UNIVOQUE BASES OF REAL NUMBERS: SIMPLY NORMAL BASES, IRREGULAR BASES AND MULTIPLE RATIONALS

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Abstract. Given a positive integer $M$ and a real number $x \in (0,1]$, we call $q \in (1, M+1]$ a univoque simply normal base of $x$ if there exists a unique simply normal sequence $(d_i) \in \{0,1,\ldots,M\}^\mathbb{N}$ such that $x = \sum_{i=1}^{\infty} d_i q^{-i}$. Similarly, a base $q \in (1, M+1]$ is called a univoque irregular base of $x$ if there exists a unique sequence $(d_i) \in \{0,1,\ldots,M\}^\mathbb{N}$ such that $x = \sum_{i=1}^{\infty} d_i q^{-i}$ and the sequence $(d_i)$ has no digit frequency. Let $U_{SN}(x)$ and $U_{Ir}(x)$ be the sets of univoque simply normal bases and univoque irregular bases of $x$, respectively.

In this paper we show that for any $x \in (0,1]$ both $U_{SN}(x)$ and $U_{Ir}(x)$ have full Hausdorff dimension. Furthermore, given finitely many rationals $x_1, x_2, \ldots, x_n \in (0,1]$ so that each $x_i$ has a finite expansion in base $M+1$, we show that there exists a full Hausdorff dimensional set of $q \in (1, M+1]$ such that each $x_i$ has a unique expansion in base $q$.

1. Introduction

Non-integer base expansions were pioneered by Rényi [35] and Parry [34]. It was extensively studied after the surprising discovery by Erdös et al. [16, 17] that for any $k \in \mathbb{N} \cup \{\aleph_0\}$ there exist $q \in (1,2]$ and $x \in [0,1/(q-1)]$ such that $x$ has precisely $k$ different $q$-expansions. This phenomenon is completely different from the integer base expansions that each $x$ has a unique expansion, except for countably many $x$ having two expansions. In the literature of non-integer base expansions there is a great interest in unique expansions due to its close connection with open dynamical systems [15, 20] and kneading theory of unimodal maps or Lorentz maps [8, 19, 31].

Given a positive integer $M$ and $q \in (1, M+1]$, a point $x \in [0,M/(q-1)]$ is called a univoque point in base $q$ if there exists a unique sequence $(d_i) \in \{0,1,\ldots,M\}^\mathbb{N}$ such that

$$x = \sum_{i=1}^{\infty} \frac{d_i}{q^i} =: ((d_i))_q.$$ (1.1)

The infinite sequence $(d_i)$ is called the (unique) expansion of $x$ in base $q$. Throughout the paper we will fix the alphabet $\{0,1,\ldots,M\}$. Let $\mathcal{U}_q$ denote the set of all univoque points in base $q$. De Vries and Komornik [11, 13] studied the topology of $\mathcal{U}_q$, and showed that $\mathcal{U}_q$ is closed if and only if $q$ does not belong to the topological closure of

$$\mathcal{U} := \{p \in (1, M+1] : 1 \in \mathcal{U}_p\}.$$
Each base \( p \in \mathcal{U} \) is called a univoque base of 1. It is known that \( \mathcal{U} \) is a Lebesgue null set of full Hausdorff dimension \([10, 11]\). Furthermore, it has a smallest element \( \min \mathcal{U} = q_{KL} \), called the Komornik-Loreti constant, which was shown to be transcendental \([3, 26]\). Its topological closure \( \overline{\mathcal{U}} \) is a Cantor set \([13, 25]\). Some local dimension properties of \( \mathcal{U} \) was studied in \([2]\). The set \( \mathcal{U} \) is also related to the bifurcation set of \( \alpha \)-continued fractions, kneading sequences of unimodal maps, and also the real slice of the boundary of the Mandelbrot set \([7]\).

Motivated by the study of \( \mathcal{U} \), Lü, Tan and Wu \([29]\) initiated the study of univoque bases of real numbers. Given \( x \geq 0 \), let
\[
\mathcal{U}(x) := \{ q \in (1, M + 1] : x \in U_q \}.
\]
Then \( \mathcal{U} = \mathcal{U}(1) \). Clearly, for \( x = 0 \) we have \( \mathcal{U}(0) = (1, M + 1] \), since 0 always has a unique expansion \( 0^\infty = 00... \) in any base \( q \in (1, M + 1] \). When \( x \in (0, 1] \) and \( M = 1 \), Lü, Tan and Wu showed in \([29]\) that \( \mathcal{U}(x) \) is a Lebesgue null set but has full Hausdorff dimension. Dajani et al. \([9]\) showed that for \( x \in (0, 1] \) the algebraic difference \( \mathcal{U}(x) - \mathcal{U}(x) \) contains an interval. The third author and his coauthors \([27]\) studied the local dimension of \( \mathcal{U}(x) \), and showed that the Hausdorff dimension of \( \mathcal{U}(x) \) is strictly smaller than one for \( x > 1 \). Recently, Allaart and the third author \([2]\) described the smallest element of \( \mathcal{U}(x) \) for all \( x > 0 \) under the condition \( M = 1 \).

In this paper we focus on \( x \in (0, 1] \), and study those bases of \( \mathcal{U}(x) \) satisfying some statistical properties. Given \( x \in (0, 1] \), a base \( q \in (1, M + 1] \) is called a univoque simply normal base of \( x \) if \( q \in \mathcal{U}(x) \) and the unique expansion \( (d_i) \in \{0, 1, \ldots, M\}^\infty \) of \( x \) in base \( q \) is simply normal. Let \( \mathcal{U}_{SN}(x) \) be the set of all univoque simply normal bases of \( x \). Then for any \( q \in \mathcal{U}_{SN}(x) \) the unique \( q \)-expansion \( (d_i) \) of \( x \) has the same digit frequency, i.e.,
\[
\text{freq}_q((d_i)) := \lim_{n \to \infty} \frac{\# \{1 \leq i \leq n : d_i = b \}}{n} = \frac{1}{M + 1} \quad \forall \ b \in \{0, 1, \ldots, M\}.
\]
Here \# \( A \) denotes the cardinality of a set \( A \).

On the other hand, a base \( q \in (1, M + 1] \) is called a univoque irregular base of \( x \) if \( q \in \mathcal{U}(x) \) and the unique expansion \( (d_i) \) of \( x \) in base \( q \) has no digit frequency. Let \( \mathcal{U}_{\ell}(x) \) be the set of all univoque irregular bases of \( x \). Then for any \( q \in \mathcal{U}_{\ell}(x) \) the unique expansion \( (d_i) \) of \( x \) in base \( q \) satisfies
\[
\liminf_{n \to \infty} \frac{\# \{1 \leq i \leq n : d_i = b \}}{n} < \limsup_{n \to \infty} \frac{\# \{1 \leq i \leq n : d_i = b \}}{n} \quad \forall \ b \in \{0, 1, \ldots, M\}.
\]
Note that a sequence \( (d_i) \) satisfying (1.2) is called essential non-normal in \([1]\).

It is clear that \( \mathcal{U}_{\ell}(x) \) and \( \mathcal{U}_{SN}(x) \) are disjoint subsets of \( \mathcal{U}(x) \). Our first result states that both \( \mathcal{U}_{SN}(x) \) and \( \mathcal{U}_{\ell}(x) \) have full Hausdorff dimension for \( 0 < x \leq 1 \).

**Theorem 1.1.** For any \( x \in (0, 1] \) we have \( \dim_H \mathcal{U}_{SN}(x) = \dim_H \mathcal{U}_{\ell}(x) = 1 \).

Note that \( \mathcal{U}(x) \) consists of all \( q \in (1, M + 1] \) such that a given \( x \) has a unique \( q \)-expansion. Then it is natural to ask if we give finitely many points \( x_1, x_2, \ldots, x_\ell > 0 \), can we find a base
q ∈ (1, M + 1] such that each x_i has a unique expansion in base q? In general this q may not exist, since for example, if x_1 > 2 then the only base such that x_1 has a unique expansion is q = 1 + M/x_1 (see [27] Theorem 1.5). On the other hand, when x_1, x_2, . . . , x_ℓ ∈ (0, 1] are all rationals with a finite expansion in base M + 1, we show that there exists a full Hausdorff dimensional set of q ∈ (1, M + 1] such that each x_i has a unique q-expansion.

More precisely, let

\[ D_M := \left\{ \sum_{i=1}^{n} \frac{d_i}{(M+1)^i} : d_i \in \{0, 1, \ldots, M\} \quad \forall 1 \leq i \leq n; \ n \in \mathbb{N} \right\}. \]

Then D_M is a dense subset of (0, 1]. Given finitely many points x_1, x_2, . . . , x_ℓ ∈ D_M, by constructing Cantor subsets of each \( U(x_i) \) and exploring their thicknesses we show that the intersection \( \cap_{i=1}^{\ell} U(x_i) \) has full Hausdorff dimension.

**Theorem 1.2.** For any x_1, . . . , x_ℓ ∈ D_M we have

\[ \dim_H \bigcap_{i=1}^{\ell} U(x_i) = 1. \]

The rest of the paper is organized as follows. In Section 2 we recall some basic properties of univoque bases of real numbers. The proof of Theorem 1.1 will be presented in Section 3. To prove Theorem 1.2 we construct in Section 4 a sequence of Cantor subsets of U(x), and therefore, by using the thickness we show in Theorem 1.1 that the algebraic sum U(x) + λU(x) contains an interval for any x ∈ (0, 1] and λ ≠ 0. The proof of Theorem 1.2 is in Section 5.

In the final section we give some remarks on Theorems 1.1 and 1.2.

### 2. Univoque bases of real numbers

The study of univoque bases relies on symbolic dynamics (cf. [28]). Given M ≥ 1, let \{0, 1, . . . , M\}^\mathbb{N} be the set of all infinite sequences \( (d_i) = d_1d_2 \ldots \) with each digit d_i from the alphabet \{0, 1, . . . , M\}. By a word we mean a finite string of digits over \{0, 1, . . . , M\}. Denote by \{0, 1, . . . , M\}^\ast the set of all finite words including the empty word \( \epsilon \). For two words c = c_1 . . . c_m and d = d_1 . . . d_n we write cd = c_1 . . . c_md_1 . . . d_n for their concatenation. In particular, for any k ∈ \mathbb{N} we denote by \( c^k \) the k-fold concatenation of c with itself, and by c^\infty the periodic sequence which is obtained by the infinite concatenation of c with itself.

Throughout the paper we will use lexicographical order \( <, \prec, \succ \) or \( \succ \) between sequences in \{0, 1, . . . , M\}^\mathbb{N}. For example, we say \( (i_n) \succ (j_n) \) if \( i_1 > j_1 \), or there exists \( n \in \mathbb{N} \) such that \( i_1 . . . i_n = j_1 . . . j_n \) and \( i_{n+1} > j_{n+1} \). We write \( (i_n) \succeq (j_n) \) if \( (i_n) \succ (j_n) \) or \( (i_n) = (j_n) \). Similarly, we say \( (i_n) \prec (j_n) \) if \( (j_n) \succ (i_n) \), and say \( (i_n) \preceq (j_n) \) if \( (j_n) \succeq (i_n) \). Equipped with the metric \( \rho \) defined by

\[ \rho((i_n), (j_n)) = (M + 1)^{-\inf\{n \geq 1 : i_n \neq j_n\}} \]

(2.1)
the symbolic space \( \{0,1,\ldots,M\}^\mathbb{N} \) becomes a compact metric space. One can verify that the induced topology by the metric \( \rho \) coincides with the order topology on \( \{0,1,\ldots,M\}^\mathbb{N} \).

Given \( x \in (0,1] \) and \( q \in (1, M + 1] \), let

\[
\Phi_x(q) = a_1(x,q)a_2(x,q)\ldots \in \{0,1,\ldots,M\}^\mathbb{N}
\]

be the lexicographically largest \( q \)-expansion of \( x \) not ending with \( 0^\infty \), called the quasi-greedy \( q \)-expansion of \( x \). In particular, for \( x = 1 \) we reserve the notation \( \alpha(q) = (\alpha_i(q)) \) for the quasi-greedy \( q \)-expansion of \( 1 \). The following property for the quasi-greedy expansion \( \Phi_x(q) \) was proven in [12, Lemma 2.3 and Lemma 2.5].

**Lemma 2.1.** Let \( x \in (0,1] \). The map \( \Phi_x : (1,M + 1] \to \{0,1,\ldots,M\}^\mathbb{N} \); \( q \mapsto \Phi_x(q) \) is left continuous under the metric \( \rho \), and is strictly increasing with respect to the lexicographical order. In particular, for \( x = 1 \) the map \( q \mapsto \alpha(q) \) is bijective from \((1,M + 1]\) to the set

\[
\left\{ (a_i) \in \{0,1,\ldots,M\}^\mathbb{N} : 0^\infty \prec a_{n+1}a_{n+2} \ldots \preceq a_1a_2 \ldots \quad \forall \ n \geq 0 \right\}.
\]

Given \( x \in (0,1] \), recall that \( \mathcal{U}(x) \) consists of all univoque bases \( q \in (1,M + 1] \) of \( x \). When \( M = 1 \), the infimum of \( \mathcal{U}(x) \) was characterized in [2]. For a general \( M \geq 1 \) we still have the following lower bound.

**Lemma 2.2.** For any \( x \in (0,1] \) we have

\[
\inf \mathcal{U}(x) \geq q_G(M) \geq \frac{M}{2} + 1,
\]

where

\[
q_G(M) := \begin{cases} 
\frac{k + 1}{2 + \sqrt{1+6k+8}} & \text{if } M = 2k, \\
\frac{k + 1}{2 - \sqrt{1+6k+8}} & \text{if } M = 2k + 1.
\end{cases}
\]

**Proof.** Note by [6] that for \( q \in (1,q_G(M)) \) we have \( \mathcal{U}_q = \{0,M/(q-1)\} \). Observe that \( q_G(M) \in \left[ \frac{M}{2} + 1,M \right) \). So, if \( x \in (0,1] \cap \mathcal{U}_q \), then we must have \( q > q_G(M) \). This implies \( \inf \mathcal{U}(x) \geq q_G(M) \) for any \( x \in (0,1] \). \( \square \)

For \( x \in (0,1] \) let

\[
\mathcal{U}(x) := \Phi_x(\mathcal{U}(x)) = \{ \Phi_x(q) : q \in \mathcal{U}(x) \}.
\]

Then \( \Phi_x \) is a bijective map from \( \mathcal{U}(x) \) to \( \mathcal{U}(x) \). Furthermore, the following property of \( \Phi_x \) on \( \mathcal{U}(x) \) was shown in [27, Proposition 3.1 and Proposition 3.3].

**Lemma 2.3.** Let \( x \in (0,1] \). Then the map \( \Phi_x : \mathcal{U}(x) \to \mathcal{U}(x) \) is locally bi-Hölder continuous under the metric \( \rho \) in \( \{0,1,\ldots,M\}^\mathbb{N} \). Furthermore, for any \( 1 < a < b < M + 1 \) we have

\[
\frac{\dim_H \Phi_x(\mathcal{U}(x) \cap (a,b))}{\log b} \leq \dim_H(\mathcal{U}(x) \cap (a,b)) \leq \frac{\dim_H \Phi_x(\mathcal{U}(x) \cap (a,b))}{\log a}.
\]
Here and throughout the paper we keep using base $M + 1$ logarithms. In view of Lemma 2.3 to study the fractal properties of $\mathcal{U}(x)$ and its subsets $\mathcal{U}_{SN}(x), \mathcal{U}_I(x)$ it suffices to study their symbolic analogues

$$U(x) = \Phi_x(\mathcal{U}(x)), \quad U_{SN}(x) = \Phi_x(\mathcal{U}_{SN}(x)) \quad \text{and} \quad U_I(x) = \Phi_x(\mathcal{U}_I(x)).$$

The following result was essentially obtained in [29, Section 4] (see also, [27, Lemma 4.2]).

**Lemma 2.4.** Given $x \in (0, 1]$, let $(\varepsilon_i) = \Phi_x(M + 1)$ be the quasi-greedy expansion of $x$ in base $M + 1$. Then there exist a word $w$, a non-negative integer $N$ and a strictly increasing sequence $\{N_j\}_{j=1}^{\infty} \subset \mathbb{N}$ such that

$$U_{N_j}(x) \subset U(x) \quad \text{for all} \quad j \geq 1,$$

where

$$U_{N_j}(x) := \{\varepsilon_1 \ldots \varepsilon_{N+N_j} w d_1 d_2 \ldots : d_{n+1} \ldots \varepsilon_{N_j} \notin \{0^{N_j}, M^{N_j}\} \quad \forall \ n \geq 0 \}.$$

In particular, if $x \in D_M$, that is $(\varepsilon_i) = \Phi_x(M + 1) = \varepsilon_1 \ldots \varepsilon_m M^\infty$ for some $m \geq 1$, then we can choose $w = \varepsilon, N = m$ and $N_j = m + j$.

### 3. Univoque simply normal bases and univoque irregular bases

Let $x \in (0, 1]$. Recall that $\mathcal{U}_{SN}(x)$ consists of all $q \in \mathcal{U}(x)$ such that $x$ has a unique $q$-expansion which is simply normal. Furthermore, we recall from (1.2) that $\mathcal{U}_I(x)$ consists of all $q \in \mathcal{U}(x)$ such that $x$ has a unique $q$-expansion with no digit frequency. Clearly, $\mathcal{U}_{SN}(x)$ and $\mathcal{U}_I(x)$ are disjoint subsets of $\mathcal{U}(x)$. Note that $\mathcal{U}(x)$ is a Lebesgue null set of full Hausdorff dimension. In this section we will prove Theorem 1.1 that $\dim H \mathcal{U}_{SN}(x) = \dim H \mathcal{U}_I(x) = \dim H \mathcal{U}(x) = 1$ for all $x \in (0, 1]$.

#### 3.1. Univoque simply normal bases

First we consider the univoque simply normal bases.

**Proposition 3.1.** For any $x \in (0, 1]$ we have $\dim_H \mathcal{U}_{SN}(x) = 1$.

Our strategy to prove Proposition 3.1 is to construct a sequence of subsets $\{\mathcal{U}_{SN,j}(x)\}_{j=1}^{\infty}$ in $\mathcal{U}_{SN}(x)$ such that $\dim_H \mathcal{U}_{SN,j}(x) \to 1$ as $j \to \infty$. In view of Lemma 2.3 we can do this construction in the symbolic space. For $j \geq 1$ let $U_{N_j}(x)$ be the subset of $U(x)$ defined as in Lemma 2.4. Without loss of generality we assume $N_1 > 6M$, since otherwise we can delete the first few terms from the sequence $\{N_j\}_{j=1}^{\infty}$. In the following we construct for each $j \geq 1$ a subset $U_{SN,j}(x)$ of $U_{N_j}(x) \cap U_{SN}(x)$.

Take $j \geq 1$. Then $N_j \geq N_1 > 6M$. For $k \geq 0$ let

$$m_k = 2^k (M + 1) \lfloor \frac{N_j}{3} \rfloor,$$
where \(|r|\) denotes the integer part of a real number \(r\). Now for \(k \geq 0\) let \(\mathcal{N}_k\) be the set of all vectors \(\vec{n}_k := (n_{k,0}, n_{k,1}, \ldots, n_{k,M})\) satisfying

\[
\sum_{b=0}^{M} n_{k,b} = m_k, \quad \text{and} \quad n_{k,b} \in \left\{ \frac{m_k}{M+1}, \frac{m_k}{M+1} - 1 \right\} \quad \forall \ 0 \leq b < M.
\]

It is easy to see that \(#\mathcal{N}_k = 2^M\) for all \(k \geq 0\). Furthermore, for any \(\vec{n}_k \in \mathcal{N}_k\) we have

\[
(3.2) \quad \left|\frac{n_{k,b}}{m_k} - \frac{1}{M+1}\right| \leq \frac{M}{m_k} \quad \forall \ b \in \{0, 1, \ldots, M\}.
\]

So \(\frac{\vec{n}_k}{m_k}\) is a \((M + 1)\)-dimension probability vector with each element approximately the same. Note that \(m_k\) and \(\mathcal{N}_k\) both depend on \(j\). In the following we define the sets \(D_{j,k}, k \geq 0\) recursively, which will be used to construct our set \(U_{SN,j}(x)\).

First we define \(D_{j,0}\). For a vector \(\vec{n}_0 = (n_{0,0}, n_{0,1}, \ldots, n_{0,M}) \in \mathcal{N}_0\) let

\[
D(\vec{n}_0) := \{d_1 \ldots d_{m_0} : \xi_b(d_1 \ldots d_{m_0}) = n_{0,b} \ \forall \ b \in \{0, 1, \ldots, M\}\},
\]

where \(\xi_b(\vec{c})\) denotes the number of digit \(b\) in the word \(\vec{c}\). Then \(D(\vec{n}_0)\) consists of all words of length \(m_0\) in which each digit \(b\) occurs precisely \(n_{0,b}\) times. The set \(D_{j,0}\) is defined by

\[
D_{j,0} := \bigcup_{\vec{n}_0 \in \mathcal{N}_0} D(\vec{n}_0).
\]

Next suppose \(D_{j,k-1}\) has been defined for some \(k \geq 1\). We define \(D_{j,k}\) recursively. Note by (3.1) that \(m_k = 2m_{k-1}\). For \(\vec{n}_k = (n_{k,0}, n_{k,1}, \ldots, n_{k,M}) \in \mathcal{N}_k\) let

\[
D(\vec{n}_k) := \{d_1 \ldots d_{m_k} \in D_{j,k-1} \times D_{j,k-1} : \xi_b(d_1 \ldots d_{m_k}) = n_{k,b} \ \forall \ b \in \{0, 1, \ldots, M\}\},
\]

and set

\[
D_{j,k} := \bigcup_{\vec{n}_k \in \mathcal{N}_k} D(\vec{n}_k).
\]

Since \(\mathcal{N}_k\) consists of \(2^M\) vectors, one can verify that

\[
(3.3) \quad \#D_{j,k} \geq \sum_{\vec{n}_k \in \mathcal{N}_k} \#D(\vec{n}_k) \geq 2^M \min_{\vec{n}_k \in \mathcal{N}_k} \#D(\vec{n}_k) \geq 2^M \left( \min_{\vec{n}_0 \in \mathcal{N}_0} \#D(\vec{n}_0) \right)^{2^k} = 2^M \left( \min_{\vec{n}_0 \in \mathcal{N}_0} \left( \frac{m_0}{\vec{n}_0} \right) \right)^{2^k},
\]

where \(\left( \frac{m_0}{\vec{n}_0} \right) = \left( \frac{m_0}{n_{0,0}, n_{0,1}, \ldots, n_{0,M}} \right)\) is a multinomial coefficient, and the second inequality holds because each block in \(D(\vec{n}_k)\) belongs to \((D_{j,0})^{2^k}\).

Given \(x \in (0, 1]\), let \((\varepsilon_i) = \Phi_x(M + 1)\). Based on the sets \(D_{j,k}, k \geq 0\) we define

\[
U_{SN,j}(x) := \{\varepsilon_1 \ldots \varepsilon_{N+N_j} \mathbf{w} \mathbf{d}_0 \mathbf{d}_1 \ldots : \mathbf{d}_k \in D_{j,k} \ \forall \ k \geq 0\},
\]

where \(N, N_j\) and \(\mathbf{w}\) are defined as in Lemma (2.4).
Lemma 3.2. Let $x \in (0, 1]$. Then for any $j \geq 1$ we have $U_{SN,j}(x) \subset U_{N_j}(x) \cap U_{SN}(x)$.

Proof. Note by our construction that each word $d_k \in D_{j,k}$ has length $m_k = 2^k m_0$ and $d_k \in (D_{j,0})^{2^k}$. Furthermore, observe that for each $d_0 \in D_{j,0}$ the lengths of consecutive zeros and consecutive Ms in $d_0$ are both bounded by $\lceil N_j/3 \rceil + M$, which is strictly smaller than the length of $d_0$. So, the lengths of consecutive zeros and consecutive Ms in each $d_k \in D_{j,k}$ should be bounded by

$$2 \left( \lceil N_j/3 \rceil + M \right) \leq \frac{2N_j}{3} + 2M < N_j,$$

where the last inequality holds since $N_j > 6M$. Therefore, by Lemma 2.4 it follows that $U_{SN,j}(x) \subset U_{N_j}(x)$.

To complete the proof we only need to show that each sequence in $U_{SN,j}(x)$ has equal digit frequency. Note that the digit frequency of a sequence is determined by its tail sequences. So, by using (3.2) and that $m_k \to \infty$ as $k \to \infty$ one can verify that $U_{SN,j}(x) \subset U_{SN}(x)$. □

Proof of Proposition 3.1. Let $\gamma_j = \max \gamma(N_j)$. Note by Lemma 2.4 that $\Phi_x(\gamma_j) \not\prec \Phi_x(M + 1)$ as $j \to \infty$. Then by Lemmas 2.1 and 2.3 it gives that $\gamma_j \not\prec M + 1$ as $j \to \infty$. By Lemma 2.3, Lemma 3.2 and [18] Theorem 2.1 it follows that

$$\dim_H U_{SN}(x) \geq \frac{\dim_H U_{SN}(x)}{\log \gamma_j} \geq \frac{\dim_H U_{SN,j}(x)}{\log \gamma_j},$$

$$= \liminf_{n \to \infty} \frac{\log \prod_{k=0}^n \#D_{j,k}}{\sum_{k=0}^n m_k \log \gamma_j} = \liminf_{n \to \infty} \frac{\sum_{k=0}^n \log \#D_{j,k}}{m_0 \sum_{k=0}^n 2^k \log \gamma_j} \geq \frac{(n + 1) \log 2^M + \sum_{k=0}^n 2^k \log \left( \frac{m_0}{\bar{n}_0^*} \right)}{m_0 \sum_{k=0}^n 2^k \log \gamma_j} = \frac{\log \left( \frac{m_0}{\bar{n}_0^*} \right)}{m_0 \log \gamma_j},$$

where the last inequality follows by (3.3) and $\left( \frac{m_0}{\bar{n}_0} \right) := \min_{i \in \delta} \left( \frac{m_0}{\bar{n}_0^*} \right)$. Note that $\left( \frac{m_0}{\bar{n}_0^*} \right) = \mathop{\text{min}}_{(n_0,0), (n_0,1), \ldots, (n_0,M)}$ and $m_0 = m_0(j) = (M + 1) \left\lfloor \frac{N_j}{3} \right\rfloor \to \infty$ as $j \to \infty$. By using $\sum_{b=0}^M n_{0,b}^* = m_0$ and the Stirling’s formula that $\log n! = n \log n - n + O(\log n)$ as $n \to \infty$, it follows that

$$\frac{\log \left( \frac{m_0}{\bar{n}_0^*} \right)}{m_0 \log \gamma_j} = \frac{\log(m_0!) - \sum_{b=0}^M \log(n_{0,b}^*)}{m_0 \log \gamma_j},$$

$$= \frac{m_0 \log m_0 - \sum_{b=0}^M n_{0,b}^* \log n_{0,b}^* + O(\log m_0)}{m_0 \log \gamma_j} = \frac{1}{\log \gamma_j} \left( - \sum_{b=0}^M \frac{n_{0,b}^*}{m_0} \log \frac{n_{0,b}^*}{m_0} + O \left( \frac{\log m_0}{m_0} \right) \right).$$
Observe by (3.2) that for any \( b \in \{0, 1, \ldots, M\} \),
\[
\left| \frac{n^{*}_{0,b}}{m_{0}} - \frac{1}{M+1} \right| \leq \frac{M}{m_{0}} = \frac{M}{(M+1)\left\lfloor \frac{N}{M} \right\rfloor} \to 0 \quad \text{as} \quad j \to \infty.
\]
Furthermore, \( \gamma_{j} \to M + 1 \) as \( j \to \infty \). So by (3.3) and (3.5) we conclude that
\[
\dim_{H} \mathcal{U}_{SN}(x) \geq \frac{1}{\log \gamma_{j}} \left( -\sum_{b=0}^{M} \frac{n^{*}_{0,b}}{m_{0}} \log \frac{n^{*}_{0,b}}{m_{0}} + O\left( \frac{\log m_{0}}{m_{0}} \right) \right) \to 1 \quad \text{as} \quad j \to \infty.
\]
This completes the proof. \( \square \)

### 3.2. Univoque irregular bases

Now we consider the univoque irregular bases.

**Proposition 3.3.** For any \( x \in (0,1] \) we have \( \dim_{H} \mathcal{U}_{I_{r}}(x) = 1 \).

To prove Proposition 3.3 we will construct a sequence of subsets of \( \mathcal{U}_{I_{r}}(x) \) whose Hausdorff dimension can be arbitrarily close to one. In view of Lemma 2.3 it suffices to construct subsets in \( U_{I_{r}}(x) = \Phi_{x}(\mathcal{U}_{I_{r}}(x)) \). Recall from Lemma 2.4 that for any \( j \geq 1 \),
\[
U_{N_{j}}(x) = \{ \varepsilon_{1} \ldots \varepsilon_{N+M_{j}}w_{d_{1}}d_{2} \ldots : d_{n+1} \ldots d_{n+M_{j}} \notin \{0^{N_{j}}, M^{N_{j}}\} \quad \forall \ n \geq 0 \}
\]
is a subset of \( U(x) \).

First we assume \( M \geq 2 \). For \( k \geq 0 \) let \( \Delta_{j,k} \) be the set of all length \( 2^{k}(M+1)N_{j}(N_{j}+1) \) words of the form
\[
(3.6) \quad c_{1} \ldots c_{2^{k}(M+1)N_{j}^{2}}(0^{N_{j}-1})^{2^{k}}(1^{N_{j}})^{2^{k}} \ldots ((M-1)^{N_{j}})^{2^{k}}(M^{N_{j}-1}(M-1))^{2^{k}},
\]
where \( c_{i} \in \{0,1,\ldots, M\} \) for all \( i \), and \( c_{i} \notin \{0,M\} \) if \( i = N_{j}n \) for some \( n \in \mathbb{N} \). Then each block in \( \Delta_{j,k} \) has neither \( N_{j} \) consecutive zeros nor \( N_{j} \) consecutive ones. Furthermore,
\[
(3.7) \quad \# \Delta_{j,k} = (M+1)^{2^{k}(M+1)N_{j}(N_{j}-1)}(M-1)^{2^{k}(M+1)N_{j}} \quad \forall \ k \geq 0.
\]
This is because each digit \( c_{i} \) has \( M-1 \) choices if the index \( i \) is a multiple of \( N_{j} \), and otherwise \( c_{i} \) has \( M \) choices.

Now we define the subset \( U_{I_{r},j}(x) \) of \( U_{N_{j}}(x) \) by
\[
(3.8) \quad U_{I_{r},j}(x) := \{ \varepsilon_{1} \ldots \varepsilon_{N+M_{j}}w_{b_{0}}b_{1}b_{2} \ldots : b_{k} \in \Delta_{j,k} \quad \forall \ k \geq 0 \},
\]
where each \( \Delta_{j,k} \) is defined in (3.6). Since \( M \geq 2 \), each block \( b_{k} \) ends with \( M-1 \notin \{0,M\} \). Thus, each sequence \( b_{0}b_{1} \ldots \) contains neither \( N_{j} \) consecutive zeros nor \( N_{j} \) consecutive ones. So, \( U_{I_{r},j}(x) \) is indeed a subset of \( U_{N_{j}}(x) \).

**Lemma 3.4.** Let \( x \in (0,1] \) and \( M \geq 2 \). Then for any \( j \geq 1 \) we have \( U_{I_{r},j}(x) \subset U_{I_{r}}(x) \).

**Proof.** Note by Lemma 2.4 that \( U_{I_{r},j}(x) \subset U_{N_{j}}(x) \subset U(x) \). So it suffices to prove that any sequence in \( U_{I_{r},j}(x) \) does not have a digit frequency. Taking a sequence \( \varepsilon_{1} \ldots \varepsilon_{N+M_{j}}w_{b_{0}}b_{1} \ldots \in \)
Suppose on the contrary that \( \lim_{n \to \infty} \xi_b(n) = 0 \). For \( n \in \mathbb{N} \) let
\[
\xi_b(n) := \xi_b(d_1 \ldots d_n) = \# \{ 1 \leq i \leq n : d_i = b \}.
\]
We will show that the limit of the sequence \( \{ \xi_b(n) / n \} \) does not exist.

Observe by (3.6) and (3.8) that each block \( b_k \) has length \( 2^k(M+1)N_j(N_j+1) \) and can be written as
\[
b_k = c_k (0^{N_j-1}1)^{2^k} (1^{N_j})^{2^k} \cdots ((M-1)^{N_j})^{2^k} (M^{N_j-1}(M-1))^{2^k},
\]
where
\[
c_k = c_1c_2 \cdots c_{2^k(M+1)N_j^2} \quad \text{with } c_i \notin \{0,M\} \text{ if } i = N_jn \text{ for some } n \in \mathbb{N}.
\]
Let \( (\ell_k) \) and \( (n_k) \) be two subsequences of \( \mathbb{N} \) such that \( \ell_k = \ell_k(b) \) and \( n_k = n_k(b) \) are the lengths of blocks \( b_0 \ldots b_{k-1}c_k(0^{N_j-1})^{2^k} (1^{N_j})^{2^k} \cdots ((b-1)^{N_j})^{2^k} \) and \( b_0 \ldots b_{k-1}c_k(0^{N_j-1})^{2^k} (1^{N_j})^{2^k} \cdots ((b-1)^{N_j})^{2^k} (b^{N_j})^{2^k} \), respectively. Then
\[
\ell_k = 2^k(M+1)N_j^2 + 2^kN_jb + \sum_{i=0}^{k-1} 2^i(M+1)N_j(N_j+1),
\]
\[
n_k = 2^k(M+1)N_j^2 + 2^kN_j(b+1) + \sum_{i=0}^{k-1} 2^i(M+1)N_j(N_j+1).
\]
Furthermore, let \( \theta_b(k) \) be the number of digit \( b \) appearing in the block \( c_0c_1 \ldots c_k \). By our definition of \( c_i \) we must have
\[
\theta_b(k) \leq \sum_{i=0}^{k} 2^i(M+1)N_j(N_j-1) = (2^{k+1} - 1)(M+1)N_j(N_j-1) \quad \text{if } b \in \{0,M\};
\]
\[
\theta_b(k) \leq \sum_{i=0}^{k} 2^i(M+1)N_j^2 = (2^{k+1} - 1)(M+1)N_j^2 \quad \text{if } b \in \{1,2,\ldots,M-1\}.
\]

In view of our construction of \( (d_i) = b_0b_1 \ldots \), we will finish our proof by considering the following three cases: (I) \( b \in \{0,M\} \); (II) \( b \in \{1,M-1\} \); (III) \( b \in \{2,3,\ldots,M-2\} \).

Case (I). \( b \in \{0,M\} \). Then by (3.9) and the definition of \( (d_i) = b_0b_1 \ldots \) it follows that
\[
\xi_b(\ell_k) = \theta_b(k) + \sum_{i=0}^{k-1} 2^i(N_j-1) = \theta_b(k) + (N_j-1)(2^k - 1).
\]
Suppose on the contrary that \( \lim_{n \to \infty} \xi_b(n)/n \) exists. Then by (3.9) and (3.11) the following limit
\[
\lim_{k \to \infty} \xi_b(\ell_k) = \lim_{k \to \infty} \frac{\theta_b(k) + (N_j-1)(2^k - 1)}{2^k(M+1)N_j^2 + 2^kN_jb + (2^k - 1)N_j(M+1)(N_j+1)}
\]
\[
= \lim_{k \to \infty} \frac{\theta_b(k) + N_j - 1}{(M+1)(2N_j+1) + b}
\]
exists, which implies that the limit
\[ \lim_{k \to \infty} \frac{\theta_b(k)}{2^k N_j} =: A_b \text{ exists for } b \in \{0, M\}. \]

Similarly,
\[ \xi_b(n_k) = \theta(b) + \sum_{i=0}^{k} 2^i (N_j - 1) = \theta_b(k) + (N_j - 1)(2^{k+1} - 1), \]
and then by (3.9) and (3.12) it follows that
\[ A_b + \frac{N_j - 1}{N_j} \frac{\xi_b(\ell_k)}{\ell_k} = \lim_{k \to \infty} \frac{\xi_b(n_k)}{n_k} = \frac{\theta_b(k) + (N_j - 1)(2^{k+1} - 1)}{(M + 1)(2N_j + 1) + b + 1}. \]

which implies that
\[ A_b = \frac{N_j - 1}{N_j} \left[ (M + 1)(2N_j + 1) + b - 1 \right] \geq \frac{N_j - 1}{N_j} \left[ (M + 1)(2N_j + 1) - 1 \right]. \]
This leads to a contradiction, since by (3.10) we have \( A_b \leq 2(M + 1)(N_j - 1) \) for \( b \in \{0, M\} \).

Case (II). \( b \in \{1, M - 1\} \). First we assume \( M \geq 3 \). Then by (3.9) it follows that
\[ \xi_1(\ell_k) = \theta_1(k) + \sum_{i=0}^{k} 2^i (N_j + 1) + 2^k, \quad \xi_{M-1}(\ell_k) = \theta_{M-1}(k) + \sum_{i=0}^{k-1} 2^i (N_j + 1); \]
and
\[ \xi_1(n_k) = \theta_1(k) + \sum_{i=0}^{k} 2^i (N_j + 1), \quad \xi_{M-1}(n_k) = \theta_{M-1}(k) + \sum_{i=0}^{k} 2^i (N_j + 1) - 2^k. \]
Suppose the limit \( \lim_{n \to \infty} \frac{\xi_b(n)}{n} \) exists for \( b \in \{1, M - 1\} \). Then the limit \( A_b := \lim_{k \to \infty} \frac{\theta_b(k)}{2^k N_j} \) also exists. By (3.9) and the same argument as in Case (I) it follows that
\[ A_1 + \frac{N_j+2}{N_j} \frac{\xi_1(\ell_k)}{\ell_k} = \lim_{k \to \infty} \frac{\xi_1(n_k)}{n_k} = \frac{A_1 + \frac{N_j+2}{N_j} + 1}{(M + 1)(2N_j + 1) + 2} \]
and
\[ A_{M-1} + \frac{N_j+1}{N_j} \frac{\xi_{M-1}(\ell_k)}{\ell_k} = \lim_{k \to \infty} \frac{\xi_{M-1}(n_k)}{n_k} = \frac{A_{M-1} + \frac{N_j+1}{N_j} + 1}{(M + 1)(2N_j + 1) + M}. \]
This leads to a contradiction with (3.10) that \( A_b \leq 2(M + 1)N_j \) for \( b \in \{1, M - 1\} \).

Next we consider \( M = 2 \). Then \( b = 1 \). By (3.9) it follows that
\[ \xi_1(\ell_k) = \theta_1(k) + \sum_{i=0}^{k-1} 2^i (N_j + 2) + 2^k, \quad \xi_1(n_k) = \theta_1(k) + \sum_{i=0}^{k} 2^i (N_j + 2) - 2^k. \]
By (3.9) and the same argument as above it follows that
\[
\frac{A_1 + \frac{N_j+3}{N_j}}{(M+1)(2N_j + 1) + 1} = \lim_{k \to \infty} \frac{\xi_1(\ell_k)}{\ell_k} = \lim_{k \to \infty} \frac{\xi_1(n_k)}{n_k} = \frac{A_1 + \frac{N_j+3}{N_j}}{(M+1)(2N_j + 1) + 2}.
\]
Again this leads to a contradiction with (3.10) that \( A_1 \leq 2(M+1)N_j \).

Case (III). \( b \in \{2,3,\ldots,M-2\} \). Then \( M \geq 4 \). By (3.9) we obtain that
\[
\xi_b(\ell_k) = \theta_b(k) + \sum_{i=0}^{k-1} 2^i N_j, \quad \xi_b(n_k) = \theta_b(k) + \sum_{i=0}^{k} 2^i N_j.
\]
Suppose on the contrary that the limit \( \lim_{n \to \infty} \frac{\xi_b(n)}{n} \) exists for \( b \in \{2,3,\ldots,M-2\} \). Then the limit \( A_b := \lim_{k \to \infty} \frac{\theta_b(k)}{2^k N_j} \) exists. By the same argument as in Case (I) and using (3.9) we obtain that
\[
\frac{A_b + 1}{(M+1)(2N_j + 1) + b} = \lim_{k \to \infty} \frac{\xi_b(\ell_k)}{\ell_k} = \lim_{k \to \infty} \frac{\xi_b(n_k)}{n_k} = \frac{A_b + 2}{(M+1)(2N_j + 1) + b + 1},
\]
which leads to a contradiction, since by (3.10) we have \( A_b \leq 2(M+1)N_j \) for \( b \in \{2,\ldots,M-2\} \).

Therefore, by Cases (I)–(III) we conclude that the frequency of digit \( b \) in \( (d_i) = b_0 b_1 b_2 \ldots \) does not exist for any \( b \in \{0,1,\ldots,M\} \). This completes the proof.

**Proof of Proposition 3.3** First we consider \( M \geq 2 \). Let \( x \in (0,1] \) and let \( (\varepsilon_i) = \Phi_x(M+1) \). Suppose \( U_{N_j}(x), j = 1,2,\ldots \) are the subsets of \( U(x) \) defined as in Lemma 2.4. For \( j \geq 1 \) let \( \gamma_j \) be the largest element of \( \mathcal{U}_{N_j}(x) := \Phi_x^{-1}(U_{N_j}(x)) \). Then \( \Phi_x(\gamma_j) = \varepsilon_1 \varepsilon_2 \varepsilon_3 \ldots \varepsilon_{N_j+1} w(M^{N_j-1}(M-1))^\infty \) for \( j \to \infty \). So by Lemmas 2.1 and 2.3 it follows that \( \gamma_j \nearrow M+1 \) as \( j \to \infty \).

Note by Lemma 3.4 that \( U_{I_r,j}(x) \subset U_{I_r}(x) \). So, by Lemma 2.3 it follows that
\[
\dim_H \mathcal{U}_{I_r, j}(x) \geq \dim_H \mathcal{U}_{I_r}(x) \geq \frac{\dim_H U_{I_r,j}(x)}{\log \gamma_j}
\]
for all \( j \geq 1 \). Note that \( \{0,1,\ldots,M\} \) is a compact metric space, where \( \rho \) is defined in (2.1). Then by (3.6), (3.7) and [18] Theorem 2.1 it follows that
\[
\dim_H U_{I_r,j}(x) = \liminf_{n \to \infty} \frac{\log \prod_{k=0}^{N_j} |b_k| \log(M+1)}{\sum_{k=0}^{N_j} 2^{k(N_j+1)}(N_j - 1) \log(M+1)}
\]
\[
= \liminf_{n \to \infty} \frac{\sum_{k=0}^{N_j} 2^{k(N_j+1)}(N_j - 1) \log(M+1)}{\sum_{k=0}^{N_j} 2^{k(N_j+1)}(N_j - 1) \log(M+1)}
\]
\[
= \frac{N_j - 1}{N_j + 1} \log(M+1) + \frac{\log(M-1)}{(N_j+1) \log(M+1) \log \gamma_j}.
\]
Since \( \gamma_j \to M+1 \) and \( N_j \to \infty \) as \( j \to \infty \), by (3.13) this implies that
\[
\dim_H \mathcal{U}_{I_r}(x) \geq \frac{N_j - 1}{(N_j + 1) \log \gamma_j} + \frac{\log(M-1)}{(N_j+1) \log(M+1) \log \gamma_j} \to 1 \quad \text{as} \quad j \to \infty.
\]
Here we emphasize that the logarithm is in base \( M+1 \).
Now we consider $M = 1$. The proof is similar. We modify the definition of $U_{I_i,j}(x)$ as

$$\tilde{U}_{I_i,j}(x) = \left\{ \varepsilon_1 \ldots \varepsilon_{N+j} \omega_0 b_1 \ldots : b_k \in \tilde{\Delta}_{j,k} \quad \forall \ k \geq 0 \right\},$$

where each $\tilde{\Delta}_{j,k}$ consists of all length $2^{k+1} N_j (N_j + 1) + 2$ blocks of the form

$$c_1 \ldots c_{2^{k+1} N_j^2} (0^{N_j - 1})^{2^k} (01^{N_j - 1})^{2^k} 01$$

with each $c_i \in \{0, 1\}$, and $c_{i-1} c_i = 01$ if $i = N_j n$ for some $n \in \mathbb{N}$. Then each sequence $\omega_0 b_1 \ldots \in \prod_{k=0}^{\infty} \tilde{\Delta}_{j,k}$ contains neither $N_j$ consecutive zeros nor $N_j$ consecutive ones. So, $\tilde{U}_{I_i,j}(x) \subset U_{N_j}(x)$. By the same argument as in the proof of Lemma 3.4 one can verify that $\tilde{U}_{I_i,j}(x) \subset U_{I_i}(x)$. Hence, by similar argument as above we can prove that $\dim_H U_{I_i}(x) = 1$ for $M = 1$.

\[ \square \]

**Proof of Theorem 4.1** The theorem follows by Propositions 3.1 and 3.3. \[ \square \]

4. Cantor subsets of $U(x)$ and thickness

In this section we will show that $U(x) + \lambda U(x)$ contains an interval for any $x \in (0, 1]$ and $\lambda \neq 0$, which generalizes the main result of [3] where they proved this result only for $M = 1$.

**Theorem 4.1.** If $f : \mathbb{R}^2 \to \mathbb{R}$ is $C^1$ such that the partial derivatives are not vanishing in $(M + 1 - \delta, M + 1)^2$ for some $\delta > 0$, then for any $x \in (0, 1]$ the set

$$U_f(x) := \{ f(p, q) : p, q \in U(x) \}$$

contains an interval.

**Remark 4.2.**

(i) If $f(x, y) = x + \lambda y$ for some $\lambda \neq 0$, then $f$ is $C^1$ with its partial derivatives not vanishing.

So, by Theorem 4.1 it follows that $U(x) + \lambda U(x)$ contains an interval for any $x \in (0, 1]$.

(ii) For possible extension of this theorem we refer to a recent paper [22] and the references therein.

4.1. Thickness of a Cantor set in $\mathbb{R}$.

The thickness of a Cantor set in $\mathbb{R}$ was introduced by Newhouse [32], and it has been applied in dynamical systems and number theory (cf. [5]). Let $E \subset \mathbb{R}$ be a Cantor set with the convex hull $E_0$. Then the complement $E_0 \setminus E = \bigcup_{i=1}^{\infty} O_i$ is the union of countably many disjoint open intervals. The sequence $\mathcal{O} = (O_1, O_2, O_3, \ldots)$ is called a derived sequence of $E$. If the lengths of these open intervals are in a non-increasing order, i.e., $|O_i| \geq |O_{i+1}|$ for all $i \geq 1$, then we call the sequence $\mathcal{O}$ an ordered sequence.

Let $E_n := E_0 \setminus \bigcup_{k=1}^{n} O_k$. Then for any $n \geq 1$ the open interval $O_n$ must belong to a unique connected component $C$ of $E_0 \setminus E$. In this case, $C \setminus O_n$ is the union of two disjoint closed intervals $L_\mathcal{O}(O_n)$ and $R_\mathcal{O}(O_n)$. Hence, the thickness of $E$ with respect to the derived sequence $\mathcal{O}$ is defined by

$$\tau_\mathcal{O}(E) := \inf_{n \geq 1} \min \left\{ \frac{|L_\mathcal{O}(O_n)|}{|O_n|}, \frac{|R_\mathcal{O}(O_n)|}{|O_n|} \right\};$$
and the thickness of $E$ is then defined by

$$
\tau(E) := \sup \{ \tau_{\vartheta}(E) : \vartheta(\theta) \text{ is a permutation of } \theta \}.
$$

Note by [5] that the supremum in (4.1) is attainable. Indeed, for any ordered sequence $\theta$ we have $\tau(E) = \tau_\theta(E)$.

The following result for the relationship between the thickness of a Cantor set and its Hausdorff dimension was given by Newhouse [32] (see also, [33]).

**Lemma 4.3.** Let $E \subset \mathbb{R}$ be a Cantor set. Then

$$
\dim_H E \geq \frac{\log 2}{\log (2 + 1/\tau(E))}.
$$

From Lemma 4.3 it follows that if the thickness of a Cantor set $E$ is very large, then its Hausdorff dimension is close to 1. The next result, which can be derived from [21], describes how the thickness can be used to study the intersection of two Cantor sets.

**Lemma 4.4.** Let $E$ and $F$ be two Cantor sets in $\mathbb{R}$ having the same maximum point $\xi$. If $\xi$ is an accumulation point of $E \cap F$, and their thicknesses $\tau(E) \geq t$ and $\tau(F) \geq t$ for some large $t > 0$, then there exists a Cantor subset $K \subset E \cap F$ such that $\max K = \xi$ and $\tau(K) \geq C \sqrt{t}$ for some $C > 0$.

The following result on the image of two Cantor sets $E$ and $F$ can be deduced from [30] and [36] (see also, [22]).

**Lemma 4.5.** Let $E$ and $F$ be two Cantor sets in $\mathbb{R}$ with $\tau(E) \tau(F) > 1$. If $f : \mathbb{R}^2 \to \mathbb{R}$ is a $C^1$ function with non-vanishing partial derivatives, then the set $\{ f(x, y) : x \in E, y \in F \}$ contains an interval.

4.2. **Proof of Theorem 4.1.** Fix $x \in (0, 1]$ let $(\varepsilon_i) = \Phi_x(M + 1)$. Recall from Lemma 2.4 that $\{ U_{N_j}(x) \}_{j \geq 1}$ is a sequence of subsets in $U(x)$. Then for any $j \geq 1$, each sequence of $U_{N_j}(x)$ ends neither with $N_j$ consecutive zeros nor $N_j$ consecutive Ms. Set

$$
\mathcal{U}_{N_j}(x) = \{ q \in (1, M + 1] : \Phi_x(q) \in U_{N_j}(x) \}.
$$

Then by Lemma 2.1 it follows that $\Phi_x : \mathcal{U}_{N_j}(x) \to U_{N_j}(x)$ is an increasing homeomorphism. Observe that each $U_{N_j}(x)$ is a Cantor set with respect to the metric $\rho$ defined in (2.1). This implies that $\mathcal{U}_{N_j}(x)$ is a Cantor subset of $(1, M + 1]$, and so it can be geometrically constructed by successively removing a sequence of open intervals from a closed interval.

To describe the geometrical construction of $\mathcal{U}_{N_j}(x)$ we first define a sequence of symbolic intervals. For $j \geq 1$ let

$$
\Omega^*_j(x) := \bigcup_{n=0}^{\infty} \Omega^*_j(x),
$$

where

$$
\Omega^*_j(x) := \{ \varepsilon_1 \ldots \varepsilon_{N_j} w d_1 d_2 \ldots d_n : d_{i+1} \ldots d_{i+N_j} \not\in \{ 0^{N_j}, M^{N_j} \} \quad \forall 0 \leq i \leq n - N_j \}.
$$
Lemma 4.6. Let \( x \in (0, 1) \) and \( j \geq 1 \). Take \( \omega = \varepsilon_1 \ldots \varepsilon_{N+N_j} \omega \in \Omega_j^*(x) \).

(i) If \( \omega \) ends with neither 0 nor \( M \), then
\[
I_\omega = [\omega(0^{N_j-1})^\infty, \omega(M^{N_j-1}(M-1))^{\infty}] .
\]

(ii) If \( \omega \) ends with \( 0^k \) for some \( k \in \{1, 2, \ldots, N_j - 1\} \), then
\[
I_\omega = [\omega 0^{N_j-1-k}(10^{N_j-1})^\infty, \omega(M^{N_j-1}(M-1))^{\infty}] .
\]

(iii) If \( \omega \) ends with \( M^k \) for some \( k \in \{1, 2, \ldots, N_j - 1\} \), then
\[
I_\omega = [\omega(0^{N_j-1})^\infty, \omega M^{N_j-1-k}(M-1)M^{N_j-1})^{\infty}] .
\]

Now we describe the geometrical construction of \( \mathcal{Y}_{N_j}(x) \) in terms of the symbolic intervals \( \{I_\omega : \omega \in \Omega_j^*(x)\} \). For a symbolic interval \( I_\omega = [(a_i), (b_i)] \) with \( \omega \in \Omega_j^*(x) \) we define the associated interval \( I_\omega = [p, q] \) by
\[
\Phi_x(p) = (a_i) \quad \text{and} \quad \Phi_x(q) = (b_i) .
\]

Since \( \Phi_x \) is an increasing homeomorphism from \( \mathcal{Y}_{N_j}(x) \) to \( U_{N_j}(x) \), it follows that the convex hull of \( \mathcal{Y}_{N_j}(x) \) is \( I_{\varepsilon_1 \ldots \varepsilon_{N+N_j} \omega} \). Furthermore, for any \( n \geq 0 \) and any \( \omega \in \Omega_j^n(x) \) the intervals \( I_{\omega} \) and \( \omega \in \Omega_j^{n+1}(x) \) are pairwise disjoint subintervals of \( I_\omega \). It turns out that the set
\[
\{I_\omega : \omega \in \Omega_j^*(x)\}
\]
of basic intervals has a tree structure. Therefore,
\[
\mathcal{Y}_{N_j}(x) = \bigcap_{n=0}^{\infty} \bigcup_{\omega \in \Omega_j^n(x)} I_\omega .
\]

Each closed interval \( I_\omega \) with \( \omega \in \Omega_j^n(x) \) is called an \( n \)-level basic interval. We emphasize that the endpoints of each \( n \)-level basic interval belong to \( \mathcal{Y}_{N_j}(x) \). Now for a \( n \)-level basic interval \( I_\omega \) we define the \((n+1)\)-level gaps associated to \( I_\omega \) as follows: suppose \( I_{\omega d} \) and \( I_{\omega(d+1)} \) are two consecutive \((n+1)\)-level basic intervals, then the gap between them, denoted by \( G_{\omega d} \), is a \((n+1)\)-level gap (see Figure [1]). By Lemma 4.6 it follows that the number of \((n+1)\)-level gaps associated to \( I_\omega \) is either \( M \) or \( M-1 \), and the later case refers to items (ii) and (iii) in Lemma 4.6.

Based on the geometrical construction of \( \mathcal{Y}_{N_j}(x) \) in (4.3) it is convenient to define its thickness according to the basic intervals \( I_\omega \) and gaps \( G_{\omega} \) for \( \omega \in \Omega_j^*(x) \). Let
\[
\tau_*(\mathcal{Y}_{N_j}(x)) = \inf_{n \geq 0} \min_{\omega \in \Omega_j^n(x)} \left\{ \frac{|I_\omega|}{|G_{\omega}|}, \frac{|I_{\omega+}|}{|G_{\omega}|} \right\} ,
\]
Then (4.6) which together with (4.5) implies
\[ G \]

Then the gap \( G_{\omega d} \), \( d = 0, \ldots, M - 1 \) implies to the \( n \)-level basic interval \( I_\omega \) in the construction of \( \mathcal{V}_{N_j}(x) \).

\[ \begin{array}{cccccc}
I_\omega & G_{\omega 0} & I_{\omega 1} & \cdots & G_{\omega d} & I_{\omega (d+1)} & \cdots & G_{\omega (M-1)} & I_{\omega M} \\
I_{\omega 0} & & & & & & & & \\
\end{array} \]

Figure 1. The \((n + 1)\)-level gaps \( G_{\omega d}, d = 0, \ldots, M - 1 \) associated to the \( n \)-level basic interval \( I_\omega \).

where for \( \omega \in \Omega_j^n(x) \), if \( \omega^+ \notin \Omega_j^n(x) \) we set \( \frac{|I_{\omega^+}|}{|G_{\omega^+}|} = +\infty \). Here for a word \( c = c_1 \ldots c_k \) we set \( c^+ := c_1 \ldots c_k-1(c_k + 1) \). By (4.1) and (4.4) it follows that \( \tau(\mathcal{V}_{N_j}(x)) \geq \tau_*(\mathcal{V}_{N_j}(x)) \).

**Proposition 4.7.** For any \( x \in (0, 1] \) we have \( \tau_*(\mathcal{V}_{N_j}(x)) \rightarrow \infty \) as \( j \rightarrow \infty \).

**Proof.** Let \( x \in (0, 1] \). Then \( (\varepsilon_i) = \Phi_x(M + 1) > 0^{\infty} \), and there exists \( \ell \in \mathbb{N} \) such that \( \varepsilon_1 \ldots \varepsilon_\ell \ni 0^{\ell-1}1 \). So, we can take \( \ell \in \mathbb{N} \) large enough such that

\[ (4.5) \quad \varepsilon_1 \ldots \varepsilon_{N+N_j} w 0^{\infty} > 0^{\ell-1}10^{\infty}. \]

Then each sequence in \( U_{N_j}(x) \) is lexicographically larger than \( 0^{\ell-1}10^{\infty} \). For \( n \geq 0 \) let \( \omega = \varepsilon_1 \ldots \varepsilon_{N+N_j} w d_1 \ldots d_n \in \Omega_j^n(x) \). For brevity we write \( n_j := |\omega| = N + N_j + |w| + n \) for its length. Suppose \( \omega_d, \omega(d + 1) \in \Omega_j^{n+1}(x) \). Write

\[ I_{\omega d} = [q_1, q_2], \quad I_{\omega (d+1)} = [q_3, q_4]. \]

Then the gap \( G_{\omega d} = (q_2, q_3) \). By Lemma 4.6 it follows that

\[ \begin{align*}
\omega d(0^{N_j-1})^\infty &\leq \Phi_x(q_1) \leq \omega d(10^{N_j-1})^\infty, \\
\Phi_x(q_2) &= \omega d(M^{N_j-1}(M - 1))^\infty, \\
\Phi_x(q_3) &= \omega (d + 1)(0^{N_j-1})^\infty, \\
\omega (d + 1)((M - 1)M^{N_j-1})^\infty &\leq \Phi_x(q_4) \leq \omega (d + 1)(M^{N_j-1}(M - 1))^\infty.
\end{align*} \]

Then

\[ (\omega d(M^{N_j-1}(M - 1))^\infty)_{q_2} = x = (\omega (d + 1)(0^{N_j-1})^\infty)_{q_3}, \]

which together with (4.5) implies

\[ (0^{N_j}1(0^{N_j-1}1)^\infty)_{q_3} - (0^{N_j+1}(M^{N_j-1}(M - 1))^\infty)_{q_2} \]

\[ = (\omega d0^\infty)_{q_2} - (\omega d0^\infty)_{q_3} \geq \frac{1}{q_2} - \frac{1}{q_3} \geq \frac{q_3 - q_2}{q_2 q_3} \geq \frac{q_3 - q_2}{q_3^{\ell+1}}. \]
On the other hand,

\[
(0^{n_j+1}(0^{N_j}1^1)^{\infty})_{q_3} - (0^{n_j+1}(M^{N_j}1^1(M - 1))^{\infty})_{q_2} \\
\leq (0^{n_j+1}(0^{N_j}1^1)^{\infty})_{q_3} - (0^{n_j+1}(M^{N_j}1^1(M - 1))^{\infty})_{q_2} \\
\leq (0^{n_j+1}(M^{N_j-1}(M + 1))^{\infty})_{q_3} - (0^{n_j+1}(M^{N_j-1}(M - 1))^{\infty})_{q_3} \\
= \frac{2}{q_3^{n_j+N_j+1}(1 - 1/q_3^{N_j})} < \frac{4}{q_3^{n_j+N_j+1}},
\]

where the second inequality follows by \((0^{n_j+1}M^{\infty})_{q_3} \geq (0^{n_j+1}10^{\infty})_{q_3}\), and the last inequality holds since \(1/q_3^{N_j} < 1/2\) for large \(j\). Hence, by (4.7) and (4.8) we obtain an upper bound on the length of \(G_{\omega d}\):

\[
|G_{\omega d}| = q_3 - q_2 < \frac{4}{q_3^{n_j+N_j-\ell}}.
\]

In the following we consider the lower bounds on the lengths of \(I_{\omega d}\) and \(I_{\omega (d+1)}\). To do this we need the following inequalities.

**Claim:** for all sufficiently large \(j\) we have

\[
(0^{n_j+1}(10^{N_j}1^1)^{\infty})_{q_3} \leq (0^{n_j+1}M(M - 1)(M^{N_j-3}(M - 1)}\)^{\infty})_{q_2}, \\
(0^{n_j+1}(0^{N_j}10^{2})^{\infty})_{q_3} \leq (0^{n_j+1}((M - 1)^{N_j-1})^{\infty})_{q_3}.
\]

Since the proofs of the two inequalities in (4.10) are similar, we only prove the first inequality. Note by (4.6) that

\[
(\omega d(0^{N_j}1^1)^{\infty})_{q_1} \leq (\omega d(M^{N_j-1}(M - 1))^{\infty})_{q_2}.
\]

This together with (4.5) implies that

\[
(0^{n_j+1}M^{\infty})_{q_2} \geq (0^{n_j+1}(M^{N_j-1}(M - 1))^{\infty})_{q_2} - (0^{n_j+1}(0^{N_j}1^1)^{\infty})_{q_1} \\
\geq (\omega d0^{\infty})_{q_1} - (\omega d0^{\infty})_{q_2} \geq \frac{1}{q_1^{\ell}} - \frac{1}{q_2^{\ell}} \geq \frac{q_2 - q_1}{q_1 q_2^{\ell}}.
\]

Whence,

\[
\frac{q_2 - q_1}{q_1} \leq \frac{M}{q_2^{n_j+1-\ell}(q_2 - 1)} \leq \frac{M}{q_2^{n_j-\ell}}.
\]
where the last inequality holds since \( q_i \geq q_G = q_G(M) \) by Lemma 2.2. Therefore, the first inequality of (4.10) can be deduced as follows:

\[
(0^{n_j+1}(10^{N_j-1})^\infty)_{q_1} = \left(1 + \frac{q_2 - q_1}{q_1}\right)^{n_j+1} \frac{((10^{N_j-1})^\infty)_{q_1}}{q_2^{n_j+1}} \leq \left(1 + \frac{M}{q_G^{n_j+1}}\right)^{n_j+1} \frac{((10^{N_j-1})^\infty)_{q_G}}{q_2^{n_j+1}} \leq \frac{(M(M-1)(M^{N_j-3}(M-1)M^2)^\infty)_{M+1}}{q_2^{n_j+1}} \leq (0^{n_j+1} M(M-1)(M^{N_j-3}(M-1)M^2)^\infty)_{q_2},
\]

where the first inequality follows by (4.12), and the second inequality holds for all sufficiently large \( j \) since

\[
\lim_{j \to \infty} \left(1 + \frac{M}{q_G^{n_j}}\right)^{n_j+1} = 1 < \lim_{j \to \infty} \frac{(M(M-1)(M^{N_j-3}(M-1)M^2)^\infty)_{M+1}}{((10^{N_j-1})^\infty)_{q_G}}.
\]

This proves the claim.

Now by (4.6) we have

\[
(\omega d(10^{N_j-1})^\infty)_{q_1} \geq x = (\omega d(M^{N_j-1}(M-1))^\infty)_{q_2},
\]

which implies that

\[
(0^{n_j+1}(M^{N_j-1}(M-1))^\infty)_{q_2} - (0^{n_j+1}(10^{N_j-1})^\infty)_{q_1} \leq (\omega d0^\infty)_{q_1} - (\omega d0^\infty)_{q_2} \leq (M^\infty)_{q_1} - (M^\infty)_{q_2} = \frac{M}{(q_1 - 1)(q_2 - 1)}(q_2 - q_1) \leq \frac{M}{(q_G - 1)^2}(q_2 - q_1).
\]

On the other hand, by the first inequality of (4.10) it follows that

\[
(0^{n_j+1}(M^{N_j-1}(M-1))^\infty)_{q_2} - (0^{n_j+1}(10^{N_j-1})^\infty)_{q_1} \geq (0^{n_j+1}(M^{N_j-1}(M-1))^\infty)_{q_2} - (0^{n_j+1} M(M-1)(M^{N_j-3}(M-1)M^2)^\infty)_{q_2} = (0^{n_j+2}10^\infty)_{q_2} = \frac{1}{q_2^{n_j+3}}.
\]

So, by (4.13) and (4.14) we obtain a lower bound on the length of \( I_{\omega d} \):

\[
|I_{\omega d}| = q_2 - q_1 \geq \frac{(q_G - 1)^2}{M q_2^{n_j+3}} > \frac{(q_G - 1)^2}{M q_3^{n_j+3}}.
\]

Similarly, by (4.6) we also have

\[
(\omega(d + 1)(10^{N_j-1})^\infty)_{q_3} = x \geq (\omega(d + 1)((M-1)M^{N_j-1})^\infty)_{q_4},
\]
which implies that
\begin{equation}
(0^{n_j+1}(M - 1)M^{N_j-1})_{q_4} - (0^{n_j+1}(0^{N_j-1})_{q_3} - (\omega(d + 1)0^\infty)_{q_4} - (\omega(d + 1)0^\infty)_{q_3},
\end{equation}
On the other hand, by the second inequality of (4.10) it follows that
\begin{equation}
(0^{n_j+1}(M - 1)N_j-1)_{q_4} - (0^{n_j+1}(0^{N_j-1})_{q_3} \geq (0^{n_j+1}10^2)_{q_4} - (0^{n_j+1}(0^{N_j-1})_{q_3} = \frac{1}{q_3^j+3}.
\end{equation}
So, by (4.16) and (4.17) we obtain a lower bound on the length of $I_{\omega(d+1)}$:
\begin{equation}
|I_{\omega(d+1)}| = q_4 - q_3 \geq \frac{(qG - 1)^{2}}{Mq_3^{j+3}}.
\end{equation}
Hence, by (4.9), (4.15) and (4.18) we conclude that
\begin{equation}
\min \left\{ \frac{|I_{\omega(d+1)}|}{|G_{\omega(d+1)}|}, \frac{|I_{\omega(d+1)}|}{|G_{\omega(d+1)}|} \right\} \geq \frac{(qG - 1)^{2}}{4Mq_3^{N_j-1}} \to \infty \quad \text{as } j \to \infty.
\end{equation}
Since $\omega$ was taken from $\Omega^\alpha$ (arbitrarily), this completes the proof by (4.4).

Proof of Theorem 4.1. Take $x \in (0, 1]$. Note by Proposition 4.7 that
\begin{equation}
\tau(U_{N_j}(x)) \geq \tau(U(x)) \to \infty \quad \text{as } j \to \infty.
\end{equation}
Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a $C^1$ function with partial derivatives not vanishing on $(M + 1 - \delta, M + 1]^2$ for some $\delta > 0$. Observe that $U_{N_j}(x) \subset (M + 1 - \delta, M + 1]$ for all large $j$. Then by Lemma 4.5 it follows that $\mathcal{B}_f(x) = \{f(p, q) : p, q \in \mathcal{B}(x)\}$ contains an interval.

5. Univoque bases of multiple rationals

In this section we will prove Theorem 1.2. Recall that $D_M$ consists of all rationals in $[0, 1]$ with a finite expansion in base $M + 1$. Given $x_1, x_2, \ldots, x_t \in D_M$, we will show that the intersection $\bigcap_{i=1}^{t} \mathcal{B}(x_i)$ has full Hausdorff dimension.

Take $x \in D_M$. Then $\Phi_x(M + 1) = \varepsilon_1 \ldots \varepsilon_m M^\infty$ for some $\varepsilon_m < M$ with $m \geq 1$. Let $j \geq m$. Then $2^j > m$, and by Lemma 2.4 it follows that
\begin{equation}
U_{2^j}(x) \subset U(x),
\end{equation}
where
\begin{equation}
U_{2^j}(x) = \left\{ \varepsilon_1 \ldots \varepsilon_m M^{2^j}d_1d_2 \ldots d_{n+1} \ldots d_{n+2^j} \notin 0^{2^j}M^{2^j} \right\} \quad \forall \; n \geq 0.
\end{equation}
Accordingly, $\mathcal{B}_{2^j}(x) = \Phi_x^{-1}(U_{2^j}(x)) \subset \mathcal{B}(x)$. Let $[\alpha_j, \beta_j]$ be the convex hull of $\mathcal{B}_{2^j}(x)$. In the following we show that these subintervals $[\alpha_j, \beta_j], j \geq m$ are pairwise disjoint and converge to $\{M + 1\}$ under the Hausdorff metric (see Figure 2).
follows that Proposition 5.2. is also a Cantor set.

\[ \alpha_j < \beta_j < \alpha_{j+1} \quad \forall \ j \geq m, \quad \text{and} \quad \alpha_j \not\succ M + 1 \quad \text{as} \ j \to \infty. \]

**Proof.** Take \( j \geq m \). By the definition of \( \mathcal{U}_j(x) \) it follows that

\[
\Phi_x(\alpha_j) = \varepsilon_1 \cdots \varepsilon_m M^{2^j} (0^{2^j-1}).
\]

(5.1)

Then \( \Phi_x(\alpha_j) < \Phi_x(\beta_j) < \Phi_x(\alpha_{j+1}) \), and thus \( \alpha_j < \beta_j < \alpha_{j+1} \) by Lemma 2.1. Furthermore, \( \Phi_x(\alpha_j) \) increasingly converges to \( \varepsilon_1 \cdots \varepsilon_m M^\infty = \Phi_x(M + 1) \) under the metric \( \rho \). By Lemma 2.1 and Lemma 2.3 it follows that \( \alpha_j \not\succ M + 1 \) as \( j \to \infty. \)

Note that each \( \mathcal{U}_j(x) \) is a Cantor set, i.e., a nonempty compact set has neither interior nor isolated points. So, by Lemma 5.1 it follows that for any \( k \geq m \),

\[ E_k(x) := \bigcup_{j=k}^\infty \mathcal{U}_j(x) \cup \{ M + 1 \} \]

is also a Cantor set.

**Proposition 5.2.** For any \( x \in D_M \) we have \( \tau(E_k(x)) \to \infty \) as \( k \to \infty. \)

**Proof.** Since \( E_k(x) = \bigcup_{j=k}^\infty \mathcal{U}_j(x) \cup \{ M + 1 \} \) is a Cantor set, by (5.1) it suffices to prove that for each \( k \in \mathbb{N} \) there exists a derived sequence \( \mathcal{O}_k \) of \( E_k(x) \) such that \( \tau_{\mathcal{O}_k}(E_k(x)) \to \infty \) as \( k \to \infty. \) Observe that

\[ \text{conv}(E_k(x)) \setminus E_k(x) = \bigcup_{j=k}^\infty \cap_{j=k}^\infty \bigcup_{t=1}^\infty \mathcal{O}_{j(t)}, \]

where \( \mathcal{O}^{(j)} = (O_1^{(j)}, O_2^{(j)}, O_3^{(j)}, \ldots) \) is an ordered sequence of \( \mathcal{U}_j(x) \). By Proposition 4.7 it follows that

(5.2) \[ \tau(\mathcal{U}_j(x)) = \tau_{\mathcal{O}^{(j)}}(\mathcal{U}_j(x)) \to \infty \quad \text{as} \ j \to \infty. \]
Now we define a derived sequence $O_k$ of $E_k(x)$ which consists of open intervals ordered in the following way:

$$(\beta_k, \alpha_{k+1}), \quad O_1^{(k)};$$
$$(\beta_{k+1}, \alpha_{k+2}), \quad O_2^{(k)};$$
$$(\beta_{k+2}, \alpha_{k+3}), \quad O_3^{(k)};$$
$$\cdots;$$
$$(\beta_{k+n-1}, \beta_{k+n}), \quad O_n^{(k)}; \quad O_n^{(k+1)}, \quad \cdots; \quad O_1^{(k+n-1)}, \quad \cdots.$$

So, by our definition of the derived sequence $O_k$ and (5.2) it suffices to prove that

$$\min \left\{ \alpha_j - \beta_j, \frac{M + 1 - \alpha_{j+1}}{\alpha_{j+1} - \beta_j} \right\} \to \infty \quad \text{as} \quad j \to \infty. \quad (5.3)$$

First we give an upper bound of $\alpha_{j+1} - \beta_j$. Note by (5.1) that

$$\langle \varepsilon_1 \ldots \varepsilon_m M^{2^j} (M^{2^{j-1}} - 1)^\infty \rangle_{\alpha_{j+1}} = \langle \varepsilon_1 \ldots \varepsilon_m M^{2^{j+1}} (0^{2^{j+1}-1})^\infty \rangle_{\alpha_{j+1}}.$$

Then

$$\langle \varepsilon_1 \ldots \varepsilon_m M^{2^{j+1}} - (M^{2^{j-1}} - 1)^\infty \rangle_{\alpha_{j+1}} = \langle \varepsilon_1 \ldots \varepsilon_m M^{2^{j+1}} - (M^{2^{j-1}} - 1)^\infty \rangle_{\alpha_{j+1}} \geq \frac{M}{\beta_{j+1}^{m+1}} \geq \frac{M}{\alpha_{j+1}^{m+1}} (\alpha_{j+1} - \beta_j).$$

This implies that

$$\alpha_{j+1} - \beta_j \leq \frac{\alpha_{j+1}^{m+2}}{M} \left[ \langle \varepsilon_1 \ldots \varepsilon_m M^{2^{j+1}} - (M^{2^{j-1}} - 1)^\infty \rangle_{\alpha_{j+1}} - \langle \varepsilon_1 \ldots \varepsilon_m M^{2^{j+1}} - (M^{2^{j-1}} - 1)^\infty \rangle_{\alpha_{j+1}} \right] \leq \frac{\alpha_{j+1}^{m+2}}{\alpha_{j+1}^{m+2}} \langle \varepsilon_1 \ldots \varepsilon_m M^{2^{j+1}} - (M^{2^{j-1}} - 1)^\infty \rangle_{\alpha_{j+1}}.$$

Thus, by Lemma 2.2 we obtain an upper bound of $\alpha_{j+1} - \beta_j$:

$$\alpha_{j+1} - \beta_j \leq \frac{\langle 20^{2^j - 1} \rangle_{\alpha_{j+1}}}{\alpha_{j+1}^{m+2}} \leq \frac{C_0}{\alpha_{j+1}^{m+2}}, \quad (5.4)$$

where $C_0 := \frac{2(M+1)^3}{x_0^{2^j-1}}$.

Next we consider the lower bounds of $\beta_j - \alpha_j$ and $M + 1 - \alpha_{j+1}$, respectively. These are based on the following two inequalities.
This proves the claim.

Claim: for all sufficiently large $j$ we have

$$
(0^{m+2j}(0^{2j-1}1)\infty)_{\alpha_j} \leq (0^{m+2j}0(M^{2j-2}(M-1)M)\infty)_{\beta_j},
$$

$$
(0^{m+2j+1}M(0^{2j+1-2}10)\infty)_{\alpha_{j+1}} \leq (0^{m+2j+1}M^\infty)_{M+1}.
$$

Since the proof of the first inequality in (5.5) is similar to the proof of (4.10), we only prove the second inequality. Note by (5.1) that

$$(\varepsilon_1\ldots\varepsilon_mM^\infty)_{M+1} = (\varepsilon_1\ldots\varepsilon_mM^{2j+1}(0^{2j+1-1}1)\infty)_{\alpha_{j+1}}.$$ 

Then

$$(0^{m+2j+1}M^\infty)_{M+1} \geq (0^{m+2j+1}M^\infty)_{M+1} - (0^{m+2j+1}(0^{2j+1-1}1)\infty)_{\alpha_{j+1}}$$

$$= (\varepsilon_1\ldots\varepsilon_mM^{2j+1}0^\infty)_{\alpha_{j+1}} - (\varepsilon_1\ldots\varepsilon_mM^{2j+1}0^\infty)_{M+1}$$

$$\geq \frac{1}{\alpha_{j+1}} - \frac{1}{(M+1)^{m+1}} \geq \frac{M+1 - \alpha_{j+1}}{\alpha_{j+1}(M+1)^{m+1}}.$$ 

Whence,

$$
(0^{m+2j+1}M(0^{2j+1-2}10)\infty)_{\alpha_{j+1}} = \left(1 + \frac{M+1 - \alpha_{j+1}}{\alpha_{j+1}}\right)^{m+2j+1} \frac{(M(0^{2j+1-2}10)\infty)_{\alpha_{j+1}}}{(M+1)^{m+2j+1}}$$

$$\leq \left(1 + \frac{1}{(M+1)^{m+2j+1}}\right)^{m+2j+1} \frac{(M(0^{2j+1-2}10)\infty)_{\alpha_{j+1}}}{(M+1)^{m+2j+1}}$$

$$\leq \frac{1}{(M+1)^{m+2j+1}} = (0^{m+2j+1}M^\infty)_{M+1},$$

where the first inequality follows by (5.6), and the second inequality holds for all sufficiently large $j$ since

$$
\lim_{j \to \infty} \left(1 + \frac{1}{(M+1)^{2j+1}-1}\right)^{m+2j+1} = 1 < \lim_{j \to \infty} \frac{1}{(M(0^{2j+1-2}10)\infty)_{\alpha_{j+1}}}.
$$

This proves the claim.

Note by (5.1) that

$$(\varepsilon_1\ldots\varepsilon_mM^{2j}(M^{2j-1}(M-1))\infty)_{\beta_j} = x = (\varepsilon_1\ldots\varepsilon_mM^{2j}(0^{2j-1}1)\infty)_{\alpha_j}.$$ 

Then

$$
(0^{m+2j}(M^{2j-1}(M-1))\infty)_{\beta_j} - (0^{m+2j}(0^{2j-1}1)\infty)_{\alpha_j}
$$

$$= (\varepsilon_1\ldots\varepsilon_mM^{2j}0^\infty)_{\alpha_j} - (\varepsilon_1\ldots\varepsilon_mM^{2j}0^\infty)_{\beta_j}$$

$$\leq (M^\infty)_{\alpha_j} - (M^\infty)_{\beta_j} = \frac{M}{(\alpha_j-1)(\beta_j-1)}(\beta_j - \alpha_j) \leq \frac{M}{(qG-1)^2}(\beta_j - \alpha_j).$$
This implies that
\[
\beta_j - \alpha_j \geq \left( \frac{q_G - 1}{M} \right)^2 \left[ (0^{m+2j}M^j - 1) \right)_{\beta_j} - (0^{m+2j}2^{j-1}1)_{\alpha_j} \right] \geq \left( \frac{q_G - 1}{M} \right)^2 \left[ (0^{m+2j}M^j - 1) \right)_{\beta_j} - (0^{m+2j}0M^{j-2}2^{j-1}1)_{\beta_j} \right]
\]
for sufficiently large \( j \), where the second inequality follows by the first inequality in (5.5).
Therefore,
\[
\beta_j - \alpha_j \geq \left( \frac{q_G - 1}{M} \right)^2 \left[ (0^{m+2j}M^j - 1) \right)_{\beta_j} - (0^{m+2j}2^{j-1}1)_{\alpha_j} \right] \geq \frac{C_1}{\beta_j^{\alpha_j + 1}},
\]
where \( C_1 := \left( \frac{q_G - 1}{M} \right)^{m+1} \).

Now we turn to a lower bound of \( M + 1 - \alpha_{j+1} \). Note by (5.1) that
\[
(\varepsilon_1 \ldots \varepsilon_m M^{2^{j+1}} (0^{2^{j+1}-1}1))_{\alpha_{j+1}} = x = (\varepsilon_1 \ldots \varepsilon_m M^\infty)_{M+1}.
\]
Then
\[
(0^{m+2^{j+1}}M^\infty)_{M+1} - (0^{m+2^{j+1}}(0^{2^{j+1}-1}1))_{\alpha_{j+1}} = (\varepsilon_1 \ldots \varepsilon_m M^{2^{j+1}}0^\infty)_{\alpha_{j+1}} - (\varepsilon_1 \ldots \varepsilon_m M^{2^{j+1}}0^\infty)_{M+1} 
\]
\[
\leq (M^\infty)_{\alpha_{j+1}} - (M^\infty)_{M+1} \leq \frac{1}{q_G - 1}(M + 1 - \alpha_{j+1}).
\]
This implies that
\[
M + 1 - \alpha_{j+1} \geq (q_G - 1) \left[ (0^{m+2^{j+1}}M^\infty)_{M+1} - (0^{m+2^{j+1}}(0^{2^{j+1}-1}1))_{\alpha_{j+1}} \right] \geq (q_G - 1) \left[ (0^{m+2^{j+1}}M(0^{2^{j+1}-2}10))_{\alpha_{j+1}} - (0^{m+2^{j+1}}(0^{2^{j+1}-1}1))_{\alpha_{j+1}} \right]
\]
for \( j \) sufficiently large, where the second inequality follows by the second inequality in (5.5).
So,
\[
M + 1 - \alpha_{j+1} \geq \frac{M(q_G - 1)}{\alpha_{j+1}^{m+2^{j+1}+1}} \geq \frac{C_2}{\alpha_{j+1}^{2^{j+1}+1}},
\]
where \( C_2 := \frac{M(q_G - 1)}{(M+1)^{m+1}} \).

Hence, by (5.4), (5.7) and (5.8) it follows that
\[
\min \left\{ \frac{\beta_j - \alpha_j}{\alpha_{j+1} - \alpha_j}, \frac{M + 1 - \alpha_{j+1}}{\alpha_{j+1} - \beta_j} \right\} \geq \min \left\{ \frac{C_1}{C_0^{2^{j+1}}}, \frac{C_2}{C_0^{2^{j+1}}} \right\} \to \infty
\]
as \( j \to \infty \). This proves (5.3) and then completes the proof. \( \square \)

**Proof of Theorem 1.2.** Let \( x_1, x_2, \ldots, x_\ell \in D_M \). Then there exists \( m \in \mathbb{N} \) such that for each \( x_i \) the sequence \( \Phi_{x_i}(M + 1) = d_1 d_2 \ldots \) satisfies \( d_{m+1}d_{m+2} \ldots = M^\infty \). By our construction each set \( \mathcal{W}(x_i) \cup \{ M + 1 \} \) contains a sequence of Cantor subsets \( E_k(x_i) = \bigcup_{j=k}^{\infty} \mathcal{W}_j(x_i) \cup \{ M + 1 \}, k \geq m \). By Proposition 5.2 the thickness \( \tau(E_k(x_i)) \to \infty \) as \( k \to \infty \). Furthermore, each \( E_k(x_i) \) has a maximum value \( M + 1 \). So, by Lemma 4.3 and Lemma 5.1 it follows that for
any \( k \geq m \) and any \( i_1, i_2 \in \{1, 2, \ldots, \ell\} \) the intersection \( E_k(x_{i_1}) \cap E_k(x_{i_2}) \) contains a Cantor subset \( E_k(x_{i_1}, x_{i_2}) \) such that

\[
\max E_k(x_{i_1}, x_{i_2}) = M + 1, \quad \text{and} \quad \tau(E_k(x_{i_1}, x_{i_2})) \to \infty \quad \text{as} \quad k \to \infty.
\]

Proceeding this argument for all \( x_1, x_2, \ldots, x_\ell \in D_M \) we obtain that for any \( k \geq m \) the intersection \( \bigcap_{i=1}^{\ell} E_k(x_i) \) contains a Cantor subset \( E_k(x_1, \ldots, x_\ell) \) satisfying

\[
\max E_k(x_1, x_2, \ldots, x_\ell) = M + 1, \quad \text{and} \quad \tau(E_k(x_1, x_2, \ldots, x_\ell)) \to \infty \quad \text{as} \quad k \to \infty.
\]

Hence, by Lemma 4.3 we conclude that

\[
\dim_H \bigcap_{i=1}^{\ell} U(x_i) \geq \dim_H \bigcap_{i=1}^{\ell} E_k(x_i) \geq \dim_H E_k(x_1, \ldots, x_\ell) \geq \frac{\log 2}{\log \left(2 + \frac{1}{\tau(E_k(x_1, \ldots, x_\ell))}\right)} \to 1 \quad \text{as} \quad k \to \infty.
\]

This completes the proof. \( \square \)

6. Final remarks

In the way of proving Theorem 1.1 we can also obtain that

\[
\sup U_{SN}(x) = \sup U_I(x) = M + 1 \quad \text{for any} \quad x \in (0, 1].
\]

On the other hand, for \( M = 1 \) and \( x = 1 \) the smallest element of \( U(1) \) is \( q_{KL} \approx 1.78723 \), and \( \Phi_1(q_{KL}) \) is the shift of the Thue-Morse sequence (cf. [23]). Note that the Thue-Morse sequence is simply normal (cf. [3]). So, \( \min U_{SN}(1) = q_{KL} \) for \( M = 1 \). However, when \( M > 1 \) the smallest element of \( U(1) \) is not univoque simply normal (cf. [24]). Then it is natural to ask what \( \inf U_{NS}(1) \) is for \( M > 1 \). In general, for \( x \in (0, 1) \) and \( M \geq 1 \) it is interesting to determine \( \inf U_{SN}(x) \). Also, it might be interesting to investigate \( \inf U_I(x) \) for a general \( x \in (0, 1] \) and \( M \geq 1 \).

In Theorem 1.2 we show that the intersection \( \bigcap_{i=1}^{\ell} U(x_i) \) has full Hausdorff dimension for any \( x_1, \ldots, x_\ell \in D_M \). However, if some \( x_i \) does not belong to \( D_M \), then our method does not work. So it is worth exploring whether Theorem 1.2 holds for any finitely many \( x_1, x_2, \ldots, x_\ell \in (0, 1] \).

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