The Oscillation Frequency of $B$ and $\bar{B}$ Mesons in a QCD Potential Model with Relativistic Effects

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Wavefunction at the origin, with the incorporation of a relativistic effect, leads to singularity in a specific potential model. To regularize the wavefunction, we introduced a short distance scale and used it to estimate the mass and decay constants of $B_d$ and $B_s$ mesons within the QCD potential model. These values were then used to compute the oscillation frequency, $\Delta m_{BS}$, of $B_d$ and $B_s$ mesons. The values were found to be in good agreement with experimental and other theoretical values.

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Investigations of weak decays of mesons composed of a heavy quark and an antiquark give us an important insight into heavy-quark dynamics. Research into the mixing and decay constants of $B$ meson also provides us with useful information about the dynamics of quarks and gluons at the hadronic scale. The weak eigenstates of neutral mesons are different from their mass eigenstates. This leads to the phenomenon of mixing, whereby neutral mesons oscillate between their matter and antimatter states. This was first observed in the Kaon sector, and subsequently in $B_d$ and $B_s$ mesons. The mass difference $\Delta m_{BS}$ is a measure of the frequency of the change from a $B$ into a $\bar{B}$, and is called the oscillation frequency. The decay constants of heavy mesons, one of the input parameters for oscillation frequency, are crucial for interpreting data on particle-antiparticle mixing in the neutral $B$ meson system, and for anticipating and interpreting new signatures for CP violation.

If the CKM element is well known from other measurements, then the pseudoscalar decay constant $f_p$ can be well measured. If, on the other hand, the CKM element is less well or poorly measured, having theoretical input on $f_p$ can allow a determination of the CKM element.$^1$ A measurement of the decay constant $f_B$ is difficult, since $B^+ \rightarrow l^+\nu_l$ is cabibo-suppressed in the standard model. Hence $f_B$ has to be provided from theory.

In this Letter, we calculate the pseudoscalar masses $M_{Bs}$ and pseudoscalar decay constants $f_{Bs}$ to compute the oscillation frequency $\Delta m_{Bs}$, $q = d, s$ within the framework of a potential model.$^{[2-4]}$ To incorporate the relativistic effect, the necessity of a short distance scale, in analogy to QED, is also pointed out.

For the light heavy flavor bound system of $qQ$ or $\bar{q}Q$, the hamiltonian can be written as

$$H = -\frac{\nabla^2}{2\mu} + V(r),$$  \hspace{1cm} (1)

where $V(r)$ is the spin-independent quark-antiquark potential.

$$V(r) = V_{\text{conf}}(r) + V_{\text{conf}}(r),$$  \hspace{1cm} (2)

where $V_{\text{conf}}$ represents the coulombic part of the potential and $V_{\text{conf}}$ represents the confining potential. The vector and scalar confining potentials in the non-relativistic limit reduce to$^{[5,6]}$

$$V_{\text{conf}}(r) = (1 - \epsilon_1)(br + C),$$ \hspace{1cm} (3)

$$V_{\text{conf}}(r) = \epsilon_1(br + C),$$ \hspace{1cm} (4)

reproducing

$$V_{\text{conf}}(r) = V_{\text{conf}}^{s}(r) + V_{\text{conf}}^{v}(r) = br + C,$$ \hspace{1cm} (5)

$$V_{\text{conf}}(r) = -\alpha_c/r,$$ \hspace{1cm} (6)

where $\alpha_c = \frac{4}{3}\alpha_s$ with $\alpha_s$ being the strong running coupling constant, $\epsilon_1$ the mixing coefficient, and $b$ and $C$ the potential parameters as used in our previous work.$^{[3,4]}$

Considering the linear part of the potential as perturbation, the coulombic part as the parent, and then using the dalgarino method, the wavefunction in the ground state is obtained,$^{[2-4]}$ i.e.

$$\psi_{\text{rel+conf}}(r) = \frac{N'}{\sqrt{\pi a_0}} e^{-r/a_0} \left(C' - \frac{\mu a_0 r^2}{2} \right) \left(\frac{r}{a_0}\right)^{-\epsilon},$$ \hspace{1cm} (7)

$$N' = 2^{1/2} \left\{ \left(2\mu\Gamma(3-2\epsilon)C'^2 - \frac{1}{4}\mu b a_0^3 \Gamma(5-2\epsilon)C' \right. \right.$$
$$+ \frac{1}{64}\mu^2 b^2 a_0^6 \Gamma(7-2\epsilon)) \right\}^{-1/2},$$ \hspace{1cm} (8)

$$C' = 1 + c a_0 \sqrt{\pi a_0^3},$$ \hspace{1cm} (9)

$$\mu = \frac{m_j m_j}{m_i + m_j},$$ \hspace{1cm} (10)

$$a_0 = \left(\frac{4}{3}\mu a_s\right)^{-1},$$ \hspace{1cm} (11)

$$\epsilon = 1 - \sqrt{1 - \left(\frac{4}{3}\alpha_s\right)^2}.$$ \hspace{1cm} (12)
Here $A_0$ is the undetermined factor appearing in the series solution of the Schrödinger equation, $\mu$ is the reduced mass, and $m_i$ and $m_j$ are the constituent quark masses. The term $(r/a_0)^{-\epsilon}$ is due to the relativistic effects. The strong running coupling constant appearing in the potential $V(r)$ in turn is related to the quark mass parameter as \[ \alpha_s(\mu^2) = \frac{4\pi}{(11 - 2\alpha_s^2)\ln\left(\frac{\mu^2 + M_F^2}{\Lambda^2}\right)} \tag{13} \]

where $n_f$ is the number of flavors, $\mu$ is the renormalization scale related to the constituent quark masses as $\mu = \frac{2m_i m_j}{m_i + m_j}$ and $\Lambda$ is the QCD scale taken here as 0.200 GeV. $M_B$ is the background mass related to the confinement term of the potential as $M_B = 2.24 \times b^{1/2} = 0.95$ GeV. In this calculation we have taken $n_f = 3$ as in Refs. [5,7] and the input mass parameters as $m_d = 0.36$ GeV, $m_s = 0.46$ GeV and $m_b = 4.95$ GeV. With these values we calculate $\alpha_s = 0.40$ for $B_d$ mesons and $\alpha_s = 0.37$ for $B_s$ mesons. However, we have also computed the corresponding results for $n_f = 4$, as listed in Tables 1 and 2.

One of the problems faced in the model to study masses and decay constants is the incorporation of relativistic effects in the wavefunction at the origin, since a singularity develops at $r = 0$ (Eq. (7)). However, singularities at $r = 0$ in relativistic and non-relativistic approaches of the quark model [6,9] are at present new, and different regularization of singularities has been discussed.

In this work we use another way to regularize the wavefunction at the origin, which has the quantum mechanical origin in QED. It is well known that the relativistic wavefunction of the hydrogen atom also has such singularities. However, such an effect is noticeable only for a tiny region, \[ z m_{\text{zo}} \leq e^{-\left(\frac{z^2}{2\alpha_s^2}\right)} \leq e^{-\frac{z^2}{2\alpha_s^2}} \sim 10^{-15}\text{m}, \tag{14} \]

where $z$ is the atomic number, $m$ is the reduced mass of the hydrogen atom, $\alpha$ is the electromagnetic coupling constant, and $\gamma = \sqrt{1 - z^2}\alpha^2$. Using such hydrogen-like properties in QCD, $m$, $\alpha$ and $1 - \gamma$ are to be replaced by $\mu$, $\frac{2}{3}\alpha_s$ and $\epsilon$, respectively. Here $\alpha_s$ is the strong coupling constant, $\epsilon = 1 - \sqrt{1 - (\frac{2}{3}\alpha)^2}$ and $(m_{\text{zo}})\gamma^{-1}$ changes to $(r/a_0)^{-\epsilon}$, leading to a cut-off parameter $r_0$, up to which the model can be extrapolated ($r \geq r_0$).

In analogy to the QED calculation (Eq. (14)), using the typical length scale for the relativistic term $(r/a_0)^{-\epsilon} \leq 1/\epsilon$, we obtain the cut-off parameter \[ r_0 \sim a_0 e^{-\frac{1}{2\epsilon}}. \tag{15} \]

Unlike QED, it is flavor dependent on the flavors of the quark masses (Eq. (11)).

The decay constants of mesons are important parameters in the study of leptonic or non-leptonic weak decay processes, and in the neutral $B - \bar{B}$ mixing process. In the non-relativistic limit, the decay constant can be expressed through the ground state wavefunction at the origin $\psi(0)$ by the Van–Royen–Weisskopf formula. [10] Although most of the models predict the mesonic mass spectrum successfully, there are disagreements in the predictions of the decay constants. Thus we re-examine the predictions of the decay constants with a new short distance scale.

We consider the non-relativistic expression for $f_p$ as \[ f_p = \sqrt{\frac{12}{M_p}(\psi(0))^2}, \tag{16} \]

where $M_p$ is the pseudoscalar mass of mesons, calculated using the relation \[ M_p = m_i + m_j - \frac{8\pi\alpha_s}{3m_i m_j} (\psi(0))^2. \tag{17} \]

| Table 1. Values of $r_0$ and $M_p$ for $B_d$ and $B_s$ mesons. |
|---|---|
| $B_d$ | 0.141 | 0.009 | 5.273 |
|      | 0.162 | 0.021 | 5.256 |
|      | 5.279 | 5.285 | 5.279 |
|      | 0.178 | 0.002 | 5.370 |
|      | 0.207 | 0.007 | 5.349 |
|      | 5.369 | 5.373 | 5.375 |

$^a$Our work with $n_f = 3$. $^b$Our work with $n_f = 4$.

The values of $r_0$ and the corresponding wavefunction at the origin for $B_d$ and $B_s$ are calculated to study the pseudoscalar masses and are shown in Table 1. Using Eq. (16), we compute the decay constants for $B_d$ and $B_s$ mesons and put them in Table 2. The results are found to be in good agreement with the experimental data.

The neutral $B_d$ and $B_s$ mesons can mix with their antiparticles by means of a box diagram involving the exchange of a pair of W bosons and intermediate $u, c, t$ quarks, leading to oscillations between the mass eigenstates. This mass oscillation is parametrized as the oscillation frequency or mixing mass parameter ($\Delta m$), as given by Refs. [16,17].

\[ \Delta m_B = \frac{G_F^2 m_t^2 m_B g(x_1)\eta_h|V_{ts}^2 V_{tb}|^2 B}{8\pi}, \tag{18} \]

where $\eta_h$ is the gluonic correction to the oscillation and is taken as 0.55, similar to Ref. [16]. The last factor $B$ is the bag parameter that represents the correction to the vacuum insertion, and is taken as 1.34. [16] The
function $g(x_t)$ is given by\textsuperscript{[18]}
\[
g(x_t) = \frac{1}{4} + \frac{9}{4(1 - x_t)} - \frac{3}{2(1 - x_t)^2} - \frac{3x_t^2}{2(1 - x_t)^3},
\]
where $x_t = m_t^2/M_b^2$. The values $m_b(174 \text{ GeV})$, $m_W(80.43 \text{ GeV})$, and the CKM matrix elements $|V_{ub}|(1)$, $|V_{td}|(7.4 \times 10^{-3})$, $|V_{ts}|(40.6 \times 10^{-3})$ are taken from the particle data group.\textsuperscript{[14,13]}

We use the estimated values of masses and decay constants to compute the mixing mass parameters and put them in Table 2. The decay constants are found to be comparable with the other theoretical values, and the mixing mass parameters are found to be in good agreement with the experimental data.

Table 2. Decay constant and oscillation frequency of B mesons.

| Mesons | $f_B$ (GeV) | $\Delta m_B$ (ps$^{-1}$) |
|--------|-------------|------------------------|
| $B_d$  | 0.213$^a$   | 0.55$^a$                |
|        | 0.240$^b$   | 0.74$^b$                |
|        | 0.189$^{[2]}$ | 0.50$^{[12]}$          |
|        | 0.196$^{(2)}$ | 0.547$^{[20]}$         |
|        | 0.190$^{(7)}$ | 0.515$^{[2]}$          |
|        | 0.210$^{(9)}$ | 0.515$^{[2]}$          |
|        | 0.216$^{(9)(19)(6)^{[26,27]}}$ | $17.34^{a}$          |
| $B_s$  | 0.265$^a$   | 0.34$^a$                |
|        | 0.311$^b$   | 0.238$^b$               |
|        | 0.218$^{[2]}$ | 0.17$^{[12]}$          |
|        | 0.216$^{[2]}$ | 0.177$^{[12]}$         |
|        | 0.217$^{(6)}$ | 0.177$^{[2]}$          |
|        | 0.244$^{(21)}$ | 0.177$^{[2]}$         |
|        | 0.259$^{(32)}$ | 0.177$^{[2]}$          |

$^a$Our work with $n_f = 3$. $^b$Our work with $n_f = 4$.

In summary, we incorporated relativistic effects to wavefunction at the origin in a QCD potential model. We introduced a short distance scale $r_0$ in analogy to QED. Unlike QED, in QCD, such a short distance scale becomes flavor-dependent. As expected, its magnitude is far larger than its QED counterpart, but still far smaller than the measure of the finite size of hadrons or its constituents. It is, however, well within the reach of the LHC,\textsuperscript{[20]} where the distance down to a scale as small as $5 \times 10^{-20} - 10^{-21} \text{ m}$ will be explored. Theoretically, this short distance scale $r_0$ can be roughly associated with the ultraviolet regularization scale of QCD.

We found from the calculation that generally the masses are not very sensitive to the running coupling constants, since the heavy quark constituent masses dominant here, but the decay constants are very sensitive to the running coupling constants and therefore a slight change in $\alpha_s$, as well as $r_0$, deviates the results significantly.

In Table 2 we presented our results for the pseudoscalar decay constants and oscillation frequency ($\Delta m_B$) of $B_d$ and $B_s$ mesons with the recent predictions of the quark model\textsuperscript{[22]} three-flavor lattice QCD\textsuperscript{[27,28]} collaboration\textsuperscript{[19–21]} and available experimental data.\textsuperscript{[18]} In our calculation we found $\frac{f_{B_d}}{f_{B_s}}=1.24$, which is in accordance with the lattice result $f_{B_d}/f_{B_s}=-1.20(3)(1)$.\textsuperscript{[26,27]} The results in Tables 1 and 2 for $B_d$ and $B_s$ mesons with the short distance scale are found to be in good agreement with each other for all the presented theoretical predictions, as well as the experimental data for $n_f = 3$, and the corresponding results with $n_f = 4$ are found to overshoot the theoretical predictions and experimental data.

References

\begin{enumerate}
\item Rosner J and Stone S arXiv:1002.1655 [hep-ph] (expanded version of a review prepared for PDG 2010)
\item Choudhury D K and Das P 1996 Pramana J. Phys. 46 349
\item Choudhury D K and Bordoloi N S 2002 MPLA 17 1909
\item Choudhury D K and Bordoloi N S 2009 MPLA 24 443
\item Ebert D, Faustov R N and Galkin V O 2009 Phys. Rev. D 79 114029
\item Ebert D, Faustov R N and Galkin V O 2006 Phys. Lett. B 635 93
\item Deogharia S and Chakraborty S 1989 Nucl. Part. Phys. G 15 1213
\item Deogharia S and Chakraborty S 1990 Nucl. Part. Phys. G 16 1825
\item Deogharia S and Chakraborty S 1992 Z. Phys. C: Particles and Fields 53 293
\item Hadizadeh M R and Tomio Lauro arXiv:1104.3891v1 [hep-ph]
\item Itzykson C and Zuber J 1986 Quantum Field Theory (Singapore: McGraw Hill) p 79
\item Van Royen R et al 1967 Nuovo Cimento A 50 517
\item Rai A K, Parmer R H and Vinod Kumar P C 2002 J. Phys. G: Nucl. Part. Phys. 28 2275
\item Halzen F and Martin A D 1984 Quarks and Leptons (New Work John Willey and Sons)
\item Yao W M et al (particle data group) 2006 J. Phys. G 33 1
\item Ebert D et al 2003 Phys. Rev. D 67 014027
\item Pierro M Di et al 2001 Phys. Rev. D 64 114004
\item Buras A 2003 Phys. Lett. B 566 115
\item Patel B, Rai A K and Vinodkumar P C 2008 arXiv:0811.3817 [hep-ph]
\item Inami, T and Lim C S 1981 Prog. Theor. Phys. 65 297 [Erratum-ibid. 66 1772]
\item Abulencia A et al [CDF collaboration] 2006 Phys. Rev. Lett. 97 242003
\item Abe F et al (CDF Collaboration) 1999 Phys. Rev. D 60 051101
\item Abe F et al (CDF Collaboration) 1999 Phys. Rev. D 60 072003
\item Ebert D et al 2006 Phys. Lett. B 634 214
\item Gvetic G et al 2004 Phys. Lett. B 596 84
\item Bernard C et al 2002 Phys. Rev. D 66 094501
\item Jamin M et al 2002 Phys. Rev. D 65 056005
\item Buras A J 2010 arXiv:1009.1303v1 [hep-ph]
\item Aubin C et al 2005 Phys. Rev. Lett. 95 122002
\item Grey A et al 2005 Phys. Rev. Lett. 95 212001
\end{enumerate}