CONTINUED FRACTIONS, STATISTICS, AND GENERALIZED PATTERNS

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Abstract

Recently, Babson and Steingrimsson (see [BS]) introduced generalized permutations patterns that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation.

Following [BCS], let $e_k\pi$ (respectively; $f_k\pi$) be the number of the occurrences of the generalized pattern 12-3-...-k (respectively; 21-3-...-k) in $\pi$. In the present note, we study the distribution of the statistics $e_k\pi$ and $f_k\pi$ in a permutation avoiding the classical pattern 1-3-2.

Also we present an applications, which relates the Narayana numbers, Catalan numbers, and increasing subsequences, to permutations avoiding the classical pattern 1-3-2 according to a given statistics on $e_k\pi$, or on $f_k\pi$.

1. Introduction

Permutation patterns: let $\alpha \in S_n$ and $\tau \in S_k$ be two permutations. We say that $\alpha$ contains $\tau$ if there exists a subsequence $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $(\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k})$ is order-isomorphic to $\tau$; in such a context $\tau$ is usually called a pattern or (classical pattern). We say $\alpha$ avoids $\tau$, or is $\tau$-avoiding, if such a subsequence does not exist. The set of all $\tau$-avoiding permutations in $S_n$ is denoted $S_n(\tau)$. For an arbitrary finite collection of patterns $T$, we say that $\alpha$ avoids $T$ if $\alpha$ avoids any $\tau \in T$; the corresponding subset of $S_n$ is denoted $S_n(T)$.

While the case of permutations avoiding a single pattern has attracted much attention, the case of multiple pattern avoidance remains less investigated. In particular, it is natural, as the next step, to consider permutations avoiding pairs of patterns $\tau_1$, $\tau_2$. This problem was solved completely for $\tau_1, \tau_2 \in S_3$ (see [SS]), for $\tau_1 \in S_3$ and $\tau_2 \in S_4$ (see [W]), and for $\tau_1, \tau_2 \in S_4$ (see [B, K] and references therein). Several recent papers [CW, MV1, Kr, MV2, MV3] deal with the case $\tau_1 \in S_3, \tau_2 \in S_k$ for various pairs $\tau_1, \tau_2$. Another natural question is to study permutations avoiding $\tau_1$ and containing $\tau_2$ exactly $t$ times. Such a problem for certain $\tau_1, \tau_2 \in S_3$ and $t = 1$ was investigated in [R], and for certain $\tau_1 \in S_3, \tau_2 \in S_k$ in [RW2, MV1, Kr, MV2]. The tools involved in these papers include continued fractions, Chebyshev polynomials, and Dyck paths.

Generalized permutation patterns: in [BS] Babson and Steingrimsson introduced generalized permutation patterns that allow the requirement that two
adjacent letters in a pattern must be adjacent in the permutation. This idea in introducing these patterns was study of Mahonian statistics.

We write a classical pattern with dashes between any two adjacent letters of the pattern, say 1324, as 1-3-2-4, and if we write, say 24-1-3, then we mean that if this pattern occurs in permutation \( \pi \), then the letters in the permutation \( \pi \) that correspond to 2 and 4 are adjacent (see [C]). For example, the permutation \( \pi = 35421 \) has only two occurrences of the pattern 23-1, namely the subsequences 352 and 351, whereas \( \pi \) has four occurrences of the pattern 2-3-1, namely the subsequences 352, 351, 342 and 341.

Claesson [C] gave a complete answer for the number of permutations avoiding any single 3-letters generalized pattern with exactly one adjacent pair of letters. Later, Claesson and Mansour [CM] presented a complete solution for the number of permutations avoiding any double 3-letters generalized patterns with exactly one adjacent pair of letters. Besides, Kitaev [Ki] investigate simultaneous avoidance of two or more 3-letters generalized patterns without internal dashes.

On the other hand, Robertson, Wilf and Zeilberger [RWZ] showed a simple continued fraction that records the joint distribution of the patterns 1-2 and 1-2-3 on 1-3-2-avoiding permutations. Recently, generalization of this theorem given, by Mansour and Vainshtein [MV1], by Krattenthaler [Kr], by Jani and Rieper [JR], and by Brändén, Claesson and Steingrimsson [BCS].

Mansour [M, Th. 2.1 and Th. 2.9] presented an analog of this theorem ([MV1]) by replace generalized patterns with classical patterns.

In the present note, we generalize [M, Th. 2.1] and [M, Th. 2.9] and give an analog for [BCS] by replace generalized patterns with classical patterns. This by use the same argument proof of the main results in [BCS] with a simple changes. In the last section, we present an applications for our results.

2. MAIN RESULTS

For all \( k \geq 1 \), we denote by \( a_k(\pi) \) the number of the occurrences of the pattern 1-2-3-\ldots-k in \( \pi \). In [BCS, Th. 1] proved the following.

**Theorem 2.1.** (P. Brändén, A. Claesson, and E. Steingrimsson, [BCS Th. 1])

The \( \sum_{\pi \in S(1-3-2)} \prod_{k \geq 1} a_k(\pi) \) is given by the following continued fraction:

\[
\frac{1}{1 - \frac{x_1^{(0)}}{x_1^{(1)} x_2^{(1)}} \frac{1}{1 - \frac{x_1^{(0)} x_2^{(0)}}{x_1^{(2)} x_2^{(2)} x_3^{(2)}} \cdots}}
\]

in which the \((n+1)st\) numerator is \( \prod_{k=0}^{n} x_{k+1}^{(n)} \).

For all \( k \geq 3 \), we denote by \( e_k(\pi) \) (respectively; \( f_k(\pi) \)) the number of the occurrences of the generalized pattern 12-3-\ldots-k (respectively; 21-3-\ldots-k) in \( \pi \). Besides,
\( e_2(\pi), f_2(\pi), \) and \( e_1(\pi) = f_1(\pi) \) are denote the number of occurrences of the pattern 12, 21 and 1, respectively, and extended that by \( e_0(\pi) = f_0(\pi) = 1. \)

Now let us give an analog for Theorem 2.1 by used generalized pattern with simply changes in the proof of Theorem 2.1.

**Theorem 2.2.** \( \sum_{\pi \in S(1\rightarrow 3\rightarrow 2)} \prod_{k \geq 1} x_e(\pi) \) is given by the following continued fraction:

\[
\frac{1}{1 - x_1 + x_1 x_2^{(0)} - \frac{x_1 x_2^{(0)}}{1 - x_1 + x_1 x_2^{(1)} x_3^{(1)} - \frac{x_1 x_2^{(1)} x_3^{(1)}}{1 - x_1 + x_1 x_2^{(2)} x_3^{(2)} x_4^{(2)} - \frac{x_1 x_2^{(2)} x_3^{(2)} x_4^{(2)}}{\ddots}}}}
\]

in which the \((n + 1)st\) numerator is \( x_1 \cdot \prod_{k=0}^{n} x_e(k+2). \)

**Proof.** Let \( \pi \) any noneempty permutation avoiding 1-2-3 such that \( \pi = (\pi', n, \pi'') \) and \( j = \pi^{-1}(n); \) so every element in \( \pi' \) is greater than every element in \( \pi'' \). Thus \( \pi', \pi'' \in S(1\rightarrow 3\rightarrow 2), \) so

\[
e_k(\pi) = \begin{cases} 
  e_k(\pi') + e_{k-1}(\pi') + e_k(\pi''), & \text{for all } k \geq 3; \\
  e_2(\pi') + e_2(\pi'') + \delta_{\pi',\emptyset}, & k = 2; \\
  e_1(\pi') + e_1(\pi'') + 1, & k = 1
\end{cases}
\]

where \( \delta_{\pi',\emptyset} \) is the Kronecker delta.

It follows that the generating function

\[
C(x_1, x_2, \ldots) = \sum_{\pi \in S(1\rightarrow 3\rightarrow 2)} \prod_{k \geq 1} x_e(\pi)
\]

satisfies (see [M, Th. 2.1])

\[
C(x_1, x_2, \ldots) = 1 + x_1 C(x_1, x_2, \ldots) + x_1 x_2 C(x_1, x_2, \ldots) (C(x_1, x_2 x_3, x_3 x_4, \ldots) - 1),
\]

which means that

\[
C(x_1, x_2, x_3, x_4, \ldots) = \frac{1}{1 - x_1 + x_1 x_2 - x_1 x_2 C(x_1, x_2 x_3, x_3 x_4, \ldots)}
\]

and the theorem follows by induction. \( \square \)

Similarly, it is easy to see

\[
f_k(\pi) = \begin{cases} 
  f_k(\pi') + f_{k-1}(\pi') + f_k(\pi''), & \text{for all } k \geq 3; \\
  f_2(\pi') + f_2(\pi'') + \delta_{\pi',\emptyset}, & k = 2; \\
  f_1(\pi') + f_1(\pi'') + 1, & k = 1
\end{cases}
\]

where \( \delta_{\pi',\emptyset} \) is the Kronecker delta. So by using [M, Th. 2.9] and the argument proof of Theorem 2.1 we get the following.
Theorem 2.3. The \( \sum_{\pi \in S(1-3-2)} \prod_{k \geq 1} x_k^{(\pi)} \) is given by the following continued fraction:

\[
1 - \frac{x_1}{x_1 x_2 - \frac{x_1}{x_1 x_2 x_3 - \frac{x_1}{x_1 x_2 x_3 x_4 - \ddots}}}.
\]

Following [BCS], we define \( \mathcal{A} \) be the ring of all infinite matrices with finite number of non zero entries in each row, that is,

\[
\mathcal{A} = \{ A : \mathcal{N} \times \mathcal{N} \to \mathbb{Z} \mid \forall n(A(n, k) = 0 \text{ for almost every } k) \},
\]

with multiplication defined by \( (AB)(n, k) = \sum_{i \geq 1} A(n, i)B(i, k) \). With each \( A \in \mathcal{A} \) we now associate a family of statistics \( \{<q, A_k>\}_{k \geq 1} \) defined on \( S(1-3-2) \), where \( q = (q_1, q_2, q_3, \ldots) \) and

\[
<q, A_k> = \sum_{i \geq 1} A(i, k)q_i.
\]

Following [BCS], let us define mathematical objects with respect \( A \) as follows. Let \( q = (q_1, q_2, \ldots) \) where the \( q_i \) are indeterminates; for each \( A \in \mathcal{A} \) and \( \pi \in S(1-3-2) \) we define three objects as follows:

The weight \( \eta(\pi, A; q) \), the weight \( \mu(\pi, A; q) \), and the weight \( \nu(\pi, A; q) \) of \( \pi \) with respect \( A \), by

\[
\eta(\pi, A; q) = \prod_{k \geq 1} q_{k}^{<a, A_k>^\pi}, \quad \mu(\pi, A; q) = \prod_{k \geq 1} q_{k}^{<e, A_k>^\pi}, \quad \nu(\pi, A; q) = \prod_{k \geq 1} q_{k}^{<f, A_k>^\pi},
\]

respectively, where \( a = (a_1, a_2, \ldots) \), \( e = (e_1, e_2, \ldots) \) and \( f = (f_1, f_2, \ldots) \).

The generating function with respect \( A \) of the, statistics \( \{<a, A_k>\}_{k \geq 1} \), statistics \( \{<e, A_k>\}_{k \geq 1} \), and statistics \( \{<f, A_k>\}_{k \geq 1} \) by

\[
F_A(q) = \sum_{\pi \in S(1-3-2)} \eta(\pi, A; q), \quad G_A(q) = \sum_{\pi \in S(1-3-2)} \mu(\pi, A; q), \quad H_A(q) = \sum_{\pi \in S(1-3-2)} \nu(\pi, A; q),
\]

respectively.

The continued fractions with respect \( A \), by

\[
C_A(q) = \frac{1}{\prod_{k \geq 0} q_{k+1}^{A(1,k)}}, \quad
D_A(q) = \frac{1}{1 - \prod_{k \geq 0} q_{k+1}^{A(2,k)}},
\]

\[
1 - q_1 + q_1 \prod_{k \geq 1} q_{k+1}^{A(1,k)} - \frac{q_1 \prod_{k \geq 1} q_{k+1}^{A(1,k)}}{1 - q_1 + q_1 \prod_{k \geq 1} q_{k+1}^{A(2,k)}} - \ddots.
\]

\[
1 - q_1 + q_1 \prod_{k \geq 1} q_{k+1}^{A(1,k)} - \frac{q_1 \prod_{k \geq 1} q_{k+1}^{A(1,k)}}{1 - q_1 + q_1 \prod_{k \geq 1} q_{k+1}^{A(2,k)}} - \ddots.
\]
and by

\[ E_A(q) = 1 - \frac{q_1 \prod_{k \geq 1} q_k^{A(1,k)} - 1}{q_1 \prod_{k \geq 1} q_k^{A(2,k)} - 1}. \]

As a remark (see [BCS, p. 3]): definition of the ring \( A \), and the fact that
\( a_i(\pi) = c_i(\pi) = f_i(\pi) = 0 \) for all \( i = |\pi| + 1, |\pi| + 2, \ldots \) yields the product in the weight is finite.

The second step in [BCS] proved the following.

**Theorem 2.4.** (P. Brändén, A. Claesson, and E. Steingrimsson, [BCS, Th. 2])

For \( A \in A \), \( F_A(q) = C_{BA}(q) \) where \( B = \left[ \binom{i}{j} \right] \), and conversely \( C_A(q) = F_{B^{-1}}A(q) \).

By above definitions with using the proof of Theorem 2.4 we get an analog of Theorem 2.5 for the statistics \( e_k \) and \( f_k \) as follows.

**Theorem 2.5.** Let \( A \in A \); then

\[ G_A(q) = D_{BA}(q), \quad H_A(q) = E_{BA}(q), \]
\[ D_A(q) = G_{B^{-1}}A(q), \quad E_A(q) = H_{B^{-1}}A(q), \]

where \( B = \left[ \binom{i}{j} \right], (\binom{n}{k}) = 0 \) for all \( k > 0 \) or \( n > k \).

**Proof.** By using definitions we get that (here, we using the same argument proof in [BCS, Th. 2])

\[ \mu(\pi, A; q) = \prod_{k \geq 1} q_k^{A(e_k)\pi} \]
\[ = \prod_{k \geq 1} \prod_{j \geq 1} q_k^{A(j,k)e_j(\pi)} \]
\[ = \prod_{j \geq 1} \left( \prod_{k \geq 1} q_k^{A(j,k)} \right)^{e_j(\pi)}. \]

Let \( x_{j+1} = \prod_{k \geq 1} q_k^{A(j,k)} \); Theorem 2.2 yields

\[ \prod_{j \geq 1} \binom{\pi - 1}{j+1} = \prod_{j \geq 1} \left( \prod_{k \geq 1} q_k^{A(j,k)} \right)^{\binom{n-1}{j-1}} \]
\[ = \prod_{k \geq 1} q_k^{\binom{n-1}{0}, \binom{n-1}{1}, \ldots}.A_k >. \]

so, again, by definitions \( G_A(q) = D_{BA}(q) \). Observing that \( B^{-1} = \left[ (-1)^{i-j} \binom{i}{j} \right] \in A \) we also obtain \( D_A(q) = G_{B^{-1}}A(q) \).

Similarly, we have \( E_{BA}(q) = H_A(q) \) and \( E_A(q) = H_{B^{-1}}A(q) \). \( \square \)

**Remark 2.6.** The general approach which described in [BCS] on the statistics \( a_k(\pi) \) its work with others statistics. Its work on the statistics \( e_k(\pi) \) and on the statistics \( f_k(\pi) \). So the natural question to ask is the following: If there any descriptions for all the statistics such this approach will work? Here we failed to give an answer for this.

### 3. Application

In the current section, we present an examples for application of Theorem 2.5. Some of these examples are related known continued fraction to the statistics \( e_k \) or \( f_k \), but others relate these statistics to others combinatorial objects.
3.1. Narayana numbers. Let \( N(n, k) = \frac{1}{n} \binom{n}{k} \binom{n}{k+1} \) be the Narayana numbers, and the corresponding generating function for the Narayana numbers we denote by \( N(x, t) \). Then
\[
N(x, t) := \sum_{n,k \geq 0} N(n, k) x^k t^n = 1 + xtN^2(x, t) - xtN(x, t) + tN(x, t).
\]
This allows us to express \( N(x, t) \) as a continued fraction:
\[
N(x, t) = \frac{1}{1 - \frac{tx}{1 - \frac{tx}{1 - \frac{tx}{\ddots}}}}.
\]

Proposition 3.1. The number \( N(n, k) \) equals the number of permutations \( \pi \in S_n(132) \) with \( e_2(\pi) = k \).

Proof. Let \( A(n, k) = \delta(n, k), (1,1) + \delta(n, k), (2,2) \) where \( \delta \) is Kronecker delta, so by applying Theorem 2.3 we get that
\[
N'(x, t) := \sum_{\pi \in S(132)} x^{e_2(\pi)} t^{e_1(\pi)} = 1 - \frac{tx}{1 - \frac{tx}{1 - \frac{tx}{\ddots}}};
\]
so \( N'(x, t) \) satisfies the same functional equation as \( N(x, t) \), hence \( N'(x, t) = N(x, t) \).

Again, the same argument works also to statistics on \( f_k \) as follows.

Proposition 3.2. The number \( N(n, k) \) equals the number of permutations \( \pi \in S_n(132) \) with \( f_2(\pi) = k \).

Proof. Let \( A(n, k) = \delta(n, k), (1,1) + \delta(n, k), (2,2) \) where \( \delta \) is Kronecker delta, so by applying Theorem 2.3 we get that
\[
N''(x, t) := \sum_{\pi \in S(132)} x^{f_2(\pi)} t^{f_1(\pi)} = 1 - \frac{tx}{1 - \frac{tx}{1 - \frac{tx}{\ddots}}};
\]
so \( N''(x, t) \) satisfies the same functional equation as \( N(x, t) \), hence \( N''(x, t) = N(x, t) \).

3.2. Increasing subsequences. As consequence of [BCS], we define as follows. The subsequence \( \pi_i, \pi_{i+1}, \pi_{i+2} \ldots, \pi_{i_k} \) \( (k \geq 2) \) of \( \pi \) is called a 2-increasing subsequence if \( \pi_{i_j} < \pi_{i_{j+1}}, i_j < i_{j+1} \) and \( i_j + 1 < i_{j+2} \). Hence, the total number of 2-increasing subsequences in a permutation is counting by \( e_2 + e_3 + \ldots \). An application of
Theorem 2.5 gives the following continued fraction for the distribution of $e_2 + e_3 + \ldots$:

$$
\sum_{\pi \in S(1\text{-}3\text{-}2)} x^{e_2(\pi) + e_3(\pi) + \ldots |\pi|} = \frac{1}{1 - t(1 - x) - \frac{x^2 t}{1 - t(1 - x^2) - \frac{x^4 t}{1 - t(1 - x^4) - \frac{x^8 t}{\ddots}}}}.
$$

The subsequence $\pi_{i_1} \pi_{i_1+1} \pi_{i_2} \ldots \pi_{i_k} \ (k \geq 2)$ of $\pi$ is called *almost 2-increasing subsequence* if $\pi_{i_j} < \pi_{i_j+1}, \ i_j < i_{j+1}$ for $j = 2, 3, \ldots, k - 1, \ i_1 + 1 < i_2$ and $\pi_{i_1+1} < p_1$. Hence, the total number of almost 2-increasing subsequences in a permutation is counting by $f_2 + f_3 + \ldots$. An application of Theorem 2.5 gives the following continued fraction for the distribution of $f_2 + f_3 + \ldots$:

$$
\sum_{\pi \in S(1\text{-}3\text{-}2)} x^{f_2(\pi) + f_3(\pi) + \ldots |\pi|} = 1 - \frac{t}{1 - \frac{1}{x} - \frac{1}{x^2 - \frac{1}{x} - \frac{1}{x^4 - \frac{1}{x} - \frac{1}{x^8 - \ddots}}}}.
$$

### 3.3. Catalan numbers

The $n$th Catalan number is given by $C_n = \frac{1}{n+1} \binom{2n}{n}$, and the corresponding generating function is given by $C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$. Theorem 2.4 yields for the statistic $s = 0 \ (s(\pi) = 0 \ for \ all \ \pi)$ the following. The generating function $C(x)$ for the number of permutations avoiding 1-3-2 can be expressed, again, in terms of continued fractions:

$$
C'(x) := \frac{1}{1 - \frac{x}{1 - \frac{x}{x - \frac{x}{\ddots}}}}, \quad C''(x) := 1 - \frac{x}{1 - \frac{x}{1 - \frac{x}{x - \frac{x}{\ddots}}}}.
$$

In the above two cases $C'(x) = C''(x) = C(x)$.

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