The bisimulation problem for equational graphs of finite out-degree.

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Abstract. The bisimulation problem for equational graphs of finite out-degree is shown to be decidable. We reduce this problem to the \( \eta \)-bisimulation problem for deterministic rational (vectors of) boolean series on the alphabet of a dpda \( M \). We then exhibit a complete formal system for deducing equivalent pairs of such vectors.

Keywords: bisimulation; equational graphs; deterministic pushdown automata; rational languages; finite dimensional vector spaces; matrix semi-groups; complete formal systems.
1 Introduction

1.1 Motivations

Processes In the context of concurrency theory, several notions of “behaviour of a process” and “behavioural equivalence between processes” have been proposed. Among them, the notion of bisimulation equivalence seems to play a prominent role (see [Mil89]). The question of whether this equivalence is decidable or not for various classes of infinite processes has been the subject of many works in the last ten years (see for example [BBK87, Cau90, CHM92, GH93, HJM94, CHS95, Cau95, Sti96, Jan97, Sen98]).

The aim of this work is to show decidability of the bisimulation equivalence for the class of all processes defined by pushdown automata whose \( \epsilon \)-transitions are deterministic and decreasing (of course, we assume that \( \epsilon \)-transitions are not visible, which implies that the graphs of the processes considered here, might have infinite in-degree). This problem was raised in [Cau95] (see Problem 6.2 of this reference) and is a significant subcase of the problem raised in [Sti96] (as the bisimulation-problem for processes “of type -1”).

Infinite graphs A wide class of graphs enjoying interesting decidability properties has been defined in [Cou89, Bau91, Bau92] (see [Cou90] for a survey). In particular it is known that the problem

“are \( \Gamma, \Gamma' \) isomorphic?”

is decidable for pairs \( \Gamma, \Gamma' \) of equational graphs. It seems quite natural to investigate whether the problem

“are \( \Gamma, \Gamma' \) bisimilar?”

is decidable for pairs \( \Gamma, \Gamma' \) of equational graphs. We show here that this problem is decidable for equational graphs of finite out-degree.

Formal languages Another classical equivalence relation between processes is the notion of language equivalence. The decidability of language equivalence for deterministic pushdown automata has been recently established in [Sen97b] (see also in [Sen97a, Sen97c] shorter expositions of this result). It was first noticed in [BBK87] that, in the case of deterministic processes, language equivalence and bisimulation equivalence are identical. Moreover deterministic pushdown automata can always be normalized (with preservation of the language) in such a way that \( \epsilon \)-transitions are all decreasing. Hence the main result of this work is a generalisation of the decidability of the equivalence problem for dpda’s.
Mathematical generality  More precisely, the present work extends the notions developed in [Sén97b] so as to obtain a more general result. As a by-product of this extension, we obtain a deduction system which, in the deterministic case, seems simpler than the one presented in [Sén97b] (see system $B_3$ in §10). The present work can also be seen as a common generalization of 3 different results: the results of [Sti96,Jan97] establishing decidability of the bisimulation equivalence in two non-deterministic sub-classes of the class considered here, and the result of [Sén97b] dealing only with deterministic pda’s (or processes).

Logics  Our solution consists in constructing a complete formal system, in the general sense taken by this word in mathematical logics i.e.: it consists of a set of well-formed assertions, a subset of basic assertions, the axioms, and a set of deduction rules allowing to derive new assertions from assertions which are already generated. The well-formed assertions we are considering are pairs $(S,T)$ of rational boolean series over the non-terminal alphabet $V$ of some strict-deterministic grammar $G = \langle X, V, P \rangle$. Such an assertion is true when the two series $S, T$ are bisimilar. Several simple formal systems generating all the identities between boolean rational expressions have been the subject of many works ([Sal66,Bof90,Kro91]; the case of bisimilar rational expressions has been also addressed in [Mil84,Koz91]. A tableau proof-system generating all the bisimilar pairs of words with respect to a given context-free grammar in Greibach normal form was also given in [HS91]. Our complete formal systems can be seen as participating in this general research stream (see in [Sén00] an overview of this subject, in the context of equivalence problems for pushdown automata).

1.2 Results

The main results of this work are the following theorems.

Theorem 107: The bisimulation problem for rooted equational 1-graphs of finite out-degree is decidable.

Theorem 1014: $B_3$ is a complete deduction system.

where $B_3$ is a formal system whose elementary rules just express the basic algebraic properties of bisimulation: the fact that it is an equivalence relation, that it is compatible with right and left (matricial) product, that Arden’s lemma remains true modulo bisimulation and at last, its link with one-step derivation (rule R34). Completeness means here that all pairs of bisimilar rational “deterministic” boolean series are generated by this formal system.

1.3 Main tools

We re-use here the notions developed in [Sén97b] (1-4) and introduce new ideas (5-7):
1. the deduction systems (which were in turn inspired by \cite{Cou83a}).
2. the deterministic boolean series (which were in turn inspired by \cite{HHY79}).
3. the deterministic spaces (which were elaborated around Meitus notion of linear independence \cite{Mei89,Mei92}).
4. the analysis of the proof-trees generated by a suitable strategy (which was somehow similar with the analysis of the parallel computations, interspersed with replacement-moves, done in \cite{Val74,Rom85,Oya87}).
5. the notion of \(\eta\)-bisimulation over deterministic row-vectors of boolean series (which, in some sense, translates the usual notion of bisimulation to the \(d\)-space of row-vectors of series).
6. the notion of oracle, which is a choice of bisimulation for every pair of bisimilar vectors; the notion of triangulation of systems of linear equations is now “parametrized” by such an oracle \(O\) (see §5); as well, the strategies are now parametrized by an oracle too.
7. an elimination argument: roughly speaking, this argument shows that, in a proof-tree \(t\), if we take into account not only the branch ending at a node \(x\), but also the whole proof-tree, then the meta-rule \(R5\)

\[
\{(p, S, S')\} \vdash (p + 2, S \circledast x, S' \circledast x')
\]

is not needed to show that \(\text{im}(t) \vdash \{t(x)\}\). A nice (and unexpected) by-product of this elimination is that the weights can be removed from the equations (see systems \(B_2, B_3\) in §10).

The proof exposed here is an updated version of the full proof given in \cite{Sen97a} and exposed in a concise way in \cite{Sen98}. Some simplifications of \cite{Sen97a} found by C. Stirling \cite{Sti99} were taken into account in this proof too:
- the technical notion of “\(N\)-stacking sequence” is replaced by the slightly simpler notion of “\(B\)-stacking sequence”
- the analysis of section 8 uses a choice of “generating set” which is simpler than the choice given in \cite{Sen97a,Sen98}.
- a main simplification linked with this more clever choice, is that we can restrict ourselves to the case of a proper, reduced strict-deterministic grammar (as is done in \cite{Sti99}), while in \cite{Sen97a,Sen98} we could not assume this restriction.
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2 Preliminaries

2.1 Graphs

Let $X$ be a finite alphabet. We call graph over $X$ any pair $\Gamma = (V_\Gamma, E_\Gamma)$ where $V_\Gamma$ is a set and $E_\Gamma$ is a subset of $V_\Gamma \times X \times V_\Gamma$. For every integer $n \in \mathbb{N}$, we call an $n$-graph every $n+2$-tuple $\Gamma = (V_\Gamma, E_\Gamma, v_1, \ldots, v_n)$ where $(V_\Gamma, E_\Gamma)$ is a graph and $(v_1, \ldots, v_n)$ is a sequence of distinguished vertices: they are called the sources of $\Gamma$.

A 1-graph $(V,E,v_1)$ is said to be rooted iff $v_1$ is a root of $(V,E)$ and $V \neq \{v_1\}$.

A 2-graph $(V,E,v_1,v_2)$ is said bi-rooted iff $v_1$ is a root, $v_2$ is a co-root of $(V,E)$, $v_1 \neq v_2$ and there is no edge going out of $v_2$ (this last technical condition will be useful for reducing the bisimilarity notion for graphs to an analogous notion on series, see §2.1, §2.3 and §3.2).

The equational graphs are the least solutions (in a suitable sense) of the systems of (hyperedge) graph-equations (see in [Cou90a] precise definitions). Let us mention that the equational graphs of finite degree are exactly the context-free graphs defined in [MS85].

Bisimulations

Definition 21 Let $\Gamma = (V_\Gamma, E_\Gamma, v_1, \ldots, v_n), \Gamma' = (V_{\Gamma'}, E_{\Gamma'}, v'_1, \ldots, v'_n)$ be two $n$-graphs over an alphabet $X$. Let $R$ be some binary relation $R \subseteq V_\Gamma \times V_{\Gamma'}$.

$R$ is a simulation from $\Gamma$ to $\Gamma'$ iff

1. $\text{dom}(R) = V_\Gamma$,
2. $\forall i \in [1,n], (v_i, v'_i) \in R$,
3. $\forall v \in V_\Gamma, w \in V_\Gamma, v' \in V_{\Gamma'}, x \in X, \text{ such that } (v, x, w) \in E_\Gamma \text{ and } vRv'$, there exists $w' \in V_{\Gamma'}$ such that $(v', x, w') \in E_{\Gamma'} \text{ and } wRw'$.

$R$ is a bisimulation from $\Gamma$ to $\Gamma'$ iff $R$ is a simulation from $\Gamma$ to $\Gamma'$ and $R^{-1}$ is a simulation from $\Gamma'$ to $\Gamma$.

This definition corresponds to the standard one ([Par81, Mil89, Cou95]) in the case where $n = 0$. The $n$-graphs $\Gamma, \Gamma'$ are said bisimilar, which we denote by $\Gamma \sim \Gamma'$, iff there exists a bisimulation $R$ from $\Gamma$ to $\Gamma'$.

Let us extend now this definition by means of a relational morphism between free monoids.

Definition 22 Let $X, X'$ be two alphabets. A binary relation $\eta \subseteq X^* \times X'^*$ is called a strong relational morphism from $X^*$ to $X'^*$ iff

1. $\eta$ is a submonoid of $X^* \times X'^*$
2. $\text{dom}(\eta) = X^*, \text{im}(\eta) = X'^*$
3. $\eta$ is generated (as a submonoid) by the subset $\eta \cap (X \times X')$. 

One can easily check that s.r. morphisms are preserved by inversion, composition and that any surjective map \( \eta : X \to X' \) induces a s.r. morphism from \( X^* \) to \( X'^* \). Let \( \Gamma = (V_{\Gamma}, E_{\Gamma}, v_1, \ldots, v_n) \) be an n-graph over the alphabet \( X \), \( \Gamma' = (V_{\Gamma'}, E_{\Gamma'}, v'_1, \ldots, v'_n) \) be an n-graph over the alphabet \( X' \). Let \( \eta \subseteq X^* \times X'^* \) be some s.r. morphism, and let \( \mathcal{R} \) be some binary relation \( \mathcal{R} \subseteq V_{\Gamma} \times V_{\Gamma'} \).

**Definition 23** \( \mathcal{R} \) is a \( \eta \)-simulation from \( \Gamma \) to \( \Gamma' \) iff

1. \( \text{dom}(\mathcal{R}) = V_{\Gamma} \),
2. \( \forall i \in [1, n], (v_i, v'_i) \in \mathcal{R}, \)
3. \( \forall v, w \in V_{\Gamma}, v' \in V_{\Gamma'}, x \in X, \text{ such that } (v, x, w) \in E_{\Gamma} \text{ and } vRv' \),

\[
\exists w' \in V_{\Gamma'}, x' \in \eta(x) \text{ such that } (v', x', w') \in E_{\Gamma'} \text{ and } wRw'.
\]

\( \mathcal{R} \) is a \( \eta \)-bisimulation iff \( \mathcal{R} \) is a \( \eta \)-simulation and \( \mathcal{R}^{-1} \) is a \( \eta^{-1} \)-simulation.

For every \( v \in V_{\Gamma}, v' \in V_{\Gamma'} \), we denote by \( v \sim v' \) the fact that there exists some \( \eta \)-bisimulation \( \mathcal{R} \) from \( \Gamma \) to \( \Gamma' \) such that \( (v, v') \in \mathcal{R} \). In all this work, the composition of binary relations is denoted by \( \circ \) and defined by: if \( \mathcal{R}_1 \subseteq E \times F \) and \( \mathcal{R}_2 \subseteq F \times G \) then,

\[
\mathcal{R}_1 \circ \mathcal{R}_2 = \{(x, z) \in E \times G | \exists y \in F, (x, y) \in \mathcal{R}_1, (y, z) \in \mathcal{R}_2\}.
\]

**Fact 24**

1. if \( \mathcal{R} \) is a \( \eta \)-bisimulation, then \( \mathcal{R}^{-1} \) is a \( \eta^{-1} \)-bisimulation
2. if \( \mathcal{R}_1 \) is a \( \eta_1 \)-bisimulation and \( \mathcal{R}_2 \) is a \( \eta_2 \)-bisimulation, then \( \mathcal{R}_1 \circ \mathcal{R}_2 \) is a \( \eta_1 \circ \eta_2 \)-bisimulation
3. if for every \( i \in I, \mathcal{R}_i \) is a \( \eta \)-bisimulation, then \( \bigcup_{i \in I} \mathcal{R}_i \) is a \( \eta \)-bisimulation.

### 2.2 Pushdown automata

A **pushdown automaton** on the alphabet \( X \) is a 7-tuple \( \mathcal{M} = < X, Z, Q, \delta, q_0, z_0, F > \) where \( Z \) is the finite stack-alphabet, \( Q \) is the finite set of states, \( q_0 \in Q \) is the initial state, \( z_0 \) is the initial stack-symbol, \( F \) is a finite subset of \( QZ^* \), the set of final configurations, and \( \delta \), the transition function, is a mapping \( \delta : QZ \times (X \cup \{\epsilon\}) \to P_f(QZ^*) \).

Let \( q, q' \in Q, \omega, \omega' \in Z^*, z \in Z, f \in X^* \) and \( a \in X \cup \{\epsilon\} \); we note \((q\omega, af) \rightarrow_M (q'\omega', f)\) if \( q'\omega' \in \delta(qz, a) \). \( \rightarrow_M^* \) is the reflexive and transitive closure of \( \rightarrow_M \).

For every \( q\omega, q'\omega' \in QZ^* \) and \( f \in X^*, \) we note \( q\omega \rightarrow_M^f q'\omega' \text{ iff } (q\omega, f) \rightarrow_M^* (q'\omega', \epsilon) \).
\( \mathcal{M} \) is said **deterministic** iff, for every \( z \in Z, q \in Q, x \in X \):

\[
\text{Card}(\delta(qz, \epsilon)) \in \{0, 1\} \quad (2)
\]

\[
\text{Card}(\delta(qz, \epsilon)) = 1 \Rightarrow \text{Card}(\delta(qz, x)) = 0, \quad (3)
\]

\[
\text{Card}(\delta(qz, \epsilon)) = 0 \Rightarrow \text{Card}(\delta(qz, x)) \leq 1. \quad (4)
\]

\( \mathcal{M} \) is said **real-time** iff, for every \( q \in Q, z \in Z, \text{Card}(\delta(qz, \epsilon)) = 0. \)

A configuration \( q\omega \) of \( \mathcal{M} \) is said \( \epsilon \)-bound iff there exists a configuration \( q'\omega' \) such that \( (q\omega, \epsilon) \xrightarrow{M} (q'\omega', \epsilon) \); \( q\omega \) is said \( \epsilon \)-free iff it is not \( \epsilon \)-bound.

A pda \( \mathcal{M} \) is said **normalized** iff, it fulfills conditions (2), (3) (see above) and (5), (6), (7):

\[
q_0z_0 \text{ is } \epsilon - \text{ free} \quad (5)
\]

and for every \( q \in Q, z \in Z, x \in X \):

\[
q'\omega' \in \delta(qz, x) \Rightarrow |\omega'| \leq 2, \quad (6)
\]

\[
q'\omega' \in \delta(qz, \epsilon) \Rightarrow |\omega'| = 0 \quad (7)
\]

All the pda considered here are assumed to fulfill condition (5). A pda \( \mathcal{M} \) will be said **bi-rooted** iff it fulfills (8) and (9):

\[
\exists \bar{q} \in Q, F = \{\bar{q}\} \text{ and } \forall q \in Q, \omega \in Z^*, f \in X^*, q_0z_0 \xrightarrow{f}{\mathcal{M}} q\omega \Rightarrow \exists g \in X^*, q\omega \xrightarrow{g}{\mathcal{M}} \bar{q}. \quad (9)
\]

The **language recognized** by \( \mathcal{M} \) is

\[
L(\mathcal{M}) = \{w \in X^* | \exists c \in F, q_0z_0 \xrightarrow{w}{\mathcal{M}} c\}.
\]

It is a “folklore” result that, given a deterministic pda \( \mathcal{M} \), one can effectively compute another dpda \( \mathcal{M}' \) which is normalized and fulfills:

\[
L(\mathcal{M}) = L(\mathcal{M}') - \{\epsilon\}.
\]

### 2.3 Graphs and pushdown automata

**Equational graphs and pushdown automata** We call **transition-graph** of a pda \( \mathcal{M} \), denoted \( T(\mathcal{M}) \), the 0-graph:

\[
T(\mathcal{M}) = (V_T(\mathcal{M}), E_T(\mathcal{M})) \text{ where } V_T(\mathcal{M}) = \{q\omega | q \in Q, \omega \in Z^*, q\omega \text{ is } \epsilon - \text{ free}\}
\]

and

\[
E_T(\mathcal{M}) = \{(c, x, c') \in V_T(\mathcal{M}) \times V_T(\mathcal{M}) | c \xrightarrow{\epsilon}{\mathcal{M}} c'\}. \quad (10)
\]

We call **computation 1-graph** of the pda \( \mathcal{M} \), denoted \((C(\mathcal{M}), v_\mathcal{M})\), the subgraph of \( T(\mathcal{M}) \) induced by the set of vertices which are accessible from the vertex \( q_0z_0 \), together with the source \( v_\mathcal{M} = q_0z_0 \). In the case where \( \mathcal{M} \) is bi-rooted, we call **computation 2-graph** of the pda \( \mathcal{M} \), denoted \((C(\mathcal{M}), v_\mathcal{M}, \bar{v}_\mathcal{M})\), the graph \( C(\mathcal{M}) \) defined just above, together with the sources \( v_\mathcal{M} = q_0z_0, \bar{v}_\mathcal{M} = \bar{q} \).
Theorem 25 Let $\Gamma = (\Gamma_0, v_0)$ be a rooted 1-graph over $X$. The following conditions are equivalent:

1. $\Gamma$ is equational and has finite out-degree.
2. $\Gamma$ is isomorphic to the computation 1-graph $(C(M), v_M)$ of some normalized pushdown automaton $M$.

The formal proof of this theorem is quite technical and is omitted here. (See the annex for a sketch of proof).

Corollary 26 Let $\Gamma = (\Gamma_0, v_0, \bar{v})$ be a bi-rooted 2-graph over $X$. The following conditions are equivalent:

1. $\Gamma$ is equational and has finite out-degree.
2. $\Gamma$ is isomorphic to the computation 2-graph $(C(M), v_M, \bar{v}_M)$ of some bi-rooted normalized pushdown automaton $M$.

Bisimulation for non-deterministic (versus deterministic) graphs

In this paragraph, we reduce the classical notion of bisimulation for equational graphs to the notion of $\eta$-bisimulation for deterministic equational graphs, where $\eta$ has been suitably chosen (see definition 23).

Lemma 27 Let $\Gamma_1$ be some rooted equational 1-graph over a finite alphabet $Y_1$ and let $\#$ be a new letter $\# \not\in Y_1$. Then one can construct an equational bi-rooted 2-graph $\Gamma$ over the alphabet $Y = Y_1 \cup \{\#\}$ such that,

1. $V_{\Gamma_1} \subseteq V_{\Gamma}$,
2. for every $v, v' \in V_{\Gamma_1}$, $(v, v' \text{ are bisimilar in } \Gamma_1)$ iff $(v, v' \text{ are bisimilar in } \Gamma)$,
3. $\Gamma_1$ has finite out-degree iff $\Gamma$ has finite out-degree.

Sketch of proof: Let us define $\Gamma$ from $\Gamma_1$ by:

$$V_{\Gamma} = V_{\Gamma_1} \cup \{\bar{v}\}, \ E_{\Gamma} = E_{\Gamma_1} \cup \{(w, \#, \bar{v}) \mid w \in V_{\Gamma_1}\}, \ \Gamma = (\Gamma_1, \bar{v}),$$

where $\bar{v}$ is a new vertex $\bar{v} \not\in V_{\Gamma_1}$. One can easily check that $\Gamma$ is equational iff $\Gamma_1$ is equational and that, provided $\Gamma_1$ is rooted, $\Gamma$ is bi-rooted. Points (1) and (3) of the lemma are clear. One can check that the mapping $R \mapsto R \cup \{(\bar{v}, \bar{v})\}$ is a bijection from the set of all the bisimulations over $\Gamma_1$ (i.e. from $\Gamma_1$ to $\Gamma_1$) to the set of all the bisimulations over $\Gamma$. Hence point (2) is true. □

Let us consider finite alphabets $X, Y$, a length-preserving homomorphism $\psi : X^* \rightarrow Y^*$ and the s.r. morphism $\hat{\psi} = \psi \circ \psi^{-1} \subseteq X^* \times X^*$. A $n$-graph $\Gamma$ over $X$ will be said $\hat{\psi}$-saturated iff, for every $v \in V_{\Gamma}$, for every $(x, x') \in \hat{\psi}$,

$$(\exists v_1 \in V_{\Gamma}, (v, x, v_1) \in E_{\Gamma}) \Leftrightarrow (\exists v'_1 \in V_{\Gamma}, (v, x', v'_1) \in E_{\Gamma}).$$
Lemma 28  Let $\Gamma$ be an equational bi-rooted 2-graph of finite out-degree over an alphabet $Y$. One can construct a finite alphabet $X$, a surjective length-preserving homomorphism $\psi : X^* \to Y^*$ and an equational, bi-rooted 2-graph $\Gamma$ over the alphabet $X$, such that

1. $\Gamma$ is deterministic,
2. $\Gamma$ is $\bar{\psi}$-saturated,
3. $V_\Gamma = V_{\Gamma_1}$,
4. $\text{Id} : V_\Gamma \to V_{\Gamma_1}$ is a $\psi$-bisimulation from $\Gamma$ to $\Gamma_1$.

**Sketch of proof:** By lemma 26, we can suppose that $\Gamma_1$ is the computation 2-graph $(C(M_1), v_{M_1}, \bar{v}_{M_1})$ of some bi-rooted normalized pushdown automaton $M_1 =< Y, Z, Q, \delta_1, q_0, z_0, \{\bar{q}\} >$. Let us consider the following integers:

$\forall q \in Q, \forall z \in Z, \forall y \in Y, t_1(qz, y) = \text{Card}(\delta_1(qz, y)), \quad \bar{t}_1 = \max\{t_1(qz, y) \mid q \in Q, z \in Z, y \in Y\}$.

Let $X = Y \times [1, \bar{t}_1]$ and let $\psi : X \to Y$ be the first projection.
Let $\rho : QZ \times X \times \mathbb{N} \to QZ^*$ such that $\text{dom}(\rho) = \bigcup_{q \in Q, z \in Z, y \in Y} \{qz\} \times \{y\} \times [1, t_1(qz, y)]$ and

$\rho(qz, y, *) : \{qz\} \times \{y\} \times [1, t_1(qz, y)] \to \delta_1(qz, y)$

is a bijection (for every $q, z, y$). We then define $M =< X, Z, Q, \delta, q_0, z_0, \{\bar{q}\} >$ by: for every $q \in Q, z \in Z, y \in Y, i \in [1, \bar{t}_1]$

$\delta(qz, \epsilon) = \delta_1(qz, \epsilon)$ if $qz$ is $\epsilon$-bound.

$\delta(qz, (y, i)) = \{q' \omega'\}$ if $\rho(qz, y, i) = q' \omega'$ or $(1 \leq t_1(qz, y) < i \leq \bar{t}_1$ and $\rho(qz, y, 1) = q' \omega'$).

The 2-graph $\Gamma = (C(M), v_M, \bar{v}_M)$ fulfills the required properties. □

Let us remark that, by point (4) and by composition of $\eta$-bisimulations, for every $v, v' \in V_\Gamma$, $v, v'$ are $\psi$-bisimilar (w.r.t. $\Gamma$) iff $v, v'$ are bisimilar (w.r.t. $\Gamma_1$).

### 2.4 Deterministic context-free grammars

Let $M$ be some deterministic pushdown automaton (we suppose here that $M$ is normalized). The **variable alphabet** $V_M$ associated to $M$ is defined as:

$V_M = \{[p, z, q] \mid p, q \in Q, z \in Z\}$.

The **context-free grammar** $G_M$ associated to $M$ is then

$G_M =< X, V_M, P_M >$

where $P_M$ is the set of all the pairs of one of the following forms:

$([p, z, q], x[p', z_1, p''][p'', z_2, q])$ (11)
where \( p, q, p', p'' \in Q, x \in X, p'z_1z_2 \in \delta(pz, x) \)

\[
([p, z, q], x[p', z', q])
\] (12)

where \( p, q, p' \in Q, x \in X, p'z' \in \delta(pz, x) \)

\[
([p, z, q], a)
\] (13)

where \( p, q, \epsilon \in X \cup \{\epsilon\}, q \in \delta(pz, a) \). \( G_M \) is a strict-deterministic grammar (see definition ?? below). A general theory of this class of grammars is exposed in [Har78] and used in [HHY79].

### 2.5 Free monoids acting on semi-rings

**Semi-ring** \( \mathbb{B}\langle\langle W \rangle\rangle \) Let \((\mathbb{B}, +, 0, 1)\) where \( \mathbb{B} = \{0, 1\} \) denote the semi-ring of “booleans”. Let \( W \) be some alphabet. By \((\mathbb{B}\langle\langle W \rangle\rangle, +, \cdot, 0, 1)\) we denote the semi-ring of boolean series over \( W \): the sum and product are defined as usual; each word \( w \in W^* \) can be identified with the element of \( \mathbb{B}\langle\langle W \rangle\rangle \) mapping the word \( w \) on 1 and every other word \( w' \neq w \) on 0; every boolean series \( S \in \mathbb{B}\langle\langle W \rangle\rangle \) can then be written in a unique way as:

\[
S = \sum_{w \in W^*} S_w \cdot w,
\]

where, for every \( w \in W^* \), \( S_w \in \mathbb{B} \).

The **support** of \( S \) is the language

\[
\text{supp}(S) = \{ w \in W^* | S_w \neq 0 \}.
\]

In the particular case where the semi-ring of coefficients is \( \mathbb{B} \) (which is the only case considered in this article) we sometimes identify the series \( S \) with its support. A series \( S \in \mathbb{B}\langle\langle W \rangle\rangle \) is called a boolean polynomial over \( W \) if and only if its support is finite. The set of all boolean polynomials over \( W \) is denoted by \( \mathbb{B}(W) \).

The usual ordering \( \leq \) on \( \mathbb{B} \) extends to \( \mathbb{B}\langle\langle W \rangle\rangle \) by:

\[
S \leq S' \text{ iff } \forall w \in W^*, S_w \leq S'_w.
\]

We recall that for every \( S \in \mathbb{B}\langle\langle W \rangle\rangle \), \( S^* \) is the series defined by:

\[
S^* = \sum_{0 \leq n} S^n.
\] (14)

Given two alphabets \( W, W' \), a map \( \psi : \mathbb{B}\langle\langle W \rangle\rangle \to \mathbb{B}\langle\langle W' \rangle\rangle \) is said \( \sigma \)-additive iff it fulfills: for every denumerable family \( (S_i)_{i \in \mathbb{N}} \) of elements of \( \mathbb{B}\langle\langle W \rangle\rangle \),

\[
\psi\left( \sum_{i \in \mathbb{N}} S_i \right) = \sum_{i \in \mathbb{N}} \psi(S_i).
\] (15)

A map \( \psi : \mathbb{B}\langle\langle W \rangle\rangle \to \mathbb{B}\langle\langle W' \rangle\rangle \) which is both a semi-ring homomorphism and a \( \sigma \)-additive map is usually called a substitution.
**Actions of monoids** Given a semi-ring \((S, +, \cdot, 0, 1)\) and a monoid \((M, \cdot, 1_M)\), a map \(\circ : S \times M \to S\) is called a *right-action* of the monoid \(M\) over the semi-ring \(S\) iff, for every \(S, T \in S, m, m' \in M\):

\[
0 \circ m = 0, \quad S \circ 1_M = S, \quad (S + T) \circ m = (S \circ m) + (T \circ m) \quad \text{and} \quad S \circ (m \cdot m') = (S \circ m) \cdot (S \circ m').
\]

(16)

In the particular case where \(S = B(\langle W \rangle)\), \(\circ\) is said to be a \(\sigma\)-right-action if it fulfills the additional property that, for every denumerable family \((S_i)_{i \in \mathbb{N}}\) of elements of \(S\) and \(m \in M\):

\[
(\sum_{i \in \mathbb{N}} S_i) \circ m = \sum_{i \in \mathbb{N}} (S_i \circ m).
\]

(17)

**The action of \(W^*\) on \(B(\langle W \rangle)\)** We recall the following classical \(\sigma\)-right-action \(\cdot\) of the monoid \(W^*\) over the semi-ring \(B(\langle W \rangle)\): for all \(S, S' \in B(\langle W \rangle), u \in W^*\)

\[
S \cdot u = S' \iff \forall w \in W^*, (S_w' = S_{u \cdot w}),
\]

(i.e. \(S \cdot u\) is the *left-quotient* of \(S\) by \(u\), or the *residual* of \(S\) by \(u\)).

For every \(S \in B(\langle W \rangle)\) we denote by \(Q(S)\) the set of residuals of \(S\):

\[
Q(S) = \{S \cdot u \mid u \in W^*\}.
\]

We recall that \(S\) is said rational iff the set \(Q(S)\) is finite. We define the *norm* of a series \(S \in B(\langle W \rangle)\), denoted \(\|S\|\) by:

\[
\|S\| = \text{Card}(Q(S)) \in \mathbb{N} \cup \{\infty\}.
\]

**The reduced grammar \(G\)** The classical reduced and \(\epsilon\)-free grammar associated with \(G_M\) is \(G_0 = <X, V_0, P_0>\) where:

\[
V_0 = \{v \in V_M \mid \exists w \in X^+, v \rightarrow^*_{P_M} w\},
\]

(18)

\[
\varphi_0 : B(\langle V \rangle) \to B(\langle V_0 \rangle)
\]

is the unique substitution such that, for every \(v \in V\):

\[
\varphi_0(v) = v \ (\text{if} \ v \in V_0), \quad \varphi_0(v) = \epsilon \ (\text{if} \ v \rightarrow^*_{P_M} \epsilon), \quad \varphi_0(v) = \emptyset \ (\text{otherwise}),
\]

(19)

\[
P_0 = \{(v, w') \in V_0 \times (X \cup V_0)^+ \mid \exists v \in V_0, \exists w \in (X \cup V_M)^*, (v, w) \in P_M, w' = \varphi_0(w)\}.
\]

\(G_0\) is the *reduced* and \(\epsilon\)-free form of \(G_M\). It is well-known that, for all \(v \in V_0\):

\[
\exists w \in X^+, v \rightarrow^*_{P_0} w \quad \text{and} \quad \{w \in X^*, v \rightarrow^*_{P_M} w\} = \{w \in X^*, v \rightarrow^*_{P_0} w\}.
\]

For technical reasons (which will be made clear in section [5]), we introduce an alphabet of “marked variables” \(\bar{V}_0\) together with a fixed bijection: \(v \mapsto \bar{v}\) from
$V_0$ to $\bar{V}_0$. Let $V = V_0 \cup \bar{V}_0$. We denote by $\rho_e$ (letter $e$ stands here for “erasing the marks”) the literal morphism $V^* \to V_0^*$ defined by: for every $v \in V_0$,

$$\rho_e(v) = v, \quad \rho_e(\bar{v}) = v.$$ 

Similarly, $\bar{\rho}_e$ is the literal morphism $V^* \to \bar{V}_0^*$ defined by: for every $v \in V_0$,

$$\bar{\rho}_e(v) = \bar{v}, \quad \bar{\rho}_e(\bar{v}) = \bar{v}.$$ 

We denote also by $\rho_e, \bar{\rho}_e$ the unique substitutions extending these monoid homomorphisms.

At last, the grammar $G$ is defined by, $G = \langle X, V, P \rangle$ where

$$P = P_0 \cup \{(\bar{\rho}_e(v), \bar{\rho}_e(w)) \mid (v, w) \in P_0\}.$$ 

In other words, the rules of $G$ consist of the rules of the usual proper and reduced grammar associated with $M$ together with their marked copies.

The action of $X^*$ on $B\langle\langle V \rangle\rangle$ Let us fix now a deterministic (normalized) pda $M$ and consider the associated grammar $G$. We define a $\sigma$-right-action $\odot$ of the monoid $X^*$ over the semi-ring $B\langle\langle V \rangle\rangle$ by: for every $v \in V, \beta \in V^*, x \in X$

$$(v \cdot \beta) \odot x = (\sum_{(v, h) \in P} h \bullet x) \cdot \beta,$$ 

$$\epsilon \odot x = \emptyset.$$ 

Let us consider the unique substitution $\varphi : B\langle\langle V \rangle\rangle \to B\langle\langle X \rangle\rangle$ fulfilling: for every $v \in V$,

$$\varphi(v) = \{u \in X^* \mid v \xrightarrow{\ast} P u\},$$

(in other words, $\varphi$ maps every subset $L \subseteq V^*$ on the language generated by the grammar $G$ from the set of axioms $L$).

Lemma 29 For every $S \in B\langle\langle V \rangle\rangle, u \in X^*, \varphi(S \odot u) = \varphi(S) \bullet u$ (i.e. $\varphi$ is a morphism of right-actions).

Proof: Let $v \in V, \beta \in V^*, x \in X$. Recall that $G$ is in Greibach normal form (i.e. $P \subseteq V \times X \cdot V^*$). One can then check on formulas (??) that:

$$\varphi(\epsilon \odot x) = \varphi(\epsilon) \bullet x \quad \text{and} \quad \varphi((v \cdot \beta) \odot x) = \varphi(v \cdot \beta) \bullet x.$$ 

By induction on $|w|$, it follows that, $\forall w \in V^*$,

$$\varphi(w \odot x) = \varphi(w) \bullet x.$$ 

By $\sigma$-additivity of $\varphi$, it follows that, $\forall S \in B\langle\langle V \rangle\rangle$,

$$\varphi(S \odot x) = \varphi(S) \bullet x.$$
By induction on $u$, it follows that, $\forall u \in X^*$,

$$\varphi(S \circ u) = \varphi(S) \bullet u.$$ 

$\Box$

We denote by $\equiv$ the kernel of $\varphi$ i.e.: for every $S, T \in B\langle V \rangle$,

$$S \equiv T \iff \varphi(S) = \varphi(T).$$
3 Series and matrices

3.1 Deterministic series, vectors and matrices

We introduce here a notion of deterministic series which, in the case of the alphabet \( V \) associated to a dpda \( M \), generalizes the classical notion of configuration of \( M \). The main advantage of this notion is that, unlike for configurations, we shall be able to define nice algebraic operations on these series (see, in particular, §3.3). Let us consider a pair \((W,\sim)\) where \( W \) is an alphabet and \( \sim \) is an equivalence relation over \( W \). We call \((W,\sim)\) a structured alphabet. The two examples we have in mind are:

- the case where \( W = V_M \), the variable alphabet associated to \( M \) and \([p, z, q] \sim [p', z', q']\) iff \( p = p' \) and \( z = z' \) (see [Har78])
- the case where \( W = X \), the terminal alphabet of \( M \) and \( x \sim y \) holds for every \( x, y \in X \) (see [Har78]).

Definitions

Definition 31 Let \( S \in B\langle\langle W \rangle\rangle \). \( S \) is said left-deterministic iff either

(1) \( S = \emptyset \) or
(2) \( S = \epsilon \) or
(3) \( \exists i_0 \in [1, m], S_{i_0} \neq \emptyset \) and \( \forall w, w' \in W^* \),

\[ S_w = S_{w'} = 1 \Rightarrow \exists A, A' \in W, w_1, w_1' \in W^*, A \sim A', w = A \cdot w_1 \text{ and } w' = A' \cdot w_1' \].

A left-deterministic series \( S \) is said to have the type \( \emptyset \) (resp. \( \epsilon \), \([A]_{\sim}\)) if case (1) (resp. (2), (3)) occurs.

Definition 32 Let \( S \in B\langle\langle W \rangle\rangle \). \( S \) is said deterministic iff, for every \( u \in W^* \), \( S \cdot u \) is left-deterministic.

This notion is the straightforward extension to the infinite case of the notion of (finite) set of associates defined in [HY79, definition 3.2 p. 188].

We denote by \( DB\langle\langle W \rangle\rangle \) the subset of deterministic boolean series over \( W \). Let us denote by \( B_{n,m}\langle\langle W \rangle\rangle \) the set of \((n,m)\)-matrices with entries in the semi-ring \( B\langle\langle W \rangle\rangle \).

Definition 33 Let \( m \in \mathbb{N}, S \in B_{1,m}\langle\langle W \rangle\rangle \) : \( S = (S_1, \cdots, S_m) \). \( S \) is said left-deterministic iff either

(1) \( \forall i \in [1, m], S_i = \emptyset \) or
(2) \( \exists i_0 \in [1, m], S_{i_0} = \epsilon \) and \( \forall i \neq i_0, S_i = \emptyset \) or
(3) \( \forall w, w' \in W^*, \forall i, j \in [1, m], (S_i)_w = (S_j)_{w'} = 1 \Rightarrow \exists A, A' \in W, w_1, w_1' \in W^*, A \sim A', w = A \cdot w_1 \text{ and } w' = A' \cdot w_1' \).
A left-deterministic row-vector $S$ is said to have the type $\emptyset$ (resp. $(\epsilon,i_0), [A]_-$) if case (1) (resp. (2), (3)) occurs.

The right-action $\cdot$ on $B\langle\langle W \rangle\rangle$ is extended componentwise to $B_{n,m}\langle\langle W \rangle\rangle$: for every $S = (s_{i,j}), u \in W^*$, the matrix $T = S \cdot u$ is defined by

$$t_{i,j} = s_{i,j} \cdot u.$$ 

The ordering $\leq$ on $B$ is also extended componentwise to $B_{n,m}\langle\langle W \rangle\rangle$.

**Definition 34** Let $S \in B_{1,m}\langle\langle W \rangle\rangle$. $S$ is said deterministic iff, for every $u \in W^*$, $S \cdot u$ is left-deterministic.

We denote by $DB_{1,m}\langle\langle W \rangle\rangle$ the subset of deterministic row-vectors of dimension $m$ over $B\langle\langle W \rangle\rangle$.

**Definition 35** Let $S \in B_{n,m}\langle\langle W \rangle\rangle$. $S$ is said deterministic iff, for every $i \in [1,n], S_{i,\cdot}$ is a deterministic row-vector.

Let us notice first some easy facts about deterministic matrices.

**Fact 36** Let $S \in DB\langle\langle W \rangle\rangle$. For every $T \in B\langle\langle W \rangle\rangle, u \in W^*$

1. $T \leq S \Rightarrow T \in DB\langle\langle W \rangle\rangle$
2. $T = S \cdot u \Rightarrow T \in DB\langle\langle W \rangle\rangle$

**Norm** Let us generalize the classical definition of rationality of series in $B\langle\langle W \rangle\rangle$ to matrices. Given $M \in B_{n,m}\langle\langle W \rangle\rangle$ we denote by $Q(M)$ the set of residuals of $M$:

$$Q(M) = \{M \cdot u \mid u \in W^*\}.$$ 

Similarly, we denote by $Q_r(M)$ the set of row-residuals of $M$:

$$Q_r(M) = \bigcup_{1 \leq i \leq n} Q(M_{i,\cdot}).$$

$M$ is said rational iff the set $Q(M)$ is finite. One can check that it is equivalent to the property that every coefficient $M_{i,j}$ is rational, or to the property that $Q_r(M)$ is finite. We denote by $RB_{n,m}\langle\langle W \rangle\rangle$ (resp. $DRB_{n,m}\langle\langle W \rangle\rangle$) the set of rational (resp. deterministic, rational) matrices over $B\langle\langle W \rangle\rangle$. For every $M \in RB_{n,m}\langle\langle W \rangle\rangle$, we define the norm of $M$ as:

$$\|M\| = \text{Card}(Q_r(M)).$$
Grammars

**Definition 37** Let $G = \langle X, V, P \rangle$ be a context-free grammar in Greibach normal form. $G$ is said strict-deterministic iff there exists an equivalence relation $\sim$ over $V$ fulfilling the following condition: for every $E \in V, x \in X$, if $(E_k)_{1 \leq k \leq m}$ is a bijection $[1, m] \rightarrow [E]_\sim$, and $H_k = \sum_{(E_k, h) \in P} h \cdot x$, then

$$(H_1, H_2, \ldots, H_m)$$ is a deterministic vector.

Any equivalence $\sim$ satisfying the above condition is said to be a strict equivalence for the grammar $G$.

This definition is a reformulation of [Har78, Definition 11.4.1 p.347] adapted to the case of a Greibach normal-form.

**Theorem 38** Let $G_1 = \langle X, V_1, P_1 \rangle$ be a strict-deterministic grammar. Then its reduced form $G_0 = \langle X, V_0, P_0 \rangle$, as defined in formulas (18, 19), is strict-deterministic too. Moreover, if $\sim$ is a strict equivalence for $G_1$, its restriction over $V_0$ is a strict equivalence for $G_0$.

The proof would consist in slightly extending the proof of [Har78, Theorem 11.4.1 p.350].

It is known that, given a dpda $M$, its associated grammar $G_M$ is strict-deterministic. By theorem $G_0$ is strict-deterministic too. Let us consider the minimal strict equivalence $\sim$ for $G_0$ and extend it to $V$ by, $\forall v, v' \in V_0$:

$$v \sim v' \iff v \sim v'; \quad \bar{v} \neq v'. $$

Then $\sim$ is a strict equivalence for $G$ (the grammar $G$ is defined in §2.3). This ensures that $G$ is strict-deterministic.

**Residuals**

**Lemma 39** Let $S \in \text{DB}(\langle W \rangle), T \in \text{B}(\langle W \rangle), u \in W^*$. If $S \bullet u \neq \emptyset$ then $(S \cdot T) \bullet u = (S \bullet u) \cdot T$.

**Proof:** Let $S \in \text{DB}(\langle W \rangle), T \in \text{B}(\langle W \rangle), u \in W^*$, such that $S \bullet u \neq \emptyset$. Let $u', u'' \in W^*$ such that $u = u' \cdot u'', u'' \neq \epsilon$ and let $w \in \text{supp}(S)$. If $w \bullet u' = \epsilon$ then $S \bullet u' = \epsilon$ (because $S \bullet u'$ is left-deterministic), hence $S \bullet u = \epsilon \bullet u'' = \emptyset$, which would contradict the hypothesis. It follows that

$$\forall u' \prec u, \forall w \in \text{supp}(S), w \bullet u' \neq \epsilon.$$ 

Hence

$$\forall w_1 \in \text{supp}(S), \forall w_2 \in \text{supp}(T), (w_1 \cdot w_2) \bullet u = (w_1 \bullet u) \cdot w_2.$$ 

This proves that $(S \cdot T) \bullet u = (S \bullet u) \cdot T$. □
Lemma 310  Let $S \in \mathcal{DB}(\langle W \rangle), T \in \mathcal{B}(\langle W \rangle), u \in W^*$ and $U = S \cdot T$. Exactly one of the following cases is true:

1. $S \cdot u \neq \emptyset$;
   in this case $U \cdot u = (S \cdot u) \cdot T$.
2. $S \cdot u = \emptyset, \exists u', u''$, $u = u' \cdot u''$, $S \cdot u' = \epsilon$;
   in this case $U \cdot u = T \cdot u''$.
3. $S \cdot u = \emptyset$, $\forall u' \preceq u, S \cdot u' \neq \epsilon$;
   in this case $U \cdot u = \emptyset = (S \cdot u) \cdot T$.

Proof:  Clearly, one of the hypotheses (1-3) must occur. Let us examine each one of these cases.

In case (1), by lemma 39, $U \cdot u = (S \cdot u) \cdot T$.
In case (2), $U \cdot u = (U \cdot u') \cdot u''$ and by case (1), $U \cdot u' = (S \cdot u') \cdot T$. It follows that $U \cdot u = T \cdot u''$.
In case (3), if $S = \emptyset$, the conclusion of the lemma is clearly true. Let us suppose now that $S \neq \emptyset$ and let $u' \prec u$ be the maximum prefix of $u$ such that $S \cdot u' \neq \emptyset$.
Then, there exist some $A \in W, u'' \in W^*$ such that $u = u' \cdot A \cdot u''$ and there exist some $B_1, \ldots, B_q \in W, S_1, \ldots, S_q \in \mathcal{B}(\langle W \rangle) - \{\emptyset\}$ such that $S \cdot u' = \sum_{1 \leq i \leq q} B_q \cdot S_q$ and $B_1 \prec \cdots \prec B_i \prec \cdots \prec B_q$ (because $S \cdot u'$ is left-deterministic). By maximality of $u'$, $A$ does not belong to $\{B_1, \ldots, B_q\}$, hence

$$U \cdot u = ((\sum_{1 \leq i \leq q} B_i \cdot S_i \cdot T) \cdot A) \cdot u'' = \emptyset \cdot u'' = \emptyset.$$  \hfill\qed

Lemma 311  Let $S \in \mathcal{DB}_{1,m}(\langle W \rangle), T \in \mathcal{B}_{m,1}(\langle W \rangle), u \in W^*$ and $U = S \cdot T$. Exactly one of the following cases is true:

1. $\exists j, S_j \cdot u \notin \{\emptyset, \epsilon\}$;
   in this case $U \cdot u = (S \cdot u) \cdot T$.
2. $\exists j_0, \exists u', u'' \ni u = u' \cdot u''$, $S_{j_0} \cdot u' = \epsilon$;
   in this case $U \cdot u = T_{j_0} \cdot u''$.
3. $\forall j, S_j \cdot u = \emptyset$,
   $\forall u' \preceq u, S_j \cdot u' \neq \epsilon$;
   in this case $U \cdot u = \emptyset = (S \cdot u) \cdot T$.

Proof:  Let us note $S = (S_j)_{1 \leq j \leq m}, T = (T_j)_{1 \leq j \leq m}$. Clearly, one of the hypotheses (1-3) must occur. Let us examine each one of these cases.

In case (1), every 3-tuple $(S_j, T_j, u)$ fulfills case (1) or (3) of lemma 310, hence $(S_j \cdot T_j) \cdot u = (S_j \cdot u) \cdot T_j$. Hence

$$U \cdot u = \sum_{1 \leq j \leq m} (S_j \cdot T_j) \cdot u = \sum_{1 \leq j \leq m} (S_j \cdot u) \cdot T_j = (S \cdot u) \cdot T.$$  \hfill\qed

In case (2), $S \cdot u'$ must be left-deterministic of type $(\epsilon, j_0)$, hence $\forall j \neq j_0, S_j \cdot u' = \emptyset$. It follows that

$$U \cdot u = T_{j_0} \cdot u'.$$
In case (3), every 3-tuple \((S_j, T_j, u)\) fulfills case (3) of lemma 310, hence \((S_j \cdot T_j) \cdot u = \emptyset = (S_j \cdot u) \cdot T_j\). It follows that

\[ U \cdot u = \emptyset = (S \cdot u) \cdot T. \]

\[ \square \]

**Lemma 312** Let \(S \in \mathbb{B}_{1,m}(\langle W \rangle), T \in \mathbb{B}_{m,s}(\langle W \rangle), u \in W^*\) and \(U = S \cdot T\). Exactly one of the following cases is true:

1. \(\exists j, S_j \cdot u \notin \{\emptyset, \epsilon\}\) in this case \(U \cdot u = (S \cdot u) \cdot T\).
2. \(\exists j_0, \exists u', u'', u = u' \cdot u'', S_{j_0} \cdot u' = \epsilon\); in this case \(U \cdot u = T_{j_0} \cdot u''\).
3. \(\forall j, \forall u' \leq u, S_j \cdot u = \emptyset, S_j \cdot u' \neq \epsilon\); in this case \(U \cdot u = \emptyset = (S \cdot u) \cdot T\).

**Proof:** Let us notice that for every \(k \in [1,s]\):

\[ U_k = S \cdot T_{*,k}, \]

and that the hypothesis of the 3 cases considered in lemma 311 depend on the vector \(S\) and the word \(u\) only (but not on the integer \(k \in [1,s]\)). In case (1), by lemma 311, \(\forall k \in [1,s] U_k \cdot u = (S \cdot u) \cdot T_{*,k}\), hence \(U \cdot u = (S \cdot u) \cdot T\). Cases 2,3 can be treated in the same way. \(\square\)

**Lemma 313** For every \(S \in \mathbb{B}_{n,m}(\langle W \rangle), T \in \mathbb{B}_{m,s}(\langle W \rangle),\) if \(S\) and \(T\) are both left-deterministic, then \(S \cdot T\) is left-deterministic.

**Lemma 314** For every \(S \in \mathbb{DB}_{n,m}(\langle W \rangle), T \in \mathbb{B}_{m,s}(\langle W \rangle), S \cdot T \in \mathbb{DB}_{n,s}(\langle W \rangle)\).

**Proof:** As the notion of deterministic matrix is defined row by row, it is sufficient to prove this lemma in the particular case where \(n = 1\). Let us note \(U = S \cdot T\). Let \(u \in W^*\). Let us show that \(U \cdot u\) is left-deterministic. Let us consider every one of the 3 cases considered in lemma 312. In case (1) or (3),

\[ U \cdot u = (S \cdot u) \cdot T, \]

and in case (2),

\[ U \cdot u = T \cdot u''. \]

In both cases, by lemma 313 \(U \cdot u\) is left-deterministic. \(\square\)
Lemma 315 Let $A \in DB_{n,m}\langle\langle W \rangle\rangle, B \in B_{m,s}\langle\langle W \rangle\rangle$. Then $\|A \cdot B\| \leq \|A\| + \|B\|$.

**Proof:** Let $A = (a_{i,k}), B = (b_{k,j}), C = A \cdot B, C = (c_{i,j})$. Let $1 \leq i \leq n, H \in Q(C_{i,*})$. Let $u \in W^*$ such that

$$H = C_{i,*} \bullet u = (A_{i,*} \cdot B) \bullet u.$$  

We apply lemma 312 to $S = A_{i,*}$ and $T = B$. If case (1) or (3) of lemma 312 is realized then

$$H = (A_{i,*} \cdot u) \cdot B.$$  

If case (2) of lemma 312 is realized then

$$H = B_{k_0,*} \bullet u''.$$  

The number of residuals $H$ obtained by case (1) is less or equal than $\|A\|$ and the number obtained by case (2) is less or equal than $\|B\|$. This proves the inequality. $\Box$

\[W=V\]  

Let $(W, \sim)$ be the structured alphabet $(V, \sim)$ associated with $M$ and let us consider a bijective numbering of the elements of $Q$: $(q_1, q_2, \ldots, q_{n_Q})$. Let us define here handful notations for some particular vectors or matrices. Let us use the Kronecker symbol $\delta_{i,j}$ meaning $\epsilon$ if $i = j$ and $\emptyset$ if $i \neq j$. For every $1 \leq n, 1 \leq i \leq n$, we define the row-vector $\epsilon^n_i$ as:

$$\epsilon^n_i = (\epsilon^n_{i,j})_{1 \leq j \leq n} \text{ where } \forall j, \epsilon^n_{i,j} = \delta_{i,j}.$$  

We call unit row-vector any vector of the form $\epsilon^n_i$.

For every $1 \leq n$, we denote by $\emptyset^n \in DB_{1,n}\langle\langle V \rangle\rangle$ the row-vector:

$$\emptyset^n = (0, \ldots, 0).$$  

For every $\omega \in Z^*, p, q \in Q$, $[p \omega q]$ is the deterministic series defined inductively by:

$$[p \omega q] = \emptyset \text{ if } p \neq q, [p \omega q] = \epsilon \text{ if } p = q,$$

$$[p \omega q] = \sum_{r \in Q} [p, z, r] \cdot [r \omega' q] \text{ if } \omega = z \cdot \omega' \text{ for some } z \in Z, \omega' \in Z^*.$$  

Let us define

$$K_0 = \max\{\|E_1, E_2, \ldots, E_n\) \circ x\| \mid (E_i)_{1 \leq i \leq n} \text{ is a bijective numbering of some class in } V/\sim, x \in X\}. \quad (23)$$

Lemma 316 For every $S \in DB_{1,\lambda}\langle\langle V \rangle\rangle, u \in X^*$,
\( (1) \ S \odot u \in \text{DB}_{1,\lambda}\langle\langle V \rangle\rangle \)

\( (2) \ \|S \odot u\| \leq \|S\| + K_0 \cdot |u| \).

**Proof:** We treat first the case where \( u \) is just a letter.

Let \( S \in \text{DB}_{1,\lambda}\langle\langle V \rangle\rangle \) and \( x \in X \). If \( S = \emptyset^\lambda \) or \( S = \varepsilon^\lambda_j \) (for some \( j \in [1, \lambda] \)), then \( S \odot x = \emptyset^\lambda \) and points (1)(2) are both true.

Otherwise

\[
S = \sum_{k=1}^{q} E_k \cdot \Phi_k
\]

for some \( q \in \mathbb{N}, \Phi_k \in \text{DB}_{1,\lambda}\langle\langle V \rangle\rangle, (E_k)_{1 \leq k \leq q} \) bijective numbering of some class of \( V/\sim \).

By equation (20), which defines the right-action \( \odot \),

\[
S \odot x = \sum_{k=1}^{q} (E_k \odot x) \cdot \Phi_k,
\]

hence \( S \odot x \) has the form \( H \cdot \Phi \) where \( H \in \text{DB}_{1,q}\langle\langle V \rangle\rangle \) (see definition 37), \( \|H\| \leq K_0 \) (see inequation (23)) and \( \Phi \in \text{DB}_{q,\lambda}\langle\langle V \rangle\rangle \).

By lemma 314, \( H \cdot \Phi \) is deterministic and by lemma 315 \( \|H \cdot \Phi\| \leq \|\Phi\| + K_0 \).

As every \( \Phi_k \in Q_r(S) \) we obtain:

\[
\|H \cdot \Phi\| \leq \|\Phi\| + K_0 \leq \|S\| + K_0.
\]

Both points (1)(2) are proved.

The general case where \( u \) is any word of \( X^* \) can be deduced by induction on \( |u| \) from this particular case. \( \square \)

**Lemma 317** Let \( \lambda \in \mathbb{N} - \{0\}, S \in \text{DB}_{1,\lambda}\langle\langle V \rangle\rangle, u \in X^* \). One of the three following cases must occur:

1. \( S \odot u = \emptyset^\lambda \),
2. \( S \odot u = \varepsilon^\lambda_j \) for some \( j \in [1, \lambda] \),
3. \( \exists u_1, u_2 \in X^*, v_1 \in V^*, q \in \mathbb{N}, E_1, \ldots, E_k, \ldots, E_q \in V, \Phi \in \text{DB}_{q,\lambda}\langle\langle V \rangle\rangle \) such that

\[
u = u_1 \cdot u_2, S \odot u_1 = S \cdot v_1 = \sum_{k=1}^{q} E_k \cdot \Phi_k, \ S \odot u = \sum_{k=1}^{q} (E_k \odot u_2) \cdot \Phi_k, \text{ and} \]

\[
\forall k \in [1, q], E_k \sim E_1, E_k \odot u_2 \notin \{\varepsilon, \emptyset\}.
\]

**Proof:** Let \( u \in X^* \). Let us prove the lemma by induction on \( |u| \).

\( u = \varepsilon \):

if \( S \in \emptyset^\lambda \cup \{\varepsilon^\lambda_j | 1 \leq j \leq \lambda\} \) then clearly the conclusion of case (1) or (2) is realized.

Otherwise, as \( S \) is left-deterministic, \( S \) has a decomposition as \( S = \sum_{k=1}^{q} E_k \cdot \Phi_k \) such that the conclusion of case (3) is realized with \( u_1 = u_2 = \varepsilon, v_1 = \varepsilon \), the given
integer $q$ and the letters $E_1 \cdots E_q \in V$.

Let us consider the $u_1, u_2, v_1, q, (E_k)_{1 \leq k \leq q}, (\Phi_k)_{1 \leq k \leq q}$ given by the induction hypothesis on $u_0$.

$$(S \odot u) \circ a = \left( \sum_{k=1}^{q} (E_k \odot u_2) \odot \Phi_k \right) \circ a \quad \text{and} \quad \forall k \in [1, q], \|E_k \odot u_2\| \geq 3.$$ 

**Case 1:** \(\forall k \in [1, q], \|E_k \odot u_2\| \geq 3.\)

Then $S \odot u a = \sum_{k=1}^{q} (E_k \odot u_2) \cdot \Phi_k$. Hence conclusion (3) of the lemma is fulfilled by $u_1' = u_1, u_2' = u_2 a, v_1' = v_1, q' = q, E_k' = E_k, \Phi_k' = \Phi_k$.

**Subcase 1:** $\Phi_r = \{\emptyset \} \cup \{\epsilon_j \} 1 \leq j \leq \lambda$.

Conclusion (1) or (2) of the lemma is then realized.

**Subcase 2:** $\Phi_r = \sum_{\ell=1}^{r'} F_\ell \cdot \Psi_\ell$ for some $r' \in \mathbb{N}, F_1 \cdots F_{r'} \in V, \Psi \in \text{DRB}_{r'}(\{ V \})$.

Then

$$S \odot u a = \sum_{\ell=1}^{r'} F_\ell \cdot \Psi_\ell; \quad S \bullet (v_1 E_r) = \Phi_r = \sum_{\ell=1}^{r'} F_\ell \cdot \Psi_\ell.$$ 

Conclusion (3) of the lemma is then realized by $u_1' = u a, u_2' = \epsilon, v_1' = v_1 E_r, q' = r', E_k' = F_k, \Phi' = \Psi$.

**Case 3:** \(\forall k \in [1, q], \|E_k \odot u_2\| = 1.\)

This means that $E \odot u_2 a = \emptyset$, hence that case (1) is realized. \(\Box\)

We give now an adaptation of lemma 312 to the action $\odot$ in place of $\bullet$.

**Lemma 318** Let $S \in \text{DB}_{1,m}(\{ V \}), T \in \mathcal{B}_{m,s}(\{ V \}), u \in X^*$ and $U = S \cdot T$. Exactly one of the following cases is true:

1. $S \odot u \not\in \{0^m\} \cup \{\epsilon_j^m | 1 \leq j \leq m\}$
   - in this case $U \odot u = (S \odot u) \cdot T$.
2. $\exists j_0, \exists u', u''$, $u = u' \cdot u'', S \odot u' = \epsilon^m_{j_0}$;
   - in this case $U \odot u = T_{j_0} \odot u''$.
3. $\forall j, \forall u' \not\preceq u, S \odot u = \emptyset^m$ and $S \odot u' \not\preceq \epsilon^m_j$;
   - in this case $U \odot u = \emptyset^s = (S \odot u) \cdot T$.

**Proof:** The arguments used in the proofs of lemma 31, 311, 311, 312 can be adapted to $\odot$ in place of $\bullet$. The only non-trivial adaptation is that of lines 6-7 of the proof of lemma 31: let us suppose that $u \in X^*$ is such that

$$\forall u' \prec u, S \odot u' \not\preceq \epsilon,$$ 

(24)
and let us prove that
\[(S \cdot T) \circ u = (S \circ u) \cdot T.\] (25)

We prove by induction on \(|u|\) that (24) implies (25).

\(|u| = 0\): by definition of a right-action, \(\forall S' \in DB\langle V \rangle, S' \circ \epsilon = S'\). Hence conclusion (25) is true.

\(u = u_0 \cdot a\), where \(u_0 \in X^*, a \in X\):
Hypothesis (24) is fulfilled by \(u_0\) too, hence, by induction hypothesis,
\[(S \cdot T) \circ u_0 = (S \circ u_0) \cdot T.\] (26)

If \(S \circ u_0 = \emptyset\), then , by the above equality \((S \cdot T) \circ u_0 = \emptyset\) too, hence
\[(S \cdot T) \circ u_0 a = \emptyset = (S \circ u_0 a) \cdot T,\]
hence (25) is true.

Otherwise, by hypothesis (24) \(S \circ u_0 \notin \{\emptyset, \epsilon\}\), hence there exists \(q \in \mathbb{N}, E_1 \leftarrow \ldots \leftarrow E_q \in V, \Phi \in DB_{m,s}\langle V \rangle\) such that
\[S \circ u_0 = \sum_{k=1}^{q} E_k \cdot \Phi_k.\] (27)

By definition (20) and the fact that \(\circ\) is a \(\sigma\)-action:
\[(E_k \cdot \Phi_k) \circ a = (E_k \circ a) \cdot \Phi_k,\]

hence, by \(\sigma\)-additivity ,
\[\left(\sum_{k=1}^{q} E_k \cdot \Phi_k\right) \circ a = \sum_{k=1}^{q} (E_k \circ a) \cdot \Phi_k\]
and by product by \(T\):
\[(S \circ u_0 a) \cdot T = \sum_{k=1}^{q} (E_k \circ a) \cdot \Phi_k \cdot T.\] (28)

Let us examine now \((ST) \circ u_0 a\). By (24):
\[(S \cdot T) \circ u_0 = \sum_{k=1}^{q} E_k \cdot \Phi_k \cdot T.\] (29)

By definition (20) and the fact that \(\circ\) is a \(\sigma\)-action:
\[(E_k \cdot \Phi_k \cdot T) \circ a = (E_k \circ a) \cdot \Phi_k \cdot T,\]

hence, by \(\sigma\)-additivity ,
\[\left(\sum_{k=1}^{q} E_k \cdot \Phi_k \cdot T\right) \circ a = \sum_{k=1}^{q} (E_k \circ a) \cdot \Phi_k \cdot T\]
Using (29) this last equality can be read:

\[(ST) \odot u_0 a = \sum_{k=1}^{q} (E_k \odot a) \cdot \Phi_k \cdot T.\]  

(30)

As equalities (30), (28) have the same right-hand-side, we conclude that (25) is true. □

**Marks** A word \(w \in V^*\) is said **marked** iff \(w \in V^* \cdot \bar{V}_0 \cdot V^*\); it is said **fully marked** iff \(w \in \bar{V}_0^*\).

A series \(S \in \mathbb{B}\langle\langle V \rangle\rangle\) is said **marked** iff \(\exists w \in \text{supp}(S), w \) is marked; it is said **fully marked** iff \(\forall w \in \text{supp}(S), w \) is fully marked. It is said **unmarked** iff it is not marked. A matrix \(S \in \mathbb{B}_{m,n}\langle\langle V \rangle\rangle\) is said marked (resp. fully marked, unmarked) iff, for every \(i \in [1, m]\), the series \(\sum_{j=1}^{n} S_{i,j}\) is marked (resp. fully marked, unmarked).

**Definition 319** Let \(d \in \mathbb{N}\). A vector \(S \in \mathbb{D}B_{1,\lambda}\langle\langle V \rangle\rangle\) is said **\(d\)-marked** iff there exists \(q \in \mathbb{N}, \alpha \in \mathbb{D}RB_{1,q}\langle V \rangle, \Phi \in \mathbb{D}RB_{q,\lambda}\langle\langle V \rangle\rangle\) such that

\[S = \sum_{k=1}^{q} \alpha_k \cdot \Phi_k \text{ and } \|\alpha\| \leq d,\]

and \(\Phi\) is unmarked.

**Lemma 320** For every \(S \in \mathbb{D}B_{1,\lambda}\langle\langle V \rangle\rangle\)

1. \(\rho_{e}(S) \in \mathbb{D}B_{1,\lambda}\langle\langle V \rangle\rangle\)
2. \(\|\rho_{e}(S)\| \leq \|S\|\).

**Sketch of proof:**

(1)-Let us notice that the homomorphism \(\rho_{e} : V^* \rightarrow V^*\) preserves the equivalence \(\sim\): for every \(v, v' \in V\), if \(v \sim v'\) then \(\rho_{e}(v) \sim \rho_{e}(v')\). It follows that the corresponding substitution \(\rho_{e}\) preserves determinism.

(2)-Let \(S \in \mathbb{D}B_{1,\lambda}\langle\langle V \rangle\rangle\). For every \(v \in V_0\)

\[\rho_{e}(S) \circ v = \rho_{e}(S \circ v) \text{ or } \rho_{e}(S) \circ v = \rho_{e}(S \circ \bar{v})\]

according to the fact that the leftmost letters of the monomials of \(S\) are in \([v]_{\sim}\) or in \([\bar{v}]_{\sim}\); both formulas are true when \(S\) is null or is a unit.

By induction on the length, it follows that, for every \(w \in V_0^*\), there exists \(w' \in V^*\) such that:

\[\rho_{e}(w') = w \text{ and } \rho_{e}(S) \circ w = \rho_{e}(S \circ w').\]
Moreover, for every \( w \in V^*V_0^* \),
\[
\rho_e(S) \cdot w = \emptyset^\lambda,
\]
but in this case too, there exists some \( w' \in V^* \) such that \( \rho_e(S) \cdot w = \rho_e(S \cdot w') \). The map \( T \mapsto \rho_e(T) \) is then a surjective map from \( Q(S) \) onto \( Q(\rho_e(S)) \), which proves that \( \|\rho_e(S)\| \leq \|S\| \). □

**Operations on row-vectors**

Let us introduce two new operations on row-vectors and prove some technical lemmas about them.

Given \( A, B \in B_{1,m}(\langle W \rangle) \) and \( 1 \leq j_0 \leq m \) we define the vector \( C = A \nabla_{j_0} B \) as follows:

\[
\text{if } A = (a_1, \ldots, a_j, \ldots, a_m), B = (b_1, \ldots, b_j, \ldots, b_m) \text{ then } C = (c_1, \ldots, c_j, \ldots, c_m)
\]

where
\[
c_j = a_j + a_{j_0} \cdot b_j \text{ if } j \neq j_0, \quad c_j = \emptyset \text{ if } j = j_0.
\]

**Lemma 321** Let \( A, B \in B_{1,m}(\langle W \rangle) \) and \( 1 \leq j_0 \leq m \).

1. if \( A, B \) are left-deterministic, then \( A \nabla_{j_0} B \) is left-deterministic.
2. if \( A, B \) are deterministic, then \( A \nabla_{j_0} B \) is deterministic.
3. if \( A, B \) are deterministic, then \( \| A \nabla_{j_0} B \| \leq \| A \| + \| B \| \).

**Proof:**

Let \( C = A \nabla_{j_0} B \).

1. Let us prove first that if \( A, B \) are both left-deterministic, then \( C \) is left-deterministic too.

   If \( A \) is left-deterministic of type \([pz]\) , then \( C \) is left-deterministic of the same type.

   If \( A \) is left-deterministic of type \((\epsilon,j_1)\) with \( j_1 \neq j_0 \), then \( C = A \), hence \( C \) is left-deterministic.

   If \( A \) is left-deterministic of type \((\emptyset,j_0)\), then \( C \leq B \), hence \( C \) is left-deterministic.

   If \( A \) is left-deterministic of type \((\emptyset)\), then \( C = \emptyset \), hence \( C \) is left-deterministic.

2. Let us suppose now that \( A \) is deterministic and let us examine a residual \( C \cdot u \)
   , for some \( u \in W^* \). Lemma 310 applies on \( S = a_{j_0} \) and \( T = b_j \) for every \( j \neq j_0 \).

   But the case of the lemma fulfilled by \((S,T_j,u)\) depends on \((S,u)\) only.

   Suppose \( a_{j_0} \cdot u \neq \emptyset \) (case 1); in this case

\[
C \cdot u = (A \cdot u) \nabla_{j_0} B \quad (31)
\]

   Suppose \( a_{j_0} \cdot u = \emptyset, \exists u', u'', u = u' \cdot u'', a_{j_0} \cdot u' = \epsilon \) (case 2); in this case

\[
C \cdot u = <(B \cdot u'')|\emptyset_{j_0}^m> \quad (32)
\]
where \( \theta^n \) is the row vector \( e^n_{\mathbf{a})} \) in which \( \theta \) and \( \epsilon \) have been exchanged and \( <*,* \) is the “scalar product” defined by \( <S,T> = \sum_{j=1}^{m} S_j \cdot T_j \).

Suppose \( a_{j_0} \cdot u = \emptyset, \forall u' \leq u, a_{j_0} \cdot u' \neq \epsilon \) (case 3); in this case, equation (33) is true again. When equation (32) is true, \( C \cdot u \) is left-deterministic by part (1) of this proof, and when equation (31) is true, \( C \cdot u \) is left-deterministic because \( B \) is assumed deterministic. We have proved that \( C \in DB_{1,m}((W)) \).

3 The number of residuals of the form (31) is bounded above by \( \|A\| \) and the number of residuals of the form (32) is bounded above by \( \|B\| \). Hence \( \|C\| \leq \|A\| + \|B\| \).

\[ u \]

Proof: Let us examine a residual \( A' \cdot u \), for some \( u \in W^* \). Let \( u^* = \max\{v \leq u | v \in a^*_{j_0}\} \). Let \( u'' \in W^* \) such that \( u = u' \cdot u'' \). One can check that for every \( S,T \in B((W)) \)

\[ (S \cdot T) \cdot u = (S \cdot u) \cdot T + \sum_{u=\mathbf{a}_1 \cdot u_2, \epsilon \in S \bullet u_1} a_j \cdot u_2. \]

Applying this formula to \( S = a^*_{j_0} \) and \( T = a_j \), with \( j \neq j_0 \) we obtain

\[ a_j \cdot u = (a_{j_0} \cdot u) \cdot a_j + \sum_{u=\mathbf{a}_1 \cdot u_2, \epsilon \in a^*_{j_0} \bullet u_1} a_j \cdot u_2. \] (33)

As \( a_{j_0} \) is deterministic, one can check that

\[ a_{j_0} \cdot u = (a_{j_0} \cdot u'') \cdot a^*_{j_0}. \]

As \( A \) is deterministic, if \( u_2 \) has some prefix \( u'_{2} \) in \( a_{j_0} \), then \( a_j \cdot u'_{2} = \emptyset \) so that \( a_j \cdot u_2 = \emptyset \). Hence

\[ \sum_{u=\mathbf{a}_1 \cdot u_2, \epsilon \in a^*_{j_0} \bullet u_1} a_j \cdot u_2 = a_j \cdot u''. \]

Plugging the two last equations into (33) we obtain

\[ a_j' \cdot u = (a_{j_0} \cdot u'') \cdot a^*_{j_0} \cdot a_j + a_j \cdot u'' \] (for \( j \neq j_0 \)), \quad a_j' \cdot u = \emptyset \] (for \( j = j_0 \)),

which can be rewritten as

\[ A' \cdot u = (A \cdot u'') \nabla_{j_0} A' \] (34)
Let us show that $A'$ is left-deterministic. If $A$ is left-deterministic of type $[pz]$, then $A'$ is left-deterministic of the same type. If $A$ is left-deterministic of type $(\varepsilon,j_1)$ with $j_1 \neq j_0$, then $A' = A$ (notice that $\emptyset^* = \varepsilon$), hence $A'$ is left-deterministic.

By point (1) of lemma 321, the fact that $A \bullet w''$ and $A'$ are both left-deterministic implies that $(A \bullet w'')^{\overline{j}_0} = A'$ is left-deterministic too. By formula (34), $A' \bullet u$ is left-deterministic. We have proved that $A' \in DB_{1,m}(\langle W \rangle)$.

Moreover, by formula (4), $\text{Card}(Q(A')) \leq \text{Card}(Q(A))$, i.e. $\|A'\| \leq \|A\|$. $\square$

### 3.2 Bisimulation of series

Up to the end of this section, we consider the structured alphabet $V$ associated with a dpda $M$ over $X$. We suppose a s.r. morphism $\eta \subseteq X^* \times X^*$ is given (see definition 22).

**Series, words and graphs** Let us give first a slight adaptation of definition 21 to the $n$-graph $(\text{DRB}_{1,n}(\langle V \rangle), \circ, (\varepsilon_i^n)_{1 \leq i \leq n})$.

**Definition 323** Let $\mathcal{R}$ be some binary relation $\mathcal{R} \subseteq \text{DRB}_{1,n}(\langle V \rangle) \times \text{DRB}_{1,n}(\langle V \rangle)$. $\mathcal{R}$ is a $\sigma - \eta$-bisimulation iff

1. $\forall (S, S') \in \mathcal{R}, \forall x \in X$, $\exists x' \in \eta(x), (S \circ x, S' \circ x') \in \mathcal{R}$ and $\exists x'' \in \eta^{-1}(x), (S \circ x'', S' \circ x) \in \mathcal{R}$,

2. $\forall (S, S') \in \mathcal{R}, \forall i \in [1,n], (S = \varepsilon_i^n \Leftrightarrow S' = \varepsilon_i^n)$.

We denote by $S \sim S'$ the fact that there exists some $\sigma - \eta$-bisimulation $\mathcal{R}$ such that $(S, S') \in \mathcal{R}$. One can notice that $\sim$ is the greatest $\sigma - \eta$-bisimulation (with respect to the inclusion ordering) over $\text{DRB}_{1,n}(\langle V \rangle)$. The $\sigma$-bisimulation relations can be conveniently expressed in terms of word-bisimulations.

**Definition 324** Let $S, S' \in \text{DRB}_{1,n}(\langle V \rangle)$ and $\mathcal{R} \subseteq X^* \times X^*$. $\mathcal{R}$ is a $w - \eta$-bisimulation with respect to $(S, S')$ iff $\mathcal{R} \subseteq \eta$ and

1. **totality:** $\text{dom}(\mathcal{R}) = X^*$, $\text{im}(\mathcal{R}) = X^*$,
2. **extension:** $\forall (u, u') \in \mathcal{R}, \forall x \in X$, $\exists x' \in \eta(x), (u \cdot x, u' \cdot x') \in \mathcal{R}$ and $\exists x'' \in \eta^{-1}(x), (u \cdot x'', u' \cdot x) \in \mathcal{R}$.
3. **coherence:** $\forall (u, u') \in \mathcal{R}, \forall i \in [1,n], (S \circ u = \varepsilon_i^n) \Leftrightarrow (S' \circ u' = \varepsilon_i^n)$,
4. **prefix:** $\forall (u, u') \in X^* \times X^*, \forall (x, x') \in X \times X, (u \cdot x, u' \cdot x') \in \mathcal{R} \Rightarrow (u, u') \in \mathcal{R}$.


(Condition (1) can be equivalently replaced by “\((\epsilon, \epsilon) \in \mathcal{R}\)”. \(\mathcal{R}\) is said to be a \(w - \eta\)-bisimulation of order \(m\) with respect to \((S, S')\) if it fulfills conditions (3-4) above and the modified conditions

\begin{enumerate}
\item[(1')] \text{dom}(\mathcal{R}) = X^{\leq m}, \text{im}(\mathcal{R}) = X^{\leq m},
\item[(2')] \forall (u, u') \in \mathcal{R} \cap (X^{\leq m-1} \times X^{\leq m-1}), \forall x \in X,
\quad \exists x' \in \eta(x), (u \cdot x, u' \cdot x') \in \mathcal{R} \text{ and } \exists x'' \in \eta^{-1}(x), (u \cdot x'', u' \cdot x) \in \mathcal{R}.
\end{enumerate}

The \(w - \eta\)-bisimulations are also called \(w - \eta\)-bisimulations of order \(\infty\). The two next lemmas are relating the notions of \(w\)-induction on the integer \(m\) and the definition of \(\mathcal{R}\) -bisimulation (on series), and \(\sigma - \eta\)-bisimulation (on the vertices of the computation 2-graph of \(\mathcal{M}\)).

**Lemma 325** Let \(S, S' \in \mathcal{DRB}_{1,n}((V))\). The following properties are equivalent:

\begin{enumerate}
\item[(i)] \(S \sim S'\)
\item[(ii)] there exists \(\mathcal{R} \subseteq X^* \times X^*\) which is a \(w - \eta\)-bisimulation w.r.t. \((S, S')\)
\item[(iii)] \(\forall m \in \mathbb{N},\) there exists \(\mathcal{R}_m \subseteq X^{\leq m} \times X^{\leq m}\) which is a \(w - \eta\)-bisimulation of order \(m\) w.r.t. \((S, S')\).
\end{enumerate}

**Proof:**

\(\Rightarrow (\text{iii})\): Suppose that \(\mathcal{S}\) is a \(\sigma - \eta\)-bisimulation w.r.t. \((S, S')\). Let us prove by induction on the integer \(m\), the following property \(P(m)\):

\[\forall u, u' \in \mathcal{R}_m, (S \circ u, S' \circ u') \in \mathcal{S}. \quad (35)\]

\(m = 0\): Let \(\mathcal{R}_0 = \{((\epsilon, \epsilon))\}.\) \(\mathcal{R}_0\) clearly fulfills points (1'),(2'),(4) of the above definition. Moreover, as \((S, S') \in \mathcal{S}\) where \(\mathcal{S}\) fulfills condition (2) of definition [22], \(\mathcal{R}_0\) fulfills point (3) of definition [22].

\(m = m + 1\): Let \(\mathcal{R}_{m'}\) be some \(w - \eta\)-bisimulation of order \(m'\) w.r.t. \((S, S')\). Let us define \(\mathcal{R}_m = \mathcal{R}_{m'} \cup \{(u \cdot x, u' \cdot x') \mid (u, u') \in \mathcal{R}_{m'}, (S \circ u, S' \circ u') \in \mathcal{S}\) and \((x, x') \in \eta\}. Property (1) of \(\mathcal{S}\) and property (1') of \(\mathcal{R}_{m'}\) imply that

\[\text{dom}(\mathcal{R}_m) = X^{\leq m}, \text{im}(\mathcal{R}_m) = X^{\leq m}. \quad (36)\]

Property (1) of \(\mathcal{S}\) and property (2') of \(\mathcal{R}_{m'}\) imply that \(\forall (u, u') \in \mathcal{R}_m \cap (X^{\leq m-1} \times X^{\leq m-1}), \forall x \in X,
\quad \exists x' \in \eta(x), (u \cdot x, u' \cdot x') \in \mathcal{R}_m\) and \(\exists x'' \in \eta^{-1}(x), (u \cdot x'', u' \cdot x) \in \mathcal{R}_m. \quad (37)\]

Property (2) of \(\mathcal{S}\) and property (3) of \(\mathcal{R}_{m'}\) imply that

\[\forall (u, u') \in \mathcal{R}_m, \forall i \in [1, n], (S \circ u = \epsilon_i^u) \leftrightarrow (S' \circ u' = \epsilon_i^{u'}). \quad (38)\]

Property (4) of \(\mathcal{R}_{m'}\) and the definition of \(\mathcal{R}_m\) imply that

\[\forall (u, u') \in X^* \times X^*, \forall (x, x') \in X \times X, (u \cdot x, u' \cdot x') \in \mathcal{R}_m \Rightarrow (u, u') \in \mathcal{R}_m. \quad (39)\]
Property (35) for $\mathcal{R}_{m'}$ and the definition of $\mathcal{R}_m$ imply that (35) is fulfilled by $\mathcal{R}_m$ too. Equations (36,37,38,39) prove that $\mathcal{R}_m$ is a $w$-$\eta$-bisimulation of order $m$ w.r.t. $(S,S')$, hence $P(m)$ is proved.

(iii) $\Rightarrow$ (ii): Let us notice that, as the alphabet $X$ is finite, for every $w$-$\eta$-bisimulation $\mathcal{R}$ of order $m$ w.r.t. $(S,S')$, 

$$\text{Card}\{\mathcal{R}' \subseteq X^* \times X^* \mid \mathcal{R} \subseteq \mathcal{R}'\} < \infty.$$ 

Hence, by Koenig’s lemma, if (iii) is true, then there exists an infinite sequence $(\mathcal{R}_m)_{m \in \mathbb{N}}$ such that for every $m \in \mathbb{N}$, $\mathcal{R}_m$ is a $w$-$\eta$-bisimulation of order $m$ w.r.t. $(S,S')$ and $\mathcal{R}_m \subseteq \mathcal{R}_{m+1}$. Let us define then

$$\mathcal{R} = \bigcup_{m \geq 0} \mathcal{R}_m.$$ 

$\mathcal{R}$ is a $w$-$\eta$-bisimulation of order $\infty$ w.r.t. $(S,S')$.

(ii) $\Rightarrow$ (i): Let $\mathcal{R}$ be a $w$-$\eta$-bisimulation of order $\infty$ w.r.t. $(S,S')$. Let us define a relation $S$ by:

$$S = \{(S \circ u, S' \circ u') \mid (u, u') \in \mathcal{R}\}.$$ 

The totality property of $\mathcal{R}$ implies that $(S, S') \in S$. The extension property of $\mathcal{R}$ implies that $S$ fulfills condition (1) of definition 323 and the coherence property of $\mathcal{R}$ implies $S$ fulfills condition (2).

Lemma 325 leads naturally to the following

**Definition 326** Let $\lambda \in \mathbb{N} - \{0\}, S, S' \in \text{DRB}_1(\langle V \rangle)$. We define the divergence between $S$ and $S'$ as:

$$\text{Div}(S, S') = \inf\{n \in \mathbb{N} \mid B_n(S, S') = \emptyset\}.$$ 

(It is understood that $\inf(\emptyset) = \infty$).

Let us suppose that the dpda $\mathcal{M} = \langle X, Z, Q, \delta, q_0, z_0, \{\bar{q}\} \rangle$ is normalized and bi-rooted. Let $\psi : X^* \rightarrow Y^*$ be a monoid homomorphism such that $\psi(X) \subseteq Y$ and let $\overline{\psi} = \psi \circ \psi^{-1}$ ( $\overline{\psi}$, the kernel of $\psi$, is a s.r. morphism which is also an equivalence relation; this additional property will be used in the sequel). Let $\Gamma$ be the computation 2-graph of $\mathcal{M}$ and let us suppose $\Gamma$ is $\bar{\psi}$-saturated. Let $\theta : V_{\Gamma} \rightarrow \text{DRB}(\langle V \rangle)$ the mapping defined by: $\forall q \in Q, \forall \omega \in Z^*$, such that $q\omega \in V_{\Gamma}$,

$$\theta(q\omega) = \varphi_0([q\omega\bar{q}]).$$

For every $q\omega \in V_{\Gamma}, S \in \text{DRB}(\langle V \rangle)$ we also define:

$$L(q\omega) = \{u \in X^*, q\omega \xrightarrow{u} \bar{q}\}; \ L(S) = \{u \in X^*, S \circ u = \epsilon\}.$$ 

**Lemma 327** For every $q\omega \in V_{\Gamma}, L(q\omega) = L(\varphi_0([q\omega\bar{q}])).$
This lemma follows from the classical result that the language recognized by $M$ with starting configuration $q_\omega$ and final configuration $\bar{q}$ is exactly the language generated by $G_M$ from the polynomial $[q_\omega \bar{q}]$ which, in turn, is equal to the language generated by $G_0$ from the polynomial $\varphi_0([q_\omega \bar{q}])$. At last, $G$ and $G_0$ generate the same language from any given polynomial over $V_0$.

**Lemma 328** Let $v, v'$ be vertices of $\Gamma$. Then $v \sim v'$, in the sense of definition 21 if $\theta(v) \sim \theta(v')$, in the sense of definition 323.

**Proof:** In this proof we denote by $\odot_{\Gamma}$ the right-action of $X^*$ over $V_{\Gamma} \cup \{\perp\}$ defined by: for every $v, v' \in V_{\Gamma}, u \in X^*$,

- $v \odot_{\Gamma} u = v'$ if $v \xrightarrow{u}_{\Gamma} v'$,
- $v \odot_{\Gamma} u = \perp$ if there is no $v'$, such that $v \xrightarrow{u}_{\Gamma} v'$.

1-Let us suppose that $(v, v') \in R$, where $R$ is some $\bar{\psi}$-bisimulation over $\Gamma$.

Let $S = \{(\theta(v) \odot u, \theta(v') \odot u') | (u, u') \in \bar{\psi}, (v \odot_{\Gamma} u, v' \odot_{\Gamma} u') \in R\} \cup \{(\emptyset, \emptyset)\}$.

Let us show that $S$ is a $\sigma - \bar{\psi}$-bisimulation.

Let us consider some pair of series in $S$. If the given pair is $(\emptyset, \emptyset)$, points (1)(2) of definition 323 are clearly fulfilled.

1.1-Let $x \in X$.

**case 1.1.1:** $\theta(v) \odot ux \neq \emptyset$.

$$L(\theta(v) \odot ux) \neq \emptyset$$

(because the grammar $G$ is reduced), hence, using lemma 27

$$L(v \odot_{\Gamma} ux) = L(v) \bullet ux = L(\theta(v)) \bullet ux \neq \emptyset.$$ 

It follows that

$$v \odot_{\Gamma} ux \neq \perp.$$ 

As $R$ is a $\bar{\psi}$-simulation, there must exists some $x' \in \bar{\psi}(x)$ such that

$$(v \odot_{\Gamma} ux, v' \odot_{\Gamma} u' x') \in R.$$ 

Hence

$$(\theta(v) \odot ux, \theta(v') \odot u' x') \in S.$$ 

**case 1.1.2:** $\theta(v) \odot ux = \emptyset$.

In this case, by lemma 27 and the fact that $\Gamma$ is bi-rooted, $v \odot_{\Gamma} ux$ must be equal to $\perp$. As $\Gamma$ is $\bar{\psi}$-saturated, it follows that

$$\forall x' \in \bar{\psi}(x), v \odot_{\Gamma} ux' = \perp.$$ 

As $R^{-1}$ is a $\bar{\psi}^{-1}$-simulation, it must also be true that

$$\forall x' \in \bar{\psi}(x), v' \odot_{\Gamma} u' x' = \perp.$$
choosing some particular $x' \in \tilde{\psi}(x)$, and using again lemma 327 we obtain:

$$\theta(v') \odot u x' = \emptyset$$

In both cases, as $v, v'$ are playing symmetric roles, property (1) of definition 323 has been verified. If the starting pair in $S$ is $(\emptyset, \emptyset)$, property (1) is again verified.

1.2- Let us suppose that $\theta(v) \odot u = \epsilon$.

This means that

$$L(\theta(v)) \bullet u = \epsilon,$$

hence, using lemma 327 that

$$L(v \odot \Gamma u) = \epsilon,$$

hence

$$v \odot \Gamma u = \tilde{q}.$$ 

As $\Gamma$ is bi-rooted, $\tilde{q}$ is the only vertex having no outgoing edge (see §2.1). As $R$ is a $\tilde{\psi}$-bisimulation, $v' \odot \Gamma u'$ we must also have no outgoing edge, hence

$$v' \odot \Gamma u' = \tilde{q},$$

and by the same arguments, used backwards now,

$$L(\theta(v')) \bullet u' = \epsilon,$$

which, as the grammar $G$ is proper and reduced, implies

$$\theta(v') \odot u' = \epsilon.$$ 

As $(v, v')$ are playing symmetric roles, property (2) of definition 323 has been verified.

2- Let us suppose that $(\theta(v), \theta(v')) \in S$, where $S$ is some $\sigma - \tilde{\psi}$-bisimulation. Let $R = \{(v \odot \Gamma u, v' \odot \Gamma u') \mid (u, u') \in \tilde{\psi}, (\theta(v) \odot u, \theta(v') \odot u') \in S - \{(\emptyset, \emptyset)\}\} \cup \{(c, c) \mid c \in V_{\Gamma}\}$. We show that $R$ is a $\tilde{\psi}$-bisimulation over $\Gamma$.

2.1- Using lemma 327, we obtain:

$$\theta(v) \odot u \neq \emptyset \Rightarrow v \odot \Gamma u \neq \bot.$$ 

Hence

$$\text{dom}(R) \subseteq V_{\Gamma}.$$ 

Conversely, due to the term $\{(c, c) \mid c \in V_{\Gamma}\},$

$$\text{dom}(R) \supseteq V_{\Gamma}.$$ 

At end, point (1) of definition 21 is fulfilled.

2.2- Due to the term $\{(c, c) \mid c \in V_{\Gamma}\}$, point (2) of definition 21 is fulfilled.

2.3- Let us consider some pair of configurations in $R$. It must have the form $(v \odot \Gamma u, v' \odot \Gamma u')$, where $(u, u') \in \tilde{\psi}$ and $(\theta(v) \odot u, \theta(v') \odot u') \in S - \{(\emptyset, \emptyset)\}$. 

By the same arguments as in case 1.1.1 above, one can show that, for every \( x \in X \), such that 
\[ v \circ_{T} u x \neq \perp, \]
there exists some \( x' \in \bar{\psi}(x) \) such that 
\[ v' \circ_{T} u' x' \neq \perp. \]
Hence \( R \) fulfills the three points of definition 21. By same means, \( R^{-1} \) fulfills them too, so that \( R \) is a \( \bar{\psi} \)-bisimulation over the graph \( \Gamma \). ∎

Extension to matrices

Let \( \delta, \lambda \in \mathbb{N} - \{0\} \). We extend the binary relation \( \sim \) from vectors in \( \text{DRB}_{1,\lambda}( \langle V \rangle ) \) to matrices in \( \text{DRB}_{\delta,\lambda}( \langle V \rangle ) \) as follows: for every \( T, T' \in \text{DRB}_{\delta,\lambda}( \langle V \rangle ) \),
\[ T \sim T' \iff \forall i \in [1, \delta], T_{i,\ast} \sim T'_{i,\ast}. \]

We call \( w-\eta\)-bisimulation of order \( n \in \mathbb{N} \cup \{\infty\} \) with respect to \( (T, T') \) every \( R = (R_i)_{i \in [1, \delta]} \) such that \( \forall i \in [1, \delta], R_i \in B_n(T_{i,\ast}, T'_{i,\ast}) \).

We denote by \( B_n(T, T') \) the set of \( w-\eta\)-bisimulations of order \( n \) w.r.t. \( (T, T') \).

Some algebraic properties of this extended relation \( \sim \) will be established in corollary 46.

Operations on \( w \)-bisimulations

The following operations on word-\( \bar{\psi} \)-bisimulations turn out to be useful.

**right-product:**

Let \( \delta, \lambda \in \mathbb{N} - \{0\} \), \( S, S' \in \text{DRB}_{1,\lambda}( \langle V \rangle ) \), \( T \in \text{DRB}_{\delta,\lambda}( \langle V \rangle ) \). For every \( n \in \mathbb{N} \cup \{\infty\} \) and \( R \in B_n(S, S') \) we define:
\[ <S|R> = \left\{ (u, u') \in R \mid \forall v \leq u, \forall i \in [1, \delta], S \circ v \neq \epsilon_i^\delta \right\} \]
\[ \cup \left\{ (u \cdot w, u' \cdot w') \mid (u, u') \in R, w \in X^*, \exists i \in [1, \delta], S \circ u = \epsilon_i^\delta \right\} \cap X^{\leq n} \times X^{\leq n}. \]

One can check that \( <S|R> \in B_n(S \cdot T, S' \cdot T) \).

**left-product:**

Let \( \delta, \lambda \in \mathbb{N} - \{0\} \), \( S \in \text{DRB}_{1,\lambda}( \langle V \rangle ) \), \( T, T' \in \text{DRB}_{\delta,\lambda}( \langle V \rangle ) \). For every \( n \in \mathbb{N} \cup \{\infty\} \) and \( R \in B_n(T, T') \) we define:
\[ <S, R> = \left\{ (u, u) \mid u \in X^*, \forall v \leq u, \forall i \in [1, \delta], S \circ v \neq \epsilon_i^\delta \right\} \]
\[ \cup \left\{ (u \cdot w, u' \cdot w') \mid u \in X^*, \exists i \in [1, \delta], S \circ u = \epsilon_i^\delta, (w, w') \in R_i \right\} \cap X^{\leq n} \times X^{\leq n}. \]

One can check that \( <S, R> \in B_n(S \cdot T, S \cdot T') \).

**star:**
Let $\lambda \in \mathbb{N} - \{0\}, S_1 \in \text{DRB}_{1,1}(\langle V \rangle), S_1 \neq \epsilon, (S_1, S) \in \text{DRB}_{1,\lambda+1}(\langle V \rangle), T \in \text{DRB}_{1,\lambda}(\langle V \rangle)$. For every $n \in \mathbb{N} \cup \{\infty\}$ and $R \in B_n(S_1 \cdot T + S, T)$ we define:

$$R_0 = R$$ (45)
$$S_0 = \begin{pmatrix} R_0 \\ \vdots \\ R_0 \end{pmatrix}$$ (46)

$$\forall k \geq 0, R_{k+1} = (S_1, S)_k \circ R_0$$ (47)
$$S_k = \begin{pmatrix} R_k \\ \vdots \\ R_k \end{pmatrix}$$ (48)

and finally

$$R^{<S_1,*>} = \bigcup_{k \geq 0} R_k \cap X^{\leq k} \times X^{\leq k}.$$ (49)

One can check that, for every $k \geq 0$:

$$R_k \in B_n(S_1^{k+1} + \sum_{i=0}^k S_1^i \cdot S, T)$$ (50)
$$S_k \in B_n((S_1^{k+1} + \sum_{i=0}^k S_1^i \cdot S), (T, T_\lambda)),$$ (51)

and finally $R^{<S_1,*>} \in B_n(S_1^* \cdot S, T)$.

**Remark 329** In fact operations could be more adequately defined on “pointed” w-bisimulations, i.e. on binary relations with sets of “terminal pairs of words” of type $i \in [1, \delta]$ corresponding to the pairs $(u, u')$ such that $S \circ u = \epsilon^i, S' \circ u' = \epsilon^i$. The two different external operations $<S, R>, <S|R>$ could then be replaced by only one binary operation $<R_1, R_2>$ over “pointed” w-bisimulations.

### 3.3 Deterministic spaces

We adapt here the key-idea of [Mei89, Mei92] to bisimulation of vectors.

**Definitions** Let $(W, \sim)$ be some structured alphabet. A vector $U = \sum_{i=1}^n \gamma_i \cdot U_i$ where $\gamma \in \text{DRB}_{1,n}(\langle W \rangle), U_i \in \text{DRB}_{1,\lambda}(\langle W \rangle)$ is called a linear combination of the $U_i$’s. We call deterministic space of rational vectors (d-space for short) any subset $V$ of $\text{DRB}_{1,\lambda}(\langle W \rangle)$ which is closed under finite linear combinations. Given any set $G = \{U_i|i \in I\} \subseteq \text{DRB}_{1,\lambda}(\langle W \rangle)$, one can check that the set $V$ of all (finite) linear combinations of elements of $G$ is a d-space (by lemma 314) and that it is the smallest d-space containing $G$. Therefore we call $V$ the d-space generated by $G$ and we call $G$ a generating set of $V$ (we note $V = V(\{U_i|i \in I\})$).

(Similar definitions can be given for families of vectors).
Linear independence

We let now $W = V$. Following an analogy with classical linear algebra, we develop now a notion corresponding to a kind of linear independence of the classes (mod $\sim$) of the given vectors. Let us extend the equivalence relation $\sim$ to d-spaces by: if $V_1, V_2$ are d-spaces,

$$V_1 \sim V_2 \iff \forall i, j \in \{1, 2\}, \forall S \in V_i, \exists S' \in V_j, S \sim S'.$$

**Lemma 330** Let $S_1, \ldots, S_j, \ldots, S_m \in \text{DRB}_1(\langle V \rangle)$. The following are equivalent

1. $\exists \alpha, \beta \in \text{DRB}_1, \langle V \rangle, \alpha \neq \beta$, such that
   $$\sum_{j=1}^{m} \alpha_j \cdot S_j \sim \sum_{j=1}^{m} \beta_j \cdot S_j$$

2. $\exists j_0 \in [1, m], \exists \gamma \in \text{DRB}_1, \langle V \rangle, \gamma \neq \epsilon_{j_0}^m$, such that
   $$S_{j_0} \sim \sum_{j=1}^{m} \gamma_j \cdot S_j$$

3. $\exists j_0 \in [1, m], \exists \gamma' \in \text{DRB}_1, \langle V \rangle, \gamma'_{j_0} \sim \emptyset$, such that
   $$\sum_{j=1}^{m} \gamma_j' \cdot S_j$$

4. $\exists j_0 \in [1, m]$, such that
   $$V((S_j)_{1 \leq j \leq m}) \sim V((S_j)_{1 \leq j \leq m, j \neq j_0}).$$

The equivalence between (1), (2) and (3) was first proved in [Mei89, Mei92], in the case where the $S_j$’s are configurations $q_j \omega$, with the same $\omega$ and $\bar{\psi} = \text{Id}_{X^*}$, hence $\sim$ is just the language equivalence relation $\equiv$. This is the key-idea around which we have developed the notion of d-spaces.

**Proof:** (1) $\Rightarrow$ (2):

Let us consider $R \in B_{\infty}(\alpha \cdot S, \beta \cdot S), \nu = \text{Div}(\alpha, \beta)$ and

$$(u, v) = \min\{(w, w') \in R \cap X^{\leq \nu} \times X^{\leq \nu} \mid \exists j \in [1, m], (\alpha \odot w = \epsilon_j^m) \Leftrightarrow (\beta \odot w' \neq \epsilon_j^m)\}. \quad (52)$$

Let us suppose, for example, that $\alpha \odot u = \epsilon_{j_0}^m$ while $\beta \odot v \neq \epsilon_{j_0}^m$ and let $\gamma = \beta \odot u$. As $(u, v) \in R \in B_{\infty}(\alpha \cdot S, \beta \cdot S)$

$$(\alpha \cdot S) \odot u \sim (\beta \cdot S) \odot v. \quad (53)$$

Using lemma [318] we obtain:

$$(\alpha \cdot S) \odot u = S_{j_0}. \quad (54)$$
Let us examine now the righthand-side of equality (53). Let \((u', v') \preceq (u, v)\) with \(|u'| = |v'|\). By condition (4) in definition 324 \((u', v') \in R\) and by minimality of \(v\), \(\beta \circ v'\) is a unit if \(\alpha \circ u'\) is a unit. But if \(\alpha \circ u'\) is a unit, then \(\alpha \circ u = \emptyset\), which is false. Hence \(\beta \circ v'\) is not a unit. Hence, \(\forall v' \prec v, \beta \circ v'\) is not a unit.

By lemma 318

\[(\beta \cdot S) \circ v = (\beta \circ v) \cdot S. \tag{55}\]

Let us plug equalities (54) and (55) in equivalence (53) and let us define \(\gamma = \beta \circ v\).

We obtain:

\[S_{j_0} \sim \gamma \cdot S, \gamma \neq \epsilon_{j_0}.\]

(2) \(\Rightarrow\) (3):

\[S_{j_0} \sim \gamma_{j_0} \cdot S_{j_0} + \left( \sum_{j \neq j_0} \gamma_j \cdot S_j \right), \quad \gamma_{j_0} \neq \epsilon.\]

By corollary 16 point C1, we can deduce that

\[S_{j_0} \sim \sum_{j \neq j_0} \gamma^*_{j_0} \cdot S_j = \nabla^*_j (\gamma) \cdot S.\]

Taking \(\gamma' = \nabla^*_j (\gamma)\) we obtain

\[S_{j_0} \sim \gamma' \cdot S \text{ where } \gamma'_{j_0} = \emptyset.\]

(3) \(\Rightarrow\) (4):

Let us denote by \(\hat{S}\) the vector \((S_1, \ldots, S_{j_0-1}, \emptyset, S_{j_0+1}, \ldots, S_m) \in DB_{1,\lambda}(\langle V \rangle)\).

If \(T = \alpha \cdot S\) then \(T \sim (\alpha \nabla_{j_0} \gamma') \cdot \hat{S}\).

(4) \(\Rightarrow\) (1):

Let us suppose (4) is true for some integer \(j_0\). The element \(S_{j_0}\) is clearly equivalent (mod \(\sim\)) to two linear combinations of the \(S_j\)'s with non-equivalent vectors of coefficients (mod \(\sim\)). Hence (1) is true.

\[\square\]

### 3.4 Derivations

For every \(u \in X^*\) we define the binary relation \(\uparrow (u)\) over \(DB_{1,\lambda}(\langle V \rangle)\) by: for every \(S, S' \in DB_{1,\lambda}(\langle V \rangle), S \uparrow (u)S' \iff \exists q \in \mathbb{N}, \exists E_1, \ldots, E_k, E_q \in V, \Phi \in DB_{q,\lambda}(\langle V \rangle)\) such that

\[S = \sum_{k=1}^{q} E_k \cdot \Phi_k, S' = \sum_{k=1}^{q} (E_k \circ u) \cdot \Phi_k,\]

and \(\forall k \in [1, q], E_1 \sim E_k, E_k \circ u \notin \{\emptyset, \epsilon\}\).

It is clear that if \(S \uparrow (u)S'\) then \(S \circ u = S'\) and that the converse is not true.

---

1. here is the main place where this condition (4) is used.
in general. A sequence of deterministic row-vectors \( S_0, S_1, \ldots, S_n \) is a derivation iff there exist \( x_1, \ldots, x_n \in X \) such that \( S_0 \circ x_1 = S_1, \ldots, S_{n-1} \circ x_n = S_n \). The length of this derivation is \( n \). If \( u = x_1 \cdot x_2 \cdot \ldots \cdot x_n \) we call \( S_0, S_1, \ldots, S_n \) the derivation associated with \( (S, u) \). We denote this derivation by \( S_0 \xrightarrow{u} S_n \).

A derivation \( S_0, S_1, \ldots, S_n \) is said to be stacking iff it is the derivation associated to a pair \( (S, u) \) such that \( S = S_0 \) and \( S_0 \uparrow (u) S_n \). A derivation \( S_0, S_1, \ldots, S_n \) is said to be a sub-derivation of a derivation \( S'_0, S'_1, \ldots, S'_m \) iff there exists some \( i \in [0, m] \) such that, \( \forall j \in [1, n], S_j = S'_{i+j} \).

**Definition 331** A vector \( S \in \text{DRB}_{1,\lambda}(\langle V \rangle) \) is said loop-free if and only if for every \( v \in V^+ \), \( S \cdot v \neq S \).

Let us notice that every polynomial is loop-free. The two following lemmas give other examples of loop-free vectors.

**Lemma 332** Let \( \alpha \in \text{DB}_{1,n}(V), \Phi \in B_{n,\lambda}(\langle V \rangle) \), such that \( \infty > \|\alpha \cdot \Phi\| > \|\Phi\| \). Then \( \alpha \cdot \Phi \) is loop-free.

**Proof:** Let \( \alpha, \Phi \) fulfill the hypothesis of the lemma and suppose, for sake of contradiction, that there exists some \( v \in V^+ \) such that:

\[
(\alpha \cdot \Phi) \cdot v = \alpha \cdot \Phi
\]

By induction, for every \( n \geq 0 \):

\[
(\alpha \cdot \Phi) \cdot v^n = \alpha \cdot \Phi
\]

(56)

As \( \alpha \) is a polynomial, there exists some \( n_0 \geq 0 \) such that \( |v^{n_0}| \) is greater than the greatest length of a monomial of \( \alpha \). Using lemma 311, equality (56) for such an integer \( n_0 \) means that there exists some \( k \in [1, n], v'' \) suffix of \( v^{n_0} \) such that:

\[
\Phi_k \cdot v'' = \alpha \cdot \Phi
\]

(57)

Using the hypothesis of the lemma we conclude that:

\[
\|\Phi\| \geq \|\Phi_k \cdot v''\| = \|\alpha \cdot \Phi\| > \|\Phi\|
\]

which is contradictory. \( \Box \)

**Lemma 333** Let \( S \in \text{DRB}_{1,\lambda}(\langle V \rangle), u \in X^* \), such that \( \|S \circ u\| > \|S\| \). Then \( S \circ u \) is loop-free.

**Proof:** Let us consider \( S, u \) fulfilling the hypothesis of the lemma and let us consider the 3 possible forms of \( S \circ u \) proposed by lemma 317. The forms (1) or (2) are incompatible with the inequality \( \|S \circ u\| > \|S\| \). Hence \( S \circ u \) has the form (3):

\[
u = u_1 \cdot u_2, S \circ u_1 = S \cdot v_1 = \sum_{k=1}^{q} E_k \cdot \Phi_k, S \circ u = \sum_{k=1}^{q} (E_k \circ u_2) \cdot \Phi_k\]

and
Lemma 316:  
\[ | \parallel A_2 | = k \parallel S | \parallel S = \parallel S | = k \parallel S | \parallel u_2 \notin \{ \epsilon, \emptyset \}. \]

Hence \( S \odot u = \alpha \cdot \Phi \) for some polynomial \( \alpha \in \text{DRB}_{1,q}(V) \). As for every \( k \), \( \Phi_k = S \odot (v_1 E_k) \), we obtain that \( \parallel S \parallel \geq \parallel \Phi \parallel \). Finally

\[ \infty > \parallel S \odot u \parallel = \parallel \alpha \cdot \Phi \parallel > \parallel S \parallel \geq \parallel \Phi \parallel, \]

and by lemma 332, \( S \odot u \) is loop-free. \( \Box \)

Lemma 334: Let \( S \in \text{DRB}_{1,\lambda}(\{V\}), w \in X^* \), such that
1. \( S \) is loop-free
2. \( \forall u \leq w, \parallel S \odot u \parallel \geq \parallel S \parallel \). Then the derivation \( S \xrightarrow{w} S \odot w \) is stacking.

Proof: \( S \) is left-deterministic. If it has type \( \emptyset \) or \( (\epsilon, j) \), the lemma is trivially true. Otherwise

\[ S = \sum_{k=1}^{q} E_k \cdot \Phi_k, \]

for some class of letter \( [E_1]_\omega = \{E_1, \ldots, E_q \} \) and some matrix \( \Phi \in \text{DRB}_{q,\lambda}(\{V\}) \).
Suppose that for some prefix \( u \leq w \) and \( k \in [1, q] \),

\[ E_k \odot u = \epsilon. \] \hspace{1cm} (58)

Then, \( S \odot u = \Phi_k \) so that \( \parallel S \odot u \parallel \leq \parallel \Phi \parallel \leq \parallel S \parallel \) which shows that \( S = S \odot u \) while \( u \neq \epsilon \). This would contradict the hypothesis that \( S \) is loop-free, hence (58) is impossible.

Let us apply now lemma 318 to the expression \( (E \cdot \Phi) \odot w \): case (2) is impossible, hence

\[ (E \cdot \Phi) \odot w = (E \odot w) \cdot \Phi, \]

which is equivalent to

\[ S \uparrow (w)S \odot w. \]

\( \Box \)

Lemma 335: Let \( S \in \text{DRB}_{1,\lambda}(\{V\}), w \in X^*, k \in \mathbb{N} \), such that
\( \parallel S \odot w \parallel \geq \parallel S \parallel + k \cdot K_0 + 1 \).

Then the derivation \( S \xrightarrow{w} S \odot w \) contains some stacking sub-derivation of length \( k \).

Sketch of proof: Let \( S = S_0, \ldots, S_i, \ldots, S_n \) be the derivation associated to \((S, w)\). Let \( i_0 = \max\{i \in [0, n] \mid \parallel S_i \parallel = \min\{\parallel S_j \parallel \mid 0 \leq j \leq n\} \} \) and \( i_1 = \max\{i \in [i_0 + 1, n] \mid \parallel S_i \parallel = \min\{\parallel S_j \parallel \mid i_0 + 1 \leq j \leq n\} \}. \) Let \( w = w_0 w_1 w' \) where \( |w_0| = i_0, |w_0 w_1| = i_1 \).

As \( \parallel S \odot w_0 w_1 \parallel > \parallel S \odot w_0 \parallel \), by lemma 333 \( S \odot w_0 w_1 = S_{i_k} \) is loop-free. Using lemma 316,

\[ \parallel S_{\alpha} \parallel - \parallel S_{i_k} \parallel \geq \parallel S_{\alpha} \parallel - \parallel S_{i_0} \parallel - (\parallel S_{i_1} \parallel - \parallel S_0 \parallel) \geq (k - 1) \cdot K_0 + 1. \]
Using lemma 316 we must have $|w'| \geq k$. Let $w' = w_2w_3$ with $|w_2| = k$. By definition of $i_1, \forall i \in [i_1 + 1, i_1 + k], \|S_i\| \geq \|S_{i_1}\| + 1$. By lemma 334, the sub-derivation $S_{i_1}, \ldots, S_{i_1+k}$ (associated to $(S_{i_1}, w_2)$) is stacking. □

**Lemma 336** Let $S, S' \in \text{DRB}_{1,\lambda}(\langle V \rangle), w \in X^*, k, d, d' \in \mathbb{N}$, such that $S$ is $d$-marked and:

1. the derivation $S \xrightarrow{w} S'$ contains no stacking sub-derivation of length $k$.
2. $|w| \geq d \cdot k$.

Then $S'$ is unmarked.

**Proof:** By hypothesis

$$S = \sum_{k=1}^{q} \alpha_k \cdot \Phi_k$$

for some $\alpha \in \text{DRB}_{q}(\langle V \rangle), \Phi \in \text{DRB}_{q,\lambda}(\langle V \rangle), \|\alpha\| \leq d, \Phi$ unmarked.

Let $S \xrightarrow{w} S' = (S_0, \ldots, S_n)$. By induction on $\ell$, using hypothesis (1) and lemma 334 (on polynomials, which are particular cases of loop-free series) one can show that: for every $\ell \in [0, d]$, there exists some prefix $w_\ell$ of $w$, with length $|w_\ell| \leq k \cdot \ell$ such that either

$$S \odot w_\ell = \sum_{k=1}^{q} (\alpha_k \odot w_\ell) \cdot \Phi_k, \text{ with } \|\alpha \odot w_\ell\| < \|\alpha\| - \ell \quad (59)$$

or there exists an integer $k \in [1, q]$ such that

$$S \odot w_\ell = \Phi_k. \quad (60)$$

Let us apply this property to $\ell = d$: inequality (59) is not possible for this value of $\ell$ because, by hypothesis (2) of the lemma $\|\alpha\| - \ell \leq 0$. Hence (59) is true and, as $\Phi$ is unmarked, $\Phi_k$ is unmarked, so that $S \odot w$ is unmarked. □
4 Deduction systems

4.1 General formal systems

We follow here the general philosophy of [HHY79,Cou83a]. Let us call formal system any triple $\mathcal{D} = (\mathcal{A}, H, \vdash)$ where $\mathcal{A}$ is a denumerable set called the set of assertions, $H$, the cost function is a mapping $\mathcal{A} \to \mathbb{N} \cup \{\infty\}$ and $\vdash$, the deduction relation is a subset of $\mathcal{P}_f(\mathcal{A}) \times \mathcal{A}$; $\mathcal{A}$ is given with a fixed bijection with $\mathbb{N}$ (an “encoding” or “Gödel numbering”) so that the notions of recursive subset, recursively enumerable subset, recursive function, ... over $\mathcal{A}, \mathcal{P}_f(\mathcal{A}), ...$ are defined, up to this fixed bijection; we assume that $\mathcal{D}$ satisfies the following axiom:

\[(A_1) \forall (P,A) \in \vdash, (\min \{H(p), p \in P\} < H(A)) \text{ or } (H(A) = \infty).\]

(We let $\min(\emptyset) = \infty$). We call $\mathcal{D}$ a deduction system iff $\mathcal{D}$ is a formal system satisfying the additional axiom:

\[(A_2) \vdash \text{ is recursively enumerable.}\]

In the sequel we use the notation $P \vdash A$ for $(P,A) \in \vdash$. We call proof in the system $\mathcal{D}$, relative to the set of hypotheses $H \subseteq \mathcal{A}$, any subset $P \subseteq \mathcal{A}$ fulfilling:

$$\forall p \in P, (\exists Q \subseteq P, Q \vdash p) \text{ or } (p \in H).$$

We call $P$ a proof iff $$\forall p \in P, (\exists Q \subseteq P, Q \vdash p)$$ (i.e. iff $P$ is a proof relative to $\emptyset$).

Let us define the total map $\chi : \mathcal{A} \to \{0,1\}$ and the partial map $\overline{\chi} : \mathcal{A} \to \{0,1\}$ by:

$$\chi(A) = 1 \text{ if } H(A) = \infty, \chi(A) = 0 \text{ if } H(A) < \infty,$$

$$\overline{\chi}(A) = 1 \text{ if } H(A) = \infty, \overline{\chi} \text{ is undefined if } H(A) < \infty.$$

($\chi$ is the “truth-value function”, $\overline{\chi}$ is the “1-value function”).

**Lemma 41** Let $P$ be a proof relative to $H \subseteq H^{-1}(\infty)$ and $A \in P$. Then $\chi(A) = 1$.

In other words: if an assertion is provable from true hypotheses, then it is true.

**Proof:** Let $P$ be a proof. We prove by induction on $n$ that,

$$\mathcal{P}(n) : \forall p \in P, H(p) \geq n.$$ 

It is clear that, $\forall p \in P, H(p) \geq 0$. Suppose that $\mathcal{P}(n)$ is true. Let $p \in P - H : \exists Q \subseteq P, Q \vdash p$. By induction hypothesis, $\forall q \in Q, H(q) \geq n$ and by (A1), $H(p) \geq n + 1$. It follows that : $\forall p \in P - H, H(p) = \infty$. But by hypothesis, $\forall p \in H, H(p) = \infty$. \(\square\)

A formal system $\mathcal{D}$ will be said complete iff, conversely, $\forall A \in \mathcal{A}, \chi(A) = 1 \implies$ there exists some finite proof $P$ such that $A \in P$. (In other words, $\mathcal{D}$ is complete iff every true assertion is “finitely” provable).
Lemma 42: If $D$ is a complete deduction system, $\chi$ is a recursive partial map.

Proof: Let $i \mapsto P_i$ be some recursive function whose domain is $\mathbb{N}$ and whose image is $\mathcal{P}_f(A)$. Let $h : (\mathcal{P}_f(A) \times A \times \mathbb{N}) \rightarrow \{0, 1\}$ be a total recursive function such that:

$$P \vdash A \text{ iff } \exists n \in \mathbb{N}, h(P, A, n) = 1$$

(such an $h$ exists, because the r.e. sets are the projections of the recursive sets, see [Rog67]).

The following (informal) semi-algorithm computes $\chi$ on the assertion $A$ :

1. $i := 0$ ; $n := 0$ ; $s := i + n$;
2. $P := P_i$;
3. $b := \min_{p \in P} \{\max_{Q \subseteq P} \{h(Q, p, n)\}\}$;
4. $c := (A \in P)$;
5. if $(b \wedge c)$ then $(\chi(A) = 1$ ; stop);
6. if $i = 0$ then $(i := s + 1$ ; $n := 0$ ; $s := i + n)$
   else $(i := i - 1$ ; $n := n + 1)$ ;
7. goto 2 ;

In order to define deduction relations from more elementary ones, we set the following definitions.

Let $\vdash \subseteq \mathcal{P}_f(A) \times A$. For every $P, Q \in \mathcal{P}_f(A)$ we set :

- $P \vdash^{[0]} Q$ iff $P \supseteq Q$
- $P \vdash^{[1]} Q$ iff $\forall q \in Q, \exists R \subseteq P, R \vdash q$
- $P \vdash^{<0>} Q$ iff $P \vdash^{[0]} Q$
- $P \vdash^{<1>} Q$ iff $\forall q \in Q, (\exists R \subseteq P, R \vdash q) \text{ or } (q \in P)$
- $P \vdash^{<n+1>} Q$ iff $\exists R \in \mathcal{P}_f(A), P \vdash^{<1>} R$ and $R \vdash^{<n>} Q$ (for every $n \geq 0$).
- $\vdash = \bigcup_{n \geq 0} \vdash^{<n>}$. 

Given $\vdash_1, \vdash_2 \subseteq \mathcal{P}_f(A) \times \mathcal{P}_f(A)$, for every $P, Q \in \mathcal{P}_f(A)$ we set :

$$P(\vdash_1 \circ \vdash_2)Q \text{ iff } \exists R \subseteq A, (P \vdash_1 R) \wedge (R \vdash_2 Q).$$
4.2 System $B_0$

Let us define here a particular formal system $B_0$ “Taylored for the $\sigma$-$\bar{\psi}$-bisimulation problem for deterministic series”.

Let us fix two finite alphabets $X, Y$, a surjection $\psi : X \to Y$ (which induces a surjection $X^* \to Y^*$ denoted by the same symbol $\bar{\psi}$) and its kernel $\bar{\psi} = \text{Ker}\psi \subseteq X^* \times X^*$ (see section 3.2). We also fix a dpda $M$ over the terminal alphabet $X$ and consider the variable alphabet $V$ associated to $M$ (see section 3.1) and the sets $\text{DRB}_{\delta,\lambda}\langle\langle V \rangle\rangle$ (the sets of Deterministic Rational Boolean matrices over $V^*$, with $\delta$ rows and $\lambda$ columns). The set of assertions is defined by:

$$\mathcal{A} = \bigcup_{\lambda \geq 1} \mathbb{N} \times \text{DRB}_{1,\lambda}\langle\langle V \rangle\rangle \times \text{DRB}_{1,\lambda}\langle\langle V \rangle\rangle$$

i.e. an assertion is here a weighted equation over $\text{DRB}_{1,\lambda}\langle\langle V \rangle\rangle$ for some integer $\lambda$.

For every $n \geq 0$ we define

$$\bar{B}_n = \{ R \subseteq \bar{\psi} | R \text{ fulfills conditions (1'), (2') and (4) of definition 324} \}. \quad (61)$$

We call the elements of $\bar{B}_n$ the admissible relations of order $n$ over $X^* \times X^*$. For every pair $(S, S') \in \text{DRB}_{1,\lambda}\langle\langle V \rangle\rangle \times \text{DRB}_{1,\lambda}\langle\langle V \rangle\rangle$, and $n \in \mathbb{N} \cup \{\infty\}$ we define:

$$\mathcal{B}_n(S, S') = \{ R \subseteq \bar{\psi} | R \text{ is a } w - \bar{\psi} \text{-bisimulation of order } n \text{ w.r.t. } (S, S') \}. \quad (62)$$

The “cost-function” $H : \mathcal{A} \to \mathbb{N} \cup \{\infty\}$ is defined by:

$$H(n, S, S') = n + 2 \cdot \text{Div}(S, S'),$$

where $\text{Div}(S, S')$ is the divergence between $S$ and $S'$ (definition 326). We recall it is defined by:

$$\text{Div}(S, S') = \inf\{ n \in \mathbb{N} | \mathcal{B}_n(S, S') = \emptyset \}.$$

(We recall $\inf(\emptyset) = \infty$).

Let us notice that, by lemma 323:

$$\chi(n, S, S') = 1 \iff S \sim S'.$$

We define a binary relation $\vdash \subseteq \mathcal{P}_f(\mathcal{A}) \times \mathcal{A}$, the elementary deduction relation, as the set of all the pairs having one of the following forms:

**(R0)**

$$\{(p, S, T)\} \vdash (p + 1, S, T)$$

for $p \in \mathbb{N}, \lambda \in \mathbb{N} - \{0\}, S, T \in \text{DRB}_{1,\lambda}\langle\langle V \rangle\rangle$,

**(R1)**

$$\{(p, S, T)\} \vdash (p, T, S)$$

for $p \in \mathbb{N}, \lambda \in \mathbb{N} - \{0\}, S, T \in \text{DRB}_{1,\lambda}\langle\langle V \rangle\rangle$,
(R2) \[ \{(p, S, S'), (p, S', S'')\} \models (p, S, S'') \]
for \( p \in \mathbb{N}, \lambda \in \mathbb{N} - \{0\}, S, T \in \text{DRB}_{1,\lambda}(\langle V \rangle), \)
(R3) \[ \emptyset \models (0, S, S) \]
for \( S \in \text{DRB}_{1,\lambda}(\langle V \rangle), \)
(R'3) \[ \emptyset \models (0, S, \rho_e(S)) \]
for \( S \in \text{DRB}_{1,\lambda}(\langle V \rangle), \)
(R4) \[ \{(p + 1, S \odot x, T \odot x') \mid (x, x') \in \mathcal{R}_1\} \models (p, S, T) \]
for \( p \in \mathbb{N}, \lambda \in \mathbb{N} - \{0\}, S, T \in \text{DRB}_{1,\lambda}(\langle V \rangle), (S \neq \epsilon \land T \neq \epsilon) \) and \( \mathcal{R}_1 \in \mathcal{B}_1, \)
(R5) \[ \{(p, S, S')\} \models (p + 2, S \odot x, S' \odot x') \]
for \( p \in \mathbb{N}, \lambda \in \mathbb{N} - \{0\}, S, T \in \text{DRB}_{1,\lambda}(\langle V \rangle), (x, x') \in \mathcal{P}_1, S \sim S' \land S \odot x \sim S' \odot x', \)
(R6) \[ \{(p, S_1 \cdot T + S, T)\} \models (p, S_1 \cdot S, T) \]
for \( p \in \mathbb{N}, \lambda \in \mathbb{N} - \{0\}, S_1 \in \text{DRB}_{1,\lambda}(\langle V \rangle), S_1 \neq \epsilon, (S_1, S) \in \text{DRB}_{1,\lambda+1}(\langle V \rangle), T \in \text{DRB}_{1,\lambda}(\langle V \rangle), \)
(R7) \[ \{(p, S, S')\} \models (p, S, T, S' \cdot T) \]
for \( p \in \mathbb{N}, \delta, \lambda \in \mathbb{N} - \{0\}, S, S' \in \text{DRB}_{1,\delta}(\langle V \rangle), T \in \text{DRB}_{1,\delta}(\langle V \rangle), \)
(R8) \[ \{(p, T_i \cdot S, T_i' \cdot S) \mid 1 \leq i \leq \delta\} \models (p, S \cdot T, S' \cdot T') \]
for \( p \in \mathbb{N}, \delta, \lambda \in \mathbb{N} - \{0\}, S \in \text{DRB}_{1,\delta}(\langle V \rangle), T, T' \in \text{DRB}_{1,\delta}(\langle V \rangle). \)

Remark 43. We do not claim that this formal system is recursively enumerable: due to rule (R5), establishing this property is as difficult as to solve the general bisimulation problem for equational graphs of finite out-degree. This difficulty will be overcome in section [4] by an elimination lemma.

Lemma 44. Let \( P \in \mathcal{P}_f(A), A \in \mathcal{A} \) such that \( P \models A \). Then \( \min\{H(p) \mid p \in P\} \leq H(A) \).

Let us introduce a notation: for every \( n \in \mathbb{N} \cup \{\infty\}, \lambda \in \mathbb{N} - \{0\}, S, S' \in \text{DRB}_{1,\lambda}(\langle V \rangle), \)
\[ S \sim_n S' \Leftrightarrow \mathcal{B}_n(S, S') \neq \emptyset. \]

Proof: Let us check this property for every type of rule.
R0: \( p + 2 \cdot \text{Div}(S, T) \leq p + 1 + 2 \cdot \text{Div}(S, T). \)
**R1:** $p + 2 \cdot \text{Div}(S, T) = p + 2 \cdot \text{Div}(T, S)$.

**R2:** as the weight $p$ is the same in all the considered equations, we are reduced to prove that:

$\forall n \in \mathbb{N}, S \sim_n S' \wedge S' \sim_n S'' \implies S \sim_n S''$. This is true because, if $R \in B_n(S, S')$ and $R' \in B_n(S', S'')$, then $R \circ R' \in B_n(S, S'')$.

**R3:** Let us notice that $\text{Id}_X \subseteq \bar{\psi}$. It follows that $\infty = \text{Div}(S, S)$.

**R'3:** The definition of $G$ from $G_0$ is such that, $S \equiv \rho_e(S)$, hence $S \sim \rho_e(S)$ and $\infty = \text{Div}(S, \rho_e(S))$.

**R4:** Let $n \in \mathbb{N}$ such that:

$\forall (x, x') \in R_1$, $n \leq \text{Div}(S \circ x, S' \circ x')$.

Let us choose, for every $(x, x') \in R_1$, some $R_{x, x'} \in B_n(S \circ x, S' \circ x')$. Let us define then

$$R = \bigcup_{(x, x') \in R_1} (x, x') \cdot R_{x, x'}.$$ 

$R$ belongs to $B_{n+1}(S, S')$. It follows that

$$\min\{\text{Div}(S \circ x, S' \circ x') \mid (x, x') \in R_1\} + 1 \leq \text{Div}(S, S')$$

hence that

$$\min\{H(p + 1, S \circ x, S' \circ x') \mid (x, x') \in R_1\} \leq H(p, S, S') - 1.$$

**R5:** By hypothesis, $H(p + 2, S \circ x, S' \circ x') = \infty$.

**R6:** Let $n \in \mathbb{N}$ such that:

$$n \leq \text{Div}(S_1 \cdot T + S, T).$$

Let $R \in B_n(S_1 \cdot T + S, T)$. Let $R' = R^{<S_1,>} \cdot (x, x') \cdot R_{x, x'}$. Let us consider $R \subseteq B_n(S_1 \cdot T + S, T)$, we get the inequality:

$$\text{Div}(S_1 \cdot T + S, T) \leq \text{Div}(S_1 \cdot T, S).$$

**R7:** Let $n \leq \text{Div}((S, S')$ and $R \in B_n(S, S')$. Let us consider: $R' = R^{<S_1,>} \cdot (x, x') \cdot R_{x, x'}$. As we have $R' \in B_n(S \cdot T, S' \cdot T)$, the required inequality is proved.

**R8:** Let $n \leq \min\{\text{Div}(T_{i, *}, T'_{i, *}) \mid 1 \leq i \leq \delta\}$ and, for every $i \in [1, \delta]$, let $R_i \in B_n(T_{i, *}, T'_{i, *})$. Let us consider $R' = R^{<S_1,>} \cdot (x, x') \cdot R_{x, x'}$. As we know that $R' \in B_n(S \cdot T, S' \cdot T')$, the required inequality is proved. \(\square\)

Let us define $\models$ by: for every $P \in \mathcal{P}_f(A), A \in \mathcal{A}$,

$$P \models A \iff P \models [1] \circ [0, 3, 4] \models \{A\}.$$
where $\vdash_{0,3,4}$ is the relation defined by $R_0, R_3, R'_3, R_4$ only. We let

$$B_0 = \langle A, H, \vdash >.$$  

**Lemma 45:** $B_0$ is a formal system.

**Proof:** Using lemma 44, one can show by induction on $n$ that:

$$P \vdash^n Q \quad \Rightarrow \quad \forall q \in Q, \min\{H(A) \mid A \in P\} \leq H(q).$$

The proof of lemma 44 also reveals that:

$$P \vdash (0,3,4) q \quad \Rightarrow \quad (\min\{H(p) \mid p \in P\} < H(q)) \text{ or } H(q) = \infty.$$

It follows that, for every $m, n \geq 0$:

$$P \vdash^n Q \vdash^1 R \vdash^m q \quad \Rightarrow \quad (\min\{H(p) \mid p \in P\} < H(q)) \text{ or } H(q) = \infty.$$

Hence axiom $(A1)$ is fulfilled. □

Let us remark the following algebraic corollaries of lemma 44.

**Corollary 46**

(C1) $\forall \lambda \in \mathbb{N} - \{0\}, S_1 \in \text{DRB}_{1,1}(\langle V \rangle), S_1 \not= \epsilon, (S_1, S) \in \text{DRB}_{1,\lambda+1}(\langle V \rangle), T \in \text{DRB}_{1,\lambda}(\langle V \rangle),$

$$S_1 \cdot T + S \sim T \quad \Rightarrow \quad S_1^* \cdot S \sim T$$

(C2) $\forall \delta, \lambda \in \mathbb{N} - \{0\}, S, S' \in \text{DRB}_{\delta,\lambda}(\langle V \rangle), T \in \text{DRB}_{\delta,\lambda}(\langle V \rangle),$

$$S \sim S' \Rightarrow S \cdot T \sim S' \cdot T$$

(C3) $\forall \lambda \in \mathbb{N} - \{0\}, S, S' \in \text{DRB}_{1,\lambda}(\langle V \rangle), T \in \text{DRB}_{1,\lambda}(\langle V \rangle),$

$$[S \cdot T \sim S' \cdot T \text{ and } T \not= \emptyset] \Rightarrow S \sim S'$$

(C4) $\forall \delta, \lambda \in \mathbb{N} - \{0\}, S \in \text{DRB}_{1,\delta}(\langle V \rangle), T, T' \in \text{DRB}_{\delta,\lambda}(\langle V \rangle),$

$$T \sim T' \Rightarrow S \cdot T \sim S \cdot T'.$$

**Proof:** Statement (Ci) (for $1 \leq i \leq 4$) is a direct corollary of the fact that the value of $H$ at the left-hand side of some rule (Rj) is smaller or equal to the value of $H$ at the right-hand side of rule (Rj): (C1) is justified by (R6), (C2) by (R7), (C4) by (R8).
Let us prove (C3): suppose that \( \lambda \in \mathbb{N} - \{0\} \), \( S, S' \in DRB_{1,1}(\langle V \rangle) \), \( T \in DRB_{1,\lambda}(\langle V \rangle) \) and
\[ S \cdot T \sim S' \cdot T \text{ and } S \not\sim S'. \] (63)

Let \( R \in B_\infty(S \cdot T, S' \cdot T) \) and let
\[ (u, u') = \min \{(v, v') \in R \mid (\rho_e(S \odot v) = \epsilon) \Leftrightarrow (\rho_e(S' \odot v') \neq \epsilon)\}. \]

From the hypothesis that \( R \in B_\infty(S \cdot T, S' \cdot T) \), we get that
\[ \forall (v, v') \in R, (S \cdot T) \odot v \sim (S' \cdot T) \odot v', \]
and by the choice of \( (u, u') \) we obtain that:
\[ T \sim (S' \odot u') \cdot T \text{ or } (S \odot u) \cdot T \sim T, \]
which, by C1, implies:
\[ T \sim (S' \odot u')^* \cdot \emptyset^\lambda \text{ or } (S \odot u)^* \cdot \emptyset^\lambda \sim T, \]
i.e. \( T \sim \emptyset^\lambda \), which implies (because \( G \) is a reduced grammar) that
\[ T = \emptyset^\lambda. \] (64)

We have proved that (63) implies (64), hence (C3). \( \square \)

4.3 Congruence closure

Let us consider the subset \( C \) of the rules of \( B_0 \), consisting of all the instances of the metarules \( R0, R1, R2, R3, R'3, R6, R7, R8 \). We also denote by \( \models_c \subseteq P_f(\mathcal{A}) \times \mathcal{A} \) the set of all instances of these meta-rules. We are interested here (and later in section 10.1) in special subsets of \( \mathcal{A} \) which express an ordinary weighted equation \((p, S, S')\) together with an admissible binary relation \( R \) of finite order (which is a candidate to be a \( w-\psi \)-bisimulation w.r.t. \((S, S')\)).

For every \( p, n \in \mathbb{N}, \lambda \in \mathbb{N} - \{0\} \), \( S, S' \in DRB_{1,\lambda}(\langle V \rangle) \), \( R \in \bar{B}_n \), we use the notation:
\[ [p, S, S', R] = \{(p + |u|, S \odot u, S' \odot u') \mid (u, u') \in R\}. \] (65)

One can check the following properties.

**composition:**
for every \( p, n \in \mathbb{N}, \lambda \in \mathbb{N} - \{0\} \), \( S, T \in DRB_{1,\lambda}(\langle V \rangle) \), \( R_1, R_2 \in \bar{B}_n \),
\[ [p, S, S', R_1] \cup [p, S', S'', R_2] \models_c [p, S, S'', R_1 \circ R_2] \]

**star:**
for every \( p, n \in \mathbb{N}, \lambda \in \mathbb{N} - \{0\} \), \( S_1 \in DRB_{1,1}(\langle V \rangle) \), \( S_1 \not\equiv_c (S_1, S) \in DRB_{1,\lambda + 1}(\langle V \rangle) \), \( T \in DRB_{1,\lambda}(\langle V \rangle) \), \( R \in \bar{B}_n \),
\[ [p, S_1 \cdot T + S, T, R] \models_c [p, S_1^* \cdot S, T, R^{S_1,*}] \]
right-product:
for every \( p, n \in \mathbb{N}, \delta, \lambda \in \mathbb{N} - \{0\}, S, S' \in \text{DRB}_1, T \in \text{DRB}_\delta, R \in \bar{B}_n, \)
\[
[p, S, S', R] \quad <\quad [p, S \cdot T, S', T, <S|R>] 
\]

left-product:
for every \( p, n \in \mathbb{N}, \delta, \lambda \in \mathbb{N} - \{0\}, S \in \text{DRB}_1, T, T' \in \text{DRB}_\delta, R_1, \ldots, R_\delta \in \bar{B}_n, \)
\[
\bigcup_{1 \leq i \leq \delta} [p, T_i, T_i', R_i] \quad <\quad [p, S \cdot T, S \cdot T', <S,R>].
\]

Given a subset \( P \in \mathcal{P}(A), \) we call congruence closure of \( P, \) denoted by \( \text{Cong}(P), \)
the set:
\[
\text{Cong}(P) = \{ A \in A \mid P <\quad A \}\quad (66)
\]
As well, for every integer \( q \geq 0 \) we define:
\[
\text{Cong}_q(P) = \{ A \in A \mid P <q\quad A \}\quad (67)
\]

4.4 Strategies

One key-step of this work is the statement that \( B_0 \) is complete (theorem 106).
We prove this completeness result by exhibiting a “strategy” \( S \) which, for every true assertion \( (p, S, S'), \)
constructs a finite \( B_0 \)-proof of this assertion. Let \( D = \langle A, H, |\quad > \rangle \) be a formal system. We call a strategy for \( D \) any map \( S : A^+ \rightarrow \mathcal{P}(A^*) \) such that:
(S1) if \( B_1 \cdots B_m \in S(A_1 A_2 \cdots A_n) \) then \( \exists Q \subseteq \{ A_i \mid 1 \leq i \leq n - 1 \} \) such that
\[
\{ B_j \mid 1 \leq j \leq m \} \cup Q |\quad A_n,
\]
(S2) if \( B_1 \cdots B_m \in S(A_1 A_2 \cdots A_n) \) then
\[
\min\{ H(A_i) \mid 1 \leq i \leq n \} = \infty \implies \min\{ H(B_j) \mid 1 \leq j \leq m \} = \infty.
\]

Remark 47 It may happen that \( \epsilon \in S(A_1 A_2 \cdots A_n) \) (and correspondingly, that \( m = 0 \) in the above conditions): it just means that \( \{ A_1, \ldots, A_{n-1} \} |\quad A_n. \) It may also happen that \( S(A_1 A_2 \cdots A_n) = \emptyset \): it means, intuitively, that \( S \) “does not know” how to extend a proof (with hypothesis), with the only information that the given proof contains the assertions \( A_1, A_2, \cdots A_n. \)

Remark 48 Axiom (A1) on systems is similar to the “monotonicity” condition of \([HHY74]\) or axiom (2.4.2) of \([Cou83a]\). Axiom (S2) on strategies is similar to the “validity” condition of \([HHY74]\) or property (2.4.1') of \([Cou83a]\).
Given a strategy $S$, we define $T(S,A)$, the set of proof-trees associated to the strategy $S$ and the assertion $A$ as the set of all the trees $t$ fulfilling the following properties:

$$
\varepsilon \in \text{dom}(t), \quad t(\varepsilon) = A,
$$

(68)

and, for every path $x_0x_1\cdots x_{n-1}$ in $t$, with labels $t(x_i) = A_{i+1}$ (for $0 \leq i \leq n-1$) if $x_{n-1}$ has $m$ sons $x_{n-1} \cdot 1, \cdots, x_{n-1} \cdot m \in \text{dom}(t)$ with labels $t(x_{n-1} \cdot j) = B_j$ (for $1 \leq j \leq m$) then

$$(B_1 \cdots B_m) \in S(A_1 \cdots A_n) \text{ or } m = 0.
$$

(69)

The proof-tree $t$ is said closed iff it fulfills the additional condition: for every path $x_0x_1\cdots x_{n-1}$ in $t$, with labels $t(x_i) = A_{i+1}$ (for $0 \leq i \leq n-1$) if $x_{n-1}$ has $m$ sons $x_{n-1} \cdot 1, \cdots, x_{n-1} \cdot m \in \text{dom}(t)$ with labels $t(x_{n-1} \cdot j) = B_j$ (for $1 \leq j \leq m$) then

$$
m = 0 \Rightarrow ((\exists i \in [1,n-1], A_i = A_n) \text{ or } (\varepsilon \in S(A_1 \cdots A_n)))
$$

(70)

A node $x \in \text{dom}(t)$ is said closed iff it is an internal node or it is a leaf fulfilling property (70) above.

The proof-tree $t$ is said repetition-free iff, for every $x, x' \in \text{dom}(t)$,

$$
[x \preceq x' \text{ and } t(x) = t(x')] \Rightarrow x = x' \text{ or } x' \text{ is a leaf}.
$$

(71)

For every tree $t$ let us define:

$$
\mathcal{L}(t) = \{t(x)|\forall y \in \text{dom}(t), x \preceq y \Rightarrow x = y\}, \quad \mathcal{I}(t) = \{t(x)|\exists y \in \text{dom}(t), x \prec y\}.
$$

(Here $\mathcal{L}$ stands for “leaves” and $\mathcal{I}$ stands for “internal nodes”).

**Lemma 49** If $S$ is a strategy for the deduction-system $D$ then, for every true assertion $A$ and every $t \in T(S,A)$

1. the set of labels of $t$ is a $D$-proof, relative to the set $\mathcal{L}(t) - \mathcal{I}(t)$.
2. every label of a leaf is true.

**Proof:** Let us suppose that $H(A) = \infty$. Let $t \in T(S,A), P = \text{im}(t)$ (the set of labels of $t$), $\mathcal{H} = \mathcal{L}(t) - \mathcal{I}(t)$.

Using (S2), one can prove by induction on the depth of $x \in \text{dom}(t)$ that, $H(t(x)) = \infty$. Point (2) is then proved. Let $x$ be an internal node of $t$, with sons $x \cdot 1, x \cdot 2, \cdots, x \cdot m (m \geq 1)$, and with ancestors $y_1, y_2, \cdots, y_{n-1}, y_n = x (n \geq 1)$, such that

$$
t(y_1) \cdots t(y_n) = A_1 \cdots A_n, \quad t(x_1) \cdots t(x_m) = B_1 \cdots B_m.
$$

By definition of $T(S,A),

$$
B_1 \cdots B_m \in S(A_1 \cdots A_n)
$$
and by condition (S1):

$$\exists Q \subseteq \{A_i \mid i \leq n - 1\}, \text{ such that } \{B_j \mid 1 \leq j \leq m\} \bigcup Q \vdash A_n.$$ 

It follows that for every $p \notin H$, $\exists R \subseteq P, R \vdash p$, hence

$$\forall p \in P, (\exists R \subseteq P, R \vdash p) \text{ or } p \in H.$$ 

Point (1) is proved. □

For every $D$-strategy $S$, we use the notation:

$$T(S) = \bigcup_{A \in H^{-1}(\infty)} T(S, A).$$

We call a global strategy w.r.t. $S$ any total map $\hat{S} : T(S) \to T(S)$ such that:

$$\forall t \in T(S), t \preceq \hat{S}(t). \quad (71)$$

$\hat{S}$ is a terminating global strategy iff:

$$\forall A_0 \in H^{-1}(\infty), \exists n_0 \in \mathbb{N}, \hat{S}^{n_0}(A_0) = \hat{S}^{n_0+1}(A_0), \quad (72)$$

$\hat{S}$ is a closed global strategy iff:

$$\forall A_0 \in H^{-1}(\infty), \forall n \in \mathbb{N}, \hat{S}^n(A_0) \text{ is closed } \iff \hat{S}^{n}(A_0) = \hat{S}^{n+1}(A_0), \quad (73)$$

(where the assertion $A_0$ is identified with the tree reduced to one node whose label is $A_0$).

**Lemma 410**: Let $D$ be a formal system, $S$ a strategy for $D$ and $\hat{S}$ a global strategy w.r.t. $S$. If $\hat{S}$ is terminating and $\hat{S}$ is closed, then $D$ is complete.

**Proof**: Let $A_0 \in A$. Under the hypothesis of the lemma, $\exists n_0 \in \mathbb{N}$ such that (72) and (73) are both true. Hence $t_\infty = \hat{S}^{n_0}(A_0)$ is a closed proof-tree for $S$. By lemma 49, $\text{im}(t_\infty)$ is a $D$-proof relative to the set $\mathcal{L}(t_\infty) - \mathcal{I}(t_\infty)$. Let $x$ be a leaf such that $t_\infty(x) \in \mathcal{L}(t_\infty) - \mathcal{I}(t_\infty)$. Let $A_0, A_1, \ldots, A_n = t_\infty(x)$ be the word labelling the path from the root to $x$. As $x$ is closed and $t_\infty(x) \in \mathcal{L}(t_\infty) - \mathcal{I}(t_\infty)$ by (70), $\varepsilon \in S(A_1 \cdots A_n)$ hence $\{A_1, \ldots, A_{n-1}\} \vdash t_\infty(x)$. It follows that $\text{im}(t_\infty)$ is a $D$-proof. □
5 Triangulations

Let $S_1, S_2, \ldots, S_d$ be a family of deterministic row-vectors over the structured alphabet $V$ (i.e. $S_i \in \text{DRB}_{1,\lambda}((V))$ where $\lambda \in \mathbb{N} \setminus \{0\}$). We recall $V$ is the alphabet associated with some dpda $\mathcal{M}$ as defined in section 2.4.

Let us consider a sequence $\mathcal{S}$ of $n$ “weighted” linear equations:

\[
(\mathcal{E}_i) : p_i \sum_{j=1}^{d} \alpha_{i,j} S_j, \sum_{j=1}^{d} \beta_{i,j} S_j
\]

(74)

where $p_i \in \mathbb{N} \setminus \{0\}$, and $A = (\alpha_{i,j})$, $B = (\beta_{i,j})$ are deterministic rational matrices of dimension $(n, d)$, with indices $m \leq i \leq m + n - 1$, $1 \leq j \leq d$.

For any weighted equation, $\mathcal{E} = (p, S, S')$, we recall the “cost” of this equation is: $H(\mathcal{E}) = p + 2 \cdot \text{Div}(S, S')$.

Let us define an oracle on deterministic vectors as a mapping $\mathcal{O} : \bigcup_{\lambda \geq 1} \text{DRB}_{1,\lambda}((V)) \times \text{DRB}_{1,\lambda}((V)) \rightarrow \mathcal{P}(X^* \times X^*)$ such that:

\[
\forall (S, S') \in \text{DRB}_{1,\lambda}((V)) \times \text{DRB}_{1,\lambda}((V)), S \sim S' \Rightarrow \mathcal{O}(S, S') \in \mathcal{B}_\infty(S, S').
\]

In other words, an oracle is a choice of $w$-$\tilde{\nu}$-bisimulation for every pair of equivalent vectors (modulo $\sim$). Let us denote by $\Omega$ the set of all oracles. Let us fix an oracle $\Omega$ throughout this section.

We associate to every system 2 another equation, $\text{INV}^{(\mathcal{O})}(\mathcal{S})$, which translates the equations of $\mathcal{S}$ into equations over the coefficients $(\alpha_{i,j}, \beta_{i,j})$ only.

The general idea of the construction of $\text{INV}^{(\mathcal{O})}$ consists in iterating the transformation used in the proof of (1) $\Rightarrow$ (2) $\Rightarrow$ (3) in lemma 330, i.e. the classical idea of triangulating a system of linear equations. Of course we must deal with the weights and relate the construction with the deduction system $\mathcal{B}_0$.

We assume here that

\[
\forall j \in [1, d], S_j \neq \emptyset^\lambda.
\]

Let us define $\text{INV}^{(\mathcal{O})}(\mathcal{S}), \text{W}^{(\mathcal{O})}(\mathcal{S}) \in \mathbb{N} \cup \{\bot\}$, $D^{(\mathcal{O})}(\mathcal{S}) \in \mathbb{N}$ by induction on $n$. $\text{W}^{(\mathcal{O})}(\mathcal{S})$ is the weight of $\mathcal{S}$, $D^{(\mathcal{O})}(\mathcal{S})$ is the weak codimension of $\mathcal{S}$.

Case 1: $\alpha_{m,*} \sim \beta_{m,*}$.

\[
\text{INV}^{(\mathcal{O})}(\mathcal{S}) = (\text{W}^{(\mathcal{O})}(\mathcal{S}), \alpha_{m,*}, \beta_{m,*}), \text{W}^{(\mathcal{O})}(\mathcal{S}) = p_m - 1, D^{(\mathcal{O})}(\mathcal{S}) = 0.
\]

Case 2: $\alpha_{m,*} \not\sim \beta_{m,*}, n \geq 2, p_{m+1} - p_{m} \geq 2 \cdot \text{Div}(\alpha_{m,*}, \beta_{m,*}) + 1$.

Let us consider $\mathcal{R} = \mathcal{O}(\sum_{j=1}^{d} \alpha_{m,j} S_j, \sum_{j=1}^{d} \beta_{m,j} S_j), \nu = \text{Div}(\alpha_{m,*}, \beta_{m,*})$ and

\[
\langle u, u' \rangle = \min\{(v, v') \in \mathcal{R} \cap X^\leq \nu \times X^\leq \nu \mid \exists j \in [1, d], (\alpha_{m,*} \circ v = \epsilon_j^\lambda) \Leftrightarrow (\beta_{m,*} \circ v' \neq \epsilon_j^\lambda)\}.
\]

The function $\text{INV}$ defined in [Sen97b] was an “elaborated version” of the inverse systems defined in [Mei89, Mei92] in the case of a single equation. We consider here a relativization of this notion to some oracle $\mathcal{O}$. 

\[2\]
Let us consider the integer $j_0 \in [1, d]$ such that $(\alpha_{m, s} \odot u = \epsilon_{j_0} \odot u) \leftrightarrow (\beta_{m, s} \odot u' \neq \epsilon_{j_0} \odot u)$.

**Subcase 1:** $\alpha_{m, j_0} \odot u = \epsilon, \beta_{m, j_0} \odot u' \neq \epsilon$.

Let us consider the equation

$$\langle \xi_m' \rangle : p_m + 2 \cdot |u|, S_{j_0}, \sum_{j \neq j_0}^d (\beta_{m, j_0} \odot u')^* (\beta_{m, j} \odot u') S_j$$

and define a new system of weighted equations $S' = (\xi_i')_{m+1 \leq i \leq m+n-1}$ by:

$$\langle \xi_i' \rangle : p_i, \sum_{j \neq j_0} [(\alpha_{i,j} + \alpha_{i,j_0} (\beta_{m,j_0} \odot u')^* (\beta_{m,j} \odot u') S_j, \sum_{j \neq j_0} [(\beta_{i,j} + \beta_{i,j_0} (\beta_{m,j_0} \odot u')^* (\beta_{m,j} \odot u') S_j$$

where the above equation is seen as an equation between two linear combinations of the $S_i$'s, $1 \leq i \leq d$, where the $j_0$-th coefficient is $\emptyset$ on both sides. We then define:

$$\text{INV}^{(O)}(S) = \text{INV}^{(O)}(S'), W^{(O)}(S) = W^{(O)}(S'), D^{(O)}(S) = D^{(O)}(S') + 1.$$  

**Subcase 2:** $\alpha_{m, j_0} \odot u \neq \epsilon, \beta_{m, j_0} \odot u' = \epsilon$.

(analogous to subcase 1).

**Case 3:** $\alpha_{m, s} \neq \beta_{m, s}, n = 1$.

We then define:

$$\text{INV}^{(O)}(S) = \bot, W^{(O)}(S) = \bot, D^{(O)}(S) = 0,$$

where $\bot$ is a special symbol which can be understood as meaning “undefined”.

**Case 4:** $\alpha_{m, s} \neq \beta_{m, s}, n \geq 2, p_{m+1} - p_m \leq 2 \cdot \text{Div}(\alpha_{m, s}, \beta_{m, s})$.

We then define:

$$\text{INV}^{(O)}(S) = \bot, W^{(O)}(S) = \bot, D^{(O)}(S) = 0.$$

**Lemma 51:** Let $S$ be a system of weighted linear equations with deterministic rational coefficients. If $\text{INV}^{(O)}(S) \neq \bot$ then, $\text{INV}^{(O)}(S)$ is a weighted linear equation with deterministic rational coefficients.

**Proof:** Follows from lemmas [21][22] and the formula defining $S'$ from $S$. □

From now on, and up to the end of this section, we simply write “linear equation” to mean “weighted linear equations with deterministic rational coefficients”.

**Lemma 52:** Let $S$ be a system of weighted linear equations with deterministic rational coefficients. If $\text{INV}^{(O)}(S) \neq \bot$ then:

1. $\{\text{INV}^{(O)}(S)\} \cup \{\xi_i | m \leq i \leq m + D^{(O)}(S) - 1\} \vdash \xi_{m+D^{(O)}(S)}$

2. $\min\{H(\xi_i) | m \leq i \leq m + D^{(O)}(S)\} = \infty \implies H(\text{INV}^{(O)}(S)) = \infty.$
\textbf{Proof:} See on figure \[ \] the “graph of the deductions” we use for proving point (1). Let us prove by induction on $D^{(O)}(\mathcal{S})$ the following strengthened version of point (1):

$$\{\text{INV}^{(O)}(\mathcal{S})\} \cup \{\mathcal{E}_i \mid m \leq i \leq m + D^{(O)}(\mathcal{S}) - 1\} \vdash^{\ast \ast} \tau^{-1}(\mathcal{E}_{m+D^{(O)}(\mathcal{S})})$$

(77)

where, for every integer $k \in \mathbb{Z}$, $\tau_k : \{(p, S, S') \in \mathcal{A} \mid p \geq -k\} \rightarrow \mathcal{A}$ is the translation map on the weights: $\tau_k(p, S, S') = (p + k, S, S')$.

if $D^{(O)}(\mathcal{S}) = 0$ : as $\text{INV}^{(O)}(\mathcal{S}) \neq \bot$, $\mathcal{S}$ must fulfill the hypothesis of case 1.

$$\mathcal{E}_m = (p_m, \sum_{j=1}^{d} \alpha_{m,j}S_j, \sum_{j=1}^{d} \beta_{m,j}S_j) = \mathcal{E}_{m+D^{(O)}(\mathcal{S})}$$

$$\text{INV}^{(O)}(\mathcal{S}) = (p_m - 1, \alpha_{m,*}, \beta_{m,*}).$$

Using rules (R7) we obtain :

$$\text{INV}^{(O)}(\mathcal{S}) \vdash^{\ast \ast} (p_m - 1, \sum_{j=1}^{d} \alpha_{m,j}S_j, \sum_{j=1}^{d} \beta_{m,j}S_j) = \tau^{-1}(\mathcal{E}_m).$$

\textbf{if} $D^{(O)}(\mathcal{S}) = n + 1, n \geq 0$ : $\mathcal{S}$ must fulfill case 2.

\textbf{Suppose case 2, subcase 1 occurs.}

As the relation $\mathcal{R}$ used in the construction of $\mathcal{E'}_m$ from $\mathcal{E}_m$ is a $w$-$\bar{\psi}$-bisimulation w.r.t. the pair of sides of equation $\mathcal{E}_m$, using (R5) and then (R6), (this is possible because $\beta_{m,j_0} \odot u' \neq \epsilon$), we obtain a deduction :

$$\mathcal{E}_m \vdash^{2:|u|+1} \mathcal{E'}_m.$$

(78)

Using (R2,R8) we get that, for every $i \in [m + 1, m + D^{(O)}(\mathcal{S})]$

$$\{\mathcal{E}_i, \mathcal{E'}_m\} \vdash^{\ast \ast} (\max\{p_i, p_m + 2 \mid u\}, \sum_{j \neq j_0} (\alpha_{i,j} + \alpha_{i,j_0}(\beta_{m,j} \odot u'))S_j, \sum_{j \neq j_0} (\beta_{i,j} + \beta_{i,j_0}(\beta_{m,j} \odot u'))S_j)$$

but the hypothesis of case 2 implies that $\max\{p_m + 1, p_m + 2 \mid u\} = p_m + 1$ and the fact that $\text{INV}^{(O)}(\mathcal{S'})$ is defined implies that $\forall i \in [m + 1, m + D^{(O)}(\mathcal{S})], p_i \geq p_m + 1$, hence, $\max\{p_i, p_m + 2 \mid u\} = p_i$ and the right-hand side of the above deduction is exactly $\mathcal{E'}_i$. Hence,

$$\forall i \in [m + 1, m + D^{(O)}(\mathcal{S})], \{\mathcal{E}_i, \mathcal{E'}_m\} \vdash^{\ast \ast} \mathcal{E'}_i.$$  

(79)

Using deductions (78) and (79), we obtain that:

$$\{\mathcal{E}_i \mid m \leq i \leq m + D^{(O)}(\mathcal{S}) - 1\} \vdash^{\ast \ast} \{\mathcal{E'}_i \mid m \leq i \leq m + D^{(O)}(\mathcal{S}) - 1\}. $$

(80)

By induction hypothesis :

$$\text{INV}^{(O)}(\mathcal{S'}) \cup \{\mathcal{E'}_i \mid m + 1 \leq i \leq m + 1 + D^{(O)}(\mathcal{S'}) - 1\} \vdash^{\ast \ast} \tau^{-1}(\mathcal{E'}_{m+1+D^{(O)}(\mathcal{S'})})$$
which is equivalent to

\[ \text{INV}^{(O)}(S) \cup \{ E'_i \mid m + 1 \leq i \leq m + D^{(O)}(S) - 1 \} \models \tau_1(E'_{m + D^{(O)}(S)}). \quad (81) \]

As \( p_m + 2 \cdot |u| \leq p_{m+1} - 1 \leq p_{m+D^{(O)}(S)} - 1 \), we have also the following inverse deduction (which is similar to deduction (79)):

\[ \{ E'_m, \tau_1(E'_{m + D^{(O)}(S)}) \} \models \tau_1(E_{m + D^{(O)}(S)}). \quad (82) \]

Combining together deductions (78) (79) and (82), we have proved (77). Using rule (R0), this last deduction leads to point (1) of the lemma.

• Suppose now that case 2, subcase 2 occurs.

This case can be treated in the same way as subcase 1, just by exchanging the roles of \( \alpha, \beta \).

Let us prove statement (2) of the lemma.

We prove by induction on \( D^{(O)}(S) \) the statement:

\[ \min \{ H(E_i) \mid m \leq i \leq m + D^{(O)}(S) \} = \infty \implies H(\text{INV}^{(O)}(S)) = \infty. \quad (83) \]

If \( D^{(O)}(S) = 0 \) : as \( \text{INV}^{(O)}(S) \neq \bot \), case 1 must occur. \( \alpha_{m,*} \sim \beta_{m,*} \) implies that \( H(\text{INV}^{(O)}(S)) = \infty \), hence the statement is true.

If \( D^{(O)}(S) = p + 1, p \geq 0 \) : as \( D^{(O)}(S) \geq 1 \) and \( \text{INV}^{(O)}(S) \neq \bot \), case 2 must occur.

Using deductions (78) and (79) established above we obtain that:

\[ \{ E_i \mid m \leq i \leq m + D^{(O)}(S) \} \models \{ E'_i \mid m + 1 \leq i \leq m + 1 + D^{(O)}(S') \}, \]

which proves that

\[ \min \{ H(E_i) \mid m \leq i \leq m + D^{(O)}(S) \} \leq \min \{ H(E'_i) \mid m + 1 \leq i \leq m + 1 + D^{(O)}(S') \}. \quad (84) \]

As \( D^{(O)}(S') = D^{(O)}(S) - 1 \), we can use the induction hypothesis:

\[ \min \{ H(E'_i) \mid m + 1 \leq i \leq m + 1 + D(S') \} = \infty \implies H(\text{INV}^{(O)}(S')) = \infty. \quad (85) \]

As \( \text{INV}^{(O)}(S) = \text{INV}^{(O)}(S') \), (84) (85) imply statement (82). \( \square \)

**Lemma 53**: Let \( S \) be a system of linear equations satisfying the hypothesis of case 2. Then, \( \forall i \in [m + 1, m + n - 1] \),

\[ \parallel \alpha'_{i,*} \parallel \leq \parallel \alpha_{i,*} \parallel + \parallel \beta_{m,*} \parallel + K_0 \mid u \parallel, \parallel \beta'_{i,*} \parallel \leq \parallel \beta_{i,*} \parallel + \parallel \beta_{m,*} \parallel + K_0 \mid u \parallel. \]

**Proof**: The formula defining \( S' \) from \( S \) show that:

\[ \alpha'_{i,*} = \alpha_{i,*} \circ j_0 (\Box_{j_0}^* (\beta_{m,*} \circ u )); \quad \beta'_{i,*} = \beta_{i,*} \circ j_0 (\Box_{j_0}^* (\beta_{m,*} \circ u')). \]
From these equalities and lemmas \([321,322,316]\) the inequalities on the norm follow. □

Let us consider the function \(F\) defined by :

\[
F(d, n) = \max \{ \text{Div}(A, B) \mid A, B \in \text{DRB}_{1,d}(V), \| A \| \leq n, \| B \| \leq n, A \neq B \}.
\]

For every integer parameters \(K_0, K_1, K_2, K_3, K_4 \in \mathbb{N} - \{0\}\), we define integer sequences \((\delta_i, \ell_i, L_i, s_i, S_i, \Sigma_i)_{m \leq i \leq m + n - 1}\) by :

\[
\begin{align*}
\delta_m &= 0, \ell_m = 0, L_m = K_2, s_m = K_3 \cdot K_2 + K_4, S_m = 0, \Sigma_m = 0, \\
\delta_{i+1} &= 2 \cdot F(d, s_i + \Sigma_i) + 1 \\
\ell_{i+1} &= 2 \cdot \delta_{i+1} + 3 \\
L_{i+1} &= K_1 \cdot (L_i + \ell_i + 1) + K_2 \\
s_{i+1} &= K_3 \cdot L_i + K_4 \\
S_{i+1} &= s_i + \Sigma_i + K_0 F(d, s_i + \Sigma_i) \\
\Sigma_{i+1} &= \Sigma_i + S_{i+1}
\end{align*}
\]

for \(m \leq i \leq m + n - 2\).

These sequences are intended to have the following meanings when \(K_0, K_1, K_2, K_3, K_4\) are chosen to be the constants defined in section \([35]\) and the equations \((\mathcal{E}_i)\) are labelling nodes of a B-stacking sequence (see section \([82]\)):

\[
\begin{align*}
\delta_{i+1} &\leq \text{increase of weight between } \mathcal{E}_i, \mathcal{E}_{i+1} \\
\ell_{i+1} &\geq \text{increase of depth between } \mathcal{E}_i, \mathcal{E}_{i+1} \\
L_{i+1} &\geq \text{increase of depth between } \mathcal{E}_m, \mathcal{E}_{i+1} \\
s_{i+1} &\geq \text{size of the coefficients of } \mathcal{E}_{i+1} \\
S_{i+1} &\geq \text{size of the coefficients of } \mathcal{E}_{i+1}^{(i+1-m)} \text{ ( these systems are introduced below in the proof of lemma } [54] \text{) } \\
\Sigma_{i+1} &\geq \text{increase of the coefficients between } \mathcal{E}_k^{(i-m)}, \mathcal{E}_k^{(i+1-m)} \text{ ( for } k \geq i + 1 \text{).}
\end{align*}
\]

For every linear equation \(\mathcal{E} = (p, \sum_{j=1}^d \alpha_j S_j, \sum_{j=1}^d \beta_j S_j)\), we define

\[
\| \| \mathcal{E} \| = \max \{ \| (\alpha_1, \ldots, \alpha_d) \|, \| (\beta_1, \ldots, \beta_d) \| \}.
\]

**Lemma 54** Let \(\mathcal{S} = (\mathcal{E}_i)_{m \leq i \leq m + d - 1}\) be a system of \(d\) linear equations such that \(H(\mathcal{E}_i) = \infty\) ( for every \(i\) ) and :

1. \(\forall i \in [m, m + d - 1], \| \| \mathcal{E}_i \| \leq s_i\)
2. \(\forall i \in [m, m + d - 2], W(\mathcal{E}_{i+1}) - W(\mathcal{E}_i) \geq \delta_{i+1}\).

Then

1. \(\text{INV}^{(\alpha)}(\mathcal{S}) \neq \perp\),
2. \(\text{D}^{(\alpha)}(\mathcal{S}) \leq d - 1\),
3. \(\| \| \text{INV}^{(\alpha)}(\mathcal{S}) \| \| \leq \Sigma_{m+d} + s_{m+d} \text{ (\alpha)}(\mathcal{S}).\)
Using now the hypothesis that $H$, we get that:

- $E_k^i = E_k$ for $m \leq k \leq m + d - 1$
- If case 1 or case 3 or case 4 is realized, $D^{(\mathcal{O})}(S) = 0$, hence $S^{(i-m)}$ is well-defined for $m \leq i \leq m + D^{(\mathcal{O})}(S)$
- If case 2 is realized then we set: $\forall i \geq m + 1, E_k^{(i-m)} = (E_k')^{(i-m-1)}$, for $m + 1 \leq k \leq m + d - 1$.

Let us prove by induction on $i \in [m, m + D^{(\mathcal{O})}(S)]$ that, $\forall k \in [i, m + d - 1]$:

$$\| E_k^{(i-m)} \| \leq s_k + \Sigma_i.$$  \hspace{1cm} (88)

$i = m$ : in this case

$$\| E_k^{(i-m)} \| = \| E_k \| \leq s_k = s_k + \Sigma_m.$$  

$i + 1 \leq m + D^{(\mathcal{O})}(S)$: in this case, by lemma 53

$$\| E_k^{(i+1-m)} \| \leq \| E_k^{(i-m)} \| + \| E_i^{(i-m)} \| + K_0 | u |$$

where $\mathcal{R} = \mathcal{O}(\sum_{j=1}^d \alpha_{i,j}^{(i-m)} \mathcal{S}_j, \sum_{j=1}^d \beta_{i,j}^{(i-m)} \mathcal{S}_j), \nu = \text{Div}(\alpha_{i,*}^{(i-m)} \beta_{i,*}^{(i-m)}), \ (u, u') = \min \{ (v, v') \in \mathcal{R} \cap X \leq \nu \times X \leq \nu \mid \exists j \in [1, d], (\alpha_{i,*}^{(i-m)} \otimes v = \epsilon_i^j) \leftrightarrow (\beta_{i,*}^{(i-m)} \otimes v' \neq \epsilon_i^j) \}$.

By definition of $F$ and the induction hypothesis:

$$| u | \leq F(d, \| E_i^{(i-m)} \|) \leq F(d, s_i + \Sigma_i).$$

Hence

$$\| E_k^{(i+1-m)} \| \leq (s_k + \Sigma_i) + (s_i + \Sigma_i) + K_0 F(d, s_i + \Sigma_i) = (s_k + \Sigma_i) + S_{i+1} = s_k + \Sigma_{i+1}.$$  

Let us notice that $D^{(\mathcal{O})}(S)$ is always an integer and that this proof is valid for $m \leq i \leq m + D^{(\mathcal{O})}(S)$, $i \leq k \leq m + d - 1$.

Let us prove now that $\text{INV}^{(\mathcal{O})}(S) \neq \bot$. Let us consider the system $(E_k^{(\mathcal{O})(S)})_{m + D^{(\mathcal{O})}(S) \leq k \leq m + d - 1}$.

If $D^{(\mathcal{O})}(S) = d - 1$, $(E_k^{(\mathcal{O})(S)})$ fulfills either case 1 or case 3 of the definition of $\text{INV}^{(\mathcal{O})}$ (just because this system consists of a single equation).

Using the successive deductions 53 established in the proof of lemma 52 we get that:

$$\{ E_i \mid m \leq i \leq m + d - 1 \} \xrightarrow{\ast} \{ E_{m+d-1}^{(d-1)} \}.$$  

Using now the hypothesis that $H(E_i) = \infty$ (for $m \leq i \leq m + d - 1$), we obtain:

$$H(\{ E_{m+d-1}^{(d-1)} \}) = \infty.$$  \hspace{1cm} (89)
For any system of equations \( S \), let us define the support of the system as

\[
\text{supp}(S) = \{ j \in [1, d] \mid \sum_{i=m}^{m+n-1} \alpha_{i,j} + \beta_{i,j} \neq \emptyset \}.
\]

Let us consider \( \delta = \text{Card}(\text{supp}(S^{(d-1)})) \). One can prove by induction on \( i \) that:

\[
\text{Card}(\text{supp}(S^{(i-m)})) \leq d - i + m,
\]

hence

\[
\delta = \text{Card}(\text{supp}(S^{(d-1)})) \leq d - (d - 1) = 1.
\]

- If \( \delta = 1, \text{supp}(S^{(d-1)}) = \{ j_0 \} \), for some \( j_0 \in [1, d] \).
  By corollary 46 point C3 and hypothesis (79), the implication

\[
[(\alpha^{(d-1)}_{m+d-1,j_0} S_{j_0} \sim \beta^{(d-1)}_{m+d-1,j_0} S_{j_0}) \implies \alpha^{(d-1)}_{m+d-1,j_0} \sim \beta^{(d-1)}_{m+d-1,j_0}]
\]

holds. Hence, by (80), \( \alpha^{(d-1)}_{m+d-1,*} \sim \beta^{(d-1)}_{m+d-1,*} \), i.e. \( S^{(d-1)} \) fulfills case 1, so that

\[
\text{INV}^{(O)}(S) = \text{INV}^{(O)}(S^{(d-1)}) \neq \perp.
\]

- If \( \delta = 0, \text{supp}(S) = \emptyset \).
  Then \( \alpha^{(d-1)}_{m+d-1,*} = \beta^{(d-1)}_{m+d-1,*} = \emptyset^d \). Here also \( S^{(d-1)} \) fulfills case 1.

If \( D^{(O)}(S) < d - 1 \), by hypothesis:

\[
W(\mathcal{E}_{m+D^{(O)}(S)+1}^{(O)}) - W(\mathcal{E}_{m+D^{(O)}(S)}^{(O)}) \geq \delta_{m+D^{(O)}(S)+1} = 2F(d, s_{m+D^{(O)}(S)} + \Sigma_{m+D^{(O)}(S)}) + 1.
\]

If \( \alpha^{D^{(O)}(S)}_{m+D^{(O)}(S),*} \sim \beta^{D^{(O)}(S)}_{m+D^{(O)}(S),*} \), then \( \mathcal{E}^{(D^{(O)}(S))}_{m+D^{(O)}(S)} \) fulfills case 1 of the definition of \( \text{INV}^{(O)} \), hence \( \text{INV}^{(O)}(S) \neq \perp \).

Otherwise, let us consider:

\[
\mathcal{R} = O\left( \sum_{j=1}^{d} \alpha^{(D^{(O)}(S))}_{m+D^{(O)}(S),j} S_j, \sum_{j=1}^{d} \beta^{(D^{(O)}(S))}_{m+D^{(O)}(S),j} S_j \right),
\]

\[
\nu = \text{Div}(\alpha^{(D^{(O)}(S))}_{m+D^{(O)}(S),*}, \beta^{(D^{(O)}(S))}_{m+D^{(O)}(S),*}),
\]

\[
(u, u') = \min\{ (v, v') \in \mathcal{R} \cap X^{\leq \nu} \times X^{\leq \nu} \mid \exists j \in [1, d], (\alpha^{(D^{(O)}(S))}_{m+D^{(O)}(S),*} \odot v = \epsilon^j) \iff (\beta^{(D^{(O)}(S))}_{m+D^{(O)}(S),*} \odot v' \neq \epsilon^j) \}.
\]

By definition of \( F \) and inequality (85),

\[
| u | \leq F(d, ||\mathcal{E}^{(D^{(O)}(S))}_{m+D^{(O)}(S)}||) \leq F(d, s_{m+D^{(O)}(S)} + \Sigma_{m+D^{(O)}(S)}).
\]
Hence \( p_{m+D^{(O)}(S)+1} - p_{m+D^{(O)}(S)} \geq 2 \mid u \mid + 1 \) i.e. the hypothesis of case 2 is realized. This proves that \( D^{(O)}(S^{(D^{(O)}(S))}) \geq 1 \) while in fact, \( D^{(O)}(S^{(D^{(O)}(S))}) = 0 \). This contradiction shows that this last case \( (D^{(O)}(S) < d - 1 \text{ and } E_{m+D^{(O)}(S)}^{(D^{(O)}(S))} \) not fulfilling case 1 of definition of \( \text{INV}^{(O)} \) is impossible. We have proved point (3) of the lemma. \( \Box \)
Fig. 1. Proof of lemma 5.2
\[ \mathcal{E}_m = \mathcal{E}_m^{(0)} \]

\[ (\mathcal{E}_m^{(0)})' \]

\[ \mathcal{E}_{m+1} = \mathcal{E}_{m+1}^{(0)} \quad \mathcal{E}_{m+1}^{(1)} \quad \cdots \]

\[ \vdots \quad \vdots \quad \cdots \quad (\mathcal{E}_{i-1}^{(i-1-m)})' \]

\[ \mathcal{E}_i = \mathcal{E}_i^{(0)} \quad \mathcal{E}_i^{(1)} \quad \cdots \quad \mathcal{E}_i^{(i-m)} \quad \cdots \]

\[ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad (\mathcal{E}_{m-1+D})' \]

\[ \mathcal{E}_{m+d-1} = \mathcal{E}_{m+d-1}^{(0)} \quad \mathcal{E}_{m+d-1}^{(1)} \quad \cdots \quad \mathcal{E}_{m+d-1}^{(i-m)} \quad \mathcal{E}_{m+d-1}^{(D)} \]

Fig. 2. Proof of lemma 5.4
6 Constants

Let us fix a birooted dpda $M$, a s.r. morphism $\psi$ and an initial equation $A_0 = (\Pi_0, S_0^-, S_0^+) \in \mathbb{N} \times DRB_{1,\lambda_0}(\langle V \rangle) \times DRB_{1,\lambda_0}(\langle V \rangle)$ in the corresponding set of assertions. This short section is devoted to the definition of some integer constants: these integers are constant in the sense that they are depending only on this triple : $(M, \psi, A_0)$. The motivation of each of these definitions will appear later on, in different places for the different constants. The equations below provide merely an overview of the dependencies between these constants and allow to check that the definitions are sound (i.e. there is no hidden loop in the dependencies).

$$k_0 = \max \{\nu(v) \mid v \in V\}, \quad k_1 = \max \{2k_0 + 1, 3\}, \quad (90)$$

$$K_0 = \max \{\| (E_1, E_2, \ldots, E_n) \circ x \| \mid (E_i)_{1 \leq i \leq n} \text{ is a bijective numbering of some class in } V/\sim, x \in X\}. \quad (91)$$

$K_0$ serves as an upper-bound on the possible increase of norm under the right-action of a single letter $x \in X$, see lemma 314.

$$D_1 = k_0 \cdot K_0 + |Q| + 2, \quad k_2 = D_1 \cdot k_1 \cdot K_0 + 2 \cdot k_1 \cdot K_0 + K_0. \quad (92)$$

$k_1$ is used in the definition of strategy $T_B$ (section 7), $D_1$ appears as an upper-bound on the marked part of series and $k_2$ is used in lemma 84.

$$k_3 = k_2 + k_1 \cdot K_0, \quad k_4 = (k_3 + 1) \cdot K_0 + k_1. \quad (93)$$

$k_3$ appears in in lemma 85, $k_4$ is used in the definition of the d-space $V_0$.

$$K_1 = k_1 \cdot K_0 + 1, \quad K_2 = k_1^2 \cdot D_1 \cdot K_0 + k_1^2 \cdot K_0 + 2 \cdot k_1 \cdot K_0 + D_1 \cdot k_1 + 2 \cdot k_1 + 4. \quad (94)$$

These constants $K_1, K_2$ appear in lemma 87.

$$K_3 = k_0 |Q|, \quad K_4 = D_1. \quad (95)$$

These constants $K_3, K_4$ appear in lemma 88.

$$d_0 = \text{Card}(X^{\leq k_4}). \quad (96)$$

$d_0$ appears as an upper-bound on the dimension of the d-space $V_0$ defined by equation 124 and used in lemma 87. We consider now the integer sequences $(\delta_i, \ell_i, L_i, s_i, S_i, \Sigma_i)_{m \leq i \leq m+n-1}$ defined by the relations 87 of section 6 where the parameters $K_0, K_1, \ldots, K_4$ are chosen to be the above constants and $m = 1, n = d = d_0$. Equivalently, they are defined by:

$$\delta_1 = 0, \quad \ell_1 = 0, \quad L_1 = K_2, \quad s_1 = K_3 \cdot K_2 + K_4, \quad S_1 = 0, \quad \Sigma_1 = 0, \quad (97)$$
\[
\begin{align*}
\delta_{i+1} &= 2 \cdot F(d_0, s_i + \Sigma_i) + 1 \\
\ell_{i+1} &= 2 \cdot \delta_{i+1} + 3 \\
L_{i+1} &= K_1 \cdot (L_i + \ell_{i+1}) + K_2 \\
s_{i+1} &= K_3 \cdot L_{i+1} + K_4 \\
S_{i+1} &= s_i + K_0 \cdot F(d_0, s_i + \Sigma_i) \\
\Sigma_{i+1} &= \Sigma_i + S_{i+1}
\end{align*}
\]

for \(1 \leq i \leq d_0 - 1\). The function \(F\) is defined in section 3 and depends on the pair \((\mathcal{M}, \bar{\psi})\) only.

\(D_2 = \max\{\Sigma_{d_0} + s_{d_0}, \|S_0^-\|, \|S_0^+\|\}\), \hspace{1cm} (99)

\(\Sigma_{d_0} + s_{d_0}\) appears in the conclusion of lemma 54 when we take \(d = d_0\) in the hypothesis and suppose that \(D^{(O)}(S)\) has its maximal possible value i.e. \(D^{(O)}(S) = d_0 - 1\). It is used as an upper-bound on the norm of vectors at the root of the trees \(\tau\) analysed in part 3 (inequation (112)).

\[\lambda_2 = \max\{\lambda_0, d_0\}\], \hspace{1cm} (100)

The integer \(\lambda_2\) is used as an upper-bound on the length of vectors at the root of the trees \(\tau\) analysed in part 3 (inequation (113)).

\[N_0 = 1 + k_3 + D_2\], \hspace{1cm} (101)

\(N_0\) appears as a lower bound for the norm in the definition of a B-stacking sequence (section 3.2 condition (116)).
7 Strategies for $B_0$

Let us define strategies for the particular system $B_0$.

7.1 Strategies

We shall define first auxiliary strategies $T_{cut}, T_{\emptyset}, T_{c}$, and then for every oracle $O \in \Omega$ auxiliary strategies $T_A^{(O)}, T_B^{(O)}, T_C^{(O)}$, we define the strategies $T_A, T_B, T_C$ and finally the “compound” strategies $S_{AB}^{(O)}, S_{ABC}^{(O)}, S_{AB}, S_{ABC}$. Let us fix here some total ordering on $X : x_1 < x_2 < \cdots < x_\alpha$.

$T_{cut}$:

$B_1 \cdots B_m \in T_{cut}(A_1 \cdots A_n)$ iff \( \exists i \in [1, n-1], \exists S, T, \)

\[ A_i = (p_i, S, T), A_n = (p_n, S, T), p_i < p_n \text{ and } m = 0. \]

$T_{\emptyset}$:

$B_1 \cdots B_m \in T_{\emptyset}(A_1 A_2 \cdots A_n)$ iff \( \exists S, T, \)

\[ A_n = (p, S, T), p \geq 0, S = T = \emptyset^n \text{ and } m = 0. \]

$T_{c}$:

$B_1 \cdots B_m \in T_{c}(A_1 \cdots A_n)$ iff

\[ A_n = (p, S, T), p \geq 0, S = T = \emptyset^n \text{ and } m = 0. \]

Let us consider an oracle $O \in \Omega$.

$T_A^{(O)}$:

$B_1 \cdots B_m \in T_A^{(O)}(A_1 \cdots A_n)$ iff

\[ A_n = (p, S, T), |X| \leq m \leq |X|^2, B_1 = (p+1, S \circ x_1, T \circ x_1'), \ldots, B_m = (p+1, S \circ x_m, T \circ x_m'), \]

where $S \neq \varepsilon, T \neq \varepsilon, O(S, T) \cap X \times X = \{(x_1, x_1'), \ldots, (x_i, x_i'), \ldots, (x_m, x_m')\}$.

$T_B^{(O)}$:

$B_1 \cdots B_m \in T_B^{(O)}(A_1 \cdots A_n)$ iff \( n \geq k_1 + 1, A_{n-k_1} = (\pi, U', \emptyset^n), \) (where $U'$ is unmarked)

\[ U' = \sum_{k=1}^{q} E_k \cdot \Phi_k \text{ for some } q \in \mathbb{N}, E_k \in V, \]

$$(E_k)_{1 \leq k \leq} \text{ bijective numbering of a class in } V/ \sim, \Phi_k \in \text{DRB}_{1, \lambda}(\{V\})$$

$A_{i} = (\pi + k_1 + i - n, U_i, U_i')$ for $n - k_1 \leq i \leq n$, $(U_i)_{n-k_1 \leq i \leq n}$ is a derivation, $(U_i')_{n-k_1 \leq i \leq n}$ is a “stacking derivation” (see definitions in §3.4),

\[ U_n' = \sum_{k=1}^{q} (E_k \circ u) \cdot \Phi_k, \text{ for some } u \in X^*, \]

\(^3\text{i.e. } B_1 \cdots B_m = \varepsilon\)
Lemma 71: We then set, for every \( V' \) such that, \( V' = \sum_{k=1}^{q} \rho_k(E_k \odot u) \cdot (\overline{U} \odot u_k) \)

where \( \forall k \in [1, q], u'_k = \min(\varphi(E_k)) \), and if \( R = O(S,T), \forall k \in [1, q], u_k = \min\{R^{-1}(u'_k)\} \).

\( T_B^{(O),+} \) is defined in the same way as \( T_B^{(O)} \) by exchanging the left series \((S^-)\) and right series \((S^+)\) in every assertion \((p, S^-, S^+)\).

\( T_C^{(O)} \): \( B_1 \cdots B_m \in T_C^{(O)} (A_1 \cdots A_n) \) iff there exists \( d \in [1, d_0], D \in [0, d-1], \lambda \in \mathbb{N} - \{0\}, S_1, S_2, \cdots, S_d \in \text{DBR}_{1, \lambda} (\{ V \}) - \{\emptyset^{\lambda}\}, 1 \leq \kappa_1 < \kappa_2 < \cdots < \kappa_{D+1} = n \), such that,

\( C1 \) every equation \( E_i = A_{\kappa_i} = (p_{\kappa_i}, S^{-}_{\kappa_i}, S^{+}_{\kappa_i}) \) is a weighted equation over \( S_1, S_2, \cdots, S_d, \) with \( p_{\kappa_i} \geq 1 \),

\( C2 \) \( D^{(O)}(S) = D \) (where \( S = (E_i)_{1 \leq i \leq D+1} \)),

\( C3 \) \( \text{INV}^{(O)}(S) \neq \perp, \| \| \text{INV}^{(O)}(S) \| \leq \Sigma_{d_0} + s_{d_0} \),

\( C4 \) \( m = 1 \) and \( B_1 = \rho_e(\text{INV}^{(O)}(S)) \) (where \( \rho_e \) is the obvious extension of \( \rho_e \) to weighted pairs of deterministic row-vectors; in other words the result of \( T_C^{(O)} \) is \( \text{INV}^{(O)}(S) \) where the marks have been removed).

We then set, for every \( W \in A^+ \):

\[ T_A(W) = \bigcup_{O \in \Omega} T_A^{(O)}(W), \]

\[ T_B^{+}(W) = \bigcup_{O \in \Omega} T_B^{(O),+}(W), \quad T_B^{-}(W) = \bigcup_{O \in \Omega} T_B^{(O),-}(W), \]

\[ T_C(W) = \bigcup_{O \in \Omega} T_C^{(O)}(W). \]

Lemma 71: \( T_{\text{cut}}, T_0, T_e, T_A \) are \( B_0 \)-strategies.

Proof:

\( T_{\text{cut}} \): (S1) is true by rule R0. (S2) is trivially true.

\( T_0 \): (S1) is true by rule R’3. (S2) is trivially true.

\( T_e \): (S1) is true by rule R’3. (S2) is trivially true.

\( T_A \): by rule (R4), \( \{B_j \mid 1 \leq j \leq m\} \vdash \vdash 4 A_n \), which proves (S1). Suppose \( H(A_n) = \infty \) i.e. \( S \sim T \). Then, \( \forall j \in [1, m], S \odot x_j \sim T \odot x_j, \) so that \( \min(H(B_j) \mid 1 \leq j \leq m) = \infty \). (S2) is proved.

\( \square \)
Lemma 72: \( T_B^+ , T_B^- \) are \( B_0 \)-strategies.

Proof: Let us show that \( T_B^+ \) is a \( B_0 \)-strategy. Let us use the notation of the definition of \( T_B^{(\sigma)} \). Let \( \mathcal{H} = \{ (\pi, \overline{U}, U'), (\pi + k_1 - 1, V, V') \} \). Let us show that

\[
\mathcal{H} \vdash_{B_0} (\pi + k_1 - 1, U_n, U'_n).
\]

(102)

Using rule (R5) we obtain: \( \forall k \in [1, q] \),

\[
\{ (\pi, \overline{U}, U') \} \vdash_{B_0} (\pi + 2 \cdot k \cdot \overline{u}, U_n, U'_n, u_k).
\]

(103)

Using rule (R3),

\[
\emptyset \vdash_{R3} (0, (\rho_e(E_1 \circ u), \ldots, \rho_e(E_q \circ u)), (E_1, \ldots, E_q)).
\]

(104)

Using (104),(103) and rules (R3),(R7),(R8), we obtain:

\[
\{ (\pi, \overline{U}, U') \} \vdash_{B_0} (\pi + 2 \cdot k_0, U_n, U'_n, V').
\]

(105)

Let us recall that \( U_n = V \). Hence, by (R0, R1, R2)

\[
\{ (\pi + k_1 - 1, V, V'), (\pi + 2 \cdot k_0, U'_n, V') \} \vdash_{C} (\pi + k_1 - 1, U_n, U'_n).
\]

(106)

By (102),(106),(102) is proved. Using now (102) and rule (R0), we obtain:

\[
\mathcal{H} \vdash_{B_0} (\pi + k_1 - 1, U_n, U'_n) \vdash_{R0} (\pi + k_1, U_n, U'_n).
\]

(107)

i.e. \( T_B^+ \) fulfills (S1). Let us suppose now that \( \forall i \in [n - k_1, n], U_i \sim U'_i \). Then, by (103), \( U'_n \sim V' \) and by hypothesis \( V = U_n \sim U'_n \). Hence \( V \sim V' \). This shows that \( T_B^+ \) fulfills (S2). An analogous proof can obviously be written for \( T_B^- \). \( \Box \)

Lemma 73 Let \( (p, S, S') \) be a weighted equation, i.e. \( p \in \mathbb{N}, \lambda \in \mathbb{N} - \{ 0 \}, S, S' \in \text{DRB}_{1, \lambda}(\langle V \rangle) \). Then \( \{ (p, S, S') \} \vdash_{C} \{ (p, \rho_e(S), \rho_e(S')) \} \) and \( \{ (p, \rho_e(S), \rho_e(S')) \} \vdash_{C} \{ (p, S, S') \} \).

Proof: Follows easily from (R1),(R2),(R3). \( \Box \)
Lemma 74 For every \( O \in \Omega \), \( T^O_C \) is a \( B_0 \)-strategy.

Proof: By lemma 52, point (1), combined with lemma 73, (S1) is proved. By lemma 52, point (2), combined with lemma 73, (S2) is proved. \( \Box \)

Let us define the strategy \( S_{ABC} \) by: for every \( W = A_1A_2\cdots A_n \),

(0) if \( T_{\text{cut}}(W) \neq \emptyset \), then \( S_{ABC}(W) = T_{\text{cut}}(W) \)
(1) elsif \( T_{\emptyset}(W) \neq \emptyset \), then \( S_{ABC}(W) = T_{\emptyset}(W) \)
(2) elsif \( T_{\varepsilon}(W) \neq \emptyset \), then \( S_{ABC}(W) = T_{\varepsilon}(W) \)
(3) elsif \( T_B^+(W) \cup T_B^-(W) \neq \emptyset \), then \( S_{ABC}(W) = T_B^+(W) \cup T_B^-(W) \cup T_C(W) \)
(4) else \( S_{ABC}(W) = T_A(W) \cup T_C(W) \)

The strategy \( S_{AB} \) is obtained from \( S_{ABC} \) by removing the occurrence of \( T_C \) in cases (3)/(4).

7.2 Global strategy

Let us define a global strategy \( \hat{S}_{ABC} \) w.r.t. the strategy \( S_{ABC} \). Let us fix (until the end of this article) a total well-ordering \( \sqsubseteq \) over the set of oracles \( \Omega \). We need now three technical definitions.

Definition 75 Let \( P \in \mathcal{P}_f(A), O \in \Omega \) and \( \pi \in \mathbb{N} \cup \{\infty\} \). \( O \) is said \( \pi \)-consistent with \( P \) iff, for every \( (\pi,S,S') \in \text{Cong}(P) \), and every \( n \in \mathbb{N} \), if

\[
\pi + n - 1 < \pi
\]

then, the binary relation \( \mathcal{R}_n = O(S,S') \cap X^{\leq n} \times X^{\leq n} \) fulfills

\[
[\pi,S,S',\mathcal{R}_n] \subseteq \text{Cong}(P).
\]

We use the notation:

\[
\Omega(\pi,P) = \{ O \in \Omega \mid O \text{ is } \pi - \text{consistent with } P \}.
\]

Definition 76 Let \( P \) be a finite subset of \( A \), and let \( \pi \in \mathbb{N} \cup \{\infty\} \). \( P \) is said \( \pi \)-consistent iff, there exists some oracle \( O \in \Omega \), which is \( \pi \)-consistent with \( P \).

For every proof tree \( t \in \mathcal{T}(S_{ABC}) \), we denote by \( \bar{\Pi}(t) \) the integer:

\[
\bar{\Pi}(t) = \min\{ \pi \in \mathbb{N} \mid \exists x \in \text{dom}(t), x \text{ is not closed for } S_{ABC}, \exists S,S',t(x) = (\pi,S,S') \}.
\] (108)

( we admit here that \( \min(\emptyset) = \infty \).

Definition 77 Let \( t \) be a finite proof-tree for the strategy \( S_{ABC} \), \( t \in \mathcal{T}(S_{ABC}) \). \( t \) is said consistent iff, \( \text{im}(t) \) is \( \bar{\Pi}(t) \)-consistent.
Let us consider some tree $t \in T(S_{ABC})$ which is consistent and not closed. Let $\bar{\pi} = \bar{P}(t)$, let $x$ be the smallest unclosed node of weight $\bar{\pi}$. Let
\[ W = A_1 \cdots A_n \]  
be the word labelling the path from the root to $x$ in $t$. (One can notice that, as $x$ is not closed, $T_{\text{cut}}(W) \cup T_{\emptyset}(W) \cup T_x(W) = \emptyset$). We define a tree of height one, $\hat{\Delta}(t)$ as follows:

1. if $\exists O \in \Omega(\bar{\pi}, \text{im}(t)), T^{(0)}_C(W) \neq \emptyset$ then
   \[ O_0 = \min\{O \in \Omega(\bar{\pi}, \text{im}(t)), T^{(0)}_C(W) \neq \emptyset\}, \quad \hat{\Delta}(t) = A_n(T^{(0)}_C(W)) \]

2. if $T^{(1)}_B(W) \neq \emptyset$ then
   \[ O_0 = \min(O \in \Omega(\bar{\pi}, \text{im}(t))), \quad \hat{\Delta}(t) = A_n(T^{(1)}_B(W)) \]

3. if $T^{(2)}_B(W) \neq \emptyset$ then
   \[ O_0 = \min(O \in \Omega(\bar{\pi}, \text{im}(t))), \quad \hat{\Delta}(t) = A_n(T^{(2)}_B(W)) \]

( In the above definition by $A(W')$, where $A \in \mathcal{A}$, $W' \in \mathcal{A}^+$ we mean the tree of height one with root labelled by $A$ and whose sequence of leaves is the word $W'$).

\[ \hat{S}_{ABC}(t) = t[\hat{\Delta}(t)/x], \]  

i.e. $\hat{S}_{ABC}(t)$ is obtained from $t$ by substituting $\hat{\Delta}(t)$ at the leaf $x$.

**Lemma 78** For every $t \in T(S_{ABC})$, if $t$ is consistent, then $\hat{\Delta}(t)$ is defined.

**Proof:** By the definition of consistency the oracle $O_0$ is always defined ( i.e. $\Omega(\bar{\pi}, \text{im}(t)) \neq \emptyset$), and for the word $W$ defined above $T_x(W) = \emptyset \Rightarrow \forall O \in \Omega, T^{(1)}_A(W') \neq \emptyset$, hence one of cases (0-3) must occur. \( \square \)

If $t$ is not consistent or is closed then we define:
\[ \hat{S}_{ABC}(t) = t. \]  

**Lemma 79** $\hat{S}_{ABC}$ is a global strategy for $S_{ABC}$.

**Sketch of proof:** By lemma $\hat{S}_{ABC}$ is defined on every $t \in T(S_{ABC})$. It suffices to check that, in every case, the word constituted by the leaves of $\hat{\Delta}(t)$ belongs to $S_{ABC}(W)$ (where $W$ is the word considered in (109)). \( \square \)
8 Tree analysis

This section is devoted to the analysis of the proof-trees \( \tau \) produced by the strategy \( S_{AB} \) defined in section 3. The main results are lemma 89 and 810 whose combination asserts that if some branch of \( \tau \) is infinite, then there exists some finite prefix on which \( T_C \) has a non-empty value. This key technical result will ensure termination of the global strategy \( \hat{S}_{ABC} \) (see section 9).

We fix throughout this section a tree \( \tau \in T(S_{AB}, (s_0, u_0^-, u_0^+)) \) (i.e. \( \tau \) is a proof tree associated to the assertion \((s_0, u_0^-, u_0^+)\)) by the strategy \( S_{AB} \). We suppose that

\[
\|U_0^-\| \leq D_2, \quad \|U_0^+\| \leq D_2, \quad U_0^-, U_0^+ \text{ are both unmarked ,} \quad (112)
\]

\[
U_0^-, U_0^+ \in DRB_{1,\lambda}(\langle V \rangle) \quad \text{with } \lambda \leq \lambda_2. \quad (113)
\]

\[
U_0^- \equiv U_0^+ \quad (114)
\]

We recall that, formally, \( \tau \) is a map \( \text{dom}(\tau) \to \mathbb{N} \times DRB_{1,\lambda}(\langle V \rangle) \times DRB_{1,\lambda}(\langle V \rangle) \) such that \( \text{dom}(\tau) \subseteq \{1, \ldots, |X|^2\}^* \) is closed under prefix and under “left-brother” (i.e. \( w \cdot (i+1) \in \text{dom}(\tau) \Rightarrow w \cdot i \in \text{dom}(\tau) \)). We denote by \( pr_{2,3} : \mathbb{N} \times DRB_{1,\lambda}(\langle V \rangle) \times DRB_{1,\lambda}(\langle V \rangle) \to DRB_{1,\lambda}(\langle V \rangle) \times DRB_{1,\lambda}(\langle V \rangle) \) the projection \((\pi, U, U') \mapsto (U, U')\). By \( \tau_s \) we denote the tree obtained from \( \tau \) by forgetting the weights: \( \tau_s = \tau \circ pr_{2,3} \).

8.1 Depth and weight

In this paragraph we check that the weight and the depth of a given node are closely related. Let us say that the strategy \( T \) “occurs at” node \( x \) iff,

\[
\tau(x) \in T(\tau(x[0]) \cdot \tau(x[1]) \cdots \tau(x[|x| - 1])),
\]

i.e. the label of \( x \) belongs to the image of the path from \( \epsilon \) (included) to \( x \) (excluded) by the strategy \( T \).

Lemma 81 Let \( \alpha \in \{-, +\}, A_1, \ldots, A_n \in \mathcal{A} \) such that \( T_B^{\alpha}(A_1 \cdots A_n) \neq \emptyset \). Then, \( \forall i \in [n-k_1+1, n], A_i \notin T_B(A_1 \cdots A_{i-1}) \).

In other words: if \( T_B \) occurs at node \( x \) of \( \tau \), it cannot occur at any of its \( k_1 \) above immediate ancestors.

Proof:

Suppose that \( \exists i \in [n-k_1+1, n], A_i \in T_B(A_1 \cdots A_{i-1}). \) Hence \( \pi_i = \pi_{i-1} - 1 < \pi_{n-k_1} + i \), contradicting one of the hypothesis under which \( T_B(A_1 \cdots A_n) \) is not empty. □

Lemma 81 ensures that, in every branch \((x_i)_{i \in I}\) and for every interval \([n+1, n+4] \subseteq I\), at most one integer \( j \) is such that \( T_B \) occurs at \( j \).

Lemma 82 Let \( \tau \) be a proof-tree associated to the strategy \( S_{AB} \). Let \( x, x' \in \text{dom}(\tau), x \leq x' \). Then \( |W(x') - W(x)| \leq |x'| - |x| \leq 2 \cdot (W(x') - W(x)) + 3. \)
(We recall the depth of a node $x$ is just its length $|x|$. We denote by $W(x)$ the weight of $x$ which we define as the first component of $\tau(x)$ i.e. the weight of the equation labelling $x$.)

**Proof:** Let $x, x'$ be such that $|x'| = |x| + 1$. Then $W(x') - W(x) \in \{-1, +1\}$, hence the inequality $|W(x') - W(x)| \leq |x'| - |x|$ is fulfilled by such nodes. The general case follows by induction on $(|x'| - |x|)$.

Let us prove now the other inequality. We distinguish two cases.

**Case 1:** $|x'| - |x| \leq 3$.
Then $|x'| - |x| \leq 2 \cdot (W(x') - W(x)) + 3$ (because there is at most one $T_B$ step in a sequence of length $\leq 3$).

**Case 2:** $|x'| - |x| \geq 4$.
Let $x = x_0, x_1, \cdots, x_q, x'$ be the sequence of nodes such that $|x' - x| = 4 \cdot q + r, 0 \leq r < 4$ and $\forall i \in [0, q - 1], |x_i+1| - |x_i| = 4$.

By lemma [3], in every set $\{y \in \text{dom}(\tau) \mid x_i < y \leq x_{i+1}\}$ at most one node $z$ is such that $T_B$ occurs at $z$. Hence $W(x_{i+1}) - W(x_i) \geq 2$.

It follows that:

\[
|x'| - |x| = \sum_{i=0}^{q-1} (|x_{i+1}| - |x_i|) + |x' - x_q| \\
\leq \sum_{i=0}^{q-1} (2(W(x_{i+1}) - W(x_i)) + |x' - x_q|) \\
\leq 2(W(x_q) - W(x)) + 2(W(x') - W(x_q)) + 3 \text{ (by the first case)} \\
\leq 2(W(x') - W(x)) + 3.
\]

\[\square\]

Let us recall the values of some constants (defined in section [8]):

\[
k_0 = \max\{\nu(v) \mid v \in V\}, \quad k_1 = \max\{2k_0 + 1, 3\}, \quad D_1 = k_0 \cdot K_0 + |Q| + 2, \quad k_2 = D_1 \cdot k_1 \cdot K_0 + 2 \cdot k_1 \cdot K_0 + K_0, \quad k_3 = k_2 + k_1 \cdot K_0, \quad k_4 = (k_3 + 1) \cdot K_0 + k_1, \quad d_0 = \text{Card}(X^{\leq k_1}), \quad N_0 = 1 + k_3 + D_2.
\]

### 8.2 $B$-stacking sequences

We establish here that every infinite branch must contain an infinite suffix (a “$B$-stacking sequence”) where at least $d_0$ labels $(U, U')$ are belonging to the same d-space $V_0$ of dimension $\leq d_0$ with coordinates not greater than $s_{d_0}$ (over some fixed generating family of cardinality $\leq d_0$).

Let $\sigma = (x_i)_{i \in I}$ be a path in $\tau$, where $I = [i_0, \infty]$ and let $(x_i)_{i \geq 0}$ be the unique branch of $\tau$ containing $\sigma$. Let us note $\tau(x_i) = (\pi_i, U_i^{-}, U_i^{+})$.

We call $\sigma$ a $B$-stacking sequence iff: there exists some $\alpha_0 \in \{-, +\}$ such that $T_B^{\alpha_0}$ occurs at $x_{i_0 + k_1 + 1}$  

(115)
and, for every \( i \in I, \alpha \in \{-, +\} \), if \( T_B^\alpha \) occurs at \( x_{i+k_1+1} \) then

\[
\|U_{i}^{-\alpha}\| \geq \|U_{i_0}^{-\alpha_0}\| \geq N_0. \tag{116}
\]

From now on and until lemma 810, we fix a B-stacking sequence \( \sigma = (x_i)_{i \in I} \) and we denote by \( S_0 \) the series \( U_{i_0}^{-\alpha_0} \).

**Lemma 83** There exists some word \( u_0 \in X^* \) and some sign \( \alpha'_0 \in \{-, +\} \) such that \( S_0 = U_{0}^{\alpha'_0} \circ u_0 \).

**Proof:** One can prove by induction on \( i \in \mathbb{N} \) that, for every \( \alpha \in \{-, +\} \), \( U_i^\alpha \) has one of the two following forms:

1. \( U_i^\alpha = U_0^\alpha \circ u \) for some \( \alpha' \in \{-, +\} \), \( |u| \leq i \),

2. \( U_i^\alpha = \sum_{k=1}^{i} \beta_k \cdot (U_0^{\alpha'} \circ uu_k) \),

for some deterministic rational vector \( \beta, \alpha' \in \{-, +\}, |u \cdot u_k| \leq i, |u_k| \leq k_0. \)

**Lemma 84** Suppose that \( i_0 \leq j < i \), no \( T_B \) occurs in \([j + 1, i]\), \( U_j^{-\alpha} \) is \( D_1 \)-marked and \( U_j^\alpha \) is unmarked. Then, for every \( j' \in [j, i] \), \( \|U_{j'}^\alpha\| \geq \|U_{i}^\alpha\| - k_2 \).

**Proof:** Let \( i, j \) fulfill the hypothesis of the lemma.

1. Let us treat first the case where \( j' = j \).

If \( (i-j) \leq (D_1 + 1)k_1 \) then, by lemma 316

\[
\|U_i^\alpha\| \leq \|U_j^\alpha\| + (D_1 + 1) \cdot k_1 \cdot K_0 \leq k_2
\]

hence the lemma is true.

Let suppose now that \( (i-j) \geq (D_1 + 1)k_1 + 1 \). We can then define the integers \( j < i_1 < i_2 < i \) by:

\[
i_1 = j + D_1 \cdot k_1, i_2 = i - k_1 - 1.
\]

By lemma 316 we know that:

\[
\|U_{i_1}^\alpha\| \leq \|U_{i}^\alpha\| + D_1 \cdot k_1 \cdot K_0 \quad \text{and} \quad \|U_{i_2}^\alpha\| \leq \|U_{i}^\alpha\| + (k_1 + 1) \cdot K_0. \tag{117}
\]

If there was some stacking subderivation of length \( k_1 \) in \( U_{i_1}^{-\alpha} \rightarrow U_{i_1}^{-\alpha} \), as all the \( U_k^\alpha \) (for \( k \in [j, i] \)) are unmarked, \( T_B \) would occur at some integer in \([j + k_1 + 1, i_1 + 1]\), which is untrue. Hence there is no such stacking subderivation, and by lemma 336 \( U_{i_1}^{-\alpha} \) is unmarked.

If there was some stacking subderivation of length \( k_1 \) in \( U_{i_1}^\alpha \rightarrow U_{i_2}^\alpha \), as all the \( U_k^{-\alpha} \) (for \( k \in [i_1, i] \)) are unmarked, \( T_B \) would occur at some integer in \([i_1 + k_1 + 1, i]\), which is untrue. Hence there is no such stacking subderivation, and by lemma 335

\[
\|U_{i_2}^\alpha\| \leq \|U_{i_1}^\alpha\| + k_1 \cdot K_0. \tag{118}
\]

Adding inequalities (117, 118) we obtain:

\[
\|U_i^\alpha\| \leq \|U_j^\alpha\| + (D_1 \cdot k_1 + 2 \cdot k_1 + 1) \cdot K_0 = \|U_j^\alpha\| + k_2,
\]
which was to be proved.

2. Let us suppose now that \( j \leq j' \leq i \).

If \((i - j) \leq (D_1 + 1)k_1\), the same inequality is true for \( i - j' \) and the conclusion is true for \( j' \).

Otherwise, if \( j' \leq i_1 \), (117, 118) are still true for \( j' \) instead of \( j \), hence the conclusion too.

Otherwise, by the arguments of part 1, \( U_{j'}^\alpha, U_i^\alpha \) are both unmarked. Hence the hypothesis of part 1 are met by \((j', i)\) instead of \((j, i)\), hence the conclusion is met too. (We illustrate our argument on figure 3). \( \square \)

\[ \| U_i^\alpha \| - k_2 \]

\[ \| U_j^\alpha \| \]

\[ T_B \text{ should occur here} \]

\[ U_j^\alpha \]

\[ U_i^\alpha \]

Fig. 3. \( \| U_j^\alpha \| \) too small is impossible.
Lemma 85  Let \( i \in I, \alpha \in \{-, +\} \) such that \( T^\alpha_B \) occurs at \( i + k_1 + 1 \). Then, there exists \( u \in X^*, |u| \leq (i - i_0), U^{\alpha \leftarrow} = S_0 \odot u \) and, for every prefix \( w \leq u \),
\[
\| S_0 \odot w \| \geq \| S_0 \| - k_3.
\]

Proof: We prove the lemma by induction on \( i \in [i_0, \infty[ \).

Basis: \( i = i_0 \).

Choosing \( u = \epsilon \), the lemma is true.

Induction step: \( i_0 \leq i' < i \), \( T^\alpha_B \) occurs at \( i' + k_1 + 1 \), \( T^\beta_B \) occurs at \( i + k_1 + 1 \) and \( T_B \) does not occur in \([i' + k_1 + 2, i + k_1]\).

By induction hypothesis, there exists some \( u' \in X^*, |u'| \leq (i' - i_0) \) fulfilling
\[
U^{-\alpha'} = S_0 \odot u', \quad \forall w' \preceq u', \| S_0 \odot w' \| \geq \| S_0 \| - k_3.
\]

Let us define \( j = i' + k_1 + 1 \).

Let \( \bar{u} \in X^* \) be the word such that
\[
U^{-\alpha} \xrightarrow{\bar{u}} U^{-\alpha}_i
\]
is the derivation described by the \(-\alpha\) component of the path from \( x_j \) to \( x_i \).

Case 1: \( \alpha' = \alpha \).

\[
U^{-\alpha}_j = U^{-\alpha'}_i \odot u_1
\]
for some \( u_1 \in X^*, |u_1| = k_1 \) and \( U^\alpha_j \) is \( D_1 \)-marked. Let us choose \( u = u' \cdot u_1 \cdot \bar{u} \).

Hence
\[
U^{-\alpha}_i = S_0 \odot u.
\]

Let us consider some prefix \( w \) of \( u \).

subcase 1: \( w \preceq u' \).

By (120) we know that \( \| S_0 \odot w \| \geq \| S_0 \| - k_3 \).

subcase 2: \( w = u' \cdot u_1 \cdot u'' \), for some \( u'' \preceq \bar{u} \).

By lemma 84 we know that \( \| S_0 \cdot w \| \geq \| U^\alpha_i \| - k_2 \), and by definition of a B-stacking sequence we also know that \( \| U^\alpha_i \| \geq \| S_0 \| \).

Hence
\[
\| S_0 \odot w \| \geq \| S_0 \| - k_2.
\]

subcase 3: \( w = u' \cdot u'_1 \), where \( u'_1 \) is a prefix of \( u_1 \).

Then, by lemma 316 and the above inequality we get:
\[
\| S_0 \odot w \| \geq \| S_0 \odot u'_1 \| - k_1 \cdot K_0 \geq \| S_0 \| - k_3.
\]

Case 2: \(-\alpha' = \alpha \).

\[
U^{-\alpha}_j = \sum_{k=1}^{q} \beta_k \cdot (U^\alpha_{i'} \odot u_k)
\]
where $\beta$ is a polynomial which is fully marked and every $|u_k| \leq k_0$. By lemma 318, either $U_i^{-\alpha} = \sum_{k=1}^{q}(\beta_k \odot \bar{u}) \cdot (U_{i'}^\alpha \odot u_k)$ or there exists a decomposition

$$\bar{u} = \bar{u}_1 \cdot \bar{u}_2$$

and an integer $k \in [1,q]$ such that

$$U_i^{-\alpha} = U_{i'}^\alpha \odot u_k\bar{u}_2.$$  

But, as $U_i^{-\alpha}$ is unmarked (by definition of $T_B^\alpha$), the first formula is impossible unless $\beta \odot \bar{u}$ is unitary or nul. Hence (123) is the only possibility. Let us consider some prefix $w$ of $u$.

**Subcase 1:** $w \preceq u'$.

Same arguments as in case 1, subcase 1.

**Subcase 2:** $w = u' \cdot u_k \cdot u''$, for some $u'' \preceq \bar{u}_2$.

By lemma 317 applied on the interval $[j + |\bar{u}_1| + 1, i]$, we can conclude that

$$\|S_0 \odot w\| \geq \|S_0\| - k_3.$$  

**Subcase 3:** $w = u' \cdot u_k'$, where $u_k'$ is a prefix of $u_k$.

Same arguments as in case 1, subcase 3.

Let us define now the following families of vectors and $d$-spaces of vectors

$$G_0 = \{S_0 \odot u | u \in X^*, |u| \leq k_4\},$$  

$$V_0 = \mathcal{V}(G_0).$$  

**Lemma 86** Let $i \geq i_0$ such that $T_B$ occurs at $i$. Then, $U_i^-, U_i^+ \in V_0$.

**Proof:** Let us suppose that $T_B^\alpha$ occurs at $i$. By lemma 317, $U_i^{-\alpha} = S_0 \odot u$ and, for every prefix $w \leq u$,

$$\|S_0 \odot w\| \geq \|S_0\| - k_3.$$  

By lemma 317, $\exists u_1, u_2 \in X^*, v_1 \in V^+, E_1, \ldots, E_k \in V, E_1 \sim E_2 \ldots \sim E_k, \Phi \in \text{DRB}_{q,\lambda}(\langle V \rangle)$, such that $u = u_1 \cdot u_2$,

$$S_0 \odot u_1 = S_0 \bullet v_1 = \sum_{k=1}^{q} E_k \cdot \Phi_k$$

$$S_0 \odot u = \sum_{k=1}^{q} (E_k \odot u_2) \cdot \Phi_k.$$  

Without loss of generality, we can suppose that $v_1$ is a minimal word realizing the equality (127). Let us notice that, as $G$ is a reduced grammar, for every $v \preceq v_1$, there exists some $\bar{v} \in X^*$, such that $S_0 \bullet v = S_0 \odot \bar{v}$. Hence, for every $v \preceq v_1$,

$$S_0 \bullet v = T_{0}^{\alpha'} \odot u_0 \cdot \bar{v} \text{ and } \|U_0^{\alpha'} \odot u_0 \cdot \bar{v}\| \geq \|S_0 \odot u_1\| > D_2 = \|U_0^{\alpha'}\|.$$  


By lemma 333, all the vectors $S_0 \cdot v$ for $v \leq v_1$ are loop-free. It follows that, for every $v \leq v' \leq v_1$
\[v < v' \Rightarrow \|S_0 \cdot v\| > \|S_0 \cdot v'\|,\]
hence
\[|v_1| \leq \|S_0\| - \|S_0 \cdot v_1\| \leq k_3.\]
The formula (128) can be rewritten
\[U_{i-k_1-1}^{-\alpha} = \sum_{k=1}^{q} (E_k \odot u_2) \cdot (S_0 \cdot v_1 E_k) = \sum_{k=1}^{q} (E_k \odot u_2) \cdot (S_0 \odot \bar{u}_k)\]
where $\bar{u}_k \in X^*, |\bar{u}_k| \leq (k_3 + 1) \cdot K_0.$
Using lemmas 318 and 314 we can deduce from the above form of $U_{i-k_1-1}^{-\alpha}$ that
\[U_i^\alpha \in V(\{S_0 \odot w \mid w \in X^*, |w| \leq (k_3+1) \cdot K_0 + k_0\}), \quad U_{i-k_1}^{-\alpha} \in V(\{S_0 \odot w \mid w \in X^*, |w| \leq (k_3+1) \cdot K_0 + k_1\}),\]
hence that both $U_{i-k_1}^{-\alpha}, U_i^\alpha$ belong to $V_0.$ □
We recall that:
\[K_1 = k_1 \cdot K_0 + 1, \quad K_2 = k_1^2 \cdot D_1 \cdot K_0 + k_0^2 \cdot K_0 + 2 \cdot k_1 \cdot K_0 + D_1 \cdot k_1 + 2 \cdot k_1 + 4.\]

**Lemma 87** For every $L \geq 0$ there exists $i \in [i_0 + L, i_0 + K_1 \cdot L + K_2]$ such that, $U_i^-, U_i^+ \in V_0.$

**Proof:** Let us establish that
\[\exists i \in [i_0 + L, i_0 + K_1 \cdot L + K_2 - k_1 - 1], \exists \alpha \in \{-, +\}, T_B^\alpha \text{ occurs at } i + k_1 + 1. \quad (129)\]
Let $L \geq 0$ and let $i' \geq i_0$ be the greatest integer in $[i_0, i_0 + L]$ such that $T_B$ occurs at $i' + k_1 + 1$. Let $j = i' + k_1 + 1.$ We then have:
\[U_{j'}^\alpha = \sum_{k=1}^{q} \beta_k \cdot (U_{i-k_1}^{-\alpha'} \odot u_k)\]
where $\|\beta\| \leq D_1$ and $U_{j'}^{-\alpha'}$ is unmarked.

**Case 1:** there exists $i \in [j, j + k_1 \cdot D_1]$, such that $T_B$ occurs at $i + k_1 + 1.$
In this case the small constants $K_1 = 0, K_2 = k_1 \cdot D_1 + k_1 + 1$ would be sufficient to satisfy (129). A fortiori the given constants satisfy (129).

**Case 2:** there exists no $i \in [j, j + k_1 \cdot D_1]$, such that $T_B$ occurs at $i + k_1 + 1.$
Then, there is no stacking subderivation of length $k_1$ in $U_{j'}^{\alpha'} \rightarrow U_{j+k_1-D_1}^{\alpha'}$. By lemma 336 it follows that both $U_{j+k_1-D_1}^{\alpha}$ are unmarked.
1-Let $j_1 = j + D_1 \cdot k_1$ and let us show that there exists some $i \geq j_1$ such that $T_B$ occurs at $i + k_1 + 1.$
If such an $i$ does not exist then, for every $\alpha \in \{-, +\}$, the infinite derivation
\[U_{j_1}^{\alpha} \rightarrow U_{j_1+1}^{\alpha} \rightarrow \ldots\]
does not contain any stacking sequence of length $k_1$. By lemma 335 we would have:

$$\forall k \geq j_1, \|U_{\alpha}^k\| \leq \|U_{\alpha}^{j_1}\| + k_1 \cdot K_0.$$  

As the set $\{\|U_{\alpha}^k\|, k \geq j_1, \alpha \in \{-, +\}\}$ is finite, there would be a repetition

$$(U_{k}^-, U_{k}^+ ) = (U_{k'}^-, U_{k'}^+)$$

with $j_1 \leq k < k'$ and $\pi_k < \pi_{k'}$,

so that $T_{cut}$ would have been defined on some finite prefix of the branch, contradicting the hypothesis that the branch is infinite.

2- Let $i > i'$ be the smallest integer (in $[j_1, \infty)$) fulfilling point 1 above and suppose that $T_{\alpha B}$ occurs at $i + k_1 + 1$.

By lemma 84,

$$\forall \ell \in [j_1, i], \|U_{\alpha}^-\| \geq N_0 - k_2 > D_2.$$  

Using lemma 83, lemma 333 and inequality (112) we conclude that

$$\forall \ell \in [j_1, i], U_{\alpha}^- \text{ is loop-free}.$$  

By an argument analogous to that used in lemma 83 we see that

$$U_{\alpha}^- = S_0 \circ u \text{ for some } |u| \leq (j_1 - i_0),$$

and by lemma 316 we get

$$\|U_{\alpha}^-\| \leq (j_1 - i_0) \cdot K_0 + \|S_0\|.$$  

(130)

We also know that:

$$\|S_0\| \leq \|U_{\alpha}^-\| \leq \|U_{\alpha}^-\| + K_0.$$  

(131)

As the derivation $U_{\alpha}^- \rightarrow U_{\alpha}^- \rightarrow U_{\alpha}^- \rightarrow \ldots$ contains no stacking sub-derivation of length $k_1$ and consists of loop-free series only, by lemma 334 we obtain:

$$\|U_{\alpha}^-\| \leq \|U_{\alpha}^-\| - (i - j_1 - 2)/k_1.$$  

(132)

Combining the three inequalities (130, 131, 132) we get successively:

$$\|S_0\| \leq \|S_0\| + (j_1 - i_0 + 1) \cdot K_0 - (i - j_1 - 2)/k_1,$$

$$i - j_1 + 1 \leq (j_1 - i_0 + 1) \cdot k_1 K_0.$$  

(133)

3- By the choice of $i', i$, we know that $i' \leq i_0 + L \leq i$. Using (133) we obtain:

$$i' = i' + (K_1 - 1)(i' - i_0) + K_2 - k_1 - 1$$

$$i \leq i_0 + K_1 \cdot L + K_2 - k_1 - 1.$$  

Assertion (129) is now established for case 2 as well as for case 1.
From (129) and lemma 86 the lemma follows. 

(We illustrate our argument on figure 4). 

Let us give now a stronger version of lemma 87 where we analyze the size of the coefficients of the linear combinations whose existence is proved in lemma 87. We recall that: 

\[ K_3 = K_0 |Q|, \quad K_4 = D_1. \]

Let us fix a total ordering on \( G_0 \): 

\[ G_0 = \{ \theta_1, \theta_2, \ldots, \theta_d \}, \]

where \( d = \text{Card}(G_0) \). 

Let us remark that 

\[ d \leq \text{Card}(X \leq k_4) = d_0. \]

**Lemma 88** Let \( L \geq 0 \). There exists \( i \in [i_0 + L, i_0 + K_1 \cdot L + K_2] \) and, for every \( \alpha \in \{-, +\} \), there exists a deterministic rational family \((\beta_{i, \alpha}^{\mathbf{\mathfrak{1}}, j})_{1 \leq j \leq d}\) fulfilling

\[
\begin{align*}
(1) & \quad U_{i, \alpha} = \sum_{j=1}^{d} \beta_{i, \alpha}^{\mathbf{\mathfrak{1}}, j} \cdot \theta_j, \\
(2) & \quad \|\beta_{i, \alpha}^{\mathbf{\mathfrak{1}}, \mathfrak{1}}\| \leq K_3 \cdot (i - i_0) + K_4.
\end{align*}
\]

**Proof:** By lemma 87 there exists \( i \in [i_0 + L, i_0 + K_1 \cdot L + K_2] \) and \( \alpha \in \{-, +\} \) such that \( T_B^\alpha \) occurs at \( i \). Let us use the notation of the proof of lemma 86 and compute upper-bounds on the coefficients of \( U_{i, -\alpha} \), \( U_{i, \alpha} \) expressed as linear combinations of the vectors of \( G_0 \). 

**Coefficients of \( U_{i, -\alpha} \):**

\( U_{i, -\alpha} = U_{i, -k_1 - 1} \odot u' \), for some \( u' \in X^*, |u'| = k_1 \). By lemma 318, \( U_{i, -\alpha} \) can be expressed in one of the following forms:

\[
U_{i, -\alpha} = S_0 \odot (\bar{u}_k \cdot u''), \quad \text{where} \ u'' \text{ is a suffix of } u',
\]

\[
U_{i, -\alpha} = \sum_{k=1}^{q} (E_k \odot u_2 u') \cdot (S_0 \odot \bar{u}_k).
\]

In case (134) we can choose as vector of coordinates: \( \beta_{i, -\alpha}^{\mathbf{\mathfrak{1}}, \mathfrak{1}} = e_{j_0}^d \). We then have \( \|\beta_{i, \alpha}^{\mathbf{\mathfrak{1}}, \mathfrak{1}}\| = 2 \leq K_4 \).

In case (135), we can choose: \( \beta_{i, -\alpha}^{\mathbf{\mathfrak{1}}, \mathfrak{1}} = E \odot u_2 u' \) (completed with \( \emptyset \) in all the columns \( j \) not corresponding to some vector \( S_0 \odot \bar{u}_k \) of \( G_0 \)). We then have:

\[ \|\beta_{i, \alpha}^{\mathbf{\mathfrak{1}}, \mathfrak{1}}\| = \|E \odot u_2 u'\| \leq K_0 \cdot (i - i_0) \leq K_3 \cdot (i - i_0). \]

**Coefficients of \( U_{i, \alpha} \):**

By definition of \( T_B^\alpha \)

\[
U_{i, \alpha} = \sum_{\ell=1}^{r} \tau_\ell \cdot (U_{i, -k_1 - 1} \odot \bar{w}_\ell),
\]

where \( \|\tau\| \leq D_1, |\bar{w}_\ell| \leq k_0 \).

Replacing \( u' \) by \( \bar{w}_\ell \) in the above analysis, we get:

\[
\forall \ell \in [1, r], \ U_{i, -k_1 - 1} \odot \bar{w}_\ell = \sum_{j=1}^{d} \gamma_{\ell, j} \cdot \theta_j,
\]

where \( \gamma_{\ell, j} \).
with \( \|\gamma_{\ell,*}\| \leq K_0 \cdot (i - i_0) \).

Equalities (136, 137) show that:

\[
U_\alpha^i = \tau \cdot \gamma \cdot \theta,
\]

where \( \tau, \gamma, \theta \) are deterministic rational matrices of dimensions respectively \((1, r), (r, d), (d, 1)\).

Let us choose \( \beta_{i,*} = (\tau \cdot \gamma) \).

\[
\|\beta_{i,*}\| \leq \|\tau\| + \|\gamma\| \leq D_1 + r \cdot K_0 \cdot (i - i_0)
\]

\[
\leq D_1 + |Q| \cdot K_0 \cdot (i - i_0) = K_3 \cdot (i - i_0) + K_4.
\]

Lemma 89 There exists \( i_0 \leq \kappa_1 < \kappa_2 < \ldots < \kappa_d \) and deterministic rational vectors \((\beta_i^a)_{1 \leq j \leq d} \) (for every \( i \in [1, d] \)) such that

\[
(0) \ W(\kappa_1) \geq 1
\]

\[
(1) \ \forall \iota, \forall \alpha, U^a_{\kappa_1} = \sum_{j=1}^{d} \beta_{i,j}^a \theta_j \in V_0
\]

\[
(2) \ \forall \iota, \forall \alpha, \|\beta_{i,*}^a\| \leq s_i
\]

\[
(3) \ \forall \iota, W(\kappa_{i+1}) - W(\kappa_i) \geq \delta_{i+1}
\]

where the sequences \((\delta_i, \ell_i, L_i, s_i, S_i, \sigma_i)\) are those defined by relations (97, 98) in section 6.

Proof: Let us consider the additional property

\[
(4) \ \kappa_i - i_0 \leq L_i.
\]

We prove by induction on \( i \) the conjunction \((1) \wedge (2) \wedge (3) \wedge (4)\).

\( i = 1 \): By lemma 88, there exists \( \kappa_1 \in [i_0, i_0 + K_2] \) such that \( \forall \alpha \in \{-, +\}, \exists \) a deterministic vector \((\beta_i^a)_{1 \leq j \leq d}\), such that

\[
U^a_{\kappa_1} = \sum_{j=1}^{d} \beta_{i,j}^a \theta_j
\]

and in addition \( \|\beta_{1,*}^a\| \leq K_3 K_2 + K_4 = s_1 \).

\( i \rightarrow 1+1 \):

Suppose that \( \kappa_1 < \kappa_2 < \ldots < \kappa_i \) are fulfilling \((1) \wedge (2) \wedge (3) \wedge (4)\). By lemma 88, there exists \( \kappa_{i+1} \in [i_0 + L_i + \ell_{i+1}, i_0 + K_1 (L_i + \ell_{i+1}) + K_2] \) such that \( \forall \alpha \in \{-, +\}, \exists \) a deterministic vector \((\beta_i^a)_{1 \leq j \leq d}\), such that

\[
U^a_{\kappa_{i+1}} = \sum_{j=1}^{d} \beta_{i+1,j}^a \theta_j
\]

and in addition

\[
\|\beta_{i+1,*}^a\| \leq K_3 (K_1 (L_i + \ell_{i+1}) + K_2) + K_4 = K_3 L_{i+1} + K_4
\]

\[
= s_{i+1}
\]
By lemma 82

\[ 2(W(\kappa_{i+1}) - W(\kappa_i)) + 3 \geq \kappa_{i+1} - \kappa_i \geq \ell_{i+1} = 2\delta_{i+1} + 3 \]

hence

\[ W(\kappa_{i+1}) - W(\kappa_i) \geq \delta_{i+1}. \quad (140) \]

At last

\[ \kappa_{i+1} - i_0 \leq K_1(L_i + l_{i+1}) + K_2 = L_{i+1}. \quad (141) \]

The above properties \(138-139-140-141\) prove the required conjunction.

It remains to prove point (0): the integer \(\kappa_1\) introduced by lemma 88 is such that \(T_B\) occurs at \(\kappa_1\), hence

\[ W(\kappa_1) = W(\kappa_1 - k_1 - 1) + k_1 - 1 \geq W(\kappa_1 - k_1 - 1) + 2 \geq 1. \]

□

**Lemma 810** Let \((x_i)_{i \in \mathbb{N}}\) be an infinite branch of \(\tau\). Then there exists some \(i_0 \in \mathbb{N}\) such that \((x_i)_{i \geq i_0}\) is a B-stacking sequence.

**Proof:** Let us distinguish, a priori, several cases, and see that only the case where \(\tau\) admits a B-stacking sequence is possible.

**Case 1:** \(T_B\) occurs finitely often on \(\tau\).

Let \(j\) be the largest integer such that \(T_B\) occurs at \(j\). By the arguments used in the proof of lemma 29 Case 2, we know that \(U_{j+k_1}^+, U_{j+k_1}^-\) are both unmarked, and that

\[ \forall k \geq j + k_1 \cdot D_1, \forall \alpha \in \{-, +\}, \|U_k^\alpha\| \leq \|U_{j+k_1}^\alpha\| + k_1 \cdot K_0. \]

This would imply that the branch contains a finite prefix on which \(T_{\text{cut}}\) is defined, which is impossible on an infinite branch.

**Case 2:** For some sign \(\alpha\), there are infinitely many integers \(i\) such that \([T_B^\alpha]\) occurs at \(i + k_1 + 1\) and \(\|U_i^\alpha\| < N_0\).

In this case there would exist an infinite sequence of integers \(i_1 < i_2 < \ldots < i_\ell < \ell\) such that

\[ \forall \ell \geq 0, U_{i_\ell}^{-\alpha} = U_{i_\ell}^{+\alpha}. \]

For a given \(U_i^{-\alpha}\), only a finite number of values are possible for the pair \((U_{i+k_1+1}^- , U_{i+k_1+1}^+)\). Hence there exist integers \(\ell < \ell'\) such that

\[ \ell < \ell', \pi_{\ell} < \pi_{\ell'} \text{ and } (U_{\ell+k_1+1}^- , U_{\ell+k_1+1}^+) = (U_{\ell'+k_1+1}^- , U_{\ell'+k_1+1}^+). \]

Here again \(T_{\text{cut}}\) would have a non-empty value on some prefix of \(\tau\), which is impossible.

**Case 3:** \(T_B\) occurs infinitely often on \(\tau\) and, for every sign \(\alpha\), there are only finitely many integers \(i\) such that \([T_B^\alpha]\) occurs at \(i + k_1 + 1\) and \(\|U_i^{-\alpha}\| < N_0\).
Let us consider the set $I_0$ of the integers $i$ such that there exists a sign $\alpha_i$ such that $T_B^{\alpha_i}$ occurs at $i + k_1 + 1$ and $\|U_i^{-\alpha_i}\| \geq N_0$.

By the hypothesis of case 3, $I_0 \neq \emptyset$. Let $i_0$ such that

$$\|U_{i_0}^{-\alpha_{i_0}}\| = \min\{\|U_i^{-\alpha_i}\| \mid i \in I_0\}.$$

Then $(x_i)_{i \geq i_0}$ is a B-stacking sequence. $\square$
Fig. 4. Two successive $T_B$. 
9 Termination

Lemma 91: \( \mathcal{S}_{ABC} \) is terminating on every unmarked assertion \( A_0 \): if \( A_0 \in \mathcal{A} \) is unmarked, then, \( \exists n_0 \in \mathbb{N}, \mathcal{S}_{ABC}^{n_0}(A_0) = \mathcal{S}_{ABC}^{n_0+1}(A_0) \).

**Proof:** Suppose \( A_0 \in \mathcal{A} \), \( A_0 \) is true, \( A_0 \) is unmarked and

\[ \forall n \in \mathbb{N}, \mathcal{S}_{ABC}^n(A_0) \prec \mathcal{S}_{ABC}^{n+1}(A_0). \] (142)

Let us consider all the constants associated to this precise \( A_0 \), the equivalence \( \bar{\psi} \) and the dpda \( \mathcal{M} \) in section 6. Let us note:

\[ t_n = \mathcal{S}_{ABC}^n(A_0) \quad \text{(for every} \ n \in \mathbb{N}) \]

and let

\[ t_\infty = \text{l.u.b.}\{t_n \mid n \in \mathbb{N}\}. \]

Let us notice that, by definition (111), the strict inequality (142) implies that

\[ \forall n \in \mathbb{N}, t_n \text{ is consistent.} \] (143)

Let us denote by \( x_n \) the node of \( t_n \) such that \( t_{n+1} = \mathcal{S}_{ABC}^{\Delta(t_n)}(x_n) \). Let us notice that, as every \( x_n \) is unclosed in \( t_n \), one can prove by induction that every \( t_n \) is repetition-free. Hence \( t_\infty \) is repetition-free.

By Koenig’s lemma, \( t_\infty \) contains an infinite branch \( y_0y_1 \cdots y_s \cdots \) whose (infinite) labelling word is \( A_0A_1 \cdots A_s \cdots \) (where \( A_s = t_\infty(y_s) \)).

Condition (C3) in the definition of \( T_C^{(C)} \), combined with lemma 320, shows that every equation \( (\pi,T,U) \) produced by \( T_C \) has size

\[ \max\{||T||,||U||\} \leq D_2, \] (145)

hence that the number of possible unweighted equations produced by \( T_C \) is finite. Hence \( T_C \) occurs only a finite number of times on this branch (because \( t_\infty \) is repetition-free (144) and \( T_{cut} \) cannot occur on an infinite branch). Let \( n_0 \) be the last point where \( T_C \) occurs (or \( n_0 = 0 \) if \( T_C \) never occurs on this branch).

\( (y_{n_0+i})_{i \geq 0} \) is a branch of a tree \( t' \in \mathcal{T}(\mathcal{S}_{AB},A_{n_0}) \). Let us notice also that

- every equation produced by \( T_C \) is unmarked,

\( \text{(by condition (C4) in the definition of } T_C^{(C)} \text{, see section 6)}, \)

and

- every equation produced by \( T_C \) has a length \( \lambda \leq \lambda_2 \),

\( \text{(by definition (112) in section 3.4 and assumed in section 8).} \)

As \( \mathcal{S}_{ABC} \) is a strategy for \( B_0 \) and \( A_0 \) is true, \( A_{n_0} \) is also true, hence hypothesis (114) assumed in section 8 is fulfilled. We may apply now the results obtained
in §8.2.
By lemma 810, the branch \((y_{n_0+i})_{i \geq 0}\) must contain an infinite B-stacking sequence. Let us remark that, as \(T_0\) does not occur (otherwise the branch would be finite) every equation \((\pi, U^-, U^+)\) labelling this branch is such that \(U^- \neq \emptyset, U^+ \neq \emptyset\). By lemma 80 such a B-stacking sequence contains a subsequence \((A_{\kappa_1}, A_{\kappa_2}, \cdots, A_{\kappa_d})\) with \(d \leq d_0\), fulfilling hypotheses (1,2) of lemma 54, and by the above remark it fulfills hypothesis (75) of section 5 too. Let \(n_i \in \mathbb{N}\) such that \(x_{n_i} = y_{\kappa_i}\), for \(1 \leq i \leq d\). By (143), \(\Omega(\bar{\Pi}(t_{n_0}), \text{im}(t_{n_0})) \neq \emptyset\). Let us consider some
\[
\mathcal{O} \in \Omega(\bar{\Pi}(t_{n_0}), \text{im}(t_{n_0})).
\]
Let \(S_d = (A_{\kappa_i})_{1 \leq i \leq d}\) and \(D = D^{(\mathcal{O})}(S_d)\). By lemma 54
\[
\text{INV}^{(\mathcal{O})}(S_d) \neq \perp, D \in [0, d-1] \text{ and } \|\text{INV}^{(\mathcal{O})}(S_d)\| \leq \Sigma+d_0+s_{d_0}.
\]
(148)
Let \(S_{D+1} = (A_{\kappa_i})_{1 \leq i \leq D+1}\). By hypothesis (2) of lemma 54 (we established that this hypothesis is true),
\[
\bar{\Pi}(t_{n_{D+1}}) \leq \bar{\Pi}(t_{n_0}),
\]
and it is straightforward that
\[
\text{im}(t_{n_{D+1}}) \subseteq \text{im}(t_{n_0}),
\]
hence,
\[
\mathcal{O} \in \Omega(\bar{\Pi}(t_{n_0}), \text{im}(t_{n_0})).
\]
(149)
Let \(W_{D+1} = A_0 \cdot A_1 \cdots A_{\kappa_1} \cdots A_{\kappa_{D+1}}\) (the word from the root to \(y_{\kappa_{D+1}}\)). Let us notice that
\[
D^{(\mathcal{O})}(S_{D+1}) = D^{(\mathcal{O})}(S_d) = D, \text{INV}^{(\mathcal{O})}(S_{D+1}) = \text{INV}^{(\mathcal{O})}(S_d).
\]
(150)
By (148), (150),
\[
\rho_c(\text{INV}^{(\mathcal{O})}(S_{D+1})) \in T^{(\mathcal{O})}_C(W_{D+1}),
\]
(151)
By (149), (151), the set \(\{\mathcal{O} \in \Omega(\bar{\Pi}(t_{n_{D+1}}), \text{im}(t_{n_{D+1}})), T^{(\mathcal{O})}_C(W_{D+1}) \neq \emptyset\}\) is not empty, so that case (0) of the definition of \(\tilde{\Delta}(t)\) (see section 5) is fulfilled and
\[
\tilde{\Delta}(t_{n_{D+1}}) = A_{n_{D+1}}(T^{(\mathcal{O})}_C(W_{D+1})),
\]
i.e. \(T_C\) occurs at \(y_{\kappa_{D+1}+1}\). This is a contradiction with the minimality of \(n_0\). We have proved that hypothesis (142) is impossible. Hence the lemma is proved. ∎
10 Elimination

10.1 System $\mathcal{B}_1$

We prove here that the new formal system $\mathcal{B}_1$ obtained by elimination of meta-rule (R5) in $\mathcal{B}_0$ is recursively enumerable and complete. The decidability of the bisimulation problem follows.

Let $\mathcal{B}_1 = \langle A, H, \vdash_B \rangle$ where $A, H$ are the same as in $\mathcal{B}_0$, but the elementary deduction relation $\vdash_{\mathcal{B}_1}$ is the relation generated by the subset of metarules $R_0, R_1, R_2, R_3, R_3', R_4, R_6, R_7, R_8$, i.e. all the metarules of $\mathcal{B}_0$ except $R_5$. The deduction relation $\vdash_{\mathcal{B}_1}$ is now defined by:

$\vdash_{\mathcal{B}_1} = \vdash_{\mathcal{B}_1} \circ [1] \vdash_{\mathcal{B}_1} \circ [1] R_0, R_3, R_3', R_4 \circ \vdash_{\mathcal{B}_1}$.

Lemma 101: $\mathcal{B}_1$ is a deduction system.

**Sketch of proof:** As $\vdash_{\mathcal{B}_1} \subseteq \vdash_{\mathcal{B}_0}$, property (A1) is fulfilled by $\vdash_{\mathcal{B}_1}$. By the well-known decidability properties for finite-automata, rules $R_0, R_1, R_2, R_3, R_3', R_4, R_6, R_7, R_8$ are recursively enumerable. Hence property (A2) is fulfilled by $\mathcal{B}_1$. □

Completeness

**Definition 102** Let $P$ be a finite subset of $A$ and let $\bar{\pi} \in \mathbb{N}$. $P$ is said locally $\bar{\pi}$-consistent iff, for every $(\pi, S, S') \in P$, if

$\pi < \bar{\pi}$,

then, there exists $R_1 \in \bar{B}_1$ such that

$[\pi, S, S', R_1] \subseteq \text{Cong}(P)$.

**Lemma 103** Let $P$ be a finite subset of $A$ and let $\bar{\pi} \in \mathbb{N}$. If $P$ is locally $\bar{\pi}$-consistent, then $P$ is $\bar{\pi}$-consistent.

**Proof:** Let us consider, for every integers $n \geq 0, p \geq 0$, the following property $Q(n, p)$: $\forall \pi \in \mathbb{N}, \lambda \in \mathbb{N} - \{0\}, S, S' \in \text{DRB}_1, (\Pi V')$,

$(\pi, S, S') \in \text{Cong}_p(P)$ and $\pi + n - 1 < \bar{\pi}$ ⇒

$\exists R_n \in \mathcal{B}_n(S, S'), [\pi, S, S', R_n] \subseteq \text{Cong}(P)$.

(152)

Let us prove by induction on $(n, p)$ that

$\forall (n, p) \in \mathbb{N} \times \mathbb{N}, Q(n, p)$.

(153)
\( n = 0, p = 0:\)

The only possible value of \( R_0 \in B_0(S, S') \) is \( R_0 = \{ (\epsilon, \epsilon) \} \), and \( [\pi, S, S', R_0] = \{ (\pi, S, S') \} \subseteq \text{Cong}_0(P) \).

\( p > 0:\)

There exists a subset \( Q \subseteq P_f(A) \), such that

\[
P \vdash_{<p-1>} Q \quad \text{and} \quad Q \vdash_{<1>} \{ (\pi, S, S') \}.
\]

As every rule of \( B_0 \) increases the weight, we can suppose that every assertion of \( Q \) has a weight \( \leq \pi \). Hence, by induction hypothesis,

\[
\forall (\pi', T, T') \in Q, \exists R_n \in B_n(T, T'), [\pi', T, T', R_n] \subseteq \text{Cong}(P). \quad (154)
\]

Let us consider the type of rule used in the last step, \( Q \vdash_{<1>} \{ (\pi, S, S') \} \), of the above deduction.

**R0:** \((\pi - 1, S, S') \in Q.

By (154), \( \exists R_n \in B_n(S, S') \),

\[
[\pi - 1, S, S', R_n] \subseteq \text{Cong}(P).
\]

As \( [\pi - 1, S, S', R_n] \vdash_{<1>} [\pi, S, S', R_n] \),

\[
[\pi, S, S', R_n] \subseteq \text{Cong}(P).
\]

**R1:** \((\pi, S', S) \in Q.

(analogous to the above case)

**R2:** \((\pi, S, T), (\pi, T, S') \in Q.

By (154), \( \exists R_n \in B_n(S, T), R_n' \in B_n(T, S') \),

\[
[\pi, S, T, R_n] \subseteq \text{Cong}(P), [\pi, T, S', R_n'] \subseteq \text{Cong}(P).
\]

Using the properties mentioned in section 4.3, we get that:

\[
[\pi, S, S', R_n \circ R_n'] \subseteq \text{Cong}(P).
\]

**R3:**

In this case, \( R_n = \text{Id} \cap X^{\leq n} \times X^{\leq n} \in B_n(S, S') \), and

\[
[\pi, S, S', R_n] \subseteq \{ (\pi, S, S') \} \cup \{ (\pi + k, T, T), 1 \leq k \leq n, T \in \text{DRB}_{1, \lambda}(\langle V \rangle) \} \subseteq \text{Cong}(P).
\]

**R3:**

In this case, \( R_n = \text{Id} \cap X^{\leq n} \times X^{\leq n} \in B_n(S, S') \), and

\[
[\pi, S, S', R_n] = \{ (\pi + k, S \circ u, \rho_e(S) \circ u) \mid 0 \leq k \leq n, u \in X^k \} \subseteq \text{Cong}(P),
\]

(because \( \rho_e(S) \circ u = \rho_e(S \circ u) \)).
R6: $(\pi, S_1 \cdot S' + U, S') \in Q, S = S_1^* \cdot U$.
By (154), $\exists R_n \in B_n(S_1 \cdot S' + U, S')$,
\[ [\pi, S_1 \cdot S' + U, S', R_n] \subseteq \text{Cong}(P). \]
Using the properties mentioned in section 4.3, we get that:
\[ [\pi, S_1, S', R_n^<S_1,\cdot> = [\pi, S_1^*, U, S', R_n^<S_1,\cdot>] \]
\[ \subseteq \text{Cong}[\pi, S_1 \cdot S' + U, S', R_n] \]
\[ \subseteq \text{Cong}(Q) \subseteq \text{Cong}(P). \]

R7: $(\pi, S_1, S_1') \in Q, S = S_1 \cdot T, S' = S_1' \cdot T$.
By (154), $\exists R_n \in B_n(S_1, S_1')$,
\[ [\pi, S_1, S_1', R_n] \subseteq \text{Cong}(P). \]
Using the properties mentioned in section 4.3, we get that:
\[ [\pi, S, S', < S_1|R_n>] = [\pi, S_1 \cdot T, S_1' \cdot T, < S_1|R_n>] \]
\[ \subseteq \text{Cong}(\langle [\pi, S_1, S_1', R_n]\rangle) \]
\[ \subseteq \text{Cong}(Q) \subseteq \text{Cong}(P). \]

R8: $\forall \delta \in [1, \delta], (\pi, T_{i*,}, T_{i*,}') \in Q, S = S_1 \cdot T, S' = S_1 \cdot T'$.
By (154), $\exists R_{1,n}, \ldots, R_{\delta,n} \in B_n(T_{i*,}, T_{i*,}')$, such that
\[ [\pi, T_{i*,}, T_{i*,}', R_{i,n}] \subseteq \text{Cong}(P). \]
Using the properties mentioned in section 4.3, we get that:
\[ [\pi, S, S', < S, R_{*,n}>] = [\pi, S_1 \cdot T, S_1 \cdot T', < S, R_{*,n}>] \]
\[ \subseteq \text{Cong}(\bigcup_{1 \leq i \leq \delta} [\pi, T_{i*,}, T_{i*,}', R_{i,n}]) \]
\[ \subseteq \text{Cong}(Q) \subseteq \text{Cong}(P). \]

In all cases $Q(n, p)$ has been established.

$n > 0, p = 0: (\pi, S, S') \in P$.

As $P$ is locally $\bar{\pi}$-consistent and $\pi \leq \pi + n - 1 < \bar{\pi}$, there exist $R_1 \in B_1(S, S')$, $q \in \mathbb{N}$ such that:
\[ [\pi, S, S', R_1] \subseteq \text{Cong}_q(P). \] (155)

As $(n-1, q) < (n, 0)$, by induction hypothesis, $\forall (x, x') \in R_1 \cap X \times X, \exists R_{x,x',n-1} \in B_{n-1}(S \circ x, S' \circ x')$ such that
\[ [\pi + 1, S \circ x, S' \circ x', R_{x,x',n-1}] \subseteq \text{Cong}(P). \] (156)

Let us consider $R_n = \{(e, e)\} \bigcup_{(x,x') \in R_1 \cap X \times X} \{(x, x')\} \cdot R_{x,x',n-1}$. One can check that $R_n \in B_n(S, S')$ and, by (155, 156) we obtain:
\[ [\pi, S, S', R_n] = \{(\pi, S, S')\} \bigcup_{(x,x') \in R_1 \cap X \times X} [S \circ x, S' \circ x', R_{x,x',n-1}] \subseteq \text{Cong}(P). \]
Let us define now an oracle $O \in \Omega$ which is $\bar{\pi}$-consistent with $P$. For every $(S, S') \in \bigcup_{\lambda \geq 1} \text{DRB}_1,\lambda(\langle V \rangle)$ occurring in $\text{Cong}(P)$ (i.e. as the projection on $\bigcup_{\lambda \geq 1} \text{DRB}_1,\lambda(\langle V \rangle)$ of an assertion in $\text{Cong}(P)$), let us note

$$W(S, S') = \min(\{\pi \in \mathbb{N} \mid (\pi, S, S') \in \text{Cong}(P)\}).$$

$$D(S, S') = \max\{\bar{\pi} - W(S, S'), 0\}.$$

$$C(S, S') = \min\{R \in B_{\text{D}(S, S')}(S, S') \mid [W(S, S'), S, S', R] \subseteq \text{Cong}(P)\}.$$

Notice that $C(S, S')$ is well-defined, owing to property (153). We then define $O$ by: for every $(S, S')$ occurring in $\text{Cong}(P)$,

$$O(S, S') = \min\{R \in B_{\infty}(S, S') \mid C(S, S') = R \cap (X^{\leq \text{D}(S, S')} \times X^{\leq \text{D}(S, S')})\},$$

and for every $(S, S')$ not occurring in $\text{Cong}(P)$,

$$O(S, S') = \min\{R \in B_{\infty}(S, S') \mid (S \sim S'), O(S, S') = \text{Id}_{X} \cdot (S \not\sim S').\}$$

One can check that, by the choice of $C(S, S')$, $O$ is $\bar{\pi}$-consistent with $P$. □

**Lemma 104** Let $A_0 \in A$ such that $H(A_0) = \infty$. Let us consider the sequence of trees $t_n = S_{ABC}^{\pi_n}(A_0)$. For every integer $n \geq 0$, $t_n$ is consistent.

Let us say that the strategy $T$ “applies on” node $x$ iff, $x$ has exactly $m$ sons $x \cdot 1, x \cdot 2, \ldots, x \cdot m$ and

$$\tau(x) \cdot \tau(x \cdot 2) \cdots \tau(x \cdot m) \in T(\tau(x[0]) \cdot \tau(x[1]) \cdots \tau(x[|x|])),$$

i.e. the word consisting of the labels of the sons of $x$ belongs to the image of the path from $\epsilon$ (included) to $x$ (included) by the strategy $T$.

**Proof:** For every $k \in \mathbb{N}$ we define

$$\bar{\pi}_k = \bar{\Pi}(t_k).$$

We prove by induction on $(n, \pi)$ the following property $R(n, \pi)$:

$$\forall x \in \text{dom}(t_n), \text{ if } t_n(x) = (\pi, S, S') \text{ with } \pi < \bar{\pi}_n, \text{ then } \exists R_1 \in B_1(S, S'), [\pi, S, S', R_1] \subseteq \text{Cong}(\text{im}(t_n)).$$

(159)

At every step of our proof by induction, we consider some node $x$ of $t_n$ fulfilling hypothesis (159) and we show that it must fulfill (160). Let us notice that, if $x$ is not closed, then hypothesis (159) cannot be true, by minimality of $\bar{\pi}_n$. Let us notice also that, if $x$ is closed, but there is some $x' < x$ such that $t_n(x') = t_n(x)$, then (160) on $x$ is the same property as (160) for $x'$. Hence, in the sequel, we can suppose that $x$ is closed and that it is minimal (w.r.t. to $\leq$):

$$x = \min_{\leq}\{y \in \text{dom}(t_n) \mid t_n(y) = t_n(x)\}.$$  

(161)
\[ n = 0, \pi = 0: \text{dom}(t_0) = \{\epsilon\}, t_0(\epsilon) = A_0. \] If \( \epsilon \) is not closed, then \( \bar{\pi}_0 = \pi = 0 \), hence there is no node \( x \) fulfilling hypothesis (150). Otherwise, \( \bar{\pi}_0 = \infty \) and \( x = \epsilon \) is closed: either \( T_\theta(A_0) = \{\epsilon\} \) or \( T_\epsilon(A_0) = \{\epsilon\}. \] Let us choose

\[ R_1 = \text{Id}_{X^*} \cap X^{\leq 1} \times X^{\leq 1}. \] (162)

If we note \( A_0 = (\pi, S_0^-, S_0^+) \), then

\[ [\pi, S_0^-, S_0^+, R_1] = \{(\pi, S_0^-, S_0^+)\} \cup \{(\pi + 1, S_0^- \circ x, S_0^+ \circ x) \mid x \in X\}, \]

where \( \forall x \in X, S_0^- \circ x \equiv S_0^+ \circ x \equiv \emptyset \). Using rule \( R'3 \), we see that

\[ [\pi, S, S', R_1] \subseteq \text{Cong}(\emptyset) \subseteq \text{Cong}(\text{im}(t_n)). \] (163)

\( n > 0, \pi = 0 \); Let \( x \) be some node of \( t_n \) such that \( \exists S, S', t_n(x) = (\pi, S, S') \) and \( \pi < \bar{\pi}_n \). Let us denote by \( W_x \) the word labelling the path from the root of \( t_n \) (including) to \( x \) (including).

**case 1:** \( \exists x' \in \text{dom}(t_n), x' \) internal node, such that \( t_n(x') = t_n(x) \).

As \( \pi = 0 \), the sons \( x' \cdot 1, x' \cdot 2, \ldots, x' \cdot m \) of \( x' \) are such that \( t_n(x' \cdot 1) \cdot t_n(x' \cdot 2) \cdots t_n(x' \cdot m) \in T_A^{(0)}(W_{x'}) \), for some oracle \( O \). Let us choose

\[ R_1 = O(S, S') \cap X^{\leq 1} \times X^{\leq 1}. \] (164)

Then

\[ [\pi, S, S', R_1] \subseteq \text{im}(t_n). \] (165)

**case 2:** \( T_\theta(W_x) = \{\epsilon\} \) or \( T_\epsilon(W_x) = \{\epsilon\} \).

In this case the choice \( R_1 = \text{Id}_{X^*} \cap X^{\leq 1} \times X^{\leq 1} \) satisfies again (163).

\( \pi > 0 \):

Let \( x \) fulfilling hypothesis (150). As \( t_n \) is a proof-tree for \( S_{ABC} \), and as we suppose \( x \) is closed and minimal (161), one of the following cases must occur.

**case 1:** \( T_{\text{cut}} \) applies on \( x \).

There exists \( x' \in \text{dom}(t_n), \exists \pi' \in \mathbb{N} \), such that

\[ t_n(x') = (\pi', S, S') \] and \( \pi' < \pi \).

By induction hypothesis

\[ \exists R_1 \in B_1(S, S'), [\pi', S, S', R_1] \subseteq \text{Cong}(\text{im}(t_n)), \]

and by means of rule \( R0 \):

\[ [\pi, S, S', R_1] \subseteq \text{Cong}([\pi', S, S', R_1]). \]

Hence (160) is true.

**case 2:** \( T_\theta \) or \( T_\epsilon \) applies on \( x \).

Here again, the choice (163) fulfills property (163).

In the remaining cases we use the following notation: for every \( k \in \mathbb{N} \) such that \( t_k \) is not closed,

\[ x_k = \min\{x \in \text{dom}(t_k), x \text{ is not closed for } S_{ABC} \text{ and } \exists S, S', t(x) = (\bar{\pi}_k, S, S')\}. \]
If $\exists k < n \mid t_k$ is not consistent or is closed, then by (111), $t_k = t_{k+1} = \cdots = t_n$, hence $R(n, \pi) \Leftrightarrow R(k, \pi)$, and this last property is true by induction hypothesis. Let us suppose now that $\forall k < n$, $t_k$ is consistent and unclosed. According to formula (110),

$$t_{k+1} = t_k[e_{k+1}/x_k],$$

for some tree of depth one, $e_{k+1}$.

Let $k \in [0, n-1], x = x_k, \pi = \bar{\pi}_k$ (such a $k$ must exist because $x$ is internal). Let $x \cdot 1, \ldots, x \cdot \mu$ be the sequence of sons of $x$.

**case 3:** $T_A$ applies on $x$.

Hence there exists some oracle $O$ such that $T_A^{(O)}$ applies on $x$. The choice (164) fulfills property (165).

**case 4:** $T_B^\alpha$ applies on $x$ (for some $\alpha \in \{-, +\}$).

Let us suppose $\alpha = +$. Let $x' = x(|x| - k_1)$ (the prefix of $x$ having length $|x| - k_1$), $t_n(x') = (\pi', \bar{U}, U')$. By definition of $S_{ABC}$, there exists some oracle $O$ which is $\bar{\pi}_k$-consistent with $im(t_k)$ and such that:

$$\mu = 1 \text{ and } t_n(x \cdot 1) = T_B^{(O),+}(W_x).$$

Let us look at the proof of lemma 72 in the particular case of this oracle $O$: as the pairs $(u_\ell, u'_\ell)$ belong to $O(\bar{U}, U')$ (for every $\ell \in [1, q]$) and $\pi' + |u_\ell| - 1 < \pi' + k_0 \leq \pi' + 2 \cdot k_0 < \bar{\pi}_k$, deduction (103) can be obtained just by using rules in $C$. As deduction (103) is the only one (in the proof of lemma 72) using rules in $B_0 - C$ we conclude that deduction (102) can be replaced by:

$$\{t_n(x'), t_n(x \cdot 1)\} \cup im(t_k) \vdash \tau_{-1}(t_n(x)).$$

(We recall $\tau_{-1}$ consists in replacing the weight of a given weighted equation into its predecessor). Deduction (166) implies that

$$\exists p \in \mathbb{N}, (\pi - 1, S, S') \in \text{Cong}_p(im(t_n)).$$

By induction hypothesis, as $\pi - 1 < \bar{\pi}_n$, $im(t_n)$ is locally $\pi - 1$-consistent, hence, by lemma (103), $im(t_n)$ is $\pi - 1$-consistent. Hypothesis (167) implies that

$$\exists R_1 \in B_1(S, S'), [\pi - 1, S, S', R_1] \subseteq \text{Cong}(im(t_n)),$$

hence, using $R_0$, that

$$\exists R_1 \in B_1(S, S'), [\pi, S, S', R_1] \subseteq \text{Cong}(im(t_n)).$$

**case 5:** $T_C$ applies on $x$.

By definition of $S_{AB}$, there exists some oracle $O$ which is $\bar{\pi}_k$-consistent with $im(t_k)$ and such that:

$$\mu = 1 \text{ and } t_n(x \cdot 1) = T_C^{(O)}(W_x).$$
Let \( W_x = A_1 \cdots A_{|x|+1}, \ \kappa_1 < \cdots < \kappa_i < \kappa_{i+1} < \cdots \kappa_{D+1} = |x| + 1, \ S = (\mathcal{E}_i)_{1 \leq i \leq D+1}, \) where, for every \( 1 \leq i \leq d, \)

\[
\mathcal{E}_i = A_{\kappa_i} = (\pi_i, \sum_{j=1}^{d} \alpha_{i,j} S_j, \sum_{j=1}^{d} \beta_{i,j} S_j)
\]

and

\[
T_C^{(\mathcal{O})}(W_x) = \rho_e(\text{INV}^{(\mathcal{O})}(S)), \ \text{W}^{(\mathcal{O})}(S) \neq \bot, \ D^{(\mathcal{O})}(S) = D \leq d - 1.
\]

Let us look at the proof of lemma \( \square \) in the particular case of this oracle \( \mathcal{O} \): the only place where a rule in \( \mathcal{B}_0 - C \) is used, is in deduction \( \square \), when case 2, subcase 1 (or case 2, subcase 2), of the recursive definition of \( \text{INV}^{(\mathcal{O})}(S) \) occurs. Let us recall that the pair \((u, u')\) chosen by the oracle \( \mathcal{O} \) is such that:

\[
\mathcal{R} = \mathcal{O}(\sum_{j=1}^{d} \alpha_{1,j} S_j, \sum_{j=1}^{d} \beta_{1,j} S_j),
\]

\[
\nu = \text{Div}(\alpha_{1,*}, \beta_{1,*}), \ \mathcal{R}_{(\nu)} = \mathcal{R} \cap X^{\leq \nu} \times X^{\leq \nu}, \ (u, u') \in \mathcal{R}_{(\nu)}.
\]

Let us notice that \( \pi_1 + \nu - 1 < \pi_1 + 2 \cdot \nu < \pi_2 \leq \text{W}^{(\mathcal{O})}(S) + 1 = \pi = \pi_k. \) As \( \mathcal{O} \) is \( \pi_k \)-consistent with \( \text{im}(t_k) \), we conclude that:

\[
(\pi_1 + |u|, (\sum_{j=1}^{d} \alpha_{i,j} S_j) \oplus u, (\sum_{j=1}^{d} \beta_{i,j} S_j) \oplus u') \in [\pi_1, \sum_{j=1}^{d} \alpha_{i,j} S_j, \sum_{j=1}^{d} \beta_{i,j} S_j, \mathcal{R}_{(\nu)}]
\]

\[
\subseteq \text{Cong}(\text{im}(t_k)).
\]

Hence deduction \( \square \) can be replaced by:

\[
\mathcal{E}'_1 \in \text{Cong}(\text{im}(t_k)). \quad (168)
\]

Similarly, for every \( i \in [2, D], \) as \( \pi_i + 2 \cdot \text{Div}(\alpha_{i,*}, \beta_{i,*}) < \pi_{i+1} \leq \text{W}^{(\mathcal{O})}(S) + 1 = \pi = \pi_k, \) and \( \mathcal{E}'_i \in \text{Cong}(\text{im}(t_k)), \)

\[
(\mathcal{E}'_i)^{i-1} \in \text{Cong}(\text{im}(t_k)). \quad (169)
\]

It follows that deduction \( \square \) can be replaced by:

\[
\{\text{INV}^{(\mathcal{O})}(S)\} \cup \text{im}(t_k) \quad \overset{<s>}{\vdash} \ \tau_1(t_n(x)). \quad (170)
\]

using the facts that \( \rho_e(\text{INV}^{(\mathcal{O})}(S)) \overset{<s>}{\vdash} \tau_1(t_n(S)) \) and \( \text{im}(t_k) \subseteq \text{im}(t_n) \) we may conclude that:

\[
\{t_n(x \cdot 1)\} \cup \text{im}(t_n) \quad \overset{<s>}{\vdash} \ \tau_1(t_n(x)) = (\pi - 1, S, S'). \quad (171)
\]
From (171) and the induction hypothesis, we can conclude, as in case 4, that
\[ \exists R_1 \in B_1(S, S'), [\pi, S, S', R_1] \subseteq \text{Cong}(\text{im}(t_n)). \]
(End of the induction).

By the above induction, for every \( n \in \mathbb{N} \), \( \text{im}(t_n) \) is \( \bar{\pi}_n \)-consistent i.e. \( t_n \) is consistent.

**Lemma 105** \( \hat{S}_{ABC} \) is closed.

**Proof:** Let \( A_0 \in \mathcal{A} \). By lemma [104], \( \forall n \in \mathbb{N}, \hat{S}_{ABC}^n(A_0) \) is consistent. If \( \hat{S}_{ABC}^n(A_0) \) is consistent and is not closed, then, by definition [110],

\[ \hat{S}_{ABC}^n(A_0) \neq \hat{S}_{ABC}^{n+1}(A_0); \]

if \( \hat{S}_{ABC}^n(A_0) \) is consistent and is closed, then, by definition [111],

\[ \hat{S}_{ABC}^n(A_0) = \hat{S}_{ABC}^{n+1}(A_0). \]

Hence the equivalence [13], which defines the notion of closed global strategy, is fulfilled by \( \hat{S}_{ABC} \). \( \square \)

**Theorem 106** : \( B_0, B_1 \) are complete formal systems.

**Proof:** By lemma [71], \( \hat{S}_{ABC} \) is terminating on every unmarked assertion and by lemma [103] \( \hat{S}_{ABC} \) is closed. Let \( A_0 \) be some unmarked true assertion. According to the proof of lemma [110], \( \exists n_0 \in \mathbb{N} \) such that \( t_\infty = \hat{S}_{ABC}^{n_0}(A_0) \) is a proof-tree which is closed, hence such that \( H(t_\infty) = \infty \). By lemma [107], \( t_\infty \) is consistent, i.e. \( \text{im}(t_\infty) \) is \( \infty \)-consistent: \( \forall (\pi, S, S') \in \text{im}(t_\infty), \)

\[ \exists R_1 \in B_1(S, S'), [\pi, S, S', R_1] \subseteq \text{Cong}(\text{im}(t_\infty)), \]

hence,

\[ \text{im}(t_\infty) \vdash_{<\leftrightarrow} [\pi, S, S', R_1] \vdash_{R4} (\pi, S, S'). \quad (172) \]

As the rules of \( C \) and \( R4 \) are rules of \( B_1 \), deduction [172] shows that

\[ \text{im}(t_\infty) \vdash_{B_1} (\pi, S, S'). \quad (173) \]

i.e. \( \text{im}(t_\infty) \) is a \( B_1 \)-proof.

In the general case where \( A_0 = (\pi_0, U_0^-, U_0^+) \) might be marked, we observe that, owing to rules (R1)(R2)(R3):

\[ \{ \rho_c(A_0) \} \vdash_{<\leftrightarrow} \{ A_0 \}. \]

This deduction combined with some \( B_1 \)-proof of \( \rho_c(A_0) \) gives a \( B_1 \)-proof of \( A_0 \). \( \square \)
Theorem 107  The bisimulation problem for rooted equational 1-graphs of finite out-degree is decidable.

Proof: Let us consider the sequence of statements: lemma 27, lemma 28, corollary 26 and lemma 328. By means of the above statements, the bisimulation problem for rooted equational 1-graphs of finite out-degree reduces to the following decision problem (we call it the bisimulation problem for deterministic vectors):

INSTANCE: a bi-rooted, normalized dpda $\mathcal{M}$, its terminal alphabet $X$, a surjective litteral morphism $\psi : X^* \to Y^*$ (we denote its kernel by $\bar{\psi}$), and $\lambda \in \mathbb{N} - \{0\}, S, S' \in \text{DRB}_{1,\lambda}(\langle V \rangle)$ (where $V$ is the structured alphabet associated with $\mathcal{M}$).

QUESTION: $S \sim S'$? (where $\sim$ is the $\bar{\psi}$-bisimulation relation).

Let us consider $\mathcal{M}, X, V, \bar{\psi}$ given by some instance. The equivalence relation $\sim$ on $\text{DRB}_{1,\lambda}(\langle V \rangle)$ has a recursively enumerable complement (this is well-known). By theorem 106 and lemma 42, relation $\sim$ is recursively enumerable too. Hence $\sim$ is recursive.

But the function associating to every $\mathcal{M}, X, V, \bar{\psi}$ the corresponding deduction system $B_1$ is recursive. Hence the bisimulation problem for deterministic vectors is decidable. $\square$

10.2 System $B_2$

We exhibit here a deduction system $B_2$ which is simpler than $B_1$ and is still complete.

Elementary rules Let us eliminate the weights in the rules of $B_1$: we define a new set of assertions, $\mathcal{A}_2$ by

$$\mathcal{A}_2 = \bigcup_{\lambda \in \mathbb{N} - \{0\}} \text{DRB}_{1,\lambda}(\langle V \rangle) \times \text{DRB}_{1,\lambda}(\langle V \rangle).$$

We define a binary relation $|\vdash | \subseteq \mathcal{P}(\mathcal{A}_2) \times \mathcal{A}_2$, the elementary deduction relation, as the set of all the pairs having one of the following forms:

(R21) $$\{(S, T)\} \vdash (T, S)$$

for $\lambda \in \mathbb{N} - \{0\}, S, T \in \text{DRB}_{1,\lambda}(\langle V \rangle)$,

(R22) $$\{(S, S'), (S'', S''')\} \vdash (S, S'')$$

for $\lambda \in \mathbb{N} - \{0\}, S, T \in \text{DRB}_{1,\lambda}(\langle V \rangle)$,

(R23) $$\emptyset \vdash (S, S)$$

for $S \in \text{DRB}_{1,\lambda}(\langle V \rangle)$,
We define $\mathcal{B}_2$ by: for every $P \in \mathcal{P}_f(A_2)$, $A \in A_2$,

$$P \vdash A \iff P \vdash_{23,24}^\circ \vdash_{23,24} \vdash_{23,24} \vdash \{ A \}.$$ 

where $\vdash_{23,24}$ is the relation defined by $R_23, R'23, R24$ only.

We define a simpler cost function $H_2 : A_2 \rightarrow \mathbb{N} \cup \{ \infty \}$ by:

$$\forall (S, S') \in A_2, H_2(S, S') = \text{Div}(S, S').$$

We let

$$\mathcal{B}_2 = \ll A_2, H_2, \vdash_{\mathcal{B}_2} \gg.$$ 

**Lemma 108**: $\mathcal{B}_2$ is a deduction system.

**Completeness**

Let us denote by $\mathcal{C}_2$ the subset of rules of $\mathcal{B}_2$ obtained by removing the weights in the rules of $\mathcal{C}$.

**Definition 109** Let $P \in \mathcal{P}_f(A_2)$. $P$ is said to be self-generating iff, for every $(S, S') \in P$,

1. either $S = S' = \epsilon$
2. or $\exists R_1 \in \bar{B}_1(S, S'), (x, x') \in R_1, P \vdash_{c_2} (S \odot x, S' \odot x').$

(See in remark 1013 below, the origins of this notion).
Lemma 1010 Let $A \in A_2$ such that $A$ is unmarked. Then $H(A) = \infty$ iff there exists a finite self-generating set $P \subseteq A_2$ such that $A \in P$.

Proof: Owing to metarules $R_{23}, R_{24}$ it is clear that every self-generating set $P \in P_f(A_2)$ is a $B_2$-proof. Hence, if $A$ belongs to some self-generating set, then $H(A) = \infty$.

Let us suppose now that $H_2(A) = \infty$. Let us consider the closed proof-tree $t_\infty$ obtained by applying the global strategy $\hat{S}_{ABC}$ on the assertion $(0, A)$. By lemma $t_\infty$ is finite and by lemma $t_\infty$ is consistent, which means that $\text{im}(t_\infty)$ is $\infty$-consistent. Let

$$P = \text{pr}_{2,3}(\text{im}(t_\infty)),$$

(where $\text{pr}_{2,3} : A \to A_2$ is the map erasing the weights).

As $\text{im}(t_\infty)$ is $\infty$-consistent, $P$ is self-generating and $A \in P$. □

Theorem 1011: $B_2$ is a complete deduction system.

Proof: We already noticed that every self-generating set is a $B_2$-proof. Hence lemma proves that every true, unmarked assertion possesses some finite $B_2$-proof.

Let $A$ be any true assertion. $\rho_e(A)$ has a finite proof $P$. Owing to rules (R1)(R2)(R’3), $Q = P \cup \{A\}$ is a $B_2$-proof of $A$. □

10.3 System $B_3$

We exhibit here a deduction system $B_3$ which is even simpler than $B_2$ and is still complete. Let us consider $B_3 =< A_3, H_3, \vdash_{B_3} >$, where

$$A_3 = \bigcup_{\lambda \in \mathbb{N} - \{0\}} \text{DRB}_{1,\lambda}(\langle V_0 \rangle) \times \text{DRB}_{1,\lambda}(\langle V_0 \rangle).$$

$H_3 = H_2 | A_3$ and $\vdash_{B_3}$ is defined below: the metarules of $B_3$ are essentially those of $B_2$, but restricted to the unmarked vectors.

(R31) $$\{(S, T)\} \vdash (T, S)$$

for $\lambda \in \mathbb{N} - \{0\}, S, T \in \text{DRB}_{1,\lambda}(\langle V_0 \rangle),$

(R32) $$\{(S, S'), (S', S'')\} \vdash (S, S'')$$

for $\lambda \in \mathbb{N} - \{0\}, S, T \in \text{DRB}_{1,\lambda}(\langle V_0 \rangle),$

(R33) $$\emptyset \vdash (S, S)$$

for $S \in \text{DRB}_{1,\lambda}(\langle V_0 \rangle),$. 
\( \{ (S \odot x, T \odot x') \mid (x, x') \in R_1 \} |\rightarrow (S, T) \)

for \( \lambda \in \mathbb{N} - \{0\}, S, T \in \text{DRB}_{1,\lambda}(\langle V_0 \rangle), (S \not\equiv \epsilon \land T \not\equiv \epsilon) \) and \( R_1 \in \overline{B}_1 \),

\( \{(S_1 \cdot T + S, T)\} |\rightarrow (S_1^* \cdot S, T) \)

for \( \lambda \in \mathbb{N} - \{0\}, S_1 \in \text{DRB}_{1,\lambda}(\langle V_0 \rangle), S_1 \not\equiv \epsilon, (S_1, S) \in \text{DRB}_{1,\lambda+1}(\langle V_0 \rangle), T \in \text{DRB}_{1,\lambda}(\langle V_0 \rangle) \),

\( \{(S, S')\} |\rightarrow (S \cdot T, S' \cdot T) \)

for \( \delta, \lambda \in \mathbb{N} - \{0\}, S, S' \in \text{DRB}_{1,\delta}(\langle V_0 \rangle), T \in \text{DRB}_{\delta,\lambda}(\langle V_0 \rangle) \).

We then define \( |\rightarrow_{B_3} \) by: for every \( P \in \mathcal{P}_f(A_3), A \in A_3 \),

\[ P |\rightarrow_{B_3} A \iff P \xrightarrow{<\rightarrow}_{B_3} |\rightarrow_{33,34} \circ |\rightarrow_{B_3} \{ A \}. \]

where \( |\rightarrow_{33,34} \) is now the relation defined by \( R_33, R_34 \) only.

As \( |\rightarrow_{B_3} \subseteq |\rightarrow_{B_2} \) \( , H_3 = H_2 \), it is clear that \( B_3 \) is a deduction system.

Completeness

Let us call \( C_3 \) the intersection of set of the rules of \( C \) with the set of rules of \( B_3 \) (it is also equal to the set of instances of \( R_31, R_32, R_33, R_35, R_36, R_37 \). Let us call now \( P \in \mathcal{P}_f(A_3) \) a \( C_2 \)-self-generating set iff it fulfills definition \( 109 \) and a self-generating set iff it fulfills definition \( 109 \) but where \( C_2 \) is replaced by \( C_3 \).

Remark 1012

1-This notion of “self-generating set (of pairs)” is a straightforward adaptation to our d-space of vectors of the notion of “self-proving set of pairs” defined in \([\text{Cou87}], p.162\) for the magma \( M(F \cup \Phi, V) \).

2-The notion of “self-bisimulation” (introduced in \([\text{Cou90}]\) and also used in \([\text{HS91, HJM94}]\) was also such an adaptation, but in the context of a monoid-structure. The notion we use in this work can be seen, as well, as a generalisation of this notion of self-bisimulation: when every class in \( V_0/\sim \) has just one element, the only “rational deterministic boolean series” over \( V_0 \) are the words; in this case the self-bisimulations are exactly the self-generating sets.

Lemma 1013 Let \( A \in A_3 \). Then \( H_3(A) = \infty \) iff there exists a finite self-generating set \( P \subseteq A_3 \) such that \( A \in P \).
Proof: Owing to metarules R33 and R34, every self-generating set is a $B_3$-proof. Let $A \in A_3$ such that $H_3(A) = \infty$. By lemma 1010, there exists some $C_2$-self-generating set $P$ such that $A \in P$. 

Let us consider $Q = \{ \rho_c(B) \mid B \in P \}$. One can check that, $\rho_c$ maps the set of rules of $C_2$ is into the set of rules of $C_3$. One can also check that $\rho_c$ and $\odot$ are commuting (i.e. $\rho_c(S \odot u) = \rho_c(S) \odot u$). Hence $Q$ is such that, for every $(S, S') \in Q$,

1. either $S = S' = \epsilon$
2. or $\exists R_1 \in \bar{B}_1(S, S'), \forall (x, x') \in R_1, Q \vdash c_3 (S \odot x, S' \odot x')$.

i.e. $Q$ is self-generating. □

Theorem 1014 : $B_3$ is a complete deduction system.

Proof: Lemma 1013 implies the completeness property. □

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ANNEX

Let us sketch here a proof of theorem 25.

Lemma 1015 Let $\Gamma = (I_0, v_0)$ be the computation $I$-graph $(C(M), v_M)$ of some normalized pushdown automaton $M$. Then $\Gamma$ is equational and has finite out-degree.

Proof: Let $M = \langle X, Z, Q, \delta, q_0, z_0, F \rangle$ be a normalized pda. Let us consider a new letter $e \notin X$ and build the real-time pda $M_e = \langle X \cup \{ e \}, Z, \delta_e, q_0, z_0, F \rangle$ obtained by setting that, for every $x \in X$ and $q \in Q, z \in Z$:

$$\delta_e(qz, x) = \delta(qz, x); \quad \delta_e(qz, e) = \delta(qz, e).$$

By [MS85, theorem 2.6 p.62], the computation-graph $C(M_e)$ is context-free and by [Bau92, theorem 6.3 p. 187] every context-free graph is equational. Hence $C(M_e)$ is equational. Let us remark that $C(M)$ is obtained from this graph just by contracting all the edges labelled by $e$. Let us contract the edges labelled by $e$ in some system of equations $S_e$ defining $C(M_e)$: we obtain a system of equations $S$ defining $C(M)$. □

We use now the notation of [Con90b]. Given a system of graph equations $S = \langle u_i = H_i; i \in [1, n] \rangle$, by $G(S, u_i)$ we denote the $i$-th component of the canonical solution of $S$.

Definition 1016 Let $S = \langle u_i = H_i; i \in [1, n] \rangle$ be a system of graph equations. It is said standard iff it fulfills the conditions

1. for every $i \in [1, n]$ and every distinct integers $k, \ell \in [1, \tau(H_i)]$, the sources $src(H_i, k)$, $src(H_i, \ell)$ are distinct vertices of $H_i$,
2. for every $i \in [1, n]$ and every hyperedge $h$ of $H_i$ which is labelled by some unknown, all the vertices of $h$ are distinct,
3. for every $i \in [1, n], k \in [1, \tau(u_i)], \lambda \in \mathbb{N}$, if there exist $\lambda$ edges going out of $src(G(S, u_i), k)$, inside the graph $G(S, u_i)$ then there exists also $\lambda$ edges going out of $src(H_i, k)$, inside the graph $H_i$.

Lemma 1017 Let $S = \langle u_i = H_i; i \in [1, n] \rangle$ be a system of graph equations where the unknown $u_i$ has type 1. One can compute from $S$ a standard system of graph equations $S' = \langle u'_i = H'_i; i \in [1, n'] \rangle$ such that the canonical solution of $S'$ has a first component $G(S', u'_1) = G(S, u_1)$.

Proof: From $S$ one can construct a first system $S_1$ which generates the same first component $G(S_1, u_1) = G(S, u_1)$ and such that restrictions (1)(2) of the lemma are fulfilled: this follows from [Con90b, proposition 2.10 p.209],(notice that the condition “separated” in this reference is exactly the conjunction (1) ∧ (2)).

Let $S_1 = \langle v_i = K_i; i \in [1, m] \rangle$. Let us replace every right-hand side $K_i$ by a finite hypergraph $L_i$ obtained by unfolding the graph $K_i$, according to the rules $v_j \rightarrow K_j$, as many times as necessary in order that every source $src(K_i, k)$ gets
Lemma 1018 Let $\Gamma = (\Gamma_0, v_0)$ be a rooted 1-graph over $X$ which is the first component of the canonical solution of some standard system of graph equations. Then, $\Gamma$ is isomorphic to the computation 1-graph $(C(M), v_M)$ of some normalized pushdown automaton $M$.

Sketch of proof: Let $S = < u_i = H_i; i \in [1,n] >$ be a standard system of graph equations such that $\Gamma = G(S, u_1)$.

Let us define $M = < X, Z, Q, \delta, q_0, Z_0, F >$ as follows. In every right-hand side $H_i$ we number bijectively all the unknown hyperedges: $\{h_{1,i}, \ldots, h_{j,i}, \ldots, h_{n,i}\}$ and all the vertices: $\{v_{1,i}, \ldots, v_{j,i}, \ldots, v_{n,i}\}$. We note $\beta(j,i) = label(h_{j,i})$.

$Z = \{[j,i] | 1 \leq i \leq n, 1 \leq j \leq n\} \cup \{[1,0]\}.$

(We extend $\beta$ by defining $\beta(1,0) = 1$).

Intuitively every symbol $[j,i]$ describes the situation of a vertex which belongs to a component which has been glued on the $j$-th unknown hyperedge of $H_i$.

Let $Q = [1,N]$ where $N$ is the maximum number of vertices in the graphs $H_i$.

Intuitively, the transitions of $M$ starting from a mode $q[j,i]$ describe the edges starting from the $q$-th vertex of $H_{\beta(j,i)}$. Let us define precisely the transitions starting from a mode $q[j,i]$:

**case 1:** $q$ is strictly larger than the number of vertices of $H_{\beta(j,i)}$.

Then there is no transition starting from $q[j,i]$.

**case 2:** vertex number $q$ of $H_{\beta(j,i)}$ is a source of $H_{\beta(j,i)}$ and $i \neq 0$.

Then

$q[j,i] \xrightarrow{\delta} q',$

where $q'$ is the number of the vertex of $H_i$ on which it is glued (it is some vertex of $h_{j,i}$).

**case 3:** vertex number $q$ of $H_{\beta(j,i)}$ is not a source of $H_{\beta(j,i)}$ or $i = 0$.

**internal edges:**

For every edge $(v_{q,\beta(j,i)}, x, v_{q',\beta(j,i)})$, we add the transition

$q[j,i] \xrightarrow{x} q'[j,i].$

**external edges:**

Let $k = \beta(j,i).$ For every $\ell$ such that $v_{q,\beta(j,i)}$ is a vertex of $h_{\ell,k}$ and every edge $(v_{r,\beta(\ell,k)}, x, v_{q',\beta(\ell,k)})$ where the vertex $v_{r,\beta(\ell,k)}$ of $H_{\beta(\ell,k)}$ is glued on the vertex $v_{q,\beta(j,i)}$ by the rewriting rule $u_{\beta(\ell,k)} \rightarrow H_{\beta(\ell,k)}$, we add the transition:

$q[j,i] \xrightarrow{x} q'[[k][j,i]].$

The starting configuration is $1[1,0]$ (i.e $q_0 = 1, z_0 = [1,0]$).

This pda is normalized (this is easy to check) and has a computation graph.
whose isomorphism-class is exactly $G(S, u_1)$ (this would be much more tedious to prove formally). □

Theorem 25 clearly follows from these three lemmas.
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