Creation probabilities of hierarchical trees

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Abstract. We consider both analytically and numerically creation conditions of diverse hierarchical trees. A connection between the probabilities to create hierarchical levels and the probability to associate these levels into united structure is found. We argue a consistent probabilistic picture requires making use of the deformed algebra. Our consideration is based on study of main types of hierarchical trees, among which both regular and degenerate ones are studied analytically, while the creation probabilities of the Fibonacci and free-scale trees are determined numerically. We find a general expression for the creation probability of an arbitrary tree and calculate the sum of terms of deformed geometrical progression that appears at consideration of the degenerate tree.

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1. Introduction

The problem of the origin of hierarchy and its implications into physical, biological, economical, ecological, social and other complex systems has a long history which can be found in Refs. [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13]. Along this line, one of the most striking manifestations of hierarchy gives complex networks [14]. As is shown in considerations of diverse systems, ranged from the World Wide Web [15] to biological [16], [17], [18], [19] and social [20], [21], [22] networks, real networks are governed by strict organizing principles displayed in the following properties: i) most networks have a high degree of clustering; ii) many networks have been found to be scale-free [23], [24] that means the probability distribution over node degrees, being the set of numbers of links with neighbors, follows the power law. Moreover, many networks are modular: one can easily identify groups of nodes that are highly interconnected with each other, but have only a few or no links to nodes outside of the group to which they belong (in society such modules represent groups of friends or coworkers [25], in the WWW denote communities with shared interests [26], in the actor network they characterize specific genres or simply individual movies). This clearly identifiable modular organization is at the origin of the high clustering coefficient seen in many real networks. In order to bring modularity, the high degree of clustering and the scale-free topology under a single roof, we need to assume that modules combine with each other in a hierarchical manner.

Formal basis of the theory of hierarchical structures is known by the fact that hierarchically constrained objects are related to an ultrametric space whose geometrical image is the Cayley tree with nodes and branches corresponding to elementary cells and their links [27]. One of the first theoretical pictures [28] has been devoted to consideration of a diffusion process on either uniformly or randomly multifurcating trees. Consequent study of the hierarchical structures has shown [29] their evolution is reduced to anomalous diffusion process in ultrametric space that arrives at a steady-state distribution over hierarchical levels, which represents the Tsallis power law inherent in non-extensive systems [30]. A principle peculiarity of the Tsallis statistics is known to be governed by a deformed algebra [31]. Our work is devoted to consideration of creation conditions of great deal of variety of hierarchical trees on the basis of methods developed initially at study of quantum groups [32].

The outline of the paper is as follows. In Section 2, we state a connection between probabilities to find hierarchical levels with given set of effective energies and the probability to associate these levels into united structure. We argue a consistent probabilistic picture requires making use of the deformed algebra, whose main rules are stated in Appendix A. Further consideration is based on study of main types of hierarchical trees depicted in Fig. 1. Sections 3 and 4 are devoted to analytical definition of the creation probabilities of both regular and degenerate trees, while in Section 5 we find these for both Fibonacci and free-scale trees numerically. The case of an arbitrary tree is considered in Section 6 and Section 7 is devoted to discussion of obtained results.
2. Defining creation probability of hierarchical structure

Let us consider a hierarchical structure comprising of $n > 1$ levels $l = 0, 1, \ldots, n$ characterized by energy barrier heights $\epsilon_l$ and total height $\epsilon_n$ connected with the natural additivity assumption

$$\epsilon_n := \sum_{l=0}^{n} \epsilon_l.$$  (1)
The principle peculiarity of hierarchical ensembles is known to be presented with the Tsallis’ thermostatistics [30] where the \( l \)th hierarchical level is related to the probability

\[ p_l = \exp_q \left( -\frac{\epsilon_l}{\Delta} \right) \]  

characterized by the deformed exponential (A.1) with dispersion \( \Delta \) and height of energy barrier \( \epsilon_l \). Self-consistent probabilistic picture of hierarchical ensembles is reached if one proposes that, in contrast to the additivity rule (1), the normalization condition

\[ p_0 \oplus_q p_1 \oplus_q \ldots \oplus_q p_n = 1 \]  

is deformed to fix the top level probability \( p_0 \) according to the summation rule (A.3).

Along this line, one should set the probability \( P_n \) related to the \( n \)-level hierarchical structure determines the total height of energy barrier \( \epsilon_n = -\Delta \ln_q(P_n) \) through the deformed logarithm (A.1). Then, the condition (1) arrives at the additivity of these logarithms:

\[ \ln_q(P_n) = \sum_{l=0}^{n} \ln_q(p_l). \]  

In accordance with the first rule (A.5), this equation means the probability relation

\[ P_n := p_0 \oplus_q p_1 \oplus_q p_2 \oplus_q \ldots \oplus_q p_n. \]  

Thus, in contrast to ordinary statistical systems, the creation probability \( P_n \) of a hierarchical structure equals to the deformed production of specific probabilities \( p_l \) related to levels \( l = 0, 1, \ldots, n \). As the production definition (A.2) shows, growth of the deformation parameter \( q > 1 \) increases essentially the probability (5) in comparison with the usual value at \( q = 1 \). From physical point of view, above deformation of the factorization rule for independent probabilities recovers the additivity condition (1) for corresponding heights of the energy barriers within the Tsallis’ thermostatistics.

With accounting (A.1), Eq. (4) arrives at the explicit form of the creation probability of a hierarchical structure:

\[ P_n = \exp_q \left[ \sum_{l=0}^{n} p_l^{1-q} - (n + 1) \right] = \left( \sum_{l=0}^{n} p_l^{1-q} - n \right)^{\frac{1}{1-q}}. \]  

Here, the last expression follows directly from the deformed production [5] with accounting the rule (A.2). The relations (6) mean the decrease of the creation probability with growing hierarchical tree in accordance with the difference equation

\[ P_{n-1}^{1-q} - P_n^{1-q} = 1 - p_n^{1-q}. \]  

In non-deformed limit \( q \to 1 \), relations (5) and (6) are reduced to the ordinary rule \( P_n = \prod_{l=0}^{n} p_l \) (respectively, Eq. (7) reads \( P_n/P_{n-1} = p_n \)), while at \( q = 2 \) the creation probability (6) takes a maximal value.

A principle peculiarity of the above scheme is that level energies \( \epsilon_l \) remain to be additive values because creation of hierarchical structure does not break the law of
energy conservation. However, the hierarchy deforms essentially the probability relations (3), (5), (6) and (7) due to appearance of coupling between level probabilities $p_l$.

According to Eq. (6) the consequent step in definition of the creation probability $P_n$ of a hierarchical structure comes to determination of set of probabilities $\{p_l\}_0^n$ related to different hierarchical levels. Let us consider first the simplest case of the regular tree depicted in Fig. 1(a).

3. Regular tree

Let us consider a regular tree whose nodes multifurcate on certain level $l$ with constant branching index $b > 1$ to generate a set of the $N_l = b^l$ nodes determined with inherent probabilities $\pi = p_0/N_l = p_0 b^{-l}$ where $p_0$ is their top magnitude being normalization constant. Within naive proposition, one could permit additivity of the node probabilities to arrive at the total probability of the $l$ level realization to be $p_l := N_l \pi = p_0$. Thus, within the condition of additivity of the node probabilities, related values $p_l = p_0 = (n + 1)^{-1}$ for all levels appear to be non-dependent of their numbers $l = 0, 1, \ldots, n$.

To escape such trivial situation we propose to replace above additive connection of the level probability $p_l$ with the node value $\pi$ by the following deformed equalities:

$$
\frac{p_l}{p_0} := \frac{\pi}{p_0} \oplus q \frac{\pi}{p_0} \oplus q \frac{\pi}{p_0} \oplus q \cdots \oplus q \frac{\pi}{p_0} \equiv N_l \odot_q \frac{\pi}{p_0} = b^l \odot_q b^{-1}.
$$

(8)

Presenting here the deformed sum of $N_l$ identical terms with help of the formula (A.4), one obtains the level distribution required in the binomial form

$$
p_l = p_0 \left[1 + \frac{(1 - q) b^{-l}}{1 - q} - 1\right].
$$

(9)

In the limit $q \to 1$, it is simplified into expression

$$
p_l \simeq p_0 \left[1 + \frac{1 - q}{2} (1 - b^{-l})\right]
$$

(10)

that shows exponentially fast variation with growth of the level number $l \geq 1$. In the cases $l \gg 1$ or $b \gg 1$, the probability (9) reaches the limit value

$$
p_\infty = \frac{e^{1-q} - 1}{1 - q} p_0 = p_0 \ln_q e
$$

(11)

being $p_\infty > p_0$ at deformation $q < 1$ and $p_\infty < p_0$ at deformation $q > 1$. Inserting Eq. (11) into Eq. (6) leads to the following expression for the creation probability of a regular tree:

$$
P_n = \left[p_0^{-q} \left(1 + n (\ln_q e)^{1-q}\right) - n\right]^{1-q}.
$$

(12)

Respectively, the deformed normalization condition related to the limit $b \gg 1$ takes the form

$$
p_0 \oplus_q [n \odot_q (p_0 \ln_q e)]
= \left[\frac{1 + p_0 (1 - q)}{1 - q}\right] \frac{[1 + p_0 (1 - q) \ln_q e]^n - 1}{1 - q} = 1.
$$

(13)
In accordance with above consideration, Fig. 2(a) shows the probability distribution over hierarchical levels of the regular tree as function of the level number at: (a) $b = 2$, $n = 10$ and $q = 10^{-4}, 0.5, 1.0, 1.5, 1.9, 1.99, 1.999$ (curves 1-7, respectively); (b) $b = 2$, $q = 1.5$ and $n = 1, 2, \ldots, 10$ (curves top-down, respectively); (c) $q = 1.9999$, $n = 5$ and $b = 2, 4, 100$ (curves 1-3, respectively).

**Figure 2.** Probability distribution over hierarchical levels of the regular tree as function of the level number at: (a) $b = 2$, $n = 10$ and $q = 10^{-4}, 0.5, 1.0, 1.5, 1.9, 1.99, 1.999$ (curves 1-7, respectively); (b) $b = 2$, $q = 1.5$ and $n = 1, 2, \ldots, 10$ (curves top-down, respectively); (c) $q = 1.9999$, $n = 5$ and $b = 2, 4, 100$ (curves 1-3, respectively).
increases with growing number \( l \) of hierarchical level at \( q < 1 \) and decays at \( q > 1 \). From physical point of view, the creation probability of a deeper hierarchical level should be less than this for upper levels, so that one ought to conclude that the case \( q > 1 \) is meaningful only.

In this case, with growing total number of hierarchical levels \( n \), the probability distribution (9) normalized with the condition (3) decays as it is shown in Fig. 2(b). Characteristically, the form of this distribution depends very slightly on both deformation parameter \( q \) and branching index \( b \) excluding the domain \( 2 - q \ll 1 \). According to Fig. 2(c), within this domain, the probability distribution over hierarchical levels decays not so sharply at small values of the branching index \( b \). With large growing the parameter \( b \gg 1 \), the dependence \( p_l \) decreases more sharply to reach exponentially fast the minimum value (11) that is independent of the branching index \( b \).

As numerical calculations show, the creation probability (6) takes meaningful values \( P_0 \leq 1 \) for deformation parameters \( q > 1 \) only. According to Fig. 3(a) the dependence of this probability on the whole number of tree levels has monotonically slowing down form whose decaying rate decreases considerably only near the limit value \( q = 2 \). On
the other hand, Fig. 3(b) shows that variation of the branching index \( b \gg 1 \) affects appreciably the dependence of the creation probability only for moderate numbers of tree levels within the domain \( 2 - q \ll 1 \).

Above data indicate distinctive feature in behavior of the regular hierarchical tree near the limit value \( q = 2 \) where the dependence (9) has not any singularity. This feature is corroborated with the dependence of the top level probability on the deformation parameter depicted in Fig. 4. It is seen, regardless of both total number of levels \( n \) and branching index \( b \), this probability increases monotonically with the \( q \)-growth to reach sharply the limit value \( p_0 = 1 \) in the point \( q = 2 \). Obviously, this means anomalous increasing probabilities \( p_l \) for the whole set of hierarchical levels (type of shown in Fig. 2(a) with the curve 7). Though, within the domain \( 2 - q \ll 1 \), the ordinary normalization condition \( \sum_{l=0}^{n} p_l = 1 \) is violated appreciably, the definition (A.3) shows the deformed normalization condition (3) can be recovered at large parameter \( q \). However, with overcoming the border \( q = 2 \) this condition is not satisfied at all. As a result, we arrive at the conclusion that physically meaning values of the deformation parameter are concentrated within the domain \( q \in [1, 2] \).

\[ \text{Figure 4. Top level probability of the regular tree as function of the deformation parameter at: (a) } b = 2 \text{ and } n = 2, 4, 10 \text{ (curves 1-3, respectively); (b) } n = 5 \text{ and } b = 2, 10^3 \text{ (curves 1,2, respectively).} \]
4. Degenerate tree

As shown in Fig. 4.1, the difference between regular and degenerate trees is that all nodes multifurcate on each level in the former case, while the only one node branches in the latter. In this sense, the degenerate tree can be considered as an antipode of the regular one to be studied analytically.

According to Fig. 4.1(b), on the \( l = 1 \) level, branching process with index \( b > 1 \) creates \( N_1 = b \) nodes with equal probabilities \( b^{-1} \). Next, on the \( l = 2 \) level, \( b - 1 \) nodes out of \( N_2 = 2(b - 1) + 1 \) ones have the same probabilities \( b^{-2} \), while the \( b \) rest nodes relate to the smaller value \( b^{-2} \). On the \( l = 3 \) level, out of \( N_3 = 3(b - 1) + 1 \) nodes one has \( b \) nodes with probabilities \( b^{-3} \), \( b - 1 \) with \( b^{-2} \) and \( b - 1 \) with \( b^{-1} \). Hence, on the \( l \) level \( N_l = l(b - 1) + 1 \) nodes are partitioned into \( l \) groups, among which \( l - 1 \) ones contain \( b - 1 \) nodes with probabilities \( b^{-1} \), \( b^{-2} \), \( \ldots \), \( b^{-(l-1)} \), while the last group has \( b \) nodes with equal probabilities \( b^{-l} \). With accounting such a partitioning, the creation probability of the \( l \)th hierarchical level is expressed with the following relations:

\[
\frac{p_l}{p_0} = \left[ (b - 1) \odot_q b^{-1} \right] \oplus_q \left[ (b - 1) \odot_q b^{-(l-1)} \right] \oplus_q \left( b \odot_q b^{-l} \right) \nonumber \]

\[
= \left[ (b - 1) \odot_q b^{-1} \right] \oplus_q \left[ (b - 1) \odot_q b^{-(l-1)} \right] \oplus_q b^{-l} \nonumber \]

\[
:= \left[ (b - 1) \odot_q (S_{l+1} \ominus_q 1) \right] \odot_q b^{-l}. \tag{14} \]

Here, in the last equation the sum of the deformed geometrical series

\[
S_l := 1 \oplus_q b^{-1} \oplus_q b^{-2} \oplus_q \ldots \oplus_q b^{-(l-1)} \tag{15} \]

is introduced. As shows related consideration in Appendix B, this sum is expressed by the power series

\[
S_l = \sum_{k=0}^{l-1} C_k^{l+1} (b)(1 - q)k b^{-\frac{k(k+1)}{2}} \tag{16} \]

with the deformed binomial coefficients \[32\]

\[
C_k^l (b) \equiv \prod_{m=0}^{k-1} \frac{1 - b^{-(l-m)}}{1 - b^{-(m+1)}}. \tag{17} \]

Inserting Eq. (16) into the last relation (14), one obtains the final expression for the \( l \)th level creation probability

\[
p_l = \frac{\left[ 1 + (1 - q)b^{-l} \right] \left[ 1 + (1 - q)\Sigma_l^{k-1} - 1 \right]}{1 - q} \tag{18} \]

where one denotes

\[
\Sigma_l \equiv S_{l+1} \ominus_q 1 = \frac{1}{2 - q} \sum_{k=1}^{l} C_k^{l+1} (b)(1 - q)k b^{-\frac{k(k+1)}{2}}. \tag{19} \]
Creation probabilities of hierarchical trees

Within production representation

\[ S_t = \frac{1}{1-q} \left\{ \prod_{m=0}^{t-1} \left[ 1 + (1-q)b^{-m} \right] - 1 \right\}, \] (20)

one has

\[ \Sigma_t = \frac{1}{1-q} \left\{ \prod_{m=0}^{t-1} \left[ 1 + (1-q)b^{-m} \right] - \frac{1}{2-q} \right\}. \] (21)

Then, the probability \( p_t \) takes the explicit form

\[ p_t = \frac{1+(1-q)b^{-l}}{(2-q)^{b-l}} \prod_{m=0}^{t-1} \left[ 1 + (1-q)b^{-m} \right]^{b-1} - 1 \] (22)

In spite of apparent differences between the formulas (9) and (22), direct calculations show actually coincident forms of the probability distributions over hierarchical levels for both regular and degenerate trees. Therefore, we postpone numerical study of the creation probability for the degenerate tree before the following section where consideration of the free-scale tree allows to compare all the results obtained analytically.

5. Free-scale tree

Above, we have considered two conceptual examples of hierarchical trees with self-similar structure – regular and degenerate trees depicted in Fig. 1. In this section, we shall study a free-scale tree whose structure is rather random, but the probability distribution over hierarchical levels tends to the power-law form inherent in self-similar statistical systems [33].

In this case, the probability distribution over tree levels is determined by the discrete difference equation [29]

\[ p_{l+1} - p_l = \frac{-p_l^q}{\Delta}, \quad l = 0, 1, \ldots, n \] (23)

accompanied with the deformed normalization condition [3] (\( \Delta \) being a distribution dispersion). It is easily to show that in continual limit \( l \to \infty \) the equation (23) arrives at the power-law dependence [13]

\[ p_l = \left( p_0^{1-q} + \frac{q-1}{\Delta} l \right)^{-\frac{1}{q-1}} \] (24)

where the top level probability is \( p_0 = \left( \frac{2}{\Delta} \right)^{\frac{1}{2-q}} \) for trees with total number of levels \( n \gg 1 \).

In figures [5] we compare the probability distributions over hierarchical levels of free-scale, regular and degenerate trees at different values of the deformation parameter. It is seen at all \( q \)-values the form of these distributions is actually equal for regular and degenerate trees, but differs appreciably for free-scale tree, where the level probability falls down much more strong, than for both other trees. In accordance with such
a behavior, the creation probabilities depicted in Figs. 6 decays faster for the free-scale tree, than in the case of the regular and degenerate ones. Characteristically, this difference appears only within the domain $2 - q \ll 1$ of the deformation parameter variation.

As shown in the end of the section 3, such a behavior is stipulated by the singular
dependence of the top level probability $p_0$ on the deformation parameter near the point $q = 2$. According to Fig. 7 this singularity is inherent in all considered hierarchical trees.

6. Arbitrary tree

Now, we are in position to consider an arbitrary hierarchical tree, over whose levels $l = 0, 1, \ldots, n$, $n \geq 1$ are distributed $N_l$ nodes $i_0i_1 \ldots i_l$ with the probabilities $p_{i_0i_1 \ldots i_l} \^\dagger$.

The main peculiarity of hierarchical trees is known to be a clustered structure, whose fragment is depicted in Fig. 8: nodes $i_0 \ldots i_l-1i_l$ of the $l$-level form a cluster $i_0 \ldots i_l-1$ on the $(l - 1)$-level; in turn, clusters $i_0 \ldots i_l-2i_l-1$ form supercluster $i_0 \ldots i_l-2$ on the following level $l - 2$, et cetera. Above clustering process spreads over upper levels $l - 3$,

\^\dagger In accordance with Ref.[27], a node coordinate of a hierarchical tree represents so names $p$-adic number $i_0i_1 \ldots i_n$ where the first digit $i_0 = 1$ relates to the major ancestor on the uppermost level $l = 0$, the second $i_1$ numbers its sons on the lower level $l = 1$, and so on – up to the last digit $i_n$ numbering the lowest descendants on the bottom level $l = n$.
Figure 7. Top level probabilities for the free-scale, regular, Fibonacci and degenerate trees (curves 1-4, respectively) as function of the deformation parameter at $\Delta = 2$, $b = 2$ and $n = 5$.

Figure 8. Node parametrization within a hierarchical cluster.

$l - 4, \ldots$ up to the pair of the top levels $l = 1$ and $l = 0$ where $N_{i_0}$ nodes $i_1$ form the superior node $i_0$. Along this way, the node probabilities on hierarchical levels ranged bottom-up are as follows: $p_{i_0 \ldots i_{n-1} i_n}$, $p_{i_0 \ldots i_{n-1}}$, $p_{i_0 \ldots i_1}$, $\ldots$, $p_{i_0}$; $p_{i_0} \equiv p_0$. Let us calculate these probabilities considering hierarchical levels top-down.

On the uppermost level $l = 0$, one has a single node $i_0 = 1$ related to the probability $p_0 \equiv p_0$. With passage down to the level $l = 1$, this node multifurcates into a cluster comprising of $N_{i_0}$ nodes $i_1$. Because of the identity of this nodes, they are characterized by the equal probabilities

$$p_{i_0 i_1} = p_0 N_{i_0}^{-1}.$$  \hspace{1cm} (25)

In similar manner, on the following level $l = 2$ one obtains the node probabilities

$$p_{i_0 i_1 i_2} = p_{i_0 i_1} N_{i_0}^{-1} = p_0 (N_{i_0} N_{i_0 i_1})^{-1}.$$  \hspace{1cm} (26)

Iteration of this procedure down to an arbitrary level $l$ yields the required result

$$p_{i_0 \ldots i_l} = p_0 \left( \prod_{m=0}^{l-1} N_{i_0 \ldots i_m} \right)^{-1}$$  \hspace{1cm} (27)
where \( N_{i_{0}...i_{m}} \) is the node number within the cluster \( i_{0}...i_{m} \).

Generalization of the first equality (8) arrives at the expression of the creation probability of an arbitrary level \( l \) through a set of related node probabilities. This expression is reduced to the following \( l \)-fold deformed sum:

\[
\frac{p_l}{p_0} = \frac{N_{i_0}}{\biguplus_{i_1=1}^{i_{i_0}}} \ldots \frac{N_{i_0...i_{l-1}}}{\biguplus_{i_l=1}^{i_{i_{l-1}}}} \frac{p_{i_0...i_l}}{p_0}, \quad l \neq 0.
\] (28)

Respectively, the normalization condition (3) takes the form

\[
p_0 \bigoplus_q p_0 = \biguplus_{i_1=1}^{N_{i_0}} \ldots \biguplus_{i_n=1}^{N_{i_0...i_{n-1}}} \frac{p_{i_0...i_n}}{p_0} = 1.
\] (29)

Above, we have used the notation of the deformed sum of \( n \) terms:

\[
\biguplus_{i=1}^{n} a_i \equiv a_1 \bigoplus_q a_2 \bigoplus_q \ldots \bigoplus_q a_n.
\] (30)

It is worth noting the characteristic peculiarity of above consideration: the node probabilities (27) are determined with making use of non-deformed algebra, while the definition (28) of the level probabilities \( p_l \) is based on the use of deformed summation (30). A ground of such a partitioning is that the former of these probabilities relates to the configuration of hierarchical trees, while the latter describes their statistical properties.

In conclusion, we consider two examples of applying above theory, among which the former concerns the Fibonacci tree (Fig. 1(b)), while the latter relates to the schematic evolution tree shown in Fig. 9 (in the last case, nodes identify substantial stages in evolution of life, e.g., human is situated on the 24th level). Using the formulas (27) and (28) for the node and level probabilities, obeying the normalization condition (29), we show that probability distributions of the Fibonacci tree depicted in Figs. 5-7 does not differ actually from related dependencies for both regular and degenerate trees. What about the evolution tree, its probability distributions (Fig. 10) show that a presence of the stopped branches (type of two rightmost ones in Fig. 9) considerably decreases
creation probability of new hierarchical level. Particularly, the probability of human appearance takes values more than $10^{-4}$ only at the deformation parameter $q = 1.9999$.

7. Concluding remarks

To escape ambiguities we are worthwhile to stress that our consideration concerns rather the probabilistic picture of creation of hierarchical trees themselves, than hierarchical phenomena and processes evolving on these trees (for example, hierarchically constrained statistical ensembles [13], diffusion processes on multifurcating trees [28], et cetera). Among others we have studied analytically both regular and degenerate trees to confirm the coincidence of both analytical and numerical results following from the developed scheme being applicable to an arbitrary tree.

A principle peculiarity of the probabilistic picture elaborated is a partitioning deformed and non-deformed values. So, effective energies of hierarchical levels in Eq. (1) are non-deformed values because creation of hierarchical structure does not break the conservation law of the energy being additive value. Moreover, the node probabilities are determined with making use of non-deformed relation (27) because these probabilities relate to the configuration of hierarchical tree itself (in other words, they are determined by geometrical, but not probabilistic reasons). At the same time, the hierarchy appearance deforms essentially the probability relations (3), (29), (5), (6) and (7) due to coupling level probabilities $p_l$. Similarly, the definition (28) of these probabilities through corresponding node values is based on the use of deformed summation (30).

Making use of deformed algebra shows increase of probabilities $p_l$ for the whole set of hierarchical levels to take anomalous character near the point $q = 2$. The deformed normalization condition (3) is fulfilled only at $q \leq 2$, while it is broken with overcoming the border $q = 2$. As a result, physically meaning values of the deformation parameter
belong to the domain $q \in [1, 2]$.

Comparison of the probability distributions over hierarchical levels of free-scale, regular, Fibonacci and degenerate trees shows (Fig. 5) the form of these distributions differs appreciably at all $q$-values only for free-scale tree where the level probability falls down much more strong. In accordance with such a behavior, the creation probabilities depicted in Fig. 6 decays faster for the free-scale tree, than for the rest ones. Characteristically, this difference appears within the condition $2 - q \ll 1$ only.

Expression (27) – (29) and (6) are a basis for numerical studies of arbitrary hierarchical structures, for example complex defect structures of solids subject to intensive external influence type of rigid radiation treatment. Unlike the amorphous systems, the number of structure levels of a real crystal is rather not large: usually, among different spatial scales, it is accepted to distinguish micro-, meso- and macroscopic levels [35]. To study a real structure, one needs first to distribute the whole ensemble of defects over hierarchical levels $l = 0, 1, \ldots, n$; then, one calculates on each of them a number of defects $N_{i_0i_1\ldots i_{l-1}}$ belonging to the cluster $i_0i_1\ldots i_{l-1}$, $i_0i_1\ldots i_{l-2}\ldots i_0i_1\ldots i_{l-1}N_{i_1i_2\ldots i_{l-1}}$ and attributes the probability $p_{i_0\ldots i_l}$ to this cluster in accordance with Eq. (27). Next, the level probabilities $p_l$ are calculated according to definition (28) where the top value $p_0$ is fixed by the normalization condition (29). Finally, the creation probabilities $P_n$ of hierarchical trees are determined by the equality (6).

Appendix A. Main rules of deformed algebra

Following [31], let us present the main equations of the deformed algebra. Related formalism is known to be based on the generalized definition of the logarithm and exponential functions

\[
\ln_q(x) := \frac{x^{1-q} - 1}{1 - q},
\]

\[
\exp_q(x) := [1 + (1 - q)x]^{\frac{1}{1-q}}
\]

being characterized by a deformation parameter $q \geq 0$ with the notion $[y]_+ \equiv \max(0, y)$.

For some numbers $x, y > 0$, deformed product and ratio are defined with the following relations:

\[
x \otimes_q y := \left[x^{1-q} + y^{-q} - 1\right]^{\frac{1}{1-q}}_+,
\]

\[
x \oslash_q y := \left[x^{1-q} - y^{-q} + 1\right]^{\frac{1}{1-q}}_+.
\]

Respectively, deformed sum and difference read

\[
x \oplus_q y := x + y + (1 - q)xy,
\]

\[
x \ominus_q y := \frac{x - y}{1 + (1 - q)y}
\]
where the condition \( y \neq -\frac{1}{1-q} \) is implied. The \( n \)-fold deformed sum of identical terms is defined as follows:

\[
 n \odot_q x \equiv x \oplus_q x \oplus_q \ldots \oplus_q x := \frac{\left(1 + (1 - q)x\right)^n - 1}{1 - q}.
\] (A.4)

The rules (A.2), (A.3) ensure the following properties of the \( q \)-logarithm and the \( q \)-exponential (A.1):

\[
\ln_q (x \otimes_q y) = \ln_q x + \ln_q y,
\]

\[
\ln_q (x \oslash_q y) = \ln_q x - \ln_q y;
\]

\[
\exp_q (x) \otimes_q \exp_q (y) = \exp_q (x + y),
\]

\[
\exp_q (x) \oslash_q \exp_q (y) = \exp_q (x - y).
\] (A.5)

**Appendix B. Deformed sum of terms of a geometrical progression**

Let a geometrical sequence \( a, ar, ar^2, \ldots, ar^{n-1} \) is determined by the common ratio \( r \), the scale factor \( a \) and the term number \( n \). Within deformed summation rule (A.3), direct calculations at lower numbers \( n = 2, 3, \ldots \) show the sum of terms of a geometrical progression

\[
S_n := a \oplus_q ar \oplus_q ar^2 \oplus_q \ldots \oplus_q ar^{n-1}
\] (B.1)

can be written as the series

\[
S_n := a \sum_{m=0}^{n-1} \sigma^m_n [(1 - q)a]^m
\] (B.2)

with unknown coefficients \( \sigma^m_n \). Iteration of Eq. (B.1) yields the chain of the following relations:

\[
S_{n+1} := S_n \oplus_q (ar^n) = (S_n + ar^n) + (1 - q)S_n(ar^n)
\]

\[
= \left\{ a \left( \sigma^0_n + r^n \right) + a \sum_{m=1}^{n-1} \sigma^m_n [(1 - q)a]^m \right\}
\]

\[
+ (1 - q)a(ar^n) \sum_{m=0}^{n-1} \sigma^m_n [(1 - q)a]^m
\]

\[
= a\sigma^0_{n+1} + a \sum_{l=0}^{n-2} \sigma^{l+1}_n [(1 - q)a]^{l+1}
\]

\[
+ a[(1 - q)a]r^n \sum_{m=0}^{n-1} \sigma^m_n [(1 - q)a]^m
\]

\[
= a\sigma^0_{n+1} + a \sum_{l=0}^{n-2} \left( \sigma^{l+1}_n + r^n \sigma^n_n \right) [(1 - q)a]^{l+1}
\]

\[
+ a[(1 - q)a]r^n \sigma^{n-1}_n [(1 - q)a]^{n-1}
\]

\[
= a\sigma^0_{n+1} + a \sum_{m=1}^{n-1} \left( \sigma^m_n + r^n \sigma^{m-1}_n \right) [(1 - q)a]^m
\]

\[
+ a\sigma^{n-1}_n [(1 - q)a]^n r^n.
\] (B.3)
Here, the in the first line takes into account the definition \[ \text{(A.3)} \]; in the second line, the series \[ \text{(B.2)} \] is applied to single out the term related to \( m = 0 \) within the braces; in the fourth line, the first term is written in accordance with the definition \[ \text{(B.2)} \] related to the term \( m = 0 \), while the summation index \( l = m - 1 \) is introduced in the second term; in the sixth line, the second term contains both sums over \( l \) and \( m \) of the previous line, the last term relates to the index \( m = n - 1 \); in the eighth line, we return to the summation index \( m = l + 1 \). As a result, the series \[ \text{(B.2)} \] takes the form

\[
S_n = a s_n + a \sum_{m=1}^{n-2} \left( \sigma_{n-1}^m + r^{n-1} \sigma_{n-1}^{m-1} \right) [(1 - q)a]^m \\
+ a \sigma_{n-1}^{-2}(1 - q)a]^{n-1}r^{-1}
\]  

(B.4)

where the sum

\[
s_n \equiv \sum_{m=0}^{n-1} r^m = \frac{1 - r^n}{1 - r}
\]  

(B.5)

of the ordinary geometrical progression \( 1, r, r^2, \ldots, r^{n-1} \) was used. Comparison of the terms of Eqs. \[ \text{(B.2)} \] and \[ \text{(B.4)} \] related to the equal \( m \) indexes arrives at the following iteration relations:

\[
\sigma_n^0 = s_n;
\]  

(B.6)

\[
\sigma_n^m = \sigma_{n-1}^m + \sigma_{n-1}^{m-1}r^{n-1}, \quad m \in [1, n - 2];
\]  

(B.7)

\[
\sigma_n^{n-1} = \sigma_{n-1}^{n-2}r^{n-1}.
\]  

(B.8)

The first of these terms gives the explicit expression of the lowest power coefficient in the series \[ \text{(B.2)} \]. It is easily to convince the regression \[ \text{(B.7)} \] is satisfied with the insertion

\[
\sigma_n^m = \sum_{l=0}^{n-1} \sigma_l^{m-1}r^l
\]  

(B.9)

whose iteration yields

\[
\sigma_n^m = \sum_{l=0}^{n-1} \sigma_l^{m-1}r^l = \sum_{l=0}^{n-1} \sum_{k=0}^{l-1} \sigma_k^{m-k}l^k = \ldots
\]

\[
= \sum_{l_{m-1}=0}^{n-1} \sum_{l_{m-2}=0}^{l_{m-1}-1} \sum_{l_0=0}^{l_{m-2}-1} \sigma_0^0 r^{l_0}.
\]  

(B.10)

However, the last expression is inconvenient for direct calculations because it contains connected exponents of the ratio \( r \) with the upper limits of the consequent sums. Hence, let us calculate explicitly the coefficients \[ \text{(B.9)} \] for small indexes \( m \):

\[
\sigma_n^1 = \sum_{l=0}^{n-1} \sigma_0^0 r^l = \sum_{l=0}^{n-1} s_l r^l = \sum_{l=0}^{n-1} \frac{1 - r^n}{1 - r} r^l
\]

\[
= r \frac{(1 - r^n)(1 - r^{n-1})}{(1 - r)(1 - r^2)};
\]

\[
\sigma_n^2 = \sum_{l=0}^{n-1} \sigma_l^1 r^l = r \sum_{l=0}^{n-1} \frac{(1 - r^l)(1 - r^{l-1})}{(1 - r)(1 - r^2)} r^l
\]
Creation probabilities of hierarchical trees

\[ m \]

with coefficients of a geometrical progression (B.1):

\[ \frac{m}{m} \text{ insertion into Eq. (B.2) arrives at the final expression of the sum of the terms } \]

\[ m \]

Thus, one can conclude the proposition (B.12) is applicable for all indexes \( m \in [0, n - 1] \) and its insertion into Eq. (B.2) arrives at the final expression of the sum of the terms of a geometrical progression (B.1):

\[ a \sum_{m=0}^{n-1} C_n^{m+1}(r)r^{m(m+1)/2}[(1-q)a]^m \]

with coefficients

\[ C_n^m(r) \equiv \prod_{l=0}^{m-1} \frac{1 - r^{n-l}}{1 - r^{l+1}}. \]
The expression (B.15) can be written within the production representation according to the relations

\[ S_n = \frac{1}{1-q} \left\{ \sum_{m=0}^{n} C_m^n (r) r^{\frac{m(m-1)}{2}} [(1-q)a]^m - 1 \right\} \]

\[ = \frac{1}{1-q} \left\{ \prod_{m=0}^{n-1} \left[ 1 + a(1-q)r^m \right] - 1 \right\}, \]  

(B.17)

the second of which expresses the deformed Gauss polynomials 32, 36

\[ \sum_{m=0}^{n} C_m^n (r) r^{\frac{m(m-1)}{2}} [(1-q)a]^m \]

\[ = \prod_{m=0}^{n-1} \left[ 1 + a(1-q)r^m \right]. \]  

(B.18)

With accounting the definition (A.4), one obtains at \( r = 1 \)

\[ S_n = \frac{[1 + (1-q)a]^n - 1}{1-q} = n \odot_q a. \ ]  

(B.19)

In non-deformed limit \( q \to 1 \), this relation takes the trivial form \( S_n = na. \ )

Rewriting the definition (B.16) in the forms

\[ C_m^n (r) = \frac{n!}{\prod_{l=1}^{m} (1-r^l)} \]

\[ = \frac{n!}{\prod_{l=1}^{m} (1-r^l) \prod_{l=1}^{n-m} (1-r^l)}, \]  

(B.20)

one can see that coefficients (B.16) are reduced to the deformed binomial coefficients 32, 36

\[ C_m^n (r) \equiv \frac{[n]_r!}{[m]_r![n-m]_r!}, \]  

(B.21)

determined with the deformed factorial

\[ [n]_r! \equiv [1]_r[2]_r \ldots [n]_r \]

\[ = \frac{(r-1)(r^2-1)\ldots(r^n-1)}{(r-1)^n}, \]

\[ [n]_r \equiv 1 + r + r^2 + \ldots + r^{n-1} = \frac{r^n - 1}{r - 1} \]  

(B.22)

According to the formula (B.21), the deformed binomial coefficients obey the usual property

\[ C_m^n (r) = C_n^{n-m}(r). \]  

(B.23)

After replacing index \( m + 1 \) by \( m \) in Eq. (B.7) with accounting Eq. (B.12), we arrive at the deformed Pascal identity

\[ C_m^n (r) = C_{n-1}^m (r) + C_{n-1}^{m-1}(r)r^{n-m} \]  

(B.24)
Creation probabilities of hierarchical trees that forms the deformed Pascal triangle

\[
\begin{array}{ccccccc}
1 & & & & & & \\
1 & 1 & C_2^1(r) & 1 & & & \\
1 & C_3^1(r) & C_3^1(r) & 1 & & & \\
1 & C_4^1(r) & C_4^2(r) & C_4^1(r) & 1 & & \\
1 & C_5^1(r) & C_5^2(r) & C_5^2(r) & C_5^1(r) & 1 & \\
& & & & & & \ldots
\end{array}
\]

where we put \(C_0^0(r) = 1\). On the other hand, iteration of the relation (B.7) yields the sum rule

\[
\begin{align*}
C_m^n(r) &= C_m^{n-1}(r) + \left[ C_{n-2}^{m-1}(r) + C_{n-2}^{m-2}(r) r^{n-m} \right] r^{n-m} \\
&= \ldots = \sum_{l=0}^{m} C_{n-(l+1)}^{m-l}(r) r^{(n-m)l}.
\end{align*}
\]

Finally, in limit \(r \to 1\) the relation (B.21) takes ordinary form:

\[
\begin{align*}
\lim_{r \to 1} C_m^n(r) &= \lim_{r \to 1} \prod_{l=0}^{m} \frac{1 - r^{n-l}}{1 - r^{l+1}} \\
&= \prod_{l=0}^{m-1} \frac{n - l}{l + 1} = \frac{n!}{m!(n-m)!}.
\end{align*}
\]
Creation probabilities of hierarchical trees

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