ON SUMMATION OF NON-HARMONIC FOURIER SERIES

YURI BELOV, YURI LYUBARSKII

Abstract. Let a sequence $\Lambda \subset \mathbb{C}$ be such that the corresponding system of exponential functions $E(\Lambda) := \{e^{i\lambda t}\}_{\lambda \in \Lambda}$ is complete and minimal in $L^2(-\pi, \pi)$ and thus each function $f \in L^2(-\pi, \pi)$ corresponds to a non-harmonic Fourier series in $E(\Lambda)$. We prove that if the generating function $G$ of $\Lambda$ satisfies Muckenhoupt ($A_2$) condition on $\mathbb{R}$, then this series admits a linear summation method. Recent results show that ($A_2$) condition cannot be omitted.

1. Introduction and main results

Let $\Lambda \subset \mathbb{C}$ be a discrete sequence and let the corresponding sequence of exponential functions $E(\Lambda) := \{e^{i\lambda t}\}_{\lambda \in \Lambda}$ be complete and minimal in $L^2(-\pi, \pi)$. By $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ we denote the biorthogonal system: $(e^{i\mu t}, \varphi_\lambda) = \delta_{\lambda,\mu}$, $\lambda, \mu \in \Lambda$. Then to each function $f \in L^2(-\pi, \pi)$ one can associate its non-harmonic Fourier series in $E(\Lambda)$

$$f \sim \sum_{\lambda \in \Lambda} (f, \varphi_\lambda)e^{i\lambda t}.$$  

It was shown by R. Young in [11] that the biorthogonal system $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ is also complete, hence, the Fourier coefficients $(f, \varphi_\lambda)$ determine the function $f$ uniquely. In this article we study reconstruction of $f$ from series (1.1) by a linear summation method.

Definition 1.1. The series (1.1) admits a linear summation method if there exists a matrix $\{w(\lambda, n)\}_{\lambda \in \Lambda, n \in \mathbb{N}}$ such that $\lim_{n \to \infty} w(\lambda, n) = 1$, $\lim_{\lambda \to \infty} w(\lambda, n) \to 0$, and

$$\sum_{\lambda \in \Lambda} w(\lambda, n)(f, \varphi_\lambda)e^{i\lambda t} \to f, \quad n \to \infty,$$

for any $f \in L^2(-\pi, \pi)$. We assume that the series in (1.2) converges for any $n$.

It is well-known (see, e.g. [6, Lecture 17]) that for complete and minimal system $E(\Lambda)$ there exists the generating function,

$$G(z) := G(0) \lim_{R \to \infty} \prod_{\lambda \in \Lambda, |\lambda| < R} \left(1 - \frac{z}{\lambda}\right),$$

and that $G$ is of exponential type $\pi$ in both half-planes $\mathbb{C}_{\pm}$.

The first author was supported by the Chebyshev Laboratory (St. Petersburg State University) under RF Government grant 11.G34.31.0026, by JSC "Gazprom Neft", and by RFBR grant 12-01-31492. The second author is partly supported by the Norwegian Research Council project DIMMA #213638. Part of this work was done while the authors were staying at the Center for Advanced Study, Norwegian Academy of Science, and they would like to express their gratitude to the institute for the hospitality.
We study the summability properties of series (1.1). In this introduction we assume that all points in $\Lambda$ belong to a shifted upper half-plane, $\Lambda \subset \mathbb{C}_\delta := \{ \zeta : \Im \zeta > \delta \}$ for some $\delta > 0$. This is done in order to present results in the introduction as simple as possible. In the main text of the article we prove the results in full generality, i.e. not assuming $\Lambda \subset \mathbb{C}_\delta$.

The simplest summability procedure corresponds to the case when $\mathcal{E}(\Lambda)$ forms an unconditional basis in $L^2(-\pi, \pi)$. Such systems are characterised by B. Pavlov ([9], see also [7] for another proof).

**Theorem A.** Let $\Lambda \subset \mathbb{C}_\delta$, $\delta > 0$. In order that the system $\mathcal{E}(\Lambda)$ be an unconditional basis in $L^2(-\pi, \pi)$ it is necessary and sufficient that

1. $|G(x)|^2$ satisfies the Muckenhoupt condition $(A_2)$:

$$
\sup_{I=[a,b]} \frac{1}{|I|^2} \int_I |G(x)|^2 dx \int_I |G(x)|^{-2} dx < \infty,
$$

2. The set $\Lambda$ satisfies the Carleson type condition $(C)$:

$$
\sup_{\lambda \in \Lambda} \sum_{\mu \in \Lambda, \mu \neq \lambda} \frac{(1 + |\Im \lambda|)(1 + |\Im \mu|)}{|\lambda - \mu|^2} < \infty.
$$

This theorem was proved by Pavlov under the additional restriction $\sup_{\lambda \in \Lambda} |\Im \lambda| < \infty$ and by Nikolski assuming only $\inf_{\lambda \in \Lambda} \Im \lambda > -\infty$. Finally, Minkin [8] got rid of all priori assumptions on $\Lambda$.

One of motivations for our article is to trace ”collaboration” between the two conditions. It seems that the $(A_2)$ condition provides existence of some linear summation method, while the choice of this method depends upon how far is the sequence $\Lambda$ from the condition $(C)$.

It is almost straightforward (see e.g. [7, Section 4]) that if the generating function $G$ is of exponential type $\pi$ in both halfplanes $\mathbb{C}_\pm$ and also $|G(x)|^2 \in (A_2)$, then the system $\mathcal{E}(\Lambda)$ is complete and minimal.

**Theorem 1.1.** Let $\Lambda$ be a sequence such that the generating function $G$ is of exponential type $\pi$ in both halfplanes $\mathbb{C}_\pm$ and also satisfies Muckenhoupt condition (1.3). Then $\mathcal{E}(\Lambda)$ admits a linear summation method.

**Remark 1.1.** A weaker reconstruction property was proved in [4, Theorem 2.5], namely if $|G(x)|^2 \in (A_2)$, then

$$
f \in \text{span}\{ (f, \varphi_\lambda)e^{i\lambda t} \} \text{ for any } f \in L^2(-\pi, \pi).
$$

This statement follows of course from the summation theorem above, yet in Section 4.10 we give a simple direct proof. Techniques of this proof may be useful in other problems of this kind.

A short historical remark may be pertinent. Complete and minimal systems in (generally speaking) Banach spaces with total biorthogonal system are called strong $M$-basis or hereditarily complete systems if each element of the space is uniquely determined by its Fourier coefficients. Existence of linear summation method may be considered as an
extreme case of hereditary completeness. Other examples of such systems have been considered by Markus in connection with spectral synthesis for linear operators, Lambrou, and Papadakis in connection with reflexive algebras. Nikolski, Dovbysh, and Sudakov found a parametrization of all nonhereditarily complete systems. We refer the reader to [1, 2] and references therein.

Remark 1.2. It is shown in the recent article [1], that one cannot omit \((A_2)\) condition, generally speaking: there exist \(\Lambda \in (C)\) and \(f \in L^2(-\pi, \pi)\) such that the system \(\mathcal{E}(\Lambda)\) is complete and minimal in \(L^2(-\pi, \pi)\) and,

\[
(1.6) \quad f \not\in \overline{\text{span}\{(f, \varphi_\lambda)e^{i\lambda t}\}}.
\]

On the other hand it was recently proved (3) that there exists hereditarily complete system \(\mathcal{E}(\Lambda), \Lambda \subset \mathbb{C}_\delta\), such that \(|G(x)|^2 \not\in (A_2)\). The authors wonder if the following statement (converse to Theorem 1.1) is true:

Let a sequence \(\Lambda \subset \mathbb{C}_\delta, \delta > 0\) be such that the system \(\mathcal{E}(\Lambda)\) is complete and minimal and the series (1.1) admits the linear summation method. Then the generating function satisfies the Muckenhoupt condition: \(|G(x)|^2 \in (A_2)\).

We reformulate our problem in the Paley-Wiener space of entire functions

\[\mathcal{PW}_\pi := \{F: F(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t)e^{izt}dt, \quad g \in L^2(-\pi, \pi)\}\]

The Fourier transform \(\mathcal{F}\) acts unitarily from \(L^2(-\pi, \pi)\) onto \(\mathcal{PW}_\pi\), after this transition the exponential functions become the reproducing kernels in \(\mathcal{PW}_\pi\),

\[\mathcal{F}e^{-i\lambda t} = \frac{\sin(\pi(z - \bar{\lambda}))}{\pi(z - \bar{\lambda})} =: k_\lambda(z),\]

while the elements of the biorthogonal system become

\[\mathcal{F}(\varphi_\lambda) =: G_\lambda = \frac{G(z)}{G'(\lambda)(z - \lambda)}\]

Simple duality reasonings show that Theorem 1.1 is equivalent to

Theorem 1.2. Under conditions of Theorem 1.1 the Lagrange interpolating series

\[
(1.7) \quad F(z) \sim \sum_{\lambda \in \Lambda} F(\lambda)\frac{G(z)}{G'(\lambda)(z - \lambda)}
\]

admits a linear summation method for all \(F \in \mathcal{PW}_\pi\).

The compactwise summability of series (1.1) holds under even weaker hypothesis:

Theorem 1.3. Let \(G\) be an entire function of exponential type \(\pi\) in both halfplanes \(\mathbb{C}_\pm\) and

\[
(1.8) \quad \int_{\mathbb{R}} \frac{|G(x)|^2}{1 + |x|^2}dx < \infty, \quad \int_{\mathbb{R}} \frac{dx}{|G(x)|^2(1 + |x|^2)} < \infty.
\]
Then there exists a linear summation method \( \{w(\lambda, n)\} \) with compactwise convergence, namely,
\[
\lim_{n \to \infty} \sum_{\lambda \in \Lambda} w(\lambda, n) F(\lambda) \frac{G(z)}{G'(\lambda)(z - \lambda)} = F(z), \text{ uniformly for } z \in K
\]
for all \( F \in \mathcal{P}W_\pi \) and \( K \subseteq \mathbb{C} \).

The conditions (1.8) are weaker than Muckenhoupt condition (1.3). But they also imply that \( \mathcal{E}(\Lambda) \) is a complete and minimal system in \( L^2(-\pi, \pi) \).

We construct two concrete summation methods for the series (1.1) and (1.7). The first method is adjusted for concrete choice of \( \Lambda \) and reflects "how far" is \( \Lambda \) from condition (C). This method is a development of ideas in [10]. In order to construct this method we introduced \textit{weighted model spaces} which may be of independent interest.

The second method stems from Abel–Poisson and Cesàro methods. It also can be used for the proof of Theorem 1.3. The construction of this method is universal for all exponential systems under consideration.

**Structure of the article.** Theorems 1.1-1.2 are proved in Section 3 using the projection method. The preliminary facts for the construction is given in Section 2. The universal summation method for the series (1.1), (1.7) is given in Section 4.

Given positive quantities \( U(x), V(x) \), the notation \( U(x) \lesssim V(x) \) (or, equivalently, \( V(x) \gtrsim U(x) \)) means that there is a constant \( C \) such that \( U(x) \leq CV(x) \) holds for all \( x \) in the set in question. We write \( U(x) \simeq V(x) \) if both \( U(x) \lesssim V(x) \) and \( V(x) \lesssim U(x) \).

2. Weighted model subspaces

Assume that \( \Lambda \subseteq \mathbb{C}_+ \). Consider the Blaschke products
\[
B(z) = \prod_{\lambda \in \Lambda} \frac{\bar{\lambda} z - \lambda}{\lambda z - \lambda}, \quad B_n(z) = \prod_{\lambda \in \Lambda, |\lambda| < n} \frac{\bar{\lambda} z - \lambda}{\lambda z - \lambda}, \quad \beta_n(z) = B(z)/B_n(z).
\]

**Theorem 2.1.** Let the generating function \( G \) be such that \( |G(x)|^2 \in (A_2) \). Then
\[
\sum_{\lambda} \beta_n(\lambda) (f, \varphi_\lambda) e^{i\lambda t} L^2(-\pi, \pi) \to f, \text{ as } n \to \infty, \text{ for any } f \in L^2(-\pi, \pi),
\]
\[
\sum_{\lambda} \beta_n(\lambda) F(\lambda) \frac{G(z)}{G'(\lambda)(z - \lambda)} \xrightarrow{PW_\pi} F, \text{ as } n \to \infty, \text{ for any } F \in \mathcal{P}W_\pi.
\]

In this section we consider weighted model subspaces which are used in the proof of this theorem.

2.1. Definition of subspaces. Given an outer function \( \omega(z) \) in \( \mathbb{C}_+ \) such that \( |\omega(x)|^2 \in (A_2) \) we consider weighted spaces \( L^2(\mathbb{R}, |\omega|^{-2}) \) with the norms
\[
\|f\|_{\omega}^2 = \int_{-\infty}^{\infty} |f|^2 |\omega|^2 d\xi; \quad \text{and} \quad \|f\|_{\omega}^{-2} = \int_{-\infty}^{\infty} |f|^2 |\omega|^{-2} d\xi,
\]
and the \textit{weighted} Hardy spaces

\begin{equation}
\mathcal{H}^+_\omega = \frac{1}{\omega} H^2(\mathbb{C}_+), \quad \mathcal{H}^-_{\omega, -1} = \omega H^2(\mathbb{C}_+), \quad \mathcal{H}^-_\omega = \frac{1}{\omega^#} H^2(\mathbb{C}_-), \quad \mathcal{H}^-_{\omega, -1} = \omega^# H^2(\mathbb{C}_-),
\end{equation}

where $H^2(\mathbb{C}_\pm)$ denote the classical Hardy spaces in $\mathbb{C}_\pm$ respectively and, given a function $f(z), z \in \mathbb{C}_+$ we set

$$f^#(z) = \overline{f(\bar{z})}, \quad z \in \mathbb{C}_-.$$ 

In what follows we do not distinguish functions in $H^2(\mathbb{C}_\pm)$ and their boundary values on $\mathbb{R}$. The projectors $\mathcal{P}_\pm : L^2(\mathbb{R}) \mapsto H^2(\mathbb{C}_\pm)$ have the form

$$\mathcal{P}_\pm : f \mapsto \pm \frac{1}{2} f(x) + \frac{1}{\pi i} \int_{\mathbb{R}} \frac{f(\xi)}{x - \xi} d\xi,$$

and since $|\omega|^2 \in (A_2)$ they are bounded in the spaces $L^2(\mathbb{R}, |\omega|^2)$. Respectively we have

$$L^2(\mathbb{R}, |\omega|^2) = \mathcal{H}^+_\omega + \mathcal{H}^-_\omega \quad \text{and} \quad L^2(\mathbb{R}, |\omega|^{-2}) = \mathcal{H}^-_{\omega, -1} + \mathcal{H}^+_{\omega, -1}.$$ 

Therefore the spaces $\mathcal{H}^+_\omega, \mathcal{H}^-_{\omega, -1}, \mathcal{H}^+_{\omega, -1}, \mathcal{H}^-_\omega$ are mutually conjugated with respect to the coupling

\begin{equation}
\langle f_+, f_- \rangle = \int_{-\infty}^{\infty} f_+(\xi) f_-(\xi) d\xi, \quad f_\pm \in L^2(\mathbb{R}, |\omega|^2),
\end{equation}

i.e. $(\mathcal{H}^+_\omega)^* = \mathcal{H}^-_{\omega, -1}$ etc.

We will also use the subspaces

$$K_+(\omega) = \overline{\text{span}}_\omega \left\{ \frac{B(z)}{z - \lambda} \right\}_{\lambda \in \Lambda}, \quad M_+(\omega) = B^+ \mathcal{H}^+_\omega, \quad L_+(\omega) = \overline{\text{span}}_\omega \left\{ \frac{1}{z - \lambda} \right\}_{\lambda \in \Lambda}.$$

$$K_+(\omega^{-1}) = \overline{\text{span}}_{\omega, -1} \left\{ \frac{B(z)}{z - \lambda} \right\}_{\lambda \in \Lambda}, \quad M_+(\omega^{-1}) = B^+ \mathcal{H}^+_{\omega, -1}, \quad L_+(\omega^{-1}) = \overline{\text{span}}_{\omega, -1} \left\{ \frac{1}{z - \lambda} \right\}_{\lambda \in \Lambda}.$$

$$K_-(\omega) = \overline{\text{span}}_\omega \left\{ \frac{\#(z)}{z - \lambda} \right\}_{\lambda \in \Lambda}, \quad M_-(\omega) = B^- \mathcal{H}^-_\omega, \quad L_-(\omega) = \overline{\text{span}}_\omega \left\{ \frac{1}{z - \lambda} \right\}_{\lambda \in \Lambda}.$$

$$K_-(\omega^{-1}) = \overline{\text{span}}_{\omega, -1} \left\{ \frac{\#(z)}{z - \lambda} \right\}_{\lambda \in \Lambda}, \quad M_-(\omega^{-1}) = B^- \mathcal{H}^-_{\omega, -1}, \quad L_-(\omega^{-1}) = \overline{\text{span}}_{\omega, -1} \left\{ \frac{1}{z - \lambda} \right\}_{\lambda \in \Lambda}.$$

Here $\overline{\text{span}}_\omega$ and $\overline{\text{span}}_{\omega, -1}$ stay for the closure of the linear span in the norms $\| \cdot \|_\omega$ and $\| \cdot \|_{\omega, -1}$ respectively.
2.2. Weighted model spaces.

Lemma 2.1.

\[(2.5) \quad L_+(\omega) = M_- (\omega^{-1})^\perp, \quad L_+ (\omega^{-1}) = M_- (\omega)^\perp, \]

\[(2.6) \quad L_-(\omega) = M_+ (\omega^{-1})^\perp, \quad L_- (\omega^{-1}) = M_+ (\omega)^\perp, \]

Here orthogonality is considered with respect to coupling \(2.4\).

Indeed that \(L_+(\omega) \subset M_- (\omega^{-1})^\perp\) follows just from the reproducing kernel property of the Cauchy kernel. On the other hand

\[ f \in M_- (\omega^{-1})^\perp \iff f \perp B\mathcal{H}_{\omega,1}^- \iff B^\# f \perp \mathcal{H}_{\omega,1}^- \iff B^\# f \in \mathcal{H}_{\omega}^- . \]

Therefore \(f\) can be extended in \(\mathbb{C}_-\) as a meromorphic function with poles in \(\overline{\Lambda}\) which yields \(f \in L_+(\omega)\). Thus we have \(L_+(\omega) = M_- (\omega^{-1})^\perp\). The rest of relations \((2.5)-(2.6)\) can be proved similarly.

Lemma 2.2.

\[ L_+(\omega) = B\mathcal{H}_{\omega}^- \cap \mathcal{H}_{\omega}^+, \quad L_+ (\omega^{-1}) = B\mathcal{H}_{\omega,1}^- \cap \mathcal{H}_{\omega,1}^+, \]

\[ L_-(\omega) = B^\# \mathcal{H}_{\omega}^+ \cap \mathcal{H}_{\omega}^-, \quad L_- (\omega^{-1}) = B^\# \mathcal{H}_{\omega,1}^+ \cap \mathcal{H}_{\omega,1}^- . \]

Proof. We restrict ourselves just to the first of these relations. Clearly \(L_+(\omega) \subset \mathcal{H}_{\omega}^+\). We also have

\[ f \in L_+(\omega) \iff f \perp B^\# \mathcal{H}_{\omega,1}^- \iff B^\# f \perp \mathcal{H}_{\omega,1}^- \iff B^\# f \in \mathcal{H}_{\omega}^- \iff f \in B\mathcal{H}_{\omega}^- . \]

Lemma 2.3.

\[ K_+ (\omega^{-1}) = L_+ (\omega^{-1}), \quad K_+ (\omega) = L_+ (\omega), \quad K_- (\omega) = L_- (\omega), \quad K_- (\omega^{-1}) = L_- (\omega^{-1}). \]

Proof. Again we restrict ourselves to the first of these relations:

\[ K_+ (\omega^{-1}) = \overline{\text{span}}_{\omega,1} \left\{ \frac{B(z)}{z - \lambda} \right\}_{\lambda \in \Lambda} = B(z) \overline{\text{span}}_{\omega,1} \left\{ \frac{1}{z - \lambda} \right\}_{\lambda \in \Lambda} = B(z) L_- (\omega^{-1}) = B(z) (B^\# \mathcal{H}_{\omega,1}^+ \cap \mathcal{H}_{\omega,1}^-) = L_+ (\omega^{-1}) . \]

\[ 2.3. \quad \text{Projection operators.} \]

Lemma 2.4.

\[(2.7) \quad K_+ (\omega) + M_+ (\omega) = \mathcal{H}_{\omega}^+, \quad K_- (\omega) + M_- (\omega) = \mathcal{H}_{\omega}^-, \]

\[(2.8) \quad K_+ (\omega^{-1}) + M_+ (\omega^{-1}) = \mathcal{H}_{\omega,1}^+, \quad K_- (\omega^{-1}) + M_- (\omega^{-1}) = \mathcal{H}_{\omega,1}^- . \]
Proof. We restrict ourselves to the first relation only. Observe that, if \( g \in \mathcal{F}_w \) and \( g \perp (K_+ (\omega) \cup M_+ (\omega)) \) then \( g = 0 \). Indeed \( g \perp K_+ (\omega) \Rightarrow g|_\lambda = 0 \Rightarrow g = B^# g_1 \), for some \( g_1 \in \mathcal{F}_w \). Further \( g \perp M_+ (\omega) \Rightarrow \langle B^# g_1, B f \rangle = 0 \) for all \( f \in \mathcal{F}_w^+ \), therefore \( g_1 = 0 \), hence \( K_+ (\omega) + M_+ (\omega) \) is dense in \( \mathcal{F}_w^+ \).

In order to complete the proof it suffices to mention that the operator

\[
P_+ = I - B P_+ B^# : \mathcal{F}_w^+ \to K_+ (\omega)
\]

is the projector onto \( K_+ (\omega) \) annihilating \( M_+ (\omega) \). Boundedness of \( P_+ \) follows from the inclusion \(|\omega|^2 \in A_2\). □

Remark 2.1. Representations \((2.7)-(2.8)\) are analogs of the classical representation \( H^2 = K_\Theta \Theta H^2 \), where \( \Theta \) is an inner function and \( K_\Theta = H^2 \Theta H^2 \) is the standard model space.

3. Projection summation method

3.1. Summation in \( K_+ (\omega) \). Given any \( f \in K_+ (\omega) \) we consider the interpolating series

\[
f \sim \sum_{\lambda \in \Lambda} f(\lambda) \frac{B(z)}{B'(\lambda)(z - \lambda)}
\]

and look for summation method for this series.

We use the method constructed in \([10]\) for the classical model spaces. For each \( n > 0 \) we define \( \Lambda_n = \{ \lambda \in \Lambda, |\lambda| < n \} \) and let \( B_n(z), \beta_n(z) \) be defined in \((2.1)\).

Consider the subspaces

\[
K_{+,n}(\omega) = \text{span}_{\omega} \left\{ \frac{1}{z - \lambda} \right\}_{\lambda \in \Lambda_n} = \text{span}_{\omega} \left\{ \frac{B_n(z)}{z - \lambda} \right\}_{\lambda \in \Lambda_n} \subset \mathcal{F}_w^+,
\]

\[
M_{+,n}(\omega) = B_n \mathcal{F}_w^+, \quad K_{-,n}(\omega^{-1}) = \text{span}_{\omega^{-1}} \left\{ \frac{1}{z - \lambda} \right\}_{\lambda \in \Lambda_n}
\]

and the corresponding projection operators \( P_{+,n} = I - B_n P_+ B^# : \mathcal{F}_w^+ \to K_{+,n}(\omega) \).

We have

\[
(3.1) \quad K_{+,n}(\omega) \nearrow K_+ (\omega), \quad P_{+,n} |_{K_{+,n}(\omega)} \xrightarrow{\sim} I |_{K_{+,n}(\omega)} \quad \text{as} \quad n \to \infty.
\]

The biorthogonality relations

\[
\left\langle \frac{1}{B_n'(\lambda)} \frac{B_n(z)}{z - \mu}, \frac{1}{z - \lambda} \right\rangle_{\in K_{+,n}(\omega)} = \delta_{\lambda,\mu}, \quad \lambda, \mu \in \Lambda_n
\]

allow one to express \( P_{+,n} \) explicitly:

\[
P_{+,n} f = \sum_{\lambda \in \Lambda_n} \left\langle f, \frac{1}{z - \lambda} \right\rangle \frac{B_n(z)}{B_n'(\lambda)(z - \lambda)} = \sum_{\lambda \in \Lambda_n} f(\lambda) \frac{B_n(z)}{B_n'(\lambda)(z - \lambda)}
\]
Relation (3.1) now takes the form
\[ P_{+n}f = \frac{1}{\beta_n(z)} \sum_{\lambda \in \Lambda_n} f(\lambda) \beta_n(\lambda) \frac{B(z)}{B'(\lambda)(z - \lambda)} \rightarrow f, \ n \to \infty, \ f \in K_+(\omega). \]

Since \(|\beta_n(x)| = 1\) and \(\beta_n(x) \to 1\) as \(n \to \infty\) we obtain
\[ S_{+n}f = \sum_{\lambda \in \Lambda} f(\lambda) \beta_n(\lambda) \frac{B(z)}{B'(\lambda)(z - \lambda)} \rightarrow f, \ n \to \infty, \ f \in K_+(\omega). \]

This is the desired summation method in \(K_+(\omega)\).

3.2. Summation in \(PW_\pi\). The generating function \(G\) admits the natural factorization
\[ G(z) = \begin{cases} 
\omega(z) B(z) e^{-i\pi z}, & \text{if } z \in \mathbb{C}_+ \\
\omega^#(z) e^{i\pi z}, & \text{if } z \in \mathbb{C}_-,
\end{cases} \]

here \(\omega\) is an outer function in \(\mathbb{C}_+\) and \(B\) is the Blaschke product with \(\Lambda\) as the zero set.

Given \(F \in PW_\pi\) consider the function
\[ \Phi(z) = \frac{F(z) e^{i\pi z}}{\omega(z)} \in \mathcal{H}_\omega^+. \]

Then convergence/summability of the interpolation series (1.7) in \(L^2(\mathbb{R})\) is equivalent to convergence/summability of the series
\[ \Phi(z) \sim B(z) \sum_{\lambda \in \Lambda} \Phi(\lambda) \frac{B(z)}{B'(\lambda)(z - \lambda)} \]
in \(|\cdot|_\omega\) norm. In order to apply the previous results, and thus prove Theorem 2.1 it suffices to prove that \(\Phi \in K_+(\omega)\).

Indeed, it follows from (3.2) that
\[ \omega(x) = \omega^#(x) e^{2i\pi x} B^#(x), \ x \in \mathbb{R}, \]
therefore
\[ \Phi(x) = \frac{F(x) e^{-i\pi x}}{\omega^#(x) B(x)}, \ x \in \mathbb{R}. \]

It is now straightforward that \(\Phi \in M_-(\omega^{-1})^\perp = K_+(\omega)\).

3.3. The general case. We briefly describe changes to be introduced in the case when the sequence \(\Lambda\) does not belong to the upper(lower) halfplane. From condition \(|G(x)|^2 \in (A_2)\) we conclude that \(\Lambda \cap \mathbb{R} = \emptyset\). Denote \(\Lambda^\pm = \Lambda \cap \mathbb{C}_\pm\);
\[ B^\pm(z) = \prod_{\lambda \in \Lambda^\pm} \frac{\lambda z - \lambda}{\lambda z - \lambda}, \ B^\pm_n(z) = \prod_{\lambda \in \Lambda^\pm, |\lambda| < n} \frac{\lambda z - \lambda}{\lambda z - \lambda}, \ \beta^\pm_n(z) = B^\pm(z)/B^\pm_n(z). \]
Theorem 3.1. Let the generating function $G$ be such that $|G(x)|^2 \in (A_2)$. Then

$$\sum_{\lambda \in \Lambda^+} \beta^+_n(\lambda)(f, \varphi_\lambda)e^{i\lambda x} + \sum_{\lambda \in \Lambda^-} \beta^-_n(\lambda)(f, \varphi_\lambda)e^{i\lambda x} \xrightarrow{L^2(-\pi, \pi)} f,$$

as $n \to \infty$, for any $f \in L^2(-\pi, \pi)$,

$$\sum_{\lambda \in \Lambda^+} \beta^+_n(\lambda)F(\lambda) \frac{G(z)}{G'(\lambda)(z - \lambda)} + \sum_{\lambda \in \Lambda^-} \beta^-_n(\lambda)F(\lambda) \frac{G(z)}{G'(\lambda)(z - \lambda)} \xrightarrow{P W_\pi} F,$$

as $n \to \infty$, for any $F \in P W_\pi$.

The proof goes in a natural way, one has to replace representation (3.2) by

$$G(z) = \begin{cases} 
\omega(z)B^+(z)e^{-iz}, & \text{if } z \in \mathbb{C}_+
\omega^\#(z)B^-(z)e^{iz}, & \text{if } z \in \mathbb{C}_-
\end{cases},$$

and literally repeat the argument from the previous sections considering the two summands in (3.6) separately.

4. Universal summation method

As in the previous section Fourier transform $\mathcal{F}$ reset our problem into the Paley-Wiener space $P W_\pi$ and we have to construct a summation method for the series $\sum_\lambda (F, G_\lambda)k_\lambda$ for each $F \in P W_\pi$.

Given any $w := \{w(\lambda, n)\}$ satisfying Definition 1.2 we clearly have

$$\sum_{\lambda \in \Lambda} w(\lambda, n)(F, G_\lambda)k_\lambda \to F$$

for the dense set of finite linear combinations of $\{k_\lambda\}_{\lambda \in \Lambda}$. Therefore in order to prove that $w$ generates a linear summation method it suffices to prove that the operators

$$T_n : F \to \sum_{\lambda \in \Lambda} w(\lambda, n)(F, G_\lambda)k_\lambda$$

are uniformly bounded.

We will prove uniform boundedness of the adjoint operators

$$T^*_n : F \to \sum_{\lambda \in \Lambda} w(\lambda, n)(F, k_\lambda)G_\lambda = \sum_{\lambda \in \Lambda} w(\lambda, n)F(\lambda) \frac{G(z)}{G'(\lambda)(z - \lambda)}.$$ 

This yields the summation method for the Lagrange series (1.7).
4.1. **Summation matrix.** Let

\[ w_n(z) = \begin{cases} [\Gamma(1 - i\alpha_n z)]^{-1}, & z \in \mathbb{C}_+ \\ [\Gamma(1 + i\alpha_n z)]^{-1}, & z \in \mathbb{C}_- \end{cases}, \]

where \( \Gamma \) is the Gamma function. We will choose an increasing sequence of finite subsets \( \Lambda_n, \cup_n \Lambda_n = \Lambda \), such that for an appropriate \( \{\alpha_n\}, \alpha_n > 0 \) (their choice will be specified in Subsection 4.5), a linear summation method for the series (1.1), (1.7) is generated by the matrix

\[ w = \begin{cases} w_n(\lambda), & \lambda \in \Lambda_n \\ 0, & \lambda \not\in \Lambda_n \end{cases}. \]

4.2. **Operators** \( T_n^* \). We start with assuming that \( \Lambda \subset \mathbb{C}_+ \). Later in Subsection 4.6 we indicate the changes needed in the general case.

It suffices to estimate \( \| T_n F \|_{L^2(\mathbb{R})} \). Chose a sequence of contours \( C_n \)

\[ C_n = [-l_n, l_n] \cup R_n \cup L_n, \quad R_n = \{ l_n - t + ic_n t : t \in [0, l_n] \}, \]

\[ L_n = \{ -l_n + t + ic_n t : t \in [0, l_n] \}, \]

(4.1)

where the real numbers \( l_n \to \infty \) and \( c_n \in [1, 10] \) will be specified later. So, \( C_n \) is the boundary of the triangle with vertices \( -l_n, l_n, il_n c_n \). Take \( \Lambda_n = \Lambda \cap \operatorname{int} C_n \). We have

\[ T_n^* F(x) = G(x) \sum_{\chi \in \Lambda_n} \frac{w_n(\lambda) F(x)}{G'(\lambda)(x - \lambda)} = \frac{G(x)}{2\pi i} \int_{C_n} \frac{F(\zeta) w_n(\zeta)}{G(\zeta)(\zeta - x)} d\zeta + \frac{1}{2} F(x) w_n(x) \chi_{[-l_n, l_n]}(x) \]

\[ = \frac{G(x)}{2\pi i} \left[ \int_{[-l_n, l_n]} + \int_{R_n} + \int_{L_n} \right] + \frac{1}{2} F(x) w_n(x) \chi_{[-l_n, l_n]}(x) \]

\[ = I_{1,n}(x) + I_{2,n}(x) + I_{3,n}(x) + \frac{1}{2} F(x) w_n(x) \chi_{[-l_n, l_n]}(x). \]

The last summand is obviously bounded in \( L^2 \)-norm by \( \| F \|^2 \). Each of \( I_{j,n}, j = 1, 2, 3 \) will be estimated separately.

4.3. **Estimate of** \( I_{1,n} \) **is straightforward.** It follows from the fact that \( w_n F G^{-1} \) belongs to the weighted space \( L^2(\mathbb{R}, |G|^2) := \{ H : \int_{\mathbb{R}} |H(x)|^2 |G(x)|^2 < \infty \} \). Since \( |G|^2 \in (A_2) \) the Hilbert transform is bounded in \( L^2(\mathbb{R}, |G|^2) \).

4.4. **Inner-outer factorization and choice of contours** \( C_n \). Let \( \omega \) be an outer function in \( \mathbb{C}_+ \) such that \( |\omega(x)| = |G(x)|, \ x \in \mathbb{R} \) and let \( B \) be the Blaschke product with \( \Lambda \) as the zero set. In order to estimate \( I_{2,n} \) and \( I_{3,n} \) we use the inner-outer factorization

\[ e^{i\pi z} G(z) = c \omega(z) B(z), \quad |c| = 1. \]

Using the estimate for the upper density of \( \Lambda \) and formula \( B'(t) = 2 \sum_{\lambda} \frac{\Delta \lambda}{|t - \lambda|^2} \) we get

\[ |\arg B(x) - \arg B(0)| \leq 2\pi \cdot \# [\Lambda \cap \{|z| < 2|x|\}] + o(|x|) \lesssim 1 + |x|. \]
Lemma 4.1. Let $B$ be a Blaschke product such that $|\arg B(x)| \lesssim 1 + |x|$. Then there exists a sequence of contours $C_n$ as above such that $1 \leq c_n \leq 10$ and
\begin{equation}
- \log |B(\lambda)| \lesssim \Im \lambda + 1, \quad \lambda \in \bigcup_n C_n.
\end{equation}

We postpone proof of this lemma until Subsection 4.8 and complete the construction of universal summation method.

Let $\{C_n\}$ be the contours defined by (4.1) chosen in Lemma 4.1. With this choice we complete the estimation of $I_{2,n}$, the summand $I_{3,n}$ can be estimated similarly.

4.5. Estimate of $I_{2,n}$. We fix $\alpha_n = [\log \log l_n]^{-1}$. We have
\begin{equation}
\|I_{2,n}\|_{L^2(\mathbb{R})} = \sup_{g \in L^2, \|g\| \leq 1} \left| \int_{\mathbb{R}} g(x) I_{2,n}(x) dx \right|.
\end{equation}
Let
\begin{equation}
A(\zeta) = \int_{\mathbb{R}} \frac{g(x)G(x)}{x - \zeta} dx.
\end{equation}
So,
\begin{equation}
J = \frac{1}{2\pi i} \int_{R_n} \frac{F(\zeta)A(\zeta)}{G(\zeta)} w_n(\zeta) d\zeta = \int_{R_n} e^{-i\pi \zeta} F(\zeta) \cdot A(\zeta) \cdot \frac{\omega(\zeta)}{B(\zeta)} d\zeta.
\end{equation}
Using inequalities $\log |\Gamma(z)| \geq \frac{1}{2} |z| \log |z|$, $\Re z \geq 10$ and $\alpha_n \simeq [\log \log |\zeta|]^{-1}$, $\zeta \in R_n$ and Lemma 4.1 we get, for some $c > 0$,
\begin{equation}
\frac{|w_n(\zeta)|}{|B(\zeta)|} \lesssim e^{-1/2|\alpha_n| \log |\alpha_n|} \lesssim 1, \quad \zeta \in R_n.
\end{equation}
We conclude that
\begin{equation}
|J|^2 \lesssim \int_{R_n} |e^{i\pi \zeta} F(\zeta)|^2 |d\zeta| \cdot \int_{R_n} \frac{|A(\zeta)|^2}{|\omega(\zeta)|^2} |d\zeta|.
\end{equation}
From the boundedness of Hilbert transform with weight $|\omega|^2$ we get $\frac{A(\zeta)}{\omega(\zeta)} \in H^2(\mathbb{C}_+)$ and
\begin{equation}
\left\| \frac{A(\zeta)}{\omega(\zeta)} \right\|_{H^2(\mathbb{C}_+)} \lesssim \|g\| \leq 1. \quad \text{On the other hand,} \quad |d\zeta|_{R_n} \text{ is a Carleson measure for } H^2(\mathbb{C}_+).
\end{equation}
Finally,
\begin{equation}
|J| \lesssim \|e^{i\pi \zeta} F(\zeta)\|_{H^2(\mathbb{C}_+)} \cdot \left\| \frac{A(\zeta)}{\omega(\zeta)} \right\|_{H^2(\mathbb{C}_+)} \lesssim \|F\|_{PW_*}.
\end{equation}

4.6. General case. If $\Lambda$ has points in both halfplanes $\mathbb{C}_\pm$, then we have to choose analogous contours $C_n^\pm$ for the points in the lower halfplane $\mathbb{C}_-$ and construct the sequence of operators $T_{n}^-$. Put $T_n = T_n^+ + T_n^-$. By the same arguments the sequence of operators $T_{n}^-$ is uniformly bounded and so $T_n$ is uniformly bounded and $T_n F \to F$ for dense set of $F$. \hfill $\Box$

4.7. The proof of Theorem 1.3 is in the same spirit as of Theorem 1.1.
4.8. Proof of Lemma 4.1. Let \( \{z_l\} \) be a zero set of Blaschke product \( B \), \( z_l = x_l + iy_l \). We choose the points \( \{l_n\}_{n=1}^{\infty} \subset \mathbb{R} \) so that \( \text{sup}_n(\arg B)'(\pm l_n) \lesssim 1 \). Hence,
\[
\sup_n(\arg B)'(\pm l_n) = \sup_n \sum_l \frac{2y_l}{(\pm l_n - x_l)^2 + y_l^2} < \infty.
\]
Hence, the sequences \( \Lambda \pm l_n \) uniformly satisfy Blaschke condition. Now we use Hayman theorem (see, [6, Lecture 15, Theorem 1]) which, in particular, state that \( |\log |B(z)|| = o(|z|) \) for all \( z \in \mathbb{C}_+ \) except some set of finite(small) view. Namely, for any \( \varepsilon > 0 \) there exists a family of disks \( D_m(w_m, r_m) := \{|z - w_m| < r_m\} \) such that
\[
\sum_m \frac{r_m}{|w_m|} < \varepsilon, \quad |\log |B(z)|| \leq c|z|, \quad z \notin \bigcup_mD_m(w_m, r_m),
\]
where \( c \) depends only on the sum of Blaschke series \( \sum_n \frac{x_n}{x_n^2 + y_n^2} \). We apply Hayman theorem to \( B(\pm l_n) \): there exists \( c_n \in [1, 10] \) so that,
\[
|B(\pm l_n)| \gtrsim e^{-c|\pm l_n|}, \quad z \in \{t \neq ic_nt, t > 0\}.
\]
If \( z \in \{t \neq ic_nt, t > 0\} \), then \( |\pm l_n| \approx 2z \) and we get the required estimate for \( B \) and contours \( C_n \) as in [4.1]. \( \square \)

4.9. Universal method. So, in this section we already proved the following result

**Theorem 4.1.** Let \( C_n^\pm(= C_n^\pm(-l_n, l_n, \pm ic_nl_n)) \) be the set of triangle contours from Lemma 4.1 and Blaschke products with zero sets \( \Lambda \cap \mathbb{C}^\pm \) respectively. Then the matrix
\[
w(n, \lambda) = \begin{cases} 
\Gamma(1 \mp i\lambda[\log \log |l_n|^{-1}])^{-1}, & \lambda \in \mathbb{C}^\pm, \ \lambda \in \text{int } C_n^\pm \\
0, & \text{otherwise}
\end{cases}
\]
generates a linear summation method for the series \( (1.1), (1.7) \).

It should be noted that usually the sequence of contours \( C_n \) is not very sparse, e.g. if \( \Lambda \subset \mathbb{C}^+ \) and corresponding Blaschke product \( B \) satisfies \( (\arg B)'(x) \lesssim 1 \), then the points \( l_n \in C_n \) can be chosen as an arbitrary sequence tending to infinity.

4.10. Hereditary completeness of \( \mathcal{E}(\Lambda) \). Here we give a direct proof of the hereditary completeness of \( \mathcal{E}(\Lambda) \) under the assumptions of Theorem 1.1. The idea of this proof comes from [1].

Assume that there exists \( f \) as in (1.6). Then there exists a non-zero \( h \) such that
\[
(h, e_{\lambda})(\varphi_{\lambda}, h) = 0, \ \text{for any } \lambda \in \Lambda.
\]
We rewrite this equation in Paley-Wiener space, \( H = \mathcal{F}h \),
\[
(H, k_{\lambda})(\varphi_{\lambda}, H) = 0, \ \text{for any } \lambda \in \Lambda.
\]
We will use the Shannon-Kotelnikov-Whittaker formula for \( H \),
\[
H(z) = \sin(\pi z) \sum_{n \in \mathbb{Z}} \frac{H(n)(-1)^n}{\pi(z - n)} = \sum_{n \in \mathbb{Z}} H(n)k_n(z).
\]
So, \((G_{\lambda}, H) = \frac{1}{\sigma(\lambda)} \sum_{n \in \mathbb{Z}} \frac{G(n)H(n)}{\lambda - n}\). Let us consider the meromorphic function
\[
H(z) \sum_n \frac{H(n)G(n)}{z - n}.
\]
this function vanishes at the points \(\Lambda\). So,
\[
(4.3) \quad H(x) \cdot \sin(\pi x) \sum_n \frac{H(n)G(n)}{x - n} = G(x)S(x)
\]
for some entire function \(S\). It is known (see, [1, Lemma 2.2]) that
\[
S(z) = \sin(\pi z) \sum |H(n)|^2 \frac{z}{z - n} + c_0 \sin(\pi z).
\]
We have
\[
H(x) \int_{\mathbb{R}} \frac{H(t)G(t)d\mu_0(t)}{x - t} = G(x) \left( c_0 + \int_{\mathbb{R}} \frac{|H(t)|^2d\mu_0(t)}{x - t} \right),
\]
where \(\mu_0 = \sum_{n \in \mathbb{Z}} \delta_n\). On the other hand we can write the same identity for any measure \(\mu_\alpha = \sum_{n \in \mathbb{Z}} \delta_{n+\alpha}, \alpha \in [0,1]\), and, hence, the analogous equation is true for the Lebesgue measure \(dx = \int_0^1 \mu_\alpha d\alpha\):
\[
H(x) \int_{\mathbb{R}} \frac{H(t)G(t)dt}{x - t} = G(x) \left( c + \int_{\mathbb{R}} \frac{|H(t)|^2dt}{x - t} \right).
\]
Using the boundedness of Hilbert transform with the weight \(|G(x)|^2\) we get that left hand side belongs to \(L^1(\mathbb{R}, |G|)\). On the other hand it is well known that \(m\{x : \int_{\mathbb{R}} \frac{|H(t)|^2dt}{x - t} > s\} \gtrsim \frac{1}{s}, \ s \to 0\), where \(m\) is Lebesgue measure. So, the function \(c + \int_{\mathbb{R}} \frac{|H(t)|^2dt}{x - t}\) is not summable even if \(c = 0\). We arrive to a contradiction.

References
[1] A. Baranov, Y. Belov, A. Borichev, Hereditary completeness for systems of exponentials and reproducing kernels, Adv. Math. 235 (2013), 525–554.
[2] A. Baranov, Y. Belov, A. Borichev, Strong M-basis property for systems of reproducing kernels in de Branges spaces, http://arxiv.org/abs/1309.6915
[3] A. Borichev, private communications.
[4] G. Gubreev, A. Tarasenko, Spectral decomposition of model operators in de Branges spaces, (Russian) Mat. Sb. 201 (2010), no. 11, 41–76; translation in Sb. Math. 201 (2010), no. 11-12, 1599–1634.
[5] S.V. Hruscev, N.K. Nikolskii, B.S Pavlov, Unconditional bases of exponentials and of reproducing kernels, Complex analysis and spectral theory (Leningrad, 1979/1980), pp. 214–335, Lecture Notes in Math., 864, Springer, Berlin-New York, 1981.
[6] B.Ya. Levin, Lectures on Entire Functions, Translations of Mathematical Monographs, vol. 150, AMS, Providence, RI, 1996.
[7] Yu. Lyubarskii, K. Seip, Complete interpolating sequences for Paley-Wiener spaces and Muckenhoupt’s \((A_p)\) condition, Rev. Mat. Iberoamericana, 13 (1997), no. 2, 361–376.
[8] A.M. Minkin, Reflection of exponents and unconditional bases of exponentials, St. Petersburg Math. J. 3 (1992), 1043–1068.
[9] B. S. Pavlov, *The basis property of a system of exponentials and the condition of Muckenhoupt.* (Russian) Dokl. Akad. Nauk SSSR 247 (1979), 37–40. English transl. in Soviet Math. Dokl. 20 (1979).

[10] V. Vasjunin, *Bases from eigensubspaces, and nonclassical interpolation problems,* Funkcional. Anal. i Prilozhen. (Russian), 9, (1975), no. 4, 65–66.

[11] R. Young, *On complete biorthogonal system,* Proc. Amer. Math. Soc. 83 (1981), no. 3, 537–540.

Yuri Belov,
Chebyshev Laboratory, St.Petersburg State University, St.Petersburg, Russia,
juri.belov@mail.ru

Yuri Lyubarskii,
Dept. Mathematical Sciences, Norwegian University of Science and Technology, NO-7491 Trondheim, Norway,
yura@math.ntnu.no