Quantum Diagonalization Method in the Tavis–Cummings Model

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Abstract

To obtain the explicit form of evolution operator in the Tavis–Cummings model we must calculate the term \( e^{-itg(S_+ \otimes a + S_- \otimes a^\dagger)} \) explicitly which is very hard. In this paper we try to make the quantum matrix \( A \equiv S_+ \otimes a + S_- \otimes a^\dagger \) diagonal to calculate \( e^{-itgA} \) and, moreover, to know a deep structure of the model.

For the case of one, two and three atoms we give such a diagonalization which is first nontrivial examples as far as we know, and reproduce the calculations of \( e^{-itgA} \) given in quant-ph/0404034. We also give a hint to an application to a noncommutative differential geometry.

However, a quantum diagonalization is not unique and is affected by some ambiguity arising from the noncommutativity of operators in quantum physics.

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Our method may open a new point of view in Mathematical Physics or Quantum Physics.

1 Introduction

The purpose of this paper is to give a new insight to the Tavis–Cummings model ([1]) and to obtain the explicit form of evolution operator by the new method in the case of some atoms.

This model is a very important one in Quantum Optics and (maybe even) in Mathematical Physics, and has been studied widely, see [2] as general textbooks in quantum optics.

We are studying a quantum computation and therefore want to study the model from this point of view, namely the quantum computation based on atoms of laser–cooled and trapped linearly in a cavity. We must in this model construct the controlled NOT gate or other controlled unitary gates to perform the quantum computation, see [3] as a general introduction to this subject.

For that aim we need the explicit form of evolution operator of the model in the case of (at least) one, two and three atoms. As to the model of one atom or two atoms it is more or less known (see [4]), while as to the case of three atoms it was given by [5]. However, the method is not clear enough in a mathematical sense ¹.

In this paper we present a quantum diagonalization method which is a quantum version of classical diagonalization and obtain the explicit form of evolution operator obtained in [5].

However, the quantum diagonalization is not unique and is affected by some ambiguity due to the noncommutativity of operators in Quantum Physics. This may be related to the so–called operator ordering problem, see for example [6] on this topics.

The Quantum Diagonalization Method is completely new and may be applied to a noncommutative differential geometry (for example, the noncommutative chiral models) because we can construct (quantum) unitary matrices explicitly. However, this is beyond our scope of the paper.

¹We used Mathematica in the process of calculation
2 Tavis–Cummings Model and Evolution Operator

We make a review of [5] within our necessity. The Tavis–Cummings model (with $n$–atoms) that we will treat in this paper can be written as follows (we set $\hbar = 1$ for simplicity).

$$H = \omega_1 L \otimes a^\dagger a + \frac{\Delta}{2} \sum_{i=1}^{n} \sigma_i^{(3)} \otimes 1 + g \sum_{i=1}^{n} \left( \sigma_i^{(+)} \otimes a + \sigma_i^{(-)} \otimes a^\dagger \right),$$

where $\omega$ is the frequency of radiation field, $\Delta$ the energy difference of two level atoms, $a$ and $a^\dagger$ are annihilation and creation operators of the field, and $g$ a coupling constant, and $L = 2^n$. Here $\sigma_i^{(+)}$, $\sigma_i^{(-)}$ and $\sigma_i^{(3)}$ are given as

$$\sigma_i^{(s)} = 1_2 \otimes \cdots \otimes 1_2 \otimes \sigma \otimes 1_2 \otimes \cdots \otimes 1_2 \ (i \text{– position}) \in M(L, \mathbb{C})$$

where $s$ is $+$, $-$ and $3$ respectively and

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \frac{1}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$  \hspace{1cm} (3)

Here let us rewrite the hamiltonian (1). If we set

$$S_+ = \sum_{i=1}^{n} \sigma_i^{(+)}, \quad S_- = \sum_{i=1}^{n} \sigma_i^{(-)}, \quad S_3 = \frac{1}{2} \sum_{i=1}^{n} \sigma_i^{(3)},$$

then (1) can be written as

$$H = \omega_1 L \otimes a^\dagger a + \Delta S_3 \otimes 1 + g \left( S_+ \otimes a + S_- \otimes a^\dagger \right) \equiv H_0 + V,$$

which is very clear. We note that $\{S_+, S_-, S_3\}$ satisfy the $su(2)$–relation

$$[S_3, S_+] = S_+, \quad [S_3, S_-] = -S_-, \quad [S_+, S_-] = 2S_3.$$ \hspace{1cm} (6)

However, the representation $\rho$ defined by $\rho(\sigma_+) = S_+$, $\rho(\sigma_-) = S_-$, $\rho(\sigma_3/2) = S_3$ is a reducible representation of $su(2)$.

We would like to solve the Schrödinger equation

$$i \frac{d}{dt} U = HU = (H_0 + V) U,$$ 

3
where $U$ is a unitary operator (called the evolution operator). We can solve this equation by using the **method of constant variation**. The result is well–known to be

$$U(t) = \left(e^{-it\omega S_3} \otimes e^{-it\omega a \dagger} a\right)e^{-itg(S_+ \otimes a + S_- \otimes a \dagger)}$$  \hspace{1cm} (8)

under the resonance condition $\Delta = \omega$, where we have dropped the constant unitary operator for simplicity. Therefore we have only to calculate the term (8) explicitly, which is however a very hard task \footnote{The situation is very similar to that of the paper quant-ph/0312060 in [7]}. In the following we set

$$A = S_+ \otimes a + S_- \otimes a \dagger$$  \hspace{1cm} (9)

for simplicity. We can determine $e^{-itgA}$ for $n = 1$ (one atom case), $n = 2$ (two atoms case) and $n = 3$ (three atoms case) completely.

**One Atom Case** In this case $A$ in (9) is written as

$$A_1 = \begin{pmatrix} 0 & a \\ a \dagger & 0 \end{pmatrix} \equiv B_{1/2}.$$  \hspace{1cm} (10)

By making use of the relation

$$A_1^2 = \begin{pmatrix} aa \dagger & 0 \\ 0 & a \dagger a \end{pmatrix} = \begin{pmatrix} N+1 & 0 \\ 0 & N \end{pmatrix}$$  \hspace{1cm} (11)

with the number operator $N$ we have

$$e^{-itgA_1} = \begin{pmatrix} \cos\left(tg\sqrt{N+1}\right) & -i\frac{\sin\left(tg\sqrt{N+1}\right)}{\sqrt{N+1}}a \\ -id\frac{\sin\left(tg\sqrt{N+1}\right)}{\sqrt{N+1}} & \cos\left(tg\sqrt{N}\right) \end{pmatrix}.$$  \hspace{1cm} (12)

We obtained the explicit form of solution. However, this form is more or less well–known, see for example the second book in [2].

**Two Atoms Case** In this case $A$ in (9) is written as

$$A_2 = \begin{pmatrix} 0 & a & a & 0 \\ a \dagger & 0 & 0 & a \\ a \dagger & 0 & 0 & a \\ 0 & a \dagger & a \dagger & 0 \end{pmatrix}.$$  \hspace{1cm} (13)
Our method is to reduce the $4 \times 4$–matrix $A_2$ in (13) to a $3 \times 3$–matrix $B_1$ in the following to make our calculation easier. For that aim we prepare the following matrix

$$T = \begin{pmatrix}
0 & 1 & 0 & 0 \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\
-\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},$$

then it is easy to see

$$T^\dagger A_2 T = \begin{pmatrix}
0 & 0 & \sqrt{2}a & 0 \\
\sqrt{2}a^\dagger & 0 & \sqrt{2}a & 0 \\
0 & \sqrt{2}a^\dagger & 0 & 0
\end{pmatrix} \equiv \begin{pmatrix}
0 \\
0 \\
B_1
\end{pmatrix}$$

(14)

where $B_1 = J_+ \otimes a + J_- \otimes a^\dagger$ and $\{J_+, J_-\}$ are just generators of (spin one) irreducible representation of (3). We note that this means a well–known decomposition of spin $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$.

Therefore to calculate $e^{-itgA_2}$ we have only to do $e^{-itgB_1}$. Noting the relation

$$B_1^3 = \begin{pmatrix}
2(2N+3) \\
2(2N+1) \\
2(2N-1)
\end{pmatrix} B \equiv DB_1,$$

we obtain

$$e^{-itgB_1} = \begin{pmatrix}
1 + \frac{2N+2}{2N+3} f(N+1) & -ih(N+1)a & \frac{2}{2N+3} f(N+1)a^2 \\
-ia^\dagger h(N+1) & 1 + 2f(N) & -ih(N)a \\
(a^\dagger)^2 \frac{2}{2N+3} f(N+1) & -ia^\dagger h(N) & 1 + \frac{2N}{2N-1} f(N-1)
\end{pmatrix}$$

(15)

where

$$f(N) = \frac{-1 + \cos \left( tg \sqrt{2(2N+1)} \right)}{2}, \quad h(N) = \frac{\sin \left( tg \sqrt{2(2N+1)} \right)}{\sqrt{2N+1}}.$$
Three Atoms Case  In this case $A$ in (9) is written as

$$A_3 = \begin{pmatrix}
0 & a & a & 0 & a & 0 & 0 & 0 \\
a^\dagger & 0 & 0 & a & 0 & a & 0 & 0 \\
a^\dagger & 0 & 0 & a & 0 & 0 & a & 0 \\
0 & a^\dagger & a^\dagger & 0 & 0 & 0 & 0 & a \\
a^\dagger & 0 & 0 & 0 & 0 & a & a & 0 \\
0 & a^\dagger & 0 & 0 & a^\dagger & 0 & 0 & a \\
0 & 0 & a^\dagger & 0 & a^\dagger & 0 & 0 & a \\
0 & 0 & 0 & a^\dagger & 0 & a^\dagger & a^\dagger & 0 \\
\end{pmatrix}, \quad (16)$$

We would like to look for the explicit form of solution like (12) or (15). If we set

$$T = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{3}} & \sqrt{3} & 0 & 0 & \frac{1}{\sqrt{3}} \\
0 & 0 & \frac{-1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{6}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 \\
0 & \frac{-1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{6}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}, \quad (17)$$

then it is not difficult to see

$$T^\dagger A_3 T = \begin{pmatrix}
0 & a \\
a^\dagger & 0 \\
0 & a \\
a^\dagger & 0 \\
0 & \sqrt{3}a & 0 & 0 \\
\sqrt{3}a^\dagger & 0 & 2a & 0 \\
0 & 2a^\dagger & 0 & \sqrt{3}a \\
0 & 0 & \sqrt{3}a^\dagger & 0 \\
\end{pmatrix} \equiv \begin{pmatrix}B_{1/2} \\
B_{1/2} \\
\end{pmatrix}. \quad (17)$$
This means a decomposition of spin $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{3}{2}$. Therefore we have only to calculate $e^{-itgB_{3/2}}$, which is however not easy. The result is

$$e^{-itgB_{3/2}} = \begin{pmatrix}
    f_2(N + 2) & -\sqrt{3}iF_1(N + 2)a & 2\sqrt{3}h_1(N + 2)a^2 & -6iH_0(N + 2)a^3 \\
    -\sqrt{3}iF_1(N + 1)a^\dagger & f_1(N + 1) & -2iH_1(N + 1)a & 2\sqrt{3}h_1(N + 1)a^2 \\
    2\sqrt{3}h_1(N)(a^\dagger)^2 & -2iH_1(N)a^\dagger & f_0(N) & -\sqrt{3}iF_0(N)a \\
    -6iH_0(N - 1)(a^\dagger)^3 & 2\sqrt{3}h_1(N - 1)(a^\dagger)^2 & -\sqrt{3}iF_0(N - 1)a^\dagger & f_{-1}(N - 1)
\end{pmatrix}$$

(18)

where

$$f_2(N) = \left\{ v_+(N)\cos(tg\sqrt{\lambda_+(N)}) - v_-(N)\cos(tg\sqrt{\lambda_-(N)}) \right\} / (2\sqrt{d(N)}),$$

$$f_1(N) = \left\{ w_+(N)\cos(tg\sqrt{\lambda_+(N)}) - w_-(N)\cos(tg\sqrt{\lambda_-(N)}) \right\} / (2\sqrt{d(N)}),$$

$$f_0(N) = \left\{ v_+(N)\cos(tg\sqrt{\lambda_-(N)}) - v_-(N)\cos(tg\sqrt{\lambda_+(N)}) \right\} / (2\sqrt{d(N)}),$$

$$f_{-1}(N) = \left\{ w_+(N)\cos(tg\sqrt{\lambda_-(N)}) - w_-(N)\cos(tg\sqrt{\lambda_+(N)}) \right\} / (2\sqrt{d(N)}),$$

$$h_1(N) = \left\{ \cos(tg\sqrt{\lambda_+(N)}) - \cos(tg\sqrt{\lambda_-(N)}) \right\} / (2\sqrt{d(N)}),$$

$$F_1(N) = \left\{ \frac{w_+(N)}{\sqrt{\lambda_+(N)}}\sin(tg\sqrt{\lambda_+(N)}) - \frac{w_-(N)}{\sqrt{\lambda_-(N)}}\sin(tg\sqrt{\lambda_-(N)}) \right\} / (2\sqrt{d(N)}),$$

$$F_0(N) = \left\{ \frac{v_+(N)}{\sqrt{\lambda_+(N)}}\sin(tg\sqrt{\lambda_+(N)}) - \frac{v_-(N)}{\sqrt{\lambda_-(N)}}\sin(tg\sqrt{\lambda_-(N)}) \right\} / (2\sqrt{d(N)}),$$

$$H_1(N) = \left\{ \sqrt{\lambda_+(N)}\sin(tg\sqrt{\lambda_+(N)}) - \sqrt{\lambda_-(N)}\sin(tg\sqrt{\lambda_-(N)}) \right\} / (2\sqrt{d(N)}),$$

$$H_0(N) = \left\{ \frac{1}{\sqrt{\lambda_+(N)}}\sin(tg\sqrt{\lambda_+(N)}) - \frac{1}{\sqrt{\lambda_-(N)}}\sin(tg\sqrt{\lambda_-(N)}) \right\} / (2\sqrt{d(N)})$$

and

$$\lambda_\pm(N) = 5N \pm \sqrt{d(N)}, \ v_\pm(N) = -2N - 3 \pm \sqrt{d(N)}, \ w_\pm(N) = 2N - 3 \pm \sqrt{d(N)},$$

$$d(N) = 16N^2 + 9.$$
3 Quantum Diagonalization Method

First of all we explain the method which we call a Quantum Diagonalization Method (QDM).

To calculate $e^{-itgA}$ for $A$ in (9)

$$A = S_+ \otimes a + S_- \otimes a^\dagger$$

we would like to diagonalize it like $A = UD_AU^\dagger$ ($D_A$ is a diagonal matrix) if possible. This is a well-known classsical procedure. However, in our case it is impossible because we cannot determine the eigenvalues by making use of its characteristic equation $f(\lambda) = \det(\lambda I - A)$. In the quantum case there is no meaning on determinant function. For example, which is correct

$$f(\lambda) = \begin{vmatrix} \lambda & -a \\ -a^\dagger & \lambda \end{vmatrix} \neq \begin{vmatrix} \lambda^2 - aa^\dagger \\ -a^\dagger a^\dagger \end{vmatrix}$$

Therefore we have no general method to make $A$ diagonal. However, we have a very skillful method for $A$ whose procedure goes like

**Classicalization $\rightarrow$ Quantization $\rightarrow$ Classicalization.**

The (quantum) matrix $A$ above can be decomposed as

$$T^\dagger AT = \sum \oplus B_j$$

(a direct sum of quantum matrices of spin $j$)

by an orthogonal matrix $T$ \footnote{To find $T$ in the general case is not easy} like in the preceeding section, where $B_j$ is given by

$$B_j = \begin{pmatrix} 0 & \sqrt{(J-1)a} \\ \sqrt{(J-1)a^\dagger} & 0 & \sqrt{(J-2)a} \\ & \sqrt{(J-2)a^\dagger} & 0 & \sqrt{(J-3)a} \\ & & \vdots & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{(J-k)a} \delta_{k,i-1} + \sqrt{(J-i)a^\dagger} \delta_{k-1,i} \\ \sqrt{(J-k)a} \delta_{k,i-1} + \sqrt{(J-i)a^\dagger} \delta_{k-1,i} & 0 & \sqrt{1(J-1)a} \\ & 0 & \sqrt{1(J-1)a^\dagger} & 0 \end{pmatrix}$$

(19)
with \( J = 2j + 1 \). In the following we set \( B = B_j \) for simplicity.

**(i) Classicalization**  We replace \( a \to z \) and \( a^\dagger \to \bar{z} \) in \( B \) and set

\[
C = \begin{pmatrix}
0 & \sqrt{(J-1)z} & \sqrt{(J-2)z} \\
\sqrt{(J-1)\bar{z}} & 0 & \sqrt{(J-3)\bar{z}} \\
\sqrt{(J-2)\bar{z}} & \sqrt{(J-3)\bar{z}} & 0 \\
\vdots & \vdots & \vdots \\
\sqrt{2(J-2)\bar{z}} & \sqrt{2(J-3)\bar{z}} & \sqrt{2(J-4)\bar{z}} \\
\sqrt{1(J-1)\bar{z}} & \sqrt{1(J-2)\bar{z}} & 0 \\
0 & \sqrt{1(J-1)\bar{z}} & 0
\end{pmatrix} = \left( \sqrt{(J-k)k}z\delta_{k,i} + \sqrt{(J-k)\bar{z}\delta_{k-1,i}} \right).
\]

We must diagonalize \( C \). The eigenvalues are

\[
\{(J-1)|z|, (J-3)|z|, \ldots, (J-2i+1)|z|, \ldots, -(J-3)|z|, -(J-1)|z|\}
\]

and corresponding orthonormal eigenvectors are

\[
|(J-2i+1)|z\rangle = \begin{pmatrix} x_{ki} \bar{z}^{k-1} \end{pmatrix}_{k=1,\cdots,J} \text{ for } i = 1 \sim J
\]

where \( x_{ki} \) are defined as

\[
x_{ki} = \frac{y_{ki}}{\sqrt{\sum_{i=1}^{J} y_{li}^2}}
\]

with \( y_{ki} (k = 1 \sim J) \) defined by the recursion relation

\[
y_{1i} = 1, \quad \sqrt{(J-k+1)(k-1)y_{k-1,i} + (J-k)ky_{k+1,i}} = (J-2i+1)y_{ki}.
\]

For example,

\[
y_{1i} = 1, \quad y_{2i} = \frac{J-2i+1}{\sqrt{(J-1)i}}, \quad y_{3i} = \frac{(J-2i+1)^2 - (J-1)i}{\sqrt{(J-1)(J-2)i}}, \quad \text{etc.}
\]
We note that the matrix \( X = (x_{ki}) \) is an (real) orthonormal one, namely \( X^TX = XX^T = 1_J \).

For example, when \( j = 1 \) (\( J = 3 \)) it is easy to show

\[
X = \begin{pmatrix}
\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\
\frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2}
\end{pmatrix}.
\]

If we set

\[
W = \left( x_{ki} \frac{z^{k-1}}{|z|^{k-1}} \right) \tag{21}
\]

then \( W \) is a unitary matrix (\( W^*W = WW^* = 1_J \)) and \( C \) is diagonalized by \( W \) as

\[
C = WD_C W^* \tag{22}
\]

where \( D_C \) is a diagonal matrix consisting of the eigenvalues \( \{(J - 2i + 1)|z| \mid i = 1 \sim J\} \).

We note that the unitary matrix \( W \) is not defined at \( z = 0 \), see (21).

(ii) **Quantization** Next we consider a quantization of \( W \): namely we want to find a (quantum) unitary matrix \( U_1 \) arising from \( W \) above. After some trial and errors we set

\[
U_1 = \left( \frac{x_{ki}}{\sqrt{N(N-1) \cdots (N-k+2)}} (a^\dagger)^{k-1} \right), \tag{23}
\]

then it is not difficult to check

\[
U_1 U_1^\dagger = U_1^\dagger U_1 = 1_J
\]

on the representation space \( \mathcal{H} \oplus \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_{J-1} \) where \( \mathcal{H}_k = \text{Vect}_\mathbb{C}\{|k\rangle, |k+1\rangle, \cdots \} \) is the subspace of the Fock space \( \mathcal{H} \equiv \mathcal{H}_0 \) generated by \( \{a, a^\dagger, N\} \). We note that \( U_1 \) is not defined on the whole space.\(^4\)

A comment is in order. Noting

\[
(a^\dagger)^l a^l = N(N-1) \cdots (N-l+1), \quad a^l (a^\dagger)^l = (N+l)(N+l-1) \cdots (N+1),
\]

\[
aN = (N+1)a, \quad a^\dagger N = (N-1)a^\dagger
\]

\(^4U_1 \) is a partial isometry on \( \mathcal{H} \oplus \mathcal{H} \oplus \cdots \oplus \mathcal{H} \) in the mathematical terminology
for \( l \geq 1 \) we have

\[
\frac{1}{\sqrt{a^l}} (a^l)^l = \frac{1}{\sqrt{N(N-1)\cdots(N-l+1)}} (a^l)^l
\]

\[
= (a^l)^l \frac{1}{\sqrt{(N+l)(N+l-1)\cdots(N+1)}} = (a^l)^l \frac{1}{\sqrt{a^l(a^l)^l}}
\]

(24)

and

\[
(a^l)^l \frac{1}{a^l(a^l)^l} a^l = (a^l)^l \frac{1}{(N+1)\cdots(N+l-1)(N+l)} a^l = 1 \quad \text{on} \quad \mathcal{H}_l.
\]

(25)

For example, when \( j = 1 \) (\( J = 3 \)) we have

\[
U_1 = \begin{pmatrix}
\frac{1}{2} & 1 & -\frac{1}{2} \\
-\frac{1}{2\sqrt{N}} a^\dagger & 0 & -\frac{1}{2\sqrt{N}} a^\dagger \\
\frac{1}{2\sqrt{N(N-1)}} (a^\dagger)^2 & -\frac{1}{\sqrt{2}} \sqrt{N(N-1)} (a^\dagger)^2 & \frac{1}{2\sqrt{N(N-1)}} (a^\dagger)^2
\end{pmatrix}
\]

(iii) Classicalization

Here we consider a diagonalization of \( B \) by \( U_1 \) above. Some calculation leads

\[
U_1^\dagger B U_1 = R = (r_{ki})
\]

(26)

where

\[
r_{ki} = \sum_{l=2}^{J} \sqrt{(J-l+1)(l-1)} (y_{l-1,k} y_{l,i} + y_{l,k} y_{l-1,i}) \sqrt{N+l-1} = r_{ik}.
\]

We see that the matrix \( R \) is hermitian and its entries consist of some functions of the number operator \( N \), so \( R \) is a kind of classical matrix. Therefore we can make \( R \) diagonal \(^5\) like

\[
R = U_2 D_R U_2^\dagger.
\]

(27)

For example, when \( j = 1 \) (\( J = 3 \)) we have

\[
R = \begin{pmatrix}
\sqrt{N+1} + \sqrt{N+2} & -\sqrt{N+2}\sqrt{N+1} & 0 \\
-\sqrt{N+2}\sqrt{N+1} & 0 & \sqrt{N+2}\sqrt{N+1} \\
0 & \sqrt{N+2}\sqrt{N+1} & -(\sqrt{N+1} + \sqrt{N+2})
\end{pmatrix}
\]

\(^5\)To obtain \( U_2 \) explicitly is not easy or almost impossible
and

\[
U_2 = \begin{pmatrix}
\frac{-\sqrt{2(2N+3)} + \sqrt{N+2 + \sqrt{N+1}}}{2\sqrt{2(2N+3)}} & \frac{\sqrt{N+2} - \sqrt{N+1}}{\sqrt{2}\sqrt{2(2N+3)}} & \frac{-\sqrt{2(2N+3)} - \sqrt{N+2 - \sqrt{N+1}}}{2\sqrt{2(2N+3)}} \\
\frac{\sqrt{N+2} - \sqrt{N+1}}{\sqrt{2}\sqrt{2(2N+3)}} & \frac{\sqrt{N+2} + \sqrt{N+1}}{2(2N+3)} & \frac{-\sqrt{2(2N+3)} - \sqrt{N+2 + \sqrt{N+1}}}{2\sqrt{2(2N+3)}} \\
\frac{-\sqrt{2(2N+3)} - \sqrt{N+2 - \sqrt{N+1}}}{2\sqrt{2(2N+3)}} & \frac{-\sqrt{2(2N+3)} - \sqrt{N+2 + \sqrt{N+1}}}{2\sqrt{2(2N+3)}} & \frac{\sqrt{N+2} + \sqrt{N+1}}{2(2N+3)}
\end{pmatrix}
\]

\[
D = \begin{pmatrix}
\sqrt{2(2N+3)} & 0 & -\sqrt{2(2N+3)} \\
0 & -\sqrt{2(2N+3)}
\end{pmatrix}
\]

As a result we finally obtain

\[B = (U_1 U_2) D_R (U_1 U_2)^\dagger \equiv UDU^\dagger.\] (28)

This is just the diagonal form of B that we are looking for. We note that all entries of \(U = U_1 U_2\) consist of \(N\) and \(a^\dagger\) (not contain \(a\)). From this we have

\[e^{-itgB} = U e^{-itgD} U^\dagger.\] (29)

This is a kind of “normal ordered” diagonal expression of the evolution operator. See Appendix for another diagonal expression. In the following we give an explicit expression in the case of one, two and three atoms.

Let us list the results.

**One Atom Case** For \(A_1\) in (10) we have

\[A_1 = U D U^\dagger\] (30)

where

\[U = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 1 \\
\frac{1}{\sqrt{N}} a^\dagger & -\frac{1}{\sqrt{N}} a^\dagger
\end{pmatrix}, \quad D = \begin{pmatrix}
\sqrt{N+1} & \\
-\sqrt{N+1}
\end{pmatrix}.\] (31)

Then it is easy to see

\[e^{-itgA_1} = U e^{-itgD} U^\dagger = \text{the right hand side of (12)}.\] (32)
Two Atoms Case  For $B_1$ in (14) we have

$$ B_1 = UDU^\dagger $$  

(33)

where

$$ U = \begin{pmatrix}
-\frac{\sqrt{N+1}}{\sqrt{2(2N+3)}} & \frac{\sqrt{2N+2}}{\sqrt{2(2N+3)}} & \frac{\sqrt{N+1}}{\sqrt{2(2N+3)}} \\
-\frac{1}{\sqrt{2N}} a^\dagger & 0 & -\frac{1}{\sqrt{2N}} a^\dagger \\
-\frac{1}{\sqrt{N+1}}(a^\dagger)^2 & -\frac{\sqrt{2}}{\sqrt{N}\sqrt{2(2N-1)}}(a^\dagger)^2 & \frac{1}{\sqrt{N+1}}(a^\dagger)^2
\end{pmatrix} $$  

(34)

and

$$ D = \begin{pmatrix}
\sqrt{2(2N+3)} & 0 \\
0 & -\sqrt{2(2N+3)}
\end{pmatrix}. $$  

(35)

Then it is not difficult to see

$$ e^{-itgB_1} = U e^{-itgD} U^\dagger = \text{the right hand side of (15)}. $$  

(36)

Three Atoms Case  For $B_{3/2}$ in (17) we have 6

$$ B_{3/2} = UDU^\dagger $$  

(37)

where

$$ U = \begin{pmatrix}
u_{11} & u_{12} & u_{13} & u_{14} \\
u_{21} & u_{22} & u_{23} & u_{24} \\
u_{31} & u_{32} & u_{33} & u_{34} \\
u_{41} & u_{42} & u_{43} & u_{44}
\end{pmatrix} $$  

(38)

where

\[ u_{11} = \frac{1}{4} \left\{ \sqrt{2} (1 - \beta \gamma) + \sqrt{6} (\beta + \gamma) \right\} \frac{1}{\sqrt{(1 + \beta^2) (1 + \gamma^2)}}, \]

\[ u_{12} = \frac{1}{4} \left\{ \sqrt{6} (1 - \beta \gamma) - \sqrt{2} (\beta + \gamma) \right\} \frac{1}{\sqrt{(1 + \beta^2) (1 + \gamma^2)}}, \]

\[ u_{13} = u_{12}, \quad u_{14} = -u_{11}, \]

6The calculation in this case is interesting and hard, so the full details will be given in [8].
\[ u_{21} = \frac{1}{4} \left\{ \frac{\sqrt{6}}{\sqrt{N}} a^\dagger (1 + \beta \gamma) + \frac{\sqrt{2}}{\sqrt{N}} a^\dagger (\beta - \gamma) \right\} \frac{1}{\sqrt{(1 + \beta^2)(1 + \gamma^2)}}, \]

\[ u_{22} = \frac{1}{4} \left\{ \frac{\sqrt{2}}{\sqrt{N}} a^\dagger (1 + \beta \gamma) - \frac{\sqrt{6}}{\sqrt{N}} a^\dagger (\beta - \gamma) \right\} \frac{1}{\sqrt{(1 + \beta^2)(1 + \gamma^2)}}, \]

\[ u_{23} = -u_{22}, \quad u_{24} = u_{21}, \]

\[ u_{31} = \frac{1}{4} \left\{ -\frac{\sqrt{6}}{\sqrt{N(N-1)}} (a^\dagger)^2 (1 - \beta \gamma) - \frac{\sqrt{2}}{\sqrt{N(N-1)}} (a^\dagger)^2 (\beta + \gamma) \right\} \frac{1}{\sqrt{(1 + \beta^2)(1 + \gamma^2)}}, \]

\[ u_{32} = -\frac{1}{4} \left\{ \frac{\sqrt{2}}{\sqrt{N(N-1)}} (a^\dagger)^2 (1 - \beta \gamma) + \frac{\sqrt{6}}{\sqrt{N(N-1)}} (a^\dagger)^2 (\beta + \gamma) \right\} \frac{1}{\sqrt{(1 + \beta^2)(1 + \gamma^2)}}, \]

\[ u_{33} = u_{32}, \quad u_{34} = -u_{31}, \]

\[ u_{41} = \frac{1}{4} \left\{ \frac{\sqrt{2}}{\sqrt{N(N-1)(N-2)}} (a^\dagger)^3 (1 + \beta \gamma) - \frac{\sqrt{6}}{\sqrt{N(N-1)(N-2)}} (a^\dagger)^3 (\beta - \gamma) \right\} \]
\[ \times \frac{1}{\sqrt{(1 + \beta^2)(1 + \gamma^2)}}, \]

\[ u_{42} = -\frac{1}{4} \left\{ \frac{\sqrt{6}}{\sqrt{N(N-1)(N-2)}} (a^\dagger)^3 (1 + \beta \gamma) + \frac{\sqrt{2}}{\sqrt{N(N-1)(N-2)}} (a^\dagger)^3 (\beta - \gamma) \right\} \]
\[ \times \frac{1}{\sqrt{(1 + \beta^2)(1 + \gamma^2)}}, \]

\[ u_{43} = -u_{42}, \quad u_{44} = u_{41} \quad (39) \]

and

\[ \beta = \frac{\mu - \nu - (x - y)}{2b}, \quad \gamma = \frac{\mu + \nu - (x + y)}{2c}, \]

\[ x = 3\sqrt{N + 1} + 6\sqrt{N + 2} + 3\sqrt{N + 3}, \quad y = 3\sqrt{N + 1} - 2\sqrt{N + 2} + 3\sqrt{N + 3}, \]

\[ b = 2\sqrt{3} \left( \sqrt{N + 1} - \sqrt{N + 3} \right), \quad c = \sqrt{3} \left( \sqrt{N + 1} - 2\sqrt{N + 2} + \sqrt{N + 3} \right), \quad (40) \]

and

\[ \mu = 4\sqrt{5(N + 2) + \sqrt{16(N + 2)^2 + 9}} = 4\sqrt{\lambda_+(N + 2)}, \]

\[ \nu = 4\sqrt{5(N + 2) - \sqrt{16(N + 2)^2 + 9}} = 4\sqrt{\lambda_-(N + 2)} \quad (41) \]
and

\[D = \frac{1}{4} \begin{pmatrix} \mu & \nu \\ \nu & -\nu \\ -\nu & -\mu \end{pmatrix} = \begin{pmatrix} \sqrt{\lambda_+(N+2)} & \sqrt{\lambda_-(N+2)} & -\sqrt{\lambda_-(N+2)} & -\sqrt{\lambda_+(N+2)} \end{pmatrix}. \quad (42)\]

Then we obtain

\[e^{-itgB_{3/2}} = U e^{-itgD} U^\dagger = \text{the right hand side of (18)}. \quad (43)\]

However, the proof is not easy, see Appendix.

Last we make one comment. We would like to perform a diagonalization for the case of more than three atoms, however it is not easy at the present. One of main difficulties is the step (iii). That is, to perform a diagonalization to the (classical hermite) matrix we must determine its eigenvalues by solving the characteristic equation and give orthonormal eigenvectors explicitly (not abstractly). The characteristic equation is in general algebraic one of degrees more than four, which is impossible to solve in an algebraic manner by the famous Galois theory. Even for algebraic equations of degrees three and four we must use the Cardano and Ferrarri formulas (see for example [9]) which make hard to determine all orthonormal eigenvectors explicitly.

4 U(1) Ambiguity

In this section we discuss a problem of U(1) ambiguity in the quantum diagonalization method.

The classical diagonalization \(C = WD_C W^\dagger\) in (22) has the following U(1) “invariance”, namely

\[C = WD_C W^\dagger = W U_0(U_0^\dagger D_C U_0) U_0^\dagger W^\dagger = (W U_0) D_C (W U_0)^\dagger\]
where $U_0$ is a diagonal matrix defined by

$$
U_0 = \begin{pmatrix}
    f_1(z, \bar{z}) \\
    f_2(z, \bar{z}) \\
    \vdots \\
    f_{J-1}(z, \bar{z}) \\
    f_J(z, \bar{z})
\end{pmatrix} \in U(J)
$$

and $f_j(z, \bar{z})$ is an element in $U(1)$, $|f_j(z, \bar{z})|^2 = 1$. For example, $f_j(z, \bar{z}) = \frac{z_j}{|z|^1}$. Namely, the diagonal matrix $D_C$ is invariant under the change of unitary matrix $W \rightarrow WU_0$.

However, this is not kept in the process of quantization. From (28)

$$
B = UD\bar{U} = (U\bar{U_0})(\bar{U_0}D\bar{U_0})(U\bar{U_0})^\dagger \equiv \tilde{U}\tilde{D}\tilde{U}^\dagger
$$

where $\tilde{U}_0$ is a quantum “diagonal” matrix defined by

$$
\tilde{U}_0 = \begin{pmatrix}
    f_1(a, a^\dagger) \\
    f_2(a, a^\dagger) \\
    \vdots \\
    f_{J-1}(a, a^\dagger) \\
    f_J(a, a^\dagger)
\end{pmatrix}
$$

and $f_j(a, a^\dagger)$ is an element satisfying $f_j(a, a^\dagger)^\dagger f_j(a, a^\dagger) = f_j(a, a^\dagger)f_j(a, a^\dagger)^\dagger = 1$. We note that $\tilde{U}_0$ is not defined on the whole space, which changes a domain and a range of $\tilde{U}$.

Here we restrict each $f_j(a, a^\dagger)$ to one satisfying a relation

$$
f_j(a, a^\dagger)^\dagger N f_j(a, a^\dagger) = g_j(N)
$$

for some function $g_j$. For example, $f_j(a, a^\dagger) = \frac{1}{\sqrt{(N+j-1)(N+j-2)\cdots(N+1)}}a^{j-1}$. In this case

$$
D \neq \tilde{D} = \tilde{U}_0^\dagger D\tilde{U}_0
$$

because $f_j(a, a^\dagger)$ and the number operator $N$ don’t commute in general.
Let us show this with an example. For $B_1$ in the two atoms case we consider a very simple case

$$\tilde{U}_0 = \begin{pmatrix}
\frac{1}{\sqrt{N+1}} a \\
\frac{1}{\sqrt{N+1}} a \\
\frac{1}{\sqrt{N+1}} a
\end{pmatrix},$$

then it is easy to see

$$B_1 = \tilde{U} \tilde{D} \tilde{U}^\dagger$$

where

$$\tilde{U} = \begin{pmatrix}
-\frac{1}{\sqrt{2(2N+3)}} a \\
-\frac{1}{\sqrt{2(2N-1)}} a^\dagger \\
-\frac{1}{\sqrt{2(2N-1)}} a^\dagger
\end{pmatrix}$$

and

$$\tilde{D} = \begin{pmatrix}
\sqrt{2(2N+1)} \\
0 \\
-\sqrt{2(2N+1)}
\end{pmatrix}.$$

Compare this with (33). Here we note that

$$\tilde{U}^\dagger \tilde{U} = 1_{\mathcal{H} \oplus \mathcal{H}_1 \oplus \mathcal{H}} \quad \text{and} \quad \tilde{U} \tilde{U}^\dagger = 1_{\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}_1}.$$

In the two expressions $B_1 = UDU^\dagger = \tilde{U} \tilde{D} \tilde{U}^\dagger$, each domain and range of $U$ and $\tilde{U}$ is different.

The diagonal part $D$ of $B$ changes according to $\tilde{U}_0$, which is unavoidable due to the non-commutativity of operators in quantum physics. We call this phenomenon a $U(1)$ ambiguity.

5 Discussion

We introduced the quantum diagonalization method and applied it to the (quantum) matrix $B$ in (33) (or $A$ in (31)) emerging from the Tavis–Cummings model and (re)obtained the explicit form of evolution operator for the one, two and three atoms case. To get the general case is not easy because of some technical reasons (numerical techniques are of course applicable).
Therefore, there are many applications to quantum optics or mathematical physics, see for example [4]. We can also apply the result to a quantum computation based on atoms of laser–cooled and trapped linearly in a cavity, see [10].

We also make a comment on an application to a noncommutative differential geometry. From (28) we have a (quantum) unitary matrix $U$ which gives the Maurer–Cartan forms

$$L_U \equiv U^{-1}\hat{d}U, \quad R_U \equiv \hat{d}UU^{-1},$$

where $\hat{d}$ is some differential with respect to $a$ and $a^\dagger$. These are fundamental objects in non-commutative chiral models. For the case of one, two and three atoms we can calculate the Maurer-Cartan forms exactly. Such a study is however beyond our scope of this paper. We expect that some researchers will develop the subject.

We conclude this paper by making a comment. The Tavis–Cummings model is based on (only) two energy levels of atoms. However, an atom has in general infinitely many energy levels, so it is natural to use this possibility. We are also studying a quantum computation based on multi–level systems of atoms (a qudit theory) [7]. Therefore we would like to extend the Tavis–Cummings model based on two–levels to a model based on multi–levels. This is a very challenging task.

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Appendix

Proof of (43)

Here we show (43). The (1,1)–entry of $Ue^{-itgDU^\dagger}$ is

$$e^{-itg_\mu/4}u_{11}^2 + e^{-itg_\nu/4}u_{12}^2 + e^{itg_\nu/4}u_{13}^2 + e^{itg_\mu/4}u_{14}^2 = 2u_{11}^2\cos\left(tg_\mu\frac{\mu}{4}\right) + 2u_{12}^2\cos\left(tg_\nu\frac{\nu}{4}\right).$$

because $u_{13}^2 = u_{12}^2$ and $u_{14}^2 = u_{11}^2$. Let us calculate both $u_{11}^2$ and $u_{12}^2$.

$$2u_{11}^2 = \frac{\sqrt{2}(1-\beta\gamma) + \sqrt{6}(\beta + \gamma)}{8(1+\beta^2)(1+\gamma^2)}.$$
\[
\frac{1}{8} \left\{ 2 + 4 \sqrt{3} \left( \frac{1}{\beta} - \beta + \frac{1}{\gamma} - \gamma \right) + \frac{\beta}{\gamma} + \frac{\gamma}{\beta} + 2 \right\} \\
= \frac{1}{8} \left\{ 2 + 4 \sqrt{3} \right\} \left\{ c(x - y) + b(x + y) \right\} + \frac{\mu^2 - \nu^2}{\mu^2 - \nu^2} - \frac{x^2 - y^2}{2} + 2bc \right\} \\
= \frac{1}{8} \left\{ 2 + \frac{-8N - 28 + 2\sqrt{16(N + 2)^2 + 9}}{\sqrt{16(N + 2)^2 + 9}} \right\} \\
= \frac{-2N - 7 + \sqrt{16(N + 2)^2 + 9}}{2\sqrt{16(N + 2)^2 + 9}} = \frac{v_+(N + 2)}{2\sqrt{d(N + 2)}},
\]

where we have used the relations

\[
\begin{align*}
\beta + \frac{1}{\beta} &= \frac{\mu - \nu}{b}, & \frac{1}{\beta} - \beta &= \frac{x - y}{b}, & \gamma + \frac{1}{\gamma} &= \frac{\mu + \nu}{c}, & \frac{1}{\gamma} - \gamma &= \frac{x + y}{c}, \\
\frac{\beta}{\gamma} + \frac{\gamma}{\beta} &= \frac{1}{2bc} \left\{ (\mu^2 - \nu^2) - (x^2 - y^2) \right\}
\end{align*}
\]

and (40) and (41). Similarly, we have

\[
2u_{11}^2 = \frac{2N + 7 + \sqrt{16(N + 2)^2 + 9}}{2\sqrt{16(N + 2)^2 + 9}} = \frac{-v_-(N + 2)}{2\sqrt{d(N + 2)}},
\]

so that

\[
2u_{11}^2 \cos \left( t \frac{\nu}{4} \right) + 2u_{12}^2 \cos \left( t \frac{\mu}{4} \right) = \frac{v_+(N + 2)}{2\sqrt{d(N + 2)}} \cos \left( t \sqrt{\lambda_+(N + 2)} \right) - \frac{v_-(N + 2)}{2\sqrt{d(N + 2)}} \cos \left( t \sqrt{\lambda_-(N + 2)} \right) = f_2(N + 2).
\]

The remaining 9 entries become more complicated because they contain \( a \) and \( a^\dagger \). See [8] in detail.

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