SURGERY FORMULA FOR THE RENORMALIZED EULER CHARACTERISTIC OF HEEGAARD FLOER HOMOLOGY

RAIF RUSTAMOV

ABSTRACT. We prove a surgery formula for renormalized Euler characteristic of Ozsváth and Szabó. The equality \( \hat{\chi} = SW \) between this Euler characteristic and the Seiberg-Witten invariant follows for rational homology three-spheres.

1. Introduction

In [13] and [11] topological invariants for closed oriented three manifolds and cobordisms between them were defined by using a construction from symplectic geometry. The resulting Floer homology package has many properties of a topological quantum field theory.

Another such Floer homology package comes from Seiberg-Witten theory [5], [7]. Similarity of properties of the Ozsváth-Szabó and Seiberg-Witten theories and also calculations heavily support the conjecture that these invariants are equivalent.

In this paper we will concentrate on a numerical invariant of rational homology spheres obtained from the Heegaard Floer homology package - the renormalized Euler characteristic, \( \hat{\chi} \). It is already known that for integral homology spheres \( \hat{\chi} \) is equal to Casson’s invariant [14], which is also the case for the Seiberg-Witten invariant of integral homology spheres [6]. Calculations of [9] push this equivalence further to the Lens spaces and Seifert manifolds. Thus, it is tempting to establish this equality in its whole generality. To this end we prove a surgery formula for \( \hat{\chi} \). This formula and several other properties of \( \hat{\chi} \) and the related invariant \( \chi^{\text{trunc}} \) together fit into the framework of [10] to give equivalence between \( \hat{\chi} \) and the Reidemester-Turaev torsion normalized by the Casson-Walker invariant. This also implies the equality \( \hat{\chi} = SW \).

The organization of the paper is as follows: the required preliminaries are presented in Section 2. The surgery theorem is formulated and its applications are given in Section 3. The paper finishes with the proof of the surgery formula in Section 4.

Acknowledgments I am pleased to thank my advisor Zoltán Szabó whose kind attitude and outstanding teaching made this work very enjoyable. I would like to express my gratitude to Paul Seymour for his support and encouragement.

Key words and phrases. Floer homology, surgery.
 Correction terms and Euler characteristics Let $Y$ be a rational homology sphere, $t$ be a Spin$^c$ structure on it. We can consider Heegaard Floer homology group $HF^+(Y, t)$. This is a $\mathbb{Q}$ graded module over $\mathbb{Z}[U]$. We can also consider a simpler version, $HF^\infty(Y, t)$, for which one can prove
\[
HF^\infty(Y, t) \cong \mathbb{Z}[U, U^{-1}]
\]
for each $t$ structure. There is a natural $\mathbb{Z}[U]$ equivariant map
\[
\pi : HF^\infty(Y, t) \to HF^+(Y, t)
\]
which is zero in sufficiently negative degrees and an isomorphism in all sufficiently positive degrees. $HF^\text{red}_+(Y, t)$ is defined as
\[
HF^\text{red}_+(Y, t) = HF^+(Y, t) / \text{Im} \pi.
\]
Let $d(Y, t)$ be the correction term defined as the minimal degree of any non-torsion class of $HF^+(Y, t)$ lying in the image of $\pi$. Main object of our study, the renormalized Euler characteristic $\hat{\chi}(Y, t)$ is defined by
\[
\hat{\chi}(Y, t) = \chi(HF^\text{red}_+(Y, t)) - \frac{1}{2}d(Y, t).
\]
When $Y$ is a rational homology $S^1 \times S^2$ there is a related numerical invariant $\chi^\text{trunc}$ as follows. Define $\chi^\text{trunc}(Y, t) = \chi(HF^+(Y, t))$ for non-torsion $t$. If $t$ is torsion then let $d(Y, t)$ be the minimal degree of any non-torsion class of $HF^+(Y, t)$ coming from $HF^\text{ev}_+(Y, t)$. The structure of $HF^\infty$ for homology $S^1 \times S^2$ implies that $\chi(HF^\leq d(Y, t)+2N+1(Y, t))$ is independent of $N$ for sufficiently large $N$. We let $\chi^\text{trunc}(Y, t)$ denote the value of this Euler characteristic.

One can express $\chi^\text{trunc}$ in terms of Turaev torsion function $\tau_Y : \text{Spin}^c(Y) \to \mathbb{Z}$. It is proved in [12] that for any $t$,
\[
\chi^\text{trunc}(Y, t) = -\tau(Y, t).
\]
For the precise statement and the sign issues for Turaev function we refer to Proposition 10.14 of [12].

In what follows, $\lambda$ denotes the Casson-Walker invariant normalized by $\lambda(\Sigma(2, 3, 5)) = -1$, where $\Sigma(2, 3, 5)$ is oriented as the boundary of the negative definite $E_8$ plumbing.

Surgery Here we set up our framework for surgeries. We directly follow [16]. Let $X$ be an oriented three-manifold with a torus boundary and $H_1(X; \mathbb{R}) \cong \mathbb{R}$. The map $H_1(\partial X; \mathbb{Z}) \to H_1(X; \mathbb{Z})$ has one-dimensional kernel. Let $\ell'$ denote a generator for the kernel, $d(X) > 0$ denote its divisibility, and let $\ell$ be the element $\ell'/d$. We call $\ell$ the longitude.

Fix a homology class $m \in H_1(\partial X)$ with $m \cdot \ell = 1$. For a pair of relatively prime integers $(p, q)$, the manifold $Y_{p/q}$ is obtained from $X$ by attaching a $S^1 \times D$ with
\[ \partial D = pm + q\ell, \text{ and let } Y = Y_{1/0}. \text{ Note that in general } Y_{p/q} \text{ depends on a choice of } m, \text{ but } Y_0 = Y_{0/1} \text{ does not. Note also that } Y_0 \text{ is a rational homology } S^1 \times S^2, \text{ while all the other } Y_{p/q} \text{ are rational homology spheres.} \]

There is a short exact sequence
\[ 0 \longrightarrow \mathbb{Z} \longrightarrow \text{Spin}^c(Y_0) \longrightarrow \text{Spin}^c(X) \longrightarrow 0, \]
by which we mean that the subgroup \( \mathbb{Z} \subset H^2(Y_0; \mathbb{Z}) \) generated by the Poincaré dual to \( m \) (viewed as a subset of \( Y_0 \)) acts freely on \( \text{Spin}^c(Y_0) \), and its quotient is naturally identified (under restriction to \( X \subset Y_0 \)) with \( \text{Spin}^c(X) \).

Thus, each \( \text{Spin}^c \) structure \( \mathfrak{s} \) on \( X \) has a natural level \( y = y(\mathfrak{s}) \in \mathbb{Z}/d\mathbb{Z} \) defined as follows. Let \( \mathfrak{b} \) be any \( \text{Spin}^c \) structure on \( Y_0 \) whose restriction is \( \mathfrak{a} \), and consider its image in
\[ \overline{\text{Spin}^c(Y_0)}/\mathbb{Z}(\text{PD}[m]) \cong \mathbb{Z}/d\mathbb{Z}, \]
where \( \text{Spin}^c(Y_0) \) is the group of \( \text{Spin}^c \) structures modulo the action of the torsion subgroup of \( H^2(Y_0; \mathbb{Z}) \).

Furthermore, for any of the \( Y_{p/q} \), the map \( \text{Spin}^c(Y_{p/q}) \) to \( \text{Spin}^c(X) \) is surjective, and its fibers consist of orbits by a cyclic group generated by the Poincaré dual to the knot which is the core of the complement \( Y_{p/q} - X \) (for \( Y = Y_{1/0} \), this fiber has order \( d = d(X) \)). For a fixed \( \text{Spin}^c \) structure \( \mathfrak{a} \) on \( X \), let \( \text{Spin}^c(Y_{p/q}; \mathfrak{a}) \) denote the set of \( \text{Spin}^c \) structures \( \mathfrak{b} \in \text{Spin}^c(Y_{p/q}) \) whose restriction to \( X \) is \( \mathfrak{a} \).

3. Surgery formula and its applications

Our main theorem is the following surgery formula for the Euler characteristic.

**Theorem 3.1.** For integers \( p, q, d, y \) with \( p \neq 0 \), \( p \) and \( q \) relatively prime, \( d > 0 \) and \( 0 \leq y < d \), there is quantity \( \epsilon(p, q, d, y) \in \mathbb{Q} \) with the following property. Let \( X \) be an oriented rational homology \( S^1 \times D \), with divisibility \( d(X) = d \), and choose \( m, \ell \) as described in the previous section. Fixing any \( \text{Spin}^c \) structure \( \mathfrak{a} \) over \( X \) with level \( y(\mathfrak{a}) = y \), we have the relation:
\[
\sum_{b \in \text{Spin}^c(Y_{p/q}; \mathfrak{a})} \hat{\chi}(Y_{p/q}, b) = p \left( \sum_{c \in \text{Spin}^c(Y; \mathfrak{a})} \hat{\chi}(Y, c) \right) - q \left( \sum_{d \in \text{Spin}^c(Y_0; \mathfrak{a})} \chi^{\text{trunc}}(Y_0, d) \right) + \epsilon(p, q, d, y).
\]

**Corollary 3.2.** For \( X \) as above,
\[
\left( \sum_{b \in \text{Spin}^c(Y_{p/q})} \hat{\chi}(Y_{p/q}, b) \right) = p \left( \sum_{c \in \text{Spin}^c(Y)} \hat{\chi}(Y, c) \right) + q \left( \sum_{i=1}^{\infty} a_i \ell^2 \right) + |\text{Tors}H_1(X; \mathbb{Z})| \epsilon(p, q, d),
\]
where $d = d(X)$, $a_i$ are the coefficients of the symmetrized Alexander polynomial of $Y_0$, normalized so that

$$A(1) = |\text{Tors}H^2(Y_0; \mathbb{Z})|,$$

and

$$\epsilon(p, q, d) = \sum_{y=0}^{d-1} \frac{\epsilon(p, q, d, y)}{d}.$$

**Proof.** This follows from the surgery formula and the fact that $\chi^{\text{trunc}}(Y_0, t) = -\tau(Y_0, t)$. \hfill \Box

**Theorem 3.3.** For any rational homology three-sphere $M$ we have

$$\sum_{t \in \text{Spin}^c(M)} \hat{\chi}(M, t) = |H_1(M; \mathbb{Z})| \lambda(M),$$

where $\lambda(M)$ is the Casson-Walker invariant of $M$.

**Proof.** We already have a surgery formula for $\sum_{t \in \text{Spin}^c(Y)} \hat{\chi}(Y, t)$. The scaled Casson-Walker invariant

$$\lambda'(Y) = |H_1(Y; \mathbb{Z})| \lambda(Y)$$

satisfies a similar formula with possibly different constants $\epsilon'(p, q, d)$, see [16]. In fact, we have

$$\lambda'(Y_{p/q}) = p\lambda'(Y) + q \left( \sum_{j \geq 1} a_j j^2 \right) + |\text{Tors}H_1(X; \mathbb{Z})| \left( \frac{q(d^2 - 1)}{24d} - \frac{pd \cdot s(q, p)}{2} \right),$$

i.e. $\epsilon'(p, q, d) = \left( \frac{q(d^2 - 1)}{24d} - \frac{pd \cdot s(q, p)}{2} \right)$. Thus, it remains to show that

$$\epsilon(p, q, d) = \epsilon'(p, q, d).$$

For $d = 1$ we can use a model calculation on $Y = S^3$ with the surgery made on the unknot. Since $S^3_{p/q} = L(-p, q)$, by [9] (see also [17]) we have

$$\sum_{t \in \text{Spin}^c(L(p,q))} d(L(-p, q), t) = p \cdot s(q, -p) = p \cdot s(q, p).$$

Taking into account that $\text{HF}^+_{\text{red}}(L(-p, q)) \cong 0$ it follows that in this case

$$\sum_{t \in \text{Spin}^c(S^3_{p/q})} \hat{\chi}(S^3_{p/q}, t) = -\frac{p \cdot s(p, q)}{2}.$$ 

Plugging this into the surgery formula we get

$$\epsilon(p, q, 1) = -\frac{p \cdot s(p, q)}{2}$$

as needed.
To complete the proof, one shows that \( \epsilon(p, q, d) \) is determined by the surgery formula and the values of \( \epsilon(p, q, 1) \). This is done by considering the Seifert manifold \( M(n, 1, -n, 1, q, -p) \). It can be obtained from \( M(n, 1, -n, 1, 0, 1) \) by \( (p, q, n) \) surgery. On the other hand, it is possible to show that this manifold can be obtained by a sequence of surgeries on knots with \( d \)'s less than \( n \), see [16] for details.

Now let us formulate the connection between the renormalized Euler characteristic and Turaev torsion. For rational homology three-sphere \( M \) and a Spin\(^c\) structure \( t \) on it define

\[
\hat{\tau}(M, t) = -\tau(M, t) + \lambda(M).
\]

**Theorem 3.4.** For any rational homology three-sphere \( M \) and a Spin\(^c\) structure \( t \) on it we have

\[
\hat{\chi}(M, t) = \hat{\tau}(M, t) = SW(M, t).
\]

**Proof.** The proof follows using the framework of [10]. According to it, there are several conditions on \( \hat{\chi} \) and \( \chi_{\text{trunc}} \) that guarantee the sought equality. We list them as follows:

- The surgery formula of Theorem 3.1 is satisfied. Note that we have a negative sign in front of the second term, but it can be made positive by switching from \( \chi_{\text{trunc}} \) to \( -\chi_{\text{trunc}} \).
- For any three-manifold \( M \) with \( b_1(M) = 1 \) and a Spin\(^c\) structure \( t \) on it

\[
-\chi_{\text{trunc}}(M, t) = \tau(M, t).
\]
- For any rational homology sphere \( M \),

\[
\sum_{t \in \text{Spin}^c(M)} \hat{\chi}(M, t) = |H_1(M; \mathbb{Z})| \lambda(M).
\]
- For any integral homology sphere \( M \)

\[
\hat{\chi}(M, t_0) = \hat{\tau}(M, t_0),
\]

where \( t_0 \) is the unique Spin\(^c\) structure on \( M \).
- When \( M \) is a Lens space

\[
\hat{\chi}(M, t) = \hat{\tau}(M, t),
\]

for any Spin\(^c\) structure \( t \) on \( M \).
- If \( M_1 \) and \( M_2 \) satisfy \( \hat{\chi} = \hat{\tau} \) then so does \( M_1 \# M_2 \).

The first three facts have already been mentioned, while the fourth item is Theorem 5.1 of [14], the fifth condition is satisfied by [9]. The last statement follows from additivity of \( d \), see Theorem 4.3 of [14] and from a Kunneth type formula, see Corollary 6.3 of [12]. The theorem follows. □
4. PROOF OF THE SURGERY FORMULA

Let $\theta^c$ denote the three-dimensional Spin$^c$ homology bordism group, defined as the set of equivalence classes of pairs $(M, t)$ where $M$ is a rational homology three-sphere, and $t$ is a Spin$^c$ structure over $M$, with the equivalence given as follows. Namely $(M_1, t_1) \sim (M_2, t_2)$ if there is a (connected, oriented, smooth) cobordism $N$ from $M_1$ to $M_2$ with $H_i(N, \mathbb{Q}) = 0$ for $i = 1$ and 2, which can be endowed with a Spin$^c$ structure $s$ whose restrictions to $M_1$ and $M_2$ are $t_1$ and $t_2$ respectively. The connected sum operation makes this set an Abelian group (whose unit is $S^3$ with its unique Spin$^c$ structure). The invariant $d(M, t)$ gives a group homomorphism

$$d : \theta^c \to \mathbb{Q}.$$ 

It is proved in [14] that $d$ is a lift of the classical homomorphism

$$\rho : \theta^c \to \mathbb{Q}/2\mathbb{Z}$$

(see [1]) defined as follows. Let $N$ be any four-manifold equipped with a Spin$^c$ structure $s$ with $\partial N \cong M$ and $s|\partial N \cong t$ then

$$\rho(M, t) \equiv \frac{c_1(s)^2 - \text{sgn}(N)}{4} \pmod{2\mathbb{Z}}$$

where $\text{sgn}(N)$ denotes the signature of the intersection form of $N$.

Going back to our surgery notation, let $W$ be the standard cobordism between $Y$ and $Y_{p/q}$ obtained by 2-handle additions. Let $\rho'(Y, t) \equiv \rho(Y, t) \pmod{2\mathbb{Z}}$ such that $\rho'(Y, t) \in [0, 2)$. For the manifold $Y_{p/q}$ and a Spin$^c$ structure $t$ on it consider any $s$ on $W$ with $s|Y_{p/q} = t$. We define $\rho'(Y_{p/q}, t) = \rho'(Y, s|Y)$.

For any constant $k$ define

$$HF_{\leq k}^+(Y_{p/q}, [a]) = \bigoplus_{t \in \text{Spin}^c(Y_{p/q}; a)} \bigoplus_{\{d \in \mathbb{Q} | \rho'(Y_{p/q}, t), \rho'(Y_{p/q}, t) \leq k\}} HF_d^+(Y_{p/q}, t).$$

$Y_0$ is not a rational homology sphere, if $t$ is torsion Spin$^c$ structure on $Y_0$ one can still define $\rho'(Y_0, t)$ similarly to above. It is useful to note that equivalence

$$d(Y_0, t) \equiv 1 + \frac{c_1(s)^2 + \text{sgn}(W)}{4} + \rho'(Y_0, t) \pmod{2\mathbb{Z}}$$

holds for any $s \in \text{Spin}^c(W)$ satisfying $s|Y_0 = t$, this follows from the grading shift formula for maps induced by cobordisms. One should look at both absolute $\mathbb{Q}$ and $\mathbb{Z}/2\mathbb{Z}$ grading shifts.

Let $\mathcal{T}$ be the subset of torsion Spin$^c$ structures of Spin$^c(Y_0)$. Now set

$$HF_{\leq k}^+(Y_0, [a]) = \bigoplus_{t \in \text{Spin}^c(Y_0; a) \setminus \mathcal{T}} HF^+(Y_0, t) \bigoplus_{t \in \text{Spin}^c(Y_0; a) \cap \mathcal{T}} \bigoplus_{\{d \in \mathbb{Q} | d \leq k + \rho'(Y_0, t)\}} HF_d^+(Y_0, t).$$
Lemma 4.1. For integers $p, q, d, y$ with $p$ and $q$ relatively prime, $d > 0$ and $0 \leq y < d$, there is quantity $k(p, q, d, y)$ with the following property. Let everything be as in Theorem 3.1, then
\[
\chi(HF_{\leq 2N}(Y_{p/q}, [a])) - N \cdot |\text{Spin}^c(Y_{p/q}; a)| =
\sum_{b \in \text{Spin}^c(Y_{p/q}; a)} \chi(Y_{p/q}, b) + p \sum_{c \in \text{Spin}^c(Y; a)} \frac{\rho'(Y, c)}{2} + k(p, q, d, y).
\]

Proof. (cf. lemma 4.8 in [15].) For sufficiently large $N$, $HF_{\text{red}}^+(Y_{p/q}, [a])$ is contained in $HF_{\leq 2N}^+(Y_{p/q}, [a])$. Over $\mathbb{Z}$, we have a splitting
\[
HF_{\leq 2N}^+(Y_{p/q}, [a]) \cong HF_{\text{red}}^+(Y_{p/q}, [a]) \oplus (\text{Im} \cap HF_{\leq 2N}^+(Y_{p/q}, [a])).
\]
But it follows readily from the structure of $HF^\infty(Y_{p/q})$ (c.f. Equation (1)) that
\[
\chi(\text{Im} \cap HF_{\leq 2N}^+(Y_{p/q}, [a])) =
\sum_{b \in \text{Spin}^c(Y_{p/q}; a)} \# \left\{ [d(Y_{p/q}, b), 2N + \rho'(Y_{p/q}, b)] \cap (d(Y_{p/q}, b) + 2\mathbb{Z}) \subset \mathbb{Q} \right\}
\sum_{\{b \in \text{Spin}^c(Y_{p/q}; a)\}} \left( N + 1 - \left\lfloor \frac{d(Y_{p/q}, b) - \rho'(Y_{p/q}, b)}{2} \right\rfloor \right),
\]
where here $\lfloor x \rfloor$ denotes the smallest integer greater than or equal to $x$. Thus we get that
\[
\chi(HF_{\leq 2N}^+(Y_{p/q}, [a])) - N \cdot |\text{Spin}^c(Y_{p/q}; a)| =
\sum_{b \in \text{Spin}^c(Y_{p/q}; a)} \left( HF_{\text{red}}^+(Y_{p/q}, b) - \left\lfloor \frac{d(Y_{p/q}, b) - \rho'(Y_{p/q}, b)}{2} \right\rfloor + 1 \right)
\sum_{b \in \text{Spin}^c(Y_{p/q}; a)} \left( HF_{\text{red}}^+(Y_{p/q}, b) - \frac{d(Y_{p/q}, b)}{2} \right)
+ \sum_{b \in \text{Spin}^c(Y_{p/q}; a)} \left( \frac{d(Y_{p/q}, b)}{2} - \left\lfloor \frac{d(Y_{p/q}, b) - \rho'(Y_{p/q}, b)}{2} \right\rfloor + 1 \right).
\]
To complete the proof we have to show that the difference
\[
k = \sum_{b \in \text{Spin}^c(Y_{p/q}; a)} \left( \frac{d(Y_{p/q}, b)}{2} - \left\lfloor \frac{d(Y_{p/q}, b) - \rho'(Y_{p/q}, b)}{2} \right\rfloor + 1 \right) - p \cdot \sum_{c \in \text{Spin}^c(Y; a)} \frac{\rho'(Y, c)}{2}
\]
depends only on $p, q, d, y$. Clearly
\[
k = \sum_{b \in \text{Spin}^c(Y_{p/q}; a)} \left( \frac{d(Y_{p/q}, b) - \rho'(Y_{p/q}, b)}{2} - \left\lfloor \frac{d(Y_{p/q}, b) - \rho'(Y_{p/q}, b)}{2} \right\rfloor + 1 \right).
\]
This in turn depends only on \( d(Y_{p/q}, b) - \rho'(Y_{p/q}, b) \) (mod 2\( \mathbb{Z} \)) which is completely determined by the collection of all \( c_1(s)^2 \) (mod 8\( \mathbb{Z} \)) with \( s \in \text{Spin}^c(W) \) satisfying \( s|Y \in \text{Spin}^c(Y; a) \). This follows from the definitions and the fact that \( \rho \) is a homomorphism. Hence, the proof is concluded by the following lemma.

**Lemma 4.2.** Let \( W \) be the standard cobordism between \( Y \) and \( Y_{p/q} \). The collection with repetitions of all \( c_1(s)^2 \) satisfying \( s \in \text{Spin}^c(W) \) and \( s|Y \in \text{Spin}^c(Y; a) \) is completely determined by the values of \( p, q, d \) and \( y \).

**Lemma 4.3.** For integers \( d, y \) with \( d > 0 \) and \( 0 \leq y < d \), there is quantity \( r(d, y) \) with the following property. Let everything be as in Theorem 3.1, then

\[
\chi(HF^+_{\leq 2N}(Y_0, [a])) = \sum_{b \in \text{Spin}^c(Y_0; a)} \chi^\text{trunc}(Y_0, b) + r(d, y).
\]

**Proof.** The idea of the proof is the same with the previous one. We do not have any terms involving \( N \) because of the different structure of \( HF^\infty \) for manifolds with \( b_1 = 1 \).

**Lemma 4.4.** For integers \( p, q, d, y \) with \( p \neq 0 \), \( p \) and \( q \) relatively prime, \( d > 0 \) and \( 0 \leq y < d \), there is quantity \( c(p, q, d, y) \) with the following property. Let everything be as in Theorem 3.1, then

\[
\chi(HF^+_{\leq 2N}(Y_{p/q}, [a])) = p \cdot \chi(HF^+_{\leq 2N}(Y, [a])) - q \cdot \chi(HF^+_{\leq 2N}(Y_0, [a])) + c(p, q, d, y),
\]

provided that \( N \) is sufficiently large.

**Proof.** The proof is a generalization of the argument of lemma 4.9 in [15]. Let us use induction on \( p + q \). The base of induction is the case when \( p + q = 1, 2 \), which reduces to \( (p, q) = (1, 0) \) or \( (1, 1) \). The lemma clearly holds for the first combination; we will discuss the second case in the end of the proof.

For a pair \( (p, q) \) of relatively prime, non-negative integers with \( p + q > 2 \), one can select two pairs of non-negative, relatively prime integers \( (p_0, q_0) \) and \( (p_2, q_2) \), with \( p_0, p_2 \neq 0 \) satisfying

\[
p_0 \cdot q - p \cdot q_0 = -1 \tag{4}
\]

\[
(p, q) = (p_0, q_0) + (p_2, q_2) \tag{5}
\]

Consider the manifolds \( Y_{p_0/q_0} \), \( Y_{p/q} \) and \( Y_{p_2/q_2} \). There are standard 2-handle cobordisms between these manifolds. Let \( W_0 \) denote the cobordism between \( Y_{p_0/q_0} \) and \( Y_{p/q} \), \( W_1 \) the cobordism between \( Y_{p/q} \) and \( Y_{p_2/q_2} \), \( W_2 \) between \( Y_{p_2/q_2} \) and \( Y_{p_0/q_0} \). We can write down the following long exact sequence

\[
\cdots \longrightarrow HF^+(Y_{p_0/q_0}, [a]) \xrightarrow{f_0} HF^+(Y_{p/q}, [a]) \xrightarrow{f_1} HF^+(Y_{p_2/q_2}, [a]) \xrightarrow{f_2} \cdots,
\]
where the maps are induced by the corresponding cobordisms. Note that $W_0$ and $W_1$ are both negative definite, but $W_2$ is not.

By inductive hypothesis the lemma holds for $(p_0, q_0)$ and $(p_2, q_2)$. When $N$ is sufficiently large, the image of the restriction $g_0$ of $f_0$ to $HF^+_{\leq 2N+\frac{1}{4}}(Y_{p_0/q_0}, [a])$ is contained in $HF^+_{\leq 2N+\frac{1}{4}}(Y_{p/q}, [a])$, the restriction $g_1$ of $f_1$ to $HF^+_{\leq 2N+\frac{1}{4}}(Y_{p/q}, [a])$ is contained in $HF^+_{\leq 2N+\frac{1}{4}}(Y_{p_2/q_2}, [a])$, and finally, the restriction $g_2$ of $f_2$ to $HF^+_{\leq 2N+\frac{1}{4}}(Y_{p_2/q_2}, [a])$ is contained in $HF^+_{\leq 2N}(Y_{p_0/q_0}, [a])$. This follows at once from the definition of $\rho^i$ which appears in the expression for $HF^+_{\leq k}$ and the grading shift formula: we have that $\chi(W_i) = 1$ and $\sigma(W_i) = -1$ for $i = 0, 1$; while the cobordism $W_2$ induces the trivial map on $HF^\infty$ since $b_2^2(W_2) = 1$.

Choosing $N$ as above, consider the diagram

\[
\begin{array}{ccccccc}
0 & & 0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \xrightarrow{g_2} & HF^+_{\leq 2N}(Y_{p_0/q_0}, [a]) & \xrightarrow{g_0} & HF^+_{\leq 2N+\frac{1}{4}}(Y_{p/q}, [a]) & \xrightarrow{g_1} & HF^+_{\leq 2N+\frac{1}{4}}(Y_{p_2/q_2}, [a]) & \xrightarrow{g_2} \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \xrightarrow{f_2} & HF^+(Y_{p_0/q_0}, [a]) & \xrightarrow{f_0} & HF^+(Y_{p/q}, [a]) & \xrightarrow{f_1} & HF^+(Y_{p_2/q_2}, [a]) & \xrightarrow{f_2} \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \xrightarrow{h_2} & HF^+_{\geq 2N}(Y_{p_0/q_0}, [a]) & \xrightarrow{h_0} & HF^+_{\geq 2N+\frac{1}{4}}(Y_{p/q}, [a]) & \xrightarrow{h_1} & HF^+_{\geq 2N+\frac{1}{4}}(Y_{p_2/q_2}, [a]) & \xrightarrow{h_2} \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0
\end{array}
\]

where the columns are exact. Note that the first and the third rows are not necessarily exact, while the middle one is exact. Let us think of these three rows as chain complexes. We denote these three rows by $R_1$, $R_2$, and $R_3$. Since $R_2$ is exact, it follows that $H^*(R_1) \cong H^*(R_3)$.

Now let us show that $H^*(R_3)$ is determined by $p, q, d$ and $y$ for $N$ sufficiently large. This is established using the structure of maps on $HF^\infty$, lemma 4.2 and the diagram

\[
\begin{array}{ccccccc}
\cdots & \xrightarrow{h_2^\infty} & HF^\infty_{\geq 2N}(Y_{p_0/q_0}, [a]) & \xrightarrow{h_0^\infty} & HF^\infty_{\geq 2N+\frac{1}{4}}(Y_{p/q}, [a]) & \xrightarrow{h_1^\infty} & HF^\infty_{\geq 2N+\frac{1}{4}}(Y_{p_2/q_2}, [a]) & \xrightarrow{h_2^\infty} \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \xrightarrow{h_2} & HF^+_{\geq 2N+\frac{1}{2}}(Y_{p_0/q_0}, [a]) & \xrightarrow{h_0} & HF^+_{\geq 2N+\frac{1}{4}}(Y_{p/q}, [a]) & \xrightarrow{h_1} & HF^+_{\geq 2N+\frac{1}{4}}(Y_{p_2/q_2}, [a]) & \xrightarrow{h_2} \cdots \\
\end{array}
\]

where here $h_0$ is the sum over all $s \in \text{Spin}^c(W_0)$ of the projections of the induced maps on $HF^\infty$; e.g. letting

\[\Pi_{\geq 2N+\frac{1}{2}}: HF^\infty_{\geq 2N+\frac{1}{2}}(Y_{p/q}, [a]) \to HF^\infty_{\geq 2N+\frac{1}{4}}(Y_{p/q}, [a])\]
denote the projection, we let \( h_0^\infty \) be the restriction to \( HF_{\geq 2N}(Y_{p_0/q_0}, [a]) \) of
\[
\sum_{s \in \text{Spin}^c(W_0)} \Pi_{\geq 2N+\frac{1}{s}} \circ F^\infty_{W_0,s}.
\]
The maps \( h_i^\infty \) are defined similarly. Note that \( h_2^\infty = 0 \), since the map induced by \( W_2 \) has \( b_+^i(W_2) = 1 \).

So far we have established that for all sufficiently large \( N \),
\[
\chi(H_*(R_1)) = \chi(HF_{\leq 2N}(Y_{p_0/q_0}, [a]) - \chi(HF_{\leq 2N+1}(Y_{p/q}, [a])) + \chi(HF_{\leq 2N+1}(Y_{p_2/q_2}, [a]))
\]
is completely determined by \( p, q, d, y \). It is also clear that for sufficiently large \( N \),
\[
\chi(HF_{\leq 2N+\frac{1}{s}}(Y_{p_2/q_2}, [a])) = \chi(HF_{\leq 2N}(Y_{p_2/q_2}, [a])) + c_3
\]
and
\[
\chi(HF_{\leq 2N+\frac{1}{s}}(Y_{p_2/q_2}, [a])) = \chi(HF_{\leq 2N}(Y_{p_2/q_2}, [a])) + c_4,
\]
with constants \( c_3 \) and \( c_4 \) again depending only on \( p_0, q_0, d, y \) and \( p_2, q_2, d, y \) respectively, lemma 4.2. Combining all the constants, we establish the inductive step in the case where \( p_0 \) is non-zero.

When \((p, q) = (1, 1)\), the above argument works with slight modification. In this case, we consider the manifolds \( Y, Y_0, Y_1 \). The dimension shifts work differently: \( \sigma(W_0) = \sigma(W_1) = 0 \) and hence, we compare \( HF_{\leq 2N}(Y, [a]), HF_{\leq 2N+1}(Y_0, [a]), \) and \( HF_{\leq 2N+1}(Y_1, [a]) \). To see that the map \( f_2 \) induced by \( W_2 \) carries \( HF_{\leq 2N+1}(Y_1, [a]) \) into \( HF_{\leq 2N}(Y, [a]) \) for sufficiently large \( N \), remember that the kernel of the map \( f_0 \) induced by \( W_0 \) is finitely generated. Some parities change under these maps, so the Euler characteristic is given as follows
\[
\chi(H_*(R_1)) = \chi(HF_{\leq 2N}(Y, [a])) - \chi(HF_{\leq 2N+\frac{1}{s}}(Y_0, [a])) - \chi(HF_{\leq 2N+\frac{1}{s}}(Y_1, [a]))
\]
compare with Proposition 5.3 in [14].

\[\square\]

**Proof of Theorem 5.3.1** When \( p \) and \( q \) are non-negative, this is a combination of Lemmas 4.1 and 4.4. The remaining case can be proved by running the induction from Lemma 4.4 to show that it still holds in the case where \( p > 0 \) and \( q \leq 0 \). \[\square\]

**References**

[1] M F Atiyah, V K Patodi, I M Singer, *Spectral asymmetry and Riemannian geometry. II*, Math. Proc. Cambridge Philos. Soc. 78(3) 1975 405–432

[2] S K Donaldson, *Floer homology groups in Yang-Mills theory*, volume 147 of *Cambridge Tracts in Mathematics*, Cambridge University Press (2002), with the assistance of M Furuta and D Kotschick

[3] K A Frøyshov, *The Seiberg-Witten equations and four-manifolds with boundary*, Math. Res. Lett 3 (1996) 373–390

[4] R E Gompf, A I Stipsicz, *4-manifolds and Kirby calculus*, volume 20 of *Graduate Studies in Mathematics*, American Mathematical Society (1999)
[5] P B Kronheimer, T S Mrowka, I.P. Lecture (1996)
[6] Y Lim, The equivalence of Seiberg-Witten and Casson invariants for homology 3-spheres, Math. Research Letters, 195 (2000) 179–204
[7] M Marcolli, B L Wang, Equivariant Seiberg-Witten Floer homology (1996), arXiv:DG.ga/9606003
[8] J W Morgan, The Seiberg-Witten Equations and Applications to the Topology of Smooth Four-Manifold, Mathematical Notes 44, Princeton University Press (1996)
[9] A Némethi, On the Ozsváth-Szabó invariant of negative definite plumbed 3-manifolds (2003), arXiv:math.GT/0310083
[10] L I Nicolaescu, Seiberg-Witten invariants of rational homology 3-spheres (2001), arXiv:math.GT/0103020
[11] P S Ozsváth, Z Szabó, Holomorphic triangles and invariants for smooth four-manifolds, arXiv:math.SG/0110169
[12] P S Ozsváth, Z Szabó, Holomorphic disks and three-manifold invariants: properties and applications (2001), arXiv:math.SG/0105202, to appear in Annals of Math.
[13] P S Ozsváth, Z Szabó, Holomorphic disks and topological invariants for closed three-manifolds (2001), arXiv:math.SG/0101206, to appear in Annals of Math.
[14] P S Ozsváth, Z Szabó, Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary, Advances in Mathematics 173 (2003) 179–261
[15] P S Ozsváth, Z Szabó, Knots with unknotting number one and Heegaard Floer homology (2004), arXiv:math.GT/0401426
[16] P S Ozsváth, Z Szabó, The theta divisor and the Casson-Walker invariant (2000), arXiv:math.GT/0006194
[17] J Rasmussen, Lens space surgeries and a conjecture of Goda and Teragaito (2004), arXiv:math.GT/0405114
[18] V Turaev, Torsion invariants of Spin^c structures on 3-manifolds, Math. Research Letters 4 (1997) 679–695
[19] E Witten, Monopoles and Four-Manifolds, Math. Research Letters 1 (1994) 769–796

THE PROGRAM IN APPLIED AND COMPUTATIONAL MATHEMATICS, Princeton University, New Jersey 08540, USA

E-mail address: rustamov@princeton.edu