QUASICONFORMAL HARMONIC MAPPINGS BETWEEN UNIT BALL AND SPATIAL DOMAIN WITH $C^{1,\alpha}$ BOUNDARY

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ABSTRACT. We prove the following. If $f$ is a harmonic quasiconformal mapping between the unit ball in $\mathbb{R}^n$ and a spatial domain with $C^{1,\alpha}$ boundary, then $f$ is Lipschitz continuous in $B$. This generalizes some known results for $n = 2$ and improves some others in higher dimensional case.

1. Introduction

For $n > 1$, let $\mathbb{R}^n$ be the standard Euclidean space with the norm $|x| = (x_1^2 + \ldots + x_n^2)^{\frac{1}{2}}$, where $x = (x_1, \ldots, x_n)$. We denote the unit ball $\{x \in \mathbb{R}^n: |x| < 1\}$ by $B$, and its boundary, the unit sphere $\{x \in \mathbb{R}^n: |x| = 1\}$ by $S$.

Let $U \subset \mathbb{R}^n$ be a domain. We say $f = (f_1, \ldots, f_n) : U \to \mathbb{R}^n$ is a harmonic mapping if the functions $f_j$ are harmonic real mappings, i.e. satisfy the $n$-dimensional Laplace equation

$$\Delta u = \sum_{i=1}^{n} D_i f_j = 0.$$ 

Let

$$P(x, \xi) = \frac{1 - |x|^2}{|x - \xi|^n}$$

be the Poisson kernel for $B$, where $x \in B, \xi \in S$, and

$$P[u](x) = \int_{S} P(x, \xi) u(\xi) d\sigma(\xi)$$

the Poisson integral of continuous function $u$ on $S$, where $\sigma$ denotes the normalized surface-area measure on $S$. Then $P[u](x)$ is continuous on $\overline{B}$ and harmonic on $B$. Since we will focus on continuous function $u$ on $\overline{B}$, that are harmonic on $B$, then we will usually express them using the Poisson integral as

$$u = P[u]|_{S}(x).$$
A homeomorphism \( f : U \to V \), where \( U, V \) are domains in \( \mathbb{R}^n \), will be called \( K \) quasiconformal (see [32]) \((K \geq 1)\) if \( f \) is absolutely continuous on lines (i.e. absolutely continuous in almost every segment parallel to some of the coordinate axes and there exist partial derivatives which are locally \( L^n \) integrable in \( U \)) and

\[
|\nabla f(x)| \leq Kl(\nabla f(x)),
\]

for all points \( x \in U \), where

\[
l(\nabla f(x)) = \inf\{|f'(x)h| : |h| = 1\}.
\]

A function \( \Phi : U \subset \mathbb{R}^n \to \mathbb{R} \) is said to be \( \mu \)-Hölder continuous, \( \Phi \in C^\mu(U) \) if

\[
\sup_{x,y \in U, x \neq y} \frac{|\Phi(x) - \Phi(y)|}{|x - y|^{\mu}} < \infty.
\]

Similarly, one defines the class \( C^{1,\mu}(U) \) to consist of all functions \( \Phi \in C^{1}(U) \) such that \( \nabla \Phi \in C^\mu(U) \). The above two definitions extends in a natural way to the case of vector-valued mappings.

We say that \( \Omega \subset \mathbb{R}^n \) has a \( C^{1,\alpha} \) boundary if it is the image of the unit ball \( B \subset \mathbb{R}^n \) under a \( C^{1,\alpha} \) diffeomorphism up to the boundary.

Pavlović in [30] showed that harmonic quasiconformal mappings of the unit disk in \( \mathbb{R}^2 \) onto itself are bi-Lipschitz mappings. From then, several important results have been obtained regarding harmonic quasiconformal mappings in \( \mathbb{R}^2 \) and the Lipschitz continuity. The second author in [11] proved that, every quasiconformal harmonic mapping between Jordan domains with \( C^{1,\alpha} \) boundaries is Lipschitz continuous on the closure of domain.

The result in [11] was extended in [12] for Jordan domains with only Dini’s smooth boundaries. Lately, in [16] it was proved the Hölder continuity (but in general, Lipschitz continuity does not hold) of a harmonic quasiconformal mapping between two Jordan domains having only \( C^1 \) boundaries. Other important results for \( n = 2 \) with different conditions and settings can be found in [2], [5], [15], [19], [22], [28] and in their references.

For higher dimensional case there are some important results also (see e.g. [3], [20], [25], [13]). In [13] it was proven that a quasiconformal mapping of the unit ball onto a domain with \( C^2 \) smooth boundary, satisfying Poisson differential inequality, is Lipschitz continuous. This implies that harmonic quasiconformal mappings from unit ball \( B \) to \( \Omega \) with \( C^2 \) boundary are Lipschitz continuous. This was also proved by Astala and Manojlovic in [3] using a slight modification of the following statement also proved there: a harmonic \( K \)-quasiconformal mapping from \( B \) to \( B \) is Lipschitz with the Lipschitz constant depending on the value of \( K \), dimension of \( n \) and \( \text{dist}(f(0), S) \).

Our main result generalizes the result in [11] and improves the mentioned corollaries in [3] and [13]. It reads as follow.
Theorem 1.1. Let \( f : B \rightarrow \mathbb{R}^n \) be a quasiconformal harmonic (qch) mapping, \( u(B) = \Omega \), and \( \partial \Omega \in C^{1,\alpha} \). Then \( f \) is Lipschitz continuous in \( B \).

The proof of the corresponding result for 2-dimensional case in [11] uses conformal mappings, however conformal mappings in higher-dimensional setting are very rigid, and this is why we need to find another way to deal with the proof of Theorem 1.1. The initial idea lies on the following simple approach. Let \( \eta \in S \) and \( f(\eta) = q \in \partial \Omega \). We can suppose that \( q = 0 \) and the tangent plane of \( q \) at \( \partial \Omega \) is \( x_n = 0 \). This can be obtained in the following way: Using a isometry \( L \) we can postcompose \( f \) such that we get a function \( \tilde{f} \) from \( B \) to \( \Omega' \), \( \tilde{f}(\eta) = 0 \) and the tangent plane of this point on \( \partial \Omega' \) is \( x_n = 0 \). Observe that \( \tilde{f} \) is also harmonic and quasiconformal, because it is composed by a isometry. The Lipschitz continuity for function \( \tilde{f} \) would yield the proof of this property for the function \( f \) also, because the isometry preserves the distances.

The proof is given in Section 3. It uses an iteration procedure. A similar procedure has been used in [26] and in [17] for similar purpose but different setting. Before that, in next section, we give some basic preparations through Theorems 2.1

2. Auxiliary results

The next theorem is of general interest; on the other side it plays an important role in proving Theorem 1.1. Some versions of it, for \( n = 2 \), can be found in [10] and [26].

Theorem 2.1. Let \( u : \overline{B} \subset \mathbb{R}^n \rightarrow \mathbb{R} \) be a real harmonic function, \( \eta \in S \). Assume that \( |u(\xi) - u(\eta)| \leq M|\xi - \eta|^{\mu}, \forall \xi \in S \), for some \( \mu \in (0, 1) \). Then we have \( C = C(M, \mu, n) \) such that

\[
|\nabla u(x)|(1 - |x|)^{1-\mu} \leq C,
\]

where \( x = r\eta, \ r \in [0, 1) \).

Proof. Throught the proof, the constant \( C \) can change its value. Using the Poisson integral formula we have

\[
u(x) = \int_S \frac{1 - |x|^2}{(1 + |x|^2 - 2\langle \xi, x \rangle)^{\frac{n}{2}}} u(\xi) d\sigma(\xi).
\]

Observe that

\[
\nabla u(x) = \int_S Q(x, \xi) u(\xi) d\sigma(\xi), \tag{2.1}
\]
where

\[
Q(x, \xi) = \frac{(-2x)(1 + |x|^2 - 2\langle \xi, x \rangle) - n(1 - |x|^2)(1 + |x|^2 - 2\langle \xi, x \rangle)}{(1 + |x|^2 - 2\langle \xi, x \rangle)^n} \left(1 - |x|^2\right)
\]

\[
= \frac{(-2x)(1 + |x|^2 - 2\langle \xi, x \rangle) - n(1 - |x|^2)(x - \xi)}{(1 + |x|^2 - 2\langle \xi, x \rangle)^{n+1}} \cdot \frac{1}{(1 + |x|^2 - 2\langle \xi, x \rangle)^{n/2}}.
\]

Let \(h \in \mathbb{R}^n\) be an arbitrary vector. Then

\[
\langle \nabla u(x), h \rangle = \int_S \langle Q(x, \xi), h \rangle u(\xi) d\sigma(\xi).
\]

As

\[
1 = \int_S \frac{1 - |x|^2}{(1 + |x|^2 - 2\langle \xi, x \rangle)^{n/2}} d\sigma(\xi),
\]

we get

\[
0 = \int_S Q(x, \xi) u(\eta) d\sigma(\xi),
\]

which, together with (2.3), gives us

\[
\langle \nabla u(x), h \rangle = \int_S \langle Q(x, \xi), h \rangle [u(\xi) - u(\eta)] d\sigma(\xi).
\]

On the other side

\[
\left| \frac{-2(\langle \xi, x \rangle)(1 + |x|^2 - 2\langle \xi, x \rangle) - n(1 - |x|^2)(x - \xi, h)}{(1 + |x|^2 - 2\langle \xi, x \rangle)} \right| \leq 2|x||h| + n \frac{(1 - |x|^2)|x - \xi||h|}{|x - \xi|^2} \leq 2|x||h| + 2n|h| \frac{1 - |x|}{|x - \xi|} \leq (2 + 2n)|h|.
\]

In the last inequality it is used the fact that \(1 - |x| \leq |x - \xi|\), which is obviously true from the geometrical point of view, but is also equivalent to \(\langle \xi, x \rangle \leq |x|\) (Cauchy-Schwartz inequality).

From (2.2), (2.6), (2.7) we get

\[
|\langle \nabla u(x), h \rangle| \leq (2n + 2)|h| \int_S \frac{|u(\xi) - u(\eta)|}{(1 + |x|^2 - 2\langle \xi, x \rangle)^{n/2}} d\sigma(\xi)
\]
As $h$ was taken arbitrary, then

$$|\nabla u(x)| \leq (2n + 2) \int_S \frac{|u(\xi) - u(\eta)|}{(1 + |x|^2 - 2\langle \xi, x \rangle)^{\frac{1}{2}}} d\sigma(\xi),$$

which is equivalent to

$$|\nabla u(r\eta)| \leq (2n + 2) \int_S \frac{|u(\xi) - u(\eta)|}{(1 + r^2 - 2r(\xi, \eta))^\frac{1}{2}} d\sigma(\xi),$$

$$= (2n + 2) \int_S \frac{|u(\xi) - u(\eta)|}{((1 - r)^2 + r^2|\xi - \eta|^2)^{\frac{1}{2}}} d\sigma(\xi),$$

where $x = r\eta$, $r = |x| \in [0, 1)$.

Using the condition of the theorem we get

$$|\nabla u(r\eta)| \leq M(2n + 2) \int_S \frac{|\xi - \eta|^\mu}{((1 - r)^2 + r^2|\xi - \eta|^2)^{\frac{1}{2}}} d\sigma(\xi).$$

1st case: $|x| = r \geq \frac{1}{2}$.

$$|\nabla u(r\eta)| \leq C \int_S \frac{|\xi - \eta|^\mu}{((1 - r)^2 + \frac{1}{2}|\xi - \eta|^2)^{\frac{1}{2}}} d\sigma(\xi).$$

Because of the symmetry, it is enough to show the required inequality for $\eta = (1, 0, \ldots, 0)$. We use spherical coordinates:

$$y_1 = \cos \phi_1$$
$$y_2 = \sin \phi_1 \cos \phi_2$$
$$\vdots$$
$$y_{n-1} = \sin \phi_1 \sin \phi_2 \cdots \sin \phi_{n-2} \cos \phi_{n-1}$$
$$y_n = \sin \phi_1 \sin \phi_2 \cdots \sin \phi_{n-2} \sin \phi_{n-1},$$

where $\phi_1, \ldots, \phi_{n-2} \in [0, \pi]$ and $\phi_{n-1} \in [0, 2\pi)$ and the area element is given by

$$dS V = \sin^{n-2} \phi_1 \sin^{n-3} \phi_2 \cdots \sin^2 \phi_{n-3} \sin \phi_{n-2}.$$

Elementary calculations show that $|\xi - \eta| = 2\sin \frac{\phi_1}{2}$, where $\xi = (y_1, \ldots, y_n)$. So we have

$$|\nabla u(r\eta)| \leq C \int_0^{2\pi} \int_0^{\pi} \int_0^{\pi} \frac{(2 \sin \frac{\phi_1}{2})^\mu}{((1 - r)^2 + 2 \sin^2 \frac{\phi_1}{2})^{\frac{1}{2}}} dS V d\phi_1 \cdots d\phi_{n-1}$$

$$= C \int_0^{2\pi} d\phi_{n-1} \int_0^\pi \sin \phi_{n-2} d\phi_{n-2} \cdots \int_0^\pi \frac{(2 \sin \frac{\phi_1}{2})^\mu}{((1 - r)^2 + 2 \sin^2 \frac{\phi_1}{2})^{\frac{1}{2}}} \sin^{n-2} \phi_1 d\phi_1.$$
As the first (left to right) $n - 2$ integrals are finite we have
\[
|\nabla u(r\eta)| \leq C \int_0^{\pi} \frac{(2 \sin \frac{\phi_1}{2})^\mu (2 \sin \frac{\phi_1}{2} \cos \frac{\phi_1}{2})^{n-2}}{(1 - r)^2 + 2 \sin^2(\frac{\phi_1}{2})((1 - r)^2 + 2 \sin^2(\frac{\phi_1}{2}))^{\frac{n-2}{2}}} d\phi_1.
\]
It is easily seen that
\[
\frac{(2 \sin \frac{\phi_1}{2} \cos \frac{\phi_1}{2})^{n-2}}{((1 - r)^2 + 2 \sin^2(\frac{\phi_1}{2}))^{\frac{n-2}{2}}} \leq 2^{\frac{n-2}{2}},
\]
so
\[
|\nabla u(r\eta)| \leq C \int_0^{\pi} \frac{\sin \frac{\phi_1}{2}^\mu}{(1 - r)^2 + 2 \sin^2(\frac{\phi_1}{2})} d\phi_1
\]
(2.13)
\[
= C \int_0^1 \frac{t^\mu}{(1 - r)^2 + 2t^2} \sqrt{1 - t^2} dt \leq C \int_0^1 \frac{t^\mu}{(1 - r)^2 + t^2} dt.
\]
Further,
\[
\int_0^1 \frac{t^\mu}{(1 - r)^2 + t^2} dt = (1 - r)^{\mu-1} \int_0^{1/r} \frac{s^\mu}{1 + s^2} ds \leq (1 - r)^{\mu-1} \int_0^\infty \frac{s^\mu}{1 + s^2} ds.
\]
As the last integral converges we finally have
\[
|\nabla u(r\eta)| (1 - r)^{1-\mu} \leq C, \quad r \in \left[\frac{1}{2}, 1\right),
\]
where $C$ depends on $M, \mu$ and $n$ only.

**2nd case** $r = |x| < \frac{1}{2}$

As
\[
\frac{|\xi - \eta|^{\mu(1 - r)^{1-\mu}}}{((1 - r)^2 + r|\xi - \eta|^2)^{\frac{\mu}{2}}} < \frac{2^{\mu} 2^{1-\mu}}{(\frac{1}{2})^n} = 2^{n+1},
\]
using (2.11) we get
\[
|\nabla u(r\eta)| (1 - r)^{1-\mu} \leq M(2n + 2)2^{n+1}.
\]

We conclude that the inequality is true for all $r \in (0, 1)$, with the final $C$ being the larger of the obtained constants on the RHS of (2.14) and (2.16).

The idea of the proof in section 3 will be based on obtaining locally the $C^\mu$ condition of $f$ on the unit sphere for $\mu < 1$, by increasing $\mu$. In relation to a fixed point $\eta = S$ this will, in one moment, give us a similar inequality as the one from Theorem 2.1 but for $\mu > 1$. So, on this step, we need a
different version of the previous statement which is given in the following theorem. However, the proof of it is very similar to the proof of the previous one.

**Theorem 2.2.** Let $u : \overline{B} \subset \mathbb{R}^n \rightarrow \mathbb{R}$, be a harmonic function, $\eta \in S$. Assume that $|u(\xi) - u(\eta)| \leq M|\xi - \eta|^\mu$, $\forall \xi \in S$, for some $\mu > 1$. Then we have $C = C(M, \mu, n)$ such that

$$|\nabla u(r\eta)| \leq C,$$

for every $r \in [0, 1)$.

**Proof.** The proof of the theorem for $r \in [\frac{1}{2}, 1)$ is identical to the previous theorem until (2.13).

$$\int_0^1 \frac{t^\mu}{(1 - r)^2 + t^2} dt \leq \int_0^1 t^{\mu-2} dt = \frac{1}{\mu - 1}$$

shows that the inequality is valid.

For $r \in [0, \frac{1}{2})$, similar to (2.15) we see that

$$\frac{|\xi - \eta|^\mu}{((1 - r)^2 + r|\xi - \eta|^2)^{\frac{\mu}{2}}}$$

is bounded, so therefore again from (2.11) we have our inequality. \qed

The next celebrated theorem will also be used. The proof can be found in [8].

**Theorem 2.3. (Mori’s theorem)** Let $g$ be a $K$-quasiconformal mapping of $B$ onto $B$, $n \geq 2$, with $g(0) = 0$. Then

$$|g(x) - g(y)| \leq M(n, K)|x - y|^{\beta},$$

for all $x, y \in B$, where $\beta = K^{-\frac{n}{2}}$.

We collect now the following useful result. The proof can be found in [29]. We will formulate it in the form which corresponds to our notation and use.

**Theorem 2.4.** Let $u$ be a real harmonic function on $\overline{B}$ and $\mu \in (0, 1)$ such that

(2.17) $||u(r\eta)| - |u(\eta)|| \leq C(1 - r)^{\mu}$, $\forall r \in [0, 1), \eta \in S,$

where $C$ is independent of $r$ and $\eta$, then $u$ is $\mu$-Hölder continuous in $\overline{B}$, i.e.:

$$|u(x) - u(y)| \leq M|x - y|^\mu,$$

for all $x, y \in \overline{B}$.

Using the previous theorem we can easily prove the following lemma.
Corollary 2.5. Let \( u \) be a real harmonic function on \( \overline{B} \) and \( \mu \in (0, 1) \) such that
\[
|\nabla u(r\eta)| \leq C(1 - r)^{\mu - 1}, \quad \forall r \in (0, 1), \eta \in S,
\]
where \( C \) does not depend on \( r \) and \( \eta \), then \( u \) is \( \mu \)-Hölder continuous in \( \overline{B} \).

Proof. In order to prove this lemma, based on the Theorem 2.4 and the relation (2.17), it is sufficient to prove
\[
(2.18) \quad |u(r\eta) - u(\eta)| \leq C(1 - r)^{\mu}, \quad \forall r \in [0, 1), \eta \in S.
\]

We have
\[
(2.19) \quad u(r\eta) - u(\eta) = \int_{\gamma_r} D_1 u\,dx_1 + \ldots + D_n u\,dx_n,
\]
where \( \gamma_r \) is the radial segment with endpoints \( r\eta \) and \( \eta \).

Therefore, we have
\[
|u(r\eta) - u(\eta)| \leq \int_{r}^{1} |\langle \nabla u(t\eta), \eta \rangle|\,dt
\]
\[
\leq C \int_{r}^{1} (1 - t)^{\mu - 1}\,dt
\]
\[
\leq C \frac{(1 - r)^{\mu}}{\mu}.
\]

\( \Box \)

3. Proof of the main result - Theorem 1.1

Proof. First, let we prove the Hölder continuity of \( f \). Indeed, let \( G \) be a quasiconformal diffeomeorphism (recall that \( \Omega \) has a \( C^{1,\alpha} \) boundary) from \( B^n \) to \( \Omega \) which is Lipschitz continuous mapping up to the boundary, such that \( G(0) = f(0) \). Then the mapping \( g = G^{-1} \circ f \) is a \( K' \) quasiconformal mapping (as a composition of two quasiconformal mappings) of \( B \) onto \( B \), where \( g(0) = 0 \). According to Mori’s theorem 2.3, there exist a constant \( M_1(n, K') \) such that
\[
|g(x) - g(y)| \leq M_1(n, K') |x - y|^{K'/1-n},
\]
for all \( x, y \in B^n \).

As \( f = G \circ g \), then \( f \) satisfies a similar inequality, being a composition of Lipschitz and Hölder continuous functions:
\[
(3.1) \quad |f(x) - f(y)| \leq C_1 |x - y|^\beta,
\]
for all \( x, y \in \overline{B}^n \), where \( \beta \in (0, 1) \), and the constant \( C_1 \) depends on \( M_1 \) and the Lipschitz constant of \( G \).
In view of the remark after the formulation of Theorem 3.1 there exists a neighbourhood $\mathcal{O}$ of the origin in $\mathbb{R}^{n-1}$ which is the projection of $\partial \Omega \cap B(0, \rho)$ in $\mathbb{R}^{n-1}$ and a $C^{1,\alpha}$ function $\Phi : \mathcal{O} \to \mathbb{R}$ such that $\partial \Omega \cap B(0, \rho)$ can be expressed as the graphic of the following function:

\begin{equation}
\mathcal{O} \ni (\zeta_1, \ldots, \zeta_{n-1}) \mapsto (\zeta_1, \ldots, \zeta_{n-1}, \Phi(\zeta_1, \ldots, \zeta_{n-1})).
\end{equation}

The function $\Phi$ has the properties $\Phi(0, \ldots, 0) = 0$ and $D_j \Phi(0, \ldots, 0) = 0$, for all $j \in \{1, 2, \ldots, n-1\}$, and

\begin{equation}
|\nabla \Phi(\zeta) - \nabla \Phi(\omega)| \leq C_2|\zeta - \omega|^{\alpha}.
\end{equation}

The constant $C_2$ is the same for all points $q \in \partial \Omega$, because of the $C^{1,\alpha}$ condition of $\partial \Omega$.

Also,

\begin{equation}
|\Phi(\zeta) - \Phi(\omega)| = |\langle \nabla \Phi(c), \zeta - \omega \rangle| \leq |\nabla \Phi(c)||\zeta - \omega|,
\end{equation}

where $c$ belongs to the segment $[\zeta, \omega]$.

Using (3.3) we get

\begin{equation}
|\nabla \Phi(c)| \leq |\nabla \Phi(\zeta)| + |\nabla \Phi(\omega)|
\leq C_2 (|\zeta|^{\alpha} + |c - \zeta|^{\alpha}) \leq C_2 (|\zeta|^{\alpha} + |\zeta - \omega|^{\alpha}),
\end{equation}

\begin{equation}
|\nabla \Phi(c)| \leq |\nabla \Phi(c) - \nabla \Phi(\omega)|
\leq C_2 (|\omega|^{\alpha} + |c - \omega|^{\alpha}) \leq C_2 (|\omega|^{\alpha} + |\zeta - \omega|^{\alpha}),
\end{equation}

which yields to

\begin{equation}
|\nabla \Phi(c)| \leq C_2 \min\{|\zeta|^{\alpha}, |\omega|^{\alpha}\} + |\zeta - \omega|^{\alpha}.
\end{equation}

Therefore, from (3.3) we have:

\begin{equation}
|\Phi(\zeta) - \Phi(\omega)| \leq C_2|\zeta - \omega|\min\{|\zeta|^{\alpha}, |\omega|^{\alpha}\} + |\zeta - \omega|^{\alpha},
\end{equation}

for all $\zeta, \omega$ in $\mathcal{O}$.

Let $F = (F_1, \ldots, F_n) = f|_S$ or $P[F] = f$. Notice that $F$ is also $C^\beta$ in $S$.

We will use the notation $\tilde{F}(\xi) = (F_1(\xi), \ldots, F_{n-1}(\xi))$. $\tilde{F}$, as $F$, also satisfies (3.1). In view of (3.2) we have that in a small neighbourhood of $\eta$ in $S$, $F_n$ is of the form

\[ F_n(\xi) = \Phi(F_1(\xi), \ldots, F_{n-1}(\xi)). \]

We may also assume that this neighbourhood of $\eta$ is of the form $V(\eta) = B(\eta, \delta) \cap S$, where $\delta$ is small enough positive constant for all $q \in \partial \Omega$. Indeed, let $\tilde{U}(q) = B(q, r_q) \cap \partial \Omega$ be the neighbourhood of $q$ in $\partial \Omega$ such that after the isometry $L_q$ (the one that sends $q$ to 0 and which makes the plane $x_n = 0$ the tangent plane of $\partial \Omega$ at point $0$), $L_q(\tilde{U}(q))$ is the neighbourhood of 0 which is the graphic of a function as in (3.2). Furthermore, we can choose $r_q$ small enough, such that for every point $p \in \tilde{U}(q)$, the image of $\tilde{U}(q)$ under the respective isometry $L_p$ is a graphic of a function.
Observe now $U(q) = B(q, \frac{r_1}{2}) \cap \partial \Omega$. The collection $\{U(q)\}_{q \in \partial \Omega}$ is a cover of $\partial \Omega$. As $\partial \Omega$ is compact, there exists a finite subcollection $\{U(q_k)\}_{k=1}^m$ which covers $\partial \Omega$. Let $\rho = \min\{\frac{r_1}{2}, \ldots, \frac{r_m}{2}\}$. As $F$ is continuous on a compact, then there exists a $\delta > 0$ such that if $|\xi_1 - \xi_2| < \delta$, $\xi_1, \xi_2 \in S$, then $|F(\xi_1) - F(\xi_2)| < \frac{\rho}{2}$.

This ensures that the image of every $V(\eta) = B(\eta, \delta) \cap S$ under $F$ is contained in a $B(q_j, r_{q_j}) \cap \partial \Omega = \tilde{U}(q_j)$, and further, after the mentioned isometry is done, this image is the graphic of a function as in (3.2).

We get back to our fixed $\eta$, such that $f(\eta) = 0$. Now

$$|F_n(\xi) - F_n(\eta)| = |\Phi(\tilde{F}(\xi)) - \Phi(0)|$$

(3.8)

$$\leq C_2|\tilde{F}(\xi)||\min\{|\tilde{F}(\xi)|^\alpha, 0\} + |\tilde{F}(\xi) - 0|^\alpha|$$

$$= C_2|\tilde{F}(\xi)|^{1+\alpha} \leq C_1^{1+\alpha} C_2|\xi - \eta|^{(1+\alpha)\beta},$$

for all $\xi \in V(\eta)$. The function $F_n$ is bounded, because $F = f|_S$ is bounded ($|F(\xi)| \leq \tilde{M}$, for all $\xi \in S$), so if $\xi \in S \setminus V(\eta)$ then

$$|F_n(\xi) - F_n(\eta)| \leq 2\tilde{M} \leq \frac{2m}{\delta(1+\alpha)^\beta} |\xi - \eta|^{(1+\alpha)\beta}. $$

(3.9)

Taking $M = \max\{C_1^{1+\alpha} C_2, \frac{2\tilde{M}}{\delta(1+\alpha)^\beta}\}$ we get

$$|F_n(\xi) - F_n(\eta)| \leq M|\xi - \eta|^{(1+\alpha)\beta},$$

(3.10)

for all $\xi \in S$.

Now, from Theorem 2.1 we have

$$|\nabla f_n(r\eta)| \leq C(1 - r)^{(1+\alpha)\beta - 1}, \quad \forall r \in [0, 1).$$

As $f$ is quasiconformal mapping then

$$\max_{|h_1|=1} \frac{|f'(x)|h_1|}{\min_{|h_2|=1}} \leq K < \infty, \quad \forall x \in B.$$

Taking, $h_1 = e_j$ and $h_2 = e_n$, for $x = r\eta$ we have

$$|\nabla f_j(r\eta)| \leq K|\nabla f_n(r\eta)| \leq K \cdot C(1 - r)^{(1+\alpha)\beta - 1},$$

for all $j \in \{1, \ldots, n - 1\}$. This implies

$$|\nabla f_j(r\eta)| \leq C(1 - r)^{(1+\alpha)\beta - 1},$$

(3.11)

where $C$ is a new global constant for all $j \in \{1, \ldots, n\}$, and all $r \in [0, 1)$.

We want to prove (3.11) in $B$. Let $\eta_1 \neq \eta$ be an arbitrary point on $S$ and $f(\eta_1) = q_1$. Let $L_{q_1}$ be the isometry that maps $q_1$ to 0, with $x_n = 0$ the tangent plane of $L_{q_1}(\partial \Omega)$ at $L_{q_1}(q_1) = 0$. 
Let $L_q \circ f = \tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_n)$. Then $\tilde{f}$ has all the properties of the function $f$ with $\eta_1$ in place of $\eta$: at $\tilde{f}(\eta_1) = 0$ the tangent plane of the surface $L_q(\partial \Omega)$ is $x_n = 0$ and $\tilde{f}(\eta_1)$ has a neighbourhood in $L_q(\partial \Omega)$ which can be presented as a part of a graphic of the form (3.2). Using the same procedure, we conclude that

$$|\tilde{f}_j (r \eta_1)| \leq C (1 - r)^{(1+\alpha)\beta - 1},$$

for all $j \in \{1, \ldots, n\}$, and all $r \in [0, 1)$. Constant $C$ is universal and it does not depend on $\eta_1$, because $\delta$ and $M$ are independent of the choice of $\eta \in S$. As $f = L_q^{-1} \tilde{f}$, ($L_q^{-1}$ is also an isometry) we get

$$f_j (\xi) = b_j + \sum_{i=1}^{n} a_{i,j} \tilde{f}_j (\xi),$$

$j \in \{1, \ldots, n\}$, so

$$\nabla f_j (\xi) = \sum_{i=1}^{n} a_{i,j} \nabla \tilde{f}_j (\xi),\quad (3.12)$$

where $\{a_{i,j}\}$ is an orthogonal matrix. From (3.12) we have:

$$|\nabla f_j (\xi)| \leq \sum_{i=1}^{n} |a_{i,j}| |\nabla \tilde{f}_j (\xi)|$$

$$\leq \left( \sum_{i=1}^{n} |\nabla \tilde{f}_j (\xi)|^2 \right)^{\frac{1}{2}}.\quad (3.13)$$

In the last inequality it is used the Cauchy-Schwartz inequality and the orthogonality of matrix $\{a_{i,j}\}_{i,j=1}^{n}$. Taking $\xi = r \eta_1$ we get

$$|\nabla f_j (r \eta_1)| \leq \sqrt{n} C (1 - r)^{(1+\alpha)\beta - 1}.$$  

As the point $\eta_1$ was arbitrary we conclude

$$|\nabla f_j (x)| \leq C (1 - r)^{(1+\alpha)\beta - 1}, \quad r = |x|,$$

for all $x \in B$.

From Lemma 2.5 it follows that $f_j \in C^{(1+\alpha)\beta} (\overline{B})$, for all $j \in \{1, \ldots, n\}$ and so $f \in C^{(1+\alpha)\beta} (\overline{B})$.

We could have chosen $\beta < \frac{1}{2}$ (by decreasing it, if necessary) so the numbers $(1+\alpha)^k \beta \neq 1$, for every $k$. As $1 + \alpha > 1$ there exists a unique integer $k_0$ such that $(1+\alpha)^{k_0} \beta < 1$ and $(1+\alpha)^{k_0+1} \beta > 1$. Repeating the procedure, we now get that $f \in C^{(1+\alpha)^{k_0} \beta} (\overline{B}), \ldots, C^{(1+\alpha)^{k_0+1} \beta} (\overline{B})$. Similar to (3.8) it follows that

$$|F_n (\xi) - F_n (\eta)| \leq M |\xi - \eta|^{(1+\alpha)^{k_0+1} \beta}, \quad \forall \xi \in S.$$  

This time, using Theorem 2.2 we obtain

$$|\nabla f_n (r \eta)| \leq C, \quad \forall r \in [0,1).$$
Using the same order of implications, first we get the same inequality for every $f_k$ on points $rη$. Then, using the isometries, we get the inequality on every point of $B$ for a global constant $C$. This implies trivially, by mean value inequality, the Lipschitz continuity of function $f$ in $B$.

\[\square\]

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