STABLY NEWTON NON-DEGENERATE SINGULARITIES

JAN STEVENS

Abstract. We discuss one of Arnold’s problems, whether every function is stably equivalent to one which is non-degenerate for its Newton diagram. The answer depends on the characteristic of the ground field. In characteristic $p$ the function $x^p$ is not stably equivalent to a non-degenerate function.

On the other hand, in characteristic zero it seems always possible to find a suitable suspension. We argue that irreducible plane curves with an arbitrary number of Puiseux pairs are stably non-degenerate. As the suspension involves many variables, it becomes very difficult to determine the Newton diagram in general, but the form of the equation indicates that it is non-degenerate. We also discuss some reducible curve singularities and some surface singularities of Lê-Yomdin type.

The reader is invited to send possible counterexamples to test our methods on.

Introduction

Many invariants of a hypersurface singularity can be computed from its Newton diagram, if the singularity is non-degenerate. Although almost all singularities with a given diagram are non-degenerate, most singularities are degenerate in every coordinate system. Sometimes it is possible to find suitable coordinates after a suspension with a quadratic form in new variables, and invariants computed from the Newton diagram of the suspension allow conclusions about the original singularity. A successful case is the study of Luengo’s example [Lu] of a non-smooth $\mu$-const stratum in [St]. The fact that one can make a singularity non-degenerate by adding variables was observed by Arnold, who raised the question whether it is always possible.

Problem 3 of Arnold’s list [Arn] in the Arcata volume reads:

Is every function stably equivalent to a $\Gamma$-non-degenerate function (in a neighbourhood of a critical point of finite multiplicity)?

The answer is no, and an example is provided by the simplest degenerate function in finite characteristic, the polynomial $x^p$ in char $p$. This is immediate for a somewhat modified concept of non-degeneracy, originally due to Wall [Wa] and studied in char $p$ by Boubakri, Greuel and Markwig [BGM]. But also for the classical notion of non-degeneracy the polynomial $x^p$ is a counterexample.

This negative answer does not extend to the case of real or complex functions. Although I have basically only one trick to make a function less degenerate, it carries a long way, and for every specific function I tried I succeeded. The purpose
of the present note is to put a number of non-trivial cases on record. I consider irreducible plane curve singularities, with an arbitrary number of Puiseux pairs. The number of variables is rapidly increasing, making it difficult to check non-degeneracy. Therefore I in fact leave this as a conjecture. Degeneracy means that there are relations between the coefficients of the occurring monomials. In our examples there are no longer any obvious relations, and it seems possible to make the coefficients generic. I refer to this situation as seemingly non-degenerate.

Also some surface singularities of Lê-Yomdin type are discussed. Further cases are Brzostowski’s examples of degenerate plane curve singularities [Br].

It is my feeling that in characteristic zero it is always possible to make a polynomial non-degenerate after suspension. Each specific example requires an analysis of the singular locus. A general proof will not be easy. I invite the reader to send me possible counterexamples to try my methods on.

1. NON-DEGENERATE FUNCTIONS

We recall the standard definitions of non-degeneracy, given by Kouchnirenko [Kou], and the related concepts of Wall [Wa].

Let \( f \in k[[x_1, \ldots, x_n]] \) be a formal power series over a field \( k \), with algebraic closure \( K \). Write (in multi-index notation) \( f = \sum a_m x^m \) and let \( \Gamma_+(f) \) be the convex hull of the set \( \bigcup_{m: a_m \neq 0} (m + \mathbb{R}^n_+) \subset \mathbb{R}^n \). The Newton diagram \( \Gamma(f) \) of \( f \) is the union of all compact faces of \( \Gamma_+(f) \). The union \( \Gamma_-(f) \) of all segments connecting the origin and the Newton diagram is the Newton polytope. The series \( f \) is convenient if for every \( 1 \leq i \leq n \) there is a \( m_i \) such that the monomial \( x_i^{m_i} \) occurs with non-zero coefficient, that is, the Newton diagram of \( f \) has a vertex on each coordinate axis.

Let \( \Delta \) be a face of \( \Gamma(f) \). One denotes the polynomial \( \sum_{m \in \Delta} a_m x^m \) by \( f_\Delta \). The principal part of \( f \) is the polynomial \( f_\Gamma = \sum_{m \in \Gamma(f)} a_m x^m \).

**Definition 1.1.** The series \( f \) is non-degenerate if for every closed face \( \Delta \subset \Gamma(f) \) the polynomials

\[
 x_1 \frac{\partial f_\Delta}{\partial x_1}, \ldots, x_n \frac{\partial f_\Delta}{\partial x_n}
\]

have no common zero on the torus \((K^*)^n\).

This condition depends only on the principal part of the series \( f \).

If \( f \) is non-degenerate, many invariants can be computed from the Newton diagram. We concentrate here on the Milnor number \( \mu(f) = \dim_k k[[x]]/(\frac{\partial f}{\partial x}) \). Note that \( \mu(f) \) can be infinite.

For any compact polytope \( S \) in \( \mathbb{R}^n_+ \) with the origin as vertex we denote by \( V_k(S) \) the sum of the \( k \)-dimensional volumes of the intersections of \( S \) with the \( k \)-dimensional coordinate subspaces of \( \mathbb{R}^n \), and we define its Newton number to
be
\[ \nu(S) = \sum_{k=0}^{n} (-1)^{n-k} k! V_k(S). \]

The Newton number \( \nu(f) \) of \( f \) is the Newton number of \( \Gamma_{-}(f) \).

The main result of Kouchnirenko [Kou] is:

**Theorem 1.2.** For every series \( f \) one has \( \mu(f) \geq \nu(f) \). Equality holds if \( f \) is convenient and nondegenerate.

For non-degenerate holomorphic function germs, which are not necessarily convenient, the meaning of the number \( \nu(f) \) is given by a theorem of Varchenko [Va]:

**Theorem 1.3.** For a non-degenerate series \( f \in \mathbb{C}\{x_1, \ldots, x_n\} \) the Newton number \( \nu(f) \) is equal to \( (-1)^{n-1}(\chi(F) - 1) \), where \( \chi(F) \) is the Euler characteristic of the Milnor fibre.

For non-degeneracy one can as well require that the functions \( \frac{\partial f_{\Delta}}{\partial x_1}, \ldots, \frac{\partial f_{\Delta}}{\partial x_n} \) have no common zero on \((K^*)^n\). In finite characteristic this condition is not the same as that the Tjurina ideal \((f_{\Delta}, \frac{\partial f_{\Delta}}{\partial x})\) has no common zeroes. The function \( f \) is weakly non-degenerate if this latter condition is satisfied for every facet (i.e., top-dimensional face) of the Newton diagram.

To treat isolated singularities, which are not convenient, Wall [Wa] introduced a somewhat different notion of non-degeneracy, which allows to extend the Newton filtration of the given diagram to the whole power series ring. The prototype of this situation is the case of semi-quasihomogeneous functions, where the Newton diagram may have more compact faces than its quasihomogeneous part, but one still works with the filtration coming from the quasihomogeneous weights. One starts from a diagram \( \Gamma \), which shares the property of the Newton diagram of a convenient function, that the closed region \( \Gamma_+ \) on and above it is convex and that central projection onto the unit simplex is a bijection. The intersection points with the coordinate axes need not be lattice points. We call such a diagram a \( C \)-diagram. A face \( \Delta \) is an inner face of \( \Gamma \) if it is not contained in any coordinate hyperplane. The non-degeneracy condition is stronger, but will be required for less faces.

Let \( Q = (q_1, \ldots, q_n) \in K^n \) be a common zero of \( \frac{\partial f_{\Delta}}{\partial x_1}, \ldots, \frac{\partial f_{\Delta}}{\partial x_n} \). We set \( I_Q = \{ i \mid q_i = 0 \} \). For an arbitrary subset \( I \subset \{1, \ldots, n\} \) we denote the coordinate subspace \( \{ (r_1, \ldots, r_n) \in \mathbb{R}^n \mid r_i = 0 \text{ if } i \notin I \} \) by \( \mathbb{R}^I \).

**Definition 1.4.** The series \( f \) is strictly non-degenerate with respect to a \( C \)-diagram \( \Gamma \) if for every inner face \( \Delta \) the following holds: \( \Delta \cap \mathbb{R}^I_Q = \emptyset \) for each common zero \( Q \) of the ideal \((\frac{\partial f_{\Delta}}{\partial x_1}, \ldots, \frac{\partial f_{\Delta}}{\partial x_n})\).

We then have:
Theorem 1.5 (Wall). If the series $f$ is strictly non-degenerate w.r.t. a $C$-diagram $\Gamma$, then $\mu(f) < \infty$ and

$$\mu(f) = \nu(\Gamma_-(f)) = \nu(\Gamma_-).$$

As observed by Boubakri, Greuel and Markwig [BGM] (whose terminology we follow), the results of Wall hold also in finite characteristic. For a comparison of the different concepts of non-degeneracy we refer to their paper.

We can ask the question about stably equivalence for the different notions of non-degeneracy.

The converse of Theorem 1.2 does not hold in general: for degenerate series it can be that $\mu(f) = \nu(f)$. The simplest example is the function $(y + x)^2 + xz + z^2$ [Kou, Remarque 1.21]. Counterexamples also exist for two variables in finite characteristic: $f = xy + x^p + y^p$ has $\mu(f) = \nu(f) = 1$, but is degenerate in char $p$ [GN, Example 2.1]. The precise consequences of the condition $\mu(f) = \nu(f)$ for functions of two variables are investigated in [GN]. The result is that $f$ is strongly non-degenerate w.r.t. a $C$-diagram. In characteristic zero it follows that $f$ is non-degenerate. In that case the function can only degenerate on edges and the result follows from the fact that $f$ is equisingular to a non-degenerate function and Newton's method to parametrise branches. Note that our definition of $\nu(f)$ is for non-convenient series not the same as that of Kouchirenko, who takes the supremum of $\nu(f + \sum x_i^m)$. Under our definition $\mu(f) = \nu(f)$ implies that $\mu(f)$ is finite. Our definition of $\mu(f)$ is the same, so in particular $\mu(f) = \infty$ for non-isolated singularities, and it is not related to the Euler characteristic of the Milnor fibre.

A method to compute $\nu(f)$ without determining the faces of the Newton diagram is to compute the Milnor number of a general enough function with the same Newton diagram, say with SINGULAR [DGPS]. Making all coefficients equal to 1 might not be general enough; in my experience a good choice is to use the coefficients 1, 2, 3, ..., $k$, if there are $k$ monomials. A first test for non-degeneracy is that $\nu(f) = \mu(f)$, but this is not sufficient. The problem is that $\mu(f)$ is related to the multiplicity of Jacobian ideal $J(f) = (\frac{\partial f}{\partial x})$, whereas the Newton diagram has to do with the ideal $I(f) = (x \frac{\partial f}{\partial x})$. In fact, by using this last ideal one gets a necessary and sufficient condition [Bi]. If the ideal $I(f)$ has finite codimension (implying in particular that $f$ is convenient), then $f$ is non-degenerate if and only if the multiplicity of $I(f)$ is to $n!V_n(\Gamma_-(f))$, see [Bi] Thm 4.1, and $n!V_n(\Gamma_-(f))$ can be computed as the multiplicity of $I(g)$ for a general enough function $g$ with the same convenient Newton diagram.

2. Finite characteristic

In finite characteristic it is no longer true that the Milnor number is invariant under contact equivalence. The simplest example is the function $f(x) = x^p$ in
characteristic $p$ with $\mu(f) = \infty$, while $\mu(g) = p$ for the contact equivalent function $g(x) = (1 + x)f(x) = x^p + x^{p+1}$.

**Proposition 2.1.** The function $x^p$, char $k = p$, is not stably equivalent to a strictly non-degenerate function.

**Proof.** As the Milnor number does not change by suspension, this follows from Theorem 1.5 saying that the Milnor number of a strictly non-degenerate function is finite. \qed

Examples with functions of two variables are easy to make: $x^p + y^a$ will do.

**Theorem 2.2.** The function $x^p$, char $k = p$, is not stably equivalent to a non-degenerate function.

**Proof.** Suppose that the function $f = x^p \in k[[x_1]]$, char $k = p$, is stably equivalent to a non-degenerate function $g \in K[[x_1, \ldots, x_n]]$. We claim that $\mu(\tilde{g}) = \infty$ for any function $\tilde{g}$, in any characteristic, with the same Newton diagram as $g$.

Consider functions of the form $g + \sum a_i x_i^m$, with $m$ not divisible by $p$. For large $m$ and generic $a_i$ such a function is non-degenerate and its Milnor number is equal to its Newton number. As $\mu(g) = \infty$, also $\lim_{m \to \infty} \mu(g + \sum a_i x_i^m) = \infty$. Then $\lim_{m \to \infty} \mu(\tilde{g}_m) = \infty$ for generic $\tilde{g}_m$ with the same Newton diagram. By semi-continuity of the Milnor number this is only possible if $\mu(\tilde{g}) = \infty$.

By assumption the series $g$ is equivalent to $f + Q$ with $Q$ a quadratic form of rank $n - 1$. Therefore the corank of $g$ is at most 1. So for a generic function $\tilde{g}$ in characteristic zero with the same diagram the corank is also at most one. By the splitting lemma such a function is right-equivalent to $x_1^l + x_2^2 + \cdots + x_n^2$ for some $l \leq \infty$, and as $\mu(\tilde{g}) = \infty$, we have in fact $l = \infty$. This implies that $\tilde{g}$ is right-equivalent to its 2-jet. The equivalence can be constructed one order at a time (cf. the proof of the splitting lemma in [GLS Thm I.2.47]). The same construction then works in characteristic $p$ and gives that the generic $\tilde{g}$ is right-equivalent to its 2-jet. But this should then also hold for the original function $g$, which is right-equivalent to a suspension of $x^p$. This contradiction shows that no such non-degenerate $g$ can exist. \qed

3. The basic trick

Let the given function be of the form $f = g + m\varphi^k$, where $m$ is any function, but preferably a monomial. Then we can remove the term $m\varphi^k$ by a double suspension:

**Lemma 3.1.** The function $f = g + m\varphi^k$ is stably equivalent to $-uv + u\varphi + mv^k + g$.

**Proof.**

$$- \left( u - m \frac{v^k - \varphi^k}{v - \varphi} \right) (v - \varphi) + m\varphi^k = -uv + u\varphi + mv^k.$$ \qed
This formula includes the special case $k = 1$: $g + m\varphi$ is stably equivalent to $-uv + u\varphi + vm + g$.

**Corollary 3.2.** Every polynomial is stably equivalent to a polynomial of degree 3.

**Proof.** A product $m\varphi$ of degree $d + e$ with $e - 1 \leq d \leq e$ can be replaced by $-uv + u\varphi + vm$ with summands of degrees 2, $e+1$ and $d+1$, which are less than $d+e$ except when $d = 1$ and $e = 2$. □

We note also the case $m = 1$ and $k = 2$, so we have $f = g + \varphi^2$. The basis trick gives $-uv + u\varphi + v^2 + g$ to which we apply the coordinate transformation $v = \bar{v} + \frac{1}{2}u$, yielding $\bar{v}^2 + \frac{1}{2}u^2 + u\varphi + g$, so $f$ is also stably equivalent $\frac{1}{4}u^2 + u\varphi + g$, which is the obvious way to treat this case.

**Example 3.3.** If $f = g + m_1\varphi^{k_1} + m_2\varphi^{k_2}$, we can apply our basic trick twice to get

$$-u_1v_1 - u_2v_2 + (u_1 + u_2)\varphi + m_1v_1^{k_1} + m_2v_2^{k_2} + g$$

after which we make $u_1 + u_2$ into a new variable, say by replacing $u_2$ by $u_2 - u_1$, giving

$$-u_1v_1 + u_1v_2 - u_2v_2 + u_2\varphi + m_1v_1^{k_1} + m_2v_2^{k_2} + g$$

This example generalises to more terms.

As one sees, the number of variables increases rapidly, making it difficult to determine the faces of the Newton diagram. In the example above the polynomial $\varphi$, which was responsible for degeneracy, occurs in the final result on its own, only multiplied with a monomial. Changing the coefficients of $\varphi$ presumably does not influence the Milnor number. We refer to this situation as seemingly non-degenerate and formulate this concept in a rather imprecise definition.

**Definition 3.4.** We say that a series is seemingly non-degenerate if changing arbitrarily the coefficients in the formula leads to arbitrary changes of the coefficients of the monomials on the Newton diagram.

Admittedly, as we typically do not vary the 2-jet of the series, we cannot be sure without actually doing the computation that our series is general enough for its Newton diagram. So the use of our rather vague term actually implies a conjecture, that the series really is non-degenerate.

Our strategy is to remove a face on which the function degenerates. Terms above the original Newton diagram can now end up on the new diagram, and we have to take care of new degeneracies. This process might never stop. As a simple example, consider a plane curve with equation of the form $f = \sum_{i=1}^{\infty} \varphi_i^2$, where the $\varphi_i$ have pairwise no common divisors. Each term $\varphi_i$ can be replaced by $-z_i^2 + 2z_i\varphi_i$, but we need infinitely many new variables. However, under our assumptions on the $\varphi_i$ we actually have an isolated singularity, so $f$ is finitely determined and right-equivalent to a polynomial of the form $\sum_{i=1}^{N} \varphi_i^2$ and therefore stably equivalent to $\sum_{i=1}^{N} -z_i^2 + 2z_i\varphi_i$. 

4. IRREDUCIBLE PLANE CURVE SINGULARITIES

We describe equations for irreducible plane curve singularities following Teissier [Te], see also [C-N]. We look at algebroid curves over an algebraically closed field $K$ of characteristic zero. We can parametrise the curves with Puiseux series, but the starting point is the semi-group.

So let $\Gamma = \langle \beta_0, \ldots, \beta_g \rangle$ be the semigroup of the curve. Define numbers $n_i$ by $e_i = \gcd(\beta_0, \ldots, \beta_i)$ and $e_{i-1} = n_i e_i$. The condition that $\Gamma$ comes from a plane curve singularity, is that $n_i \beta_i \in \langle \beta_0, \ldots, \beta_{i-1} \rangle$ and $n_i \beta_i < \beta_{i+1}$.

**Remark 4.1.** The semigroup $\Gamma$ determines the Puiseux characteristic

$$(\beta_0; \beta_1, \ldots, \beta_g),$$

where $\beta_0 = n = \bar{\beta}_0$ is the multiplicity of the curve, by $\beta_1 = \bar{\beta}_1$ and the formula $\beta_i - \beta_{i-1} = \bar{\beta}_i - n_{i-1} \bar{\beta}_{i-1}$. Putting $\beta_i = m_i e_i$ gives the Puiseux pairs $(m_i, n_i)$, $i = 1, \ldots, g$.

Teissier showed that every plane curve singularity with semigroup $\Gamma$ occurs in the positive weight part of versal deformation of the monomial curve $C_\Gamma$ with this semigroup. Embed $C_\Gamma$ in $K^{g+1}$ by $u_i = t^{\bar{\beta}_i}$. Write

$$n_i \beta_i = \ell^{(i)}_0 \beta_0 + \ell^{(i)}_1 \beta_1 + \cdots + \ell^{(i)}_{i-1} \beta_{i-1}.$$

The curve $C_\Gamma$ is a complete intersection with equations

$$f_1 = u_1^{n_1} - u_0^{\ell^{(1)}_0} = 0,$$
$$f_2 = u_2^{n_2} - u_0^{\ell^{(2)}_0} u_1^{\ell^{(2)}_1} = 0,$$
$$\vdots$$
$$f_g = u_g^{n_g} - u_0^{\ell^{(g)}_0} \cdots u_{g-1}^{\ell^{(g)}_{g-1}} = 0.$$

A particular simple deformation of positive weight is given by $f_i + \varepsilon u_{i+1}$, and we may even take $\varepsilon = 1$. It is then easy to eliminate the $u_i$ with $i \geq 2$ to obtain an equation of a plane curve. Cassou-Nogues [C-N] has shown that one can write the whole equisingular deformation of this particular curve as $f_i + \varepsilon u_{i+1}$, where $f_i$ only depends on the coordinates $u_0, \ldots, u_i$, so it is possible to do the same elimination for the whole stratum. However, as the curve is no longer quasi-homogeneous it is no longer clear whether every plane curve occurs in this family.

The easiest elimination occurs when $\ell^{(i)}_j = 0$ for all $j \geq 2$ and all $i$. Such semi-groups exist for all $g$. They can be constructed inductively. Given $\langle \beta_0, \ldots, \beta_{g-1} \rangle$ with $\gcd(\beta_0, \ldots, \beta_{g-1}) = 1$ and such that $\ell^{(i)}_j = 0$ for $j \geq 2$, take a semigroup $\langle n_g \beta_0, \ldots, n_g \beta_{g-1}, \beta_g \rangle$ with $\gcd(n_g, \beta_g) = 1$, $\beta_g > n_{g-1} n_g \beta_{g-1}$ and $\bar{\beta}_g \in \langle \beta_0, \bar{\beta}_1 \rangle$.

**Lemma 4.2.** The deformed curve $f_i + u_{i+1}$, with $\ell^{(i)}_j = 0$ for all $j \geq 2$, is stably equivalent to a seemingly non-degenerate singularity.
Proof. In this case the equation of the plane curve is
\[
\left( \ldots \left( u_1^{n_1} - u_0^{(1)} u_1^{(1)} \right) n_2 - u_0^{(2)} u_1^{(2)} \right) n_3 - u_0^{(g-1)} u_1^{(g-1)} \right)^n - u_0^{(g)} u_1^{(g)} = 0
\]
This is of the form \( \varphi_g^{n_g} - u_0^{(g)} u_1^{(g)} = 0 \), and \( \varphi_g = \varphi_g^{n_g-1} - u_0^{(g-1)} u_1^{(g-1)} \) is itself of the same form. The principal part is a complete \( n_g \)-th power. We apply the basic trick (Lemma 3.1) and write
\[
-v_g w_g + v_g \varphi_g = \left( \varphi_g^{n_g-1} - u_0^{(g-1)} u_1^{(g-1)} \right),
\]
so remember that \( v_g \varphi_g = v_g \left( \varphi_g^{n_g-1} - u_0^{(g-1)} u_1^{(g-1)} \right) \), so we apply the basic trick once more, now to \( v_g \varphi_g^{n_g-1} \), and obtain
\[
-v_g w_g - v_g \varphi_g - v_g w_g^{n_g-1} + v_g w_g^{n_g-1} + w_g^{n_g} - v_g u_0^{(g-1)} u_1^{(g-1)} = - u_0^{(g)} u_1^{(g)}.
\]
The next step takes care of \( v_g \varphi_g^{n_g-1} \) and we continue inductively. The final result is
\[
-v_g w_g - \cdots - v_2 w_2 + v_2 (u_1^{n_1} - u_0^{(1)}) + v_3 w_2^{n_2} + \cdots + w_g^{n_g}
\]
\[
- v_3 u_0^{(2)} u_1^{(2)} - \cdots - v_g u_0^{(g)} u_1^{(g)} - u_0^{(g)} u_1^{(g)}.
\]

Conjecture 4.3. The final function above is non-degenerate, as all facets of the Newton diagram are simplices.

We show this in the case \( g = 3 \). In this case there are eight monomials, \( v_3 w_3, v_2 w_2, v_2 u_1^{n_1}, v_2 u_0^{(1)} , v_3 w_2^{n_2}, w_3, v_3 u_0^{(2)}, u_1^{(2)}, u_0^{(3)} u_1^{(3)} \). The facets containing both \( v_2 u_1^{n_1} \) and \( v_2 u_0^{(1)} \) are rather easy to describe.

The first facet contains \( v_3 w_3, v_2 w_2, v_2 u_1^{n_1}, v_2 u_0^{(1)}, v_3 w_2^{n_2} \) and \( w_3. \) We have \( \text{wt}(w_3) = \frac{1}{n_3}, \text{wt}(v_3) = 1 - \frac{1}{n_3}, \text{wt}(w_2) = \frac{1}{n_2 n_3}, \text{wt}(v_2) = 1 - \frac{1}{n_2 n_3}, \text{wt}(u_1) = \frac{1}{n_1 n_2 n_3} \) and \( \text{wt}(u_0) = \frac{1}{n_1 n_2 n_3}. \) The other two points lie above this plane. To see this, remember that \( \beta_0 = n_1 n_2 n_3, \beta_1 = l_0^{(1)} n_2 n_3 \) and \( n_1 \bar{\beta}_1 = l_0^{(1)} \bar{\beta}_0 + l_1^{(1)} \bar{\beta}_1 \) for \( i = 2, 3 \) so
\[
\frac{l_0^{(1)} n_1 + l_1^{(1)} n_1}{l_0^{(1)} n_1} = \frac{n_1 \beta_i}{n_1 \bar{\beta}_i}.
\]
Furthermore \( \bar{\beta}_{i+1} > n_i \bar{\beta}_i \).

Now suppose that a facet contains \( v_3 w_3, v_2 w_2, v_2 u_1^{n_1}, v_2 u_0^{(1)} \). Let \( \text{wt}(w_3) = \alpha \geq \frac{1}{n_3} \), then \( \text{wt}(v_3) = 1 - \alpha \), and let \( \text{wt}(w_2) = \beta \geq \frac{1}{n_2} \), so \( \text{wt}(v_2) = 1 - \beta \). Then
wt\(u_1\) = \(\frac{\beta}{n_1}\) and wt\(u_0\) = \(\frac{\beta}{l_0}\). We compute

\[
wt(v_3u_0^{(2)}u_1^{(2)}) = (1 - \alpha) + \beta \left( \frac{l_0^{(2)}}{l_0^{(1)}} + \frac{l_1^{(2)}}{n_1} \right) = (1 - \alpha) + \frac{\beta n_2 \beta_2}{n_1 \beta_1} > (1 - \alpha) + \beta n_2 \geq 1,
\]

\[
wt(u_0^{(3)}u_1^{(3)}) = \beta \frac{l_0^{(3)}n_1 + l_1^{(3)}l_0^{(1)}}{l_0^{(1)}n_1} = \beta \frac{n_3 \beta_3}{n_1 \beta_1} > 1.
\]

This shows that indeed the points lie above the plane, and that the facet given above is the only one containing \(v_3w_3\), \(v_2w_2\), \(v_2u_1\) and \(v_2u_0^{(1)}\).

Suppose that \(v_3w_3\) lies above a facet containing the other five points from the first facet. Then wt\(v_3\) = \(\frac{1}{n_3}\), wt\(v_2\) = \(1 - \alpha \), wt\(u_1\) = \(\frac{\alpha}{n_1n_2}\), and wt\(u_0\) = \(\frac{\alpha}{l_0 n_2}\). As before we find

\[
wt(v_3u_0^{(2)}u_1^{(2)}) > 1, \quad \text{but now } wt(u_0^{(3)}u_1^{(3)}) = \frac{\alpha n_3 \beta_3}{n_2 n_1 \beta_1}, \quad \text{so } \alpha = \frac{n_2 n_1 \beta_1}{n_3 \beta_3}.
\]

For the facet not containing \(v_2w_2\) we find wt\(v_3\) = \(\frac{1}{n_3}\), wt\(v_2\) = \(1 - \frac{n_1 \beta_1}{n_3 n_2 \beta_2}\), wt\(u_1\) = \(\frac{n_1 \beta_1}{n_3 n_2 \beta_2}\), and wt\(u_0\) = \(\frac{n_1 \beta_1}{l_0 n_3 n_2 \beta_2}\). Now

\[
wt(v_3u_0^{(2)}u_1^{(2)}) > 1. \quad \text{Finally there is a facet not containing } v_2w_2 \text{ and } v_3w_3. \quad \text{For it wt}(u_3) = \frac{1}{n_3}, \quad \text{wt}(v_3) = 1 - \frac{n_2 \beta_2}{n_3 \beta_3}, \quad \text{wt}(u_2) = \frac{n_2 \beta_2}{n_3 \beta_3}, \quad \text{wt}(v_2) = 1 - \frac{n_2 \beta_2}{n_3 \beta_3}, \quad \text{wt}(u_1) = \frac{n_2 \beta_2}{n_3 \beta_3}
\]

and wt\(u_0\) = \(\frac{n_2 \beta_2}{l_0 n_3 n_2 \beta_2}\).

The remaining facets, on which only one of \(v_2u_1\) and \(v_2u_0^{(1)}\) lies, are more difficult to describe, as they depend on the values of \(l_0^{(i)}\). We note that such a facet also contains exactly six points: one has to leave out at two points from the first simplex. Suppose on the contrary that there is a facet containing \(v_3w_3\), \(v_2w_2\), \(v_3u_2\), \(u_3\) and say \(v_2u_1\). Then wt\(v_3\) = \(\frac{1}{n_3}\), wt\(v_2\) = \(1 - \frac{1}{n_3}\), wt\(u_1\) = \(\frac{1}{n_1 n_2 n_3}\), and wt\(u_0\) = \(\frac{1}{l_0 n_2 n_3}\). But then wt\(v_0^{(3)}u_1^{(3)}\) > 1 and wt\(v_3u_0^{(2)}u_1^{(2)}\) > 1. So indeed every facet is a simplex.

**Example 4.4.** Without the assumption \(l_0^{(i)} = 0\) for all \(j \geq 2\) the situation is more complicated and we only give the case \(g = 4\). The equation is now

\[
\left( (u_1^{n_1} - u_0^{l_1^{(1)}})^{n_2} - u_1^{l_1^{(2)}} u_0^{l_1^{(2)}} \right)^{n_3} - \left( (u_1^{n_1} - u_0^{l_1^{(3)}})^{l_2^{(3)}} u_1^{l_0^{(3)}} u_0^{l_0^{(3)}} \right)^{n_4} - \left( (u_1^{n_1} - u_0^{l_1^{(4)}})^{l_2^{(4)}} u_1^{l_0^{(4)}} u_0^{l_0^{(4)}} \right).
\]
We start with one application of the basic trick (Lemma 3.3) to get
\[
-v_4 w_4 + w^{n_4} + v_4 \left( (u_1^{n_1} - u_0^{(1)})^{n_2} - u_1^{(2)} u_0^{(2)} \right) \left( u_1^{(3)} u_0^{(3)} \right) - v_4 \left( u_1^{n_1} - u_0^{(1)} \right)^2 u_1^{(3)} u_0^{(3)} - \\
\left( (u_1^{n_1} - u_0^{(1)})^{n_2} - u_1^{(2)} u_0^{(2)} \right) \left( u_1^{n_1} - u_0^{(1)} \right)^2 u_1^{(3)} u_0^{(3)} .
\]

Let \( \varphi_3 = (u_1^{n_1} - u_0^{(1)})^{n_2} - u_1^{(2)} u_0^{(2)} \). Then we have two terms involving a power of \( \varphi_3 \), so we apply the basic trick twice, followed by a coordinate transformation as in Example 3.3 to get
\[
-v_4 w_4 - v_3.1 w_3.1 + v_3.1 w_3.2 - v_3.2 w_3.2 + w^{n_4} + v_4 w^{n_3} + v_3.2 (u_1^{n_1} - u_0^{(1)})^{n_2} - v_3.2 u_1^{(2)} u_0^{(2)} \\
- v_4 (u_1^{n_1} - u_0^{(1)})^{n_2} u_1^{(3)} u_0^{(3)} - w_3.2 (u_1^{n_1} - u_0^{(1)})^{n_2} u_1^{(3)} u_0^{(3)}. 
\]

Finally we introduce six new variables to handle the powers of \( \varphi_2 = u_1^{n_1} - u_0^{(1)} \).
\[
-v_4 w_4 - v_3.1 w_3.1 + v_3.1 w_3.2 - v_3.2 w_3.2 - v_2.1 w_2.1 - v_2.2 w_2.2 + v_2.1 w_2.3 + v_2.2 w_2.3 - v_2.3 w_2.3 \\
+ w^{n_4} + v_4 w^{n_3} + v_3.2 w^{2.3} + v_2.3 (u_1^{n_1} - u_0^{(1)}) - v_3.2 u_1^{(2)} u_0^{(2)} \\
- v_4 w^{2.1} u_1^{(3)} u_0^{(3)} - w^{2.2} w^{2.3} u_1^{(4)} u_0^{(4)}. 
\]

5. Some Surface Singularities

We look at Lê-Yomdin singularities of the type
\[
f = f_d + l^k
\]
where \( f_d \) defines a projective hypersurface with one singular point. Suppose \( f_d \) is degenerate, but is stably equivalent to \( q + f_d \), with the coordinate transformations of the type considered earlier, that is, not changing the original coordinates. Then \( q + f + l^k \) will degenerate on the face defined by \( l^k \). This can be remedied by the basic trick, giving \( q + f - uv + ul + v^k \). The suspension is not necessary, if \( l \) itself is a coordinate function. In practice this will be the case. Otherwise we can start with a coordinate transformation, but one has to be careful not to make \( f_d \) too complicated. This said, we will concentrate on making \( f_d \) non-degenerate.

We consider with projective curves with only ordinary double points. If there are at most three in general position, we can place them at the vertices of the coordinate triangle, and then the function is non-degenerate.

We can also apply our tricks.

Example 5.1 (A cubic curve with one double point). In this case the Lê-Yomdin singularity is just \( T_{3,3,k} \), which is non-degenerate in its standard normal form, where the double point lies at the origin of the affine \((x,y)\) chart.
Let the double point be a general point. Write its tangent cone as \( m_1^2 - m_2^2 \), where \( m_1 \) and \( m_2 \) are linear forms. There linear forms \( l_1 \) and \( l_2 \) such that the equation has the form

\[
f_3 = l_1 m_1^2 + l_2 m_2^2 .
\]

Indeed, by a coordinate transformation we achieve that the tangent cone is \( x^2 - y^2 \), so in affine coordinates the equation has the form \( x^2 - y^2 + g_3(x, y) \), which can be written as \( l_1' x^2 + l_2' y^2 \); now transform back. The polynomial \( f_3 \) is stably equivalent to

\[
- u_1 v_1 - u_2 v_2 + u_1 m_1 + u_2 m_2 + v_1^2 l_1 + v_2^2 l_2,
\]

which for general \( l_1, l_2, m_1 \) and \( m_2 \) is non-degenerate.

**Example 5.2 (A cubic with three double points).** The equation has the form \( f_3 = l_1 l_2 l_3 \). We apply the basic trick, first once:

\[
- u_1 v_1 + u_1 l_1 + v_1 l_2 l_3
\]

and then once again:

\[
- u_1 v_1 - u_2 v_2 + u_1 l_1 + u_2 l_2 + v_1 v_2 l_3.
\]

**Example 5.3 (A quartic with four double points).** Now it is no longer possible to place the double points at the vertices of the coordinate triangle. The four points are a complete intersection of two conics and the equation has just the form \( f_4 = q_1 q_2 \), where \( q_1 \) and \( q_2 \) are nonsingular. This is stably equivalent to

\[
- uv + u q_1 + v q_2 .
\]

**Example 5.4 (A quintic with four double points).** Let the four points again be given by \( q_1 \) and \( q_2 \). The general quintic with nodes at the four points can be written as

\[
f_5 = l_1 q_1^2 + l_2 (q_1 + q_2)^2 + l_3 q_2^2
\]

with the \( l_i \) linear forms. Now we first form

\[
- u_1 v_1 - u_2 v_2 - u_3 v_3 + u_1 q_1 + u_2 (q_1 + q_2) + u_3 q_3 + v_1^2 l_1 + v_2^2 l_2 + v_3^2 l_3
\]

and then we replace \( u_1 \) by \( u_1 - u_2 \) and \( u_3 \) by \( u_3 - u_2 \), resulting in

\[
- u_1 v_1 + u_2 v_1 - u_2 v_2 + u_2 v_3 - u_3 v_3 + u_1 q_1 + u_3 q_2 + v_1^2 l_1 + v_2^2 l_2 + v_3^2 l_3 .
\]

This case can also be treated in another way. Write

\[
f_5 = h_1 q_1 + h_2 q_2,
\]

where now the \( h_i \) are cubic forms. This polynomial is stably equivalent to

\[
- u_1 v_1 - u_2 v_2 + u_1 h_1 + u_2 q_2 + v_2 h_2 .
\]

Also this function is seemingly non-degenerate.
6. SOME DEGENERATE REDUCIBLE CURVES

Examples of degenerate singularities with unexpected properties have been given by Brzostowski [Br]. They are
\[ f = ((x - y)^2 - x^6)(x^2 + y^4), \]
\[ g = f + 2xy^4. \]
Both functions have the same Newton diagram with \( \nu = 11 \). The function \( f \) degenerates on only one face of the diagram, whereas \( g \) degenerates on both. The striking property is that \( \mu(f) = 15 > \mu(g) = 13 \). On second thoughts this is maybe not so strange, as the Milnor number of a degenerate function depends on the monomials above the supporting hyperplanes.

Both \( f \) and \( g \) are in fact right-equivalent to non-degenerate functions [Br]: for \( f \) use the transformation \( (x, y) \mapsto (x, x + y) \) and for \( g \) the transformation \( (x, y) \mapsto (x - y^2, x + y) \). We will not use this, but make the functions non-degenerate by suspension and coordinate transformations, which leave \( x \) and \( y \) invariant.

6.1. The function \( f \). We remove the monomials on the lower face of the diagram:
\[ f_1 = -z^2 + 2zx(x - y) + y^4(x - y)^2 - x^6(x^2 + y^4). \]
This function is still degenerate, which can be seen by writing it in the following way:
\[ f_1 = -z^2 + (x - y)(2zx + y^4(x - y)) - x^6(x^2 + y^4). \]
It is stably equivalent to
\[ f_2 = -z^2 - uv + v(x - y) + u(2zx + y^4(x - y)) - x^6(x^2 + y^4). \]
After the transformation \( v \mapsto v - uy^4 \) we obtain

\[
f_3 = -z^2 - uv + v(x - y) + 2uxz + u^2y^4 - x^6(x^2 + y^4).
\]

Now \( \nu(f_3) = \mu(f_3) = 15 \). The easiest way to compute \( \nu(f_3) \) is to compute the Milnor number of a general enough function with the same Newton diagram with \textsc{Singular} \cite{DGPS}. But still \( f_3 \) is degenerate. To make degeneracy evident, we complete the square involving \( z \) and write

\[
f_3 = -(z - ux)^2 - uv + v(x - y) + (u^2 - x^6)(x^2 + y^4).
\]

We leave out \((z - ux)^2\) and introduce two new variables to arrive at our final function

\[
f_4 = -uv - st + v(x - y) + s(u^2 - x^6) + t(x^2 + y^4).
\]

With hindsight we could have arrived at \( f_4 \) more directly, as \( f \) was given as a product, which we first split, and then take care of the multiple factor in the term \( s(x - y)^2 \).

**Lemma 6.1.** The function \( f_4 \) is non-degenerate for its Newton diagram.

**Proof.** As the number of variables does not exceed six, we can use the program \textsc{Germènes} \cite{Mo} to determine the Newton diagram. There are four facets, and in total 143 compact faces, which all are simplicial. Therefore the function is non-degenerate.

We remark that the approach using the multiplicity of the ideal \( I(f_4^{(c)}) \) for a suitable convenient function, and a general enough function with the same diagram, does not work here, as the computation takes too long. \( \square \)

6.2. The function \( g \). We first remove the monomials on the upper edge of the diagram:

\[
g_1 = -z^2 + 2zy(x + y^2) + (y^4 + x^2)(-2xy + x^2) - x^6(x^2 + y^4).
\]

The degeneracy on the other edge of course survives and manifests itself on the lower quadrangle:

\[
-z^2 + 2zyx - 2yx^3 + x^4 = (x^2 - 2yx + z)(x^2 - z).
\]

By our basic trick \( g \) is stably equivalent to

\[
g_2 = -uv + u(x^2 - z) + v(x^2 - 2xy + z) + 2zy3 - 2xy^5 - y^4x^2 - x^6(x^2 + y^4).
\]

**Lemma 6.2.** The function \( g_2 \) is non-degenerate for its Newton diagram.

**Proof.** We again use the program \textsc{Germènes} \cite{Mo}. There are seven compact facets, and in total 133 compact faces. Most of them are simplicial. For the other ones it is easy to check the function is non-degenerate. \( \square \)
7. Final Remarks

The strategy in the above examples is to place the singular locus in the newly introduced coordinate hyperplanes, thereby achieving non-degeneracy. This requires a detailed study of this singular locus and of the equation under consideration. This makes a general proof of a positive answer to the question difficult. On the other hand, I see no restrictions for succeeding in every particular case. I welcome suggestions for possible counterexamples.

References

[Arn] V.I. Arnold, Some open problems in the theory of singularities. In: Singularities, Proc. Symp. Pure Math. 40, Part 1, pp. 57–69 (1983).

[Bi] Carles Bivià-Ausina, Jacobian ideals and the Newton non-degeneracy condition. Proc. Edinb. Math. Soc., II. Ser. 48 (2005), 21–36.

[BGM] Yousra Boubakri, Gert-Martin Greuel and Thomas Markwig, Invariants of hypersurface singularities in positive characteristic. Rev. Mat. Complut. 25 (2012), 61–85.

[Br] Szymon Brzostowski, Degenerate singularities and their Milnor numbers. Univ. Iagel. Acta Math., 49 (2011), 37–44.

[C-N] P. Cassou-Noguès, Courbes de semi-groupe donné. Rev. Mat. Univ. Complutense Madr. 4 (1991), 13–44.

[DGPS] W. Decker, G.-M. Greuel, G. Pfister and H. Schönemann. SINGULAR 3-1-6 — A computer algebra system for polynomial computations. [http://www.singular.uni-kl.de](http://www.singular.uni-kl.de) (2012).

[GLS] G-M. Greuel, C. Lossen and E. Shustin, Introduction to singularities and deformations. Springer Monographs in Mathematics. Berlin: Springer (2007).

[GN] Gert-Martin Greuel and Nguyen Hong Duc. Some remarks on the planar Kouchnirenko’s theorem Rev. Mat. Complutense bf 25 (2012), 557–579.
[Kou] A.G. Kouchnirenko, *Polyèdres de Newton et nombres de Milnor*. Invent. Math. **32** (1976), 1–31.

[Lu] Ignacio Luengo, *The $\mu$-constant stratum is not smooth*. Invent. Math. **90** (1987), 139–152.

[Mo] Ángel Montesinos, GÉRMINES. [http://www.uv.es/montesin/](http://www.uv.es/montesin/)

[St] Jan Stevens, *On the $\mu$-constant stratum and the $V$-filtration: an example*. Math. Z. **201** (1989), 139–144.

[Te] Bernard Teissier, Appendix to: O. Zariski, Le problème des modules pour les branches planes. Course given at the Centre de Mathématiques de l'Ecole Polytechnique, Paris, October–November 1973. Second edition. Hermann, Paris, 1986. English translation by Ben Lichtin, University Lecture Series, 39 (2006).

[Va] A.N. Varchenko, *Zeta-function of monodromy and Newton’s diagram*. Invent. Math. **37** (1976), 253–262.

[Wa] C.T.C. Wall, *Newton polytopes and non-degeneracy*. J. Reine Angew. Math. **509** (1999), 1–19.

Department of Mathematical Sciences, Chalmers University of Technology and University of Gothenburg. SE 412 96 Gothenburg, Sweden

E-mail address: stevens@chalmers.se