Aggregation of identical mechanical systems with oscillations

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Abstract. We consider identical mechanical systems described by Lagrange’s equations of the second kind and subject to the action of positional forces. It is assumed that a separate system allows for single-frequency oscillation. The problem of aggregating the set of systems into a coupled system with an attractive cycle close to the oscillation of uncoupled systems is solved. For this purpose, weak universal coupling controls are found.

1. Introduction
Aggregation of individual systems is performed to obtain a coupled system with a given property. This procedure should be accompanied by the study of the coupled system.

Usually, aggregation is performed after decomposition. Methods for aggregating complex systems by the Lyapunov method are given in [1]. It is proposed to study the set of systems in the framework of a model containing weakly coupled subsystems [2]. The idea of [2] was implemented for autonomous and periodic coupled systems described by general differential equations in [3-6]. In this paper, we consider identical mechanical systems.

Coupled systems appear in various fields of knowledge. A classic example in mechanics is the Sommerfeld sympathetic pendulum. For more examples, see [7-11].

2. Problem statement
Consider \( k \) degree-of-freedom identical mechanical systems described by Lagrange’s equation of the second kind under the action of positional forces. Each system is assumed to allow for a single-frequency oscillation.

The phase portrait of an individual mechanical system is symmetrical with respect to the fixed space \( M_l = \{ q_l, \dot{q}_l : \dot{q}_l = 0 \} \), where \( q_l \) is the generalized coordinate, \( l = 1, \ldots, k \). The generalized velocity \( \dot{q}_l \) turns into zero on \( M_l \), while the periodic motion is symmetrical with respect to \( M_l \) and is called the symmetrical periodic motion (SPM). The set of SPMs form \( h \)-parametric families \( \Sigma_i(h) \).

We consider families \( \Sigma_i(h) \) that contain only non-degenerate SPMs, i.e. SPMs with the period depending on \( h \) strictly monotonically.

SPMs cannot be stabilized in the framework of the considered mechanical model. In order to stabilize an SPM, it is necessary to apply a force that violates the symmetry of the phase portrait. If the applied force is \( \epsilon \)-small, then the oscillation of the controlled mechanical system, which is stabilized, is \( \epsilon \)-close to the SPM. The appropriate controlled mechanical system that possesses an orbitally asymptotically stable cycle is constructed in [12,13]. A small force becomes a weak coupling control for a set of mechanical systems.

Stated is the problem of aggregation of considered identical mechanical systems such that the resulting system allows for an attracting cycle, which is close to the oscillations of uncoupled systems. At that, universal coupling controls (i.e. suitable for any mechanical systems) are found.

3. Aggregation technique
The proposed technique interprets a set of uncoupled identical systems as a single mechanical system. This system, according to the problem statement, allows for a family \( \Sigma(h) = \bigcup \Sigma_i(h) \) of nondegenerate SPMs. A controlled mechanical system is constructed, where an \( \epsilon \)-small control is
applied. The existence of such a control is justified in [12,13]. The further investigations are carried out in the framework of the controlled mechanical system

\[
\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_s} - \frac{\partial T}{\partial q_s} = Q_s(q) + \varepsilon u_s(q, \dot{q}), \quad s = 1, \ldots, m,
\]

where \( T \) is kinetic energy, \( Q_s(q) \) is a positional force, \( m = kn \), and \( u = (u_1, \ldots, u_m) \) is chosen in the general form

\[
u_s = \sigma \left[ 1 - K(h^*) b(q, \dot{q}) \right] \sum_{j=1}^{m} r_{sj} \dot{q}_j, \quad b > 0, \quad r_{sj} = \text{const}, \quad s = 1, \ldots, m.
\]

For \( k \) identical mechanical systems with \( n \) degrees of freedom, the dimension of system (1) is \( m = kn \). The value of \( h^* \) corresponds to the attractive cycle; the choice of \( \sigma \) is described below.

Function \( K(h) \) is the characteristics of the family of SPMs. It is defined by the identity

\[
\int_{0}^{\tau(h)} \left[ 1 - K(h) b(\varphi(h,t), \dot{\varphi}(h,t)) \right] \sum_{s,j=1}^{m} r_{sj} \dot{\varphi}_j(h,t) \psi_s(h,t) dt = 0.
\]

As a result, we obtain

\[
K(h) = \left( \int_{0}^{\tau(h)} \theta(h,t) dt \right)^{-1}, \quad \theta = \sum_{s,j=1}^{m} r_{sj} \dot{\varphi}_j(h,t) \psi_s(h,t).
\]

In [13] it is proved that an orbitally asymptotically stable cycle always occurs in system (1),(2) at the proper choice of matrix \( \|r_{sj}\| \). In (2), multiplier

\[
\sigma = \begin{cases} 
1, & \text{if } \chi < 0 \\
-1, & \text{if } \chi > 0
\end{cases},
\]

where \( \chi = \frac{dK(h^*)}{dh} \) is applied. Therefore, the problem of stabilization of \( k \) mechanical systems can be solved in principle, provided that appropriate function \( b \) and matrix \( \|r_{sj}\| \) in (2) are found.

4. Cycle of the coupled mechanical system

Necessary and sufficient conditions for a \( \tau \)-periodic SPM to exist are expressed as follows

\[
\dot{q}_s(q_0, \ldots, q_n, t) = 0, \quad t = 0, \tau / 2; \quad s = 1, \ldots, n,
\]

where \( q_0 \) is the initial point at \( t = 0 \). This means that SPMs constitute a \( \tau \)-parametric family.

Recall that we consider the family \( \Sigma(h) \) of nondegenerate SPMs, so that the period depends on the numerical parameter strictly monotonously. Denote the trajectory on \( \Sigma(h) \) by \( q = \varphi(h,t + \gamma) \). Here \( h \) is the family parameter, and \( \gamma \) is the shift of the initial point along the trajectory. The solution is assumed to be in fixed set \( M \) at \( t=0 \), so put \( \gamma = 0 \). Family \( \Sigma \) fills the invariant two-dimensional manifold \( \hat{\Sigma} \) in the phase space. This fact, as well as the monotonous behaviour of function \( \tau = \tau(h) \) is established in [14].

Since (1) is satisfied on \( \hat{\Sigma} \), the following lemma holds.
Lemma. A family of nondegenerate SPMs is described by a one-degree-of-freedom conservative system.

Proof. Equation (3) implies that the following linear equation holds.

\[ d\xi = b_1(q^0,t) dq_1 + \cdots + b_m(q^0,t) dq_m + c_s(q^0,t) dt = 0, \]
\[ B = \begin{bmatrix} b_{sj}(q^0,t) \end{bmatrix}, \quad b_{sj}(q^0,t) = \frac{\partial q_s}{\partial q_j}, \]
\[ C = \begin{bmatrix} c_s(q^0,t) \end{bmatrix}, \quad c_s = \frac{\partial q_s}{\partial \dot{q}_s}(q^0,t), \quad s,j = 1,\ldots,n. \]

(4)

The above equations become identity on \( \hat{\Sigma} \). Since the SPMs are nondegenerate, conditions \( \det B_n = \det C_n \neq 0 \) are satisfied, therefore there exists a linear transform \( \eta = P \xi, \quad \xi = (\xi_1,\ldots,\xi_n) \), where \( P = [p_{sj}] \) is a constant matrix, such that \( \eta_2 = 0,\ldots,\eta_n = 0 \). Since the transform holds for any point \( (q^0,t) \), the first coordinate \( \eta_1 \) is chosen in \( \hat{\Sigma} \), and (4) can be rearranged as follows.

\[ d\eta_1 = \sum_{j=1}^{m} \hat{b}_1(q^0,t) dq_j + \hat{c_1}(q^0,t) dt = 0, \quad d\eta_k(q^0,t) = 0, \quad k = 2,\ldots,n, \quad \hat{c}_1 = \sum_{j=1}^{n} p_{1,j} c_j(q^0,t). \]

(5)

For \( \Sigma \), the existence of the linear transform mentioned above means that there exist coordinates \( w_1,\ldots,w_n \) such that \( w_2 = 0,\ldots,w_n = 0 \) on \( \Sigma \). The initial point \( q^0 \) on \( \Sigma \) is the function of a single parameter. Take \( w_1^0 \) as this parameter to obtain the following equation from the first equation of (5).

\[ \ddot{w}_1 + \ddot{c}_1 w_1 = 0. \]

Therefore, the dynamics on \( \hat{\Sigma} \) is described by a one-degree-of-freedom conservative system, q.e.d.

According to Lemma, a one-degree-of-freedom conservative system is a key subsystem of the multi-degree-of-freedom system that implement the family \( \Sigma(h) \). The whole system can be described in the neighborhood of \( \hat{\Sigma} \) by the generalized coordinates \( (x,y) \), where \( x = w_1, \quad y = (w_2,\ldots,w_n) \). Since \( y = 0 \) is satisfied on \( \hat{\Sigma} \), we obtain \( x^2 = \sum_{j=1}^{m} q_j^2 \). Following [13], implement the universal control

\[ (1-Kx^2)\dot{x} \]

(\( K \) is a constant value) on \( \hat{\Sigma} \). Thus, function \( b \) in (2) can be found on \( \hat{\Sigma} \).

Control \( \sigma(1-Kx^2)\dot{\omega} \) in the transformed mechanical system is given by positive definite form

\[ \sum_{s=1}^{m} w_s^2 \].

The linear transform in Lemma is implemented with the constant nondegenerate matrix, so it does not change the signature of the quadratic form. So the positive definite form

\[ R = \frac{1}{2} \sum_{s,j=1}^{m} r_{sj} q_s q_j > 0, \quad r_{sj} = \text{const} \]

of initial coordinates is implemented to establish control (2).

Let us recall that \( m = kn \) in the case of \( k \) identical \( n \)-degree-of-freedom mechanical systems. Besides, the characteristics is calculated as \( K(h)/k \), where \( K(h) \) is the characteristics of the individual system. As a result, we obtain the following theorem that gives the solution to the problem of aggregation.
Theorem Consider the set of \( k \) identical \( n \)-degree-of-freedom mechanical systems under the action of positional forces. Let the systems have a family of nondegenerate symmetrical periodic motions. Let this set be coupled by the following coupling controls

\[
 u_s = \sigma \left[ 1 - \frac{K(h^*)}{k} \rho \right] \sum_{j=1}^{kn} r_{sj} \dot{q}_j, \quad \rho = \sum_{j=1}^{kn} \dot{q}_j^2, \quad r_{sj} = \text{const}, \quad s = 1, \ldots, kn, \tag{6}
\]

where \( r_{sj} \) are coefficients of a positive definite quadratic form. Then an orbitally asymptotically stable cycle close to the oscillation of uncoupled systems at \( h = h^* \) occurs.

Remark 1 Coupling controls (6) ensure the natural stabilization of the cycle in the coupled mechanical system.

Remark 2 Given Theorem, oscillations of conservative systems become synchronized in frequency and phase.

Remark 3 For conservative mechanical systems, matrix \( r_{sj} \) is block-diagonal with \( k \) blocks.

5. Example
Consider the set of \( k \) identical mathematical pendulums \( \ddot{x} + \sin x = 0 \). Take total mechanical energy as \( h \). Characteristics \( K(h) \) of a single pendulum is shown in figure 1.

![Figure 1. Characteristics of a single pendulum.](image)

As it can be seen, \( dK(h)/dh < 0 \) for all \( h \in (-1;1) \). Calculate the characteristics for the set of \( k \) pendulums as \( \bar{K}(h) = K(h)/k \). So \( d\bar{K}(h)/dh < 0 \), and \( \sigma = 1 \). By the aggregation with the aid of coupling controls we obtain the controlled system

\[
 \ddot{x}_s + \sin x_s = \varepsilon \left[ 1 - \bar{K}(h^*) \sum_{j=1}^{k} x_j^2 \right] \sum_{j=1}^{k} r_{sj} \dot{x}_j, \quad s = 1, \ldots, k, \tag{7}
\]

where \( r_{sj} \) are coefficients of the positive definite quadratic form. Note that, according to Remark 3, the matrix of this quadratic form is block-diagonal, so equation (7) is rearranged as
\[
\ddot{x}_s + \sin x_s = \varepsilon \left[ 1 - \bar{K}(h^*) \sum_{j=1}^{k} x_j^2 \right] \dot{x}_s, \quad s = 1, \ldots, k.
\]  

System (8) has an attracting cycle, which is \( \varepsilon \)-close to the oscillation of uncoupled pendulums with energy \( h = h^* \).

6. Conclusion

Single-frequency oscillations of mechanical systems with positional forces are symmetric and form families. They cannot be stabilized in the framework of this model. The aggregation problem is stated in order to obtain a coupled system with an attracting cycle. The problem is solved by implementing appropriate weak coupling controls.

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