Gentile statistics with a large maximum occupation number

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Abstract

In Gentile statistics the maximum occupation number can take on unrestricted integers: \(1 < n < \infty\). It is usually believed that Gentile statistics will reduce to Bose-Einstein statistics when \(n\) equals the total number of particles in the system \(N\). In this paper, we will show that this statement is valid only when the fugacity \(z < 1\); nevertheless, if \(z > 1\) the Bose-Einstein case is not recovered from Gentile statistics as \(n\) goes to \(N\). Attention is also concentrated on the contribution of the ground state which was ignored in related literature. The thermodynamic behavior of a \(\nu\)-dimensional Gentile ideal gas of particle of dispersion \(E = \frac{p^\nu}{2m}\), where \(\nu\) and \(s\) are arbitrary, is analyzed in detail. Moreover, we provide an alternative derivation of the partition function for Gentile statistics.

I. Introduction

Quantum statistics is classified as either Bose-Einstein or Fermi-Dirac statistics. As a generalization, Gentile \[1\] proposed a kind of intermediate statistics, in which the maximum number of particles in any quantum state is neither 1 nor \(\infty\), but equal to a finite number \(n\). A particularly interesting case of Gentile statistics is \(n \to N\), where \(N\) is the total number of particles in the system. Some authors, e.g., Ref. \[2\], argue that in this case there is no difference between Bose-Einstein and Gentile statistics. However, this statement is valid only when the fugacity \(z < 1\). In the present paper, we will discuss the case of \(n \to N\) with \(z > 1\) and show that in this case the Bose-Einstein case can not be recovered from Gentile statistics.

The thermodynamic behavior of an ideal gas obeying Gentile statistics has been discussed in some literature \[1, 3\]. Nevertheless, in these works the contribution from the ground state was ignored. It is known that such a contribution can be safely ignored in fermion systems; however, in boson systems it plays an important role especially in the case of low temperatures and high densities. In Gentile statistics, the maximum occupation number \(n\) can take on unrestricted integers: \(1 < n < \infty\). When \(n\) is large, especially \(n \to N\), we will show that the contribution from the ground state becomes very important.

In the present paper, we will calculate the thermodynamic quantities in \(\nu\)-dimensional space with the dispersion relation \(E = \frac{p^\nu}{2m}\), where \(\nu\) and \(s\) are arbitrary. A special case of \(\nu = 3\) and \(s = 2\), without the ground state contribution, has been discussed in Ref. \[1, 3\]. Our result shows that, the

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thermodynamic behavior of Gentile statistics, especially the contribution of the ground state for the
case \( \nu > s \), depends sensitively on the maximum occupation number \( n \).

In Sec. II, we first provide an alternative way to calculate the partition function for Gentile
statistics. Special attention is concentrated on the case of \( n \rightarrow N \) with \( z > 1 \). In this case, Gentile
statistics will not reduce to Bose-Einstein statistics. In Sec. III, we calculate the thermodynamic
quantities in which the contribution from the ground state is reckoned in. In Sec. IV, we discuss
the low temperature and high density result for the internal energy, the specific heat, the magnetic
susceptibility and other thermodynamic quantities. In this case, the ground state contribution plays
an important role. A summary and conclusions are given in Sec. V.

II. The partition function and the case of \( n \rightarrow N \) with \( z > 1 \)
in Gentile statistics

For completeness, we first calculate the partition function for Gentile statistics in a way which is
somewhat different from that in literature. The approach we shall employ is based on multinomial
theorem.

Starting with counting the number of states, we express the grand partition function as follows:

\[
\Xi = \prod_{\ell} \Xi_\ell = \prod_{\ell} \sum_{\alpha_\ell} \Omega_\ell e^{-(\alpha + \beta \varepsilon_\ell) a_\ell}.
\]

(1)

Here

\[
\Xi_\ell = \sum_{\alpha_\ell} \Omega_\ell e^{-(\alpha + \beta \varepsilon_\ell) a_\ell},
\]

(2)

and \( \Omega_\ell \) is the number of quantum states of \( a_\ell \) particles distributed in one level \( \varepsilon_\ell \) (the \( \ell \)-th level) with
a degeneracy \( \omega_\ell \). In this occasion, one state can be occupied by at most \( n \) particles. It means

\[
\Omega_\ell = \sum_{\alpha_0} \cdots \sum_{\alpha_n} \frac{\omega_\ell!}{\prod_{i=0}^{n} \alpha_i!},
\]

(3)

where \( \alpha_i \) (\( i = 0, 1, \cdots, n \)) denotes the number of states which contains \( i \) particles, and the summation
runs over all possible values. The set of \( \{\alpha_i\} \) must satisfy the conditions

\[
\sum_i \alpha_i = \omega_\ell, \quad \sum_i i \alpha_i = a_\ell.
\]

(4)

To obtain the grand partition function, we need the multinomial theorem \[4\]: The coefficient \( \Omega \)
of \( x^a \) \((0 \leq a \leq n \omega)\) in the expansion of \((1 + x + x^2 + \cdots + x^n)^\omega\) is

\[
\Omega = \sum_{\alpha_0} \cdots \sum_{\alpha_n} \frac{\omega!}{\prod_{i=0}^{n} \alpha_i!},
\]

where \( \alpha_i \) (\( i = 0, 1, \cdots, n \)) satisfy \( \sum_{i=0}^{n} \alpha_i = \omega, \sum_{i=0}^{n} i \alpha_i = a \), and the summation runs over all possible
values.

Based on such a relation, we can obtain the analytic form of the \( \Xi_\ell \). The \( \Omega_\ell \) in Eq.(3) (constrained
by Eq.(4)) is the coefficient of the term \( e^{-(\alpha + \beta \varepsilon_\ell) a_\ell} \) in the expansion of

\[
[1 + e^{-(\alpha + \beta \varepsilon_\ell)} + e^{-2(\alpha + \beta \varepsilon_\ell)} + \cdots + e^{-n(\alpha + \beta \varepsilon_\ell)}]^{\omega_\ell}.
\]

(5)
As a result, Eq.(2) can be calculated exactly:
\[
\Xi^{\ell} = \left[ \frac{e^{-(n+1)(\alpha + \beta \epsilon^{\ell})} - 1}{e^{-(\alpha + \beta \epsilon^{\ell})} - 1} \right]^{\omega^{\ell}},
\]
(6)

The result agrees with that found by Gentile [1].

Obviously, when \( n = \infty \) the necessary condition on obtaining Eq.(6) from Eq.(5) is the fugacity \( z < 1 \) \( (z = e^{-\alpha}) \); otherwise, \( \Xi^{\ell} \) will diverge. This means that Gentile statistics will return to Bose case only when \( z < 1 \). Many authors, e.g., Ref. [2], however, argue that it does not make any difference whether one take \( n = N \) or \( n = \infty \); in other words, Bose-Einstein statistics is recovered when \( n = N \). Nevertheless, Eq.(6) will hold so long as \( n \neq \infty \) even for \( n \gg 1 \) and \( z > 1 \). The behavior of the statistics corresponding to \( z > 1 \) and \( n \gg 1 \) will be very different from Bose-Einstein statistics. In Sec. IV, we will discuss such statistics in detail.

III. Thermodynamics with the contribution of the ground state

In this section, we will discuss the thermodynamics corresponding to Gentile statistics in detail. Specifically, we will study the contribution of the ground state to the thermodynamic quantities, which is ignored in Ref. [1, 3].

The grand partition function for Gentile statistics is
\[
\Xi = \prod_{\ell} \left[ \frac{z^{n+1}e^{-(n+1)\beta \epsilon^{\ell}} - 1}{ze^{-\beta \epsilon^{\ell}} - 1} \right]^{\omega^{\ell}},
\]
(7)

Replacing the summation over \( \ell \) by the corresponding integral and carrying out the integral by parts, we have
\[
\ln \Xi = \sum_{\ell} \ln \Xi^{\ell} = \frac{1}{\lambda^{\nu}} \frac{2gV}{\nu \Gamma(\frac{\nu}{2})} \int_0^\infty \left[ \frac{1}{z^{1}e^{\xi} - 1} - \frac{n + 1}{z^{-(n+1)}e^{(n+1)\xi} - 1} \right] \xi^{\frac{s}{2}} d\xi + \ln \frac{1}{1 - z^{n+1}},
\]
(8)

while we have used the asymptotic formula in \( \nu \) dimensions with the dispersion relation \( E = \frac{\epsilon^{s}}{2m} \):
\[
\sum_{\ell} \longrightarrow g \frac{V}{h^{\nu}} \int \frac{2\pi^{\nu/2}}{\Gamma(\frac{\nu}{2})} \frac{1}{s(2m)^{\frac{s}{2}}\epsilon^{\frac{s}{2}}} d\epsilon,
\]
and introduced \( \epsilon = kT \xi \). It should be emphasized that, the parameter \( m \) in the dispersion relation can be explained as the mass of particles only when \( s = 2 \). Here \( \lambda \) is the thermal wavelength
\[
\lambda = \left( \frac{h^{\frac{s}{2}}}{2\pi^{\nu/2} mkT} \right)^{1/s},
\]
(9)

and \( g \) is a weight factor that arises from the internal structure of the particles (the number of internal degrees of freedom). We have split off the term in Eq.(8) corresponding to \( \epsilon = 0 \), which we explicitly retain because, when \( \nu > s \), the density of states is zero. The physical interpretation of such a term is similar to the Bose case: Sometimes the single term \( \epsilon = 0 \) may be as important as the entire sum. The term \( \epsilon = 0 \) represents the contribution of the ground state of the system. The contribution of a state is proportional to the number of particles which are accommodated in the state. As we know,
particles tend to occupy the lower state as possible as they can. At the low temperature limit the particles all want to occupy the ground state; thus the particle number of the ground state nearly equals to the maximum occupation number of the state. Therefore, when \( T \to 0 \) the contribution of the ground state will be proportional to the maximum occupation number \( n \). For fermions, the contribution from the ground state can be ignored since \( n = 1 \); however, for bosons, \( n = \infty \), the contribution from the ground state becomes very important in the low-temperature limit. Indeed, when \( \nu > s \), the term \( \epsilon = 0 \) diverges in the Bose case and causes the phase transition — Bos-Einstein condensation. In Gentile statistics the maximum occupation number \( n \) is finite, so the term \( \epsilon = 0 \) does not diverge and there are no phase transitions like Bose-Einstein condensation. However, when \( n \) is large the influence of such a term may be very important. In the following we will show this influence on the thermodynamic quantities. Moreover, to obtain the result corresponding to \( \nu \leq s \), we only need to drop the term \( \epsilon = 0 \).

In the theory of Bose-Einstein and Fermi-Dirac systems we come across integrals of the type

\[
g_{\sigma}(z) = \frac{1}{\Gamma(\sigma)} \int_{0}^{\infty} \frac{x^{\sigma-1}}{z^{-1}e^{x} - 1} \, dx \quad \text{and} \quad f_{\sigma}(z) = \frac{1}{\Gamma(\sigma)} \int_{0}^{\infty} \frac{x^{\sigma-1}}{z^{-1}e^{x} + 1} \, dx.
\]

To deal with the systems with the grand potential Eq.(8), we can introduce the function

\[
h_{\sigma}(z) = \frac{1}{\Gamma(\sigma)} \int_{0}^{\infty} \left[ \frac{1}{z^{-1}e^{x} - 1} - \frac{n + 1}{z^{-(n+1)}e^{(n+1)x} - 1} \right] x^{\sigma-1} \, dx.
\]

The function \( h_{\sigma}(z) \) will return to Bose-Einstein or Fermi-Dirac integrals when \( n \) equals to 1 or \( \infty \), respectively. A simple process of differentiation brings out the relationship between \( h_{\sigma}(z) \) and \( h_{\sigma-1}(z) \):

\[
h_{\sigma-1}(z) = z \frac{\partial}{\partial z} h_{\sigma}(z).
\]

In terms of the function \( h_{\sigma}(z) \) we can express the grand potential Eq.(8) in the following form:

\[
\ln \Xi = \sum_{\sigma} \ln \Xi_{\sigma} = \frac{1}{\lambda T} \frac{2gV}{\nu \Gamma(\frac{\nu}{2})} \Gamma(\frac{\nu}{2} + 1) h_{\frac{\nu}{2} + 1}(z) + \ln \frac{z^{n+1} - 1}{z - 1}.
\]

The last term describes the contribution from the ground state. For bosons, this term returns to \( -\ln(1 - z) \) with \( n \to \infty \) and \( z < 1 \), which is in connection with the Bose-Einstein condensation; for fermions, this term may be ignored since the ground state of a fermion system can contain only one particle. In Gentile statistics \( 1 < n < \infty \), the contribution of the ground state depends on the value of \( n \). When \( n \) is close to 1, like the Fermi case, the contribution of the ground state can be ignored; however, for \( \nu > s \), when \( n \) is comparable in magnitude to the total number of particles, the ground state contribution becomes important. Notice that, although there are no divergences, consequently there are no phase transitions, the influence which is related to \( n \) also can be very large. Ideally, phase transitions display only when infinities appear. In practice, however, infinity only means very large, so when \( n \to N \), though there are no divergences, the behavior of the system will be very similar to phase transitions.

Since the grand potential Eq.(12) involves the contribution corresponding to \( \epsilon = 0 \), from here onward, the thermodynamic quantities which reckon in the influence of the ground state follow straightforwardly.

Eliminating \( z \) between the two equations

\[
\begin{align*}
\frac{P}{kT} &= \frac{1}{V} \ln \Xi = \frac{2g \Gamma(\frac{s}{2} + 1)}{\lambda \nu \Gamma(\frac{s}{2})} h_{\frac{s}{2} + 1}(z) + \frac{1}{V} \ln \frac{z^{n+1} - 1}{z - 1},
\frac{1}{v} &= \frac{N}{V} \frac{1}{z} \frac{\partial}{\partial z} \ln \Xi = \frac{2g \Gamma(\frac{s}{2} + 1)}{\lambda \nu \Gamma(\frac{s}{2})} h_{\frac{s}{2}}(z) + \frac{N_{0}}{V},
\end{align*}
\]

(13)
we obtain the equation of state of ideal Gentile gases. Here, the last terms of these two equations
describe the contribution from the ground state, and

\[ N_0 = n - \left( \frac{1}{z - 1} - \frac{n + 1}{z^{n+1} - 1} \right) \] (14)

is the average occupation number of the ground state. Easily to see that when \( n = \infty \) and \( z < 1 \)
the occupation number \( N_0 = \frac{1}{z} \). This is just the case of bosons. In the case of fermions, \( N_0 = \frac{1}{z} \)
and its contribution is negligible compared to the first term in any cases. In Gentile statistics
\( 1 < n < \infty \), the influence of such a term depends on the value of \( n \). When \( n \) is large this influence
becomes important.

The internal energy of the ideal gas is given by

\[ \frac{U}{N} = -\frac{1}{N} \frac{\partial}{\partial \beta} \ln \Xi = (1 - \frac{N_0}{N}) \frac{\nu}{s} kT \frac{h_{\nu + 1}(z)}{h_{\nu}(z)}. \] (15)

For the Helmholtz free energy of the gas, we get

\[ F = N\mu - PV = NkT \left[ \ln z - (1 - \frac{N_0}{N}) \frac{h_{\nu + 1}(z)}{h_{\nu}(z)} - \frac{1}{N} \ln \frac{z^{n+1} - 1}{z - 1} \right], \] (16)

and for the entropy

\[ S = \frac{U - F}{T} = Nk \left( 1 - \frac{N_0}{N} \right) \left( \frac{\nu}{s} + 1 \right) \frac{h_{\nu + 1}(z)}{h_{\nu}(z)} - \ln z + \frac{1}{N} \ln \frac{z^{n+1} - 1}{z - 1} \]. (17)

The specific heat capacity is given by

\[ C_V = \frac{\partial U}{\partial T} = \frac{\nu}{s} Nk (1 - \frac{N_0}{N}) \left( \frac{\nu}{s} + 1 \right) \frac{h_{\nu + 1}(z)}{h_{\nu}(z)} \]

\[ - \frac{\nu}{s} \left( 1 + \frac{1}{N - N_0} \left( \frac{z}{(z - 1)^2} - \frac{(n + 1)^2 z^{n+1}}{(z^{n+1} - 1)^2} \right) \frac{h_{\nu}(z)}{h_{\nu - 1}(z)} \right)^{-1} \frac{h_{\nu}(z)}{h_{\nu - 1}(z)}. \] (18)

In the derivation of \( C_V \) we have used the relation

\[ \frac{\partial z}{\partial T} = -\frac{\nu}{s} \frac{h_{\nu}(z)}{h_{\nu - 1}(z)} \left( 1 + \frac{1}{N - N_0} \left( \frac{z}{(z - 1)^2} - \frac{(n + 1)^2 z^{n+1}}{(z^{n+1} - 1)^2} \right) \frac{h_{\nu}(z)}{h_{\nu - 1}(z)} \right)^{-1}, \] (19)

which can be obtained from the expression of \( N \) in Eq. (13). The existence of ground state term
makes the formulas somewhat complex.

Comparing with the thermodynamic quantities given by Ref. [1, 3], we can see that there is an
extra factor \( 1 - N_0/N \) in our result. The addition of such a factor to the thermodynamic quantities
brings in important information: the influence of the ground state. If we ignore the contribution
from the ground state or, equivalently, take \( N_0 = 0 \), the above thermodynamic quantities will return
to the results given by Ref. [1, 3]. However, since in Gentile statistics \( N_0 \) can take any value, the
influence of the ground state must be reckoned in. For instance, when \( z \gg 1 \) and \( N_0 \approx N \) so that
the factor \( 1 - N_0/N \to 0 \), the thermodynamic quantities, such as the internal energy, will approach 0.
The ratio \( N_0/N \) reflects the difference of the various intermediate statistics corresponding to different
\( n \). Notice that, the factor \( 1 - N_0/N \) comes from the term corresponding to \( \epsilon = \frac{1}{n} \). Such a term is needed
only when \( \nu > s \).

For studying the properties of thermodynamic quantities in detail, we first need to discuss the
behavior of \( z \) as determined by the second equation of Eq. (13), namely:
\[ \lambda' = \frac{1}{v} \frac{1}{1 - \frac{N_0}{N}} 2g \frac{\Gamma(\frac{\nu}{2} + 1)}{\nu \Gamma(\frac{\nu}{2})} h_{\frac{\nu}{2}}(z). \]  

(20)

Instead of the Bose-Einstein and the Fermi-Dirac integrals \( g_{\frac{\nu}{2}}(z) \) and \( f_{\frac{\nu}{2}}(z) \), there is a function \( h_{\frac{\nu}{2}}(z) \). To determine the range of the fugacity \( z \), we rewrite the factor in function \( h_{\frac{\nu}{2}}(z) \) in the following form:

\[ \frac{1}{\xi - 1} - \frac{n + 1}{\xi^{n+1} - 1} = \frac{\xi^{n-1} + 2\xi^{n-2} + \cdots + n}{\xi^n + \xi^{n-1} + \cdots + 1}, \]

(21)

where \( \xi = z^{-1}e^x \). It is obvious that for any real values of \( z \), \( h_{\frac{\nu}{2}}(z) \) is a bounded, positive function of \( z \).

From Eq. (21), one can learn that if we expect \( h_{\frac{\nu}{2}}(z) \) to return to the Bose-Einstein integral the necessary condition \( z < 1 \) is needed so that \( h_{\frac{\nu}{2}}(z) \) is bounded. In other words, if we expect the above formulas to return to the Bose case when \( n = \infty \), we need an additional restriction \( z < 1 \). For comparison we recall that \( 0 \leq z < \infty \) in Fermi-Dirac statistics and Gentile statistics.

IV. Low temperatures and high densities

In this section, we will focus on the behavior of the statistics corresponding to \( z \gg 1 \). For \( \lambda' / v \gg 1 \) the average thermal wavelength is much greater than the average interparticle separation. It is easy to see that this is equivalent to the requirement \( z \gg 1 \). It can be expected that in this case the contribution from the ground state will become important. It is noteworthy that, in this case, the result will not return to the Bose case even when \( n \to N \). To obtain Bose-Einstein statistics from Gentile statistics, one needs to not only perform the limit \( n \to N \), but also restrict the fugacity \( z \leq 1 \). Thus the asymptotic results of the thermodynamic functions for low temperatures and high densities given in this section will not return to the Bose case since the asymptotic expansion is based on \( z \gg 1 \). (We need not to consider the case of \( n \to N \) with \( z < 1 \) because this is just the Bose case [2].)

To study the behavior of \( h_{\nu}(z) \), which contains two integrals, for large \( z \), we introduce the variable \( t = \ln z \) so that the first integral in Eq. (10) can be expanded in powers of \( t \):

\[ I = \int_0^\infty \frac{\xi^{\frac{\nu}{2}}}{z^{-1}e^\xi - 1} d\xi \]

\[ = \int_0^\infty \frac{\xi^{\frac{\nu}{2}}}{e^{\xi-t} - 1} d\xi \]

\[ \simeq -\frac{t^{\frac{\nu}{2}+1}}{\frac{\nu}{2} + 1} + 2 \sum_{j=1,3,5} \left( \frac{\nu}{j} \right) t^{\frac{\nu}{2} - j} \Gamma(1 + j) \zeta(1 + j), \]

(22)

while the lower limit \( t \) in one of the integral is approximately replaced by \( \infty \). By the same procedure, we can derive the second integral:

\[ I' = \int_0^\infty \frac{(n + 1)\xi^{\frac{\nu}{2}}}{z^{-n}e^{(n+1)\xi} - 1} d\xi \]

\[ = \int_0^\infty \frac{(n + 1)\xi^{\frac{\nu}{2}}}{e^{(n+1)(\xi-t)} - 1} d\xi \]

\[ \simeq -(n + 1)\frac{t^{\frac{\nu}{2}+1}}{\frac{\nu}{2} + 1} + 2 \sum_{j=1,3,5} \left( \frac{\nu}{j} \right) t^{\frac{\nu}{2} - j} \frac{1}{(n + 1)^j} \Gamma(1 + j) \zeta(1 + j). \]

(23)
Then we obtain:

\[
    h_\sigma(z) = \frac{1}{\Gamma(\sigma+1)} t^\sigma n \left[ 1 + \frac{\pi^2}{3} \frac{1}{n+1} \sigma(\sigma-1)t^{-2} + \frac{\pi^4}{45} \frac{1}{(n+1)^3} \sigma(\sigma-1)(\sigma-2)(\sigma-3)t^{-4} + \cdots \right].
\]

(24)

Substituting Eq. (24) into Eq. (13), we obtain

\[
    \frac{N}{V} = \frac{1}{\lambda^\nu \nu \Gamma(\nu) / \Gamma(\nu/2)} t^{\frac{\nu}{2}} n \left[ 1 + \frac{\pi^2}{3} \frac{1}{n+1} \frac{\nu}{s} (\nu - 1)t^{-2} + \frac{\pi^4}{45} \frac{1}{(n+1)^3} \frac{(\nu - 1)(\nu - 2)(\nu - 3)}{s} t^{-4} + \cdots \right].
\]

(25)

To obtain the thermodynamic functions for low temperatures and high densities we first write down the expansion for the chemical potential from Eq. (25):

\[
    \mu = \epsilon_F \left[ 1 - \frac{\pi^2}{3} \frac{1}{n+1} \left( \frac{\nu}{s} - 1 \right) (\frac{kT}{\epsilon_F})^2 + \cdots \right].
\]

(26)

The expansion parameter is \( \frac{kT}{\epsilon_F} \). This expression implies that when \( \nu = s \) the chemical potential \( \mu \) is equal to the Fermi energy \( \epsilon_F \) at an arbitrary temperature. A special case is free particles in two dimensions, in which \( \nu = s = 2 \). The relation between \( \epsilon_F \) and the Fermi energy \( \epsilon_F^{\text{fermion}} \) in Fermi-Dirac statistics is

\[
    \epsilon_F = \epsilon_F^{\text{fermion}} \left( \frac{N - n}{N n} \right)^{\frac{\nu}{2}},
\]

(27)

where

\[
    \epsilon_F^{\text{fermion}} = \frac{\hbar^s}{2m g} \left[ \frac{2^\nu \pi^{\nu/2}}{\Gamma(\nu)} \frac{1}{(\nu/2 + 1)N} \right]^{\frac{1}{\nu}}.
\]

(28)

In other words, \( \epsilon_F \) can be regarded as an analogue of the Fermi energy in Gentile statistics, and \( \epsilon_F \) will return to \( \epsilon_F^{\text{fermion}} \) when \( n = 1 \).

With the help of Eq. (26), the asymptotic expansion of the internal energy is then given by

\[
    \frac{U}{N} = \frac{N - n}{N} \frac{1}{\nu + 1} \epsilon_F \left[ 1 + \frac{\pi^2}{3} \frac{1}{n+1} \left( \frac{\nu}{s} + 1 \right) (\frac{kT}{\epsilon_F})^2 + \cdots \right].
\]

(29)

The pressure of the gas is:

\[
    P = \frac{1}{\nu + 1} \frac{N - n}{V} \epsilon_F \left[ 1 + \frac{\pi^2}{3} \frac{1}{n+1} \left( \frac{\nu}{s} + 1 \right) (\frac{kT}{\epsilon_F})^2 + \cdots \right] + \frac{n}{V} \epsilon_F \left[ 1 - \frac{\pi^2}{3} \frac{1}{n+1} \left( \frac{\nu}{s} - 1 \right) (\frac{kT}{\epsilon_F})^2 + \cdots \right].
\]

(30)

The free energy of the system follows directly from Eq. (29) and Eq. (30)

\[
    \frac{F}{N} = \mu - \frac{PV}{N} = \frac{N - n}{N} \frac{1}{\nu + 1} \epsilon_F \left[ 1 - \frac{\pi^2}{3} \frac{1}{n+1} \left( \frac{\nu}{s} + 1 \right) (\frac{kT}{\epsilon_F})^2 + \cdots \right],
\]

(31)
whence we obtain the entropy of the system:

\[
\frac{S}{Nk} = \frac{N - n}{N} \frac{2\pi^2}{3} \frac{1}{n + 1} \frac{\nu kT}{\epsilon_F} + \ldots.
\] (32)

From Eq. (29), the specific heat of the gas can be obtained

\[
\frac{C_V}{Nk} = \frac{N - n}{N} \frac{2\pi^2}{3} \frac{1}{n + 1} \frac{\nu kT}{\epsilon_F} + \ldots.
\] (33)

It is easy to see that there is a factor \((N - n)\) appearing in each expression of the thermodynamic quantities, where \(N\) is the total number of particles and \(n\) is the maximum occupation number of a state in Gentile statistics. When \(n\) is of the order of \(N\), the values of the thermodynamic quantities will be strongly suppressed by this factor. It is worth comparing \(C_V\) to the specific heat of the ideal Fermi gas \(C_{V,\text{fermion}}\) so as to discuss the difference between Gentile statistics and Fermi-Dirac statistics:

\[
C_V = \eta_1 C_{V,\text{fermion}},
\] (34)

where

\[
\eta_1 = \left(\frac{N - n}{N}\right)^{1-\frac{\nu}{s}} \frac{2n^{\frac{s}{n+1}}}{n+1}.
\] (35)

Similarly, the magnetic susceptibility per unit volume of a system is given by

\[
\chi = \eta_2 \chi_{\text{fermion}},
\] (36)

where

\[
\eta_2 = \left(\frac{N - n}{Nn}\right)^{1-\frac{\nu}{s}}
\] (37)

and \(\chi_{\text{fermion}}\) is the magnetic susceptibility of a fermion system:

\[
\chi_{\text{fermion}} = \frac{g}{\hbar \nu} \frac{2\pi^{\nu/2}}{s \Gamma(\frac{\nu}{2})} (2m)^{\frac{s}{2}} (\epsilon_{F,\text{fermion}})^{\frac{s}{2}-1}.
\] (38)

Obviously, the influence of the ground state depends sensitively on the maximum occupation number \(n\) when \(\nu > s\).

V. Conclusions and outlook

In summary, we first discuss the case \(n \to N\). In some literature, e.g., Ref. [2], the authors argue that such a case is just the Bose case. In this paper we point out that this argument is valid only when the fugacity \(z < 1\). When \(z > 1\), Gentile statistics with \(n \to N\) does not return to Bose-Einstein statistics and in such a case the contribution from ground state plays a very important role. Moreover, we discuss the contribution of the ground state in Gentile statistics which is ignored in the previous literature. The result shows that the ground state contribution becomes important when the maximum occupation number \(n\) is large, especially in the case of low temperatures and high densities. The results given by current literature focus only on the 3-dimensional space with the dispersion relation \(E = \frac{p^2}{2m}\). In the present paper, we give the thermodynamic quantities for an arbitrary dispersion relation \(E = \frac{p^\nu}{2m}\) in arbitrary \(\nu\)-dimensional space and find that the result depends sensitively on the maximum occupation number \(n\) when \(\nu > s\).
As a generalization of Bose-Einstein and Fermi-Dirac statistics, fractional statistics has been discussed for many years [5, 6]. The particles obeying fractional statistics must not be real particles. As an effective method, however, the introducing of such imaginary particles can be used to deal with some complex interaction systems [7]. Generally speaking, there are two ways to achieve the fractional statistics: (1) One is based on counting the number of many-body quantum states, i.e., generalizing the Pauli exclusion principle [1, 8, 9]. The most direct generalization is to allow more than one particles occupying one state, and such an approach gives Gentile statistics [1]. Another way is to introduce a parameter $\alpha$ valued from 0 to 1 in the expression of the number of quantum states [9]. Bose-Einstein or Fermi-Dirac statistics becomes its limit case when $\alpha$ is 0 or 1, respectively.

(2) Alternatively, the fractional statistics can be achieved by analyzing the symmetry properties of the wave function. As is well known, the wave function will change a phase factor when two identical particles exchanges. The phase factor can be +1 or $-1$ related to bosons or fermions, respectively. When this result is generalized to an arbitrary phase factor $e^{i\theta}$, the concept of anyon is obtained [6]. Comparing these two approaches to fractional statistics a question arises naturally: Are there any connections between them? To answer this question we need to build a bridge between the generalized Pauli principle (in which the maximum occupation number is extended to an arbitrary integer $n$) and the exchange symmetry of identical particles. We will discuss this problem elsewhere [10].

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