A proof of the uniqueness of the limit cycle of a quasi-homogeneous system

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Abstract

A. Gasull shared a list of 33 open problems in low dimensional dynamical systems in his work in 2021. The second part of Problem 3 is about whether the limit cycle of a quasi-homogeneous system \( \dot{x} = y, \quad \dot{y} = -x^3 + \alpha x^2 y + y^3 \) is unique. In this paper, we give a positive answer to this question by analysing the uniqueness of the heteroclinic separatrix at infinity.

Keywords: Limit cycle; quasi-homogeneous system; Poincaré transformation; comparison theorem.

MSC: 34C07, 34C25, 37C27

1. Introduction

Consider the uniqueness of the limit cycle of the following concrete system with a real parameter \( \alpha \):

\[
\begin{align*}
\frac{dx}{dt} &= y, \\
\frac{dy}{dt} &= -x^3 + \alpha x^2 y + y^3.
\end{align*}
\]

(1.1)

This problem is put forward by A. Gasull in [3, Problem 3(ii)]. The origin of the system, which is the unique and fixed equilibrium, is a stable focus for \( \alpha < 0 \), while it is an unstable focus for \( \alpha \geq 0 \), see [1, Corollary 5]. Since for
any $xy \neq 0$,

$$\begin{vmatrix} y & -x^3 + \alpha x^2 y + y^3 \\ \frac{\partial}{\partial \alpha} y & \frac{\partial}{\partial \alpha} (-x^3 + \alpha x^2 y + y^3) \end{vmatrix} = x^2 y^2 > 0$$

and

$$\frac{dy}{dx} = \frac{y^3 - x^3}{y} + \alpha x^2 \to \pm \infty \quad \text{as} \quad \alpha \to \pm \infty,$$

the system is a semi-complete family of rotated vector fields (mod $xy = 0$) with respect to $\alpha$, which is defined in [7, Definitions 2, 3]. From the properties of rotated vector fields, an unstable limit cycle bifurcates from the origin for $\alpha < 0$ and $\alpha$ sufficiently close to 0, which is also a generalized Hopf bifurcation. For the number of the limit cycle of the system, the following results are already known.

**Lemma 1.1** ([4]). (i) System (1.1) has no limit cycles when $\alpha \geq 0$ or $\alpha < -2.679$.

(ii) System (1.1) has at most one limit cycle when $0 > \alpha > -3/\sqrt{2} \approx -2.381$. The limit cycle is hyperbolic and unstable if it exists.

**Lemma 1.2** ([5]). System (1.1) has at least one limit cycles when $0 > \alpha > -3 \sqrt[3]{6/3} - 9/\sqrt{4} \approx -2.1103$.

Besides, it is also mentioned in [4] that the existence range of limit cycles seems to be $(-2.198, 0)$ by a numerical computation.

In this paper, we first analyse the behavior of system (1.1) at infinity by the Poincaré transformation. As one of the most important results, we estimate the behavior of the separatrices of two saddles at $\alpha = -3/\sqrt{2} \approx -2.3811$. Then with the help of known results, we show that there is a unique $\alpha^* \in (-2.3811, -2.1103]$ such that system (1.1) has a heteroclinic separatrix at infinity. Furthermore, the system has exactly one limit cycle for any $\alpha \in (\alpha^*, 0)$ and no cycles for the else region.

2. Phase portrait at infinity

We first study the phase portrait of system (1.1) at infinity. It is not hard to see that $(\pm \infty, 0)$ are not critical points at infinity. Then, by the Poincaré transformation

$$x = \frac{v}{z}, \quad y = \frac{1}{z},$$
system (1.1) is transformed to

\[
\begin{aligned}
\frac{dv}{d\tau} &= v f(v, \alpha) + z^2, \\
\frac{dz}{d\tau} &= zf(v, \alpha),
\end{aligned}
\]  
(2.1)

where \(d\tau = dt/z^2\) and

\[f(v, \alpha) = v^3 - \alpha v^2 - 1.\]

It’s easy to see that for any \(\alpha < 0\) and \(\mu \in (-1, f(2\alpha/3, \alpha))\), the equation \(f(v, \alpha) = \mu\) has three different real roots for \(v\). In ascending order, they are denoted by

\[v_i^\mu(\alpha), i = 1, 2, 3.\]  
(2.2)

\(v_i^\mu(\alpha)\) is a continuous function with respect to \(\alpha\) in some suitable range, and we have \(v_1^\mu(\alpha) < 2\alpha/3 < v_2^\mu(\alpha) < 0 < v_3^\mu(\alpha)\). Since \(f(2\alpha/3, \alpha) = -4\alpha^3/27 - 1\) monotonically decreases as \(\alpha\) decreases to \(-\infty\), it is not hard to verify the following result.

**Lemma 2.1.** As \(\alpha\) decreases to \(-\infty\), \(v_1^\mu(\alpha)\) monotonically decreases to \(-\infty\), \(v_2^\mu(\alpha)\) monotonically increases to 0 and \(v_3^\mu(\alpha)\) monotonically decreases to 0.

Before the last lemma in this section, i.e., Lemma 2.5, we only consider the case \(\mu = 0\), for which \(v_i^\mu(\alpha)\) is the abscissa of the equilibrium of system (2.1). Precisely, when \(\alpha < -3/\sqrt{4}\), system (2.1) has four equilibria, which are respectively denoted by

\[P_0 = (0, 0), \ P_1(\alpha) = (v_1^0(\alpha), 0), \ P_2(\alpha) = (v_2^0(\alpha), 0), \ P_3(\alpha) = (v_3^0(\alpha), 0).\]

Throughout this section, we always assume \(\alpha < -3/\sqrt{4} \approx -1.8899.\)

### 2.1. Behavior of the trajectories for \(\alpha < -3/\sqrt{4}\)

Since system (2.1) is symmetric with respect to \(v\)-axis, we only need to consider the upper half plane. Divide the upper half plane into six domains in each of which \(dv/d\tau\) and \(dz/d\tau\) do not change signs (see Figure 1):

\[D_1(\alpha) = (-\infty, v_1^0(\alpha)] \times \mathbb{R}^+,\]
\[D_2(\alpha) = \{(v, z) \in \mathbb{R} \times \mathbb{R}^+ \mid v_1^0(\alpha) < v < v_2^0(\alpha), \ v f(v, \alpha) + z^2 > 0\},\]
\[D_3(\alpha) = \{(v, z) \in \mathbb{R} \times \mathbb{R}^+ \mid v_2^0(\alpha) < v < v_3^0(\alpha), \ v f(v, \alpha) + z^2 > 0\},\]
\[D_4(\alpha) = [v_3^0(\alpha), +\infty) \times \mathbb{R}^+,\]
\[A^- (\alpha) = \{(v, z) \in \mathbb{R} \times \mathbb{R}^+ \mid v_1^0(\alpha) < v < v_2^0(\alpha), \ v f(v, \alpha) + z^2 < 0\},\]
\[A^+ (\alpha) = \{(v, z) \in \mathbb{R} \times \mathbb{R}^+ \mid 0 < v < v_3^0(\alpha), \ v f(v, \alpha) + z^2 < 0\},\]
and let

\[ D(\alpha) = D_1(\alpha) \cup D_2(\alpha) \cup D_3(\alpha) \cup D_4(\alpha), \quad A(\alpha) = A^-(\alpha) \cup A^+(\alpha). \]

Denote the curves by

\[ K^-(\alpha) = \{ (v, \sqrt{-vf(v, \alpha)}) \mid v_1^0(\alpha) < v < v_2^0(\alpha) \}, \]
\[ K^+(\alpha) = \{ (v, \sqrt{-vf(v, \alpha)}) \mid 0 < v < v_3^0(\alpha) \}. \]

We see that \( dv/d\tau > 0 \) in \( D \), \( dv/d\tau < 0 \) in \( A \) and \( dv/d\tau = 0 \) on \( K^- \) and \( K^+ \). Moreover, \( K^- \) and \( K^+ \) are two traversals of system (2.1). The tangent vectors on \( K^- \) direct from \( A^- \) to \( D_2 \), and the ones on \( K^+ \) direct from \( D_3 \) to \( A^+ \) (also see Figure 1). Note that \( A^- \) is bounded by the traversal \( K^- \), the equilibria \( P_1 \) and \( P_2 \) and the trajectory \( \overline{P_2P_1} \). If a trajectory intersects \( K^- \) at a point \( P \), then the negative part from \( P \) of this trajectory entirely lies in \( A^- \). Similarly, if a trajectory intersects \( K^+ \) at a point \( P \), then the positive part from \( P \) of this trajectory entirely lies in \( A^+ \).

![Figure 1](image_url)

Figure 1: Division of domains in the upper half plane and approximate directions of tangent vectors in each domain.

**Proposition 2.1.** Assume that \( \alpha < -3/\sqrt{4}. \)

(i) \( P_0(\alpha) \) is a hyperbolic stable node.
(ii) $P_1(\alpha)$ is a semi-hyperbolic saddle, whose stable separatrices entirely lies on $v$-axis and unstable separatrices are tangent to $v = v_1^0(\alpha)$ at $P_1(\alpha)$. Moreover, the unstable separatrix in the upper half plane has no common with $D_1(\alpha)$, $A^-(\alpha)$ or $K^-(\alpha)$.

(iii) $P_2(\alpha)$ is a semi-hyperbolic stable node.

(iv) $P_3(\alpha)$ is a semi-hyperbolic saddle, whose unstable separatrices entirely lies on $v$-axis and stable separatrices are tangent to $v = v_3^0(\alpha)$ at $P_3(\alpha)$. Moreover, the stable separatrix in the upper half plane has no common with $D_4(\alpha)$, $A^+(\alpha)$ or $K^+(\alpha)$.

Proof. One can easily see that $P_0$ is a hyperbolic equilibrium and that the linearized matrix at $P_0$ has two negative eigenvalues. From the Perron’s Theorem, $P_0$ is a stable node.

By a translation $\nu = v - v_i^0$ with $i \geq 1$, $P_i$ is translated to the origin and system $(2.1)$ is transferred to

$$
\frac{d\nu}{d\tau} = (3v_i^0 + 2\alpha) (v_i^0)^2 \nu + Q(\nu, z),
$$

$$
\frac{dz}{d\tau} = R(\nu, z),
$$

where

$$
Q(\nu, z) = \left[ 3 \left( 2v_i^0 + \alpha \right) v_i^0 \right] \nu^2 + \left( 4v_i^0 + \alpha \right) \nu^3 + \nu^4 + z^2,
$$

$$
R(\nu, z) = \left[ \left( 3v_i^0 + 2\alpha \right) v_i^0 \nu + \left( 3v_i^0 + \alpha \right) \nu^2 + \nu^3 \right] z.
$$

Since $(3v_i^0 + 2\alpha) (v_i^0)^2 \neq 0$, the origin of system $(2.3)$ is a semi-hyperbolic equilibrium introduced in many books, see [2] for example. From the Implicit Function Theorem, the equation $(3v_i^0 + 2\alpha) (v_i^0)^2 \nu + Q(\nu, z) = 0$ has a unique and analytic solution

$$
\nu = \nu(z) = -\frac{2z^2}{(3v_i^0 + 2\alpha) (v_i^0)^2} + O(z^3)
$$

in a small neighborhood of the origin. Substituting it into $R(\nu, z)$, we have

$$
R(\nu(z), z) = -\frac{2z^3}{v_i^0} + O(z^5).
$$

From [2, Theorem 2.19 and Remark 2.20], there is an invariant analytic curve tangent to the $\nu$-axis at the origin, which is $\nu$-axis itself in our system.
Besides, the qualitative properties of the origin is determined by the signs of the terms \((3v^0_1 + 2\alpha)(v^0_1)^2\) and \(-2/v^0_1\), which states as follows:

- for \(v^0_1\), we have \((v^0_1 + 2\alpha)(v^0_1)^2 < 0\) and \(-2/v^0_1 > 0\). Then, the origin of system \((2.3)\) is a topological saddle, whose unstable separatrices are tangent to the \(z\)-axis at the origin;

- for \(v^0_2\), we have \((v^0_2 + 2\alpha)(v^0_2)^2 > 0\) and \(-2/v^0_2 > 0\). Then, the origin of system \((2.3)\) is an unstable topological node, whose trajectories near the origin except for the two lying on the \(\nu\)-axis are tangent to the \(z\)-axis at the origin;

- for \(v^0_3\), we have \((v^0_3 + 2\alpha)(v^0_3)^2 > 0\) and \(-2/v^0_3 < 0\). Then, the origin of system \((2.3)\) is a topological saddle, whose stable separatrices are tangent to the \(z\)-axis at the origin.

Moreover, if any trajectory \(L\) passes through a point \((v_0, z_0)\) in \(D_1\) in which \(dv/d\tau > 0\) and \(dz/d\tau \leq 0\), the \(\alpha\)-limit set of \(L\) is in \(\{v < v_0, z \geq z_0\} \subset D_1\) which doesn’t contain \(P_1\). On the other hand, recall that \(A^-\) is bounded by the traversal \(K^-\), the equilibria \(P_1\) and \(P_2\) and the trajectory \(P_2P_1\). If any trajectory \(L\) passes through a point on \(K^-\) or in \(A^-\) in which \(dv/d\tau < 0\) \(dz/d\tau > 0\), \(L\) must be negatively approaches \(P_2\). Therefore, the unstable separatrix of \(P_1\) in the upper half plane has no common with \(D_1, A^-\) or \(K^-\).

By analogous analysis, the stable separatrix of \(P_3\) in the upper half plane has no common with \(D_4, A^+\) or \(K^+\).

\(\square\)

Denote the solution of system \((2.1)\) by

\[L(v_0, z_0, \alpha) := (v(\tau; v_0, z_0, \alpha), z(\tau; v_0, z_0, \alpha)),\]

or \((v(\tau), z(\tau))\) for short, with initial condition \(v(0; v_0, z_0, \alpha) = v_0\) and \(z(0; v_0, z_0, \alpha) = z_0\). Assume that \((v_0, z_0) \in D(\alpha)\). Let

\[\tau(v_0, z_0, \alpha) = \inf \left\{ \tau < 0 \mid v(\tau_1; v_0, z_0, \alpha)f(v(\tau_1; v_0, z_0, \alpha), \alpha) + (z(\tau_1; v_0, z_0, \alpha))^2 > 0, \forall \tau_1 \in [\tau, 0] \right\},\]

\[\bar{\tau}(v_0, z_0, \alpha) = \sup \left\{ \tau > 0 \mid v(\tau_1; v_0, z_0, \alpha)f(v(\tau_1; v_0, z_0, \alpha), \alpha) + (z(\tau_1; v_0, z_0, \alpha))^2 > 0, \forall \tau_1 \in [0, \tau] \right\} .\]
Since $d\nu(\tau; v_0, z_0, \alpha)/d\tau > 0$ on $(\tau, \tau')$, let
\[
\overline{\nu}(v_0, z_0, \alpha) = \lim_{\tau \to \Sigma(v_0, z_0, \alpha)} \nu(\tau; v_0, z_0, \alpha), \quad \overline{\nu}(v_0, z_0, \alpha) = \lim_{\tau \to \Sigma(v_0, z_0, \alpha)} \nu(\tau; v_0, z_0, \alpha).
\]

Then, we see that the trajectory $L$ on $(\tau, \tau')$ is also the solution curve of the
first-order differential equation
\[
\frac{dz}{d\nu} = \frac{zf(v, \alpha)}{vf(v, \alpha) + z^2}
\]
(2.4)
on $(\nu, \nu)$ with initial point $(v_0, z_0)$. Denote the corresponding solution by
\[
z = \phi(v; v_0, z_0, \alpha),
\]
or $z = \phi(v)$ for short, on $(\nu, \nu)$ with initial condition $z_0 = \phi(v_0; v_0, z_0, \alpha)$.

**Remark 2.1.** In fact, $(\nu, \nu)$ is the maximal existence interval of the solution
of equation (2.4) with initial point $(v_0, z_0)$.

In the upper half plane, $dz/d\nu$ changes signs when the trajectory crosses
the lines $v = v_i$ for $i = 1, 2, 3$. Note that if a trajectory $L$ passes through
a point $(v_0, z_0) \in D$, $v(\tau)$ monotonically depends on $\tau$ on $(\tau, \tau')$. There is
some $\tau^*$ such that the segment of $L$ on $(\tau^*, \tau)$ lies entirely in some domain
$D_i$, in which $dz/d\nu$ does not change sign. Hence, $z(\tau)$ monotonically varies
when $\tau$ eventually approaches $\tau$. It follows that $\phi(v)$ monotonically varies
when $v$ eventually approaches $\tau$. Similarly, $\phi(v)$ monotonically varies when
$v$ approaches $\nu$. Moreover, we have the following result.

**Lemma 2.2.** Assume that $\alpha < -3/\sqrt{4}$ and that the solution (or the trajec-
tory) $L(v_0, z_0, \alpha)$ of system (2.1) passes through a point $(v_0, z_0) \in D(\alpha)$. If
$\overline{\nu}(v_0, z_0, \alpha) < +\infty$ (resp. $\underline{\nu}(v_0, z_0, \alpha) > -\infty$), then
\[
\lim_{\tau \to \Sigma(v_0, z_0, \alpha)} z(\tau; v_0, z_0, \alpha) < +\infty \quad \left( \text{resp.} \quad \lim_{\tau \to \Sigma(v_0, z_0, \alpha)} z(\tau; v_0, z_0, \alpha) < +\infty \right).
\]

**Proof.** We only consider the case for $\overline{\nu}$ and the analysis on the case for $\underline{\nu}$ is
analogous.

From the previous analysis, we see that the limit of $z(\tau)$ at $\overline{\nu}$ exists or is
equal to $+\infty$. Assume that $\lim_{\tau \to \overline{\nu}} z(\tau) = +\infty$. It follows that
\[
\lim_{v \to \overline{\nu}} \phi(v) = +\infty,
\]
where $\phi(v)$ is the solution of (2.1) on $(\bar{\nu}, \bar{\nu})$ with initial condition $z_0 = \phi(v_0)$. Since $\nu < +\infty$, there is a sufficiently large $M$ such that for some $\epsilon_0 > 0$

$$\left| \frac{dz}{dv} \right| = \left| \frac{zf(v, \alpha)}{vf(v, \alpha) + z^2} \right| < 1, \quad \forall (v, z) \in (\nu - \epsilon_0, \nu) \times (M, +\infty),$$

and that for some $\epsilon_1 \in (0, \epsilon_0)$

$$\phi(v) > M, \quad \forall v \in (\nu - \epsilon_1, \nu).$$

Take some $\tilde{v} \in (\nu - \epsilon_1, \nu)$, then we have

$$\lim_{v \to \nu} \phi(v) = \lim_{\nu \to \nu_0} \int_{\tilde{v}}^{\nu} d\phi + \phi(\tilde{v}) \leq \lim_{\nu \to \nu_0} \int_{\tilde{v}}^{\nu} \left| \frac{\phi(v)f(v, \alpha)}{vf(v, \alpha) + (\phi(v))^2} \right| dv + \phi(\tilde{v}) < \phi(\tilde{v}) + \epsilon_1 < +\infty,$$

which is a contradiction. \hfill \square

Given a trajectory $L$ passing a point $(v_0, z_0) \in D$, we have two claims:

(i) $L$ approaches the curve $\mathcal{K}^+$ or the equilibrium $P_0$ or $P_3$ as $\tau \to \tau$ if $\nu < +\infty$, and $\lim_{\tau \to +\infty} z(\tau) = +\infty$ if $\nu = +\infty$;

(ii) $L$ approaches the curve $\mathcal{K}^-$ or the equilibrium $P_1$ or $P_2$ as $\tau \to \tau$ if $\nu > -\infty$, and $\lim_{\tau \to -\infty} z(\tau) = +\infty$ if $\nu = -\infty$.

For the case $\nu < +\infty$, since from Lemma 2.2 $\lim_{\tau \to \tau} z(\tau)$ always exists, $L$ must approach some point as $\tau \to \tau$. Precisely, if $\nu < +\infty$ and $\tau < +\infty$, then the regular point $(v(\tau), z(\tau))$ is obviously on $\mathcal{K}^+$; if $\nu < +\infty$ and $\tau = +\infty$, then the limit point is the equilibrium, i.e., $P_0$ or $P_3$. For the case $\nu = +\infty$ which implies that $\tau = +\infty$, there is a $\tau^*$ such that the segment of $L$ on $(\tau^*, +\infty)$ lies in $\mathcal{D}_4$ in which $dz/d\tau \geq f(\tilde{v}, \alpha)z$ for any $v \geq \tilde{v} \gg 0$, and hence,

$$z(\tau) \geq z(\tau^*) \exp \left[ f(\tilde{v}, \alpha)(\tau - \tau^*) \right] \to +\infty \quad \text{as} \quad \tau \to +\infty.$$

The discussion on $\nu$ is analogous.
2.2. Behavior of the separatrices of $P_1(\alpha)$ and $P_3(\alpha)$

Proposition 2.1 implies that the unstable separatrix of $P_1(\alpha)$ in the upper half plane, denoted by $L^-(\alpha)$, passes through $D_2(\alpha)$, and the stable separatrix of $P_3(\alpha)$ in the upper half plane, denoted by $L^+(\alpha)$, passes through $D_3(\alpha)$. Choose two points $(v_1, z_1) \in D_2(\alpha) \cap L^-(\alpha)$ and $(v_2, z_2) \in D_3(\alpha) \cap L^+(\alpha)$, then we have

$$L^-(\alpha) = L(v_1, z_1, \alpha), \quad L^+(\alpha) = L(v_2, z_2, \alpha).$$

Let

$$v^-(\alpha) = \nu(v_1, z_1, \alpha), \quad v^+(\alpha) = \nu(v_2, z_2, \alpha),$$

and

$$\phi^-(v; \alpha) = \phi(v; v_1, z_1, \alpha), \quad \phi^+(v; \alpha) = \phi(v; v_2, z_2, \alpha).$$

Note that these notations are independent of the choices of $(v_1, z_1)$ and $(v_2, z_2)$. Since $L^-(\alpha)$ negatively approaches $P_1(\alpha)$, we have $\nu(v_1, z_1, \alpha) = v^0_1(\alpha)$, and hence, $\phi^-(v; \alpha)$ is defined on $v \in (v^0_1(\alpha), v^-(-\alpha))$. Similarly, $\phi^+(v; \alpha)$ is defined on $v \in (v^+(\alpha), v^0_3(\alpha))$.

In terms of Proposition 2.1 and the claims after Lemma 2.2, we can obtain the possible behavior of $L^-(\alpha)$ and $L^+(\alpha)$, which are entirely related to the possible values of $v^-(\alpha)$ and $v^+(\alpha)$.

**Proposition 2.2.** Assume that $\alpha < -3/\sqrt{4}$. $v^-(\alpha)$ and $v^+(\alpha)$ admit one of the following cases:

(i) $v^-(\alpha) = 0$ and $v^+(\alpha) = -\infty$, see Figure 2(a).

(ii) $0 < v^-(\alpha) < v^0_3(\alpha)$ and $v^+(\alpha) = -\infty$, see Figure 2(b).

(iii) $v^-(\alpha) = v^0_3(\alpha)$ and $v^+(\alpha) = v^0_1(\alpha)$, see Figure 2(c).

(iv) $v^-(\alpha) = +\infty$ and $v^0_1 < v^+(\alpha) < v^0_2(\alpha)$, see Figure 2(d).

**Remark 2.2.** From Proposition 2.2, one can see that $\lim_{v \to 0} \phi^-(v; \alpha)$ and $\lim_{v \to 0} \phi^+(v; \alpha)$ always exist, the former of which is written as $\phi^-(0; \alpha)$ even though $v^-(\alpha) = 0$, i.e., the case shown in Figure 2(a), while the latter of which is always larger than 0. Then the four cases listed in Proposition 2.2 correspond to $\phi^-(0; \alpha) = 0$, $0 < \phi^-(0; \alpha) < \phi^+(0; \alpha)$, $\phi^-(0; \alpha) = \phi^+(0; \alpha)$ and $\phi^-(0; \alpha) > \phi^+(0; \alpha)$, respectively.
Figure 2: Possible cases for the two separatrices $L^-$ and $L^+$ of the saddles $P_1$ and $P_3$ with $\alpha < -3/\sqrt{4}$, respectively.

It is well known that any trajectory of system (2.1) with fixed initial point continuously depends on $\alpha$ in some suitable range. That is to say, $v(\tau; v_0, z_0, \alpha)$ and $z(\tau; v_0, z_0, \alpha)$ continuously depend on $\alpha$ in some suitable range at any finite $\tau$. However, this property is not much easy to describe and confirm for the separatrices $L^-(\alpha)$ and $L^+(\alpha)$, because we only know that their limit points $P_1(\alpha)$ and $P_3(\alpha)$ continuously depend on $\alpha$. Therefore, we first give a proof of the continuity with respect to $\alpha$ of these two separatrices.

**Lemma 2.3.** Assume that $\alpha_0 < -3/\sqrt{4}$. For any $v_0 \in (v_0^1(\alpha_0), 0]$, $\phi^-(v_0; \alpha)$ continuously depends on $\alpha$ at $\alpha_0$. For any $v_0 \in [0, v_0^3(\alpha_0))$, $\phi^+(v_0; \alpha)$ continuously depends on $\alpha$ at $\alpha_0$.

**Proof.** We only consider the case for $\phi^-$, and the analysis on the case for $\phi^+$ is analogous.

First consider the case $v^-(\alpha_0) > 0$, for which $\phi^-(v; \alpha_0) > 0$ on $(v_0^1(\alpha_0), 0]$ (see Figure 2(b) to 2(d)). For any $v_0 \in (v_0^1(\alpha_0), 0]$, since $v_1^0(\alpha)$ is continuous with respect to $\alpha < -3/\sqrt{4}$, there must be a $\delta_0 > 0$ such that $v_1^0(\alpha) < v_0$ on $U_{\delta_0}(\alpha_0)$, where $U_{\delta}(P)$ denotes the $\delta$-neighborhood of a point $P$. Hence, $\phi^-(v_0; \alpha)$ makes sense for any $\alpha \in U_{\delta_0}(\alpha_0)$.

Let $z_0 = \phi^-(v_0; \alpha_0)$. Since the point $(v_0, z_0)$ belongs to the open set $\mathcal{D}(\alpha_0)$, in which $dv/d\tau > 0$, for any sufficiently small $\varepsilon > 0$ we have $U_{2\varepsilon}(v_0, z_0) \subset$
Consider the trajectory starts from \((v_0, z_0 + \varepsilon)\), i.e., \(L(v_0, z_0 + \varepsilon, \alpha_0)\). From Lemma 2.2 and Proposition 2.1(ii), we see that \(L(v_0, z_0 + \varepsilon, \alpha_0)\) negatively goes into the inner of \(D_1(\alpha_0)\). That is to say, there is a \(\tau_1 < 0\) such that

\[
v(\tau_1; v_0, z_0 + \varepsilon, \alpha_0) < v_1^0(\alpha_0), \quad z(\tau_1; v_0, z_0 + \varepsilon, \alpha_0) > 0.
\]

Since \(v(\tau_1; v_0, z_0 + \varepsilon, \alpha), z(\tau_1; v_0, z_0 + \varepsilon, \alpha)\) and \(v_1^0(\alpha)\) are continuous respect to \(\alpha\), there is a \(\delta_1 > 0\) such that

\[
(v(\tau_1; v_0, z_0 + \varepsilon, \alpha), z(\tau_1; v_0, z_0 + \varepsilon, \alpha)) \in D_1(\alpha), \quad \forall \alpha \in U_{\delta_1}(\alpha_0). \quad (2.5)
\]

On the other hand, consider the trajectory starts from \((v_0, z_0 - \varepsilon)\), i.e., \(L(v_0, z_0 - \varepsilon, \alpha_0)\). From Proposition 2.1(ii) and (iii), we see that \(L(v_0, z_0 - \varepsilon, \alpha_0)\) negatively goes into \(A_1(\alpha_0)\) and eventually approaches \(P_2(\alpha_0)\). That is to say, there is a \(\tau_2 < 0\) such that

\[
v_1^0(\alpha_0) < v(\tau_2; v_0, z_0 - \varepsilon, \alpha_0) < v_2^0(\alpha_0),
\]

\[0 < z(\tau_2; v_0, z_0 - \varepsilon, \alpha_0) < \sqrt{-v(\tau_2; v_0, z_0 - \varepsilon, \alpha_0)} f(v(\tau_2; v_0, z_0 - \varepsilon, \alpha_0), \alpha_0).
\]

Since \(v(\tau_2; v_0, z_0 - \varepsilon, \alpha), z(\tau_2; v_0, z_0 - \varepsilon, \alpha), v_1^0(\alpha), v_2^0(\alpha)\) and \(K^- (\alpha)\) are continuous respect to \(\alpha\), there is a \(\delta_2 > 0\) such that

\[
(v(\tau_2; v_0, z_0 - \varepsilon, \alpha), z(\tau_2; v_0, z_0 - \varepsilon, \alpha)) \in A_1(\alpha), \quad \forall \alpha \in U_{\delta_2}(\alpha_0). \quad (2.6)
\]

Now take \(\delta_3 = \min\{\delta_0, \delta_1, \delta_2\}\). In terms of (2.5), (2.6) and Proposition 2.1(ii), we see that \(L^- (\alpha)\) is always bounded by \(L(v_0, z_0 + \varepsilon, \alpha)\) and \(L(v_0, z_0 - \varepsilon, \alpha)\) for any \(\alpha \in U_{\delta_3}(\alpha_0)\). Hence, we have

\[z_0 - \varepsilon < \phi^-(v_0; \alpha) < z_0 + \varepsilon, \quad \forall \alpha \in U_{\delta_3}(\alpha_0),
\]

which completes the proof for the case \(v^- (\alpha_0) > 0\).

If \(v^- (\alpha_0) = 0\), then \(\phi^- (0; \alpha_0) = 0\) (see Figure 2(a)). We only need to consider \(v_0 = 0\). For any \(\varepsilon > 0\), there is a \(\tilde{v}(< 0)\) sufficiently close to \(0\) such that

\[
\phi^- (\tilde{v}; \alpha_0) < \frac{\varepsilon}{2}.
\]

From the first case, there is \(\delta > 0\) such that

\[
|\phi^- (\tilde{v}; \alpha) - \phi^- (\tilde{v}; \alpha_0)| < \frac{\varepsilon}{2}, \quad \forall \alpha \in U_{\delta}(\alpha_0).
\]
Recall that \( \frac{dz}{d\tau} < 0 \) and \( \frac{dv}{d\tau} > 0 \) in the inner of \( D_3(\alpha) \). It follows that \( \frac{d\phi^- (v; \alpha)}{dv} < 0 \) on \( (v_0^0(\alpha), 0) \). Then, we have

\[
\phi^- (0; \alpha) < \phi^- (\bar{v}; \alpha) \leq \left| \phi^- (\bar{v}; \alpha) - \phi^- (\bar{v}; \alpha_0) \right| + \phi^- (\bar{v}; \alpha) < \varepsilon, \quad \forall \alpha \in U_3(\alpha_0).
\]

The proof is completed. \( \square \)

**Remark 2.3.** Given a smooth curve \( C \), it is known that the intersection of a trajectory \( L(v, z, \alpha) \) and \( C \) varies continuously on \( C \) when \( v, z \) and \( \alpha \) vary in some suitable ranges. The continuity of \( L^- (\alpha) \) with respect to \( \alpha \) is described in the same manner and it can be proved by applying Lemma 2.3 on \( L^- (\alpha) = L(v_0, \phi^- (v_0; \alpha), \alpha) \) with some fixed \( v_0 \in (v_0^0(\alpha_0), 0) \). Similar statements hold for \( L^+ (\alpha) \). However, we only need to consider the continuity of \( \phi^- (0; \alpha) \) and \( \phi^+ (0; \alpha) \) with respect to \( \alpha \) in this paper.

For the main purposes of this section, we are going to show (in Lemma 2.4) that \( \phi^- (0; \alpha) \) (if larger than 0) and \( \phi^+ (0; \alpha) \) vary monotonically as \( \alpha \) varies, and (in Lemma 2.5) that they behave as Figure 2(d) when \( \alpha = -3/\sqrt{2} \). Before this, let’s introduce a useful theorem which is a modified version for the basic comparison theorem in [6].

**Theorem 2.1 (Comparison theorem).** Let \( \xi(x) \) and \( \eta(x) \) be continuously differentiable functions on \( (x_0, x_1) \), and \( g(x, y) \) be a continuous function on \( D \subset \mathbb{R}^2 \), where \( D \) contains the set \( \{(x, \xi(x))|x_0 < x < x_1\} \cup \{(x, \eta(x))|x_0 < x < x_1\} \).

Assume that

\[
\frac{d\xi}{dx} - g(x, \xi(x)) < \frac{d\eta}{dx} - g(x, \eta(x)), \quad \forall x \in (x_0, x_1),
\]

and either

\[
\limsup_{x \to x_0} (\xi(x) - \eta(x)) < 0 \tag{2.8}
\]

or

\[
\limsup_{x \to x_0} (\xi(x) - \eta(x)) = 0, \quad \limsup_{x \to x_0} \left( \frac{d\xi}{dx} - \frac{d\eta}{dx} \right) < 0. \tag{2.9}
\]

Then \( \xi(x) < \eta(x) \) on \( (x_0, x_1) \). Similar statements hold for \( x_1 < x_0 \).

**Proof.** Assume that \( \xi(x) \geq \eta(x) \) for some \( x \in (x_0, x_1) \). We can take

\[
p^* = \inf \{ x \in (x_0, x_1) \mid \xi(x) \geq \eta(x) \}.
\]
If \( p^* > x_0 \), then from the continuity of \( \xi \) and \( \eta \) on \((x_0, x_1)\), we have \( \xi(p^*) = \eta(p^*) \). However, from assumption (2.7), we have

\[
\frac{d\xi}{dx} \bigg|_{x=p^*} - g(p^*, \xi(p^*)) < \frac{d\eta}{dx} \bigg|_{x=p^*} - g(p^*, \eta(p^*))
\]

and hence,

\[
\frac{d\xi}{dx} \bigg|_{x=p^*} < \frac{d\eta}{dx} \bigg|_{x=p^*}.
\]

Thus, there is a \( x \in (x_0, p^*) \) such that \( \xi(x) > \eta(x) \), which contradicts to the definition of \( p^* \).

Then, we have \( p^* = x_0 \). From the definition of \( p^* \), there is a sequence \( \{p_n\} \) on \((x_0, x_1)\) such that

\[
p_{n+1} < p_n, \quad \lim_{n \to \infty} p_n = x_0, \quad \zeta(p_n) \geq 0,
\]

where \( \zeta(x) = \xi(x) - \eta(x) \). It follows that

\[
\limsup_{x \to x_0} \zeta(x) \geq \limsup_{n \to \infty} \zeta(p_n) \geq \liminf_{n \to \infty} \zeta(p_n) \geq 0,
\]

which contradicts to condition (2.8). Thus, we must have \( \limsup_{x \to x_0} \zeta(x) = 0 \), and hence,

\[
\lim_{n \to \infty} \zeta(p_n) = 0. \tag{2.10}
\]

On the other hand, the latter part of condition (2.9) implies that there is a sufficiently small \( \epsilon > 0 \) such that

\[
\frac{d\zeta}{dx} < 0, \quad \forall x \in (x_0, x_0 + \epsilon).
\]

Then, we have

\[
\zeta(p_{n+1}) - \zeta(p_n) = (p_{n+1} - p_n) \frac{d\zeta}{dx} \bigg|_{x=q_n} > 0
\]

for some \( q_n \in [p_{n+1}, p_n] \) and any sufficiently large \( n \) such that \( p_n < x_0 + \epsilon \). Therefore, when \( p_n < x_0 + \epsilon \), we have

\[
0 \leq \zeta(p_n) < \zeta(p_{n+1}) < \zeta(p_{n+2}) < \cdots,
\]

which contradicts to (2.10). The disproof of the theorem is completed.

Similar statements hold for \( x_1 < x_0 \) by considering \(-x \in (-x_0, -x_1)\). \qed
Remark 2.4. In particular, if \( \xi(x) \) and \( \eta(x) \) are solutions of two first-order differential equations \( dy/dx = g_1(x, y) \) and \( dy/dx = g_2(x, y) \), respectively, and if \( g_1(x, \eta(x)) \) (resp. \( g_2(x, \xi(x)) \)) makes sense on \( (x_0, x_1) \), then condition (2.7) in Theorem 2.1 reduces to
\[
g_1(x, \eta(x)) < g_2(x, \eta(x)) \quad \text{(resp. } g_1(x, \xi(x)) < g_2(x, \xi(x)) \text{)}
\]
by taking \( g = g_1 \) (resp. \( g = g_2 \)).

Lemma 2.4. Assume that \( b < a < -3/\sqrt{4} \). For any \( v \in (v_0^1(a), 0) \), we have
\[
\phi^-(v; a) \leq \phi^-(v; b),
\]
where the equal sign holds if and only if \( v = v^- (b) = 0 \) (which is equivalent to \( \phi^- (0; b) = 0 \) from Remark 2.2). For any \( v \in [0, v_0^3(b)] \), we have
\[
\phi^+ (v; a) > \phi^+ (v; b).
\]

Proof. We only consider \( \phi^- \) and the analysis on \( \phi^+ \) is analogous.

First, consider the case \( v \in (v_1^0(a), 0) \). Recall that \( v_0^1(a) > v_0^1(b) \) from Lemma 2.1 which implies that
\[
\lim_{v \to v_0^1(a)} \phi^- (v; a) = 0 < \phi^- (v_1^0(a); b). \tag{2.11}
\]

Note that when \( z > 0 \) and \( vf(v, \alpha) + z^2 \neq 0 \), we have
\[
\frac{\partial}{\partial \alpha} \frac{zf(v, \alpha)}{vf(v, \alpha) + z^2} = -\frac{v^2z^3}{(vf(v, \alpha) + z^2)^2} < 0,
\]
and that \( 0 < (\phi^- (v; b))^2 + vf(v, b) \leq (\phi^- (v; b))^2 + vf(v, \alpha) \) for any \( \alpha \in [b, a] \), which implies that \( (v, \phi^- (v; b)) \in D(\alpha) \) for any \( \alpha \in [b, a] \). It follows that
\[
\frac{zf(v, a)}{vf(v, a) + z^2} \leq \frac{zf(v, b)}{vf(v, b) + z^2} \tag{2.12}
\]
In terms of (2.11), (2.12) and applying Theorem 2.1 and Remark 2.4 on \( (v_1^0(a), 0) \), we have \( \phi^- (v; a) < \phi^- (v; b) \) for any \( v \in (v_1^0(a), 0) \).

Next, consider the case \( v = 0 \) but \( v^- (b) > 0 \) which implies that \( \phi^- (0; b) > 0 \) from Remark 2.2. From the first case in this proof, we have
\[
\phi^- (0; a) = \lim_{v \to 0, v < 0} \phi^- (0; a) \leq \lim_{v \to 0, v < 0} \phi^- (0; b) = \phi^- (0; b).
\]
Assume that $\phi^{-}(0; a) = \phi^{-}(0; b)$. Then $(0, \phi^{-}(0; a))$ is a regular point on both $L^{-}(a)$ and $L^{-}(b)$. Let $\hat{D}$ be the open domain bounded by $L^{-}(a), L^{-}(b)$ and the line segment $\hat{P}_{1}(b)\hat{P}_{1}(a)$. Let $\hat{L}$ be a trajectory of system (2.1) with $\alpha = a$ passing through $\hat{D}$. Since both $L^{-}(a)$ and $\hat{P}_{1}(b)\hat{P}_{1}(a)$ are trajectories of system (2.1) with $\alpha = a$, $\hat{L}$ must cross $L^{-}(b)$ from $\hat{D}$ to $R^{2} - \hat{D}$ at a point $\hat{P} \in L^{-}(b) \cap \{v < 0\}$. Together with $\frac{dv}{d\tau}|_{\hat{P}} > 0$, i.e., $\hat{P} \in D(\alpha)$, for any $\alpha \in [b, a]$, we have

$$\frac{dz}{dv}|_{\hat{P}, \alpha = a} \geq \frac{dz}{dv}|_{\hat{P}, \alpha = b} \quad \text{but} \quad \frac{\partial}{\partial \alpha} \frac{dz}{dv}|_{\hat{P}} < 0, \forall \alpha \in [b, a],$$

which is a contradiction.

At last, from the aforementioned analysis, the equal sign only occurs in the case $\phi^{-}(0; a) = \phi^{-}(0; b) = 0$, which is equivalent to $v^{-}(a) = v^{-}(b) = 0$ from Remark 2.2.

Next we show that when $\alpha = -3/\sqrt{2}$ the two separatrices behave as the case shown in Figure 2(d).

**Lemma 2.5.** $\phi^{-}(0; -3/\sqrt{2}) > \phi^{+}(0; -3/\sqrt{2})$.

**Proof.** We estimate $\phi^{-}(0; \alpha)$ and $\phi^{+}(0; \alpha)$ by constructing two curves such that one lies below $L^{-}(\alpha)$ and the other above $L^{+}(\alpha)$. For convenience, if $\alpha$ is arbitrary, $\phi^{-}(v; \alpha)$, $\phi^{+}(v; \alpha)$ and $f(v, \alpha)$ are abbreviated to $\phi^{-}(v)$, $\phi^{+}(v)$ and $f(v)$, respectively.

Let $\alpha < -3/\sqrt{4}$ and $0 < \mu < f(2\alpha/3) = -4\alpha^{3}/27 - 1$. Consider the following piecewise differential equation

$$\frac{dz}{dv} = G^{\mu}(v, z) \triangleq \begin{cases} \frac{\mu z}{\mu v + z^{2}}, & v \in [v_{1}^{\mu}, v_{2}^{\mu}), \\ 0, & v \in [v_{2}^{\mu}, v_{0}^{\mu}), \\ \frac{z}{v - z^{2}}, & v \in [v_{0}^{\mu}, 0], \end{cases}$$

where $v_{i}^{\mu}$ is defined by (2.2). Take the point $(\nu_{1}^{\mu}, z_{1}^{\mu})$ between the curve $\mu v + z^{2} = 0$ and $L^{-}$, i.e.,

$$\sqrt{-\mu \nu_{1}^{\mu}} < z_{1}^{\mu} < \phi^{-}(\nu_{1}^{\mu}). \quad (2.13)$$

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Let \( z = \varphi^\mu(v) \) be the solution of equation \( \frac{dz}{dv} = G^\mu(v, z) \) with initial point \((v_1^\mu, z_1^\mu)\), i.e., \( z_1^\mu = \varphi^\mu(v_1^\mu) \). It is not hard to show that \( \varphi^\mu(v) \) exists on \([v_1^\mu, 0] \).
In fact, it can be explicitly formulated by

\[
\varphi^\mu(v) = \begin{cases} 
\frac{1}{2} \left[ z_1^\mu - \frac{v_1^\mu}{z_1^\mu} + \sqrt{\left( z_1^\mu + \frac{v_1^\mu}{z_1^\mu} \right)^2 + 4\mu (v - v_1^\mu)} \right], & v \in [v_1^\mu, v_2^\mu), \\
y_2^\mu, & v \in [v_2^\mu, 0), \\
\frac{1}{2} \left[ z_2^\mu + \frac{v_2^0}{z_2^\mu} + \sqrt{\left( z_2^\mu + \frac{v_2^0}{z_2^\mu} \right)^2 - 4v} \right], & v \in [v_2^0, 0],
\end{cases}
\]

where

\[
z_2^\mu = \frac{1}{2} \left[ z_1^\mu - \frac{v_1^\mu}{z_1^\mu} + \sqrt{\left( z_1^\mu + \frac{v_1^\mu}{z_1^\mu} \right)^2 + 4\mu (v_2^\mu - v_1^\mu)} \right]
\geq \sqrt{\mu} \left( \sqrt{v_1^\mu} + \sqrt{v_2^\mu - v_1^\mu} \right).
\]

Notice

\[
f(v) > \begin{cases} 
\mu, & v \in (v_1^\mu, v_2^\mu), \\
0, & v \in [v_2^\mu, 0), \\
-1, & v \in [v_2^0, 0],
\end{cases}
\]

\( f(v) < 0 \) on \((v_2^0, 0)\) and

\[
0 < v + \frac{(\phi^-(v))^2}{f(v)} < v + \frac{(\phi^-(v))^2}{\mu}, \quad \forall v \in (v_1^\mu, v_2^\mu),
\]

\[
v + \frac{(\phi^-(v))^2}{f(v)} < v - (\phi^-(v))^2 < 0, \quad \forall v \in (v_2^0, 0).
\]

It follows that

\[
G^\mu(v, \phi^-(v)) < \frac{\phi^-(v)}{v + (\phi^-(v))^2 (f(v))^{-1}} = \frac{\phi^-(v) f(v)}{v f(v) + (\phi^-(v))^2},
\]

\( \forall v \in (v_1^\mu, v_2^\mu) \cup (v_2^\mu, v_2^0) \cup (v_2^0, 0) \).

In terms of (2.13), (2.14) and applying Theorem 2.1 and Remark 2.4 on \((v_1^\mu, v_2^\mu)\), we obtain \( \varphi^\mu(v) < \phi^-(v) \) on \((v_1^\mu, v_2^\mu)\). Then, since \( \phi^-(v) \) and \( \varphi^\mu(v) \)
continuously depend on $v$ and $\phi^{-}(v)$ strictly increases for $v \in [v_2^0, v_2^0]$, we have
\[
\varphi^\mu(v_2^0) = \varphi^\mu(v_2^0) \leq \phi^{-}(v_2^0) < \phi^{-}(0).
\] (2.15)
In terms of (2.14), (2.15) and applying Theorem 2.1 and Remark 2.4 on $(v_2^0, 0)$, we obtain $\varphi^\mu(v) < \phi^{-}(v)$ for any $v \in (v_2^0, 0)$, and hence,
\[
\varphi^\mu(0) \leq \phi^{-}(0).
\]
See Figure 3(a) and 3(b) for possible behavior of $\varphi^\mu(v)$.

One the other hand, let $\lambda \in (-1, 0)$ and consider the following piecewise differential equation
\[
\frac{dv}{dz} = H^\lambda(v, z) \triangleq \begin{cases} 
\frac{v}{z} + \frac{z}{\lambda}, & z \in (0, z^\lambda_3], \\
\frac{v}{z} - z, & z \in (z^\lambda_3, +\infty),
\end{cases}
\]
where
\[
z^\lambda_3 = \sqrt{-\lambda} \left( \sqrt{v_3^0} + \sqrt{v_3^0 - v_3^\lambda} \right),
\]
with initial point $(v, z) = \left(v_3^0, \sqrt{-\lambda v_3^0}\right)$. Its solution $v = \psi^\lambda(z)$ is explicitly formulated by
\[
\psi^\lambda(z) = \begin{cases} 
\frac{z^2}{\lambda} + 2\sqrt{-v_3^0} z, & z \in \left[\sqrt{-\lambda v_3^0}, z^\lambda_3\right], \\
-z^2 + \left(\frac{v_3^\lambda}{z^\lambda_3} + z^\lambda_3\right) z, & z \in (z^\lambda_3, +\infty).
\end{cases}
\]
Note that
\[
v_3^\lambda = \psi^\lambda(z^\lambda_3) = \lim_{z \to z^\lambda_3^+} \psi^\lambda(z) = \lim_{z \to z_3^\lambda} \left[-z^2 + \left(\frac{v_3^\lambda}{z_3^\lambda} + z_3^\lambda\right) z\right],
\]
and that the equation $v_3^\lambda = -z^2 + \left(\frac{v_3^\lambda}{z_3^\lambda} + z_3^\lambda\right) z$ has two real roots (counting with multiplicity) the larger of which is denoted by $\bar{z}^\lambda$ and satisfies $\sqrt{v_3^\lambda} \leq \bar{z}^\lambda < v_3^\lambda/z_3^\lambda + z_3^\lambda$. Letting $I^\lambda = \left(\sqrt{-\lambda v_3^0}, z_3^\lambda\right) \cup \left(\bar{z}^\lambda, v_3^\lambda/z_3^\lambda + z_3^\lambda\right)$, One can see that
\[
H^\lambda(\psi^\lambda(z), z) = \frac{d\psi^\lambda}{dz} < 0, \quad \forall z \in I^\lambda.
\]
Letting \( \omega^\lambda(v) \) be the inverse of \( \psi^\lambda(z) \) on \( I^\lambda \), we have
\[
\lim_{v \to v_3^\lambda} \omega^\lambda(v) = \sqrt{-\lambda v_3^0} > 0 = \lim_{v \to v_3^\lambda} \phi^+(v)
\] (2.16)
and \( \lim_{v \to v_3^\lambda, v < v_3^\lambda} \omega^\lambda(v) = \bar{z}^\lambda \geq z^\lambda = \lim_{v \to v_3^\lambda, v > v_3^\lambda} \omega^\lambda(v) \). Noting that
\[
0 > f(v) > \begin{cases} 
-1, & v \in (0, v_3^\lambda], \\
\lambda, & v \in (v_3^\lambda, v_3^0],
\end{cases}
\]
we have for any \( v \in (0, v_3^\lambda) \cup (v_3^\lambda, v_3^0) = \psi^\lambda(I^\lambda) \),
\[
\frac{v}{\omega^\lambda(v)} + \frac{\omega^\lambda(v)}{f(v)} = \frac{\psi^\lambda(z)}{z} + \frac{z}{f(\psi^\lambda(z))} < H^\lambda(\psi^\lambda(z), z) = \frac{d\psi^\lambda}{dz} < 0,
\]
and hence,
\[
\frac{\omega^\lambda(v)f(v)}{vf(v, \alpha) + (\omega^\lambda(v))^2} > \frac{1}{H^\lambda(v, \omega^\lambda(v))} = \frac{d\omega^\lambda}{dv}, \quad \forall v \in (0, v_3^\lambda) \cup (v_3^\lambda, v_3^0).
\] (2.17)
In terms of (2.16), (2.17) and applying Theorem 2.1 and Remark 2.4 on \((v_3^\lambda, v_3^0)\), we obtain \( \phi^+(v) < \omega^\lambda(v) \) on \((v_3^\lambda, v_3^0)\), and hence,
\[
\phi^+(v_3^\lambda) \leq \lim_{v \to v_3^\lambda, v > v_3^\lambda} \omega^\lambda(v) = z_3^\lambda \leq \bar{z}^\lambda = \lim_{v \to v_3^\lambda, v < v_3^\lambda} \omega^\lambda(v).
\] (2.18)
If \( \phi^+(v_3^\lambda) = z_3^\lambda = \bar{z}^\lambda \), then
\[
\frac{v_3^\lambda}{\phi^+(v_3^\lambda)} + \frac{\phi^+(v_3^\lambda)}{f(v_3^\lambda)} = \frac{v_3^\lambda}{z_3^\lambda} + \frac{z_3^\lambda}{f(v_3^\lambda)} < \frac{v_3^\lambda}{z_3^\lambda} - z_3^\lambda = \lim_{z \to z_3^\lambda, z > z_3^\lambda} H(\psi^\lambda(z), z) \leq 0.
\]
Hence,
\[
\frac{d\phi^+}{dv} \bigg|_{v = v_3^\lambda} > \lim_{z \to z_3^\lambda, z > z_3^\lambda} \frac{1}{H(\psi^\lambda(z), z)} = \lim_{v \to v_3^\lambda, v < v_3^\lambda} \frac{1}{H^\lambda(v, \omega^\lambda(v))} = \lim_{v \to v_3^\lambda, v < v_3^\lambda} \frac{d\omega^\lambda}{dv} \quad \text{if} \quad \phi^+(v_3^\lambda) = z_3^\lambda = \bar{z}^\lambda.
\] (2.19)
In terms of (2.17), (2.18), (2.19) and applying Theorem 2.1 and Remark 2.4 on \((0, v_3^\lambda)\), we obtain \( \phi^+(v) < \omega^\lambda(v) \) for any \( v \in (0, v_3^\lambda) \), and hence,
\[
\phi^+(0) \leq \lim_{v \to 0} \omega^\lambda(v) = z_3^\lambda + \frac{v_3^\lambda}{z_3^\lambda}.
\]
(a) $\varphi^\mu(0) = 0 \geq z_2^\mu + v_2^0/z_2^\mu$

(b) $\varphi^\mu(0) = z_2^\mu + v_2^0/z_2^\mu > 0$

(c) $z_3^\lambda = \bar{z}$

(d) $z_3^\lambda < \bar{z}$

Figure 3: Possible cases for $\varphi^\mu$ and $\psi^\lambda$ with different $\mu$ and $\lambda$, respectively.

See Figure 3(c) and 3(d) for possible behavior of $\psi^\lambda(z)$.

Now from the structures of $\varphi^\mu(v)$ and $\psi^\lambda(z)$ ($\omega^\lambda(v)$), we see that

$$
\phi^- (0; \alpha) \geq \varphi^\mu(0) = \max \left\{ z_2^\mu + \frac{v_2^0}{z_2^\mu}, 0 \right\}
$$

$$
\geq \max \left\{ \sqrt{\mu} \left( \sqrt{-v_1^\mu + \sqrt{v_1^\mu - v_1^0}} + \frac{v_2^0}{\sqrt{\mu} \left( \sqrt{-v_1^\mu + \sqrt{v_1^\mu - v_1^0}} \right)} \right), 0 \right\},
$$

$$
\phi^+ (0; \alpha) \leq \omega^\lambda(0) = z_3^\lambda + \frac{v_3^\lambda}{z_3^\lambda}
$$

$$
= \sqrt{-\lambda} \left( \sqrt{v_3^0 + \sqrt{v_3^0 - v_3^\lambda}} + \frac{v_3^\lambda}{\sqrt{-\lambda} \left( \sqrt{v_3^0 + \sqrt{v_3^0 - v_3^\lambda}} \right)} \right).
$$

When $\alpha = -3/\sqrt{2}$, the roots of $f(v) = 0$ are

$$
v_1^0 \left( -\frac{3}{\sqrt{2}} \right) = -\frac{\sqrt{3} + 1}{\sqrt{2}}, \quad v_2^0 \left( -\frac{3}{\sqrt{2}} \right) = -\frac{1}{\sqrt{2}}, \quad v_3^0 \left( -\frac{3}{\sqrt{2}} \right) = \frac{-\sqrt{3} - 1}{\sqrt{2}}.
$$
By choosing $\mu = \frac{11}{16}$ ($< f(2\alpha/3) = 1$) and $\lambda = \frac{-9}{16}$, we have

$$v^\frac{11}{16}_1 \left( -\frac{3}{\sqrt{2}} \right) = -\frac{3\sqrt{5} + 3}{4\sqrt{2}}, \quad v^\frac{11}{16}_2 \left( -\frac{3}{\sqrt{2}} \right) = -\frac{3}{2\sqrt{2}}, \quad v^{-\frac{9}{16}}_3 \left( -\frac{3}{\sqrt{2}} \right) = \frac{1}{2\sqrt{2}}.$$ 

Then from (2.20), we have

$$\phi^- \left( 0; -\frac{3}{\sqrt{2}} \right) \geq \sqrt{\mu} \left( \sqrt{-v^\mu_1} + \sqrt{v^\mu_2} - v^\mu_1 \right) + \frac{v^0_2}{\sqrt{\mu} \left( \sqrt{-v^\mu_1} + \sqrt{v^\mu_2} - v^\mu_1 \right)} \bigg|_{\mu = \frac{11}{16}, \alpha = -\frac{3}{3\sqrt{2}}}$$

$$= \sqrt{2\sqrt{33}} \left( \frac{\sqrt{\sqrt{5} + 2}}{8} - \frac{4}{33} \sqrt{\sqrt{5} - 2} \right) \approx 1.4358$$

and

$$\phi^+ \left( 0; -\frac{3}{\sqrt{2}} \right) \leq \sqrt{-\lambda} \left( \sqrt{v^\lambda_3} + \sqrt{v^\lambda_2} - v^\lambda_3 \right) + \frac{v^\lambda_3}{\sqrt{-\lambda} \left( \sqrt{v^\lambda_3} + \sqrt{v^\lambda_2} - v^\lambda_3 \right)} \bigg|_{\lambda = \frac{-9}{16}, \alpha = -\frac{3}{3\sqrt{2}}}$$

$$= \left( \frac{25}{12} - \frac{7}{24} \sqrt{3 - \sqrt{3}} \right) \frac{\sqrt{3 - 1}}{\sqrt{2}} \approx 1.3377.$$ 

Hence,

$$\phi^- \left( 0; -\frac{3}{\sqrt{2}} \right) > \phi^+ \left( 0; -\frac{3}{\sqrt{2}} \right). \quad (2.21)$$

\[ \square \]

3. Uniqueness and existence range of the limit cycle

We first establish the relation between the limit cycle of system (1.1) and the behavior of the separatrices of system (2.1).

**Lemma 3.1.** Assume that $\alpha < -\frac{3}{\sqrt{4}}$.

(i) If the separatrices in the transferred system (2.1) satisfy $\phi^- \left( 0; \alpha \right) < \phi^+ \left( 0; \alpha \right)$, i.e., the case shown in Figure 2(a) or 2(b), then system (1.1) has at least one limit cycle.

(ii) If system (1.1) has at least one limit cycle and the outermost one is externally unstable, then $\phi^- \left( 0; \alpha \right) \leq \phi^+ \left( 0; \alpha \right)$, i.e., the case shown in Figure 2(a), 2(b) or 2(c).
Proof. (i) From Proposition 2.2 we have \( v^+ = -\infty \). For system (2.1) in the \((v, z)\) plane, \( L^+ \) positively approaches \( P_3 \) along the line \( v = v^0_3 \), crosses positive \( z \)-axis and negatively approaches infinity in the second quadrant. Then for system (1.1) in the \((x, y)\) plane, \( L^+ \) positively approaches infinity along the line \( y = v^0_3 x \) in the upper half plane, crosses positive \( y \)-axis, and negatively approaches negative \( x \)-axis (see Figure 4(a)). Since \((\pm \infty, 0)\) are not critical points of system (1.1) and the system is central symmetry, \( L^+ \) must spiral outward the origin. On the other hand, the origin is a stable focus. Therefore, the \( \alpha \)-limit set of \( L^+ \) must be a limit cycle.

(ii) Assume that \( \phi^-(0; \alpha) > \phi^+(0; \alpha) \) though the system has at least one limit cycle and the outermost one is externally unstable. From Proposition 2.2 we have \( v^- = +\infty \). For system (2.1) in the \((v, z)\) plane, \( L^- \) negatively approaches \( P_1 \) along the line \( v = v^0_1 \), crosses positive \( z \)-axis and positively approaches infinity in the first quadrant. Then for system (1.1) in the \((x, y)\) plane, \( L^- \) negatively approaches infinity along the line \( y = v^0_1 x \) in the upper half plane, crosses positive \( y \)-axis, and positively approaches positive \( x \)-axis (see Figure 4(b)). Since \((\pm \infty, 0)\) are not critical points of system (1.1) and the system is central symmetry, \( L^- \) must spiral toward the outermost and externally unstable limit cycle, which is a contradiction. Hence, \( \phi^-(0; \alpha) \leq \phi^+(0; \alpha) \).
Then with the help of known results, we immediately obtained that system (2.1) has a unique heteroclinic separatrix in the upper half plane when $\alpha$ varies.

**Theorem 3.1.** There is a unique $\alpha^* \in (-\infty, -3/\sqrt{4})$ such that $\phi^-(0;\alpha^*) = \phi^+(0;\alpha^*)$, i.e. the case shown in Figure 2(c). Furthermore, we have

$$-3 \sqrt{2} (\approx -2.3811) < \alpha^* \leq \frac{-3 \sqrt{6 \sqrt{3} - 9}}{\sqrt{4}} (\approx -2.1103).$$

**Proof.** Recall that from Lemma 1.1(ii) and Lemma 1.2, system (1.1) has a unique and unstable limit cycle when $0 > \alpha > -3 \sqrt{6 \sqrt{3} - 9}/\sqrt{4} \approx -2.1103$. Since Lemma 2.3 gives the continuity of $\phi^-$ and $\phi^+$ with respect to $\alpha$, then from Lemma 3.1(ii) we have

$$\phi^-(0; -3 \sqrt{6 \sqrt{3} - 9}/4) \leq \phi^+(0; -3 \sqrt{6 \sqrt{3} - 9}/4).$$

Together with Lemma 2.5 which says

$$\phi^-(0; -3/\sqrt{2}) > \phi^+(0; -3/\sqrt{2}),$$

there is a $\alpha^* \in (-3 \sqrt{2}, -3 \sqrt{6 \sqrt{3} - 9}/\sqrt{4})$ such that

$$\phi^-(0; \alpha^*) = \phi^+(0; \alpha^*).$$

The uniqueness of $\alpha^*$ on $(-\infty, -3/\sqrt{4})$ is deduced from Lemma 2.4. \qed

**Remark 3.1.** By a numerical computation, we have $\alpha^* \approx -2.198$.

At last, we give the uniqueness of the limit cycle of system (1.1) after introducing two lemmas.

**Lemma 3.2 ([4, Lemma 4.1(iv)])**. If for some $\bar{\alpha} < 0$ system (1.1) has no limit cycles, the same holds for any $\alpha \leq \bar{\alpha}$.

Since system (1.1) is a semi-complete family of rotated vector fields (mod $xy = 0$) with respect to $\alpha$, we have the following result which is deduced from [7, Lemma 4 and Remark 1].

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Lemma 3.3. Given any negatively oriented, externally unstable limit cycle \( \Gamma(\alpha_0) \) of system (1.1) with \( \alpha = \alpha_0 \), there exists an outer neighborhood \( N \) of \( \Gamma(\alpha_0) \) such that through each point of \( N \) there passes a limit cycle \( \Gamma(\alpha) \) of the system with \( \alpha < \alpha_0 \).

Theorem 3.2. System (1.1) has a unique limit cycle when \( \alpha \in (\alpha^*, 0) \), where \( \alpha^* \) is given in Theorem 3.1, while it has no limit cycles when \( \alpha \in (-\infty, \alpha^*) \cup [0, +\infty) \).

Proof. Since from Lemma 1.2 the system has a limit cycle for \( \alpha \in \left( -\frac{3}{\sqrt{6}} - \frac{9}{\sqrt{4}} \right) \), we can denote \( \bar{\alpha} \) by

\[
\bar{\alpha} = \inf \{ \alpha \mid \text{system (1.1) has at least one limit cycle} \}.
\]

Lemma 3.2 implies that the system has at least one limit cycle for any \( \alpha \in (\bar{\alpha}, 0) \). Apparently, there are three cases: \( \bar{\alpha} > \alpha^* \), \( \bar{\alpha} < \alpha^* \) and \( \bar{\alpha} = \alpha^* \). Recall that from Theorem 3.1, \( \alpha^* \) is unique and satisfies \( -3/\sqrt{2} (\approx -2.3811) < \alpha^* \leq -3/\sqrt{3} - 9/\sqrt{4} (\approx -2.1103) \).

If \( \bar{\alpha} > \alpha^* \), take \( \beta \in (\alpha^*, \bar{\alpha}) \). From Lemma 2.4, we have \( \phi^- (0; \beta) < \phi^- (0; \alpha^*) = \phi^+ (0; \alpha^*) < \phi^+ (0; \beta) \). Then form Lemma 3.1(i), system (1.1) with \( \alpha = \beta \) has at least one limit cycle, which contradicts to the definition of \( \bar{\alpha} \).

If \( \bar{\alpha} < \alpha^* \), take \( \beta \in \left( \max \{ \bar{\alpha}, -3/\sqrt{2} \}, \alpha^* \right) \). From Lemma 2.4, we have \( \phi^- (0; \beta) > \phi^- (0; \alpha^*) = \phi^+ (0; \alpha^*) > \phi^+ (0; \beta) \). From Lemma 1.1(ii) and the definition of \( \bar{\alpha} \), system (1.1) with \( \alpha = \beta \) has a unique and unstable limit cycle. Then from Lemma 3.1, we have \( \phi^- (0; \beta) \leq \phi^+ (0; \beta) \), which is a contradiction.

From the above discussions, we have \( \bar{\alpha} = \alpha^* \). At last, we show that the system has no limit cycles when \( \alpha = \alpha^* \). Otherwise, the limit cycle, denoted by \( \Gamma \), must be unique and unstable from Lemma 1.1(ii), and it is negatively oriented. From Lemma 3.3, system (1.1) with some \( \alpha < \alpha^* \) has a limit cycle passing through some outer neighborhood of \( \Gamma \), which contradicts to the definition of \( \bar{\alpha} = \alpha^* \).

Since from Lemma 1.1(i) system (1.1) has no limit cycles when \( \alpha \geq 0 \), it has a unique limit cycle when \( 0 > \alpha > \bar{\alpha} = \alpha^* \approx -2.198 \), while it has no limit cycles for the else region.

\[ \square \]
References

[1] A. Cima, A. Gasull, F. Mañosas, Cyclicity of a family of vector fields, J. Math. Anal. Appl. 196 (3) (1995) 921–937.

[2] F. Dumortier, J. Llibre, J. C. Artés, Qualitative theory of planar differential systems, Springer, 2006.

[3] A. Gasull, Some open problems in low dimensional dynamical systems, SeMA Journal (2021) 1–37.

[4] A. Gasull, H. Giacomini, Upper bounds for the number of limit cycles through linear differential equations, Pacific J. Math. 226 (2) (2006) 277–296.

[5] H. Giacomini, M. Grau, Transversal conics and the existence of limit cycles, J. Math. Anal. Appl. 428 (1) (2015) 563–586.

[6] A. McNabb, Comparison theorems for differential equations, J. Math Anal. Appl. 119 (1-2) (1986) 417–428.

[7] L. M. Perko, Rotated vector fields, J. Differ. Equ. 103 (1) (1993) 127–145.