HOW TO CONSTRUCT ALL METRIC $f$-$K$-CONTACT MANIFOLDS

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Abstract. We show that any compact metric $f$-$K$-contact, respectively $S$-manifold is obtained from a compact $K$-contact, respectively Sasakian manifold by an iteration of constructions of mapping tori, rotations, and type II deformations.

1. Introduction

The class of metric $f$-$K$-contact manifolds generalizes the class of $(K)$-contact manifolds. They are $f$-manifolds (namely smooth manifolds together with a $(1,1)$-tensor $f$ of constant rank and such that $f^3 + f = 0$, \cite{8}) endowed with a Riemannian metric, $s$ (Killing) vector fields $\xi_i$ and $s$ one-forms dual to the $\xi_i$'s, satisfying some compatibility conditions (see Section 2).

In addition to generalizing almost complex and almost contact manifolds (which we have in case the $f$-structure $f$ has maximal rank and the dimension of the manifold is even or respectively odd), $f$-manifolds appear naturally when studying the hypersurfaces of almost contact manifolds, see \cite{8} for details.

In the present paper we look at metric $f$-contact manifolds from a dual perspective: motivated by the unusual property of metric $f$-(K-)contact and $S$-manifolds that their geometric structure is inherited by mapping tori with respect to automorphisms of the structure (see \cite{4}), we ask whether a given metric $f$-$K$-contact or $S$-manifold is the mapping torus of a lower-dimensional such manifold. The main result of this paper reads:

**Theorem 1.1.** Any compact connected metric $f$-$K$-contact (resp. $S$-)manifold is obtained from a compact $K$-contact (resp. Sasakian) manifold by a finite iteration of the following operations:

1. construction of the mapping torus with respect to an automorphism
2. rotation
3. type II deformation

We will recall the mapping torus construction and introduce (anti-)rotation and type II deformation of a metric $f$-(K-)contact or $S$-manifold in Section \cite{8} As in the Sasakian setting \cite{4}, type II deformation does not

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modify the characteristic vector fields, while it changes the one-forms of the structure by the addition of other one-forms. Rotation and anti-rotation do not have an analogue in the contact setting: they are a type of deformation in which one applies an appropriate (constant) base change to the characteristic vector fields.

2. Preliminaries

Let us review the notion of a metric $f$-contact manifold. We assume that we are given linearly independent one-forms $\eta_1, \ldots, \eta_s$ on a smooth manifold $M^{2n+s}$, as well as vector fields $\xi_1, \ldots, \xi_s$ satisfying $\eta_i(\xi_j) = \delta_{ij}$. In this setting the tangent bundle $TM$ decomposes as the sum of the parallelizable subbundle given by the span of the $\xi_i$, and the intersection of the kernels of the one-forms $\eta_i$. In addition, let $f$ be a $(1,1)$-tensor on $M$ satisfying

$$f(\xi_i) = 0, \quad \text{Im } f = \bigcap_{i=1}^s \ker(\eta_i), \quad f^2|_{\text{Im } f} = -\text{id}|_{\text{Im } f}.$$

Note that $f$ is then of constant rank and satisfies $f^3 + f = 0$, i.e., it defines an $f$-structure in the sense of Yano [8]. For $s = 0$ one recovers the notion of an almost complex, and for $s = 1$ that of an almost contact manifold.

A Riemannian metric $g$ on $M$ satisfying

$$g(fX, fY) = g(X, Y) - \sum_{i=1}^s \eta_i(X)\eta_i(Y)$$

is called compatible with the $f$-structure, and in this situation one speaks of a metric $f$-manifold $(M, f, \eta_i, \xi_i, g)$. The fundamental 2-form of the metric $f$-manifold $M$ is given by

$$\omega(X, Y) = g(X, fY),$$

for $X, Y \in TM$. One calls $M$ a metric $f$-contact manifold if

$$d\eta_i = \omega$$

for all $i$.

Let $(M, f, \xi_i, \eta_i, g)$ be a metric $f$-contact manifold. We will refer to the vector fields $\xi_i$ as the characteristic vector fields of the structure, and denote by $\mathcal{F}$ the characteristic foliation, i.e., the foliation spanned by the characteristic vector fields. We recall that by [5 Equation (2.4)], the characteristic vector fields commute, i.e.,

$$[\xi_i, \xi_j] = 0. \quad (2.1)$$

By [5 Theorem 2.6], $\xi_i$ is Killing if and only if

$$\mathcal{L}_{\xi_i} f = 0. \quad (2.2)$$

We call $M$ a metric $f$-$K$-contact manifold if all characteristic vector fields are Killing vector fields.
A metric $f$-contact manifold satisfying the normality condition

$$[f, f] + 2 \sum_{\alpha=1}^{s} d\eta_\alpha \otimes \xi_\alpha = 0,$$

is called an $S$-manifold; here $[f, f]$ denotes the Nijenhuis torsion of $f$, i.e.,

$$[f, f](X, Y) = f^2[X, Y] + [fX, fY] - f[fX, Y] - f[X, fY],$$

where $X, Y$ are arbitrary vector fields on $M$. By [2, Theorem 1.1], the characteristic vector fields of an $S$-manifold are Killing, i.e., $S$-manifolds are metric $f$-$K$-contact.

3. Deformation of metric $f$-$K$-contact manifolds

In this section we describe four ways to construct new metric $f$-$K$-contact manifolds out of old ones. We explain how the well-known type II deformation generalizes from the Sasakian setting to metric $f$-$K$-contact manifolds, introduce new constructions called rotation and anti-rotation that do not exist for Sasakian manifolds, and recall the construction of the mapping torus of a metric $f$-$K$-contact manifold from [6].

We begin with rotation and anti-rotation. In these deformations, the characteristic foliation $\mathcal{F}$ and the $(1,1)$-tensor $f$ remain the same; essentially, one applies an appropriate (constant) base change to the characteristic vector fields $\xi_i$.

**Lemma 3.1.** Let $(M, f, \xi_1, \ldots, \xi_s, \eta_1, \ldots, \eta_s, g)$ be a metric $f$-$(K\cdot)$contact manifold (resp. an $S$-manifold), and $A = (a_{ij}) \in O(s)$ an orthogonal $s \times s$ matrix, such that $e_i := \sum_{j=1}^{s} a_{ij} \neq 0$ for every $i \in \{1, \ldots, s\}$. Then, the tensor fields

$$\eta'_i := \sum_{t=1}^{s} a_{ti} c_t \eta_t, \quad \xi'_i := \sum_{t=1}^{s} \frac{1}{c_t} a_{ti} \xi_t, \quad i \in \{1, \ldots, s\},$$

together with

$$g' := g - \sum_{\alpha=1}^{s} \eta_\alpha \otimes \eta_\alpha + \sum_{\alpha=1}^{s} \eta'_\alpha \otimes \eta'_\alpha, \quad f' := f,$$

(3.1)

determine a new metric $f$-$(K\cdot)$contact structure (resp. an $S$-structure) on $M$, that we call a rotation of $(f, \xi_i, \eta_i, g)$.

The tensor fields

$$\tilde{\eta}_i := \frac{1}{c_i} \sum_{t=1}^{s} a_{it} \eta_t, \quad \tilde{\xi}_i := c_i \sum_{t=1}^{s} a_{it} \xi_t, \quad i \in \{1, \ldots, s\},$$

(3.2)

determine a new metric $f$-$(K\cdot)$contact structure (resp. an $S$-structure) on $M$, that we call an anti-rotation of $(f, \xi_i, \eta_i, g)$. 
Proof. We prove only the statements for the rotation; the result for the anti-rotation is entirely analogous. From (3.1) it follows that the fundamental 2-form $\omega'$ of $(f', \xi'_i, \eta'_i, g')$ coincides with the fundamental 2-form $\omega$ of $(f, \xi_i, \eta_i, g)$, and that

$$g'(f'X, f'Y) = g(fX, fY) = g(X, Y) - \sum_{\alpha=1}^{s} \eta_{\alpha}(X)\eta_{\alpha}(Y)$$

$$= g'(X, Y) - \sum_{\alpha=1}^{s} \eta'_{\alpha}(X)\eta'_{\alpha}(Y)$$

for all vector fields $X, Y$ on $M$. Moreover, for all $i, k \in \{1, \ldots, s\}$,

$$\eta'_i(\xi'_j) = \sum_{t,k=1}^{s} a_{kj}a_{ti} c_{t} \delta_{tk} = \delta_{ij}, \quad d\eta'_i = \sum_{t,j=1}^{s} a_{ti} a_{tj} d\eta_j = \sum_{j=1}^{s} \delta_{ij} d\eta_j = \omega',$$

$$g'(\xi'_i, \xi'_j) = g(\xi'_i, \xi'_j) - \sum_{\alpha=1}^{s} \eta_{\alpha}(\xi'_i)\eta_{\alpha}(\xi'_j) + \delta_{ij}$$

$$= \sum_{t,k=1}^{s} \frac{a_{ti}a_{kj}}{c_{t} c_{k}} \delta_{tk} - \sum_{\alpha=1}^{s} \frac{a_{\alpha i}a_{\alpha j}}{c_{\alpha}^{2}} + \delta_{ij} = \delta_{ij},$$

and

$$- id + \sum_{\alpha=1}^{s} \eta'_\alpha \otimes \xi'_\alpha = - id + \sum_{\alpha=1}^{s} \left( \sum_{t=1}^{s} a_{t\alpha}c_{t} \eta_t \right) \otimes \left( \sum_{j=1}^{s} \frac{1}{c_{j}}a_{j\alpha} \xi_j \right)$$

$$= - id + \sum_{t,j=1}^{s} \frac{c_{t}}{c_{j}} \left( \sum_{\alpha=1}^{s} a_{t\alpha}a_{j\alpha} \right) \eta_t \otimes \xi_j$$

$$= - id + \sum_{t,j=1}^{s} \delta_{tj} \eta_t \otimes \xi_j = - id + \sum_{j=1}^{s} \eta_j \otimes \xi_j = f^2 = f'^2.$$ 

hence $(f', \xi'_i, \eta'_i, g')$ is a metric $f$-contact structure. Moreover, since for every $i \in \{1, \ldots, s\}$,

$$\mathcal{L}_{\xi'_i}f' = \sum_{t=1}^{s} \frac{1}{c_{t}} a_{ti} \mathcal{L}_{\xi_t}f,$$

and, because of $d\eta'_\alpha = \omega = d\eta_\alpha$ and $\sum_{i=1}^{s} \xi'_i = \sum_{i=1}^{s} \xi_i$,

$$[f', f'] + 2 \sum_{\alpha=1}^{s} d\eta'_\alpha \otimes \xi'_\alpha = [f, f] + 2 \sum_{j=1}^{s} d\eta_j \otimes \xi_j,$$

the rotation $(f', \xi'_i, \eta'_i, g')$ of a metric $f$-$K$-contact or of an $S$-structure on $M$ are metric $f$-contact structures of the same type. \qed

Remark 3.2. Given a metric $f$-contact manifold $(M, f, \xi_i, \eta_i, g)$, the operations of rotation and anti-rotation with respect to an orthogonal matrix $A$ as in Lemma 3.1 are inverse to each other.
We define now type II deformations in the context of metric $f$-contact manifolds. The following definition generalizes the corresponding definition from the Sasakian setting, see [4], p. 240.

**Lemma 3.3.** Let $(M^{2n+s}, f, \xi_1, \ldots, \xi_s, \eta_1, \ldots, \eta_s, g)$ be a metric $f$-(K-)contact (resp. S-) manifold. Let $\theta_1, \ldots, \theta_s$ be closed, $\mathcal{F}$-basic one-forms on $M$. Then, the one-forms

$$\bar{\eta}_i := \eta_i + \theta_i, \quad i \in \{1, \ldots, s\},$$

together with the tensors

$$\xi_1, \ldots, \xi_s, \quad \bar{g} := g + \sum_{i=1}^{s} \eta_i \otimes \theta_i + \theta_i \otimes \bar{\eta}_i, \quad \bar{f} = f - \sum_{i=1}^{s} (\theta_i \circ f) \otimes \xi_i,$$

determine a new metric $f$-K-contact (resp. S-)structure on $M$, that we call a type II deformation of $(f, \xi_1, \ldots, \xi_s, \eta_1, \ldots, \eta_s, g)$.

**Proof.** Since the 1-forms $\theta_1, \ldots, \theta_s$ are closed and $\mathcal{F}$-basic,

$$\bar{\eta}_i(\xi_j) = \delta_{ij}, \quad d\bar{\eta}_i = d\eta_i =: \omega, \quad i, j \in \{1, \ldots, s\}.$$

The kernel of the endomorphism $\bar{f}$ equals that of $f$, i.e., the span of the vector fields $\xi_i$, and

$$\bar{f}^2 X = f(\bar{f}X) - \sum_{i=1}^{s} (\theta_i \circ f)(\bar{f}X)\xi_i = f^2 X - \sum_{i=1}^{s} (\theta_i \circ f)(fX)\xi_i$$

$$=- X + \sum_{i=1}^{s} \eta_i(X)\xi_i + \sum_{i=1}^{s} \theta_i(X)\xi_i = - X + \sum_{i=1}^{s} \bar{\eta}_i(X)\xi_i$$

for every vector field $X$ on $M$. Moreover, for vector fields $X, Y$ on $M$, we compute

$$\bar{g}(X, \bar{f}Y) = g(X, \bar{f}Y) + \sum_{i=1}^{s} \left( \bar{\eta}_i(X)\theta_i(\bar{f}Y) + \theta_i(X)\bar{\eta}_i(\bar{f}Y) \right)$$

$$= g(X, fY) - \sum_{j=1}^{s} \theta_j(fY)g(X, \xi_j) + \sum_{i=1}^{s} \eta_i(X)\theta_i(fY)$$

$$= g(X, fY) = \omega(X, Y) = d\bar{\eta}_i(X, Y)$$
and

\[ \bar{g}(\bar{f}X, \bar{f}Y) = g(\bar{f}X, fY) = g(fX - \sum_{i=1}^{s} \theta_i(fX)\xi_i, fY) = g(fX, fY) = g(X, Y) - \sum_{i=1}^{s} \eta_i(X)\eta_i(Y) = \bar{g}(X, Y) - \sum_{i=1}^{s} \eta_i(X)\eta_i(Y) = \bar{g}(X, Y) - \sum_{i=1}^{s} \bar{\eta}_i(X)\bar{\eta}_i(Y), \]

and hence \((\bar{f}, \xi_i, \bar{\eta}_i, \bar{g})\) is a metric \(f\)-contact structure on \(M\). By definition of \(\bar{f}\) we obtain

\[ (3.3) \quad \mathcal{L}_{\xi_j}\bar{f} = \mathcal{L}_{\xi_j}f - \sum_{i=1}^{s} \mathcal{L}_{\xi_j}((\theta_i \circ f) \otimes \xi_i) = \mathcal{L}_{\xi_j}f - \sum_{i=1}^{s} (\mathcal{L}_{\xi_j}(\theta_i \circ f)) \otimes \xi_i. \]

If \((f, \xi_i, \eta_i, g)\) is a metric \(f\)-\(K\)-contact structure, then \(\mathcal{L}_{\xi_j}f = 0\) for every \(j \in \{1, \ldots, s\}\) by Equation (2.1), and Equation (3.3) becomes

\[ (\mathcal{L}_{\xi_j}\bar{f})X = -\sum_{i=1}^{s} (\mathcal{L}_{\xi_j}(\theta_i \circ f))(X)\xi_i = \sum_{i=1}^{s} (-\mathcal{L}_{\xi_j}(\theta_i(fX)) + \theta_i(f\xi_j, X))\xi_i = \sum_{i=1}^{s} (-\mathcal{L}_{\xi_j}(\theta_i(fX)) + \theta_i([\xi_j, fX])\xi_i = -\sum_{i=1}^{s} (\mathcal{L}_{\xi_j}\theta_i)(fX)\xi_i = 0, \]

where \(X\) is any vector field on \(M\). Then, by [3] Theorem 2.6, \((\bar{f}, \xi_i, \bar{\eta}_i, \bar{g})\) is a metric \(f\)-\(K\)-contact structure on \(M\).

By straightforward computations using Equation (2.1) and the fact that the \(\theta_i\)'s are basic and closed, we obtain

\[ [\bar{f}, f](X, Y) = [f, f](X, Y) - \sum_{i=1}^{s} \theta_i([f, f](X, Y))\xi_i - \sum_{i=1}^{s} \theta_i(fX)(\mathcal{L}_{\xi_j}f)Y + \sum_{i=1}^{s} \theta_i(fY)(\mathcal{L}_{\xi_j}f)X. \]

(3.4)

So if we assume that \((f, \xi_i, \eta_i, g)\) is an \(S\)-structure on \(M\), then \(\mathcal{L}_{\xi_j}f = 0\) (see Section 2), and from (3.4) we get

\[ [\bar{f}, f](X, Y) + 2 \sum_{i=1}^{s} d\bar{\eta}_i(X, Y)\xi_i = [f, f](X, Y) - \sum_{i=1}^{s} \theta_i([f, f](X, Y))\xi_i + 2 \sum_{i=1}^{s} d\eta_i(X, Y)\xi_i = 0; \]

hence \((\bar{f}, \xi_i, \bar{\eta}_i, \bar{g})\) is again an \(S\)-structure on \(M\). \qed
Remark 3.4. Type II deformation, rotation and anti-rotation do not change the characteristic foliation $F$ of a metric $f$-contact manifold.

Rotations and type II deformations commute with each other in the sense of the following lemma.

Lemma 3.5. Let $(M, f, \xi_1, \ldots, \xi_s, \eta_1, \ldots, \eta_s, g)$ be a metric $f$-contact manifold. Let $\theta_i, i = 1, \ldots, s$, be closed, $F$-basic, one-forms on $M$, $A = (a_{ij}) \in \text{O}(s)$, and $\theta_i := \frac{1}{c_i} \sum_{k=1}^{s} a_{ik} \theta_k$, with $c_i := \sum_{j=i}^{s} a_{ij} \neq 0, i \in \{1, \ldots, s\}$.

The operations in (1) and (2) lead to the same metric $f$-contact structure on $M$:

1. Rotation of $(f, \xi_i, \eta_i, g)$ with respect to $A$ and then type II deformation with respect to $\theta_1, \ldots, \theta_s$.
2. Type II deformation of $(f, \xi_i, \eta_i, g)$ with respect to $\tilde{\theta}_1, \ldots, \tilde{\theta}_s$ and then rotation with respect to $A$.

Proof. We denote by $(f', \xi_i', \eta_i', g')$ and by $(\tilde{f}, \xi_i', \tilde{\eta}_i, \tilde{g})$ the metric $f$-contact structures on $M$ obtained from $(f, \xi_i, \eta_i, g)$ after performing a rotation with respect to $A$, respectively a type II deformation with respect to $\tilde{\theta}_i$. A type II deformation of $(f', \xi_i', \eta_i', g')$ with respect to $\theta_1, \ldots, \theta_s$ gives a new metric $f$-contact structure with the following structure tensors:

$$\hat{\eta}_i = \eta_i' + \theta_i, \quad \hat{\xi}_i = \xi_i', \quad \hat{f} = f - \sum_{i=1}^{s} (\theta_i \circ f) \otimes \xi_i,$$

$$\hat{g} = g' + \sum_{i=1}^{s} \eta_i' \otimes \theta_i + \theta_i \otimes \hat{\eta}_i = g - \sum_{i=1}^{s} \eta_i \otimes \eta_i + \sum_{i=1}^{s} \eta_i' \otimes \eta_i' + \sum_{i=1}^{s} \eta_i \otimes \theta_i + \sum_{i=1}^{s} \theta_i \otimes (\eta_i' + \theta_i) = g - \sum_{i=1}^{s} \eta_i \otimes \eta_i + \sum_{i=1}^{s} \hat{\eta}_i \otimes \hat{\eta}_i.$$

We check that the metric $f$-contact structure $(f'', \xi_i'', \eta_i'', g'')$ obtained from a rotation of $(\tilde{f}, \xi_i', \tilde{\eta}_i, \tilde{g})$ with respect to $A$ coincides with $(\hat{f}, \xi_i', \hat{\eta}_i, \hat{g})$:

$$\eta_i'' = \sum_{i=1}^{s} c_i a_{ii} \tilde{\eta}_i = \eta_i' + \sum_{i=1}^{s} a_{ii} a_{ik} \theta_k = \eta'_i + \theta_i = \hat{\eta}_i, \quad \xi_i'' = \hat{\xi}_i,$$

$$f'' = \tilde{f} = f - \sum_{i=1}^{s} (\tilde{\theta}_i \circ f) \otimes \xi_i = f - \sum_{i=1}^{s} (\theta_k \circ f) \otimes \sum_{i=1}^{s} \frac{1}{c_i} a_{ik} \xi_i = \hat{f},$$

$$g'' = \tilde{g} - \sum_{i=1}^{s} \tilde{\eta}_i \otimes \tilde{\eta}_i + \sum_{i=1}^{s} \eta_i'' \otimes \eta_i'' = g + \sum_{i=1}^{s} \eta_i \otimes \tilde{\theta}_i + \sum_{i=1}^{s} \tilde{\theta}_i \otimes \tilde{\eta}_i - \sum_{i=1}^{s} \tilde{\eta}_i \otimes \tilde{\eta}_i + \sum_{i=1}^{s} \hat{\eta}_i \otimes \hat{\eta}_i = \hat{g}.$$
Remark 3.6. The metric $f$-$K$-contact structure obtained from $(f, \xi_i, \eta_i, g)$ after a type II deformation with respect to $\theta_i + \theta'_i$ coincides with the metric $f$-$K$ contact structure obtained performing first a type II deformation with respect to $\theta$ of $(f, \xi, \eta, g)$ and then a type II deformation with respect to $\theta'_i$. In particular, the inverse operation of a type II deformation with respect to $\theta_i$ is a type II deformation with respect to $-\theta_i$.

Remark 3.7. As the inverse operation of a rotation is an anti-rotation, cf. Remark 3.2, and the inverse operation of a type II deformation is again a type II deformation, anti-rotations and type II deformations commute in a similar way as in Lemma 3.5.

We recall from [6, Section 3] how a metric $f$-structure induces in a natural way the same structure on the mapping torus of an automorphism. Let $(M, f, \xi_i, \eta_i, g)$ be a metric $f$-contact manifold and $\phi: M \to M$ an automorphism of the structure. The tensors $(f, \xi_i, \eta_i, g)$ on $M$ induce the following natural metric $f$-contact structure on $M \times \mathbb{R}$:

\[
\bar{f}(X) = f(X), \quad \bar{f}\left(\frac{d}{dt}\right) = 0, \quad \bar{\eta}_\alpha(X) = \eta_\alpha(X), \quad \bar{\eta}_\alpha\left(\frac{d}{dt}\right) = 0,
\]

\[
\bar{\eta}_{s+1}(X) = \frac{1}{s}(\eta_1(X) + \cdots + \eta_s(X)), \quad \bar{\eta}_{s+1}\left(\frac{d}{dt}\right) = 1,
\]

\[
\bar{\xi}_{s+1} := \frac{d}{dt}, \quad \bar{\xi}_\alpha := \xi_\alpha - \frac{1}{s}\frac{d}{dt}, \quad \alpha = 1, \ldots, s,
\]

for each $X \in TM$ and where $\frac{d}{dt}$ denotes the standard coordinate vector field on $\mathbb{R}$,

\[
\bar{g}(X,Y) = g(X,Y), \quad \bar{g}(X,\bar{\xi}_\alpha) = 0, \quad \bar{g}(\bar{\xi}_\alpha, \bar{\xi}_\beta) = \delta_\alpha^\beta,
\]

for each $X,Y \in \text{im}(f)$ and $\alpha, \beta \in \{1, \ldots, s+1\}$. This structure is invariant under the $\mathbb{Z}$-action on $M \times \mathbb{R}$ determined by $\phi$:

\[
m \cdot (p, t) \mapsto (\phi^m(p), t + mt_0),
\]

where $t_0 \in \mathbb{R}$, $t_0 \neq 0$, and descends to the quotient of the $\mathbb{Z}$-action, i.e., the mapping torus $M_\phi$ of $(M, \phi)$, making it a metric $f$-contact manifold. In [6] we also computed that $M_\phi$ is a metric $f$-$K$-contact (resp. an $S$-)manifold if and only if $M$ is.

4. Main result

In this section we prove our main result, which states that any compact metric $f$-$K$-contact manifold is obtained from a compact $K$-contact manifold by successively applying rotations, type II deformations, and constructions of mapping tori. The main idea of the proof is to apply our cohomological splitting theorem for compact metric $f$-$K$-contact manifolds from [6] in order to find a suitable deformation (i.e., type II deformation, combined with anti-rotation) whose characteristic foliation has closed leaves, and then exhibit this deformed structure as a mapping torus.
Lemma 4.1. Let $M^{2n+s}$ be a compact connected metric $f$-$K$-contact manifold. Then

$$H^1(M) = \text{span}_\mathbb{R}\{[\eta_1 - \eta_s], \ldots, [\eta_{s-1} - \eta_s]\} \oplus H^1(M, \mathcal{F}).$$

Proof. By [6, Theorem 4.5] we have an isomorphism

$$H^s(M) \cong \Lambda(\mathbb{R}^{s-1}) \otimes H^s(M, \mathcal{F}_{s-1}),$$

where $\Lambda(\mathbb{R}^{s-1})$ embeds in $H^s(M)$ as the exterior algebra over the cohomology classes $[\eta_i - \eta_s]$. We thus obtain

$$H^1(M) = \text{span}_\mathbb{R}\{[\eta_1 - \eta_s], \ldots, [\eta_{s-1} - \eta_s]\} \oplus H^1(M, \mathcal{F}_{s-1})$$

and are left with showing $H^1(M, \mathcal{F}_{s-1}) = H^1(M, \mathcal{F})$.

To show this we make use of one of the exact sequences from [6, Proposition 4.4]:

$$0 \rightarrow H^1(M, \mathcal{F}) \rightarrow H^1(M, \mathcal{F}_{s-1}) \rightarrow H^0(M, \mathcal{F}) \xrightarrow{\delta} H^2(M, \mathcal{F}) \rightarrow \cdots,$$

where the connecting homomorphism $\delta$ is given by $\delta([\sigma]) = [\omega \wedge \sigma]$. Since the fundamental 2-form $\omega$ is a non-zero element of $H^2(M, \mathcal{F})$ [6, Lemma 6.3], the map $\delta : H^0(M, \mathcal{F}) \rightarrow H^2(M, \mathcal{F})$ is injective and thus $H^1(M, \mathcal{F}) \cong H^1(M, \mathcal{F}_{s-1})$. \hfill $\square$

Lemma 4.2. Let $(M, f, \xi_1, \ldots, \xi_s, \eta_1, \ldots, \eta_s, g)$ be a compact connected metric $f$-$K$-contact manifold, where $s \geq 2$. Then there exists an anti-rotation $(\tilde{f}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ of $(f, \xi_i, \eta_i, g)$ and a closed, $\mathcal{F}$-basic, 1-form $\theta$ on $M$, such that the cohomology class $[\tilde{\eta}]$, where

$$\eta := \tilde{\eta}_s - \frac{1}{s-1}(\tilde{\eta}_1 + \cdots + \tilde{\eta}_{s-1}) + \theta,$$

is a real multiple of an integer class.

Moreover, the closed 1-form $\eta$ is nowhere vanishing and determines a codimension-one foliation $\mathcal{F}_\eta := \ker \eta$ with compact leaves.

Proof. We consider the open set

$$U := \{A = (a_{ij}) \in \text{O}(s) \mid \sum_{t=1}^s a_{it} \neq 0, i = 1, \ldots, s\}$$

of $\text{O}(s)$, and the map $h : U \rightarrow \mathbb{R}^s$, defined by

$$h(A) = A^t \begin{pmatrix} \frac{1}{c_1(A)} & 0 & \cdots & 0 \\ 0 & \frac{1}{c_2(A)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{c_s(A)} \end{pmatrix} \begin{pmatrix} \frac{1}{s-1} \\ \vdots \\ \frac{1}{s-1} \\ 1 \end{pmatrix},$$

where $A^t$ is the transpose of the matrix $A$, and

$$c_k : U \rightarrow \mathbb{R}; \ (a_{ij}) \mapsto \sum_{j=1}^s a_{kj},$$

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for every $k \in \{1, \ldots, s\}$. Observe that $h(A)$ gives the coordinates of the vector
\[
\eta := \bar{\eta}_s - \frac{1}{s-1} (\bar{\eta}_1 + \cdots + \bar{\eta}_{s-1}) \in \text{span}_\mathbb{R} \{\eta_1, \ldots, \eta_s\},
\]
with respect to the basis $\{\eta_1, \ldots, \eta_s\}$, where $\bar{\eta}_i = \frac{1}{c_i(A)} \sum_{t=1}^s a_{it} \eta_t$ are the one-forms on $M$ obtained from $(f, \xi_i, \eta_i, g)$ after an anti-rotation with respect to $A$. Moreover, $h$ maps to the codimension-one subspace of $\mathbb{R}^s$,
\[
V := \{(u_1, \ldots, u_s) \mid \sum_{i=1}^s u_i = 0\} \simeq \text{span}_\mathbb{R} \{\eta_1 - \eta_s, \ldots, \eta_{s-1} - \eta_s\}.
\]

To prove the first part of this lemma, by Lemma 4.1, it suffices to show that $\text{im} \ h$ contains an open neighborhood of $\eta$ in $V$. For this purpose, we show that the image of the differential $(dh)_I : \mathfrak{o}(s) \to \mathbb{R}^s$ contains $V$. A direct computation shows that, for every $X \in \mathfrak{o}(s)$,
\[
(dh)_I (X) = X^t \begin{pmatrix}
\frac{1}{s-1} & c_1(X) & 0 & \cdots & 0 \\
0 & c_2(X) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & c_s(X) \\
\frac{1}{s-1} & 0 & \cdots & 1
\end{pmatrix}
\]
In particular, for every element of the orthogonal Lie algebra $\mathfrak{o}(s)$ of type
\[
\bar{X} = \begin{pmatrix}
a_1 & \cdots & a_{s-1} \\
a_1 & \cdots & a_{s-1} \\
-1 & \cdots & -a_{s-1} \\
0 & \cdots & 0
\end{pmatrix},
\]
we have that
\[
(dh)_I (\bar{X}) = \left(\frac{1}{s-1} - 1\right) \begin{pmatrix}
a_1 \\
\vdots \\
a_{s-1} \\
-\sum_{i=1}^{s-1} a_i
\end{pmatrix}.
\]
Then, $\text{im} (dh)_I \supset V$.

The second part of the lemma follows directly from [1, Lemma 3.5].

**Proposition 4.3.** Let $M$ be a compact, connected, metric $f$-$K$-contact (resp. $S$-) manifold. Then, there exists a compact, metric $f$-$K$-contact (resp. $S$-) manifold $N$, so that $M$ is isomorphic to the mapping torus of $N$ with respect to an automorphism of the structure, up to a rotation and a type II deformation.

**Proof.** Let $(f, \xi_1, \ldots, \xi_{s+1}, \eta_1, \ldots, \eta_{s+1}, g)$ be the metric $f$-$K$-contact structure on $M$, and consider the 1-form
\[
\eta := \eta_{s+1} - \frac{1}{s} (\eta_1 + \cdots + \eta_s),
\]
which is nowhere vanishing and closed, since $\eta(\xi_{s+1}) = 1$ and $d\eta_1 = \cdots = d\eta_s$. By Lemma 4.2 up to performing an anti-rotation and a type II
deformation of the structure, we can assume that \([\eta]\) is a real multiple of an integer class, and the leaves of the foliation \(\mathcal{F}_\eta\) defined by \(\ker \eta\) are compact. We will show that in this situation \(M\) is isomorphic to the mapping torus of a compact metric \(f\)-\(K\)-contact manifold \(N\) with respect to an automorphism of \(N\); as rotation and anti-rotation are inverse to each other by Remark 3.2, this will imply the proposition.

We observe that, for every \(X \in \text{im} f\),
\[
0 = 2d\eta(\xi_{s+1}, X) = \xi_{s+1}(\eta(X)) - X\eta(\xi_{s+1}) - \eta[\xi_{s+1}, X];
\]
thus \([\xi_{s+1}, X] \in TF_\eta\) and, as the \(\xi_i\) commute with each other, we obtain \([\xi_{s+1}, TF_\eta] \subset TF_\eta\). Then, by Proposition 2.2 of [7], the flow \(\phi\) of \(\xi_{s+1}\) leaves \(\mathcal{F}_\eta\) invariant, namely it carries leaves of \(\mathcal{F}_\eta\) to leaves.

Let \(N\) be any leaf of \(\mathcal{F}_\eta\). We observe that there exists \(t_0 \in \mathbb{R}\) such that \(\varphi := \phi_{t_0}\) maps \(N\) to itself: the map
\[
N \times \mathbb{R} \to M; \ (p, t) \mapsto \phi(p, t),
\]
is a local diffeomorphism and hence in particular \(\phi(N \times \mathbb{R}) \subset M\) is an open, \(\mathcal{F}_\eta\)-saturated subset. This implies that \(\phi|_{N \times \mathbb{R}} : N \times \mathbb{R} \to M\) is surjective (otherwise \(M \setminus \phi(N \times \mathbb{R}) \neq \emptyset\) would be an open, \(\mathcal{F}_\eta\)-saturated set, contradicting the fact that \(M\) is connected). Thus, by the compactness of \(M\), \(\phi|_{N \times \mathbb{R}} : N \times \mathbb{R} \to M\) is not injective. Let \((p, t), (q, s) \in N \times \mathbb{R}\) such that \((p, t) \neq (q, s)\) and \(\phi(p, t) = \phi(q, s)\). Since \(\phi\) maps leaves to leaves and \(\phi_{t-s}(p) = q\), we have that the map \(\phi_{t-s}\) maps \(N\) to \(N\).

We construct on \(N\) a metric \(f\)-\(K\)-contact structure such that \(\varphi : N \to N\) is an automorphism of that structure. For every \(i = 1, \ldots, s\), we denote by \(\tilde{\eta}_i\) the pullback one-form of \(\eta_i\) on \(N\), via the immersion \(j : N \hookrightarrow M\), and by \(\xi_i\) the vector field on \(N\), \(j\)-related to the vector field \(\xi_i + \frac{1}{s}\xi_{s+1}\) on \(M\) tangent to \(\mathcal{F}_\eta\) (observe that \(\eta\left(\xi_i + \frac{1}{s}\xi_{s+1}\right) = 0\)). We define moreover a Riemannian metric \(\tilde{g}\) and a \((1, 1)\)-tensor \(\tilde{f}\) on \(N\) by:
\[
\tilde{g}(X, Y) = g(X, Y), \quad \tilde{g}(X, \xi_i) = 0, \quad \tilde{g}(\xi_i, \xi_j) = \delta_{ij},
\]
\[
\tilde{f}(\xi_i) = 0, \quad \tilde{f}(X) = f(X),
\]
for every \(i = 1, \ldots, s\), and \(X, Y \in \ker(\tilde{\eta}_1)_p \cap \cdots \cap \ker(\tilde{\eta}_s)_p = (\text{im} \tilde{f})_p = (\text{im} f)_p\), \(p \in N\). It is easy to check that \((\tilde{f}, \xi_i, \tilde{\eta}_i, \tilde{g})\) is a metric \(f\)-\(K\)-contact structure on \(N\), and the diffeomorphism \(\varphi : N \to N\) preserves the tensors \(\xi_i, \tilde{\eta}_i, \tilde{g}, \tilde{f}\). If \((f, \xi_1, \ldots, \xi_{s+1}, \eta_1, \ldots, \eta_{s+1}, g)\) is an \(S\)-structure on \(M\), then for all local vector fields \(X, Y\) on \(M\) tangent to \(\mathcal{F}_\eta\),
\[
([\tilde{f}, \tilde{f}] + 2 \sum_{\alpha=1}^s d\tilde{\eta}_\alpha \otimes \xi_\alpha)(X, Y) = [f, f](X, Y) + 2 \sum_{\alpha=1}^s d\eta_\alpha (X, Y) \left(\xi_\alpha + \frac{1}{s}\xi_{s+1}\right)
\]
\[
= [f, f](X, Y) + 2 \sum_{\alpha=1}^{s+1} d\eta_\alpha (X, Y) \xi_\alpha = 0,
\]
and hence \((N, \tilde{f}, \tilde{\xi}_i, \tilde{\eta}_i, \tilde{g})\) is an \(S\)-manifold.

By [6, Section 3], and as recalled in Section 3, the metric \(f\)-\(K\)-contact (resp. \(S\)-)structure on \(N\) determines a metric \(f\)-\(K\)-contact (resp. \(S\)-)structure \((\tilde{f}, \tilde{\xi}_i, \tilde{\eta}_i, \tilde{g})\) on the mapping torus \(N_{\varphi^{-1}}\). Finally, observe that the map

\[
\Psi : N_{\varphi^{-1}} \to M; \quad [(p, t)] \mapsto \phi(p, t),
\]
is well defined, injective and a local diffeomorphism. Moreover, since \(N_{\varphi^{-1}}\) and \(M\) are compact, \(\Psi\) is a diffeomorphism, which by construction preserves the given structure tensors: for every \(X, Y \in \text{im} f\), and \(\alpha \in \{1, \ldots, s\},
\[
(d\phi)_{(p, t)} \tilde{\xi}_\alpha = (d\phi_p)_p (\xi_\alpha + \frac{1}{s} \xi_{s+1}) - \frac{1}{s} (d\phi_p)_t \frac{dt}{dt} \Big|_{t=\xi_\alpha} = \xi_\alpha,
\]
\[
(d\phi)_{(p, t)} \tilde{\xi}_{s+1} = (d\phi_p)_t \frac{dt}{dt} \Big|_{t=\xi_{s+1}} = \xi_{s+1},
\]
\[
\tilde{g}(X, Y) = \tilde{g}(X, Y) = g(X, Y) = g((d\phi_t)_p X, (d\phi_t)_p Y) = (\phi^* g)(X, Y),
\]
\[
g((d\phi_t)_p X, (d\phi_t)_p \tilde{\xi}_i) = g((d\phi_t)_p X, \tilde{\xi}_i) = g(X, (d\phi_{-t})_p \tilde{\xi}_i) = 0 = \tilde{g}(X, \tilde{\xi}_i),
\]
\[
f((d\phi_t)_p X) = f(X) = \tilde{f}(X) = f(X).
\]
\[\square\]

**Theorem 4.4.** Any compact connected metric \(f\)-\(K\)-contact (resp. \(S\)-) manifold is obtained from a compact \(K\)-contact (resp. Sasakian) manifold by a finite iteration of the following operations:

1. construction of the mapping torus with respect to an automorphism
2. rotation
3. type II deformation

**Proof.** This follows directly by applying Proposition 4.3 inductively, as in each step the dimension of the manifold decreases by one. \(\square\)

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