DIFFERENTIAL-GEOMETRIC AND TOPOLOGICAL STRUCTURE OF MULTIDIMENSIONAL DELSARTE TRANSMUTATION OPERATORS

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Abstract. A differential geometrical and topological structure of Delsarte transmutation operators in multidimension is studied, the relationships with De Rham-Hodge-Skrypnik theory of generalized differential complexes is stated.

1. Introduction

Consider the Hilbert space $H = L_2(\mathbb{R}^m; \mathbb{C}^N)$, $m, N \in \mathbb{Z}_+$, with the scalar semi-linear form on $H^* \times H$

\begin{equation}
(\varphi, \psi) := \int_{\mathbb{R}^m} \tilde{\varphi}(x)^\dagger \psi(x) dx
\end{equation}

for any pair $(\varphi, \psi) \in H^* \times H$, where, evidently $H^* \simeq H$, sign $\dagger$ is the usual matrix transposition. Take also $H_0$ and $\tilde{H}_0$ being some two closed subspaces of $H$ and correspondingly two linear operators $L$ and $\tilde{L}$ acting from $H$ into $H$.

Definition 1.1. (J. Delsarte and J. Lions [2]) A linear invertible operator $\Omega$ defined on the whole $H$ and acting from $H_0$ onto $\tilde{H}_0$ is called a Delsarte transmutation operator for a pair of linear operators $L$ and $\tilde{L}$ if the following two conditions hold:

- the operator $\Omega$ and its inverse $\Omega^{-1}$ are continuos in $H$;
- the operator identity

\begin{equation}
\tilde{L}\Omega = \Omega L
\end{equation}

is satisfied.

Such transmutation operators were for the first time introduced in [1, 2] for the case of one-dimensional second order differential operators. In particular, for the Sturm-Liouville and Dirac operators the complete structure of the corresponding Delsarte transmutation operators was described in [8, 9], where also the extensive applications to spectral theory were given.

As there was become clear just recently, some special cases of the Delsarte transmutation operators were constructed much before by Darboux and Crum (see [10]).

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A special generalization of the Delsarte-operators for the two-dimensional Dirac operators was done for the first time in [11], where its applications to inverse spectral theory and solving some nonlinear two-dimensional evolution equations were also presented.

Recently some progress in this direction was made in [12, 13] due to analyzing a special operator structure of Darboux type transformations which appeared in [14].

In this work we give in some sense a complete description of multi-dimensional Delsarte transmutation operators based on a natural generalization of the differential-geometric approach devised in [13], and discuss how one can apply these operators to studying spectral properties of linear multi-dimensional differential operators.

2. The differential-geometric structure of a generalized Lagrangian identity

Let a multi-dimensional linear differential operator $L : \mathcal{H} \to \mathcal{H}$ of order $n(L) \in \mathbb{Z}_+$ be of the form

$$L := \sum_{|\alpha| = 0}^{n(L)} a_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha},$$

where, as usually, $\alpha \in \mathbb{Z}_m^+$ is a multi-index, $x \in \mathbb{R}^m$, and for brevity one assumes that coefficients $a_\alpha \in S(\mathbb{R}^m; \text{End}\mathbb{C}^N)$. Consider the following easily derivable generalized Lagrangian identity for the differential expression (2.1):

$$< L^* \varphi, \psi > - < \varphi, L \psi > = \sum_{i=1}^{m} (-1)^{i+1} \frac{\partial}{\partial x_i} Z_i[\varphi, \psi],$$

where $(\varphi, \psi) \in \mathcal{H}^* \times \mathcal{H}$, mappings $Z_i : \mathcal{H}^* \times \mathcal{H} \to \mathbb{C}$, $i = 1, m$, are semilinear due to the construction and $L^* : \mathcal{H}^* \to \mathcal{H}^*$ is the corresponding formally conjugated to (2.1) differential expression, that is

$$L^* := \sum_{|\alpha| = 0}^{n(L)} (-1)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x^\alpha} \cdot \overline{a_\alpha^T(x)}.$$

Having multiplied the identity (2.2) by the usual oriented Lebesgue measure $dx = \wedge_{j=1}^{m} dx_j$, we get that

$$< L^* \varphi, \psi > dx - < \varphi, L \psi > dx = dZ^{(m-1)}[\varphi, \psi]$$

for all $(\varphi, \psi) \in \mathcal{H}^* \times \mathcal{H}$, where

$$Z^{(m-1)}[\varphi, \psi] := \sum_{i=1}^{m} dx_1 \wedge dx_2 \wedge \ldots \wedge dx_{i-1} \wedge Z_i[\varphi, \psi] dx_{i+1} \wedge \ldots \wedge dx_m$$

is an $(m-1)$-differential form on $\mathbb{R}^m$.

Consider now all such pairs $(\varphi(\lambda), \psi(\mu)) \in \mathcal{H}_0^* \times \mathcal{H}_0$, $\lambda, \mu \in \Sigma$, where $\Sigma \in \mathbb{C}^p$, $p \in \mathbb{Z}_+$, is some fixed measurable space of parameters endowed with a bounded Lebesgue measure $\rho$, that the differential form (2.4) is exact, that is there exists such
the set of $(m-2)$-differential forms $\Omega^{(m-2)}[\varphi(\lambda), \psi(\mu)] \in \Lambda^{m-2}(\mathbb{R}^m; \mathbb{C}), \lambda, \mu \in \Sigma$, on $\mathbb{R}^m$ satisfying the condition

\begin{equation}
Z^{(m-1)}[\varphi(\lambda), \psi(\mu)] = d\Omega^{(m-2)}[\varphi(\lambda), \psi(\mu)].
\end{equation}

Assume also that for any fixed element $\varphi(\lambda) \in \mathcal{H}_0, \lambda \in \Sigma$, the set $\mathcal{H}_\varphi \subset \mathcal{H}_0$ of functions $\psi(\mu) \in \mathcal{H}_0, \mu \in \Sigma$, satisfying the condition (2.5) is dense in $\mathcal{H}$, that is $\mathcal{H}_\varphi = \mathcal{H}$. Since the relationship (2.5) is semilinear in $(\varphi(\lambda), \psi(\mu)) \in \mathcal{H}_0^* \times \mathcal{H}_0, \lambda, \mu \in \Sigma$, one gets easily that it holds for any pair $(\varphi(\lambda), \psi(\mu)) \in \mathcal{H}_0^* \times \mathcal{H}_0, \lambda, \mu \in \Sigma$. Thus, taking into account that $d^2 = 0$, one follows from (2.3) by integration over $\mathbb{R}^m$ that for any pair $(\varphi(\lambda), \psi(\mu)) \in \mathcal{H}_0^* \times \mathcal{H}_0$ the identity $<L^* \varphi(\lambda), \psi(\mu) > = (\varphi(\lambda), L\psi(\mu) >), \lambda, \mu \in \Sigma$, holds, that is the operator (2.1) possesses its adjoint $L^*$ in $\mathcal{H}^*$. Another way to realize this condition is to take spaces $\mathcal{H}_0^*$ and $\mathcal{H}_0$ as solutions to the following linear differential equations:

\begin{equation}
\mathcal{H}_0 : = \{ \psi(\lambda) \in \mathcal{H}_- : L\psi(\lambda) = 0, \; \psi(\lambda)|_{x \in \Gamma} = 0, \; \lambda \in \Sigma \},
\end{equation}

\begin{equation}
\mathcal{H}_0^* : = \{ \varphi(\lambda) \in \mathcal{H}_+^* : L^* \varphi(\lambda) = 0, \; \varphi(\lambda)|_{x \in \Gamma^*} = 0, \; \lambda \in \Sigma \},
\end{equation}

where we have introduced following [7] a corresponding Hilbert-Schmidt rigged chain of Hilbert spaces

\begin{equation}
\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-,
\end{equation}

allowing to determine properly a set of generalized eigenfunctions of extended operators $L, L^* : \mathcal{H}_- \to \mathcal{H}_+$, and $\Gamma, \Gamma^* \subset \mathbb{R}^m$ are some $(n-1)$-dimensional piece-wise smooth hypersurfaces imbedded into the configuration space $\mathbb{R}^m$. Let now $S(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)})$ denote an $(m-1)$-dimensional piece-wise smooth hypersurface imbedded into $\mathbb{R}^m$ such that its boundaries $\partial S(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)}) = \sigma_x^{(m-2)} - \sigma_{x_0}^{(m-2)}$, where $\sigma_x^{(m-2)}$ and $\sigma_{x_0}^{(m-2)} \in H_{m-2}(\mathbb{R}^m; \mathbb{C})$ are some $(m-2)$-dimensional homological cycles from the homology group $H_{m-2}(\mathbb{R}^m; \mathbb{C})$ of $\mathbb{R}^m$, parametrized formally by means of two points $x, x_0 \in \mathbb{R}^m$ and related in some way with the chosen above hypersurfaces $\Gamma$ and $\Gamma^* \subset \mathbb{R}^m$. Then from (2.5) based on the general Stokes theorem [16, 17] one correspondingly gets easily that

\begin{equation}
\int_{S(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)})} Z^{(m-1)}[\varphi(\lambda), \psi(\mu)] - \int_{\partial S(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)})} \Omega^{(m-2)}[\varphi(\lambda), \psi(\mu)] =
\end{equation}

\begin{equation}
\int_{S(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)})} \bar{Z}^{(m-1), \tau}[\varphi(\lambda), \psi(\mu)] - \int_{\partial S(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)})} \bar{\Omega}^{(m-2), \tau}[\varphi(\lambda), \psi(\mu)] =
\end{equation}

\begin{equation}
\int_{S(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)})} \bar{O}^{(m-2), \tau}[\varphi(\lambda), \psi(\mu)] - \int_{\partial S(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)})} \bar{\Omega}^{(m-2), \tau}[\varphi(\lambda), \psi(\mu)] =
\end{equation}

for the set of functions $(\varphi(\lambda), \psi(\mu)) \in \mathcal{H}_* \times \mathcal{H}, \lambda, \mu \in \Sigma$, with kernels $\Omega_x[\varphi(\lambda), \psi(\mu)], \Omega_x^*[\varphi(\lambda), \psi(\mu)]$ and $\Omega_{x_0}[\varphi(\lambda), \psi(\mu)], \Omega_{x_0}^*[\varphi(\lambda), \psi(\mu)], \lambda, \mu \in \Sigma$, acting naturally in
the Hilbert space $L^2(\Sigma; \mathbb{C})$, are assumed further to be nondegenerate in $L^2(\Sigma; \mathbb{C})$ and satisfying the regularity conditions

$$\lim_{x \to x_0} \Omega_{\varphi(x), \psi(x)} = \Omega_{x_0, [\varphi(x), \psi(x)]}, \quad \lim_{x \to x_0} \Omega_{\varphi(x), \psi(x)}^* = \Omega_{x_0}^* [\varphi(x), \psi(x)].$$

Define now actions of the following two linear Delsarte permutations operators $\Omega : \mathcal{H} \to \mathcal{H}$ and $\Omega^* : \mathcal{H}^* \to \mathcal{H}^*$ still upon a fixed set of functions $(\varphi(x), \psi(x)) \in \mathcal{H}_0^* \times \mathcal{H}_0, \lambda, \mu \in \Sigma$,

$$(2.8) \quad \hat{\varphi}(\lambda) = \Omega^* [\varphi(\lambda), \psi(\lambda)] := \int d\rho(\eta) \int d\rho(\mu) [\varphi(\eta), \psi(\mu)] [\varphi(\mu), \psi(\lambda)],$$

Making use of the expressions (2.8), based on arbitrariness of the chosen set of functions $(\varphi(x), \psi(x)) \in \mathcal{H}_0^* \times \mathcal{H}_0, \lambda, \mu \in \Sigma$, we can easily retrieve the corresponding operator expressions for operators $\Omega$ and $\Omega^* : \mathcal{H}_0 \to \mathcal{H}_0$, forcing the kernels $\Omega_{x_0} [\varphi(x), \psi(x)]$ and $\Omega_{x_0}^* [\varphi(x), \psi(x)]$, $\lambda, \mu \in \Sigma$, to variate:

$$\hat{\psi}(\lambda) = \Omega(\psi(\lambda)) := \int d\rho(\eta) \int d\rho(\mu) [\varphi(\eta), \psi(\mu)] [\varphi(\mu), \psi(\lambda)] - \int d\rho(\eta) \int d\rho(\mu) \int d\rho(\psi) [\varphi(\eta), \psi(\mu)] [\varphi(\mu), \psi(\psi)] \times$$

$$\times \int S(\sigma_2^{(m-2)}, \sigma_2^{(m-2)}) Z^{(m-1)} [\varphi(\mu), \psi(\psi)] = \psi(\lambda) - \int d\rho(\eta) \int d\rho(\mu) \int d\rho(\psi) \int d\rho(\xi) [\varphi(\eta), \psi(\mu)] [\varphi(\mu), \psi(\xi)] \times$$

$$\times \int S(\sigma_2^{(m-2)}, \sigma_2^{(m-2)}) Z^{(m-1)} [\varphi(\xi), \psi(\mu)] = \psi(\lambda) - \int d\rho(\eta) \int d\rho(\mu) [\varphi(\eta), \psi(\mu)] \int S(\sigma_2^{(m-2)}, \sigma_2^{(m-2)}) Z^{(m-1)} [\varphi(\mu), \psi(\lambda)]$$

$$= (1 - \int d\rho(\eta) \int d\rho(\mu) [\varphi(\eta), \psi(\mu)] \times$$

$$\times \int S(\sigma_2^{(m-2)}, \sigma_2^{(m-2)}) Z^{(m-1)} [\varphi(\mu), \psi(\lambda)]) \hat{\psi}(\lambda) := \Omega \cdot \psi(\lambda);$$

$$\hat{\varphi}(\lambda) = \int d\rho(\eta) \int d\rho(\mu) [\varphi(\eta), \psi(\mu)] \int S(\sigma_2^{(m-2)}, \sigma_2^{(m-2)}) Z^{(m-1)} [\varphi(\mu), \psi(\lambda)]$$

$$- \int d\rho(\eta) \int d\rho(\mu) [\varphi(\eta), \psi(\mu)] \int S(\sigma_2^{(m-2)}, \sigma_2^{(m-2)}) Z^{(m-1)} [\varphi(\mu), \psi(\lambda)]$$
\[ \varphi(\lambda) = \int d\rho(\eta) \int d\rho(\xi) \int d\rho(\mu) \int d\rho(\nu) \varphi(\eta) \Omega_x^{-1}[\varphi(\nu), \psi(\eta)] \times \]
\[ \times \Omega_x^* \left[ \varphi(\xi), \psi(\mu) \right] \Omega_x^{*-1} \left[ \varphi(\mu), \psi(\xi) \right] \int_{S(\sigma_x^{(m-2)}, \sigma_x^{(m-2)})} Z^{(m-1)}[\varphi(\lambda), \psi(\mu)] \]
\[(2.9) = (1 - \int d\rho(\eta) \int d\rho(\mu) \tilde{\varphi}(\eta) \Omega_x^{-1}[\varphi(\eta), \psi(\mu)] \int_{S(\sigma_x^{(m-2)}, \sigma_x^{(m-2)})} Z^{(m-1)}[\varphi(\mu), \psi(\eta)]] \times \]
\[ \times \int_{S(\sigma_x^{(m-2)}, \sigma_x^{(m-2)})} Z^{(m-1)}[\varphi(\mu), \psi(\eta)] \psi(\lambda) = \Omega^* \cdot \varphi(\lambda), \]
where, by definition,
\[ \Omega := 1 - \int d\rho(\eta) \int d\rho(\mu) \tilde{\varphi}(\eta) \Omega_x^{-1}[\varphi(\eta), \psi(\mu)] \int_{S(\sigma_x^{(m-2)}, \sigma_x^{(m-2)})} Z^{(m-1)}[\varphi(\mu), \psi(\eta)] \]
\[(2.10) \Omega^* := 1 - \int d\rho(\eta) \int d\rho(\mu) \tilde{\varphi}(\eta) \Omega_x^{*-1}[\varphi(\eta), \psi(\mu)] \int_{S(\sigma_x^{(m-2)}, \sigma_x^{(m-2)})} Z^{(m-1)}[\varphi(\mu), \psi(\eta)] \]
are of Volterra type multidimensional integral operators. It is to be noted here that now elements \((\varphi(\lambda), \psi(\mu)) \in H^*_0 \times H_0^*\) and \((\tilde{\varphi}(\lambda), \tilde{\psi}(\mu)) \in H^*_0 \times H_0^*, \lambda, \mu \in \Sigma, \) inside the operator expressions (2.10) are arbitrary but fixed. Therefore, the operators (2.10) realize an extension of their actions (2.8) on a fixed pair of functions \((\varphi(\lambda), \psi(\mu)) \in H^*_0 \times H_0^*, \lambda, \mu \in \Sigma, \) upon the whole functional space \(H^* \times H.\)

Due to the symmetry of expressions (2.8) and (2.10) with respect to two sets of functions \((\varphi(\lambda), \psi(\mu)) \in H^*_0 \times H_0^*\) and \((\tilde{\varphi}(\lambda), \tilde{\psi}(\mu)) \in H^*_0 \times H_0^*, \lambda, \mu \in \Sigma,\) it is very easy to state the following lemma.

**Lemma 2.1.** Operators (2.10) are bounded and invertible of Volterra type expressions in \(H^* \times H\) whose inverse are given as follows:
\[(2.11) \Omega^{-1} := 1 - \int d\rho(\eta) \int d\rho(\mu) \psi(\eta) \tilde{\Omega}_x^{-1}[\varphi(\eta), \psi(\mu)] \int_{S(\sigma_x^{(m-2)}, \sigma_x^{(m-2)})} Z^{(m-1)}[\varphi(\mu), \psi(\eta)] \]
\[\Omega^{*-1} := 1 - \int d\rho(\eta) \int d\rho(\mu) \varphi(\eta) \tilde{\Omega}_x^{*-1}[\varphi(\eta), \psi(\mu)] \int_{S(\sigma_x^{(m-2)}, \sigma_x^{(m-2)})} Z^{(m-1)}[\varphi(\mu), \psi(\eta)] \]

where two sets of functions \((\varphi(\lambda), \psi(\mu)) \in H^*_0 \times H_0^*\) and \((\tilde{\varphi}(\lambda), \tilde{\psi}(\mu)) \in H^*_0 \times H_0^*, \lambda, \mu \in \Sigma,\) are taken arbitrary but fixed.

For the expressions (2.11) to be compatible with mappings (2.8) the following actions must hold:
\[\psi(\lambda) = \Omega^{-1} \cdot \tilde{\psi}(\lambda) = \int d\rho(\eta) \int d\rho(\mu) \psi(\eta) \tilde{\Omega}_x^{-1}[\varphi(\eta), \psi(\mu)] \Omega_x[\varphi(\mu), \psi(\lambda)],\]
\[(2.12) \varphi(\lambda) = \Omega^{*-1} \cdot \tilde{\varphi}(\lambda) = \int d\rho(\eta) \int d\rho(\mu) \varphi(\eta) \tilde{\Omega}_x^{*-1}[\varphi(\eta), \psi(\mu)] \Omega_x^*[\varphi(\mu), \psi(\lambda)].\]
where for any two sets of functions \((\tilde{\varphi}(\lambda), \tilde{\psi}(\mu)) \in \mathcal{H}_0^\times \mathcal{H}_0\) and \((\tilde{\varphi}(\lambda), \tilde{\psi}(\mu)) \in \mathcal{H}_0^\times \mathcal{H}_0, \lambda, \mu \in \Sigma\), the next relationship is satisfied:

\[
(\langle L^* \tilde{\varphi}(\lambda), \tilde{\psi}(\mu) \rangle - (\langle \tilde{\varphi}(\lambda), L\tilde{\psi}(\mu) \rangle) dx = d(\tilde{Z}^{(m-1)}[\tilde{\varphi}(\lambda), \tilde{\psi}(\mu)]),
\]

\[
\tilde{Z}^{(m-1)}[\tilde{\varphi}(\lambda), \tilde{\psi}(\mu)] = d\Omega^{(m-2)}[\tilde{\varphi}(\lambda), \tilde{\psi}(\mu)].
\]

(2.13)

Moreover, the expressions \(\tilde{L} : \mathcal{H} \to \mathcal{H}\) and \(\tilde{L}^* : \mathcal{H}^* \to \mathcal{H}^*\) must in the result be differential too. Since this condition determines properly Delsarte transmutation operators (2.11), we need to state the following theorem.

**Theorem 2.2.** The pair of operator expressions \(\tilde{L} := \Omega L \Omega^{-1}\) and \(\tilde{L}^* := \Omega^* L^* \Omega^{*-1}\) is purely differential on the whole space \(\mathcal{H}^\times \mathcal{H}\) for any suitably chosen hyper-surface \(S(\mathcal{H})^{(m-2)} \subset \mathbb{R}^m\).

**Proof.** For proving the theorem it is necessary to show that the formal pseudo-differential expressions corresponding to operators \(\tilde{L}\) and \(\tilde{L}^*\) contain no integral elements. Making use of an idea devised in [13, 11], one can formulate such a lemma.

**Lemma 2.3.** A pseudo-differential operator \(L : \mathcal{H} \to \mathcal{H}\) is purely differential if
the following equality

\[
(\langle h, (L \partial^{[\alpha]} \partial^{-\alpha}) f \rangle) = \langle h, L \partial^{[\alpha]} \partial^{-\alpha} f \rangle,
\]

holds for any \(|\alpha| \in \mathbb{Z}_+\) and all \((h, f) \in \mathcal{H}^\times \mathcal{H}\), that is the condition (2.14) is equivalent to the equality \(L_+ = L\), where, as usually, the sign "\(\langle \cdot, \cdot \rangle_+\)" means the purely differential part of the corresponding expression inside the bracket.

Based now on this Lemma and exact expressions of operators (2.10), similarly to calculations done in [13], one shows right away that operators \(\tilde{L}\) and \(\tilde{L}^*\), depending correspondingly only both on the homological cycles \(\mathcal{H}_0^{(m-2)} \subset \mathbb{R}^m\), marked by points \(x, x_0 \in \mathbb{R}^m\), and on two sets of functions \((\tilde{\varphi}(\lambda), \tilde{\psi}(\mu)) \in \mathcal{H}_0^\times \mathcal{H}_0\) and \((\tilde{\varphi}(\lambda), \tilde{\psi}(\mu)) \in \mathcal{H}_0^\times \mathcal{H}_0, \lambda, \mu \in \Sigma,\) are purely differential, thereby finishing the proof.

3. The general differential-geometric and topological structure of Delsarte transmutation operators

Let \(M := \mathbb{R}^m\) denote a suitably compactified metric space of dimension \(m = \text{dim} M \in \mathbb{Z}_+\) (without boundary) and define some finite set \(\mathcal{L}\) of smooth commuting to each other linear differential operators

\[
L_j(x; \partial) := \sum_{|\alpha| = 0}^{n(L_j)} a^{(j)}_\alpha(x) \partial^{[\alpha]} / \partial x^{\alpha},
\]

(3.1)

\(x \in M,\) with Schwartz coefficients \(a^{(j)}_\alpha \in \mathcal{S}(M; \text{End} \mathbb{C}^N),\) \(|\alpha| = 0, n(L_j), n(L_j) \in \mathbb{Z}_+, j = 1, m,\) and acting in the Hilbert space \(\mathcal{H} := L_2(M; \mathbb{C}^N)\). It is assumed that domains \(D(L_j) := D(\mathcal{L}) \subset \mathcal{H}, j = 1, m,\) are dense in \(\mathcal{H}\).
Consider now a generalized external differentiation operator \( d_L : \Lambda(M; \mathcal{H}) \rightarrow \Lambda(M; \mathcal{H}) \) acting in the Grassmann algebra \( \Lambda(M; \mathcal{H}) \) as follows: for any \( \beta^{(k)} \in \Lambda^k(M; \mathcal{H}), k = 0, m, \)

\[
(3.2) \quad d_L \beta^{(k)} := \sum_{j=1}^m dx_j \wedge L_j(x; \partial) \beta^{(k)} \in \Lambda^{k+1}(M; \mathcal{H}).
\]

It is easy to see that the operation (3.2) in the case \( L_j(x; \partial) := \partial / \partial x_j, j = 1, m, \) coincides exactly with the standard external differentiation \( d = \sum_{j=1}^m dx_j \wedge \partial / \partial x_j \) on the Grassmann algebra \( \Lambda(M; \mathcal{H}) \). Making use of the operation (3.2) on \( \Lambda(M; \mathcal{H}) \), one can construct the following generalized de Rham complex

\[
(3.3) \quad \mathcal{H} \rightarrow \Lambda^0(M; \mathcal{H}) \xrightarrow{d_L} \Lambda^1(M; \mathcal{H}) \xrightarrow{d_L} \ldots \xrightarrow{d_L} \Lambda^m(M; \mathcal{H}) \xrightarrow{d_L} 0.
\]

The following important property concerning the complex (3.3) holds.

**Lemma 3.1.** The co-chain complex (3.3) is exact.

**Proof.** It follows easily from the equality \( d_L d_L = 0 \) holding due to the commutation of operators (3.1).

Below we will follow the ideas developed before in [3]. A differential form \( \beta \in \Lambda(M; \mathcal{H}) \) will be called \( d_L \)-closed if \( d_L \beta = 0 \), and a form \( \gamma \in \Lambda(M; \mathcal{H}) \) will be called \( d_L \)-homological to zero if there exists on \( M \) such a form \( \omega \in \Lambda(M; \mathcal{H}) \) that \( \gamma = d_L \omega \).

Consider now the standard algebraic Hodge star-operation

\[
(3.4) \quad : \Lambda^k(M; \mathcal{H}) \rightarrow \Lambda^{m-k}(M; \mathcal{H}),
\]

\( k = 0, m, \) as follows [4]: if \( \beta \in \Lambda^k(M; \mathcal{H}) \), then the form \( \ast \beta \in \Lambda^{m-k}(M; \mathcal{H}) \) is such that:

i) \( (m-k) \)-dimensional volume \( |\ast \beta| \) of the form \( \ast \beta \) equals \( k \)-dimensional volume \( |\beta| \) of the form \( \beta \);

ii) the \( m \)-dimensional measure \( \bar{\beta}^T \wedge \ast \beta > 0 \) under the fixed orientation on \( M \).

Define also on the space \( \Lambda(M; \mathcal{H}) \) the following natural scalar product: for any \( \beta, \gamma \in \Lambda^k(M; \mathcal{H}), k = 0, m, \)

\[
(3.5) \quad \langle \beta, \gamma \rangle := \int_M \bar{\beta}^T \wedge \ast \gamma.
\]

Subject to the scalar product (3.5) we can naturally construct the corresponding Hilbert space

\[
\mathcal{H}_\Lambda(M) := \bigoplus_{k=0}^m \mathcal{H}^k_{\Lambda}(M)
\]

well suitable for our further consideration. Notice also here that the Hodge star \( \ast \)-operation satisfies the following easily checkable property: for any \( \beta, \gamma \in \mathcal{H}^k_{\Lambda}(M), k = 0, m, \)

\[
(3.6) \quad \langle \beta, \gamma \rangle = \langle \ast \beta, \ast \gamma \rangle,
\]

that is the Hodge operation \( \ast : \mathcal{H}_\Lambda(M) \rightarrow \mathcal{H}_\Lambda(M) \) is isometry and its standard adjoint with respect to the scalar product (3.5) operation \( \ast' = (\ast)^{-1} \).

Denote by \( d'_L \) the formally adjoint expression to the external weak differential operation \( d_L : \mathcal{H}_\Lambda(M) \rightarrow \mathcal{H}_\Lambda(M) \) in the Hilbert space \( \mathcal{H}_\Lambda(M) \). Making now use of
the operations $d_L'$ and $d_L$ in $H_A(M)$ one can naturally define [4] the generalized
Laplace-Hodge operator $\Delta_L : H_A(M) \to H_A(M)$ as
\[ \Delta_L := d_L'^2 + d_L d_L'. \]

Take a form $\beta \in H_A(M)$ satisfying the equality
\[ \Delta_L \beta = 0. \]

Such a form is called harmonic. One can also verify that a harmonic form $\beta \in H_A(M)$ satisfies simultaneously the following two adjoint conditions:
\[ d_L \beta = 0, \quad d_L' \beta = 0, \]
easily stemming from (3.7) and (3.9).

It is not hard to check that the following differential operation in $H_A(M)$
\[ d_L^* := \ast d_L(\ast)^{-1} \]
defines the usual [16, 17] external anti-differential operation in $H_A(M)$. The corre-
sponding dual to (3.3) complex
\[ H \to \Lambda^0(M; H) \xrightarrow{d_L} \Lambda^1(M; H) \xrightarrow{d_L} \cdots \xrightarrow{d_L} \Lambda^m(M; H) \xrightarrow{d_L} 0 \]
is evidently exact too, as the property $d_L^* d_L = 0$ holds due to the definition (3.7).

Denote further by $H_{k(L)}^k(M), k = 0, m$, the cohomology groups of $d_L$-closed and
by $H_{k(L)}^k(M), k = 0, m$, the cohomology groups of $d_L^*$-closed differential forms,
correspondingly, and by $H_{k(L)}^k(M), k = 0, m$, the abelian groups of harmonic
differential forms from the Hilbert sub-spaces $H_A^k(M), k = 0, m$. Before formulat-
ing next results, define the standard Hilbert-Schmidt rigged chain [7] of positive and
negative Hilbert spaces of differential forms
\[ H_A^{k,+}(M) \subset H_A^k(M) \subset H_A^{k,-}(M) \]
and the corresponding rigged chains of Hilbert sub-spaces for harmonic
\[ H_{A(L),+}(M) \subset H_{A(L)}^k(M) \subset H_{A(L),-}(M), \]
and cohomology groups:
\[ H_{k(L),+}(M) \subset H_{A(L)}^k(M) \subset H_{A(L),-}(M), \]
\[ H_{k(L),+}(M) \subset H_{A(L),+}^k(M) \subset H_{A(L),-}^k(M), \]
for any $k = 0, m$. Now by reasonings similar to those in [4, 17] one can formulate
the following a little generalized de Rham-Hodge theorem.

**Theorem 3.2.** The groups of harmonic forms $H_{A(L)}^k(M), k = 0, m$, are, cor-
respondingly, isomorphic to the cohomology groups $(H^k(M; \mathbb{C}))^\Sigma, k = 0, m$, where
$H^k(M; \mathbb{C})$ is the $k$–th cohomology group of the manifold $M$ with complex coef-
ficients, $\Sigma \subset \mathbb{C}^p$ is a set of suitable "spectral" parameters marking the linear space
of independent $d_L^*$-closed 0-forms from $H_{A(L),-}^0(M)$ and, moreover, the following
direct sum decompositions
\[ H_{A(L),-}^k(M) \oplus \Delta H_{A(L)}^k(M) = H_{A,L}^{k,-}(M) = H_{A(L),-}(M) \oplus d_L H_{A,L}^{k-1}(M) \oplus d_L H_{A,L}^{k+1}(M) \]
hold for any $k = 0, m$. 
Another variant of the statement similar to that above was formulated in [3] and reads as the following generalized de Rham-Hodge-Skrypnik theorem.

**Theorem 3.3.** (See Skrypnik I.V. [3]) The generalized cohomology groups \( H^k_{\Lambda(L^*)}(-,M), k = 0, m \), are isomorphic, correspondingly, to the cohomology groups \( H^k(M;\mathbb{C}) \Sigma \), \( k = 0, m \).

A proof of this theorem is based on some special sequence [3] of differential Lagrange type identities. Define the following closed subspace

\[
H^*_\Phi := \{ \varphi(\lambda) \in H^0_{\Lambda(L^*)}(-,M) : d^*_L \varphi(\lambda) = 0, \varphi(\lambda)|_{\Gamma^*} = 0, \lambda \in \Sigma \}
\]

for some smooth \((m-1)\)-dimensional hypersurface \( \Gamma^* \subset M \) and \( \Sigma \subset (\sigma(L^*) \cap \sigma(L^*)) \times \Sigma_\sigma \subset \mathbb{C}^p \), where \( H^0_{\Lambda(L^*)}(-,M) \) is, as above, a suitable Hilbert-Schmidt rigged [7] zero-order cohomology group Hilbert space from the chain given by (3.15), \( \sigma(L^*) \) and \( \sigma(L^*) \) are, correspondingly, mutual spectra of the sets of operators \( \tilde{L} \) and \( L^* \). Thereby the dimension \( \dim H^*_\Phi = \text{card} \Sigma \) is assumed to be known.

The next lemma stated by Skrypnik I.V. [3] being fundamental for the proof holds.

**Lemma 3.4.** (See Skrypnik I.V. [3]) There exists a set of differential \( k \)-forms \( Z^{(k+1)}[\varphi(\lambda), d_L \psi] \in \Lambda^{k+1}(M;\mathbb{C}), k = 0, m \), and a set of \( k \)-forms \( Z^{(k)}[\varphi(\lambda), \psi] \in \Lambda^k(M;\mathbb{C}), k = 0, m \), parametrized by a set \( \Sigma \ni \lambda \) and semilinear in \((\varphi(\lambda), \psi) \in H^*_\Phi \times H^k_{\Lambda(L^*)}(-,M), \lambda \in \Sigma \).

**Proof.** A proof is based on the following generalized Lagrange type identity holding for any pair \((\varphi(\lambda), \psi) \in H^*_\Phi \times H^k_{\Lambda(L^*)}(-,M) :\)

\[
0 = \langle d^*_L \varphi(\lambda), *(\psi \wedge \tilde{\gamma}) \rangle = \langle d^*_L (\ast)^{-1} \varphi(\lambda), \ast (\psi \wedge \tilde{\gamma}) \rangle > 0
\]

where

\[
Z^{(k+1)}[\varphi(\lambda), d_L \psi] \wedge \tilde{\gamma} = \ast^{-1} \varphi(\lambda), d_L \psi \wedge \tilde{\gamma} + dZ^{(k)}[\varphi(\lambda), \psi] \wedge \tilde{\gamma}
\]

and

\[
Z^{(k)}[\varphi(\lambda), \psi] \in \Lambda^{k+1}(M;\mathbb{C}), k = 0, m, \text{ and } Z^{(k)}[\varphi(\lambda), \psi] \in \Lambda^{k+1}(M;\mathbb{C}), k = 0, m, \text{ and }
\]

Based on this Lemma 3.3 one can construct the cohomology group isomorphism claimed in the Theorem 3.2 formulated above. Namely, following [3], let us take some simplicial partition \( K(M) \) of the manifold \( M \) and introduce linear mappings \( B^k_{\Lambda} : H^k_{\Lambda}(-,M) \to C_k(M), k = 0, m, \lambda \in \Sigma, \text{ where } C_k(M), k = 0, m, \text{ are the } \text{free abelian groups over the field } \mathbb{C} \text{ generated, correspondingly, by all } k \text{-chains of simplexes } S^{(k)} \in C_k(M), k = 0, m, \text{ of the simplicial [17] complex } K(M) \text{ as follows:}\)

\[
B^k_{\Lambda}(\psi) := \sum_{S^{(k)} \in C_k(M)} S^{(k)} \int_{S^{(k)}} Z^{(k)}[\varphi(\lambda), \psi]
\]

with \( \psi \in H^k_{\Lambda}(M), k = 0, m. \) The following theorem based on mappings (3.19) holds.
Theorem 3.5. (See Skrypnik I.V. [3]) The set of operations (3.19) parametrized by \( \lambda \in \Sigma \) realizes the cohomology groups isomorphism formulated in the Theorem 3.2.

Proof. A proof of this theorem one can get passing over in (3.19) to the corresponding cohomology \( \mathcal{H}_{k(\lambda,\ell)}^{-}(M) \) and homology \( \mathcal{H}_{k}(M;\mathbb{C}) \) of \( M \) for every \( k = \overline{0, m} \). If one to take an element \( \psi := \psi(\mu) \in \mathcal{H}_{k(\lambda,\ell)}^{-}(M), k = \overline{0, m} \), solving the equation \( d_L \psi(\mu) = 0 \) with \( \mu \in \Sigma_k \), being some set of the related "spectral" parameters marking elements of the subspace \( \mathcal{H}_{k(\lambda,\ell)}^{-}(M) \), then one finds easily from (3.19) and the identity (3.18) that

\[
(3.20) \quad d\mathcal{Z}^{(k)}[\varphi(\lambda), \psi(\mu)] = 0
\]

for all pairs \( (\lambda, \mu) \in \Sigma \times \Sigma_k, k = \overline{0, m} \). This, in particular, means due to the Poincare lemma [16, 17] that there exist differential \((k - 1)\)-forms \( \Omega^{(k-1)}[\varphi(\lambda), \psi(\mu)] \in \Lambda^{k-1}(M;\mathbb{C}), k = \overline{0, m} \), such that

\[
(3.21) \quad Z^{(k)}[\varphi(\lambda), \psi(\mu)] = d\Omega^{(k-1)}[\varphi(\lambda), \psi(\mu)]
\]

for all pairs \( (\varphi(\lambda), \psi(\mu)) \in \mathcal{H}_{k(\lambda,\ell)}^{*} \times \mathcal{H}_{k(\lambda,\ell)}^{-}(M) \) parametrized by \( (\lambda, \mu) \in \Sigma \times \Sigma_k, k = \overline{0, m} \). As a result of passing on the right-hand side of (3.19) to the homology groups \( \mathcal{H}_{k}(M;\mathbb{C}), k = \overline{0, m} \), one gets due to the standard Stokes theorem [16] that the mappings

\[
(3.22) \quad B^{(k)}_{\lambda} : \mathcal{H}_{k(\lambda,\ell)}^{-}(M) \Rightarrow \mathcal{H}_{k}(M;\mathbb{C})
\]

are isomorphisms for every \( \lambda \in \Sigma \). Making further use of the Poincare duality [17] between the homology groups \( \mathcal{H}_{k}(M;\mathbb{C}), k = \overline{0, m} \), and the cohomology groups \( \mathcal{H}^{k}(M;\mathbb{C}), k = \overline{0, m} \), correspondingly, one obtains finally the statement claimed in theorem 3.5, that is \( \mathcal{H}_{k(\lambda,\ell)}^{-}(M) \simeq (\mathcal{H}^{k}(M;\mathbb{C}))^{\Sigma} \).

Take now such a fixed pair \( (\varphi(\lambda), \psi(\mu)) \in \mathcal{H}_{k(\lambda,\ell)}^{*} \times \mathcal{H}_{k(\lambda,\ell)}^{-}(M) \), parametrized by \( (\lambda, \mu) \in \Sigma \times \Sigma_k, k = \overline{0, m} \), for which due to both Theorem 3.3 and the Stokes theorem [16, 17] the equality

\[
(3.23) \quad B^{(k)}_{\lambda}(\psi(\mu)) = s^{(k)}_{x} \int_{\partial S^{(k)}_{x}} \Omega^{(k-1)}[\varphi(\lambda), \psi(\mu)],
\]

holds, where \( s^{(k)}_{x} \in \mathcal{H}_{k}(M;\mathbb{C}), k = \overline{0, m} \), are some arbitrary but fixed elements parametrized by an arbitrarily chosen point \( x \in M \). Consider next the integral expressions

\[
(3.24) \quad \Omega^{(k-1)}_{x}(\lambda, \mu) := \int_{\partial S^{(k)}_{x}} \Omega^{(k-1)}[\varphi(\lambda), \psi(\mu)], \quad \Omega^{(k-1)}_{x_{0}}(\lambda, \mu) := \int_{\partial S^{(k)}_{x_{0}}} \Omega^{(k-1)}[\varphi(\lambda), \psi(\mu)]
\]

and interpret them as the corresponding kernels [7] of the integral invertible operators of Hilbert-Schmidt type \( \Omega^{(k-1)}_{x_{0}} \), \( \Omega^{(k-1)}_{x_{0}} : \mathcal{L}^{(0)}(\Sigma;\mathbb{C}) \rightarrow \mathcal{L}^{(0)}(\Sigma_{k};\mathbb{C}), k = \overline{0, m} \), where \( \rho \) and \( \rho_{k}, k = \overline{0, m} \), are some Lebesgue measures on the parameter sets \( \Sigma \) and \( \Sigma_{k} \), correspondingly. It assumes also above for simplicity that boundaries \( \partial S^{(k)}_{x} \) and \( \partial S^{(k)}_{x_{0}}, k = \overline{0, m} \), are taken homological to each other as \( x \rightarrow x_{0} \in M \). Define now the expressions

\[
(3.25) \quad \Omega^{(k)} : \psi(\eta) \rightarrow \tilde{\psi}(\eta)
\]
for $\psi(\eta) \in H^k_{\lambda(\xi),-}(M)$ and some $\tilde{\psi}(\eta) \in H^k_{\lambda,-}(M)$, $k = 0, m$, where, by definition
\begin{equation}
(3.26) \quad \tilde{\psi}(\eta) := (\psi \Omega_x^{(k-1)}, \xi \Omega_x^{(k-1)})(\eta) = \sum_{\eta_k(0), \underbrace{0, \ldots, 0, m}} \int_{\Sigma_k} d\rho_k(\mu) \psi(\mu) \int_{\Sigma} d\rho(\xi) \Omega_x^{(k-1)}(\mu, \xi) \Omega_x^{(k-1)}(\xi, \eta) \end{equation}
for any $\eta \in \Sigma_k$, $k = 0, m$.

Suppose now that the elements (3.26) are ones being related to some another Delsarte transformed cohomology groups $H^k_{\lambda(\xi),-}(M)$, $k = 0, m$, that is the following condition
\begin{equation}
(3.27) \quad d\tilde{\psi}(\eta) = 0
\end{equation}
for $\tilde{\psi}(\eta) \in H^k_{\lambda(\xi),-}(M)$, $\eta \in \Sigma_k$, $k = 0, m$, and some new external antidifferentiation operation in $H_{\lambda,-}(M)$
\begin{equation}
(3.28) \quad d\tilde{L} := \sum_{j=1}^m dx_j \wedge \tilde{L}_j(x, \partial).
\end{equation}
hold. Here, by definition, we will put
\begin{equation}
(3.29) \quad \tilde{L}_j := \Omega L_j \Omega^{-1}
\end{equation}
for each $j = 1, m$, where $\Omega : \mathcal{H} \to \mathcal{H}$ is the corresponding Delsarte transmutation operator.
Since all of operators $L_j : \mathcal{H} \to \mathcal{H}$, $j = 1, m$, were taken commuting, the same property also holds for the transformed operators (3.29), that is $[\tilde{L}_j, \tilde{L}_k] = 0$, $k, j = 0, m$. The latter is, evidently, equivalent due to (3.28) to the following general expression:
\begin{equation}
(3.30) \quad d\tilde{L} = \Omega d\tilde{L} \Omega^{-1}.
\end{equation}
For the condition (3.30) and (3.27) to be satisfied, let us consider the corresponding to (3.23) expressions
\begin{equation}
(3.31) \quad \hat{B}^{(k)}(\tilde{\psi}(\eta)) = S_x^{(k)}(\tilde{\psi}(\eta))
\end{equation}
related with the corresponding external differentiation (3.30), where $S_x^{(k)} \in H_{k}(M; \mathbb{C})$ and $(\lambda, \eta) \in \Sigma \times \Sigma_k$, $k = 0, m$. Assume further that there is also defined a mapping
\begin{equation}
(3.32) \quad \Omega^* \phi(\lambda) := \hat{\phi}(\lambda), \quad \Omega \psi(\eta) := \tilde{\psi}(\eta),
\end{equation}
with $\Omega^* : \mathcal{H}^* \to \mathcal{H}^*$ being an operator associated (but not necessary adjoint!) with the basic Delsarte transmutation operator $\Omega : \mathcal{H} \to \mathcal{H}$ satisfying the standard relationships $\tilde{L}_j^* := \Omega^* L_j \Omega^* \Omega^* \Omega^* = 1, m$. The corresponding Delsarte type operators $\Omega^{(k)} : H^k_{\lambda(\xi),-}(M) \to H^k_{\lambda(\xi),-}(M)$, $k = 0, m$, are related with the action (3.26) under the conditions
\begin{equation}
(3.33) \quad d\tilde{L} \tilde{\psi}(\eta) = 0, \quad d\tilde{L}^* \hat{\phi}(\lambda) = 0,
\end{equation}
needed to be satisfied, meaning evidently that the elements $\hat{\phi}(\lambda) \in H^0_{\lambda(\xi),-}(M)$, $\lambda \in \Sigma$, and elements $\tilde{\psi}(\eta) \in H^k_{\lambda(\xi),-}(M)$, $\eta \in \Sigma_k$, $k = 0, m$. Now we need to formulate a lemma being important for the conditions (3.33) to hold.
Lemma 3.6. The following invariance property
\[(3.34) \quad \tilde{Z}^{(k)} = \Omega_{x_0}^{(k-1)}\Omega^{(k-1), -1}_{\tilde{Z}}Z^{(k)}\Omega_{x_0}^{(k-1), -1}\]
holds for any \(k = 0, m\).

As a result of (3.34) and the symmetry invariance between cohomology spaces \(\mathcal{H}_0^0(\mathcal{L}^\ast,-)(M)\) and \(\mathcal{H}_0^0(\mathcal{L},-)(M)\) one obtains the following pairs of related mappings:
\[(3.35) \quad \psi = \tilde{\psi}\Omega_{x_0}^{(k-1), -1}\tilde{\Omega}_{x_0}^{(k-1)}, \quad \varphi = \tilde{\varphi}\Omega_{x_0}^{(k-1), -1}\tilde{\Omega}_{x_0},
\]
where the integral operator kernels defined as
\[(3.36) \quad \Omega^\ast_{x}(\lambda, \mu) := \int_{\partial S^{(m-1)}} \tilde{\Omega}^{(m-2), \tau[\varphi(\lambda), \psi(\mu)]},
\]
\[\tilde{\Omega}^\ast_{x}(\lambda, \mu) := \int_{\partial S^{(m-1)}} \tilde{\Omega}^{(m-2), \tau[\varphi(\lambda), \psi(\mu)]}\]
for all \((\lambda, \eta) \in \Sigma \times \Sigma_k, k = 0, m\), giving rise to proper Delsarte transmutation operators ensuring the pure differential nature of the transformed expressions (3.29).

Note here also that due to (3.34) and (3.35) the following operator property
\[(3.37) \quad \Omega_{x_0}^{(k-1)}\Omega_{x_0}^{(k-1), -1}\Omega_{x_0}^{(k-1), -1}\Omega_{x_0}^{(k-1), -1} = 0\]
holds for every \(k = 0, m\), meaning that \(\tilde{\Omega}_{x_0}^{(k-1)} = -\Omega_{x_0}^{(k-1)}\).

Take now \(k = m - 1\); then one can define similar to (3.16) the additional closed and dense in \(\mathcal{H}\) three subspaces
\[(3.38) \quad \mathcal{H}_0 := \{\psi(\mu) \in \mathcal{H}_0^0(\mathcal{L},-)(M): d_{\mathcal{L}}\psi(\mu) = 0, \quad \psi(\mu)|_{\Gamma} = 0, \quad \mu \in \Sigma\},
\]
\[\tilde{\mathcal{H}}_0 := \{\tilde{\psi}(\mu) \in \mathcal{H}_0^0(\mathcal{L},-)(M): d_{\mathcal{L}}\tilde{\psi}(\mu) = 0, \quad \tilde{\psi}(\mu)|_{\tilde{\Gamma}} = 0, \quad \mu \in \Sigma\},
\]
\[\mathcal{H}_{\ast}^\ast := \{\tilde{\varphi}(\eta) \in \mathcal{H}_0^0(\mathcal{L}^\ast,-)(M): d_{\mathcal{L}}\tilde{\varphi}(\eta) = 0, \quad \tilde{\varphi}(\eta)|_{\tilde{\Gamma}} = 0, \quad \eta \in \Sigma\},
\]
where \(\Gamma\) and \(\tilde{\Gamma}\) are some smooth \((m - 2)\)-dimensional hypersurfaces, and construct the actions
\[(3.39) \quad \mathbf{\Omega}: \psi \rightarrow \tilde{\psi} := \psi\Omega_{x}^{\ast,-1}\Omega_{x_0}, \quad \mathbf{\Omega}^\ast: \varphi \rightarrow \tilde{\varphi} := \varphi\Omega_{x_0}^{\ast,-1}\Omega_{x_0}^\ast\]
on arbitrary but fixed pairs of elements \((\tilde{\varphi}(\lambda), \tilde{\psi}(\mu)) \in \mathcal{H}_0^0 \times \mathcal{H}_0\), parametrized by the set \(\Sigma\), where by definition, one needs that all obtained pairs \((\tilde{\varphi}(\lambda), \tilde{\psi}(\mu))\) belong to \(\mathcal{H}_0^0(\mathcal{L}^\ast,-)(M) \times \mathcal{H}_0^0(\mathcal{L},-)(M)\). Here for all \((\lambda, \eta) \in \Sigma \times \Sigma\) we defined, as usually,
by expressions
\[(3.40) \quad \Omega_{x}(\lambda, \mu) := \int_{\partial S^{(m-1)}} \Omega^{(m-2)}[\varphi(\lambda), \psi(\mu)], \quad \Omega_{x}^\ast(\lambda, \mu) := \int_{\partial S^{(m-1)}} \tilde{\Omega}^{(m-2), \tau[\varphi(\lambda), \psi(\mu)]},
\]
\[\Omega_{x_0}(\lambda, \mu) := \int_{\partial S^{(m-1)}} \Omega^{(m-2)}[\varphi(\lambda), \psi(\mu)], \quad \Omega_{x_0}^\ast(\lambda, \mu) := \int_{\partial S^{(m-1)}} \tilde{\Omega}^{(m-2), \tau[\varphi(\lambda), \psi(\mu)]}\]
the corresponding kernels of integral operators acting in the Hilbert space \(L^2(\Sigma; \mathbb{C})\) of measurable functions with respect to some Borel measure \(\rho\) on Borel subsets of
the set Σ. The related operator property (3.37) can be compactly written down as follows:

(3.41) \[ \tilde{\Omega}_x = \tilde{\Omega}_{x_0} \Omega_x^{-1} \Omega_{x_0} = -\Omega_{x_0} \Omega_x^{-1} \Omega_{x_0}. \]

Construct now from the expressions (3.40) the following operator quantities in the Hilbert space \( L^2_\rho(\Sigma; \mathbb{C}) \):

(3.42) \[
\Omega - \Omega_{x_0} = \int_{\partial S_{x_0}^{(m-1)}} \Omega^{(m-2)}[\varphi(\lambda), \psi(\mu)] - \int_{\partial S_{x_0}^{(m-1)}} \Omega^{(m-2)}[\varphi(\lambda), \psi(\mu)]
\]

\[
= \int_{S_{x_0}^{(m-1)}(\sigma_{x_0}^{(m-2)}, \sigma_{x_0}^{(m-2)})} d\tilde{\Omega}^{(m-2)}[\varphi(\lambda), \psi(\mu)] - \int_{S_{x_0}^{(m-1)}(\sigma_{x_0}^{(m-2)}, \sigma_{x_0}^{(m-2)})} Z^{(m-1)}[\varphi(\lambda), \psi(\mu)],
\]

\[
\Omega^*_x - \Omega^*_{x_0} = \int_{\partial S_{x_0}^{(m-1)}} \Omega^{(m-2)}[\varphi(\lambda), \psi(\mu)] - \int_{\partial S_{x_0}^{(m-1)}} \Omega^{(m-2)}[\varphi(\lambda), \psi(\mu)]
\]

\[
= \int_{S_{x_0}^{(m-1)}(\sigma_{x_0}^{(m-2)}, \sigma_{x_0}^{(m-2)})} d\tilde{\Omega}^{(m-2), \tau}[\varphi(\lambda), \psi(\mu)] - \int_{S_{x_0}^{(m-1)}(\sigma_{x_0}^{(m-2)}, \sigma_{x_0}^{(m-2)})} Z^{(m-1), \tau}[\varphi(\lambda), \psi(\mu)],
\]

where, by definition, an \((m - 1)\)-dimensional surface \( S_{x_0}^{(m-1)}(\sigma_{x_0}^{(m-2)}, \sigma_{x_0}^{(m-2)}) \subset M \) is spanned smoothly between two homological cycles \( \sigma_{x_0}^{(m-2)} := \partial S_{x_0}^{(m-1)} \) and \( \sigma_{x_0}^{(m-2)} := \partial S_{x_0}^{(m-1)} \in H_{m-1}(M; \mathbb{C}) \).

Since the integral operator expressions \( \Omega_{x_0}, \Omega^*_{x_0} : L^2_\rho(\Sigma; \mathbb{C}) \rightarrow L^2_\rho(\Sigma; \mathbb{C}) \) are at a fixed point \( x_0 \in M \) evidently constant and assumed to be invertible, for extending the actions given (3.39) on the whole Hilbert space \( \mathcal{H} \times \mathcal{H}^* \) one can apply to them the classical constants variation approach, making use of the expressions (3.42). As a result, we obtain easily the following Delsarte transmutation integral operator expressions

(3.43) \[
\Omega = 1 - \int_{\Sigma \times \Sigma} \frac{d\rho(\xi)d\rho(\eta)}{S_{x_0}^{(m-1)}(\sigma_{x_0}^{(m-2)}, \sigma_{x_0}^{(m-2)})} \tilde{\Omega}(x; \xi) \Omega_{x_0}^{-1} [\tilde{\varphi}(\lambda), \tilde{\psi}(\mu)](\xi|\eta) \times Z^{(m-1)}[\varphi(\eta), \mu]
\]

\[
\Omega^* = 1 - \int_{\Sigma \times \Sigma} \frac{d\rho(\xi)d\rho(\eta)}{S_{x_0}^{(m-1)}(\sigma_{x_0}^{(m-2)}, \sigma_{x_0}^{(m-2)})} \tilde{\Omega}(x; \eta) \Omega_{x_0}^{-1} [\tilde{\varphi}(\lambda), \tilde{\psi}(\mu)](\eta|\xi) \times Z^{(m-1), \tau}[\varphi(\xi), \psi(\xi)]
\]

for fixed pairs \((\tilde{\varphi}(\lambda), \tilde{\psi}(\mu))\) and \((\tilde{\varphi}(\lambda), \tilde{\psi}(\mu)) \in \tilde{\mathcal{H}}_0 \times \tilde{\mathcal{H}}_0\), being bounded invertible integral operators of Volterra type on the whole space \( \mathcal{H} \times \mathcal{H}^* \). Applying the same arguments as in Section 1, one can show also that correspondingly transformed sets of operators \( \tilde{L}_j := \Omega L_j \Omega^{-1}, j = 1, m \), and \( \tilde{L}_k^* := \Omega^* L_k^* \Omega^{*-1}, k = 1, m \), appear to be purely differential too. Thereby, one can formulate the following final theorem.
Theorem 3.7. The expressions (3.43) are bounded invertible Delsarte transmutation integral operators of Volterra type onto $\mathcal{H} \times \mathcal{H}^*$, transforming, correspondingly, given commuting sets of operators $L_j$, $j = 1, m$, and their formally adjoint ones $L^*_k$, $k = 1, m$, into the pure differential sets of operators $\tilde{L}_j := \Omega L_j \Omega^{-1}$, $j = 1, m$, and $\tilde{L}^*_k := \Omega^* L^*_k \Omega^{*-1}$, $k = 1, m$. Moreover, the suitably constructed closed subspaces $\mathcal{H}_0 \subset \mathcal{H}$ and $\tilde{\mathcal{H}}_0 \subset \mathcal{H}$, such that $\Omega : \mathcal{H}_0 \rightleftharpoons \tilde{\mathcal{H}}_0$, depend strongly on the topological structure of the generalized cohomology groups $\mathcal{H}^0_{\Lambda,\mathcal{L},\mathcal{C}}(M)$ and $\mathcal{H}^0_{\Lambda,\mathcal{L},\mathcal{C}}(M)$, parametrized by points $x, x_0 \in M$.

4. Discussion.

Consider a differential operator $L : \mathcal{H} \rightarrow \mathcal{H}$ in the form (2.1) and assume that its spectrum $\sigma(L)$ consists of the discrete $\sigma_d(L)$ and continuous $\sigma_c(L)$ parts. By means of the general form of the Delsarte transmutation operators (3.43) one can construct a transformed more complicated differential operator $\tilde{L} := \Omega L \Omega^{-1}$ in $\mathcal{H}$, such that its continuous spectrum $\sigma_c(\tilde{L}) = \sigma_c(L)$ but $\sigma_d(L) \neq \sigma_d(\tilde{L})$. Thereby these Delsarte transformed operators can be effectively used for both studying spectral properties of differential operators [7, 5, 8, 9] and constructing a wide class of nontrivial differential operators with a prescribed spectrum as it was done [8, 6] in one dimension.

As was shown before in [5, 11] for the two-dimensional Dirac and three-dimensional perturbed Laplace operators, the kernels of the corresponding Delsarte transmutation operator satisfy some special of Fredholm type linear integral equations called the Gelfand-Levitan-Marchenko ones, which are of very importance for solving the corresponding inverse spectral problem and having many applications in modern mathematical physics. Such equations can be naturally constructed for our multidimensional case too, thereby making it possible to pose the corresponding inverse spectral problem for describing a wide class of multidimensional operators with a priori given spectral characteristics. The mentioned problem appears (see [7]) to be strongly related with that of spectral representation of kernels commuting in some sense with a given pair of differential operators. Also, similar to [11, 15], one can use such results for studying so called completely integrable nonlinear evolution equations, especially for constructing by means of special Darboux transformations [10, 12] their exact solutions like solitons and many others. Such an activity is now in progress and the corresponding results will be published later.

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References

[1] Delsarte J. Sur certaines transformations fonctionnelles relative aux equations lineaires aux derives partielles du second ordre. C.R. Acad. Sci. Paris, 1938, v. 206, p. 178-182
[2] Delsarte J. and Lions J. Transmutations d’operateurs differentiels dans le domaine complexe. Comment. Math. Helv., 1957, v. 52, p. 113-128
[3] Skrypnik I.V. Periods of A-closed forms. Proceedings of the USSR Academy of Sciences, 1965, v. 160, N4, p. 772-773 (in Russian)
[4] Chern S.S. Complex manifolds. Chicago University Publ., USA, 1956
[5] Faddeev L.D. Quantum inverse scattering problem. II. in Modern problems of mathematics, M: VINITY Publ., 1974, v.3, p. 93-180 (in Russian)
[6] Novikov S.P. (Editor) Theory of solitons. Moscow, Nauka Publ., 1980 (in Russian)
[7] Berezansky Yu. M. Eigenfunctions expansions subject to differential operators. Kiev, Nauk.Dumka Publ., 1965 (in Russian)
[8] Marchenko V.A. Spectral theory of Sturm-Liouville operators. Kiev, Nauk. Dumka Publ., 1972 (in Russian)
[9] Levitan B.M. and Sargsian I.S. Sturm-Liouville and Dirac operators. Moscow, Nauka Publ., 1988. (in Russian)
[10] Matveev V.B. and Salle M.I. Darboux-Backlund transformations and applications. NY, Springer, 1993.
[11] Nizhnik L.P. Inverse scattering problems for hyperbolic equations. Kiev, Nauk. Dumka Publ., 1991 (in Russian)
[12] Samoilenko A.M., Prykarpatsky Y.A. and Samoylenko V.G. The structure of Darboux-type binary transformations and their applications in soliton theory. Ukr. Mat. Zhurnal, 2003, v. 55, N12, p.1704-1723 (in Ukrainian)
[13] Samoilenko A.M. and Prykarpatsky Y.A. Algebraic-analytic aspects of completely integrable dynamical systems and their perturbations. Kyiv, NAS, Inst. Mathem. Publisher, 2002, v.41. (in Ukrainian)
[14] Nimmo J.C.C. Darboux transformations from reductions of the KP-hierarchy. Preprint of the Dept. of Mathem. at the University of Glasgow, November 8, 2002, 11 p.
[15] Prykarpatsky A.K. and Mykytiuk I.V. Algebraic integrability of nonlinear dynamical systems on manifolds: classical and quantum aspects. Kluwer Acad. Publishers, the Netherlands, 1998
[16] Godbillon C. Geometrie differentielle et mechanique analytique. Paris, Hermann, 1969.
[17] Teleman R. Elemente de topologie si varietati differentiabile. Bucuresti Publ., Romania, 1964

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