HIGHER FRIEZE PATTERNS

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Abstract. Frieze patterns have an interesting combinatorial structure, which has proven very useful in the study of cluster algebras. We introduce \((k, n)\)-frieze patterns, a natural generalisation of the classical notion. A generalisation of the bijective correspondence between frieze patterns of width \(n\) and clusters of Plücker coordinates in the cluster structure of the Grassmannian \(\text{Gr}(2, n + 3)\) is obtained.

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1. Introduction

Frieze patterns were introduced by Coxeter in 1971 [7], and a bijection between frieze patterns of width \(n\) and triangulations of \((n + 3)\)-gons was established by Conway and Coxeter shortly thereafter [4][5]. This impressive result received wider interest with the advent of cluster algebras, introduced by Fomin and Zelevinsky in [10][11]. Frieze patterns satisfy many nice combinatorial properties, such as being invariant under a glide reflection, and are therefore periodic. There already exist many generalisations of frieze patterns, see [17] for a comprehensive introduction to the subject.

A first motivating example of a cluster algebra was given by the homogeneous coordinate ring \(\mathbb{C}[\text{Gr}(2, n + 3)]\) of the Grassmannian of 2-dimensional subspaces in \(\mathbb{C}^{n+3}\), the ring being generated by Plücker coordinates, and subject to Plücker relations. Each cluster of Plücker coordinates in the cluster structure of \(\mathbb{C}[\text{Gr}(2, n + 3)]\) corresponds to a cluster-tilted algebra of Dynkin type \(A_n\) [11 Prop 12.7]. In particular, each cluster in this cluster algebra corresponds to a triangulation of an \((n + 3)\)-gon, just as each frieze pattern of width \(n\) does. This connection was formalised in [3], where the Caldero-Chapoton formula was introduced for clusters and found to determine the entries in the corresponding frieze pattern.
However, while cluster algebras for Grassmannians $\text{Gr}(k, n)$ are well understood in the case that $k = 2$, they are by no means limited to this case. In [22], Scott proved that the homogeneous coordinate ring $\mathbb{C}[\text{Gr}(k, n)]$ is a cluster algebra for all values of $2 \leq k \leq n/2$. We denote the set of Plücker coordinates in the cluster structure of $\mathbb{C}[\text{Gr}(k, n)]$ by $\mathcal{A}_{k,n}$ and throughout this paper we will consider $k$ and $n$ to be integers with $2 \leq k \leq n/2$.

The motivation for this paper is to fully describe the clusters in $\mathcal{A}_{k,n}$ with $k \geq 3$ in terms of frieze patterns. To this purpose, we introduce $(k, n)$-frieze patterns or higher frieze patterns. However it turns out that there are significantly more $(k, n)$-frieze patterns than clusters in $\mathcal{A}_{k,n}$. Therefore, we primarily consider geometric $(3, n)$-frieze patterns, which we will show are precisely those higher frieze patterns that correspond to a cluster in $\mathcal{A}_{3,n}$, and moreover satisfy a generalised version of the unimodular rule that defines the classical frieze patterns.

In the classical case, a frieze of width $n$ corresponds to a cluster-tilting object of type $\mathbb{A}_n$; this leads to a more general definition of frieze patterns as functions on a repetition quiver, see [1] or [17] for details. A similar approach may be considered for higher frieze patterns using the higher Auslander-Reiten theory introduced by Iyama in [12], [13]. In particular, a higher frieze pattern may be seen as a function on the cylinder of a higher Auslander algebra of type $A$, invariant under a higher-dimensional glide reflection. To find where higher frieze patterns lie in the pantheon of generalisations of frieze patterns, we show that each $(k, n)$-frieze pattern determines an $\text{SL}_k$-frieze pattern of width $n - k - 1$. This complements a result of [18, Proposition 3.2.1], that each $\text{SL}_3$-frieze pattern of width $n - k - 1$ is found to determine a point in the Grassmannian $\text{Gr}(k, n)$.

We remark here that Oppermann and Thomas [20] found a generalisation of the bijection between triangulations of $(n + 3)$-gons and cluster-tilting objects of type $\mathbb{A}_n$ for higher cluster-tilting theory using triangulations of cyclic polytopes. In a sequel paper [16], we will instead associate a particular class of clusters in $\mathcal{A}_{k,n}$ with superimposed triangulations. This illustrates contrasting combinatorics for these two generalisations of the combinatorial model for cluster algebras of type $A$.

2. Coxeter’s Frieze Patterns

In the sense of Coxeter, a frieze pattern [7] is an array of numbers satisfying the following conditions:

- The array has finitely many rows (though infinitely many columns are needed)
- The two uppermost and bottommost rows are fixed such the first and final rows consist of only 0’s; the second and penultimate rows are rows consisting of only 1’s.
- Consecutive rows are displayed with a shift, and every diamond

\[
\begin{align*}
& a \\
& \downarrow \quad \downarrow \\
& b & c \\
& \downarrow \quad \downarrow \\
& d
\end{align*}
\]

satisfies the unimodular rule: $bc - ad = 1$.

Note that we omit, by convention, the top and bottom rows of zeroes from the frieze pattern. Such an array satisfying instead that every $k \times k$-minor has determinant one, in place of the unimodular rule, is called an $\text{SL}_k$-frieze pattern [6] (see also [17]). An $\text{SL}_2$-frieze pattern is another name for one of Coxeter’s frieze patterns.

Example 2.1. Examples of frieze patterns (in the sense of Coxeter) include:

\[
\begin{align*}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{align*}
\]
A frieze pattern is said to be of width $n$ if it has $n$ rows strictly between the border rows of ones at the top and bottom.

3. Background

3.1. Plücker Relations. Recall that the Grassmannian of all $k$-dimensional subspaces of $\mathbb{C}^n$, $\text{Gr}(k,n)$, can be embedded into the projective space $\mathbb{P}(\wedge^k(\mathbb{C}^n))$ via the Plücker embedding. The coordinates of $\wedge^k(\mathbb{C}^n)$ are called the Plücker coordinates and are indexed by the $k$-multisets $I = \{i_1, i_2, \cdots, i_k\}$ with elements from $\{1, \cdots, n\}$. The coordinate defined by $\{i_1, i_2, \cdots, i_k\}$ will be denoted $p_{i_1 i_2 \cdots i_k}$.

In general, the ordering of elements in a Plücker coordinate may not be known; the definition of the Plücker coordinates may be extended to allow for this. By convention, the Plücker coordinates possess an antisymmetry:

$$p_{i_1 \cdots i_r i_s \cdots i_k} = -p_{i_1 \cdots i_s i_r \cdots i_k}.$$ (1)

In particular, if $i_r = i_s$ then $p_{i_1 \cdots i_r i_s \cdots i_k} = 0$. The Plücker embedding satisfies the determinantal Plücker relations. In the case $k = 2$, the Plücker relations are

$$p_{ac}p_{bd} = p_{ab}p_{cd} + p_{ad}p_{bc},$$

where $1 \leq a < b < c < d \leq n$. Now, let $p_{i_1 i_2 \cdots i_r}$ be the Plücker coordinate for the subset $I \cup \{i_1\} \cup \{i_2\} \cup \cdots \cup \{i_r\}$. Then, more generally, the Plücker relations for $k > 2$ are

$$p_{Iac}p_{Ibd} = p_{Iab}p_{Icd} + p_{Iad}p_{Ibc},$$

where $I$ is a $(k-2)$-subset of $\{1, \cdots, n\}$ with $\{a, b, c, d\} \cap I = \emptyset$. Another set of generating relations for the $(k,n)$-Plücker relations is the set of relations

$$\sum_{r=0}^{k} (-1)^r p_{i_1 i_2 \cdots i_{r-1} j_r} p_{j_0 j_1 \cdots j_{k-r}} = 0,$$ (2)

where $1 \leq i_0 < i_1 < \cdots < i_{k-1} \leq n$ and $1 \leq j_0 < j_1 < \cdots j_k \leq n$. See for example [15] for a reference on Plücker relations.
3.2. Higher-Dimensional Auslander-Reiten Theory. In the context of generalising classical Auslander-Reiten theory to higher dimensions, Iyama introduced in [13] the notion of a higher Auslander algebra. For a quiver $Q$ of Dynkin type, there is an explicit description of the quiver of the $n$-Auslander algebra of the path algebra $\mathbb{C}Q$. Let $Q$ be the following quiver:

$$1 \rightarrow 2 \rightarrow \cdots \rightarrow d$$

Set $e_i$ to be the $\mathbb{Z}^n$ vector with 1 in the $i$-th coordinate, and 0 in every other coordinate. Then let

$$v_i = \begin{cases} -e_i & i = 1, \\ e_{i-1} - e_i & 2 \leq i \leq n. \end{cases}$$

The quiver of the $m$-Auslander algebra $A_d^{(m)}$ can be described as follows.

- The vertices are
  $$(Q_d^{(m)})_0 = \{(l_1, l_2, \ldots, l_{m+1}) \in \mathbb{Z}^{m+1} | l_1 \geq 1; l_2, \ldots, l_{m+1} \geq 0; l_1 + l_2 + \cdots + l_{m+1} \leq d\}.$$

- For each vertex $\underline{l} \in (Q_d^{(m)})_0$, there is an arrow $\underline{l} \rightarrow \underline{l} + v_i$ wherever $\underline{l} + v_i$ is in $(Q_d^{(m)})_0$.

For example, the quiver of the 1-Auslander algebra $A_4^{(1)}$ is:

```
13 \rightarrow 22 \rightarrow 31 \rightarrow 40 \rightarrow 30 \rightarrow 20 \rightarrow 10
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The quiver of the 2-Auslander algebra $A_4^{(2)}$ is:

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130 \rightarrow 220 \rightarrow 310 \rightarrow 400 \rightarrow 300 \rightarrow 200 \rightarrow 100
```

4. Higher Frieze Patterns

In the two-dimensional case, a glide reflection is a reflection about a line, composed with a translation along that line. An important property of frieze patterns is that they are invariant under a glide reflection and hence periodic. Pictorially, this means points on a frieze pattern of width $n$ can be parameterised by 2-subsets of $\{1, 2, \ldots, n + 3\}$, where each subset $\{i, j\}$ has at
most four neighbours: \( \{i-1,j\}, \{i,j-1\}, \{i+1,j\} \) and \( \{i,j+1\} \), where addition is modulo \( n + 3 \). The case where \( n = 3 \) is shown below:

\[
\begin{array}{cccccccc}
12 & 23 & 34 & 45 & 56 & 16 & 12 & 23 \\
14 & 25 & 36 & 14 & 25 & 36 & 14 & 25 \\
16 & 26 & 13 & 24 & 35 & 46 & 15 & 26 \\
\end{array}
\]

To define a frieze pattern, choose values \( \{p_I\} \) for each 2-subset \( I \) of \( \{1,2,\ldots,n+3\} \) with the conditions that for any for any \( i \leq n \), \( p_i = 0 \) and \( p_{i+1} = 1 \) (addition modulo \( n + 3 \)). In this perspective, the unimodular rule is equivalent to the Plücker relations: any diamond has values given by

\[
P_{x(y+1)} = P_{xy} P_{(x+1)(y+1)}
\]

So the Plücker relations

\[
P_{xy} P_{(x+1)(y+1)} = P_{x(y+1)} P_{(x+1)y} + P_{x(y+1)} P_{y(y+1)}
\]

imply the unimodular rule, using \( P_{x(y+1)} P_{y(y+1)} = 1 \) as above. It then makes sense to generalise frieze patterns in the following sense.

**Definition 4.1.** Let \( k \) and \( n \) be positive integers such that \( 2 \leq k \leq n/2 \). A \((k,n)\)-frieze pattern, \( \mathcal{P} \), is a map from the \( k \)- multisets of \( \{1,2,\ldots,n\} \) to the non-negative integers (sending \( I \) to \( p_I \)) such that:

- Each \( k \)-multiset \( I \) with elements from \( \{1,2,\ldots,n\} \), is associated a non-negative integer value \( p_I \).
- Each interval subset, that is a subset \( I = \{i,i+1,\ldots,i+k-1\} \) for a given \( 1 \leq i \leq n \) and under addition modulo \( n \), satisfies \( p_I = 1 \).
- The set of values \( \{p_I\} \) satisfy the \((k,n)\)-Plücker relations.

A \((2,n)\)-frieze pattern is just a Coxeter frieze pattern of width \( n - 3 \). The first and final rows consist of only zeroes - these (omitted) rows consist of the coordinates corresponding to a 2-multiset \( \{i,i\} \), for some \( 1 \leq i \leq n \), and Equation (1) implies that each \( p_{ii} = 0 \).

**Definition 4.2.** For \( k \geq 3 \), we call a \((k,n)\)-frieze pattern a higher frieze pattern. Let \( S \) be the set of \( k \)-subsets of \( \{1,2,\ldots,n\} \). The underlying graph of a \((k,n)\)-frieze pattern for \( k \geq 3 \) is the graph with vertices indexed by the elements of \( S \times \mathbb{Z} \) and consisting of edges between \( (I,m) \) and \( (J,m) \) wherever \( I \setminus \{i\} = J \setminus \{i+1\} \) for some \( 1 \leq i < n \) as well as edges between \( (I,m) \) and \( (J,m+1) \) wherever \( I \setminus \{n\} = J \setminus \{1\} \). The coordinate label of a vertex \((I,m) \in S \times \mathbb{Z} \), where \( I = \{i_1,i_2,\ldots,i_k\} \), in the underlying graph of a higher frieze pattern is \( i_1 i_2 \cdots i_k \). A higher frieze pattern is displayed on its underlying graph by setting each vertex \((I,m) \in S \times \mathbb{Z} \) to have value \( p_I \).

The fundamental domain of a \((k)\)-frieze pattern is given by the collection of values \( \{p_I\} \) indexed by the \( k \)-subsets \( I \subset \{1,2,\ldots,n\} \). The underlying graph of the fundamental domain of a \((k)\)-frieze pattern is the graph with vertices indexed by the elements of \( S \) and consisting of edges between \( I \) and \( J \) wherever \( I \setminus \{i\} = J \setminus \{i+1\} \) for some \( 1 \leq i < n \).

One complicating factor in the study of higher frieze patterns is that they should be visualised...
in a higher dimension. In contrast, any Plücker exchange relation should still be visualised on a two dimensional plane. We introduce a convention for \((3, n)\)-frieze patterns to make sense of this disparity.

**Definition 4.3.** For a given \((3, n)\)-frieze pattern and element \(x \in \{1, 2, \cdots, n\}\), the restriction of the underlying graph of the frieze pattern to the coordinates \(\{(I, 0) \in S \times \mathbb{Z} | x \in I\}\) is the **cross-sectional triangle at** \(x\). An example of a cross-sectional triangle is given in Figure\[\text{FIGURE 1.}\] There is an order on the elements \(\{1, 2, \cdots, n\} \setminus \{x\}\) given by \(x + 1 < x + 2 < \cdots < x - 1\); denote this order by \(i \prec x j\).

Now we can generalise diamond relations to higher dimensions.
Figure 2. The generalised diamond relation, and two examples coming from a cross-sectional triangle from the $(3,9)$-frieze pattern in Figure 4. The generalised diamond relation depicted above shows $3 \times 21 - 5 \times 9 = 3 \times 6$ and the generalised diamond relation depicted below shows $6 \times 2 - 3 \times 1 = 1 \times 9$.

**Definition 4.4.** Let $\{x, i, j\}$ be a coordinate in the cross-sectional triangle at $x$ with $i \leq x \leq j$, and let $l$ and $m$ be integers such that $i + m < x < j - l$. An $l \times m$-diamond (or generalised diamond) is formed by the points 

$$A = p_{x(i+1)}, B = p_{x(i-l)}, C = p_{x(i+m)j} \text{ and } D = p_{x(i+m)(j-1)}$$

as indicated in Figure 2.

Setting $E = p_{x(i+m)}$ and $F = p_{x(j-l)j}$, then the generalised diamond relation holds:

$$BC - AD = EF.$$

The generalised diamond relation is illustrated in Figure 2.

**Proposition 4.1.** The underlying graph of the fundamental domain of a $(k,n)$-frieze pattern has the same underlying graph as $A_{n-k+1}^{k-1}$.

**Proof.** The fundamental domain of $(k,n)$-frieze patterns consists of the $k$-subsets of $\{1,2,\cdots,n\}$. The proof follows from the observation that the set of vertex labels of a $A_{n-k+1}^{k-1}$ is in bijection with the $k$-subsets of $\{1,2,\cdots,n\}$ under the map

$$\phi : (l_1, l_2, \cdots, l_k) \mapsto \{l_k + 1, l_k + l_{k-1} + 2, \cdots, l_k + l_{k-1} + \cdots + l_1 + k - 1, l_k + l_{k-1} + \cdots + l_1 + k - 1 \}.$$ 

The condition $l_1 + l_2 + \cdots + l_k \leq n - k + 1$ means

$$l_k + l_{k-1} + \cdots + l_1 + k - 1 \leq n.$$

So the map sends each vertex of $A_{n-k+1}^{k-1}$ to a $k$-subset of $\{1,2,\cdots,n\}$ (each element of the subset must be distinct as each $l_1 \geq 1$). Conversely, every $k$-subset $I \subset \{1,2,\cdots,m\}$, where
\[ I = \{x_1, x_2, \ldots, x_k\}, \] defines a unique vertex of \( A_{n-k+1}^{k-1} \) by setting \( l_k = x_1 - 1 \), \( l_1 = x_k - x_{k-1} \) and \( l_{k-i} = x_{i+1} - x_i - 1 \) for \( 2 \leq i \leq k-1 \).

We further claim that the edges in the underlying graph of the fundamental domain of a \((k, n)\)-frieze pattern coincide with the arrows of \( A_{n-k+1}^{k-1} \). Recall that there is an arrow from \( \bar{l} \) to \( \bar{m} \) in \( A_{n-k+1}^{k-1} \) if \( \bar{l} = \bar{m} + v_i \) for \( 1 \leq i \leq k \). It is straightforward to check that \( \phi(\bar{m}) \setminus \{i\} = \phi(\bar{l}) \setminus \{i+1\} \) if and only if \( \bar{l} = \bar{m} + v_{k-i} \).

Proposition \ref{prop:higher-frieze} can be rephrased to say that the underlying graph of the fundamental domain of a \((k, n)\)-frieze pattern has the underlying graph of a higher Auslander algebra when \( k > 2 \). It is for this reason that we call a \((k, n)\)-frieze pattern for \( k > 2 \) a higher frieze pattern.

Remark. By definition, the generalised diamond relations in a \((3, n)\)-frieze pattern are in bijection with a set of generating relations for the \((3, n)\)-Plücker relations. It follows that we may combinatorially define a \((3, n)\)-frieze pattern as a map from the cylinder of the Auslander algebra of an \( A_{n-2} \) quiver to the positive integers that satisfies the generalised diamond relations.

5. Geometric Frieze Patterns

5.1. Connection to \( SL_k \)-Frieze Patterns. It has been observed in \cite[Section 3.2]{18}, see also \cite[Section 3.4]{17}, that any \( SL_k \)-frieze of width \( n-k-1 \) determines a point on the Grassmannian \( Gr(k, n) \). This observation reveals the connection between \( SL_k \)-frieze patterns and \((k, n)\)-frieze patterns.
Figure 4. An example of a \((3, 9)\)-frieze pattern that is not geometric. We have omitted the edges between different cross-sectional triangles.
Firstly, any \((k, n)\)-frieze pattern determines an \(\text{SL}_k\)-frieze pattern of width \(n - k - 1\). Given a \((k, n)\)-frieze pattern \(\mathcal{P}\), we obtain an \(\text{SL}_k\)-frieze pattern by taking as the \(i^{th}\)-column in the array to be
\[\{P_i(i+1)\cdots(i+k-2)(i+k-1), P_i(i+1)\cdots(i+k-2)(i+k)\cdots, P_i(i+1)\cdots(i+k-2)(i-1)\}\]
This has width \(n - k - 1\). Any \(k \times k\)-matrix in the array is of the following form:
\[M_{ij} := \begin{bmatrix}
P(i+1)\cdots(i+k-2)j & P(i+1)\cdots(i+k-2)(j+1) & \cdots & P(i+1)\cdots(i+k-2)(j+k-1) \\
P(i+1)(i+2)\cdots(i+k-1)(j+1) & P(i+1)(i+2)\cdots(i+k-1)(j+k-1) & \cdots & P(i+1)(i+2)\cdots(i+k-1)(j+k-1) \\
\cdots & \cdots & \cdots & \cdots \\
P(i+k-1)(i+k)\cdots(i+2k-3)j & P(i+k-1)(i+k)\cdots(i+2k-3)(j+1) & \cdots & P(i+k-1)(i+k)\cdots(i+2k-3)(j+k-1)
\end{bmatrix}\]
It has been shown by \(\cite{2}\) that \(\det(M_{ij}) = 1\). Conversely, a \((k, n)\)-frieze pattern is obtained from its induced \(\text{SL}_k\)-frieze pattern in the following fashion. Let
\[
\begin{array}{cccccc}
1 & 1 & 1 & 1 & \cdots & \cdots \\
d_{i+1,j+1} & d_{i+2,j+2} & \cdots & d_{i+k,j+k} & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
d_{i+n-k-1,j+1} & d_{i+n-k,j+2} & \cdots & d_{i+n-2,j+k}
\end{array}
\]
be part of an \(\text{SL}_4\)-frieze of width \(n - k - 1\) that was determined by a \((k, n)\)-frieze pattern. Then the \(k \times n\)-matrix
\[
\begin{bmatrix}
1 & d_{i+1,j+1} & d_{i+2,j+1} & \cdots & d_{i+n-k-1,j+1} & 1 & 0 & \cdots & 0 & 0 \\
0 & 1 & d_{i+2,j+2} & \cdots & d_{i+n-k-1,j+2} & d_{i+n-k-1} & 1 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & 1 & d_{i+k,j+k} & \cdots & \cdots & \cdots & d_{i+n-2,j+k} & 1
\end{bmatrix}
\]
is a (non-unique) point on the Grassmannian \(\text{Gr}(k, n)\). Label the columns sequentially with the leftmost column labelled by 1 and the rightmost column labelled \(n\). Then the \((k, n)\)-frieze pattern is obtained by setting \(p_I\) to be the \(k \times k\)-minor based on columns with indices in \(I\). For more information, see the surveys \(\cite{13, 17}\).

5.2. Positive Grassmannian. Two \(k\)-subsets \(I\) and \(J\) of \(\{1, 2, \cdots, n\}\) are said to be non-crossing (sometimes referred to as weakly separated, see for example \(\cite{9, 19}\)) if there do not exist distinct elements \(s < t < u < v\) (ordered modulo \(n\)) where \(s, u \in I \setminus J\) and \(t, v \in J \setminus I\). A cluster of Plücker coordinates in \(\mathcal{A}_{k,n}\) is a maximal collection of pairwise non-crossing \(k\)-subsets of \(\{1, 2, \cdots, n\}\). It was proven in \(\cite{8, 19}\) that every cluster of Plücker coordinates in the cluster structure of \(\text{Gr}(k, n)\) has \((k - 1)(n - k - 1) + n\) members.

A \((k, n)\)-frieze pattern determines a point on the Grassmannian. In particular, by definition it determines a point on the Grassmannian such that each Plücker coordinate has a positive value. This means that a \((k, n)\)-frieze pattern determines a point on the positive Grassmannian, as defined in \(\cite{21}\). For a \((k, n)\)-frieze pattern, an arrangement of smallest minors in \(\text{Gr}^+(k, n)\), as defined in \(\cite{9}\), is a collection of Plücker coordinates \(\mathcal{J}\) such that \(p_I = 1\) for all \(I \in \mathcal{J}\).
Theorem 5.1. \[9, \text{Theorem 5.6}\] For \( k \leq 3 \), a collection of \( k \)-subsets of \( \{1, 2, \cdots, n\} \) is an arrangement of smallest minors in \( \text{Gr}^+(k, n) \) if and only if it is a collection of pairwise non-crossing \( k \)-subsets of \( \{1, 2, \cdots, n\} \).

This description does not hold in general, see Remark 12.12 of \[9\].

5.3. Main Result. We now turn our attention towards cluster algebras.

Definition 5.1. A \((k, n)\)-frieze pattern is geometric if the collection of \( k \)-subsets \( I \subset \{1, 2, \cdots, n\} \) that satisfy \( p_I = 1 \) forms a cluster in \( A_{k,n} \).

Not every frieze pattern is geometric - to the point that many \((k, n)\)-frieze patterns do not contain a non-consecutive subset \( I \) with \( p_I = 1 \). For an example of a \((3, 9)\) frieze pattern where this happens, see Figure 4.

Theorem 5.2. For \( 2 \leq k \leq n/2 \), a cluster in \( A_{k,n} \) determines a unique, geometric \((k, n)\)-frieze pattern. If further \( k \leq 3 \), this restricts to a bijection between geometric \((k, n)\)-frieze patterns and clusters in \( A_{k,n} \).

Proof. When \( k = 2 \), this is well known, see for example \[17\, \text{Section 1.5}\]. In the case \( k = 3 \), we will actually prove that there is a bijection between maximal collections of pairwise non-crossing \( k \)-subsets of \( \{1, 2, \cdots, n\} \), \( C \), and geometric \((k, n)\)-frieze patterns, \( P \).

Given a \((3, n)\)-frieze pattern \( P \), and any two \( 3 \)-subsets \( I \) and \( J \) with \( p_I = p_J = 1 \), then Theorem \[5.1\] implies that \( I \) and \( J \) are non-crossing. So each geometric \((3, n)\)-frieze pattern \( P \) determines a maximal collection of pairwise non-crossing \( 3 \)-subsets of \( \{1, 2, \cdots, n\} \), \( C = \{ I \mid p_I = 1 \} \).

We are left to show that any maximal set of pairwise non-crossing \( k \)-subsets of \( \{1, 2, \cdots, n\} \), \( C \) generates a geometric \((k, n)\)-frieze pattern, \( P \). Such a maximal set of pairwise non-crossing \( k \)-subsets of \( \{1, 2, \cdots, n\} \) determines a cluster in \( A_{k,n} \). In other words, a unique Laurent polynomial \( f_I(\underline{x}) \) over the indeterminates \( \underline{x} = (x_i \mid i \in C) \) is associated to each \( k \)-subset \( I \subset \{1, 2, \cdots, n\} \). Moreover we must have that \( p_I = f_I(\underline{x}) \) for any choice of values of the \( x_i \). Simply set \( x_i = 1 \) for all \( i \in C \) and it is now a consequence of Theorem \[5.1\] that this determines a geometric \((k, n)\)-frieze pattern. \( \square \)

6. Further Directions

An early observation is that a (Coxeter) frieze pattern of width \((n - 3)\) is completely determined by its first non-trivial row, and that this row consists of a sequence of \( n \) integers repeated periodically. A sequence of \( n \) integers that induces a frieze pattern (of width \((n - 3)\)) is called a quiddity sequence of order \( n \). Conway and Coxeter \[4\] used quiddity sequences to proof that the frieze patterns of width \((n - 3)\) are in bijection with the triangulations of an \( n \)-gon. The proof is elementary, yet insightful. In a sequel paper \[16\], we determine the class of \( \text{SL}_3 \)-frieze patterns (alternatively \((3, n)\)-frieze patterns) for which quiddity sequences have properties analogous to the classical notion.

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Figure 5. A geometric (3, 6)-frieze pattern expressed via its Laurent polynomials. The frieze pattern on the left of Figure 3 can be obtained by setting \( x_1 = x_2 = x_3 = x_4 = 1 \)

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