COMPACTNESS OF RIESZ TRANSFORM COMMUTATOR ON STRATIFIED LIE GROUPS

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Abstract. Let $G$ be a stratified Lie group and $\{X_j\}_{1 \leq j \leq n}$ a basis for the left-invariant vector fields of degree one on $G$. Let $\Delta = \sum_{j=1}^{n} X_j^2$ be the sub-Laplacian on $G$. The $j$th Riesz transform on $G$ is defined by $R_j := X_j(\Delta)^{-\frac{1}{2}}$, $1 \leq j \leq n$. In this paper, we provide a concrete construction of the “twisted truncated sector” which is related to the pointwise lower bound of the kernel of $R_j$ on $G$. Then we obtain the characterisation of compactness of the commutators of $R_j$ with respect to VMO, the space of functions with vanishing mean oscillation on $G$.

1. Introduction and statement of main results

A central topic of modern harmonic analysis is to study singular integral operators and their applications in characterizing function spaces. In [2], Calderón introduced the commutator of a singular integral operator $T$ with a symbol $b$ as

$$[b, T](f) := bT(f) − T(bf).$$

When $T$ is the Riesz transform $R_j = \partial / \partial x_j \Delta^{-\frac{1}{2}}$ on the Euclidean space $\mathbb{R}^n$, Coifman, Rochberg and Weiss [5] showed that the commutator $[b, R_j]$ is bounded on $L^p(\mathbb{R}^n)$ with $1 < p < \infty$ if and only if $b \in BMO(\mathbb{R}^n)$, which is the space of functions with bounded mean oscillation. Uchiyama [25] then showed that $[b, R_j]$ is compact on $L^p(\mathbb{R}^n)$ with $1 < p < \infty$ if and only if $b \in VMO(\mathbb{R}^n)$, the space of functions with vanishing mean oscillation on $\mathbb{R}^n$. Later on and recently, there has been an intensive study of the compactness of commutators of singular integrals in many different settings, such as the Riesz transform associated with Bessel operator on the positive real line, the Cauchy’s integrals on the real line, the Calderón–Zygmund operator associated with homogeneous kernels $\frac{\Omega(x)}{|x|^n}$ on $\mathbb{R}^n$, and the multilinear Riesz transforms, see for example [4, 9, 13, 17, 18, 19, 23] and related references therein.

Beyond these operators in the Euclidean setting, it is natural to ask whether this characterisation of compactness of commutators also holds for Riesz transforms associated with the sub-Laplacian on Heisenberg groups $\mathbb{H}^n$, which is the boundary of the Siegel upper half space in $\mathbb{C}^n$. Recall that $\mathbb{H}^n$ and the Siegel upper half space are holomorphically equivalent to the unit sphere and unit ball in $\mathbb{C}^n$, and hence the role of Riesz transform associated with the sub-Laplacian on $\mathbb{H}^n$ is similar to the role of Hilbert transform on the real line.

We note that to study the boundedness and compactness of Riesz transform commutator, one only needs the upper bound of the Riesz transform kernel and its derivative (see [10], [22]). However, a full characterisation of the Riesz transform commutator would also require the kernel lower bound. Recently, in [7, 8] the authors studied the pointwise lower bound of the kernel of Riesz transform associated with the sub-Laplacian on stratified Lie groups, and then they established the characterisation of commutator of Riesz transforms with respect to the BMO space (see Theorem 1.2 in [7] and Theorems 1.2–1.5 in [8]), which extends the classical result of Coifman–Rochberg–Weiss [5] to the setting of stratified Lie groups. One
of the important examples of stratified Lie groups is the Heisenberg group $\mathbb{H}^n$. And the pointwise kernel lower bound obtained in [8] is as follows.

**Theorem A** ([8], Theorem 1.1). Suppose that $\mathcal{G}$ is a stratified Lie group with homogeneous dimension $Q$ and that $j \in \{1, 2, \ldots, n\}$. There exist a large positive constant $r_o$ and a positive constant $C$ such that for every $g \in \mathcal{G}$ there exists a “twisted truncated sector” $G \subset \mathcal{G}$ such that $
exists g'(g, g') = r_o$, and that for every $g_1 \in B_{r_o}(g, 1)$ and $g_2 \in G$, we have

$$|K_j(g_1, g_2)| \geq C \rho(g_1, g_2)^{-Q}, \quad |K_j(g_2, g_1)| \geq C \rho(g_1, g_2)^{-Q},$$

and all $K_j(g_1, g_2)$ as well as all $K_j(g_2, g_1)$ have the same sign.

Moreover, this “twisted truncated sector” $G$ is regular, in the sense that $|G| = \infty$ and that for any $R > 2r_o$,

$$|B_{r_o}(g, R) \cap G| \approx R^Q,$$

where the implicit constants are independent of $g$ and $R$.

Here and in what follows, $\rho$ is the homogeneous norm on $\mathcal{G}$, and for $g \in \mathcal{G}$, $r > 0$, $B_{r_o}(g, r)$ is the ball defined via $\rho$, and $K_j(g_1, g_2)$ is the kernel of the $j$th Riesz transform $R_j$. For the details of the notation, we refer to Section 2 below.

The aim of this paper is to establish the characterisation of the compactness of the commutator of Riesz transforms associated with sub-Laplacian on stratified Lie groups via the VMO space, which is defined as the closure of the $C_0^\infty$ functions (functions with arbitrary order of derivatives and with compact support) under the norm of the BMO space. For the precise definition of $C_0^\infty$ functions, the properties of Riesz transforms and the BMO, Hardy spaces on stratified Lie groups, we refer to Folland–Stein [10], see also Saloff-Coste [22]. See Theorem 1.1 for the equivalent characterisation of VMO as the space of functions with vanishing mean oscillation.

However, to establish the characterisation of compactness of Riesz commutators, we point out that the condition (1.1) for the “twisted truncated sector” $G \subset \mathcal{G}$ related to the pointwise lower bound of the kernel of $R_j$ is not enough, since we need to know more about the behaviour of this twisted truncated sector $G$ in each annuli that intersects with $G$.

Thus, the main results of this paper are twofold. First, we give a particular construction of the “twisted truncated sector” $G \subset \mathcal{G}$ related to the pointwise lower bound of the kernel of $R_j$, which is regular in each annuli that intersects with $G$, while the previous version in Theorem A only states that the “twisted truncated sector” is regular in each large ball that intersects with $G$. Second, by using this kernel lower bound and the more explicit information on $G$, we establish the characterisation of the compactness of Riesz commutators via functions in VMO space on $\mathcal{G}$, where a characterisation of the VMO space is needed.

To be more precise, we have the following results.

**Theorem 1.1.** Suppose that $\mathcal{G}$ is a stratified Lie group with homogeneous dimension $Q$ and that $j \in \{1, 2, \ldots, n\}$. There exist a large positive constant $r_o$ and a positive constant $C$ such that for every $g \in \mathcal{G}$ there exists a “twisted truncated sector” $G_g \subset \mathcal{G}$ satisfying that

$$\inf_{g' \in G_g} \rho(g, g') = r_o$$

and that for every $g_1 \in B_{r_o}(g, 1)$ and $g_2 \in G_g$, we have

$$|K_j(g_1, g_2)| \geq C \rho(g_1, g_2)^{-Q}, \quad |K_j(g_2, g_1)| \geq C \rho(g_1, g_2)^{-Q},$$

and all $K_j(g_1, g_2)$ as well as all $K_j(g_2, g_1)$ have the same sign.

Moreover, this “twisted truncated sector” $G_g$ is regular, in the sense that $|G_g| = \infty$ and that for any $R_2 > R_1 > 2r_o$,

$$|\left(B_{r_o}(g, R_2) \setminus B_{r_o}(g, R_1)\right) \cap G_g| \approx \left|B_{r_o}(g, R_2) \setminus B_{r_o}(g, R_1)\right|,$$

where the implicit constants are independent of $g$ and $R_1, R_2$. 
Here we point out that the set $G_g$ that we constructed in Theorem 1.1 above is a connected open set spreading out to infinity, which plays the role of the “truncated sector centred at a fixed point” in the Euclidean setting. The shape of $G_g$ here may not be the same as the usual sector since the norm $\rho$ on $G$ is different from the standard Euclidean metric. However, such a kind of twisted sector always exists.

Second, based on the property of the Riesz transform kernel, we establish the following commutator theorem on stratified Lie group via providing the characterisation of the VMO space, following the approach of Uchiyama [25]. In what follows we use $A_p(G)$ to denote the Muckenhoupt type weighted class on stratified Lie groups, whose precise definition will be given in Section 2.

**Theorem 1.2.** Let $1 < p < \infty$, $w \in A_p(G)$, $b \in L^1_{\text{loc}}(G)$. Then $b \in \text{VMO}(G)$ if and only if for some $\ell \in \{1, \cdots, n\}$, Riesz transform commutator $[b, R_\ell]$ is compact on $L^p_w(G)$.

This paper is organised as follows. In Section 2 we recall necessary preliminaries on stratified nilpotent Lie groups $G$. In Section 3 we provide a particular construction of the twisted truncated sector and then obtain the pointwise lower bound of the Riesz transform kernels, and then prove Theorem 1.1. In Section 4, by using the kernel lower bound that we established, we prove Theorem 1.2 the characterisation of compactness of the Riesz commutator. In the end, in the appendix, we provide the characterisation of the VMO space following the approach of Uchiyama [25].

**Notation:** Throughout this paper, $\mathbb{N}$ will denote the set of all nonnegative integers. For a real number $a$, $[a]$ means the largest integer no greater than $a$. In what follows, $C$ will denote positive constant which is independent of the main parameters, but it may vary from line to line. By $f \lesssim g$, we shall mean $f \leq Cg$ for some positive constant $C$. If $f \lesssim g$ and $g \lesssim f$, we then write $f \approx g$.

## 2. Preliminaries on stratified Lie groups $G$

Recall that a connected, simply connected nilpotent Lie group $G$ is said to be stratified if its left-invariant Lie algebra $g$ (assumed real and of finite dimension) admits a direct sum decomposition

$$g = \bigoplus_{i=1}^s V_i, \quad [V_1, V_i] = V_{i+1}, \quad \text{for} \ i \leq s - 1 \ \text{and} \ [V_1, V_s] = 0.$$ 

$s$ is called the step of the group $G$.

For $i = 1, \cdots, s$, let $n_i = \dim V_i$ and $m_i = n_1 + \cdots + n_i$, $m_0 = 0$ and $m_s = N$. We choose once and for all a basis $\{X_1, \cdots, X_N\}$ for $g$ adapted to the stratification, that is, such that $\{X_{m_{j-1}+1}, \cdots, X_{m_j}\}$ is a basis of $V_j$ for each $j = 1, \cdots, s$. One identifies $g$ and $G$ via the exponential map

$$\exp : g \to G,$$

which is a diffeomorphism, i.e., any $g \in G$ can be written in a unique way as $g = \exp(x_1 X_1 + \cdots + x_N X_N)$. Using these exponential coordinates, we identify $g$ with the $N$-tuple $(x_1, \cdots, x_N) \in \mathbb{R}^N$ and identify $G$ with $([\mathbb{R}^N, \circ])$, where the group operation $\circ$ is determined by the Campbell-Hausdorff formula (c.f. [1] Section 2.2.2). We fix once and for all a (bi-invariant) Haar measure $dx$ on $G$ (which is just the lift of Lebesgue measure on $g$ via $\exp$).

There is a natural family of dilations on $g$ defined for $r > 0$ as follows:

$$\delta_r \left( \sum_{i=1}^s v_i \right) = \sum_{i=1}^s r^i v_i, \quad \text{with} \ v_i \in V_i.$$
This allows the definition of dilation on $\mathcal{G}$, which we still denote by $\delta_r$ (see Section 3).

Denote by $n = n_1$, for the basis $\{X_1, \ldots, X_n\}$ of $V_1$, we consider the sub-Laplacian $\Delta = \sum_{j=1}^n X_j^2$. Observe that $X_j (1 \leq j \leq n)$ is homogeneous of degree 1 with respect to the dilations, and $\Delta$ of degree 2 in the sense that:

$$X_j (f \circ \delta_r) = r (X_j f) \circ \delta_r, \quad 1 \leq j \leq n, \quad r > 0, \quad f \in C^1,$$

$$\delta^2 \circ \Delta \circ \delta_r = r^2 \Delta, \quad \forall r > 0.$$ 

Let $Q$ denote the homogeneous dimension of $\mathcal{G}$, namely,

$$Q = \sum_{i=1}^s i \dim V_i.$$

And let $p_h (h > 0)$ be the heat kernel (that is, the integral kernel of $e^{h\Delta}$) on $\mathcal{G}$. For convenience, we set $p_h (g) = p_h (g, 0)$ (that is, in this note, for a convolution operator, we will identify the integral kernel with the convolution kernel) and $p (g) = p_1 (g)$.

Recall that (c.f. for example [10])

$$p_h (g) = h^{-\frac{n}{2}} p (\frac{\delta}{\sqrt{h} \sigma} (g)), \quad \forall h > 0, \quad g \in \mathcal{G}. \quad (2.2)$$

The kernel of the $j$th Riesz transform $X_j (-\Delta)^{-\frac{1}{2}} \quad (1 \leq j \leq n)$ is written simply as $K_j (g, g') = K_j (g' \circ \delta_r \circ g)$. It is well-known that

$$K_j \in C^\infty (\mathcal{G} \setminus \{0\}), \quad K_j (\delta_r (g)) = r^{-Q} K_j (g), \quad \forall g \neq 0, \quad r > 0, \quad 1 \leq j \leq n, \quad (2.3)$$

which also can be explained by (2.2) and the fact that

$$K_j (g) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} h^{-\frac{n}{2}} X_j p_h (g) \, dh = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} h^{-\frac{n}{2} - 1} (X_j p) (\delta_{\frac{1}{\sqrt{h} \sigma}} (g)) \, dh.$$

Next we recall the homogeneous norm $\rho$ (see for example [10]) on $\mathcal{G}$ which is defined to be a continuous function $g \to \rho (g)$ from $\mathcal{G}$ to $[0, \infty)$, which is $C^\infty$ on $\mathcal{G} \setminus \{0\}$ and satisfies

(a) $\rho (g^{-1}) = \rho (g)$;
(b) $\rho (\delta_r (g)) = r \rho (g)$ for all $g \in \mathcal{G}$ and $r > 0$;
(c) $\rho (g) = 0$ if and only if $g = 0$.

For the existence (also the construction) of the homogeneous norm $\rho$ on $\mathcal{G}$, we refer to [10], Chapter 1, Section A. For convenience, we set

$$\rho (g, g') = \rho (g' \circ g) = \rho (g^{-1} \circ g'), \quad \forall g, g' \in \mathcal{G}.$$ 

Recall that (see [10]) this defines a quasi-distance in sense of Coifman-Weiss, namely, there exists a constant $C_\rho > 0$ such that

$$\rho (g_1, g_2) \leq C_\rho \left( \rho (g_1, g') + \rho (g', g_2) \right), \quad \forall g_1, g_2, g' \in \mathcal{G}. \quad (2.4)$$

In the sequel, we fix a homogeneous norm $\rho$ on $\mathcal{G}$ (see Section 3).

We now denote by $d$ the Carnot–Carathéodory metric associated to $\{X_j\}_{1 \leq j \leq n}$, which is equivalent to $\rho$ in the sense that: there exist $C_{d_1}, C_{d_2} > 0$ such that for every $g_1, g_2 \in \mathcal{G}$ (see [11]),

$$C_{d_1} \rho (g_1, g_2) \leq d (g_1, g_2) \leq C_{d_2} \rho (g_1, g_2). \quad (2.5)$$

We point out that the Carnot–Carathéodory metric $d$ even on the most special stratified Lie group, the Heisenberg group, is not smooth on $\mathcal{G} \setminus \{0\}$.

In the sequel, to avoid confusion we will use $B_\rho (g, r)$ and $S_\rho (g, r)$ to denote the open ball and the sphere with center $g$ and radius $r$ defined by $\rho$, respectively. And we will use $B (g, r)$ and $S (g, r)$ to denote the open ball and the sphere defined by $d$, respectively. In the
following, $B$ is always a ball defined by $d$ and $r_B$ is its radius. For any $\alpha > 0$, denote by $\alpha B(g, r) = B(g, \alpha r)$.

**Definition 2.1.** The bounded mean oscillation space $\text{BMO}(\mathcal{G})$ is defined to be the space of all locally integrable functions $f$ on $\mathcal{G}$ such that

$$\|f\|_{\text{BMO}(\mathcal{G})} := \sup_{B \subset \mathcal{G}} M(f, B) := \sup_{B \subset \mathcal{G}} \frac{1}{|B|} \int_B |f(g) - f_B| \, dg < \infty,$$

where

$$f_B = \frac{1}{|B|} \int_B f(g) \, dg.$$

**Definition 2.2.** We define $\text{VMO}(\mathcal{G})$ as the closure of the $C_0^\infty$ functions on $\mathcal{G}$ under the norm of the BMO space.

For the Folland–Stein BMO space $\text{BMO}(\mathcal{G})$, note that we have an equivalent norm, which is defined by

$$\|b\|_{\text{BMO}(\mathcal{G})} = \sup_{B \subset \mathcal{G}} \inf_c \frac{1}{|B|} \int_B |b(g) - c| \, dg.$$

For a ball $B$, the infimum above is attained and the constants where this happens can be found among the median values.

**Definition 2.3.** A median value of $b$ over a ball $B$ will be any real number $m_b(B)$ that satisfies simultaneously

$$|\{x \in B : b(g) > m_b(B)\}| \leq \frac{1}{2}|B|$$

and

$$|\{x \in B : b(g) < m_b(B)\}| \leq \frac{1}{2}|B|.$$

Following the standard proof in [24, p.199], we can see that the constant $c$ in the definition of $\|b\|_{\text{BMO}(\mathcal{G})}$ can be chosen to be a median value of $b$. And it is easy to see that for any ball $B \subset \mathcal{G}$,

$$M(b, B) \approx \frac{1}{|B|} \int_B |f(g) - m_b(B)| \, dg,$$

where the implicit constants are independent of the function $b$ and the ball $B$.

The theory of $A_p$ weight was first introduced by Muckenhoupt in the study of weighted $L^p$ boundedness of Hardy-Littlewood maximal functions in [21]. For $A_p$ weights on the stratified Lie group (which is an example of spaces of homogeneous type in the sense of Coifman and Weiss [6]) one can refer to [15]. By a weight, we mean a non-negative locally integrable function on $\mathcal{G}$.

**Definition 2.4.** Let $1 < p < \infty$, a weight $w$ is said to be of class $A_p(\mathcal{G})$ if

$$[w]_{A_p} := \sup_{B \subset \mathcal{G}} \left( \frac{1}{|B|} \int_B w(g) \, dg \right)^{p-1} \left( \frac{1}{|B|} \int_B w(g)^{-1/(p-1)} \, dg \right)^{p-1} < \infty.$$

A weight $w$ is said to be of class $A_1(\mathcal{G})$ if there exists a constant $C$ such that for all balls $B \subset \mathcal{G}$,

$$\frac{1}{|B|} \int_B w(g) \, dg \leq C \inf_{x \in B} w(g).$$

For $p = \infty$, we define

$$A_\infty(\mathcal{G}) = \bigcup_{1 \leq p < \infty} A_p(\mathcal{G}).$$
Note that $w$ is doubling when it is in $A_p$, i.e. there exists a positive constant $C$ such that $w(2B) \leq Cw(B)$ for every ball $B$.

Recall that the Muckenhoupt weights have some fundamental properties. A close relation to $A_\infty(G)$ is the reverse Hölder condition. If there exist $r > 1$ and a fixed constant $C$ such that
\[
\left( \frac{1}{|B|} \int_B w(g)^r \, dg \right)^{1/r} \leq \frac{C}{|B|} \int_B w(g) \, dg
\]
for all balls $B \subset G$, we then say that $w$ satisfies the reverse Hölder condition of order $r$ and write $w \in RH_r(G)$. According to [10 Theorem 19 and Corollary 21], $w \in A_\infty(G)$ if and only if there exists some $r > 1$ such that $w \in RH_r(G)$.

For any $w \in A_\infty(G)$ and any Lebesgue measurable set $E$, denote by $w(E) := \int_E w(g) \, dg$. By the definition of $A_p$ weight and Hölder’s inequality, we can easily obtain the following standard properties.

**Lemma 2.5.** Let $w \in A_p(G) \cap RH_r(G), p \geq 1$ and $r > 1$. Then there exist constants $C_1, C_2 > 0$ such that
\[
C_1 \left( \frac{|E|}{|B|} \right)^p \leq \frac{w(E)}{w(B)} \leq C_2 \left( \frac{|E|}{|B|} \right)^{(r-1)/r}
\]
for any measurable subset $E$ of a ball $B$. Especially, for any $\lambda > 1$,
\[
w(B(g_0, \lambda R)) \leq C\lambda^{Qp} w(B(g_0, R)),
\]
where $Q$ is the homogeneous dimension of $G$.

### 3. LOWER BOUND FOR KERNEL OF RIESZ TRANSFORM $R_j := X_j(-\Delta)^{-\frac{1}{2}}$ AND PROOFS OF THEOREMS 1.1

In this section, we study a suitable version of the lower bound for kernel of Riesz transform $R_j := X_j(-\Delta)^{-\frac{1}{2}}, j = 1, \ldots, n$, on stratified Lie group $G$. Here we will use the Carnot–Carathéodory metric $d$ associated to $\{X_j\}_{1 \leq j \leq n}$ to study the lower bound, and we also make good use of the dilation structure on $G$. It is not clear whether one can obtain similar lower bounds for the Riesz kernel on general nilpotent Lie groups which is not stratified.

To begin with, we first recall that by the classical estimates for heat kernel and its derivations on stratified Lie groups (see for example [22, 20]), it is well-known that for any $1 \leq j \leq n$ and $g \neq g'$
\[
|K_j(g, g')| + d(g, g') \sum_{i=1}^n (|X_{i,g}K_j(g, g')| + |X_{i,g'}K_j(g, g')|) \lesssim d(g, g')^{-Q}, \tag{3.1}
\]
where $X_{i,g}$ denotes the derivation with respect to $g$.

Two important families of diffeomorphisms of $G$ are the translations and dilations of $G$. For any $g \in G$, the (left) translation $\tau_g : G \rightarrow G$ is defined as
\[
\tau_g(g') = g \circ g'.
\]
For any $\lambda > 0$, the dilation $\delta_\lambda : G \rightarrow G$, is defined as
\[
\delta_\lambda(g) = \delta_\lambda(x_1, x_2, \ldots, x_N) = (\lambda^{\alpha_1} x_1, \lambda^{\alpha_2} x_2, \ldots, \lambda^{\alpha_N} x_N), \tag{3.2}
\]
where $\alpha_j = i$ whenever $m_i - 1 < j \leq m_i$, $i = 1, \ldots, s$. Therefore, $1 = \alpha_1 = \cdots = \alpha_{n_1} < \alpha_{n_1+1} = 2 \leq \cdots \leq \alpha_n = s$.

Before proving Theorem 1.1 we need the following elementary properties of the group operation (see for example [20, [1 Proposition 2.2.22]).
Lemma 3.1. The group law of $G$ has the form

$$g \circ g' = g + g' + P(g, g'), \quad \forall \ g, g' \in \mathbb{R}^N,$$

where $P = (P_1, P_2, \cdots, P_N): \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ and each $P_j$ is a homogeneous polynomial of degree $\alpha_j$ with respect to the intrinsic dilations of $G$ defined in (3.2), i.e.,

$$P_j(\delta(g), \delta(g')) = \lambda^{\alpha_j} P_j(g, g'), \quad \forall \ g, g' \in G.$$

Moreover,

(i) $P$ is anti-symmetric, i.e., for any $g, g' \in G$, $P_j(g, g') = -P_j(-g', g)$.

(ii) For any $g, g' \in G$, $P_1(g, g') = \cdots = P_n(g, g') = 0$.

(iii) For $n < j \leq N$, $P_j(g, 0) = P_j(0, g')$, $P_j(g, g) = P_j(g, -g) = 0$.

(iv) For any $g = (x_1, \cdots, x_N)$ and $g' = (y_1, \cdots, y_N)$, if $j \leq m_i$, $1 \leq i \leq s$, $P_j(g, g') = P_j(x_1, \cdots, x_{m_i-1}, y_1, \cdots, y_{m_i-1})$.

(v) $P_j(g, g') = \sum_{l,h} R_{l,h}^j (g, g')(x_l y_h - x_h y_l)$, where the functions $R_{l,h}^j$ are polynomials, homogeneous of degree $\alpha_i - \alpha_j - \alpha_h$ with respect to group dilations, and the sum is extended to all $l, h$ such that $\alpha_i - \alpha_h \leq \alpha_j$.

Remark 3.2. It follows from Lemma 3.1 that $\delta_\lambda : G \to G$ is an automorphism of the group, i.e.,

$$\delta_\lambda(g) \circ \delta_\lambda(g') = \delta_\lambda(g \circ g').$$

And the unit element of $G$ is the origin $0 = (0, \cdots, 0) \in \mathbb{R}^N$. Consequently, the inverse $g^{-1}$ of an element $g = (x_1, \cdots, x_N) \in G$ has the form

$$g^{-1} = (-x_1, \cdots, -x_N).$$

Define

$$|g|_G = \left( \sum_{j=1}^{s} |x^{(j)}|^{2^{\alpha_j}} \right)^{1/2}, \quad g = (x^{(1)}, \cdots, x^{(n)}) \in G,$$

where $|x^{(j)}|$ denotes the Euclidean norm on $\mathbb{R}^{n_j}$. Then $| \cdot |_G$ is a homogeneous norm on $G$ (see for example [II Section 5.1]). In what follows, we will use $\rho(g)$ to denote $|g|_G$ for any $g \in G$.

In [II] Lemma 3.1, the authors proved the following property for the Riesz kernel $K_j$.

Lemma 3.3. For all $1 \leq j \leq n$, we have $K_j \not\equiv 0$ in $G \setminus \{0\}$.

Now we prove Theorem 1.1.

Proof of Theorem 1.1. For any fixed $j \in \{1, \cdots, n\}$, by Lemma 3.3 and the scaling property of $K_j$ (c.f. (2.3)), there exists a compact set $\Omega$ on the unit sphere $S_\sigma(0, 1)$ with $\sigma(\Omega) > 0$, where $\sigma$ is the Radon measure on $S_\rho(0, 1)$, satisfying

$$\rho(\tilde{g}) = 1 \quad \text{and} \quad K_j(\tilde{g}) \not\equiv 0, \quad \forall \ \tilde{g} \in \Omega,$$

and all the values $K_j(\tilde{g})$ on $\Omega$ have the same sign.

We claim that there exist $0 < \varepsilon_0 \ll 1$ and $C(K_j)$ such that for any $0 < \eta < \varepsilon_0$, any $\tilde{g} \in \Omega$ and for all $g \in G$ and $r > 0$,

$$|K_j(g_1, g_2)| \geq C(K_j) r^{-Q}, \quad |K_j(g_2, g_1)| \geq C(K_j) r^{-Q},$$

for any $g_1 \in B_\rho(g, \eta r), g_2 \in B_\rho(g \circ \delta_r(\tilde{g}), \eta r)$. Moreover, all $K_j(g_1, g_2)$ and all $K_j(g_2, g_1)$ have the same sign.

In fact, for any fixed $\tilde{g} \in \Omega$, since $K_j$ is a $C^\infty$ function in $G \setminus \{0\}$, there exists $0 < \varepsilon_{\tilde{g}} \ll 1$ such that

$$K_j(\tilde{g}) \not\equiv 0 \quad \text{and} \quad |K_j(\tilde{g})| > \frac{1}{2} |K_j(\tilde{g})|$$
for all \( g' \in B_\rho(\tilde{g}, 4C^2_\rho \varepsilon_{\tilde{g}}) \), where \( C_\rho \geq 1 \) is the constant from (2.4). To be more specific, we have that for all \( g' \in B_\rho(\tilde{g}, 4C^2_\rho \varepsilon_{\tilde{g}}) \), the values \( K_j(g') \) and \( K_j(\tilde{g}) \) have the same sign.

Since \( \Omega \) is compact, and

\[
\bigcup_{\tilde{g} \in \Omega} B_\rho(\tilde{g}, \varepsilon_{\tilde{g}}) \supset \Omega,
\]
we have a finite subcover, say \( B_\rho(\tilde{g}_1, \varepsilon_{\tilde{g}_1}), \ldots, B_\rho(\tilde{g}_m, \varepsilon_{\tilde{g}_m}) \). Then for any \( \tilde{g} \in \Omega \), there exists \( 1 \leq l \leq m \) such that

\[
\tilde{g} \in B_\rho(\tilde{g}_l, \varepsilon_{\tilde{g}_l}).
\]

For any fixed \( g \in \mathcal{G} \), let

\[
g_* = g \circ \delta_r(\tilde{g}^{-1}).
\]

Then

\[
\rho(g, g_*) = \rho(g, g \circ \delta_r(\tilde{g}^{-1})) = r.
\]

Let \( \varepsilon_\rho = \min_{1 \leq l \leq m} \{ \varepsilon_{\tilde{g}_l} \} \), for every \( \eta \in (0, \varepsilon_\rho) \), we consider the two balls \( B_\rho(g, \eta r) \) and \( B_\rho(g_*, \eta r) \). It is clear that for every \( g_1 \in B_\rho(g, \eta r) \), we can write

\[
g_1 = g \circ \delta_r(g'_1),
\]
where \( g'_1 \in B_\rho(0, \eta) \). Similarly, for every \( g_2 \in B(g_*, \eta r) \), we can write

\[
g_2 = g_* \circ \delta_r(g'_2),
\]
where \( g'_2 \in B_\rho(0, \eta) \).

As a consequence, we have

(3.6)

\[
K_j(g_1, g_2) = K_j(g \circ \delta_r(g'_1), g_* \circ \delta_r(g'_2))
= K_j(g \circ \delta_r(g'_1), g \circ \delta_r(\tilde{g})^{-1} \circ \delta_r(g'_2))
= r^{-Q}K_j((g'_2)^{-1} \circ \tilde{g} \circ g'_1).
\]

Similarly,

(3.7)

\[
K_j(g_2, g_1) = r^{-Q}K_j((g'_1)^{-1} \circ \tilde{g} \circ g'_2).
\]

Next, we note that

\[
\rho((g'_2)^{-1} \circ \tilde{g} \circ g'_1, \tilde{g}_1) = \rho(\tilde{g} \circ g'_1, g_2 \circ \tilde{g}_1)
\leq C^2_\rho \left[ \rho(\tilde{g} \circ g'_1, \tilde{g}) + \rho(\tilde{g}, \tilde{g}_1) + \rho(g_2 \circ \tilde{g}_1, \tilde{g}) \right]
\leq 3C^2_\rho \varepsilon_{\tilde{g}_1},
\]
and also

\[
\rho((g'_1)^{-1} \circ \tilde{g} \circ g'_2, \tilde{g}_2) \leq 3C^2_\rho \varepsilon_{\tilde{g}_2},
\]
which shows that \((g'_2)^{-1} \circ \tilde{g} \circ g'_1\) and \((g'_1)^{-1} \circ \tilde{g} \circ g'_2\) are contained in the ball \( B_\rho(\tilde{g}_1, 4C^2_\rho \varepsilon_{\tilde{g}_1}) \) for all \( g'_1 \in B_\rho(0, \eta) \) and for all \( g'_2 \in B_\rho(0, \eta) \).

Thus, from (3.6), we obtain that

(3.8)

\[
|K_j((g'_2)^{-1} \circ \tilde{g} \circ g'_1)| > \frac{1}{2}|K_j(\tilde{g}_1)|, \quad |K_j((g'_1)^{-1} \circ \tilde{g} \circ g'_2)| > \frac{1}{2}|K_j(\tilde{g}_2)|,
\]
for all \( g'_1 \in B_\rho(0, \eta) \) and for all \( g'_2 \in B_\rho(0, \eta) \). Moreover, \( K_j((g'_2)^{-1} \circ \tilde{g} \circ g'_1) \), \( K_j((g'_1)^{-1} \circ \tilde{g} \circ g'_2) \) and \( K_j(\tilde{g}_1) \) have the same sign.

Now combining (3.6), (3.7) and (3.8), we obtain that

(3.9)

\[
|K_j(g_1, g_2)| > \frac{1}{2}r^{-Q}|K_j(\tilde{g}_1)|, \quad |K_j(g_2, g_1)| > \frac{1}{2}r^{-Q}|K_j(\tilde{g}_2)|
\]
for every $g_1 \in B_\rho(g, \eta r)$ and for every $g_2 \in B_\rho(g_*, \eta r)$, where $K_j(g_1, g_2)$, $K_j(g_2, g_1)$ and $K_j(\tilde{g}_l)$ have the same sign. Here $K_j(\tilde{g}_l)$ is a fixed constant independent of $\eta, r, g, g_1$ and $g_2$. Set

$$C(K_j) = \frac{1}{2} \min_{1 \leq l \leq m} \{|K_j(\tilde{g}_l)|\}.$$

From the lower bound (3.9) above, we further obtain that for every $\eta \in (0, \varepsilon_0)$,

$$|K_j(g_1, g_2)| > C(K_j) r^{-Q}, \quad |K_j(g_2, g_1)| > C(K_j) r^{-Q}$$

for every $g_1 \in B_\rho(g, \eta r)$, every $g_2 \in B_\rho(g \circ \delta_r(\tilde{g}^{-1}), \eta r)$ and every $\tilde{g} \in \Omega$. Moreover, since $K_j(\tilde{g}_l)$, $1 \leq l \leq m$, have the same sign, we can see, for any $\tilde{g} \in \Omega$, the sign of $K_j(g_1, g_2)$ and of $K_j(g_2, g_1)$ are invariant, respectively, for every $g_1 \in B_\rho(g, \eta r)$ and every $g_2 \in B_\rho(g \circ \delta_r(\tilde{g}^{-1}), \eta r)$.

It can be checked that there exists $r_* = r_*(s) > \frac{10}{\varepsilon_0}$ such that for $r > r_*$, we have

$$\max_{1 \leq \nu \leq \rho} \{ (r^{\nu} - 1)^{\frac{1}{\nu}} \} = (r^* - 1)^{\frac{1}{s}} \quad \text{and} \quad \min_{1 \leq \nu \leq \rho} \{ (r^{\nu} + 1)^{\frac{1}{\nu}} \} = (r^* + 1)^{\frac{1}{s}}.$$

Step 1, take $r_1 = r_*$, we can have $\eta_1 < \varepsilon_0$ such that $\eta_1 r_1 = 1$. Let

$$E_1 := \{ g' \in \tau_q(\delta_\rho(\tilde{\Omega})) : (r^*_1 - 1)^{\frac{1}{s}} < g < (r^*_1 + 1)^{\frac{1}{s}} \},$$

where $\tilde{\Omega} = \{ g^{-1} : g \in \Omega \}$ and $\tau_q(\delta_\rho(\tilde{\Omega})) = \{ g \circ \delta_\rho(\tilde{g}^{-1}) : \tilde{g} \in \Omega \}$. Recall that for any $g \in \mathcal{G}$, $g^{-1} = -g$, then we have

$$|E_1| = \frac{\sigma(\Omega)}{Q} \left[ (r^*_2 + 1)^{\frac{Q}{s}} - (r^*_1 + 1)^{\frac{Q}{s}} \right].$$

For any $g' \in E_1$, there exists $\tilde{g} = (x^{(1)}, \ldots, x^{(s)}) \in \Omega$ such that $g' = g \circ \delta_\rho(\tilde{g}^{-1})$, where $x^{(\nu)} \in \mathbb{R}^{\nu}, \nu = 1, \ldots, s$. Then by Lemma 3.1 and 3.3, we have

$$\rho(g', g \circ \delta_\rho(\tilde{g}^{-1})) = \rho(g \circ \delta_\rho(\tilde{g}^{-1}), g \circ \delta_\rho(\tilde{g}^{-1})) = \rho(\delta_\rho(\tilde{g}^{-1}), \delta_\rho(\tilde{g}^{-1}))$$

$$= \left( \sum_{\nu=1}^{s} \left| (g^{(\nu)} - r^*_1)^{(\nu)} \right| \left( \frac{\| \cdot \|^{\frac{1}{\nu}}}{\| \cdot \|} \right)^{\frac{\nu}{s}} \right)^{\frac{1}{s}}$$

$$\leq \max_{1 \leq \nu \leq s} |g^{(\nu)} - r^*_1|^{\frac{1}{s}} < 1,$$

which implies that $g' \in B_\rho(g \circ \delta_\rho(\tilde{g}^{-1}), 1)$. Therefore, by our claim, for any $g_1 \in B_\rho(g, 1)$,

$$|K_j(g_1, g')| \geq C(K_j) r_1^{-Q} \geq C(K_j, Q) \rho(g_1, g')^{-Q},$$

and also

$$|K_j(g', g_1)| \geq C(K_j, Q) \rho(g_1, g')^{-Q}.$$

Moreover, for every $g' \in E_1$ and every $g_1 \in B_\rho(g, 1)$, all $K_j(g_1, g')$ and all $K_j(g_1, g')$ have the same sign.

Step 2, take $r_2 = (r^*_1 + 2)^{\frac{1}{s}}$, we can choose $\eta_2 < \varepsilon_0$ such that $\eta_2 r_2 = 1$. Let

$$E_2 := \{ g' \in \tau_q(\delta_\rho(\tilde{\Omega})) : (r^*_2 - 1)^{\frac{1}{s}} < g < (r^*_2 + 1)^{\frac{1}{s}} \}.$$

Then

$$|E_2| = \frac{\sigma(\Omega)}{Q} \left[ (r^*_2 + 1)^{\frac{Q}{s}} - (r^*_2 - 1)^{\frac{Q}{s}} \right].$$

Moreover, $E_2 \cap E_1 = \emptyset$. By the same discussion as above, for any $g' \in E_2$ and any $g_1 \in B_\rho(g, 1)$, we have

$$|K_j(g_1, g')| \geq C(K_j, Q) \rho(g_1, g')^{-Q}, \quad |K_j(g', g_1)| \geq C(K_j, Q) \rho(g_1, g')^{-Q},$$

and all $K_j(g_1, g')$ and all $K_j(g', g_1)$ have the same sign as those when $g' \in E_1$. 

In general, for $l \geq 2$, take $r_l = (r_{l-1}^s + 2)^{\frac{1}{\nu}}$ and let

$$E_l := \{ g' \in \tau_g(\delta_\rho(\overline{\Omega})) : (r_l^s - 1)^{\frac{1}{\nu}} < \rho < (r_l^s + 1)^{\frac{1}{\nu}} \}.$$ 

Then

$$|E_l| = \frac{\sigma(\Omega)}{Q} \left[ (r_l^s + 1)^{\frac{Q}{\nu}} - (r_l^s - 1)^{\frac{Q}{\nu}} \right].$$

Moreover, $E_l \cap E_{l-1} = \emptyset$. By the same discussion as above, for any $g' \in E_l$ and any $g_1 \in B(g, 1)$, we have

$$|K_j(g_1, g')| \geq C(K_j, Q)\rho(g_1, g')^{-Q}, \quad |K_j(g, g_1)| \geq C(K_j, Q)\rho(g_1, g)^{-Q},$$

and all $K_j(g_1, g')$ as well as all $K_j(g', g_1)$ have the same sign as those when $g' \in \cup_{\nu=1}^{l-1} E_\nu$.

Set

$$G_g = \bigcup_{l=1}^{\infty} E_l,$$

and $r_o := (r_*^s - 1)^s$, then $\inf_{g \in G} \rho(g, g') = r_o$, and for every $g_1 \in B(g, 1)$ and $g_2 \in G_g$, we have

$$|K_j(g_1, g_2)| \geq C(K_j, Q)\rho(g_1, g_2)^{-Q}, \quad K_j(g_2, g_1) \geq C(K_j, Q)\rho(g_1, g_2)^{-Q},$$

and all $K_j(g_1, g_2), K_j(g_2, g_1)$ have the same sign. Moreover, $|G_g| = \infty$ and for any $R_2 > R_1 > 2r_o$,

$$|(B_{\rho}(g, R_2) \setminus B_{\rho}(g, R_1)) \cap G_g| \approx \frac{\sigma(\Omega)}{Q} (R_2^Q - R_1^Q) \approx |B_{\rho}(g, R_2) \setminus B_{\rho}(g, R_1)|,$$

where the implicit constants are independent of $g, R_1$ and $R_2$. This completes the proof of Theorem 1.1. \hfill \Box

By performing minor modification in the above proof, we can also get the similar result for any ball $B(g, R_0)$.

**Corollary 3.4.** Suppose that $G$ is a stratified nilpotent Lie group with homogeneous dimension $Q$ and that $j \in \{1, 2, \ldots, n\}$. Now let $C_{d_j}$ be the constant appeared in (2.8). There exist a large positive constant $r_*$ and a positive constant $C$ depending on $K_j, Q$ and $C_{d_j}$ such that for every $g \in G$ there exists a set $G_g \subset G$ such that $\inf_{g' \in G_g} \rho(g, g') = r_o R_0$ and that for every $g_1 \in B_{\rho}(g, R_0)$ and $g_2 \in G_g$, we have

$$|K_j(g_1, g_2)| \geq C d(g_1, g_2)^{-Q}, \quad |K_j(g_2, g_1)| \geq C d(g_1, g_2)^{-Q},$$

all $K_j(g_1, g_2)$ as well as all $K_j(g_2, g_1)$ have the same sign.

Moreover, the set $G_g$ is regular, in the sense that $|G_g| = \infty$ and that for any $R_2 > R_1 > 2r_o R_0$,

$$|(B_{\rho}(g, R_2) \setminus B_{\rho}(g, R_1)) \cap G_g| \approx |B_{\rho}(g, R_2) \setminus B_{\rho}(g, R_1)|,$$

where the implicit constants are independent of $g, R_1$ and $R_2$.

**Proof.** By taking $r_1 = r_o R_0, r_j = (r_{j-1}^s + 2R_0^s)^{\frac{1}{\nu}}$ for $j \in \mathbb{N}$ and $j \geq 2$, and

$$E_l := \{ g' \in \tau_g(\delta_\rho(\overline{\Omega})) : (r_l^s - R_0^s)^{\frac{1}{\nu}} < \rho < (r_l^s + R_0^s)^{\frac{1}{\nu}} \}, \quad l \in \mathbb{N},$$

in the proof of Theorem 1.1, we can see that for any $g' \in E_l$ and any $g_1 \in B(g, R_0)$, we have

$$|K_j(g_1, g')| \geq C(K_j, Q)\rho(g_1, g')^{-Q} \geq C(K_j, Q, C_{d_j}) d(g_1, g')^{-Q},$$

$$|K_j(g, g_1)| \geq C(K_j, Q)\rho(g_1, g)^{-Q} \geq C(K_j, Q, C_{d_j}) d(g_1, g)^{-Q},$$

as well as all $K_j(g_1, g')$ as well as all $K_j(g', g_1)$ have the same sign as those when $g' \in \cup_{\nu=1}^{l-1} E_\nu$. \hfill \Box
and all \(K_j(g_1, g')\) as well as all \(K_j(g', g_1)\) have the same sign as those when \(g' \in \bigcup_{\nu=1}^{l-1} E_{\nu} \). Set

\[
G_g = \bigcup_{l=1}^{\infty} E_l,
\]

then \(\inf_{g' \in G} \rho(g, g') = r_o R_o\), and for every \(g_1 \in B_\rho(g, R_0)\) and \(g_2 \in G_g\), we have

\[
|K_j(g_1, g_2)| \geq C(K_j, Q, C_{d_2}) d(g_1, g_2)^{-Q}, \quad K_j(g_2, g_1)| \geq C(K_j, Q, C_{d_2}) d(g_1, g_2)^{-Q},
\]

and all \(K_j(g_1, g_2), K_j(g_2, g_1)\) have the same sign. The rest part of the proof is the same as that of Theorem 1.1. \(\square\)

### 4. Compactness of Riesz Transform Commutator

In this section, we will give the proof of Theorem 1.2. We need the following upper and lower bounds for integrals of \([b, R_\ell]\).

**Lemma 4.1.** Assume that \(b \in BMO(G)\) with \(\|b\|_{BMO(G)} = 1\) and there exist \(\delta > 0\) and a sequence \(\{B_j\}_{j=1}^\infty := \{B(g_j, r_j)\}_{j=1}^\infty\) of balls such that for each \(j\),

\[
M(b, B_j) > \delta.
\]

Then there exist functions \(\{f_j\} \subset L_0^\infty(G)\) with \(\|f_j\|_{L_0^\infty(G)} = 1\), positive constants \(\beta_1 > C_{\ell^2_1 r_o, \beta_2, \beta_3}\) such that for any integers \(k \geq \lceil \log_2 \beta_1 \rceil\) and \(j\),

\[
\int_{(2^{k+1}B_j \setminus 2^kB_j) \cap G_{g_j}} \|\langle b, R_\ell \rangle f_j(g)\|^p w(g)dg \geq \beta_2 \delta^{p/2} 2^{-Qk} \frac{w(2^{k+1}B_j)}{w(B_j)},
\]

and

\[
\int_{2^{k+1}B_j \setminus 2^{k}B_j} \|\langle b, R_\ell \rangle f_j(g)\|^p w(g)dg \leq \beta_3 2^{-Qk} \frac{w(2^{k+1}B_j)}{w(B_j)},
\]

where \(C_{d_1} \) and \(C_{d_2}\) are in (2.5), \(r_o\) and \(G_{g_j}\) are the same as those in Theorem 1.1.

**Proof.** For every \(j \in \mathbb{N}\), we define \(f_j\) as follows. By the definition of median value, we can find disjoint subsets \(E_{j1}, E_{j2} \subset B_j\) such that

\[
E_{j1} \supset \{g \in B_j : b(g) \geq m_b(B_j)\}, \quad E_{j2} \supset \{g \in B_j : b(g) \leq m_b(B_j)\},
\]

and \(|E_{j1}| = |E_{j2}| = \frac{1}{2} |B_j|\). Define \(f_j(g) = w(B_j)^{-\frac{1}{p}} (\chi_{E_{j1}}(g) - \chi_{E_{j2}}(g))\). Then \(f_j\) satisfies \(\text{supp } f_j \subset B_j\) and for every \(g \in B_j\),

\[
|f_j(g)| = w(B_j)^{-\frac{1}{p}}, \quad f_j(g) (b(g) - m_b(B_j)) \geq 0.
\]

Moreover,

\[
\int_{B_j} f_j(g)dg = 0, \quad \|f_j\|_{L_0^\infty(G)} = 1.
\]

Note that \([b, R_\ell]f = R_\ell((b - m_b(B_j))f) - (b - m_b(B_j))R_\ell(f)\). For any \(g \in G \setminus (2B_j)\), by (3.1) and (4.5), we have

\[
\|(b - m_b(B_j))R_\ell(f_j)(g)\| = |b(g) - m_b(B_j)||\int_{B_j} (K_\ell(g, g') - K_\ell(g, g_j)) f_j(g')dg'| \leq |b(g) - m_b(B_j)| \int_{B_j} |K_\ell(g, g') - K_\ell(g, g_j)||f_j(g')|dg' \leq |b(g) - m_b(B_j)| \int_{B_j} d(g', g_j)^{Q+1} |f_j(g')|dg'
\]
and (4.6), we have

$$\int_{2^j+1 B} |b(g) - m_b(B)|^p \, dg \lesssim \int_{2^j+1 B} |b(g) - m_b(2^{j+1}B)|^p \, dg + |2^j+1B| |m_b(2^{j+1}B) - m_b(B)|^p$$

(4.6)

By John-Nirenberg inequality (c.f. [3]), for each \(l \in \mathbb{N} \) and \(B \subset \mathcal{G} \),

$$\int_{2^l+1 B} |b(g) - m_b(B)|^p \, dg \lesssim \int_{2^l+1 B} |b(g) - m_b(2^{l+1}B)|^p \, dg + |2^l+1B| |m_b(2^{l+1}B) - m_b(B)|^p$$

Since \(w \in A_p(\mathcal{G}) \), there exists \(r > 1 \) such that \(w \in RH_r(\mathcal{G}) \). Then by Hölder’s inequality and (4.6), we have

$$\int_{2^k+1 B \setminus 2^k B_j} \left| (b - m_b(B_j)) \mathcal{R}_\ell(f_j)(g) \right|^p w(g) \, dg \lesssim \frac{r^p |B_j|^p}{w(B_j)} \int_{2^k+1 B \setminus 2^k B_j} \frac{1}{d(g, g_j)^{p(Q+1)}} |b(g) - m_b(B_j)|^p w(g) \, dg$$

(4.7)

$$\lesssim \frac{1}{2^{kp(Q+1)}w(B_j)} \int_{2^{k+1} B \setminus 2^k B_j} |b(g) - m_b(B_j)|^p w(g) \, dg$$

$$\lesssim \frac{1}{2^{kp(Q+1)}w(B_j)} \left( \int_{2^{k+1} B \setminus 2^k B_j} |b(g) - m_b(B_j)|^{pr'} \, dg \right)^{\frac{p}{r'}} \left( \int_{2^{k+1} B_j} w(g)^r \, dg \right)^{\frac{1}{r'}}$$

$$\lesssim \frac{k^p}{2^{kp(Q+1)}w(B_j)} |2^{k+1} B_j|^{\frac{1}{p'}} \left| 2^{k+1} B_j \right|^{\frac{1}{p'}} \left( \frac{1}{|2^{k+1} B_j|} \int_{2^{k+1} B_j} w(g)^r \, dg \right)^{\frac{1}{r'}}$$

$$\leq \beta_4 \frac{k^p}{2^{kp(Q+1)}w(B_j)} |w(2^{k+1} B_j)| \frac{w(2^{k+1} B_j)}{w(B_j)}$$

For \(g \in (\mathcal{G} \setminus 2^k B_j) \cap G_{g_j}, k > \log_2 \left( \frac{C_{\mathcal{G}}}{\epsilon_{d_1} r_0} \right) \), by Corollary 3.4 and (3.1), we have

$$|\mathcal{R}_\ell((b - m_b(B_j))f_j(g))| = \left| \int_{E_{g_1 \cup E_{g_2}}} K_{\ell}(g, g') \left| (b - m_b(B_j))f_j(g') \right| \, dg' \right|$$

$$\lesssim \left| \int_{E_{g_1 \cup E_{g_2}}} \frac{1}{d(g, g_j)^Q} |b(g') - m_b(B_j)| \, dg' \right|$$

$$\lesssim \frac{1}{w(B_j)^{\frac{1}{Q}} d(g, g_j)^Q} \int_{B_j} |b(g') - m_b(B_j)| \, dg'$$

$$\lesssim \frac{1}{w(B_j)^{\frac{1}{Q}} d(g, g_j)^Q}$$

Using Corollary 3.4 again and Lemma 2.5, we have

$$\int_{(2^{k+1} B_j \setminus 2^k B_j) \cap G_{g_j}} \left| \mathcal{R}_\ell((b - m_b(B_j))f_j(g)) \right|^p w(g) \, dg$$

(4.8)

$$\lesssim \frac{\delta^p |B_j|^p}{w(B_j)} \int_{(2^{k+1} B_j \setminus 2^k B_j) \cap G_{g_j}} \frac{1}{d(g, g_j)^{Qp}} w(g) \, dg$$

$$\lesssim \frac{\delta^p}{2^{2kpw(B_j)}} \int_{(2^{k+1} B_j \setminus 2^k B_j) \cap G_{g_j}} w(g) \, dg$$

$$\lesssim \frac{\delta^p}{2^{2kpw(B_j)}} \left( \frac{|(2^{k+1} B_j \setminus 2^k B_j) \cap G_{g_j}|}{|2^{k+1} B_j|} \right)^p \frac{w(2^{k+1} B_j)}{w(2^{k+1} B_j)}$$
\[ \sqrt{\frac{\delta^p w(2^{k+1}B_j)}{2^{Qp}w(B_j)}} \geq \beta_2 \frac{\delta^p w(2^{k+1}B_j)}{2^{Qp}w(B_j)}. \]

Therefore, \[ \beta_2 \frac{\delta^p}{2^{Qp} + p - 1} - \beta_4 \frac{k^p}{2^{Qp} + 1} \geq \beta_2 \frac{\delta^p}{2^{Qp} + p}. \]

Then by (4.7) and (4.8), for any \( k \geq \log_2 \beta_1 \),

\[ \beta_2 \frac{\delta^p}{2^{Qp} + p - 1} - \beta_4 \frac{k^p}{2^{Qp} + 1} \geq \beta_2 \frac{\delta^p}{2^{Qp} + p}. \]

On the other hand, for \( g \in G \setminus 2B_j \), we have

\[ |\mathcal{R}_\ell((b - m_b(B_j))^f_j(g))| \leq \frac{1}{w(B_j)} \int_{B_j} |K_\ell(g, g')| |b(g') - m_b(B_j)| dg' \]

\[ \leq \frac{1}{w(B_j)} \int_{B_j} \frac{1}{d(g, g')^Q} |b(g') - m_b(B_j)| dg' \]

\[ \leq \frac{1}{w(B_j)^{\frac{1}{Q}}} d(g, g')^Q \int_{B_j} |b(g') - m_b(B_j)| dg' \]

\[ \leq \frac{1}{w(B_j)^{\frac{1}{Q}}} d(g, g')^Q |B_j|. \]

Therefore,

\[ \int_{2^{k+1}B_j \setminus 2^kB_j} |\mathcal{R}_\ell((b - m_b(B_j))^f_j(g))|^p w(g) dg \leq \frac{|B_j|^p}{w(B_j)} \int_{2^{k+1}B_j \setminus 2^kB_j} \frac{1}{d(g, g')^p} w(g) dg \]

\[ \leq \frac{|B_j|^p}{w(B_j)} \frac{1}{(2^k r_j)^p} w(2^{k+1}B_j \setminus 2^kB_j) \]

\[ \leq \frac{1}{2^{kpQ}} \frac{w(2^{k+1}B_j)}{w(B_j)}. \]

Take \( k \) large enough such that \( \frac{k}{p} < 1 \), then by (4.7), we can obtain

\[ \int_{2^{k+1}B_j \setminus 2^kB_j} |[b, \mathcal{R}_\ell f_j(g)]|^p w(g) dg \leq \int_{2^{k+1}B_j \setminus 2^kB_j} |\mathcal{R}_\ell((b - m_b(B_j))^f_j(g))|^p w(g) dg \]

\[ + \int_{2^{k+1}B_j \setminus 2^kB_j} |(b - m_b(B_j))^f_j(g)|^p w(g) dg \]

\[ \leq \left( \frac{1}{2^{kpQ}} + \frac{k^p}{2^{kp(Q+1)}} \right) \frac{w(2^{k+1}B_j)}{w(B_j)}. \]
\[ \leq \beta \frac{1}{2^k p q} \frac{w(2^{k+1}B_j)}{w(B_j)}. \]

This completes the proof of Lemma 4.1. \(\square\)

Recall that in [8], we have established the Bloom-type two weight estimates for the commutators \([b, R_j]\). From this result, for the case of one weight, we have the following estimates.

**Lemma 4.2** ([8], Theorem 1.2). Suppose \(w \in A_p(G)\) and \(j \in \{1, \ldots, n\}\). Then

(i) if \(b \in \text{BMO}(G)\), then

\[ \| [b, R_j] (f) \|_{L^p_w(G)} \lesssim \| b \|_{\text{BMO}(G)} \| f \|_{L^p_w(G)}. \]

(ii) for every \(b \in L^1_{\text{loc}}(G)\), if \([b, R_j]\) is bounded on \(L^p_w(G)\), then \(b \in \text{BMO}(G)\) with

\[ \| b \|_{\text{BMO}(G)} \lesssim \| [b, R_j] \|_{L^p_w(G) \rightarrow L^p_w(G)}. \]

Górka and Macios established the Riesz-Kolmogorov theorem on doubling measure spaces [11, Theorem 1]. Since \(A_p\) weights are doubling, we have the following corresponding result.

**Lemma 4.3.** Let \(1 < p < \infty\), \(g_0 \in G\). Then the subset \(F\) of \(L^p_w(G)\) is relatively compact in \(L^p_w(G)\) if and only if the following conditions are satisfied:

(i) \(F\) is bounded.

(ii) \(\lim_{R \to \infty} \int_{G \setminus B(g_0, R)} |f(g)|^p w(g) dg = 0\)

uniformly for \(f \in F\).

(iii) \(\lim_{r \to 0} \int_G |f(g) - f_{B(g, r)}|^p w(g) dg = 0\)

uniformly for \(f \in F\).

For the proof of Theorem 1.2, we also need to establish the characterisation of \(\text{VMO}(G)\). We will give its proof in Appendix. For the Euclidean case one can refer to [25].

**Theorem 4.4.** Let \(f \in \text{BMO}(G)\). Then \(f \in \text{VMO}(G)\) if and only if \(f\) satisfies the following three conditions.

(i) \(\lim_{a \to 0} \sup_{r_B = a} M(f, B) = 0\).

(ii) \(\lim_{a \to \infty} \sup_{r_B = a} M(f, B) = 0\).

(iii) \(\lim_{r \to \infty} \sup_{B \subset G \setminus B(0, r)} M(f, B) = 0\).

Now we prove Theorem 1.2.

**Proof of Theorem 1.2.** **Sufficient condition:** Assume that \([b, R_d]\) is compact on \(L^p_w(G)\), then \([b, R_d]\) is bounded on \(L^p_w(G)\). By Lemma 4.2, we have \(b \in \text{BMO}(G)\). Without loss of generality, we may assume that \(\| b \|_{\text{BMO}(G)} = 1\). To show \(b \in \text{VMO}(G)\), we may use a contradiction argument via Theorem 4.3. Suppose that \(b \notin \text{VMO}(G)\), then \(b\) does not satisfy at least one of the three conditions in Theorem 4.4. We will consider these three cases separately.

**Case (i).** Suppose \(b\) does not satisfy (i) in Theorem 4.4. Then there exist \(\delta > 0\) and a sequence \(\{B_j\}_{j=1}^\infty := \{B(x_j, r_j)\}_{j=1}^\infty\) of balls such that \(M(f, B_j) > \delta\) and \(r_j \to 0\) as \(j \to \infty\).
Let $f_j, \beta_1 > \frac{C_{d_2}}{r_{d_1}} r_o, \beta_2, \beta_3$ be as in Lemma 4.1 and $C_1, C_2$ in Lemma 2.5 and $\gamma_1 > \beta_1$ large enough such that

\begin{equation}
\gamma_2^p := \frac{\beta_2}{C_2} 2^{pQ(1-\frac{1}{r_o} - 3\gamma_1)} \beta_1^Q \frac{1}{1 - 2^{-Q\sigma}} \geq \frac{\beta_3}{C_1} 2^{pQ(p\sigma - [\log_2 \gamma_1] \sigma)}
\end{equation}

where $\sigma$ is in (4.12).

Since $r_j \to 0$ as $j \to \infty$, we may choose a subsequence $\{B_{j_1}^{(1)}\}$ of $\{B_j\}$ such that

\begin{equation}
\frac{|B_{j_1}^{(1)}|}{|B_j^{(1)}|} \leq \frac{1}{\gamma_1^Q}.
\end{equation}

For fixed $i, m \in \mathbb{N}$, denote

$$\Omega := \gamma_1 B_{j_i}^{(1)} \setminus \beta_1 B_{j_i}^{(1)}, \quad \Omega_1 := \Omega \setminus \gamma_1 B_{j_i+m}^{(1)}, \quad \Omega_2 := \mathcal{G} \setminus \gamma_1 B_{j_i+m}^{(1)}.$$ 

It is clear that

$$\Omega_1 \subset \gamma_1 B_{j_i}^{(1)} \cap \Omega_2, \quad \Omega_1 = \Omega \setminus (\Omega \setminus \Omega_2).$$

Then we have

$$\left| \{b, \mathcal{R}_d \}(f_{j_i}) - \{b, \mathcal{R}_d \}(f_{j_{i+m}}) \right|_{L_p^p(G)} \geq \left( \int_{\Omega_1} \left| \{b, \mathcal{R}_d \}(f_{j_i})(g) - \{b, \mathcal{R}_d \}(f_{j_{i+m}})(g) \right|^p w(g)dg \right)^{\frac{1}{p}}$$

$$\geq \left( \int_{\Omega_1} \left| \{b, \mathcal{R}_d \}(f_{j_i})(g) \right|^p w(g)dg \right)^{\frac{1}{p}} - \left( \int_{\Omega_2} \left| \{b, \mathcal{R}_d \}(f_{j_{i+m}})(g) \right|^p w(g)dg \right)^{\frac{1}{p}}$$

$$= \left( \int_{\Omega \setminus (\Omega \setminus \Omega_2)} \left| \{b, \mathcal{R}_d \}(f_{j_i})(g) \right|^p w(g)dg \right)^{\frac{1}{p}} - \left( \int_{\Omega_2} \left| \{b, \mathcal{R}_d \}(f_{j_{i+m}})(g) \right|^p w(g)dg \right)^{\frac{1}{p}}$$

$$=: I_1 - I_2.$$

We first consider the term $I_1$. Assume that $\Gamma_{j_i} := \Omega \setminus \Omega_2 \neq \emptyset$, then $\Gamma_{j_i} \subset \gamma_1 B_{j_i+m}^{(1)}$. Hence, by (4.10), we have

$$|\Gamma_{j_i}| \leq |\gamma_1 B_{j_i+m}^{(1)}| = \gamma_1^Q |\Gamma_{j_i+m}| \leq |B_{j_i}^{(1)}|.$$ 

Now for each $k \geq [\log_2 \beta_1]$,

$$2^{k+1} B_{j_i}^{(1)} \setminus 2^k B_{j_i}^{(1)} = |B(0,1)| \left( 2^{(k+1)Q - 2^{kQ}} r_{j_i}^Q \right) r_{j_i}^Q > |2^k B_{j_i}^{(1)}| \geq |\Gamma_{j_i}|.$$ 

From this fact, it follows that there exist at most two rings, $2^{k_0+2} B_{j_i}^{(1)} \setminus 2^{k_{0+1}} B_{j_i}^{(1)}$ and $2^{k_{0+1}} B_{j_i}^{(1)} \setminus 2^{k_0} B_{j_i}^{(1)}$ such that $\Gamma_{j_i} \subset (2^{k_0+2} B_{j_i}^{(1)} \setminus 2^{k_{0+1}} B_{j_i}^{(1)}) \cup (2^{k_{0+1}} B_{j_i}^{(1)} \setminus 2^{k_0} B_{j_i}^{(1)})$. Then by (4.2) and Lemma 2.5 we have

\begin{equation}
P_k = \int_{\Omega \setminus (\Omega \setminus \Omega_2)} \left| \{b, \mathcal{R}_d \}(f_{j_i})(g) \right|^p w(g)dg \geq \sum_{k=[\log_2 \beta_1] + 1, k \neq k_0, k_0+1} |\log_2 \gamma_1| \int_{2^{k+1} B_{j_i}^{(1)} \setminus 2^k B_{j_i}^{(1)}} \left| \{b, \mathcal{R}_d \}(f_{j_i})(g) \right|^p w(g)dg \geq \beta_2 \delta^p \sum_{k=[\log_2 \beta_1] + 1, k \neq k_0, k_0+1} 2^{-kQ} \frac{w(2^{k+1} B_{j_i}^{(1)})}{w(B_{j_i}^{(1)})}.
\end{equation}
Thus not compact on \( \delta > 0 \).

Then we have

\[
\begin{align*}
\beta_2 & \delta^p \sum_{k=\left[\log_2 \beta_1 \right]+1}^{\left[\log_2 \beta_1 \right]+1, k \neq k_0, k_1+1} 2^{-k Q p_2 (k+1) Q (1-\frac{1}{p})} \\
& \geq \beta_2 \delta^p Q (1-\frac{1}{p}) \sum_{k=\left[\log_2 \beta_1 \right]+3}^{\left[\log_2 \beta_1 \right]} 2^{-k Q (p-1+\frac{1}{p})} \\
& \geq \beta_2 \delta^p Q (4-\frac{1}{p}-3p) \beta^1_1 Q (1-p-\frac{1}{p}) =: \gamma_2^p.
\end{align*}
\]

If \( \Omega \setminus \Omega_2 = \emptyset \), the inequality above still holds.

For \( I_2 \), by (4.3) in Lemma 4.1 we have

\[
I_2^p = \int_{G \setminus B_{j+1}^{(1)}} \left|[b, \mathcal{R}_\ell](f_{j+1})(g)\right|^p w(g) \, dg \\
\leq \sum_{k=\left[\log_2 \gamma_1 \right]}^{\infty} \int_{G \setminus B_{j+1}^{(1)}} \left|[b, \mathcal{R}_\ell](f_{j+1})(g)\right|^p w(g) \, dg \\
\leq \beta_3 \sum_{k=\left[\log_2 \gamma_1 \right]}^{\infty} 2^{-k Q p} \frac{w(2^{k+1} B_{j+1}^{(1)})}{w(1)} \\
\leq \frac{\beta_3}{C_1} \sum_{k=\left[\log_2 \gamma_1 \right]}^{\infty} 2^{-k Q p} Q (k+1) Q (p-\sigma) \leq \frac{\beta_3}{C_1} 2^{Q (p-\sigma-\left[\log_2 \gamma_1 \right])} \leq \frac{\gamma_2^p}{2}.
\]

Consequently,

\[
\left\| [b, \mathcal{R}_\ell](f_j) - [b, \mathcal{R}_\ell](f_{j+1}) \right\|_{L_p^w(G)} \geq \frac{\gamma_2^p}{2}.
\]

Therefore, \( \{[b, \mathcal{R}_\ell](f_j)\}_{j=1}^{\infty} \) is not relatively compact in \( L_p^w(G) \), which implies that \( [b, \mathcal{R}_\ell] \) is not compact on \( L_p^w(G) \). Thus \( b \) satisfies condition (i).

**Case (ii).** If \( b \) does not satisfy (ii) in Theorem 4.4 then there also exist \( \delta > 0 \) and a sequence \( \{B_j\}_{j=1}^{\infty} \) of balls such that \( M(f, B_j) > \delta \) and \( r_j \to \infty \) as \( j \to \infty \). We take a subsequence \( \{B_{j(2)}\} \) of \( B_j \) such that

\[
\left| B_{j(2)} \right| \leq \frac{1}{\gamma_1^j}.
\]

The method in this case is very similar to that in case (i), we just redefine our sets in a reversed order, i.e. for fixed \( i \) and \( m \), let

\[
\tilde{\Omega}_i := \gamma_1 B_{j+1}^{(2)} \cap \beta_1 B_{j+i}^{(2)}, \quad \tilde{\Omega}_i := \Omega \setminus \gamma_1 B_{j+i}^{(2)}, \quad \tilde{\Omega}_2 := G \setminus \gamma_1 B_{j+i}^{(2)}.
\]

Then we have

\[
\tilde{\Omega}_1 \subset \gamma_1 B_{j+i}^{(2)} \cap \tilde{\Omega}_2, \quad \tilde{\Omega}_1 = \tilde{\Omega} \setminus \left( \tilde{\Omega} \setminus \tilde{\Omega}_2 \right).
\]

Like in case (i), by Lemma 4.1 and (4.13), we can see that \( [b, \mathcal{R}_\ell] \) is not compact on \( L_p^w(G) \). Thus \( b \) satisfies condition (ii) of Theorem 4.4.

**Case (iii).** Assume that condition (iii) in Theorem 4.4 does not hold for \( b \). Then there exists \( \delta > 0 \) such that for any \( r > 0 \), there exists \( B \subset G \setminus B(0, r) \) with \( M(b, B) > \delta \).
We claim that for the $\delta$ above, there exists a sequence $\{B_j\}_j$ of balls such that for any $j$,
\begin{equation}
M(b, B_j) > \delta, \tag{4.14}
\end{equation}
and for any $i \neq m$,
\begin{equation}
\gamma_1 B_i \cap \gamma_1 B_m = \emptyset. \tag{4.15}
\end{equation}
To see this, let $C_\delta > 0$ to be determined later. Then for $R_1 > C_\delta$, there exists a ball $B_1 := B(g_1, r_1) \subset G \setminus \{0, R_1\}$ such that (4.14) holds. Similarly, for $R_j := |g_j - 1| + 4\gamma_1 C_\delta$, $j = 2, 3, \ldots$, there exists $B_j := B(g_j, r_j) \subset G \setminus \{0, R_j\}$ satisfying (4.14). Repeating this procedure, we can obtain a sequence of balls $\{B_j\}_j$ with each $B_j$ satisfying (4.14). Moreover, since $b$ satisfies condition (ii) in Theorem 4.4, for the $\delta$ above, there exists a constant $C_\delta > 0$ such that $M(b, B) < \delta$ for any ball $B$ satisfying $r_B > C_\delta$. This together with the choice of $\{B_j\}$ implies that $r_j \leq C_\delta =: C_\delta$ for all $j$. Therefore, for each $j$,
\begin{equation}
\gamma_1 r_j < \gamma_1 C_\delta < 4\gamma_1 C_\delta.
\end{equation}
Thus, for all $i \neq m$, without loss of generality, we may assume $i < m$,
\begin{equation}
d(\gamma_1 B_i, \gamma_1 B_m) \geq R_{i+1} - (|g_i| + \gamma_1 r_i - \gamma_1 r_{i+1}) \geq 4\gamma_1 C_\delta - 2\gamma_1 C_\delta = 2\gamma_1 C_\delta,
\end{equation}
which implies the claim.

We define
\begin{equation}
\hat{\Omega}_1 := \gamma_1 B_j \setminus \beta_1 B_j, \quad \hat{\Omega}_2 := G \setminus \gamma_1 B_{j+m}.
\end{equation}
Observe that $\hat{\Omega}_1 \subset \hat{\Omega}_2$. Therefore,
\begin{equation}
\begin{aligned}
\| [b, R\xi] (f_j) - [b, R\xi] (f_{j+m}) \|_{L^p_v(G)} &\geq \left( \int_{\hat{\Omega}_1} | [b, R\xi] (f_j)(g) - [b, R\xi] (f_{j+m})(g) |^p w(g) dg \right)^{\frac{1}{p}} \\
&\geq \left( \int_{\hat{\Omega}_1} | [b, R\xi] (f_j)(g) |^p w(g) dg \right)^{\frac{1}{p}} - \left( \int_{\hat{\Omega}_2} | [b, R\xi] (f_{j+m})(g) |^p w(g) dg \right)^{\frac{1}{p}} \\
&=: I_1 - I_2.
\end{aligned}
\end{equation}
By the similar estimates of $I_1$ and $I_2$ in case (i) and the definition of $\gamma_2$ in (4.11), we can deduce that $I_1^p \geq \gamma_2^p$ and $I_2^p \leq \frac{1}{2} \gamma_2^p$. Consequently,
\begin{equation}
\| [b, R\xi] f_j - [b, R\xi] f_{j+m} \|_{L^p_v(G)} \geq \gamma_2,
\end{equation}
which contradicts to the compactness of $[b, R\xi]$ on $L^p_v(G)$, thereby $b$ also satisfies condition (iii) in Theorem 4.4. This finishes the proof of the sufficiency of Theorem 1.2.

**Necessary condition:** Assume that $b \in \text{VMO}(G)$, we will show that $[b, R\xi]$ is compact on $L^p_v(G)$. Since $b \in \text{VMO}(G)$, for any $\epsilon > 0$, there exists $b_\epsilon \in C_0^\infty(G)$ such that
\begin{equation}
\| b - b_\epsilon \|_{\text{BMO}(G)} < \epsilon.
\end{equation}
By Lemma 4.2, we can see
\begin{equation}
\| [b, R\xi] (f) - [b_\epsilon, R\xi] (f) \|_{L^p_v(G)} = \| [b - b_\epsilon, R\xi] (f) \|_{L^p_v(G)} \leq \| b - b_\epsilon \|_{\text{BMO}(G)} \| f \|_{L^p_v(G)}.
\end{equation}
Therefore,
\begin{equation}
\| [b, R\xi] - [b_\epsilon, R\xi] \|_{L^p_v(G) \to L^p_v(G)} \leq \| b - b_\epsilon \|_{\text{BMO}(G)}.
\end{equation}
Thus, it suffices to show that $[b, R\xi]$ is a compact operator for $b \in C_0^\infty(G)$.

Suppose $b \in C_0^\infty(G)$, to show $[b, R\xi]$ is compact on $L^p_v(G)$, it suffices to show that for every bounded subset $E \subset L^p_v(G)$, the set $[b, R\xi] E$ is precompact. Thus, we only need to show that $[b, R\xi] E \subset L^p_v(G)$ satisfies (i)-(iii) in Lemma 4.3. Firstly, by Lemma 4.2, $[b, R\xi]$ is bounded on $L^p_v(G)$, which implies that $[b, R\xi] E$ satisfies (i) in Lemma 4.3.
Next we will show that $[b,\mathcal{R}_t]E$ satisfies (ii) in Lemma 4.3. We may assume that $b \in C_0^\infty(G)$ with $\text{supp} b \subset B(0, R)$. Then for $t > 2$, we have
\[ \| [b, \mathcal{R}_t] f \|_{L^p(E \setminus B(0, tR))} \]
\[ = \left( \int_{G \setminus B(0, tR)} |b(g)\mathcal{R}_t(f)(g) - \mathcal{R}_t(bf)(g)|^p w(g)dg \right)^{\frac{1}{p}} \]
\[ \leq \left( \int_{G \setminus B(0, tR)} |b\mathcal{R}_t(f)(g)|^p w(g)dg \right)^{\frac{1}{p}} + \left( \int_{G \setminus B(0, tR)} |\mathcal{R}_t(bf)(g)|^p w(g)dg \right)^{\frac{1}{p}}. \]

Since $\text{supp} b \subset B(0, R)$ and $B(0, R) \cap (G \setminus B(0, tR)) = \emptyset$, the first term on the right hand side of the above inequality is zero. For $g \in G \setminus B(0, tR)$, by (4.11), Hölder’s inequality, the definition of $A_p$ weights and the fact that $b \in C_0^\infty(G)$, we have
\[ |\mathcal{R}_t(bf)(g)| \leq \int_G |K_t(g, g')||b(g')||f(g')|dg' \lesssim \int_{B(0, R)} \frac{1}{d(g, g')^q} |b(g')||f(g')|dg' \]
\[ \lesssim \frac{1}{d(g, g')^q} \int_{B(0, R)} |f(g')|w(g')^{\frac{2}{p}} w(g')^{-\frac{1}{p}} dg' \]
\[ \lesssim \frac{1}{d(g, g')^q} \|f\|_{L^p(G)} \left( \int_{B(0, R)} w(g')^{-\frac{1}{p}} dg' \right)^{1-\frac{1}{p}} \]
\[ \lesssim \frac{1}{d(g, g')^q} \|f\|_{L^p(G)} \frac{|B(0, R)|}{w(B(0, R))} \frac{1}{d(g, g')^q} \]
Therefore,
\[ \| [b, \mathcal{R}_t] f \|_{L^p(E \setminus B(0, tR))} \lesssim \|f\|_{L^p(G)} \frac{|B(0, R)|}{w(B(0, R))} \left( \int_{G \setminus B(0, tR)} \frac{1}{d(g)^{2q}} w(g)dg \right)^{\frac{1}{p}} \]
\[ \lesssim \|f\|_{L^p(G)} \frac{|B(0, R)|}{w(B(0, R))} \left( \sum_{k=\log_2 t}^{\infty} \int_{2^{k+1}B(0, R) \setminus 2^kB(0, R)} \frac{1}{d(g)^{2q}} w(g)dg \right)^{\frac{1}{p}} \]
\[ \lesssim \|f\|_{L^p(G)} \frac{|B(0, R)|}{w(B(0, R))} \left( \sum_{k=\log_2 t}^{\infty} \frac{1}{(2^k R)^2 q^p} w(2^{k+1}B(0, R) \setminus 2^kB(0, R)) \right)^{\frac{1}{p}} \]
\[ \lesssim \|f\|_{L^p(G)} \left( \sum_{k=\log_2 t}^{\infty} \frac{1}{2^{kq} q^p} \right)^{\frac{1}{p}} \lesssim \|f\|_{L^p(G)} \left( \sum_{k=\log_2 t}^{\infty} \frac{1}{2^{kq} q^p} \right)^{\frac{1}{p}} \]
\[ \lesssim \left( \frac{2^q}{1 - 2^{-q^p}} \right)^{\frac{1}{p}} \|f\|_{L^p(G)} t^{-\frac{q^p}{p}}, \]
which approaches to zero as $t$ goes to infinity. This proves condition (ii) in Lemma 4.3.

At last, we prove that $[b, \mathcal{R}_t]E$ satisfies (iii) in Lemma 4.3. By a change of variables, we have
\[ [b, \mathcal{R}_t] f(g) - ([b, \mathcal{R}_t] f)_{B(g, r)} = \frac{1}{|B(g, r)|} \int_{B(g, r)} [b, \mathcal{R}_t] f(g) - [b, \mathcal{R}_t] f(g')dg' \]
\[ = \frac{1}{|B(0, r)|} \int_{B(0, r)} [b, \mathcal{R}_t] f(g) - [b, \mathcal{R}_t] f(\tilde{g})d\tilde{g}. \]
Take an arbitrary $\varepsilon \in (0, \frac{1}{2})$, $r \in \mathbb{R}_+$ and $\tilde{g} \in B(0, r)$. Then for any $g \in G$,

$$[b, \mathcal{R}_\ell]f(g) - [b, \mathcal{R}_\ell]f(\tilde{g}g)$$

$$= \int_{G} K_\ell(g, g') \left( b(g) - b(g') \right) f(g') dg' - \int_{G} K_\ell(\tilde{g}g, g') \left( b(\tilde{g}g) - b(g') \right) f(g') dg'$$

$$= \int_{d(g, g') > \varepsilon^{-1}d(\tilde{g}, 0)} K_\ell(g, g') \left( b(g) - b(\tilde{g}g) \right) f(g') dg'$$

$$+ \int_{d(g, g') > \varepsilon^{-1}d(\tilde{g}, 0)} (K_\ell(g, g') - K_\ell(\tilde{g}g, g')) \left( b(\tilde{g}g) - b(g') \right) f(g') dg'$$

$$+ \int_{d(g, g') \leq \varepsilon^{-1}d(\tilde{g}, 0)} K_\ell(g, g') \left( b(g) - b(g') \right) f(g') dg'$$

$$- \int_{d(g, g') \leq \varepsilon^{-1}d(\tilde{g}, 0)} K_\ell(\tilde{g}g, g') \left( b(\tilde{g}g) - b(g') \right) f(g') dg'$$

$$=: L_1 + L_2 + L_3 + L_4.$$

Let us first consider the term $L_2$. By (3.1), we have

$$|L_2| \leq \int_{d(g, g') > \varepsilon^{-1}d(\tilde{g}, 0)} \left| K_\ell(g, g') - K_\ell(\tilde{g}g, g') \right| \left| b(\tilde{g}g) - b(g') \right| |f(g')| dg'$$

$$\lesssim d(\tilde{g}, 0) \sum_{k = \lfloor \log_2 \varepsilon^{-1} \rfloor}^{\infty} \int_{d(g, g') \leq 2^{k+1}d(\tilde{g}, 0)} \frac{1}{d(g, g')^{Q+1}} |f(g')| dg'$$

$$\leq d(\tilde{g}, 0) \sum_{k = \lfloor \log_2 \varepsilon^{-1} \rfloor}^{\infty} \int_{d(g, g') \leq 2^{k+1}d(\tilde{g}, 0)} \frac{1}{(2^k d(\tilde{g}, 0))^{Q+1}} |f(g')| dg'$$

$$\lesssim \sum_{k = \lfloor \log_2 \varepsilon^{-1} \rfloor}^{\infty} \frac{1}{2^k} M(f)(g)$$

$$\lesssim \varepsilon M(f)(g),$$

where $M(f)$ is the Hardy-Littlewood maximal operator on $G$.

For $L_3$, by the mean value theorem and (3.1), we have

$$|L_3| = \left| \int_{d(g, g') \leq \varepsilon^{-1}d(\tilde{g}, 0)} K_\ell(g, g') \left( b(g) - b(g') \right) f(g') dg' \right|$$

$$\lesssim \int_{d(g, g') \leq \varepsilon^{-1}d(\tilde{g}, 0)} \frac{1}{d(g, g')^{Q-1}} |f(g')| dg'$$

$$\lesssim \sum_{k = -\infty}^{\infty} \int_{2^k \varepsilon^{-1}d(\tilde{g}, 0) < d(g, g') \leq 2^{k+1} \varepsilon^{-1}d(\tilde{g}, 0)} \frac{1}{d(g, g')^{Q-1}} |f(g')| dg'$$

$$\lesssim \sum_{k = -\infty}^{\infty} \frac{1}{(2^k \varepsilon^{-1}d(\tilde{g}, 0))^{Q-1}} \int_{d(g, g') \leq 2^{k+1}d(\tilde{g}, 0)} |f(g')| dg'$$

$$\lesssim \varepsilon^{-1}d(\tilde{g}, 0) M(f)(g) \sum_{k = -\infty}^{\infty} 2^k \lesssim \varepsilon^{-1}d(\tilde{g}, 0) M(f)(g).$$
For $L_4$, again by (3.1), we can obtain

$$|L_4| = \left| \int_{d(g,g') \leq \varepsilon^{-1}d(\tilde{g},0)} K_\varepsilon(\tilde{g}g, g') (b(\tilde{g}g) - b(g')) f(g') dg' \right|$$

$$\lesssim \int_{d(g,g') \leq \varepsilon^{-1}d(\tilde{g},0)} \frac{d(\tilde{g}g, g')}{d(\tilde{g}g, g')} f(g') dg' \lesssim \int_{d(\tilde{g}g,g') \leq \varepsilon^{-1}d(\tilde{g},0) + d(\tilde{g},0)} \frac{1}{d(\tilde{g}g, g')^{Q-1}} f(g') dg'$$

$$\lesssim (\varepsilon^{-1} d(\tilde{g},0) + d(\tilde{g},0)) M(f)(g)$$

$$\lesssim \varepsilon^{-1} d(\tilde{g},0) M(f)(g).$$

For $L_1$,

$$|L_1| = \left| \int_{d(g,g') > \varepsilon^{-1}d(\tilde{g},0)} K_\varepsilon(g, g') (b(g) - b(\tilde{g}g)) f(g') dg' \right|$$

$$= |b(g) - b(\tilde{g}g)| \int_{d(g,g') > \varepsilon^{-1}d(\tilde{g},0)} K_\varepsilon(g, g') f(g') dg'$$

$$\leq |b(g) - b(\tilde{g}g)| \sup_{t>0} \int_{d(g,g') > t} K_\varepsilon(g, g') f(g') dg' = |b(g) - b(\tilde{g}g)| R^*_\varepsilon(f)(g)$$

$$\lesssim d(\tilde{g},0) R^*_\varepsilon(f)(g).$$

Then by all the above estimates, we have

$$\left( \int_G \left[ |b, R_\varepsilon f(g) - ([b, G_\varepsilon^1] f)_{B_\varepsilon(g,r)} |^p w(g) dg \right]^\frac{1}{p} \right)^{\frac{1}{p}}$$

$$= \left( \int_G \frac{1}{B(0,r)} \int_{B_\varepsilon(r)} |b, R_\varepsilon f(g) - ([b, G_\varepsilon^1] f(\tilde{g}g) |^p w(g) dg \right)^\frac{1}{p}$$

$$\lesssim \int_G \frac{1}{B(0,r)} \left( |L_1| + |L_2| + |L_3| + |L_4| \right) w(g) dg \right)^\frac{1}{p}$$

$$\lesssim \left( \int_G \left( r R^*_\varepsilon(f)(g) + (\varepsilon + \varepsilon^{-1} r) M(f)(g) \right)^p w(g) dg \right)^\frac{1}{p}$$

$$\lesssim r \left( \int_G (R^*_\varepsilon(f)(g))^p w(g) dg \right)^\frac{1}{p} + (\varepsilon + \varepsilon^{-1} r) \left( \int_G (M(f)(g))^p w(g) dg \right)^\frac{1}{p}$$

$$\lesssim r \| f \|_{L^p_\varepsilon(G)} + (\varepsilon + \varepsilon^{-1} r) \| f \|_{L^p_\varepsilon(G)},$$

where the first term of the last inequality comes from [14] Theorem 1.3 together with [12] Theorem 9.4.5. Thus if we take $r < \varepsilon^2$, then

$$\left( \int_G \left[ |b, R_\varepsilon f(g) - ([b, G_\varepsilon^1] f)_{B_\varepsilon(g,r)} |^p w(g) dg \right]^\frac{1}{p} \right)^{\frac{1}{p}} \lesssim \varepsilon.$$

This shows that $[b, R_\varepsilon] E$ satisfies condition (iii) in Lemma 4.3. Hence, $[b, R_\varepsilon]$ is a compact operator. This finishes the proof of Theorem 1.2.

5. Appendix: Characterisation of VMO($G$)

In this section, we provide the characterisation of VMO space on stratified Lie groups by giving the proof of Theorem 1.4. We point out that the main frame of the proof is similar to that in the Euclidean spaces from [23]. However, the technique in the proof in [23] there depends heavily on the decomposition cubes, and on general stratified Lie groups, there is
no such convenient tools. We have balls with respect to the metric instead of the cubes. Hence, the main contribution of our proof of Theorem 4.4 is to use balls to replace cubes in the Euclidean setting, which relies on the fact that the metric here is geometrically doubling and gives rise to the technique of coverings.

We point out that our proof is written for stratified Lie groups but it also works for general space of homogeneous type in the sense of Coifman and Weiss with the modification that we change $C_0^\infty(G)$ to the Lipschitz function space on space of homogeneous type.

Proof of Theorem 4.4 In the following, for any integer $m$, we use $B^m$ to denote the ball $B(0, 2^m)$.

**Necessary condition:** Assume that $f \in \text{VMO}(G)$. If $f \in C_0^\infty(G)$, then (i)-(iii) hold. In fact, by the uniform continuity, $f$ satisfies (i). Since $f \in L^1(G)$, $f$ satisfies (ii). By the fact that $f$ is compactly supported, $f$ satisfies (iii). If $f \in \text{VMO}(G) \setminus C_0^\infty(G)$, by definition, for any given $\varepsilon > 0$, there exists $f_\varepsilon \in C_0^\infty(G)$ such that $\|f - f_\varepsilon\|_{\text{BMO}(G)} < \varepsilon$. Since $f_\varepsilon$ satisfies (i)-(iii), by the triangle inequality of BMO($G$) norm, we can see (i)-(iii) hold for $f$.

**Sufficient condition:** In this proof for $j = 1, 2, \ldots, 8$, the value $\alpha_j$ is a positive constant depending only on $Q$ and $\alpha_i$ for $1 \leq i < j$. Assume that $f \in \text{BMO}(G)$ and satisfies (i)-(iii). To prove that $f \in \text{VMO}(G)$, it suffices to show that there exist positive constants $\alpha_1, \alpha_2$ such that, for any $\varepsilon > 0$, there exists $\phi_\varepsilon \in \text{BMO}(G)$ satisfying

\begin{equation}
\inf_{h \in C_0^\infty(G)} \|\phi_\varepsilon - h\|_{\text{BMO}(G)} < \alpha_1 \varepsilon,
\end{equation}

and

\begin{equation}
\|\phi_\varepsilon - f\|_{\text{BMO}(G)} < \alpha_2 \varepsilon.
\end{equation}

By (i), there exist $i_\varepsilon \in \mathbb{N}$ such that

\begin{equation}
\sup \{M(f, B) : r_B \leq 2^{-i_\varepsilon + 1}\} < \varepsilon.
\end{equation}

By (iii), there exists $j_\varepsilon \in \mathbb{N}$ such that

\begin{equation}
\sup \{M(f, B) : B \cap B^{j_\varepsilon} = \emptyset\} < \varepsilon.
\end{equation}

We first establish a cover of $G$. Observe that

\[B^{j_\varepsilon} = B^{-i_\varepsilon} \cup \left( \bigcup_{\nu = 1}^{2^{j_\varepsilon} - i_\varepsilon} B\left(0, (\nu + 1)2^{-i_\varepsilon}\right) \right) =: \bigcup_{\nu = 0}^{2^{j_\varepsilon} - i_\varepsilon - 1} \mathcal{R}_{\nu, -i_\varepsilon}^{j_\varepsilon} \]

For $m > j_\varepsilon$,

\[B^m \setminus B^{m-1} = \bigcup_{\nu = 0}^{2^{j_\varepsilon} + i_\varepsilon - 1 - 1} B\left(0, 2^m - (\nu + 1)2^{m-j_\varepsilon-i_\varepsilon}\right) \setminus B\left(0, 2^{m-1} + \nu 2^{m-j_\varepsilon-i_\varepsilon}\right)
\]

\[= \bigcup_{\nu = 0}^{2^{j_\varepsilon} + i_\varepsilon - 1 - 1} \mathcal{R}_{\nu, m-j_\varepsilon-i_\varepsilon}^m.
\]

For each $\mathcal{R}_{\nu, -i_\varepsilon}^{j_\varepsilon}$, $\nu = 1, 2, \ldots, 2^{j_\varepsilon} + i_\varepsilon - 1$, let $\mathcal{B}_{\nu, -i_\varepsilon}^{j_\varepsilon}$ be an open cover of $\mathcal{R}_{\nu, -i_\varepsilon}^{j_\varepsilon}$ consisting of open balls with radius $2^{-i_\varepsilon}$ and center on the sphere $S(0, (\nu + 2^{-1})2^{-i_\varepsilon})$. Let $\mathcal{B}_{0, -i_\varepsilon}^{j_\varepsilon} = \{B(0, 2^{-i_\varepsilon})\}$ and $\mathcal{B}_{\nu, -i_\varepsilon}^{j_\varepsilon}$ be the finite subcover of $\mathcal{B}_{\nu, -i_\varepsilon}^{j_\varepsilon}$. Similarly, for each $m > j_\varepsilon$ and $\nu = 0, 1, \ldots, 2^{j_\varepsilon} + i_\varepsilon - 1 - 1$, let $\mathcal{B}_{\nu, m-j_\varepsilon-i_\varepsilon}^m$ be the finite cover of $\mathcal{R}_{\nu, m-j_\varepsilon-i_\varepsilon}^m$ consisting of open balls with radius $2^{m-j_\varepsilon-i_\varepsilon}$ and center on the sphere $S(0, (2^{m-1} + (\nu + 2^{-1})2^{m-j_\varepsilon-i_\varepsilon})$.

We define $B_g$ as follows. If $g \in \mathcal{B}_{\varepsilon}^{j_\varepsilon}$, then there is $\nu \in \{0, 1, \ldots, 2^{j_\varepsilon} + i_\varepsilon - 1\}$ such that $g \in \mathcal{R}_{\nu, -i_\varepsilon}^{j_\varepsilon}$, let $B_g$ be a ball in $\mathcal{B}_{\nu, -i_\varepsilon}^{j_\varepsilon}$ that contains $g$. If $g \in B^m \setminus B^{m-1}$, $m > j_\varepsilon$, then there
is \( \nu \in \{0, 1, \ldots, 2^{k+i_1}-1\} \) such that \( g \in \mathcal{R}_{\nu,m-j_k-i_2} \), let \( B_g \) be a ball in \( \mathcal{B}_{\nu,m-j_k-i_2} \) that contains \( g \). We can see that if \( \overline{B}_g \cap \overline{B}_{g'} \neq \emptyset \), then
\[
(5.5) \quad \text{either } r_{B_g} \leq 2 r_{B_{g'}} \text{ or } r_{B_{g'}} \leq 2 r_{B_g}.
\]
In fact, if \( r_{B_g} > 2r_{B_{g'}} \), then there is \( m_0 \in \mathbb{N} \) such that \( g \in B^{m_0+2} \setminus B^{m_0+1} \) and \( g' \in B^{m_0} \), thus
\[
d(g,g') \geq d(0,g) - d(0,g') \geq 2^{m_0+1} - 2^{m_0} > 2^{m_0+2-j_k-i_2} + 2^{m_0-j_k-i_2} = r_{B_g} + r_{B_{g'}},
\]
which is contradict to the fact that \( \overline{B}_g \cap \overline{B}_{g'} \neq \emptyset \).

Now we define \( \phi_\varepsilon \). By (ii), there exists \( m_\varepsilon > j_\varepsilon \) large enough such that when \( r_B > 2^{m_\varepsilon - j_\varepsilon - j_k} \), we have
\[
M(f,B) < 2g(-i_k-j_\varepsilon - 1)^{-1}\varepsilon.
\]
Define
\[
\phi_\varepsilon(g) = \begin{cases} f_{B_g}, & \text{if } g \in B^{m_\varepsilon}, \\ f_{B^{m_\varepsilon} \setminus B^{m_\varepsilon-1}}, & \text{if } g \in \mathcal{G} \setminus B^{m_\varepsilon}, \end{cases}
\]
where \( f_B \) is defined in (2.6).

We claim that there exists a positive constant \( \alpha_3, \alpha_4 \) such that if \( \overline{B}_g \cap \overline{B}_{g'} \neq \emptyset \) or \( g, g' \in \mathcal{G} \setminus B^{m_\varepsilon}, \)
\[
(5.7) \quad |\phi_\varepsilon(g) - \phi_\varepsilon(g')| < \alpha_3\varepsilon.
\]
And if \( 2B_g \cap 2B_{g'} \neq \emptyset \), then for any \( g_1 \in B_g, g_2 \in B_{g'} \), we have
\[
(5.8) \quad |\phi_\varepsilon(g_1) - \phi_\varepsilon(g_2)| < \alpha_4\varepsilon.
\]
Assume (5.7) and (5.8) at the moment, we now continue to prove the sufficiency of Theorem 4.4.

Now we show (5.1). Let \( \tilde{h}_\varepsilon(g) := \phi_\varepsilon(g) - f_{B^{m_\varepsilon} \setminus B^{m_\varepsilon-1}} \). By definition of \( \phi_\varepsilon \), we can see that
\[
\tilde{h}_\varepsilon(g) = 0 \text{ for } g \in \mathcal{G} \setminus B^{m_\varepsilon} \text{ and } \|\tilde{h}_\varepsilon - \phi_\varepsilon\|_{BMO(\mathcal{G})} = 0.
\]
Observe that \( \text{supp}(\tilde{h}_\varepsilon) \subset B^{m_\varepsilon} \) and there exists a function \( h_\varepsilon \in C_c(\mathcal{G}) \) such that for any \( g \in \mathcal{G}, |\hat{h}_\varepsilon(g) - h_\varepsilon(g)| < \varepsilon \). Let \( \psi \in C^\infty_c(\mathcal{G}) \) be a positive valued function with \( \int_{\mathcal{G}} \psi = 1 \), then by [10] Proposition 1.20], \( \psi_t \ast h_\varepsilon(g) \) approaches to \( h_\varepsilon(g) \) uniformly for \( g \in \mathcal{G} \) as \( t \) goes to 0.

Since
\[
\|\psi_t \ast h_\varepsilon - \phi_\varepsilon\|_{BMO(\mathcal{G})} \leq \|\psi_t \ast h_\varepsilon - h_\varepsilon\|_{BMO(\mathcal{G})} + \|h_\varepsilon - \tilde{h}_\varepsilon\|_{BMO(\mathcal{G})} + \|\tilde{h}_\varepsilon - \phi_\varepsilon\|_{BMO(\mathcal{G})} \\
\leq \|\psi_t \ast h_\varepsilon - h_\varepsilon\|_{BMO(\mathcal{G})} + 2\varepsilon,
\]
we can obtain (5.1) by letting \( t \) go to 0 and by taking \( \alpha_1 = 2 \).

Now we show (5.2). To this end, we only need to prove that for any ball \( B \subset \mathcal{G} \),
\[
M(f - \phi_\varepsilon, B) < \alpha_2\varepsilon.
\]
We first prove that for every \( B_g \) with \( g \in B^{m_\varepsilon}, \)
\[
(5.9) \quad \int_{B_g} |f(g') - \phi_\varepsilon(g')| \, dg' \leq \alpha_5\varepsilon|B_g|.
\]
In fact,
\[
\int_{B_g} |f(g') - \phi_\varepsilon(g')| \, dg' = \int_{B_g \cap B^{m_\varepsilon}} |f(g') - f_{B_{g'}}| \, dg' + \int_{B_g \cap (\mathcal{G} \setminus B^{m_\varepsilon})} |f(g') - f_{B^{m_\varepsilon} \setminus B^{m_\varepsilon-1}}| \, dg'.
\]
When \( g \in B(0, 2^{m_\varepsilon} - 2^{m_\varepsilon-j_k}), \) then \( B_g \subset B^{m_\varepsilon} \), thus
\[
\int_{B_g} |f(g') - \phi_\varepsilon(g')| \, dg' = \int_{B_g} |f(g') - f_{B_{g'}}| \, dg' \leq \int_{B_g} |f(g') - f_{B_g}| \, dg' + \int_{B_g} |f_{B_g} - f_{B_{g'}}| \, dg'.
\]
and (5.7), we have

$$M(f, B_g) = \int_B |f| B_g |f| B_g' \, dg'.$$

Note that if $g' \in B_g$, then $B_g \cap B_g' \neq \emptyset$. Therefore, If $B_g \cap \mathcal{B}^{i\varepsilon} = \emptyset$, by (5.4) and (5.7), we have

$$\int_{B_g} |f(g') - \phi_\varepsilon(g')| \, dg' < (\varepsilon + \alpha_3 \varepsilon) |B_g|.$$

If $B_g \cap \mathcal{B}^{i\varepsilon} \neq \emptyset$, then $r_{B_g} \leq 2^{-i\varepsilon + 1}$, then by (5.3) and (5.7),

$$\int_{B_g} |f(g') - \phi_\varepsilon(g')| \, dg' < (\varepsilon + \alpha_3 \varepsilon) |B_g|.$$

When $g \in \mathcal{B}^{m\varepsilon} \setminus B(0, 2^{m\varepsilon} - 2^{m\varepsilon - j\varepsilon - i\varepsilon})$, it is clear that $B_g \cap \mathcal{B}^{j\varepsilon} = \emptyset$, then by (5.3), (5.6) and (5.7), we have

$$\int_{B_g} |f(g') - \phi_\varepsilon(g')| \, dg' \leq \int_{B_g \cap \mathcal{B}^{m\varepsilon}_g} |f(g') - f_{B_g}| \, dg' + \int_{B_g \cap \mathcal{B}^{m\varepsilon}_g} |f| B_g - f_{B_g'}| \, dg'$$

$$+ \int_{B_g \cap (\mathcal{G} \setminus \mathcal{B}^{m\varepsilon}_g)} |f(g') - f_{B_g} + 1| \, dg' + \int_{B_g \cap (\mathcal{G} \setminus \mathcal{B}^{m\varepsilon}_g)} |f| B_{m\varepsilon} \setminus B_{m\varepsilon - 1}| \, dg'$$

$$\leq |B_g| M(f, B_g) + \alpha_3 \varepsilon |B_g| + |\mathcal{B}^{m\varepsilon} \setminus B_{m\varepsilon} + 1| M(f, B_{m\varepsilon} + 1) + \frac{|\mathcal{B}^{m\varepsilon} \setminus B_{m\varepsilon - 1}| M(f, B_{m\varepsilon} + 1)}{|\mathcal{B}^{m\varepsilon} \setminus B_{m\varepsilon - 1}| M(f, B_{m\varepsilon} + 1)}$$

$$< (2\varepsilon + \alpha_3 \varepsilon) |B_g|.$$

Then (5.9) holds by taking $\alpha_5 = (2 + \alpha_3)$.

Let $B$ be an arbitrary ball in $\mathcal{G}$, then $M(f - \phi_\varepsilon, B) \leq M(f, B) + M(\phi_\varepsilon, B)$. If $B \subset \mathcal{B}^{m\varepsilon}$ and $\max\{r_{B_g} : B_g \cap B \neq \emptyset\} > 8r_B$, then

$$\min\{r_{B_g} : B_g \cap B \neq \emptyset\} > 2r_B.$$

In fact, assume that $r_{B_{g_0}} = \max\{r_{B_g} : B_g \cap B \neq \emptyset\}$ and $g_0 \in B^{l_0} \setminus B^{l_0 - 1}$ for some $l_0 \in \mathbb{Z}$. Then $B \subset B^{l_0} \cap 2B_{g_0}$. If $l_0 \leq j\varepsilon$, then (5.10) holds. If $l_0 > j\varepsilon$, then $r_{B_{g_0}} = 2^{l_0 - j\varepsilon - i\varepsilon}$, and

$$r_B < \frac{1}{8} r_{B_{g_0}} = 2^{l_0 - j\varepsilon - i\varepsilon - 3}.$$

Since for any $g' \in 2B_{g_0}$,

$$d(0, g') \geq d(0, g_0) - d(g_0, g') \geq 2^{l_0 - 1} - \frac{3}{2} 2^{l_0 - j\varepsilon - i\varepsilon} > 2^{l_0 - 1} - 2^{l_0 - j\varepsilon - i\varepsilon + 1},$$

we have

$$\text{dist}(0, \frac{3}{2} B_{g_0}) := \inf_{g' \in \frac{3}{2} B_{g_0}} d(0, g') > 2^{l_0 - 1} - 2^{l_0 - j\varepsilon - i\varepsilon + 1}.$$

Thus $B \subset B^{l_0} \setminus \frac{3}{2} B^{l_0 - 2}$. Therefore, if $B_g \cap B \neq \emptyset$, then $g \in B^{l_0} \setminus B^{l_0 - 2}$, which implies that

$$r_{B_g} \geq 2^{l_0 - 2 - j\varepsilon - i\varepsilon} > 2r_B.$$

From (5.10) we can see that if $B_{g_i} \cap B \neq \emptyset$ and $B_{g_j} \cap B \neq \emptyset$, then $2B_{g_i} \cap 2B_{g_j} \neq \emptyset$. Then by (5.8), we can get

$$M(\phi_\varepsilon, B) \leq \frac{1}{|B|} \int_B \frac{1}{|B|} \int_B \int_{B_{g_i} \cap B \neq \emptyset} \int_{B_{g_j} \cap B \neq \emptyset} |\phi_\varepsilon(g) - \phi_\varepsilon(g')| \, dg' \, dg = \frac{1}{|B|^2} \sum_{i : B_{g_i} \cap B \neq \emptyset} \sum_{j : B_{g_j} \cap B \neq \emptyset} \int_{B_{g_i} \cap B} \int_{B_{g_j} \cap B} |\phi_\varepsilon(g) - \phi_\varepsilon(g')| \, dg' \, dg$$
\[
< \alpha_4 \varepsilon \frac{1}{|B|^2} \left( \sum_{i:B_{g_i} \cap B \neq \emptyset} |B_{g_i} \cap B| \right) \left( \sum_{i:B_{g_i} \cap B \neq \emptyset} |B_{g_i} \cap B| \right) < \alpha_4 \alpha_0^2 \varepsilon. 
\]

Moreover, if \( B \cap B_{j^*} \neq \emptyset \), then by (5.10), \( r_B < 2^{-i} \), thus by (5.3), we have \( M(f, B) < \varepsilon \). If \( B \cap B_{j^*} = \emptyset \), then by (5.4), \( M(f, B) < \varepsilon \). Consequently,

\[
M(f - \phi_e, B) \leq M(f, B) + M(\phi_e, B) < (1 + \alpha_4 \alpha_0^2) \varepsilon. 
\]

If \( B \subset B^{m_e} \) and \( \max\{|r_{B_g} : B_g \cap B \neq \emptyset\} \leq 8r_B \), since the number of \( B_g \) with \( g \in B^{m_e} \) that covers \( B \) is bounded by \( \alpha_7 \), by (5.9), we have

\[
M(f - \phi_e, B) \leq \frac{2}{|B|} \int_B |f(g) - \phi_e(g)| \, dg \leq \frac{2}{|B|} \sum_{i:B_{g_i} \cap B \neq \emptyset} \int_{B_{g_i}} |f(g) - \phi_e(g)| \, dg 
\]

\[
\leq \frac{2}{|B|} \alpha_5 \varepsilon \sum_{i:B_{g_i} \cap B \neq \emptyset} |B_{g_i}| \leq \frac{2}{|B|} \alpha_5 \alpha_7 \varepsilon |B| = 2^{3Q+1} \alpha_5 \alpha_7 \varepsilon. 
\]

If \( B \subset \mathcal{G} \setminus B^{m_e-1} \), then \( B \cap B_{j^*} = \emptyset \), from (5.6) we can see \( M(f, B) < \varepsilon \). By (5.4),

\[
M(\phi_e, B) \leq \frac{1}{|B|} \int_B \frac{1}{|B|} \int_B |\phi_e(g) - \phi_e(g')| \, dg \, dg < \alpha_3 \varepsilon. 
\]

Therefore,

\[
M(f - \phi_e, B) \leq M(f, B) + M(\phi_e, B) < (1 + \alpha_3) \varepsilon. 
\]

If \( B \cap (\mathcal{G} \setminus B^{m_e}) \neq \emptyset \) and \( B \cap B^{m_e-1} \neq \emptyset \). Let \( p_B \) be the smallest integer such that \( B \subset B^{p_B} \), then \( p_B > m_e \). If \( p_B = m_e + 1 \), then \( r_B > \frac{1}{2}(2^{m_e} - 2^{m_e-1}) = 2^{m_e-2} \). If \( p_B > m_e + 1 \), then \( r_B > \frac{1}{2}(2^{p_B-1} - 2^{m_e-1}) \). Thus

\[
\frac{|B^{p_B}|}{|B|} \leq 2^{3Q}. 
\]

Therefore,

\[
M(f - \phi_e, B) \leq \frac{1}{|B^{p_B}|} \int_{B^{p_B}} |f(g) - \phi_e(g) - (f - \phi_e)_{B^{p_B}}| \, dg + (f - \phi_e)_{B^{p_B}} - (f - \phi_e)_{B} 
\]

\[
\leq \frac{2}{|B|} \frac{1}{|B^{p_B}|} \int_{B^{p_B}} |f(g) - \phi_e(g) - (f - \phi_e)_{B^{p_B}}| \, dg 
\]

\[
\leq 2^{3Q+1} \left( M(f, B^{p_B}) + M(\phi_e, B^{p_B}) \right) \leq 2^{3Q+1} (\varepsilon + M(\phi_e, B^{p_B})) , 
\]

where the last inequality comes from (5.6). By definition,

\[
M(\phi_e, B^{p_B}) \leq \frac{1}{|B^{p_B}|} \int_{B^{p_B}} |\phi_e(g) - (\phi_e)_{B^{p_B} \setminus B^{m_e}}| \, dg + |(\phi_e)_{B^{p_B} \setminus B^{m_e}} - (\phi_e)_{B^{p_B}}| 
\]

\[
\leq \frac{2}{|B^{p_B}|} \int_{B^{p_B}} |\phi_e(g) - (\phi_e)_{B^{p_B} \setminus B^{m_e}}| \, dg. 
\]

By (5.4), (5.9) and the fact that \( \phi_e(g) = f_{B^{m_e} \setminus B^{m_e-1}} \) if \( g \in \mathcal{G} \setminus B^{m_e} \), we have

\[
\int_{B^{p_B}} |\phi_e(g) - (\phi_e)_{B^{p_B} \setminus B^{m_e}}| \, dg \leq \int_{B^{p_B} \setminus B^{m_e}} \frac{1}{|B^{p_B} \setminus B^{m_e}|} \int_{B^{p_B} \setminus B^{m_e}} |\phi_e(g) - \phi_e(g')| \, dg' \, dg 
\]

\[
= \int_{B^{m_e}} |\phi_e(g) - f_{B^{m_e} \setminus B^{m_e-1}}| \, dg 
\]

\[
\leq \int_{B^{m_e}} |\phi_e(g) - f(g)| \, dg + \int_{B^{m_e}} |f(g) - f_{B^{m_e}}| \, dg + |B^{m_e}| |f_{B^{m_e}} - f_{B^{m_e} \setminus B^{m_e-1}}| 
\]

\[
\leq \sum_{i:B_{g_i} \cap B^{m_e} \neq \emptyset, g_i \in B^{m_e}} \int_{B_{g_i}} |\phi_e(g) - f(g)| \, dg + \left( |B^{m_e}| + \frac{|B^{m_e}|^2}{|B^{m_e} \setminus B^{m_e-1}|} \right) M(f, B^{m_e}) 
\]
\[
< \alpha_5 \varepsilon \sum_{i:B_{g_i} \cap B^{m_e} \neq \emptyset, g_i \in B^{m_e}} |B_{g_i}| + 3 \varepsilon |B^{m_e}| < (\alpha_5 \alpha_8 + 3) \varepsilon |B^{m_e}|.
\]

Therefore,
\[
M(f - \phi_\varepsilon, B) \leq 2^{3Q+1} (\varepsilon + M(\phi_\varepsilon, B^{P_\epsilon})) \leq 2^{3Q+1} \left( \varepsilon + \frac{2|B^{m_e}|}{|B^{P_\epsilon}|} (\alpha_5 \alpha_8 + 3) \varepsilon \right)
\]
\[
< 2^{3Q+1} (2\alpha_5 \alpha_8 + 7) \varepsilon.
\]

Then (5.2) holds by taking \(\alpha_2 = \max\{1 + \alpha_4 \alpha_6^2, 1 + \alpha_3, 2^{3Q+1}(2\alpha_5 \alpha_8 + 7)\}\). This finishes the proof of Theorem 4.4

**Proof of (5.7):**

We first claim that
\[
(5.11) \quad \sup \left\{ \left| f_{B_g} - f_{B_{g'}} \right| : g, g' \in B^{m_e} \setminus B^{m_e-1} \right\} < \varepsilon.
\]

By (6.6), for any \(g \in B^{m_e} \setminus B^{m_e-1}\), we have
\[
\left| f_{B_g} - f_{B^{m_e+1}} \right| \leq \frac{|B^{m_e+1}|}{|B_g|} \frac{1}{|B^{m_e+1}|} \int_{B^{m_e+1}} \left| f(g') - f_{B^{m_e+1}} \right| \, dg' = \frac{2^{Q(m_e+1)}}{2^{Q(m_e-j_\epsilon+i_\epsilon)}} M(f, B^{m_e+1}) < \frac{\varepsilon}{2}.
\]

Similarly, for any \(g' \in B^{m_e} \setminus B^{m_e-1}, \left| f_{B_g} - f_{B^{m_e+1}} \right| < \frac{\varepsilon}{2}\). Consequently, (5.11) holds.

For the case \(g, g' \in \mathcal{G} \setminus B^{m_e-1}\), firstly, if \(g, g' \in \mathcal{G} \setminus B^{m_e}\), then by definition
\[
\left| \phi_\varepsilon(g) - \phi_\varepsilon(g') \right| = 0.
\]

Secondly, if \(g, g' \in B^{m_e} \setminus B^{m_e-1}\), then by (5.11), we have
\[
\left| \phi_\varepsilon(g) - \phi_\varepsilon(g') \right| < \varepsilon.
\]

Thirdly, without loss of generality, we may assume that \(g \in B^{m_e} \setminus B^{m_e-1}\) and \(g' \in \mathcal{G} \setminus B^{m_e}\), then by (5.6), we have
\[
\left| \phi_\varepsilon(g) - \phi_\varepsilon(g') \right| = \left| f_{B_g} - f_{B^{m_e+1}} \right| \leq \left| f_{B_g} - f_{B^{m_e+1}} \right| + \left| f_{B^{m_e+1}} - f_{B^{m_e} \setminus B^{m_e-1}} \right|
\]
\[
\leq \frac{|B^{m_e+1}|}{|B_g|} M(f, B^{m_e+1}) + \frac{|B^{m_e}|}{|B^{m_e} \setminus B^{m_e-1}|} M(f, B^{m_e+1})
\]
\[
\leq \left( \frac{2^{Q(m_e+1)}}{2^{Q(m_e-j_\epsilon+i_\epsilon)}} + \frac{2^{Q(m_e)}}{2^{Q(m_e-j_\epsilon+i_\epsilon)}} \right) M(f, B^{m_e+1})
\]
\[
< \left( 2^{Q(1+j_\epsilon+i_\epsilon)} + 2^{Q+1} \right) M(f, B^{m_e+1}) < \varepsilon.
\]

For the case \(\overline{B_g} \cap \overline{B_{g'}} \neq \emptyset\) and \(g, g' \in B^{m_e-1}\), we may assume \(B_g \neq B_{g'}\) and \(r_{B_g} < r_{B_{g'}}\). By (5.3), \(B_{g'} \subset 5B_g \subset 15B_{g'}\). If \(g' \in B^{j_\epsilon+1}\), then by (5.3), we have
\[
\left| \phi_\varepsilon(g) - \phi_\varepsilon(g') \right| = \left| f_{B_g} - f_{B_{g'}} \right| \leq \left| f_{B_g} - f_{3B_{g'}} \right| + \left| f_{3B_{g'}} - f_{3B_{g'}} \right|
\]
\[
\leq \left( \frac{3|B_{g'}|}{|B_g|} + \frac{|B_{g'}|}{|B'_{g'}|} \right) M(f, 3B_{g'}) \leq (15^Q + 3^Q) M(f, 3B_{g'}) \leq (15^Q + 3^Q) \varepsilon.
\]

If \(g' \notin B^{j_\epsilon+1}\), then \(3B_{g'} \cap B^{j_\epsilon} = \emptyset\), by (5.4), we have
\[
\left| \phi_\varepsilon(g) - \phi_\varepsilon(g') \right| \leq (3^Q + 15^Q) M(f, 3B_{g'}) \leq (15^Q + 3^Q) \varepsilon.
\]

Therefore, (5.7) holds by taking \(\alpha_3 = 15^Q + 3^Q\).
Proof of (5.8):

Since \( g_1 \in B_{g}, g_2 \in B'_{g} \), we have \( B_{g_1} \cap B_{g} \neq \emptyset \) and \( B_{g_2} \cap B'_{g} \neq \emptyset \), by (5.7),

\[
|\phi_{\varepsilon}(g_1) - \phi_{\varepsilon}(g_2)| \leq |\phi_{\varepsilon}(g_1) - \phi_{\varepsilon}(g)| + |\phi_{\varepsilon}(g) - \phi_{\varepsilon}(g')| + |\phi_{\varepsilon}(g') - \phi_{\varepsilon}(g_2)| \\
\leq 2\alpha_3 \varepsilon + |\phi_{\varepsilon}(g) - \phi_{\varepsilon}(g')|.
\]

We may assume \( B_{g} \neq B'_{g} \) and \( r_{B_{g}} \leq r_{B_{g}'} \). If \( g, g' \in G \setminus B^{m_{e}-1} \), then (5.8) follows from (5.7).

If \( g, g' \in B^{m_{e}-1} \), when \( g' \in B^{i_{e}+1} \), then \( 2^{-i_{e}} \leq r_{B_{g}} \leq r_{B_{g}'} \leq 2^{-i_{e}+1} \), thus \( B_{g} \subset 10B_{g} \subset 60B_{g} \), by (5.3), we have

\[
|\phi_{\varepsilon}(g) - \phi_{\varepsilon}(g')| \leq \left| f_{B_{g}} - f_{6B_{g}'} \right| + \left| f_{B_{g}'} - f_{6B_{g}} \right| = \left( \frac{|6B_{g'}|}{|B_{g}|} + \frac{|6B_{g}|}{|B_{g}'}| \right) M(f, 6B_{g}') \\
\leq (60^Q + 6^Q) M(f, 6B_{g}') \leq (60^Q + 6^Q) \varepsilon.
\]

When \( g' \notin B^{i_{e}+1} \), then there exist \( \tilde{m}_0 \in \mathbb{N} \) and \( \tilde{m}_0 \geq i_{e} + 2 \) such that \( g' \in B^{\tilde{m}_0} \setminus B^{\tilde{m}_0-1} \). Since \( 2B_{g} \cap 2B_{g} \neq \emptyset \), we have \( B_{g} \subset 6B_{g} \). Note that \( 6B_{g'} \cap B^{\tilde{m}_0-2} = \emptyset \) (in fact, for any \( \tilde{g} \in 6B_{g'} \), \( d(0, \tilde{g}) \geq d(0, g') - d(g', \tilde{g}) \geq 2^{\tilde{m}_0-1} - 6 \cdot 2^{\tilde{m}_0-1-i_{e}} > 2^{\tilde{m}_0-2} \)), thus \( B_{g} \cap B^{\tilde{m}_0-2} = \emptyset \) and then \( \frac{r_{B_{g}}}{r_{B_{g}'} = 2^{\tilde{m}_0-1-i_{e}} \leq r_{B_{g}} \leq 2^{\tilde{m}_0-1-i_{e}} = r_{B_{g}'} \). Therefore, \( B_{g} \subset 10B_{g} \). Then by (5.4), we have

\[
|\phi_{\varepsilon}(g) - \phi_{\varepsilon}(g')| \leq (60^Q + 6^Q) M(f, 6B_{g}') < (60^Q + 6^Q) \varepsilon.
\]

If \( g \in B^{m_{e}-1} \) and \( g' \in G \setminus B^{m_{e}-1} \), since \( 2B_{g} \cap 2B_{g} \neq \emptyset \), by the construction of \( B_{g} \) we can see that \( g \in B^{m_{e}-1} \setminus B^{m_{e}-2} \) and \( g' \in B^{m_{e}} \setminus B^{m_{e}-1} \). Thus, \( B_{g} \subset 10B_{g} \subset 40B_{g} \). Then by (5.6), we have

\[
|\phi_{\varepsilon}(g) - \phi_{\varepsilon}(g')| \leq (40^Q + 4^Q) M(f, 4B_{g}') < (40^Q + 4^Q) \varepsilon.
\]

Taking \( \alpha_4 = 60^Q + 6^Q + 2\alpha_3 \), then (5.8) holds. \( \square \)

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