Demazure formula for $A_n$ Weyl polytope sums

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Abstract

The weights of finite-dimensional representations of simple Lie algebras are naturally associated with Weyl polytopes. Representation characters decompose into multiplicity-free sums over the weights in Weyl polytopes. The Brion formula for these Weyl polytope sums is remarkably similar to the Weyl character formula. Moreover, the same Lie characters are also expressible as Demazure character formulas. This motivates a search for new expressions for Weyl polytope sums, and we prove such a formula involving Demazure operators. It applies to the Weyl polytope sums of the simple Lie algebras $A_n$, for all dominant integrable highest weights and all ranks $n$. 
1. Introduction

The Brion formula \[3, 4\] is a general expression for an exponential sum over lattice points in a polytope. That sum is sometimes called the integer-point transform of the polytope.

In the weight lattice of a simple Lie algebra, a Weyl polytope has the weights in an orbit of the Weyl group as its vertices. Applied to Weyl polytopes, the Brion theorem yields a formula that is remarkably similar to the Weyl character formula \[6, 14, 11\]. As a consequence, the polytope expansion of Weyl characters in terms of integer-point transforms is natural and useful \[14, 11, 13, 15, 6, 12\].

Here we explore further the relation between Lie characters and integer-point transforms. Other formulas exist for the characters, and following \[14\], we are interested in finding similar expressions for the integer-point transforms of Weyl polytopes, herein called Weyl polytope sums. We focus on the Demazure character formula \[5, 1, 7\] and use the Demazure operators involved to write expressions for the Weyl polytope sums. We have obtained results for the simple Lie algebras \(A_n\), for all ranks \(n \in \mathbb{N}\).

In the following section, we review the initial motivation for the present work, in part to establish our notation. We describe the similarity between the Weyl character formula and the Brion formula, and the polytope expansion that exploits it. Section 3 is a quick account of the Demazure character formula. Our new formula for \(A_n\) Weyl polytope sums is presented and proved in Section 4. The final section offers a short conclusion.

2. Polytope expansion of Lie characters

Let \(X_n\) denote a simple Lie algebra of rank \(n\) (\(X\) is a letter from \(A\) to \(G\)). The sets of fundamental weights and simple roots are denoted by \(F := \{\Lambda^i \mid i = 1, \ldots, n\}\) and \(S := \{\alpha_i \mid i = 1, \ldots, n\}\), respectively. The corresponding weight and root lattices are \(P := \mathbb{Z}F\) and \(Q := \mathbb{Z}S\). The set of dominant integrable weights is \(P^+ := \mathbb{N}_0 F\), and we write \(R (R_+, R_-)\) for the set of (positive, negative) roots of \(X_n\).

2.1. Weyl character formula. Consider a finite-dimensional irreducible module \(L(\lambda)\) over \(X_n\) of highest weight \(\lambda \in P_+.\) The formal character of \(L(\lambda)\) is defined as

\[
\text{ch}_\lambda := \sum_{\mu \in P} \mult_{\lambda}(\mu) \, e^{\mu} = \sum_{\mu \in P(\lambda)} \mult_{\lambda}(\mu) \, e^{\mu},
\]

where \(\mult_{\lambda}(\mu)\) is the multiplicity of the weight \(\mu\) in the module \(L(\lambda)\), while \(P(\lambda)\) is the set of weights of \(L(\lambda)\):

\[
P(\lambda) = \{ \mu \in P \mid \mult_{\lambda}(\mu) > 0 \}.
\]

The formal exponentials of weights obey \(e^\mu \, e^\nu = e^{\mu+\nu}\). With

\[
e^{\mu}(\sigma) := e^{(\mu, \sigma)}, \quad \sigma \in P,
\]

where \((\mu, \sigma)\) is the inner product of weights \(\mu\) and \(\sigma\), the formal exponential \(e^\mu\) simply stands for \(e^{(\mu, \sigma)}\) before a choice of weight \(\sigma\) is made. A choice of \(\sigma\) fixes a conjugacy class of elements in the Lie group \(\exp(X_n)\), and the formal character becomes a true character: \(\text{ch}_\lambda(\sigma)\), the trace, in the irreducible
highest-weight representation of highest weight \( \lambda \), of elements of \( \exp(X_n) \) in the conjugacy class labelled by \( \sigma \).

The celebrated Weyl character formula is

\[
\text{ch}_\lambda = \frac{\sum_{w \in W} (\det w) e^{w(\lambda + \rho) - \rho}}{\prod_{\alpha \in R_+} (1 - e^{-\alpha})} ,
\]

where \( \rho := \sum_{i=1}^n \Lambda^i \). The Weyl invariance of the character can be made manifest, as

\[
\text{ch}_\lambda = \sum_{w \in W} e^{w\lambda} \prod_{\alpha \in R_+} (1 - e^{-w\alpha})^{-1} ,
\]

where

\[
(1 - e^\beta)^{-1} = \begin{cases} 
1 + e^\beta + e^{2\beta} + \ldots, & \beta \in R_-; \\
-(e^{-\beta} + e^{-2\beta} + \ldots), & \beta \in R_.
\end{cases}
\]

For brevity, we sometimes write \( G_\beta := (1 - e^{-\beta})^{-1} \), for \( \beta \in R \). For each \( w \in W \), we also define

\[
w(e^\mu) := e^{w\mu}, \quad w e^\mu := e^{w\mu} w ,
\]

meaning that \( w \) acts on an explicitly indicated argument only or, if no argument is given, on everything to its right. We can then rewrite (5) as

\[
\text{ch}_\lambda = \mathcal{C}(e^\lambda) ,
\]

with

\[
\mathcal{C} := \sum_{w \in W} \left( \prod_{\alpha \in R_+} G_{\alpha \lambda} \right) w = \sum_{w \in W} w \prod_{\alpha \in R_+} G_{\alpha} .
\]

2.2. Brion formula. A polytope is the convex hull of finitely many points in \( \mathbb{R}^d \). A polytope’s vertices form such a set of points, with minimum cardinality. A lattice polytope has all its vertices in an integral lattice in \( \mathbb{R}^d \). The corresponding (formal) integer-point transform of the polytope is the sum of terms \( e^\varphi \) over the lattice points \( \varphi \) in the polytope.

Brion \cite{Br03,Br04} found a general formula for these integer-point transforms. For \( \lambda \in P \), let the Weyl polytope \( \text{Pt}_\lambda \) be the polytope with vertices given by the Weyl orbit \( W\lambda \). Consider the integer-point transform

\[
\text{B}_\lambda := \sum_{\mu \in (\lambda + Q) \cap \text{Pt}_\lambda} e^\mu ,
\]

where the relevant lattice is the \( \lambda \)-shifted root lattice \( \lambda + Q \) of the algebra \( X_n \). We refer to these integer-point transforms as Weyl polytope sums. Since

\[
\text{B}_\lambda = \sum_{\mu \in P(\lambda)} e^\mu ,
\]

the Weyl polytope sum has an interpretation as a “multiplicity-free” character \cite{Br06}. By \cite{Br06}, it is obtained from the character \cite{Br09} by putting \( \text{mult}_\lambda(\mu) \rightarrow 1 \) for all \( \mu \in P(\lambda) \).

Applied to a Weyl polytope, the Brion formula yields

\[
\text{B}_\lambda = \sum_{w \in W} e^{w\lambda} \prod_{\alpha \in S} (1 - e^{-w\alpha})^{-1} .
\]

Following \cite{Br03} and \cite{Br04}, we rewrite (12) as

\[
\text{B}_\lambda = \mathcal{B}(e^\lambda) ,
\]
with
\[ B := \sum_{w \in \mathcal{W}} \left( \prod_{\alpha \in \mathcal{S}} G_{w\alpha} \right) w = \sum_{w \in \mathcal{W}} w \prod_{\alpha \in \mathcal{S}} G_{\alpha}. \]  

(14)

2.3. Polytope expansion. The Brion formula (12) is remarkably similar to the Weyl character formula (5) [14, 11, 6]. It is therefore natural and fruitful to consider the polytope expansion of Lie characters [14, 6, 15]:
\[ \text{ch}_{\lambda} = \sum_{\mu \leq \lambda} A_{\lambda, \mu} B_{\mu}. \]  

(15)

The expansion coefficients \( A_{\lambda, \mu} \) are integers. They were dubbed polytope multiplicities and denoted \( \text{polyt}_{\lambda}(\mu) \) in [15], in analogy with the weight multiplicities \( \text{mult}_{\lambda}(\mu) \) appearing in the expansion (1).

For type \( A \), they were shown in [9] to be non-negative. However, other examples have been found that are negative [9], so “multiplicity” appears to be a misnomer.

We do not consider the polytope expansion further in this note. Instead, we focus on the striking relationship between characters and Weyl polytope sums.

3. Demazure character formula

Here we show that expressions similar to the Demazure character formula can be written for the Weyl polytope sums in (10).

Let us first sketch the Demazure character formula. The Weyl group \( \mathcal{W} \) is generated by the reflections \( r_{\beta} \) in weight space across the hyperplanes normal to the corresponding roots \( \beta \in \mathcal{R} \):
\[ r_{\beta}(\lambda) := \lambda - (\lambda, \beta^\vee) \beta, \]

(16)

where \( \beta^\vee := 2\beta/(\beta, \beta) \). In fact, the Weyl group is generated by the primitive (simple-root) reflections, \( r_i \equiv r_{\alpha_i} \).

For each primitive reflection \( r_i \), we define the Demazure operator
\[ D_i := \frac{1 - e^{-\alpha_i} r_i}{1 - e^{-\alpha_i}}. \]  

(17)

For \( \lambda \in \mathcal{P} \), we set \( r_i(e^\lambda) := e^{\alpha_i} \) and thus have
\[ D_i(e^\lambda) = \begin{cases} 
  e^\lambda + e^{\lambda-\alpha_i} + e^{\lambda-2\alpha_i} + \ldots + e^{\alpha_i}, & (\lambda, \alpha_i^\vee) \geq 0; \\
  0, & (\lambda, \alpha_i^\vee) = -1; \\
  -(e^{\lambda+\alpha_i} + e^{\lambda+2\alpha_i} + \ldots + e^{\alpha_i(\lambda+\alpha_i)}), & (\lambda, \alpha_i^\vee) < -1.
\end{cases} \]  

(18)

We will also use the modified Demazure operators
\[ d_i := D_i - 1 = \frac{e^{-\alpha_i}(1 - r_i)}{1 - e^{-\alpha_i}}. \]  

(19)

For every \( w \in \mathcal{W} \), a Demazure operator \( D_w \) can be defined: In a reduced decomposition of \( w \), replace the factors \( r_j \) with \( D_j \). Demazure has shown [5] that the resulting operator \( D_w \) is independent of which reduced decomposition is used (see also [7]). Accordingly, the Demazure operators obey relations encoded in the Coxeter-Dynkin diagrams of \( \mathcal{X}_{\lambda} \). To illustrate, let \( w_L \in \mathcal{W} \) denote the longest element of the Weyl group. For \( A_2 \), for example, we have \( w_L = r_1 r_2 r_1 = r_2 r_1 r_2 \), and the associated Demazure operator can be written in the two ways \( D_{w_L} = D_1 D_2 D_1 = D_2 D_1 D_2 \).
The Demazure character formula takes the form (4) with
\[ C = D_{w_L}, \quad \text{i.e.} \quad \chi_\lambda = D_{w_L}(e^\lambda). \]  

(20)

4. \( A_n \) weight-polytope formulas of Demazure type

We will now restrict attention to the simple Lie algebras \( A_n \). When appropriate, the superscript \((n)\) will be used to indicate the dependence on the rank \( n \).

With \( G_i \equiv G_{\alpha_i} \), rewriting
\[ D_i = (1 + r_i) G_i \]  
will be useful. For \( k, m \in \{1, \ldots, n\} \) with \( k \neq m \), we also define
\[ G_{k,m} := (1 - e^{-\alpha_k - \alpha_m})^{-1}. \]  
(22)

From
\[ G_k r_m = (1 - e^{-\alpha_k})^{-1} r_m = r_m (1 - e^{-r_m \alpha_k})^{-1}, \]  
(23)
it follows that
\[ G_{k+1} r_k = r_k G_{k+1, k}, \quad G_{k-1} r_k = r_k G_{k-1, k}, \]
\[ G_{k, k+1} r_k = r_k G_{k+1, k}, \quad G_{k-1, k} r_k = r_k G_{k-1, k}. \]  
(24)

Here and henceforth we use the standard numbering of \( A_n \) simple roots, so that \( (\alpha_k, \alpha_k^\vee) = -1 \) for all \( k \in \{1, \ldots, n-1\} \).

Let us define
\[ s_{i,j} := r_j r_{j-1} \cdots r_i, \quad 1 \leq i \leq j \leq n, \]  
(25)
and
\[ w_{i,j} := \sum_{k=i}^{j} s_{k,j} + 1 = s_{i,j} + s_{i+1,j} + \ldots + s_{j,j} + 1. \]  
(26)

The following expression is reminiscent of the Poincaré series discussed by Macdonald in [10].

Lemma. For the simple Lie algebras \( A_n \),
\[ \sum_{w \in W^{(n)}} w = w_{1,1} w_{1,2} \cdots w_{1,n}. \]  
(27)

Proof. Since \( W^{(n)} \cong S_{n+1} \), the result follows by showing that the sum in (27) acting on \((1, \ldots, n+1)\) produces all possible permutations thereof, where \( r_j(1, \ldots, j-1, j, j+1, j+2, \ldots, n+1) := (1, \ldots, j-1, j+1, j, j+2, \ldots, n+1) \), for \( j = 1, \ldots, n \). Our proof is by induction on \( n \). For \( n = 1 \), the result is trivial; for \( n = 2 \), it is readily verified. Applying \( w_{1,n} \) to \((1, \ldots, n+1)\) produces \( n+1 \) terms, one for each possible value of the last entry. By the induction hypothesis, the relation (27) holds for \( n \) replaced by \( n-1 \), so all permutations of \((1, \ldots, n+1)\) result. ■

Our main result is given in (33) below and may be viewed as the Demazure analogue of this Lemma.

For \( 1 \leq i \leq j \leq n \), we now define the \emph{generalized Demazure operators}
\[ D_{i,j} := r_j r_{j-1} \cdots r_{i+1} d_i + r_j r_{j-1} \cdots r_{i+2} d_{i+1} + \ldots + d_j + 1, \]  
(28)
noting that \( D_{i,i} = d_i + 1 = D_i \).

**Lemma.** For \( 1 \leq i \leq j \leq n \),
\[
D_{i,j} = s_{i,j}G_i + s_{i+1,j}G_i G_{i+1}/G_{i+1} + s_{i+2,j}G_{i+1} G_{i+2}/G_{i+1,i+2} + \ldots + s_{j,j}G_{j-1} G_j/G_{j-1,j} + G_j .
\] (29)

**Proof.** Use (21) and substitute
\[
d_i = (r_i + 1)G_i - 1
\] into (28) to obtain
\[
D_{i,j} = s_{i,j}G_i + s_{i+1,j}(G_i - 1 + G_{i+1}) + s_{i+2,j}(G_{i+1} - 1 + G_{i+2}) + \ldots + s_{j,j}(G_{j-1} - 1 + G_j) + G_j .
\] (31)

Now notice that
\[
G_k - 1 + G_m = G_k G_m (1 - e^{-\alpha_k - \alpha_m}) = G_k G_m / G_{k,m} ,
\] (32)
from which the result follows. 

With the similarity between (28) and (26), the Weyl group algebra relation (27) motivates our main result, the following Theorem.

**Theorem.** For the simple Lie algebras \( A_n \),
\[
\mathcal{B}^{(n)} = D_{1,1} D_{1,2} \cdots D_{1,n} .
\] (33)

**Proof.** Our proof is by induction on the rank \( n \). For \( n = 1 \), the characters and Weyl polytope sums coincide, and \( w_L = r_1 \), so
\[
\mathcal{B}^{(1)} = C^{(1)} = D_{w_L} = D_1 = D_{1,1} .
\] (34)

For the induction step, we assume
\[
D_{1,1} D_{1,2} \cdots D_{1,n-1} = w_{1,1} w_{1,2} \cdots w_{1,n-1}(G_1 G_2 \cdots G_{n-1}) ,
\] (35)
and prove that
\[
D_{1,1} \cdots D_{1,n-1} D_{1,n} = w_{1,1} \cdots w_{1,n-1} w_{1,n}(G_1 \cdots G_{n-1} G_n) .
\] (36)

By (35), the left-hand-side of (36) is
\[
w_{1,1} w_{1,2} \cdots w_{1,n-1}(G_1 G_2 \cdots G_{n-1}) D_{1,n} .
\] (37)

To complete the proof, we establish
\[
(G_1 G_2 \cdots G_{n-1}) D_{1,n} = w_{1,n}(G_1 G_2 \cdots G_{n-1} G_n) .
\] (38)

By (29), we have
\[
D_{1,n} = s_{1,n}G_1 + s_{2,n}G_1 G_2/G_1,2 + s_{3,n}G_2 G_3/G_2,3 + \ldots + s_{n,n}G_{n-1} G_n/G_{n-1,n} + G_n .
\] (39)

Using (24), we see that
\[
(G_1 \cdots G_{n-1}) s_{j,n} = s_{j,n}(G_1 \cdots G_{j-2} G_{j-1,j} G_{j+1} \cdots G_n) .
\] (40)

The relation (38) now follows, thus completing the proof. 

5. Conclusion

Our main result is the formula involving (modified) Demazure operators for the weight-polytope lattice sums for the Lie algebras $A_n$. It is valid for all ranks $n \in \mathbb{N}$ and all dominant integrable highest weights.

In [16], formulas are written for the rank-2 weight-polytope lattice sums. It is interesting to note that these formulas are easily recast into a form that is very similar to the one we have found for $A_n$. Apart from $A_2$, $C_2 \cong B_2$ and $G_2$ are the only (up to isomorphism) rank-2 simple Lie algebras. In both cases, let $\alpha_1$ denote the short root. For $C_2$, we then find

$$B = (1 + d_2) (1 + d_1 + r_1 d_2 + r_1 r_2 d_1) ,$$

while for $G_2$, we obtain

$$B = (1 + d_2) (1 + d_1 + r_1 d_2 + r_1 r_2 d_1 + r_1 r_2 r_1 d_2 + r_1 r_2 r_1 r_2 d_1) .$$

We believe this indicates that we are on track toward a general form, valid for all simple Lie algebras. Furthermore, we hope that such a formula might lead to one that applies beyond the Lie context, as the Brion formula does, to polytopes besides the Weyl polytopes.

To finish, let us mention some interesting related work. The polytope expansion of Lie characters is highly reminiscent of the early work of Antoine and Speiser [2] and the recursive formulas found by Kass [8]. Recent work generalizes the context significantly. Dhillon and Khare [6] thus report results for all simple highest-weight modules over Kac-Moody algebras. In [9] by Lecouvey and Lenart, a connection with the atomic decomposition of characters is described, along with $(q$- or $t$-)deformations of the structures described herein.

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Data sharing is not applicable to this article as no new data were created or analyzed in this study.