CHARACTERIZATIONS OF ORTHONORMAL SCALE FUNCTIONS: A PROBABILISTIC APPROACH

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Abstract.

The construction of a multiresolution analysis starts with the specification of a scale function. The Fourier transform of this function is defined by an infinite product. The convergence of this product is usually discussed in the context of $L^2(\mathbb{R})$. Here, we treat the convergence problem by viewing the partial products as probabilities, converging weakly to a probability defined on an appropriate sequence space. We obtain a sufficient condition for this convergence, which is also necessary in the case where the scale function is continuous. These results extend and clarify those of A. Cohen, and Hernández, Wang, and Weiss. The method also applies to more general dilation schemes that commute with translations by $\mathbb{Z}^d$.

Introduction.

We will say that a function $\phi(x), x \in \mathbb{R}$ is a scaling (or scale) function if $\phi(x) \in L^2(\mathbb{R})$, and

(a') the function $\hat{\phi}(2\xi) = m(\xi)\hat{\phi}(\xi)$ with $m(\xi)$ a $2\pi$-periodic function in $L^2(\mathbb{R})$;

(b') $\sum_{k \in \mathbb{Z}} |\hat{\phi}(\xi + 2\pi k)|^2 = 1$ a.e., $\xi \in \mathbb{R}$;

(c') $\lim_{j \to \infty} |\hat{\phi}(\xi/2^j)| = 1$ a.e.

This characterization of scaling functions is given in Chapter 7 of [5]. The conditions (a') and (b') mean that

(a) the function $\frac{1}{2}\phi\left(\frac{x}{2}\right) = \sum m_j \phi(x - j)$ with $\sum m_j^2 < \infty$;

(b) the translates of $\phi(x - k), k \in \mathbb{Z}$, form an orthonormal sequence in $L^2(\mathbb{R}, dx/2\pi)$.

The condition (c') is independent of (a') and (b'). However, the “garden variety” scale functions are integrable, with integral one, so that $\hat{\phi}(\xi)$ is continuous, and $\hat{\phi}(0) = 1$. In these cases, (c') is satisfied. Therefore, let us assume, for purposes of this introduction, that $\phi$ is integrable with integral one.

How do we construct (or recognize) such functions $\phi$? Certain features are easily discernable. Since $\hat{\phi}(\xi)$ is continuous, and $\hat{\phi}(0) = 1$, the two-scale equation (a') tells us that $\hat{\phi}(\xi) = \prod_{j=1}^\infty m(\xi/2^j)$, so that the properties of $\phi(x)$ (or $\hat{\phi}(\xi)$) are determined by $m(\xi)$. If we divide the sum in (b') into two parts, according to the parity of the indices $k$, ...
and use the variable $2\xi$, we find

$$\sum_{k \in \mathbb{Z}} |\hat{\phi}(2\xi + 2\pi k)|^2$$

$$= \sum_{k \in 2\mathbb{Z}} |\hat{\phi}(2\xi + 2\pi k)|^2 + \sum_{k \in 2\mathbb{Z}+1} |\hat{\phi}(2\xi + 2\pi k)|^2$$

$$(*)$$

$$= |m(\xi)|^2 \sum_{j \in \mathbb{Z}} |\hat{\phi}(\xi + 2\pi j)|^2 + |m(\xi + \pi)|^2 \sum_{j \in \mathbb{Z}} |\hat{\phi}(\xi + \pi + 2\pi j)|^2$$

$$= |m(\xi)|^2 + |m(\xi + \pi)|^2$$

$$= 1, \text{ a.e.}$$

Therefore, in addition to $2\pi$-periodicity, the function $m(\xi)$ must satisfy the identity expressed in the last two lines; since $|\hat{\phi}(0)|^2 = 1$, it also follows that $|m(0)|^2 = 1$. However, this identity is not sufficient to insure that (b′) is satisfied. If (b′) and (c′) together are satisfied, then $m(\xi)$ is called a low-pass filter. Many authors have considered the problem of finding sufficient (and necessary) conditions on $m(\xi)$ so that the infinite product $\hat{\phi}(\xi)$ is a scale function. When $m(\xi)$ is a polynomial, two such sufficient conditions have been proposed, one by Mallat [6] and the other by Daubechies (see [3], page 182). Mallat’s condition requires

$$\inf_{|\xi| \leq \pi/2} |m(\xi)| > 0$$

and Daubechies:

$$m(\xi) = \left[(1 + e^{i\xi})/2\right]^N \mathcal{L}(\xi)$$

with $\sup_{\xi} |\mathcal{L}(\xi)| \leq 2^{N-1/2}$. The first necessary and sufficient conditions were found by Cohen [2], in the case where $m(\xi)$ is a polynomial; he later extended his considerations to the case where $m(\xi)$ is $C^1(\mathbb{R})$. The problem for more general $m(\xi)$ was considered by Hernández, Wang, and Weiss [4]. They obtained a necessary and sufficient condition when $|m(\xi)|$ takes the values 0 and 1. In the notes of Chapter 7 of the recent text by Hernández and Weiss [5], the authors propose the problem of finding necessary and sufficient conditions in the case when $m(\xi)$ is not necessarily $C^1(\mathbb{R})$. The principle purpose of this paper is to address this question. Our results are inspired by Cohen’s ideas; however, we have translated his ideas into probabilistic terms. This approach seems to us to be very natural for the problem at hand, and allows us to obtain necessary and sufficient conditions in a
very general context. In particular, the results of Cohen and Hernández, Wang, and Weiss are unified as special cases of a general theorem. The method applies, as well, to more general dilations schemes in $\mathbb{R}^d$. These are treated in brief, in a separate section.

To motivate the probabilistic approach, let us summarize the problem as it is usually presented. (See, for example, Daubechies ([3], Chapter 6.3) or Hernández and Weiss ([4], Chapter 7.4).) Given a candidate $2\pi$ periodic function $m(\xi)$ with $|m(\xi)|^2 + |m(\xi + \pi)|^2 = 1$, and $|m(0)|^2 = 1$, we form the sequence of partial products

$$|\hat{\phi}_N(\xi + 2\pi k)|^2 = \prod_{j=1}^{N} \left| m\left( \frac{\xi + 2\pi k}{2^j} \right) \right|^2.$$ 

Then $\lim_{N \to \infty} |\hat{\phi}_N(\xi + 2\pi k)|^2 := |\hat{\phi}(\xi + 2\pi k)|^2$; this limit is well defined a.e. since $|\hat{\phi}_N(\xi + 2\pi k)|^2$ decreases with increasing $N$. The convergence in $L^2(\mathbb{R})$ is a different matter since the function $\hat{\phi}_N(\xi)$ is $(2\pi)2^{-N}$-periodic. Therefore, except for trivial cases, $|\hat{\phi}_N(\xi)|^2$ is never integrable as a function of $\xi \in \mathbb{R}$. An obvious remedy for this defect is to restrict $\hat{\phi}_N(\xi)$ to the interval $[-2^N \pi, 2^N \pi]$. Thus, if

$$\hat{\phi}^*_N(\xi) = \begin{cases} \hat{\phi}_N(\xi) & \text{if } |\xi| \leq 2^N \pi \\ 0 & \text{otherwise,} \end{cases}$$

then $|\hat{\phi}^*_N(\xi)|^2$ also converges to $|\hat{\phi}(\xi)|^2$ pointwise a.e. To verify property (b’) in the definition of a scale function it turns out that it is enough to show that $\hat{\phi}^*_N(\xi)$ also converges in $L^2(\mathbb{R})$. Here matters become delicate. The $L^2$ convergence is complicated by the fact that there is no obvious domination. This is the point where Cohen’s ideas come into play. He suggested that one should modify $\hat{\phi}_N(\xi)$ by multiplying by $\chi_K(\xi)$, rather than $\chi_{[-\pi, \pi]}(\xi)$, where $K$ is a finite union of intervals forming a compact set that is congruent to $[-\pi, \pi]$ in a sense described below. When such a $K$ exists, the sequence $\hat{\phi}^{**}_N(\xi) = \hat{\phi}_N(\xi) \cdot \chi_K(\xi/2^N)$ may be shown to converge in $L^2(\mathbb{R})$. With this convergence established, the convergence in $L^2(\mathbb{R})$ of the original sequence $\hat{\phi}^*_N(\xi)$ may also be proved. It was this feature of Cohen’s approach that provoked our effort to find another perspective where Cohen’s condition would appear in a more transparent fashion. For smooth $m(\xi)$, Cohen’s condition requires that

$$\inf_{j>0} \inf_{\xi \in K} |m(\xi/2^j)| > 0$$

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where $K$ is a compact set that is a finite union of intervals, one of which contains 0 as an interior point, such that $K$ is congruent to $[-\pi, \pi]$ in the following sense:

(a) the Lebesgue measure of $K$ is $2\pi$;

(b) for every $\xi \in [-\pi, \pi]$, there is a $k \in \mathbb{Z}$ such that $\xi + 2\pi k \in K$.

Notice that Cohen’s condition is equivalent to a restriction on the partial products $\hat{\phi}_N(\xi)$ for $\xi \in K$: Since $m(\xi)$ is smooth in a neighborhood of the origin, the partial products converge uniformly on any compact subset of $\mathbb{R}$; therefore, the condition may be stated as

$$\inf_{N \geq 1} \inf_{\xi \in K} |\hat{\phi}_N(\xi)| \geq \delta > 0,$$

for some $\delta > 0$. In fact, a more succinct way to formulate the condition would be to omit the mention of $K$ altogether. As we shall see, what is important is the existence of a lower bound $\delta$ for the infinite product. Furthermore, it is not the topological, but the measure-theoretic character of $K$ that is important: it is enough to require the lower bound to hold almost everywhere in the following sense:

Either

(1) $|\hat{\phi}(\xi)| \geq \delta > 0$ almost everywhere in $[0, 2\pi]$

or

(2) $\sup_{k \in \mathbb{Z}} |\hat{\phi}(\xi + 2\pi k)| \geq \delta > 0$ almost everywhere in $[0, 2\pi]$.

Now let us drop the requirement that $\phi(x)$ is integrable, and focus on properties (a') and (b') for a function in $L^2(\mathbb{R})$. Given $m(\xi)$ as specified above, the conditions (1) and (2) are sufficient for the function $\hat{\phi}(\xi)$, satisfying (a') to also satisfy (b'). When $\hat{\phi}(\xi)$ is continuous, then conditions (1)-(2) are necessary and sufficient for $\hat{\phi}(\xi)$ to satisfy (b') everywhere, rather than almost everywhere. In fact, we show that it is possible for $\hat{\phi}(\xi)$ to be continuous, such that (b') holds except at two points. Furthermore, any example where (b') holds almost everywhere, but not everywhere, is such that (1)-(2) fails for any $\delta > 0$.

The authors would like to express their gratitude to Professors G. Weiss and H. Šikić of Washington University, for many valuable remarks on the subject of this paper. (In particular, see the section “The encounter...”)

**The probabilistic approach.** In summary, we interpret the function $|m(\xi)|^2$ as a conditional probability defined on a space of infinite sequences. The partial products
define a consistent family of probabilities on this sequence space which converge, in the usual (Kolmogorov) sense, to a probability. The existence of a scale function is equivalent to the “tightness” of this family of probabilities on “finite” sequences.

The probability space. Let \( M(\xi) = |m(2\pi \xi)|^2 \). Notice that \( M(\xi) \) is a one-periodic function that satisfies \( M(\xi) + M(\xi + 1/2) = 1 \), and \( M(0) = 1 \). The basic probability space \( \Omega \) for our discussion is the disjoint union of two spaces of infinite sequences \( \omega \) with coordinates \( \omega_i = 0 \) or \( 1 \). We will represent elements of \( \Omega \) by \({0,1}\} \times \{0,1\}^N; \Omega^+ \) and \( \Omega^- \) will denote sequences starting with 0 and 1, respectively. We identify integers with a subset of \( \Omega \) in the following way. A positive integer \( k \) with dyadic expansion

\[
k = \sum_{i=1}^{\infty} \omega_i(k) 2^{i-1}
\]

is represented by the sequence

\[
(0, \omega_1(k), \omega_2(k), \ldots).
\]

The integer zero is identified with the sequence that is identically zero. A negative integer \( k \) is represented by coefficients of dyadic expansion of \(-(k + 1)\) preceded by 1 (thus, for example, the sequence \(1, 0, 0, \ldots\) represents \(-1\). We denote the sequences corresponding to nonnegative integers as \( \mathbb{Z}^+ \), and those corresponding to negative integers as \( \mathbb{Z}^- \). Fix \( k \in \mathbb{Z} \) and let \( k_N = \{\omega : \omega_i = \omega_i(k), 0 \leq i \leq N\} \) be the \( N \) dimensional \( \Omega^+ \)-cylinder that contains \( \omega(k) \). For each \( \xi \in [0,1] \) we define a probability \( Q^N_\xi \), \( 0 \leq \xi < 1 \), on the set of all such cylinders by the following prescription. For \( 0 \leq k \leq 2^N - 1 \), we set

\[
Q^N_\xi(k) = \prod_{j=1}^{N} M\left(\frac{\xi + k}{2^j}\right).
\]

We then have

\[
\sum_{0 \leq k < 2^N} \prod_{j=1}^{N} M\left(\frac{\xi + k}{2^j}\right) = 1,
\]

where we used the basic fact that \( M(\xi) + M(\xi + 1/2) = 1 \). In the language of (conditional) probability,

\[
M\left(\frac{\xi + k}{2^j}\right) = Q_\xi(\omega_j(k) | \omega_{j-1}, \ldots, \omega_1),
\]
and the above sum is computed by the standard successive conditioning procedure.

With this interpretation of $M(\frac{\xi + k}{2^j})$, we see that the product defines a probability on cylinders of $\Omega^+$, and that

$$Q_\xi^N(k_N) = Q_\xi^{N+1}(k_N),$$

where $k_N$ is the $N$-dimensional cylinder corresponding to $0 \leq k \leq 2^N - 1$. In order to define corresponding probabilities on $\Omega^-$ let us consider a “reflected” filter

$$\widetilde{M}(\xi) = M(-\xi).$$

This filter may also be used to construct a probability on the positive integers $0 \leq k < 2^N$ in the same fashion, by setting for $0 \leq \eta < 1$ and $0 \leq \ell < 2^N$

$$\widetilde{Q}_\eta^N(\ell) = \prod_{j=1}^{N} \widetilde{M}(\eta + \ell).$$

We now define measures $P_\xi^N$ on cylinders in $\Omega$ by setting

$$P_\xi^N(k) = \begin{cases} Q_\xi^{N+1}(k), & \text{if } 0 \leq k < 2^N; \\ \widetilde{Q}_\eta^{N+1}(1 - (k + 1)), & \text{if } -2^N \leq k < 0. \end{cases}$$

Notice that there is a double reflection, on the function, and on the argument, and that $P_\xi^N$ corresponds to $N + 1$ products in $Q$’s. This specification shows that $P_\xi^N, N \geq 0$, is a consistent family ($P_\xi^N(k) = P_\xi^{N+1}(k)$ for each fixed $k$), since each of the families $Q_\xi^N, N \geq 1$, and $\widetilde{Q}_{1-\xi}^N, N \geq 1$ are consistent. To see that $P_\xi^N$ defines a probability on the integers $-2^N \leq k < 2^N$, notice that

$$\sum_{-2^N \leq k < 2^N} P_\xi^N(k) = \sum_{0 \leq k < 2^N} Q_\xi^{N+1}(k) + \sum_{-2^N \leq k < 0} \widetilde{Q}_\xi^{N+1}(1 - (k + 1))$$

$$= \sum_{0 \leq k < 2^N} Q_\xi^{N+1}(k) + \sum_{-2^N \leq k < 0} Q_\xi^{N+1}(2^N + 1 + k)$$

$$= \sum_{0 \leq k < 2^{N+1}} Q_\xi^{N+1}(k)$$

$$= 1.$$
The encounter at Washington University. The initial version of this paper contained an error in the formulation of the definition of the family $P^N_\xi(\cdot)$, $N \geq 0$. We are extremely grateful to H. Šikić, of Washington University, for showing us this error. The discussion with Šikić occurred during a visit by one of us, to St. Louis, and resulted in a radical adjustment in the definition of $P^N_\xi$. It is remarkable that the conclusions of Theorem 2 survived this adjustment with minimal changes.

Now, we can restate the problem concerning the existence of a scaling function in a very succinct fashion:

**Theorem 1.** The function $m(\xi)$ is a low-pass filter for a scaling function whose Fourier transform is $\hat{\phi}(\xi)$ if and only if

(b′′) the probability $P_\xi$ is concentrated on finite sequences for almost every $\xi$, $0 \leq \xi < 1$. We denote this by saying $P_\xi(Z) = 1$ a.e.;

(c′′) there exists a set $L_+ \subset [0, 1)$ of positive measure such that for $\xi \in L_+$,

$$
\lim_{j \to \infty} |\hat{\phi}((\xi + k)/2^j)|^2 = 1
$$

for all $k \geq 0$, and a set of positive measure $L_- \subset [0, 1)$ such that for $\xi \in L_-$,

$$
\lim_{j \to \infty} |\hat{\phi}((\xi + k)/2^j)|^2 = 1
$$

for all $k \leq -1$.

**Proof of Theorem 1.** If $m(\xi)$ is a low-pass filter, then $\hat{\phi}(\xi)$ satisfies (b′) which implies (b′′). Conversely, the condition (b′′) is another way of stating (b′).

Now we must show that (c′′) and (c′) are equivalent. We use the following proposition.

**Proposition 1.** Let $f(\cdot)$ be a function defined on $\mathbb{R}_+$. For $0 \leq \xi < 1$, and $k \geq 0$, consider the set

$$
L = \left\{ \xi : \lim_{j \to \infty} f((\xi + k)/2^j) = 1 \right\}
$$

for all $k \geq 0$. This set has measure one or zero.

**Proof of Proposition 1.** This is a special case of the Kolmogorov zero-one law. The set $L$ is a “tail set” in the sense that it is invariant under all transformations $\xi \to (\xi + k)/2^n$ for fixed $k$ and $n$, $0 \leq k < 2^n$. Such invariant sets have measure zero or one.
Now, if \((c')\) holds then \((c'')\) holds. Conversely, if the (apparently) weaker condition \((c'')\) holds, the stronger condition \((c')\) holds by Proposition 1. That is, the sets \(L^\pm\) have measure one.

**Remarks.** The formulation of the second part of Theorem 1 was inspired by Theorem 3.16 of Papadakis, Šikić, and Weiss [7]. They propose a characterization of nonnegative periodic functions \(m(\xi)\) that are low-pass filters; this characterization assumes that the infinite product \(\hat{\phi}(\xi)\) satisfies \((c')\), and they require that the partial products, suitably truncated, converge in \(L^2(\mathbb{R})\) to the limit \(\hat{\phi}(\xi)\).

This requirement is equivalent to our \((b'')\) in Theorem 1. Rather than simply assume \((c')\) as they did, we chose to state it in the present form for the following reason. Papadakis, Šikić, and Weiss exhibit an example attributed to M. Paluszynski, where \((b')\) holds (that is, the partial products converge in \(L^2(\mathbb{R})\)) but \((c')\) fails. The example is simply \(\hat{\phi}(\xi) = \chi_{[0,1)}(\xi)\). Clearly, \(P_\xi(Z^-) = 0\) for all \(\xi\), \(0 \leq \xi < 1\), and the condition \((c')\) does not hold: \(\lim_{j \to \infty} \hat{\phi}(\xi/2^j) = 0\) if \(\xi < 0\). Therefore, in search of minimal conditions, one might suggest that there exist sets of positive measure \(L^+\) and \(L^-\) such that \(P_\xi(Z^+) > 0\) for \(\xi \in L^+\) and \(P_\xi(Z^-) > 0\) for \(\xi \in L^-\). These conditions are implied by \((c'')\). That is, they are necessary conditions, but, in fact, fail to be sufficient. If the suggested necessary condition is strengthened to \(P_\xi(Z^+) > 0\) for almost every \(\xi\), \(0 \leq \xi < 1\), the condition fails to be necessary. Consider the Shannon filter,

\[
m(\xi) = \chi_{[0,1/4)}(\xi) + \chi_{[3/4,1)}(\xi),
\]

extended periodically. Here \(\hat{\phi}(\xi) = \chi_{[-1/2,1/2)}(\xi)\) and \(P_\xi(Z^+) = P_\xi(0) = 1\), with \(P_\xi(Z^-) = 0\) if \(0 \leq \xi < 1/2\); also \(P_\xi(Z^-) = P_\xi(-1) = 1\) with \(P_\xi(Z^+) = 0\), if \(1/2 \leq \xi < 1\). Therefore, the qualification “on a set of positive measure” is necessary. With this qualification, the suggested condition is not sufficient to imply \((c'')\). We can perturb the Shannon filter so that on a set of positive measure \(E\), such that

\[
\lim |\hat{\phi}(\xi/2^j)|^2 = 0
\]

for \(\xi \in E\), but \(P_\xi(Z) = 1\) a.e. (We omit the details of this example.) The upshot of all of this is that we must have \(P_\xi(Z^+) > 0\) on a set of positive measure, and \(P_\xi(Z^-) > 0\) on a
set of positive measure, as well as an almost everywhere dyadic continuity at zero. These
two requirements are captured in condition (c$''$).

The condition (c$''$) has a probabilistic interpretation in terms of the underlying Markov
chains associated with the functions $M(\xi)$. However, we introduced the probability no-
tions as a tool, and our interest in the details of the probability structure are secondary.
Therefore, we chose not to express condition (c$''$) in purely probabilistic terms, as we did for (b$''$).

In order to show that $P_{\xi}(Z) = 1$, we will use Prokhorov’s criterion of tightness for a
sequence of probability measures.

**Definition:** The sequence $P_{\xi}^N$ is said to be tight on $Z$ in $\Omega$, if for every $\epsilon > 0$, there
is an $n(\epsilon) = n(\epsilon, \xi) > 0$, such that

$$\sum_{n(\epsilon) \leq |k_N|} P_{\xi}^N(k_N) \leq \epsilon \text{ for all } N \geq 0.$$ 

Here $|k_N|$ is the index $i$ with largest absolute value such that $\omega_i(k) = 1$.

In terms of the integers $k \in \mathbb{Z}$, we may write this tightness condition as

$$\sum_{n(\epsilon) \leq k < 2^N} P_{\xi}^N(k) + \sum_{n(\epsilon) \leq -k \leq 2^N} P_{\xi}^N(-k) \leq \epsilon.$$

We note that

$$\lim_{N \to \infty} P_{\xi}^N(k_N) = P_{\xi}(\omega(k)) = |\hat{\phi}(\xi + k)|^2,$$

or less formally,

$$\lim_{N \to \infty} P_{\xi}^N(k) = P_{\xi}(k).$$

Finally, we write

$$P_{\xi}(Z) = 1$$

if $\hat{\phi}(\xi)$ satisfies (b$'$).

**Criterion:** $P_{\xi}(Z) = 1$ if and only if $P_{\xi}^N$ are tight.

We omit the details of this argument. (See Billingsley [1].)

We are now in a position to state the principal result.
Theorem 2. (i) A sufficient condition for $P_\xi(Z) = 1$ almost everywhere is the following (condition (C)):

Suppose that for almost every $\xi$, $0 \leq \xi \leq 1$, there exists a $\delta > 0$ and an integer $k(\xi)$, such that $P_\xi(k(\xi)) \geq \delta$.

(ii) Let $\xi \mapsto P_\xi(k)$ be continuous for each $k \in Z$. (In other words, $|\hat{\phi}(\theta)|$ is continuous for all $\theta \in R$, so that $\hat{\phi}$ satisfies the condition $(c')$ for a scale function.) If condition (C) is satisfied, then $P_\xi(Z) = 1$ for every $\xi$, $0 \leq \xi \leq 1$. (That is, there is no exceptional set.) Conversely, if $P_\xi(Z) = 1$ for every $\xi$, $0 \leq \xi \leq 1$, then condition (C) holds with no exceptional set.

(iii) There exists a function $M(\xi)$ infinitely differentiable at $\xi = 0$ such that $P_\xi(k)$ is continuous for each $k$, and such that $P_\xi(Z) = 1$, except at two points $\xi$, $0 < \xi < 1$. At these exceptional points, $P_\xi(Z) = 0$. In particular, condition (C) fails to hold for any $\delta > 0$.

Remark 1. At first, the distinction between “almost everywhere” and “everywhere” in the above theorem may seem somewhat fastidious. However, these distinctions are crucial for the following reasons. If $\hat{\phi}(\xi)$ is the Fourier transform of a scale function, then the equation (b') holds almost everywhere. The circumstances where (b') holds everywhere are of secondary interest. In the same spirit, the natural assumption of the theorem concerns the behavior of $P_\xi$ almost everywhere. If, however, we require $P_\xi(k)$ to be continuous in $\xi$ for each $k$, then the sufficient condition (a.e.) gives the conclusion $P_\xi(Z) = 1$ everywhere. Conversely, if $P_\xi(Z) = 1$ everywhere, then the sufficient condition (C) holds everywhere. Thus, when $P_\xi(k)$ is supposed to be continuous, the sufficient condition (C) becomes necessary, but with a blemish: the natural necessary condition should read, “If $P_\xi(Z) = 1$ almost everywhere, then condition (C) holds almost everywhere.” However part (iii) states that this cannot hold in general, even when $P_\xi(k)$ is continuous. In particular, there are low-pass filters of class $C^0(R)$ generating continuous scale functions that do not satisfy Cohen’s condition.

Remark 2. When $P_\xi(k)$ is continuous for each $k$, the condition (C) is equivalent to that given by Cohen. Since $P_0(0) = P_1(1) = 1$ and $P_\xi(0)$ ($P_\xi(1)$) is continuous, there are one-sided neighborhoods of zero and one such that $P_\xi(0) \geq \delta > 0$, $0 \leq \xi < \alpha$, and
P_ξ(1) \geq \delta > 0, \ 1 - \alpha \leq \xi \leq 1. \text{ In other words, } |\hat{\phi}(\xi)|^2 \geq \delta > 0 \text{ for } |\xi| \leq \alpha. \ \text{ Thus, the first condition for a Cohen set is satisfied. With each } \xi_0 \text{ we can associate an interval, } 
\{\xi : |\xi - \xi_0| < \epsilon\} \text{ centered at } \xi_0, \text{ such that } P_\xi((k(\xi_0)) \geq \delta/2 \text{ for every } \xi \text{ in the interval. Then, we find a finite subcollection } \xi_i, k(\xi_i), \ i = 0, 1, \ldots, N, \text{ such that the corresponding union of intervals covers the unit interval. The compact set specified by Cohen may be constructed using translations by } k_i(\xi), i = 0, 1, \ldots, N. \}

Now suppose that a compact set \(K\), with Cohen’s specifications, exists. We will show that the probabilities \(P_\xi^N(\cdot)\), \(N \geq 1\) are tight. Choose \(n(\epsilon)\) large enough so that \[
\sum_{n(\epsilon) \leq |k|} P_\xi(k) \leq \epsilon.
\]
Now estimate \(P_\xi^N(k), n(\epsilon) \leq k < 2^N\) as follows:

\[
P_\xi^N(k) = P_\xi^N(k + 2^{N+1}j)
\]

where \(j = j((\xi + k)/2^{N+1})\) is the integer such that \(j + (\xi + k)/2^{N+1} \in K\). If \(j \geq 0\),

\[
P_\xi^N(k) \leq \frac{1}{\delta} P_\xi^N(k + 2^{N+1}j) P_{(\xi+k)/2^{N+1}}(j)
\]

and \(n(\epsilon) \leq k \leq k + 2^{N+1}j\). On the other hand, if \(j < 0\), and \(n(\epsilon) \leq k < 2^N\), then

\[
P_\xi^N(k) \leq \frac{1}{\delta} P_\xi^N(-2^{N+1}|j| - k))
\]

and \(n(\epsilon) \leq 2^{N+1}|j| - k\) if \(n(\epsilon) \leq 2^N(2|j| - 1)\). Therefore, either

\[
P_\xi^N(k) \leq \frac{1}{\delta} P_\xi^N(k + 2^{N+1}|j|)
\]
or

\[
P_\xi^N(k) \leq \frac{1}{\delta} P_\xi^N(-2^{N+1}|j| - k)).
\]

In the first case,

\[
\sum_{n(\epsilon) \leq k < 2^N} P_\xi^N(k) \leq \frac{1}{\delta} \sum_{n(\epsilon) \leq n} P_\xi(n) \leq \epsilon/\delta;
\]
in the second case,

\[
\sum_{n(\epsilon) \leq k < 2^N} P_\xi^N(k) \leq \frac{1}{\delta} \sum_{2^N(2|j| - 1) \leq |n|} P_\xi(-n).
\]
Now we choose $N$ large enough so that $2^N + 1 \geq n(\epsilon)$. A similar argument may be made for $k < 0$, with $n(\epsilon) \leq |k| \leq 2^N$. This shows that $P_\xi^N, N \geq N(\epsilon)$ is tight, and therefore, that the entire collection $P_\xi^N$ is tight.

**Remark 3.** Hernández, Wang, and Weiss [4] treated the case where $M(\xi)$ is a $C^1(\mathbb{R})$ function, as well as the case when $M(\xi)$ is a function taking only values 0 and 1. In the latter case $P_\xi(k)$ also takes values 0 and 1, and the condition (C) of the theorem becomes

$$P_\xi(k) = 1 \text{ for some } k = k(\xi)$$

for almost every $\xi$, $0 \leq \xi \leq 1$. This condition is obviously necessary for $P_\xi(\mathbb{Z}) = 1$ a.e., in this case. Furthermore, the “almost everywhere” cannot be altered.

**Proof of (i).** Suppose $P_\xi(\cdot)$ satisfies the condition (C). Let us call the sequence of points $\{\xi' : \xi' = (\xi + k)/2^N \mod 1, N > 0, k \in \mathbb{Z}\}$ the orbit of $\xi \in [0, 1]$. We want all the probabilities $P_{\xi'}$ to satisfy condition (C), where $\xi'$ belongs to the orbit of $\xi$. The set of “good” points $G$, where condition (C) holds has full measure, and the translates of $G$ by dyadically rational points, $G_k$ also have full measure. So, we take the set $\tilde{G} = \bigcap G_k$, of full measure, of points $\xi$ that satisfy our requirement.

Now we turn to the proof of the tightness of the sequence $P_\xi^N$, for $\xi \in \tilde{G}$. Let $k_N$ denote an $N$ cylinder corresponding to the integer $k$, as specified earlier. Let $\xi' = (\xi + k)/2^{N+1} \mod 1$ and $k(\xi')$ be an integer such that $P_{\xi'}(k(\xi')) \geq \delta$. Then the $\omega$-sequence corresponding to $k + 2^{N+1}k(\xi')$ belongs to the $N$ cylinder $k_N$, and

$$P_\xi(k + 2^{N+1}k(\xi')) = P_{k_N}(k_N)P_{\xi'}(k(\xi')).$$

Therefore,

$$P_{k_N}(k_N) \leq \delta^{-1}P_\xi(k + 2^{N+1}k(\xi')).$$

(This estimation is simply a transcription of Cohen’s calculation.) Now, observe that the probability $P_\xi(\cdot)$ always satisfies the condition for tightness on $\mathbb{Z}$. That is, for $\epsilon > 0$, there exists an $n = n(\epsilon, \xi)$ such that

$$\sum_{n \leq |k|} P_{\xi}(k) \leq \epsilon$$

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where $|k|$ is the largest (or smallest) index in the sequence $\omega(k)$ such that $\omega_i(k) = 1$. (This is always true since $P_\xi(Z) \leq 1$.) Therefore
\[
\sum_{n \leq |k| N} P^N_\xi(k_N) \leq \delta^{-1} \sum_{n \leq |k| \leq N} P_\xi(k + 2^{N+1} k(\xi')) \\
\leq \delta^{-1} \sum_{n \leq |k|} P_\xi(k) \\
\leq \delta^{-1} \cdot \epsilon.
\]
This proves that the condition of part (i) is sufficient for tightness, and so proves that $P_\xi(Z) = 1$ for $\xi \in G$.

**Proof of (ii).** Now we assume the condition (C) of the theorem, and that $P_\xi(k)$ is continuous on $[0, 1]$ for each $k$. We wish to show that $P_\xi(Z) = 1$ for all $\xi$ in $[0, 1]$. By part (i), the equality holds almost everywhere. If it fails at some point $\xi_1$, $0 < \xi_1 < 1$, then there must be a point $\xi_0$ where $P_{\xi_0}(Z) = 0$. (Consider the possibility that $P_\xi(Z) > 0$ for every $\xi$. Then, for every $\xi$, there exist $k(\xi)$ such that $P_\xi(k(\xi)) > 0$. By the continuity of $\xi \to P_\xi(k)$, the sets $\{\xi : P_\xi(k) > 0\}$ are an open cover of $[0, 1]$. Therefore, there exists a $\delta > 0$ such that $P_\xi(Z) \geq \delta > 0$ for every $\xi$. Then the condition of part (i) holds everywhere, and the argument given in part (i) shows that $P_\xi(Z) = 1$ everywhere. Therefore, there exist points $\xi_0$ where $P_{\xi_0}(Z) = 0$.) Since $P_\xi(0)$ is continuous, and tends to one as $\xi$ tends to zero, $P_{\xi_0}(Z) = 0$ if and only if for each $k \in Z$, there exists $N = N(k)$ such that $M((\xi_0 + k)/2^N) = 0$. This “sudden death syndrome” is inconsistent with the hypothesis of the theorem: we will prove that, given any $\epsilon > 0$, there exists an open set of points where
\[
\max_k P_\xi(k) \leq \epsilon.
\]
To this end, let $\xi_0$ be a point in $[0, 1]$ such that $P_{\xi_0}(Z) = 0$. Now consider a neighborhood of $\xi_0$,
\[
N_\eta(\xi_0) = \{\xi : |\xi - \xi_0| < \eta\},
\]
where $\eta$ is chosen so that
\[
|M(\xi) - M(\xi')| \leq \epsilon
\]
for any two points $(\xi, \xi')$ such that $|\xi - \xi'| < \eta$ (mod 1). We claim that
\[
\max_k P_\xi(k) \leq \epsilon
\]
in the neighborhood $N_\eta(\xi_0)$. If $\xi = \xi_0 + \Delta$, where $|\Delta| < \eta$, then $M\left((\xi + k)/2^N\right) \leq \epsilon$ for $N = N(k)$. This implies

$$P_\xi(k) \leq \epsilon$$

for every $\xi \in N_\eta(\xi_0)$. That is, this contradicts the assumption if we choose $\epsilon < \delta$.

Now let us prove the necessity of the condition (C). Suppose that $P_\xi(Z) = 1$ for every $\xi$ in the unit interval. This implies that there exists a finite set of integers $Z_\xi$ such that

$$P_\xi(Z_\xi) \geq \delta(\xi) > 0.$$ 

By the assumption that $P_\xi(k)$ is continuous for each $k$, the fact that $P_\xi(Z) = 1$ for every $\xi$, and the compactness of $[0, 1]$, we can find a finite set of integers $Z_0$, independent of $\xi$, and a fixed $\delta > 0$, such that for $\xi, 0 \leq \xi \leq 1$,

$$P_\xi(Z_0) \geq \delta.$$ 

This implies that

$$\max_{k \in Z_0} P_\xi(k) \geq \delta/\text{card}(Z_0).$$

Thus, we have shown that condition (C) holds for every $\xi, 0 \leq \xi \leq 1$.

**Proof of (iii).** The statement of part (ii) would be vacuous if it were not possible to construct a family $P_\xi$, continuous in $\xi$ for each $k$, such that $P_\xi(Z) = 1$ almost everywhere, but not everywhere. The following is such a construction, inspired by an example given by Cohen [2].

Let $M(\xi)$ be a continuous periodic function, with period one, such that $M(0) = 1$ and $M(\xi)$ is infinitely differentiable in neighborhoods of zero, and one half. The condition $M(\xi) + M(\xi + 1/2) = 1$ is imposed, as usual. The function $M(\xi)$ is to have only three zeros in $0 < \xi < 1$: $M(1/2) = 0$ (dictated by the usual condition), and $M(1/6) = M(5/6) = 0$. The latter two zeros mean that $M(1/3) = M(2/3) = 1$. At this point, we have the example due to Cohen, cited above. However, we insist that the function $M(\xi)$ should have cusps at the points $\xi = 1/3$ and $\xi = 2/3$, so that

$$\sum_{k=1}^{\infty} \left(1 - M(1/3 \pm \epsilon/2^k)\right) = \infty$$
and
\[ \sum_{k=1}^{\infty} \left(1 - M(2/3 \pm \epsilon/2^k)\right) = \infty, \]
for any \( \epsilon, 0 < \epsilon < 1 \). (For example, we may take
\[ M(\xi) \approx 1 - \left(\log(|1/3 - \xi|)\right)^{-1} \]
for \( \xi \) in a neighborhood of 1/3, with a similar specification around 2/3.)

The probability \( P_\xi \), constructed using this \( M(\xi) \), has the following properties:

(a) For any integer \( k \in \mathbb{Z} \), \( P_\xi(k) \) is continuous in \( \xi \), since the infinite product converges uniformly in \( \xi \). (\( M(\xi) \) is smooth in a neighborhood of zero.)

(b) \( P_\xi(Z) = 0 \) at \( \xi = 1/3 \) and at \( \xi = 2/3 \). In fact, at the point \( \xi = 1/3 \), \( P_\xi \) is concentrated on the single sequence \( \omega \) such that \( \omega_0 = 1, \omega_1 = 1, \omega_2 = 0, \ldots (\omega_{2i} = 0, \omega_{2i+1} = 1, i \geq 0) \); at the point \( \xi = 2/3 \), \( P_\xi \) is concentrated on \( \omega = (0, 0, 1, 0, 1, \ldots) \) (i.e., \( \omega_0 = 0, \omega_{2i-1} = 0, \omega_{2i} = 1, i \geq 1 \)). On the other hand, if \( \xi \neq 1/3, \xi \neq 2/3 \), the divergence of the above sums implies that \( P_\xi(\omega) = 0 \) for the two sequences described above. (Notice that if the cusps were placed at \( \xi = 0 \) and \( \xi = 1 \), rather than at \( \xi = 1/3 \) and \( \xi = 2/3 \), then \( P_\xi(Z) \equiv 0 \) for all \( \xi, 0 < \xi < 1 \).)

(c) \( P_\xi(Z) = 1 \) for all other points in the unit interval.

To prove (c) we must show that the sequence \( P_{\xi}^N \) is tight.

To ease the burden of subscript notation, we will denote the cylinder \( k_N \) by \( k \). With this convention, we must show that
\[ \sum_{n(\epsilon) \leq |k| \leq 2^N} P_{\xi}^N(k) \leq \epsilon \]
for some integer \( n(\epsilon) \) and all \( N \geq 0 \). Now, to find the integer \( n(\epsilon) \) in the definition of tightness, we make a finite number of choices, starting the process by finding \( m(\epsilon) \) such that
\[ \sum_{m(\epsilon) \leq |k|} P_{\xi}(k) \leq \epsilon. \]

Now choose \( \delta \) small enough so that the interval \((1/3 - \delta, 1/3 + \delta)\) is strictly contained in \([0, 1/2]\). Let
\[ A_{\delta} = (1/3 - \delta, 1/3 + \delta) \cup (-1/3 - \delta, -1/3 + \delta), \]
and
\[ A = A(\xi, \delta) = \{ k : |k| \leq 2^N, (\xi + k)/2^{N+1} \in A_\delta \}. \]

Notice that if \( \xi' = (\xi + k)/2^{N+1}, k \in \mathbb{Z}, |k| \leq 2^N, \) and \( \xi' \notin A_\delta, \) then the probability \( P_{\xi'}(0), \xi' > 0 \) (or \( P_{1+\xi'}(0), \xi' < 0 \)) is uniformly bounded away from zero. In the sequel, the subscripts \(-1 < \xi' < 0\) are to be interpreted as \(1 + \xi'.\) Thus, with this notation, we have just stated that
\[ \inf_{\xi' \notin A_\delta} P_{\xi'}(0) := C^{-1}(\delta) > 0. \]

Since \( P_{\xi}(k) = P_{\xi'}^N(k)P_{\xi'}(0), \) we may estimate as we did in part (ii) of the proof, to obtain
\[ \sum_{\substack{k \notin A \\text{m(\epsilon) \leq |k| \leq 2^N}}} P_{\xi'}^N(k) \leq C(\delta) \sum_{\substack{k \notin A \\text{m(\epsilon) \leq |k|}}} P_{\xi}(k). \]

We increase \( m(\epsilon) \) to \( p(\epsilon) \) if necessary, so that
\[ \sum_{p(\epsilon) \leq |k|} P_{\xi}(k) \leq C^{-1}(\delta)\epsilon. \]

Therefore,
\[ \sum_{p(\epsilon) \leq |k|} P_{\xi'}^N(k) \leq \epsilon. \]

The “real work” is to estimate the sum for \( k \in A, |k| \geq p(\epsilon). \) If \( k \) satisfies these restrictions and \( k > 0, \) then \( \omega_0(k) = 0 \) and \( \omega_N(k) = 1, \omega_{N-1}(k) = 0, \omega_{N-2}(k) = 1, \ldots \) with this alternating pattern continuing for a least \( J \) steps. The alternating pattern is dictated by the fact that \( k/2^{N+1} \) is approximately \( 1/3, \) which has the alternating pattern in its dyadic expansion. The fact that the approximation is \( \delta \)-close \( (k \in A) \) means that the alternating pattern continues for at least \( J = J(\delta) \) steps, with \( \omega_{N-J}(k) = 1. \)

If \( -2^N \leq -k < 0, \) then our convention dictates that \( \omega_0(-k) = 1 \) and
\[
-k = -\left(1 + \sum_{i=1}^{N} \omega_i(-k)2^{i-1}\right) \\
= -\left(1 + \sum_{i=1}^{N} \omega_i(k-1)2^{i-1}\right).
\]
We wish to compute the probability

\[ P_N^\xi (-k) = \prod_{j=1}^{N+1} M\left(\frac{(\xi - k)/2^j}{\omega_{N+1}}\right) \]

\[ = \prod_{j=1}^{N+1} M\left(\frac{(\xi + 2^{N+1} - k)/2^j}{\omega_{N+1}}\right) \]

for \(-k \in A\). This restriction \(-k \in A\) means that \(\omega_N(-k) = 1, \omega_{N-1}(-k) = 0, \ldots\) with the alternating ones and zeros continuing for at least \(J = J(\delta)\) steps.

In any case, the restriction \(|k| \geq p(\epsilon)\) means that \(N \geq \lceil \log_2 p(\epsilon) \rceil := L\), where \([x]\) is the integer part of \(x\). To prove tightness for the entire sequence \(P_N^\xi\), \(N \geq 1\), it suffices to prove tightness for \(P_N^\xi\), \(N \geq N(\epsilon)\). Therefore, we can restrict our attention to \(N\) such that \(N - J > L\).

With this pattern in mind, we can decompose \(k \in A\), \(0 < k < 2^n\), into two integers:

\[ k = t_\ell + b_\ell \]

where \(t_\ell\) is the “top” of \(k\)

\[ t_\ell = \sum_{j=\ell+1}^{N} \omega_j 2^{j-1} \]

where the sequence \(\omega_j, j = \ell, \ldots, N\), is alternately 0 and 1, as specified above. The index \(\ell\) is determined by the following rule: We observe the sequence \(\omega_{N-j}, j = 0, 1, \ldots, \ell\) which alternates between 1 and 0, starting at \(\omega_N = 1\); we stop at the index \(\ell\) where the coefficient \(\omega_\ell = 1\) and the pattern is broken for coefficients smaller than \(\ell\). (Thus, \((\omega_{\ell-1} = 0, \omega_{\ell-2} = 0)\) and \(\omega_{\ell-1} = 1\) are the two possibilities when \(\ell > 0\). If the pattern is not broken, then \(\ell = 0\) and \(b_\ell = 0\).) As we remarked above \(0 \leq \ell \leq N - J\). This means that the “bottom” part of \(k\),

\[ b_\ell = \sum_{j=1}^{\ell} \omega_j 2^{j-1} \]

has arbitrary coefficients \(\omega_j\) for \(j < \ell\), and \(\omega_\ell = 1\). Also, we note that \((\xi + b_\ell)/2^{\ell+1} \notin A_\delta\) (or \(b_\ell \notin A\)) for any \(0 \leq \ell \leq N - J\), and \(b_\ell \geq p(\epsilon)\) if \(\ell > L\).

If \(k \in A\), \(2^N \leq k < 0\) we may carry out a similar decomposition for the positive integer \(-(k + 1)\). As we have noted, \(k \in A\) implies that \(-(k + 1)/2^{N+1}\) is approximately
1/3. In terms of the above notation,

\[ k = -(1 + b\ell + t\ell) \]

and

\[ P_N^{\xi}(k) = \tilde{Q}_{1-\xi}^{N+1}(- (k + 1)) \]

\[ = \tilde{Q}_{1-\xi}^{N+1}(b\ell + t\ell). \]

In this way, we see that the estimation of \( P_N^{\xi}(k) \), for \( k < 0 \), may be carried out in the same way as for \( k > 0 \) by using the reflected filter to define probabilities on nonnegative integers.

Now suppose that \( k > 0 \); we may write

\[ \sum_{k \in A} P_N^{\xi}(k) = \sum_{\ell=0}^{N-J} \sum_b P_N^{\xi}(b + t). \]

(Here we have omitted the subscript \( \ell \), so that \( b = b\ell, t = t\ell \).) Write the sum on \( \ell \) in two parts

\[ \sum_{\ell=0}^{N-J} \sum_b P_N^{\xi}(b + t) = \sum_{\ell=0}^{L} \sum_b P_N^{\xi}(b + t) + \sum_{\ell=L+1}^{N-J} \sum_b P_N^{\xi}(b + t). \]

To estimate the first sum, we write each term

\[ P_N^{\xi}(b + t) = P_{\xi}^{\ell}(b) P_{(\xi+b)/2^{\ell+1}}^{N-\ell-1}(t') \]

where \( t' = t/2^{\ell+1} \). Notice that \( t' \) is an integer, and that the coefficients of \( t' \) satisfy \( \omega_j(t') = \omega_{\ell+1+j}(t), j = 0, 1, \ldots, N - \ell - 2 \). This means that \( t' \) has the same pattern as \( t \). Since the infinite sequences of alternating zeros and ones are assigned probability zero unless \( \xi = 1/3 \) or \( 2/3 \) (property (b)), we have

\[ P_{(\xi+b)/2^{\ell+1}}^{N-\ell-1}(t') = o(1) \]

as \( N \) tends to infinity when \( \ell \leq L \), uniformly in \( b = b\ell \notin A \). Therefore,

\[ \sum_{\ell=0}^{L} \sum_b P_N^{\xi}(b + t) = \sum_{\ell=0}^{L} \sum_b P_{\xi}^{\ell}(b) P_{(\xi+b)/2^{\ell+1}}^{N-\ell-1}(t') \]

\[ \leq (L + 1)) \cdot o(1) = o(1) \]
as $N$ tends to infinity. Recall here that neither $L$ nor $J$ depend on $N$. That is, the above sum can be made less than $\epsilon$ if $N \geq N(\epsilon)$. This imposes another restriction on the $n(\epsilon)$ we are seeking, and we incorporate this into the calculation without further mention.

Now we estimate

$$\sum_{\ell=L+1}^{N-J} \sum_{b} P_\xi^N (b + t) \leq \sum_{\ell=L+1}^{N-J} \sum_{b} P_\xi^\ell (b).$$

Recall that $b \notin A$, and $p(\epsilon) \leq b$ so that

$$P_\xi^\ell (b) \leq C(\delta) P_\xi (b).$$

Consequently

$$\sum_{\ell=L+1}^{N-J} \sum_{b} P_\xi^N (b) \leq C(\delta) \sum_{p(\epsilon) \leq b} P_\xi (b) \leq \epsilon.$$

In summary, we have shown that there exists $n(\epsilon) = \max \{ p(\epsilon), N(\epsilon) \}$ such that

$$\sum_{n(\epsilon) \leq |k| \leq 2^N} P_\xi^N (k) \leq 3\epsilon$$

for all $N$. This is sufficient and concludes the proof of part (iii) of the theorem.

The multidimensional case. The construction of scale functions corresponding to more general dilation schemes may be accomplished in much the same manner as described above for the case of dyadic dilations. Cohen’s criterion may be applied without essential change. The class of dilation schemes most frequently considered are implemented by a matrix $A$ that maps $\mathbb{Z}^d$, the integer lattice, into itself. We assume that $A$ is strictly expansive in the sense that all eigenvalues $\lambda_i$ are such that $|\lambda_i| > 1$. Here, a scale function $\phi(x), x \in \mathbb{R}^d$ is a function that belongs to $L^2(\mathbb{R}^d/(2\pi)^d)$ such that

(a’)$ \hat{\phi}(A^\ast \xi) = m(\xi) \hat{\phi}(\xi)$

for $\xi \in \mathbb{R}^d$, with $m(\xi)$ periodic on the $2^d$-dimensional torus $(2\pi)^d$ and $m(0) = 1$;

(b’)$ \sum_{k \in \mathbb{Z}^d} |\hat{\phi}(\xi + 2\pi k)|^2 = 1$ a.e.

These assumptions are not enough to insure that $\phi$ corresponds to a multiresolution analysis since $\hat{\phi}(\xi)$ is not assumed to be continuous at zero; however, this requirement is not
relevant to the present discussion. (See Theorem 1.7, Chapter 2 of [5].) For a discussion of multiresolution analyses in this generality, see Wojtaszczyk ([9], Chapter 5). In particular, see Proposition 5.21 op. cit. for a statement of Cohen’s theorem.

Given a function \( \hat{\phi}(\xi) \), satisfying (b'), we have a probability \( P_\xi(\cdot) \) defined from \( \hat{\phi}(\xi) \), that is concentrated on the lattice \( \mathbb{Z}^d \), for almost every \( \xi \) in any set that is congruent to \( (2\pi)^d \). (Such sets are often called fundamental domains for the action of \( (2\pi)^d \mathbb{Z} \) on \( \mathbb{R}^d \); we shall use this term also.) The question arises: When (b') holds, does \( \hat{\phi}(\xi) \) correspond to a probability on a space containing \( \mathbb{Z}^d \) in a manner similar to the case when \( A = 2 \), acting on \( \mathbb{R}^d \)? The “enveloping probability space” is certainly not canonical, and, the construction for the case \( A = 2 \) has an ad hoc character. This being so, can we describe a procedure for constructing this probability space that applies to any dilation? The general case presents certain technical problems associated with the fact that we do not know of a fundamental domain that is invariant under the action of \( (A^{-1})^* \). As a consequence, we failed in our attempts to describe a universal sequence space \( \Omega \) which is independent of \( \xi \). However, if we restrict attention to the class of transformations that are similarities, we can carry out a construction that generalizes the case \( A = 2 \), and looks somewhat less impromptu than that described above. We hope that it illuminates what was done in that case. A similarity is a matrix \( A \) such that the eigenvalues \( \lambda_i \) have constant modulus; in our case \( |\lambda_i| \equiv c > 1 \). The fundamental lemma for this construction is a result due to Strichartz ([8], Lemma 5.1). We quote the lemma and include its proof for completeness.

**Lemma 1** (Strichartz). Let \( B \) be a strictly expansive similarity defined on \( \mathbb{R}^d \), such that \( B(\mathbb{Z}^d) \subset \mathbb{Z}^d \). Suppose that the common value of the modulus of any eigenvalue \( \lambda \) satisfies \( |\lambda| > 1 + d^{1/2} \). Then there exists a set of coset representatives \( r_1, r_2, \ldots, r_q \) (\( q = |\lambda|^d \)) for the group \( \mathbb{Z}^d / B(\mathbb{Z}^d) \) such that every element \( k \in \mathbb{Z}^d \) has a finite expansion

\[
k = r_{i_0} + Br_{i_1} + \cdots + B^nr_{i_n}.
\]

**Proof.** The choice of coset representatives is chosen as the set

\[
\mathbb{Z}^d \cap B((-1/2, 1/2]^d).
\]

This is possible since the unit cube, centered at the origin, is a fundamental domain for
\( Z^d \) acting on \( R^d \). The element \( k \in Z^d \) has the coset representation

\[
k = r_{i_0} + Br_{i_1} + \cdots + B^{n-1}r_{i_{n-1}} + B^n\tilde{r}_n
\]

for some \( \tilde{r}_n \in Z^d \). We must show that \( \tilde{r}_n \in B((-1/2, 1/2]^d) \) for some \( n \geq 0 \). Since \( B \) is a similarity, \( B \) maps the ball of radius \( 1/2 \) centered at the origin, onto a ball of radius \(|\lambda|/2\), centered at the origin, contained in \( B((-1/2, 1/2]^d) \). We will prove that \( \|\tilde{r}_n\| < |\lambda|/2 \) (that is, \( \tilde{r}_n \) lies in the centered ball of radius \(|\lambda|/2\), and so is one of the coset representatives. Since \( \|B\| = |\lambda| \) and \( |r_i| \leq |\lambda|d^{1/2}/2 \), we have

\[
\|B^n(\tilde{r}_n)\| \leq |k| + \left( \sum_{i=0}^{n-1} |\lambda|^i d^{1/2} \right) \left( |\lambda|/2 \right)
\]

\[
< |k| + \left[ |\lambda|^n d^{1/2}/(|\lambda| - 1) \right] \left( |\lambda|/2 \right).
\]

Therefore, if we take \( B^{-n} \) on the left-hand side, we obtain

\[
\|\tilde{r}_n\| < |k|/|\lambda|^n + \left[ d^{1/2}/(|\lambda| - 1) \right] \left( |\lambda|/2 \right),
\]

so that \( \|\tilde{r}_n\| < |\lambda|/2 \) for some \( n \), as we wished to show.

Armed with the above lemma, Strichartz proved the following theorem, using the facts about tilings of \( R^d \).

**Theorem 3 (Strichartz [8])**. Let \( B \) be a strictly expansive similarity transformation such that \( B(Z^d) \subset Z^d \). Suppose that the (common) value of the modulus of any eigenvalue is greater than \( 1 + d^{1/2} \). Let \( \{r_1, r_2, \ldots, r_q\} = \mathcal{R} \) be the set of coset representatives specified in Lemma 1. Then the set \( T \subset R^d \) defined by the equation

\[
B(T) = \sum_{r_i \in \mathcal{R}} (T + r_i)
\]

tiles \( R^d \). That is, the Lebesgue measure of \((T + k) \cap (T + j)\) is zero if \( k \neq j \) and \( \bigcup_{k \in Z^d} (T + k) = R^d \).

We refer the reader to Strichartz’s paper [8], and the references there, for a proof.

Now let us consider the problem of constructing a sequence space \( \Omega \), and an embedding of \( Z^d \to \Omega \), given a strictly expansive similarity matrix \( A \) mapping \( Z^d \) into itself, and a candidate function \( m(2\pi \xi) \), periodic with period one, for \( \xi \in R^d \).
A necessary (but not sufficient) condition for \( M(\xi) := |m(2\pi \xi)|^2 \) to be associated with a scale function (that is, a function \( \hat{\phi} \) satisfying (a') and (b')) is that
\[
\sum_{i=1}^{q} M(\xi + (A^*)^{-1}r_i) = 1 \text{ a.e.}
\]
where the integers \( r_i, i = 1, 2, \ldots, q \) are coset representatives of the group \( \mathbb{Z}^d/A^*(\mathbb{Z}^d) \).
This follows from properties (a') and (b') by an argument very similar to the one given above for the case when \( A = A^* = 2 \), acting on \( \mathbb{Z} \). Thus, for each fixed \( \xi \), we have a probability measure concentrated on \( q \) points in \( \mathbb{Z}^d \). It is important to note that the measure is invariant under changes of coset representatives. That is, if \( r_i \) is replaced by \( \tilde{r}_i = r_i + A^*(k), i = 1, 2, \ldots, q \) for some \( k \in \mathbb{Z}^d \), then, since \( M(\xi) \) is periodic,
\[
M(\xi + (A^*)^{-1}r_i) \equiv M(\xi + (A^*)^{-1}\tilde{r}_i)
\]
for \( i = 1, 2, \ldots, q \).

We have assumed that \( A \) is a strictly expansive similarity. Although \( A \) does not necessarily satisfy the condition of Lemma 1, that \( |\lambda| > 1 + d^{1/2} \), there is a (smallest) integer \( p \) such that \( A^p \) does fulfill this condition. The subsequence of partial products
\[
P_N(\xi) := \prod_{j=1}^{pN} M((A^*)^{-j}(\xi + k)),
\]
where \( p \) is fixed and \( N = 1, 2, \ldots \) defines a sequence of probabilities on \( \mathbb{Z}^d \). Each of these probabilities may be considered as a probability on a sequence space \( \Omega \) whose coordinates are integers that form a complete set of coset representatives for the group \( \mathbb{Z}^d/(A^*)^p(\mathbb{Z}^d) \).
The parameter set containing \( \xi \) is taken to be the tile \( T \) generated by \( (A^*)^p \).

To be more specific, given the candidate function \( M(\xi) \) we define \( \widetilde{M}(\xi) \) by the product
\[
\widetilde{M}(\xi) = \prod_{j=0}^{p-1} M((A^*)^j\xi).
\]
Now set \( B = (A^*)^p \) and consider \( \widetilde{M}(\xi) \) as a candidate function with the dilation matrix \( B \). Notice that \( \widetilde{M}(\xi) \) is one-periodic and
\[
\prod_{j=1}^{\infty} \widetilde{M}(B^{-j}\xi) = \prod_{j=1}^{\infty} M((A^*)^{-j}\xi).
\]
We may summarize this equality be saying that $\tilde{M}(\xi)$ is the square of the modulus of a low-pass filter for $\hat{\phi}(\xi)$ corresponding to the dilation $B^*$. The necessary condition given above for $\hat{\phi}(\xi)$ to be a scale function, expressed in terms of $B$ and $\tilde{M}$, becomes

$$\sum_{i=1}^{q^p} \tilde{M}(\xi + B^{-1}r_i) = 1 \text{ a.e.}$$

where $r_i, \ i = 1, 2, \ldots, q^p$ is any collection of coset representatives for the group $\mathbb{Z}^d/B(\mathbb{Z})$. This equality follows from its predecessor for $A^*$. Now we are in a position to specify $\Omega$ as a sequence space with coordinates $\omega_j(k) = r_j$ where vectors $r_j$ are the coset representatives of $\mathbb{Z}^d/B(\mathbb{Z})$ that appear in the expansion

$$k = r_0 + Br_1 + \cdots + B^n r_n$$

where $n = n(k)$ is the maximal exponent in the finite expansion provided by Lemma 1. We let $\xi$ be the generic point in the tile $T$ generated by $B$. For each such $\xi$, the partial products

$$P_N^\xi(k) = \prod_{j=1}^{N} \tilde{M}(B^{-j}(\xi + k))$$

define a sequence of consistent measures on the cylinder of $\Omega$, as described in the one dimensional case, and the limiting measure $P_\xi$ is defined on the $\sigma$-field generated by the cylinders. It is important to note that $P_N^\xi$ defines a measure concentrated on finite sequences $\omega(k) \in \Omega$ with $\omega_j(k) = r_j$ and $\omega_{n+j}(k) \equiv 0$ for some $n$, all $j > 0$, defined by the expansion given in Lemma 1:

$$k = \sum_{j=0}^{n} B^j r_j, \quad n = n(k).$$

Furthermore, the sets

$$Z_N = \{ k : n(k) = N \}$$

are nested ($Z_N \subset Z_{N+1}$) and $\mathbb{Z}^d = \lim Z_N$. The limiting measure $P_\xi(\mathbb{Z}^d) = 1$ if and only if the sequence $P_N^\xi$ is tight in the sense that given $0 < \epsilon < 1$, there exists a set $Z_{N(\epsilon)}$ such that

$$P_N^{\epsilon+j}(Z_{N(\epsilon)}) \geq 1 - \epsilon$$
for all $j > 0$.

Cohen’s condition: There exists a compact set $K$ containing a neighborhood of the origin, and congruent to $(1/2, 1/2]^d$ such that for $\xi \in K$, $M((A^*)^{-n}\xi) > 0$ for all $n \geq 1$. The following more general condition is equivalent to Cohen’s condition when $P_\xi(k)$ is continuous for each $k \in \mathbb{Z}^d$: There exists a $\delta > 0$ and $k = k(\xi) \in \mathbb{Z}^d$ such that

$$(\text{Condition C}) \quad P_\xi(k(\xi)) \geq \delta > 0$$

for almost every $\xi \in T$.

The proof that Condition C is sufficient for tightness is similar to the reasoning for the case $A = 2$: Given $\epsilon > 0$, find $Z_{N(\epsilon)}$ such that $P_\xi(Z_{N(\epsilon)}^c) \leq \delta \epsilon$. Then

$$P_\xi(Z_{N(\epsilon)}^c + j(Z_{N(\epsilon)} + j) \cap Z_{N(\epsilon)}^c) \leq \delta^{-1} P_\xi(Z_{N(\epsilon)}^c) \leq \epsilon,$$

since for $k \in Z_{N(\epsilon)} + j$ and $\ell \in \mathbb{Z}^d$

$$P_\xi(k + B_{N(\epsilon)}(\ell)) = P_\xi(k + \ell) = P_{B_{-(N(\epsilon) + j)(\xi + k)}(\ell)}.$$
References

1. Billingsley, P. Convergence of Probability Measures, Wiley, New York, NY, 1968.

2. Cohen, A. Ondelettes, analyses multirésolutions, et filtres miroir en quadrature, 
Ann. Inst. H. Poincaré, Anal. nonlinéaire, 7, 439-459, 1990.

3. Daubechies, I. Ten Lectures on Wavelets, (CBMS-NSF regional conference series in applied mathematics, 61) SIAM, Philadelphia, PA, 1992.

4. Hernández, E., Wang, X., and Weiss, G., Smoothing minimally supported frequency wavelets: part II, J. Fourier Anal. Appl., 3 (1), 23-41, 1997.

5. Hernández, E. and Weiss, G. A First Course on Wavelets, CRC Press, Inc., Boca Raton, FL, 1996.

6. Mallat, S. Multiresolution approximation and wavelets, Trans. Amer. Math. Soc., 315, 69-88, 1989.

7. Papadakis, M., Šikić, H., and Weiss, G., The characterization of low-pass filters and some basic properties of wavelets, scaling functions, and related concepts, J. Fourier Anal. and Appl., 5 (5), 495-521, 1999.

8. Strichartz, R. Wavelets and self-affine tilings, Constr. Approx., 9, 327-346, 1993.

9. Wojtasczyk, P. A Mathematical Introduction to Wavelets, London Math. Soc. Student Texts 37, Cambridge Univ. Press, Cambridge, U.K., 1997.

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