Measurement theory in classical mechanics

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We investigate measurement theory in classical mechanics in the formulation of classical mechanics by Koopman and von Neumann (KvN), which uses Hilbert space. We show a difference between classical and quantum mechanics in the “relative interpretation” of the state of the target of measurement and the state of the measurement device. We also derive the uncertainty relation in classical mechanics.

Subject Index A00, A60

1. Introduction

In order to discuss the crucial difference between quantum and classical mechanics, it is essential to compare the two using the same formalism. Bohm described a quantum system in terms of classical mechanics [1], and he found that the presence or absence of quantum potential characterizes the difference between a quantum system and a classical system.¹

However, there is a clear difference between a comparison in the quantum system and a comparison in the classical system. The difference is the non-commutative nature of the operator. Unlike classical mechanics, quantum mechanics is described in terms of q-numbers, i.e., non-commutative physical quantities. This difference is apparent in measurement theories, such as Heisenberg’s uncertainty principle [2].

In other words, quantum mechanics is a world of q-numbers. On the other hand, classical mechanics is a world of c-numbers, i.e., a commutative world. Such an argument, however, is obscured by the fact that the two are in different formalisms.

To clarify this, it is important to rewrite classical mechanics into quantum mechanics and compare the two.

Koopman and von Neumann [3,4], at the early stages of quantum mechanics, rewrote the classical system in the form of quantum mechanics using the KvN equations. Later, Gozzi and Mauro studied this formalism in detail [5–7]. Here, we will refer to the quantum-theoretical description of classical mechanics by KvN equations as KvN formalism. Sudarshan discussed the KvN formalism as a model of quantum-classical interaction [8].

¹ Quantum potential is defined as follows. Rewriting \( \psi \) to \( \psi = R \exp(iS) \), we obtain a Hamilton–Jacobi-like equation,

\[
\frac{\partial S}{\partial t} = -\frac{1}{2m} \left( \frac{\partial S}{\partial x} \right)^2 + V(x) - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \log R(x, t),
\]

where \(-\frac{\hbar^2}{2m}(\partial^2/\partial x^2)(\log R)\) is called the quantum potential.
In the KvN formalism, classical mechanics is described using non-commutative operators, as in quantum mechanics. There, classical mechanics is given by new variables that are non-commutative with respect to position and momentum, and time evolution is performed by unitary operators consisting of position and momentum and these new variables.

If the new variable introduced is not included in the Hamiltonian, this formalism is equivalent to classical mechanics. Under such an operator formalism, the difference between classical mechanics and quantum mechanics is not in the non-commutative nature of the operators, but in the form of the commutations relations.

To summarize, the normal case can be illustrated by

\[
\begin{array}{cc}
\text{QM} & \text{CM} \\
q\text{-numbers} & c\text{-numbers}
\end{array}
\]  
\tag{1.1}

but in Bohm’s argument it is

\[
\begin{array}{cc}
\text{QM(Bohm)} & \text{CM} \\
c\text{-numbers} & c\text{-numbers}
\end{array}
\]  
\tag{1.2}

and it is

\[
\begin{array}{cc}
\text{QM} & \text{CM(KvN)} \\
q\text{-numbers} & q\text{-numbers}
\end{array}
\]  
\tag{1.3}

in KvN formalism.

In this paper, we apply quantum measurement theory, which was originally formulated for quantum systems, to classical systems and investigate how we describe measurements of classical systems. We show a difference between classical and quantum mechanics in the “relative interpretation” of the state of the target of measurement and the state of the measurement device.

Next, we derive the uncertainty relation in classical mechanics. Until now, measurement in classical mechanics has not been sufficiently considered.\(^2\) Jens, Wilkens, and Lewenstein found that the formalism of quantum mechanics is useful to more than quantum mechanics [9]. It makes us expect that the KvN formalism has a new meaning and application in classical mechanics.

The structure of this paper is as follows. First, we review the KvN formalism in Sect. 2. Next, we extend the KvN formalism to quantum mechanics and show that this is equivalent to quantum theory. After reviewing the observation problem using the von Neumann model in Sect. 4, we discuss in Sect. 5 the measurement theory in classical mechanics. In Sect. 6, we construct the classical mechanics’ version of the uncertainty relation. The last chapter will give a summary and discussion.

The appendices include the following. In Appendix A, we comment that the KvN formalism for free particles can be regarded as a von Neumann model. Next, in Appendix B, we discuss the von Neumann model in a formalism that extends the KvN formalism to quantum mechanics, which is discussed in Sect. 3. In Appendix C, we introduce the Kraus operator. In Appendix D, we discuss the case where the initial condition is the only known probability. In Appendix E, we comment on

\(^2\)Classical measurement theory has been studied in part as classical information theory, and the main result is known as Shannon’s coding theorem. This extension to quantum mechanics is still studied [10–14], and reinterpreting Shannon’s coding theory in terms of the KvN formalism is essential in comparing quantum information theory with classical information theory.
the Planck operator. In Appendix F, we describe in detail the evolution of time in the von Neumann model.

2. The KvN formalism

This section briefly introduces the KvN formalism. In quantum mechanics, the commutation relation between the position and momentum operator of a particle is given by,

$$[\hat{x}, \hat{p}] = i\hbar. \quad (2.1)$$

A state can be written $|\psi\rangle$ as an expansion using position and momentum eigenvalue states,

$$|\psi\rangle = \int dx |x\rangle \langle x|\psi\rangle = \int dp |p\rangle \langle p|\psi\rangle. \quad (2.2)$$

The time evolution of the state is described using the Hamiltonian operator $H(\hat{x}, \hat{p})$ as

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = \hat{H} |\psi\rangle. \quad (2.3)$$

The wave function (probability amplitude) $\psi(x)$ is a function of $x$ only, and $\psi(p)$ is its Fourier transform. Thus it is not a function on the phase space.

The essence of the KvN formalism is to introduce operators $\hat{\pi}_x, \hat{\pi}_p$ in addition to $\hat{x}, \hat{p}$ and require non-commutability between $\hat{x}, \hat{p}$ and $\hat{\pi}_x, \hat{\pi}_p$ while $\hat{x}$ and $\hat{p}$ are made commutative.

$$[\hat{x}, \hat{p}] = [\hat{\pi}_x, \hat{\pi}_p] = 0, [\hat{x}, \hat{\pi}_x] = [\hat{p}, \hat{\pi}_p] = i. \quad (2.4)$$

Since position and momentum are commutative, the state $|\psi\rangle$ can be expanded by simultaneous eigenstates of position and momentum as

$$|\psi\rangle = \int dx dp |x, p\rangle \langle x, p|\psi\rangle. \quad (2.5)$$

That is, in the KvN formalism, the wave function (probability amplitude) $\psi(x, p) = \langle x, p|\psi\rangle$ is a complex function in phase space.

It should be noted that $\psi(x, p)$ is not a pseudo-probability like a Wigner function [15] or a Husimi function [16], but a probability amplitude.

By the Fourier transform, $|\psi\rangle$ can be described as

$$|\psi\rangle = \int dx d\pi_x |x, \pi_x\rangle \langle x, \pi_x|\psi\rangle \quad (2.6)$$

$$= \int d\pi_x dp |\pi_x, p\rangle \langle \pi_x, p|\psi\rangle = \int d\pi_x d\pi_p |\pi_x, \pi_p\rangle \langle \pi_x, \pi_p|\psi\rangle. \quad (2.7)$$

In addition, the Liouvilian will be introduced in correspondence with the Hamiltonian

$$\hat{L} = \frac{\partial H}{\partial p} \hat{\pi}_x - \frac{\partial H}{\partial x} \hat{\pi}_p. \quad (2.8)$$

The KvN equation corresponding to the Schrödinger equation is introduced as

$$i \frac{\partial}{\partial t} |\psi\rangle = \hat{L} |\psi\rangle. \quad (2.9)$$

Note, the Hamiltonian does not depend on $\hat{\pi}_x, \hat{\pi}_p$. 

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By applying \( \langle x, p | \) from the left,

\[
\frac{i}{\partial t} \psi(x, p, t) = \frac{i}{\partial p} \frac{\partial H}{\partial x} \psi(x, p, t) - \frac{i}{\partial x} \frac{\partial H}{\partial p} \psi(x, p, t)
\]  

(2.10)

is obtained.

This form is the same as the Liouville equation, but in this case there is a difference in that \( \psi \) is a complex function. In the KvN formalism, as with quantum mechanics, \( \psi(x, p) \) is regarded as the probability amplitude, and \(|\psi(x, p)|^2\) is the probability density in the phase space.

As an example, let us consider a free particle \( H(x, p) = \frac{p^2}{2m} \) [5]. Using the Liouvillian

\[
\hat{L} = \frac{\hat{p}}{m} \hat{\pi}_x,
\]

(2.11)

the KvN equation yields

\[
\frac{\partial}{\partial t} |\psi\rangle = \frac{\hat{p}}{m} \hat{\pi}_x |\psi\rangle.
\]

(2.12)

Now applying \( \langle \pi_x, p | \) from the left, we obtain

\[
\frac{i}{\partial t} \langle \pi_x, p | \psi \rangle = \frac{p}{m} \pi_x \langle \pi_x, p | \psi \rangle.
\]

(2.13)

Using proportionality factor \( A \), the solution is given by

\[
\langle \pi_x, p | \psi \rangle = A e^{\frac{p}{m} \pi_x t}.
\]

(2.14)

By the Fourier transform for \( \pi_x \), we obtain

\[
\langle x, p | \psi \rangle = \int d\pi_x A e^{\frac{p}{m} \pi_x t - i\pi x t} = A \delta \left( x - \frac{p}{m} t \right).
\]

(2.15)

Here, if the initial state is \( |x_0, p_0 \rangle \), it is \( A = 1 \) and

\[
\langle x, p | \psi \rangle = \langle x, p | e^{i\hat{L} t} |x_0, p_0 \rangle = \delta \left( x - \frac{p}{m} t \right).
\]

(2.16)

This solution reproduces the linear orbit of a free particle in classical mechanics.

Since KvN is a rewrite of classical mechanics to the quantum mechanics formalism, it is natural that the Hamiltonian \( H \) only includes \( x \) and \( p \). However, this does not mean that they are two independent free particles. This can be understood from the form of Louvillian \( \hat{L} \):

\[
\hat{L} = \frac{\partial H}{\partial \hat{p}} \hat{\pi}_x - \frac{\partial H}{\partial \hat{x}} \hat{\pi}_p.
\]

If \( H \) contains a term like \( \hat{\pi}_x \), then \( \hat{L} \) will contain a term like \( \hat{\pi}_x^2 \). This represents fluctuations, as is often the case with operator forms in thermodynamics [17].

In quantum mechanics, this fluctuation is equivalent to adding a quantum effect. Our argument is to discuss classical mechanics in the form of quantum mechanics, and does not include such a term.

However, expressing the quantum effect by adding \( \pi_x, \pi_p \) is interesting as a way of discussing the boundary region between classical mechanics and quantum mechanics, and is being studied as generalized classical mechanics [7].
3. Relation to quantum mechanics

Now we consider the relationship between the KvN formalism and quantum mechanics.

In quantum mechanics, position $\hat{x}_q$ and momentum $\hat{p}_q$ satisfy the canonical commutation relation,

$$[\hat{x}_q, \hat{p}_q] = i\hbar.$$  \hfill (3.1)

The same algebra can be constructed using $\hat{x}, \hat{\pi}_x, \hat{p}, \hat{\pi}_p$. If we define operators $\hat{x}_\hbar$ and $\hat{p}_\hbar$ as

$$\hat{x}_\hbar = \hat{x} - \frac{1}{2}\hbar \hat{\pi}_p,$$  \hfill (3.2)

$$\hat{p}_\hbar = \hat{p} + \frac{1}{2}\hbar \hat{\pi}_x,$$  \hfill (3.3)

then we obtain

$$[\hat{x}_\hbar, \hat{p}_\hbar] = i\hbar.$$  \hfill (3.4)

A similar algebra is discussed in Ref. [18].\textsuperscript{4,5}

Then we obtain quantum wave functions in phase space,

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = H \left( \hat{x} + h \frac{1}{2} \hat{\pi}_p, \hat{p} + h \frac{1}{2} \hat{\pi}_x \right) |\psi\rangle.$$  \hfill (3.5)

\textsuperscript{4} These equations satisfy the Weyl relation,

$$\hat{U}(t) \hat{V}(s) \hat{U}(t') \hat{V}(s') = \hat{V}(s') \hat{U}(t') \hat{V}(s) \hat{U}(t), \quad \hat{V}(s) = e^{i\hat{\pi}_x s / \hbar}, \quad \hat{V}(s) = e^{i\hat{\pi}_p s / \hbar} = e^{i\hat{p} - h \frac{1}{2} \hat{\pi}_x s / \hbar}.$$

Then, from the Stone-von Neumann theorem, there exists a unitary transformation $\hat{U}$ such that

$$\hat{U} e^{i\hat{x}_\hbar t / \hbar} \hat{U}^\dagger = e^{i\hat{\pi}_p t / \hbar} \hat{U},$$

$$\hat{U} e^{i\hat{p}_\hbar t / \hbar} \hat{U}^\dagger = e^{i\hat{\pi}_p t / \hbar} \hat{U}.$$  

Then, $\hat{x}_\hbar, \hat{p}_\hbar$ describe quantum mechanics exactly.

\textsuperscript{5} We also obtain

$$\hat{x}_{s,h} = \hat{x} + \frac{1}{2}h \hat{\pi}_p,$$

$$\hat{p}_{s,h} = \hat{p} - \frac{1}{2}h \hat{\pi}_x,$$

and should discuss for the back reaction from these operators. However, these $\hat{x}_{s,h}, \hat{p}_{s,h}$ are commutative with $\hat{x}_\hbar, \hat{p}_\hbar$.

$$[\hat{x}_\hbar, \hat{x}_{s,h}] = 0,$$

$$[\hat{x}_\hbar, \hat{p}_{s,h}] = 0,$$

$$[\hat{p}_\hbar, \hat{x}_{s,h}] = 0,$$

$$[\hat{p}_\hbar, \hat{p}_{s,h}] = 0.$$  

Therefore, it is understood that the back reaction from $\hat{x}_{s,h}, \hat{p}_{s,h}$ does not need to be considered.
To recover the original KvN formalism, we expand the right-hand side of Eq. (3.5) in the power of $\hbar$,

$$i\frac{\partial}{\partial t}\vert\psi\rangle = \frac{1}{\hbar}H\vert\psi\rangle + \frac{1}{2}\left(-\frac{\partial H(\hat{x},\hat{p})}{\partial \hat{x}} \hat{p} + \frac{\partial H(\hat{x},\hat{p})}{\partial \hat{p}} \hat{x}\right)$$

$$- \frac{\hbar}{2 \cdot 2^2}\left(\frac{\partial^2 H(\hat{x},\hat{p})}{\partial \hat{x}^2} \hat{p}^2 + \frac{\partial^2 H(\hat{x},\hat{p})}{\partial \hat{p}^2} \hat{x}^2 + 2\frac{\partial^2 H(\hat{x},\hat{p})}{\partial \hat{x}\partial \hat{p}} \hat{x}\hat{p}\right) + \cdots.$$  

(3.6)  

(3.7)

Then, in the limit $\hbar \to 0$ this equation returns to the KvN formalism.

We comment that the state $\vert x_q\rangle$ in quantum mechanics corresponds to $\vert x, \pi \rangle$, not $\vert x, p \rangle$. Therefore, $\vert x, \pi \rangle$ and $\vert p, \pi \rangle$ in the KvN formalism have a connection with quantum theory in spite of classical mechanics.

4. The von Neumann model as a measurement theory of quantum mechanics

4.1. The von Neumann model

In this section, we introduce the von Neumann model as a simple example of the measurement model [19].

The system consists of a measurement target and a measurement device, and the corresponding physical quantities $\{\hat{x}, \hat{p}\}$, $\{\hat{X}, \hat{P}\}$ satisfy the canonical commutation relations

$$[\hat{x}, \hat{p}] = i\hbar,$$  

(4.1)

$$[\hat{X}, \hat{P}] = i\hbar,$$  

(4.2)

and the other commutators of $\hat{x}, \hat{p}, \hat{X}, \hat{P}$ are 0.

As an interaction between the measurement target and the measurement device, we introduce a Hamiltonian

$$\hat{H} = \hat{x}\hat{P}$$  

(4.3)

and the free Hamiltonian part is not considered for the sake of simplicity. Also, we take $t = 1$.

Then, the time evolution operator is given by

$$\hat{U} = e^{-ix\hat{P}}.$$  

(4.4)

We take the initial state as

$$\vert\psi\rangle = \vert\phi\rangle \otimes \vert\eta\rangle,$$  

(4.5)

where $\vert\phi\rangle$ is the initial state of the measurement target and $\vert\eta\rangle$ is the initial state of the measurement device.

The time evolution of state is expressed as

$$\vert\psi_{\text{after}}\rangle \equiv \hat{U}\vert\psi\rangle = \int dx \vert x\rangle \langle x|\phi\rangle \otimes e^{-ix\hat{P}}\vert\eta\rangle,$$  

(4.6)

and we expand $\vert\eta\rangle$ using $\vert X\rangle$ to obtain

$$\vert\psi_{\text{after}}\rangle = \int dxdX \langle x|\phi\rangle \langle X - x|\eta\rangle \otimes \vert X\rangle.$$  

(4.7)
Next, we perform a projective measurement on the measurement device. The probability that the measurement device (or needle) obtains $x_0$ is

$$
\int dx \left| \langle x | \phi \rangle \langle x - x_0 | \eta \rangle \right|^2 .
$$

(4.8)

If the initial state of the measurement device is $|0 \rangle_X$, this probability is

$$
\int dx \left| \langle x | \phi \rangle \langle x - x_0 | 0 \rangle_X \right|^2 = |\phi(x_0)|^2 .
$$

(4.9)

This equation is consistent with the results of the projective measurement of the measurement target [20].

4.2. Relative state

Because the projection hypothesis is an inherent problem in quantum mechanics, it is conceptually difficult to consider a measurement theory that includes the projection hypothesis in classical mechanics. Therefore, in this section, we will introduce the relative state.6

We use the notation

$$
|\eta[x]\rangle \equiv e^{-ix\hat{P}} |\eta\rangle.
$$

(4.10)

Equation (4.6) is then rewritten as

$$
|\psi_{\text{after}}\rangle = \int dx \langle x | \phi \rangle |x\rangle \otimes |\eta[x]\rangle.
$$

(4.11)

$|x\rangle \otimes |\eta[x]\rangle$ is called the relative state. In relative state interpretation, $|x\rangle \otimes |\eta[x]\rangle$ is interpreted as the measurement device observing its position as $x$.7

On the other hand, $|\psi_{\text{after}}\rangle$ can be expanded as follows,

$$
|\psi_{\text{after}}\rangle = \int dP \langle P | \eta \rangle |\phi(P)\rangle \otimes |P\rangle.
$$

(4.12)

This is different from the previous one, and it can be interpreted that the measurement target observed the momentum of the measurement device as $P$.

Note that these two propositions do not hold in relative state at the same time.

In contrast, these two propositions will hold in relative state at the same time in measurement theory in classical mechanics.

5. Measurement theory in classical mechanics

In this section, we discuss measurement theory in classical mechanics using the KvN formalism and the von Neumann model.

As in the previous section, the system consists of a measurement target and a measurement device, and the corresponding physical quantities $\{\hat{x}, \hat{p}, \hat{\pi}_x, \hat{\pi}_p\}, \{\hat{X}, \hat{P}, \hat{\pi}_X, \hat{\pi}_P\}$ satisfy the canonical

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6 This state is discussed by Everett [21].

7 In our discussion, we do not consider decoherence [22] because we do not adopt the many-worlds interpretation.
commutation relations

\[ [\hat{x}, \hat{p}] = 0, [\hat{x}, \hat{\pi}_x] = i, [\hat{p}, \hat{\pi}_p] = i, \quad (5.1) \]

\[ [\hat{X}, \hat{P}] = 0, [\hat{X}, \hat{\pi}_X] = i, [\hat{P}, \hat{\pi}_P] = i. \quad (5.2) \]

Note that the dimension of \( \hat{\pi}_x \) and \( \hat{\pi}_p \) is \([x^{-1}] \) and \([p^{-1}] \).

From the von Neumann model’s Hamiltonian \( \hat{H} = \hat{x}\hat{P} \), we obtain the Liouvillian

\[ \hat{L} = \frac{\partial \hat{H}}{\partial \hat{p}} \hat{\pi}_p - \frac{\partial \hat{H}}{\partial \hat{x}} \hat{\pi}_x \quad (5.3) \]

\[ = \hat{P} \hat{\pi}_p - \hat{x} \hat{\pi}_x \quad (5.4) \]

Then, the time evolution operator is given by\(^8\)

\[ \hat{U} = e^{-i(\hat{P} \hat{\pi}_p - \hat{x} \hat{\pi}_x)}. \quad (5.5) \]

As with quantum mechanics, the time evolution of state is obtained as

\[ |\psi_{\text{after}}\rangle = e^{-i(\hat{P} \hat{\pi}_p - \hat{x} \hat{\pi}_x)} |\phi\rangle \otimes |\eta\rangle \quad (5.6) \]

\[ = \int dx \, dp |\phi(p \rightarrow p - P)\rangle \otimes |\eta(X \rightarrow X + x)\rangle \quad (5.7) \]

\[ = \int dp \, dx \, dX \, dp \phi(x, p) \eta(X, P) |x, p[-P]\rangle \otimes |X[x], P\rangle, \quad (5.8) \]

where

\[ |\phi(p \rightarrow p - P)\rangle \equiv \int dp \phi(x, p)|x, p[-P]\rangle, \quad (5.9) \]

\[ |\eta(X \rightarrow X + x)\rangle \equiv \int dX \eta(X, P)|X[x], P\rangle. \quad (5.10) \]

The important difference from quantum mechanics is that, in relative state interpretation, two propositions

(1) the measurement device observed the position of the measurement target as \( x \),

(2) the measurement target observed the momentum of the measurement device as \( P \),

hold in relative state at the same time.\(^9\) \(^10\)

\(^8\) The time evolution is set to \( t = 1 \), but the situation is the same for Neumann’s measurement model in quantum theory. See Appendix F for details of this model.

\(^9\) As a more modern approach, we discuss the Kraus operator in classical mechanics in Appendix C. It has not been discussed in KvN formalism until now.

\(^10\) In classical mechanics, the Hamiltonian \( H \) of the von Neumann model is

\[ H = xP, \]

which definitely has a non-trivial interaction between \( x \) and \( P \). Therefore, the measurement target and the measurement device have a certain interaction with each other. If it uses quantum mechanical formalism, the Liouvillian \( \hat{L} \) of the von Neumann model is

\[ \hat{L} = \hat{P} \hat{\pi}_p - \hat{x} \hat{\pi}_x, \]

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6. Uncertainty relations in classical mechanics

6.1. Uncertainty relation in classical mechanics with $\hat{x}$ and $\hat{p}$

Here, we investigate the relationship between error and disturbance. The Ozawa’s inequality is a relational expression for error and disturbance [23]. We discuss how we can obtain the Ozawa’s inequality in classical mechanics.

We introduce error operator $\hat{N}(t) = \hat{X}(t) - \hat{x}$ and disturbance operator $\hat{D}(t) = \hat{p}(t) - \hat{p}$. The error $\epsilon$ and disturbance $\eta$ are defined by

$$\epsilon = \sqrt{\langle \hat{N}^2 \rangle} \geq \sigma(\hat{N}),$$

$$\eta = \sqrt{\langle \hat{D}^2 \rangle} \geq \sigma(\hat{D}),$$

where $\sigma(\hat{A}) \equiv \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2$. Using Kennard–Robertson uncertainty relations [24,25]

$$\sigma(\hat{N})\sigma(\hat{D}) \geq \frac{1}{2} \langle [\hat{N}, \hat{D}] \rangle,$$

we get

$$\epsilon \eta \geq \frac{1}{2} \langle [\hat{N}, \hat{D}] \rangle. \quad (6.4)$$

Next, $[\hat{N}(t), \hat{D}(t)]$ is calculated as

$$[\hat{N}(t), \hat{D}(t)] = [\hat{X}(t), \hat{p}(t)] - [\hat{x}, \hat{p}(t)] + [\hat{x}, \hat{p}]. \quad (6.5)$$

Under the reasonable assumption $[\hat{X}(t), \hat{p}(t)] = 0$, this equation yields

$$\langle [\hat{N}, \hat{D}] \rangle + \langle [\hat{X}(t), \hat{p}] + [\hat{x}, \hat{p}(t)] \rangle = \langle [\hat{x}, \hat{p}] \rangle. \quad (6.6)$$

Since $\hat{x}$ and $\hat{p}$ commute in classical mechanics, we get

$$\langle [\hat{N}, \hat{D}] \rangle + \langle [\hat{X}(t), \hat{p}] \rangle + \langle [\hat{x}, \hat{p}(t)] \rangle = 0 \quad (6.7)$$

and its time differentiation

$$\frac{d}{dt} \langle [\hat{N}, \hat{D}] \rangle + \langle [\dot{\hat{X}}(t), \hat{p}] \rangle + \langle [\dot{\hat{x}}, \hat{p}(t)] \rangle = 0. \quad (6.8)$$

Because the Liouvillian is given by

$$\hat{L} = \frac{\partial \hat{H}}{\partial \hat{p}} \hat{\pi}_x - \frac{\partial \hat{H}}{\partial \hat{x}} \hat{\pi}_p + \frac{\partial \hat{H}}{\partial \hat{p}} \hat{\pi}_x - \frac{\partial \hat{H}}{\partial \hat{x}} \hat{\pi}_p, \quad (6.9)$$

we get

$$\langle [\dot{\hat{X}}(t), \hat{p}] \rangle + \langle [\dot{\hat{x}}, \hat{p}(t)] \rangle = 0. \quad (6.10)$$

as if there is no interaction between $\hat{x}$ and $\hat{P}$, and as a result, simultaneous observation is possible. This is a non-trivial result specific to the KvN formalism.
After all, we get

$$\langle [\hat{N}, \hat{p}] \rangle + \langle [\hat{x}, \hat{D}] \rangle = -C,$$

$$\langle [\hat{N}, \hat{D}] \rangle = C \tag{6.12}$$

where $C$ is some time-invariant constant. Then, for $t = 0$,

$$\langle [\hat{N}(0), \hat{D}(0)] \rangle = 0 = C. \tag{6.13}$$

Then we obtain

$$\langle [\hat{N}, \hat{p}] \rangle = \langle [\hat{D}, \hat{x}] \rangle, \langle [\hat{N}, \hat{D}] \rangle = 0. \tag{6.14}$$

Note that

$$\epsilon \eta \geq 0, \epsilon \sigma(\hat{p}) + \eta \sigma(\hat{x}) \geq 0 \tag{6.15}$$

are trivial inequalities. Now, we discuss the condition of the equal sign.

From the Heisenberg equation for $\hat{L} = \hat{P} \hat{\pi}_p - \hat{x} \hat{\pi}_X$, we obtain

$$\frac{d\hat{x}}{dt} = 0, \frac{d\hat{p}}{dt} = -\hat{P}, \tag{6.16}$$

$$\frac{d\hat{X}}{dt} = \hat{x}, \frac{d\hat{P}}{dt} = 0, \tag{6.17}$$

$$\frac{d\hat{\pi}_x}{dt} = \hat{\pi}_X, \frac{d\hat{\pi}_p}{dt} = 0, \tag{6.18}$$

$$\frac{d\hat{\pi}_X}{dt} = 0, \frac{d\hat{\pi}_P}{dt} = -\hat{\pi}_p. \tag{6.19}$$

Therefore, with the initial conditions of $\hat{x}(0) = \hat{x}_0, \hat{p}(0) = \hat{p}_0, \hat{\pi}_x(0) = \hat{\pi}_{x0}, \hat{\pi}_p(0) = \hat{\pi}_{p0}, \hat{X}(0) = \hat{X}_0, \hat{P}(0) = \hat{P}_0, \hat{\pi}_X(0) = \hat{\pi}_{X0}$ and $\hat{\pi}_P(0) = \hat{\pi}_{P0}$, we obtain

$$\hat{x}(t) = \hat{x}_0, \hat{p}(t) = \hat{p}_0 - t\hat{P}_0, \tag{6.20}$$

$$\hat{X}(t) = \hat{X}_0 + t\hat{x}_0, \hat{P}(t) = \hat{P}_0, \tag{6.21}$$

$$\hat{\pi}_x(t) = \hat{\pi}_{x0} + t\hat{\pi}_{X0}, \hat{\pi}_p(t) = \hat{\pi}_{p0}, \tag{6.22}$$

$$\hat{\pi}_X(t) = \hat{\pi}_{X0}, \hat{\pi}_P(t) = \hat{\pi}_{P0} - t\hat{\pi}_{P0}. \tag{6.23}$$

Using these results, $\hat{N}$ and $\hat{D}$ are expressed as

$$\hat{N}(t) = \hat{X}_0 + (t - 1)\hat{x}_0, \tag{6.24}$$

$$\hat{D}(t) = -t\hat{P}_0. \tag{6.25}$$

These equations are the same as the result obtained by the von Neumann model of quantum mechanics.

Then,

$$\langle x, p, X, P | \hat{N}(t) | x, p, X, P \rangle = X + (t - 1)x, \tag{6.26}$$

$$\langle x, p, X, P | \hat{D}(t) | x, p, X, P \rangle = -tP. \tag{6.27}$$
Therefore, the unbiased condition is given by

\[ X = (1 - t)x, P = 0. \]  
(6.28)

Under this condition

\[ \langle x, p, X, P | \hat{N}(t)^2 | x, p, X, P \rangle = (X + (t - 1)x)^2 = 0, \]  
(6.29)

\[ \langle x, p, X, P | \hat{D}(t)^2 | x, p, X, P \rangle = t^2 P^2 = 0. \]  
(6.30)

Then, in this condition, we get

\[ \epsilon = \eta = 0, \]  
(6.31)

\[ \sigma(p) = \sigma(x) = 0. \]  
(6.32)

If the unbiased condition is not satisfied, at \( t = 1 \) we get

\[ \epsilon = X, \eta = P. \]  
(6.33)

This represents the initial calibration of the device. In such a case, \( X = 0 \) or \( P = 0 \) is the condition of the equal sign of Eq. (6.15). The case where the initial condition is the only known probability is discussed in Appendix D.

6.2. Uncertainty relations in classical mechanics with \( \hat{\pi}_x \) and \( \hat{\pi}_p \)

We clarify the role of \( \hat{\pi}_x \) and \( \hat{\pi}_p \) in classical mechanics. Although \( \hat{\pi}_x \) and \( \hat{\pi}_p \) are hidden variables in classical mechanics, they can be expressed as physical quantities by combining quantum mechanics and classical mechanics.

As confirmed in Sect. 3, \( \hat{\pi}_x \) and \( \hat{\pi}_p \) are described by

\[ \hat{\pi}_x = -\frac{2}{\hbar} (\hat{p}_h - \hat{p}), \]  
(6.34)

\[ \hat{\pi}_p = \frac{2}{\hbar} (\hat{x}_h - \hat{x}). \]  
(6.35)

In these relational expressions, we can determine \( \hat{\pi}_x \) and \( \hat{\pi}_p \) by using both classical and quantum observables.\(^{11}\)

Therefore, in addition to the usual disturbance \( \hat{D} \), we should consider another disturbance,

\[ \hat{D}_{\pi_x}(t) = \hat{\pi}_x(t) - \hat{\pi}_x. \]  
(6.36)

By a similar argument such as

\[ [\hat{N}(t), \hat{D}_{\pi_x}(t)] = [\hat{X}(t), \hat{\pi}_x(t)] - [\hat{X}(t), \hat{\pi}_x] - [\hat{x}, \hat{\pi}_x(t)] + [\hat{x}, \hat{\pi}_x], \]  
(6.37)

using Kennard-Robertson uncertainty relations, non-commutativity of \( \hat{x} \) and \( \hat{\pi}_x \) gives an Ozawa-like inequality:

\[ \epsilon \eta_{\pi_x} + \epsilon \sigma(\hat{\pi}_x) + \sigma(\hat{x}) \eta_{\pi_x} \geq \frac{1}{2}, \]  
(6.38)

\(^{11}\) Note that these equations can also be regarded as differentiation by Planck’s constant. This helps us understand the relationship between KvN formalism and quantum theory by treating the derivative of Planck’s constant as an operator. See Appendix E.
where $\eta_{\pi_x} = \sqrt{\langle \hat{D}_{\pi_x} \rangle}$. Since the Planck constant does not appear in this inequality, it holds even in the classical mechanical limit.

7. Discussion

We constructed the measurement theory of classical mechanics.

In contrast to quantum mechanics, we have found two propositions hold in relative state at the same time.

1. The measurement device observed the position of the measurement target as $x$.
2. The measurement target observed the momentum of the measurement device as $P$.

This difference in simultaneity corresponds to the result of the discussion of uncertainty relations, and Ozawa’s inequality becomes trivial in classical mechanics. If the initial state is not well known, we can obtain a relational expression about the error and the disturbance in the von Neumann model.

We extended the KvN formalism to quantum theory and determined $\hat{\pi}_x$ and $\hat{\pi}_p$ using both classical and quantum observables. Then, we also introduced another disturbance on $\hat{\pi}_x$ and obtained an Ozawa-like uncertainty relation. Since this relation is independent of Planck’s constant, it holds in classical mechanics. Treating $\hat{\pi}_x$ and $\hat{\pi}_p$ as observables is known as generalized classical mechanics [7]. In this case, the Louvillian becomes an observable, and its eigenvalues are closely related in the conditions in ergodic theory [26]. As already mentioned, $\hat{\pi}_x$ and $\hat{\pi}_p$ are closely related to the effect of quantum fluctuation. This uncertainty relation may be significant in the theory of the intermediate scale between classical theory and quantum theory.

The application of these relational expressions to behavioral economics in recent years is astonishing. Through these applications, the role of phase in classical mechanics may be newly understood.

As an application of measurement theory in classical mechanics, it is possible to analytically formulate thought experiments in classical mechanics such as Maxwell’s demon and Einstein’s optical clock [27]. Although many have discussed these in the past, our study can contribute to the conceptual discussion of science. Further research will reveal these.

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Appendix A. Measurement interpretation of classical mechanics

We comment that the KvN formalism for a free particle can be regarded as a von Neumann model.

In such a case, the Liouvillian is give by

$$\hat{L} = -\frac{\hat{p}}{m} \hat{\pi}_x.$$  \hspace{1cm} (A.1)

If we regard $|p,x\rangle$ as a composite of the measurement target $|p\rangle$ and the measurement device $|x\rangle$, the time evolution of the free particle

$$e^{-i\frac{\hat{L}}{\hbar}t}|p\rangle|x\rangle = |p\rangle|x + \frac{p}{m}t\rangle$$  \hspace{1cm} (A.2)
can be regarded as the observation of position \( x \) by momentum \( p \).

Note that the disturbance in this case is \( \hat{D}_{\hat{\pi}_p} = \hat{\pi}_p(t) - \hat{\pi}_p. \hat{D}_{\hat{\pi}_p} \) is defined in Sect. 6.

**Appendix B. Measurement theory in the extended KvN formalism**

We discuss the von Neumann model in a formalism that extends the KvN formalism to quantum mechanics, which is discussed in Sect. 3.

The Hamiltonian of the von Neumann model is

\[
\hat{H} = \hat{x}\hat{P}_h = \left( \hat{x} + \frac{\hbar}{2}\hat{\pi}_p \right) \left( \hat{P} - \frac{\hbar}{2}\hat{x}_X \right) = \hat{x}\hat{P} + \frac{\hbar}{2} \left( \hat{\pi}_p \hat{P} - \hat{x}\hat{\pi}_X \right) - \frac{\hbar^2}{4}\hat{\pi}_p\hat{\pi}_X. \tag{B.1}
\]

We take \( |x, \pi_p \rangle \otimes |X, P \rangle \) as the initial condition. We assume in this initial condition that the measurement target is quantum, and the measurement device is classical.

The time evolution of state is

\[
e^{i\hat{H}t/\hbar}|x, \pi_p \rangle \otimes |X, P \rangle = e^{ixP_t/\hbar + i\frac{1}{2}(\pi_p\hat{P}_X - x\hat{\pi}_X)t - \frac{\hbar}{2}\pi_p\hat{\pi}_X}|x, \pi_p \rangle \otimes |X, P \rangle \tag{B.2}
\]

\[
= e^{ix + i\frac{1}{2}\pi_p P_t}|x, \pi_p \rangle \otimes |X + \frac{1}{2}(x + \frac{\hbar}{2}\pi_p) t, P \rangle \tag{B.3}
\]

\[
= e^{ixP_t}|x, \pi_p \rangle \otimes |X + \frac{1}{2}x_P t, P \rangle. \tag{B.4}
\]

Next, we take \( |x, \pi_p \rangle \otimes |\pi_X, P \rangle \) as the initial condition. We assume in this initial condition that the measurement target is quantum, and the measurement device is also quantum.

The time evolution of state is

\[
e^{i\hat{H}t/\hbar}|x, \pi_p \rangle \otimes |X, \pi_p \rangle = e^{ixP_t/\hbar + i\frac{1}{2}(\pi_p\hat{P}_X - x\hat{\pi}_X)t - \frac{\hbar}{2}\pi_p\hat{\pi}_X}|x, \pi_p \rangle \otimes |X, \pi_p \rangle \tag{B.5}
\]

\[
= e^{ix + i\frac{1}{2}\pi_p P_t}|x, \pi_p \rangle \otimes |X + \frac{1}{2}(x + \frac{\hbar}{2}\pi_p) t|\pi_p + \left( x + \frac{1}{2}P t \right)/h \rangle \tag{B.6}
\]

\[
= e^{ixP_t}|x, \pi_p \rangle \otimes |X + \frac{1}{2}x_P t, \pi_p + x_P/h \rangle. \tag{B.7}
\]

Note that \( |X + \frac{1}{2}x_P t, \pi_p + x_P/h \rangle \) is an eigenstate of \( \hat{\pi}_p \):

\[
\hat{\pi}_p|X + \frac{1}{2}x_P t, \pi_p + x_P/h \rangle = \left( \hat{\pi}_p + \frac{\hbar}{2}\hat{\pi}_p \right)|X + \frac{1}{2}x_P t, \pi_p + x_P/h \rangle. \tag{B.8}
\]

\[
= x_P|X + \frac{1}{2}x_P t, \pi_p + x_P/h \rangle \tag{B.9}
\]

As described above, the quantum measurement also affects the \( \hat{\pi}_p \) side.

**Appendix C. The Kraus operator**

We discuss the Kraus operator in classical mechanics, which is used in more modern measurement theory.\(^{12}\) It has not been discussed in KvN formalism until now.

The Kraus operator is obtained by integrating out the measurement device.

\(^{12}\) Notation follows Sect. 5.
In quantum mechanics, a state \(|\psi(t)\rangle\) is given by

\[
|\psi(t)\rangle = \int dX |X\rangle \langle X| \hat{U}(t)|\phi\rangle |\eta\rangle = \int dX \hat{M}(X, t)|\phi\rangle |X\rangle,
\] (C.1)

where

\[
\hat{M}(X, t) \equiv \langle X| \hat{U}(t)|\eta\rangle = \int dx \langle X - x|\eta\rangle |x\rangle.
\] (C.2)

The Positive Operator Valued Measure (POVM) \(\hat{E}\) is constructed as

\[
\hat{E}(X, t) = \hat{M}^\dagger(X, t)\hat{M}(X, t).
\] (C.3)

The probability of projective measurement of the measurement device is

\[
\text{Pr}(X) = \langle \psi(t)|X\rangle \langle X|\psi(t)\rangle = \langle \psi(t)|\hat{E}(X, t)|\psi(t)\rangle.
\] (C.4)

The state after measurement is

\[
\hat{M}(X, t)|\phi\rangle |X\rangle.
\] (C.5)

On the other hand, integrating out with \(P\) gives another \(\hat{M}\),

\[
\hat{M}(P, t) \equiv \langle P| \hat{U}|\eta\rangle = e^{-i\hat{p}P} \langle P|\eta\rangle
\] (C.6)

In classical mechanics, a state \(|\psi(t)\rangle\) is given by

\[
|\psi(t)\rangle = \int dXdP |X, P\rangle \langle X| \hat{U}(t)|\phi\rangle |\eta\rangle = \int dXdP \hat{M}(X, P, t)|\phi\rangle |X, P\rangle,
\] (C.7)

where

\[
\hat{M}(X, P, t) \equiv \langle X, P| \hat{U}(t)|\eta\rangle = \int dx dp \langle X - x, P|\eta\rangle |x, p\rangle |x, p - P\rangle.
\] (C.8)

In the same way,

\[
\hat{M}(X, \pi P, t) = \int dxd\pi p \langle X + x, \pi P - \pi p|\eta\rangle |x, \pi p\rangle |x, \pi p|,
\] (C.9)

\[
\hat{M}(\pi X, P, t) = e^{iP\hat{p}P - i\hat{x}\pi X} \langle \pi X, P|\eta\rangle,
\] (C.10)

\[
\hat{M}(\pi X, \pi P, t) = \int dxd\pi p \langle \pi X, P - \pi p|\eta\rangle |x, \pi p| |x + \pi X, \pi p|.
\] (C.11)

From the form of these expressions, \(\hat{M}(X, t)\) in quantum mechanics corresponds to \(\hat{M}(X, \pi p, t)\) in classical mechanics and \(\hat{M}(P, t)\) in quantum mechanics corresponds to \(\hat{M}(\pi X, P, t)\).

There is no counterpart to \(\hat{M}(X, P, t), \hat{M}(\pi X, \pi P, t)\) in quantum mechanics.

**Appendix D. The case of probability**

Consider the case where the initial condition is the only known probability:

\[
|\psi(0)\rangle = |\phi\rangle |\eta\rangle,
\] (D.1)
\[
\langle \phi, \eta | \hat{N} | \phi, \eta \rangle = \int dx \, dp \, dX \, dP \, (X + (t - 1)x)|\phi(x, p)|^2 |\eta(X, P)|^2
\]
\[
= \langle X \rangle \eta + (t - 1)x \phi, \quad \text{(D.2)}
\]
\[
\langle \phi, \eta | \hat{P} | \phi, \eta \rangle = \int dx \, dp \, dX \, dPtP \, (X + (t - 1)x)|\phi(x, p)|^2 |\eta(X, P)|^2 = t \langle P \rangle \eta.
\]
\[
\text{(D.4)}
\]
Then, the unbiased condition is
\[
\langle X \rangle \eta = (1 - t) \langle x \rangle \phi, \quad \langle P \rangle \eta = 0.
\]
\[
\text{(D.5)}
\]
Under this condition,
\[
\epsilon = \eta = 0, \quad \text{(D.6)}
\]
\[
\sigma(p) = \sigma(x) = 0.
\]
\[
\text{(D.7)}
\]
If the unbiased condition is not satisfied, at \( t = 1 \) we get
\[
\epsilon = \langle X \rangle \eta, \quad \eta = \langle P \rangle \eta, \quad \text{(D.8)}
\]
\[
\langle X \rangle \eta \sigma(p) + \langle P \rangle \eta \sigma(x) = 0.
\]
\[
\text{(D.9)}
\]
**Appendix E. Planck Operator**

In quantum mechanics, \( \hat{x}_q \) and \( \hat{p}_q \) are commutatively related;
\[
[\hat{x}_q, \hat{p}_q] = i \hbar, \quad \text{(E.1)}
\]
and the time development of state is
\[
i \hbar \frac{d}{dt} |\psi\rangle = \hat{H} |\psi\rangle, \quad \text{(E.2)}
\]
where \( \hat{H} \) is Hamiltonian. Usually, the Planck constant \( \hbar \) is constant. We suggest new quantum algebra
\[
[\hat{x}_q, \hat{p}_q] = i \hat{\hbar}, \quad \text{(E.3)}
\]
where \( \hat{\hbar} \) is operator. We call \( \hat{\hbar} \) the Planck operator.

We introduce \( \hat{\hbar} \)'s conjugate operator \( \hat{I} \), then
\[
[\hat{\hbar}, \hat{I}] = i. \quad \text{(E.4)}
\]
We consider Planck's constant as an operator because it naturally introduces the relation between quantum mechanics and classical mechanics.

We introduce classical mechanics operators \( \hat{x}, \hat{p}, \hat{x}_x, \hat{x}_p \), and commutative relations
\[
[\hat{x}, \hat{p}] = 0, \quad \text{(E.5)}
\]
\[
[\hat{x}, \hat{x}_x] = i, \quad \text{(E.6)}
\]
\[
[\hat{p}, \hat{x}_p] = i. \quad \text{(E.7)}
\]
These constitute the KvN formalism of classical mechanics.
In classical mechanics, the time development of state is
\[ \dot{i} \frac{d}{dt} |\psi_c\rangle = \hat{L} |\psi_c\rangle, \] (E.8)
where \( \hat{L} \) is Liouvillian.

Now, we introduce commutative relations of \( \hat{I} \) and \( \hat{x}_q \) such as
\[ [\hat{I}, \hat{x}_q] = \hat{\pi}_p, \] (E.9)
\[ [\hat{I}, \hat{p}_q] = -\hat{\pi}_x, \] (E.10)
\[ [\hat{I}, \hat{x}] = [\hat{I}, \hat{p}] = [\hat{I}, \hat{\pi}_x] = [\hat{I}, \hat{\pi}_p] = 0. \] (E.11)

We obtain
\[ \hat{x}_q = \hat{x} + \hbar \hat{\pi}_p \] (E.12)
\[ \hat{p}_q = \hat{p} + \hbar \hat{\pi}_x \] (E.13)
\[ \hat{x}_q = e^{-\hbar \hat{\pi}_p \hat{x} e^{+\hbar \hat{\pi}_p \hat{x}}} \] (E.14)
\[ \hat{p}_q = e^{-\hbar \hat{\pi}_p \hat{x} e^{+\hbar \pi_p \hat{x}}} \] (E.15)

These relations introduce a natural relation of \( \hat{L} \) and \( \hat{H} \), such as
\[ \hat{L} = e^{\hbar \hat{\pi}_p \hat{x}} [\hat{I}, \hat{H}] e^{-\hbar \hat{\pi}_p \hat{x}}, \] (E.16)

because
\[ e^{\hbar \hat{\pi}_p \hat{x}} [\hat{I}, \hat{H}(\hat{x}_q, \hat{p}_q)] e^{-\hbar \hat{\pi}_p \hat{x}} = e^{\hbar \hat{\pi}_p \hat{x}} \left( \frac{\partial \hat{H}}{\partial \hat{x}_q} [\hat{I}, \hat{x}_q] + \frac{\partial \hat{H}}{\partial \hat{p}_q} [\hat{I}, \hat{p}_q] \right) e^{-\hbar \hat{\pi}_p \hat{x}} \] (E.17)
\[ = e^{\hbar \hat{\pi}_p \hat{x}} \left( \frac{\partial \hat{H}}{\partial \hat{x}_q} \hat{x}_q - \frac{\partial \hat{H}}{\partial \hat{p}_q} \hat{p}_q \right) e^{-\hbar \hat{\pi}_p \hat{x}} \] (E.18)
\[ = \frac{\partial \hat{H}(\hat{x}, \hat{p})}{\partial \hat{x}} \hat{p}_q - \frac{\partial \hat{H}(\hat{x}, \hat{p})}{\partial \hat{p}} \hat{x}_q = \hat{L}. \] (E.19)

We may also introduce thermal fluctuation. We may describe thermal fluctuation and quantum fluctuation at once.

**Appendix F. Time evolution of the von Neumann model**

Here, we discuss the time dependence of the von Neumann model explicitly following Ref. [28].
\[ V = \epsilon g(t) x P, \] (F.1)
and if the interaction occurs only at \( t = t_1 \), \( g(t) \) can be approximated as
\[ g(t) = \delta(t - t_1). \] (F.2)

Therefore in quantum theory, using
\[ \dot{V}(t) = \epsilon g(t) \dot{x} \dot{P}, \] (F.3)
we obtain the Hamiltonian as
\[ \hat{H} = \hat{H} + \hat{V}(t), \]
and the time evolution as
\[ i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle. \]
We set the initial state to
\[ |\psi(0)\rangle = |\phi\rangle |\eta\rangle. \]
Therefore we describe the time evolution of the state as
\[ |\psi\rangle_I = \hat{U}_0(t) |\psi(0)\rangle \equiv \hat{U}_f(t) |\psi(0)\rangle, \]
\[ \hat{U}_f(t) = \hat{U}_0(t) \hat{U}(t), \]
\[ i\hbar \frac{\partial}{\partial t} \hat{U}_f(t) = \hat{V}_f(t) \hat{U}_f(t), \]
\[ \hat{V}_f(t) = \hat{U}_0^\dagger(t) \hat{V}(t) \hat{U}_0(t), \]
\[ \hat{V}_f(t) = \epsilon \delta(t - t_1) \hat{x}(t) \hat{P}(t) \]
in the interaction picture.
At \( t > t_1, \) \( \hat{U}(t) \) is
\[ \hat{U}_f(t) = e^{-\frac{i}{\hbar} \int_0^t \hat{V}_f(t') dt'} = e^{-\frac{i}{\hbar} \epsilon \hat{x}(t) \hat{P}(t_1)} = \hat{U}_0(t_1) e^{-\frac{i}{\hbar} \epsilon \hat{x} \hat{p}} \hat{U}_0(t_1) = \hat{U}_{f1}. \]
Thus, if we convert this to a Schrödinger picture, we obtain
\[ \hat{U}_f = \hat{U}_0(t) \hat{U}_{f1} = \hat{U}_0(t - t_1) e^{-\frac{i}{\hbar} \epsilon \hat{x} \hat{p}} \hat{U}_0(t_1). \]
We can understand \( \hat{U}_f \) that the free motion until \( t = t_1, \) then the interaction \( t = t_1 \) and finally the free motion is performed.
This discussion can be repeated in classical mechanics of KvN formalism.
In this case, the Liouvillian \( \hat{L} \) is
\[ \hat{L} = \hat{L}_0 + \hat{L}_1, \]
\[ \hat{L}_0 = \frac{\partial H_0}{\partial \hat{p}} \hat{\pi}_x - \frac{\partial H_0}{\partial \hat{x}} \hat{\pi}_p + \frac{\partial H_0}{\partial \hat{p}} \hat{\pi}_x - \frac{\partial H_0}{\partial \hat{x}} \hat{\pi}_p, \]
\[ \hat{L}_1 = \frac{\partial V}{\partial \hat{p}} \hat{\pi}_x - \frac{\partial V}{\partial \hat{x}} \hat{\pi}_p + \frac{\partial V}{\partial \hat{p}} \hat{\pi}_x - \frac{\partial V}{\partial \hat{x}} \hat{\pi}_p = \epsilon \delta(t - t_1) \left( \hat{P} \hat{\pi}_p - \hat{\pi}_x \hat{x} \right). \]
By the same argument as quantum mechanics, we obtain
\[ \hat{U}_f = \hat{U}_0(t) \hat{U}_{f1} = \hat{U}_0(t - t_1) e^{-\frac{i}{\hbar} \epsilon \hat{x} \hat{p}} \hat{U}_0(t_1). \]
This shows that, like quantum mechanics, we can regard \( \hat{U}_f \) as performing free motion until \( t = t_1, \) interacting at \( t = t_1, \) and after that performing the free motion.
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