Dispersive deformations of Hamiltonian systems of hydrodynamic type in $2 + 1$ dimensions

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Dedicated to Professor Boris Dubrovin on the occasion of his 60th birthday

Abstract

We develop a theory of integrable dispersive deformations of $2 + 1$ dimensional Hamiltonian systems of hydrodynamic type following the scheme proposed by Dubrovin and his collaborators in $1 + 1$ dimensions. Our results show that the multi-dimensional situation is far more rigid, and generic Hamiltonians are not deformable. As an illustration we discuss a particular class of two-component Hamiltonian systems, establishing the triviality of first order deformations and classifying Hamiltonians possessing nontrivial deformations of the second order.

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1 Introduction

Deformation theory of 1+1 dimensional Hamiltonian systems has been thoroughly investigated by Dubrovin and his collaborators in [8, 9, 11, 12, 13]: given a Hamiltonian system of hydrodynamic type,

\[ u_i^t = \{ u_i, H_0 \} = P^{ij} \frac{\delta H_0}{\delta u_j}, \tag{1} \]

\( i, j = 1, \ldots, n, \) where \( P^{ij} = \epsilon^i \delta^{ij} \frac{d}{dx} \) is the Hamiltonian operator and \( H_0 = \int h(u) \, dx \) is a Hamiltonian with the density \( h(u) \), one looks for deformations of the form

\[ H = H_0 + \epsilon H_1 + \epsilon^2 H_2 + \ldots \tag{2} \]

where the density of \( H_i \) is assumed to be a homogeneous polynomial of degree \( i \) in the \( x \)-derivatives of \( u \). Here the Hamiltonian operator \( P^{ij} \) can be assumed undeformed due to the general results of [19, 5]. Deformation (2) is called integrable (to the order \( \epsilon^m \)) if any hydrodynamic Hamiltonian \( F_0 = \int f(u) \, dx \) commuting with \( H_0 \) can be deformed in such a way that \( \{ H, F \} = 0 \mod \epsilon^{m+1} \). It is assumed that \( H_0 \) generates an integrable system of hydrodynamic type [23, 24, 7]: any system of this kind possesses an infinity of commuting Hamiltonians \( F_0 \) parametrised by \( n \) arbitrary functions of one variable. The classification of integrable deformations is performed modulo canonical transformations of the form

\[ H \rightarrow H + \epsilon \{ K, H \} + \frac{\epsilon^2}{2} \{ K, \{ K, H \} \} + \ldots \tag{3} \]

where \( K \) is any functional of the form (2). The richness of this deformation scheme is due to the following basic facts:

- The variety of integrable ‘seed’ Hamiltonians \( H_0 \) is parametrised by \( n(n-1)/2 \) arbitrary functions of two variables;
- For a fixed integrable Hamiltonian \( H_0 \), the deformation procedure introduces extra arbitrary functions of one variable known, in bi-Hamiltonian context, as ‘central invariants’. One should point out that it is still an open problem to extend a deformation, for arbitrary values of these functions, to all orders in the deformation parameter \( \epsilon \).

The main goal of this paper is to discuss the analogous deformation scheme in 2+1 dimensions. One again starts with the Hamiltonian system (1) where \( P^{ij} \) is a 2-dimensional Hamiltonian operator of hydrodynamic type, see [6, 21, 22] for the general theory and classification results. In the two-component case there exist only three types of such operators: the first two of them can be reduced to constant-coefficient forms,

\[ P = \begin{pmatrix} \frac{d}{dx} & 0 \\ 0 & \frac{d}{dy} \end{pmatrix}, \quad P = \begin{pmatrix} 0 & \frac{d}{dx} \\ \frac{d}{dx} & \frac{d}{dy} \end{pmatrix}, \]

while the third one is essentially non-constant,

\[ P = \begin{pmatrix} 2v & w \\ w & 0 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} 0 & v \\ v & 2w \end{pmatrix} \frac{d}{dy} + \begin{pmatrix} w_x & w_y \\ v_x & v_y \end{pmatrix}, \]

here \( v, w \) are the dependent variables. We will refer to them as Hamiltonian operators of type I, II and III, respectively. To be specific, we will concentrate on case II. The corresponding Hamiltonian systems take the form

\[ \begin{pmatrix} v \\ w \end{pmatrix}_t = \begin{pmatrix} 0 & \frac{d}{dx} \\ \frac{d}{dx} & \frac{d}{dy} \end{pmatrix} \begin{pmatrix} \delta H_0/\delta v \\ \delta H_0/\delta w \end{pmatrix}, \tag{4} \]
\[ H_0 = \int h(v, w) \, dx dy, \] or, explicitly,
\[ v_t = (h_w)_x, \quad w_t = (h_v)_x + (h_w)_y. \]

We will be looking at deformations of the form (2) where the density of \( H_i \) is a homogeneous polynomial of degree \( i \) in the \( x \)- and \( y \)-derivatives of \( v \) and \( w \). The Hamiltonian operator will be assumed undeformed (although we are not aware of any results establishing the triviality of Poisson cohomology in higher dimensions). Since a system of the form (4) does not possess any nontrivial conservation laws of hydrodynamic type other than the Casimirs and the Hamiltonian, the definition of integrability needs to be modified. Thus, the ‘seed’ system (4) will be called integrable if it possesses infinitely many hydrodynamic reductions \([20, 14]\). This requirement imposes strong constraints on the Hamiltonian density \( h(v, w) \), providing an efficient classification criterion (see Sect. 2 for more details).

Following \([17, 18]\), a deformation of \( H_0 \) will be called integrable (to the order \( \epsilon^m \)) if it inherits all hydrodynamic reductions of the seed system (4) to the same order (the deformation procedure is outlined in Sect. 3). The main features of the 2+1 dimensional deformation scheme can be summarised as follows:

- The variety of integrable ‘seed’ Hamiltonians \( H_0 \) is finite dimensional.
- \( Generic \) integrable Hamiltonians \( H_0 \) possess no nontrivial deformations.

Nevertheless, there exist deformable (non-generic) Hamiltonians.

**Example 1.** Let \( H_0 = \int \frac{w^2}{2} + f(v) \, dx dy \). In this case the integrability conditions reduce to a single fourth order ODE, \( f''''(f'') = f'''^2(3f'' - 2\beta^2) \). Modulo canonical transformations, this Hamiltonian possesses a unique integrable dispersive deformation of the form
\[
H = \int \frac{w^2}{2} + f(v) - \frac{\epsilon^2}{3!} f''v_x^2 + O(\epsilon^4) \, dx dy.
\]
For \( f(v) = e^v \) it can be rewritten in the equivalent form
\[
H = \int \frac{w^2}{2} + e^v + \frac{\epsilon^2}{3!} e^v v_{xx} + O(\epsilon^4) \, dx dy.
\]
It is quite remarkable that this deformation can be extended to all orders in the deformation parameter \( \epsilon \), providing a Hamiltonian formulation of the 2D Toda system,
\[
H = \int \frac{w^2}{2} + \exp \left( v + \frac{\epsilon^2}{3!} v_{xx} + \frac{\epsilon^4}{9!} v_{xxxx} + \ldots \right) \, dx dy,
\]
see Sect. 6 for further details.

**Example 2.** Let \( H_0 = \int \alpha \frac{v^2}{2} + \beta vw + f(w) \, dx dy \). Here the integrability conditions reduce to a single fourth order ODE, \( f'''(\alpha f'' - \beta^2) = f'''(3\alpha f'' - 2\beta^2) \). Modulo canonical transformations, this Hamiltonian possesses a unique integrable dispersive deformation of the form
\[
H = \int \alpha \frac{v^2}{2} + \beta vw + f(w) + \epsilon^2 f'' \left( -\frac{\alpha^2}{f''} v_x^2 + \frac{\beta^2}{f''} v_y^2 + 2\alpha w_x^2 + 2\beta v_y w_y + f'' w_y^2 \right) + O(\epsilon^4) \, dx dy.
\]
Although for $\beta = 0$ the Hamiltonian $H_0$ gives rise to the dispersionless KP (dKP) equation, the deformation presented here is not equivalent to the full KP equation: see Sect. 7 for further discussion.

It will be demonstrated (Theorem 2 of Sect. 5) that, modulo certain equivalence transformations, these two examples exhaust the list of Hamiltonians of type II which possess nontrivial integrable deformations to the order $\epsilon^2$. In Sect. 4 we prove the triviality of $\epsilon$-deformations. The structure of $\epsilon^2$-deformations is analysed in Sect. 5. Deformations of Hamiltonians of type II/III are discussed in Sect. 8/9.

2 Classification of integrable Hamiltonian densities of type II

In this section we review the classification of integrable Hamiltonian systems of the form (4). Following [14, 16] we require the existence of $N$-phase solutions of the form

$$v = v(R^1, R^2, \ldots, R^N), \quad w = w(R^1, R^2, \ldots, R^N),$$

(5)

where the phases $R^i(x, y, t)$ satisfy the commuting equations

$$R^i_t = \lambda^i(R) R^i_x, \quad R^i_\mu = \mu^i(R) R^i_x;$$

(6)

recall that the assumption of commutativity imposes the following restrictions on the characteristic speeds $\lambda^i$ and $\mu^i$:

$$\frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} = \frac{\partial_j \mu^i}{\mu^j - \mu^i},$$

(7)

$\partial_j = \partial/\partial R^j, \ i \neq j$, see [24]. Equations (6) are said to define an $N$-component hydrodynamic reduction of the original system (4). It was observed in [14] that the requirement of the existence of such reductions imposes strong constraints on the original system (4), and provides an efficient classification criterion. Recall that the key property is the existence of three-component reductions: in this case one also has $N$-component reductions for arbitrary $N$. This property is reminiscent of the three-soliton condition in the theory of integrable systems. On the contrary, the existence of one- or two-component reductions is a common phenomenon which is not generally related to the integrability (at least for two-component systems as in the present paper).

As shown in [16], the requirement of existence of three-component reductions leads to a system of fourth order PDEs for the Hamiltonian density $h(v, w)$, which constitute the integrability conditions:

$$h_{ww}(h_{ww} h_{ww} - h_{ww}^2) h_{www} = -6h_{ww}^2 h_{ww}^2 + 3h_{ww} h_{ww} h_{ww}^2 + 4h_{ww}^2 h_{ww}^2,$$

$$h_{ww}(h_{ww} h_{ww} - h_{ww}^2) h_{www} = -3h_{ww}^2 h_{ww} h_{ww}^2 + 3h_{ww} h_{ww} h_{ww} h_{ww} h_{ww} + h_{ww}^2 h_{ww} h_{ww} h_{ww},$$

$$h_{ww}(h_{ww} h_{ww} - h_{ww}^2) h_{www} = -3h_{ww} h_{ww}^2 h_{www} + h_{ww} h_{ww} h_{ww} h_{ww} + h_{ww}^2 h_{ww} h_{ww} h_{ww},$$

$$h_{ww}(h_{ww} h_{ww} - h_{ww}^2) h_{www} = -3h_{ww}^2 h_{ww} h_{ww} h_{www} + 2h_{ww} h_{ww} h_{ww} h_{www} + h_{ww}^2 h_{ww} h_{ww} h_{www},$$

(8)

$$h_{ww}(h_{ww} h_{ww} - h_{ww}^2) h_{www} = -2h_{ww} h_{ww}^2 h_{www} + 3h_{ww} h_{ww} h_{ww} h_{ww} h_{www} + h_{ww}^2 h_{ww} h_{ww} h_{www},$$

$$h_{ww}(h_{ww} h_{ww} - h_{ww}^2) h_{www} = -2h_{ww} h_{ww} h_{ww} h_{www} h_{www} + 3h_{ww} h_{ww} h_{ww} h_{ww} h_{www} + h_{ww}^2 h_{ww} h_{ww} h_{www} h_{www},$$

$$h_{ww}(h_{ww} h_{ww} - h_{ww}^2) h_{www} = -2h_{ww} h_{ww} h_{ww} h_{www} + h_{ww} h_{ww} h_{ww} h_{ww} h_{www} + h_{ww}^2 h_{ww} h_{ww} h_{www} h_{www},$$

$$h_{ww}(h_{ww} h_{ww} - h_{ww}^2) h_{www} = -2h_{ww} h_{ww} h_{ww} h_{www} + 3h_{ww} h_{ww} h_{ww} h_{ww} h_{www} + 4h_{ww}^2 h_{ww} h_{ww} h_{www}.$$
This system is in involution, and is invariant under the 9-parameter group of Lie-point symmetries,

\[ v \rightarrow av + b, \]
\[ w \rightarrow pv + cw + d, \]
\[ h \rightarrow ah + \beta v + \gamma w + \delta. \]

These transformations form the equivalence group of the problem. They preserve the Hamiltonian structure, and will be used to simplify the classification results. Under the Legendre transformation,

\[ V = h_v, \quad W = h_w, \quad H = vh_v + wh_w - h, \quad H_V = v, \quad H_W = w, \]

the integrability conditions \( \mathcal{S} \) simplify to

\[
\begin{align*}
H_{VVVV} & = \frac{2H^2_{VVV}}{H_{VV}}, \\
H_{VVVW} & = \frac{2H_{VVV}H_{VVV}}{H_{VV}}, \\
H_{VWVW} & = \frac{2H^2_{VVW}}{H_{VV}}, \\
H_{VVWW} & = \frac{3H_{VVW}H_{VVW} - H_{WWW}H_{VVV}}{H_{VV}}, \\
H_{WWWW} & = \frac{6H^2_{WWW} - 4H_{WWWWW}H_{VVW}}{H_{VV}}.
\end{align*}
\]

These equations were explicitly solved in \([16]\), leading to the following classification result:

**Theorem 1** Modulo the natural equivalence group, the generic integrable potential \( H(V,W) \) of type II is given by the formula

\[ H = V \ln \frac{V}{\sigma(W)}, \]

where \( \sigma \) is the Weierstrass sigma-function: \( \sigma'/\sigma = \zeta, \quad \varphi^2 = 4\varphi^3 - g_3 \). Its degenerations correspond to

\[ H = V \ln \frac{V}{W}, \quad H = V \ln V, \quad H = \frac{V^2}{2W} + \alpha W^7, \]

as well as the following polynomial potentials:

\[ H = \frac{V^2}{2} + \frac{VW^2}{2} + \frac{W^4}{4}, \quad H = \frac{V^2}{2} + \frac{W^3}{6}. \]

Taking the inverse Legendre transform, one can obtain a complete list of integrable Hamiltonian densities \( h(v,w) \). Just to mention a few of them, one gets

\[ h(v,w) = \frac{w^2}{2} + e^v, \quad h(v,w) = \frac{v^2}{2} + w^{3/2}, \quad h(v,w) = \frac{1}{2}(w + v^2/2)^2, \quad h(v,w) = \frac{1}{2}(w + e^v)^2, \]

etc. However, we would prefer to avoid case-by-case considerations, and work with the full set of integrability conditions \( \mathcal{S} \).
3 Dispersive deformations in $2+1$ dimensions

Given a Hamiltonian system of the form (4), its deformation $H = H_0 + \epsilon H_1 + \cdots + \epsilon^m H_m + O(\epsilon^{m+1})$ will be called integrable (to the order $\epsilon^m$) if both equations (5) and (6) defining $N$-phase solutions can be deformed to the same order in $\epsilon$, in other words, the deformed dispersive system is required to ‘inherit’ all hydrodynamic reductions of its dispersionless limit [17, 18]. More precisely, we require the existence of expansions

$$v = v(R^1, R^2, \ldots, R^N) + \epsilon v_1 + \cdots + \epsilon^m v_m + O(\epsilon^{m+1}),$$

$$w = w(R^1, R^2, \ldots, R^N) + \epsilon w_1 + \cdots + \epsilon^m w_m + O(\epsilon^{m+1}),$$

where $v_i$ and $w_i$ are assumed to be homogeneous polynomials of degree $i$ in the $x$-derivatives of $R$’s (thus, both $R^i_{xx}$ and $R^i_x R^i_x$ have degree two, etc). Similarly, hydrodynamic reductions (6) are deformed as

$$R^i_t = \lambda^i(R) R^i_x + \epsilon a_1 + \cdots + \epsilon^m a_m + O(\epsilon^{m+1}),$$

$$R^i_y = \mu^i(R) R^i_x + \epsilon b_1 + \cdots + \epsilon^m b_m + O(\epsilon^{m+1}),$$

where $a_i$ and $b_i$ are assumed to be homogeneous polynomials of degree $i + 1$ in the $x$-derivatives of $R$’s. We require that the substitution of (9), (10) into the deformed system (4) satisfies the equations up to the order $O(\epsilon^{m+1})$. This requirement proves to be very restrictive indeed, and imposes strong constraints on the structure of the deformed Hamiltonian $H$.

**Remark.** Expansions (9), (10) are invariant under Miura-type transformations of the form

$$R^i \to R^i + \epsilon r_1 + \epsilon^2 r_2 + \ldots,$$

where $r_i$ denote terms which are polynomial of degree $i$ in the $x$-derivatives of $R$’s. These transformations can be used to simplify calculations. For instance, working with one-phase solutions one can assume that $v$ remains undeformed. Similarly, working with two-phase solutions one can assume that both $v$ and $w$ remain undeformed. For three-phase solutions this normalisation still leaves some extra Miura-freedom which can be used to simplify expressions for $a_i$ and $b_i$ (to the best of our knowledge there exist no general theory of normal forms under Miura-type transformations).

4 Triviality of $\epsilon$-deformations

In this section we prove that all $\epsilon$-deformations are trivial and can be eliminated by an appropriate canonical transformation. Thus, we consider deformations of the form

$$\begin{pmatrix} v \\ w \end{pmatrix}_t = \begin{pmatrix} 0 & d/dx \\ d/dx & d/dy \end{pmatrix} \begin{pmatrix} \delta H/\delta v \\ \delta H/\delta w \end{pmatrix}$$

where

$$H = \int h(v, w) + \epsilon (av_x + bw_y + pw_x + qw_y) + O(\epsilon^2) \ dx dy.$$

Here $a, b, p, q$ are functions of $v$ and $w$. We require that all $N$-phase solutions (5) can be extended to the order $\epsilon$,

$$v = v(R^1, R^2, \ldots, R^N) + \epsilon v_1 + O(\epsilon^2), \quad w = w(R^1, R^2, \ldots, R^N) + \epsilon w_1 + O(\epsilon^2),$$

(12)
where $v_1$ and $w_1$ are polynomials of order one in the $x$-derivatives of $R$’s. Similarly, hydrodynamic reductions \((12)\) are deformed as
\[
R^i_t = \lambda^i(R)R^i_x + \epsilon a_1 + O(\epsilon^2), \quad R^i_y = \mu^i(R)R^i_x + \epsilon b_1 + O(\epsilon^2),
\]
where $a_1$ and $b_1$ are polynomials of order two in the $x$-derivatives of $R$’s. We thus require that relations \((12), (13)\) satisfy the original system \((11)\) up to the order $O(\epsilon^2)$.

It was verified by a direct calculation that all one- and two-component reductions can be deformed in this way, for any $a, b, p, q$ and any density $h(v, w)$, not necessarily integrable. On the contrary, the requirement of the inheritance of three-component reductions (recall that the existence of three-component reductions forces $h(v, w)$ to satisfy the integrability conditions \((5)\), is nontrivial, and leads to the following single relation:
\[
\left( \frac{h_{vw}N_w - h_{ww}(M_w - N_v)}{h_{vv}h_{ww} - h_{vw}^2} \right)_w = \left( \frac{h_{vw}N_w - h_{ww}(M_w - N_v)}{h_{vv}h_{ww} - h_{vw}^2} \right)_v,
\]
\[(14)\]

here $M = (a_w - p_v)/h_{ww}$, $N = (b_w - q_v)/h_{ww}$. It remains to show that the relation \((14)\) is necessary and sufficient for the existence of a canonical transformation of the form
\[
H \to H + \epsilon\{K, H\} + O(\epsilon^2),
\]
with $K = \int k(v, w) \, dx dy$, which eliminates all $\epsilon$-terms. Since the density of the functional $H + \epsilon\{K, H\}$ is given by the formula
\[
h(v, w) + \epsilon(aw_x + bw_y + pw_x + qw_y) + \epsilon(k_v, k_w) \left( \begin{array}{c} 0 \\ d/dx \\ d/dy \end{array} \right) \left( \begin{array}{c} h_v \\ h_w \end{array} \right) + O(\epsilon^2) =
\]
\[
h(v, w) + \epsilon(Av_x + Bv_y + Pw_x + Qw_y) + O(\epsilon^2),
\]
where
\[
A = a + k_v h_{vw} + k_w h_{vw}, \quad B = b + k_w h_{vw}, \quad P = p + k_v h_{ww} + k_w h_{vw}, \quad Q = q + k_w h_{ww},
\]
the conditions that $\epsilon$-terms are trivial (form a total derivative), take the form $A_w = P_v$, $B_w = Q_v$. This leads to the following linear system for $k(v, w)$:
\[
k_{vw}h_{ww} - k_{ww}h_{vw} = a_w - p_v, \quad k_{vw}h_{ww} - k_{ww}h_{vw} = b_w - q_v.
\]
The compatibility conditions of these equations for $k$ can be obtained by introducing the auxiliary variable $p$ via the relation $k_{ww} = ph_{ww}$, and solving for the remaining second order derivatives of $k$,
\[
k_{vw} = M + ph_{vw}, \quad k_{vw} = N + ph_{vw}, \quad k_{ww} = ph_{ww}.
\]
Cross-differentiating and solving for $p_v$ and $p_w$ we obtain
\[
p_v = \frac{h_{vw}N_w - h_{ww}(M_w - N_v)}{h_{vv}h_{ww} - h_{vw}^2}, \quad p_w = \frac{h_{vw}N_w - h_{ww}(M_w - N_v)}{h_{vv}h_{ww} - h_{vw}^2}.
\]
Ultimately, the compatibility condition $p_{vw} = p_{ww}$ gives the required relation \((14)\), thus finishing the proof.
5 Reconstruction of $\epsilon^2$-deformations

In this section we analyse the structure of $\epsilon^2$-deformations. The result of the previous section allows us to set all $\epsilon$-terms equal to zero. Thus, we consider deformations of the form

$$
\begin{pmatrix}
  v \\
  w
\end{pmatrix}_t = 
\begin{pmatrix}
  0 & d/dx \\
  d/dx & d/dy
\end{pmatrix}
\begin{pmatrix}
  \delta H/\delta v \\
  \delta H/\delta w
\end{pmatrix}
$$

where

$$
H = \int h(v, w) + \epsilon^2 h_2(v, w, v_x, w_x, v_y, w_y) + O(\epsilon^3) \, dx dy.
$$

Here $h_2$ is assumed to be of second order in the $x$- and $y$-derivatives of $v$ and $w$,

$$
h_2 = f_1 v_x^2 + f_2 v_y^2 + f_3 w_x^2 + f_4 w_y^2 + f_5 v_x w_x + f_6 (v_x v_y + v_y w_x) + f_7 v_y w_y + f_8 v_x v_y + f_9 w_x w_y,
$$

where $f_1, \ldots, f_9$ are functions of $v$ and $w$. Note that all terms which are linear in the second order derivatives of $v$ and $w$ can be removed via integration by parts. Furthermore, any expression of the form $f(v, w)(v_x w_y - v_y w_x)$ can be omitted, since its variational derivative is identically zero. We require that all $N$-phase solutions \([4]\) can be extended to the order $\epsilon^2$,

$$
v = v(R^1, R^2, \ldots, R^N) + \epsilon^2 v_2 + O(\epsilon^3), \quad w = w(R^1, R^2, \ldots, R^N) + \epsilon^2 w_2 + O(\epsilon^3),
$$

where $v_2$ and $w_2$ are polynomials of order two in the $x$-derivatives of $R$’s. Similarly, hydrodynamic reductions \([6]\) are deformed as

$$
R^i_t = \lambda^i(R) R^i_x + \epsilon^2 a_2 + O(\epsilon^3), \quad R^i_y = \mu^i(R) R^i_x + \epsilon^2 b_2 + O(\epsilon^3),
$$

where $a_2$ and $b_2$ are polynomials of order three in the $x$-derivatives of $R$’s. We thus require that relations \([16], [17]\) satisfy the deformed system \([15]\) up to the order $O(\epsilon^3)$. The classification is performed modulo canonical transformations of the form

$$
H \to H + \epsilon \{ K, H \} + O(\epsilon^3),
$$

here $K = \epsilon \int (aw_x + bw_y + pw_x + qw_y) \, dx dy$. Note that the density of the functional $\epsilon \{ K, H \}$ is given by the following formula (set $m = a_w - p_v$, $n = b_w - q_v$):

$$
\epsilon^2 (\delta K/\delta v, \delta K/\delta w) \begin{pmatrix}
  0 & d/dx \\
  d/dx & d/dy
\end{pmatrix} \begin{pmatrix}
  h_v \\
  h_w
\end{pmatrix} + O(\epsilon^3) =
$$

$$
\epsilon^2 m(h_v v_x^2 + h_w v_x v_y + h_{vw} v_x w_y - h_{ww} w_x^2) + \epsilon^2 n(h_v v_x v_y + h_{vw} v_x^2 + h_{ww} v_x w_y - h_{ww} w_x w_y) + O(\epsilon^3).
$$

Our calculations demonstrate that generic integrable Hamiltonians $H_0$ do not possess nontrivial dispersive deformations. To be precise, these deformations are parametrised by two arbitrary functions, analogous to $m$ and $n$ above, which can be eliminated by a canonical transformation. There are cases, however, where dispersive deformations are parametrised by two arbitrary functions and a constant. It is exactly this extra constant which gives rise to a non-trivial deformation. We emphasize that canonical transformations can be used from the very beginning to bring the deformation to a ‘normal form’: since $h_{ww} \neq 0$ one can set, say, $f_6 = f_9 = 0$. This normalisation simplifies all subsequent calculations. Our results can be summarised as follows.
Theorem 2 A Hamiltonian \( H_0 = \int h(v,w) \, dx dy \) of type II possesses a nontrivial integrable deformation to the order \( \epsilon^2 \) if and only if, along with the integrability conditions (8), it satisfies the additional differential constraints

\[
\begin{align*}
&h_{vww}h_{vww} - h^2_{vww} = 0, \quad h_{vww}h_{wvw} - h_{vww}h_{vww} = 0, \quad h_{www}h_{vww} - h^2_{vww} = 0,
&\text{that is,}
&\text{rank } \begin{pmatrix} h_{vww} & h_{vww} & h_{vww} \\ h_{wvw} & h_{vww} & h_{wvw} \\ h_{wvw} & h_{wvw} & h_{www} \end{pmatrix} = 1. 
\end{align*}
\]  

(18)

Modulo equivalence transformations, this gives two types of deformable densities:

\[
h(v, w) = \frac{w^2}{2} + e^v, \quad h(v, w) = \frac{\alpha v^2}{2} + \beta vw + f(w),
\]

where \( f(w) \) satisfies the integrability condition \( f'''f''(\alpha f'' - \beta^2) = f'''^2(3\alpha f'' - 2\beta^2) \).

Proof:

In contrast to the case of \( \epsilon \)-corrections where all constraints were coming from deformations of three-component reductions, at the order \( \epsilon^2 \) the main constraints appear at the level of one-component reductions already. Furthermore, it was verified by a direct calculation that multi-component reductions impose no extra conditions. Since the third order derivative \( h_{www} \) appears as a factor in all deformation formulae, there are two cases to consider.

Case 1: \( h_{www} = 0 \). Then the integrability conditions imply \( h_{vww} = 0 \). The further analysis shows that one has to impose an extra condition, namely \( h_{vww} = 0 \), otherwise all deformations are trivial. Notice that conditions \( h_{www} = h_{vww} = h_{wvw} = 0 \) clearly imply (18). Modulo equivalence transformations, this is the case of the Hamiltonian density \( h(v, w) = \frac{w^2}{2} + e^v \). Its dispersive deformation is given in Example 1 of the Introduction.

Case 1: \( h_{www} \neq 0 \). In this case one gets a system of equations for the coefficients \( f_1, \ldots, f_9 \) which contains \( f_4 \) as a factor. If \( f_4 \) equals zero, all deformations are trivial. In the case \( f_4 \neq 0 \) one can express \( f_1, f_2, f_3, f_5, f_7, f_8 \) in terms of \( f_4, f_6, f_9 \). What is left will be a system of two compatible first order PDEs for \( f_4 \), and a system of additional differential constraints for \( h(v, w) \) which coincides with (18). Solving equations for \( f_4 \) we obtain a constant of integration which is responsible for non-trivial dispersive deformations. To find integrable Hamiltonian densities satisfying (18) we set \( h_{www} = q, \ h_{vww} = pq \). Then the remaining third order derivatives of \( h \) can be parametrised as

\[
h_{www} = q, \quad h_{vww} = pq, \quad h_{vww} = p^2 q, \quad h_{vww} = p^3 q.
\]

Calculating the compatibility conditions we obtain \( p_e = pp_w, \ q_v = (pq)_w \). With this ansatz the integrability conditions (6) imply \( p = \text{const} \) so that \( q = F(w + pv + c) \) where \( f \) is a function of one variable. Thus, \( h \) can be represented in the form \( h(v, w) = f(w + pv + c) + Q(v, w) \), where \( Q(v, w) \) is an arbitrary quadratic form. Modulo the equivalence group any such density can be written in the form \( h(v, w) = \alpha \frac{w^2}{2} + \beta vw + f(w) \), and the substitution into (6) gives a fourth order ODE for \( f \). The dispersive deformation of this Hamiltonian is presented in Example 2. We believe that both Hamiltonians from Theorem 2 can be deformed to all orders in \( \epsilon \).
6 Example 1: deformation of the Boyer-Finley equation

In this section we discuss the key example where dispersive deformations can be reconstructed explicitly at all orders of the deformation parameter $\epsilon$. Let us consider system (4) with the Hamiltonian density $h = \frac{w^2}{2} + e^v$,

$$v_t = w_x, \quad w_t = e^v v_x + w_y.$$ 

On the elimination of $w$, it reduces to the Boyer-Finley equation [3],

$$v_{tt} - v_{ty} = (e^v)_{xx},$$

(the left hand side can be put into the standard form $v_{ty}$ by a linear transformation of $t$ and $y$).

An integrable dispersive deformation of this example is closely related to the 2D Toda equation, see [1] for an equivalent construction based on the central extension procedure. Let us introduce the auxiliary Hamiltonian system

$$\begin{pmatrix} u \\ w \end{pmatrix}_t = \begin{pmatrix} 0 & \frac{1}{\epsilon} \sinh(\epsilon d/dx) \\ \frac{1}{\epsilon} \sinh(\epsilon d/dx) & d/dy \end{pmatrix} \begin{pmatrix} h_u \\ h_w \end{pmatrix},$$

where the Hamiltonian density $h$ is the same as above, $h(u, w) = \frac{w^2}{2} + e^u$ (the exact relation between $u$ and $v$ is specified below). Explicitly, this gives

$$u_t = \frac{1}{\epsilon} \sinh(\epsilon d/dx) w, \quad w_t = \frac{1}{\epsilon} \sinh(\epsilon d/dx) e^u + w_y,$$

which, on elimination of $w$, leads to the integrable 2D Toda equation,

$$u_{tt} - u_{ty} = \frac{1}{\epsilon^2} (\sinh(\epsilon d/dx))^2 e^u = \frac{1}{4\epsilon^2} \left( e^{u(x+2\epsilon)} + e^{u(x-2\epsilon)} - 2e^{u(x)} \right).$$

Introducing the change of variables $u \leftrightarrow v$ by the formula

$$u = \frac{1}{\epsilon} (d/dx)^{-1} \sinh(\epsilon d/dx)v = (d/dx)^{-1} \left( \frac{v(x + \epsilon) - v(x - \epsilon)}{2\epsilon} \right) = v + \frac{\epsilon^2}{3!} v_{xx} + \frac{\epsilon^4}{5!} v_{xxxx} + \ldots,$$

one can verify that the Hamiltonian operator in (19) transforms into the Hamiltonian operator in (4), while the Hamiltonian density $h(u, w) = \frac{w^2}{2} + e^u$ takes the form

$$h(v, w) = \frac{w^2}{2} + \exp \left( \frac{1}{\epsilon} (d/dx)^{-1} \sinh(\epsilon d/dx)v \right) = \frac{w^2}{2} + \exp \left( v + \frac{\epsilon^2}{3!} v_{xx} + \frac{\epsilon^4}{5!} v_{xxxx} + \ldots \right) = \frac{w^2}{2} + e^v \left( 1 + \frac{\epsilon^2}{3!} v_{xx} + \frac{\epsilon^4}{5!} (v_{xxxx} + \frac{5}{3} v_{xx}^2) + \ldots \right).$$

This provides the required integrable deformation for the Hamiltonian density $h = \frac{w^2}{2} + e^v$. The $1 + 1$ dimensional $y$-independent limit of this construction was discussed in [10].
Without any loss of generality one can set $f = \frac{v^2}{2} + f(w)$ where $f'' f''' = 3f''^2$. The corresponding deformation assumes the form

$$H = \int \frac{v^2}{2} + f(w) + \epsilon^2 f''' \left(-\frac{1}{f}v_x^2 + 2w_x^2 + f''w_y^2\right) + O(\epsilon^4) \; dx dy.$$

Without any loss of generality one can set $f(w) = \frac{2\sqrt{2}}{3}w^{3/2}$. In this case the dispersionless system takes the form

$$v_t = (\sqrt{2}w)_x, \quad w_t = v_x + (\sqrt{2}w)_y.$$ 

Introducing the new variable $u = \sqrt{2}w$ one obtains

$$v_t = u_x, \quad uu_t = v_x + u_y,$$

which, on elimination of $v$, leads to the dKP equation $(u_y - uu_t)_t + u_{xx} = 0$, with ‘non-standard’ notation for the independent variables. The corresponding KP equation, $(u_y - uu_t)_t + u_{xx} + \epsilon^2 u_{tttt} = 0$, gives rise to the following integrable deformation of the original system:

$$v_t = (\sqrt{2}w)_x, \quad w_t = v_x + (\sqrt{2}w)_y + \epsilon^2(\sqrt{2}w)_{ttt}.$$

This, however, is clearly outside the class of Hamiltonian deformations.

### 8 Deformable Hamiltonians of type I

In this section we summarise our results on deformations of Hamiltonian systems of the form

$$\left(\begin{array}{c} v \\ w \end{array}\right)_t = \left(\begin{array}{cc} d/dx & 0 \\ 0 & d/dy \end{array}\right) \left(\begin{array}{c} h_v \\ h_w \end{array}\right),$$

or, explicitly,

$$v_t = (h_v)_x, \quad w_t = (h_w)_y.$$

The integrability conditions constitute a system of fourth order PDEs for the Hamiltonian density $h(v, w)$ \[13\]:

$$h_{vw}(h_{ww}^2 - h_{vw}h_{ww}h_{vww})h_{vww} = 4h_{vw}h_{vww}(h_{vw}h_{vww} - h_{vww}h_{vw}^2)$$

$$+ 3h_{vw}h_{vw}^2 - 2h_{vw}h_{vw}h_{vww} - h_{vw}h_{vw}h_{vww}^2,$$

$$h_{vv}(h_{ww}^2 - h_{vw}h_{ww})h_{vww} = -h_{vww}h_{vww}(h_{ww}h_{vww} + h_{vw}h_{ww}h_{vww})$$

$$+ 3h_{vw}^2h_{ww}^2 - 2h_{vw}h_{vw}h_{vww}h_{vww} + h_{vw}h_{vw}h_{vww}^2,$$

$$h_{ww}(h_{vw}^2 - h_{vw}h_{ww})h_{vww} = 4h_{vw}^2h_{vw}h_{vww}$$

$$- h_{vww}h_{vww}(h_{ww}h_{vww} + h_{vw}h_{ww}h_{vww}) - h_{ww}h_{vww}(h_{vw}h_{vww} + h_{ww}h_{vw}^2),$$

$$h_{vw}(h_{ww}^2 - h_{vw}h_{ww})h_{vww} = -h_{vww}h_{vww}(h_{vw}h_{vww} + h_{ww}h_{vww})$$

$$+ 3h_{vw}^2h_{ww}^2 - 2h_{vw}h_{vw}h_{vww}h_{vww} + h_{vw}h_{vw}h_{vww}^2,$$

$$h_{vw}(h_{ww}^2 - h_{vw}h_{ww})h_{vww} = 4h_{vw}h_{vww}(h_{vw}h_{vww} - h_{ww}h_{vw}^2)$$

$$+ 3h_{vw}h_{vw}^2 - 2h_{vw}h_{vw}h_{vww}h_{vww} - h_{vw}h_{vw}h_{vww}^2.$$
This system is in involution, and is invariant under the 8-parameter group of Lie-point symmetries,

\[
\begin{align*}
v &\rightarrow av + b, \\
w &\rightarrow cw + d, \\
h &\rightarrow ah + \beta v + \gamma w + \delta,
\end{align*}
\]

which constitute the equivalence group of the problem. Furthermore, there is an obvious symmetry corresponding to the interchange of \(v\) and \(w\). Particular solutions include

\[
\begin{align*}
h(v, w) &= vw + \alpha v^3, & h(v, w) &= w\sqrt{v} + \alpha v^{5/2}, & h(v, w) &= \frac{1}{2}(v + w)^2 + e^v, & h(v, w) &= (w + a(v))^2,
\end{align*}
\]

where \(a(v)\) solves the ODE \(a' a''' a'''' = 2a''^2 a''' + a' a'''^2\), etc., see [15] for the general discussion. As before, generic Hamiltonians are not deformable. Although Case I turns out to be considerably more complicated from computational point of view, our calculations support the conjecture that deformable densities of type I are characterised by exactly the same additional constraints as in case II:

**Conjecture** A Hamiltonian \(H_0 = \int h(v, w) \, dxdy\) of type I possesses a nontrivial integrable deformation to the order \(\epsilon^2\) if and only if, along with the integrability conditions (20), it satisfies the additional differential constraints

\[
\begin{align*}
h_{vvv}h_{wvw} - h_{vvw}^2 &= 0, & h_{vww}h_{wwv} - h_{vww}h_{vwv} &= 0, & h_{wwv}h_{vww} - h_{wvw}^2 &= 0,
\end{align*}
\]

or, equivalently,

\[
\text{rank} \begin{pmatrix} h_{vww} & h_{vww} & h_{vww} \\ h_{wvw} & h_{wwv} & h_{www} \end{pmatrix} = 1.
\]

Modulo equivalence transformations, this gives three types of deformable densities:

\[
\begin{align*}
h(v, w) &= vw + \alpha v^3, & h(v, w) &= \frac{1}{2}(v + w)^2 + e^v, & h(v, w) &= \alpha v^2 + \beta vw + \frac{\gamma}{2} w^2 + f(v + w),
\end{align*}
\]

where \(f\) satisfies the integrability condition

\[
(\beta + f'')\Delta f''' = f'''''[3\Delta + (\beta - \alpha)(\beta - \gamma)],
\]

here \(\Delta = (\beta + f'')^2 - (\alpha + f'')(\gamma + f'')\). Dispersive deformations of these Hamiltonians are given by the following formulae (we use the normalisation \(f_8 = f_9 = 0\) which can always be achieved by a canonical transformation; furthermore, all \(\epsilon\)-deformations are trivial, and have been set equal to zero):

\[
\begin{align*}
H &= \int vw + \alpha v^3 + \epsilon^2(6\alpha v_x^2 + v_x w_x) + O(\epsilon^4) \, dxdy, \\
H &= \int \frac{1}{2}(v + w)^2 + e^v + \epsilon^2 e^v (2(1 + e^v)v_x^2 - 2w_x^2 - 2e^v v_x w_x + v_x w_y + v_y w_x) + O(\epsilon^4) \, dxdy.
\end{align*}
\]

The third case is somewhat more complicated:

\[
H = \int \frac{\alpha}{2} v^2 + \beta vw + \frac{\gamma}{2} w^2 + f(v + w) + \epsilon^2 h_2 + O(\epsilon^4) \, dxdy,
\]
where \( h_2 = f_1 v_x^2 + f_2 v_y^2 + f_3 w_x^2 + f_4 w_y^2 + f_5 v_x w_x + f_6 (v_x v_y + v_y w_x) + f_7 v_y w_y \), and the coefficients \( f_1 - f_7 \) are defined as follows:

\[
\begin{align*}
  f_1 &= (\alpha + f''')(\beta + f'')^2 \Delta, \\
  f_2 &= (4\beta - \alpha + 3f'')(\beta + f'')^2 \Delta, \\
  f_3 &= (4\beta - \gamma + 3f'')(\beta + f'')^2 \Delta, \\
  f_5 &= (\beta + f'') \Delta[\Delta + 4(\alpha + f'')(\beta + f'')], \\
  f_7 &= (\beta + f'') \Delta[\Delta + 4(\gamma + f'')(\beta + f'')], \\
  f_6 &= \frac{1}{2}(\beta + f'') \Delta[2\Delta + (2\beta - \alpha - \gamma)^2].
\end{align*}
\]

We conjecture that Hamiltonians from Theorem 3 can be deformed to all orders in \( \epsilon \).

9 Deformable Hamiltonians of type III

In this section we consider Hamiltonian systems of the form

\[
\left( \begin{array}{c} v \\ w \end{array} \right)_t = \left[ \begin{array}{cc} 2v & w \\ w & 0 \end{array} \right] \frac{d}{dx} + \left[ \begin{array}{cc} 0 & v \\ v & 2w \end{array} \right] \frac{d}{dy} + \left( \begin{array}{cc} v_x & v_y \\ w_x & w_y \end{array} \right) \left( \begin{array}{c} h_v \\ h_w \end{array} \right),
\]

or, explicitly,

\[
\begin{align*}
  v_t &= (2vh_x + wh_y - h)_x + (vh_w)_y, \\
  w_t &= (wh_x)_x + (2wh_w + vh_y - h)_y.
\end{align*}
\]

The integrability conditions constitute a system of fourth order PDEs for the Hamiltonian density \( h(v, w) \) which is not presented here due to its complexity. We have verified in [16] that this system is in involution, and its solution space is 10-dimensional. It is invariant under an 8-dimensional group of Lie-point symmetries,

\[
\begin{align*}
  v &\rightarrow av + bw, \\
  w &\rightarrow cv + dw, \\
  h &\rightarrow ah + \beta v + \gamma w + \delta,
\end{align*}
\]

which constitute the equivalence group of the problem. Particular integrable Hamiltonian densities include

\[
h(v, w) = \frac{w}{v} + \alpha v^2, \quad h(v, w) = \frac{w}{v} + (v + c) \ln(v + c), \quad h(v, w) = (vw)^{1/3}.
\]

A more complicated example has the form \( h(v, w) = wf(v/w^2) \) where the function \( f(y) \) solves the ODE

\[
y^2 f' f'' (f' + 2y f'') f''' = 2y^2 f'(f' + 3y f'') f'''
\]

\[
+ 2y f'' (2f'^2 + 9y f' f'' + 4y^2 f''^2) f''' + 3f''^2 (2f'^2 + 8y f' f'' + 5y^2 f''^2).
\]

However, calculations suggest that none of these examples are deformable:

**Conjecture** For Hamiltonians of type III, all deformations of the order \( \epsilon^2 \) are trivial.
10 Concluding remarks

In this paper we discuss, in the spirit of [8] – [13], the deformation theory of 2 + 1 dimensional Hamiltonian systems of hydrodynamic type, defined by local Poisson brackets and local Hamiltonians. Our results demonstrate that, already at the order $\epsilon^2$, the requirement of the existence of nontrivial dispersive deformations is very restrictive so that ‘generic’ integrable Hamiltonians are not deformable. The main reason for this is apparently the assumption that all higher order dispersive corrections are local expressions in the dependent variables $v, w$ and $x, y$-derivatives thereof. It would be of interest to extend this scheme to the case of nonlocal brackets/Hamiltonians, see [2] for particular examples obtained via Dirac reduction.

Furthermore, to the best of our knowledge the theory of deformations of multi-dimensional Poisson brackets of hydrodynamic type has not been constructed: is it true that all such deformations are trivial, as in the 1 + 1 dimensional case?

Finally, calculations leading to Example 1 of Sect. 2 show that any Hamiltonian of the form $H_0 = \int \frac{w^2}{2} + f(v) \, dx dy$, where the function $f$ is arbitrary, possesses a unique dispersive deformation of the form

$$H = \int \frac{w^2}{2} + f(v) + \frac{\epsilon^2}{3!} f''' v^2 x + \mathcal{O}(\epsilon^4) \, dx dy,$$

which inherits all one-phase solutions to the order $\epsilon^2$. Thus, one can speak of ‘partial integrability’ of a certain kind. However, already the requirement of the inheritance of two-phase solutions forces $f$ to satisfy the integrability condition $f''' f'' = f'''^2$.

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