ASYMPTOTIC STABILITY FOR A CLASS OF METRIPLECTIC SYSTEMS

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Abstract

Using the framework of metriplectic systems on $\mathbb{R}^n$ we will describe a constructive geometric method to add a dissipation term to a Hamilton-Poisson system such that any solution starting in a neighborhood of a nonlinear stable equilibrium converges towards a certain invariant set. The dissipation term depends only on the Hamiltonian function and the Casimir functions.

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1 Introduction

In an attempt for an unification of the conservative and nonconservative (or dissipative) dynamics, A.N. Kaufman [4] has introduced the notion of metriplectic system. More exactly, let $\{\cdot, \cdot\}$ be a Poisson structure on $\mathbb{R}^n$ with $\Pi$ the associated matrix i.e., $\Pi = \{x_i, x_j\}$ and $C_1, \ldots, C_k \in C^\infty(\mathbb{R}^n, \mathbb{R})$ a complete set of functionally independent Casimir functions. Let $G$ be a smooth function from $\mathbb{R}^n$ to the vector space of symmetric matrices of type $n \times n$.

Definition 1.1. ([4]) A metriplectic system on $\mathbb{R}^n$ is a system of differential equations of the following type:

$$\dot{x} = \Pi(x) \cdot \nabla H(x) + G(x) \cdot \nabla \varphi(C_1, \ldots, C_k)(x) \quad (1.1)$$

where $H \in C^\infty(\mathbb{R}^n, \mathbb{R})$ and $\varphi \in C^\infty(\mathbb{R}^k, \mathbb{R})$ such that the following conditions hold:

(M1) $\Pi \cdot \nabla C_i = 0, i = 1, k$, i.e. $C_i$ is a Casimir of our Poisson configuration $(\mathbb{R}^n, \{\cdot, \cdot\})$.

(M2) $G \cdot \nabla H = 0$.

(M3) $(\nabla \varphi(C_1, \ldots, C_k))^t \cdot G \cdot \nabla \varphi(C_1, \ldots, C_k) \leq 0$. 

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Remark 1.1. It is not hard to see that:

(i) The derivative of $H$ along the solutions of (1.1) is $\frac{dH}{dt} = 0$, i.e. $H$ is a conserved quantity of the dynamics (1.1).

(ii) The derivative of $\varphi(C_1, \ldots, C_k)$ along the solutions of (1.1) is $\frac{d\varphi(C_1, \ldots, C_k)}{dt} \leq 0$, i.e. $\varphi(C_1, \ldots, C_k)$ plays the role of the "entropy function" for the dynamics (1.1).

\[ \square \]

Remark 1.2. The dynamics (1.1) can be viewed as a perturbation of the Hamilton-Poisson system:

\[ \dot{x} = \{x, H\}, \]

with the dissipative term $G \cdot \nabla \varphi(C_1, \ldots, C_k)$.

\[ \square \]

Metriplectic systems have been extensively studied in connection with mathematical physics problems, see for instance [2], [4], [5], [6] and [8]. In [1] was proven that a certain type of dissipation induces instability. The goal of our paper is to introduces a new type of dissipation, using the formalism of metriplectic systems, in such a way that any solution of (1.1) starting in a small neighborhood of a nonlinear stable equilibrium converges towards a certain invariant set containing the equilibrium. The dissipative part can be interpreted as a set of controls added to the conservative part.

2 A class of metriplectic systems on $\mathbb{R}^n$

Let $(\mathbb{R}^n, \{\cdot, \cdot\}, H)$ be an Hamilton-Poisson system on $\mathbb{R}^n$ and $C_1, \ldots, C_k \in C^\infty(\mathbb{R}^n, \mathbb{R})$ a complete set of functionally independent Casimir functions of the Poisson vector space $(\mathbb{R}^n, \{\cdot, \cdot\})$. The dynamic is described by the following set of differential equations:

\[ \dot{x} = \Pi \cdot \nabla H. \tag{2.1} \]

Our goal is to construct explicitly a dissipative perturbation, i.e., to determine effectively a matrix $g = [g^i_j]$ such the perturbed system:

\[ \dot{x} = \Pi \cdot \nabla H + G \cdot \nabla \varphi(C_1, \ldots, C_k) \]

to be a metriplectic system.
Denote with $\partial_i H \overset{def}{=} \frac{\partial H}{\partial x_i}$ and $\partial_i C_j \overset{def}{=} \frac{\partial C_j}{\partial x_i}$. The matrix $G = [g^{ij}]$ given below satisfies all the conditions from the definition of a metriplectic system

$$
G = \begin{bmatrix}
- \sum_{i=1, i \neq 1}^{n} (\partial_i H)^2 & \partial_1 H \partial_2 H & \ldots & \partial_1 H \partial_n H \\
\partial_1 H \partial_2 H & - \sum_{i=1, i \neq 2}^{n} (\partial_i H)^2 & \ldots & \partial_2 H \partial_n H \\
\vdots & \vdots & \ddots & \vdots \\
\partial_1 H \partial_n H & \partial_2 H \partial_n H & \ldots & - \sum_{i=1, i \neq n}^{n} (\partial_i H)^2
\end{bmatrix} \tag{2.2}
$$

Indeed, we have that the $j$-component of the vector field $G \cdot \nabla H$ is given by

$$(G \cdot \nabla H)_j = \sum_{i \neq j}^{n} (\partial_i H)^2 \partial_j H + (- \sum_{i \neq j}^{n} (\partial_i H)^2) \partial_j H = 0.$$ 

Consequently, condition $(M2)$ is satisfied.

In the case when in the dissipation term we take only one Casimir function we have the following computation,

$$(\nabla C) \cdot G \cdot \nabla C = \sum_{j=1}^{n} \partial_j C (G \cdot \nabla C)_j = \sum_{j=1}^{n} \partial_j \left( \sum_{i \neq j}^{n} (\partial_i H \partial_j H - (\partial_i H)^2 \partial_j C) \right) = \partial_1 C \partial_2 H (\partial_2 C \partial_1 H - \partial_1 C \partial_2 H) + \cdots + \partial_i C \partial_n H (\partial_n C \partial_i H - \partial_i C \partial_n H) + \partial_2 C \partial_1 H (\partial_1 C \partial_2 H - \partial_2 C \partial_1 H) + \cdots + \partial_2 C \partial_n H (\partial_n C \partial_2 H - \partial_2 C \partial_n H) + \cdots + \partial_n C \partial_1 H (\partial_1 C \partial_n H - \partial_n C \partial_1 H) + \cdots + \partial_n C \partial_{n-1} H (\partial_{n-1} C \partial_n H - \partial_n C \partial_{n-1} H).$$

Regrouping the terms we obtain the desired inequality which proves that condition $(M3)$ is also satisfied,

$$(\nabla C) \cdot G \cdot \nabla C = -(\partial_1 C \partial_2 H - \partial_2 C \partial_1 H)^2 \quad - (\partial_1 C \partial_3 H - \partial_3 C \partial_1 H)^2 \quad \ldots \quad - (\partial_1 C \partial_n H - \partial_n C \partial_1 H)^2$$

$$\quad -(\partial_2 C \partial_3 H - \partial_3 C \partial_2 H)^2 \quad \ldots \quad - (\partial_2 C \partial_n H - \partial_n C \partial_2 H)^2$$

$$\quad \cdots$$

$$\quad - (\partial_{n-1} C \partial_n H - \partial_n C \partial_{n-1} H)^2 \\ \leq 0. \tag{2.3}$$
Remark 2.1. The above inequality is an equality iff \( dH \wedge dC = 0 \). Consequently, we obtain equality in \( \tilde{C} := \varphi(C_1,\ldots,C_k) \) of the complete set of functionally independent Casimir functions \( C_1,\ldots,C_k \) we obtain a new Casimir function \( \tilde{C} \) and consequently,

\[
(\nabla \varphi(C_1,\ldots,C_k))^t \cdot G \cdot \nabla \varphi(C_1,\ldots,C_k) = (\nabla \tilde{C})^t \cdot G \cdot \nabla \tilde{C} \leq 0
\]

with equality iff \( \nabla H \) and \( \nabla \varphi(C_1,\ldots,C_k) \) are linearly dependent. \( \square \)

Let us consider now the metriplectic system (1.1) where \( G \) is given by the relation (2.2). Then we can define in a canonical way two vector fields on \( \mathbb{R}^n \), namely:

\[
\xi_\Pi = \Pi \cdot \nabla H
\]

and

\[
\xi = \Pi \cdot \nabla H + G \cdot \nabla \varphi(C_1,\ldots,C_k).
\]

Proposition 2.1. Let \( (\mathbb{R}^n, \Pi, H) \) be an Hamilton-Poisson system. If \( x_0 \in \mathbb{R}^n \) is an equilibrium state of the vector field \( \xi \), i.e., \( \xi(x_0) = 0 \) then \( x_0 \) is an equilibrium state of the vector field \( \xi_\Pi \).

Proof. Indeed, \( \xi(x_0) = 0 \) implies that

\[
(\nabla \varphi(C_1,\ldots,C_k)(x_0))^t \xi(x_0) = 0,
\]

and so

\[
(\nabla \varphi(C_1,\ldots,C_k)(x_0))^t \Pi(x_0) \nabla H(x_0) + (\nabla \varphi(C_1,\ldots,C_k)(x_0))^t G(x_0) \nabla \varphi(C_1,\ldots,C_k)(x_0) = 0.
\]

This is equivalent (since \( \nabla \varphi(C_1,\ldots,C_k) \) is a Casimir we have \( (\nabla \varphi(C_1,\ldots,C_k)(x_0))^t \Pi(x_0) \nabla H(x_0) = 0 \) with

\[
(\nabla \varphi(C_1,\ldots,C_k)(x_0))^t G(x_0) \nabla \varphi(C_1,\ldots,C_k)(x_0) = 0.
\]

This leads us immediately via Remark 2.1 to

\[
\nabla H(x_0) = \lambda \nabla \varphi(C_1,\ldots,C_k)(x_0)
\]

for some \( \lambda \in \mathbb{R}^n \). Therefore

\[
\xi_\Pi(x_0) = \Pi(x_0) \nabla H(x_0) = \lambda \Pi(x_0) \nabla \varphi(C_1,\ldots,C_k)(x_0) = 0
\]

as required. \( \square \)
**Proposition 2.2.** Let $(\mathbb{R}^n, \Pi, H)$ be an Hamilton-Poisson system. Let $x_0 \in \mathbb{R}^n$ be an equilibrium point of the vector field $\xi_\Pi$. If there exists a function $\varphi \in C^\infty(\mathbb{R}^k, \mathbb{R})$ such that $\nabla \varphi(C_1, \ldots, C_k)(x_0)$ and $\nabla H(x_0)$ are linear dependent then $x_0$ is an equilibrium point of the vector field
\[
\xi = \Pi \cdot \nabla H + G \nabla \varphi(C_1, \ldots, C_k).
\]

**Proof.** Indeed, if
\[
\xi_\Pi(x_0) = 0,
\]
then we have also that
\[
\Pi(x_0) \cdot \nabla H(x_0) = 0
\]
and consequently we have two possibilities:

(i) $\nabla H(x_0) = 0$. This implies (see the construction of $G$) the equality
\[
G(x_0) = 0
\]
and then
\[
G(x_0) \cdot \nabla \varphi(C_1, \ldots, C_k)(x_0) = 0.
\]
Therefore
\[
\xi(x_0) = 0.
\]

(ii) $\Pi(x_0) \cdot \nabla H(x_0) = 0$ and $\nabla H(x_0) \neq 0$. By hypothesis we obtain
\[
G(x_0) \cdot \nabla \varphi(C_1, \ldots, C_k)(x_0) = \lambda G(x_0) \nabla H(x_0)
\]
\[
= 0
\]
and we can conclude that
\[
\xi(x_0) = 0
\]
as required.

**Corollary 2.3.** The set $E := \{x \in \mathbb{R}^n | \nabla H(x) \text{ and } \nabla \varphi(C_1, \ldots, C_k)(x) \text{ are linearly dependent}\}$ is a set of equilibrium points for both vector fields $\xi_\Pi$ and $\xi$.

**Proof.** For an arbitrary point $y \in E$ we have
\[
\Pi(y) \cdot \nabla H(y) = \lambda \Pi(y) \cdot \nabla \varphi(C_1, \ldots, C_k)(y) = 0,
\]
which shows that $y$ is a equilibrium point for the vector field $\xi_\Pi$. Proposition 2.2 implies that $y$ is also an equilibrium point for the vector field $\xi$. 

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3 Metriplectic systems and asymptotic stability

For the beginning let us briefly recall some definitions of stability for a dynamical system on $\mathbb{R}^n$

$$\dot{x} = f(x),$$ (3.1)

where $f \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$.

**Definition 3.1.** An equilibrium point $x_e$ is stable if for any small neighborhood $U$ of $x_e$ there is a neighborhood $V$ of $x_e$, $V \subset U$ such that if initially $x(0)$ is in $V$, then $x(t) \in U$ for all $t > 0$. If in addition we have:

$$\lim_{t \to \infty} x(t) = x_e$$

then $x_e$ is called asymptotically stable.

For studying more complicated asymptotic behavior we need to introduce the notion of $\omega$-limit set. Let $\phi^t$ be the flow defined by equation (3.1). The $\omega$-limit set of $x$ is $\omega(x) := \{y \in \mathbb{R}^n | \exists t_1, t_2, \ldots \to \infty \text{ s.t. } \phi(t_k, x) \to y \text{ as } k \to \infty\}$. The $\omega$-limit sets have the following properties that we will use later. For more details see Robinson [9].

(i) If $\phi(t, x) = y$ for some $t \in \mathbb{R}$, then $\omega(x) = \omega(y)$.

(ii) $\omega(x)$ is a closed subset and both positively and negatively invariant (contain complete orbits).

Next we will prove a version of LaSalle’s Theorem. For a more general result see the original work of LaSalle [2].

**Theorem 3.1.** Let $x_0$ be an equilibrium point for (3.1) and $U$ a small compact neighborhood of $x_0$. Suppose there exists $L : U \to \mathbb{R}$ a $C^1$ differentiable function with $L(x) > 0$, $L(x_0) = 0$ and $\dot{L}(x) \leq 0$. Let $E$ be the set of all points in $U$ where $\dot{L}(x) = 0$. Let $M$ be the largest invariant set in $E$. Then there exists a small neighborhood $V \subset U$ such that $\omega(x) \subset M$ for all $x \in V$.

**Proof.** The conditions of the theorem ensures the stability of $x_0$. There is a small compact neighborhood $U$ of $x_0$ and a smaller neighborhood $V \subset U$ such that $\phi(t, x) \in U$ for any $x \in V$ and $t \geq 0$. As $U$ is closed we also have $\omega(x) \subset U$.

Let $x \in V$ and $y \in \omega(x)$ be arbitrarily chosen. Since $\dot{L}(\phi(t, x)) \leq 0$ we have that $L(\phi(t, x))$ is a nonincreasing function of $t$. Because $L$ is a positive bounded function on $U$ and $\phi(t, x)$ remains for all time in $U$ we have $\lim_{t \to \infty} L(\phi(t, x)) = l$, where $0 \leq l < \infty$. As $y \in \omega(x)$ and $L$ is continuous we obtain
that $L(y) = l$. The invariance of $\omega(x)$ shows that $L(\phi(t,y)) = l$ and consequently $\dot{L}(\phi(t,y)) = 0$ for all $t \in \mathbb{R}$. Hence $y \in E$ and so $\omega(x) \subset E$. As $\omega(x)$ is invariant implies that $\omega(x) \subset M$. 

The following is the main result of this paper.

**Theorem 3.2.** Let $(\mathbb{R}^n, \Pi, H)$ be a Hamilton-Poisson system and $x_0 \in \mathbb{R}^n$ an equilibrium state for the dynamic

$$
\dot{x} = \Pi(x) \cdot \nabla H(x).
$$

(3.2)

Suppose that there exists a function $\varphi \in C^\infty(\mathbb{R}^k; \mathbb{R})$, where $k$ equals the number of functionally independent Casimirs for the Poisson structure $\Pi$, such that

(i) $\delta H_{\varphi}(x_e) = 0$

(ii) $\delta^2 H_{\varphi}(x_e)$ is positive definite,

where

$$H_{\varphi}(x) = H(x) + \varphi(C_1(x), \ldots, C_k(x))$$

with $C_1, \ldots, C_k \in C^\infty(\mathbb{R}^n; \mathbb{R})$ a set of functionally independent Casimirs of $\Pi$.

Let $G$ be the matrix defined by (2.2) then there exist a small closed and bounded neighborhood $U$ of the equilibrium state $x_e$ of the corresponding metriplectic system

$$
\dot{x} = \Pi(x) \cdot \nabla H(x) + G(x) \cdot \nabla \varphi(C_1, \ldots, C_k)(x)
$$

(3.3)

and a neighborhood $V \subset U$ such that every solution of (3.3) starting in $V$ approaches $U \cap E$ as $t \to \infty$, where $E := \{x \in \mathbb{R}^n| \nabla H(x) \text{ and } \nabla \varphi(C_1, \ldots, C_k)(x) \text{ are linearly dependent}\}$.

**Proof.** First we have to prove that $x_e$ is an equilibrium point for the dynamics (3.3). This is guarantied by Proposition 2.2.

Next we will prove that under the hypothesis of the theorem the function $L \in C^\infty(\mathbb{R}^n, \mathbb{R})$ given by

$$L(x) \overset{def}{=} H_{\varphi}(x) - H_{\varphi}(x_e)$$

is a Lyapunov function for the dynamic (3.3). More exactly, using the hypothesis and Remark 2.1 we obtain that there exists a closed and bounded neighborhood $U$ of the critical point $x_e$ such that

(i) $L(x_e) = 0$.

(ii) $L(x) > 0$, $(\forall) x \in U, x \neq x_e$
(iii) \( \dot{L}(x) \leq 0, \forall x \in U, \)

which implies that \( x_e \) is a stable equilibrium point for (3.3).

By Remark 2.1 we have that \( E \) equals the set of all points where \( \dot{L}(x) = 0 \). Using Corollary 2.3 we have that the largest invariant subset in \( E \) for (3.3) equals \( E \).

We showed that all the conditions of the Theorem 3.1 are satisfied and so we obtain the desired result.

\[ \square \]

**Remark 3.1.** The above result tells us in fact how to built in an effective way a set controls which locally asymptotically stabilize a nonlinear stable equilibrium state of a given Hamilton-Poisson dynamics.

\[ \square \]

### 4 Examples

It is well known that Euler’s angular momentum equations of the free rigid body can be written on \( \mathbb{R}^3 \) in the following form:

\[
\begin{align*}
\dot{x}_1 &= a_1 x_2 x_3 \\
\dot{x}_2 &= a_2 x_1 x_3 \\
\dot{x}_3 &= a_3 x_1 x_2
\end{align*}
\]

where

\[
a_1 = \frac{1}{I_3} - \frac{1}{I_2}; \quad a_2 = \frac{1}{I_3} - \frac{1}{I_1}; \quad a_3 = \frac{1}{I_2} - \frac{1}{I_1};
\]

\( I_1, I_2, I_3 \) being the components of the inertia tensor and we suppose as usually that

\[ I_1 > I_2 > I_3 > 0. \]

The equations (4.1) have the following Hamilton-Poisson realization:

\[
((so(3))^* \approx \mathbb{R}^3, \{\cdot, \cdot\}_-, H)
\]

where \( \{\cdot, \cdot\}_- \) is minus-Lie-Poisson structure on \((so(3))^* \approx \mathbb{R}^3\), generated by the matrix:

\[
\mathbf{\Pi}_- = \begin{bmatrix}
0 & -x_3 & x_2 \\
x_3 & 0 & -x_1 \\
-x_2 & x_1 & 0
\end{bmatrix}
\]

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and the Hamiltonian $H$ is given by:

$$H(x_1, x_2, x_3) = \frac{1}{2} \left( \frac{x_2^2}{I_1} + \frac{x_2^2}{I_2} + \frac{x_3^2}{I_3} \right). \quad (4.2)$$

It is not hard to see that the function $C \in C^\infty(\mathbb{R}^3, \mathbb{R})$ given by:

$$C(x_1, x_2, x_3) = \frac{1}{2} (x_1^2 + x_2^2 + x_3^2) \quad (4.3)$$

is a Casimir of our configuration ($(so(3))^* \approx \mathbb{R}^3, \{\cdot, \cdot\}_\mathbb{L}$).

Let $(M_0, 0, 0)$ be an equilibrium point for (4.1). The function $H_\varphi(x) = H(x) + \varphi(C(x))$, where $\varphi$ is given for instance by

$$\varphi(s) = \left( s - \frac{1}{2}M_0^2 \right)^2 - \frac{s}{I_1},$$

satisfies the conditions (i) and (ii) of Theorem 3.2. In this case the perturbed system (3.3) is given by

$$\begin{cases}
\dot{x}_1 = a_1 x_2 x_3 + x_1 (x_2^2 + x_2^2 + x_3^2) - c (\frac{a_3}{I_2} x_2^2 + \frac{a_2}{I_3} x_3^2) \\
\dot{x}_2 = a_2 x_1 x_3 + x_2 (x_1^2 + x_2^2 + x_3^2) - c (\frac{a_3}{I_1} x_1^2 - \frac{a_1}{I_3} x_3^2) \\
\dot{x}_3 = a_3 x_1 x_2 + x_3 (x_1^2 + x_2^2 + x_3^2) - c (\frac{a_2}{I_1} x_1^2 + \frac{a_1}{I_2} x_2^2)
\end{cases} \quad (4.4)$$

where $c = M_0^2 - \frac{1}{I_1}$.

The set of points in $\mathbb{R}^3$ where $\nabla H(x)$ and $\nabla \varphi(C(x))$ are linearly dependent is given by

$$E = \{(\lambda, 0, 0)|\lambda \in \mathbb{R}\} \cup \{(0, \lambda, 0)|\lambda \in \mathbb{R}\} \cup \{(0, 0, \lambda)|\lambda \in \mathbb{R}\}.$$

By Theorem 3.2 we obtain that there exists a small closed and bounded neighborhood $U \subset \mathbb{R}^3$ around the equilibrium point $(M_0, 0, 0)$ and $V \subset U$ such that any solution of (4.1) starting in $V$ approaches the set $\{(M_0 + \lambda, 0, 0)|\lambda \in [-\epsilon, \epsilon] \subset \mathbb{R}\}$ as $t \to \infty$.

A similar result with obvious modifications can be also obtain for the equilibrium state $(0, 0, M_0)$, $M_0 \in \mathbb{R}$.

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