Quaternionic and Octonionic Spinors

Francesco Toppan

CCP/CBPF, Rua Dr. Xavier Sigaud 150, cep 22290-180 Rio de Janeiro (RJ), Brazil

Abstract. Quaternionic and octonionic spinors are introduced and their fundamental properties (such as the space-times supporting them) are reviewed. The conditions for the existence of their associated Dirac equations are analyzed. Quaternionic and octonionic supersymmetric algebras defined in terms of such spinors are constructed. Specializing to the $D = 11$-dimensional case, the relation of both the quaternionic and the octonionic supersymmetries with the ordinary $M$-algebra are discussed.

INTRODUCTION

The division algebras are responsible for many important mathematical structures of interest for physicists (as an example one can cite the Hopf fibrations).

In this talk we review at first the well-known, see e.g. [1], [2] connection between division algebras and Clifford algebras, explaining also in which sense we can extend the (associative) notion of Clifford algebra in order to accommodate an alternative (i.e. non-associative) structure as the one given by the octonions. This motivates us to introduce quaternionic and octonionic spinors following [3] for later studying their free dynamics (namely, their associated Dirac-type of equations) for any space-time supporting quaternionic or octonionic spinors.

The potentially most interesting physical applications of the formalism here discussed concern supersymmetry. There are several reasons for that. Division algebras, including quaternions and octonions, naturally enter the classification of supersymmetry, [4]. Octonions (for a review on them one can consult [5]), which on mathematical side are the most interesting structure (taking into account that they are the maximal division algebra and are associated with the existence of the exceptional Lie algebras, [6]), so far have not found any concrete application in physics, despite many attempts to introduce them in several different contexts (e.g. in order to explain the strong interactions and the confinement of quarks, [7]). On the other hand, see [8] and references therein, one very appealing possibility could be found at the very heart of the unification program of the interactions which goes under the name of $M$-Theory. Indeed in the past [9] octonionic-valued superstrings have been described. More recently, [10] and [11], it was shown that an octonionic-valued version of the 11-dimensional $M$-algebra can be constructed and admits very peculiar properties. This opens the way to a possible octonionic formulation of the $M$-theory, which could correspond to some suggestions put forward in [8] and [12].

For what concerns the quaternionic supersymmetry, its mathematical aspects have been clarified and classified (see [13]). It must be said that no concrete physical imple-
mentation has yet been investigated. The most closely related relevant application so far concerns the analytic continuation of the $M$-algebra to the Euclidean \cite{14}. It is based on complex spinors living on a quaternionic spacetime.

**ON CLIFFORD ALGEBRAS AND DIVISION ALGEBRAS**

The basic relation defining a Clifford algebra is given by

$$\Gamma^\mu \Gamma^\nu + \Gamma^\nu \Gamma^\mu = 2\eta^{\mu\nu}, \quad (1)$$

with $\eta^{\mu\nu}$ being a diagonal matrix of $(p,q)$ signature (i.e. $p$ positive, $+1$, and $q$ negative, $-1$, diagonal entries).

On the other hand the four division algebra of real ($\mathbb{R}$) and complex ($\mathbb{C}$) numbers, quaternions ($\mathbb{H}$) and octonions ($\mathbb{O}$) possess respectively 0, 1, 3 and 7 imaginary elements $e_i$ satisfying the relations

$$e_i \cdot e_j = -\delta_{ij} + C_{ijk} e_k, \quad (2)$$

($i, j, k$ are restricted to take the value 1 in the complex case, 1, 2, 3 in the quaternionic case and 1, 2, ..., 7 in the octonionic case; furthermore, the sum over repeated indices is understood), with $C_{ijk}$ the totally antisymmetric division-algebra structure constants. The octonionic division algebra is the maximal, since quaternions, complex and real numbers can be obtained as its restriction. Its totally antisymmetric octonionic structure constants can be expressed as

$$ C_{123} = C_{147} = C_{165} = C_{246} = C_{257} = C_{354} = C_{367} = 1 \quad (3) $$

(and vanishing otherwise).

The octonions are the only non-associative, however alternative (see \cite{5}), division algebra.

It is therefore clear, due to the antisymmetry of $C_{ijk}$, that (1) can be realized, for the $(0,3)$ and the $(0,7)$ signatures, in terms of, respectively, the imaginary quaternions and the imaginary octonions.

With an abuse of language (due to their non-associativity) when in the following we will speak about “octonionic Clifford algebra” we will always have in mind the above connection.

For our later purposes it is of particular importance the notion of division-algebra principal conjugation. Any element $X$ in the given division algebra can be expressed through the sum

$$X = x_0 + x_i e_i, \quad (4)$$

where $x_0$ and $x_i$ are real, the summation over repeated indices is understood and the positive integral $i$ are restricted up to 1, 3 and 7 in the $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ cases respectively. The principal conjugate $X^*$ of $X$ is defined to be

$$X^* = x_0 - x_i e_i, \quad (5)$$
It allows introducing the division-algebra norm through the product $X^*X$. The normed-one restrictions

$$X^*X = 1 \quad (6)$$

select the three parallelizable spheres $S^1$, $S^3$ and $S^7$ in association with C, H and O respectively.

Further comments on the division algebras and their relations with Clifford algebras can be found in [3].

The connection between division algebras and Clifford algebras can be extended to other signature spacetimes as well. The two following algorithms can be used to lift $d$-dimensional Gamma matrices (denoted as $\gamma$) of a $D = p + q$ spacetime with $(p,q)$ signature into $2d$-dimensional $D+2$ Gamma matrices (denoted as $\Gamma_j$) of a $D+2$ spacetime, produced according to either

$$\Gamma_j \equiv \begin{pmatrix} 0 & \gamma \\ \gamma & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \mathbf{1}_d \\ -\mathbf{1}_d & 0 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{1}_d & 0 \\ 0 & -\mathbf{1}_d \end{pmatrix}$$

$$(p,q) \mapsto (p+1,q+1). \quad (7)$$

or

$$\Gamma_j \equiv \begin{pmatrix} 0 & \gamma \\ -\gamma & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \mathbf{1}_d \\ \mathbf{1}_d & 0 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{1}_d & 0 \\ 0 & -\mathbf{1}_d \end{pmatrix}$$

$$(p,q) \mapsto (q+2,p). \quad (8)$$

The relation (1) can therefore be realized, for specific spacetimes, in terms of quadratic matrices with either quaternionic or octonionic entries. The spacetimes supporting such Clifford realizations can be easily computed. In the octonionic case, up to $D = 13$, we obtain the following list of octonionic spacetimes

| $D$  | $(p,q)$                              |
|------|--------------------------------------|
| 7    | (0,7), (7,0)                         |
| 8    | (0,8), (8,0)                         |
| 9    | (0,9), (9,0), (1,8), (8,1)           |
| 10   | (1,9), (9,1)                         |
| 11   | (1,10), (10,1), (2,9), (9,2)         |
| 12   | (2,10), (10,2)                       |
| 13   | (3,10), (10,3), (2,11), (11,2)       |

(9)

An analogous list can be produced in the quaternionic case too.

**A comment on the octonionic realization**

One should be aware of the properties of the non-associative realizations of the relation (1), in terms of Gamma-matrices with octonionic-valued entries. In the octonionic
case the commutators

\[ \Sigma_{\mu\nu} = [\Gamma_\mu, \Gamma_\nu] \]  

(10)

no longer correspond, as in the associative case, to the generators of the Lorentz group. They correspond instead to the generators of the coset \( SO(p,q)/G_2 \), being \( G_2 \) the 14-dimensional exceptional Lie algebra of automorphisms of the octonions. This point can be easily illustrated with the basic example of the Euclidean 7-dimensional case expressed by the imaginary octonions. Their commutators give rise to \( 7 = 21 - 14 \) generators, isomorphic to the imaginary octonions. Indeed

\[ [e_i, e_j] = 2G_{ijk}e_k. \]  

(11)

The alternativity property satisfied by the octonions implies that the seven-dimensional commutator algebra among imaginary octonions is not a Lie algebra, the Jacobi identity being replaced by a weaker condition that endorses (11) with the structure of a Malcev algebra (see [5]).

Such an algebra admits a nice geometrical interpretation [15, 3]. Indeed, the normed 1 unitary octonions \( X = x_0 + x_ie_i \) satisfying the (6) condition describe the seven-sphere \( S^7 \). The latter is a parallelizable manifold with a quasi (due to the lack of associativity) group structure.

On the seven sphere, infinitesimal homogeneous transformations which play the role of the Lorentz algebra can be introduced through

\[ \delta X = a \cdot X, \]  

(12)

with \( a \) an infinitesimal constant octonion. The requirement of preserving the unitary norm (6) implies the vanishing of the \( a_0 \) component, so that \( a \equiv a_ie_i \). Therefore, the above commutator algebra (11), generated by the seven \( e_i \), can be interpreted as the algebra of “quasi” Lorentz transformations acting on the seven sphere \( S^7 \). At least in this specific example we discovered a nice geometrical setting underlining the use of the octonionic realization of the Clifford relation (1) with \( (0,7) \) signature. Indeed, while the associative representation (realized by \( 8 \times 8 \) real matrices) of the seven dimensional Clifford algebra is required for describing the Euclidean 7-dimensional flat space, the non-associative realization describes the geometry of \( S^7 \).

The Weyl condition

Spinors can simply be introduced as column-vectors with entries valued in the given division algebra and carrying a representation of the Lorentz-algebra generators \( \Sigma_{\mu\nu} \) introduced in (10) (Octonionic spinors, as discussed in the previous subsection, carry a representation of the \( G_2 \) coset).

A particular case arises for those space-time whose associated Gamma-matrices can be chosen to be block-antidiagonal, i.e. of the form

\[ \Gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix} \]  

(13)
The corresponding signatures can be easily recovered from the introduced algorithmic constructs (7) and (8). We will call the Gamma-matrices satisfying (13) as “generalized Weyl matrices”. The generators $\Sigma_{\mu\nu}$ of (10) in this case carry fundamental spinor-representations, realized by either upper or lower Weyl spinors, whose number of components is only half of their size.

In the Weyl case two projectors $P_{\pm}$ can be introduced through

$$P_{\pm} = \frac{1}{2}(1_{2d} \pm \Gamma),$$
$$\Gamma = \begin{pmatrix} 1_d & 0 \\ 0 & -1_d \end{pmatrix}$$

and the corresponding chiral (upper components) and antichiral (lower components) Weyl spinors, whose number of components is half of the size of the corresponding $\Gamma$-matrices, can be defined as satisfying

$$\Psi_{\pm} = P_{\pm}\Psi.$$  

**QUATERNIONIC AND OCTONIONIC DIRAC EQUATIONS**

In the previous section we have introduced all the necessary ingredients (division-algebra valued Clifford algebras and the associated spinors) to define the quaternionic and octonionic versions of the Dirac equation.

In this section we introduce such equations and provide the full classification [3] of the spacetimes supporting them (and under which condition, i.e. the possible presence of massive or pseudomassive terms, Weyl spinors, etc.). The results will be presented in a series of tables.

Let us introduce at first the needed conventions. A matrix $A$, given by the product of all time-like Gamma matrices and generalizing the role of $\Gamma^0$ in the Minkowskian case is used to define barred spinors ($\overline{\psi} = \psi^\dagger A$). (Pseudo)-kinetic and (pseudo)-massive terms can be introduced for full (NW) and Weyl (W) spinors according to the following prescriptions (in order to avoid unnecessary repetitions, it is sufficient to list here the octonionic case, the quaternionic one being easily recovered from the given formulas).

The (pseudo)-kinetic terms are given according to

$$K_X = \frac{1}{2}tr[(\Psi^\dagger A\Gamma^\mu X)\partial_\mu \Psi] + \frac{1}{2}tr[\Psi^\dagger (A\Gamma^\mu X \partial_\mu \Psi)]$$
$$K_{//X} = \frac{1}{2}tr[(\Psi_+^\dagger A\Gamma^\mu X)\partial_\mu \Psi_+] + \frac{1}{2}tr[\Psi_+^\dagger (A\Gamma^\mu X \partial_\mu \Psi_+)]$$
$$K_{\perp X} = \frac{1}{2}tr[(\Psi_-^\dagger A\Gamma^\mu X)\partial_\mu \Psi_-] + \frac{1}{2}tr[\Psi_-^\dagger (A\Gamma^\mu X \partial_\mu \Psi_-)] + \frac{1}{2}tr[(\Psi_{\perp}^\dagger A\Gamma^\mu X)\partial_\mu \Psi_+] + \frac{1}{2}tr[\Psi_{\perp}^\dagger (A\Gamma^\mu X \partial_\mu \Psi_+)].$$

Some remarks are in order. The first line refers to full spinors, while the suffices “//” and “\perp” are used to denote bilinear terms constructed with Weyl spinors of, respectively,
same and opposite chiralities. Please notice that, due to the non-associativity of the octonions, we need to specify the correct order in which the operations are taken (this is not necessary in the quaternionic case). The symbol “X” denotes the possibility of introducing, depending on the given space-time, external, extra-type of Gamma matrices, which could be either of time-like, or of space-like nature. More specifically, in the tables below, X will be denoted as T, S, J or F whether it will be associated with external time-like Gamma matrices (T), space-like (S) ones, the product of two of them (J) or, finally, of three of them (F). The presence of an extra-number specifies how many inequivalent choices for the introduction of such matrices can be given.

It should also be noticed that, in the octonionic case, the symbol “tr” introduced on the r.h.s. denotes the projection over the octonionic identity (while in the quaternionic case it coincides with the usual definition of the trace).

In full analogy with the (pseudo)-kinetic terms, (pseudo)-massive terms in a lagrangian action can be introduced through

\[ M_X = tr(\Psi^\dagger AX\Psi), \]
\[ M_{//X} = tr(\Psi^\dagger AX\Psi_+), \]
\[ M_{\perp X} = tr(\Psi_+^\dagger AX\Psi_+ + \Psi_-^\dagger AX\Psi_+), \]

Due to the anticommuting character of the spinors and of their basic components, the non-vanishing (pseudo)kinetic and (pseudo)massive terms are only allowed in given spacetimes. In the next two subsection we report the full classification of the allowed Dirac equations in, respectively, the quaternionic and octonionic cases.

### The quaternionic Dirac equations

We present here the tables of the allowed free (pseudo)-kinetic and (pseudo)-massive terms for quaternionic spinors. The columns are labeled by \( t \mod 4 \) and the rows by \( t - s \mod 8 \) \((t,s\) denoting the number of time-like and space-like directions of the given space-time\), while the symbols used in the entries have been explained above.

For full spinors \((NW)\) case we have

|   | 0         | 1         | 2         | 3         |
|---|-----------|-----------|-----------|-----------|
| 0 | \(K_J, K_F, M_{S_j}, M_{J_j}\) | \(K, K_F, M_{J_j}\) | \(K, K_{S_j}, M, M_F\) | \(K_{S_j}, K_J, M, M_{S_j}, M_F\) |
| 5 | \(K\)     | \(K, M\)  | \(M\)     |           |
| 6 | \(M_S\)   | \(K\)     | \(K, K_{S_j}, M\) | \(K_{S_j}, M, M_S\) |
| 7 | \(K_J, M_{S_i}, M_J\) | \(K, M_J\) | \(K, K_{S_j}, M\) | \(K_{S_i}, K_J, M, M_{S_i}\) |
while for Weyl spinors (W case) we get

\[
\begin{array}{c|c|c|c|c}
1 & K_{//T_i}, K_{\perp T_i}, & K_{//T_i}, K_{\perp T_i}, & K_{//F_i}, K_{\perp F_i}, & K_{\perp F_i}, K_{//J_j}, \\
& M_{//J_j} & M_{//J_j}, M_{\perp J_j} & M_{//F_i}, M_{//J_j}, M_{\perp J_j} & M_{//F_i}, M_{//J_j}, M_{\perp J_j} \\
2 & K_{//T_i}, K_{\perp T_i}, & K_{//T_i}, K_{\perp T_i}, & K_{\perp}, & K_{//J_j}, \\
& M_{//J_j} & M_{//J_j}, M_{\perp J_j} & M_{//J_j}, M_{\perp J_j} & M_{//J_j}, M_{\perp J_j} \\
3 & K_{//T} & K_{//T}, K_{\perp T}, & K_{\perp}, & M_{//T_j} \\
& M_{//T_j} & M_{//T_j}, M_{\perp T_j} & M_{//T_j}, M_{\perp T_j} & M_{//T_j}, M_{\perp T_j} \\
4 & K_{//} & K_{//} & K_{\perp}, & M_{//} \\
& M_{//} & M_{//}, M_{\perp} & M_{//}, M_{\perp} & M_{//}, M_{\perp} \\
\end{array}
\]

Please notice that in the two tables above the suffix “\(j\)” denotes the existence of three inequivalent choices for the corresponding matrices (e.g., the three distinct space-like matrices \(S_j\)), while the suffix “\(i\)” denotes the existence of two inequivalent choices.

The octonionic Dirac equations.

In full analogy with the previous case, we are able to produce the tables corresponding to the allowed (pseudo)-kinetic and (pseudo)-massive terms in octonionic Dirac equations. We get, in the (NW) case

\[
\begin{array}{c|c|c|c|c}
1 & K & K, M & M \\
& K & K, M & K, M, M \\
2 & M_S & K, M & K, M, M, M \\
& K, M_S, M_J & K, M, M, M & K, M, M, M \\
& K, M_J & K, M, M, M & K, M, M, M \\
4 & K_{//J_i}, K_{//F_i}, M_{//J_j} & K, K_{//F_i}, M_{//J_j}, M_{//F_i} & K, K_{//F_i}, M_{//J_j}, M_{//F_i} & K, K_{//F_i}, M_{//J_j}, M_{//F_i} \\
& M_{//J_j} & M_{//J_j}, M_{\perp J_j} & M_{//J_j}, M_{\perp J_j} & M_{//J_j}, M_{\perp J_j} \\
\end{array}
\]
while in the $W$ (Weyl) case we obtain

|   |   |   |   |   |
|---|---|---|---|---|
|   | 0 | 1 | 2 | 3 |
| 0 | $K_{//}$ | $K_{\perp}$, $M_{//}} | $M_{//}}$ |
| 5 | $K_{//}}T_i, K_{\perp}J_i$, $M_{//}}J_i, M_{//}}F$ | $K_{//}}, K_{\perp}T_i$, $M_{//}}, M_{\perp}J_i$ | $K_{\perp}, K_{//}}F$, $M_{//}}, M_{//}}F$ | $K_{//}}J_i, K_{\perp}F$, $M_{\perp}, M_{//}}F$ |
| 6 | $K_{//}}T_i, K_{\perp}J_i$, $M_{//}}J_i$, $M_{//}}T_i$ | $K_{//}}, K_{\perp}T_i$, $M_{//}}, M_{\perp}J_i$ | $K_{\perp}, K_{//}}T_i$, $M_{//}}, M_{\perp}T_i$ | $K_{//}}J_i, M_{\perp}$ |
| 7 | $K_{//}}T$ | $K_{//}}, K_{\perp}T$, $M_{//}}T$ | $K_{\perp}, M_{//}}, M_{\perp}T$ | $M_{//}}$ |

The introduced symbols have the same meaning as before.

**QUATERNIONIC AND OCTONIONIC SUPERSYMMETRIES**

It comes to no surprise that the most potentially interesting physical applications of the formalism and results previously introduced concern supersymmetry. After all, supersymmetry is nowadays (in the superstring/M-theory scenario) a necessary ingredient for our present understanding of fundamental interactions.

If octonions should play any role at all in physics, this is quite likely being in relation with the unification of all interactions realized by supersymmetry in higher-dimensional spacetimes. The mere fact that an octonionic formulation of the $M$-algebra is available \[10\] gives us some hopes that this program could one day be carried out.

In this section we will briefly review how generalized supersymmetries in higher-dimensional space-times (which are regarded as the scenarios for the unification of interactions and must be supplemented by some dimensional-reduction mechanism such as the Kaluza-Klein compactifications) can be constructed in terms of quaternionic and octonionic spinors (depending on the spacetime, either within or without a Weyl projection). The scheme of this section is therefore the following, at first the necessary ingredients to define (division-algebra valued) generalized supersymmetries are introduced. Next, the two cases of quaternionic and octonionic supersymmetries are more closely analyzed. We especially focus on results concerning the $D = 11$-dimensional spacetimes, mostly because this is the space-time dimensionality where the supposed $M$-theory should live.

In terms of $n$-component real spinors $Q_{a}$, the most general real supersymmetry algebra is represented by

$$\{Q_{a}, Q_{b}\} = \mathcal{Z}_{ab},$$

(19)

where the matrix $\mathcal{Z}$ appearing in the r.h.s. is the most general $n \times n$ symmetric matrix with total number of $\frac{n(n+1)}{2}$ components. For any given space-time we can easily compute its associated decomposition of $\mathcal{Z}$ in terms of the antisymmetrized products of
\( k\)-Gamma matrices, namely

\[
\mathcal{Z}^{\sigma}_{ab} = \sum_k (A \Gamma_{[\mu_1\ldots\mu_k]} )_{ab} Z^{[\mu_1\ldots\mu_k]} ,
\]

where the values \( k \) entering the sum in the r.h.s. are restricted by the symmetry requirement for the \( a \leftrightarrow b \) exchange and are specific for the given spacetime. The coefficients \( Z^{[\mu_1\ldots\mu_k]} \) are the rank-\( k \) abelian tensorial central charges.

In the Minkowskian \((10,1)\) space-time, supporting 32-component real spinors, the bosonic r.h.s. is split into the \( 11 + 55 + 462 = 528 \) bosonic components sectors \( M_1 + M_2 + M_5 \), where the \( k \) in \( M_k \), for \( k = 1,2,5 \), specifies the level of the rank-\( k \) antisymmetric tensors.

When the the fundamental spinors entering the supersymmetry algebra belong to a division algebra other than the real one (this is evidently true in the quaternionic and octonionic cases), an extra possibility is available. The most general supersymmetry algebra can be expressed in terms of anticommutators among the fundamental spinors \( Q_a \) and their conjugate \( Q^\ast_\dot{a} \), where the conjugation refers to the principal conjugation in the given division algebra. One should remember that the principal conjugation, restricted to real spinors, acts as the identity, see (5). In the quaternionic and octonionic (as well as complex) cases we have

\[
\{Q_a, Q_b\} = \mathcal{Z}^{\sigma}_{ab} \quad , \quad \{Q^\ast_\dot{a}, Q^\ast_\dot{b}\} = \mathcal{Z}^{\ast*}_{\dot{a}\dot{b}},
\]

(21)

together with

\[
\{Q_a, Q^\ast_\dot{b}\} = \mathcal{W}^{\ast*}_{ab},
\]

(22)

where the matrix \( \mathcal{Z}^{\sigma}_{ab} \) (\( \mathcal{Z}^{\ast*}_{\dot{a}\dot{b}} \) is its conjugate and does not contain new degrees of freedom) is symmetric, while \( \mathcal{W}^{\ast*}_{ab} \) is hermitian.

Two big classes of subalgebras, respecting the Lorentz-covariance, can be obtained from (21) and (22) by imposing division-algebra constraints, obtained by setting identically equal to zero either \( \mathcal{Z} \) or \( \mathcal{W} \), namely \( \mathcal{Z}^{\sigma}_{ab} \equiv \mathcal{Z}^{\ast*}_{\dot{a}\dot{b}} \equiv 0 \), so that the only bosonic degrees of freedom enter the hermitian matrix \( \mathcal{W}^{\ast*}_{ab} \) or, conversely, \( \mathcal{W}^{\ast*}_{ab} \equiv 0 \), so that the only bosonic degrees of freedom enter \( \mathcal{Z}^{\sigma}_{ab} \) and its conjugate matrix \( \mathcal{Z}^{\ast*}_{\dot{a}\dot{b}} \). The first type of constraint will be referred as the one giving rise to the “hermitian” generalized supersymmetries, while the generalized supersymmetries satisfying the second constraint will be referred to as “holomorphic” generalized supersymmetries.

Several other constraints can be imposed, for instance one can consistently set, for complex spinors, the matrix \( Z \) entering (21) to be real. However, for our purposes, it is enough to concentrate on hermitian and holomorphic supersymmetries.

**Quaternionic supersymmetries**

Both the hermitian and holomorphic quaternionic supersymmetries can be classified with the help of tables specifying the number and type of bosonic elements (abelian tensorial central charges of rank \( k \)) entering the r.h.s. It is worth noticing that the
results do not depend on the signature of the associated space-time, but only on its
dimensionality $D$, provided of course that the associated spacetime is actually carrying
quaternionic spinors.

For what concerns the quaternionic hermitian supersymmetry we get

| spacetime | bosonic sectors | bosonic components |
|-----------|----------------|-------------------|
| $D = 3$   | $M_0$          | 1                 |
| $D = 4$   | $M_0$          | 1                 |
| $D = 5$   | $M_0 + M_1$    | $1 + 5 = 6$       |
| $D = 6$   | $M_1$          | 6                 |
| $D = 7$   | $M_1 + M_2$    | $7 + 21 = 28$     |
| $D = 8$   | $M_2$          | 28                |
| $D = 9$   | $M_2 + M_3$    | $36 + 84 = 120$   |
| $D = 10$  | $M_3$          | 120               |
| $D = 11$  | $M_0 + M_3 + M_4$ | $1 + 165 + 330 = 496$ |
| $D = 12$  | $M_0 + M_4$    | $1 + 495 = 496$   |
| $D = 13$  | $M_0 + M_1 + M_4 + M_5$ | $1 + 13 + 715 + 1287 = 2016$ |

The last column denotes the number of bosonic components entering the rank-$k$ decomposition. It is worth noticing that the hermitian quaternionic supersymmetry saturates the bosonic sector.

This property is not hold by the holomorphic quaternionic supersymmetry. The reason can be traced to the fact that if we try implementing transposition on imaginary quaternions we are in conflict with their product since, e.g., $(e_1 \cdot e_2)^T = e_2^T \cdot e_1^T = -e_3 \neq e_3^T$. Indeed the only consistent operation respecting the composition law would correspond to setting $e_i^T = -e_i$, but this in fact coincides with the principal conjugation employed in the construction of quaternionic hermitian matrices and quaternionic hermitian super-
symmetries. The holomorphic analog of the previous table is given by

| spacetime $D$ | bosonic sectors $M_0 + M_1$ | bosonic components $1 + 3$ |
|---------------|-----------------------------|---------------------------|
| $D = 3$       | $M_0 + M_1$                 | $1 + 3 = 4$               |
| $D = 4$       | $M_1$                       | $4$                       |
| $D = 5$       | $M_1$                       | $5$                       |
| $D = 6$       | $-$                         | $-$                       |
| $D = 7$       | $-$                         | $-$                       |
| $D = 8$       | $-$                         | $-$                       |
| $D = 9$       | $M_0$                       | $1$                       |
| $D = 10$      | $M_0 + M_1$                 | $1 + 10 = 11$             |
| $D = 11$      | $M_0 + M_1$                 | $1 + 11 = 12$             |
| $D = 12$      | $M_1$                       | $12$                      |
| $D = 13$      | $M_1$                       | $13$                      |

The above results can be interpreted as follows: quaternionic holomorphic supersymmetry cannot admit bosonic tensorial central charges of rank $k \geq 2$. At most a single bosonic central charge ($M_0$), depending on the dimensionality of the space-time, can exist. In some dimensions, no consistent quaternionic holomorphic supersymmetry can be defined.

From physical point of view, so far, the most interesting application of supersymmetries of quaternionic spacetimes does not directly concern the quaternionic supersymmetry, but the complex holomorphic supersymmetry which can be realized with the quaternionic spinors entering the 11-dimensional quaternionic spacetime $(0,11)$. The bosonic components correspond to the 528 bosonic components of the real $M$ algebra and the 11$D$ complex Euclidean holomorphic supersymmetry can be regarded as the Euclideanized version of the $M$ algebra, see [14].

### Octonionic supersymmetries

Let us discuss now the peculiar features of the octonionic supersymmetries which are consequences of the non-associativity of octonions. The octonionic supersymmetries exist for the spacetimes entering the (9) table. It is worth mentioning that here we limit ourselves to consider only “hermitian” octonionic supersymmetries.

In a $D$-dimensional spacetime described in terms of the octonions, $D − 7$ Clifford Gamma matrices are purely real, while the remaining 7 of them are given by the imaginary octonions $e_i$, $i = 1, 2, \ldots, 7$, multiplying a common real matrix. In describing the antisymmetric product of $k > 2$ octonionic $\Gamma$-matrices a correct prescription must be specified to take into account the non-associativity of the octonions. As a matter of fact, the correct prescription can be induced by assuming a given prescription for the antisymmetrized product of $k > 2$ imaginary octonions $e_i$. The correct prescription can
be uniquely specified by assuming the validity of the Hodge duality and an irreducibility requirement, namely that the rank-\(k\) antisymmetric product of \(k\) imaginary octonions are either proportional to the octonionic identity or to the imaginary octonions. In full generality, this prescription corresponds at taking the following antisymmetrized product of \(k\) octonionic Gamma matrices

\[
[\Gamma_1 \cdot \Gamma_2 \cdot \ldots \cdot \Gamma_k] \equiv \frac{1}{k!} \sum_{perm.} (-1)^{\varepsilon_{i_1 \ldots i_k}} (\Gamma_{i_1} \cdot \Gamma_{i_2} \cdot \ldots \cdot \Gamma_{i_k}),
\]

(25)

where \((\Gamma_1 \cdot \Gamma_2 \cdot \ldots \cdot \Gamma_k)\) denotes the symmetric product

\[
(\Gamma_1 \cdot \Gamma_2 \cdot \ldots \cdot \Gamma_k) \equiv \frac{1}{2}((\Gamma_1 \Gamma_2)\Gamma_3 \ldots )\Gamma_k + \frac{1}{2}(\Gamma_1(\Gamma_2(\ldots \Gamma_k))).
\]

(26)

The usefulness of this prescription is due to the fact that the product

\[
A[\Gamma_1 \cdot \Gamma_2 \cdot \ldots \cdot \Gamma_k],
\]

(27)

(where \(A\) is the matrix, product of the time-like Gamma matrices, already introduced at the beginning of this section) has a definite (anti)-hermiticity property. The different tensors, for different choices of the Gamma’s, are all hermitian or antihermitian, depending only on the value of \(k\) and not of the \(\Gamma\)’s themselves. In odd-dimensions \(D\) we get the table, whose columns are labeled by the antisymmetric tensors rank \(k\), specifying the number of independent bosonic components in each rank-\(k\) antisymmetric product (25).

| \(D = 7\) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|----------|---|---|---|---|---|---|---|---|---|---|----|----|----|----|
| \(D = 9\) | 1 | 1 | 1 | 1 | 7 | 7 | 7 | 1 |   |   |    |    |    |    |
| \(D = 11\) | 1 | 11 | 41 | 75 | 76 | 52 | 52 | 76 | 75 | 41 | 11 | 1 |    |    |
| \(D = 13\) | 1 | 13 | 64 | 168 | 267 | 279 | 232 | 232 | 279 | 267 | 168 | 64 | 13 | 1 |

(28)

An analogous table can be produced in even-dimensional spacetimes as well.

In the above table the \(k\) sectors corresponding to hermitian matrices (and therefore entering the r.h.s. of a generalized supersymmetry) are underlined.

The table above shows the existence of identities relating higher-rank antisymmetric octonionic tensors. Let us discuss the \(D = 11\) example. The 52 independent components of an octonionic hermitian \((4 \times 4)\) matrix can be expressed either as a rank-5 antisymmetric tensors (simbolically denoted as “\(M5\)”), or as the combination of the 11 rank-1 \((M1)\) and the 41 rank-2 \((M2)\) tensors. The relation between \(M1 + M2\) and \(M5\) can be made explicit as follows. The 11 vectorial indices \(\mu\) are split into 4 real indices, labeled
by \( a, b, c, \ldots \) and 7 octonionic indices labeled by \( i, j, k, \ldots \). We get, on one side,

\[
\begin{align*}
4 & : M1_a \\
7 & : M1_i \\
6 & : M2_{[ab]}
\end{align*}
\]

\[4 \times 7 = 28\]

\[M2_{[ai]} \equiv M2_i\]

while, on the other side,

\[
\begin{align*}
7 & : M5_{[abcdi]} \equiv M5_i \\
4 \times 7 = 28 & : M5_{[abci]} \equiv M5_{[ai]} \\
6 & : M5_{[abijk]} \equiv M5_{[ab]} \\
4 & : M5_{[aijkl]} \equiv M5_a \\
7 & : M5_{[ijklm]} \equiv M5_i
\end{align*}
\]

which shows the equivalence of the two sectors, as far as the tensorial properties are concerned. Please notice that the correct total number of 52 independent components is recovered

\[
52 = 2 \times 7 + 28 + 6 + 4.
\]

(29)

The octonionic equivalence of different antisymmetric tensors can be symbolically expressed, in odd space-time dimensions, through

\[
\begin{array}{|c|c|}
\hline
D = 7 & M0 \equiv M3 \\
D = 9 & M0 + M1 \equiv M4 \\
D = 11 & M1 + M2 \equiv M5 \\
D = 13 & M2 + M3 \equiv M6 \\
D = 15 & M3 + M4 \equiv M0 + M7 \\
\hline
\end{array}
\]

(30)

The octonionic \( M \)-algebra

We are now in the position to introduce the octonionic \( M \) algebra [10].

It corresponds to replace the real supersymmetry algebra in the \((10, 1)\) spacetime, given by

\[
\{Q_a, Q_b\} = (A \Gamma_\mu)_{ab} P^\mu + (A \Gamma_{[\mu \nu]})_{ab} Z^{[\mu \nu]} + (A \Gamma_{[\mu_1 \ldots \mu_5]})_{ab} Z^{[\mu_1 \ldots \mu_5]}
\]

(31)

with its two octonionic-valued variants, given by 4-component octonionic spinors \( Q_a \) (together with their conjugate spinors \( Q^*_b \)) and the 52 octonionic-valued \( 4 \times 4 \) hermitian matrices which can be expressed, either as the 11 + 41 bosonic generators entering

\[
\{Q_a, Q^*_b\} = P^\mu (A \Gamma_\mu)_{ab} + Z^{\mu \nu}_O (A \Gamma_{\mu \nu})_{ab},
\]

(32)
or as the 52 bosonic generators entering

\[
\{ Q_a, Q^*_b \} = Z^{[\mu_1...\mu_5]}_O (A \Gamma_{\mu_1...\mu_5})_{ab}.
\] (33)

Associated to the above octonionic \( M \) algebra, its superconformal extension, given by the \( Osp(1,8|\mathbb{O}) \) superalgebra, can be constructed \cite{11}.

**CONCLUSIONS**

In this paper we have quickly reviewed the fundamental issues concerning the employment of quaternionic and octonionic spinors. In particular we have described a general construction which allows us to specify the spacetimes supporting quaternionic and octonionic spinors. The specific problems raised by the non-associativity of the octonions have been discussed and clarified. It was shown, in particular, that octonionic spinors naturally encode the geometry of the Euclidean seven-sphere \( S^7 \).

With our tools we were able to construct and classify all free Dirac-type equations involving quaternionic and octonionic spinors. The concepts of Weyl spinors, the presence of (pseudo)-kinetic and (pseudo)-massive terms in association with different space-times have been fully investigated and the complete list of results has been reported.

In the last part of this talk we took a further step. In view of studying the possible physical consequences of the formalism here introduced we applied the above investigation to the construction and the classification of the generalized supersymmetries supported by quaternionic and octonionic spinors. Some recent results on this subject have been reported, like the notion of division-algebra constrained (in the quaternionic case) hermitian and holomorphic supersymmetries.

By far the most intriguing possible application of the ideas related with the octonionic spinors concern the \( M \)-theory investigations, as suggested in \cite{10} and \cite{11}. Indeed, it is quite remarkable that an octonionic structure can be introduced, instead of the standard real structure, in defining the (octonionic version of) the \( M \)-algebra. Peculiar identities, relating different rank-\( k \) antisymmetric tensors of the bosonic sectors, are an absolute novel feature of the octonionic formulation, finding no counterparts in the standard formulation. It is worth noticing that, in a somewhat different context, octonions have been suggested \cite{8} to be linked to a possible exceptional formulation \cite{12} for a single unifying theory of all interactions.

The investigations concerning the dynamics of octonionic spinors are a necessary preliminary step to unveil this challenging and fascinating present area of research.

**ACKNOWLEDGMENTS**

The present paper reports results based on a series of works done in collaboration with J. Lukierski and with H.L. Carrion and M. Rojas, who I am pleased to acknowledge.
REFERENCES

1. Porteous, I.R., *Clifford Algebras and the Classical Groups*, Cambridge Un. Press, 1995.
2. Okubo, S., *J. Math. Phys.* **32**, 1657 (1991); *ibid.* **32**, 1669 (1991).
3. Carrion, H.L., Rojas, M. and Toppan, F., JHEP04 (2003) 040.
4. Kugo, T. and Townsend, P., *Nucl. Phys.* **B 221**, 357 (1983).
5. Baez, J., *The Octonions*, math.RA/0105155.
6. Barton, C.A. and Sudbery, T., math.RA/0203010.
7. Günaydin, M. and Güreşy, F., *Lett. Nuovo Cim.* **6** (1973), 401.
8. Boya, L., *Octonions and M-theory*, hep-th/0301037.
9. Fairlie, D.B. and Manogue, A.C., *Phys. Rev.* **D 34**, 1832 (1986).
10. Lukierski, J. and Toppan, F., *Phys. Lett.* **B 539**, 266 (2002).
11. Lukierski, J. and Toppan, F., *Phys. Lett.* **B 567**, 125 (2003).
12. Ramond, P., *Algebraic Dreams*, hep-th/0112261.
13. Toppan, F., JHEP09 (2004) 016.
14. Lukierski, J. and Toppan, F., *Phys. Lett.* **B 584**, 315 (2004).
15. Lukierski, J. and Minnaert, P., *Phys. Lett.* **B 129** (1983), 392.