Solutions of Dirac Equation for Symmetric Generalized Woods-Saxon Potential by the Hypergeometric Method

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Abstract

The Dirac equation is solved approximately for the Hulthen potential with the pseudospin symmetry for any spin-orbit quantum number \( \kappa \) in the position-dependent mass background. Solutions are obtained reducing the Dirac equation into a Schrödinger-like differential equation by using an appropriate coordinate transformation. The Nikiforov-Uvarov method is used in the calculations to get energy eigenvalues and the corresponding wave functions.

Keywords: Hypergeometric method, Woods-Saxon potential, Dirac equation

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I. INTRODUCTION

In the past few years there has been considerable work on non-Hermitian Hamiltonians. Among this kind of Hamiltonians, much attention has been focused on the investigation of properties of so-called $\mathcal{PT}$-symmetric Hamiltonians. Following the early studies of Bender et al. [1], the $\mathcal{PT}$-symmetry formulation has been successfully utilized by many authors [2-8]. The $\mathcal{PT}$-symmetric but non-Hermitian potentials could have real spectra even it is non-Hermitian. Non-Hermitian Hamiltonians with real or complex spectra have also been analyzed by using different methods [3-6,9]. Non-Hermitian but $\mathcal{PT}$-symmetric models have applications in different fields, such as nuclear physics [10], condensed matter [11] and population biology [12].

When a particle is in a strong potential field, the relativistic effect must be considered, which gives the correction for non-relativistic quantum mechanics [13]. Taking the relativistic effects into account, a particle in a potential field should be described with the Klein-Gordon (KG) and Dirac equations. In recent years, there has been an increased interest in finding exact solutions to Schrödinger, K-G, Dirac and Salpeter equations for various potential schemes [13–27]. The problems that can be exactly solved for the KG and/or Dirac equations are seldom except a few examples, such as hydrogen atom and electrons in a uniform magnetic field. Recently some authors solved such relativistic equations for some potentials. Şimşek and Eğrifes [15] have presented the bound-state solutions of the one-dimensional ($1D$) Klein-Gordon (KG) equation for $\mathcal{PT}$-symmetric potentials with real and complex generalized Hulthén potential. Moreover, Eğrifes and Sever [16] investigated the bound state solutions of the $1D$ Dirac equation with $\mathcal{PT}$-symmetric real and complex forms of generalized Hulthén potential. Yi et al. [19] obtained the energy equations in the KG theory with equally mixed vector and scalar Rosen-Morse-type potentials. In recent works, we have solved the spinless $1D$ Salpeter equation analytically for its exact bound state spectra and wavefunctions with real and complex forms of the $\mathcal{PT}$-symmetric generalized Hulthén potential [20]. We have also investigated the bound state solutions of the $1D$ KG equation with real and complex forms of the generalized Woods-Saxon (WS) potential [21].

In the present work, we present a new procedure to construct solution to the $(1+1)$ dimensional Dirac equation with gauge invariant (minimal) vector coupling. Using the Nikiforov-
Uvarov (NU) method [28], we are set to obtain the bound states solutions (relativistic energy spectrum and the two-component spinor wave functions) for particles trapped in a spherically symmetric, generalized Woods-Saxon potential [21,22] which possesses a \( \mathcal{PT} \)-symmetry as well. Changing the values of the potential parameters from real to pure imaginary, we obtain Hamiltonians that may or may not be \( \mathcal{PT} \)-symmetric. Further, by making the same parameter change in the energy spectrum and two-components spinor wave functions, we obtain solutions for the new trigonometric and periodic potential (real or complex) forms. The corresponding energy spectra are either real or complex. The Dirac problem with a \( \mathcal{PT} \)-symmetric or a non-\( \mathcal{PT} \)-symmetric imaginary generalized Woods-Saxon (WS) potential is mapped into the exactly solvable problem in which one may apply the NU method to generate possible real energy spectra [28].

The paper is structured as follows: In Section II we briefly introduce the basic concepts of the hypergeometric method (NU). Section III is devoted to the solution of the Dirac problem to obtain the exact bound state energy spectrum for real and complex cases of generalized WS potentials and the lower and upper spinor components eigenfunction by applying the NU method. The results of this relativistic study are discussed in Section ??.

II. THE HYPERGEOMETRIC METHOD

The basic equations of the theoretical background are similar to those given in [15,16,20-25]. According to a brief description of this method [28], the Dirac equation can be transformed to the following generalized equation of hypergeometric type after employing an appropriate coordinate transformation, \( s = s(r) \),

\[
\psi''_n(s) + \frac{\tau(s)}{\sigma(s)} \psi'_n(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)} \psi_n(s) = 0, \tag{1}
\]

where \( \sigma(s) \) and \( \tilde{\sigma}(s) \) are polynomials, at most of second-degree, and \( \tau(s) \) is a first-degree polynomial. In order to find a particular solution to Eq. (1), we use the following wave function

\[
\psi_n(s) = \phi_n(s) y_n(s). \tag{2}
\]

This reduces Eq. (1) to an equation of a hypergeometric type
\[ \sigma(s)y''_n(s) + \tau(s)y'_n(s) + \lambda y_n(s) = 0, \quad (3) \]

which demands that the following conditions be satisfied:

\[ \sigma(s) = \pi(s) \frac{\phi(s)}{\phi'(s)}, \quad (4) \]
\[ \tau(s) = \tilde{\tau}(s) + 2\pi(s), \quad \tau'(s) < 0, \quad (5) \]

and \( \lambda \) is thus a new eigenvalue equation for the second-order differential equation becomes

\[ \lambda = \lambda_n = -n\tau'(s) - \frac{n(n-1)}{2}\sigma''(s), \quad n = 0, 1, 2, \ldots. \quad (6) \]

The polynomial \( \tau(s) \) with the parameter \( s \) and prime factors show the differentials at first degree be negative. It is worthwhile to note that \( \lambda \) or \( \lambda_n \) are obtained from a particular solution of the form \( y(s) = y_n(s) \) which is a polynomial of degree \( n \). The other part \( y_n(s) \) of the wavefunction (2) is the hypergeometric-type function whose polynomial solutions are given by Rodrigues relation

\[ y_n(s) = B_n \frac{d^n}{ds^n} [\sigma^n(s)\rho(s)], \quad (7) \]

where \( B_n \) is the normalization constant with the weight function \( \rho(s) \) must satisfy the following condition

\[ w'(s) - \left( \frac{\tau(s)}{\sigma(s)} \right) w(s) = 0, \quad w(s) = \sigma(s)\rho(s). \quad (8) \]

On the other hand, in order to find the eigenfunctions, \( \phi_n(s) \) and \( y_n(s) \) in Eqs. (4) and (7) and eigenvalues \( \lambda_n \) in Eq. (6), we need to calculate the functions:

\[ \pi(s) = \frac{\sigma'(s) - \tilde{\tau}(s)}{2} \pm \sqrt{\left( \frac{\sigma'(s) - \tilde{\tau}(s)}{2} \right)^2 - \tilde{\sigma}(s) + k\sigma(s)}, \quad (9) \]
\[ \lambda = k + \pi'(s). \quad (10) \]

In principle, since \( \pi(s) \) has to be a polynomial of degree at most one, the expression under the square root sign in Eq. (9) can be arranged to be the square of a polynomial of first degree [28]. This is possible only if its discriminant is zero. Thus, the equation for \( k \) obtained from the solution of Eq. (9) can be further substituted in Eq. (10). In addition, the energy eigenvalues are obtained from Eqs. (6) and (10).
III. SOLUTIONS OF THE QUANTUM SYSTEM

The exact solution of the relativistic wave equations is of much concern in quantum mechanics. Many works appeared in the recent years in that direction towards obtaining exact solution of some relativistic wave equations for certain potentials of physical interest (cf. [15,16,20-22] and references therein). For a spinless particle of rest mass $m$ and total energy $E_{nl}$, the 1D time-independent Dirac equation with any given interaction potential $V(x)$ in the vector coupling scheme (choosing the natural atomic units $\hbar = c = 1$) is

$$\begin{align*}
\left\{ i \frac{d}{dx} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + [E_{nl} - V(x)] \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \psi_{nq}(x) &= 0,
\end{align*}$$

where the spinor radial wave function $\psi_{nq}(x) = \begin{pmatrix} u_{nq}(x) \\ w_{nq}(x) \end{pmatrix}$ is normalized with $u_{nq}(x)$ ($w_{nq}(x)$), the upper (lower) spinor component (the $x$ range is from $-\infty$ to $\infty$ on the full-line problem). The interaction among nuclei is commonly described by using a potential which consists of the Coulomb and the nuclear potentials. It is usually taken in the form of Woods-Saxon (WS) potential. Here we take the following Hermitian real-valued 1D generalized WS potential which is specified by the shape (deformation) parameter, $q$, [22,32,33]

$$V_q(x) = -V_0 \frac{e^{-\alpha x}}{1 + q e^{-\alpha x}}, \quad \alpha = 1/a, \quad r - R_0 \to x, \quad q \geq 0, \quad R_0 \gg a,$n

where $r$ refers to the center-of-mass distance between the projectile and the target nuclei (the $r$ range is from 0 to $\infty$). The relevant parameters of the inter-nuclear potential are given as follows: $R_0 = r_0 A^{1/3}$ is to define the confinement barrier position value of the corresponding spherical nucleus or the width of the potential, $A$ is the target mass number, $r_0$ is the radius parameter, the field strength $V_0$ controls the barrier height of the Coulombic part, $a$ is the surface diffuseness parameter has to control its slope, which is usually adjusted to the experimental values of ionization energies. Note further, $q$ is a shape (deformation) parameter, the strength of the exponential part other than unity, set to determine the shape of potential and is arbitrarily taken to be a real constant within the potential. It should be noted that the spatial coordinates in the potential are not deformed and thus the potential still remains spherical.
It is worth to state that under radial coordinate transformation, \( r \rightarrow r + \Delta \), then the generalized WS potential in Eq. (12) changes into the standard WS potential \( (q = 1) \) but with the displacement parameter \( \Delta \) satisfies the expression \( \exp(\Delta/a) = q \) and with a field strength \( V_0' = V_0 \exp(-\Delta/a) \) [32]. The sense of generalization or deformation of the potential becomes clear. For completeness, it could be stated that if \( \Delta \) is positive (corresponding to \( q > 1 \)) then one may need to impose the condition on the choice of \( \Delta \), that is, \( |\Delta| \ll R_0 \).

Obviously, for some specific \( q \) values this potential reduces to the well-known types, such as for \( q = 0 \) to the exponential potential and for \( q = -1 \) and \( a = \delta^{-1} \) to the generalized Hulthén potential (cf. [20,34] and the references therein).

For the given generalized WS potential, Eq. (11) decomposes into the following two components:

\[
\left( \frac{i}{d} \frac{d}{dx} + E_{nq} + V_0 e^{-\alpha x} \right) u_{nq}(x) = mw_{nq}(x), \tag{13}
\]

\[
\left( -\frac{i}{d} \frac{d}{dx} + E_{nq} + V_0 e^{-\alpha x} \right) w_{nq}(x) = mu_{nq}(x). \tag{14}
\]

It should also be noted that the coupled differential equations allow finite bound states (real) solutions as \( u_{nq}(r \rightarrow \infty) \rightarrow 0 \) and \( w_{nq}(r \rightarrow \infty) \rightarrow 0 \). The above set of coupled equations can be reduced to a second-order differential equation. Hence, combining the last two equations, this provides the following second-order Schrödinger-type equation for upper (lower) components, respectively, as

\[
u''_{nq}(x) + \left[ \tilde{E}_{nq} + V_1 \frac{e^{-2\alpha x}}{(1 + qe^{-\alpha x})^2} + V_2 \frac{e^{-\alpha x}}{1 + qe^{-\alpha x}} \right] u_{nq}(x) = 0, \tag{15}\]

\[
w''_{nq}(x) + \left[ \tilde{E}_{nq} + V_1^* \frac{e^{-2\alpha x}}{(1 + qe^{-\alpha x})^2} + V_2^* \frac{e^{-\alpha x}}{1 + qe^{-\alpha x}} \right] w_{nq}(x) = 0, \tag{16}\]

with the following definitions

\[
\tilde{E}_{nq} = E_{nq}^2 - m^2, \ V_1 = V_0^2 - iqV_0/a, \ V_2 = 2E_{nq}V_0 + iV_0/a, \tag{17}\]

and \( V_1^* (V_2^*) \) is the complex conjugate of \( V_1 (V_2) \), respectively.

Using the NU method, we are set to obtain bound states solutions (relativistic energy spectrum and spinor wavefunctions) of a spin-zero particle for a three parameter \( \{V_0, q, \alpha\} \) generalized WS potential. We employ the following dimensionless transformation parameter, \( s(x) = (1 + qe^{-\alpha x})^{-1} \), which maintains the transformed wavefunctions finite, on
the boundary conditions (i.e., \(0 \leq r < \infty \rightarrow -\infty \leq x < \infty \rightarrow 0 \leq s \leq 1\)) [21,22]. Hence, Eq. (15) is reduced into the generalized equation of hypergeometric type which is given by Eq. (1):

\[
d\frac{d^2 u_{nq}(s)}{ds^2} + \frac{1 - 2s}{(s - s^2)^2} \frac{du_{nq}(s)}{ds} + \frac{a^2}{(s - s^2)^2} \left[ \tilde{E}_{nq} + \frac{V_1}{q^2} (1 - s)^2 + \frac{V_2}{q} (1 - s) \right] u_{nq}(s) = 0,
\]

where we have set \(u_{nq}(x) = u_{nq}(s)\). Therefore, with the dimensionless definitions

\[
-\epsilon^2 = a^2 \tilde{E}_{nq} \geq 0, \quad \beta^2 = a^2 V_2/q, \quad \gamma^2 = a^2 V_1/q^2, \quad (E_{nq}^2 \leq m^2, \quad \beta^2 > 0, \quad \gamma^2 > 0),
\]

for bound states (i.e., real \(\epsilon^2\)), one can arrive at the simple hypergeometric equation given by

\[
d''_{nq}(s) + \frac{1 - 2s}{(s - s^2)^2} d'_{nq}(s) + \frac{s^2 \gamma^2 - s (\beta^2 + 2 \beta \gamma \epsilon^2) + \beta^2 + \gamma^2 - \epsilon^2}{(s - s^2)^2} u_{nq}(s) = 0.
\]

Before further proceeding, it is necessary to compare the last equation with Eq. (1) to obtain the following polynomials:

\[
\tilde{\tau}(s) = 1 - 2s, \quad \sigma(s) = (s - s^2), \quad \bar{\sigma}(s) = s^2 \gamma^2 - s (\beta^2 + 2 \gamma \epsilon^2) + \beta^2 + \gamma^2 - \epsilon^2.
\]

The substitution of the above expressions into Eq. (9), together with \(\sigma'(s) = 1 - 2s\), gives

\[
\pi(s) = \pm i \sqrt{s^2 (\gamma^2 + k) - s (\beta^2 + 2 \gamma^2 + k) + \beta^2 + \gamma^2 - \epsilon^2}.
\]

It is taken into consideration that the discriminant of the square root sign has to be zero. Hence, the second-order equation under the expected roots are obtained as \(k_{1,2} = \beta^2 - 2 \epsilon^2 \pm 2 \epsilon b\), where \(b = \sqrt{\epsilon^2 - \beta^2 - \epsilon^2} = -n + 1 + ia V_0/q\) - \(\epsilon\) with \(n = 0, 1, 2, \cdots\). In this case, substituting these values for each \(k\) into Eq. (22), the possible solutions are obtained for \(\pi(s)\) are:

\[
\pi(s) = \pm \begin{cases} 
(b - \epsilon) s - b; & \text{for } k_1 = \beta^2 - 2 \epsilon^2 + 2 \epsilon b, \\
(b + \epsilon) s - b; & \text{for } k_2 = \beta^2 - 2 \epsilon^2 - 2 \epsilon b.
\end{cases}
\]

For bound state solutions, it is necessary to choose one of the four possible forms in the last equation. Therefore, the most suitable form is established by
\[ \pi(s) = -(b + \epsilon)s + b, \quad k = \beta^2 - 2\epsilon^2 - 2\epsilon b. \quad (24) \]

The following track in this selection is to achieve the condition \( \tau(s) < 0 \) in Eq. (5), which can be obtained as
\[ \tau(s) = -2(1 + b + \epsilon)s + 1 + 2b, \]
\[ \tau(s) = -2(1 + b + \epsilon) = 2(n + iaV_0/q). \quad (25) \]

A particular solution can be calculated by using Eqs. (6) and (10). Consequently, this solution is obtained as
\[ \lambda = -\gamma^2 - (b + \epsilon)(b + \epsilon + 1), \]
\[ \lambda_n = n^2 + n + 2n(\epsilon + b). \quad (26) \]

After setting \( \lambda_n = \lambda \) and solving for \( E_{nq} \), we find the Dirac exact energy spectra as
\[ E_{nq} = -\left[ \frac{V_0}{2q} \pm \kappa_n(q, a, V_0) \sqrt{\frac{m^2}{V_0^2 + \kappa_n^2(q, \alpha, V_0) - \frac{1}{4q^2}}} \right], \]
\[ \kappa_n(q, \alpha, V_0) = -[iV_0 + qa(n + 1)], \quad n = 0, 1, 2, \ldots \quad (27) \]

We should point out that Eğrifes and Sever [16] have recently obtained a similar expression to Eq. (27) for the case of the Hulthén potential. Let us now find the corresponding wavefunctions. We have seen that the energy expression has a complex form for the potential under study. We look for the complex generalized WS potential forms that have real spectrum.

Moreover, the restriction which gives the critical coupling value leads to the result
\[ n \leq \frac{1}{qa} \left( \sqrt{4q^2m^2 - V_0^2} - iV_0 \right) - 1, \quad (28) \]
that is, there are only finitely many eigenvalues. In order that at least one level might exist, it is necessary that the inequality
\[ qa + iV_0 \leq \sqrt{4q^2m^2 - V_0^2}, \quad (29) \]
is fulfilled. As can be seen from Eq. (28), there are only two lower-lying states for the Dirac particle of mass unity when the parameters $\alpha = 1$, $q = \pm 1$ for any given $V_0$:

$$n \leq \pm \left(\sqrt{4 - V_0^2} - iV_0\right) - 1. \quad (30)$$

(i) Choosing $q = 1$, the potential form (12) is reduced to the shifted WS potential:

$$V(x) = -V_0 + \frac{V_0}{1 + e^{-\alpha x}}. \quad (31)$$

and then its energy spectra yield

$$E_n = -\frac{V_0}{2} \pm [iV_0 + \alpha(n + 1)] \sqrt{\frac{m^2}{V_0^2 + [iV_0 + \alpha(n + 1)]^2} - \frac{1}{4}}, \quad n = 0, 1, 2, \cdots \quad (32)$$

(ii) Choosing $q = -1$, the potential form (12) is reduced to the shifted Hulthén potential:

$$V(x) = V_0 - \frac{V_0}{1 - e^{-\alpha x}}. \quad (33)$$

and then the resulting energy eigenvalues become

$$E_n = \frac{V_0}{2} \pm [iV_0 - \alpha(n + 1)] \sqrt{\frac{m^2}{V_0^2 + [iV_0 - \alpha(n + 1)]^2} - \frac{1}{4}}, \quad n = 0, 1, 2, \cdots \quad (34)$$

(iii) For the case $q \to 0$, the potential expression (12) is reduced to the exponential potential:

$$V(x) = -V_0 e^{-\alpha x}, \quad (35)$$

the eigenvalues expression (27) does not give an explicit form, i.e., the NU method is not applicable to the exponential potential (34).

Note that for this potential there is no explicit form of the energy expression of bound states for Schrödinger [18], KG [35] and also Dirac [17] equations.

In addition, it can be seen easily that while the field strength $V_0 \to 0$, the energy eigenvalues yield:

$$E_n = \pm \frac{1}{2} \sqrt{4m^2 - (n + 1)^2 \alpha^2}, \quad n = 0, 1, 2, \cdots \quad (36)$$

Note that in the above equation there exist bound states for the ground and excited states ($n = 0, 1$) which are $E_0 = \pm \sqrt{3m}/2$ and $E_1 = 0$, respectively, for positive $q$ values and $a = \lambda_c$, where $\lambda_c = 1/m$ denotes the Compton wavelength of the Dirac particle. Otherwise, there are no bound states for $n \geq 2$ states.
On the other hand, for the same value of $\alpha$ and negative $q$ values when $V_0 \to 0$, all energy eigenvalues go to zero. If the value of $q$ is increasing, all positive bound states go to zero, from (27), asymptotically.

An inspection of the energy expression given by Eq. (27), for any given $\alpha$, shows that we deal with a family of generalized WS potentials. The sign of $V_0$ does not effect the bound states. The spectrum consists of complex eigenvalues depending on $q$. As we shall see the role played by the range parameter $\alpha$ is very crucial in this regard. Of course, it is clear that by imposing appropriate changes in the parameters $\{\alpha, V_0, q\}$, the energy spectrum in Eq. (27) for any modified parameter can be also calculated by resolving Dirac equation for every parameter change.

Let us calculate the wavefunctions. Inserting, $\pi(s)$ and $\sigma(s)$ in Eq. (4) and consequently solving the resulting first-order differential equation, we find

$$\phi_n(s) = s^b(1 - s)^c.$$

In addition, to find the function, $y_{nq}(s)$, which is the polynomial solution of hypergeometric-type equation, we multiply Eq. (3) by $\rho(s)$ so that it can be written in self-adjoint form [28]

$$(\sigma(s)\rho(s)y_{nq}'(s))' + \lambda\rho(s)y_{nq}(s) = 0,$$

where $\rho(s)$ satisfies Eq. (8), which gives

$$\rho(s) = s^2 b(1 - s)^{2c}.$$  

The second eigenfunction can be obtained by Eq. (7) as

$$y_{nq}(s) = D_{nq}s^{-2b}(1 - s)^{-2c} \frac{d^n}{ds^n} \left[s^{n+2b}(1 - s)^{n+2c}\right],$$

where $D_{nq}$ is a normalization constant. In the limit $q \to 1$, the polynomial solutions of $y_n(s)$ are expressed in terms of Jacobi Polynomials, which is one of the classical orthogonal polynomials, with weight function given by Eq. (39) for $s \in [0, 1]$, giving $y_n(s) \simeq P_n^{(2b,2c)}(1 - 2s)$. Obviously, the radial wave function $u_{nq}(s)$ for the s-wave can be obtained by substituting Eqs. (37) and (40) into Eq. (2) as
\[ u_n(s) = N_{nq} s^b (1 - s)^c P_n^{(2b,2c)}(1 - 2s), \] (41)

where \( s(r) = (1 + e^{-(r-R_0)/a})^{-1} \), \( R_0 \gg a \) and \( N_{nq} \) is a new normalization constant. Further, by using the differential and recursion properties of the Jacobi polynomials \([36]\), the lower spinor component can be obtained from Eq. (13) as

\[ mw_{nq}(s) = N_{nq} s^b (1 - s)^c \left\{ E_{nq} - i\alpha + \left[ \frac{V_0}{q} - i\alpha \left(n + 1 + \frac{iV_0}{\alpha q}\right)(1 - s)\right] \right\} P_n^{(2b,2c)}(1 - 2s) + N_{nq} i\alpha \left(n + 1 + \frac{2iV_0}{q\alpha}\right)s^b (1 - s)^c P_n^{(2b+1,2c+1)}(1 - 2s). \] (42)

Notice the well behavior of the wave function at infinity. As an example, the ground state wave function behaves like

\[ u_0(s \to 0) \to 0, \quad u_0(s \to 1) \to 0, \] (43)

and thus representing a truly bound-state solution. Further, we make use of the fact that the Jacobi polynomials can be explicitly written in two different ways \([36]\):

\[ P_n^{(\rho,\nu)}(z) = 2^{-n} \sum_{p=0}^{n} (-1)^{n-p} \binom{n + \rho}{p} \binom{n + \nu}{n - p} (1 - z)^{n-p} (1 + z)^p, \] (44)

\[ P_n^{(\rho,\nu)}(z) = \frac{\Gamma(n + \rho + 1)}{n! \Gamma(n + \rho + \nu + 1)} \sum_{r=0}^{n} \binom{n}{r} \frac{\Gamma(n + \rho + \nu + r + 1)}{\Gamma(r + \rho + 1)} \left(\frac{z-1}{2}\right)^r, \] (45)

where \( \binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{\Gamma(n+1)}{\Gamma(r+1)\Gamma(n-r+1)}. \) Using Eqs. (44) and (45), we obtain the following two explicit expressions:

\[ P_n^{(2b,2c)}(1 - 2s) = (-1)^n \Gamma(n + 2b + 1) \Gamma(n + 2c + 1) \]

\[ \times \sum_{p=0}^{n} \frac{(-1)^p q^{n-p}}{p!(n-p)! \Gamma(p + 2c + 1) \Gamma(n + 2b - p + 1)} s^{n-p} (1 - s)^p, \] (46)

\[ P_n^{(2b,2c)}(1 - 2s) = \frac{\Gamma(n + 2b + 1)}{\Gamma(n + 2b + 2c + 1)} \sum_{r=0}^{n} \frac{(-1)^r q^r \Gamma(n + 2b + 2c + r + 1)}{r!(n-r)! \Gamma(2b + r + 1)} s^r. \] (47)

\[ 1 = N_{nq}^2 (-1)^n \frac{\Gamma(n + 2c + 1) \Gamma(n + 2b + 1)^2}{\Gamma(n + 2\epsilon + 2b + 1)} \left\{ \sum_{p=0}^{n} \frac{(-1)^p q^{n-p}}{p!(n-p)! \Gamma(p + 2\epsilon + 1) \Gamma(n + 2b - p + 1)} \right\}. \]
\[
\times \left\{ \sum_{r=0}^{n} \frac{(-1)^r q^r \Gamma(n + 2\epsilon + 2b + r + 1)}{r!(n-r)!\Gamma(2b + r + 1)} \right\} I_{nq}(p, r),
\]  

where

\[
I_{nq}(p, r) = \int_0^1 s^{n+2b+r-p}(1-s)^{p+2\epsilon} ds.
\]

Using the following integral representation of the hypergeometric function [36]

\[
\int_0^1 s^{\alpha_0-1}(1-s)^{\gamma_0-\alpha_0-1}(1-qs)^{-\beta_0} ds = 2F_1(\alpha_0, \beta_0; \gamma_0; q) \frac{\Gamma(\alpha_0)\Gamma(\gamma_0 - \alpha_0)}{\Gamma(\gamma_0)},
\]

\[
[\text{Re}(\gamma_0) > \text{Re}(\alpha_0) > 0, \ |\text{arg}(1-q)| < \pi],
\]

which gives

\[
2F_1(\alpha_0, \beta_0 : \alpha_0 + 1; q)/\alpha_0 = \int_0^1 s^{\alpha_0-1}(1-qs)^{-\beta_0} ds,
\]

\[
2F_1(\alpha_0, \beta_0; \gamma_0; q) = \frac{\Gamma(\gamma_0)\Gamma(\gamma_0 - \alpha_0 - \beta_0)}{\Gamma(\gamma_0 - \alpha_0)\Gamma(\gamma_0 - \beta_0)},
\]

\[
[\text{Re}(\gamma_0 - \alpha_0 - \beta_0) > 0, \ \text{Re}(\gamma_0) > \text{Re}(\beta_0) > 0,
\]

for \( q = 1 \). Setting \( \alpha_0 = n + 2b + r - p + 1, \ \beta_0 = -p - 2\epsilon, \) and \( \gamma_0 = \alpha_0 + 1, \) one gets

\[
I_{nq}(p, r) = \frac{2F_1(\alpha_0, \beta_0; \gamma_0; q)}{\alpha_0} = \frac{(n + 2b + r - p + 1)!(p + 2\epsilon)!}{(n + 2b + r - p + 1)(n + 2\epsilon + r + 2b + 1)!}.
\]

In view of the above complex energy spectra (27), it will be of interest to see how complex potential form of Eq. (12) would effect this result. Therefore, we shall change the values of the potential parameters \((\alpha, q, V_0)\) from real to pure imaginary (complex) to obtain Hamiltonians that may or may not be \(\mathcal{PT}\)-symmetric. Hence, we also make the same parameter change in the energy spectra (27) and the upper and lower spinor components of the wavefunction Eqs. (41) and (42) respectively. The resulting non-Hermitian complex potential could have real energy spectra. To this end we consider the complexified forms of the generalized WS potential.
A. Non-Hermitian $\mathcal{PT}$–symmetric new trigonometric and periodic potential form

Let us consider the case where at least one of the potential parameters be complex. In this case, $\alpha$ is taken to be a complex parameter (i.e., $\alpha \to i\alpha$). Consequently, the potential in Eq. (12) transforms into the form

$$V_q(x) = -\frac{V_0}{q^2 + 2q\cos(\alpha x) + 1} [q + \cos(\alpha x) - i \sin(\alpha x)] = V_q^*(-x),$$

(54)

which is a $\mathcal{PT}$–symmetric but non-Hermitian. We note that the transformed potential in Eq. (54) has a trigonometric and periodic form. Obviously, the last form of potential forms have no physical relation whatsoever with the WS potential in (12), any of it’s generalizations, it’s well-known behavior or properties. As we have seen, simple mathematical manipulations have not to cloud our intuitive judgement and conceptual physical understanding. From a mathematical prospective, it might be possible that one can use calculus to think of the hyperbolic or exponential functions as another form of trigonometric ones. However, physically these functions, if considered as potentials, are dramatically different. We should not refer to either potential in Eq. (54) as WS-type. Nevertheless, this new complex potential embodies their periodic, trigonometric and $\mathcal{PT}$–symmetric behaviors. Hence, this type of potentials (54) has real spectrum given by

$$E_{nq} = -\frac{V_0}{2q} \pm \frac{\sqrt{1 - \frac{m^2}{V_0^2 - (V_0 + q(n + 1)/a)^2}}}{V_0^2 - (V_0 + q(n + 1)/a)^2},$$

(55)

if the following restriction $4q^2m^2 \leq V_0^2 - (V_0 + q(n + 1)/a)^2$ being achieved. The critical coupling value is

$$V_0 \leq -\frac{q(n + 1)}{2a} - \frac{2qam^2}{n + 1},$$

which leads to the following condition:

$$\frac{V_0 + q\alpha - \sqrt{V_0^2 - 4q^2m^2}}{q\alpha} \leq n \leq \frac{V_0 + q\alpha + \sqrt{V_0^2 - 4q^2m^2}}{q\alpha},$$

meaning that the number of real eigenvalues are finite. Further, the corresponding radial wave function $u_{nq}(s)$ for the s-wave could be determined as

$$u_{nq}(s) = N_{nq} s^{c}(1 - s)^{i\epsilon} P_n^{(2c,2i\epsilon)}(1 - 2s),$$

(56)
\[ m \omega_{nq}(s) = N_{nq}s^c(1-s)^i \epsilon \left\{ E_{nq} + i\alpha \epsilon + \left[ \frac{V_0}{q} + \alpha \left( n + 1 + \frac{V_0}{\alpha q} \right) (1-s) \right] \right\} P_n^{(2c,2i\epsilon)}(1-2s) \]

\[ - N_{nq}\alpha \left( n + 1 + \frac{2V_0}{q\alpha} \right) s^c(1-s)^i \epsilon P_n^{(2c+1,2i\epsilon+1)}(1-2s), \]  

(57)

where \( c = -(n + 1 + aV_0/q) - i\epsilon, \) and \( s(r) = (1 + qe^{-i(r-R_0)/a})^{-1}. \)

For the sake of comparing the relativistic and non-relativistic binding energies, we need to solve the 1D Schrödinger equation for the complex form of the generalized WS potential given by Eq. (54). Employ a convenient transformation given by \( s(r) = (1 + qe^{-i(r-R_0)/a})^{-1}, \)

\[ 0 \leq r \leq \infty \rightarrow 0 \leq s(r) \leq 1, \]

we obtain [21]

\[ \psi''_{nq}(s) + \frac{1-2s}{(s-s^2)} \psi'_{nq}(s) + \frac{[-\beta^2 s + \beta^2 - \epsilon^2]}{(s-s^2)^2} \psi_{nq}(s) = 0, \]  

(58)

for which

\[ \tau(s) = 1 - 2s, \quad \sigma(s) = s - s^2, \quad \sigma'(s) = -\beta^2 s + \beta^2 - \epsilon^2, \]

\[ \epsilon^2 = \frac{2ma^2}{\hbar^2} E_{nq}, \quad \beta^2 = -\frac{2ma^2}{\hbar^2 q} V_0 \quad (E_{nq} < 0, \beta^2 > 0). \]  

(59)

The function \( \tau(s) \) could be obtained as

\[ \tau(s) = -2(1 + d + \epsilon)s + (1 + 2d), \quad d = \sqrt{\epsilon^2 - \beta^2}, \]  

(60)

if \( \pi(s) = -(d + \epsilon)s + d \) is chosen for \( k_- = -(d + \epsilon)^2. \) We can also find the eigenvalues from Eqs. (6) and (10) as

\[ \lambda = -(d + \epsilon)(d + \epsilon + 1), \quad \lambda_n = 2n(d + \epsilon + 1) + n(n - 1). \]  

(61)

Thus, solving Eq. (61) for the energy eigenvalues, we obtain

\[ E_{nq}(V_0, i\alpha) = \frac{\hbar^2}{2ma^2} \left[ \frac{n+1}{2} - \frac{\gamma}{(n+1)} \right]^2, \quad \gamma = \frac{ma^2 V_0}{\hbar^2 q}, \quad 0 \leq n < \infty. \]  

(62)

On the other hand, the radial wavefunctions in the present case become

\[ \psi_{nq}(s) = N_{nq}s^d(1-s)^i P_n^{(2d,2i\epsilon)}(1-2s), \]  

(63)

with \( s(r) = (1 + qe^{-i(r-R_0)/a})^{-1} \) and \( N_{nq} \) is a new normalization constant determine by
\[ 1 = N_{nq}^2(-1)^n \frac{(n + 2\epsilon)! \Gamma(n + 2d + 1)^2}{\Gamma(n + 2d + 2 + 1)} \left\{ \sum_{p=0}^{n} \frac{(-1)^p q^{n-p}}{p!(n-p)!(2\epsilon + p)! \Gamma(n + 2d - p + 1)} \right\} \times \left\{ \sum_{r=0}^{n} \frac{(-1)^r q^r \Gamma(n + 2d + r + 2\epsilon + 1)}{r!(n-r)! \Gamma(2d + r + 1)} \right\} I_{nq}(p,r), \] (64)

where the integral \( I_{nq}(p,r) = \int_0^1 s^{n+2d+r-p}(1 - qs)^p\epsilon ds \) is given by

\[ I_{nq}(p,r) = _2 F_1(n + 2c + r - p + 1, -p - 2\epsilon : n + 2c + r - p + 2 : 1) B(n + 2c + r - p + 1, 1), \] (65)

Figures 1 and 2 show the variation of the ground-state (i.e., \( n = 0 \)) as a function of the coupling constant \( V_0 \) for different positive and negative \( q \), and \( a = \lambda_c \). Obviously, in Figure 1, the non-Hermitian \( PT \)-symmetric generalized WS potential generates real and negative bound-states for \( q > 0 \), it generates real and positive bound-states for the same value of \( \alpha \) when \( q < 0 \) (Figure 2). Further, Figures 3 and 4 show the variation of the first three energy eigenstates as a function of \( \alpha \) for (a) \( q = 1.0 \) and (b) \( q = -1.0 \) with \( V_0 = 2.5m \). Obviously, for the given \( V_0 \), as seen from Figures 3 and 4 all possible eigenstates have negative (positive) eigenenergies if the parameter \( q \) is positive (negative). It is almost notable that there are some crossing points of the relativistic energy eigenvalues for some \( V_0 \) values.

B. Non-Hermitian non-\( PT \)-symmetric generalized Woods-Saxon potential

In this part, we consider two parameters \( \{V_0, q\} \) to be complex parameters (i.e., \( V_0 \rightarrow iV_0 \), \( q \rightarrow iq \)). Consequently, the potential in Eq. (12) transforms to the following form

\[ V_q(x) = V_0 \frac{2 \cosh^2(\alpha x) - \sinh(2\alpha x) - 1}{1 + q^2 \left[ 2 \cosh^2(\alpha x) - \sinh(2\alpha x) - 1 \right]} - i \left[ \cosh(\alpha x) - \sinh(\alpha x) \right], \] (66)

which is a non-\( PT \)-symmetric but non-Hermitian. The complex energy eigenvalues of Eq. (66) are given by

\[ E_{nq} = -\frac{V_0}{2q} \pm i (iV_0 + q(n + 1)/a) \sqrt{\frac{1}{4q^2} - \frac{m^2}{V_0^2 + (iV_0 + q(n + 1)/a)^2}}, \] (67)

On the other hand, the corresponding radial wave functions \( u_{nq}(s) \) for the s-wave could be determined as
\[ u_{nq}(s) = N_{nq} s^b (1 - s)^c P_n^{(2b,2c)}(1 - 2s), \] (68)

and the lower spinor component \( w_{nq}(s) \) is given by Eq. (42) with \( s(r) = (1 + i q e^{-(r-R_0)/a})^{-1} \).

The integral \( I_{nq}(p, r) = \int_0^1 s^{n+2b+r-p} (1 - s)^{p+2c} ds \) is given by

\[ I_{nq}(p, r) = _2 F_1 (n + 2b + r - p + 1, -p - 2c : n + 2b + r - p + 2; i) B(n + 2b + r - p + 1, 1). \] (69)

C. Pseudo-Hermiticity and \( \mathcal{PT} \)-symmetric new trigonometric and periodic potential form

Finally, in this part, when all the parameters \( \{V_0, \alpha, q\} \) are complex parameters (i.e., \( V_0 \rightarrow i V_0, \alpha \rightarrow i \alpha, q \rightarrow iq \)), from Eq. (12) we obtain

\[ V_q(x) = -\frac{V_0}{q^2 + 2q \sin(\alpha x) + 1} [q + \sin(\alpha x) + i \cos(\alpha x)] = V_q^{*}\left(\frac{\pi}{\alpha} - x\right). \] (70)

Note that the transformed potential in Eq. (70) has a trigonometric and periodic form. As we remarked earlier, the above potential forms do not have any of the WS potential properties. Nonetheless, they have periodic, trigonometric and \( \mathcal{PT} \)-symmetric behaviors. The potential in Eq. (70) is a pseudo-Hermitian potential [37,38] having a \( \pi/\alpha \) phase difference with respect to the potential (I), it is also a \( \mathcal{PT} \)-symmetric, \( \eta = P \)-pseudo-Hermitian (i.e., \( PT V_q(x)(PT)^{-1} = V_q(x) \)), with \( P = \eta : x \rightarrow \frac{\pi}{2 \alpha} - x \) and \( T : i \rightarrow -i \) but non-Hermitian having real spectrum given by

\[ E_{nq} = -\frac{V_0}{2q} \pm (V_0 + q(n + 1)/a) \frac{1}{\sqrt{\frac{1}{4q^2} - \frac{m^2}{V_0^2 - (V_0 + q(n + 1)/a)^2}}} \] (71)

if the same restrictions after Eq. (55) are achieved.

On the other hand, the corresponding radial wave functions \( u_{nq}(s) \) for the s-wave could be determined as

\[ \psi_{nq}(s) = N_{nq} s^c (1 - s)^{i e} P_n^{(2c,2ie)}(1 - 2s), \] (72)

and the lower spinor component \( w_{nq}(s) \) is given by Eq. (48) with \( s(r) = (1 + i q e^{i(r-R_0)/a})^{-1} \).

The integral \( I_{nq}(p, r) = \int_0^1 s^{n+2c+r-p} (1 - s)^{p+2ie} ds \) is given by
\[ I_{nq}(p, r) = _2F_1(n + 2c + r - p + 1, -p - 2i\epsilon; n + 2c + r - p + 2; i)B(n + 2c + r - p + 1, 1). \] (73)

IV. THE SOLUTION OF THE GENERALIZED WS POTENTIAL FOR \( q = 0 \)

We have obtained the bound state solutions of the generalized WS potential with \( q \neq 0 \) and the explicit form of the eigenvalues and the spinor wavefunctions. In addition, we start finding solutions for \( q = 0 \) case with the definition of new variable \( s = e^{-\alpha x} \). Hence, Eq. (18) is reduced to the generalized equation of hypergeometric type:

\[
\frac{d^2 u_n(s)}{ds^2} + \frac{1}{s} \frac{du_n(s)}{ds} + \frac{1}{s^2} \left[ -\epsilon^2 + \beta s + \gamma s^2 \right] u_n(s) = 0. \quad (74)
\]

Further, we define the following dimensionless expressions:

\[
\epsilon^2 = -\frac{1}{\alpha^2}(E^2 - m^2), \quad \beta = \frac{V_0}{\alpha} + \frac{2EV_0}{\alpha^2}, \quad \gamma = \frac{V_0^2}{\alpha^2},
\]

\[
\tilde{\tau}(s) = 1, \quad \sigma(s) = s, \quad \tilde{\sigma}(s) = -\epsilon^2 + \beta s + \gamma s^2,
\]

with real \( \epsilon^2 > 0 \) \((E^2 < m^2)\) for bound states. The substitution of the above expressions into Eq. (9), together with \( \sigma'(s) = 1 \), gives

\[
\pi(s) = \pm i\sqrt{\gamma s^2 - (k - \beta)s - \epsilon^2}. \quad (77)
\]

Substituting the value for each \( k \) into the last equation, we obtain

\[
\pi(s) = \pm \begin{cases} 
  i\delta s + \epsilon; & \text{for } k = \beta + 2i\epsilon, \\
  i\delta s - \epsilon; & \text{for } k = \beta - 2i\epsilon,
\end{cases} \quad (78)
\]

where \( \delta = \sqrt{\gamma} = \frac{V_0}{\alpha} \). Therefore, the most suitable form is established by

\[
\pi(s) = -i\delta s + \epsilon, \quad k = \beta - 2i\epsilon. \quad (79)
\]

The following track in this selection is to achieve the condition \( \tau(s) < 0 \) in Eq. (5), which can be obtained as

\[
\tau(s) = -2i\delta s + 1 + 2\epsilon, \quad \lambda = \beta - 2i\epsilon - i\delta. \quad (80)
\]
Let us calculate the wavefunctions. Inserting, $\pi(s)$ and $\sigma(s)$ in Eq. (4) and consequently solving the resulting first-order differential equation, we find

$$\phi_n(s) = s^\epsilon e^{-iV_0 s/\alpha}. \quad \text{(81)}$$

In addition, to find the function, $y_{nq}(s)$, which is the polynomial solution of hypergeometric-type equation, we multiply Eq. (3) by $\rho(s)$ so that it can be written in self-adjoint form [28]

$$sy_n'' + [1 + 2\epsilon - \nu s]y_n' - \left[\left(\epsilon + \frac{1}{2}\right) \nu - \beta\right] y_n = 0, \ \nu = 2i\delta. \quad \text{(82)}$$

The solution can be written in terms of confluent hypergeometric function as follows:

$$y_n(s) = \text{$_1F_1$} \left(\epsilon + \frac{1}{2} + \frac{i\beta}{2V_0}, 1 + 2\epsilon, \frac{2iV_0}{\alpha} s\right), \quad \text{(83)}$$

and consequently the upper spinor

$$u(s) = A_1 \text{$_1F_1$} \left(\epsilon + \frac{1}{2} + \frac{i\beta}{2V_0}, 1 + 2\epsilon, \frac{2iV_0}{\alpha} s\right) s^\epsilon e^{-iV_0 s/\alpha}. \quad \text{(84)}$$

Finally, the lower spinor is found by

$$mw(s) = \left[i \frac{d}{dx} + E + V_0 s\right] A_1 \text{$_1F_1$} \left(\epsilon + \frac{1}{2} + \frac{i\beta}{2V_0}, 1 + 2\epsilon, \frac{2iV_0}{\alpha} s\right) s^\epsilon e^{-iV_0 s/\alpha}. \quad \text{(85)}$$

V. RESULTS AND CONCLUSIONS

In this work, we have seen that the s-wave Dirac equation with the generalized WS potential can be solved exactly for its bound states using the hypergeometric method. The relativistic bound states energy spectrum and the corresponding wave functions for the generalized WS potential have been obtained by the hypergeometric method. Some interesting results including the $\mathcal{PT}$-symmetric, non-$\mathcal{PT}$-symmetric non-Hermitian, and non-$\mathcal{PT}$-symmetric $P$-pseudo-Hermitian versions of the generalized WS potential have also been discussed for bound states. In addition, we have discussed the relation between the non-relativistic and relativistic solutions and the possibility of existence of bound states for complex parameters.
VI. ACKNOWLEDGMENTS

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FIG. 1: The ground-state \((n = 0)\) energy, in a non-Hermitian \(\mathcal{PT}\)-symmetric potential given by Eq. (54), as a function of the coupling constant \(V_0\) for three positive shape parameters \(q\) with \(a = \lambda_c\).

FIG. 2: The ground-state \((n = 0)\) energy, in a non-Hermitian \(\mathcal{PT}\)-symmetric potential given by Eq. (54), as a function of the coupling constant \(V_0\) for three negative shape parameters \(q\) with \(a = \lambda_c\).

FIG. 4: The first three energy eigenstates, in a non-Hermitian \(\mathcal{PT}\)-symmetric potential given by Eq. (54), as a function of the range parameter \(\alpha\) for a positive shape parameter \((q = 1.0)\) with a coupling constant \(V_0 = 2.5m\).

FIG. 3: The first three energy eigenstates, in a non-Hermitian \(\mathcal{PT}\)-symmetric potential given by Eq. (54), as a function of the range parameter \(\alpha\) for a negative shape parameter \((q = -1.0)\) with a coupling constant \(V_0 = 2.5m\).
