How the mass of a scalar field influences resonance Casimir-Polder interaction in Schwarzschild spacetime

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Abstract – Resonance Casimir-Polder interaction (RCPI) occurs in nature when two or more atoms are in their excited states and the exchange of a real photon occurs between them due to vacuum fluctuations of the quantum fields. In recent times, many attempts have been made to show that some curved spacetimes can be differentiated from a thermal Minkowski spacetime by using the RCPI. Motivated by these ideas, here we study the RCPI between two atoms that interact with a massive scalar field in Schwarzschild spacetime provided the atoms are placed in the near-horizon region. In this context we use the tool of the open quantum system and calculate the Lamb shift of the atomic energy level of the entangled states. We show that the behavior of RCPI changes depending on the mass of the scalar field. In the high mass limit, the interaction becomes short-ranged and eventually disappears beyond a characteristic length scale of $1/m$, where $m$ is the mass of the scalar field.

Introduction. – The phenomenon of Casimir-Polder interaction (CPI) occurs between an atom and a conducting plate due to vacuum fluctuation of the quantized fields [1–3]. It has major applications from the field of quantum electrodynamics (QED) to the signature of gravity [4–6]. CPI mimics the van der Waals force in the non-relativistic regime where a dispersive force is acting in few nanometer length scale [7,8]. In physical chemistry it has also huge implications in atomic adsorption and desorption in the thermally excited surfaces [9]. An experimental realization exists for the motion of the Bose-Einstein condensate (BEC) under surface CPI [10]. Other successful efforts have been made to probe more complicated contexts like detecting spacetime curvature [6,11–14], Unruh effect [15–17], Hawking radiation of a black hole and checking thermal and non-thermal scaling in a black hole spacetime [18,19]. One can show that the background spacetime and the relativistic motion of interacting systems can modify the CPI [6,13–24].

RCPI is another manifestation of CPI, where the vacuum fluctuations generate correlations between multiple quantum systems [25–27]. Two uncorrelated atoms also can be entangled if they both share a common environment [28]. The dissipative dynamics of a quantum system in a common environment are successfully described by the Lindblad master equation [29]. The main features of the dynamical equations are the presence of second-order individual system-field coupling cross-terms which show such a unique non-dissipative behavior. The second-order imaginary terms are known as dynamical Lamb shifts which also consist of those cross-terms [30]. The non-zero expectation values of the Lamb shift Hamiltonian in maximally entangled Bell state clearly describe the formation of inter-atomic correlation. On the other hand, it has been shown that a uniformly accelerated observer in Minkowski spacetime associates a thermal bath to the vacuum state of an inertial observer (thermalization theorem) [31]. Therefore one can also apply this formalism for a two-atom system weakly coupled with a scalar field in a Schwarzschild spacetime [32]. Consequently, the Lamb shift produces the inter-atomic correlation in the background spacetime and the correlation is also dependent on the vacuum fluctuation of the quantum field. Hence these inter-atomic correlations are the source of RCPI in background spacetime [6,33–35]. In this paper, following the recent works, [6,33], we theoretically investigate the RCPI between two atoms that interact with a massive scalar field in Schwarzschild spacetime.
spacetime. Two atoms are initially uncorrelated and interact individually with the scalar field. We explicitly describe the common-environment effect [29] and show how the inter-atomic correlations between the atoms depend on the mass of the scalar field by calculating the Lamb shift. Recently it has been shown that, for Schwarzschild spacetime, beyond a characteristic length scale which is proportional to the inverse of the surface gravity, the RCPI between two entangled atoms is characterized by the shift factor. Now $\rho(0)$ is the initial density matrix of the system. For a closed system, the “von Neumann-Liouville” equation is used to describe the full dynamics,

$$\frac{d\rho(t)}{dt} = -i[H^0 + \lambda \rho_{\text{Lamb}}, \rho(t)].$$  (5)

Here, $\rho(t)$ is the total density matrix. Starting from eq. (5) we perturbatively expand the time evolution operator ($U(t, t_0) = T \exp(-i \int_{t_0}^{t} dt' [H^0 + \lambda \rho_{\text{Lamb}} + \mathcal{L}(\rho_s(t))])$ to the second order of coupling Hamiltonian [30] and taking trace over field degrees of freedom, we get the quantum master equation, which is also known as GKSL equation [38–40]. The quantum master equation in the Zeeman basis and evolved under proper time $\tau$ can be expressed as [30]

$$\frac{d\rho_s(\tau)}{d\tau} = -i[H^0 + \lambda \rho_{\text{Lamb}}, \rho_s(\tau)] + \mathcal{L}(\rho_s(\tau)).$$  (6)

Here, we assume that two static atoms are arbitrarily close so that they should not experience any relative curvature effect [33]. Otherwise, the proper time carried by the clocks of the atoms will differ by a gravitational redshift factor. Now $\mathcal{H}_{\text{Lamb}}$ is the Lamb shift Hamiltonian that leads to the renormalization of the Zeeman Hamiltonian and $\mathcal{L}(\rho_s(\tau))$ is the dissipator of the master equation. These two can be written as [32]

$$\mathcal{L}(\rho_s) = \sum_{a,b=1}^{2} \sum_{j,k=1}^{3} \gamma_{jk}^{ab} (\sigma_{a}^{j} \rho_s \sigma_{b}^{k} - \frac{1}{2} \{\sigma_{a}^{j} \sigma_{b}^{k}, \rho_s\}),$$  (7)

$$\mathcal{H}_{\text{Lamb}} = -\frac{i}{2} \sum_{a,b=1}^{2} \sum_{j,k=1}^{3} S_{jk}^{ab} \sigma_{a}^{j} \sigma_{b}^{k},$$  (8)

where $S_{jk}^{ab}$ and $\gamma_{jk}^{ab}$ depend on the Fourier transforms of the two-point correlation functions of the scalar field. The second-order terms of atom-field coupling Hamiltonian contain both the real and imaginary parts. The shift terms are the imaginary parts and the dissipator comes from the real parts and they are Kramers-Kronig pairs to each other [36]. As we are interested in the shift between the energy levels of the two-atom system, our main focus is to demonstrate the Lamb shift term and not to think about real parts. However, the dissipator is responsible for driving the two-atom system to the thermal
equilibrium state, the equilibrium temperature is defined as *Bekenstein-Hawking temperature* [31]. Lamb shift can be calculated from the Hilbert transforms of the Fourier transforms of the two-point functions of the scalar field which are shown below,

\[ K^{ab}(\omega_0) = \frac{P}{\pi i} \int_{-\infty}^{\infty} d\omega \frac{G^{ab}(\omega)}{\omega - \omega_0}. \]  

(9)

Here \( P \) denotes the principal value. \( G^{ab}(\omega) \) are the Fourier transforms of the two-point correlation functions of the scalar field. The Fourier transforms of the two-point functions of the scalar field are given by

\[ G^{ab}(\omega) = \int_{-\infty}^{\infty} d\Delta \tau \, e^{i\omega \Delta \tau} \, G^{ab}(\Delta \tau). \]  

(10)

Here we assume that \( \chi^A_\mu \) satisfies \( \sum_{a=1}^{N} \chi^A_\mu (\chi^A_\mu)^\dagger = \delta_{\mu\nu}. \) So the field correlation functions are diagonal, i.e., \( G^{ab}_\nu(x - y) = \delta_{ij} G^{ab}(x - y), \) which are also defined as

\[ G^{ab}(\Delta \tau) = \langle \Phi(\tau, x_\mu) \Phi(\tau', x_\nu) \rangle, \]  

(11)

where \( \Delta \tau = (\tau - \tau'). \) The exact form of \( S^{ij}_{jk} \) can be written by

\[ S^{ij}_{jk} = A^{ab} \delta_{jk} - i B^{ab}(\mu, \nu) \delta_{ij} - A^{ab} \delta_{ij} \delta_{ik}, \]  

(12)

where the terms \( A^{ab} \) and \( B^{ab} \) are given by

\[ A^{ab} = \frac{\lambda^2}{4} \left[ K^{ab}(\omega_0) + K^{ab}(-\omega_0) \right], \]  

(13)

\[ B^{ab} = \frac{\lambda^2}{4} \left[ K^{ab}(\omega_0) - K^{ab}(-\omega_0) \right]. \]  

(14)

Inter-atomic correlation between two atoms through a massive scalar field. Two atoms are initially uncorrelated but due to interaction with the massive scalar field, the energy levels become correlated. So, there is a formation of field-induced interaction between the atoms which depends on the two-point correlation functions [29]. The two-point functions of the scalar field can be computed along the trajectories of the atoms thus they depend on the spacetime background. The matrix form of \( \mathcal{H}{\text{amb}} \) is the same as the secular part of the dipolar coupling Hamiltonian [41]. The off-diagonal elements of the matrix are present, which denotes that the expectation values of the symmetric and anti-symmetric state are non-zero. Although the entanglement generation of the equilibrium state is initial-value-dependent at a longer time [42], this dynamical Lamb shift can create a quantum correlation between them in the intermediate time regime. The symmetric and anti-symmetric states of a two-atom system are given by [41]

\[ |E\rangle = \frac{|e_1\rangle|g_2\rangle + |g_1\rangle|e_2\rangle}{\sqrt{2}}; \quad |A\rangle = \frac{|e_1\rangle|g_2\rangle - |g_1\rangle|e_2\rangle}{\sqrt{2}}. \]

Along with the separable states \((|e_1\rangle|e_2\rangle, |g_1\rangle|g_2\rangle)\), two maximally entangled Bell states also can form an alternative choice of proper eigenbasis of the system. The expressions for energy level shifts are given by [6]

\[ \delta E_{S_{LS}} = \langle E| \mathcal{H}{\text{amb}} | E \rangle \]  

\[ = \frac{i}{2} \left[ \sum_{j=1}^{3} (S_{j1}^{12} + S_{j1}^{21} + S_{jj}^{11} + S_{jj}^{22}) - 2(S_{55}^{11} + S_{55}^{22}) \right], \]  

(10)

\[ \delta E_{A_{LS}} = \langle A| \mathcal{H}{\text{amb}} | A \rangle \]  

\[ = \frac{i}{2} \left[ \sum_{j=1}^{3} (S_{j1}^{12} + S_{j1}^{21} - S_{jj}^{11} - S_{jj}^{22}) \right]. \]  

(15)

Here \( \delta E_{S_{LS}} \) stands for the second-order energy level shift of the symmetric state and \( \delta E_{A_{LS}} \) stands for the corresponding energy level shift of the anti-symmetric state. In the next section we compute these shifts of the energy level in Schwarzschild spacetime when the atoms are located close to the horizon.

Schwarzschild metric in the near-horizon region. Here we consider a \((3+1)\)-dimensional Schwarzschild spacetime to compute the *RCP* between two entangled atoms interacting with a massive scalar field when the atoms are located close to the horizon. The Schwarzschild spacetime is described by the line element,

\[ ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2, \]  

(16)

where \( f(r) = (1 - r_s/r) \) and \( r_s = 2GM \) is the Schwarzschild radius related to the line element. It is well known that, close to the event horizon, the Schwarzschild coordinates \( t \) and \( r \) behave as Rindler spacetime coordinates [33,43–46]. Now a proper distance \( l \) from the horizon can be defined in terms of the radial distance \( r \) by the formula [43]

\[ l = \sqrt{r(r - r_s)} + r_s \sin^{-1}\left(\frac{r}{r_s} - 1\right). \]  

(17)

In terms of \( l \), the Schwarzschild metric (16) becomes

\[ ds^2 = -f(r)dt^2 + dl^2 + r^2(l)\, d\theta^2 + r^2(l)\, \sin^2\theta d\phi^2, \]  

(18)

where \( f(r) = (1 - r_s/r(l)) \). Now, near the horizon, where \( r = r_s + \delta \), \( l = 2\sqrt{r_s}\beta \) and on a small angular region which is around \( \theta \sim 0 \), the new coordinates have been defined as [33]

\[ X_1 = l \cosh \frac{r}{2r_s}; \quad T = l \sinh \frac{r}{2r_s}; \]  

\[ X_2 = r_s \theta \cos \phi; \quad X_3 = r_s \theta \sin \phi. \]  

(19)

Using this coordinates (19), the line element (16) becomes [33]

\[ ds^2 = -dT^2 + dX_1^2 + dX_2^2 + dX_3^2. \]  

(20)

This line element (20) exactly matches with the line element of Minkowski spacetime [33,43–46].
Two-point correlation function for the massive scalar field. In the position space, using the coordinates of the inertial metric (20), the two-point function for a massive scalar field of mass $m$ can be expressed as [37]

$$G(x, x') = \langle 0 | \hat{\Phi}(x) \hat{\Phi}(x') | 0 \rangle = \langle 0 | \hat{\Phi}(T, x) \hat{\Phi}(T', x') | 0 \rangle = \int \frac{d^3k}{(2\pi)^3} \delta(k^2 - m^2)e^{-ik(x-x')}.$$ \hspace{1cm} (21)

After integrating the above expression, the two-point function for the scalar field can be written as [47]

$$G(x, x') = \frac{im}{4\pi^2} K_1 \left( im \sqrt{(T-T' - i\epsilon)^2 - (x-x')^2} \right) \sqrt{(T-T')^2 - (x-x')^2}. \hspace{1cm} (22)$$

This is called positive-frequency Wightmann function. Here $i\epsilon$ is a small constant in the complex plane and acts as a regulator to avoid the divergence and ensures the analytic property of the function over the chosen complex plane [48]. $K_1$ is the modified Bessel function of the second kind, $(x-x')^2 = (X_1 - X_1')^2 + (X_2 - X_2')^2 + (X_3 - X_3')^2.$ Here we define $R = \sqrt{(T-T')^2 - (x-x')^2}$ and choose $\epsilon \to 0$. In the limit $mR \ll 1$, the function (22) behaves like the two-point correlation function for a massless scalar field case [49]. In this limit, the expression of the two-point function is shown below,

$$G(x-x') = -\frac{1}{4\pi^2 R^2}. \hspace{1cm} (23)$$

On the other hand in the high mass limit, $mR \gg 1$, the correlation function (22) is given by [49]

$$G(x-x') = \frac{1}{4\pi R} \left( \frac{im}{2\pi R} \right) \frac{1}{e^{-imR}}. \hspace{1cm} (24)$$

Using eq. (19), $R$ can be written as [33]

$$R = 2l \sinh \left( \frac{\tau - \tau'}{2l} \right). \hspace{1cm} (25)$$

Now to avoid divergence at $(\tau - \tau' = 0)$, $R$ is transformed by the following transformation $(\tau$ is the proper time):

$$R = 2l \sinh \left( \frac{\tau - \tau'}{2l} \right) + i\epsilon. \hspace{1cm} (26)$$

Here $1/l$ behaves as the acceleration of the system and $\epsilon$ is a small positive quantity. So,

$$|G(x-x')| = \left( \frac{m}{32\pi^3 |R|^3} \right)^{\frac{1}{2}} e^{-m\epsilon},$$

where $|R|^2 = \epsilon^2 + \left( 2l \sinh \left( \frac{\tau - \tau'}{2l} \right) \right)^2. \hspace{1cm} (27)$

Using Lebesgue’s bounded convergence theorem the Fourier transform of the correlation function goes to zero for the large mass limit [31,49], which also indicates that in the high mass limit RCPI may disappear.

RCPI for the two-atom system. – We consider two static atoms placed near the horizon with coordinates $(r, \theta, \phi)$ and $(r', \theta', \phi')$. Here $\theta$ and $\theta'$ are assumed to be small. The response function for a detector (atom) per unit time ($T_0$) is given by [47]

$$\mathcal{F}(\omega) = \frac{1}{T_0} = \int_0^\infty \frac{dk}{2\pi} \beta_k |\omega - k^2|^2. \hspace{1cm} (28)$$

$|\beta_k|^2$ is the number of particles with mode $k$ of the isotropic bath with temperature $T_0 = \frac{1}{2\pi}$ in the static space time [31]. Following the above discussion, the Fourier transforms of the two-point correlation functions (10) for these two spacetime points can be written as

$$G^{11}(\omega) = G^{22}(\omega) = \frac{1}{\Omega(\omega, m)} - e^{-2\pi l \Omega(\omega, m)}.$$ \hspace{1cm} (29)

Here we define $\Omega(\omega, m) = \sqrt{\omega^2 - m^2}$ and $\beta_k = \frac{\sin(2\Omega(\omega, m) \sin^{-1}(z/l))}{2\Omega(\omega, m) \sqrt{1+z^2/l^2}}$. For $m = 0$ the expression exactly matches with that in ref. [33]. From the mathematical point of view, it is shown that $\omega$ is just replaced with $\sqrt{\omega^2 - m^2}$ for this case. We also define $g(\Omega(\omega, m), \omega) = \frac{\sin(2\Omega(\omega, m) \sin^{-1}(z/l))}{2\Omega(\omega, m) \sqrt{1+z^2/l^2}}$. For $m = 0$ the expression exactly matches with that in ref. [33]. The temperatures felt by the individual atoms would not be the same if they were at different fixed positions. Calculating the Hilbert transforms (9) of the above functions (29) the coefficients (13), (14) can be written as

$$A_1 = \frac{\lambda^2 P}{8\pi^2 i} \int_{-\infty}^{\infty} d\omega \Omega(\omega, m) \left( \frac{1}{\omega - \omega_0} + \frac{1}{\omega + \omega_0} \right)$$

$$B_1 = \frac{\lambda^2 P}{8\pi^2 i} \int_{-\infty}^{\infty} d\omega \Omega(\omega, m) \left( \frac{1}{\omega - \omega_0} - \frac{1}{\omega + \omega_0} \right)$$

$$A_2 = \frac{\lambda^2 P}{8\pi^2 i} \int_{-\infty}^{\infty} d\omega \Omega(\omega, m) \left( \frac{1}{\omega - \omega_0} + \frac{1}{\omega + \omega_0} \right)$$

$$B_2 = \frac{\lambda^2 P}{8\pi^2 i} \int_{-\infty}^{\infty} d\omega \Omega(\omega, m) \left( \frac{1}{\omega - \omega_0} - \frac{1}{\omega + \omega_0} \right) \hspace{1cm} (30)$$

Here, $A_{11} = A_{22} = A_1, A_{12} = A_{21} = A_2, B_{11} = B_{22} = B_1, B_{12} = B_{21} = B_2$. If we put the coefficients (30) in...
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eqs. (12) we get

\[ S_{j_k}^{11} = S_{j_k}^{12} = A_1 \delta_{j_k} - i B_1 \epsilon_{j_k} \delta_{\mu} - A_1 \delta_{j_k}, \]
\[ S_{j_k}^{11} = S_{j_k}^{12} = A_2 \delta_{j_k} - i B_2 \epsilon_{j_k} \delta_{\mu} - A_2 \delta_{j_k}, \] (31)

the shifts of the energy level of the symmetric state and the anti-symmetric state (15) of the two-atom system are given by

\[ \delta E_{SLS} = \frac{\lambda^2 \mathcal{P}}{4\pi^2} \int_0^\infty d\omega \Omega(\omega, m) \left( \frac{1}{\omega - \omega_0} + \frac{1}{\omega + \omega_0} \right) \times [g(\Omega(\omega, m), L/2) + 1], \]
\[ \delta E_{ALS} = \frac{\lambda^2 \mathcal{P}}{4\pi^2} \int_0^\infty d\omega \Omega(\omega, m) \left( \frac{1}{\omega - \omega_0} + \frac{1}{\omega + \omega_0} \right) \times [g(\Omega(\omega, m), L/2) - 1]. \] (32)

These shifts depend on the proper length \( L \). To calculate the Casimir-Polder force between the atoms we need to take a derivative with respect to \( L \) [6,13]. So, we neglect the terms which do not depend on \( L \). The modulus of the energy shift value for symmetric and anti-symmetric states then becomes the same. From the above discussion, the Lamb shift of the symmetric and anti-symmetric state of the two-atom system is then given below.

\[ \delta E = \frac{\lambda^2 \mathcal{P}}{4\pi^2} \int_0^\infty d\omega \Omega(\omega, m) \left( \frac{1}{\omega - \omega_0} + \frac{1}{\omega + \omega_0} \right) \times g(\Omega(\omega, m), L/2), \] (33)

where we define \(-\delta E_{SLS} = \delta E_{ALS} \equiv \delta E\). After substituting \( \omega^2 - m^2 = z^2 \), the integral form looks like

\[ \mathcal{I} = \frac{\lambda^2 \mathcal{P}}{2\pi^2} \int_0^\infty dz \frac{z}{z^2 + m^2 - \omega_0^2} \eta \sin(\alpha z), \] (34)

where \( \eta = \frac{1}{2L^{1/2} \sqrt{1 + \frac{L^2}{4\pi^2}}} \) and \( \alpha = 2l \sinh^{-1}(L/2l) \). Previously, from the description of the response function per unit time of a scalar field (29), the term \( \theta(\omega - m) \) suggested that momentum cannot be imaginary but now in the expression of the integrand (34) there is no restriction of theta function on \( (m^2 - \omega_0^2) \), it can be either positive or negative or zero. The contour is chosen in the upper half of the complex plane to evaluate the integral (eq. (34)) in the following limits.

\text{Case I, } \omega_0^2 > m^2. \ \text{For case I, the poles of the integrand (34) lie on the real line. Therefore, the expressions for Lamb shift of two atoms are given by}

\[ \delta E = \frac{\lambda^2}{4\pi L} \cos \left( 2\sqrt{\omega_0^2 - m^2} \sinh^{-1}(L/2l) \right), \] (35)

We also note that there exists a characteristic length scale \( l \), which up to the leading order approximation is equal to \( \frac{1}{\kappa} \), where \( \kappa = \frac{1}{2\pi} \) is the surface gravity \[33\].

To analyze the behavior of the RCP, here we also consider both the limits of the proper distance between the atoms which are either larger or smaller than the characteristic length scale. From eq. (35), it is shown that in the limit \( L \gg l \) (or \( \Delta \theta \gg 2 \sqrt{\frac{2}{3}} \)) \[33\], the expressions for Lamb shift and resonance Casimir-Polder force (RCPF) are

\[ \delta E = \frac{\lambda^2}{2\pi L^2} \cos \left( 2\sqrt{\omega_0^2 - m^2} \log(L/l) \right), \]
\[ \delta \mathcal{F} = \frac{\lambda^2}{\pi} \left[ L \cos \left( \frac{2\sqrt{\omega_0^2 - m^2} \log(L/l)}{L^3} \right) \right. \]
\[ + \left. l^2 \sqrt{\omega_0^2 - m^2} \sin \left( \frac{2\sqrt{\omega_0^2 - m^2} \log(L/l)}{L^3} \right) \right], \] (37)

where we define RCPF as \( \delta \mathcal{F} = -d(\delta E)/dl \). Now in the limit \( L \ll l \) (or \( \Delta \theta \ll 2 \sqrt{\frac{3}{7}} \)),

\[ \delta E = \frac{\lambda^2}{4\pi L} \cos \left( \sqrt{\omega_0^2 - m^2} \right), \]
\[ \delta \mathcal{F} = \frac{\lambda^2}{4\pi} \left[ \cos \left( \frac{\sqrt{\omega_0^2 - m^2} L}{L^2} \right) \right. \]
\[ + \left. \sqrt{\omega_0^2 - m^2} \sin \left( \frac{\sqrt{\omega_0^2 - m^2} L}{L^2} \right) \right]. \] (39)

\text{Case II, } \omega_0^2 < m^2. \ For case II, the poles in the integrand (34) lie on the imaginary axis, the expressions for RCP are given by

\[ \delta E = \frac{\lambda^2}{4\pi L} \left( e^{-2\sqrt{m^2 - \omega_0^2} \sinh^{-1}(L/2l)} \right), \] (40)

The expressions have an exponential decaying term, which indicates the same kind of result of the two-point correlation function at \( mR \gg 1 \). Therefore, the correlation function and RCP vanish in the large mass limit. From eq. (40), in the limit \( L \gg l \) (or \( \Delta \theta \gg 2 \sqrt{\frac{3}{7}} \)), the expressions for RCP and RCPF are given by

\[ \delta E = \frac{\lambda^2}{2\pi L^2} \left( e^{-2\sqrt{m^2 - \omega_0^2} \log(L/l)} \right), \]
\[ \delta \mathcal{F} = \frac{\lambda^2}{\pi} \left[ \frac{l}{L^3} e^{-2\sqrt{m^2 - \omega_0^2} \log(L/l)} \right. \]
\[ + \left. l^2 \sqrt{m^2 - \omega_0^2} e^{-2\sqrt{m^2 - \omega_0^2} \log(L/l)} \right]. \] (42)
and in the limit $L \ll l$ (or $\Delta \theta \ll 2 \sqrt{\frac{\delta}{r_0}}$),

\[
\delta E = \frac{\lambda^2}{4\pi L} \left( e^{-\sqrt{m^2 - \omega_0^2} L} - e^{-\sqrt{m^2 - \omega_0^2} \frac{2}{L^2} L} + \frac{\sqrt{m^2 - \omega_0^2}}{L} e^{-\sqrt{m^2 - \omega_0^2} L} \right).
\]

(43)

\[
\delta F = \frac{\lambda^2}{4\pi} \left[ e^{-\sqrt{m^2 - \omega_0^2} L} + \sqrt{m^2 - \omega_0^2} e^{-\sqrt{m^2 - \omega_0^2} \frac{2}{L^2} L} \right].
\]

(44)

**Case III**, $\omega_0^2 = m^2$. For case III, the pole of the integrand (34) is at the origin, $z = 0$. The expressions for RCPI are shown below,

\[
\delta E = \frac{\lambda^2}{4\pi L \sqrt{1 + \frac{l^2}{L^2}}},
\]

(45)

From eq. (45), in the limit $L \gg l$ (or $\Delta \theta \gg 2 \sqrt{\frac{\delta}{r_0}}$), the expressions for RCPI and RCPF are given by

\[
\delta E = \frac{\lambda^2 l}{2 \pi L^2},
\]

(46)

\[
\delta F = \frac{\lambda^2 l}{\pi L^2}.
\]

(47)

and in the limit $L \ll l$ (or $\Delta \theta \ll 2 \sqrt{\frac{\delta}{r_0}}$),

\[
\delta E = \frac{\lambda^2}{4\pi L},
\]

(48)

\[
\delta F = \frac{\lambda^2}{4\pi L^2}.
\]

(49)

However, the length scale limit beyond a characteristic value ($L \gg l$) for the previous three cases is not compatible with the local flatness of the spacetime [33].

**Discussions.** From the analogy of equivalence principle, we can say that the trajectory of a particle in a gravitational field is indistinguishable locally from the trajectory of a free particle viewed from an accelerated frame and from the thermalization theorem we find a connection between an accelerating atom and a quantum system connected to a thermal bath. Therefore, we use the quantum master equation approach to describe the dynamics of a two-atom system interacting with a massive scalar field in Schwarzschild spacetime. We find the expressions for the second-order shift term for the system-field coupling Hamiltonian. In the presence of vacuum fluctuations of the field, the effective shift terms are responsible for the inter-atomic correlations between the atoms. Here we calculate the RCPI for the Schwarzschild spacetime in the near-horizon region. The recent studies show that the interaction has a length scale dependence. For $L \ll l$, it is always possible to find a local inertial frame where the results should be the same as those obtained in the Minkowski spacetime. In that spacetime, in the other limit, $L \gg l$, the metric shows a strong non-inertial behavior and henceforth the shifts calculated there come out to be different as compared to the Minkowski spacetime. The expression of the energy level shifts suggests that the RCPI in that limit will follow a $1/L^2$ power law. However, the limit is not compatible with the locally flat spacetime. Because the motion of the atoms is not followed through the same radial vector, as $\Delta \theta$ is changing and the relative proper acceleration between them is not negligible, so the system becomes non-inertial and the thermalization theorem does not hold. This large relative acceleration term can also be explained in terms of the tidal effect, which appears due to the curvature of the spacetime [51,52]. Therefore, the local inertial approximation is violated for $L \gg l$. For the massive scalar field, we also find the same kind of behavior. With this characteristic length scale, we also introduce a mass dependence of RCPI, which serves another independent length scale in the dynamics. The key point we have addressed here is that both the limits ($\omega^2 > m^2, \omega^2 < m^2$) merge to a constant value which is independent of the mass of scalar field and the frequency of Zeeman levels. There is a competition between the frequency of Zeeman levels and the mass of the scalar field. The energy shifts become maximum when they are equal. This particular value exactly shows the behavior of a two-atom system whose frequency of Zeeman levels is zero in a massless scalar field. In summary, we have shown that in the low mass limit it exactly follows the behavior reported in ref. [33] with the frequency modulated by a mass term and, in the high mass limit, the behavior of RCPI becomes short-ranged and eventually disappears beyond a characteristic length scale of $1/m$. So, the atoms become uncorrelated in the large mass limit. We can also uncorrelate the atom by putting $L \rightarrow \infty$. As the distance between them becomes large, the probability of inter-atomic correlation is very low. The length scale dependence for massive and massless cases is exactly the same for a fixed mass in RCPF. The mass of the scalar field imposes extra features in the RCPF. We can also classify it into three categories. Interestingly the force depends on $|\omega_0^2 - m^2|$. For a low value of $|\omega_0^2 - m^2|$, the first term dominates; so the RCPF varies with $1/L^2$ in the $L \ll l$ limit and for a high value, the second term dominates, therefore it varies with $1/L$, see eqs. (39), (44). Similarly one can find two different dependences on the value of $|\omega_0^2 - m^2|$ in the opposite length scale limit in eqs. (37), (42). We do not consider the limit as the local inertial approximation is broken down for $L \gg l$. Another important fact is that for the symmetric and anti-symmetric states, the force is acting differently. In the symmetric state, the atoms act like identical bosons, thus the RCPF is attractive, similarly, in the anti-symmetric state, the atoms act like identical fermions, hence the force is repulsive.

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