SOME NEW FANO VARIETIES WITH A MULTIPLICATIVE CHOW–KÜNNETH DECOMPOSITION

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ABSTRACT. Let $Y$ be a smooth dimensionally transverse intersection of the Grassmannian $\text{Gr}(2,n)$ with 3 Plücker hyperplanes. We show that $Y$ admits a multiplicative Chow–Künneth decomposition, in the sense of Shen–Vial. As a consequence, a certain tautological subring of the Chow ring of powers of $Y$ injects into cohomology.

1. INTRODUCTION

Given a smooth projective variety $Y$ over $\mathbb{C}$, let $A^i(Y) := CH^i(Y)_Q$ denote the Chow groups of $Y$, i.e. the groups of codimension $i$ algebraic cycles on $Y$ with $\mathbb{Q}$-coefficients, modulo rational equivalence. Let us write $A^i_{\text{hom}}(Y)$ and $A^i_{\text{AJ}}(Y)$ for the subgroups of homologically trivial (resp. Abel–Jacobi trivial) cycles. Intersection product defines a ring structure on $A^*(Y) = \bigoplus_i A^i(Y)$, the Chow ring of $Y$ [16]. In the case of K3 surfaces, this ring structure has a peculiar property:

**Theorem 1.1** (Beauville–Voisin [3]). Let $S$ be a K3 surface. The $\mathbb{Q}$-subalgebra

$$R^*(S) := \langle A^1(S), c_j(S) \rangle \subset A^*(S)$$

injects into cohomology under the cycle class map.

Inspired by the remarkable behaviour of K3 surfaces and of abelian varieties, Beauville [2] has famously conjectured that for certain special varieties, the Chow ring should admit a multiplicative splitting. To make concrete sense of Beauville’s elusive “splitting property conjecture”, Shen–Vial [42] have introduced the concept of multiplicative Chow–Künneth decomposition; let us abbreviate this to “MCK decomposition”.

What can one say about the class of special varieties admitting an MCK decomposition? This class is not yet well-understood. Varieties with $A^1_{\text{hom}}(Y) = 0$ (i.e. varieties with trivial Chow groups) admit an MCK decomposition, for trivial reasons. The question becomes interesting for varieties with $A^*_\text{AJ}(Y) = 0$ (conjecturally, these are exactly the varieties with Hodge level at most 1, i.e. the Hodge numbers $h^{p,q}$ are zero for $|p - q| > 1$). It is known that hyperelliptic curves have an MCK decomposition [42 Example 8.16], but the very general curve of genus $\geq 3$ does not have an MCK decomposition [13 Example 2.3] (for more details, cf. subsection 2.1 below). Also, there exist Fano threefolds that do not admit an MCK decomposition. On the positive side,
here are some higher-dimensional varieties with Hodge level 1 that are known to have an MCK decomposition:

- cubic threefolds and cubic fivefolds [7], [13];
- Fano threefolds of genus 8 [28];
- complete intersections of 2 quadrics [27];
- Gushel–Mukai fivefolds [26].

The goal of the present note is to add some new varieties with Hodge level 1 to this list:

**Theorem (Theorem 3.7)**. Let $Y$ be a smooth dimensionally transverse intersection

\[ Y := \text{Gr}(2, n) \cap H_1 \cap H_2 \cap H_3 \subset \mathbb{P}^{(n)-1}, \]

where $\text{Gr}(2, n)$ denotes the Grassmannian of 2-dimensional linear subspaces of a fixed $n$-dimensional vector space, and the $H_j$ are Plücker hyperplanes. Then $Y$ has an MCK decomposition.

In case $n$ is odd, a variety $Y$ as in Theorem 3.7 has trivial Chow groups and so the statement is vacuously true. In case $n$ is even, there is a curve $C$ naturally associated to $Y$, and one has a relation of Chow motives

\[ h(Y) \cong h(C)((1 - \dim Y)/2) \oplus \bigoplus_1 (\ast) \quad \text{in } \mathcal{M}_{\text{rat}} \]  

(cf. Theorem 3.2). The relation between $Y$ and $C$ has previously been studied on the level of Hodge theory in [8], and on the level of derived categories in [20], [21]. As a result of independent interest, we prove here (Theorem 3.2) that the relation (1) also holds on the level of Chow motives.

The existence of an MCK decomposition has profound intersection-theoretic consequences. This is exemplified by the following corollary, which is about a certain tautological subring of the Chow ring of powers of $Y$:

**Corollary (Corollary 4.1)**. Let $Y$ be as in Theorem 3.7 and $m \in \mathbb{N}$. Let

\[ R^\ast(Y^m) := \langle (p_i)^\ast \text{Im}(A^\ast(\text{Gr}(2, n)) \to A^\ast(Y)), (p_{ij})^\ast(\Delta_Y) \rangle \subset A^\ast(Y^m) \]

be the $\mathbb{Q}$-subalgebra generated by (pullbacks of) cycles coming from the Grassmannian and the diagonal $\Delta_Y \in A^\ast(Y \times Y)$. (Here $p_i$ and $p_{ij}$ denote the various projections from $Y^m$ to $Y$ resp. to $Y \times Y$). The cycle class map induces injections

\[ R^\ast(Y^m) \hookrightarrow H^\ast(Y^m, \mathbb{Q}) \quad \text{for all } m \in \mathbb{N}. \]

Corollary 4.1 is somewhat surprising, because the corresponding statement for the associated curve $C$ is false: in general it is not true that the $\mathbb{Q}$-subalgebra

\[ \langle (p_i)^\ast(h), (p_{ij})^\ast(\Delta_C) \rangle \subset A^\ast(C^m) \]

injects into cohomology (cf. Proposition 4.3 for the precise statement). This means that the injection

\[ A^\ast(C^m) \hookrightarrow A^\ast(Y^m) \]

induced by (1) does not send tautological cycles to tautological cycles!
Let us end this introduction with an open question. In view of Theorem 3.7, one might ask whether more generally smooth complete intersections of Grassmannians $\text{Gr}(k, n)$ with an arbitrary number of Plücker hyperplanes have an MCK decomposition. This concerns in particular the Debarre–Voisin 20folds

$$\text{Gr}(3, 10) \cap H,$$

which are Fano varieties of K3 type [6], and also the Fano eightfolds

$$\text{Gr}(2, 8) \cap H_1 \cap \cdots \cap H_4,$$

which are again of K3 type [41], [10]. Such varieties (being of Hodge level $> 1$) are out of scope of the argument of the present note.

Conventions. In this note, the word variety will refer to a reduced irreducible scheme of finite type over $\mathbb{C}$. A subvariety is a (possibly reducible) reduced subscheme which is equidimensional.

All Chow groups will be with rational coefficients: we denote by $A_j(Y)$ the Chow group of $j$-dimensional cycles on $Y$ with $\mathbb{Q}$-coefficients; for $Y$ smooth of dimension $n$ the notations $A_j(Y)$ and $A^{n-j}(Y)$ are used interchangeably. The notations $A_0^{\text{hom}}(Y)$ and $A_1^{\text{AJ}}(Y)$ will be used to indicate the subgroup of homologically trivial (resp. Abel–Jacobi trivial) cycles.

The contravariant category of Chow motives (i.e., pure motives with respect to rational equivalence as in [39], [34]) will be denoted $\mathcal{M}_{\text{rat}}$.

2. Preliminaries

2.1. MCK decomposition.

**Definition 2.1** (Murre [33]). Let $X$ be a smooth projective variety of dimension $n$. We say that $X$ has a CK decomposition if there exists a decomposition of the diagonal

$$\Delta_X = \pi_X^0 + \pi_X^1 + \cdots + \pi_X^n \in A^n(X \times X),$$

such that the $\pi_X^i$ are mutually orthogonal idempotents and $(\pi_X^i)_*H^*(X, \mathbb{Q}) = H^i(X, \mathbb{Q})$.

(NB: “CK decomposition” is shorthand for “Chow–Künneth decomposition”.)

**Remark 2.2.** The existence of a CK decomposition for any smooth projective variety is part of Murre’s conjectures [33], [17].

**Definition 2.3** (Shen–Vial [42]). Let $X$ be a smooth projective variety of dimension $n$. Let $\Delta_X^{\text{sm}} \in A^{2n}(X \times X \times X)$ be the class of the small diagonal

$$\Delta_X^{\text{sm}} := \{(x, x, x) \mid x \in X\} \subset X \times X \times X.$$

An MCK decomposition is a CK decomposition $\{\pi_X^i\}$ of $X$ that is multiplicative, i.e. it satisfies

$$\pi_X^i \circ \Delta_X^{\text{sm}} \circ (\pi_X^j \times \pi_X^j) = 0 \quad \text{in } A^{2n}(X \times X \times X) \quad \text{for all } i + j \neq k.$$

(NB: “MCK decomposition” is shorthand for “multiplicative Chow–Künneth decomposition”.)

**Remark 2.4.** The small diagonal (seen as a correspondence from $X \times X$ to $X$) induces the multiplication morphism

$$\Delta_X^{\text{sm}} : h(X) \otimes h(X) \to h(X) \quad \text{in } \mathcal{M}_{\text{rat}}.$$
Let us assume $X$ has a CK decomposition

$$h(X) = \bigoplus_{i=0}^{2n} h^i(X) \quad \text{in } \mathcal{M}_{\text{rat}}.$$  

By definition, this decomposition is multiplicative if for any $i, j$ the composition

$$h^i(X) \otimes h^j(X) \to h(X) \otimes h(X) \xrightarrow{\Delta_X^{\text{pp}}} h(X) \quad \text{in } \mathcal{M}_{\text{rat}}$$

factors through $h^{i+j}(X)$.

If $X$ has an MCK decomposition, then setting $A^i_j(X) := (\pi_X^{2i-j})_* A^i(X)$, one obtains a bigraded ring structure on the Chow ring: that is, the intersection product sends $A^i_j(X) \otimes A^i_{j'}(X)$ to $A^{i+j}_0(X)$.

It is expected that for any $X$ with an MCK decomposition, one has $A^i_j(X) = 0$ for $j < 0$, $A^i_0(X) \cap A^i_{\text{hom}}(X) = 0$; this is related to Murre’s conjectures B and D, that have been formulated for any CK decomposition [33].

The property of having an MCK decomposition is restrictive, and is closely related to Beauville’s “splitting property conjecture” [2]. To give an idea: hyperelliptic curves have an MCK decomposition [42, Example 8.16], but the very general curve of genus $\geq 3$ does not have an MCK decomposition [13, Example 2.3]. As for surfaces: a smooth quartic in $\mathbb{P}^3$ has an MCK decomposition, but a very general surface of degree $\geq 7$ in $\mathbb{P}^3$ should not have an MCK decomposition [13, Proposition 3.4]. There are examples of Fano threefolds that do not admit an MCK decomposition [13, Example 1.11].

For more detailed discussion, and examples of varieties with an MCK decomposition, we refer to [42, Section 8], as well as [48], [43], [14], [22], [32], [23], [24], [27], [30], [13].

2.2. The Franchetta property.

**Definition 2.5.** Let $\mathcal{Y} \to B$ be a smooth projective morphism, where $\mathcal{Y}, B$ are smooth quasi-projective varieties. We say that $\mathcal{Y} \to B$ has the Franchetta property in codimension $j$ if the following holds: for every $\Gamma \in A^j(\mathcal{Y})$ such that the restriction $\Gamma|_{Y_b}$ is homologically trivial for the very general $b \in B$, the restriction $\Gamma|_b$ is zero in $A^j(Y_b)$ for all $b \in B$.

We say that $\mathcal{Y} \to B$ has the Franchetta property if $\mathcal{Y} \to B$ has the Franchetta property in codimension $j$ for all $j$.

This property is studied in [37], [4], [11], [12].

**Definition 2.6.** Given a family $\mathcal{Y} \to B$ as above, with $Y := Y_b$ a fiber, we write

$$GDA_B(Y) := \text{Im} \left( A^j(\mathcal{Y}) \to A^j(Y) \right)$$

for the subgroup of generically defined cycles. In a context where it is clear to which family we are referring, the index $B$ will often be suppressed from the notation.
With this notation, the Franchetta property amounts to saying that $GDA_B^*(Y)$ injects into cohomology, under the cycle class map, for every fiber $Y$.

There is some flexibility with respect to the base $B$:

**Lemma 2.7.** Let $\mathcal{Y} \to B$ be a smooth projective family, and $B_0 \subset B$ the intersection of a countable number of dense open subsets. Then $\mathcal{Y} \to B$ has the Franchetta property if and only if $\mathcal{Y} \to B_0$ has the Franchetta property.

**Proof.** This follows from a well-known spread lemma [50, Lemma 3.2].

2.3. A Franchetta-type result.

**Proposition 2.8.** Let $M$ be a smooth projective variety with trivial Chow groups. Let $L_1, \ldots, L_r \to M$ be very ample line bundles, and let $\mathcal{Y} \to B$ be the universal family of smooth dimensionally transverse complete intersections of type

$$Y = M \cap H_1 \cap \cdots \cap H_r, \quad H_j \in |L_j|.$$  

Assume the fibers $Y = Y_b$ have $H^1_{\text{et}}(Y, \mathbb{Q}) = 0$. There is an inclusion

$$\ker \left( GDA^\text{dim}_B(Y \times Y) \to H^2_{\text{dim}}(Y \times Y, \mathbb{Q}) \right) \subset \left\langle (p_1)^*GDA^*_B(Y), (p_2)^*GDA^*_B(Y) \right\rangle.$$

**Proof.** This is essentially Voisin’s “spread” result [49, Proposition 1.6] (cf. also [31, Proposition 5.1] for a reformulation of Voisin’s result). We give a proof which is somewhat different from [49]. Let $\bar{B} := \mathbb{P} H^0(M, L_1 \oplus \cdots \oplus L_r)$ (so $B \subset \bar{B}$ is a Zariski open), and let us consider the projection

$$\pi : \mathcal{Y} \times_B \mathcal{Y} \to M \times M.$$  

Using the very ampleness assumption, one finds that $\pi$ is a $\mathbb{P}^s$-bundle over $(M \times M) \setminus \Delta_M$, and a $\mathbb{P}^r$-bundle over $\Delta_M$. That is, $\pi$ is what is termed a stratified projective bundle in [11]. As such, [11, Proposition 5.2] implies the equality

$$GDA^*_B(Y \times Y) = \text{Im} \left( A^*(M \times M) \to A^*(Y \times Y) \right) + \Delta_*GDA_B^*(Y),$$  

where $\Delta : Y \to Y \times Y$ is the inclusion along the diagonal. As $M$ has trivial Chow groups, $A^*(M \times M)$ is generated by $A^*(M) \otimes A^*(M)$. Base-point freeness of the $L_j$ implies that

$$GDA_B^*(Y) = \text{Im} \left( A^*(M) \to A^*(Y) \right).$$

The equality (2) thus reduces to

$$GDA^*_B(Y \times Y) = \left\langle (p_1)^*GDA^*_B(Y), (p_2)^*GDA^*_B(Y), \Delta_Y \right\rangle$$  

(where $p_1, p_2$ denote the projection from $S \times S$ to first resp. second factor). The assumption that $Y$ has non-zero transcendental cohomology implies that the class of $\Delta_Y$ is not decomposable in cohomology. It follows that

$$\text{Im} \left( GDA_{\text{dim}}^B(Y \times Y) \to H^2_{\text{dim}}(Y \times Y, \mathbb{Q}) \right) = \text{Im} \left( \text{Dec}_{\text{dim}}(Y \times Y) \to H^2_{\text{dim}}(Y \times Y, \mathbb{Q}) \right) \oplus \mathbb{Q}[\Delta_Y],$$

where $\text{Dec}$ denotes the decomposition map.
where we use the shorthand
\[ \text{Dec}^j(Y \times Y) := \left( (p_1)^* \text{GDA}^j_B(Y), (p_2)^* \text{GDA}^j_B(Y) \right) \cap A^j(Y \times Y) \]
for the decomposable cycles. We now see that if \( \Gamma \in \text{GDA}^{\dim Y}(Y \times Y) \) is homologically trivial, then \( \Gamma \) does not involve the diagonal and so \( \Gamma \in \text{Dec}^{\dim Y}(Y \times Y) \). This proves the proposition. \( \square \)

**Corollary 2.9.** Let \( Y \to B \) be as in Proposition 2.8. Assume that \( Y \to B \) has the Franchetta property. Then for any fiber \( Y \) the cycle class map induces an injection
\[ \text{GDA}^{\dim Y}(Y \times Y) \hookrightarrow H^{2 \dim Y}(Y \times Y, \mathbb{Q}) \].

**Proof.** This is immediate from Proposition 2.8: the Franchetta property for \( Y \to B \), combined with the Künneth decomposition in cohomology, implies that the right-hand side of Proposition 2.8 injects into cohomology. \( \square \)

### 2.4. A CK decomposition.

**Lemma 2.10.** Let \( M \) be a smooth projective variety with trivial Chow groups. Let \( Y \subset M \) be a smooth complete intersection of dimension \( \dim Y = d \) defined by ample line bundles. The variety \( Y \) has a self-dual CK decomposition \( \{ \pi^j_Y \} \) with the property that
\[ h^j(Y) := (Y, \pi^j_Y, 0) = \oplus \mathbb{1}(*) \text{ in } M_{\text{rat}} \quad \forall j \neq d. \]

Moreover, this CK decomposition is generically defined: writing \( Y \to B \) for the universal family (of complete intersections of the type of \( Y \)), there exist relative projectors \( \pi^j_Y \in A^d(Y \times B Y) \) such that \( \pi^j_Y = \pi^j_Y|_b \) (where \( Y = Y_b \) for \( b \in B \)).

**Proof.** This is a standard construction, one can look for instance at [38] (in case \( d \) is odd, which will be the case in this note, the “variable motive” \( h(Y)_{\text{var}} \) of [38] Theorem 4.4 coincides with \( h^d(Y') \)). \( \square \)

### 3. Main results

#### 3.1. An isomorphism of motives.

**Definition 3.1.** Let \( V \) be a vector space of dimension \( n \), and let
\[ \text{Gr}(2, n) := \text{Gr}(2, V) \subset \mathbb{P}(\wedge^2 V) \]
be the Grassmannian (parametrizing 2-dimensional subspaces of \( V \)) in its Plücker embedding. Assuming \( n \) is even, let
\[ \text{Pf} \subset \mathbb{P}(\wedge^2 V) \]
denote the projective dual of \( \text{Gr}(2, n) \subset \mathbb{P}(\wedge^2 V) \), called the Pfaffian. (The Pfaffian \( \text{Pf} \) is a hypersurface of degree \( n/2 \) and singular locus of codimension 7.)

Assume \( n \) is even. Given a linear subspace \( U \subset \wedge^2 V \) of codimension 3, one can define varieties by intersecting on the Grassmannian side and on the Pfaffian side:
\[ Y = Y_U := \text{Gr}(2, V) \cap \mathbb{P}(U) \subset \mathbb{P}(\wedge^2 V), \]
\[ C = C_U := \text{Pf} \cap \mathbb{P}(U^\perp) \subset \mathbb{P}(\wedge^2 V). \]
We say that $Y$ and $C$ are dual. For $U$ generic, the intersections $Y$ and $C$ are smooth and dimensionally transverse, of dimension $2(n − 2) − 3$ resp. 1.

**Theorem 3.2.** Let $Y$ be a smooth dimensionally transverse intersection  

$$Y := \text{Gr}(2, n) \cap H_1 \cap H_2 \cap H_3,$$

where the $H_j$ are Plücker hyperplanes.

(i) Assume $n$ is odd. Then $A_{\text{hom}}^*(Y) = 0$.

(ii) Assume that $n$ is even, and that $Y$ has a smooth dual curve $C$. There is an isomorphism  

$$h^d(Y) \cong h^1(C)((1 − d)/2) \quad \text{in} \quad M_{\text{rat}},$$

where $d := \dim Y$ and $h^d(Y)$ is as in Lemma (2.10).

**Proof.** This is a special case of [29, Theorem 3.17]. Since this is crucial to the present note, let us include a (sketch of) proof.

With notation as in Definition 3.1, let us consider  

$$Q := \left\{ (T, \mathbb{C}\omega) \in \text{Gr}(2, V) \times \mathbb{P}(U^\perp) \mid \omega|_T = 0 \right\} \subset \text{Gr}(2, V) \times \mathbb{P}(U^\perp),$$

the so-called Cayley hypersurface. There is a diagram  

$$\begin{array}{ccc}
Q_Y & \hookrightarrow & Q & \twoheadrightarrow & Q_C \\
\downarrow & & \downarrow^p & & \downarrow^q \\
Y & \hookrightarrow & \text{Gr}(2, V) & \twoheadrightarrow & \mathbb{P}(U^\perp) & \twoheadrightarrow & C
\end{array}$$

Here, $C$ is defined to be the empty set for $n$ is odd, and the dual curve $C \subset \mathbb{P}^f$ in case $n$ is even. The morphisms $p$ and $q$ are induced by the natural projections, and the closed subvarieties $Q_Y, Q_C \subset Q$ are defined as $p^{-1}(Y)$ resp. $q^{-1}(C)$.

The restriction of $p$ to $Q \setminus Q_Y$ is trivial with fibre $Q_u \cong \mathbb{P}^1$, while the restriction of $p$ to $Q_Y$ is Zariski locally trivial with fibre $Q_{Y,Y} \cong \mathbb{P}^2$. This allows us to relate the motives of $Q$ and $Y$: an application of the “motivic Cayley trick” [18, Corollary 3.2] gives an isomorphism  

$$h(Q) \cong h(Y)(−2) \oplus h(\text{Gr}(2, n))) \oplus h(\text{Gr}(2, n))(−1)$$

$$\cong h(Y)(−2) \oplus \bigoplus I(\ast) \quad \text{in} \quad M_{\text{rat}}. \quad (4)$$

The restriction of $q$ to $Q_C$ is piecewise trivial (in the sense of [40, Section 4.2]) with constant fiber $F_1$, while the restriction of $q$ to $Q \setminus Q_C$ is piecewise trivial with constant fiber $F_2$. The fibers $F_1$ and $F_2$ are explicitly known; they have only algebraic cohomology [29, Lemma 3.5]. This allows to relate $Q$ and $C$ on the level of the Grothendieck ring of varieties, and hence also on the level of cohomology:

$$h(Q) \cong h(C)(−2 − n) \oplus \bigoplus I(\ast) \quad \text{in} \quad M_{\text{hom}}. \quad (5)$$

(Here the convention is that $h(C) = 0$ in case $n$ is odd.)

Combining (4) and (5), we find a split injection of homological motives  

$$h^d(Y) \hookrightarrow h^1(C)((1 − d)/2) \quad \text{in} \quad M_{\text{hom}}. \quad (6)$$
Let us now consider things family-wise. Writing $B_0 \subset \bar{B} := \mathbb{P}H^0(\mathbb{P}(\wedge^2 V), O(1)^{\oplus 3})$ for the dense open parametrizing sections such that both $Y_b := \text{Gr}(2, n) \cap H_1^b \cap H_2^b \cap H_3^b$ and the dual curve $C_b \subset \text{Pf}^c$ are smooth and dimensionally transverse (and in addition $C_b$ is contained in the non-singular locus $\text{Pf}^c \subset \text{Pf}$), we have universal families
\[ \mathcal{Y} \to B_0, \quad C \to B_0. \]

The above construction can be performed for every fiber $Y = Y_b$ of the family $\mathcal{Y} \to B_0$. A Hilbert schemes argument \cite[Proposition 2.11]{29} then allows to find generically defined correspondences (with respect to $B_0$) inducing the split injection (6). Then, the Franchetta-type result (Proposition \[2.8\]) allows to lift the split injection (6) to an injection of Chow groups:
\[ A^*_{\text{hom}}(Y) = A^*(h^d(Y)) \hookrightarrow A^*_{\text{hom}}(h^1(C)((1 - d)/2)) = A^1_{\text{hom}}(C). \]

We conclude from (7) that $A^*_A(Y) = 0$ and so $Y$ is Kimura finite-dimensional (i.e. $h(Y)$ is finite-dimensional in the sense of \[19\]). Combining (4) and (5), we find a numerical equality $\dim h^d(Y, \mathbb{Q}) = \dim h^1(C, \mathbb{Q})$ and so the injection (6) is actually an isomorphism of homological motives. Using Kimura finite-dimensionality of both sides, it follows that (6) is also an isomorphism of Chow motives:
\[ h^d(Y) \cong h^1(C)((1 - d)/2) \quad \text{in } \mathcal{M}_{\text{rat}}. \]

This proves the theorem.

\[ \square \]

3.2. Some instances of the Franchetta property.

**Notation 3.3.** Let $\bar{B}$ and $B_0$ be as in the proof of Theorem \[3.2\] and let $B \supset B_0$ be the set parametrizing smooth dimensionally transverse intersections $Y_b = \text{Gr}(2, n) \cap H_1 \cap H_2 \cap H_3$; there is a universal family
\[ \mathcal{Y} \to B. \]

Assuming $n$ is even, let us write
\[ C \to B_0 \]
for the universal family of smooth dual curves $C_b \subset \text{Pf}^c$, as in the proof of Theorem \[3.2\].

**Proposition 3.4.** The following families have the Franchetta property:
(i) the family $\mathcal{Y} \to B$;
(ii) the family $C \to B_0$.

**Proof.** For (i), let us note that the statement is vacuously true in case $n$ is odd, because then each fiber $Y_b$ has trivial Chow groups (Theorem \[3.2\](i)). Let us now assume that $n$ is even, say $n = 2m$. We observe that the projection
\[ \tilde{\mathcal{Y}} \to \text{Gr}(2, n) \]
is a projective bundle, and so (reasoning with the projective bundle formula, or directly applying \cite[Proposition 5.2]{11}) one finds that for any fiber $\tilde{Y} := Y_b$ there is equality
\[ GDA^i(\tilde{Y}) = \text{Im} \left( A^i(\text{Gr}(2, n)) \to A^i(\tilde{Y}) \right). \]
We know from Theorem 3.2(ii) that the only non-trivial Chow group of $Y$ is
\[ A^{\frac{d+1}{2}}(Y) = A^{\frac{2(n-2)-3+1}{2}}(Y) = A^{2m-3}(Y), \]
and so we only need to prove that $GDA^{2m-3}(Y)$ injects into cohomology. The Chow ring of the Grassmannian is
\[ A^*(\text{Gr}(2, n)) = \langle h, c \rangle, \]
where $c := c_2(Q) \in A^2(\text{Gr}(2, n))$ is the second Chern class of the tautological quotient bundle $Q$, and so
\[ A^{2m-4}(\text{Gr}(2, n)) \overset{h}{\longrightarrow} A^{2m-3}(\text{Gr}(2, n)) \]
is surjective (and hence an isomorphism, by hard Lefschetz). Let $\tau: Y \rightarrow \text{Gr}(2, n)$ denote the inclusion morphism. The normal bundle formula tells us that the composition
\[ A^{2m-4}(\text{Gr}(2, n)) \overset{h}{\longrightarrow} A^{2m-3}(\text{Gr}(2, n)) \overset{\tau^*}{\longrightarrow} A^{2m-3}(Y) \overset{\tau_\ast}{\longrightarrow} A^{2m}(\text{Gr}(2, n)) \]
is a non-zero multiple of
\[ A^{2m-4}(\text{Gr}(2, n)) \overset{h^4}{\longrightarrow} A^{2m}(\text{Gr}(2, n)). \]
This last map is the same as
\[ H^{4m-8}(\text{Gr}(2, n), \mathbb{Q}) \overset{h^4}{\longrightarrow} H^{4m}(\text{Gr}(2, n), \mathbb{Q}), \]
which is an isomorphism thanks to hard Lefschetz for the $(4m-4)$-dimensional variety $\text{Gr}(2, n)$. This proves the required injectivity of $GDA^{2m-3}(Y)$ into cohomology.

As for (ii), one can either prove this directly, or can reduce to (i) via the generically defined isomorphism
\[ A^4_{\text{hom}}(C) \overset{\cong}{\longrightarrow} A^{2m-3}(Y) \]
given by Theorem 3.2(ii). \qed

**Proposition 3.5.** The following families have the Franchetta property:
(i) the family $C \times_{B_0} C \rightarrow B_0$;
(ii) the family $\mathcal{Y} \times_{\mathcal{Y}} \mathcal{Y} \rightarrow B$.

**Proof.** (i) Let $\bar{C} \subset \text{Pf} \times \bar{B}$ denote the projective closure of $C$, and let us consider the projection
\[ \pi: \bar{C} \times_B \bar{C} \rightarrow \text{Pf} \times \text{Pf}. \]
This is a stratified projective bundle (in the sense of [11]). As such, [11] Proposition 5.2] implies the equality
\[ (8) \quad GDA^i_{B_0}(C \times C) = \text{Im} \left( A^i(\text{Pf}^\circ \times \text{Pf}^\circ) \rightarrow A^*(C \times C) \right) + \Delta \cdot GDA^i_{B_0}(C), \]
where $\Delta: C \rightarrow C \times C$ is the inclusion along the diagonal, and $\text{Pf}^\circ \subset \text{Pf}$ denotes the non-singular locus of the Pfaffian. As $\text{Pf}^\circ$ has the Chow–Künneth property [29] Example 2.7], $A^*(\text{Pf}^\circ \times \text{Pf}^\circ)$ is generated by $A^*(\text{Pf}^\circ) \otimes A^*(\text{Pf}^\circ)$. The equality (8) thus simplifies to
\[ (9) \quad GDA^i_{B_0}(C \times C) = \langle (p_j)^* \text{Im} \left( A^j(\text{Pf}^\circ) \rightarrow A^*(C) \right), \Delta_C \rangle. \]

We now proceed to check that $GDA^i_{B_0}(C \times C)$ injects into cohomology:
In case \( j = 1 \), we know that \( \Delta_C \) is linearly independent from the decomposable classes
\[
\left\langle (p_j)^* \text{Im}(A^j(\text{Pf}^c) \to A^*(C)) \right\rangle
\]
in cohomology (indeed, we may assume that \( C \) has genus \( > 0 \), for otherwise the statement is vacuously true). The required injectivity then reduces to Proposition 3.4(ii).

In case \( j = 2 \), we know that \( A^1(\text{Pf}^c) \) is 1-dimensional, generated by a hyperplane class \( H \) (cf. Lemma 3.6 below). Since \( C \subset \mathbb{P}^2 \) is a plane curve, clearly we have an equality
\[
\Delta_C \cdot (p_i)^*(H) = \sum_{r=0}^2 \frac{1}{\deg C} (p_1)^*(H^r) \cdot (p_2)^*(H^{2-r}) \quad \text{in } A^2(C \times C),
\]
and so
\[
GDA^2_{B_0}(C \times C) = \left\langle (p_j)^* \text{Im}(A^j(\text{Pf}^c) \to A^*(C)) , \Delta_C \right\rangle \cap A^2(C \times C)
\]
\[
= \left\langle (p_j)^* \text{Im}(A^j(\text{Pf}^c) \to A^*(C)) \right\rangle \cap A^2(C \times C).
\]
The required injectivity then reduces to Proposition 3.4(ii).

In the above, we have used the following lemma:

**Lemma 3.6.** Let \( \text{Pf}^c \subset \text{Pf} \) denote (as above) the non-singular locus of the Pfaffian. We have
\[
A^1(\text{Pf}^c) \cong \mathbb{Q}[H].
\]

**Proof.** (of the lemma.) We consider
\[
\tilde{\text{Pf}} := \left\{ (\omega, K) \in \text{Pf} \times \text{Gr}(2, n) \mid K \subset \ker \omega \right\} \subset \text{Pf} \times \text{Gr}(2, n).
\]
The projection \( \tilde{\text{Pf}} \to \text{Gr}(2, n) \) is a projective bundle (and so \( \tilde{\text{Pf}} \) is smooth), and the projection \( \tilde{\text{Pf}} \to \text{Pf} \) is an isomorphism over the non-singular locus (and so \( \tilde{\text{Pf}} \to \text{Pf} \) is a resolution of singularities).

Being a projective bundle over a Grassmannian, \( \tilde{\text{Pf}} \) has Picard number 2:
\[
A^1(\tilde{\text{Pf}}) = \mathbb{Q}^2.
\]
The complement of (the isomorphic pre-image of) \( \text{Pf}^c \) inside \( \tilde{\text{Pf}} \) is an irreducible divisor \( D \) (it is a partial flag variety). The localization sequence
\[
A_*(D) \to A^1(\tilde{\text{Pf}}) \to A^1(\text{Pf}^c) \to 0
\]
then gives the result.

(ii) Again, we may assume that \( n \) is even (for otherwise the statement is vacuously fulfilled). In view of Lemma 2.7, it will suffice to prove the Franchetta property for \( Y \times_{B_0} Y \to B_0 \). Thanks to Theorem 3.2(ii), for any fiber \( Y = Y_b \) with \( b \in B_0 \) we have split injections
\[
A^j(Y \times Y) \hookrightarrow A^{j+1-d}(C \times C) \oplus \bigoplus A^*(C) \oplus \mathbb{Q}^n.
\]
The isomorphism of Theorem 3.2 being generically defined, there are also split injections
\[
GDA^j(Y \times Y) \hookrightarrow GDA^{j+1-d}(C \times C) \oplus \bigoplus GDA^*(C) \oplus \mathbb{Q}^n.
\]
The required injectivity now follows from (i) and Proposition 3.4(ii).

3.3. MCK.

Theorem 3.7. Let $Y$ be a smooth dimensionally transverse intersection

$$Y := \text{Gr}(2, n) \cap H_1 \cap H_2 \cap H_3,$$

where the $H_j$ are Plücker hyperplanes. Then $Y$ has an MCK decomposition.

Proof. In case $n$ is odd, $Y$ has trivial Chow groups (Theorem 3.2(i)) and so the statement is vacuously true. In case $n = 4$, $Y$ is a rational curve and again the statement is vacuously true. We may thus suppose that $n$ is even and $\geq 6$. We have the following general result:

Proposition 3.8. Let $\mathcal{Y} \rightarrow B$ be a family of smooth projective varieties, verifying

(a1) the fibers $Y_b$ are of odd dimension $d \geq 5$ and

$$A^j_{\hom}(Y_b) = 0 \quad \forall j > (d + 1)/2 \quad \forall b \in B;$$

(a2) the fibers $Y_b$ have a generically defined Künneth decomposition, i.e. there exist $\{\pi^j_b\} \in A^d(\mathcal{Y} \times_B \mathcal{Y})$ such that the fiberwise restriction $\pi^j_b := \pi^j_b|_{Y_b} \in A^d(Y_b \times Y_b)$ is a Künneth decomposition for all $b \in B$;

(a3) the family $\mathcal{Y} \times_B \mathcal{Y} \rightarrow B$ has the Franchetta property.

Then $\{\pi^j_b\}$ is an MCK decomposition for any $b \in B$.

Proof. (of Proposition 3.8) Condition (a1) implies (via the Bloch–Srinivas argument, cf. [5]) that for every fiber $Y_b$ there exists a curve $C_b$ and a split injection of motives

$$h(Y_b) \hookrightarrow h(C_b)((1 - d)/2) \oplus \bigoplus I(*) \quad \text{in } \mathcal{M}_{\text{rat}}.$$ (10)

Condition (a3) implies that the Künneth decomposition $\{\pi^j_{Y_b}\}$ of (a2) is a self-dual CK decomposition. Let $h(Y_b) = \oplus_j h^j(Y_b)$ denote the corresponding decomposition of the motive of $X$. Using the injection (10), one finds that $h^j(Y_b) = \oplus I(*)$ for all $j \neq d$, while for $j = d$ one finds a split injection

$$h^d(Y_b) \hookrightarrow h^1(C_b)((1 - d)/2) \quad \text{in } \mathcal{M}_{\text{rat}}.$$ (11)

Let us now establish that the CK decomposition $\{\pi^j_{Y_b}\}$ is MCK. By definition, what we need to check is that the cycle

$$\Gamma_{ijk} := \pi^k_{Y_b} \circ \Delta_{Y_b} \circ (\pi^i_{Y_b} \times \pi^j_{Y_b}) \in A^{2d}(Y_b \times Y_b \times Y_b)$$

is zero for all $i + j \neq k$.

Let us assume at least one of the integers $i, j, k$ is different from $d$. In this case, there is an injection

$$\Gamma_{ijk} \subset (\pi^{2d-i}_{Y_b} \times \pi^{2d-j}_{Y_b} \times \pi^k_{Y_b})_+ A^{2d}(Y_b \times Y_b \times Y_b) \hookrightarrow \bigoplus A^*(Y_b \times Y_b),$$

and this injection sends generically defined cycles to generically defined cycles. But $\Gamma_{ijk}$ is generically defined and homologically trivial, and so the Franchetta property for $\mathcal{Y} \times_B \mathcal{Y} \rightarrow B$ gives the required vanishing $\Gamma_{ijk} = 0$. 

Next, let us assume \( i = j = k = d \). In this case, the injection of motives \((\Pi)\) induces an injection of Chow groups

\[
\Gamma_{ijk} \in (\pi^d_{Y_b} \times \pi^d_{Y_b} \times \pi^d_{Y_b})_* A^{2d}(Y_b \times Y_b) \hookrightarrow A^{(d+3)/2}(C_b \times C_b \times C_b).
\]

But the right-hand side vanishes for dimension reasons for any \( d \geq 5 \), and so \( \Gamma_{ijk} = 0 \). \(\square\)

Let us now consider the family \( Y \rightarrow B \) of all smooth complete intersections \( \text{Gr}(2, n) \cap H_1 \cap H_2 \cap H_3 \), where \( n \geq 6 \) is even. Each fiber \( Y_b \) has a generically defined CK decomposition \( \{ \pi^j_{Y_b} \} \) (Lemma 2.10). To check that \( \{ \pi^j_{Y_b} \} \) is MCK, it suffices to do this over a dense open of \( B \); for instance we may take \( B_0 \subset B \) the locus as before where \( Y_b \) has a smooth dual curve \( C_b \) contained in \( \text{Pf}^\circ \). Let us check that \( Y \rightarrow B_0 \) verifies the conditions of Proposition 3.8. Condition (a1) is immediate from Theorem 3.2(ii). Condition (a2) is fulfilled by the \( \{ \pi^j_{Y_b} \} \). As for condition (a3), this is Proposition 3.5(ii). This ends the proof. \(\square\)

4. THE TAUENTIAL RING

4.1. A positive result.

**Corollary 4.1.** Let \( Y \) be as in Theorem 3.7 and \( m \in \mathbb{N} \). Let

\[
R^*(Y^m) := \left\langle (p_i)^* \text{Im}(A^*(\text{Gr}(2, n)) \to A^*(Y)), (p_{ij})^*(\Delta_Y) \right\rangle \subset A^*(Y^m)
\]

be the \( \mathbb{Q} \)-subalgebra generated by \( \text{pullbacks of cycles coming from } \text{Gr}(2, n) \) and \( \text{pullbacks of the diagonal } \Delta_Y \in A^d(Y \times Y) \). (Here \( p_i \) and \( p_{ij} \) denote the various projections from \( Y^m \) to \( Y \) resp. to \( Y \times Y \)). The cycle class map induces injections

\[
R^*(Y^m) \hookrightarrow H^*(Y^m, \mathbb{Q}) \quad \text{for all } m \in \mathbb{N}.
\]

**Proof.** This is inspired by the analogous result for cubic hypersurfaces [12, Section 2.3], which in turn is inspired by analogous results for hyperelliptic curves [44], [45] (cf. Remark 4.2 below) and for K3 surfaces [51].

The Chow ring \( A^*(\text{Gr}(2, n)) \) is generated by the Plücker polarization \( h \in A^1(\text{Gr}(2, n)) \) and the Chern class \( c_2(Q) \in A^2(\text{Gr}(2, n)) \), where \( Q \to \text{Gr}(2, n) \) is the universal quotient bundle [9]. As in [12, Section 2.3], let us write

\[
o := \frac{1}{\deg(h^d)} h^d \in A^d(Y), \quad c := c_2(Q)|_Y \in A^2(Y),
\]

and

\[
\tau := \pi^d_Y = \Delta_Y - \sum_{j \neq d} \pi^j_Y \in A^d(Y \times Y),
\]

where the \( \pi^j_Y \) are as above, and \( d := \dim Y \).
Moreover, for any $1 \leq i < j \leq m$ let us write
\[
\begin{align*}
o_i &:= (p_i)^*(o) \in A^d(Y^{m}) , \\
h_i &:= (p_i)^*(h) \in A^1(Y^{m}) , \\
c_i &:= (p_i)^*(c) \in A^2(Y^{m}) , \\
\tau_{ij} &:= (p_{ij})^*(\tau) \in A^d(Y^{m}) .
\end{align*}
\]
Note that (by definition) we have
\[
R^*(Y^{m}) = \left< o_i, h_i, c_i, \tau_{ij} \right> \subset A^*(Y^{m}) .
\]
Let us now define the $\mathbb{Q}$-subalgebra
\[
\bar{R}^*(Y^{m}) := \left< o_i, h_i, c_i, \tau_{ij} \right> \subset H^*(Y^{m}, \mathbb{Q})
\]
(where $i$ ranges over $1 \leq i \leq m$, and $1 \leq i < j \leq m$); this is the image of $R^*(Y^{m})$ in cohomology. One can prove (just as [12, Lemma 2.11] and [51, Lemma 2.3]) that the $\mathbb{Q}$-algebra $\bar{R}^*(Y^{m})$ is isomorphic to the free graded $\mathbb{Q}$-algebra generated by $o_i, h_i, c_i, \tau_{ij}$, modulo the following relations:
\begin{align}
(12) \quad &h_i \cdot o_i = c_i \cdot o_i = 0, \quad c_i^{(d+1)/2} = 0, \quad c_i^{(d-1)/2} = \lambda h_i^{d-1}, \quad \ldots, \quad h_i^d = \deg(h^d) o_i ; \\
(13) \quad &\tau_{ij} \cdot o_i = \tau_{ij} \cdot h_i = \tau_{ij} \cdot c_i = 0, \quad \tau_{ij} \cdot \tau_{ij} = -b_{d} o_i \cdot o_j ; \\
(14) \quad &\tau_{ij} \cdot \tau_{jk} = \tau_{jk} \cdot o_i ; \\
(15) \quad &\sum_{\sigma \in \mathfrak{S}_{d+2}} \prod_{i=1}^{b_{d}/2+1} \tau_{\sigma(2i-1),\sigma(2i)} = 0 .
\end{align}
Here $\lambda \in \mathbb{Q}$, and the dots “…” in $(12)$ indicate certain relations of type $c_i^{m_j} h_i^{n_j} = \lambda h_i^{2m_j+n_j}$.

By definition, $b_d := \dim H^d(Y, \mathbb{Q})$ and $\mathfrak{S}_r$ denotes the symmetric group on $r$ elements.

To prove Corollary 4.1, it suffices to check that all these relations are verified modulo rational equivalence. The relations $(12)$ take place in $R^*(Y)$ and so they follow from the Franchetta property for $Y$ (Proposition 3.4). The relations $(13)$ take place in $R^*(Y^2)$. The last relation is trivially verified, because ($Y$ being Fano) $A^{2d}(Y^2) = \mathbb{Q}$. As for the other relations of $(13)$, these follow from the Franchetta property for $X \times Y$ (Proposition 3.5).

Relation $(14)$ takes place in $R^*(Y^3)$ and follows from the MCK decomposition. Indeed, we have
\[
\Delta_Y^\text{sm} \circ (\pi_Y^d \times \pi_Y^d) = \pi_Y^{2d} \circ \Delta_Y^\text{sm} \circ (\pi_Y^d \times \pi_Y^d) \quad \text{in } A^{2d}(Y^3) ,
\]
which (using Lieberman’s lemma) translates into
\[
(\pi_Y^d \times \pi_Y^d \times \Delta_Y)_* \Delta_Y^\text{sm} = (\pi_Y^d \times \pi_Y^d \times \pi_Y^{2d})_* \Delta_Y^\text{sm} \quad \text{in } A^{2d}(Y^3) ,
\]
which means that
\[
\tau_{13} \cdot \tau_{23} = \tau_{12} \cdot o_3 \quad \text{in } A^{2d}(Y^3) .
\]
Finally, relation (15), which takes place in $R^*(Y^{b_d+2})$, is related to the Kimura finite-dimensionality relation [19]: relation (15) expresses the vanishing
\[ \text{Sym}^{b_d+2} H^d(Y, \mathbb{Q}) = 0, \]
where $H^d(Y, \mathbb{Q})$ is seen as a super vector space. This relation is also verified modulo rational equivalence, (i.e., relation (15) is also true in $A^d(Y^{b_d+2})$): relation (15) involves a cycle in
\[ A^*(\text{Sym}^{b_d+2} H^d(Y)), \]
and $\text{Sym}^{b_d+2} H^d(Y)$ is 0 because $Y$ has Kimura finite-dimensional motive (Theorem 3.2).

This ends the proof. \hfill \Box

**Remark 4.2.** Given any curve $C$ and an integer $m \in \mathbb{N}$, one can define the tautological ring
\[ R^*(C^m) := \langle (p_i)^*(K_C), (p_{ij})^*(\Delta_C) \rangle \subset A^*(C^m) \]
(where $p_i, p_{ij}$ denote the various projections from $C^m$ to $C$ resp. $C \times C$). Tavakol has proven [45, Corollary 6.4] that if $C$ is a hyperelliptic curve, the cycle class map induces injections
\[ R^*(C^m) \hookrightarrow H^*(C^m, \mathbb{Q}) \quad \text{for all } m \in \mathbb{N}. \]

On the other hand, there exist curves for which the tautological ring $R^*(C^3)$ does not inject into cohomology, cf. Proposition 4.3 below.

**4.2. A negative result.**

**Proposition 4.3.** Let
\[ Y := \text{Gr}(2, n) \cap H_1 \cap H_2 \cap H_3 \]
be a very general intersection of the Grassmannian with 3 Plücker hyperplanes, where $n$ is even and $8 \leq n \leq 2000$. Let $C \subset \text{Pf}$ be the curve dual to $Y$ (Definition 5.7). The $\mathbb{Q}$-subalgebra
\[ R^*(C^m) := \langle (p_i)^*(K_C), (p_{ij})^*(\Delta_C) \rangle \subset A^*(C^m) \]
does not inject into cohomology for $m = 3$.

**Proof.** The point is that $C$ is a plane curve of degree $n/2$, and that the general plane curve of degree $n/2$ arises in this way [11]. Using the spread lemma (Lemma 2.7), it follows that the assumption that $R^*(C^m)$ injects into cohomology for the very general $C$ as in Proposition 4.3 would imply that $R^*(C^m)$ injects into cohomology for every plane curve of degree $n/2$. Taking $m = 3$, this would mean that every plane curve of degree $n/2$ has a self-dual MCK decomposition. As explained in [15, Proposition 7.1] and [13, Remark 2.4], this would imply that for every plane curve $C$ of degree $n/2$ the Ceresa cycle
\[ C - [-1]_*(C) \in A_1(\text{Jac}(C)) \]
is algebraically trivial. But this is known to be false for the Fermat curve of degree between 4 and 1000, cf. [36]. \hfill \Box

**Acknowledgments.** Thanks to Lie Fu and Charles Vial for lots of inspiring exchanges around MCK.
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