Multi-User Multi-Armed Bandits for Uncoordinated Spectrum Access

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Abstract

A stochastic multi-user multi-armed bandit framework is used to develop algorithms for uncoordinated spectrum access. In contrast to prior work, the number of users is assumed to be unknown to each user and can possibly exceed the number of channels. Also, in contrast to prior work, it is assumed that rewards can be non-zero even under collisions. The proposed algorithm consists of an estimation phase and an allocation phase. It is shown that if every user adopts the algorithm, the system wide regret is constant with time with high probability. The regret guarantees hold for any number of users and channels, i.e., even when the number of users is less than the number of channels. The algorithm is extended to the dynamic case where the number of users in the system evolves over time and our algorithm leads to sub-linear regret.

I. INTRODUCTION

The existing spectrum management paradigm treats frequency spectrum as a fixed commodity, which leads to spectrum under utilization. Cognitive radio has emerged as a useful strategy to increase spectrum utilization. The existing literature on cognitive radio has largely been focused on the primary/secondary user paradigm, where secondary users need to detect vacant spectrum when available and vacate the occupied spectrum when a primary user wants to transmit.

We focus on a different type of spectrum sharing system in which there is no distinction between users, and in which there is no coordination among the users. The collective performance across all users is more important than that of individual users. This is in contrast to the typical primary/secondary user paradigm in which secondary users bear the responsibility for ensuring priority-based spectrum sharing. We model this system using a stochastic multi-user multi-armed bandit framework [1]. Our goal is to design an efficient channel access mechanism by managing interference in the system by means of a decentralized policy across the users.

Multi-armed bandit problems have been studied in the context of cognitive radio using different formulations. A Markovian channel model for a two-user two-channel system was considered in [2] where the probability transitions were assumed to be known. Coordination between users was considered in the schemes of [3], [4], [5].

More relevant to our study is the stochastic multi-armed bandit model with no communication between the users. Multi-arm bandit formulations in multi-user cognitive radios without user coordination were considered in [6], [7], [8] and [9]. The algorithm in [6] is based on a time-division fair sharing (TDFS) of the best arms between users. Although the algorithm achieves order optimal regret, it requires pre-agreement among users and it is assumed that the number of users is fixed and known to all users. The algorithm in [7] does not require any coordination between users and achieves optimal regret, but assumes that the number of users is known. The algorithm in [8] combines an $\epsilon$-greedy learning rule with a collision avoidance mechanism, and [9] considers a musical chairs algorithm. Both of these approaches achieve sub-linear regret and do not require knowledge of the number of users.

All the existing approaches including [8] and [9] focus on the primary/secondary user paradigm in the scenario where the reward distribution for a user is unknown but fixed. In particular, when multiple users access the same channel they receive zero reward. Hence, all these approaches fail when the number of users is greater than the number of channels. In our model, all users are treated equally and the reward obtained by each user largely depends on the actions of other users. When multiple users access the same channel, we allow for a non-zero reward with the assumption that the reward for each user decreases as the number of users on the channel increases. Thus we include the case where there are more users than channels.

We assume that the reward on the channel depends on the number of users on the channel and is drawn i.i.d from a distribution depending on the number of users on the channel. The degradation of the reward as a function of number of users depends on the system, e.g., the distance between the users, the protocol used for transmission (e.g., hybrid ARQ) and is captured through a reward distribution that depends on the number of users on the channel.

We assume that the number of users is unknown and that there is no communication between the users. However, we make the mild assumption that the users have access to a shared clock for time synchronization (see also, [9], [10], [11]). We propose an algorithm and show that if each user employs the algorithm, the system wide regret is $O(1)$ in time, with high probability. The algorithm can be used for any number of users or channels. We extend our algorithm to the dynamic case, and show that

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with minor restrictions on the rate at which users enter and leave the system, we can achieve sub-linear regret. To the best of our knowledge, these are the first algorithms with such regret guarantees without user coordination when the number of users is greater than the number of channels.

The system model for the cognitive radio as a multi-user multi-armed bandit is described in Section III. The proposed algorithm is described in Section IV. The regret guarantees for the algorithm are discussed in Section V. An extension to the dynamic case is presented in Section VI. Some simulation results are discussed in VII. Finally some concluding remarks are given in Section VIII.

II. System Model

Let $K$ be the number of users in the system. We assume that the users have unlimited data for transmission for our initial investigation. In a more realistic setting, users may become active or inactive depending on their transmission needs. Each user can choose one among $M$ channels for transmission, where we first consider $K \geq M$. With $M$ channels and $K$ users attempting to access the spectrum, we assume that each user has prior knowledge of $M$, but not of $K$. The assumption of known $M$ is reasonable if the spectrum partition is enforced and fixed. On the other hand, it is not realistic to assume the knowledge of $K$ in an uncoordinated network.

In the absence of a primary user, we want to ensure that a user does not take over a channel for a long time and we therefore impose the following condition. For each user, transmission on a particular channel takes place for a maximum of $T_\epsilon$ seconds after which the user releases the channel some time before attempting to access the same channel.

We model the system as a multi-player multi-armed bandit system with $K$ users and $M$ arms (channels). In each time-slot $t$, let $A_{t,k}$ denote the set of channels available to user $k$. User $k$ chooses a channel $a_{t,k} \in A_{t,k}$ based on the previous reward history according to a certain policy and receives a reward. The reward observed is inversely proportional to the number of users transmitting on the same channel. For example, the reward could be the rate achieved by the user on the channel which reduces due to interference from other users accessing the channel. Specifically, the reward on each arm depends on the number of users who have chosen the arm. Let $f_t = [f_t(1), \ldots, f_t(M)]$ denote the number of users on each channel at time $t$, where $\sum_{m=1}^{M} f_t(m) = K$. Thus, the reward $r_{t,k}(a_{t,k}, f_t(a_{t,k}))$ received by user $k$ at time $t$ is a function of the channel chosen $a_{t,k}$ and the number of users on the channel $f_t(a_{t,k})$. Without loss of generality let $r_{t,k} \in [0, 1]$. The maximum achievable reward of one is achieved when a single user is transmitting on the channel. Let $\mu(m, f(m))$ denote the mean reward on channel $m$ when the number of users on the channel is $f(m)$. We assume that each user chooses a channel according to the same policy. We assume that $\mu(m, f(m))$ decays to zero for some $f(m) = \beta$, where $\beta$ depends on the system. This restricts the number of users in the system as $K/M \leq \beta$.

We define the expected regret in the system in the following way,

$$\mathbb{E}[R(T)] = T \sum_{i=1}^{M} f^*(i) \mu(i, f^*(i)) - \sum_{t,k} \mathbb{E}[r_{t,k}(a_{t,k}, f_t(a_{t,k}))]$$

where $f^*$ is the solution to $\max_{f} \sum_{i=1}^{M} f(i) \mu(i, f(i))$ and corresponds to the optimal number of users on each channel. We wish to find a policy with sub-linear regret as a function of $T$.

To estimate the means on each channel as a function of number of users, we need to impose some separability condition. We consider the following separability condition for the means of the distributions on each channel. For any $m \in [M]$ and $r, s \in [\beta]$ and some $\epsilon_2 \in (0, 1)$,

$$|\mu(m, r) - \mu(m, s)| \geq 4Mc \exp\left(\frac{K - 1}{M - 1}\right) \sqrt{\sigma^2 + \epsilon_2},$$  \hspace{1cm} (1)

where $\sigma^2$ is the variance of the distributions and $c$ is a constant.

III. Algorithm

We now propose an algorithm which when employed by all the users leads to constant system wide regret with time. The algorithm has two phases. The first is an estimation phase during which we estimate the number of users $K$ and $\mu(m, f(m))$, the average mean reward on each channel as a function of the number of users on the channel. The second is an allocation phase where the users arrange themselves in a way that minimizes system regret.

We estimate the number of users by keeping track of the number of collisions similar to [12], with the estimate given by $\hat{K} = \min\{1 + \text{round}(\ln(\frac{\ln(\frac{M}{1 - \beta})}{\ln(\frac{M}{1 - \beta})})/\beta M\}$.

We estimate $\mu(m, n)$ separately for each channel based on the reward $x(m)$ observed on the corresponding channel. This is done by clustering the samples using the $k$-means algorithm. We do this using algorithm Cluster (see Algorithm 2) inspired by [12]. We are interested in finding the centroids of the clusters rather than the correct classification of all the samples. Hence,
we use an \( \alpha \)-approximation algorithm with a run time \( T \) to find the estimates the centroids of the cluster and show that we get good estimates with high probability. We consider the approximation algorithm in [13] with a run time \( T \).

After obtaining the estimates for \( \hat{\mu}(m, f) \) and \( \hat{K} \), the optimal number of users on each channel \( f^* \) can be calculated. We use Alloc (see Algorithm 3) to ensure that each user settles or 'fixes' on a channel \( m \) with number of users less than \( f^*(m) \). That is, on finding a channel \( m \) with \( \mu(m, f(m)) \leq \mu(m, f^*(m)) \), the user keeps transmitting on it. The system incurs regret until all users have settled on some channels, and we call this duration the fixing time. Once all the users have settled on their channels the system does not incur regret. However in our system model, a user can transmit on a channel for at most \( T_x \) seconds, after which the user must switch. We assume that \( T_x \) is fixed for all the users but can vary with time. We use Permute (see Algorithm 4) to have efficient allocation such that the regret does not grow with time. In order to avoid system-wide regret every time users have to switch, we fix the ordering of each user after \( N_0 \) epochs; this can be done for any \( N_0 \geq 2 \). Our goal is to have each user transmit on all the channels. This is the coupon collector problem with each user having to collect \( M \) channels with the expected number of trials \( N_0 \sim O(M \log M) \). If any of the \( N_0 \) epochs does not result in successful fixing of all the users, we incur regret each time the epoch is repeated. Hence, if a user does not settle or fix during any epoch or observes more number of users on his channel consistently throughout the epoch, the user discards the epoch. When \( K \leq M \), in order to have efficient allocation so that the regret does not grow with time, after the first epoch, each user switches to the next channel among the set of \( K \) best channels.

We fix the epoch size to be \( T_x \). Let \( T_f \) be the expected time taken for a user to fix on a channel. We assume that \( T_f \leq T_x \) and \( 2 \max_m f^*(m) \leq \sum_m f^*(m) \) to ensure that after every transmitting for \( T_x \) seconds, each user has other available channels. Note that our algorithm works even when \( K \leq M \), in which case it reduces to a modified version of the algorithm in [9].

IV. ANALYSIS

We first consider the case where \( K > M \). We show that if all the users in the system use Algorithm 1, with high probability the expected regret is \( O(1) \).
Algorithm 3 Alloc
1: Input: $A,T$
2: for $t = 1$ to $T$ do
3: $a_t \sim U(A)$
4: if $\mu(a_t, f(a_t)) \geq \mu(a_t, f^*(a_t))$ then
5: Choose action $a_t = a_t$, $\forall t \geq t$
6: end if
7: end for

Algorithm 4 Permute
1: Input: $N_0, T_x, T_1$
2: $t = 1$, $A_i = [M]$
3: for $i = 1$ to $N_0$ do
4: $p(i) = \text{Alloc}(A_i)$
5: $p(i) = \text{Alloc}(A_i, T_x)$; Discard $p(i)$ if user unfixed or observes more than optimal no. of users consistently
6: $A_i \leftarrow [M]\{p(i)\}$
7: end for
8: $t = 0$, $i = 0$
9: while $t \leq T_1$ do
10: $i = t \mod N_0$
11: Choose $p(i)$ for next $\min\{T_1, i(T_x + 1) - 1\}$ rounds
12: end while

Theorem 1: For any fixed $\epsilon$ and $\delta \in (0, 1)$, with probability greater than $1 - \delta$, the expected regret of $K$ users using Algorithm 1 with $M$ arms for $T$ rounds, with parameter $T_0 = \frac{32\exp(K\beta)}{\epsilon^2}M\ln\frac{2MK\beta(\beta+1)}{\delta}$, $T_c \sim O(T_0)$ and any $N_0$, is given by
\[
\mathbb{E}[R(T)] \leq K(T_0 + T_c) + N_0K^2M\exp\left(\frac{K-1}{M-1}\right),
\]
i.e., $\mathbb{E}[R(T)] \sim O(1)$.

The first term $K(T_0 + T_c)$ corresponds to the regret accumulated during the estimation phase. We show that by running the estimation phase for $T_0 + T_c$ seconds, we have the correct estimates for $\mu(m, f(m))$ and $K$ with high probability. Here $T_c$ denotes the time used for running the $\alpha$-approximation algorithm for clustering. In the allocation phase, the regret in the system is accrued only during the $N_0$ number of fixing phases. In Theorem 4, we show that the regret in each fixing phase is $K^2\exp\left(\frac{K-1}{M-1}\right)$.

A. Estimation phase
We now show that, with high probability, we have the correct estimates for $\mu(m, f(m))$. More precisely, we find estimates $\hat{\mu}^k(m, n)$ such that $|\hat{\mu}^k(m, n) - \mu(m, n)| \leq \epsilon$ with high probability.

Theorem 2: For any fixed $\epsilon, \delta$, player $k$, channel $m$ and number of users on the channel $n \leq \beta$ the estimate $\hat{\mu}^k(m, n)$ obtained after running the algorithm for $T_0 = \frac{32\exp(K\beta)}{\epsilon^2}M\ln\frac{2MK\beta(\beta+1)}{\delta}$, and the $\alpha$ approximation algorithm for $T_c \sim O(T_0)$ rounds, we have with probability at least $1 - \delta$,
\[
|\hat{\mu}^k(m, n) - \mu(m, n)| \leq \epsilon.
\]

Proof: Let $A_1$ denote the event that there is at least one combination $k, m, n$ such that $|\hat{\mu}^k(m, n) - \mu(m, n)| \geq \epsilon$ and $A_2$ denote the event that each player has more than $\frac{16\exp(K\beta)}{\epsilon^2}\ln\frac{2MK\beta(\beta+1)}{\delta}$ observations from distribution with mean $\mu(m, n)$ for each $m, n$.
\[
\Pr(A_1) = \Pr(A_1|A_2)\Pr(A_2) + \Pr(A_1|A_2^c)\Pr(A_2^c)
\leq \Pr(A_1|A_2) + \Pr(A_2^c).
\]
It suffices to show that $\Pr(A_1|A_2) \leq \frac{4}{\delta}$ and $\Pr(A_2^c) \leq \frac{\delta}{2}$. From Theorem 7, we have $\Pr(A_2^c) \leq \frac{\delta}{2}$. We will show that $\Pr(A_1|A_2) \leq \frac{4}{\delta}$.
\[
\Pr(A_1|A_2) \leq \sum_{k,m,n} \Pr(|\hat{\mu}^k(m, n) - \mu(m, n)| \geq \epsilon|A_2),
\]
where the inequality follows from union bound. To show that \( \Pr(A_1|A_2) \leq \frac{\delta}{2} \), it suffices to show that \( \Pr(\mu^K(m,n) - \mu(m,n) \geq \epsilon | A_2) \leq \frac{\delta}{2MK^{(\beta+1)}} \) which follows from Theorem [1] with \( \delta \leftarrow \frac{\epsilon}{2MK^{(\beta+1)}} \) for \( n \geq 2 \) and follows from Hoeffding’s inequality for \( n = 1 \). ■

**Theorem 3:** For any \( \delta \), if we run the estimation phase of the algorithm for \( T_0 \geq \lceil \frac{M^2 \exp(2(\frac{K-1}{49}) \ln(\frac{2}{\delta}))}{\epsilon^2} \rceil \) rounds, then with probability at least \( 1 - \delta \), we have \( \hat{K} = K \).

**Proof:** Probability of collision for a user is given by

\[
p = 1 - \Pr(\text{No collision}) = 1 - \sum_{\text{channels}} \frac{1}{M}(1 - \frac{1}{M})^{K-1} = 1 - (1 - \frac{1}{M})^{K-1}.
\]

Let \( \hat{p} = \sum_{t} \frac{1}{\text{collision at time } t} \). We have \( E[\hat{p}] = p \) and we can use Hoeffding’s inequality since collision at each time-slot is independent. Thus if \( t \geq \frac{\ln(\frac{2}{\delta})}{2\epsilon^2} \), with probability greater than \( 1 - \delta \), we have \( |\hat{p} - p| \leq \epsilon_2 \).

We have \( \hat{K} = \text{round}\left(\frac{\ln(1-\frac{1}{\hat{p}})}{\ln(1-\frac{1}{M})}\right) + 1 \) and \( K = \frac{\ln(1-p)}{\ln(1-\frac{1}{M})} \). In order to show \( \hat{K} = K \), it suffices to show

\[
|\hat{K} - K| = \left| \frac{\ln(1-\frac{1}{\hat{p}})}{\ln(1-\frac{1}{M})} \right| \leq 0.49,
\]

which is equivalent to showing

\[
(1-p)(1 - (1 - \frac{1}{M})^{-0.49}) \leq \hat{p} - p \leq (1-p)(1 - (1 - \frac{1}{M})^{0.49}).
\]

It suffices to show

\[
\epsilon_2 \leq (1-p) \min\{||1 - (1 - \frac{1}{M})^{-0.49}||, ||1 - (1 - \frac{1}{M})^{0.49}||\}.
\]

We have

\[
||1 - (1 - \frac{1}{M})^{-0.49}|| = (1 + \frac{1}{M - 1})^{0.49} - 1 \geq \frac{0.49}{M - 1}
\]

and

\[
(1 - (1 - \frac{1}{M})^{0.49}) \geq \frac{0.49}{M}
\]

where the inequalities follow from the Bernoulli inequality, \((1 + x)^r \leq 1 + xr\) for \( 0 \leq r \leq 1 \) and \( x \geq -1 \).

We have from \((1 + \frac{1}{x})^{x-1} \geq \frac{1}{\exp(1)}\) for \( x \geq 1 \),

\[
1 - p = (1 - \frac{1}{M})^{K-1} \geq \frac{1}{\exp(\frac{K-1}{M-1})}.
\]

Hence, we choose \( \epsilon_2 \leq \frac{0.49}{M \exp(\frac{K-1}{M-1})} \). ■

**B. Allocation phase**

We now find bounds on the expected regret during each fixing phase, given that the estimates of \( \mu(m,f(m)) \) and \( K \) are accurate.

**Theorem 4:** The expected regret accumulated by the system during a fixing phase \( R_f \) is bounded as follows,

\[
\mathbb{E}[R_f] \leq K^2 M \exp(\frac{K-1}{M-1}).
\]

**Proof:** Let \( U_t \) denote the set of unfixed arms in the system at time \( t \). Probability of any player \( k \) being fixed at some time \( t \) is given by,
Pr(Player $k$ being fixed) 

$$= \sum_{m \in U_t} \Pr(\text{Choosing arm } m) \Pr(\text{Being fixed|arm } m)$$

$$= \sum_{m \in U_t} \frac{1}{M} \Pr(\text{At most } f^*_m - 1 \text{ users choose arm } m)$$

$$= \sum_{m \in U_t} \frac{1}{M} \sum_{i=0}^{f^*_m-1} \binom{K-1}{i} \left( \frac{1}{M} \right)^i \left( 1 - \frac{1}{M} \right)^{K-1-i}$$

$$\geq \frac{1}{M} \left( 1 - \frac{1}{M} \right)^{K-1} = \frac{1}{M} \left( 1 - \frac{1}{M} \right)^{(K-1)+M-1}/(M-1)$$

$$\geq \frac{1}{M} \exp \left( \frac{K-1}{M-1} \right)$$

where (a) follows because we only consider one term in the each of the summations with $i = 0$, and (b) follows from $(1 - \frac{1}{x})^{x-1} \geq \frac{1}{\exp(x)}$ for $x \geq 1$. Thus for any player $k$, the expected time to get fixed is given by

$$T_f = \mathbb{E}[t^k_f] = \frac{1}{\Pr(\text{Player } k \text{ being fixed})} \leq M \exp \left( \frac{K-1}{M-1} \right)$$

and thus the system-wide regret $\mathbb{E}[R_f]$ during the fixing phase is given by,

$$\mathbb{E}[\sum_{k} \max_{t=t_0+1} t^k_f] \leq \mathbb{E}[\text{Kmax } t^k_f] \leq \mathbb{E}[\text{Ksum } t^k_f] \leq K^2 T_f,$$

where $R_{k,t}$ denotes the regret incurred by player $k$ at time $t$ and we have $R_{k,t} \leq 1$ by our assumption on the reward distribution.

**Analysis for $K \leq M$**

For the case where $K \leq M$, there is no need for clustering. We only need the estimates for $\mu(m, 1)$ and all users individually choose the best $K$ channels. This reduces to the musical chairs algorithm and the analysis can be found in [9]. After fixing on a channel during the first allocation time, after every $T_x$ seconds, each user switches to the next channel among the $K$ best channels.

**V. Dynamic Case**

In this section, we extend the results to a dynamic system with a changing number of users. We run Algorithm 1 repeatedly in epochs. However, in order to obtain a sub-linear regret bound, we need to impose some restrictions on the number of epochs and on the way users enter or leave the system. It is easy to see that the number of epochs $N$ must be sub-linear in time to have sub-linear regret in the system. We restrict the number of users entering and leaving the system $\kappa$ to be $O(T^4)$ where $\zeta < \frac{1}{2}$. We note that this is different from [9] where the time horizon is fixed and known, and there is also a restriction on when users can enter or leave the system. In our model, the dynamic scenario also includes the case where $K_t$ can go from greater than $M$ to less than $M$ and vice-versa.

Let $K_t$ denote the number of active users at time $t$, where $\frac{K_t}{M} \leq \beta$. Note that all the theorems in Section IV follow for the dynamic case with $K_t \leq M \beta$. We consider $t_1$ where $t_1$ is the starting epoch length with $t_1 \geq T_0(1) + T_c(1) + N_0 T_x$. We run Algorithm 1 for time $t_1$. Then we run it for $2t_1$, then for $3t_1$ and so on. The algorithm is given below.

**Algorithm 5 Dynamic Allocation**

1: Input: Parameter $t_1$
2: for $t_1 \frac{(r+1)}{2} \leq \tau \leq t_1 \frac{(r+1)(r+2)}{2}$ do
3: Run Algorithm 1 with $\delta \leftarrow \frac{\delta}{\tau^{r+1}}$
4: end for

**Theorem 5:** With a probability greater than $1 - \delta$, the expected regret accumulated by the system after running the algorithm Dynamic Allocation for $T$ rounds where $t_1 \frac{(r+1)}{2} \leq T \leq t_1 \frac{(r+1)(r+2)}{2}$ is given by

$$\mathbb{E}[R(T)] \leq M \beta [N \tau^2 T_0(\tau) + T_c(r) + M \beta T_f + \kappa T^2]$$
i.e., \( \mathbb{E}[R(T)] \sim O(\kappa T^{3}) \).

**Proof:** We have \( t_{1} \frac{r(r+1)}{2} \leq T \) which gives us \( r \leq \left( \frac{2T}{t_{1}} \right)^{\frac{1}{2}} \). The epoch length is changing with time. The total number of epochs \( N_{e} \) until time \( T \) is \( N_{e} \leq (r+1) \sim O(T^{\frac{3}{2}}) \).

We consider Theorem 1 with \( \delta \) set as in the algorithm Dynamic allocation. Thus we have \( T^{(r)}_{0} \sim O(\ln r) \sim O(\ln T) \). Using union bound we show that with a probability at least \( 1 - \delta \), we have good estimates for \( \mu \) and \( K_{t} \) over all epochs.

\[
\Pr(\exists \text{epoch with wrong estimate}) \leq \sum_{\text{epochs}} \Pr(\text{wrong estimate}) \leq \delta \sum_{i=1}^{r+1} \frac{1}{2^{i}} \leq \delta.
\]

In epochs with fixed or static users, regret follows from Theorem 1, and in epochs with dynamic users, the system incurs regret during the entire epoch.

\[
\mathbb{E}[R(T)] \leq N_{e}(\text{Static case regret}) + K_{t} \sum_{\text{Epoch length}} \leq M\beta[N_{e}(T^{(r)}_{0} + T^{(r)}_{c} + M\beta T_{f})] + \kappa rt_{1}]
\]

Thus \( \mathbb{E}[R(T)] \sim O(\kappa T^{3}) \), and if \( \kappa \) is \( O(T^{\zeta}) \) where \( \zeta < \frac{1}{2} \), we have sub-linear regret.

**VI. EXPERIMENTS**

In this section, our goal is to validate the performance of the estimation phase in the algorithm and show that the performance in the allocation phase does not suffer due to use of the estimated values i.e., the regret does not grow with time in the allocation phase.

We consider a system with \( K = 10 \) users and \( M = 6 \) channels and the non dynamic case. We set \( T_{0} = 1000, T_{x} = 1000 \) time units and \( N_{0} = 5 \) and repeat the experiment 100 times and consider the average accumulated regret. We set \( \beta = 3 \) and reward distributions are chosen to be uniform with a variance of 0.01, and means between 0 and 1 given below,

\[
\mu = \begin{bmatrix}
1 & 0.49 & 0.1 & 0.005 \\
0.98 & 0.42 & 0.13 & 0.002 \\
0.97 & 0.5 & 0.12 & 0.009 \\
1 & 0.48 & 0.009 & 0.008 \\
0.92 & 0.43 & 0.1 & 0.001 \\
0.9 & 0.44 & 0.1 & 0.001
\end{bmatrix}
\]

We compare the performance of Algorithm 1 with the estimated values of \( \mu \) and \( K \) with Algorithm 1 with the true parameter values. We also show how the estimates change with number of iterations in the estimation phase \( T_{0} \). We used the in-built MATLAB kmeans function for clustering.

![Fig. 1: Accumulated regret as a function of time.](image)

From Figure 1, we see that the accumulated regret grows with time during the estimation phase and remains constant during the allocation phase. We also notice that there is no noticeable difference between Algorithm 1 with the true parameter values and the one with the estimated values. This follows because the estimates of \( K \) and the mean converge to the true values with few iterations as shown in Figure 2 and Figure 3.
VII. CONCLUSION

We considered a spectrum allocation problem modeled as a multi-user multi-armed bandit with no communication among the users. We proposed a new algorithm that achieves constant regret with high probability when the number of users are fixed. We extended our algorithm to the dynamic case and achieved sub-linear regret. We provided simulation results to show that the algorithm performs well in practice when the number of users is fixed.

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[7] A. Anandkumar, N. Michael, A. K. Tang, and A. Swami, “Distributed algorithms for learning and cognitive medium access with logarithmic regret,” IEEE Journal on Selected Areas in Communications, vol. 29, no. 4, pp. 731–745, 2011.
Theorem 6: The separability condition is satisfied with high probability when \( N = T_0 = \frac{32 \exp(\frac{K-1}{\epsilon^2}) M}{\epsilon^2} \ln \frac{2MK\beta(\beta+1)}{\delta} \) if for any \( r, s \),
\[
|\mu_r - \mu_s| \geq 4M \exp(\frac{K-1}{M-1}) \sqrt{\sigma^2 + \epsilon_2},
\]
where \( \sigma \) is the variance of the distribution.

Proof:
From Hoeffdings, we have
\[
\Pr \left[ \frac{1}{N} \sum_{i \in [N]} (x_i - E[x_i])^2 - \sigma^2 \geq \epsilon_2 \right] \leq \exp(-2N\epsilon_2^2)
\]
i.e., with probability greater than \( 1 - \frac{\delta}{2MK\beta(\beta+1)} \), we have \( \sum_{i \in [N]} (x_i - E[x_i])^2 \leq N(\sigma^2 + \epsilon_2) \).

We have \( ||x||_1 \leq \sqrt{N}||x||_2 \).

\[
\left( \frac{1}{n_r} + \frac{1}{n_s} \right) \sum_{i \in [N]} |x_i - E[x_i]| \leq \left( \frac{1}{n_r} + \frac{1}{n_s} \right) \sqrt{N} \sqrt{\sum_{i \in [N]} (x_i - E[x_i])^2} \\
\leq \left( \frac{1}{n_r} + \frac{1}{n_s} \right) N \sqrt{\sigma^2 + \epsilon_2} \\
\leq 4M \exp(\frac{K-1}{M-1}) \sqrt{\sigma^2 + \epsilon_2}
\]
where the last inequality follows because from Theorem 7 we have \( n_s \geq \frac{16}{\epsilon^2} \ln \frac{2MK\beta(\beta+1)}{\delta} \).

Theorem 7: If \( T_0 = \left[ \frac{32 \exp(\frac{K-1}{\epsilon^2}) M}{\epsilon^2} \ln \frac{2MK\beta(\beta+1)}{\delta} \right] \), then all users using Algorithm 1 have at least \( \frac{16}{\epsilon^2} \ln \frac{2MK\beta(\beta+1)}{\delta} \) observations of each reward distribution on each arm with probability greater than \( 1 - \frac{\delta}{7} \).

Proof: Let \( A_{k,m,n}(t) = I \) \{player \( k \) observed arm \( m \) with \( n \) users at round \( t \)\}. Note that for any round \( t \) and any \( k, m, n \) we have that
\[
\Pr( A_{k,m,n}(t) = 1 ) = \frac{1}{M} \left( K - 1 \right) \left( 1 - \frac{1}{M} \right)^{K-n} \left( \frac{1}{M} \right)^{n-1}
\]
\[
\Rightarrow \mathbb{E}[ A_{k,m,n}(t) ] = \frac{1}{M} \left( K - 1 \right) \left( 1 - \frac{1}{M} \right)^{K-n} \left( \frac{1}{M} \right)^{n-1} \geq \frac{1}{M} \left( 1 - \frac{1}{M} \right)^{K-1} \geq \frac{1}{M \exp(\frac{\beta}{M+1})} \text{ for all } M > 1.
\]
where the last inequality follows from \( (1 - \frac{1}{x})^{-1} \geq \frac{1}{\exp(1)} \) for \( x \geq 1 \).

We have,
\[
\Pr \left( \exists k, m, n \text{ s.t. } \sum_{t=1}^{T_0} A_{k,m,n}(t) \leq \frac{1}{2} T_0 \mathbb{E}[ A_{k,m,n}(t) ] \right) \leq \sum_k \sum_m \sum_n \Pr \left( \sum_{t=1}^{T_0} A_{k,m,n}(t) \leq \frac{1}{2} T_0 \mathbb{E}[ A_{k,m,n}(t) ] \right)
\]
\[
\leq \sum_k \sum_m \sum_n \exp \left( -\frac{1}{4} T_0 \mathbb{E}[ A_{k,m,n}(t) ] \right)
\]
\[
= K(\beta+1) M \exp \left( -\frac{1}{4} T_0 \mathbb{E}[ A_{k,m,n}(t) ] \right)
\]
where the first inequality follows from union bound and the second inequality follows from Chernoff bound. Note that for a particular \( k, m \) and \( n \), \( A_{k,m,n} \) is i.i.d across \( t \), since all users are choosing channels uniformly at random.

VIII. APPENDIX
In order for this probability to be upper bounded by \( \frac{\delta}{2} \) we need:

\[
K(\beta + 1)M \exp \left( -\frac{1}{2} T_0 E [A_{k,m,n}(t)] \right) < \frac{\delta}{2}
\]

\[\implies T_0 > \frac{1}{8E(A_{k,m,n}(t))} \ln \left( \frac{2K(\beta + 1)M}{\delta} \right) .\]

We have shown that if \( T_0 > \frac{1}{8E(A_{k,m,n}(t))} \ln \left( \frac{2K(\beta + 1)M}{\delta} \right) \) then w.p. \( \geq 1 - \frac{\delta}{2} \) we have \( \forall k,m,n \) the number of observations player \( k \) has of arm \( m \) with \( n \) users, \( \sum_{t=1}^{T_0} A_{k,m,n}(t) > \frac{1}{2} T_0 E [A_{k,m,n}(t)] . \)

We also need the total number of observations each player has of each arm to be at least \( \frac{16}{\epsilon^2} \ln \frac{2MK\beta(\beta + 1)}{\delta} \), i.e.

\[
\sum_{t=1}^{T_0} A_{k,m,n}(t) > \frac{1}{2} T_0 E [A_{k,m,n}(t)] \geq \frac{16}{\epsilon^2} \ln \frac{2MK\beta(\beta + 1)}{\delta} \]

\[\implies T_0 \geq \frac{2}{E[A_{k,m,n}(t)]} \frac{16}{\epsilon^2} \ln \frac{2MK\beta(\beta + 1)}{\delta} .\]

So we have two constraints on \( T_0 \), which gives us:

\[
T_0 = \lceil \max \left\{ \frac{1}{8E[A_{k,m,n}(t)]} \ln \left( \frac{2K(\beta + 1)M}{\delta} \right), \frac{2}{E[A_{k,m,n}(t)]} \frac{16}{\epsilon^2} \ln \frac{2MK\beta(\beta + 1)}{\delta} \right\} \rceil .
\]

which can be further simplified to

\[
T_0 = \left\lceil \frac{32 \exp \left( \frac{K-1}{K-1} \right) M}{\epsilon^2} \ln \frac{2MK\beta(\beta + 1)}{\delta} \right\rceil .
\]

\[\blacksquare\]

A. Clustering

Let \( N \) points \( \{x_i, \ldots, x_N\} \) be drawn independently from \( \beta \) distributions with mean \( \mu_r \) where \( r \in [\beta] \). Let number of samples drawn from distribution with mean \( \mu_r \) be denoted by \( n_r \) and the separability condition \( [\text{1}] \) is satisfied. We now present some lemmas that are useful for proving that after clustering, the centroids are closer to the means. Additional notation used is introduced in Table \( \text{I} \).

| \( \Delta_s \) | \( |\mu_s - \nu_s|\) |
| --- | --- |
| \( T_{s, s \in [\beta]} \) | \( \max_{r \neq s} \frac{\alpha}{n_r - n_s} \) |
| \( \gamma \) | \( \gamma \) |
| \( \{T_s\} s \in [\beta] \) | True partition of the samples \( X \) |
| \( n_s \) | \# of samples |
| \( \phi_s \) | \( |T_s| \) |
| \( g(S) \) | \( \frac{1}{|S|} \sum_{x \in S} x \) |
| \( \rho_{in}^s \) | Fraction of points misclassified as cluster \( s \) |
| \( \rho_{out}^s \) | Fraction of misclassified points in cluster \( s \) |

**TABLE I: Notation.**

**Theorem 8:** If the separability condition \( [\text{1}] \) is satisfied. After using Cluster algorithm, we have that for any fixed \( \epsilon, \delta \) and \( n_s \geq N_{c, \delta} = \lceil \frac{16}{\epsilon^2} \ln \left( \frac{2}{\delta} \right) \rceil \), with probability greater than 1 - \( \delta \),

\[
|\hat{\mu}_s - \mu_s| = |g(S_s) - \mu_s| \leq \epsilon .
\]

**Proof:**

From Lemma \( [5] \) after the \( \alpha \) approximation algorithm, we have \( \Delta_s \leq 2(\alpha + 1) \frac{\omega}{n_s} \) and \( \gamma < \frac{2(\alpha + 1)\omega}{\epsilon} \). If we want \( \gamma \leq \frac{1}{8} \) which gives \( \alpha < \frac{\epsilon}{\omega n} - 1 \). From Lemma \( [3] \) \( \rho_{in}^s + \rho_{out}^s \leq \frac{8}{\epsilon} \) which we need to be less than \( \frac{1}{2} \) this giving us \( c > 16 \). From this and Lemma \( [2] \) the conditions for Lemma \( [5] \) are satisfied and \( \gamma < \frac{1}{8} \). Thus we have,

\[
|g(S_s) - \mu_s| \leq 2(1 - \rho_{out}^s) |g(S_s \cap T_s) - \mu_s| + 4 \sum_{r \neq s: \rho_{in}^s(r) \neq 0} \rho_{in}^s(r) |g(S_s \cap T_r) - \mu_r| .
\]
For each $r \in [\beta]$, $S_s \cap T_r$ denotes independently drawn bounded random variables from reward distribution with mean $\mu_r$. We use Hoeffding’s lemma.

$$\Pr(\exists r \text{ s.t. } |g(S_s \cap T_r) - \mu_r| \geq \epsilon) \leq \sum_{r \in \beta} \Pr(|g(S_s \cap T_r) - \mu_r| \geq \epsilon) \leq (a) \exp(-2n_s(1 - \rho_{out}^s)(\frac{\epsilon}{4})^2) + \sum_{r \neq s; \rho_{in}^r(r) \neq 0} \exp(-2n_s\rho_{in}^s(r)(\frac{\epsilon}{4})^2) \leq (b) \exp(-2n_s\rho_{in}^s(\frac{\epsilon}{4})^2) + \sum_{r \neq s; \rho_{in}^r(r) \neq 0} \exp(-2n_s\epsilon^2) \leq (c) \beta \exp(-2n_s\epsilon^2) \leq \delta,$$

where $c_1 = \min_{r,s} \rho_{in}^s(r) : \rho_{out}^s(r) \neq 0$. Inequality (a) follows from Hoeffding’s lemma, inequality (b) from $1 - \rho_{out}^s \geq \rho_{in}^s$ and inequality (c) from the definition of $c_1$.

For $\beta \exp(-2n_s\epsilon^2) \leq \delta$, we need $n_s \geq \frac{8}{c_1\epsilon^2} \ln(\frac{\beta}{\delta})$. Since $c_1 < \frac{1}{2}$, we have $n_s \geq \frac{16}{\epsilon^2} \ln(\frac{\beta}{\delta})$.

Thus, with probability greater than $1 - \delta$, we have

$$|g(S_s) - \mu_s| \leq \frac{2\epsilon}{4} + 4 \sum_{r \neq s; \rho_{in}^r(r) \neq 0} \rho_{in}^s(r) \frac{\epsilon}{4} \leq \frac{\epsilon}{2} + \rho_{in}^s \epsilon \leq \epsilon.$$

**Lemma 1:** An $\alpha$ approximation algorithm returns the set of centroids $\{\nu_1, \ldots, \nu_\beta\}$ where $C(x)$ returns the centroid of the cluster to which $x$ belongs. We have $\forall T_s \exists \nu_s$ such that $|\nu_s - \mu_s| \leq 2(\alpha + 1)\frac{\phi_T}{n_s}$ and $\gamma < \frac{2(\alpha + 1)}{c}$.

**Proof:**

We first show that $\forall s, \Delta_s \leq (\alpha + 1)\frac{\phi_T}{n_s}$ where $\phi_T = \sum_{s=1}^\beta \sum_{x \in T_s} |x - g(T_s)|$. Assume the contrary that for some $T_s, |\nu_r - \mu_s| > (\alpha + 1)\frac{\phi_T}{n_s} \forall r \in [\beta]$.

$$\sum_{x \in T_s} |x - C(x)| \geq \sum_{x \in T_s} |C(x) - g(T_s)| - |x - g(T_s)| \geq |T_s|\frac{(\alpha + 1)\phi_T}{|T_s|} - \sum_{x \in T_s} |x - g(T_s)| \geq (\alpha + 1)\phi_T - \phi_T = a\phi_T,$$

which is a contradiction. We now show that $\phi_T \leq 2\phi_s$ which proves that $\Delta_s \leq 2(\alpha + 1)\frac{\phi_s}{n_s}$.

$$\phi_T = \sum_{s=1}^\beta \sum_{x \in T_s} |x - g(T_s)| \leq \sum_{s=1}^\beta \sum_{x \in T_s} |g(T_s) - \mu_s| + \sum_{s=1}^\beta \sum_{x \in T_s} |x - \mu_s| \geq \sum_{s=1}^\beta |T_s||g(T_s) - \mu_s| + \phi_s = \sum_{s=1}^\beta \sum_{x \in T_s} |x - \mu_s| + \phi_s \leq \sum_{s=1}^\beta \sum_{x \in T_s} |x - \mu_s| + \phi_s = 2\phi_s.$$

Now we show that $\gamma \leq \frac{2(\alpha + 1)}{c}$. For any $s, r$,

$$\frac{2(\alpha + 1)}{c} |\mu_r - \mu_s| \geq \frac{2(\alpha + 1)}{c} c\phi_s \left(\frac{1}{n_s} + \frac{1}{n_r}\right) \geq \Delta_s.$$
Since this is true for all \( r, s \), we have
\[
\gamma \leq \frac{2(\alpha + 1)}{c}.
\]

**Lemma 2:** If \( \gamma < \frac{1}{4} \), the following results hold \( \forall x \in S_r \),
1) \( |x - \mu_s| \geq (\frac{1}{2} - 2\gamma)|\mu_r - \mu_s| \), \( \forall s \neq r \).
2) \( |x - \mu_r| \leq \frac{1}{1 - 2\gamma}|x - \mu_s| \).

**Proof:** (1)
\[
|\nu_r - \nu_s| = |\nu_r - \mu_r + \mu_r - \mu_s + \mu_s - \nu_s| \\
\geq |\mu_r - \mu_s| - |\nu_r - \mu_r| - |\mu_s - \nu_s| \\
\geq (1 - 2\gamma)|\mu_r - \mu_s|,
\]
where the last inequality follows from the definition of \( \gamma \).
\[
|x - \mu_s| \geq |x - \nu_s| - |\mu_s - \nu_s| \\
\geq \frac{1}{2}|\nu_r - \nu_s| - |\mu_s - \nu_s| \\
\geq (\frac{1}{2} - \gamma)|\mu_r - \mu_s| - |\mu_s - \nu_s| \\
\geq (\frac{1}{2} - \gamma)|\mu_r - \mu_s| - \gamma|\mu_r - \mu_s| \\
= (\frac{1}{2} - 2\gamma)|\mu_r - \mu_s|,
\]
where the second inequality follows from \( x \in S_r \) and the last from the definition of \( \gamma \).
(2)
\[
|x - \mu_r| \leq |\mu_r - \nu_r| + |x - \nu_r| \\
\leq |\mu_r - \nu_r| + |x - \nu_s| \\
\leq |\mu_r - \nu_r| + |x - \mu_s| + |\mu_s - \nu_s|.
\]
Note that the first statement with the definition of \( \gamma \) also implies for \( l = r, s \)
\[
\frac{1 - 4\gamma}{2\gamma} |\mu_l - \nu_l| \leq |x - \mu_s|,
\]
which gives us
\[
|x - \mu_r| \leq (1 + \frac{4\gamma}{1 - 4\gamma})|x - \mu_s| \\
= \frac{1}{1 - 4\gamma}|x - \mu_s|.
\]

**Lemma 3:** If \( \gamma < \frac{1}{4} \) and \( |\mu_r - \mu_s| \geq c\frac{\phi_s}{n_s} \), we have \( \rho_{in}^s \leq \frac{2}{(1 - 4\gamma)c} \) and \( \rho_{out}^s \leq \frac{2}{(1 - 4\gamma)c} \).

**Proof:** From the separability condition, we have \( |\mu_r - \mu_s| \geq c\frac{\phi_s}{n_s} \).
\[
n_s\rho_{out}^s(\frac{1}{2} - 2\gamma)c\frac{\phi_s}{n_s} \leq \sum_{r \neq s} |T_s \cap S_r|((\frac{1}{2} - 2\gamma)|\mu_s - \mu_r|) \\
\leq \sum_{r \neq s} \sum_{x_i \in T_s \cap S_r} ((\frac{1}{2} - 2\gamma)|\mu_s - \mu_r|) \\
\leq \sum_{r \neq s} \sum_{x_i \in T_s \cap S_r} |x_i - \mu_s| \\
\leq \phi_s,
\]
where the first and second inequalities follow from the separability condition and Lemma 2, respectively. This gives us \( \rho_{out}^s \leq \frac{2}{(1 - 4\gamma)c} \) and similarly we also have \( \rho_{in}^s \leq \frac{2}{(1 - 4\gamma)c} \).
Lemma 4: If \((a)\rho_{\text{in}}^s + \rho_{\text{out}}^s < \frac{1}{2}\) and \((b)|g(S_s \cap T_r) - \mu_r| \geq (1 - 4\gamma)|g(S_s \cap T_r) - \mu_s|\) we have,

\[
|g(S_s) - \mu_s| \leq 2(1 - \rho_{\text{out}}^s)|g(S_s \cap T_s) - \mu_s| + \frac{2}{1 - 4\gamma} \sum_{r \neq s} \rho_{\text{in}}^s(r)|g(S_s \cup T_r) - \mu_r|.
\]

Proof: \(|g(S_s) - \mu_s|\)

\[
= \left| \frac{|S_s \cap T_s|g(S_s \cap T_s) + \sum_{r \neq s}|S_s \cap T_r|g(S_s \cap T_r)}{|S_s|} - \mu_s \right|
\]

\[
= \frac{|n_s(1 - \rho_{\text{out}}^s)(g(S_s \cap T_s) - \mu_s) + \sum_{r \neq s} n_s \rho_{\text{in}}^s(r)(g(S_s \cap T_r) - \mu_s)|}{|S_s|}
\]

\[
\leq (a) 2(1 - \rho_{\text{out}}^s)|g(S_s \cap T_s) - \mu_s| + 2 \sum_{r \neq s} n_s \rho_{\text{in}}^s(r)(g(S_s \cap T_r) - \mu_s)|
\]

\[
\leq 2[(1 - \rho_{\text{out}}^s)|g(S_s \cap T_s) - \mu_s| + \sum_{r \neq s} n_s \rho_{\text{in}}^s(r)(g(S_s \cap T_r) - \mu_s)|
\]

\[
\leq (b) 2(1 - \rho_{\text{out}}^s)|g(S_s \cap T_s) - \mu_s| + \frac{2}{1 - 4\gamma} \sum_{r \neq s} n_s \rho_{\text{in}}^s(r)(g(S_s \cap T_r) - \mu_r)|.
\]