Live Functional Programming with Typed Holes

CYRUS OMAR, University of Chicago
IAN VOSEY, Carnegie Mellon University
RAVI CHUGH, University of Chicago
MATTHEW A. HAMMER, University of Colorado Boulder

Live programming environments aim to provide programmers (and sometimes audiences) with continuous feedback about a program’s dynamic behavior as it is being edited. The problem is that programming languages typically assign dynamic meaning only to programs that are complete, i.e. syntactically well-formed and free of type errors. Consequently, live feedback presented to the programmer has temporal or perceptive gaps, i.e. it flickers in and out or it goes stale whenever the program becomes incomplete.

This paper confronts this “gap problem” from type-theoretic first principles by developing a dynamic semantics for incomplete functional programs, starting from the static semantics recently developed by Omar et al. [2017a]. We model incomplete functional programs as expressions with holes, with empty holes standing for missing expressions or types, and non-empty holes operating as membranes around static and dynamic type inconsistencies. Rather than aborting when evaluation encounters any of these holes as in several existing systems, evaluation proceeds around holes, tracking the closure around each hole instance as it flows through the remainder of the program. Editor services can report information from these hole closures to help the programmer decide how to fill the holes in the program. Hole closures also enable a fill-and-resume operation that avoids the need to restart evaluation after edits that amount to hole filling. Formally, the semantics borrows machinery from both gradual type theory (to handle type holes) and contextual modal type theory (which provides a logical basis for hole closures), combining these and developing additional machinery necessary to continue evaluation past holes while maintaining important metatheoretic properties.

We have mechanized the metatheory of this core calculus, called Hazelnut Live, using the Agda proof assistant. We have also implemented these ideas into the Hazel programming environment. The implementation inserts holes automatically to guarantee that every editor state has some (possibly incomplete) type, following the Hazelnut edit action calculus of Omar et al. [2017a]. Taken together with this paper’s type safety property, the result is a proof-of-concept live programming environment where rich dynamic feedback is truly available without gaps, i.e. it is available for every possible editor state.

1 Introduction

Programmers typically shift back and forth between program editing and program evaluation many times before converging upon a program that behaves as intended. Live programming environments aim to support this workflow by interleaving editing and evaluation so as to narrow what Burckhardt et al. [2013] call the “temporal and perceptive gap” between these activities.

For example, read-evaluate-print loops (REPLs) and derivatives thereof, like the IPython/Jupyter lab notebooks popular in data science [Pérez and Granger 2007], allow the programmer to edit and immediately execute program fragments organized into a sequence of cells. Spreadsheets are live functional dataflow environments, with cells organized into a grid [Wakeling 2007]. More specialized examples include live direct manipulation programming environments like SuperGlue [McDirmid 2007], Sketch-n-Sketch [Chugh et al. 2016; Hempel and Chugh 2016], and the tools demonstrated by Victor [2012] in his lectures; live user interface frameworks [Burckhardt et al. 2013]; live image processing languages [Tanimoto 1990]; and live visual and auditory dataflow languages [Burnett et al. 1998], which can support live coding as a performance art. Editor-integrated debuggers [McCauley et al. 2008] and other systems that support editing run-time state, like Smalltalk environments [Goldberg and Robson 1983], are also live programming environments.

Authors’ addresses: Cyrus Omar, University of Chicago, comar@cs.uchicago.edu; Ian Voysey, Carnegie Mellon University, iev@cs.cmu.edu; Ravi Chugh, University of Chicago, rchugh@cs.uchicago.edu; Matthew A. Hammer, University of Colorado Boulder, matthew.hammer@colorado.edu.
The problem at the heart of this paper is that programming languages typically assign meaning only to complete programs, i.e. programs that are syntactically well-formed and free of static type and binding errors. A program editor, however, frequently encounters incomplete, and therefore meaningless, states. As a result, live feedback either “flickers out”, creating a temporal gap, or it “goes stale”, i.e. it relies on the most recent complete editor state, creating a perceptive gap because the feedback may not accurately reflect what the programmer is seeing in the editor.

In some cases these gaps are momentary, like while the programmer is entering a short expression. In other cases, these gaps can persist over substantial lengths of time, such as when there are many branches of a case analysis whose bodies are initially left blank or when the programmer makes a mistake. Novice programmers, of course, make more mistakes [Fitzgerald et al. 2008; McCauley et al. 2008]. The problem is particularly pronounced for languages with rich static type systems where certain program changes, such as a change to a type definition, can cause type errors to propagate throughout the program. Throughout the process of addressing these errors, the program text remains formally meaningless. Overall, about 40% of edits performed by Java programmers using Eclipse left the program text malformed [Omar et al. 2017a; Yoon and Myers 2014] and some additional number, which could not be determined from the data, were well-formed but ill-typed.

In recognition of this “gap problem”—that incomplete programs are formally meaningless—Omar et al. [2017a] develop a static semantics (i.e. a type system) for incomplete functional programs, modeling them formally as typed expressions with holes in both expression and type position. Empty holes stand for missing expressions or types, and non-empty holes operate as “membranes” around static type inconsistencies (i.e. they internalize the “red underline” that editors commonly display under a type inconsistency). Omar et al. [2017a,b] discuss several ways to determine an incomplete expression from the editor state. When the editor state is a text buffer, error recovery mechanisms can insert holes implicitly [Aho and Peterson 1972; Charles 1991; Graham et al. 1979; Kats et al. 2009]. Alternatively, the language might provide explicit syntax for holes, so that the programmer can insert them either manually or semi-automatically via a code completion service [Amorim et al. 2016]. For example, GHC Haskell supports the notation _u for empty holes, where u is an optional hole name [Jones et al. 2014]. Structure editors insert explicitly represented holes fully automatically [Omar et al. 2017a]; we say more about structure editors in Sec. 3.5.

For the purposes of live programming, however, a static semantics does not suffice—we also need a corresponding dynamic semantics that specifies how to evaluate expressions with holes.

The simplest approach would be to define a dynamic semantics that aborts with an error when evaluation reaches a hole. This mirrors a workaround that programmers commonly deploy: raising an exception as a placeholder, e.g. raise Unimplemented. GHC Haskell supports this mode of evaluation for programs with holes using the -fdefer-typed-holes flag. Although better than nothing, this “exceptional approach” to expression holes has limitations within a live programming environment because (1) it provides limited information about the dynamic state of the program where the exception occurs (typically only a stack trace); (2) it provides no information about the behavior of the remainder of program, parts of which may not depend on the missing or erroneous expression (e.g. later cells in a lab notebook, or tests of those components of the program that are already complete); and (3) it provides no means by which to resume evaluation after filling a hole. Furthermore, exceptions can appear only in expressions, but we might also like to be able to evaluate programs that have type holes. Again, existing approaches do not support this situation well—GHC supports type holes, but compilation fails if type inference cannot automatically fill them [Jones et al. 2014]. The static semantics developed by Omar et al. [2017a] derives the machinery

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1 Without this flag, holes cause compilation to fail with an error message that reports information about each hole’s type and typing context. Proof assistants like Agda [Norell 2007, 2009] and Idris [Brady 2013] also respond to holes in this way.
for reasoning statically about type holes from gradual type theory, identifying the type hole with the unknown type [Siek and Taha 2006; Siek et al. 2015a]. As such, we might look to the dynamic semantics from gradual type theory, which selectively inserts dynamic casts as necessitated by missing type information to maintain type safety. However, when a cast fails, evaluation again stops with an exception and so the traditional gradual typing approaches still leave the live programming environment unable to provide rich, continuous feedback for the three reasons just enumerated.

**Contributions.** This paper develops a dynamic semantics for incomplete functional programs, starting from the static semantics developed by Omar et al. [2017a], that addresses these limitations of the exceptional approach. In particular, rather than stopping when evaluation encounters an expression hole, evaluation continues “around” the hole so that dynamic feedback about other parts of the program remains available. For programs with unfillable type holes, casts are inserted as in gradual type theory [Siek et al. 2015a]. Evaluation proceeds around failed casts in the same way.

The system tracks the closure around each expression hole instance as evaluation proceeds so that the live programming environment can feed relevant information from the hole closures to the programmer as they work to fill the holes in the program. Then, when the programmer performs an edit that fills an empty expression hole or that replaces a non-empty hole with a type-correct expression, evaluation can resume from the paused, i.e. indeterminate, evaluation state. We call this operation *fill-and-resume*. We observe that empty hole closures are closely related to metavariable closures from contextual modal type theory (CMTT) [Nanevski et al. 2008] (which, by its Curry-Howard interpretation, provides a logical basis for reasoning about and operating on hole closures). These connections to well-established systems, together with our mechanized proofs, serve to support our claim that this approach is theoretically well-grounded.

We are integrating this approach into the Hazel programming environment being developed by Omar et al. [2017b]. Hazel features a live context inspector that integrates static information with hole closure information and interactively presents recursive hole closures, which arise from incomplete recursive functions. The editor component of this implementation defines a language of structured edit actions, based on the Hazelnut structure editor calculus developed by Omar et al. [2017a], that inserts holes automatically to guarantee that every editor state has some, possibly incomplete, type. The type safety invariant that we establish then guarantees that every editor state has dynamic meaning. Taken together, the result is a full solution to the gap problem, i.e. a proof-of-concept live functional programming environment that provides rich static and dynamic feedback that never suffers from temporal or perceptive gaps.

**Paper Outline.** We begin in Sec. 2 by detailing the approach informally, with several example programming tasks, in the setting of the Hazel design, which is based closely on well-established existing designs (in particular, on the Elm programming language and the Jupyter user interface).

Sec. 3 then abstracts away the inessential details of the language and user interface and makes the intuitions developed in Sec. 2 formally precise by detailing the primary contribution of this paper: a core calculus, Hazelnut Live, that supports evaluating incomplete expressions and tracking hole closures. Sec. 3.4 outlines our Agda-based mechanization of Hazelnut Live, which is included in the anonymized supplement. Sec. 3.5 provides some additional details on the implementation of these ideas into Hazel, which is based on the Hazelnut edit action calculus [Omar et al. 2017a], and formally states the continuity invariant, i.e. the solution to the gap problem, as a corollary of the primary theorems of Hazelnut and Hazelnut Live.

Sec. 4 defines the fill-and-resume operation, which is rooted in the contextual substitution operation from CMTT. We establish the correctness of fill-and-resume with a commutativity theorem. We also discuss how the fill-and-resume operation allows us to semantically interpret the act of editing and evaluating cells in a REPL or Jupyter-like live lab notebook environment.
(a) Evaluating an incomplete functional program past the first hole

(b) The live context inspector communicates relevant static and dynamic information about variables in scope.

Fig. 1. Example 1: Grades

Sec. 5 describes related work in detail and simultaneously discusses limitations and directions for future work. Sec. 6 briefly concludes.

The appendix (1) provides some straightforward auxiliary definitions and proofs that were omitted from the paper for concision; and (2) defines some simple extensions to the core calculus (namely, numbers, and sum types), together with a brief exposition on defining further extensions.

2 Live Functional Programming with Typed Holes in Hazel

This section gives an example-driven overview of our approach as implemented in Hazel, a live programming environment being developed by Omar et al. [2017b]. The Hazel user interface is based roughly on IPython/Jupyter [Pérez and Granger 2007], with results appearing below cells containing code, and the Hazel language is tracking toward feature parity with Elm (elm-lang.org) [Czaplicki 2012, 2018], a popular pure functional programming language similar to "core ML", with which we assume familiarity. Hazel is intended initially for use by students and instructors in introductory functional programming courses (where Elm has been successful [D’Alves et al. 2017]). For the sake of exposition, we have post-processed the screenshots in this section after generating them in Hazel to make use of some "syntactic and semantic sugar" from Elm that was not available in Hazel as of this writing, e.g. pattern matching in function arguments, list notation, and record labels (currently there are only tuples). These conveniences are orthogonal to the contributions of this paper; all of the user interface features demonstrated in this section have been implemented.
2.1 Example 1: Evaluating Past Holes and Hole Closures

Consider the perspective of a teacher in the midst of developing a Hazel notebook to compute final student grades at the end of a course. Fig. 1a depicts the cell containing the incomplete program that the teacher has written so far (we omit irrelevant parts of the UI).

At the top of this program, the teacher defines a record type, `Student`, for recording a student’s course data—here, the student’s name, of type `string`, and, for simplicity, three grades, each of type `float`. Next, the teacher constructs a list of student records, binding it to the variable `students`. For simplicity, we include only three example students. At the bottom of the program, the teacher maps a function `weighted_average` over this student data (map is the standard map function over lists, not shown), intending to compute a final weighted average for each student. However, the program is incomplete because the teacher has not yet completed the body of the `weighted_average` function.

This pattern is quite common: programmers often consume a function before implementing it. Thusfar in the body of `weighted_average`, the teacher has decomposed the function argument into variables by record destructuring, then multiplied the homework grade, `hw`, by `30.0` and finally inserted the `+` operator. The cursor now sits at an empty hole, as indicated by the vertical bar and the green background. Each hole has a unique name (generated automatically in Hazel), here simply `1`.

Let us briefly digress: in a conventional “batch” programming system, writing `30.0*hw +` by itself would simply cause parsing to fail and there would be no static or dynamic feedback. In response, the programmer might encode a hole by raising an exception. This would cause typechecking to succeed. However, evaluation would proceed only as far as the first `map` iteration, which would call into `weighted_average` and then fail when attempting to evaluate the hole.

Hazel does not take this “exceptional” interpretation of holes. Instead, evaluation continues past the hole, treating it as an opaque expression of the appropriate type. The result, shown at the bottom of Fig. 1a, is a list of length 3, confirming that `map` does indeed behave as expected in this regard despite the teacher having provided an incomplete argument. Furthermore, each element of the resulting list has been evaluated as far as possible, i.e. the arithmetic expression `30.0*hw` has been evaluated for each corresponding value of `hw`, as expected. Evaluation cannot proceed any further because holes appear as addends. We say that each of these addition expressions is an indeterminate sub-expression, and the result as a hole is therefore also indeterminate, because it is not yet a value, nor can it take a step due to holes in elimination positions.

At this point, the teacher might take notice of the magnitude of the numbers being computed, e.g. `2640.0` and `2280.0`, and realize immediately that a mistake has been made: the teacher wants to compute a weighted average between `0.0` and `100.0`, and so the correct constant is `0.30`, not `30.0`.

Although these observations might save only a small amount of time in this case, it demonstrates the broader motivations of live programming: continuous feedback about the dynamic behavior of the program can help confirm the mental model that the programmer has developed (in this case, regarding the behavior of `map`), and also help quickly dispel misconceptions about the actual behavior of the program (in this case, the magnitude of the arithmetic expressions being computed).

Going further, Hazel helps programmers reason about the dynamic behavior of expressions bound to variables in scope at a hole via the live context inspector, normally displayed as a sidebar but shown disembodied in three states in Fig. 1b. In all three states, the live context inspector displays the names and types of the variables that are in scope. New to our approach are the values associated with bindings, which come from the environment associated with the selected hole instance in the result. We call a hole instance paired with an environment a hole closure, by analogy to function closures (and also due to the logical connection detailed in Sec. 3). In this case, there are three instances of hole `1` in the result, numbered sequentially `1:1`, `1:2` and `1:3`, arising from the

\[\text{In a lazy language, like Haskell, the result would be much the same because the environment forces the result for printing.}\]
2.2 Example 2: Recursive Functions

Let us now consider a second more sophisticated example: an incomplete implementation of the recursive quicksort function, shown in Fig. 2a. So far, the programmer (perhaps a student, or a lecturer using Hazel as a presentational aid) has filled in the base case, and in the recursive case, partitioned the remainder of the list relative to the head, and made the two recursive calls. A hole appears in return position as the programmer contemplates how to fill the hole with an appropriate expression of list type, as indicated by the type inspector in Fig. 2b.

At the bottom of the cell in Fig. 2a, the programmer has applied `qsort` to an example list. However, the indeterminate result of this function application is simply an instance of hole 1, which serves only to confirm that evaluation went through the recursive case of `qsort`. More interesting is the live context inspector, shown in three states in Fig. 2c, which provides feedback about the values of the variables in scope at hole 1 from the the various instances of hole 1 that appear in the result,
either immediately or within an outer closure. For example, in its initial state (Fig. 2c, left) it shows the closure at the instance of hole 1 that appears immediately in the result due to the outermost application of \texttt{qsort}. From this, the programmer can confirm (or the lecturer can visually point out) that the lists \texttt{smaller} and \texttt{bigger} computed by the call to \texttt{partition} are appropriately named, and observe that they are not yet themselves sorted.

The results from the subsequent recursive calls, \texttt{r_smaller} and \texttt{r_bigger}, are again hole instances, 1:2 and 1:3. The programmer can click on either of these hole instances to reveal the associated closures from the corresponding recursive calls. For example, clicking on 1:3 reveals the hole closure from the \texttt{r_bigger} recursive call as shown in Fig. 2c (middle). From there, the programmer can click another hole closure, e.g. 1:6 to reveal the hole closure from the subsequent \texttt{r_smaller} recursive call as shown in Fig. 2c (right). Notice in each case that the path from the result to the selected hole closure is reported as shown at the bottom of the context inspector in Fig. 2c. In exploring these paths rooted at the result, the programmer can develop concrete intuitions about the recursive structure of the computation (e.g. by following through to the base case as shown in Fig. 2c) even before the program is complete.

\subsection{Example 3: Live Programming with Static Type Errors}

The previous examples were incomplete because of \textit{missing} expressions. Now, we discuss programs that are incomplete, and therefore conventionally meaningless, because of \textit{type inconsistencies}. Let us return to the quicksort example just described, but assume that the programmer has filled in the previous hole as shown in Fig. 3. In Sec. 4, we discuss how the programming environment might avoid restarting evaluation after such edits, but for small examples like this, restarting works fine.

The programmer appears to be on the right track conceptually in recognizing that the pivot needs to appear between the smaller and bigger elements. However, the types do not quite work out: the \texttt{@} operator here performs list concatenation, but the pivot is an integer. Most compilers and editors will report a static error message to the programmer in this case, and Hazel follows suit in the type inspector (shown inset in Fig. 3). However, our argument is that the presence of a static type error should not cause all feedback about the dynamic behavior of the program to "flicker out" or "go stale" – after all, there are perfectly meaningful parts of the program (both nearby and far away from the error) whose dynamic behavior may be of interest. Evaluation can also assist the programmer in understanding the type error by presenting concrete values [Seidel et al. 2016].

Our approach, following the prior work of Omar et al. [2017a], is to semantically internalize the “red outline” around type inconsistencies, representing it as a \textit{non-empty hole}. Evaluation safely proceeds past a non-empty hole just as if it were an empty hole. The semantics also associates an environment with each instance of a non-empty hole, so we can use the live context inspector
Fig. 4. Example 4: Type Holes and Dynamic Type Errors

```plaintext
f :: a -> b -> string
f simple x =
  if simple then string_of_int x else "Value: " ^ x
(f True 1, f False 2, f 3 True)
```

RESULT OF TYPE: (string, string, string)

{"1",
  "Value: " ^ Z(int = __ = string),
  if 3(int = __ = bool) then ... else ...
}

2.4 Example 4: Type Holes and Dynamic Type Errors

In Hazel, the program can also be incomplete because holes appear in types. Omar et al. [2017a] confirmed that the literature on gradual type systems [Siek and Taha 2006; Siek et al. 2015a] is directly relevant to the problem of reasoning with type holes, by identifying the type hole with the unknown type. Indeed, the purpose of gradual typing is to be able to run programs that are not yet sufficiently annotated with types by inserting casts only where necessary. As such, let us consider only a small synthetic example to demonstrate what is unique to our approach.

Fig. 4 defines a simple function, \( f \), of two arguments. The type annotation on the first line leaves the type of those arguments unknown. As such, the Hazel type system, following the gradual typing approach, allows the body of the function to use those two arguments at any type (that is, the hole type is universally consistent). Here, the first argument, \( \text{simple} \), is used at one type, \( \text{bool} \), and the second argument, \( x \), is used at two different types in the two branches (perhaps because the programmer made a mistake), \( \text{int} \) and \( \text{string} \) (\(^*\) is string concatenation). Although Hazel supports only local type inference as of this writing, a system that uses ML-style type reconstruction to fill type holes statically, like GHC Haskell, would only be able to fill the first hole. Leaving the second hole unfilled is a parsimonious alternative to arbitrarily or heuristically choosing one of the possibilities and marking the other uses of \( x \) as ill-typed (see [Chen and Erwig 2018]).

At the bottom of the cell in Fig. 4, we have three example applications of \( f \), tupled together for concision. All three are statically well-typed, again because the hole type is universally consistent. The result at the bottom of Fig. 4 demonstrates that the first application of \( f \) is dynamically unproblematic. This allows the programmer to confirm that the first branch operates as intended without the need to address the typing problems in the other branch [Bayne et al. 2011].

The second application of \( f \), in contrast, causes a dynamic type error because the second argument, \( 2 \), is an \( \text{int} \) but evaluation takes the branch where it is used as a \( \text{string} \). Rather than aborting evaluation when this occurs, as in existing gradual type systems, the problematic term becomes a failed cast term, shown shaded in red, which can be read "\( 2 \) is an \( \text{int} \) that was used through a variable of hole type (\( ? \)) as a \( \text{string} \)". A failed cast acts much like a non-empty hole as a membrane around a problematic term. The surrounding concatenation operation becomes indeterminate, but evaluation can continue on to the third application of \( f \), which is also problematic, this time because the first argument is not a \( \text{bool} \) (perhaps because the programmer had an incorrect understanding of the argument order). Again, this causes a failed cast to appear, this time in guard position. Like a hole in guard position, evaluation cannot determine which branch to take so the whole conditional becomes indeterminate. The pretty printer hides the two branches behind ellipses for concision.
We distinguish between external expressions, \( e \), and internal expressions, \( d \). We write \( x \) to range over variables, \( u \) over hole names, and \( \sigma \) over finite substitutions (i.e., environments) which map variables to internal expressions, written \( d_1/x_1, \ldots, d_n/x_n \) for \( n \geq 0 \).

\[
\Gamma \vdash e \Rightarrow \tau \quad e \text{ synthesizes type } \tau
\]

\[
\begin{align*}
\text{SConst} & \\
\Gamma \vdash c & \Rightarrow b \\
\text{SVar} & \\
\Gamma, x : \tau & \vdash_\Gamma e \Rightarrow \tau \\
\text{SLam} & \\
\Gamma, x : \tau, t_1 & \vdash_\Gamma e \Rightarrow \tau_1 \\
\text{SEHole} & \\
\Gamma & \vdash \langle e \rangle \Rightarrow \langle \rangle \\
\text{SNEHole} & \\
\Gamma & \vdash \langle e \rangle^u \Rightarrow \langle \rangle \\
\end{align*}
\]

\[
\begin{align*}
\text{SAp} & \\
\Gamma & \vdash e_1 \Rightarrow \tau_1 \\
\Gamma & \vdash e_2 \Leftarrow \tau_2 \quad \tau_1 \xrightarrow{\Gamma} \tau_2 \Rightarrow \tau \\
\text{SAsc} & \\
\Gamma & \vdash e \Leftarrow \tau \\
\Gamma & \vdash e : \tau \Rightarrow \tau
\end{align*}
\]

\[
\begin{align*}
\text{ALam} & \\
\tau & \xrightarrow{\Gamma} \tau_1 \Rightarrow \tau_2 \\
\Gamma, x : \tau_1 & \vdash e \Leftarrow \tau_2 \\
\Gamma & \vdash \lambda x.e \Leftarrow \tau
\end{align*}
\]

\[
\begin{align*}
\text{ASubsume} & \\
\Gamma & \vdash e \Rightarrow \tau \\
\tau & \sim \tau' \\
\Gamma & \vdash e \Leftarrow \tau'
\end{align*}
\]

Fig. 6. Bidirectional Typing of External Expressions

3 Hazelnut Live

We will now make the intuitions developed in the previous section formally precise by specifying a core calculus, which we call Hazelnut Live, and developing its metatheory.

Overview. The syntax of the core calculus given in Fig. 5 consists of types and expressions with holes. We distinguish between external expressions, \( e \), and internal expressions, \( d \). External expressions correspond to programs as entered by the programmer (see Sec. 1 for discussion of implicit, manual, semi-automated and fully automated hole entry methods). Each well-typed external expression (see Sec. 3.1 below) expands to a well-typed internal expression (see Sec. 3.2) before it is evaluated (see Sec. 3.3). We take this approach, notably also taken in the “redefinition” of Standard ML by Harper and Stone [2000], because (1) the external language supports type inference and explicit type ascriptions, \( e : \tau \), but it is formally simpler to eliminate ascriptions and specify a type assignment system when defining the dynamic semantics; and (2) we need additional syntactic machinery during evaluation for tracking hole closures and dynamic type casts. This machinery is inserted by the expansion step, rather than entered explicitly by the programmer. In this regard, the internal language is analogous to the cast calculus in the gradually typed lambda calculus [Siek and Taha 2006; Siek et al. 2015a], though as we will see the Hazelnut Live internal language goes beyond the cast calculus in several respects. We have mechanized these formal developments using the Agda proof assistant [Norell 2007, 2009] (see Sec. 3.4). Rule names in this section, e.g. SVar, correspond to variable names used in the mechanization. The Hazel implementation substantially follows the formal specification of Hazelnut (for the editor component) and Hazelnut Live (for the semantics), and we can formally state a continuity invariant for a putative combined calculus (see Sec. 3.5).
τ₁ ∼ τ₂  τ₁ is consistent with τ₂

\[ \begin{align*}
\text{TCHole1} & : (\emptyset) \sim \tau \\
\text{TCHole2} & : \tau \sim (\emptyset) \\
\text{TCRefl} & : \tau \sim \tau \\
\text{TCArr} & : \tau_1 \sim \tau_1' \quad \tau_2 \sim \tau_2' \\
\end{align*} \]

τ → τ₁ → τ₂  τ has matched arrow type τ₁ → τ₂

\[ \begin{align*}
\text{MAHole} & : (\emptyset) \triangleright (\emptyset) \rightarrow (\emptyset) \\
\text{MAArr} & : \tau_1 \rightarrow \tau_2 \triangleright \tau_1 \rightarrow \tau_2 \\
\end{align*} \]

Fig. 7. Type Consistency and Matching

3.1 Static Semantics of the External Language

We start with the type system of the Hazelnut Live external language, which closely follows the Hazelnut type system [Omar et al. 2017a]; we summarize the minor differences as they come up.

Fig. 6 defines the type system in the bidirectional style with two mutually defined judgements [Chlipala et al. 2005; Christiansen 2013; Dunfield and Krishnaswami 2013; Pierce and Turner 2000]. The type synthesis judgement \( \Gamma \vdash e \Rightarrow \tau \) synthesizes a type \( \tau \) for external expression \( e \) under typing context \( \Gamma \), which tracks typing assumptions of the form \( x : \tau \) in the usual manner [Harper 2016; Pierce 2002]. The type analysis judgement \( \Gamma \vdash e \Leftarrow \tau \) checks expression \( e \) against a given type \( \tau \). Algorithmically, analysis accepts a type as input, and synthesis gives a type as output. We start with synthesis for the programmer’s “top level” external expression.

The primary benefit of specifying the Hazelnut Live external language bidirectionally is that the programmer need not annotate each hole with a type. An empty hole is written simply \( (\emptyset) \), where \( u \) is the hole name, which we tacitly assume is unique (holes in Hazelnut were not named). Rule \( \text{SEHole} \) specifies that an empty hole synthesizes hole type, written \( (\emptyset) \). If an empty hole appears where an expression of some other type is expected, e.g., under an explicit ascription (governed by Rule \( \text{SAsc} \)) or in the argument position of a function application (governed by Rule \( \text{SAp} \), discussed below), we apply the subsumption rule, Rule \( \text{ASubsume} \), which specifies that if an expression \( e \) synthesizes type \( \tau \), then it may be checked against any consistent type, \( \tau' \).

Fig. 7 specifies the type consistency relation, written \( \tau \sim \tau' \), which specifies that two types are consistent if they differ only up to type holes in corresponding positions. The hole type is consistent with every type, and so, by the subsumption rule, expression holes may appear where an expression of any type is expected. The type consistency relation here coincides with the type consistency relation from gradual type theory by identifying the hole type with the unknown type [Siek and Taha 2006]. Type consistency is reflexive and symmetric, but it is not transitive. This stands in contrast to subtyping, which is anti-symmetric and transitive; subtyping may be integrated into a gradual type system following Siek and Taha [2007].

Non-empty expression holes, written \( (e)^n \), behave similarly to empty holes. Rule \( \text{SNEHole} \) specifies that a non-empty expression hole also synthesizes hole type as long as the expression inside the hole, \( e \), synthesizes some (arbitrary) type. Non-empty expression holes therefore internalize the “red underline/outline” that many editors display around type inconsistencies in a program.

For the familiar forms of the lambda calculus, the rules again follow prior work. For simplicity, the core calculus includes only a single base type \( b \) with a single constant \( c \), governed by Rule \( \text{SConst} \) (i.e., \( b \) is the unit type). By contrast, Omar et al. [2017a] instead defined a number type with a single operation. That paper also defined sum types as an extension to the core calculus. We follow suit on both counts in the appendix.
Rule SVar synthesizes the corresponding type from \( \Gamma \). For the sake of exposition, Hazelnut Live includes “half-annotated” lambdas, \( \lambda x : \tau . e \), in addition to the unannotated lambdas, \( \lambda x . e \), from Hazelnut. Half-annotated lambdas may appear in synthetic position according to Rule SLam, which is standard [Chlipala et al. 2005]. Unannotated lambdas may only appear where the expected type is known to be either an arrow type or the hole type, which is treated as if it were \( \{ \} \rightarrow \{ \} \). To avoid the need for two separate rules, Rule ALam uses the matching relation \( \tau \rightarrow_{\text{t1}} \tau_{\text{t2}} \) defined in Fig. 7, which produces the matched arrow type \( \{ \} \rightarrow \{ \} \) given the hole type, and operates as the identity on arrow types [Garcia and Cimini 2015; Siek et al. 2015a].

The rule governing function application, Rule SAp, similarly treats an expression of hole type in function position as if it were of type \( \{ \} \rightarrow \{ \} \) using the same matched arrow type judgement.

We do not formally need an explicit fixpoint operator because this calculus supports general recursion due to type holes, e.g. we can express the Y combinator as \( (\lambda x : \{ \} . x(x))((\lambda x : \{ \} . x(x)) \). More generally, the untyped lambda calculus can be embedded as described by Siek and Taha [2006].

### 3.2 Expansion

Each well-typed external expression \( e \) expands to a well-typed internal expression \( d \), for evaluation. Fig. 8 specifies expansion, and Fig. 9 specifies type assignment for internal expressions.

As with the type system for the external language (above), we specify expansion bidirectionally [Ferreira and Pientka 2014]. The synthetic expansion judgement \( \Gamma \vdash e \Rightarrow \tau \leadsto d + \Delta \) produces an expansion \( d \) and a hole context \( \Delta \) when synthesizing type \( \tau \) for \( e \). We describe hole contexts, which serve as “inputs” to the type assignment judgement \( \Gamma \vdash d : \tau \), further below. The analytic expansion judgement \( \Gamma \vdash e \Leftarrow \tau \leadsto d : \tau' + \Delta \), produces an expansion \( d \) of type \( \tau' \), and a hole context \( \Delta \), when checking \( e \) against \( \tau \). The following theorem establishes that expansions are well-typed and in the analytic case that the assigned type, \( \tau' \), is consistent with provided type, \( \tau \).

**Theorem 3.1 (Typed Expansion).**

(1) If \( \Gamma \vdash e \Rightarrow \tau \leadsto d + \Delta \) then \( \Gamma \vdash d : \tau \).

(2) If \( \Gamma \vdash e \Leftarrow \tau \leadsto d : \tau' + \Delta \) then \( \tau \sim \tau' \) and \( \Gamma \vdash d : \tau' \).

The reason that \( \tau' \) is only consistent with the provided type \( \tau \) is because the subsumption rule permits us to check an external expression against any type consistent with the type that the expression actually synthesizes, whereas every internal expression can be assigned at most one type, i.e. the following standard unicity property holds of the type assignment system.

**Theorem 3.2 (Type Assignment Unicity).** If \( \Delta ; \Gamma \vdash d : \tau \) and \( \Delta ; \Gamma \vdash d : \tau' \) then \( \tau = \tau' \).

Consequently, analytic expansion reports the type actually assigned to the expansion it produces. For example, we can derive that \( \Gamma \vdash c \Leftarrow \{ \} \leadsto c : b + \{ \} \).

Before describing the rules in detail, let us state two other guiding theorems. The following theorem establishes that an expansion exists for every well-typed external expression. The mechanization also establishes that when an expansion exists, it is unique (not shown).

**Theorem 3.3 (Expandability).**

(1) If \( \Gamma \vdash e \Rightarrow \tau \) then \( \Gamma \vdash e \Rightarrow \tau \leadsto d + \Delta \) for some \( d \) and \( \Delta \).

(2) If \( \Gamma \vdash e \Leftarrow \tau \leadsto d : \tau' + \Delta \) for some \( d \) and \( \tau' \) and \( \Delta \).

The following theorem establishes that expansion generalizes external typing.

**Theorem 3.4 (Expansion Generalality).**

(1) If \( \Gamma \vdash e \Rightarrow \tau \leadsto d + \Delta \) then \( \Gamma \vdash e \Rightarrow \tau \).

(2) If \( \Gamma \vdash e \Leftarrow \tau \leadsto d : \tau' + \Delta \) then \( \Gamma \vdash e \Leftarrow \tau \).

---

3 A system supporting ML-style type reconstruction [Damas and Milner 1982] might also include a synthetic rule for unannotated lambdas, e.g. as outlined by Dunfield and Krishnaswami [2013], but we stick to this simpler “Scala-style” local type inference scheme in this paper [Odersky et al. 2001; Pierce and Turner 2000].
\[ \Gamma \vdash e \Rightarrow \tau \leadsto d + \Delta \]  

\( e \) synthesizes type \( \tau \) and expands to \( d \)

\[
\begin{align*}
\text{ESVar} & : \quad x : \tau \in \Gamma \\
\Gamma \vdash c & \Rightarrow b \leadsto c + \emptyset \\
\Gamma \vdash x & \Rightarrow \tau \leadsto x + \emptyset \\
\text{ESLam} & : \quad \Gamma, x : \tau_1 \vdash e \Rightarrow \tau_2 \leadsto d + \Delta \\
\Gamma & \vdash \lambda x : \tau_1. e \Rightarrow \Delta \leadsto \lambda x : \tau_1. d + \Delta \\
\end{align*}
\]

\[
\begin{align*}
\text{ESAp} & : \quad \\
\Gamma \vdash e_1 & \Rightarrow \tau_1 \\
\tau_1 & \Rightarrow \tau_2 \Rightarrow \tau \\
\Gamma \vdash e_1 & \Leftarrow \tau_2 \Rightarrow d_1 : \tau_1' + \Delta_1 \\
\Gamma & \vdash e_2 \Leftarrow \tau_2 \Rightarrow d_2 : \tau_2' + \Delta_2 \\
\Gamma \vdash e_1(e_2) & \Rightarrow \tau \leadsto (d_1(\tau_1' \Rightarrow \tau_2 \Rightarrow \tau))((d_2(\tau_2' \Rightarrow \tau_2)) + \Delta_1 \cup \Delta_2) \\
\end{align*}
\]

\[
\begin{align*}
\text{ESEHole} & : \quad \\
\Gamma & \vdash \emptyset \Rightarrow \emptyset \leadsto \emptyset + \Delta \\
\Gamma & \vdash \emptyset \Rightarrow \emptyset \Rightarrow \emptyset + \Delta \\
\end{align*}
\]

\[
\begin{align*}
\text{ESNEHole} & : \quad \\
\Gamma & \vdash \emptyset \Rightarrow \emptyset \Rightarrow \emptyset + \Delta \\
\Gamma & \vdash \emptyset \Rightarrow \emptyset \Rightarrow \emptyset + \Delta \\
\end{align*}
\]

\[
\begin{align*}
\text{ESAsc} & : \quad \\
\Gamma & \vdash e \Leftarrow \tau \Rightarrow \tau \Rightarrow d : \tau' + \Delta \\
\Gamma & \vdash e \Leftarrow \tau \Rightarrow \tau \Rightarrow d : \tau' + \Delta \\
\end{align*}
\]

\[
\begin{align*}
\text{EALam} & : \quad \\
\tau & \Rightarrow \tau' \Rightarrow \tau_2 \\
\Gamma, x : \tau_1 & \vdash e \Leftarrow \tau_2 \Rightarrow d : \tau_1' + \Delta \\
\Gamma & \vdash \lambda x : \tau_1.e \Rightarrow \tau_2 \Rightarrow \tau_1' + \Delta \\
\end{align*}
\]

\[
\begin{align*}
\text{EASubsume} & : \quad \\
e & \not \equiv \emptyset^u \\
\Gamma & \vdash e \Rightarrow \tau' \Rightarrow d + \Delta \\
\Gamma & \vdash e \Rightarrow \tau' \Rightarrow d + \Delta \\
\end{align*}
\]

\[
\begin{align*}
\text{EASubsume} & : \quad \\
e & \not \equiv \emptyset^u \\
\Gamma & \vdash e \Rightarrow \tau' \Rightarrow d + \Delta \\
\Gamma & \vdash e \Rightarrow \tau' \Rightarrow d + \Delta \\
\end{align*}
\]

\[
\begin{align*}
\text{EANEd} & : \quad \\
\Gamma & \vdash \emptyset \Rightarrow \emptyset \Rightarrow \emptyset + \Delta \\
\Gamma & \vdash \emptyset \Rightarrow \emptyset \Rightarrow \emptyset + \Delta \\
\end{align*}
\]

\[
\begin{align*}
\text{EANEd} & : \quad \\
\Gamma & \vdash \emptyset \Rightarrow \emptyset \Rightarrow \emptyset + \Delta \\
\Gamma & \vdash \emptyset \Rightarrow \emptyset \Rightarrow \emptyset + \Delta \\
\end{align*}
\]

\[
\begin{align*}
\text{TACast} & : \quad \\
\Delta & \vdash d : \tau_1 \Rightarrow \tau_2 \Rightarrow \tau_1 \Rightarrow \tau_2 \\
\Delta & \vdash d : \tau_1 \Rightarrow \tau_2 \Rightarrow \tau_1 \Rightarrow \tau_2 \\
\end{align*}
\]

Fig. 8. Expansion

\[
\begin{align*}
\Delta ; \Gamma & \vdash d : \tau \\
\text{d is assigned type } \tau \\
\end{align*}
\]

\[
\begin{align*}
\text{TACast} & : \quad \\
\Delta ; \Gamma & \vdash d : \tau_1 \Rightarrow \tau_2 \Rightarrow \tau_1 \Rightarrow \tau_2 \\
\Delta ; \Gamma & \vdash d : \tau_1 \Rightarrow \tau_2 \Rightarrow \tau_1 \Rightarrow \tau_2 \\
\end{align*}
\]

Fig. 9. Type Assignment for Internal Expressions
The rules governing expansion of constants, variables and lambda expressions — Rules ESConst, ESVar, ESLam and EALam — mirror the corresponding type assignment rules — Rules TACst, TAVar and TALam — and in turn, the corresponding bidirectional typing rules from Fig. 6. To support type assignment, all lambdas in the internal language are half-annotated—Rule EALam inserts the annotation when expanding an unannotated external lambda based on the given type. The rules governing hole expansion, and the rules that perform cast insertion—those governing function application and type ascription—are more interesting. Let us consider each of these two groups of rules in turn in Sec. 3.2.1 and Sec. 3.2.2, respectively.

3.2.1 Hole Expansion. Rules ESEHole, ESNEHole, EAEHole and EANEHole govern the expansion of empty and non-empty expression holes to empty and non-empty hole closures, \( \emptyset^u \) and \( d^u \). The hole name \( u \) on a hole closure identifies the external hole to which the hole closure corresponds. While we assume each hole name to be unique in the external language, once evaluation begins, where it appears in the original expression.

Formally, we define this relation in terms of type assignment as follows:

\[ \Delta \vdash \text{id}(\Gamma) = \Delta ; \Gamma \vdash d : \tau \]

\[ \text{we have that } d/x \in \sigma \text{ and } \Delta ; \Gamma \vdash d : \tau. \]

It is easy to verify that the identity substitution satisfies this requirement, i.e. that \( \Delta ; \Gamma \vdash \text{id}(\Gamma) : \Gamma \).

Empty hole closures, \( \emptyset^u \), correspond to the metavariable closures (a.k.a. deferred substitutions) from CMTT, \( \text{clo}(u, \sigma) \). Sec. 3.3 defines how these closure environments evolve during evaluation. Non-empty hole closures \( d^u \) have no direct correspondence with a notion from CMTT (see Sec. 4).

---

4 We use a hole context, rather than recording the typing context and type directly on each hole closure, to ensure that all closures for a hole name have the same typing context and type.
3.2.2 **Cast Insertion.** Holes in types require us to defer certain structural checks to run time. To see why this is necessary, consider the following example: \((\lambda x: \top. x(c))(c)\). Viewed as an external expression, this example synthesizes type \(\top\), since the hole type annotation on variable \(x\) permits applying \(x\) as a function of type \(\top \to \top\), and base constant \(c\) may be checked against type \(\top\), by subsumption. However, viewed as an internal expression, this example is not well-typed—the type assignment system defined in Fig. 9 lacks subsumption. Indeed, it would violate type safety if we could assign a type to this example in the internal language, because beta reduction of this example viewed as an internal expression would result in \(c(c)\), which is clearly not well-typed. The difficulty arises because leaving the argument type unknown also leaves unknown how the argument is being used (in this case, as a function).\(^5\) By our interpretation of hole types as unknown types from gradual type theory, we can address the problem by performing cast insertion.

The cast form in Hazelnut Live is \(d(\tau_1 \Rightarrow \tau_2)\). This form serves to “box” an expression of type \(\tau_1\) for treatment as an expression of a consistent type \(\tau_2\) (Rule \(\text{TACast}\) in Fig. 9).\(^6\)

Expansion inserts casts at function applications and ascriptions. The latter is more straightforward: Rule \(\text{ESAsc}\) in Fig. 8 inserts a cast from the assigned type to the ascribed type. Theorem 3.1 inductively ensures that the two types are consistent. We include ascription for exposition purposes—this form is derivable by using application together with the half-annotated identity, \(e : \tau = (\lambda x: \tau. x)(e)\); as such, application expansion, discussed below, is more general.

Rule \(\text{ESAp}\) expands function applications. To understand the rule, consider the expansion of external expression \((\lambda x: \top. x(c))(c)\), the example discussed above:

\[
(\lambda x: \top. x(\top)) \Rightarrow (\top \to \top)(c(b \Rightarrow \top))(\top \to \top \Rightarrow \top \Rightarrow \top)(c(b \Rightarrow \top))
\]

Consider the (indicated) function body, where expansion inserts a cast on both the function expression \(x\) and its argument \(c\). Together, these casts for \(x\) and \(c\) permit assigning a type to the function body according to the rules in Fig. 9, where we could not do so under the same context without casts. We separately consider the expansions of \(x\) and of \(c\).

First, consider the function position of this application, here variable \(x\). Without any cast, the type of variable \(x\) is the hole type \(\top\); however, the inserted cast on \(x\) permits treating it as though it has arrow type \(\top \to \top\). The first three premises of Rule \(\text{ESAp}\) accomplish this by first synthesizing a type for the function expression, here \(\top\), then by determining the matched arrow type \(\top \to \top\), and finally, by performing analytic expansion on the function expression with this matched arrow type. The resulting expansion has some type \(\tau_1^*\) consistent with the matched arrow type. In this case, because the subexpression \(x\) is a variable, analytic expansion goes through subsumption so that type \(\tau_1^*\) is simply \(\top\). The conclusion of the rule inserts the corresponding cast. We go through type synthesis, then analytic expansion, so that the hole context records the matched arrow type for holes in function position, rather than the type \(\top\) for all such holes, as would be the case in a variant of this rule using synthetic expansion for the function expression.

Next, consider the application’s argument, here constant \(c\). The conclusion of Rule \(\text{ESAp}\) inserts the cast on the argument’s expansion, from the type it is assigned by the final premise of the rule (type \(b\) here), to the argument type of the matched arrow type of the function expression (type \(\top\) here).

The example’s second, outermost application goes through the same application expansion rule. In this case, the cast on the function is the identity cast for \(\top \Rightarrow \top\). For simplicity, we do not

---

\(^5\)In a system where type reconstruction is first used to try to fill in type holes, we could express a similar example by using \(x\) at two or more different types, thereby causing type reconstruction to fail.

\(^6\)In the earliest work on gradual type theory, the cast form only gave the target type \(\tau_2\) [Siek and Taha 2006], but it simplifies the dynamic semantics substantially to include the assigned type \(\tau_1\) in the syntax [Siek et al. 2015a].
attempt to avoid the insertion of identity casts in the core calculus; these will simply never fail during evaluation. However, it is safe in practice to eliminate such identity casts during expansion, and some formal accounts of gradual typing do so by defining three application expansion rules, including the original account of Siek and Taha [2006].

### 3.3 Dynamic Semantics

To recap, the result of expansion is a well-typed internal expression with appropriately initialized hole closures and casts. This section specifies the dynamic semantics of Hazelnut Live as a “small-step” transition system over internal expressions equipped with a meaningful notion of type safety even for incomplete programs, i.e. expressions typed under a non-empty hole context, $\Delta$. We establish that evaluation does not stop immediately when it encounters a hole, nor when a cast fails, by precisely characterizing when evaluation does stop. We also establish that an essentially standard notion of type safety holds when running complete programs.

It is perhaps worth stating at the outset that a dynamic semantics equipped with these properties does not simply “fall out” from the observations made above that (1) empty hole closures correspond to metavariable closures from CMTT [Nanevski et al. 2008] and (2) casts also arise in gradual type theory [Siek et al. 2015a]. We say more in Sec. 5.

Figures 10-13 define the dynamic semantics. Most of the cast-related machinery closely follows the cast calculus from the “refined” account of the gradually typed lambda calculus by Siek et al. [2015a], which is known to be theoretically well-behaved. In particular, Fig. 10 defines the judgement $\tau$ ground, which distinguishes the base type $b$ and the least specific arrow type $\langle \rangle \rightarrow \langle \rangle$ as ground types; this judgement helps simplify the treatment of function casts, discussed below.

Fig. 11 defines the judgement $d$ final, which distinguishes the final, i.e. irreducible, forms of the transition system. The two rules distinguish two classes of final forms: (possibly-)boxed values and indeterminate forms. The judgement $d$ boxedval defines (possibly-)boxed values as either ordinary values (Rule BVVal), or one of two cast forms: casts between disequal function types and casts from a ground type to the hole type. In each case, the cast must appear inductively on a boxed value. These forms are irreducible because they represent values that have been boxed but have never flowed into a corresponding “unboxing” cast, discussed below. The judgement $d$ indet defines indeterminate forms, so named because they are rooted at expression holes and failed casts, and so, conceptually, their ultimate value awaits programmer action (see Sec. 4). The first two rules specify that empty hole closures are always indeterminate, and that non-empty hole closures are indeterminate when they consist of a final inner expression. Below, we describe failed casts and the remaining indeterminate forms simultaneously with the corresponding transition rules.

Figures 12-13 define the transition rules. Top-level transitions are steps, $d \rightarrow d'$, governed by Rule Step in Fig. 13, which (1) decomposes $d$ into an evaluation context, $E$, and a selected sub-term, $d_0$; (2) takes an instruction transition, $d_0 \rightarrow d_0'$, as specified in Fig. 12; and (3) places $d_0'$ back at the selected position, indicated in the evaluation context by the mark, $\circ$, to obtain $d'$. This approach was originally developed in the reduction semantics of Felleisen and Hieb [1992] and is the predominant style of operational semantics in the literature on gradual typing. Because we distinguish final forms judgementally, rather than syntactically, we use a judgemental formulation of this approach called a contextual dynamics by Harper [2016]. It would be straightforward to construct an equivalent structural operational semantics [Plotkin 2004] by using search rules instead of evaluation contexts (Harper [2016] relates the two approaches).

---

7 Most accounts of the cast calculus distinguish ground types and values with separate grammars together with an implicit identification convention. Our judgemental formulation is more faithful to the mechanization and cleaner for our purposes, because we are distinguishing several classes of final forms.

8 We say “mark”, rather than the more conventional “hole”, to avoid confusion with the (orthogonal) holes of Hazelnut Live.
The rules maintain the property that final expressions truly cannot take a step.

**Theorem 3.7 (Finality).** There does not exist \(d\) such that both \(d\) final and \(d \rightarrow d'\) for some \(d'\).

### 3.3.1 Application and Substitution

Rule ITLam in Fig. 12 defines the standard beta reduction transition. The bracketed premises of the form \([d\) final] in Fig. 12-13 may be included to specify an eager, left-to-right evaluation strategy, or excluded to leave the evaluation strategy and order unspecified. In our metatheory, we exclude these premises, both for the sake of generality, and to support the specification of the fill-and-resume operation (see Sec. 4).

Substitution, written \([d/x]d'\), operates in the standard capture-avoiding manner [Harper 2016] (see the appendix for the full definition). The only cases of special interest arise when substitution reaches a hole closure:

\[
\begin{align*}
[d/x] []^u_\sigma &= []^u_\sigma | [d/x] \sigma \\
[d/x] (d'^u_\sigma) &= (d'^u_\sigma) | [d/x] d'^u_\sigma \sigma
\end{align*}
\]
In both cases, we write \([d/x]\sigma\) to perform substitution on each expression in the hole environment \(\sigma\), i.e. the environment "records" the substitution. For example, \((\lambda x: \tau. b). (\lambda y: \tau'. c) \mapsto \lambda y: \tau'. b. (\lambda x: \tau''/y). (\lambda y: \tau'''). (c)\) \mapsto \lambda y: \tau'. b. (\lambda x: \tau''/y). (\lambda y: \tau'''). \) Beta reduction can duplicate hole closures. Consequently, the environments of different closures with the same hole name may differ, e.g., when a reduction applies a function with a hole closure body multiple times as in Fig. 1. Hole closures may also appear within the environments of other hole closures, giving rise to the closure paths described in Sec. 2.2.

The ITLam rule is not the only rule we need to handle function application, because lambdas are not the only final form of arrow type. Two other situations may also arise. First, the expression in function position might be a cast between arrow types, in which case we apply the arrow cast conversion rule, Rule ITApCast, to rewrite the application form, obtaining an equivalent application where the expression \(d_1\) under the function cast is exposed. We know from inverting the typing rules that \(d_1\) has type \(\tau_1 \rightarrow \tau_2\), and that \(d_2\) has type \(\tau'_1 \rightarrow \tau'_2\), where \(\tau_1 \sim \tau'_1\). Consequently, we maintain type safety by placing a cast on \(d_2\) from \(\tau'_1\) to \(\tau_1\). The result of this application has type \(\tau_2\), but the original cast promised that the result would have consistent type \(\tau'_2\), so we also need a cast on the result from \(\tau_2\) to \(\tau'_2\).

Second, the expression in function position may be indeterminate, where arrow cast conversion is not applicable, e.g. \((\lambda y. c)\). In this case, the application is indeterminate (Rule ITAp in Fig. 11), and the application reduces no further.

### 3.3.2 Casts

Rule ITCastId strips identity casts. The remaining instruction transition rules assign meaning to non-identity casts. As discussed in Sec. 3.2.2, the structure of a term cast to hole type is statically obscure, so we must await a use of the term at some other type, via a cast away from hole type, to detect the type error dynamically. Rules ITCastSucceed and ITCastFail handle this situation when the two types involved are ground types (Fig. 10). If the two ground types are equal, then the cast succeeds and the cast may be dropped. If they are not equal, then the cast fails and the failed cast form, \(d(\tau_1 \Rightarrow \tau_2)\), arises. Rule IFailedCast specifies that a failed cast is well-typed exactly when \(d\) has ground type \(\tau_1\) and \(\tau_2\) is a ground type disjunct to \(\tau_1\). Rule IFailedCast specifies that a failed cast operates as an indeterminate form (once \(d\) is final), i.e. evaluation does not stop. For simplicity, we do not include blame labels as found in some accounts of gradual typing [Siek et al. 2015a; Wadler and Findler 2009], but it would be straightforward to do so by recording the blame labels from the two constituent casts on the two arrows of the failed cast.
The key to establishing the progress theorem under a non-empty hole context is to explicitly account for indeterminate forms, i.e. those rooted at either a hole closure or a failed cast. The proof relies on a preservation lemma for instruction transitions and a standard substitution lemma stated in the appendix. Hole closures can disappear during evaluation, so we rely on simple weakening of \( \Delta \) rules just described only operate at ground type. The two remaining instruction transition rules, Rule ITGround and ITExpand, insert intermediate casts from non-ground type to a consistent ground type, and \textit{vice versa}. These rules serve as technical devices, permitting us to restrict our interest exclusively to casts involving ground types and type holes elsewhere. Here, the only non-ground types are the arrow types, so the grounding device, permitting us to restrict our interest exclusively to casts involving ground types and vice versa.

\textbf{Theorem 3.9} (Progress). If \( \Delta; \emptyset \vdash d : \tau \) then \( \Delta; \emptyset \vdash d' : \tau \).

\textbf{Theorem 3.10} (Preservation). If \( \Delta; \emptyset \vdash d : \tau \) then either (a) \( d \rightsquigarrow d' \) or (b) \( d \) boxedval or (c) \( d \) indet.

The key to establishing the progress theorem under a non-empty hole context is to explicitly account for indeterminate forms, i.e. those rooted at either a hole closure or a failed cast. The proof relies on canonical forms lemmas stated in the appendix.
Complete Programs. Although this paper focuses on running incomplete programs, it helps to know that the necessary machinery does not interfere with running complete programs, i.e. those with no type or expression holes. The appendix defines the predicates e complete, \( \tau \) complete, \( \Gamma \) complete and \( d \) complete. Of note, failed casts cannot appear in complete internal expressions. The following theorem establishes that expansion preserves program completeness.

**Theorem 3.11** (Complete Expansion). If \( \Gamma \) complete and \( e \) complete and \( \Gamma \vdash e \Rightarrow \tau \leadsto d + \Delta \) then \( \tau \) complete and \( d \) complete and \( \Delta = \emptyset \).

The following preservation theorem establishes that stepping preserves program completeness.

**Theorem 3.12** (Complete Preservation). If \( \Delta, \emptyset \vdash d : \tau \) and \( d \) complete and \( d \leftrightarrow d' \) then \( \Delta, \emptyset \vdash d' : \tau \) and \( d' \) complete.

The following progress theorem establishes that evaluating a complete program always results in classic values, not boxed values nor indeterminate forms.

**Theorem 3.13** (Complete Progress). If \( \Delta, \emptyset \vdash d : \tau \) and \( d \) complete then either \( d \leftrightarrow d' \) or \( d \) val.

### 3.4 Agda Mechanization

The supplemental material includes our Agda mechanization [Aydemir et al. 2005; Norell 2007, 2009] of the semantics and metatheory of Hazelnut, including all of the theorems stated above and necessary lemmas. We take as assumptions only a few standard properties of general hypothetical judgements, substitution and finite maps. Our approach is otherwise standard: we model judgements as inductive datatypes, and rules as dependently typed constructors of these judgements. We adopt Barendregt’s convention for bound variables [Barendregt 1984; Urban et al. 2007] and hole names and encode typing contexts, hole contexts and finite substitutions using metafunctions. To support this encoding choice, we postulate function extensionality (which is independent of Agda’s axioms) [Awodey et al. 2012]. The documentation provided with the mechanization has more details.

### 3.5 Implementation and Continuity

The Hazel implementation demonstrated in Sec. 2 includes an unoptimized interpreter, written in OCaml, that implements the semantics as described in this section, albeit for an Elm-like language that goes somewhat beyond the simple calculus here. As with many full-scale systems, there is not currently a formal specification for the full Elm or Hazel language, but the appendix discusses how the standard approach for deriving a “gradualized” version of a language construct provides most of the necessary scaffolding [Cimini and Siek 2016], and provides some examples.

The editor component of the Hazel implementation is derived from the structure editor calculus of Hazelnut, but with support for more natural cursor-based movement and infix operator sequences (the details of which are beyond the scope of this paper). It exposes a language of structured edit actions that automatically insert empty and non-empty holes as necessary to guarantee that every edit state has some (possibly incomplete) type. This corresponds to the top-level Sensibility invariant established for the Hazelnut calculus by Omar et al. [2017a], reproduced below:

**Proposition 3.14** (Sensibility). If \( \Gamma \vdash \hat{e} \Rightarrow \tau \) and \( \Gamma \vdash \hat{e} \Rightarrow \tau \xrightarrow{\alpha} \hat{e}' \Rightarrow \tau' \) then \( \Gamma \vdash \hat{e}^\circ \Rightarrow \tau' \).

Here, \( \hat{e} \) is an editor state (an expression with a cursor), and \( \hat{e}^\circ \) drops the cursor, producing an expression (\( e \) in this paper). So in words, “if, ignoring the cursor, the editor state, \( \hat{e}^\circ \), initially has type \( \tau \) and we perform an edit action \( \alpha \) on it, then the resulting editor state, \( \hat{e}^\circ \), will have type \( \tau' \).”

By composing this Sensibility property with the Expandability, Typed Expansion, Progress and Preservation properties from this section, we establish a uniquely powerful Continuity invariant:

**Corollary 3.15** (Continuity). If \( \emptyset \vdash \hat{e} \Rightarrow \tau \) and \( \emptyset \vdash \hat{e} \Rightarrow \tau \xrightarrow{\alpha} \hat{e}' \Rightarrow \tau' \) then \( \emptyset \vdash \hat{e}^\circ \Rightarrow \tau' \leadsto d + \Delta \) for some \( \hat{d} \) and \( \Delta \) such that \( \Delta, \emptyset \vdash \hat{d} : \tau' \) and either (a) \( \hat{d} \leftrightarrow \hat{d}' \) for some \( \hat{d}' \) such that \( \Delta, \emptyset \vdash \hat{d}' : \tau' \); or (b) \( d \) boxedval or (c) \( d \) indet.
When the programmer performs one or more edit actions to fill in a hole in the program, a new which we call fill-and-resume operation would not operate as expected under this interpretation.

Fig. 14. Hole Filling

This addresses the gap problem: every editor state has a static meaning (so editor services like the type inspector from Fig. 2b are always available) and a non-trivial dynamic meaning (a result is always available, evaluation does not stop when a hole or cast failure is encountered, and editor services that rely on hole closures, like the live context inspector from Fig. 1b, are always available).

In settings where the editor does not maintain this Sensibility invariant, our approach still helps to reduce the severity of the gap problem, i.e. more editor states are meaningful, even if not all are.

4 A Contextual Modal Interpretation of Fill-and-Resume

When the programmer performs one or more edit actions to fill in a hole in the program, a new result must be computed, ideally quickly [Tanamoto 1990, 2013]. Naïvely, the system would need to compute the result “from scratch” on each such edit. For small exploratory programming tasks, recomputation is acceptable, but in cases where a large amount of computation might occur, e.g. in data science tasks, a more efficient approach is to resume evaluation from where it left off after an edit that amounts to hole filling. This section develops a foundational account of this feature, which we call fill-and-resume. This approach is complementary to, but distinct from, incremental computing (which is concerned with changes in input, not code insertions) [Hammer et al. 2014].

Formally, the key idea is to interpret hole environments as delayed substitutions. This is the same interpretation suggested for metavariable closures in contextual modal type theory (CMTT) by Nanevski et al. [2008]. Fig. 14 defines the hole filling operation $[d/u]d’$ based on the contextual substitution operation of CMTT. Unlike usual notions of capture-avoiding substitution, hole filling imposes no condition on the binder when passing into the body of a lambda expression—the expression that fills a hole can, of course, refer to variables in scope where the hole appears. When hole filling encounters an empty closure for the hole being instantiated, $[d/u]()$, the result is $[d/u]d’$. That is, we apply the delayed substitution to the fill expression $d$ after first recursively filling any instances of hole $u$ in $\sigma$. Hole filling for non-empty closures is analogous, where it discards the previously-enveloped expression. This case shows why we cannot interpret a non-empty hole as an empty hole of arrow type applied to the enveloped expression—the hole filling operation would not operate as expected under this interpretation.

The following theorem characterizes the static behavior of hole filling.

**Theorem 4.1 (Filling).** If $\Delta, u :: \tau'[\Gamma']$; $\Gamma \vdash d : \tau$ and $\Delta; \Gamma' \vdash d' : \tau'$ then $\Delta; \Gamma \vdash [d'/u]d : \tau$. 
Dynamically, the correctness of fill-and-resume depends on the following commutativity property: if there is some sequence of steps that go from \( d_1 \) to \( d_2 \), then one can fill a hole in these terms at either the beginning or at the end of that step sequence. We write \( d_1 \red→^* d_2 \) for the reflexive, transitive closure of stepping (see the appendix).

**Theorem 4.2 (Commutativity).** If \( \Delta, u :: \tau' | \Gamma' ; \emptyset \vdash d_1 : \tau \) and \( \Delta ; \Gamma' \vdash d' : \tau' \) and \( d_1 \red→^* d_2 \) then \( [d'/u]d_1 \red→^* [d'/u]d_2 \).

Critically, this property relies on the version of the semantics from Sec. 3 where evaluation order is unspecified (i.e. where we omit the bracketed premises). In general, resuming from \([d'/u]d_2\) will not reduce sub-expressions in the same order as a “fresh” left-to-right reduction sequence starting from \([d'/u]d_1\). In other words, this notion of fill-and-resume only works for languages where evaluation order “does not matter”. (There are various standard ways to formalize this intuition. For the sake of space, we review these in the appendix.)

We describe the proof, which is straightforward but involves a number of lemmas and definitions, in the appendix. In particular, care is needed to handle the situation where a now-filled non-empty hole had taken a step in the original evaluation.

We do not separately define hole filling in the external language (i.e. we consider a change to an external expression to be a hole filling if the new expansion differs from the previous expansion up to hole filling). In practice, it may be useful to cache more than one recent edit state to take full advantage of hole filling. As an example, consider two edits, the first filling a hole \( u \) with the number \( 2 \), and the next applying operator \(+\), resulting in \( 2 + \langle \rangle \sigma \). This second edit is not a hole filling edit with respect to the immediately preceding edit state, \( 2 \), but it can be understood as filling hole \( u \) from two states back with \( 2 + \langle \rangle \sigma \).

Hole filling also allows us to give a contextual modal interpretation to lab notebook cells like those of Jupyter/IPython [Pérez and Granger 2007] (and read-eval-print loops as a restricted case where edits to previous cells are impossible). Each cell can be understood as a series of \texttt{let} bindings ending implicitly in a hole, which is filled by the next cell. The live environment in the subsequent cell is exactly the hole environment of this implicit trailing hole. Hole filling when a subsequent cell changes avoids recomputing the environment from preceding cells, without relying on mutable state. Commutativity provides a reproducibility guarantee missing from Jupyter/IPython, where editing and executing previous cells can cause the state to differ substantially from the state that would result when attempting to run the notebook from the top.

## 5 Related and Future Work

This paper defined a dynamic semantics for incomplete functional programs, building on the static semantics developed by Omar et al. [2017a]. That paper, as well as a subsequent “vision paper” introducing Hazel, suggested as future work a corresponding dynamic semantics for the purposes of live programming without semantic gaps [Omar et al. 2017b]. Others have extensively argued the value of assigning meaning even to incomplete and erroneous programs [Bayne et al. 2011]. This paper delivers conceptual and type-theoretic details necessary to further advance this vision.

**Gradual Type Theory.** The semantics borrows machinery related to type holes from gradual type theory [Siek and Taha 2006; Siek et al. 2015a], as discussed at length in Sec. 3. The main innovation relative to this prior work is the treatment of cast failures like holes, rather than errors. Many of the methods developed to make gradual typing more expressive and practical are directly relevant to future directions for Hazel and other implementations of the ideas herein [Takikawa et al. 2015]. For example, there has been substantial work on the problem of implementing casts efficiently [Garcia 2013; Herman et al. 2010; Siek and Wadler 2010], and on integrating gradual typing with polymorphism [Devriese et al. 2018; Igarashi et al. 2017; Xie et al. 2018], and with refinement types.
Another direction that we leave to future work is the integration of these ideas into dependently typed proof assistants, where evaluation is primarily at the service of equational reasoning [Abel and Pientka 2010]. Several modern proof assistants, e.g. Agda [Norell 2007, 2009], Idris [Brady 2013] and Beluga [Pientka 2010] already provide syntactic and static support for holes (see Sec. 1) and interactive, hole-driven development is standard practice.

Another interesting future direction would be to move beyond pure functional programming and carefully integrate imperative features, e.g. ML-style references. Siek and Taha [2006] and Siek et al. [2015b] show how to incorporate such features into gradual type theory (by treating ref(τ) as invariant with respect to consistency); we expect that this approach would also work for the semantics of Sec. 3, i.e. it would conserve the type safety properties established there, with suitable modifications to account for a store. However, the commutativity property we establish in Sec. 4 will not hold for a language that supports non-commutative effects. We leave to future work the task of defining more restricted special cases of the fill-and-resume operation that respects a suitable commutativity property in effectful settings (perhaps by checkpointing).

Going beyond references to incorporate external effects, e.g. IO effects, raises some additional practical concerns, however—we do not want to continue past a hole or error and in so doing haphazardly trigger an unintended effect. In this setting, it is likely better to explicitly ask the programmer whether to continue evaluation past a hole or cast failure. The small step specification in this paper is suitable as the basis for a step-based evaluator like this. Whitington and Ridge [2017] discuss some unresolved usability issues relevant to single steppers for functional languages.

Contextual Modal Type Theory. The other major pillar of related work is the work on contextual modal type theory (CMTT) by Nanevski et al. [2008], which we also discussed at length throughout the paper. To reiterate, there is a close relationship between expression holes in this paper and metavariables in CMTT. Hole contexts correspond to modal contexts. Empty hole closures relate to the concept of a metavariable closure in CMTT, which consists of a metavariable paired with a substitution for all of the variables in the typing context associated with that metavariable. Hole filling relates to contextual substitution.

These connections with gradual typing and CMTT are instructive, but our contributions do not neatly fall out from this prior work. The problem is first that Nanevski et al. [2008] defined only the logical reductions for CMTT, viewing it as a proof system for intuitionistic contextual modal logic via the propositions-as-types (Curry-Howard) principle. The paper therefore proved only a subject reduction property (which is closely related to type preservation). This is not a full dynamic semantics, and in particular, there is no notion of progress, i.e. that well-typed terms cannot get “stuck” in an undefined state [Wright and Felleisen 1994]. In any case, a conventional dynamic semantics for CMTT would not be immediately relevant to our goal of evaluating incomplete programs because, by our interpretation of hole closures, we would need a dynamic semantics for terms with free metavariables (and, of course, with casts, i.e. gradual contextual modal type theory).

Nanevski et al. [2008] sketched an interpretation of CMTT into the simply-typed lambda calculus with sums under permutation conversion, which has been studied by de Groote [2002]. Permutation conversions are necessary to encode the commuting reductions of CMTT, which in turn are necessary to prove a strong normalization property. These issues are not relevant in Hazelnut Live because, as in gradual type theory, type holes admit non-termination [Siek and Taha 2006]. In any case, under this interpretation an analogous problem arises—metavariables become variables of a function type, so we cannot rely on the standard notion of progress on closed terms.

While there has been some formal work on reduction of open terms, leading to programs in a weak head normal form [Abel and Pientka 2010; Abramsky 1990; Barendregt 1984], there is no clear progress theorem in this setting: in the setting of an evaluator for a programming language,
we do not want to evaluate under arbitrary binders but rather only around holes when they would otherwise have been evaluated [Blanc et al. 2005]. Another important distinction is that simple evaluation of open terms will not support hole closure tracking, which is an essential component of our approach. CMTT makes the appropriate distinctions between variables and metavariables, and our notion of an indeterminate form allows us to prove a progress theorem of familiar form (where there are no free variables). Weak head normal forms are more useful when using evaluation to perform optimizations throughout a program, e.g. when using supercompilation-by-evaluation [Bolingbroke and Jones 2010] or symbolic evaluation [Baldoni et al. 2018; King 1976], or when using evaluation in the service of equational reasoning (e.g. in Beluga [Abel and Pientka 2010]).

It is also worth emphasizing that we use the machinery borrowed from CMTT only extralinguistically. A key feature of CMTT that we have not yet touched on is the internalization of metavariable binding and contextual substitution via the contextual modal types, $\Gamma \tau$, which are introduced by the operation box($\Gamma.d$) and eliminated by the operation letbox($d_1,u.d_2$). A hole filling can be interpreted as a value of contextual modal type, and the act of hole filling followed by evaluation to the next possibly-indeterminate edit state as evaluation under the binder of a suitable letbox construct, which is enabled by the dynamic semantics in Sec. 3. This interpretation could be generalized to allow us to compute a hole filling, rather than simply stating it, by specifying a non-trivial expression, rather than just a value, of modal type. This could, in turn, serve as the basis for a live computational hole refinement system, extending the capabilities of purely static hole refinement systems like those available in some proof assistants, e.g. the elaborator reflection system of Idris [Brady 2013; Christiansen and Brady 2016] and the refinement system of Beluga [Pientka 2010; Pientka and Cave 2015]. Each applied hole filling serves as a boundary between dynamic edit stages. This contextual modal interpretation of live staged hole refinement mirrors the modal interpretation of staged argument evaluation [Davies and Pfenning 2001]. This parallel neatly explains the difference between our dynamics and existing work on staging and partial evaluation—existing staging and partial evaluation systems are focused on partial evaluation with respect to an input that sits outside of a function [Jones et al. 1993], whereas holes have contexts.

There are various other systems similar in certain ways to CMTT in that they consider the problem of reasoning about metavariables. For example, McBride’s OLEG is another system for reasoning contextually about metavariables [McBride 2000], and it is the conceptual basis of certain hole refinement features in Idris [Brady 2013]. Geuvers and Jojgov [2002] discuss similar ideas. CMTT is unique relative to prior approaches in that it has a clear Curry-Howard interpretation and that it was designed with commutativity properties in mind. This paper bears out that the basic constructs of CMTT are ideally suited for a purpose previously not considered: live functional programming with typed expression holes, with commutative edits.

Work on explicit substitutions confronts problems similar to the problem of tracking the environment around a closure using delayed substitutions [Abadi et al. 1991, 1990; Lévy and Maranget 1999]. Abel and Pientka [2010] have developed a theory of explicit substitutions for CMTT, which, following other work on explicit substitutions and environmental abstract machines [Curien 1991], would be useful in implementing Hazelnut Live more efficiently by delaying both standard and contextual substitutions until needed during evaluation. We leave this and other questions of fast implementation (e.g. using thunks to encapsulate indeterminate sub-expressions) to future work.

**Type Error Messages.** A key feature of our semantics is that it permits evaluation not only of terms with empty holes, but also terms with non-empty holes, i.e. reified static type inconsistencies. DuctileJ [Bayne et al. 2011] and GHC [Vytiniotis et al. 2012] have also considered this problem, but have taken an “exceptional approach” — these systems can defer static type errors until run-time, but do not continue further once the term containing the error has been evaluated.
Understanding and debugging static type errors is notoriously difficult, particularly for novices. A variety of approaches have been proposed to better localize and explain type errors [Chen and Erwig 2014, 2018; Lerner et al. 2006; Pavlinovic et al. 2015; Zhang et al. 2017]. One of these approaches, by Seidel et al. [2016], uses symbolic execution and program synthesis to generate a dynamic witness that demonstrates a run-time failure that could be caused by the static type error. Hazelnut Live has similar motivations in that it can run programs with type errors and provide concrete feedback about the values that erroneous terms take during evaluation (Sec. 2.3-2.4). However, no attempt is made to synthesize examples that do not already appear in the program.

**Coroutines.** The fill-and-resume interaction is reminiscent of the interactions that occur when using coroutines [Kahn and MacQueen 1977] and related mechanisms (e.g. algebraic effects) – a coroutine might yield to the caller until a value is sent back in and then continue in the suspended environment. The difference is that the hole filling can make use of the context around the hole.

**Dynamic Error Propagation.** Hritcu et al. [2013] consider the problem of dynamic violations of information-flow control (IFC) policies. An exceptional approach here is problematic in practice, because it would lead critical systems to shutdown entirely. Instead, the authors develop several mechanisms for propagating errors through subsequent computations (in a manner that preserves non-interference properties). The most closely related is the NaV-lax approach, which turns errors into special "not-a-value" (NaV) terms that consume other terms that they interact with (like floating point NaN). Our approach differs in that terms are not consumed. Furthermore, we track hole closures and consider static and dynamic type errors, but not exception propagation.

**Debuggers.** Our approach is reminiscent of the workflow that debuggers make available using breakpoints [Fitzgerald et al. 2008; Tolmach and Appel 1995], visualizers of program state [Guo 2013; Nelson et al. 2017], and a variety of logging and tracing capabilities. Debuggers do not directly support incomplete programs, so the programmer first needs to insert suitable dummy exceptions as discussed in Sec. 1. Beyond that, there are two main distinctions. First, evaluation does not stop at each hole and so it is straightforward to explore the space of values that a variable takes. Second, breakpoints, logs and tracing tools convey the values of a variable at a position in the evaluation trace. Hole closures, on the other hand, convey information from a syntactic position in the result of evaluation. The result is, by nature, a simpler object than the trace.

Some debuggers support "edit-and-continue", e.g. Visual Studio [Jones 2017] and Flutter [Flutter Developers 2017], based on the dynamic software update (DSU) capabilities of the underlying run-time system [Hayden et al. 2012; Hicks and Nettles 2005; Stoyle et al. 2007]. These features do not come with any guarantee that rerunning the program will produce the same result.

**Structure Editors.** Holes play a prominent role in structure editors, and indeed the prior work on Hazelnut was primarily motivated by this application [Omar et al. 2017a]. Most work on structure editors has focused on the user interfaces that they present. This is important work—presenting a fluid user interface involving exclusively structural edit actions is a non-trivial problem that has not yet been fully resolved, though recent studies have started to show productivity gains for blocks-based structure editors like Scratch for novice programmers [Resnick et al. 2009; Weintrop et al. 2018; Weintrop and Wilensky 2015], and for keyboard-driven structure editors like mbeddr in professional programming settings [Asenov and Müller 2014; Voelter et al. 2012, 2014]. Some structure editors support live programming in various ways, e.g. Lamdu is a structure editor for a typed functional language that displays variable values inline based on recorded execution traces (see above) [Lotem and Chuchem 2016]. However, they do not generally support execution of incomplete programs, which was our focus in this paper. Scratch will execute a program with holes by simply skipping over incomplete commands, but this is a limited protocol because rerunning
the program after filling the hole may produce a result unrelated to initial result. Though in some situations, skipping over problematic commands has been observed to work surprisingly well [Rinard 2012], this paper contributes a semantically sound approach (Sec. 4).

We make no empirical claims regarding the usability of the particular user interface presented in this paper. The Hazel user interface as presented is a proof of concept that demonstrates (1) a comprehensive solution to the gap problem, as described in Sec. 3.5; and (2) one possible user interface for presenting hole closures, including recursive hole closures, to the programmer (Fig. 2c).

For larger programs, there are some limitations that call for further UI refinement. Indeterminate results can get to be quite large, particularly when a hole or failed cast appears in guard position as in Fig. 4. Although the pretty printer elides irrelevant branches, the semantics in Sec. 3 would also allow for evaluation to pause when such a form is produced, continuing only if requested.

In larger programs, there can be many dozens of variables in scope, so search and sorting tools in the live context inspector would be useful. It would also be useful to support the evaluation of arbitrary “watched” expressions in the selected closure. There are other ways to display the variable values, e.g. intercalated into the code [Lotem and Chuchem 2016; Sulír and Porubán 2018].

Finally, it is not always helpful to continue evaluation as far as possible. In particular, when there is a hole or cast failure in the argument position of a function application, it may often be more helpful to stop before beta reduction, to avoid exposing the internals of the applied function to the client. The semantics in Sec. 3 leaves evaluation order undefined, so various heuristic approaches are possible (e.g. we might wish avoid beta reduction for functions from imported libraries).

Program Slicing. Hole-like constructs also appear in work on program slicing [Perera et al. 2012; Ricciotti et al. 2017], where empty expression holes arise as a technical device to determine which parts of a complete program do not impact a selected part of the result of evaluating that program. In other words, the focus is on explaining a result that is presumed to exist, whereas the work in this paper is focused on producing results where they would not have otherwise been available. Combining the strengths of these approaches is another fruitful avenue for future research.

Program Synthesis. Expression holes also often appear in the context of program synthesis, serving as placeholders in templates [Srivastava et al. 2013] or sketches [Solar-Lezama 2009] to be filled in by an expression synthesis engine. We leave to future work the exciting possibility of combining these approaches. For example, one can imagine running an incomplete program as described in this paper and then adding tests or assertions about the value that a variable should take on in the different hole closures where it appears, via the live context inspector. These could serve as constraints for use by a type-and-example-driven synthesis engine [Frankle et al. 2016].

6 Conclusion

“[H]ow truly sad it is that just at the very moment when the computer has something important to tell us, it starts speaking gibberish.”

— Gerald Weinberg, The Psychology of Computer Programming (1971)

Weinberg’s sentiment applies to countless situations that programmers face as they work to develop and critically investigate the mental model of the program they are writing—at the very moment where rich feedback might be most helpful, e.g. when there is an error in the program or when the programmer is unsure how to fill a hole, the computer often has comparatively little feedback to offer (perhaps just a parser error message, or an explanation of one type error). It is our hope that the well-behaved type-theoretic foundations developed here will enable not only Hazel but live programming tools of a wide variety of other designs to further narrow the temporal and perceptive gap and provide meaningful feedback to programmers at these important moments.
REFERENCES

Martin Abadi, Luca Cardelli, Pierre-Louis Curien, and Jean-Jacques Lévy. 1991. Explicit Substitutions. J. Funct. Program. 1, 4 (1991), 375–416. https://doi.org/10.1017/S0956796800000186

M. Abadi, L. Cardelli, P.-L. Curien, and J.-J. Levy. 1990. Explicit substitutions. In POPL ’90: Proceedings of the 17th ACM SIGPLAN-SIGACT symposium on Principles of programming languages. 31–46.

Andreas Abel and Brigitte Pientka. 2010. Explicit Substitutions for Contextual Type Theory. In Proceedings 5th International Workshop on Logical Frameworks and Meta-languages: Theory and Practice, LFMTT 2010, Edinburgh, UK, 14th July 2010. (EPTCS), Karl Crary and Marino Miculan (Eds.), Vol. 34. 5–20. https://doi.org/10.4204/EPTCS.34.3

S. Abramsky. 1990. The Lazy Lambda Calculus. In Research Topics in Functional Programming. 65–116.

Alfred V. Aho and Thomas G. Peterson. 1972. A Minimum Distance Error-Correcting Parser for Context-Free Languages. SIAM J. Comput. 1, 4 (1972), 305–312. https://doi.org/10.1137/0201022

Luis Eduardo de Souza Amorim, Sebastian Erdweg, Guido Wachsmuth, and Elco Visser. 2016. Principled Syntactic Code Completion Using Placeholders. In ACM SIGPLAN International Conference on Software Language Engineering (SLE).

Dimitar Asenov and Peter Müller. 2014. Envision: A fast and flexible visual code editor with fluid interactions (Overview). In IEEE Symposium on Visual Languages and Human-Centric Computing, VL/HCC 2014.

Steve Awodey, Nicola Gambino, and Kristina Sojakova. 2012. Inductive types in homotopy type theory. In Proceedings of the 2012 27th Annual ACM/SIGLOG Conference on Logic in Computer Science. IEEE Computer Society, 95–104.

Brian E. Aydemir, Aaron Bohannon, Matthew Fairbairn, J. Nathan Foster, Benjamin C. Pierce, Peter Sewell, Dimitrios Vytiniotis, Geoffrey Washburn Stephanie Weirich, and Steve Zdancewic. 2005. Mechanized Metatheory for the Masses: The POPLmark Challenge. (2005).

Roberto Baldoni, Emilio Coppa, Daniele Cono D’Elia, Camil Demetrescu, and Irene Finocchi. 2018. A Survey of Symbolic Execution Techniques. ACM Comput. Surv. 51, 3. Article 50 (2018).

H.P. Barendregt. 1984. The Lambda Calculus. Vol. 103.

Michael Bayne, Richard Cook, and Michael D. Ernst. 2011. Always-available Static and Dynamic Feedback. In Proceedings 33rd International Conference on Software Engineering (ICSE ’11). ACM, New York, NY, USA, 521–530. https://doi.org/10.1145/1985793.1985864

Tomasz Blanc, Jean-Jacques Lévy, and Luc Maranget. 2005. Sharing in the Weak Lambda-Calculus. In Processes, Terms and Cycles: Steps on the Road to Infinity, Essays Dedicated to Jan Willem Klop, on the Occasion of His 60th Birthday (Lecture Notes in Computer Science), Aart Middeldorp, Vincent van Oostrom, Femke van Raamsdonk, and Roel C. de Vrijer (Eds.), Vol. 3838. Springer, 70–87. https://doi.org/10.1007/11601548_7

Maximilian C. Bolingbroke and Simon L. Peyton Jones. 2010. Supercompilation by evaluation. In Proceedings of the 3rd ACM SIGPLAN Symposium on Haskell, Haskell 2010, Baltimore, MD, USA, 30 September 2010, Jeremy Gibbons (Ed.). ACM, 135–146. https://doi.org/10.1145/1863523.1863540

Edwin Brady. 2013. Idris, A General-Purpose Dependent Typedly Programming Language: Design and Implementation. Journal of Functional Programming 23, 05 (2013), 552–593.

Sebastian Burckhardt, Manuel Fähndrich, Peli de Halleux, Sean McDermid, Michal Moskal, Nikolai Tillmann, and Jun Kato. 2013. It’s alive! continuous feedback in UI programming. In Conference on Programming Language Design and Implementation (PLDI).

Margaret M. Burnett, John W. Atwood Jr., and Zachary T. Welch. 1998. Implementing Level 4 Liveness in Declarative Visual Programming Languages. In IEEE Symposium on Visual Languages.

Naim Çagman and J. Roger Hindley. 1998. Combinatory Weak Reduction in Lambda Calculus. Theor. Comput. Sci. 198, 1-2 (1998), 239–247. https://doi.org/10.1016/S0304-3975(97)00250-8

Philippe Charles. 1991. A Practical Method for Constructing Efficient LLR(K) Parsers with Automatic Error Recovery. Ph.D. Dissertation. New York, NY, USA. UMI Order No. GAX91-34651.

Sheng Chen and Martin Erwig. 2014. Counter-Factual Typing for Debugging Type Errors. In Symposium on Principles of Programming Languages (POPL).

Sheng Chen and Martin Erwig. 2018. Systematic identification and communication of type errors. J. Funct. Program. 28 (2018), e2.

Adam Chlipala, Leaf Petersen, and Robert Harper. 2005. Strict bidirectional type checking. In TLDI ’05: Proceedings of the 2005 ACM SIGPLAN international workshop on Types in languages design and implementation. 71–78.

David Raymond Christiansen. 2013. Bidirectional Typing Rules: A Tutorial. http://davidchristiansen.dk/tutorials/bidirectional.pdf. (2013).

David R. Christiansen and Edwin Brady. 2016. Elaborator reflection: extending Idris in Idris. In Proceedings of the 21st ACM SIGPLAN International Conference on Functional Programming, ICFP 2016, Nara, Japan, September 18-22, 2016, Jacques Garrigue, Gabriele Keller, and Eijiro Sumii (Eds.). ACM, 284–297. https://doi.org/10.1145/2951913.2951932

Ravi Chugh, Brian Hempel, Mitchell Spradlin, and Jacob Albers. 2016. Programmatic and Direct Manipulation, Together at Last. In Conference on Programming Language Design and Implementation (PLDI).
VSTTE 2012, Philadelphia, PA, USA, January 28-29, 2012. Proceedings (Lecture Notes in Computer Science), Rajeev Joshi, Peter Müller, and Andreas Podeski (Eds.), Vol. 7152, Springer, 278–293. https://doi.org/10.1007/978-3-642-27705-4_22
Brian Hempel and Ravi Chugh. 2016. Semi-Automated SVG Programming via Direct Manipulation. In Symposium on User Interface Software and Technology (UIST).
David Herman, Aaron Tomb, and Cormac Flanagan. 2010. Space-efficient gradual typing. Higher-Order and Symbolic Computation 23, 2 (2010), 167.
Michael W. Hicks and Scott Nettles. 2005. Dynamic software updating. ACM Trans. Program. Lang. Syst. 27, 6 (2005), 1049–1096. https://doi.org/10.1145/1108970.1108971
Catalin Hritcu, Michael Greenberg, Ben Karel, Benjamin C. Pierce, and Greg Morrisett. 2013. All Your IFCEException Are Belong to Us. In IEEE Symposium on Security and Privacy. IEEE Computer Society, 3–17.
Gérard Huet. 1980. Confluent Reductions: Abstract Properties and Applications to Term Rewriting Systems: Abstract Properties and Applications to Term Rewriting Systems. J. ACM 27, 4 (1980), 797–821.
Yuu Igarashi, Taro Sekiyama, and Atsushi Igarashi. 2017. On Polymorphic Gradual Typing. Proc. ACM Program. Lang. 1, ICFP, Article 40 (Aug. 2017), 29 pages. https://doi.org/10.1145/3110284
Mike Jones. 2017. Edit code and continue Debugging in Visual Studio (C#, VB, C++). (2017). https://docs.microsoft.com/en-us/visualstudio/debugger/edit-and-continue. Retrieved Apr. 27, 2018.
Neil D. Jones, Carsten K.gomard, and Peter Sestoft. 1993. Partial evaluation and automatic program generation.
Simon Peyton Jones, Sean Leather, and Thijs Alkemade. 2014. Language options — Glasgow Haskell Compiler 8.4.1 User’s Guide (Typed Holes). http://downloads.haskell.org/~ghc/latest/docs/html/users_guide/glasgow_exts.html. Retrieved Apr 16, 2018.. (2014)
Gilles Kahn and David B. MacQueen. 1977. Coroutines and Networks of Parallel Processes. In IFIP Congress. 993–998.
Lennart C. L. Kats, Maartje de Jonge, Emma Nilsson-Nyman, and Eelco Visser. 2009. Providing rapid feedback in generated modular language environments: adding error recovery to scannerless generalized-LR parsing. In ACM SIGPLAN Conference on Object-Oriented Programming, Systems, Languages, and Applications (OOPSLA).
James C. King. 1976. Symbolic Execution and Program Testing. Commun. ACM 19, 7 (July 1976), 385–394. https://doi.org/10.1145/360248.360252
Nico Lehmann and Éric Tanter. 2017. Gradual refinement types. In Proceedings of the 44th ACM SIGPLAN Symposium on Principles of Programming Languages, POPL 2017, Paris, France, January 18-20, 2017, Giuseppe Castagna and Andrew D. Gordon (Eds.). ACM, 775–788. http://dl.acm.org/citation.cfm?id=3008956
Benjamin Lerner, Dan Grossman, and Craig Chambers. 2006. Seminal: Searching for ML Type-error Messages. In Workshop on ML.
Jean-Jacques Lévy and Luc Maranget. 1999. Explicit substitutions and programming languages. In International Conference on Foundations of Software Technology and Theoretical Computer Science. Springer, 181–200.
Eyal Lotem and Yair Chuchem. 2016. Project Lamdu. http://www.lamdu.org//. (2016). Accessed: 2016-04-08.
Conor McBride. 2000. Dependently typed functional programs and their proofs. Ph.D. Dissertation. University of Edinburgh, UK. http://hdl.handle.net/1842/374
Renée McCauley, Sue Fitzgerald, Gary Lewandowski, Laurie Murphy, Beth Simon, Lynda Thomas, and Carol Zander. 2008. Debugging: a review of the literature from an educational perspective. Computer Science Education 18, 2 (2008), 67–92.
Sean McDirmid. 2007. Living It Up with a Live Programming Language. In ACM SIGPLAN Conference on Object-Oriented Programming, Systems, Languages, and Applications (OOPSLA).
Aleksandar Nanevski, Frank Pfennig, and Brigitte Pientka. 2008. Contextual modal type theory. ACM Trans. Comput. Log. 9, 3 (2008). https://doi.org/10.1145/1352582.1352591
Greg L. Nelson, Benjamin Xie, and Andrew J. Ko. 2017. Comprehension First: Evaluating a Novel Pedagogy and Tutoring System for Program Tracing in CS1. In Proceedings of the 2017 ACM Conference on International Computing Education Research, ICER 2017, Tacoma, WA, USA, August 18-20, 2017, 2–11.
Ulf Norell. 2007. Towards a practical programming language based on dependent type theory. Ph.D. Dissertation. Department of Computer Science and Engineering, Chalmers University of Technology, SE-412 96 Göteborg, Sweden.
Ulf Norell. 2009. Dependently typed programming in Agda. In Advanced Functional Programming. Springer, 230–266.
Martin Odersky, Christoph Zenger, and Matthias Zenger. 2001. Colored local type inference. In POPL ’01: Proceedings of the 28th ACM SIGPLAN-SIGACT symposium on Principles of programming languages. 41–53.
Cyrus Omar, Ian Voysey, Michael Hilton, Jonathan Aldrich, and Matthew Hammer. 2017a. Hazelnut: A Bidirectionally Typed Structure Editor Calculus. In Principles of Programming Languages (POPL).
Cyrus Omar, Ian Voysey, Michael Hilton, Joshua Sunshine, Claire Le Goues, Jonathan Aldrich, and Matthew A. Hammer. 2017b. Toward Semantic Foundations for Program Editors. In Summit on Advances in Programming Languages (SNAPL).
Zvonimir Pavlinovic, Tim King, and Thomas Wies. 2015. Practical SMT-Based Type Error Localization. In International Conference on Functional Programming (ICFP).
Live Functional Programming with Typed Holes

Roly Perera, Umut A. Acar, James Cheney, and Paul Blain Levy. 2012. Functional programs that explain their work. In ACM SIGPLAN International Conference on Functional Programming (ICFP).

Fernando Pérez and Brian E. Granger. 2007. IPython: a System for Interactive Scientific Computing. Computing in Science and Engineering 9, 3 (May 2007), 21–29. https://doi.org/10.1109/MCSE.2007.53

Brigitte Pientka. 2010. Beluga: Programming with Dependent Types, Contextual Data, and Contexts. In International Symposium on Functional and Logic Programming (FLOPS).

Brigitte Pientka and Andrew Cave. 2015. Inductive Beluga: Programming Proofs. In International Conference on Automated Deduction. Springer, 272–281.

Benjamin C. Pierce. 2002. Types and Programming Languages. MIT Press.

Benjamin C. Pierce and David N. Turner. 2000. Local type inference. ACM Trans. Program. Lang. Syst. 22, 1 (2000), 1–44.

Gordon D. Plotkin. 2004. A structural approach to operational semantics. J. Log. Algebr. Program. 60-61 (2004), 17–139.

Mitchel Resnick, John Maloney, Andrés Monroy-Hernández, Natalie Rusk, Evelyn Eastmond, Karen Brennan, Amon Millner, Eric Rosenbaum, Jay Silver, Brian Silverman, and Yasmin Kafai. 2009. Scratch: Programming for All. Commun. ACM 52, 11 (Nov. 2009), 60–67.

Wilmer Ricciotti, Jan Stolarek, Roly Perera, and James Cheney. 2017. Imperative functional programs that explain their work. PACMPL 1, ICFP (2017), 14:1–14:28. https://doi.org/10.1145/3110258

Martin C. Rinard. 2012. Obtaining and reasoning about good enough software. In The 49th Annual Design Automation Conference 2012. DAC ’12, San Francisco, CA, USA, June 3–7, 2012, Patrick Groeneveld, Donatella Sciuto, and Soha Hassoun (Eds.). ACM, 930–935. https://doi.org/10.1145/2228360.2228526

Eric L. Seidel, Ranjit Jhala, and Westley Weimer. 2016. Dynamic Witnesses for Static Type Errors (or, Ill-typed Programs Usually Go Wrong). In International Conference on Functional Programming (ICFP).

Jeremy G. Siek and Walid Taha. 2006. Gradual Typing for Functional Languages. In Scheme and Functional Programming Workshop. http://scheme2006.cs.uchicago.edu/13-siek.pdf

Jeremy G. Siek and Walid Taha. 2007. Gradual Typing for Objects. In ECOOP 2007, Vol. 4609. 2–27.

Jeremy G. Siek, Michael M. Vitousek, Matteo Cimini, and John Tang Boyland. 2015a. Refined Criteria for Gradual Typing. In 1st Summit on Advances in Programming Languages, SNAPL 2015, May 3-6, 2015, Asilomar, California, USA. 274–293. https://doi.org/10.4230/LIPIcs.SNAPL.2015.274

Jeremy G. Siek, Michael M. Vitousek, Matteo Cimini, Sam Tobin-Hochstadt, and Ronald Garcia. 2015b. Monotonic References for Efficient Gradual Typing. In Programming Languages and Systems - 24th European Symposium on Programming, ESOP 2015, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2015, London, UK, April 11-18, 2015. Proceedings (Lecture Notes in Computer Science), Jan Vitek (Ed.), Vol. 9032. Springer, 432–456. https://doi.org/10.1007/978-3-662-46609-8_18

Jeremy G. Siek and Philip Waldier. 2010. Threesomes, With and Without Blame. In Proceedings of the 37th Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL ’10). ACM, New York, NY, USA, 365–376. https://doi.org/10.1145/1706299.1706342

Armando Solar-Lezama. 2009. The Sketching Approach to Program Synthesis. In APLAS, Vol. 5904. Springer, 4–13.

Saurabh Srivastava, Sumit Gulwani, and Jeffrey S Foster. 2013. Template-based program verification and program synthesis. International Journal on Software Tools for Technology Transfer 15, 5-6 (2013), 497–518.

Gareth Paul Stoyle, Michael W. Hicks, Gavin M. Bierman, Peter Sewell, and Iulian Neamtiu. 2007. Mutatis Mutandis: Safe and predictable dynamic software updating. ACM Trans. Program. Lang. Syst. 29, 4 (2007), 22. https://doi.org/10.1145/1255450.1255455

Matúš Sulír and Jaroslav Porubčan. 2018. Augmenting Source Code Lines with Sample Variable Values. CoRR abs/1806.07449 (2018).

Asumu Takikawa, Daniel Feltey, Earl Dean, Matthew Flatt, Robert Bruce Findler, Sam Tobin-Hochstadt, and Matthias Felleisen. 2015. Towards Practical Gradual Typing. In 29th European Conference on Object-Oriented Programming (ECOOP 2015) (Leibniz International Proceedings in Informatics (LIPIcs)), John Tang Boyland (Ed.), Vol. 37. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, Dagstuhl, Germany, 4–27. https://doi.org/10.4230/LIPIcs.ECOOP.2015.4

Steven L. Tanimoto. 2013. VIVA: A visual language for image processing. J. Vis. Lang. Comput. 1, 2 (1990), 127–139. https://doi.org/10.1016/S1045-926X(05)80012-6

Steven L. Tanimoto. 2013. A perspective on the evolution of live programming. In International Workshop on Live Programming (LIVE).

Andrew P. Tolmach and Andrew W. Appel. 1995. A Debugger for Standard ML. J. Funct. Program. 5, 2 (1995), 155–200. https://doi.org/10.1017/S0956796800001313

Christian Urban, Stefan Berghofer, and Michael Norrish. 2007. Barendregt’s Variable Convention in Rule Inductions. In Conference on Automated Deduction (CADE). 16.

B Victor. 2012. Inventing on principle, Invited talk at the Canadian University Software Engineering Conference (CUSEC). (2012).
Markus Voelter, Daniel Ratiu, Bernhard Schaezt, and Bernd Kolb. 2012. Mbeddr: An Extensible C-based Programming Language and IDE for Embedded Systems. In SPLASH (2012).
Markus Voelter, Janet Siegmund, Thorsten Berger, and Bernd Kolb. 2014. Towards User-Friendly Projectional Editors. In International Conference on Software Language Engineering (SLE).
Dimitrios Vytiniotis, Simon L. Peyton Jones, and José Pedro Magalhães. 2012. Equality proofs and deferred type errors: a compiler pearl. In ICFP. ACM, 341–352.
Philip Wadler and Robert Bruce Findler. 2009. Well-Typed Programs Can’t Be Blamed. In Programming Languages and Systems, 18th European Symposium on Programming, ESOP 2009, Held as Part of the Joint European Conferences on Theory and Practice of Software, ETAPS 2009, York, UK, March 22-29, 2009. Proceedings (Lecture Notes in Computer Science), Giuseppe Castagna (Ed.), Vol. 5502. Springer, 1–16. https://doi.org/10.1007/978-3-642-00590-9_1
David Wakeling. 2007. Spreadsheet functional programming. J. Funct. Program. 17, 1 (2007), 131–143. https://doi.org/10.1017/S0956796806006186
David Weintrop, Afsoon Afzal, Jean Salac, Patrick Francis, Boyang Li, David C. Shepherd, and Diana Franklin. 2018. Evaluating CoBlox: A Comparative Study of Robotics Programming Environments for Adult Novices. In Proceedings of the 2018 CHI Conference on Human Factors in Computing Systems, CHI 2018, Montreal, QC, Canada, April 21-26, 2018, Regan L. Mandryk, Mark Hancock, Mark Perry, and Anna L. Cox (Eds.). ACM, 366. https://doi.org/10.1145/3173574.3173940
David Weintrop and Uri Wilensky. 2015. To block or not to block, that is the question: students’ perceptions of blocks-based programming. In Proceedings of the 14th International Conference on Interaction Design and Children, IDC ’15, Medford, MA, USA, June 21–25, 2015, Marina Umaschi Bers and Glenda Revelle (Eds.). ACM, 199–208. https://doi.org/10.1145/2771839.2771860
John Whitington and Tom Ridge. 2017. Visualizing the Evaluation of Functional Programs for Debugging. In SLATE.
Andrew K. Wright and Matthias Felleisen. 1994. A syntactic approach to type soundness. Information and Computation 115, 1 (1994), 38–94.
Ningning Xie, Xuan Bi, and Bruno C. d. S. Oliveira. 2018. Consistent Subtyping for All. In Programming Languages and Systems - 27th European Symposium on Programming, ESOP 2018, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2018, Thessaloniki, Greece, April 14-20, 2018, Proceedings (Lecture Notes in Computer Science), Amal Ahmed (Ed.), Vol. 10801. Springer, 3–30. https://doi.org/10.1007/978-3-319-89884-1_1
Y. S. Yoon and B. A. Myers. 2014. A longitudinal study of programmers’ backtracking. In IEEE Symposium on Visual Languages and Human-Centric Computing (VL/HCC).
Danfeng Zhang, Andrew C. Myers, Dimitrios Vytiniotis, and Simon Peyton-Jones. 2017. SHErrLoc: A Static Holistic Error Locator. Transactions on Programming Languages and Systems (TOPLAS) (2017).
A  Additional Definitions for Hazelnut Live

A.1 Substitution

\[
\begin{align*}
[d/x]c &= c \\
[d/x]x &= \tau' \\
[d/x]y &= y \quad \text{when } x \neq y \\
[d/x]λy:τ.d' &= λy:τ.[d/x]d' \quad \text{when } x \neq y \text{ and } y \notin \text{FV}(d) \\
[d/x]d_1(d_2) &= ([d/x]d_1) ([d/x]d_2) \\
[d/x]\emptyset^u &= \emptyset^u \quad \text{when } x \neq y \\
[d/x][d']^u_\sigma &= \left((d/x)[d']^u\right)_\sigma \\
[d/x]d'(τ_1 \Rightarrow τ_2) &= ([d/x]d')(τ_1 \Rightarrow τ_2) \\
[d/x]d'(τ_1 \Rightarrow τ_2) &= \left((d/x)[d']\right)'(τ_1 \Rightarrow τ_2) \\
[d/x]· &= \cdot \\
[d/x]σ, d/y &= [d/x]σ, [d/x]d/y
\end{align*}
\]

Lemma A.1 (Substitution).

1. If \(Δ; Γ, x : τ' \vdash d : τ \text{ and } Δ; Γ \vdash [d'/x]d : τ\).
2. If \(Δ; Γ, x : τ' \vdash σ : Γ' \text{ and } Δ; Γ \vdash [d'/x]σ : Γ'\).

Proof. By rule induction on the first assumption in each case. The conclusion follows from the definition of substitution in each case. □

A.2 Canonical Forms

Lemma A.2 (Canonical Value Forms). If \(Δ; Γ, x : τ \vdash d : τ \text{ and } d \text{ val then } τ \neq\langle\rangle\) and

1. If \(τ = b\) then \(d = c\).
2. If \(τ = \tau_1 \rightarrow \tau_2\) then \(d = λx:τ_1.d'\) where \(Δ; x : τ_1 \vdash d' : τ_2\).

Lemma A.3 (Canonical Boxed Forms). If \(Δ; Γ, x : τ \vdash d : τ \text{ and } d \text{ boxedval then}

1. If \(τ = b\) then \(d = c\).
2. If \(τ = \tau_1 \rightarrow \tau_2\) then either
   i. \(d = λx:τ_1.d'\) where \(Δ; x : τ_1 \vdash d' : τ_2\), or
   ii. \(d = d'(τ'_1 \rightarrow τ'_2 \Rightarrow τ_1 \rightarrow τ_2)\) where \(τ'_1 \rightarrow τ'_2 \neq τ_1 \rightarrow τ_2 \text{ and } Δ; Γ, 0 \vdash d' : τ'_1 \rightarrow τ'_2\).
3. If \(τ = \langle\rangle\) then \(d = d'(τ' \Rightarrow \langle\rangle)\) where \(τ'\) ground and \(Δ; Γ \vdash d' : τ'\).

Lemma A.4 (Canonical Indeterninate Forms). If \(Δ; Γ, x : τ \vdash d : τ \text{ and } d \text{ indet then either}

1. \(d = \langle\rangle^u_\tau\) and \(u :: τ[\Gamma'] \in Δ\), or
2. \(d = \langle\rangle^u_\tau\) and \(d'\) final and \(Δ; 0 \vdash d' : τ'\) and \(u :: τ[\Gamma'] \in Δ\), or
3. \(d = d_1(d_2) \text{ and } Δ; 0 \vdash d_1 : τ_2 \rightarrow τ \text{ and } Δ; 0 \vdash d_2 : τ_2 \text{ and } d_1 \text{ indet and } d_2 \text{ final and } d_1 \neq d_2 \text{ and } d_3 (τ_3 \rightarrow τ_4 \Rightarrow τ'_3 \rightarrow τ'_4)\), or
4. \(τ = b\) and \(d = d'(x' \Rightarrow \langle\rangle)\) and \(d' \neq d''(τ' \Rightarrow \langle\rangle)\), or
5. \(τ = b\) and \(d = d'(τ' \Rightarrow \langle\rangle) \Rightarrow b\) and \(τ'\) ground and \(τ' \neq b\) and \(Δ; 0 \vdash d' : τ'\), or
6. \(τ = τ_1 \rightarrow τ_2\) and \(d = d'(τ_1 \rightarrow τ_2 \Rightarrow τ_1 \rightarrow τ_2)\) and \(d' \text{ indet and } τ_1 \rightarrow τ_2 \neq τ_1 \rightarrow τ_2\), or
7. \(τ = \langle\rangle \Rightarrow \langle\rangle\) and \(d = d'(τ' \Rightarrow \langle\rangle) \Rightarrow \langle\rangle\) and \(d' \text{ indet and } d' \neq d''(τ' \Rightarrow \langle\rangle)\), or
8. \(τ = \langle\rangle \Rightarrow \langle\rangle\) and \(d = d'(τ' \Rightarrow \langle\rangle) \Rightarrow \langle\rangle \Rightarrow \langle\rangle\) and \(τ' \neq τ\) and \(τ'\) ground and \(d' \text{ indet and } Δ; 0 \vdash d' : τ'\), or
9. \(τ = \langle\rangle \Rightarrow \langle\rangle\) and \(d = d'(τ' \Rightarrow \langle\rangle) \Rightarrow \langle\rangle \Rightarrow \langle\rangle\) and \(τ' \neq τ\) and \(τ'\) ground and \(d' \text{ indet.}\)
The proofs for all three of these theorems follow by straightforward rule induction.
A.3 Complete Programs

\[
\begin{array}{c}
\text{\(\tau\) complete} & \tau \text{ is complete} \\
\text{TCBase} & b \text{ complete} \\
\hline
\text{TCArr} & \begin{array}{c}
\tau_1 \text{ complete} \\
\tau_2 \text{ complete}
\end{array} \\
\tau_1 \rightarrow \tau_2 \text{ complete}
\end{array}
\]

\[
\begin{array}{c}
\text{\(e\) complete} & e \text{ is complete} \\
\hline
\text{ECVar} & x \text{ complete} \\
\text{ECConst} & c \text{ complete} \\
\text{ECLam1} & \begin{array}{c}
\tau \text{ complete} \\
e \text{ complete}
\end{array} \\
\lambda x : \tau. e \text{ complete}
\end{array}
\]

\[
\begin{array}{c}
\text{ECLam2} & e \text{ complete} \\
\hline
\text{ECAsc} & \begin{array}{c}
e \text{ complete} \\
\tau \text{ complete}
\end{array} \\
e : \tau \text{ complete}
\end{array}
\]

\[
\begin{array}{c}
\text{ECAp} & \begin{array}{c}
e_1 \text{ complete} \\
e_2 \text{ complete}
\end{array} \\
e_1(e_2) \text{ complete}
\end{array}
\]

\[
\begin{array}{c}
\text{DCCast} & \begin{array}{c}
\tau_1 \text{ complete} \\
\tau_2 \text{ complete}
\end{array} \\
d \langle \tau_1 \Rightarrow \tau_2 \rangle \text{ complete}
\end{array}
\]

Fig. 15. Complete types, external expressions, and internal expressions

We define \(\Gamma\) complete as follows.

**Definition A.5** (Typing Context Completeness). \(\Gamma\) complete iff for each \(x : \tau \in \Gamma\), we have \(\tau\) complete.

When two types are complete and consistent, they are equal.

**Lemma A.6** (Complete Consistency). If \(\tau_1 \sim \tau_2\) and \(\tau_1\) complete and \(\tau_2\) complete then \(\tau_1 = \tau_2\).

**Proof.** By straightforward rule induction. \(\square\)

This implies that in a well-typed and complete internal expression, every cast is an identity cast.

A.4 Multiple Steps

\[
\begin{array}{c}
\text{\(d \multistep d'\)} & \text{\(d\) multi-steps to \(d'\)} \\
\hline
\text{MultiStepRef1} & \\
\text{MultiStepSteps} & \\
\hline
\text{d multisteps to d''} \\
d \multistep d' \\
d' \multistep d''
\end{array}
\]

Fig. 16. Multi-Step Transitions
A.5 Hole Filling

**Lemma A.7** (Filling).

(1) If \( \Delta, u : \tau'[\Gamma']; \Gamma + d : \tau \) and \( \Delta; \Gamma' + d' : \tau' \) then \( \Delta; \Gamma + \left[ \left[ d'/u \right] d \right] : \tau \).

(2) If \( \Delta, u : \tau'[\Gamma']; \Gamma + \sigma : \Gamma'' \) and \( \Delta; \Gamma' + d' : \tau' \) then \( \Delta; \Gamma + \left[ \left[ d'/u \right] \sigma \right] : \Gamma'' \).

**Proof.** In each case, we proceed by rule induction on the first assumption, appealing to the Substitution Lemma as necessary.

We need the following auxiliary definitions, which lift hole filling to evaluation contexts taking care to consider the special situation where the mark is inside the hole that is being filled, to prove the Commutativity theorem.

- **inhole** The mark in \( \mathcal{E} \) is inside non-empty hole closure \( u \)

| InHoleNEHole | InHoleAp1 | InHoleAp2 |
|---------------|-----------|-----------|
| inhole(\( u; \mathcal{E} \)) | inhole(\( u; \mathcal{E}(d) \)) | inhole(\( u; \mathcal{E}(d) \)) |
| InHoleCast | InHoleFailedCast |
| inhole(\( u; \mathcal{E}(\tau_1 \Rightarrow \tau_2) \)) | inhole(\( u; \mathcal{E}(\tau_1 \Rightarrow (\tau_1 \Rightarrow \tau_2)) \)) |

**[\( d/u \)\( \mathcal{E} = \mathcal{E}' \)]** \( \mathcal{E}' \) is obtained by filling hole \( u \) in \( \mathcal{E} \) with \( d \)

| EFillMark | EFillAp1 | EFillAp2 |
|-----------|----------|----------|
| \( [d/u] \circ = \circ \) | \( [d/u] \mathcal{E} = \mathcal{E}' \) | \( [d/u] \mathcal{E} = \mathcal{E}' \) |
| \( [d/u] \mathcal{E}(d_2) = \mathcal{E}' \left( [d/u] d_2 \right) \) | \( [d/u] d_1(\mathcal{E}) = ([d/u] d_1)(\mathcal{E}') \) | |
| EFillNEHole | EFillCast |
| \( [d/u] \mathcal{E} = \mathcal{E}' \) | \( [d/u] \mathcal{E} = \mathcal{E}' \) |
| \( [d/u] \mathcal{E}(\tau_1 \Rightarrow \tau_2) = \mathcal{E}'(\tau_1 \Rightarrow \tau_2) \) | \( [d/u] \mathcal{E}(\tau_1 \Rightarrow (\tau_1 \Rightarrow \tau_2)) = \mathcal{E}'(\tau_1 \Rightarrow (\tau_1 \Rightarrow \tau_2)) \) |

**Fig. 17. Evaluation Context Filling**

**Lemma A.8** (Substitution Commutativity). If

(1) \( \Delta, u : \tau'[\Gamma']; x : \tau_2 + d_1 : \tau \) and

(2) \( \Delta, u : \tau'[\Gamma']; \emptyset + d_2 : \tau_2 \) and

(3) \( \Delta; \Gamma' + d' : \tau' \) then \( \left[ \left[ d'/u \right] [d_2/x] d_1 \right] = \left[ \left[ d'/u \right] d_2/x \right] d_1 \).

**Proof.** We proceed by structural induction on \( d_1 \) and rule induction on the typing premises, which serve to ensure that the free variables in \( d' \) are accounted for by every closure for \( u \). □

**Lemma A.9** (Instruction Commutativity). If

(1) \( \Delta, u : \tau'[\Gamma']; \emptyset + d_1 : \tau \) and

(2) \( \Delta; \Gamma' + d' : \tau' \) and
(3) \( d_1 \rightarrow d_2 \)
\[
\text{then } \llbracket d'/u \rrbracket d_1 \rightarrow \llbracket d'/u \rrbracket d_2.
\]

**Proof.** We proceed by cases on the instruction transition assumption (no induction is needed). For Rule ITLam, we defer to the Substitution Commutativity lemma above. For the remaining cases, the conclusion follows from the definition of hole filling. □

**Lemma A.10** (Filling Totality). Either inhole\( (u; E) \) or \( \llbracket d/u \rrbracket E = E' \) for some \( E' \).

**Proof.** We proceed by structural induction on \( E \). Every case is handled by one of the two judgements. □

**Lemma A.11** (Discarding). If

1. \( d_1 = E\{d'_1\} \) and
2. \( d_2 = E\{d'_2\} \) and
3. inhole\( (u; E) \)

then \( \llbracket d/u \rrbracket d_1 = \llbracket d/u \rrbracket d_2. \)

**Proof.** We proceed by structural induction on \( E \) and rule induction on all three assumptions. Each case follows from the definition of instruction selection and hole filling. □

**Lemma A.12** (Filling Distribution). If \( d_1 = E\{d'_1\} \) and \( \llbracket d/u \rrbracket E = E' \) then \( \llbracket d/u \rrbracket d_1 = E'\{\llbracket d/u \rrbracket d'_1\}. \)

**Proof.** We proceed by rule induction on both assumptions. Each case follows from the definition of instruction selection and hole filling. □

**Theorem A.13** (Commutativity). If

1. \( \Delta, u : \tau'[\Gamma'] : \emptyset \vdash d_1 : \tau \) and
2. \( \Delta; \Gamma' \vdash p' : \tau' \) and
3. \( d_1 \overset{\ast}{\rightarrow} d_2 \)

then \( \llbracket d'/u \rrbracket d_1 \overset{\ast}{\rightarrow} \llbracket d'/u \rrbracket d_2. \)

**Proof.** By rule induction on assumption (3). The reflexive case is immediate. In the inductive case, we proceed by rule induction on the stepping premise. There is one rule, Rule Step. By Filling Totality, either inhole\( (u; E) \) or \( \llbracket d/u \rrbracket E = E' \). In the former case, by Discarding, we can conclude by MultiStepRef1. In the latter case, by Instruction Commutativity and Filling Distribution we can take a Step, and we can conclude via MultiStepSteps by applying Filling, Preservation and then the induction hypothesis. □

We exclude these proofs and definitions from the Agda mechanization for two reasons. First, fill-and-resume is merely an optimization, and unlike the meta theory of Sec. 3, these properties are generally not conserved by certain reasonable extensions of the core calculus (e.g., reference cells and other non-commuting effects). Second, to properly encode the hole filling operation, such a mechanization requires a significantly more complex representation of hole environments; unfortunately, Agda cannot be easily convinced that the definition of hole filling is well-founded (Nanevski et al. [2008] establish that it is in fact well-founded). By contrast, the developments in Sec. 3 do not require these more complex (and somewhat problematic) representations.
A.6 Confluence, and Friends

There are various ways to encode the intuition that “evaluation order does not matter”. One way to do so is by establishing a confluence property (which is closely related to the Church-Rosser property [Church and Rosser 1936]).

The most general confluence property does not hold for the dynamic semantics in Sec. 3 for the usual reason: We do not reduce under binders (Blanc et al. [2005] discuss the standard counterexample). We could recover confluence by specifying reduction under binders, either generally or in a more restricted form where only closed sub-expressions are reduced [Blanc et al. 2005; Çagman and Hindley 1998; Lévy and Maranget 1999]. However, reduction under binders conflicts with the standard implementation approaches for most programming languages [Blanc et al. 2005]. A more satisfying approach considers confluence modulo equality [Huet 1980]. The simplest such approach restricts our interest to top-level expressions of base type that result in values, in which case the following special case of confluence does hold (trivially when the only base type has a single value, but also more generally for other base types).

**Lemma A.14** (Base Confluence). If $\Delta; \emptyset \vdash d : b$ and $d \rightarrow^* d_1$ and $d_1 \text{ val}$ and $d \rightarrow^* d_2$ then $d_2 \rightarrow^* d_1$.

We can then prove the following property, which establishes that fill-and-resume is sound.

**Theorem A.15** (Resumption). If $\Delta, u :: \tau \lbrack \Gamma \rbrack; \emptyset \vdash d : b$ and $\Delta, \Gamma \vdash d' : \tau'$ and $d \rightarrow^* d_1$ and $[d'/u]d \rightarrow^* d_2$ and $d_2 \text{ val}$ then $[d'/u]d_1 \rightarrow^* d_2$.

**Proof.** By Commutativity, $[d'/u]d \rightarrow^* [d'/u]d_1$. By Base Confluence, we can conclude. □
B Extensions to Hazelnut Live

We give two extensions here, numbers and sum types, to maintain parity with Hazelnut as specified by Omar et al. [2017a].

It is worth observing that these extensions do not make explicit mention of expression holes. The “non-obvious” machinery is almost entirely related to casts. Fortunately, there has been excellent work of late on generating “gradualized” specifications from standard specifications [Cimini and Siek 2016, 2017]. The extensions below, and other extensions of interest, closely follow the output of the gradualizer: http://cimini.info/gradualizerDynamicSemantics/ (which, like our work, is based on the refined account of gradual typing by [Siek et al. 2015a]). The rules for indeterminate forms are mainly where the gradualizer is not sufficient. We leave to future work the related question of automatically generating a hole-aware static and dynamic semantics from a standard language specification.
B.1 Numbers

We extend the syntax as follows:

\[
\begin{align*}
\text{HTyp} & \quad \tau := \cdots | \text{num} \\
\text{HExp} & \quad e := \cdots | n | e + e \\
\text{IHExp} & \quad d := \cdots | n | d + d \\
\end{align*}
\]

\[\Gamma \vdash e \Rightarrow \tau \sim d : \Delta \quad \text{e synthesizes type } \tau \text{ and expands to } d\]

\[\Gamma \vdash n \Rightarrow \text{num} \sim n + \cdot \quad \Gamma \vdash e_1 + e_2 \Rightarrow \text{num} \sim d_1 + d_2 : \Delta_1 \cup \Delta_2 \]

\[\Delta; \Gamma \vdash d : \tau \quad d \text{ is assigned type } \tau\]

\[\Delta; \Gamma \vdash n : \text{num} \quad \Delta; \Gamma \vdash d_1 : \text{num} \quad \Delta; \Gamma \vdash d_2 : \text{num}\]

\[d \text{ val} \quad d \text{ is a value}\]

\[n \text{ val}\]

\[\tau \text{ ground} \quad \tau \text{ is a ground type}\]

\[\text{num ground}\]

\[d \text{ indet} \quad d \text{ is indeterminate}\]

\[\frac{d_1 \neq n \quad d_1 \text{ indet} \quad d_2 \text{ final}}{d_1 + d_2 \text{ indet}} \quad \frac{d_2 \neq n \quad d_1 \text{ final} \quad d_2 \text{ indet}}{d_1 + d_2 \text{ indet}}\]

\[\text{EvalCtx} \quad E := \cdots | E + d_2 | d_1 + E\]

\[d = E\{d'\} \quad d \text{ is obtained by placing } d' \text{ at the mark in } E\]

\[\frac{d_1 = E\{d'_1\}}{d_1 + d_2 = (E + d_2)\{d'_1\}} \quad \frac{d_2 = E\{d'_2\}}{d_1 + d_2 = (d_1 + E)\{d'_2\}}\]

\[d_1 \rightarrow d_2 \quad d_1 \text{ steps to } d_2\]

\[\frac{n_1 + n_2 = n_3}{n_1 + n_2 \rightarrow n_3}\]
B.2 Sum Types

We extend the syntax for sum types as follows:

\[
\begin{align*}
\text{HTyp} & \quad \tau ::= \cdots \mid \tau + \tau \\
\text{HExp} & \quad e ::= \cdots \mid \text{inl}(e) \mid \text{inr}(e) \mid \text{case}(e, x.e, y.e) \\
\text{LHExp} & \quad d ::= \cdots \mid \text{inl}_\tau(d) \mid \text{inr}_\tau(d) \mid \text{case}(d, x.d, y.d)
\end{align*}
\]

Types \(\tau_1\) and \(\tau_2\) join (consistently), forming type \(\tau_3\)

\[
\begin{align*}
\text{join}(\tau_1, \tau_2) &= \tau_3 \\
\text{join}(\tau, \tau) &= \tau \\
\text{join}(\emptyset, \tau) &= \tau \\
\text{join}(\tau, \emptyset) &= \tau \\
\text{join}(\tau_1 \to \tau_2, \tau_1 \to \tau_2) &= \text{join}(\tau_1, \tau_2) \to \text{join}(\tau_1, \tau_2) \\
\text{join}(\tau_1 + \tau_2, \tau_1 + \tau_2) &= \text{join}(\tau_1, \tau_2) + \text{join}(\tau_1, \tau_2)
\end{align*}
\]

**Theorem B.1** (Joins). If \(\text{join}(\tau_1, \tau_2) = \tau\) then types \(\tau_1, \tau_2\) and \(\tau\) are pair-wise consistent, i.e., \(\tau_1 \sim \tau_2, \tau_1 \sim \tau\) and \(\tau_2 \sim \tau\).

**Proof.** By induction on the derivation of \(\text{join}(\tau_1, \tau_2) = \tau\). \(\square\)

\[
\begin{align*}
\tau \triangleright+ \tau_1 + \tau_2 & \quad \text{Type } \tau \text{ matches the sum type } \tau_1 + \tau_2 \\
\tau_1 + \tau_2 \triangleright+ \tau_1 + \tau_2 & \quad \emptyset \triangleright+ \emptyset + \emptyset
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash e & \iff \tau_1 \sim d : \tau_2 \downarrow \Delta \\
\tau \triangleright+ \tau_1 + \tau_2 & \quad \Gamma \vdash e \iff \tau_1 \sim d : \tau_1' + \Delta \\
\Gamma \vdash \text{inl}(e) & \iff \tau \sim \text{inl}_{\tau_1}(d) : \tau_1' + \tau_2 + \Delta \\
\Gamma \vdash \text{inr}(e) & \iff \tau \sim \text{inr}_{\tau_2}(d) : \tau_1 + \tau_2' + \Delta \\
\text{join}(\tau_2, \tau_3) &= \tau' \\
\Gamma, x : \tau_1 + \tau_2 & \vdash e \iff \tau \sim d : \tau_2 + \Delta_2 \\
\Gamma, y : \tau_1 + \tau_2 & \vdash e \iff \tau \sim d : \tau_3 + \Delta_3 \\
\Gamma \vdash \text{case}(e_1, x.e_2, y.e_3) & \iff \tau \sim \text{case}(d_1(\tau_1 \Rightarrow \tau_1 + \tau_2), x.d_2(\tau_2 \Rightarrow \tau'), y.d_3(\tau_3 \Rightarrow \tau')) : \tau' + \Delta_1 + \Delta_2 + \Delta_3
\end{align*}
\]

\[
\begin{align*}
\Delta; \Gamma \vdash d : \tau & \quad d \text{ is assigned type } \tau \\
\Delta; \Gamma \vdash \text{inl}_{\tau_1}(d) : \tau_1 + \tau_2 & \quad \Delta; \Gamma \vdash \text{inr}_{\tau_2}(d) : \tau_1 + \tau_2 \\
\Delta; \Gamma \vdash \text{case}(d_1, x.d_2, y.d_3) : \tau
\end{align*}
\]

\[
\begin{align*}
d \text{ val} & \quad d \text{ is a value} \\
\text{inl}_{\tau}(d) \text{ val} & \quad \text{inr}_{\tau}(d) \text{ val}
\end{align*}
\]

\(\tau\) ground \quad \tau \text{ is a ground type}
\[ \langle \lambda \rangle + \langle \lambda \rangle \text{ ground} \]

- **boxedval**
  - \( d \text{ is a boxed value} \)
  - \( \text{inl}_\tau(d) \text{ boxedval} \)
  - \( \text{inr}_\tau(d) \text{ boxedval} \)
  - \( \tau_1 + \tau_2 \neq \tau'_1 + \tau'_2 \text{ boxedval} \)

- **indet**
  - \( d \text{ is indeterminate} \)
  - \( \text{inl}_\tau(d) \text{ indet} \)
  - \( \text{inr}_\tau(d) \text{ indet} \)
  - \( \tau_1 + \tau_2 \neq \tau'_1 + \tau'_2 \text{ indet} \)

- **boxedval**
  - \( \text{boxedval} \)
  - \( \text{boxedval} \)
  - \( \tau_1 + \tau_2 \Rightarrow \tau'_1 + \tau'_2 \text{ boxedval} \)

- **indet**
  - \( \text{indet} \)
  - \( \text{indet} \)
  - \( \tau_1 + \tau_2 \Rightarrow \tau'_1 + \tau'_2 \text{ indet} \)

**EvalCtx**

\[ E ::\cdots | \text{inl}_\tau(E) | \text{inr}_\tau(E) | \text{case}(E, x. d_1, y. d_3) \]

- **boxedval**
  - \( \text{boxedval} \)
  - \( \text{boxedval} \)
  - \( \tau_1 + \tau_2 \Rightarrow \tau'_1 + \tau'_2 \text{ boxedval} \)

- **indet**
  - \( \text{indet} \)
  - \( \text{indet} \)
  - \( \tau_1 + \tau_2 \Rightarrow \tau'_1 + \tau'_2 \text{ indet} \)

**boxedval**

\[ \text{boxedval} \]

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