AN ANALOGUE OF SIEGEL’S DETERMINANT

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Abstract. Siegel-Shidlovskii theory of E-functions involves a non-vanishing proof for the determinants attached to the linear forms $D^k R(t)$, derivatives of an auxiliary function $R(t)$. Let a non-zero function $F(t)$ satisfy $m$th order linear differential equation which we shall write using the differential operator $\Delta = tD$ and let $L(t)$ be any non-zero linear form of the derivatives $\Delta^i F(t)$ ($i = 0, \ldots, m - 1; m \geq 2$). The determinants $\det A_k$ attached to the linear forms $\Delta^k L(t)$ have certain simple properties that allow us to give a short proof for the non-vanishing of $\det A_k$ for a class of differential equations including a subclass of hypergeometric differential equations.

1. Introduction

Let $\mathcal{F}$ be a differential field of characteristic zero, say $\mathcal{F} = K((t))$, where $K$ is the field of constants with respect to the differential operator $D$. If $K = \mathbb{C}$ or $K = \mathbb{C}_p$, then let $D = \frac{d}{dt}$ be the usual derivative operator. In the following we suppose that a non-zero function $F: K \to K$ satisfies the linear homogeneous differential equation

$$TD^m F(t) = Q_1 D^{m-1} F(t) + \ldots + Q_m D^0 F(t)$$

of order $m \geq 2$, where $T, Q_1, \ldots, Q_m \in K[t]$ are polynomials with degrees $S' = \deg T(t), r_j' = \deg Q_j(t)$.

In studying linear and algebraic independence of numbers, Siegel [8] used Thue’s lemma to construct an auxiliary function

$$R(t) = B_{0,1} D^{m-1} F(t) + \ldots + B_{0,m} D^0 F(t)$$

with polynomials $B_{0,1}, \ldots, B_{0,m} \in K[t]$. Siegel also introduced the following linear forms

$$R_n = (TD)^n R(t) = B_{n,1} D^{m-1} F(t) + \ldots + B_{n,m} D^0 F(t) \quad \forall n \in \mathbb{N},$$

where $B_{n,1}, \ldots, B_{n,m} \in K[t]$. A crucial tool in Siegel’s method is a non-vanishing proof of the $m \times m$ determinants

$$\det B_k = \begin{vmatrix} B_{k,1} & \ldots & B_{k,m} \\ \vdots & \ddots & \vdots \\ B_{m+k-1,1} & \ldots & B_{m+k-1,m} \end{vmatrix}, \quad k \in \mathbb{N}. $$

Thereafter the approach shown by Siegel [5], [9] has widely been applied in proving algebraic independence of the values of $E$-functions, see [4]. Shidlovskii [7] made major progress by proving the non-vanishing of the determinants [4] for a large class of functions including Siegel $E$-functions. However, the non-vanishing proofs of the determinants [4] which use Shidlovskii’s lemma have the disadvantage that they have not generally yielded effective independence measures. Further advance came from Beukers-Brownawell-Heckman [2] who proved Siegel’s normality condition for a large class of meromorphic hypergeometric functions

$$(5) \quad A_{FB} \left( a_1, \ldots, a_A \mid b_1, \ldots, b_B \right) t = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_A)_n}{n!(b_1)_n \cdots (b_B)_n} t^n,$$

thus giving a possibility to achieve effective quantitative algebraic independence results for the values of the functions [5].
Here we will use the differential operator $\Delta = tD$ and write the differential equation (1) in an equivalent form

$N\Delta^m F(t) = P_1\Delta^{m-1} F(t) + \ldots + P_m\Delta^0 F(t)$  \hspace{1cm} (6)

where $N, P_1, \ldots, P_m \in K[t]$ and $S = \deg N(t), r_j = \deg P_j(t)$. Let

$L(t) = A_{0,1}\Delta^{m-1} F(t) + \ldots + A_{0,m}\Delta^0 F(t)$,

where $A_{0,1}, \ldots, A_{0,m} \in K[t]$, be an arbitrary linear form from which we will construct the following linear forms

$L_n = (N\Delta)^n L(t) = A_{n,1}\Delta^{m-1} F(t) + \ldots + A_{n,m}\Delta^0 F(t)$ \hspace{1cm} (7) \hspace{1cm} \forall n \in \mathbb{N},

where again $A_{n,1}, \ldots, A_{n,m} \in K[t]$. So, let

\[
\det A_k = \begin{vmatrix}
A_{k,1} & \ldots & A_{k,m} \\
\vdots & \ddots & \vdots \\
A_{k+m-1,1} & \ldots & A_{k+m-1,m}
\end{vmatrix}, \quad k \in \mathbb{N},
\]

be the determinants analogous to Siegel’s determinants (4). Under the conditions

$(A_{0,1}, \ldots, A_{0,m}) \neq (0, \ldots, 0)$

and

$r_m = S + 1, \quad r_j < r_m \quad \forall j = 1, \ldots, m - 1$

we will prove in Theorem 3.1 the non-vanishing of the determinant $\det A_k$ directly by studying the degrees of the polynomials $A_{i,j}$. Consequently our approach will imply effective quantitative linear independence results for a class of functions $F(t)$ satisfying (6), (15) and (16) including e.g. the hypergeometric series

\[
_{0}F_{B}
\begin{pmatrix}
\ast \\
 b_1, \ldots, b_B
\end{pmatrix}
(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!(b_1)_n \cdots (b_B)_n}.
\]

Because the determinant (8) is non-zero for any non-zero linear form $L(t)$, we get immediately an irreducibility criterion, in Corollary 4.1 for the differential operator (6).

It is interesting to note that, if the method of Theorem 4.1 is used with the operator $D$ and the corresponding classical linear forms (3), then we have the assumptions of Theorem 5.2 which are similar to the assumption (16). However, the assumptions used in Theorem 5.2 do not correspond to those interesting cases, say (9), which are covered by Theorem 5.1.

The method of Theorem 5.1 is applied also in [1] where some linear independence results are proved for the solutions of certain $q$-functional equations analogous to (1). Here, it is interesting that these solutions include $q$-analogues of the series (9).

2. THE LINEAR FORMS

Let

$L_0 = L(t) = A_{0,1}\Delta^{m-1} F(t) + \ldots + A_{0,m}\Delta^0 F(t)$  \hspace{1cm} (10)

be any linear form, where $A_{0,1}, \ldots, A_{0,m} \in K[t]$. Then we define the linear forms $L_n$ recursively by

$L_{n+1} = N\Delta L_n \quad \forall n \in \mathbb{N}$  \hspace{1cm} (11)

and we denote

$L_n = A_{n,1}\Delta^{m-1} F(t) + \ldots + A_{n,m}\Delta^0 F(t) \quad \forall n \in \mathbb{N}$.  \hspace{1cm} (12)

The definitions (11) and (12) imply directly the following recurrences.
Lemma 2.1. The polynomials $A_{n,1}(t), \ldots, A_{n,m}(t)$ satisfy the linear recurrences

\begin{align}
A_{k+1,j} &= P_j A_{k,1} + N \Delta A_{k,j} + N A_{k,j+1} \quad \forall j = 1, \ldots, m - 1, \\
A_{k+1,m} &= P_m A_{k,1} + N \Delta A_{k,m}
\end{align}

for all $k \in \mathbb{N}$.

3. A NON-VANISHING PROOF

Here we suppose that the linear form \((10)\) is arbitrary.

Theorem 3.1. Let the linear form \((10)\) satisfy

\begin{align}
(A_{0,1}, \ldots, A_{0,m}) &\neq (0, \ldots, 0)
\end{align}

and assume that

\begin{align}
S &\geq 0, \quad r_m = S + 1, \quad r_j < r_m \quad \forall j = 1, \ldots, m - 1.
\end{align}

Then

\begin{align}
\det A_k = \begin{vmatrix}
A_{k,1} & \ldots & A_{k,m} \\
\ddots & \ddots & \ddots \\
A_{k+m-1,1} & \ldots & A_{k+m-1,m}
\end{vmatrix} \neq 0, \quad \forall k \in \mathbb{N}.
\end{align}

Proof. Denote $\deg A_{k,j} = d_j$, and let us suppose

\begin{align}
d_1 = A, \ldots, d_{l-1} = B < d_l = C \geq D = d_{l+1}, \ldots, d_m = F
\end{align}

for some $1 \leq l \leq m$ (if $l = 1$, then $A = C$, and if $l = m$, then $F = C$). (Here we put $\deg 0(x) = -\infty$, where $0(x)$ is the zero polynomial.)

Let first $l \geq 2$. Then

\begin{align}
\deg A_{k+1,l-1} &= \deg(P_{l-1} A_{k,1} + N \Delta A_{k,l-1} + N A_{k,l}) \\
&= \max\{r_{l-1} + d_l; \ S + d_{l-1}; \ S + d_l\} = d_l + S.
\end{align}

If $h \leq l - 2$, then

\begin{align}
\deg A_{k+1,h} &\leq \max\{r_h + d_1; \ S + d_h; \ S + d_{h+1}\} \leq d_l + S - 1.
\end{align}

If $l \leq h \leq m$, then

\begin{align}
\deg A_{k+1,h} &\leq \max\{r_h + d_1; \ S + d_h; \ S + d_{h+1}\} \leq d_l + S.
\end{align}

Let $l = 1$, then

\begin{align}
\deg A_{k+1,h} &\leq \max\{r_h + d_1; \ S + d_h; \ S + d_{h+1}\} \leq d_1 + S < \\
&d_1 + S + 1 = \max\{r_m + d_1; \ S + d_m\} = \deg (P_m A_{k,1} + N \Delta A_{k,m}) = \deg A_{k+1,m},
\end{align}

for all $h \leq m - 1$.

Proceeding by induction we shall meet the situation

\begin{align}
\deg A_{k+l-1,1} &\geq \deg A_{k+l-1,2}, \ldots, \deg A_{k+l-1,m}
\end{align}

in the $l$th row of the determinant

\begin{align}
\det A_k = \begin{vmatrix}
A_{k,1} & \ldots & A_{k,l-1} & A_{k,l} & A_{k,l+1} & \ldots & A_{k,m} \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
A_{k+m-1,1} & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
A_{k+l-1,1} & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{vmatrix}.
\end{align}
By (22) the process starts in a similar fashion from the \((l + 1)\)th row and consequently the degrees will behave in the following manner

\[
\begin{array}{ccccccccc}
A & \ldots & B & < C & \geq D & \ldots & F \\
. & \ldots & < C + S & \geq \ldots & A + S + 1 \\
\vdots & & C + (l - 1)S & \geq \ldots & \vdots \\
A' & \ldots & < C + lS + 1 & \ldots & \vdots \\
\end{array}
\]

(25)

Next we shall permute the rows in the determinant \(\det A_k\) in such a way that we have the maximum degree polynomials in the anti-diagonal which shows that the degree of determinant \(\det A_k\) is

\[
md_l + (m^2 - m)S/2 + m - l
\]

and thus

\[
\det A_k \neq 0. \quad \square
\]

Next we use the differential equation satisfied by \(F(t)\) in the form (1). Also here we suppose that the linear form (2) is arbitrary i.e. without any assumption for the order of \(R(t)\). Then the method used in proof of Theorem 3.1 shows the following result.

**Theorem 3.2.** Let

\[
(B_{0,1}, \ldots, B_{0,m}) \neq (0, \ldots, 0),
\]

and

\[
S' \geq 0, \quad r'_m = S' + 1, \quad r'_j < r'_m \quad \forall j = 1, \ldots, m - 1
\]

or

\[
r'_m \geq 0, \quad S' = r'_m + 1, \quad r'_j \leq r'_m \quad \forall j = 1, \ldots, m.
\]

Then

\[
\det B_k \neq 0 \quad \forall k \in \mathbb{N}.
\]

Note, that in the proof of Theorem 3.2 we use recurrences which we get just by replacing the operator \(\Delta\) by \(D\) in (13) and (14). Thus we may prove Theorem 3.2 by the assumption (28) but a minor difference in the step which corresponds to (19) gives the possibility for the extra assumption (29).

4. Applications

Let us recall some basic facts we need of homogeneous linear differential equations. It is known that

\[
t^n D^n = \sum_{k=0}^{n} s_1(n,k) \Delta^k \quad \forall n \in \mathbb{N},
\]

(31)

\[
\Delta^n = \sum_{k=0}^{n} S_2(n,k) t^k D^k \quad \forall n \in \mathbb{N},
\]

(32)

where \(s_1(n,k)\) and \(S_2(n,k)\) are Stirling numbers of the first and second kind, respectively.

Thus any linear homogeneous differential equation may be written as

\[
E_1(t, D)y = 0
\]

(33)
or equivalently as
\[(34)\]
\[E_2(t, \Delta)y = 0,\]
where \(E_i(t, x) \in K[t, x]\) and \(\text{deg}_x E_1(t, x) = \text{deg}_x E_2(t, x)\), the order of the differential equation.

Put \(\Lambda_1 = D, \Lambda_2 = \Delta\) and consider the following conditions:

1* The differential equation
\[(35)\]
\[E_i(t, \Lambda)y = 0\]
for a particular solution \(F\), is of the minimum order \(m\), which is denoted by
\[\text{deg}_{\Lambda_i}(F) = m \quad i = 1, 2.\]

2* The derivatives
\[(36)\]
\[\Lambda_i^0 F, ..., \Lambda_i^{m-1} F\]
of a particular solution \(F\) of \((35)\) are linearly independent over \(K(t)\) for \(i = 1, 2\).

A* The differential equation \((35)\) is homogeneously linearly irreducible i.e.
\[\text{deg}_{\Lambda_i}(F) = m\]
for all solutions of \((35)\) and \(i = 1, 2\).

B* The derivatives
\[(37)\]
\[\Lambda_i^0 F, ..., \Lambda_i^{m-1} F\]
are linearly independent over \(K(t)\) for all solutions \(F\) of \((35)\) and \(i = 1, 2\).

C* The differential operator \(E(\Lambda_i, t)\) is irreducible for \(i = 1, 2\).

The equivalence between the cases \(i = 1\) and \(i = 2\) follows from the fact that the connection matrix \(S\) between the sets \((36)\), \(i = 1, 2\), is subdiagonal having a non-zero determinant.

The cases 1* and 2* are equivalent and further from Nesterenko (see \([5]\) Lemma 3.1.) it follows that that the conditions A*, B* and C* are equivalent. Clearly A* implies 1*.

**Corollary 4.1.** Assume the condition \((16)\) of Theorem 3.1. Then the differential operator \(E(\Delta, t)\) corresponding to the equation \((6)\) is irreducible.

**Proof.** Let \(F\) be any non zero solution of \((6)\). Let us suppose on the contrary that
\[(38)\]
\[L_0 = A_{0,1} \Delta^{m-1} F + ... + A_{0,m} \Delta^0 F = 0\]
with some \((A_{0,1}, ..., A_{0,m}) \in K[t]^m \setminus \{(0, ..., 0)\}\). Now \((11)\) and \((38)\) imply
\[(39)\]
\[L_n = A_{n,1} \Delta^{m-1} F + ... + A_{n,m} \Delta^0 F = 0\]
for all \(n \in \mathbb{N}\) and thus
\[(40)\]
\[A_0(\Delta^{m-1} F, ..., \Delta^0 F)^T = (0, ..., 0)^T.\]
By \((17)\) we have a contradiction with the assumption that \(F(t)\) is a non-zero function.

Thus the differential operator \(E(\Delta, t)\) is irreducible by the equivalence of \(B*\) and \(C*\). \(\square\)

In a similar fashion we may prove a corresponding result for the differential operator \(D\).

**Corollary 4.2.** Assume the condition \((28)\) or \((29)\) of Theorem 3.2. Then the differential operator \(E(D, t)\) corresponding to the equation \((1)\) is irreducible.

On the other hand, due the folklore the above corollaries have converse statements. Namely, if the differential operator is irreducible, then Corollary X implies Theorem X. For the completeness a sketch of the proof for this fact is given starting from the basic facts of differential modules.

Let \(\mathcal{D} = \mathcal{F}[\partial]\) be a ring of differential operators, where, say, \(\partial = D\) or \(\partial = \Delta\). If \(L \in \mathcal{D}\), then \(\mathcal{D}L\) denotes the left submodule of \(\mathcal{D}\) generated by \(L\). By these notations we may state the following lemma which may be found e.g. from \([10]\).

**Lemma 4.3.** \(L \in \mathcal{D}\) is irreducible \(\iff\) \(\mathcal{D}/\mathcal{D}L\) is a simple left \(\mathcal{D}\)-module.
Now we are ready to show how Corollary 4.1 implies Theorem 3.1. Put
\[ W = \mathcal{D}/\mathcal{D}E(\Delta, t) \]
and let \( W^* \) be its dual. Then
\[ m = \dim_{K(t)} W = \dim_{K(t)} W^*. \]
Define then \( V^* = \mathcal{D}L \) be a left \( \mathcal{D} \)-submodule of \( W^* \) generated by a non zero \( L = L_0 \in W^* \). Suppose now that \( E(\Delta, t) \) is irreducible. Hence \( W \) and consequently \( W^* \) are simple implying that \( V^* = W^* \). But
\[ \det \mathcal{A}_k = 0 \iff \dim_{K(t)} V^* < m \]
see, e.g. [7]. Thus the irreducibility of the differential operator \( E(\Delta, t) \) shows the non-vanishing of the determinant \( \mathcal{A}_k \).

**Theorem 4.4.** Theorem 3.1 is satisfied by the series
\[ F(t) = \sum_{n=0}^{\infty} \frac{t^n}{\prod_{k=0}^{n-1} P(k)}, \quad \deg P(x) = m \geq 2, \quad P(-1) = 0. \]

Proof. When one writes \( P(x) = (x+1)P_1(x+1) \), then the series (41) satisfies the differential equation
\[ [\Delta P_1(\Delta) - t]F(t) = 0 \]
where \( S = r_j = 0 \) for all \( j = 1, \ldots, m - 1 \) and \( r_m = 1 \). Thus Theorem 3.1 is valid for \( F(t) \).

Here we note that in Lemma 4.2 of [2] the authors verified the condition \( 2^* \) for the hypergeometric functions
\[ \sum_{n=0}^{\infty} \frac{(a_1)n \cdots (a_p)n}{n!(b_1)n \cdots (b_q-1)n} t^n, \quad p < q, \]
where \( a_i - b_j \notin \mathbb{Z} \) for all \( i, j \) with \( b_q = 1 \). Note also the similar results of Shalikhov, see e.g. [6].

5. **Appendix by an anonymous referee**

5.1. **An irreducibility criterion.** As in the introduction of the text, let \( K = \mathbb{C} \) or \( \mathbb{C}_p \), and let \( \mathcal{K} = K((z)) \) be the field of formal power series in the variable \( z \), endowed with the derivations
\[ \partial = d/dz, \theta = z\partial. \]

Let further \( v \) be the \( z \)-adic valuation on \( \mathcal{K} \): for any non-zero \( f \in \mathcal{K} \), \( v(f) \) is the order of \( f \) at 0. Finally, let \( \mathcal{D} = \mathcal{K}[\partial] = \mathcal{K}[\theta] \) be the ring of differential operators with coefficients in \( \mathcal{K} \). A non zero element \( L \) of \( \mathcal{D} \) is called irreducible if it cannot be written as the product \( L_1L_2 \) of two elements of \( \mathcal{D} \) of orders strictly smaller than that of \( L \).

Let
\[ L = q_m(z)\partial^m + \ldots + q_1(z)\partial + q_0(z) \in \mathcal{D} \]
be a \( m \)-th order differential operator, with coefficients \( q_m \neq 0, q_{m-1}, \ldots, q_0 \) in \( \mathcal{K} \). We can alternatively write \( L \) in terms of \( \theta \), as follows;
\[ L = a_m(z)\theta^m + \ldots + a_1(z)\theta + a_0(z), \]
where the coefficients \( a_j \) can be computed in terms of the coefficients \( q_i, i \geq j \). From the classical Formula (31) of the text and its "converse" (expressing \( t^nD^n \) in terms of the \( \Delta^k \)'s), one deduces that
\[ \forall i = 0, \ldots, m : \min_{j=i, \ldots, m} v(a_j) = \min_{j=i, \ldots, m} (v(q_i) - i). \]

The Newton polygon \( \mathcal{N}(L) \) of \( L \) is the convex hull of the set \( \cup_{0 \leq i \leq m} \{ (x, y) \in \mathbb{R}^2, 0 \leq x \leq i, y \geq v(q_i) - i \} \). By the preceding relations, it coincides with the convex hull of the set \( \cup_{0 \leq i \leq m} \{ (x, y) \in \mathbb{R}^2, 0 \leq x \leq i, y \geq v(a_i) \} \). The slopes of its non-vertical bordering lines are called the *slopes* of \( L \) (at the point 0). In particular, for any non zero \( q \in \mathcal{K} \), the differential
operators $L$ and $qL$ have the same slopes. Clearly, the slopes of $L$ are rational numbers, and we have:

**Lemma 5.1.** (Katz [3]). Let $L \in \mathcal{D}$ be a differential operator of order $m > 0$. Assume that $L$ has only one slope $\lambda$, and that the exact denominator of the rational number $\lambda$ is equal to $m$. Then, $L$ is irreducible in the ring $\mathcal{D}$.

**Proof.** Since $L$ is irreducible if and only if $\mathcal{D}/DL$ is a simple $\mathcal{D}$-module, this is just a rephrasing of the Irreducibility Criterion 2.2.8 of Katz [3]. See also [5], bottom of p.79.

5.2. **An alternative proof of Corollary 4.1.** We here consider the field $K = K(t)$, and recall the notations

$$D = d/dt, \Delta = tD$$

from the text. Setting $z = 1/t$, we can view $K$ as a subfield of $K = K((z))$. The valuation induced on $K$ by the valuation $v$ of $K$ then corresponds to the place $t = \infty$. In particular, for any polynomial $A \in K[t] \subset K$, we have:

$$v(A) = -\deg(A).$$

Consider the operator of order $m$

$$E(\Delta, t) = N(t)\Delta^m - P_1(t)\Delta^{m-1} - \ldots - P_m(t)\Delta^0$$

from Formula (6) of the text, and assume that its (polynomial) coefficients satisfy Condition (15) of Theorem 4.1. Setting $z = 1/t$, so that $\theta = td/dt = -zd/dz = -\Delta$, we have $E(\Delta, t) = L(\theta, z) = a_m(z)\theta^m + a_{m-1}\theta^{m-1} + \ldots + a_0(z)$, where

$$a_m(z) = (-1)^mN(\frac{1}{z}), \quad a_{m-1}(z) = \pm P_1(\frac{1}{z}), \quad \ldots, \quad a_0(z) = \pm P_m(\frac{1}{z}).$$

In the notation of the text for the degrees of the polynomials $N, P_i$, we therefore have:

$$v(a_m) = -S, \quad v(a_{m-1}) = -r_1, \quad \ldots, \quad v(a_1) = -r_{m-1}, \quad v(a_0) = -r_m.$$

Conditions (10) now read:

$$v(a_0) = v(a_m) - 1, \quad \text{and} \quad \forall j = 1, \ldots, m - 1, \quad v(a_{m-j}) > v(a_0).$$

Therefore, all the points $(x = j, y = v(a_j)) \in \mathbb{R}^2, j = 1, \ldots, m - 1$, lie on or above the line $y = v(a_0) + 1$, and the Newton polygon of $L = L(\theta, z)$ is bordered by a unique non-vertical line, joining the points $(0, v(a_0))$ and $(m, v(a_m) = v(a_0) + 1)$. In particular, $\mathcal{N}(L)$ has a unique slope

$$\lambda = \frac{v(a_m) - v(a_0)}{m} = \frac{1}{m}.$$ 

Since its denominator is $m$, we deduce from the criterion that $L$ is irreducible in the ring $K[\theta]$. A fortiori, $L$ is irreducible in the ring $K[\theta] = K(t)[D]$, and so is $E = E(\Delta, t)$, as announced in Corollary 4.1.

5.3. **An alternative proof of Corollary 4.2.** With the same notations as above, let us now consider the operator of order $m$

$$E(D, t) = Q_0(t)D^m - Q_1(t)D^{m-1} - \ldots - Q_m(t)D^0$$

from Formula (1) (where we have set : $Q_0(t) := T(t)$), and assume the degrees $r'_0 := S' = \deg(Q_0), r'_j = \deg(Q_j), j = 1, \ldots, m$ of its (polynomial) coefficients satisfy either Condition (28) or Condition (29) of Theorem 3.2. We shall then prove:

**Corollary 5.2.** ( = Corollary 4.2): under Condition (28) or (29), the differential operator $E$ is irreducible in the ring $K(t)[D]$. 

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Using Formula \(31\), and then setting \(z = 1/t\), we can rewrite
\[
E(D, t) = \frac{Q_0(t)}{t^m} (t^m D^m) - \frac{Q_1(t)}{t^{m-1}} (t^{m-1} D^{m-1}) + ... - Q_m(t)
\]
\[= \tilde{Q}_0(t) \Delta^m + \tilde{Q}_1(t) \Delta^{m-1} + ... + \tilde{Q}_m(t),
\]
\[= \tilde{a}_m(z) \theta^m + \tilde{a}_{m-1}(z) \theta^{m-1} + ... + \tilde{a}_0(z) := \tilde{L}(\theta, z).
\]
A computation similar to that of §1 (using both \(31\) and its converse) shows that the coefficients \(\tilde{a}_j(z) = \pm \tilde{Q}_{m-j}(\frac{1}{2})\) of \(\tilde{L}\) satisfy \(v(\tilde{a}_m) = -\deg(Q_0) + m\), and more generally:
\[
\forall i = 0, ..., m, \ \min_{j=i, ..., m} v(\tilde{a}_j) = \min_{j=i, ..., m} (-\deg(Q_{m-j}) + j).
\]
Therefore, the Newton polygon of \(\tilde{L}\) is given by the convex hull of the points:
\[(0, -r_m'), (1, -r_{m-1}' + 1), ..., (m - 1, -r_1' + m - 1), (m, v(\tilde{a}_m) = -S' + m).
\]
Let us now study this Newton polygon \(\mathcal{N}(\tilde{L})\).

**Case of Condition \(28\)**: in this case, the first point is \(P_0 = (0, -r_m' = -(S' + 1))\), the last point is \(P_m = (m, -S' + m)\), and since \(r_i' < r_m'\), i.e. \(r_i' \leq S'\) for \(1 \leq i \leq m - 1\), all the other points \((x = i, y = -r_{m-i}' + i \geq -S'+i)\) lie on or above the line of slope 1 passing through the point \(P_m\), while \(P_0\) lies below this line. Therefore, \(\mathcal{N}(\tilde{L})\) has only one non vertical side, given by \([P_0P_m]\), whose slope is equal to
\[
\lambda = \frac{(-S' + m) + (S' + 1)}{m} = \frac{m + 1}{m}.
\]
Since the exact denominator of this number is \(m\), the irreducibility Criterion applies.

**Case of Condition \(29\)**: in this case, the first point is \(P_0 = (0, -r_m' = -(S' - 1))\), the last point is \(P_m = (m, -S' + m)\), the slope of the line \((P_0P_m)\) is
\[
\lambda = \frac{(-S' + m) + (S' - 1)}{m} = \frac{m - 1}{m} = 1 - \frac{1}{m},
\]
and since \(r_i' \leq r_m' < S'\) for \(1 \leq i \leq m - 1\), all the other points \(P_i = (i, -r_{m-i}' + i)\) lie above this line: indeed, the slope of \((P_iP_m)\) is given by
\[
\frac{(-S' + m) - (-r_{m-i}' + i)}{m - i} = 1 - \frac{S' - r_{m-i}'}{m - i} < 1 - \frac{1}{m} = \lambda.
\]
Therefore, \(\lambda\) is the unique slope of \(\mathcal{N}(\tilde{L})\), and since its denominator is equal to \(m\), the irreducibility Criterion again applies.

### 5.4. Corollary X implies Theorem X.

In each case, Corollary X says that a certain \(\mathcal{D}\)-module \(V\) is irreducible. In each case, Theorem X concerns a certain \(\mathcal{D}\)-module \(W\), as well as the natural structure of \(\mathcal{D}\)-module that its dual \(W^*\) acquires, and says the \(\mathcal{D}\)-submodule of \(W^*\) generated by any non-zero linear form \(\ell\) on \(W\) fills up \(W^*\). More precisely, the determinant considered in Formula \(3\) (resp. \(4\)) vanishes if and only if the dimension over \(K\) of the \(\mathcal{D}\)-submodule generated by \(\ell = L\) (resp. \(\ell = R\)) is strictly smaller than \(m = \text{dim}_K W\). So, Theorem X exactly says that \(W^*\) is irreducible (and this occurs if and only if \(W\) itself is irreducible).

To prove the equivalence of Corollary X and Theorem X, it therefore suffices to prove that \(V\) is isomorphic to \(W\) or to \(W^*\). This follows from the fact that in each case \(X = 1\), resp. 2, both \(V\) and \(W\) are isomorphic to \(\mathcal{D}/DE\) (or to its dual), for the same differential operator \(E\).
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