INTERIOR SCHAUDER ESTIMATES FOR THE FOURTH ORDER HAMILTONIAN STATIONARY EQUATION IN TWO DIMENSIONS

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Abstract. We consider the Hamiltonian stationary equation for all phases in dimension two. We show that solutions that are $C^{1,1}$ will be smooth and we also derive a $C^{2,\alpha}$ estimate for it.

1. Introduction

In this paper, we study the regularity of the Lagrangian Hamiltonian stationary equation, which is a fourth order nonlinear PDE. Consider the function $u : B_1 \to \mathbb{R}$ where $B_1$ is the unit ball in $\mathbb{R}^2$. The gradient graph of $u$, given by $\{(x, Du(x)) | x \in B_1\}$ is a Lagrangian submanifold of the complex Euclidean space. The function $\theta$ is called the Lagrangian phase for the gradient graph and is defined by

$$\theta = F(D^2 u) = \text{Im} \log \det (I + iD^2 u)$$

or equivalently,

$$(1.1) \quad \theta = \sum_i \arctan(\lambda_i)$$

where $\lambda_i$ represents the eigenvalues of the Hessian.

The nonhomogenous special Lagrangian equation is given by the following second order nonlinear equation

$$(1.2) \quad F(D^2 u) = f(x).$$

The Hamiltonian stationary equation is given by the following fourth order nonlinear PDE

$$(1.3) \quad \Delta_g \theta = 0$$

where $\Delta_g$ is the Laplace-Beltrami operator, given by:

$$\Delta_g = \sum_{i,j=1}^{2} \frac{\partial_i (\sqrt{\det g} g^{ij} \partial_j)}{\sqrt{\det g}}$$
and $g$ is the induced Riemannian metric from the Euclidean metric on $\mathbb{R}^4$, which can be written as

$$g = I + (D^2 u)^2.$$

Recently, Chen and Warren [CW16] proved that in any dimension, a $C^{1,1}$ solution of the Hamiltonian stationary equation will be smooth with uniform estimates of all orders if the phase $\theta \geq \delta + (n - 2)\pi/2$, or, if the bound on the Hessian is small. In the two dimensional case, using [CW16]'s result, we get uniform estimates for $u$ when $|\theta| \geq \delta > 0$ (by symmetry). In this paper, we consider the Hamiltonian stationary equation for all phases in dimension two without imposing a smallness condition on the Hessian or on the range of $\theta$, and we derive uniform estimates for $u$, in terms of the $C^{1,1}$ bound which we denote by $\Lambda$. We write $||u||_{C^{1,1}(B_1)} = ||Du||_{C^{0,1}(B_1)} = \Lambda$. Our main results are the following:

**Theorem 1.1.** Suppose that $u \in C^{1,1}(B_1) \cap W^{2,2}(B_1)$ and satisfies (1.3) on $B_1 \subset \mathbb{R}^2$. Then $u$ is a smooth function with interior Hölder estimates of all orders, based on the $C^{1,1}$ bound of $u$.

**Theorem 1.2.** Suppose that $u \in C^{1,1}(B_1) \cap W^{2,2}(B_1)$ and satisfies (1.2) on $B_1 \subset \mathbb{R}^2$. If $\theta \in C^\alpha(B_1)$, then there exists $R = R(2, \Lambda, \alpha) < 1$ such that $u \in C^{2,\alpha}(B_R)$ and satisfies the following estimate

$$|D^2 u|_{C^{\alpha}(B_R)} \leq C_1(||u||_{L^\infty(B_1)}, \Lambda, |\theta|_{C^{\alpha}(B_1)}).$$

Our proof goes as follows: we start by applying the De Giorgi-Nash theorem to the uniformly elliptic Hamiltonian stationary equation (1.3) on $B_1$ to prove that $\theta \in C^\alpha(B_{1/2})$. Next we consider the non-homogenous special Lagrangian equation (1.2) where $\theta \in C^\alpha(B_{1/2})$. Using a rotation of Yuan [Yua02] we rotate the gradient graph so that the new phase $\tilde{\theta}$ of the rotated gradient graph satisfies $|\tilde{\theta}| \geq \delta > 0$. Now we apply [CC01] to the new potential $\tilde{u}$ of the rotated graph to obtain a $C^{2,\alpha}$ interior estimate for it. On rotating back the rotated gradient graph to our original gradient graph, we see that our potential $u$ turns out to be $C^{2,\alpha}$ as well. A computation involving change of co-ordinates gives us the corresponding $C^{2,\alpha}$ estimate, shown in (1.4). Once we have a $C^{2,\alpha}$ solution of (1.3), smoothness follows by [CW16, Corollary 5.1].

In two dimensions, solutions to the second order special Lagrangian equation

$$F(D^2 u) = C$$
enjoy full regularity estimates in terms of the potential \( u \) \cite{WY09}. For higher dimensions, such estimates fail \cite{WY13} for \( \theta = C \) with \( |C| < (n - 2)\pi/2 \).

2. Proof of theorems:

We first prove Theorem 1.2, followed by the proof of Theorem 1.1. We prove Theorem 1.2 using the following lemma.

**Lemma 2.1.** Suppose that \( u \in C^{1,1}(B_1) \cap W^{2,2}(B_1) \) satisfies (1.2) on \( B_1 \subset \mathbb{R}^2 \). Suppose

\[
0 \leq \theta(0) < \frac{\pi}{2} - \arctan \Lambda \leq \frac{\pi}{4}.
\]

If \( \theta \in C^{\alpha}(B_1) \), then there exists \( 0 < \alpha < \bar{\alpha} \) and \( C_0 \) such that

\[
|D^2u(x) - D^2u(0)| \leq C_0(\|u\|_{L^{\infty}(B_1)}, \Lambda, \|\theta\|_{C^{\alpha}(B_1)}) \cdot |x|^\alpha.
\]

**Proof.** Consider the gradient graph \( \{(x, Du(x))| x \in B_1\} \) where \( u \) has the following Hessian bound

\[-\Lambda I_n \leq D^2 u \leq \Lambda I_n \]

a.e. where it exists.

Define \( \delta \) as

\[
\delta = \frac{(\pi/2 - \arctan \Lambda)}{2} > 0.
\]

Since by (2.1) we have \( 0 \leq \theta(0) < \delta/2 \), there exists \( R'(\delta, \|\theta\|_{C^{\alpha}}) > 0 \) such that

\[
|\theta(x) - \theta(0)| < \delta/2
\]

for all \( x \in B_{R'} \subseteq B_1 \). This implies for every \( x \) in \( B_{R'} \) for which \( D^2u \) exists, we have

\[
\delta > \theta > \theta(0) - \delta/2.
\]

So now we rotate the gradient graph \( \{(x, Du(x))| x \in B_{R'}\} \) downward by an angle of \( \delta \).

Let the new rotated co-ordinate system be denoted by \( (\bar{x}, \bar{y}) \) where

\[
\bar{x} = \cos(\delta)x + \sin(\delta)Du(x),
\]

\[
\bar{y} = -\sin(\delta)x + \cos(\delta)Du(x).
\]

On differentiating \( \bar{x} \) (2.2) with respect to \( x \) we see that

\[
\frac{d\bar{x}}{dx} = \cos(\delta)I_n + \sin(\delta)D^2u(x) \leq \cos(\delta)I_n + \Lambda \sin(\delta)I_n.
\]

Thus

\[
\cos(\delta)I_n - \Lambda \sin(\delta)I_n \leq \frac{d\bar{x}}{dx} \leq \cos(\delta)I_n + \Lambda \sin(\delta)I_n.
\]
To obtain Lipschitz constants so that
\[
\frac{1}{L_2} I_n \leq \frac{d\bar{x}}{dx} \leq L_1 I_n
\]
let
\[
L_1 = \cos(\delta) + \Lambda \sin(\delta)
\]
\[
L_2 = \max \left\{ \left| \frac{1}{\cos(\delta) I_n + D^2 u(x) \sin(\delta)} \right| \middle| x \in B_{R'} \right\}.
\]
To find the value of $L_2$, we see that in $B_{R'}$ we have the following:
let $\min \{ \theta_1, \theta_2 \} \geq -A$ where $A = \arctan \Lambda$.
\[
\cos(\delta) I_n + \sin(\delta) D^2 u(x) \geq \cos(\delta) - \sin(\delta) \tan(A)
\]
\[
= \cos(\delta)(1 - \tan(\delta) \tan(A))
\]
\[
= \cos(\delta) \frac{\tan(\delta) + \tan(A)}{\tan(\delta + A)}
\]
\[
= \cos(\delta) \frac{\tan(\delta) + \tan(A)}{\tan(\pi/2 - A)}
\]
\[
= \cos(\delta) \frac{\tan(\delta) + \tan(A)}{\tan(\pi/2 - \delta)}.
\]
This shows that
\[
\frac{1}{L_2} = \cos(\delta) \frac{\tan(\delta) + \tan(A)}{\tan(\pi/2 - \delta)}.
\]
Clearly $1/L_2$ is positive.

Now, by [CW16, Prop 4.1] we see that there exists a function $\bar{u}$ such that
\[
\bar{y} = D_x \bar{u} (\bar{x})
\]
deﬁnes $\bar{u}$ implicity in terms of $\bar{x}$ (since $\bar{x}$ is invertible). Here $\bar{x}$ refers to the rotation map (2.2).

Note that
\[
\bar{\theta}(\bar{x}) - \bar{\theta}(\bar{y}) = \theta(x) - \theta(y)
\]
which implies that $\bar{\theta}$ is also a $C^\alpha$ function
\[
\frac{|\bar{\theta}(\bar{x}_1) - \bar{\theta}(\bar{x}_2)|}{|\bar{x}_1 - \bar{x}_2|^\alpha} = \frac{|\theta(x_1) - \theta(x_2)|}{|x_1 - x_2|^\alpha} \ast \frac{|x_1 - x_2|^{\frac{\alpha}{2}}}{|\bar{x}_1 - \bar{x}_2|^\alpha}
\]
thus,
\[ |\bar{\theta}|_{C^\alpha(B_{r_0})} \leq L_2^\delta |\theta|_{C^\alpha(B_{r_0'})}. \]

Let \( \Omega = \bar{x}(B_{r_0'}) \). Note that \( B_{r_0} \subset \Omega \) where \( r_0 = R'/2L_2 \). So our new gradient graph is \( \{(\bar{x}, D_\bar{x} \bar{u}(\bar{x}))|\bar{x} \in \Omega\} \). The function \( \bar{u} \) satisfies the equation

\[ F(D_\bar{x}^2 \bar{u}) = \bar{\theta}(\bar{x}) \]

in \( B_{r_0} \) where \( \bar{\theta} \in C^\alpha(B_{r_0}) \). Observe that on \( B_{r_0} \) we have

\[ \bar{\theta} = \theta - 2\delta < \delta - 2\delta = -\delta < 0 \]

as \( \theta < \delta \) on \( B_{R'} \).

**Claim 2.2.** : If \( |\bar{\theta}| > \delta \), then \( F(D_\bar{x}^2 \bar{u}) = \bar{\theta} \) is a solution to a uniformly elliptic concave equation.

*Proof.* The proof follows from [CPW17, lemma 2.2] and also from [CW16, pg 24]. \( \square \)

Now using [CC01, Corollary 1.3] we get interior Schauder estimates for \( \bar{u} \):

(2.6) \[ |D_\bar{x}^2 \bar{u}(\bar{x}) - D_\bar{x}^2 \bar{u}(0)| \leq C(|\bar{\theta}|_{L^\infty(B_{r_0/2})} + |\bar{\theta}|_{C^\alpha(B_{r_0/2})}) \]

for all \( \bar{x} \) in \( B_{r_0/2} \) where \( C = C(\Lambda, \alpha) \). This is our \( C^{2,\alpha} \) estimate for \( \bar{u} \).

Next, in order to show the same Schauder type inequality as (2.6) for \( u \) in place of \( \bar{u} \), we establish relations between the following pairs:

(i) oscillations of the Hessian of \( D_\bar{x}^2 u \) and \( D_\bar{x}^2 \bar{u} \)
(ii) oscillations of \( \theta \) and \( \bar{\theta} \)
(iii) the supremum norms of \( u \) and \( \bar{u} \).

We rotate back to our original gradient graph by rotating up by an angle of \( \delta \) and consider again the domain \( B_{R'}(0) \). This gives us the following relations:

\[ x = \cos(\delta)\bar{x} - \sin(\delta)D_\bar{x} \bar{u}(\bar{x}) \]
\[ y = \sin(\delta)\bar{x} + \cos(\delta)D_\bar{x} \bar{u}(\bar{x}) . \]

(2.7) \[ \frac{dx}{d\bar{x}} = \cos(\delta)I_n - \sin(\delta)D_\bar{x}^2 \bar{u}(\bar{x}) \]
\[ D_\bar{x} y = \sin(\delta)I_n + \cos(\delta)D_\bar{x}^2 \bar{u}(\bar{x}) . \]

This gives us:
So we have
\[ D_x^2 u(x) = D_x^2 \frac{d\bar{x}}{dx} = [\sin(\delta) I_n + \cos(\delta) D_x^2 \bar{u}(\bar{x})] [\cos(\delta) I_n - \sin(\delta) D_x^2 \bar{u}(\bar{x})]^{-1}. \]

The above expression is well defined everywhere because \( D_x^2 \bar{u}(\bar{x}) < \cot(\delta) I_n \) for all \( \bar{x} \in B_{r_0} \).

Note that we have \( \cos(\delta) I_n - D_x^2 \bar{u}(\bar{x}) \sin(\delta) \geq \frac{1}{L_1} \), since
\[ \frac{dx}{d\bar{x}} = \cos(\delta) I_n - \sin(\delta) D_x^2 \bar{u}(\bar{x}) = \left( \frac{d\bar{x}}{dx} \right)^{-1} \geq \frac{1}{L_1} I_n \]
by (2.4).

Next,
\[ D_x^2 u(x) - D_x^2 u(0) = [\sin(\delta) I_n + \cos(\delta) D_x^2 \bar{u}(\bar{x})] [\cos(\delta) I_n - \sin(\delta) D_x^2 \bar{u}(\bar{x})]^{-1} - [\sin(\delta) I_n + \cos(\delta) D_x^2 \bar{u}(0)] [\cos(\delta) I_n - \sin(\delta) D_x^2 \bar{u}(0)]^{-1}. \]

(2.8)

For simplification of notation we write
\[ D_x^2 \bar{u}(\bar{x}) = A \]
\[ D_x^2 \bar{u}(0) = B \]
\[ \cos(\delta) = c, \sin(\delta) = s. \]

Noting that \([sI_n + cA]\) and \([cI_n - sA]^{-1}\) commute with each other we can write (2.8) as the following equation
\[ D_x^2 u(x) - D_x^2 u(0) = [cI_n - sB]^{-1}[sI_n + cA][cI_n - sA]^{-1} - [cI_n - sB]^{-1}[sI_n + cB][cI_n - sA][cI_n - sA]^{-1}. \]

Again we see that
\[ [cI_n - sB][sI_n + cA] - [sI_n + cB][cI_n - sA] = A - B. \]
This means
\[ D_x^2 u(x) - D_x^2 u(0) = [cI_n - sB]^{-1}[A - B][cI_n - sA]^{-1}. \]

We have already shown that
\[ |cI_n - sA| \geq \frac{1}{L_1} \]
which implies
\[ |cI_n - sA|^{-1} \leq L_1. \]
Thus we get
\[ |D^2_x u(x) - D^2_x u(0)| \leq L_1^2 |\bar{D}^2_x \bar{u}(\bar{x}) - \bar{D}^2_x \bar{u}(0)|. \]
\[ \leq CL_1^2 (||\bar{u}||_{L^\infty(B_{\rho_0/2})} + ||\bar{\theta}||_{C^\alpha(B_{\rho_0/2})}) ||\bar{x}|^\alpha \]
(2.9)
\[ \leq CL_1^{\alpha+2} (||\bar{u}||_{L^\infty(B_{\rho_0/2})} + ||\bar{\theta}||_{C^\alpha(B_{\rho_0/2})}) ||\bar{x}|^\alpha \]
where \( L_1 \) is the Lipschitz constant of the co-ordinate change map. This implies
\[ \frac{1}{L_1^{\alpha+2}} |D^2_x u(x)|_{C^\alpha(B_R)} \leq |D^2_x u(\bar{x})|_{C^\alpha(B_{\rho_0/2})}. \]
(2.10)
Recall from (2.5) that
\[ \bar{u}(x) = u(x) + g(x). \]
This shows
\[ ||\bar{u}(\bar{x})||_{L^\infty(B_{\rho_0/2})} = ||\bar{u}(x)||_{L^\infty(\bar{x}^{-1}(B_{\rho_0/2}))} \leq ||\bar{u}(x)||_{L^\infty(B_{R^*})} \]
\[ \leq ||u(x)||_{L^\infty(B_{R^*})} + ||g||_{L^\infty(B_{R^*})}. \]
(2.11)
Note that
\[ ||g||_{L^\infty(B_R)} \leq R||Du||_{L^\infty(B_R)} + \frac{1}{2} [R^2 + ||Du||_{L^\infty(B_R)}^2] \]
(2.12)
and combining (2.10), (2.11), (2.12) with (2.9) we get
\[ |D^2_x u(x) - D^2_x u(0)| \]
\[ \leq CL_1^{\alpha+2} \left\{ R||Du||_{L^\infty(B_{R^*})} + \frac{1}{2} [R^2 + ||Du||_{L^\infty(B_R)}^2] \right\} ||\bar{x}|^\alpha. \]
This proves the Lemma.

Proof of Theorem 1.2. First note that the lemma gives a Hölder norm on any interior ball, by a rescaling of the form
\[ u_\rho(x) = \frac{u(\rho x)}{\rho^2} \]
for values of \( \rho > 0 \) and translation of any point to the origin. Consider the gradient graph \( \{(x, Du(x))|x \in B_1\} \) where \( u \) satisfies
\[ F(D^2 u) = \theta \]
on \( B_1 \) and \( \theta \in C^\alpha(B_1) \). Then there exists a ball of radius \( r \) inside \( B_1 \) on which \( osc \theta < \delta/4 \) where \( \delta \) is as defined in Lemma 2.1.
Now this means that either we have \( \theta(x) < \delta/2 \) in which case, by the above lemma we see that \( u \in C^{2,\alpha}(B_r) \) satisfying the given estimates; or we have \( \theta(x) > \delta/4 \) in which case \( u \in C^{2,\alpha}(B_r) \) with uniform estimates,
by claim (2.2) and [CC01, Corollary 1.3].

Proof of Theorem 1.1. Since $u \in C^{1,1}(B_1) \cap W^{2,2}(B_1)$ satisfies the uniformly elliptic equation
\[ \Delta_g \theta = 0, \]
by the De Giorgi-Nash Theorem we have that $\theta \in C^\alpha(B_{1/2})$. This means that $u$ satisfies
\[ F(D^2u) = \theta. \]
By Theorem 1.2 we see that $u \in C^{2,\alpha}(B_r)$ where $r < 1/2$. Smoothness follows by [CW16, Corollary 5.1].

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