Regularizing Property of the Maximal Acceleration Principle in Quantum Field Theory

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Abstract

It is shown that the introduction of an upper limit to the proper acceleration of a particle can smooth the problem of ultraviolet divergencies in local quantum field theory. For this aim, the classical model of a relativistic particle with maximal proper acceleration is quantized canonically by making use of the generalized Hamiltonian formalism developed by Dirac. The equations for the wave function are treated as the dynamical equations for the corresponding quantum field. Using the Green’s function connected to these wave equations as propagators in the Feynman integrals leads to an essential improvement of their convergence properties.

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I. INTRODUCTION

During the last years, strong evidences have arose in different areas of theoretical physics, that the proper acceleration of elementary particles (in general case, of any physical object) cannot be arbitrary large, but it should be superiorly limited by some universal value \( A_m \) (maximal proper acceleration). For instance, in string theory \([1,2]\), seeking to present a unified description of all fundamental interactions, including gravity, it was derived that string acceleration must be less than some critical value, determined by the string tension and its mass \([3–7]\). Otherwise in string dynamics the Jeans–like instabilities arise, which lead to unlimited grow of the string length \([3–5]\).

On the other hand the theories of the fundamental and hadronic strings unambiguously predict an upper limit to the temperature in the thermodynamical ensemble of the strings. It is due to an extremely fast (exponential) grow of the number of levels in the string spectrum when the energy or mass raise \([2,8]\). At the critical temperature, the statistical weight of the energy levels completely suppresses the Boltzmann factor \( \exp \left( -\frac{E_n}{T} \right) \) and, as a result, the statistical sum of the string ensemble proves to be infinite. In hadronic physics this critical temperature is the Hagedorn temperature \([9]\). Further, there is a well–known relation between the acceleration of an observer, \( a \), and the temperature of photons bath detected, \( a = 2\pi T \) (Unruh’s effect \([10]\)). Thus the Hagedorn critical temperature gives rise to an upper limiting value of the proper acceleration.

In the framework of an absolutely different approach, a conjecture about the existence of a maximal proper acceleration of an elementary particle was introduced by E.R. Caianiello \([11]\). In these papers a new geometric setting for the quantum mechanics has been developed in which the quantization was interpreted as introduction of a curvature in the relativistic eight dimensional space–time tangent bundle \( TM = M_4 \otimes TM_4 \), that incorporates both the space–time manifold \( M_4 \) and the momentum space \( TM_4 \). The standard operators of the Heisenberg algebra, \( \hat{q} \) and \( \hat{p} \) are represented as the covariant derivatives in \( TM \), the quantum commutation relations being treated as the components of the curvature tensor. It is remarkable, that the line element in \( TM_8 \) intrinsically involves an upper limit on the proper acceleration of the particle \([12,13]\).

The existence of the upper bound on the proper acceleration is intrinsically connected with the extended nature of the particle or string. Therefore one can expect that quantum field theory, involving maximal proper acceleration, could be free of ultraviolet divergencies, originated by the point–like character of the particles in local quantum field theory, or, at least, that the degree of these divergencies could be lower.

Without claiming to solve the problem of constructing a new quantum field theory inglobing the principle of maximal acceleration, this note seeks to present some convincing

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1 We are using the unit system where \( \hbar = c = 1 \). Therefore the dimensions of acceleration and mass are the same.

2 R. Hagedorn \([9]\) derived this critical temperature in the framework of bootstrap description of the hadron dynamics, before the development of the hadronic string models.
evidences that, in fact, the bond on the maximal acceleration can at least smooth the problem of divergencies. The starting point of our consideration will be the classical model of a relativistic particle with maximal proper acceleration \cite{12,13}. Upon quantizing this model the equations for the wave functions can be treat as the dynamical equations for the corresponding field function. Finally we are interested to the Green’s function for this new field equations, specifically, to the behaviour of this function in momentum space. It will be shown that using this Green’s function as the propagator in the Feynman integrals leads to an essential improvement of the convergence properties of the latter.

The layout of this paper is as follows. In Sect. II the classical dynamics of the relativistic particle with maximal proper acceleration is presented both in the Lagrangian and in the Hamiltonian forms. Sect. III is devoted to the quantum theory of this model. In Sect. IV (Conclusion) the arguments presented in favor of the regularizing role of the maximal proper acceleration are shortly discussed.

II. RELATIVISTIC PARTICLE WITH MAXIMAL PROPER ACCELERATION

Let \(x^{\mu}(s), \mu = 0, 1, \ldots, D - 1\) be the world trajectory of a particle in the \(D\) dimensional Minkowski space-time with the Lorentz signature \((+, -, \ldots, -)\). We are using here the natural parameterization of the particle trajectory, \(ds^2 = dx_\mu dx^\mu\). From the geometrical point of view the proper acceleration of the particle is nothing else as the curvature of its trajectory \cite{15}

\[
k^2(s) = -\frac{d^2x_\mu}{ds^2} \frac{d^2x^\mu}{ds^2}.
\]

(2.1)

In a complete analogy with the action for an usual spinless relativistic particle

\[
S_0 = -m \int ds
\]

(2.2)

the action of a particle with upper bonded acceleration is given by the formula \cite{12,14}

\[
S = -\mu_0 \int \sqrt{\mathcal{A}_m^2 - k^2(s)} \, ds.
\]

(2.3)

Here \(\mu_0 = m/\mathcal{A}_m\) and \(\mathcal{A}_m\) is the maximal proper acceleration of the particle. When \(\mathcal{A}_m \to \infty\) the action (2.3) reduces to (2.2).

In paper \cite{14} the classical dynamics generated by the action (2.3) has been investigated completely in the framework of a new method for integrating the equations of motion for the Lagrangians with arbitrary dependence on the particle proper acceleration. It was shown, in particularly, that the particle acceleration in this model always obeys the condition \(k^2(s) < \mathcal{A}_m^2\).

In order to quantize this model, one has to develop the Hamiltonian formalism. For this purpose an arbitrary parameterization of the particle trajectory \(x^{\mu}(\tau)\) should be considered, where the evolution parameter \(\tau\) is subjected only to the condition \(\dot{x}^2 > 0\). Dot means differentiation with respect \(\tau\). In the \(\tau\)-parameterization the proper acceleration of the particle is determined by the formula
\[ k^2(\tau) = \frac{(\dot{x}\ddot{x})^2 - \dot{x}^2 \ddot{x}^2}{(x^2)^3}. \]  

When passing from the natural parameterization \((s)\) to the arbitrary evolution parameter \((\tau)\), the relations \([14]\)

\[
\frac{d}{d\tau} = \sqrt{\dot{x}^2} \frac{d}{ds}, \quad \frac{d^2 x_\mu}{ds^2} = \frac{\ddot{x}_\mu - (\dot{x}\ddot{x}) x_\mu}{(\dot{x}^2)^2}, \quad k \frac{\partial k}{\partial \dot{x}_\mu} = -\frac{1}{\dot{x}^2} \frac{d^2 x_\mu}{ds^2},
\]

prove to be useful. According to Ostrogradskii \([17, 18]\), the Hamiltonian description of the model in question requires introduction of the following canonical variables

\[
q_{1\mu} = x_\mu, \quad q_{2\mu} = \dot{x}_\mu, \quad \quad p_{1\mu} = -\frac{\partial L}{\partial \dot{x}_\mu} - \frac{dp_{2\mu}}{d\tau}, \quad p_{2\mu} = -\frac{\partial L}{\partial \ddot{x}_\mu},
\]

where \(L\) is the Lagrange function in the \(\tau\)-parameterization.

The invariance of the action \((2.3)\) under the transformations \(x^\mu \rightarrow x^\mu + a^\mu\), \(a^\mu = \text{const}\) entails the conservation of the energy–momentum vector \(p_{1\mu}^1\). Hence in our consideration the mass should be defined by

\[
M^2 = p_{1\mu}^2. \quad (2.7)
\]

In contrast to the usual relativistic particle with the action \((2.2)\) the mass of the particle with restricted proper acceleration, in general case, does not coincide with the parameter \(m\) entered its action \((2.3)\).

The invariance of the action \((2.3)\) under the Lorentz transformations leads to conservation of the angular momentum tensor

\[
M_{\mu\nu} = \sum_{a=1}^{2} (q_{a\mu} p_{a\nu} - q_{a\nu} p_{a\mu}). \quad (2.8)
\]

Usually the tensor \(M_{\mu\nu}\) is used for constructing the spin variable \(S\). In the case of the \(D\)-dimensional space–time \(S\) is defined by \([19]\)

\[
S^2 = \frac{W}{M^2}, \quad (2.9)
\]

where

\[
W = \frac{1}{2} M_{\mu\nu} M^{\mu\nu} p_1^2 - (M_{\mu\sigma} p_1^\mu)^2, \quad M^2 = p_1^2.
\]

For \(D = 4\) the invariant \(-W\) is the squared Pauli–Lubanski vector

\[
W = -w_\mu w^\mu, \quad w_\mu = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} M^{\nu\rho} p_1^\sigma. \quad (2.10)
\]
Obviously spin $S$ is also a conserved quantity.

An essential distinction of the Lagrangians with higher derivatives, like (2.3), is the following [14]. The mass $M$ and the spin $S$ are not expressed in terms of the parameters of the Lagrangian, but are integrals of motion, whose specific values should be determined by the initial conditions for the equations of motion $\ddot{x}$. This point considerably complicates transition to the secondary quantized theory (quantum field theory) starting just from the action (2.3). Usually the wave equation for quantum field uniquely specifies the mass and the spin of the particles described by this field [20].

The action (2.3), as well as (2.2), is invariant under the reparameterization $\tau \rightarrow f(\tau)$ with an arbitrary function $f(\tau)$. As a result there should be the constraints in the phase space [21]. Using the definition of $p_2$ in (2.6) and formulae (2.5) one easily deduces the primary constraint

$$\phi(q,p) = p_2^\mu q_2^\mu \approx 0,$$

where $\approx$ means a weak equality [16]. There are no other primary constraints in the model under consideration because the rank of the Hessian matrix

$$\frac{\partial^2 L}{\partial \dot{x}_\mu \partial \dot{x}_\nu}$$

equals $D - 1$. The canonical Hamiltonian

$$H = -p_1^\mu \dot{x}_\mu - p_2^\mu \dot{x}_\mu - L,$$

rewritten in terms of the canonical variables $q_a, p_a$, $a = 1, 2$, becomes

$$H = -p_1 q_2 + A_m \sqrt{q_2^2 (\mu_0^2 - q_2^2 p_2^2)}.$$

The equations of motion in the phase space

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, H_T\}$$

are generated by the total Hamiltonian

$$H_T = H + \lambda(\tau) \phi(q,p).$$

Here $f$ is an arbitrary function of the canonical variables and of the evolution parameter $\tau$, $\lambda(\tau)$ is the Lagrange multiplier. The Poisson brackets are defined in a standard way

$$\{f, g\} = \sum_{a=1}^2 \left( \frac{\partial f}{\partial p_a^\mu} \frac{\partial g}{\partial q_a^\mu} - \frac{\partial f}{\partial q_a^\mu} \frac{\partial g}{\partial p_a^\mu} \right).$$

3The spin $S$ of the usual relativistic particle with the action (2.3), calculated by the formulae (2.9) or (2.10), identically equals zero because in this case there is only one pair of the canonical variables $q, p$. 5
Requirement of the stationarity of the primary constraint \((2.11)\)
\[
\frac{d\phi}{d\tau} = \{\phi, H_T\} \approx 0
\] (2.17)
leads to the secondary constraint
\[
\{\phi, H_T\} = \{\phi, H\} = H \approx 0 .
\] (2.18)
As one could anticipate, the canonical Hamiltonian vanishes in a weak sense. It is a direct consequence of the reparameterization invariance of the initial action \((2.3)\) \([16]\). Obviously, at this step the procedure of the constraint generation terminates. Thus in this model there are only two constraints \((2.11)\) and \((2.18)\), and they are of the first class because
\[
\{\phi, H\} \approx 0 .
\] (2.19)
For the spin variable \(S\) we deduce from Eqs. \((2.9)\) or \((2.10)\)
\[
S^2 p_1^2 = p_1^2 p_2^2 q_2^2 + 2(p_1 p_2)(p_1 q_2)(p_2 q_2) - (p_1 p_2)^2 q_2^2 - p_1^2 (p_2 q_2)^2 - p_2^2 (p_1 q_2)^2.
\] (2.20)
The energy–momentum vector squared \(p_1^2\) and the spin squared \(S^2\) are the integrals of motion \([14]\). In the Hamiltonian dynamics this implies
\[
\frac{dp_1^2}{d\tau} = \{p_1^2, H_T\} \approx 0 ,
\] (2.21)
\[
\frac{dS^2}{d\tau} = \{S^2, H_T\} \approx 0 .
\] (2.22)
Specifying the values of \(p_1^2\) and \(S^2\)
\[
p_1^2 = M^2 ,
\] (2.23)
\[
S^2(p, q) = S^2
\] (2.24)
we pick a certain sector in the classical dynamics of the model, which is invariant with respect to the evolution in time.

Further, it is very convenient to use the natural parameterization of the particle trajectory putting
\[
\dot{x}^2 = q_2^2 = m^{-2} .
\] (2.25)
It is remarkable that this condition is in accordance with the Hamiltonian equations of motion
\[
\{q_2^2, H_T\} \approx 0 ,
\] (2.26)
the Lagrange multiplier \(\lambda(\tau)\) being undetermined. Therefore Eq. \((2.25)\), on the same footing as Eqs. \((2.23)\) and \((2.24)\), should be treated as an invariant relation rather than a gauge condition \([22]\). In the parameterization \((2.25)\) the canonical momenta \(p_1\) and \(p_2\) prove to be proportional to \(\dot{x} = q_2\) and \(\ddot{x}\), respectively. Hence, we have now
\[
p_1 p_2 \approx 0
\] (2.27)
in view of \((2.11)\) or \((2.25)\).
III. QUANTUM THEORY

Transition to quantum theory is accomplished in a standard way by imposing the commutators

$$[\hat{A}, \hat{B}] = i\{A, B\}, \tag{3.1}$$

where \{\ldots, \ldots\} are the Poisson brackets (2.16), \(A\) and \(B\) are arbitrary functions of the canonical variables \(q_a, p_a, \ a = 1, 2\). All the constraints in the model concerned are of first–class and the gauge is not fixed, thus there is no need to use the Dirac brackets in our consideration [16].

In quantum theory we are interested in Eqs. (2.23) and (2.24) which should be imposed as the conditions on the physical state vectors \(|\psi\rangle\)

$$(p_1^2 - M^2)|\psi\rangle = 0, \tag{3.2}$$

$$S^2|\psi\rangle = S(S + D - 1)|\psi\rangle = 0, \ S = 0, 1, 2, \ldots, \tag{3.3}$$

where \(D\) is the dimension of space–time. The classical value of the spin variable \(S^2\) has been substituted here by the eigenvalues of the spin operator \(S^2\) in the \(D\)-dimensional space-time, \(S(S + D - 1)\). These equations fix the values of the Casimir operators of the Poincare group specifying the irreducible representation of this group in terms of the state vectors \(|\psi\rangle\) and providing in this way the basis for consistent description of the elementary particles in the relativistic quantum theory.

As was noted in Introduction we shall treat equations (3.2) and (3.3) in two-fold way, as the equations for the wave function \(|\psi\rangle\) in the quantum mechanics of the model in hand and as the wave equations for the corresponding relativistic quantum field \(\Phi(q_1, q_2)\) (the secondary quantized theory). This field, as well as the wavefunction \(|\psi\rangle\), should depend on two arguments \(q_\mu^1 = x^\mu\) and \(q_\mu^2 = \dot{x}^\mu\) which are treated as the canonical coordinates in the Ostrogradskii formalism [17,18].

Before proceeding further we have to take into account the constraints. Substituting Eqs. (2.11), (2.18), (2.25) and (2.27) into Eq. (2.20) we obtain

$$S^2 = \frac{p_2^2}{m^2} \left[1 - \left(\frac{m^2}{M^2} - \frac{A^2_{\mu
u} p_2^\mu p_2^\nu}{M^2 m^2}\right)\right]. \tag{3.4}$$

Now the second field equation (3.3) can be rewritten in the form

$$(\Box_2 + m_1^2) (\Box_2 + m_2^2) \Phi(q_1, q_2) = 0, \tag{3.5}$$

where

$$\Box_a \equiv \partial_\alpha \partial^\alpha, \quad \partial_{\alpha a} = \frac{\partial}{\partial q_\alpha^a}, \quad a = 1, 2. \tag{3.6}$$

There is no summation with respect to \(a\) in Eq. (3.6). The dependence of the quantum field \(\Phi(q_1, q_2)\) on \(q_1\) is determined by the Klein–Gordon equation (3.2) with the mass \(M\).
\[(\Box + M^2)\Phi(q_1, q_2) = 0.\]  

(3.7)

For \(M \neq 0\) the parameters \(m_a^2, \ a = 1, 2\) in Eq. (3.3) are given by

\[
m_{1,2}^2 = \frac{m^2}{2} \left[ 1 - \mu^2 \pm \sqrt{(1 - \mu^2)^2 + 4S(S + D - 1)\frac{A_m^2}{M^2}} \right],
\]

(3.8)

where \(\mu = m/M\). In the massless case \((M = 0)\), we have

\[m_1^2 = 0, \quad m_2^2 = -\mu_0^2 m^2.\]  

(3.9)

Thus in general case in Eq. (3.3) there are one real ‘mass’ \(m_1^2 \geq 0\) and one tachyonic ‘mass’ \(m_2^2 < 0\). The latter is not dangerous here because the physical mass of the quanta described by the field \(\Phi(q_1, q_2)\) is equal to \(M\) according to Eq. (3.7).

From Eqs. (3.3) and (3.7) we deduce the Green function in quantum field theory concerned

\[G(k_1, k_2) = \frac{1}{(k_1^2 - M^2)(k_2^2 - m_1^2)(k_2^2 - m_2^2)}.\]

(3.10)

The regularizing property of this propagator can be elucidated by considering the one–loop Feynman diagram in such bilocal field theory

\[
\int \int G^2(k_1, k_2) d^4k_1 d^4k_2 \simeq \int \frac{d^4k_1}{k_1^4} \int \frac{d^4k_2}{k_2^8} \sim \int \frac{d^4k_1}{k_1^4} \int \frac{d^4k_2}{k_2^8(k_1)} \sim \int \frac{d^4k_1}{k_1^4} \frac{1}{k_1^4}. \quad (3.11)
\]

Here we have taken into account that the momenta \(k_1\) and \(k_2\) will be, as usually, ‘mixed’ in this diagram describing the interacting field \(\Phi(q_1, q_2)\), and the integration over \(dk_2\) has been carried out at first. As a result, we obtained in Eq. (3.11) an additional factor \(k_1^{-4}\) in comparison with the usual one–loop Feynman diagram with the propagator \((k^2 - m^2)^{-1}\).

This gives evidence that the maximal acceleration can really provide natural regularization in corresponding quantum field theory.

IV. CONCLUSION

In comparison with other physical regulators of quantum field theory, for example, elementary length, which also originates in extended nature of elementary particles, the principle of maximal acceleration is obviously favorable because it preserves the continuous space–time.

In order to put the arguments in favor of the regularizing property of the maximal acceleration principle onto a more rigorous footing, it is required to develop the theory of bilocal quantum field introduced above. A special attention should be paid here on the interaction mechanism in this formalism. However all these problems are far beyond the scope of this short note.

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