Holomorphic Vector Field and Topological Sigma Model on \( \mathbb{C}P^1 \) World Sheet

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Abstract

Witten suggested that fixed-point theorems can be derived by the supersymmetric sigma model on a Riemann manifold \( M \) with potential term induced from Killing vector on \( M \) \cite{3}. One of the well-known fixed-point theorem is the Bott residue formula \cite{9} which represents intersection number of Chern classes of holomorphic vector bundles on a Kähler manifold \( M \) as sum of contributions from fixed point sets of a holomorphic vector field \( K \) on \( M \). In this paper, we derive the Bott residue formula by using topological sigma model (A-model) that describes dynamics of maps from \( \mathbb{C}P^1 \) to \( M \), with potential term induced from the vector field \( K \). Our strategy is to restrict phase space of path integral to maps homotopic to constant maps. As an effect of adding a potential term to topological sigma model, we are forced to modify BRST symmetry of the original topological sigma model. Our potential term and BRST symmetry are closely related to the idea used in the paper by Beasley and Witten \cite{2} where potential terms induced from holomorphic section of a holomorphic vector bundle and corresponding supersymmetry are considered.

1 Introduction

1.1 Background

Various topological indices of a Riemann manifold \( M \), such as Euler number, Hirzebruch signature, Atiyah-Singer index etc., are computed by using path integral of supersymmetric sigma model with target space \( M \) \cite{11, 13, 14}. In \cite{8}, Witten considered the supersymmetric sigma model with various potential
terms. Especially, he suggested that fixed-point formula for signature of even-dimensional \(M\) can be obtained by using this model with a potential term induced from a Killing vector field on \(M\). On the other hand, various fixed-point formulas, such as Duistermaat-Heckman formula etc., have been derived by using this kind of potential terms [11][12].

Our aim of this paper is to derive the Bott residue formula [9], that describes intersection number of Chern classes of holomorphic vector bundles on a Kähler manifold \(M\) as sum of contributions from fixed point sets of a holomorphic tangent vector field \(K\) on \(M\), by using topological sigma model (A-model) from \(CP^1\) to \(M\) with a potential term induced from the vector field \(K\). In order to extend the BRST-symmetry to the A-model with the potential, we have to use half of the supersymmetry transformations used in constructing the BRST-transformation of topological sigma model (A-model) from the \(N = (2,2)\) supersymmetric sigma model. The new BRST symmetry is still conserved in the original Lagrangian of the A-model, and it can be extended to the A-model with the potential term. As a result, BRST-closed observables correspond to Dolbeault cohomology of \(M\) instead of De Rham cohomology of \(M\). But Chern classes of holomorphic vector bundles on \(M\) are given by \((i,i)\) forms of \(M\), and they are automatically Dolbeault cohomology classes of \(M\). Therefore, change of BRST-symmetry causes no problem for our purpose in this paper. Beasley and Witten considered supersymmetry closely related to our new BRST-transformation for \((0,2)\) linear sigma models with a potential term induced from holomorphic section of a holomorphic vector bundle [2]. Our new BRST-symmetry seems to be closely related to their idea applied to the case when the holomorphic vector bundle is given by the holomorphic tangent vector bundle \(T'M\). Of course, they derive a kind of fixed-point formula, but their result is different from our goal in this paper: “deriving the Bott-residue formula by using the topological sigma model (A-model) with the potential term induced from holomorphic tangent vector field”.

1.2 Main result

The purpose of this paper is to provide a derivation of the Bott residue formula by using topological sigma model (A-model) with potential terms induced from holomorphic tangent vector field. First, we introduce assertion of the Bott residue formula [9].

The Bott Residue Formula

\[
\int_M \varphi(E) = \sum_\alpha \int_{N_\alpha} \varphi(\Lambda_\alpha) \left| \frac{\det(\theta_{\nu}^{\alpha} + tR_{\nu}^{\alpha})}{t^{m}} \right|_{t=\frac{i\pi}{2}}.
\]

Here, \(M\) is a compact Kähler manifold with \(\text{dim}_\mathbb{C}(M) = m\). Let \(E\) be a holomorphic vector bundle over \(M\) with \(\text{rank}_\mathbb{C}E = q\) and \(\varphi(E)\) is a wedge product of Chern classes of \(E\) represented by symmetric homogeneous polynomial \(\varphi(x_1, \ldots, x_q)\) of degree \(m\) (explicit definition will be introduced later). Let \(K\) be a holomorphic tangent vector field on \(M\) and \(\{N_\alpha\}\) be set of connected components of the zero set of \(K\). \(\varphi(\Lambda_\alpha)\) is a cohomology class of \(N_\alpha\) which will be defined later. \(\theta_{\nu}^{\alpha}\) is the automorphism induced by action of \(K\) on the normal bundle of \(N_\alpha\). \(R_{\nu}^{\alpha}\) is the curvature \((1,1)\)-form of the normal bundle of \(N_\alpha\).
Next, we introduce the topological sigma model (A-model). We use the topological sigma model (A-model) from $\mathbb{CP}^1$ to the Kähler manifold $M$ with potential terms induced from the holomorphic tangent vector field $K = K^i \frac{\partial}{\partial z^i}$. Fields that appear in the model is given as follows.

- $\phi^i \bar{\phi}^j$: bosonic fields given as $C^\infty$-map from $\mathbb{CP}^1$ to $M$.
- $\chi^i$: fermionic fields that takes values in $C^\infty$ section of $\phi^{-1}T'M$
- $\bar{\psi}^i_z$: fermionic fields that takes values in $C^\infty$ section of $\phi^{-1}T'M$
- $\psi^i_z$: fermionic fields that takes values in $C^\infty$ section of $T'\mathbb{CP}^1 \otimes \phi^{-1}T'M$

Let $g_{ij}$ be Kähler metric of $M$ and $R_{ijkl}$ be its curvature tensor. Then the Lagrangian of our model is given as follows [7].

$$L + V = \int_{\mathbb{CP}^1} dz d\bar{z} \left[ \frac{t}{2} g_{ij} (\partial_z \phi^i \partial_{\bar{z}} \phi^j + \partial_{\bar{z}} \phi^i \partial_z \phi^j) + \sqrt{t} i g_{ij} \bar{\psi}^i_z D_z \chi^j + \sqrt{t} i g_{ij} \psi^i_z D_{\bar{z}} \chi^j - R_{ijkl} \bar{\psi}^i_z \psi^j_z \chi^k \chi^l + ts^2 g_{ij} K^i \bar{K}^j + ts g_{ij} \nabla_{\mu} K^i \bar{\chi}^j + s^2 g_{ij} \nabla_{\mu} K^i \bar{\psi}^j_z \psi^j_z \psi^j_z \right]$$

(1.1)

We set $\beta = 2\pi i$. Covariant derivatives are given by,

$$D_z \chi^i = \partial_z \chi^i + \Gamma^i_{\mu \nu} \partial_z \phi^\mu \chi^\nu$$

$$D_{\bar{z}} \chi^i = \partial_{\bar{z}} \chi^i + \Gamma^i_{\mu \nu} \partial_{\bar{z}} \phi^\mu \chi^\nu.$$  

(1.2)

(1.3)

Our BRST-transformation for this model is given as follows. ($\alpha$ is a fermionic variable.)

$$\delta \phi^i = i\beta \chi^i$$

$$\delta \bar{\psi}^i_z = -i\alpha \Gamma^i_{\mu \nu} \chi^\mu \psi^\nu_z$$

$$\delta \psi^i_z = -\sqrt{t} \alpha \partial_z \phi^i$$

$$\delta \bar{\chi}^i = i s \beta K^i$$

$$\delta \bar{\psi}^i_z = 0$$

(1.4)

The above potential terms and BRST-symmetry are closely related to the supersymmetry used in [2]. In this paper, we prove the following proposition that play important roles in our derivation.

Proposition 1 Correlation functions of BRST-closed observables are invariant under variation of $s$.

Then we can derive the Bott residue formula by evaluating correlation function of degree zero map both in the limit $s \to 0$ and $s \to \infty$.

1.3 Organization of the Paper

This paper is organized as follows.

In section 2, we introduce the Bott residue formula and notations used in this paper.

In section 3, we introduce our topological sigma model. First, we review outline of the topological sigma model (A-model) without potential terms and introduce our BRST-symmetry that uses half of the supersymmetry used in the
usual BEST-symmetry of the original topological sigma model. We show that
our new BRST-symmetry conserves the Lagrangian of the topological sigma
model without potential terms. Under the new BRST-symmetry, BRST-closed
observables become elements of Dolbeaut cohomology of the target Kähler man-
ifold. Next, we include potential terms induced from a holomorphic tangent
vector field and extend the new BRST-symmetry to this case. Then BRST-
closed observables become elements of equivariant Dolbeaut cohomology under
the action of the holomorphic vector field. Mathematically relationship between
the Bott residue formula [8] and this cohomology is discussed in [14], [12].

Section 4 is the main section of this paper. We evaluate the degree 0 (i.e.,
homotopic to constant maps) correlation function of our model. It corresponds
to the correlation function represented by the Bott residue formula. Proposition
1 claims that the correlation function is invariant under change of the parameter
s. Hence we can compare the results evaluated under the $s \to 0$ limit and the
$s \to \infty$ limit.

In the $s \to 0$ limit, the degree 0 correlation function turns into classical
integration on $M$ of differential forms that represent Chen classes by the stan-
dard argument of weak coupling limit of the topological sigma model. In the
$s \to \infty$ limit, evaluation of the degree 0 correlation function reduces to eval-
uation of contributions of from connected components of the zero set of the
holomorphic tangent vector field $K$. This follows from localization principle.
Then we perform standard localization computation. The result of evaluation
from each connected component turns out to be the contribution in the Bott
residue formula from the same connected component.

By equating these two results, we obtain the desired Bott residue formula.
In appendix, we prove the proposition 1 proposed in the previous subsection.

2 Notation and The Bott Residue Formula

2.1 Action of Holomorphic Tangent Vector Field

We introduce here our basic notations.

$M$: a compact Kähler manifold $\dim \mathbb{C}(M) = m$, \quad $K$: a holomorphic tangent
vector field on $M$.

$E$: a holomorphic vector bundle on $M$ with rank$c(E) = q$,

$\{e_i | i = 1 \cdots q\}$: local holomorphic frame of $E$,

$\Gamma(E)$: $C^\infty$ section of $E$.

$\Lambda: \Gamma(E) \to \Gamma(E)$ is a differential operator which acts on $fs$ ($f: C^\infty$ a function
on $M$, $s \in \Gamma(E)$) which satisfies the following conditions,

$$\Lambda(fs) = (Kf) \cdot s + f \Lambda(s), \quad \bar{\partial} \Lambda = \Lambda \bar{\partial},$$

where $\bar{\partial}$ is anti-holomorphic part of the exterior derivative operator $d$ of $M$. We
call $\Lambda$ action of $K$ on $E$. In the case when $E = T'M$, $\Lambda$ is given by bracket
operation $\theta(K)$ define by $\theta(K)(Y) = [K,Y]$. 

4
2.2 Notations for Characteristic Classes

We introduce here notations to describe characteristic classes of $E$.

\[ \varphi(x_1, \cdots, x_q) \]: a symmetric homogeneous polynomial in $x_1, \cdots, x_q$ with complex coefficients of homogeneous degree $m = \dim \mathbb{C}(M)$.

Next, we define $\varphi(A)$ where $A$ is an endomorphism $A : V \to V$. ($V$: a complex $q$-dimensional vector space). Let $\lambda_i$ $(I = 1, \cdots, q)$ be eigenvalues of $A$. Then it is defined by,

\[ \varphi(A) := \varphi(\lambda_1, \cdots, \lambda_q). \]  \hspace{1cm} (2.6)

We then regard $x_1, \cdots, x_q$ as Chern roots of $E$ defined by,

\[ \prod_{i=1}^q (1 + tx_i) := 1 + tc_1(E) + t^2c_2(E) + \cdots + t^qc_q(E). \]  \hspace{1cm} (2.7)

With this set-up, a characteristic class $\varphi(E)$ is defined as follows.

\[ \varphi(E) := \varphi(x_1, \cdots, x_q). \]  \hspace{1cm} (2.8)

Let $\{N_{\alpha}\}$ be set of connected components of zero set of $K$. We assume that each $N_{\alpha}$ is a compact Kähler submanifold of $M$. In the following, we define $\varphi(\Lambda_{\alpha})$, which is given as a cohomology class of $N_{\alpha}$. Let $\Lambda_{\alpha}$ be $\Lambda|_{N_{\alpha}}$, i.e., restriction of $\Lambda$ to $N_{\alpha}$. By the first condition in (2.5), $\Lambda_{\alpha}$ becomes an endomorphism of $E_{\alpha} := E|_{N_{\alpha}}$. We say that $\Lambda$ is constant type iff eigenvalues of $\Lambda_{\alpha}$ are constant on each connected component $N_{\alpha}$. In this paper, we assume that $\Lambda$ is constant type. Then we introduce the following notations:

\[ \{ \lambda_{i\alpha}^\alpha | i = 1, \cdots, r \} \]: distinct eigenvalues of $\Lambda_{\alpha}$ ($r \leq q$),

\[ m_{i\alpha}^\alpha \]: multiplicity of $\lambda_{i\alpha}^\alpha$ ($\sum_{i=1}^r m_{i\alpha}^\alpha = q$),

\[ E_{\alpha}(\lambda_{i\alpha}^\alpha) \]: the largest sub-bundle of $E_{\alpha}$ on which $(\Lambda_{\alpha} - \lambda_{i\alpha}^\alpha)$ is nilpotent, $(\text{rank}_\mathbb{C}(E_{\alpha}(\lambda_{i\alpha}^\alpha)) = m_{i\alpha}^\alpha$).

Then, $E_{\alpha}$ canonically decomposes into a direct sum, $E_{\alpha} = \bigoplus_{i=1}^r E_{\alpha}(\lambda_{i\alpha}^\alpha)$. Let $c_i(E_{\alpha}(\lambda_{i\alpha}^\alpha))$ be the $i$-th Chern class of $E_{\alpha}(\lambda_{i\alpha}^\alpha)$ and $x_j(\lambda_{i\alpha}^\alpha)$ ($j = 1, \cdots, m_{i\alpha}^\alpha$) be Chern roots of $E_{\alpha}(\lambda_{i\alpha}^\alpha)$ defined by,

\[ \prod_{j=1}^{m_{i\alpha}^\alpha} (1 + tx_j(\lambda_{i\alpha}^\alpha)) := 1 + tc_1(E_{\alpha}(\lambda_{i\alpha}^\alpha)) + t^2c_2(E_{\alpha}(\lambda_{i\alpha}^\alpha)) + \cdots + t^{m_{i\alpha}^\alpha}c_{m_{i\alpha}^\alpha}(E_{\alpha}(\lambda_{i\alpha}^\alpha)). \]  \hspace{1cm} (2.9)

With these set-up’s, the cohomology class $\varphi(\Lambda_{\alpha})$ is defined by,

\[ \varphi(\Lambda_{\alpha}) := \varphi(\lambda_1^1 + x_1(\lambda_1^1), \cdots, \lambda_1^n + x_{m_1}(\lambda_1^n), \lambda_2^1 + x_1(\lambda_2^1), \cdots, \lambda_2^n + x_{m_2}(\lambda_2^n), \cdots, \lambda_r^1 + x_1(\lambda_r^1), \cdots, \lambda_r^n + x_{m_r}(\lambda_r^n)). \]  \hspace{1cm} (2.10)

This is the original definition of $\varphi(\Lambda_{\alpha})$ used in [8]. Let $F_{\alpha}$ be curvature $(1,1)$-form (valued in $\text{End}(E|_{N_{\alpha}})$) of $E|_{N_{\alpha}}$. If we regard $\Lambda_{\alpha} + \frac{i}{2\pi} F_{\alpha}$ as $\text{End}(E|_{N_{\alpha}})$ valued form on $N_{\alpha}$, $\varphi(\Lambda_{\alpha})$ is rewritten by using (2.6) as follows.

\[ \varphi(\Lambda_{\alpha}) = \varphi(\Lambda_{\alpha} + \frac{i}{2\pi} F_{\alpha}). \]  \hspace{1cm} (2.11)
2.3 The Bott residue formula

We assume that the endomorphism \( \theta|_{N_\alpha} \), induced by the action of \( K \) on the holomorphic tangent bundle \( T'M|_{N_\alpha} \) has precisely \( T'N_\alpha \) for its kernel; i.e., the sequence

\[
0 \to T'N_\alpha \to T'M|_{N_\alpha} \xrightarrow{\theta|_{N_\alpha}} T'M|_{N_\alpha}
\]

is exact. From the above exact sequence, \( \text{Im}(\theta|_{N_\alpha}) \cong T'M|_{N_\alpha}/T'N_\alpha \) follows. Hence we obtain an automorphism \( \theta_\nu|_{N_\alpha} : T'M|_{N_\alpha}/T'N_\alpha \to T'M|_{N_\alpha}/T'N_\alpha \). Let \( R_\nu^{\alpha} \) be curvature (1,1)-form of the holomorphic normal bundle \( T'M|_{N_\alpha}/T'N_\alpha \). Then the Bott residue formula is given as follows.

\[
\int_M \varphi(E) = \sum_\alpha \int_{N_\alpha} \frac{\varphi(\Lambda_\alpha)}{\det(\theta_\nu^{\alpha} + \frac{i}{2\pi} R_\nu^{\alpha})} = \sum_\alpha \int_{N_\alpha} \frac{\varphi(\Lambda_\alpha + tF_\alpha)}{\det(\theta_\nu^{\alpha} + tR_\nu^{\alpha})}
\]

(2.12)

We will derive the Bott Residue formula in the form of the second line of the above equality.

3 The Model in This Paper

3.1 Base Model (Topological Sigma Model (A-Model) with Half BRST-Symmetry)

We introduce our base model (topological sigma model (A-model)). Lagrangian of the model is given as follows [7].

\[
L = \int_{\mathbb{C}P^1} dz \bar{dz} \left[ \frac{i}{2} g_{ij} \partial_z \phi^i \partial_{\bar{z}} \phi^j + \partial_z \phi^i \partial_{\bar{z}} \phi^j \right] + \sqrt{i} t g_{ij} \bar{\psi}_j^i \bar{z} \chi^j + \sqrt{i} t g_{ij} \psi^i_{\bar{z}} D_z \chi^j - R_{ijkl} \bar{\psi}_j^i \bar{\psi}_l^k \chi^j \bar{\chi}^l \right].
\]

(3.14)

Fields in the Lagrangian and covariant derivatives are the same as the ones introduced in Section 1.

Original BRST-symmetry of this model is given in [7].

\[
\begin{align*}
\delta \phi^i &= i \alpha \chi^i , \\
\delta \psi^i_{\bar{z}} &= -\sqrt{i} t \alpha \partial_{\bar{z}} \phi^i - i \alpha \Gamma^i_{\mu \nu} \chi^\mu \psi^\nu_{\bar{z}} , \\
\delta \psi^i_{z} &= -\sqrt{i} t \alpha \partial_z \phi^i - i \alpha \Gamma^i_{\mu \nu} \chi^\mu \psi^\nu_z , \\
\delta \chi^j &= \delta \chi^j = 0,
\end{align*}
\]

(3.15)

where \( \alpha \) is a fermionic parameter. In order to include potential terms induced from holomorphic tangent vector field \( K \), we have to change the BRST-symmetry in the following way (we observed that this change is inevitable to extend BRST-symmetry to the Lagrangian with potential terms).

\[
\begin{align*}
\delta \phi^i &= i \alpha \chi^i , \\
\delta \psi^i_{\bar{z}} &= -i \alpha \Gamma^i_{\mu \nu} \chi^\mu \psi^\nu_{\bar{z}} , \\
\delta \psi^i_{z} &= -\sqrt{i} t \alpha \partial_z \phi^i , \\
\delta \phi^i &= \delta \chi^j = \delta \chi^j = 0.
\end{align*}
\]

(3.16)
where $\bar{\alpha}$ is also a fermionic parameter. In the next subsection, we prove that the Lagrangian \((3.14)\) remains invariant under this new BRST-symmetry. Let $Q$ be the generator of this transformation defined via the relation $\delta X =: i\alpha \{Q, X\}$ ($X$ is a field that appears in the theory). We can check nilpotency of $Q$, i.e., $\{Q, \{Q, X\}\} = 0$. The most non-trivial part comes from deriving $\{Q, \{Q, \psi^i_z\}\} = 0$ In this case, we have $\{Q, \psi^i_z\} = -\Gamma^i_{\mu\nu} \chi^\mu \psi^\nu_z$. By using the following relation that holds for the curvature tensor of Kähler manifold,

$$R^i_{\alpha\mu\nu} = \partial_i \Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\mu\beta} \Gamma^\beta_i \Gamma_{\nu}^\beta - \partial_\mu \Gamma^\alpha_i \Gamma_{\nu}^\beta + \Gamma^\beta_i \Gamma^\alpha_{\mu\nu} = 0$$

we can show $\delta(-\Gamma^i_{\mu\nu} \chi^\mu \psi^\nu_z) = 0$. Hence $\{Q, \{Q, \psi^i_z\}\} = 0$ holds. Check for other fields is straightforward.

### 3.2 Proof of $\delta L = 0$

In this subsection, we check invariance of the Lagrangian in \((3.14)\) under the BRST-transformation given in \((3.16)\), i.e., the equality $\delta L = 0$. We first evaluate variation of $g_{ij} \partial_z \phi^i \partial_z \phi^j$.

$$\delta(tg_{ij} \partial_z \phi^i \partial_z \phi^j) = t\delta(g_{ij}) \delta \phi^i \partial_z \phi^j + tg_{ij} \partial_z \phi^i \partial_z (\delta \phi^j) = it\alpha(g_{ij} \Gamma^j_i \partial_z \phi^i \chi^j + g_{ij} \partial_z \phi^i \partial_z \chi^j)$$

$$= it\alpha g_{ij} \partial_z \phi^i (\partial_z \chi^j + \Gamma^j_{\mu\nu} \partial_z \phi^\mu \chi^\nu)$$

By integration by parts, we obtain the following (we neglect total differential).

$$\int \Sigma dz \bar{\delta}(\frac{1}{2} g_{ij} \partial_z \phi^i \partial_z \phi^j)$$

$$= it\alpha \int \Sigma dz \bar{\delta}(\frac{1}{2} (g_{ij} \partial_z \phi^i \partial_z \chi^j + g_{ij} \partial_z \phi^i \partial_z \chi^j))$$

$$= it\alpha \int \Sigma dz \bar{\delta}(\frac{1}{2} (-\partial g_{ij} \partial_z \phi^i \partial_z \chi^j - \partial g_{ij} \partial_z \phi^i \partial_z \chi^j)$$

$$- g_{ij} \partial_z \phi^i \partial_z \chi^j + \partial_z \partial_z \phi^i \partial_z \chi^j)$$

$$= it\alpha \int \Sigma dz \bar{\delta}(\frac{1}{2} [g_{ij} \partial_z \phi^i (\partial_z \chi^j + \Gamma^j_{\mu\nu} \partial_z \phi^\mu \chi^\nu)])$$

$$= it\alpha \int \Sigma dz \bar{\delta}(\frac{1}{2} (g_{ij} \partial_z \phi^i D_z \chi^j)$$

Variation of other terms are evaluated as follows.

$$\delta(\frac{1}{2} tg_{ij} \partial_z \phi^i \partial_z \phi^j) = it\alpha \frac{1}{2} (g_{ij} \Gamma^j_i \partial_z \phi^j + tg_{ij} \partial_z \phi^i \partial_z \chi^j)$$

$$\delta(\frac{1}{2} tg_{ij} \partial_z \phi^i \partial_z \phi^j) = it\alpha \frac{1}{2} g_{ij} \partial_z \phi^i D_z \chi^j$$

$$\delta((\sqrt{\bar{\alpha}}tg_{ij} \psi^i_z D_z \chi^j) = -\sqrt{\bar{\alpha}} R_{i\mu\nu} \partial_z \phi^i \psi^j_z \chi^\nu \chi^j$$

$$\delta((\sqrt{\bar{\alpha}}tg_{ij} \psi^i_z D_z \chi^j) = \sqrt{\bar{\alpha}} g_{ij} (\partial_l \Gamma^j_l - \Gamma^j_{\mu\nu} \Gamma^3_{\mu\nu}) \psi^i_z \partial_z \phi^\mu \chi^\nu - t\sqrt{\bar{\alpha}} g_{ij} \partial_z \phi^i D_z \chi^j$$

(3.22)
By using the following equality that holds for curvature of Kähler manifolds:

$$R^i_{\ell \mu \nu} = \partial_l \Gamma^i_{\mu \nu} - \partial_\mu \Gamma^i_{l \nu} + \Gamma^j_{\mu \nu} \Gamma^i_{l j} - \Gamma^j_{\mu l} \Gamma^i_{\nu j} = 0. \quad (3.23)$$

and Kähler condition $\Gamma^\beta_{\mu \nu} = \Gamma^\beta_{\nu \mu}$, we obtain,

$$\delta \left( \sqrt{2} \xi g_{ij} \partial_i j D_z \chi \right) = -i \tilde{a} g_{ij} \partial_i j \phi^j D_z \chi. \quad (3.24)$$

Next, we use the following formula of covariant derivative of the curvature,

$$\nabla_i R^j_{kl} = \partial_i R^j_{kl} - R^j_{i \beta k} \Gamma^\beta_{\lambda j} - R^j_{i \lambda k} \Gamma^\lambda_{\beta j} \quad (3.25)$$

and Bianchi's identity,

$$\nabla_i R^i_{jk} = \nabla_j R^i_{ik}. \quad (3.26)$$

Then we obtain,

$$\delta \left( - R^j_{i \lambda k} \partial_i j \psi^j_k \chi^l \right) = -i \tilde{a} \partial_i j R^j_{i \lambda k} \psi^j_k \chi^l \lambda + \sqrt{2} \tilde{a} R^j_{i \lambda k} \partial_i j \phi^j \psi^j_k \chi^l. \quad (3.27)$$

As a result, all the variations cancel each other. Therefore, we have shown the equality: $\delta L = 0 \Leftrightarrow \{Q, L\} = 0$.

### 3.3 BRST-Closed Observables of the Base Model

In this subsection, we consider BRST-closed observable of this model, i.e., observable $O$ that satisfies $\{Q, O\} = 0$. Here we restrict observables that are obtained from differential forms on $M$. Let $W$ be a $(p, q)$-form on $M$:

$$W = \frac{1}{p! q!} W_{i_1 i_2 \ldots i_p j_1 j_2 \ldots j_q} (z^1, \ldots, z^n) dz^{i_1} dz^{i_2} \ldots dz^{i_p} d\bar{z}^{j_1} d\bar{z}^{j_2} \ldots d\bar{z}^{j_q}, \quad (3.28)$$

then we consider the following observable $O_W$:

$$O_W = \frac{1}{p! q!} W_{i_1 i_2 \ldots i_p j_1 j_2 \ldots j_q} (\phi) \chi^{i_1} \chi^{i_2} \ldots \chi^{i_p} \chi^{j_1} \chi^{j_2} \ldots \chi^{j_q}. \quad (3.29)$$

Variation of $O_W$ under the BRST-transformation:

$$\{Q, \phi^i\} = 0, \{Q, \phi^j\} = \chi^j, \{Q, \chi^i\} = 0, \{Q, \chi^j\} = 0,$n

is given by,

$$\delta O_W = \frac{1}{p! q!} \partial_{\bar{z}} W_{i_1 i_2 \ldots i_p j_1 j_2 \ldots j_q} \delta \phi^j \chi^{i_1} \chi^{i_2} \ldots \chi^{i_p} \chi^{j_1} \chi^{j_2} \ldots \chi^{j_q} = i \tilde{a} \frac{1}{p! q!} \partial_{\bar{z}} W_{i_1 i_2 \ldots i_p j_1 j_2 \ldots j_q} \chi^{i_1} \chi^{i_2} \chi^{j_1} \chi^{j_2} \ldots \chi^{j_q}. \quad (3.30)$$

Let $\tilde{\partial}$ be the anti-holomorphic part of the exterior derivative $d$. Then, the above result is summarized as follows.

$$\delta O_W = i \tilde{a} \tilde{\partial} O_W = i \tilde{a} \{Q, O_W\}. \quad (3.31)$$
\( \{ Q, \mathcal{O}_W \} = \mathcal{O}_{\overline{\partial}W} \)

Therefore, a BRST-closed observable \( \mathcal{O}_W \) is obtained from a differential form \( W \) that satisfies \( \overline{\partial}W = 0 \). By standard discussion of topological field theory, correlation function of BRST-closed observables with insertion of a observable of type \( \{ Q, \mathcal{O}_W \} = \mathcal{O}_{\overline{\partial}W} \) automatically vanishes. Hence, physical observables of the base model correspond to elements of Dolbeault cohomology. Let us recall Dolbeault’s theorem and Hodge’s decomposition theorem.

**Theorem 1** (Dolbeault’s theorem)

\[
H^q(M, \wedge^p T^\ast M) \cong H^p_{\overline{\partial}}(M). \tag{3.32}
\]

**Theorem 2** (Hodge’s decomposition theorem) \(^{13}\)

\[
H^r(M, \mathbb{C}) \cong \bigoplus_{p+q=r} H^{p, q}(M) \tag{3.33}
\]

\[
H^r(M, \overline{\partial}^\ast)(M) \cong H^r(M, \wedge^p T^\ast M). \tag{3.34}
\]

If \( W \) is a Chern class of a holomorphic vector bundle of \( M \), it is given as a closed \((i, i)\)-form on \( M \). Therefore \( \partial W = \overline{\partial}W = 0 \) follows from \( dW = (\partial + \overline{\partial})W = 0 \) and \( \mathcal{O}_W \) is a BRST-closed observable of the base model.

### 3.4 Potential Terms Induced from Holomorphic Tangent Vector Field

In this subsection, we include potential terms induced from the holomorphic tangent vector field \( K \). The potential terms are given as follows (we use a parameter \( \beta \) that equals \( 2\pi i \) for brevity).

\[
V = \int \Sigma \frac{dxdz}{2\pi i} [s^2 \beta g_{ij} K^i \overline{K}^j + tsg_{ij} \nabla_{\mu} \overline{K}^j \chi^i_{\mu} + s\beta g_{ij} \nabla_{\mu} K^i \psi^\mu_{\overline{\psi}_2} \overline{\psi}_2]. \tag{3.35}
\]

\( V \) contains a parameter \( s \in \mathbb{R} \) that controls scale of the vector field \( K \). We can extend the BRST-transformation to the new Lagrangian \( L + V \) as follows.

\[
\delta \phi^i = i\alpha \chi^i \quad \delta \psi^i = -i\alpha \Gamma^i_{\mu\overline{\nu}} \chi^\mu \overline{\psi}_2 \quad \delta \psi^\mu_{\overline{\nu}} = -2\sqrt{t} \alpha \partial_{\overline{\nu}} \phi^i \tag{3.36}
\]

\[
\delta \chi^i = i\alpha \beta K^i \quad \delta \phi^i = \delta \chi^i = 0 \tag{3.37}
\]

Let \( Q \) be a generator of this transformation whose action is defined via the relation \( \delta X =: i\alpha \{ Q, X \} \) (\( X \) is a field that appears in the theory). We check nilpotency of this generator \( \{ Q, \{ Q, X \} \} = 0 \). Non-trivial parts caused by appearance of \( K \) in \( (3.37) \) is given as follows.

\[
\delta \phi^i = i\alpha \{ Q, \phi^i \} \Rightarrow \{ Q, \phi^i \} = \chi^i \\
\delta \chi^i = i\alpha \{ Q, \phi^i \} \Rightarrow \{ Q, \phi^i \} = 0 \\
\delta \psi^i = i\alpha \{ Q, \chi^i \} \Rightarrow \{ Q, \chi^i \} = \beta s K^i \\
\delta K^i = i\alpha \{ Q, \chi^i \} \Rightarrow \{ Q, \chi^i \} = 0
\]

Hence the relation \( \{ Q, \{ Q, X \} \} = 0 \) also holds in this case.
3.5 Proof of $\delta(L + V) = 0$

In this subsection, we check invariance of the Lagrangian $L + V$ under $\delta$, i.e., the relation $\delta(L + V) = 0$. Let us recall some properties of covariant derivatives of $K$.

\[\nabla_\mu K^i = 0, \quad \nabla_\mu \bar{K}^i = 0, \quad \partial_i (g_{ij} \nabla_\mu \bar{K}^j) = g_{ij} \nabla_i \nabla_\mu \bar{K}^j + g_{ij} \Gamma^0_{\mu} \nabla_\sigma \bar{K}^j, \]
\[\partial_i (g_{ij} \nabla_\mu K^i) = g_{ij} \Gamma^0_{\mu} \nabla_\sigma K^i + R_{ij\mu} K^i. \]

We use the above formulas and standard property of Kähler metric. Then additional terms that appear in checking $\delta(L + V)$ are given as follows.

\[\delta \left( \sqrt{t s} \bar{\alpha} \beta g_{ij} \bar{\psi}^i \bar{\psi}^j D_2 \chi \right) = \sqrt{t s} \bar{\alpha} \beta g_{ij} \bar{\psi}^i \nabla_1 \bar{K}^i \partial_2 \phi \]
\[\delta (-R_{ijkl} \bar{\psi}^i \bar{\psi}^j \chi^k \bar{\chi}^l) = -s \bar{\alpha} \beta i R_{ijkl} \bar{\psi}^i \bar{\psi}^j K^k \bar{\chi}^l \]
\[\delta (t^2 \bar{\alpha} \beta g_{ij} K^i \bar{K}^j) = t s^2 i \bar{\alpha} \beta g_{ij} K^i \nabla_1 \bar{K}^j \bar{\chi} \]
\[\delta (t s g_{ij} \nabla_\mu \bar{K}^j \chi^l \bar{\chi}^l) = -t i s^2 \bar{\alpha} \beta g_{ij} \nabla_\mu \bar{K}^j \chi^l \bar{\chi}^l \]
\[\delta (s \bar{\alpha} \beta g_{ij} \bar{\psi}^i \bar{\psi}^j \bar{\psi}^2) = s \bar{\alpha} \beta i R_{ijkl} \bar{\psi}^i \bar{\psi}^j K^k \chi^l - \sqrt{t s} \bar{\alpha} \beta g_{ij} \nabla \mu K^i \partial_2 \phi \partial_2 \bar{\psi}^2 \]

From the above results, we can conclude that $\delta(L + V) = 0$ holds. From now on, we only consider the model given by $L + V$.

3.6 BRST-Closed Observable of the Model

In this subsection, we construct BRST-closed observable of the model with potential terms. In the same way as the previous discussions, we restrict our selves to observables obtained from a $(p, q)$-form $W$ on $M$. It is represented in the following form.

\[O_W = \frac{1}{p! q!} W_{i_1, i_2, \ldots, i_p, j_1, j_2, \ldots, j_q} \chi^{i_1} \chi^{i_2} \ldots \chi^{i_p} \chi^{j_1} \chi^{j_2} \ldots \chi^{j_q} \]

Variation $\delta O_W$ under the BRST-transformation is given by,

\[\delta O_W = \frac{1}{p! q!} \partial_i W_{i_1, i_2, \ldots, i_p, j_1, j_2, \ldots, j_q} \delta \phi^j \chi^{i_1} \chi^{i_2} \ldots \chi^{i_p} \chi^{j_1} \chi^{j_2} \ldots \chi^{j_q} \]
\[+ \frac{1}{(p - 1)! q!} W_{i_1, i_2, \ldots, i_{p-1}, j_1, j_2, \ldots, j_q} \delta \chi^l \chi^{i_1} \chi^{i_2} \ldots \chi^{i_{p-1}} \chi^{j_1} \chi^{j_2} \ldots \chi^{j_q} \]
\[= i \bar{\alpha} \left[ \frac{1}{p! q!} \partial_i W_{i_1, i_2, \ldots, i_p, j_1, j_2, \ldots, j_q} \chi^{i_1} \chi^{i_2} \ldots \chi^{i_p} \chi^{j_1} \chi^{j_2} \ldots \chi^{j_q} \right] \]
\[+ \frac{s \beta}{(p - 1)! q!} W_{i_1, i_2, \ldots, i_{p-1}, j_1, j_2, \ldots, j_q} \chi^{i_1} \chi^{i_2} \ldots \chi^{i_{p-1}} \chi^{j_1} \chi^{j_2} \ldots \chi^{j_q} \]

Let $i(K)$ be the inner-product operator by $K$. Then the above result is rewritten as follows.

\[\delta O_W = i \bar{\alpha} O_{(\partial + \beta i(sK))W} =: i \bar{\alpha} \{Q, O_W\}. \]

Hence BRST-closed observable is obtained from a differential form $W$ that satisfies $(\partial + \beta i(sK))W = 0$. Note that this condition reduces to $\partial W = 0$ if $s = 0$. 

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We comment on mathematical background to this condition. In general, differential form \( \omega \) on \( M \) is graded by the following operators:

\[
F_A \omega = (p + q) \omega, \quad F_V \omega = (q - p) \omega.
\]

This says that \( \omega \) is a \((p, q)\)-form on \( M \). For the operator \( \bar{\partial} + \beta i(sK) \), we adopt \( F_V \) as the grading operator and consider the following vector space:

\[
A^{(k)} = \bigoplus_{q-p=k} \Omega^{p,q}(M)
\]

where \( k \) ranges from \(-m\) to \( m \). From the condition (3.48), we can see that observables correspond to elements of cohomology of the complex \((A^{(k)}, \bar{\partial} + \beta i(sK))\). This complex is called Liu’s complex. In [14], it is shown that cohomology of this complex is independent of \( s \) \((s \neq 0)\). This property is closely related to Proposition 1.

4 Derivation of the Bott Residue Formula

4.1 Overview

In this subsection, we explain our strategy of deriving the Bott residue formula by using the topological sigma model with potential terms.

Since the Bott residue formula is a fixed point formula for integration of Chern classes of the holomorphic vector bundle \( E \), we consider observable that corresponds to wedge product of Chern classes in the \( s \to 0 \) limit. We first construct observable \( O_W \) that satisfies \((\bar{\partial} + \beta i(sK))W = 0\) and \( \lim_{s \to 0} W = \varphi(E) \). Let us recall Proposition 1 introduced in Subsection 1.2.

**Proposition 1**

Correlation functions of BRST-closed observables are invariant under variation of \( s \).

Assuming its proposition, we evaluate degree 0 correlation function \( \langle O_W \rangle_0 \) both in the \( s \to 0 \) limit and \( s \to \infty \) limit.

As for the \( s \to 0 \) limit, the observable becomes \( O_{\varphi(E)} \) and we can use standard weak coupling limit. Moreover, the potential terms vanish in this limit. Then we expand each field around the solution of the classical equations of motion \((\phi = \phi_0 \text{ (constant map)}, \chi = \chi_0 \text{ (constant solution)}, \psi = 0)\) and perform Gaussian integration of oscillation modes. We show that contributions from Gaussian integral is trivial. Hence \( \langle O_W \rangle_0 \) turns out to be classical integration of \((m, m)\)-differential form \( \varphi(E) \) on \( M \), i.e., the l.h.s of (2.13).

In the \( s \to \infty \) limit, path integral is localized around neighborhood of zero set \( \{M_\alpha\} \) of the holomorphic vector field \( K \). We also expand each field around the solution of the classical equations of motion \((\phi = \phi_0 \in M_\alpha \text{ (constant map)}, \chi = \chi_0 \text{ (constant solution)}, \psi = 0)\). In this case, we carefully discuss integration measure of oscillation modes by using eigenvalue decomposition by Laplacian for differential forms on \( \mathbb{C}P^1 \). Since contributions from oscillation modes do
not affect correlation functions, contribution from a connected component \( M_\alpha \) turns out to be integration of differential form on \( M_\alpha \) given in the r.h.s of (2.13). Summing up all the connected components, \( \langle \mathcal{O}_W \rangle_0 \) becomes the r.h.s of (2.13).

We can equate these two results by using Proposition 1 and obtain the Bott residue formula.

4.2 A Remark on Action of \( K \) on Holomorphic Tangent Bundle \( T'M \)

In Subsection 2.1, we introduced the operator \( \Lambda : \Gamma(E) \rightarrow \Gamma(E) \) that describes action of \( K \) on section of the holomorphic vector bundle \( E \). Its property is given in (2.5).

In the case when \( E \) is the holomorphic tangent bundle \( T'M \), \( \Lambda \) is explicitly given by holomorphic Lie derivative of the holomorphic tangent vector field \( Y \) by \( K \):

\[
\theta(K) : Y \rightarrow [K, Y]. \tag{4.49}
\]

We can check that \( \theta(Y) \) satisfies the condition (2.5).

\[
\theta(K)(fY) = [K, fY] = K^i \frac{\partial}{\partial z^i} (fY^j) \frac{\partial}{\partial z^j} - fY^j \frac{\partial}{\partial z^j} (K^i \frac{\partial}{\partial z^i}) = K^i \frac{\partial f}{\partial z^i} (Y^j) \frac{\partial}{\partial z^j} + f[K, Y] = (Kf)Y + f[K, Y].
\]

If we rescale \( K \) into \( sK \), \( \theta(K) \) is also rescaled to \( s\theta(K) \). In general, \( \Lambda \) is also rescaled to \( s\Lambda \).

4.3 The BRST-Closed Observable Used for Derivation

In this subsection, we construct the observable \( \mathcal{O}_W \) that satisfies \( \lim_{s \rightarrow 0} W = \varphi(E) \) and \( (\bar{\partial} + \beta(K))W = 0 \). We assume that the holomorphic vector bundle \( E \) has canonical connection compatible with Hermitian metric. Let \( \bar{\nabla} \) be canonical connection on \( E \) and \( \Omega^{p,q}(E) \) be complex vector space of \( E \)-valued \((p,q)\)-forms. We also introduce the exterior holomorphic covariant derivative \( D' : \Omega^{p,q}(E) \rightarrow \Omega^{p+1,q}(E) \). Then the canonical connection is decomposed into \( \bar{\nabla} = D' + \bar{\partial} \). Let \( \{e_a\} \) be local holomorphic frame of \( E \). Then the following relation holds.

\[
\bar{\partial}e_a = 0, \quad \bar{\nabla}e_a = D'e_a = \Theta_{ak}^b dz^k e_a,
\]

where \( \Theta_{ak}^b dz^k \) is the connection (1,0)-form of \( E \). Curvature (1,1)-form \( F^b_{ak} \) is given by \( F^b_{ak}dz^k \wedge dz^l \). Then we define \( L = (L_a^b : \Gamma(E) \rightarrow \Gamma(E) \) by,

\[
L_a^b e_b := s\Lambda_b^a e_b - s\Theta_{ak}^b K^k e_b,
\]
where we rescale $K$ into $sK$. Let us note that the following relations hold.

\[
\begin{align*}
\bar{\partial}(sK)\bar{\nabla}e_b &= s\partial_i(\Theta_{ab}^b)K^b dz^i e_b = -i(sK)(F^b_{aik}dz^k \wedge d\bar{z}^i e_b) \\
\bar{\partial}(L_a^b) &= \bar{\partial}(s\Lambda_a^b) - s\partial_i(\Theta_{ab}^b)K^b dz^i e_b = -s\partial_i(\Theta_{ab}^b)K^b dz^i e_b = i(sK)(F^b_{aik}dz^k \wedge d\bar{z}^i e_b) \\
\bar{\partial}(F^a_{bkl}) &= 0 \\
i(sK)(L_a^b e^b) &= 0
\end{align*}
\]

By using the above relations, we obtain $(\bar{\partial} + \beta i(sK))(L_a^b e_b + \frac{i}{2\pi} F_a^b e_b) = \bar{\partial}(L_a^b e_b) - i(sK)(F^b_{aik}dz^k \wedge d\bar{z}^i e_b) = 0.$

If we define matrix valued form:

\[
A = (A_i^a), \quad A = L + \frac{i}{2\pi} F,
\]

it says $(\bar{\partial} + \beta i(sK))A = 0$. Therefore $(\bar{\partial} + \beta i(sK))tr(A^m) = 0$ holds for arbitrary positive integer $m$. Let $U$ be a linear automorphism of a complex vector space $V$ with $\text{dim}_\mathbb{C} = \text{rank}(E) = q$. It is well-known that $\varphi(U)$ defined in Subsection 2.2 can be represented in the following way.

\[
\varphi(U) = \sum_{m_i \geq 0, \sum m_i = m} \alpha_{m_1, m_2, \ldots} \text{tr}(U^{m_1})\text{tr}(U^{m_2}) \cdots \text{tr}(U^{m_i}),
\]

where $m$ is the complex dimension of $M$. Therefore, $\varphi(A) = \varphi(L + \frac{i}{2\pi} F)$ is annihilated by the operator $\bar{\partial} + \beta i(sK)$. Obviously, $\varphi(L + \frac{i}{2\pi} F)$ reduces to $\varphi(\frac{i}{2\pi} F) = \varphi(E)$ under the $s \to 0$ limit. In this way, we have constructed the operator $\mathcal{O}_\varphi(L + \frac{i}{2\pi} F)$ that is used in derivation of the Bott residue formula. From now on, we simply denote it by $\varphi$ for brevity.

### 4.4 Degree 0 Correlation Function and Integral Measure in the $s \to 0$ Limit

From now on, we consider the degree 0 correlation function $\langle \varphi \rangle_0$. First, we rewrite the Lagrangian in the following form.

\[
L + V = \frac{t}{2} \int_{\Sigma} \phi^*(\omega) + L' + V',
\]

\[
L' + V' := \int_{\mathbb{C}P^1} dz d\bar{z} \left[ t g_{ij} \partial_z \phi^i \partial_{\bar{z}} \phi^j + \sqrt{t} i g_{ij} \psi_2^i D_z \chi^j + \sqrt{t} i g_{ij} \psi_1^i D_{\bar{z}} \bar{\chi}^j \right.
\]

\[
- R_{ijkl} \psi_1^i \psi_2^j \chi^{k \ell} + t s^2 \beta g_{ij} K^{ij} \bar{K}^j \chi^i + t s g_{ij} \nabla_{\mu} K^{ij} \bar{\chi}^i + s \beta g_{ij} \nabla_{\mu} K^{ij} \psi_2^i \psi_2^j
\]

where $\omega = g_{ij} dz^i \wedge d\bar{z}^j$ is the Kähler form of $M$. $\int_{\Sigma} \phi^*(\omega)$ is a topological term that gives mapping degree of $\phi : \mathbb{C}P^1 \to M$. Since we focus on the degree 0 correlation function, we use the Lagrangian $L' + V'$ instead of $L + V$. By using Proposition 1, we obtain the following equality.

\[
\lim_{s \to 0} \langle \varphi \rangle_0 = \lim_{s \to 0} < \varphi >_0.
\]
In this subsection, we focus on the left hand side. In the \( s \to 0 \) limit, the Lagrangian \( L + V \) becomes Lagrangian of the usual topological sigma model (A-model). Moreover, \( \varphi \) turns into \( \mathcal{O}_E(\varphi) \), which is also a standard BRST-closed observable of the A-model. Then we can apply standard result of the weak coupling limit \( t \to \infty \) \cite{7}. It says that path integral reduces to Gaussian integration around the constant map \( \phi(z, \bar{z}) = \phi_0 \in M \). Then the correlation function becomes,

\[
\lim_{s \to 0} < \varphi >_0 = \int_M d\phi_0 d\chi_0 \mathcal{O}_E(\varphi) = \int_M \varphi(E) =: [\varphi(E)]_M,
\]

where \( d\phi_0 d\chi_0 \) is the measure for integration of position of \( \phi_0 \in M \) and the corresponding zero-mode of \( \chi \), that can be interpreted as integration of \((m,m)\)-form on \( M \).

### 4.5 Integration Measure for the Degree 0 Correlation Function in the \( s \to \infty \) Limit

We discuss integration measure in evaluating the correlation function in the \( s \to \infty \) limit with fixed \( t \). Since we are considering degree 0 correlation function, the map \( \phi \) is homotopic to a constant map \( \phi(z, \bar{z}) = \phi_0 \in M \). Therefore, we expand the fields \( \phi, \chi \) and \( \psi \) around the constant map \( \phi_0 \). Then \( \chi \) (resp. \( \psi \)) becomes section of \( \phi_0^{-1}(T'M) \) (resp. \( \phi_0^{-1}(T'M) \otimes \overline{\mathcal{O}} \mathbb{C}(\mathbb{P}^1) \)) and its complex conjugate. But \( \phi_0^{-1}(T'M) \) is isomorphic to trivial bundle \( \mathbb{C} \mathbb{P}^1 \times \mathbb{C}^m \). Hence we can simply regard \( \chi^i, \chi^i \) (resp. \( \psi^i, \psi^i \)) as \((0,0)\)-form (resp. \((0,1)\)-form, \((1,0)\)-form) on \( \mathbb{C} \mathbb{P}^1 \). By using standard Kähler metric of \( \mathbb{C} \mathbb{P}^1 \), we can apply eigenvalue decomposition by Laplacian for differential forms on \( \mathbb{C} \mathbb{P}^1 \) to the expansion. The Laplacian is represented as follows († means adjoint defined by Hodge operator of \( \mathbb{C} \mathbb{P}^1 \)).

\[
\Delta := dd^\dagger + d^\dagger d \quad \Delta_\beta := \partial \bar{\partial} + \bar{\partial} \partial \quad \Delta_{\beta} := \bar{\partial} \bar{\partial} + \bar{\partial} \partial
\]

\[
\Delta = 2\Delta_\beta = 2\Delta_{\beta}
\]

Let \( \Delta^{(p,q)} \) be restriction to \( \Omega^{(p,q)}(\mathbb{C} \mathbb{P}^1) \). Vector space of \((p,q)\)-forms with zero eigenvalue is known as \( H^{p,q}(\mathbb{C} \mathbb{P}^1) \): the vector space of \((p,q)\) harmonic forms. The following result is well-known.

\[
\dim_{\mathbb{C}}(H^{0,0}(\mathbb{C} \mathbb{P}^1)) = 1, \quad \dim_{\mathbb{C}}(H^{1,0}(\mathbb{C} \mathbb{P}^1)) = \dim_{\mathbb{C}}(H^{0,1}(\mathbb{C} \mathbb{P}^1)) = 0. \quad (4.57)
\]

Let \( \{ E_n \mid n > 0 \} \) be set of positive eigenvalues of \( \frac{1}{2} \Delta^{(0,0)} \) ordered as follows.

\[
0 < E_1 \leq E_2 \leq E_3 \leq \cdots . \quad (4.58)
\]

Then we denote by \( f_n(z, \bar{z}) \) the \((0,0)\)-form that satisfy,

\[
\frac{1}{2} \Delta^{(0,0)} f_n(z, \bar{z}) = E_n f_n(z, \bar{z}). \quad (4.59)
\]

**Lemma 1** Sets of positive eigenvalues of \( \frac{1}{2} \Delta^{(1,0)} \) and \( \frac{1}{2} \Delta^{(0,1)} \) are both given by \( \{ E_n \mid n > 0 \} \) and \((1,0)\) and \((0,1)\) forms with eigenvalue \( E_n \) are given by \( \partial f_n(z, \bar{z}) \) and \( \bar{\partial} f_n(z, \bar{z}) \) respectively.
Proof) Since $\frac{1}{2}\Delta$ equals $(\partial\bar{\partial} + \bar{\partial}\partial)$, we obtain,

$$
\frac{1}{2} \Delta \partial f_n(z, \bar{z}) = (\partial\bar{\partial} + \bar{\partial}\partial)\partial f_n(z, \bar{z})
$$

$$
= \partial\bar{\partial} f_n(z, \bar{z})
$$

$$
= \partial(\partial\bar{\partial} + \bar{\partial}\partial)f_n(z, \bar{z})
$$

$$
= \frac{1}{2} \partial \Delta f_n(z, \bar{z})
$$

$$
= E_n \partial f_n(z, \bar{z}). \tag{4.60}
$$

Hence $\partial f_n(z, \bar{z})$ is $(1, 0)$-form with eigenvalue $E_n$. On the contrary, let $\omega$ be $(1, 0)$ form with eigenvalue $E$. Then $(0, 0)$ form $\partial^i \omega$ satisfy,

$$
\frac{1}{2} \partial \omega = (\partial\bar{\partial} + \bar{\partial}\partial)\partial^i \omega
$$

$$
= \partial\bar{\partial} \partial^i \omega
$$

$$
= \partial^i(\partial\bar{\partial} + \bar{\partial}\partial)\omega
$$

$$
= \frac{1}{2} \partial^i \Delta \omega
$$

$$
= E \partial^i \omega. \tag{4.61}
$$

Therefore, $\partial^i \omega$ must coincide some $f_n(z, \bar{z})$. This completes proof for $\Delta^{(1, 0)}$. Proof for $\frac{1}{2} \Delta^{(0, 1)}$ goes in the same way by using the equality $\frac{1}{2} \Delta = (\partial\bar{\partial} + \bar{\partial}\partial)$.

Variation $\delta \phi$ from the constant map $\phi_0$ can also be regard as $(0, 0)$ form on $\mathbb{CP}^1$. Combining

$$
\phi^i = \phi_0^i + \sum_{n>0} \phi_n^i f_n(z, \bar{z}), \quad \phi^j = \phi_0^j + \sum_{n>0} \phi_n^j f_n(z, \bar{z}). \tag{4.62}
$$

$$
\chi^i = \chi_0^i + \sum_{n>0} \chi_n^i f_n(z, \bar{z}), \quad \chi^j = \chi_0^j + \sum_{n>0} \chi_n^j f_n(z, \bar{z}). \tag{4.63}
$$

$$
\psi^j = \sum_{n>0} \psi_n^j \frac{1}{\sqrt{E_n}} \partial z f_n(z, \bar{z}), \quad \psi^i = \sum_{n>0} \psi_n^i \frac{1}{\sqrt{E_n}} \partial \bar{z} f_n(z, \bar{z}). \tag{4.64}
$$

We set the volume of $\mathbb{CP}^1$ to 1. Since $\Delta$ is Hermitian, $\{f_0(z, \bar{z}) = 1, f_1(z, \bar{z}), f_2(z, \bar{z}), \cdots\}$ can be considered as orthonormal basis of $\Omega^{(0, 0)}$.

$$
(f_n, f_m) := \int_{\mathbb{CP}^1} f_n(z, \bar{z}) f_m(z, \bar{z}) dz \wedge d\bar{z} = \delta_{n,m} \quad (n, m \geq 0). \tag{4.65}
$$

Since $(f_n, (\partial^i \bar{\partial} + \bar{\partial}\partial^i) f_m) = E_m \delta_{n,m} = (f_n, \partial^i \partial \bar{\partial} f_m) = (\partial f_n, \bar{\partial} f_m)$, we obtain,

$$
\int_{\mathbb{CP}^1} \partial \bar{z} f_n(z, \bar{z}) \partial z f_m(z, \bar{z}) dz \wedge d\bar{z} = E_m \delta_{n,m} \quad (n, m > 0). \tag{4.66}
$$
This forces us to adopt \( \{ \frac{1}{\sqrt{E_n}} \partial_z f_n(z, \bar{z}) \mid n > 0 \} \) and \( \frac{1}{\sqrt{E_n}} \partial_{\bar{z}} f_n(z, \bar{z}) \mid n > 0 \) as expansion basis of \( \psi \). Therefore, integration measure for path-integral is given by,

\[
\mathcal{D}\phi \mathcal{D}\chi \mathcal{D}\psi = \left( \prod_{i=1}^{m} d\phi_i d\phi_i \right) \left( \prod_{i=1}^{m} d\chi_i d\chi_i \right) \left( \prod_{n=1}^{\infty} \frac{d\psi_n d\bar{\psi}_n}{2\pi i} \right).
\]

4.6 The case when \( E = T'M \)

In this subsection, we discuss the case when the vector bundle \( E \) equals \( T'M \) and the zero set of \( K \) is given by a finite set of discrete points \( \{ p_1, \cdots, p_N \} \). In this case, the action \( \Lambda \) on \( T'M \) is given by \( \theta(sK) : Y \rightarrow [sK, Y] \). On the zero set of \( K \), it is explicitly given as follows.

\[
\theta(sK)(\frac{\partial}{\partial z^j}) = [sK^i \frac{\partial}{\partial z^j}] = sK^i \frac{\partial}{\partial z^j} - \frac{\partial}{\partial z^j} (sK^i \frac{\partial}{\partial z^j})
\]

\[
= -s\partial_j K^i \frac{\partial}{\partial z^j}.
\]

Hence we set \( A_j^i = -s\partial_j K^i \) in this subsection.

4.6.1 Explicit Construction of the BRST-closed Observable

First, we construct the observable \( \varphi \). For this purpose, we have only to determine the explicit form of \( A_j^i \). Since the local holomorphic frame \( \{ e_a \} \) is given by \( \{ \frac{\partial}{\partial z^a} \} \), we don’t distinguish subscripts of local frame from ones of local coordinate. Then canonical connection becomes \( \nabla \frac{\partial}{\partial z^j} = \Gamma^i_{jk} d\bar{z}^k \frac{\partial}{\partial z^j} \). Then we obtain

\[
i(sK) \nabla \frac{\partial}{\partial z^j} = sK^i \Gamma^j_{ik} \frac{\partial}{\partial z^j} = sK^i \Gamma^j_{ik} \frac{\partial}{\partial z^j}
\]

\[
F^j_j = \partial (\Gamma^i_{jk} d\bar{z}^k \frac{\partial}{\partial z^j}) = \partial \Gamma^i_{jk} d\bar{z}^k \wedge d\bar{z}^j = R^i_{jk} d\bar{z}^k \wedge d\bar{z}^j = R^i_{jk} d\bar{z}^k \wedge d\bar{z}^j
\]

and

\[
A_j^i \frac{\partial}{\partial z^j} := L_j^i \frac{\partial}{\partial z^j} + \frac{i}{2\pi} F^j_j = \Lambda^i_j \frac{\partial}{\partial z^j} - i(sK) \nabla \frac{\partial}{\partial z^j} - F^j_j \frac{\partial}{\partial z^j}
\]

\[
= (-s\partial_j K^i - s\Gamma^i_{jk} K^j) + \frac{i}{2\pi} R^i_{jk} d\bar{z}^k \wedge d\bar{z}^j \frac{\partial}{\partial z^j}.
\]

This is the formula we use in this subsection.

4.6.2 Expansion of the Lagrangian up to the Second Order

In the \( s \rightarrow \infty \) limit, path-integral is localized on neighborhood of \( p_a \). Therefore, we use expansion of the fields in (4.62), (4.63) and (4.64) with \( \phi_0 = p_a \). Then we expand the Lagrangian up to the second order of the expansion variables. In the neighborhood of \( p_a \), \( K \) is expanded in the form: \(-K_{\alpha} z^\alpha + \cdots \). We can
also assume that $a_{ij} = \delta_{ij}$ and $\Gamma^k_{ij} = \Gamma^k_{ji} = 0$. Therefore, in expanding the Lagrangian, we can use the following simplification.

$$D_z \rightarrow \partial_z \quad D_z \rightarrow \partial_z \quad \nabla_j \rightarrow \partial_j \quad \nabla_j \rightarrow \partial_j \quad (4.69)$$

We also have $\nabla_\mu \vec{R}^\mu = -\vec{R}^\mu_{\alpha \beta} + \cdots$. Then expansion of the Lagrangian up to the second order is given as follows.

$$(L + V)_{2nd} := L_0^\alpha + L'^\alpha,$$

$$L_0^\alpha := t[\beta s^2 \delta_{ij} K_{\alpha \mu j} \vec{K}^\mu_{\alpha \rho} \phi_0^\rho - s \delta_{ij} \vec{K}^\mu_{\alpha \mu j} \lambda_0^\lambda],$$

$$L'^\alpha := \sum_{n>0}[t \delta_{ij} \phi_n^\mu \phi_n^\nu E_n + \sqrt{t} \delta_{ij} (\psi_n^\mu \lambda_n + \psi_n^\nu \chi_n)] \sqrt{E_n}$$

$$+ \sum_{n>0} \{t \beta s^2 \delta_{ij} K_{\alpha \mu j} \vec{K}^\mu_{\alpha \rho} \phi_n^\rho - t s \delta_{ij} \vec{K}^\mu_{\alpha \mu j} \chi_n - s \beta \delta_{ij} K_{\alpha \mu j} \psi_n^\mu \nu_n^\nu\},$$

where $L_0$ is zero mode part and $L'$ is oscillation mode part.

### 4.6.3 Evaluation of $\lim_{s \rightarrow \infty} \varphi > 0$

We represent the correlation function in the following form:

$$\lim_{s \rightarrow \infty} \varphi > 0 = \sum_{\alpha} \lim_{s \rightarrow \infty} \int \mathcal{D} \phi \mathcal{D} \chi \mathcal{D} \psi \mid_{p_0} e^{-L_0^\alpha - L'^\alpha}, \quad (4.70)$$

On $p_\alpha$, $A^j = -s \partial_j K^i - s \Gamma^k_{ij} K^\mu + \frac{i}{\sqrt{s}} R^i_{jkl} dz^k \wedge dz^l$ becomes $s K_{\alpha j}^i - \frac{i}{\sqrt{s}} R_{jkl}^i dz^k \wedge dz^l$. Let $K_{\alpha j}$ be $m \times m$ matrix defined by $K_{\alpha j}^i$. Then we have $\varphi \mid_{p_\alpha} = s^m \varphi(K_{\alpha j} - \frac{i}{\sqrt{s}} R_{jkl}^i) = : s^m \varphi(p_\alpha, s)$. With this set-up, we evaluate the contribution from $p_\alpha$. First, we integrate oscillation modes. $\varphi \mid_{p_\alpha}$ does not contain $\psi$ oscillation modes and we neglect the third and higher order terms that contain oscillation modes. The part of oscillation modes integration is given as follows.

$$\lim_{s \rightarrow \infty} \int \mathcal{D} \phi \mathcal{D} \chi \mathcal{D} \psi \mid_{p_\alpha} e^{-L'^\alpha} = \lim_{s \rightarrow \infty} \varphi \mid_{p_\alpha} \int \prod_{i=1}^m \prod_{n=1}^\infty \frac{d\phi_n^\mu d\phi_n^\nu}{2\pi i} d\chi_n^\mu d\chi_n^\nu d\psi_n^\mu d\psi_n^\nu e^{-L'^\alpha}. \quad (4.70)$$

At this stage, we transform integration variables in the following way ($n > 0$).

$$\phi_n^\mu = \frac{1}{\sqrt{s}} \phi_n^\mu \quad \phi_n^\nu = \frac{1}{\sqrt{s}} \phi_n^\nu$$

$$\chi_n^\mu = \frac{1}{\sqrt{s}} \chi_n^\mu \quad \chi_n^\nu = \frac{1}{\sqrt{s}} \chi_n^\nu$$

$$\psi_n^\mu = \frac{1}{\sqrt{s}} \psi_n^\mu \quad \psi_n^\nu = \frac{1}{\sqrt{s}} \psi_n^\nu \quad \lambda_n^\mu = \frac{1}{\sqrt{s}} \lambda_n^\mu \quad \lambda_n^\nu = \frac{1}{\sqrt{s}} \lambda_n^\nu \quad (4.71)$$

$$\lambda_n^\mu = \frac{1}{\sqrt{s}} \lambda_n^\mu \quad \lambda_n^\nu = \frac{1}{\sqrt{s}} \lambda_n^\nu \quad (4.72)$$

Then integral measures of oscillation modes are invariant under the transformation and $L'^\alpha$ is transformed in the following form.

$$\prod_{i=1}^m \prod_{n=1}^\infty \frac{d\phi_n^\mu d\phi_n^\nu}{2\pi i} d\chi_n^\mu d\chi_n^\nu d\psi_n^\mu d\psi_n^\nu = \prod_{i=1}^m \prod_{n=1}^\infty \frac{d\phi_n^\mu d\phi_n^\nu}{2\pi i} d\chi_n^\mu d\chi_n^\nu d\psi_n^\mu d\psi_n^\nu, \quad (4.73)$$

$$L'^\alpha := \sum_{n>0} \left\{ t \beta \delta_{ij} K_{\alpha \mu j} \vec{K}^\mu_{\alpha \rho} \phi_n^\rho - t \delta_{ij} \vec{K}^\mu_{\alpha \mu j} \chi_n - \beta \delta_{ij} K_{\alpha \mu j} \psi_n^\mu \nu_n^\nu \right\}. \quad (4.74)$$

$$+ \sum_{n>0} \left\{ t \beta \delta_{ij} K_{\alpha \mu j} \vec{K}^\mu_{\alpha \rho} \phi_n^\rho - t \delta_{ij} \vec{K}^\mu_{\alpha \mu j} \chi_n + \beta \delta_{ij} K_{\alpha \mu j} \psi_n^\mu \nu_n^\nu \right\}. \quad (4.75)$$
We neglect $O(s^{-1})$ part since we take the $s \to \infty$ limit. As a result, integration of oscillation modes is given by,

$$
\lim_{s \to \infty} \varphi|_{p_m} \int m \prod_{n=1}^\infty \frac{d\phi_n^i d\phi_n^j}{2\pi i} d\chi_n^i d\psi_n^i d\psi_n^j \nonumber 
$$

$$
\times \exp \left\{ \sum_{n>0} \left\{ -t\beta \delta_{ij} K_{\alpha\mu}^i K_{\alpha\nu}^j \phi_n^\alpha \phi_n^\nu + t\delta_{ij} K_{\alpha\mu}^i \chi_n^\alpha + \beta \delta_{ij} K_{\alpha\mu}^i \psi_n^\alpha \psi_n^\nu \right\} \right\},
$$

(4.76)

$$
= \lim_{s \to \infty} \varphi|_{p_m} \prod_{n=1}^\infty \frac{1}{(2\pi i)^m} \int m \prod_{i=1}^\infty \frac{d\phi_n^i d\phi_n^j}{2\pi i} d\chi_n^i d\psi_n^i d\psi_n^j 
$$

$$
\times \exp \left\{ -t\beta \delta_{ij} K_{\alpha\mu}^i K_{\alpha\nu}^j \phi_n^\alpha \phi_n^\nu + t\delta_{ij} K_{\alpha\mu}^i \chi_n^\alpha + \beta \delta_{ij} K_{\alpha\mu}^i \psi_n^\alpha \psi_n^\nu \right\}.
$$

(4.77)

By using the following integral formulas,

$$
\int d\phi \exp \left\{ -u M_{ij} \phi_i^0 \phi_j^0 \right\} = \left( \frac{-2\pi i}{u} \right)^m (\det M)^{-1},
$$

(4.78)

$$
\int d\chi_0 \exp \left\{ M_{ij} \chi_0^i \chi_0^j \right\} = \det M,
$$

(4.79)

we proceed as follows.

$$
\lim_{s \to \infty} \varphi|_{p_m} \prod_{n=1}^\infty \frac{1}{(2\pi i)^m} \int m \prod_{i=1}^\infty \frac{d\phi_n^i d\phi_n^j}{2\pi i} d\chi_n^i d\psi_n^i d\psi_n^j 
$$

$$
\times \exp \left\{ -t\beta \delta_{ij} K_{\alpha\mu}^i K_{\alpha\nu}^j \phi_n^\alpha \phi_n^\nu + t\delta_{ij} K_{\alpha\mu}^i \chi_n^\alpha + \beta \delta_{ij} K_{\alpha\mu}^i \psi_n^\alpha \psi_n^\nu \right\}.
$$

(4.80)

$$
= \lim_{s \to \infty} \varphi|_{p_m} \prod_{n=1}^\infty \left( \frac{-1}{t\beta} \right)^m (-t\beta)^m \frac{\det (\delta_{ij} K_{\alpha\mu}^j)}{\det (\delta_{ij} K_{\alpha\mu}^j)}
$$

(4.81)

$$
= \lim_{s \to \infty} \varphi|_{p_m}.
$$

(4.82)

Contribution from oscillation modes turn out to be 1. Next, we calculate integral of zero mode part.

$$
\lim_{s \to \infty} \int d\phi_0 d\chi_0 s^m \varphi(p_\alpha, s) e^{-L_0^2} 
$$

(4.83)

$$
= \lim_{s \to \infty} s^m \varphi(p_\alpha, s) \frac{1}{\pi^m} \int d\phi_0 d\chi_0 \exp \left\{ -t (s^2 \beta \delta_{ij} K_{\alpha\mu}^i K_{\alpha\nu}^j \phi_0^\alpha \phi_0^\nu - s \delta_{ij} K_{\alpha\mu}^j \chi_0^\alpha) \right\}
$$

$$
= \lim_{s \to \infty} s^m \varphi(p_\alpha, s) \int d\phi_0 \exp \left\{ -t s^2 \beta \delta_{ij} K_{\alpha\mu}^i K_{\alpha\nu}^j \phi_0^\alpha \phi_0^\nu \right\}
$$

(4.84)

$$
\times \int d\chi_0 \exp \left\{ t s \delta_{ij} K_{\alpha\mu}^j \chi_0^{\alpha} \chi_0^{\nu} \right\}
$$

$$
= \lim_{s \to \infty} s^m \varphi(p_\alpha, s) \frac{2 \pi i m}{(ts^2 \beta)^m} \frac{\det (\delta_{ij} K_{\alpha\mu}^j)}{\det (\delta_{ij} K_{\alpha\mu}^j)}
$$

(4.85)
where we used $\beta = 2\pi i$. Since $\lim_{s \to \infty} \varphi(p_\alpha, s) = \varphi(K_\alpha - \frac{1}{s} R|_{p_\alpha}) = \varphi(K_\alpha)$, we obtain,

$$\lim_{s \to \infty} \int D\phi_0 D\chi_0 \varphi(p)e^{-L_0} = \frac{\varphi(K_\alpha)\det(\delta_{ij} K_{\alpha\beta}^2)}{\det(K_{\alpha\beta})} = \frac{\varphi(K_\alpha)}{\det(K_{\alpha\beta})}. \tag{4.85}$$

Since we already know $\theta|_{p_\alpha} = K_{\alpha\beta}$, the above result is rewritten by,

$$\lim_{s \to \infty} <\varphi> = \sum_{\alpha=1}^{N} \frac{\varphi(K_\alpha)}{\det(\theta|_{p_\alpha})}. \tag{4.86}$$

By combining Proposition 1 and the result in the $s \to 0$ limit, we obtain the Bott residue formula in the case of this subsection.

$$\varphi(T' M)[M] = \sum_{\alpha=1}^{N} \frac{\varphi(K_\alpha)}{\det(\theta|_{p_\alpha})}. \tag{4.87}$$

Let us assume that $E$ is a general holomorphic vector bundle on $M$ and that zero set of $K$ is given by a discrete point set $\{p_1, \ldots, p_N\}$. In the same way as the discussion of this subsection, we can derive the Bott residue formula:

$$\varphi(E)[M] = \sum_{\alpha=1}^{N} \frac{\varphi(A|_{p_\alpha})}{\det(\theta|_{p_\alpha})}. \tag{4.88}$$

This result corresponds to the example given in [9].

### 4.7 Derivation in General Case

In this subsection, we derive general case of the Bott residue formula, i.e., zero set of $K$ is given by $\{N_\alpha\}$ where $N_\alpha$ is a connected compact Kähler submanifold of $M$. For simplicity, we focus on one connected component $N := N_\alpha$ in the following discussion. We set $\text{codim}_C(N) = \nu$. In the $s \to \infty$ limit, path integral is localized to neighborhood of $N$, we apply expansion given in subsection 4.5 around the constant map $\phi_0 \in N$. But one subtlety occurs in this case. Since $N$ is a Kähler submanifold of $M$, local coordinates around $\phi_0 \in N$ can be taken in the following form:

$$\left( z_1^{\perp}, \ldots, z_{\nu}^{\perp}, z_1^{\parallel}, \ldots, z_{m}^{\parallel} \right),$$

where points in $N$ is described by the condition $z_1^{\perp} = \cdots = z_{\nu}^{\perp} = 0$. Then fields $\phi$, $\chi$ and $\psi$ are also decomposed into $\phi^{\perp} + \phi^{\parallel}$, $\chi^{\perp} + \chi^{\parallel}$ and $\psi^{\perp} + \psi^{\parallel}$ respectively. From now on, we use alphabets for $\perp$ directions and Greek characters for $\parallel$.
isfies following Taylor expansion around $p$

$$
\phi^i_\perp = \phi^i_\perp^{(0)} + \sum_{n>0} \phi^i_\perp^{(n)} f_n(z, \bar{z}), \quad \phi^i_\parallel = \phi^i_\parallel^{(0)} + \sum_{n>0} \phi^i_\parallel^{(n)} f_n(z, \bar{z}),
$$

$$
\chi^i_\perp = \chi^i_\perp^{(0)} + \sum_{n>0} \chi^i_\perp^{(n)} f_n(z, \bar{z}), \quad \chi^i_\parallel = \chi^i_\parallel^{(0)} + \sum_{n>0} \chi^i_\parallel^{(n)} f_n(z, \bar{z}),
$$

$$
\psi^i_\perp = \sum_{n>0} \frac{1}{\sqrt{E_n}} \partial_z \psi^i_\perp f_n(z, \bar{z}), \quad \psi^i_\parallel = \sum_{n>0} \frac{1}{\sqrt{E_n}} \partial_\bar{z} \psi^i_\parallel f_n(z, \bar{z}),
$$

(4.89)

In other words, we decompose the integration measure as follows,

$$
\int_N \mathcal{D}\phi^0 \mathcal{D} \chi^i \mathcal{D} \psi^i \int_{N_\perp} \mathcal{D}\phi^0 \mathcal{D} \chi^i \mathcal{D} \psi^i \int_{N_\parallel} \mathcal{D}\phi^0 \mathcal{D} \chi^i \mathcal{D} \psi^i
$$

4.7.1 Expansion of $L$ up to the Second Order

By using the orthonormal relation (4.65) and (4.66), expansion $L$ up to the second order is given by,

$$
L = \sum_{n>0} \left[ tE_n \delta_{ij} \phi^i_\perp^{(1)} \phi^j_\perp^{(1)} + \sqrt{t} i \sqrt{E_n} \delta_{ij} \psi^i_\perp^{(1)} \psi^j_\perp^{(1)} + \sqrt{E_n} \delta_{ij} \psi^i_\perp^{(1)} \psi^j_\perp^{(1)} + \sqrt{E_n} \delta_{ij} \psi^i_\perp^{(1)} \psi^j_\perp^{(1)} \right]
$$

$$
+ \sum_{n>0} \left[ tE_n \delta_{ij} \phi^i_\parallel^{(1)} \phi^j_\parallel^{(1)} + \sqrt{t} i \sqrt{E_n} \delta_{ij} \psi^i_\parallel^{(1)} \psi^j_\parallel^{(1)} + \sqrt{E_n} \delta_{ij} \psi^i_\parallel^{(1)} \psi^j_\parallel^{(1)} + \sqrt{E_n} \delta_{ij} \psi^i_\parallel^{(1)} \psi^j_\parallel^{(1)} \right],
$$

(4.91)

where we used local coordinates that make $g_{ij}$ and $\Gamma^k_{ij}(\Gamma^k_{ij})$ into $\delta_{ij}$ and 0 respectively.

4.7.2 Expansion of the Potential $V$

In this subsection, we expand the potential term $V$ around neighborhood of $p \in N$.

$$
V = \int_{\mathbb{R}^m} dz d\bar{z} \left[ \sum_{I,J} g_{IJ} K^I K^J \gamma^{IJ} + \sum_{I} \nabla \gamma^{I} \nabla \gamma^{I} + \sum_{I,J} \gamma^{IJ} \nabla \gamma^{I} \gamma^{I} \right].
$$

(4.92)

where subscripts $I, J, M, \cdots$ run through all the directions $1, 2, \cdots, m$.

First, we consider expansion of the second term of $V$. Let us consider the following Taylor expansion around $p$. We use the coordinate system that satisfies

$$
g_{IJ}(p) = \delta_{IJ} + \cdots \quad (g_{ij}(p) = \delta_{ij}, g_{\mu\nu}(p) = \delta_{\mu\nu}, g_{\psi}(p) = 0),
$$

$$
\Gamma^I_{I\mu} = R^I_{IJ\mu} \phi^J + \partial_N \Gamma^I_{IJ\mu}(p) \phi^J + \cdots.
$$

(4.93)

(4.94)
Moreover, $p$ is in the zero set of $K$, we can use $K^i = -K^i_{\perp} + \ldots$. We neglect second order or higher term each.

$$s_{I\mu J} \bar{\nabla}_I K^J = s_{I\mu J}(\bar{\nabla}_I K^J + \Gamma^J_{MI} K^I)$$

$$= -\delta_{IJ} K^J_{m} - sR_{I\mu J} K^J_{\phi\|\phi\|^N} - \delta_{IJ} \bar{\nabla}_I K^J_{\phi\|\phi\|^N} \ldots$$

$$= -\delta_{IJ} K^J_{m} - sR_{I\mu J} K^J_{\phi\|\phi\|^N} - sR_{I\mu J\nu} K^J_{\phi\|\phi\|^N} + \ldots.$$  

Since the last term doesn’t contain holomorphic part of $\phi$ and contain $\bar{\phi}_\perp$, it does not contribute to Gaussian integration of $\bar{\phi\|}_0$ that will be done later. Hence we neglect this term. Then we obtain,

$$s_{I\mu J} \bar{\nabla}_I K^J = -\delta_{IJ} K^J_{m} - sR_{I\mu J} K^J_{\phi\|\phi\|^N} - sR_{I\mu J\nu} K^J_{\phi\|\phi\|^N}.$$  

(4.95)

and

$$s_{I\mu J} \bar{\nabla}_I K^J = -\delta_{IJ} K^J_{m} - sR_{I\mu J} K^J_{\phi\|\phi\|^N} - sR_{I\mu J\nu} K^J_{\phi\|\phi\|^N}.$$

(4.96)

Since $\chi = \chi\| + \chi\perp$, we decompose,

$$\chi_{\|} = \chi\| + \chi\perp + \chi\perp + \chi\perp.$$

Then the part that corresponds to (4.96) is rewritten as follows.

$$\int_{\mathbb{C}^p} dzd\bar{z} \left\{ s_{I\mu J} \bar{\nabla}_I K^J_{\chi\|} \chi^J_{\chi\|} \right\}$$

$$= \int_{\mathbb{C}^p} dzd\bar{z} \left\{ -s\delta_{IJ} K^J_{m} \chi_{\|} \chi_{\perp} - sR_{I\mu J} K^J_{\phi\|\phi\|^N} \chi_{\|} \chi^J_{\perp} \right\}$$

Then we use the expansion (4.89) and (4.90).

$$\int_{\mathbb{C}^p} dzd\bar{z} \left\{ -s\delta_{IJ} K^J_{m} \chi_{\|} \chi_{\perp} \right\}$$

$$= -s\delta_{IJ} K^J_{m} \chi_{\|} \chi_{\perp} - \sum_{n \in \mathbb{Z}(n \neq 0)} \chi_{\perp n} \chi^J_{\perp n}$$

$$\int_{\mathbb{C}^p} dzd\bar{z} \left\{ sR_{I\mu J} K^J_{\phi\|\phi\|^N} \chi_{\|} \chi^J_{\perp} \right\}$$

$$= \int_{\mathbb{C}^p} dzd\bar{z} \left\{ -sR_{I\mu J} K^J_{\phi\|\phi\|^N} \chi_{\perp} \chi^J_{\perp} \right\}$$

$$= -sR_{I\mu J} K^J_{\phi\|\phi\|^N} \chi_{\perp} \chi_{\perp} - sR_{I\mu J} K^J_{\phi\|\phi\|^N} \chi_{\perp} \chi_{\perp}$$

$$- sR_{I\mu J} K^J_{\phi\|\phi\|^N} \chi_{\perp} \chi_{\perp} - sR_{I\mu J} K^J_{\phi\|\phi\|^N} \chi_{\perp} \chi_{\perp}$$
From the above result, oscillation mode part is the same as the one in the zero mode part. We summarize the result of expansion of \( L \) previous subsection. So, new things that we have to consider is integration of result, we obtain the following form.

We can neglect the third term by applying the same rule for expansion. As a result, we obtain the following form.

In this expansion, we neglect third and higher terms that contain oscillation modes. Next, we expand the first term by using the same rule.

\[
\int_{\mathcal{C}_p} dzd\bar{z} \left\{ s^2 \beta g_{ij} K^i \bar{K}^j \right\} = \int_{\mathcal{C}_p} dzd\bar{z} \left\{ s^2 \beta \delta_{ij} K^i_m \bar{K}^j_\mu \phi^{\mu}_{\perp,0} \right\}
\]

We can neglect the third term by applying the same rule for expansion. As a result, we obtain the following form.

\[
\int_{\mathcal{C}_p} dzd\bar{z} \left\{ s^2 \beta g_{ij} K^i \bar{K}^j + s g_{ij} \nabla_n K^i_j \chi^M \chi^I \right\} = s^2 \beta \delta_{ij} K^i_m \bar{K}^j_\nu \phi^{\nu}_{\perp,0} + s \sum_{n>0} \beta \delta_{ij} K^i_m \bar{K}^j_\mu \phi^{\mu}_{\perp,n}
\]

From the above result, oscillation mode part is the same as the one in the previous subsection. So, new things that we have to consider is integration of zero mode part. We summarize the result of expansion of \( L + V \) in the following form.

\[
L + V = L_0 + L_1' + L_2',
\] (4.97)
4.7.3 Evaluation of $L_0$ and $L_\perp$

For the measure, the transformation rule is given by Berezinian as follows.

\[
L_0 := \sum_{n>0} \left[ t E_n \delta x_i \phi_{in}^{0} + \sqrt{t E_n \delta x_i \psi_{in}^{0}} \right]
\]

\[
L_\perp := \sum_{n>0} \left[ (t E_n \delta x_i \phi_{in}^{m} + \sqrt{t E_n \delta x_i \psi_{in}^{m}} ) \right]
\]

Let us focus on the measure:

\[
\phi_{n0} = \phi_{0}^{m}, \quad \phi_{01} = \frac{1}{s} \phi_{10}^{0}, \quad \phi_{00} = \frac{1}{s} \phi_{00}^{0}, \quad \phi_{10} = \frac{1}{s} \phi_{00}^{0}
\]

\[
\chi_{n0} = \sqrt{s} \chi_{0}^{m}, \quad \chi_{01} = \sqrt{s} \chi_{0}^{m}, \quad \chi_{00} = \sqrt{s} \chi_{0}^{m}, \quad \chi_{10} = \frac{1}{s} \chi_{00}^{0}.
\]

\[
\phi_{n1} = \phi_{1}^{m}, \quad \phi_{11} = \phi_{1}^{m}, \quad \phi_{11} = \frac{1}{s} \phi_{11}^{0}, \quad \phi_{11} = \frac{1}{s} \phi_{11}^{0},
\]

\[
\chi_{n1} = \chi_{1}^{m}, \quad \chi_{11} = \chi_{1}^{m}, \quad \chi_{11} = \frac{1}{s} \chi_{11}^{0}, \quad \chi_{11} = \frac{1}{s} \chi_{11}^{0}.
\]

\[
\psi_{n1} = \psi_{1}^{m}, \quad \psi_{11} = \psi_{1}^{m}, \quad \psi_{11} = \frac{1}{s} \psi_{11}^{0}, \quad \psi_{11} = \frac{1}{s} \psi_{11}^{0}.
\]

Let us focus on the measure:

\[
d\phi_{n0} d\chi_{00} = d\phi_{n0}^{0} d\phi_{00}^{0} \cdots d\phi_{n0}^{m} d\phi_{00}^{m} d\chi_{00}^{m} d\chi_{00}^{m} \cdots d\chi_{n0}^{m} d\chi_{00}^{m}.
\]

Transformation rule of the measure is given by Berezinian as follows.

\[
d\phi_{00} d\chi_{00} = \frac{1}{s^{m-n}} d\phi_{00}^{0} d\chi_{00}^{0}.
\]

As for the measure:

\[
d\phi_{00} d\chi_{00} = d\phi_{00}^{0} d\phi_{00}^{0} \cdots d\phi_{00}^{0} d\phi_{00}^{0} d\chi_{00}^{0} d\chi_{00}^{0} \cdots d\chi_{00}^{0} d\chi_{00}^{0},
\]

transformation rule is given by,

\[
d\phi_{00} d\chi_{00} = \frac{1}{s^{m-n}} d\phi_{00}^{0} d\chi_{00}^{0}.
\]
Integral measures of oscillation modes are defined as follows.

\[
\mathcal{D}\phi'_{\parallel}\mathcal{D}\chi'_{\parallel}\mathcal{D}\psi'_{\parallel} = \prod_{n>0} \frac{1}{(2\pi)^{m-\nu}} d\phi_{\parallel n}^{\nu+1} d\phi_{\perp n}^{\nu} \cdots d\phi_{\parallel n}^{m} d\phi_{\perp n}^{m} d\chi_{\parallel n}^{\nu+1} d\chi_{\perp n}^{\nu} \cdots d\chi_{\parallel n}^{m} d\chi_{\perp n}^{m}
\]

\[
\times d\psi_{\parallel n}^{\nu+1} d\psi_{\perp n}^{\nu} \cdots d\psi_{\parallel n}^{m} d\psi_{\perp n}^{m}
\]

\[
\mathcal{D}\phi'_{\perp}\mathcal{D}\chi'_{\perp}\mathcal{D}\psi'_{\perp} = \prod_{n>0} \frac{1}{(2\pi)^{\nu}} d\phi_{\perp n}^{\nu} d\phi_{\perp n}^{\nu} \cdots d\phi_{\perp n}^{m} d\phi_{\perp n}^{m} d\chi_{\perp n}^{\nu+1} d\chi_{\perp n}^{\nu} \cdots d\chi_{\perp n}^{m} d\chi_{\perp n}^{m}
\]

\[
\times d\psi_{\perp n}^{\nu+1} d\psi_{\perp n}^{\nu} \cdots d\psi_{\perp n}^{m} d\psi_{\perp n}^{m}
\]

These are invariant under the transformation.

\[
\mathcal{D}\phi'_{\parallel}\mathcal{D}\chi'_{\parallel}\mathcal{D}\psi'_{\parallel} = \mathcal{D}\phi''_{\parallel}\mathcal{D}\chi''_{\parallel}\mathcal{D}\psi''_{\parallel},
\]

\[
\mathcal{D}\phi'_{\perp}\mathcal{D}\chi'_{\perp}\mathcal{D}\psi'_{\perp} = \mathcal{D}\phi''_{\perp}\mathcal{D}\chi''_{\perp}\mathcal{D}\psi''_{\perp}.
\]

In sum, transformation of the whole integral measure is given by,

\[
\int_{N} \mathcal{D}\phi'_{\parallel}\mathcal{D}\chi'_{\parallel}\mathcal{D}\psi'_{\parallel} \int_{N} \mathcal{D}\phi'_{\perp}\mathcal{D}\chi'_{\perp}\mathcal{D}\psi'_{\perp}
\]

\[
= \frac{1}{s^m} \int_{N} \mathcal{D}\phi''_{\parallel}\mathcal{D}\chi''_{\parallel}\mathcal{D}\psi''_{\parallel} \int_{N} \mathcal{D}\phi''_{\perp}\mathcal{D}\chi''_{\perp}\mathcal{D}\psi''_{\perp}.
\]

Let us consider \(\varphi\) at \(p \in N\). By using the above variables, \(A_n^b\) is expanded in the following form:

\[
A_n^b = sA_n^b + \frac{i}{2\pi} F_{\parallel}^b \parallel J \chi^J \chi^J = s(A_n^b + \frac{i}{2\pi} F_{\parallel}^b \psi^\mu_0 \psi_0^\mu) + \sqrt{s} \{\ldots\} + \ldots + s^{-1}\{\ldots\}
\]

Since we neglect the third and higher terms that contain oscillation modes, \(F_{\parallel}\) does not depend on \(\varphi_{\parallel}\). Then we expand \(\varphi(p)\) in the form:

\[
\varphi(p) = \sum_{k=-2m}^{2m} s^k \varphi_k(p).
\]

Note that \(\varphi_{2m}(p)\) is written as \(\varphi(A_n^b + \frac{i}{2\pi} F_{\parallel}^b \psi^\mu_0 \psi_0^\mu)\). We then look back at the expansion of the potential term. Since we neglect terms that have negative powers in \(s\), it is represented as follows.

\[
L_0 = t\{\beta \delta_{ij} \bar{K}_m \bar{K}_i \bar{K}_j \phi_{\parallel 0}^{\mu} \phi_{\perp 0}^{\mu} - \delta \bar{K}_m \bar{K}_i \phi_{\perp 0}^{\mu} \phi_{\perp 0}^{\mu}
\]

\[-sR_{ijkl} \bar{K}_m \phi_{\parallel 0}^{\mu} \phi_{\parallel 0}^{\mu} \phi_{\perp 0}^{\mu} - R_{ijkl} \bar{K}_m \phi_{\perp 0}^{\mu} \phi_{\perp 0}^{\mu} \phi_{\perp 0}^{\mu}
\]

\[-R_{ijkl} \bar{K}_m \phi_{\parallel 0}^{\mu} \phi_{\perp 0}^{\mu} \phi_{\perp 0}^{\mu} - R_{ijkl} \bar{K}_m \phi_{\perp 0}^{\mu} \phi_{\perp 0}^{\mu} \phi_{\perp 0}^{\mu}\}
\]

\[
L'_0 = \sum_{n>0} \left[tE_n \delta_{\tau\tau} \phi_{\parallel 0}^{\mu} \phi_{\parallel 0}^{\mu} + \sqrt{tE_n} \delta_{\tau\tau} \phi_{\parallel 0}^{\mu} \phi_{\perp 0}^{\mu} \phi_{\perp 0}^{\mu} + \sqrt{tE_n} \delta_{\tau\tau} \phi_{\parallel 0}^{\mu} \phi_{\perp 0}^{\mu} \phi_{\perp 0}^{\mu} \right],
\]

\[
L'_0 = \sum_{n>0} \left[t\beta \delta_{ij} \bar{K}_m \bar{K}_i \bar{K}_j \phi_{\parallel 0}^{\mu} \phi_{\perp 0}^{\mu} \phi_{\perp 0}^{\mu} - t \delta \bar{K}_m \bar{K}_i \phi_{\parallel 0}^{\mu} \phi_{\perp 0}^{\mu} \phi_{\perp 0}^{\mu} - \beta \delta_{ij} \bar{K}_m \phi_{\parallel 0}^{\mu} \phi_{\perp 0}^{\mu} \phi_{\perp 0}^{\mu} \right].
\]
Let us evaluate the contribution from the component $N$ to $\lim_{\phi \to 0} < \phi >$. First, we integrate oscillation modes in $\|\cdot\|$-part. Since each $\varphi_k(p)$ doesn’t contain $\psi_m$ and $\phi''_m$, we have only to perform simple Gaussian integral.

$$\int \mathcal{D}\phi''_m \mathcal{D}\chi''_m \mathcal{D}\psi''_m \varphi(p) e^{-L_1'}$$

$$= \varphi'(p) \int \mathcal{D}\phi''_m \mathcal{D}\chi''_m \mathcal{D}\psi''_m \exp \left\{ - \sum_{n > 0} \left[ tE_n \delta_{ij} \phi''_n \phi''_m + \sqrt{tE_n} i\delta_{ij} \psi''_n \chi''_n + \sqrt{tE_n} i\delta_{ij} \psi''_m \chi''_n \right] \right\}$$

$$= \varphi'(p) \prod_{n > 0} \frac{1}{(2\pi)^{m-\nu}} \int d\phi''_n d\chi''_n d\psi''_n$$

$$\times \exp \left\{ - \left[ tE_n \delta_{ij} \phi''_n \phi''_m + \sqrt{tE_n} i\delta_{ij} \psi''_n \chi''_n + \sqrt{tE_n} i\delta_{ij} \psi''_m \chi''_n \right] \right\}$$

$$= \varphi'(p) \prod_{n > 0} \left( \frac{tE_n}{tE_n} \right)^{m-\nu} = \varphi'(p).$$

We mean by $\varphi'(p)$ the operator obtained from removing $\chi''_m$ from $\varphi(p)$. Next, we integrate oscillation modes in $\perp$-part. We expand $\varphi'(p) = \sum_{k=-2m}^{2m} s^p \varphi_k(p)$, $(\varphi_m(p) = \varphi_{2m}(p) = \varphi(N^b_a + \frac{i}{2\pi} \frac{F^{b}_\mu}{\Gamma^{b}_\nu} \chi_0 ^{m\mu\nu} \phi_0 ^{m\mu\nu}))$. Then we obtain,

$$\int \mathcal{D}\phi''_m \mathcal{D}\chi''_m \mathcal{D}\psi''_m \varphi'(p) \exp(-L_1')$$

$$= \int \mathcal{D}\phi''_m \mathcal{D}\chi''_m \mathcal{D}\psi''_m \varphi'(p) \exp \left\{ - \sum_{n > 0} \left[ t\beta_{ij} K_m \tilde{K}^j m \phi''_n \phi''_m - t\delta_{ij} \tilde{K}_m \chi''_n \chi''_n \right] \right\}$$

$$- \beta_{ij} K^i m \phi''_n \phi''_m \left\{ - \beta_{ij} K^i m \phi''_n \phi''_m \right\}$$

$$= \sum_{n > 0} \left[ \beta_{ij} K^i m \tilde{K}^j m \phi''_n \phi''_m \right] + \left( \text{terms of lower power in } s \right)$$

$$= \sum_{n > 0} \left[ \beta_{ij} K^i m \tilde{K}^j m \phi''_n \phi''_m \right] + \left( \text{terms of lower power in } s \right)$$

At this stage, we can represent the contribution from $N$ in the following form.

$$\int_{N\alpha} d\phi''_0 d\chi''_0 \varphi_{2m}(p) \int_{N\perp} d\phi''_1 d\chi''_1 \exp \{ -L_0 \} + \left( \text{terms of lower power in } s \right)$$

Finally, we integrate out zero modes in $\perp$-part.

$$\int_{N\perp} d\phi''_0 d\chi''_0 \exp \{ -L_0 \}$$

$$= \int_{N\perp} \left[ \beta_{ij} K^i m \tilde{K}^j m \phi''_n \phi''_m \right] + \left( \text{terms of lower power in } s \right)$$

$$= \int_{N\perp} \left[ \beta_{ij} K^i m \tilde{K}^j m \phi''_n \phi''_m \right] + \left( \text{terms of lower power in } s \right)$$

$$= \int_{N\perp} \left[ \beta_{ij} K^i m \tilde{K}^j m \phi''_n \phi''_m \right] + \left( \text{terms of lower power in } s \right)$$

$$= \int_{N\perp} \left[ \beta_{ij} K^i m \tilde{K}^j m \phi''_n \phi''_m \right] + \left( \text{terms of lower power in } s \right)$$

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\[ \times \int d\chi_0'' \exp \left\{ \delta_{ij} K_i'' K_j'' \right\} \]
\[ = \int_N d\nu'' \exp \left\{ -t \nu'' \delta_{ij} K_i'' K_j'' \right\} \]
\[ \times \left\{ -t \nu'' \varepsilon \det \left( \delta_{ij} K_i'' \right) \right\}
+ \int d\chi_0'' \exp \left\{ (R_{ijm} K_i'' K_j'' \nu'' m_0 \nu'' m_0) \right\} \]

Then we preform Gaussian integral of \( \exp(-t \nu'' \delta_{ij} K_i'' K_j'' \nu'' m_0 \nu'' m_0) \).

Note that terms except for \( (t)^\nu \varepsilon \det(\delta_{ij} K_i'' K_j'') \) include grassmann variables. Hence by expanding exponential, we only have to consider polynomial correlation function of \( \phi_{\perp 0} \) for these terms. But the matrix \( \left\{ \delta_{ij} K_i'' K_j'' \nu'' m_0 \nu'' m_0 \right\} \)

\[ \times \left\{ -t \nu'' \varepsilon \det \left( \delta_{ij} K_i'' \right) \right\} \]
\[ \times \left\{ -t \nu'' \varepsilon \det \left( \delta_{ij} K_i'' \right) \right\}
+ \int d\chi_0'' \exp \left\{ (R_{ijm} K_i'' K_j'' \nu'' m_0 \nu'' m_0) \right\} \]

From \( R_{ijm} = R_{mij} = -\delta_{ij} R_{m''} \),
\[ \int_N d\nu'' \exp \left\{ -t \nu'' \varepsilon \det \left( \delta_{ij} K_i'' \right) \right\} \]

We remark \( \varphi_{2m}^\perp (p) = \varphi(\Lambda_a^t + \frac{1}{2\pi} F_{\nu''} \chi_0'') \varphi(\Lambda_a) \). By adding up contributions from all the connected components, we obtain the correlation function in the following form.

\[ \lim_{s \to \infty} < \varphi >_0 = \lim_{s \to \infty} \sum_{\alpha} \left[ \int_{N_0} \frac{\varphi(\Lambda_a)}{\det( \delta_{ij} K_i'' + \frac{2\pi}{\nu''} R_{m''} \chi_0'') \nu'' m_0 \nu'' m_0) \right] \]
\[ + \text{(terms of negative power in } s) \]
\[ = \sum_{\alpha} \int_{N_0} \frac{\varphi(\Lambda_a)}{\det( \delta_{ij} K_i'' + \frac{2\pi}{\nu''} R_{m''} \chi_0'') \nu'' m_0 \nu'' m_0) \}
\]

Lastly, we rewrite the determinant in denominator. \( K_i'' \) corresponds to the map \( \theta_{\nu''}^\perp : T'M|_N/T'N' \to T'M|_N/T'N \) and \( R_{m''} \chi_0'' \chi_0'') \) is nothing but the curvature \( (1,1) \)-form of \( T'M|_N/T'N' \) \( ( R_i'' = R_{m''}^t ) \). As a result, we can rewrite the above result into the form:

\[ \lim_{s \to \infty} < \varphi >_0 = \sum_{\alpha} \int_{N_0} \frac{\varphi(\Lambda_a)}{\det( \theta_{\nu''}' + \frac{2\pi}{\nu''} R_{m''} )} \]

By combining the above result with Proposition 1, we finally obtain the Bott residue formula:

\[ \varphi(E)|_M = \sum_{\alpha} \int_{N_0} \frac{\varphi(\Lambda_a)}{\det( \theta_{\nu''}' + \frac{2\pi}{\nu''} R_{m''} )} \]

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Appendix A  Proof of Proposition 1

We prove the correlation function is independent of parameter $s$. The basic idea comes from [3] and [5]. Sigma model has two charge for fermion $F_A$ and $F_V$. These charge acts on the operator $O_\omega$ obtained from $(p,q)$-form $\omega$ as follows.

$$F_A O_\omega = (p + q) O_\omega \quad F_V O_\omega = (-p + q) O_\omega$$

The symmetry $F_A$ is broken by the potential term. So, observables are graded by $F_V$. However, since $F_A$ is counting total degree of differential forms, we can use it for taking conjugation of operators (this idea was used in the discussion on Landau-Ginzburg model in [5]). Let us consider $e^{\lambda F_A}$ ($\lambda \in \mathbb{R}$). Then $e^{-\lambda F_A} Q_s e^{\lambda F_A}$ is evaluated as follows.

$$e^{-\lambda F_A} Q_s e^{\lambda F_A} O_\omega = e^{-\lambda} Q_{sc^{2\lambda}} O_\omega.$$ 

Next, we focus on the observable $\varphi_s$. We decompose observable $\varphi_s$ into $\varphi_s = \sum_{k=0}^{m} s^k \varphi_{m-k}$. Since $\varphi_{m-k}$ corresponds to $(m-k, m-k)$-form, we can compute $e^{-\lambda F_A} \varphi_s e^{\lambda F_A} O_\omega$.

$$e^{-\lambda F_A} s^k \varphi_{m-k} e^{\lambda F_A} O_\omega = e^{-2m\lambda} (se^{2\lambda})^k \varphi_{m-k} O_\omega.$$ 

Hence we obtain,

$$e^{-\lambda F_A} \varphi_s e^{\lambda F_A} = e^{-2m\lambda} \varphi_{sc^{2\lambda}}.$$ 

Let us introduce vacuum vector $|0\rangle$ and its dual $<0|$. Then we can represent the correlation function $<\varphi_s>$ as $<0|\varphi_s|0\rangle$. By using the relation (??), we obtain,

$$<\varphi_s> = <0|\varphi_s|0\rangle = <0|e^{\lambda F_A} e^{-\lambda F_A} \varphi_s e^{\lambda F_A} e^{-\lambda F_A} |0\rangle = <0|e^{\lambda F_A} \varphi_{sc^{2\lambda}} e^{-\lambda F_A} |0\rangle e^{-2m\lambda}.$$ 

Since our theory has $2m$ fermion zero modes $\chi_i^0$ and $\chi_i^\dagger$ ($i = 1, \cdots, m$), it is anomalous. Therefore, if we assign $|0\rangle$ charge (0, 0), we have to assign $<0|$ charge $(m, m)$. Therefore, we have $F_A|0\rangle = 0$ and $<0|F_A = 2m < 0$. Hence we obtain,

$$<\varphi_s> = <0|e^{\lambda F_A} \varphi_{sc^{2\lambda}} e^{-\lambda F_A} |0\rangle e^{-2m\lambda} = e^{2m\lambda} <0|\varphi_{sc^{2\lambda}} |0\rangle e^{-2m\lambda} = <0|\varphi_{sc^{2\lambda}} |0\rangle = <\varphi_{sc^{2\lambda}} > .$$ 

This completes proof of the proposition.

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