ON A CONJECTURE OF HARVEY AND LAWSON

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1. Introduction

Let \( \gamma \) be a smooth simple closed curve in complex projective space \( \mathbb{P}^n \).

**Question:** Under what conditions on \( \gamma \) does there exist a 1-complex dimensional analytic variety \( V \) in \( \mathbb{P}^n \) such that \( \gamma \) is the boundary of \( V \)?

Dolbeault and Henkin in [1], and Harvey and Lawson in [2] have studied this problem. Harvey and Lawson introduced the notion of the projective hull \( \hat{K} \) of a compact set \( K \) in \( \mathbb{P}^n \), which is defined as follows: Fix a point \( x \) in \( \mathbb{P}^n \) with homogeneous coordinates \( [Z] = [Z_0, \ldots, Z_n] \) Let \( P \) be a homogeneous polynomial on \( \mathbb{C}^{n+1} \) of degree \( d \). Define
\[
||P(x)|| = \frac{|P(Z)|}{||Z||^d}, \text{ where } ||Z||^2 = \sum |Z_j|^2.
\]

Fix a compact set \( K \) in \( \mathbb{C}^n \). We define the set \( \hat{K} \) as the collection of points \( x \) in \( \mathbb{P}^n \) such that there exists a constant \( C_x \) such that
\[
(1) \quad ||P(x)|| \leq C_x \cdot \sup_K ||P||
\]

for each homogeneous polynomial \( P \) on \( \mathbb{C}^{n+1} \) of degree \( d \), and for all \( d \).

It follows from this definition that if \( x \) is a point in \( \mathbb{C}^n \), then \( x \in \hat{K} \) if and only if there exists a constant \( c \) such that
\[
(2) \quad |p(x)| \leq c^d \cdot \sup_K |p|
\]

for every polynomial \( p \) in \( C[z_1, \ldots, z_n] \) of degree \( \leq d \).

Harvey and Lawson made the following Conjecture:

"If \( \gamma \) is a real-analytic closed curve in \( \mathbb{C}^n \), and if \( \hat{\gamma} \neq \gamma \) then \( \hat{\gamma} \setminus \gamma \) is a 1-complex dimensional analytic subvariety of \( \mathbb{P}^n \setminus \gamma \)."

If this holds, then \( \hat{\gamma} \) is either an algebraic curve which contains \( \gamma \) or \( \hat{\gamma} \) is a variety having \( \gamma \) as its boundary.

The motivation for requiring real-analyticity of \( \gamma \), rather than merely smoothness, is given in [2].

Let next \( X \) be a complex manifold, and denote by \( H(X) \) the space of all holomorphic functions on \( X \). Let \( K \) be a compact subset of \( X \). The **hull of \( K \) in \( X \)**, denoted \( h_X(K) \), is defined as the set of points \( x \) in \( X \) such that
\[
(3) \quad |F(x)| \leq \sup_K |F| \quad \text{for all } F \in H(X)
\]
Theorem 1.1. Let $\gamma$ be a smooth closed curve in $\mathbb{C}^n$. Assume

(i) $\hat{\gamma}$ is closed in $\mathbb{P}^n$, and

(ii) $\Omega$ is a Stein domain in $\mathbb{P}^n$ with $\hat{\gamma}$ contained in $\Omega$.

Then $\hat{\gamma} = h_\Omega(\gamma)$

2. Proof of Theorem 1.1

Proof. By hypothesis, there exists a Stein domain $\Omega$ in $\mathbb{P}^n$ with $\hat{\gamma}$ contained in $\Omega$. Also, $\hat{\gamma}$ is closed by hypothesis, and hence compact.

We now define $A$ as the uniform closure on $\hat{\gamma}$ of $H(\Omega)$, restricted to $\hat{\gamma}$. Since $\Omega$ is Stein, $H(\Omega)$ separates points of $\hat{\gamma}$, and so $A$ is a uniform algebra on $\hat{\gamma}$.

Let $y_0$ be a peak-point of $A$ on $\hat{\gamma}$, i.e., $y_0$ is a point of $\hat{\gamma}$ such that there exists $F^* \in A$, with $F^*(y_0) = 1$ and $|F^*| < 1$ on $\hat{\gamma} \setminus y_0$.

We claim that $y_0$ is in $\gamma$. Suppose not. Then we can choose an open neighborhood $U$ of $y_0$ in $\mathbb{P}^n$ with $U$ compact and $U \cap \gamma = \emptyset$. Without loss of generality, $U$ is contained in an affine subspace $W$ of $\mathbb{P}^n$ and $\hat{\gamma}$ is polynomially convex in $W$.

Theorem 12.8 in [2] now yields

\begin{equation}
\hat{\gamma} \cap U \text{ is contained in the polynomial hull of } \hat{\gamma} \cap \delta U
\end{equation}

It follows that if $P$ is a polynomial on $W$, then

$$|P(y_0)| \leq \max |P| \text{ over } \hat{\gamma} \cap \delta U.$$ 

Since $\tilde{U}$ is polynomially convex in $W$, every $F$ in $H(\Omega)$ is uniformly approximable on $\tilde{U}$ by polynomials on $W$. So for $F$ in $H(\Omega)$, we have

\begin{equation}
|F(y_0)| \leq \max |F| \text{ over } \hat{\gamma} \cap \delta U.
\end{equation}

Our function $F^*$ above satisfies $F^*(y_0) = 1$ and $|F^*| < 1$ on $\hat{\gamma} \cap \delta U$. We choose $F$ in $H(\Omega)$ so close to $F^*$ on $\hat{\gamma}$ that

\begin{equation}
|F(y_0)| > \max |F| \text{ over } \hat{\gamma} \cap \delta U.
\end{equation}

Assertions (5) and (6) are in contradiction. So $y_0$ is in $\gamma$, as claimed.

Choose now an element $F$ in $A$. By Th. 12.10 in [3], there exists a peak-point $p$ of $A$ such that $\max |F|$ over $\hat{\gamma}$ equals $|F(p)|$.

By the preceding, $p$ is in $\gamma$. Hence, $\max |F|$ over $\hat{\gamma}$ $\leq \max |F|$ over $\gamma$. This holds in particular for $F$ in $H(\Omega)$. So we have

\begin{equation}
\hat{\gamma} \subset h_\Omega(\gamma).
\end{equation}

To prove Theorem 1.1 we need to prove the reverse inclusion. Fix a point $x$ in $h_\Omega(\gamma)$, We choose a complex hyperplane $l$ of $\mathbb{P}^n$ such that $x$ is not in $l$. Let $z_1, \ldots, z_n$ be affine coordinates on the affine space $\mathbb{P}^n \setminus l$. Each $z_j$ extends as a meromorphic function to $\mathbb{P}^n$, with pole set $l$.

Since $\Omega$ is a Stein manifold, there exists a holomorphic function $\Lambda$ on $\Omega$ such that $\Lambda$ vanishes on $l \cap \Omega$ and $\Lambda(x) \neq 0$. It follows that, for all $j$, $\Lambda \times z_j$ is holomorphic on $l \cap \Omega$, and hence is holomorphic on all of $\Omega$.

Let $J$ denote the multi-index $(j_1, \ldots, j_n)$, and let $z^J$ denote the product of the monomials $z_r^j$ for $r = 1, \ldots, n$. Let $P$ be the polynomial which is the sum of terms $c_J z^J$ taken over the multi-indices $J$, where $c_J$ is a scalar. Let $d = \text{deg} P$. Then if $c_J \neq 0$, we have $\sum_{s=1}^n j_s \leq d$. Hence

$$\Lambda^d \times P = \sum c_J \Lambda^d z^J = \sum c_J (\Lambda z_1)^{j_1} \cdots (\Lambda z_n)^{j_n} \Lambda^d \cdot S,$$
where \( S = \sum_{i=1}^{n} j_i \). Hence \( \Lambda^d \times P \) is holomorphic on \( \Omega \). Also, \( \Lambda(x) \neq 0 \). Since \( x \) is in \( h_{\Omega}(\gamma) \), we have

\[
(8) \quad |(\Lambda^d P)(x)| \leq \max |\Lambda^d P| \text{ over } \gamma.
\]

We now argue as in [2], proof of Proposition 2.3: It follows from (8) that

\[
|\Lambda(x)|^d \times |P(x)| \leq (\max |\Lambda|)^d \times \max |P|,
\]

where the maxima are taken over \( \gamma \). We now put

\[
C_x = \frac{\max |\Lambda|}{|\Lambda(x)|}.
\]

Then

\[
(9) \quad |P(x)| \leq C_x^d \max |P(x)|
\]

Since (9) holds for all \( P \), we have that \( x \) is in \( \hat{\gamma} \). Thus \( h_{\Omega}(\gamma) \subset \hat{\gamma} \). So \( h_{\Omega}(\gamma) = \hat{\gamma} \), and we are done.

\[\square\]

### 3. The Hull of a Curve in a Stein Manifold

**Theorem 3.1.** Let \( X \) be a Stein manifold, and let \( \beta \) be a real-analytic closed curve in \( X \). Then

\[
h_X(\beta) = \beta \cup V
\]

where \( V \) is a 1-complex dimensional subvariety of \( X \setminus \beta \), \( \beta \) and \( V \) are disjoint, and \( \beta \) is the boundary of \( V \).

**Proof.** Theorem 3.1 follows from the fact that it holds when \( X = \mathbb{C}^n \) ([4]), together with the following well-known properties of a Stein manifold \( X \).

(a) \( X \) admits a biholomorphic embedding \( \Phi \) into \( \mathbb{C}^N \) for some \( N \).

(b) Every holomorphic submanifold \( Y \) of \( \mathbb{C}^N \) is the zero set of some vector-valued entire function on \( \mathbb{C}^N \), and

(c) Every holomorphic function on \( Y \) admits a holomorphic extension to an entire function on \( \mathbb{C}^N \).

\[\square\]

**Note 1:** If the Conjecture is true, then conditions (i) and (ii) in Theorem 1.1 are satisfied by \( \gamma \). We see this as follows:

Put \( V = \hat{\gamma} \setminus \gamma \). Assume that \( V \) is a subvariety of \( \mathbb{P}^n \setminus \gamma \), with boundary \( \gamma \). Then \( \hat{\gamma} = V \cup bdV \), and so \( \hat{\gamma} \) is closed in \( \mathbb{P}^n \). So (i) holds.

Since \( \gamma \) is real-analytic, \( \gamma \) lies on some Riemann surface “collar” \( S \), and \( S \) fits together with \( V \) to form a holomorphic subvariety \( V^* \) of some open subset \( O \) of \( \mathbb{P}^n \), with \( V^* \) a relatively closed subset of \( O \). Then \( V^* \) is a Stein subspace of \( O \). Hence by a result of Siu, [6], \( V^* \) admits a Stein neighborhood \( \Omega \) in \( O \). Then \( \hat{\gamma} = V \cup \gamma \subset V^* \subset \Omega \), so (ii) holds.

**Note 2:** In Theorem 3.1, with \( \beta \) assumed real-analytic, \( \beta \) is the boundary of \( V \) in the sense of “manifold with boundary”. If \( \beta \) is merely assumed smooth, \( \beta \) is the boundary of \( V \) in a more general sense. (See, [5], Th. 7.2).
References

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