On state versus channel quantum extension problems: exact results for $U \otimes U \otimes U$ symmetry

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Abstract

We develop a framework which unifies seemingly different extension (or ‘joinability’) problems for bipartite quantum states and channels. This includes known extension problems such as optimal quantum cloning and quantum marginal problems as special instances. Central to our generalization is a variant of the Jamiołkowski isomorphism between bipartite states and linear transformations, which we term the homocorrelation map: in contrast to the better-known Choi isomorphism which emphasizes the preservation of the positivity constraint, use of the Jamiołkowski isomorphism allows one to characterize the preservation of the statistical correlations of bipartite states and quantum channels. The resulting homocorrelation map thus acquires a natural operational interpretation. We define and analyze state-joining, channel-joining, and local-positive-joining problems in three-party settings with collective $U \otimes U \otimes U$ symmetry, obtaining exact analytical characterizations in low dimensions. We find that bipartite quantum states are limited in the degree to which their measurement outcomes may agree, whereas quantum channels are limited in the degree to which their measurement outcomes may disagree. Loosely speaking, quantum mechanics enforces an upper bound on the strength of positive correlation across two subsystems at a single time, as well as on the strength of negative correlation between the state of a single system across two instants of time. We argue that these general statistical bounds inform the quantum joinability limitations, and show that they are in fact sufficient for the three-party $U \otimes U \otimes U$-invariant setting.

Keywords: quantum correlations, quantum entanglement, quantum channel-state duality

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1. Introduction

It has long been appreciated that many of the intuitive features of classical probability theory do not translate to quantum theory. For instance, every classical probability distribution has a unique decomposition into extremal distributions, whereas a general density operator does not admit a unique decomposition in terms of extremal operators (pure states). Entanglement is responsible for another distinctive trait of quantum theory: as vividly expressed by Schrödinger back in 1935 [1], ‘the best possible knowledge of a total system does not necessarily include total knowledge of all its parts,’ in striking contrast to the classical case. Certain features of classical probability theory do, nonetheless, carry over to the quantum domain. While it is natural to view these distinguishing features as a consequence of quantum theory being a non-commutative generalization of classical probability theory in an appropriate sense, thoroughly understanding how and the extent to which the purely quantum features of the theory arise from its mathematical structure remains a longstanding central question across quantum foundations, mathematical physics, and quantum information processing (QIP), see e.g. [2–5].

In this paper, we investigate a QIP-motivated setting which allows us to directly compare and contrast features of quantum theory with classical probability theory, namely, the relationship between the parts (subsystems) of a composite quantum system and the system as a whole. Specifically, building on our earlier work [6], we develop and investigate a general framework for what we refer to as **quantum joinability**, which addresses the compatibility of different statistical correlations among quantum measurements on different systems. Arguably, the most familiar case of joinability is provided by the ‘quantum marginal’ (aka ‘local consistency’) problem [7, 8]. In this case, we ask whether there exists a joint quantum state compatible with a given set of reduced states on (typically non-disjoint) groupings of subsystems. The quintessential example of a failure of joinability is the fact that two pairs of two-level systems (qubits), say, Alice–Bob (A–B) and Alice–Charlie (A–C), cannot both simultaneously be described by the singlet state, \( |\psi^-(\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \). A seminal exploration of this observation was carried out by Coffman, Kundu, and Wootters [9] and later dubbed the ‘monogamy of entanglement’ [10]. In classical probability theory, a necessary and sufficient condition for marginal probability distributions on A–B and A–C to admit a joint probability distribution (or ‘extension’) on A–B–C is that the marginals over A be equal [7, 11]. The analogous compatibility condition is indeed necessary in quantum theory, but, as demonstrated by the above example, is clearly no longer sufficient. The identification of necessary and sufficient conditions in general settings with overlapping marginals remains an actively investigated open problem [6, 12, 13].

Physically, standard **state-joinability problems** as formulated above for density operators, may be regarded as characterizing the compatibility of statistical correlations of two (or more) different subsystems at a given time. However, correlations between a single system before and after the action of time evolution—most generally, a quantum channel, i.e. a completely positive trace-preserving (CPTP) dynamical map—may also be considered, for example, in order to characterize the ‘location’ of quantum information that one subsystem may carry about another [14] and/or the causal structure of the events on which probabilities are defined [4, 15]. The work in [15] thoroughly explores the idea of placing kinematic and dynamic correlations on equal footing, by introducing a formalism of ‘quantum conditional states’ to represent the correlations of either bipartite quantum states or quantum channels as bipartite
operators. With these ideas in mind, one may want to formulate a quantum marginal problem for quantum channels (see also [16]). For example, given two quantum channels \( \mathcal{M}_{AB} : B(H_A) \to B(H_B) \) and \( \mathcal{M}_{AC} : B(H_A) \to B(H_C) \) (with \( B(H) \) denoting the space of bounded linear operators on \( H \)), one may ask whether there exists a quantum channel \( \mathcal{M}_{ABC} : B(H_A) \to B(H_B \otimes H_C) \), whose reduced channels are \( \mathcal{M}_{AB} \) and \( \mathcal{M}_{AC} \), respectively.

A motivation for considering such channel-joinability problems is that questions regarding the optimality of paradigmatic QIP tasks such as quantum cloning [17, 18] or broadcasting [19] may be naturally recast as such. A fundamental tool here is the Choi isomorphism [20, 21], which may be used to translate optimal cloning problems into quantum marginal problems [22, 23], and vice-versa [6]. Both monogamy of entanglement and the no-cloning theorem [24] have significant implications for the behavior of quantum systems: the former effectively constrains the kinematics of a multipartite quantum system, while the latter constrains the dynamics of a quantum system (composite or not). As both of these fundamental concepts are closely related to respective quantum joinability problems, we are prompted to explore in more depth their similarities and differences. Identifying a general joinability framework, able to encompass all such quantum marginal problems, is one of our main aims here.

The content is organized as follows. In section 2, we revisit another canonical correspondence on bipartite operator space, the Jamiołkowski isomorphism, and use it to introduce what we term the homocorrelation map as our main tool for representing quantum channels as bipartite operators. Despite the different motivation, this representation shares suggestive points of contact with the conditional-state formalism introduced in [15]. Formally, we show how this isomorphism enables a notion of quantum joinability that incorporates all joinability problems of interest, and discuss ways in which different joinability problems may be mapped into one another. In section 3, we obtain a complete analytical characterization of some archetypal examples of few-system quantum joinability problems. Namely, we address three-party joinability of quantum states, quantum channels, and block-positive (or ‘local-positive’) operators, in the case that the relevant operators are invariant under the group of collective unitary transformations, that is, under the action of arbitrary transformations of the form \( U \otimes U \otimes U \). In keeping with a main motivation of our previous work [6], these examples allow us to gain further insight into the stricter joinability limitations that quantum theory imposes, beyond the limitations that simply stem from the requirement of consistency with classical probability theory. Further to that, these example also allow us to directly contrast the joinability properties of quantum channels versus states. In section 4, we investigate a possible source for the stricter joinability bounds in quantum theory, as compared to classical probability theory. We introduce the notion of degree of agreement (disagreement), that is, the probability that a random local collective measurement yields the same (different) outcomes on each system, as given by an appropriate two-value POVM. We find that quantum theory places different bounds on the degree of agreement arising from quantum states than it does on that of quantum channels: while quantum states are limited in their degree of agreement, quantum channels are limited in their degree of disagreement. The differences in these bounds point to a crucial distinction between quantum channels and states. At least in the examples of section 3 and a few others, these limitations suffice in fact to determine the bounds of joinability exactly. Possible implications of such bounds with regards to joinability properties of general quantum states and channels are also discussed, and final remarks conclude in section 5.
2. General quantum joinability framework

We begin by reviewing the standard state-joinability (quantum marginal) problem, framing it in a language suitable for generalization. Given a composite Hilbert space $\mathcal{H}^{(N)} = \bigotimes_{i=1}^{N} \mathcal{H}_i$, a joinability scenario is defined by a list of partial traces $\{\text{tr}_\ell\}$, with each $\ell \subseteq \{1, \ldots, N\}$, along with a convex set of allowed ‘joining operators’, $W$, which in this case is the set of positive trace-one operators acting on $\mathcal{H}^{(N)}$; accordingly, we may associate a joinability scenario with a 2-tuple $(W, \{\text{tr}_\ell\})$. For any list of states $\{\rho_\ell\} \in \{R_\ell\}$, the following definition then applies:

**Definition 2.1. (State-joinability)** Given a joinability scenario described by the pair $(W, \{\text{tr}_\ell\})$, the reduced states $\rho_\ell \in \{R_\ell\}$ are joinable if there exists a joining state $w \in W$ such that $\rho_\ell = \text{tr}_\ell(w)$ for all $\ell$.

The first step toward achieving the intended generalization of the above definition to quantum channels is to represent quantum channels as bipartite operators. In the following subsection, we establish a tool to achieve this and highlight its broader utility.

2.1. Homocorrelation map

The most familiar identification between linear maps on operator space and bipartite operators is the Choi isomorphism [21, 25]. Despite having useful properties, the Choi isomorphism will not be suitable for the identification we seek. Instead, we shall develop an operationally motivated variant of the related Jamiołkowski isomorphism [20], that will serve as our bridge between linear maps and bipartite operators.

Let $\mathcal{L}(\mathcal{H}_A, \mathcal{H}_B)$ be the set of linear maps, or ‘superoperators’, from $\mathcal{B}(\mathcal{H}_A)$ to $\mathcal{B}(\mathcal{H}_B)$. The Choi isomorphism, here denoted $C$, identifies each map $M \in \mathcal{L}(\mathcal{H}_A, \mathcal{H}_B)$ with the state resulting from the map’s action on one subsystem of a maximally entangled state:

$$C(M) \equiv M_C = [I_A \otimes M]([\Phi^+]) = \frac{1}{d_A} \sum_{ij} |i\rangle \otimes M(|j\rangle \langle j|),$$

where $I_A$ is the identity map on $B(\mathcal{H}_A)$, $|\Phi^+\rangle = \sum_i |ii\rangle/\sqrt{d_A}$ and $d_A = \dim(\mathcal{H}_A)$. We note that $d_A |\Phi^+\rangle \langle \Phi^+| = T_A$, where $T_A$ denotes partial transposition on subsystem $A$ and $V$ is the swap operator on $\mathcal{H}_A \otimes \mathcal{H}_A$, defined by $V = \sum_{ij} |ij\rangle \langle ji|$ with respect to any orthonormal basis $\{|l\rangle\}$ on $\mathcal{H}_A$. The transformation is an isomorphism in that it preserves the positivity of the objects it maps to and from; namely, quantum channels (CPTP maps) are mapped to quantum states (positive trace-one operators). Consequently, $C$ is a useful diagnostic tool for determining whether a map is CP.

The Choi map, however, inherits the local basis dependence of the state $|\Phi^+\rangle$. In fact, the structure of the isomorphism in equation (1) naturally identifies a continuous family of maps generated by $(I \otimes U)|\Phi^+\rangle$ and parameterized by the set of unitary transformations. For this reason, one should rather understand $C$ to connect each CP linear map to a whole family of positive bipartite operators, and analogously for non-CP linear maps. Thus, while the Choi isomorphism is an appropriate tool for characterizing the preservation of the positivity.

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constraint, we seek a bonafide identification which bears an operational significance—linking, more strongly, the properties of the linear map with those of the resulting bipartite operator.

The Jamiołkowski isomorphism, often conflated with the Choi isomorphism, is another means of identifying linear maps on operator space with bipartite operators. It differs from the Choi isomorphism in that the swap operator on $H_A \otimes H_A$ is used in place of $|\Phi^+\rangle\langle\Phi^+|$ to generate the identification. That is:

$$J(M) \equiv M_J = [I_A \otimes M](V) = \sum_j |j\rangle \langle j| \otimes M(|j\rangle \langle j|).$$

(2)

Notably, the difference between $C$ and $J$ has been stressed in [15]; wherein, the authors use the Jamiołkowski isomorphism to translate any quantum channel $M_{BIA} : B(H_A) \to B(H_B)$ into a causal conditional state $\rho_{BIA} \equiv J(M_{BIA}) \in B(H_A \otimes H_B)$.

The desideratum for the identification we seek calls for an operational definition of states and channels exhibiting the same statistical correlations. In the case of a bipartite state, we consider such correlations to be given by the set of probabilities $p(i,j)$ for the joint outcomes of all local POVM pairs $\{E_i^A\}$ and $\{F_j^B\}$, on $A$ and $B$, respectively. The analogous notion for a channel is slightly more involved. Clearly, the POVMs on system $A$ and system $B$ must correspond to measurements before and after the application of the channel, respectively. But, to generate a joint probability distribution with these, we must introduce some initial, pre-measurement state $\rho_A$ of $A$ informing the measurement statistics of $E_i^A$. The only unbiased choice is $\rho_A = 1/d_A$, corresponding to no prior information about the input system. Putting this together, we have the following:

**Definition 2.2.** A bipartite quantum state $\rho \in B(H_A \otimes H_B)$ and a quantum channel $M : B(H_A) \to B(H_B)$ exhibit the same correlations if for every pair of local POVMs, $\{E_i^A\}$ and $\{F_j^B\}$, the likelihood of each joint outcome $p(i,j)$ on $\rho$ is equal to the likelihood of obtaining outcome $i$ given state $1/d_A$ on $A$ and outcome $j$ after $M$ has acted.

We now show that a rescaled version of $J$ provides the desired identification:

**Proposition 2.3.** A bipartite state $\rho \in B(H_A \otimes H_B)$ and a quantum channel $M : B(H_A) \to B(H_B)$ exhibit the same correlations if and only if $J(M)/d_A = \rho$.

**Proof.** First we calculate the statistical correlations for $\rho$ and then $M$. For the state $\rho$, the statistical correlations for the local POVMs $\{E_i^A\}$ and $\{F_j^B\}$ are simply

$$p(i,j) = \text{tr} \left( E_i^A \otimes F_j^B \rho \right).$$

(3)

In the channel case, we calculate the statistical correlations using the conditional probability, $p(i,j) = p(i) p(j|i)$. The first factor is $p(i) = \text{tr} \left( E_i^A 1/d_A \right) = \text{tr} \left( E_i^A \right)/d_A$. To calculate the second factor, we need an expression for the state of the system conditioned on outcome $i$. Most generally, the post-measurement state may be written as

$$\rho_i = \frac{\sqrt{E_i^A U 1/d_A U^+} E_i^A}{\text{tr} \left( \sqrt{E_i^A U 1/d_A U^+} E_i^A \right)} = \frac{E_i^A}{\text{tr} \left( E_i^A \right)}.$$
from which it follows that

\[ p(j|i) = \text{tr} \left[ F^B_j \mathcal{M}(\rho_j) \right] = \text{tr} \left[ F^B_j \mathcal{M}(E^A_j) \right] / \text{tr} \left( E^A_i \right). \]

Putting these together, the statistical correlations for a channel are given by

\[ p(i, j) = \frac{1}{d_A} \text{tr} \left[ F^B_j \mathcal{M}(E^A_j) \right]. \]

This above expression may be rewritten using the Jamiołkowski map as follows:

\[
\frac{1}{d_A} \text{tr} \left[ F^B_j \mathcal{M}(E^A_j) \right] = \frac{1}{d_A} \sum_{i,j} \langle j | E^A_j | i \rangle \text{tr} \left[ F^B_j \mathcal{M}(\langle j | \langle i |) \right] \\
= \frac{1}{d_A} \sum_{i,j} \text{tr} \left[ F^B_j (\langle i | E^A_j ) \otimes (\mathcal{M}(\langle i | \langle j |) \right] \\
= \frac{1}{d_A} \sum_{i,j} \text{tr} \left[ F^B_j (\langle i | \otimes (\mathcal{M}(\langle i | \langle j |)E^A_j ) \right] \\
= \text{tr} \left[ E^A_i \otimes F^B_j \mathcal{J}(\mathcal{M})/d_A \right]. \tag{4} \]

Comparing equations (3) and (4), the ‘if’ of the proposition follows immediately. For the ‘only if’ direction, we use the fact that the real span of all joint POVM elements is the Hermitian sector of the space \( B(\mathcal{H}_A \otimes \mathcal{H}_B) \), and hence

\[ \text{tr} \left( E^A_i \otimes F^B_j \rho \right) = \text{tr} \left[ E^A_i \otimes F^B_j \mathcal{J}(\mathcal{M})/d_A \right] \]

can hold for all POVMs and their elements only if \( \rho = \mathcal{J}(\mathcal{M})/d_A \). \( \square \)

Because of its operational significance and use in this paper, we provide a name for the above correspondence. Formally, we define the homocorrelation map, \( \mathcal{H} \), by letting
From equation (4), we may take the defining property of the homocorrelation map to be

$$\text{tr} [\mathcal{H}(\mathcal{M})A \otimes B] = \frac{1}{d_A} \text{tr} [\mathcal{M}(A)B], \quad \forall A \in B(\mathcal{H}_A), B \in B(\mathcal{H}_B).$$

(6)

The relationships among these various transformations are depicted in figure 1.

Making contact with related work, we point out that the approach to remove basis-dependence by replacing the reference state with the (normalized) swap operator also builds upon an identification that was introduced for the special case of qubits in [27]. Furthermore, in terms of the conditional state formalism of [15], it is interesting to observe that the channel operator $M_H$ resulting from the homocorrelation map applied to a CPTP map $\mathcal{M}$ is precisely the causal joint state obtained by conditioning the causal conditional state of $\mathcal{M}$ on the input $I_dA$. Thus, the homocorrelation map may also be seen as the composition of the Jamiołkowski isomorphism with the conditioning of the completely mixed state:

$$\mathcal{H}(\mathcal{M}) = \sqrt{\lambda_3/d_A} \mathcal{J}(\mathcal{M}) \sqrt{\lambda_3/d_A}.$$

Example. A simple example may illustrate the operational relevance of $\mathcal{H}$. Consider the one-parameter family of qudit depolarizing channels [28], defined as

$$D_\eta(\rho) = (1 - \eta) \text{tr} (\rho) \frac{1}{d} + \eta \rho.$$

The action of this channel commutes with all unitary channels in that $D_\eta(U\rho U^\dagger) = UD_\eta(\rho)U^\dagger$. Under the homocorrelation map, the depolarizing channels are taken to operators with $\otimes$ symmetry, namely,

$$\mathcal{H}(D_\eta) = (1 - \eta) \mathbb{1} \otimes \mathbb{1} + \eta \frac{V}{d}.$$

(7)

Positive semi-definite operators of this form are the well-known Werner states [29]. Imagine that an observer does not know a priori whether her two measurements are made on distinct subsystems in a Werner state or on the same system before and after a depolarizing channel has been applied. If presented with a Werner state or depolarizing channel having $\eta \in [-\frac{1}{d^2-1}, \frac{1}{d^2+1}]$ (see also section 3.1 for the meaning of this parameter range), the observer will not be able to distinguish between the two cases. The homocorrelation map makes this operational identification explicit. To contrast, the Choi map takes the depolarizing channels to so-called isotropic states [30],

$$C(D_\eta) = (1 - \eta) \frac{\mathbb{1} \otimes \mathbb{1}}{d^2} + \eta |\Phi^+\rangle\langle \Phi^+|,$$

(8)

where as before $|\Phi^+\rangle$ is the maximally entangled state. The isotropic states are defined by their symmetry with respect to $U \otimes U^\dagger$ transformations. An observer in the scenario above would certainly be able to distinguish between the correlations of the depolarizing channel and the isotropic states, as long as $\eta \neq 0$.

The distinction between the maps $C$ and $\mathcal{H}$ may be further appreciated by contrasting the sets of operators they produce. The set of CP maps forms a cone in the space of super-operators $\mathcal{L}(\mathcal{H}_A, \mathcal{H}_B)$. Both $C$ and $\mathcal{H}$ are cone-preserving maps (by linearity) from $\mathcal{L}(\mathcal{H}_A, \mathcal{H}_B)$ to $B(\mathcal{H}_A \otimes \mathcal{H}_B)$. While in the case of $C$ the resulting cone is exactly the cone of bipartite states, in the case of $\mathcal{H}$ the corresponding cone is distinct from the cone spanned by
states. One of our main findings here is that the correlations exhibited by bipartite states and the ones exhibited by quantum channels need not be equivalent. Furthermore, we find that this difference plays a role in their distinct joinability properties. The homocorrelation representation of channels provides us with a natural framework for exploring this difference: a channel and a state with differing correlations will be represented as distinct operators in the same operator space. These notions and their use in joinability are fleshed out in the remainder of this section.

### 2.2. Positive cone structures

The cone of positive operators plays a central role in defining joinability of quantum states. Analogously, the cone of homocorrelation-mapped channels will play a central role in defining joinability of quantum channels. These two cones are commonly known as positive-semidefinite operators (PSD) and positive-partial transpose operators (PPT), respectively. Here, we adopt an alternative terminology that emphasizes their physical significance, and refer to these cones and the operators they contain as state-positive and channel-positive, respectively. Formally:

**Definition 2.4. (State-positivity (PSD))** An operator $M \in B(H)$ is state-positive if $\text{tr}(MP) \geq 0$ for all Hermitian projectors $P = P^1 = P^2 \in B(H)$. We note this condition as $M \succeq_{\text{st}} 0$ and emphasize that the resulting set is a self-dual cone.

Recall that a map $\mathcal{M}$ is a quantum channel if $\text{tr}(C(\mathcal{M})P) \geq 0$ for all $P = P^1 = P^2 \in B(H_A \otimes H_B)$ [21]. Using the relationships of figure 1, we translate this condition to one on the homocorrelation-mapped operator $\mathcal{M}H_a^\mathcal{M}$. Specifically, we define:

**Definition 2.5. (Channel-positivity (PPT))** An operator $M \in B(H_A \otimes H_B)$ is channel-positive with respect to the $A$–$B$ bipartition if $\text{tr}(MP^{B_A}) \geq 0$ for all Hermitian projectors $P = P^1 = P^2 \in B(H_A \otimes H_B)$. We note this condition $M \succeq_{\text{ch}} 0$, and emphasize that the resulting set is, again, a self-dual cone.

A characterization of the intersection of the state- and channel-positive cones reveals a connection between quantum kinematics and dynamics.

**Proposition 2.6.** If a bipartite state $\rho \in B(H_A \otimes H_B)$ and a quantum channel $\mathcal{M} : B(H_A) \rightarrow B(H_B)$ exhibit the same correlations (in the sense of definition 2.2), then $\rho$ is PPT and $H(\mathcal{M})$ is PSD.

**Proof.** Assume $\rho$ and $\mathcal{M}$ exhibit the same correlations. Then, by proposition 2.3, $\rho = H(\mathcal{M})$. Then $H(\mathcal{M})$ is PSD because $\rho$ is. The analogous statement for PPT follows from $H(\mathcal{M})$ being PPT, but we find it instructive to give this calculation explicitly. Since $C$ is related to $H$ by a partial trace, we also have $H(\mathcal{M})^{\mathcal{M}} = C(\mathcal{M})/d_A$. By the positivity preservation of $C$, $\mathcal{M}$ being CPTP implies that $C(\mathcal{M})$ is a positive operator. Thus, we have that $\rho^{\mathcal{M}} = C(\mathcal{M})/d_A$ is positive, making $\rho$ PPT as claimed. $\square$
The above result may be used to directly connect statistical correlations in quantum states and channels to separability and, respectively, entanglement-breaking properties [32] of the channel:

**Corollary 2.7.** If the correlations of a bipartite state $\rho \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ cannot be exhibited by a quantum channel, then the state is not separable. Similarly, if the correlations of a quantum channel $\mathcal{M} : \mathcal{B}(\mathcal{H}_A) \rightarrow \mathcal{B}(\mathcal{H}_B)$ cannot be exhibited by a bipartite state, then the channel is not entanglement breaking.

**Proof.** Since the correlations of $\rho$ cannot be exhibited by a quantum channel, the operator is not PPT, by proposition 2.6. Then, by the Peres–Horodecki criterion [31], the state is necessarily entangled. For the second part, since the correlations of the channel cannot be exhibited by a bipartite state, the homocorrelation mapped channel $\mathcal{H}(\mathcal{M})$ is not PSD, by proposition 2.6. Recall that an entanglement breaking channel $\mathcal{E} : \mathcal{B}(\mathcal{H}_A) \rightarrow \mathcal{B}(\mathcal{H}_B)$ can always be written in the ‘Holevo form’ [32, 33],

$$\mathcal{E}(\rho) = \sum_k \tau_k \text{tr}(\sigma_k \rho),$$

where $\tau_k \in \mathcal{B}(\mathcal{H}_B)$ and $\sigma_k \in \mathcal{B}(\mathcal{H}_A)$ are PSD operators normalized to ensure that $\mathcal{E}$ is TP. If $\mathcal{M}$ were entanglement breaking, then by using the definition of the Jamiołkowski isomorphism in equation (2), $\mathcal{H}(\mathcal{M})$ would have the form

$$\mathcal{H}(\mathcal{M}) = \frac{1}{d_A} \sum_k \frac{1}{\text{tr}(\sigma_k)} \sum_{i,j} \langle j | \otimes \tau_k \text{tr}_2(\sigma_k | j \rangle \langle j |)$$

$$= \frac{1}{d_A} \sum_k \text{tr}_2[(\mathbb{1}_A \otimes \sigma_k)V] \otimes \tau_k = \frac{1}{d_A} \sum_k \sigma_k \otimes \tau_k,$$
where the subscripts 1 (2) refer to the first (second) copy in $\mathcal{H}_A \otimes \mathcal{H}_A$ respectively. Thus, $\mathcal{H}(M)$ is a (separable) bipartite quantum state. Since $\mathcal{H}(M)$ is not PSD, and hence, not a quantum state, $M$ is not entanglement breaking.

A pictorial representation of the geometry of state- and channel-positive cones is shown in figure 2 for two-qudit Werner operators as in equation (7). This will aid the understanding of agreement bounds for quantum states versus channels (see section 4). It is also worth noting that recent work has also emphasized the difference between the set of state-positive (positive) and channel-positive (PPT) operators [34], in the context of showing the extent to which local measurements on quantum systems may be used to distinguish between causal-influence and common-cause relationships. Loosely speaking, statistics consistent with a non-PPT operator indicate some degree of common cause, while those consistent with a non-positive operator indicate some degree of causal influence—hence implying that entanglement also provides a quantum advantage for causal inference as compared to the analogous classical setting.

2.3. Generalization of joinability

We are now poised to use the homocorrelation representation to formulate joinability problems for quantum channels. The channel-positive operators provide an alternative set with which to define the allowed joining operators $W$. As a warm-up, we rephrase the channel-joinability problem that was posed in the Introduction. Consider quantum channels from $\mathcal{H}_A$ to $\mathcal{H}_B \otimes \mathcal{H}_C$. Under the homocorrelation map, these correspond to tripartite operators lying in the channel-positive cone, notated $W_{ABC}$. The partial traces $\text{tr}_C$ and $\text{tr}_B$ take channel-positive operators in $W_{ABC}$ to channel-positive operators in $W_{AB}$ and $W_{AC}$, respectively; that is, operators in $W_{AB}$ and $W_{AC}$ correspond to valid quantum channels via the homocorrelation map. The corresponding channel-joining scenario is then defined as $(W_{ABC}, [\text{tr}_C, \text{tr}_B])$. A channel-joinability (or extension) problem presents two channel operators $M_{AB} \in W_{AB}$ and $M_{AC} \in W_{AC}$ and seeks to determine the existence of a channel operator $M_{ABC} \in W_{ABC}$ which reduces to the two channel operators in question. In general, we thus have the following:

**Definition 2.8. (Channel-joinability)** Given a joinability scenario described by the pair $(W \geq_{ch} 0, \{\text{tr}_f\})$, the reduced operators $\{M_f\} \in \{R_f\}$ are **joinable** if there exists a joint operator $M \in W$ such that $\text{tr}_f(M) = M_f$ for all $f$.

A channel joinability problem can in principle be stated using the Choi instead of the homocorrelation map, as done in [16]. However, as we argued, the latter provides a platform to **directly** compare the joinability of states and channels of equivalent correlations. For instance, it will allow us to **simultaneously** compare the joinability of local-unitary-invariant states and channels, and consequently to compare these both to the joinability of analogous classical probability distributions (cf figure 5).

Before proceeding to the general notion of joinability, we remark that allowed joining operators in $W$ have thus far been considered to be either state-positive or channel-positive. However, from a mathematical standpoint, a viable joinability problem **only needs $W$ to be a convex cone**. To investigate this generalization and to further meld state and channel joining, we consider a third type of positivity that we call local-positivity. This notion is equivalent to both block-positivity and to map-positivity (not necessarily CP) [35, 36], in that by representing linear maps using $\mathcal{H}$, the cone of (transformed) positive maps is equal to the cone of
bipartite block-positive operators. In fact, it was precisely the correspondence between block-positive operators and positive maps that led Jamiołkowski to use this isomorphism in [20]. Formally:

**Definition 2.9. (Local-positivity)** An operator $M \in B(\mathcal{H}_A \otimes \mathcal{H}_B)$ is local-positive with respect to the $A-B$ factorization if $\text{tr}(MP_A \otimes P_B) \geq 0$ for all pure states $P_A = P_A^1 \in B(\mathcal{H}_A)$ and $P_B = P_B^2 \in B(\mathcal{H}_B)$. We notate this condition $M \geq_{\text{loc}} 0$.

Local-positivity is readily generalized to more than two systems. The set of channel-positive operators and state-positive operators are each sub-cones of the local-positive operators, as local-positivity clearly is a weaker condition. In particular, joinability with respect to the state-positive or channel-positive cone implies joinability with respect to the local-positive cone. Local-positive operators are directly relevant to QIP, in particular because they may serve as entanglement witnesses [37].

Compared to state- and channel-positivity, the notion of local-positivity yields a less-strict definition of quantum joinability, which allows for closer comparison to joinability limitations derived from classical probability. We take a brief detour to explain how classical probability theory limits quantum joinability. Consider three bipartite quantum states $\rho_{AB}, \rho_{AC}$, and $\rho_{BC}$. By choosing a product basis for the three systems $\{|i\rangle | j\rangle | k\rangle\}_{ABC}$, we may correspond each density operator to a probability distribution via $\rho_{AB} \mapsto \langle \langle i | | j \rangle | | k \rangle \rangle_{AB}$, etc. Assume the three density operators could be joined by the state $\rho_{ABC}$. Then, to $\rho_{ABC}$, we could associate a joint probability distribution $\langle \langle i | | j \rangle | | k \rangle \rangle_{ABC}$ whose marginals would necessarily be $\langle \langle i | | j \rangle \rangle_{AB}$, etc. In other words, if $\rho_{AB}$, $\rho_{AC}$, and $\rho_{BC}$ were joinable, then their corresponding classical distributions would be joinable. The contrapositive of this statement describes how classical probability limits quantum joinability: if ever the $\langle \langle i | | j \rangle \rangle_{AB}$, etc are not joinable with one another, then the corresponding density operators $\rho_{AB}$, etc are not be joinable with one another. Thus, classical probability theory provides necessary conditions for quantum joinability. This reasoning holds for channel-joinability scenarios and local-positive joinability scenarios alike: the positivity of the derived classical distributions (e.g. $\langle \langle i | | j \rangle \rangle_{ABC}$) is ensured by channel-positivity and by local-positivity. In general, given a choice of local bases $\{|i\rangle | j\rangle | k\rangle\}_{ABC}$, the joining operator is constrained by

$$p(i|j\rangle | k\rangle | m_E\rangle \equiv \text{tr}(\rho_{ABC}... | i\rangle \langle i | | j \rangle | | k \rangle \rangle \otimes ... | m_E\rangle \langle m_E |) \geq 0.$$  

Local-positivity is equivalent to requiring the above to hold with respect to all choices of basis. In order to distinguish between local-positive constraints and those of classical probability theory we consider only classical probability distributions derived from local bases that are identified with one another ($\{|i\rangle\} \equiv \{|i\rangle\} \equiv \{|i\rangle\} = ...$). Then, local-positivity constraints are more general (hence stricter) than classical probability constraints in that the local projectors of the former may be oblique (i.e. neither orthogonal nor parallel) with respect to one another. In section 3.2 we show how Werner operators provide an example of this strict inclusion.

Another way of viewing the various definitions of positivity is to understand the subscript on the inequality to indicate the dual cone from which inner products with $M$ must be positive. For $M \geq_{\text{st}} 0, M \geq_{\text{ch}} 0$, and $M \geq_{\text{loc}} 0$, the respective dual cones are the positive span of rank-one projectors, the positive span of partially-transposed projectors, and the positive span of product projectors (from which the trace-one condition confines to the set of separable states). We note that the first two cones are self-dual (and, furthermore, symmetric [38]), while the local-positive cone is not. With several relevant examples of positivity
established, each being a different convex set with which to define $W$, we are in a position to finally give the following:

**Definition 2.10. (General quantum joinability)** Let $W$ be a convex cone in $B(H^{(N)})$, and $\{\text{tr}_{\ell}\}$ be partial traces with $\ell \in \mathbb{Z}_N$. Given the joinability scenario $(W, \{\text{tr}_{\ell}\})$, the operators $\{M_{\ell}\} \in \{R_{\ell}\}$ are joinable if there exists a joining operator $w \in W$ such that $\text{tr}_{\ell}(w) = M_{\ell}$ for all $\ell$.

This general definition encompasses the various joinability problems referenced in the Introduction. Specifically, if $W$ is the set of quantum states on a multipartite system, the joinability problem reduces to the quantum marginal problem, whereas if $W$ consists of channel-positive operators describing quantum channels from one multipartite system to another, one recovers the channel-joining problem instead. Specific instances of this problem are the optimal asymmetric cloning problem [17, 18, 39], the symmetric cloning problem [40, 41], and the $k$-extendibility problem for quantum maps [42]. In addition to providing a unified perspective, our approach has the important advantage that different classes of joinability problems may be mapped into one another, in such a way that a solution to one provides a solution to another. This is made formal in the following:

**Proposition 2.11.** Let $W$ and $W'$ be two cones of operators acting on the space $H^{(N)}$, $\{\text{tr}_{\ell}\}$ a set of partial traces that apply to both cones, and $\phi: B(H^{(N)}) \to B(H^{(N)})$ a linear cone-preserving map (i.e. $\phi(W) \subseteq W'$), which permits reduced actions $\phi_{\ell}$ satisfying $\phi_{\ell} \circ \text{tr}_{\ell} = \text{tr}_{\ell} \circ \phi$. If $\{M_{\ell}\} \in \{\text{tr}_{\ell}(W)\}$ is joinable with respect to $W$, then $\{\phi_{\ell}(M_{\ell})\} \in \{\text{tr}_{\ell}(W')\}$ is joinable with respect to $W'$.

**Proof.** Assume that $w$ is a valid joining operator for the operators $\{M_{\ell}\} \in \{\text{tr}_{\ell}(W)\}$. Then, the set of operators $\{\phi_{\ell}(M_{\ell})\} \in \{\text{tr}_{\ell}(W')\}$ is joined by the operator $\phi(w)$, since $\text{tr}_{\ell}[\phi(w)] = \phi_{\ell}(\text{tr}_{\ell}[w]) = \phi_{\ell}(M_{\ell})$ and $\phi(w) \in W'$.

This is shown in the commutative diagram of figure 3. In what follows, a stronger corollary of the above result will be employed:

**Corollary 2.12.** Let $\phi: B(H^{(N)}) \to B(H^{(N)})$ be an invertible linear map satisfying $\phi(W) = W'$, with invertible reduced actions $\phi_{\ell}$ satisfying $\phi_{\ell} \circ \text{tr}_{\ell} = \text{tr}_{\ell} \circ \phi$ (and similarly...
for their inverses). Then a set of operators \( \{M_t\} \in \{\text{tr}_I(W)\} \) is joinable if and only if the set of operators \( \{\phi_I(M_t)\} \) is joinable.

**Proof.** The forward implication follows from proposition 2.11, while the backwards implication follows from the fact that \( \phi \) and the \( \phi_I \) are invertible, along with the contrapositive of proposition 2.11.

The joinability-problem isomorphism we make use of is the partial transpose map. This transformation permits a natural reduced action, namely, partial transpose on the subsystems that remain after partial trace (see also figure 3). As explained, the partial transpose is a positivity-preserving bijection between states and channel operators (up to normalization). Thus, by determining the joinable–unjoinable demarcation for a class of states, we will also determine the joinable–unjoinable demarcation for a corresponding class of channel-operators.

### 3. Three-party joinability settings with collective invariance

In this section, we narrow our scope and investigate the joinability of Werner operators in the three-party setting. By taking advantage of the special symmetry properties these operators enjoy, namely, collective unitary invariance under transformations of the form \( U \otimes U \otimes U \), we obtain an exact analytical characterization of state-joining, channel-joining, and local-positive joining problems. That is, we determine what trios of bipartite Werner operators \( M_{AB} \), \( M_{AC} \), and \( M_{BC} \) may be joined by a valid joining operator \( w_{ABC} \) for each type of problem. Bipartite Werner states are characterized by a single parameter, which we may interpret as a degree of agreement between collective measurements on the two systems. Operationally, this degree of agreement \( \alpha_{XY} \) is the likelihood that any collective projective measurement made on the systems \( X, Y \) will produce an agreeing outcome (e.g. \( |00\rangle \langle 00|, |11\rangle \langle 11| \), etc). Of course, we may define such a degree of agreement for classical joint probability distributions of \( d \)-nary random variables. As shown in [6], for a probability distribution of three \( d \)-nary random variables \( i_A, j_B \), and \( k_C \), the degrees of agreement \( \alpha_{AB}, \alpha_{AC}, \) and \( \alpha_{BC} \) satisfy:

\[
\begin{align*}
-\alpha_{AB} + \alpha_{AC} + \alpha_{BC} &\leq 1, \\
\alpha_{AB} - \alpha_{AC} + \alpha_{BC} &\leq 1, \\
\alpha_{AB} + \alpha_{AC} - \alpha_{BC} &\leq 1,
\end{align*}
\]

and, in the case of \( d = 2 \), also

\[
\alpha_{AB} + \alpha_{AC} + \alpha_{BC} \geq 1, \quad \text{(12)}
\]

where each constraint is derived by requiring the likelihood of a particular set of outcomes to be non-negative. Assume three Werner operators are joined by a valid joining operator \( w_{ABC} \), to which we associate a classical joint probability distribution \( p(ijk) \equiv \text{tr}(w_{ABC} |ijk\rangle \langle ijk|) \). Then, each Werner operator’s degree of agreement is derivable as a classical degree of agreement from this valid joint probability distribution, and hence, must satisfy the above inequalities. As the latter provide necessary but not sufficient conditions for three-party Werner operator joinability, one of the aims of this section is to investigate the way in which quantum joinability limitations are stricter.

We begin with state joinability, whereby the bipartite operators along with the joining tripartite operator are state-positive. The next case considered is ‘1–2 channel joinability’: here,
we specify a bipartition (say, \(A|BC\)) and consider the bipartite operators which cross the bipartition (\(M_{AB}\) and \(M_{AC}\)), along with the joining operator, to be channel-positive with respect to the bipartition, while the remaining bipartite operator (\(M_{BC}\)) is state-positive. Since each of the three possible bipartition choices (\(A|BC\), \(B|AC\), and \(C|AB\)) constitutes a different channel-joinability scenario, a total of four possibilities arise for three-party state/channel joinability. Lastly, motivated by the suggestive symmetry arising from the state and channel joinability results and their relation to classical joining, we consider the weaker notion of local-positive joining, in which all operators involved are only required to be local-positive.

3.1. Joinability limitations from state-positivity and channel-positivity

Bipartite Werner operators obey collective unitary invariance. By standard Shur–Weyl duality [43], these operators have the form given in equation (7), which for convenience we repeat here,

\[
\rho(\eta) = (1 - \eta) \frac{1}{d^2} + \eta \frac{V}{d},
\]

with \(V\) being the swap operator defined earlier. The \(\pm 1\) eigenvalues of \(V\) dictate the state positive range of \(\rho(\eta)\) to be \(-\frac{1}{d^2-1} \leq \eta \leq \frac{1}{d^2-1}\), corresponding to the qudit Werner states we already mentioned. This parameterization is chosen so that \(\eta\) is a 'correlation' measure: if \(d = 2\), then \(\eta = -1\) corresponds to the singlet state, \(\eta = 0\) to the maximally mixed state. The correlation parameter is related to the ‘degree of agreement’, discussed above, via

\[
\eta = \frac{d}{d - 1} \left( \alpha - \frac{1}{d} \right).
\]

We choose to present our results in terms of \(\eta\) because it simplifies the expressions. Note that the value \(\eta = 1\) (\(\alpha = 1\)) does not correspond to a valid quantum state, but rather to a valid quantum channel operator that expresses perfect correlation for all possible collective measurements. As seen in the example of section 2.1, channel-positive operators with \(\otimes U\)-invariance correspond to depolarizing channels, with \(\eta = 1\) labeling the identity channel. It is known that channel-positivity of the depolarizing map is ensured provided that \(-\frac{1}{d^2-1} \leq \eta \leq 1\) [44]. We find it instructive to re-establish state- and channel-positivity bounds using the Choi isomorphism.

To this end, we enlarge the above class of Werner operators to include the operators which are obtained from partial transposition of Werner operators. From equation (8), such operators are the \(U^* \otimes U\)-invariant isotropic operators. The span of operators exhibiting either \(U \otimes U\) or \(U^* \otimes U\) invariance will be invariant under the action of collective orthogonal transformations, that is operators that may be written as \(O \otimes O \equiv U \otimes U = U^* \otimes U\) for some unitary \(U\). This extended class of operators provides a useful test bed in that it expresses familiar states as well as channel operators of well-known quantum channels, while being non-trivially closed under partial transpose. A general operator in this space may be parameterized as follows [46]:

\[3\] Note that in the basis defined for conjugation, the entries of \(O\) are real. The set of collective orthogonal transformations constitutes the Brauer algebra [45–47]. Irreducible representations of the Brauer algebra have been recently characterized in [48]. The Brauer algebra acting on \(N\) \(d\)-dimensional Hilbert spaces is spanned by representations of subsystem permutations \(\{V_i|_{i \in S_k}\}\), along with their partial transpositions with respect to groupings of the subsystems \(\{V_i^T|_{i \in S_k}, \{1, \ldots, N\}\}\). In terms of tensor network diagrams, each element of this basis is represented by a set of disjoint pairings of \(2N\) vertices, with the vertices arranged in two rows of \(N\).
\[ \rho(\eta, \beta) = (1 - \eta - \beta) \frac{1}{d^2} + \frac{V}{d} + \beta V T_2 \]  

(15)

In particular, the operator \( \rho(0, 1) \) is a generic Bell state on two qudits, \( \rho(0, -1/(d-1)) \) is the maximally entangled Werner state (e.g., the singlet state for \( d = 2 \)), \( \rho(1, 0) \) is the identity channel, and \( \rho(0, 0) \) is the completely mixed state (or the completely depolarizing channel). We can then establish the following:

**Proposition 3.1.** A bipartite operator \( \rho(\eta) \) with collective unitary invariance is channel-positive if and only if \( -\frac{1}{d^2 - 1} \leq \eta \leq 1 \).

**Proof.** By definition, an operator is channel-positive if it is PPT. Furthermore, the Brauer operators in equation (15) satisfy the property \( \rho(\eta, \beta) = \rho(\beta, \eta) \). Thus, we need only determine the state-positivity of the operators \( \rho(\beta, \eta) \) to determine the bounds which channel-positivity places on \( \eta \) and \( \beta \). The eigenspaces of any Brauer operator are the anti-symmetric subspace, the one-dimensional space spanned by \( \Phi^+ \), and the space spanned by vectors \( |y\rangle \) satisfying \( \langle y| (y')^+ = 0 \), for which we label the projectors \( P_A, P_r, \) and \( P_Y \), respectively. For the operator \( \rho(\beta, \eta) \), where \( \eta \) and \( \beta \) have been switched by the partial transpose, the eigenvalues are as follows:

\[
\rho(\beta, \eta) = \begin{cases} 
(1 - \eta - \beta)/d^2 - \beta/d & P_A, \\
(1 - \eta - \beta)/d^2 + \beta/d + \eta & P_r, \\
(1 - \eta - \beta)/d^2 + \beta/d & P_Y.
\end{cases}
\]

Hence, channel-positivity of the bipartite Brauer operators is ensured by

\[
1 \geq (d + 1)\beta + \eta, \quad 1 \geq -(d - 1)\beta - (d^2 - 1)\eta, \quad 1 \geq -(d - 1)\beta + \eta.
\]

In particular, we recover that the channel-positive range for \( U \otimes U \)-invariant operators \( (\beta = 0) \) is \( -\frac{1}{d^2 - 1} \leq \eta \leq 1 \), whereas the state-positive range (obtained by swapping \( \eta \) and \( \beta \) in the above inequalities) is \( -\frac{1}{d^2 - 1} \leq \eta \leq \frac{1}{d^2 - 1} \).

Reasoning in a similar manner, we can also obtain the ranges of local-positivity:

**Proposition 3.2.** A bipartite operator \( \rho(\eta) \) with collective unitary invariance is local-positive if and only if \( -\frac{1}{d^2 - 1} \leq \eta \leq 1 \).

**Proof.** Local positivity is ensured by the non-negativity of expectation values with respect to product vectors, say, \( \langle \phi \psi | \rho(\eta, \beta) | \phi \psi \rangle \). Explicitly, we have

\[
\frac{1 - \eta - \beta}{d^2} + \frac{\eta}{d} |\langle \phi \psi | \phi \psi \rangle|^2 + \frac{\beta}{d} |\langle \phi \psi | \psi \rangle|^2 \geq 0,
\]

with \( 0 \leq |\langle \phi | \psi \rangle|^2, |\langle \phi | (\psi')^+ \rangle|^2 \leq 1 \). We now show that all four extremal value pairings of these factors may be achieved, leading to four inequalities whose satisfaction guarantees that of all others. Consider \( |x\rangle \) and \( |y\rangle \) satisfying \( |x\rangle^\dagger = |x\rangle \) and \( |y\rangle^\dagger = |y\rangle \), where the bar indicates a

---

4 As noted, both the definition of \( |\phi^+\rangle \) and the use of complex conjugation are *basis-dependent* notions. It is understood that all usages of either refer to the same (arbitrary) choice of basis.
vector orthogonal to the original vector. Setting \( |\psi\rangle = |x\rangle \) achieves \( \langle \phi | \psi \rangle^2 = 0 \), \( \langle \phi | (i |\psi\rangle)^\dag \rangle^2 = 1 \); setting \( |\psi\rangle = |y\rangle \) achieves \( \langle \phi | \psi \rangle^2 = 0 \), \( \langle \phi | (i |\psi\rangle)^\dag \rangle^2 = 1 \); setting \( |\psi\rangle = |y\rangle \) achieves \( \langle \phi | \psi \rangle^2 = 1 \), \( \langle \phi | (i |\psi\rangle)^\dag \rangle^2 = 0 \); and setting \( |\psi\rangle = |x\rangle \) achieves \( \langle \phi | \psi \rangle^2 = 1 \), \( \langle \phi | (i |\psi\rangle)^\dag \rangle^2 = 1 \). Thus, the following four extremal inequalities bound the local-positive region of the \( \eta-\beta \) space:

\[
0 \leq \langle \rho | (\eta, \beta) \rangle_{cx} = (1 - \eta - \beta)/d^2 + \eta/d + \beta/d,
0 \leq \langle \rho | (\eta, \beta) \rangle_{ct} = (1 - \eta - \beta)/d^2,
0 \leq \langle \rho | (\eta, \beta) \rangle_{ty} = (1 - \eta - \beta)/d^2 + \eta/d,
0 \leq \langle \rho | (\eta, \beta) \rangle_{ty} = (1 - \eta - \beta)/d^2 + \beta/d.
\]

More compactly, we may write

\[
-\frac{1}{d - 1} \leq \eta + \beta \leq 1, \quad -(d - 1)\eta + \beta \leq 1, \quad \eta - (d - 1)\beta \leq 1,
\]

whereby the desired result follows\(^5\).

A pictorial summary of the three positivity bounds provided by propositions 3.1 and 3.2 is presented in figure 4.

Having characterized all types of positivity for the (bipartite) operators to be joined, we now turn to characterize the positivity for the (tripartite) joining operators. For each positive tripartite set \( (W \geq_{\text{st}} 0, W \geq_{\text{ch}} 0, \text{and } W \geq_{\text{loc}} 0) \), we obtain the trios of joinable bipartite operators by simply applying the three partial traces \( \Tr_A, \Tr_B, \Tr_C \) to each positive operator. Our approach is to obtain an expression for the positivity boundary of the tripartite operators in terms of operator space coordinates, and then re-express this boundary in terms of parameters for the reduced operators (the three Werner parameters in this case). It is important to

\(^5\) We see that, within the set of bipartite Brauer operators, local-positive operators coincide with the set obtained from convex combinations of state- and channel-positive operators. An operator which is a convex combination of state- and channel-positive operators is called decomposable [35]. Interestingly, such a coincidence also holds for arbitrary bipartite qubit states [49].
realize that the Werner symmetry, satisfied by the bipartite reduced operators, can be exploited to narrow down the set of joining operators to consider. Namely, if a trio of bipartite Werner operators is joined by some $w$ with given positivity properties, then the trio is also joined by a ‘twirled’ operator $\tilde{w}$ with the same positivity properties, given by (cf equation (13) in [6]):

$$\tilde{w} = \int d\mu(U) U \otimes U \otimes U \otimes U (U \otimes U \otimes U)^\dagger.$$ 

Thus, a trio of bipartite Werner operators is joinable if and only if it is joinable by some tripartite Werner operator. For state- and channel-positivity, the desired characterization follows directly from the analysis reported in our previous work [6].

**Corollary 3.3.** With reference to the parameterization of equation (13), we have that:

(i) Three qudit Werner states with parameters $(\eta_{AB}, \eta_{AC}, \eta_{BC})$ are joinable relative to the $(W_{ABC} \geq_d 0, \{\text{tr}_A, \text{tr}_B, \text{tr}_C\})$ scenario if and only if

$$\begin{cases}
\frac{1}{2} \left(1 - \eta_{AB} - \eta_{AC} - \eta_{BC}\right) \geq \left|\eta_{AB} + \omega \eta_{AC} + \omega^2 \eta_{BC}\right|, & \omega \equiv e^{\pm 2\pi/3}, \\
\eta_{AB} + \eta_{AC} + \eta_{BC} \geq -1,
\end{cases}$$

for $d = 2$, while for $d \geq 3$ they need only satisfy

$$\frac{3}{2d(d+1)} \left(1 \pm (d \mp 1)(\eta_{AB} + \eta_{AC} + \eta_{BC})\right) \geq \left|\eta_{AB} + \omega \eta_{AC} + \omega^2 \eta_{BC}\right|.$$

(ii) Three qudit channel-positive Werner operators with parameters $(\eta_{AB}, \eta_{AC}, \eta_{BC})$ are channel-joinable relative to the $(W_{ABC} \geq_ch 0, \{\text{tr}_A, \text{tr}_B, \text{tr}_C\})$ scenario if and only if

$$\frac{1}{d-1} + \eta_{AB} + \eta_{AC} - \eta_{BC} \geq \frac{2}{d-1} + d\eta_{BC} + \sqrt{\frac{2d}{d-1}} \left(e^{\pm \eta_{AB}} + e^{\pm \eta_{AC}}\right),$$

$e^{\pm \eta} \equiv \sqrt{(d-1)/2d} \pm i \sqrt{(d+1)/2d}$.

The channel-joinability limitations in the other two scenarios $B \setminus AC$ and $C \setminus AB$ may be obtained by permuting the $\eta$'s accordingly.

**Proof.** Result (i) above corresponds to theorem 3 in [6], re-expressed in terms of the parametrization of equation (13) (with reference to the notation of equations (15)–(17) in [6], one has $\eta_i = (d/d^2 - 1)(\Psi^2 - 1/2)$, $l = AB, AC, BC$).

In order to establish (ii), note that $C$ may be used to translate any $U^* \otimes U$-invariant state-positive joinability problem into a $U \otimes U$-invariant channel-positive joinability problem, drawing on corollary 2.12. Explicitly, under $C$ (partial transpose in the case of operators), the $U^* \otimes U \otimes U$-invariant state-positive operators $W_{ABC}^*$ are in one-to-one correspondence with the $U \otimes U \otimes U$-invariant channel-positive operators $W_{ABC}$. Hence, by the joinability isomorphism induced by the partial transpose, the solution to a joinability problem of the scenario $(W_{ABC}^*, \{\text{tr}_A, \text{tr}_B, \text{tr}_C\})$ gives a solution to a corresponding joinability problem of $(W_{ABC}, \{\text{tr}_A, \text{tr}_B, \text{tr}_C\})$. Thus, to obtain the depolarizing channel joinability boundaries, we simply translate the isotropic state parameters of equations (20)–(21) in [6] into $\eta$ parameters. □
The joinability limitations of all four scenarios are depicted in figure 5. In the qubit case we are granted some liberty in perspective. In the space depicting the joinability limitations, the coordinates are truly the reduced-state Werner parameters; however, we may also view them as coordinates in operator space of the tripartite joining operator, with the axes corresponding to the orthogonal operator basis \( \{ \mathbb{1}_{AB}, \mathbb{1}_{AC}, \mathbb{1}_{BC} \} \). This justifies an abuse of terminology, identifying the space of reduced state coordinates with the space of relevant joining operators.

By rewriting equations (9)–(12) in terms of the \( \eta \)-parametrization by using equation (14), we determine the tetrahedra given in figures 5(a)–(b). In the qubit case, it is intriguing that the inclusion of the quantum channel-joinability limitations allows one to regain the tetrahedral symmetry imposed by the classical limitations; whereas each scenario on its own expresses a continuous rotational symmetry\(^6\) that is not reflected classically. In other words, if we consider the joinability scenario defined by

\[
\left( \text{span}\{ W_{ABC}, W_{AB}, W_{BC}, W_{AC}, \{ \text{tr}_A, \text{tr}_B, \text{tr}_C \} \} \right).
\]

\(^6\) We note that in the state-joining case this rotation is generated by the action \( e^{i(V_{AB} - V_{AC})} \), which is a continuous interpolation among the cycles of the three systems, and hence, an action on the tensor product structure.
the joinable bipartite operators respect the tetrahedral symmetry of the classical joinability bounds. This amounts to ask: What trios of bipartite correlations—as derivable from either quantum states or channels, or from probabilistic combinations of the two—may be obtained from the measurements on three systems? Though the result expresses the tetrahedral symmetry of the classical joinability limitations, these classical joinability limitations do not suffice to enforce the stricter quantum limitations, as manifest in the fact that the corners of the classical joinability tetrahedron are not reached by the quantum boundaries. We diagnose such limitations as strictly quantum features that do not have classical analogues—as we will discuss later in this work.

3.2. Joinability limitations from local-positivity

We now explore how local-positive joinability (a weaker restriction, as noted) relates to the state/channel-joinability limitations above, as well as to the underlying classical limitations. As of yet, we only know that the local-positive limitations will lie between the classical and the quantum boundaries. Since obtaining a simple analytical characterization for arbitrary subsystem dimension \(d\) appears challenging in the local-positive setting, and useful insight may already be gained in the lowest-dimensional (qubit) setting, we focus on \(d = 2\) in this section. Our main result is the following:

**Theorem 3.4.** With reference to equation (13), three qubit local-positive Werner operators with parameters \((\eta_{AB}, \eta_{AC}, \eta_{BC})\) are joinable by a local-positive tripartite operator if and only if the following conditions hold:

\[
\begin{align*}
1 + \eta_{AB} + \eta_{AC} + \eta_{BC} &\geq 0, \\
1 - \eta_{AB} + \eta_{AC} - \eta_{BC} &\geq 0, \\
1 - \eta_{AB} - \eta_{AC} + \eta_{BC} &\geq 0,
\end{align*}
\]

and

\[
2\eta_{AB}\eta_{AC} - \eta_{AB}^2 - \eta_{AC}^2 - \eta_{BC}^2 \geq 0.
\]

We note that the first four linear inequalities delimit the tetrahedral classical joinability boundary determined in [6], once expressed in terms of the \(\eta\) variables via equation (14). The
proof is rather lengthy and deferred to a separate appendix. The resulting boundary is depicted in figure 6; this shape and its determining equation are recognized as the convex hull of the Roman surface (aka Steiner surface) \[35, 50\]. Comparing with the classical joinability limitations (tetrahedron), we see that, still, the quantum joinability limitations arising from from local-positivity are more strict. However, the Roman surface encloses the region of state/channel joinability from figure 5(a), as expected, since both state-positive and channel-positive imply local-positive. To shed light on the cause of the stricter quantum boundary (i.e., the local-positive joining boundary) here, we can explicitly construct a product-state projector, whose likelihood in a measurement would be negative if joinability outside of this shape were allowed. The family of joining states \(w\) that we need to consider (see appendix) may be parameterized in terms of the bipartite reduced state Werner parameters as

\[
w(\eta_{AB}, \eta_{AC}, \eta_{BC}) = \frac{1}{8} + \frac{\eta_{AB}}{4}(V_{AB} - \mathbb{I}/2) + \frac{\eta_{AC}}{4}(V_{AC} - \mathbb{I}/2) + \frac{\eta_{BC}}{4}(V_{BC} - \mathbb{I}/2).
\]

Consider, in particular, the following state on \(A\text{--}B\text{--}C\):

\[
|\psi\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} \cos 2\pi/3 \\ \sin 2\pi/3 \end{bmatrix} \otimes \begin{bmatrix} \cos 4\pi/3 \\ \sin 4\pi/3 \end{bmatrix},
\]

which corresponds to the pure product state with the local Bloch vectors pointing ‘as anti-parallel with one another as possible’. Computing its expectation with respect to \(w(\eta_{AB} = \eta_{AC} = \eta_{BC} \equiv \eta)\), the largest value of \(\eta\) that admits a non-negative value is \(\eta = 2/3\). Hence, local-positivity limits the joinability of these Werner operators to a maximum of \(\eta = 2/3\). The operational interpretation of this result deserves attention. Consider local projective measurements made on each of three qubit systems, and consider the three systems to have a collective unitary symmetry, in the sense that there are no preferred local bases. In our general picture, the systems need not be three distinct systems—they may also be systems at two different points in time. Then, local positivity enforces the rule that ‘all probabilities arising from such measurements must be non-negative’. In the example above (i.e. \(\eta_{AB} = \eta_{AC} = \eta_{BC}\)), this implies that the three equal correlations (as measured by the \(\eta_{s}\)) can never exceed 2/3. The classical joinability constraints are derived from projectors which are parallel or orthogonal to one another. Thus, the ‘relative obliqueness’ of the three projectors in this example is what allows for the local-positive constraint to be stricter than classical constraints. Notwithstanding, both limitations express a tetrahedral symmetry, that is, symmetry with respect to individually inverting two axes. In contrast, the state-joining and channel-joining scenarios manifest a preference towards the negative axis (anticorrelation) and the positive axis (correlation) of the \(\eta_{s}\), respectively.

Before concluding this section, we connect the above discussion to the relationship between local-positivity and separability. As mentioned earlier, the cone of local positive operators and the cone of separable operators are dual to one another. The operator subspace we are dealing with is spanned by the orthonormal operators \(\frac{1}{\sqrt{6}}(V_{AB} - \mathbb{I}/2)\), \(\frac{1}{\sqrt{6}}(V_{AC} - \mathbb{I}/2)\), and \(\frac{1}{\sqrt{6}}(V_{BC} - \mathbb{I}/2)\) with coordinates \(\frac{\sqrt{3}}{\sqrt{2}}\eta_{AB}\), \(\frac{\sqrt{3}}{\sqrt{2}}\eta_{AC}\), and \(\frac{\sqrt{3}}{\sqrt{2}}\eta_{BC}\), respectively. In theorem 3.4, we determined the algebraic surface bounding the local positive operators; hence, the dual to this surface will bound the separable operators within this space. The dual to the Roman surface is known as Cayley’s cubic surface \[51\], which, for a given scale parameter \(w\) is characterized by
We first set $x = \sqrt{\frac{2}{3}} \eta_{AB}$, $y = \sqrt{\frac{2}{3}} \eta_{AC}$, and $z = \sqrt{\frac{2}{3}} \eta_{BC}$. Then we must set $w$ so that the Cayley surface delimits the separable states. For each extremal separable state in our space, there is a corresponding local-positive operator acting as an entanglement witness; a state is separable if the inner product with its entanglement witness is nonnegative.

Consider the extremal local-positive operator $\eta_{AB} = \eta_{AC} = \eta_{BC} = 2/3$ that we made use of previously. This operator will act as an entanglement witness for another operator with $\eta_{AB} = \eta_{AC} = \eta_{BC} = \sigma$. We obtain $\sigma$ by solving

$$\begin{vmatrix} w & x & y \\ x & w & z \\ y & z & w \end{vmatrix} = 0.$$ 

We first set $x = \sqrt{\frac{2}{3}} \eta_{AB}$, $y = \sqrt{\frac{2}{3}} \eta_{AC}$, and $z = \sqrt{\frac{2}{3}} \eta_{BC}$. Then we must set $w$ so that the Cayley surface delimits the separable states. For each extremal separable state in our space, there is a corresponding local-positive operator acting as an entanglement witness; a state is separable if the inner product with its entanglement witness is nonnegative.

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$$\begin{pmatrix} 1 \\ \sqrt{\frac{8}{3}} \\ \sqrt{\frac{8}{3}} \\ \sqrt{\frac{8}{3}} \\ \sqrt{\frac{8}{3}} \\ \sqrt{\frac{8}{3}} \end{pmatrix} = 0,$$

$$\begin{pmatrix} 1 \\ \sqrt{\frac{8}{3}} \\ \sqrt{\frac{8}{3}} \\ \sqrt{\frac{8}{3}} \\ \sqrt{\frac{8}{3}} \\ \sqrt{\frac{8}{3}} \end{pmatrix} = 0,$$

We first set $x = \sqrt{\frac{2}{3}} \eta_{AB}$, $y = \sqrt{\frac{2}{3}} \eta_{AC}$, and $z = \sqrt{\frac{2}{3}} \eta_{BC}$. Then we must set $w$ so that the Cayley surface delimits the separable states. For each extremal separable state in our space, there is a corresponding local-positive operator acting as an entanglement witness; a state is separable if the inner product with its entanglement witness is nonnegative.

Consider the extremal local-positive operator $\eta_{AB} = \eta_{AC} = \eta_{BC} = 2/3$ that we made use of previously. This operator will act as an entanglement witness for another operator with $\eta_{AB} = \eta_{AC} = \eta_{BC} = \sigma$. We obtain $\sigma$ by solving

$$\begin{pmatrix} \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} \end{pmatrix} = 0,$$

$$\begin{pmatrix} \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} \end{pmatrix} = 0,$$

We first set $x = \sqrt{\frac{2}{3}} \eta_{AB}$, $y = \sqrt{\frac{2}{3}} \eta_{AC}$, and $z = \sqrt{\frac{2}{3}} \eta_{BC}$. Then we must set $w$ so that the Cayley surface delimits the separable states. For each extremal separable state in our space, there is a corresponding local-positive operator acting as an entanglement witness; a state is separable if the inner product with its entanglement witness is nonnegative.

Consider the extremal local-positive operator $\eta_{AB} = \eta_{AC} = \eta_{BC} = 2/3$ that we made use of previously. This operator will act as an entanglement witness for another operator with $\eta_{AB} = \eta_{AC} = \eta_{BC} = \sigma$. We obtain $\sigma$ by solving

$$\begin{pmatrix} \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} \end{pmatrix} = 0,$$

$$\begin{pmatrix} \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} \end{pmatrix} = 0,$$

We first set $x = \sqrt{\frac{2}{3}} \eta_{AB}$, $y = \sqrt{\frac{2}{3}} \eta_{AC}$, and $z = \sqrt{\frac{2}{3}} \eta_{BC}$. Then we must set $w$ so that the Cayley surface delimits the separable states. For each extremal separable state in our space, there is a corresponding local-positive operator acting as an entanglement witness; a state is separable if the inner product with its entanglement witness is nonnegative.

Consider the extremal local-positive operator $\eta_{AB} = \eta_{AC} = \eta_{BC} = 2/3$ that we made use of previously. This operator will act as an entanglement witness for another operator with $\eta_{AB} = \eta_{AC} = \eta_{BC} = \sigma$. We obtain $\sigma$ by solving

$$\begin{pmatrix} \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} \end{pmatrix} = 0.$$
This inequality may also be obtained using theorem 1 in [52]. The shape of the separable states is depicted in figure 7.

Several remarks may be made. First, the set of separable states exhibits the tetrahedral symmetry shared by the classical joinability boundary and the local-positive joinability boundary. Thus, among the various boundaries we have considered in this three dimensional Euclidean space, the state- and channel-positive boundaries are the only ones not obeying tetrahedral symmetry. However, both the convex hull and the intersection of the state- and channel-positive cones bound regions which recover this symmetry. Indeed, it is a curious observation that the convex hull of these cones is ‘nearly’ the local-positive region, while the intersection is ‘nearly’ the set of separable states. Earlier we found, in the two-qudit case, that local-positivity coincides with the union of the state- and channel-positive regions, as well as that the separable region was their intersection. Here we consider the analog for three qubits. The result is that (i) the convex hull of state- and channel-positive operators is strictly contained in the set of local-positive operators; and (ii) the intersection of the state- and channel-positive operators is strictly contained in the set of separable states.

We may further interpret the latter result in terms of PPT considerations. The operators which result from a homocorrelation-mapped channel are necessarily PPT. Corollary 1 in [52] states that Werner operators which are positive and PPT are bi-separable. Thus, any state-positive operator which is also a homocorrelation mapped channel is necessarily bi-separable. Hence, the intersection of the four cones will be the set of states which are bi-separable with respect to any of the three partitions. This set clearly contains the set of tri-separable states. These observations illuminate the relationships among entanglement, quantum states, and quantum channels. Specifically, the homocorrelation map allows us to place quantum channels in the same arena as quantum states, and hence to directly compare and contrast them. Finding that the tri-separable operators are a proper subset of the bi-separable ones, we wonder what features these strictly bi-separable operators possess, and what does bi-separability imply for the states or channels supporting such correlations.

4. Agreement bounds for quantum states and channels

In what remains, we illustrate some crucial differences between channel- and state-positive operators, which also inform the nature of their respective joinability limitations. We focus here on operators in \( B(H_d \otimes H_d) \). Qualitatively, state-positive operators are restricted in the degree to which they can bear agreeing outcomes, whereas channel-positive operators are restricted in the degree to which they can bear disagreeing outcomes. To quantify this, we define the degree of agreement to be the likelihood of a certain locally realizable (hence separable) POVM element. Consider a local projective measurement \( M = \{|ij\rangle \langle ij|\} \). We can coarse-grain this into a two-element projective measurement with the bipartition into ‘agreeing’ outcomes, \( E_A = \sum |ii\rangle \langle ii| \), and ‘disagreeing’ outcomes, \( E_D = \sum_{\neq} |ij\rangle \langle ij| \), respectively. Lastly, so that these outcomes are basis-independent, we can ‘twirl’ \( E_A \) and \( E_D \) as follows:

\[
1 + 54\eta_{AB}\eta_{AC}\eta_{BC} - 9\left(\eta_{AB} + \eta_{AC} + \eta_{BC}\right)^2 + 18\left(\eta_{AB}\eta_{AC} + \eta_{AB}\eta_{BC} + \eta_{AC}\eta_{BC}\right) \geq 0.
\]
\[ E_A = \int d\mu(U) U \otimes U \left( \sum_i |ii\rangle \langle ii| \right) U^\dagger \otimes U^\dagger, \]
\[ E_D = \int d\mu(U) U \otimes U \left( \sum_{i \neq j} |jj\rangle \langle jj| \right) U^\dagger \otimes U^\dagger, \]

where \( d\mu(U) \) denotes integration with respect to the invariant (Haar) measure. It is simple to see that these two operators yield a resolution of identity and hence form a POVM. We can compute these two operators explicitly as follows. By the invariance of the Haar measure, we can rewrite \( E \) as

\[ E_A = d \int d\mu(\psi) |\psi\rangle \langle \psi| \otimes 1. \]

for which the above integral is proportional to the projector onto \( \mathcal{H}_+^2 \subset \mathcal{H}^2 \) of dimension \( d^2 \) \cite{53}. Explicitly:

\[ E_A = \frac{d d^2}{2}, \quad d_2^+ = \left( \frac{2 + d - 1}{2} \right) = \frac{d(d+1)}{2}, \] (17)

\[ E_D = 1 - E_A. \] (18)

We define the degree of agreement to be the likelihood of \( E_A \) and, similarly, the degree of disagreement to be the likelihood of \( E_D \). Operationally, these values are the probability that, for a randomly chosen local projective measurement made collectively, the local outcomes will agree or, respectively, disagree. It is worth noting that the normalization of the operator \( E_A \), which is determined by the above operational construction, is crucial in making the POVM \( \{ E_A, E_D \} \) locally realizable, as demanded for a consistent comparison with a classical scenario \(^7\).

We proceed to show how quantum channels differ from quantum states in their allowed range of agreement likelihood. In the case of a bipartite density operator, \( \rho \in B(\mathcal{H}_d \otimes \mathcal{H}_d) \), we are familiar with computing this agreement probability as \( tr(E_A \rho) \). To carry out the same computation for a channel operator, the homocorrelation map becomes expedient. Given a quantum channel \( \mathcal{M}: B(\mathcal{H}_d) \rightarrow B(\mathcal{H}_d) \), we wish to determine the probability that the outcome of a randomly chosen projective measurement (made on the completely mixed state) will agree with the outcome of the same measurement after the application of \( \mathcal{M} \). Assume the outcome was \( |i\rangle \) from an orthogonal basis \( \{|i\rangle\} \). Then the post-channel state is \( \mathcal{M} |i\rangle \langle i| \), and the likelihood that the post-channel measurement will also correspond to \( |i\rangle \) is \( E_A = E_D \). Lastly, if we want to average this likelihood of agreement over all choices of basis, we integrate with respect to the Haar measure:

\(^7\) Our POVM is in fact on the cusp of being non-locally realizable. Consider the adapted POVM \( \{ E_A \equiv p E_A, E_B = 1 - E_A \} \), with \( 0 \leq p \leq (d+1)/2 \) to ensure positivity of the POVM elements. Then one may see that \( E_B \) would be unseparable for \( p > 1 \) for all \( d \); and non-separability of any POVM element implies the measurement is not locally realizable \cite{54}. The inequality follows from relating \( E_B \) to a Werner operator of the form given in equation (13), \( E_B = (1 - \frac{p}{d+1}) I - \frac{p}{d+1} V \). Separability is determined by the ratio of the coefficients in front of \( I \) and \( V \). We obtain the bounds for this ratio, and hence on \( p \), by equating the \( p \)-involved ratio to the \( \eta \)-involved ratio from equation (13) and rewriting the separability inequalities for \( \eta \rightarrow 1/(d^2-1) \leq \eta \leq 1/(d+1) \), in terms of \( p \).
\[ p(\text{agree}) = \int d\mu(U) \text{tr} \left( \mathcal{M}(U |i\rangle \langle i| U^\dagger)U |i\rangle \langle i| U^\dagger \right) \]
\[ = \int d\mu(\psi) \text{tr} \left( \mathcal{M}(|\psi\rangle \langle \psi| |\psi\rangle \langle \psi| \right). \]

If we wish to find the bounds on this value, the above form does not make transparent the fact that we are performing an optimization problem in a convex cone. But, recalling the namesake property of the homocorrelation map, equation (6), we may rewrite

\[ p(\text{agree}) = \text{tr} \left[ \mathcal{H}(\mathcal{M})d \int d\mu(\psi)|\psi\rangle \langle \psi| \otimes |\psi\rangle \langle \psi| \right] = \text{tr} \left[ \mathcal{H}(\mathcal{M})E_A \right]. \]

Accordingly, the likelihood of agreement is calculated for channel operators in the homocorrelation representation in just the same way as it is for bipartite density operators.

With the stage set, the desired bounds are described in the following theorem:

**Theorem 4.1.** Let \( w \) be an operator in \( \mathcal{B}(\mathcal{H}_d \otimes \mathcal{H}_d) \), and consider a POVM with operation elements \( \{E_A, E_D\} \) as in equations (17)–(18). Then the degree of agreement for \( w \geq_{st} 0 \) as calculated by the likelihood of \( E_A \) is bounded by

\[ 0 \leq \text{tr} \left( wE_A \right) \leq \frac{d}{d^*_+} = \frac{2}{d + 1}, \]

while the degree of agreement for \( w \geq_{ch} 0 \) is bounded by

\[ \frac{1}{d + 1} \leq \text{tr} \left( wE_A \right) \leq 1. \]

**Proof.** For the state-positive case, \( w \) ranges over positive normalized operators and hence the bounds of \( \text{tr} (wE_A) \) are determined by the eigenvalues of \( E_A \). Since \( E_A \) is proportional to a projector (see equation (17)), we have that \( 0 \leq \text{tr} (wE_A) \leq d/d^*_+ \).

In the channel-positive case, \( w \) ranges over normalized operators that are positive under partial transposition. Partial transpose preserves trace. Thus, evaluating the bounds of \( \text{tr} (wE_A) \) over normalized channel-positive \( w \) is equivalent to evaluating the bounds of \( \text{tr} (wT_A E_A) \) for a density operator \( \rho = wT_A \). As before, the bounds are thus determined by extremal eigenvalues, only this time for \( E_A T_A \). Explicitly, we have

\[ \frac{d}{d^*_+} = \frac{1}{d + 1} + \frac{V_{T_A}}{2}. \]

Using the fact that \( V_{T_A}/d \) is a projector, we obtain the bounds of \( \frac{d}{d^*_+} = \text{tr} (wE_A) \leq \frac{d}{d^*_+} + \frac{d}{2} \), which simplify to those of equation (4.1).

By virtue of the homocorrelation map, the above result may be understood geometrically. The objects involved are the agreement/disagreement POVM operators \( E_A \) and \( E_D \), and the state- and channel-positive cones \( W_{st} \) and \( W_{ch} \), respectively. Theorem 4.1 places an upper bound on the inner product between trace-one operators in \( W_{st} \) and \( E_A \), and, similarly, on the inner product between trace-one operators in \( W_{ch} \) and \( E_D \). This geometric understanding is further aided by the example of Werner operators shown in figure 2.

We have constructed the above quantum degree of agreement so that it is subject to the classical joinability limitations of equations (9)–(12). As discussed previously, choosing a
local basis \(|\langle ii\rangle\rangle\), we associate any tripartite density operator to a probability distribution via
\[ w_{ijk} \Rightarrow p(ijk) \equiv \text{tr}(w_{ABC} \langle ijk\rangle\langle ijk\rangle). \]
Then, for each pair of systems we obtain the corresponding classical degree of agreement as
\[ \sum_{ijk} p(ijk) = \text{tr}(\rho_{AB} E_A), \quad \text{etc.} \]
Since \( \text{tr}(\rho_{AB} E_A), \quad \text{tr}(\rho_{AC} E_A), \quad \text{and} \quad \text{tr}(\rho_{BC} E_A) \) must satisfy equations (9)–(12), by linearity, so must their averages with respect to all local basis choices. Thus, we obtain the necessary condition that \( \rho_{AB}, \quad \rho_{AC}, \quad \text{and} \quad \rho_{BC} \) are joinable only if
\[ \text{tr}(\rho_{AB} E_A), \quad \text{tr}(\rho_{AC} E_A), \quad \text{and} \quad \text{tr}(\rho_{BC} E_A) \]
satisfy equations (9)–(12).

Theorem 4.1 may be combined with the fact that the quantum degrees of agreement are subject to classical joinability bounds to derive some general joinability limitations. Specifically, we recover the symmetric qudit cloning bound and the qubit Werner 1–2 sharability bound (see [6]), implying a common cause for these distinct notions. Symmetric quantum cloning and 1–2 sharability both impose
\[ \rho_{AB} \equiv \text{tr}(E_{AB}^A), \quad \rho_{AC} \equiv \text{tr}(E_{AC}^A), \quad \text{and} \quad \rho_{BC} \equiv \text{tr}(E_{BC}^A) \]
satisfy equations (10)–(13) because the POVMs are realized locally. For symmetric quantum cloning, we insert the quantum degrees of agreement into the classical bound of equation (11),
\[ \text{tr}(\rho_{AB} E_A) + \text{tr}(\rho_{AC} E_A) - \text{tr}(\rho_{BC} E_A) \leq 1. \]
Saturating \( \text{tr}(\rho_{BC} E_A) \leq \frac{2}{d+1} \) and imposing B–C symmetry gives
\[ \alpha \leq \frac{d + 3}{2(d + 1)}, \]
which corresponds precisely to the optimal bound for qudit cloning [55] (cf equation (21) therein, where their \( F \) coincides with our \( \alpha \)). For 1–2 sharability of qubit Werner states, equation (12) applies. Following the above approach, we obtain \( \alpha \geq \frac{1}{5} \), which is the exact condition established in theorem 6 of [6] (recall that \( \alpha = (\Psi + 1)/3 \)).

While obtaining a full generalization of theorem 4.1 to multiparty systems would entail a detailed understanding of representation theory for Brauer algebras which is beyond our current purpose, a partial generalization is nevertheless possible. Let the relevant POVM be defined in analogy to equations (17)–(18), namely
\[ E_A = \frac{d}{d_N} 1_N, \quad d_N^+ = \binom{N + d - 1}{N}, \quad E_D = I - E_A. \]
We then have the following:

**Theorem 4.2.** Let \( w \in B(\mathcal{H}^N) \), and consider a POVM with operation elements \( \{E_A, E_D\} \) as defined above. Then the degree of agreement for \( w \geq_\alpha 0 \) as calculated by the likelihood of \( E_A \) is bounded by
\[ 0 \leq \text{tr}(wE_A) \leq \frac{d}{d_N} = \frac{d}{d - 1 + N}. \]

**Proof.** As before, since the extremization is taken over normalized positive operators \( w \), the bounds of \( \text{tr}(wE_A) \) are achieved by the extremal eigenvalues of \( E_A \). Writing \( E_A = d/d_N^+ \), we see that the eigenvalues, and hence the bounds, are as given. \( \square \)
From the above multiparty bound, one may attempt to recover, for instance, the known bounds on 1-n sharability of Werner states [6]. However, thus far we have not been successful in this endeavor. In the tripartite qudit setting, such bounds were found to be sufficient, but this might be a special feature of this particular case. Therefore, it remains an open question to determine whether there exists a simple principle (or simple principles) which govern joinability limitations beyond the tripartite setting.

5. Conclusion

In this paper we have developed a unifying framework for the concept of quantum joinability. Many problems regarding the part–whole relationship in multiparty quantum settings, such as the quantum marginal problem, the asymmetric cloning problem, and various quantum extension problems, are encapsulated by this framework. An important step was to revisit the Jamiołkowski isomorphism and recognize that a simple (rescaled) variant, which we have termed the homocorrelation map, provides a natural way to represent quantum channels with bipartite operators, making them statistically and geometrically comparable to quantum states. Using this tool, it is possible to directly contrast the joinability properties of quantum states with those of quantum channels. In particular, applying the general framework to the simple tractable case of $U \otimes U \otimes U$-invariant operators, we found that the state and channel joinability bounds work in tandem to exhibit the symmetry inherent in the limitations of classical joinability. In addition, we derived the local-positivity joinability bounds in this setting. Though less strict than state- or channel-joinability bounds, we found that the local positivity joinability bounds are still more strict than purely classical ones, and provided an operational interpretation of this fact.

The Choi isomorphism illuminates a duality and hence a similarity between bipartite quantum states and quantum channels. In contrast, a main point of this work was to highlight a crucial difference between states and channels which manifests in the statistical (measurement) correlations that are obtainable from each. Namely, bipartite quantum states are limited in their possible degree of agreement, whereas quantum channels are limited in their possible degree of disagreement, defined in terms of an appropriate, locally realizable POVM. Again, this difference is made explicit by representing quantum channels with the homocorrelation map. We showed how these differences, expressed in terms of agreement bounds, in turn inform the joinability properties of channels versus states. In view of their general nature, both the homocorrelation representation and these agreement bounds may have further implications yet to be discovered.

In closing, we note that throughout our analysis we have only considered scenarios with a pre-defined tensor product structure, and consequently all operator reductions are obtained via the usual partial-trace construction. However, it is important to appreciate that this was not a necessary restriction. Following [56], one may also consider a more general notion of a reduced state, which results from appropriately restricting the global state, viewed as a linear functionals on operators, to a distinguished operator subspace. Such a notion of reduction is operationally motivated in situations where a tensor product structure is not uniquely or naturally afforded on physical grounds (notably, systems of indistinguishable particles or operational quantum theory, see e.g. [57]). This naturally points to a further extension of the present joinability framework ‘beyond subsystems’, which we plan to address in future investigation.
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Appendix. Local positivity of Werner operators

We present here a detailed proof of theorem 3.4. As mentioned before corollary 3.3 in the main text, the first step is to realize that, thanks to the symmetry properties of Werner operators, it suffices to restrict the tripartite joining operators to tripartite Werner operators. An arbitrary tripartite Werner operator may be parametrized as

\[ w = aI + bV_{AB} + cV_{AC} + dV_{BC} + e(V_{ABC} + V_{CBA})/2 + f(V_{ABC} - V_{CBA})/2, \]

where \( a, b, c, d, e, f \in \mathbb{R} \) and normalization is left arbitrary for now. However, in the two-dimensional case, the six permutation representation operators are not independent, since \( I - (V_{AB} + V_{AC} + V_{ABC}) + V_{ABC} + V_{CBA} = 0 \). Consequently, we may absorb the \( V_{ABC} + V_{CBA} \) contribution into the first four terms, leaving us with

\[ w = aI + bV_{AB} + cV_{AC} + dV_{BC} + f(V_{ABC} - V_{CBA})/2. \]

With \( |\psi_{\text{loc}}\rangle \equiv |\psi_1\rangle |\psi_2\rangle |\psi_3\rangle \), local positivity of \( w \) is guaranteed by \( \langle \psi_{\text{loc}} | w |\psi_{\text{loc}}\rangle \geq 0 \), holding for all \( |\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle \in \mathcal{H} \). Writing

\[ \langle \psi_{\text{loc}} | w |\psi_{\text{loc}}\rangle = a + b |\langle \psi_1 | \psi_2 \rangle|^2 + c |\langle \psi_1 | \psi_3 \rangle|^2 + d |\langle \psi_2 | \psi_3 \rangle|^2 + f \text{ Im} \left( \langle \psi_1 | \psi_2 \rangle \langle \psi_2 | \psi_3 \rangle \langle \psi_3 | \psi_1 \rangle \right) \geq 0, \]

each choice of \( |\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle \) enforces a linear inequality on \( a, b, c, d, e, f \). However, certain \( |\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle \) may result in an inequality whose satisfaction is guaranteed by a stricter inequality corresponding to a different set of product vectors. For each choice of \( a, \ldots, f \), there will be an extremal (set of) product vector(s) \( |\psi_{\text{loc}}'\rangle \) for which \( \langle \psi_{\text{loc}}' | w |\psi_{\text{loc}}'\rangle \geq 0 \) implies \( \langle \psi_{\text{loc}} | w |\psi_{\text{loc}}\rangle \geq 0 \) for all \( |\psi_{\text{loc}}\rangle \). We seek to obtain such extremal product vectors, and write their inner products (e.g. \( |\langle \psi_1 | \psi_2 \rangle|^2 \), etc) in terms of \( a, \ldots, f \).

For Werner states, the local-positivity condition is invariant under a collective unitary transformation of \( |\psi_{\text{loc}}\rangle \). Such a transformation corresponds to a rotation on the Bloch sphere. Thus, given \( |\psi_1\rangle |\psi_2\rangle |\psi_3\rangle \), we may perform a collective unitary which takes this state to \( |1\rangle_z \otimes (\cos \theta |1\rangle_z + \sin \theta |1\rangle_z) \otimes (\cos \Omega |1\rangle_z + e^{i\phi} \sin \Omega |1\rangle_z) \). Without loss of generality, this will be our representative \( |\psi_{\text{loc}}\rangle \). This allows us to rewrite the expression of local-positivity as

\[ F = a + b \cos^2 \theta + c \cos^2 \Omega + d \left( \cos^2 \theta \cos^2 \Omega + \frac{1}{2} \cos \phi \sin 2\theta \sin 2\Omega + \sin^2 \theta \sin^2 \Omega \right) + f \sin \phi \sin 2\theta \sin 2\Omega \geq 0. \]

Our goal is to determine the set of bipartite Werner operator trios that can be joined by a local-positive state \( w \). These reduced states on \( A-B, A-C, \) and \( B-C \) are each characterized by the
single parameter $\alpha_{AB} = \text{tr} (V_{AB} w)$, $\alpha_{AC} = \text{tr} (V_{AC} w)$, and $\alpha_{BC} = \text{tr} (V_{BC} w)$, respectively. In the next step, we show that if the local-positive state $w$ joins reduced Werner states with $\alpha_{AB}$, $\alpha_{AC}$, and $\alpha_{BC}$, then $w' = w|_{f=0}$ is local-positive and also joins them.

First, note that the bipartite reduced states $V_{w_{AB}}$, etc, do not depend on $f$; hence, if three bipartite states are local-positive-joinable by some $w$ with $f \neq 0$, then $w' = w|_{f=0}$ will reduce to the same bipartite states as $w$. It remains to show that $w'$ is local-positive. Specifically, we want to show that if $F \geq 0$ for all $\theta, \Omega, \phi$, then $F(f = 0) \geq 0$ for all $\theta, \Omega, \phi$. This follows from the fact that, independent of all else, the factor of $\sin \phi$ may determine the sign of its corresponding term; thus, for a given $a, ..., f$, the angles which minimize $F$ must be such that the term containing $f$ is non-positive. In this case, setting $f = 0$ cannot decrease $F$.

We have thus shown that a sufficient joining state is of the form

$$w = a \mathbf{1} + b V_{AB} + c V_{AC} + d V_{BC},$$

and, in terms of the parameterization of the product state $|\psi_{\text{loc}}\rangle$, local positivity is ensured by requiring that

$$F = a + b \cos^2 \theta + c \cos^2 \Omega + d \cos^2 \Omega$$

$$+ \left( \cos^2 \theta \cos^2 \Omega + \frac{1}{2} \cos \phi \sin 2\theta \sin 2\Omega + \sin^2 \theta \sin^2 \Omega \right) \geq 0,$$

for all $\theta, \Omega \in [0, \pi], \phi \in [0, 2\pi]$. It remains to determine the extremal angles $\theta, \Omega$, and $\phi$, for a given $a, b, c, d$. With respect to the $\phi$ dependence, $F$ is extremized by setting $\cos \phi = \pm 1$, which determines the sign of the corresponding term. However, the sign of this term is also determined by the sign of $\theta$ or $\Omega$, which does not alter the remainder of the expression for $F$. Thus, we absorb this choice of $\cos \phi = \pm 1$ into the sign of $\theta$, say. This allows us to further simplify our expression to

$$F = a + b \cos^2 \theta + c \cos^2 \Omega + d \cos^2 (\theta - \Omega).$$

The interpretation of this simplification is that it suffices to consider states $|\psi_1\rangle$, $|\psi_2\rangle$, $|\psi_3\rangle$ all lying in an equatorial plane of the Bloch sphere. We make a final simplification by enforcing the normalization $\text{tr} (w) = 1$. This removes $a$ by $a = \frac{1}{8} - \frac{1}{2} (b + c + d)$, giving

$$F = \frac{1}{2} \left( \frac{1}{4} + b \cos \theta + c \cos \Omega + d \cos (\theta + \Omega) \right),$$

(A.1)

where we have replaced $2\theta \rightarrow \theta$ and $-2\Omega \rightarrow \Omega$ without loss of generality.

Now in order to find the desired extremal inequalities, we take partial derivatives with respect to the remaining two angles, namely:

$$\frac{\partial F}{\partial \theta} = -b \sin (\theta) - d \sin (\theta + \Omega) = 0,$$

$$\frac{\partial F}{\partial \Omega} = -c \sin (\Omega) - d \sin (\theta + \Omega) = 0.$$

Assuming $b, c, d \neq 0$, the zeros of the gradient of $F$ are given by either

$$\sin \theta = \sin \Omega = \sin (\theta + \Omega) = 0,$$

or

$$\frac{b}{d} = -\frac{\sin (\theta + \Omega)}{\sin \theta}, \quad \frac{c}{d} = -\frac{\sin (\theta + \Omega)}{\sin \Omega}.$$

The first set of solutions correspond to $\theta = n\pi$ and $\Omega = m\pi$. There are four inequalities derived from these
\[
\frac{1}{4} + b + c + d \geq 0, \quad \frac{1}{4} + b - c - d \geq 0, \quad (A.2)
\]
\[
\frac{1}{4} - b + c - d \geq 0, \quad \frac{1}{4} - b - c + d \geq 0. \quad (A.3)
\]
Satisfaction of these is certainly necessary for \( w \) to be locally positive, but it is not sufficient.

Although they do not minimize \( F \) for all \( b, c, d \), the solutions \( \sin \theta = \sin \Omega = \sin (\theta + \Omega) = 0 \) allow us to obtain four equalities
\[
\cos x = \frac{\cos x \sin y}{\sin y} = \frac{\sin (x + y) - \sin (x - y)}{2 \sin y},
\]
\[
\sin (x + y) \sin (x - y) = \sin^2 x - \sin^2 y.
\]
Putting these together we have
\[
\cos x = \frac{1}{2} \left[ \sin (x + y) + \sin (x - y) - \frac{\sin^2 x}{\sin (x + y)} \right].
\]
Thus, we can write each of the \( \cos \) terms in terms of \( b, c, d \)
\[
\cos \theta = \frac{1}{2} \left[ \frac{c}{d} + \frac{d}{c} - \frac{cd}{b^2} \right],
\]
\[
\cos \Omega = \frac{1}{2} \left[ \frac{b}{d} + \frac{d}{b} - \frac{bd}{c^2} \right],
\]
\[
\cos (\theta + \Omega) = \frac{1}{2} \left[ b + c + \frac{bc}{d^2} \right].
\]
Substituting these into equation \( (A.1) \), we obtain
\[
\frac{1}{2} - \frac{(bc)^2 + (bd)^2 + (cd)^2}{bcd} \geq 0 \quad (A.4)
\]
as the remaining necessary condition for local positivity. The above condition, along with equations \( (A.2) \)–\( (A.3) \) ensure the local positivity of the relevant states. As a final step, note that the Werner parameters of equation \( (13) \) are related to \( b, c, d \) via
\[
b = \frac{1}{4} \eta_{AB}, \quad c = \frac{1}{4} \eta_{AC}, \quad d = \frac{1}{4} \eta_{BC}.
\]
Upon re-expressing equations \( (A.2) \)–\( (A.4) \) in terms of Werner parameters \( \eta_s \), the result quoted in theorem 3.4 is established. \( \square \)

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