Energy distribution of solutions to defocusing semi-linear wave equation in two dimensional space

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Abstract
We consider finite-energy solutions to the defocusing nonlinear wave equation in two dimensional space. We prove that almost all energy moves to the infinity at almost the light speed as time tends to infinity. In addition, the inward/outward part of energy gradually vanishes as time tends to positive/negative infinity. These behaviours resemble those of free waves. We also prove some decay estimates of the solutions if the initial data decay at a certain rate as the spatial variable tends to infinity. As an application, we prove a couple of scattering results for solutions whose initial data are in a weighted energy space. Our assumption on decay rate of initial data is weaker than previous known scattering results.

Mathematics Subject Classification 35L05 · 35L71

1 Introduction

In this work we consider the defocusing nonlinear wave equation in 2-dimensional space

\[
\begin{align*}
\partial_t^2 u - \Delta u &= -|u|^{p-1}u, \quad (x, t) \in \mathbb{R}^2 \times \mathbb{R}; \\
u|_{t=0} &= u_0; \\
\partial_t u|_{t=0} &= u_1.
\end{align*}
\]

(CP1)

The conserved energy is defined by

\[
E = \int_{\mathbb{R}^2} \left( \frac{1}{2} |\nabla u(x, t)|^2 + \frac{1}{2} |u_t(x, t)|^2 + \frac{1}{p+1} |u(x, t)|^{p+1} \right) dx.
\]
1.1 Background

Defocusing nonlinear wave equations

$$\partial_t^2 u - \Delta u = -|u|^{p-1}u, \quad (x, t) \in \mathbb{R}^d \times \mathbb{R}$$

have been extensively studied in the past few decades. This equation is invariant under a natural rescaling. Namely, if $u$ is a solution with initial data $(u_0, u_1)$, then

$$u_\lambda(x, t) = \lambda^{-2/(p-1)}u(x/\lambda, t/\lambda)$$

is another solution to the same equation with initial data

$$(u_{0,\lambda}, u_{1,\lambda}) = \left(\lambda^{-2/(p-1)}u_0(\cdot/\lambda), \lambda^{-2/(p-1)-1}u_1(\cdot/\lambda)\right).$$

A basic calculation shows that $(u_{0,\lambda}, u_{1,\lambda})$ share the same $\dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^d)$ norm as the original initial data if we choose $s_p = d/2 - 2/(p-1)$. Thus the space $\dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^d)$ is usually called the critical Sobolev space of this equation.

Local theory

The existence and uniqueness of solutions to semi-linear wave equation like (CP1) follows a combination of suitable Strichartz estimates and a fixed-point argument. Readers may refer to, for example, Kapitanski [21] and Lindblad and Sogge [26] for more details. In this work we mainly consider finite-energy solutions in the 2-dimensional case. The global well-posedness of these solutions has been proved in Ginibre and Velo [13]. Therefore we focus on global, especially asymptotic behaviours of solutions to (CP1). We first give a brief review on previously known results concerning the global behaviour of solutions to defocusing semi-linear wave equations, both in higher dimensions $d \geq 3$ and dimension 2.

Previous results in higher dimensions

The global behaviour of solutions with energy critical nonlinearity $p_e = 1 + 4/(d - 2)$ in 3 or higher dimensional space has been well understood. It was proved in the last few decades of 20th century that any solution with a finite energy must exist globally in time and scatter in both time directions. Please see, for instance, Bahouri and Gérard [1], Bahouri and Shatah [2], Ginibre et al. [12], Grillakis [17, 18], Kapitanski [20], Nakanishi [29, 30], Pecher [33], Shatah and Struwe [35] and Struwe [41]. The energy sub-critical case $p < p_e$ and energy super-critical case $p > p_e$ have also been discussed. For example, there are many conditional scattering results, in different dimensions and for different ranges of $p$, proving that if the critical Sobolev norm of a solution is uniformly bounded in the whole lifespan, then this solution must be a global solution and scatter. Please see, for instance, Bulut [3], Dodson and Lawrie [6], Dodson et al. [7], Duyckaerts et al. [8], Kenig and Merle [22], Killip and Visan [23, 24], Rodriguez [34] and Shen [36]. There are also scattering results only depending on the information about initial data. For example, Ginibre and Velo [14] proved the scattering of solutions by conformal conservation law in the energy sub-critical range if the initial is contained in a weighted Sobolev space. Recently the second author [38, 39] introduced an inward/outward
energy theory and proved the scattering of solutions for initial data in a larger weighted Sobolev space, by considering the energy distribution properties of solutions and/or weighted Morawetz estimates.

**Previous results in dimension 2** Since the energy critical exponent is \( p = +\infty \) in dimension 2, the equation (CP1) is always energy sub-critical for all \( p \in (1, +\infty) \). In general, there are less results available in dimension 2. Theoretically speaking, it is more difficult to obtain scattering result in dimension 2 than higher dimensions. It is because the dispersive rate of linear wave equation depends on the dimension. It is well known that if the initial data are smooth and compactly supported, then the corresponding solution to linear homogenous wave equation satisfies \(|u(x, t)| \lesssim t^{-(d-1)/2}\). The decay rate is lower in dimension 2 than higher dimensions. Many previously known scattering results in dimension 2 are based on the conformal conversation laws. Please see Ginibre and Velo [14], Glassey and Pecher [16], Hidano [19] and Wei and Yang [44], for example. These works assume that the initial data satisfy

\[
\int_{\mathbb{R}^2} \left( |x|^2 + 1 \right) \left( |\nabla u_0(x)|^2 + |u_1(x)|^2 + |u_0(x)|^2 \right) dx < \infty;
\]
or

\[
\int_{\mathbb{R}^2} \left( |x|^2 + 1 \right) \left( |\nabla u_0(x)|^2 + |u_1(x)|^2 + |u_0(x)|^{p+1} \right) dx < +\infty.
\]

In a joint work [37] with Staffilani, the second author applies a transformation (introduced in Tataru [42]) between wave equation in Euclidean space and shifted wave equation in hyperbolic space and proves the scattering of solutions to the quintic equation \( p = 5 \), if the initial data are radial and satisfy

\[|\nabla u_0(x)|, |u_1(x)| \lesssim (1 + |x|)^{-3/2-\varepsilon}, \quad |u_0(x)| \lesssim (1 + |x|)^{-1/2-\varepsilon}, \quad \varepsilon > 0.\]

Tsutaya [43] proves the same result for small but possibly non-radial initial data with a similar decay rate.

### 1.2 Main idea

In this work we discuss the energy distribution properties of solutions and then investigate their global behaviour in dimension 2. We start by introducing a few notations for convenience. These notation will be used throughout this work.

**Notations** In this work we will use the notation \( e(x, t) \) for the energy density

\[e(x, t) = \frac{1}{2} |\nabla u(x, t)|^2 + \frac{1}{2} |u_t(x, t)|^2 + \frac{1}{p + 1} |u(x, t)|^{p+1}.\]

We use \( u_r, \nabla u \) for the derivative in the radial direction and the covariant derivative on the circle centred at the origin, respectively.
\[ u_r(x, t) = \frac{x}{|x|} \cdot \nabla u(x, t); \quad \nabla u = \nabla u - u_r \frac{x}{|x|}; \quad |\nabla u|^2 = |u_r|^2 + |\nabla u|^2 \]

We also define the weighted energy
\[ E_\kappa(u_0, u_1) = \int_{\mathbb{R}^2} (|x|^\kappa + 1) \left( \frac{1}{2} |\nabla u_0(x)|^2 + \frac{1}{2} |u_1(x)|^2 + \frac{1}{p + 1} |u_0(x)|^{p+1} \right) dx. \]

Please note that this is no longer a conserved quantity in general.

**Morawetz estimates** The main tool of this work is still a Morawetz-type estimate. This kind of estimates were first found by Morawetz [28] for wave/Klein–Gordon equations. Lin and Strauss [25] then generalized Morawetz estimates to Schrödinger equations. Colliander et al. [5] introduced interaction Morawetz estimates for Schrödinger equations. Nowadays the Morawetz estimate has been one of the most important tools in the study of dispersive equations. In the case of 2D wave equation, there are also previously known Morawetz estimates
\[ \int_{1}^\infty \int_{\mathbb{R}^2} \left( \frac{r^2 + |x|^2}{(r^2 + |x|^2)^{3/2}} \right) |\nabla u|^2 + |ru_t + tu_r|^2 + r^2 |u|^{p+1} \] \( \lesssim C(E). \) (1)

Nakanishi [29] first introduced this kind of Morawetz estimate for non-coercive energy critical wave equations in dimension \( d \geq 3 \) by using the multiplier \( \frac{(-t, x)}{\sqrt{r^2 + |x|^2}} \), then generalized it to Klein–Gordon equation in lower dimensions \( d = 1, 2 \) in a subsequent work [31]. A similar argument shows that this kind of Morawetz estimate holds for 2D wave equation as well. For the convenience of readers, we give a brief proof of (1) in the appendix by the method of Nakanishi.

**Challenge and idea in 2D** The global integral estimate (1) does give some information on the asymptotic behaviour of solutions. For example, since \( 1/t \) is not integrable, the inequality (1) implies that the \( L^{p+1} \) norm of \( u(\cdot, t) \) must tend to zero (in the average sense) as \( t \to +\infty \). In order to gain more useful information about asymptotic behaviour, we try to use another multiplier \( \nabla \Psi \) and auxiliary function \( \varphi \)
\[ \nabla \Psi = \begin{cases} x, & \text{if } |x| \leq R; \\ Rx/|x|, & \text{if } |x| > R. \end{cases} \quad \varphi = \begin{cases} 1/2, & \text{if } |x| \leq R; \\ 0, & \text{if } |x| > R. \end{cases} \]

i.e. the same multiplier as we used in the higher dimensional case \( d \geq 3 \). We define
\[ E(t) = \int_{\mathbb{R}^2} u_t(x, t) \left( \nabla u(x, t) \cdot \nabla \Psi + u(x, t) \left( \frac{\Delta \Psi}{2} - \varphi \right) \right) dx \]
and utilize the basic theorem of calculus to obtain
\[ (E + E(t_2)) + (E - E(t_1)) - \int_{t_1}^{t_2} E'(t) dt = 2E. \]
We then calculate each of the three terms in the left hand side in details. In three or higher dimensional case the left hand side come with all positive terms. It enables us to let $t_1 \to -\infty$, $t_2 \to +\infty$ and obtain a global space-time integral estimate, i.e. the Morawetz estimate. However, a similar argument can no longer give a global space-time integral estimate in dimension 2, due to the presence of two negative terms in the left hand side. (Details can be found in Proposition 3.1) We have to move these two terms to the right hand side and rewrite

$$
\text{Positive terms} = 2E + \int_{t_1}^{t_2} \int_{|x| > R} \frac{|u|^2}{4|x|^3} dx dt + \sum_{i=1,2} \int_{|x| > R} \frac{|u(x, t_i)|^2}{8|x|^2} dx.
$$

If we choose $t_1 = -R$ and $t_2 = R$, then one of the terms in the left hand side is almost $2E$. We may move this major term to the right hand and obtain

$$
\text{Positive terms} \leq 2 \int_{\mathbb{R}^2} \min\{|x|/R, 1\} e(x, 0) dt + \int_{-R}^{R} \int_{|x| > R} \frac{|u|^2}{4|x|^3} dx dt
$$

$$
+ \sum_{\pm} \int_{|x| > R} \frac{|u(x, \pm R)|^2}{8|x|^2} dx.
$$

Here the positive terms in the left hand side contain important information about the asymptotic behaviours of the solutions. The remaining work is to prove that these terms decay as $R \to +\infty$ and to figure out their minimal decay rate if possible. It suffices to consider the decay of the right hand side. The first integral in the right hand side converges to zero as $R \to +\infty$. In addition, if the initial data satisfy $E_\kappa(u_0, u_1) < +\infty$ for a constant $\kappa \in (0, 1)$, then this integral decays faster than $R^{-\kappa}$. The other two terms in the right hand side can be dealt with in two different ways:

- We observe that the left hand side contains the following terms:

$$
\int_{t_1}^{t_2} \int_{|x| > R} \frac{|u|^{p+1}}{|x|} dx dt, \quad \int_{|x| > R} |u(x, t_i)|^{p+1} dx.
$$

If $|u| \gg |x|^{-2/(p-1)}$, then we have $|u|^2/|x|^3 \ll |u|^{p+1}/|x|$ and $|u|^2/|x|^2 \ll |u|^{p+1}$. In this case the additional terms in the right hand side can be absorbed by the left hand side. On the other hand, if $|u| \lesssim |x|^{-2/(p-1)}$, then the additional terms themselves are already small if $|x|$ is large. In summary we can always remove these terms at a reasonable cost of $R^{-4/(p-1)}$.

- If the initial data decay sufficiently fast as the spatial variable tends to infinity, we may find a better estimate of these additional terms by finite speed of propagation and a weighted Hardy inequality.

We will calculate as carefully as we can when we work on a Morawetz-type estimate. A few positive terms, which were simply neglected in most previous works, turn out to be very useful ones when we discuss the energy distribution of solutions.
1.3 Main results

Now we give the main results of this work. The first result is about the energy distribution of solutions to (CP1).

**Theorem 1.1** Assume $p > 3$. Let $u$ be a solution to (CP1) with a finite energy. Then

(a) The following limits hold as time tends to infinity

$$\lim_{t \to \pm \infty} \int_{|x| < |t|} \frac{|t| - |x|}{|t|} e(x, t) dx = 0.$$

(b) The inward/outward part of energy vanishes as time tends positive/negative infinity.

$$\lim_{t \to \pm \infty} \int_{\mathbb{R}^2} \left(|u_r \pm u_t|^2 + |\nabla u|^2 + |u|^{p+1}\right) dx = 0.$$

(c) Furthermore, if the initial data satisfy $E_\kappa(u_0, u_1) < +\infty$ for a constant

$$\kappa \in \begin{cases} (0, 1), & \text{if } p \geq 5; \\ (0, \frac{p-3}{2}), & \text{if } 3 < p < 5; \end{cases}$$

then we have the following decay estimates

$$\lim_{t \to \pm \infty} \int_{|x| < |t|} \frac{|t| - |x|}{|t|^{1-\kappa}} e(x, t) dx = 0;$$

$$\lim_{t \to \pm \infty} |t|^\kappa \int_{\mathbb{R}^2} \left(|u_r \pm u_t|^2 + |\nabla u|^2 + |u|^{p+1}\right) dx = 0.$$

**Remark 1.2** The results in Theorem 1.1 look similar to those in the higher dimensional case (see [38, 39]), but it is distinct in the following aspect: In the higher dimensional case $d \geq 3$, we have the Sobolev embedding

$$\int_{\mathbb{R}^d} |u(x)|^{\frac{2d}{d-2}} dx \lesssim \left(\int_{\mathbb{R}^d} |\nabla u(x)| dx\right)^{\frac{p+1}{p+1 - \frac{d}{2}}}.$$

In addition, a finite energy free wave $u$ satisfies $\|u(\cdot, t)\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} \to 0$ as $t \to \pm \infty$. These facts are especially useful in the energy critical case. The nonlinear equation can usually be viewed as a perturbation of the linear equation. In the 2-dimensional case, however, the $H^1$ norm can not dominate any $L^q$ norm. The $L^q$-type estimates are only available for defocusing equations. This distinction in lower dimensions is more remarkable in the one dimensional case. A typical solution to free one-dimensional wave equation does not possess any $L^q$ decay as $t \to +\infty$, even if the initial data are smooth and compactly supported. In the defocusing case, however, both the $L^\infty$ and $L^{p+1}$ norms of a finite energy solution must converge to zero as time tends to infinity, as shown in Lindblad and Tao [27] and Wei and Yang [45].
As an application of our theory on energy distribution, we also have the following scattering results.

**Theorem 1.3** Assume \( p > 5 \). Let \( u \) be a solution to \((CP1)\) whose initial data satisfy \( E_{\kappa}(u_0, u_1) < +\infty \) for a constant \( \kappa > \frac{3p+5}{4p} \). Then the solution \( u \) scatters in both two time directions. More precisely, there exist \((u^+_0, u^+_1) \in (H^1 \cap \dot{H}^{s_p}(\mathbb{R}^2)) \times (L^2 \cap \dot{H}^{s_p-1}(\mathbb{R}^2))\), so that

\[
\lim_{t \to \pm \infty} \left\| \begin{pmatrix} u(\cdot, t) \\ u_t(\cdot, t) \end{pmatrix} - S_L(t) \begin{pmatrix} u^+_0 \\ u^+_1 \end{pmatrix} \right\|_{\dot{H}^s \times \dot{H}^{s_p-1}(\mathbb{R}^2)} = 0.
\]

holds for all \( s \in [s_p, 1] \). Here \( S_L \) is the linear wave propagation operator.

**Theorem 1.4** Assume \( p > 1 + 2\sqrt{3} \approx 4.464 \). Let \( u \) be a radial solution to \((CP1)\) with a finite energy. Then

(a) There exist two radial finite-energy free waves \( \tilde{u}^+, \tilde{u}^- \), so that

\[
\lim_{t \to \pm \infty} \int_{|x| > |t| - \eta} \left( |\nabla \tilde{u}^+(x, t) - \nabla u(x, t)|^2 + |\tilde{u}^+_t(x, t) - u_t(x, t)|^2 \right) dx = 0,
\]

\( \forall \eta \in \mathbb{R} \).

(b) If the initial data \((u_0, u_1)\) satisfy \( E_{\frac{4}{p+1}}(u_0, u_1) < +\infty \), then the solution \( u \) scatters in both two time directions.

\[
\lim_{t \to \pm \infty} \|\begin{pmatrix} (\tilde{u}^+(\cdot, t) - u(\cdot, t), \tilde{u}^+_t(\cdot, t) - u_t(\cdot, t) \end{pmatrix} \|_{\dot{H}^1 \times L^2(\mathbb{R}^2)} = 0.
\]

**Remark 1.5** We may call Part (a) of Theorem 1.4 exterior scattering. The free waves \( \tilde{u}^\pm = S_L(u^\pm_0, u^\pm_1) \) are the radiation part of solutions \( u \).

**Remark 1.6** The method of conformal conservation laws assumes that \( E_{2}(u_0, u_1) < +\infty \). The initial data in works [37, 43] satisfy \( E_{1+\varepsilon}(u_0, u_1) < +\infty \). Please note that \( \kappa = \frac{4}{p+1} \) is the minimal value of \( \kappa \) so that the potential is dominated by the weighted Sobolev norm

\[
\left( \int_{\mathbb{R}^2} |u(x)|^{p+1} dx \right)^{\frac{2}{p+1}} \lesssim \int_{\mathbb{R}^2} (|x|^\kappa + 1)|\nabla u(x)|^2 dx.
\]

The decay rate of initial data in the scattering results of this work is lower than these previously known results. The initial data in Theorem 1.4 are not necessary contained in the critical Sobolev space.

**The structure of this work** This paper is organized as follows. We first give a few preliminary results and technical lemmata in Sect. 2. Then in Sect. 3, we discuss our Morawetz-type estimates in details. Section 4 is devoted to the proof of energy distribution properties. Finally we discuss scattering theory of non-radial and radial solutions in Sects. 5 and 6.
2 Preliminary results

Lemma 2.1 If $u \in \dot{H}^1(\mathbb{R}^2) \cap L^{p+1}(\mathbb{R}^2)$ is a radial function, then we have the pointwise estimate

$$|u(r)| \lesssim r^{-\frac{2}{p+1}} \|u\|_{\dot{H}^1}^{\frac{2}{p+1}} \|u\|_{L^{p+1}}^{\frac{p+1}{p+3}}, \quad r > 0.$$ 

Proof We start by

$$|u(r_2) - u(r_1)| = \left| \int_{r_1}^{r_2} u_r(r)dr \right| \leq \left( \int_{r_1}^{r_2} r^{-1}dr \right)^{1/2} \left( \int_{r_1}^{r_2} r|u_r|^2dr \right)^{1/2} \leq \left( \frac{r_2 - r_1}{r_1} \right)^{1/2} \|u\|_{\dot{H}^1}.$$ 

Thus we have

$$|u(r') - u(r)| \leq |u(r)|/2 \Rightarrow |u(r')| \geq |u(r)|/2, \quad \forall r \leq r' \leq r + \frac{r|u(r)|^2}{4\|u\|^2_{\dot{H}^1}}.$$ 

Combining this estimate with $L^{p+1}$ norm, we obtain

$$\|u\|_{L^{p+1}}^{p+1} \geq \int_{r<|x|<r+\frac{r|u(r)|^2}{4\|u\|^2_{\dot{H}^1}}} |u(x)|^{p+1}dx \geq 2\pi r \cdot \frac{r|u(r)|^2}{4\|u\|^2_{\dot{H}^1}} \cdot \left( \frac{|u(r)|}{2} \right)^{p+1}.$$ 

This immediately finishes the proof. $\Box$

Proposition 2.2 (Strichartz estimates, see Proposition 3.1 of Ginibre and Velo [15])

Let $2 \leq q_1, q_2 \leq \infty$, $2 \leq r_1, r_2 < \infty$ and $\rho_1, \rho_2, s \in \mathbb{R}$ be constants with

$$\frac{2}{q_i} + \frac{1}{r_i} \leq \frac{1}{2}, \quad i = 1, 2; \quad \frac{1}{q_1} + \frac{2}{r_1} = 1 + \rho_1 - s; \quad \frac{1}{q_2} + \frac{2}{r_2} = \rho_2 + s.$$ 

Assume that $u$ is the solution to the linear wave equation

$$\begin{cases} 
\partial_t u - \Delta u = F(x, t), \quad (x, t) \in \mathbb{R}^2 \times [0, T]; \\
|u|_{t=0} = u_0 \in \dot{H}^s; \\
\partial_t|u|_{t=0} = u_1 \in \dot{H}^{s-1}. 
\end{cases}$$
Then we have

\[
\sup_{t \in [0, T]} \left\| (u(\cdot, t), \partial_t u(\cdot, t)) \right\|_{\dot{H}^{s} \times \dot{H}^{s-1}} + \| D_{x}^{\rho_{1}} u \|_{L^{q_{1}} L^{r_{1}}([0, T] \times \mathbb{R}^{2})} \\
\leq C \left( \left\| (u_{0}, u_{1}) \right\|_{\dot{H}^{s} \times \dot{H}^{s-1}} + \| D_{x}^{-\rho_{2}} F \|_{L^{\hat{q}_{2}} L^{\hat{r}_{2}}([0, T] \times \mathbb{R}^{2})} \right).
\]

Here \( \hat{q}_{2}, \hat{r}_{2} \) are the conjugates of \( q_{2}, r_{2} \) respectively, i.e. \( 1/q_{2} + 1/\hat{q}_{2} = 1, 1/r_{2} + 1/\hat{r}_{2} = 1 \). The fractional differential operator \( D_{x}^{\rho} \) is defined by the Fourier multiplier \( |\xi|^\rho \).

The constant \( C \) does not depend on \( T \) or \( u \).

**Remark 2.3** We call \( (q_{1}, r_{1}) \) an \( s \)-admissible pair if \( q_{1}, r_{1}, s \) and \( \rho_{1} = 0 \) satisfy the conditions given above.

**Chain rule** We also need the following “chain rule” for fractional derivatives. Please refer to Christ and Weinstein [4], and Taylor [40] for more details.

**Lemma 2.4** Assume that a function \( F \in C^{2}(\mathbb{R}) \) satisfies \( F(0) = F'(0) = 0 \) and

\[
|F'(a + b)| \leq C(|F'(a)| + |F'(b)|), \quad |F''(a + b)| \leq C(|F''(a)| + |F''(b)|),
\]

for all \( a, b \in \mathbb{R} \). Then we have

\[
\|D^{\alpha} F(u)\|_{L^{p}(\mathbb{R}^{2})} \leq C\|D^{\alpha} u\|_{L^{p_{1}}(\mathbb{R}^{2})}\|F'(u)\|_{L^{p_{2}}(\mathbb{R}^{2})}
\]

for \( 0 < \alpha < 1 \) and \( 1/p = 1/p_{1} + 1/p_{2}, 1 < p, p_{1}, p_{2} < \infty \).

**Proposition 2.5** Assume \( p \geq 5 \). Let \( u \) be a solution to \((CP1)\) defined for all \( t \geq 0 \) with \((u_{0}, u_{1}) \in \dot{H}^{s_{p}} \times \dot{H}^{s_{p}-1}(\mathbb{R}^{2}) \). If \( u \) satisfies \( u \in L^{3p/(p-1)} \times L^{6p/(p-1)}(0, T] \times \mathbb{R}^{2} \) for all \( T > 0 \) and scatters in the critical Sobolev space, i.e. there exists \((u_{0}^{+}, u_{1}^{+}) \in \dot{H}^{s_{p}} \times \dot{H}^{s_{p}-1}(\mathbb{R}^{2}) \) so that

\[
\lim_{t \to +\infty} \left\| \begin{pmatrix} u(\cdot, t) \\ \partial_{t} u(\cdot, t) \end{pmatrix} - S_{L}(t) \begin{pmatrix} u_{0}^{+} \\ u_{1}^{+} \end{pmatrix} \right\|_{\dot{H}^{s_{p}} \times \dot{H}^{s_{p}-1}(\mathbb{R}^{2})} = 0,
\]

then \( u \in L^{q} L^{r}([0, \infty) \times \mathbb{R}^{2}) \) for all \( s_{p} \)-admissible pairs \((q, r)\).

**Proof** The proof is similar to the higher dimensional case. For convenience of readers, we stretch a proof here. First of all, given any time \( t_{0} \) and time interval \( I \) containing \( t_{0} \), we introduce the \( Y(I) \) norm

\[
\|u\|_{Y(I)} = \|u\|_{L^{3p/(p-1)} \times L^{6p/(p-1)}(I \times \mathbb{R}^{2})}.
\]

By Strichartz estimates, the solution \( u \) to the linear equation \( \partial_{t}^{2} u - \Delta u = F \) with initial data \((u_{0}, u_{1})\) at time \( t_{0} \) satisfies
\[
\|u\|_{Y(I)} \leq \|S_L(t - t_0)(u_0, u_1)\|_{Y(I)} + C_1 \|F\|_{L^{3(p-1)} L^{\frac{6(p-1)}{3p+5}}(I \times \mathbb{R}^2)};
\]
\[
\|S_L(t - t_0)(u_0, u_1)\|_{Y(I)} \leq C_2 \|(u_0, u_1)\|_{\dot{H}^p \times \dot{H}^{p-1}}.
\]

In addition, we have
\[
\| - |u|^{p-1} u \|_{L^{\frac{3(p-1)}{3p-5}} L^{\frac{6(p-1)}{3p+5}}(I \times \mathbb{R}^2)} \leq \|u\|_{Y(I)}^p;
\]
\[
\|v|^{p-1} v - |u|^{p-1} u \|_{L^{\frac{3(p-1)}{3p-5}} L^{\frac{6(p-1)}{3p+5}}(I \times \mathbb{R}^2)} \leq C_3 \|u - v\|_{Y(I)} (\|u\|_{Y(I)}^{p-1} + \|v\|_{Y(I)}^{p-1}).
\]

Given \( u \in Y(I) \), we define \( Tu \) to be the solution \( v \) to linear wave equation \( \partial_t^2 v - \Delta v = -|u|^{p-1} u \) with initial data \((u_0, u_1)\) at time \( t_0 \). Then the inequalities above imply
\[
\|Tu\|_{Y(I)} \leq \|S_L(t - t_0)(u_0, u_1)\|_{Y(I)} + C_1 \|u\|_{Y(I)}^p;
\]
\[
\|Tu - Tv\|_{Y(I)} \leq C_3 (\|u\|_{Y(I)}^{p-1} + \|v\|_{Y(I)}^{p-1}) \|u - v\|_{Y(I)}.
\]

Thus there exists a constant \( \delta = \delta(p) > 0 \), so that if initial data \((u_0, u_1)\) satisfy
\[
\|S_L(t - t_0)(u_0, u_1)\|_{Y(I)} \leq \delta,
\]
then the operator \( T \) becomes a contraction map on the metric space \( \{ u : \|u\|_{Y(I)} < 2\delta \} \). As a result there exists a unique solution \( u \) to (CP1) in the time interval \( I \) with \( \|u\|_{Y(I)} \leq 2\delta \). In addition, we have
\[
\|S_L(t - T)(u(\cdot, T), u_T(\cdot, T))\|_{Y(I, +\infty)} \leq \|u^+\|_{Y(I, +\infty)} + \|S_L(t - T)(u(\cdot, T) - u^+(\cdot, T), u_T(\cdot, T) - u_T^+(\cdot, T))\|_{Y(I, +\infty)}
\]
\[
\leq \|u^+\|_{Y(I, +\infty)} + C_2 \|(u(\cdot, T) - u^+(\cdot, T), u_T(\cdot, T) - u_T^+(\cdot, T))\|_{\dot{H}^p \times \dot{H}^{p-1}}.
\]

Here \( u^+ = S_L(t)(u_0^+, u_1^+) \) is the free wave with initial data \((u_0^+, u_1^+)\). We utilize the scattering assumption and the Strichartz estimates \( \|u^+\|_{Y(\mathbb{R}^+)} < +\infty \) to conclude
\[
\lim_{T \to +\infty} \|S_L(t - T)(u(\cdot, T), u_T(\cdot, T))\|_{Y(I, +\infty)} = 0.
\]

Thus there exists a large time \( T > 0 \) so that
\[
\|S_L(t - T)(u(\cdot, T), u_T(\cdot, T))\|_{Y(I, +\infty)} \leq \delta,
\]
According to the local theory given above, we obtain
\[
\|u\|_{Y(I, +\infty)} < 2\delta.
\]
Therefore we have $\|u\|_{Y([0, +\infty))} < +\infty$. Finally we apply the Strichartz estimates and conclude

$$\|u\|_{L^q L^r([0, \infty) \times \mathbb{R}^2)} \lesssim \|S_L(t)(u_0, u_1)\|_{L^q L^r([0, \infty) \times \mathbb{R}^2)} + \|\Delta u\|_{L^{3(p-1)} L^{6(p-1)}([0, \infty) \times \mathbb{R}^2)}$$

$$\lesssim \|(u_0, u_1)\|_{H^s_p \times H^{s_p-1}([\mathbb{R}^2])} + \|u\|_{Y([0, +\infty))} < +\infty$$

for any given $s_p$-admissible pair $(q, r)$. □

**Theorem 2.6** (Radiation field) Let $u$ be a solution to the free wave equation $\partial_t^2 u - \Delta u = 0$ with initial data $(u_0, u_1) \in \dot{H}^1 \times \dot{L}^2(\mathbb{R}^2)$. Then

$$\lim_{t \to +\infty} \int_{\mathbb{R}^2} |\nabla u(x, t)|^2 \, dx = 0$$

and there exists a function $G_+ \in L^2(\mathbb{R} \times S^1)$ so that

$$\lim_{t \to +\infty} \int_0^\infty \int_{S^1} \left| r^{1/2} \partial_r u (r \Theta, t) - G_+ (r - t, \Theta) \right|^2 \, d\Theta \, dr = 0;$$

$$\lim_{t \to +\infty} \int_0^\infty \int_{S^1} \left| r^{1/2} \partial_r u (r \Theta, t) + G_+ (r - t, \Theta) \right|^2 \, d\Theta \, dr = 0.$$

In addition, the map $(u_0, u_1) \to \sqrt{2} G_+$ is a bijective isometry form $\dot{H}^1 \times \dot{L}^2(\mathbb{R}^2)$ to $L^2(\mathbb{R} \times S^1)$.

**Remark 2.7** Here the space $\dot{H}^1(\mathbb{R}^2)$ is the completion of $C^\infty_0(\mathbb{R}^2)$ equipped with the norm $\|\nabla u\|_{L^2}$. Please note that an element $u \in \dot{H}^1(\mathbb{R}^2)$ is not necessarily a tempered distribution, i.e. we may have $f \notin S'$, although $\nabla u \in L^2(\mathbb{R}^2)$ is always well-defined. However, this does not introduce any troubles when we consider free waves with initial data $(u_0, u_1) \in \dot{H}^1 \times \dot{L}^2$. Because we can always use smooth approximation techniques by the linearity and energy conservation law. In the case of defocusing wave equation discussed in this work, the solutions $u(x, t)$ are always functions defined almost everywhere in $\mathbb{R}^2$ for any given time $t$. Because our assumption on the initial data and energy conservation law imply that $(u, u_1) \in (\dot{H}^1 \cap L^{p+1}) \times \dot{L}^2$ for any time $t$.

The 3-dimensional version of radiation field has been known many years ago. For example, please see Friedlander [10, 11]. Duyckaerts et al. [9] gives a proof for all dimensions $d \geq 3$. The 2-dimensional case can be proved in almost the same way. In fact, only one of the ingredients in Duychaerts–Kenig–Merle’s proof is unique in 3 or higher dimensions: the limit

$$\lim_{t \to +\infty} \int_{\mathbb{R}^d} \frac{|u(x, t)|^2}{|x|^2} \, dx = 0$$
holds for all free waves \( u \). In general this is not true in the 2-dimensional case. In fact the integral above is not necessarily meaningful. Nevertheless, a careful review of the proof given in [9] shows that we may substitute the limit above by a weaker version:

**Lemma 2.8** Let \( u \) be a solution to 2-dimensional free wave equation \( \partial_t^2 u - \Delta u = 0 \) with smooth and compactly supported initial data. Then we have

\[
\lim_{t \to +\infty} \int_{|x| > t - \eta} \frac{|u(x, t)|^2}{|x|^2} \, dx = 0, \quad \forall \eta \in \mathbb{R}.
\]

**Proof** This immediately follows the well-known dispersive estimate: if the initial data is smooth and compactly supported, then we have \( |u(x, t)| \lesssim |t|^{-1/2} \). \( \square \)

The following result can be proved by Poisson’s formula and smooth approximation techniques. The authors would like to mention that although this result seems to be a direct consequence of radiation fields given above, it is actually an ingredient of the proof of radiation fields.

**Proposition 2.9** Let \( u \) be a free wave with finite energy. Then we always have

\[
\lim_{\eta \to +\infty} \sup_{t \geq \eta} \int_{|x| < t - \eta} \left( |\nabla u(x, t)|^2 + |u_t(x, t)|^2 \right) \, dx = 0.
\]

### 3 Morawetz identity and estimates

#### 3.1 Morawetz identity

**Proposition 3.1** (Morawetz identity) Let \( u \) be a solution to (CP1) with a finite energy \( E \). Then the following identity holds for any radius \( R > 0 \) and time \( t_1 < t_2 \). Here \( \sigma_R \) is the regular line measure of the circle \( |x| = R \) in \( \mathbb{R}^2 \).

\[
\frac{1}{2R} \int_{t_1}^{t_2} \int_{|x| < R} \left( |\nabla u|^2 + |u_t|^2 + \frac{p - 3}{p + 1} |u|^{p+1} \right) \, dx \, dt + \frac{1}{4R^2} \int_{t_1}^{t_2} \int_{|x| = R} |u|^2 \, d\sigma_R(x) \, dt
\]

\[
+ \int_{t_1}^{t_2} \int_{|x| > R} \left( \frac{|\nabla u|^2}{|x|} + \frac{p - 1}{2(p + 1)} \frac{|u|^{p+1}}{|x|} - \frac{1}{4} \frac{|u|^2}{|x|^3} \right) \, dx \, dt
\]

\[
+ \sum_{i=1,2} \int_{|x| < R} \left( \frac{R^2 - |x|^2}{2R^2} |u_t|^2 + \frac{1}{2} \frac{|x|^2}{R^2} u_t + \frac{u}{2R} + (-1)^i u_{1i} \right) \, dx
\]

\[
+ \sum_{i=1,2} \int_{|x| > R} \left( \frac{1}{2} |u_t + \frac{u}{2|x|} + (-1)^i u_{1i}|^2 + \frac{|\nabla u|^2}{2} + \frac{|u|^{p+1}}{p + 1} - \frac{|u|^2}{8|x|^2} \right) \bigg|_{t=t_i} \, dx = 2E.
\]
Remark 3.2 If $u$ is a solution to (CP1) with a finite energy $E$, then we have
\[
\int_{|x| > R'} \left| \frac{u}{|x|} \right|^2 \, dx \leq \left( \int_{|x| > R'} (|u|^2)^{p+1} \, dx \right)^{\frac{1}{p+1}} \left( \int_{|x| > R'} (|u|^{-2})^{\frac{p+1}{p-1}} \, dx \right)^{\frac{p-1}{p+1}}
\]
\[
\lesssim_p (R')^{-\frac{4}{p+1}} \left( \int_{|x| > R'} |u|^{p+1} \, dx \right)^{\frac{1}{p+1}}
\]
\[
\lesssim_p (R')^{-\frac{4}{p+1}} E^\frac{2}{p+1}.
\]

Thus the integrals in the inequality above are all finite.

**Proof** We follow a similar argument to the one given by Perthame and Vega in the final section of their work [32]. Some terms in 2-dimensional case come with a different sign from the higher dimensional case thus we have to work more carefully. Let us first consider solutions with compact support. We will calculate as though the solutions are sufficiently smooth, otherwise smooth approximation techniques can be applied.

Given a positive constant $R$, we define two radial functions $\Psi$ and $\varphi$ by
\[
\nabla \Psi = \begin{cases} x, & \text{if } |x| \leq R; \\
Rx/|x|, & \text{if } |x| \geq R; \end{cases} \quad \varphi = \begin{cases} 1/2, & \text{if } |x| \leq R; \\
0, & \text{if } |x| > R. \end{cases}
\]

Since $u$ is defined for all time $t \in \mathbb{R}$, we may also define a function on $\mathbb{R}$
\[
\mathcal{E}(t) = \int_{\mathbb{R}^2} u_t(x, t) \left( \nabla u(x, t) \cdot \nabla \Psi + u(x, t) \left( \frac{\Delta \Psi}{2} - \varphi \right) \right) \, dx.
\]

We may differentiate $\mathcal{E}$, utilize the equation $u_{tt} - \Delta u = -|u|^{p-1}u$, apply integration by parts and obtain
\[
-\mathcal{E}'(t) = \int_{\mathbb{R}^2} \left( \sum_{i,j=1}^2 u_i \Psi_{ij} u_j - \varphi |\nabla u|^2 + \Psi |u_t|^2 \right) \, dx + \frac{1}{4} \int_{\mathbb{R}^2} \nabla (|u|^2) \cdot \nabla (\Delta \Psi - 2\varphi) \, dx
\]
\[
+ \int_{\mathbb{R}^2} |u|^{p+1} \left( \frac{p-1}{2(p+1)} \Delta \Psi - \varphi \right) \, dx = I_1 + I_2 + I_3.
\]

Here we have
\[
\Psi_{ij} = \begin{cases} \delta_{ij}, & \text{if } |x| < R; \\
\frac{R \delta_{ij}}{|x|^3} - \frac{Rx_i x_j}{|x|^3}, & \text{if } |x| > R; \end{cases} \quad \Delta \Psi = \begin{cases} 2, & \text{if } |x| < R; \\
R/|x|, & \text{if } |x| > R; \end{cases}
\]
\[
\Delta \Psi - 2\varphi = \begin{cases} 1, & \text{if } |x| \leq R; \\
R/|x|, & \text{if } |x| > R; \end{cases} \in C(\mathbb{R}^2).
\]

when $|x| > R$, we may calculate
\[
\sum_{i,j=1}^2 u_i \Psi_{ij} u_j = \sum_{i,j=1}^2 u_i \left( \frac{R \delta_{ij}}{|x|^3} - \frac{Rx_i x_j}{|x|^3} \right) u_j = \frac{R}{|x|} |\nabla u|^2 - \frac{R}{|x|^3} |\nabla u|^2 = \frac{R}{|x|} |\nabla u|^2.
\]
Thus we have

\[ I_1 = \frac{1}{2} \int_{|x| < R} (|∇u|^2 + |u_t|^2) \, dx + R \int_{|x| > R} \frac{|∇u|^2}{|x|} \, dx. \]

A basic computation shows

\[ I_3 = \frac{p - 3}{2(p + 1)} \int_{|x| < R} |u|^{p+1} \, dx + \frac{(p - 1)R}{2(p + 1)} \int_{|x| > R} \frac{|u|^{p+1}}{|x|} \, dx. \]

Finally let us calculate \( I_2 \) carefully

\[ I_2 = \frac{1}{4} \int_{R^2} \nabla(|u|^2) \cdot \nabla (\Delta \Psi - 2\varphi) \, dx \]
\[ = \frac{1}{4} \int_{|x| > R} \nabla(|u|^2) \cdot \frac{-Rx}{|x|^3} \, dx \]
\[ = \frac{1}{4} \int_{|x| > R} \left[ \text{div} \left( |u|^2 \cdot \frac{-Rx}{|x|^3} \right) - \frac{R}{|x|^3} |u|^2 \right] \, dx \]
\[ = \frac{1}{4R} \int_{|x| = R} |u|^2 dσ_R(x) - \frac{R}{4} \int_{|x| > R} \frac{|u|^2}{|x|^3} \, dx. \]

Since \(-\mathcal{E}'(t) = I_1 + I_2 + I_3\), we have

\[ \int_{t_1}^{t_2} (I_1 + I_2 + I_3) \, dt = \mathcal{E}(t_1) - \mathcal{E}(t_2). \] (2)

We may rewrite \( \mathcal{E}(t_1) \) in the form of

\[ R\mathcal{E}(t_1) = \frac{1}{2} \int_{R^2} \left( R^2 |u_t(x, t_1)|^2 + |∇u(x, t_1) \cdot ∇\Psi + u(x, t_1) \left( \frac{Δ\Psi}{2} - \varphi \right) \right)^2 \, dx \]
\[ - \frac{1}{2} \int_{R^2} \left| ∇u(x, t_1) \cdot ∇\Psi + u(x, t_1) \left( \frac{Δ\Psi}{2} - \varphi \right) - Ru_t(x, t_1) \right|^2 \, dx \]
\[ = J_1 - J_2 \]

We then calculate \( J_1, J_2 \) carefully

\[ J_1 = \frac{1}{2} \int_{R^2} \left( R^2 |u_t|^2 + |∇u \cdot ∇\Psi|^2 + \left( \frac{Δ\Psi}{2} - \varphi \right) |∇(u|^2| \cdot ∇\Psi + \left( \frac{Δ\Psi}{2} - \varphi \right)^2 |u|^2 \right) \, dx \]
\[ = \frac{1}{2} \int_{R^2} \left[ R^2 |u_t|^2 + |∇u \cdot ∇\Psi|^2 - \text{div} \left( \left( \frac{Δ\Psi}{2} - \varphi \right) \nabla |u|^2 + \left( \frac{Δ\Psi}{2} - \varphi \right)^2 |u|^2 \right) \right] \, dx \]
A basic calculation shows

$$\text{div} \left( \left( \frac{\Delta \Psi}{2} - \varphi \right) \nabla \Psi \right) = \begin{cases} 1, \text{ if } |x| < R; \\ 0, \text{ if } |x| > R; \end{cases}$$

Thus we have

$$J_1 = \frac{1}{2} \int_{|x| < R} \left[ R^2 |u_t|^2 + |x \cdot \nabla u|^2 - \frac{3}{4} |u|^2 \right] dx + \frac{1}{2} \int_{|x| > R} \left[ R^2 |u_t|^2 + R^2 |u_r|^2 + \frac{R^2 |u|^2}{4|x|^2} \right] dx$$

$$= R^2 E - R^2 \int_{|x| < R} \left[ \frac{R^2 - |x|^2}{2R^2} |u_r|^2 + \frac{3}{8R^2} |u|^2 + \frac{1}{2} \frac{\nabla u^2}{|x|^2} + \frac{1}{p+1} |u|^{p+1} \right] dx$$

$$+ \frac{R^2}{8} \int_{|x| > R} \frac{|u|^2}{|x|^2} dx - R^2 \int_{|x| > R} \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{p+1} |u|^{p+1} \right) dx$$

In addition we have

$$J_2 = \frac{1}{2} \int_{|x| < R} \left| x \cdot \nabla u + \frac{1}{2} u - Ru_t \right|^2 dx + \frac{R^2}{2} \int_{|x| > R} \left| \frac{x}{|x|} \cdot \nabla u + \frac{u}{2|x|} - u_t \right|^2 dx.$$ 

Combining $J_1$, $J_2$, we obtain

$$RE(t_1) = R^2 E - R^2 \int_{|x| > R} \left( \frac{1}{2} |u_r + \frac{u}{2|x|} - u_t|^2 + \frac{|\nabla u|^2}{2} + \frac{|u|^{p+1}}{p+1} - \frac{|u|^2}{8|x|^2} \right) dx$$

$$- R^2 \int_{|x| < R} \left( \frac{R^2 - |x|^2}{2R^2} |u_r|^2 + \frac{1}{2} \left| \frac{|x|}{R} u_r + \frac{u}{2R} - u_t \right|^2 + \frac{|\nabla u|^2}{2} + \frac{|u|^{p+1}}{p+1} + \frac{3|u|^2}{8R^2} \right) dx.$$ 

Finally we may find a similar expression of $-RE(t_2)$

$$-RE(t_2) = R^2 E - R^2 \int_{|x| > R} \left( \frac{1}{2} |u_r + \frac{u}{2|x|} + u_t|^2 + \frac{|\nabla u|^2}{2} + \frac{|u|^{p+1}}{p+1} - \frac{|u|^2}{8|x|^2} \right) dx$$

$$- R^2 \int_{|x| < R} \left( \frac{R^2 - |x|^2}{2R^2} |u_r|^2 + \frac{1}{2} \left| \frac{|x|}{R} u_r + \frac{u}{2R} + u_t \right|^2 + \frac{|\nabla u|^2}{2} + \frac{|u|^{p+1}}{p+1} + \frac{3|u|^2}{8R^2} \right) dx.$$ 

then plug all the expressions of $I_1$, $I_2$, $I_3$ and $\mathcal{E}(t_1)$, $\mathcal{E}(t_2)$ into the integral identity (2) to finish the proof if the solution is compactly supported. In order to deal with general solutions $u$, we fix a smooth radial cut-off function $\phi : \mathbb{R}^2 \to [0, 1]$ so that

$$\phi(x) = \begin{cases} 1, \text{ if } |x| \leq 1; \\ 0, \text{ if } |x| > 2; \end{cases}$$

define initial data $(u_{0,R'}(x), u_{1,R'}(x)) = \phi(x/R')(u(x, t_1), u_r(x, t_1))$ and consider the corresponding solution $u_{R'}$ to (CP1). The argument above shows that $u_{R'}$ satisfies the Morawetz identity. We observe

- The identity $u_{R'}(x, t) = u(x, t)$ holds if $|x| < R' + t_1 - t$ by finite speed of propagation;
These facts enable us to take the limit $R' \to +\infty$ and prove the Morawetz identity for general solutions $u$ without compact support.

### 3.2 Morawetz inequalities

A combination of Morawetz identity and finite speed of propagation gives a few useful inequalities, which lead to our main results in subsequent sections. The key observation here is that if $R$ is large, the first term in the Morawetz identity is almost $2E$ when $t_1 \leq -R$ and $t_2 \geq R$, thus all other terms must be small.

**Corollary 3.3** Let $u$ be a solution to (CP1) with initial data $(u_0, u_1) \in (\dot{H}^1(\mathbb{R}^2) \cap L^{p+1}(\mathbb{R}^2)) \times L^2(\mathbb{R}^2)$. Given any $R > 0$, $r \geq 0$, $0 \leq \mu_1 \leq \frac{2(p-1)}{p+1}$ and $0 \leq \mu_2 \leq \frac{1}{p+1}$, we have

$$
\sum_{j=1}^6 M_j \leq \int_{\mathbb{R}^2} \min\{|x|/R, 1\} \left( |\nabla u_0|^2 + |u_1|^2 + \frac{2}{p+1}|u_0|^{p+1}\right) dx
\quad + \sum_{\pm} \int_{|x| > R} \left( \frac{9|u|^2}{8|x|^2} - \lambda_2 |u|^{p+1}\right) |_{t=\pm(R+r)} dx
\quad + \int_{-R-r}^{R+r} \int_{|x| > R} \left( \frac{5|u|^2}{4|x|^3} - \frac{\lambda_1 |u|^{p+1}}{|x|}\right) dx dt.
$$

The notations $M_j$ represent

- $M_1 = \frac{1}{2R} \int_{R-|t| < R+r} \int_{|x| < R} \left( |\nabla u|^2 + |u_t|^2 + \frac{p-3}{p+1}|u|^{p+1}\right) dx dt$;
- $M_2 = \frac{p-5}{2(p+1)R} \int_{R-|t| < R+r} \int_{|x| < R} |u|^{p+1} dx dt$;
- $M_3 = \frac{1}{4R^2} \int_{R-|t| < R+r} \int_{|x| = R} |u|^2 d\sigma_R(x) dt$;
- $M_4 = \int_{-R-r}^{R+r} \int_{|x| > R} \left( \frac{|\nabla u|^2}{|x|} + \frac{\mu_1 |u|^{p+1}}{|x|^2} + \frac{|u|^2}{|x|^3}\right) dx dt$;
- $M_5 = \sum_{\pm} \int_{|x| < R} \left( \frac{R^2-|x|^2}{2R^2} |u_r|^2 + \frac{1}{2} \left| \frac{|x|}{R} u_r + \frac{u}{2R} \pm u_t \right|^2 + \frac{3|u|^2}{8R^2} \frac{|\nabla u|^2}{2} + \frac{|u|^{p+1}}{p+1}\right) |_{t=\pm(R+r)} dx$;
- $M_6 = \sum_{\pm} \int_{|x| > R} \left( \frac{1}{2} |u_r + \frac{u}{2|x|} \pm u_t|^2 + \frac{|\nabla u|^2}{2} + \mu_2 |u|^{p+1} + \frac{|u|^2}{|x|^2}\right) |_{t=\pm(R+r)} dx.$
The constants $\lambda_i$’s are defined by $\lambda_1 = \frac{2(p-1)}{p+1} - \mu_1$ and $\lambda_2 = \frac{1}{p+1} - \mu_2$. Moreover, if $\lambda_1, \lambda_2 > 0$, then there exists a constant $C = C(p, \mu_1, \mu_2)$ so that

$$
\sum_{j=1}^{6} M_j \leq \int_{\mathbb{R}^2} \min\{|x|/R, 1\} \left(|\nabla u_0|^2 + |u_1|^2 + \frac{2}{p+1}|u_0|^{p+1}\right) dx + C(R + r) R^{-1 - \frac{4}{p-1}}. \tag{3}
$$

**Remark 3.4** If $p \geq 5$ is conformal or super-conformal, then all the terms $M_j$ are nonnegative. If $3 \leq p < 5$, then all $M_j$’s except for $M_2$ are nonnegative. The terms $M_5$ and $M_6$ are the most relevant to the energy distribution at large time and will be used in later sections. Some of $M_j$’s will not be used in this work but we still give them here for completeness.

**Proof** We first choose $t_1 = -R - r$, $t_2 = R + r$ in the Morawetz identity, add

$$
\int_{-R-r}^{R+r} \int_{|x| \geq R} \left( \frac{5|u|^2}{4|x|^3} - \frac{\lambda_1 |u|^{p+1}}{|x|} \right) dx dt + \sum_{i=1,2} \int_{|x| \geq R} \left( \frac{9|u(x, t_i)|^2}{8|x|^2} - \lambda_2 |u(x, t_i)|^{p+1} \right) dx
$$

in both sides (here $\lambda_1 = \frac{2(p-1)}{p+1} - \mu_1 \geq 0$, $\lambda_2 = \frac{1}{p+1} - \mu_2 \geq 0$), and obtain

$$
\frac{1}{2R} \int_{-R-r}^{R+r} \int_{|x| < R} \left( |\nabla u|^2 + |u_t|^2 + \frac{p-3}{p+1}|u|^{p+1} \right) dx + \sum_{j=3}^{6} M_j \leq 2E + \sum_{i=1,2} \int_{|x| \geq R} \left( \frac{9|u|^2}{8|x|^2} - \lambda_2 |u|^{p+1} \right) \bigg|_{i=\pm(R+r)} dx
$$

The first term above can be written as a sum of three terms

$$
\frac{1}{2R} \int_{-R-r}^{R+r} \int_{|x| < R} \left( |\nabla u|^2 + |u_t|^2 + \frac{p-3}{p+1}|u|^{p+1} \right) dx = M_1 + \frac{1}{2R} \int_{-R}^{R} \int_{|x| < R} \left( |\nabla u|^2 + |u_t|^2 + \frac{p-3}{p+1}|u|^{p+1} \right) dx
$$

$$
= M_1 + M_2 + \frac{1}{2R} \int_{-R}^{R} \int_{|x| < R} \left( |\nabla u|^2 + |u_t|^2 + \frac{2}{p+1}|u|^{p+1} \right) dx
$$
Thus

\[
\frac{1}{2R} \int_{-R}^{R} \int_{|x|<R} \left( |\nabla u|^2 + |u_t|^2 + \frac{2}{p+1} |u|^{p+1} \right) \, dx \, dt
+ \sum_{j=1}^{6} M_j = 2E + \sum_{|x|>R} \left( \frac{9|u|^2}{8|x|^2} - \lambda_2 |u|^{p+1} \right) \bigg|_{t=\pm(R+r)} 
+ \int_{|x|>R} \left( \frac{5|u|^2}{4|x|} - \frac{\lambda_1 |u|^{p+1}}{|x|} \right) \, dx.
\]

In order to prove the first inequality we only need to show

\[
I = 2E - \frac{1}{2R} \int_{-R}^{R} \int_{|x|<R} \left( |\nabla u|^2 + |u_t|^2 + \frac{2}{p+1} |u|^{p+1} \right) \, dx \, dt
\leq \int_{\mathbb{R}^2} \min\{ |x|/R, 1 \} \left( |\nabla u_0|^2 + |u_1|^2 + \frac{2}{p+1} |u_0|^{p+1} \right) \, dx.
\]

This immediately follows energy conservation law and finite speed of propagation of energy

\[
I = \frac{1}{R} \int_{-R}^{R} \int_{|x|>R} \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{2} |u_t|^2 + \frac{1}{p+1} |u|^{p+1} \right) \, dx \, dt
\leq \frac{1}{R} \int_{-R}^{R} \int_{|x|>|t|} \left( \frac{1}{2} |\nabla u_0|^2 + \frac{1}{2} |u_1|^2 + \frac{1}{p+1} |u_0|^{p+1} \right) \, dx \, dt
= \frac{1}{R} \int_{\mathbb{R}^2} \min\{ |x|, R \} \left( |\nabla u_0|^2 + |u_1|^2 + \frac{2}{p+1} |u_0|^{p+1} \right) \, dx.
\]

Finally if \( \lambda_1, \lambda_2 > 0 \), we need to find an upper bound of the integrals

\[
\sum_{\pm} \int_{|x|>R} \left( \frac{9|u|^2}{8|x|^2} - \lambda_2 |u|^{p+1} \right) \bigg|_{t=\pm(R+r)} \, dx
+ \int_{|x|>R} \left( \frac{5|u|^2}{4|x|} - \frac{\lambda_1 |u|^{p+1}}{|x|} \right) \, dx.
\]

If \( u \gg |x|^{-2/(p-1)} \), then \( |u|^2/|x|^2 \ll |u|^{p+1} \), Thus we always have

\[
\frac{9|u|^2}{8|x|^2} - \frac{\lambda_2 |u|^{p+1}}{p+1} \lesssim_{p, \mu_2} |x|^{-2-\frac{4}{p-1}}; \quad \frac{5|u|^2}{4|x|^3} - \frac{\lambda_1 |u|^{p+1}}{|x|} \lesssim_{p, \mu_1} |x|^{-3-\frac{4}{p-1}}.
\]
Plugging these upper bounds in the integrals above, we obtain

\[
\sum_{\pm} \int_{|x| > R} \left( \frac{|9u|^2}{8|x|^2} - \lambda_2 |u|^{p+1} \right)_{t = \pm (R + r)} \, dx
\]

\[
+ \int_{-R-r}^{R+r} \int_{|x| > R} \left( \frac{5|u|^2}{4|x|^3} - \frac{\lambda_1 |u|^{p+1}}{|x|} \right) \, dx
\]

\[
\lesssim_{p, \mu_1, \mu_2} (R + r) R^{-1 - \frac{4}{p+1}}.
\]

This immediately proves inequality (3) and finishes the proof. \(\square\)

4 Energy distribution

In this section we prove Theorem 1.1. It suffices to consider the positive time direction \(t > 0\), since the wave equation is time-reversible. We start with the conformal and super-conformal case \(p \geq 5\).

4.1 Conformal and super-conformal case

The proof mainly relies on

**Lemma 4.1** Assume \(p \geq 5\). Let \(u\) be a finite-energy solution to \((CP1)\). Then the following inequalities hold for large time \(t > 0\).

\[
\int_{|x| < t} \frac{t - |x|}{t} e(x, t) \, dx + \int_{\mathbb{R}^2} \left( |u_r + u_t|^2 + |\nabla u|^2 + |u|^{p+1} \right) \, dx
\]

\[
\lesssim_p \int_{\mathbb{R}^2} \min\{|x|/t, 1\} e(x, 0) \, dx + \sum_{\pm} \int_{|x| > t} \frac{|u(x, \pm t)|^2}{|x|^2} \, dx
\]

\[
+ \int_{-t}^{t} \int_{|x| > t} \left( \frac{|u(x, t')|^2}{|x|^3} \right) \, dx \, dt';
\]

\[
\int_{|x| < t} \frac{t - |x|}{t} e(x, t) \, dx + \int_{\mathbb{R}^2} \left( |u_r + u_t|^2 + |\nabla u|^2 + |u|^{p+1} \right) \, dx
\]

\[
\lesssim_p \int_{\mathbb{R}^2} \min\{|x|/t, 1\} e(x, 0) \, dx + t^{-\frac{4}{p+1}}.
\]

**Proof** We choose \(R = t, r = 0, \mu_1 = \frac{2(p-1)}{p+1}\) and \(\mu_2 = \frac{1}{p+1}\) in Corollary 3.3, then we have (the two integrals below are evaluated at time \(t\))

\[
\int_{|x| < t} \left( \frac{t^2 - |x|^2}{2t^2} |u_r|^2 + \frac{1}{2} \left| \frac{x}{t} u_r + \frac{u}{2t} + u_t \right|^2 + \frac{3|u|^2}{8t^2} + \frac{1}{2} \frac{|\nabla u|^2}{t^2} + \frac{1}{p+1} |u|^{p+1} \right) \, dx
\]

\[
+ \int_{|x| > t} \left( \frac{1}{2} |u_r + \frac{u}{2|x|} + u_t|^2 + \frac{1}{2} \frac{|\nabla u|^2}{t} + \frac{1}{p+1} |u|^{p+1} + \frac{|u|^2}{|x|^2} \right) \, dx \lesssim_p RH1.
\]
Here RH1 represents the right hand side of the first inequality in Lemma 4.1. Thus we may verify the first inequality by observing

\[
|\nabla u|^2 + |u_t|^2 \lesssim_1 |u_r|^2 + \left| \frac{|x|}{t} u_r + \frac{u}{2t} + u_t \right|^2 + \frac{|u|^2}{t^2} + |\nabla u|^2, \quad \text{if } |x| < t;
\]

and

\[
|u_r + u_t|^2 \lesssim_1 \left( \frac{t - |x|}{t^2} \right)^2 |u_r|^2 + \left| \frac{|x|}{t} u_r + \frac{u}{2t} + u_t \right|^2 + \frac{|u|^2}{t^2}, \quad \text{if } |x| < t;
\]

\[
|u_r + u_t|^2 \lesssim_1 \left| u_r + \frac{u}{2|x|} + u_t \right|^2 + \frac{|u|^2}{|x|^2}, \quad \text{if } |x| > t.
\]

The second inequality can be proved in the same way by choosing \( R = t, r = 0, \mu_1 = \frac{p-1}{p+1}, \mu_2 = \frac{1}{2(p+1)} \) in Corollary 3.3 and using inequality (3).

\[ \square \]

**Proof of Part (a)(b)** It is clear that Part (a) and (b) immediately follow the second inequality given in Lemma 4.1. We only need to let \( t \to +\infty \) and apply the dominated convergence theorem.

**Proof of part (c), smaller decay rate** If \( \kappa \in (0, \frac{4}{p-1}) \) and \( E_\kappa(u_0, u_1) < +\infty \), then we may apply Lemma 4.1 again

\[
t^\kappa \int_{|x| < t} \frac{t - |x|}{t} e(x, t) dx + t^\kappa \int_{\mathbb{R}^2} \left( |u_r + u_t|^2 + |\nabla u|^2 + |u|^{p+1} \right) dx \\
\lesssim_p \int_{\mathbb{R}^2} \min\{|x|^{t^\kappa-1}, t^\kappa\} e(x, 0) dx + t^{\kappa - \frac{4}{p-1}}.
\]

We observe the facts

\[
\min\{|x|^{t^\kappa-1}, t^\kappa\} \leq |x|^\kappa, \quad \lim_{t \to +\infty} |x|^{t^\kappa-1} = 0,
\]

then apply the dominated convergence theorem again and conclude that the upper bound found above converges to zero as \( t \to +\infty \). This finishes the proof of part (c).

**Remark 4.2** We may slightly improve the decay estimate if decay rate \( \kappa \in (0, \frac{4}{p-1}) \). By Lemma 4.1 we have

\[
\int_1^\infty t^{\kappa-1} \left( \int_{|x| < t} \frac{t - |x|}{t} e(x, t) dx + \int_{\mathbb{R}^2} \left( |u_r + u_t|^2 + |\nabla u|^2 + |u|^{p+1} \right) dx \right) dt \\
\lesssim_p \int_1^\infty \left( \int_{\mathbb{R}^2} \min\{|x|^{t^{\kappa-2}}, t^{\kappa-1}\} e(x, 0) dx + t^{\kappa - \frac{4}{p-1} - 1} \right) dt \\
\lesssim_p, \kappa \int_{\mathbb{R}^2} |x|^\kappa e(x, 0) dx + 1 < +\infty.
\]
Proof of Part (c), higher decay rate
Now we assume \( \kappa \in [\frac{4}{p-1}, 1) \) and \( E_\kappa(u_0, u_1) < +\infty \). We utilize the first inequality given in Lemma 4.1 and obtain
\[
\int_{|x|<t} \frac{t-|x|}{t} e(x, t) dx + \int_{\mathbb{R}^2} \left(|u_r + u_t|^2 + |\nabla u|^2 + |u|^{p+1}\right) dx \\
\lesssim_p \int_{\mathbb{R}^2} \min(|x|/t, 1) e(x, 0) dx + \sum_{\pm} \int_{|x|>t} \frac{|u(x, \pm t)|^2}{|x|^2} dx \\
+ \int_{-t}^{t} \int_{|x|>t} \frac{|u(x, t')|^2}{|x|^3} dx dt' \tag{5}
\]
We have already known how to deal with the first integral in the right hand side above. Thus we still need to deal with the last two integrals. The upper bounds of these two integrals can be found by a combination of finite speed of propagation and a weighted Hardy inequality. We first prove two lemmata.

Lemma 4.3 (Weighted Hardy inequality) Let \( \kappa \in (0, 1) \). If \( v \in \dot{H}^1(\mathbb{R}^2) \) satisfies
\[
\int_{\mathbb{R}^2} (|x|^\kappa + 1)|\nabla v(x)|^2 dx < +\infty,
\]
then we may find a locally integrable function \( v \) whose weak derivative is exactly \( \nabla v \) so that
\[
\int_0^{2\pi} |v(r, \theta)|^2 d\theta \lesssim_{\kappa} r^{-\kappa} \int_{|x|>r} (|x| - r)^\kappa |\nabla v(x)|^2 dx, \quad \forall r > 0. \tag{6}
\]
Here we use the polar coordinates \((r, \theta)\). In addition, if \( 0 \leq R_1 \leq R \), then we have
\[
\int_{|x|>R} \frac{|v(x)|^2}{|x|^2} dx \lesssim_{\kappa} \int_{|x|>R} (R^{-\kappa} - |x|^{-\kappa})(|x| - R_1)^\kappa |\nabla v(x)|^2 dx.
\]

Remark 4.4 In general we can not talk about the value of \( v(x) \) at a point \( x \) if \( v \) is merely in the space \( \dot{H}^1(\mathbb{R}^2) \). But if \( v \in \dot{H}^1 \cap L^{p+1}(\mathbb{R}^2) \) satisfies the conditions above, then the locally integrable function we define above (we temporarily call it \( v_1 \)) must coincides with the original \( v \). This is because there exists a constant \( C \) so that \( v = v_1 + C \) almost everywhere since they share the same weak derivatives. We then conclude that \( C = 0 \) by the assumption \( v \in L^{p+1} \) and the fact
\[
\lim_{r \to +\infty} \int_0^{2\pi} |v_1(r, \theta)|^2 d\theta = 0.
\]
A similar argument shows that the locally integrable function in Lemma 4.3 is unique. Thus an \( \dot{H}^1(\mathbb{R}^2) \) function satisfying \( \|\nabla\|_{L^2(\mathbb{R}^2; (|x|^{\kappa+1})dx)} < +\infty \) is actually a locally integrable function.
Proof It suffices to prove the first inequality (6). Because the second inequality immediately follows that first one:

\[
\int_{|x|>R} \frac{|v(x)|^2}{|x|^2} dx = \int_{R}^{\infty} \int_{0}^{2\pi} r^{-1} |v(r, \theta)|^2 d\theta dr \\
\lesssim \kappa \int_{R}^{\infty} \left( r^{-1-\kappa} \int_{|x|>r} (|x|-r)^\kappa |\nabla v(x)|^2 dx \right) dr \\
\lesssim \kappa \int_{R}^{\infty} \left( r^{-1-\kappa} \int_{|x|>r} (|x|-R_1)^\kappa |\nabla v(x)|^2 dx \right) dr \\
\lesssim \kappa \int_{|x|>R} (R^{-\kappa} - |x|^{-\kappa})(|x|-R_1)^\kappa |\nabla v(x)|^2 dx.
\]

Now we prove (6) in details. We first choose a nonnegative function \( \varphi \in C_0^\infty(\mathbb{R}^2) \) so that

\[
\int_{\mathbb{R}^2} \varphi(x) dx = 1.
\]

We define \( \varphi_n(x) = 2^{2n} \varphi(2^n x) \) and consider \( v_n = v * \varphi_n \in \dot{H}^1(\mathbb{R}^2) \). It immediately follows that

\[
\lim_{n \to +\infty} \int_{\mathbb{R}^2} (|x|^\kappa + 1)|\nabla v(x) - \nabla v_n(x)|^2 dx = 0.
\]

Now \( \nabla v_n \in C^\infty(\mathbb{R}^2) \), thus we define a function \( v_n \in C^\infty(\mathbb{R}^2) \) whose derivative is exactly \( \nabla v_n \). We observe that

\[
\int_{0}^{2\pi} \int_{1}^{\infty} |\partial_r v_n(r, \theta)| dr d\theta \\
\leq \left( \int_{0}^{2\pi} \int_{1}^{\infty} r^{1+\kappa} |\partial_r v_n(r, \theta)|^2 dr d\theta \right)^{1/2} \left( \int_{0}^{2\pi} \int_{1}^{\infty} r^{-1-\kappa} dr d\theta \right)^{1/2} \\
\leq \left( 2\pi \kappa^{-1} \int_{|x|>1} |x|^\kappa |\nabla v_n(x)|^2 dx \right)^{1/2}
\]

is finite. Therefore for almost everywhere \( \theta \in [0, 2\pi] \), the function \( \partial_r v_n(r, \theta) \) is integrable in \([1, +\infty) \). As a result, the limit

\[
v_n(+\infty, \theta) = \lim_{r \to +\infty} v_n(r, \theta)
\]

exists almost everywhere \( \theta \in [0, 2\pi] \). In addition, if \( v_n(+\infty, \theta_1) \) and \( v_n(+\infty, \theta_2) \) are both well-defined, then
\[
\int_1^\infty \frac{|v_n(r, \theta_2) - v_n(r, \theta_1)|^2}{r} dr \leq \int_1^\infty r^{-1} \left( \int_{\theta_1}^{\theta_2} |\partial_\theta v_n(r, \theta)| d\theta \right)^2 dr \\
\leq (\theta_2 - \theta_1) \int_1^\infty r^{-1} \left( \int_{\theta_1}^{\theta_2} |\partial_\theta v_n(r, \theta)|^2 d\theta \right) dr \\
\leq (\theta_2 - \theta_1) \int_{|x|>1} |\nabla v_n(x)|^2 dx < +\infty.
\]

Unless \(v_n(+\infty, \theta_1) = v_n(+\infty, \theta_2)\) the inequality above can not hold. Thus \(v_n(+\infty, \theta)\) is a constant \(C_n\) almost everywhere. We then substitute \(v_n\) by \(v_n - C_n\) so that \(v_n(+\infty, \theta) = 0\) holds almost everywhere \(\theta \in [0, 2\pi]\). As a result, we have

\[
\int_0^{2\pi} |v_n(r, \theta)|^2 d\theta \leq \int_0^{2\pi} \left( \int_r^{\infty} |\partial_r v_n(r', \theta)| dr' \right)^2 d\theta \\
\leq \int_0^{2\pi} \left( \int_r^{\infty} (r' - r)^{\kappa} |\partial_r v_n(r', \theta)|^2 dr' \right) \left( \int_r^{\infty} (r' - r)^{-\kappa} dr' \right) d\theta \\
\leq_{\kappa} \int_0^{2\pi} r^{-\kappa} \left( \int_r^{\infty} (r' - r)^{\kappa} |\partial_r v_n(r', \theta)|^2 dr' \right) d\theta \\
\leq_{\kappa} r^{-\kappa} \int_{|x|>r} (|x| - r)^{\kappa} |\nabla v_n(x)|^2 dx
\]

In other words, the inequality (6) holds for each \(v_n\). Similarly we have

\[
\lim_{m,n \to +\infty} \int_0^{2\pi} |v_m(r, \theta) - v_n(r, \theta)|^2 d\theta \leq_{\kappa} \lim_{m,n \to +\infty} r^{-\kappa} \\
\times \int_{|x|>r} (|x| - r)^{\kappa} |\nabla v_m(x) - \nabla v_n(x)|^2 dx = 0.
\]

Namely \(v_n(r, \theta)\) is a Cauchy sequence in the space \(C([R, +\infty); L^2([0, 2\pi]))\) for any \(R > 0\). Finally we let \(v\) to be the limit of \(v_k\) and finish the proof. \(\square\)

**Lemma 4.5** (Finite speed of propagation) Let \(a(r)\) be an absolutely continuous and increasing function defined on \([R, +\infty)\). If \(u\) is a finite-energy solution to (CP1), then for any time \(t' \in [-R, R]\) we have

\[
\int_{|x|>R} a(|x|) e(x, t') dx \leq \int_{|x|>R-|t'|} a(|x| + |t'|) e(x, 0) dx.
\]

**Proof** This is a combination of finite speed of propagation and Fubini’s theorem. In fact we have

\[
\int_{|x|>R} a(|x|) e(x, t') dx = \int_R^\infty a'(r) \left( \int_{|x|>r} e(x, t') dx \right) dr + a(R) \int_{|x|>R} e(x, t') dx \\
\leq \int_R^\infty a'(r) \left( \int_{|x|>r-|t'|} e(x, 0) dx \right) dr + a(R) \int_{|x|>R-|t'|} e(x, 0) dx
\]
\[
= \int_{|x| > R - |r'|} a(|x| + |r'|) e(x, 0) dx.
\]

Finally we may complete the proof by dominated convergence theorem.

Now we are ready to find the upper bounds of the last two inequalities in (5). We apply the two lemmata above and obtain (|r'| \leq t)

\[
\int_{|x| > t} \frac{|u(x, t')|^2}{|x|^2} dx \lesssim \kappa \int_{|x| > t} (t^{-\kappa} - (|x| - |r'|)^{-\kappa}) \|\nabla u(x, t')\|^2 dx
\]

\[
\lesssim 1 \int_{|x| > t} (t^{-\kappa} - (|x| - |r'|)^{-\kappa}) e(x, t') dx
\]

\[
\leq \int_{|x| > t - |r'|} (t^{-\kappa} - (|x| + |r'|)^{-\kappa}) |x|^\kappa e(x, 0) dx.
\]

Thus we have

\[
\sum_{\pm} \int_{|x| > t} \frac{|u(x, \pm t)|^2}{|x|^2} dx \lesssim \kappa \int_{\mathbb{R}^2} (t^{-\kappa} - (|x| + t)^{-\kappa}) |x|^\kappa e(x, 0) dx;
\]

and

\[
\int_{-t}^{t} \int_{|x| > t} \frac{|u(x, t')|^2}{|x|^3} dx dt' \leq t^{-1} \int_{-t}^{t} \int_{|x| > t} \frac{|u(x, t')|^2}{|x|^2} dx dt'
\]

\[
\lesssim \kappa t^{-1} \int_{-t}^{t} \left( \int_{|x| > t - |r'|} (t^{-\kappa} - (|x| + |r'|)^{-\kappa}) |x|^\kappa e(x, 0) dx \right) dt'
\]

\[
\lesssim 1 \int_{\mathbb{R}^2} (t^{-\kappa} - (|x| + t)^{-\kappa}) |x|^\kappa e(x, 0) dx.
\]

Plugging these two upper bounds in (5), we obtain

\[
\int_{|x| < t} \frac{t - |x|}{t} e(x, t) dx + \int_{\mathbb{R}^2} \left( |u_r + u_t|^2 + |\nabla u|^2 + |u|^{p+1} \right) dx
\]

\[
\lesssim_{p, \kappa} \int_{\mathbb{R}^2} \left[ \min\{|x|/t, 1\} + (t^{-\kappa} - (|x| + t)^{-\kappa}) |x|^\kappa \right] e(x, 0) dx.
\]

Therefore

\[
t^\kappa \int_{|x| < t} \frac{t - |x|}{t} e(x, t) dx + t^\kappa \int_{\mathbb{R}^2} \left( |u_r + u_t|^2 + |\nabla u|^2 + |u|^{p+1} \right) dx
\]

\[
\lesssim_{p, \kappa} \int_{\mathbb{R}^2} \left[ \min\{|x|/t^{\kappa-1}, t^\kappa\}
\right.

\[
+ \left( 1 - \left( \frac{t}{t + |x|} \right)^\kappa \right) |x|^\kappa \right] e(x, 0) dx.
\]

Finally we may complete the proof by dominated convergence theorem.

\[\square\] Springer
4.2 Sub-conformal case

In this subsection, we consider the case $3 < p < 5$. In general, the argument is almost the same as in the super-conformal case. The only difference is the presence of an additional term in the upper bounds:

$$\frac{5 - p}{2(p + 1)R} \int_{-R}^{R} \int_{|x| < R} |u|^{p+1} dx dt$$

Thus we have to find a good upper bound of it first. We choose $R = t$, $r = 0$ in Corollary 3.3, follow the same argument as in the case $p \geq 5$ and obtain

$$\mu_2 \int_{\mathbb{R}^2} (|u(x, t)|^{p+1} + |u(x, -t)|^{p+1}) dx$$

$$+ c_1 \sum_{\pm} \int_{|x| < t} \frac{t - |x|}{t} \left( |\nabla u(x, \pm t)|^2 + |u_t(x, \pm t)|^2 \right) dx$$

$$+ c_2 \sum_{\pm} \int_{\mathbb{R}^2} \left( |u_r \pm u_t)(x, \pm t)|^2 + |\nabla u(x, \pm t)|^2 \right) dx$$

$$\leq \frac{5 - p}{2(p + 1)t} \int_{-t}^{t} \int_{|x| < t} |u(x, t')|^{p+1} dx dt'$$

$$+ 2 \int_{\mathbb{R}^2} \min\{|x|/t, 1\} e(x, 0) dx + C_{p, \mu_2} t^{-\frac{4}{p+1}}.$$
Proof of part (a)(b) We may rewrite the recurrence formula as
\[ Q(t) \leq \frac{\lambda}{t} \int_0^t Q(t') dt' + o(1), \]
We may take upper limits of both sides and obtain an inequality
\[ \limsup_{t \to +\infty} Q(t) \leq \limsup_{t \to +\infty} \frac{\lambda}{t} \int_0^t Q(t') dt' \leq \lambda \limsup_{t \to +\infty} Q(t). \]
We recall the fact \( \lambda \in (0, 1) \) and observe that \( Q(t) \lesssim E \) is uniformly bounded, therefore we have
\[ \limsup_{t \to +\infty} Q(t) = 0. \]
This verifies (a)(b).

Proof of part (c) Now we assume that the initial data satisfy additional decay assumption \( E_\kappa(u_0, u_1) < +\infty \) and prove the decay estimates in part (c). We start by multiplying both sides of the recurrence formula by \( t^{\kappa - 1} \) and integrate from \( t = 1 \) to \( t = T \), utilize our assumption on initial data, then obtain
\[
\int_1^T t^{\kappa - 1} Q(t) dt \\
\leq \int_1^T t^{\kappa - 1} \left( \frac{\lambda}{t} \int_0^t Q(t') dt' \right) dt + C_\kappa \int_{\mathbb{R}^2} \min(|x|, |x|^{\kappa}) e(x, 0) dx + C_{p, \mu_2, \kappa} \\
\leq \frac{\lambda}{1 - \kappa} \int_0^T \min\{t^{\kappa - 1}, 1\} Q(t') dt' + C_\kappa \int_{\mathbb{R}^2} \min(|x|, |x|^{\kappa}) e(x, 0) dx + C_{p, \mu_2, \kappa} \\
\leq \frac{\lambda}{1 - \kappa} \int_1^T (t')^{\kappa - 1} Q(t') dt' + C_\kappa \int_{\mathbb{R}^2} \min(|x|, |x|^{\kappa}) e(x, 0) dx + C_\kappa E + C_{p, \mu_2, \kappa}.
\]
We may choose \( \mu_2 \) sufficiently close to \( \frac{1}{p+1} \) so that the constant
\[
\frac{\lambda}{1 - \kappa} = \frac{5 - p}{2(p + 1) \mu_2 (1 - \kappa)} < 1,
\]
since we have assumed \( \kappa < \frac{p-3}{2} \). Therefore we have
\[
\int_1^T t^{\kappa - 1} Q(t) dt \lesssim_{p, \mu_2, \kappa} \int_{\mathbb{R}^2} \min(|x|, |x|^{\kappa}) e(x, 0) dx + E + 1.
\]
Because neither the right hand side nor the implicit constant here depends on \( T \), we may make \( T \to +\infty \) to conclude
\[
\int_1^\infty t^{\kappa - 1} Q(t) dt < +\infty. \tag{7}
\]
Combining this with the fact $Q(t) \lesssim E$, we have

$$
\int_0^\infty t^{\kappa - 1} Q(t) dt < +\infty.
$$

We may multiply both sides of the recurrence formula by $t^\kappa$:

$$
t^\kappa Q(t) \leq \lambda \int_0^t t^{\kappa - 1} Q(t') dt' + 2 \int_{\mathbb{R}^2} \min\{|x|t^{\kappa - 1}, t^\kappa\} e(x, 0) dx + C_{p, \mu_2} t^{\kappa - \frac{4}{p - 1}}.
$$

Finally we apply dominated convergence theorem to finish the proof.

### 5 Scattering theory of non-radial solutions

In this section we prove Theorem 1.3. First of all, energy conservation law gives

$$
\| S_L(-t_1) \begin{pmatrix} u(\cdot, t_1) \\ u_t(\cdot, t_1) \end{pmatrix} - S_L(-t_2) \begin{pmatrix} u(\cdot, t_2) \\ u_t(\cdot, t_2) \end{pmatrix} \|_{\dot{H}^1 \times L^2(\mathbb{R}^2)} \lesssim 1 E^{1/2}.
$$

In addition, we may apply Theorem 1.1 and energy conservation law to obtain

$$
\int_{\mathbb{R}^2} |u(x, t)|^{p+1} dx \lesssim \min\{1, |t|^{-\kappa}\}.
$$

This implies that $u \in L^q L^{p+1}(\mathbb{R} \times \mathbb{R}^2)$ for all $q > (p + 1)/\kappa$. Since we have assumed $\kappa > \frac{3(p+5)}{4p}$, thus $u \in L^{\frac{4p(p+1)}{3p+5}} L^{p+1}(\mathbb{R} \times \mathbb{R}^2)$. As a result, we may choose $s' = \frac{p+7}{4(p+1)} \in \left(1, \frac{1}{2}\right)$, apply Strichartz estimates and obtain

$$
\limsup_{t_1, t_2 \to +\infty} \| S_L(-t_1) \begin{pmatrix} u(\cdot, t_1) \\ u_t(\cdot, t_1) \end{pmatrix} - S_L(-t_2) \begin{pmatrix} u(\cdot, t_2) \\ u_t(\cdot, t_2) \end{pmatrix} \|_{\dot{H}^{s'} \times \dot{H}^{s'-1}(\mathbb{R}^2)} = 0.
$$

An interpolation between the spaces $\dot{H}^1 \times L^2$ and $\dot{H}^{s'} \times \dot{H}^{s'-1}$ then gives ($s' < 1/2 < s_p < 1$)

$$
\limsup_{t_1, t_2 \to +\infty} \| S_L(-t_1) \begin{pmatrix} u(\cdot, t_1) \\ u_t(\cdot, t_1) \end{pmatrix} - S_L(-t_2) \begin{pmatrix} u(\cdot, t_2) \\ u_t(\cdot, t_2) \end{pmatrix} \|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^2)} = 0.
$$

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A similar argument shows

$$\sup_{t \in \mathbb{R}} \left\| S_L (-t) \begin{pmatrix} u (\cdot, t) \\ u_t (\cdot, t) \end{pmatrix} - \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \right\|_{\dot{H}^s \times \dot{H}^{s-1} (\mathbb{R}^2)} < +\infty. \quad (8)$$

Next we utilize Sobolev embedding

$$\| (u_0, u_1) \|_{\dot{H}^s \times \dot{H}^{s-1}} \lesssim \| (u_0, u_1) \|_{W^{1, \frac{2(p-1)}{p+1}} \times L^{\frac{2(p-1)}{p+1}}} \lesssim \left\| (1 + |x|)^{\kappa/2} (\nabla u_0, u_1) \right\|_{L^2} \left\| (1 + |x|)^{-\kappa/2} \right\|_{L^{p-1}} \lesssim E_\kappa (u_0, u_1)^{1/2} \left( \int_{\mathbb{R}^2} (1 + |x|)^{-\frac{(p-1)\kappa}{2}} dx \right)^{1/(p-1)} < +\infty.$$ 

Here $$\frac{(p-1)\kappa}{2} > \frac{(p-1)(3p+5)}{8p} > 2$$. Combining this with (8), we obtain $$(u (\cdot, t), u_t (\cdot, t)) \in \dot{H}^s \times \dot{H}^{s-1}$$ for all time $$t \in \mathbb{R}$$. By completeness of the space $$\dot{H}^s \times \dot{H}^{s-1}$$ we conclude that there exists $$(u_0^+, u_1^+) \in \dot{H}^s \times \dot{H}^{s-1}$$, so that

$$\limsup_{t \to +\infty} \left\| S_L (-t) \begin{pmatrix} u (\cdot, t) \\ u_t (\cdot, t) \end{pmatrix} - \begin{pmatrix} u_0^+ \\ u_1^+ \end{pmatrix} \right\|_{\dot{H}^s \times \dot{H}^{s-1} (\mathbb{R}^2)} = 0.$$ 

Thus we obtain the scattering of solutions in the critical Sobolev space

$$\limsup_{t \to +\infty} \left\| \begin{pmatrix} u (\cdot, t) \\ u_t (\cdot, t) \end{pmatrix} - S_L (t) \begin{pmatrix} u_0^+ \\ u_1^+ \end{pmatrix} \right\|_{\dot{H}^s \times \dot{H}^{s-1} (\mathbb{R}^2)} = 0.$$ 

In addition, we have $$u \in L^{\frac{3p(p-1)}{3p-5}} L^{\frac{6p(p-1)}{3p+5}} ([0, T] \times \mathbb{R}^2)$$ for any $$T > 0$$ by the embedding $$\dot{H}^1 (\mathbb{R}^2) \cap L^{p+1} (\mathbb{R}^2) \subset L^{\frac{6p(p-1)}{3p+5}} (\mathbb{R}^2)$$, since the inequality $$\frac{6p(p-1)}{3p+5} > p + 1$$ holds if $$p > 5$$. We then apply the scattering criterion (Proposition 2.5) and obtain $$u \in L^{\frac{2}{5}(p-1)} L^{\frac{2}{5}(p-1)} (\mathbb{R}^+ \times \mathbb{R}^2)$$. Next we show that the scattering also happens in the energy space. We start by applying Strichartz estimates and a fractional chain rule (Lemma 2.4) to obtain
We then apply Strichartz estimates and obtain
\[
\|D^{1/2}u\|_{L^6 L^6([t_1, t_2] \times \mathbb{R}^2)} \leq C \|(u(\cdot, t_1), u_t(\cdot, t_1))\|_{\dot{H}^1 \times L^2} + C_1 \|D^{1/2}(-|u|^{p-1}u)\|_{L^{6/5}L^{6/5}([t_1, t_2] \times \mathbb{R}^2)}
\]
\[
\leq 2^{1/2} C E^{1/2} + C_2 \|u\|^{p-1}_{L^2(p-1)L^{3(p-1)}([t_1, t_2] \times \mathbb{R}^2)}
\]
\[
\times \|D^{1/2}u\|_{L^6 L^6([t_1, t_2] \times \mathbb{R}^2)}
\]  (9)

Please note that \(\|D^{1/2}u\|_{L^6 L^6([t_1, t_2] \times \mathbb{R}^2)} < +\infty\) for all \(t_2 > t_1 \geq 0\). Because

\[
\|D^{1/2}u\|_{L^6 L^6([t_1, t_2] \times \mathbb{R}^2)} 
\leq C \|(u(\cdot, t_1), u_t(\cdot, t_1))\|_{\dot{H}^1 \times L^2} + \|u\|^{p-1}_{L^1 L^2([t_1, t_2] \times \mathbb{R}^2)}
\]
\[
\leq 2^{1/2} C E^{1/2} + (t_2 - t_1) \|u\|^{p-1}_{L^\infty L^2([t_1, t_2] \times \mathbb{R}^2)}
\]
\[
\leq 2^{1/2} C E^{1/2} + (t_2 - t_1) \|u\|^{p}_{L^\infty L^{2p}([t_1, t_2] \times \mathbb{R}^2)} < +\infty
\]

Here we apply Strichartz estimates and use the embedding \(\dot{H}^1 \cap L^{p+1} = L^{2p} \hookrightarrow L^{2p} \hookrightarrow L^p \). Since \(u \in L^{3(p-1)} \times L^2(p-1)\), we may find a large time \(t_1\) so that

\[
C_2 \|u\|^{p-1}_{L^{3(p-1)} L^{3(p-1)}([t_1, +\infty) \times \mathbb{R}^2)} < 1/2.
\]

A combination of this inequality with (9) immediately gives

\[
\|D^{1/2}u\|_{L^6 L^6([t_1, t_2] \times \mathbb{R}^2)} \leq 2^{3/2} C E^{1/2},
\]

for all \(t_2 > t_1\). Therefore we have

\[
\|D^{1/2}u\|_{L^6 L^6([t_1, +\infty) \times \mathbb{R}^2)} < +\infty \Rightarrow \|D^{1/2}(-|u|^{p-1}u)\|_{L^{6/5}L^{6/5}([t_1, +\infty) \times \mathbb{R}^2)} < +\infty.
\]

We then apply Strichartz estimates and obtain

\[
\lim_{t_1, t_2 \to +\infty} \|S_L(-t_1) \begin{pmatrix} u(\cdot, t_1) \\ u_t(\cdot, t_1) \end{pmatrix} - S_L(-t_2) \begin{pmatrix} u(\cdot, t_2) \\ u_t(\cdot, t_2) \end{pmatrix}\|_{\dot{H}^1 \times L^2} = 0.
\]

Therefore the convergence

\[
S_L(-t) \begin{pmatrix} u(\cdot, t) \\ u_t(\cdot, t) \end{pmatrix} \to \begin{pmatrix} u_0^+ \\ u_1^+ \end{pmatrix}
\]

happens not only in the space \(\dot{H}^{s_p} \times \dot{H}^{s_p-1}\) but also in the space \(\dot{H}^1 \times L^2\). Finally an interpolation between these two spaces implies the scattering of solutions in all spaces \(\dot{H}^s \times \dot{H}^{s-1} = \mathbb{R}^2\) with \(s \in [s_p, 1]\).
6 Scattering theory of radial solutions

In this section we prove scattering theory of radial finite-energy solutions to (CP1). The general idea is the same as in higher dimensions, see [38]. We combine a method of characteristic lines with energy distribution properties of solutions.

6.1 Method of characteristic lines

Reduction to one-dimensional case Let \( u \) be a radial solution to (CP1) with a finite energy \( E \). Please note that we will work as though the solutions are sufficiently smooth, otherwise we may apply standard smooth approximation techniques. We start by defining

\[
\begin{align*}
  w(r, t) &= r^{1/2} u(r, t); \\
  v_+(r, t) &= w_t(r, t) - w_r(r, t); \\
  v_-(r, t) &= w_t(r, t) + w_r(r, t).
\end{align*}
\]

The function \( w(r, t) \) satisfies a one-dimensional wave equation

\[
w_{tt} - w_{rr} = f(r, t) = \frac{1}{4} r^{-3/2} u - r^{1/2} |u|^{p-1} u.
\]

A simple calculation shows that \( v_+ (\cdot, t), v_-(\cdot, t) \in L^2_{loc}(\mathbb{R}^+) \) satisfy

\[
(\partial_t \pm \partial_r) v_\pm (r, t) = w_{tt} - w_{rr} = f(r, t).
\]

This immediately gives variation of \( v_\pm \) along characteristic lines \( t \pm r = \text{Const} \).

\[
\begin{align*}
  v_+(t_2 - \eta, t_2) - v_+(t_1 - \eta, t_1) &= \int_{t_1}^{t_2} f(t - \eta, t) dt, \quad t_2 > t_1 > \eta; \\
  v_-(s - t_2, t_2) - v_-(s - t_1, t_1) &= \int_{t_1}^{t_2} f(s - t, t) dt, \quad t_1 < t_2 < s.
\end{align*}
\]

Next we give the upper bounds of the integrals of \( f \) above. According to Lemma 2.1, we have

\[
\begin{align*}
  \int_{t_1}^{t_2} (t - \eta)^{-3/2} u(t - \eta, t) dt &\lesssim_E \int_{t_1}^{t_2} (t - \eta)^{-\frac{3}{2} - \frac{2}{p+3}} dt \lesssim_E (t_1 - \eta)^{\frac{1}{2} - \frac{2}{p+3}}. \\
  \int_{t_1}^{t_2} (t - \eta)^{1/2} |u|^{p-1} u(t - \eta, t) dt &\leq \left( \int_{t_1}^{t_2} (t - \eta) |u(t - \eta, t)|^{p+1} dt \right)^{\frac{p}{p+1}} \\
  &\times \left( \int_{t_1}^{t_2} (t - \eta)^{-\frac{p-1}{2}} dt \right)^{\frac{1}{p+1}}
\end{align*}
\]
Combining these two upper bounds together we have
\[
\left| \int_{t_1}^{t_2} f(t - \eta, t) dt \right| \lesssim_{p,E} (t_2 - t_1)^{-\frac{p-3}{2(p+1)}}.
\]

Thus there exists a function \( g_+(\eta) \in L^2_{\text{loc}}(\mathbb{R}) \), so that \( v_+(t - \eta, t) \) converges to \( g_+(\eta) \) in \( L^2_{\text{loc}}(\mathbb{R}) \) as \( t \to +\infty \). In fact we have
\[
\int_{\eta_1}^{\eta_2} |v_+(t - \eta, t) - g_+(\eta)|^2 d\eta \lesssim_{p,E} (\eta_2 - \eta_1)(t - \eta_2)^{-\frac{p-3}{p+1}}, \quad t > \eta_2 + 1.
\]

We apply change of variables \( \eta = t - r \) and rewrite this inequality in the form
\[
\int_{r_1}^{r_2} |v_+(r, t) - g_+(t - r)|^2 dr \lesssim_{p,E} (r_2 - r_1)r_1^{-\frac{p-3}{p+3}}, \quad r_1 > 1.
\] (10)

A basic calculation shows that
\[
v_+(r, t) = r^{1/2} (u_t(r, t) - u_r(r, t)) - (1/2)r^{-1/2} u(r, t).
\]

We may apply Lemma 2.1 and obtain
\[
\int_{r_1}^{r_2} |r^{-1/2} u(r, t)|^2 dr \lesssim_{p,E} (r_2 - r_1)r_1^{-\frac{p-3}{p+3}}, \quad r_1 > 1.
\]

A combination of this inequality with (10) gives
\[
\int_{r_1}^{r_2} r^{1/2} (u_t(r, t) - u_r(r, t)) - g_+(t - r) \right) |^2 dr \lesssim_{p,E} (r_2 - r_1)r_1^{-\frac{p-3}{p+3}}, \quad r_1 > 1.
\]

Next we show \( g_+ \in L^2(\mathbb{R}) \). Given any \( \eta_1 < \eta_2 \), we may utilize the inequality above with large time \( t \), \( r_1 = t - \eta_2 \), \( r_2 = t - \eta_1 \) and consider the limit \( t \to +\infty \):
\[
\int_{\eta_1}^{\eta_2} |g_+(\eta)|^2 d\eta = \lim_{t \to +\infty} \int_{t-\eta_2}^{t-\eta_1} r |u_t(r, t) - u_r(r, t)|^2 dr \lesssim_{1} E.
\]

This implies \( g_+ \in L^2(\mathbb{R}) \) since \( \eta_1 < \eta_2 \) are arbitrary constants. In summary we have

**Lemma 6.1** If \( u \) is a radial solution to (CP1) with a finite energy \( E \), then there exist two functions \( g_+, g_- \in L^2(\mathbb{R}) \) so that
\[
\int_{r_1}^{r_2} r^{1/2} (u_t(r, t) - u_r(r, t)) - g_+(t - r) \right) |^2 dr \lesssim_{p,E} (r_2 - r_1)r_1^{-\frac{p-3}{p+3}}, \quad r_1 > 1;
\]
\[
\int_{r_1}^{r_2} \left| r^{1/2} (u_t(r, t) + u_r(r, t)) - g_-(t + r) \right|^2 dr \lesssim_{p,E} (r_2 - r_1) r_1^{-\frac{p-3}{p+1}}, \quad r_1 > 1.
\]

### 6.2 Exterior scattering

Given any \( \eta \in \mathbb{R} \) and large positive number \( R \), we may apply Lemma 6.1 with \( r_1 = t - \eta, r_2 = t + R \) and obtain

\[
\int_{t-\eta}^{t+R} \left| r^{1/2} (u_t(r, t) - u_r(r, t)) - g_+(t - r) \right|^2 dr \lesssim_{p,E} (R + \eta)(t - \eta)^{-\frac{p-3}{p+1}}, \quad t \gg 1.
\]

We may let \( t \to +\infty \):

\[
\lim_{t \to +\infty} \int_{t-\eta}^{t+R} \left| r^{1/2} (u_t(r, t) - u_r(r, t)) - g_+(t - r) \right|^2 dr = 0. \tag{11}
\]

Similarly we have

\[
\lim_{t \to +\infty} \int_{t-\eta}^{t+R} \left| r^{1/2} (u_t(r, t) + u_r(r, t)) - g_-(t + r) \right|^2 dr = 0.
\]

Since \( g_- \in L^2(\mathbb{R}) \), we may discard \( g_-(t + r) \) above and write

\[
\lim_{t \to +\infty} \int_{t-\eta}^{t+R} \left| r^{1/2} (u_t(r, t) + u_r(r, t)) \right|^2 dr = 0.
\]

We combine this with (11) and obtain

\[
\lim_{t \to +\infty} \int_{t-\eta}^{t+R} \left( \left| r^{1/2} u_t(r, t) - \frac{1}{2} g_+(t - r) \right|^2 + \left| r^{1/2} u_r(r, t) + \frac{1}{2} g_+(t - r) \right|^2 \right) dr = 0 \tag{12}
\]

We then observe

- Finite speed of propagation of energy implies

\[
\lim_{R \to +\infty} \sup_{t \geq 0} \int_{t+R}^{\infty} r \left( |u_t(r, t)|^2 + |u_r(r, t)|^2 \right) dr \\
\leq \lim_{R \to +\infty} \int_{R}^{\infty} r (|u_1(r)|^2 + |\partial_r u_0(r)|^2) dr = 0.
\]

- Since \( g_+ \in L^2(\mathbb{R}) \), we have

\[
\lim_{R \to +\infty} \int_{t+R}^{\infty} |g_+(t - r)|^2 dr = \lim_{R \to +\infty} \int_{-\infty}^{-R} |g_+(\eta)|^2 d\eta = 0.
\]
Combining these two limits with (12), we obtain
\[
\lim_{t \to +\infty} \int_{t-\eta}^{+\infty} \left( \left| r^{1/2} u_t(r, t) - \frac{1}{2} g_+(t-r) \right|^2 + \left| r^{1/2} u_r(r, t) + \frac{1}{2} g_+(t-r) \right|^2 \right) dr = 0
\]
(13)

By radiation fields, there exists a radial free wave \( \tilde{u}^+ \), so that
\[
\lim_{t \to +\infty} \int_0^{+\infty} \left( \left| r^{1/2} \tilde{u}^+_t(r, t) - \frac{1}{2} g_+(t-r) \right|^2 + \left| r^{1/2} \tilde{u}^+_r(r, t) + \frac{1}{2} g_+(t-r) \right|^2 \right) dr = 0.
\]

Therefore we have
\[
\lim_{t \to +\infty} \int_{t-\eta}^{+\infty} r \left( \left| u_t(r, t) - \tilde{u}^+_t(r, t) \right|^2 + \left| u_r(r, t) - \tilde{u}^+_r(r, t) \right|^2 \right) dr = 0.
\]

This proves the exterior scattering of solutions, i.e. scattering outside any forward light cone.

6.3 Scattering by energy decay

In this subsection we prove Part (b) of Theorem 1.4. Let \( t > 0 \) be sufficiently large and \( c \in (0, 1) \) be a constant. We may choose \( r_1 = t - c \cdot t^{\frac{p-3}{p+1}} \) and \( r_2 = t \) in Lemma 6.1 and obtain
\[
\limsup_{t \to +\infty} \int_{t-c \cdot t^{\frac{p-3}{p+1}}}^{t} \left( \left| r^{1/2} (u_t(r, t) - u_r(r, t)) - g_+(t-r) \right|^2 \right) dr \lesssim_p, E c;
\]
\[
\limsup_{t \to +\infty} \int_{t-c \cdot t^{\frac{p-3}{p+1}}}^{t} \left( \left| r^{1/2} (u_t(r, t) + u_r(r, t)) - g_-(t+r) \right|^2 \right) dr \lesssim_p, E c.
\]

Following the same argument as in last subsection, we have
\[
\limsup_{t \to +\infty} \int_{t-c \cdot t^{\frac{p-3}{p+1}}}^{t} r \left( \left| u_t(r, t) - \tilde{u}^+_t(r, t) \right|^2 + \left| u_r(r, t) - \tilde{u}^+_r(r, t) \right|^2 \right) dr \lesssim_p, E c.
\]

We may combine this with the regular exterior scattering and obtain
\[
\limsup_{t \to +\infty} \int_{t-c \cdot t^{\frac{p-3}{p+1}}}^{+\infty} r \left( \left| u_t(r, t) - \tilde{u}^+_t(r, t) \right|^2 + \left| u_r(r, t) - \tilde{u}^+_r(r, t) \right|^2 \right) dr \lesssim_p, E c.
\]
(14)
Next we consider the region \( \{ x : |x| < t - c \cdot t^{\frac{p-3}{p+1}} \} \). We utilize the conclusion of Theorem 1.4, part (c) and obtain
\[
\lim_{t \to +\infty} \int_{|x| < t - c \cdot t^{\frac{p-3}{p+1}}} e(x, t) dx \lesssim c \lim_{t \to +\infty} t^{\frac{4}{p+1}} \int_{|x| < t} \frac{t - |x|}{t} e(x, t) dx = 0.
\]

Please note that in the sub-conformal range our assumption \( p > 1 + 2\sqrt{3} \) guarantees that \( \frac{4}{p+1} < \frac{p-3}{2} \). By Lemma 2.9, we also have
\[
\lim_{t \to +\infty} \int_{|x| < t - c \cdot t^{\frac{p-3}{p+1}}} (|\nabla \tilde{u}^+ (x, t)|^2 + |\tilde{u}^+_t (x, t)|^2) dx = 0.
\]

Combining these two limits we obtain
\[
\lim_{t \to +\infty} \int_{|x| < t - c \cdot t^{\frac{p-3}{p+1}}} \left( |\nabla \tilde{u}^\pm (x, t) - \nabla u(x, t)|^2 + |\tilde{u}^\pm_t (x, t) - u_t(x, t)|^2 \right) dx = 0.
\]

We may combine this with stronger exterior scattering (14) to conclude
\[
\limsup_{t \to +\infty} \int_{\mathbb{R}^2} \left( |\nabla \tilde{u}^\pm (x, t) - \nabla u(x, t)|^2 + |\tilde{u}^\pm_t (x, t) - u_t(x, t)|^2 \right) dx. \lesssim_{p, E} c.
\]

Finally we make \( c \to 0^+ \) and finish the proof.

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Declarations

Conflict of interest No potential conflict of interest was reported by the authors.

7 Appendix

In this appendix we give a brief proof of a Morawetz estimate for solutions to 2D wave equation. This kind of Morawetz estimates were first introduced by Nakanishi. For convenience we use the same notation as in Nakanishi [31]:

\[
\lambda = \sqrt{t^2 + r^2}; \quad \Theta = \left( \frac{-t, x}{\lambda} \right); \quad g = \frac{d - 1}{2\lambda} + \frac{t^2 - r^2}{2\lambda^3};
\]

\[
m_h = \Theta \cdot (u_t, \nabla u) + ug; \quad l(u) = \frac{|\nabla u|^2}{2} - \frac{|u_t|^2}{2} + \frac{|u|^{p+1}}{p+1}; \quad \Box = \partial^2_t - \Delta;
\]
and $(\partial_0, \partial_1, \ldots, \partial_d) = (-\partial^0, \partial^1, \ldots, \partial^d) = (\partial_t, \nabla)$. Then we have an identity

$$(\Box u + |u|^{p-1}u)m_h = \sum_{\alpha=0}^{d} \partial_\alpha \left( -m_h \partial^\alpha u + l(u) \Theta_\alpha + \frac{|u|^2}{2} \partial^\alpha g \right)$$

$$+ \left( \frac{\nabla u}{\lambda} \right)^2 + \frac{|ru_t + tu_r|^2}{\lambda^3} + \frac{p-1}{p+1} |u|^p g + \frac{|u|^2}{2} \Box g.$$ A basic calculation shows that

$$\Box g = \frac{(d-3)(d+3)}{2\lambda^3} + 3(d-1) \frac{t^2 - r^2}{\lambda^5} + 15 \frac{(t^2 - r^2)^2}{2\lambda^7}.$$ Thus we may integrate in the region $\mathbb{R}^2 \times [1, T]$ and obtain

$$\int_{\mathbb{R}^2} \left( -m_h u_t + \frac{t}{\lambda} l(u) + \frac{|u|^2}{2} g_t \right) dt \bigg|_{t=1}^{t=T} = \int_1^T \int_{\mathbb{R}^2} \left( \frac{\nabla u}{\lambda} + \frac{|ru_t + tu_r|^2}{\lambda^3} + \frac{p-1}{p+1} \frac{t^2 |u|^{p+1}}{\lambda^3} + \frac{|u|^2}{2} \Box g \right) dx dt. \ (15)$$ In order to deal with the terms involving $|u|^2$, we need to apply Hölder’s inequality

$$\int_{\mathbb{R}^2} \left| u(x, t) \right|^2 \frac{2}{\lambda^2} dx \leq \left( \int_{\mathbb{R}^2} |u|^{p+1} dx \right)^{\frac{2}{p+1}} \left( \int_{\mathbb{R}^2} \lambda^{-\frac{2(p+1)}{p-1}} dx \right)^{\frac{p-1}{p+1}} \lesssim_p t^{-\frac{4}{p+1}} E^2 \frac{2}{p+1}.$$ Next we observe the facts $|g_t| \lesssim \lambda^{-2}$, $|\Box g| \lesssim \lambda^{-3}$ and obtain

$$\left| \int_{\mathbb{R}^2} \left( -m_h u_t + \frac{t}{\lambda} l(u) + \frac{|u|^2}{2} g_t \right) dx \right| \lesssim_p \int_{\mathbb{R}^2} \left( \left| \nabla u \right|^2 + |u_t|^2 + |u|^{p+1} + \frac{|u|^2}{\lambda^2} \right) dx \lesssim_p E + E \frac{2}{p+1},$$

and

$$\int_1^T \int_{\mathbb{R}^2} \frac{|u|^2}{2} \Box g dx dt \leq \int_1^T \int_{\mathbb{R}^2} \frac{|u|^2}{t \lambda^2} dx dt \lesssim_p E \frac{2}{p+1} \int_1^T t^{-\frac{4}{p+1}} dt \lesssim_p E^2 \frac{2}{p+1}.$$ We plug these upper bound in (15) and conclude

$$\int_1^T \int_{\mathbb{R}^2} \left( \frac{\nabla u}{\lambda} + \frac{|ru_t + tu_r|^2}{\lambda^3} + \frac{t^2 |u|^{p+1}}{\lambda^3} \right) dx dt \lesssim_p E + E \frac{2}{p+1}.$$
References

1. Bahouri, H., Gérard, P.: High frequency approximation of solutions to critical nonlinear equations. Am. J. Math. 121, 131–175 (1999)
2. Bahouri, H., Shatach, J.: Decay estimates for the critical semilinear wave equation. Anna. l’Inst. Henri Poincaré Anal. Non Linéaire 15, 783–879 (1998)
3. Bulut, A.: Global well-posedness and scattering for the defocusing energy-supercritical cubic nonlinear wave equation. J. Funct. Anal. 263, 1609–1660 (2012)
4. Christ, M., Weinstein, M.: Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation. J. Funct. Anal. 100, 87–109 (1991)
5. Colliander, J., Keel, M., Staffilani, G., Takaoka, H., Tao, T.: Global existence and scattering for rough solutions to a nonlinear Schrödinger equation in \( \mathbb{R}^3 \). Commun. Pure Appl. Math. 57, 987–1014 (2004)
6. Dodson, B., Lawrie, A.: Scattering for the radial 3d cubic wave equation. Anal. PDE 8, 467–497 (2015)
7. Dodson, B., Lawrie, A., Mendelson, D., Murphy, J.: Scattering for defocusing energy subcritical nonlinear wave equations. Anal. PDE 13, 1995–2090 (2020)
8. Duyckaerts, T., Kenig, C.E., Merle, F.: Scattering for radial, bounded solutions of focusing supercritical wave equations. Int. Math. Res. Not. 20, 224–258 (2014)
9. Duyckaerts, T., Kenig, C.E., Merle, F.: Scattering profile for global solutions of the energy-critical wave equation. J. Eur. Math. Soc. 21, 2117–2162 (2019)
10. Friedlander, F.G.: On the radiation field of pulse solutions of the wave equation. Proc. R. Soc. Ser. A 269, 53–65 (1962)
11. Friedlander, F.G.: Radiation fields and hyperbolic scattering theory. Math. Proc. Camb. Philos. Soc. 88, 483–515 (1980)
12. Ginibre, J., Soffer, A., Velo, G.: The global Cauchy problem for the critical nonlinear wave equation. J. Funct. Anal. 110, 96–130 (1992)
13. Ginibre, J., Velo, G.: The global Cauchy problem for the nonlinear Klein–Gordon equation. Math. Z. 189(4), 487–505 (1982)
14. Ginibre, J., Velo, G.: Conformal invariance and time decay for nonlinear wave equations”. Ann. l’inst. Henri Poincaré (A) Phys. Théorique 47, 221–276 (1987)
15. Ginibre, J., Velo, G.: Generalized Strichartz inequality for the wave equation. J. Funct. Anal. 133, 50–68 (1995)
16. Glassey, R., Pecher, H.: Time decay for nonlinear wave equations in two space dimensions. Manuscr. Math. 38, 387–400 (1982)
17. Grillakis, M.: Regularity and asymptotic behaviour of the wave equation with critical nonlinearity. Ann. Math. 132, 485–509 (1990)
18. Grillakis, M.: Regularity for the wave equation with a critical nonlinearity. Commun. Pure Appl. Math. 45, 749–774 (1992)
19. Hidano, K.: Conformal conservation law, time decay and scattering for nonlinear wave equation. J. Anal. Math. 91, 269–295 (2003)
20. Kapitanski, L.: Global and unique weak solutions of nonlinear wave equations. Math. Res. Lett. 1, 211–223 (1994)
21. Kapitanski, L.: Weak and yet weaker solutions of semilinear wave equations. Commun. Partial Differ. Equ. 19, 1629–1767 (1994)
22. Kenig, C.E., Merle, F.: Nondispersive radial solutions to energy supercritical non-linear wave equations, with applications. Am. J. Math. 133(4), 1029–1065 (2011)
23. Killip, R., Visan, M.: The defocusing energy-supercritical nonlinear wave equation in three space dimensions. Trans. Am. Math. Soc. 363, 3893–3934 (2011)
24. Killip, R., Visan, M.: The radial defocusing energy-supercritical nonlinear wave equation in all space dimensions. Proc. Am. Math. Soc. 139, 1805–1817 (2011)
25. Lin, J., Strauss, W.: Decay and scattering of solutions of a nonlinear Schrödinger equation. J. Funct. Anal. 30, 245–263 (1978)
26. Lindblad, H., Sogge, C.: On existence and scattering with minimal regularity for semi-linear wave equations. J. Funct. Anal. 130, 357–426 (1995)
27. Lindblad, H., Tao, T.: Asymptotic decay for a one-dimensional nonlinear wave equation. Anal. PDE 5, 411–422 (2012)
28. Morawetz, C.S.: Time decay for the nonlinear Klein–Gordon equations. Proc. R. Soc. Lond. Ser. A 306, 291–296 (1968)
29. Nakanishi, K.: Unique global existence and asymptotic behaviour of solutions for wave equations with non-coercive critical nonlinearity. Commun. Partial Differ. Equ. 24, 185–221 (1999)
30. Nakanishi, K.: Scattering theory for nonlinear Klein–Gordon equations with Sobolev critical power. Int. Math. Res. Not. 1, 31–60 (1999)
31. Nakanishi, K.: Energy scattering for nonlinear Klein–Gordon and Schrödinger equations in spatial dimensions 1 and 2. J. Funct. Anal. 169, 201–225 (1999)
32. Perthame, B., Vega, L.: Morrey–Campanato estimates for Helmholtz equations. J. Funct. Anal. 164, 340–355 (1999)
33. Pecher, H.: Nonlinear small data scattering for the wave and Klein–Gordon equation. Math. Z. 185, 261–270 (1984)
34. Rodriguez, C.: Scattering for radial energy-subcritical wave equations in dimensions 4 and 5. Commun. Partial Differ. Equ. 42, 852–894 (2017)
35. Shatah, J., Struwe, M.: Well-posedness in the energy space for semilinear wave equations with critical growth. Int. Math. Res. Not. 7, 303–309 (1994)
36. Shen, R.: On the energy subcritical, nonlinear wave equation in \(\mathbb{R}^3\) with radial data. Anal. PDE 6, 1929–1987 (2013)
37. Shen, R., Staffilani, G.: A semi-linear shifted wave equation on the hyperbolic spaces with application on a quintic wave equation on \(\mathbb{R}^2\). Trans. Am. Math. Soc. 368, 2809–2864 (2016)
38. Shen, R.: Energy distribution of radial solutions to energy subcritical wave equation with an application on scattering theory. Trans. Am. Math. Soc. 374, 3827–3857 (2021)
39. Shen, R.: Inward/outward energy theory of non-radial solutions to 3D semi-linear wave equation. Adv. Math. 374, 107384 (2020)
40. Taylor, M.: Tools for PDE. Pseudo Differential Operators, Paradifferential Operators and Layer Potentials. Mathematical Surveys and Monographs, vol. 81. AMS, Providence (2000)
41. Struwe, M.: Globally regular solutions to the \(u^5\) Klein–Gordon equation. Ann. Scuola Norm. Super. Pisa Classe Sci. 15, 495–513 (1988)
42. Tataru, D.: Strichartz estimates in the hyperbolic space and global existence for the similinear wave equation. Trans. Am. Math. Soc. 353, 795–807 (2000)
43. Tsutaya, K.: Scattering theory for semilinear wave equations with small data in two space dimensions. Trans. Am. Math. Soc. 342(2), 595–618 (1994)
44. Wei, D., Yang, S.: On the global behaviours for defocusing semilinear wave equation in \(\mathbb{R}^{1+2}\). Anal. PDE 15(1):245–272 (2022)
45. Wei, D., Yang, S.: Asymptotic decay for defocusing semilinear wave equations in \(\mathbb{R}^{1+1}\). Ann. PDE 7, 9 (2021)

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