Geometric Scaling in the Quantum Hall System

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Abstract

The transitions between neighbouring plateaux in the quantum Hall system are observed to follow “anti-holomorphic” scaling with “superuniversal” scaling exponents, showing that the system contains an emergent sub-modular discrete symmetry and a holomorphic structure at low energies. We identify a class of effective scaling models consistent with this data, which is parametrized by the complex structure of a torus with a special spin structure, in which only the number of fermions ($c$) remains undetermined. For $c = 2$ this gives the superuniversal anti-holomorphic scaling potential previously inferred from data, with scaling exponent $\nu \approx 2.6$, in reasonable agreement with available scaling data.

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It has previously been pointed out that the fixed point structure and unusual scaling behaviour observed in the quantum Hall system suggests a unified approach to the integer and fractional quantum Hall effects [1]. Scaling is observed in the transition between neighbouring plateaux, and the scaling exponents appear to be independent of which transition is examined (“superuniversality”). The conductivities $\sigma^{xx}$ and $\sigma^{xy}$ are the effective coupling constants in the low-energy effective field theory (EFT) of quantum charge transport in an external magnetic field in a disordered medium. We can therefore treat these scaling phenomena as critical points of the renormalization group (RG) beta-functions $\beta^{xx} = d\sigma^{xx}/dt$ and $\beta^{xy} = d\sigma^{xy}/dt$, where $t$ is the dominant scale parameter [1, 2, 3].

Superuniversality is automatic if the EFT possesses a kind of “complexified Kramers-Wannier duality”, identified in [1] as the symmetry group $\Gamma_0(2)$, which is generated by translations $T : \sigma = \sigma^{xy} + i\sigma^{xx} \to \sigma + 1$ and the “complexified duality transformation for fermions” $ST^2S : \sigma \to \sigma/(1 - 2\sigma)$. The complexified beta-function $\beta^\sigma = d\sigma/dt = \beta^{xy} + i\beta^{xx}$ transforms as a contravariant vector under $\Gamma_0(2)$-transformations [4], but this is not sufficient information usefully to constrain the scaling theory. Recently it was pointed out that the scaling data provide additional information [5]. When the scaling exponents for $\sigma^{xy}$ and $\sigma^{xx}$ are defined geometrically they appear to be of equal magnitude, which strongly suggests that the effective scaling theory contains a holomorphic structure. Taken together these constraints are so strong that the beta-function in the scaling region was found to be unique up to normalization: $\beta_\sigma \propto \partial_\sigma \varphi(\sigma)$, where

$$\varphi(\sigma) = \ln \Delta(2\sigma) - \ln \Delta(\sigma), \quad (1)$$

and $\Delta(\sigma)$ is the modular cusp form of weight 12. This scaling form is a complex analytic (holomorphic), everywhere non-singular and non-vanishing function scalar (weightless) under $\Gamma_0(2)$. It plays the role of Zamolodchikov’s RG-potential (C-function) in his celebrated C-theorem [6], which spells out what needs to be done in simple geometrical terms: we need a model containing sufficient dynamical information to yield the covariant gradient flow physically normalized, $\beta_\sigma = \partial_\sigma C$, which together with the physical parameter space metric $G_{\sigma\sigma}$ gives the physical (contravariant) beta-function: $\beta^{\text{phys}} = \beta^\sigma = G^{\sigma\sigma} \beta_\sigma / 12$.

While a mathematical analysis of eq. (1) is beyond the scope of this letter, we pause here briefly to remark on the simplicity and uniqueness of this result, since this will be helpful when we construct a scaling model below. Given that we are studying holomorphic gradient
flows automorphic under $\Gamma_0(2)$, up to additive and multiplicative constants this is probably the only form the potential can take. Holomorphic scalar forms are scarce, and in fact no such function exists for the full modular group. The obvious way to build one is by taking a weightless ratio of cusp forms, but for SL$(2,\mathbb{Z})$ all cusp forms are built from $\Delta(\sigma)$, so that there is at most one linearly independent cusp form at each weight, and no non-trivial scalar can be constructed. Sub-modular flows are a different matter. In particular, for $\Gamma_0(2)$ there are two linearly independent weight 12 cusp forms: the oldform $\Delta(\sigma)$ inherited from SL$(2,\mathbb{Z})$ and the newform $\Delta(2\sigma)$, and consequently $\ln(\Delta(2\sigma)/\Delta(\sigma))$ is a viable potential\footnote{We believe that any scalar form is generated by this one.}.

This is as far as the “phenomenological” approach advocated in \cite{1} can take us. What remains is to determine the critical exponents and for this it is necessary to identify an EFT in the universality class of the quantum Hall system. It is a general result from conformal perturbation theory that all scaling behaviour is determined by conformal data, i.e. by the scale invariant (conformal) EFT which exists at the critical point. It is therefore sufficient to work with scale-invariant theories to extract the data we need: the RG-potential $C$ and the Zamolodchikov metric $G$. In general no way is known of extracting the C-function and metric directly from scaling data, but in our case the constraints (the symmetry and analyticity in the scaling region) are so strong that everything is determined up to constants: once a candidate EFT has been identified our task is reduced to determining normalizations.

The identification of an appropriate effective scaling theory is helped enormously by the need to have a theory invariant under the sub-modular group $\Gamma_0(2)$. Our experience with conformal field theories immediately suggests that we consider free fermions on a torus with complex structure $\tau$. If we do not endow the manifold with any additional structure, all choices of the lattice defining the torus which are related by fractional linear transformations in SL$(2,\mathbb{Z})$ are equivalent, in the sense that these tori have the same complex structure. The additional structure we need is a choice of spin structure, of which there are four on the torus. These are the four possible combinations of boundary conditions that can be imposed on a fermion along the two cycles of the torus: PP, PA, AP, and AA, where P denotes periodic and A denotes anti-periodic.

The physical interpretation of these spin structures may be seen from a standard gauge argument \cite{7} which forces the Hall conductivities to have fractional odd denominator values.
at the IR fixed points. The group which respects this parity structure is \( \Gamma_0(2) \), since it groups the fractions into two equivalence classes: those with odd denominators and those with even denominators. In other words, the standard (fully spin-polarized) Hall system contains only one type of anyons, and by exploiting the symmetry we have grouped the quantum Hall phases, and their anyonic excitations, into a single equivalence class. Both the PP and PA spin structures are invariant under \( \Gamma_0(2) \), but PP is also invariant under the full modular group, so only PA respects the statistics of the quantum Hall anyons, and is therefore the extra data we need for our EFT. Since the space of complex structures (space of inequivalent \( \tau \)) of this torus model coincides with the space of conductivities (\( \sigma \)) observed in experiments, it is clear that we should identify these spaces and set \( \tau = \sigma \).

Finally, both the disorder and the observed antiholomorphic scaling in the QHS strongly suggests that the scaling should derive from a model with broken supersymmetry. This has the immediate benefit that we can avail ourselves of results from superstring theory, where it was shown [10] that Zamolodchikov’s physical metric on the moduli space of complex structures that appears in the low-energy geometric limit of supersymmetric sigma-models with Calabi-Yau targets, coincides with the mathematical Weil-Petersson metric. The simplest Calabi-Yau space is the torus, and in this case the Weil-Petersson metric reduces to the Poincare metric on the upper half plane. Since this is the hyperbolic metric of constant negative curvature we have \( C^{\tau \bar{\tau}} = (\text{Im} \sigma)^2 \delta^{\tau \bar{\tau}} \). We cannot expect this metric to be correct away from the critical region, but it is sufficient for the purpose of computing the critical exponents.

In summary, we have arrived at a simple EFT which plausibly encodes the correct anyonics and concommitant symmetries for the quantum Hall system. As is often the case for effective theories, the problem of deriving the EFT from microscopics has been sidestepped by exploiting the low-energy symmetries. We can now calculate the critical exponents in these models, and if they agree with experiments we will have strong evidence that it is in the universality class of the quantum Hall system.

We extract the scaling part of the C-function by computing the vacuum energy \( F \) of the EFT, because all scalar potentials, including the real potentials \( F \) and \( C \), must be functions of the unique potential \( \varphi \) invariant under the symmetry \( \Gamma_0(2) \), and hence \( C = C(F) \).

At criticality both the free energy \( F \) and the central charge \( c = C(\sigma_\infty, \overline{\sigma}_\infty) \) count degrees
of freedom. Since this must be true for an arbitrary number of degrees of freedom this implies that $C(F) \propto F$, so that near the critical point we have $C \approx cF/F_\odot$.

The vacuum energy $F_{PA} = - \ln Z_{PA} = - \ln \text{Det}_{PA}$ of PA-twisted fermions on a torus is well known. The determinant factorizes into holomorphic and anti-holomorphic pieces \cite{8}, so we have $F_{PA} \propto f_{PA} + \bar{f}_{PA}$ with $f_{PA}(\sigma) = - \ln \text{Pf}_{PA}(\sigma)$, where the Pfaffian $\text{Pf}_{PA}(\sigma)$ is the functional “determinant” for a single PA-twisted spinor on the torus. It is a “polynomial” in $q = \exp(2\pi i \sigma)$ of infinite degree which can be extracted from the literature on conformal field theories \cite{8}:

$$\text{Pf}_{PA}(\sigma) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 + q^n)^2.$$ \hfill (2)

Recalling the definition of the Dirichlet eta-function:

$$\eta(\sigma) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \hfill (3)$$

we find that $f_{PA}(\sigma) = \ln(2\sigma) - \ln \eta(\sigma)$. Furthermore, since $\Delta(\sigma) = \eta(\sigma)^{24}$ we have $f_{PA} = \varphi/24$ and $F_{PA} \propto \text{Re} \varphi$, which gives

$$C(\sigma, \bar{\sigma}) \approx c \ \text{Re} \varphi(\sigma)/\text{Re} \varphi(\sigma_\odot). \hfill (4)$$

Putting all this together and expanding the beta-function around the critical (saddle) point at $\sigma_\odot = (1 + i)/2$, we find:

$$\beta_{\text{phys}} \approx \frac{1}{12} (\text{Im} \sigma_\odot)^2 \partial \! C(\sigma, \bar{\sigma}) = \frac{1}{\nu} (\bar{\sigma} - \sigma_\odot) + \ldots \hfill (5)$$

with the superuniversal anti-holomorphic scaling exponent $\nu \approx 5.2/c$. With two twisted complex fermions ($c = 2$) we obtain $\nu = 2.6$ which agrees with scaling data within experimental error \cite{5,10,11}.

In conclusion we have identified a discrete set of EFTs, one of which may be in the same universality class as the quantum Hall system; i.e. which possesses the same symmetry properties and scaling behaviour as the low energy transport theory of the quantum Hall slab.

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2 This can also be inferred from a standard finite size scaling argument combined with the constraints of automorphy.

3 Note that the determinant for the PP-torus vanishes due to the existence of a zero mode, which is consistent with our previous observation that there is no scalar potential for the full modular group due to the paucity of cusp forms.
Using the Zamolodchikov form for the beta-function, which is consistent with the observed anti-holomorphic scaling, we have calculated the partition function of these effective theories and extracted the critical exponents. The fact that one of these models, with a reasonable value of $c$, agrees with the observed values of the exponents suggests that this effective scaling theory provides an accurate description of the quantum Hall system near criticality.

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