Covariant formulation of non-Abelian gauge theories without anticommuting variables.

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Abstract

A manifestly Lorentz invariant effective action for Yang-Mills theory depending only on commuting fields is constructed. This action possesses a bosonic symmetry, which plays a role analogous to the BRST symmetry in the standard formalism.

1 Introduction

A peculiar feature of non-Abelian gauge theories is the appearance of anticommuting scalar fields (Faddeev-Popov ghosts) in a Lorentz covariant local effective action [1]. It is believed to be an unavoidable price to be paid for the construction of a manifestly Lorentz invariant local effective action. Locality of the effective action is the essential ingredient of the proof of unitarity in the Lorentz covariant quantization scheme for nonabelian gauge fields (BRST quantization) [2]. However in this approach there is one problem which, to my knowledge, has never been seriously discussed. It is known that Fermi-Dirac quantization of scalar fields contradicts local commutativity. Therefore it is by no means evident that a theory including anticommuting scalar fields respects a causality condition. Of course it is not a serious problem for the Yang-Mills theory, as in this case one can prove the equivalence of the formulation based on the Lorentz invariant action and the Coulomb gauge formulation, in which causality and unitarity is manifest. However, if one applies BRST quantization to more complicated models, where a true Hamiltonian formulation is not at hand, this problem deserves special investigation. For this reason it would be desirable to have a Lorentz covariant formulation which does not involve anticommuting scalar fields.

In this paper we shall show that using a bosonization procedure proposed by us earlier [3], one can construct an alternative effective action which includes only commuting fields. This action describes a five dimensional constrained system, which in the physical sector is equivalent to the usual Yang-Mills theory. Our bosonic effective action possesses a symmetry which plays a role similar to the BRST symmetry [5] in the usual approach. In particular this symmetry generates the relations between the Green functions which are equivalent to Generalized Ward Identities [6], [7].

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2 Local bosonic effective action.

Our Lorentz covariant formulation is based on the following effective action

\[
S = \int_{-L}^{L} dt \int d^4x \left\{ \frac{1}{2L} \left[ -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} - \frac{1}{2\alpha}(\partial_{\mu}A^a_{\mu})^2 \right] - \frac{i}{2} \left( \frac{\partial c^a}{\partial t} + M^+ M c^a + c^a \right) \right\}
\]

(1)

Here \( F^a_{\mu\nu} \) is the usual Yang-Mills curvature tensor and

\[
M = \partial_{\mu}D^a_{\mu}
\]

(2)

where \( D^a_{\mu} \) is the usual covariant derivative

\[
D^a_{\mu} = (\delta^a_{\mu} - g t^{abc} A^c_{\mu})
\]

(3)

The gauge fields \( A_{\mu}(x) \) and the Lagrange multiplier \( \chi \) depend on four dimensional coordinates \( x_{\mu} \), whereas the scalar fields \( \bar{c}(x,t) \), \( c(x,t) \) depend also on extra variable \( t \), which acquires values in the interval \(-L \leq t \leq L\). The fields \( A_{\mu} \) and \( \chi \) are Hermitean, whereas \( \bar{c} \) and \( c \) are conjugate to each other. All the fields are commuting.

We claim that in the limit \( L \to \infty \) the Yang-Mills field Green functions generated by the effective action \( \{1\} \) coincide with the corresponding Green functions in the standard Faddeev-Popov formalism. More precisely

\[
Z(J_{\mu}) = \lim_{L \to \infty} \int \exp\{i[S + \int d^4x J^a_{\mu}(x) A^a_{\mu}(x)]\} d\bar{c} dc dA_{\mu}
\]

\[
= \exp\{i \int d^4x \left[ -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} - \frac{1}{2\alpha}(\partial_{\mu}A^a_{\mu})^2 + J^a_{\mu} A^a_{\mu} \right] \} \det M dA_{\mu}
\]

(4)

where \( \det M \) is the usual Faddeev-Popov determinant. The integration in the eq.\{4\} goes over the fields \( \bar{c}(x,t) \), \( c(x,t) \) satisfying the following boundary conditions

\[
c(x,L) = B(x), \quad \bar{c}(x,L) = B^+(x)
\]

(5)

where \( B(x) \) is an arbitrary, but fixed function, bounded in the limit \( L \to \infty \). It may depend on other fields, but the simplest choice is \( B = 0 \). The boundary conditions in fourdimensional coordinates \( x_{\mu} \) are the usual ones (see \{5\})

To prove the equation \{4\} we calculate the integral over \( \bar{c}, c \) explicitly. First of all we note that the operator \( M^+ M \) is Hermitean and we can introduce a complete system of it’s eigenfunctions

\[
M^+ M \phi_{\alpha} = m_{\alpha} \phi_{\alpha}
\]

(6)

In this basis the integral over \( \bar{c}, c \) may be written in the form

\[
Z_c = \lim_{L \to \infty} \int \exp\left\{ i \int_{-L}^{L} dt \sum_{\alpha} \left[ -\frac{i}{2} \frac{\partial c^\alpha}{\partial t} + m_{\alpha} c^\alpha \right] + \chi_{\alpha}(\bar{c}^\alpha + c^\alpha) \right\} dc^\alpha d\bar{c}^\alpha
\]

(7)

Making the change of variables

\[
\bar{c}^\alpha(t) \to \exp\{2im_{\alpha}t\} \bar{c}^\alpha(t)
\]

\[
c^\alpha(t) \to \exp\{-2im_{\alpha}t\} c^\alpha(t)
\]

(8)

we can rewrite it in the form

\[
Z_c = \lim_{L \to \infty} \int_{-L}^{L} dt \sum_{\alpha} \left[ -\frac{i}{2} \frac{\partial c^\alpha}{\partial t} - \bar{c}^\alpha \frac{\partial \bar{c}^\alpha}{\partial t} \right] + \chi_{\alpha}(\exp\{2im_{\alpha}t\} c^\alpha + \bar{c}^\alpha \exp\{2im_{\alpha}t\}) dc^\alpha d\bar{c}^\alpha
\]

(9)
The integral over \( \bar{c}, c \) is saturated by the stationary values of \( \bar{c}, c \), defined by the classical equations:

\[
\begin{align*}
\frac{\partial c^\alpha}{\partial t} + \exp\{2im_\alpha t\}\chi^\alpha &= 0 \\
-\frac{\partial \bar{c}^\alpha}{\partial t} + \exp\{-2im_\alpha t\}\chi^\alpha &= 0
\end{align*}
\]

\[c^\alpha(L) = B^\alpha \exp\{-2im_\alpha L\}; \quad \bar{c}^\alpha(L) = B^\alpha \exp\{2im_\alpha L\}\]  \(\text{(10)}\)

The solution of these equations is

\[c^\alpha = \frac{\chi^\alpha}{2m_\alpha} \exp\{2im_\alpha t\} + A_\alpha \]

\[A_\alpha = B_\alpha \exp\{-2im_\alpha L\} - \frac{\chi^\alpha}{2m_\alpha} \exp\{2im_\alpha L\}\]

\[\bar{c}^\alpha = (c^\alpha)^+ \]  \(\text{(11)}\)

Substituting these solutions into the integral \(\text{(9)}\) and integrating over \( t \) one gets

\[Z_c = \lim_{L \to \infty} \int \exp\{i \sum_\alpha \left[ (\chi^\alpha)^2 \frac{L}{m_\alpha} + \frac{\chi^\alpha}{m_\alpha} \sin(2m_\alpha L)(A_\alpha + A_\alpha^+) \right]\} d\chi\]  \(\text{(12)}\)

Renormalizing the fields \( \chi^\alpha \), \( \chi^\alpha \rightarrow L^{-1/2} \chi^\alpha \) and integrating over \( \chi^\alpha \) one sees, that

\[Z_c = \prod_\alpha \sqrt{m_\alpha} = \det M\]  \(\text{(13)}\)

where we used the fact that \( \det M = \det M^+ \). It completes the proof of the eq. \((4)\).

### 3 Symmetry of the classical action

In the previous section we proved that in the limit \( L \to \infty \) the Green functions generated by the effective action \((3)\) coincide with the usual Yang-Mills Green functions in covariant \( \alpha \)-gauges. Therefore they satisfy Generalized Ward Identities \([6], [8] \). So one expects that our effective action possesses some symmetry which plays a role of BRST symmetry in the usual formalism. Below we shall show that such a symmetry indeed exists. It is interesting to note that although our construction reproduces the usual Yang-Mills theory only in the limit \( L \to \infty \), the symmetry exists even for a finite \( L \). In the limit \( L \to \infty \) the symmetry transformations simplify considerably.

The symmetry of the action \((3)\) can be found by trial and error method. Firstly one notes that transformation of the gauge field \( A_\mu \) must leave invariant the Yang-Mills Lagrangian. So it is natural to start with the special gauge transformation of the fields \( A_\mu, \bar{c}, c \):

\[
\begin{align*}
\delta A_\mu^\alpha &= (D_\mu \chi)^\alpha \epsilon \\
\delta_1 \bar{c}^\alpha &= t^{ab} \bar{c}^b \chi^d \epsilon \\
\delta_1 c^\alpha &= t^{ab} c^b \chi^d \epsilon
\end{align*}\]  \(\text{(14)}\)

Here \( \epsilon \) is a constant infinitesimal parameter. We note that although transformations \((14)\) are special gauge transformations, they of course represent a global symmetry.

The transformations \((14)\) change the gauge fixing term and quadratic form for the fields \( \bar{c}, c \):

\[
\delta_1 S = \int d^4x \partial_\mu \chi^\alpha(x) \int_{-L}^L dt f_\mu^\alpha(x, t) \epsilon = \int d^4x \partial_\mu (\chi^\alpha F_\mu^\alpha(x)) - \int d^4x \chi^\alpha \int_{-L}^L dt \partial_\mu f_\mu(x, t) \epsilon
\]  \(\text{(15)}\)
Here the first term at the r.h.s. is a total divergency and the second term may be compensated by the appropriate variation of fields $\bar{c}, c$:

$$\delta_2 \bar{c}^a = \delta_2 c^a = \frac{1}{2L} \int_{-L}^{L} dt \partial_\mu f_\mu(x, t)$$  \hspace{1cm} (16)

The explicit form of the function $f_\mu$ is

$$f_\mu^a = \alpha^{-1}(D_\mu (\partial_\nu A_\nu)^a + + i^{ab}[ (D_\mu \bar{c})^b (\partial_\nu D_\nu c)^a + h.c.]$$  \hspace{1cm} (17)

The transformation (16) compensates the variation $\delta_1 S$, but changes the quadratic form for the fields $\bar{c}, c$:

$$\delta_2 S = -\frac{i}{2} \int d^4x \int_{-L}^{L} dt \frac{\partial \bar{c}^a}{\partial t} + h.c. + \int d^4x \int_{-L}^{L} dt \bar{c}^a M^+ M \bar{c}^a + h.c.$$  \hspace{1cm} (18)

In this equation the first term is a total derivative over $t$ and the second term is proportional to

$$\int_{-L}^{L} dt (\bar{c}^a (x, t) + c^a (x, t))$$  \hspace{1cm} (19)

and vanishes at the constraint surface. To provide a manifest invariance of the action one can eliminate the variation (18) by the following change of Lagrange multiplier $\chi$:

$$\delta \chi^a = -\frac{i}{2} MM^+ \delta_2 \bar{c}^a + h.c.$$  \hspace{1cm} (20)

Combining the equations (14, 16, 20) we get the complete symmetry transformation of the bosonic effective action (1). These transformations change the effective Lagrangian by a total derivative and via Neuther theorem lead to the existence of a five-dimensional conserved current

$$\partial_t j_t + \partial_\mu j_\mu = 0$$  \hspace{1cm} (21)

However for physical applications we are interested in the existence of a conserved charge

$$Q = \int_{-L}^{L} dt \int d^3x j_0$$  \hspace{1cm} (22)

Integrating the conservation law (21) over $dt$ and $d^3x$ we see that the integral of $\partial_t j_t$ vanishes by the usual reasonings due to fast decreasing of fields at spatial infinity. The integral of $\partial_\mu j_\mu$ gives

$$\int_{-L}^{L} dt \partial_t j_t = j_t (L) - j_t (-L)$$  \hspace{1cm} (23)

where

$$j_t = \frac{\delta L}{\delta \bar{c}_t} \delta \bar{c} - \frac{i}{2} \bar{c}^a \delta_2 c^a + h.c.$$  \hspace{1cm} (24)

The conservation law (21) is fulfilled due to the classical equations of motion. These equations look as follows

$$i \frac{\partial \bar{c}}{\partial t} + M^+ M \bar{c} + \chi = 0$$

$$-i \frac{\partial c}{\partial t} + M^+ M c + \chi = 0$$

$$c(L) = B, \hspace{0.5cm} \bar{c}(L) = B^+$$  \hspace{1cm} (25)
It is easy to see that solutions of these equations satisfy the relation

$$\tilde{c}(L) = c(-L)$$  \hspace{1cm} (26)

Due to this property the r.h.s. of eq. (23) is zero and

$$\partial_b \int_{-L}^{L} dt \int d^3 x j_0(x, t) = 0$$  \hspace{1cm} (27)

For a finite \(L\) the symmetry transformations are rather complicated. In particular they are nonlocal in extra variable \(t\). However for physical applications we are interested in the limiting case \(L \to \infty\). One can show that in this limit all nonlocal terms give vanishing contributions to relevant quantities and one can use the asymptotic symmetry transformations

$$\delta A^a_{\mu} = (D_{\mu} \chi)^a \epsilon$$

$$\delta \tilde{c}^a = [\mathbf{t}^{abcd} \tilde{c}^b \chi^d + \frac{1}{2\alpha L} (D_{\mu} \partial_{\nu}(\partial_{\mu} A_{\mu}))] \epsilon$$

$$\delta c^a = [\mathbf{t}^{abcd} c^b \chi^d + \frac{1}{2\alpha L} (D_{\mu} \partial_{\nu}(\partial_{\mu} A_{\mu}))] \epsilon$$

$$\delta \chi^a = \frac{1}{2\alpha L} [\mathbf{M} \mathbf{M}^+ M (\partial_{\mu} A_{\mu})]^a \epsilon$$  \hspace{1cm} (28)

The asymptotic symmetry (28) do not leave the action invariant, but transforms it to another action \(S_L \to \tilde{S}_L\), which in the limit \(L \to \infty\) leads to the same physical conclusions. In the next section we shall show how using he asymptotic symmetry (28) one can derive Generalized Ward Identities.

### 4 Generalized Ward Identities.

To derive Generalized Ward Identities we consider the following generating functional

$$Z(J_\mu, \eta) = \lim_{L \to \infty} \int \exp\{iS + \int d^4 x [J^a_{\mu}(x) A^a_{\mu}(x) + \eta^a(x) \chi^a(x)]\} dA_\mu d\tilde{c} dc d\chi$$  \hspace{1cm} (29)

Let us make in this integral the change of variables, which is the asymptotic symmetry transformation (28). As the change of variables does not change the integral, we can put

$$\frac{dZ}{de} = 0$$  \hspace{1cm} (30)

which is, as we shall see, the equation, generating Generalized Ward Identities. Taking into account that the Jacobian of this transformation is trivial, we get in this way

$$\lim_{L \to \infty} \int \exp\{iS + \int d^4 x [J^a_{\mu}(x) A^a_{\mu}(x) + \eta^a(x) \chi^a(x)]\} \{ \int d^4 x [J^a_{\mu}(x) (D_{\mu} \chi(x))^a + (2\alpha L)^{-1} \eta^a(x)(\mathbf{M} \mathbf{M}^+ M \partial_{\mu} A_{\mu})^a(x)] + O\} d\tilde{c} dc d\chi dA_\mu = 0$$  \hspace{1cm} (31)

The term \(O = O(1/L)\) arises from noninvariance of the quadratic form of the fields \(\tilde{c}, c\) under asymptotic symmetry transformations (28). Using the eq. (10) one can see that the variation of the action is quadratic in \(\tilde{c}, c\) and proportional to \(L^{-1}\). These terms have to be compared with the quadratic form in the action (11) which is of order \(O(1)\). It is easy to verify that these new terms produce the corrections vanishing in the limit \(L \to \infty\).

Performing the integration over \(\tilde{c}, c\) in the eq. (33), we get

$$\lim_{L \to \infty} \int \exp\{i \int d^4 x [-1/4 F_{\mu \nu} F_{\mu \nu} - 1/(2\alpha)(\partial_{\mu} A_{\mu})^2 + J^a_{\mu} A^a_{\mu} + \eta^a \chi^a - \chi^a 2L(\mathbf{M} \mathbf{M}^+)^{-1} \chi^b] \} \int d^4 y [J^a_{\mu}(y) (D_{\mu} \chi(y))^a + (2\alpha L)^{-1} \eta^a(y)(\mathbf{M} \mathbf{M}^+ M \partial_{\mu} A_{\mu})^a(y)] dA_\mu d\chi = 0$$  \hspace{1cm} (32)
Differentiating this equation with respect to \( \eta \) and putting \( \eta = 0 \) we arrive to the identity

\[
\lim_{L \to \infty} \int \exp \{ i [S_{Y M} + \int d^4 x (J^a_\mu A^a_\mu - \chi^a 2L (MM^+)^{-1}_{ab} \chi^b)] \} \times \\
\left[ \int d^4 y J^a_\mu (y) (D_\mu \chi(y))^a \chi^b(z) + (2\alpha L)^{-1} [(MM^+M)(\partial_\mu A_\mu)]^b \chi^a(z) \right] dA_\mu d\chi = 0 \quad (33)
\]

Finally, performing integration over \( \chi \) and applying the operator \((MM^+M)^{-1}\) we get the standard Generalized Ward Identity in the form [6], [8].

\[
\int \exp \{ i(S_{YM} + \int d^4 x J^a_\mu A^a_\mu) \} \det M \times \\
\left[ \int d^4 y J^a_\mu (y) (D_\mu \chi(y))^a \chi^b(z) + 1/\alpha (\partial_\mu A_\mu)^a(y) \right] = 0 \quad (34)
\]

5 Discussion

We showed above that a purely bosonic Lorentz covariant formulation of Non-Abelian gauge theories is possible provided one introduces the dependence of auxiliary fields on extra variable. Due to locality of the effective action and absence of anticommuting integer spin fields, no problem with causality arises in this approach. Of course a complete formulation must also include the discussion of regularization and renormalization procedure. It can be done but will not be discussed here. An interesting feature of our construction is the existence of the symmetry, which replaces in our approach BRST symmetry. It is important to note that this symmetry exists even for finite values of \( L \) and therefore may be used for the construction of a Lorentz covariant quantization procedure alternative to BRST quantization. There are several interesting problems which remain open in this approach. At present it is not clear what is a geometrical meaning of the new symmetry. It would be interesting to try to generalize this construction to supersymmetric theories.

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