On the $C^{8/3}$-regularisation of simultaneous binary collisions in the planar four-body problem

Nathan Duignan$^1$,∗ and Holger R Dullin$^2$

$^1$ Department of Applied Mathematics, University of Colorado, Boulder, CO 80309-0526, United States of America
$^2$ School of Mathematics and Statistics, University of Sydney, Camperdown, 2006 NSW, Australia

E-mail: nathan.duignan@colorado.edu

Received 11 February 2021, revised 9 May 2021
Accepted for publication 13 May 2021
Published 25 June 2021

Abstract
The dynamics of the four-body problem allows for two binary collisions to occur simultaneously. It is known that in the collinear four-body problem this simultaneous binary collision (SBC) can be block-regularised, but that the resulting block map is only $C^{8/3}$ differentiable. In this paper, it is proved that the $C^{8/3}$ differentiability persists for the SBC in the planar four-body problem. The proof uses several geometric tools, namely, blow-up, normal forms, dynamics near normally hyperbolic manifolds of equilibrium points, and Dulac maps.

Keywords: normally hyperbolic, blow-up, collisions, regularisation, celestial mechanics, normal forms
Mathematics Subject Classification numbers: 70F16, 70F10.

1. Introduction
The $N$-body problem has always driven progress in dynamical systems. This is spectacularly evidenced by Poincaré’s discovery of chaos in the three-body problem, see, e.g., Chenciner’s recent review [6]. The extraordinary richness of the dynamics of the three-body problem, as proved by Moeckel and Montgomery [26], can be traced in part to the dynamics near the triple collision. While the binary collision has been known classically [21] to be regularisable, McGehee showed [25] that the triple collision cannot be regularised. Consequently, it serves as a building block of complicated dynamics. A new type of collision appears in the four-body problem, where two binary collisions can occur in separate locations in space, but at the same
time. The regularisability of this simultaneous (or double) binary collision (SBC) in the planar four-body problem is the subject of this paper. It has been conjectured by Simó and Martínez [23, 33] that the SBC can be regularised, but, only in such a way that the regularisation is finitely smooth, precisely $C^{8/3}$.

Initial investigations by Simó and Lacomba [34], as well as Belbruno [3], revealed that the SBC is topologically regularisable. Based on numerical evidence Simó and Martínez [23] conjectured the regularisation was only $C^{8/3}$ and, in a paper by the same authors [24], they proved that this $C^{8/3}$ differentiability holds in the collinear four-body problem. More precisely, the finite differentiability means that an isolating block, as introduced by Conley and Easton [7, 13], can be constructed around the SBC and the consequent block map can be extended to a map that is at most, and at least, $C^{8/3}$ differentiable at orbits going to collision. For further context on the problem and details on relevant theory see [8].

The primary achievement of this paper is to extend the $C^{8/3}$ result to the planar four-body problem in theorem 4.12. Whilst it is immediate from the collinear result that collinear collision orbits in the planar problem will exhibit this finite differentiability, we show that the block map at non-collinear collision orbits is still at least $C^{8/3}$.

Before describing in more detail the strategy of the proof we would like to explain the essence of the mechanism in a toy model first introduced in [10]. How can non-smoothness arise in a smooth (or even polynomial) planar dynamical system? The answer to this question as described in [10] is: through a block map past a degenerate equilibrium point. A degenerate equilibrium point can, e.g., be such that the negative $x$-axis is the stable manifold and the positive $x$-axis is the unstable manifold. In this case, a Poincaré map between transversal sections to the $x$-axis can be defined. In [10] it was shown that this map generically cannot be extended to $y = 0$ in a smooth way. The reason is that orbits passing the equilibrium with $y > 0$ may be sufficiently different from those passing the equilibrium with $y < 0$. This difference can be precisely characterised after replacing the degenerate point with a copy of $S^1$; a so called blow-up. In the collinear problem the fixed point is replaced with a manifold of fixed points, which are in turn replaced by copies of $S^1$ [11]. The story is similar in the planar problem, however, the dimensionality of the problem requires the blow up to replace points by a three-dimensional manifold instead of $S^1$.

A precursor of this work is [11], where the SBC in the collinear four-body problem is treated. A key difference to the first proof given in [24] is that the proof of [11] is more geometric and allows for a generalisation to the planar problem. The idea behind this previous work and the proof for the planar problem is the same. After regularising the simple binary collisions, scaling time, and using the (approximately constant) energy of the binaries as a coordinate, the consequent vector field of the four-body problem has a manifold of degenerate equilibria corresponding to SBC for which the linear part of the vector field vanishes. A normal form procedure and a blow-up allows the construction of an asymptotic expansion of the block map. However, due to the higher codimension of the manifold of equilibria in the planar compared to the collinear problem, some nontrivial extensions of the normal form theory, blow-up procedure, and asymptotics of the transition map have been made.

The paper is structured as follows. In section 2 we introduce generalised Levi-Civita coordinates which include the energy of the binaries and two complex coordinates $\zeta_1, \zeta_2$ whose imaginary part is the (approximately constant) angular momentum of the binaries. This generalises work of Elbialy [16] on the collinear problem.

To resolve the dynamics near the degenerate equilibria $\mathcal{C}$ (the set of collision points) a blow-up is performed in section 3, which augments $\mathcal{C}$ to obtain a smooth collision manifold $\mathcal{C}$. In prior proofs of the $C^0$-regularity, McGehee type coordinates were used to blow-up the
singularity [15, 23]. Essentially, in these coordinates \(C\) is augmented with copies of \(S^3\) to produce the collision manifold. We will instead use a blow-up procedure which augments \(C\) with copies of \(\mathbb{RP}^3\). This will be shown to have several benefits; the regularisation of isolated binary collisions, a relatively simple argument proving the union of the collision and ejection orbits forms a smooth manifold, and that the flow on the collision manifold is integrable. Ultimately, this blow-up procedure shows that the collision manifold \(C\) is foliated by the homoclinic connections of a manifold of normally hyperbolic saddle singularities. A new proof of the \(C^0\)-regularity follows.

To simplify the analysis in section 4, a nonlinear formal normal form is computed at an arbitrary degenerate equilibrium point. Since the equilibrium points are degenerate, a non-standard type of normal form due to [37] is used. The homological operator is not an automorphism in this case as the leading order terms are quadratic instead of linear. Moreover, we use a restriction of the quadratic part to the collinear problem to construct the homological operator which ultimately produces approximate integrals near SBC. The final step is the computation of the block map through the composition of three simpler maps obtained in the blown up system. The computation of these so called Dulac maps is based on work of [12, 27, 29] and its recent extension from fixed points in the plane to manifolds of fixed points [9] in \(\mathbb{R}^n\).

The road-map just described provides insight into why the smoothness is a peculiar 8/3. The blow up generates hyperbolic equilibria that have resonant eigenvalues 1 and 3. The corresponding Dulac maps have components whose leading order consists of powers 1/3 and 3. If the transition map describing the dynamics along the homoclinic connection of the hyperbolic point would be the identity map these powers would cancel and the composed map would be \(C^\infty\) smooth. Instead, the first terms in the normal form that cannot be removed appear at order 9, and this results in a term of order 8 in the transition map. The composition of these maps hence gives a map that is only \(C^{8/3}\). The appearance of the resonant terms in normal form can also be interpreted as the absence of an invariant foliation of a neighbourhood of the manifold of equilibria.

2. Coordinates near simultaneous binary collision

2.1. The planar four-body problem

Suppose there are four bodies in the plane consisting of two binaries undergoing collision in different regions of configuration space at precisely the same time \(t_c\). Further, suppose that the bodies with mass \(m_1\) and \(m_2\) undergo one of the binary collisions and bodies with masses \(m_3\) and \(m_4\) undergo the other. We will call these two binaries the distressed binaries. Let the difference vector between the bodies in each binary be given by \(Q_1, Q_2 \in \mathbb{C}\) respectively and let \(x \in \mathbb{C}\) be the difference vector between the two centre of masses of the binaries. The coordinates are depicted in figure 1. If \(P_1, P_2, y \in \mathbb{C}^*\) are the conjugate momenta of \(Q_1, Q_2, x\), the dynamics is given by the Hamiltonian,

\[
H(Q, x, P, y) = \sum_{j=1}^{2} \left( \frac{1}{2M_j} |P_j|^2 - k_j |Q_j|^{-1} \right) + \frac{1}{2} \mu |y|^2 - \hat{K}(Q_1, Q_2, x),
\tag{2.1}
\]

with symplectic form \(\omega = d\Theta\), and \(\Theta\) is the tautological one-form on \(T^* \mathbb{C}^3\), \(\Theta = \text{Re} (P_1 dQ_1 + P_2 dQ_2 + y dx)\). The function \(\hat{K}\) contains the potential terms coupling the two binaries and is smooth in a neighbourhood of SBC which occurs at \(Q_1 = Q_2 = 0\). Each of
Figure 1. The configuration variables near SBC.

$M_j, k_j, \mu, d_j, c_j > 0$ are constant functions of the masses given explicitly:

\[
\hat{K} = \frac{d_1}{|x + c_2 Q_1 - c_4 Q_2|} + \frac{d_2}{|x + c_2 Q_1 + c_3 Q_2|} + \frac{d_3}{|x - c_1 Q_1 - c_4 Q_2|} + \frac{d_4}{|x - c_1 Q_1 + c_3 Q_2|},
\]

\[
M_1 = \frac{m_1 m_2}{m_1 + m_2}, \quad M_2 = \frac{m_3 m_4}{m_3 + m_4}, \quad k_1 = m_1 m_2, \quad k_2 = m_3 m_4, \quad (2.2)
\]

\[
\mu = \frac{m_1 + m_2 + m_3 + m_4}{(m_1 + m_2)(m_3 + m_4)},
\]

\[
d_1 = m_1 m_3, \quad d_2 = m_1 m_4, \quad d_3 = m_2 m_3, \quad d_4 = m_2 m_4,
\]

\[
c_1 = m_1^{-1} M_1, \quad c_2 = m_1^{-1} M_1, \quad c_3 = m_2^{-1} M_2, \quad c_4 = m_3^{-1} M_2.
\]

This choice of coordinates achieves a reduction of the system by translational symmetry. The coordinates have been chosen so that the mass metric is diagonal.

It is more convenient to work with rescaled variables $\tilde{Q}_j, \tilde{P}_j$ via a symplectic transformation

\[
\tilde{Q}_j = 4k_j M_j Q_j, \quad \tilde{P}_j = (4k_j M_j)^{-1} P_j. \quad (2.3)
\]

The rescaled Hamiltonian is then

\[
H(\tilde{Q}, x, \tilde{P}, y) = \sum_{j=1}^{2} \frac{1}{2} a_j \left( |\tilde{P}_j|^2 - \frac{1}{2} |\tilde{Q}_j|^{-1} \right) + \frac{1}{2} \mu |y|^2 - \tilde{K}(\tilde{Q}, x), \quad (2.4)
\]

where $a_j = 16k_j^2 M_j$ and $\tilde{K}(\tilde{Q}, x) = \tilde{K} \left( (4k_j M_j)^{-1} \tilde{Q}, x \right)$.

The system has rotational symmetry and hence the total angular momentum $\text{Im}(Q_1 P_1) + \text{Im}(Q_2 P_1) + \text{Im}(xy)$ is conserved. Moreover, the usual scaling symmetry of the four-body problem in these variables reads

\[
Q_i \rightarrow s Q_i, \quad x \rightarrow s x, \quad P_i \rightarrow s^{-1/2} P_i, \quad y \rightarrow s^{-1/2} y, \quad H \rightarrow H/s. \quad (2.5)
\]
2.2. Levi-Civita regularisation of binary collisions

The next stage of transformations regularise the binary collision of each distressed binary. This is done by passing to the Levi-Civita variables via the symplectic map

\[ \tilde{Q}_j = \frac{1}{2} \tilde{z}_j^2, \quad \tilde{P}_j = \tilde{z}_j^{-1} u_j \]

resulting in the partially regularised, translation reduced Hamiltonian,

\[ H(\tilde{z}, x, y) = \sum_{j=1}^{2} \frac{1}{2} |u_j| \tilde{z}_j^{-2} (|u_j|^2 - 1) + \frac{1}{2} \mu |y|^2 - \tilde{K} \tilde{z}_1, \tilde{z}_2; x). \tag{2.6} \]

with \( \tilde{K}(\tilde{z}_1, \tilde{z}_2, x) := K(\frac{1}{2} \tilde{z}_1^2, \frac{1}{2} \tilde{z}_2^2 - x) \).

Time can be rescaled to \( dt = |\tilde{z}_1|^2 |\tilde{z}_2|^2 d\tau \) by using the Poincaré trick of moving to extended phase space and restricting to a constant energy surface. That is, by constructing the new Hamiltonian

\[ H(\hat{z}, x, y) = |\tilde{z}_1|^2 |\tilde{z}_2|^2 (H(\tilde{z}, x, y) - h) \]

\[ = \frac{1}{2} \mu |y|^2 - \tilde{K} \tilde{z}_1, \tilde{z}_2; x) \tag{2.7} \]

and restricting to a constant energy surface in the original Hamiltonian, \( H = h \), yielding \( H = 0 \). The flow on \( H = 0 \) is equivalent to the flow on \( H = h \) up to time rescaling. The Hamiltonian \( H \) is regular at \( \tilde{z}_1 = 0 \) or \( \tilde{z}_2 = 0 \) and so the binary collision singularities have been regularised. However, the set of SBCs \( \tilde{z}_1 = \tilde{z}_2 = 0 \), denoted by \( \mathcal{C} \), is a critical point of \( \mathcal{H} \) and hence the associated Hamiltonian vector field has fixed points at \( \mathcal{C} \). Essentially, in rescaling time to regularise the binary collisions, the vector field was over scaled near \( \mathcal{C} \), slowing down orbits as they approach the singularity and creating a manifold of equilibria. In the new time \( \tau \), collision (resp. ejection) orbits approach \( \mathcal{C} \) as \( \tau \to \infty \) (resp. \( \tau \to -\infty \)).

The Hamiltonian (2.7) is canonical and regular at collision. To distinguish orbits in the collision limit we need to introduce the intrinsic energies of the distressed binaries as non-canonical coordinates. This is done in the next section. As a result the further analysis will be done directly on the level of the vector field instead of on the level of the Hamiltonian. Before we do so, let us recall a classical result about the collision limit which is best stated in the canonical coordinates of this section.

Let \( \hat{C} \) be the set of unit complex numbers and denote by \( \Gamma_j = \exp(i \text{ arg}(u_j)) \in \hat{C} \) the angular component of \( u_j \). The following proposition gives the asymptotic behaviour of collision and ejection orbits. Analogous propositions have been proved in many works, for example, \([14, 23]\).

**Proposition 2.1.** Suppose that \( (\tilde{z}_1(\tau), \tilde{z}_2(\tau), x(\tau), u_1(\tau), u_2(\tau), y(\tau)) \) is a collision (resp. ejection) orbit. Let \( x^*, y^* \in C \), \( \Gamma_1^*, \Gamma_2^* \in \hat{C} \) and denote by \( |\tilde{z}|^2 = |\tilde{z}_1|^2 + |\tilde{z}_2|^2 \). Then,

\[ u_j \to \Gamma_j^*, \quad \frac{\tilde{z}_j}{|\tilde{z}|} \to a_j^{1/2} \Gamma_j^*, \quad x \to x^*, \quad y \to y^*, \]

as \( \tau \to \infty \) (resp. \( \tau \to -\infty \)). Moreover, the set of orbits asymptotic to any choice of \( x^*, y^*, \Gamma_1^*, \Gamma_2^* \) forms a real three-dimensional manifold.
A geometrical proof can be constructed using the methods of blow-up and desingularisation. In fact, it follows immediately from proposition 3.2 in section 3.

2.3. Generalised Levi-Civita coordinates

It has been argued in [11] that the Levi-Civita coordinates are not ideal for analysis of orbits local to a SBC in the collinear four-body problem. The primary issue arises from the dimensionality of the asymptotic orbits given in proposition 2.1. In order for \( \mathcal{C} \) to be \( C^k \)-regularisable, there must exist a map \( \pi \) taking each collision orbit to a unique ejection orbit. However, from proposition 2.1, each point in \( \mathcal{C} \) has a three-dimensional manifold of asymptotic orbits. Consequently, uniqueness can not be upheld. In [11], the so called generalised Levi-Civita coordinates were introduced to resolve this problem. In that work, it became clear that taking approximate integrals as coordinates was advantageous to understanding the dynamics near collision. In particular, for the collinear problem, the intrinsic energy of each binary was taken as a coordinate. This can be done analogously for the planar problem. Additionally, the intrinsic angular momentum of each binary \( \frac{1}{2} \text{Im}(\bar{u}_j \tilde{z}_j) \) is an approximate integral, thus a good choice of coordinate.

Introduce the intrinsic energy of each distressed binary

\[
\tilde{h}_j = \frac{1}{2} a_j |\tilde{z}_j|^{-2} (|u_j|^2 - 1),
\]

and the variables \( \tilde{\zeta}_j = \tilde{u}_j \tilde{z}_j \). Consider the rescaling

\[
\tilde{\zeta}_j = a_j^{-1/3} \zeta_j, \quad h_j = 2 a_j^{-1/3} \tilde{h}_j.
\] (2.9)

Then, the generalised Levi-Civita coordinates are given by \((\zeta_1, \zeta_2, h_1, h_2, \Gamma_1, \Gamma_2, x, y) \in \mathbb{C}^2 \times \mathbb{R}^2 \times \mathbb{C}^2 \times \mathbb{C}^2\). Notice that these are not canonical coordinates.

The Hamiltonian in the generalised Levi-Civita coordinates is,

\[
H = \frac{1}{2} a_1^{1/3} h_1 + \frac{1}{2} a_2^{1/3} h_2 + \frac{1}{2} |y|^2 - K(\zeta_1, \zeta_2, h_1, h_2, \Gamma_1, \Gamma_2, x),
\] (2.10)

\[
K(\zeta_1, \zeta_2, h_1, h_2, \Gamma_1, \Gamma_2, x) = \tilde{K} \left( a_1^{1/3} U_1(\zeta_1, h_1)^{-1} \Gamma_1 \zeta_1, a_2^{1/3} U_2(\zeta_2, h_2)^{-1} \Gamma_2 \zeta_2, x \right),
\]

and \( U_j(\zeta_j, h) \) is the unique solution to

\[
h_j = |\zeta_j|^{-2} U_j^2(U_j^2 - 1),
\] (2.11)

satisfying \( U_j(0, h) = 1 \). Note that \( U_j = |u_j| \) is the magnitude of the canonical momenta so that \( u_j = U_j \Gamma_j \). The condition \( U_j(0, h) = 1 \) is nothing more than ensuring \( |u_j| \to 1 \) as \( \tau \to \infty \) for a collision orbit as per proposition 2.1. Defining \( U_j \) through (2.11) is only invertible when \( U_j > 1/\sqrt{2} \). We restrict to a sufficiently small neighbourhood of the SBC \( \zeta_1 = \zeta_2 = 0 \) so this inequality holds.

The pullback of the symplectic form under the transformation to the intrinsic energies is

\[
\text{Pullback of symplectic form}
\]
\[ \omega = \text{Re}(d\bar{y} \wedge dx) + \sum_{j=1}^{2} \frac{-a_j^{1/3}}{2U_j^2(2U_j^2 - 1)} \left( |\zeta_j|^2 \text{Re} \left(d\zeta_j \wedge d\bar{\zeta}_j\right) \right) \\
+ i\hbar \text{Im} \left( \zeta_j \right) \left(d\zeta_j \wedge d\bar{\zeta}_j\right) + \frac{1}{2} a_j^{1/3} \Gamma_j \left( d\zeta_j - d\bar{\zeta}_j \right) \wedge d\Gamma_j \]

(2.12)

The associated Poisson bracket is found as

\[ \Pi = 4 \text{Re} \left( \partial_x \wedge \partial_y \right) + \sum 4a_j^{-1/3} U_j^{2} \left(2U_j^2 - 1 \right) \text{Re} \left( \partial_{\zeta_j} \wedge \partial_{\bar{\zeta}_j} \right) \\
+ 2ia_j^{-1/3} \Gamma_j \left( \frac{\hbar_j}{|\zeta_j|^2} \text{Im} \left( \zeta_j \right) \partial_{\zeta_j} \wedge \partial_{\bar{\zeta}_j} + \text{Im} \left( \partial_{\zeta_j} \right) \wedge \partial_{\bar{\zeta}_j} \right) \]

(2.13)

Hamilton’s equations \( X_H = \Pi(\cdot, H) \) gives the Hamiltonian vector field \( X_H \) for the planar four-body problem in the generalised Levi-Civita coordinates. Explicitly, \( X_H \) is given by

\[ \hat{\zeta}_j = \frac{U_j^2}{|\zeta_j|^2} + 2\hbar_j + 2ia_j^{-1/3} \text{Im} \left( \zeta_j \partial_{\zeta_j} K \right) \]

\[ \hat{\hbar}_j = \frac{4a_j^{-1/3} U_j^4}{|\zeta_j|^2} \text{Re} \left( \partial_{\zeta_j} K \right) \]

\[ \hat{\Gamma}_j = -\frac{i}{|\zeta_j|^2} \text{Im} \left( \hbar_j \zeta_j - 2a_j^{-1/3} |\zeta_j|^2 \partial_{\zeta_j} K \right) \]

(2.14)

By scaling \( X_H \) through a space dependent time rescaling \( dt = |\zeta_1|^2 |\zeta_2|^2 d\tau \), the single binary collisions at \( \zeta_j = 0 \) are regularised. Denote by \( X = |\zeta_1|^2 |\zeta_2|^2 X_H \) the rescaled vector field and let \( \dot{\cdot} \) denote a derivative with respect to the fictitious time \( \tau \). The rescaled vector field \( X \) is regular when \( \zeta_j \to 0 \). However, time is ‘over-scaled’, which means that the whole vector field \( X \) vanishes in the collision limit. What is worse, the family of equilibrium points with \( \zeta_1 = \zeta_2 = 0 \) has vanishing linear part. Thus, in the next section, blow-up is used to study the collision limit.

The following technical lemma is crucial in obtaining the form of \( X_H \) given in (2.14).

**Lemma 2.2.** The following relations hold.

\[ \text{Re} \left( \zeta_j \partial_{\zeta_j} K \right) = 0 \]

\[ \Gamma_j \partial_{\zeta_j} K = U_j \partial_{\zeta_j} K \]

\[ \Gamma_j \partial_{\bar{\zeta}_j} K = 2\zeta_j \partial_{\zeta_j} K \]

(2.15)

**Proof.** The proof is a computation. One method of attack is to introduce the intermediate transformation \( (z_j, u_j, x, y) \to (z_j, \hbar_j, \Gamma_j, x, y) \). In these coordinates we have that \( \partial_{\Gamma_j} K = \partial_{\hbar_j} K = 0 \). These two conditions in the \( (\zeta_1, \hbar_j, \Gamma_j, x, y) \) variables gives the first and last relations in the lemma. The second relation comes from pulling back the vector \( \partial_{\zeta_j} \) under the coordinate transformation and applying this to \( K \).

**Remark 2.3.** All the terms coupling the two distressed binaries from the potential can be removed by setting \( K = 0 \). In doing so, the dynamics of two uncoupled Kepler problems is
recovered. From (2.14), the equations of motion for the uncoupled problem are given by
\[\begin{align*}
\dot{\zeta}_j &= \frac{U_j^2}{|\zeta_j|^2} + 2h_j \\
\dot{h}_j &= 0 \\
\dot{\Gamma}_j &= \frac{i}{|\zeta_j|^2} \Gamma_j \text{Im} (h_j \zeta_j) \\
\dot{x} &= \mu y \\
\dot{y} &= 0.
\end{align*}\]
(2.16)

We recover from this that the (scaled) intrinsic energy \(h_j\), angular momentum \(\text{Im}(\zeta_j)\), and momenta \(y\) between the two binaries are conserved in this limit.

**Remark 2.4.** The collision limit described in proposition 2.1 expressed in the new variables reads \(\zeta_i \to |\tilde{\zeta}_i|\). In particular, \(\zeta_1 \to \zeta_2\) in the collision limit and that the \(\zeta_i\) are real. One way to permanently achieve real \(\tilde{\zeta}_i\) is to restrict to the collinear problem in which all imaginary parts vanish. Since \(\tilde{\zeta}_i = 2\tilde{Q}_i\tilde{P}_i = \tilde{Q}_i\tilde{P}_i\) another possibility to achieve real \(\zeta_i\) is to require that \(\tilde{Q}_i \perp \tilde{P}_i\). This ensures the angular momentum \(\text{Im} \, \tilde{Q}_i\tilde{P}_i\) vanishes even thought the dynamics is not collinear. The latter method occurs in the rectangular four-body problem. A more general invariant sub-problem is the so-called Caledonian problem [30, 31, 36]. There \(\tilde{Q}_1 = -\tilde{Q}_2\) and \(\tilde{P}_1 = -\tilde{P}_2\) and hence \(\tilde{\zeta}_1 = \tilde{\zeta}_2\). Moreover, the masses are pairwise equal so that the coefficients \(a_1 = a_2\) are the same.

### 3. \(C^0\)-regularity of block map

The primary aim of this section is to provide a proof of theorem 3.8 on the \(C^0\)-regularisation of SBCs. Blow-up and desingularisation on the set of collisions \(\mathcal{C}\) is implemented and a study of the flow on the resultant collision manifold \(\mathcal{C}\) provides the desired proof. In the process, the flow on the collision manifold is revealed to possess integrals that are related to the intrinsic angular momentum of each distressed binary, and to the original time variable. The blown up collision manifold \(\mathcal{C}\) has dimension 11, and the flow on it is not Hamiltonian. The non-constant dynamics takes place on \(\mathbb{RP}^3\), and we will show that there are two additional integrals that make the flow on \(\mathbb{RP}^3\) integrable.

#### 3.1. Blow-up and desingularisation

The use of blow-up in celestial mechanics was first introduced by McGehee [25] in his study of the triple collision. Later it was implemented in investigations of the SBC by Elbialy [14] and Martinez and Simó [23]. Both of these papers perform the blow-up by introducing a set of coordinates akin to the usual McGehee coordinates. Essentially, the McGehee coordinates replace each SBC with a copy of \(S^3\). A study of the resulting flow on each \(S^3\) provides topological properties of the flow near collision. However, in both works, the McGehee coordinates are taken before Levi-Civita regularisation. Consequently, each binary collision is not regularised and one needs to perform a more trying analysis of the flow on each \(S^3\) to prove statements like the \(C^0\) regularity. Instead of using the McGehee coordinates, a blow-up procedure after Levi-Civita regularisation will be implemented. Further, the more algebraic route of blowing up each collision with a copy of \(\mathbb{RP}^3\) is followed; see [20] for more details. The advantage of
blowing up with $\mathbb{RP}^3$ instead of $S^3$ is both the avoidance of trig functions and the relatively simple proof of corollary 3.3 which shows the set of collision and ejection orbits is a smooth manifold.

Before introducing the blow-up for the SBC, it will be useful to first describe the blow-up procedure for a point in $C^2 \cong \mathbb{R}^4$. Take $(\zeta_1, \zeta_2) \in C^2$, let $\zeta_1 = \zeta_11 + i\zeta_{12}, \zeta_2 = \zeta_{21} + i\zeta_{22}$, and denote $\zeta := (\zeta_{11}, \zeta_{12}, \zeta_{21}, \zeta_{22}) \in \mathbb{R}^4$, $\alpha := [\alpha, \beta, \gamma, \delta] \in \mathbb{R}^3$. Define the four-dimensional manifold,

$$B := \{ (\zeta, \alpha) \in \mathbb{R}^4 \times \mathbb{RP}^3 \mid \exists r \in \mathbb{R} \text{ s.t. } \zeta = r\alpha \},$$

the blow-up space or blow-up of $\mathcal{C}$.

There is a natural projection

$$\Psi : B \to \mathbb{R}^4, \quad (\zeta, \alpha) \to \zeta.$$ (3.1)

As $\Psi$ is differentiable the push forward $\Psi_* : TB \to T\mathbb{R}^4$ induces a vector field $X_B$ on the blow-up space $B$ if we require $\Psi_* (X_B) = X$. Note that $\Psi$ is an isomorphism outside of $0 \in \mathbb{R}^4$ and loses injectivity on the manifold $\mathcal{C} := \Psi^{-1}(0) \cong \mathbb{RP}^3$, often referred to as the exceptional divisor. The primary achievement of the blow-up is to replace the point at $0 \in \mathbb{R}^4$ with the manifold $\mathbb{RP}^3$ whilst maintaining diffeomorphic conjugacy between the flow of $X$ on $\mathbb{R}^4 \setminus \{0\}$ and of $X_B$ on $B\setminus \mathcal{C}$.

The blow-up method just described can easily be extended to construct a blow-up of the manifold of SBCs $\mathcal{C} \equiv \mathbb{R}^2 \times \hat{C}^2 \times C^2 := \mathcal{M}$ with coordinates $(h_1, h_2) \in \mathbb{R}^2$, $(\Gamma_1, \Gamma_2) \in \hat{C}^2$ and $(x, y) \in C^2$. It is done by performing the map $\Psi$ for each point in $\mathcal{C}$ through the map

$$\Psi \times Id : B \times \mathcal{M} \to \mathbb{R}^4 \times \mathcal{M}.$$ (3.2)

As above, the push forward of $\Psi \times Id$ gives a vector field $X_B$ from the vector field in generalised Levi-Civita variables $X$. Traditionally in celestial mechanics, the exceptional divisor

$$\mathcal{C} := (\Psi \times Id)^{-1}(\mathcal{C})$$

is named the collision manifold. The blow-up $\Psi \times Id$ replaces the set of collisions $\mathcal{C} \equiv \mathcal{M}$ with the higher dimensional collision manifold $\mathcal{C} \equiv \mathbb{RP}^3 \times \mathcal{M}$. Yet, the flow of $X_B$ off the collision manifold is still conjugate to the flow of $X$ off the collision set $\mathcal{C}$.

The flow on $\mathcal{C}$ is fictitious in that it does not correspond to physical orbits of the system. Nevertheless, due to the continuity of the vector field $X_B$, orbits near collision are shadowed by orbits on $\mathcal{C}$. By studying the flow on $\mathcal{C}$ a topological picture of a neighbourhood of $\mathcal{C}$ can be formed.

The flow on $\mathcal{C}$ can be given explicitly by considering charts. Take an atlas $\mathcal{A}$ consisting of the four charts $\psi_\alpha, \psi_\beta, \psi_\gamma, \psi_\delta$ defined on open sets $U_\alpha, U_\beta, U_\gamma, U_\delta \subset B$ which are given by

$$\psi_\alpha ((\zeta_{11}, \zeta_{12}, \zeta_{21}, \zeta_{22}), [\alpha, \beta, \gamma, \delta]) = (\zeta_{11}, \beta/\alpha, \gamma/\alpha, \delta/\alpha), \quad U_\alpha = B \cap \{ \alpha \neq 0 \}$$

$$\psi_\beta ((\zeta_{11}, \zeta_{12}, \zeta_{21}, \zeta_{22}), [\alpha, \beta, \gamma, \delta]) = (\zeta_{12}, \alpha/\beta, \gamma/\beta, \delta/\beta), \quad U_\beta = B \cap \{ \beta \neq 0 \}$$

$$\psi_\gamma ((\zeta_{11}, \zeta_{12}, \zeta_{21}, \zeta_{22}), [\alpha, \beta, \gamma, \delta]) = (\zeta_{21}, \alpha/\gamma, \beta/\gamma, \delta/\gamma), \quad U_\gamma = B \cap \{ \gamma \neq 0 \}$$

$$\psi_\delta ((\zeta_{11}, \zeta_{12}, \zeta_{21}, \zeta_{22}), [\alpha, \beta, \gamma, \delta]) = (\zeta_{22}, \alpha/\delta, \beta/\delta, \gamma/\delta), \quad U_\delta = B \cap \{ \delta \neq 0 \}$$

(3.4)
Figure 2. Commutative diagram to generate $\zeta_{ij}$ blow-ups.

Then, the $\zeta_{ij}$-directional blow-ups $P_{ij}$ are obtained by asserting that the diagram in figure 2 commutes. Explicitly one computes

$$
P_{11}(r_{\alpha}, \beta_{\alpha}, \gamma_{\alpha}, \delta_{\alpha}) = r_{\alpha}(1, \beta_{\alpha}, \gamma_{\alpha}, \delta_{\alpha})
$$

$$
P_{12}(r_{\beta}, \alpha_{\beta}, \gamma_{\beta}, \delta_{\beta}) = r_{\beta}(\alpha_{\beta}, 1, \gamma_{\beta}, \delta_{\beta})
$$

$$
P_{21}(r_{\gamma}, \alpha_{\gamma}, \beta_{\gamma}, \delta_{\gamma}) = r_{\gamma}(\alpha_{\gamma}, \beta_{\gamma}, 1, \delta_{\gamma})
$$

$$
P_{22}(r_{\delta}, \alpha_{\delta}, \beta_{\delta}, \gamma_{\delta}) = r_{\delta}(\alpha_{\delta}, \beta_{\delta}, \gamma_{\delta}, 1).
$$

Then the notation used for coordinates in each chart is designed for quick computation of inverses and transitions between charts. To see this, identify $(\zeta_{11}, \zeta_{12}, \zeta_{21}, \zeta_{22})$ with $(\alpha, \beta, \gamma, \delta)$ and let $\xi, \eta, \nu \in \{\alpha, \beta, \gamma, \delta\}$. Then the following formulas hold,

$$
r_{\eta} = \eta, \quad \xi_{\eta} = \xi/\eta, \quad r_{\eta}\xi_{\eta} = \xi, \quad \xi_{\eta} = \xi_{\nu}/\eta_{\nu}
$$

(3.6)

where $\xi_{\eta} = 1$ if $\xi = \eta$. For example, $r_{\eta} = \alpha$ which in turn can be identified with $\zeta_{11}$, that is, $r_{\alpha} = \zeta_{11}$. Similarly, $\beta_{\alpha} = \beta/\alpha = \zeta_{12}/\zeta_{11}$.

Each chart $\psi_{\eta} \in A$ is naturally extended to a chart on $B \times M$. Making a distinction between the extension and the original $\psi_{\eta}$ will not be needed so the extension is denoted by the same symbol.

The explicit vector fields $X_{\alpha}, X_{\beta}, X_{\gamma}, X_{\delta}$ calculated respectively by the pull-back of $X$ under $P_{11}, P_{12}, P_{21}, P_{22}$ can now be given. In the blow-up procedure one must also desingularise the vector fields [20]. This is done by considering the rescaled vector fields $1/r_{\eta}X_{\eta}$, $\eta \in \{\alpha, \beta, \gamma, \delta\}$, which for convenience will be renamed as $X_{\eta}$. We give only the components of $X_{\eta}$ coming from the $(\zeta_{1}, \zeta_{2})$ components of $X$ as the $(h_{1}, h_{2}, \Gamma_{1}, \Gamma_{2}, x, y)$ components are computed by mere replacement of $(\zeta_{1}, \zeta_{2})$ by the corresponding $P_{ij}$. Moreover, only the lowest order terms in $r_{\eta}$ are given as this is all that is required for the analysis in this section.

At last, we obtain the blow-up systems (3.7)–(3.10). We have set $\Gamma_{j} = \Gamma_{j1} + i\Gamma_{j2}$. The collision manifold $C$ in each chart is given by $r_{\eta} = 0$ for each $\eta \in \{\alpha, \beta, \gamma, \delta\}$ and is invariant under each of the flows.

$$
r'_{\alpha} = r_{\alpha} (\gamma_{\alpha}^{2} + \delta_{\alpha}^{2}) + O(r_{\alpha})
$$

$$
\beta'_{\alpha} = -\beta_{\alpha} (\gamma_{\alpha}^{2} + \delta_{\alpha}^{2}) + O(r_{\alpha})
$$

$$
\gamma'_{\alpha} = (\beta_{\alpha}^{2} + 1) - \gamma_{\alpha} (\gamma_{\alpha}^{2} + \delta_{\alpha}^{2}) + O(r_{\alpha})
$$

$$
\delta'_{\alpha} = -\delta_{\alpha} (\gamma_{\alpha}^{2} + \delta_{\alpha}^{2}) + O(r_{\alpha})
$$

(3.7)
\[ r'_\beta = 0 + O(r_\beta^3) \]
\[ \alpha'_\beta = (\gamma^2_\beta + \delta^2_\beta) + O(r_\beta) \]
\[ \gamma'_\beta = (\alpha^2_\beta + 1) + O(r_\beta) \]
\[ \delta'_\beta = 0 + O(r_\beta) \]
\[ r'_\gamma = r_\gamma (\alpha^2_\gamma + \beta^2_\gamma) + O(r_\gamma^2) \]
\[ \alpha'_\gamma = (\delta^2_\gamma + 1) - |u_2| \alpha_\gamma (\alpha^2_\gamma + \beta^2_\gamma) + O(r_\gamma) \]
\[ \beta'_\gamma = - \beta_\gamma (\alpha^2_\gamma + \beta^2_\gamma) + O(r_\gamma) \]
\[ \delta'_\gamma = - \delta_\gamma (\alpha^2_\gamma + \beta^2_\gamma) + O(r_\gamma) \]
\[ r'_3 = 0 + O(r_3^2) \]
\[ \alpha'_3 = (\gamma^2_3 + 1) + O(r_3) \]
\[ \beta'_3 = 0 + O(r_3) \]
\[ \delta'_3 = (\alpha^2_3 + \beta^2_3) + O(r_3) \]  

(3.8)  

(3.9)  

(3.10)

One must be careful in dealing with the different desingularisations \( X_\eta \). Even though each \( X_\eta \) is analytic, the different rescaling \( 1/r_\eta \) in each chart prevent them from forming a compatible set of vector fields on \( \mathcal{C} \). However, we are not concerned with the exact time parameterisation of each orbit, merely the orbit itself. In this sense, the various desingularisations form compatible integral curves on \( \mathcal{C} \). More precisely, instead of considering \( X_\eta \) as inducing vector fields on \( \mathcal{C} \), we can think of each \( X_\eta \) as inducing line fields on \( \mathcal{C} \). A line field is given by taking a line \( l_p \) in the tangent space \( T_p \mathbb{R}^4 \) for each point in a given chart. We can take natural line fields \( X_\eta \) induced by the vector fields \( X_\eta \) through \( X_\eta(p) = \text{Span} X_\eta(p) \). Each line field creates the analog of integral curves by defining orbits of the line field as curves whose tangent at each point is in the line field at that point. The desingularised vector fields \( X_\eta \) induce a compatible line field on \( \mathcal{C} \).

Another issue with the desingularisation procedure is the fact that the rescaling \( r_\eta^{-1} \) is not strictly positive, forcing the time reversal of the vector fields when \( r_\eta^{-1} < 0 \). Hence, in order to get a picture of the direction of orbits near \( \mathcal{C} \), it is essential to keep track of this time reversal.

As an example of when one must be careful, consider the simpler example of the blow-up of \( 0 \in \mathbb{R}^2 \) through the map \( \Psi_2 : B_2^2 \subset \mathbb{R}^2 \times \mathbb{R}P \rightarrow \mathbb{R}^2 \) defined analogously to the \( \mathbb{R}P^3 \) case already discussed. One takes charts by considering the \( x \) and \( y \)-directional blow-ups \( (x,y) = (r_\alpha, r_\beta, \beta_\alpha) \) and \( (x,y) = (r_\beta, \alpha_\beta, r_\gamma) \) denoted by \( P_x, P_y \) respectively (in this paragraph \( x,y \) are coordinates in the plane unrelated to \( (x,y) \) in the planar problem. For more details on this example see [11]). An orbit near collision occurring at \( (x,y) = (0,0) \), the image of the orbit in the two directional blow-ups, and some intermediate sections \( \Sigma_0, \Sigma^+_1, \Sigma^+_2, \Sigma_3 \) are plotted in figure 3.

The example orbit starts with \( r_\alpha > 0 \) and closely follows the collision manifold at \( r_\alpha = 0 \) in forward time, heading toward \( \beta_\alpha = \infty \). The shadowing flow on \( r_\alpha = 0 \) is given the correct direction by taking \( X_\alpha \). When one swaps charts from \( X_\alpha \) to \( X_\beta \) in order to follow the orbit through \( \beta_\alpha = \infty \), the orbit passes from \( \Sigma^+_1 \) with \( \alpha_\beta < 0 \) to \( \Sigma^+_2 \) with \( \alpha_\beta > 0 \). Note that \( r_\beta > 0 \), so the compatible flow on \( r_\beta = 0 \) is given by \( X_\beta \). Finally, when mapped back to the \( \psi_\alpha \) chart, \( r_\alpha \) is now negative and the compatible flow is given by \(-X_\alpha\), a time reversed system.
If \((r_\alpha, \beta_\alpha) = (0, 0)\) is a hyperbolic saddle, then the eigenvalues will swap sign between \(X_\alpha\) and \(-X_\alpha\). In the \(+X_\alpha\) flow, orbits are pulled toward the collision manifold at \(r_\alpha = 0\) and in the \(-X_\alpha\) flow pulled away. This is compatible with the true orbit in \(\mathbb{R}^2\) as it approaches 0 when \(x > 0\) \((r_\alpha > 0)\) and leaves 0 when \(x < 0\) \((r_\alpha < 0)\).

A different representation of \(\mathcal{B}_2\) is shown in figure 4. The blow-up space \(\mathcal{B}_2\) can be thought of as the tautological line bundle over \(\mathbb{R}P\), which in turn is diffeomorphic to a Möbius band. This is represented by embedding the band in \(\mathbb{R}^3\). From the resulting figure it can be seen that \(\mathcal{B}_2\), considered as a line bundle over \(\mathbb{R}P\), is in fact a Möbius bundle. Consequently, as an orbit follows around the collision manifold \(C \cong \mathbb{R}P \cong S^1\), the line bundle changes orientation. In the \(P_x\) chart, this implies \(r_\alpha \to -r_\alpha\) as an orbit passes around \(C\).

In the example figure 4, \(\mathcal{N}\) is a hyperbolic saddle. However, due to the required time reversing, the directions of orbits are not continuous in a neighbourhood of \(\mathcal{N}\). The following definition is useful for discussing dynamical objects when the exact directions of time are irrelevant.

**Definition 3.1.** Call a singular point \(p\) an **orbital hyperbolic saddle** if, in a neighbourhood of \(p\), the flow is orbitally equivalent to the flow of a hyperbolic saddle. Similarly define **orbital heteroclinic connection**, **orbital focus**, etc.

As an example of this definition, in figure 4, one would say \(\mathcal{N}\) is an orbital hyperbolic saddle and \(C\) constitutes an orbital homoclinic connection of \(\mathcal{N}\).

After these preliminary consideration in a low-dimensional example where \(\mathcal{N}\) is a point instead of a manifold of fixed points and the blow-up is of \(\mathbb{R}^2\) instead of \(\mathbb{C}^2\) we are now going to describe the dynamics near the SBC in the four-body problem.
3.2. Dynamics on the collision manifold

In this section, the consequences of the blow-up and desingularisation procedure are harvested. The flow on the collision manifold \( \mathcal{C} \) is given by setting \( r_\eta = 0 \) in each \( X_\eta \). Note that in each chart the \((h_1, h_2, \Gamma_1, \Gamma_2, x, y)\) components of the vector field can be factored by \( r_\eta \). Each of these components consequently vanish on the collision manifold, that is, \((h_1, h_2, \Gamma_1, \Gamma_2, x, y)\) are integrals on \( \mathcal{C} \). Geometrically it can be said that the collision manifold \( \mathcal{C} \) is foliated by invariant \( \mathbb{RP}^3 \). The flow on each \( \mathbb{RP}^3 \) depends on \( \Gamma_1^*, \Gamma_2^* \), but is invariant under a choice of values of \((h_1^*, h_2^*, x^*, y^*)\). A first study of the flow on \( \mathcal{C} \) produces the following proposition on its topological structure.

**Proposition 3.2.** The collision manifold \( \mathcal{C} \) is an orbital homoclinic connection of a normally hyperbolic manifold of fixed points, \( \mathcal{N} \), that is diffeomorphic to \( \mathcal{M} = \mathbb{R}^2 \times \hat{\mathbb{C}}^2 \times \mathbb{C}^2 \). The following properties of \( \mathcal{N} \) hold:

(a) \( \mathcal{N} \) is given by the graph \((\zeta, [\alpha, \beta, \gamma, \delta]) = (0, [1, 0, 1, 0])\).
(b) The normal bundle of \( \mathcal{N} \) is four-dimensional in the \((r, [\zeta_{11}, \zeta_{12}, \zeta_{21}, \zeta_{22}])\) directions.
(c) The orbital homoclinic connection is foliated by invariant \( \mathbb{RP}^3 \).
(d) In the normal bundle \( \mathcal{N} \) is an orbital, resonant hyperbolic saddle with (un)stable manifolds depending on the direction of time. In one choice, each point in \( \mathcal{N} \) has a three-dimensional unstable manifold and a one-dimensional stable manifold. The dimensions are swapped in the alternative choice. Denote this one-dimensional manifold at a point \( p \in \mathcal{N} \) by \( E_p \).
(e) The three-dimensional manifold of a point in \( \mathcal{N} \) coincides with the homoclinic connection whilst the one-dimensional manifold \( E_p \) is normal to \( \mathcal{C} \).
(f) In the three-dimensional unstable case, the eigenvalues are of the form \( \lambda_1 < 0 < \lambda_2 = \lambda_3 < \lambda_4 \) with ratios of hyperbolicity \(-\lambda_1 : \lambda_j\) a constant \(1 : 3, 1 : 1, 1 : 1\) for any fixed point on \( \mathcal{N} \).
Proof. The desingularised vector field in the chart $\psi_\alpha$ is given by setting $r_\alpha = 0$ in the vector field $X_\alpha$. Note when $r_\alpha = 0$ each of $r_\alpha^\prime = h_1^\prime = h_2^\prime = \Gamma_1^\prime = \Gamma_2^\prime = x^\prime = y^\prime = 0$. This immediately gives the result that $C$ is foliated by invariant $\mathbb{RP}^3$.

Now, for each choice of constants $(h_1^\prime, h_2^\prime, \Gamma_1^\prime, \Gamma_2^\prime, x^\prime, y^\prime) \in \mathbb{R}^2 \times \dot{\mathbb{C}}^2 \times \mathbb{C}^2$, we have precisely one equilibrium of $X_\alpha$ at $(\beta_\alpha, \gamma_\alpha, \delta_\alpha) = (0, 1, 0)$. This equilibrium point maps to the point $$(\zeta, [\alpha, \beta, \gamma, \delta], h_1, h_2, \Gamma_1, \Gamma_2, x, y) = (0, [1, 0, 1, 0], h_1^\prime, h_2^\prime, \Gamma_1^\prime, \Gamma_2^\prime, x^\prime, y^\prime) =: p^\prime$$ on the collision manifold $\mathcal{C}$. Hence, we have a manifold of fixed points $\mathcal{N}$ given by the graph $$(\zeta, [\alpha, \beta, \gamma, \delta]) = (0, [1, 0, 1, 0]).$$ The Jacobian at each point $p^\prime$ in the $\psi_\alpha$ chart is given by,

$$DX_\alpha|_{p^\prime} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$ (3.11)

Consequently, at each point $p^\prime \in \mathcal{N}$ we have a center manifold in the $h_1, h_2, \Gamma_1, \Gamma_2, x, y$ direction, that is, coinciding with the tangent space of $\mathcal{N}$. In the normal bundle of $\mathcal{N}$, there are four non-zero eigenvalues, $\{1, -1, -3, -1\}$, giving the normal hyperbolicity of $\mathcal{N}$ and the saddle structure. Eigenvectors are given by the basis vectors. It can be concluded, for this choice of direction, each point $p^\prime \in \mathcal{N}$ has a one-dimensional unstable manifold normal to $\mathcal{N}$ (in the direction of $r_\alpha$ for the $\psi_\alpha$ chart), and a three-dimensional stable manifold tangent to the invariant $\mathbb{RP}^3$ corresponding to $p^\prime$ (in the $(\beta_\alpha, \gamma_\alpha, \delta_\alpha)$ directions).

The ratios of the eigenvalues are precisely $1 : 1, 1 : 3, 1 : 1$ as asserted.

Lastly, because each invariant $\mathbb{RP}^3$ is compact with a single equilibrium whose stable manifold is three-dimensional and tangent to $\mathbb{RP}^3$, the stable manifold, considered only as a line field, must in fact be all of $\mathbb{RP}^3$. That is, every orbit in the collision manifold is a homoclinic connection of a point $p^\prime \in \mathcal{N}$. \hfill \Box

With some effort we can picture a lower dimensional version of proposition 3.2. To get this lower dimensional version suppose instead that $\mathcal{M} = \mathbb{R}$ and that the blow-up is only two-dimensional so that the entire blown-up manifold is $\mathbb{B}_2 \times \mathbb{R}$. As each $\mathbb{B}_2$ is the M"{o}bius bundle as seen in figure 4 we are guaranteed that $\mathbb{B}_2 \times \mathbb{R}$ is a non-orientable three-manifold, thus, cannot be embedded in $\mathbb{R}^3$ without self intersections. However, if we cut $\mathbb{B}_2$ along $C$ in figure 4, we obtain the orientable $S^1 \times \mathbb{R}^+$ with $C$ double covered by $S^1 \times \{0\}$. The 3D analog of proposition 3.2 is now represented in figure 5. In this analog, the critical manifold is given by a cylinder $S^1 \times \mathbb{R}$, it is foliated by invariant $\mathbb{RP}^1 \cong S^1$ and there is a normally hyperbolic invariant manifold $\mathcal{N}$ on $C$. This manifold $\mathcal{N}$ in figure 5 is represented twice due to the double cover.

In proposition 3.2, the fact the one-dimensional manifold $\mathcal{E}_p$ at each point $p \in \mathcal{N}$ is normal to the collision manifold $C$ provides some crucial information. Let $\mathcal{U}$ be a tubular neighbourhood of $\mathcal{N}$. Due to the time reversal, $\mathcal{E}_p$ on one side of $C$ in $\mathcal{U}$ is asymptotically approaching $p$ in forward time, whilst the other side is approaching $p$ in negative time. Denote these two halves by $\mathcal{E}_p^+$ and $\mathcal{E}_p^-$ respectively. As the orbit $\mathcal{E}_p^+$ is approaching the collision manifold in forward time it must in fact be a unique collision orbit with asymptotic values $p = (h_1^\prime, h_2^\prime, \Gamma_1^\prime, \Gamma_2^\prime, x^\prime, y^\prime)$. Likewise, $\mathcal{E}_p^-$ is an ejection orbit with the same asymptotic values $p$. Hence, we call the entire $\mathcal{E}_p$ a collision-ejection orbit.

By the standard stable manifold theorem for normally hyperbolic invariant manifolds, not only is each $\mathcal{E}_p$ a smooth manifold, but the bundle $\mathcal{E} := \cup_p \mathcal{E}_p^+ \cup \mathcal{E}_p^-$, called the collision-ejection manifold, is smooth [18, 19]. When projecting $\mathcal{E} \subset \mathcal{B} \times \mathcal{M}$ to $\mathbb{R}^4 \times \mathcal{M}$ via $\Psi \times Id$, the conjugacy
outside of $\mathcal{C}$ yields a true manifold with the same properties. The result is summarised in the following corollary. It first appears in [17] albeit with a different proof.

**Corollary 3.3.** Each collision orbit is connected to a unique ejection orbit with the same asymptotic values. Moreover, the union of the collision and ejection orbits forms a smooth invariant manifold $\mathcal{E}$.

The corollary is somewhat visualised in figures 3 and 4. In the figures, the union of the ingoing and outgoing asymptotic orbit is given by a one-dimensional manifold emanating from $\mathcal{N}$ and normal to $\mathcal{C}$. The manifold is smooth and so too is it projection into $\mathbb{R}^2$.

Corollary 3.3 leads to another neat consequence of proposition 3.2. Once the existence of a $C^0$ block map $\pi$ is established in theorem 3.8 below, the smoothness of $\mathcal{E}$ guarantees that this $\pi$, restricted to $\mathcal{E}$, is smooth. The following corollary is immediate.

**Corollary 3.4.** Consider a sub-problem $\mathcal{P}$ of the planar four-body problem which is entirely contained within the collision-ejection manifold $\mathcal{E}$. Inside of $\mathcal{P}$, the set of SBCs are $C^\infty$-regularisable.

In particular, the rhomboidal and symmetric collinear sub-problems are $C^\infty$-regularisable. For details on these sub-problems see, for instance, [1]. This observation agrees with the regularisation results of [2, 32, 35].

### 3.3. Integrability of the flow on the collision manifold

In the $\zeta_1, \zeta_2$ coordinates some key properties of the flow on the collision manifold become clear. Note that $\beta, \delta$ are local integrals of the flow on the collision manifold in the respective charts $\psi_1, \psi_2$. As remarked in proposition 3.2, so too are $h_1, h_2, \Gamma_1, \Gamma_2, x, y$. In fact, the flow in each desingularised vector field $X_\eta$ is integrable.

To see this, recall that $L_j := \text{Im}(\zeta_j) = \frac{1}{2}a_j^{1/3} \text{Im}(\bar{u}_j\bar{z}_j)$ is proportional to the intrinsic angular momentum of the $j$th distressed binary, $(\frac{1}{2} \text{Im}(\bar{u}_j\bar{z}_j))$. Projecting this (scaled) intrinsic angular momentum $L_j$ onto $\mathcal{C}$ results in the projective coordinate $[0, \beta, 0, 0]$ for $j = 1$ and $[0, 0, 0, \delta]$ for $j = 2$. The ratio $L_2/L_1$ is invariant under the blow-up $\Psi$ and maps onto $\mathcal{C}$ as $\kappa_1 := \delta/\beta$. Hence $\kappa_1$ is an integral.
There is a second integral $\kappa_2$ related to the original time of the system $t$. Take the leading order terms in $(\zeta_1, \zeta_2)$ of system (2.14),

$$\dot{\zeta}_j = |\zeta_j|^{-2} \Rightarrow \dot{I}_j = (I_j^2 + L_j^2)^{-1}$$

with $\zeta_j = I_j + iL_j$.

Clearly each $L_j$ is an integral of the leading terms. This observation leads to the remarks above on $\kappa_1$. But also the equation for $I_j$ can be integrated to give $3t - 3I_0 = I_j^3 + 3I_jL_j^2$ for $j = 1, 2$. Taking the equation for $j = 1$ and subtracting for $j = 2$ yields $(I_1^3 + 3I_1L_1^2) - (I_2^3 + 3I_2L_2^2)$.

As argued above, taking the ratio with $L_1^3$ or $L_2^3$, this approximate integral descends to a true integral $\kappa_2$ on the collision manifold $C$.

After taking care of some technicalities, we will now show the integrability of the line field on the collision manifold $C$.

**Definition 3.5.** We say a manifold $\mathcal{K} \subset C$ is orbitally invariant if the image in each chart $\psi_\eta(\mathcal{K})$, $\eta \in \{\alpha, \beta, \gamma, \delta\}$ is an invariant manifold of the desingularised vector fields $X_\eta$.

**Proposition 3.6.** Define the two smooth functions $\hat{\kappa}_1, \hat{\kappa}_2 : C \cong \mathbb{RP}^3 \rightarrow \mathbb{RP}$ by,

$$\hat{\kappa}_1([\alpha, \beta, \gamma, \delta]) = [\beta, \delta]$$

$$\hat{\kappa}_2([\alpha, \beta, \gamma, \delta]) = [\beta^3, (\alpha^3 + 3\beta^2\alpha) - (\gamma^3 + 3\delta^2\gamma)].$$

Then for each $w, v \in \mathbb{RP}$, the closures of the pre-images $\hat{\kappa}_1^{-1}(w), \hat{\kappa}_2^{-1}(v)$ are smooth, orbitally invariant manifolds that intersect transversally.

**Proof.** Let $w = [\beta^*, \delta^*] \in \mathbb{RP}$ and consider $\hat{\kappa}_1^{-1}(w) = \{[\alpha, \beta, \gamma, \delta] | \beta, \delta] = [\beta^*, \delta^*]\}$. Clearly $\hat{\kappa}_1^{-1}(w)$ is a smooth submanifold of $C$. For each $\eta \in \{\alpha, \beta, \gamma, \delta\}$, $\hat{\kappa}_1^{-1}(w)$ in the $\psi_\eta$ chart is

$$\{(\alpha_\eta, \beta_\eta, \gamma_\eta, \delta_\eta) | \delta_\eta/\beta_\eta = \kappa_1 \in \mathbb{R} \cup \{\infty\}, (\beta_\eta, \delta_\eta) \neq (0, 0) \} \subset \mathbb{R}^3,$$

where $\eta_\eta = 1$ is removed from $(\alpha_\eta, \beta_\eta, \gamma_\eta, \delta_\eta)$. A quick calculation reveals these level sets of $\delta_\eta/\beta_\eta$ are invariant in each chart. Each $(\beta_\eta, \delta_\eta) = \kappa_1$ is a two-plane in $\mathbb{R}^3$ minus the line $\beta_\eta = \delta_\eta = 0$. Taking the closure, we obtain a complete two-plane which is clearly smooth.

Similarly, $\hat{\kappa}_2^{-1}(w)$ in each $\psi_\eta$ chart is given by,

$$\{(\alpha_\eta, \beta_\eta, \gamma_\eta, \delta_\eta) | (\alpha_\eta^3 + 3\beta_\eta^2\alpha_\eta) - (\gamma_\eta^3 + 3\delta_\eta^2\gamma_\eta))/\beta_\eta^3 = \kappa_2 \in \mathbb{R} \cup \{\infty\}, (\beta_\eta, \alpha_\eta^3 - \gamma_\eta^3 - 3\delta_\eta^2\gamma_\eta) \neq 0 \} \subset \mathbb{R}^3,$$

where again $\eta_\eta = 1$ is removed. A quick calculation in each $X_\eta$ reveals $\kappa_2$ is invariant. Further, define the function $F_2(\alpha_\eta, \beta_\eta, \gamma_\eta, \delta_\eta) = ((\alpha_\eta^3 + 3\beta_\eta^2\alpha_\eta) - (\gamma_\eta^3 + 3\delta_\eta^2\gamma_\eta))/\kappa_2^3$. Then

$$DF_2 = \left(3(\alpha_\eta^2 + \beta_\eta^2) \quad 6\alpha_\eta \beta_\eta - \kappa_2 \beta_\eta \quad -3(\gamma_\eta^2 + \delta_\eta^2) \quad -6\gamma_\eta \delta_\eta \right).$$

Now, at least one of $\alpha_\eta, \beta_\eta, \gamma_\eta, \delta_\eta$ must be equal to 1. Therefore, $DF_2 \neq 0$ in any of the charts. By the implicit function theorem, the closure of each $\hat{\kappa}_2^{-1}(w)$ is a smooth submanifold of $C$.

Lastly, defining $F_1(\alpha_\eta, \beta_\eta, \gamma_\eta, \delta_\eta) = \delta_\eta - \kappa_1 \beta_\eta$, computing

$$DF_1 = \left(0 \quad -\kappa_1 \quad 0 \quad 1 \right),$$
and comparing $DF_2$ to $DF_1$ along $F_1 = 0, F_2 = 0$, it is seen that $DF_1 \neq DF_2$ at any mutual point. That is, $\tilde{\kappa}_1'(w), \tilde{\kappa}_2'(v)$ intersect transversally.

Define the function $\iota : \mathbb{R}^p \to \mathbb{R} \cup \{\infty\}$ by $[\alpha, \beta] \mapsto \beta / \alpha$ and the two functions $\kappa_1 := \iota \circ \tilde{\kappa}_1, \kappa_2 := \iota \circ \tilde{\kappa}_2$. Due to proposition 3.6, each of the level sets of $\kappa_1, \kappa_2$ must define smooth invariant manifolds. With the intersection between any two level sets of $\kappa_1, \kappa_2$ transverse, we obtain the integrability of the flows as a corollary.

**Corollary 3.7.** Each desingularised vector field $X_\delta$ is integrable with first integrals given by the images of $\kappa_1, \kappa_2, h_1, h_2, \Gamma_1, \Gamma_2, x, y$ in each chart.

### 3.4. Proof of $C^0$-regularity

Finally, we are in a position to give a new proof of the $C^0$-regularity of $\mathcal{C}$.

**Theorem 3.8.** The set of SBCs $\mathcal{C}$ is at least $C^0$-regularisable in the planar four-body problem.

**Proof.** In order to show $C^0$-regularity of $\mathcal{C}$ we must first define two sections; $\Sigma_0$ transverse to the collision orbits $\mathcal{E}^+$, and $\Sigma_3$ transverse to ejection orbits $\mathcal{E}^-$. Then, by flowing points from $\Sigma_0 \setminus \mathcal{E}^+$ to $\Sigma_3 \cap \mathcal{E}^-$ we obtain a homeomorphism $\pi$. The $C^0$-regularity of $\mathcal{C}$ is guaranteed provided one can extend $\pi$ to some $\pi : \Sigma_0 \to \Sigma_3$ such that $\pi$ is unique and $C^0$. The blown-up systems $X_\delta$ will be essential to prove this statement.

In the blown-up systems $X_\delta$, take a section $\Sigma_0$ transverse to $\mathcal{E}_p^+$ for some point $p \in \mathcal{N}$. Recall from the remarks in section 3.1, the desingularisation $X_\delta$ or $-X_\delta$ that is consistent with the flow outside of $\mathcal{C}$ must be taken. In order to see which time direction to take, recall that $\mathcal{N}$ is given by the graph $(\zeta, [\alpha, \beta, \gamma, \delta] = (0, [1, 0, 1, 0])$. Hence, $\mathcal{E}_p^+$, in some tubular neighborhood of $\mathcal{C}$, is entirely contained in the chart $\psi_{\alpha}$. From proposition 3.2, $\mathcal{E}_p^+$ is the unstable manifold of a point $p \in \mathcal{N}$ which is normal to $\mathcal{C}$. It can be concluded, in order to have the orbit $\mathcal{E}_p^+$ approach the collision in forward time as collision orbits should, we must consider the desingularisation $-X_\delta$. Moreover, as one only needs to consider $-X_\delta$ for orbits with $r_\alpha < 0$, and $r_\alpha$ is mapped under $\psi_\alpha$ to $\zeta_{11}$, we must have that collision orbits approach the set of SBCs $\mathcal{C}$ with $\zeta_{11} < 0$. A symmetrical argument shows for ejection orbits $r_\alpha > 0 \Rightarrow \zeta_{11} > 0$ and the correct desingularisation is $X_\delta$.

Now, orbits on $\Sigma_0$ follow the stable manifold $\mathcal{E}_{p}^-$ until they pass through a normally hyperbolic region close to $p \in \mathcal{N}$. From proposition 3.2, this point is a normally hyperbolic saddle and, in the $-X_\delta$, desingularisation, has a one-dimensional stable manifold $\mathcal{E}_{p}^-$ and a three-dimensional unstable manifold coinciding with the three-dimensional invariant $\mathbb{R}^3$ at the fixed value $h_1^\ast, h_2^\ast, \Gamma_1, \Gamma_2, x^\ast, y^\ast$. It is known that normally hyperbolic manifolds are topologically equivalent to their linear part [28]. Hence, we can conclude that near collision orbits around $\mathcal{E}_{p}^-$ will get pulled away from $\mathcal{E}_{p}^+$ and begin to follow orbits on the unstable manifold of $p$, that is, orbits on the collision manifold $\mathcal{C}$. That is, after the hyperbolic region near $\mathcal{N}$ the near collision orbits will be shadowed by orbits on $\mathcal{C}$.

By proposition 3.2, each orbit on the collision manifold is a homoclinic connection of the point $p \in \mathcal{N}$. As the orbit on $\mathcal{C}$ passes around the manifold back to $p$, the fact that $\mathcal{C} \cong \mathbb{R}^3 \times M$ and the non-orientability of $\mathbb{R}^3$ ensure that, for the near collision orbits, $r_\alpha$ changed orientation from $r_\alpha < 0$ to $r_\alpha > 0$. That is, if an orbit passes near collision it must pass through $\Sigma_0$ where $\zeta_{11} < 0$ and exit on $\Sigma_3$ where $\zeta_{11} > 0$. When $r_\alpha > 0$, we need to consider the desingularisation $X_\delta$ where $\mathcal{E}_p^-$ is the unstable manifold of $p$. The near collision orbits then pass back through a hyperbolic region where they begin to follow the orbit $\mathcal{E}_p^+$.

Each of these orbits then intersects some $\Sigma_3$, a transverse section to the ejection orbit $\mathcal{E}_p^+$.
By limiting to the orbit $\mathcal{E}_p^\pm$ on $\Sigma_0$, we obtain near collision orbits which pass around the collision manifold and limit onto the unique ejection orbit $E_{-}^p$. Using corollary 3.3 and the topological conjugacy of the blow-up $B \times M$ outside of $\mathcal{C}$, we conclude the $C^0$ regularity of the SBCs for the planar problem.

□

4. $C^{8/3}$-regularity of the block map

This section contains the proof of the main result of the paper, theorem 4.12 on the $C^{8/3}$-regularisation of the SBC of the planar four-body problem.

4.1. Normal form and absence of foliation

In previous work on the collinear problem [11], a link was established between the $C^{8/3}$-regularity of the SBC and the inability to foliate a tubular neighbourhood of the collinear collision manifold by invariant manifolds normal to the manifold. A method which quantifies how well the collinear collision manifold admits an invariant foliation was developed. The normal form of $X$ in a neighbourhood of $\mathcal{C}$ was crucial to the quantification. Specifically, this inability to foliate the neighbourhood was shown to be a consequence of resonant terms at order 8 in the normal form of $X$ in a neighbourhood of $\mathcal{C}$. In this section, we use similar arguments to show the normal space to $\mathcal{C}$ in the planar problem also lacks a smooth foliation. The result and the proceeding calculations in section 4.2 will prove the $C^{8/3}$ regularity.

4.1.1. Normal form theory. Recall the following definitions and theorem on normal forms which combines the works of Belitskii [4, 5], and Stolovitch and Lombardi [22].

**Definition 4.1.** Decompose a vector field $X$ into its Taylor series, $X = X_0 + X_1 + \cdots$ with $X_0$ the leading order homogeneous component of degree $s$ and each $X_d \in \mathcal{H}_{d+s-1}$ in the space of degree $d+s$ homogeneous vector fields. Let $[\cdot,\cdot]$ be the usual Lie bracket for vector fields and define the cohomological operator

$$L_{X_0} := [X_0, \cdot]$$

and its restriction to $\mathcal{H}_d$ by $L_d : \mathcal{H}_d \mapsto \mathcal{H}_{d+s-1}$.

Most treatments of normal form theory are focused on the case $s = 1$, i.e. equilibrium points that have a non-vanishing linear part. Note that for the SBC in the four-body problem after the time re-scaling equilibrium points with vanishing linear part are obtained. The general case with $s = 2$ needed here is, e.g., described in [22]. The following normal form theorem is due to [4, 22] for a particular choice of $\mathcal{U}_d$ described below. We need a straightforward generalisation which allows for a more general choice of $\mathcal{U}_d$.

**Theorem 4.2.** For each $d \geq 1$, let $\mathcal{U}_d$ be a subspace of $\mathcal{H}_d$ such that $\mathcal{H}_d = \text{Im} L_d + \mathcal{U}_d$. Then there exists a formal transformation $\hat{\phi}^{-1} = I + \sum_{d \geq 1} U_d$ with $U_d \in \mathcal{H}_d$ that formally conjugates $X = X_0 + \sum_{d \geq 1} X_d$ to the normal form,

$$\hat{\phi} X = X_0 + \sum_{d \geq 1} N_d,$$  \hspace{1cm} (4.1)
such that \( N_d \in \mathcal{U}_d \).

The version of theorem 4.2 as proved in [4, 22] takes \( \mathcal{U}_d \) as a complement of \( \text{Im} \ L_d \) defined using the Fischer inner product on \( \mathcal{H}_d \). Specifically, using an inner product, the adjoint \( L_d^* \) of \( L_d \) can be defined, and a complementary subspace can be chosen as \( \ker(L_d) \). Details on the Fischer inner product are given in [5]. Our work on the collinear problem [11] follows this convention. There is no requirement to making the same choice here for the planar problem. However, if one chooses to do so one will discover at least two unfavourable outcomes.

The first unfavourable outcome is that the normal form procedure fails to compute higher order integrals of \( L_1, L_2 \). The ability to compute these integrals allows for ease in the computations of the block map in the proceeding section. The second unfavourable outcome is that the resulting normal form fails to limit to that computed in [11] when restricted to the collinear problem.

These concerns can be treated by making a non-standard choice for \( \mathcal{U}_d \), which does not happen to be complementary subspace to \( \text{Im} \ L_d \). Depending on the choice of \( \mathcal{U}_d \) this may remove fewer terms in the normal form. Depending on the situation, this may be favourable.

In our current situation, instead of taking \( \mathcal{U}_d = \ker(L_d^*) \), we will take \( \mathcal{U}_d = \ker(L_{X_c}^*) \big|_{\mathcal{H}_d} \), where \( X_c \) contains the leading order terms of \( X \) restricted to the collinear problem,

\[
X_c = I_2^2 \partial_{I_1} + I_1^2 \partial_{I_2} = X_0 |_{I_1 = I_2 = 0}.
\]

To define the adjoint we still use the Fischer inner product, so that

\[
X_c^* = I_2^2 \partial_{I_1}^* + I_1^2 \partial_{I_2}^*.
\]

The novelty in our treatment is that we do not use the adjoint of the homological operator to define a complementary subspace, but the adjoint of a different operator obtained by restriction to the invariant subspace \( L_1 = L_2 = 0 \). The particular choice of \( X_c \) is supported by the following lemma.

**Lemma 4.3.** For each \( d \leq 15 \) the space of degree \( d + 1 \) homogenous vector fields \( \mathcal{H}_d \) can be written as

\[
\mathcal{H}_d = \text{Im}(L_d) + \ker(L_{X_c}^*).
\]

**Proof.** This can be verified by explicit computation. \( \square \)

Unfortunately we do not have a general proof for all \( d \in \mathbb{N} \).

### 4.1.2. Calculation of the normal form

With a suitable choice of subspace \( \mathcal{U}_d \), we may now proceed with the calculation of the normal form. Let \( x = x_1 + ix_2, y = y_1 + iy_2 \), and \( X_\mathbb{R} \) be the real vector field associated to \( X \), the vector field near \( C \) in the generalised Levi-Civita coordinates and with fictitious time \( \tau \). In order to construct \( X_\mathbb{R} \) a choice of coordinates on \( \hat{C} \) for each \( \Gamma_j \) must also be made. The proceeding normal form calculation is independent of the choice, so we refrain from making any specifications. The realisation of \( X \) is hence \( X_\mathbb{R} \), a vector field on \( \mathbb{R}^{12} \).

The leading order term at any point in \( \mathcal{C} \) is given by \( X_0 = |\zeta|^2 \partial_{\zeta} + |\zeta|^2 \partial_{\bar{\zeta}} \). Let \( w = (w_1, \ldots, w_{12}) \in \mathcal{H}_{d+1} \) and denoting \( X_c = I_2^2 \partial_{I_1} + I_1^2 \partial_{I_2} \) the leading order vector field \( X_0 \)
restricted to the collinear subspace $L_1 = L_2 = 0$. The operator $L^*_\chi$ is given explicitly by,

$$L^*_\chi w = X^*_\chi(w) - (DX^*_\chi)^T w, \quad DX^*_\chi = \begin{pmatrix} 0 & 0 & 2\partial h_1 \\ 0 & 0 & 0 \\ 2\partial h_1 & 0 & 0 \end{pmatrix} \oplus 0_8, \quad (4.2)$$

with $0_8$ the $8 \times 8$ zero matrix.

Note that the cohomological operator decouples into an operator in $I_1, L_1, I_2, L_2$ and merely $X^*_\chi$ in each of the remaining variables. This is a consequence of the fact that $X_0$ decouples into the $\zeta_1, \zeta_2$ system and a trivial vector field in the other variables. The decoupling leads to a proof of corollary 4.6 on the inability to construct an invariant foliation of the normal space to $\mathcal{E}$ in the planar problem.

The normal form near an arbitrary SBC to degree 9 will now be computed. For this calculation we require the degree 9 Taylor expansion of the vector field $X$ around an arbitrary fixed point $p = (0, 0, h_1, h_2, \Gamma_1, \Gamma_2, x', y')$ in $\mathcal{E}$. Consequently, the potential $K(\zeta_1, \zeta_2, h_1, h_2, \Gamma_1, \Gamma_2, x)$ must be expanded to degree 8.

Firstly, there are four terms in the original $\hat{K}(Q_1, Q_2, x)$ which are of the form,

$$\frac{d_i}{|x + C_1Q_1 + C_2Q_2|} = \frac{1}{|x|} \frac{1}{1 + C_1Q_1/x + C_2Q_2/x},$$

In order to get $K$ to degree 9 we need to compute this expansion to degree 4 in $Q_1, Q_2$ and substitute the various coordinate transformations of section 2. Consider the function,

$$F(\xi) = (1 + \xi)^{-1/2}, \quad F : \mathbb{C} \to \mathbb{C}, \quad (4.3)$$

which is holomorphic away from $\xi = -1$, in particular, in a neighbourhood of 0. As it is holomorphic, it has a convergent Laurent series about 0 given by

$$F(\xi) = \sum_{j=1}^{\infty} (-1)^j \left( \frac{-1/2}{j} \right) \xi^j. \quad (4.4)$$

The potential can then be written in the form,

$$\hat{K}(Q_1, Q_2, x) = \frac{1}{|x|} \left( d_1 \left| F \left( c_2 \frac{Q_1}{x} - c_3 \frac{Q_2}{x} \right) \right|^2 + d_2 \left| F \left( c_2 \frac{Q_1}{x} + c_3 \frac{Q_2}{x} \right) \right|^2 + d_3 \left| F \left( -c_1 \frac{Q_1}{x} - c_4 \frac{Q_2}{x} \right) \right|^2 + d_4 \left| F \left( -c_1 \frac{Q_1}{x} + c_4 \frac{Q_2}{x} \right) \right|^2 \right)^{1/2}. \quad (4.5)$$

Using $|F(\xi)|^2 = F(\xi)F(\xi)$ and (4.4), we can expand $\hat{K}(Q_1, Q_2, x)$ in a neighbourhood of $Q_1, Q_2 = 0, x \neq 0$. Substituting the values of $d_i, c_i$ given in (2.2) this expansion to degree 4 of $K$ is given as,

$$\hat{K}(Q_1, Q_2, x) = \frac{1}{|x|} \left( b_0 + \hat{K}_1 \left( \frac{Q_1}{x} \right) + \hat{K}_2 \left( \frac{Q_2}{x} \right) + \hat{b}_j W_j \left( \frac{Q_1}{x}, \frac{Q_2}{x} \right) \right), \quad (4.6)$$

$$\hat{K}_j(Q) = \sum_{j=2}^{4} \hat{b}_j W_j (Q).$$
where $\hat{b}_{ij}, \hat{b}_c, b_0$ are functions of the masses, $W_j$ are homogeneous degree $j$ polynomials in $Q, \overline{Q}$ independent of the masses, and $W_c$ is a homogeneous degree 4 polynomial independent of the masses and containing all the degree 4 coupled terms between $Q_1$ and $Q_2$. The constants $\hat{b}_{ij}$ and the homogeneous polynomials are given in appendix A.

**Remark 4.4.** *A priori,* there should be coupled terms of lower order; for example the monomial $Q_1Q_2$ of order 2. Remarkably, for the particular potential $K$, all these terms vanish. The first coupled monomials are at order 4 in $b_2W_4(Q_1/Q_2)$. It is seen that these first coupling terms play a crucial role in the arrival of non-vanishing resonant terms and ultimately the finite differentiability of the block map. In the collinear problem the crucial role of the coupling terms was heuristically observed by Martinez and Simó [23].

Each of the transformations in section 2 can be carried through equation (4.6) to get expansions to order 9 of $K$. It will then take the form,

$$K(\zeta_1, \zeta_2, x) = \frac{1}{|x|} \left( b_0 + K_1 \left( \frac{U_2^{-2}I_2^2\zeta_1^2}{x} \right) + K_2 \left( \frac{U_2^{-2}I_2^2\zeta_2^2}{x} \right) \right) + b_2W_c \left( \frac{U_2^{-2}I_2^2\zeta_1^2}{x}, \frac{U_2^{-2}I_2^2\zeta_2^2}{x} \right),$$

(4.7)

$$K_i(Q) = \sum_{j=2}^4 b_{ij} W_j(Q/x),$$

with $b_{ij}, b_c$ given in appendix A.

The following result on the normal form near an arbitrary SBC can now be given. The potential is expanded under the assumption that $|x|$ is large compared the distances of the distressed binaries $|Q_i|$. In the expansion of the potential thus $x$ appears in the denominator of all terms. Due to the scaling and rotational symmetry of the Hamiltonian $H$, see (2.5), it can be assumed that $x$ has the asymptotic value $x^* = 1$. From now on we assume that the scaling and rotation that achieves this has been fixed, and hence the $x$ in the denominator of the expansion disappears.

After the Taylor expansion the terms with the smallest degree in $\zeta_i$ in the components of the vector field (2.14) have degree (2, 8, 2, 8, 5, 5, 3, 3, 4, 4, 4). As a result of the time scaling the leading order terms are always proportional to either $|\zeta_1|^2$, $|\zeta_2|^2$, or both, so that $\zeta_1 = \zeta_2 = 0$ makes the vector field vanish, irrespective of the values of $h_1, h_2, \Gamma_1, \Gamma_2, x, y$. The normal form transformation produces a vector field $X^g$ where the leading degrees are (2, 0, 2, 0, 9, 9, 0, 0, 0, 0, 0, 0), where 0 represents a vanishing component up to including order 9.

**Proposition 4.5.** The normal form $X^g$ of the vector field (2.14) in a neighbourhood of the SBC with asymptotic values $(h_1, h_2, \Gamma_1, \Gamma_2, x, y) = (h_1^*, h_2^*, \Gamma_1^*, \Gamma_2^*, 1, y^*)$ is given to degree 9 by,

$$I'_1 = |\zeta_1|^2 + R^g_1(\zeta_1, \zeta_2, h_1, h_2) + R^g_2(\zeta_1, \zeta_2, h_1, h_2) + R^g_3(\zeta_1, \zeta_2, h_1, h_2)$$

$$L'_1 = 0$$

$$I'_2 = |\zeta_1|^2 + R^g_1(\zeta_1, \zeta_2, h_1, h_2) + R^g_2(\zeta_2, \zeta_1, h_2, h_1) + R^g_3(\zeta_1, \zeta_2, h_1, h_2)$$

$$L'_2 = 0$$

(4.8)
\[ h'_1 = b_1 a_1^{-1/3} R_4^9(\zeta_1, \zeta_2; \Gamma_1^*, \Gamma_2^*) \]
\[ h'_2 = -b_2 a_2^{-1/3} R_2^9(\zeta_1, \zeta_2; \Gamma_1^*, \Gamma_2^*) \]
\[ \Gamma'_1 = \Gamma'_2 = x' = y' = 0, \]

where \( R_j^k \) are degree \( k \) real-valued, homogeneous polynomials in \( I_j, L_1, I_2, L_2 \) independent of the masses and \( R_9^6 \) is a degree 8 polynomial in \( \Gamma_1^*, \Gamma_2^* \). \( R_4^1 \) and \( R_6^1 \) are given in appendix A.

We omit the proof of the proposition as it is a huge computation that is far too unwieldy to be contained here. The transform bringing the vector field \( X \) to the normal form \( X_9 \) and the resonant terms \( R_j^8 \) can be provided upon request, the first few terms in powers of \( L_i \) are listed in the appendix (A.4). Only the degree in \( \zeta_1, \zeta_2 \), and not the exact form of the resonant terms, will be relevant to the arguments in the remainder work.

There is a lot of information to unpack from proposition 4.5. Firstly, the normal form procedure concludes with the appearance of resonant terms at degree 9 in \( I_j, L_j \) for the \( h_j \) components. Consequently, the following corollary can be proved.

**Corollary 4.6.** It is not possible to construct a smooth foliation into invariant four-planes normal to the SBC manifold \( \mathcal{C} \). Specifically, one cannot find smooth invariants diffeomorphic to \( h_i \) at order 8 in \( I_i, L_i \).

**Proof.** The result is an immediate consequence of [11, corollary 4.3]. As the collinear problem sits inside the planar problem, and the impossibility of the foliation has been shown for the collinear problem, so too it must hold for the planar case. \( \square \)

**Remark 4.7.** Note that, after blow-up, degree 9 terms become degree 8 terms due to the rescaling by \( d\tau = \tau d\tilde{r} \). Therefore, an obstacle to the foliation is occurring at degree 8 in the blow-up space. This agrees with the results on the collinear problem [11] where the degree 8 was linked to the \( C^{8/3} \)-regularity of the collinear problem.

**Remark 4.8.** Up to order 8, the normal form procedure has computed invariants given by the transformed \( L_1, L_2, x, y, \Gamma_1, \Gamma_2 \) and the additional

\[ H = a_2^{-1/3} h_1 + a_1^{-1/3} h_2. \] (4.9)

There is in fact another one. The quantity

\[ (I_1^3 + 3I_1L_1^2) - (I_2^3 + 3I_2L_2^2) \]

was used in section 3 to obtain the integral \( \kappa_2 \) of the line field on the collision manifold. The quantity is an integral of the leading order dynamics. It can be extended to an integral of \( X^9 \),

\[ \kappa = (I_1^3 + 3I_1L_1^2) - (I_2^3 + 3I_2L_2^2) + G^9(\zeta_1, \zeta_2, h_1, h_2) + G^7(\zeta_1, \zeta_2, h_1, h_2) \] (4.10)

where \( G^j \) is a homogeneous degree \( j \) polynomial in \( \zeta_1, \zeta_2 \). The full expressions are given in (A.5).

The existence of each of these integrals will play a central role in showing the \( R_8 \) and \( R_6 \) terms do not affect the \( 8/3 \) regularity of the block map.
Remark 4.9. The rectangular and the Caledonian problem mentioned in remark 2.4 are invariant sub-problems that are contained in the collision manifold. For these it is known [15] that a smooth regularisation is possible. As described in remark 2.4 the condition that defines the collision manifold is \( L_1 = L_2 = 0 \) and \( L_1 = I_2 \). When \( L_1 = L_2 = 0 \), \( \Gamma_i = 1 \) the problem reduces to the collinear problem, and thus \( R_i^2 \) also reduces to the polynomial found in the collinear problem [11] up to an overall factor that depends on \( \Gamma_i \), see equation (A.4) in the appendix. This polynomial vanishes when \( L_1 = I_2 \) and hence the term that is responsible for the \( C^{8/3} \) regularisability vanishes.

4.2. Geometric sketch of proof

A procedure for determining the finite differentiability of the block map can now be sketched. The procedure is not too dissimilar to the one developed in showing the \( C^{8/3} \)-regularity of the collinear problem [11]. The crucial difference is the higher co-dimensionality of \( \mathcal{C} \) in the planar problem in comparison to the collinear problem. Consequently, the geometrical sketch is harder to picture and more importantly, certain computations in the proof become substantially more involved and require some new theory.

Recall that proposition 3.2 gives the topological structure of the flow near \( \mathcal{C} \). Understanding this structure was instrumental in proving the \( C^0 \)-regularity of the block map \( \pi : \Sigma_0 \to \Sigma_3 \). This structure was revealed by blowing up \( \mathcal{C} \) to produce the collision manifold \( \mathcal{C}^* \) and a consequent analysis of the flow on \( \mathcal{C}^* \). The flow on \( \mathcal{C}^* \) will again be central to determining the finite differentiability.

Any tubular neighbourhood \( \mathcal{T} \) of \( \mathcal{C} \) has a natural decomposition due to proposition 3.2. Overlapping neighbourhoods \( \mathcal{U} \), a tubular neighbourhood of the normally hyperbolic invariant manifold \( \mathcal{N} \), and \( \mathcal{V} \), a tubular neighbourhood of the homoclinic connection of \( \mathcal{N} \), can be chosen so that \( \mathcal{T} = \mathcal{U} \cup \mathcal{V} \). The decomposition splits \( \mathcal{T} \) into a region \( \mathcal{U} \) where the flow is topologically equivalent to a neighbourhood of a manifold consisting entirely of saddle singularities and a region \( \mathcal{V} \) where the flow is topologically equivalent to a regular flow.

By splitting the flow into different topological regions, a geometric sketch for computing the block map \( \pi : \Sigma_0 \to \Sigma_3 \) unfolds. The idea is to introduce an intermediate section, \( \Sigma_{\text{int}} \subset \mathcal{U} \cap \mathcal{V} \), which is homeomorphic to \((-a, a) \times S^2 \times \mathbb{R}^8 \) for \( a \ll 1 \), transverse to the flow on \( \mathcal{C} \), and the intersection \( \mathcal{C} \cap \Sigma_{\text{int}} \) is homeomorphic to \( S^2 \). An example of such a \( \Sigma_{\text{int}} \) is to take the boundary of the box \([-1, 1]^3 \) on \( \mathcal{C} \) in the coordinates \((\beta_\alpha, \gamma_\alpha, \delta_\alpha)\) and take the direct product with an interval \((-a, a) \) in \( \epsilon_\alpha \) and with \( \mathbb{R}^8 \) from the remaining variables. See figure 6 for a depiction of \( \Sigma_{\text{int}} \).

The conditions on \( \Sigma_{\text{int}} \) force it to surround \( \mathcal{N} \) so that, if \( \mathcal{U} \) is taken sufficiently small, any orbit in \( \mathcal{U} \) will intersect \( \Sigma_{\text{int}} \). Then, as shown in the proof of theorem 3.8, these near collision orbits shadow an orbit on \( \mathcal{C} \). After leaving \( \mathcal{U} \), as all the orbits on \( \mathcal{C} \) are necessarily homoclinic trajectories, the near collision orbits will transition through \( \mathcal{V} \) and necessarily reenter \( \mathcal{U} \). Ultimately, the orbits intersect \( \Sigma_{\text{int}} \) again. It follows that \( \Sigma_{\text{int}} \) splits \( \pi \) over the different topological regions \( \mathcal{U}, \mathcal{V} \),

\[
\pi = D_2 \circ T \circ D_1, \\
D_1 : \Sigma_0 \to \Sigma_{\text{int}}, \quad T : \Sigma_{\text{int}} \to \Sigma_{\text{int}}, \quad D_2 : \Sigma_{\text{int}} \to \Sigma_3.
\]

On the one hand the maps \( D_1, D_2 \) are transitions near a normally hyperbolic manifold of saddle singularities. On the other hand, \( T \) is a smooth transition map. The asymptotic structure of the maps \( D_1, D_2 \) was studied in [15]. These asymptotic structures will be used in the proceeding section to prove theorem 4.12. However, before the details of the asymptotics are discussed,
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Figure 6. A depiction of the hyperbolic transitions $D_1, D_2$. For clarity, only two of the walls, $\Sigma_1, \Sigma_2$, of $\Sigma_{int}$ have been sketched. It is impossible to plot the co-dimension 4 manifold $N$ of the planar problem. Instead, a co-dimension 3 problem has been plotted. Further, only a projection into a three-dimensional slice, normal to $N$, containing $p \in N$, can be shown.

It is useful to make some assumptions to get a clearer picture of how finite differentiability of the block-map may creep in.

In this sketch of the proof, the variables $x, y, z, w \in \mathbb{R}$ are used independently of any previous meaning. It was shown in proposition 3.2 that $N$ is a manifold of normally hyperbolic saddle singularities of co-dimension 4. Assume that the one-dimensional invariant manifold leaving a point on $N$ in the normal direction is stable. Then, up to scaling, the non-zero eigenvalues of $N$ are $(-1, 3, 1, 1)$. Assume that coordinates $(x, y, z, w, u) \in \mathbb{R}^4 \times \mathbb{R}^8$ have been chosen near $N$ so that the local stable manifold is given by $y = z = w = 0$ and the local unstable manifold by $x = 0$. To leading order in $x, y, z, w, u$, the vector field in a tubular neighbourhood of $N$ is of the form,

\[
\begin{align*}
\dot{x} &= x \\
\dot{y} &= -3y \\
\dot{z} &= -z \\
\dot{w} &= -w \\
\dot{u} &= 0
\end{align*}
\] (4.11)

In general, system (4.11) may have higher order terms. One can ask whether such terms are removable, that is, whether the vector field near $N$ has a linearisation. This is a question of normal forms for $N$. This type of normal form was studied in [9]. There, it was shown that
linearisation is only guaranteed if the non-zero-eigenvalues of $N$, say $\lambda \in \mathbb{C}^4$, do not satisfy any of the resonance conditions,

$$n \cdot \lambda = \lambda, \quad n \cdot \lambda = 0,$$

where $n \in \mathbb{N}^4$ and $n \cdot \lambda$ is the usual dot product. Unfortunately, for our specific $N$, the eigenvalues $\lambda = (-1, 3, 1, 1)$ do indeed satisfy the resonance conditions, thus it cannot be assumed that the system admits a linearisation. The exact form of the normal form is given in the appendix as proposition B.1.

Regardless, assume the vector field local to $N$ can be linearised. If this is true, the system admits the integrals,

$$x^3y, \quad xz, \quad xw.$$

Now choose sections $\Sigma_0 = \{x = 1\}, \Sigma_3 = \{x = -1\}$ and take $\Sigma_{\text{int}}$ as the boundary of the box $(-a, a) \times [-1, 1]^3 \times \mathbb{R}^8$. Further, decompose $\Sigma_{\text{int}}$ into its faces,

$$\Sigma^\pm_\eta := \{\eta = \pm 1\} \cap \Sigma_{\text{int}} \quad \text{for } \eta = y, z, w.$$

In turn, $D_1, D_2$ can be decomposed into the directional transitions

$$D_{1,\eta}^+: \Sigma_0 \to \Sigma^+_\eta, \quad D_{2,\eta}^+: \Sigma^+_\eta \to \Sigma_3,$$

with $\eta = y, z, w$. Further, the domain $D_{1,\eta}^+$ of each $D_{1,\eta}^+$ is a different subset of $\Sigma_0$. It follows that the block map $\pi$ is split over these domains into $\pi^\pm_\eta$.

Splitting $T$ over the different faces of $\Sigma_{\text{int}}$ requires some technicalities. It is entirely possible that trajectories on $\Sigma^+_\eta$ do not reach $\Sigma^-\eta$, instead passing through another face of $\Sigma_{\text{int}}$. There are two ways to solve this issue; restrict the domain of $T|_{\Sigma^+_\eta}$ so that the codomain is $\Sigma^-\eta$, or modify the faces of $\Sigma_{\text{int}}$ so that the codomain is contained in $\Sigma^-\eta$. We will take the latter solution as asserting the domain rather than the codomain of $T$ is more useful for computing $\pi$.

Specifically, we can modify $\Sigma_{\text{int}}$ so that a given face $\Sigma^-\eta$ is sufficiently wide enough to capture all trajectories from $\Sigma^+_\eta$. Hence, we can split, $T_\eta : \Sigma^+_\eta \to \Sigma^-\eta$.

Now, let us first focus on $\pi^+_y$. If one takes coordinates $(-y_3, -z_3, -w_3, u_3)$ on $\Sigma_3$ and coordinates $(-x_2, -z_2, -w_2, u_2)$ on $\Sigma^-_y$ then the integrals (4.12) give $D_{2,y}^+$ as,

$$D_{2,y}^+: \quad y_3 = x_2^3, \quad z_3 = x_2z_2, \quad w_3 = x_2w_2, \quad u_3 = u_2.$$

Similarly, taking coordinates $(y_0, z_0, w_0, u_0)$ on $\Sigma_0$ and coordinates $(x_1, z_1, w_1, u_1)$ on $\Sigma^+_y$, $D_{1,y}^+$ is obtained as

$$D_{1,y}^+: \quad x_1 = y_0^{1/3}, \quad z_1 = y_0^{-1/3}z_0, \quad w_1 = y_0^{-1/3}w_0, \quad u_1 = u_0.$$

It is immediate that $D_{1,y}^+ = (D_{2,y}^-)^{-1}$. However, it will be shown in this section that this may not be true when higher order terms are considered.
Note that, analysing solutions near-collision amounts to treating \( x_2 \) and \( y_0, z_0, w_0 \) as small variables. \( D_2 \) is clearly continuous in \( x_2 \), however, \( D^+_1 \) is not continuous in \( y_0, z_0, w_0 \). This is a consequence of the fact that \( D_1 : \Sigma_0 \to \Sigma_{\text{int}} \) maps the eight dimensional surface \((0, 0, 0, \bar{u}_0)\) to the 10 dimensional surface \((0, z_1, w_1, u_1)\). However, if \( y_0, z_0, w_0 \) are rescaled by \( \varepsilon \ll 1 \) to reflect the fact they are all small, then

\[
D_{1, \varepsilon} : \begin{align*}
y_1 &= \varepsilon^{1/3} y_0^{1/3}, \\
z_1 &= \varepsilon^{2/3} y_0^{-1/3} z_0, \\
w_1 &= \varepsilon^{2/3} y_0^{-1/3} w_0, \\
u_1 &= u_0,
\end{align*}
\]

is continuous in \( \varepsilon \) on \([0, 1)\) for any choice \((y_0, z_0, w_0)\) with \( y_0 \neq 0 \). This is sufficient to compute the asymptotic expansion of \( \pi^+ \) as only an asymptotic expansion in \( \varepsilon \) is desired.

Here is the crucial point of the sketch of finite differentiability. As \( D_{1, \varepsilon}^+ = (D_{2, \varepsilon}^-)^{-1} \) then clearly their composition \( D_{2, \varepsilon}^- \circ D_{1, \varepsilon}^+ \) is the identity and hence smooth. However, when the smooth transition map \( T \) is introduced, \( \pi^+_{\varepsilon} \) becomes an asymptotic series in \( \varepsilon^{1/3} \). This is most easily conveyed by letting \( T \) be a smooth map of the form,

\[
T : (x_1, z_1, w_1, u_1) \mapsto (x_1, z_1, w_1, u_1 + a(z_1, u_1)x_1^{8}),
\]

for some smooth function \( a \). Then composing yields,

\[
\pi^+_{\varepsilon} = D_{2, \varepsilon}^- \circ T \circ D_{1, \varepsilon}^+ = (\varepsilon y_0, \varepsilon z_0, \varepsilon w_0, u_0 + a(0, 0)\varepsilon^{8/3} y_0^{8/3} + \ldots),
\]

giving the \( 8/3 \) regularity. This is the essence of the finite differentiability in the regularisation of SBCs.

It is worth remarking that the block map \( \pi \) may be smoother in certain directions. To see this, consider the component \( \pi^+_{\varepsilon} \). The hyperbolic transition maps are given by

\[
D^+_1 : (x_1, y_1, w_1, u_1) = (\varepsilon y_0, \varepsilon^{-2} y_0^{-3} y_0, z_0^{-1} w_0, u_0) \\
D^-_2 : (y_3, z_3, w_3, u_3) = (x_2 y_2, x_2, x_2 w_2, u_2)
\]

Provided \( y_0 \neq 0 \) we have, as \( \varepsilon \to 0, y_1 \to \infty \). Hence \( D^+_1 \) is not continuous in \( \varepsilon \) for \( y_0 \neq 0 \). But, \( y_1 \to \infty \) merely implies, as we consider smaller \( \varepsilon \), that points starting on \( \Sigma_0 \) will intersect a different face of \( \Sigma_{\text{int}} \). In this case, provided \( y_0 \neq 0 \), points on \( \Sigma_0 \) for sufficiently small \( \varepsilon \) will intersect \( \Sigma^+ \). The only case left to treat then is \( y_0 = 0 \).

Now assume the transition map \( T \) between \( \Sigma^+ \) and \( \Sigma^- \) is something of the form

\[
T : (x_1, 0, w_1, u_1) \mapsto (x_1, 0, w_1, u_1 + b(w_1)x_1^{8}),
\]

as before with \( b \) some smooth function. Then composing yields,

\[
\pi^+_{\varepsilon} = D_{2, \varepsilon}^- \circ T \circ D_{1, \varepsilon}^+ = (0, \varepsilon w_0, u_0 + b(w_0 z_0^{-1})\varepsilon^{8/3} z_0^{8} + \ldots).
\]

Provided \( z_0 \neq 0 \), the regularity in the \( z \)-direction is then \( 8 \), smoother than the \( 8/3 \) of the \( y \)-direction. The final direction, the \( w \)-direction, is analogous to the \( z \)-direction.
To turn this sketch into a proof, one has to deal with all the possible resonant terms in the normal form which prevent the linearisation of the vector field near \( \mathcal{N} \). Moreover, the transition map \( T \) must be shown to behave as suggested in (4.15).

4.3. Proof of \( C^{8/3} \)-regularisation

In this section the necessary theory to prove theorem 4.12 on the \( C^{8/3} \)-regularisation of SBCs is developed. The work in [9] is used in appendix B to obtain lemma 4.10, an asymptotic expansion of the hyperbolic transitions \( D_1, D_2 \) in the planar four-body problem. Then the smooth transition map \( T \) is computed to sufficiently high order using the integrals found in remark 4.8 and the result summarised in lemma 4.11. Ultimately, the asymptotic series computed for each \( D_2, T, D_1 \) are composed to prove theorem 4.12.

4.3.1. Asymptotic structure of the hyperbolic transitions

Following the geometric sketch in section 4.2, we seek the asymptotic expansions of the transition \( D_1, D_2 \) near the normally hyperbolic manifold \( \mathcal{N} \) corresponding to SBCs. Consider again the system \( X^9 \), defined in (4.8). We want to choose coordinates so that \( \mathcal{N} \) corresponds to \([\alpha, \beta, \gamma, \delta] = [1, 0, 0, 0]\). In doing so, it can be ensured that the chart \( U_\alpha \) contains \( \mathcal{N} \) whilst \( U_\beta, U_\gamma, U_\delta \) contain no singularities. Similar to the collinear problem in [11], this is done by first making the transform,

\[
J_1 = \frac{1}{2}(I_1 + I_2), \quad J_2 = \frac{1}{2}(I_1 - I_2) \tag{4.16}
\]

Then \( X^9 \) is transformed to a system of the form,

\[
J'_1 = J_1^2 + J_2^2 + \frac{1}{2}(L_1^2 + L_2^2) + \tilde{R}_1^1(J, L, h_1, h_2) \\
+ \tilde{R}_1^0(J, L, h_1, h_2) + \tilde{R}_1^0(J, L, h_1, h_2) \\
L'_1 = 0 \\
J'_2 = -2J_1J_2 - \frac{1}{2}(L_1^2 - L_2^2) + \tilde{R}_2^1(J, L, h_1, h_2) \\
+ \tilde{R}_2^0(J, L, h_1, h_2) + \tilde{R}_2^0(J, L, h_1, h_2) \tag{4.17}
\]

\[
L'_2 = 0 \\
\alpha'_1 = \alpha_1^{-1/3} \tilde{b}_r \tilde{R}_3^1(J, L, \Gamma^1) \\
\beta'_1 = -\alpha_2^{-1/3} \tilde{b}_r \tilde{R}_3^1(J, L, \Gamma^1) \\
\Gamma'_1 = \Gamma'_2 = x' = y' = 0.
\]

Let \( \tilde{X}_9 \) be the vector field associated to (4.17).

Performing the \( \alpha \)-directional blow-up on \( \tilde{X}_9 \), we obtain the system,

\[
r'_\alpha = r_\alpha \left( 1 + \gamma_\alpha^2 + \frac{1}{2}(\beta_\alpha^2 + \delta_\alpha^2) \right) + r_\alpha^2 \tilde{R}_1^1(1, \beta_\alpha, \gamma_\alpha, \delta_\alpha, h) \\
+ r_\alpha^3 \tilde{R}_2^0(1, \beta_\alpha, \gamma_\alpha, \delta_\alpha, h) + O(r_\alpha^4) \\
\beta'_\alpha = -\beta_\alpha r_\alpha^{-1} \Gamma'_\alpha \\
\gamma'_\alpha = -3\gamma_\alpha - \gamma_\alpha \left( \gamma_\alpha^2 + \frac{1}{2}(\beta_\alpha^2 + \delta_\alpha^2) \right) - \frac{1}{2}(\beta_\alpha^2 - \delta_\alpha^2) + 3r_\alpha^2 \tilde{R}_2^0(\beta_\alpha, \gamma_\alpha, \delta_\alpha, h) \tag{4.18}
\]
with

\[
\bar{R}_{3,\alpha}^{\dagger}(\beta_{\alpha}, \gamma_{\alpha}, \delta_{\alpha}, h) := \bar{R}_{3}^{\dagger}(1, \beta_{\alpha}, \gamma_{\alpha}, \delta_{\alpha}, h) - \gamma_{\alpha} \bar{R}_{1}^{\dagger}(1, \beta_{\alpha}, \gamma_{\alpha}, \delta_{\alpha}, h). \tag{4.19}
\]

\[
\kappa_{\alpha} = 6r^{\dagger}_{\alpha} \left( \gamma_{\alpha} + \frac{1}{3} \delta_{\alpha} + \frac{1}{2} \delta_{\alpha}^{2} (1 + \gamma_{\alpha}) \right) + r^{\dagger}_{\alpha} \tilde{G}^{\dagger}_{\alpha}(\beta_{\alpha}, \gamma_{\alpha}, \delta_{\alpha}) + O(\gamma_{\alpha}^{8}). \tag{4.20}
\]

Let the vector field associated to (4.18) be \( \bar{X}_{\alpha} \).

Using the approximate integral \( \kappa \) introduced in (4.10), the leading order terms of the normal form transformation for the hyperbolic singularity at \( \mathcal{N} \) can be computed. In the current coordinates \( r_{\alpha}, \beta_{\alpha}, \gamma_{\alpha}, \delta_{\alpha} \), the approximate integral is given by,

\[
\int \tilde{G}_{\alpha}(\beta_{\alpha}, \gamma_{\alpha}, \delta_{\alpha}) = \int \tilde{G}_{\alpha}(\beta_{\alpha}, \gamma_{\alpha}, \delta_{\alpha}) + O(\gamma_{\alpha}^{8}).
\]

In the vector field the equations for \( r_{\alpha}, \beta_{\alpha}, \delta_{\alpha} \) are all proportional to \( r^{\dagger}_{\alpha} / r_{\alpha} \), which can be removed by a time scaling. The equation for \( \gamma_{\alpha} \) can be simplified by observing that up to a factor \( 6r^{\dagger}_{\alpha} \) the integral to leading order is equal to \( \gamma_{\alpha} \). Thus it is used to introduce a modified \( \tilde{\gamma}_{\alpha} \) which satisfies a much simpler equation. Thus introducing the coordinate \( \tilde{\gamma}_{\alpha} \) through

\[
\tilde{\gamma}_{\alpha} = \frac{1}{6} r^{-3}_{\alpha} \kappa,
\]

provides the normal form coordinates to order 8 in \( r_{\alpha} \). With an additional time rescaling of \( d\tau = r^{-1}_{\alpha} d\tilde{\tau} \), the vector field (4.18) is put in the normal form

\[
\begin{align*}
\dot{r}'_{\alpha} & = r_{\alpha} + O(\gamma^{8}_{\alpha}) \\
\dot{\beta}'_{\alpha} & = -\beta_{\alpha} + O(\gamma^{8}_{\alpha}) \\
\dot{\gamma}'_{\alpha} & = -3\gamma_{\alpha} + O(\gamma^{8}_{\alpha}) \\
\dot{\delta}'_{\alpha} & = -\delta_{\alpha} + O(\gamma^{8}_{\alpha}) \\
\dot{h}'_{1} & = \dot{h}'_{2} = x' = y' = \Gamma'_{1} = \Gamma'_{2} = 0 + O(\gamma^{8}_{\alpha})
\end{align*}
\tag{4.22}
\]

an all the extraneous polynomial terms of degree 4 and 6 have been removed.

As argued in section 4.2, we want to compute the transitions \( D_{1} : \Sigma_{0} \to \Sigma_{\text{int}} \) and \( D_{2} : \Sigma_{\text{int}} \to \Sigma_{3} \) for some surfaces \( \Sigma_{0}, \Sigma_{3} \) transverse to the manifolds \( \mathcal{E}^{+}, \mathcal{E}^{-} \) respectively, and \( \Sigma_{\text{int}} \) homeomorphic to \( (-a, a) \times S^{2} \times \mathbb{R}^{8} \) and transverse to the unstable manifold. We will study these transitions for the specific sections,
In this section we show that Nonlinearity C ∼ D r due the analogous properties of the well studied Dulac maps near hyperbolic saddles. A good review of results known for the planar Dulac maps is given in [29]. Theory for Dulac maps near normally hyperbolic invariant manifolds is given in [9].

Studying the hyperbolic transitions \( D_1, D_2 \) is most easily done by splitting up their actions on the various faces of \( \Sigma_{\text{int}} \). Explicitly, let

\[
\Sigma^\pm_\eta := \{ \eta = \pm 1 \} \cap \Sigma_{\text{int}} \quad \text{for } \eta = \beta, \gamma, \delta,
\]

so that \( \Sigma_{\text{int}} = \bigcup_\eta \Sigma^\pm_\eta \) and define,

\[
D^+_1 : \Sigma_0 \to \Sigma^+_\eta, \quad D^+_2 : \Sigma^+_\eta \to \Sigma_3.
\]

For each \( \eta = \beta, \gamma, \delta \), the maps \( D^+_2 \) will be denoted the \( \eta \)-directional Dulac map and \( D^+_1 \) its inverse.

To prove the \( C^{8/3} \)-regularity of the block map \( \varpi \) it is sufficient to study just the maps \( D^+_1 \), \( D^+_2 \), because the factor 3 in the vector field only appears in the \( \tilde{\gamma}_\alpha \) component. Let \( (r_1, \beta_1, \delta_1, u_1), (-r_2, -\beta_2, -\delta_2, u_2) \), be coordinates on \( \Sigma^+_\beta, \Sigma^-_\delta \) respectively. From both proposition B.2 and lemma B.3 the following lemma on the asymptotic expansion of the hyperbolic transitions near \( N \) can be concluded.

**Lemma 4.10.** Scale \( (\beta_0, \gamma_0, \delta_0) = \varepsilon (\beta, \gamma, \delta) \). Then the hyperbolic transitions \( D^+_{1,5a} \) and \( D^+_{2,5a} \) have the asymptotic approximations

\[
D^+_{1,5a} : \begin{align*}
\beta_1 & \sim \varepsilon^{2/3} \gamma_0^{-1/3} \delta_0 \left( 1 + O(\varepsilon^3 \ln \varepsilon) \right) \\
\delta_1 & \sim \varepsilon^{2/3} \gamma_0^{-1/3} \delta_0 \left( 1 + O(\varepsilon^3 \ln \varepsilon) \right) \\
u_1 & \sim u_0 + O(\varepsilon^3 \ln \varepsilon)
\end{align*}
D^+_{2,5a} : \begin{align*}
\beta_3 & \sim \varepsilon^{1/3} \gamma_0 \delta_0 \left( 1 + O(\varepsilon^3 \ln \varepsilon) \right) \\
\delta_3 & \sim \varepsilon^{1/3} \gamma_0 \delta_0 \left( 1 + O(\varepsilon^3 \ln \varepsilon) \right) \\
u_3 & \sim u_2 + O(\varepsilon^3 \ln \varepsilon)
\end{align*}
\]

for \( u = (h_1, h_2, x, y, \Gamma_1, \Gamma_2) \).

There are several technical details needed to prove lemma 4.10. The proof is given in appendix B.

**4.3.2. Smooth transition \( T \).** In this section we show that \( T \) is of the form sketched in (4.15). As mentioned in section 4.2, it is useful to split \( \Sigma_{\text{int}} \) into its various faces \( \Sigma^\pm_\eta \), \( \eta = y, z, w \). In doing so, \( T \) should also split over its action on the various faces. Moreover, recall that, due to the fact that \( C \) is a Möbius bundle, \( T : \Sigma^\pm_\eta \to \Sigma^\pm_\eta \). For clarity, denote each restriction by,

\[
T^\pm_\eta : \Sigma^\pm_\eta \to \Sigma^\pm_\eta, \quad \eta = \beta, \delta, \gamma.
\]
We will focus on computing $T_{\tilde{\gamma}_a} : \Sigma^+_{\tilde{\gamma}_a} \rightarrow \Sigma^-_{\tilde{\gamma}_a}$. This can be computed from the $\gamma$-directional blow-up. In the rotated system $\tilde{X}_\gamma$, given by (4.17), the $\gamma$-directional blow-up produces the vector field $\tilde{X}_\gamma$, given as,

$$
\tilde{r}_\gamma = -r_\gamma (2\alpha_\gamma + \frac{1}{2}(\beta_\gamma^2 - \delta_\gamma^2)) + r_\gamma \tilde{R}_1^\gamma (\alpha_\gamma, \beta_\gamma, 1, \delta_\gamma, h)
$$

$$+ r_\gamma \tilde{R}_2^\gamma (\alpha_\gamma, \beta_\gamma, 1, \delta_\gamma, h) + O(r_\gamma^3)
$$

$$\alpha_\gamma' = 1 + 3\alpha_\gamma^\gamma_\gamma_\gamma + \frac{1}{2}(\beta_\gamma^2 + \delta_\gamma^2) + \frac{1}{2} \alpha_\gamma (\beta_\gamma^2 - \delta_\gamma^2) + r_\gamma^\gamma_\gamma_\gamma_\gamma_\gamma (\alpha_\gamma, \beta_\gamma, \delta_\gamma, h) + O(r_\gamma^3)
$$

$$\beta_\gamma' = -\beta_\gamma r_\gamma^{-1} r_\gamma' + O(r_\gamma^3)
$$

$$\delta_\gamma' = -\delta_\gamma r_\gamma^{-1} r_\gamma' + O(r_\gamma^3)
$$

$$h_\gamma' = b_\gamma r_\gamma^\gamma_\gamma_\gamma_\gamma_\gamma (\alpha_\gamma, \beta_\gamma, 1, \delta_\gamma) + O(r_\gamma^9)
$$

$$h_\gamma^\gamma_\gamma_\gamma_\gamma_\gamma = -b_\gamma r_\gamma^\gamma_\gamma_\gamma_\gamma_\gamma (\alpha_\gamma, \beta_\gamma, 1, \delta_\gamma) + O(r_\gamma^9)
$$

$$x' = y' = \Gamma_1' = \Gamma_2' = 0 + O(r_\gamma^3)
$$

with

$$
\tilde{R}_1^\gamma (\alpha_\gamma, \beta_\gamma, \delta_\gamma, h) := \tilde{R}_1^\gamma (\alpha_\gamma, \beta_\gamma, 1, \delta_\gamma, h) - \alpha_\gamma \tilde{R}_2^\gamma (\alpha_\gamma, \beta_\gamma, 1, \delta_\gamma, h).
$$

To aid in computing $T_{\tilde{\gamma}_a}$, we will first use the integral $\kappa$ to simplify (4.24). In the $U_\gamma$ chart we have $\kappa$ given by,

$$
\kappa_\gamma = r_\gamma \left( (2 + 6\alpha_\gamma^2 + 3\beta_\gamma^2 + 3\delta_\gamma^2 + 3\alpha_\gamma (\beta_\gamma^2 - \delta_\gamma^2)) + r_\gamma^\gamma_\gamma_\gamma_\gamma_\gamma \tilde{G}_\gamma^\gamma_\gamma_\gamma_\gamma (\alpha_\gamma, \beta_\gamma, \delta_\gamma) + O(r_\gamma^3) \right)
$$

where $\tilde{G}_\gamma$ is $\tilde{G}$ in the $U_\gamma$ chart.

Introduce the coordinate $\tilde{\kappa}_\gamma$ through

$$
\tilde{\kappa}_\gamma = \kappa_\gamma^{1/3}.
$$

Then $\tilde{\kappa}_\gamma$ will satisfy,

$$
\tilde{\kappa}_\gamma = 0 + O(r_\gamma^7).
$$

Using $\tilde{\kappa}_\gamma$ and the approximate integrals $L_1 = r_\gamma \beta_\gamma, L_2 = r_\gamma \delta_\gamma$ we are able to prove the following lemma giving the transition map $T_{\tilde{\gamma}_a}$.

**Lemma 4.11.** Take $u = h_1, h_2, x, y, \Gamma_1, \Gamma_2$ and let $(r_{\alpha_1}, \beta_{\alpha_1}, \delta_{\alpha_1}, u_1), (-r_{\alpha_2}, -\beta_{\alpha_2}, -\delta_{\alpha_2}, u_2)$ be coordinates on $\Sigma^+_{\gamma_\alpha}, \Sigma^-_{\gamma_\alpha}$ respectively. Then the transition map $T_{\tilde{\gamma}_a} : \Sigma^+_{\gamma_\alpha} \rightarrow \Sigma^-_{\gamma_\alpha}$ is,

$$
T_{\tilde{\gamma}_a} : \begin{cases}
    r_{\alpha_2} = r_{\alpha_1} + O(r_{\alpha_1}^7) & \delta_{\alpha_2} = \delta_{\alpha_1} + O(r_{\alpha_1}^7) \\
    \beta_{\alpha_2} = \beta_{\alpha_1} + O(r_{\alpha_1}^7) & u_2 = u_1 + O(r_{\alpha_1}^7)
\end{cases}
$$

**Proof.** The transition $T_{\tilde{\gamma}_a}$ in the lemma is given in the coordinates $\eta_\alpha$. First we compute the transition in the $\eta_\gamma$ coordinates before converting to the $\eta_\alpha$ coordinates. Let $\Sigma^+_{\gamma_\alpha}, \Sigma^-_{\gamma_\alpha}$ be the
For any choice of masses, the SBC is precisely \( C^{8/3} \)-regularisable in the planar four-body problem.

**Proof.** It must be shown that the block map \( \mathcal{P} \) is at least, and at most, \( C^{8/3} \) regular. In [11] it was shown that the collinear problem is precisely \( C^{8/3} \) regularisable. The collinear problem naturally embeds inside the planar problem. Hence, there is a submanifold, the collinear submanifold, for which the restriction of \( \mathcal{P} \) is known to be \( C^{8/3} \) regularisable. This automatically gives that the full map \( \mathcal{P} \) is at best \( C^{8/3} \). However, the question remains whether it is of worse regularity away from collinearity.

In what follows, let \( u = (u_1, u_2, x, y, \Gamma_1, \Gamma_2) \). We will collect the various results of the preceding section for the proof.

Take the two sections,

\[
\Sigma_0 = (1, \beta_{a0}, \gamma_{a0}, \delta_{a0}, u_0), \quad \Sigma_3 = (-1, -\beta_{a3}, -\gamma_{a3}, -\delta_{a3}, u_1)
\]

defined in the \( \alpha \)-directional blow-up. Note that \( \Sigma_0, \Sigma_3 \) is transverse to \( \mathcal{E}^+ \) and \( \mathcal{E}^- \) respectively. Further, define \( \tilde{\gamma}_\alpha \) through equation (4.21) and consider the two sections, transverse to \( \mathcal{C} = \{ r_\alpha = 0 \} \),

\[
\Sigma^+_{\tilde{\gamma}_\alpha} = \{ (r_{\alpha1}, \beta_{\alpha1}, \tilde{\gamma}_\alpha, \delta_{\alpha1}, u_1) | \tilde{\gamma}_\alpha = 1 \},
\]

\[
\Sigma^-_{\tilde{\gamma}_\alpha} = \{ (-r_{\alpha2}, -\beta_{\alpha2}, \tilde{\gamma}_\alpha, -\delta_{\alpha2}, u_2) | \tilde{\gamma}_\alpha = -1 \}.
\]

The section of the block map with \( \tilde{\gamma}_\alpha > 0 \) is then given by,
If we keep track of their removable terms and worse so, the coefficients in the vector field as computed from the normal form are fartoo planar problem the trajectories on the collision manifold are significantly more complicated collision manifold and the coefficients of the vector field near the collision manifold. In the first order linear differential equation with coefficients dependent on the trajectories on the collision manifold, on one can quickly identify that the block map is regular in the variables. Hence $\Gamma$ is at least $C^{8/3}$ even away from collinearity.

5. Concluding remarks

If we keep track of the irremovable terms $R_3$ in the $h_j$ components of the normal form $X_3$ then ultimately they produce non-zero terms at order $\varepsilon^{8/3}$ of the $h_j$ components of the block map $\pi$. That is $h_\beta$ has the form,

$$h_\beta = h_\beta + F_j(\beta_{\alpha_0}, \gamma_{\alpha_0}, \delta_{\alpha_0}, \Gamma_1^+, \Gamma_2^+)\varepsilon^{8/3} + O(\varepsilon^3)$$

with $F_j$ a $C^{8/3}$ function in $\beta_{\alpha_0}, \gamma_{\alpha_0}, \delta_{\alpha_0}$ and polynomial in $\Gamma_1^+, \Gamma_2^+$. If the function $F_j$ is zero for a given set of values $(\beta_{\alpha_0}, \gamma_{\alpha_0}, \delta_{\alpha_0}, \Gamma_1^+, \Gamma_2^+)$ the block map $\pi$ will be more than $C^{8/3}$ regular in the direction $[\beta_{\alpha_0}^*, \gamma_{\alpha_0}^*, \delta_{\alpha_0}^*, \Gamma_1^*, \Gamma_2^*]$ along $\mathcal{N}$.

For the collinear problem, $F_j$ has been explicitly computed [11]. This was done by blowing up and explicitly computing the smooth transition map to order 8 (rather than order 7 as done here) through the use of the variational equations. This involves solving a non-autonomous first order linear differential equation with coefficients dependent on the trajectories on the collision manifold and the coefficients of the vector field near the collision manifold. In the planar problem the trajectories on the collision manifold are significantly more complicated and worse so, the coefficients in the vector field as computed from the normal form are far too
unwieldy. This is why in this paper the error terms are one order less than in the collinear paper [11].

The fact that $F_j$ is polynomial in $\Gamma_1^*, \Gamma_2^*$ and, due to the fact $\pi$ is $C^{6/3}$, $F_j$ is not identically zero, it is guaranteed that the set of zeroes of $F_j$ is of measure 0. The interesting question of whether this set is empty remains. In particular, there are two questions worth pursuing:

(a) Is there a choice of asymptotic point $\Gamma_1^*, \Gamma_2^*$ for which $F_j = 0$?
(b) Is there a choice of $\beta_{i0}, \gamma_{i0}, \delta_{i0}$ and a choice of $\Gamma_1^*, \Gamma_2^*$ for which $F_j = 0$?

Of course question 2 is a weaker question than 1. An answer to either would provide regions of phase space for which the SBC is more regularisable, perhaps even $C^\infty$. In such a case, by finding sub-problems, like the trapezoidal, caledonian, collinear, etc, which are contained within these regions, one may be able to find a Levi-Civita type transformation for these sub-problems which smoothly regularise the collision.

Appendix A. Potential and normal form functions

The mass constants in (4.6) are given by

\[
\begin{align*}
    b_0 &= (m_1 + m_2)(m_3 + m_4), \\
b_{12} &= b_0 \frac{m_1 m_2}{8(m_1 + m_2)^2}, & b_{22} &= b_0 \frac{m_3 m_4}{8(m_3 + m_4)^2}, \\
b_{13} &= b_0 \frac{m_1 m_2(m_1 - m_2)}{16(m_1 + m_2)^3}, & b_{23} &= b_0 \frac{m_3 m_4(m_3 - m_4)}{16(m_3 + m_4)^3}, \\
b_{14} &= b_0 \frac{m_1 m_2(m_1^2 - m_1 m_2 + m_2^2)}{128(m_1 + m_2)^4}, & b_{24} &= b_0 \frac{m_3 m_4(m_3^2 - m_1 m_4 + m_4^2)}{128(m_3 + m_4)^4}, \\
b_c &= \frac{3}{64}M_1 M_2
\end{align*}
\]

Each of the homogeneous polynomials $W_j$ are given by,

\[
\begin{align*}
    W_2(Q) &= 3\bar{Q}^2 + 2\bar{Q}\bar{Q} + \bar{Q}^2 \\
    W_3(Q) &= 5\bar{Q}^3 + 3\bar{Q}^2\bar{Q} + 5\bar{Q}\bar{Q}^2 + 5\bar{Q}^3 = (\bar{Q} + \bar{Q})(5\bar{Q}^2 - 2\bar{Q}\bar{Q} + 5\bar{Q}^2) \\
    W_4(Q) &= 35\bar{Q}^4 + 20\bar{Q}^3\bar{Q} + 18\bar{Q}^2\bar{Q}^2 + 20\bar{Q}\bar{Q}^3 + 35\bar{Q}^4 \\
    W_c(Q_1, Q_2) &= 35\bar{Q}_1^2\bar{Q}_2^2 + 10\bar{Q}_1\bar{Q}_2^2\bar{Q}_1 + 3\bar{Q}_2^2\bar{Q}_1^2 + 10\bar{Q}_1^2\bar{Q}_2 \bar{Q}_1 \\
    &\quad + 12\bar{Q}_1\bar{Q}_2\bar{Q}_1\bar{Q}_2 + 10\bar{Q}_2\bar{Q}_1^2\bar{Q}_2 + 3\bar{Q}_1^2\bar{Q}_2^2 + 10\bar{Q}_1\bar{Q}_2\bar{Q}_1^2 \\
    &\quad + 35\bar{Q}_1^2\bar{Q}_2^2
\end{align*}
\]

The normal form functions are given by

\[
\begin{align*}
    R_c^0(\zeta_1, \zeta_2, h_1, h_2) &= \frac{4}{5}L_1^2((h_1 + h_1^*)(h_1^2 - h_2 + h_2^2)L_2^2) \\
    R^0_c(\zeta_1, \zeta_2, h_1, h_2) &= -\frac{16}{25}L_1^2L_2^2(h_1 + h_1^*)(h_2 + h_2^2) + \frac{4}{107925}(h_1 + h_1^*)^2R_{11}^6 \\
    &\quad + \frac{4}{107925}(h_2 + h_2^2)R_{12}^6
\end{align*}
\]
\[ R^6_{11}(\zeta_1, \zeta_2, h_1, h_2) = 33300l_1^4l_2^2 - 21645l_1^2l_2^2 + 34965l_1^4l_2^2 + 11285l_1^2L_1^2L_2^2 \]
\[ - 23026L_1^2L_2^2 - 59385l_1^2L_2^2 - 46990l_1L_2^4 \]
\[ R^6_{12}(\zeta_1, \zeta_2, h_1, h_2) = 6105l_1^4 + 34965l_1^2l_2^2 + 46990l_1^2L_1^2 + 18130L_1^6 + 5550l_1^4L_2^2 \]
\[ - 10545l_1L_3^2l_2^2 - 5550l_2^4 - 11285l_1L_3^2L_2^2 + 78972L_6^2 \]

(A.3)

Writing \( \alpha = 4 \arg(\Gamma) \) the first few terms in \( R_3^\alpha \) are

\[ R_3^\alpha = \frac{8}{19} (9l_1 - l_2)(l_1^2 + l_1l_2 + l_2^2)(l_1^6 - 11l_1^4l_2^2 + l_2^6) (6 + 10 \cos \alpha_1 \]
\[ + 10 \cos \alpha_2 + 35 \cos(\alpha_1 + \alpha_2) + 3 \cos(\alpha_1 - \alpha_2)) + \]
\[ - \frac{2}{209} l_1L_3^2(5l_1^4 - 35l_1^2l_2^2 + 14l_2^4) (109 (10 \cos \alpha_1 + 35 \cos(\alpha_1 + \alpha_2) \]
\[ + 3 \cos(\alpha_1 - \alpha_2)) - 86 (3 + 5 \cos \alpha_2)) + \{1 \leftrightarrow 2 \} \]
\[ + \frac{256}{13} L_3^2(l_1^6 - 5l_1^4l_2^2 + l_2^6)(10 \sin \alpha_1 + 35 \sin(\alpha_1 + \alpha_2) \]
\[ + 3 \sin(\alpha_1 - \alpha_2) - \{1 \leftrightarrow 2 \} \]  

(A.4)

plus higher order terms in \( L_4 \), where \( \{1 \leftrightarrow 2 \} \) denotes the exchange of indices in the preceding term.

The relevant functions for the integral \( \kappa \) in equation (4.10) are given by,

\[ G^2(\zeta_1, \zeta_2, h_1, h_2) = \frac{4}{5}(l_1^2 + 6l_1L_3^2 - l_2^2)((h_1 + h_2)L_1^2 - (h_2 + h_2)L_2^2) \]
\[ G^2(\zeta_1, \zeta_2, h_1, h_2) = (h_1 + h_2)^2 \left( G_1^2(\zeta_1, \zeta_2) + \frac{1}{2} G_2^2(\zeta_1, \zeta_2) \right) \]
\[ + (h_1 + h_2)(h_2 + h_2)G_3^2(\zeta_1, \zeta_2) \]
\[ - (h_2 + h_2)^2 \left( G_1^2(\zeta_2, \zeta_1) - \frac{1}{2} G_2^2(\zeta_2, \zeta_1) \right) \]
\[ G_1^2(\zeta_1, \zeta_2) = -29370604l_1^4L_1^2L_2 - 39471605l_1^4L_2 - 1522920L_4L_2 \]
\[ + 1107225l_1^4L_2^2 + 3181815l_1^2L_2^2 - 807525L_2^2 \]
\[ + 2447550l_1^2L_2^2 + 6394710l_1^2L_2^2 \]
\[ - 2692305l_2^4L_2^2 - 944468l_1^4L_2^2 - 899220l_1^2L_2^2 \]
\[ - 1503082l_1^2L_2^2 - 3800244L_2^2 \]
\[ G_2^2 = 56 (18315l_1^4L_2^2 - 27973l_1^3L_2^2 - 32115l_1^2L_2^2 \]
\[ + 27973l_1^2L_2^4 + 135723l_2^2L_2^2 \]
\[ G_3^2(\zeta_1, \zeta_2) = -\frac{16}{25} L_1^2(l_1^2 + 12l_1L_3^2 - L_2^2) L_2^2 \]
Appendix B. Proof of lemma 4.10

In this section we state and use the required technicalities from [9] to prove lemma 4.10. The theory developed in [9] is more general than what is needed in this appendix. As such, all referenced propositions have been modified for their particular application in this work.

Assume that the one-dimensional invariant manifold leaving a point on \( \mathcal{N} \) in the normal direction is stable. By proposition 3.2, the non-zero eigenvalues of \( \mathcal{N} \) are \((-1, 1, 1, 1)\) up to scaling. Assume that coordinates \((x, y, z, w, u) \in \mathbb{R}^5 \times \mathbb{R}^8\) have been chosen near \( \mathcal{N} \) so that the local stable manifold is given by \( y = z = w = 0 \) and the local unstable manifold by \( x = 0 \). Due to the fact the non-zero eigenvalues of \( \mathcal{N} \) are resonant, it is not, in general, possible to linearise the vector field in a tubular neighbourhood of \( \mathcal{N} \). The following proposition from [9] gives the ‘simplest’ representation of a vector field in a neighbourhood of \( \mathcal{N} \). This is the so called normal form.

**Proposition B.1 ([9]).** Define the monomials,

\[
U_y = x^3 y, \quad U_z = x z, \quad U_w = x w
\]

and let \( n = (n_1, n_2, n_3) \in \mathbb{N}^3 \) and \( |n| := n_1 + n_2 + n_3 \). There exists a smooth, near-identity transformation \( \Phi \) and a smooth time rescaling bringing \( X \) into the normal form \( X_N \),

\[
\dot{x} = -x
\]

\[
\dot{y} = 3y + y \sum_{|n| \geq 1} G^{(n)}_y(u) U^n_y U^n_z U^n_w + y U^{-1}_y \sum_{n_1 + n_2 \geq 3} B^{(n)}_y(u) U^n_y U^n_z
\]

\[
\dot{z} = z + z \sum_{|n| \geq 1} G^{(n)}_z(u) U^n_y U^n_z U^n_w + z U^{-1}_z \sum_{n_1 + n_2 \geq 1} B^{(n)}_z(u) U^n_y U^n_z
\]

\[
\dot{w} = w + w \sum_{|n| \geq 1} G^{(n)}_w(u) U^n_y U^n_z U^n_w + w U^{-1}_w \sum_{n_1 + n_2 \geq 1} B^{(n)}_w(u) U^n_y U^n_z
\]

\[
\dot{u} = \sum_{|n| \geq 1} G^{(n)}_u(u) U^n_y U^n_z U^n_w,
\]

where \( G^{(n)}_\eta(u), B^{(n)}_\eta(u) \) are smooth functions in \( u \) for each \( \eta = x, y, z, w, u \).

Assume that \((x, y, z, w, u)\) are the coordinates so that the vector field in a tubular neighbourhood of \( \mathcal{N} \) is given by the normal form (B.1). Consider the section \( \Sigma = [0, 1] \times [-1, 1]^3 \times \mathbb{R}^8 \) defined in the normal form coordinates and its various faces,

\[
\Sigma_x := \Sigma \cap \{ x = 1 \}, \quad \Sigma^\pm_x := \Sigma \cap \{ \eta = \pm 1 \}, \quad \eta = y, z, w.
\]

Denote the Dulac map by the continuous map,

\[
D : \Sigma^\pm_x \cup \Sigma^\pm_z \cup \Sigma^\pm_w \rightarrow \Sigma^\pm_x,
\]

and its action restricted to each face by,

\[
D^\pm_\eta : \Sigma^\pm_\eta \rightarrow \Sigma_x, \quad \eta = y, z, w.
\]
The following proposition, adapted from [9], gives the asymptotic properties of $D$.

**Proposition B.2 ([9])**. Let $(y_0, z_0, w_0, u_0, x_1, y_1, z_1, w_1, u_1)$ be coordinates on $\Sigma_\eta, \Sigma_\eta^*$, $\eta = y, z, w$ respectively, and define

$$
U_0^0 = x_3^3 y_1, \quad U_0^1 = x_1^3 z_1, \quad U_0^2 = x_1 w_1, \quad U_0^n := (t_0^n)^{n_1} (t_0^n)^{n_2} (t_0^n)^{n_3}
$$

for $n = (n_1, n_2, n_3) \in \mathbb{N}^3$. Then the components of the Dulac map have the asymptotic series,

$$
y_0 \sim x_1 \left( y_1 + y_1 \sum_{|n| \geq 1} \tilde{G}^{(1)}_y(u_1, \ln x_1) U_0^n + x_1^{-3} \right) \\
\times \sum_{n_1 + n_2 \geq 3} \tilde{B}^{(1,0)}_y(u_1, \ln x_1) (U_0^n)^{n_1} (U_0^n)^{n_2} \\
z_0 \sim x_1 \left( z_1 + z_1 \sum_{|n| \geq 1} \tilde{G}^{(1)}_z(u_1, \ln x_1) U_0^n + x_1^{-1} \right) \\
\times \sum_{n_1 + n_2 \geq 3} \tilde{B}^{(1,0)}_z(u_1, \ln x_1) (U_0^n)^{n_1} (U_0^n)^{n_2} \\
w_0 \sim x_1 \left( w_1 + w_1 \sum_{|n| \geq 1} \tilde{G}^{(1)}_w(u_1, \ln x_1) U_0^n + x_1^{-1} \right) \\
\times \sum_{n_1 + n_2 \geq 3} \tilde{B}^{(1,0)}_w(u_1, \ln x_1) (U_0^n)^{n_1} (U_0^n)^{n_2} \\
u_0 \sim u_1 + \sum_{|n| \geq 1} \tilde{G}^{(1)}_u(u_1, \ln x_1) U_0^n
$$

(B.2)

with $y_1, z_1, w_1$ set to $\pm 1$ for $D^\pm_1, D^\pm_2, D^\pm_3$ respectively. Each coefficient function $K^{(1)} = \tilde{G}^{(1)}, B^{(1)}$ are:

(a) Polynomial in $\ln x_1$ with vanishing constant term and smooth in $u_1$.
(b) Polynomial in $B^{(1)}_y(u_1), G^{(1)}_y(u_1), \eta = y, z, w, u$, with vanishing constant term for all $|n| \leq |n|$. Moreover, $B^{(1)} = 0$ (resp. $G^{(1)}_u = 0$) for $|n| \leq m$ if and only if $B^{(1)} = 0$ (resp. $G^{(1)}_u = 0$) for $|n| \leq m$.

A crucial point in proposition B.2 is the fact that, if $B^{(1)}_y(u_0), G^{(1)}_y(u_0), \eta = y, z, w, u$, vanishes for all $|n| \leq |n|$, then the coefficients, $G^{(1)}, B^{(1)}$, in the Dulac maps vanish as well. If one only knows the normal form (B.1) up to some order in $(x, y, z, w)$ then one can infer to what order $D$ is known. For example, if it is known that the normal form $X_\eta$ in (B.1) has no resonant terms of type $G^{(1)}_y(u)$ for $|n| < m$ and $B^{(1)}_y(u)$ for $n_1 + n_2 < m + 1$ and $\eta = y, z, w, u$, then the Dulac maps are known to $|n| = m$ in the $G$ summation and $n_1 + n_2 = m + 1$ in the $B$ summation.

To prove lemma 4.10 an understanding of the inverse map $D^{-1}$ is also required. The inverse is not continuous and as such, does not admit an asymptotic expansion in the usual sense.
However, by appropriately rescaling \((y_0, z_0, w_0) = \varepsilon(y_0, z_0, w_0)\), with \(\varepsilon \ll 1\), one can obtain compositional inverse of \(D\) using \(\varepsilon\) as the small parameter. The asymptotic structure of \(D^{-1}\) is not, in general, as nice as the forward map \(D\). However, if the resonant terms are known to vanish to some order, then one can quickly compute the inverse directional Dulac maps up to this order. We prove this in the following lemma.

**Lemma B.3.** Let \(m_1, m_2 > 0\) and suppose that \(G^{(n)} = 0\) for \(|n| < m_1\), \(\eta = x, y, z, w\), that \(B^{(n, n_2)} = 0\) for \(n_1 + n_2 < m_1 + 1, \eta = z, w\), and that \(G^{(n)} = 0\) for \(n_1 + n_2 < m_1 + 3\). Moreover, assume that \(G^{(n)} = 0\) for \(|n| < m_2\).

If \((\varepsilon y_0, \varepsilon z_0, \varepsilon w_0, \varepsilon u_0)\) are scaled coordinates on \(\Sigma_0\), then each of the directional Dulac maps have inverses asymptotic to,

\[
D_{1,y} : \\
\begin{aligned}
x_1 &\sim \varepsilon^{1/3} y_0^{1/3} (1 + O(\varepsilon^{m_1} \ln \varepsilon)) & w_1 &\sim \varepsilon^{2/3} y_0^{1/3} w_0 (1 + O(\varepsilon^{m_1} \ln \varepsilon)) \\
z_1 &\sim \varepsilon^{2/3} y_0^{1/3} z_0 (1 + O(\varepsilon^{m_1} \ln \varepsilon)) & u_1 &\sim u + O(\varepsilon^{m_2} \ln \varepsilon)
\end{aligned}
\]

\[
D_{1,z} : \\
\begin{aligned}
x_1 &\sim \varepsilon z_0 (1 + O(\varepsilon^{m_1} \ln \varepsilon)) & w_1 &\sim \varepsilon z_0 w_0 (1 + O(\varepsilon^{m_1} \ln \varepsilon)) \\
y_1 &\sim \varepsilon^{-2} z_0^{3} y_0 (1 + O(\varepsilon^{m_1} \ln \varepsilon)) & u_1 &\sim u + O(\varepsilon^{m_2} \ln \varepsilon)
\end{aligned}
\]

\[
D_{1,w} : \\
\begin{aligned}
x_1 &\sim \varepsilon w_0 (1 + O(\varepsilon^{m_1} \ln \varepsilon)) & w_1 &\sim \varepsilon^{-2} z_0^{3} w_0 (1 + O(\varepsilon^{m_1} \ln \varepsilon)) \\
y_1 &\sim \varepsilon^{-2} w_0^{3} y_0 (1 + O(\varepsilon^{m_1} \ln \varepsilon)) & z_1 &\sim w_0^{-1} z_0 (1 + O(\varepsilon^{m_1} \ln \varepsilon))
\end{aligned}
\]

(B.3)

**Proof.** We will prove the lemma for \(D_{1,y}^+\) only as the other cases follow analogously. By hypothesis we have \(G^{(n)} = 0\) for \(|n| < m_1\), \(\eta = x, y, z, w\), that \(B^{(n, n_2)} = 0\) for \(n_1 + n_2 < m_1 + 1, \eta = z, w\), and that \(G^{(n)} = 0\) for \(n_1 + n_2 < m_1 + 3\) in the normal form (B.1). From proposition B.2, particular the remark after ‘moreover’, the map \(D_{1}^+\) is asymptotic to,

\[
y_0 \sim x_1 \left( 1 + \sum_{|n| \geq m_1} G^{(n)}_y (u_1, \ln x_1) U_0^m + x_1^{-3} \right) \\
\times \sum_{n_1 + n_2 \geq 3 + m_1} \tilde{B}^{(n_1, n_2)} (u_1, \ln x_1) (U_0^m)^{n_1} (U_0^m)^{n_2}
\]

\[
z_0 \sim x_1 \left( z_1 + z_1 \sum_{|n| \geq m_1} \tilde{G}_z^{(n)} (u_1, \ln x_1) U_0^m + x_1^{-1} \right) \\
\times \sum_{n_1 + n_2 \geq 1 + m_1} \tilde{B}_z^{(n_1, n_2)} (u_1, \ln x_1) (U_0^m)^{n_1} (U_0^m)^{n_2}
\]

\[
u_0 \sim x_1 \left( u_1 + u_1 \sum_{|n| \geq m_1} G^{(n)}_w (u_1, \ln x_1) U_0^m + x_1^{-1} \right)
\]
\[
\times \sum_{n_1+n_2 \geq 1} \beta_{\eta, \beta_2}^{(n_1, n_2)}(u_1, \ln x_1) (U_1^0)^{n_1} (U_2^0)^{n_2} \\

u_0 \sim u_1 + \sum_{|n| \geq m_2} G_{\eta}^{(n)}(u_1, \ln x_1) U_2^n
\] (B.4)

Taking the scaled coordinates \((\varepsilon y_0, \varepsilon z_0, \varepsilon w_0, \varepsilon u_0)\) to keep track of relative size, then this form of \(D_j^+\) gives the first order asymptotics,

\[
x_1 \sim \varepsilon^{1/3} y_0^{1/3}, \quad z_1 \sim \varepsilon^{2/3} z_0 y_0^{-1/3}, \quad w_1 \sim \varepsilon^{2/3} w_0 y_0^{-1/3}, \quad u_1 \sim u_0.
\]

it follows that each of the monomials \(U_0^0, U_1^0, U_2^0\) has the leading order asymptotics,

\[
U_0^0 = \varepsilon x_1^3, \quad U_1^0 = \varepsilon x_1 z_1, \quad U_2^0 = \varepsilon x_1 w_1.
\]

Substituting these two leading order asymptotics into equation (B.4) concludes the lemma. □

At last lemma 4.10 can be proved.

**Proof of lemma 4.10.** The lemma follows provided we can show the hypothesis of lemma B.3 is satisfied for \(m_1 = 2, m_2 = 3\). That is, we need to show \(G_{\eta}^{(n)} = 0\) for \(|n| < 2, \eta = r, \beta, \gamma, \delta, \), that \(B_{\eta}^{(n_1, n_2)} = 0\) for \(n_1 + n_2 < 3, \eta = w, \) that \(G_{\eta}^{(n)} = 0\) for \(n_1 + n_2 < 5, \) and finally that \(G_{\eta}^{(0)} = 0\) for \(|n| < 3, \).

We show only the first hypothesis, \(G_{\eta}^{(n)} = 0\) for \(|n| < 2,\) as the others follows analogously. Observe that \(U_0^n = r^{n_1 + n_2} \beta^{n_1} \gamma^{n_2} \delta^{n_3}.\) Now, system (4.22) has no resonant terms with \(r_\alpha < 4,\) It follows that all resonant terms \(G_{\eta}^{(n)} U_0^n\) in the normal form (B.1) for \(|n| = 1\) must have \(G_{\eta}^{(n)} = 0\) as desired. □

**ORCID iDs**

Nathan Duignan @ https://orcid.org/0000-0002-9433-6490

Holger R Dullin @ https://orcid.org/0000-0001-6617-196X

**References**

[1] Alvarez-Ramirez M and Medina-Valdez M 2014 A review of the planar Caledonian four-body problem Astron. J. 24 1–14

[2] Bakker L F, Ouyang T, Yan D and Simmons S 2011 Existence and stability of symmetric periodic simultaneous binary collision orbits in the planar pairwise symmetric four-body problem Celest. Mech. Dyn. Astron. 110 271–90

[3] Belbruno E A 1984 On simultaneous double collision in the collinear four-body problem J. Differ. Equ. 52 415–31

[4] Belitskii G R 1979 Invariant normal forms of formal series Funct. Anal. Appl. 13 46–7

[5] Belitskii G 2002 C\(_\infty\)-normal forms of local vector fields Acta Appl. Math. 70 23–41

[6] Chenciner A 2015 Poincaré and the three-body problem Henri Poincaré, 1912–2012 (Berlin: Springer) pp 51–149

[7] Conley C and Easton R 1971 Isolated invariant sets and isolating blocks Trans. Am. Math. Soc. 158 35
[8] Duignan N 2019 On the regularisation of simultaneous binary collisions PhD Thesis University of Sydney
[9] Duignan N 2021 Normal forms for manifolds of normally hyperbolic singularities and asymptotic properties of nearby transitions Qual. Theor. Dyn. Syst. 20 26
[10] Duignan N and Dullin H R 2019 Regularisation for planar vector fields Nonlinearity 33 106–38
[11] Duignan N and Dullin H R 2020 On the $C^{3\theta}$-regularisation of simultaneous binary collisions in the collinear four-body problem J. Differ. Equ. 269 7975–8006
[12] Dumortier F and Roussarie R 2010 Smooth normal linearization of vector fields near lines of singularities Qual. Theor. Dyn. Syst. 9 39–87
[13] Easton R 1971 Regularization of vector fields by surgery J. Differ. Equ. 10 92–9
[14] Elbialy M S 1990 Collision singularities in celestial mechanics SIAM J. Math. Anal. 21 1563–93
[15] Elbialy M S 1993 On simultaneous binary collisions in the planar $n$-body problem Z. Angew. Math. Phys. 44 880–90
[16] Elbialy M S 1993 Simultaneous binary collisions in the collinear $N$-body problem J. Differ. Equ. 102 209–35
[17] Elbialy M S 1996 The flow of the $N$-body problem near a simultaneous-binary-collision singularity and integrals of motion on the collision manifold Arch. Ration. Mech. Anal. 134 303–40
[18] Fenichel N 1972 Persistence and smoothness of invariant manifolds for flows Indiana Univ. Math. J. 21 193–226
[19] Hirsch M, Pugh C C and Shub M 1977 Invariant Manifolds (Lecture Notes in Mathematics vol 583) (Berlin: Springer)
[20] Ilyashenko Y and Yakovenko S 2008 Lectures on Analytic Differential Equations (Graduate Studies in Mathematics vol 86) (Providence, RI: American Mathematical Society)
[21] Levi-Civita T 1920 Sur la résolution du problème des trois corps Acta Math. 42 99–144
[22] Lombardi E and Stolovitch L 2010 Normal forms of analytic perturbations of quasihomogeneous vector fields: rigidity, invariant analytic sets and exponentially small approximation Ann. Sci. École Norm. Sup. 43 659–718
[23] Martínez R and Simó C 1999 Simultaneous binary collisions in the planar four-body problem Nonlinearity 12 903
[24] Martínez R and Simó C 2000 The degree of differentiability of the regularization of simultaneous binary collisions in some $N$-body problems Nonlinearity 13 2107
[25] McGehee R 1974 Triple collision in the collinear three-body problem Invent Math. 27 191–227
[26] Moeckel R and Montgomery R 2015 Realizing all reduced syzygy sequences in the planar three-body problem Nonlinearity 28 1919
[27] Mourtada A 1990 Cyclique finie des polycycles hyperboliques de champs de vecteurs du plan mise sous forme normale Bifurcations of Planar Vector Fields pp 272–314
[28] Palis J and Takens F 1977 Topological equivalence of normally hyperbolic dynamical systems Topology 16 335–45
[29] Roussarie R 1998 Bifurcation of Planar Vector Fields and Hilbert's Sixteenth Problem (Progress in Mathematics vol 164) (Basel: Birkhäuser)
[30] Roy A E and Steves B A 1998 Some special restricted four-body problems—I. From Caledonia to Copenhagen Planet. Space Sci. 46 1475–86
[31] Roy A E and Steves B A 2000 The Caledonian symmetrical double binary four-body problem I: surfaces of zero-velocity using the energy integral Celest. Mech. Dyn. Astron. 78 299–318
[32] Sekiguchi M and Tanikawa K 2004 On the symmetric collinear four-body problem Publ. Astron. Soc. Japan 56 235–51
[33] Simó C 2018 Some questions looking for answers in dynamical systems Discrete Continuous Dyn. Syst. A 38 6215–39
[34] Simó C and Lacomba E A 1992 Regularization of simultaneous binary collisions in the $n$-body problem J. Differ. Equ. 98 241–59
[35] Sivasankaran A, Steves B A and Sweatman W L 2010 A global regularisation for integrating the Caledonian symmetric four-body problem Celest. Mech. Dyn. Astron. 107 157–68
[36] Steves B A and Roy A E 1998 Some special restricted four-body problems—I. Modelling the Caledonian problem Planet. Space Sci. 46 1465–74
[37] Stolovitch L 2009 Progress in normal form theory Nonlinearity 22 R77–99