Simplicial Burnside ring

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Abstract

This paper develops links between the Burnside ring of a finite group $G$ and the slice Burnside ring. The goal is to gain a better understanding of ghost maps, idempotents, prime spectrum of these Burnside rings and connections between them.

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1 Introduction

Starting from questions in representation theory and homotopy theory, the investigation of biset functors received considerable attention over the last decades. We refer to [2], which covered this subject in detail. We recall some key definitions relevant for the current paper, the presentation follows recent articles very closely. Let $G$ be a finite group. The Grothendieck ring constructed from the category of $G$-sets is denoted by $B(G)$ and is called the Burnside ring of $G$. If $X$ is a finite $G$-set, let $[X]$ be its image in $B(G)$. Additively, $B(G)$ is the free abelian group on isomorphism classes of transitive $G$-sets. Equivalently, an additive $\mathbb{Z}$-basis is given by the $[G/H]$ where $H$ runs through a set $[C(G)]$ of representatives of conjugacy classes of subgroups of $G$. The multiplication comes from the decomposition of $G/H \times G/K$ into orbits. The ring $B(G)$ is commutative with unit $[G/G]$. If $H$ is a subgroup of $G$, then there is a unique linear form $\phi_H : B(G) \to \mathbb{Z}$ such that $\phi_H([X]) = |X^H|$ for any $G$-set $X$. It is clear moreover that $\phi_H$ is a ring homomorphism, and Burnside’s theorem ([4] Chap. XII Theorem 1) is equivalent to the ghost map $\Phi = \prod_{[H\in[C(G)]]} \phi_H^G : B(G) \to \prod_{[H\in[C(G)]]} \mathbb{Z}$ being injective. The cokernel of the ghost map is finite, and has been explicitly described by Dress [5]. In particular, the ghost map $Q\Phi : QB(G) \to \prod_{[H\in[C(G)]]} \mathbb{Q}$ is an algebra isomorphism, where $QB(G) = \mathbb{Q} \otimes_\mathbb{Z} B(G)$. This shows that $QB(G)$ is a split semisimple commutative $\mathbb{Q}$-algebra. Explicit formulas for its primitive idempotents have been given by Gluck [7] and independently by Yoshida [14]. Andreas Dress proved in [5] that if $p$ is 0 or a prime and $I_{H,p} = \{X \in B(G) \mid \phi_H(X) \in p\mathbb{Z}\}$, then any prime ideal in $B(G)$ is of the form $I_{H,p}$ for some $H,p$. Moreover, a finite group is solvable if and only if the spectrum of its Burnside ring is connected (in the sense of Zariski’s topology), i.e. if and only if 0 and 1 are the only idempotents in $B(G)$. If $X$ is a finite $G$-set, the $\mathbb{Q}$-vector space $QX$ with basis $X$ has a natural $QG$-module structure, induced by the action of $G$ on $X$. The construction $X \mapsto QX$ maps disjoint unions of $G$-sets to direct sums of $QG$-modules, and so it induces a map $Ch : B(G) \to RQ(G)$. There are important connections between the Burnside ring and the permutation representations. This latter map, leads to an associated map $Spec : Spec(RQ(G)) \to Spec B(G)$
which is always injective (See [6]).

The slice Burnside ring \( \Xi(G) \) introduced by Serge Bouc, is built as the Grothendieck ring of the category of morphisms of finite \( G \)-sets, instead of the category of finite \( G \)-sets used to build the usual Burnside ring. It shares almost all properties of the Burnside ring. In particular, as already shown (see [3] for a more complete description), the slice Burnside ring is a commutative ring, which is free of finite rank as a \( \mathbb{Z} \)-module. The investigation of the slices, that is the pairs of groups \((T,S)\) such that \( S \) is a subgroup of \( T \), is a central subject in the study of the slice Burnside ring. One reason for considering slices is that if \( f : X \to Y \) is a morphism of finite \( G \)-sets then in \( \Xi(G) \)

\[
[ X \xrightarrow{f} Y ] = \sum_{x \in [G \setminus X]} [ G/G_x \xrightarrow{p} G/G_{f(x)} ],
\]

where square brackets to denote here the image of the isomorphism class of \( f \) in \( \Xi(G) \) and \( G_x \) denotes the stabilizer of \( x \). Thus, the group \( \Xi(G) \) is generated by the elements \([ G/S_1 \xrightarrow{p} G/S_0 ]\) where \((S_1, S_0)\) runs through a set \([\Pi(G)]\) of representatives of conjugacy classes of slices of \( G \). One can show that this generating set is actually a basis of the slice Burnside group. There is an analogue of Burnside's theorem. After tensoring with \( \mathbb{Q} \), the slice Burnside ring becomes a split semisimple \( \mathbb{Q} \)-algebra, and an explicit formula for his primitive idempotents can be stated. The prime spectrum of this ring has been described, and Dress's characterization of solvable groups in terms of the connectedness of the spectrum of the Burnside ring can be generalized as well.

In this paper, we introduce a ring \( B_n(G) \), for \( n \in \mathbb{N} \) such that \( B_0(G) = B(G) \) and \( B_1(G) = \Xi(G) \). We extend to \( B_n(G) \) most of the properties recalled above in the case \( n = 0 \) or \( n = 1 \).

Recall that any poset \( \Pi \) can be treated as a category in which the objects are the elements of \( \Pi \) and in which there is exactly one morphism \( x \to y \) if \( x \leq y \) and there are no other morphisms. The nerve of \( \Pi \) is then the same as the ordered simplicial complex associated (that is the vertices of \( \Pi_n \) are the objects of \( \Pi \) and the \( n \)-simplices are the chains of objects of \( \Pi \) of length \( n \), with face maps given by \( d_i(S_0, \ldots, S_n) = (S_0, \ldots, \hat{S}_i, \ldots, S_n) \), where as usual, the term \( \hat{\cdot} \) denotes a term that is being omitted and the degeneracy maps given by \( s_j(S_0, \ldots, S_n) = (\hat{S}_0, \ldots, S_j, \ldots, S_n) \). In particular, if we consider the collection of all subgroups of \( G \) ordered by inclusion, we get the nerve category \( \Pi_n(G) \) where elements in \( \Pi_n(G) \), whose are just chains

\[
S : S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n
\]

will be called \( n \)-slice, this is the simplicial complex considered in [11] for \( p \)-groups. Our ring \( B_n(G) \) has basis the set of conjugacy classes of \( n \)-slices, with multiplication given by

\[
(S_0, \ldots, S_n) \cdot (T_0, \ldots, T_n) = \sum_{g \in [G \setminus G/T_0]} (S_0 \cap g T_0, \ldots, S_n \cap g T_n).
\]

This paper is organized as follows:

In Sect. 2, we examine the Grothendieck group of the nerve of the skeleton of the category of finite \( G \)-sets (we abuse for using the term nerve of \( G\text{-Set} \)). It turns out that the obtained group \( B_n(G) \), called the \( n \)-simplicial group, is very similar to the classical Burnside group. It is worthwhile to discover whether these well-known properties of the Burnside ring characterize \( B_n(G) \), since then will known regard this ring as a "geometric realization" of some simplicial \( G \)-set. Hence in this part we deepen the links between the classical Burnside rings and the slice Burnside rings.

In Sect. 3, we establish that the \( n \)-simplicial Burnside ring embeds in a product
of copies of the integers, via a \textit{ghost map}, and this map has a finite cokernel. It turns out that $B_n(G)$ is a commutative semisimple algebra after tensoring with $\mathbb{Q}$, isomorphic to a direct sum indexed by $[\Pi_n(G)]$ of copies of $\mathbb{Q}$. In sect. 4. we give an explicit formula for the primitive idempotents of $QB_n(G)$. Sect.5 is devoted to the study of the prime spectrum of $B_n(G)$ by extending Dress’s characterization of solvable groups for $B(G)$. The last section examines the Green biset functor structure of $B_n$.

2 Simplicial Burnside group

For $G$ a finite group, let $C^G_n$ be the \textit{nerve of the category $G$Set} of finite $G$-sets (see [9] for more details, [8] P.177), that is, the simplicial set whose $n$-simplices are diagrams $$c^G_n = \{ X^f_n := X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots X_{n-1} \xrightarrow{f_n} X_n \}$$ where the $X_i$ are $G$-sets and the $f_i$ are morphisms of $G$-sets, and the degeneracy $s_i$ and face $d_i$ \footnote{Normally, we might be careful to label the face maps from $c^G_n$ to $c^G_{n-1}$ as $d^G_{n,0}, \ldots, d^G_{n,n}$, similarly for the degeneracy maps, but this is rarely done in practice.} maps are defined by including an identity $X_i \xrightarrow{id} X_i$, and leaving out $X_0$ if $i = 0$, contracting $X_{i-1} \xrightarrow{f_{i-1}} X_i \xrightarrow{f_{i+1}} X_{i+1}$ to $X_{i-1} \xrightarrow{f_{i+1} \circ f_i} X_{i+1}$ if $0 < i < n$, leaving $X_n$ if $i = n$, respectively. Then the following identities may be verified directly

\begin{align}
    d_j \circ d_i &= d_{i+1} \circ d_j \quad \text{for} \quad i \leq j \\
    s_j \circ s_i &= s_{j+1} \circ s_i \quad \text{for} \quad i \leq j \\
    d_i \circ s_j &= \begin{cases} 
        s_{j-1} \circ d_i & \text{if} \quad i < j, \\
        \text{id}_{[n]} & \text{if} \quad i = j, j + 1, \\
        s_j \circ d_{i-1} & \text{otherwise} 
    \end{cases}
\end{align}

\textbf{Notation 2.1.} Let $C^G_n$ be the category defined as follows:

\begin{itemize}
    \item The objects of $C^G_n$ are sequences $X^f_n : X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots X_{n-1} \xrightarrow{f_n} X_n$ of morphisms of $G$-sets called $(n,G)$-simplices.
    \item If $X^f_n$ and $Y^g_n$ are objects of $C^G_n$, a \textit{morphism} from $X^f_n$ to $Y^g_n$ is a family $(\mu_i : X_i \rightarrow Y_i)_{0 \leq i \leq n}$ of morphisms of $G$-sets such that $\mu_i \circ f_i = g_i \circ \mu_{i-1}$, for $i = 1, \ldots, n$.
    \item Morphisms of $(n,G)$-simplices compose in the obvious way (coordinate-wise).
\end{itemize}

\textbf{Proposition 2.2.} For a non-negative integer $n$, the category $C^G_n$ has finite products $\times$ and coproducts $\sqcup$ induced by those of the category of finite $G$-sets, respectively. It has also an initial object $\emptyset = \{ \emptyset \rightarrow \emptyset \rightarrow \cdots \rightarrow \emptyset \rightarrow \emptyset \}$.

\textit{Proof.} This is straightforward. \hfill \square

\textbf{Definition 2.3.} Let $G$ be a finite group. A $n$-slice of $G$ is a $(n+1)$-tuple $(S_0, S_1, \ldots, S_n)$ of subgroups of $G$, with $S_{i-1} \leq S_i$, $\forall i \in [1, \ldots, n]$. It is helpful to refer the tuple as $\mathcal{S}$, in which each composite $S_i$ as representing the group $S_i$. The set of all $n$-slices of $G$ will be denoted by $\Pi_n(G)$. 

Definition 2.4. For any nerve $C_\bullet$, define a pre-ordering $\preceq$ of $\text{Ob}(C_n)$ by $A \preceq B$ if $\text{Hom}_{C_n}(A,B) \neq \emptyset$ and an equivalence $\cong$ on $\text{Ob}(C_n)$ by $A \cong B$ if and only if $A \preceq B$ and $B \preceq A$. So, on $\Pi_n(G)$, we have the following relation

$$\bar{T} := (T_0, \ldots, T_n) \preceq \bar{S} := (S_0, \ldots, S_n) \iff T_i \leq S_i, \quad \forall i = 0, \ldots, n$$

Recall that in $\mathcal{G}\text{Set}$, $X$ is indecomposable if and only if $X$ is simple and any simple $G$-set is isomorphic to $G/H$ for some subgroup $H$ of $G$.

Notation 2.5.

- Any $n$-slice $\mathcal{S}$ of $G$ gives rise to an $(n, G)$-simplex

$$(G\mathcal{S}n) := (G/S_0 \xrightarrow{p_1} G/S_1 \xrightarrow{p_2} \ldots G/S_{n-1} \xrightarrow{p_n} G/S_n),$$

where $p_i$ are the projection morphisms.

The $(n, G)$-simplices $(G\mathcal{S}n)$ are indecomposable in the sense that $(G\mathcal{S}n) = \mathcal{Y}_n^G \cup \mathcal{Z}_n^G \Rightarrow \mathcal{Y}_n^G = \emptyset$ or $\mathcal{Z}_n^G = \emptyset$.

- For a $(n, G)$-simplex $\mathcal{X}_n^I$, we set

$$\phi_{\mathcal{S}}(\mathcal{X}_n^I) := |\text{Hom}_{\mathcal{C}_n}(G\mathcal{S}n, \mathcal{X}_n^I)|,$$

the number of elements in the set $\text{Hom}_{\mathcal{C}_n}(G\mathcal{S}n, \mathcal{X}_n^I)$.

Proposition 2.6. Let $\mathcal{X}_n^I, \mathcal{Y}_n^G$ be $(n, G)$-simplices

$$\phi_{\mathcal{S}}(\mathcal{X}_n^I) = \left| f_1^{-1} \left( f_2^{-1} \left( \ldots f_{n-1}^{-1} \left( f_n^{-1}(x_n^{S_n}) \right) \right) \ldots \right) \right|,$$

in particular, for any $n$-slice $\bar{T}$, one has

$$\phi_{\mathcal{S}}((G\bar{T}n)) = ||g \in G/T_0 | \bar{S} \preceq \bar{T}||.$$

Proof. Observe that for any $i = 0, \ldots, n$, there is a bijection between $\text{Hom}(G/S_i, X_i)$ and the set $\mathcal{X}_n^I := \{ x \in X_i | gx = x \text{ for all } g \in S_i \}$. Indeed, each $f$ in $\text{Hom}(G/S_i, X_i)$ maps the element $S_i \subseteq G/S_i$ onto an element $x_i \in X_i$ which is invariant by $S_i$. Further, $f$ is completely determined by $x_i$, since $f(gS_i) = gx_i$ for all $g \in G$. The correspondence $f \mapsto x_i$ gives the desired bijection. Therefore, the set $\text{Hom}(G\mathcal{S}n, \mathcal{X}_n^I)$ is in bijection with the set

$$\{(x_0, x_1, \ldots, x_n) \in X_0^{S_0} \times X_1^{S_1} \times \ldots \times X_n^{S_n} | f_1(x_0) = x_1; \ldots; f_n(x_{n-1}) = x_n\},$$

where the last equalities follow from the commutativity of the diagram

$$\begin{array}{cccc}
G/S_0 & \xrightarrow{p_1} & G/S_1 & \xrightarrow{p_2} & \ldots & \xrightarrow{p_{n-1}} & G/S_{n-1} & \xrightarrow{p_n} & G/S_n \\
X_0 & \xrightarrow{f_1} & X_1 & \xrightarrow{f_2} & \ldots & \xrightarrow{f_{n-1}} & X_{n-1} & \xrightarrow{f_n} & X_n
\end{array}$$

If we take $X_i = G/T_i$ for $i = 0, \ldots, n$, we note that the left coset $gT_i$ is $S_i$-invariant if and only if $S_i gT_i = gT_i$, that is, $g^{-1} S_i g \leq T_i$. This establishes that

$$(G/T_i)^{S_i} = \{gT_i | g \in G, g^{-1} S_i g \leq T_i\}$$
Corollary 2.7. Let $X^f_n$, $Y^g_n$ be $(n, G)$-simplices

1. If $X^f_n$ and $Y^g_n$ are isomorphic then $\phi_S(X^f_n) = \phi_S(Y^g_n)$.

2. Let $p$ be a prime and let $\mathcal{S}$ be a $n$-slice of $G$. If $P$ is a $p$-subgroup of $N_G(\mathcal{S}) = \cap_{i \neq 0} N_G(S_i)$ and $P\mathcal{S}$ denotes the $n$-slice $(PS_0, \ldots, PS_n)$, then

   $$\phi_S(X^f_n) \equiv \phi_{P\mathcal{S}}(X^f_n) \pmod{p},$$

   for any $(n, G)$-simplex $X^f_n$.

3. Two indecomposable $(n, G)$-simplices $(G/\mathcal{S})_n$ and $(G/\mathcal{T})_n$ are isomorphic if and only if the $n$-slices $\mathcal{S}$ and $\mathcal{T}$ are conjugate (we set $\mathcal{S} \equiv_G \mathcal{T}$).

4. If $\mathcal{S}$ and $\mathcal{T}$ are two $n$-slices, then $\phi_{\mathcal{S}}(X^f_n) \leq \phi_{\mathcal{T}}(X^f_n)$ for any $(n, G)$-simplex $X^f_n$ if and only if $\mathcal{T} \leq_G \mathcal{S}$.

5. For any two indecomposable $(n, G)$-simplices $(G/\mathcal{S})_n$ and $(G/\mathcal{T})_n$, one has $\phi_S((G/\mathcal{S})_n)$ divides $\phi_{\mathcal{T}}((G/\mathcal{S})_n)$.

Proof.

1. It is clear, since any isomorphism of $G$-sets $\mu_i : X_i \rightarrow Y_i$ induces a bijection $X^f_n \rightarrow Y^f_n$ on the sets of fixed points by any subgroup $S$ of $G$.

2. For any $(n, G)$-simplex $X^f_n$, the set

   $$\text{Inv}_S(X^f_n) := f_1^{-1} \left( f_2^{-1} \left( \cdots f_{i-1}^{-1} \left( f_{i+1}^{-1} \left( f_n^{-1}(X_n)^{S_{n-1}} \cdots S_1^S \right) \right) \cdots \right) \right)$$

   is invariant by $N_G(\mathcal{S})$, and so

   $$|\text{Inv}_S(X^f_n)| = |f_1^{-1} \left( \cdots f_{i-1}^{-1} \left( f_{i+1}^{-1} \left( f_n^{-1}(X_n)^{S_{n-1}} \cdots S_1^S \right) \right) \cdots \right) \pmod{p},$$

   and moreover

   $$f_1^{-1} \left( \cdots f_{i-1}^{-1} \left( f_{i+1}^{-1} \left( f_n^{-1}(X_n)^{S_{n-1}} \cdots S_1^S \right) \right) \cdots \right) \equiv \pmod{f_1^{-1} \left( \cdots f_{i-1}^{-1} \left( f_{i+1}^{-1} \left( f_n^{-1}(X_n)^{P_{S_0}} \cdots P_{S_1}^S \right) \right) \cdots \right) \pmod{p}.}

3. If $(G/\mathcal{S})_n \equiv (G/\mathcal{T})_n$ then $\text{Hom}(G/\mathcal{S}, G/\mathcal{T}) \neq \varnothing$ and so $\mathcal{S} \leq_G \mathcal{T}$. Therefore $\mathcal{S} = G \mathcal{T}$ by symmetry. Conversely if $\mathcal{S} = G \mathcal{T}$, for example $\mathcal{S} = \mathcal{T}$ with $g \in G$, then there exists an isomorphism of $(n, G)$-simplices $(\mu_i : G/T_i \rightarrow G/S_i)_{0 \leq i \leq n}$, given by

   $$\mu_i(gT_i) = \mu_i(g^{-1} S_i g) = g^i g^{-1} S_i, \text{ for } g \in G.$$

   So $(G/\mathcal{S})_n \equiv (G/\mathcal{T})_n$ as $(n, G)$-simplices if and only $\mathcal{S} = G \mathcal{T}$.
4. If $\phi_\mathcal{S}(X_n^I) \leq \phi_\mathcal{T}(X_n^I)$ for any $(n, G)$-simplex, then in particular $\phi_\mathcal{T}((G/\mathcal{S})_n) \neq 0$ since $\phi_\mathcal{S}(G/\mathcal{S}) \neq 0$, and so $\mathcal{T} \leq G \mathcal{S}$. On the other hand, if $\mathcal{T} \leq G \mathcal{S} = \mathcal{K}$, then 
\[ \phi_\mathcal{S}(X_n^I) = |\text{Inv}_\mathcal{S}(X_n^I)| = |\text{Inv}_\mathcal{K}(X_n^I)| \leq |\text{Inv}_\mathcal{T}(X_n^I)| = \phi_\mathcal{T}(X_n^I). \]

5. Consider the action of $N_G(\mathcal{S})$ on $G/S_0$ defined by 
\[ x \cdot gS_0 = gx^{-1}S_0 \]
for $x \in N_G(\mathcal{S})$ and $gS_0 \in G/S_0$. Then $S_0$ acts trivially on $G/S_0$ and so it becomes a left $N_G(\mathcal{S})/S_0$. Moreover, $N_G(\mathcal{S})/S_0$ acts freely on $G/S_0$. Note that for any $(n, G)$-slice $\mathcal{T}$, the set 
\[ \text{Inv}_\mathcal{T}((G/\mathcal{S})_n) := \{ gS_0 \mid \mathcal{T} \cdot gS = gS \} \]
is an $N_G(\mathcal{S})/S_0$-subset of $G/S_0$. So $N_G(\mathcal{S})/S_0$ acts freely on $\text{Inv}_\mathcal{T}((G/\mathcal{S})_n)$, and so $|N_G(\mathcal{S})/S_0|$ divides $|\text{Inv}_\mathcal{T}((G/\mathcal{S})_n)|$. 
\[ \Box \]

**Definition 2.8.** A $(n, G)$-simplex $X_n^I$ is called $i$-sliceable with $i$ in $[0, \ldots, n]$ if $X_i = A_i \cup B_i$ as disjoint union of two non-empty $G$-sets.

**Remark 2.9.** Note that for any $i$-sliceable $(n, G)$-simplex $X_n^I$, one has $(n, G)$-simplices 
\[ A_n^I: A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots A_i \xrightarrow{f_{i+1}} X_{i+1} \xrightarrow{f_{i+2}} \cdots X_{n-1} \xrightarrow{f_n} X_n, \]
\[ B_n^I: B_0 \xrightarrow{f_1} B_1 \xrightarrow{f_2} \cdots B_i \xrightarrow{f_{i+1}} X_{i+1} \xrightarrow{f_{i+2}} \cdots X_{n-1} \xrightarrow{f_n} X_n, \]
defined inductively by 
\[ A_{j-1} := f_{j-1}^{-1}(A_j) \text{ and } B_{j-1} := f_{j-1}^{-1}(B_j), \]
for any $1 \leq j \leq i$.

We set $[X_n^I, A_n^I, B_n^I]$, to denote the corresponding triple. If $\Gamma$ denotes the class of such triple $(X_n^I, A_n^I, B_n^I)$, then $\Gamma$ is closed by isomorphism.

**Proposition 2.10.** For any $(X_n^I, A_n^I, B_n^I) \in \Gamma$ and for a $n$-slice $\mathcal{S}$ fixed, we have 
\[ \phi_\mathcal{S}(X_n^I) = \phi_\mathcal{S}(A_n^I) + \phi_\mathcal{S}(B_n^I). \] (4)

**Proof.** Let $x_n$ be an element of $X_n^S$ and $i$ be an integer such that $X_i = A_i \cup B_i$, Then, either $f_{j+1}^{-1} \circ \cdots \circ f_n^{-1}(x_n) \in A_j$ and $f_{j+1}^{-1} \circ \cdots \circ f_n^{-1}(x_n) \in A_j$ for any $j < i$, or $f_{j+1}^{-1} \circ \cdots \circ f_n^{-1}(x_n) \in B_j$ and $f_{j+1}^{-1} \circ \cdots \circ f_n^{-1}(x_n) \in B_j$ for any $j < i$. Hence $\phi_\mathcal{S}(X_n^I) = \left| \left[ A_0 \cap f^{-1}(X_n^I) \mathcal{S} \right] \cup \left[ B_0 \cap f^{-1}(X_n^I) \mathcal{S} \right] \right| = \phi_\mathcal{S}(A_n^I) + \phi_\mathcal{S}(B_n^I).$ \( \Box \)

**Definition 2.11.** Let $G$ be a finite group.

We denote $\Omega_n(G)$ the Grothendieck group of $\mathcal{C}_n^G$ with respect to the relations $\Gamma$ that is, the quotient 
\[ \Omega_n(G) := \Omega_n(G)/\bar{\Omega}_n(G) \]
of the free abelian group $\Omega_n(G)$ on the set of isomorphism classes of $(n, G)$-simplices by the subgroup $\bar{\Omega}_n(G)$ generated by the formal differences $[X_n^I] - [A_n^I] - [B_n^I]$. 

Remark 2.12.

- The construction satisfies the following property: if \( \phi : C_n^G \to A \) is a map from \( C_n^G \) to an abelian group \( A \) given that \( \phi(\lambda^f_n) \) depends only on the isomorphism class of \( C_n^G \) and \( \phi(\lambda^f_n) = \phi(A_n^f + \phi(B_n^f)) \) for any element \( (\lambda^f_n, A_n^f, B_n^f) \) in \( \Gamma \), then there exists a unique \( \bar{\phi} : B_n(G) \to A \) such that \( \phi(\lambda^f_n) = \bar{\phi}(\lambda^f_n) \) for any \((n, G)\)-simplex \( \lambda^f_n \). Let now \((G/\bar{S})_n\) be a fixed \((n, G)\)-simplex.

The function \( \lambda^f_n \mapsto |\text{Hom}_{\bar{S}}((G/\bar{S})_n, \lambda^f_n)| \) defined on the class of \((n, G)\)-simplices (up to isomorphism), with values in \(\mathbb{Z}\) extends to a group homomorphism \( B_n(G) \to \mathbb{Z} \). So, to see that \( B_n(G) \) is non-trivial, it suffices to find \( \lambda^f_n \) with

\[
|\text{Hom}((G/\bar{S})_n, (G/\bar{S})_n)| \neq 0.
\]

we have that \( B_n(G) \neq 0 \).

- In the special where \( n = 0 \), one recovers the classical Burnside group \( B(G) \) (see [1]), and for \( n = 1 \), we have \( B_1(G) = \Xi(G) \) the slice Burnside group introduced by Bouc in [3].

Proposition 2.13. The functor \( d_j, j = 0, \ldots, n \) (resp. \( s_i, i = 0, \ldots, n-1 \)) induces a group homomorphism

\[
d_j : B_n(G) \to B_{n-1}(G) \quad (\text{resp. } s_i : B_{n-1}(G) \to B_n(G))
\]

such that the identities (1), (2), (3) hold.

Proof. Indeed, for any \((n, G)\)-simplices \( \lambda^f_n \) with decomposition \( [\lambda^f_n, A_n^f, B_n^f]_k \), i.e.

\[
A_0 \sqcup B_0 \xrightarrow{f_1} A_1 \sqcup B_1 \xrightarrow{f_2} \cdots \xrightarrow{f_k} X_{k+1} \xrightarrow{f_{k+1}} \cdots \xrightarrow{f_{n-1}} X_n
\]

then there is a decomposition of \( \hat{d}_j(\lambda^f_n) \) as \( [d_j(\lambda^f_n), d_j(A_n^f), d_j(B_n^f)]_k \) if \( k < j \) and \( [d_j(\lambda^f_n), d_j(A_n^f), d_j(B_n^f)]_{j-1} \) if \( k \geq j \). So the map which assigns an \((n, G)\)-simplex \( \lambda^f_n \) to the image of \( \hat{d}_j(\lambda^f_n) \) in \( B_{n-1}(G) \) induces, by the universal property, a well-defined map from \( B_n(G) \) to \( B_{n-1}(G) \). Similarly, there is a decomposition of \( s_j(\lambda^f_n) \) as \( [s_j(\lambda^f_n), s_j(A_n^f), s_j(B_n^f)]_k \) if \( k < j \) and \( [s_j(\lambda^f_n), s_j(A_n^f), s_j(B_n^f)]_{k+1} \) if \( k \geq j \). So \( s_j \) maps the defining relations \( B_{n-1}(G) \) to those of \( B_n(G) \) and therefore gives a well-defined map (which is a group homomorphism) from \( B_{n-1}(G) \) to \( B_n(G) \). The equalities (1), (2), (3) may be verified easily.

\[\square\]

Notation 2.14.

- Let \( \pi(\lambda^f_n) \) denote the image in \( B_n(G) \) of the isomorphism class of \( \lambda^f_n \).

- We set \( (\bar{G})_n := \pi((G/\bar{G})_n) \), for any \( n \)-slice \( \bar{G} = (G_0, G_1, \ldots, G_n) \).

- For any \((n, G)\)-simplex \( \lambda^f_n \), we denote by \( f(\lambda^f_n) \) the \((n, G)\)-simplex

\[
X_0 \xrightarrow{f_1} f_1(X_0) \xrightarrow{f_2} \cdots \xrightarrow{f_n} f_n(f_{n-1}(f_{n-2}(\ldots f_1(X_0))\ldots))
\]

Then,

\[\text{Note that the maps } d_j \text{ and } s_i \text{ should more appropriately be labeled, but we stick common practice and use } d_j \text{ and } s_i \text{ for face and degeneracy wherever we find them.} \]
Lemma 2.15. Let $X^f_n$ be a $(n, G)$-simplex. Then in the group $B_n(G)$, we have

$$\pi(X^f_n) = \sum_{x \in [G \hat{\times} X]} \langle G^f_x \rangle_G,$$

where $G^f_x$ denotes the $n$-slice $(G, G_{f_1}, \ldots, G_{f_n})$ and $G_e$ denotes the stabilizer of the element $e$.

Proof. Note first that in the group $B_n(G)$ we have that $\pi(X^f_n) = \pi(f(X^f_n))$. Indeed writing $X_n = X_n \cup \emptyset$, we get by the defining relations of $B_n(G)$ that

$$\pi(X^f_n) = \pi(\emptyset) + \pi(f^{-1}(X_1) \to f_1^{-1}(X_2) \to \cdots \to f_{n-1}^{-1}(X_n) \to f_n X_n)$$

Further,

$$\pi(f(X^f_n)) = \pi(\prod_{x \in [G \hat{\times} X]} \mathcal{O}_x \to f_1 \to \cdots \to f_{n-1} \to f_n)$$

where the last equality follows from the defining relations of $B_n(G)$. Now, the $(n, G)$-simplex $X^f_n$ is obviously isomorphic to the indecomposable $(n, G)$-simplex

$$G/G_x \to G/G_{f_1} \to \cdots \to G/G_{f_n}.$$

Since two indecomposable $(n, G)$-simplices $(G/\tilde{S})_n$ and $(G/\tilde{T})_n$ are isomorphic if and only if the $n$-slices $\tilde{S}$ and $\tilde{T}$ are conjugate we have the following corollary:

Corollary 2.16. The group $B_n(G)$ is generated by the elements $\langle \tilde{S} \rangle_G$ where $\tilde{S}$ runs through a set $[\Pi_n(G)]$ of representatives of conjugacy classes of $n$-slices of $G$.

Proposition 2.17. The product of $(n, G)$-simplices induces a commutative ring structure on $B_n(G)$ with identity $e_n := \bullet \cdots \bullet \bullet$, where $\bullet$ is a $G$-set of cardinality 1. This ring is called the $n$-simplicial ring of the finite group $G$.

Moreover, the morphisms $d_j$ ($j = 1, \ldots, n$) and $s_i$ ($i = 0, \ldots, n - 1$) in Proposition 2.13 define morphisms of rings.

Proof. We have to show that the product of $(n, G)$-simplices induces a well-defined bilinear product $B_2(G) \times B_2(G) \to B_2(G)$. Let $X^f_n$ be a $(n, G)$-simplex such that there is a decomposition $[X^f_n, A^f_n, B^f_n]$, and let $Y^g_n$ be any $(n, G)$-simplex. We set $Z^h_n := X^f_n \times Y^g_n$. Then the $(n, G)$-simplex $Z^h_n$ can be visualized as follows

$$(A_0 \times Y_0) \cup (B_0 \times Y_0) \to \cdots \to (A_1 \times Y_1) \cup (B_1 \times Y_1) \to \cdots \to (A_n \times Y_n) \cup (B_n \times Y_n).$$
since \((A_i \sqcup B_i) \times Y_i = (A_i \times Y_i) \sqcup (B_i \times Y_i)\). Hence
\[
\pi(Z^g_n) = \pi(C_0 \cup D_0 \xrightarrow{h_0} \cdots C_i \cup D_i \xrightarrow{h_i} X_{i+1} \times Y_{i+1} \cdots h_{n-1} \xrightarrow{X_n} X_n \times Y_n))
\]
\[
= \pi(C_0 \xrightarrow{h_0} \cdots C_i \xrightarrow{h_i} X_{i+1} \times Y_{i+1} \cdots h_{n-1} \xrightarrow{X_n} X_n \times Y_n))
\]
\[
+ \pi(D_0 \xrightarrow{h_0} \cdots D_i \xrightarrow{h_i} X_{i+1} \times Y_{i+1} \cdots h_{n-1} \xrightarrow{X_n} X_n \times Y_n))
\]
\[
= \pi(f_1^{-1}(A_1) \times g_1^{-1}(Y_1) \xrightarrow{h_0} f_2^{-1}(A_2) \times g_2^{-1}(Y_2) \cdots h_i \xrightarrow{X_i} A_i \times Y_i \cdots h_{n-1} \xrightarrow{X_n} X_n \times Y_n))
\]
\[
+ \pi(f_1^{-1}(B_1) \times g_1^{-1}(Y_1) \xrightarrow{h_0} f_2^{-1}(B_2) \times g_2^{-1}(Y_2) \cdots h_i \xrightarrow{X_i} A_i \times Y_i \cdots h_{n-1} \xrightarrow{X_n} X_n \times Y_n))
\]
Clearly, we have (it suffices to set \(Y_i = \emptyset\))
\[
\pi(g_1^{-1}(Y_1) \xrightarrow{g_0} g_2^{-1}(Y_2) \cdots g_i \xrightarrow{g_i} Y_i \cdots g_{n-1} \xrightarrow{g_n} Y_n) = \pi(Y_n^g).
\]
So \(\pi(Z^g_n) = \pi(A_n \times Y_n^g) + \pi(B_n \times Y_n^g)\), this shows that the product preserves the defining relations of \(B_n(G)\). Now it is obvious to see that the product is associative, commutative and admits \(e_n\) as an identity element.

Let \(Y_n^G\) be \((n,G)\)-simplex. Since \((f_{j+1} \circ f_j) \times (g_{j+1} \circ g_j) = (f_{j+1} \times g_{j+1}) \circ (f_j \times g_j)\), we have that
\[
d_j(X_n^f \times Y_n^g) = d_j(X_n^f) \times d_j(Y_n^g), \quad s_j(X_n^f \times Y_n^g) = s_j(X_n^f) \times s_j(Y_n^g)
\]
and so the induced maps \(d_j\) and \(s_j\) are ring homomorphisms. Furthermore \(d_j(e_n) = e_{n-1}\) and \(s_j(e_n) = e_{n+1}\).

**Remark 2.18.** Letting \(\Delta\) the simplicial category that is \(\Delta\) has as objects all finite ordinal numbers \([n] = \{0, 1, \ldots, n\}\) and as morphisms \(f : [n] \to [m]\) all monotone maps; that is, the maps \(f\) such that \(f(i) \leq f(j)\) if \(i < j\).

Then an induced functor \(c_n^{G,G} \to c_m^{G,G}\) for any finite \(G\) and an induced \(G\)-equivariant map \(\Pi_n(G) \to \Pi_m(G)\) and a group homomorphism \(B_n(G) \to B_m(G)\) given by,
\[
X_n \mapsto s_t^{-1} \cdots s_{i_k}^{-1} d_{i_k}^{n-k} d_{i_1}^{m-k} \cdots d_{i_h}^{m}
\]
where \(k + h = m\),

- \(n + k - h = m\),
- \(0 \leq t, \ldots, k \leq m\),
- \(0 \leq j_1, \ldots, j_h \leq n\),
- \(i_1, \ldots, i_k\) are the elements of \([m]\) not in the image of \(f\),
- \(j_1, \ldots, j_h\) are the elements of \([n]\) at which \(f\) does not increase.

**Proposition 2.19.** Using the generators of \(B_n(G)\), the multiplication is given by
\[
\langle S \rangle_G \langle T \rangle_G = \sum_{g \in [S_n \cap G/T_n]} \langle S \cap gT \rangle_G
\]
Proof. Note that each $G$-orbit of the $G$-set $(G/S_0) \times (G/T_0)$ determines a double coset $S_0gT_0$, in the following way:

$$\mathcal{O}(xS_0yT_0) \rightarrow S_0x^{-1}yT_0.$$ 

Conversely, the $G$-orbit of $(G/S_0) \times (G/T_0)$ corresponding to $S_0aT_0$ consists of all distinct pairs in the collection $\{(xS_0yT_0) \mid x^{-1}y \in S_0aT_0\}$. The stabilizer of the pair $(S_0, gT_0)$ is precisely $S_0 \cap gT_0$. Therefore, the orbit of $(G/S_0) \times (G/T_0)$ corresponding to $S_0gT_0$ is isomorphic (as $G$-set) to $G/(S_0 \cap gT_0)$, and therefore

$$(G/S_0) \times (G/T_0) \cong \bigsqcup_{g \in [S_0G/T_0]} G/(S_0 \cap gT_0)$$

On the other hand, the image of $(S_i, gT_j)$ by the map

$$(G/S_i) \times (G/T_j) \rightarrow (G/S_{i+1}) \times (G/T_{j+1})$$

is the pair $(S_{i+1}, gT_{j+1})$, whose stabilizer is $S_{i+1} \cap gT_{j+1}$. Hence inductively the result follows from Lemma 2.15. \hfill \Box

## 3 Ghost maps

In this section, we examine a map $\Phi^G_n$ that this intrinsically related with the $n$-simplicial ring $B_n(G)$ in the sense that $\Phi^G_n$ may be discovered from the ring structure of $B_n(G)$.

**Proposition 3.1.**

1. For a $n$-slice $\mathcal{S}$ fixed, the correspondence $\mathcal{X}_n^f \mapsto \phi_\mathcal{S}(\mathcal{X}_n^f)$ extends to a ring homomorphism still denoted by

$$\phi_\mathcal{S} : B_n(G) \rightarrow \mathbb{Z}.$$ 

2. (An analogue of Burnside’s Theorem) We let

$$\Phi^G_n = (\phi_\mathcal{S}) : B_n(G) \rightarrow \bigsqcup_{\mathcal{S} \in [\Pi_n(G)]} \mathbb{Z} = C_n(G)$$

be the product of the $\phi_\mathcal{S}$. Then $\Phi^G_n$ is an injective ring homomorphism, with finite cokernel as morphism of abelian groups. The set

$$\{(\mathcal{S})_G \mid \mathcal{S} \in [\Pi_n(G)]\}$$

form a basis of $B_n(G)$.

3. Let $R$ be an integral domain, $\phi : B_n(G) \rightarrow R$ any ring homomorphism, and

$$T(\phi) := \{\mathcal{S} \in [\Pi_n(G)] \mid \phi((\mathcal{S})_G) \neq 0\}.$$ 

Then, there exists exactly one element $\hat{K} \in [\Pi_n(G)]$ that is minimal with respect to $\leq$ in $T(\phi)$. Moreover, one has $\phi(x) = \phi_\hat{K}(x) \cdot 1_R$ for all $x \in B_n(G)$ and this minimal $\hat{K}$ in $T(\phi)$.

**Proof.** 1. Corollary 2.10 shows that $\phi_\mathcal{S}$ the defining relations of $B_n(G)$ are mapped to $0$ by $\phi_\mathcal{S}$. So it induces a well defined map from $B_n(G)$ to $\mathbb{Z}$. Now, since the product of $(n, G)$-simplices is the product of the category of $(n, G)$-simplices, it follows that

$$\phi_\mathcal{S}(\mathcal{X}_n^f, \mathcal{Y}_n^g) = \phi_\mathcal{S}(\mathcal{X}_n^f) \phi_\mathcal{S}(\mathcal{Y}_n^g).$$

The image of the identity $e_n$ by $\phi_\mathcal{S}$ is obviously $1$. 


2. By definition $\Phi^G_n$ is a ring homomorphism. Suppose that $u \neq 0$ is in the kernel of $\Phi^G_n$. We write $u$ in terms of the generators

$$u = \sum_{S \in [\Pi_n(G)]} a_S(\bar{S})_G.$$ 

We have a partial ordering on the $\langle S_0 \ldots S_n \rangle_G$ induced by $\leq$. Let $\langle \bar{S} \rangle_G$ be maximal among the generators with $a_S \neq 0$. Then $\phi_S((\bar{T})_G) \neq 0$ implies that $\langle \bar{S} \rangle_G \leq \langle \bar{T} \rangle_G$. Hence

$$0 = \phi_S(u) = a_S \phi_S(\langle \bar{S} \rangle_G) = 0,$$

a contradiction.

Now by the first part of the proof, $B_n(G)$ has $\mathbb{Z}$-rank $\|\Pi_n(G)\| = \text{rank}_2 C_n(G)$, and since $\Phi^G_n$ is injective, then $\text{Im}(\Phi^G_n)$ and $C_n(G)$ have the same rank, and so, the cokernel of $\Phi^G_n$ is finite.

3. The set $T(\phi)$ is not empty because $e_n \notin \text{Ker} \phi$.

Let $\bar{S}$ be minimal in $T(\phi)$ with respect to the relation $\leq$. Then by Proposition 2.19, for any $n$-slice $\bar{T}$

$$\langle \bar{S} \rangle_G, ((\bar{T})_G) = \sum_{g \in [\Pi_n(G/T_0)]} \langle S_0 \cap ^g T_0, \ldots, S_n \cap ^g T_n \rangle_G$$

Since $\phi(\langle \bar{S} \rangle_G, (\bar{T})_G) = \phi(\langle \bar{S} \rangle_G, (\bar{T})_G) \neq 0$ in $R$, there exists $g \in S_0 \setminus G/T_0$ such that $\phi(\bar{S} \cap ^g \bar{T}) \neq 0$. Hence, by minimality of $\bar{S}$, we have that those $g$ ranges over the set $\{g \in G/T_0 \mid S_n \leq ^g T_n \ldots \ldots ^0 S_0 \leq ^g T_0\}$. So $\phi(\langle \bar{S} \rangle_G), \phi((\bar{T})_G) = \phi(\bar{S}((\bar{T})_G)) \phi(\langle \bar{S} \rangle_G)$. Since $R$ is an integral domain, we can divide both sides by $\phi(\langle \bar{S} \rangle_G)$ and we then have $\phi((\bar{T})_G) = \phi(\bar{S}((\bar{T})_G)) \cdot 1_R$. So by linearity, $\phi(x) = \phi(\bar{S}(x)) \cdot 1_R$ for all $x \in B_n(G)$. Now, if $\bar{K}$ is another minimal element, then $\phi((\bar{K})_G) = \phi(\bar{S}((\bar{K})_G))$ and $\phi((\bar{K})_G) = \phi(\bar{K}((\bar{K})_G))$. So $\bar{S} \leq_G \bar{K}$ and by symmetry $\bar{S} \leq_G \bar{K}$.

\[\square\]

Remark 3.2.

- If $R$ is an integral domain, then two homomorphisms $\phi, \phi' : B_n(G) \to R$ coincide if and only if they have the same kernel. Moreover, any homomorphism $\phi : B_n(G) \to R$ factors through some $\phi \bar{S} : B_n(G) \to \mathbb{Z}$ and the unique homomorphism $\mathbb{Z} \to R$ given by $n \mapsto n.1_R$.

- The ring $B_n(G)$ is finitely generated as $\mathbb{Z}$-module, hence is a noetherian ring. For each $n$-slice $\bar{S}$, the kernel of $\phi_{\bar{S}}$ is a prime ideal, since $\mathbb{Z}$ is an integral domain, and the intersection of all those kernels for $n$-slices $\bar{S}$ of $G$ is zero. In particular, the ring $B_n(G)$ is reduced (the intersection of all the prime ideals of $B_n(G)$ is zero).

- Since $C_n(G)$ is an integral over $B_n(G)$, the Theorem of Cohen-Seidenberg implies that their Krull dimensions coincide, and every prime ideal of $B_n(G)$ comes from $C_n(G)$.

- The point 2) of the proposition shows in particular that the set

$$\{(\bar{S})_G \mid \bar{S} \in [\Pi_n(G)]\}$$

form a basis of $B_n(G)$ and a $(1, G)$-simplicial burnside can also be seen as the lattice Burnside ring of some lattice introduced by Oda, Takegahara and Yoshida in [10].
We have the following corollary,

**Corollary 3.3.** The map

\[ \mathbb{Q} \Phi^n_G : \mathbb{Q} B_n(G) \rightarrow \prod_{S \in [\Pi_n(G)]} \mathbb{Q}, \]

where \( \mathbb{Q} \Phi^n_G \) is the rational extension of \( \Phi^n_G \), is an isomorphism of \( \mathbb{Q} \)-algebras.

Note that every \( \mathbb{Q} \)-algebra homomorphism \( \mathbb{Q} B_n(G) \rightarrow \mathbb{Q} \) is of the form \( \mathbb{Q} \phi \hat{S} \) for some \( \hat{S} \in \Pi_n(G) \). For \( \hat{S}, \hat{T} \in \Pi_n(G) \), we have \( \mathbb{Q} \phi \hat{S} = \mathbb{Q} \phi \hat{T} \) if and only \( \hat{S} \sim_G \hat{T} \).

More generally,

**Notation 3.4.** Put \( W_G(\hat{S}) = N_G(\hat{S})/S_n \). Let \( p \) be a prime, and \( \infty \) be just a symbol. For each \( \mathbb{Z} \)-module \( M \), we shall set \( M(p) = \mathbb{Z}(p) \otimes_{\mathbb{Z}} M \), where \( \mathbb{Z}(p) \) is the localisation of \( \mathbb{Z} \) at \( p \), and \( M(\infty) = M \). For a \( n \)-slice \( \hat{S} \), we denote \( W_G(\hat{S})_p \) a Sylow \( p \)-subgroup of \( W_G(\hat{S}) \), and set \( W_G(\hat{S})_\infty = W_G(\hat{S}) \). Let \( \{ \Phi^n_G \} \) or simply \( \Phi^n_G \) (if there is no risk of confusion) denote the homomorphism of \( \mathbb{Z}(p) \)-modules from \( B_n(G)_p \) to \( C_n(G)_p \) induced by \( \Phi^n_G \).

**Proposition 3.5.** Then

1. We consider \( B_n(G)_p \) and \( C_n(G)_p \) as subrings of \( \prod_{\hat{S} \in [\Pi_n(G)]} \mathbb{Q} \).

   The set \( J' = \left\{ -\frac{1}{\hat{S}} \right\} \mathbb{Q} \hat{S} : = \left( \frac{\phi \hat{S}}{\hat{S}} \right)_{\hat{S} \in [\Pi_n(G)]} \mid \hat{S} \in [\Pi_n(G)] \} \) is a basis of \( C_n(G) \).

2. If we define \( \text{Obs}(G) = \bigoplus_{\hat{S} \in [\Pi_n(G)]} \mathbb{Z}[W_G(\hat{S})][\mathbb{Z}] \), then

   \[ C_n(G)_p/\text{Im} \Phi^n_G \cong \text{Obs}(G)_p \]

3. Define a homomorphism of \( \mathbb{Z}(p) \)-modules \( \Psi^n_G : C_n(G)_p \rightarrow \text{Obs}(G)_p \) by

   \[ (x \hat{S})_{\hat{S} \in [\Pi_n(G)]} \mapsto \left( \bigoplus_{gT_0 \in W_G(\hat{T})_p} x_{g \in \hat{T}} \mod |W_G(\hat{T})_p| \right)_{\hat{T} \in [\Pi_n(G)]} \]

   Then \( \Psi^n_G \) is surjective.

4. The sequence

   \[ 0 \rightarrow B_n(G)_p \overset{\Phi^n_G}{\rightarrow} C_n(G)_p \overset{\Psi^n_G}{\rightarrow} \text{Obs}(G)_p \rightarrow 0 \]

   of \( \mathbb{Z}(p) \)-modules is exact.

**Proof.** 1. By Lemma 2.7 4), we have that \( \phi \hat{S} \) divides \( \phi \hat{T} \), and so \( \left( \frac{\phi \hat{S}}{\phi \hat{T}} \right)_{\hat{T} \in [\Pi_n(G)]} \) is an element of \( C_n(G)_p \), for any \( \hat{S} \in [\Pi_n(G)] \). Now, compare the set \( J' \) with the canonical basis

   \[ J := \{ i_{\hat{S}} = (\delta(\hat{S}, \hat{T})_{\hat{T} \in [\Pi_n(G)]} \mid \hat{S} \in [\Pi_n(G)] \} \]

   of \( C_n(G)_p \), where \( \delta \) is the Kronecker’s symbol, i.e. \( \delta(\hat{S}, \hat{T}) = \begin{cases} 1 & \text{for } \hat{S} = \hat{T} \\ 0 & \text{for } \hat{S} \neq \hat{T} \end{cases} \)

   Since \( |J| = |J'| = |\Pi_n(G)| \), it suffices to prove that each \( i_{\hat{S}} \) can be written as an
integral combination of the $\frac{1}{\varphi_S(S)}S$. This is done by induction with respect to $\leq$. If $S = (1, \ldots, 1)$ then $\frac{\varphi_S(S)}{\varphi_S(S)} = \delta(S, \hat{T})$, for any $\hat{T}$, and so, $i_S \in J'$. For arbitrary $S$, we have $\frac{\varphi_S(S)}{\varphi_S(S)} = 1$ and $\frac{\varphi_S(S)}{\varphi_S(S)} = 0$ for $\hat{T} \not\geq S$, and so, $\frac{1}{\varphi_S(S)}S = i_S + \sum_{\hat{T} \not\geq S} n_{\hat{T},S}i_{\hat{T}}$ with $n_{\hat{T},S} \in \mathbb{Z}$. 

Now by induction hypothesis, any $i_{\hat{T}}$ with $\hat{T} < S$ is an integral linear combination of the elements of $J'$. So, the same is true for $i_S$.

2. By Proposition 3.1 2), we have $\text{Im} \Phi^G((\hat{T})_G) = \bigoplus_{S \in \Pi_n(G)} \Phi^G(G)_{(\hat{T})_G}$. Hence $C_n(G)_{(p)} \cong \text{Im} \Phi^G((\hat{T})_G)$.

3. Let $\tilde{i}_S = (\delta(S, \hat{T}) \mod |W_G(S)_p|)_{\hat{T} \in \Pi_n(G)}$. Obviously, the elements $\tilde{i}_S$ for $S \in \Pi_n(G)$ form a $\mathbb{Z}^p$-basis of $\text{Ob}(G)_{(p)}$. Now set

$$R_0 = \{S \in \Pi_n(G) \mid \tilde{i}_S \notin \text{Im} \Psi^G((\hat{T})_G)\}$$

Suppose that $R_0 \neq \emptyset$, and let $S$ be a minimal element of $R_0$ with respect to $\leq_G$. Then no element $\hat{T}$ of $R_0 - \{S\}$ satisfies $\hat{T} \leq_G S$, and so $\Psi^G((i_{\hat{T}})_{\hat{T} \in \Pi_n(G)}) = (y_{\hat{T}})_{\hat{T} \in \Pi_n(G)}$, where

$$\forall \hat{T} = \begin{cases} 1 \mod |W_G(S)_p| & \text{for } \hat{T} = S \\ 0 \mod |W_G(S)_p| & \text{for } \hat{T} \in R_0 - \{S\} \end{cases}$$

But, $i_{\hat{T}} \notin \text{Im} \Psi^G((\hat{T})_G)$ for any $\hat{T} \in R_0$, which yields $i_S \notin \text{Im} \Psi^G((\hat{T})_G)$. This is a contradiction. Consequently, we have $R_0 = \emptyset$, and so $\Psi^G((\hat{T})_G)$ is surjective.

4. Let $\tilde{S} \in \Pi_n(G)$. Then

$$\Psi^G((\Phi^G(\hat{T})_G)) = \sum_{rT_0 \in W_G((\hat{T})_p)} |\text{Inv}_{<r>\hat{T}}(\hat{T})| \mod |W_G((\hat{T})_p)|_{\hat{T} \in \Pi_n(G)}$$

where $\text{Inv}_{<r>\hat{T}}(\hat{T}) = \{gS_0 \in G/S_0 \mid (r > \hat{T} \leq gS) = 1_{<r>\hat{T}}\}$. Set $W = W_G(S)_p$. Then Inv is can be viewed as a left $W$-set by the action given by $rT_0 \cdot gS_0 = gS_0$. With this action, one has $\text{Inv}_{<r>\hat{T}}(\hat{T}) = \{gS_0 \in \text{Inv}_{<r>\hat{T}}(\hat{T}) \mid rT_0 \cdot gS_0 = gS_0\}$. So

$$\sum_{rT_0 \in W} |\text{Inv}_{<r>\hat{T}}(\hat{T})| = \sum_{gS_0 \in \hat{T} \cdot W} |rT_0 \cdot gS_0 - gS_0|$$

$$= \sum_{S_0 \in \hat{T} \cdot W} |gS_0|$$

$$= \sum_{gS_0 \in \hat{T} \cdot W} |gS_0| W_{gS_0}$$

$$\equiv 0 \mod |W|,$$

and so $\text{Im} \Phi^G((\hat{T})_G) \subseteq \text{Ker} \Psi^G((\hat{T})_G)$. It remains to prove that $\text{Ker} \Psi^G((\hat{T})_G) \subseteq \text{Im} \Phi^G((\hat{T})_G)$. Now, since $\Psi^G((\hat{T})_G)$ is surjective and $\Psi^G((\hat{T})_G) \cdot \Phi^G((\hat{T})_G) = 0$, we have that $\Psi^G((\hat{T})_G)$ factorizes through
the cokernel of $\Phi^G_{(p)}$, which is isomorphic by $\text{Obs}_{(p)}$ by 2). So, we obtain a surjective map $\text{Coker} \Phi^G_{(p)} \to \text{Obs}_{(p)}$ between two isomorphic groups. This map is then an isomorphism, and so $\text{Ker} \Psi^G_{(p)}$ is equal to $\text{Im} \Phi^G_{(p)}$.

\[\square\]

4 Idempotents elements

By the isomorphism $\mathcal{Q} \Phi$, there is an element $e^G_{T} \in \mathcal{Q}B_n(G)$ for each $n$-slice $T$ of $G$ such that

\[\mathcal{Q} \Phi^G_{S}(e^G_{T}) = \begin{cases} 1 & \text{if } S = G \cdot T \\ 0 & \text{otherwise} \end{cases}\]

Clearly, the set $\{e^G_{T} \mid T \in [\Pi_n(G)]\}$ is the set of primitive idempotents of $\mathcal{Q}B_n(G)$.

In order to give the explicit formula of the primitive idempotent $e^G_{S}$, we need the Möbius function of a finite poset $(X, \leq)$. The Möbius function $\mu_X : X \times X \to \mathbb{Z}$ of a finite poset is defined inductively as follows (see [13]):

\[\mu_X(x, x) = 1, \quad \mu_X(x, y) = 0 \text{ if } x \nleq y, \quad \sum_{t \leq y} \mu_X(x, t) = 0 \text{ if } x < y.\]

Proposition 4.1. Let $\tilde{S}$ be a $n$-slice of $G$. Then the explicit formula of the primitive idempotent $e^G_{\tilde{S}}$ is given by

\[e^G_{\tilde{S}} = \frac{1}{|N_G(\tilde{S})|} \sum_{T \leq \tilde{S}} |T_0| \mu_{\Pi}(\tilde{T}, \tilde{S})(\tilde{T})_G,\]

where $\mu_{\Pi}$ is the Möbius function of the poset $(\Pi_n(G), \leq)$.

Proof. Let $\bar{K}$ be any $n$-slice of $G$. Then

\[
\mathcal{Q} \Phi^G_{S}(|G|e^G_{\bar{S}}) = \frac{|G|}{|N_G(\bar{S})|} \sum_{T \leq \bar{S}} |T_0| \mu_{\Pi}(\bar{T}, \bar{S}) (Q\Phi^G_{\bar{T}})(\bar{T})_G
\]

\[= \frac{|G|}{|N_G(\bar{S})|} \sum_{T \leq \bar{S}} |T_0| \mu_{\Pi}(\bar{T}, \bar{S}) (|g \in G/T_0 | \bar{K} \leq \bar{S} \cdot T)|\]

\[= \frac{|G|}{|N_G(\bar{S})|} \sum_{g \in G} \sum_{\bar{K} \leq \bar{T} \leq \bar{S}} \mu_{\Pi}(\bar{T}, \bar{S})
\]

\[= |G| \delta(\bar{S}, \bar{T})\]

where the second equality follows from Lemma 2.7. By the property of the Möbius function, we have that the sum $\sum_{\bar{K} \leq \bar{T} \leq \bar{S}} \mu_{\Pi}(\bar{T}, \bar{S})$ is zero unless $\bar{K} = \bar{S}$, for any $g \in G$. Therefore,

\[\phi^G_{\bar{K}}(e^G_{\bar{S}}) = \begin{cases} 1 & \text{if } \bar{K} = G \cdot \bar{S} \\ 0 & \text{otherwise} \end{cases}\]

\[\square\]

In the case $n = 0$ the formula of the primitive idempotent was given by Gluck in [7] and independently by Yoshida in [14]. The formula in the case $n = 1$ was stated by Bouc in [3]. The idempotent $e^G_{\tilde{S}}$ is the only idempotent of $\mathcal{Q}B_n(G)$ with the following property:
Proposition 4.2 (Characterization of $e^G_S$).
Let $\mathcal{S}$ be a $n$-slice of $G$. Then $X \cdot e^G_S = Q\phi_S^G(X)e^G_S$, for any $X \in \mathbb{Q}B_n(G)$.
Conversely, if $\mathcal{Y} \in \mathbb{Q}B_n(G)$ is such that $X \cdot \mathcal{Y} = Q\phi_S^G(X)\mathcal{Y}$, then $\mathcal{Y} \in \mathbb{Q}e^G_S$ (that is $\mathcal{Y}$ is a rational multiple of $e^G_S$).

Proof. 
Note that a $\mathbb{Q}$-basis of $\mathbb{Q}B_n(G)$ is given by the $e^G_T$ where $T$ runs through the set $\Pi_n(G)$ of conjugacy classes of $n$-slices of $G$. So for any $X \in \mathbb{Q}B_n(G)$, we have that

$$X = \sum_{T \in [\Pi_n(G)]} \lambda_T e^G_T,$$

where $\lambda_T$ are rational numbers. Since the elements $e^G_T$ are orthogonal, it follows that for any $n$-slice $\mathcal{S} \in [\Pi_n(G)]$ we have,

$$\phi_S^G(X) = \lambda_{\mathcal{S}} \text{ and } X \cdot e^G_S = \phi_S^G(X)e^G_S.$$

Conversely, let $\mathcal{Y}$ be an element of $\mathbb{Q}B_n(G)$ verifying $X \cdot \mathcal{Y} = \phi_S^G(X)\mathcal{Y}$ for any $X \in \mathbb{Q}B_n(G)$. Then in particular $e^G_T \cdot \mathcal{Y} = 0$ if $\mathcal{S} \not\preceq T$, thus $\mathcal{Y} = \phi_S^G(\mathcal{Y})e^G_S$ is a rational multiple of $e^G_S$.

\[\square\]

Proposition 4.3. Let $(X, \preceq)$ be a finite poset. Let $\Pi_n(X)$ denote the set of $n$-tuples $(x_0, \ldots, x_n)$ of elements of $X$ such that $x_0 \leq \cdots \leq x_n$. Define a partial order $\preceq$ on $\Pi_n(X)$ by

$$(x_0, \ldots, x_n) \preceq (y_0, \ldots, y_n) \iff x_i \leq y_i, \forall i = 0, \ldots, n$$

Then the Möbius function $\mu_{\Pi}$ of the poset $(\Pi_n(X), \preceq)$ can be computed as follows, for any $\bar{x} := (x_0, \ldots, x_n), \bar{y} := (y_0, \ldots, y_n) \in \Pi_n(X)$:

$$\mu_{\Pi}(\bar{x}, \bar{y}) = \left\{ \begin{array}{ll} \prod_{i=0}^{n} \mu_\lambda(x_i, y_i) & \text{if } x_0 \leq y_0 \leq x_1 \leq \cdots \leq x_n \leq y_n \\ 0 & \text{otherwise} \end{array} \right. \quad (5)$$

where $\mu_\lambda$ is the Möbius function of the poset $(X, \preceq)$.

Proof. Let $m(\bar{x}, \bar{y})$ denote the expression defined by the right hand side of (5). Then if $\bar{x} \preceq \bar{y}$

$$\sum_{\bar{t} \in \Pi_n(X)} m(\bar{x}, \bar{t}) = \sum_{(t_0, \ldots, t_n) \in \mathcal{P}_n} \prod_{i=0}^{n} \mu_\lambda(x_i, t_i) \prod_{(t_0, \ldots, t_n) \in \mathcal{P}_n}$$

where $\mathcal{P}_n := \{ f \in \Pi_n(X) \mid \begin{array}{l} x_i \leq t_i \leq y_i \quad \text{for all } i = 0, \ldots, n \\ x_0 \leq \cdots \leq x_n \\ y_0 \leq \cdots \leq y_n \\ t_i \leq x_{i+1} \quad \text{for all } i = 0, \ldots, n-1 \end{array} \}$.

So,

$$\sum_{(t_0, \ldots, t_n) \in \mathcal{P}_n} \prod_{i=0}^{n} \mu_\lambda(x_i, t_i) = \left( \sum_{x_n \leq t_n \leq y_n} \prod_{i=0}^{n} \mu_\lambda(x_i, t_i) \right) \left( \sum_{(t_0, \ldots, t_{n-1}) \in \mathcal{P}_{n-1}} \prod_{i=0}^{n-1} \mu_\lambda(x_i, t_i) \right) \quad (\star)$$
where $\mathcal{P}'_{n-1} := \{(t_0, \ldots, t_{n-1}) \in \Pi_{n-1}(X) \mid \left\{ \begin{array}{l} x_i \leq t_i \leq y_i \text{ for all } i = 0, \ldots, n-1 \\ t_i \leq x_{i+1} \text{ for all } i = 0, \ldots, n-1 \end{array} \right\}$.

The first factor of (⋆) is equal to 0 if $x_n \neq y_n$, and 1 if $x_n = y_n$. Hence, if $x_n = y_n$, then the second factor is equal to

$$\sum_{(t_0, \ldots, t_{n-1}) \in \mathcal{P}'_{n-1}} \mu_\chi(x_0, t_0) \cdots \mu_\chi(x_{n-1}, t_{n-1})$$

Hence inductively, we have

$$\sum_{i \in \Pi_{\chi}(\bar{x})} m(i, \bar{t}) = \prod_{i=0}^{n} \delta(x_i, y_i)$$

and the proposition follows.

\[\square\]

**Corollary 4.4.** Let $\bar{S} = (S_0, \ldots, S_n)$ and $\bar{T} = (T_0, \ldots, T_n)$ be $n$-slices of $G$. Then

$$\mu_{\Pi}(\bar{T}, \bar{S}) = \begin{cases} \prod_{i=0}^{n} \mu(T_i, S_i) & \text{if } T_0 \leq S_0 \leq T_1 \leq S_1 \cdots \leq T_n \leq S_n \\ 0 & \text{otherwise} \end{cases}$$

where $\mu$ is the Möbius function of the poset of subgroups of $G$.

In particular

$$e_\bar{S}^G = \frac{1}{|N_G(S)|} \sum_{T_0 \leq S_0 \leq \cdots \leq T_n \leq S_n} |T_0| \mu(T_0, S_0) \cdots \mu(T_n, S_n)(\bar{T})_G.$$  

## 5 Prime ideals

The aim of this part is the proof that a finite group $G$ is solvable if and only if the prime ideal spectrum of $B_n(G)$ is connected, i.e., if and only if 0 and 1 are the only idempotents in $B_n(G)$, like established by A. Dress in [5] for $B_0(G)$ and S. Bouc in [3]. Note that $C_n(G)$ is integral over $B_n(G)$, because it is generated by idempotent elements which are integral over any subring. Hence by the going-up theorem, every prime ideal of $B_n(G)$ comes from $C_n(G)$.

**Proposition 5.1.** Let $p$ denote either 0 or a prime number. If $\bar{S} \in \Pi_n(G)$, let $I_{\bar{S}, p}$ be the prime ideal of $B_n(G)$ defined as the kernel of the ring homomorphism

$$B_n(G) \xrightarrow{\phi_{\bar{S}}^G} \mathbb{Z} \xrightarrow{p} \mathbb{Z}/p\mathbb{Z}. $$

Then every prime ideal $I$ of $B_n(G)$ has the form $I_{\bar{S}, p}$ for a suitable $\bar{S} \in \Pi_n(G)$. Moreover, given a prime ideal $I$ there exists a unique $\bar{K} \in [\Pi_n(G)]$ with $I = I_{\bar{K}, p}$ and $\phi_{\bar{K}}^G((\bar{K})_G) \equiv 0 \pmod{p}$ where $p$ is the characteristic of the ring $R := B_n(G)/I$.

**Proof.** Consider the natural map $\phi : B_n(G) \to B_n(G)/I$. Then $\phi(x) = \phi_{\bar{S}}(x) \cdot 1_R$ for some $n$-slice $\bar{S}$ by Proposition 3.1. So $I = \{ x \in B_n(G) \mid \phi(x) = 0 \}$

$$= \{ x \in B_n(G) \mid \phi_{\bar{S}}(x) \cdot 1_R = 0 \}$$

$$= \{ x \in B_n(G) \mid \phi_{\bar{S}}(x) \equiv 0 \pmod{p} \} = I_{\bar{S}, p}$$
where \( p \) is the characteristic of the ring \( B_n(G)/I \).

Let \( I_n := \{ \bar{S} \in \Pi_n(G) \mid \langle \bar{S} \rangle_G \not\in I \} \). If \( \bar{T} \) and \( \bar{K} \) are both minimal in \( I \), then

\[
\langle \bar{T} \rangle_G : \langle \bar{K} \rangle_G = \sum_{g \in [I_G/K_0]} \langle \bar{T} \cap g \bar{K} \rangle_G
\]

\[\equiv \phi_T((\langle \bar{K} \rangle_G)(\langle \bar{T} \rangle_G) \pmod I)\]

(This relation must hold for any \( \langle \bar{K} \rangle_G \in I \))

Since \( \langle \bar{T} \rangle_G : \langle \bar{K} \rangle_G \in I \), we have \( \phi_T((\langle \bar{K} \rangle_G) ≠ 0 \) and \( \phi_T((\langle \bar{T} \rangle_G) ≠ 0 \) by symmetry, and so \( \bar{K} \equiv_G \bar{T} \). So the set \( I_n \) has a unique minimal element, up to isomorphism. On the other hand, the quotient ring \( B_n(G)/I \) is an integral domain, we have that either \( p = 0 \) or \( p \) is a prime number. If \( p = 0 \), then the projection \( \bar{p} : B_n(G) \to B_n(G)/I \) is equal to \( \phi_K \) and \( I = I_{G,p} \). If \( p ≠ 0 \), then \( \phi \) is equal to the reduction of \( \phi_K \) modulo \( p \).

If \( \bar{G} \) is any \( n \)-slice of \( G \) with \( I \equiv_G \bar{G} \) and \( \phi_{G'}((\langle \bar{G} \rangle_G) ≠ 0 \pmod p \) then for \( \bar{K} \) minimal in \( I_n \)

\[
\phi_{\bar{K}}((\langle \bar{K} \rangle_G) \equiv \phi_{\bar{G'}}((\langle \bar{K} \rangle_G) \pmod p.
\]

In particular \( \phi_{\bar{K}}((\langle \bar{K} \rangle_G) \equiv 0 \pmod p \) and similarly \( \phi_{\bar{G'}}((\langle \bar{K} \rangle_G) \equiv 0 \pmod p \). This can only occur if \( \bar{K} \equiv_G \bar{G} \).

\[\square\]

**Notation 5.2.** Let \( p \) be a prime number.

- Let \( (\Pi_n(G) \pmod p) \) denote the subset of \( \Pi_n(G) \) consisting of the \( n \)-slices \( \bar{S} \) such that \( N_G(\bar{S})/S_0 \) is a \( p' \)-group.

- For any \( \bar{S} \in \Pi_n(G) \), let \( \bar{S}^{\sim} \) be the unique element \( \bar{K} \) of \( (\Pi_n(G) \pmod p) \), up to conjugation, such that \( I_{\bar{S}^p} = I_{\bar{K}^p} \).

- If \( \bar{S} \) is an \( n \)-slice of \( G \), let \( \bar{S}^+_n \) denote a slice of the form \( p\bar{S} := (PS_0, \ldots, PS_n) \) of \( G \), where \( P \) is a Sylow \( p \)-subgroup of \( N_G(\bar{S}) \).

Define inductively an increasing sequence \( (\bar{S}^i) \) in \( (\Pi_n(G), ≤) \) by \( S^0 = \bar{S} \) and \( \bar{S}^{i+1} = (\bar{S}^i)^+_p \) for \( i \in \mathbb{N} \). We set \( \bar{S}^\infty \) for the largest term of the sequence \( (\bar{S}^i) \).

- By Proposition 5.1, the prime ideal of \( B_n(G) \) is parametrized by pairs \( (\bar{S}, p) \) where \( p \) is equal to 0 or a prime number and \( \bar{S} \in \Pi_n(G) \) is such that \( N_G(\bar{S})/S_0 \not\equiv 0 \pmod p \). We set \( \Theta_n(G) \) the set of \( (\bar{S}, p) \) where \( p \) is equal to 0 or a prime number and \( \bar{S} \in \Pi_n(G) \) is such that \( N_G(\bar{S})/S_0 \not\equiv 0 \pmod p \).

Note that if we define a relation “\( \bar{S} \equiv \bar{T} \)” on \( \Pi_n(G) \) by \( \bar{S} \equiv \bar{T} \) if \( \bar{S}^{\sim} =_G \bar{T}^{\sim} \), then “\( \bar{S} \equiv \bar{T} \)” is an equivalence relation.

**Proposition 5.3.** The \( n \)-slice \( \bar{S}^p \) is conjugate to \( \bar{S}^\infty \).

**Proof.** By definition, the \( n \)-slice \( \bar{S}^p \) is a minimal element \( \bar{K} \) of \( (\Pi_n(G), ≤) \) such that

\[
\phi_{\bar{S}}((\langle \bar{K} \rangle_G) := [g \in G/K_0 \mid \bar{S}^g ≤ \bar{K}] \not\equiv 0 \pmod p.
\]

Thus one can assume that \( \bar{S} ≤ \bar{K} \). Since \( \phi_{\bar{S}} \equiv \phi_{\bar{S}^p} \pmod p \) for any \( p \)-subgroup \( P \) of \( N_G(\bar{S}) \) by Corollary 2, one can also assume that \( \bar{S}^p ≤ \bar{K} \), and inductively, that \( \bar{S}^\infty ≤ \bar{K} \). Moreover \( \phi_{\bar{S}^\infty} \equiv \phi_{\bar{K}} \pmod p \). As \( N_G(\bar{S}^\infty)/S_0 \) is a \( p' \)-group, it follows that \( \bar{S}^\infty =_G \bar{K} \).

\[\square\]

**Proposition 5.4.** Let \( (\bar{S}, p), (\bar{S}', p') \) be elements of \( \Theta_n(G) \).

Then \( I_{\bar{S}^p} \subseteq I_{\bar{S}'^p} \) if and only if
• either $p' = p$ and the $n$-slices $\mathcal{S}'$ and $\mathcal{S}$ are conjugate in $G$.

• or $p' = 0$ and $p > 0$, and the $n$-slices $\mathcal{S}'_p$ and $\mathcal{S}$ are conjugate in $G$.

Proof. Assume that $I_{\mathcal{S}'_p} \subseteq I_{\mathcal{S}_p}$. Then there exists a surjective ring homomorphism $\phi$ from $B_n(G)/I_{\mathcal{S}} \cong \mathbb{Z}/p\mathbb{Z}$ to $B_n(G)/I_{\mathcal{S}_p} \cong \mathbb{Z}/p\mathbb{Z}$. Hence either $p' = p$ or $p' = 0$ and $p > 0$. If $p = p'$ then $\phi$ is a bijection, and so $I_{\mathcal{S}'_p} = I_{\mathcal{S}_p}$ which implies that $\mathcal{S}'$ and $\mathcal{S}$ are conjugate. If $p' = 0$ and $p > 0$ then $\phi^G_{\mathcal{S}}$ is the reduction modulo $p$ of $\phi^G_{\mathcal{S}}$, and so $I_{\mathcal{S}'_p} \subseteq I_{\mathcal{S}_p}$. Hence $\mathcal{S}'_p$ and $\mathcal{S}$ are conjugate in $G$, by Corollary 5.1. Conversely, if $\mathcal{S}'$ and $\mathcal{S}$ are conjugate then $\phi^G_{\mathcal{S}} = \phi^G_{\mathcal{S}}$, in particular $I_{\mathcal{S}',0} = I_{\mathcal{S},0}$. If $p$ is a prime then Proposition 5.4 implies that $\phi^G_{\mathcal{S}}(x) \equiv \phi^G_{\mathcal{S}_p}(x) \mod p$ for any $x \in B_n(G)$.

So $I_{\mathcal{S}',0} \subseteq I_{\mathcal{S}_p}$. And if $\mathcal{S}'_p$ and $\mathcal{S}$ are conjugate in $G$, then $I_{\mathcal{S}',p} = I_{\mathcal{S}_p}$. So $I_{\mathcal{S}',0} \subseteq I_{\mathcal{S}_p}$.

\[ \square \]

Remark 5.5.

• Since for a prime $p$, the prime ideals of $(B_n(G))_p$ are of the form $\mathcal{Z}_{(p)}I$ where $I$ is a prime ideal of $B_n(G)$ which does not meet $\mathbb{Z} - p\mathbb{Z}$, we have that $I = I_{\mathcal{S},p}$ or $I = I_{\mathcal{S},0}$. Hence the connected component of $(B_n(G))_p$ is indexed by $[(\Pi_n(G))_p]$. Moreover, the component indexed by the $n$-slice $\mathcal{S}$ consists of a unique maximal element $\mathcal{Z}_{(p)}I_{\mathcal{S},p}$ and of ideals $\mathcal{Z}_{(p)}I_{\mathcal{S},0}$, where $\mathcal{S}' \in \Pi_n(G)$ is such that $\mathcal{S}'_p$ is conjugate to $\mathcal{S}$.\[ \square \]

Proposition 5.6. Two ideals $I_{\mathcal{S}',p}$ and $I_{\mathcal{S}_p}$ are in the same connected component of $Spec(B_n(G))$ if and only $D^{\infty}(\mathcal{S})$ is conjugate to $D^{\infty}(\mathcal{S}')$, where $D^{\infty}(\mathcal{K}) := (D^{\infty}(\mathcal{K}_0), \ldots, D^{\infty}(\mathcal{K}_n))$ and $D^{\infty}(\mathcal{K}_i)$ denotes the last term in the derived series of $\mathcal{K}_i$. In particular, $Spec(B_n(G))$ is connected if and only if $G$ is solvable.

Proof. Recall that if $R$ is a Noetherian ring. For any prime ideal $I \in Spec(R)$, let $\overline{T} = \{P \mid P \in Spec(R), P \supseteq I\}$ be the closure of $I$ in $Spec(R)$. Then, two ideals $P$ and $P'$ are in the same connected component of $Spec(R)$ if and only if there exists a series of minimal ideals $I_1, \ldots, I_n$ with $P \subseteq \overline{I_1}, P' \subseteq \overline{I_n}$ and $I_{i+1} \cap \overline{I_1} = \emptyset$ for $i = 1, \ldots, n - 1$. Now, if $R = B_n(G)$, then $I_{\mathcal{S},0} \cap I_{\mathcal{S}',0} = \emptyset$ if and only if the $n$-slice $\mathcal{S}'_p$ is conjugate to $\mathcal{S}_p$, for some prime $p$. Hence, if $I_{\mathcal{S},p}$ and $I_{\mathcal{S}',p}$ are in the same connected component of $Spec(B_n(G))$ then $D^{\infty}(\mathcal{S})$ is conjugate to $D^{\infty}(\mathcal{S}')$. The ideals $I_{\mathcal{S},p}$ and $I_{\mathcal{S}',p}$ are in the same connected component. Indeed, one can find a series of normal subgroups of $S_0$ such that $D^{\infty}(S_0) = S_0^{(n)} \prec S_0^{(n-1)} \prec \cdots \prec S_0^{(1)} \prec S_0^{(0)} = S_0$ such that $S_0^{(i-1)}/S_0^{(i)}$ is a $p_i$-group for some prime $p_i, i = 1, \ldots, n$. Hence, letting $\mathcal{K}_i = (S_0^{(n)}, S_1, \ldots, S_n)$ for $i = 0, \ldots, n$ one obtains

\[
\mathcal{S} = \mathcal{K}_0 \leftarrow \mathcal{K}_1 \leftarrow \cdots \leftarrow \mathcal{K}_{n-1} \leftarrow \mathcal{K}_n = (D^{\infty}(S_0), S_1, \ldots, S_n),
\]

where the arrows indicate the inclusion of connected components.
and
\[ I_{\bar{S},p} \in \tilde{I}_{K_{n},0}, I_{D^\circ}(s_0), S_1,..., S_n, 0 \in \tilde{I}_{K_{n},0} \text{ and } \tilde{I}_{K_{n},0} \cap I_{\bar{K}_{n},0} \neq \emptyset. \] So, \( I_{\bar{S},p} \) and \( I_{D^\circ}(s_0), S_1,..., S_n, 0 \) are in the same connected component. By the same proceed, one prove that \( I_{(D^\circ)(s_0), (s_1),...,(s_n), 0} \) and \( I_{\bar{S},p} \) are in the same component, and so one.

\[ \square \]

### 6 Green biset functor

Let \( R \) be a commutative ring with identity. The biset category over \( R \) will be denoted by \( RC \): its objects are all finite groups, and that for finite groups \( G \) and \( H \), the hom-set \( \text{Hom}_{RC}(G, H) \) is \( R \tilde{B}(H, G) = R \otimes B(H, G) \), where \( B(H, G) \) is the Grothendieck group of the category of finite \((H, G)\)-biset. The composition of morphisms in \( RC \) is induced by \( R \)-bilinearity from the composition of bisets (see [2] Definition 3.1.1).

A good choice of a family \( G \) of finite groups and for every \( G, H \in G \), a set \( \Gamma(G, H) \) of subgroups of \( G \times H \) can lead to an important category. For example if we fix a non-empty class \( D \) of finite groups closed under subquotients and cartesian products and we denote by \( RD \) the full subcategory of \( RC \) consisting of groups in \( D \), then \( RD \) is a replete subcategory of \( RC \) in the sense of Bouc [2].

The category of biset functors, i.e. the category of \( R \)-linear functors from \( RC \) to the category \( R \text{-Mod} \) of all \( R \)-modules, will be denoted by \( F_{R} \). The category \( F_{D,R} \) of \( D \)-biset is the category of \( R \)-linear functors from \( RD \) to \( R \text{-Mod} \).

Let \( G \) be a finite group, \( H \) a subgroup of \( G \) and \( N \) be a normal subgroup of \( G \). One sets

- \( \text{Res}^G_H \) := \([H \times G]_G \) for the image in the biset Burnside group \( B(H, G) \) of the isomorphism class of \( G \) where \( G \) is viewed as an \((H, G)\)-biset via the left and right multiplication.

- \( \text{Ind}^G_H \) := \([G \times H]_G \) for the image in \( B(H, G) \) of the isomorphism class of \( G \) where \( G \) is viewed as a \((G, H)\)-biset via the left and right multiplication.

- \( \text{Inf}^G_{G/N} := [G/(G/N)]_{G/N} \) for the image in \( B(G, G/N) \) of the isomorphism class of \( G/N \) where \( G/N \) is viewed as a \((G, G/N)\)-biset via the the canonical epimorphism \( G \rightarrow G/N \) and left and right multiplication.

- \( \text{Def}^G_{G/N} := [G/N/(G/N)]_G \) for the image in \( B(G/N, G) \) of the isomorphism class of \( G/N \) where \( G/N \) is viewed as a \((G, G/N)\)-biset via the the canonical epimorphism \( G \rightarrow G/N \) and left and right multiplication.

- If \( f : G \rightarrow H \) is a group isomorphism, then we set \( \text{Iso}(f) := [H \times H]/_H \) for the image in \( B(H, G) \) of the isomorphism class of \( H \) where \( H \) is considered as an \((H, G)\)-biset via \( h \times x = h f(g) \) for \( h, x \in H \) and \( g \in G \).

It is possible to verify that all elements in \( B(H, G) \) are sums of \([H \times G/L]_L \) where \( L \) runs through subgroups of \( H \times G \) and that these satisfy the following decomposition:

\[ [H \times G/L] = \text{Ind}^H_D \circ \text{Inf}^{D/C}_C \circ \text{Iso}(f) \circ \text{Def}^B_{B/A} \circ \text{Res}^G_B \]  \hspace{1cm} (6)

where \((D, C)\) and \((B, A)\) are 1-slices of \( H \) and \( G \) respectively with the additional properties that \( C \triangleleft D \) and \( A \triangleleft B \), and \( f \) is some the group isomorphism from \( B/A \) to \( D/C \) (see [2] Lemma 2.3.26 for more details).

A Green \( D \)-biset functor is defined as a monoid in \( F_{D,R} \). This is equivalent to the following definitions:

**Definition 6.1** ([2] Definition 8.5.1). A \( D \)-biset functor \( A \) is a Green \( D \)-biset functor if it is equipped with a linear products \( A(G) \times A(H) \rightarrow A(G \times H) \) denoted by \( (a, b) \mapsto a \times b \), for groups, \( G, H \) in \( D \), and an element \( e_A \in A(1) \), satisfying the following conditions:
1. (Associativity). Let $G, H$ and $K$ be groups in $\mathcal{D}$. If we consider the canonical isomorphism from $G \times (H \times K)$ to $(G \times H) \times K$, then for any $a \in A(G), b \in A(H)$ and $c \in A(K)$

$$(a \times b) \times c = A\left(\text{Iso}_{G \times H \times K}^{G \times (H \times K)}(a \times (b \times c))\right).$$

2. (Identity element). Let $G$ be a group in $\mathcal{D}$ and consider the canonical isomorphisms $1 \times G \to G$ and $G \times 1 \to G$. Then for any $a \in A(G)$

$$a = A\left(\text{Iso}_{1 \times G}^{G}(e_A \times a)\right) = A\left(\text{Iso}_{G \times 1}^{G}(a \times e_A)\right)$$

3. (Functoriality). If $\phi : G \to G'$ and $\psi : H \to H'$ are morphisms in $RD$, then for any $a \in A(G)$ and $b \in A(H)$

$$A(\phi \times \psi)(a \times b) = A(\phi)(a) \times A(\psi)(b).$$

There is an equivalent way of defining a Green biset functor given by Romero in ([12] Lema 4.2.3):

**Definition 6.2** ([12] Definición 3.2.7). A $\mathcal{D}$-Green biset functor is an object $A \in \mathcal{F}_{\mathcal{D}, R}$ together with the datum of an $R$-algebra structure on each $A(H), H \in \mathcal{D}$, such that the following axioms are satisfied for all groups in $K$ and $G$ in $\mathcal{D}$ and all group homomorphisms $K \to G$:

1. For the $(K, G)$-biset $G_r$, the morphism $A(G_r)$ is a ring homomorphism.
2. For the $(G, K)$-biset $G_l$, the morphism $A(G_l)$ satisfies the Frobenius identities for all $b \in A(G)$ and $a \in A(K)$,

$$A(G_l)(a) \cdot b = A(G_l)\left(\text{Id} \cdot A(G_r)(b)\right),$$

$$b \cdot A(G_l)(a) = A(G_l)\left(A(G_r)(b) \cdot \text{Id}\right)$$

where $\cdot$ denotes the ring product on $A(G)$, resp. $A(K)$.

**Definition 6.3**. If $A$ and $C$ are Green $\mathcal{D}$-biset functors, a morphism of Green $\mathcal{D}$-biset functors from $A$ to $C$ is a natural transformations $f : A \to C$ such that $f_{H \times K}(a \times b) = f_H(a) \times f_K(b)$ for any groups $H$ and $K$ in $\mathcal{D}$ and any $a \in A(H), b \in A(K)$, and such that $f_1(e_A) = e_C$.

**Proposition 6.4.** The correspondence

$$G \mapsto B_n(G)$$

defines a structure of Green biset functor.

**Proof.** There are several steps:

- Any $(n, G)$ simplex $\mathcal{X}_n^f$ give rise a $(n, H)$-simplex $U \times_G \mathcal{X}_n^f$ in the following rule (following the notion of Definition 2.3.11 in [2]):

$$U \times_G \mathcal{X}_n^f : (U \times_G X_0 \xrightarrow{UF_1} U \times_G X_1 \xrightarrow{UF_2} \ldots U \times_G X_{n-1} \xrightarrow{UF_n} U \times_G X_n)$$
defined by \( Uf_i : U \times_G X_{i-1} \to U \times_G X_i, (u,x) \mapsto (u,f_i(x)) \) for any \( i = 1, \ldots, n \).

Let \( \mu = (\mu_i) \) be a morphism from \( X_n^f \) to \( Y_n^g \). Then \( \mu \) defines a morphism of \((n,H)\)-simplices from \( U \times_G X_n^f \to U \times_G Y_n^g \) given by

\[
U \times_G \mu = (U\mu_i : (u,x) \mapsto (u,\mu_i(x)))_i.
\]

Indeed, for any \( i, x_i \in X_i \) and \( u \in U \), we have

\[
(U\mu_i) \circ (Uf_i)(u,x_{i-1}) = (u,\mu_i f_i(x_{i-1})) = (u,\mu_i g_i \mu_{i-1}(x_{i-1})) = (Ug_i \circ (U\mu_i - 1))(u,x_{i-1})
\]

Therefore, the correspondence \( I_{IJ} : X_n^f \mapsto U \times_G X_n^f \) is a functor from the \((n,G)\)-simplices to \((n,H)\)-simplices. On the other hand, it is straightforward to show that the defining relations of \( B_n(G) \) are mapped to the defining relations of \( B_n(H) \). Hence, the later functor induces a homomorphism of groups \( B_n(U) : B_n(G) \to B_n(H) \).

- The correspondence \( G \mapsto B_n(G) \) defines a structure of biset functor:
  Clearly, if \( U \cong U' \) (as \((H,G)\)-bisets) then the functors \( I_{UJ} \) and \( I_{UJ'} \) are isomorphic. So, \( B_n(U) = B_n(U') \). If \( U \) has the form \( U = U_1 \cup U_2 \) (as \((H,G)\)-bisets) then \( I_{UJ} = I_{UJ} \cup I_{UJ} \), and so \( B_n(U) = B_n(U_1) + B_n(U_2) \). We may state that for any \((K,H)\)-biset \( V \), we have an isomorphism of \((n,K)\)-simplices between \( V \times_H (U \times X_n^f) \) and \( (V \times_H U) \times X_n^f \), which induces an isomorphism \( I'_{U} \circ I_{IJ} \cong I'_{V \times_H UJ} \) and so \( B_n(V) \circ B_n(U) = B_n(V \times_H U) \). Finally, since \( I_{Id_G} \cong 1 \) (the functor identity, we have \( B_n(Id_G) \cong 1 \)). This shows that we have a functor from the biset category to the category \( \mathbb{Z} \text{-Mod} \).

- Let \( Y_n^g \) be a \((n,H)\)-simplex. Then the product \( X_n^f \times Y_n^g \) is a \((n,G \times H)\)-simplex in the obvious way. It induces a well-defined bilinear

\[
\times : B_n(G) \times B_n(H) \to B_n(G \times H), \quad ((a,b)) \mapsto a \times b,
\]

and \((n,1)\)-simplex \( e \) is the identity of the product, up to identification \( G \times 1 = G \). One verifies easily that the axioms of Definition 6.1 are satisfied.

\( \square \)

**Proposition 6.5.** The functors \( d_j, j = 1, \ldots, n \) (resp. \( s_i, i = 0, \ldots, n-1 \)) induce morphisms of Green biset functors

\[
d_j : B_n \to B_{n-1} \quad \text{(resp. } s_i : B_{n-1} \to B_n \text{)}
\]

such that the identities (1), (2), (3) hold.

**Proof.** This is a simple verification. \( \square \)

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