Moduli of Contact Circles

Hansjörg Geiges and Jesús Gonzalo

Contents

1 Definitions and previous results 1
2 Deformation spaces 2
3 Cartan structures and Weil spaces 6
4 $\tilde{\text{SL}}_2$-geometry 15
5 $\tilde{\text{E}}_2$-geometry 29
6 $\text{SU}(2)$-geometry 35
7 The complex structure of Teichmüller space 40

This paper is a continuation of our study of contact circles begun in [5]. In that paper we gave a complete classification of the closed, orientable 3-manifolds that admit what we called a \textit{taut contact circle}. Below we recall the definition of this type of structure and some fundamental concepts and results from our previous paper.

The main objective of the present paper is to introduce and describe deformation spaces for taut contact circles. In [5] we had already observed that taut contact circles admit non-trivial deformations, and we had described the moduli spaces of taut contact circles on spherical 3-manifolds. Here we introduce Teichmüller and moduli spaces for taut contact circles in a more systematic fashion, and we completely determine these spaces for all 3-manifolds admitting taut contact circles.

1 Definitions and previous results

In this section we briefly summarise the basic definitions and the main result from [5]. In later sections we shall occasionally quote other results from our earlier paper without much further explanation, but on the whole we have tried to make the present paper reasonably self-contained.

\textit{2000 MSC:} 53D35 (Primary); 58D27, 32G15, 57S30 (Secondary)
Definition 1.1. A **taut contact circle** on a 3-manifold $M$ is a pair of contact forms $(\omega_1, \omega_2)$ such that the 1-form $\lambda_1 \omega_1 + \lambda_2 \omega_2$ is a contact form defining the same volume form for all $(\lambda_1, \lambda_2) \in S^1 \subset \mathbb{R}^2$. Equivalently, we require that the following equations be satisfied:

\[
\begin{align*}
\omega_1 \wedge d\omega_1 &= \omega_2 \wedge d\omega_2 \\
\omega_1 \wedge d\omega_2 &= -\omega_2 \wedge d\omega_1.
\end{align*}
\]

If the mixed terms $\omega_1 \wedge d\omega_2$ and $\omega_2 \wedge d\omega_1$ are identically zero rather than just of opposite sign, we speak of a **Cartan structure**.

The following was our main classification result in [5].

**Theorem 1.2.** Let $M$ be a closed 3-manifold. Then $M$ admits a taut contact circle if and only if $M$ is diffeomorphic to a quotient of the Lie group $\mathcal{G}$ under a discrete subgroup $\Gamma$ acting by left multiplication, where $\mathcal{G}$ is one of the following.

1. $S^3 = SU(2)$, the universal cover of $SO(3)$.
2. $\tilde{SL}_2$, the universal cover of $PSL_2\mathbb{R}$.
3. $\tilde{E}_2$, the universal cover of the Euclidean group (that is, orientation preserving isometries of the Euclidean plane).

Each of these manifolds admits a Cartan structure.

If $(\omega_1, \omega_2)$ is a taut contact circle, then so is $(v \omega_1, v \omega_2)$ for any positive function $v$, and so is $(\omega_1 \cos \theta - \omega_2 \sin \theta, \omega_1 \sin \theta + \omega_2 \cos \theta)$ for any constant angle $\theta$. When classifying taut contact circles, we shall do so up to this ambiguity. To this end we introduce the following concepts.

**Definition 1.3.** The **conformal class** (resp. **homothety class**) of a taut contact circle $(\omega_1, \omega_2)$ is the collection of all pairs $(\omega'_1, \omega'_2)$ obtained from $(\omega_1, \omega_2)$ by multiplication by some positive function $v$ (resp. multiplication by some $v$ and rotation by some $\theta$).

### 2 Deformation spaces

We now introduce Teichmüller and moduli spaces for taut contact circles in complete analogy to the corresponding definitions for complex or other geometric structures. Let $M$ be a manifold as in Theorem 1.2. Write $\text{Diff}(M)$ for the diffeomorphism group of $M$ and $\text{Diff}_0(M)$ for the group of diffeomorphisms isotopic to the identity. Let $C(M)$ be the space of homothety classes of taut contact circles on $M$. Our main objects of study in the present paper are the following spaces.

**Definition 2.1.** The **Teichmüller space** of taut contact circles on $M$ is $\mathcal{T}(M) = C(M)/\text{Diff}_0(M)$.

The **moduli space** of taut contact circles is $\mathcal{M}(M) = C(M)/\text{Diff}(M)$. 

2
In terms of this language, the results from [5, Section 6] about the moduli of taut contact circles on left-quotients of $S^3$ can be stated as follows.

**Theorem 2.2.** Let $M = \Gamma \backslash SU(2)$ be a left-quotient of $SU(2)$ under a (discrete, cocompact) subgroup of $SU(2)$. If $\Gamma$ is non-abelian, then $\mathcal{M}(M)$ consists of a single point. Otherwise, $M$ must be a lens space $L(m, m - 1)$, $m \in \mathbb{N}$, and $\mathcal{M}(L(m, m - 1))$ is the disjoint union of $\mathcal{M}_1$ and $\mathcal{M}_2$ with

$$\mathcal{M}_1 = \{a \in \mathbb{C} : 0 < \text{Re}(a) < 1\}/(a \sim 1 - a),$$

$$\mathcal{M}_2 = \{n \in \mathbb{N} : n \equiv -1 \text{ mod } m\}.$$

**Remark 2.3.** Our notational convention is that $\mathbb{N}$ stands for the natural numbers $\geq 1$.

Of course, the component $\mathcal{M}_2$ of $\mathcal{M}(L(m, m - 1))$ is in any case simply a countable set, but the values $n \equiv -1 \text{ mod } m$ have concrete geometric meaning, namely, the contact circles corresponding to $n \in \mathcal{M}_2$ are induced from the contact circle $(\omega_1, \omega_2)$ on $S^3 \subset \mathbb{C}^2$ defined by

$$\omega_1 + i\omega_2 = nz_1dz_2 - z_2dz_1 + z_2^2dz_2.$$

The contact circles corresponding to $a \in \mathcal{M}_1$ are given by

$$\omega_1 + i\omega_2 = az_1dz_2 - (1 - a)z_2dz_1.$$

As pointed out in [5] one cannot put a ‘good’ metric topology on the disjoint union $\mathcal{M}_1 \cup \mathcal{M}_2$. An analogous phenomenon was first observed by Kodaira-Spencer [15] in their study of moduli of Hopf surfaces.

The Teichmüller spaces for the spherical manifolds will be determined in Section 6 (Theorems 6.2 and 6.6).

Henceforth, $M$ will denote a closed, orientable 3-manifold of the form $\Gamma \backslash \mathcal{G}$ with $\mathcal{G}$ equal to $\tilde{SL}_2$ or $\tilde{E}_2$ and $\Gamma$ a discrete, cocompact subgroup of $\mathcal{G}$. Write $\pi = \pi_1(M) \cong \Gamma$ for the fundamental group of $M$. Consider the pair $(\pi, \mathcal{G})$, where $\mathcal{G}$ is the geometry on which $M$ is modelled. The first main result of the present paper translates the geometric definition of Teichmüller and moduli spaces into a more algebraic one. This will later allow us to arrive at explicit descriptions of these deformation spaces. First we recall the following concept, cf. [12], [16], [30].

**Definition 2.4.** The **Weil space** $\mathcal{R}(\pi, \mathcal{G})$ is the space of faithful representations $\rho$ of $\pi$ in $\mathcal{G}$ such that $\rho(\pi)$ is discrete and cocompact in $\mathcal{G}$.

Elements $a \in \mathcal{G}$ act from the left on $\mathcal{R}(\pi, \mathcal{G})$ by inner automorphisms

$$\rho \mapsto \rho^a, \quad \rho^a(u) = a\rho(u)a^{-1}.$$
Write $\text{Inn}(G)$ for the group of such inner automorphisms. The automorphism group $\text{Aut}(\pi)$ of $\pi$ acts from the right on $R(\pi, G)$; an element $\vartheta \in \text{Aut}(\pi)$ acts by

$$\rho \mapsto \rho \circ \vartheta.$$ 

Write $\text{Out}(\pi)$ for the group of outer automorphisms of $\pi$.

In the case $G = \tilde{E}_2$ we make a slight modification to the Weil space. Recall that $\tilde{E}_2$ may be regarded as $\mathbb{R}^3$ with multiplication

$$\left(\begin{array}{c} x_0 \\ y_0 \end{array}\right), \theta_0 \right) \cdot \left(\begin{array}{c} x \\ y \end{array}\right), \theta \right) = \left(\begin{array}{cc} \cos \theta_0 & -\sin \theta_0 \\ \sin \theta_0 & \cos \theta_0 \end{array}\right) \left(\begin{array}{c} x \\ y \end{array}\right) + \left(\begin{array}{c} x_0 \\ y_0 \end{array}\right), \theta_0 + \theta \right).$$

We call the $(x, y)$-component of an element in $\tilde{E}_2$ its translational part and the $\theta$-component its rotational part. Define

$$R'(\pi, \tilde{E}_2) = R(\pi, \tilde{E}_2)/\mathbb{R}^+,$$

where $\mathbb{R}^+$ acts on the Weil space by scaling of the translational parts of the $\rho(u), u \in \pi$.

**Definition 2.5.** The algebraic Teichmüller space $T^{\text{alg}}(M)$ of $M$ is defined as

$$\text{Inn}(\tilde{\text{SL}}_2) \setminus R(\pi, \tilde{\text{SL}}_2)$$

if $M$ is modelled on $\tilde{\text{SL}}_2$, and as

$$\text{Inn}(\tilde{E}_2) \setminus R'(\pi, \tilde{E}_2)$$

if $M$ is modelled on $\tilde{E}_2$.

**Theorem 2.6.** If $M$ is modelled on $\tilde{\text{SL}}_2$ or $\tilde{E}_2$, then

$$T(M) = T^{\text{alg}}(M)$$

and

$$\mathcal{M}(M) = T^{\text{alg}}(M)/\text{Out}(\pi).$$

Initially we shall treat the equal signs in this theorem merely as set-theoretic bijections. In the course of our further explicit description of these spaces we shall see that there are in fact natural topologies on these deformation spaces and the various quotients of the Weil space such that the equal signs may be read as homeomorphisms.
If $M$ is modelled on $\tilde{\text{SL}}_2$, it admits a Seifert fibration $M \to O_M$, unique up to isotopy. Write $\mathcal{T}(O_M)$ for the Teichmüller space of hyperbolic structures on the base orbifold $O_M$, and write $\mathcal{T}^{\text{alg}}(O_M)$ for the algebraic Teichmüller space of $O_M$, defined via representations of the orbifold fundamental group in $\text{PSL}_2 \mathbb{R}$. As will be explained in Section 4, a taut contact circle on $M$ induces in a natural way a hyperbolic structure on $O_M$. We shall exhibit a commutative diagram

\[
\begin{array}{ccc}
\mathcal{T}(M) & \xrightarrow{\Phi} & \mathcal{T}^{\text{alg}}(M) \\
\downarrow & & \downarrow \\
\mathcal{T}(O_M) & \xrightarrow{\phi} & \mathcal{T}^{\text{alg}}(O_M),
\end{array}
\]

where $\Phi$ and $\phi$ are homeomorphisms, and the vertical maps are coverings.

Passing to moduli spaces, our theory entails the construction of certain non-trivial branched coverings of the moduli space $\mathcal{M}(O_M)$ of hyperbolic structures on $O_M$. Some comments about this issue are made at the end of Section 4; we reserve details for a forthcoming publication [9].

The complex structure on the ‘classical’ Teichmüller space $\mathcal{T}(O_M)$ induces complex structures on the other spaces in this diagram so that all maps become local biholomorphisms. By exhibiting a universal family over $\mathcal{T}(M)$ it will be shown in Section 6 that this complex structure on $\mathcal{T}(M)$ is the natural one to consider. In the case of the geometries $S^3$ and $\tilde{E}_2$, Teichmüller or moduli space is (or will be) described explicitly as a complex space. In the $\tilde{E}_2$ case we also describe a universal family. In the $S^3$ case it is clear from the explicit description of the taut contact circles corresponding to $\mathcal{M}_1$ that the complex structure on $\mathcal{M}_1$ is naturally adapted to the classification of taut contact circles. Further justification of this will also be given in Section 6, where we describe a complex Godbillon-Vey invariant, which turns out to define a holomorphic isomorphism of $\mathcal{M}_1$ with a domain in $\mathbb{C}$.

Theorem 2.6 shows that taut contact circles on left-quotients of $\tilde{\text{SL}}_2$ or $\tilde{E}_2$ have deformation spaces closely related to the deformation spaces for geometric structures (in the sense of Thurston) on Seifert fibred 3-manifolds studied by Kulkarni-Lee-Raymond [16]. For these geometric structures, the translation from a geometric to an algebraic definition of the corresponding deformation spaces is given by the developing map. This is not made explicit in [16], where the investigation starts directly from the algebraic definition. In our situation, slightly more work is necessary for this process of translation. It needs to be pointed out that our algebraic set-up differs from that of [16] by the fact that we consider representations in $\mathcal{G}$ rather than in the isometry group of $\mathcal{G}$. Moreover, our subsequent elaboration of these deformation spaces in Sections 4 and 5 is closer in spirit to the work of Ohshika [20]. Compare also the related results of Ue [29] about deformations of geometric structures on Seifert 4-manifolds. Again, that paper deals directly with the algebraic
definitions of the relevant deformation spaces.

We should like to argue that the deformation spaces for taut contact circles introduced in the present paper may prove valuable on two counts – apart from the fact mentioned above that they give rise to interesting non-trivial branched coverings of $\mathcal{M}(O_M)$. First of all, they provide a geometric interpretation (in a slightly altered setting) of the algebraic deformation spaces studied by Kulkarni-Lee-Raymond, and thus carry structure not seen in $\mathcal{T}_{\text{alg}}(M)$. This additional structure might be of interest in getting a better understanding of the related algebraic spaces, in particular the Weil space, just as classical (geometric) Teichmüller theory, notably the analytical methods developed there, yield information about the Weil space of representations in $\text{PSL}_2\mathbb{R}$.

Secondly, our deformation spaces may be seen as a bridge between the deformation theory of 2-dimensional orbifolds and that of complex surfaces. In the case of $\tilde{\text{SL}}_2$, it also makes sense to regard our theory as some sort of desingularisation of the 2-dimensional theory.

Theorem 2.6 will be proved in Section 3. The explicit geometric descriptions of these deformation spaces – the analogues of Theorem 2.2 – constitute the other principal results of the present paper. For the geometry $\tilde{\text{SL}}_2$ this description is given in Theorem 4.11; for $\tilde{E}_2$, in Theorem 5.2.

Remark 2.7. In our definition of a taut contact circle we have opted not to make any assumptions on orientations. That is, we do not require that $\omega_i \wedge d\omega_i$ be a positive volume form for a chosen orientation on $M$. As it turns out, most of the manifolds that admit taut contact circles, e.g. left-quotients of $\tilde{\text{SL}}_2$ and non-abelian left-quotients of SU(2), do not admit any orientation reversing diffeomorphisms, see [18]. In these cases, it will be implicit in our results that all taut contact circles on a given $M$ define one and the same orientation.

By taking triples of 1-forms in place of pairs, but otherwise making requirements analogous to those in Definition 1.1, we arrive at the notion of a \textit{taut contact sphere}. As shown in [5], such structures exist exactly on the left-quotients of SU(2). The corresponding moduli problem is solved in [8], by methods quite different from those in the present paper. No higher-dimensional linear families of contact forms can exist on 3-manifolds, because our definition implies in particular the pointwise linear independence of the 1-forms spanning the linear family. Some generalisations of these concepts to higher-dimensional manifolds have been discussed in [7] and [11].

3 \hspace{1em} \textbf{Cartan structures and Weil spaces}

In this section we prove Theorem 2.6. We begin with several preliminary observations about Cartan structures. The idea is to show that, in the cases of interest to us, each conformal class of taut contact circles contains an essentially unique distinguished Cartan structure (Proposition 3.5). With the
help of this Cartan structure we can define an analogue of the developing map for geometric structures (Lemma 3.7), which will be the key to proving Theorem 2.6.

**Lemma 3.1.** Let \((\omega_1, \omega_2)\) be a Cartan structure on some 3-manifold. Then there is a unique 1-form \(\omega_3\) such that
\[
\begin{align*}
d\omega_1 &= \omega_2 \wedge \omega_3, \\
d\omega_2 &= \omega_3 \wedge \omega_1.
\end{align*}
\]

The proof of this simple lemma is left to the reader. Occasionally we write a Cartan structure as \((\omega_1, \omega_2, \omega_3)\) if we wish to fix a label for the third 1-form determined by \((\omega_1, \omega_2)\).

**Definition 3.2.** A Cartan structure \((\omega_1, \omega_2)\) is called a \(K\)-Cartan structure if the unique \(\omega_3\) of the preceding lemma satisfies
\[
d\omega_3 = K \omega_1 \wedge \omega_2.
\]

Here \(K\) may in principle be any function on \(M\) that is constant along the common kernel of \(\omega_1\) and \(\omega_2\) (since \(dK \wedge \omega_1 \wedge \omega_2 = d^2 \omega_3 = 0\)), but the two cases of interest to us will be \(K \equiv -1\) and \(K \equiv 0\). Notice that the Lie groups \(\mathcal{G}\) in Theorem 1.2 are exactly the universal covers of the group of orientation preserving isometries of the simply-connected 2-dimensional space forms. For \(K\) a constant, write \(\mathcal{G}_K\) for the corresponding Lie group. We mention without proof the following variation on Theorem 1.2, which can be derived without much trouble from the results of [5].

**Proposition 3.3.** Let \(M\) be a closed 3-manifold. Then \(M\) admits a \(K\)-Cartan structure with \(K\) constant if and only if \(M\) is diffeomorphic to a left-quotient \(\Gamma \backslash \mathcal{G}_K\).

**Remark 3.4.** A 3-manifold with a \((-1)\)-Cartan structure is equivalent to what Jacobowitz [14] calls a projective structure. These structures are defined as a 3-dimensional manifold together with a triple of independent 1-forms \(\beta_1, \beta_2, \beta_3\) satisfying the equations
\[
\begin{align*}
d\beta_1 &= -\beta_2 \wedge \beta_3, \\
d\beta_2 &= -2\beta_1 \wedge \beta_2, \\
d\beta_3 &= 2 \beta_1 \wedge \beta_3.
\end{align*}
\]

The translation from a projective structure to a \((-1)\)-Cartan structure is given by \(\omega_1 = 2\beta_1, \omega_2 = \beta_2 + \beta_3, \omega_3 = \beta_2 - \beta_3\).
Proposition 3.5. (a) Let $M$ be a (compact) left-quotient of $\widetilde{SL}_2$. In each conformal class of taut contact circles on $M$ there is one and only one $(-1)$-Cartan structure.

(b) Let $M$ be a (compact) left-quotient of $\widetilde{E}_2$. Every conformal class of taut contact circles on $M$ contains a 0-Cartan structure. This 0-Cartan structure is unique up to multiplication by a positive constant.

Proof. (a) Existence of a $(-1)$-Cartan structure in every homothety class was proved in [5, Thm. 7.4]; since the rotate of a $K$-Cartan structure is again a $K$-Cartan structure (with the same $\omega_3$) we also have existence in every conformal class. To prove uniqueness, we argue as follows. Let $(\omega_1, \omega_2)$ be a $(-1)$-Cartan structure on $M$. As shown in the cited theorem from [5], the common kernel $\ker \omega_1 \cap \ker \omega_2$ determines a Seifert fibration $M \to O_M$, with $O_M$ a hyperbolic orbifold. Lift $(\omega_1, \omega_2)$ to a finite covering $M' \to M$ corresponding to a covering $\Sigma_g \to O_M$ of $O_M$ by an honest surface. The $(-1)$-Cartan equations imply that the symmetric bilinear form $\omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2$ is invariant under the flow of $X \in \ker \omega_1 \cap \ker \omega_2$ with $\omega_3(X) = 1$, and that it induces a metric of constant curvature $-1$ on $\Sigma_g$.

Now assume that we have a further $(-1)$-Cartan structure $(v\omega_1, v\omega_2)$ on $M$. The structure equations imply $dv \wedge \omega_1 \wedge \omega_2 = 0$, so $v$ is constant along the fibres of the $S^1$-fibration, and $v^2(\omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2)$ also induces a metric of constant curvature $-1$ on $\Sigma_g$.

But there is a unique hyperbolic metric (of constant curvature $-1$) in a given conformal class of metrics on $\Sigma_g$, since conformal automorphisms of the hyperbolic plane $\mathbb{H}^2$ are actually isometries. This forces $v \equiv 1$.

(b) In the euclidean case, existence of a 0-Cartan structure in every conformal class was also proved in [5]. However, the common kernel of $\omega_1$ and $\omega_2$ will not, in general, define a Seifert fibration (cf. [5, Thm. 7.4]), so we need a substitute for the argument in (a).

Assume that $(\omega_1, \omega_2, \omega_3)$ and $(v\omega_1, v\omega_2, \omega'_3)$ are 0-Cartan structures on a given left-quotient of $\widetilde{E}_2$. As in (a) the structure equations imply $dv \wedge \omega_1 \wedge \omega_2 = 0$. Since

$$\omega_1 \wedge \omega_2 \wedge \omega_3 = \omega_1 \wedge d\omega_1 \neq 0,$$

the $\omega_i$ are pointwise linearly independent and we can write

$$dv = v_1 \omega_1 + v_2 \omega_2$$

with uniquely defined functions $v_i : M \to \mathbb{R}$. Then the structure equations for the two 0-Cartan structures yield

$$v_2 \omega_2 \wedge \omega_1 + v_3 \omega_3 \wedge \omega_3 = v \omega_2 \wedge \omega'_3,$$

$$v_1 \omega_1 \wedge \omega_2 + v_3 \omega_3 \wedge \omega_1 = v \omega'_3 \wedge \omega_1.$$
This implies
\[ \omega'_3 = \frac{v_2}{v} \omega_1 - \frac{v_1}{v} \omega_2 + \omega_3 \]
and
\[ d\left(\frac{v_2}{v} \omega_1 - \frac{v_1}{v} \omega_2\right) = d\left(\frac{v_2}{v} \omega_1 - \frac{v_1}{v} \omega_2 + \omega_3\right) = d\omega'_3 = 0. \]

Let \( * \) be the Hodge star with respect to the Riemannian metric
\[ \omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2 + \omega_3 \otimes \omega_3 \]
on \( M \) (and orientation given by \( \omega_1 \wedge \omega_2 \wedge \omega_3 \)). Then
\[ *d \log v = *dv = v \frac{\omega_2}{v} \wedge \omega_3 + v \frac{\omega_3}{v} \wedge \omega_1 = \frac{1}{v} (v_1 \omega_2 - v_2 \omega_1) \wedge \omega_3, \]
and hence \( d* d \log v = 0 \). On functions on 3-manifolds, the Laplace operator \( \Delta \) takes the form \( \Delta = - *d *d \), so we have in fact \( \Delta \log v = 0 \), i.e. \( \log v \) is harmonic. Since \( M \) is closed, this forces \( \log v \) and hence \( v \) to be constant. \( \square \)

**Remark 3.6.** There is no analogue of this result for left-quotients of \( SU(2) \), see [5, Section 6]. It is for this reason that in the spherical case there is no direct translation into algebraic Teichmüller and moduli spaces.

Fix a standard left-invariant \( K \)-Cartan structure \( (\alpha_1, \alpha_2, \alpha_3) \) on \( G \), with \( (G, K) \) equal to \( (\widetilde{SL}_2, -1) \) or \( (\tilde{E}_2, 0) \), cf. [5, Section 2]. In particular, on \( \tilde{E}_2 \) we may choose
\[
\begin{align*}
\alpha_1 &= \cos \theta \, dx + \sin \theta \, dy, \\
\alpha_2 &= -\sin \theta \, dx + \cos \theta \, dy, \\
\alpha_3 &= -d\theta
\end{align*}
\]
For each discrete, cocompact subgroup \( \Gamma \subset G \), we continue to write \( \alpha_i \) for the 1-forms induced on \( \Gamma \setminus G \). If \( M \) is a compact manifold diffeomorphic to a left-quotient of \( G \), and \( (\omega_1, \omega_2, \omega_3) \) a \( K \)-Cartan structure on \( M \), we keep the same notation for the pull-back of this \( K \)-Cartan structure to the universal cover \( \tilde{M} \).

**Lemma 3.7.** There is a diffeomorphism \( f: \tilde{M} \to G \) such that \( f^*(\alpha_i) = \omega_i \), \( i = 1, 2, 3 \), and such that \( f \) descends to a diffeomorphism \( \overline{f}: M \to \Gamma \backslash G \), for a suitable discrete, cocompact subgroup \( \Gamma \) of \( G \).

**Remark 3.8.** This diffeomorphism \( f \) is the analogue of the developing map for \( (G, X) \)-structures in the sense of Thurston [27].
Proof. According to [5, Thm. 1.6], given any taut contact circle \((\omega_1, \omega_2)\) on \(M\), there is a \(\Gamma \subset \mathcal{G}\) and a diffeomorphism \(f : M \to \Gamma \backslash \mathcal{G}\) which pulls back the standard Cartan structure \((\alpha_1, \alpha_2)\) on \(\Gamma \backslash \mathcal{G}\) to the homothety class of \((\omega_1, \omega_2)\) on \(M\). Rotations of \((\alpha_1, \alpha_2)\) on \(\mathcal{G}\) (without changing \(\alpha_3\)) can be effected by the flow of the Lie algebra element \(e_3\) dual to \(\alpha_3\) (with respect to the basis \(\alpha_1, \alpha_2, \alpha_3\)). This is the same as right multiplication by the one-parameter subgroup corresponding to \(e_3\), and so it descends to any left-quotient \(\Gamma \backslash \mathcal{G}\). This implies that we can actually find a diffeomorphism \(\tilde{f} : M \to \Gamma \backslash \mathcal{G}\) that pulls back \((\alpha_1, \alpha_2)\) to the conformal class of \((\omega_1, \omega_2)\).

Now both \((\omega_1, \omega_2)\) and \(\tilde{f}^*(\alpha_1, \alpha_2)\) are \(K\)-Cartan structures (the former by assumption, the latter by construction). For \((\mathcal{G}, K) = (\text{SL}_2, -1)\) this forces \((\omega_1, \omega_2) = \tilde{f}^*(\alpha_1, \alpha_2)\) by Proposition 3.5; for \((\mathcal{G}, K) = (\mathcal{E}_2, 0)\) we only get \((\omega_1, \omega_2) = c \tilde{f}^*(\alpha_1, \alpha_2)\) for some \(c \in \mathbb{R}^+\).

Consider the Lie group isomorphism

\[
s_c : \begin{pmatrix} E_2 \\ (x, y) \end{pmatrix}, \theta \mapsto \begin{pmatrix} E_2 \\ (c x, y), \theta \end{pmatrix}.
\]

Then \(s_c^*(\alpha_1, \alpha_2, \alpha_3) = (c\alpha_1, c\alpha_2, \alpha_3)\). Hence we obtain the desired diffeomorphism if we replace the lift \(f\) of \(\tilde{f}\) by \(s_c \circ f\), and the subgroup \(\Gamma\) by \(s_c(\Gamma)\).

This diffeomorphism \(f\) will determine a representation of \(\pi = \pi_1(M)\) in \(\mathcal{G}\). The following lemma describes the indeterminacy in this construction.

Lemma 3.9. Let \(f\) and \(f'\) be two diffeomorphisms \(\tilde{M} \to \mathcal{G}\) as in the preceding lemma, with corresponding subgroups \(\Gamma, \Gamma' \subset \mathcal{G}\). Then there is a \(w \in \mathcal{G}\) such that \(f' = L_w \circ f\) (and \(\Gamma' = w \Gamma w^{-1}\)), with \(L_w\) denoting left multiplication.

Proof. Fix a base point \(p_0 \in \tilde{M}\). There is a \(w \in \mathcal{G}\) such that \(L_w \circ f(p_0) = f'(p_0)\). The triple of 1-forms \(\omega_1 - \alpha_1, \omega_2 - \alpha_2, \omega_3 - \alpha_3\) on \(\tilde{M} \times \mathcal{G}\) (where we forgo writing the pull-back maps of the projections onto the two factors) generate a differential ideal \(\mathcal{I}\), that is, \(d\mathcal{I} \subset \mathcal{I}\). For instance,

\[
d(\omega_1 - \alpha_1) = \omega_2 \wedge (\omega_3 - \alpha_3) + (\omega_2 - \alpha_2) \wedge \alpha_3.
\]

So \(\mathcal{I}\) defines a 3-dimensional foliation on \(\tilde{M} \times \mathcal{G}\). The graphs of \(f'\) and \(L_w \circ f\) are integral submanifolds of this foliation, and since both pass through the point \((p_0, f'(p_0)) = (p_0, L_w \circ f(p_0))\), they must be identical.

Proof of Theorem 2.6. The key to the proof of this theorem is that we have the following ingredients:

(1) a developing map (constructed in the preceding lemmas), which allows to translate from taut contact circles to suitable representations of the fundamental group;

10
(2) an analogue of the Nielsen theorem, which guarantees that all automorphisms of the fundamental group are induced by diffeomorphisms, and thus ensures the surjectivity of this process of translation;

(3) an analogue of the Baer theorem, which allows to pass from homotopy to isotopy, and thus ensures injectivity.

Any geometric structure (on manifolds homotopy equivalent to Eilenberg-MacLane spaces) that provides these ingredients will have an algebraic deformation theory based on representations of the fundamental group analogous to the one we are about to construct.

Identify $\pi$ with the deck transformation group of the universal covering $\tilde{M} \to M$. Thus we regard the elements of $\pi$ as acting on $\tilde{M}$ from the left.

Given a taut contact circle on $M$, choose a $K$-Cartan structure in its homothety class. Choose a diffeomorphism $f: \tilde{M} \to G$, lifted from a diffeomorphism $\overline{f}: M \to \Gamma \backslash G$, with $f^*(\alpha_i) = \omega_i$, $i = 1, 2, 3$.

Given a deck transformation $u \in \pi$ of $\tilde{M} \to M$, we get a corresponding deck transformation

$$f \circ u \circ f^{-1}$$

of $G \to \Gamma \backslash G$. This must of course be an element of $\Gamma \subset G$. Indeed, $f \circ u \circ f^{-1}$ acts trivially on a basis $(\alpha_1, \alpha_2, \alpha_3)$ of left-invariant 1-forms, and hence on every left-invariant 1-form; therefore $f \circ u \circ f^{-1}$ is left multiplication by some element $\rho(u) \in G$.

So a choice of $K$-Cartan structure and a choice of $f$ determine a discrete, faithful representation $\rho \in R(\pi, G)$ via the equation

$$f \circ u = L_{\rho(u)} \circ f.$$ 

The following points (i)–(iii) discuss the dependence of $\rho$ on the various choices in this construction.

(i) According to Lemma 3.9, a different choice $f'$ instead of $f$ (with fixed $K$-Cartan structure) must be of the form $f' = L_w \circ f$ for some $w \in G$. The corresponding representation $\rho'$ satisfies

$$f' \circ u = L_{\rho'(u)} \circ f'$$

for all $u \in \pi$, that is,

$$L_w \circ f \circ u = L_{\rho'(u)} \circ L_w \circ f.$$ 

Hence

$$L_{\rho(u)} \circ f = f \circ u = L_w^{-1} \circ L_{\rho'(u)} \circ L_w \circ f = L_{w^{-1} \rho'(u) w} \circ f,$$
and we conclude \( w^{-1}\rho'(u)w = \rho(u) \). So different choices of \( f \) lead to representations that differ by an element of \( \text{Inn}(\mathcal{G}) \).

(ii) We can change a \( K \)-Cartan structure \((\omega_1, \omega_2, \omega_3)\) on \( M \) within its homothety class by rotating \((\omega_1, \omega_2)\) by an angle \( \theta \) and keeping \( \omega_3 \) fixed. Write \( R_\theta \) for the time \( \theta \) flow of \( e_3 \) on \( \mathcal{G} \) (in the notation of the proof of Lemma [3.7]). The map \( R_\theta \) is given by right multiplication, and it has the effect of keeping \( \alpha_3 \) fixed and pulling back \((\alpha_1, \alpha_2)\) to a \( \theta \)-rotate of itself.

So replacing \((\omega_1, \omega_2)\) by a \( \theta \)-rotate amounts to replacing \( f \) by \( f' = R_\theta \circ f \). From the defining equation for the corresponding representation \( \rho' \),

\[
R_\theta \circ f \circ u = L_{\rho'(u)} \circ R_\theta \circ f,
\]

we get

\[
L_{\rho(u)} = f \circ u \circ f^{-1} = R_\theta^{-1} \circ L_{\rho'(u)} \circ R_\theta.
\]

But left and right multiplication commute, so \( \rho'(u) = \rho(u) \).

(iii) By Proposition [3.3], the rotation discussed in (ii) is the only ambiguity in the choice of a \((-1)\)-Cartan structure within a homothety class of taut contact circles on a left-quotient of \( \tilde{\text{SL}}_2 \). For left-quotients of \( \tilde{E}_2 \) we may also scale the \( 0 \)-Cartan structure by a positive constant. If \( f: \tilde{M} \to \tilde{E}_2 \) pulls back \((\alpha_1, \alpha_2, \alpha_3)\) to \((\omega_1, \omega_2, \omega_3)\), then \( f' = s_c \circ f \), \( c \in \mathbb{R}^+ \), pulls it back to \((c\omega_1, c\omega_2, \omega_3)\). So the representation \( \rho' \) corresponding to \((c\omega_1, c\omega_2)\) is defined by

\[
s_c \circ f \circ u = L_{\rho'(u)} \circ s_c \circ f,
\]

hence

\[
L_{\rho'(u)} = s_c \circ L_{\rho(u)} \circ s_c^{-1}.
\]

If

\[
\rho(u) = \left( \begin{pmatrix} x_u \\ y_u \end{pmatrix}, \theta_u \right),
\]

then a straightforward calculation gives

\[
\rho'(u) = \left( c \begin{pmatrix} x_u \\ y_u \end{pmatrix}, \theta_u \right),
\]

so \( \rho \) and \( \rho' \) define the same element of \( \mathcal{R}'(\pi, \tilde{E}_2) \).

From (i)–(iii) we conclude that there is a well-defined map

\[
\tilde{\Phi}: C(M) \longrightarrow T^\text{alg}(M),
\]
where $T^{\text{alg}}$ is the algebraic Teichmüller space introduced in Definition 2.5.

(iv) The next step is to see that $\tilde{\Phi}$ induces a map

$$\Phi: T(M) \to T^{\text{alg}}(M).$$

To this end, suppose that we take the pull-back of a given taut contact circle $(\omega_1, \omega_2)$ on $M$ under $\varphi \in \text{Diff}_0(M)$. The corresponding $K$-Cartan structures in the respective homothety classes may also be assumed to be related by the pull-back map $\varphi^*$. The diffeomorphism $\varphi$ lifts to a diffeomorphism $\varphi$ of $\tilde{M}$. Then $f \circ \varphi: M \to \mathcal{G}$ is a lift of $\tilde{f} \circ \varphi: M \to \Gamma \setminus \mathcal{G}$ and $(f \circ \varphi)^* \alpha_i = \varphi^* \omega_i$, $i = 1, 2, 3$. So the representation $\rho'$ corresponding to $\varphi^* (\omega_1, \omega_2)$ is defined by

$$f \circ \varphi \circ u = L_{\rho'(u)} \circ f \circ \varphi.$$

But $\varphi \circ u \circ \varphi^{-1} = u$, because $\varphi$ is homotopic to the identity, so $\rho'(u) = \rho(u)$.

(v) To complete the proof of the statement in Theorem 2.6 about Teichmüller spaces, we need to show that $\Phi$ is a bijection.

$\Phi$ is surjective: To prove this we first observe that the diffeomorphism type of $\rho(\pi) \setminus \mathcal{G}$ is completely determined by the isomorphism type of $\pi$: For $\mathcal{G} = \tilde{\text{SL}}_2$ this is a consequence of the fact that $\rho(\pi) \setminus \tilde{\text{SL}}_2$ is a large Seifert manifold in the sense of Orlik [21, p. 92] with fundamental group $\pi$, see [21, p. 97]. For $\mathcal{G} = \tilde{E}_2$ this follows from the complete classification of left-quotients of $\tilde{E}_2$: up to diffeomorphism these are exactly the five torus bundles over $S^1$ with periodic monodromy, and it is easy to compute explicitly that they are distinguished by their fundamental groups, see Section 3.

Therefore any $\rho \in \mathcal{R}(\pi, \mathcal{G})$ gives rise to a diffeomorphic copy $\rho(\pi) \setminus \mathcal{G}$ of $M$. Moreover, the diffeomorphism $M \to \rho(\pi) \setminus \mathcal{G}$ can be chosen such that its induced map on fundamental groups gives the representation $\rho$, because for the manifolds under consideration any automorphism of the fundamental group can be induced by a diffeomorphism of the manifold.

For the left-quotients of $\tilde{E}_2$ this last fact was proved by Neuwirth [13]. For the left-quotients of $\tilde{\text{SL}}_2$ we argue as follows. First one observes that in this case $\pi$ contains a unique maximal normal subgroup isomorphic to $\mathbb{Z}$, generated by the class of a regular fibre of the Seifert fibration $M \to O_M$, cf. [21] and Section 4 below. Thus an automorphism of $\pi$ induces an automorphism of the orbifold fundamental group $\pi^{\text{orb}} = \pi^{\text{orb}}_1(O_M)$, i.e. the deck transformation group of the universal covering of $O_M$ (which in this case is the hyperbolic plane $\mathbb{H}^2$). By the generalised Nielsen theorem, cf. [21, Thm. 8.1], this automorphism of $\pi^{\text{orb}}$ is induced by conjugation with a diffeomorphism $\psi$ of $\mathbb{H}^2$. This $\psi$ has to send any elliptic element of $\pi^{\text{orb}}$ to another elliptic element of the same order. In other words, $\psi$ descends to an ‘orbifold diffeomorphism’ $\overline{\psi}$ of $O_M$ which preserves the set of cone points and may permute only cone points of the same order amongst each other. Moreover, the multiple fibres
over cone points that are being permuted must have the same Seifert invariants \((\alpha, \beta)\), since we started with an automorphism of \(\pi\). Therefore \(\bar{\psi}\) lifts to a diffeomorphism of \(M\), inducing the given automorphism of \(\pi\).

**Remark 3.10.** The preceding argument about lifting a diffeomorphism \(\bar{\psi}\) of \(O_M\) to a diffeomorphism \(\psi\) of \(M\) – provided \(\bar{\psi}\) corresponds to an automorphism of \(\pi_{\text{orb}}\) induced from one of \(\pi\) – is valid for all Seifert manifolds fibred over a hyperbolic orbifold. For left-quotients \(M\) of \(\tilde{\text{SL}}_2\) we shall in fact see in Section 4 that – on a fixed \(M\) – multiple fibres of the same order \(\alpha\) always have the same Seifert invariant \((\alpha, \beta)\). This is not true, in general, for quotients of \(\tilde{\text{SL}}_2\) under discrete, cocompact subgroups of the full isometry group of \(\tilde{\text{SL}}_2\), or Seifert manifolds modelled on the geometry \(H^2 \times \mathbb{R}\). As a consequence, for left-quotients \(M\) of \(\tilde{\text{SL}}_2\) any diffeomorphism of \(O_M\) lifts to one of \(M\). This causes some essential simplification as compared with the results of [16]; see in particular the proof of Lemma [4.13] below.

\(\Phi\) is injective: Suppose we have two elements in \(T(M)\) with the same image in \(T^{\text{alg}}(M)\) under \(\Phi\). By (i) and (iii) above we may choose \(K\)-Cartan structures \((\omega_1, \omega_2)\) and \((\omega'_1, \omega'_2)\) representing these two elements and corresponding diffeomorphisms \(f, f' : \tilde{M} \to G\) in such a way that we get the same image even in \(\mathcal{R}(\pi, G)\), that is, we may assume that for all \(u \in \pi\) we have

\[
\begin{align*}
  f \circ u &= L_{\rho(u)} \circ f, \\
  f' \circ u &= L_{\rho(u)} \circ f'.
\end{align*}
\]

Then

\[
  f^{-1} \circ f' \circ u = f^{-1} \circ L_{\rho(u)} \circ f' = u \circ f^{-1} \circ f',
\]

which implies that \(f^{-1} \circ f'\) descends to a diffeomorphism \(\varphi\) of \(M\) which induces the identity on \(\pi = \pi_1(M)\) and pulls back \((\omega_1, \omega_2)\) to \((\omega'_1, \omega'_2)\). Since \(M\) is an Eilenberg-MacLane space \(K(\pi, 1)\), this implies that \(\varphi\) is homotopic to the identity. By results of Waldhausen [11], Scott [25], and Boileau-Otal [2], for the spaces in question this implies that \(\varphi\) is isotopic to the identity. Thus \((\omega_1, \omega_2)\) and \((\omega'_1, \omega'_2)\) define the same element in \(T(M)\).

(vi) Next we want to prove the statement of Theorem 2.14 concerning the moduli space \(\mathcal{M}(M)\). Define

\[
\mathcal{M}^{\text{alg}}(M) = T^{\text{alg}}(M)/\text{Out}(\pi).
\]

So we have a map

\[
T(M) \longrightarrow \mathcal{M}^{\text{alg}}(M),
\]

given by composing \(\Phi\) with the quotient map \(T^{\text{alg}}(M) \to \mathcal{M}^{\text{alg}}(M)\), and analogous to point (iv) above we need to check that this descends to a map

\[
\Phi' : \mathcal{M}(M) \longrightarrow \mathcal{M}^{\text{alg}}(M).
\]
Consider a taut contact circle \((\omega_1, \omega_2)\) on \(M\) and its pull-back under \(\varphi \in \text{Diff}(M)\). As in (iv) we find for the representations \(\rho\) and \(\rho'\) corresponding to \((\omega_1, \omega_2)\) and \(\varphi^*(\omega_1, \omega_2)\), respectively, that

\[ L_{\rho'(u)} \circ f = f \circ \varphi \circ u \circ \varphi^{-1} = L_{\rho(\varphi \circ u \circ \varphi^{-1})} \circ f. \]

Hence \(\rho'(u) = \rho \circ \vartheta(u)\), where \(\vartheta \in \text{Aut}(\pi)\) is defined by \(u \mapsto \varphi \circ u \circ \varphi^{-1}\), and hence \([\rho'] = [\rho]\) in \(\mathcal{M}^{\text{alg}}(M)\).

In spite of its appearance, the automorphism \(\vartheta\) is of course, in general, not an inner automorphism of \(\pi\). Moreover, inner automorphisms of \(\pi\) do not move points in \(\mathcal{T}^{\text{alg}}(M)\), so it is appropriate to define \(\mathcal{M}^{\text{alg}}(M)\) as the quotient of \(\mathcal{T}^{\text{alg}}(M)\) under \(\text{Out}(M)\).

(vii) Finally it remains to be checked that \(\Phi'\) is bijective. Surjectivity is proved as in (v) (without having to worry about choosing a particular diffeomorphism \(M \to \rho(\pi) \setminus \mathcal{G}\)). To prove injectivity, suppose that we have two elements in \(\mathcal{M}(M)\) with the same image in \(\mathcal{M}^{\text{alg}}(M)\) under \(\Phi'\). By (i) and (iii) we may assume that these elements in \(\mathcal{M}(M)\) are represented by \(K\)-Cartan structures \((\omega_1, \omega_2)\) and \((\omega'_1, \omega'_2)\) which already have the same image in \(\mathcal{S}(\pi) = \mathcal{R}(\pi, \mathcal{G}) / \text{Aut}(\pi)\),

which is usually called the Chabauty space of \(\pi\) (or space of discrete subgroups). So we have representations \(\rho, \rho' \in \mathcal{R}(\pi, \mathcal{G})\) with \(\rho' = \rho \circ \vartheta\) for some \(\vartheta \in \text{Aut}(\pi)\), and diffeomorphisms \(f, f': \tilde{M} \to \mathcal{G}\) that pull back \((\alpha_1, \alpha_2)\) to \((\omega_1, \omega_2)\) and \((\omega'_1, \omega'_2)\), respectively, such that

\[ f \circ u = L_{\rho(u)} \circ f, \]
\[ f' \circ u = L_{\rho \circ \vartheta(u)} \circ f'. \]

It is important to observe that both \(f\) and \(f'\) are lifted from diffeomorphisms \(M \to \Gamma \setminus \mathcal{G}\) with \(\Gamma = \rho(\pi) = \rho'(\pi)\). So \(f^{-1} \circ f'\), as in (v), descends to a diffeomorphism \(\varphi\) of \(M\) that pulls back \((\omega_1, \omega_2)\) to \((\omega'_1, \omega'_2)\), so these \(K\)-Cartan structures define the same element in \(\mathcal{M}(M)\). The automorphism \(\vartheta\) of \(\pi\) is determined by the action of \(\varphi\) on \(\pi\), since

\[ f^{-1} \circ f' \circ u = \vartheta(u) \circ f^{-1} \circ f'. \]

This concludes the proof of Theorem 2.6. \(\square\)

4 \(\tilde{\text{SL}}_2\)-geometry

Let \(M\) be a left-quotient of \(\tilde{\text{SL}}_2\). Then \(M\) is Seifert fibred, and – in contrast with the other geometries we are considering – the Seifert fibration is unique up to isotopy, cf. \([24], [20, Cor. 2.3]\). In particular, the (closed, orientable) base
orbifold $O_M$ is uniquely determined by $M$ (up to diffeomorphism). Hence, if $(\omega_1, \omega_2)$ is a taut contact circle, its common kernel will induce this unique Seifert fibration (cf. the proof of Proposition 3.3). Moreover, if $(\omega_1, \omega_2)$ is the $(-1)$-Cartan representative in that conformal class, $\omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2$ will define a hyperbolic metric on $O_M$, and the coframe $(\omega_1, \omega_2)$ will induce an orientation on $O_M$. The pair $(\omega_2, \omega_1)$ would induce the same metric, but the opposite orientation. So it would seem reasonable to fix an orientation of $O_M$ once and for all, and only to consider taut contact circles on $M$ compatible with this orientation. Yet, for the translation to an algebraic deformation theory it is more convenient not to fix such an orientation.

Thus, write $\mathcal{C}(O_M)$ for the space of hyperbolic metrics (of constant curvature $-1$) on $O_M$, together with a choice of orientation. Equivalently, $\mathcal{C}(O_M)$ may be regarded as the space of complex structures on $O_M$. Teichmüller and moduli space of such metrics are defined as

$$\mathcal{T}(O_M) = \mathcal{C}(O_M)/\text{Diff}_0(O_M),$$

$$\mathcal{M}(O_M) = \mathcal{C}(O_M)/\text{Diff}(O_M).$$

These spaces come equipped with a natural topology, cf. [27]. Notice that our definition entails that $\mathcal{T}(O_M)$ has two components (corresponding to the two choices of orientation), each homeomorphic to $\mathbb{R}^{6g-6+2n}$, where $g$ is the genus and $n$ the number of cone points of $O_M$.

Again there are algebraic definitions of these deformation spaces. Write

$$\mathbb{H}^2 = \{ \tau \in \mathbb{C} : \text{Im} \tau > 0 \}$$

for the upper half-plane. Its group of orientation preserving isometries is $\text{PSL}_2\mathbb{R}$, acting by fractional linear transformations. As earlier let $\pi_{\text{orb}}$ denote the orbifold fundamental group $\pi_{\text{orb}}^0(O_M)$, i.e. the deck transformation group of the universal covering of $O_M$ (an explicit presentation of this group will be given shortly). Consider the Weil space $\mathcal{R}(\pi_{\text{orb}}, \text{PSL}_2\mathbb{R})$ of faithful representations $\pi_{\text{orb}} \to \text{PSL}_2\mathbb{R}$ with discrete, cocompact image, and set

$$\mathcal{T}^{\text{alg}}(O_M) = \text{Inn}(\text{PSL}_2\mathbb{R})\backslash \mathcal{R}(\pi_{\text{orb}}, \text{PSL}_2\mathbb{R}),$$

$$\mathcal{M}^{\text{alg}}(O_M) = \mathcal{T}^{\text{alg}}(O_M)/\text{Out}(\pi_{\text{orb}}).$$

Notice that $\mathcal{T}^{\text{alg}}(O_M)$ has two components, just as $\mathcal{T}(O_M)$. To get only one component, one would have to restrict to representations that are related to one another by conjugation by an orientation preserving homeomorphism of $\mathbb{H}^2$. This, however, would entail a not entirely obvious restriction of the allowable automorphisms of $\pi_{\text{orb}}$.

**Remark 4.1.** Let $\Pi$ be an abstract group topologised discretely, and $G$ a connected Lie group (or at least a group such that the index of the identity component $G_0$ in $G$ is finite). Write $\text{Hom}(\Pi, G)$ for the space of representations
$\Pi \to G$, equipped with the topology of pointwise convergence. A fundamental result of Weil, cf. [12], [30, p. 191], says that the Weil space $R(\Pi, G)$ is open in $\text{Hom}(\Pi, G)$. In the sequel, the algebraic Teichmüller and moduli spaces will be equipped with the quotient topology coming from the topology of $R \subset \text{Hom}$.

In classical Teichmüller theory it is shown that the maps

$$\phi : \mathcal{T}(O_M) \to \mathcal{T}^{\text{alg}}(O_M)$$

and

$$\phi' : \mathcal{M}(O_M) \to \mathcal{M}^{\text{alg}}(O_M)$$

given by the developing map are homeomorphisms, cf. [12, p. 301], [30, p. 194]. In Section 3 we employed arguments analogous to these classical ones. The key points in the classical theory is an analogue of Nielsen’s (and Baer’s) theorem for orbifolds, establishing an equivalence between isotopy classes of self-diffeomorphisms and outer automorphisms of $\pi^{\text{orb}}$ (cf. step (v) in the proof of Theorem 2.6), see [17].

We recall a few further facts about Seifert manifolds from [21], [24]. Assume that $M$ has normalised Seifert invariants

$$\{g, b, (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\}.$$  

Since we are dealing with a left-quotient of $\widetilde{SL}_2$, we know that the orbifold Euler characteristic of the base

$$\chi^{\text{orb}}(O_M) = 2 - 2g - n + \sum_{j=1}^{n} \frac{1}{\alpha_j}$$

and Euler number of the Seifert fibration

$$e = -\left(b + \sum_{j=1}^{n} \frac{\beta_j}{\alpha_j}\right)$$

satisfy

$$\chi^{\text{orb}}(O_M) < 0 \text{ and } e \neq 0.$$  

Furthermore, by a theorem of Raymond and Vasquez [22], cf. [3], there are integers $r, k_1, \ldots, k_n$ such that

$$rb = 2g - 2 - \sum_{j=1}^{n} k_j,$$

$$r\beta_j = \alpha_j - 1 + k_j\alpha_j, \quad j = 1, \ldots, n.$$  

Observe that the integer $r$ is completely determined by the normalised Seifert invariants, thanks to the formula $r = \chi^{\text{orb}}/e$, cf. Remark 4.9 below.
Remark 4.2. The $\beta_j$ satisfy, by the definition of normalised Seifert invariants, $1 \leq \beta_j < \alpha_j$, and so the equations above show that the multiple fibres with the same $\alpha_j$ also have the same $\beta_j$. This is false, in general, for Seifert manifolds which are not diffeomorphic to a left-quotient of $\widetilde{SL}_2$.

In the sequel we fix the following presentations of $\pi_{\text{orb}} = \pi_1(OM)$ and $\pi = \pi_1(M)$:

$$\pi_{\text{orb}} = \left\{ \overline{u}_1, \overline{v}_1, \ldots, \overline{u}_g, \overline{v}_g, \overline{q}_1, \ldots, \overline{q}_n : \prod_i [\overline{u}_i, \overline{v}_i] \prod_j \overline{q}_j = 1, \overline{q}_j^{\alpha_j} = 1 \right\}$$

$$\pi = \left\{ u_1, v_1, \ldots, u_g, v_g, q_1, \ldots, q_n, h : \prod_i [u_i, v_i] \prod_j q_j = h^b, \right.$$  
$$\left. q_j^{\alpha_j} h^{\beta_j} = 1, h \text{ central} \right\}.$$

Most of the work towards achieving explicit descriptions of the deformation spaces in the $\widetilde{SL}_2$-case will be contained in the proof of the following theorem about Weil spaces. Compare [16, Thm 2.5] for the corresponding statement when representations in $\text{Isom}_0(\widetilde{SL})$ are being considered.

**Theorem 4.3.** Let $M$ be a left-quotient of $\widetilde{SL}_2$ with fundamental group $\pi$. The Weil space $R(\pi, \widetilde{SL}_2)$ of $M$ is a trivial principal $\mathbb{Z}^{2g}$-bundle over the Weil space $R(\pi_{\text{orb}}, PSL_2\mathbb{R})$ of $OM$.

Remark 4.4. Our argument is largely analogous to considerations of Ohshika [20] concerning the space $R(\pi, \text{Isom}_0(\widetilde{SL}_2))$. However, he uses an invalid presentation for $\pi$. Our argument repairs that oversight.

We prepare for the proof of Theorem 4.3 by first establishing a concrete topological realisation of the orbifold fundamental group, and by describing the geometry $\widetilde{SL}_2$ in a little more detail.

**The deck transformation group $\pi_{\text{orb}}$**

Let $OM$ be a fixed topological orbifold of genus $g$ and with $n$ cone points of multiplicity $\alpha_1, \ldots, \alpha_n$. We do not yet fix a hyperbolic structure on $OM$. Choose a base point $x_0 \in OM$ distinct from all the cone points, and a lift $\tilde{x}_0 \in \tilde{OM}$ of $x_0$ in the universal covering space $\tilde{OM}$. Choose a system of $2g$ loops on $OM$, based at $x_0$, and a curve from $x_0$ to each of the cone points, such that $OM$ looks as in Figure 1 when cut open along these $2g + n$ curves. We may interpret that figure as a fundamental region in $\tilde{OM}$; it is determined (amongst all possible fundamental regions whose boundary polygon maps to the chosen system of curves) by the indicated placement of $\tilde{x}_0$ on its boundary. Notice that the sides of this polygon labelled $\varepsilon_j, \varepsilon'_j$ meet at a vertex mapping to the $j$-th cone point in $OM$; all other vertices are lifts of $x_0$. 

18
Let $u_1, v_1, \ldots, u_g, v_g, q_1, \ldots, q_n$ be the deck transformations of $\tilde{O}_M$ which effect the gluing maps of the sides of the chosen fundamental polygon as indicated in Figure 1. The deck transformation \( \prod_i [u_i, v_i] \prod_j q_j \) (read from the right as a composition of maps) fixes the point $x_0$, which is not the lift of a cone point, so we conclude \( \prod_i [u_i, v_i] \prod_j q_j = 1 \).

Similarly, we have \( q_j^{\alpha_j} = 1, \ j = 1, \ldots, n \).

So our choices provide us with a topological realisation of $\pi^{\text{orb}}$ as deck transformation group, consistent with the presentation of $\pi^{\text{orb}}$ fixed earlier on. Once $O_M$ is equipped with a hyperbolic structure and an orientation, then $\tilde{O}_M = \mathbb{H}^2$ and the $u_i, v_i, q_j$ are isometries of $\mathbb{H}^2$, i.e. elements of $\text{PSL}_{2}\mathbb{R}$. The identification of $\tilde{O}_M$ with $\mathbb{H}^2$ is uniquely determined if we specify, for instance, the lift $\tilde{x}_0 \in \mathbb{H}^2$, the initial direction of $\sigma_1$ at that point, and require that the orientation lifted from $O_M$ coincides with a chosen orientation of $\mathbb{H}^2$. In this way an oriented hyperbolic structure on $O_M$ defines an element of $\mathcal{R}(\pi^{\text{orb}}, \text{PSL}_{2}\mathbb{R})$. 

Figure 1: A fundamental domain for $O_M$. 

19
The geometry $\tilde{\text{SL}}_2$

Identify $\tilde{\text{SL}}_2$, the universal cover of the unit tangent bundle $\text{STH}^2$, with $\mathbb{H}^2 \times \mathbb{R}$ with coordinates $(z, \theta)$, cf. [32], [28], [5]. An element $\mathfrak{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2\mathbb{R}$ acts isometrically on $\mathbb{H}^2$ by the fractional linear transformation

$$\mathfrak{A}(z) = \frac{az + b}{cz + d}.$$ 

On differentiation this yields

$$\partial_z \mapsto \frac{\partial_z}{(cz + d)^2}.$$ 

By taking the argument modulo $2\pi$ of the logarithm of this action, we obtain the description of the differential of $\mathfrak{A}$ in terms of the coordinates $(z, \theta \mod 2\pi)$ for $\text{STH}^2$:

$$(z, \theta \mod 2\pi) \mapsto \left(\frac{az + b}{cz + d}, \theta - 2 \arg(cz + d) \mod 2\pi\right).$$

An element $A \in \tilde{\text{SL}}_2$ that maps to $\mathfrak{A} \in \text{PSL}_2\mathbb{R}$ then acts by left-multiplication on $\tilde{\text{SL}}_2 = \text{STH}^2$ as

$$A(z, \theta) = \left(\frac{az + b}{cz + d}, \theta - 2 \arg(cz + d)\right),$$

where the argument function depends on $z \in \mathbb{H}^2$ and the lift $A$ of $\mathfrak{A}$.

For $\mathfrak{A}$ hyperbolic, or elliptic of finite order, we can define a preferred lift $A$ as follows:

- If $\mathfrak{A}$ is a hyperbolic element, then the action of $A$ on the $\theta$-coordinate is defined by parallel translation of a unit tangent vector along the axis of $\mathfrak{A}$.

- If $\mathfrak{A}$ is an elliptic element of order $\alpha \in \mathbb{N}$, rotating by $2\pi/\alpha$ in positive direction around its fixed point (with respect to the complex orientation of $\mathbb{H}^2$), then $A$ is determined by $A^\alpha(z, \theta) = (z, \theta + 2\pi)$.

More geometrically, this means the following. Given a hyperbolic element $\mathfrak{A}$, take the path of hyperbolic elements, all with the same axis, from the identity in $\text{PSL}_2\mathbb{R}$ to $\mathfrak{A}$. If $\mathfrak{A}$ is elliptic, rotating by $2\pi/\alpha$ in positive direction, take the path of rotations about the same fixed point, starting at the identity and ending with $\mathfrak{A}$ (without completing any full turns). The lifts of these paths to $\tilde{\text{SL}}_2$, starting at the identity, end at the preferred lift $A$ of $\mathfrak{A}$.
The sign in the lift of an elliptic element is crucial for the subsequent discussion, so we give a brief justification for it. Consider the element
\[ A_t = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}. \]

Notice that, for increasing \( t \in \mathbb{R} \), the action of \( A_t \) on \( \mathbb{H}^2 \) by fractional linear transformations defines a positive (i.e. counterclockwise) rotation about \( i \in \mathbb{H}^2 \). The differential of \( A_t \) is given by
\[ (z, \theta) \mapsto (A_t(z), \theta - 2 \arg(-z \sin t + \cos t) \mod 2\pi). \]

In particular,
\[ (i, \theta) \mapsto (i, \theta - 2 \arg(-i \sin t + \cos t) \mod 2\pi). \]

Up to integer multiples of \( 2\pi \) we have \( \arg(\cos t - i \sin t) = -t \). Therefore the choice of \( -t \) for this argument is the only one for which the lifts \( A_t \) of \( A_t \) form a 1-parameter subgroup. This yields
\[ A_{-t}(z, 0) = (z, \theta + 2\pi). \]

Recall that the identity component Isom_0(\( \widetilde{SL}_2 \)) of the full isometry group of \( \widetilde{SL}_2 \) is \( \widetilde{SL}_2 \times_{\mathbb{Z}} \mathbb{R} \), where the \( \mathbb{R} \)-factor acts by translation of the \( \theta \)-coordinate, and \( \mathbb{Z} \) corresponds to the kernel of the (\( \widetilde{SL}_2 \times \mathbb{R} \))-action, generated by \( (A_x, -2\pi) \), cf. [5, p. 188] (beware that in the cited reference we worked with a negative rotation).

**Proof of Theorem 4.3.** Suppose we are given \( \rho \in R(\pi^{orb}, PSL_2 \mathbb{R}) \). Assume that the orientation of \( O_M \) is chosen in such a way that the loop
\[ \prod \sigma_i \tau_i (\sigma'_i)^{-1} (\tau'_i)^{-1} \prod \varepsilon_j (\varepsilon'_j)^{-1} \]  
(1)

(where the composition of paths is read, as usual, from the left), corresponds under the developing map to a positively oriented polygonal loop in \( \mathbb{H}^2 \). This amounts to fixing one of the two components of \( R(\pi^{orb}, PSL_2 \mathbb{R}) \), and geometrically it means that Figure 1 may be interpreted as a fundamental region inside \( \mathbb{H}^2 \) with its standard orientation. In particular, the \( \overline{q}_j \) (or more precisely: \( \overline{p}(\overline{q}_j) \)) are rotations in positive direction by \( 2\pi/\alpha_j \).

Now lift \( \overline{p} \) to a representation \( \rho_0: \pi \to \text{Isom}_0(\widetilde{SL}_2) \) as follows:

- The \( \rho_0(u_i) \) and \( \rho_0(v_i) \) are defined as the preferred lifts of \( \overline{p}(u_i) \) and \( \overline{p}(v_i) \), respectively.
- \( \rho_0(h)(z, \theta) = (z, \theta + 2\pi r) \).
- \( \rho_0(q_j) \) is the lift of \( \overline{p}(q_j) \) that satisfies \( \rho_0(q_j)^{\alpha_j} \rho_0(h)^{\beta_j} = 1 \).
(For the other choice of orientation, the sign in the definition of $\rho_0(h)$ needs to be changed).

**Remark 4.5.** The integer $r$ is what we called the *fibre index* in [3]. Geometrically it describes the number of full turns along a regular fibre of $M \to O_M$ made by the two contact planes in the standard taut contact circle on $\rho_0(\pi) \backslash \widetilde{SL}_2$.

**Lemma 4.6.** This does indeed define a representation of $\pi$, and in fact we have $\rho_0 \in R(\pi, \widetilde{SL}_2)$.

*Proof.* We first check that $\rho_0(q_j) \in \widetilde{SL}_2$; for all the other generators of $\pi$ this is clear by definition. The preferred lift of $\overline{\rho(q_j)}$ lies in $\widetilde{SL}_2$. With our choice of orientation, $\overline{\rho(q_j)}$ is an elliptic element rotating by $2\pi/\alpha_j$ in positive direction around its fixed point. So the $\alpha_j$-th power of the preferred lift of $\overline{\rho(q_j)}$ gives a shift by $2\pi$ in $\theta$-direction. Thus $\rho_0(q_j)$ differs from the preferred lift by a translation in $\theta$-direction by

$$-2\pi \frac{1 + \beta_j r}{\alpha_j} = -2\pi(1 + k_j) \in 2\pi\mathbb{Z},$$

so it also lies in $\widetilde{SL}_2$.

Provided that $\rho_0$ is a representation of $\pi$, it is clear from the geometry of $\widetilde{SL}_2$ that $\rho_0(\pi) \backslash \widetilde{SL}_2$ will define a Seifert manifold $M$ with base orbifold $O_M$. So to verify that $\rho_0 \in R(\pi, \widetilde{SL}_2)$, it only remains to be checked that our definition of $\rho_0$ is compatible with the relations in $\pi$.

Write $\rho'_0(q_j)$ for the lift of $\overline{\rho(q_j)}$ to $\text{Isom}_0(\widetilde{SL}_2)$ which satisfies $\rho'_0(q_j)^{\alpha_j} = 1$ (that is, $\rho'_0(q_j)$ is distinguished amongst all lifts of $\overline{\rho(q_j)}$ as the one which does not act as a non-trivial helicoidal motion). This $\rho'_0(q_j)$ differs from the preferred lift of $\overline{\rho(q_j)}$ by a shift in $\theta$-direction by $-2\pi/\alpha_j$; in particular, $\rho'_0(q_j) \not\in \widetilde{SL}_2 \subset \text{Isom}_0(\widetilde{SL}_2)$. The reason for introducing this lift is its geometric significance implicit in the following statement:

$$\prod_i [\rho_0(u_i), \rho_0(v_i)] \prod_j \rho'_0(q_j)(z, \theta) = (z, \theta - 2\pi \chi^{\text{orb}}(O_M)). \quad (2)$$

This equation also appears in [3], but we trust the reader will appreciate a little more elucidation than is given there. Notice that $-2\pi \chi^{\text{orb}}$ is the area of $O_M$, hence may be thought of as *minus* the integral of the constant Gauß curvature $-1$ over a fundamental region for $O_M$ in $\mathbb{H}^2$. To prove equation (2) it therefore suffices to show, by the classical Gauß-Bonnet theorem, that the total effect on the $\theta$-coordinate of the isometries on the left-hand side is equal to the parallel transport along the inverse of the loop (1).
Different choices of lifts of $\overline{\varphi}(\pi_i)$ and $\overline{\varphi}(\pi_i)$ do not affect the commutators on the left-hand side of (2), so instead of $\rho_0(u_i)$ and $\rho_0(v_i)$ we may use lifts $\rho'_0(u_i)$ and $\rho'_0(v_i)$ defined as follows (refer to Figure 2 for notation, where we drop the index $i$ for convenience, and recall that $\overline{\mathrm{SL}_2} = \overline{\mathrm{ST}\mathbb{H}^2}$).

- For $X \in \overline{\mathrm{ST}\mathbb{H}^2}$, that is, an element of $\overline{\mathrm{ST}\mathbb{H}^2}$ in the $\mathbb{R}$-fibre over $c \in \mathbb{H}^2$, we have that $\rho'_0(u)(X) \in \overline{\mathrm{ST}_b\mathbb{H}^2}$ is obtained by parallel transport of $X$ along $\tau^{-1}$, and $\rho'_0(v)(X) \in \overline{\mathrm{ST}_d\mathbb{H}^2}$ is obtained by parallel transport of $X$ along $(\sigma')^{-1}$.

Since $\overline{\varphi}(\pi)$ maps $\sigma'$ isometrically to $\sigma$ and $\overline{\varphi}(\pi)$ maps $\tau$ isometrically to $\tau'$, it is a simple matter to check that the map

$$[\rho'_0(u),\rho'_0(v)] : \overline{\mathrm{ST}_a\mathbb{H}^2} \to \overline{\mathrm{ST}_a\mathbb{H}^2}$$

is given by parallel transport along the path $\tau'\sigma'\tau^{-1}\sigma^{-1}$.

For the elliptic elements we argue similarly. The previously defined $\rho'_0(q_j)$ fixes the fibre of $\overline{\mathrm{ST}\mathbb{H}^2}$ over the corresponding cone point, and since $\overline{\varphi}(\pi_j)$ maps $\varepsilon'_j$ isometrically onto $\varepsilon_j$, we see that $\rho'_0(q_j)$ is given by parallel transport along $\varepsilon'_j\varepsilon_j^{-1}$. This proves equation (2).
We can now complete the proof of Lemma 4.6. We only need to verify that \( \rho_0 \) respects the relation \( \prod_i [u_i, v_i] \prod_j q_j = h^b \). We have
\[
\rho_0(q_j)(z, \theta) = \rho_0^0(q_j)(z, \theta) - (0, 2\pi \frac{\beta_j r}{\alpha_j}).
\]
Hence
\[
\prod_i [\rho_0(u_i), \rho_0(v_i)] \prod_j \rho_0(q_j)(z, \theta) = (z, \theta + \theta_0)
\]
with
\[
\theta_0/2\pi = -\chi_{\text{orb}}^\circ(O_M) - \sum_j \frac{\beta_j r}{\alpha_j} = -\left(2 - 2g - n + \sum_j \frac{1}{\alpha_j}\right) - \sum_j \left(1 - \frac{1}{\alpha_j} + k_j\right) = 2g - 2 - \sum_j k_j = r b,
\]
thus
\[
\prod_i [\rho_0(u_i), \rho_0(v_i)] \prod_j \rho_0(q_j) = \rho_0(h)^b,
\]
as desired. This completes the proof of Lemma 4.6. \( \square \)

We have thus found a particular lift \( \rho_0 \in \mathcal{R}(\pi, \widetilde{SL}_2) \) of \( \rho \in \mathcal{R}(\pi_{\text{orb}}, \text{PSL}_2\mathbb{R}) \).

It remains to settle the question what ambiguity there is in the choice of such a lift. The next lemma shows that for the generators \( h, q_1, \ldots, q_n \) there is no ambiguity at all.

**Lemma 4.7.** Let \( \rho \in \mathcal{R}(\pi, \widetilde{SL}_2) \) be any other lift of \( \rho \). Then \( \rho(h) = \rho_0(h) \) and \( \rho(q_j) = \rho_0(q_j) \).

**Proof.** Write
\[
\rho(h)(z, \theta) = \rho_0(h)(z, \theta) + (0, 2\pi y),
\]
\[
\rho(q_j)(z, \theta) = \rho_0(q_j)(z, \theta) + (0, 2\pi x_j).
\]
The relations \( q_j^{\alpha_j} h^b = 1 \) and \( \prod_i [u_i, v_i] \prod_j q_j = h^b \) then yield the equation
\[
C \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ y \end{pmatrix} = 0
\]

24
with
\[
C = \begin{pmatrix}
\alpha_1 & 0 & \cdots & 0 & \beta_1 \\
0 & \alpha_2 & \cdots & 0 & \beta_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \alpha_n & \beta_n \\
1 & 1 & \cdots & 1 & -b
\end{pmatrix}.
\]

Recall that the Euler number \( e \) was defined as \( e = -\left(b + \sum_j \beta_j/\alpha_j\right) \), hence
\[
\det C = \begin{vmatrix}
\alpha_1 & 0 & \cdots & 0 & \beta_1 \\
0 & \alpha_2 & \cdots & 0 & \beta_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \alpha_n & \beta_n \\
0 & 0 & \cdots & 0 & e
\end{vmatrix} = \alpha_1 \cdots \alpha_n e \\

\ne 0.
\]

This implies that the only solution is \( x_1 = \ldots = x_n = y = 0 \). \( \square \)

**Remark 4.8.** Observe that the condition \( e \neq 0 \) (that is, \( \tilde{SL}_2 \)-geometry as opposed to \( M \) being modelled on \( \mathbb{H}^2 \times \mathbb{R} \)) enters crucially in this lemma. Notice further that \( \rho(h) \) is forced to be the vertical shift by \( 2\pi r \), with \( r \) equal to the fibre index, no matter what \( \rho \) is.

For the generators \( u_i, v_i \), on the other hand, there is a certain freedom in choosing lifts. Indeed, we can define a lift \( \rho_w \in \mathcal{R}(\pi, \tilde{SL}_2) \) of the representation \( \bar{\rho} \in \mathcal{R}(\pi^{orb}, PSL_2\mathbb{R}) \) for any \( w \in \mathbb{Z}^g \) by setting
\[
\rho_w(u_i)(z, \theta) = \rho_0(u_i)(z, \theta) + (0, 2\pi w_{2i-1}),
\]
\[
\rho_w(v_i)(z, \theta) = \rho_0(v_i)(z, \theta) + (0, 2\pi w_{2i}),
\]
and any lift of \( \bar{\rho} \) to \( \mathcal{R}(\pi, \tilde{SL}_2) \) must be of this form.

Conversely, if we are given a representation \( \rho \in \mathcal{R}(\pi, \tilde{SL}_2) \), the discrete subgroup \( \rho(\pi) \subset \tilde{SL}_2 \) preserves the fibration \( \tilde{SL}_2 \to \mathbb{H}^2 \), which induces a Seifert fibration of \( \rho(\pi) \setminus \tilde{SL}_2 \). The central element \( h \in \pi \) must map to the centre of \( \tilde{SL}_2 \), which equals the kernel of the projection
\[
\text{Isom}_0(\tilde{SL}_2) \subset \tilde{SL}_2 \xrightarrow{\text{pr}} \text{Isom}_0\mathbb{H}^2 = PSL_2\mathbb{R},
\]
and \( \text{pr} \circ \rho \) defines an element of \( \mathcal{R}(\pi^{orb}, PSL_2\mathbb{R}) \), cf. [24, Thm. 4.15].

Notice that we have a free \( \mathbb{Z}^{2g} \)-action on \( \mathcal{R}(\pi, \tilde{SL}_2) \) defined by
\[
\mathbb{Z}^{2g} \times \mathcal{R}(\pi, \tilde{SL}_2) \to \mathcal{R}(\pi, \tilde{SL}_2) \quad (w', \rho_w) \mapsto \rho_{w+w'},
\]
and a section \( \rho_0 \) of the covering \( \mathcal{R}(\pi, \tilde{SL}_2) \to \mathcal{R}(\pi^{orb}, PSL_2\mathbb{R}) \).

This concludes the proof of Theorem 4.3. \( \square \)
Remark 4.9. Observe that with $C$ the matrix from the proof of Lemma \[4.7\], $|\det C| = \alpha_1 \cdots \alpha_n |e|$ equals the order of the torsion subgroup of the abelianised $\pi$. The $\alpha_j$ are recovered from $\pi$ as the orders of the different maximal cyclic subgroups (modulo conjugation) of $\pi_{\text{orb}}$, which is the quotient of $\pi$ by its centre $\langle h \rangle$. The genus $g$ of the base orbifold is determined by the rank of the abelianised $\pi$ (which equals $2g$). This allows to compute $|r| = |\chi_{\text{orb}}/e|$ from the abstract group structure of $\pi$ alone. Fixing a sign of $r$ amounts to fixing orientations of $O_M$ and the Seifert fibres; with this choice of sign the normalised Seifert invariants are recovered from $\pi$. Alternatively, the Seifert invariants and $\pi$ can be recovered from $\pi_{\text{orb}}$ and $r$.

Remark 4.10. Throughout this paper we have been thinking of $\pi$ and $\pi_{\text{orb}}$ as deck transformation groups, rather than as fundamental groups of homotopy classes of based loops. In the course of proving equation \[4.8\], however, we had to show that the action (on the fibre of $\tilde{\text{SL}}_2 \to \mathbb{H}$) of a certain word $w \in \pi$ could be computed as parallel transport along a suitable loop in $\mathbb{H}$. It is tempting to believe that this loop corresponds directly to the projected word $\bar{w} \in \pi_{\text{orb}}$ if $\pi_{\text{orb}}$ is reinterpreted in terms of homotopy classes of loops. We deem it worth pointing out that the relation between the fundamental group of a topological space $X$ interpreted as a deck transformation group (which we shall denote $\pi$) and the fundamental group interpreted in terms of loops based at $x_0$ (for which we write $\pi_1(X, x_0)$) is a little more subtle, a fact that is usually brushed over in elementary treatments. (For orbifold fundamental groups there are completely analogous statements.)

Write $p\colon \tilde{X} \to X$ for the universal covering. It is well-known that $\pi_1(X, x_0)$ acts on $p^{-1}(x_0)$ from the right, with $[\gamma] \in \pi_1(X, x_0)$ sending $y \in p^{-1}(x_0)$ to the endpoint $y\gamma$ of the lift of $\gamma$ to $\tilde{X}$ with initial point $y$. If we fix an $\tilde{x}_0 \in p^{-1}(x_0)$, there is a unique deck transformation $u_{\gamma}$ such that $u_{\gamma}(\tilde{x}_0) = \tilde{x}_0\gamma$. One checks easily that the map $[\gamma] \mapsto u_{\gamma}$ defines an isomorphism $\pi_1(X, x_0) \to \pi$, where composition in $\pi_1(X, x_0)$ is read from the left as composition of paths, in $\pi$ from the right as composition of maps.

However, this isomorphism depends crucially on the choice of $\tilde{x}_0$. If $\tilde{x}_0$ is replaced by $u(\tilde{x}_0)$ for some $u \in \pi$, one verifies that the corresponding isomorphism $\pi_1(X, x_0) \to \pi$ is given by $[\gamma] \mapsto uu_{\gamma}u^{-1}$.

This implies that the loop \[4.9\] corresponds, under the isomorphism of $\pi_{\text{orb}}(O_M, x_0)$ with $\pi_{\text{orb}}$ determined by the choice of $\tilde{x}_0$, not to the word in equation \[4.8\], but to a word where the letters have been replaced by increasingly complicated conjugate letters.

Theorem 4.11. Let $M$ be a left-quotient of $\tilde{\text{SL}}_2$. The Teichmüller space $\mathcal{T}(M)$ of taut contact circles on $M$ is a trivial principal $\mathbb{Z}^{2g}$-bundle over the Teichmüller space $\mathcal{T}(O_M)$ of hyperbolic metrics on the base orbifold $O_M$ of the unique Seifert fibration $M \to O_M$. 
The moduli space $M(M)$ is an $r^{2g}$-fold branched covering of $M(O_M)$, where $g$ is the genus of $O_M$ and the integer $r$ is determined from the Seifert bundle structure $M \to O_M$ by the equation $r = \chi_{\text{orb}}(O_M)/e$.

Proof. We have seen that the projection $\widetilde{\text{SL}}_2 \to \text{PSL}_2 \mathbb{R}$ induces a covering map $R(\pi, \widetilde{\text{SL}}_2) \to R(\pi_{\text{orb}}, \text{PSL}_2 \mathbb{R})$, and this in turn induces a map $T_{\text{alg}}(M) \to T_{\text{alg}}(O_M)$. This map is well-defined, since any inner automorphism of $\widetilde{\text{SL}}_2$ induces an inner automorphism of $\text{PSL}_2 \mathbb{R}$. Furthermore, integer shifts in fibre direction of $\widetilde{\text{SL}}_2 \to \mathbb{H}^2$ lie in the centre of $\widetilde{\text{SL}}_2$, so it follows directly from the proof of Theorem 4.3 that $T_{\text{alg}}(M)$ is a trivial principal $\mathbb{Z}^{2g}$-bundle over $T(O_M)$.

Similarly, we have a map $T(M) \to T(O_M)$, defined as follows: The common kernel $\text{ker} \omega_1 \cap \text{ker} \omega_2$ of a taut contact circle gives $M$ the structure of a Seifert fibration. The Seifert fibration of $M$ being unique up to isotopy, we can choose a representative of $[(\omega_1, \omega_2)] \in T(M)$ such that it induces the fixed Seifert fibration $M \to O_M$. The unique $(-1)$-Cartan structure in the homothety class of this representative then induces a hyperbolic metric and an orientation on $O_M$.

It is not difficult to see that these maps fit together to form a commutative diagram

\[
\begin{array}{ccc}
T(M) & \xrightarrow{\Phi} & T_{\text{alg}}(M) \\
\downarrow & & \downarrow \\
T(O_M) & \xrightarrow{\phi} & T_{\text{alg}}(O_M).
\end{array}
\]

Remark 4.12. There are natural topologies on $T(M)$ and $T_{\text{alg}}(M)$ which turn the vertical arrows into covering maps. The fact that $\phi$ is a homeomorphism then implies that the bijection $\Phi$ is a homeomorphism as well.

In order to determine the moduli space $M(M)$ we need some information about the automorphism group of $\pi$.

Lemma 4.13. There is a split short exact sequence

\[
0 \longrightarrow \mathbb{Z}^{2g} \longrightarrow \text{Aut}(\pi) \longrightarrow \text{Aut}(\pi_{\text{orb}}) \longrightarrow 1,
\]

where $c \in \mathbb{Z}^{2g}$ acts on $\pi$ by

\[
\begin{align*}
u_i & \mapsto u_i h^{c_{2i}-1}, \\
v_i & \mapsto u_i h^{c_{2i}}.
\end{align*}
\]

Proof. Except for the surjectivity of the homomorphism $\text{Aut}(\pi) \to \text{Aut}(\pi_{\text{orb}})$ and the existence of a splitting this follows easily from the explicit presentations of $\pi$ and $\pi_{\text{orb}}$. This surjectivity is a property specific to left-quotients of $\widetilde{\text{SL}}_2$. 

27
We argue geometrically, as in the proof of the surjectivity of $\Phi$ in Section 3. Given an element of $\text{Aut}(\pi^\text{orb})$, it can be realised geometrically by a diffeomorphism $\psi$ of $O_M$, which may only permute cone points of equal multiplicity. We may assume that $\psi$ fixes an additional base point, distinct from any of the cone points. Changing, if necessary, $\psi$ by an isotopy fixing the base point $x_0$ and the cone points $x_1, \ldots, x_n$, we can find disjoint disc neighbourhoods $D_0, \ldots, D_n$ of these points such that $\psi$ preserves the complement $U \subset O_M$ of this collection of discs and either fixes the boundary curve of each disc (if $\psi$ is orientation preserving) or reverses each of these curves (in the orientation reversing case). Over $U$ the Seifert bundle is trivial, so $\psi$ lifts to the diffeomorphism $\psi \times (\pm \text{id})$ of $U \times S^1$.

By Remark 4.2 the Seifert invariant $(\alpha, \beta)$ of a multiple fibre only depends on the multiplicity $\alpha$ (for a fixed $M$ with a choice of base and fibre orientations, and hence fixed $r$). Depending on whether $\psi$ preserves or reverses orientation, it is a straightforward matter to check that exactly one of $\psi \times (\pm \text{id})$ extends to a diffeomorphism $\psi$ of $M$ which induces an automorphism of $\pi$ that projects to the given automorphism of $\pi^\text{orb}$.

For a fixed choice of trivialisation $U \times S^1$, the diffeomorphism $\psi$ is determined up to isotopy by $\psi$, so it follows that the map $\text{Aut}(\pi^\text{orb}) \to \text{Aut}(\pi)$, associating to the automorphism induced by $\psi$ the one induced by $\psi$, is a splitting of the short exact sequence. Since $h$ is central in $\pi$, one sees for purely algebraic reasons that there is a $\mathbb{Z}^2g$-worth of such splittings. Geometrically, they correspond to the different choices of trivialisations $U \times S^1$.

We continue with the proof of Theorem 4.11. Before turning to the moduli space, we consider the Chabauty space

$$S(M) = \mathcal{R}(\pi, \tilde{\text{SL}}_2)/\text{Aut}(\pi).$$

By Lemma 4.7 we have $\rho(h)(z, \theta) = (z, \theta + 2\pi r)$ for any $\rho \in \mathcal{R}(\pi, \tilde{\text{SL}}_2)$, and from the definition of the principal $\mathbb{Z}^{2g}$-action on $\mathcal{R}(\pi, \tilde{\text{SL}}_2)$ we infer that $c \in \mathbb{Z}^{2g} \subset \text{Aut}(\pi)$ acts on $\mathcal{R}(\pi, \tilde{\text{SL}}_2)$ by

$$\rho_w \mapsto \rho_{w + rc}.$$ 

Hence, by first dividing out this $\mathbb{Z}^{2g}$-action, and then the $\text{Aut}(\pi^\text{orb})$-action, we get

$$S(M) = (\mathcal{R}(\pi^\text{orb}, \text{PSL}_2\mathbb{R}) \times \mathbb{Z}_r^{2g})/\text{Aut}(\pi^\text{orb}).$$

The Weil space consists of faithful representations, and so $\text{Aut}(\pi^\text{orb})$ acts freely on it. It follows that $S(M)$ is a $\mathbb{Z}_r^{2g}$-bundle, i.e. an $r^{2g}$-fold covering of

$$S(O_M) = (\mathcal{R}(\pi^\text{orb}, \text{PSL}_2\mathbb{R}))/\text{Aut}(\pi^\text{orb}).$$

The moduli space $M(M)$ may now be regarded as

$$\mathcal{T}^\text{alg}(M)/\text{Out}(\pi) = (\mathcal{T}^\text{alg}(M)/\mathbb{Z}^{2g})/\text{Out}(\pi^\text{orb}),$$
or as

$$\text{Inn}(\text{SL}_2)\backslash \mathcal{S}(M).$$

In either case, $\mathcal{M}(M)$ is obtained as the quotient under an action that may stabilise a point in the base (by which we mean $\mathcal{T}^{\text{alg}}(O_M)$ or $\mathcal{S}(O_M)$, respectively), without stabilising the points sitting over it, cf. [10, pp. 194–195]. This results in the branching of the covering $\mathcal{M}(M)$ over $\mathcal{M}(O_M)$.

Geometrically, points with non-trivial stabiliser correspond to Seifert manifolds resp. base orbifolds with non-trivial isometries.

A fundamental question about the moduli space for any kind of geometric structure is whether it is connected. The moduli space being connected amounts to there being a unique structure up to diffeomorphism and deformations.

In our situation, this issue can be addressed by studying the covering $\mathcal{T}^{\text{alg}}(M)/\mathbb{Z}^{2g} \to \mathcal{T}(O_M)$; details will appear in [4]. The fibre of this covering can be shown to be in natural one-to-one correspondence with the fibrewise $r$-fold coverings $M \to \text{STO}_M$, with $\text{STO}_M$ denoting the unit tangent bundle of $O_M$ (which is defined even in the presence of cone points). In particular, for $r = 2$ and $O_M$ a surface without cone points, this corresponds to spin structures on $O_M$. More generally, for a choice of a hyperbolic structure on $O_M$ and an $r$-fold fibrewise covering $M \to \text{STO}_M$, sections of $M \to O_M$ are roots of index $r$ of (unit) tangent vectors, just like automorphic forms of weight $1/r$ for the subgroup of $\text{PSL}_2\mathbb{R}$ defining $O_M$. The latter are traditionally required to correspond to holomorphic sections of the associated complex line bundle; in the presence of cone points, this last statement needs to be qualified.

In the case that $O_M$ is free of cone points, it turns out that $\mathcal{M}(M)$ is connected for $r$ odd, and it has exactly two components for $r$ even. For $r = 2$ this corresponds to the fact that there are exactly two spin structures on $O_M$ up to diffeomorphism, cf. [4]. It is worth pointing out that any surface of genus $g \geq 2$ (in fact, $g \geq 1$ suffices) with its two non-diffeomorphic spin structures represents the two elements in the spin cobordism group $\Omega^{\text{Spin}}_2 \cong \mathbb{Z}_2$.

We conclude that there is indeed non-trivial branching in the covering $\mathcal{M}(M) \to \mathcal{M}(O_M)$.

5 $\tilde{\mathbb{E}}_2$-geometry

The left-quotients of $\tilde{\mathbb{E}}_2$ under a discrete, cocompact subgroup $\Gamma$ are, up to diffeomorphism, exactly the five $T^2$-bundles over $S^1$ with periodic monodromy $A \in \text{SL}_2\mathbb{Z}$, cf. [22]. Table 1 gives a choice of possible monodromy matrices $A$, their periods, and the lattices in $\mathbb{C}$ generated by $1$ and $\tau$, $\text{Im} \tau > 0$, which are invariant under the action of $A \in \text{SL}_2\mathbb{Z} \subset \text{GL}_2\mathbb{R}$ (here the action is the usual linear one on $\mathbb{R}^2 = \mathbb{C}$).
We label the five left-quotients of \( \tilde{E}_2 \) as \( M_k \), with \( k \) denoting the period of the monodromy matrix defining \( M_k \), but we continue to write \( T^3 \) for the 3-torus \( M_1 \).

If the monodromy matrix is \( A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \), then the fundamental group \( \pi = \pi_1(M) \) of the total space has the presentation

\[
\pi = \left\{ s, t, b : \text{st = ts, bsb}^{-1} = s^\alpha t^\gamma, \text{btb}^{-1} = s^\beta t^\delta \right\},
\]

cf. [23]. In Table 1 we also list the first homology groups \( H_1(M) \), obtained by abelianising \( \pi_1(M) \). This gives a proof of our claim made in Section 3 that these spaces are distinguished by their fundamental groups.

We now want to study the Weil spaces \( R(\pi, \tilde{E}_2) \) for the five different groups \( \pi \). Given a representation \( \rho : \pi \rightarrow \tilde{E}_2 \), write

\[
\rho : \pi 
\rightarrow E_2 = \text{Isom}_0\mathbb{R}^2
\]

for the composition of \( \rho \) with the projection \( \tilde{E}_2 \rightarrow E_2 \).

Recall the definition of translational and rotational part of an element in \( \tilde{E}_2 \) given in Section 2. Elements with rotational part in \( 2\pi\mathbb{Z} \) are naturally identified with translations of \( \mathbb{R}^3 \).

**Lemma 5.1.** Let \( s, t \in \pi \) be commuting elements of infinite order which generate a (normal) subgroup \( \mathbb{Z} \oplus \mathbb{Z} \) of \( \pi \). Then \( \overline{\rho}(s) \) and \( \overline{\rho}(t) \) are translations of \( \mathbb{R}^2 \), hence \( \rho(s) \) and \( \rho(t) \) are translations of \( \mathbb{R}^3 \).

Normality of the subgroup generated by \( s \) and \( t \) will not be used in the following proof, but it will hold in all cases where we apply this lemma.

| \( A \) | period | \( \tau \) | \( H_1(M) \) |
|-------|--------|--------|-------------|
| \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) | 1 | any | \( \mathbb{Z}^3 \) |
| \( \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \) | 2 | any | \( \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) |
| \( \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \) | 3 | \( \exp(2\pi i/3) \) | \( \mathbb{Z} \oplus \mathbb{Z}_3 \) |
| \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) | 4 | \( i \) | \( \mathbb{Z} \oplus \mathbb{Z}_2 \) |
| \( \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \) | 6 | \( \exp(2\pi i/6) \) | \( \mathbb{Z} \) |

Table 1: \( T^2 \)-bundles over \( S^1 \).
Proof. Write
\[ \rho(s) = \left( \begin{array}{c} x_s \\ y_s \end{array} \right), \theta_s \right). \]
We want to show that \( \theta_s \in 2\pi\mathbb{Z} \). Arguing by contradiction, we assume \( \theta_s \notin 2\pi\mathbb{Z} \), so \( \overline{\rho}(s) \) is not a translation. Then \( \overline{\rho}(s) \) is rotation about a unique fixed point \( z_s \in \mathbb{R}^2 \). Since \( s \) and \( t \) commute, we get
\[ \rho(s) \rho(t)(z_s) = \rho(t) \rho(s)(z_s) = \rho(t)(z_s), \]
hence \( \rho(t)(z_s) = z_s \). So \( \rho(s), \rho(t) \) are rotations about \( z_s \in \mathbb{R}^2 \) with angle \( \theta_s, \theta_t \), respectively.

If \( \theta_t/\theta_s \in \mathbb{Q} \), then \( m\theta_s + n\theta_t = 0 \) for suitable \( m, n \in \mathbb{Z} \setminus \{0\} \). This implies that \( \rho(s^m t^n) \) is the identity element in \( \tilde{E}_2 \), so \( \rho \) would not be faithful.

If \( \theta_t/\theta_s \notin \mathbb{Q} \), then, as is well known, for any \( \varepsilon > 0 \) one can find \( m, n \in \mathbb{Z} \) such that
\[ 0 < |m\theta_s + n\theta_t| < \varepsilon. \]
So in this case \( \rho \) would not have a discrete image.

Since \( \rho \in \mathcal{R}(\pi, \tilde{E}_2) \) is, by definition, faithful and with discrete image, the assumption \( \theta_s \notin 2\pi\mathbb{Z} \) leads to a contradiction. \( \Box \)

To state the geometric description of the deformation spaces we recall some standard notation. As in Section 4 we write
\[ \mathbb{H}^2 = \{ \tau \in \mathbb{C}: \text{Im} \tau > 0 \} \]
for the upper half-plane, on which \( \text{SL}_2 \mathbb{Z} \) acts from the left by fractional linear transformations. Write \( \mathbb{H}^2 \times T^2 \) for the trivial \( T^2 \)-bundle over \( \mathbb{H}^2 \) defined as a quotient
\[ (\mathbb{H}^2 \times \mathbb{C})/\sim, \]
where \( (\tau, z) \sim (\tau', z') \) if and only if \( \tau = \tau' \) and \( z - z' \) lies in the lattice \( \langle 1, \tau \rangle \subset \mathbb{C} \) generated over \( \mathbb{Z} \) by 1 and \( \tau \). This topologically trivial \( T^2 \)-bundle is the ‘universal elliptic curve’: a bundle over the Teichmüller space \( \mathbb{H}^2 \) of elliptic curves, where the fibre over a point \( \tau \in \mathbb{H}^2 \) is exactly the elliptic curve corresponding to that element of Teichmüller space. Cf. Section 6 for the analogous construction of universal families for taut contact circles on 3-manifolds.

The natural diagonal action of \( \text{SL}_2 \mathbb{Z} \) on \( \mathbb{H}^2 \times T^2 \) is given by
\[ \text{SL}_2 \mathbb{Z} \ni \left( \begin{array}{cc} a & b \\ c & d \end{array} \right): (\tau, z \mod (1, \tau)) \mapsto \left( \begin{array}{c} a\tau + b \\ c\tau + d \end{array} \right), z \mod (1, \begin{array}{c} a\tau + b \\ c\tau + d \end{array}) \right). \]

The following theorem describes the moduli spaces of taut contact circles on the five left-quotients of \( \tilde{E}_2 \). The subsequent proof will also contain a description of the corresponding Teichmüller spaces.

31
Theorem 5.2. Let $M_k$ be the $T^2$-bundle over $S^1$ with periodic monodromy of period $k \in \{1, 2, 3, 4, 6\}$. The moduli spaces $\mathcal{M}(M_k)$ of taut contact circles are given in the following table:

| $k$   | $\mathcal{M}(M_k)$                                      |
|-------|---------------------------------------------------------|
| 1     | $\mathbb{N} \times \text{SL}_2 \mathbb{Z} / (\mathbb{H}^2 \times T^2)$ |
| 2     | $\{r \in \mathbb{N} : r \equiv 1 \text{ mod } 2\} \times \text{PSL}_2 \mathbb{Z} / \mathbb{H}^2$ |
| 3     | $\{r \in \mathbb{N} : r \equiv \pm 1 \text{ mod } 3\}$ |
| 4     | $\{r \in \mathbb{N} : r \equiv \pm 1 \text{ mod } 4\}$ |
| 6     | $\{r \in \mathbb{N} : r \equiv \pm 1 \text{ mod } 6\}$ |

Proof. (i) For $M_1 = T^3$, Lemma 5.1 applies to show that $\rho(u)$ is a translation for all $u \in \pi \cong \mathbb{Z}^3$, hence $\mathcal{R}(\mathbb{Z}^3, E_2) \subset \mathcal{R}(\mathbb{Z}^3, \mathbb{C} \times 2\pi \mathbb{Z})$ (after identifying the translational parts of elements in $E_2$ with $\mathbb{C}$). Observe that the action $\rho \mapsto \rho''$ of $w \in E_2$ by conjugation on $\rho$ is as follows: If $w$ is a translation this action is trivial; if the translational part of $w$ is 0, then $\rho''(u)$ is obtained by rotating the translational part of $\rho(u)$ by an angle equal (mod $2\pi$) to the rotational part of $w$. This implies that the left-action of $\text{Inn}(E_2)$ and the right-action of $\mathbb{R}^+$ on $\mathcal{R}(\mathbb{Z}^3, \mathbb{C} \times 2\pi \mathbb{Z})$ together constitute the standard $\mathbb{C}^*$-action on $\mathbb{C}^3$. Hence

$$\mathcal{T}(T^3) = \{([z_1 : z_2 : z_3], (r_1, r_2, r_3)) \in \mathbb{C}P^2 \times \mathbb{Z}^3 : \text{rank}_\mathbb{R}((z_1, r_1), (z_2, r_2), (z_3, r_3)) = 3\}.$$  

Notice that the points in $\mathcal{T}(T^3)$ with fixed coordinates $(r_1, r_2, r_3) \in \mathbb{Z}^3$ form the complement of a real hypersurface in $\mathbb{C}P^2$. For $(r_1, r_2, r_3) = (0, 0, 1)$, the $z_3$-coordinates of a point in $\mathcal{T}(T^3)$ are of the form $[1 : z_2 : z_3]$ with $z_2 \notin \mathbb{R}$. Hence, topologically $\mathcal{T}(T^3)$ consists of a disjoint union of copies of $\mathbb{R}^4$.

We now turn to $\mathcal{M}(T^3) = \mathcal{T}(T^3) / \text{Out}(\mathbb{Z}^3)$. Given $\rho \in \mathcal{R}(\mathbb{Z}^3, E_2)$, one can find a basis $a_1, a_2, a_3$ for $\mathbb{Z}^3$ such that the rotational parts of $\rho(a_1)$ and $\rho(a_2)$ are zero, and their translational parts form a negative basis for $\mathbb{R}^2 = \mathbb{C}$ (with respect to the complex orientation), and such that the rotational part of $\rho(a_3)$ is equal to $2\pi r$ with $r \in \mathbb{N}$. Here $r$ is the greatest common divisor of the rotational parts/2$\pi$ of the $\rho(u)$, $u \in \mathbb{Z}^3$ (all of which lie in $2\pi \mathbb{Z}$).

The right action of $\text{Out}(\mathbb{Z}^3) = \text{GL}_3 \mathbb{Z}$ allows us to make arbitrary basis changes in $\mathbb{Z}^3$, so we can fix a basis $e_1, e_2, e_3$ and assume that our representative $\rho$ of $[\rho] \in \mathcal{M}(T^3)$ maps this fixed basis as described above for $a_1, a_2, a_3$. Any further action of $\text{Out}(\mathbb{Z}^3)$ on special representatives of this kind is then given by the subgroup $\mathcal{B} \subset \text{SL}_3 \mathbb{Z}$ consisting of elements which stabilise the subgroup $\mathbb{Z}^2 \oplus 0 \subset \mathbb{Z}^3$ generated by $e_1$ and $e_2$. That is, elements of $\mathcal{B}$ are of the form

$$\begin{pmatrix} b_{11} & b_{12} & * \\ b_{21} & b_{22} & * \\ 0 & 0 & 1 \end{pmatrix},$$

32
with $B = (b_{ij}) \in \text{SL}_2 \mathbb{Z}$ (here $\text{SL}_2 \mathbb{Z}$ acts on $\mathbb{Z}^3$ from the left).

Furthermore, the $\mathbb{C}^*$-action on $\mathcal{R}(\mathbb{Z}^3, \tilde{E}_2)$ allows us to assume that our representative $\rho$ is as follows:

$$
\begin{align*}
\rho(e_1) &= (\tau, 0), \quad \tau \in \mathbb{H}^2, \\
\rho(e_2) &= (1, 0), \\
\rho(e_3) &= (z, 2\pi r), \quad r \in \mathbb{N}.
\end{align*}
$$

We call this a representation in \textit{standard form}. Thus, writing $\mathcal{M}(T^3) = \mathcal{R}''(\mathbb{Z}^3, \tilde{E}_2)/B$, and this translates into the description given in the theorem, in complete analogy with the usual description of moduli space for surfaces of genus 1. The only difference is due to the additional parameters $r$ and $z$. The fact that $-\text{id} \in \text{SL}_2 \mathbb{Z}$ acts non-trivially on $z$ means that we cannot pass from $\text{SL}_2 \mathbb{Z}$ to $\text{PSL}_2 \mathbb{Z}$. Notice that away from the fixed points of the $\text{SL}_2 \mathbb{Z}$-action on $\mathbb{H}^2$, the space $\text{SL}_2 \mathbb{Z}\backslash(\mathbb{H}^2 \times T^2)$ is the universal elliptic curve over the moduli space $\text{SL}_2 \mathbb{Z}\backslash\mathbb{H}^2$ of elliptic curves; over the fixed points the fibre is the quotient of an elliptic curve by a finite group action.

(ii) As we pass to higher values of $k$, the symmetries coming from a non-trivial monodromy mean that we lose the freedom in the parameters $z$ and $\tau$. This accounts for the loss in dimension of the corresponding moduli spaces. In all these remaining cases $k \in \{2, 3, 4, 6\}$ we fix a presentation of $\pi = \pi(k)$ as at the beginning of this section. Let $\rho$ be an element of $\mathcal{R}(\pi, \tilde{E}_2)$. From Lemma 5.1 we know that $\rho(s)$ and $\rho(t)$ are translations. The element $\rho(b)$, on the other hand, cannot be a translation; otherwise it would commute with $\rho(s)$ and $\rho(t)$, and $\rho$ would not be faithful.

Conjugation of $\rho$ by a translation in $\tilde{E}_2$ does not affect translations, and it allows to change the fixed point in $\mathbb{R}^2 = \mathbb{C}$ of a rotation. So we can choose a representative $\rho$ for $[\rho] \in \mathcal{T}(M_k)$, $k \in \{2, 3, 4, 6\}$, in such a way that $\overline{\rho}(b)$ is a rotation about $0 \in \mathbb{C}$.

Since $b^k$ does commute with $s$ and $t$, Lemma 5.1 shows that $\overline{\rho}(b^k)$ is a translation. But it fixes $0 \in \mathbb{C}$, so it must be trivial. Hence $\rho(b) = (0, 2\pi r/k)$ with $r \in \mathbb{Z} - \{0\}$.

As earlier we write $\theta_u$ for the rotational part of $\rho(u) \in \tilde{E}_2$, $u \in \pi$. We have $\theta_u = \theta_s + \theta_v$, so the relations in $\pi$ yield the following equations:

$$
\begin{align*}
\theta_s &= \alpha \theta_s + \gamma \theta_t, \\
\theta_t &= \beta \theta_s + \delta \theta_t,
\end{align*}
$$

or as a matrix equation,

$$
\begin{pmatrix}
\alpha - 1 & \gamma \\
\beta & \delta - 1
\end{pmatrix}
\begin{pmatrix}
\theta_s \\
\theta_t
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0
\end{pmatrix}.
$$

33
The determinant of this \((2 \times 2)\)-matrix is

\[(\alpha - 1)(\delta - 1) - \beta \gamma = \alpha \delta - \beta \gamma - (\alpha + \delta) + 1 = 2 - \text{trace } A.\]

For a periodic matrix \(A \in \text{SL}_2 \mathbb{Z}\) different from the identity matrix we have \(\text{trace } A \neq 2\), so the only solution to the equation above is \(\theta_s = \theta_t = 0\). Therefore, any element of \(\mathcal{T}(M_k)\) has a representative of the form

\[
\begin{align*}
\rho(s) &= (1, 0), \\
\rho(t) &= (\tau, 0), \quad \tau \in \mathbb{C} - \mathbb{R}, \\
\rho(b) &= (0, 2\pi r/k), \quad r \in \mathbb{Z} - \{0\}.
\end{align*}
\]

(As in (i), rotation of the translational parts can be effected by conjugation with a rotation \(w \in \tilde{E}_2\).)

(iii) Now consider the different values of \(k\) in turn, beginning with \(k = 2\): To get a representation for \(\pi(2)\) it is necessary and sufficient to have \(r\) odd, so

\[\mathcal{T}(M_2) = \{r \in \mathbb{Z}: r \equiv 1 \text{ mod } 2\} \times (\mathbb{C} - \mathbb{R}).\]

The group \(\pi(2)\) admits automorphisms which send any of the generators \(s, t, b\) to their inverse and fixes the others. So in \(\mathcal{M}(M_2) = \mathcal{T}(M_2)/\text{Out}(\pi(2))\) we may impose on the representative \(\rho\) for a given class the further conditions \(\text{Im } \tau > 0\) and \(r \in \mathbb{N}\).

Any further action of \(\text{Out}(\pi(2))\) on representations of this kind must (and can) be of the form

\[s \mapsto s^{a} t^{b}, \quad t \mapsto s^{c} t^{d}\]

for arbitrary \(\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \text{SL}_2 \mathbb{Z}\), where the action of \(-\text{id} \in \text{SL}_2 \mathbb{Z}\) is trivial modulo \(\text{Im}(\tilde{E}_2)\). It follows that \(\mathcal{M}(M_2)\) is as claimed.

(iv) \(k = 3, 4, 6\): The arguments in these three cases are completely analogous to one another, and we only present the one for \(k = 3\). The presentation of \(\pi(3)\) is

\[\pi(3) = \{s, t, b: \ st = ts, \ b \ b^{-1} = t, \ b \ t \ b^{-1} = s^{-1} t^{-1}\}.
\]

Since the monodromy has order 3, we must have \(r \equiv \pm 1 \text{ mod } 3\) in the description of \(\rho\). Then the relation \(b \ b^{-1} = t\) forces \(\tau = \exp(2\pi i r/3)\). This is consistent with \(b \ t \ b^{-1} = s^{-1} t^{-1}\), indeed, the given \(\tau\) satisfies \(\tau^2 = -1 - \tau\). Hence

\[\mathcal{T}(M_3) = \{r \in \mathbb{Z}: r \equiv \pm 1 \text{ mod } 3\}.
\]

The group \(\pi(3)\) admits an automorphism defined by

\[s \mapsto s, \quad t \mapsto s^{-1} t^{-1}, \quad b \mapsto b^{-1},\]

34
as can also be seen from the geometry of the torus bundle, and there are no other non-trivial outer automorphisms to consider. Thus

\[ \mathcal{M}(M_3) = \mathcal{T}(M_3)/\pm = \{ r \in \mathbb{N} : r \equiv \pm 1 \mod 3 \}. \]

This completes the proof of Theorem 5.2.

\[ \blacksquare \]

6 \quad SU(2)-geometry

In this section we return to the study of taut contact circles on left-quotients \( M \) of SU(2). Our aim is to obtain information about the Teichmüller spaces \( \mathcal{T}(M) \) of taut contact circles on these manifolds. Since \( \mathcal{M}(M) \) is known (Theorem 2.2) and

\[ \mathcal{M}(M) = \mathcal{T}(M)/(\text{Diff}(M)/\text{Diff}_0(M)), \]

we need to investigate the action of the homeotopy group

\[ \Lambda(M) = \text{Diff}(M)/\text{Diff}_0(M) \]

on taut contact circles. Notice that, as a set, this homeotopy group can be identified with the set \( \pi_0(\text{Diff}(M)) \) of path-connected components of \( \text{Diff}(M) \).

First we recall the list of finite subgroups of SU(2), see [33, Thm. 2.6.7]. We also fix a particular representation \( \rho \) of these groups in SU(2). Thus we do not distinguish between the abstract group \( \pi \) and its image \( \Gamma = \rho(\pi) \subset \text{SU}(2) \), and we identify \( u \in \pi \) both with \( \rho(u) \in \Gamma \) and its action \( L_{\rho(u)} \) by left multiplication. For our purposes this is justified since, by the cited theorem, isomorphic finite subgroups of SU(2) are actually conjugate in SU(2).

We identify SU(2) with the unit quaternions \( S^3 \subset \mathbb{H} \) by the group isomorphism

\[
\begin{pmatrix}
a_0 + ia_1 & b_0 + ib_1 \\
b_0 + ib_1 & a_0 - ia_1
\end{pmatrix} \mapsto a_0 + ia_1 + jb_0 + kb_1.
\]

- The cyclic group \( C_m \) of order \( m \), generated by \( x = \cos(2\pi/m) + i \sin(2\pi/m) \).

- The binary dihedral group of order \( 4n \),

\[ D_{4n}^* = \{ x, y : x^2 = (xy)^2 = y^n \}, \]

generated by \( x = i \) and \( y = \cos(\pi/n) + j \sin(\pi/n) \). The group

\[ D_8^* = \{ \pm 1, \pm i, \pm j, \pm k \} \]

is also called the quaternion group \( Q_8 \).
• The **binary tetrahedral group** of order 24,
\[ T^* = Q_8^{x,y} \rtimes C_3^z, \]
generated by \( x = i, y = j \), and \( z = -(1 + i + j + k)/2 \). Here \( C_3 \) acts on \( Q_8 \) by \( zxz^{-1} = y \) and \( zyz^{-1} = xy \).

• The **binary octahedral group** of order 48,
\[ O^* = T^* \rtimes C_4^w/(w^2 = x), \]
with \( w = (i - k)/\sqrt{2} \). The action of \( C_4 \) on \( T^* \) is given by \( wxw^{-1} = yx \), \( wyw^{-1} = y^{-1} \), and \( wzw^{-1} = z^{-1} \).

• the **binary icosahedral group** of order 120 with presentation
\[ I^* = \{ a, b : a^2 = (ab)^3 = b^5, a^4 = 1 \} . \]

**Remark 6.1.** The groups \( T^* \) and \( O^* \) also admit the more concise presentations
\[ \{ a, b : a^2 = (ab)^3 = b^5, a^4 = 1 \}, \beta = 3,4, \]
but the presentations above are more useful for our purposes. For \( T^* \), for instance, the isomorphism between the two presentations is given by \( a \mapsto x \) and \( b \mapsto yz \).

Let \( \overline{\varphi} \) be a diffeomorphism of \( M = \Gamma \\backslash \text{SU}(2) \). This lifts to a diffeomorphism \( \varphi \) of \( \text{SU}(2) \), which defines an automorphism \( \varphi_{\overline{\varphi}} \) of \( \Gamma \) by
\[ \varphi \circ u \circ \varphi^{-1} = \varphi_{\overline{\varphi}}(u) \ \forall u \in \Gamma. \]
Different lifts \( \varphi \) of \( \overline{\varphi} \) differ by a deck transformation (i.e., an element of \( \Gamma \)), so the equivalence class \([\varphi_{\overline{\varphi}}] \in \text{Out}(\Gamma)\) is determined by \( \overline{\varphi} \), and it is easy to check that \( \overline{\varphi} \mapsto [\varphi_{\overline{\varphi}}] \) defines a homomorphism.

If \( \overline{\varphi}_0 \) is an isotopy from \( \overline{\varphi}_0 = \overline{\varphi} \) to \( \overline{\varphi}_1 = \text{id} \), there is a lifted isotopy \( \varphi_t \) from \( \varphi_0 = \varphi \) to a deck transformation \( \varphi_1 = w \in \Gamma \). During this isotopy the automorphism \( \varphi_{\overline{\varphi}} \) of the discrete group \( \Gamma \) is forced to stay fixed. So there is a well-defined homomorphism
\[ \chi : \Lambda(M) \rightarrow \text{Out}(\Gamma) \]
\[ [\overline{\varphi}] \mapsto [\varphi_{\overline{\varphi}}]. \]

We first consider the case that \( M = \Gamma \\backslash \text{SU}(2) \) is a quotient of \( \text{SU}(2) \) under one of the non-abelian subgroups \( \Gamma \) of \( \text{SU}(2) \). We know that in this case the moduli space \( \mathcal{M}(M) \) consists of a single point. Since \( \mathcal{M}(M) = \mathcal{T}(M)/\Lambda(M) \), it follows that \( \mathcal{T}(M) \) is a discrete set on which \( \Lambda(M) \) acts transitively. So the order \(|\mathcal{T}(M)|\) of this set is equal to \(|\Lambda(M)/\Lambda_0(M)|\), where \( \Lambda_0(M) \) denotes the subgroup of \( \Lambda(M) \) stabilising a particular element of \( \mathcal{T}(M) \), say the one represented by \( \omega_1 + i\omega_2 = z_1 dz_2 - z_2 dz_1 \). In the sequel we are going to determine \(|\Lambda_0(M)|\), and with that information we obtain the following theorem.
**Theorem 6.2.** Let \( M = \Gamma \backslash \text{SU}(2) \) be a left-quotient of \( \text{SU}(2) \) under a discrete, cocompact, non-abelian subgroup \( \Gamma \). Then \( T(M) \) is a discrete set, and the order of this set is given in the following table, where we write \( |\Lambda| = |\Lambda(M)| \) for short:

| \( \Gamma \) | \( Q_8 \) | \( D_{5n}^\pm \) | \( T^* \) | \( O^* \) | \( I^* \) |
|---|---|---|---|---|---|
| \( |T(M)| \) | \( |\Lambda|/6 \) | \( |\Lambda|/2 \) | \( |\Lambda| \) | \( |\Lambda| \) | \( |\Lambda| \) |

**Remark 6.3.** We are not aware of any information about \( |\Lambda(M)| \) for the non-abelian groups \( \Gamma \).

**Lemma 6.4.** Let \( M \) be a non-abelian left-quotient of \( \text{SU}(2) \). Then

\[
\chi|_{\Lambda_0(M)} : \Lambda_0(M) \longrightarrow \text{Out}(\Gamma)
\]

is a monomorphism onto the subgroup \( \text{Out}_0(\Gamma) \) of outer automorphisms of \( \Gamma \subset \text{SU}(2) \) that are given by conjugation with a matrix in \( \text{SL}_2\mathbb{C} \).

**Proof.** Let \( \varphi \) be an element of \( \Lambda_0(M) \). This means that \( \varphi \) preserves the homothety class of \( \omega_1 + i\omega_2 = z_1dz_2 - z_2dz_1 \) up to a diffeomorphism isotopic to the identity. Hence, without changing the class \( \varphi \) we may actually assume that \( \varphi \) fixes this homothety class. Moreover, the rotation of \( (\omega_1, \omega_2) \) can be effected (on \( S^3 \)) by the flow of the left-invariant vector field \( X \) defined by \( \omega_1(X) = \omega_2(X) = 0 \) and \( d\omega_1(X,.) = \omega_1 \) (this is simply the Hopf vector field). This flow descends to \( M \), so we may assume that \( \varphi \) fixes the conformal class of \( (\omega_1, \omega_2) \), say \( \varphi^*(\omega_1, \omega_2) = \overline{\varphi}(\omega_1, \omega_2) \) with \( \overline{\varphi} : M \to \mathbb{R}^+ \). Lift \( \overline{\varphi} \) to a diffeomorphism \( \varphi \) of \( S^3 \) and \( \overline{\varphi} \) to a function \( v \) on \( S^3 \). From the construction in [5, Section 3] it follows that the diffeomorphism

\[
\phi : S^3 \times \mathbb{R} \longrightarrow S^3 \times \mathbb{R} \\
(x, t) \longmapsto (\varphi(x), t - \log v(x))
\]

is an \( \mathbb{R} \)- and \( \Gamma \)-equivariant holomorphic automorphism of \( \mathbb{C}^2 - \{(0, 0)\} \) under the natural identification of that space with \( S^3 \times \mathbb{R} \), and it preserves the holomorphic 1-form \( \omega_1 + i\omega_2 \). This \( \phi \) extends to an automorphism of \( \mathbb{C}^2 \) fixing the origin. The \( \mathbb{R} \)-equivariance implies that the holomorphic differential of \( \phi \) is bounded, and hence constant. Therefore \( \phi \) has to be a linear automorphism, and the fact that it preserves \( d(\omega_1 + i\omega_2) = dz_1 \wedge dz_2 \) forces \( \phi \in \text{SL}_2\mathbb{C} \). The \( \Gamma \)-equivariance of \( \phi \) takes the form

\[
\phi \circ u \circ \varphi^{-1} = \varphi(u) \quad \forall u \in \Gamma,
\]

where the left-hand side may now be read as matrix multiplication in \( \text{SL}_2\mathbb{C} \).

Conversely, if we start with an element of \( \text{SL}_2\mathbb{C} \) that normalises \( \Gamma \), then it induces an \( \mathbb{R} \)-equivariant diffeomorphism of \( \Gamma \backslash (\mathbb{C}^2 - \{(0, 0)\}) \cong M \times \mathbb{R} \) that
preserves $\omega_1 + i \omega_2 = z_1 dz_2 - z_2 dz_1$, and hence a diffeomorphism of $M$ preserving the conformal class of $(\omega_1, \omega_2)$. This proves that $\chi(\Lambda_0(M)) = \text{Out}_0(\Gamma)$.

It remains to prove injectivity of $\chi|_{\Lambda_0(M)}$. Since the image $\chi(\Lambda_0(M))$ consists of automorphisms given by conjugation with an element $\phi \in \text{SL}_2 \mathbb{C}$, it suffices to show that the condition

$$\phi \circ u \circ \phi^{-1} = wuw^{-1} \quad \forall u \in \Gamma,$$

with $\phi \in \text{SL}_2 \mathbb{C}$ and some $w \in \Gamma$, forces $\phi \in \Gamma$. But this follows easily from the fact that the $\mathbb{C}$-linear span of any non-abelian subgroup $\Gamma \subset \text{SU}(2)$ consists of all complex $(2 \times 2)$-matrices, so the preceding condition (which is $\mathbb{C}$-linear in $u$) does in fact imply $\phi = w$. This concludes the proof of the lemma.

Together with the following proposition, this lemma constitutes a proof of Theorem 6.2.

**Proposition 6.5.** The groups $\text{Out}_0(\Gamma)$ are as given in the following table:

| $\Gamma$ | $Q_8$ | $D_{2n+4}$ | $D_{2n+8}$ | $T^*$ | $O^*$ | $I^*$ |
|-----------|-------|------------|------------|------|------|------|
| $\text{Out}_0(\Gamma)$ | $S_3$ | $C_2$ | $1$ | $C_2$ | $1$ | $1$ |

**Proof.** (i) $Q_8$: The group $\text{Aut}(Q_8)$ contains as a normal subgroup the Klein 4-group $V_4$, corresponding to the sign changes of $i$ and $j$ (the sign of $k$ is then determined). These are inner automorphisms of $Q_8$, given by conjugation with $i$, $j$, or $k$. The quotient group $\text{Aut}(Q_8)/V_4$ is isomorphic to the symmetric group $S_3$, acting by permutations of $i$, $j$, and $k$. The transposition $(ij)$ is given by conjugation with the unit quaternion $(1 + k)/\sqrt{2}$,

$$\frac{1 + k}{\sqrt{2}} i \frac{1 - k}{\sqrt{2}} = j.$$

The 3-cycle $(ijk) = (jk)(ik)$ is given by conjugation with

$$\frac{1 + i}{\sqrt{2}} \frac{1 - j}{\sqrt{2}} = \frac{1 + i - j - k}{2}.$$

These are not inner automorphisms, but they are generated by conjugation with unit quaternions, and hence elements of $\text{SU}(2) \subset \text{SL}_2 \mathbb{C}$ under our identification of the unit quaternions with $\text{SU}(2)$. It follows that

$$\text{Out}(Q_8) = \text{Out}_0(Q_8) = S_3.$$

(ii) $D_{4n}$, $n \geq 3$: From the relation $x^2 = (xy)^2 = y^n$ one obtains $x^{-1}yx = y^{-1}$, hence

$$x^2 = x^{-1}x^2x = x^{-1}y^n x = y^{-n} = x^{-2},$$

38
so $x$ is of order 4 and $y$ of order $2n$. Since $yx = xy^{-1}$, a complete list of elements of $D_{4n}^*$ is given by
\[ \{1, y, \ldots, y^{2n-1}, x, xy, \ldots, xy^{2n-1}\}. \]
The order of each element of the form $xy^k$ is equal to 4. It follows that automorphisms of $D_{4n}^*$, $n \geq 3$, can only be of the form $x \mapsto xy^l$, $y \mapsto y^k$ with $\gcd(k, 2n) = 1$. The assumption that such an automorphism is given by conjugating $D_{4n}^* \subset SU(2)$ with an element $\phi \in SL_2\mathbb{C}$ leads, via some simple algebra, to the following list of possibilities:
\[
\begin{align*}
    x & \mapsto x, \quad y \mapsto y^{2n-1}; \\
    x & \mapsto xy^n = x^{-1}, \quad y \mapsto y; \\
    x & \mapsto xy^n = x^{-1}, \quad y \mapsto y^{2n-1}.
\end{align*}
\]
If we regard $D_{4n}^*$ as a subgroup of the unit quaternions, these automorphisms are induced by conjugation with $i$, $j$ and $k$, respectively. So the first one is always inner; the second and third one are inner if and only if $n$ is even (and they always differ by an inner automorphism).

(iii) $T^*$: This group contains $Q_8$ as a normal subgroup, and it is a straightforward check from the explicit presentations that the non-trivial elements of $Q_8 \subset T^*$ are characterised as exactly those elements of $T^*$ that have order 4. It follows that any automorphism of $T^*$ induces an automorphism of $Q_8$. So we can refer to the results in (i).

The automorphism of $Q_8$ given by conjugation with $(1+k)/\sqrt{2}$ also induces one of $T^*$, and it is not inner. Conjugation with $(1+i-j-k)/2 = zy$, on the other hand, is an inner automorphism. This implies $\text{Out}_0(T^*) = S_3/A_3 = C_2$.

(iv) $O^*$: This group contains a unique copy of $T^*$, so any automorphism of $O^*$ will induce one of $T^*$. Since $xwz = (1+k)/\sqrt{2}$, the only possible outer automorphism of $T^*$ given by conjugation with an element of $SL_2\mathbb{C}$ actually comes from an inner automorphism of $O^*$. We infer that $\text{Out}_0(O^*)$ is trivial.

(v) $I^*$: It is known that $\text{Out}(I^*) = C_2$, cf. [33, p. 195]. Moreover, $I^*$ has exactly two non-equivalent irreducible complex fixed-point free representations, both of degree 2, which differ by the non-trivial outer automorphism, see [33, Lemma 7.1.7]. So this outer automorphism cannot come from conjugation with an element in $SL_2\mathbb{C}$.

This concludes the proof of the proposition.

We now turn to the lens spaces $L(m, m - 1)$. Use the notation $\mathcal{M}_1, \mathcal{M}_2$ as in Theorem 2.2 and write $\tilde{\mathcal{M}}_1$ for the complex slab $\{a \in \mathbb{C}: 0 < \text{Re}(a) < 1\}$.

**Theorem 6.6.** For $M = L(m, m - 1)$, the Teichmüller spaces $T(M)$ are as follows:
\[
\begin{array}{c|c}
 m & T(L(m, m - 1)) \\
1, 2 & M_1 \sqcup M_1 \sqcup M_2 \sqcup M_2 \\
\geq 3 & \tilde{M}_1 \sqcup M_2 \sqcup M_2
\end{array}
\]

Proof. (i) If \( m = 1 \) or \( 2 \) (i.e. \( M = S^3 \) or \( \mathbb{RP}^3 \)), the identification of elements in \( C(L(m, m - 1)) \) corresponding to the parameter values \( a, 1 - a \) (in the continuous family of taut contact circles described in Section \ref{section2}) can be effected by the diffeomorphism of \( L(m, m - 1) \) induced by \( (z_1, z_2) \mapsto (z_2, -z_1) \). This diffeomorphism lies in \( \text{Diff}_0(M) \) in both cases. Furthermore, we have \( \Lambda(M) = C_2 \), generated by the orientation reversing diffeomorphism \( (z_1, z_2) \mapsto (\overline{z}_1, z_2) \). So \( \Lambda(M) \) acts non-trivially on \( T(M) \), and we conclude that \( T(M) \) consists of two disjoint copies of \( M \).

(ii) If \( m \geq 3 \), we still have \( \Lambda(M) = C_2 \), see \cite{3, 13}. But the diffeomorphism \( (z_1, z_2) \mapsto (z_2, -z_1) \) conjugates the action of \( x = \cos(2\pi/m) + i\sin(2\pi/m) \) to that of \( \overline{x} \). Since \( x \) and \( \overline{x} \) are not conjugate in \( C_m \) for \( m \geq 3 \), the induced diffeomorphism of \( L(m, m - 1) \) cannot be isotopic to the identity. (In particular, this orientation preserving diffeomorphism defines the non-trivial element in \( \Lambda(M) \), and \( M \) does not admit any orientation reversing diffeomorphism.) As in the proof of Lemma \ref{lemma6.4} we see that a diffeomorphism of \( L(m, m - 1) \) that preserves the homothety class of one of the standard models for a taut contact circle, or sends the model with parameter value \( a \) to that with parameter \( 1 - a \), has to be induced by conjugation with an element \( \phi \in \text{SL}_2\mathbb{C} \). This forces \( \phi \) to be diagonal or anti-diagonal. The latter happens exactly when we exchange \( a \) and \( 1 - a \) and the automorphism of \( C_m \) induced by \( \phi \) is \( x \mapsto \overline{x} \). It follows that \( T(M) \) is as claimed. \( \square \)

7 The complex structure of Teichmüller space

For manifolds \( M \) modelled on \( \tilde{\text{SL}}_2 \) we have seen in Theorem \ref{thm4.11} that \( T(M) \rightarrow T(O_M) \) is a trivial principal \( \mathbb{Z}^{2g-1} \)-bundle, so we can equip \( T(M) \) with a complex structure such that the projection onto \( T(O_M) \) becomes a local biholomorphism, where the complex structure on \( T(O_M) \) is the one from classical Teichmüller theory. For manifolds modelled on \( \text{SU}(2) \) or \( \tilde{E}_2 \) we have described the moduli space \( \mathcal{M}(M) \) explicitly as a complex space. Our aim in the present section is to show that these complex structures on \( T(M) \) or \( \mathcal{M}(M) \) are the ‘natural’ ones to consider.

For the geometry \( \text{SU}(2) \) this may be justified from the explicit description of the contact circles corresponding to points in \( \mathcal{M}_1 \) (see Theorem \ref{thm2.2}) in terms of the complex parameter \( a \). This description hinges on our theory developed in \cite{3} which relates taut contact circles on \( M \) to complex structures on \( M \times \mathbb{R} \). Here, by contrast and more intrinsically, we show that one can associate with any taut contact circle a complex analogue of the Godbillon-
Vey invariant, which essentially recovers the moduli parameter \( a \) in the case of left-quotients of \( \text{SU}(2) \). That complex Godbillon-Vey invariant is in fact an invariant of transversely conformal flows on 3-manifolds; this aspect will be pursued further in a forthcoming paper \([10]\).

For the geometries \( \widetilde{\text{SL}}_2 \) and \( \widetilde{E}_2 \) we describe universal families analogous to those of Bers \([1]\) in classical Teichmüller theory. Such a universal family consists of a holomorphic fibration over \( T(M) \), where the fibre over a point \( \sigma \in T(M) \) is the complex surface (diffeomorphic to \( M \times S^1 \)) determined – in the sense of \([5]\) – by the taut contact circle on \( M \) corresponding to \( \sigma \).

Analogous universal families for Seifert 4-manifolds are described in \([29]\).

\((1)\) In the case of the geometry \( \widetilde{E}_2 \) we only need to consider \( k = 1 \) (i.e. taut contact circles on the 3-torus). For \( k = 2 \) the construction will be similar. For the other values of \( k \) the Teichmüller spaces are discrete sets, so geometrically the construction of a universal family is not an issue, but it might be of some arithmetic interest.

For the 3-torus a universal family can best be defined over \( R^{''} (\mathbb{Z}^3, \widetilde{E}_2) \) (and it can then be pulled back to \( T(T^3) \)). Given \([\rho] \in R^{''} (\mathbb{Z}^3, \widetilde{E}_2)\), choose the unique representative \( \rho \) in the standard form described in Section \([5]\). Then define a representation \( \rho^C : \mathbb{Z}^4 \rightarrow \mathbb{C}^2 \) as follows:

\[
\begin{align*}
\rho^C(e_1) &= (\tau, 0), \\
\rho^C(e_2) &= (1, 0), \\
\rho^C(e_3) &= (z, 2\pi i r), \\
\rho^C(e_4) &= (0, 1).
\end{align*}
\]

Recall that the identification of \( \widetilde{E}_2 \times E^1 \) (where \( E^1 \) denotes the Euclidean line) with \( \mathbb{C}^2 \) is given by \((z, \theta, \lambda) \equiv (z, \lambda + i \theta) = (z, w)\), cf. \([3]\) p. 192.

Now let \( E_C \) be the quotient space of \( R^{''} (\mathbb{Z}^3, \widetilde{E}_2) \times \mathbb{C}^2 \) under the equivalence relation

\[
([\rho_1], (z_1, w_1)) \sim ([\rho_2], (z_2, w_2)) :\iff \quad [\rho_1] = [\rho_2] \text{ and } (z_2 - z_1, w_2 - w_1) \in \rho^C_1 (\mathbb{Z}^4).
\]

We have a natural complex structure on

\[
R^{''} (\mathbb{Z}^3, \widetilde{E}_2) = \mathbb{N} \times \mathbb{H}^2 \times \mathbb{C}.
\]

Using this and the identification of \([\rho]\) with \((r, \tau, z)\), the space \( E_C \) inherits a complex structure such that the projection

\[
\Psi_C : \quad E_C \quad \longrightarrow \quad R^{''} (\mathbb{Z}^3, \widetilde{E}_2)
\]

\[
[[\rho], (z, w)] \quad \longmapsto \quad [\rho]
\]

is holomorphic. The fibres of \( \Psi_C \) are complex tori, and it is immediate from the construction in \([3]\) that \( \Psi_C^{-1} ([\rho]) \) is biholomorphically equivalent to the
complex torus naturally associated to the taut contact circle on $T^3$ defined by $\rho$ (with $t_0 = 1$ in the notation of [3, p. 191]). These considerations show that the complex structure on $R''(\mathbb{Z}^3, \tilde{E}_2)$ is the one naturally adapted to the study of taut contact circles within the complex geometric framework developed in [3].

It is therefore appropriate to define a universal family $\mathcal{E} = \mathcal{E}(T^3)$ of taut contact circles on $T^3$ as the quotient space of $R''(\mathbb{Z}^3, \tilde{E}_2) \times \tilde{E}_2$ under the analogue of the equivalence relation above, with $C^2 = E_2 \times E^1$ replaced by $\tilde{E}_2$ and $\rho C$ by $\rho$. Then

$$\Psi: \mathcal{E} \rightarrow R''(\mathbb{Z}^3, \tilde{E}_2) \quad [\rho, (z, i\theta)] \mapsto [\rho]$$

is a smooth map such that $\Psi^{-1}([\rho])$ is a copy of $T^3$ equipped with the taut contact circle corresponding to $\rho$. The identification of $\Psi^{-1}([\rho])$ with $T^3$ is determined, up to an element of Diff$_0(T^3)$, by the requirement that, for a fixed set of generators of $\pi_1(T^3)$, the developing map of Section 3 give the representation $\rho$ in standard form.

The action of $B$ on $R''(\mathbb{Z}^3, \tilde{E}_2)$ can be extended to an action on $\mathcal{E}$ as follows. Let $\rho$ be a representative in standard form of an element $[\rho]$ in $R''(\mathbb{Z}^3, \tilde{E}_2)$, and let $\vartheta \in B$. Then $\vartheta$ acts on such a standard representative via

$$\rho \mapsto \mu(\rho \circ \vartheta),$$

where $\mu = \mu(\rho, \vartheta) \in \mathbb{C}^*$ is determined by the condition that $\mu(\rho \circ \vartheta)$ be again in standard form.

The fixed points of this action are exactly those $[\rho]$ for which the 2-torus $T^2 = C^2/(\mathbb{Z} \oplus \tau \mathbb{Z})$ has a symmetry, given by multiplication by some $\mu \in \mathbb{C}^*$, for which $z \in T^2$ is a fixed point. So we obtain a fibre space

$$\mathcal{E}/B \rightarrow \mathcal{M}(T^3),$$

where the fibres are copies of $M$ with a taut contact circle determined by $[\rho] \in \mathcal{M}(T^3)$, quotiented out by symmetries of the kind described. The possible symmetries are given in the following table:

| $\mu$          | $\tau$         | $z$                                  |
|---------------|----------------|-------------------------------------|
| $-1$ (2-fold) | arbitrary      | $0, 1/2, \tau/2, 1/2 + \tau/2$     |
| $\exp(2\pi i/3)$ (3-fold) | $\exp(2\pi i/6, \exp(2\pi i/3)$ | $1/2 + i/(2\sqrt{3}), 1 + i/\sqrt{3}$ |
| $i$ (4-fold)   | $i$            | $0, 1/2 + i/2$                      |

(2) In the case of the geometry $\tilde{SL}_2$ we build a universal family from the universal family of Bers [1] in analogy with the description for Seifert 4-manifolds given by Ue [23].

42
According to [1] there is a fibre space
\[ F(O_M) = \{(\sigma, z) \in T(O_M) \times \mathbb{C} : z \in D(\sigma)\} \]
over \( T(O_M) \), where \( D(\sigma) \) is a domain in \( \mathbb{C} \) on which \( \pi_{\text{orb}} = \pi_1^{\text{orb}}(O_M) \) acts according to \( \sigma \). With the complex structure on \( T(O_M) \) from classical Teichmüller theory, \( D(\sigma) \) and the group \( \sigma(\pi_{\text{orb}}) \) depend holomorphically on \( \sigma \) in a sense explained in [1]. Holomorphic dependence of \( \sigma(\pi_{\text{orb}}) \) on \( \sigma \) means that the map \( (\sigma, z) \mapsto (\sigma, \sigma(z)) \) is holomorphic in \( \sigma \in T(O_M) \) and \( z \in D(\sigma) \) for each fixed \( \sigma \in \pi_{\text{orb}} \).

So \( \sigma \) is interpreted as a representation of \( \pi_{\text{orb}} \) in the holomorphic automorphisms of \( D(\sigma) \), and we have a biholomorphism \( h_\sigma : \mathbb{H}^2 \to D(\sigma) \) and a representation \( \rho_\sigma \in \mathcal{R}(\pi_{\text{orb}}, \text{PSL}_2 \mathbb{R}) \) with \( \rho_\sigma = \sigma \in T(O_M) \) such that there is a commutative diagram
\[
\begin{array}{ccc}
\mathbb{H}^2 & \overset{\rho(\sigma)}{\longrightarrow} & \mathbb{H}^2 \\
h_\sigma \downarrow & & \downarrow h_\sigma \\
D(\sigma) & \overset{\sigma(\rho)}{\longrightarrow} & D(\sigma).
\end{array}
\]

No statement is made about the dependence of \( h_\sigma \) on \( \sigma \). Taking the quotient of \( D(\sigma) \) by \( \sigma(\pi_{\text{orb}}) \) in the fibre of \( F(O_M) \) over \( \sigma \) gives the holomorphic family \( E(O_M) \to T(O_M) \) of complex structures on \( O_M \).

To get the corresponding universal family of taut contact circles, the idea is simply to construct the logarithmic differential of the commutative diagram above. To this end, identify \( \widetilde{\text{SL}}_2 \times \mathbb{E}^1 \) with \( \mathbb{H}^2 \times \mathbb{C} \) with coordinates \((z, w)\), so that \( w = \lambda + i\theta \) corresponds to \( \log dz \), with \( \lambda \) denoting the \( \mathbb{E}^1 \)-coordinate. Given \( w \in \mathbb{Z}^2 \), write \( \sigma = \sigma_w = [\rho_w] \in T(M) \) for the corresponding lift of \( \sigma = [\rho] \in T(O_M) \). Regarding \( \mathbb{H}^2 \times \mathbb{C} \) and \( D(\sigma) \times \mathbb{C} \) as holomorphic tangent bundles, it makes sense to define a biholomorphism \( \mathbb{H}^2 \times \mathbb{C} \to D(\sigma) \times \mathbb{C} \) by
\[
h_\sigma(z, w) = (h_\sigma(z), w + \log \left( \frac{\partial h_\sigma}{\partial z} \right)(z)).
\]
Here the branch of the logarithm may be chosen arbitrarily (but holomorphically in \( z \); this is possible since \( (\partial h_\sigma/\partial z)(\mathbb{H}^2) \) is a simply connected domain in \( \mathbb{C} \) not containing 0.). With \( \pi = \pi_1(M) \) and \( \rho_w \) as above, define a representation
\[
\rho^C = \rho_w^C : \pi \times \mathbb{Z} \longrightarrow \widetilde{\text{SL}}_2 \times \mathbb{E}^1
\]
by
\[
\rho^C(u, 0) = \rho_w(u) \in \widetilde{\text{SL}}_2 \subset \widetilde{\text{SL}}_2 \times \mathbb{E}^1
\]
and, with \( e_0 \) denoting the neutral element of \( \pi \),
\[
\rho^C(e_0, 1)(z, w) = (z, w + 1).
\]
Since $h_\sigma$ commutes with translation in the $w$-coordinate and the $\rho_w(u)$ are lifts to $\tilde{\text{STH}}^2 = \mathbb{H}^2 \times \mathbb{C}$ of the differentials of the $\rho(\overline{u})$ (and hence unique up to such translations), we can define a holomorphic map

$$\sigma^C(u, n) = \sigma^C_w(u, n): D(\overline{\sigma}) \times \mathbb{C} \rightarrow D(\overline{\sigma}) \times \mathbb{C}$$

for each $(u, n) \in \pi \times \mathbb{Z}$ such that $\sigma^C(u, n)$ covers $\sigma(\overline{u})$ and such that we get the commutative diagram

$$\begin{array}{ccc}
\mathbb{H}^2 \times \mathbb{C} & \xrightarrow{\rho^C(u, n)} & \mathbb{H}^2 \times \mathbb{C} \\
\downarrow h_\sigma & & \downarrow h_\sigma \\
D(\overline{\sigma}) \times \mathbb{C} & \xrightarrow{\sigma^C(u, n)} & D(\overline{\sigma}) \times \mathbb{C}.
\end{array}$$

Notice that the choice of logarithm in the definition of $h_\sigma$ does not affect the definition of $\sigma^C(u, n)$, and that the map

$$(\sigma, z, w) \mapsto (\sigma, \sigma^C(u, n)(z, w)), \quad \sigma \in \mathcal{T}(M), \quad (z, w) \in D(\overline{\sigma}) \times \mathbb{C}$$

is holomorphic for any $(u, n) \in \pi \times \mathbb{Z}$.

Analogous to the construction in (1) we define $\mathcal{E}_C(M)$ as the quotient space of

\[ \{ (\sigma, z, w) \in \mathcal{T}(M) \times \mathbb{C} \times \mathbb{C} : z \in D(\overline{\sigma}) \} \]

under the equivalence relation

\[ (\sigma_1, z_1, w_1) \sim (\sigma_2, z_2, w_2) : \iff \sigma_1 = \sigma_2 \text{ and } (z_2, w_2) \in \sigma^C(\pi \times \mathbb{Z})(z_1, w_1). \]

With the complex structure thus induced on $\mathcal{E}_C(M)$, the projection

$$\Psi_C: \mathcal{E}_C(M) \rightarrow \mathcal{T}(M)$$

is holomorphic and the fibre over $\sigma$ is the complex surface associated (in the sense of [3]) to the taut contact circle on $M$ defined by $\sigma$. The universal family

$$\Psi: \mathcal{E}(M) \rightarrow \mathcal{T}(M)$$

of taut contact circles on $M$ is then also defined as in (1).

The action of $\text{Out}(\pi)$ on $\mathcal{T}(M)$ extends to an action on $\mathcal{E}(M)$ such that $\mathcal{E}(M)/\text{Out}(\pi)$ is a fibre space over $\mathcal{M}(M)$. Here the fibre over $[\sigma] \in \mathcal{M}(M)$ is the quotient of $M$ under the symmetries of the taut contact circle determined by $\sigma$, which correspond to the biholomorphisms of the complex surface $M \times S^1$ (defined by $\sigma^C$) that are identical in the $S^1$-factor.
We end this paper with an alternative description of the moduli of taut contact circles on $S^3$. The corresponding statements for left-quotients of SU(2) can be derived without much difficulty.

In the sequel, $(\omega_1, \omega_2)$ will always denote a pair of pointwise linearly independent 1-forms on an orientable 3-manifold $M$. Then the line field $\ker \omega_1 \cap \ker \omega_2$ is orientable and thus admits nowhere zero sections.

**Definition 7.1.** (i) The complex-valued 1-form $\omega = \omega_1 + \omega_2$ is formally integrable if $\omega \wedge d\omega \equiv 0$.

(ii) A nowhere zero vector field $Y \in \ker \omega_1 \cap \ker \omega_2$ defines a transversely conformal flow if there is a complex-valued function $f + ig$ on $M$ such that the Lie derivative of $\omega$ satisfies $L_Y \omega = (f + ig) \omega$.

Observe: (i) If $(\omega_1, \omega_2)$ is a taut contact circle, then $\omega$ is formally integrable. (ii) The condition $L_Y \omega = (f + ig) \omega$ implies that the flow of $Y$ preserves the conformal structure transverse to $Y$ defined by $\omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2$.

The two definitions just given are in fact equivalent:

**Proposition 7.2.** The complex 1-form $\omega = \omega_1 + i\omega_2$ is formally integrable if and only if any nowhere zero vector field $Y \in \ker \omega_1 \cap \ker \omega_2$ defines a transversely conformal flow.

This is immediate from the following equivalences:

$$\begin{align*}
\omega \wedge d\omega \equiv 0 & \iff i_Y (\omega \wedge d\omega) \equiv 0 \text{ for some/every nonzero } Y \in \ker \omega_1 \cap \ker \omega_2 \\
& \iff \omega \wedge L_Y \omega \equiv 0 \\
& \iff L_Y \omega = (f + ig) \omega.
\end{align*}$$

The proof of the following proposition is completely analogous to the construction of the Godbillon-Vey invariant for real codimension 1 foliations, cf. [26], and will be left to the reader.

**Proposition 7.3.** Assume $\omega = \omega_1 + i\omega_2$ is formally integrable. Then there is a complex 1-form $\gamma$ such that $d\omega = \gamma \wedge \omega$, and the complex number

$$\int_M \gamma \wedge d\gamma \in \mathbb{C}$$

depends only on the oriented line field $\ker \omega_1 \cap \ker \omega_2$ and the transverse conformal structure $\omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2$. In particular, if $(\omega_1, \omega_2)$ is a taut contact circle, then this complex number is an invariant of the homothety class of $(\omega_1, \omega_2)$.

**Definition 7.4.** We call the complex number $\int_M \gamma \wedge d\gamma$ the Godbillon-Vey invariant of the transversely conformal flow defined by $\omega$. 

45
Write $|S^3|$ for the volume of $S^3$ with respect to the volume form
\[ \sum_{i=1}^{4} x_i \partial_{x_i} (dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4). \]

For the taut contact circle corresponding to $[a] \in M_1$ in the notation of Theorem 2.2, i.e. the class of $a$ and $1 - a$, the Godbillon-Vey invariant can be computed as
\[ GV([a]) = \int_{S^3} \gamma_a \wedge d\gamma_a = -\frac{2|S^3|}{a(1-a)}. \]

For the taut contact circle corresponding to $n \in M_2$ one finds
\[ GV(n) = \int_{S^3} \gamma_n \wedge d\gamma_n = -\frac{2|S^3|}{(n+1)^2}. \]

This will be proved in [10].

Notice that the class $[a] \in M_1$ is determined by the complex number $a(1 - a)$. So the Godbillon-Vey invariant defines a biholomorphism
\[ M_1 \rightarrow \{ (x + iy \in \mathbb{C} : x \geq y^2 \} \]
\[ [a] \mapsto a(1-a) = -2|S^3|/GV([a]). \]

Other aspects of the Godbillon-Vey invariant will be discussed in [10]. For instance, for the pair of Liouville-Cartan forms on the unit cotangent bundle of a surface with a Riemannian metric, the Godbillon-Vey invariant equals the total Gauß curvature of the metric, and one can prove a generalisation of the Gauß-Bonnet theorem about the topological invariance of this total curvature, showing that under suitable assumptions the Godbillon-Vey invariant is not just an invariant of the transversely conformal structure, but in fact only depends on the oriented and cooriented common kernel foliation.

Furthermore, we shall prove there that the homothety class of a taut contact circle on $S^3$ is determined, up to diffeomorphism, by the oriented and cooriented common kernel foliation, and define a diffeomorphism invariant of such foliations that allow directly to recover the modulus of the taut contact circle.

Acknowledgements. This work was initiated during several brief visits by J. G. to the Forschungsinstitut für Mathematik (FIM) of the ETH Zürich between 1995 and 1997, while H. G. held an assistant professorship there, and a visit by H. G. to Madrid, supported by DGICYT (Dirección General de Investigación Científica y Técnica) grant no. PB95-0185. A major share of the research was done during a visit by J. G. to Leiden University from March till May 2000, supported by an NWO (Nederlandse Organisatie voor Wetenschappelijk Onderzoek) bezoekersbeurs. The final version was prepared while H. G. was a guest of the Universidad Autónoma. We thank these funding organisations and universities for their support.

We also thank Horst Knörrer for helpful discussions.
References

[1] L. Bers, Fiber spaces over Teichmüller spaces, Acta Math. 130 (1973), 89–126.

[2] M. Boileau and J.-P. Otal, Scindements de Heegaard et groupe des homéotoopies des petites variétés de Seifert, Invent. Math. 106 (1991), 85–107.

[3] F. Bonahon, Difféotopies des espaces lenticulaires, Topology 22 (1983), 305–314.

[4] L. Dabrowski and R. Percacci, Spinors and diffeomorphisms, Comm. Math. Phys. 106 (1986), 691–704.

[5] H. Geiges and J. Gonzalo, Contact geometry and complex surfaces, Invent. Math. 121 (1995), 147–209.

[6] H. Geiges and J. Gonzalo, Seifert invariants of left-quotients of 3-dimensional simple Lie groups, Topology Appl. 66 (1995), 117–127.

[7] H. Geiges and J. Gonzalo, An application of convex integration to contact geometry, Trans. Amer. Math. Soc. 348 (1996), 2139–2149.

[8] H. Geiges and J. Gonzalo, Contact spheres and quaternionic structures, preprint (2001).

[9] H. Geiges and J. Gonzalo, Generalised spin structures on surfaces and orbifolds, in preparation.

[10] H. Geiges and J. Gonzalo, Transversely conformal flows on 3-manifolds, in preparation.

[11] H. Geiges and C. B. Thomas, Hypercontact manifolds, J. London Math. Soc. (2) 51 (1995), 342–352.

[12] W. J. Harvey, Spaces of discrete groups, in: Discrete Groups and Automorphic Functions, Proceedings of an Instructional Conference held in Cambridge (1975), Academic Press, London, 1977, pp. 295–348.

[13] C. Hodgson and J. H. Rubinstein, Involutions and isotopies of lens spaces, in: Knot Theory and Manifolds (Vancouver, 1983), Lecture Notes in Math. 1144, Springer-Verlag, Berlin, 1985, pp. 60–96.

[14] H. Jacobowitz, An Introduction to CR Structures, Math. Surveys Monogr. 32, American Mathematical Society, Providence, 1990.

[15] K. Kodaira and D. C. Spencer, On deformations of complex analytic structures II, Ann. of Math. (2) 67 (1958), 403–466.
[16] R. Kulkarni, K. B. Lee and F. Raymond, Deformation spaces for Seifert manifolds, in: Geometry and Topology (College Park, 1983/84), Lecture Notes in Math. 1167, Springer-Verlag, Berlin, 1985, pp. 180–216.

[17] C. Maclachlan and W. J. Harvey, On mapping-class groups and Teichmüller spaces, Proc. London Math. Soc. (3) 30 (1975), 496–512.

[18] W. D. Neumann and F. Raymond, Seifert manifolds, plumbing, µ-invariant and orientation reversing maps, in: Algebraic and Geometric Topology (Santa Barbara, 1977), Lecture Notes in Math. 664, Springer-Verlag, Berlin, 1978, pp. 163–196.

[19] L. Neuwirth, A topological classification of certain 3-manifolds, Bull. Amer. Math. Soc. 69 (1963), 372–375.

[20] K. Ohshika, Teichmüller spaces of Seifert fibered manifolds with infinite π₁, Topology Appl. 27 (1987), 75–93.

[21] P. Orlik, Seifert Manifolds, Lecture Notes in Math. 291, Springer-Verlag, Berlin, 1972.

[22] F. Raymond and A. T. Vasquez, 3-manifolds whose universal coverings are Lie groups, Topology Appl. 12 (1981), 161–179.

[23] K. Sakamoto and S. Fukuhara, Classification of T²-bundles over T², Tokyo J. Math. 6 (1983), 311–327.

[24] P. Scott, The geometries of 3-manifolds, Bull. London Math. Soc. 15 (1983), 401–487.

[25] P. Scott, Homotopy implies isotopy for some Seifert fibre spaces, Topology 24 (1985), 341–351.

[26] I. Tamura, Topology of Foliations: An Introduction, Transl. Math. Monogr. 97, American Mathematical Society, Providence, 1992.

[27] W. P. Thurston, Three-Dimensional Geometry and Topology, Vol. 1, Princeton Math. Ser. 35, Princeton University Press, 1997.

[28] M. Ue, Geometric 4-manifolds in the sense of Thurston and Seifert 4-manifolds II, J. Math. Soc. Japan 43 (1991), 149–183.

[29] M. Ue, On the deformations of the geometric structures on the Seifert 4-manifolds, in: Aspects of Low-Dimensional Manifolds, Adv. Stud. Pure Math. 20, Kinokuniya, Tokyo, 1992, pp. 331–363.

[30] È. B. Vinberg and O. V. Shvartsman, Discrete groups of motion of spaces of constant curvature, in: Geometry II, Encyclopaedia Math. Sci. 29, Springer-Verlag, Berlin, 1993, pp. 139–248.
[31] F. Waldhausen, On irreducible 3-manifolds which are sufficiently large, Ann. of Math. (2) 87 (1968), 56–88.

[32] C. T. C. Wall, Geometric structures on compact complex analytic surfaces, Topology 25 (1986), 119–153.

[33] J. A. Wolf, Spaces of Constant Curvature, 5th edition, Publish or Perish, Houston, 1984.

[34] H. Zieschang, On the homeotopy groups of surfaces, Math. Ann. 206 (1973), 1–21.

Hansjörg Geiges, Mathematisch Instituut, Universiteit Leiden, Postbus 9512, 2300 RA Leiden, The Netherlands; e-mail: geiges@math.leidenuniv.nl

Jesús Gonzalo, Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain; e-mail: jesus.gonzalo@uam.es