Möbius invariant integrable lattice equations associated with KP and 2DTL hierarchies

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Abstract

Integrable lattice equations arising in the context of singular manifold equations for scalar, multicomponent KP hierarchies and 2D Toda lattice hierarchy are considered. These equation generate the corresponding continuous hierarchy of singular manifold equations, its Bäcklund transformations and different forms of superposition principles. They possess rather special form of compatibility representation. The distinctive feature of these equations is invariance under the action of Möbius transformation. Geometric interpretation of these discrete equations is given.

Introduction

The equations we are going to discuss here were derived by the authors in frame of analytic-bilinear approach to integrable hierarchies [1], [2]. They arise on the third level of the hierarchy, i.e. as a discrete singular manifold equations (or, in other words, in the context of the hierarchy in Schwarzian

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form). These equations are highly symmetric and possess very peculiar properties.

First, they have a very special compatibility condition representation, symmetric with respect to lattice variables and indicating duality between wave functions and potential (providing in fact equations for both objects in a similar way).

Second, they possess (even in the matrix case, corresponding to multi-component KP hierarchy) a Möbius symmetry, which is typical for equations in Schwarzian form and which is deeply connected to projective geometry. Similar equations for the KdV case were interpreted by Bobenko and Pinkall as equations of discrete isothermic surfaces. We hope that equations we discuss may also be interpreted in terms of some special classes of discrete surfaces, though the interpretation is yet unknown for the matrix case. In the scalar case, however, there are at least two ways to connect the equations to discrete geometry: as a reduction of Darboux system and in terms of the three-dimensional lattice on the complex plane with some special geometrical properties.

Third, for the basic hierarchy we have one lattice equation of the considered type, and it generates by expansion in parameters the continuous hierarchy itself, its Bäcklund transformations and different types of superposition formulae for Bäcklund transformations. Thus it encodes all the information about the hierarchy (more or less in the way of Hirota bilinear discrete equation, with which it is closely connected, but not in terms of the $\tau$-function, rather in terms of wave functions and potentials).

**Basic equations and their properties**

In this section we introduce basic equations and study some of their properties without the reference to the original scheme of derivation of these equations (a sketch of it will be given later).

In the context of the scalar KP hierarchy the basic equation reads

\[
(T_2 \Delta_1 \Phi)(T_3 \Delta_2 \Phi)(T_1 \Delta_3 \Phi) = (T_2 \Delta_3 \Phi)(T_3 \Delta_1 \Phi)(T_1 \Delta_2 \Phi).
\]  

(1)

Here

\[
\Phi = \Phi(n_1, n_2, n_3),
\]

\[
T_1 \Phi = \Phi(n_1 + 1, n_2, n_3),
\]
\[ \Delta_1 = T_1 - 1. \]

In the case of multicomponent KP hierarchy \( \Phi \) is \( N \times N \) matrix-valued function and the basic equation looks like

\[ (T_1 \Delta_3 \Phi) \cdot (T_3 \Delta_1 \Phi)^{-1} \cdot (T_3 \Delta_2 \Phi) \cdot (T_2 \Delta_3 \Phi)^{-1} \cdot (T_2 \Delta_1 \Phi) \cdot (T_1 \Delta_2 \Phi)^{-1} = 1. \quad (2) \]

And finally, for the scalar 2D Toda lattice hierarchy one has

\[
\{T_-(T_+ - 1)\Phi\} \{T(T_- T^{-1})\Phi\} \{T_+(1 - T^{-1})\Phi\} = \\
\{(T_+ - 1)\Phi\} \{T_+(T_- T^{-1})\Phi\} \{T_-(T - 1)\Phi\}. \quad (3)
\]

This equation is not symmetric with respect to all three shifts, one of them \( (T) \) plays a special role and corresponds to the original (essentially) discrete variable of the Toda lattice. In what follows we will mainly study equations (1), (2).

Equation (1) may be treated as compatibility condition for the following set of equations for the function \( f \)

\[
\frac{a_i T_i f}{a_j T_j f} = \frac{\Delta_i \Phi}{\Delta_j \Phi} \quad i, j, k \in \{1, 2, 3\}, \quad (4)
\]

where \( a_i \) are some constants. To demonstrate this, we break the symmetry of the relation (4) and distinguish one of the shifts (say, \( T_1 \)).

**Remark.** Unfortunately, we don’t know a symmetric derivation starting from relation (4), though the initial equation and the final equation are symmetric. Symmetric way of deriving (4) in another context will be given later in this paper.

Then we rewrite (4) in the form

\[
T_2 f = \frac{a_1 \Delta_2 \Phi}{a_2 \Delta_1 \Phi} T_1 f = UT_1 f,
\]

\[
T_3 f = \frac{a_3 \Delta_3 \Phi}{a_3 \Delta_1 \Phi} T_1 f = VT_1 f.
\]

In this form the system looks more familiar. Then, as usual, taking cross-shifts of the first and the second equations (which should give the same results) and using the equations themselves to get rid of the shifts \( T_2, T_3 \) acting on \( f \), one obtains the following equation (compatibility condition)

\[(T_3 U)(T_1 V) = (T_2 V)(T_1 U).\]
Substituting the expressions of $U$, $V$ through $\Phi$, one gets exactly equation (1).

It is possible to start also with the following ('dual') system to get equation (1)

$$\frac{a_i T_i^{-1} \tilde{f}}{a_j T_j^{-1} f} = \frac{\tilde{\Delta}_i \Phi}{\tilde{\Delta}_j \Phi} \quad i, j, k \in \{1, 2, 3\},$$

where

$$\tilde{\Delta}_i = T_i^{-1} - 1.$$

On the other hand, it is possible to treat the system (1) as a set of linear equations for the function $\Phi$. To do it, we rewrite the system in the form

$$\Delta_2 \Phi = \frac{a_2 T_2 f}{a_1 T_1 f} \Delta_1 f = U \Delta_1 \Phi,$$

$$\Delta_3 \Phi = \frac{a_3 T_3 f}{a_1 T_1 f} \Delta_1 f = V \Delta_1 \Phi.$$

The compatibility condition for this system gives two equations

$$(T_3 U)(T_1 V) = (T_2 V)(T_1 U),$$

$$\Delta_3 U + (T_3 U) \Delta_1 V = \Delta_2 V + (T_2 V) \Delta_1 U.$$

The first of these equations is resolved by the substitution of expressions for the functions $U$, $V$ in terms of $f$, the second gives the following equation for $f$

$$\sum_{(ijk)} \epsilon_{ijk} a_j a_k T_j \left( \frac{f}{T_1 f} \right) = 0,$$

summation here is over different permutations of indices.

Starting from the dual linear system (5), one obtains an equation for the function $\tilde{f}$

$$\sum_{(ijk)} \epsilon_{ijk} a_j a_k T_k \left( \frac{T_1 \tilde{f}}{\tilde{f}} \right) = 0,$$

Both these equation are connected with the modified KP hierarchy (the equation for $\tilde{f}$ with the mKP hierarchy itself and the equation for $f$ with the dual (adjoint) hierarchy). Similar equations were derived by Nijhoff et al [4].
The matrix case (2) can be treated the same way. Equations (5) and the dual system for this case read

\[(\Delta_j \Phi)^{-1} \Delta_i \Phi = (A_j T_j f)^{-1} (A_i T_i f),\]

\[\tilde{\Delta}_i \Phi (\tilde{\Delta}_j \Phi)^{-1} = (T_i^{-1} \tilde{f} A_i) (T_j^{-1} \tilde{f} A_j)^{-1},\]

where \(A_i\) are some diagonal matrices; in what follows we suggest that the determinant of \(A_1\) is not equal to zero.

The linear system for the function \(f\) takes the form

\[A_1^{-1} A_2 T_2 f = (T_1 f) (\Delta_1 \Phi)^{-1} \Delta_2 \Phi = (T_1 f) U,\]

\[A_1^{-1} A_3 T_3 f = (T_1 f) (\Delta_1 \Phi)^{-1} \Delta_3 \Phi = (T_1 f) V.\]

The compatibility condition for this system gives equation (2) in the form

\[(T_1 \Delta_3 \Phi) \cdot (T_3 \Delta_1 \Phi)^{-1} \cdot (T_3 \Delta_2 \Phi) = (T_1 \Delta_2 \Phi) \cdot (T_2 \Delta_1 \Phi)^{-1} \cdot (T_2 \Delta_3 \Phi).\]

Then, the linear system for the function \(\Phi\) reads

\[\Delta_2 \Phi = \Delta_1 \Phi (a_1 T_1 f)^{-1} A_2 T_2 f = \Delta_1 \Phi U,\]

\[\Delta_3 \Phi = \Delta_1 \Phi (a_1 T_1 f)^{-1} A_3 T_3 f = \Delta_1 \Phi V,\]

and a compatibility condition gives a matrix version of equation (6)

\[\sum_{ijk} \epsilon_{ijk} A_j T_j \left( f \cdot (T_i f)^{-1} \right) A_k = 0.\]

A matrix version of (7) is

\[\sum_{ijk} \epsilon_{ijk} A_k T_k \left( \tilde{f}^{-1} \cdot (T_i \tilde{f}) \right) A_j = 0.\]

**Symmetries of the basic equations**

It is easy to check that equation (1), (3) possess a Möbius symmetry transformation, i.e. if function \(\Phi\) is a solution to this equation, then function \(\Phi'\),

\[\Phi' = \frac{a \Phi + b}{c \Phi + d}.\]
where \(a, b, c, d\) are arbitrary constants, is also a solution. For the matrix equation (2) the symmetries include inversion, left and right matrix multiplication and shift

\[
\Phi' = \Phi^{-1}, \\
\Phi' = A\Phi, \quad \Phi' = \Phi B, \\
\Phi' = \Phi + C,
\]

where \(A, B, C\) are some matrices (determinant of \(A\) and \(B\) is not equal to zero).

The structure of equation (2) resembles the structure of quartic equations given by Bobenko and Pinkall [3] for the quaternion description of discrete isothermic surfaces in \(\mathbb{R}^3\). Probably equation (2) also has a geometric interpretation in terms of quaternion construction.

It is interesting to note that due to the inversion symmetry, the equation (2) admits a reduction

\[
\Phi\Phi = -1,
\]

that for \(\Phi\) belonging to the field of quaternions means that the vector in \(\mathbb{R}^3\) corresponding to the (imaginary) quaternion \(\Phi\) has a unit length, and so the equation (2) in this case defines a three-dimensional lattice on the unit sphere.

Equations (6), (7) are connected by the transformation

\[
f' = \tilde{f}^{-1}, \\
\tilde{f}' = f^{-1},
\]

that keeps in the matrix case.

**The hierarchy of continuous equations**

Let us suggest that the function \(\Phi\) in equation (1) is a function of a standard infinite set of KP variables \(\mathbf{x}, \mathbf{x} = \{x_k\}, \quad 1 \leq k < \infty\) (we will also use notations \(x = x_1, y = x_2, t = x_3\)) and the shift operators \(T_i\) are realized as

\[
T_i : \mathbf{x} \to \mathbf{x} + [a_i], \quad (8)
\]

\[
[a_i] = \{\frac{1}{k}a_i^k\}.
\]
This suggestion is justified by the explicit construction given in [1], [2]. Then, expanding (1) into the powers of \( a_1 \) and taking the first term, we get

\[
(T_2 \Phi_x)(T_3 \Delta_2 \Phi) \Delta_3 \Phi = (T_3 \Phi_x)(T_2 \Delta_3 \Phi) \Delta_2 \Phi.
\]  
(9)

The next step is to expand this equation in \( a_2 \). The first nonvanishing term arises at the second order in \( a_2 \). After simple transformations we get

\[
\frac{\partial}{\partial x} \ln \left( \frac{1}{\Phi_x} \frac{\Delta \Phi}{a} \right) = \frac{1}{2} \Delta \left( \frac{\Phi_y + \Phi_{xx}}{\Phi_x} \right),
\]  
(10)

where we are left with one discrete variable and parameter \( a \), so we omit the index 3. And finally, expanding in parameter and taking the first nonvanishing term, which corresponds to the third order in \( a_3 \) in the initial equation (1), we obtain

\[
\Phi_t = \frac{1}{4} \Phi_{xxx} + \frac{3}{8} \frac{\Phi_y^2 - \Phi_{xx}^2}{\Phi_x} + \frac{3}{4} \Phi_x W_y, \quad W_x = \frac{\Phi_y}{\Phi_x}.
\]  
(11)

This equation first arose in Painleve analysis of the KP equation as a singular manifold equation [5]. The higher terms of expansion of equation (1) will lead to the hierarchy of singular manifold equations.

So we have the following objects:
1. Lattice equation (1)
2. Equation with two discrete and one continuous variables (9)
3. Equation with one discrete and two continuous variables (10)
4. PDE (11) with three continuous variables

(we would like to emphasize that for the cases 2, 3, 4 we have an infinite hierarchy of equations). All these equations are the symmetries of each other by construction, and can be interpreted in different ways (continuous symmetries of discrete equations, discrete symmetries of continuous equations and mixed cases). The interpretation depends on the choice of the basic equation (i.e. in some sense on the point of reference).

A standard way is to take continuous equation (11) as a basic system. Then the interpretation of the other objects is the following:

3. Equation (11) defines a Bäcklund transformation for the equation (1)
2. Equation (9) is a superposition principle for two Bäcklund transformations
1. Lattice equation (1) provides an algebraic superposition principle for three Bäcklund transformations.
Derivation of equations (1), (2) from the equations for the CBA function

In [1], [2] the following equations were derived for the scalar KP Cauchy-Baker-Akhiezer (CBA) function

\[ \Delta_i \Psi(\lambda, \mu, \mathbf{x}) = a_i \tilde{\psi}(\mu, \mathbf{x}) T_i \psi(\lambda, \mathbf{x}), \]  

(12)

where \( \Psi(\lambda, \mu, \mathbf{x}) \) is a CBA function,

\[ \Psi(\lambda, \mu, \mathbf{x}) = g(\lambda, \mathbf{x}) \chi(\lambda, \mu, \mathbf{x}) g^{-1}(\mu, \mathbf{x}), \]

\[ g(\lambda, \mathbf{x}) = \exp \left( \sum_{n=1}^{\infty} x_n \lambda^{-n} \right), \]

\( \chi(\lambda, \mu, \mathbf{x}) \) is defined in the unit disc in \( \lambda, \mu, \) is analytic in these variables for \( \lambda \neq \mu \) and for \( \lambda \to \mu \) behaves like \((\lambda - \mu)^{-1}\); \( \psi(\lambda, \mathbf{x}) \) and \( \tilde{\psi}(\mu, \mathbf{x}) \) are respectively Baker-Akhiezer (BA) and dual (adjoint) BA functions

\[ \psi(\lambda, \mathbf{x}) = g(\lambda, \mathbf{x}) \chi(\lambda, 0, \mathbf{x}) \]

\[ \tilde{\psi}(\mu, \mathbf{x}) = \chi(0, \mu, \mathbf{x}) g^{-1}(\mu, \mathbf{x}), \]

the shifts \( T_i \) are realized in the form (8).

For the multicomponent KP case CBA and BA functions are \( N \times N \) matrix-valued functions, we have \( N \) infinite sets of KP variables

\[ \mathbf{x} = (x_1, \ldots, x_N), \]

and the action of three shift operators \( T_i \) is defined as

\[ T_i : \mathbf{x} \to \mathbf{x} + [\mathbf{a}_i], \]

\[ [\mathbf{a}_i] = ([a_{i(1)}], \ldots, [a_{i(N)}]), \]

\( \mathbf{a}_i \) are some constant \( N \)-dimensional vectors. The equation (12) for the multicomponent case reads

\[ \Delta_i \Psi(\lambda, \mu, \mathbf{x}) = \tilde{\psi}(\mu, \mathbf{x}) A_i T_i \psi(\lambda, \mathbf{x}), \]  

(13)

\[ A_i = \text{diag}(a_{i(1)}, \ldots, a_{i(N)}). \]
Integrating the equation (12) with two arbitrary functions $\rho(\lambda), \tilde{\rho}(\mu)$ over the unit circle in $\lambda, \mu$, we get
\[ \Delta_i \Phi(x) = a_i \tilde{f}(x) T_i f(x). \] (14)

Using this equation, it is easy to check that $\Phi$ satisfies the equation (1).
And, taking cross-differences of equations (14) and combining them, one gets equations (6), (7) in a symmetric algebraic manner, without reference to compatibility conditions. The multicomponent case (2) is analogous.

An alternative way to derive equations for $\Phi, f, \tilde{f}$ from (14) is just to reproduce equations (4), (5) (which is straightforward), and then use compatibility conditions.

Taking the weight functions on the unit circle in the form $\rho(\lambda) = \delta(\lambda_0 - \lambda)$, $\tilde{\rho}(\mu) = \delta(\mu_0 - \mu)$, one arrives to the conclusion that the CBA function itself satisfies the equation (1) (or (2) in the matrix case). Then, using the analytic properties of the CBA function and going to the limit $\lambda \to \mu$, one obtains for the function $u(x) = \Psi_r(0, 0, x)$ the discrete version of the KP hierarchy in the form of algebraic superposition principle for three Bäcklund transformations
\[ \sum_{(ijk)} \epsilon_{ijk} A_k T_k (\Delta_i u - u A_i T_i u) A_j = 0, \] (15)
where summation goes over different permutations of indices. Similar equation can be found in [4].

It is also possible to reproduce in this way equations (3), (4) taking only one weight function as $\delta$-function and going to the limit $\lambda \to 0$ or $\mu \to 0$.

Thus the equations (1), (2) encode all the three levels of generalized KP hierarchy: hierarchy of singular manifold equations, modified hierarchy and the basic KP hierarchy.

**Equation (1) and quadrilateral lattices**

A more general version of equation (12) reads (we need only scalar case here)
\[ \Psi(\lambda, \mu, x + [a]) - \Psi(\lambda, \mu, x + [b]) = (a - b) \tilde{\psi}_b(\mu, x + [b]) \psi_b(\lambda, x + [a]), \] (16)
where $\psi_b(\lambda, x), \tilde{\psi}_b(\mu, x)$ are respectively BA and dual BA function associated with the point $b$ rather then the point zero for equation (12),

\[ \psi_b(\lambda, x) = \chi(\lambda, b, x) g(\lambda, x), \]
\[
\tilde{\varphi}_b(\mu, x) = g^{-1}(\mu, x)\chi(b, \mu, x).
\]

For \(a = 0\) the equation (16) gives
\[
\Psi(\lambda, \mu, x + [b]) - \Psi(\lambda, \mu, x) = b\tilde{\varphi}_b(\mu, x + [b])\varphi_b(\lambda, x).
\]

Introducing lattice variables and taking \(b\) equal to \(a_i\), we get
\[
\Delta_i\Psi(\lambda, \mu, n) = a_i\varphi_{n_i}(\lambda, n)T_i\tilde{\varphi}_{n_i}(\mu, n).
\]

Then, recalling analytic properties of BA and CBA functions, removing the common factors \(g_j(\lambda, n_j), g_k(\mu, n_k), k \neq i \neq j \neq k\) from the equation (18),
\[
g_i(\lambda, n_i) = g(\lambda, n_i[a_i]) = \left(\frac{\lambda}{\lambda - a_i}\right)^{n_i},
\]
and taking it at the points \(\lambda = a_j, \mu = a_k\), we get a discrete Darboux equation in terms of rotation coefficients \([6]\)
\[
\Delta_i\beta_{jk} = \beta_{ji}T_i\beta_{ik}
\]
where
\[
\beta_{ij} = \sqrt{a_i a_j}\frac{g_j(a_i)g_k(a_i)}{g_i(a_j)g_k(a_j)}\chi(a_i, a_j)
\]

As it was recently discovered \([7]\), the equation (19) describes a system of planar quadrilateral lattices.

We have derived the equation (19) in some very special setting, i.e. in the context of the scalar KP hierarchy, so we could expect to have some additional constrains. Indeed, doing some algebra, from (16) one also gets the following relations
\[
\phi_{ij}\phi_{ji} = 1, \\
\phi_{ij}\phi_{jk}\phi_{ki} = \phi_{ik}\phi_{kj}\phi_{ji} = 1, \\
\phi_{ij} = T_i\beta_{ij}.
\]

These constraints from the first sight look a bit mysterious, and one wonders how it is possible to satisfy them. But recalling the expression for the \(\chi(a_i, a_j, n)\) in terms of the \(\tau\)-function \([1]\)
\[
\chi(a_i, a_j, n) = \frac{1}{a_i - a_j} T_j T_i^{-1} \frac{\tau(n)}{\tau(n)},
\]
for $\phi_{ij}$ one gets

$$\phi_{ij} = \sqrt{\frac{a_i g_j(a_i) g_k(a_i)}{a_j g_i(a_j) g_k(a_j)}} T_j T_k \tau(n).$$

It is easy to check that all the constraints are satisfied by the $\tau$-functional substitution. Moreover, if we use this substitution for Darboux equations (19), what we get is just a standard Hirota bilinear difference equation

$$a_j(a_k - a_i)(T_i T_k \tau) T_j \tau + a_k(a_i - a_j)(T_i T_j \tau) T_k \tau + a_i(a_j - a_k)(T_j T_k \tau) T_i \tau = 0.$$

**Remark 1** The existence of constraints (20) is not limited by the case of one-component KP hierarchy and discrete times labeled by the continuous parameter $(a_i)$; in fact what we really need to get these constraints is common zero of the functions $g_i(\lambda)$ corresponding to discrete shifts $T_i$ (see [4] for more details). In the case when all three discrete variables are essentially discrete, corresponding to four-component KP hierarchy (there are three independent essentially discrete variables in the four-component case) the $\tau$-functional substitution resolving the constraints looks like

$$\beta_{ij}(n) = \frac{T_j T_i^{-1} \tau(n)}{\tau(n)},$$

and using it, we get from (19) an addition formula for the four-component KP case containing only essentially discrete shifts

$$(T_i T_k \tau) T_j \tau + (T_i T_j \tau) T_k \tau + (T_j T_k \tau) T_i \tau = 0.$$  

Intermediate cases (i.e. when some of the discrete variables are essentially discrete, while others contain continuous parameter) are also possible.

**Remark 2** It is interesting to note that the set of relations (20) arises also in projective geometry in the description of coordinate systems in the incidence space ([8], p. 260-262).

There is an important question what kind of system of planar quadrilateral lattices are described by this special case (reduction) of the discrete Darboux system (corresponding to the one-component KP case). To understand it, we should recover that the function $\Phi$ satisfying the equation (11) is connected with the radius-vector of the related system of lattices. More explicitly,

$$r(n) = (\Phi_1, \Phi_2, \Phi_3),$$
where $\Phi_1, \Phi_2, \Phi_3$ are solutions to (1) obtained from the same CBA function integrated with different weight functions.

So geometrical characterization of this special system of planar quadrilateral discrete surfaces connected with the one-component KP hierarchy is the following: it is the three-dimensional lattice built of planar quadrilaterals such that the projection of the radius-vector to each coordinate axis satisfies equation (1).

The case of equations (19) + constrains (20) corresponding to the multi-component KP hierarchy will lead us to the matrix equation (2), where the matrix contains several radius-vectors as columns, and we do not have clear geometric interpretation for this case yet.

There is another geometric interpretation for equation (1) if the function $\Phi$ is complex-valued. Then this equation defines two conditions for the system of hexagons in the complex plane with the vertices

$$(T_1 \Phi, T_1 T_3 \Phi, T_3 T_2 \Phi, T_2 \Phi, T_1 T_2 \Phi).$$

These hexagons are not necessarily convex and may even have self-intersections. The first condition is that the sum of the angles at the vertices $T_1 \Phi, T_3 \Phi, T_2 \Phi$ (and also at the vertices $T_1 T_3 \Phi, T_3 T_2 \Phi, T_1 T_2 \Phi$) is $2\pi n$. The second is that

$$|T_1 \Phi, T_1 T_3 \Phi||T_3 T_2 \Phi||T_2 \Phi, T_1 T_2 \Phi| = |T_1 T_3 \Phi, T_3 \Phi||T_3 T_2 \Phi, T_2 \Phi||T_1 T_2 \Phi T_1 \Phi|.$$

Similar interpretation also exists for equation (3).

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