Anonymous HIBE with Short Ciphertexts: Full Security in Prime Order Groups*

Kwangsu Lee† Jong Hwan Park‡ Dong Hoon Lee§

Abstract

Anonymous Hierarchical Identity-Based Encryption (HIBE) is an extension of Identity-Based Encryption (IBE), and it provides not only a message hiding property but also an identity hiding property. Anonymous HIBE schemes can be applicable to anonymous communication systems and public key encryption systems with keyword searching. However, previous anonymous HIBE schemes have some disadvantages that the security was proven in the weaker model, the size of ciphertexts is not short, or the construction was based on composite order bilinear groups. In this paper, we propose the first efficient anonymous HIBE scheme with short ciphertexts in prime order (asymmetric) bilinear groups, and prove its security in the full model with an efficient reduction. To achieve this, we use the dual system encryption methodology of Waters. We also present the benchmark results of our scheme by measuring the performance of our implementation.

Keywords: Identity-based encryption, Hierarchical identity-based encryption, Anonymity, Full model security, Bilinear maps.

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†Korea University, Korea and Columbia University, USA. Email: guspin@korea.ac.kr.
‡Korea University, Korea and Sangmyung University, Korea. Email: decartian@korea.ac.kr.
§Korea University, Korea. Email: donghlee@korea.ac.kr.
Hierarchical Identity-Based Encryption (HIBE) is an extension of Identity-Based Encryption (IBE) that uses an identity as a public key. In HIBE, a user’s identity is represented as a hierarchical tree structure and an upper level user can delegate the private key generation capability to a lower level user. Horwitz and Lynn introduced the concept of HIBE to reduce the burden of the private-key generator of IBE [27]. After the introduction of HIBE, it was shown that HIBE can have various applications like identity-based signature [25], public-key broadcast encryption [20], forward-secure public key encryption [14], and chosen-ciphertext secure HIBE [15].

Recently, as a result of the increasing concern with users’ privacy, the need for cryptographic systems that protect users’ privacy also increases. Anonymous HIBE can provide users’ privacy by supporting not only the message hiding property but also the identity hiding property that hides identity information in ciphertexts. Abdalla et al. formalized the concept of anonymous HIBE [1]. After that, Boyen and Waters proposed the first secure anonymous HIBE scheme without random oracles [13]. The main applications of anonymous HIBE are anonymous communication systems that provide anonymity between a received message and a true sender and public key encryption systems with keyword searching that enable keyword searches on encrypted data [8].

The security model of anonymous HIBE is defined as a game between a challenger and an adversary. In this game, the adversary adaptively requests private keys in the private key query step and selects two hierarchical identities \( ID_0, ID_1 \) and two messages \( M_0, M_1 \) in the challenge step. Next, the adversary is given a challenge ciphertext of \( ID_{\gamma}, M_{\gamma} \) where \( \gamma \) is a random bit chosen by the challenger. The adversary wins the game if he can correctly guess \( \gamma \). The security model is divided as a selective model where the adversary should commit the target hierarchical identities in the initial step and a full model where the adversary can select the target hierarchical identities in the challenge step. Generally a selectively secure HIBE scheme is converted to a fully secure HIBE scheme, but the reduction is inefficient [5]. The efficiency of the reduction is important not only for theoretical reasons but also for practical reasons.

Let \( \text{Adv}_A \) be the advantage of an adversary \( A \) that breaks a scheme and \( \text{Adv}_B \) be the advantage of an algorithm \( B \) that breaks an assumption using the adversary \( A \). Suppose that \( \text{Adv}_A \leq L \cdot \text{Adv}_B \) where \( L \) is a reduction loss. Let \( \lambda, k \) be the security level of the scheme and the assumption, respectively. If the assumption provides the \( k \)-bit security, then it guarantees that \( \text{Adv}_B \leq 1/2^k \) for any PPT algorithm \( B \). Then we can derive \( \text{Adv}_A \leq L \cdot 1/2^k \) from two inequalities \( \text{Adv}_A \leq L \cdot \text{Adv}_B \) and \( \text{Adv}_B \leq 1/2^k \). To construct the scheme that provides the \( \lambda \)-bit security, it should be guaranteed that \( \text{Adv}_A \leq 1/2^k \) for any PPT adversary \( A \). It is easy to achieve this by setting \( L \cdot 1/2^k \leq 1/2^k \) since \( \text{Adv}_A \leq L \cdot 1/2^k \). Thus we can derive a relation \( k \geq \lambda + \log_2(L) \). This relation says that the bit size \( k \) of a group order for the assumption should be larger than \( \lambda + \log_2(L) \) to construct the scheme with the \( \lambda \)-bit security. For example, if there is a selectively secure scheme with a hierarchical depth \( l = 10 \), then we should select \( k = 880 \) since \( \lambda = 80 \) and \( L = 2^{8l} \).

To construct a fully secure HIBE scheme with an efficient reduction, the new proof methodology named the dual system encryption method was proposed by Waters [45]. In the dual system encryption method, ciphertexts and private keys can be a normal type or a semi-functional type, and the semi-functional types of ciphertexts and private keys are only used in security proofs. Additionally, the normal type and the semi-functional type are indistinguishable, and the semi-functional ciphertexts are not decrypted by using the semi-functional private keys. The proof of the dual system encryption method consists of hybrid games that change a normal ciphertext and normal private keys to a semi-functional ciphertext and semi-functional...
private keys. Using this methodology, Waters proposed a fully secure HIBE scheme with linear-size ciphertexts and a fully secure HIBE scheme with constant-size ciphertexts [33, 45]. The dual system encryption method can be used to prove the security of fully secure attribute-based encryption [31], fully secure predicate encryption [37], and leakage-resilient cryptography [32].

The first secure anonymous HIBE scheme was proposed by Boyen and Waters [13], and it was proven to be selectively secure without random oracles. After the first construction of anonymous HIBE, several anonymous HIBE schemes were presented, but they were only proved to be secure in the selective model [21, 29, 41]. Recently, De Caro et al. proposed a fully secure anonymous HIBE scheme with short ciphertexts by using the dual system encryption method [16]. However, their scheme is inefficient since the scheme is based on composite order groups where the group order is a product of four prime numbers. One may use the conversion method of Freeman [22] to construct a scheme in prime order groups from a scheme in composite order groups, but this method can not be applied to the dual system encryption method of Lewko and Waters [33] since it does not provide the parameter hiding property in composite order groups. Lewko recently devised another conversion method for the dual system encryption method and constructed a (non-anonymous) unbounded HIBE scheme with linear-size ciphertexts in prime order groups [30]. However, this method is not known to be applicable for the construction of an anonymous HIBE scheme with constant-size ciphertexts since it uses dual pairing vector spaces (DPVS) [32].

Anonymous HIBE can also be constructed from Predicate Encryption (PE) with the delegation capability. Shi and Waters constructed an anonymous HIBE scheme with linear-size ciphertexts from a delegatable Hidden Vector Encryption (dHVE) scheme [42] and Okamoto and Takashima constructed an anonymous HIBE scheme with linear-size ciphertexts from a Hierarchical Inner Product Encryption (HIPE) scheme [31, 36, 37, 39]. However, currently known anonymous HIBE schemes from PE schemes with the delegation capability only have linear-size ciphertexts. It is also possible to derive anonymous HIBE from anonymous Spatial Encryption (SE) [11, 19]. However, there is no known anonymous SE scheme with constant-size ciphertexts. Thus the construction of efficient and fully secure anonymous HIBE with short ciphertexts is an unsolved problem.

1.1 Our Contributions

Motivated by the above challenge, we propose the first fully secure and anonymous HIBE scheme with short ciphertexts in prime order (asymmetric) bilinear groups. The comparison between previous HIBE schemes and ours is given in Table 1. To construct a fully secure and anonymous HIBE scheme, we use the IBE scheme in prime order (asymmetric) bilinear groups of Lewko and Waters [33]. Note that their IBE scheme does not even converted to a (non-anonymous) HIBE scheme with short ciphertexts since it does not support private key re-randomization [19].

To construct an anonymous HIBE scheme, we should devise techniques for private key re-randomization and ciphertext anonymization. The private key re-randomization process is required in the delegation algo-

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1Lewko and Waters used the parameter hiding property of composite order groups to prove the full security of their HIBE scheme using the dual system encryption technique [33]. The parameter hiding property of composite order $N = pqr$ is stated that an exponent $Z_N$ has one-to-one correspondence with $(Z_p, Z_q, Z_r)$ because of Chinese Remainder Theorem (CRT) and $Z_q$ and $Z_r$ values are information theoretically hidden to an adversary even if $Z_p$ value is revealed to the adversary.

2The dimensions of DPVS is generally proportional to the size of an identity vector in the scheme that uses DPVS [20, 36, 39]. Thus an HIBE scheme based on DPVS that supports $l$-depth has linear-size of ciphertexts since it requires at least $l$-dimensions in DPVS. To reduce the dimensions of DPVS, one may try to use the technique of Okamoto and Takashima [38], but it only applied to non-anonymous schemes since it should reveal the identity of ciphertexts.

3To support private key re-randomization using a public key, some elements $\hat{g}, \hat{u}, \hat{h} \in \hat{G}$ in a private key should be moved to a public key. However, these elements cannot be moved to the public key since the proof of dual system encryption goes wrong.
scheme by introducing a new assumption since the simple DDH assumption is not enough for the security claim of efficiency and we measured the performance of our scheme.

Furthermore, we implemented our anonymous HIBE scheme using the PBC library to support our assumption still holds in asymmetric bilinear groups of prime order. We prove the anonymity property of our private keys. The main idea to provide ciphertext anonymity is that the Decisional Diffie-Hellman (DDH) and semi-functional type-2, and we show that it is hard to distinguish these two types of semi-functional private key.

Nevertheless, if the private re-randomization technique of Boyen and Waters that re-randomizes private keys using the private elements of private keys. However, private keys of anonymous HIBE cannot be simply re-randomized with the addition of random values in semi-functional private keys.

To resolve this difficulty, we define two types of semi-functional private keys as semi-functional type-1 and semi-functional type-2, and we show that it is hard to distinguish these two types of semi-functional private keys. The main idea to provide ciphertext anonymity is that the Decisional Diffie-Hellman (DDH) assumption still holds in asymmetric bilinear groups of prime order. We prove the anonymity property of our scheme by introducing a new assumption since the simple DDH assumption is not enough for the security proof. Furthermore, we implemented our anonymous HIBE scheme using the PBC library to support our claim of efficiency and we measured the performance of our scheme.

Table 1: Comparison between previous HIBE schemes and ours

| Scheme       | ANON | R.L. | Prime | PP Size | SK Size | CT Size | Assumption          |
|--------------|------|------|-------|---------|---------|---------|---------------------|
| GS-HIBE [25] | No   | $\Omega(q^2)$ | Yes | $O(\lambda)$ | $O(\lambda)$ | $O(\lambda)$ | BDH (ROM)         |
| BB-HIBE [3]  | No   | $\Omega(2^l\lambda)$ | Yes | $O(\lambda)$ | $O(\lambda)$ | $O(\lambda)$ | DBDH               |
| BBG-HIBE [7] | No   | $\Omega(2^l\lambda)$ | Yes | $O(\lambda)$ | $O(\lambda)$ | $2k + k_T$ | $q$-Type          |
| CS-HIBE [13] | No   | $\Omega(q^2)$ | Yes | $O(\lambda)$ | $O(\lambda)$ | $O(\lambda)$ | DBDH               |
| Waters-HIBE [45] | No | $\Omega(q^2)$ | Yes | $O(\lambda)$ | $O(\lambda)$ | $O(\lambda)$ | DBDH, DLIN        |
| LW-HIBE [33] | No   | $\Omega(q)$ | No | $O(\lambda)$ | $O(\lambda)$ | $2k + k_T$ | Static             |
| LW-HIBE [34] | No   | $\Omega(q)$ | No | $O(\lambda)$ | $O(\lambda)$ | $O(\lambda)$ | Static             |
| LL-HIBE [29] | No   | $\Omega(q)$ | Yes | $O(l^4\lambda)$ | $O(l^2\lambda)$ | $133k + k_T$ | DLIN               |
| Lewko-HIBE [30] | Yes | $\Omega(q^2)$ | Yes | $O(\lambda)$ | $O(\lambda)$ | $O(\lambda)$ | DLIN               |
| OT-HIPE [18] | Yes  | $\Omega(2^l\lambda)$ | Yes | $O(l^2\lambda)$ | $O(l^2\lambda)$ | $O(\lambda)$ | $q$-Type          |
| LW-HIBE [34] | Yes  | $\Omega(2^l\lambda)$ | Yes | $O(l^2\lambda)$ | $O(l^2\lambda)$ | $O(\lambda)$ | DLIN               |
| LL-HIBE [29] | Yes  | $\Omega(2^l\lambda)$ | Yes | $O(l^2\lambda)$ | $O(l^2\lambda)$ | $6k + k_T$ | $q$-Type          |
| DIP-HIBE [16] | Yes  | $\Omega(q)$ | No | $O(\lambda)$ | $O(\lambda)$ | $2k + k_T$ | Static             |
| LOSTW-HIPE [51] | Yes | $\Omega(q)$ | Yes | $O(l^4\lambda)$ | $O(l^3\lambda)$ | $O(\lambda)$ | $q$-Type          |
| OT-HIPE [37] | Yes  | $\Omega(l^2q)$ | Yes | $O(l^3\lambda)$ | $O(l^3\lambda)$ | $O(l^2\lambda)$ | DLIN               |
| OT-HIPE [39] | Yes  | $\Omega(q)$ | Yes | $O(l^2\lambda)$ | $O(l^2\lambda)$ | $O(\lambda)$ | DLIN               |
| Ours         | Yes  | $\Omega(q)$ | Yes | $O(\lambda)$ | $O(\lambda)$ | $6k + k_T$ | Static             |

ANON = anonymity, R.L. = reduction loss, Prime = prime order bilinear groups

$\lambda$ = security parameter, $l$ = hierarchical depth, $q$ = polynomial value, $k, k_T$ = the bit size of group $G$ and $G_T$
1.2 Related Work

IBE was introduced to solve the certificate management problem in public key encryption systems, but it additionally requires a Private-Key Generator (PKG) \[9\][10]. HIBE was invented to reduce the burden of the IBE’s PKG by re-arranging an identity as a hierarchical tree structure and by allowing the delegation of private key generation from upper level users to lower level users [27]. Gentry and Silverberg proposed the first HIBE scheme in the random oracle model [25]. Canetti et al. constructed the first HIBE scheme without random oracles and introduced a selective model to prove the security of their scheme [14]. The selective model was widely used in the security proof of IBE and HIBE even though it is weaker than the full model. For instance, Boneh and Boyen proposed an efficient HIBE scheme with linear-size ciphertexts [5][6], and Boneh et al. proposed an HIBE scheme with constant-size ciphertexts [7].

To construct a fully secure HIBE scheme, Boneh and Boyen showed that a selectively secure HIBE scheme is naturally converted to a fully secure HIBE scheme with exponential loss of a reduction efficiency [5]. However, this approach has a serious problem – that is, the efficiency of the reduction is $1/\Omega(2^{\lambda l})$ where $\lambda$ is a security parameter and $l$ is the maximum hierarchical depth. To remedy this situation, Waters proposed an HIBE scheme by extending his fully secure IBE scheme with an efficient reduction to a HIBE scheme [44], and Chatterjee and Sarkar improved the efficiency of Waters’ scheme [18]. However, these schemes also have the problem of an inefficient reduction $1/\Omega(q^l)$ in the hierarchical setting where $q$ is a polynomial value. Gentry and Halevi proposed another fully secure HIBE scheme with an efficient reduction by using complex assumptions [24]. Recently, Waters introduced the dual system encryption method that can be used to construct a fully secure HIBE scheme with an efficient reduction under simple assumptions [33][45].

Anonymous IBE is related to public key encryption with keyword search (PEKS) [8][23], and the concept of anonymous HIBE was introduced by Abdalla et al. [1] by extending the concept of anonymous IBE. Boyen and Waters proposed the first anonymous HIBE scheme without random oracles and proved its security in the selective model [13]. For the construction of anonymous HIBE, they devised a linear splitting technique for ciphertext anonymity and a private re-randomization technique for private key randomization. Seo et al. proposed the first anonymous HIBE scheme with short ciphertexts in composite order bilinear groups [41]. Ducas constructed anonymous HIBE schemes using asymmetric bilinear groups of prime order [21]. Lee and Lee proposed an efficient anonymous HIBE scheme with short ciphertexts that is secure in all types of bilinear groups of prime order [29]. De Caro et al. proposed the first fully secure and anonymous HIBE scheme with short ciphertexts using the dual system encryption method in composite order bilinear groups [16].

HIBE schemes also can be constructed from Attribute Based Encryption (ABE) schemes [26] and Predicate Encryption (PE) schemes with delegation capabilities [36][42]. PE schemes with linear-size ciphertexts that have the delegation capability include the dHVE scheme of Shi and Waters in composite order bilinear groups [42] and HIPE schemes of Okamoto and Takashima based on dual pairing vector spaces [31][36][37][39]. A non-anonymous HIPE scheme based on dual pairing vector spaces can have constant-size ciphertexts, but the ciphertext should contain a linear-size identity vector [38]. Though bilinear groups were widely used in the construction of HIBE, some HIBE schemes were designed in lattices [3][4][17].
2 Preliminaries

We define anonymous HIBE and give the formal definition of its full model security. Let \( \mathcal{I} \) be an identity space and \( \mathcal{M} \) be a message space. A hierarchical identity \( ID \) of depth \( c \) is defined as an identity vector \((I_1, \ldots, I_c) \in \mathcal{I}^c\). A hierarchical identity \( ID = (I_1, \ldots, I_c) \) of depth \( c \) is a prefix of a hierarchical identity \( ID' = (I_1', \ldots, I'_d) \) of depth \( d \) if \( c \leq d \) and for all \( i \in \{1, \ldots, c\}, I_i = I'_i \).

2.1 Anonymous HIBE

An anonymous HIBE scheme consists of five algorithms (\textbf{Setup}, \textbf{KeyGen}, \textbf{Delegate}, \textbf{Encrypt}, \textbf{Decrypt}). Formally it is defined as:

\textbf{Setup}(1^λ, l). The setup algorithm takes as input a security parameter \( 1^λ \) and a maximum hierarchical depth \( l \). It outputs a master key \( MK \) and public parameters \( PP \).

\textbf{KeyGen}(ID, MK, PP). The key generation algorithm takes as input a hierarchical identity \( ID \) of depth \( m \) where \( m \leq l \), the master key \( MK \), and the public parameters \( PP \). It outputs a private key \( SK_{ID} \) for \( ID \).

\textbf{Delegate}(ID', SK_{ID}, PP). The delegation algorithm takes as input a hierarchical identity \( ID' \) of depth \( m+1 \) where \( m+1 \leq l \), a private key \( SK_{ID} \) for a hierarchical identity \( ID \) of depth \( m \), and the public parameters \( PP \). If \( ID \) is a prefix of \( ID' \), then it outputs a delegated private key \( SK_{ID'} \) for \( ID' \).

\textbf{Encrypt}(ID, M, PP). The encryption algorithm takes as input a hierarchical identity \( ID \) of depth \( n \) where \( n \leq l \), a message \( M \in \mathcal{M} \), and the public parameters \( PP \). It outputs a ciphertext \( CT \) for \( ID \) and \( M \).

\textbf{Decrypt}(CT, SK_{ID}, PP). The decryption algorithm takes as input a ciphertext \( CT \) for a hierarchical identity \( ID' \), a private key \( SK_{ID} \) for a hierarchical identity \( ID \), and the public parameters \( PP \). If \( ID = ID' \), then it outputs an encrypted message \( M \).

The correctness property of anonymous HIBE is defined as follows: For all \( MK, PP \) generated by \textbf{Setup}, all \( ID, ID' \in \mathcal{I}^n \), any \( SK_{ID} \) generated by \textbf{KeyGen}, and any \( M \), it is required that

- If \( ID = ID' \), then \( \textbf{Decrypt}(\textbf{Encrypt}(ID', M, PP), SK_{ID}, PP) = M \).
- If \( ID \neq ID' \), then \( \textbf{Decrypt}(\textbf{Encrypt}(ID', M, PP), SK_{ID}, PP) = \bot \) with all but negligible probability.

The second condition of the correctness property is not a trivial one to satisfy since the decryption algorithm of anonymous HIBE cannot easily check whether \( ID = ID' \) or not because of anonymity. One possible relaxation is to use a computational condition instead of a statistical condition. For a computational condition, we can use weak robustness of Abdalla et al. [2].

The security property of anonymous HIBE under a chosen plaintext attack is defined in terms of the following experiment between a challenger \( C \) and a PPT adversary \( A \):

1. \textbf{Setup}: \( C \) runs \textbf{Setup}(1^λ, l) to generate a master key \( MK \) and public parameters \( PP \). It keeps \( MK \) to itself and gives \( PP \) to \( A \).

2. \textbf{Query 1}: \( A \) may adaptively request a polynomial number of private keys for hierarchical identities \( ID_1, \ldots, ID_{q_1} \) of arbitrary depths. In response, \( C \) gives the corresponding private keys \( SK_{ID_1}, \ldots, SK_{ID_{q_1}} \) to \( A \) by running \textbf{KeyGen}(ID, MK, PP).


3. **Challenge**: $\mathcal{A}$ submits two hierarchical identities $ID_0^*, ID_1^* \in \mathcal{I}^n$ and two messages $M_0^*, M_1^*$ with equal length subject to the restriction: for all $ID_i$ of private key queries, $ID_i$ is not a prefix of $ID_0^*$ and $ID_1^*$. $C$ flips a random coin $\gamma \in \{0, 1\}$ and gives the challenge ciphertext $CT^*$ to $\mathcal{A}$ by running $\text{Encrypt}(ID_\gamma^*, M_\gamma^*, PP)$.

4. **Query 2**: $\mathcal{A}$ may continue to request a polynomial number of private keys for hierarchical identities $ID_{q_1+1}, \ldots, ID_q$ subject to the restriction as before.

5. **Guess**: $\mathcal{A}$ outputs a guess $\gamma' \in \{0, 1\}$ of $\gamma$, and wins the game if $\gamma' = \gamma$.

The advantage of $\mathcal{A}$ is defined as $\text{Adv}_{\mathcal{A}}^{\text{AHIBE}}(\lambda) = \left| \Pr[\gamma = \gamma'] - 1/2 \right|$ where the probability is taken over all the randomness of the experiment. An anonymous HIBE scheme is fully secure under a chosen plaintext attack if for all PPT adversary $\mathcal{A}$, the advantage of $\mathcal{A}$ in the above experiment is negligible in the security parameter $\lambda$.

The security experiment of anonymous HIBE can be relaxed to complete one introduced by Shi and Waters [42] that traces the path of delegation. Our definition of the security experiment that does not trace the path of delegation is stronger than the complete one of Shi and Waters. Thus if an anonymous HIBE scheme is secure in the security experiment of this section, then the scheme is also secure in the complete one.

### 2.2 Asymmetric Bilinear Groups

Let $G, \hat{G}$ and $G_T$ be multiplicative cyclic groups of prime order $p$ with the security parameter $\lambda$. Let $g, \hat{g}$ be generators of $G, \hat{G}$. The bilinear map $e : G \times \hat{G} \rightarrow G_T$ has the following properties:

1. **Bilinearity**: $\forall u \in G, \forall \hat{v} \in \hat{G}$ and $\forall a, b \in \mathbb{Z}_p$, $e(u^a, \hat{v}^b) = e(u, \hat{v})^ab$.

2. **Non-degeneracy**: $\exists g, \hat{g}$ such that $e(g, \hat{g})$ has order $p$, that is, $e(g, \hat{g})$ is a generator of $G_T$.

We say that $G, \hat{G}, G_T$ are bilinear groups with no efficiently computable isomorphisms if the group operations in $G, \hat{G}$, and $G_T$ as well as the bilinear map $e$ are all efficiently computable, but there are no efficiently computable isomorphisms between $G$ and $\hat{G}$.

### 2.3 Complexity Assumptions

We introduce five assumptions under asymmetric bilinear groups of prime order. Assumptions 1 and 2 were introduced in Lewko and Waters [33], and Assumptions 3 and 4 are well-known. Assumption 5 (Asymmetric 3-Party Diffie-Hellman) is an asymmetric version of the Composite 3-Party Diffie-Hellman assumption introduced by Boneh and Waters [12] with a slight modification by augmenting one additional element, and it is secure in the generic group model.

**Assumption 1 (LW1)** Let $(p, G, \hat{G}, G_T, e)$ be a description of the asymmetric bilinear group of prime order $p$ with the security parameter $\lambda$. Let $g, \hat{g}$ be generators of $G, \hat{G}$ respectively. The assumption is that if the challenge values

$$D = ((p, G, \hat{G}, G_T, e), g, \hat{g}, g^a, \hat{g}^b, g^{ab^2}, \hat{g}^{b^2}, g^b, \hat{g}^b, g^{ac}, \hat{g}^{bc}, g^{b^2c}, \hat{g}^{b^2c}, g^b, \hat{g}^b) \text{ and } T$$

are given, no PPT algorithm $\mathcal{B}$ can distinguish $T = T_0 = g^{ab^2c}$ from $T = T_1 = g^d$ with more than a negligible advantage. The advantage of $\mathcal{B}$ is defined as $\text{Adv}_{\mathcal{B}}^{\text{LW1}}(\lambda) = \left| \Pr[\mathcal{B}(D, T_0) = 0] - \Pr[\mathcal{B}(D, T_1) = 0] \right|$ where the probability is taken over the random choice of $a, b, c, d \in \mathbb{Z}_p$. 

\[7\]
Assumption 2 (LW2) Let \((p, \mathbb{G}, \hat{\mathbb{G}}, \mathbb{G}_T, e)\) be a description of the asymmetric bilinear group of prime order \(p\) with the security parameter \(\lambda\). Let \(g, \hat{g}\) be generators of \(\mathbb{G}, \hat{\mathbb{G}}\) respectively. The assumption is that if the challenge values

\[
D = ((p, \mathbb{G}, \hat{\mathbb{G}}, \mathbb{G}_T, e), g, g^a, g^{a^2}, g^{b^y}, g^{ab}, g^{a^2x}, \hat{g}, \hat{g}^a, \hat{g}^b, \hat{g}^c) \text{ and } T
\]

are given, no PPT algorithm \(B\) can distinguish \(T = T_0 = \hat{g}^c\) from \(T = T_1 = \hat{g}^d\) with more than a negligible advantage. The advantage of \(B\) is defined as \(\text{Adv}^{\lambda}_{b} = |\text{Pr}[B(D, T_0) = 1] - \text{Pr}[B(D, T_1) = 1]|\) where the probability is taken over the random choice of \(a, b, c, d \in \mathbb{Z}_p\).

Assumption 3 (Symmetric eXternal Diffie-Hellman) Let \((p, \mathbb{G}, \hat{\mathbb{G}}, \mathbb{G}_T, e)\) be a description of the asymmetric bilinear group of prime order \(p\) with the security parameter \(\lambda\). Let \(g, \hat{g}\) be generators of \(\mathbb{G}, \hat{\mathbb{G}}\) respectively. The assumption is that if the challenge values

\[
D = ((p, \mathbb{G}, \hat{\mathbb{G}}, \mathbb{G}_T, e), g, \hat{g}, \hat{g}^a, \hat{g}^b) \text{ and } T
\]

are given, no PPT algorithm \(B\) can distinguish \(T = T_0 = g^{ab}\) from \(T = T_1 = g^c\) with more than a negligible advantage. The advantage of \(B\) is defined as \(\text{Adv}^{\lambda}_{b} = |\text{Pr}[B(D, T_0) = 1] - \text{Pr}[B(D, T_1) = 1]|\) where the probability is taken over the random choice of \(a, b, c, d \in \mathbb{Z}_p\).

Assumption 4 (Decisional Bilinear Diffie-Hellman) Let \((p, \mathbb{G}, \hat{\mathbb{G}}, \mathbb{G}_T, e)\) be a description of the asymmetric bilinear group of prime order \(p\) with the security parameter \(\lambda\). Let \(g, \hat{g}\) be generators of \(\mathbb{G}, \hat{\mathbb{G}}\) respectively. The assumption is that if the challenge values

\[
D = ((p, \mathbb{G}, \hat{\mathbb{G}}, \mathbb{G}_T, e), g, g^a, g^b, g^c, \hat{g}, \hat{g}^a, \hat{g}^b, \hat{g}^c) \text{ and } T
\]

are given, no PPT algorithm \(B\) can distinguish \(T = T_0 = e(g, \hat{g})^{abc}\) from \(T = T_1 = e(g, \hat{g})^d\) with more than a negligible advantage. The advantage of \(B\) is defined as \(\text{Adv}^{\lambda}_{b} = |\text{Pr}[B(D, T_0) = 1] - \text{Pr}[B(D, T_1) = 1]|\) where the probability is taken over the random choice of \(a, b, c, d \in \mathbb{Z}_p\).

Assumption 5 (Asymmetric 3-Party Diffie-Hellman) Let \((p, \mathbb{G}, \hat{\mathbb{G}}, \mathbb{G}_T, e)\) be a description of the asymmetric bilinear group of prime order \(p\) with the security parameter \(\lambda\). Let \(g, \hat{g}\) be generators of \(\mathbb{G}, \hat{\mathbb{G}}\) respectively. The assumption is that if the challenge values

\[
D = ((p, \mathbb{G}, \hat{\mathbb{G}}, \mathbb{G}_T, e), g, g^a, g^b, g^c, g^{ab}, g^{a^2b}, \hat{g}, \hat{g}^a, \hat{g}^b) \text{ and } T
\]

are given, no PPT algorithm \(B\) can distinguish \(T = T_0 = g^{abc}\) from \(T = T_1 = g^d\) with more than a negligible advantage. The advantage of \(B\) is defined as \(\text{Adv}^{\lambda}_{b} = |\text{Pr}[B(D, T_0) = 1] - \text{Pr}[B(D, T_1) = 1]|\) where the probability is taken over the random choice of \(a, b, c, d \in \mathbb{Z}_p\).

3 Anonymous HIBE

We construct an anonymous HIBE scheme in prime order (asymmetric) bilinear groups and prove its full model security under static assumptions.

3.1 Construction

Let \(I = \mathbb{Z}_p^*\). Our anonymous HIBE scheme is described as follows:
Setup($1^k$, $l$): This algorithm first generates the asymmetric bilinear groups $\mathbb{G}, \mathbb{G}_\text{ID}$ of prime order $p$ of bit size $\Theta(\lambda)$. It chooses random elements $g \in \mathbb{G}$ and $\hat{g} \in \mathbb{G}_\text{ID}$. It also chooses random exponents $\nu, \phi_1, \phi_2 \in \mathbb{Z}_p$ and sets $\tau = \phi_1 + \nu \phi_2$. Next, it selects random exponents $y_h, \{y_{uw}, \{y_w, \alpha \in \mathbb{Z}_p \text{ and sets } h = g^{y_h}, \hat{h} = \hat{g}^{y_h}, \{u_i = g^{y_{uw}}, \hat{u}_i = \hat{g}^{y_{uw}}\}_i = 1, \hat{w} = \hat{g}^{y_w}. It outputs a master key $MK = (\hat{g}, \hat{g}^{\alpha}, \hat{h}, \{\hat{u}_i\}_i = 1)$ and public parameters as

$$PP = \left( g, g^\nu, g^{-\tau}, h, h^\nu, h^{-\tau}, \{u_i, u_i^\nu, u_i^{-\tau}\}_i = 1, \hat{w}^{\phi_1}, \hat{w}^{\phi_2}, \hat{w}, \Omega = e(g, \hat{g})^\alpha \right).$$

KeyGen($ID, MK, PP$): This algorithm takes as input a hierarchical identity $ID = (I_1, \ldots, I_m) \in \mathbb{T}^m$ and the master key $MK$. It first selects random exponents $r_1, c_1, c_2, \{c_{3,j}\}_i = m+1 \in \mathbb{Z}_p$ and creates the decryption and delegation components of a private key as

$$K_{1,1} = \hat{g}^{\alpha} (h) \prod_{i=1}^m \hat{u}_i^{r_i} (\hat{w}^{\phi_1})^{c_i}, K_{1,2} = (\hat{w}^{\phi_2})^{c_1}, K_{1,3} = \hat{w}^{\phi_1},$$

$$K_{2,1} = \hat{g}^{\alpha} (\hat{w}^{\phi_1})^{c_2}, K_{2,2} = (\hat{w}^{\phi_2})^{c_2}, K_{2,3} = \hat{w}^{\phi_2},$$

$$\{L_{3,i,1} = \hat{u}_i^{r_1} (\hat{w}^{\phi_1})^{c_{3,i}}, L_{3,i,2} = (\hat{w}^{\phi_2})^{c_{3,i}}, L_{3,i,3} = \hat{w}^{c_{3,i}}\}_i = m+1.$$

Next, it selects random exponents $r_2, c_4, c_5, \{c_{6,i}\}_i = m+1 \in \mathbb{Z}_p$ and creates the randomization components of the private key as

$$R_{1,1} = (h) \prod_{i=1}^m \hat{u}_i^{r_2} (\hat{w}^{\phi_1})^{c_4}, R_{1,2} = (\hat{w}^{\phi_2})^{c_4}, R_{1,3} = \hat{w}^{\phi_4},$$

$$R_{2,1} = \hat{g}^{\alpha} (\hat{w}^{\phi_1})^{c_5}, R_{2,2} = (\hat{w}^{\phi_2})^{c_5}, R_{2,3} = \hat{w}^{\phi_5},$$

$$\{R_{3,i,1} = \hat{u}_i^{r_2} (\hat{w}^{\phi_1})^{c_{6,i}}, R_{3,i,2} = (\hat{w}^{\phi_2})^{c_{6,i}}, R_{3,i,3} = \hat{w}^{c_{6,i}}\}_i = m+1.$$

Finally, it outputs a private key as

$$SK_{ID} = \left( K_{1,1}, K_{1,2}, K_{1,3}, K_{2,1}, K_{2,2}, K_{2,3}, \{L_{3,i,1}, L_{3,i,2}, L_{3,i,3}\}_i = m+1, \right.$$  

$$\left. R_{1,1}, R_{1,2}, R_{1,3}, R_{2,1}, R_{2,2}, R_{2,3}, \{R_{3,i,1}, R_{3,i,2}, R_{3,i,3}\}_i = m+1 \right).$$

Delegate($ID', SK_{ID}, PP$): This algorithm takes as input a hierarchical identity $ID' = (I_1, \ldots, I_{m+1}) \in \mathbb{T}^{m+1}$ and a private key $SK_{ID'}$ for a hierarchical identity $ID = (I_1, \ldots, I_m) \in \mathbb{T}^m$ where $ID$ is a prefix of $ID'$. Let $(W_1, W_2, W_3) = (\hat{w}^{\phi_1}, \hat{w}^{\phi_2}, \hat{w})$. It first selects random exponents $\gamma_1, \delta_1, \delta_2, \{\delta_{3,j}\}_i = m+2 \in \mathbb{Z}_p$ and creates the decryption and delegation components of a delegated private key as

$$\left( K'_{1,k} = K_{1,k} L_{3,m+1,k}^{r_1} (R_{1,k} R_{3,m+1,k}^{r_2}) \right)_{i = 1 \leq k \leq 3}, \left( K'_{2,k} = K_{2,k} \cdot R_{1,k}^{\gamma_1} W_k^{\delta_1} \right)_{1 \leq k \leq 3},$$

$$\left( L'_{i,k} = L_{3,i,k} \cdot R_{3,i,k}^{r_2} W_k^{\delta_{3,i}} \right)_{1 \leq k \leq 3}.$$

Next, it selects random exponents $\gamma_2, \delta_4, \delta_5, \{\delta_{6,i}\}_i = m+2 \in \mathbb{Z}_p$ and creates the randomization components of the delegated private key as

$$\left( R'_{1,k} = (R_{1,k} R_{3,m+1,k}^{r_1}) \gamma_2 W_k^{\delta_{3,i}} \right)_{1 \leq k \leq 3}, \left( R'_{2,k} = R_{1,k} W_k^{\delta_5} \right)_{1 \leq k \leq 3}, \left( R'_{3,k} = R_{3,i,k} W_k^{\delta_{6,i}} \right)_{1 \leq k \leq 3}.$$

9
Finally, it outputs a delegated private key as

$$SK_{ID} = \left( K_{1,1}, K_{1,2}, K_{1,3}, K_{2,1}, K_{2,2}, K_{2,3}, \{ L_{i,j,l}^1, L_{i,j,l}^2, L_{i,j,l}^3 \}_{i=m+2} \right),$$

$$R_{1,1}, R_{1,2}, R_{1,3}, R_{2,1}, R_{2,2}, R_{2,3}, \{ R_{i,j,l}^1, R_{i,j,l}^2, R_{i,j,l}^3 \}_{i=m+2} \right).$$

The distribution of the delegated private key is the same as the original private key since the random values are defined as $r'_1 = r_1 + r_2 \gamma_1, r'_2 = r_2 \gamma_2$ where $r_1, r_2$ are random exponents in the private key $SK_{ID}$. Note that $c_1, c_2, \{ c_{3,j} \}, c_4, c_5, \{ c_{6,j} \}$ are perfectly re-randomized since $\hat{w}^{\phi_1}, \hat{w}^{\phi_2}, \hat{w}$ are publicly known and $\delta_1, \delta_2, \{ \delta_{3,j} \}, \delta_4, \delta_5, \{ \delta_{6,j} \}$ are chosen randomly.

**Encrypt($ID, M, PP$):** This algorithm takes as input a hierarchical identity $ID = (I_1, \ldots, I_n)$, a message $M \in \mathcal{G}_T$, and the public parameter $PP$. It selects a random exponent $t \in \mathbb{Z}_p$ and outputs a ciphertext as

$$CT = \left( C = \Omega M, C_{1,1} = g^t, C_{1,2} = (g^\tau)^t, C_{1,3} = (g^{-\tau})^t, C_{2,1} = (h \prod_{i=1}^n u_i^t)^t, C_{2,2} = (h^\nu \prod_{i=1}^n (u_i^{-\tau} R_i)^t)^t, C_{2,3} = (h^{-\tau} \prod_{i=1}^n (u_i^{-\tau} R_i)^t)^t \right).$$

**Decrypt($CT, SK_{ID}, PP$):** This algorithm takes as input a ciphertext $CT$ and a private key $SK_{ID}$ for a hierarchical identity $ID = (I_1, \ldots, I_n)$. It outputs the encrypted message as

$$M \leftarrow C \cdot \prod_{i=1}^3 e(C_{1,i}, K_{1,i})^{-1} \cdot \prod_{i=1}^3 e(C_{2,i}, K_{2,i}).$$

### 3.2 Correctness

The first condition of the correctness property can be easily checked by the following equation as

$$\prod_{i=1}^3 e(C_{1,i}, K_{1,i})^{-1} \cdot \prod_{i=1}^3 e(C_{2,i}, K_{2,i}) = e(g^t, g^a (h \prod_{i=1}^n u_i^t)^t)^{-1} \cdot e((h \prod_{i=1}^n u_i^t)^t, g^\tau)^{-a} = e(g, g)^{-\alpha t}$$

since the inner product of $(1, \nu, -\tau)$ and $(\phi_1, \phi_2, 1)$ are zero. The second condition of the correctness property can be satisfied by using the technique of Boneh and Waters [12] that uses the limited message space. If we use a computational condition instead of a statistical condition, then we can achieve weak robustness by using the transformation of Abdalla et al. [2].

### 3.3 Security Analysis

**Theorem 3.1.** The above anonymous HIBE scheme is fully secure under a chosen plaintext attack if Assumptions 1, 2, 3, 4 and 5 hold. That is, for any PPT adversary $A$, there exist PPT algorithms $B_1, B_2, B_3, B_4$, and $B_5$ such that

$$\text{Adv}_{A}^{\text{HIBE}}(\lambda) \leq \text{Adv}_{B_1}^{\text{A}}(\lambda) + q \left( \text{Adv}_{B_2}^{\text{A}}(\lambda) + \text{Adv}_{B_3}^{\text{A}}(\lambda) \right) + \text{Adv}_{B_4}^{\text{A}}(\lambda) + \text{Adv}_{B_5}^{\text{A}}(\lambda),$$

where $q$ is the maximum number of private key queries of $A$. 10
Proof. To prove the security of our scheme, we use the dual system encryption technique of \cite{33,45}. We first describe a semi-functional key generation algorithm and a semi-functional encryption algorithm. They are not used in a real system, but they are used in the security proof. For semi-functionality, we set \( f = g^y, \hat{f} = g^{\hat{y}} \) where \( y_f \) is a random exponent in \( \mathbb{Z}_p \).

**KeyGenSF-1.** The semi-functional type-1 key generation algorithm first creates a normal private key using the master key. Let \( \{ K_{1,1}', K_{1,2}', K_{1,3}' \}, \{ R_{3,i,1}', R_{3,i,2}', R_{3,i,3}' \}_{i=m+1} \) be the normal private key of a hierarchical identity \( ID = \{ I_1, \ldots, I_m \} \) with random exponents \( r_1, r_2, c_1, c_2, \{ c_3,i \}, c_4, c_5, \{ c_6,i \} \in \mathbb{Z}_p \). It selects random exponents \( s_{k,1}, z_{k,1}, \{ z_{k,2},i \}, s_{k,2} \in \mathbb{Z}_p \) and outputs a semi-functional type-1 private key as

\[
K_{1,1} = K_{1,1}'(\hat{f}^{-v})^{s_{k,1}z_{k,1}}, \quad K_{1,2} = K_{1,2}'(\hat{f}^{s_{k,1}z_{k,1}}), \quad K_{1,3} = K_{1,3}',
\]

\[
K_{2,1} = K_{2,1}'(\hat{f}^{-v})^{R_{3,i,1}'}, \quad K_{2,2} = K_{2,2}'(\hat{f}^{R_{3,i,1}'}), \quad K_{2,3} = K_{2,3}'.
\]

\[
\{ L_{i,1} = L_{i,1}'(\hat{f}^{-v})^{s_{k,1}z_{k,1}}, L_{i,2} = L_{i,2}'(\hat{f}^{s_{k,1}z_{k,1}}), L_{i,3} = L_{i,3}' \}_{i=m+1},
\]

\[
R_{1,1} = R_{1,1}'(\hat{f}^{-v})^{s_{k,1}z_{k,1}}, \quad R_{1,2} = R_{1,2}'(\hat{f}^{s_{k,1}z_{k,1}}), \quad R_{1,3} = R_{1,3}',
\]

\[
R_{2,1} = R_{2,1}'(\hat{f}^{-v})^{s_{k,1}z_{k,1}}, \quad R_{2,2} = R_{2,2}'(\hat{f}^{s_{k,1}z_{k,1}}), \quad R_{2,3} = R_{2,3}'.
\]

\[
\{ R_{3,i,1} = R_{3,i,1}'(\hat{f}^{-v})^{s_{k,1}z_{k,1}}, R_{3,i,2} = R_{3,i,2}'(\hat{f}^{s_{k,1}z_{k,1}}), R_{3,i,3} = R_{3,i,3}' \}_{i=m+1}.
\]

Note that the randomization components should contain the semi-functional part since this semi-functional part enables the correct simulation of the security proof for anonymity.

**KeyGenSF-2.** The semi-functional type-2 key generation algorithm first creates a normal private key using the master key. Let \( \{ K_{1,1}', \ldots, K_{1,3}' \}, \{ R_{3,i,1}', \ldots, R_{3,i,3}' \}_{i=m+1} \) be the normal private key of a hierarchical identity \( ID = \{ I_1, \ldots, I_m \} \). It selects random exponents \( s_{k,1}, z_{k,1}, \{ z_{k,2},j \}, s_{k,2}, z_{k,3}, \{ z_{k,4},j \}_{i=m+1} \in \mathbb{Z}_p \) and outputs a semi-functional type-2 private key the same as the semi-functional type-1 private key except that the randomization components are generated as

\[
R_{1,1} = R_{1,1}'(\hat{f}^{-v})^{s_{k,1}z_{k,1}}, \quad R_{1,2} = R_{1,2}'(\hat{f}^{s_{k,1}z_{k,1}}), \quad R_{1,3} = R_{1,3}',
\]

\[
R_{2,1} = R_{2,1}'(\hat{f}^{-v})^{s_{k,2}z_{k,2}}, \quad R_{2,2} = R_{2,2}'(\hat{f}^{s_{k,2}z_{k,2}}), \quad R_{2,3} = R_{2,3}',
\]

\[
\{ R_{3,i,1} = R_{3,i,1}'(\hat{f}^{-v})^{s_{k,1}z_{k,1}}, R_{3,i,2} = R_{3,i,2}'(\hat{f}^{s_{k,1}z_{k,1}}), R_{3,i,3} = R_{3,i,3}' \}_{i=m+1}.
\]

Note that new random exponents \( z_{k,3}, \{ z_{k,4},j \}_{i=1} \) are chosen to generate the randomization components of the semi-functional type-2 private key, whereas the same exponents \( z_{k,1}, \{ z_{k,2},j \}_{i=1} \) of the decryption and delegation components are used to generate the randomization components in the semi-functional type-1 private key.

**EncryptSF.** The semi-functional encryption algorithm first creates a normal ciphertext using the public parameters. Let \( \{ C_{1}, \ldots, C_{2} \} \) be the normal ciphertext. It selects random exponents \( s_c, z_c \in \mathbb{Z}_p \) and outputs a semi-functional ciphertext as

\[
C = C_{1}, \quad C_{1,1} = C_{1,1}', \quad C_{1,2} = C_{1,2}'f^{s_{k}z_{c}}, \quad C_{1,3} = C_{1,3}'(f^{-\phi_2})^{s_{k}z_{c}},
\]

\[
C_{2,1} = C_{2,1}', \quad C_{2,2} = C_{2,2}'f^{s_{k}z_{c}}, \quad C_{2,3} = C_{2,3}'(f^{-\phi_2})^{s_{k}z_{c}}.
\]

If we decrypt a semi-functional ciphertext by using a semi-functional type-2 private key, then the decryption fails since an additional element \( e(f, \hat{f})^{s_{k}z_{c}^2} = (s_{k,1}z_{1} + s_{k,2}z_{2}z_{3})g - (s_{k,1}z_{1} + s_{k,2}z_{2}z_{3})g \) remains. Note that the decryption
can be done after re-randomizing the private key using a random exponent $\gamma$. If $(sk_1z_{k,1} + sk_2z_{k,2}) = (sk_1 + sk_2\gamma)z_\epsilon$, then the decryption algorithm succeeds. However, the probability of this is negligible since $sk_1, sk_2, z_{k,1}, z_{k,2}, z_\epsilon, \gamma$ are randomly chosen. In case of the semi-functional type-1 private key, the additional random element can be restated as $e(f, f)^{(sk_1 + sk_2\gamma)z_\epsilon}$. If $z_{k,1} = z_\epsilon$, then the decryption algorithm succeeds. In this case, we say that the private key is nominally semi-functional type-1.

The security proof consists of a sequence of games. The first game will be the original security game and the last one will be a game such that the adversary has no advantage. We define the games as follows:

**Game $G_0$.** This game is the original security game. That is, the private keys and the challenge ciphertext are normal.

**Game $G_1$.** We first modify $G_0$ into a new game $G_1$. This game is almost identical to $G_0$ except that the challenge ciphertext is semi-functional.

**Game $G_2$.** Next, we modify $G_1$ into a game $G_2$. In this game, the private keys are semi-functional type-2 and the challenge ciphertext is semi-functional. Suppose that an adversary makes at most $q$ private key queries. For the security proof, we define a sequence of games $G_{1,0}, \ldots, G_{1,k}, G_{1,k}, \ldots, G_{1,q}$ where $G_{1,0} = G_1$. In $G_{1,k}$ and $G_{1,k}$, a normal private key is given to the adversary for all $j$-th private key queries such that $j > k$ and a semi-functional type-2 private key is given to the adversary for all $j$-th private key queries such that $j < k$. However, for $k$-th private key query, a semi-functional type-1 private key is given to the adversary in $G'_{1,k}$ where as a semi-functional type-2 private key is given in $G_{1,k}$. It is obvious that $G_{1,q}$ is equal to $G_2$.

**Game $G_3$.** We now define a new game. This game differs from $G_2$ where the challenge ciphertext component $C$ is replaced by a random element in $\mathcal{G}_T$.

**Game $G_4$.** Finally, we change $G_3$ to a new game $G_4$. In this game, the semi-functional ciphertext components $(C_{2,1}, C_{2,2}, C_{2,3})$ are formed as $(P^e, (P^e)^f f^{\lambda_2} z_\epsilon, (P^{-\beta})^f (f^{\beta_2} t z_\epsilon)$ where $P$ is a random element in $\mathcal{G}$. In this game, the challenge ciphertext gives no information about the random coin $\gamma$. Therefore, the adversary can win this game with probability at most $1/2$.

Let $\text{Adv}^G_{A,j}$ be the advantage of $A$ in $G_j$ for $j = 0, \ldots, 4$. Let $\text{Adv}^G_{A,k}$ and $\text{Adv}^G_{A,k}$ be the advantage of $A$ in $G_{1,k}$ and $G'_{1,k}$ for $k = 0, \ldots, q$. It is clear that $\text{Adv}^{\text{AHIBE}}_{A}(\lambda) = \text{Adv}^G_{A}$, $\text{Adv}^{G,0}_{A} = \text{Adv}^G_{A}$, $\text{Adv}^{G,k}_{A} = \text{Adv}^G_{A}$, and $\text{Adv}^{G,q}_{A} = 0$. From the following five Lemmas, we obtain that it is hard to distinguish $G_{i-1}$ from $G_i$ under the given assumptions. Therefore, we have that

$$
\text{Adv}^{\text{AHIBE}}_{A}(\lambda) = \text{Adv}^G_{A} + \sum_{i=1}^{3} (\text{Adv}^G_{A} - \text{Adv}^{G_i}_{A}) - \text{Adv}^{G,q}_{A} \leq \sum_{i=1}^{4} |\text{Adv}^{G_{i-1}}_{A} - \text{Adv}^{G_i}_{A}|
$$

$$
= \text{Adv}^{A_1}_{B_1}(\lambda) + \sum_{k=1}^{q} (\text{Adv}^{A_2}_{B_1}(\lambda) + \text{Adv}^{A_3}_{B_1}(\lambda)) + \text{Adv}^{A_4}_{B_1}(\lambda) + \text{Adv}^{A_5}_{B_1}(\lambda).
$$

This completes our proof of Theorem 3.1.

**Lemma 3.2.** If Assumption 1 holds, then no PPT algorithm can distinguish between $G_0$ and $G_1$ with a non-negligible advantage. That is, for any adversary $A$, there exists a PPT algorithm $B_1$ such that $|\text{Adv}^{G_0}_{A} - \text{Adv}^{G_1}_{A}| = \text{Adv}^{A_1}_{B_1}(\lambda)$. 

12
Lemma 3.3. If Assumption 2 holds, then no PPT algorithm can distinguish between $G_{1,k-1}$ and $G'_{1,k}$ with a non-negligible advantage. That is, for any adversary $A$, there exists a PPT algorithm $B_2$ such that $|Adv_{A}^{G_{1,k-1}} - Adv_{B_2}^{G'_{1,k}}| = Adv_{B_2}^{A_{2}}(\lambda)$.

Lemma 3.4. If Assumption 3 holds, then no PPT algorithm can distinguish between $G'_{1,k}$ and $G_{1,k}$ with a non-negligible advantage. That is, for any adversary $A$, there exists a PPT algorithm $B_3$ such that $|Adv_{A}^{G'_{1,k}} - Adv_{B_3}^{G_{1,k}}| = Adv_{B_3}^{A_{3}}(\lambda)$.

Lemma 3.5. If Assumption 4 holds, then no PPT algorithm can distinguish between $G_2$ and $G_3$ with a non-negligible advantage. That is, for any adversary $A$, there exists a PPT algorithm $B_4$ such that $|Adv_{A}^{G_2} - Adv_{B_4}^{G_3}| = Adv_{B_4}^{A_{4}}(\lambda)$.

Lemma 3.6. If Assumption 5 holds, then no PPT algorithm can distinguish between $G_3$ and $G_4$ with a non-negligible advantage. That is, for any adversary $A$, there exists a PPT algorithm $B_5$ such that $|Adv_{A}^{G_3} - Adv_{B_5}^{G_4}| = Adv_{B_5}^{A_{5}}(\lambda)$.

The security proof of Lemmas 3.2, 3.3, 3.4, 3.5, and 3.6 are given in Section 5.

3.4 Extensions

Relaxed Security Model. The original security experiment of anonymous HIBE requires that an adversary should select two hierarchical identities $ID_0, ID_1 \in I^n$ with equal depth $n$. One possible relaxation of the security experiment of anonymous HIBE is to allow the adversary to select two hierarchical identities $ID_0^*, ID_1^* \in I^{n_1}, ID_1^* \in I^{n_2}$ with different depths $n_1, n_2$. Our scheme is also fully secure in this relaxed security experiment since the ciphertext size is constant. The two challenge hierarchical identities with different depths only matter in the security proof that distinguishes $G_3$ from $G_4$. In that proof, we showed that the adversary cannot distinguish the challenge hierarchical identity $ID_1^*$ from a random value. Thus our scheme is secure in this relaxed experiment since the ciphertext size does not reveal the depth of the hierarchical identity.

4 Performance Analysis

In this section, we analyze the running time of our scheme, and then we measure the performance of the scheme by implementing it.

4.1 Runtime Analysis

To analyze the efficiency of our scheme, we use the abstract cost of expensive mathematical operations. In bilinear groups, the expensive operations are exponentiation operations and pairing operations. Additionally, the efficiency of exponentiations and pairings can be improved by doing $m$-term exponentiations and $m$-term pairings respectively. The abstract cost of these operations is defined as follows:

- $\text{MPairCost}(G, \hat{G}, m)$: $m$-term pairing $\prod_{i=1}^{m} e(g_i, \hat{h}_i)$ where $g_i \in G, h_i \in \hat{G}$
- $\text{PairCost}(G, \hat{G})$: pairing $e(g, \hat{h})$ where $g \in G, h \in \hat{G}$
- $\text{MExpCost}(G, m)$: $m$-term exponentiation $\prod_{i=1}^{m} g_i^a$ where $g_i \in G$
The PBC library on the test machine can compute an exponentiation of \( r \) in \( G \), and the delegation algorithm for one depth takes about 4.259 seconds and 5.257 seconds respectively. It takes about 0.936 seconds to generate the public parameters and the master key. The key generation algorithm and the delegation algorithm for one depth take about 4.259 seconds and 5.257 seconds respectively.

Let \( l \) be the maximum number of hierarchical depth and \( d \) be the depth of \( ID \). We define the abstract costs of the setup algorithm, the key generation algorithm, the delegation algorithm, the encryption algorithm, and the decryption algorithm as \( \text{SetupCost}, \text{GenCost}, \text{DelCost}, \text{EncCost}, \text{DecCost} \) respectively. The abstract costs of these algorithm are obtained as follows:

\[
\begin{align*}
\text{SetupCost}(l) & \geq (2l + 4) \cdot \text{ExpCost}(G) + 2 \cdot \text{ExpCost}(\hat{G}) + \text{PairCost}(G, \hat{G}), \\
\text{GenCost}(l, d) & \geq (4(l - d) + 4) \cdot \text{ExpCost}(\hat{G}) + (2(l - d) + 2) \cdot \text{MExpCost}(\hat{G}, 2) + \frac{d}{m} \cdot \text{MExpCost}(\hat{G}, m), \\
\text{DelCost}(l, d) & \geq (6(l - d) + 6) \cdot \text{MExpCost}(\hat{G}, 2) + 9 \cdot \text{ExpCost}(\hat{G}), \\
\text{EncCost}(d) & \geq \frac{3d}{m} \cdot \text{MExpCost}(G, m) + 6 \cdot \text{ExpCost}(G) + \text{ExpCost}(G_T), \\
\text{DecCost} & \geq 2 \cdot \text{MPairCost}(G, \hat{G}, 3).
\end{align*}
\]

In asymmetric bilinear groups, the bit size of \( \hat{G} \) and the bit size of \( G_T \) increase proportionally to the embedding degree of asymmetric bilinear groups. Thus the cost of exponentiation in \( \hat{G} \) is higher than the cost of exponentiation in \( G \). In our scheme, the cost of the key generation algorithm and the cost of the delegation algorithm are higher than the cost of other algorithm since our scheme uses group elements in \( G \) for ciphertexts and group elements in \( \hat{G} \) for private keys, and these costs decrease proportionally to the depth of \( ID \). The cost of the encryption algorithm is small since it uses \( m \)-term exponentiations in \( G \), and the cost of the decryption algorithm is constant.

### 4.2 Implementation

To show the efficiency of our scheme, we present the implementation of our scheme and analyze the performance of it. We use the Pairing Based Cryptography (PBC) library [35] to implement our scheme, and we use a notebook computer with an Intel Core i5 2.53 GHz CPU as a test machine. We select a 175-bit Miyaji-Nakabayashi-Takano (MNT) curve with embedding degree 6. In the 175-bit MNT curve, the group size of \( G \) is about 175 bits, the group size of \( \hat{G} \) is about 525 bits, and the group size of \( G_T \) is about 1050 bits. The PBC library on the test machine can compute an exponentiation of \( G \) in 1.6 ms, an exponentiation of \( \hat{G} \) in 20.3 ms, an exponentiation of \( G_T \) in 4.7 ms, and a pairing in 15.6 ms. Additionally, the PBC library can compute a three-term multi-exponentiation of \( G \) in 2.1 ms, a two-term multi-exponentiation of \( \hat{G} \) in 27.3 ms, a three-term multi-exponentiation of \( \hat{G} \) in 28.6 ms, and a three-term multi-pairing in 31.2 ms. Therefore, we can obtain the cost of our scheme using the 175-bit MNT curve on the test machine as follows:

\[
\begin{align*}
\text{GenCost}(l, d) & \geq 135.8 \ast (l - d) + 9.5 \ast d + 135.8 \text{ ms}, \\
\text{DelCost}(l, d) & \geq 163.8 \ast (l - d) + 346.5 \text{ ms}, \\
\text{EncCost}(d) & \geq 2.1 \ast d + 14.3 \text{ ms}, \\
\text{DecCost} & \geq 62.4 \text{ ms}.
\end{align*}
\]

Let \( l = 30 \). The performance results of each algorithms are described in Figure[1]. The setup algorithm takes about 0.936 seconds to generate the public parameters and the master key. The key generation algorithm and the delegation algorithm for one depth take about 4.259 seconds and 5.257 seconds respectively.
Figure 1: Performance of our HIBE scheme

One method to improve the performance of the key generation algorithm is to preprocess the public parameters and the master key. If the preprocessing method is used, then the cost of the key generation algorithm is reduced to 1/5. This method also can be used in the delegation algorithm.

5 Proof of Lemmas

In this section, we give the security proofs of Lemmas for our HIBE scheme.

5.1 Proof of Lemma 3.2 (Indistinguishability of G₀ and G₁)

In this proof, private keys are normal and the challenge ciphertext should be normal or semi-functional depending on the T value of the given assumption. The main idea of this proof is that a simulator can only create normal private keys since an element for semi-functional private keys is not given in the assumption and the simulator embeds the T element of the assumption into the challenge ciphertext.

Simulator. Suppose there exists an adversary A that distinguishes between G₀ and G₁ with a non-negligible advantage. A simulator B₁ that breaks Assumption 1 using A is given: a challenge tuple \( D = ((p, G, \hat{G}, G_T, e), k, k^a, k^b, k^{ab^2}, k^b, k^c, k^e, k^{bc}, k^{b^2c}, \hat{k}, \hat{k}^b) \) and \( T \) where \( T = T_0 = k^{ab^2}c \) or \( T = T_1 = k^{ab^2}c + d \). Then B₁ that interacts with A is described as follows: B₁ first chooses random exponents \( \phi_2, B, \{A_i\}_{i=1}^l, \alpha \in \mathbb{Z}_p \) and random blinding values \( y_{g_i}, y_{h_i}, \{y_{u_i}\}_{i=1}^l, y_w \in \mathbb{Z}_p \). It implicitly sets \( v = a, \phi_1 = b, \tau = b + a\phi_2 \) and creates the public parameters as

\[
\begin{align*}
g &= k^{b^2}k^y_s, \quad g^v = k^{ab^2}(k^a)^{y_s}, \quad g^{-\tau} = (k^{b^3}(k^b)^{y_s}(k^{ab^2})^{\phi_2}(k^a)^{y_s\phi_2})^{-1}, \\
h &= (k^{b^2}b)^{k^y_s}, \quad h^v = (k^{ab^2})^b(k^a)^{y_s}, \quad h^{-\tau} = ((k^{b^3})^b(k^b)^{y_s}(k^{ab^2})^{\phi_2}(k^a)^{y_s\phi_2})^{-1}, \\
\{u_i = (k^{b^2})^{A_i}k^{y_u}, u_i^v = (k^{ab^2})^{A_i}(k^a)^{y_u}, u_i^{-\tau} = ((k^{b^3})^{A_i}(k^b)^{y_u}(k^{ab^2})^{\phi_2}(k^a)^{y_u\phi_2})^{-1} \} \cup l, \\
w^{\phi_2} = (k^{b^2})^{\phi_2}, \quad w^{\phi_2} = k^{y_u\phi_2}, \quad w = k^{y_u}, \quad \Omega = (e(k^{b^3}, \hat{k})^b, e(k^{b^2}, \hat{k})^2y_s \cdot e(k, \hat{k})^{y_s^2})^\alpha.
\end{align*}
\]

It also implicitly sets \( \hat{g} = \hat{k}^{b^2}k^y_s, \hat{h} = \hat{k}^{ab^2}k^{y_u}, \hat{u}_i = \hat{k}^{ab^2}A_i\hat{k}^{y_u} \) for the master key, but it cannot create these elements since \( k^{b^2} \) is not given. Additionally, it sets \( f = k, \hat{f} = \hat{k} \) for the semi-functional ciphertext and
The public parameters are correctly distributed since the random blinding values $y_w, \{y_{u_i}\}$ and $\Gamma(ID) = B + \sum_{i=1}^{m} A_i I_i$ where $ID = (I_1, \ldots, I_m)$. $A$ adaptively requests a private key for $ID = (I_1, \ldots, I_m)$. To response the private key query, $B_1$ first selects random exponents $r_1, c'_1, c'_2, \{c_{3,i}\}_{i=m+1}^{1}$ $\in \mathbb{Z}_p$. It implicitly sets $c_1 = -b(\alpha + \Gamma(ID)r_1)/y_w + c'_1$, $c_2 = -br_1/y_w + c'_2$, $c_{3,i} = -bA_i r_1/y_w + c_{3,i}$ and creates the decryption and delegation components of a private key as

$$K_{1,1} = \hat{k}^{-\alpha + \Delta(ID)r_1} (\hat{w}^\phi)^{c'_1}, K_{1,2} = (K_{1,3})^{\phi_2}, K_{1,3} = (\hat{k}^b)^{-\alpha + \Gamma(ID)r_1} \hat{w}^{c'_2},$$

$$K_{2,1} = \hat{k}^{\gamma r_1} (\hat{w}^\phi)^{c'_2}, K_{2,2} = (K_{2,3})^{\phi_2}, K_{2,3} = (\hat{k}^b)^{-r_1} \hat{w}^{c'_2},$$

$$\{L_{3,i,1} = \hat{k}^{\gamma u_i r_1} (\hat{w}^\phi)^{c'_{2,i}}, L_{3,i,2} = (L_{3,i,3})^{\phi_2}, L_{3,i,3} = (\hat{k}^b)^{-A_i r_1} \hat{w}^{c'_{2,i}}\}_{i=m+1}^{1}.$$ 

It also creates the randomization components of a private key similarly by selecting random exponents $r_2, c'_4, c'_5, \{c'_{6,i}\}_{i=n+1}^{1} \in \mathbb{Z}_p$ except that $R_{1,1}$ does not have $\hat{g}^a$. We omit the detailed description of these. In the challenge step, $A$ submits two challenge hierarchical identities $ID_0 = (I_{0,1}, \ldots, I_{0,n}), ID_1 = (I_{1,1}, \ldots, I_{1,n})$ and two messages $M_0, M_1$. $B_1$ flips a random coin $\gamma \in \{0, 1\}$ internally. It implicitly sets $t = \gamma$ and creates a challenge ciphertext as

$$C = (e(k^{bc}, \hat{k}) \cdot e(k^{bc}, \hat{k})^{y_w} \cdot e(k^c, \hat{k})^2)^{\alpha} \cdot M_\gamma,$$

$$C_{1,1} = k^{bc} (k^c)^{y_w}, C_{1,2} = T(k^{ac})^{y_w}, C_{1,3} = ((k^{bc})^{y_w} (T)^{\phi_2} (k^{ac})^{y_w})^{-1},$$

$$C_{2,2} = (T)^{\Gamma(ID_\gamma)} (k^{ac})^{\Delta(ID_\gamma)}, C_{2,3} = ((k^{bc})^{t\Gamma(ID_\gamma)} (T)^{\phi_2} (k^{ac})^{\Delta(ID_\gamma)}).$$

Finally, $A$ outputs a guess $\gamma'$. If $\gamma = \gamma', B_1$ outputs 0. Otherwise, it outputs 1.

**Analysis.** We first show that the distribution of the simulation using $D, T = T_0 = k^{ac}c$ is the same as $G_0$. The public parameters are correctly distributed since the random blinding values $y_w, y_h, \{y_{u_i}\}$ are used.

The private key is correctly distributed as

$$K_{1,1} = \hat{g}^{\alpha} (\hat{h} \prod_{i=1}^{m} \hat{u}_{i,1}^{r_1} (\hat{w}^\phi)^{c'_1} = (\hat{k}^{b^2 + y_w} \alpha (k^{bc})^{y_w} \prod_{i=1}^{m} k^{(b^2 A_i + y_{u_i}) r_1} (\hat{k}^{b^2 + y_w} - b(\alpha + \Gamma(ID)r_1)/y_w + c'_1)$$

$$= (\hat{k}^{\gamma u_i + \Delta(ID)r_1} (\hat{w}^\phi)^{c'_1},$$

$$K_{2,1} = \hat{g}^{\gamma r_1} (\hat{w}^\phi)^{c'_2}, K_{2,2} = (\hat{k}^{b^2 + y_w} - b r_1/y_w + c'_2, K_{2,3} = \hat{w}^{c'_2},$$

$$L_{3,i,1} = \hat{u}_{i,1}^{r_1} (\hat{w}^\phi)^{c'_{2,i}}, L_{3,i,2} = (L_{3,i,3})^{\phi_2}, L_{3,i,3} = (\hat{k}^b)^{-A_i r_1} \hat{w}^{c'_{2,i}}\}_{i=m+1}^{1}.$$ 

Note that it can create a normal private key since $c_1, c_2, \{c_{3,i}\}, c_4, c_5, \{c_{6,i}\}$ enable the cancellation of $\hat{k}^b$, but it cannot create a semi-functional private key since $\hat{k}^a$ is not given. The challenge ciphertext is correctly distributed as

$$C_{1,1} = g^{\gamma} (k^{b^2 + y_w} c) = k^{bc} (k^c)^{y_w}, C_{1,2} = (g^\nu)^{\gamma} = k^{(b^2 + y_w) ac} = T_0 (k^{ac})^{y_w},$$

$$C_{1,3} = (g^{-\nu})^{\gamma} = (k^{b^2 + y_w} (b + \alpha \phi_2) c)^{-1} = ((k^{bc})^{y_w} (T_0)^{\phi_2} (k^{ac})^{y_w})^{-1},$$

$$C_{2,1} = (h \prod_{i=1}^{n} u_i^{T_{i,1}})^{\gamma} = (k^{b^2 + y_w} \prod_{i=1}^{n} k^{(b^2 A_i + y_{u_i}) T_{i,1}} c = (k^{bc})^{\Gamma(ID_\gamma)} (k^c)^{\Delta(ID_\gamma)},$$

$$C_{2,2} = (h^\nu \prod_{i=1}^{n} (u_i^{T_{i,1}})^{\gamma})^{\gamma} = (k^{b^2 + y_w} \prod_{i=1}^{n} k^{(b^2 A_i + y_{u_i}) u_{i,1}^{T_{i,1}}} c = (T_0)^{\Gamma(ID_\gamma)} (k^{ac})^{\Delta(ID_\gamma)},$$

16
We next show that the distribution of the simulation using \( D, T = T_1 = k^{ab^2c+d} \) is the same as \( G_1 \). We only consider the distribution of the challenge ciphertext since \( T \) is only used in the challenge ciphertext. The only difference between \( T_0 \) and \( T_1 \) is that \( T_1 \) additionally has \( k^d \). Thus \( C_{1,2}, C_{1,3}, C_{2,2}, C_{2,3} \) components that have \( T \) in the simulation additionally have \( (k^d)−θ(c), (k^d)−θ(1), (k^d)−θ(2) \) respectively. If we implicitly set \( s_c = d, z_c = Γ(ID) \), then the challenge ciphertext is semi-functional. The distribution of this semi-functional challenge ciphertext is the same as \( G_1 \) since \( B, \{ A_i \} \) for \( z_c \) are information theoretically hidden to \( \mathcal{A} \). We obtain \( \Pr[B_1(D, T_0) = 0] − 1/2 = \text{Adv}^{G_0}_\mathcal{A} \) and \( \Pr[B_1(D, T_1) = 0] − 1/2 = \text{Adv}^{G_1}_\mathcal{A} \) from the above analysis. Thus, we can easily derive the advantage of \( B \) as

\[
\text{Adv}^{G_1}_\mathcal{A}(\lambda) = \left| \Pr[B_1(D, T_0) = 0] − \Pr[B_1(D, T_1) = 0] \right| = |\text{Adv}^{G_0}_\mathcal{A} − \text{Adv}^{G_1}_\mathcal{A}|.
\]

This completes our proof.

5.2 **Proof of Lemma 3.3 (Indistinguishability of \( G_{1,k−1} \) and \( G'_{1,k} \))**

In this proof, the challenge ciphertext is semi-functional and the \( k \)-th private key should be normal or semi-functional type-1 depending on the \( T \) value of the given assumption. However, the paradox of dual system encryption occurs in this proof since a simulator can create a semi-functional ciphertext to check the type of the \( k \)-th private key by decrypting the semi-functional ciphertext using the \( k \)-the private key. The main idea to solve this paradox is to use a nominally semi-functional type-1 private key. If the \( k \)-th private key is nominally semi-functional type-1, then \( z_{c,1} \) of the nominally semi-functional private key is the same as the \( z_c \) of a semi-functional challenge ciphertext. Thus the simulator cannot distinguish the type of \( k \)-th private key since the decryption of the semi-functional ciphertext using the \( k \)-th private key always succeeds.

Before proving this lemma, we introduce Assumption 2-A as follows: Let \((p, \mathbb{G}, \hat{\mathbb{G}}, G, T, e)\) be a description of the asymmetric bilinear group of prime order \( p \). Let \( g, \hat{g} \) be generators of \( \mathbb{G}, \hat{\mathbb{G}} \) respectively. Assumption 2-A is that if the challenge values \( D = ((p, \mathbb{G}, \hat{\mathbb{G}}, G, T, e), k, k^a, k^b, k^{ab}, k^{abx}, k^{abx}, k^{ax}, k^a, k^b, k^{ay}, k^{by}, k^{by_1}, k^{by_2}, k^{by_1}, k^{by_2}) \) and \( T = (D_1, D_2) \) are given, no PPT algorithm can distinguish \( T = (\hat{k}^{by_1}, \hat{k}^{by_2}) \) from \( T = (k^x, k^y) \) with more than a negligible advantage. It is easy to show that if there exists an adversary that breaks Assumption 2-A, then an algorithm can break Assumption 2 with the same probability by setting \( k^{y_1} = (\hat{k}^b)r_1k^{y_1}, k^{y_2} = (\hat{k}^b)r_2k^{y_2}, D_1 = (T)^{(k^y)}, D_2 = (T)^{(k^y)} \) where \( k^b, k^c, T \) are given in Assumption 2 and \( r_1, r_2, s_1, s_2 \) are random exponents in \( \mathbb{Z}_p \). The simulated values are correctly distributed since there exists one-to-one correspondence between \( \{r_1, r_2, s_1, s_2\} \), \( \{y_1, y_2, d_1, d_2\} \).

**Simulator.** Suppose there exists an adversary \( \mathcal{A} \) that distinguishes between \( G_{1,k−1} \) and \( G'_{1,k} \) with a non-negligible advantage. A simulator \( B_2 \) that breaks Assumption 2-A using \( \mathcal{A} \) is given: a challenge tuple \( D = ((p, \mathbb{G}, \hat{\mathbb{G}}, G, T, e), k, k^a, k^b, k^{ab}, k^{abx}, k^{ax}, k^b, \hat{k}^{y_1}, \hat{k}^{y_2}) \) and \( T = (D_1, D_2) \) where \( T = T_0 = (D_0, D_2) = (k^{by_1}, k^{by_2}) \) or \( T = T_1 = (D_1, D_2) = (k^{by_1} + d_1, k^{by_2} + d_2) \). Then \( B_2 \) interacts with \( \mathcal{A} \) as described as follows: \( B_2 \) first chooses random exponents \( v, y_1, B, \{ A_i \}, \alpha \in \mathbb{Z}_p \) and random blinding values \( y_h, \{ y_w \} \) in \( \mathbb{Z}_p \). It implicitly sets \( \phi_1 = -vb + (a + y_1), \phi_2 = b, \tau = a + y_2 \) and creates the public parameters as

\[
g = k^a, \quad g^v = (k^a)^v, \quad g^{−v} = (k^a^{−v})^{−1},
\]

\[
h = (k^a)^{Bk^a}, \quad h^v = (k^a)^{Bk^a}, \quad h^{−v} = ((k^a)^{−Bk^a})^{−1}.
\]
\[
\begin{align*}
\{u_i = (k^a)^{A_i}k^\alpha, u_i^r = (k^a)^{A_i}k^\alpha, u_i^\tau = ((k^a)^{\gamma_i}(k^a)^{\gamma_i}k^\alpha)^{-1}\}_{i=1}^l, \\
\hat{\psi}^{\phi_i} = ((\hat{k}^b)^{\gamma_i}(\hat{k}^b)^{\gamma_i})^\alpha, \hat{\psi}^{\phi_2} = (\hat{k}^b)^{\gamma_i}, \hat{\omega} = e(k^a, \hat{k}^a)^\alpha.
\end{align*}
\]

It also sets \(\hat{\gamma} = \hat{k}^a, \hat{\psi}^{\phi_i} = (\hat{k}^a)^\alpha, \hat{h} = (\hat{k}^a)^{B_{\mathcal{ID}}}, \{u_i = (k^a)^{A_i}k^\alpha\}_{i=1}^l\) for the master key. Additionally, it sets \(f = k, \hat{f} = \hat{k}\) for the semi-functional ciphertext and private key. Let \(\Delta(ID) = y_h + \sum_{i=1}^m y_u l_i\) and \(\Gamma(ID) = B + \sum_{i=1}^m A_i l_i\) where \(ID = (I_1, \ldots, I_m)\). \(\mathcal{A}\) adaptively requests a private key for \(ID = (I_1, \ldots, I_m)\). If this is a \(j\)-th private key query, then \(B_2\) handles this query as follows:

- **Case** \(j < k\) : It creates a semi-functional private key by calling \textbf{KeyGenSF-2} since it knows the master key and the tuple \((\hat{f}^r, \hat{f}, 1)\) for the semi-functional private key.

- **Case** \(j = k\) : It first selects random exponents \(r'_1, c'_1, c'_2, \{c'_{3,i}\}_{i=m+1}^l \in \mathbb{Z}_p\). It implicitly sets \(r_1 = -y_1 + r'_1, c_1 = y_1 \Gamma(ID)/y_w + c'_1, c_2 = y_1/y_w + c'_2, \{c_{3,i} = y_1 A_i/y_w + c'_{3,i}\}_{i=m+1}^l\) and creates the decryption and delegation components of a private key as

\[
\begin{align*}
K_{1,1} &= \hat{\delta}^{\gamma_1}((\hat{k}^b)^{\gamma_1})^{-\Delta(ID)}(\hat{h}^{\sum_{i=1}^m u_i^r})^{\gamma_1}(D_1)^{-\gamma_1 \gamma(ID)}(\hat{\psi}^{\phi_i})^{c'_1}, \\
K_{1,2} &= (D_1)^{\gamma(ID)}(\hat{\psi}^{\phi_i})^{c'_1}, K_{1,3} = (\hat{k}^b)^{\gamma_1} \Gamma(ID) \hat{\psi}^{c'_1}, \\
K_{2,1} &= \hat{g}^{r'_1}(D_1)^{-\gamma_1 \gamma(ID)}(\hat{\psi}^{\phi_i})^{c'_1}, K_{2,2} = D_1(\hat{\psi}^{\phi_i})^{c'_1}, K_{2,3} = (\hat{k}^b)^{\gamma_1} \hat{\psi}^{c'_1}, \\
L_{3,1} &= (\hat{k}^b)^{\gamma_1} y_u u_i^r (D_1)^{-\gamma_1 \gamma(ID)}(\hat{\psi}^{\phi_i})^{c'_1}, L_{3,2} = (D_1)^{\gamma(ID)}(\hat{\psi}^{\phi_i})^{c'_1}, \\
L_{3,3} &= (\hat{k}^b)^{\gamma_1} \hat{\psi}^{c'_1} \}_{i=m+1}^l.
\end{align*}
\]

It also creates the randomization components of a private key similarly by selecting random exponents \(r''_2, c''_4, c''_5, \{c''_{6,i}\}_{i=m+1}^l \in \mathbb{Z}_p\) except that it uses \(\hat{k}^b, D_2\) instead of \(\hat{k}^b, D_1\). We omit the detailed description of these.

- **Case** \(j > k\) : It creates a normal private key by calling \textbf{KeyGen} since it knows the master key.

In the challenge step, \(\mathcal{A}\) submits two challenge hierarchical identities \(ID_0^* = (I_{0,1}, \ldots, I_{0,n}), ID_1^* = (I_{1,1}, \ldots, I_{1,n})\) and two messages \(M_0^*, M_1^*\). \(B_2\) flips a random coin \(\gamma \in \{0, 1\}\) internally and chooses a random exponent \(t' \in \mathbb{Z}_p\). It implicitly sets \(t = bx + t', s_e = -a^e x, z_e = \Gamma(ID_e^*)\) and creates a semi-functional ciphertext as

\[
C = e(k^{abx}, \hat{k}^a)^{\alpha} \cdot e(k^a, \hat{k}^a)^{\alpha t'} \cdot M_e^*, \\
C_{1,1} = k^{abx} s_e^{r'}, C_{1,2} = (k^{abx})^y (g^y)^{t'} (k^{ax})^{-1}, C_{1,3} = (k^{abx})^{-\gamma_e} (g^{-t})^{t'}, \\
C_{2,1} = (k^{abx})^{\Gamma(ID_e^*)^y} (k^{bx})^{\Delta(ID_e^*)} (h \prod_{i=1}^n u_i^{I_e^*})^{t'}, \\
C_{2,2} = (k^{abx})^{\Gamma(ID_e^*)^y} (k^{bx})^{\Delta(ID_e^*)} (h \prod_{i=1}^n u_i^{I_e^*})^{t'} (k^{ax})^{-\gamma_e}, \\
C_{2,3} = (k^{abx})^{-\gamma_e} (k^{bx})^{-\Delta(ID_e^*)} (h^{-\gamma_e} \prod_{i=1}^n u_i^{I_e^*})^{t'}.
\]

Finally, \(\mathcal{A}\) outputs a guess \(\gamma\). If \(\gamma = \gamma'\), \(B_2\) outputs 0. Otherwise, it outputs 1.

**Analysis.** We first show that the distribution of the simulation using \(D, T_0 = (D_1^0, D_0^0) = (\hat{k}^b x_1, \hat{k}^b y_2)\) is the same as \(G_{1,k-1}\). The public parameters are correctly distributed since the random blinding values
The $k$-th private key is correctly distributed as

$$K_{1,1} = g^a (\prod_{i=1}^{m} d_i^{r_i} (\hat{w}^\Phi)^{c_i}) = g^a (\hat{k}^{(\hat{y} a + \hat{y}_w)} \prod_{i=1}^{m} (\hat{k}^{(a A_i + y_w)}) y_i)^{y_{1,1} + c_i}$$

$$= g^a (\hat{y}_1) - \Delta(ID) (\hat{h} \prod_{i=1}^{m} d_i^{r_i} (D_1^0)^{-v + \Gamma(ID)} (\hat{k}^{(y_1)}) y_1)^{y_{1,1} + c_i} (\hat{w}^\Phi)^{c_i},$$

$$K_{2,1} = g^a (\hat{y}_1) = (\hat{k}^{(y_1)}) y_1 + c_2 = \hat{g}^{r_1} (D_1^0)^{-v + \Gamma(ID)} (\hat{k}^{(y_1)}) y_1 (\hat{w}^\Phi)^{c_2},$$

$$L_{3,i} = \hat{u}_1^{d_i} (\hat{w}^\Phi)^{c_{3,i}} = (\hat{k}^{(a A_i + y_w)}) y_1 + c_3 = \hat{g}^{r_1} (D_1^0)^{-v + \Gamma(ID)} (\hat{k}^{(y_1)}) y_1 (\hat{w}^\Phi)^{c_{3,i}}.$$ 

The semi-functional challenge ciphertext is correctly distributed as

$$C_{1,1} = g^d = (k a^{x + y})^{t - 1} = k^{a x},$$

$$C_{1,2} = (g^a)^y = (k a^{y + x}) k^{-a x} = (k a^{x})^{-1},$$

$$C_{1,3} = (g^a)^y = (k a^{y + y_1}) k^{-a x} = (k a^{x})^{-1},$$

$$C_{2,1} = (h \prod_{i=1}^{n} u_i^{f_i})^{t - 1} = (k a^{x} y_1) \prod_{i=1}^{n} (k a^{y_1}) y_1 \prod_{i=1}^{n} (u_i^{f_i})^{t - 1} k^{-a x} (h \prod_{i=1}^{n} u_i^{f_i})^{t - 1}.$$

Note that it can create the semi-functional ciphertext with only fixed $z_c = \Gamma(ID_y)$ since $s_c, z_c$ enable the cancellation of $k a^{x}$. Even though the simulator uses the fixed $z_c$, the distribution of $z_c$ is correct since $B, \{A_i\}$ for $z_c$ are information theoretically hidden to $A$. We next show that the distribution of the simulation using $D, T_1 = (D_1^0, D_2^0) = (\hat{k}^{y_1 + d_1}, \hat{k}^{y_2 + d_2})$ is the same as $G_{1,k}'$ except the $k$-th private key is nominally semi-functional. We only consider the distribution of the $k$-th private key since $T = (D_1, D_2)$ is only used in the $k$-th private key. The only difference between $T_0 = (D_1^0, D_2^0)$ and $T_1 = (D_1^1, D_2^1)$ is that $T_1 = (D_1^1, D_2^1)$ additionally has $(\hat{k}^{d_1}, \hat{k}^{d_2})$. The decryption and delegation components $K_{1,1}, K_{1,2}, K_{2,1}, K_{2,2}, \{L_{3,i}, L_{3,2}\}$ that have $D_1$ in the simulation additionally have $(\hat{k}^{d_1})^{-v + \Gamma(ID)} (\hat{k}^{d_2}) \Gamma(ID), (\hat{k}^{d_1})^{-v}, (\hat{k}^{d_2}) \Gamma(ID), \{\hat{k}^{d_1}, \hat{k}^{d_2}\} \Gamma(ID)$ respectively. The randomization components $R_{1,1}, R_{1,2}, R_{2,1}, R_{2,2}, \{R_{3,i}, R_{3,2}\}$ that have $D_2$ in the simulation also have the additional values except that $\hat{k}^{d_2}$ is used instead of $\hat{k}^{d_1}$. If we implicitly set $s_{k,1} = d_1, z_{k,1} = \Gamma(ID), \{z_{k,2,i} = A_i\}_{i=1}^{n, j=1}, s_{k,2} = d_2$, then the distribution of the $k$-th private key is the same as $G_{1,k}'$ except that the $k$-th private key is nominally semi-functional type-1.

Finally, we show that the adversary cannot distinguish the nominally semi-functional type-1 private key from the semi-functional type-1 private key. The main idea of this proof is that the adversary cannot request...
a private key for ID that is a prefix of a challenge identity ID\(^*\) in the security model. Suppose there exists an unbounded adversary, then the adversary can gather the values \(z_{k,1} = \Gamma(ID) = B + \sum_{i=1}^{m} A_i l_i, \{z_{k,2,i} = A_1 l_{T,i} \}_{i=m+1}^{l} \) from the \(k\)-th private key query for \(ID = (l_1, \ldots, l_m)\) and \(z_c = \Gamma(ID^*_T) = B + \sum_{i=1}^{n} A_i l_{T,i}^* \) from the challenge ciphertext for \(ID^*_T = (l_{T,1}^*, \ldots, l_{T,m}^*)\). In case of \(n \geq m\), the values that are revealed to the adversary are described as

\[
\begin{pmatrix}
1 & I_{T,1}^* & \cdots & I_{T,m}^* & I_{T,m+1}^* & \cdots & 0 \\
1 & I_1 & \cdots & I_m & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
B \\
A_1 \\
A_m \\
A_{m+1} \\
A_l
\end{pmatrix}
= 
\begin{pmatrix}
z_c \\
z_{k,1} \\
z_{k,2,m+1} \ \\
\vdots \\
z_{k,2,l}
\end{pmatrix}.
\]

It is easy to show that the row rank of the above \((l - m + 2) \times (l + 1)\) matrix is \(l - m + 2\) since there exists an index \(j\) such that \(I_j \neq I^*_j\). It means that the above matrix is non-singular. In case of \(n < m\), the revealed values to the adversary also can be described as a similar matrix equation as the above one. The row rank of this \((l - m + 2) \times (l + 1)\) matrix is \(l - m + 2\) since \(I_m \neq 0\). Therefore these values look random to the unbounded adversary since the matrices for two cases are non-singular and \(B, A_1, \ldots, A_l\) are chosen randomly. We obtain

\[
\Pr[B_2(D, T_0) = 0] - 1/2 = \text{Adv}^{G_1,l-1}_A \quad \text{and} \quad \Pr[B_2(D, T_1) = 0] - 1/2 = \text{Adv}^{G_1,l}_A
\]

from the above analysis. Thus, we can easily derive the advantage of \(B_2\) as

\[
\text{Adv}^{A_2}_{B_2}(\lambda) = |\Pr[B_2(D, T_0) = 0] - \Pr[B_2(D, T_1) = 0]| = |\text{Adv}^{G_1,l-1}_A - \text{Adv}^{G_1,l}_A|.
\]

This completes our proof.

5.3 Proof of Lemma 5.4 (Indistinguishability of G\(_{1,k}\) and G\(_{1,k}\))

In this proof, the challenge ciphertext is semi-functional and the \(k\)-th private key should be semi-functional type-1 or semi-functional type-2 depending on the \(T\) value of the given assumption. The main idea of this proof is to show that the semi-functional type-1 and semi-functional type-2 private keys are computationally indistinguishable using the given assumption.

Before proving this lemma, we introduce Assumption 3-A as follows: Let \((p, G, \hat{G}, G_T, e)\) be a description of the asymmetric bilinear group of prime order \(p\). Let \(g, \hat{g}\) be generators of \(G, \hat{G}\) respectively. Assumption 3-A is that if the challenge values \(D = ((p, G, \hat{G}, G_T, e), k, k_1, k_{S_1}, \tilde{k}_{z_1}, \ldots, k_{z_2}, k_{z_2})\) and \(T = (D_1, D_2, \ldots, D_2)\) are given, no PPT algorithm can distinguish \(T = T_0 = (k_{S_1}, \tilde{k}_{z_1}, \ldots, k_{z_2})\) from \(T = T_1 = (k_{S_1}, k_{S_2}, \ldots, k_{S_2})\) with more than a negligible advantage. It is easy to show that if there exists an adversary that breaks Assumption 3-A, then an algorithm can break Assumption 3 with the same probability by setting \(k_{S_1} = (k_0, \ldots, k_{S_1}), \{k_{z_2}, \ldots, k_{z_2}\}_{i=1}^{I_1}, k_i = (k_0, \ldots, k_i), D_1 = (T_{S_1}, \tilde{k}_{z_1}, \ldots, k_{z_2}), D_2 = (T_{S_2}, \ldots, k_{z_2})_{i=1}^{I_2}\) where \(k, k_0, T\) are given in Assumption 3 and \(r_1, s_1, \{r_2, s_2, \}^{I_1} = \text{random exponents in } \mathbb{Z}_p, \text{ the simulated values are correctly distributed since there exists one-to-one correspondence between } r_1, s_1, \{r_2, s_2\}^{I_1} \text{ and } \{x_1, \{x_2\}, d_1, \{d_2\}\}\).

**Simulator.** Suppose there exists an adversary \(A\) that distinguishes between \(G_{1,k}^*\) and \(G_{1,k}\) with a non-negligible advantage. A simulator \(B_3\) that breaks Assumption 3-A using \(A\) is given: a challenge tuple \(D = ((p, G, \hat{G}, G_T, e), k, \hat{k}, k_{S_1}, k_{S_2}, \ldots, k_{z_2}, \hat{k}_{z_2})\) and \(T = (D_1, \ldots, D_2)\) where \(T = T_0 = (D_0, \ldots, D_0)\) =
\( \langle \hat{k}_{x_{1}}, \hat{k}_{x_{2}}, \ldots, \hat{k}_{x_{l}} \rangle \) or \( T = T_1 = (D_1^1, \ldots, D_1^i) = (\hat{k}_{d_1}, \hat{k}_{d_2}, \ldots, \hat{k}_{d_l}) \). Then \( B_3 \) that interacts with \( A \) is described as follows: \( B_3 \) first chooses random exponents \( \nu, \phi_1, \phi_2, \alpha \in \mathbb{Z}_p \) and random binding values \( y_g, y_h \), \( \{y_{u_i}\}_{i=1}^{l}, y_{\nu} \in \mathbb{Z}_p \). It implicitly sets \( \tau = \phi_1 + \nu \phi_2 \) and sets \( g = k^{\nu}, \hat{g} = \hat{k}^\nu, h = k^\tau, \hat{h} = \hat{k}^\tau, \{u_i = k^{\nu_i}, \hat{u}_i = k^{\nu_i}\}_{i=1}^{l}, \hat{w} = k^\nu \). It creates the public parameters as

\[
PP = (g, g^\nu, g^{-\tau}, h, h^{-\tau}, \{u_i, u_i^\nu, u_i^{-\tau}\}_{i=1}^{l}, \hat{w}^\phi_1, \hat{w}^\phi_2, \hat{w}, \Omega = c(g, \hat{g})^\alpha)
\]

and the master key as \( MK = (\hat{g}, g^\nu, \hat{h}, \{u_i\}_{i=1}^{l}) \). Additionally, it sets \( f = k, \hat{f} = \hat{k} \) for the semi-functional ciphertext and private key. Let \( \Delta(ID) = y_h + \sum_{i=1}^{m} y_u I_i \) where \( ID = (I_1, \ldots, I_m) \). \( A \) adaptively requests a private key for \( ID = (I_1, \ldots, I_m) \). If this is a \( j \)-th private key query, then \( B_3 \) handles this query as follows:

- **Case** \( j < k \): It creates a semi-functional private key by calling \( \text{KeyGenSF-2} \) since it knows the master key and the tuple \( (\hat{f}^{-\nu}, \hat{f}, 1) \) for the semi-functional private key.

- **Case** \( j = k \): It first selects random exponents \( r_1, c_1, c_2, \{c_3,i\}_{i=m+1}^{l}, s_{k,1} \in \mathbb{Z}_p \). It implicitly sets \( z_{k,1} = x_1, \{z_{k,2,i} = x_2,i\}_{i=m+1}^{l} \) and creates the decryption and delegation components of a private key as

\[
K_{1,1} = \hat{g}^\alpha (\hat{h}^{\prod_{i=1}^{m} d_i^r}) (\hat{w}^\phi_1) (\hat{k}^{\tau})^{-\nu s_{k,1}}, K_{1,2} = (\hat{w}^\phi_2)^{c_1} (\hat{k}^{\tau})^{s_{k,1}}, K_{1,3} = \hat{w}^c_1,
\]

\[
K_{2,1} = \hat{g}^\tau (\hat{w}^\phi_1)^{c_2} \hat{k}^{-\nu s_{k,1}}, K_{2,2} = (\hat{w}^\phi_2)^{c_2} \hat{k}^{s_{k,1}}, K_{2,3} = \hat{w}^c_2,
\]

\[
\{L_{3,i,1} = \hat{u}_i^\nu (\hat{w}^\phi_1)^{c_3} (\hat{k}^{\tau})^{-\nu s_{k,1}}, L_{3,i,2} = (\hat{w}^\phi_2)^{c_3} (\hat{k}^{\tau})^{s_{k,1}}, L_{3,i,3} = \hat{w}^{c_3} \}_{i=m+1}^{l}.
\]

Next, it selects random exponents \( r_2, c_4, c_5, \{c_6,i\}_{i=m+1}^{l} \in \mathbb{Z}_p \). It implicitly sets \( s_{k,2} = y \) and creates the randomization components of a private key as

\[
R_{1,1} = (\hat{h}^{\prod_{i=1}^{m} d_i^r}) (\hat{w}^\phi_1)^{c_4} (D_1)^{-\nu}, R_{1,2} = (\hat{w}^\phi_2)^{c_4} D_1, R_{1,3} = \hat{w}^{c_4},
\]

\[
R_{2,1} = \hat{g}^\tau (\hat{w}^\phi_1)^{c_5} (\hat{k}^{\tau})^{-\nu}, R_{2,2} = (\hat{w}^\phi_2)^{c_5} \hat{k}^\nu, R_{2,3} = \hat{w}^{c_5},
\]

\[
\{R_{3,i,1} = \hat{u}_i^{\nu} (\hat{w}^\phi_1)^{c_6} (D_{2,i})^{-\nu}, R_{3,i,2} = (\hat{w}^\phi_2)^{c_6} D_{2,i}, R_{3,i,3} = \hat{w}^{c_6} \}_{i=m+1}^{l}.
\]

- **Case** \( j > k \): It creates a normal private key by calling \( \text{KeyGen} \) since it knows the master key.

In the challenge step, \( A \) submits two challenge hierarchical identities \( ID'_0 = (I'_0, \ldots, I'_0, n), ID'_1 = (I'_1, \ldots, I'_{1,n}) \) and two messages \( M'_0, M'_1 \). \( B_3 \) flips a random coin \( \gamma \in \{0, 1\} \) internally. It creates a semi-functional challenge ciphertext by calling \( \text{EncryptSF} \) on the message \( M \) and the hierarchical identity \( ID'_\gamma \) since it knows the tuple \( (1, f, f^{-\phi_1}) \) for the semi-functional ciphertext. Finally, \( A \) outputs a guess \( \gamma' \). If \( \gamma = \gamma' \), \( B_3 \) outputs 0. Otherwise, it outputs 1.

**Analysis.** We first show that the distribution of the simulation using \( D, T_0 = (D_0^1, \ldots, D_0^{l}) \) is the same as \( G_{1,k}' \). It is easy to check that the private key components are correctly distributed except the randomization components of the \( k \)-th private key. If we implicitly set \( z_{k,1} = x_1, \{z_{k,2,i} = x_2,i\}_{i=m+1}^{l}, s_{k,2} = y \), then the randomization components of the \( k \)-th private key have the same distribution as \( G_{1,k}' \). We next show that the distribution of the simulation using \( D, T_1 = (D_1^1, \ldots, D_1^{l}) \) is the same as \( G_{1,k}' \). We only consider the distribution of the randomization components of the \( k \)-th private key since \( T \) is only used in the randomization
components of the \( k \)-th private key. If we implicitly set \( x_k = y, z_k = d_1/y, \{ z_k, 1 \} = d_2/y \) \( i = m+1 \), then the randomization \( A \) is correctly distributed as

\[
R_{1,1} = (\hat{h} \prod_{i=1}^{m} \hat{a}_i) y (\hat{\phi}_1) e_4 (\hat{f} - v) y x_k z_k = (\hat{h} \prod_{i=1}^{m} \hat{a}_i) y (\hat{\phi}_1) e_4 (\hat{k} - v) y d_1/y = (\hat{h} \prod_{i=1}^{m} \hat{a}_i) y (\hat{\phi}_1) e_4 (D_1) - v,
\]

\[
R_{3,1} = \hat{a}_i (\hat{\phi}_1) e_6 i (\hat{k} - v) y x_k z_k = \hat{a}_i y (\hat{\phi}_1) e_6 (D_2) - v.
\]

From the above analysis, we can obtain \( \Pr[B_3(D, T_0) = 0] = 1/2 = \text{Adv}^{G_{1, k}}_A \) and \( \Pr[B_3(D, T_1) = 0] = 1/2 = \text{Adv}^{G_{1, k}}_A \). Thus, we can easily derive the advantage of \( B_3 \) as

\[
\text{Adv}^{G_{1, k}}_{B_3} (\lambda) = \left| \Pr[B_3(D, T_0) = 0] - \Pr[B_3(D, T_1) = 0] \right| = \left| \text{Adv}^{G_{1, k}}_A - \text{Adv}^{G_{1, k}}_A \right|.
\]

This completes our proof.

### 5.4 Proof of Lemma 3.5 (Indistinguishability of \( G_2 \) and \( G_3 \))

In this proof, private keys and the challenge ciphertext are semi-functional type-2 and semi-functional respectively, but a session key should be correct or random depending on the \( T \) value of the given assumption. The main idea of this proof is to enforce a simulator to solve the Computational Diffie-Hellman (CDH) problem in order to create the normal types of private keys and ciphertexts. However, the simulator can generate the semi-functional types of private keys and ciphertexts since an additional random value in semi-functional types enables the cancellation of the CDH value.

**Simulator.** Suppose there exists an adversary \( A \) that distinguishes between \( G_2 \) and \( G_3 \) with a non-negligible advantage. A simulator \( B_4 \) that breaks Assumption 4 using \( A \) is given: a challenge tuple \( D = ((p, g, \tilde{g}, G_T, e), k, k', k^b, k^c, \tilde{k}, \tilde{k}^b, \tilde{k}^c) \) and \( T \) where \( T = T_0 = e(k, \tilde{k})^{abc} \) or \( T = T_1 = e(k, \tilde{k})^d \). Then \( B_4 \) that interacts with \( A \) is described as follows: \( B_4 \) first chooses random exponents \( \phi_1, \phi_2 \in \mathbb{Z}_p \) and random blinding values \( y_{gk}, y_h, \{ y_{l_i} \} | = 1, y_w \in \mathbb{Z}_p \). It sets \( g = k^{y_g}, h = k^{y_h}, \{ u_i = k^{y_{l_i}} \} | = 1, \tilde{g} = \tilde{k}^{y_g}, \tilde{h} = \tilde{k}^{y_h}, \{ \tilde{u}_i = \tilde{k}^{y_{l_i}} \} | = 1, \tilde{w} = \tilde{k}^{y_w} \). It implicitly sets \( \nu = a, \tau = \phi_1 + a \phi_2, \alpha = ab \) and creates the public parameters as

\[
g, g^\nu = (k^a)^{\nu}, g^- = k^{-\phi_1} (k^a)^{\nu}, h, h^\nu = (k^a)^{\nu}, h^- = k^{-\phi_1} (k^a)^{\nu}, \phi_2,
\]

\[
\{ u_i, u_i^\nu = (k^a)^{\nu}, u_i^- = k^{-\phi_1} (k^a)^{\nu}, \} | = 1, \tilde{w} \phi_1, \tilde{w} \phi_2, \tilde{w}, \Omega = e(k^a, \tilde{k}^b)^{\nu}.
\]

Additionally, it sets \( f = k, \hat{f} = \tilde{k} \) for the semi-functional ciphertext and private key. Let \( \Delta(ID) = y_h + \sum_{i=1}^{m} y_{u_i} t_i \) where \( ID = (I_1, \ldots, I_m) \). \( A \) adaptively requests a private key for \( ID = (I_1, \ldots, I_m) \). To respond the private key query, \( B_4 \) first selects random exponents \( r_1, c_1, c_2, \{ c_3 \} \{ i = m+1 \}, s_{k,1}, z_{k,1}, \{ z_{k,2} \} \{ i = m+1 \} \in \mathbb{Z}_p \). It implicitly sets \( z_{k,1} = b y_g/s_{k,1} + z_{k,1} \) and creates the decryption and delegation components of a semi-functional private key as

\[
K_{1,1} = (\hat{h} \prod_{i=1}^{m} \hat{a}_i) e_4 (\hat{\phi}_1) e_4 (\hat{k}^{s_{k,1}}), K_{1,2} = (\hat{w} \phi_2) e_4 (\hat{k}^{s_{k,1}}), K_{1,3} = \hat{w} e_i,
\]

\[
K_{2,1} = \hat{g} (\hat{\phi}_2) e_4 (\hat{k}^{s_{k,1}}), K_{2,2} = (\hat{\phi}_2) e_4 (\hat{k}^{s_{k,1}}), K_{2,3} = \hat{w} e_i,
\]

\[
\{ L_{3,1,1} = \hat{a}_i e_4 (\hat{\phi}_1) e_4 (\hat{k}^{s_{k,1}}), \{ L_{3,1,2} = (\hat{\phi}_2) e_4 (\hat{k}^{s_{k,1}}), L_{3,1,3} = \hat{w} e_i \} \{ i = m+1 \}.
\]

Next, it selects random exponents \( r_2, c_4, c_5, \{ c_6 \} \{ i = m+1 \}, s_{k,2}, z_{k,2}, \{ z_{k,4} \} \{ i = m+1 \} \in \mathbb{Z}_p \) and creates the randomization components of a semi-functional private key. In the challenge step, \( A \) submits two challenge hierarchical identities \( ID_0 = (I_{0,1}, \ldots, I_{0,n}), ID_1 = (I_{1,1}, \ldots, I_{1,n}) \) and two messages \( M_0, M_1 \). \( B_4 \) flips a random coin.
γ ∈ \{0, 1\} internally and chooses random exponents \(s'_c, z'_c \in \mathbb{Z}_p\). It implicitly sets \(t = c, s_c = -ac_y + s'_c, z_c = -ac(\Delta(ID_r)/s_c + z'_c/s_c)\) and creates the semi-functional ciphertext as

\[
C = (T)^{s'_c} \cdot M^*_r, \quad C_{1,1} = (k^c)^{s'_c}, \quad C_{1,2} = k^{s'_c}, \quad C_{1,3} = (k^c)^{-\gamma_k \phi_k - \phi_{a'}}, \\
C_{2,1} = (k^c)^{\Delta(ID_r)}, \quad C_{2,2} = k^{s'_c}, \quad C_{2,3} = (k^c)^{-\Delta(ID_r) \phi_k - \phi_{a'}\gamma_k}.
\]

Finally, \(A\) outputs a guess \(\gamma'\). If \(\gamma = \gamma'\), \(B_4\) outputs 0. Otherwise, it outputs 1.

**Analysis.** We first show that the distribution of the simulation using \(D, T_0 = e(k, \hat{k})^{abc}\) is the same as \(G_2\). The public parameters are correctly distributed since the random blinding values \(y_g, y_h, \{y_u\}, y_w\) are used. The semi-functional private key is correctly distributed as

\[
K_{1,1} = g^t (\hat{h} \prod_{i=1}^{m} \hat{a}_i^{t_i})(\hat{w} \Phi_k)^{c_i} (\hat{f}^{-y_c}) (s_{k_1} + z_{k_1}) = k^{s_{k_1}} (\hat{h} \prod_{i=1}^{m} \hat{a}_i^{t_i})(\hat{w} \Phi_k)^{c_i} (\hat{k}^{-a}) (s_{k_1} + z_{k_1}) + y_{k_1} + z'_{k_1}.
\]

Note that it can only create a semi-functional private key since \(s_{k_1} = y_{k_1} + z_{k_1} + z'_{k_1}\) enables the cancellation of \(k^{ab}\). The semi-functional challenge ciphertext is correctly distributed as

\[
C = e(g, \hat{g})^{t} (\hat{h} \prod_{i=1}^{m} \hat{a}_i^{t_i})(\hat{w} \Phi_k)^{c_i} (\hat{f}^{-y_c}) (s_{k_1} + z_{k_1}) = k^{s_{k_1}} M^*_r,
\]

\[
C_{1,1} = g^t = (k^c)^{s_{k_1}}, \quad C_{1,2} = (g^t)^{s_{k_1}} = (k^c)^{s_{k_1}} = k^{s_{k_1}}, \quad C_{1,3} = (g^{-y_k})^t (\hat{f}^{-y_c}) = (k^{-y_k} - \phi_{a'})(k^{s_{k_1} + z_{k_1}}) = (k^c)^{-y_k \phi_k - \phi_{a'}},
\]

\[
C_{2,1} = (\hat{h} \prod_{i=1}^{n} (\hat{u}_i^{t_i}))^t = (k^{s_{k_1}} \prod_{i=1}^{n} \hat{u}_i^{t_i})^t = (k^c)^{\Delta(ID_r)},
\]

\[
C_{2,2} = (\hat{h} \prod_{i=1}^{n} (\hat{u}_i^{-y_c} \phi_{a'})) = (k^{s_{k_1}} \prod_{i=1}^{n} \hat{u}_i^{t_i})^t = (k^c)^{-\Delta(ID_r) \phi_k - \phi_{a'}}.
\]

Note that it can create a semi-functional ciphertext since \(s_c, z_c\) enable the cancellation of \(k^{ac}\). We next show that the distribution of the simulation using \(D, T_1 = e(k, \hat{k})^d\) is the same as \(G_3\). It is obvious that \(C\) is a random element since \(T_1 = e(k, \hat{k})^d\). From the above analysis, we obtain \(\Pr[B_4(D, T_0) = 0] - 1/2 = \text{Adv}^{G_2}_A\) and \(\Pr[B_4(D, T_1) = 0] - 1/2 = \text{Adv}^{G_3}_A\). Thus, we can easily derive the advantage of \(B_4\) as

\[
\text{Adv}^{G_4}_{B_4}(\lambda) = |\Pr[B_4(D, T_0) = 0] - \Pr[B_4(D, T_1) = 0]| = |\text{Adv}^{G_2}_A - \text{Adv}^{G_3}_A|.
\]

This completes our proof.

### 5.5 Proof of Lemma 3.6 (Indistinguishability of \(G_3\) and \(G_4\))

In this proof, private keys and the challenge ciphertext are semi-functional type-2 and semi-functional respectively, and the elements of the challenge ciphertext should be well-formed or random depending on...
the $T$ value of the given assumption. The idea to generate semi-functional type-2 private keys and semi-functional ciphertexts is similar to Lemma 3.5 but it uses a different assumption. To prove anonymity, the simulator embeds the $T$ value of the assumption into the all elements of the challenge ciphertext that contains an identity.

**Simulator.** Suppose there exists an adversary $A$ that distinguishes between $G_3$ and $G_4$ with a non-negligible advantage. A simulator $B_5$ that breaks Assumption 5 using $A$ is given: a challenge tuple $D = ((p, \mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T, e), k, k^a, k^b, k^c, k^{ab}, k^{ac}, k^{bc})$ and $T$ where $T = T_0 = k^{abc}$ or $T = T_1 = k^d$. Then $B_5$ that interacts with $A$ as follows: $B_5$ first chooses random exponents $\phi_1, \phi_2, \alpha \in \mathbb{Z}_p$ and random blinding values $y_g, y_h, \{y_{ui}\}_{i=1}^l, y_w \in \mathbb{Z}_p$. It sets $g = k^{x_1}, h = (k^{ab})^{y_h}, \{u_i = (k^{ab})^{y_{ui}}\}_{i=1}^l$, $\hat{g} = k^{x_1}, \hat{h} = k^{y_h}, \hat{g}^\alpha = \hat{k}^{y_\alpha}$. It implicitly sets $v = a, \tau = \phi_1 + a\phi_2$ and publishes the public parameters as

$$g, g^\nu = (k^a)^{y_g}, g^{-\tau} = k^{-y_\phi_1}(k^a)^{-y_\phi_2}, h, h^\nu = (k^{ab})^{y_h}, h^{-\tau} = (k^{ab})^{-y_\phi_1}(k^{ab})^{-y_\phi_2},$$

$$\{u_i, u_i^\nu = (k^{ab})^{y_{ui}}, u_i^{-\tau} = (k^{ab})^{-y_\phi_1}(k^{ab})^{-y_\phi_2}\}_{i=1}^l, \hat{w}^\phi_1, \hat{w}^\phi_2, \hat{w}, \Omega = e(k, \hat{k})^{y_\alpha}.$$ 

It also implicitly sets $\hat{h} = (k^{ab})^{y_h}, \{\hat{u}_i = (k^{ab})^{y_{ui}}\}$ for the master key, but it cannot create these values since $k^{ab}$ is not given. Additionally, it sets $f = k, \hat{f} = \hat{k}$ for the semi-functional ciphertext and private key. Let $\Delta(ID) = y_h + \sum_{i=1}^l y_{ui}l_i$ where $ID = (l_1, \ldots, l_m)$. A adaptively requests a private key for $ID = (l_1, \ldots, l_m)$. To response the private key query, $B_5$ first selects random exponents $r_1, c_1, c_2, \{c_{3,i}\}_{i=m+1}^{3m}, s_{1,1}, \{s_{1,2}\}_{i=m+1}^{3m} \in \mathbb{Z}_p$. It implicitly sets $z_{k,1} = b\Delta(ID)r_1/s_{1,1} + z_{k,1}', \{z_{k,2,i} = b\Delta(ID)r_1/s_{1,1} + z_{k,2,i}'\}_{i=m+1}^{3m}$ and creates the decryption and delegation components of a semi-functional private key as

$$K_{1,1} = \hat{g}^\alpha (\hat{w}^\phi_1)^{c_1} (\hat{k})^{s_{1,1}}, K_{1,2} = (\hat{w}^\phi_1)^{c_1} (\hat{k})^{\Delta(ID)r_1} \hat{k}^{s_{1,1}}, K_{1,3} = \hat{w}^{c_1},$$

$$K_{2,1} = \hat{g}^{c_1} (\hat{w}^\phi_2)^{c_2} \hat{k}^{s_{1,1}}, K_{2,2} = (\hat{w}^\phi_1)^{c_2} \hat{k}^{s_{1,1}}, K_{2,3} = \hat{w}^{c_2},$$

$$\{L_{3,1,1} = (\hat{w}^\phi_1)^{c_3} (\hat{k})^{s_{1,1}}, L_{3,1,2} = \hat{k}^{s_{1,2}}, L_{3,2,2} = (\hat{w}^\phi_2)^{c_3} (\hat{k})^{s_{1,1}}, L_{3,3,3} = \hat{w}^{c_3}\}_{i=m+1}^{3m}.$$ 

Next, it selects random exponents $r_2, c_4, c_5, \{c_{6,1,i} = m+1, s_{2,1,2,1} = m+1, \{z_{k,4,1}' = b\Delta(ID)r_2/s_{2,1} + z_{k,4,1}'\}_{i=m+1}^{3m}$. We omit the detailed description of these, since these are similar to the decryption and delegation components except that $R_{1,1}$ does not have $\hat{g}^\alpha$. In the challenge step, $A$ submits two challenge hierarchical identities $ID_0 = (I_{0,1}^1, \ldots, I_{0,1}^n), ID_1 = (I_{1,1}^1, \ldots, I_{1,1}^n)$ and two messages $M_0^*, M_1^*$. $B_5$ flips a random coin $\gamma \in \{0, 1\}$ internally and chooses random exponents $\delta, \delta', \delta'' \in \mathbb{Z}_p$. It implicitly sets $t = c, s_c = -acy_g + \delta', z_c = -abu_x + abc', s_c = -ac\delta' + \delta''$ and creates the semi-functional ciphertext as

$$C = \Omega^\delta \cdot M_\gamma^*, C_{1,1} = (k^c)^{y_g}, C_{1,2} = (k^a)^{\delta'}, C_{1,3} = (k^c)^{-y_\phi_1}k^{-\phi_2}g, C_{2,1} = (T)^{\Delta(ID_0)}, C_{2,2} = (T)^{\delta'}, C_{2,3} = (T)^{-\Delta(ID_1)}\phi_1 (T)^{-\phi_2}.$$ 

Finally, $A$ outputs a guess $\gamma'$. If $\gamma = \gamma', B_5$ outputs 0. Otherwise, it outputs 1.

**Analysis.** We first show that the distribution of the simulation using $D, T_0 = k^{abc}$ is the same as $G_3$. The public parameters are correctly distributed since the random blinding values are used. The semi-functional private key is correctly distributed as

$$K_{1,1} = \hat{g}^\alpha \left( \prod_{i=1}^m d_i \right) r_1 (\hat{w}^\phi_1)^{c_1} (\hat{k})^{-y_\alpha} s_{1,1}z_{1,1} = \hat{g}^\alpha (\hat{k})^{\Delta(ID)r_1} (\hat{w}^\phi_1)^{c_1} (\hat{k})^{-y_\alpha} s_{1,1}(b\Delta(ID)r_1/s_{1,1} + z_{1,1}').$$

24
\[ K_{2.1} = g^{c_1} (\tilde{\psi})^{c_2} (\tilde{f}^{-1})^{y_{k,1}} = g^{c_1} (\tilde{\psi})^{c_2} (\tilde{k})^{-s_{k,1}}, \]
\[ L_{3.1} = \tilde{u}_i^{c_1} (\tilde{\psi})^{c_2} (\tilde{f}^{-1})^{y_{k,1}} = (\tilde{k})^{-s_{k,1}} \cdot (\tilde{u}_i^{c_1} (\tilde{\psi})^{c_2} (\tilde{k})^{-s_{k,1}} (b y_{r,1} / s_{k,1} + z_{k,2,i})), \]
\[ = (\tilde{\psi})^{c_2} (\tilde{k})^{-s_{k,1}} z_{k,2,i}. \]

Note that it can only create a semi-functional type-2 private key since \( z_{k,1}, \{z_{k,2,i}\}, z_{k,3}, \{z_{k,4,i}\} \) enable the cancellation of \( \tilde{k}^{ab} \). The semi-functional challenge ciphertext is correctly distributed as

\[ C_{1.1} = g' = (k'^y)^c = (k'^y)_s, \]
\[ C_{1.2} = (g^y)^r f_{x_c} = (k'^a)^c k^{-acy_x + s'_c} = k'^c, \]
\[ C_{1.3} = (g^{-r})^t (f^{-v_2})^{x_c} = (k^{-y_x (\Phi_1 + a \Phi_2)})^{-c} k^{-\Phi_2 (-acy_x + s'_c)} = (k^c)^{-y_x \Phi_1} k^{-\Phi_2 s'_c}, \]
\[ C_{2.1} = (h \prod_{i=1}^n u_i F_{i})^{y} = (k^{ab})^{\Delta (ID_v)} c = (T_0)^{\Delta (ID_v)}, \]
\[ C_{2.2} = (h \prod_{i=1}^n u_i F_{i})^{y} = (k^{ab})^{\Delta (ID_v)} = (T_0)^{\Delta (ID_v)}, \]
\[ C_{2.3} = (h \prod_{i=1}^n u_i F_{i})^{y} \cdot (f^{-v_2})^{x_c} = (k^{ab})^{\Delta (ID_v)} c (k^{1} + a \Phi_2) c k^{-\Phi_2 s_c} (-a^2 b c \Delta (ID_v))/s_c + ab z'_c/s_c, \]
\[ = (T_0)^{-\Delta (ID_v)} \Phi_1 (T_0)^{-s'_c \Phi_2}. \]

Note that it can only create a semi-functional ciphertext since \( s_c, z_c \) enable the cancellation of \( k'^2 bc \). We next show that the distribution of the simulation using \( D, T_1 = k^d \) is the same as \( G_4 \). We only consider \( C_{2.1}, C_{2.2}, C_{2.3} \) components of the semi-functional challenge ciphertext since \( T \) is used for these components. If we implicitly sets \( P = k^{\Delta (ID_v) d/c} \) and \( z_c = -a d \Delta (ID_v)/s_c + d z'_c/s_c \), then the semi-functional challenge ciphertext is correctly distributed as

\[ C_{2.1} = P^c = (k^{\Delta (ID_v) d/c})^c = (T_1)^{\Delta (ID_v)}, \]
\[ C_{2.2} = P^{vc} f_{x_c} z_c = (k^{\Delta (ID_v) d/c})^{ac} k^{x_c} (-a^2 b c \Delta (ID_v))/s_c + d z'_c/s_c = (T_1)^{s'_c}, \]
\[ C_{2.3} = P^{-vc} (f^{-v_2})^{x_c} z_c = (k^{\Delta (ID_v) d/c})^{ac} (\Phi_1 + a \Phi_2) c k^{-\Phi_2 s_c} (-a^2 b c \Delta (ID_v))/s_c + ab z'_c/s_c, \]
\[ = (T_1)^{-\Delta (ID_v) \Phi_1} (T_1)^{-s_c \Phi_2}. \]

From the above analysis, we obtain \( \Pr[B_5(D, T_0) = 0] - 1/2 = \text{Adv}_{\lambda}^{G_3} \) and \( \Pr[B_5(D, T_1) = 0] - 1/2 = \text{Adv}_{\lambda}^{G_4} \). Thus, we can easily derive the advantage of \( B_5 \) as

\[ \text{Adv}_{B_5}^{G_5}(\lambda) = | \Pr[B_5(D, T_0) = 0] - \Pr[B_5(D, T_1) = 0] | = | \text{Adv}_{\lambda}^{G_3} - \text{Adv}_{\lambda}^{G_4} |. \]

This completes our proof.

## 6 Generic Group Model

In this section, we prove that the new assumption of this paper is secure under the generic group model. The generic group model was introduced by Shoup [43], and it is a tool for analyzing generic algorithms that work independently of the group representation. In the generic group model, an adversary is given a random encoding of a group element or an arbitrary index of a group element instead of the actual representation of a group element. Thus, the adversary performs group operations through oracles that are provided by a simulator, and the adversary only can check the equality of group elements. The detailed explanation of the generic group model is given in [7] [28].

25
6.1 Master Theorem

To analyze the new assumption of this paper, we slightly modify the master theorem of Katz et al. [28] since the new assumption is defined over asymmetric bilinear groups of prime order. Let $\mathbb{G}, \hat{\mathbb{G}}, \mathbb{G}_T$ be asymmetric bilinear groups of prime order $p$. The bilinear map is defined as $e : \mathbb{G} \times \hat{\mathbb{G}} \rightarrow \mathbb{G}_T$. In the generic group model, a random group element of $\mathbb{G}, \hat{\mathbb{G}}, \mathbb{G}_T$ is represented as a random variable $P_i, Q_i, R_i$ respectively where $P_i, Q_i, R_i$ are chosen uniformly in $\mathbb{Z}_p$. We say that a random variable has degree $t$ if the maximum degree of any variable is $t$. The generalized definition of dependence and independence is given as follows:

**Definition 6.1.** Let $P = \{P_1, \ldots, P_u\}$, $T_0, T_1$ be random variables over $\mathbb{G}$ where $T_0 \neq T_1$, let $Q = \{Q_1, \ldots, Q_w\}$ be random variables over $\hat{\mathbb{G}}$, and let $R = \{R_1, \ldots, R_v\}$ be random variables over $\mathbb{G}_T$. Let $l = \max\{u, w, v\}$. We say that $T_b$ is dependent on $P$ if there exists constants $\alpha, \{\beta_i\}$ such that

$$\alpha \cdot T_b = \sum_{i=1}^{u} \beta_i \cdot P_i$$

where $\alpha \neq 0$. We say that $T_b$ is independent of $P$ if $T_b$ is not dependent on $P$. We say that $\{e(T_b, Q_i)\}_i$ is dependent on $P \cup Q \cup R$ if there exist constants $\{\alpha_i\}, \{\beta_{i,j}\}, \{\gamma_i\}$ such that

$$\sum_{i=1}^{u} \alpha_i \cdot e(T_b, Q_i) = \sum_{i=1}^{u} \sum_{j=1}^{w} \beta_{i,j} \cdot e(P_i, Q_j) + \sum_{i=1}^{v} \gamma_i \cdot R_i$$

where $\alpha_i \neq 0$ for at least one $i$. We say that $\{e(T_b, Q_i)\}_i$ is independent of $P \cup Q \cup R$ if $\{e(T_b, Q_i)\}_i$ is not dependent on $P \cup Q \cup R$.

We can obtain the following theorem by using the above dependence and independence of random variables.

**Theorem 6.2.** Let $P = \{P_1, \ldots, P_u\}$, $T_0, T_1$ be random variables over $\mathbb{G}$ where $T_0 \neq T_1$, let $Q = \{Q_1, \ldots, Q_w\}$ be random variables over $\hat{\mathbb{G}}$, and let $R = \{R_1, \ldots, R_v\}$ be random variables over $\mathbb{G}_T$. Let $l = \max\{u, w, v\}$. Consider the following experiment in the generic group model:

An algorithm is given $P = \{P_1, \ldots, P_u\}, Q = \{Q_1, \ldots, Q_w\},$ and $R = \{R_1, \ldots, R_v\}$. A random bit $b$ is chosen, and the adversary is given $T_b$. The algorithm outputs a bit $b'$, and succeeds if $b' = b$. The algorithm’s advantage is the absolute value of the difference between its success probability and $1/2$.

If $T_b$ is independent of $P$ for all $b \in \{0, 1\}$, and $\{e(T_b, Q_j)\}_j$ is independent of $P \cup Q \cup R$ for all $b \in \{0, 1\}$, then any algorithm $\mathcal{A}$ issuing at most $q$ instructions has an advantage at most $3(q + 2l)^2t/p$.

**Proof.** The proof consists of a sequence of games. The first game will be the original experiment that is described in the theorem and the last game will be a game that the algorithm has no advantage. We define the games as follows:

**Game $G_1$.** This game is the original game. In this game, the simulator instantiates each of random variables $P, Q, R, T_b$ by choosing random values for each of the formal variables. Then it gives the handles of $P, Q, R, T_b$ to the algorithm $\mathcal{A}$. Next, $\mathcal{A}$ requests a sequence of multiplication, exponentiation, and pairing instructions, and is given the handles of results. Finally, $\mathcal{A}$ outputs a bit $b'$. 
Game $G_2$. We slightly modify $G_1$ into a new game $G_2$. In this game, the simulator never concretely instantiates the formal variables. Instead it keeps the formal polynomials themselves. Additionally, the simulator gives identical handles for two elements only if these elements are equal as formal polynomials in each of their components. That is, the simulator of this game assigns different handles for $X$ and $Y$ since these are different polynomials. Note that the simulator of $G_1$ assigned the same handle for $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_n)$ if $X_i = Y_i$ for all $i$.

To prove the theorem, we will show that the statistical distance between two games $G_1$ and $G_2$ is negligible and the advantage of the algorithm in $G_2$ is zero. Then the advantage of the algorithm in the original game is bounded by the statistical distance between two games.

We first show that the statistical distance between two games $G_1$ and $G_2$ is negligible. The only difference between two games is the case that two different formal polynomials take the same value by concrete instantiation. The probability of this event is at most $t/p$ from the Schwartz-Zippel Lemma [40]. If we consider all pairs of elements produced by the algorithm $A$, the statistical distance between two games is at most $3(q + 2l)^2t/p$ since $A$ can request at most $q$ instructions, the maximum size of handles in each group is at most $q + 2l$, and there are three different groups.

We next show that the advantage of the algorithm in $G_2$ is zero. In this game, the algorithm $A$ only can distinguish whether it is given $T_0$ or $T_1$ if it can generate a formal polynomial that is symbolically equivalent to some previously generated polynomial for one value of $b$ but not the other. In this case, we have $\alpha \cdot T_b = \sum_{i=1}^{m} \beta_i \cdot P_i$ where $\alpha \neq 0$, or else we have $\sum_{i=1}^{m} \alpha_i \cdot e(T_b, Q_i) = \sum_{i=1}^{m} \sum_{j=1}^{n} \beta_{i,j} \cdot e(P_i, Q_j) + \sum_{i=1}^{m} \gamma_i \cdot R_i$ where $\alpha_i \neq 0$ for at least one $i$ (otherwise, symbolic equality would hold for both value of $b$). However, the above equations are contradict to the independence assumptions of the theorem. Therefore, the advantage of $A$ in this game is zero. 

6.2 Analysis of Asymmetric 3-Party Diffie-Hellman

To apply the master theorem of the previous section, we only need to show the independence of $T_0, T_1$ random variables. Using the notation of previous section, Assumption 5 (Asymmetric 3-Party Diffie-Hellman) can be written as

$$P = \{1, A, B, C, AB, A^2B\}, \quad Q = \{1, A, B\}, \quad R = \{1\}, \quad T_0 = ABC, \quad T_1 = D.$$

At first, we show the independence of $T_1$. It is trivial that $T_1$ is independent of $P$ since a random variable $D$ does not exist in $P$. It is easy to show that $\{e(T_1, Q_i)\}_i$ is independent of $P \cup Q \cup R$ since $T_1$ contains a random variable $D$ that does not exist in $P, Q, R$. Next, we show the independence of $T_0$. It is easy to show that $T_0$ is independent of $P$ since the random variables with degree 3 are different. To show the independence of $\{e(T_0, Q_i)\}_i$, we can derive the sets of random variables as

$$\{e(T_0, Q_i)\}_i = \{ABC, A^2BC, AB^2C\},$$
$$\{e(P_i, Q_j)\}_{i,j} = \{1, A, B, C, AB, A^2B, AC, A^2B, B^2, BC, AB^2, A^2B^2\},$$
$$\{R_i\} = \{1\}.$$

The random variables of $\{e(T_0, Q_i)\}_i$ always contain $C$ and the degree of these random variables is greater than 3. However, the random variables of $\{e(P_i, Q_j)\}_{i,j}$ that contain $C$ have the degree at most 2. Thus $\{e(T_0, Q_i)\}_i$ is independent of $P \cup Q \cup R$. 

27
7 Conclusion

In this paper, we proposed an efficient anonymous HIBE scheme with short ciphertexts and proved its full model security under static assumptions. Though our construction is based on the IBE scheme of Lewko and Waters [33], it was not trivial to construct an anonymous HIBE scheme, since the randomization components of private keys cause a problem in the security proof of dual system encryption. We leave it as an interesting problem to construct a fully secure and anonymous HIBE scheme with short ciphertexts under standard assumptions.

References

[1] Michel Abdalla, Mihir Bellare, Dario Catalano, Eike Kiltz, Tadayoshi Kohno, Tanja Lange, John Malone-Lee, Gregory Neven, Pascal Paillier, and Haixia Shi. Searchable encryption revisited: Consistency properties, relation to anonymous ibe, and extensions. In Victor Shoup, editor, CRYPTO, volume 3621 of Lecture Notes in Computer Science, pages 205–222. Springer, 2005.

[2] Michel Abdalla, Mihir Bellare, and Gregory Neven. Robust encryption. In Daniele Micciancio, editor, TCC, volume 5978 of Lecture Notes in Computer Science, pages 480–497. Springer, 2010.

[3] Shweta Agrawal, Dan Boneh, and Xavier Boyen. Efficient lattice (h)ibe in the standard model. In Henri Gilbert, editor, EUROCRYPT, volume 6110 of Lecture Notes in Computer Science, pages 553–572. Springer, 2010.

[4] Shweta Agrawal, Dan Boneh, and Xavier Boyen. Lattice basis delegation in fixed dimension and shorter-ciphertext hierarchical ibe. In Tal Rabin, editor, CRYPTO, volume 6223 of Lecture Notes in Computer Science, pages 98–115. Springer, 2010.

[5] Dan Boneh and Xavier Boyen. Efficient selective-id secure identity-based encryption without random oracles. In Christian Cachin and Jan Camenisch, editors, EUROCRYPT, volume 3027 of Lecture Notes in Computer Science, pages 223–238. Springer, 2004.

[6] Dan Boneh and Xavier Boyen. Efficient selective identity-based encryption without random oracles. J. Cryptology, 24(4):659–693, 2011.

[7] Dan Boneh, Xavier Boyen, and Eu-Jin Goh. Hierarchical identity based encryption with constant size ciphertext. In Ronald Cramer, editor, EUROCRYPT, volume 3494 of Lecture Notes in Computer Science, pages 440–456. Springer, 2005.

[8] Dan Boneh, Giovanni Di Crescenzo, Rafail Ostrovsky, and Giuseppe Persiano. Public key encryption with keyword search. In Christian Cachin and Jan Camenisch, editors, EUROCRYPT, volume 3027 of Lecture Notes in Computer Science, pages 506–522. Springer, 2004.

[9] Dan Boneh and Matthew K. Franklin. Identity-based encryption from the weil pairing. In Joe Kilian, editor, CRYPTO, volume 2139 of Lecture Notes in Computer Science, pages 213–229. Springer, 2001.

[10] Dan Boneh and Matthew K. Franklin. Identity-based encryption from the weil pairing. SIAM J. Comput., 32(3):586–615, 2003.
[11] Dan Boneh and Michael Hamburg. Generalized identity based and broadcast encryption schemes. In Josef Pieprzyk, editor, ASIACRYPT, volume 5350 of Lecture Notes in Computer Science, pages 455–470. Springer, 2008.

[12] Dan Boneh and Brent Waters. Conjunctive, subset, and range queries on encrypted data. In Salil P. Vadhan, editor, TCC, volume 4392 of Lecture Notes in Computer Science, pages 535–554. Springer, 2007.

[13] Xavier Boyen and Brent Waters. Anonymous hierarchical identity-based encryption (without random oracles). In Cynthia Dwork, editor, CRYPTO, volume 4117 of Lecture Notes in Computer Science, pages 290–307. Springer, 2006.

[14] Ran Canetti, Shai Halevi, and Jonathan Katz. A forward-secure public-key encryption scheme. In Eli Biham, editor, EUROCRYPT, volume 2656 of Lecture Notes in Computer Science, pages 255–271. Springer, 2003.

[15] Ran Canetti, Shai Halevi, and Jonathan Katz. Chosen-ciphertext security from identity-based encryption. In Christian Cachin and Jan Camenisch, editors, EUROCRYPT, volume 3027 of Lecture Notes in Computer Science, pages 207–222. Springer, 2004.

[16] Angelo De Caro, Vincenzo Iovino, and Giuseppe Persiano. Fully secure anonymous hibe and secret-key anonymous ibe with short ciphertexts. In Marc Joye, Atsuko Miyaji, and Akira Otsuka, editors, Pairing, volume 6487 of Lecture Notes in Computer Science, pages 347–366. Springer, 2010.

[17] David Cash, Dennis Hofheinz, Eike Kiltz, and Chris Peikert. Bonsai trees, or how to delegate a lattice basis. In Henri Gilbert, editor, EUROCRYPT, volume 6110 of Lecture Notes in Computer Science, pages 523–552. Springer, 2010.

[18] Sanjit Chatterjee and Palash Sarkar. Hibe with short public parameters without random oracle. In Xuejia Lai and Kefei Chen, editors, ASIACRYPT, volume 4284 of Lecture Notes in Computer Science, pages 145–160. Springer, 2006.

[19] Jie Chen, Hoon Wei Lim, San Ling, and Huaxiong Wang. The relation and transformation between hierarchical inner product encryption and spatial encryption. Cryptology ePrint Archive, Report 2011/455, 2011. http://eprint.iacr.org/2011/455.

[20] Yevgeniy Dodis and Nelly Fazio. Public key broadcast encryption for stateless receivers. In Joan Feigenbaum, editor, Digital Rights Management Workshop, volume 2696 of Lecture Notes in Computer Science, pages 61–80. Springer, 2002.

[21] Léo Ducas. Anonymity from asymmetry: New constructions for anonymous hibe. In Josef Pieprzyk, editor, CT-RSA, volume 5985 of Lecture Notes in Computer Science, pages 148–164. Springer, 2010.

[22] David Mandell Freeman. Converting pairing-based cryptosystems from composite-order groups to prime-order groups. In Henri Gilbert, editor, EUROCRYPT, volume 6110 of Lecture Notes in Computer Science, pages 44–61. Springer, 2010.

[23] Craig Gentry. Practical identity-based encryption without random oracles. In Serge Vaudenay, editor, EUROCRYPT, volume 4004 of Lecture Notes in Computer Science, pages 445–464. Springer, 2006.
[24] Craig Gentry and Shai Halevi. Hierarchical identity based encryption with polynomially many levels. In Omer Reingold, editor, *TCC*, volume 5444 of *Lecture Notes in Computer Science*, pages 437–456. Springer, 2009.

[25] Craig Gentry and Alice Silverberg. Hierarchical id-based cryptography. In Yuliang Zheng, editor, *ASIACRYPT*, volume 2501 of *Lecture Notes in Computer Science*, pages 548–566. Springer, 2002.

[26] Vipul Goyal, Omkant Pandey, Amit Sahai, and Brent Waters. Attribute-based encryption for fine-grained access control of encrypted data. In Ari Juels, Rebecca N. Wright, and Sabrina De Capitani di Vimercati, editors, *ACM Conference on Computer and Communications Security*, pages 89–98. ACM, 2006.

[27] Jeremy Horwitz and Ben Lynn. Toward hierarchical identity-based encryption. In Lars R. Knudsen, editor, *EUROCRYPT*, volume 2332 of *Lecture Notes in Computer Science*, pages 466–481. Springer, 2002.

[28] Jonathan Katz, Amit Sahai, and Brent Waters. Predicate encryption supporting disjunctions, polynomial equations, and inner products. In Nigel P. Smart, editor, *EUROCRYPT*, volume 4965 of *Lecture Notes in Computer Science*, pages 146–162. Springer, 2008.

[29] Kwangsu Lee and Dong Hoon Lee. New techniques for anonymous hibe with short ciphertexts in prime order groups. *KSII Trans. Internet Inf. Syst.*, 4(5):968–988, 2010.

[30] Allison B. Lewko. Tools for simulating features of composite order bilinear groups in the prime order setting. In David Pointcheval and Thomas Johansson, editors, *EUROCRYPT*, volume 7237 of *Lecture Notes in Computer Science*, pages 318–335. Springer, 2012.

[31] Allison B. Lewko, Tatsuaki Okamoto, Amit Sahai, Katsuyuki Takashima, and Brent Waters. Fully secure functional encryption: Attribute-based encryption and (hierarchical) inner product encryption. In Henri Gilbert, editor, *EUROCRYPT*, volume 6110 of *Lecture Notes in Computer Science*, pages 62–91. Springer, 2010.

[32] Allison B. Lewko, Yannis Rouselakis, and Brent Waters. Achieving leakage resilience through dual system encryption. In Yuval Ishai, editor, *TCC*, volume 6597 of *Lecture Notes in Computer Science*, pages 70–88. Springer, 2011.

[33] Allison B. Lewko and Brent Waters. New techniques for dual system encryption and fully secure hibe with short ciphertexts. In Daniele Micciancio, editor, *TCC*, volume 5978 of *Lecture Notes in Computer Science*, pages 455–479. Springer, 2010.

[34] Allison B. Lewko and Brent Waters. Unbounded hibe and attribute-based encryption. In Kenneth G. Paterson, editor, *EUROCRYPT*, volume 6632 of *Lecture Notes in Computer Science*, pages 547–567. Springer, 2011.

[35] Ben Lynn. The pairing-based cryptography library. [http://crypto.stanford.edu/pbc/](http://crypto.stanford.edu/pbc/)

[36] Tatsuaki Okamoto and Katsuyuki Takashima. Hierarchical predicate encryption for inner-products. In Mitsuru Matsui, editor, *ASIACRYPT*, volume 5912 of *Lecture Notes in Computer Science*, pages 214–231. Springer, 2009.
[37] Tatsuaki Okamoto and Katsuyuki Takashima. Fully secure functional encryption with general relations from the decisional linear assumption. In Tal Rabin, editor, *CRYPTO*, volume 6223 of *Lecture Notes in Computer Science*, pages 191–208. Springer, 2010.

[38] Tatsuaki Okamoto and Katsuyuki Takashima. Achieving short ciphertexts or short secret-keys for adaptively secure general inner-product encryption. In Dongdai Lin, Gene Tsudik, and Xiaoyun Wang, editors, *CANS*, volume 7092 of *Lecture Notes in Computer Science*, pages 138–159. Springer, 2011.

[39] Tatsuaki Okamoto and Katsuyuki Takashima. Adaptively attribute-hiding (hierarchical) inner product encryption. In David Pointcheval and Thomas Johansson, editors, *EUROCRYPT*, volume 7237 of *Lecture Notes in Computer Science*, pages 591–608. Springer, 2012.

[40] Jacob T. Schwartz. Fast probabilistic algorithms for verification of polynomial identities. *J. ACM*, 27(4):701–717, 1980.

[41] Jae Hong Seo, Tetsutaro Kobayashi, Miyako Ohkubo, and Koutarou Suzuki. Anonymous hierarchical identity-based encryption with constant size ciphertexts. In Stanislaw Jarecki and Gene Tsudik, editors, *Public Key Cryptography*, volume 5443 of *Lecture Notes in Computer Science*, pages 215–234. Springer, 2009.

[42] Elaine Shi and Brent Waters. Delegating capabilities in predicate encryption systems. In Luca Aceto, Ivan Damgård, Leslie Ann Goldberg, Magnús M. Halldórsson, Anna Ingólfsdóttir, and Igor Walukiewicz, editors, *ICALP (2)*, volume 5126 of *Lecture Notes in Computer Science*, pages 560–578. Springer, 2008.

[43] Victor Shoup. Lower bounds for discrete logarithms and related problems. In Walter Fumy, editor, *EUROCRYPT*, volume 1233 of *Lecture Notes in Computer Science*, pages 256–266. Springer, 1997.

[44] Brent Waters. Efficient identity-based encryption without random oracles. In Ronald Cramer, editor, *EUROCRYPT*, volume 3494 of *Lecture Notes in Computer Science*, pages 114–127. Springer, 2005.

[45] Brent Waters. Dual system encryption: Realizing fully secure ibe and hibe under simple assumptions. In Shai Halevi, editor, *CRYPTO*, volume 5677 of *Lecture Notes in Computer Science*, pages 619–636. Springer, 2009.