Spark Deficient Gabor Frames for Inverse Problems

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Abstract—In this paper, we apply star-Digital Gabor Transform in analysis Compressed Sensing and speech denoising. Based on assumptions on the ambient dimension, we produce a window vector that generates a spark deficient Gabor frame with many linear dependencies among its elements. We conduct computational experiments on both synthetic and real-world signals, using as baseline three Gabor transforms generated by state-of-the-art window vectors and compare their performance to star-Gabor transform. Results show that the proposed star-Gabor transform outperforms all others in all signal cases.

I. INTRODUCTION

We address two ill-posed inverse problems: Compressed Sensing (CS) and speech denoising. To remedy their ill-posedness, we assume the problem is analysis sparse [1]. The optimization problems occurring under this scenario are – for CS and denoising respectively – the following:

\[
\min_{\mathbf{x} \in \mathbb{V}^L} \|\Phi \mathbf{x}\|_1 \quad \text{subject to} \quad \|A\mathbf{x} - \mathbf{y}\|_2 \leq \eta \quad (1)
\]

\[
\min_{\mathbf{x} \in \mathbb{V}^L} \|\Phi \mathbf{x}\|_1 \quad \text{subject to} \quad \|\mathbf{x} - \mathbf{y}\|_2 \leq \eta, \quad (2)
\]

\(\mathbb{V} \in \mathbb{C}^{K \times L} (\mathbb{V} = \mathbb{R} \text{ or } \mathbb{C}, K < L).\)

We turn to analysis sparsity instead of its synthesis twin [2] due to some advantages the former has, e.g., lower computational cost, the optimization algorithm used may need less measurements for perfect reconstruction etc.

A. Motivation and key contributions

We are motivated by related works proposing either analysis operators \(\Phi\) associated to full spark frames or a finite difference operator [1], [3], [4]. In a similar spirit, we employ spark deficient Gabor frames (SDGF). Our main contributions are the following: a) we generate an SDGF, associate to it a Gabor analysis operator/digital Gabor transform (DGT) called star-DGT and use it as a sparsifying transform in both analysis CS and denoising b) we compare numerically star-DGT with three other DGTs, based on famous windows of time-frequency analysis, on synthetic and real-world data. Our experiments show that our method outperforms all others, consistently for all signals, in both CS and denoising.

II. MAIN RESULTS

Definition II.1. A discrete Gabor frame \((g, a, b)\) [5] is a collection of time-frequency shifts of a window vector \(g \in \mathbb{C}^L\), expressed as

\[g_{n,m}(l) = e^{2\pi i n m l / L} g(l - na), \quad l \in [L],\]

which spans \(\mathbb{C}^L\) [6]. Here, \(a, b\) denote time and frequency parameters respectively, \(n \in [N]\) with \(N = L/a \in \mathbb{N}\) and \(m \in [M]\) with \(M = L/b \in \mathbb{N}\) denote time and frequency shift indices respectively.

The number of elements in \((g, a, b)\) according to (3) is \(P = MN = L^2/ab\) and since \((g, a, b)\) is a frame, we have \(ab < L\).

Definition II.2. The Gabor analysis operator associated to a Gabor frame is defined as

\[
\Phi_g : \mathbb{C}^L \mapsto \mathbb{C}^{M \times N} : x \mapsto \sum_{l=0}^{L-1} x(l) g(l - na)e^{-2\pi i n m l / L}, \quad (4)
\]

for \(m \in [M], n \in [N]\).

Definition II.3 ([7]). To the Zauner matrix \(U \in \mathbb{C}^L\) is the size of the smallest linearly dependent subset of \(F\). A frame is full spark if and only if every set of \(L\) of its elements is a basis, otherwise it is spark deficient.

Theorem II.5 ([7]). Let \(L \in \mathbb{Z}\) such that \(2 \nmid L, 3 \nmid L\) and \(L\) is square-free. Then, any eigenvector of the Zauner unitary matrix \(U\) generates a spark deficient Gabor frame for \(\mathbb{C}^L\).

In order to produce an SDGF and apply its associated analysis operator in (1) and (2), we first choose an ambient dimension \(L\) that fits the assumptions of Theorem II.5. Then, we perform the spectral decomposition of \(U\) in order to acquire its eigenvectors. Since all the eigenvectors of \(U\) generate SDGFs, we may choose an arbitrary one, call it star window from now on and denote it as \(g_s\). We call the analysis operator associated with such an SDGF star-DGT and denote it \(\Phi_{g_s}\).

III. NUMERICAL EXPERIMENTS

We solve (1) and (2), for synthetic (complex and real-valued) and real-world speech signals respectively, using star-DGT along with three other DGTs emerging from famous window vectors (Gaussian, Hann, Hamming or itersine). Fig. 2 presented in the next page, show a) for CS, the 4 relative reconstruction error decays as the number of measurements increases b) for speech denoising, how the 4 MSEs scale as the noise’s standard deviation increases. In all cases, star-DGT (blue line) outperforms the rest of DGTs, consistently for all signals and for different choices of ambient dimension with time-frequency parameters for each signal.

IV. CONCLUSION

In the present paper, we took advantage of a window vector to generate a spark deficient Gabor frame and introduced a (highly) redundant Gabor transform, i.e. the star-DGT, associated with this SDGF. We then applied star-DGT to analysis Compressed Sensing and analysis-sparsity-based speech denoising, along with three other DGTs generated by state-of-the-art window vectors in the field of Gabor Analysis. Our experiments confirm improved performance: star-DGT outperforms all others for both synthetic and real-world data. Future directions will include the combination of deep learning architectures with star-DGT, as well as the extension of the presented framework to largescale problems.
Fig. 1: Rate of robustness for denoising real-world speech signals, with additive zero-mean Gaussian noise, having a standard deviation $\sigma$ taking values uniformly in $[0.001, 0.01]$.

TABLE I: Signals’ details: top to bottom for both CS and denoising

| Labels      | Samples | Ambient Dimension & Lattice Parameters |
|-------------|---------|----------------------------------------|
| Denoising   | 251-138532-0014 | (L, a, b) = (33915, 51, 19) |
| Denoising   | 3752-4944-0042 | (L, a, b) = (51051, 21, 23) |
| Denoising   | 5694-64038-0013 | (L, a, b) = (51051, 33, 17) |
| CS: TwoChirp | 57       | (L, a, b) = (57, 1, 19) |
| CS: Bumps   | 45       | (L, a, b) = (45, 9, 1) |
| CS: Cusp    | 57       | (L, a, b) = (57, 1, 19) |

Fig. 2: Rate of approximate success for CS with synthetic data, using a random Gaussian $A \in \mathbb{R}^{K \times L}$ and standard deviation of Gaussian noise added $\sigma = 0.001$.

REFERENCES

[1] M. Kabanava and H. Rauhut, “Analysis $l_1$-recovery with frames and Gaussian measurements,” Acta Applicandae Mathematicae, vol. 140, no. 1, pp. 173–195, 2015.

[2] I. W. Selesnick and M. A. Figueiredo, “Signal restoration with overcomplete wavelet transforms: Comparison of analysis and synthesis priors,” in Wavelets XIII, vol. 7446. International Society for Optics and Photonics, 2009, p. 74460D.

[3] S. Nam, M. E. Davies, M. Elad, and R. Gribonval, “The cosparse analysis model and algorithms,” Applied and Computational Harmonic Analysis, vol. 34, no. 1, pp. 30–56, 2013.

[4] M. Genzel, G. Kutyniok, and M. März, “$l_1$-analysis minimization and generalized (co-) sparsity: When does recovery succeed?” Applied and Computational Harmonic Analysis, vol. 52, pp. 82–140, 2021.

[5] P. L. Søndergaard, P. C. Hansen, and O. Christensen, “Finite discrete Gabor analysis,” Ph.D. dissertation, Technical University of Denmark, 2007.

[6] R.-D. Malikiosis, “Spark deficient Gabor frames,” Pacific Journal of Mathematics, vol. 294, no. 1, pp. 159–180, 2018.

[7] H. B. Dang, K. Blanchfield, I. Bengtsson, and D. M. Appleby, “Linear dependencies in Weyl–Heisenberg orbits,” Quantum Information Processing, vol. 12, no. 11, pp. 3449–3475, 2013.