A characterization of hypergraphs that achieve equality in the Chvátal-McDiarmid Theorem

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Abstract

For $k \geq 2$, let $H$ be a $k$-uniform hypergraph on $n$ vertices and $m$ edges. The transversal number $\tau(H)$ of $H$ is the minimum number of vertices that intersect every edge. Chvátal and McDiarmid [Combinatorica 12 (1992), 19–26] proved that $\tau(H) \leq (n + \left\lfloor \frac{k}{2} \right\rfloor m) / \left\lfloor \frac{3k}{2} \right\rfloor)$. When $k = 3$, the connected hypergraphs that achieve equality in the Chvátal-McDiarmid Theorem were characterized by Henning and Yeo [J. Graph Theory 59 (2008), 326–348]. In this paper, we characterize the connected hypergraphs that achieve equality in the Chvátal-McDiarmid Theorem for $k = 2$ and for all $k \geq 4$.

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1 Introduction

In this paper we continue the study of transversals in hypergraphs. Hypergraphs are systems of sets which are conceived as natural extensions of graphs. A hypergraph $H = (V, E)$ is a finite set $V = V(H)$ of elements, called vertices, together with a finite multiset $E = E(H)$ of subsets of $V$, called hyperedges or simply edges.

A $k$-edge in $H$ is an edge of size $k$. The hypergraph $H$ is said to be $k$-uniform if every edge of $H$ is a $k$-edge. Every (simple) graph is a 2-uniform hypergraph. Thus graphs are special hypergraphs. The degree of a vertex $v$ in $H$, denoted by $d_H(v)$ or simply by $d(v)$ if $H$ is clear from the context, is the number of edges of $H$ which contain $v$. The minimum and maximum degrees among the vertices of $H$ is denoted by $\delta(H)$ and $\Delta(H)$, respectively.

Two vertices $x$ and $y$ of $H$ are adjacent if there is an edge $e$ of $H$ such that $\{x, y\} \subseteq e$. The neighborhood of a vertex $v$ in $H$, denoted $N_H(v)$ or simply $N(v)$ if $H$ is clear from the context, is the set of all vertices different from $v$ that are adjacent to $v$. Two vertices $x$ and $y$ of $H$ are connected if there is a sequence $x = v_0, v_1, v_2 \ldots, v_k = y$ of vertices of $H$ in
which \( v_{i-1} \) is adjacent to \( v_i \) for \( i = 1, 2, \ldots, k \). A connected hypergraph is a hypergraph in which every pair of vertices are connected. A maximal connected subhypergraph of \( H \) is a component of \( H \). Thus, no edge in \( H \) contains vertices from different components.

If \( H \) denotes a hypergraph and \( X \) denotes a subset of vertices in \( H \), then \( H - X \) will denote that hypergraph obtained from \( H \) by removing the vertices \( X \) from \( H \), removing all hyperedges that intersect \( X \) and removing all resulting isolated vertices, if any. If \( X = \{x\} \), we simply denote \( H - X \) by \( H - x \). We remark that in the literature this is sometimes denoted by strongly deleting the vertices in \( X \).

A subset \( T \) of vertices in a hypergraph \( H \) is a transversal (also called vertex cover or hitting set in many papers) if \( T \) has a nonempty intersection with every edge of \( H \). The transversal number \( \tau(H) \) of \( H \) is the minimum size of a transversal in \( H \). A transversal of size \( \tau(H) \) is called a \( \tau(H) \)-set. Transversals in hypergraphs are well studied in the literature (see, for example, [1, 2, 3, 4, 5, 8, 10]). Chvátal and McDiarmid [1] established the following upper bound on the transversal number of a uniform hypergraphs in terms of its order and size.

**Chvátal-McDiarmid Theorem.** For \( k \geq 2 \), if \( H \) is a \( k \)-uniform hypergraph on \( n \) vertices with \( m \) edges, then

\[
\tau(H) \leq \frac{n + \left\lceil \frac{k}{2} \right\rceil m}{\frac{3k}{2}}.
\]

As a special case of the Chvátal-McDiarmid Theorem when \( k = 3 \), we have that if \( H \) is a 3-uniform hypergraph on \( n \) vertices with \( m \) edges, then \( \tau(H) \leq (n + m)/4 \). This bound was independently established by Tuza [11] and a short proof of this result was also given by Thomassé and Yeo [10]. The extremal connected hypergraphs that achieve equality in the Chvátal-McDiarmid Theorem when \( k = 3 \) were characterized by Henning and Yeo [3]. Their characterization showed that there are three infinite families of extremal connected hypergraphs, as well as two special hypergraphs, one of order 7 and the other of order 8.

Our aim in this paper is to characterize the connected hypergraphs that achieve equality in the Chvátal-McDiarmid Theorem for \( k = 2 \) and for all \( k \geq 4 \). For this purpose we define two special families of hypergraphs.

### 1.1 Special Families of Hypergraphs

For \( k \geq 2 \), let \( E_k \) denote the \( k \)-uniform hypergraph on \( k \) vertices with exactly one edge. The hypergraph \( E_4 \) is illustrated in Figure 1.

For \( k \geq 2 \), a generalized triangle \( T_k \) is defined as follows. Let \( A, B, C \) and \( D \) be vertex-disjoint sets of vertices with \( |A| = \lfloor k/2 \rfloor \), \( |B| = |C| = \lfloor k/2 \rfloor \) and \( |D| = \lfloor k/2 \rfloor - \lfloor k/2 \rfloor \). In particular, if \( k \) is even, the set \( D = \emptyset \), while if \( k \) is odd, the set \( D \) consist of a singleton vertex. Let \( T_k \) denote the \( k \)-uniform hypergraph with \( V(T_k) = A \cup B \cup C \cup D \) and with \( E(T_k) = \{e_1, e_2, e_3\} \), where \( V(e_1) = A \cup B \), \( V(e_2) = A \cup C \), and \( V(e_3) = B \cup C \cup D \). The hypergraphs \( T_4 \) and \( T_5 \) are illustrated in Figure 1.
Figure 1: The hypergraphs $E_4$, $T_4$, and $T_5$

2 Main Result

We shall prove:

**Theorem 1** For $k = 2$ or $k \geq 4$, let $H$ be a connected $k$-uniform hypergraph on $n$ vertices and $m$ edges. Then,

$$
\tau(H) \leq n + \left\lfloor \frac{k}{2} \right\rfloor m
$$

with equality if and only if $H = E_k$ or $H = T_k$.

We proceed as follows. We first recall some important results on edge colorings of multigraphs in Section 3. Thereafter we establish a key theorem about matchings in multigraphs in Section 4. Finally in Section 5 we present a proof of Theorem 1 using an interplay between transversals in hypergraphs and matchings in multigraphs.

3 Edge Colorings of Multigraphs

Let $G$ be a multigraph. An *edge coloring* of $G$ is an assignment of colors to the edges of $G$ such that adjacent edges receive different colors. The minimum number of colors needed for an edge coloring is called the chromatic index of the multigraph, denoted $\chi'(G)$. The edge-multiplicity of an edge $e = uv$, written $\mu(e)$, is the number of edges joining $u$ and $v$.

In his study of electrical networks, Shannon [9] established the following upper bound on the chromatic index of a multigraph.

**Shannon’s Theorem.** If $G$ is a multigraph, then $\chi'(G) \leq \lfloor 3\Delta(G)/2 \rfloor$.

For $d \geq 2$, a *Shannon multigraph of degree $d$* is a multigraph on three vertices, with one pair of vertices joined by $\lceil d/2 \rceil$ edges and the other two pairs joined by $\lfloor d/2 \rfloor$ edges. Thus for fixed $d$, all Shannon multigraphs of degree $d$ are isomorphic to the multigraph $G$ with
vertex set $V(G) = \{x, y, z\}$ and with $\mu(xy) = \lfloor d/2 \rfloor$, $\mu(xz) = \lfloor d/2 \rfloor$ and $\mu(yz) = \lceil d/2 \rceil$.

A characterization of multigraphs achieving the upper bound in Shannon’s Theorem when the maximum degree is at least 4 was given by Vizing [12].

**Vizing’s Theorem.** If $G$ is a connected multigraph with $\Delta(G) \geq 4$ and $\chi'(G) = \lfloor 3\Delta(G)/2 \rfloor$, then $G$ contains a Shannon multigraph of degree $\Delta(G)$ as a submultigraph.

We remark that the maximum degree condition in Vizing’s Theorem is essential. For example, if $G$ is a connected multigraph with $\Delta(G) = 2$ and $\chi'(G) = \lfloor 3\Delta(G)/2 \rfloor = 3$, then $G$ need not contain a Shannon multigraph of degree $\Delta(G)$ as a subgraph as may be seen by simply taking $G$ to be an odd cycle of length at least 5.

**4 Matchings in Multigraphs**

Let $G$ be a multigraph. Two edges in $G$ are independent if they are not adjacent in $G$. A set of pairwise independent edges of $G$ is called a matching in $G$, while a matching of maximum cardinality is a maximum matching. The number of edges in a maximum matching of $G$ is called the matching number of $G$ which we denote by $\alpha'(G)$. Our key matching theorem characterizes connected multigraphs with small matching number determined by Shannon’s Theorem.

**Theorem 2** For $d \geq 4$, let $G$ be a connected multigraph of size $m$ with $\Delta(G) \leq d$. Then, $\alpha'(G) \geq m/\lfloor 3d/2 \rfloor$, with equality if and only if either $m = 0$ or $G$ is a Shannon multigraph of degree $d$.

**Proof.** Let $\mathcal{C}$ be an arbitrary edge coloring of the edges of $G$ using $\chi'(G)$ colors. The matching number of $G$ is at least the cardinality of a maximum edge color class in $\mathcal{C}$, and so, by Shannon’s Theorem,

$$\alpha'(G) \geq \frac{m}{\chi'(G)} \geq \frac{m}{\lfloor 3\Delta(G)/2 \rfloor} \geq \frac{m}{\lfloor 3d/2 \rfloor},$$

which establishes the desired lower bound. Suppose that $\alpha'(G) = m/\lfloor 3d/2 \rfloor$ and $m \geq 1$. Then we must have equality throughout the above inequality chain. Thus, $\Delta(G) = d$, $\chi'(G) = \lfloor 3d/2 \rfloor$ and $\alpha'(G) = m/\chi'(G)$. In particular, since $\mathcal{C}$ is an arbitrary $\chi'(G)$-edge coloring, the edge color classes in every $\chi'(G)$-edge coloring have the same cardinality. Equivalently, the edge color classes in $\mathcal{C}$ are balanced. Since $\chi'(G) = \lfloor 3d/2 \rfloor$, Vizing’s Theorem implies that $G$ contains a Shannon multigraph, $M$ say, of degree $d$ as a submultigraph.

If $d$ is even, then every vertex of $M$ has degree $d$ in $M$. Since $\Delta(G) = d$, the Shannon multigraph $M$ cannot be a proper submultigraph of the connected multigraph $G$, implying that $G = M$. Hence if $d$ is even, then $G$ is a Shannon multigraph of degree $d$. Therefore we may assume that $d$ is odd, for otherwise the desired result holds.
Since $d$ is odd, $d \geq 5$ and one pair of vertices in $M$ is joined by $(d + 1)/2$ edges and the other two pairs are joined by $(d - 1)/2$ edges. Thus two vertices in $M$ have degree $d$ in $M$ and one vertex, $x$ say, of $M$ has degree $d - 1$ in $M$. Assume that $M$ is a proper submultigraph of $G$. Since $\Delta(G) = d$, the vertex $x$ is adjacent to exactly one vertex $v \notin V(M)$. Since $xv$ is a bridge in $G$, the edge $xv$ cannot belong to a submultigraph of $G$ that is isomorphic to a Shannon multigraph of degree $d$. Thus all submultigraphs of $G$ that are isomorphic to a Shannon multigraph of degree $d$ are vertex-disjoint.

Let $G'$ be the multigraph that arises from $G$ by deleting every edge from $G$ that belongs to a submultigraph of $G$ that is isomorphic to a Shannon multigraph of degree $d$. Then, $\Delta(G') \leq \Delta(G) - 1$ and, by construction, $G'$ does not contain a submultigraph of $G$ that is isomorphic to a Shannon multigraph of degree $d$. Since $xv \in E(G')$, the multigraph $G'$ has at least one edge. By Shannon’s Theorem and Vizing’s Theorem, we deduce that $\chi'(G) < \lfloor \frac{3d}{2} \rfloor$.

Let $C'$ be a $\chi'(G')$-edge coloring of the edges of $G'$. By construction, every submultigraph of the connected multigraph $G$ that is isomorphic to a Shannon multigraph of degree $d$ contains exactly one vertex that is incident with an edge of $G'$. Since $\chi'(G) < \lfloor 3d/2 \rfloor$, the coloring $C'$ can therefore be extended to a $\lfloor 3d/2 \rfloor$-edge coloring $C^*$ of $G$. Since $C'$ colors the edges of $G'$ with fewer than $\lfloor 3d/2 \rfloor$ colors, $C^*$ is a $\chi'(G)$-edge coloring of the edges of $G$ with at least two edge color classes having different cardinality. This contradicts our earlier observation that the edge color classes in every $\chi'(G)$-edge coloring have the same cardinality. Therefore, $M$ is not a proper submultigraph of the connected multigraph $G$, implying that $G = M$. Hence if $d$ is odd, then $G$ is a Shannon multigraph of degree $d$.

Conversely, if $G$ is a Shannon multigraph of degree $d$, then $m = \lfloor 3d/2 \rfloor$ and $\alpha'(G) = 1$, implying that $\alpha'(G) = m/ \lfloor 3d/2 \rfloor$. $\square$

We close this section by recalling Hall’s Matching Theorem due to König [7] and Hall [6].

**Hall’s Matching Theorem.** Let $G$ be a bipartite graph with partite sets $X$ and $Y$. Then $X$ can be matched to a subset of $Y$ if and only if $|N(S)| \geq |S|$ for every nonempty subset $S$ of $X$.

## 5 Proof of Main Result

We shall need the following properties of special hypergraphs defined in Section 1.1.

**Observation 3** Let $k \geq 2$ and let $H = E_k$ or $H = T_k$ and let $H$ have $n$ vertices and $m$ edges. Then the following holds.

(a) If $H = E_k$, then $\tau(H) = 1$.
(b) If $H = T_k$, then $\tau(H) = 2$.
(c) $\tau(H) = (n + \lfloor \frac{n}{2} \rfloor m)/ \lfloor \frac{3n}{2} \rfloor$.
(d) Every vertex in $H$ belongs to some $\tau(H)$-set.
We are now in a position to prove our main result. Recall the statement of Theorem 1.

**Theorem** 1. For \( k = 2 \) or \( k \geq 4 \), let \( H \) be a connected \( k \)-uniform hypergraph on \( n \) vertices and \( m \) edges. Then,

\[
\tau(H) \leq n + \left\lfloor \frac{k}{2} \right\rfloor m
\]

with equality if and only if \( H = E_k \) or \( H = T_k \).

**Proof.** The upper bound on \( \tau(H) \) is a restatement of the Chvátal-McDiarmid Theorem. We only need prove that \( \tau(H) = (n + \left\lfloor \frac{k}{2} \right\rfloor m)/\left\lceil \frac{3k}{2} \right\rceil \) if and only if \( H = E_k \) or \( H = T_k \). If \( H = E_k \) or \( H = T_k \), then by Observation 3, \( \tau(H) = (n + \left\lfloor \frac{k}{2} \right\rfloor m)/\left\lceil \frac{3k}{2} \right\rceil \), as desired.

To prove the converse, suppose that \( \tau(H) = (n + \left\lfloor \frac{k}{2} \right\rfloor m)/\left\lceil \frac{3k}{2} \right\rceil \), where \( k = 2 \) or \( k \geq 4 \). We proceed by induction on the order \( n \) to show that \( H = E_k \) or \( H = T_k \). If \( m = 0 \), then \( \tau(H) = 0 < n/\left\lceil \frac{3k}{2} \right\rceil \), a contradiction. Hence \( m \geq 1 \), and so \( n \geq k \). If \( n = k \), then \( H = E_k \), and we are done. This establishes the base case. Let \( n \geq k + 1 \) and let \( H \) be a connected \( k \)-uniform hypergraph on \( n \) vertices and \( m \) edges, and assume that the desired result holds for all connected \( k \)-uniform hypergraph on fewer than \( n \) vertices.

**Claim A** \( \delta(H) \geq 1 \).

**Proof.** Suppose that \( \delta(H) = 0 \). Let \( F \) be obtained from \( H \) by deleting all isolated vertices. Let \( F \) have \( n_F \) vertices and \( m_F \) edges. Then, \( n_F \leq n - 1 \) and \( m_F = m \). Every transversal in \( H' \) is a transversal in \( H \), and so \( \tau(H) \leq \tau(H') \). By the Chvátal-McDiarmid Theorem, we have that

\[
\tau(H) \leq \tau(H') \leq n_F + \left\lfloor \frac{k}{2} \right\rfloor m_F \leq n + \left\lfloor \frac{k}{2} \right\rfloor m,
\]

a contradiction. Hence, \( \delta(H) \geq 1 \). (c)

Let \( v \) be a vertex of maximum degree \( \Delta(H) \) in \( H \) and let \( H' = H - v \) have \( n' \) vertices and \( m' \) edges. Then, \( H' \) is a \( k \)-uniform hypergraph. Every transversal in \( H' \) can be extended to a transversal in \( H \) by adding to it the vertex \( v \), and so \( \tau(H) \leq \tau(H') + 1 \). Recall that \( k = 2 \) or \( k \geq 4 \).

**Claim B** If \( k \) is even, then \( \Delta(H) \leq 2 \), while if \( k \) is odd, then \( \Delta(H) \leq 3 \).

**Proof.** Suppose first that \( k \) is even and \( \Delta(H) \geq 3 \). Then, \( n' \leq n - 1 \) and \( m' \leq m - 3 \). Since \( k \) is even, we have by the Chvátal-McDiarmid Theorem that

\[
\tau(H) \leq \tau(H') + 1 \leq n' + \left\lfloor \frac{k}{2} \right\rfloor m' + 1 \leq n + \left\lfloor \frac{k}{2} \right\rfloor m - 3 \leq n + \left\lfloor \frac{k}{2} \right\rfloor m,
\]

a contradiction. Hence if \( k \) is even, then \( \Delta(H) \leq 2 \). Suppose next that \( k \) is odd and \( \Delta(H) \geq 4 \). Then, \( n' \leq n - 1 \) and \( m' \leq m - 4 \). Since \( k \geq 5 \) is odd, we have by the
Chvátal-McDiarmid Theorem that
\[
\tau(H) \leq \tau(H') + 1 \leq \frac{n' + \lfloor \frac{k}{2} \rfloor m'}{\left\lfloor \frac{3k}{2} \right\rfloor} + 1 \leq \frac{(n-1) + \lfloor \frac{k}{2} \rfloor (m-4)}{\left\lfloor \frac{3k}{2} \right\rfloor} + 1 < \frac{n + \lfloor \frac{k}{2} \rfloor m}{\left\lfloor \frac{3k}{2} \right\rfloor},
\]
a contradiction. Hence if \( k \) is odd, then \( \Delta(H) \leq 3 \). \( \Box \)

Claim C If \( k = 2 \), then \( H \) is a generalized triangle \( T_2 \).

Proof. Suppose that \( k = 2 \), and so \( H \) is a graph and \( \tau(H) = (n + m)/3 \). By Claim A and Claim B, we have that \( \delta(H) \geq 1 \) and \( \Delta(H) \leq 2 \). Thus, \( H \) is a path or a cycle. If \( H \) is a path on \( n \geq 2 \) vertices, then \( (2n - 1)/3 = (n + m)/3 = \tau(H) = \lceil n/2 \rceil \), implying that \( n = 2 \) and \( H = E_2 \). However this contradicts the fact that \( n \geq k + 1 \). Hence, \( H \) is a cycle on \( n \geq 3 \) vertices. Thus, \( 2n/3 = (n + m)/3 = \tau(H) = \lceil n/2 \rceil \), implying that \( n = 3 \) and \( H \) is a generalized triangle \( T_2 \). \( \square \)

In what follows we may assume that \( k \geq 4 \), for otherwise the desired result follows by Claim C.

Claim D If \( \Delta(H) \leq 2 \), then \( H = T_k \).

Proof. Suppose that \( \Delta(H) \leq 2 \). For \( i = 1, 2 \), let \( n_i \) be the number of vertices of degree \( i \) in \( H \). By Claim A, \( \delta(H) \geq 1 \) and so \( n_1 + n_2 = n \). By the \( k \)-uniformity of \( H \) we have that \( n_1 + 2n_2 = km \), or, equivalently, \( n_2 = km - n \). We now consider the multigraph \( G \) whose vertices are the edges of \( H \) and whose edges correspond to the \( n_2 \) vertices of degree \( 2 \) in \( H \): if a vertex of \( H \) is contained in the edges \( e \) and \( f \) of \( H \), then the corresponding edge of \( G \) joins vertices \( e \) and \( f \) of \( G \). Since \( H \) is \( k \)-uniform and \( \Delta(H) \leq 2 \), the maximum degree in \( G \) is at most \( k \). Further since \( H \) is connected, so too is \( G \).

Let \( M \) be a maximum matching in \( G \), and so by Theorem 2 \( |M| = \alpha'(G) \geq n_2/\lfloor 3k/2 \rfloor \). Let \( S \) be the set of vertices of \( H \) that correspond to the set of edges \( M \) in \( G \). Then, \( S \) is an independent set in \( H \) and every vertex in \( S \) has degree \( 2 \) in \( H \). By the maximality of \( M \), we note that the set of edges in \( H \) that do not intersect \( S \) are vertex-disjoint. Let \( S' \) be a set of vertices in \( H \) that consists of exactly one vertex from every edge of \( H \) that does not intersect \( S \). Then, \( |S'| = m - 2|S| \) and the set \( S \cup S' \) is a transversal in \( H \). Thus, \( \tau(H) \leq |S| + |S'| = m - |S| = m - |M| \). Hence,
\[
\frac{n + \lfloor \frac{k}{2} \rfloor m}{\left\lfloor \frac{3k}{2} \right\rfloor} = \tau(H) \leq m - |M| \leq m - \frac{n_2}{\left\lfloor \frac{3k}{2} \right\rfloor} = m - \frac{km - n}{\left\lfloor \frac{3k}{2} \right\rfloor} = \frac{\lfloor \frac{k}{2} \rfloor m + n}{\left\lfloor \frac{3k}{2} \right\rfloor}.
\]
Consequently, we must have equality throughout the above inequality chain. In particular, \( \alpha'(G) = |M| = n_2/\lfloor 3k/2 \rfloor \). Thus by Theorem 2 either \( n_2 = 0 \) or \( G \) is a Shannon multigraph of degree \( k \). If \( n_2 = 0 \), then \( n = n_1 \), implying by the connectivity of \( H \) that \( H = E_k \).
However this contradicts the fact that \( n \geq k + 1 \). Hence, \( G \) is a Shannon multigraph of degree \( k \), implying that \( H \) is a generalized triangle \( T_k \). \( \Box \)

By Claim D, if \( \Delta(H) \leq 2 \), then \( H \) is a generalized triangle \( T_k \), and we are done. Hence we may assume in what follows that \( \Delta(H) \geq 3 \). By Claim B, \( k \geq 5 \) is odd and \( \Delta(H) = 3 \). We now prove a series of claims that culminate in a contradiction.\(^1\)

**Claim E** The following hold in the hypergraph \( H \).
(a) \( \tau(H) = \tau(H') + 1 \).
(b) \( n' = n - 1 \).
(c) Every component of \( H' \) is either \( E_k \) or \( T_k \).

**Proof.** Since \( \Delta(H) = 3 \), we note that \( n' \leq n - 1 \) and \( m' = m - 3 \). Since \( k \) is odd, we have by the Chvátal-McDiarmid Theorem that

\[
\tau(H) \leq \tau(H') + 1 \leq \frac{n' + \left\lfloor \frac{k}{2} \right\rfloor m'}{\left\lfloor \frac{3k}{2} \right\rfloor} + 1 \leq \frac{(n - 1) + \left\lfloor \frac{k}{2} \right\rfloor (m - 3)}{\left\lfloor \frac{3k}{2} \right\rfloor} + 1 = \frac{n + \left\lfloor \frac{k}{2} \right\rfloor m}{\left\lfloor \frac{3k}{2} \right\rfloor}.
\]

Since \( \tau(H) = \frac{(n + \left\lfloor \frac{k}{2} \right\rfloor m)}{\left\lfloor \frac{3k}{2} \right\rfloor} \), we must have equality throughout the above inequality chain, implying that \( \tau(H) = \tau(H') + 1 \), \( \tau(H') = \frac{(n' + \left\lfloor \frac{k}{2} \right\rfloor m')}{\left\lfloor \frac{3k}{2} \right\rfloor} \) and \( n' = n - 1 \). Applying the inductive hypothesis to every component of \( H' \), we have that every component of \( H' \) is either \( E_k \) or \( T_k \). \( \Box \)

By Claim E(c) every component of \( H' \) is either \( E_k \) or \( T_k \). By Observation 3 every component of \( H' \) that is \( E_k \) or \( T_k \) contributes 1 or 2, respectively, to \( \tau(H') \).

Let \( e_1, e_2, e_3 \) be the three edges that contain the vertex \( v \) in \( H \) and let \( E_v = \{e_1, e_2, e_3\} \). By Claim E(b), \( n' = n - 1 \), which implies that \( |V(e) \cap V(H')| = k - 1 \) for each edge \( e \in E_v \).

**Claim F** Let \( C \) be a component of \( H' \) that is a generalized triangle \( T_k \). If \( |V(C) \cap V(e_1)| \geq 2 \), \( |V(C) \cap V(e_2)| \geq 2 \) and \( |V(C) \cap V(e_3)| \leq k - 2 \), then \( |V(C) \cap V(e_1)| + |V(C) \cap V(e_2)| \leq (k + 1)/2 \).

**Proof.** Assume, to the contrary, that \( |V(C) \cap V(e_1)| + |V(C) \cap V(e_2)| > (k + 1)/2 \). Since \( |V(C) \cap V(e_3)| \leq k - 2 \), there is a vertex \( u_3 \in V(e_3) \setminus (V(C) \cup \{v\}) \). If there is a vertex \( u_1 \in V(C) \cap V(e_1) \cap V(e_2) \), then by Observation 3(d) there is a \( (H') \)-set \( T \) that contains both \( u_1 \) and \( u_3 \). Since \( \{u_1, u_3\} \) intersects all three edges that contain \( v \) in \( H \), the set \( T \) is a transversal of \( H \), and so \( \tau(H) \leq |T| = \tau(H') \), contradicting Claim E(a). Hence, \( V(C) \cap V(e_1) \cap V(e_2) = \emptyset \).

\(^1\)We remark that if we allow \( k = 3 \), then it is indeed possible that \( \Delta(H) = 3 \). The current proof technique therefore fails in this special case when \( k = 3 \) since we are then unable to associate a multigraph with the hypergraph \( H \) as is done in the proof of Claim D. However as remarked earlier, the special case when \( k = 3 \) has fortunately been handled in [3].
Since $|V(C) \cap V(e_1)| \geq 2$ and since there is a unique vertex of $C$ of degree 1 in $H'$, there is a vertex $u_1 \in V(C) \cap V(e_1)$ of degree 2 in $H'$. Let $f$ be the unique edge of $C$ that does not contain $u_1$. If there is a vertex $u_2 \in V(f) \cap V(e_2)$, then $\{u_1, u_2\}$ is a $\tau(C)$-set and, by Observation 3(d), there is a $\tau(H')$-set $T$ that contains the set $\{u_1, u_2, u_3\}$. Since $\{u_1, u_2, u_3\}$ intersects all three edges that contain $v$ in $H$, the set $T$ is a transversal of $H$, and so $\tau(H) \leq |T| = \tau(H')$, contradicting Claim E(a). Hence, $V(f) \cap V(e_2) = \emptyset$.

Since $|V(C) \cap V(e_2)| \geq 2$ and since there is a unique vertex of $C$ of degree 1 in $H'$, there is a vertex $x_2 \in V(C) \cap V(e_2)$ of degree 2 in $H'$. By assumption, $|V(C) \cap V(e_1)| + |V(C) \cap V(e_2)| > (k+1)/2$. As observed earlier, the edges $e_1$ and $e_2$ do not intersect in $C$ and $V(f) \cap V(e_2) = \emptyset$. Since $|V(C) \setminus V(f)| = (k+1)/2$, there is therefore a vertex $x_1 \in V(f) \cap V(e_1)$. Thus the set $\{x_1, x_2\}$ is a $\tau(C)$-set and, by Observation 3(d), there is a $\tau(H')$-set $T$ that contains the set $\{x_1, x_2, u_3\}$. Since $\{x_1, x_2, u_3\}$ intersects all three edges that contain $v$ in $H$, the set $T$ is a transversal of $H$, and so $\tau(H) \leq |T| = \tau(H')$, once again contradicting Claim E(a). Therefore, $|V(C) \cap V(e_1)| + |V(C) \cap V(e_2)| \leq (k+1)/2$. (c)

Claim G $H'$ is disconnected.

Proof. Assume, to the contrary, that $H'$ is connected. Since $\Delta(H) = 3$, the $k$-uniformity of $H$ implies that $km = \sum_{v \in V(H)} d(v) \leq 3n$. By Claim E(c), $H$ is either $E_k$ or $T_k$. Suppose first that $H = E_k$. Then, $n = k + 1$ and $m = 4$. However $k \geq 5$, and so $km = 4k \geq 3k + 5 > 3k + 3 = 3n$, a contradiction. Hence, $H = T_k$. Thus, $n = 3(k + 1)/2$ and $m = 6$. However $k \geq 5$, and so $km = 6k = 9k/2 + 3k/2 \geq 9k/2 + 15/2 > 9k/2 + 9/2 = 3n$, once again producing a contradiction. Therefore, $H'$ is disconnected. (c)

Claim H $H'$ has at least three components.

Proof. Assume, to the contrary, that $H'$ has at most two components. Then by Claim G, the hypergraph $H'$ has exactly two components which we call $C_1$ and $C_2$. As observed earlier, $|V(e) \cap V(H')| = k - 1$ for each edge $e \in E_v$. Renaming the components $C_1$ and $C_2$ if necessary, we may assume that

$$\sum_{e \in E_v} |V(C_1) \cap V(e)| \geq \frac{3}{2} (k - 1) \geq \sum_{e \in E_v} |V(C_2) \cap V(e)|$$  (1)

and that if we have equality throughout the Inequality Chain (1), then $V(C_2)$ intersects as least as many edges of $E_v$ as $V(C_1)$ does. Since $H$ is connected, the vertex $v$ is adjacent in $H$ to a vertex from $V(C_1)$ and to a vertex from $V(C_2)$.

Claim H.1 $C_1 = T_k$.

Proof. Assume, to the contrary, that $C_1 = E_k$, and so $C_1$ has $k$ vertices. By our choice of $C_1$, $\sum_{e \in E_v} |V(C_1) \cap V(e)| \geq 3(k - 1)/2$. Since $k \geq 5$, we have that $3(k - 1)/2 > k$. Hence by the pigeonhole principle, at least one vertex, $u_1$ say, of $C_1$ is contained in two edges of
which implies that the edge \( e \) intersects \( V(C_2) \), then let \( u_3 \in V(C_2) \cap V(e_3) \). By Observation 3(d), there is a \( \tau(H') \)-set \( T \) that contains the set \( \{u_1, u_3\} \). Since \( \{u_1, u_3\} \) intersects all three edges that contain \( v \) in \( H \), the set \( T \) is a transversal of \( H \), and so \( \tau(H) \leq |T| = \tau(H') \), contradicting Claim E(a). Hence the edge \( e_3 \) does not intersect \( V(C_2) \). Thus, \( V(e_3) \setminus \{v\} \subset V(C_1) \).

Suppose that both edges \( e_1 \) and \( e_2 \) intersect \( V(C_2) \). If all vertices of \( V(C_1) \cap V(e_3) \) have degree 2 in \( H \), then \( V(e_3) \setminus \{v\} = V(C_1) \setminus \{u_1\} \) and \( V(C_1) \cap V(e_1) = \{u_1\} = V(C_1) \cap V(e_2) \). Thus, \( 3(k - 1)/2 \leq \sum_{e \in E_v} |V(C_1) \cap V(e)| = k + 1 \), and so \( k \leq 5 \). Consequently, \( k = 5 \) and we have equality throughout the Inequality Chain (1). But then all three edges in \( E_v \) intersect \( V(C_1) \) but only two edges in \( E_v \) intersect \( V(C_2) \), contradicting our choice of \( C_1 \) and \( C_2 \). Therefore there is a vertex \( x_1 \in V(C_1) \cap V(e_3) \) that has degree 3 in \( H \). Renaming the edges \( e_1 \) and \( e_2 \), if necessary, we may assume that \( x_1 \in V(e_1) \). By assumption, the edge \( e_2 \) intersects \( V(C_2) \). Let \( x_2 \in V(C_2) \cap V(e_2) \). By Observation 3(d), there is a \( \tau(H') \)-set \( T \) that contains the set \( \{x_1, x_2\} \). Since \( \{x_1, x_2\} \) intersects all three edges that contain \( v \) in \( H \), the set \( T \) is a transversal of \( H \), and so \( \tau(H) \leq |T| = \tau(H') \), contradicting Claim E(a). Hence, at most one of \( e_1 \) and \( e_2 \) intersects \( V(C_2) \).

Hence renaming \( e_1 \) and \( e_2 \), if necessary, we may assume that \( V(e_1) \setminus \{v\} \subset V(C_1) \). By the pigeonhole principle, there is a vertex \( w_1 \in V(C_1) \cap V(e_1) \cap V(e_3) \). Since \( H \) is connected, the edge \( e_2 \) intersects \( V(C_2) \). Let \( w_2 \in V(C_2) \cap V(e_2) \). By Observation 3(d), there is a \( \tau(H') \)-set \( T \) that contains the set \( \{w_1, w_2\} \). Since \( \{w_1, w_2\} \) intersects all three edges that contain \( v \) in \( H \), the set \( T \) is a transversal of \( H \), and so \( \tau(H) \leq |T| = \tau(H') \), contradicting Claim E(a). Therefore, \( C_1 \) is a generalized triangle \( T_k \). \( \circ \)

By Claim H.1, the component \( C_1 \) is a generalized triangle \( T_k \). Renaming the edges \( e_1, e_2, e_3 \) if necessary, we may assume that

\[
|V(C_1) \cap V(e_1)| \geq |V(C_1) \cap V(e_2)| \geq |V(C_1) \cap V(e_3)|,
\]

which implies that

\[
|V(C_1) \cap V(e_3)| \leq \frac{1}{3} \sum_{e \in E_v} |V(C_1) \cap V(e)|.
\]

Therefore,

\[
|V(C_1) \cap V(e_1)| + |V(C_1) \cap V(e_2)| \geq \frac{2}{3} \sum_{e \in E_v} |V(C_1) \cap V(e)| \geq k - 1 > \frac{1}{2}(k + 1).
\]

If \( e_3 \) does not intersect \( V(C_2) \), then neither do the edges \( e_1 \) and \( e_2 \), implying that \( H \) is disconnected, a contradiction. Hence, \( e_3 \) intersect \( V(C_2) \), and so \( |V(C_1) \cap V(e_3)| \leq k - 2 \). If \( |V(C_1) \cap V(e_2)| \geq 2 \), then \( |V(C_1) \cap V(e_1)| \geq 2 \). But then we contradict Claim F. Therefore, \( |V(C_1) \cap V(e_2)| \leq 1 \), and so \( |V(C_1) \cap V(e_3)| \leq 1 \). Now by our choice of \( C_1 \),

\[
k + 1 \leq \frac{3}{2}(k - 1) \leq \sum_{e \in E_v} |V(C_1) \cap V(e)| \leq (k - 1) + 1 + 1 = k + 1.
\]
Consequently, we must have equality throughout the above inequality chain. In particular, 
\[ \sum_{e \in E_v} |V(C_1) \cap V(e)| = 3(k-1)/2, \ |V(C_1) \cap V(e_1)| = k - 1 \text{ and } |V(C_1) \cap V(e_2)| = |V(C_1) \cap V(e_3)| = 1. \] But then we have equality throughout the Inequality Chain (1) and all three edges in \( E_v \) intersect \( V(C_1) \) but only two edges in \( E_v \) intersect \( V(C_2) \), contradicting our choice of \( C_1 \) and \( C_2 \). Therefore, \( H' \) has at least three components. This completes the proof of Claim H. (c)

We now return to the proof of Theorem 1. By Claim H, the hypergraph \( H' \) has at least three components. Let \( F \) be a bipartite graph with partite sets \( V_1 \) and \( V_2 \), where \( V_1 = E_v = \{e_1, e_2, e_3\} \) and where the vertices in \( V_2 \) correspond to the components of \( H' \). Further the edge set of \( F \) is defined as follows: If an edge \( e \in E_v \) intersects a component \( C \) of \( H' \) in \( H \), then the vertex \( e \in V_1 \) is adjacent to the vertex \( C \in V_2 \) in \( F \).

Since \( H' \) has at least three components, \( |V_2| \geq 3 \). Since \( H \) is connected, every component in \( H' \) has a nonempty intersection with at least one edge in \( E_v \), and so every vertex in \( V_2 \) has degree at least 1 in \( F \) and \( N_F(V_1) = V_2 \). Thus if \( S = V_1 \), then \( |N_F(S)| = |V_2| \geq 3 = |S| \). Since every edge \( e \in E_v \) intersects at least one component of \( H' \) in \( H \), every vertex in \( V_1 \) has degree at least 1 in \( F \). Thus if \( S \subseteq V_1 \) and \( |S| = 1 \), then \( |N_F(S)| \geq |S| \). Hence by Hall’s Matching Theorem, either \( V_1 \) can be matched to a subset of \( V_2 \) in \( F \) or \( |N_F(S)| < |S| \) for some subset \( S \subseteq V_1 \) with \( |S| = 2 \).

Suppose that \( V_1 \) can be matched to a subset of \( V_2 \) in \( F \). Let \( M_F \) be such a matching in \( F \). We now name the components in \( H' \) so that \( M_F = \{e_1C_1, e_2C_2, e_3C_3\} \). Hence for \( i \in \{1, 2, 3\} \), the edge \( e_i \) intersects the component \( C_i \) of \( H' \) in \( H \). For \( i \in \{1, 2, 3\} \), let \( u_i \in V(C_i) \cap V(e_i) \). By Observation 3(d), there is a \( \tau(H') \)-set \( T \) that contains the set \( \{u_1, u_2, u_3\} \). Since \( \{u_1, u_2, u_3\} \) intersects all three edges that contain \( v \) in \( H \), the set \( T \) is a transversal of \( H \), and so \( \tau(H) \leq |T| = \tau(H') \), contradicting Claim E(a). Therefore, \( |N_F(S)| < |S| \) for some subset \( S \subseteq V_1 \) with \( |S| = 2 \).

Renaming the edges in \( E_v \) if necessary, we may assume that \( S = \{e_1, e_2\} \). Thus in \( H \) we have that \( V(e_1), V(e_2) \subseteq V(C) \cup \{v\} \) for some component \( C \) of \( H' \). Since \( H \) is connected, the edge \( e_3 \) intersects every component of \( H' \) different from \( C \) in \( H \). Thus, \( |V(C) \cap V(e_1)| = k - 1, \ |V(C) \cap V(e_2)| = k - 1 \text{ and } |V(C) \cap V(e_3)| \leq k - 3 \). If \( C \) is a generalized triangle \( T_k \), then we contradict Claim F. Hence, \( C = E_k \).

Let \( C' \) be an arbitrary component of \( H' \) different from \( C \), and let \( u_3 \in V(C') \cap V(e_3) \). Since \( \sum_{i=1}^{2} |V(C) \cap V(e_i)| = 2(k-1) > k \), by the pigeonhole principle at least one vertex, \( u_1 \) say, of \( C \) is contained in both edges \( e_1 \) and \( e_2 \). By Observation 3(d), there is a \( \tau(H') \)-set \( T \) that contains the set \( \{u_1, u_3\} \). Since \( \{u_1, u_3\} \) intersects all three edges that contain \( v \) in \( H \), the set \( T \) is a transversal of \( H \), and so \( \tau(H) \leq |T| = \tau(H') \), contradicting Claim E(a). This completes the proof of Theorem 1. □
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References

[1] V. Chvátal and C. McDiarmid, Small transversals in hypergraphs. *Combinatorica* 12 (1992), 19–26.

[2] M. A. Henning and C. Löwenstein, Hypergraphs with large transversal number and with edge sizes at least four. *Central European J. Math.* 10(3) (2012), 1133–1140.

[3] M. A. Henning and A. Yeo, Hypergraphs with large transversal number and with edge sizes at least three. *J. Graph Theory* 59 (2008), 326–348.

[4] M. A. Henning and A. Yeo, Strong transversals in hypergraphs and double total domination in graphs. *SIAM Journal of Discrete Mathematics*. 24(4) (2010), 1336–1355.

[5] M. A. Henning and A. Yeo, Hypergraphs with large transversal number. *Discrete Math.* 313 (2013), 959-966.

[6] P. Hall, On representation of subsets. *J. London Math. Soc.* 10 (1935), 26–30.

[7] D. König, Graphen und Matrizen. *Math. Riz. Lapok* 38 (1931), 116–119.

[8] F. C. Lai and G. J. Chang, An upper bound for the transversal numbers of 4-uniform hypergraphs. *J. Combin. Theory Ser. B* 50 (1990), 129-133.

[9] C. E. Shannon, A theorem on colouring the lines of a network. *J. Math. Phys.* 28 (1949), 148–151.

[10] S. Thomassé and A. Yeo, Total domination of graphs and small transversals of hypergraphs. *Combinatorica* 27 (2007), 473–487.

[11] Zs. Tuza, Covering all cliques of a graph. *Discrete Math.* 86 (1990), 117–126.

[12] V. G. Vizing, The chromatic class of a multigraph. *Kibernetika* (Kiev) 1 (1965), 29–39 [in Russian]. English translation: *Cybernetics* 1 (1965), 32–41. 29, 102, 103