SOME EXAMPLES OF ISOTROPIC $SL(2, \mathbb{R})$-INVARIANT SUBBUNDLES OF THE HODGE BUNDLE

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ABSTRACT. We construct some examples of origamis (square-tiled surfaces) such that the Hodge bundles over the corresponding $SL(2, \mathbb{R})$-orbits on the moduli space admit non-trivial isotropic $SL(2, \mathbb{R})$-invariant subbundles. This answers a question posed to the authors by A. Eskin and G. Forni.

CONTENTS

1. Introduction 1
2. Preliminaries 3
3. Isotropic $SL(2, \mathbb{R})$-invariant subbundles of Hodge bundle 5
Appendix A. A picture of the origami $X$ 20
Acknowledgements 21
References 21

1. INTRODUCTION

The investigation of $SL(2, \mathbb{R})$-invariant probabilities on moduli spaces of Abelian (and/or quadratic) differentials on Riemann surfaces is a fascinating subject whose applications nowadays include: the description of deviations of ergodic averages of interval exchange transformations, translation flows and billiards (see [Z1] and [F1]), the confirmation of a conjecture of the physicists J. Hardy and J. Weber on the abnormal rate of diffusion of trajectories on typical realisations of Ehrenfest’s wind-tree model of Lorenz gases (see [DHL]), and the classification of the commensurability classes of all presently known non-arithmetic ball quotients (see [KM]).

After the recent breakthrough work of A. Eskin and M. Mirzakhani [EM], we have a better understanding of the geometry of $SL(2, \mathbb{R})$-invariant probabilities on moduli spaces of Abelian differentials. Indeed, A. Eskin and M. Mirzakhani showed the “Ratner theory like statement” that such measures are always supported on affine suborbifolds of the moduli space. In
their (long) way to prove this statement, they employed several different arguments inspired from several sources such as the low entropy method of M. Einsiedler, A. Katok and E. Lindenstrauss [EKL] and the exponential drift argument of Y. Benoist and J. F. Quint [BQ]. Moreover, as an important preparatory step for the exponential drift argument, A. Eskin and M. Mirzakhani showed the semisimplicity of the so-called Kontsevich-Zorich cocycle on the Hodge bundle.

The derivation in [EM] of this semisimplicity property uses the work of G. Forni [F1] (see also [FMZ2]) and the study of symplectic and isotropic $SL(2, \mathbb{R})$-invariant subbundles of the Hodge bundle. Interestingly enough, while symplectic $SL(2, \mathbb{R})$-invariant subbundles occur in several known examples (see, e.g., [BM], [EKZ], [FMZ2] and [FMZ3]), it is not so easy to put the hands on concrete examples of isotropic $SL(2, \mathbb{R})$-invariant subbundles. In fact, the only “clue” one had so far was that isotropic $SL(2, \mathbb{R})$-invariant subbundles are confined to the so-called Forni subbundle of the Hodge bundle (see [EM]).

Partly motivated by the scenario of the previous paragraph, A. Eskin and G. Forni (independently) asked us about the actual existence of concrete examples of $SL(2, \mathbb{R})$-invariant probabilities on some moduli spaces of Abelian differentials whose Hodge bundles admit non-trivial $SL(2, \mathbb{R})$-invariant isotropic subbundles.

In this note, we answer affirmatively the question of A. Eskin and G. Forni by exhibiting concrete square-tiled surfaces (origamis) such that the Hodge bundle over their $SL(2, \mathbb{R})$-orbits have non-trivial $SL(2, \mathbb{R})$-invariant isotropic subbundles. The idea of our construction is very simple:

- we start with the so-called Eierlegende Wollmilchsau origami (see [F2] and [HS]); as it was shown in the [MY], the Kontsevich-Zorich cocycle over the $SL(2, \mathbb{R})$-orbit $SL(2, \mathbb{R}) \cdot (M_3, \omega_3)$ of the Eierlegende Wollmilchsau $(M_3, \omega_3)$ acts by a (very explicit) finite group of symplectic matrices on a certain $SL(2, \mathbb{R})$-invariant subbundle of the Hodge bundle over $\mathcal{C} = SL(2, \mathbb{R}) \cdot (M_3, \omega_3)$;
- in particular, by taking an adequate abstract finite cover $\hat{\mathcal{C}}$ of $\mathcal{C}$, one eventually gets that the lift of the Kontsevich-Zorich cocycle to $\hat{\mathcal{C}}$ acts trivially on a certain $SL(2, \mathbb{R})$-invariant subbundle $H$ of the Hodge bundle over $\hat{\mathcal{C}}$; it follows that any 1-dimensional (equivariant) subbundle of $H$ is an isotropic $SL(2, \mathbb{R})$-invariant subbundle of the Hodge bundle over $\hat{\mathcal{C}}$; however, there is no a priori reason that $\hat{\mathcal{C}}$ corresponds to the support of a $SL(2, \mathbb{R})$-invariant probability in some moduli space of Abelian differentials;
- to overcome the difficulty related to the fact that $\hat{\mathcal{C}}$, we use the results of [S] (see also [EllMcR]) to show that some of the abstract
covers \( \hat{C} \) described in the previous item can be realised as \( SL(2, \mathbb{R}) \)-
orbits of certain square-tiled surfaces.

We organise this note as follows. In the next section, we will briefly recall
the basic notions of translation surfaces, square-tiled surfaces, Kontsevich-
Zorich cocycle and affine diffeomorphisms. Then, in the last section, we
state and prove our main results, namely, Theorem 1 and 2 answering to
the question of A. Eskin and G. Forni. Finally, we depict in Appendix A
our “smallest” origami satisfying the conclusions of Theorem 1.

2. Preliminaries

The basic references for this entire section are the surveys of A. Zorich [Z2], and P. Hubert and T. Schmid [HuSc]. Also, the reader may find useful
to consult the introduction of the article [MY] for further comments on
the relationship between the Kontsevich-Zorich cocycle and the action on
homology of affine diffeomorphisms of translation surfaces.

A translation surface is the data \((M, \omega)\) of a non-trivial Abelian differential \(\omega\) on a Riemann surface \(M\). This nomenclature comes from the fact
that the local primitives of \(\omega\) outside the set \(\Sigma\) of its zeroes provides an
atlas on \(M - \Sigma\) whose changes of coordinates are all translations of the
plane \(\mathbb{R}^2\). In the literature, these charts are called translation charts and an
atlas formed by translation charts is called translation atlas or translation (surface) structure. For later use, we define the area \(a(M, \omega)\) of \((M, \omega)\) as
\[ a(M, \omega) := \frac{i}{2} \int_M \omega \wedge \bar{\omega}. \]

The Teichmüller space \(\hat{\mathcal{H}}_g\) of unit area Abelian differentials of genus \(g \geq 1\) is the set of unit area translation surfaces \((M, \omega)\) of genus \(g \geq 1\)
modulo the natural action of the group \(\text{Diff}^+_0(M)\) of orientation-preserving homeomorphisms of \(M\) isotopic to the identity, where \(M\) is a fixed topological surface. The moduli space \(\mathcal{H}_g\) of unit area Abelian differentials of genus \(g \geq 1\) is the set of unit area translation surfaces \((M, \omega)\) of genus \(g \geq 1\)
modulo the natural action of the group \(\text{Diff}^+(M)\) of orientation-preserving homeomorphisms of \(M\), where \(M\) is a fixed topological surface. In particular, \(\mathcal{H}_g = \hat{\mathcal{H}}_g / \Gamma_g\) where \(\Gamma_g := \text{Diff}^+(M)/\text{Diff}^+_0(M)\) is the mapping class group (of isotopy classes of orientation-preserving homeomorphisms of \(M\)).

The point of view of translation structures is useful because it makes clear
that \(SL(2, \mathbb{R})\) acts on the set of Abelian differentials \((M, \omega)\): indeed, given
\(h \in SL(2, \mathbb{R})\), we define \(h \cdot (M, \omega)\) as the translation surface whose
translation charts are given by post-composing the translation charts of \((M, \omega)\)
with \(h\). This action of \(SL(2, \mathbb{R})\) descends to \(\hat{\mathcal{H}}_g\) and \(\mathcal{H}_g\). The action of the
diagonal subgroup $g_t := \text{diag}(e^t, e^{-t})$ of $SL(2, \mathbb{R})$ is the so-called Teichmüller (geodesic) flow.

**Remark 2.1.** By collecting together unit area Abelian differentials with orders of zeroes prescribed by a list $\kappa = (k_1, \ldots, k_s)$ of positive integers with $\sum k_n = 2g - 2$, we obtain a subset $\mathcal{H}(\kappa)$ of $\mathcal{H}_g$ called stratum in the literature. From the definition of the $SL(2, \mathbb{R})$-action on $\mathcal{H}_g$, it is not hard to check that the strata $\mathcal{H}(\kappa)$ are $SL(2, \mathbb{R})$-invariant.

The Hodge bundle $H^1_\theta$ over $\mathcal{H}_g$ is the quotient of the trivial bundle $\hat{\mathcal{H}}_g \times H_1(M, \mathbb{R})$ by the natural action of the mapping-class group $\Gamma_g$ on both factors. In this language, the Kontsevich-Zorich cocycle $G_{t}^{KZ}$ is the quotient of the trivial cocycle $G_{t}^{KZ} : \hat{\mathcal{H}}_g \times H_1(M, \mathbb{R}) \to \hat{\mathcal{H}}_g \times H_1(M, \mathbb{R})$

$G_{t}^{KZ}(\omega, [c]) = (g_t(\omega), [c])$

by the mapping-class group $\Gamma_g$. In the sequel, we will call $G_{t}^{KZ}$ as KZ cocycle for short.

For the sake of this note, let us restrict ourselves to the class of translation surfaces $(M, \omega)$ covering (with at most one ramification point) the square flat torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ equipped with the Abelian differential induced by $dz$ on $\mathbb{C} = \mathbb{R}^2$. In the literature, these translation surfaces $(M, \omega)$ are called square-tiled surfaces or origamis.

The stabiliser $SL(M, \omega)$ – also known as Veech group – of a square-tiled surface $(M, \omega) \in \mathcal{H}_g$ with respect to the action of $SL(2, \mathbb{R})$ is commensurable to $SL(2, \mathbb{Z})$, and its $SL(2, \mathbb{R})$-orbit is a closed subset of $\mathcal{H}_g$ isomorphic to the unit cotangent bundle $SL(2, \mathbb{R})/SL(M, \omega)$ of the hyperbolic surface $\mathbb{H}/SL(M, \omega)$.

The Veech group $SL(M, \omega)$ consists of the “derivatives” (linear parts) of all affine diffeomorphisms of $(M, \omega)$, that is, the orientation-preserving homeomorphisms of $M$ fixing the set $\Sigma$ of zeroes of $\omega$ whose local expressions in the translation charts of $(M, \omega)$ are affine maps of the plane. The group of affine diffeomorphisms of $(M, \omega)$ is denoted by $\text{Aff}(M, \omega)$ and it is possible to show that $\text{Aff}(M, \omega)$ is precisely the subgroup of elements of $\Gamma_g$ stabilising $SL(2, \mathbb{R}) \cdot (M, \omega)$ in $\mathcal{H}_g$. The Veech group and the affine diffeomorphisms group are part of the following exact sequence

$\{id\} \to \text{Aut}(M, \omega) \to \text{Aff}(M, \omega) \to SL(M, \omega) \to \{id\}$

where, by definition, $\text{Aut}(M, \omega)$ is the subgroup of automorphisms of $(M, \omega)$, i.e., the subgroup of elements of $\text{Aff}(M, \omega)$ whose linear part is trivial (i.e., identity).

In this language, the KZ cocycle on the Hodge bundle over the $SL(2, \mathbb{R})$-orbit of $(M, \omega)$ is intimately related to the action on homology of $\text{Aff}(M, \omega)$. Indeed, since $\text{Aff}(M, \omega) \subset \Gamma_g$ is the stabiliser of $SL(2, \mathbb{R}) \cdot (M, \omega)$ in
\( \mathcal{H}_g = \hat{\mathcal{H}}_g / \Gamma_g \), we have that the KZ cocycle is the quotient of the trivial cocycle

\[
g_t \times id : \hat{\mathcal{H}}_g \times H_1(M, \mathbb{R}) \to \hat{\mathcal{H}}_g \times H_1(M, \mathbb{R})
\]

by \( \text{Aff}(M, \omega) \).

For later use, we observe that, given a square-tiled surface \( p : (M, \omega) \to (\mathbb{T}^2, dz) \) (where \( p \) is a finite cover ramified precisely over \( 0 \in \mathbb{T}^2 \)), the KZ cocycle, or equivalently \( \text{Aff}(M, \omega) \), preserves the decomposition

\[
H_1(M, \mathbb{R}) = H_1^\text{st} + H_1^{(0)}(M, \mathbb{R}),
\]

where \( H_1^{(0)}(M, \mathbb{R}) := \text{Ker}(p_*) \) and \( H_1^\text{st} := (p_*)^{-1}(H_1(\mathbb{T}^2, \mathbb{R})) \). Here we denote by \( (p_*)^{-1}(H_1(\mathbb{T}^2, \mathbb{R})) \) the isomorphic preimage of \( H_1(\mathbb{T}^2, \mathbb{R}) \) via \( p_* \) which is the orthogonal to \( H_1^{(0)}(M, \mathbb{R}) \) with respect to the intersection form.

Closing this preliminary section, we recall that, given a finite ramified covering \( X_1 \to X_2 \) of Riemann surfaces, the ramification data of a point \( p \in X_2 \) is the list of ramification indices of all pre-images of \( p \) counted with multiplicities.

### 3. ISOTROPIC SL(2, \mathbb{R})-INVARIANT SUBBUNDLES OF HODGE BUNDLE

This section is divided into three parts. In Subsection 3.1, we will show an “abstract” criterion (cf. Proposition 2 below) leading to square-tiled surfaces such that the Hodge bundle over their \( SL(2, \mathbb{R}) \)-orbits have \( SL(2, \mathbb{R}) \)-invariant isotropic subbundles. Then, in Subsection 3.2, we will use the “abstract” criterion to exhibit (cf. Definition 1) a square-tiled surface with genus 15 and 512 squares covering the special \( (\text{Eierlegende Wollmilchsau}) \) origami \( (M_3, \omega_{(3)}) \) such that the Hodge bundle over its \( SL(2, \mathbb{R}) \)-orbit has \( SL(2, \mathbb{R}) \)-invariant isotropic subbundles. Finally, in Subsection 3.3, we present an infinite family of origamis with this property. They come from coverings to an origami \( (M_4, \omega_{(4)}) \) also called \( \text{ornithorynque origami} \) having similar properties as \( (M_3, \omega_{(3)}) \).

#### 3.1. “Abstract” examples of \( SL(2, \mathbb{R}) \)-invariant isotropic subbundles.

Let \( (M_3, \omega_{(3)}) \) be the \( \text{Eierlegende Wollmilchsau} \) origami, that is, the translation surface associated to the Riemann surface \( M_3 \) defined by the algebraic equation

\[
\{ y^4 = x^3 - x \}
\]

and the Abelian differential \( \omega_{(3)} = c^{-1} dx/y^2 \) where \( c = \int_1^\infty \frac{dx}{\sqrt{x^3-x}} = \frac{(\Gamma(1/4))^2}{2\sqrt{2\pi}} \). Note that \( M_3 \) is a genus 3 Riemann surface and \( \omega_{(3)} \) has 4 simple zeroes at the points \( x_1, \ldots, x_4 \in M_3 \) over \( 0, 1, -1, \infty \).
The square-tiled surface structure on \((M_3, \omega_{(3)})\) becomes apparent from the natural covering \(h(x, y) = (x, y^2) = (x, z)\) mapping \((M_3, \omega_{(3)})\) into the torus (elliptic curve)

\[ T = \{ z^2 = x^3 - x \} \]
equipped with the (unit area) Abelian differential \(c^{-1}dx/z\).

The KZ cocycle over the \(SL(2, \mathbb{R})\)-orbit of \((M_3, \omega_{(3)})\), or more precisely, the homological action of the group \(\text{Aff}(M_3, \omega_{(3)})\) of affine diffeomorphisms of \((M_3, \omega_{(3)})\) was analysed in details in [MY]. Here, we will need the following fact proved there. Denote by \(\text{Aff}^1(M_3, \omega_{(3)})\) the subgroup of affine elements fixing each zero of \(\omega_{(3)}\) and consider the canonical derivative morphism from \(\text{Aff}(M_3, \omega_{(3)})\) to the Veech group \(SL(2, \mathbb{Z})\) of \((M_3, \omega_{(3)})\). In this setting, let \(\Gamma_{EW}(4)\) be the subgroup consisting of elements of \(\text{Aff}^1(M_3, \omega_{(3)})\) whose image in \(SL(2, \mathbb{Z})\) under the derivative morphism belong to the principal congruence subgroup \(\Gamma(4)\) of level 4 of \(SL(2, \mathbb{Z})\).

**Proposition 1** (cf. Lemma 2.8 of [MY]). *The elements of \(\Gamma_{EW}(4)\) act by \(\pm \text{id}\) on \(H^1_1(M_3, \mathbb{R})\).*

Using this fact, one can build examples of \(SL(2, \mathbb{R})\)-invariant isotropic subbundles of the Hodge bundle by taking adequate finite (ramified) coverings of the Eierlegende Wollmilchsau \((M_3, \omega_{(3)})\) (see Figure 1).

More precisely, the Eierlegende Wollmilchsau \((M_3, \omega_{(3)})\) can be thought as the collection of 8 unit squares \(sq(g)\) indexed by the elements \(g \in Q = \{ \pm 1, \pm i, \pm j, \pm k \}\) of quaternion group glued together by the following rule: the right-hand side of \(sq(g)\) is glued by translation to the left-hand side of \(sq(gi)\) and the top side of \(sq(g)\) is glued by translation to the bottom side of \(sq(gj)\). In this language, we have a natural projection \(\pi : (M_3, \omega_{(3)}) \to \mathbb{R}^2/\mathbb{Z}^2\) of degree 8.

Consider now \(p : (X, \omega) \to (M_3, \omega_{(3)})\) a finite ramified covering and let’s denote by \(q : (X, \omega) \to E = \mathbb{R}^2/\mathbb{Z}^2\) the natural covering \(q = \pi \circ p\).

The next proposition puts constraints on the group \(\text{Aff}(X, \omega)\) of affine diffeomorphisms of \((X, \omega)\) depending on the ramification data at certain points of \((M_3, \omega_{(3)})\).
Proposition 2. Consider the following conditions on \((X, \omega)\):

A) The covering \(q: (X, \omega) \to \mathbb{R}^2/\mathbb{Z}^2 = E\) is ramified at most over 4-division points.

B) The covering \(q\) is ramified above \((0, 0), (3/4, 0)\) and \((0, 1/4)\). The ramification data at \((0, 0), (3/4, 0)\) and \((0, 1/4)\) are distinct and different from all other ramification data.

C) The ramification data of \(p: (X, \omega) \to (M_3, \omega(3))\) at the four zeroes of \(\omega(3)\) are distinct.

It holds:

i) If \((X, \omega)\) satisfies A) and B), then any affine diffeomorphism \(f \in \text{Aff}(X, \omega)\) descends via \(p\) to an affine diffeomorphism \(g \in \text{Aff}(M_3, \omega(3))\) (i.e., \(g \circ p = p \circ f\)). Furthermore \(D(f) = D(g)\) lies in \(\Gamma(4)\).

ii) If \((X, \omega)\) in addition satisfies C), then \(g\) fixes the four zeroes of \(\omega(3)\), that is, \(g \in \text{Aff}(1)(M_3, \omega(3))\). We have in particular that \(g\) lies in \(\Gamma(M_3, \omega(3))\).

The following basic fact about ramified coverings will be crucial in the proof of Proposition 2. Let \(p: X \to Y\) be a ramified covering of topological surfaces. If \(f: X \to X\) is a homeomorphism of \(X\) which descends to \(g: Y \to Y\) via \(p\), i.e. \(g \circ p = p \circ f\), then \(f\) preserves ramification indices and \(g\) preserves ramification data.

Proof of Proposition 2 i) Suppose that A) and B) hold. Any affine diffeomorphism \(f\) of \(X\) descends via \(q\) to some affine diffeomorphism \(h: E \to E\) (see e.g. [S, Prop. 2.6] for a detailed proof). It follows from A) and B) that \(h\) fixes the point \(\infty = (0, 0), (3/4, 0)\) and \((0, 1/4)\) pointwise, thus \(D(f) = D(h)\) is in \(\Gamma(4)\).

Since \(h\) fixes \(\infty\), it is also a homeomorphism of the once-punctured torus \(E^* = E\setminus\{\infty\}\). We have from [HS, Prop. 1.2] that any affine diffeomorphism of \(E\), which restricts to one of \(E^*\), lifts to \(M_3\). More precisely, since the degree 8 covering \(M_3\to E\) is normal, there are 8 lifts. Again since this covering is normal, one of the lifts is a descend of \(f\), see e.g. [MS, Lemma 3.13]. Thus \(f\) descends via \(p\) to some affine homeomorphism \(g\) of \((M_3, \omega(3))\), and \(D(g) = D(f) = D(h)\) is in \(\Gamma(4)\).

ii) Suppose that A), B) and C) hold. The descend \(g\) found in i) is an affine homeomorphism and therefore permutes the zeroes of \(\omega(3)\). It directly follows from C) that it even fixes them pointwise and thus is in \(\Gamma_{EW}(4)\). □

By putting together Propositions 1 and 2 it is not difficult to get examples of non-trivial \(SL(2, \mathbb{R})\)-invariant isotropic subbundles of the Hodge bundle over \(SL(2, \mathbb{R})\)-orbits of origamis:
Theorem 1. Let \( p : (X, \omega) \to (M_3, \omega_{(3)}) \) be a finite ramified covering of the Eierlegende Wollmilchsau \((M_3, \omega_{(3)})\) satisfying the conditions A), B) and C) stated in Proposition \( \parallel \). Then, the Hodge bundle over the \( SL(2, \mathbb{R}) \)-orbit of \((X, \omega)\) contains non-trivial \( SL(2, \mathbb{R}) \)-invariant isotropic subbundles.

Proof. Given a finite ramified covering \( p : (X, \omega) \to (M_3, \omega_{(3)}) \), denote by \( H := p^{-1}(H_1^{(0)}(M_3, \mathbb{R})) \) the 4-dimensional subbundle of the Hodge bundle over \( SL(2, \mathbb{R}) \cdot (X, \omega) \) obtained by lifting the 4-dimensional subbundle \( H_1^{(0)}(M_3, \mathbb{R}) \) of the Hodge bundle over \( SL(2, \mathbb{R}) \cdot (M_3, \omega_{(3)}) \). Here similarly as before \( p^{-1}(H_1^{(0)}(M_3, \mathbb{R})) \) denotes the isomorphic preimage of \( H_1^{(0)}(M_3, \mathbb{R}) \) which is the intersection of the kernel of \( q_* \) with the (symplectic) orthogonal to the kernel of \( p_* \).

By definition, \( SL(2, \mathbb{R}) \) acts by post-composition with translation charts. Thus, we have that \( H \) is a \( SL(2, \mathbb{R}) \)-invariant subbundle because of the \( SL(2, \mathbb{R}) \)-invariance of \( H_1^{(0)}(M_3, \mathbb{R}) \).

Now, suppose that \( p : (X, \omega) \to (M_3, \omega_{(3)}) \) verifies the hypothesis of the theorem. By Proposition \( \parallel \) we know that any affine diffeomorphism \( f \in \text{Aff}(X, \omega) \) descends to (a unique) \( g \in \Gamma_{E_W}(4) \). Therefore, by Proposition \( \parallel \) we have that any \( f \in \text{Aff}(X, \omega) \) acts trivially (i.e., by \( \pm \text{id} \)) on the \( SL(2, \mathbb{R}) \)-invariant subbundle \( H \).

In particular, every equivariant subbundle \( E \subset H \) over \( SL(2, \mathbb{R}) \cdot (X, \omega) \) is \( SL(2, \mathbb{R}) \)-invariant. Thus, the Hodge bundle over \( SL(2, \mathbb{R}) \cdot (X, \omega) \) contains plenty \( SL(2, \mathbb{R}) \)-invariant isotropic subbundles: for instance, any 1-dimensional equivariant subbundle \( E \subset H \) has this property. \( \square \)

Remark 3.1. More examples of \( SL(2, \mathbb{R}) \)-invariant isotropic subbundles of the Hodge bundle can be produced by taking adequate finite covers \( p : (X, \omega) \to (M_4, \omega_{(4)}) \) of an origami \((M_4, \omega_{(4)})\) of genus 4 introduced in \([FM], [FMZ]\) (and sometimes called Ornithorynque in the literature). Indeed, the holomorphic action of \( \text{Aff}(M_4, \omega_{(4)}) \) was also studied in \([MY]\) where it is shown that any \( f \in \text{Aff}(M_4, \omega_{(4)}) \) with derivative in the principal congruence subgroup \( \Gamma(3) \) of level 3 of \( SL(2, \mathbb{Z}) \) acts trivially on \( H_1^{(0)}(M_4, \mathbb{R}) \) (see Subsection 3.5 and Lemma 3.3 of Subsection 3.7 of \([MY]\)).

In particular, one setup conditions (similar to A), B) and C) above) on the ramification data of \( p : (X, \omega) \to (M_4, \omega_{(4)}) \) over 3-torsion points of \((M_4, \omega_{(4)}) \to \mathbb{R}^2/\mathbb{Z}^2 \) so that any 1-dimensional equivariant subbundle \( E \) of the 6-dimensional subbundle \( H := p^{-1}(H_1^{(0)}(M_4, \mathbb{R})) \) is \( SL(2, \mathbb{R}) \)-invariant and isotropic. This will be briefly described in Subsection 5.3 below.

In the next subsection, we will use Theorem \( \parallel \) to produce an explicit origami (square-tiled surface) \((X, \omega)\) with non-trivial \( SL(2, \mathbb{R}) \)-invariant isotropic subbundle in the Hodge bundle over its \( SL(2, \mathbb{R}) \)-orbit.
3.2. A “concrete” example of $SL(2, \mathbb{R})$-invariant isotropic subbundles.
We construct in the following an example $X$ of an origami which satisfies the conditions A), B) and C) of Proposition 2. We start from the origami $(M_3, \omega_{(3)})$ shown in Figure 1. Observe that this is the same origami as shown on the left hand side of Figure 3. Its horizontal and vertical gluings are given by the two permutations

$$(3.1) \quad \sigma^M_a = (1, 6, 3, 8)(2, 5, 4, 7) \quad \text{and} \quad \sigma^M_b = (1, 2, 3, 4)(5, 6, 7, 8).$$

It has the four zeroes $\circ, \bullet, \blacksquare$ and $\square$. We subdivide each square of $(M_3, \omega_{(3)})$ into 16 subsquares. This gives an origami with 128 squares (see Figure 2).

The origami $X$ will be a degree 4 cover of it ramified over the four points $Q_1, Q_2, Q_3$ and $Q_4$ and over the four zeroes of $\omega_{(3)}$. Here $Q_i \ (i \in \{1, \ldots, 4\})$ is the end point of the segment $e_i$ shown in Figure 2 which is not a zero. The ramification over $Q_i$ will be given by the permutation $\pi_i$ with

$$\pi_1 = (1, 3)(2, 4), \quad \pi_2 = (1, 2), \quad \pi_3 = (1, 3)(2, 4) \quad \text{and} \quad \pi_4 = (1, 3, 2).$$

The origami $X$ thus will consist of $512 = 4 \cdot 8 \cdot 16$ squares.

More precisely we construct $X$ as follows (see Figure 7). We take four copies of the origami shown in Figure 2. We glue each edge to the corresponding edge in the same copy. Only the four edges $e_1, e_2, e_3$ and $e_4$ are glued to the corresponding edges $\overline{e_1}, \overline{e_2}, \overline{e_3}$ and $\overline{e_4}$, respectively, of a possibly different copy. The gluings for those are given by the permutations $\pi_1, \pi_2, \pi_3, \pi_4$, respectively. This means e.g. that the edge $e_1$ in Copy 1 is glued to $\overline{e_1}$ in Copy 3 $= \pi_1(1)$, $e_1$ in Copy 2 is glued to $\overline{e_1}$ in Copy 4 $= \pi_1(2)$, $e_1$ in Copy 3 is glued to $\overline{e_1}$ in Copy 1 $= \pi_1(3)$ and $e_1$ in Copy 4 is glued to $\overline{e_1}$ in Copy 2 $= \pi_1(4)$.

We will label the 512 squares of $X$ by tuples $(a, i, j, k) \in \{1, \ldots, 8\} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$, where $a$ denotes the number of the square in $M_3$, $(i, j)$ denotes which of the 16 subsquares it is, and $k$ is the leaf of the cover (see Figure 3).
Figure 2. Building plan for the origami $X$ from Definition 1; the point $Q_i$ respectively is the end point of $e_i$ which is not a singularity.
**Definition 1.** In the following we use the two permutations $\sigma_a^{M_3}$ and $\sigma_b^{M_3}$ in $S_8$ defined in (3.1). Let $X$ be the origami defined by the following two permutations (compare Figure 2 and Figure 3):

\[
\sigma_a : (a, i, j, k) \mapsto \begin{cases} 
(a, i, j + 1, k) & \text{if } j \in \{0, 1, 2\}, \\
(\sigma_a^{M_3}(a), i, j + 1, k) & \text{if } j = 3 \\
\text{and } (a, i, j) \notin \{(1, 0, 3), (2, 0, 3), (6, 0, 3)\}, \\
(\sigma_a^{M_3}(a), i, j + 1, \pi_1(k)) & \text{if } (a, i, j) = (1, 0, 3) \\
(\sigma_a^{M_3}(a), i, j + 1, \pi_2(k)) & \text{if } (a, i, j) = (2, 0, 3) \\
(\sigma_a^{M_3}(a), i, j + 1, \pi_3(k)) & \text{if } (a, i, j) = (6, 0, 3)
\end{cases}
\]

\[
\sigma_b : (a, i, j, k) \mapsto \begin{cases} 
(a, i + 1, j, k) & \text{if } i \in \{0, 1, 2\} \\
(\sigma_b^{M_3}(a), i + 1, j, k) & \text{if } i = 3 \\
\text{and } (a, i, j) \notin \{(4, 3, 3)\} \\
(\sigma_b^{M_3}(a), i + 1, j, \pi_4(k)) & \text{if } (a, i, j) = (4, 3, 3)
\end{cases}
\]

We now list some properties of the origami $X$.

**Lemma 3.1.** Let $X$ be the origami defined in Definition 1. Recall that $Q_i$ ($i \in \{1, 2, 3, 4\}$) is the end point of $e_i$ which is not a zero (see Figure 2). Then we have:
i) $X$ has genus $15$. It has 17 zeroes and lies in $H(5, 3, 3, 3, 2, 1, \ldots, 1)$. 

ii) The ramification data of the degree 4 cover $p : X \to M_3$ are:

- $(2, 2)$ at the zero $\bullet$, $(2, 1, 1)$ at the zero $\circ$, $(1, 1, 1, 1)$ at the zero $\blacksquare$, $(3, 1)$ at the zero $\square$.
- $(2, 2)$ at $Q_1$, $(2, 1, 1)$ at $Q_2$.
- $(2, 2)$ at $Q_3$ and $(3, 1)$ at $Q_4$.

iii) The ramification data of the degree 32 cover $q : X \to E$ are:

- $(2, 2, 2, 2, 2, 2, 2, 2)$ at the point $\infty$.
- $(2, 2, 2, 2, 1, \ldots, 1)$ at the point $(0, \frac{1}{4})$.
- $(3, 1, \ldots, 1)$ at the point $(\frac{3}{4}, 0)$.

In particular $X$ satisfies the conditions A), B) and C) from Proposition 2.

Proof. ii) We directly obtain from the construction of $X$ that the monodromy of a small positively oriented loop around $Q_i$ on $M_3$ is $\pi_i$. Thus the ramification data of $p$ at $Q_i$ is the list of the lengths of cycles of $\pi_i$. A small loop around the zero $\bullet$ on $M_3$ crosses the slit $e_3$, thus the monodromy is $\pi_3^{-1} = (1, 3)(2, 4)$. Similarly one obtains for the loops around $\circ$, $\blacksquare$ and $\square$ the monodromies $\pi_2^{-1} = (1, 2)$, the identity $id$ and $\pi_1^{-1}\pi_4^{-1} = (1, 4, 2)$, respectively. This gives the ramification data of $p$ at the zeroes.

iii) Recall that for the composition $q = (p : X \to M_3) \circ (\pi : M_3 \to E)$ we have: for each $x$ in $X$ the ramification index of $q$ at $x$ is equal to the ramification index of $p$ at $Q_i$ multiplied by the ramification index of $\pi$ at $p(x)$. Recall that $\infty$ has the four zeroes $\bullet, \circ, \blacksquare$ and $\square$ as preimages on $M_3$ and that their ramification index with respect to $\pi$ is 2. The point $(0, 1/4)$ has the three preimages $Q_1$, $Q_2$ and $Q_3$ and further five preimages over which $p$ is unramified (see Figure 2). Finally, $(3/4, 0)$ has the preimage $Q_4$ and seven preimages which are not ramification points of $p$. Now we obtain the data in iii) from the data in ii).

i) follows from iii).
$M_4$ has the three singularities $X_1 = \circ$, $Y_1 = \bullet$ and $Z_1 = \diamond$, see Figure 4, and is of genus 4. Its Veech group again is the full group $SL(2, \mathbb{Z})$, see [MY] p. 473. Similarly as before we use as a main tool that we explicitly know from [MY, Section 3] the elements of the affine group of $M_4$ that act trivially on the subbundle $H := H_1(0)(M_4, \mathbb{R})$. More precisely, we have that $f$ acts trivially on $H$ if and only if its derivative $D(f)$ lies in the principal congruence group $\Gamma(3)$ and $f$ fixes all preimages of the 2-division points on $E$ under $\pi_2$. Here we choose the point $A$ on the torus $E$ shown in Figure 6 as zero point and call the other 2-division points $X$, $Y$ and $Z$, see again Figure 6. In particular, we have $\pi_2(X_1) = X$, $\pi_2(Y_1) = Y$ and $\pi_2(Z_1) = Z$. Thus [MY] gives us:

\[ f \text{ acts trivially on } H := H_1(0)(M_4, \mathbb{R}) \iff D(f) \in \Gamma(3) \text{ and } f(P) = P \text{ for all } P \in \{A_1, A_2, A_3, X_1, Y_1, Z_1\} \]

Here $A_1$, $A_2$ and $A_3$ are the three preimages of $A$ (see Figure 4). The fact that the three unramified points $A_1$, $A_2$ and $A_3$ have to be fixed pointwise will give us some extra trouble in our construction which enforces an additional step: we will use a covering $h$ of $M_4$ which is ramified over them and takes care of this problem.

![Figure 4](image-url)

**Figure 4.** The Ornithorynque origami $M_4$: The label of an edge indicates to which square it is glued. Crossing one of the three bars from right to left leads one copy of $E$ higher (modulo 3), i.e. from Square 1 to Square 6, from Square 5 to Square 4 and from Square 3 to Square 2. $CovM_4$ has the three singularities $\circ$, $\bullet$ and $\diamond$, each of order 2.

**Proposition 3.** Consider a translation covering $q_1 : CovM_4 \to M_4$ and the following properties:

A) The ramification data of $q_1$ over the point $A_3$ is different from the ones over $A_1$ and $A_2$. 


B) The ramification data of \( q = \pi_2 \circ q_1 \) over \( A \) differ from the ramification data of \( q \) over all other points.

C) The Veech group \( \Gamma(CovM_4) \) of \( CovM_4 \) is contained in \( \Gamma(6) \).

We then have:

i) Any affine homeomorphism \( f \) of \( CovM_4 \) descends to an affine homeomorphism \( \bar{f} \) of \( M_4 \) which fixes the three singularities \( X_1, Y_1 \) and \( Z_1 \) and the three points \( A_1, A_2 \) and \( A_3 \) pointwise.

ii) The affine homeomorphisms of \( CovM_4 \) act trivially on the lift of \( H := H^{(1)}_1(M_4, \mathbb{R}) \) to \( H_1(CovM_4, \mathbb{R}) \). The Hodge bundle over the \( SL(2, \mathbb{R}) \)-orbit of \( (CovM_4, \omega) \) thus contains non-trivial \( SL(2, \mathbb{R}) \)-invariant isotropic subbundles.

**Proof.** ii) directly follows from [3.2] and i) in the same way as in Theorem[1]. To prove i) we first show that \( f \) descends to \( M_4 \). This again follows from [MS] Lemma 3.13] in the following way. By the universality of the torus, we know that \( f \) descends via \( q \) to some \( g \) on \( E \) which by B) fixes the point \( A \). Furthermore, the Veech group of \( M_4 \) is the full group \( SL(2, \mathbb{Z}) \), see [MY, p. 473]. Thus there is some affine homeomorphism \( \bar{f} \) of \( M_4 \) with derivative \( D(\bar{f}) = D(f) = D(g) \). It descends via \( \pi_2 \) to a homeomorphism of \( E \) with the same derivative. The descend preserves two-division points and thus has to fix the point \( A \), since this is the only non-ramification point with respect to \( \pi_2 \) among the two-division points. Thus the descend actually is equal to \( g \) and \( \bar{f} \) is a lift of \( g \). Now [MS Lemma 3.13] tells us that \( f \) descends via \( q_1 \) to some affine homeomorphism \( \bar{f} \) of \( M_4 \).

We now show that \( \bar{f} \) fixes the desired points: Since \( g \) fixes \( A \), we have that \( \bar{f} \) fixes \( \pi_2^{-1}(A) = \{ A_1, A_2, A_3 \} \). It follows then from A) that \( \bar{f} \) fixes the point \( A_3 \). Any affine homeomorphism of \( M_4 \) that fixes \( A_3 \) also fixes \( A_1 \) and \( A_2 \), see [MY, p.473]. Furthermore, since the Veech group is contained in \( \Gamma(2) \) by C), we obtain that \( g \) fixes \( X, Y \) and \( Z \) pointwise. Thus we have that \( \bar{f} \) fixes the singularities \( X_1, Y_1 \) and \( Z_1 \) pointwise. \( \square \)

In the following we give an explicit construction of origamis \( CovM_4 \) that have the properties required in Proposition[3] We use for this the construction of what we call **fake fibre product**, compare [N] Section 1.2], which we define in the following.

**Definition 2.** Let \( \pi_1 : X_1 \to Y \) and \( \pi_2 : X_2 \to Y \) be two ramified coverings. Let \( S \) be the union of the set of ramification points of \( \pi_1 \) and that of \( \pi_2 \). Thus we obtain unramified coverings \( \pi_1 : X_1^* \to Y^* \) and \( \pi_2 : X_2^* \to Y^* \), with \( Y^* = Y \setminus S, X_1^* = X_1 \setminus \pi_1^{-1}(S) \) and \( X_2^* = X_2 \setminus \pi_2^{-1}(S) \). Let now \( \rho_1 \) and \( \rho_2 \) be the monodromy groups of these unramified coverings, respectively. Hence they are maps from the fundamental group \( \pi_1(Y^*) \) of \( Y^* \) to the symmetric groups \( S_{d_1} \), respectively \( S_{d_2} \), where \( d_1 \) is the degree of \( \pi_1 \) and \( d_2 \) is
the degree of $\pi_2$. Let $d = d_1 \cdot d_2$ and identify $S_d$ with the symmetric group of the set $\{1, \ldots, d_1\} \times \{1, \ldots, d_2\}$. We then can consider the map

$$\rho : \pi_1(Y^*) \to S_d, w \mapsto \rho(w) \text{ with } \rho(w)(a, b) = (\rho_1(w)(a), \rho_2(w)(b)).$$

Note that $\rho$ does not have to be a transitive action. If it is transitive, it defines a finite (connected) unramified covering $q : Z^* \to Y^*$ for some punctured finite Riemann surface $Z^*$. We then may extend $q$ to a ramified covering $q : Z \to Y$ between closed Riemann surfaces. We call $q$ the fake fibre product of $\pi_1$ and $\pi_2$. If $\pi_1$ and $\pi_2$ are translation coverings, the translation atlas on $Y^*$ can be lifted to $Z^*$ and $q$ becomes a translation covering.

**Remark 3.2.** Observe that the fake fibre product naturally comes with coverings $q_1 : Z \to X_2$ and $q_2 : Z \to X_1$ such that we have a commutative diagram

$$\begin{array}{ccc}
Z & \xrightarrow{q_2} & X_1 \\
\downarrow q_1 & & \downarrow \pi_1 \\
X_2 & \xrightarrow{\pi_2} & E
\end{array}$$

For the construction of the family of origamis $CovM_4 = CovM_4(n)$, we now take as main ingredients which will be defined next:

- A covering $h : \tilde{M}_4 \to M_4$ which ramifies differently over $A_3$ than over $A_1$ and $A_2$ and is unramified over all other points. We then work with $\tilde{\pi}_2 = \pi_2 \circ h : \tilde{M}_4 \to E$.
- Coverings $\pi_1 : Y = Y(n) \to E$ which ramify pairwise differently over $A$ and the two six-division points $P$ and $Q$ shown in Figure 6.
- The fake fibre product $q : CovM_4 \to E$ of $\tilde{\pi}_2$ and $\pi_1$. We show that we indeed obtain a connected surface. We denote the projection $CovM_4 \to \tilde{M}_4$ by $\tilde{q}_1$.

The map $q_1 = h \circ \tilde{q}_1$ will then be the map which we need for Proposition 3, see also the diagram in Theorem 2.

Let $\tilde{M}_4$ be the origami shown in Figure 5 and $h : \tilde{M}_4 \to M_4$ the degree 2 covering to $M_4$ obtained from mapping square $i$ on $\tilde{M}_4$ to square $i$ mod 12 on $M_4$. You can directly read off the ramification data of $h$ from Figure 5.

**Remark 3.3.** The degree 2 covering $h : \tilde{M}_4 \to M_4$ is ramified over $A_1$ and $A_2$. All other points are unramified. Thus the map $\tilde{\pi}_2 = \pi_2 \circ h$ has the ramification data:

\[\text{Observe that this is not the fibre product of } X_1 \text{ and } X_2 \text{ in the category of algebraic curves since that may have singularities.}\]
Figure 5. Degree 2 covering \( h : \tilde{M}_4 \to M_4 \) of the Ornithorynque origami \( M_4 \)

- over \( A \): \((2, 2, 1, 1)\)
- over \( X, Y \) and \( Z \): \((3, 3)\)

For \( n \in \mathbb{N} \) define the covering \( \pi_1 = \pi_1(n) : Y = Y_n \to E \) as follows: Take \( n \) copies of the square shown in Figure 6. Glue each edge to the opposite edge in the same copy, except for the edges \( a, b, c, d \) and \( e \). Glue the edges labelled by \( a \) according to the permutation \( \sigma_a = (1, \ldots, n) \), i.e. the upper edge of Square 31 in Copy \( i \) is glued to the lower edge of Square 1 in Copy \( \sigma_a(i) \). Similarly glue the edges labelled by \( b, c, d \) and \( e \) according to the permutation \( \sigma_b = \sigma_c = \sigma_d = \sigma_e = (1, 2) \). In the case of \( e \) you glue the right edge of Square 2 in Copy \( i \) with the left edge of Square 3 in Copy \( \sigma_e(i) \). You can directly check from the construction that \( Y_n \) is connected.

Remark 3.4. The covering \( \pi_1 = \pi_1(n) \) has the following ramification data:
- over \( A \): \((n)\)
- over \( P \): \((1, n-1)\)
- over \( Q \): \((2, 1, \ldots, 1)_{n-2}\)

and is unramified over all other points.

Theorem 2. The following construction gives an infinite sequence of origamis \( Cov M_4 = Cov M_4(n) \) whose affine group act trivially on the lift of \( H := \)
Let $n \geq 5$, $n$ odd, and let $\pi_1 = \pi_1(n)$ and $h$ be the coverings defined above. Define $\pi_2 = \pi_2 \circ h$. Take now the fake fibre product $q = \pi_1 \times \pi_2 : CovM_4 \to E$ of $\pi_1$ and $\tilde{\pi}_2$. This comes by definitions with two coverings $\tilde{q}_1 : CovM_4 \to \tilde{M}_4$ and $q_2 : CovM_4 \to Y$ such that $q = \tilde{\pi}_2 \circ \tilde{q}_1 = \pi_1 \circ q_2$. We choose on $E$ the translation structure coming from the identification $E = \mathbb{C}/(6\mathbb{Z} \oplus 6\mathbb{Z}i)$ and denote its pullback to $CovM_4$ via $q$ by $\omega$.

Before giving the proof we study the ramification data of the maps which we have just defined. It follows from the definition of the fibre product that the ramification data of the map $q$ over a point $p \in E$ are obtained from the ramification data of $\pi_1$ and $\tilde{\pi}_2$ over $p$, respectively. More precisely we have that each pair of points $(p_1, p_2)$ in $\pi_1^{-1}(p) \times \tilde{\pi}_2^{-1}(p)$, where $p_1$ has ramification index $e$ with respect to $\pi_1$ and $p_2$ has ramification index $f$ with respect to $\tilde{\pi}_2$, produces $ef \text{lcm}(e, f) = \gcd(e, f)$ preimages of $p$ on $CovM_4$ with ramification index $\text{lcm}(e, f)$.

Similarly, we can read off the ramification data of the map $\tilde{q}_1$ over a point $p_2$ in the fibre $\tilde{\pi}_2^{-1}(p)$ from the ramification data of $\pi_1$ over $p$ and the ramification index $f$ of $\tilde{\pi}_2$ in $p_2$. More precisely each point $p_1$ in $\pi_1^{-1}(p)$ of
ramification index $e$ with respect to $\pi_1$ produces preimages with ramification index $\frac{\text{lcm}(e,f)}{f}$ with multiplicity $\frac{f \cdot e}{\text{lcm}(e,f)} = \gcd(e,f)$. In particular, if $\tilde{\pi}_2$ is unramified in $p_2$, then the ramification data of $\tilde{q}_1$ over $p_2$ are equal to those of $\pi_1$ over $p$.

Let finally $\hat{A}_1$ and $\hat{A}_2$ be the preimage of $A_1$ and $A_2$ under $h$, respectively. Let $\hat{A}_3^1$ and $\hat{A}_3^2$ be the two preimages of $A_3$.

**Remark 3.5.** From the previous considerations and Remark 3.3 and Remark 3.4 we obtain the ramification data for $q$, $\tilde{q}_1$ and $q_1$.

i) The map $q$ has the following ramification data:

- over $A$: $(2n, 2n, n, n)$, if $n$ is odd;
  
  $(n, n, n, n, n, n)$, if $n$ is even.

- over $X$, $Y$ and $Z$: $(3, \ldots, 3)$

- over $P$: $\left(1, \ldots, 1, n - 1, 1, \ldots, n - 1\right)$

- over $Q$: $\left(2, \ldots, 2, 1, \ldots, 1\right)$

ii) The map $\tilde{q}_1$ has the following ramification data:

- over $\hat{A}_1$ and $\hat{A}_2$: $(n)$, if $n$ is odd and $(\frac{n}{2}, \frac{n}{2})$, if $n$ is even.

- over $\hat{A}_3^1$ and $\hat{A}_3^2$: $(n)$

- over each preimage of $X$, $Y$ or $Z$:

- over each preimage of $P$: $\left(1, n - 1\right)$

- over each preimage of $Q$: $\left(2, 1, \ldots, 1\right)$, $\left(2, 2, 1, \ldots, 1\right)$

iii) We finally obtain the ramification data of $q_1 = h \circ \tilde{q}_1$:

- over $A_1$ and $A_2$: $(2n)$, if $n$ is odd and $(n, n)$, if $n$ is even.

- over $A_3$: $(n, n)$

- over the preimage of $X$, $Y$ or $Z$: $(1, \ldots, 1)$

- over each preimage of $P$: $\left(1, 1, n - 1, 1, n - 1\right)$

- over each preimage of $Q$: $\left(2, 2, 1, \ldots, 1\right)$, $\left(2, \ldots, 2, 1, \ldots, 1\right)$, $\left(2, \ldots, 2, 1, \ldots, 1\right)$.$^{\text{Lcm}}$

**Proof of Theorem 2** We first show that the fake fibre product $\text{Cov}M_3$ is connected and then that $q_1 = h \circ \tilde{q}_1$ satisfies the assumptions of Proposition 3.

**Connectedness of $\text{Cov}M_3$:** Let $\rho_1$ be the monodromy map of $\pi_1$ and $\rho_2$ that of $\tilde{\pi}_2$. To make notation easier we remove the full set of six-division points from $E$ and call the resulting surface $E^*$. Hence $\rho_1$ and $\rho_2$ are maps from
the fundamental group $\pi_1(E^*)$ to $S_n$ and $S_6$, respectively. Also for the sake of simpler notations we choose the base point of $\pi_1(E^*)$ in Square 7, see Figure 6. We label its preimages under $\pi_1$ on $Y = Y_n$ by the number of the corresponding sheet they lie in. For the map $\tilde{\pi}_2$, observe that the six preimages of the base point lie in the squares labelled by 8, 10, 12, 20, 22, 24, see Figure 5. We label these preimages by the label 1, 2, 3, . . . , 6, respectively.

We then have for the closed curves $x_6$ and $y_2x_6y_2^{-2}$ in $\pi_1(E^*)$ that $\rho_1(x_6)$ and $\rho_1(y^2x_6y_2^{-2})$ are both trivial, since both paths do not cross one of the edges $a, b, c, d$ and $e$ (see Figure 6). Furthermore, we have 

$$\rho_2(x_6) = (1, 5, 6)(2, 3, 4)$$
$$\rho_2(y^2x_6y_2^{-2}) = (1, 3, 2)(4, 6, 5).$$

In particular $\rho_2(x_6)$ and $\rho_2(y^2x_6y_2^{-2})$ act transitively on $\{1, \ldots, 6\}$. Since $\rho_1$ acts transitively on $\{1, \ldots, n\}$ and furthermore trivially on the two elements $x_6$ and $y^2x_6y_2^{-2}$, the action of the product $\rho_1 \times \rho_2$ is transitive on $\{1, \ldots, n\} \times \{1, \ldots, 6\}$. Hence $Y_n$ is connected.

Assumptions of Proposition 3. By Remark 3.5, $q_1$ satisfies the conditions A) and B) in Proposition 3. Consider an affine homeomorphism $f$ of $CovM_4$. Let $g$ be its descend to $E$ via $q$. By condition A), the map $g$ fixes the point $A$. By Remark 3.5, the ramification data of $g$ over $P$ and $Q$ are different from each other and different from all other six-division points. Therefore $g$ also fixes $P$ and $Q$ and hence $D(f) = D(g)$ lie in $\Gamma(6)$. Thus we have $\Gamma(CovM_4) \subseteq \Gamma(6)$ and also Condition C) from Proposition 3 is fulfilled.

We directly obtain from Remark 3.5 the stratum of the surfaces $CovM_4(n)$.

Remark 3.6. The origami $CovM_4(n)$ with odd $n \geq 5$ lies in the stratum

$$\mathcal{H}(2n - 1, 2n - 1, n - 1, n - 1, n - 2, \ldots, n - 2, 2, \ldots, 2, 1, \ldots, 1)$$

and its genus is $12n - 4$. 

APPENDIX A. A PICTURE OF THE ORIGAMI $X$

In Figure 7 below we show the full origami $X$ from Definition 1 which has 512 squares.

Figure 7. The origami $X$ from Definition 1
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