ENTROPY DIMENSION OF MEASURE PRESERVING SYSTEMS

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Abstract. The notion of metric entropy dimension is introduced to measure the complexity of entropy zero dynamical systems. For measure preserving systems, we define entropy dimension via the dimension of entropy generating sequences. This combinatorial approach provides us with a new insight to analyze the entropy zero systems. We also define the dimension set of a system to investigate the structure of the randomness of the factors of a system. The notion of a uniform dimension in the class of entropy zero systems is introduced as a generalization of a K-system in the case of positive entropy. We investigate the joinings among entropy zero systems and prove the disjointness property among entropy zero systems using the dimension sets. Given a topological system, we compare topological entropy dimension with metric entropy dimension.

1. Introduction

Since entropy was introduced by Kolmogorov from information theory, it has played an important role in the study of dynamical systems. Entropy measures the chaoticity or unpredictability of a system. It is well known as a complete invariant for the Bernoulli automorphism class. Properties of positive entropy systems have been studied in many different respects along with their applications. Comparing with positive entropy systems, we have much less understanding and less tools for entropy zero systems. Entropy zero systems which are called deterministic systems in the case of \( \mathbb{Z} \)-actions cover a wide class of dynamical systems exhibiting different “random” behaviors or different level of complexities. They range from irrational rotations on a circle, more generally isometry on a compact metric space, Toeplitz systems to horocycle flows. Also many of the physical systems studied recently show intermittent or weakly chaotic behavior \[19, 22\]. They have the property that a generic orbit has sequences of 0’s with density 1 and hence we would say that they have very low complexity or randomness. They do not have finite invariant measures which are physically meaningful. Hence to analyze the complexity of these systems, the notion of algorithmic information content or Kolmogorov complexity has been employed instead of the entropy. It measures the information content of generic orbits of the system.

Moreover many of general group actions like \( \mathbb{Z}^n \)—actions have entropy zero with interesting subdynamics. They exhibit diverse complexities and their non-cocompact subgroup actions show very different behavior \[2, 16, 17, 18\]. We may mention a few known examples of entropy zero with their properties in the case of \( \mathbb{Z}^2 \)—actions:

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(1) (a) \( h(\sigma^{(p,q)}) = 0 \) for \( \forall (p, q) \in \mathbb{Z}^2 \setminus \{(0, 0)\} \), 
(b) For any given \( 0 < \alpha < 2 \),
\[
\limsup_{n \to \infty} \frac{1}{n^\alpha} \mathcal{H}( \bigvee_{(i,j) \in R_n} \sigma^{-(i,j)} \mathcal{P} ) > 0;
\]

(2) (a) \( h(\sigma^{(p,q)}) = 0 \) for \( \forall (p, q) \in \mathbb{Z}^2 \), 
(b) 
\[
\lim_{n \to \infty} \frac{1}{n} \mathcal{H}( \bigvee_{(i,j) \in R_n} \sigma^{-(i,j)} \mathcal{P} ) > 0;
\]

(3) (a) \( h(\sigma^{(1,0)}) > 0 \)
(b) \( h(\sigma^{(p,q)}) = 0 \) for \( \forall (p, q) \neq (n, 0) \),
(c) 
\[
\lim_{n \to \infty} \frac{1}{n} \mathcal{H}( \bigvee_{(i,j) \in R_n} \sigma^{-(i,j)} \mathcal{P} ) > 0;
\]

where \( R_n \) denotes the square of size \( n \times n \).

The first example is by Katok and Thouvenot [12] and the second one is constructed in [18]. Although the third example is not written anywhere explicitly, it is known that the example is by Ornstein and Weiss, also independently by Thouvenot.

In his study of Cellular Automaton maps [16], J. Milnor considered the Cellular Automaton maps together with horizontal shifts as \( \mathbb{Z}^2 \)-actions of zero entropy. He introduced the notion of directional entropy and investigated the properties of the complexities of these systems via directional entropies and their entropy geometry. Boyle and Lind pursued the study of the entropy geometry further in [2]. Besides Milnor’s examples we have many examples whose directional entropies are finite and continuous in all directions including irrational directions [16, 17]. As was shown in the above example (2), there are \( \mathbb{Z}^2 \)-actions whose directional entropy does not capture the complexity of a system. We may say that the examples (2) and (3) have complexity in the order of \( n \), while positive entropy systems have complexity in the order of \( n^2 \).

Cassaigne constructed a uniformly recurrent point and hence a minimal system of a given subexponential orbit growth rate [3]. In [6], inspired by the viewpoint of “topological independence” (see [10, 11]), the authors gave the definition of topological entropy dimension to analyze the entropy zero systems. It measures the the sub-exponential but sup-polynomial topological complexity via the growth rate of orbits. Together with the examples, some of the properties of the entropy zero topological systems have been investigated. As was shown in physical models, many examples of low complexity do not carry finite invariant measure. Their meaningful invariant measures are \( \sigma \)-finite. Examples of finite measure preserving systems of subexponential growth rate were first constructed in [7]. Katok and Thouvenot introduced the notion of slow entropy for \( \mathbb{Z}^2 \)-actions to show that certain measure preserving \( \mathbb{Z}^2 \)-actions are not realizable by two commuting Lipschitz continuous maps in [12]. Since the “natural” extension of the definition of entropy to slow entropy is not an isomorphism invariant, they use the number of \( \epsilon \)-balls in the Hamming distance to define the slow...
entropy. It is clear that their definition is easily applied to \( \mathbb{Z} \)-actions to differentiate the complexity.

We introduce the notion of the entropy generating sequence and positive entropy sequence to understand the complexity of entropy zero systems. By the definition, it is clear that the entropy generating sequence is a sequence along which the system has some independence. First we define the dimension of a subset of \( \mathbb{N} \) of density zero and use the notion to define the entropy dimension of a system via the entropy generating sequence. It is clear that the properties should be further investigated to understand the structure of zero entropy systems. We hope that many of the tools developed for the study of positive entropy class are to be investigated in the class of a given entropy dimension. For example, we ask if we can have \( \alpha \)-dimension Pinsker \( \sigma \)-algebra and \( \alpha \)-dimension Bernoulli in the case that \( \alpha \)-entropy exists. We ask also if we have some kind of regularity in the size of the atoms of the iterated partition of these systems. Moreover since general group actions have many “natural” examples of entropy zero with diverse complexity, we need to extend our study to general group actions. We believe the study of entropy dimension together with the study of subgroup actions will lead us to the understanding of more challenging and interesting properties of entropy zero general group actions.

We briefly describe the content of the paper. In section 2, we introduce the notion of entropy generating sequence and positive entropy sequence which are subsets of \( \mathbb{N} \). For a given subset of \( \mathbb{N} \), we introduce the notion of the dimensions, upper and lower, of a subset to measure the “size” of the subset. This notion classifies the “size” of the subsets of density 0. We show (Proposition 2.4) the relation between the dimensions of entropy generating sequence and positive entropy sequence. For a measure-preserving system we will define the metric entropy dimension through the dimensions of entropy generating sequence and positive entropy sequence. We will study many of the basic properties of entropy dimension. In section 3, we define the dimension set of a system to understand the structure of the complexity of its factors. We also introduce the notion of uniform dimension whose dimension set consists of a singleton. Using the dimension sets, we also study the property of disjointness among entropy zero systems. We prove a theorem which is more general than the disjointness between K-mixing systems and zero entropy systems. In section 4, for a compact metric space we consider the entropy dimension of a given open cover with respect to a measure and show that the topological entropy dimension is always bigger than or equal to the metric entropy dimension of a topological system. We provide a class of examples of uniform dimension in section 5. In a rough statement, we may say that the property without a factor of smaller entropy dimension corresponds to the K-mixing property without zero entropy factors. Our construction is based on the method similar to [7], but it demands technical arguments to guarantee that no partition has smaller entropy dimension. We need to make level sets of each step “spread out” through the columns of the later towers without the increase of the sub-exponential growth rate of orbits.

We mention that we noticed recently that the entropy dimension was first introduced in [4]. Since we have started our work on the complexity of topological and metric entropy zero systems, there are several papers published in different directions in the
Clearly this is the beginning of the study of entropy zero systems with many more open questions.

2. Entropy dimension

In the case of zero entropy, we want to generalize the definition of entropy to measure the growth rate of the iterated partitions. However it has been noticed in [7] that \( \inf \{ \beta : \limsup_{n \to \infty} \frac{1}{n^\beta} H_\mu(\bigvee_{i=0}^{n-1} P) = 0 \} \) is not an isomorphic invariant. More precisely, the following was proved, if there exists a partition \( P \) such that

\[
\inf \{ \beta : \limsup_{n \to \infty} \frac{1}{n^\beta} H_\mu(\bigvee_{i=0}^{n-1} P) = 0 \} = \alpha > 0,
\]

then for any \( \alpha < \tau < 1 \) and \( \epsilon > 0 \), there exists a partition \( \tilde{P} \) such that

\[
(1) \quad |P - \tilde{P}| < \epsilon, \quad \text{and} \quad (2) \quad \inf \{ \beta : \limsup_{n \to \infty} \frac{1}{n^\beta} H_\mu(\bigvee_{i=0}^{n-1} \tilde{P}) = 0 \} = \tau.
\]

Before we introduce the notion of entropy dimension for a measure-preserving system, we define the dimension of a subset \( S \) of positive integers \( \mathbb{N} \). Let \( S = \{s_1 < s_2 < \cdots \} \) be an increasing sequence of positive integers. For \( \tau \geq 0 \), we define

\[
\overline{D}(S, \tau) = \limsup_{n \to \infty} \frac{n}{(s_n)^\tau} \quad \text{and} \quad \underline{D}(S, \tau) = \liminf_{n \to \infty} \frac{n}{(s_n)^\tau}.
\]

It is clear that \( \overline{D}(S, \tau) \leq \overline{D}(S, \tau') \) if \( \tau \geq \tau' \geq 0 \) and \( \overline{D}(S, \tau) \notin \{0, +\infty\} \) for at most one \( \tau \geq 0 \). We define the upper dimension of \( S \) by

\[
\overline{D}(S) = \inf \{ \tau \geq 0 : \overline{D}(S, \tau) = 0 \} = \sup \{ \tau \geq 0 : \overline{D}(S, \tau) = \infty \}.
\]

Similarly, \( \underline{D}(S, \tau) \leq \underline{D}(S, \tau') \) if \( \tau \geq \tau' \geq 0 \) and \( \underline{D}(S, \tau) \notin \{0, +\infty\} \) for at most one \( \tau \geq 0 \). We define the lower dimension of \( S \) by

\[
\underline{D}(S) = \inf \{ \tau \geq 0 : \underline{D}(S, \tau) = 0 \} = \sup \{ \tau \geq 0 : \underline{D}(S, \tau) = \infty \}.
\]

Clearly \( 0 \leq \underline{D}(S) \leq \overline{D}(S) \leq 1 \). When \( \overline{D}(S) = \underline{D}(S) = \tau \), we say \( S \) has dimension \( \tau \). For example, if \( S \) has positive density, then \( \overline{D}(S) = \underline{D}(S) = 1 \) and if \( S = \{n^2 : n = 1, 2, \cdots\} \), then clearly \( \overline{D}(S) = \underline{D}(S) = \frac{1}{2} \).

In the following, we will investigate the dimension of a special kind of sequences, which is called the entropy generating sequence.

Let \((X, \mathcal{B}, \mu, T)\) be a measure-theoretical dynamical system (MDS, for short) and \( \alpha \in \mathcal{P}_X \), where \( \mathcal{P}_X \) denotes the collection of finite measurable partitions of \( X \). We say an increasing sequence \( S = \{s_1 < s_2 < \cdots\} \) of \( \mathbb{N} \) is an entropy generating sequence of \( \alpha \) if

\[
\liminf_{n \to \infty} \frac{1}{n} H_\mu(\bigvee_{i=1}^n T^{-s_i} \alpha) > 0.
\]
We say $S = \{s_1 < s_2 < \cdots \}$ of $\mathbb{N}$ is a **positive entropy sequence** of $\alpha$ if the **sequence entropy** of $\alpha$ along the sequence $S$, which is defined by

$$h^S_\mu(T, \alpha) := \limsup_{n \to \infty} \frac{1}{n} H_\mu(\bigvee_{i=1}^n T^{-s_i} \alpha),$$

is positive.

Denote $\mathcal{E}_\mu(T, \alpha)$ by the set of all entropy generating sequences of $\alpha$, and $\mathcal{P}_\mu(T, \alpha)$ by the set of all positive entropy sequences of $\alpha$. Clearly $\mathcal{P}_\mu(T, \alpha) \supset \mathcal{E}_\mu(T, \alpha)$.

**Definition 2.1.** Let $(X, \mathcal{B}, \mu, T)$ be a MDS and $\alpha \in \mathcal{P}_X$. We define

$$\mathcal{D}^e_\mu(T, \alpha) = \begin{cases} \sup_{S \in \mathcal{E}_\mu(T, \alpha)} \mathcal{D}(S) & \text{if } \mathcal{E}_\mu(T, \alpha) \neq \emptyset \\ 0 & \text{if } \mathcal{E}_\mu(T, \alpha) = \emptyset \end{cases},$$

$$\mathcal{D}^p_\mu(T, \alpha) = \begin{cases} \sup_{S \in \mathcal{P}_\mu(T, \alpha)} \mathcal{D}(S) & \text{if } \mathcal{P}_\mu(T, \alpha) \neq \emptyset \\ 0 & \text{if } \mathcal{P}_\mu(T, \alpha) = \emptyset \end{cases}.$$  

Similarly, we define $\mathcal{D}^e_\mu(T, \alpha)$ and $\mathcal{D}^p_\mu(T, \alpha)$ by changing the upper dimension into lower dimension.

**Definition 2.2.** Let $(X, \mathcal{B}, \mu, T)$ be a MDS. We define

$$\mathcal{D}^e_\mu(X, T) = \sup_{\alpha \in \mathcal{P}_X} \mathcal{D}^e_\mu(T, \alpha), \quad \mathcal{D}^p_\mu(X, T) = \sup_{\alpha \in \mathcal{P}_X} \mathcal{D}^p_\mu(T, \alpha),$$

$$\mathcal{D}^e_\mu(X, T) = \sup_{\alpha \in \mathcal{P}_X} \mathcal{D}^e_\mu(T, \alpha), \quad \mathcal{D}^p_\mu(X, T) = \sup_{\alpha \in \mathcal{P}_X} \mathcal{D}^p_\mu(T, \alpha).$$

Since the sequence entropies along a given sequence are the same for mutually conjugated systems, we can deduce that these four quantities are also conjugacy invariants. But the following proposition shows that $\mathcal{D}^p_\mu(X, T)$ can only take trivial values $0$ and $1$. We recall that a MDS $(X, \mathcal{B}, \mu, T)$ is called null if $h^S_\mu(T, \alpha) = 0$ for any sequence $S$ of $\mathbb{N}$ and $\alpha \in \mathcal{P}_X$.

**Proposition 2.3.** Let $(X, \mathcal{B}, \mu, T)$ be a MDS. Then

$$\mathcal{D}^p_\mu(T, \alpha) = \begin{cases} 1 & \text{if } \mathcal{P}_\mu(T, \alpha) \neq \emptyset \\ 0 & \text{if } \mathcal{P}_\mu(T, \alpha) = \emptyset \end{cases} \quad \text{for } \alpha \in \mathcal{P}_X.$$  

Moreover, $\mathcal{D}^p_\mu(X, T) = 0$ or $1$, and $\mathcal{D}^p_\mu(X, T) = 0$ if and only if $(X, \mathcal{B}, \mu, T)$ is null.

**Proof.** When $\mathcal{P}_\mu(T, \alpha) = \emptyset$, $\mathcal{D}^p_\mu(T, \alpha) = 0$. Now assume $\mathcal{P}_\mu(T, \alpha) \neq \emptyset$, thus there exists $S = \{s_1 < s_2 < \cdots \} \subset \mathbb{N}$ such that

$$\limsup_{n \to +\infty} \frac{1}{n} H_\mu(\bigvee_{i=1}^n T^{-s_i} \alpha) = a > 0.$$
Next we take $1 \leq n_1 < n_2 < n_3 < \cdots$ such that $n_{j+1} \geq 2s_{n_j}$ for each $j \in \mathbb{N}$ and at the same time \( \limsup_{j \to +\infty} \frac{1}{n_j} H_\mu(\bigvee_{i=1}^{n_j} T^{-s_i} \alpha) = a \). Then put

\[
F = S \cup \{1, 2, \cdots, n_1\} \cup \bigcup_{i=1}^{\infty} \{s_{n_i} + 1, s_{n_i} + 2, \cdots, n_{i+1}\}.
\]

For simplicity, we write $F = \{f_1 < f_2 < \cdots \}$. Notice that

\[
F \cap [1, s_{n_j}] \subset [1, n_j] \cup (F \cap [n_j + 1, s_{n_j}]) \subset [1, n_j] \cup \{s_1, s_2, \cdots, s_{n_j}\},
\]

hence $|F \cap [1, s_{n_j}]| \leq 2n_j$. So we have

\[
\limsup_{n \to +\infty} \frac{1}{n} H_\mu(\bigvee_{i=1}^{n_j} T^{-s_{i}} \alpha) \geq \limsup_{j \to +\infty} \frac{H_\mu(\bigvee_{i=1}^{n_j} T^{-s_{i}} \alpha)}{|F \cap [1, s_{n_j}]|} \geq \limsup_{j \to +\infty} \frac{H_\mu(\bigvee_{i=1}^{n_j} T^{-s_{i}} \alpha)}{2n_j} = \frac{a}{2} > 0,
\]

therefore $F \in \mathcal{P}_\mu(T, \alpha)$. Since $n_{j+1} \geq 2s_{n_j}$ for each $j \in \mathbb{N}$, it is easy to see that $\mathcal{D}(F) = 1$. This implies $\mathcal{D}_\mu^e(T, \alpha) = 1$ as $F \in \mathcal{P}_\mu(T, \alpha)$.

In the following, we investigate the relations among these dimensions.

**Proposition 2.4.** Let $(X, \mathcal{B}, \mu, T)$ be a MDS and $\alpha \in \mathcal{P}_X$. Then

\[
\mathcal{D}_\mu^c(T, \alpha) \leq \mathcal{D}_\mu^e(T, \alpha) = \mathcal{D}_\mu^p(T, \alpha) \leq \mathcal{D}_\mu^p(T, \alpha).
\]

**Proof.** 1). $\mathcal{D}_\mu^c(T, \alpha) \leq \mathcal{D}_\mu^e(T, \alpha)$ and $\mathcal{D}_\mu^p(T, \alpha) \leq \mathcal{D}_\mu^p(T, \alpha)$ are obvious by Definition 2.1.

2). We will show that $\mathcal{D}_\mu^e(T, \alpha) \leq \mathcal{D}_\mu^p(T, \alpha)$. If $\mathcal{D}_\mu^e(T, \alpha) = 0$, then it is obvious that $\mathcal{D}_\mu^e(T, \alpha) \leq \mathcal{D}_\mu^p(T, \alpha)$. Now we assume that $\mathcal{D}_\mu^e(T, \alpha) > 0$, and $\tau \in (0, \mathcal{D}_\mu^e(T, \alpha))$ is given.

There exists $S = \{s_1 < s_2 < \cdots \} \in \mathcal{E}_\mu(T, \alpha)$ with $\mathcal{D}(S) > \tau$, i.e. \( \limsup_{n \to +\infty} \frac{n}{s_n^\tau} = +\infty \). Hence

\[
(2.1) \quad \limsup_{n \to +\infty} \frac{n}{n + s_n^\tau} = 1.
\]

Next we put $F = S \cup \{[n^\frac{1}{2}] : n \in \mathbb{N}\}$, where $[r]$ denotes the largest integer less than or equal to $r$. Clearly $\mathcal{D}(F) \geq \tau$.

Let $F = \{f_1 < f_2 < \cdots \}$. Then for each $n \in \mathbb{N}$ there exists unique $m(n) \in \mathbb{N}$ such that $s_n = f_{m(n)}$. Since

\[
\{s_1, s_2, \cdots, s_n\} \subseteq \{f_1, f_2, \cdots, f_{m(n)}\} \subseteq \{s_1, s_2, \cdots, s_n\} \cup \{[k^\frac{1}{2}] : k \leq s_n^\tau\},
\]
we have $n \leq m(n) \leq n + s_n^\tau$. Combing this with \((2.1)\), we get

\begin{equation}
\limsup_{n \to +\infty} \frac{n}{m(n)} = 1.
\end{equation}

Now we have

\[
\limsup_{m \to +\infty} \frac{H_\mu(\bigvee_{i=1}^{m} T^{-f_i} \alpha)}{m} \geq \limsup_{n \to +\infty} \frac{H_\mu(\bigvee_{i=1}^{m(n)} T^{-f_i} \alpha)}{m(n)}
\]
\[
\geq \limsup_{n \to +\infty} \frac{H_\mu(\bigvee_{i=1}^{n} T^{-s_i} \alpha)}{n} \cdot \frac{n}{m(n)}
\]
\[
= \liminf_{n \to +\infty} \frac{H_\mu(\bigvee_{i=1}^{n} T^{-s_i} \alpha)}{n} \cdot \limsup_{n \to +\infty} \frac{n}{m(n)} \quad \text{(by \((2.2)\))}
\]
\[
> 0 \quad \text{(since } S \in \mathcal{E}_\mu(T, \alpha)).
\]

This implies $F \in \mathcal{P}_\mu(T, \alpha)$. Hence $D_p^\mu(T, \alpha) \geq D(F) \geq \tau$. Since $\tau$ is arbitrary, we have $D_p^\mu(T, \alpha) \leq D_p^\mu(T, \alpha)$.

3). We need to prove that $D_p^\mu(T, \alpha) \leq D_e^\mu(T, \alpha)$. If $D_p^\mu(T, \alpha) = 0$, then it is obvious that $D_p^\mu(T, \alpha) \leq D_e^\mu(T, \alpha)$. Now we assume that $D_p^\mu(T, \alpha) > 0$ and $\tau \in (0, D_p^\mu(T, \alpha))$ is given.

In the following, we show that

**Fact A.** There exist a sequence $F = \{f_1 < f_2 < \cdots\}$ of natural numbers and a real number $d > 0$ such that $D(F) \geq \tau$ and for any $1 \leq m_1 \leq m_2$,

\begin{equation}
H_\mu(\bigvee_{i=1}^{m_2} T^{-f_i} \alpha) \geq (m_2 + 1 - m_1)d.
\end{equation}

Moreover by \((2.3)\) we know $F \in \mathcal{E}_\mu(T, \alpha)$. Hence $D_e^\mu(T, \alpha) \geq D(F) \geq \tau$. Finally since $\tau$ is arbitrary, we have $D_e^\mu(T, \alpha) \geq D_p^\mu(T, \alpha)$.

Now it remains to prove Fact A. First, there exists $S = \{s_1 < s_2 < \cdots\} \in \mathcal{P}_\mu(T, \alpha)$ with $D(S) > \tau$, i.e. $\liminf_{n \to +\infty} \frac{n}{s_n} = +\infty$. Hence there exists $a > 0$ such that

\begin{equation}
an \geq s_n^\tau
\end{equation}

for all $n \in \mathbb{N}$.

Since $S \in \mathcal{P}_\mu(T, \alpha)$, there exist an increasing sequence $\{n_1 < n_2 < \cdots < n_k < \cdots\}$ of positive integers and $0 < b < 4$ such that $H_\mu(\bigvee_{i=1}^{n_k} T^{-s_i} \alpha) \geq n_kb$ for all $k \in \mathbb{N}$.
\[ N. \text{ Without loss of generality(if necessary we choose a subsequence), we assume that } n_{k+1} \geq \frac{4(H_\mu^{(\alpha)}+1)}{b} \sum_{j=1}^{k} n_j \text{ for all } k \in \mathbb{N}. \] Let \( c = \frac{b}{4(H_\mu^{(\alpha)}+1)} \) and \( n_0 = 0. \) Then \( 0 < c < 1 \) and we have

**Claim:** For each \( k \in \mathbb{N}, \) there exist \( l_k \in \mathbb{N} \) and

\[ F_k := \{ i_1^k < i_2^k < \cdots < i_{l_k}^k \} \subseteq \{ n_{k-1} + 1, n_{k-1} + 2, \ldots, n_k \} \]

such that \( cn_k \leq l_k \leq n_k - n_{k-1} \) and \( H_\mu(\bigvee_{i \in F_k} T^{-s_i} \alpha) \geq \frac{|F_k|^2}{4} \) for each \( \emptyset \neq F_k \subseteq F_k. \)

**Proof of claim.** Assume that the claim is not true. Then there exist \( w \in \mathbb{N} \) and \( E_1, E_2, \ldots, E_w \subseteq \{ n_{k-1} + 1, n_{k-1} + 2, \ldots, n_k \} \) such that \( 1 \leq |E_1|, |E_2|, \ldots, |E_w| < cn_k, \)
\( E_i \cap E_j = \emptyset \) for any \( 1 \leq i < j \leq w \) and \( \bigcup_{i=1}^{w} E_i = \{ n_{k-1} + 1, n_{k-1} + 2, \ldots, n_k \} \) and for \( 1 \leq j \leq w - 1, \) \( H_\mu(\bigvee_{t \in E_j} T^{-s_t} \alpha) < \frac{|E_j|^2}{4}. \)

This implies that

\[
H_\mu^{n_k} \bigg( \bigvee_{i=1}^{n_{k-1}} T^{-s_i} \alpha \bigg) \leq H_\mu^{n_{k-1}} \bigg( \bigvee_{i=1}^{w} E_i \bigg) + \sum_{j=1}^{w} H_\mu^{n_k} \bigg( \bigvee_{t \in E_j} T^{-s_t} \alpha \bigg) \leq n_{k-1} H_\mu^{(\alpha)} + \sum_{j=1}^{w-1} |E_j| \frac{b}{4} + |E_w| H_\mu^{(\alpha)} \leq \frac{b}{4} n_k + \frac{b}{4} (n_k - n_{k-1}) + cn_k H_\mu^{(\alpha)} \leq \frac{b}{4} n_k + \frac{b}{4} n_k + \frac{b}{4} n_k \leq bn_k,
\]

a contradiction. This completes the proof of the Claim.

Let \( F = \bigcup_{k=1}^{\infty} \{ s_i : i \in F_k \}. \) For simplicity, we write \( F = \{ f_1 < f_2 < \cdots \}. \) Then

\[
\bar{D}(F, \tau) = \limsup_{m \to +\infty} \frac{m}{f_m} \geq \limsup_{v \to +\infty} \frac{\sum_{k=1}^{v} l_k}{(f_{\sum_{k=1}^{v} l_k})^r} = \limsup_{v \to +\infty} \frac{\sum_{k=1}^{v} l_k}{(s_{(s_{1}^{v})})^r} \geq \limsup_{v \to +\infty} \frac{l_v}{(s_{1}^{v})^r} \geq \limsup_{v \to +\infty} \frac{l_v}{a n_v} \geq \frac{c}{a} > 0.
\]

Hence \( \bar{D}(F) \geq \tau. \)

For a given \( m \in \mathbb{N}, \) there exists a unique \( k(m) \in \mathbb{N} \) such that \( \sum_{k=0}^{k(m)-1} l_k < m \leq \sum_{k=1}^{k(m)} l_k, \)
where \( l_0 = 0. \) Set \( r(m) = m - \sum_{k=0}^{k(m)-1} l_k. \) Then \( f_m = s_{r(m)} \). Now for \( 1 \leq m_1 \leq m_2, \)
there are three cases.
Case 1: $k(m_1) = k(m_2)$. Then
\[
H_\mu(\bigvee_{i=m_1}^{m_2} T^{-f_i} \alpha) = H_\mu(\bigvee_{j=r(m_1)}^{r(m_2)} T^{-s_{k(m_1)}} \alpha) \geq \frac{b}{4}(r(m_2) + 1 - r(m_1)) \quad \text{(by Claim)}
\]
\[
= \frac{b}{4}(m_2 - m_1 + 1).
\]

Case 2: $k(m_2) = k(m_1) + 1$. Then
\[
H_\mu(\bigvee_{i=m_1}^{m_2} T^{-f_i} \alpha) = H_\mu(\bigvee_{j=r(m_1)}^{l_{k(m_1)}} T^{-s_{k(m_1)}} \alpha \vee \bigvee_{j=1}^{r(m_2)} T^{-s_{k(m_2)}} \alpha)
\geq \frac{1}{2} \left( H_\mu(\bigvee_{j=r(m_1)}^{l_{k(m_1)}} T^{-s_{k(m_1)}} \alpha) + H_\mu(\bigvee_{j=1}^{r(m_2)} T^{-s_{k(m_2)}} \alpha) \right)
\geq \frac{b}{8}((l_{k(m_1)} + 1 - r(m_1)) + r(m_2)) = \frac{b}{8}(m_2 - m_1 + 1).
\]

Case 3: $k(m_2) \geq k(m_1) + 2$. Then
\[
H_\mu(\bigvee_{i=m_1}^{m_2} T^{-f_i} \alpha) \geq H_\mu(\bigvee_{j=1}^{l_{k(m_2)-1}} T^{-s_{k(m_2)-1}} \alpha \vee \bigvee_{j=1}^{r(m_2)} T^{-s_{k(m_2)}} \alpha)
\geq \frac{1}{2} \left( H_\mu(\bigvee_{j=1}^{l_{k(m_2)-1}} T^{-s_{k(m_2)-1}} \alpha) + H_\mu(\bigvee_{j=1}^{r(m_2)} T^{-s_{k(m_2)}} \alpha) \right)
\geq \frac{b}{8}(l_{k(m_2)-1} + r(m_2)) \geq \frac{b}{8}(cn_{k(m_2)-1} + r(m_2)) \quad \text{(by Claim)}
\geq \frac{b}{8}(c \sum_{j=1}^{k(m_2)-1} l_j + cr(m_2)) \quad \text{(by Claim)}
\geq \frac{bc}{8}m_2 \geq \frac{bc}{8}(m_2 - m_1 + 1).
\]

Let $d = \frac{bc}{8}$. Then \(2.3\) follows from the above three cases. \(\square\)

The following Theorem is a direct application of Proposition 2.4.

**Theorem 2.5.** Let \((X, \mathcal{B}, \mu, T)\) be a MDS, then $\overline{D}_\nu(X, T) = D_\nu(X, T)$.

By Proposition 2.4 and Theorem 2.5, we have the following definitions.

**Definition 2.6.** Let \((X, \mathcal{B}, \mu, T)\) be a MDS and $\alpha \in \mathcal{P}_X$. We define
\[
\overline{D}_\mu(T, \alpha) := \overline{D}_\mu^\nu(T, \alpha) = D_\mu(T, \alpha),
\]
which is called the upper entropy dimension of \( \alpha \). And we define
\[
D_\mu(T, \alpha) := D_\mu^u(T, \alpha)
\]
to be the lower entropy dimension of \( \alpha \). When \( D_\mu(T, \alpha) = D_\mu^u(T, \alpha) \), we note this quantity \( D_\mu(T, \alpha) \), the entropy dimension of \( \alpha \).

**Definition 2.7.** Let \((X, \mathcal{B}, \mu, T)\) be a MDS. We define
\[
D_\mu(X, T) = \sup_{\alpha \in \mathcal{P}_X} D_\mu(T, \alpha),
\]
which is called the upper metric entropy dimension of \((X, \mathcal{B}, \mu, T)\). And we define
\[
\overline{D}_\mu(X, T) = \sup_{\alpha \in \mathcal{P}_X} D_\mu(T, \alpha),
\]
which is called the lower metric entropy dimension of \((X, \mathcal{B}, \mu, T)\). When \( D_\mu(X, T) = \overline{D}_\mu(X, T) \), we denote the quantity by \( D_\mu(X, T) \) and call it the metric entropy dimension of \((X, \mathcal{B}, \mu, T)\).

By Proposition 2.3 and 2.4, we have

**Theorem 2.8.** Let \((X, \mathcal{B}, \mu, T)\) be a null MDS. Then \( D_\mu(X, T) = 0 \).

In the following, we study the basic properties of entropy dimension of measure-preserving system. But since in general the upper dimension and the lower dimension are not identical, most of the properties hold only for the upper dimension.

**Proposition 2.9.** Let \((X, \mathcal{B}, \mu, T)\) and \((Y, \mathcal{D}, \nu, S)\) be two MDS’s and \(\alpha, \beta \in \mathcal{P}_X, \eta \in \mathcal{P}_Y\). Then

1. If \(\alpha \leq \beta\), then \(\overline{D}_\mu(T, \alpha) \leq \overline{D}_\mu(T, \beta)\), where by \(\alpha \leq \beta\) we mean that every atom of \(\beta\) is contained in one of the atoms of \(\alpha\).
2. For any \(0 \leq m \leq n\), \(\overline{D}_\mu(T, \alpha) = \overline{D}_\mu(T, \bigvee_{i=m}^n T^{-i} \alpha)\).
3. \(\overline{D}_\mu(T, \alpha \vee \beta) = \max\{\overline{D}_\mu(T, \alpha), \overline{D}_\mu(T, \beta)\}\).
4. \(\overline{D}_\mu(X, T) = \sup\{\overline{D}_\mu(T, \alpha) : \alpha \in \mathcal{P}_X^2\}\), where \(\mathcal{P}_X^2\) denotes the set of all partitions by two measurable sets of \(X\).
5. \(\overline{D}_{\mu \times \nu}(T \times S, \alpha \times \eta) = \max\{\overline{D}_\mu(T, \alpha), \overline{D}_\nu(S, \eta)\}\).

Statements (1) and (2) also hold for lower dimensions and dimensions when they exist.

**Proof.** By the definition, (1) and (2) are obvious. Also (5) follows from (3). For (3), firstly we have \(\overline{D}_\mu(T, \alpha \vee \beta) \geq \max\{\overline{D}_\mu(T, \alpha), \overline{D}_\mu(T, \beta)\}\) by (1). Secondly, if \(\overline{D}_\mu(T, \alpha \vee \beta) = 0\), then it is clear that \(\overline{D}_\mu(T, \alpha \vee \beta) = \max\{\overline{D}_\mu(T, \alpha), \overline{D}_\mu(T, \beta)\}\).

Now we assume that \(0 < \overline{D}_\mu(T, \alpha \vee \beta)\). For any \(\tau \in (0, \overline{D}_\mu(T, \alpha \vee \beta))\). There exists \(S = \{s_1 < s_2 < \cdots\} \in \mathcal{P}_\mu(T, \alpha \vee \beta)\) with \(\overline{D}(S) > \tau\).

Since \(S \in \mathcal{P}_\mu(T, \alpha \vee \beta)\), \(\limsup_{n \to +\infty} \frac{1}{n} H_\mu(\bigvee_{i=1}^n T^{-s_i} (\alpha \vee \beta)) > 0\). This implies
\[
\limsup_{n \to +\infty} \frac{1}{n} H_\mu(\bigvee_{i=1}^n T^{-s_i} \alpha) > 0 \text{ or } \limsup_{n \to +\infty} \frac{1}{n} H_\mu(\bigvee_{i=1}^n T^{-s_i} \beta) > 0,
\]
that is, $S \in \mathcal{P}_\mu(T, \alpha)$ or $S \in \mathcal{P}_\mu(T, \beta)$. Hence $\tau \leq \overline{D}(S) \leq \max\{\overline{D}_\mu(T, \alpha), \overline{D}_\mu(T, \beta)\}$. As $\tau$ is arbitrary, we get $\overline{D}_\mu(T, \alpha \vee \beta) = \max\{\overline{D}_\mu(T, \alpha), \overline{D}_\mu(T, \beta)\}$.

Now we are to show (4). Clearly, $\overline{D}_\mu(X, T) \geq \sup\{\overline{D}_\mu(T, \alpha) : \alpha \in \mathcal{P}_X^2\}$. Conversely, for any $\alpha = \{A_1, \cdots, A_k\} \in \mathcal{P}_X$, let $\alpha_i = \{A_i, A_i'\}$ for $i = 1, 2, \cdots, k$. Then $\bigvee_{i=1}^k \alpha_i \geq \alpha$.

Hence by (1) and (3), we have

$$\overline{D}_\mu(T, \alpha) \leq \max\{\overline{D}_\mu(T, \alpha_i) : 1 \leq i \leq k\} \leq \sup\{\overline{D}_\mu(T, \alpha) : \alpha \in \mathcal{P}_X^2\}.$$ 

Finally, since $\alpha$ is arbitrary, we get (4).

\begin{lemma}
Let $(X, \mathcal{B}, \mu, T)$ be a MDS and $\alpha = \{A_1, A_2, \cdots, A_k\} \in \mathcal{P}_X$. Then for any $\epsilon > 0$, there exists $\delta > 0$ such that if $\beta = \{B_1, B_2, \cdots, B_k\} \in \mathcal{P}_X$ and $\mu(\beta \Delta \alpha) := \sum_{i=1}^k \mu(B_i \Delta A_i) < \delta$, then

1. $\overline{D}_\mu(T, \beta) > \overline{D}_\mu(T, \alpha) - \epsilon$,
2. $\overline{D}_\mu(T, \beta) > \overline{D}_\mu(T, \alpha) - \epsilon$ and
3. $\overline{D}_\mu(T, \beta) > \overline{D}_\mu(T, \alpha) - \epsilon$ when the dimensions exist.

\end{lemma}

\begin{proof}
We only prove for upper dimension. If $\overline{D}_\mu(T, \alpha) = 0$, it is obvious. Now assume that $\overline{D}_\mu(T, \alpha) > 0$. For any $\epsilon > 0$, there exists $S = \{s_1 < s_2 < s_3 < \cdots\} \in \mathcal{E}_\mu(T, \alpha)$ with $\overline{D}(S) > \overline{D}_\mu(T, \alpha) - \epsilon$. There exists $\delta > 0$ such that if $\beta = \{B_1, B_2, \cdots, B_k\} \in \mathcal{P}_X$ and $\mu(\beta \Delta \alpha) := \sum_{i=1}^k \mu(B_i \Delta A_i) < \delta$ then

$$H_\mu(\alpha \| \beta) + H_\mu(\beta \| \alpha) < \frac{1}{2} \liminf_{n \to \infty} \frac{1}{n} H_\mu\left(\bigvee_{i=1}^n T^{-s_i} \alpha\right)$$

(see Lemma 4.15 in [20]).

For any $\beta = \{B_1, B_2, \cdots, B_k\} \in \mathcal{P}_X$ and $\mu(\beta \Delta \alpha) := \sum_{i=1}^k \mu(B_i \Delta A_i) < \delta$,

$$\liminf_{n \to +\infty} \frac{1}{n} H_\mu\left(\bigvee_{i=1}^n T^{-s_i} \beta\right)$$

$$\geq \liminf_{n \to +\infty} \frac{1}{n} \left(H_\mu\left(\bigvee_{i=1}^n T^{-s_i} (\alpha \vee \beta)\right) - H_\mu\left(\bigvee_{i=1}^n T^{-s_i} \beta\right)\right)$$

$$\geq \liminf_{n \to +\infty} \frac{1}{n} \left(H_\mu\left(\bigvee_{i=1}^n T^{-s_i} \alpha\right) - n H_\mu(\beta \| \alpha)\right)$$

$$\geq \frac{1}{2} \liminf_{n \to +\infty} \frac{1}{n} H_\mu\left(\bigvee_{i=1}^n T^{-s_i} \alpha\right) > 0,$$

that is, $S \in \mathcal{E}_\mu(T, \beta)$. Hence $\overline{D}_\mu(T, \beta) \geq \overline{D}(S) > \overline{D}_\mu(T, \alpha) - \epsilon$.

\end{proof}

\begin{theorem}
Let $(X, \mathcal{B}, \mu, T)$ be a MDS.

1. If $\{\alpha_i\}_{i \in \mathbb{N}} \subset \mathcal{P}_X$ and $\alpha \in \mathcal{P}_X$ satisfying $\alpha \preceq \bigvee_{i \in \mathbb{N}} \alpha_i$, then $\overline{D}_\mu(T, \alpha) \leq \sup_{i \geq 1} \overline{D}_\mu(T, \alpha_i)$.

\end{theorem}
(2) If \( \{\alpha_i\}_{i \in \mathbb{N}} \subset \mathcal{P}_X \) and \( \alpha_i \not\sim \mathcal{B}(\text{mod } \mu) \), then \( \overline{D}_\mu(X, T) = \lim_{i \to +\infty} \overline{D}_\mu(T, \alpha_i) \). Moreover, if \( \alpha \) is a generating partition, i.e. \( \bigvee_{i=0}^\infty T^{-i} \alpha = \mathcal{B}(\text{mod } \mu) \), then \( \overline{D}_\mu(X, T) = \overline{D}_\mu(T, \alpha) \).

Proof. It is obvious that (1) implies (2). Now we are to show (1). Let \( \alpha = \{A_1, A_2, \cdots, A_k\} \in \mathcal{P}_X \) and fix \( \epsilon > 0 \). By Lemma 2.10 there exists \( \delta > 0 \) such that if \( \beta = \{B_1, B_2, \cdots, B_k\} \in \mathcal{P}_X \) and \( \mu(\beta \Delta \alpha) := \sum_{i=1}^k \mu(B_i \Delta A_i) < \delta \), then \( \overline{D}_\mu(T, \beta) > \overline{D}_\mu(T, \alpha) - \delta \).

Since the above inequality is true for any \( \alpha \) and \( \epsilon > 0 \), there exist \( N \in \mathbb{N} \) and \( \gamma = \{C_1, C_2, \cdots, C_k\} \leq \bigvee_{i=1}^N \alpha_i \) such that \( \mu(\gamma \Delta \alpha) < \delta \). Thus \( \overline{D}_\mu(T, \gamma) \geq \overline{D}_\mu(T, \alpha) - \epsilon \) and so

\[
\sup_{i \geq 1} \overline{D}_\mu(T, \alpha_i) \geq \max\{\overline{D}_\mu(T, \alpha_i) | i = 1, 2, \cdots, N\} = \overline{D}_\mu(T, \bigvee_{i=1}^N \alpha_i) \geq \overline{D}_\mu(T, \gamma) \geq \overline{D}_\mu(T, \alpha) - \epsilon.
\]

Since the above inequality is true for any \( \epsilon > 0 \), we get \( \sup_{i \geq 1} \overline{D}_\mu(T, \alpha_i) \geq \overline{D}_\mu(T, \alpha) \).

\(\Box\)

**Proposition 2.12.** Let \( (X, \mathcal{B}, \mu, T) \) be a MDS and \( \alpha \in \mathcal{P}_X \).

(1) For \( k \in \mathbb{N} \), we have \( \overline{D}_\mu(T^k, \alpha) = \overline{D}_\mu(T, \alpha) \). Moreover, \( \overline{D}_\mu(X, T^k) = \overline{D}_\mu(X, T) \) and this is also true for lower dimensions and dimensions whenever the dimensions exist.

(2) When \( T \) is invertible, \( \overline{D}_\mu(T, \alpha) = \overline{D}_\mu(T^{-1}, \alpha) \). Moreover \( \overline{D}_\mu(X, T) = \overline{D}_\mu(X, T^{-1}) \).

Proof. (1). We only prove for the upper dimension. Given \( k \in \mathbb{N} \). Let \( S \in \mathcal{E}_\mu(T^k, \alpha) \).

Then \( kS = \{ks : s \in S\} \in \mathcal{E}_\mu(T, \alpha) \). Since \( \overline{D}(kS) = \overline{D}(S), \overline{D}_\mu(T, \alpha) \geq \overline{D}(S) \). Finally since \( S \) is arbitrary, \( \overline{D}_\mu(T, \alpha) \geq \overline{D}(T^k, \alpha) \).

Conversely, let \( S = \{s_1 < s_2 < \cdots \} \in \mathcal{E}_\mu(T, \alpha) \). Without loss of generality, we assume \( s_1 \geq k \). Set \( S_1 = \{\lfloor \frac{s_i}{k} \rfloor : i \in \mathbb{N}\} \). For simplify, we write \( S_1 = \{t_1 < t_2 < \cdots \} \). Then \( \lfloor \frac{t_j}{k} \rfloor \leq j \leq \lfloor \frac{s_j}{k} \rfloor \) for all \( j \in \mathbb{N} \).

Now

\[
\liminf_{n \to +\infty} \frac{H_\mu\left(\bigvee_{j=1}^n T^{-kt_j}, \alpha\right)}{n} \geq \liminf_{n \to +\infty} \frac{H_\mu\left(\bigvee_{j=1}^{k-1} T^{-\left(kt_j+i\right)}, \alpha\right)}{kn} \geq \liminf_{n \to +\infty} \frac{H_\mu\left(\bigvee_{j=1}^n T^{-s_j}, \alpha\right)}{kn} > 0,
\]

as \( S \in \mathcal{E}_\mu(T, \alpha) \). This implies that \( S_1 \in \mathcal{E}_\mu(T^k, \alpha) \). Since \( \overline{D}(S) = \overline{D}(S_1), \overline{D}_\mu(T^k, \alpha) \geq \overline{D}(S_1) = \overline{D}(S) \). Finally since \( S \) is arbitrary, \( \overline{D}_\mu(T^k, \alpha) \geq \overline{D}_\mu(T, \alpha) \).

(2). Let \( T \) be invertible. By symmetry of \( T \) and \( T^{-1} \), it is sufficient to show \( \overline{D}_\mu(T^{-1}, \alpha) \geq \overline{D}_\mu(T, \alpha) \). If \( \overline{D}_\mu(T, \alpha) = 0 \), this is obvious. Now we assume that \( \overline{D}_\mu(T, \alpha) > 0 \). Given \( \tau \in (0, \overline{D}_\mu(T, \alpha)) \).
By Fact A in the proof of Proposition 2.4, we know that there exists a sequence 
\( S = \{s_1 < s_2 < \cdots \} \subset \mathbb{N} \) and \( a > 0 \) such that 
\( \overline{D}(S) > \tau \) and for any \( 1 \leq m_1 \leq m_2 \),

\[
(2.5) \quad H_\mu \left( \bigvee_{i=m_1}^{m_2} T^{-s_i} \alpha \right) \geq (m_2 + 1 - m_1)a.
\]

Since \( \overline{D}(S) > \tau \), there exists a sequence \( \{n_1 < n_2 < \cdots \} \) such that \( n_1 \geq 2 \), \( n_{i+1} \geq 1 + 2 \sum_{j=1}^{i} n_i \) and \( n_i \geq s_{n_i}^* \) for all \( i \in \mathbb{N} \). Let \( n_0 = 0 \), \( s_0 = 0 \), \( f_0 = 0 \) and \( f_m = s_{n_j} - s_{n_{j-m}} \) if \( n_{j-1} < m \leq n_j \) for some \( j \in \mathbb{N} \). Put \( F = \{f_m : m \in \mathbb{N}\} \).

Set \( n_{-1} = 0 \). Given \( n \in \mathbb{N} \) with \( n \geq n_1 + 1 \), there exists \( j \geq 2 \) such that \( n_{j-1} < n \leq n_j \). Now

\[
H_\mu \left( \bigvee_{m=1}^{n} T^{f_m} \alpha \right) \geq \max \{H_\mu \left( \bigvee_{m=n_j-2+1}^{n_j-1} T^{f_m} \alpha \right), H_\mu \left( \bigvee_{m=n_{j-1}+1}^{n} T^{f_m} \alpha \right) \}
\]

\[
= \max \{H_\mu \left( \bigvee_{m=n_j-2+1}^{n_j-1} T^{-s_{n_{j-1}}-s_{n_{j-1}}-m} \alpha \right), H_\mu \left( \bigvee_{m=n_{j-1}+1}^{n} T^{-s_{n_{j}}-s_{n_{j}}-m} \alpha \right) \}
\]

\[
= \max \{H_\mu \left( \bigvee_{m=n_j-2+1}^{n_j-1} T^{-s_{n_{j-1}}-m} \alpha \right), H_\mu \left( \bigvee_{m=n_{j-1}+1}^{n} T^{-s_{n_{j}}-m} \alpha \right) \}
\]

\[
\geq \max \{H_\mu \left( \bigvee_{i=1}^{n_j-1-n_{j-2}-1} T^{-s_i} \alpha \right), H_\mu \left( \bigvee_{i=n_j-n}^{n} T^{-s_i} \alpha \right) \}
\]

\[
\geq \max \{a(n_{j-1} - n_{j-2} - 1), a(n - n_{j-1}) \} \quad \text{(by (2.5))}
\]

\[
\geq \max \{a \frac{n_{j-1}}{2}, a \frac{n - n_{j-1}}{2} \} \geq a \frac{(n_{j-1} + n - n_{j-1})}{4} \geq a \frac{n}{4}
\]

that is,

\[
(2.6) \quad H_\mu \left( \bigvee_{m=1}^{n} T^{f_m} \alpha \right) \geq \frac{a}{4} n.
\]

Since (2.6) is true for any \( n \geq n_1 + 1 \), \( F \in \mathcal{E}_\mu(T^{-1}, \alpha) \). Now note that \( f_{n_j} = s_{n_j} \) for \( j \in \mathbb{N} \), one has

\[
\overline{D}(F, \tau) = \limsup_{m \to +\infty} \frac{m}{f_m^\tau} \geq \limsup_{j \to +\infty} \frac{n_j}{s_{n_j}^\tau} \geq 1.
\]

Hence \( \overline{D}(F) \geq \tau \). Moreover \( \overline{D}_\mu(T^{-1}, \alpha) \geq \overline{D}(F) \geq \tau \) as \( F \in \mathcal{E}_\mu(T^{-1}, \alpha) \). Finally since \( \tau \) is arbitrary, we have \( \overline{D}_\mu(T^{-1}, \alpha) \geq \overline{D}_\mu(T, \alpha) \).
3. FACTORS AND JOININGS VIA ENTROPY DIMENSIONS.

In this section, we will introduce several notions like dimension sets, dimension \( \sigma \)-algebras and uniform dimension systems to understand the structure of entropy zero systems.

When the metric entropy dimension of a MDS exists, the entropy dimensions of its factors still may not exists. One of the examples is a product system, one system has entropy dimension but the other system has no entropy dimension. So in this section we are only to consider the upper entropy dimension.

**Definition 3.1.** We define the dimension set of a MDS \((X, \mathcal{B}, \mu, T)\) by

\[
\text{Dims}_\mu(X, T) = \{ \overline{D}_\mu(T, \{ A, X \setminus A \}) : A \in \mathcal{B} \text{ and } 0 < \mu(A) < 1 \}.
\]

**Remark 3.2.** It is clear that \(\text{Dims}_\mu(X, T) = \emptyset\) if and only if \((X, \mathcal{B}, \mu, T)\) is a trivial system, i.e., \(\mathcal{B} = \{ \emptyset, X \}\) (mod \(\mu\)). We suppose \(\text{sup} \{ \tau \in \text{Dims}_\mu(X, T) \} = 0\) when \(\text{Dims}_\mu(X, T) = \emptyset\). Thus \(\overline{D}_\mu(X, T) = \text{sup} \{ \tau \in \text{Dims}_\mu(X, T) \} \).

Let \((X, \mathcal{B}, \mu, T)\) be a MDS. For \(\tau \in [0, 1)\), we define

\[
P^\tau_\mu(T) := \{ A \in \mathcal{B} : \overline{D}_\mu(T, \{ A, X \setminus A \}) \leq \tau \}.
\]

It is clear that \(P^\tau_\mu(T) \subseteq P^{\tau_1}_\mu(T) \subseteq P_\mu(T)\) for any \(0 \leq \tau_1 \leq \tau_2 < 1\), where \(P_\mu(T)\) is the Pinsker \(\sigma\)-algebra of \((X, \mathcal{B}, \mu, T)\).

**Theorem 3.3.** Let \((X, \mathcal{B}, \mu, T)\) be a MDS and \(\tau \in [0, 1)\). Then

1. \(P^\tau_\mu(T)\) is a sub \(\sigma\)-algebra of \(\mathcal{B}\).
2. \(T^{-1} P^\tau_\mu(T) = P^\tau_\mu(T)\) (mod \(\mu\)).
3. For \(k \geq 1\), \(P^{\tau k}_\mu(T) = P^\tau_\mu(T^k)\). If \(T\) is invertible, then \(P^\tau_\mu(T) = P^\tau_\mu(T^{-1})\).

**Proof.** (1). Clearly, \(\emptyset, X \in P^\tau_\mu(T)\). Let \(A, B \in P^\tau_\mu(T)\). Since \(X \setminus (X \setminus A) = A\), \(X \setminus A \in P^\tau_\mu(T)\).

Let \(A_i \in P^\tau_\mu(T)\), \(i \in \mathbb{N}\). Now we are to show that \(\bigcup_{i=1}^\infty A_i \in P^\tau_\mu(T)\), i.e.,

\[
\overline{D}_\mu(T, \{ \bigcup_{i=1}^\infty A_i, X \setminus \bigcup_{i=1}^\infty A_i \}) \leq \tau.
\]

Since \(\{ \bigcup_{i=1}^\infty A_i, X \setminus \bigcup_{i=1}^\infty A_i \} \subseteq \bigvee_{i=1}^\infty \{ A_i, X \setminus A_i \}\), using Theorem 2.11 (1) we get

\[
\overline{D}_\mu(T, \{ \bigcup_{i=1}^\infty A_i, X \setminus \bigcup_{i=1}^\infty A_i \}) \leq \text{sup}_{i \in \mathbb{N}} \overline{D}_\mu(T, \{ A_i, X \setminus A_i \}) \leq \tau.
\]

Hence \(\bigcup_{i=1}^\infty A_i \in P^\tau_\mu(T)\). This shows \(P^\tau_\mu(T)\) is a sub \(\sigma\)-algebra of \(\mathcal{B}\).

(2). Since for \(A \in \mathcal{B}\),

\[
\overline{D}_\mu(T, \{ A, X \setminus A \}) = \overline{D}_\mu(T, T^{-1} \{ A, X \setminus A \}) = \overline{D}_\mu(T, \{ T^{-1} A, X \setminus T^{-1} (A) \}),
\]

we have \(T^{-1} P^\tau_\mu(T) \subseteq P^\tau_\mu(T)\).

Conversely, let \(A \in P^\tau_\mu(T)\). Then \(A \in P^\tau_\mu(T) \subseteq P_\mu(T) = T^{-1} P_\mu(T)\). Hence there exists \(B \in \mathcal{B}\) such that \(A = T^{-1} B\). Now note that

\[
\overline{D}_\mu(T, \{ B, X \setminus B \}) = \overline{D}_\mu(T, \{ T^{-1} B, T^{-1} (X \setminus B) \}) = \overline{D}_\mu(T, \{ A, X \setminus A \}) \leq \tau,
\]
we have $B \in P^r_\mu(T)$, i.e., $A \in T^{-1}P^r_\mu(T)$. Therefore $P^r_\mu(T) = T^{-1}P^r_\mu(T)$.

(3). For $\alpha \in \mathcal{P}_X$ and $k \in \mathbb{N}$, we have $\overline{D}_\mu(T^k, \alpha) = \overline{D}_\mu(T, \alpha)$ (see Proposition 2.12). Hence $\alpha \subset P^r_\mu(T)$ if and only if $\alpha \subset P^r_\mu(T^k)$. This implies $P^r_\mu(T) = P^r_\mu(T^k)$. Finally, $P^r_\mu(T) = P^r_\mu(T^{-1})$ by Proposition 2.12.

We call $P^r_\mu(T)$ the $\tau^r$-dimension sub $\sigma$-algebra of $(X, \mathcal{B}, \mu, T)$, and if $P^r_\mu(T) = \mathcal{B}$, we call $(X, \mathcal{B}, \mu, T)$ a $\tau^r$-dimension system.

The following theorem states that the entropy dimension set must be right closed.

**Theorem 3.4.** Let $(X, \mathcal{B}, \mu, T)$ be an invertible ergodic MDS. If $r_i \in \text{Dims}_\mu(X, T)$, $i \in \mathbb{N}$ and $r \in [0, 1]$ such that $r_i \nearrow r$, then $r \in \text{Dims}_\mu(X, T)$.

**Proof.** For $i \in \mathbb{N}$, let $P_i = \{A_i, X \setminus A_i\}$ for some $A_i \in \mathcal{B}$ with $0 < \mu(A_i) < 1$ such that $r_i = \overline{D}_\mu(T, P_i) \in \text{Dims}_\mu(X, T)$. We denote by $\mathcal{B}_i = \bigvee_{n=-\infty}^{\infty} T^{-n}P_i$, the $\sigma$-algebra generated by $P_i$. Let $\mathcal{D} = \bigvee_{i=1}^{\infty} \mathcal{B}_i$. For each $i \in \mathbb{N}$, $h_\mu(T, P_i) = 0$ since $r_i < 1$. Thus $h_\mu(T, \mathcal{D}) = 0$, moreover by Krieger’s generator theorem, we have a partition $P = \{A, X \setminus A\}$ such that $\mathcal{D} = \bigvee_{n=-\infty}^{\infty} T^{-n}P$. Then we have $\overline{D}_\mu(T, P_i) \geq r_i$ for all $i = 1, 2, \ldots$, and hence $\overline{D}_\mu(T, P) = r$ by Proposition 2.9 (2) and Theorem 2.11. This shows $r \in \text{Dims}_\mu(X, T)$.

In the following we will give a disjointness theorem via entropy dimension. Let’s recall the related notions first.

Let $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{D}, \nu, S)$ be two MDS’s. A probability measure $\lambda$ on $(X \times Y, \mathcal{B} \times \mathcal{D})$ is a joining of $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{D}, \nu, S)$ if it is $T \times S$-invariant, and has $\mu$ and $\nu$ as marginals; i.e. $\text{proj}_X(\lambda) = \mu$ and $\text{proj}_Y(\lambda) = \nu$. We let $J(\mu, \nu)$ be the space of all joinings of $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{D}, \nu, S)$. We say $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{D}, \nu, S)$ are disjoint if $J(\mu, \nu) = \{\mu \times \nu\}$. More generally if $\{(X_i, \mathcal{B}_i, \mu_i, T_i)\}_{i \in I}$ is a collection of MDS’s, a probability measure $\lambda$ on $(\prod_{i \in I} X_i, \prod_{i \in I} \mathcal{B}_i)$ is a joining of $\{(X_i, \mathcal{B}_i, \mu_i, T_i)\}$ if it is $\prod_{i \in I} T_i$-invariant, and has $\mu_i$ as marginals; i.e. $\text{proj}_{X_i}(\lambda) = \mu_i$ for every $i \in I$. We let $J(\{\mu_i\}_{i \in I})$ be the spaces of all these joinings. When $(X_i, \mathcal{B}_i, \mu_i, T_i) = (X, \mathcal{B}, \mu, T)$ for $i \in I$, we write $J(\{\mu_i\}_{i \in I})$ as $J(\mu; I)$ and call $\lambda \in J(\mu; I)$ I-fold self-joinings.

**Lemma 3.5.** Let $(X, \mathcal{B}, \mu, T)$ be a MDS. If $\eta \in J(\mu; \mathbb{Z})$, then $\overline{D}_\eta(X^\mathbb{Z}, T^\mathbb{Z}) = \overline{D}_\mu(X, T)$.

**Proof.** First there exists $\{\alpha_i\}_{i=1}^{\infty} \subseteq \mathcal{P}_X$ such that $\alpha_1 \leq \alpha_2 \leq \alpha_3 \cdots$ and $\bigvee_{i=1}^{\infty} \alpha_i = \mathcal{B}$ (mod $\mu$). Then for $i \in \mathbb{Z}$ let $\pi_i : X^\mathbb{Z} \to X$ be the $i$-th coordinate projection. Let $\beta^i_j = \pi_i^{-1}(\alpha_j)$ for $i \in \mathbb{N}$ and $j \in \mathbb{Z}$. Then $\beta^i_j \in \mathcal{P}_{X^\mathbb{Z}}$. It is clear that $\bigvee_{i \in \mathbb{N}, j \in \mathbb{Z}} \beta^i_j = \mathcal{B}^\mathbb{Z}$ (mod $\lambda$) and $\overline{D}_\mu(T, \alpha_i) = \overline{D}_\eta(T, \beta^i_j)$ for $i \in \mathbb{N}, j \in \mathbb{Z}$. Hence by Theorem 2.11 (1),

$$
\overline{D}_\eta(X^\mathbb{Z}, T^\mathbb{Z}) = \sup_{i, j} \overline{D}_\eta(T^\mathbb{Z}, \beta^i_j) = \sup_{i \in \mathbb{N}} \overline{D}_\mu(T, \alpha_i) = \overline{D}_\mu(X, T).
$$

This finishes the proof of Lemma.
Theorem 3.6. Let \((X, \mathcal{B}, \mu, T)\) be an invertible MDS and \((Y, \mathcal{D}, \nu, S)\) be an ergodic MDS. If \(\text{Dims}_\mu(X, T) > \overline{\text{Dims}}_\nu(Y, S)\) (i.e. for any \(\tau \in \text{Dims}_\nu(X, T), \tau > \overline{\text{Dims}}_\nu(Y, S)\)), then \((X, \mathcal{B}, \mu, T)\) is disjoint from \((Y, \mathcal{D}, \nu, S)\).

Proof. We follow the arguments in the proof of Theorem 1 in \([8]\). Let \(\lambda\) be a joining of \((X, \mathcal{B}, \mu, T)\) and \((Y, \mathcal{D}, \nu, S)\). Let

\[
\lambda = \int_X \delta_x \times \lambda_x d\mu(x)
\]

be the disintegration of \(\lambda\) over \(\mu\) and define probability measure \(\lambda_\infty\) on \(X \times Y^Z\) and \(\nu_\infty\) on \(Y^Z\) by:

\[
\lambda_\infty = \int_X \delta_x \times (\cdots \times \lambda_x \times \lambda_x \cdots) d\mu(x) \quad \text{and} \quad \nu_\infty = \int_X (\cdots \times \lambda_x \times \lambda_x \cdots) d\mu(x).
\]

Since \(\lambda\) is \(T \times S\)-invariant,

\[
\lambda = (T \times S)\lambda = \int_X \delta_{Tx} \times S\lambda_x d\mu(x) = \int_X \delta_x \times S\lambda_{T^{-1}x} d\mu(x).
\]

By uniqueness of disintegration we have \(\lambda_x = S\lambda_{T^{-1}x}\) for \(\mu\)-a.e. \(x \in X\), i.e., \(S\lambda_x = \lambda_{Tx}\) for \(\mu\)-a.e. \(x \in X\). Moreover

\[
S^Z\nu_\infty = \int_X (\cdots \times S\lambda_x \times S\lambda_x \cdots) d\mu(x) = \int_X (\cdots \times \lambda_x \times \lambda_{Tx} \cdots) d\mu(x) = \nu_\infty
\]

This implies \(\nu_\infty \in J(\nu, \mathbb{Z})\) since \(\int_X \lambda_x d\mu(x) = \nu\). It is also clear that \(\lambda_\infty \in J(\{\mu, \nu_\infty\})\), i.e., \(\lambda_\infty\) is a joining of \((X, \mathcal{B}, \mu, T)\) and \((Y^Z, \mathcal{D}^Z, \nu_\infty, S^Z)\).

Let \(E = \{E \in \mathcal{B} : \exists F \in \mathcal{D}^Z, \lambda_\infty((E \times Y^Z) \Delta (X \times F)) = 0\}\) and

\[
\mathcal{F} = \{F \in \mathcal{D}^Z : \exists E \in \mathcal{B}, \lambda_\infty((E \times Y^Z) \Delta (X \times F)) = 0\}.
\]

Then \(\mathcal{E}\) is a \(T\)-invariant sub-\(\sigma\)-algebra and \(\mathcal{F}\) is a \(S^Z\)-invariant sub-\(\sigma\)-algebra.

Now for any \(E \in \mathcal{E}\) there exists \(F \in \mathcal{D}^Z\) such that \(\lambda_\infty((E \times Y^Z) \Delta (X \times F)) = 0\). Now

\[
\overline{\text{Dims}}_\mu(X, T) > \overline{\text{Dims}}_\nu(Y, S), \text{ we have } \mu(E) = 0 \text{ or } 1. \quad \text{Hence } \mathcal{E} = \{\emptyset, X\} \pmod{\mu} \quad \text{and so } \mathcal{F} = \{\emptyset, Y^Z\} \pmod{\nu_\infty}.
\]

Define a transformation \(R : X \times Y^Z \to X \times Y^Z\) by \(R(x, y) = (x, \sigma y)\) where \(y = \{y_i\}_{i \in \mathbb{Z}} \in Y^\mathbb{Z}\) and \(\sigma\) is the left shift on \(Y^\mathbb{Z}\). Now if \(f(x, y)\) is an \(R\)-invariant measurable function on \(X \times Y^Z\) then for every \(x \in X\) the function \(f_x(y) = f(x, y)\) is a \(\sigma\)-invariant function on the Bernoulli \(\mathbb{Z}\)-system \((Y^\mathbb{Z}, \lambda^Z, \sigma)\), hence a constant, \(\lambda^Z\)-a.e.; i.e., \(f(x, y) = f(x), \lambda_\infty\)-a.e.. Thus every \(R\)-invariant function on \(X \times Y^Z\) is \(\mathcal{B} \times Y^Z\)-measurable.
For any $F \in D^Z$ with $\nu_\infty(\sigma^{-1} F \Delta F) = 0$, let $f(x, y) = 1_F(y)$ for $\lambda_\infty$-a.e. $(x, y) \in X \times Y^Z$. Then $f$ is $R$-invariant and so $f$ is $B \times Y^Z$-measurable. Thus there exists $E \in B$ such that $f(x, y) = 1_E(x)$ for $\lambda_\infty$-a.e. $(x, y) \in X \times Y^Z$ since $f$ is a characteristic function. This implies $F \in F = \{0, Y^Z\} \mod \nu_\infty$ so $\nu_\infty(F) = 0$ or $1$. Hence $(Y^Z, D^Z, \nu_\infty, \sigma)$ is ergodic.

Moreover since $(Y^Z, D^Z, \lambda^Z, \sigma)$ is ergodic for $\mu$-a.e. $x \in X$ and $\nu_\infty = \int_X \lambda^Z d\mu(x)$, we have $\lambda^Z = \nu_\infty$ for $\mu$-a.e. $x \in X$. Hence $\lambda = \mu \times \nu$. Then it follows that $(X, B, \mu, T)$ is disjoint from $(Y, D, \nu, S)$.

In the following, we consider some special case for dimension set.

Let $\tau \in (0, 1]$, we call $(X, B, \mu, T)$ a $\tau$−uniform entropy dimension system ($\tau$−u.d. system for short) if $\text{Dims}_\mu(X, T) = \{\tau\}$ and call $(X, B, \mu, T)$ a $\tau^+$− entropy dimension system ($\tau^+−d.$ system for short) if $\text{Dims}_\mu(X, T) \subset [\tau, 1]$. If $0 \notin \text{Dims}_\mu(X, T)$, we will say $(X, B, \mu, T)$ has strictly positive entropy dimension.

The following result is also obvious.

**Theorem 3.7.** Let $\pi : (X, B, \mu, T) \to (Y, D, \nu, S)$ be a factor map between two MDS’s. Then $\text{Dims}_\mu(X, T) \supseteq \text{Dims}_\nu(Y, S)$. In particular, the dimension set is invariant under measurable isomorphism, and so is the entropy dimension.

By Definition 3.1 and Theorem 3.7 we have

**Proposition 3.8.** Let $\tau \in (0, 1]$. Then

1. A nontrival factor of a $\tau$−u.d. system is also a $\tau$−u.d. system.
2. A nontrival factor of a $\tau^+−d.$ system is also a $\tau^+−d.$ system.
3. If a system has strictly positive entropy dimension, then any nontrival factor of this system also has strictly positive entropy dimension.

**Lemma 3.9.** Let $(X, B, \mu, T)$ be a MDS. If $(X, B, \mu, T)$ has strictly positive entropy dimension, then $(X, B, \mu, T)$ is weakly mixing.

**Proof.** It is well known that if $(X, B, \mu, T)$ is not weakly mixing, then there exists a nontrivial null factor $(Y, D, \nu, S)$ of $(X, B, \mu, T)$. By Theorem 2.8 $D_\nu(Y, S) = 0$, a contradiction with Proposition 3.8 (3). \qed

As a direct application of Theorem 3.6, we have

**Corollary 3.10.** 1. $\alpha$−u.d. invertible MDS’s are disjoint from ergodic $\beta$−u.d. MDS’s when $1 \geq \alpha > \beta \geq 0$.

2. An invertible MDS which has strictly positive entropy dimension is disjoint from all ergodic 0-entropy dimension MDS’s.

The motivation that we consider the u.d. systems comes from the K-mixing systems. We can view the u.d. systems as the analogy of the K-mixing properties in zero entropy situation. The following example shows that two systems with the same entropy dimension may also have disjointness property.
Example 3.11. Choose $0 < r_i < 1$ and let $(X_i, B_i, \mu_i, T_i)$ be $r_i$-u.d. invertible MDS (in section 5 we will show existence of such MDS’s). Let $(X, B, \mu, T)$ be the product system, i.e. $(X, B, \mu, T) = (\prod_{i=1}^{\infty} X_i, \prod_{i=1}^{\infty} B_i, \prod_{i=1}^{\infty} \mu_i, \prod_{i=1}^{\infty} T_i)$. Then $D(\mu, X, T) = 1$ by Theorem [2.9]. Also, since each $(X_i, B_i, \mu_i, T_i)$ is weakly mixing, so is $(X, B, \mu, T)$, and hence ergodic. Since $h_{\mu}(X, T) = 0$, any K-automorphism MDS (which is also a 1-u.d. system) is disjoint from the ergodic MDS $(X, B, \mu, T)$.

4. Metric entropy dimension of an open cover

By a topological dynamical system (TDS for short) $(X, T)$ we mean a compact metrizable space $X$ together with a surjective continuous map $T$ from $X$ to itself. Let $(X, T)$ be a TDS and $\mu \in M(X, T)$, where $M(X, T)$ denotes the collection of invariant probability measures of $(X, T)$. Denote by $C_X$ the set of finite covers of $X$ and $C_X^0$ the set of finite open covers of $X$. For a $U \in C_X$, we define

$$H_{\mu}(U) = \inf \{ H_{\mu}(\alpha) : \alpha \in P_X \text{ and } \alpha \geq U \},$$

where by $\alpha \geq U$ we mean that every atom of $\alpha$ is contained in one of the elements of $U$. We say an increasing sequence $S = \{ s_1 < s_2 < \cdots \}$ of $\mathbb{N}$ is an entropy generating sequence of $U$ w.r.t. $\mu$ if

$$\lim_{n \to \infty} \frac{1}{n} H_{\mu}(\bigvee_{i=1}^{n} T^{-s_i} U) > 0.$$ 

We say $S = \{ s_1 < s_2 < \cdots \}$ of $\mathbb{N}$ is a positive entropy sequence of $U$ w.r.t. $\mu$ if

$$h_{\mu}^S(T, U) := \limsup_{n \to \infty} \frac{1}{n} H_{\mu}(\bigvee_{i=1}^{n} T^{-s_i} U) > 0.$$ 

Denote $E_{\mu}(T, U)$ by the set of all entropy generating sequences of $U$, and $P_{\mu}(T, U)$ by the set of all positive entropy sequences of $U$. Clearly $P_{\mu}(T, U) \supset E_{\mu}(T, U)$.

Definition 4.1. Let $(X, T)$ be a TDS, $\mu \in M(X, T)$ and $U \in C_X$. We define

$$\bar{D}_{\mu}(T, U) = \begin{cases} \sup_{S \in E_{\mu}(T, U)} \bar{D}(S) & \text{if } E_{\mu}(T, U) \neq \emptyset \\ 0 & \text{if } E_{\mu}(T, U) = \emptyset \end{cases},$$

$$\bar{D}_{\mu}^p(T, U) = \begin{cases} \sup_{S \in P_{\mu}(T, U)} \bar{D}(S) & \text{if } P_{\mu}(T, U) \neq \emptyset \\ 0 & \text{if } P_{\mu}(T, U) = \emptyset \end{cases}.$$ 

Similarly, we can define $D_{\mu}(T, U)$ and $D_{\mu}^p(T, \alpha)$ by changing the upper dimension into the lower dimension.

Similar to Proposition [2.4] we have

Proposition 4.2. Let $(X, T)$ be a TDS, $\mu \in M(X, T)$ and $U \in C_X$. Then

$$D_{\mu}(T, U) \leq D_{\mu}^p(T, U) = D_{\mu}^p(T, U) \leq \bar{D}_{\mu}(T, U).$$

By Proposition [4.2] we have
Definition 4.3. Let \( (X, T) \) be a TDS, \( \mu \in M(X, T) \) and \( U \in \mathcal{C}_X \). We define
\[
\overline{D}_\mu(T, U) := \overline{D}_\mu(T, U) = D_\mu(T, U),
\]
which is called the upper entropy dimension of \( U \). Similarly, we have the definition of lower dimension and dimension.

Theorem 4.4. Let \( (X, T) \) be a TDS and \( \mu \in M(X, T) \). Then
\[
\overline{D}_\mu(X, T) = \sup_{U \in \mathcal{C}_X} \overline{D}_\mu(T, U).
\]

Proof. Let \( U = \{U_1, U_2, \ldots, U_n\} \in \mathcal{C}_X \). For any \( s = (s(1), \ldots, s(n)) \in \{0, 1\}^n \), set \( U_s = \bigcap_{i=1}^n U_i(s(i)) \), where \( U_i(0) = U_i \) and \( U_i(1) = X \setminus U_i \). Let \( \alpha = \{U_s : s \in \{0, 1\}^n\} \). Then \( \alpha \) is the Borel partition generated by \( U \) and \( \overline{D}_\mu(X, T) \geq \overline{D}_\mu(T, \alpha) \geq \overline{D}_\mu(T, U) \).

Since \( U \) is arbitrary, we get
\[
\overline{D}_\mu(X, T) = \sup_{U \in \mathcal{C}_X} \overline{D}_\mu(T, U).
\]

For the other direction, let \( \alpha = \{A_1, \ldots, A_k\} \in \mathcal{P}_X \). If \( \overline{D}_\mu(T, \alpha) = 0 \), it is obvious \( \overline{D}_\mu(T, \alpha) \leq \sup_{U \in \mathcal{C}_X} \overline{D}_\mu(T, U) \). Now assume that \( \overline{D}_\mu(T, \alpha) > 0 \). For any \( \epsilon > 0 \), there exists \( S = \{s_1 < s_2 < s_3 < \cdots\} \in \mathcal{P}_\mu(T, \alpha) \) with \( \overline{D}(S) > \overline{D}(T, \alpha) - \epsilon \).

Let \( a := \frac{h^S(T, \alpha)}{2} > 0 \). We have

Claim. There exists \( U \in \mathcal{C}_X \) such that \( H_\mu(T^{-i} \alpha) \leq a \) if \( i \in \mathbb{Z}_+ \) and \( \beta \in \mathcal{P}_X \) satisfying \( \beta \geq T^{-i}U \).

Proof of Claim. By [20] Lemma 4.15, there exists \( \delta_1 = \delta_1(k, \epsilon) > 0 \) such that if \( \beta_i = \{B_i^1, \ldots, B_i^k\} \in \mathcal{P}_X, i = 1, 2 \) satisfy \( \sum_{i=1}^k \mu(B_i^1 \Delta B_i^2) < \delta_1 \) then \( H_\mu(\beta_1|\beta_2) \leq a \). Since \( \mu \) is regular, we can take closed subsets \( B_i \subseteq A_i \) with \( \mu(A_i \setminus B_i) < \frac{\delta_1}{2k^2}, i = 1, \ldots, k \). Let \( B_0 = X \setminus \bigcup_{i=1}^k B_i \) and \( \mu(B_0) < \frac{\delta_1}{2k} \) and \( U = \{U_1, \ldots, U_k\} \in \mathcal{C}_X \).

Let \( i \in \mathbb{Z}_+ \). If \( \beta \in \mathcal{P}_X \) is finer than \( T^{-i}U \), then we can find \( \beta' = \{C_1, \ldots, C_k\} \in \mathcal{P}_X \) satisfying \( C_j \subseteq T^{-i}U_j, j = 1, \ldots, k \) and \( \beta \geq \beta' \), and so \( H_\mu(T^{-i} \alpha|\beta') \leq H_\mu(T^{-i} \alpha|\beta) \).

For each \( j = 1, \ldots, k \), as \( T^{-i}U_j \supseteq C_j \supseteq X \setminus \bigcup_{i \neq j} T^{-i}U_i \supseteq T^{-i}B_j \) and \( T^{-i}A_j \supseteq T^{-i}B_j \), one has
\[
\mu(C_j \Delta T^{-i}A_j) \leq \mu(T^{-i}A_j \setminus T^{-i}B_j) + \mu(T^{-i}B_0) = \mu(A_j \setminus B_j) + \mu(B_0)
\]
\[
< \frac{\delta_1}{2k} + \frac{\delta_1}{2k^2} \leq \frac{\delta_1}{k}.
\]

Thus \( \sum_{j=1}^k \mu(C_j \Delta T^{-i}A_j) < \delta_1 \). It follows that \( H_\mu(T^{-i} \alpha|\beta') \leq a \) and so \( H_\mu(T^{-i} \alpha|\beta) \leq \frac{\delta_1}{k} \).

\[ \square \]
For \( n \in \mathbb{N} \), if \( \beta_n \in \mathcal{P}_X \) with \( \beta_n \geq \bigvee_{i=1}^{n} T^{-s_i}U \) then \( \beta_n \geq T^{-s_i}U \) for each \( i \in \{1, 2, \ldots, n\} \), and so using the above Claim one has
\[
H_\mu(\bigvee_{i=1}^{n} T^{-s_i}a) \leq H_\mu(\beta_n) + H_\mu(\bigvee_{i=1}^{n} T^{-s_i}a | \beta_n)
\]
\[
\leq H_\mu(\beta_n) + \sum_{i=1}^{n} H_\mu(T^{-s_i}a | \beta_n) \leq H_\mu(\beta_n) + na.
\]
Moreover, since the above inequality is true for any \( \beta_n \in \mathcal{P}_X \) with \( \beta_n \geq \bigvee_{i=1}^{n} T^{-s_i}U \), one has \( H_\mu(\bigvee_{i=1}^{n} T^{-s_i}a) \leq H_\mu(\bigvee_{i=1}^{n} T^{-s_i}U) + na \). That is, \( \frac{1}{n} H_\mu(\bigvee_{i=1}^{n} T^{-s_i}U) \geq a \). Now letting \( n \) converges to infinity one has \( h^S(T, U) \geq a > 0 \). Hence \( \overline{D}_\mu(T, U) \geq D(S) \geq \underline{D}_\mu(T, a) - \epsilon \). This implies \( \sup_{U \in \mathcal{C}^o_X} \overline{D}_\mu(T, U) \geq \overline{D}_\mu(T, a) - \epsilon \). Finally, since \( a \) and \( \epsilon \) are arbitrary, we get \( \sup_{U \in \mathcal{C}^o_X} \overline{D}_\mu(T, U) = \overline{D}_\mu(X, T) \).

Now let us recall the corresponding notions in topological settings, which appeared in [6].

Let \((X, T)\) be a TDS and \(U \in \mathcal{C}^o_X\). We recall [6] that an increasing sequence \(S = \{s_1 < s_2 < \cdots\}\) of \(\mathbb{N}\) is an *entropy generating sequence* of \(U\) if
\[
\liminf_{n \to \infty} \frac{1}{n} \log N(\bigvee_{i=1}^{n} T^{-s_i}U) > 0,
\]
and \(S = \{s_1 < s_2 < \cdots\}\) of \(\mathbb{N}\) is a *positive entropy sequence* of \(U\) if
\[
h^S_{\text{top}}(T, U) := \limsup_{n \to \infty} \frac{1}{n} \log N(\bigvee_{i=1}^{n} T^{-s_i}U) > 0.
\]

Denote by \(\mathcal{E}(T, U)\) the set of all entropy generating sequences of \(U\), and by \(\mathcal{P}(T, U)\) the set of all positive entropy sequences of \(U\). Clearly \(\mathcal{P}(T, U) \supset \mathcal{E}(T, U)\).

Let \((X, T)\) be a TDS and \(U \in \mathcal{C}^o_X\). We define
\[
\overline{D}_e(T, U) = \begin{cases} \sup_{S \in \mathcal{E}(T, U)} \overline{D}(S) & \text{if } \mathcal{E}(T, U) \neq \emptyset \\ 0 & \text{if } \mathcal{E}(T, U) = \emptyset \end{cases},
\]
\[
\underline{D}_e(T, U) = \begin{cases} \sup_{S \in \mathcal{P}(T, U)} \underline{D}(S) & \text{if } \mathcal{P}(T, U) \neq \emptyset \\ 0 & \text{if } \mathcal{P}(T, U) = \emptyset \end{cases}.
\]

It is similar to the proof of Proposition 2.4 we have \(\overline{D}_e(T, U) = \underline{D}_e(T, U)\) for any \(U \in \mathcal{C}^o_X\). Hence we define
\[
\mathcal{D}(T, U) := \overline{D}_e(T, U) = \underline{D}_e(T, U) \quad \text{and} \quad \mathcal{D}(X, T) = \sup_{U \in \mathcal{C}^o_X} \mathcal{D}(T, U).
\]
We call $D(X, T)$ the upper entropy dimension of $(X, T)$. Similarly, we have the definition of lower dimension and dimension.

Using Theorem 4.4 we have

**Theorem 4.5.** Let $(X, T)$ be a TDS and $\mu \in M(X, T)$. Then

$$D_\mu(X, T) \leq D(X, T).$$

**Example 4.6.** For any $\tau \in (0, 1]$, there exists a minimal system $(X, T)$ satisfying $D(X, T) = \tau$ and $D_\mu(X, T) = 0$ for any $\mu \in M(X, T)$.

**Proof.** Let $(X, T)$ be the system generated by Cassaigne’s model \[3\] (the uniformly recurrent one), then it is minimal. By taking $\phi(n) = \frac{n}{\log n}$ in this construction, we get $D(X, T) = 1$. By taking $\phi(n) = n^\tau$ in this construction, we get $D(X, T) = \tau$, for any $0 < \tau < 1$. In \[1\], it is shown that $(X, T)$ is uniquely ergodic and with respect to the unique ergodic invariant measure $\mu$, $D_\mu(X, T) = 0$. \[\square\]

**Definition 4.7.** An invertible TDS $(X, T)$ is doubly minimal if for all $x, y \in X$, $y \in \{T^n x\}_{n \in \mathbb{Z}}$, $\{(T^j x, T^j y)\}_{j \in \mathbb{Z}}$ is dense in $X \times X$.

The following results is Theorem 5 in \[21\].

**Lemma 4.8.** Any ergodic system $(Y, \mathcal{C}, \nu, S)$ with $h_\nu(S) = 0$ has a uniquely ergodic topological model $(X, T)$ that is doubly minimal.

It is well known that any doubly minimal system has zero entropy (see \[21\]). Though we have

**Example 4.9.** There exists a doubly minimal system with positive entropy dimension.

**Proof.** This comes directly from Lemma 4.8 and Theorem 4.5 since there exists an ergodic system with metric entropy dimension $0 < \tau < 1$ (see section 5). \[\square\]

A TDS $(X, T)$ with metric $d$ is called distal, if $\inf_{n \geq 0} d(T^n x, T^n y) > 0$ for every $x \neq y \in X$. Let $(X, \mathcal{B}, \mu, T)$ be an invertible ergodic MDS. A sequence $A_1 \supset A_2 \supset A_3 \cdots$ of sets in $\mathcal{B}$ with $\mu(A_n) > 0$ and $\mu(A_n) \to 0$, is called a separating sieve if there exists a subset $X_0 \subset X$ with $\mu(X_0) = 1$ such that for every $x, x' \in X_0$ the condition for every $n \in \mathbb{N}$ there exists $k \in \mathbb{Z}$ with $T^k x, T^k x' \in A_n$ implies $x = x'$. We say that the invertible ergodic MDS $(X, \mathcal{B}, \mu, T)$ is measure distal if either $(X, \mathcal{B}, \mu, T)$ is finite or there exists a separating sieve. In \[14\] E. Lindenstrauss shows that every invertible ergodic measure distal MDS can be represented as a minimal topologically distal system.

It is well known that a distal TDS has zero topological entropy, and an invertible ergodic measure distal MDS has zero measure entropy. To end this section let us ask the following questions:

**Question 4.10.**

1. Is the entropy dimension of a minimal distal TDS zero?
2. Is the entropy dimension of an invertible ergodic measure distal MDS zero?
5. The existence of u.d. MDS’s

In this section, our aim is to show that for every \( \tau \in (0, 1) \), there exists a MDS \((X, B, \mu, T)\) having the property of \( \tau \)-u.d.. We mention that a \( K \)–mixing system is of u.d. for \( \tau = 1 \) and an irrational rotation is of u.d. for \( \tau = 0 \).

Through a sequence of cutting and stacking steps, we successively construct a tower with a given entropy dimension \( \tau \in (0, 1) \), by controlling the heights of independent and repetition steps. However we need to be careful as in most of these constructive methods not to generate some kind of “rotation” factors. That is, we need to show that any non-trivial partition \( P = \{A, A^c\} \) has an entropy generating sequence with the same dimension. We will use three kinds of operations in our construction. The first kind operation is the independent cutting and stacking, which generates most of the complexity. The second kind is the repetition cutting and stacking, which won’t increase the complexity. Two sequences of integers \( \{e_n\} \) and \( \{r_n\} \) will be chosen to be the sizes of operation in each step such that the complexity is controlled by dimension \( \tau \). We can see clearly from the construction what the entropy generating sequence is. The third kind is to add proper spacers while we do independent cutting and stacking in some steps, which will eliminate the rotation factors.

Let \( X \) be the interval \([0, 1)\). In the construction, \([0, 1)\) will be cut into many subintervals and all of them are left closed and right open. Let \( B_i \subset [0, 1), 1 \leq i \leq h, \) be \( h \) disjoint intervals with the same length. A column \( C \) is the ordered set of these intervals, i.e.

\[
C = \{B_1, B_2, \cdots , B_h\} = \{B_i : 1 \leq i \leq h\}.
\]

We say \( C \) has base \( B_1 \), top \( B_h \), height \( h(C) = h \) and width \( w(C) = |B_i| \), the length of \( B_i \). Denote \(|C| = \bigcup_{i=1}^{h} B_i \). We call each \( B_i \) a level set of \( C \). A tower \( W \) is a finite collection of columns, which generally have different height. In this paper, all the columns of a tower will have the same height. The width of tower \( W \) is

\[
w(W) = \sum_{C \text{ is a column of } W} w(C).
\]

The cardinality of a tower \( W \), denoted by \( \#W \), is the number of its columns. Denote by \(|W|\) the union of all the level sets of its columns. The base of the tower \( W \), which is denoted by \( base(W) \), is the union of all the bases of its columns.

We divide \([0, 1)\) into three parts: \( P_0 = [0, \frac{\xi}{2}) \), \( P_1 = [\frac{\xi}{2}, \xi) \) and \( P_s = [\xi, 1) \), where we will decide \( \xi \) later and \( s \) stands for “spacer”. This will be our initial tower and any level set of any other tower will be a subset of \( P_0, P_1, \) or \( P_s \). Due to the initial tower, we can refer any level set \( B \) a name “a” if \( B \subset P_a, a = 0, 1, s \). The name of a column \( C = \{B_1, B_2, \cdots, B_h\} \) is a word \( b = b_1 b_2 \cdots b_h \in \{0, 1, s\}^h \), where \( b_i \) is the name of \( B_i \). The name of a tower is the collection of names of its columns. By \( N(W) \) we denote the number of different names of columns of the tower \( W \). We say two columns are isomorphic if they have the same name. We say two towers are isomorphic if they have the same name and scaling factor in columns. A segment \( S \) is a collection of consecutive level sets of a column. By a \( W \)–segment, where \( W \) is a tower, we mean a segment of a column which is isomorphic to a column of \( W \) up to rescaling.
5.1. Three kinds of operations.

Now we will describe the three kinds of operations we need.

1. Independent cutting and stacking.

Let $W^1$ and $W^2$ be two towers with the same width $w$. Assume $W^j$ has $c_j$—many columns $C_j^1, C_j^2, \ldots, C_j^l$ for $j = 0, 1$. Divide each column $C_1^i$ into $c_2$—many subcolumns $C_{i,k}^1$ with width $w(C_{i,k}^1) = w(C_1^i) = \frac{w(C_1^1)}{w}$, $i = 1, 2, \ldots, c_1, k = 1, 2, \ldots, c_2$. Divide each column $C_{k,i}^2$ into $c_1$—many subcolumns $C_{k,i}^2$ with width $w(C_{k,i}^2) = w(C_2^k) = \frac{w(C_1^1)}{w}$, $i = 1, 2, \ldots, c_1, k = 1, 2, \ldots, c_2$. Then stack each $C_{k,i}^2$ on top of $C_{i,k}^1$ to form a new column $C_{i,k}^1 * C_{k,i}^2$ since they have the same width. We denote the new tower $\{C_{i,k}^1 * C_{k,i}^2\}$ by $W^1 * W^2$.

For a tower $W$ and an integer $e \geq 1$, we equally divide $W$ into $e$—many subtower $W^1, W^2, \ldots, W^e$. We divide each column of $W$ into $e$—many subcolumns equally and take all the $i$—th subcolumn to make the tower $W^i$). We call the tower $\text{Ind}(W, e) = W^1 * W^2 * \ldots * W^e$ the $e$—many independent cutting and stacking of $W$. We note that $\#\text{Ind}(W, e) = (\#W)^e$, $h(\text{Ind}(W, e)) = eh(W)$. In fact we can cut each column of $W$ into $c(\#W)^e$—many subcolumns equally and then choose these subcolumns from different $e$—many combinations of columns of $W$ to stack to form $\#\text{Ind}(W, e)$.

2. Repetition cutting and stacking.

For a tower $W = \{C_1, C_2, \ldots, C_c\}$ and an integer $r \geq 1$, we equally divide each column $C_i$ of $W$ into $r$—many subcolumns $C_{i,1}, C_{i,2}, \ldots, C_{i,r}$ and stack them one by one to make a new column $C_{i,1} * C_{i,2} * \ldots * C_{i,r}$. Then we call the tower $\text{Rep}(W, r) = \{C_{i,1} * C_{i,2} * \ldots * C_{i,r} : i = 1, 2, \ldots, c\}$ the $r$—many repetition cutting and stacking of $W$. We note that $\#\text{Rep}(W, r) = \#W$.

3. Inserting spacers while independent cutting and stacking.

Let $W$ be a tower with columns $\{C_1, C_2, \ldots, C_c\}$ and $e, h^* \geq 1$ be two integers. Due to the definition of $\text{Ind}(W, e)$, we can assume tower $\text{Ind}(W, e)$ is formed by columns $\overline{C}_{i_1} * \overline{C}_{i_2} * \ldots * \overline{C}_{i_c}$, for $i_1, i_2, \ldots, i_c \in \{1, 2, \ldots, c\}$, where $\overline{C}_{i}$ is a subcolumn of $C_i$. Cut each column $\overline{C}_{i_1} * \overline{C}_{i_2} * \ldots * \overline{C}_{i_c}$ of $\text{Ind}(W, e)$ into $e$—many subcolumns equally, which we denote by $(\overline{C}_{i_1} * \overline{C}_{i_2} * \ldots * \overline{C}_{i_c})_{i_{e+1}}$, $i_{e+1} = 1, 2, \ldots, c$. Then the new tower has $e^{c+1}$—many columns but each column is stacked by $e$—many $W$—segments (we still denote by $\overline{C}_{i,k}, k = 1, 2, \ldots, e$). Now we insert totally $e \cdot h^*$—many spacers between these segments of $(\overline{C}_{i_1} * \overline{C}_{i_2} * \ldots * \overline{C}_{i_c})_{i_{e+1}}$, each spacer is an interval cut from $P_t$. We write $(\overline{C}_{i_1} * \overline{C}_{i_2} * \ldots * \overline{C}_{i_c})_{i_{e+1}}$ as $\overline{C}_{i_1}^{e+1} * \overline{C}_{i_2}^{e+1} * \ldots * \overline{C}_{i_c}^{e+1}$. For $k = 1, 2, \ldots, e$, let $\ell = i_k + (\text{mod } h^*)$, $0 \leq \ell \leq h^* - 1$, we insert $\ell$—many spacers before the $k$—th $W$—segment $\overline{C}_{i_k}$ and $(h^* - \ell)$—many spacers after. In other words, we change each $\overline{C}_{i_k}^{e+1}$ into $s^\ell \overline{C}_{i_k}^{e+1}$, $s^{h^*-\ell}$, where we identify a segment with its name. We denote the new tower by $\text{Ins}(W, e, h^*)$. We should notice here that some columns of $\text{Ins}(W, e, h^*)$ may have the same name. Furthermore,

\[
(5.1) \quad (N(W))^e \leq N(\text{Ins}(W, e, h^*)) \leq \#\text{Ins}(W, e, h^*) = (\#W)^{e+1}.
\]
If we collect all the segments \( s^tC_{i_k}s^{h^*-\ell} \) to form a tower \( \overline{W} \), then each column of \( Ins(W, e, h^*) \) is formed by \( e \)—many \( \overline{W} \)—segments. The probability of all the columns the \( k \)–th \( \overline{W} \)–segments of which begin with \( \ell \)–many spacers, say \( p_\ell \), is either \( \frac{h^*}{e} \) or \( \frac{h^*+1}{e} \). Since

\[
\frac{|p_\ell - \frac{h^*}{e}|}{h^*} \leq \frac{h^*}{e},
\]

we may say that the number of beginning spacers of \( \overline{W} \)–segments is uniformly distributed on \( \{0, 1, \ldots, h^*-1\} \) within \( \frac{h^*}{e} \)–error.

5.2. The choice of the parameters.

To construct a MDS \((X, \mathcal{B}, \mu, T)\) with \( \tau \)-u.d. for fixed \( \tau \in (0, 1) \), we need to determine integer parameters \( 1 < r_1 < r_2 < \cdots \), \( 1 < e_0 < e_1 < e_2 < \cdots \) and \( 1 < l_1 < n_1 < l_2 < n_2 < \cdots \).

Given \( \tau \in (0, 1) \), we let \( r_n = C_\tau n^2 \), where \( C_\tau \) is an integer such that \( C_\tau^{1+\tau} > 2 \). Let \( 1 < l_1 < n_1 < l_2 < n_2 < \cdots \) be any sequence of integers. Then we put \( e_0 = 2, h_0 = 1, w_0 = 1 \) and \( h_1 = e_0 \).

Next we inductively construct \( \tilde{h}_n(w_n, e_n) \). For \( n \geq 1 \), put

\[
\tilde{h}_n = h_n r_n, \quad w_n = \begin{cases} \tilde{h}_n & \text{if } n \notin \{n_1, n_2, \ldots\} \\ \tilde{h}_n + h_t & \text{if } n = n_t \text{ for some } t \end{cases}
\]

\[(5.2)\]

and \( h_{n+1} = w_n e_n \), where

\[
e_n \geq \left[ \left( \frac{(w_{n-1}e_{n-1}r_n)^\tau}{e_0e_1 \cdots e_{n-1}} \right)^{\frac{1}{1+\tau}} \right] = \left[ \left( \frac{(w_{n-1}e_{n-1})^\tau}{e_0e_1 \cdots e_{n-1}} \right)^{\frac{1}{1+\tau}} \cdot \left( r_n \right)^{\frac{1}{1+\tau}} \right] \geq \left[ \left( r_n \right)^{\frac{1}{1+\tau}} \right] \geq \left[ C_\tau^{\frac{1}{1+\tau}} \right] \geq 2.
\]

Thus it is clear that \( \lim_{n \to +\infty} e_n = +\infty \) and so

\[(5.3)\]

\[
\lim_{n \to +\infty} \frac{e_0e_1 \cdots e_n}{(w_n e_n)^\tau} = 1
\]

by \((5.2)\). Moreover note that \( \lim_{n \to +\infty} \frac{w_n}{\tilde{h}_n} = 1 \), one has

\[
e_n = \left[ \left( \frac{(w_n)^\tau}{e_0e_1 \cdots e_{n-1}} \right)^{\frac{1}{1+\tau}} \right] = \left[ \left( \frac{w_n}{\tilde{h}_n} \right)^{\frac{1}{1+\tau}} \cdot \left( \frac{(w_{n-1}e_{n-1})^\tau}{e_0e_1 \cdots e_{n-1}} \right)^{\frac{1}{1+\tau}} \right] \]

\[
= \left[ \left( \frac{w_n}{\tilde{h}_n} \right)^{\frac{1}{1+\tau}} \cdot \left( \frac{(w_{n-1})^\tau}{e_0e_1 \cdots e_{n-2}} \right)^{\frac{1}{1+\tau}} \cdot \left( r_n \right)^{\frac{1}{1+\tau}} \right] \sim \left( r_n \right)^{\frac{1}{1+\tau}} = (C_\tau n^2)^{\frac{1}{1+\tau}}
\]

\[(5.4)\]
which deduces that \(e_n\) increases to infinity when \(n\) is sufficiently large. Moreover, from (5.2) we have that

\[
(5.5) \quad w_n \geq (e_0 e_1 \cdots e_{n-1})^{ \frac{1}{2}}.
\]

5.3. The construction.

Let \(W_0 = \hat{W}_0 = \{P_0, P_1\}\), we emphasize here that \(W_0\) and \(\hat{W}_0\) contain no level sets from \(P_0\). The construction consists of a sequence of steps, step \(n\) and step \(\hat{n}\), \(n \in \mathbb{N}\).

At step 1, we do \(e_0\)—many independent cutting and stacking of \(\hat{W}_0\) to construct the first tower \(W_1\) of height \(h_1 = e_0\), i.e., \(W_1 = Ind(\hat{W}_0, e_0)\). We note that we have \(2^{e_0}\)—many columns of all possible sequences of 0’s and 1’s as their names of equal width and height in \(W_1\). Suppose after step \(n\) we have obtained the tower \(W_n\) of height \(h_n\). Then at step \(\hat{n}\), we do \(r_n\)—many repetition cutting and stacking of \(W_n\), i.e. if we denote the tower after this step by \(\hat{W}_n\), then \(\hat{W}_n = Rep(W_n, r_n)\). At step \((n + 1)\), if \(n \notin \{n_1, n_2, \cdots\}\), we do \(e_n\)—many independent cutting and stacking of \(\hat{W}_n\), i.e. \(W_{n+1} = Ind(\hat{W}_n, e_n)\); otherwise if \(n = n_t\) for some \(t \geq 1\), we inserting spacers while doing independent cutting and stacking, we let \(W_{n+1} = Ins(\hat{W}_{n_t}, e_{n_t}, h_{t_l})\).

Then we get an ergodic invertible MDS and denote it by \((X, \mathcal{B}, \mu, T)\), where \(\mu\) is the Lebesgue measure on \(X\), \(\mathcal{B}\) is a \(\sigma\)—algebra of \(X\) generated by the level sets of tower \(\hat{W}_n\) and \(T\) is the obvious map.

5.4. List of the parameters and notations. We remind the following notations.

- \(e_n\) — we do \(e_n\)—many independent stackings at Step \(n + 1\).
- \(r_n\) — we do \(r_n\)—many repetition stackings at Step \(n\).
- \(W_n(\hat{W}_n)\) — towers after Step \(n(\hat{n})\).
- \(h_n(\hat{h}_n)\) — height of columns of the tower \(W_n(\hat{W}_n)\).
- \(c_n(\hat{c}_n)\) — the total number of the columns of \(W_n(\hat{W}_n)\).
- \(n_t, l_t\) — at setp \(n_t + 1\) for \(t \geq 1\), we add spacers while we do independent cutting and stacking, \(h_{t_l}\) is the parameter related with the number of the spacers.
- \(\xi_n(\hat{\xi}_n)\) — the total Lebesgue measure of the level sets in the tower \(W_n(\hat{W}_n)\).

Since at each step \((n_t + 1)\) we add spacers of measure \(\xi_{n_t} \cdot \frac{h_{t_l}}{h_{n_t}}\), the measures \(\xi_n\)’s of the tower \(W_n\)’s satisfy the following,

\[
(5.6) \quad \xi_1 = \xi_2 = \cdots = \xi_{n_1} = \xi, \\
\xi_{n_t + 1} = \xi_{n_t} \frac{w_{n_t}}{h_{n_t}} = \xi_{n_t}(1 + \frac{h_{t_l}}{h_{n_t}}), \\
\xi_{n_t + 1} = \xi_{n_t + 1} = \cdots = \xi_{n_t + 1}, t \geq 1.
\]

Due to the choice of \(r_n\), \(\sum_{t=1}^{\infty} \frac{1}{r_{n_t}}\) converges. So \(\sum_{t=1}^{\infty} \frac{h_{t_l}}{h_{n_t}} < \sum_{t=1}^{\infty} \frac{1}{r_{n_t}}\) converges. Let

\[
\xi = \prod_{t=1}^{\infty} (1 + \frac{h_{t_l}}{h_{n_t}})^{-1},
\]
we have $0 < \xi < 1$ and $\lim_{n \to +\infty} \xi_n = 1$.

5.5. **The upper bound of the entropy dimension.**

For convenience, for a finite collection $\mathcal{A}$ consisting of measurable sets in $\mathcal{B}$ (need not to be a partition), we denote

$$H_\mu(\mathcal{A}) = \sum_{A \in \mathcal{A}} -\mu(A) \log \mu(A).$$

For $\beta \in \mathcal{P}_X$ and $E \subseteq X$, we define

$$\beta \cap E := \{ B \cap E : B \in \beta \}.$$ 

**Lemma 5.1.** $\overline{D}_\mu(X, T) \leq \tau$.

**Proof.** Given $k \in \mathbb{N}$, let $E$ be a level set in $W_k$ and $\alpha = \{ E, X \setminus E \}$. In the following we are to estimate $H_\mu(\bigvee_{i=0}^{n-1} \alpha)$.

Given $n \in \mathbb{N}$ and $K \gg n$, let

$$U_K = |W_K| \setminus \left( \bigcup_{i=0}^{n-1} T^{h_{K-i}}(base(W_K)) \right).$$

Then

$$H_\mu\left( \bigvee_{i=0}^{n-1} T^{-i} \alpha \right) \leq H_\mu\left( \bigvee_{i=0}^{n-1} T^{-i} \alpha \left\{ U_K, X \setminus U_K \right\} \right) = H_\mu\left( \bigvee_{i=0}^{n-1} T^{-i} \alpha \cap U_K \right) + H_\mu\left( \bigvee_{i=0}^{n-1} T^{-i} \alpha \cap (X \setminus U_K) \right) \leq -\mu(U_K) \log \frac{\mu(U_K)}{\#(\bigvee_{i=0}^{n-1} T^{-i} \alpha \cap U_K)} - \mu(X \setminus U_K) \log \frac{\mu(X \setminus U_K)}{\#(\bigvee_{i=0}^{n-1} T^{-i} \alpha)}$$

$$= \mu(U_K) \log \#(\bigvee_{i=0}^{n-1} T^{-i} \alpha \cap U_K) + \mu(X \setminus U_K) \log \#(\bigvee_{i=0}^{n-1} T^{-i} \alpha)$$

(5.7)

Since $\mu(U_K) = \xi_K \cdot (1 - \frac{n}{h_K})$ and $\#(\bigvee_{i=0}^{n-1} T^{-i} \alpha) \leq 2^n$, when $K$ is large enough, we can make

$$\mu(X \setminus U_K) \log \#(\bigvee_{i=0}^{n-1} T^{-i} \alpha) - \mu(U_K) \log \mu(U_K) - \mu(X \setminus U_K) \log \mu(X \setminus U_K) \leq 1.$$

**Claim:** $\bigvee_{i=0}^{n-1} T^{-i} \alpha \cap U_K \leq p(n)2^n$, where $p(n)$ is a polynomial of $n$. 
Proof of claim. Any element in the collection $\bigvee_{i=0}^{n-1} T^{-i} \alpha \cap U_K$ is a union of some level sets in $W_K$. We can refer each of these level sets contained in $U_K$, say $E$, an $(n, \alpha)$-name $b = b_0b_1 \cdots b_{n-1} \in \{0, 1\}^n$ by $b_i = 0$, if $T^i E \subset E$, $b_i = 1$, if $T^i E \subset (X \setminus E)$. Then $\#(\bigvee_{i=0}^{n-1} T^{-i} \alpha \cap U_K)$ is no more than the total number of these different $(n, \alpha)$-names, which is bounded by $p(n)2^n$, where $p(n)$ is a polynomial of $n$. □

So by the claim above

$$H_{\mu}(\bigvee_{i=0}^{n-1} T^{-i} \alpha) \leq \log \left( p(n)2^n \right) + 1.$$ 

Now let $S = \{s_1 < s_2 < \cdots \}$ be an increasing sequence of positive integers such that $D(S) = \tau' > \tau$, then $\liminf_{n \to \infty} \frac{n}{(s_n)^{\tau'}} = \infty$. Using above estimation,

$$\limsup_{n \to \infty} \frac{1}{n} H_{\mu}(\bigvee_{i=1}^{n} T^{-s_i} \alpha) \leq \limsup_{n \to \infty} \frac{1}{n} H_{\mu}(\bigvee_{i=0}^{s_n} T^{-i} \alpha) \leq \limsup_{n \to \infty} \frac{1}{n} \left( \log \left( p(s_n)2^{(s_n)^{\tau'}} \right) + 1 \right) \leq \limsup_{n \to \infty} \frac{1}{n} \cdot (s_n)^{\tau'} = 0.$$ 

Hence the lower dimension of all the positive entropy sequence of $\alpha$ is no more than $\tau$. Since

$$\bigvee_{k=1}^{\infty} \left( \bigvee_{E \text{ is a level set of } W_k} \{E, X \setminus E\} \right) = B \pmod{\mu},$$

we have $D_{\mu}(X, T) \leq \tau$ by Theorem 2.11. □

5.6. The lower bound.

For $A, B \subset \mathbb{Z}$, let $A + B \triangleq \{a + b : a \in A, b \in B\}$ and $|A|$ the number of integers in $A$. Recall that $w_n$ is given in (5.2). For $t \geq 1$, let

$$F^t_0 = \{0, w_n, 2w_n, \cdots, (e_n - 1)w_n\},$$

$$F^t_k = F^t_{k-1} + \{0, w_n + k, 2w_n + k, \cdots, (e_n + k - 1)w_n + k\}$$

for $k \geq 1$, and

$$F^t = \bigcup_{k=0}^{\infty} F^t_k.$$ 

Then

$$|F^t_k| = e_ne_{n+1} \cdots e_{n+k}.$$ 

Lemma 5.2. For any $t \geq 1$, one has $D(F^t) = \tau$.

Proof. Given $t \geq 1$, let $F^t = \{t_1 < t_2 < \cdots \}$. For any $n \in \mathbb{N}$, there exists a unique $k = k(n)$ such that $t_n \in F^t_{k+1} \setminus F^t_k$. Then

$$e_ne_{n+1} \cdots e_{n+k} < n \leq e_ne_{n+1} \cdots e_{n+k+1}$$

and

$$h_{n+k+1} < t_n \leq h_{n+k+2}.$$
So for any $0 \leq \tau' < \tau$,

$$D(F^t, \tau') = \liminf_{n \to \infty} \frac{n}{(t_n)^{\tau'}} \geq \liminf_{k \to \infty} \frac{e_n e_{n+1} \cdots e_{n+k}}{(h_{n+k+1})^{\tau'}} = \liminf_{k \to \infty} \frac{e_n e_{n+1} \cdots e_{n+k+1}}{(w_{n+k+1} e_{n+k+1})^{\tau'-\tau}} \geq \liminf_{k \to \infty} \frac{e_n e_{n+1} \cdots e_{n+k+1}}{e_0 e_1 \cdots e_{n-1} (w_{n+k} e_{n+k})^{\tau'-\tau}}.$$

(5.9)

Applying (5.4) and (5.5),

$$\frac{(w_{n+k} e_{n+k})^{\tau'-\tau}}{e_0 e_1 \cdots e_{n-1}} \to \infty.$$

Combining this fact with (5.9) and equation (5.3), one gets $D(F^t, \tau') = \infty$. Hence $D(F^t) \geq \tau'$. Since the inequality is true for any $\tau' \in [0, \tau)$, one has $D(F) \geq \tau$.

For any $\tau < \tau' < 1$,

$$\overline{D}(F^t, \tau') = \limsup_{n \to \infty} \frac{n}{(t_n)^{\tau'}} \leq \limsup_{k \to \infty} \frac{e_n e_{n+1} \cdots e_{n+k+1}}{(h_{n+k+1})^{\tau'}} = \limsup_{k \to \infty} \frac{e_n e_{n+1} \cdots e_{n+k+1}}{(w_{n+k} e_{n+k+1})^{\tau'-\tau}} \leq \limsup_{k \to \infty} \frac{e_n e_{n+1} \cdots e_{k+1}}{e_0 e_1 \cdots e_{n-1} (w_{n+k} e_{n+k})^{\tau'-\tau}}.$$

(5.10)

Applying (5.4) and (5.5) again,

$$\frac{e_{n+k+1}}{e_0 e_1 \cdots e_{n-1} (w_{n+k} e_{n+k})^{\tau'-\tau}} \to 0.$$

Combining this fact with (5.10) and equation (5.3), one gets $\overline{D}(F^t, \tau') = 0$. Hence $\overline{D}(F^t) \leq \tau'$. Since the inequality is true for any $\tau' \in [\tau, 1)$, one has $\overline{D}(F^t) \leq \tau$. Hence $D(F^t) = \tau$.

Lemma 5.3. Given $t > 0$ and $k \geq 0$, let $B \subset F^t_k$ and $E_b$ be a level set in $W_t$, for $b \in B$ ($E_b's$ need not to be different), then

$$\mu(\bigcap_{b \in B} T^{-b} E_b) \leq (1 + \frac{h_t}{c_t}) |B| \cdot \left(\frac{1}{\xi_t}\right)^{|B|} \prod_{b \in B} \mu(E_b).$$

(5.11)

Moreover, let $A_b$ be a union of finite level sets in $W_t$, for $b \in B$, then

$$\mu(\bigcap_{b \in B} T^{-b} A_b) \leq (1 + \frac{h_t}{c_t}) |B| \cdot \left(\frac{1}{\xi_t}\right)^{|B|} \prod_{b \in B} \mu(A_b).$$

(5.12)

Proof. We just consider the case that $|B| = 2$. Without loss of generality, we may assume $B = \{0, b\}$. Then

$$\mu(E_0) = \mu(E_b) = \frac{\xi_t}{c_t} \cdot \frac{1}{h_t},$$

since they are level sets in $W_t$. 


Notice that the level sets \( E_0 \) and \( E_b \) in \( W_t \) are both spread out into many small level sets after some sufficiently large step. For the small level sets \( A_0 \)'s from \( E_0 \), to ensure the level set \( T^b A_0 \) is from \( E_b \), the \( W_t \)-segment which contains \( T^b A_0 \) must be isomorphic with the column that contains \( E_b \) in \( W_t \). This situation happens with probability \( \frac{\alpha}{c(t)} \). Also we need the height of \( E_0 \) in \( W_t \) coincides with the height of \( E_0 \) in \( W_t \) after inserting spacers. Since the numbers of beginning spacers of \( W_t \)-segments are uniformly distributed on \( \{0, 1, \cdots, h_t - 1\} \) within \( \frac{b_t}{c(t)} \)-error, at most \( \frac{1}{b_t} + \frac{1}{c(t)} \) of them coincide. So

\[
\mu(E_0 \bigcap T^{-b} E_b) \leq \mu(E_0) \cdot \frac{1}{c(t)} \cdot \left( \frac{1}{b_t} + \frac{1}{c(t)} \right)
\]

\[
= (1 + \frac{b_t}{c(t)}) \cdot \left( \frac{1}{\xi_t} \mu(E_0) \mu(E_b) \right)
\]

\[
\leq (1 + \frac{b_t}{c(t)})^2 \cdot \left( \frac{1}{\xi_t} \right)^2 \mu(E_0) \mu(E_b).
\]

By the similar discussion, for the case \(|B| > 2\), the same conclusion holds.

\[\square\]

**Remark 5.4.** From the above lemma, for any \( p \in \mathbb{Z}^+ \) and any two level sets \( E \) and \( \tilde{E} \), there exists \( n > 0 \) with \( \mu(T^{-np} E \cap \tilde{E}) > 0 \). We note that the \( \sigma \)-algebra \( B \) is generated by the level sets. Approximated by the union of these level sets, for any two sets \( A \) and \( \tilde{A} \) with positive measures, there also exists \( n > 0 \) with \( \mu(T^{-np} A \cap \tilde{A}) > 0 \). This implies that \( \mu \) is an ergodic measure under \( T^p \) for any \( p \).

In the following we will prove the u.d. property for the partition \( \{A, A^c\} \) where \( A \) is the union of level sets in \( W_t \) for some \( \ell \).

**Lemma 5.5.** Let \( A \) be a union of finite level sets in \( W_\ell \) for some \( \ell \in \mathbb{N} \) with \( 0 < \mu(A) \leq \frac{1}{2} \xi_\ell \), \( \alpha = \{A, A^c\} \). Then for sufficiently large \( t \),

\[
\liminf_{n \to \infty} \frac{1}{n} \mu(T^{-t_1} \alpha \cdots T^{-t_n} \alpha) \geq -\frac{1}{2} \mu(A) \log \frac{\mu(A)}{1 - \mu(A)} > 0,
\]

where \( F^t = \{t_1 < t_2 < \cdots\} \) is given by (5.8). Hence \( F^t \) is an entropy generating sequence of \( \alpha \). Moreover, \( D_\mu(T, \alpha) = \tau \).

**Proof.** Since \( 0 < \mu(A) \leq \frac{1}{2} \xi_\ell < \frac{1}{2} \), \( -\mu(A) \log \frac{\mu(A)}{1 - \mu(A)} > 0 \). We can take \( t \) sufficiently large such that \( l_t \geq \ell \) and

\[
\log \left( \left(1 + \frac{b_t}{c(t)}\right) \cdot \frac{1}{\xi_t} \right) < -\frac{1}{2} \mu(A) \log \frac{\mu(A)}{1 - \mu(A)}.
\]

For convenience, let \( A_0 = A, A_1 = A^c \). For any finite subset \( B \) of \( F^t \) and any \( s = (s_b)_{b \in B} \in \{0, 1\}^B \), let \( B_0(s) = \{b \in B : s_b = 0\} \) and \( B_1(s) = \{b \in B : s_b = 1\} \). Since \( A \)
is also the union of level sets in $W_t$, by Lemma 5.3 we have

\begin{equation}
\mu\left(\bigcap_{b \in B} T^{-b} A_{s_0}\right) \leq \mu\left(\bigcap_{b \in B_0(s)} T^{-b} A_0\right) \\
\leq \left(1 + \frac{h_{t_2}}{c_{nt}}\right) \cdot \left(\frac{1}{\xi_{t_2}}\right)^{|B_0(s)|} \prod_{b \in B_0(s)} \mu(A_0) \\
= \prod_{b \in B_0(s)} \mu(A_0) \cdot \prod_{b \in B_1(s)} \mu(A_1) \cdot \left(1 + \frac{h_{t_2}}{c_{nt}}\right) \cdot \left(\frac{1}{\xi_{t_2}}\right)^{|B_0(s)|} \cdot \left(\frac{1}{\mu(A_1)}\right)^{|B_1(s)|}
\end{equation}

So

\begin{equation}
H_{\mu}\left(\bigvee_{i=1}^{m} T^{-t_i} \alpha\right) = \sum_{s \in \{0,1\}^{m}} -\mu\left(\bigcap_{i=1}^{m} T^{-t_i} A_{s_i}\right) \log \left(\mu\left(\bigcap_{i=1}^{m} T^{-t_i} A_{s_i}\right)\right) \\
\geq \sum_{s \in \{0,1\}^{m}} -\mu\left(\bigcap_{i=1}^{m} T^{-t_i} A_{s_i}\right) \log \left(\prod_{i=1}^{m} \mu(A_{s_i}) \cdot (1 + \frac{h_{t_2}}{c_{nt}})^m \cdot \left(\frac{1}{\xi_{t_2}}\right)^m \cdot \left(\frac{1}{\mu(A_1)}\right)^m\right).
\end{equation}

Since

\begin{equation}
\sum_{s \in \{0,1\}^{m}} -\mu\left(\bigcap_{i=1}^{m} T^{-t_i} A_{s_i}\right) \log \left(\prod_{i=1}^{m} \mu(A_{s_i})\right) = \sum_{i=1}^{m} \sum_{s \in \{0,1\}} -\mu\left(\bigcap_{i=1}^{m} T^{-t_i} A_{s_i}\right) \log \left(\mu(A_{s_i})\right)
\end{equation}

we have

\begin{equation}
H_{\mu}\left(\bigvee_{i=1}^{m} T^{-t_i} \alpha\right) \geq m H_{\mu}(\alpha) - \log \left(1 + \frac{h_{t_2}}{c_{nt}}\right)^m \cdot \left(\frac{1}{\xi_{t_2}}\right)^m \cdot \left(\frac{1}{\mu(A_1)}\right)^m
\end{equation}

\begin{equation}
> m \left( H_{\mu}(\alpha) + \frac{1}{2} \mu(A) \log \frac{\mu(A)}{1 - \mu(A)} + \log (1 - \mu(A)) \right) \\
= m \left( -\frac{1}{2} \mu(A) \log \frac{\mu(A)}{1 - \mu(A)} \right).
\end{equation}

And hence

\begin{equation}
\liminf_{n \to \infty} \frac{1}{n} H_{\mu}\left(\bigvee_{i=1}^{n} T^{-t_i} \alpha\right) \geq -\frac{1}{2} \mu(A) \log \frac{\mu(A)}{1 - \mu(A)} > 0.
\end{equation}

By Lemma 5.1 and 5.2 we have $D_{\mu}(T, \alpha) = \tau$. \qed

**Theorem 5.6.** $(X, \mathcal{B}, \mu, T)$ is a $\tau$-u.d. system.
Proof. Given any \( \beta = \{B, X \setminus B\} \in \mathcal{P}_X^2 \) with \( 0 < \mu(B) < \frac{1}{2} \). Then
\[
c(\beta) = \frac{1}{2} \mu(B) \log \frac{\mu(B)}{1 - \mu(B)} > 0.
\]
Take \( \epsilon > 0 \) with \( \epsilon < \frac{1}{2} c(\beta) \). So by Lemma 4.15 of [20], we can choose \( \delta > 0 \) small enough such that \( H_\mu(\beta|\gamma) + H_\mu(\gamma|\beta) < \epsilon \), whenever \( \gamma = \{E, X \setminus E\} \in \mathcal{P}_X^2 \) with \( \mu(B\Delta E) + \mu((X \setminus B)\Delta (X \setminus E)) < \delta \). Now there is a subset \( A \) of \( X \) which is a union of level sets in \( W_\ell \) for sufficiently large \( \ell \), such that \( \mu(A\Delta B) < \frac{\delta}{2} \) and \( \mu((X \setminus A)\Delta (X \setminus B)) < \frac{\delta}{2} \). Let \( \alpha = \{A, X \setminus A\} \), then \( H_\mu(\alpha|\beta) + H_\mu(\beta|\alpha) < \epsilon \). Moreover, we can make \( c(\alpha) > \frac{1}{2} c(\beta) \) when \( \delta \) is sufficiently small. By Lemma 5.2 and Lemma 5.5 there exists \( F = \{t_1 < t_2 < \cdots\} \) such that
\[
\liminf_{n \to \infty} \frac{1}{n} H_\mu \left( \bigvee_{i=1}^{n} T^{-t_i} \alpha \right) \geq c(\alpha).
\]
Moreover by (5.15), we have
\[
\liminf_{n \to +\infty} \frac{1}{n} H_\mu \left( \bigvee_{i=1}^{n} T^{-t_i} \beta \right)
= \liminf_{n \to +\infty} \frac{1}{n} \left( H_\mu \left( \bigvee_{i=1}^{n} T^{-t_i} (\alpha \lor \beta) \right) - H_\mu \left( \bigvee_{i=1}^{n} T^{-t_i} \alpha \bigg| \bigvee_{i=1}^{n} T^{-t_i} \beta \right) \right)
\geq \liminf_{n \to +\infty} \frac{1}{n} \left( H_\mu \left( \bigvee_{i=1}^{n} T^{-t_i} \alpha \right) - n H_\mu(\alpha|\beta) \right)
\geq c(\alpha) - \frac{1}{2} c(\beta) > 0,
\]
which means that \( F \) is also an entropy generating sequence of \( \beta \). Hence \( D_\mu(T, \beta) \geq D(F) \geq \tau \). Combining with Lemma 5.1, \( D_\mu(T, \beta) = \tau \).

Now we suppose that for some \( \beta = \{B, X \setminus B\} \) with \( \mu(B) = \frac{1}{2} \), \( D_\mu(T, \beta) < \tau \). Since \( \mu \) is ergodic under both \( T \) and \( T^2 \) (Remark 5.4), \( \mu(B \cap T^{-1}B) \neq 0, \frac{1}{2} \). Notice that \( \{B \cap T^{-1}B, X \setminus (B \cap T^{-1}B)\} \leq \beta \land T^{-1} \beta \), applying (2) of Proposition 2.9, we have
\[
D_\mu \left( T, \{B \cap T^{-1}B, X \setminus (B \cap T^{-1}B)\} \right)
\leq D_\mu(T, \beta) \lor T^{-1} \beta
= D_\mu(T, \beta) < \tau,
\]
which leads a contradiction.

Since \( \beta \) is arbitrary, we conclude that \( (X, \mathcal{B}, \mu, T) \) is a \( \tau \)-u.d. system. \( \square \)

Remark 5.7. By the similar method, we can also choose suitable parameters such that \( (X, \mathcal{B}, T, \mu) \) is a 1−u.d. system with zero entropy.

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