}(\alpha)(n, k) = \alpha^{n-k} L(n, k). \] (21)

Proof. Since \( \langle j \rangle_n = (j + n - 1)_n \), then
\[
\hat{w}(\alpha)(n, k) = \frac{\alpha^{n-k}}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (j + n - 1)_n
\]
\[ = \frac{\alpha^{n-k} n!}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \binom{n}{n-j}. \]

From [9, Identity 5.24], it is known that the binomial coefficients satisfy the following useful identity:
\[ \sum_{j} \binom{\ell}{m+j} \binom{s+j}{n} (-1)^{j} = (-1)^{\ell+m} \binom{s-m}{n-\ell}. \] (22)

Hence, with \( m = 0, \ell = k \) and \( s = n - 1 \), we obtain
\[ \hat{w}(\alpha)(n, k) = \alpha^{n-k} \frac{n!}{k!} \binom{n-1}{n-k}. \] (23)

This completes the proof. \( \square \)

Corollary 2.1.2. The translated Whitney–Lah numbers satisfy the following exponential generating function:
\[ \sum_{n=k}^{\infty} \hat{w}(\alpha)(n, k) \frac{t^n}{n!} = \frac{1}{k!} \left( \frac{t}{1-\alpha t} \right)^k. \] (24)

Proof. Applying (20), and both the binomial and negative binomial expansions,
\[
\sum_{n=k}^{\infty} \hat{w}(\alpha)(n, k) \frac{t^n}{n!} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \sum_{n=k}^{\infty} (\alpha t)^n \binom{n}{j+n-1}
\]
\[ = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (1-\alpha t)^{-j}
\]
\[ = \frac{1}{k!} \left[ (1-\alpha t)^{-1} - 1 \right]^k
\]
\[ = \frac{1}{k!} \left( \frac{t}{1-\alpha t} \right)^k. \] \( \square \)

Clearly, the results shown in the previous corollaries give back identities (1) and (3) for the classical Lah numbers when \( \alpha = 1 \). The binomial identity in (22) can also be utilized to derive another interesting formula for the translated Whitney–Lah numbers. By setting \( s = n, \ell = k - 1 \) and \( m = -1 \),
\[ \sum_{j=1}^{k} \binom{k-1}{j-1} \binom{n+j}{n} (-1)^{j} = (-1)^{k-2} \binom{n+1}{n-k+1}. \]

Multiplying both sides by \( k! \) gives
\[ \sum_{j=1}^{k} \binom{k-1}{j-1} \binom{n+j}{n} (-1)^{j} = \sum_{j=1}^{k} \hat{w}(\alpha)(k, j) \frac{(n+j)!}{n! \alpha^{k-j}} (-1)^{j} \]
in the left-hand side after using (23).
On the other hand, the right-hand side simply becomes
\[
(-1)^{k-2} \binom{n+1}{n-k+1} = (-1)^k \frac{(n+1)!}{(n-k+1)!}.
\]
Thus, we have derived the following theorem:

**Theorem 2.2.** For \( k \geq 2 \) and \( n \geq k - 1 \), the translated Whitney–Lah numbers satisfy
\[
\sum_{j=1}^{k} (-\alpha)^j \tilde{w}(\alpha)(k, j)(n + j)! = (-\alpha)^k \frac{n!(n+1)!}{(n-k+1)!}.
\]

When \( \alpha = 1 \), we immediately recognize
\[
\sum_{j=1}^{k} (-1)^j L(k, j)(n + j)! = (-1)^k \frac{n!(n+1)!}{(n-k+1)!},
\]
an identity for the classical Lah numbers which was proved using six different methods by Guo and Qi [10]. A more direct approach in establishing (25) is as follows.

**Alternative proof of Theorem 2.2.** The generating function in (16) may be rewritten as
\[
(-\alpha)^k (-t)_k = \sum_{j=0}^{k} \alpha^j \tilde{w}(\alpha)(k, j)(t)_j.
\]
Since \((-n-1)_j n! = (-1)^j (n + j)!\), then replacing \( t \) with \(-n-1\) in the previous equation gives
\[
(-\alpha)^k n!(n + 1)_k = \sum_{j=0}^{k} (-\alpha)^j \tilde{w}(\alpha)(k, j)(n + j)!
\]
as desired.

We now proceed to deriving a generalization of the Bell number formula in (10). In the paper of Qi [21], two methods to prove (10) are presented. The first one employs the Faa di Bruno’s formula and the \( n \)-th derivative of the exponential function \( e^{\pm1/x} \) given by
\[
(e^{\pm1/x})^{(n)} = (-1)^n e^{\pm1/x} \sum_{k=1}^{n} (\pm1)^k L(n, k) \frac{1}{tn+k}
\]
found in the paper of Daboud et al. [7]. The second is less complicated and requires only the use of the inverse relation
\[
f_n = \sum_{j=0}^{n} \left[ \frac{n}{j} \right] g_j \iff g_n = \sum_{j=0}^{n} (-1)^{n-j} \left\{ \frac{n}{j} \right\} f_j.
\]
To obtain our next objective, we adopt a process that is similar to the latter since by using the orthogonal relations [13, Corollary 4.2]
\[
\sum_{j=m}^{n} (-1)^{j-m} \tilde{W}(\alpha)(n, j) \tilde{w}(\alpha)(j, m) = \sum_{j=m}^{n} (-1)^{n-j} \tilde{w}(\alpha)(n, j) \tilde{W}(\alpha)(j, m) = \delta_{m,n},
\]
it can be easily shown that the following inverse relation for the translated Whitney numbers of the first kind is valid:
\[
f_n = \sum_{j=0}^{n} \tilde{w}(\alpha)(n, j) g_j \iff g_n = \sum_{j=0}^{n} (-1)^{n-j} \tilde{w}(\alpha)(n, j) f_j.
\]
Now, taking \( g_j = \tilde{W}_{(\alpha)}(j, k) \) and \( f_n = \hat{w}_{(\alpha)}(n, k) \), we can apply the above inverse relation to (17) to get

\[
\tilde{W}_{(\alpha)}(n, k) = \sum_{j=0}^{n} (-1)^{n-j} \tilde{W}_{(\alpha)}(n, j) \hat{w}_{(\alpha)}(j, k).
\] (30)

We then recall that the translated Dowling numbers [15], denoted by \( D_{(\alpha)}(n) \), are defined as the sum of the translated Whitney numbers of the second kind, i.e.

\[
D_{(\alpha)}(n) = \sum_{k=0}^{n} \tilde{W}_{(\alpha)}(n, k).
\] (31)

So by summing both sides of (30) up to \( n \) and applying (31),

\[
D_{(\alpha)}(n) = \sum_{k=0}^{n} \sum_{j=0}^{n} (-1)^{n-j} \tilde{W}_{(\alpha)}(n, j) \hat{w}_{(\alpha)}(j, k).
\]

Thus, we have proved the result in the next theorem.

**Theorem 2.3.** The translated Dowling numbers satisfy the explicit formula given by

\[
D_{(\alpha)}(n) = \sum_{j=0}^{n} (-1)^{n-j} \left( \sum_{k=0}^{j} \hat{w}_{(\alpha)}(j, k) \right) \tilde{W}_{(\alpha)}(n, j).
\] (32)

To close this section, notice that by (21), we may write

\[
D_{(\alpha)}(n) = \sum_{j=0}^{n} (-1)^{n-j} \left( \sum_{k=0}^{j} \alpha^{j-k} L(j, k) \right) \tilde{W}_{(\alpha)}(n, j).
\]

Since it is known that [13, 15] \( \tilde{W}_{(1)}(n, j) = \{n\} \) and \( D_{(1)}(n) = B_n \), it means that the formula in (32) reduces to the one in (10) when \( \alpha = 1 \). Moreover, we acknowledge a generalization of (32) that can be seen in the paper of Corcino et al. [6]. The result in the said paper involves an explicit formula for the \((r, \beta)\)-Bell numbers (or \(r\)-Dowling numbers). Readers are also directed to another paper by Corcino et al. [5] which contain more related results.

### 3 Translated \(q\)-Whitney–Lah numbers

Let \([n]_q\) denote the \(q\)-analogue of an integer \(n\) defined by

\[
[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + q^2 + \ldots + q^{n-1}
\]

and let \([t|\alpha]_n\) denote the product

\[
[t|\alpha]_n = \prod_{i=0}^{n-1} [t - i\alpha]_q.
\]

The translated \(q\)-Whitney numbers of the first and second kinds [14], denoted by \(w_{(\alpha)}^1[n, k]_q\) and \(w_{(\alpha)}^2[n, k]_q\), respectively, are defined in terms of the following horizontal generating functions:
Various combinatorial properties of the numbers $w_1(\alpha)[n,k]_q$ and $w_2(\alpha)[n,k]_q$ and a certain combinatorial interpretation in the context of A-tableaux have already been established in the same paper. The properties include the inverse relation \[ f_n = \sum_{j=0}^{n} w_1(\alpha)[n,j]_q g_j \quad \iff \quad g_n = \sum_{j=0}^{n} w_2(\alpha)[n,j]_q f_j. \] (35)

In general, the term “$q$-analogue” refers to a mathematical expression in terms of a parameter $q$ such that as $q \to 1$, it reduces to a known identity or formula. For instance, it is clear that \[ \lim_{q \to 1} [n]_q! = n. \]

Other examples are the $q$-binomial coefficient \[ \binom{n}{k}_q = \prod_{j=1}^{k} \frac{q^{n-j+1} - 1}{q^j - 1} = \frac{[n]_q!}{[k]_q![n-k]_q!}, \] and the $q$-falling factorial of $n$ of order $k \[ [n]_{q,k} = \prod_{j=0}^{k-1} \frac{q^{n-j} - 1}{q - 1} = \frac{[n]_q!}{[n-k]_q!}, \]

where $[n]_q! = \prod_{i=1}^{n} [i]_q$ is the $q$-factorial of $n$. See for instance the following limits which are easy to verify:

\[ \lim_{q \to 1} [n]_q! = n!, \quad \lim_{q \to 1} \binom{n}{k}_q = \binom{n}{k}, \quad \lim_{q \to 1} [n]_{q,k} = (n)_k. \]

The book of Kac and Cheung [11] is a rich source for further discussions on $q$-analogues. The study of $q$-analogues of mathematical identities has been the interest of many mathematicians over a long period of time. For the case of the Lah numbers, Lindsay et al. [12] defined a $q$-analogue $L_q(n,k)$ in terms of the following relation:

\[ t(t + [1]_q) \cdots (t + [n-1]_q) = \sum_{k=0}^{n} L_q(n,k) t(t - [k]_q) \cdots (t - [k - 1]_q). \] (36)

An earlier $q$-analogue of the Lah numbers can be attributed to Garsia and Remmel [8] who defined the $q$-Lah numbers, denoted by $L_q(n,k)$, as

\[ [t]_q[t + 1]_q \cdots [t + n - 1]_q = \sum_{k=0}^{n} L_q(n,k)[t]_q[t - 1]_q \cdots [t - k + 1]_q \] (37)

with the recurrence relation

\[ L_q(n+1,k) = q^{n+k-1} L_q(n,k - 1) + [n + k]_q L_q(n,k) \] (38)
and explicit formula
\[ L_q(n, k) = \binom{n}{k} \frac{[n-1]!}{[k-1]!} q^{k(k-1)}. \] (39)

A more general notion was also introduced in [14, Equation 15] called the translated \( q \)-Whitney numbers of the third kind, denoted by \( L_{(\alpha)}[n, k]_q \), which are defined as coefficients in the expansion of

\[ [t] - \alpha \] = \sum_{k=0}^{n} L_{(\alpha)}[n, k]_q [t] \alpha[k]. \] (40)

These numbers can be computed recursively using the formula [14, Equation 31]

\[ L_{(\alpha)}[n + 1, k]_q = q^{\alpha(n + k - 1)} L_{(\alpha)}[n, k - 1]_q + [\alpha(n + k)]_q L_{(\alpha)}[n, k]_q. \] (41)

Looking at equations (38) and (41), it is easy to see that \( L_{(1)}[n, k]_q = L_q(n, k) \).

**Theorem 3.1.** The numbers \( L_{(\alpha)}[n, k]_q \) satisfy the following:

\[ L_{(\alpha)}[n, k]_q = \sum_{j=0}^{n} w^1_{(-\alpha)}[n, j]_q w^2_{(\alpha)}[j, k]_q. \] (42)

**Proof.** Putting \(-\alpha\) in place of \(\alpha\) in (33) and by applying (34),

\[ [t] - \alpha \] = \sum_{k=0}^{n} w^1_{(-\alpha)}[n, k]_q [t]_q^k

\[ = \sum_{j=0}^{n} \left\{ \sum_{k=j}^{n} w^1_{(-\alpha)}[n, k]_q w^2_{(\alpha)}[k, j]_q \right\} [t]_q^j. \]

Comparing the coefficients of \([t]_q^j\) in the last equation with that of (40) gives the desired result. \(\square\)

The identity in the previous theorem suggests that the numbers \( L_{(\alpha)}[n, k]_q \) may be referred to as the translated \( q \)-Whitney–Lah numbers. To establish an explicit formula, we will use a method different from the one used in the previous section. We start by rewriting (40) into the form

\[ [\alpha k] - \alpha \] = \sum_{j=0}^{n} L_{(\alpha)}[n, j]_q [\alpha k]_q \alpha[j] \]

\[ = \sum_{j=0}^{k} \binom{k}{j} q^\alpha \left\{ \frac{L_{(\alpha)}[n, j]_q [\alpha k]_q \alpha[j]}{\binom{k}{j} q^\alpha} \right\}. \]

Since the well-known \( q \)-binomial inversion formula can be expressed as

\[ f_k = \sum_{j=0}^{k} \binom{k}{j} g_j \iff g_k = \sum_{j=0}^{k} (-1)^{k-j} q^{\alpha(k-j)} \binom{k}{j} q^\alpha f_j, \] (43)

then with \( f_k = [\alpha k] - \alpha \) and \( g_j = \frac{L_{(\alpha)}[n, j]_q [\alpha k]_q \alpha[j]}{\binom{k}{j} q^\alpha} \), we get

\[ [\alpha k]_q L_{(\alpha)}[n, k]_q = \sum_{j=0}^{k} (-1)^{k-j} q^{\alpha(k-j)} \binom{k}{j} q^\alpha [\alpha j] - \alpha[n], \]

the result in the next theorem.
Theorem 3.2. The translated $q$-Whitney–Lah numbers satisfy the following explicit formula:

$$L_{(\alpha)}[n, k]_q = \frac{1}{[n]_{q^n}![\alpha]_q^k} \sum_{j=0}^{k} (-1)^{k-j} q^\alpha \binom{k}{j} \binom{k}{j} [\alpha j]! - [\alpha]_q^k. \tag{44}$$

Formula (44) is a $q$-analogue of the explicit formula in (20) since

$$\lim_{q \to 1} [k]_{q^n}! = k!, \quad \lim_{q \to 1} [\alpha j]_n = [\alpha]_q^n \binom{j}{n}. \tag{45}$$

and

$$\lim_{q \to 1} L_{(\alpha)}[n, k]_q = \lim_{q \to 1} \left( \frac{1}{[k]_{q^n}! [\alpha]_q^k} \sum_{j=0}^{k} (-1)^{k-j} q^\alpha \binom{k}{j} \binom{k}{j} [\alpha j]! - [\alpha]_q^k \right) = \frac{\alpha^{n-k}}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \binom{j}{n}. \tag{46}$$

Furthermore, we may use the above explicit formula in establishing a kind of exponential generating function for the numbers $L_{(\alpha)}[n, k]_q$. But before proceeding, we first mention the following useful identities:

$$[\alpha j]! - [\alpha]_q^n [j + n - 1]_{q^n}! \cdot \frac{[j + n - 1]_{q^n}!}{[n]_{q^n}!} = \binom{j + n - 1}{n}_{q^n}. \tag{47}$$

and

$$\prod_{k=0}^{n-1} \frac{1}{1 - q^k t} = \sum_{k=0}^{\infty} \binom{n + k - 1}{k}_{q^n} t^k. \tag{48}$$

Corollary 3.2.1. The translated $q$-Whitney–Lah numbers satisfy the following exponential generating function:

$$\sum_{n=0}^{\infty} L_{(\alpha)}[n, k]_q \frac{t^n}{[n]_{q^n}!} = \frac{1}{[k]_{q^n}! [\alpha]_q^k} \sum_{j=0}^{k} (-1)^{k-j} q^\alpha \binom{k}{j} \binom{k}{j} [\alpha j]! \sum_{n=0}^{\infty} \frac{[j + n - 1]_{q^n}!}{[n]_{q^n}!} \binom{j + n - 1}{n}_{q^n}. \tag{49}$$

Proof. From equations (44) and (45), we have

$$\sum_{n=0}^{\infty} L_{(\alpha)}[n, k]_q \frac{t^n}{[n]_{q^n}!} = \frac{1}{[k]_{q^n}! [\alpha]_q^k} \sum_{j=0}^{k} (-1)^{k-j} q^\alpha \binom{k}{j} \binom{k}{j} [\alpha j]! \sum_{n=0}^{\infty} \frac{[j + n - 1]_{q^n}!}{[n]_{q^n}!} \binom{j + n - 1}{n}_{q^n}. \tag{50}$$

The result is obtained by applying (46) in the second summation. \hfill \Box

By taking the limit of (47) as $q \to 1$,

$$\lim_{q \to 1} \sum_{n=0}^{\infty} L_{(\alpha)}[n, k]_q \frac{t^n}{[n]_{q^n}!} = \frac{1}{\alpha^k k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \left( \frac{1}{1 - \alpha t} \right)^j,$$

which in turn simplifies to (24). On the other hand, the next theorem contains a $q$-analogue of (25).

Theorem 3.3. The translated $q$-Whitney–Lah numbers satisfy the following:

$$\sum_{j=0}^{k} (-[\alpha]_q j) q^{-nj-\binom{j+1}{2}} L_{(\alpha)}[k, j]_q [n + j]_{q^n}! = \frac{(-[\alpha]_q)^k [n]_{q^n}![n + 1]_{q^n}!}{[n - k + 1]_{q^n}!}. \tag{51}$$
Proof. The proof is somewhat parallel to the alternative proof of Theorem 2.2. We proceed by rewriting (40) as
\[
-\alpha k q^k = \sum_{j=0}^{k-1} [\alpha]_q^j L_{(\alpha)}[k, j] q^j \prod_{i=0}^{j-1} [t - i]_q^{\alpha}.
\] (49)
We put \(-n - 1\) in place of \(t\) and multiply both sides by \([n]_q\) so that the left-hand side becomes
\[
-\alpha k q^k \prod_{i=0}^{k-1} [n + 1 - i]_q^n [n]_q! = -\alpha k [n]_q! [n + 1]_q^k
\]
while the right-hand side is
\[
\sum_{j=0}^{k} [\alpha]_q^j L_{(\alpha)}[k, j] [n]_q^n \prod_{i=0}^{j-1} [t - i]_q^{\alpha} = \sum_{j=0}^{k} (-1)^j L_{(\alpha)}[n, j]_q [n + j]_q^j.
\]
where the identity \(j(n + 1) + \binom{j}{2} = nj + \binom{j+1}{2}\) is used. Combining these equations give the desired result.

The corollary below is a direct consequence of (48) when we set \(\alpha = 1\). This formula is a \(q\)-analogue of Guo and Qi’s [10] identity in (26) which can easily be verified by taking the limit as \(q \to 1\).

Corollary 3.3.1. The \(q\)-Lah numbers satisfy
\[
\sum_{j=0}^{k} (-1)^j q^{-nj - \binom{j+1}{2}} L_{q}(k, j) [n + j]_q! = \frac{(-1)^k [n]_q! [n + 1]_q}{[n - k + 1]_q}.
\] (50)

The translated \(q\)-Dowling numbers [14], denoted by \(D_{(\alpha)}[n]_q\), are defined by the following sum:
\[
D_{(\alpha)}[n]_q = \sum_{k=0}^{n} w_{(\alpha)}^2[n, k]_q.
\] (51)
The last theorem presents a \(q\)-analogue of the explicit formula in (32).

**Theorem 3.4.** The translated \(q\)-Dowling numbers satisfy the following explicit formula
\[
D_{(\alpha)}[n]_q = \sum_{j=0}^{n} \left( \sum_{j=0}^{k} L_{(\alpha)}[j, k]_q \right) w_{(\alpha)}^2[n, j]_q.
\] (52)

**Proof.** We put \(-\alpha\) in place of \(\alpha\), and set \(g_j = w_{(\alpha)}^2[j, k]_q\) and \(f_n = L_{(\alpha)}[n, k]_q\) in the inverse relation in (35) so that when the resulting relation is applied to (42),
\[
w_{(\alpha)}^2[n, k]_q = \sum_{j=0}^{n} w_{(-\alpha)}^2[n, j]_q L_{(\alpha)}[j, k]_q.
\]
The desired result is obtained by summing over up to \(n\). □
The explicit formula [15, Equation 10]
\[ \tilde{W}(\alpha)(n, k) = \frac{1}{\alpha^k k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (\alpha^j)^n \]
shows that \( \tilde{W}(-\alpha)(n, k) = (-1)^{n-k} \tilde{W}(\alpha)(n, k) \). Hence,
\[
\lim_{q \to 1} D_{\alpha}(n) = \lim_{q \to 1} \sum_{j=0}^{n} \left( \sum_{j=0}^{k} L_{(\alpha)}[j, k]_q \right) w_{(\alpha)}^2 [n, j] q^{n-j} = \sum_{j=0}^{n} (-1)^{n-j} \left( \sum_{k=0}^{j} \hat{w}_{(\alpha)}(j, k) \right) \tilde{W}(\alpha)(n, j)
\]
which is precisely (32). A similar formula for a \( q \)-analogue of the \( r \)-Dowling numbers can be seen in the paper of Cillar and Corcino [3]. However, since the definitions of their \( q \)-analogue and ours are distinctly motivated, it is difficult to say that their result is a generalization of the one in Theorem 3.4.

As we end, it may be worthwhile to say that the present paper was not able to express the explicit formula of \( L_{(\alpha)}[n, k]_q \) in a way similar to that of (23) for the case of \( \hat{w}_{(\alpha)}(n, k) \). Perhaps this can be done by establishing a \( q \)-analogue of the binomial identity in (22) and use it to simplify the right-hand side of the explicit formula in (44).

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