LECTURE 16

QUANTUM FIELD THEORY IN SINGULAR LIMITS *

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1. INTRODUCTION

It has been suggested [1] to approximate geometry by dense Feynman graphs of the same topology, taking the number of vertices to infinity. The dynamically triangulated random surfaces summed on different topologies are then viewed as the manifold for string propagation. Summing up Feynman graphs of the O(N) matrix model in d dimensions results in a genus expansion and it provides, in some sense, a nonperturbative treatment of string theory when the double scaling limit is enforced [2]. If these efforts could be extended to d > 1 dimensions, then a major progress would have been achieved in studying a long lasting problem in elementary particle theory. Namely, the relation between d dimensional quantum field theory and its possible formulation in terms of strings. The possibility of reaching a "stringy" representation of SU(N) and U(N) quantum field theory in a correlated singular limit was proposed some time ago in [3].

O(N) symmetric vector models represent discretized filamentary surfaces - randomly branched polymers in the double scaling limit, in the same manner in which matrix models, in their double scaling limit, provide representations

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of dynamically triangulated random surfaces. Though matrix theories attract most attention, a detailed understanding of these theories exists only for dimensions $d \leq 1$. On the other hand, in many cases, the double scaling limit of the $O(N)$ vector models\cite{4, 5} can be successfully studied also in dimensions $d > 1$ thus, providing intuition for the search for a possible description of quantum field theory in terms of extended objects, string-like excitations, in four dimensions. This stimulated new studies\cite{5} of the phase structure of $O(N)$ vector quantum field theories that offered a new look into this long lasting problem. Most often, quantum field theories are described by their point-like quanta and a direct relation to a string-like structure is difficult to visualize; if there is a correspondence to an hadronic string, it must be of a less direct kind\cite{3}.

2. SINGULAR LIMITS OF $O(N)$ SYMMETRIC MODELS

2.1. The Double Scaling Limit in $O(N)$ Matrix Models

In $O(N)$ matrix models the double scaling limit is enforced in the calculation of the partition function

$$Z_N(g) = \int D\hat{\Phi} \exp\{-\beta \int d^d x \text{Tr}\{\hat{\Phi}(x)\hat{K}\hat{\Phi}(x) + V(\hat{\Phi}(x))}\} \quad [1]$$

where $\hat{\Phi}(x)$ is an $N \times N$ Hermitian matrix and $V(\hat{\Phi}(x))$ is the potential depending on the coupling(s) constant(s) $\{g_i\}$.

In zero dimensions, after performing the integration on the angular variables, one is left with the integration on the eigenvalues $\lambda_i$

$$Z_N(g) = \Omega_N \int d\lambda_i \ exp\{2 \sum_{i,j} \ln|\lambda_i - \lambda_j| - \beta \sum_i U(\lambda_i)\}. \quad [2]$$

In Eq. [2] one notes a Pauli repulsion between the eigenvalues, and a critical point $\{g_i\} = \{g_iC\}$ is found when the Fermi level reaches the extremum of the potential. The weak coupling limit, namely, $\frac{1}{N} \sim \frac{1}{\beta} \to 0$, gives a one-dimensional frozen Dyson gas and the planar graphs dominate. As $N \to \infty$ and $\frac{N}{\beta} \to 1$ (or $\{g_i\} \to \{g_iC\}$), the “melting” of the gas starts and the non-planar graphs become important. The genus ($G$) expansion of the free energy of the system is given by ($S$ is the area which is proportional to the order of the Feynman graphs):

$$F = \ln Z_N = a + b \ln \beta + \sum_{G,S} N^{2(1-G)} \left(\frac{N}{\beta}\right)^S F_S \sim \sum_G \left(\frac{1}{N}\right)^{2G-2} \mathcal{A}_G\{g_i\} \quad [3]$$

This topological series is not Borel summable due to the factorial growth of the positive $\mathcal{A}_G\{g_i\}$ with the genus $G$, and a nonperturbative approach is
needed. As the set of coupling constants \( \{ g_i \} \rightarrow \{ g_C \} \) approaches a set of critical values, the loop expansion at a given topology diverges. At a given topology, \( A_G\{g\} \) has a finite radius of convergence when expanded in powers of the coupling constant \( g \). The limit \( g \rightarrow g_c \) emphasizes the higher order terms in this expansion, namely, the denser Feynman graphs. Typically, for a potential \( V(\Phi) \) with a single coupling,

\[
A_G\{g\} \sim (g - g_c)^{-\kappa_G} \rightarrow \infty \quad \{ \text{as } g \rightarrow g_c \} \tag{4}
\]

In the suitable singular limit (the double scaling limit), all terms in the topological expansion in Eq. [3] are of equal importance. The nonperturbative framework needed here should be capable of reproducing the topological series as an asymptotic expansion but should also be a framework for nonperturbative calculations[2]. In the limit \( \beta = N \rightarrow \infty \) \( g \rightarrow g_c \) in a correlated manner, the powers of \( \frac{1}{N} \) in Eq. [3] are compensated by the growing \( A_G\{g\} \) in Eq.[4]. In this limit, Eq.[3] turns into an expansion in the string coupling constant since all genera are relevant now. The physical meaning of this formal limit will be soon clarified in the O(N) vector quantum field theory.

The model belongs to the same universality class of two dimensional conformally invariant matter field in gravitational background and thus, the critical exponents of this model are calculable from Eq. [3]. One may, indeed, find it quite remarkable that these relatively simple integrals possess the critical behavior of two dimensional gravity coupled to conformal matter.

2.2. The Double Scaling Limit in O(N) Vector Models

The double scaling limit of the O(N) vector models follows an analogous procedure,

\[
Z_N(g) = \int D\Phi exp\{-\beta \int d^d x \{ \tilde{\Phi}(x) \tilde{K} \tilde{\Phi}(x) + V(\tilde{\Phi}^2(\tilde{\Phi})) \} \} \tag{5}
\]

where \( \tilde{\Phi} \) is an O(N) vector. In zero dimensions, this turns into a simple N dimensional integral over the components of \( \tilde{\Phi} \). The expansion in terms of Feynman graphs in the large N limit resembles now an expansion of "randomly branched polymers"[3]. The graphical realization is obtained when the dual diagrams are defined from the Feynman graphs by interchanging the vertex and propagator in the Feynman graph by the bond and "molecule" in the dual graph, respectively, which describes now the branched polymer.

\[
\ln Z_N = \sum_{h,b} N^{(1-h)} \left( \frac{N}{\beta} \right)^b F_b \sim \sum_b \left( \frac{1}{N} \right)^{h-1} A_b\{g_i\} \tag{6}
\]

Following here the suitable scaling procedure, Eq. [5] turns into an expansion in a "polymer" coupling constant. For \( V(\tilde{\Phi}^2) \sim g\tilde{\Phi}^2(x)^2 \) in the correlated limit \( N \rightarrow \infty \) and \( g \rightarrow g_C \), the negative powers of N are compensated by
$A_h \{ g \} \sim (g - g_C)^{-\gamma_h} \to \infty \quad \text{as } g \to g_c$. Criticality comes from the tuning of the potential in order to balance the centrifugal barrier. Whereas matrix models in dimensions $d > 1$ are very difficult to study, even in the leading large $N$ limit or the double scaling limit, vector models are relatively easy to analyze in these limits even at $d \geq 2$.

There are several interesting physical questions that can be answered when the formal double scaling limit is enforced in $d \geq 2$ O(N) vector theories. A possible list of questions follows:

1. What is the phase structure, spectrum and symmetries of the quantum field theory in this singular limit?
2. Does the flow $g \to g_c$ agree with the renormalization group flow?
3. How are the ultraviolet divergences taken into account at $d>2$? (Needless to say that these questions were not raised in matrix models in $d \leq 1$).
4. Taking into account that we are discussing a renormalizable quantum field theory, what is the nature of the divergences needed for compensating the powers of $N^{-h}$ in Eq. [6] so that all orders in $N^{-1}$ are of equal importance?

These issues have been discussed in [5]-[7], while considering as an example the large $N$ limit of the self interacting scalar O(N) symmetric vector model in $0 \leq d < 4$ Euclidian dimensions defined by the functional integral in Eq.[5] with $\hat{K} = \partial^2 D(\frac{\partial^2}{\Lambda^2})$, (where $D(z)$ is a positive non-vanishing polynomial with $D(0)=1$). The following results have been found [6, 7]:

1. The physical meaning of the double scaling limit can be phrased as tuning the force between the O(N) quanta so that a singlet massless bound state is created in the spectrum. This is a singular limit in the sense that while $N \to \infty$ the coupling constant is often tuned to a negative value $g \to g_C$. The physical spectrum consists of the propagating O(N) quanta of small mass $m$ in addition to the massless O(N) singlet.
2. The flow of $g \to g_c$ is consistent with the renormalization group flow, provided the limits $N \to \infty$ and the cutoff $\Lambda \to \infty$ are appropriately correlated. This point was made also in Ref.[3].
3. The ultraviolet divergences are dictating the effective field theory obtained in the singular limit of the double scaling limit. They enforce a detailed relation between $\Lambda$ and $N$, mentioned in (2) above, $\Lambda = \Lambda(N)$.
4. The divergences that compensate the decreasing powers of $N^{-h}$ in Eq. [6] and make all orders in $N^{-1}$ of equal importance, are infrared divergences. Thus, in the double scaling limit, the tuning of the forces that produce a massless singlet excitation, produce infrared divergences which are the essential ingredient of this limit.
5. Following (4) above, in order that the compensating infrared singularities will show up, the effective field theory of the singlet bound state is super-renormalizable.
6. In the critical dimensions (e.g., $\Phi^6$ in $d=3$) the massless O(N) singlet excitation is the Goldston boson of spontaneous breaking of scale invariance - the dilaton [8].
In the conventional treatment of the large N limit, Eq. [5] is expressed by:

$$Z_N = \int D\rho \int D\lambda \exp\{-N \int d^d x [V(\rho) - \frac{1}{2} \lambda \rho] + \frac{1}{2} N \text{tr} \ln(-\partial^2 + \lambda)\}. \quad [7]$$

For $N$ large the integral is evaluated by the steepest descent. The saddle point value $\lambda$ is the $\vec{\Phi}$-field mass squared $\lambda = m^2$ and $\rho = \rho_s$ at the saddle point. The matrix of the second partial derivatives of the effective action is:

$$N \left( \begin{array}{cc} V''(\rho) & -\frac{1}{2} \\ -\frac{1}{2} B_2(p;m^2) & \frac{1}{4} \end{array} \right), \quad [8]$$

where $B_2(p;m^2)$ is the appropriate "bubble graph". Since the integration contour for $\lambda = m^2$ should be parallel to the imaginary axis, a necessary condition for stability is that the determinant remains negative. For Pauli-Villars type regularization, the function $B_2(p;m^2)$ is decreasing so that this condition is implied by the condition at zero momentum

$$\text{det} M < 0 \iff 2V''(\rho)B_2(0;m^2) + 1 > 0 \quad [9].$$

For $m$ small

$$B_2(0;m^2) = \frac{1}{2}(d-2)K(d)m^{d-4} - a(d)\Lambda^{d-4} + O\left(m^{d-2}\Lambda^{-2}, m^2\Lambda^{d-6}\right). \quad [10]$$

$V''(\rho)$ can be expanded now around the critical $\rho$. From the saddle point condition:

$$\rho - \rho_c = -K(d)m^{d-2} + a(d)m^2\Lambda^{d-4} + O\left(m^d\Lambda^{-2}, m^2\Lambda^{d-6}\right). \quad [11]$$

$$\rho_c = \frac{1}{(2\pi)^d} \int d^d k \frac{1}{k^2}, \quad K(d) = \frac{\Gamma(1-d/2)}{(4\pi)^{d/2}}, \quad a(d) = \frac{1}{(2\pi)^d} \int d^d k \left(1 - \frac{1}{D^2(k^2)}\right).$$

The constant $a(d)$ depends on the cut-off procedure, and one finds in Eq. [9] for a multicritical point:

$$(-1)^n \frac{d-2}{(n-2)!} K^{n-1}(d)m^{n(d-2)-d}V^{(n)}(\rho_c) + 1 > 0. \quad [12]$$

This condition is satisfied by a normal critical point since $V''(\rho_c) > 0$. For $n$ even it is always satisfied while for $n$ odd Eq.[12] is always satisfied above the critical dimension and never below. At the upper-critical dimension $2/(n-1) = d-2$ we find a condition on the value of $V^{(n)}(\rho_c)$.

The mass-matrix has a zero eigenvalue which corresponds to the appearance of a new massless excitation other than the $\vec{\Phi}$ quanta (which has a mass $m$). Then

$$\text{det} M = 0 \iff 2V''(\rho)B_2(0;m^2) + 1 = 0. \quad [13]$$
In the two-space, the corresponding eigenvector has components \((1/2, V''(ρ))\). In the small \(m\) limit \(V''(ρ)\) must be small and we are close to a multicritical point and

\[ (-1)^{n-1} \frac{d - 2}{(n - 2)!} K^{n-1}(d) m^{n(d-2)} - d V^{(n)}(ρ_c) = 1. \]  

[14]

This equation has solutions only for \(n(d-2) = d\), i.e., at the critical dimension. The compatibility then fixes the value of \(V^{(n)}(ρ_c) = Ω_c\). If we take into account the leading correction to the small \(m\) behavior we find instead:

\[ V^{(n)}(ρ_c)Ω_c^{-1} - 1 \sim (2n - 3) \frac{a(d)}{K(d)} \left( \frac{m}{Λ} \right)^{4-d}. \]  

[15]

This means that when \(a(d) > 0\) there exists a small region \(V^{(n)}(ρ_c) > Ω_c\) where the vector field is massive with a small mass \(m\) and the O(N) singlet bound-state is massless. The value \(Ω_c\) is a fixed point value. The analysis can be extended to a situation where the scalar field has a small but non-vanishing mass \(M\) and \(m\) is still small. In particular, the neighborhood of the special point \(V^{(n)}(ρ_c) = Ω_c\) can be explored. The vanishing of the determinant in Eq.[8] implies \(1 + 2V''(ρ)B_2(iM; m^2) = 0\). Because \(M\) and \(m\) are small, this equation still implies that \(ρ\) is close to a point \(ρ_c\) where \(V''(ρ)\) vanishes. Since reality imposes \(M < 2m\), it is easy to verify that this equation has also solutions for only the critical dimension.

Of particular interest is the \(η_0(\vec{Φ}^2)^3\) theory in three dimensions, discussed in the past in Ref.[8]. Taking the limit \(N → ∞, Λ → ∞\) in a correlated manner with \(η_0 → Ω_c\) one encounters a manifestation of dimensional transmutation at a nontrivial ultraviolet fixed point. In the massive phase described above, scale invariance is broken only spontaneously. Indeed, one finds that the trace of the energy momentum tensor stays zero at \(η_0 = Ω_c\):

\[ < P'\mid Θ_{\mu\nu}\mid P > = P'_μ P_ν + P'_ν P_μ - g_μν P'P + g_μν m^2 + (g_μν q_ν - q^2 g_μν) \left( \frac{m^2}{q^2} + \frac{1}{4} \right) \]  

[16]

where \(q = P' - P\). A massless dilaton - the Goldston boson associated with this spontaneous breaking - appears in the ground state spectrum as a reflection of the Goldston realization of scale symmetry. The normal ordering of \(\vec{Φ}^6\) induces a \(\vec{Φ}^4\) interaction which guarantees the appearance of the dilaton pole in the physical amplitudes as \(η_0 → Ω_c\).

The double scaling limit results here in a theory with an ultraviolet fixed point and the following properties are found: (a) There is a dynamical mass \(m \neq 0\) for the scalar \(Φ\) particles. (b) The \(Φ - Φ\) bound state pseudo-Goldston boson (dilaton) has a finite small mass and the interaction term of these scalars is calculable. This is an interesting mechanism to produce a low mass scalar in a spectrum with possible phenomenological implications. (c) The trace of the energy momentum tensor \(Θ_{μν}^\Lambda\) is finite. (d) An induced \(Φ^4\) coupling appears in the theory.
3. DISCUSSION

This is a short summary of the phase structure of $O(N)$ symmetric quantum field theories in a singular limit, the double scaling limit. The main point emphasized here is that this formal singular limit, recently discussed mainly in $d = 0$ $O(N)$ matrix models, has an intriguing physical meaning in $d \geq 2$ $O(N)$ vector theories. In this limit all orders in $\frac{1}{N}$ are of equal importance since at each order infrared divergences compensate for the decrease in powers of $\frac{1}{N}$.

The infrared divergences are due to the tuning of the strength of the force ($g \to g_c$) between the $O(N)$ quanta so that a massless $O(N)$ singlet appears in the spectrum. At critical dimension an interesting phase structure is revealed, the massless excitation has the expected physical meaning: it is the Goldstone boson of spontaneous breaking of scale invariance - the dilaton.

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