The mean number of 2-torsion elements in the class groups of 
$n$-monogenized cubic fields

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Abstract
We prove that, on average, the monogenicity or $n$-monogenicity of a cubic field has an altering effect on the behavior of the 2-torsion in its class group.

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1 Introduction

The seminal works of Cohen–Lenstra [17] and Cohen–Martinet [18] provide heuristics that predict, for suitable “good primes” $p$, the distribution of the $p$-torsion subgroups $\text{Cl}_p(K)$ of the class groups of number fields $K$ of degree $d$. To date, only two cases of these conjectures have been proven; they concern the mean sizes of the 3-torsion subgroups of the class groups of quadratic fields, and the 2-torsion subgroups of the class groups of cubic fields:

**Theorem 1** (Davenport–Heilbronn [21, Theorem 3]) Let $K$ run through all isomorphism classes of quadratic fields ordered by discriminant. Then:

(a) The average size of $\text{Cl}_3(K)$ over real quadratic fields $K$ is $4/3$;

(b) The average size of $\text{Cl}_3(K)$ over complex quadratic fields $K$ is $2$.

**Theorem 2** ([4, Theorem 5]) Let $K$ run through all isomorphism classes of cubic fields ordered by discriminant. Then:

(a) The average size of $\text{Cl}_2(K)$ over totally real cubic fields $K$ is $5/4$;

(b) The average size of $\text{Cl}_2(K)$ over complex cubic fields $K$ is $3/2$.

These two theorems have had a number of important applications. For example, they imply that, when ordered by discriminant, a positive proportion of quadratic fields have class number indivisible by 3, and a positive proportion of cubic fields have class number indivisible by 2 – among numerous other consequences relating to the asymptotics and ranks of elliptic curves, the density of discriminants of number fields of given degree, the existence of units in number fields with given signatures, rational points on surfaces, the statistics of Galois representations and modular forms of weight one, and more (see, e.g., [25], [37], [23], [8], [4], [12], [15], [24], [7], [33]).

The averages occurring in Theorems 1 and 2 are remarkably robust. In [11, Corollary 4] and [12, Theorem 1], it was shown that the averages in these two theorems remain unchanged even when one ranges over quadratic and cubic fields, respectively, that satisfy any specified set of local splitting conditions at finitely many primes, or even suitable sets of local conditions at infinitely many primes.

A natural next question that arises is: how stable are the averages in Theorems 1 and 2 if more global conditions are imposed on the fields being considered? One natural such global condition on a field is *monogenicity*, i.e., that the ring of integers in the field is generated by one element. While monogenicity is a global condition that automatically holds for all quadratic fields, it is a nontrivial condition for cubic fields. The purpose of this paper is to show, surprisingly (at least to the authors!), that monogenicity does have a nontrivial effect on the behavior of class groups of cubic fields and, in particular, the condition of monogenicity does change the averages occurring in Theorem 2.

1.1 The stability of averages in Theorem 2 when ordering cubic fields by height

Recall that a number field $K$ is called *monogenic* if its ring $\mathcal{O}_K$ of integers is generated by one element as a $\mathbb{Z}$-algebra, i.e., $\mathcal{O}_K = \mathbb{Z}[\alpha]$ for some element $\alpha \in \mathcal{O}_K$; such an $\alpha$ is then called a *monogenizer* of $K$ or of $\mathcal{O}_K$. The asymptotic number of monogenic cubic fields of bounded discriminant is not known, and it is therefore more convenient to order these fields by the heights of their monic defining polynomials.
A pair \((K, \alpha)\) is called a **monogenized cubic field** if \(K\) is a cubic field and \(\mathcal{O}_K = \mathbb{Z}[\alpha]\) for some \(\alpha \in \mathcal{O}_K\). More generally, we define an **n-monogenized cubic field** to be a pair \((K, \alpha)\) where \(K\) is a cubic field, \(\alpha \in \mathcal{O}_K\) is primitive in \(\mathcal{O}_K/\mathbb{Z}\), and \([\mathcal{O}_K : \mathbb{Z}[\alpha]] = n\); such an \(\alpha\) is then called an **n-monogenizer** of \(K\) or of \(\mathcal{O}_K\). Two \(n\)-monogenized cubic fields \((K, \alpha)\) and \((K', \alpha')\) are **isomorphic** if \(K\) and \(K'\) are isomorphic as cubic fields and, under such an isomorphism, \(\alpha\) is mapped to \(\alpha' + m\) for some \(m \in \mathbb{Z}\).

We next define an isomorphism-invariant height on the set of \(n\)-monogenized cubic fields. Let \((K, \alpha)\) be an \(n\)-monogenized cubic field, and suppose that \(f(x)\) is the characteristic polynomial of \(\alpha\). Since \((K, \alpha)\) and \((K, \alpha + m)\) are isomorphic for every \(m \in \mathbb{Z}\), we may define two invariants \(I(f)\) and \(J(f)\) on the space of monic cubic polynomials \(f(x) = x^3 + ax^2 + bx + c\), by setting

\[
I(f) := a^2 - 3b,
J(f) := -2a^3 + 9ab - 27c.
\]

These invariants satisfy \(I(f(x)) = I(f(x + m))\) and \(J(f(x)) = J(f(x + m))\) for all \(m \in \mathbb{Z}\). We then define the **height** \(H\) of an \(n\)-monogenized cubic field \((K, \alpha)\) by

\[
H(K, \alpha) := n^{-2}H(f) := n^{-2}\max\{|I(f)|^3, J(f)^2/4\}.
\]

Since the discriminant \(\Delta(K)\) of \(K\) is described in terms of the invariants of \(f\) as

\[
27\Delta(K) = n^{-2}(4I(f)^3 - J(f))^2,
\]

we see that the height of an \(n\)-monogenized cubic field is comparable with its discriminant.

We note that all cubic fields \(K\) are \(n\)-monogenized for some \(n \ll |\Delta(K)|^{1/4}\); see Remark 3.3. Thus we may expect that the average size of the 2-torsion subgroup of the class group over all cubic fields \(K\) ordered by absolute discriminant is the same as the average over all \(n\)-monogenized cubic fields \((K, \alpha)\) ordered by height \(H(K, \alpha)\) with \(n \ll H(K, \alpha)^{1/4}\). This is indeed the case, as we will prove in Theorem 3.2. We will also prove in §3, that the number of \(n\)-monogenized cubic fields \((K, \alpha)\) with height \(H(K, \alpha) < X\) and \(n \ll H(K, \alpha)^{1/4}\) grows as \(\asymp X\), which are the same asymptotics as for the number of cubic fields having absolute discriminant bounded by \(X\).

We may also consider, for every \(\delta \in (0, 1/4]\), the average over the much thinner family of \(n\)-monogenized cubic fields \((K, \alpha)\) satisfying \(H(K, \alpha) < X\) and \(n \leq H(K, \alpha)^{\delta}\); the number of such \(n\)-monogenized fields grows as \(\asymp X^{5/6+2\delta/3} = o(X)\) (see Theorem 3.6). In §3, we will prove the following theorem.

**Theorem 3** Let \(0 < \delta \leq 1/4\) and \(c > 0\) be real numbers, and let \(F(\leq cH^{\delta}, X)\) denote the set of isomorphism classes of \(n\)-monogenized cubic fields \((K, \alpha)\) with height \(H(K, \alpha) < X\) and \(n \leq cH(K, \alpha)^{\delta}\). Then, as \(X \to \infty\):

(a) The average size of \(\text{Cl}_2(K)\) over totally real cubic fields \(K\) in \(F(\leq cH^{\delta}, X)\) approaches \(5/4\);

(b) The average size of \(\text{Cl}_2(K)\) over complex cubic fields \(K\) in \(F(\leq cH^{\delta}, X)\) approaches \(3/2\).

Furthermore, these values remain the same if we instead average over those \(n\)-monogenized cubic fields satisfying any given set of local splitting conditions at finitely many primes.

Theorem 3 shows that the averages in Theorem 2 remain very stable, both under the imposition of local conditions and under changing the ordering of the fields from discriminant to height.
1.2 The effect of \( n \)-monogenicity on the averages in Theorem 2 for fixed \( n \)

We consider next the limiting behavior of Theorem 3 as \( \delta \) approaches 0. First, we consider the family of monogenized cubic fields, ordered by height. In this case, we find that the average values appearing in Theorems 2 and 3 do change. We have the following theorem:

**Theorem 4** Let \( K \) run through all isomorphism classes of monogenized cubic fields ordered by height. Then:

(a) The average size of \( \text{Cl}_2(K) \) over totally real cubic fields \( K \) is \( 3/2 \);

(b) The average size of \( \text{Cl}_2(K) \) over complex cubic fields \( K \) is \( 2 \).

Furthermore, these values remain the same if we instead average over those monogenized cubic fields satisfying any given set of local splitting conditions at finitely many primes.

Surprisingly, the values in Theorem 4 are different from those in Theorems 2 and 3. In particular, these class groups do not appear to be random groups in the same sense as Cohen–Lenstra–Martinet. It therefore appears that monogenicity has an altering effect on the class groups of cubic fields! To be more precise, it appears that on average, monogenicity has a doubling effect on the nontrivial part of the 2-torsion subgroups of the class groups of cubic fields.

More generally, we may ask what happens when we restrict Theorem 3 to \( n \)-monogenized cubic fields for any fixed positive integer \( n \). Once again, the averages occurring in Theorems 2 and 3 change as a rather interesting function of \( n \):

**Theorem 5** Fix a positive integer \( n \) and write \( n = m^2k \) where \( k \) is squarefree. Let \( \sigma(k) \) denote the sum of the divisors of \( k \). Let \( (K, \alpha) \) run through all isomorphism classes of \( n \)-monogenized cubic fields ordered by height. Then:

(a) The average size of \( \text{Cl}_2(K) \) over totally real cubic fields \( K \) is \( \frac{5}{4} + \frac{1}{\sigma(k)} \);

(b) The average size of \( \text{Cl}_2(K) \) over complex cubic fields \( K \) is \( \frac{3}{2} + \frac{1}{2\sigma(k)} \).

Furthermore, these values remain the same if we instead average over those \( n \)-monogenized cubic fields satisfying any given set of local splitting conditions at finitely many primes.

We also prove that the average sizes of \( \text{Cl}_2(K) \) in Theorem 5 surprisingly do change further if the \( n \)-monogenized cubic fields \( (K, \alpha) \) being averaged over satisfy specified local conditions at primes dividing \( k \), where \( n = m^2k \) and \( k \) is squarefree. In Theorem 7, we compute the average size of \( \text{Cl}_2(K) \) as \( K \) varies over a family of \( n \)-monogenized cubic fields satisfying any finite set (and certain infinite sets) of local conditions. One simple consequence of Theorem 7 is:

**Theorem 6** Let \( n \) be a fixed positive integer, and write \( n = m^2k \) where \( k \) is squarefree. Let \( (K, \alpha) \) run through all isomorphism classes of \( n \)-monogenized cubic fields having discriminant prime to \( k \), ordered by height. Then:

(a) The average size of \( \text{Cl}_2(K) \) over totally real cubic fields \( K \) is \( 3/2 \) if \( k = 1 \) and \( 5/4 \) otherwise;

(b) The average size of \( \text{Cl}_2(K) \) over complex cubic fields \( K \) is \( 2 \) if \( k = 1 \) and \( 3/2 \) otherwise.

Furthermore, these values remain the same if we instead average over such monogenized cubic fields satisfying any given set of local splitting conditions at finitely many primes.

Note that the averages occurring in Theorem 6 agree with the larger values in Theorem 4 when \( n \) is a square, but with the smaller values in Theorems 2 and 3 otherwise. Furthermore, Theorems 5
and 6 imply that for a positive integer $n$, and a prime $p$ dividing $n$ to odd order, ramification at $p$ causes, on average, an increasing effect on the 2-torsion in the class groups of $n$-monogenic cubic fields.

Theorems 4, 5, and 6 all follow from our main result, Theorem 7 below, which determines the average sizes of $\text{Cl}_2$ and $\text{Cl}_2^+$, the 2-torsion subgroups of class groups and narrow class groups, respectively, in any “large” family of $n$-monogenic cubic fields defined by local conditions at primes.

For a prime $p$, let $T_p$ denote the set of all isomorphism classes of $n$-monogenic étale cubic extensions of $\mathbb{Q}_p$, i.e., the set of all isomorphism classes of pairs $(\mathcal{K}_p, \alpha_p)$, where $\mathcal{K}_p$ is an étale cubic extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_p$, and $\alpha_p \in \mathcal{O}_p$ is primitive in $\mathcal{O}_p/\mathbb{Z}_p$, such that the $p$-part of the index of $\mathbb{Z}_p[\alpha_p]$ in $\mathcal{O}_p$ is equal to the $p$-part of $n$; here, two pairs $(\mathcal{K}_p, \alpha_p)$ and $(\mathcal{K}_p', \alpha_p')$ are isomorphic if $\mathcal{K}_p$ and $\mathcal{K}_p'$ are isomorphic as $\mathbb{Q}_p$-algebras and, under such an isomorphism, $\alpha_p$ is mapped to $\alpha_p' + m$ for some $m \in \mathbb{Z}_p$. In Remark 2.20, we will see that the set $T_p$ naturally injects into a closed subset of the set of cubic polynomials over $\mathbb{Z}_p$ with leading coefficient $n$, which equips $T_p$ with a topology and a measure.

For each prime $p$, let $\Sigma_p \subset T_p$ be an open and closed set whose boundary has measure 0. We say that $\Sigma = (\Sigma_p)_p$ is a large collection of local specifications for $n$ if, for all but finitely many primes $p$, the set $\Sigma_p$ contains all pairs $(\mathcal{K}_p, \alpha_p)$ such that $\mathcal{K}_p$ is not a totally ramified cubic extension of $\mathbb{Q}_p$. Let $F(n, X)$ denote the set of isomorphism classes of all $n$-monogenic cubic fields $(K, \alpha)$ such that $H(K, \alpha) < X$. Given a large collection $\Sigma = (\Sigma_p)_p$ of local specifications, let $F_\Sigma(n, X)$ denote the subset of $F(n, X)$ consisting of pairs $(K, \alpha)$ such that for all primes $p$, we have $(K \otimes \mathbb{Q}_p, \alpha) \in \Sigma_p$.

For a prime $p$ dividing $n$, we say that an element $(\mathcal{K}_p, \alpha_p) \in T_p$ is sufficiently ramified if one of the following two conditions is satisfied:

(a) $\mathcal{K}_p$ is a totally ramified cubic extension of $\mathbb{Q}_p$;

(b) $\mathcal{K}_p = \mathbb{Q}_p \times F$, where $F$ is a ramified quadratic extension of $\mathbb{Q}_p$, and $\mathbb{Z}_p[\alpha_p] = \mathbb{Z}_p \times \mathcal{O}$, where $\mathcal{O}$ is an order in $F$.

For a prime $p$ dividing $n$, we define the local sufficiently-ramified density $\rho_p(\Sigma_p)$ of a large collection $\Sigma = (\Sigma_p)_p$ to be density of the set of sufficiently-ramified elements within $\Sigma_p$. The global sufficiently-ramified density $\rho(\Sigma)$ with respect to $n$ is then the product of $\rho_p(\Sigma_p)$ over all primes $p$ that divide $n$ to an odd power. Our main theorem is as follows:

**Theorem 7 (Main $n$-monogenic theorem)** Let $n$ be a positive integer, and let $\Sigma$ be a large collection of local specifications for $n$. Then, as $X \to \infty$:

(a) The average size of $\text{Cl}_2(K)$ over totally real cubic fields $K$ in $F_\Sigma(n, X)$ approaches $\frac{5}{2} + \frac{1}{4}\rho(\Sigma)$;

(b) The average size of $\text{Cl}_2(K)$ over complex cubic fields $K$ in $F_\Sigma(n, X)$ approaches $\frac{3}{2} + \frac{1}{2}\rho(\Sigma)$;

(c) The average size of $\text{Cl}_2^+(K)$ over totally real cubic fields $K$ in $F_\Sigma(n, X)$ approaches $2 + \frac{1}{2}\rho(\Sigma)$.

Theorem 4 follows from Theorem 7 by noting that there are no primes dividing $n = 1$ to odd order, and so $\rho(\Sigma) = 1$ for every large collection of local specifications $\Sigma$. Theorem 5 follows from Theorem 7 by Corollary 4.36, which states that the local sufficiently-ramified density $\rho_p(T_p)$ is equal to $1/(p+1)$ for each $p | k$. Finally, Theorem 6 follows from Theorem 7 by noting that when $\Sigma_p \subset T_p$ contains no extensions ramified at primes dividing $k$, then $\rho_p(\Sigma_p) = 0$.

Theorem 7 also implies an analogous increasing effect of $n$-monogenicity on the 2-torsion subgroup of the narrow class group, and this increase on average is from 2 (cf. [12, Theorem 1(c)]) to 2.5, in the case when $n$ is a square.
Taken together, Theorems 2, 5, 6, and 7 thus paint the following picture. For integers \( n \) that are perfect squares, \( n \)-monogenicity has a doubling effect on the nontrivial part of the 2-torsion in the class groups of cubic fields. For nonsquare integers \( n = m^2 k \) with \( k > 1 \) squarefree, \( n \)-monogenicity still has an increasing effect on the nontrivial part of the 2-torsion in the class groups of cubic fields. This increase goes to 0 as \( k \) tends to infinity; moreover, the increase is concentrated on those \( n \)-monogenized cubic fields that are sufficiently ramified at every prime dividing \( k \). The analogous phenomena also occur for the 2-torsion in narrow class groups of these fields.

1.3 Method of proof

In \( \S 2 \), we prove parametrizations of \( n \)-monogenized cubic fields and index 2 subgroups of class groups of \( n \)-monogenic cubic fields, respectively, in terms of suitable spaces of forms. Namely, in \( \S 2.1 \), we begin by proving that there is a natural bijection between

(a) \( n \)-monogenized cubic fields \((K, \alpha)\), and
(b) certain cubic polynomials \( f \) with integer coefficients whose leading coefficient is \( n \).

In \( \S \S 2.2–2.3 \), we use the parametrization of quartic rings in [3] to give a natural bijection between:

(a) index 2 subgroups of narrow class groups of \( n \)-monogenized cubic fields \((K, \alpha)\) whose corresponding cubic polynomial is \( f(x) \), and
(b) certain \( \text{SL}_3(\mathbb{Z}) \)-orbits on pairs \((A, B)\) of integer-coefficient ternary quadratic forms such that \( 4 \det(Ax - B) = f(x) \).

Our main results are then proved by counting the relevant elements in (b) having bounded height in both of the above parametrizations, and then computing the limiting ratios.

More precisely, Theorem 3 is proven in \( \S 3 \) by counting \( \text{SL}_3(\mathbb{Z}) \)-orbits of pairs \((A, B)\) of integer-coefficient ternary quadratic forms such that \( f(x) := 4 \det(Ax - B) \) satisfies \( H(f) < X \) and \( n = 4 \det(A) < cH(f)^\delta \). The technique to obtain this count is a suitable adaptation of the averaging and sieving methods developed in [4, 5] for counting by discriminant, but modified to allow imposing constraints on two different parameters (the height \( H \) and index \( n \)).

Next, Theorem 7 is proved in \( \S 4 \) by counting pairs \((A, B)\) such that \( f(x) := 4 \det(Ax - B) \) satisfies \( H(f) < X \), but where \( 4 \det(A) = n \) is fixed. This is much more subtle than the count in Theorem 3 where \( n \) varies. It requires counting pairs \((A, B)\) in a fundamental domain for the action of the group \( \text{SL}_3(\mathbb{Z}) \) on the hypersurface defined by \( \det(A) = n/4 \). To carry out this count, we sum the total number of \( B \) in a fundamental domain for the action of \( \text{SO}_4(\mathbb{Z}) \) on the space of real ternary quadratic forms, where \( A \) runs over a set of representatives for the distinct \( \text{SL}_3(\mathbb{Z}) \)-classes of integer-coefficient ternary quadratic forms having determinant \( n/4 \). This calculation is intimately related to the mass calculations for quadratic forms of a given determinant in [29, 30]; see also the work of Ibukiyama and Saito [32] on Shintani zeta functions associated to spaces of quadratic forms. This count, and the analogous counts with suitable congruence conditions, enable the completion of the proof of the general result, Theorem 7, implying in particular Theorems 4, 5, and 6.

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2 Parametrizations involving quartic rings and $n$-monogenized cubic rings

The purpose of this section is to describe the connection between 2-torsion subgroups in the class groups of $n$-monogenized cubic fields and pairs $(A, B)$ of integer-coefficient ternary quadratic forms with $4 \det(A) = n$.

In §2.1, we adapt the Delone–Faddeev parametrization [22] of cubic rings to obtain a parametrization of isomorphism classes of $n$-monogenized cubic rings by integer-coefficient binary cubic forms having leading coefficient $n$. In §2.2, we recall the parametrization of quartic rings by pairs of integer-coefficient ternary quadratic forms, as developed in [3], and use these two parametrizations (in conjunction with class field theory) in §2.3 to parametrize index 2 subgroups of class groups of $n$-monogenized cubic fields. Finally, in §2.4, we then discuss versions of these parametrization results where $\mathbb{Z}$ is replaced by a principal ideal domain $R$.

2.1 Parametrization of $n$-monogenized cubic rings

Definition 2.1 A cubic ring (resp. quartic ring) is a ring that is free of rank 3 (resp. rank 4) as a $\mathbb{Z}$-module.

The works of Levi [35], Delone–Faddeev [22], and Gan–Gross–Savin [27] give a parametrization of cubic rings by $GL_2(\mathbb{Z})$-orbits of integer-coefficient binary cubic forms.

Theorem 2.2 ([35],[22],[27]) There is a bijection between isomorphism classes of cubic rings $C$ with a chosen basis $\langle \bar{\omega}, \bar{\theta} \rangle$ of $C/\mathbb{Z}$, and integer-coefficient binary cubic forms $f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$. The bijection is given by

$$(C, \langle \bar{\omega}, \bar{\theta} \rangle) \mapsto 1 \wedge (x\omega + y\theta) \wedge (x\omega + y\theta)^2 = (ax^3 + bx^2y + cxy^2 + dy^3)(1 \wedge \omega \wedge \theta) \in \wedge^3 C$$

where $\omega, \theta \in C$ denote any lifts of $\bar{\omega}, \bar{\theta} \in C/\mathbb{Z}$.

See also [9, §2] for a concise proof of Theorem 2.2.

Notation 2.3 For any ring $R$, an element $\gamma \in GL_2(R)$ acts on the space $U(R)$ of binary cubic forms $f$ with coefficients in $R$ via the twisted action

$$\gamma \cdot f(x, y) := \det(\gamma)^{-1}f((x, y)\gamma)$$

where we view $(x, y)$ as a row vector.

We then have the following immediate corollary of Theorem 2.2.

Corollary 2.4 There is a bijection between isomorphism classes of cubic rings and $GL_2(\mathbb{Z})$-orbits on the space $U(\mathbb{Z})$ of integer-coefficient binary cubic forms.

Corollary 2.4 follows from Theorem 2.2 by noting that the action of $GL_2(\mathbb{Z})$ on the basis $\langle \omega, \theta \rangle$ of $C/\mathbb{Z}$ leads to the action (4) on the corresponding binary cubic form $f$.

We observe that the leading coefficient $a$ of the binary cubic form in the bijection of Theorem 2.2 is equal to the (signed) index of $\mathbb{Z}[\omega]$ in $C$, because setting $x = 1$ and $y = 0$ in (3) yields $1 \wedge \omega \wedge \omega^2 = a(1 \wedge \omega \wedge \theta)$. Hence cubic rings having a monogenic subring of index $n$ may be classified in terms of binary cubic forms having (leading) $x^3$-coefficient $n$. 

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Definition 2.5 A pair \((C, \alpha)\) is an \(n\)-monogenized cubic ring if \(C\) is a cubic ring, \(\alpha\) is a primitive element of \(C/\mathbb{Z}\), and \(|C : \mathbb{Z}[\alpha]| = n\). Two \(n\)-monogenized cubic rings \((C, \alpha)\) and \((C', \alpha')\) are isomorphic if there is a ring isomorphism \(C \to C'\) that maps \(\alpha\) to \(\alpha' + m\) for some \(m \in \mathbb{Z}\).

Thus an \(n\)-monogenized cubic ring is a cubic ring \(C\) equipped with a basis \(\langle \bar{\omega}, \bar{\theta} \rangle\) of \(C/\mathbb{Z}\), but where \((C, \langle \bar{\omega}, \bar{\theta} \rangle)\) and \((C, \langle \bar{\omega}, k\bar{\omega} + \bar{\theta} \rangle)\) are considered isomorphic, as only the basis element \(\bar{\omega}\) is relevant in defining the monogenic subring \(\mathbb{Z}[\omega]\) of index \(n\) in \(C\). The change-of-basis \(\langle \bar{\omega}, \bar{\theta} \rangle \mapsto \langle \bar{\omega}, k\bar{\omega} + \bar{\theta} \rangle\) corresponds to the transformation \(f(x, y) \mapsto f(x + ky, y)\) on the associated binary cubic form.

Notation 2.6 For a ring \(R\), we let \(M(R) \subset \text{GL}_2(R)\) denote the subgroup of lower triangular unipotent matrices
\[
M(R) := \left\{ \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} : k \in R \right\}. \tag{5}
\]
For \(n \in R\), let \(U_n(R) \subset U(R)\) denote the subset of binary cubic forms having (leading) \(x^3\)-coefficient \(n\).

Then we have proven the following theorem.

Theorem 2.7 There is a bijection between isomorphism classes of \(n\)-monogenized cubic rings and \(M(\mathbb{Z})\)-orbits on the set \(U_n(\mathbb{Z})\) of integer-coefficient binary cubic forms having leading coefficient \(n\).

2.2 Parametrization of quartic rings having \(n\)-monogenic cubic resolvent rings

In this section, we recall the parametrization in [3] of pairs \((Q, C)\), where \(Q\) is a quartic ring and \(C\) is a cubic resolvent ring of \(Q\).

Notation 2.8 For any ring \(R\), let \(V(R)\) denote the space of pairs of ternary quadratic forms with coefficients in \(R\). We represent an element in \(V(R)\) as \((A, B)\), where \(A(x_1, x_2, x_3) = \sum_{1 \leq i \leq j \leq 3} a_{ij} x_i x_j\) and \(B(x_1, x_2, x_3) = \sum_{1 \leq i \leq j \leq 3} b_{ij} x_i x_j\) with \(a_{ij}, b_{ij} \in R\). When 2 is not a zero divisor in \(R\), we may also represent an element \((A, B) \in V(R)\) as a pair of \(3 \times 3\) symmetric matrices with entries in \(R[1/2]\) via the Gram identification
\[
(A, B) = \left( \frac{1}{2} \begin{bmatrix} 2a_{11} & a_{12} & a_{13} \\ a_{12} & 2a_{22} & a_{23} \\ a_{13} & a_{23} & 2a_{33} \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 2b_{11} & b_{12} & b_{13} \\ b_{12} & 2b_{22} & b_{23} \\ b_{13} & b_{23} & 2b_{33} \end{bmatrix} \right).
\]

The group \(G(R) \subset \text{GL}_2(R) \times \text{GL}_3(R)\) defined by
\[
G(R) := \{(g_2, g_3) \in \text{GL}_2(R) \times \text{GL}_3(R) : \det(g_2) \det(g_3) = 1\}
\]
acts naturally on \(V(R)\) as follows:
\[
(g_2, g_3) \cdot (A, B) = (g_3 A g_3^t, g_3 B g_3^t) \cdot g_2^t. \tag{6}
\]
Given an element \((A, B) \in V(R)\), we define its cubic resolvent form \(\text{Res}(A, B) \in U(R)\) by
\[
\text{Res}(A, B) := 4 \det(Ax - By). \tag{7}
\]
Since the action of \(G(R)\) on \(V(R)\) in (6) and the resolvent map \(\text{Res} : V(R) \to U(R)\) in (7) are
defined by integer-coefficient polynomials in the entries of \((g_2, g_3) \in G(R)\) and the coefficients \(a_{ij}\) of \(A\) and \(b_{ij}\) of \(B\), we see that the action of \(G(R)\) on \(V(R)\) and the definitions of \(4 \det(A)\) and \(\text{Res}(A, B)\) for \((A, B) \in V(R)\) make sense for arbitrary rings \(R\).

The following theorem is proved in [3].

**Theorem 2.9 ([3])** There is a bijection between pairs \((A, B) \in V(\mathbb{Z})\) of integer-coefficient ternary quadratic forms and isomorphism classes of pairs \(((Q, \langle \bar{\alpha}, \bar{\beta}, \bar{\gamma} \rangle), (C, \langle \bar{\omega}, \bar{\theta} \rangle))\), where \(Q\) is a quartic ring with a chosen basis \(\langle \bar{\alpha}, \bar{\beta}, \bar{\gamma} \rangle\) of \(Q/\mathbb{Z}\) and \(C\) is a cubic resolvent ring of \(Q\) with a chosen basis \(\langle \bar{\omega}, \bar{\theta} \rangle\) of \(C/\mathbb{Z}\). Furthermore, under this bijection, \((C, \langle \bar{\omega}, \bar{\theta} \rangle)\) is the data corresponding to the cubic resolvent form of \((A, B)\) under Theorem 2.2.

A complete description of the construction of \(((Q, \langle \bar{\alpha}, \bar{\beta}, \bar{\gamma} \rangle), (C, \langle \bar{\omega}, \bar{\theta} \rangle))\) from \((A, B)\) can be found in [3, §3.2, §3.3]. Theorem 2.9 has the following immediate corollary.

**Corollary 2.10** There is a bijection between \(G(\mathbb{Z})\)-orbits on the space \(V(\mathbb{Z})\) of pairs of integer-coefficient ternary quadratic forms and isomorphism classes of pairs \((Q, C)\), where \(Q\) is a quartic ring and \(C\) is a cubic resolvent ring of \(Q\).

Corollary 2.10 follows from Theorems 2.2 and 2.9 by noting that the action of \(G \subset GL_2(\mathbb{Z}) \times GL_3(\mathbb{Z})\) on the bases \(\langle \bar{\alpha}, \bar{\beta}, \bar{\gamma} \rangle\) of \(Q/\mathbb{Z}\) and \(\langle \bar{\omega}, \bar{\theta} \rangle\) of \(C/\mathbb{Z}\) leads to the action (6) on the corresponding pair \((A, B)\) of ternary quadratic forms.

Finally, the identical reasoning now yields the following parametrization of quartic rings having \(n\)-monogenized cubic rings.

**Corollary 2.11** There is a bijection between \(M(\mathbb{Z}) \times SL_3(\mathbb{Z})\)-orbits on the set \(V_n(\mathbb{Z})\) of pairs \((A, B)\) of integer-coefficient ternary quadratic forms with \(\det(A) = n\) and isomorphism classes of pairs \((Q, (C, \alpha))\), where \(Q\) is a quartic ring and \((C, \alpha)\) is an \(n\)-monogenized cubic resolvent ring of \(Q\).

### 2.3 Parametrization of index 2 subgroups of class groups of cubic fields

With the parametrizations of quartic rings having \(n\)-monogenized cubic rings established, we are now in a position to parametrize index 2 subgroups in the class groups of \(n\)-monogenized cubic fields. This parametrization will be useful to us because the number of elements of order 2 in a finite abelian group \(A\) is equal to the number of index 2 subgroups of \(A\).

We will need the following definition.

**Definition 2.12** For a maximal quartic ring \(Q\) and a prime \(p\), we say that \(Q\) is overramified at \(p\) if the ideal \(p\mathbb{Z}\) factors in \(Q\) as either \(P^4\), \(P^2\), or \(P_1^2P_2^2\), where \(P\), \(P_1\), and \(P_2\) are prime ideals of \(Q\). A maximal quartic ring \(Q\) is overramified at \(\infty\) if \(Q \otimes \mathbb{R} \cong \mathbb{C}^2\) as \(\mathbb{R}\)-algebras. A maximal quartic ring \(Q\) is nowhere overramified if it is not overramified at any (finite or infinite) place.

The significance of being nowhere overramified comes from the following theorem of Heilbronn.

**Theorem 2.13 ([31])** Let \(K_4\) be a totally real \(S_4\)-quartic field, and \(K_3\) a cubic (resolvent) field inside \(K_{24}\), the Galois closure of \(K_4\). Let \(K_6\) be the non-Galois sextic field in \(K_{24}\) containing \(K_3\). Then the quadratic extension \(K_6/K_3\) is unramified precisely when the quartic field \(K_4\) is nowhere overramified. Conversely, every unramified quadratic extension \(K_6/K_3\) of a cubic \(S_3\)-field \(K_3\) lies in the Galois closure of a nowhere overramified quartic field \(K_4\) which is unique up to conjugacy.
**Notation 2.14** For the maximal order $C$ in a cubic field $K_3$, let $\text{Cl}_2(C)$ and $\text{Cl}_2^+(C)$ denote the 2-torsion subgroups of the class group and narrow class group of $C$, respectively. Let $\text{Cl}_2(C)^*$ and $\text{Cl}_2^+(C)^*$ denote the groups dual to $\text{Cl}_2(C)$ and $\text{Cl}_2^+(C)$, respectively. Then the set of nontrivial elements of $\text{Cl}_2(C)^*$ (resp. $\text{Cl}_2^+(C)^*$) are in bijection with the set of index two subgroups of $\text{Cl}(C)$ (resp. $\text{Cl}^+(C)$) simply by mapping a character to its kernel.

Theorems 2.9 and 2.13, together with class field theory, now immediately yield a parametrization of index 2 subgroups of the class groups and narrow class groups of cubic fields.

**Theorem 2.15** Let $C$ be the maximal order in an $S_3$-cubic field $K_3$, and let $f(x,y)$ be a binary cubic form corresponding to $C$ under Theorem 2.2.

(a) If $\Delta(K_3) > 0$, then there is a canonical bijection between elements of $\text{Cl}_2^+(C)^*$ and $\text{SL}_3(\mathbb{Z})$-orbits on $\text{Res}^{-1}(f) \subset V(\mathbb{Z})$.

Under this bijection, elements of $\text{Cl}_2(C)^* \subset \text{Cl}_2^+(C)^*$ correspond to $\text{SL}_4(\mathbb{Z})$-orbits on pairs $(A,B) \in V(\mathbb{Z})$ such that $A(x,y,z) = B(x,y,z) = 0$ has a nonzero solution over $\mathbb{R}$.

(b) When $\Delta(K_3) < 0$, there is a canonical bijection between elements of $\text{Cl}_2^+(C)^* = \text{Cl}_2(C)^*$ and $\text{SL}_3(\mathbb{Z})$-orbits on $\text{Res}^{-1}(f) \subset V(\mathbb{Z})$.

**Proof:** By class field theory, the nontrivial elements of $\text{Cl}_2(C)^*$ (resp. $\text{Cl}_2^+(C)^*$) correspond to quadratic extensions $K_6/K_3$ that are unramified at all places (resp. unramified at all finite places). These quadratic extensions $K_6/K_3$, by Theorem 2.13, in turn correspond to quartic fields $K_4$ whose maximal orders $Q$ are nowhere overramified (resp. not overramified at all finite places). In this scenario, we have the equality of discriminants $\Delta(Q) = \Delta(C)$ (by [31]), and so $C$ is the unique cubic resolvent ring of $Q$ ([3, Def. 8]). The bijections in (a) and (b), for nontrivial elements of $\text{Cl}_2^+(K_3)^*$ and $\text{SL}_3(\mathbb{Z})$-orbits on $\text{Res}^{-1}(f) \subset V(\mathbb{Z})$ corresponding to integral domains $Q$, now follow from Theorem 2.9. The bijections in both (a) and (b) of Theorem 2.15 are completed by sending the identity element in $\text{Cl}_2^+(K_3)^*$ to the unique $\text{SL}_3(\mathbb{Z})$-orbit on $\text{Res}^{-1}(f) \subset V(\mathbb{Z})$ corresponding to the quartic ring $Q = \mathbb{Z} \times C$, whose unique cubic resolvent ring is $C$ as well. □

### 2.4 Parametrizations over principal ideal domains

**Definition 2.16** Let $R$ be a principal ideal domain. A cubic ring (resp. quartic ring) over $R$ is an $R$-algebra that is free of rank 3 (resp. rank 4) as an $R$-module.

The following theorem, proved by Gross and Lucianovic [28] and in [10], generalizes the parametrization of cubic and quartic rings over $\mathbb{Z}$ in [22] and [3], respectively, to the setting of cubic and quartic rings over a principal ideal domain. For a formulation and proof in the vastly more general case where $\mathbb{Z}$ is replaced by an arbitrary ring, or even an arbitrary base scheme, see Wood [38].

**Theorem 2.17** ([28, Proposition 2.1], [10, Theorem 5]) Let $R$ be a principal ideal domain.

(a) There is a natural bijection between isomorphism classes of cubic rings over $R$ and $\text{GL}_2(R)$-orbits on $U(R)$. Under this bijection, the group of automorphisms of a cubic ring over $R$ is isomorphic to the stabilizer in $\text{GL}_2(R)$ of the corresponding binary cubic form in $U(R)$.

(b) There is a natural bijection between isomorphism classes of pairs $(Q,C)$, where $Q$ is a quartic ring over $R$ and $C$ is a cubic resolvent ring of $Q$, and $G(R)$-orbits on $V(R)$. Under this bijection, the group of automorphisms of a quartic ring $Q$ over $R$ is isomorphic to the stabilizer in $G(R)$ of the corresponding pair of ternary quadratic forms in $V(R)$. 

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Definition 2.18 Let $R$ be a principal ideal domain. For $n \in R$, an $n$-monogenized cubic ring over $R$ is a pair $(C, \alpha)$, where $C$ is a cubic ring over $R$, the element $\alpha \in C$ is primitive in $C/R$, and the $R$-ideal $(C : R[\alpha]) := \{ r \in R : r \wedge C \subseteq \wedge^3 R[\alpha] \}$ as $R$-modules is generated by $n$. Two $n$-monogenized cubic rings $(C, \alpha)$ and $(C', \alpha')$ over $R$ are isomorphic if there is an $R$-algebra isomorphism $C \rightarrow C'$ that maps $\alpha$ to $\alpha' + m$ for some $m \in R$.

Note that if $R = \mathbb{Z}$ or $R = \mathbb{Z}_p$, then $(C : R[\alpha])$ is the ideal generated by the usual index $[C : R[\alpha]]$.

The proofs of Theorems 2.7 and 2.17 have the following consequence.

Corollary 2.19 Let $R$ be a principal ideal domain. There is a bijection between isomorphism classes of $n$-monogenized cubic rings over $R$ and $M(R)$-orbits on the set $U_n(R)$.

Remark 2.20 When $R = \mathbb{Z}_p$, Corollary 2.19 provides a bijection between isomorphism classes of $n$-monogenized cubic rings over $\mathbb{Z}_p$ and the set

$$\{ f(x, y) = nx^3 + bx^2y + cxy^2 + dy^3 : b \in \mathbb{Z}, \ c, d \in \mathbb{Z}_p, \ 0 \leq b < \gcd(p, 3n) \}$$

which is a fundamental domain for the action of $M(\mathbb{Z}_p)$ on $U_n(\mathbb{Z}_p)$. The set (8) can be identified with $\mathbb{Z}_p \times \mathbb{Z}_p \times \{ 0, 1, \ldots, \gcd(p, 3n) - 1 \}$. We then equip the set of isomorphism classes of $n$-monogenized cubic rings over $\mathbb{Z}_p$ with the product topology and measure, via the usual topology and additive Haar measure on $\mathbb{Z}_p$ and the discrete topology and counting measure on $\{ 0, 1, \ldots, \gcd(p, 3n) - 1 \}$.

By Lemma 4.33, the set $T_p$ of isomorphism classes of étale cubic extensions of $\mathbb{Q}_p$ can be identified with an open and closed subset of $\mathbb{Z}_p \times \mathbb{Z}_p \times \{ 0, 1, \ldots, \gcd(p, 3n) - 1 \}$, namely, by identifying $T_p$ with the subset of (8) corresponding to $n$-monogenized cubic rings $(\mathcal{O}_p, \alpha_p)$ over $\mathbb{Z}_p$, where $\mathcal{O}_p$ is the maximal order in an étale cubic extension of $\mathbb{Q}_p$. The set $T_p$ also thereby inherits a topology and a measure.

We will have occasion to use Theorem 2.17 with $R$ specialized to $\mathbb{R}$, $\mathbb{Z}_p$, and $\mathbb{F}_p$. In these cases, it will be convenient to use the language of splitting types, which we now define.

Definition 2.21 Let $f$ be a binary cubic form in $U(\mathbb{R})$, $U(\mathbb{Z}_p)$, or $U(\mathbb{F}_p)$. When $f$ belongs to $U(\mathbb{R})$, assume that $f$ has three roots (counted with multiplicity) in $\mathbb{P}^1(\mathbb{C})$, and when $f$ belongs to $U(\mathbb{Z}_p)$ or $U(\mathbb{F}_p)$, assume that $f$ has three roots (counted with multiplicity) in $\mathbb{P}^1(\mathbb{F}_p)$. We define the splitting type of $f$ to be $(f_1^e_1, f_2^e_2, \ldots)$, where the $f_i$ are the degrees over $\mathbb{R}$ or $\mathbb{F}_p$ of the fields of definition of these roots and the $e_i$ are their respective multiplicities.

Similarly, let $(A, B)$ be an element in $V(\mathbb{R})$, $V(\mathbb{Z}_p)$, or $V(\mathbb{F}_p)$. When $(A, B)$ belongs to $V(\mathbb{R})$, assume that the intersection of the conics defined by $A$ and $B$ consists of four points (counted with multiplicity) in $\mathbb{P}^2(\mathbb{C})$. When $(A, B)$ belongs to $V(\mathbb{Z}_p)$ or $V(\mathbb{F}_p)$, assume that the intersection of the conics defined by $A$ and $B$ consists of four points (counted with multiplicity) in $\mathbb{P}^2(\mathbb{F}_p)$. We define the splitting type of $(A, B)$ to be $(f_1^e_1, f_2^e_2, \ldots)$, where the $f_i$ are the degrees over $\mathbb{R}$ or $\mathbb{F}_p$ of the fields of definition of these points and the $e_i$ are their respective multiplicities.

We conclude this section with a discussion of local versions of Theorem 2.15. Let $p$ be a prime and let $K_m$ be an étale degree $m$ extension of $\mathbb{Q}_p$. We write $K_m = \prod_{i=1}^k L_i$ as a product of fields $L_i$. An unramified degree 2 extension $K_{2m}$ of $K_m$ is a product $\prod_{i=1}^k L_i'$, where $L_i'$ is either $L_i \times L_i$ (i.e., $L_i$ splits) or the (unique) quadratic unramified extension of $L_i$ (i.e., $L_i$ is inert). Let $\text{Aut}_{K_m}(K_{2m})$ denote the group of automorphisms of $K_{2m}$ fixing $K_m$ pointwise. As there are $2^k$ different unramified extensions $K_{2m}$ of $K_m$, and each such extension has automorphism group $(\mathbb{Z}/2\mathbb{Z})^k$, we have

$$\sum_{[K_{2m}:K_m] = 2}^{\text{unramified}} \frac{1}{|\text{Aut}_{K_m}(K_{2m})|} = 1.$$
The norm of the discriminant of each $K_{2m}/K_m$ is either a square or nonsquare in $\mathbb{Z}_p^\times$. Indeed,

$$N_{K_m/Q_p}\Delta(K_{2m}/K_m) = \prod_{i=1}^{k} N_{L_i/Q_p}\Delta(L_i'/L_i).$$

Let $e_i$ denote the ramification degree of $L_i$; then $N_{K_m/Q_p}\Delta(K_{2m}/K_m)$ is a square if and only if there are an even number of $i$ for which both $L_i$ is inert and $e_i$ is odd.

We now focus on the case $m = 3$. The arguments in [1, §2] imply the following.

**Theorem 2.22 ([1])** There is a bijection between étale non-overramified quartic extensions $K_4/Q_p$ with cubic resolvent $K_3$ and étale unramified quadratic extensions $K_6/K_3$ such that $N_{K_3/Q_p}\Delta(K_6/K_3)$ is a square in $\mathbb{Z}_p^\times$.

**Remark 2.23** The bijection of Theorem 2.22 can be made explicit using the correspondences of Theorem 2.17. Let $K_3$ be an étale cubic algebra corresponding to a binary cubic form $f(x, y)$. Let $P = \{P_1, P_2, P_3\}$ denote the set of roots of $f(x, y)$ in $\mathbb{P}^1(\overline{\mathbb{Q}}_p)$. Let $K_4$ be an étale non-overramified quartic extension of $Q_p$ with resolvent $K_3$ corresponding to a pair $(A, B)$ of ternary quadratic forms. Let $Q = \{Q_1, Q_2, Q_3, Q_4\}$ denote the set of common zeros of $A$ and $B$ in $\mathbb{P}^2(\overline{\mathbb{Q}}_p)$. These two sets $P$ and $Q$ come equipped with an action of the absolute Galois group $G_{Q_p}$ of $Q_p$. Furthermore, a set of three points (resp. four points) together with an action of $G_{Q_p}$ uniquely determines an étale cubic (resp. quartic) extension of $Q_p$, and in this manner $P$ corresponds to the étale cubic extension $K_3$ and $Q$ corresponds to the étale quartic extension $K_4$ of $Q_p$.

Let $P'$ denote the set of pairs of lines $(L_1, L_2)$ where $L_1$ passes through two of the points $Q_i$ and $L_2$ passes through the other two $Q_i$. Then $P'$ has three elements and we have a natural bijection $P \to P'$ that is $G_{Q_p}$-equivariant. Indeed, we simply associate to $P_i = (x_i, y_i)$ the zero set of $Ax_i - By_i$, which is a pair of lines in $\mathbb{P}^2(\overline{\mathbb{Q}}_p)$ since $4\det(Ax_i - By_i) = f(x_i, y_i) = 0$. Moreover, since the four points in $Q$ are in general position, both of these lines must pass through exactly two points in $Q$. We may thus naturally identify the Galois sets $P$ and $P'$. Let $L$ denote the set of six lines passing through each choice of two points in $Q$. The Galois action on $Q$ induces one on $L$, which in turn yields an étale sextic extension $K_6$ of $Q_p$; this is indeed the unramified quadratic extension of $K_3$ with discriminant of square norm corresponding to $K_4$ in Theorem 2.22.

### 3 The mean number of 2-torsion elements in the class groups of $n$-monogenized cubic fields ordered by height with varying $n$

The purpose of this section is to prove a more general version of Theorem 3 where we also allow our $n$-monogenized cubic fields to satisfy certain infinite sets of congruence conditions. We fix real numbers $c$ and $\delta$ satisfying $c > 0$ and $0 < \delta \leq 1/4$.

**Definition 3.1** For each prime $p$, let $S_p \subseteq U(\mathbb{Z}_p)$ be an open and closed nonempty subset whose boundary has measure 0. Then $S := (S_p)_p$ is a **collection of cubic local specifications**. The collection $S = (S_p)_p$ is **large** if, for all but finitely many primes $p$, the set $S_p$ contains all elements $f \in U(\mathbb{Z}_p)$ with $p^2 \mid \Delta(f)$. To each $S_p$, we associate the set $\Sigma_p$ of pairs $(K_p, \alpha_p)$, up to isomorphism, where $K_p$ is an étale cubic extension of $Q_p$ with ring of integers $O_p$, $\alpha_p$ is an element of $O_p$ that is primitive in $O_p/Q_p$, and the pair $(O_p, \alpha_p)$ corresponds to some $f(x, y) \in S_p$. The collection $\Sigma := (\Sigma_p)_p$ is called **large** if $S$ is large. For a large collection $\Sigma$, let $F_{\Sigma}(\leq cH^k, X)$ denote the set of isomorphism classes of $n$-monogenized cubic fields $(K, \alpha)$ such that $n \leq cH(K, \alpha)^{\delta}$, $H(K, \alpha) < X$, and $(K \otimes Q_p, \alpha) \in \Sigma_p$ for all primes $p$.

The main result of this section is then the following theorem.
Theorem 3.2 Let notation be as above. As \( X \to \infty \), we have:

(a) The average size of \( \text{Cl}_2(K) \) over totally real cubic fields \( K \) in \( F_\Sigma(\leq cH^\delta, X) \) approaches \( 5/4 \);
(b) The average size of \( \text{Cl}_2(K) \) over complex cubic fields \( K \) in \( F_\Sigma(\leq cH^\delta, X) \) approaches \( 3/2 \);
(c) The average size of \( \text{Cl}_2^+(K) \) over totally real cubic fields \( K \) in \( F_\Sigma(\leq cH^\delta, X) \) approaches \( 2 \).

That is, the averages in Theorem 2 and [12, Theorem 1] remain the same even when ordering cubic fields by height, restricting \( n \) to slowly growing ranges relative to the height, and imposing quite general local conditions on the cubic fields.

Remark 3.3 We always take \( \delta \leq 1/4 \) because every cubic field \( K \) is \( n \)-monogenized for some \( n \ll |\Delta(K)|^{1/4} \). Indeed, the covolume of \( \mathcal{O}_K \) in \( \mathcal{O}_K \otimes \mathbb{R} \) is \( |\Delta(K)|^{1/2} \), and so the length of the second successive minimum \( \alpha \in \mathcal{O}_K \) is \( \ll |\Delta(K)|^{1/4} \) (the first successive minimum being \( 1 \in \mathcal{O}_K \)). Therefore,

\[
|\Delta(\mathbb{Z}[\alpha])| = |\Delta(\langle 1, \alpha, \alpha^2 \rangle)| \ll (1 |\Delta(K)|^{1/4}|\Delta(K)|^{1/2})^2 \ll |\Delta(K)|^{3/2}.
\]

Hence

\[
|\mathcal{O}_K : \mathbb{Z}[\alpha]| = |\Delta(\mathbb{Z}[\alpha])/\Delta(\mathcal{O}_K)|^{1/2} \ll (|\Delta(K)|^{3/2}/|\Delta(K)|)^{1/2} = |\Delta(K)|^{1/4}.
\]

This section is organized as follows. In §3.1, we give asymptotics for the number of \( n \)-monogenized cubic rings of bounded height \( H \) and \( n < cH^\delta \) in terms of local volumes of certain sets of binary cubic forms. In §3.2, we determine asymptotics for the number of quartic rings with \( n \)-monogenized cubic resolvent rings, where these resolvent rings again have bounded height and bounded \( n \). In §3.3, we then prove uniformity estimates that allow us to impose conditions of maximality on these counts. The leading constants for these asymptotics are expressed as a product of local volumes of sets in \( U(R) \) and \( V(R) \), where \( R \) ranges over \( \mathbb{R} \) and \( \mathbb{Z}_p \) for all primes \( p \). In §3.4, we prove certain mass formulas relating étale quartic and cubic extensions of \( \mathbb{Q}_p \). Finally, in §3.5, we use these mass formulas to compute the required local volumes, concluding the proofs of Theorem 3 and Theorem 3.2.

3.1 The number of \( n \)-monogenized cubic rings of bounded height \( H \) and \( n < cH^\delta \)

In this subsection, we determine the asymptotic number of \( n \)-monogenized cubic rings of bounded index and height. By Theorem 2.7, such rings are parametrized by \( M(\mathbb{Z}) \)-orbits on binary cubic forms in \( U(\mathbb{Z}) \) of bounded height \( H \) whose \( x^3 \)-coefficient is positive and less than \( cH^\delta \).

Definition 3.4 For a binary cubic form \( f(x,y) = ax^3 + bx^2y + cxy^3 + dy^3 \in U(\mathbb{R}) \), we define the index \( \text{ind} \), the \( F \)-invariants \( I \) and \( J \), height \( H \), and discriminant \( \Delta \) of \( f \) by:

\[
\text{ind}(f) := a; \\
I(f) := b^2 - 3ac; \\
J(f) := -2b^3 + 9abc - 27a^2d; \\
H(f) := \alpha^{-2}\max\{|I(f)|^3, |J(f)|^2/4\}; \\
\Delta(f) := b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd.
\]

If the \( M(\mathbb{Z}) \)-orbit of an element \( f \in U(\mathbb{Z}) \) corresponds to an \( n \)-monogenized ring \((C, \alpha)\) by the bijection of Theorem 2.7, then:

\[
\text{ind}(f) = n; \quad I(f) = I(C, \alpha); \quad J(f) = J(C, \alpha); \quad H(f) = H(C, \alpha); \quad \Delta(f) = \Delta(C).
\]
Construction 3.5 Define the sets \( F_U^+ \) and \( F_U^-(\leq cH^\delta, X) \) as follows:
\[
F_U^+ := \{ f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3 \in U(\mathbb{R}) : \pm \Delta(f) > 0, 0 < a, 0 \leq b < 3a \};
\]
\[
F_U^-(\leq cH^\delta, X) := \{ f(x, y) \in F_U^+ : \text{ind}(f) \leq cH(f)^\delta, H(f) < X \}.
\]

Then \( F_U^+ \) is a fundamental domain for the action of \( M(\mathbb{Z}) \) on the set of binary cubic forms \( f(x, y) \in U(\mathbb{R}) \) such that \( \pm \Delta(f) > 0 \) and the \( x^3 \)-coefficient of \( f(x, y) \) is positive. (In this paper, the symbol “±” always refers to two distinct statements, one for + and one for −.)

In this subsection, we prove the following theorem.

**Theorem 3.6** Let \( N^+_3(\leq cH^\delta, X) \) denote the number of isomorphism classes of \( n \)-monogenized \( S_3 \)-orders \((C, \alpha)\) such that \( \pm \Delta(C) > 0, n \leq cH(C, \alpha)^\delta \), and \( H(C, \alpha) < X \). Then
\[
N^+_3(\leq cH^\delta, X) = \text{Vol}(F_U^+(\leq cH^\delta, X)) + O(X^{5/6}) = kX^{5/6+28/3} + O(X^{5/6}),
\]
where \( k \) and the implied \( O \)-constant depend only on \( n, c, \delta \).

We begin by counting \( M(\mathbb{Z}) \)-equivalence classes of integer-coefficient binary cubic forms, with bounded index and height, that satisfy any finite set of congruence conditions. We use the following result on counting lattice points in regions due to Davenport.

**Proposition 3.7** ([20]) Let \( R \) be a bounded, semi-algebraic multiset in \( \mathbb{R}^n \) having maximum multiplicity \( m \) that is defined by at most \( k \) polynomial inequalities each having degree at most \( \ell \). Let \( R' \) denote the image of \( R \) under any (upper or lower) triangular unipotent transformation of \( \mathbb{R}^n \). Then the number of integer lattice points (counted with multiplicity) contained in the region \( R' \) is
\[
\text{Vol}(R) + O(\max\{\text{Vol}(R), 1\}),
\]
where \( \text{Vol}(R) \) denotes the greatest \( d \)-dimensional volume of any projection of \( R \) onto a coordinate subspace obtained by equating \( n - d \) coordinates to zero, where \( d \) takes all values from 1 to \( n - 1 \). The implied constant in the second summand depends only on \( n, m, k, \) and \( \ell \).

**Notation 3.8** Let \( L \subset U(\mathbb{Z}) \) denote an \( M(\mathbb{Z}) \)-invariant set defined by congruence conditions modulo some positive integer, and let \( \nu(L) \) denote the volume of the closure of \( L \) in \( U(\hat{\mathbb{Z}}) \), where \( \hat{\mathbb{Z}} := \prod_p \mathbb{Z}_p \). For any subset \( L \subset U(\mathbb{R}) \), let \( L^\pm \) denote the set of elements \( f \in L \) with \( \pm \Delta(f) > 0 \). For real numbers \( T, X > 0 \) such that \( T = O(X^{1/4}) \), define the sets
\[
F_U^+(T; X) := \{ f(x, y) \in F_U^+ : \text{ind}(f) < 2T, X \leq H(f) < 2X \}.
\]

**Proposition 3.9** We have
\[
\#\{ f \in M(\mathbb{Z}) \setminus L^\pm : \text{ind}(f) < 2T, X \leq H(f) < 2X \} = \nu(L)\text{Vol}(F_U^+(T; X)) + O(X^{5/6}/T^{1/3}).
\]

**Proof:** The height and leading-coefficient bounds on an element \( f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3 \in F_U^+(T; X) \) imply that we have:
\[
|a| \ll T; \ |b| \ll T; \ |c| \ll X^{1/3}/T^{1/3}; \ |d| \ll X^{1/2}/T.
\]
Applying Proposition 3.7 yields
\[
\#(F_U^+(T; X) \cap L) = \nu(L)\text{Vol}(F_U^+(T; X)) + O(X^{5/6}/T^{1/3}).
\]
Since \( F_U^+ \) is a fundamental domain for the action of \( M(\mathbb{Z}) \) on \( U(\mathbb{R})^\pm \), the proposition follows. □

The main term in Proposition 3.9 grows as \( X^{5/6}T^{2/3} \). Hence the error term is smaller than the main term whenever \( T \gg X^\epsilon \).
Definition 3.10 An element $f \in U(\mathbb{Z})$ is generic if the cubic ring corresponding to $f$ is an order in an $S_3$-cubic field. Similarly, an element $v \in V(\mathbb{Z})$ is generic if the quartic ring corresponding to $v$ is an order in an $S_4$-quartic field. For any subset $L$ of $U(\mathbb{Z})$ (resp. $V(\mathbb{Z})$), we denote the set of generic elements in $L$ by $L^{\text{gen}}$.

Lemma 3.11 We have \( \#((U(\mathbb{Z}) \setminus U(\mathbb{Z})^{\text{gen}}) \cap \mathcal{F}_T^+ (T; X)) = O(X^{1/2+\varepsilon} T) \).

Proof: This follows immediately from the coefficient bounds (11) along with the fact that when $a$ and $d$ are nonzero, the coefficients $a$, $b$, and $d$ of an integer-coefficient binary cubic form $ax^3 + bx^2y + cxy^2 + dy^3 \in U(\mathbb{Z}) \setminus U(\mathbb{Z})^{\text{gen}}$ determine $c$ up to $O(\sqrt{X})$ choices (as in the proofs of [9, Lemmas 21–22]). $\square$

Proof of Theorem 3.6: Theorem 3.6 immediately follows by breaking the intervals $[1, X]$ and $[1, cX^\delta]$ into dyadic ranges, and then applying Proposition 3.9 and Lemma 3.11 to each pair of dyadic ranges. $\square$

3.2 The number of quartic rings having $n$-monogenized cubic resolvent rings of bounded height $H$ and $n < cH^\delta$

In this subsection, we determine the asymptotic number of quartic rings having an $n$-monogenized cubic resolvent ring with bounded height $H$ and $n < cH^\delta$.

Theorem 3.12 For $i \in \{0, 1, 2\}$, let $N_i(Q) \leq cH^\delta, X$ denote the number of isomorphism classes of pairs $(Q, (C, \alpha))$, where $Q$ is an order in an $S_4$-quartic field with $4-2i$ real embeddings and $(C, \alpha)$ is an $n$-monogenized cubic resolvent ring of $Q$ with $n \leq cX^\delta$ and $H < X$. Then

\[
N_i(Q) \leq cH^\delta, X = \frac{1}{m_i} \text{Vol} (\mathcal{F}_{\text{SL}_3} \cdot \mathcal{F}_V(Q) \leq cH^\delta, X)) + O(X^{5/6+\delta/3}),
\]

where $\mathcal{F}_{\text{SL}_3}$ is a fundamental domain for the action of $\text{SL}_3(\mathbb{Z})$ on $\text{SL}_3(\mathbb{R})$, the sets $\mathcal{F}_V(Q) \leq cH^\delta, X)$ and the constants $m_i$ are defined immediately after Proposition 3.21.

Notation 3.13 Let $V(\mathbb{Z})_+ := \{(A, B) \in V(\mathbb{Z}) : n = \det(A) > 0\} = V(\mathbb{Z}) \cap V(\mathbb{R})_+$.

To prove Theorem 3.12, we use the natural bijection of Corollary 2.11 between the sets

\[
\{(Q, (C, \alpha))\} \leftrightarrow (M(\mathbb{Z}) \times \text{SL}_3(\mathbb{Z})) \setminus V(\mathbb{Z})_+,
\]

where $Q$ is a quartic ring, $C$ is a cubic resolvent ring of $Q$, and $\alpha$ is an $n$-monogenizer of $C$ for some $n \geq 1$. Given $(A, B) \in V(\mathbb{Z})_+$ corresponding to $(Q, (C, \alpha))$, the resolvent binary cubic form $\text{Res}(A, B) = f(x, y) = 4\det(Ax - By)$ corresponds to the $n$-monogenized cubic ring $(C, \alpha)$ under the bijection of Theorem 2.7.

Definition 3.14 We use the resolvent map $\text{Res}$ to define the functions $\text{ind}$, $I$, $J$, $H$, and $\Delta$ on $V(\mathbb{Z})_+ := \{(A, B) \in V(\mathbb{R}) : \det(A) > 0\}$. For $v \in V(\mathbb{R})_+$, we set

\[
\text{ind}(v) := \text{ind}(\text{Res}(v)); \quad I(v) := I(\text{Res}(v)); \quad J(v) := J(\text{Res}(v)); \quad H(v) := H(\text{Res}(v)); \quad \Delta(v) := \Delta(\text{Res}(v)).
\]

These functions are invariants for the action of $M(\mathbb{R}) \times \text{SL}_3(\mathbb{R})$ on $V(\mathbb{R})_+$.

In the rest of this subsection, to prove Theorem 3.12, we determine the number of $M(\mathbb{Z}) \times \text{SL}_3(\mathbb{Z})$-orbits on $V(\mathbb{Z})_+$ having bounded height and index.
### 3.2.1 Reduction theory for the action of $M(\mathbb{Z}) \times SL_3(\mathbb{Z})$ on $V(\mathbb{R})_+$

**Notation 3.15** For $i \in \{0, 1, 2\}$, let $V(\mathbb{R})^{(i)} \subset V(\mathbb{R})_+$ denote the set of elements corresponding to the $R$-algebra $\mathbb{R}^{4-2i} \times \mathbb{C}^i$. That is, $V(\mathbb{R})^{(i)}$ consists of elements having splitting type $(1111)$ when $i = 0$, $(112)$ when $i = 1$, and $(22)$ when $i = 2$.

**Lemma 3.16** The size of the stabilizer in $SL_3(\mathbb{R})$ of an element $(A, B) \in V(\mathbb{R})$ is 4 if $\Delta(A, B) > 0$ and 2 if $\Delta(A, B) < 0$.

**Proof:** As described in Remark 2.23, a nondegenerate element $(A, B) \in V(\mathbb{R})$ gives rise to a set $Q$ of four points in $\mathbb{P}^2(\mathbb{C})$, equipped with the action of $Gal(C/R) = \mathbb{Z}/2\mathbb{Z}$. Let $P'$ denote the set of pairs of lines $(L_1, L_2)$, where $L_1$ passes through two of the points in $Q$, and $L_2$ passes through the other two. The set $P'$ inherits an action of $Gal(C/R)$ from $Q$. Theorem 2.17 now implies that the stabilizer in $SL_3(\mathbb{R})$ of $(A, B)$ is isomorphic to the set of Galois-invariant permutations of $Q$ which induce the trivial permutation on $P'$.

If $\Delta(A, B) > 0$, then $Q$ either consists of four points defined over $\mathbb{R}$ or consists of two pairs of complex conjugate points. In either case, the nontrivial permutations of $Q$ which induce the trivial permutation on $P'$ are the double transpositions. If $\Delta(A, B) < 0$, then $Q$ consists of two points $Q_1$ and $Q_2$ defined over $\mathbb{R}$ and a pair $Q_3$ and $Q_4$ of complex conjugate points. In this case, the only nontrivial permutation of $Q$ which induces the trivial permutation on $P'$ is the one switching $Q_1$ with $Q_2$ and $Q_3$ with $Q_4$. This concludes the proof of the lemma. $\square$

**Lemma 3.17** Let $f(x, y) \in U(\mathbb{R})$ be a binary cubic form with $\Delta(f) \neq 0$ and positive $x^3$-coefficient.

(a) If $\Delta(f) > 0$, then the set of elements in $V(\mathbb{R})_+$ with resolvent $f$ consists of one $SL_3(\mathbb{R})$-orbit with splitting type $(1111)$ and three $SL_3(\mathbb{R})$-orbits with splitting type $(22)$.

(b) If $\Delta(f) < 0$, then the set of elements in $V(\mathbb{R})_+$ with resolvent $f$ consists of a single $SL_3(\mathbb{R})$-orbit with splitting type $(112)$.

**Proof:** In Case (a), Theorem 2.17 implies that the set of $SL_3(\mathbb{R})$-orbits on $V(\mathbb{R})_+$ with resolvent $f$ are in bijection with the set of étale quartic extensions of $\mathbb{R}$ having cubic resolvent algebra $\mathbb{R}^3$ along with a choice of basis for $\mathbb{R}^3$. Theorem 2.22 implies that the latter set is in bijection with the set of étale quadratic extensions of $\mathbb{R}$ whose discriminants have square norm. There are four such extensions: first, each $\mathbb{R}$-factor can split, yielding the algebra $\mathbb{R}^6$ corresponding to elements in $V(\mathbb{R})$ with splitting type $(1111)$. Second, exactly one of the $\mathbb{R}$-factors can split, yielding three different extensions $\mathbb{R}^2 \times \mathbb{C}^2$ each corresponding to elements in $V(\mathbb{R})$ with splitting type $(22)$.

Similarly, in Case (b), the set of $SL_3(\mathbb{R})$-orbits on $V(\mathbb{R})_+$ with resolvent $f$ are in bijection with étale quadratic extensions of $\mathbb{R} \times \mathbb{C}$ whose discriminants have square norm in $\mathbb{R}$. There is only one such extension, namely, $\mathbb{R}^2 \times \mathbb{C}^2$ corresponding to elements in $V(\mathbb{R})$ with splitting type $(112)$. $\square$

**Remark 3.18** When $\Delta(f) > 0$, we can describe the three $SL_3(\mathbb{R})$-orbits in $V(\mathbb{R})_+$ with splitting type $(22)$ as follows. An element $(A, B) \in V(\mathbb{R})_+$ has splitting type $(22)$ if and only if $A$ and $B$ have no common zeros in $\mathbb{P}^2(\mathbb{R})$. This can occur in three ways:

(i) $A$ has no zeros in $\mathbb{P}^2(\mathbb{R})$, i.e., $A$ is anisotropic.

(ii) $A$ has zeros in $\mathbb{P}^2(\mathbb{R})$, i.e., $A$ is isotropic, and $B$ takes only positive values on the zeros of $A$ in $\mathbb{P}^2(\mathbb{R})$.

(iii) $A$ has zeros in $\mathbb{P}^2(\mathbb{R})$, i.e., $A$ is isotropic, and $B$ takes only negative values on the zeros of $A$ in $\mathbb{P}^2(\mathbb{R})$.  

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Note that the conditions (ii) and (iii) are disjoint because, if $B$ took both positive and negative values on the zeros of $A$, then by the intermediate value theorem, $A$ and $B$ would have a common zero in $\mathbb{P}^2(\mathbb{R})$—a contradiction.

**Notation 3.19** The conditions (i), (ii), and (iii) on $(A, B) \in V(\mathbb{R})_+$ in Remark 3.18 correspond exactly to the three orbits in Lemma 3.17(a) having splitting type $(22)$; we denote these three orbits in $V(\mathbb{R})_+$ by $V(\mathbb{R})^{(2\#)}$, $V(\mathbb{R})^{(2+)}$, and $V(\mathbb{R})^{(2-)}$, respectively. Thus $V(\mathbb{R})^{(2)} = V(\mathbb{R})^{(2\#)} \cup V(\mathbb{R})^{(2+)} \cup V(\mathbb{R})^{(2-)}$. For any $i \in \{0, 1, 2, 2\#, 2+, 2-\}$ and any subset $L \subset V(\mathbb{R})_+$, let $L^{(i)}$ denote the set $L \cap V(\mathbb{R})^{(i)}$.

**Construction 3.20** Suppose $2T < cX^\delta$, and recall the sets $\mathcal{F}_U^\pm(T; X) \subset U(\mathbb{R})$ defined in (10). Let

$$\kappa = \kappa(T; X) := \lceil X^{1/6}/T^{2/3} \rceil.$$ 

Since $2T < cX^\delta \ll X^{1/4}$, we have $\kappa \gg 1$. Let $\mathcal{F}_U^\pm(T; X)$ denote the $\kappa$-fold cover

$$\mathcal{F}_U^\pm(T; X) := \bigcup_{0 \leq k \leq \kappa} \left( \frac{1}{k} \right) \cdot \mathcal{F}_U^\pm(T; X)$$

of $\mathcal{F}_U^\pm(T; X)$. The coefficients of elements $f \in \mathcal{F}_U^\pm(T; X)$ satisfy the bounds (11). It follows that the coefficients of an element $ax^3 + bx^2y + cxy^2 + dy^3 \in \mathcal{F}_U^\pm(T; X)$ satisfy:

$$|a| \ll T; \quad 0 \leq b \ll X^{1/6}T^{1/3}; \quad |c| \ll X^{1/3}/T^{1/3}; \quad |d| \ll X^{1/2}/T. \quad (13)$$

We now describe a fundamental set for the action of $\text{SL}_3(\mathbb{R})$ on elements in $V(\mathbb{R})_+$ with resolvent in $\mathcal{F}_U^\pm(T; X)$.

**Proposition 3.21** There exist continuous maps

$$s : U(\mathbb{R})^\pm \rightarrow V(\mathbb{R})^{(i)}$$

for $i \in \{0, 2\#, 2+, 2-\}$, and $i = 1$,

satisfying:

(a) The resolvent cubic form of $s(f)$ is $f$, i.e., $s$ gives a section of the cubic resolvent map $V(\mathbb{R})^{(i)} \rightarrow U(\mathbb{R})^\pm$.

(b) If $f \in \mathcal{F}_U^\pm(T; X)$, and $s(f) = (A, B)$ with $A = (a_{ij})$ and $B = (b_{ij})$, then

$$|a_{ij}| \leq T^{1/3} \quad \text{and} \quad |b_{ij}| \leq X^{1/6}/T^{1/3}. \quad (14)$$

**Proof:** Lemma 3.17 implies that sections $s : U(\mathbb{R})^\pm \rightarrow V(\mathbb{R})^{(i)}$ exist, and it suffices to prove that the section $s$ can be chosen to satisfy the bounds of (b).

Let $g(x, y) = f(T^{-1/3}x, X^{-1/6}T^{-1/3}y)$. Then by (13), the absolute values of all coefficients of $g$ are $\leq 1$. A section $t \in V(\mathbb{R})$ can be constructed on elements $h \in U(\mathbb{R})$ having absolutely bounded coefficients so that all the entries of $t(h)$ have size $O(1)$; for example, when $i = 0$ or $i = 1$, we may take the section:

$$t : ax^3 + bx^2y + cxy^2 + dy^3 \mapsto \begin{pmatrix} 1/2 & -a \\ -a & 1/2 \\ 1/2 & c/2 & -d \end{pmatrix}.$$ 

Writing $t(g) = (A, B)$, we set $s(f) = (T^{1/3}A, X^{1/6}T^{-1/3}B)$; then $s(f)$ satisfies the bounds (14). □
Construction 3.22 For \( i \in \{0, 1, 2\#, 2+, 2-\} \), let \( \mathcal{F}_V^i(T; X) \) (resp. \( \mathcal{F}_V^{\pm}(T; X) \), \( \mathcal{F}_V^i(\leq cH^\delta, X) \)) denote the image of \( \mathcal{F}_{U_i}^+(T; X) \) (resp. \( \mathcal{F}_{U_i}^+(T; X) \), \( \mathcal{F}_{U_i}^\pm(\leq cH^\delta, X) \)) under the map \( s : U(\mathbb{R})^\pm \to V(\mathbb{R})^{(i)} \) of Proposition 3.21. Let

\[
\mathcal{F}_V^2(T; X) := \mathcal{F}_V^{(2-)}(T; X) \cup \mathcal{F}_V^{(2-)}(T; X) \cup \mathcal{F}_V^{(2#)}(T; X); \\
\mathcal{F}_V^{\pm}(T; X) := \mathcal{F}_V^{(2+)}(T; X) \cup \mathcal{F}_V^{(2+)}(T; X) \cup \mathcal{F}_V^{(2#)}(T; X); \\
\mathcal{F}_V^i(\leq cH^\delta, X) := \mathcal{F}_V^{(2+)}(\leq cH^\delta, X) \cup \mathcal{F}_V^{(2-)}(\leq cH^\delta, X) \cup \mathcal{F}_V^{(2#)}(\leq cH^\delta, X).
\]

Let \( \mathcal{F}_{SL_3} \) be a fundamental domain for the action of \( SL_3(\mathbb{Z}) \) on \( SL_3(\mathbb{R}) \) by left multiplication, and let

\[
m_0 = m_2 = m_{2\pm} = m_{2\#} = 4; \quad m_1 = 2.
\]

Proposition 3.23 For \( i \in \{0, 1, 2\#, 2+, 2-\} \), the multiset \( \mathcal{F}_{SL_3} \cdot \mathcal{F}_V^i(T; X) \) is an \( m_i\kappa \)-fold cover of a fundamental domain for the action of \( M(\mathbb{Z}) \times SL_3(\mathbb{Z}) \) on the set of elements \( v \in V(\mathbb{R})^{(i)} \) with \( T \leq \text{ind}(v) < 2T \) and \( X \leq H(v) < 2X \).

Construction 3.24 We now fix \( \mathcal{F}_{SL_3} \) to lie in a Siegel domain as in [14], i.e., every element \( \gamma \in \mathcal{F}_{SL_3} \) can be expressed in Iwasawa coordinates in the form \( \gamma = ntk \), where \( n \) is a lower triangular unipotent matrix with coefficients bounded by 1 in absolute value, \( k \) belongs to the compact group \( SO_3(\mathbb{R}) \), and \( t = t(s_1, s_2) \) is a diagonal matrix having diagonal entries \( s_1^{-2}s_2^{-1}, s_1s_2^{-1}, s_1s_2^2 \), with \( s_1, s_2 \gg 1 \). A Haar measure \( d\gamma \) on \( SL_3(\mathbb{R}) \) in these coordinates is \( (s_1s_2)^{-6}d\nu ds_1d^6s_2dk \) (for a proof, see [26, Proposition 1.5.3]).

3.2.2 Averaging and cutting off the cusp

Let \( L \subset V(\mathbb{Z}) \) be a finite union of translates of lattices that is \( M(\mathbb{Z}) \times SL_3(\mathbb{Z}) \)-invariant. We now determine the asymptotic number of \( M(\mathbb{Z}) \times SL_3(\mathbb{Z}) \)-orbits on \( L^{\text{gen}} \) having bounded height \( H \) and index bounded by \( cH^\delta \).

Notation 3.25 Let \( N^i(L; T; X) \) denote the number of \( M(\mathbb{Z}) \times SL_3(\mathbb{Z}) \)-orbits on the set of elements \( v \in L^{\text{gen}} \cap V(\mathbb{R})^{(i)} \) such that \( T \leq \text{ind}(v) < 2T \) and \( X \leq H(v) < 2X \).

Then we prove the following theorem.

Theorem 3.26 We have

\[
N^i(L; T; X) = \frac{1}{m_i\kappa} \nu(L) \text{Vol}(\mathcal{F}_{SL_3} \cdot \mathcal{F}_V^i(T; X)) + O_\epsilon(X^{5/6+\epsilon}T^{1/3}).
\]

Proof: By Proposition 3.23, it follows that

\[
N^i(L; T; X) = \frac{1}{m_i\kappa} \#\{\mathcal{F}_{SL_3} \cdot \mathcal{F}_V^i(T; X) \cap L^{\text{gen}}\}.
\]

Let \( G_0 \subset SL_3(\mathbb{R}) \) be a nonempty open bounded set. Then, using the identical averaging argument as in [8, Theorem 2.5], we have

\[
\#\{\mathcal{F}_{SL_3} \cdot \mathcal{F}_V^i(T; X) \cap L^{\text{gen}}\} = \frac{1}{\text{Vol}(G_0)} \int_{\gamma \in \mathcal{F}_{SL_3}} \#\{\gamma G_0 \cdot \mathcal{F}_V^i(T; X) \cap L^{\text{gen}}\} d\gamma. \tag{17}
\]

The remainder of the proof now follows exactly the arguments of [4, §3]. First, proceeding as in the
proves of [4, Lemma 11] and [4, Lemmas 12–13], we obtain the following two estimates, respectively:

\[ \int_{\gamma \in \mathcal{F}_{\text{SL3}}} \# \{ (A, B) \in \gamma G_0 \cdot \tilde{F}^{(i)}_V (T; X) \cap L^{\text{gen}} : a_{11} = 0 \} d\gamma \ll X/T^{1/3}; \]

\[ \int_{\gamma \in \mathcal{F}_{\text{SL3}}} \# \{ (A, B) \in \gamma G_0 \cdot \tilde{F}^{(i)}_V (T; X) \cap (L \setminus L^{\text{gen}}) : a_{11} \neq 0 \} d\gamma \ll \epsilon X^{1+\epsilon}/T^{1/3}. \]

Next, since Proposition 3.21 implies that the coefficients \( a_{ij} \) of elements in \( \tilde{F}^{(i)}_V (T; X) \) are bounded by \( O(T^{1/3}) \), it follows that the set

\[ \{ (A, B) \in \gamma G_0 \cdot \tilde{F}^{(i)}_V (T; X) \cap V(Z) : a_{11} \neq 0 \} \]

is empty unless \( s_1^{-4} s_2^{-2} T^{1/3} \gg 1 \), or equivalently, \( s_1^4 s_2^2 \ll T^{1/3} \). Carrying out the integral in (17), cutting it off when \( s_1^4 s_2^2 \ll T^{1/3} \), and applying Proposition 3.7 to estimate the number of lattice points in \( \gamma G_0 \cdot \tilde{F}^{(i)}_V (T; X) \), we obtain

\[ \# \{ \mathcal{F}_{\text{SL3}} \cdot \tilde{F}^{(i)}_V (T; X) \cap L^{\text{gen}} \} = \frac{\nu(L)}{\text{Vol}(G_0)} \int_{\gamma \in \mathcal{F}_{\text{SL3}}} \text{Vol}(\gamma G_0 \cdot \tilde{F}^{(i)}_V (T; X)) d\gamma + O_\epsilon (X^{1+\epsilon} T^{-1/3}) \]

\[ = \frac{\nu(L)}{\text{Vol}(G_0)} \int_{\gamma \in G_0} \text{Vol}(\mathcal{F}_{\text{SL3}} \gamma \cdot \tilde{F}^{(i)}_V (T; X)) d\gamma + O_\epsilon (X^{1+\epsilon} T^{-1/3}) \]

\[ = \nu(L) \cdot \text{Vol}(\mathcal{F}_{\text{SL3}} \cdot \tilde{F}^{(i)}_V (T; X)) + O_\epsilon (X^{1+\epsilon} T^{-1/3}). \tag{18} \]

Finally, we note that by Proposition 3.23, the multiset \( \mathcal{F}_{\text{SL3}} \cdot \tilde{F}^{(i)}_V (T; X) \) contains exactly \( m_i \kappa \) representatives of an \( M(Z) \times \text{SL}_3(Z) \)-orbit \( v \in L^{\text{gen}} \cap V(R)^{(i)} \). Dividing the first and last terms of (18) by \( m_i \kappa \) thus yields Theorem 3.26. □

Proof of Theorem 3.12: Theorem 3.12 now follows immediately by breaking the intervals [1, \( X \)] and [1, \( cX^\delta \)] into dyadic ranges, and then applying Theorem 3.26 to each pair of dyadic ranges. □

### 3.3 Uniformity estimates and squarefree sieves

In this subsection, we count \( M(Z) \)-orbits on \( U(Z) \) and \( M(Z) \times \text{SL}_3(Z) \)-orbits on \( V(Z) \) of bounded height and index satisfying certain infinite sets of congruence conditions.

**Notation 3.27** For a large collection \( S = (S_p)_p \) of cubic local specifications, let \( U(Z)_S \) denote the set of elements \( f \in U(Z) \) such that \( f \) belongs to \( S_p \) for all \( p \), and let \( V(Z)_S \) denote the set of elements in \( V(Z) \) whose cubic resolvent forms are in \( U(Z)_S \).

We prove the following theorem:

**Theorem 3.28** For a set \( L \subset U(Z) \) (resp. \( L \subset V(Z) \)) defined via congruence conditions, let \( \nu(L) \) denote the volume of the closure of \( L \) in \( U(\mathbb{Z}) \) (resp. \( V(\mathbb{Z}) \)). Then

\[ N^+(U(Z)_S; T; X) = \nu(U(Z)_S) \text{Vol}(\mathcal{F}^+_U (T; X)) + o(X^{5/6} T^{2/3}); \]

\[ N^{(i)}(V(Z)_S; T; X) = \frac{1}{m_i} \nu(V(Z)_S) \text{Vol}(\mathcal{F}_{\text{SL3}} \cdot \tilde{F}^{(i)}_V (T; X)) + o(X^{5/6} T^{2/3}). \]

To prove the first claim of Theorem 3.28, we must impose infinitely many congruence conditions on \( U(Z) \). Namely, we must impose the congruence conditions of maximality at every prime as well as the congruence conditions forced by \( \Sigma \). For this, we require estimates on the number of \( M(Z) \)-orbits in \( U(Z) \) having bounded height and index with discriminant divisible by the square of some
prime \( p > M \). For a prime \( p \) and an element \( f \in U(\mathbb{Z}) \), note that \( p^2 \mid \Delta(f) \) precisely when the cubic ring corresponding to \( f \) is totally ramified (i.e., has splitting type \((1^3)\)) at \( p \) or is non-maximal at \( p \).

Similarly, to prove the second claim of Theorem 3.28, we require estimates on the number of \( M(\mathbb{Z}) \times \text{SL}_3(\mathbb{Z}) \)-orbits on \( V(\mathbb{Z}) \) having bounded height and index with discriminant divisible by the square of some prime \( p > M \). This time, note that \( p^2 \mid \Delta(v) \) for \((A, B) \in V(\mathbb{Z})\) precisely when \((A, B)\) corresponds to a quartic ring that is either extra ramified (i.e., has splitting type \((1^3)\), \((1^22)\), \((2^2)\), or \((1^4)\)) at \( p \) or is nonmaximal at \( p \).

**Notation 3.29** For a prime \( p \), let \( \mathcal{W}_p^{(1)}(U) \) denote the set of integer-coefficient binary cubic forms with splitting type \((1^3)\) at \( p \), and \( \mathcal{W}_p^{(2)}(U) \) the set of integer-coefficient binary cubic forms that are nonmaximal at \( p \). Similarly, let \( \mathcal{W}_p^{(1)}(V) \) denote the set of elements in \( V(\mathbb{Z}) \) with splitting type \((1^3)\), \((1^22)\), \((2^2)\), or \((1^4)\) at \( p \), and \( \mathcal{W}_p^{(2)}(V) \) the set of elements in \( V(\mathbb{Z}) \) that are nonmaximal at \( p \). Finally, set \( \mathcal{W}_p(U) := \mathcal{W}_p^{(1)}(U) \cup \mathcal{W}_p^{(2)}(U) \) and \( \mathcal{W}_p(V) := \mathcal{W}_p^{(1)}(V) \cup \mathcal{W}_p^{(2)}(V) \).

In the language of [6, §1.5], the sets \( \mathcal{W}_p^{(1)}(U) \) and \( \mathcal{W}_p^{(1)}(V) \) consist of elements in \( U(\mathbb{Z}) \) and \( V(\mathbb{Z}) \), respectively, whose discriminants are divisible by \( p^2 \) for “mod \( p^i \) reasons”.

We begin by bounding the number of \( n \)-monogenized cubic rings (resp. quartic rings with \( n \)-monogenized cubic resolvent rings) that are totally ramified (resp. extra ramified) at some prime \( p > M \).

**Proposition 3.30** For any positive real numbers \( X, T, \) and \( M \) with \( T \ll X^{1/4} \), we have

\[
\sum_{p > M} N^\pm(\mathcal{W}_p^{(1)}(U); T; X) = O_T(X^{5/6}T^{2/3}/M^{1-\epsilon}) + O(X^{5/6+\epsilon});
\]

\[
\sum_{p > M} N(\mathcal{W}_p^{(1)}(V); T; X) = O_T(X^{5/6}T^{2/3}/M^{1-\epsilon}) + O(X^{5/6}T^{1/3}).
\]

**Proof:** Theorem 3.30 follows immediately by [6, Theorem 3.3] and our counting results (Theorems 3.6 and 3.12) of the previous two subsections. □

Next, we obtain bounds on the number of orbits on \( U(\mathbb{Z}) \) and \( V(\mathbb{Z}) \) of bounded height and index that correspond to nonmaximal rings.

**Notation 3.31** For a cubic number field \( K \), we define the following **height** function \( H \) on its ring of integers \( \mathcal{O}_K \); for \( \beta \in \mathcal{O}_K \), let \( H(\beta) := \max_v \{ |\beta'|_v \} \), where \( v \) ranges over the archimedean valuations of \( K \) and \( \beta' \) denotes the unique \( \mathbb{Z} \)-translate of \( \beta \) such that the absolute trace \( \text{Tr}(\beta') \in \{0,1,2\} \).

Let \( \alpha \in \mathcal{O}_K \). Since either \( \alpha', \alpha' - 1/3 \), or \( \alpha' - 2/3 \) satisfies the equation

\[
x^3 - (I(f)/3)x - J(f)/27 = 0,
\]

we have the estimate

\[
H(\alpha) \ll \max\{|I(f)|^3, J(f)^2/4\}^{1/6} = (n^2H(f))^{1/6} = \text{ind}(f)^{1/3}H(f)^{1/6}.
\]

Indeed, if \( H(\alpha) \gg \text{ind}(f)^{1/3}H(f)^{1/6} \), then the left-hand side of (19) could not equal zero as the \( x^3 \)-term would dominate the other two terms.

**Definition 3.32** Let \( C \) be a nondegenerate cubic ring, and consider \( C \) embedded as a lattice in \( C \otimes \mathbb{R} \cong \mathbb{R}^3 \) with covolume \( \sqrt{\Delta(C)} \). Let \( 1 \leq \ell_1(C) \leq \ell_2(C) \) denote the three successive minima of \( C \). We define the **skewness** of a cubic ring \( C \) to be

\[
\text{sk}(C) := \ell_2(C)/\ell_1(C).
\]

As in [36, Lemma 4.7], the number of integers \( \alpha \in C \) with bounded height can be controlled by the discriminant and skewness of \( C \).
Lemma 3.33 Let $C$ be a cubic ring with discriminant $D = \Delta(C)$ and skewness $Z = \sk(C)$. Then the number of elements $\alpha \in C \setminus Z$ with $\Tr(\alpha) \in \{0, 1, 2\}$ and $H(\alpha) < H$ is

$$
\ll \begin{cases} 
0 & \text{if } H < \ell_1(C) \asymp D^{1/4}/|Z|^{1/2}; \\
HZ^{1/2}/D^{1/4} & \text{if } \ell_1(C) < H < \ell_2(C) \asymp D^{1/4}Z^{1/2}; \\
H^2/D^{1/2} & \text{if } \ell_2(C) < H.
\end{cases}
$$

Proof: The definition of $\sk(C)$ and the fact that $\ell_1(C)\ell_2(C) \asymp \sqrt{D}$ imply that $\ell_1(C) \asymp D^{1/4}/Z^{1/2}$ and $\ell_2(C) \asymp D^{1/4}Z^{1/2}$. When $H < \ell_1(C)$, there are no elements $\alpha \in C \setminus Z$ with $H(\alpha) < H$. When $\ell_1(C) < H < \ell_2(C)$, the only elements $\alpha \in C \setminus Z$ with $\Tr(\alpha) \in \{0, 1, 2\}$ are (appropriately translated) multiples of the element in $C$ of length $\ell_1(C)$. When $H > \ell_2(C)$, the number of such $\alpha$ is then bounded by the volume of the region $\{\theta \in C \otimes \mathbb{R} : H(\theta) < H \text{ and } \Tr(\theta) \in \{0, 1, 2\}\}$ divided by $\sqrt{D}$, the covolume of $C$. □

Proposition 3.34 The number of cubic orders $C$ (resp. pairs $(Q, C)$, where $Q$ is a quartic order and $C$ is a cubic resolvent order of $Q$) satisfying $D \leq |\Delta(C)| < 2D$ and $\sk(C) \gg Z$ is $O(D/Z)$.

Proof: Proposition 3.34 follows from the proofs of [36, Theorem 4.1 and Proposition 4.4]. □

Corollary 3.35 The number of pairs $(C, \alpha)$, where $C$ is a cubic order (resp. triples $(Q, C, \alpha)$, where $Q$ is a quartic order and $C$ is a cubic resolvent order of $Q$) satisfying $|\Delta(C)| < X$, $\alpha \in C \setminus Z$ with $\Tr(\alpha) \in \{0, 1, 2\}$, and $H(\alpha) \ll H$ is $O_e(H^2X^{1/2+\epsilon})$.

Proof: We break up the discriminant range $[0, X]$ of $C$ into $O(\log X)$ dyadic ranges. For each such dyadic range $[D, 2D]$, we break up the range of possible skewness of $C$ into dyadic ranges $[Z, 2Z]$, where clearly $Z \ll D^{1/2}$. For a cubic order $C$ with $|\Delta(C)| \asymp D$ and $\sk(C) \asymp Z$, Lemma 3.33 implies that the number of elements $\alpha \in C \setminus Z$ with $\Tr(\alpha) \in \{0, 1, 2\}$ and $H(\alpha) \ll H$ is bounded by

$$
\ll \begin{cases} 
H^2/D^{1/2} + HZ^{1/2}/D^{1/4} & \text{if } Z \gg D^{1/2}/H^2; \\
0 & \text{otherwise}.
\end{cases}
$$

Adding up over the $O(D/Z)$ orders with discriminant and skewness in this range, gives us the following bound on the number of pairs $(C, \alpha)$ (resp. triples $(Q, C, \alpha)$):

$$
\ll \begin{cases} 
D^{1/2}H^2/Z + HD^{3/4}/Z^{1/2} & \text{if } Z \gg D^{1/2}/H^2; \\
0 & \text{otherwise}.
\end{cases}
$$

In either case, the bound is $O(H^2D^{1/2})$. Adding up over the dyadic ranges yields the result. □

We are now ready to prove the other required uniformity estimate.

Proposition 3.36 For any positive real numbers $X$, $T$, and $M$ with $T \ll X^{1/4}$, we have

$$
\sum_{p > M} N^{(i)}(W_p^{(2)}(U); T; X) = O_e\left(X^{5/6}T^{2/3}/M^{1-\epsilon}\right);
$$

$$
\sum_{p > M} N^{(i)}(W_p^{(2)}(V); T; X) = O_e\left(X^{5/6}T^{2/3}/M^{1-\epsilon}\right).
$$

Proof: If a prime $p$ satisfies $p \ll X^{1/8}$ in the first sum, then we have the following upper bound on the corresponding summand:

$$
N^{(i)}(W_p^{(2)}(U); T; X) = \# \{ F_U^\pm(T; X) \cap W_p^{(2)}(U) \} \ll X^{5/6}T^{2/3}/p^2.
$$
This can be seen by simply partitioning $W_p^{(2)}(U)$ into $O(p^6)$ translates of $p^2U(\mathbb{Z})$ and counting integer points of bounded height in each translate using Proposition 3.7, noting that the last variable $d$ takes a range of length at least $X^{1/2}/T \gg X^{1/4} \gg p^2$. The sum of the bound (21) over all primes $p$ with $M < p < X^{1/8}$ is less than the bound of Proposition 3.36. We may thus assume that $M \gg X^{1/8}$ for the purpose of proving the first estimate of the proposition. Similarly, since the variable $b_{22}$ takes a range of at least $X^{1/6}/T^{1/3} \gg X^{1/12}$ in the proof of Theorem 3.26, we may assume that $M \gg X^{1/24}$ for the purpose of proving the second estimate.

Let $f$ be an irreducible element of

$$
\bigcup_{p>M} \{f_U^+(T; X) \cap W_p^{(2)}(U)\}.
$$

(22)

Then $f$ corresponds to a pair $(C, \alpha)$, where $C$ is a cubic ring and $\alpha$ is an element of $C$. Furthermore, $C$ is nonmaximal at some prime $p > M$. Let $C_1$ be the (unique) order containing $C$ with index $p$. The treatment of the case $n = 3$ in [6, §4.2] implies that $C_1$ comes from at most 3 such rings $R$. It thus follows from (20) that the set (22) maps into the set

$$
\{ (C_1, \alpha) : |\Delta(C_1)| \ll X/M^2, \quad H(\alpha) \asymp X^{1/6}T^{1/3}\},
$$

(23)

where $K$ is a cubic field and $\alpha$ is an element in $O_K$ with $\text{Tr}(\alpha) \in \{0, 1, 2\}$. Moreover, the sizes of the fibers of this map are bounded by 3. Therefore, we have

$$
\sum_{p>M} N^{(i)}(W_p^{(2)}(U); T; X) \ll \# \{(C_1, \alpha) : |\Delta(C_1)| \ll X/M^2, \quad H(\alpha) \asymp X^{1/6}T^{1/3}\}.
$$

(24)

Similarly, let $v \in V(\mathbb{Z})$ be an element contributing to the count of the left-hand side of the second line of the proposition. Then the cubic resolvent of $v$ belongs to (22), and thus $v$ corresponds to a triple $(Q, C, \alpha)$, where $Q$ is a quartic ring, $C$ is a cubic resolvent of $Q$ nonmaximal at a prime $p > M$, and $\alpha$ is an element of $C$. Let $C_1$ be as in the previous paragraph. The treatment of the case $n = 4$ in [6, §4.2] implies that the set of quartic rings with resolvent $C$ maps to the set of quartic rings with resolvent $C_1$, where the sizes of the fibers of this map are bounded by 6. Therefore,

$$
\sum_{p>M} N^{(i)}(W_p^{(2)}(V); T; X) \ll \# \{(Q_1, C_1, \alpha) : |\Delta(C_1)| \ll X/M^2, \quad H(\alpha) \asymp X^{1/6}T^{1/3}\},
$$

(25)

where $Q_1$ is a quartic ring with resolvent $C_1$ and $\alpha \in C_1$ has trace in $\{0, 1, 2\}$.

Corollary 3.35 implies that the right-hand sides of (24) and (25) are bounded by

$$
O_i(X^{5/6+\epsilon}T^{2/3}/M).
$$

This concludes the proof of the lemma, since $M \gg X^{1/24}$. □

**Proof of Theorem 3.28:** Theorem 3.28 follows immediately from the counting results of Proposition 3.9 and Theorem 3.26, in conjunction with the tail estimates in Propositions 3.30 and 3.36, by applying a standard squarefree sieve. See [9, §8.5] for an identical proof. □

### 3.4 Local mass formulas

To obtain the volumes in Theorem 3.28, we prove certain mass formulas relating étale quartic and cubic algebras over $\mathbb{Q}_p$. Let $K_3$ be an étale cubic extension of $\mathbb{Q}_p$. Let $K_4$ be an étale quartic extension of $\mathbb{Q}_p$ with cubic resolvent $K_3$ that corresponds to the unramified quadratic extension $K_6/K_3$ under the bijection of Theorem 2.22. By Remark 2.23, the étale $\mathbb{Q}_p$-algebras $K_3$, $K_4$, and $K_6$
correspond to sets $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{L}$, respectively, equipped with actions of $G_{\mathbb{Q}_p}$. The automorphism groups $\text{Aut}(K_3)$, $\text{Aut}(K_4)$, and $\text{Aut}(K_6)$ can be identified with the groups of $G_{\mathbb{Q}_p}$-equivariant permutations of $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{L}$, respectively. A $G_{\mathbb{Q}_p}$-equivariant permutation of $\mathcal{Q}$ induces $G_{\mathbb{Q}_p}$-equivariant permutations of $\mathcal{P}$ and $\mathcal{L}$.

**Notation 3.37** Let $\text{Aut}_{K_3}(K_4)$ denote the subgroup of $\text{Aut}(K_4)$ consisting of automorphisms of $K_4$ that induce the trivial automorphism of $K_3$ (equivalently, the subgroup consisting of Galois-equivalent permutations of $\mathcal{Q}$ that induce the trivial permutation of $\mathcal{P}$).

We have the following result.

**Theorem 3.38** Let $K_3$ be an étale cubic extension of $\mathbb{Q}_p$, and let $\mathcal{R}(K_3)$ denote the set of étale non-overramified quartic extensions $K_4$ of $\mathbb{Q}_p$, up to isomorphism, with resolvent $K_3$. Then

$$\sum_{K_4 \in \mathcal{R}(K_3)} \frac{1}{|\text{Aut}_{K_3}(K_4)|} = 1.$$ 

**Proof:** Recall that $\text{Aut}_{K_3}(K_6)$ is the subgroup of $\text{Aut}(K_6)$ consisting of automorphisms of $K_6$ that fix every element in $K_3$. Let $\text{Aut}_{K_3}^+(K_6)$ denote the index 2 subgroup of $\text{Aut}_{K_3}(K_6)$ consisting of even permutations of $\mathcal{L}$. We may check that the map from Galois-equivalent permutations of $\mathcal{Q}$ to the Galois-equivalent permutations of $\mathcal{L}$ induces an isomorphism $\text{Aut}_{K_3}(K_4) \cong \text{Aut}_{K_3}^+(K_6)$.

Exactly half of the unramified quadratic extensions $K_6/K_3$ have discriminant whose norm is a square in $\mathbb{Z}_p$. Moreover, all of these quadratic extensions $K_6/K_3$ have isomorphic automorphism groups. Therefore, we have

$$\sum_{K_4 \in \mathcal{R}(K_3)} \frac{1}{|\text{Aut}_{K_3}(K_4)|} = \frac{1}{2} \sum_{[K_6:K_3]=2 \text{ unramified}} \frac{1}{|\text{Aut}_{K_3}^+(K_6)|} = \sum_{[K_6:K_3]=2 \text{ unramified}} \frac{1}{|\text{Aut}_{K_3}(K_6)|} = 1,$$

where the last equality follows from (9). □

Finally, we translate Theorem 3.38 into the language of binary cubic forms.

**Definition 3.39** For $f \in U(\mathbb{Z}_p)$, define the local mass $\text{Mass}_p(f)$ of $f$ by

$$\text{Mass}_p(f) := \sum_{v \in \text{Res}^{-1}(f)} \frac{1}{\#\text{Stab}_{\text{SL}_3(\mathbb{Z}_p)}(v)},$$

where $\text{Res}^{-1}(f)$ is a set of representatives for the action of $\text{SL}_3(\mathbb{Z}_p)$ on $\text{Res}^{-1}(f)$.

Since the stabilizers in $M(\mathbb{Q}_p)$ and $G(\mathbb{Q}_p)$ of maximal elements in $U(\mathbb{Z}_p)$ and $V(\mathbb{Z}_p)$ actually lie in $M(\mathbb{Z}_p)$ and $G(\mathbb{Z}_p)$, respectively, we obtain the following consequence of Theorems 2.17 and 3.38:

**Corollary 3.40** Suppose $f \in U(\mathbb{Z}_p)$ corresponds to a maximal cubic ring over $\mathbb{Z}_p$. Then

$$\text{Mass}_p(f) = 1.$$ 

### 3.5 Volume computations and proof of Theorem 3

In this subsection, we prove Theorem 3.2, from which Theorem 3 will be shown to follow. To compute the volumes of sets in $V(\mathbb{R})$ and $V(\mathbb{Z}_p)$, we use the following Jacobian change of variables.
Proposition 3.41 Let \( K \) be \( \mathbb{R} \) or \( \mathbb{Z}_p \), let \( | \cdot | \) denote the usual normalized absolute value on \( K \), and let \( s : U(K) \to V(K) \) be a continuous map such that \( \text{Res}(s(f)) = f \), for each \( f \in U(K) \). Then there exists a constant \( J \in \mathbb{Q}_N \), independent of \( K \) and \( s \), such that for any measurable function \( \phi \) on \( V(K) \), we have:

\[
\begin{align*}
\int_{\text{SL}_3(K) \cdot s(U(K))} \phi(v) dv &= |J| \int_{f \in U(K)} \int_{g \in \text{SL}_3(K)} \phi(g \cdot s(f)) \omega(g) df, \\
\int_{V(K)} \phi(v) dv &= |J| \int_{f \in U(K)} \sum_{\text{Disc}(f) \neq 0} \frac{1}{\# \text{Stab}_{\text{SL}_3(K)}(v)} \int_{g \in \text{SL}_3(K)} \phi(g \cdot v) \omega(g) df,
\end{align*}
\]

(27)

where \( \text{Res}^{-1}(f) \) is a set of representatives for the action of \( \text{SL}_3(K) \) on \( \text{Res}^{-1}(f) \).

Proof: Proposition 3.41 follows immediately from the proofs of [8, Propositions 3.11 and 3.12] (see also [8, Remark 3.14]). □

Corollary 3.42 Let \( S_p \) be an open and closed subset of \( U(\mathbb{Z}_p) \) such that the boundary of \( S_p \) has measure 0 and every element in \( S_p \) is maximal. Consider the set \( V(\mathbb{Z})_{S_p} := \text{Res}^{-1}(S_p) \). Then

\[
\text{Vol}(V(\mathbb{Z})_{S_p}) = |J|_p \text{Vol}(\text{SL}_3(\mathbb{Z}_p)) \text{Vol}(S_p).
\]

(28)

Proof: We set \( \phi \) to be the characteristic function of \( V(\mathbb{Z})_{S_p} \) in the second line of (27) to obtain

\[
\text{Vol}(V(\mathbb{Z})_{S_p}) = |J|_p \text{Vol}(\text{SL}_3(\mathbb{Z}_p)) \int_{f \in S_p} \text{Mass}_p(f) df.
\]

The result then follows from Corollary 3.40. □

Proof of Theorem 3.2: Let \( \Sigma \) be associated to a large collection \( S = (S_p)_p \) of local specifications, where we may assume that every element \( f(x, y) \in S_p \) is maximal. For any finite set \( T \) of \( n \)-monogenized cubic fields, let \( \text{Avg}(\text{Cl}_2, T) \) (resp. \( \text{Avg}(\text{Cl}_2^+, T) \)) denote the average size of the 2-torsion in the class group (resp. narrow class group) of the fields in \( T \). Let \( F_3^\Sigma(\leq cH^\delta, X) \) denote the set of elements \( (K, \alpha) \in F_3(\leq cH^\delta, X) \) with \( +\Delta(K) > 0 \). By Theorem 3.28 and Corollary 3.42,

\[
\text{Avg}(\text{Cl}_2, F_3^\Sigma(\leq cH^\delta, X)) = 1 + \frac{1}{m_0} \frac{\nu(V(\mathbb{Z})_S) \text{Vol}(\mathcal{F}_{\text{SL}_3} \cdot \mathcal{F}_V(\leq cH^\delta, X))}{\text{Vol}(\mathcal{F}_T(\leq cH^\delta, X)) \nu(V(\mathbb{Z})_S)} + o(1)
\]

\[
= 1 + \frac{1}{m_0} |J| \text{Vol}(\mathcal{F}_{\text{SL}_3}) \prod_p \left[ |J|_p \text{Vol}(\text{SL}_3(\mathbb{Z}_p)) \text{Vol}(S_p) \right] + o(1)
\]

\[
= 1 + \frac{1}{m_0} + o(1) = \frac{5}{4} + o(1),
\]

since \( \text{Vol}(\mathcal{F}_{\text{SL}_3}) \cdot \prod_p \text{Vol}(\text{SL}_3(\mathbb{Z}_p)) \) is equal to 1, the Tamagawa number of \( \text{SL}_3 \). Similarly,

\[
\text{Avg}(\text{Cl}_2, F_3^\Sigma(\leq cH^\delta, X)) = 1 + \frac{1}{m_1} + o(1) = \frac{3}{2} + o(1);
\]

\[
\text{Avg}(\text{Cl}_2^+, F_3^\Sigma(\leq cH^\delta, X)) = 1 + \frac{1}{m_0} + \frac{1}{m_{2+}} + \frac{1}{m_{2-}} + \frac{1}{m_{2#}} + o(1) = 2 + o(1). \quad \square
\]

Proof of Theorem 3: Theorem 3 now follows immediately from Theorem 3.2 by letting \( \Sigma_p \) consist of all pairs \( (\mathcal{K}_p, \alpha_p) \), where \( \mathcal{K}_p \) is an étale cubic extension of \( \mathbb{Q}_p \) satisfying the splitting conditions prescribed in Theorem 3. □
4 The mean number of 2-torsion elements in the class groups of $n$-monogenized cubic fields ordered by height and fixed $n$

In this section we fix a positive integer $n$ throughout, and prove Theorem 7 by computing, for such a fixed $n$, the average size of the 2-torsion subgroups in the class groups of $n$-monogenized cubic fields when these fields are ordered by height.

In §4.1, we determine asymptotics for the number of $n$-monogenized cubic rings of bounded height. In §4.2, we determine asymptotics for the number of quartic rings having an $n$-monogenic cubic resolvent of bounded height. In §4.3, we prove uniformity estimates that enable us to impose conditions of maximality on these counts of cubic and quartic rings. The leading constants for these asymptotics are expressed as products of volumes of sets in $U_n(R)$ and $V(R)$, where $R$ ranges over $\mathbb{R}$ and $\mathbb{Z}_p$ for all primes $p$. In §4.4, we prove certain mass formulas relating étale quartic and cubic algebras over $\mathbb{Q}_p$. Finally, in §4.5, we use these mass formulas to compute the necessary local volumes in order to prove our main results.

4.1 The number of $n$-monogenized cubic rings of bounded height

In this subsection, we determine the number of primitive $n$-monogenized cubic rings having bounded height that are orders in $S_3$-number fields. Specifically, we prove the following result.

**Theorem 4.1** Let $N_3^+(n, X)$ (resp. $N_3^-(n, X)$) denote the number of isomorphism classes of $n$-monogenized cubic orders in $S_3$-cubic fields with positive (resp. negative) discriminant and height bounded by $X$. Then

(a) $N_3^+(n, X) = \frac{8}{135n^{1/3}}X^{5/6} + O_\epsilon(X^{1/2+\epsilon});$

(b) $N_3^-(n, X) = \frac{32}{135n^{1/3}}X^{5/6} + O_\epsilon(X^{1/2+\epsilon}).$

Theorem 4.1 is proved by using the parametrizations of §2.1 in conjunction with a count of elements in $U_n(\mathbb{Z})$ having bounded height.

**Notation 4.2** For any ring $R$ and integers $n$ and $b$, let $U_{n,b}(R) \subset U_n(R)$ denote the set of binary cubic forms $f(x, y) \in U_n(R)$ such that the $x^2y$-coefficient of $f$ is $b$. For a subset $S \subset U_n(\mathbb{R})$, let $S^{\pm}$ denote the set of elements $f \in S$ with $\pm \Delta(f) > 0$. For a subset $L \subset U_n(\mathbb{Z})$, let $N^\pm(L; n, X)$ denote the number of generic $M(\mathbb{Z})$-equivalence classes $f$ in $L^\pm$ such that $H(f) < X$.

**Lemma 4.3** The number of non-generic $M(\mathbb{Z})$-orbits in $U_n(\mathbb{Z})$ of height less than $X$ is $O_\epsilon(X^{1/2+\epsilon}).$

**Proof:** An $M(\mathbb{Z})$-orbit on $U_n(\mathbb{Z})$ has a unique representative with $x^2y$-coefficient in $[0, 3n]$. Let $f(x) = nx^3 + bx^2y + cxy^2 + dy^3$ be such an element with height less than $X$. Then we have the bounds $c \ll X^{1/3}$ and $d \ll X^{1/2}$. Furthermore, the proofs of [9, Lemmas 21, 22] imply that if $f(x, y)$ is either reducible or corresponds to an order in a $C_3$-number field, then the value of $d$ determines the value of $c$ up to $O_\epsilon(X^{1/2})$ choices, proving the lemma. □

**Proposition 4.4** Let $b \in [0, 3n)$ be an integer, and let $L \subset U_{n,b}(\mathbb{Z})$ be defined by finitely many congruence conditions. Then

(a) $N^+(L; n, X) = \frac{8}{405n^{1/3}}\nu(L)X^{5/6} + O_\epsilon(X^{1/2+\epsilon}),$
(b) \( N^-(L; n, X) = \frac{32}{405n^{4/3}} \nu(L)X^{5/6} + O_c(X^{1/2+\epsilon}), \)

where \( \nu(L) \) denotes the volume of the closure of \( L \) in \( U_{n,b}(\hat{\mathbb{Z}}) \).

**Proof:** By Lemma 4.3, it suffices to estimate the number of elements in \( L^\pm \) that have height bounded by \( X \). By Proposition 3.7, this is equal to \( \nu(L) \) times the volume of \( U_{n,b}(\mathbb{R})^\pm_{H<X} \), up to an error of \( O(X^{1/2}) \), since the largest projection of \( U_{n,b}(\mathbb{R})^\pm_{H<X} \) is onto the \( y^3 \)-coefficient of elements in \( U_{n,b}(\mathbb{R}) \) and this projection has size \( O(X^{1/2}) \).

The volume of \( U_{n,0}(\mathbb{R})^+_{H<X} \) is

\[
\int_{c=0}^{X^{1/3}/(3n^{1/3})} \int_{d=-2^{3/2}c^{3/2}n^{1/2}}^{2^{3/2}c^{3/2}n^{1/2}} \frac{4c^{3/2}}{3^{3/2}n^{1/2}} dc \, dd = \frac{8}{405n^{4/3}} X^{5/6},
\]

and the volume of \( U_{n,0}(\mathbb{R})^-_{H<X} \) is

\[
\frac{8}{81n^{4/3}} X^{5/6} - \text{Vol}(U_{n,0}(\mathbb{R})^+_{H<X}) = \frac{32}{405n^{4/3}} X^{5/6}.
\]

To compute the volumes of \( U_{n,b}(\mathbb{R})^\pm_{H<X} \), note that there exists a bijective unipotent invariant-preserving map \( U_{n,0}(\mathbb{R})^\pm_{H<X} \to U_{n,b}(\mathbb{R})^\pm_{H<X} \) defined by \( g(x, y) \mapsto g(x + by/(3n), y) \), and the Jacobian determinant of this map is equal to 1. Thus the volume of \( U_{n,b}(\mathbb{R})^\pm_{H<X} \) is equal to that of \( U_{n,0}(\mathbb{R})^\pm_{H<X} \), concluding the proof of Proposition 4.4. \( \square \)

**Proof of Theorem 4.1:** Theorem 4.1 follows from the parametrization of \( n \)-monogenized cubic orders in Theorem 2.7, together with Proposition 4.4 applied to \( U_{n,b}(\mathbb{Z}) \) for each \( b \) with \( 0 \leq b < 3n \) and then summing over these \( b \). \( \square \)

### 4.2 The number of quartic rings having \( n \)-monogenized cubic resolvent rings of bounded height

In this subsection, we determine asymptotics for the number of pairs \((Q, (C, \alpha))\), where \( Q \) is an order in an \( S_4 \)-quartic field and \((C, \alpha)\) is an \( n \)-monogenized cubic resolvent ring of \( Q \) having bounded height. This requires determining asymptotics for the number of \( M(\mathbb{Z}) \times \text{SL}_3(\mathbb{Z})\)-orbits \((A, B)\) on \( V(\mathbb{Z}) \) of bounded height that satisfy \( \det(A) = n/4 \). Fixing the determinant of \( A \) imposes a cubic polynomial condition on elements in \( V(\mathbb{R}) \); however, counting integer points on a cubic hypersurface is in general a difficult question.

Instead, we proceed as follows. We use the action of \( \text{SL}_3(\mathbb{Z}) \) to transform any element \((A, B) \in V(\mathbb{Z})\), with \( \det(A) = n/4 \), to an element \((A', B')\), where \( A' \) belongs to a fixed finite set of representatives for the action of \( \text{SL}_n(\mathbb{Z}) \) on integer-coefficient ternary quadratic forms of determinant \( n/4 \).

**Notation 4.5** For a ring \( R \) and an element \( n \in R \), let \( Q_n(R) \) denote the space of ternary quadratic forms \( A \) with coefficients in \( R \) such that \( 4 \det(A) = n \). For a fixed \( A \), let \( V_A(R) \) denote the set of pairs \((A, B) \in V(R)\) where \( B \) is arbitrary. If \( A \in Q_n(R) \), then the resolvent map \( \text{Res} \) sends \( V_A(R) \) to \( U_{n}(R) \). For a fixed \( b \in R \), let \( V_{A, b}(R) \) denote the subset of \( V_A(R) \) mapping to \( U_{n,b}(R) \) under \( \text{Res} \). The set \( V_{A,b}(R) \) is defined by linear conditions on \( V_A(R) \), since \( b \) is linear in the coefficients of \( B \).

Instead of counting \( M(\mathbb{Z}) \times \text{SL}_3(\mathbb{Z})\)-orbits \((A', B')\) with \( \det(A') = n/4 \), it suffices to count \( \text{SO}_4(\mathbb{Z})\)-orbits on \( V_A(\mathbb{Z}) \), where \( A \) ranges over a fixed set of representatives for \( \text{SL}_3(\mathbb{Z}) \setminus Q_n(\mathbb{Z}) \). We note that these representations are different \( \mathbb{Z} \)-forms of the same representation over \( \mathbb{C} \).
**Remark 4.6** One such \( \mathbb{Z} \)-form of the representation of \( \text{SO}_A(\mathbb{Z}) \) on \( V_A(\mathbb{Z}) \) is the representation of \( \text{PGL}_2 \) on the space of binary quartic forms. Indeed, when \( n = 1 \), the set \( Q_1(\mathbb{Z}) \) consists of a single \( \text{SL}_3(\mathbb{Z}) \)-orbit, and one of the representatives of this orbit is the \( 3 \times 3 \) symmetric matrix \( A_1 \) given by

\[
A_1 := \begin{pmatrix}
1/2 & 1/2 & 1/2 \\
1/2 & -1 & 1/2 \\
1/2 & 1/2 & 1/2 \\
\end{pmatrix}.
\] (31)

The algebraic group \( \text{PGL}_2 \) is isomorphic to \( \text{SO}_A_1 \) via the map

\[
\tau : \text{PGL}_2(\mathbb{Z}) \to \text{SL}_3(\mathbb{Z}), \quad \text{given explicitly by}
\]

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{1}{ad-bc} \begin{pmatrix} d^2 & cd & c^2 \\ 2bd & ad+bc & 2ac \\ b^2 & ab & a^2 \end{pmatrix}.
\] (32)

Furthermore, we have a map from the space of binary quartic forms to the space of pairs \((A_1, B)\) given by

\[
\phi : ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 \mapsto \left( \begin{array}{ccc} 1/2 & 1/2 & 1/2 \\ -1 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 \end{array} \right), \quad \left( \begin{array}{ccc} a & b/2 & 0 \\ b/2 & c & d/2 \\ 0 & d/2 & e \end{array} \right).
\] (33)

The above two maps (with the latter map slightly modified) are considered in work of Wood [39], who then shows that the representation of \( \text{PGL}_2 \) on the space of binary quartic forms is isomorphic over \( \mathbb{Z} \) to the representation of \( M(\mathbb{Z}) \times \text{SO}_A_1(\mathbb{Z}) \) on the space \( V_{A_1} \).

Asymptotics for the number of \( \text{PGL}_2(\mathbb{Z}) \)-orbits on the set of integer-coefficient binary quartic forms with bounded height were determined in [8]. In this section, we determine analogous asymptotics for other \( \mathbb{Z} \)-forms of this representation. As a consequence, we prove the following theorem.

**Theorem 4.7** For \( i \in \{0, 1, 2\} \), let \( N_{4i}^i(n, X) \) denote the number of isomorphism classes of pairs \((Q, (C, \alpha))\), where \( Q \) is an order in a quartic \( \mathbb{Z} \)-field having \( i \) complex embeddings, and \((C, \alpha)\) is an \( n \)-monogenicized cubic resolvent ring of \( Q \) with height bounded by \( X \). Then

\[
N_{4i}^i(n, X) = \frac{3n}{m_i} \sum_{A \in \text{SL}_3(\mathbb{Z}) \setminus Q_n(\mathbb{Z})} \text{Vol}(\mathcal{F}_A \cdot \mathcal{F}_{V_{A,b}}^i(X)) + o(X^{5/6}),
\]

where \( \mathcal{F}_A \) is a fundamental domain for the action of \( \text{SO}_A(\mathbb{Z}) \) on \( \text{SO}_A(\mathbb{R}) \), and the sets \( \mathcal{F}_{V_{A,b}}^i(X) \) are defined in §4.2.1 prior to Proposition 4.9.

In §4.2.1, we construct fundamental domains for the action of \( \text{SO}_A(\mathbb{Z}) \) on \( V_{A,b}(\mathbb{R}) \) for any \( 3 \times 3 \) symmetric matrix \( A \) having determinant \( n/4 \), and any \( b \) with \( 0 \leq b < 3n \). In §4.2.2, using these fundamental domains, we count the number of generic \( \text{SO}_A(\mathbb{Z}) \)-orbits on \( V_{A,b}(\mathbb{Z}) \) having bounded height. Summing over \( A \in \text{SL}_3(\mathbb{Z}) \setminus Q_n(\mathbb{Z}) \) and \( 0 \leq b < 3n \) will then immediately yield Theorem 4.7.

### 4.2.1 Reduction theory

Let \( A \) be the Gram matrix of an integer-coefficient ternary quadratic form with \( \det(A) = n/4 \). In this subsubsection, we construct finite covers of fundamental domains for the action of \( \text{SO}_A(\mathbb{Z}) \) on \( V_{A,b}(\mathbb{R}) \). We start by constructing fundamental sets for the action of \( \text{SO}_A(\mathbb{R}) \) on \( V_{A,b}(\mathbb{R}) \).
Construction 4.8 First, suppose that $A$ is isotropic over $\mathbb{R}$. If $A = A_1$, where $A_1$ is defined in (31), then by the discussion above we may construct our fundamental sets $\mathcal{F}^{(i)}_{V_{A,b}}$ for $i \in \{0, 1, 2+, 2-\}$, using analogous fundamental sets constructed for the action of $\text{PGL}_2$ on binary quartic forms. Namely, let

$$R_1^{(i)} := \phi(\mathbb{R}_{>0} \cdot L^{(i)}),$$

where $\phi$ is defined in (33) and the sets $L^{(i)}$ are defined in [8, Table 1]. If $A$ is a general integer-coefficient ternary quadratic form that is isotropic over $\mathbb{R}$, then we simply translate the sets $R_1^{(i)}$ using the group action of $\text{GL}_3(\mathbb{R})$ on the space of ternary quadratic forms. Namely, let $g_3 \in \text{GL}_3(\mathbb{R})$ be the element such that $g_3 \cdot A_1 := g_3 A_1 g_3^T = A$, and let

$$R^{(i)} := g_3.\phi(\mathbb{R}_{>0} \cdot L^{(i)}).$$

Then, for $A$ isotropic, $i \in \{0, 1, 2+, 2-\}$, and any integer $b \in [0, 3n]$, let

$$\mathcal{F}^{(i)}_{V_{A,b}} := \{(A, c_B A + B) : (A, B) \in R^{(i)}\},$$

where $c_B$ is the unique real number such that $\text{Res}(A, c_B A + B)$ has $x^2y$-coefficient $b$.

Next, suppose that $A$ is anisotropic over $\mathbb{R}$. We begin with the case $b = 0$. Let

$$\mathcal{F}^{(2\#)}_{V_{A,0}} := \{g_3 \cdot (\text{Id}, c_B f) : c \in \mathbb{R}_{>0}, f \in U_{n,0}(\mathbb{R})^+, H(f) = 1\},$$

where $g_3 \in \text{GL}_3(\mathbb{R})$ is the matrix taking the identity matrix $\text{Id}$ to $A$, and $Bf$ corresponds to the diagonal matrix whose entries are the (real) roots of $f(x,1)$ in ascending order. For a general integer $b \in [0, 3n]$, let

$$\mathcal{F}^{(2\#)}_{V_{A,b}} := \{(A, B + bA/3) : (A, B) \in \mathcal{F}^{(2\#)}_{V_{A,0}}\}. \quad (35)$$

If $A$ is isotropic over $\mathbb{R}$ and $i = 2\#$, or $A$ is anisotropic over $\mathbb{R}$ and $i \in \{0, 1, 2+, 2-\}$, then let

$$\mathcal{F}^{(i)}_{V_{A,b}} = \emptyset. \quad (36)$$

Let

$$\mathcal{F}^{(2)}_{V_{A,b}} := \mathcal{F}^{(2+)}_{V_{A,b}} \cup \mathcal{F}^{(2-)}_{V_{A,b}} \cup \mathcal{F}^{(2\#)}_{V_{A,b}}. \quad (37)$$

Finally, for $X > 0$, let $\mathcal{F}^{(i)}_{V_{A,b}}(X)$ denote the subset of elements in $\mathcal{F}^{(i)}_{V_{A,b}}$ having height less than $X$.

Proposition 4.9 If $A$ is isotropic over $\mathbb{R}$, then the sets $\mathcal{F}^{(0)}_{V_{A,b}}, \mathcal{F}^{(1)}_{V_{A,b}},$ and $\mathcal{F}^{(2)}_{V_{A,b}} = \mathcal{F}^{(2+)}_{V_{A,b}} \cup \mathcal{F}^{(2-)}_{V_{A,b}}$ are fundamental sets for the action of $\text{SO}_A(\mathbb{R})$ on $V_{A,b}(\mathbb{R})^{(0)}, V_{A,b}(\mathbb{R})^{(1)},$ and $V_{A,b}(\mathbb{R})^{(2)}$, respectively. If $A$ is anisotropic over $\mathbb{R}$, then the set $\mathcal{F}^{(2\#)}_{V_{A,b}}$ is a fundamental set for the action of $\text{SO}_A(\mathbb{R})$ on $V_{A,b}(\mathbb{R})$. Moreover, for $i \in \{0, 1, 2, 2+, 2-, 2−\}$, all entries of elements in $\mathcal{F}^{(i)}_{V_{A,b}}(X)$ are $O(X^{1/6})$.

Proof: When $A$ is anisotropic over $\mathbb{R}$, $A$ can be translated via $\text{GL}_3(\mathbb{R})$ into the identity matrix $\text{Id}$. In that case, every element in $V_{\text{Id},0}(\mathbb{R})$ has splitting type (22) over $\mathbb{R}$. Furthermore, by the spectral theorem, every element in $V_{\text{Id},0}(\mathbb{R})$ is $\text{SO}_3(\mathbb{R})$ equivalent to $(\text{Id}, B)$, where $B$ is diagonal and $b_{11} < b_{22} < b_{33}$. Then $\mathcal{F}^{(2\#)}_{V_{A,b}}$ gives the desired fundamental set. By [8, 2.1], the $\mathcal{F}^{(i)}_{V_{A,b}}$ are fundamental sets for $i \neq (2\#)$.

Regarding the heights of elements in $\mathcal{F}^{(i)}_{V_{A,b}}$, note that the set of elements in $\mathcal{F}^{(i)}_{V_{A,b}}$ having height 1 have absolutely bounded coefficients. The final assertion now follows since $H$ is a homogeneous function on $V_{\text{Id},0}(\mathbb{R})$ of degree 6. \(\square\)

Theorem 4.10 Let $\mathcal{F}_A$ be a fundamental domain for the action of $\text{SO}_A(\mathbb{Z})$ on $\text{SO}_A(\mathbb{R})$. Then $\mathcal{F}_A \cdot \mathcal{F}^{(i)}_{V_{A,b}}$ is an $m_i$-fold cover of a fundamental domain for the action of $\text{SO}_A(\mathbb{Z})$ on $V(\mathbb{R})^{(i)}$.

Proof: If $(A, B) \in V(\mathbb{R})^{(i)}$, then by Lemma 3.16 the stabilizer of $(A, B)$ in $\text{SL}_3(\mathbb{R})$ has size $m_i$. 28
Since every element of this stabilizer must belong to $\text{SO}_A(\mathbb{R})$, it follows that the size of the stabilizer of $(A, B)$ in $\text{SO}_A(\mathbb{R})$ is also $m_i$. The result now follows from Proposition 4.9. □

We will describe explicit constructions of $\mathcal{F}_A$ in the next subsection.

4.2.2 Counting $\text{SO}_A(\mathbb{Z})$-orbits on $V_A(\mathbb{Z})$

In this subsection, we fix positive integers $n$ and $b \in \{0, 3n\}$.

**Notation 4.11** Let $A$ be an integer-coefficient ternary quadratic form of determinant $n/4$. For a subset $L \subset V_{A,b}(\mathbb{Z})$ defined by congruence conditions and $i \in \{0, 1, 2, 2\pm 2\#\}$, let $N^{(i)}(L; A, X)$ denote the number of generic $\text{SO}_A(\mathbb{Z})$-orbits on $L^{(i)}$ having height less than $X$. Let $\nu(L)$ denote the volume of the closure of $L$ in $V_{A,b}(\mathbb{Z})$.

**Theorem 4.12** Let $L \subset V_{A,b}(\mathbb{Z})$ be defined by finitely many congruence conditions, and let $\mathcal{F}_A$ denote a fundamental domain for the action of $\text{SO}_A(\mathbb{Z})$ on $\text{SO}_A(\mathbb{R})$. Then

$$N^{(i)}(L; A, X) = \frac{1}{m_i} \nu(L) \text{Vol}(\mathcal{F}_A \cdot \mathcal{F}^{(i)}_{V_{A,b}}(X)) + o(X^{5/6}),$$

where $i = 2\#$ if $A$ is isotropic over $\mathbb{R}$ and $i \in \{0, 1, 2+, 2-\}$ if $A$ is anisotropic over $\mathbb{R}$.

Since the truth of Theorem 4.12 is independent of the choice of $\mathcal{F}_A$, we begin by constructing convenient fundamental domains $\mathcal{F}_A$ for the action of $\text{SO}_A(\mathbb{Z})$ on $\text{SO}_A(\mathbb{R})$.

**Construction 4.13** Suppose that $A$ is anisotropic over $\mathbb{Q}$. Then we may choose $\mathcal{F}_A$ to be a compact fundamental domain for the left action of $\text{SO}_A(\mathbb{Z})$ on $\text{SO}_A(\mathbb{R})$. By Theorem 4.10, the multiset $\mathcal{F}_A \cdot \mathcal{F}^{(i)}_{V_{A,b}}$ is an $m_i$-fold cover of a fundamental domain for the action of $\text{SO}_A(\mathbb{Z})$ on $V_{A,b}(\mathbb{R})^{(i)}$.

Next, suppose that $A$ is isotropic over $\mathbb{Q}$. Then there exists $g_A \in \text{SL}_3(\mathbb{Q})$ such that $g_A A g_A^{-1}$ is

$$A_n := \begin{pmatrix} 1/2 & 1/2 & -n \\ 1/2 & & \\ & & 1/2 \end{pmatrix}.$$  \hspace{1cm} (38)

For $F = \mathbb{R}$ or $\mathbb{Q}$, consider the maps

$$\sigma_A : V_{A,b}(F) \rightarrow V_{A_n,b}(F),$$
$$\sigma_G : \text{SO}_A(F) \rightarrow \text{SO}_{A_n}(F),$$

given by $\sigma_A(A, B) = (A_n, g_A B g_A^{-1})$ and $\sigma_G(g) = g A g A^{-1}$. The map $\sigma_A$ is height preserving and $\sigma_A(g \cdot v) = \sigma_G(g) \cdot \sigma_A(v)$. Let $L_n \subset V_{A_n,b}(\mathbb{R})$ denote the lattice $\sigma_{\text{A}}(L)$ and $\Gamma \subset \text{SO}_{A_n}(\mathbb{R})$ the subgroup $\sigma_{G}(\text{SO}_A(\mathbb{Z}))$. Then $\Gamma$ is commensurable with $\text{SO}_{A_n}(\mathbb{Z})$. By [13, Example 2.5], there exists a fundamental domain $\mathcal{F}$ for the action of $\Gamma$ on $\text{SO}_{A_n}(\mathbb{R})$ that is contained in a finite union $\bigcup_j g_j \mathcal{S}$, where $g_j \in \text{SO}_{A_n}(\mathbb{Q})$ and $\mathcal{S}$ is a Siegel domain. We choose $\mathcal{S}$ to be $N'T'K$, where

$$N' := \left\{ \begin{pmatrix} 1 & 1 & 2nu \\ 2nu & u & 1 \\ & & u \end{pmatrix} : u \in [-1/2, 1/2] \right\}, \quad T' := \left\{ \begin{pmatrix} t^{-2} & 1 \\ & t^2 \end{pmatrix} : t \geq 1/2 \right\},$$

and $K$ is a maximal compact subgroup of $\text{SO}_{A_n}(\mathbb{R})$. We set $\mathcal{F}_A := \sigma_{G}^{-1}(\mathcal{F})$. Then $\mathcal{F}_A$ is a fundamental domain for the action of $\text{SO}_A(\mathbb{Z})$ on $\text{SO}_A(\mathbb{R})$.

**Proof of Theorem 4.12:** First, we consider the case that $A$ is anisotropic. By Proposition 4.9, the entries of elements in $\mathcal{F}_A \cdot \mathcal{F}^{(i)}_{V_{A,b}}(X)$ are each $O(X^{1/6})$. An argument identical to the proof of [5, Lemma 14] implies that the number of non-generic integral points in $\mathcal{F}_A \cdot \mathcal{F}^{(i)}_{V_{A,b}}(X)$ is $o(X^{5/6})$. 29
Thus, to prove Theorem 4.12, it suffices to estimate the number of elements in $\mathcal{F}_A \cdot \mathcal{F}_{\mathcal{V}_{A,b}}^{(i)}(X) \cap L$, and this is immediate from Proposition 3.7.

Next, we assume that $A$ is isotropic. Theorem 4.10 implies that

$$N^{(i)}(L; A, X) = \frac{1}{m_i} \# (\mathcal{F} \cdot \sigma_A(\mathcal{F}_{\mathcal{V}_{A,b}}^{(i)}(X)) \cap L^\text{gen}_n),$$

where $L^\text{gen}_n$ denotes the subset $v \in L_n$ such that $\sigma_A^{-1}(v)$ is generic. Let $G_0$ be an open nonempty bounded subset of $\text{SO}_{A_n}(\mathbb{R})$. Then, by an averaging argument as in [8, Theorem 2.5], we obtain

$$N^{(i)}(L; A, X) = \frac{1}{m_i \text{Vol}(G_0)} \int_{h \in \mathcal{F}} \# (hG_0 \cdot \sigma_A(\mathcal{F}_{\mathcal{V}_{A,b}}^{(i)}(X)) \cap L^\text{gen}_n) \text{d}h,$$

(40)

where the volume of $G_0$ is computed with respect to any fixed Haar measure $dh$.

**Lemma 4.14** If $h = utk \in \mathcal{S} = N'T'K$ as above, $t \gg X^{1/24}$, and $g \in \text{SO}_{A_n}(\mathbb{Q})$, then

$$hG_0 \cdot \sigma_A(\mathcal{F}_{\mathcal{V}_{A,b}}^{(i)}(X)) \cap g^{-1}L^\text{gen}_n = \emptyset.$$

**Proof:** The entries of elements in $G_0 \cdot \sigma_A(\mathcal{F}_{\mathcal{V}_{A,b}}^{(i)}(X))$ are all $O(X^{1/6})$. Since $h = utk \in \mathcal{S}$, the entries of $u$ and $k$ are bounded. Hence the six coefficients of elements in $hG_0(\sigma_A(\mathcal{F}_{\mathcal{V}_{A,b}}^{(i)}(X)))$, considered as a subset of $\mathcal{V}_{A_n}(\mathbb{R})$, satisfy:

$$b_{11} \ll t^{-4}X^{1/6}; \quad b_{12} \ll t^{-2}X^{1/6}; \quad b_{13}, b_{22} \ll X^{1/6}; \quad b_{23} \ll t^2X^{1/6}; \quad b_{33} \ll t^4X^{1/6}. \quad (41)$$

The subset $\mathcal{V}_{A_n,b}(\mathbb{R})$ of $\mathcal{V}_{A_n}(\mathbb{R})$ is cut out by one linear equation, involving only $b_{13}$ and $b_{22}$. Hence $b_{11}, b_{12}, b_{22}, b_{23}, b_{33}$ form a complete set of variables for $\mathcal{V}_{A_n,b}(\mathbb{R})$, and elements in $hG_0(\sigma_A(\mathcal{F}_{\mathcal{V}_{A,b}}^{(i)}(X)))$ satisfy the same coefficient bounds as in (41).

Since $g^{-1}L_n$ is a lattice commensurable to $\mathcal{V}_{A_n,b}(\mathbb{Z})$, the denominators of elements in $g^{-1}L$ are absolutely bounded. Therefore, if $t \gg X^{1/24}$, then elements $(A_n, B) \in hG_0 \cdot \sigma_A(\mathcal{F}_{\mathcal{V}_{A,b}}^{(i)}(X)) \cap g^{-1}L_n$ satisfy $b_{11} = 0$, and are reducible since $A_n$ and $B$ have a common zero in $\mathbb{P}^2(\mathbb{Q})$. □

**Lemma 4.15** We have

$$\int_{t \ll X^{1/24}} \# (hG_0 \cdot \sigma_A(\mathcal{F}_{\mathcal{V}_{A,b}}^{(i)}(X)) \cap (L_n \backslash L^\text{gen}_n)) \text{d}h = o(X^{5/6}).$$

**Proof:** The proof is identical to that of [5, Lemma 14]. □

By Lemma 4.14, the integral on the right-hand side of (40) can be restricted to $h \in \mathcal{F}'$, where $\mathcal{F}' = \{h = gi \in \mathcal{F} : t \ll X^{1/24}\}$. Lemma 4.15 implies that replacing $L^\text{gen}_n$ by $L_n$ on the right-hand side of (40) introduces an error of at most $o(X^{5/6})$.

Using Proposition 3.7 to estimate $\# (hG_0 \sigma_A(\mathcal{F}_{\mathcal{V}_{A,b}}^{(i)}(X)) \cap L_n)$ for $i = 0, 1, 2+, \text{ and } 2-$, we obtain

$$N^{(i)}(L; A, X) = \frac{1}{m_i \text{Vol}(G_0)} \int_{h \in \mathcal{F}} \# (hG_0 \sigma_A(\mathcal{F}_{\mathcal{V}_{A,b}}^{(i)}(X)) \cap L_n) \text{d}h + o(X^{5/6})$$

$$= \frac{1}{m_i \text{Vol}(G_0)} \int_{h \in \mathcal{F}} \text{Vol}'(hG_0 \cdot \sigma_A(\mathcal{F}_{\mathcal{V}_{A,b}}^{(i)}(X))) + O(t^4X^{4/6}) \text{d}h + o(X^{5/6}),$$

where the volumes $\text{Vol}'$ of sets in $\mathcal{V}_{A_n,b}(\mathbb{R})$ are computed with respect to Euclidean measure normalized so that $L_n$ has covolume 1.

The measure $dn\, d\alpha \, t \, dk / t^2$ is a Haar measure on $\text{SO}_{A_n,b}(\mathbb{R})$. The integral over $\mathcal{F}$ of $t^4$ is $O(X^{1/12})$.
while the volume of $F \setminus F'$ is $O(X^{-1/12})$. It follows that
\[
N^{(i)}(L; A, X) = \frac{1}{m_i \text{Vol}(G_0)} \int_{h \in F} \text{Vol}'(hG_0 \cdot \sigma_A(F^{(i)}_{V_{A,b}}(X)))dh + O(X^{3/4}) + o(X^{5/6})
\]
\[
= \frac{1}{m_i} \text{Vol}'(F \cdot \sigma_A(F^{(i)}_{V_{A,b}}(X))) + o(X^{5/6})
\]
\[
= \frac{1}{m_i} \nu(L) \text{Vol}(F_A \cdot F^{(i)}_{V_{A,b}}(X)) + o(X^{5/6}),
\]
where the second equality follows from the analogue of Proposition 3.41 for $V$, stated as Proposition 4.27 in §4.5 below, and the volume of $F_A \cdot F^{(i)}_{V_{A,b}}(X)$ is computed with respect to Euclidean measure on $V_{A,b}(\mathbb{R})$ normalized so that $V_{A,b}(\mathbb{Z})$ has covolume 1. We have proven Theorem 4.12. \qquad \Box

Proof of Theorem 4.7: By Corollary 2.11, we have
\[
N^{(i)}_4(n, X) = \sum_{A \in SL_3(\mathbb{Z}) \setminus \mathbb{Q}_n(\mathbb{Z})} \sum_{b=0}^{3n-1} N^{(i)}(V_{A,b}(\mathbb{Z}); A, X) = \frac{3n}{m_i} \sum_{A \in SL_3(\mathbb{Z}) \setminus \mathbb{Q}_n(\mathbb{Z})} \text{Vol}(F_A \cdot F^{(i)}_{V_{A,b}}(X)) + o(X^{5/6}),
\]
where the final equality follows from Theorem 4.12. \qquad \Box

4.3 Uniformity estimates and squarefree sieves

As before, $n \geq 1$ is a fixed integer. Throughout this subsection, we also fix an integer $b \in [0, 3n)$. We begin with the following definition.

Definition 4.16 A collection $S = (S_p)_p$, where $S_p$ is an open and closed subset of $U_{n,b}(\mathbb{Z}_p)$ whose boundary has measure 0, is called a **collection of cubic local specifications**. Such a collection $S$ is **large** if, for all but finitely many primes $p$, the set $S_p$ contains all elements $f \in U_{n,b}(\mathbb{Z}_p)$ with $p^2 \nmid \Delta(f)$. We associate to $S$ the set $U_{n,b}(\mathbb{Z})_S \subset U_{n,b}(\mathbb{Z})$ of integer-coefficient binary cubic forms, where $f \in U_{n,b}(\mathbb{Z})_S$ if and only if $f \in S_p$ for all primes $p$. We also associate to $S$ the set $V_{A,b}(\mathbb{Z})_S \subset V_{A,b}(\mathbb{Z})$, where $v \in V_{A,b}(\mathbb{Z})_S$ if $\text{Res}(v) \in S_p$ for all primes $p$.

In this subsection, we deduce the following theorem.

Theorem 4.17 Let $S$ be a large collection of local specifications on $U_{n,b}$, and let $A$ be an integer-coefficient ternary cubic form with determinant $n/4$. For $i \in \{0, 1, 2\# , 2+, 2-\}$, we have
\[
N^{(i)}(U_{n,b}(\mathbb{Z})_S; n, X) = \nu(U_{n,b}(\mathbb{Z})_S) \text{Vol}(U_{n,b}(\mathbb{R})^{(i)}(X)) + o(X^{5/6});
\]
\[
N^{(i)}(V_{A,b}(\mathbb{Z})_S; A, X) = \frac{1}{m_i} \nu(V_{A,b}(\mathbb{Z})_S) \text{Vol}(F_A \cdot F^{(i)}_{V_{A,b}}(X)) + o(X^{5/6}).
\]

Note that we have $\nu(U_{n,b}(\mathbb{Z})_S) = \prod_p \text{Vol}(S_p)$. The determination of $\nu(V_{A,b}(\mathbb{Z})_S)$ in terms of $S$ is more subtle, and we devote most of the next subsection to it.

To prove Theorem 4.17, we require the following tail estimate.

Proposition 4.18 For a prime $p$, let $W_p(U_{n,b})$ (resp. $W_p(V_{A,b})$) denote the set of elements in $U_{n,b}(\mathbb{Z})$ (resp. $V_{A,b}(\mathbb{Z})$) whose discriminants are divisible by $p^2$. Then for any $M > 0$, we have
\[
\sum_{p > M} N(W_p(U_{n,b}); n, X) = O_e(X^{5/6+\epsilon}/M) + O(X^{1/3}),
\]
\[
\sum_{p > M} N(W_p(V_{A,b}); A, X) = O_e(X^{5/6+\epsilon}/M) + O(X^{19/24}),
\]
where the implied constants are independent of $M$ and $X$. 31
Proof: We write $\mathcal{W}_p(U_{n,b})$ (resp. $\mathcal{W}_p(V_{A,b})$) as the disjoint union $\mathcal{W}_p^{(1)}(U_{n,b}) \cup \mathcal{W}_p^{(2)}(U_{n,b})$ (resp. $\mathcal{W}_p^{(1)}(V_{A,b}) \cup \mathcal{W}_p^{(2)}(V_{A,b})$), where $\mathcal{W}_p^{(1)}(U_{n,b})$ (resp. $\mathcal{W}_p^{(1)}(V_{A,b})$) consists of elements in $\mathcal{W}_p(U_{n,b})$ (resp. $\mathcal{W}_p(V_{A,b})$) whose discriminants are divisible by $p$ for mod $p$ reasons (in the sense of [6, §1.5]). The bounds of (42), with $\mathcal{W}_p(U_{n,b})$ and $\mathcal{W}_p(V_{A,b})$ replaced by $\mathcal{W}_p^{(1)}(U_{n,b})$ and $\mathcal{W}_p^{(1)}(V_{A,b})$, respectively, follow from [6, Theorem 3.5] in conjunction with the proofs of Theorems 4.4 and 4.12.

The proof of (42), with $\mathcal{W}_p(U_{n,b})$ and $\mathcal{W}_p(V_{A,b})$ replaced by $\mathcal{W}_p^{(2)}(U_{n,b})$ and $\mathcal{W}_p^{(2)}(V_{A,b})$, respectively, follows from Proposition 3.36 by taking $T = X^e$. □

Proof of Theorem 4.17: Theorem 4.17 is an immediate consequence of the tail estimate in Proposition 4.18, together with an application of a squarefree sieve identically as in the proof of [8, Theorem 2.21]. □

4.4 Local mass formulas

To compute the volumes in Theorem 4.17 in a manner analogous to those computed in §3.4, we prove certain mass formulas relating étale quartic and cubic algebras over $\mathbb{Q}_p$.

Definition 4.19 Let $n$ be a positive integer, and let $p$ be a prime dividing $n$. Let $(\mathcal{O}, \alpha)$ be an $n$-monogenized cubic ring over $\mathbb{Z}_p$, where $\mathcal{O}$ is the ring of integers of an étale cubic extension $K$ of $\mathbb{Q}_p$. We define the algebra at infinity $A_\infty(\alpha)$ of $(\mathcal{O}, \alpha)$ in two equivalent ways.

Let $f(x, y) \in U_n(\mathbb{Z}_p)$ be a binary cubic form corresponding to $(\mathcal{O}, \alpha)$, so that $K = \mathbb{Q}_p[x]/(f(x, 1))$. Then $K = \prod L_i$ as a product of fields, where $L_i := \mathbb{Q}_p[x]/(f_i(x, y))$ and $f = \prod f_i$ as a product of irreducible factors over $\mathbb{Z}_p$. Without loss of generality, we may assume that the leading coefficient of $f_1$ is $n$, and that the other $f_i$’s have leading coefficient 1. (Indeed, if two factors $f_1$ and $f_2$ both have leading coefficients that are multiples of $p$, then $\prod f_i$ would not correspond to a maximal ring by Lemma 4.33.) Then we define $A_\infty(\alpha)$ to be $L_1$, the factor of $K$ corresponding to $f_1$.

Equivalently, we can also define $A_\infty(\alpha)$ intrinsically in terms of the data $(\mathcal{O}, \alpha)$. If $\mathbb{Z}_p[\alpha] \subset \mathcal{O}$ factors as $\mathbb{Z}_p \times S$ for some quadratic ring $S$ over $\mathbb{Z}_p$, then $K \cong \mathbb{Q}_p \times (S \otimes \mathbb{Q}_p) = L_1 \times L_2$, and we define $A_\infty(\alpha) := L_1 \cong \mathbb{Q}_p$. Otherwise, if $\mathbb{Z}_p$ is not a factor of $\mathbb{Z}_p[\alpha]$, then we define $A_\infty(\alpha)$ as the (unique) factor $L_1$ of $K$ that is a ramified field extension of $\mathbb{Q}_p$.

Remark 4.20 To see the equivalence of the two definitions of the algebra at infinity $A_\infty(\alpha)$, note that if $f(x, y) = \prod_{i \geq 1} f_i(x, y)$ as in Definition 4.19, then the characteristic polynomial of $\alpha$ is $g(x) = (f_1(x, n)/n) \prod_{i \geq 1} f_i(x, n)$ when expressed as a product of monic polynomials. If $f_1(x, y)$ has degree 1, then $\mathcal{O}$ has a factor of $\mathbb{Z}_p$ corresponding to $f_1(x, y)$; this is because $f_1(x, n)/n$ is a linear factor of $g(x)$ sharing no common root modulo $p$ with $\prod_{i \geq 1} f_i(x, n) \equiv x^2 \pmod{p}$. Otherwise, $f_1(x, y)$ has degree at least 2 with a root at infinity modulo $p$. It follows that $f_1(x, y)$ corresponds to a ramified factor of $K$. Hence our two definitions of $A_\infty(\alpha)$ agree.

Notation 4.21 For an étale cubic extension $K_3$ of $\mathbb{Q}_p$, let $\mathcal{R}(K_3)$ denote again the set of étale non-overramified quartic extensions of $\mathbb{Q}_p$, up to isomorphism, with cubic resolvent $K_3$. For an $n$-monogenized étale cubic extension $(K_3, \alpha)$ of $\mathbb{Q}_p$, let $\mathcal{R}^+(K_3, \alpha)$ (resp. $\mathcal{R}^-(K_3, \alpha)$) consist of those $K_4 \in \mathcal{R}(K_3)$ such that $A_\infty(\alpha)$ splits (resp. stays inert) in the unramified quadratic extension $K_6/K_3$ corresponding to $K_4$.

Theorem 4.22 Let $(K_3, \alpha)$ be an $n$-monogenized étale cubic extension of $\mathbb{Q}_p$. If $(K_3, \alpha)$ is sufficiently ramified, then

$$\sum_{K_4 \in \mathcal{R}^+(K_3, \alpha)} \frac{1}{|\text{Aut}_{K_3}(K_4)|} = 1; \quad \sum_{K_4 \in \mathcal{R}^-(K_3, \alpha)} \frac{1}{|\text{Aut}_{K_3}(K_4)|} = 0.$$  \hspace{1cm} (43)
Otherwise, 
\[ \sum_{K_4 \in \mathcal{R}^+(K_3, \alpha)} \frac{1}{|\text{Aut}_{K_3}(K_4)|} = \frac{1}{2}; \sum_{K_4 \in \mathcal{R}^-(K_3, \alpha)} \frac{1}{|\text{Aut}_{K_3}(K_4)|} = \frac{1}{2}. \] (44)

**Proof:** First, assume that \((K_3, \alpha)\) is sufficiently ramified; then, by definition, either:

(a) \(K_3\) is a totally ramified cubic extension of \(\mathbb{Q}_p\), in which case \(A_{\infty}(\alpha) = K_3\); or

(b) \(K_3 = \mathbb{Q}_p \times F\), where \(F\) is a ramified quadratic extension of \(\mathbb{Q}_p\), and \(A_{\infty}(\alpha) = \mathbb{Q}_p\).

Let \(K_6\) be an unramified extension of \(K_3\) such that \(N_{K_3/\mathbb{Q}_p}\Delta(K_6/K_3)\) is a square in \(\mathbb{Z}_p^+\). In Case (a) above, \(K_6/K_3\) must split for \(N_{K_3/\mathbb{Q}_p}\Delta(K_6/K_3)\) to be a square. In Case (b), note that \(N_{F/\mathbb{Q}_p}\Delta(F'/F)\) is a square for every unramified extension \(F'/F\); hence, for \(N_{K_3/\mathbb{Q}_p}\Delta(K_6/K_3)\) to be a square, the component \(\mathbb{Q}_p\) of \(K_3\) must split in \(K_6\). Thus (43) follows from Theorem 3.38.

Next assume that \((K_3, \alpha)\) is not sufficiently ramified. Then \(K_3\) has a factor \(F \neq A_{\infty}(\alpha)\) that is not a ramified quadratic extension of \(\mathbb{Q}_p\). Now \(N_{F/\mathbb{Q}_p}\Delta(F'/F)\) is a square for an unramified quadratic extension \(F'/F\) if and only if it is split; hence the sizes of \(\mathcal{R}^+(K_3, \alpha)\) and \(\mathcal{R}^-(K_3, \alpha)\) are equal. Since the automorphism group of every \(K_4 \in \mathcal{R}(K_3)\) is the same, (44) follows. \(\square\)

**Definition 4.23** Let \(p\) be a prime. A ternary quadratic form \(A \in \mathcal{Q}_3(\mathbb{Z}_p)\) is said to be **good at \(p\)** if the conic in \(\mathbb{P}^2(\mathbb{F}_p)\) given as the zero set of the reduction of \(A\) modulo \(p\) is either smooth or a union of two distinct lines. In the latter case, if each line is defined over \(\mathbb{F}_p\), then we define \(\kappa_p(A) := 1\) and say that \(A\) is **residually hyperbolic**. Otherwise, the two lines are a pair of conjugate lines each defined over \(\mathbb{F}_p^2\); we then define \(\kappa_p(A) := -1\) and say that \(A\) is **residually nonhyperbolic**.

**Lemma 4.24** Let \(p\) be a fixed prime and \(n\) a fixed integer.

(a) If \(p \nmid n\), then \(\mathcal{Q}_n(\mathbb{Z}_p)\) is a single \(\text{SL}_3(\mathbb{Z}_p)\)-orbit.

(b) If \(p \mid n\), then the set of good elements in \(\mathcal{Q}_n(\mathbb{Z}_p)\) breaks up into two \(\text{SL}_3(\mathbb{Z}_p)\)-orbits consisting of elements having \(\kappa_p = 1\) and \(\kappa_p = -1\), respectively.

Finally, if \(A\) is a ternary quadratic form over \(\mathbb{Z}_p\) that is not good at \(p\), then for every ternary quadratic form \(B\) over \(\mathbb{Z}_p\), \(\text{Res}(A, B)\) corresponds to a ring that is nonmaximal at \(p\).

**Proof:** Suppose \(p \neq 2\). Then any \(m\)-ary quadratic form over \(\mathbb{Z}_p\) can be diagonalized via \(\text{SL}_m(\mathbb{Z}_p)\)-transformations (see, e.g., [16, Chapter 8, Theorem 3.1]). Furthermore, a diagonal binary quadratic form \([u_1, u_2]\) over \(\mathbb{Z}_p\) with unit determinant (i.e., \(u_1u_2 \in \mathbb{Z}_p^+\)) is \(\text{SL}_2(\mathbb{Z}_p)\)-equivalent to the diagonal form \([1, u_1u_2]\). Indeed, if at least one of \(u_1\) or \(u_2\) is a unit square, then this is immediate; if both \(u_1\) and \(u_2\) are unit nonsquares in \(\mathbb{Z}_p\), then it suffices to check that a unit nonsquare in \(\mathbb{Z}_p\) can be expressed as a sum of two unit squares, and this is true since the smallest nonsquare \(m\) in \(\mathbb{Z}/p\mathbb{Z}\) is the sum \(1 + (m - 1)\) of two unit squares.

Part (a) now follows because any form \(A \in \mathcal{Q}_n(\mathbb{Z}_p)\) with \(n\) a unit in \(\mathbb{Z}_p\) is seen to be \(\text{SL}_3(\mathbb{Z}_p)\)-equivalent to the diagonal form \([1, 1, n/4]\). To deduce Part (b), note that the reduction modulo \(p\) of a good form \(A \in \mathcal{Q}_n(\mathbb{Z}_p)\) with \(p \mid n\) must have \(\mathbb{F}_p\)-rank 2. Therefore, \(A\) is \(\text{SL}_3(\mathbb{Z}_p)\)-equivalent to either \([1, 1, n/4]\) or \([1, m, n/(4m)]\), where \(m\) is a unit nonsquare; these two forms are \(\text{SL}_3(\mathbb{Z}_p)\)-inequivalent, as one is residually hyperbolic while the other is not.

Next suppose \(p = 2\). A binary quadratic form over \(\mathbb{Z}_2\) with unit discriminant is \(\text{GL}_2(\mathbb{Z}_2)\)-equivalent to either \(S(x, y) := xy\) or \(U(x, y) := x^2 + xy + y^2\) (see, e.g., [16, Chapter 8, Lemma 4.1]).
Since a good ternary quadratic form \( A \) over \( \mathbb{Z}_2 \) is not diagonalizable (for otherwise \( A \) modulo 2 would be the square of a linear form), it must have a summand that is either \( S \) or \( U \). Thus every good form \( A \in \mathcal{Q}_n(\mathbb{Z}_2) \) is equivalent to either \( B := S \oplus [-n] \) or \( B' := U \oplus [n/3] \). It remains to prove that \( B \) is \( \text{SL}_3(\mathbb{Z}_2) \)-equivalent to \( B' \) if and only if \( n \) is odd. Since \( x^2 + xy + y^2 \) represents every odd square class in \( \mathbb{Z}_2 \), it follows that when \( n \) is odd, the form \( B' \) represents 0 and is thus equivalent to \( B \). When \( n \) is even, \( B \) and \( B' \) are inequivalent since \( B \) is residually hyperbolic while \( B' \) is not.

(For quadratic forms of arbitrary dimension, see [16, Chapter 8] and [19, Chapter 15, §7] for the general theory over \( \mathbb{Z}_p \), and [29, Theorem 2.6] for a proof of (a) over local fields.)

Finally, if \( A \) is a ternary quadratic form over \( \mathbb{Z}_p \) that is not good, and \( B \) is any ternary quadratic form over \( \mathbb{Z}_p \), then the \( x^3 \) - and \( x^2y \)-coefficients of \( \text{Res}(A, B) \) are divisible by \( p^2 \) and \( p \), respectively. The last assertion of the lemma then follows from [2, Lemma 2.10] (stated below in Lemma 4.33), which asserts that a binary cubic form whose \( x^3 \)-coefficient is divisible by \( p^2 \) and whose \( x^2y \)-coefficient is divisible by \( p \) is nonmaximal at \( p \). □

The above lemma motivates the definition of the following partial mass formulas.

**Definition 4.25** Let \( n \) be a positive integer, let \( p \) a prime dividing \( n \), and let \( f \) be an element in \( U_n(\mathbb{Z}_p) \) corresponding to a maximal ring. Then we define the \( \kappa \)-mass of \( f \) as

\[
\text{Mass}_p^{\kappa}(f) := \frac{1}{\# \text{Stab}_{\text{SL}_3}(A, B)},
\]

where \( \frac{\text{Res}^{-1}(f)}{\text{SL}_3(\mathbb{Z}_p)} \) is a set of representatives for the action of \( \text{SL}_3(\mathbb{Z}_p) \) on \( \text{Res}^{-1}(f) \).

**Corollary 4.26** Let \( n \) be a positive integer, let \( p \) be a prime dividing \( n \), and let \( (A, B) \) be an element in \( U_n(\mathbb{Z}_p) \) corresponding to a maximal ring. Then

\[
\text{Mass}_p^{+}(f) = 1 \quad \text{and} \quad \text{Mass}_p^{-}(f) = 0 \quad \text{if} \ f \ \text{is sufficiently ramified};
\]

\[
\text{Mass}_p^{+}(f) = \frac{1}{2} \quad \text{and} \quad \text{Mass}_p^{-}(f) = \frac{1}{2} \quad \text{otherwise}.
\]

**Proof:** Let \( (A, B) \) be an element of \( V(\mathbb{Z}_p) \) with resolvent \( f \). Then \( (A, B) \) corresponds to an étale quartic algebra \( K_4 \) with cubic resolvent \( K_3 \), and \( K_4 \) yields an unramified quadratic extension \( K_6/K_3 \). It follows that \( A \) splits (resp. stays inert) in \( K_6 \) if and only if \( A \) is residually hyperbolic (resp. residually nonhyperbolic). □

### 4.5 Volume computations and proof of the main \( n \)-monogenic theorem (Theorem 7)

Let \( \Sigma \) denote a large collection of \( n \)-monogenized cubic fields. We may assume that every \( (K, \alpha) \) arising from \( \Sigma \) satisfies \( \text{Tr}(\alpha) = b \in [0, 3n) \), as the general case follows by summing over all such \( b \). The large collection \( \Sigma \) of \( n \)-monogenized cubic fields gives rise to a large collection \( S = (S_p)_p \) of binary cubic forms where \( S_p \subset U_{n,b}(\mathbb{Z}_p) \) for every \( p \) and every \( f \in S_p \) is maximal. To compute \( \nu(V_{A,b}(\mathbb{Z}_p)) \), we use the following Jacobian change of variables.

**Proposition 4.27** Let \( K \) be \( \mathbb{R} \) or \( \mathbb{Z}_p \), let \( | \cdot | \) denote the usual absolute value on \( K \), and let \( s : U_{n,b}(K) \to V_{A,b}(K) \) be a continuous map such that \( \text{Res}(s(f)) = f \), for each \( f \in U_{n,b}(K) \). Then
there exists a rational nonzero constant $J$, independent of $K$ and $s$, such that for any measurable function $\phi$ on $V_{A,b}(K)$, we have

$$\int_{SO_A(K) \cdot s(U_{n,b}(K))} \phi(v) dv = |J| \int_{f \in U_{n,b}(K)} \int_{g \in SO_A(K)} \phi(g \cdot s(f)) \omega(g) df,$$

$$\int_{V_{A,b}(K)} \phi(v) dv = |J| \int_{f \in U_{n,b}(K)} \sum_{v \in \frac{V_{A,b}(K) \cap \text{Res}^{-1}(f)}{SO_A(K)}} \frac{1}{\# \text{Stab}_{SO_A(Z_p)}(v)} \int_{g \in SO_A(K)} \phi(g \cdot v) \omega(g) df,$$

where $\frac{V_{A,b}(K) \cap \text{Res}^{-1}(f)}{SO_A(K)}$ is a set of representatives for the action of $SO_A(K)$ on $V_{A,b}(K) \cap \text{Res}^{-1}(f)$.

**Proof:** Proposition 4.27 follows immediately from the proofs of [8, Propositions 3.11–12] (see also [8, Remark 3.14]). □

**Corollary 4.28** Let $S_p \subset U_{n,b}(Z_p)$ be a closed subset whose boundary has measure 0. Consider the set $V_{A,b}(Z)_{S,p}$ defined by $V_{A,b}(Z)_{S,p} := V_{A,b}(Z_p) \cap \text{Res}^{-1}(S_p)$. Then

$$\text{Vol}(V_{A,b}(Z)_{S,p}) = \begin{cases} |J|_p \text{Vol}(SO_A(Z_p)) \text{Vol}(S_p) & \text{when } p \nmid n; \\ \frac{1}{2} |J|_p \text{Vol}(SO_A(Z_p)) \text{Vol}(S_p)(1 + \kappa_p(A) \rho_p(\Sigma)) & \text{when } p \mid n. \end{cases} \quad (45)$$

**Proof:** By Proposition 4.27 (setting $\phi$ to be the characteristic function of $V(Z)_{S,p}$), we have

$$\text{Vol}(V_{A,b}(Z)_{S,p}) = |J|_p \text{Vol}(SO_A(Z_p)) \int_{f \in S_p} \sum_{v \in \frac{V_{A,b}(Z_p) \cap \text{Res}^{-1}(f)}{SO_A(Z_p)}} \frac{1}{\# \text{Stab}_{SL_3(Z_p)}(v)} df, \quad (46)$$

since the stabilizers of $v$ in $SO_A(Z_p)$ and $SL_3(Z_p)$ are the same, being equal to $SO_A(Z_p) \cap SO_B(Z_p)$.

If $p \nmid n$, then $Q_n(Z_p)$ is a single $SL_3(Z_p)$-orbit by Lemma 4.24(a). Hence there is a one-to-one correspondence between the sets $SO_A(Z_p) \setminus (V_{A,b}(Z_p) \cap \text{Res}^{-1}(f))$ and $SL_3(Z_p) \setminus \text{Res}^{-1}(f)$, and so the integrand on the right-hand side of (46) is equal to $\text{Mass}_p(f)$.

If $p \mid n$, then $Q_n(Z_p)$ consists of two $SL_3(Z_p)$-orbits by Lemma 4.24(b), having $\kappa_p = 1$ and $\kappa_p = -1$, respectively. Thus there is a one-to-one correspondence between $SO_A(Z_p) \setminus (V_{A,b}(Z_p) \cap \text{Res}^{-1}(f))$ and the set of elements $(A_1, B_1) \in SL_3(Z_p) \setminus \text{Res}^{-1}(f)$ with $\kappa_p(A_1) = \kappa_p(A) = \pm 1$. Hence the integrand on the right-hand side of (46) is $\text{Mass}_p^\pm(f)$. By Corollaries 3.40 and 4.26, we then have

$$\int_{f \in S_p} \text{Mass}_p(f) = 1; \quad \int_{f \in S_p} \text{Mass}_p^\pm(f) = \frac{1 \pm \rho_p(\Sigma)}{2},$$

as desired. □

**Proposition 4.29** Let $G$ be a genus of elements in $Q_n(Z)$ that are good at $p$ for all $p \mid n$. Then

$$\sum_{A \in SL_3(Z) \setminus G} N(i)(V_{A,b}(Z)_{S}; A, X) \prod_{i \mid n} \frac{1}{\rho_p(\Sigma)} = \frac{1}{m_i} \prod_{p \mid n} \frac{1 + \kappa_p(A) \rho_p(\Sigma)}{2} + o(1),$$

where: $i \in \{0, 2+, 2, -2, 2\#\}$ if $\pm = +$; $i = 1$ if $\pm = -$ and $A$ is isotropic over $\mathbb{R}$; and $i = 2\#$ and $\pm = +$ if $A$ is anisotropic over $\mathbb{R}$.

**Proof:** From Theorem 4.17, we obtain

$$\frac{N(i)(V_{A,b}(Z)_{S}; A, X)}{N^\pm(U_{n,b}(Z)_{S}; n, X)} = \frac{1}{m_i} \frac{\nu(V_{A,b}(Z)_{S}) \text{Vol}(F_{A} \cdot F_{V_{A,b}}^{(i)}(X))}{\nu(U_{n,b}(Z)_{S}) \text{Vol}(U_{n,b}(R)^\pm)} + o(1).$$

We evaluate $\text{Vol}(F_A \cdot F_{V_{A,b}}^{(i)}(X))$ and $\nu(V_{A,b}(Z)_{S,p})$ using Proposition 4.27 (with $\phi$ being the char-
acteristic function of $F_A \cdot F_{V_{A,b}}^{(i)}(X))$ and Corollary 4.28, respectively, yielding
\[
\frac{1}{m_i} \nu(V_{A,b}(\mathbb{Z})) \text{Vol}(F_A \cdot F_{V_{A,b}}^{(i)}(X)) = \frac{1}{\nu(U_n,b(\mathbb{Z})) \text{Vol}(U_n,b(\mathbb{R}))} \text{Vol}(F_A) \prod_p \text{Vol}(SO_A(\mathbb{Z}_p)) \prod_{p | n} \frac{1 + \kappa_p(A) \rho_p(\Sigma)}{2}.
\]
Since
\[
\sum_{A \in \text{SL}_3(\mathbb{Z}) \setminus \mathcal{G}} \text{Vol}(F_A) \prod_p \text{Vol}(SO_A(\mathbb{Z}_p)) = 2,
\]
the Tamagawa number of $SO_3$, the proposition follows. □

The following lemma will be used repeatedly in the proof of Theorem 7.

**Lemma 4.30 ([29, Lemma 4.17])** Suppose $\mathbb{T}$ is a nonempty finite set, $X_i$ and $Y_i$ are indeterminates for each $i \in \mathbb{T}$, and $\mu_N$ is the set of all $N^{th}$ roots of unity in $\mathbb{C}$. Then for any $c \in \mu_N$, we have the polynomial identity
\[
\sum_{(\epsilon_i)_{i \in \mathbb{T}}} \prod_{i \in \mathbb{T}} (X_i + \epsilon_i Y_i) = N^{[\mathbb{T}]-1} \left[ \prod_{i \in \mathbb{T}} X_i + c \prod_{i \in \mathbb{T}} Y_i \right].
\]  (47)

We will also require the following notation.

**Notation 4.31** For a genus $\mathcal{G}$ of elements in $\mathbb{Q}_n(\mathbb{Z})$ that are good at $p$ for all $p$, define $\kappa_p(\mathcal{G}) := \kappa_p(A)$ for $A \in \mathcal{G}$. Let $T := T_n$ denote the set of primes $p$ dividing $n$, and let $T_{\text{even}} := T_{n,\text{even}}$ (resp. $T_{\text{odd}} := T_{n,\text{odd}}$) denote the set of $p \in T$ such that the order of $p$ dividing $n$ is even (resp. odd).

**Proof of Theorem 7:** Let $F_{\Sigma}^+(n,X)$ denote the set of fields $K$ in $F_{\Sigma}(n,X)$ with $\pm \Delta(K) > 0$.

**(a):** In the case of 2-class groups of totally real cubic fields, by Theorem 2.15 we have
\[
\text{Avg}(\text{Cl}_2, F_{\Sigma}^+(n,X)) = 1 + \frac{\#(\text{SL}_3(\mathbb{Z}) \setminus (V(\mathbb{Z})^{(0),\text{gen}} \cap \text{Res}^{-1}(U_n,b(\mathbb{Z})_S^+; X)))}{N^+(U_n,b(\mathbb{Z})_S; n,X)} + o(1)
\]
\[
= 1 + \sum_{A \in \text{SL}_3(\mathbb{Z}) \setminus \mathcal{Q}_n(\mathbb{Z})} \frac{N^{(0)}(V_{A,b}(\mathbb{Z})_S; A,X)}{N^+(U_n,b(\mathbb{Z})_S; n,X)} + o(1).
\]

We claim that there is a (unique) genus $\mathcal{G}$ of good ternary quadratic forms with $\kappa_p(\mathcal{G}) = \epsilon_p$ for all $p \in T$ if and only if the ordered tuple $\epsilon \in \{\pm 1\}^T$ satisfies $\prod_{T_{\text{odd}}} \epsilon_p = \kappa_\infty(\mathcal{G})$; here we define $\kappa_\infty(\mathcal{G})$ to be 1 if the forms in $\mathcal{G}$ are isotropic over $\mathbb{R}$ and $-1$ otherwise. Indeed, if $c_v(A)$ denotes the Hasse invariant of $A$ (see [16, p. 55]) and $(a,b)_v$ denotes the Hilbert symbol of $a$ and $b$ over $\mathbb{Q}_v$, then one checks that
\[
c_p(A) = (-1, -n)_p \cdot \kappa_p(A)^{\text{ord}_p(n)}
\]  (48)
for any prime $p$ and any $A \in \mathcal{Q}_n(\mathbb{Z})$ that is good at all primes. Since $n$ is positive, we have $\kappa_\infty(A) = -c_\infty(A)$. Thus the product formula for the Hilbert symbol translates directly to
\[
\prod_{p \in T_{\text{odd}}} \kappa_p(A) = \kappa_\infty(A).
\]

The claim now follows from [16, Theorem 1.3, p. 77] and [16, Theorem 1.2, p. 129].
By Proposition 4.29, we therefore have

$$\text{Avg} (\text{Cl}_2, F^+(n, X)) = 1 + \frac{1}{2} \sum_{G: \kappa_{\infty}(G) = 1} \prod_{p | n} \left( 1 + \frac{\kappa_p(G) \rho_p(\Sigma)}{2} \right) + o(1)$$

$$= 1 + \frac{1}{2} \sum_{(\epsilon_p) \in \{\pm 1\}^{T_{\text{odd}}}} \prod_{T_{\text{odd}}} \left( 1 + \frac{\epsilon_p \rho_p(\Sigma)}{2} \right) + o(1)$$

$$= 1 + \frac{1}{2} \left( 1 + \prod_{T_{\text{odd}}} \rho_p(\Sigma) \right) + o(1)$$

$$= \frac{5}{4} + \frac{1}{4} \rho(\Sigma) + o(1), \quad (49)$$

where the penultimate equality follows from Lemma 4.30 with $N = 2$, $c = 1$, and $T = T_{\text{odd}}$.

(b): In the case of complex cubic fields, we similarly have

$$\text{Avg} (\text{Cl}_2, F^-_S(n, X)) = 1 + \frac{\#(\text{SL}_3(\mathbb{Z}) \setminus (V(\mathbb{Z})^{(1) \text{gen}} \cap \text{Res}^{-1}(U_{n,b}(\mathbb{Z})_{S,X})))}{N^{-}(U_{n,b}(\mathbb{Z})_{S}; n, X)} + o(1)$$

$$= 1 + \frac{A \in \text{SL}_3(\mathbb{Z}) \cap \mathbb{Q}_n(\mathbb{Z})}{N^{-}(U_{n,b}(\mathbb{Z})_{S}; n, X)} + o(1)$$

$$= 1 + \frac{1}{2} \left( 1 + \prod_{T_{\text{odd}}} \rho_p(\Sigma) \right) + o(1)$$

$$= \frac{3}{2} + \frac{1}{2} \rho(\Sigma) + o(1), \quad (50)$$

where we again use Lemma 4.30 with $N = 2$, $c = 1$, and $T = T_{\text{odd}}$.

(c): Finally, in the case of narrow 2-class groups of totally real cubic fields, we have

$$\text{Avg} (\text{Cl}^+_2, F^+_S(n, X)) = 1 + \sum_{A \in \text{SL}_3(\mathbb{Z}) \setminus \mathbb{Q}_n(\mathbb{Z})} \frac{N^{(0)}(V_{A,b}(\mathbb{Z})_{S}; A, X) + N^{(2)}(V_{A,b}(\mathbb{Z})_{S}; A, X)}{N^{+}(U_{n,b}(\mathbb{Z})_{S}; n, X)} + o(1).$$

The contributions of $N^{(2+)}(V_{A,b}(\mathbb{Z})_{S}; A, X)$ and $N^{(2-)}(V_{A,b}(\mathbb{Z})_{S}; A, X)$ are the same as that of $N^{(0)}(V_{A,b}(\mathbb{Z})_{S}; A, X)$, yielding a total contribution of

$$\frac{3}{4} \left( 1 + \prod_{T_{\text{odd}}} \rho_p(\Sigma) \right) + o(1)$$

to the average size of $\text{Cl}^+_2$ from the splitting types 0, 2+, and 2−. On the other hand, since $\kappa_{\infty}(A) = -1$ for pairs $(A, B) \in V(\mathbb{Z})^{(2\#)}$, the contribution of $N^{(2\#)}(V_{A,b}(\mathbb{Z})_{S}; A, X)$ is

$$\frac{1}{4} \left( 1 - \prod_{T_{\text{odd}}} \rho_p(\Sigma) \right) + o(1),$$

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this time using Lemma 4.30 with \( N = 2, c = -1, \) and \( T = T_{\text{odd}}. \) Summing up, we obtain

\[
\text{Avg}(\text{Cl}_2^+, F_\Sigma^+(n, X)) = 2 + \frac{1}{2} \rho(\Sigma) + o(1).
\] (51)

We have proven Theorem 7. □

### 4.6 Deduction of Theorems 4, 5, and 6

**Proof of Theorem 4:** Theorem 4 follows from Theorem 7 by noting that there are no primes dividing \( n = 1 \) to odd order, and so \( \rho(\Sigma) = 1 \) for every collection \( \Sigma. \) □

**Proof of Theorem 6:** Theorem 6 follows from Theorem 7 by noting that when \( \Sigma_p \subset T_p \) contains no extensions ramified at primes dividing \( k, \) then \( \rho_p(\Sigma_p) = 0. \) □

To deduce Theorem 5 from Theorem 7, we determine the probability that an \( n \)-monogenized cubic field is sufficiently ramified at a prime dividing \( n. \) We use the following lemmas.

**Lemma 4.32** Let \( n \) be a positive integer, and let \( p \) be a prime dividing \( n. \) Suppose \( f(x, y) \in U_n(\mathbb{Z}_p) \) corresponds to a maximal \( n \)-monogenized cubic ring \( (C_p, \alpha_p). \) Then the pair \( (C_p \otimes \mathbb{Q}_p, \alpha_p) \) is sufficiently ramified if and only if either:

(a) \( f(x, y) \) has a triple root in \( \mathbb{P}^1(\mathbb{F}_p); \) or

(b) \( f(x, 1) \) has a double root in \( \mathbb{F}_p.

**Proof:** If \( f(x, y) \) has distinct roots in \( \mathbb{P}^1(\overline{\mathbb{F}}_p), \) then \( \mathcal{K}_p := C_p \otimes \mathbb{Q}_p \) is unramified and so \( (\mathcal{K}_p, \alpha_p) \) is not sufficiently ramified. If \( f(x, y) \) has a triple root modulo \( p, \) then \( \mathcal{K}_p \) is a totally ramified cubic extension of \( \mathbb{Q}_p, \) and hence \( (\mathcal{K}_p, \alpha_p) \) is sufficiently ramified.

We now assume that the reduction of \( f(x, y) \) modulo \( p \) has a double root but not a triple root. Then the pair \( (C_p, \alpha_p) \) is sufficiently ramified when \( \mathbb{Z}_p[\alpha] \otimes \mathbb{F}_p \) is isomorphic to \( \mathbb{F}_p \times \mathbb{F}_p[t]/(t^2), \) and not sufficiently ramified when \( \mathbb{Z}_p[\alpha] \otimes \mathbb{F}_p \) is isomorphic to \( \mathbb{F}_p[t]/(t^3). \) Write \( f(x, y) = nx^3 + bx^2y + cxy^2 + dy^3. \) Then the characteristic polynomial of \( \alpha_p \) is equal to \( x^3 + bx^2y + nxy^2 + n^2dy^3. \) Hence \( \mathbb{Z}_p[\alpha] \otimes \mathbb{F}_p \equiv \mathbb{F}_p[x]/(x^3 + bx^2). \) and so \( (C_p, \alpha_p) \) is sufficiently ramified if and only if \( p \nmid d. \) Thus, when the reduction of \( f(x, y) \) modulo \( p \) has a double root but not a triple root, the pair \( (C_p, \alpha_p) \) is sufficiently ramified if and only if the reduction of \( f(x, 1) \) modulo \( p \) has a double root in \( \mathbb{F}_p. \) □

The next result determines when \( f \in U(\mathbb{Z}) \) corresponds to a cubic ring nonmaximal at \( p. \)

**Lemma 4.33 (Lemma 2.10)** A cubic ring corresponding to a binary cubic form \( f(x, y) \) fails to be locally maximal at \( p \) if and only if either: (a) \( f \) is a multiple of \( p, \) or (b) there is some \( \text{GL}_2(\mathbb{Z}) \)-transformation of \( f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3 \) such that \( d \) is a multiple of \( p^2 \) and \( c \) is a multiple of \( p. \)

**Proposition 4.34** The density \( \mu_{\text{max}}(U_n(\mathbb{Z}_p)) \) of elements in \( U_n(\mathbb{Z}_p) \) that are maximal is given by

\[
\mu_{\text{max}}(U_n(\mathbb{Z}_p)) = \begin{cases} 
\frac{(p^2 - 1)/p^2}{(p - 1)^2(p + 1)/p^3} & \text{if } p^2 \nmid n; \\
\frac{(p - 1)^2(p + 1)}{p^3} & \text{if } p^2 \mid n.
\end{cases}
\]

**Proof:** We calculate the probability of nonmaximality in \( U_n(\mathbb{Z}_p) \) using conditions (a) and (b) of Lemma 4.33. Regarding (a), the probability that an element of \( U_n(\mathbb{Z}_p) \) is imprimitive is 0 if \( p \nmid n \) and is \( 1/p^3 \) otherwise.

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Now (b) implies that a primitive binary cubic form \( f(x, y) = nx^3 + bx^2y + cxy^2 + dy^3 \in U_n(\mathbb{Z}_p) \) is nonmaximal if and only if: \( p \not| b \) and \( p^2 \not| n \) (Condition \( C_\infty \)); or for some \( r \in \{0, 1, \ldots, p - 1\} \), the binary cubic form \( f(x + ry, y) = nx^3 + b_r x^2y + c_r xy^2 + d_r y^3 \) satisfies \( p \not| c_r \) and \( p^2 \not| d_r \) (Condition \( C_r \)). We determine next the probabilities that primitivity and nonmaximality occur due to Condition \( C_r \) for \( r = 0, 1, \ldots, p - 1, \infty \), and then sum.

- If \( p \nmid n \), then for finite \( r \), primitivity and Condition \( C_r \) hold when \( p \not| c_r \) and \( p^2 \not| d_r \), which has probability \( 1/p^3 \); Condition \( C_\infty \) never holds in this case.

- If \( p \mid n \), then for finite \( r \), primitivity and Condition \( C_r \) hold when \( p \nmid b_r \), \( p \not| c_r \), and \( p^2 \not| d_r \), which has probability \( (p - 1)/p \times 1/p^3 = (p - 1)/p^4 \); Condition \( C_\infty \) never holds in this case.

- If \( p^2 \mid n \), then for finite \( r \), primitivity and Condition \( C_r \) hold (just as when \( p \mid n \)) with probability \( (p - 1)/p^4 \); moreover, primitivity and Condition \( C_\infty \) hold when \( p \nmid b \) and either \( p \mid c \) or \( p \mid d \), which occurs with probability \( 1/p \times (p - 1)/p^2 = (p - 1)/p^3 \).

Thus the probability of nonmaximality is given by: \( p \times 1/p^3 = 1/p^2 \) if \( p \nmid n \); \( 1/p^3 + p \times (p - 1)/p^4 = 1/p^2 \) if \( p \mid n \); and \( 1/p^2 + (p - 1)/p^3 = (p^2 + p - 1)/p^3 \) if \( p^2 \mid n \). \( \Box \)

**Proposition 4.35** The density \( \mu_{\text{max, suff}}(U_n(\mathbb{Z}_p)) \) of elements in \( U_n(\mathbb{Z}_p) \) that are maximal and sufficiently ramified is given by

\[
\mu_{\text{max, suff}}(U_n(\mathbb{Z}_p)) = \begin{cases} 
\frac{(p - 1)}{2p^2} & \text{if } p^2 \nmid n; \\
\frac{(p - 1)^2}{2p^3} & \text{if } p^2 \mid n.
\end{cases}
\]

**Proof:** Here we use Lemma 4.32 and Lemma 4.33, and also the same notation and representatives \( r \) for \( \mathbb{P}^1(\mathbb{F}_p) \) as in the proof of Proposition 4.34.

- If \( p \nmid n \), then for finite \( r \), we have that \( f \) is maximal with a multiple root at \( r \) precisely when \( p \mid c_r \) and \( p \mid d_r \), which has probability \( 1/p \times (p - 1)/p^2 = (p - 1)/p^3 \); a multiple root at \( \infty \) cannot occur in this case.

- If \( p \mid n \), then for finite \( r \), we have that \( f \) is maximal with a multiple root at \( r \) precisely when \( p \nmid c_r \), \( p \not| c_r \) and \( p \not| d_r \), which has probability \( (p - 1)/p \times 1/p \times (p - 1)/p^2 = (p - 1)^2/p^3 \); maximality and a triple root occur at \( \infty \) when \( p \not| b \), \( p \not| c \), and \( p \nmid d \), which has probability \( 1/p \times 1/p \times (p - 1)/p = (p - 1)/p^3 \).

- If \( p^2 \mid n \), then for finite \( r \), we have that \( f \) is maximal with a multiple root at \( r \) (just as when \( p \mid n \)) with probability \( (p - 1)/p^4 \); maximality with a multiple root at \( \infty \) cannot occur. So the probability that \( f \) is maximal and sufficiently ramified is: \( p \times (p - 1)/p^3 = (p - 1)/p^2 \) if \( p \nmid n \); \( p \times (p - 1)^2/p^4 + (p - 1)/p^3 = (p - 1)/p^2 \) if \( p \mid n \); and \( p \times (p - 1)^2/p^4 = (p - 1)^2/p^3 \) if \( p^2 \mid n \). \( \Box \)

Propositions 4.34 and 4.35 therefore imply the following.

**Corollary 4.36** For any positive integer \( n \) and prime \( p \), the relative density of sufficiently-ramified elements in \( U_n(\mathbb{Z}_p) \) among the maximal elements in \( U_n(\mathbb{Z}_p) \) is

\[
\frac{\mu_{\text{max, suff}}(U_n(\mathbb{Z}_p))}{\mu_{\text{max}}(U_n(\mathbb{Z}_p))} = \frac{1}{p + 1}.
\]

**Proof of Theorem 5:** Theorem 5 follows immediately from Theorem 7 and Corollary 4.36. \( \Box \)
### Index of notation

| Notation          | Description                                                                                   | In       |
|-------------------|---------------------------------------------------------------------------------------------|----------|
| $A_n$             | The symmetric antidiagonal matrix with antidiagonal $(1/2, -n, 1/2)$.                       | C4.13    |
| $\mathcal{A}_K(\alpha)$ | The algebra at infinity of the $n$-monogenized cubic ring $(\mathcal{O}, \alpha)$. | D4.19    |
| $\text{Aut}_{K_3}(K_4)$ | The subgroup of $\text{Aut}(K_4)$ inducing the trivial element of $\text{Aut}(K_3)$. | N3.37    |
| $(C, \alpha)$     | An $n$-monogenized cubic ring.                                                              | D2.5     |
| $\text{Cl}_2(C)$, $\text{Cl}_2(C)^*$, $\text{Cl}_2(C)^{**}$ | The 2-torsion subgroups of the class group, the narrow class group, and their respective dual groups. | N2.14    |
| $\Delta(f)$       | $b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd$, where $f(x, y) = ax^4 + bx^2y + cxy^3 + dy^4 \in U(R)$. | D3.4     |
| $d_\gamma$        | A Haar measure on $\text{SL}_3(\mathbb{R})$.                                               | C3.24    |
| $F(n, X)$         | The set of $n$-monogenized cubic fields $(K, \alpha)$ with $H(K) < X$.                    | §1.2     |
| $F(\leq cH^8, X)$ | The set of $n$-monogenized cubic fields $(K, \alpha)$ such that $n \leq cH(K, \alpha)^8$, $H(K, \alpha) < X$, and $(K \otimes \mathbb{Q}_p, \alpha) \in \Sigma_p$ for all primes $p$. | T3       |
| $F_{\Sigma}(n, X)$ | The set of $n$-monogenized cubic fields $(K, \alpha)$ with local conds. of $\Sigma$ and $H(K, \alpha) < X$. | §1.2     |
| $F_{\Sigma}(\leq cH^8, X)$ | The set of $n$-monogenized cubic fields $(K, \alpha)$ with local conds. of $\Sigma$, $n \leq cH(K, \alpha)^8$, and $H(K, \alpha) < X$. | D3.1     |
| $F_A$             | A fundamental domain for the action of $\text{SO}_A(\mathbb{Z})$ on $\text{SO}_A(\mathbb{R})$. | C4.13    |
| $F_{\text{SL}_3}$ | A fundamental domain for the action of $\text{SL}_3(\mathbb{Z})$ on $\text{SL}_3(\mathbb{R})$.  | C3.22    |
| $F_U$            | A fundamental domain for the action of $\text{SO}_A(\mathbb{Z})$ on $\text{SO}_A(\mathbb{R})$ with $\pm \Delta(f) > 0$ and $\text{ind}(f) > 0$. | C3.5     |
| $F_U^{(\leq cH^8, X)}$ | The set $\{f \in F_U^{(\leq cH^8, X)}$ with $\text{ind}(f) \leq cH(f)^8$ and $H(f) < X$. | C3.5     |
| $F_U^{(T; X)}$    | The set $\{f \in F_U^{(\leq cH^8, X)}$ with $T \leq \text{ind}(f) < 2T$ and $X \leq H(f) < 2X \}$. | N3.8     |
| $\kappa(T; X)$    | $\kappa(T; X)$-fold cover of $F_U^{(\leq cH^8, X)}(T; X)$.                                  | C3.20    |
| $F_V^{(\leq cH^8, X)}$ | A fundamental set for the action of $\text{SL}_3(\mathbb{R})$ on the set of elements in $V(\mathbb{R})^{(i)}$ with resolvent in $F_U^{(\leq cH^8, X)}$. | C3.22    |
| $F_V^{(i)}(T; X)$ | A fundamental set for the action of $\text{SL}_3(\mathbb{R})$ on the set of elements in $V(\mathbb{R})^{(i)}$ with resolvent in $F_U^{(i)}(T; X)$ | C3.22    |
| $\kappa(T; X)$-fold cover of $F_V^{(i)}(T; X)$. | $\kappa(T; X)$-fold cover of $F_V^{(i)}(T; X)$. | C3.22    |
| $F_{V_{A,B}}^{(i)}$ | A fundamental set for the action of $\text{SO}_A(\mathbb{R})$ on $V_{A,B}(\mathbb{R})^{(i)}$. | C4.8     |
| $F_{V_{A,B}}^{(i)}(X)$ | The set of elements in $F_{V_{A,B}}^{(i)}$ having height less than $X$. | C4.8     |
| $g_A$             | An element in $\text{SL}_3(\mathbb{Q})$ such that $g_AAg_A^t = A_n$.                       | C4.13    |
| $G$               | A genus of integer-coefficient ternary quadratic forms.                                       | N4.31    |
| $G(R)$            | The subgroup $\{(g_2, g_3) \in \text{GL}_2(R) \times \text{GL}_3(R) : \text{det}(g_2)\text{det}(g_3) = 1\}$. | N2.8     |
| $(g_2, g_3) \cdot (A, B)$ | The action of $(g_2, g_3) \in G(R)$ on $(A, B) \in V(R)$. | N2.8     |
| $\gamma : f(x, y)$ | The twisted action of $\gamma \in \text{GL}_2$ on a binary cubic form $f(x, y)$.           | N2.3     |
| generic           | Corresponds to an order in an $S_3$-cubic field or an $S_4$-quartic field.                  | D3.10    |
| good at $p$       | The associated conic in $\mathbb{P}^2(\mathbb{F}_p)$ is either smooth or a union of two lines. | D4.23    |
| $H(\beta)$       | $\max_v \{|\beta_v^\prime|\}$, where $\beta^\prime - \beta \notin \mathbb{Z}$ and the absolute trace $\text{Tr}(\beta^\prime) \in \{0, 1, 2\}$. | N3.31    |
| $H(f)$           | $a^2 - \max \{|H(f)|^3, |J(f)|^2/4\}$, where $f \in U(R)$.                                   | D3.4     |
| $I(f)$           | $b^2 - 3ac$, where $f(x, y) = ax^4 + bx^2y + cxy^3 + dy^4 \in U(R)$.                          | D3.4     |
| Notation | Description |
|----------|-------------|
| \(\text{ind}(f)\) | \(a\), where \(f(x, y) = ax^3 + bx^2y + cxy^3 + dy^3 \in U(R)\). |
| \(J(f)\) | \(-2b^3 + 9abc - 27a^2d\), where \(f(x, y) = ax^3 + bx^2y + cxy^3 + dy^3 \in U(R)\). |
| \(\kappa = \kappa(T; X)\) | \(\lceil X^{1/6}/T^{2/3} \rceil\). |
| \(\kappa_p := \kappa_p(A)\) | (For \(A\) having \(\mathbb{F}_p\)-rank two.) \(1\) if residually hyperbolic, \(-1\) otherwise. |
| \(\kappa_p(G)\) | The \(\kappa\)-invariant \(\kappa_p(A)\) for any \(A \in G\). |
| \(L_{\text{gen}}\) | The set of generic elements in \(L\). |
| \(L^\pm\) | The set of elements \(f \in L\) with \(\pm \Delta(f) > 0\). |
| \(L^{(i)}\) | \(L \cap V(\mathbb{R})^{(i)}\). |
| \(\text{large}\) | \(S := (S_p)\) is large if for all but finitely many \(p\), the set \(S_p\) contains all \(f \in U(\mathbb{Z}_p)\) (resp. \((A, B) \in V(\mathbb{Z}_p)\)) with \(p^2 \mid \Delta(f)\) (resp. \(p^2 \mid \Delta(A, B)\)). |
| \(M(R)\) | The group of \(2 \times 2\) lower triangular unipotent matrices over \(R\). |
| \(\text{Mass}_p(f)\) | The local mass of binary cubic form \(f \in U(\mathbb{Z}_p)\) at the prime \(p\). |
| \(\text{Mass}^\pm_p(f)\) | The local \(\kappa\)-mass of the binary cubic form \(f \in U(\mathbb{Z}_p)\) at the prime \(p\). |
| \(m_i\) | \(m_0 = m_2 = m_{2\pm} = m_{2\#} = 4; \quad m_1 = 2\). |
| \(N'\) | A compact subset of unipotent lower triangular \(3 \times 3\) matrices over \(\mathbb{R}\). |
| \(N^3_3(n, X)\) | Number of \(n\)-monogenized \(S_3\)-orders \((C, \alpha)\) with \(\pm \Delta(C) > 0\) and \(H(C, \alpha) < X\). |
| \(N^3_3(\leq cH^3, X)\) | Number of \(n\)-monogenized \(S_3\)-orders \((C, \alpha)\) with \(\pm \Delta(C) > 0\), \(n \leq cH(C, \alpha)^3\), and \(H(C, \alpha) < X\). |
| \(N^{(i)}_4(n, X)\) | Number of pairs \((Q, (C, \alpha))\) where \(Q\) is an \(S_4\)-order, \(Q \otimes \mathbb{R} \cong \mathbb{R}^{4-2i} \times \mathbb{C}^i\), and \((C, \alpha)\) is an \(n\)-monogenized cubic resolvent of \(Q\) with \(H(C, \alpha) < X\). |
| \(N^{(i)}_4(\leq cH^3, X)\) | Number of pairs \((Q, (C, \alpha))\) where \(Q\) is an \(S_4\)-order, \(Q \otimes \mathbb{R} \cong \mathbb{R}^{4-2i} \times \mathbb{C}^i\), and \((C, \alpha)\) is an \(n\)-monogenized cubic resolvent of \(Q\) with \(n \leq cH(C, \alpha)^3\) and \(H(C, \alpha) < X\). |
| \(N^\pm(L; n, X)\) | The number of generic \(M(\mathbb{Z})\)-orbits \(f \in L^\pm\) with \(H(f) < X\). |
| \(N^{(i)}(L; A, X)\) | The number of generic \(SO_4(\mathbb{Z})\)-orbits \(v \in L \cap V_A(\mathbb{Z})^{(i)}\) with \(H(v) < X\). |
| \(N^{(i)}(L; T; X)\) | The number of generic \(M(\mathbb{Z}) \times \text{SL}_3(\mathbb{Z})\)-orbits \(v \in L^{(i)}\) with \(T \leq \text{ind}(v) < 2T\) and \(X \leq H(v) < 2X\). |
| nowhere overramified | Not overramified at any finite or infinite place. |
| \(\nu(L)\) | The volume of the closure of \(L\) in \(U(\mathbb{Z})\). |
| overramified at \(p\) | The prime ideal \(p\mathbb{Z}\) factors as \(P_1^4, P_2^2\), or \(P_1^2P_2^2\). |
| \(Q_n(R)\) | The set of ternary quadratic forms with coefficients in \(R\) and \(4 \text{ det } = n\). |
| \(\mathcal{R}(K_3)\) | The set of étale non-overramified quartic extensions of \(\mathbb{Q}_p\) with cubic resolvent \(K_3\). |
| \(\mathcal{R}^+(K_3), \mathcal{R}^-(K_3)\) | The elements of \(\mathcal{R}(K_3)\) that are respectively split or inert in the unramified quadratic extension \(K_6/K_3\) corresponding to \(K_4\). |
| \(\text{Res}\) | The resolvent map \(V(R) \to U(R)\) defined by \((A, B) \to 4 \text{ det } (Ax - By)\). |
| residually hyperbolic at \(p\) | The associated conic in \(\mathbb{P}^2(\mathbb{F}_p)\) is a union of two lines defined over \(\mathbb{F}_p\). |
| residually nonhyperbolic at \(p\) | The associated conic in \(\mathbb{P}^2(\mathbb{F}_p)\) is a union of two lines not defined over \(\mathbb{F}_p\). |
### Notation

| Notation | Description                                                                                                                                                                                                 | In       |
|----------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------|
| $S := (S_p)_p$ | A collection of local cubic specifications $S_p$ indexed by primes $p$.                                                                                                                                 | D3.1,    |
| $\sigma_A$ | The map $V_{A,b}(F) \to V_{A_n,b}(F)$ given by $(A, B) \mapsto (A_n, g_{AB} B^g_A)$.                                                                                                                                 | C4.13    |
| $\sigma_G$ | The map $\text{SO}_A(F) \to \text{SO}_{A_n}(F)$ given by $g \mapsto g_A g_A^{-1}$.                                                                                                                                 | C4.13    |
| $\Sigma$ | $\Sigma := (\Sigma_p)_p$.                                                                                                                                                                                                                                           | D3.1     |
| $\Sigma_p$ | A set of pairs $(K_p, \alpha_p)$ where $K_p$ is an étale cubic extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_p$, $\alpha_p$ is an element of $\mathcal{O}_p$ that is primitive in $\mathcal{O}_p/\mathbb{Z}_p$, and the pair $(\mathcal{O}_p, \alpha_p)$ corresponds to some $f(x, y) \in S_p$. | D3.1     |
| $\text{sk}(C)$ | The skewness of the cubic ring $C$.                                                                                                                                                                           | D3.32    |
| $T$ | A subset of the diagonal $3 \times 3$ matrices over $\mathbb{R}$ with det = 1.                                                                                                                               | C4.13    |
| $T_{\text{even}}, T_{\text{odd}}$ | The set of primes $p$ dividing $n$ to even or odd order, respectively.                                                                                                                                         | N4.31    |
| $U(R)$ | The set of binary cubic forms $f(x, y)$ over $R$.                                                                                                                                                            | N2.3     |
| $U_n(R)$ | The set of binary cubic forms $f(x, y) \in U(R)$ with $x^3$-coefficient $n$.                                                                                                                                   | N2.6     |
| $U_{n,b}(R)$ | The set of binary cubic forms $f(x, y) \in U_n(R)$ with $x^2y$-coefficient $b$.                                                                                                                              | N4.2     |
| $U(\mathbb{Z})_S$ | The set of elements in $U(\mathbb{Z})$ satisfying the local prescriptions of $S$.                                                                                                                            | N3.27    |
| $U_{n,b}(\mathbb{Z})_S$ | The set of elements in $U_{n,b}(\mathbb{Z})$ satisfying the local prescriptions of $S$.                                                                                                                      | D4.16    |
| $V(R)$ | The set of pairs of ternary quadratic forms $(A, B)$ over $R$.                                                                                                                                               | N2.8     |
| $V_{A,b}(R)$ | The subset of $(A, B) \in V(R)$ with fixed $A$.                                                                                                                                                              | N4.5     |
| $V_{A,b}(R)$ | The subset of $(A, B) \in V_{A}(R)$ with resolvent in $U_{n,b}(R)$ for some $n$.                                                                                                                            | N4.5     |
| $V(\mathbb{Z})_S$ | The set of elements $v \in V(\mathbb{Z})$ such that Res$(v) \in U(\mathbb{Z})_S$.                                                                                                                        | N3.27    |
| $V_{A,b}(\mathbb{Z})_S$ | The set of elements $v \in V_{A,b}(\mathbb{Z})$ such that Res$(v) \in U_{n,b}(\mathbb{Z})_S$.                                                                                                              | D4.16    |
| $V(\mathbb{R})_+$ | The set of elements $(A, B) \in V(\mathbb{R})$ with $\text{det}(A) > 0$.                                                                                                                                 | D3.14    |
| $V(\mathbb{Z})_+$ | $V(\mathbb{Z}) \cap V(\mathbb{R})_+$.                                                                                                                                                                      | N3.13    |
| $V(\mathbb{R})^{(i)}$ | $A G(\mathbb{R})$-orbit in $V(\mathbb{R})_+$ specified by the index $i \in \{0, 1, 2, 2\#, 2+, 2-\}$.                                                                                                    | D3.14,   |
| $V(\mathbb{Z})^{(i)}$ | $V(\mathbb{Z}) \cap V(\mathbb{R})^{(i)}$.                                                                                                                                                                  | N3.19    |
| $\mathcal{W}_p(U), \mathcal{W}_p(V)$ | Subsets of $U(\mathbb{Z}), V(\mathbb{Z})$ where $p^2 \mid \Delta$.                                                                                                                                          | N3.29    |
| $\mathcal{W}_p^{(t)}(U), \mathcal{W}_p^{(t)}(V)$ | Subsets of $U(\mathbb{Z}), V(\mathbb{Z})$ where $p^2 \mid \Delta$ for “mod $p^t$ reasons”.                                                                                                                 | N3.29    |

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