Three-Loop Results on the Lattice

B. Allésa, M. Campostrinib, A. Feo and H. Panagopoulosb,c

aDepartamento de Física Teórica y del Cosmos, Facultad de Ciencias
Universidad de Granada, 18071-Granada, Spain

bDipartimento di Fisica dell’Università and I.N.F.N.
Piazza Torricelli 2, 56126-Pisa, Italy

cDepartment of Physics, University of Cyprus, Nicosia, Cyprus

We present some new three-loop results in lattice gauge theories, for the Free Energy and for the Topological Susceptibility. These results are an outcome of a scheme which we are developing (using a symbolic manipulation language), for the analytic computation of renormalization functions on the lattice.

1. INTRODUCTION

The computation of vacuum expectation values of composite operators is one of the most important tasks in lattice gauge theories. Examples of such expectation values in QCD are the Condensates which are essential tools for the SVZ sum rules [1] or the Topological Susceptibility which is necessary to solve the so-called $U_A(1)$ problem and to understand the $\eta'$ mass [2–4]. In this talk we present some results related to the computation on the lattice of the Gluon Condensate

$$G_2 \equiv \langle 0 | g^2 \varepsilon_{\mu\nu} G_{\mu\nu} G_{\rho\sigma} | 0 \rangle,$$

(1)

and the Topological Susceptibility $\chi$

$$\chi \equiv \int d^4x \langle 0 | T(Q(x)Q(0)) | 0 \rangle,$$

(2)

where $Q(x)$ is the Topological Charge density

$$Q(x) \equiv g^2 \frac{1}{64\pi^2} \varepsilon_{\mu\nu\rho\sigma} G_{\mu\nu}^b G_{\rho\sigma}^b.$$

(3)

In these equations, $g$ is the coupling constant of the QCD lagrangian and $G_{\mu\nu}^b$, the strength tensor for the gluon fields.

To evaluate one of these quantities on the lattice, e. g. $\langle 0 | A | 0 \rangle$, one defines first a lattice version $A_L$ of $A$ in such a way that $A_L \xrightarrow{a=d} a^d A + O(a^{d+1})$ where $d$ is the mass dimension of the operator $A$ and $a$ is the lattice spacing. For the Gluon Condensate and Topological Susceptibility respectively, our choices for these lattice versions are

$$G_2^L \equiv \langle 0 | 1 - \Pi_{\mu\nu} | 0 \rangle,$$

(4)

$$\chi_L \equiv \langle 0 | \sum x Q^L(x)Q^L(0) | 0 \rangle,$$

where $Q^L(x)$ is a lattice version of the Topological Charge density

$$Q^L(x) = g^2 \frac{1}{2\tau_8} \sum_{\pm} \varepsilon_{\mu\nu\rho\sigma} \sum_{\mu,\nu,\rho,\sigma = \pm 1}^{\pm 4} \epsilon_{\mu\nu\rho\sigma} \times \langle 0 | \Pi_{\mu\nu}(x) \Pi_{\rho\sigma}(0) | 0 \rangle.$$

(5)

In these equations, $\Pi_{\mu\nu}$ is the usual plaquette in the $\mu - \nu$ plane.

Once the lattice version of the operator is defined, one can perform a Monte Carlo simulation. The Monte Carlo data will give the physical continuum value of $\langle 0 | A | 0 \rangle$ modified by some $a-$dependent renormalizations. These expressions for the Gluon Condensate and Topological Susceptibility can be written in the following way

$$G_2^L = \frac{\pi^2}{12N} Z_G a^4 G_2 + \sum_{n \geq 1} \frac{c_n}{\beta^n},$$

(6)

$$\chi_L = Z_Q a^4 \chi + a^4 G_2 \sum_{n \geq 1} \frac{b_n}{\beta^n} + \sum_{n \geq 3} \frac{d_n}{\beta^n},$$

(7)

Presented the talk.
where $N$ is the number of colors and as usual $\beta = 2N/g^2$. The lattice spacing $a$ and $\beta$ are related by the renormalization group equation

$$a\Lambda_L = \left( \frac{2Nr_0}{\beta} \right)^{-r_1/2r_0^2} \exp\left( -\frac{\beta}{4Nr_0} \right), \quad (8)$$

where $\Lambda_L$ is the renormalization group invariant mass parameter of QCD and $r_0$ and $r_1$ the first two coefficients of the $\beta$ function, $\beta(g) = -r_0g^3 - r_1g^5 - ...$

The last terms in these expressions (those proportional to $c_n$ and $d_n$) are the perturbative tails and represent mixings with the unity operator. The second term in Eq.(7) is a mixing with the Gluon Condensate. Finally, $Z_G$ and $Z_Q$ are multiplicative finite renormalizations which relate the lattice and the continuum definitions of the respective operators. All of these coefficients can be computed in perturbation theory and theirknowledge is essential to extract the physical values $G_2$ and $\chi$ from the Monte Carlo data and Eq.(6-7). The values of these coefficients also depend on the lattice versions used for the operators and the lattice action chosen. Throughout this work we have used the lattice versions shown in Eq.(4) and the Wilson action.

2. RENORMALIZATION CONSTANTS

The first coefficients in the renormalization terms of Eq.(6-7) can be calculated with rather small effort because they involve very few Feynman diagrams. The result for $SU(3)$ is

$$Z_G = 1 + O(1/\beta^2), \quad Z_Q = 1 - 5.45/\beta + O(1/\beta^2)$$

$$b_2 = 6.32 \times 10^{-3} \quad d_3 = 3.58 \times 10^{-3} \quad (9)$$

$$c_1 = 2.0 \quad c_2 = 1.22.$$ 

However, the next coefficients involve many more diagrams. For instance, $c_1$ and $c_2$ are calculated with three diagrams, but the next order, $c_3$, needs the computation of 30 three-loop Feynman diagrams! To evaluate these Feynman diagrams we have developed an algebraic computer program. Major tasks in this algorithm are:

i) Computing $n$-point vertices.

ii) Producing a list of relevant diagrams, with the corresponding weights.

iii) Performing the contractions for each diagram, using up all existing symmetries to produce a compact result.

iv) Extracting the analytic dependence of each diagram on its external momenta $p$, in the limit $ap \to 0$.

v) Producing the optimized code for the numerical calculation of the loop integrals.

Regarding the first step, some difficulties inherent to the lattice are: The existence of vertices with an arbitrary number of gluons, a plethora of “tensor” structures (due to lack of rotational invariance) and a great proliferation in the size of vertices (a 6-point vertex in its most compact form may typically require some dozens of output pages). This in turn necessitates, in the third step, simplifying all intermediate expressions as much as possible, by devising algorithms which use up the symmetries of the diagram under exchange of external legs, under allowed redefinitions of momenta and under permutation of the (numerous) dummy indices.

The fourth step is necessary for computing multiplicative renormalizations. The evaluation of these form factors is in progress.

Now the final expression for the diagram can be integrated. With the code produced by the algorithm, we can calculate the numerical value of the diagram for rather small lattices and then extrapolate for larger lattices. This extrapolation is performed by assuming the following dependence of the diagram on the lattice size $L$

$$\text{diagram} = A + B \frac{1}{L^n}. \quad (10)$$

The explicit values for $A$, $B$ as well as that of $n$ (which may vary for different diagrams) are determined by the extrapolation. The result is then confronted with the results for an infinite lattice, which we also compute. We will explain this computation for an infinite lattice with an example. Let us consider the following integral

$$I = \int_{-\pi}^{+\pi} \frac{d^4p}{p^2 q^2} \frac{d^4q}{p + q} \frac{1}{(2\pi)^4} \frac{1}{(2\pi)^4} \quad (11)$$

which is the value of a two-loop diagram for an infinite lattice. In this expression, $p^2 = 2 \sum_{\mu} (1 - ...$
3. RESULTS

Summing up the individual contributions of all diagrams we obtain for the gauge group $SU(N)$ and an infinite lattice

\[ d_4 = N^4(N^2-1)(1.73N^2-10.83 + \frac{73.83}{N^2})10^{-7}. \tag{15} \]

c_3 = N^2(N^2-1)(7.28N^2-27.35 + \frac{46.15}{N^2})10^{-3}. \tag{16} \]

The exact values for the coefficients obtained with our scheme can be compared with the values extracted by a best fit of all Monte Carlo data with Eqs.(6-9). Both values are shown in table 1 for the coefficients $d_4$ and $c_3$ and for an infinite lattice.

The agreement between fitted and exact results is manifest for $c_3$. The data for the Topological Susceptibility had a rather low statistics, therefore and within the errors the numbers shown in table 1 for $d_4$ are in acceptable agreement.

Finally, we can add the exact values of $c_3$ and $d_4$ to those of Eq.(9) and perform again the best fits. The values extracted for $G_2$ and $\chi$ are in complete agreement with those reported in refs. 6-8, 12.

4. ACKNOWLEDGMENTS

We wish to thank Adriano Di Giacomo for useful conversations and the spanish-italian “Acción Integrada/Azione Integrata” number A17 for financial support. B. A. also acknowledges a spanish CICYT contract.

REFERENCES

1. M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B147 (1979) 385, 448, 519.
2. G. t’Hooft, Phys. Rev. Lett. 37 (1976) 8; Phys. Rev. D14 (1976) 3432.
3. E. Witten, Nucl. Phys. B156 (1979) 269.
4. G. Veneziano, Nucl. Phys. B159 (1979) 213.
5. P. Di Vecchia, K. Fabricius, G. C. Rossi and...
6. M. Campostrini, G. Curci, A. Di Giacomo and G. Paffuti, Z. Phys. C32 (1986) 377.
7. M. Campostrini, A. Di Giacomo and Y. Gündüç, Phys. Lett. B225 (1989) 393.
8. M. Campostrini, A. Di Giacomo, H. Panagopoulos and E. Vicari, Nucl. Phys. B329 (1990) 683.
9. B. Allés and M. Gianetti, Phys. Rev. D44 (1991) 513.
10. M. Campostrini, A. Di Giacomo and H. Panagopoulos, Phys. Lett. B212 (1988) 206.
11. B. Allés, M. Campostrini, A. Feo and H. Panagopoulos, Lattice Perturbation Theory Done Symbolically: A Three-Loop Result for the Topological Susceptibility, Pisa preprint IFUP-TH-31/92; The Three-Loop Lattice Free Energy, Pisa preprint IFUP-TH-32/92.
12. M. Campostrini, A. Di Giacomo, Y. Gündüç, M. P. Lombardo, H. Panagopoulos and R. Tripiccione, Phys. Lett. B252 (1990) 436.