On certain supercuspidal representations of $SL_n(F)$ associated with tamely ramified extensions: the formal degree conjecture and the root number conjecture

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1 Introduction

1.1 Let $F/\mathbb{Q}_p$ be a finite extension with $p \neq 2$ whose integer ring $O_F$ has unique maximal ideal $\mathfrak{p}_F$ which is generated by $\pi_F$. The residue class field $F = O_F/\mathfrak{p}_F$ is a finite field of $q$-elements. The Weil group of $F$ is denoted by $W_F$ which is a subgroup of the absolute Galois group $\text{Gal}(\overline{F}/F)$ where $\overline{F}$ is a fixed algebraic closure of $F$ in which we will take the algebraic extensions of $F$.

Let $G$ be a connected semi-simple linear algebraic group defined over $F$. For the sake of simplicity, we will assume that $G$ splits over $F$. Then the $L$-group $^L G$ of $G$ is equal to the dual group $G^\vee$ of $G$. An admissible representation $\phi : W_F \times SL_2(\mathbb{C}) \to ^L G$

of the Weil-Deligne group of $F$ is called a discrete parameter of $G$ over $F$ if the centralizer $A_\phi = Z_{^L G}(\text{Im} \phi)$ of the image of $\phi$ in $^L G$ is a finite group. Let us denote by $\mathcal{D}_F(G)$ the $G^\vee$-conjugacy classes of the discrete parameters of $G$ over $F$. The conjectural parametrization of $\text{Irr}_2(G)$ (resp. $\text{Irr}_s(G)$), the set of the equivalence classes of the irreducible admissible square-integrable (resp. supercuspidal) representations of $G$, by $\mathcal{D}_F(G)$ is (see [8, p.483, Conj.7.1] for the details)

**Conjecture 1.1.1** For every $\phi \in \mathcal{D}_F(G)$, there exists a finite subset $\Pi_\phi$ of $\text{Irr}_2(G)$ such that

1) $\text{Irr}_2(G) = \bigsqcup_{\phi \in \mathcal{D}_F(G)} \Pi_\phi$,

2) there exists a bijection of $\Pi_\phi$ onto the equivalence classes $\hat{A}_\phi$ of the irreducible complex linear representations of $A_\phi$.

3) $\Pi_\phi \subset \text{Irr}_s(G)$ if $\phi|_{SL_2(\mathbb{C})} = 1$.

The finite set $\Pi_\phi$ is called a $L$-packet of $\phi$. 

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According to this conjecture, any \( \pi \in \text{Irr}_2(G) \) is determined by \( \varphi \in \mathcal{D}_F(G) \) and \( \sigma \in \mathcal{A}_\varphi \). So the formal degree of \( \pi \) should be determined by these data. The formal degree conjecture due to Hiraga-Ichino-Ikeda [9] is (with the formulation of [8])

**Conjecture 1.1.2** The formal degree \( d_\pi \) of \( \pi \) with respect to the absolute value of the Euler-Poincaré measure (see [14, §3] for the details) on \( G(F) \) is equal to

\[
\frac{\dim \sigma}{|\mathcal{A}_\varphi|} \cdot \frac{\gamma(\varphi, \text{Ad}, \psi, d(x), 0)}{\gamma(\varphi_0, \text{Ad}, \psi, d(x), 0)}.
\]

Here

\[
\gamma(\varphi, \text{Ad}, \psi, d(x), s) = \varepsilon(\varphi, \text{Ad}, d(x), s) \cdot \frac{L(\varphi^\vee, \text{Ad}, 1-s)}{L(\varphi, \text{Ad}, s)}
\]

is the gamma-factor associated with the \( \varphi \) combined with the adjoint representation \( \text{Ad} \) of \( G \) on its Lie algebra \( \mathfrak{g} \), and a continuous additive character \( \psi \) of \( F \) such that \( \{ x \in F \mid \psi(xO_F) = 1 \} = O_F \) and the Haar measure \( d(x) \) on the additive group \( F \) such that \( \int_{O_F} d(x) = 1 \). See [8, pp.440-441] for the details.

\( \varphi_0 : W_F \times SL_2(\mathbb{C}) \xrightarrow{\text{proj.}} SL_2(\mathbb{C}) \rightarrow G^- \)

is the principal parameter (see [8, p.447] for the definition).

The formal degree conjecture concerns with the absolute value of the epsilon-factor

\[
\varepsilon(\varphi, \text{Ad}, d(x), s) = w(\varphi, \text{Ad}) \cdot q^{a(\varphi, \text{Ad})(\frac{1}{2} - s)}
\]

where \( a(\varphi, \text{Ad}) \) is the Artin-conductor and \( w(\varphi, \text{Ad}) \) is the root number.

In order to state the root number conjecture, we need some notations. Let \( T \subseteq G \) be a maximal torus split over \( F \) with respect to which the root datum

\[
(X(T), \Phi(T), X^\vee(T), \Phi^\vee(T))
\]

is defined. Then the dual group \( G^- \) is, by the definition, the connected reductive complex algebraic group with a maximal torus \( T^- \) with which its root datum is isomorphic to

\[
(X^\vee(T), \Phi^\vee(T), X(T), \Phi(T)).
\]

Put \( 2 \cdot \rho = \sum_{0 < \alpha \in \Phi^\vee(T)} \alpha \), then \( \epsilon = 2 \cdot \rho(-1) \in T \) is a central element of \( G \). Now the root number conjecture says that

**Conjecture 1.1.3** [8, p.493, Conj.8.3]

\[
\frac{w(\varphi, \text{Ad})}{w(\varphi_0, \text{Ad})} = \pi(\epsilon)
\]

where \( \epsilon \) is the central element of \( G \) defined above (see [8] p.492, (65) for the details).

Since \( G \) is assumed to be splits over \( F \), we have \( w(\varphi_0, \text{Ad}) = 1 \) (see [8, p.448]).
1.2 In this paper, we will construct quite explicitly supercuspidal representations of $G(F) = SL_n(F)$ associated with a tamely ramified extension $K/F$ of degree $n$ (Theorem 2.3.1). When $K/F$ is normal, we will also give candidates of Langlands parameters of the supercuspidal representations (the section 3), and will verify the validity of the formal degree conjecture (Theorem 4.3.3) and the root number conjecture (Theorem 5.2.1) with them.

Our supercuspidal representations, denoted by $\pi_{\beta,\theta}$, are given by the compact induction $\text{ind}_{G(F)}^{G(O_F)} \delta_{\beta,\theta}$ from irreducible unitary representations $\delta_{\beta,\theta}$ of the hyperspecial compact subgroup $G(O_F) = SL_n(O_F)$. Here $\pi_{\beta,\theta}$ and $\delta_{\beta,\theta}$ are characterized each other by the conditions

1) $\delta_{\beta,\theta}$ factors through the canonical morphism $G(O_F) \to G(O_F/p^r_F)$ with $r \geq 2$, and the multiplicity of $\delta_{\beta,\theta}$ in $\pi_{\beta,\theta}|_{G(O_F)}$ is one,

2) any irreducible unitary representation $\delta$ of $G(O_F)$ which factors through the canonical morphism $G(O_F) \to G(O_F/p^r_F)$, and a constituent of $\pi_{\beta,\theta}|_{G(O_F)}$, then $\delta = \delta_{\beta,\theta}$.

The parameters $\beta$ and $\theta$ are associated with the tamely ramified extension $K/F$, that is, $O_K = O_F[\beta]$ and $\theta$ is a certain continuous unitary character of $U_{K/F} = \{x \in K^\times \mid N_{K/F}(x) = 1\}$ (see the subsection 2.2 for the precise definitions). We have the irreducible representation $\delta_{\beta,\theta}$ by the general theory given by [18].

The candidate of Langlands parameter is given by the method of Kaletha [11]. Regard the compact group $U_{K/F}$ as the group of $F$-rational points of an elliptic torus of $SL_n$. Then, by the local Langlands correspondence of tori (see [21]) and the Langlands-Schelstad procedure ([13]) gives a group homomorphism $\varphi$ of the Weil group $W_F$ of $F$ to the dual group $G^\vee = PGL_n(\mathbb{C})$ of $SL_n$ over $F$.

1.3 The section 2 is devoted to the construction of the supercuspidal representation $\pi_{\beta,\theta}$ of $SL_n(F)$. After recalling, in the subsection 2.1 the general theory of the regular irreducible representations of the finite group $G(O_F/p^r_F)$ ($r \geq 2$) given by [19], we will define the irreducible unitary representation $\delta_{\beta,\theta}$ of $SL_n(O_F)$ in the subsection 2.2. The construction of the supercuspidal representation $\pi_{\beta,\theta}$ is given in the subsection 2.3.

The candidate of Langlands parameter is given in the section 3. The local Langlands correspondence of elliptic torus (Proposition 3.1.1), the Langlands-Schelstad procedure (the subsection 3.2) are given quite explicitly. In particular, the candidate of Langlands parameter is given by

$$\varphi : W_F \to W_{K/F} \xrightarrow{(*)} GL_n(\mathbb{C}) \to PGL_n(\mathbb{C})$$

where $(*) = \text{Ind}_{W_K/F}^{W_{K/F}} \tilde{\vartheta}$ is the induced representation from a character $\tilde{\vartheta}$ of $K^\times$ to the relative Weil group $W_{K/F} = W_F/[W_K,W_K]$.

Using the explicit description of the parameter $\varphi$, we will verify the formal degree conjecture in the section 4 and the root number conjecture in the section 5.

Several basic facts on the local factor associated with representations of the Weil group are given in the appendix A.
2 Supercuspidal representations of $SL_n(F)$

2.1 Regular irreducible characters of hyperspecial compact subgroup

Let us recall the main results of [15].

Fix a continuous unitary additive character $\psi : F \to \mathbb{C}^1$ such that

$$\{ x \in F \mid \psi(xO_F) = 1 \} = O_F.$$

Let $G = SL_n$ be the $O_F$-group scheme such that, for any $O_F$-algebra $R$, the group of the $A$-valued point is $G(A) = SL_n(A)$. Let $\mathfrak{g}$ be the Lie algebra scheme of $G$ which is a closed affine $O_F$-subscheme of $\mathfrak{gl}_n$, the Lie algebra scheme of $GL_n$ defined by

$$\mathfrak{g}(R) = \{ X \in \mathfrak{gl}_n(R) \mid \text{tr}(X) = 0 \}$$

for all $O_F$-algebra $R$. Let

$$B : \mathfrak{gl}_n \times_{O_F} \mathfrak{gl}_n \to \mathfrak{h}_{O_F}$$

be the trace form, that is $B(X,Y) = \text{tr}(XY)$ for all $X,Y \in \mathfrak{gl}_n(R)$ with any $O_F$-algebra $R$. Since $G$ is smooth $O_F$-group scheme, we have a canonical isomorphism

$$\mathfrak{g}(O_F)/\mathfrak{z}^r \mathfrak{g}(O_F) \cong \mathfrak{g}(O_F/\mathfrak{p}^r) = \mathfrak{g}(O_F) \otimes_{O_F} O_F/\mathfrak{p}^r$$

([13] Chap.II, §4, Prop.4.8]) and the canonical group homomorphism $G(O_F) \to G(O_F/\mathfrak{p}^r)$ is surjective, due to the formal smoothness [13] p.111, Cor. 4.6], whose kernel is denoted by $K_r(O_F)$. For any $0 < l < r$, let us denote by $K_l(O_F/\mathfrak{p}^r)$ the kernel of the canonical group homomorphism $G(O_F/\mathfrak{p}^r) \to G(O_F/\mathfrak{p}^r)$ which is surjective.

Throughout this paper, let us assume that $p$ is prime to $n$. Then the following basic assumptions on $G$ are satisfied

\begin{enumerate}
  \item I) $B : \mathfrak{g}(F) \times \mathfrak{g}(F) \to F$ is non-degenerate,
  \item II) for any integers $r = l + l'$ with $0 < l' \leq l$, we have a group isomorphism
    $$\mathfrak{g}(O_F/\mathfrak{p}^{l'}) \cong K_l(O_F/\mathfrak{p}^r)$$
    defined by $X (\text{mod } \mathfrak{p}^{l'}) \mapsto 1 + \mathfrak{z}^r X (\text{mod } \mathfrak{p}^r)$,
  \item III) if $r = 2l - 1 \geq 3$ is odd, then we have a mapping
    $$\mathfrak{g}(O_F) \to K_{l-1}(O_F/\mathfrak{p}^r)$$
    defined by $X \mapsto (1 + \mathfrak{z}^{l-1}X + 2^{-1} \mathfrak{z}^{2l-2}X^2) (\text{mod } \mathfrak{p}^r)$.
\end{enumerate}

The condition I) implies that $B : \mathfrak{g}(O_F/\mathfrak{p}^l) \times \mathfrak{g}(O_F/\mathfrak{p}^l) \to O_F/\mathfrak{p}^l$ is non-degenerate for all $l > 0$, and so $B : \mathfrak{g}(O_F) \times \mathfrak{g}(O_F) \to O_F$ is also non-degenerate. By the condition II), $K_l(O_F/\mathfrak{p}^r)$ is a commutative normal subgroup of $G(O_F/\mathfrak{p}^r)$, and its character is

$$\chi_{\beta}(1 + \mathfrak{z}^r X (\text{mod } \mathfrak{p}^r)) = \psi(\mathfrak{z}^{-l'} B(X, \beta)) \quad (X (\text{mod } \mathfrak{p}^{l'}) \in \mathfrak{g}(O_F/\mathfrak{p}^{l'}))$$

\footnote{In this paper, an $O_F$-algebra means an unital commutative $O_F$-algebra.}
with $\beta \pmod{p^r} \in g(O_F/p^r)$.

Since any finite dimensional complex continuous representation of the compact group $G(O_F)$ factors through the canonical group homomorphism $G(O_F) \to G(O_F/p^r)$ for some $0 < r < \mathbb{Z}$, we want to know the irreducible complex representations of the finite group $G(O_F/p^r)$. Let us assume that $r > 1$ and put $r = l + l'$ with the minimal integer $l$ such that $0 < l' \leq l$, that is

$$l' = \begin{cases} l & : r = 2l, \\ l-1 & : r = 2l-1. \end{cases}$$

Let $\delta$ be an irreducible complex representation of $G(O_F/p^r)$. The Clifford’s theorem says that the restriction $\delta|_{K_i(O_F/p^r)}$ is a sum of the $G(O_F/p^r)$-conjugates of characters of $K_i(O_F/p^r)$:

$$\delta|_{K_i(O_F/p^r)} = \left( \bigoplus_{\beta \in \Omega} \chi_{\beta} \right)^m$$

with an adjoint $G(O_F/p^r)$-orbit $\Omega \subset g(O_F/p^r)$. In this way the irreducible complex representations of $G(O_F/p^r)$ correspond to adjoint $G(O_F/p^r)$-orbits in $g(O_F/p^r)$.

Fix an adjoint $G(O_F/p^r)$-orbit $\Omega \subset g(O_F/p^r)$ and let us denote by $\Omega^\sim$ the set of the equivalence classes of the irreducible complex representations of $G(O_F/p^r)$ correspond to $\Omega$. Then [18] gives a parametrization of $\Omega^\sim$ as follows:

**Theorem 2.1.1** Take a representative $\beta \pmod{p^r} \in \Omega (\beta \in g(O_F))$ and assume that

1) the centralizer $G_\beta = Z_G(\beta)$ of $\beta \in g(O_F)$ in $G$ is smooth over $O_F$,

2) the characteristic polynomial $\chi(t) = \det(t \cdot 1_n - \overline{\beta})$ of $\overline{\beta} = \beta \pmod{p} \in g(F) \subset g_n(F)$ is the minimal polynomial of $\overline{\beta} \in M_n(F)$.

Then there exists a bijection $\theta \mapsto \delta_{\beta, \theta}$ of the set

$$\{ \theta \in G_\beta(O_F/p^r)^{\sim} \text{ s.t. } \theta = \chi_{\beta} \text{ on } G_\beta(O_F/p^r) \cap K_i(O_F/p^r) \}$$

onto $\Omega^\sim$.

The correspondence $\theta \mapsto \delta_{\beta, \theta}$ is given by the following procedure. The second condition in the theorem implies

$$G_\beta(O_F/p^r) = G(O_F/p^r) \cap (O_F/p^r)[\beta \pmod{p^r}],$$

in particular $G_\beta(O_F/p^r)$ is commutative. So $G_\beta(O_F/p^r)^{\sim}$ means the character group of $G_\beta(O_F/p^r)$.

$\Omega^\sim$ consists of the irreducible complex representations whose restriction to $K_i(O_F/p^r)$ contains the character $\chi_{\beta}$. Then the Clifford’s theory says the followings: put

$$G(O_F/p^r; \beta) = \{ g \in G(O_F/p^r) | \chi_{\beta}(g^{-1}hg) = \chi_{\beta}(h) \forall h \in K_i(O_F/p^r) \}$$

$$= \{ g \in G(O_F/p^r) | \text{Ad}(g)\beta \equiv \beta \pmod{p^r} \}$$
and let us denote by \( \text{Irr}(G(O_F/p^r; \beta), \chi_\beta) \) the set of the equivalence classes of the irreducible complex representations \( \sigma \) of \( G(O_F/p^r; \beta) \) such that the restriction \( \sigma|_{\text{Ki}(O_F/p^r)} \) contains the character \( \chi_\beta \). Then \( \sigma \mapsto \text{Ind}_{G(O_F/p^r; \beta)}^{G(O_F/p^r)} \sigma \) gives a bijection of \( \text{Irr}(G(O_F/p^r; \beta), \chi_\beta) \) onto \( \Omega^r \).

Since \( G_\beta \) is smooth over \( O_F \), the canonical homomorphism \( G_\beta(O_F/p^r) \to G_\beta(O_F/p^r) \) is surjective. Hence we have

\[
G(O_F/p^r; \beta) = G_\beta(O_F/p^r) \cdot K_i(O_F/p^r).
\]

If \( r = 2l \) is even, then \( l' = l \) and, for any character \( \theta \in G_\beta(O_F/p^r) \) such that \( \theta = \chi_\beta \) on \( G_\beta(O_F/p^r) \cap K_i(O_F/p^r) \), the character

\[
\sigma_{\theta, \beta}(gh) = \theta(g) \cdot \chi_\beta(h) \quad (g \in G_\beta(O_F/p^r), h \in K_i(O_F/p^r))
\]

of \( G(O_F/p^r; \beta) \) is well-defined, and \( \theta \mapsto \sigma_{\theta, \beta} \) is a surjection onto \( \text{Irr}(G(O_F/p^r; \beta), \chi_\beta) \). Hence

\[
\theta \mapsto \delta_{\theta, \beta} = \text{Ind}_{G(O_F/p^r; \beta)}^{G(O_F/p^r)} \sigma_{\theta, \beta}
\]

is the bijection of Theorem 2.1.1.

If \( r = 2l - 1 \) is odd, then \( l' = l - 1 \). Let us denote by \( g_\beta = \text{Lie}(G_\beta) \) the Lie algebra \( O_F \)-scheme of the smooth \( O_F \)-group scheme \( G_\beta \). Then

\[
\mathbb{V}_\beta = g(F)/g_\beta(F)
\]

is a symplectic \( F \)-space with a symplectic \( F \)-form

\[
D_\beta(X, Y) = B([X, Y], \overline{\mathbb{F}}) \in F \quad (X, Y \in g(F)).
\]

Let \( H_\beta = \mathbb{V}_\beta \times \mathbb{C}^1 \) be the Heisenberg group associated with \((\mathbb{V}_\beta, D_\beta)\) and \((\sigma^\beta, L^2(\mathbb{W}'))\) the Schrödinger representation of \( H_\beta \) associated with a polarization \( \mathbb{V}_\beta = \mathbb{W}' \oplus \mathbb{W} \). More explicitly the group operation of \( H_\beta \) is defined by

\[
(u, s) \cdot (v, t) = (u + v, st \cdot \hat{\psi}(2^{-1}D_\beta u, v))
\]

where \( \hat{\psi}(\overline{x}) = \psi(x^{-1}x) \) for \( \overline{x} = x \pmod{p} \in F \), and the action of \( h = (u, s) \in H_\beta \) on \( f \in L^2(\mathbb{W}') \) (a complex-valued function on \( \mathbb{W}' \)) is defined by

\[
(\sigma^\beta(h)f)(w) = s \cdot \hat{\chi} \left(2^{-1}D_\beta(u-, u+) + D_\beta(w, u+)\right) \cdot f(w + u-)
\]

where \( u = u_- + u_+ \in \mathbb{V}_\beta = \mathbb{W}' \oplus \mathbb{W} \).

Take a character \( \theta : G_\beta(O_F/p^r) \to \mathbb{C}^\times \) such that

\[
\theta = \chi_\beta \quad \text{on} \quad G_\beta(O_F/p^r) \cap K_i(O_F/p^r).
\]

Then an additive character \( \rho_\theta : g_\beta(F) \to \mathbb{C}^\times \) is defined by

\[
\rho_\theta(X \pmod{p}) = \chi \left(-x^{-1}B(X, \beta)\right) \cdot \theta \left(1 + x^{l-1}X + 2^{-1} \omega x^{2l-2}X^2 \pmod{p^r}\right)
\]

with \( X \in g_\beta(O_F) \). Fix a \( F \)-vector subspace \( V \subset g(F) \) such that \( g(F) = V \oplus g_\beta(F) \). Then an irreducible representation \((\sigma_\beta^\beta, L^2(\mathbb{W}'))\) of \( K_{l-1}(O_F/p^r) \) is defined by the following proposition:
Proposition 2.1.2 Take a \( g = 1 + \varpi(T \bmod p^r) \in K_{l-1}(O_F/p^r) \) with \( T \in g_l_n(O_F) \). Then we have \( T \bmod p^{l-1} \in g(O_F/p^{l-1}) \) and

\[
\sigma^{\beta, \theta}(g) = \tau (\varpi^{-1} B(T, \beta) - 2^{-1} \varpi^{-1} B(T^2, \beta)) \cdot \rho_0(Y) \cdot \sigma^3(v, 1)
\]

where \( T = [v] + Y \in g(F) \) with \( v \in V_\beta \) and \( Y \in g_\beta(F) \).

Then main result shown in [18], under the assumptions of Theorem 2.1.1, is that there exists a group homomorphism (not unique)

\[
U : G_{\beta}(O_F/p^r) \rightarrow GL_C(L^2(W'))
\]

such that

1) \( \sigma^{\beta, \theta}(h^{-1}gh) = U(h)^{-1} \circ \sigma^{\beta, \theta}(g) \circ U(h) \) for all \( h \in G_{\beta}(O_F/p^r) \) and \( g \in K_{l-1}(O_F/p^r) \), and
2) \( U(h) = 1 \) for all \( h \in G_{\beta}(O_F/p^r) \cap K_{l-1}(O_F/p^r) \).

Now an irreducible representation \( (\sigma_{\beta, \theta}, L^2(W')) \) is defined by

\[
\sigma_{\beta, \theta}(hg) = \theta(h) \cdot U(h) \circ \sigma^{\beta, \theta}(g)
\]

for \( h \in G(O_F/p^r; \beta) = G_{\beta}(O_F/p^r) \cdot K_{l-1}(O_F/p^r) \) with \( h \in G_{\beta}(O_F/p^r) \) and \( g \in K_{l-1}(O_F/p^r) \), and \( \theta \mapsto \sigma_{\beta, \theta} \) is a surjection onto \( \text{Irr}(G(O_F/p^r; \beta), \chi_\beta) \). Then

\[
\theta \mapsto \delta_{\beta, \theta} = \text{Ind}_{G(O_F/p^r; \beta)}^{G(O_F/p^r)} \sigma_{\beta, \theta}
\]

is the bijection of Theorem 2.1.1.

Because the connected \( O_F \)-group scheme \( G = SL_n \) is reductive, that is, the fibers \( G \otimes_K F \) are reductive \( K \)-algebraic groups, the dimension of a maximal torus in \( G \otimes_K F \) is independent of \( K \) which is denoted by \( \text{rank}(G) \).

For any \( \beta \in g(O_F) \) we have

\[
\dim_K g_{\beta}(K) = \dim g_{\beta} \otimes_K O_F K \geq \dim G_{\beta} \otimes_K O_F K \geq \text{rank}(G). \tag{2.2}
\]

We say \( \beta \) is smoothly regular over \( K \) if \( \dim_K g_{\beta}(K) = \text{rank}(G) \) (see [16, (5.7)]). In this case \( G_{\beta} \otimes_K O_F K \) is smooth over \( K \).

In our case of \( G = SL_n \), the following two statements are equivalent for a \( \beta \in g(O_F) \):

1) \( \overline{\beta} \in g(K) \) is smoothly regular over \( K \),
2) the characteristic polynomial of \( \overline{\beta} \in g(K) \subset g_l_n(K) \) is equal to its minimal polynomial

where \( \overline{\beta} \in g(K) \) is the image of \( \beta \in g(O_F) \) by the canonical morphism \( g(O_F) \rightarrow g(K) \) with \( K = F \) or \( F \).

Since we have canonical isomorphisms

\[
g(F) \rightarrow K_{m-1}(O_F/p^m), \quad g_{\beta}(F) \rightarrow G_{\beta}(O_F/p^m) \cap K_{m-1}(O_F/p^m)
\]

and the canonical morphism \( G_{\beta}(O_F) \rightarrow G_{\beta}(O_F/p^m) \) is surjective for any \( m > 1 \), we have

\[
|G(O_F/p^m)| = |G(F)| \cdot q^{(m-1)\dim G}, \quad |G_{\beta}(O_F/p^m)| = |G_{\beta}(F)| \cdot q^{(m-1)\text{rank} G}
\]
for all \( m > 0 \). Then we have

\[
\sharp \Omega = \sharp \{ \theta \in G_\beta(\mathcal{O}_F/p^e) \mid \text{s.t. } \theta = \psi_\beta \text{ on } G_\beta(\mathcal{O}_F/p^e) \cap K_i(\mathcal{O}_F/p^e) \}
\]

\[
= (G_\beta(\mathcal{O}_F/p^e) : G_\beta(\mathcal{O}_F/p^e) \cap K_i(\mathcal{O}_F/p^e)) = |G_\beta(\mathcal{O}_F/p^e)|
\]

\[
= |G_\beta(\mathcal{F})| \cdot q^{(l-1)\dim G} = \frac{|G(\mathcal{F})|}{\sharp \Omega} \cdot q^{(l-1)\dim G}
\]

where \( \sharp \Omega \subset g(\mathcal{F}) \) is the image of \( \Omega \subset g(\mathcal{O}_F/p^e) \) under the canonical morphism \( g(\mathcal{O}_F/p^e) \rightarrow g(\mathcal{F}) \). On the other hand we have

\[
\dim \sigma_{\beta,\theta} = \begin{cases} 
1 & \text{if } r \text{ is even,} \\
\frac{1}{2} \dim g(\mathcal{F})/g_\beta(\mathcal{F}) = q^{(\dim G - \dim \Gamma)}/2 & \text{if } r \text{ is odd,}
\end{cases}
\]

so we have

\[
\dim \delta_{\beta,\theta} = (G(\mathcal{O}_F/p^e) : G(\mathcal{O}_F/p^e; \beta) \cdot \dim \sigma_{\beta,\theta}
\]

\[
= \frac{\sharp \Omega}{2} q^{(r-2)(\dim G - \dim \Gamma)/2}.
\]

\( \text{(2.3)} \)

**Remark 2.1.3** The assumption in Theorem 2.1.1 that the centralizer \( G_\beta \) to be smooth \( \mathcal{O}_F \)-group scheme can be replaced by the surjectivity of the canonical morphisms

\[
G_\beta(\mathcal{O}_F) \rightarrow G_\beta(\mathcal{O}_F/p^e), \quad g_\beta(\mathcal{O}_F) \rightarrow g_\beta(\mathcal{O}_F/p^e),
\]

for all \( l > 0 \).

### 2.2 Regular irreducible character associated with tamely ramified extensions

Let \( K/F \) be a field extension of degree \( n > 1 \). Let

\[
e = e(K/F), \quad f = f(K/F)
\]

be the ramification index and the inertial degree of \( K/F \) respectively so that we have \( ef = n \). Since we assume that \( p \) is prime to \( n \), the extension \( K/F \) is tamely ramified.

Let \( K_0/F \) be the maximal unramified subextension of \( K/F \). Then \( K_0/F \) is a cyclic Galois extension whose Galois group is generated by the geometric Frobenius automorphism \( \text{Fr} \) which induces the inverse of the Frobenius automorphism \( [x \mapsto x^p] \) of the residue field \( K_0 \) over \( F \). Since \( K/K_0 \) is totally ramified, there exists a prime element \( \wp_K \) of \( K \) such that \( \wp_K^e \in K_0 \). Then \( \{1, \wp_K, \wp_K^2, \cdots, \wp_K^{e-1}\} \) is an \( O_{K_0} \)-basis of \( O_K \). The following two propositions are proved by Shintani [15, Lemma 4-7, Cor.1, Cor.2, pp.545-546]:

**Proposition 2.2.1** Put \( \beta = \sum_{i=0}^{e-1} a_i \wp_K^i \in O_K \) (\( a_i \in O_{K_0} \)). Then \( O_K = O_F[\beta] \) if and only if the following two conditions are satisfied:

1) \( a_0_F \neq a_0 (\text{mod } \wp_{K_0}) \) if \( f > 1 \),

2) \( a_1 \in O_{K_0}^e \) if \( e > 1 \).
Proposition 2.2.2 Let $\chi_\beta(t) \in O_F[t]$ be the characteristic polynomial of $\beta \in O_K \subset M_n(O_F)$ via the regular representation with respect to an $O_F$-basis of $O_K$. If $O_K = O_F[\beta]$, then

1) $\chi_\beta(t) \pmod{p_F} \in F[t]$ is the minimal polynomial of $\beta \in M_n(F)$,

2) $\chi_\beta(t) \pmod{p_F} = p(t)^e$ with an irreducible polynomial $p(t) \in F[t]$,

3) if $e > 1$, then $\chi_\beta(t) \pmod{p_F^2}$ is irreducible over $O_F/p_F^2$.

We can prove the following

Proposition 2.2.3 Take a $\beta \in M_n(O_F)$ whose the characteristic polynomial be

$$\chi_\beta(t) = t^n - a_n t^{n-1} - \cdots - a_2 t - a_1.$$ 

If $\chi_\beta(t) \pmod{p_F} \in F[t]$ is the minimal polynomial of $\beta \pmod{p_F} \in M_n(F)$, then

1) $\{X \in M_n(O_F) \mid [X, \beta] = 0\} = O_F[\beta]$,

2) for any $m > 0$, put $\overline{\beta} = (\beta \pmod{p_F^m}) \in M_n(O_F/p_F^m)$, then

$$\{X \in M_n(O_F/p_F^m) \mid [X, \overline{\beta}] = 0\} = O_F/p_F^m$$

3) there exists a $g \in GL_n(O_F)$ such that

$$g \beta g^{-1} = \begin{bmatrix} 0 & 0 & \cdots & 0 & a_1 \\ 1 & 0 & \cdots & 0 & a_2 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & a_{n-1} \\ 1 & 0 & \cdots & a_n & 1 \end{bmatrix}.$$ 

Now our $O_F$-group scheme $G = SL_n$ is defined based on the free $O_F$-module $O_K$ with a fixed $O_F$-basis of $O_K$. Take a $\beta \in O_K$ such that $O_K = O_F[\beta]$ and $T_{K/F}(\beta) = 0$. Identify the $F$-algebra $K$ with a $F$-subalgebra of $End_n(F)$ by the regular representation with respect to the fixed $O_F$-basis of $O_K$. By Proposition 2.2.2, the characteristic polynomial of $\overline{\beta} = (\beta \pmod{p_F}) \in M_n(F)$ is equal to its minimal polynomial. Then, by Proposition 2.2.3, we have

$$\{X \in M_n(O_F) \mid [X, \beta] = 0\} = O_F[\beta] = O_K$$

and

$$\{X \in M_n(O_F/p_F^m) \mid [X, \overline{\beta}] = 0\} = O_F/p_F^m = O_K/p_K^m$$

for any $m > 0$. Put $U_{K/F} = \{\varepsilon \in O_K \mid N_{K/F}(\varepsilon) = 1\}$. Then we have

$G_\beta(O_K) = G(O_F) \cap O_K = U_{K/F}$.

We have also

$g_\beta(O_F) = g(O_F) \cap O_K = \{X \in O_K \mid T_{K/F}(X) = 0\}$.
\[ G_\beta(O_F/p_F^l) = \{ \tau \in (O_K/p_K^\infty)^\times | N_{K/F}(\epsilon) \equiv 1 \pmod{p_F^l} \}, \]
\[ g_\beta(O_F/p_F^l) = \{ \chi \in O_K/p_K^\infty | T_{K/F}(X) \equiv 0 \pmod{p_F^l} \} \]

for all \( l > 0 \). Then the canonical morphisms

\[ G_\beta(O_F) \rightarrow G_\beta(O_F/p_F^l), \quad g_\beta(O_F) \rightarrow g_\beta(O_F/p_F^l) \]

are surjective for all \( l > 0 \). In fact, take an \( \epsilon \in O_K^\times \) such that \( N_{K/F}(\epsilon) \equiv 0 \pmod{p_F^l} \). Since \( K/F \) is tamely ramified, we have \( N_{K/F}(1 + p_K^l) = 1 + p_F^l \), that is, there exists \( \eta \in 1 + p_K^l \) such that \( N_{K/F}(\eta) = N_{K/F}(\epsilon) \). Then \( \eta = \eta^{-1}\epsilon \in U_{K/F} \) such that \( \xi \equiv \epsilon \pmod{p_K^l} \). Take a \( x \in O_K \) such that \( T_{K/F}(x) \equiv 0 \pmod{p_F^l} \), or \( T_{K/F}(x) = a \cdot \omega_F^l \) with \( a \in O_F \). Since \( K/F \) is tamely ramified, we have \( T_{K/F}(O_K) = O_F \), that is, there exists a \( y \in O_K \) such that \( T_{K/F}(y) = a \).

Then \( z = x - y \omega_F^l \in O_K \) such that \( T_{K/F}(z) = 0 \) and \( z \equiv x \pmod{p_K^l} \).

Due to Remark 2.1.3 we can apply the general theory of subsection 2.1 to our \( \beta \in g(O_F) \). Take an integer \( r > 1 \) and put \( r = l + l' \) with minimal integer \( l \) such that \( 0 < l' \leq l \). Let \( \Omega \subset g(O_F/p_F^{l'}) \) be the adjoint \( G(O_F/p_F^{l'}) \)-orbit of \( \beta \pmod{p_F^{l'}} \in g(O_F/p_F^{l'}) \), and \( \Omega^\ast \) the set of the equivalent classes of the irreducible representations of \( G(O_F/p_F^l) \) corresponding to \( \Omega \) via Clifford’s theory described in subsection 2.1. Then we have a bijection \( \theta \mapsto \delta_{\beta, \theta} \) of the continuous unitary character \( \theta \) of \( U_{K/F} \) such that

1) \( \theta \) factors through the canonical morphism \( U_{K/F} \rightarrow (O_K/p_K^\infty)^\times \),

2) for an \( \epsilon \in U_{K/F} \) such that \( \epsilon \equiv 1 + \omega_F^l x \pmod{p_K^l} \) with \( x \in O_K \) and \( T_{K/F}(x) \equiv 0 \pmod{p_F^l} \), we have \( \theta(\epsilon) = \psi \left( \omega_F^{-l'} T_{K/F}(x\beta) \right) \).

onto \( \Omega^\ast \). Here \( \psi : F \rightarrow \mathbb{C}^\times \) is a continuous unitary character of the additive group \( F \) such that \( \{ x \in F | \psi(xO_F) = 1 \} = O_F \). Then we have

**Proposition 2.2.4**

\[ \dim \delta_{\beta, \theta} = \frac{q^{r-n(n-1)/2}}{1 - q^{-r}} \cdot \frac{1}{(O_F^\times : N_{K/F}(O_K^\times))} \cdot \prod_{k=1}^{n} \left( 1 - k^{-k} \right). \]

**[Proof]** For the dimension formula (2.24), we have

\[ \dim G = n^2 - 1, \quad \text{rank } G = n - 1, \quad \sharp \Omega = \frac{|G(F)|}{|G_\beta(F)|} \]

and

\[ |G(F)| = |SL_n(F)| = q^{n^2-1} \prod_{k=2}^{n} (1 - q^{-k}). \]

On the other hand \( G_\beta(F) \) is the kernel of

\[ (\ast) : (O_K/p_K)^\times \rightarrow (O_F/p_F)^\times \quad (\epsilon \pmod{p_K^l}) \mapsto N_{K/F}(\epsilon \pmod{p_F^l}). \]
Since $K/F$ is tamely ramified extension, we have

$$1 + p_F = N_{K/F}(1 + p_K^e) \subset N_{K/F}(O_K^\times) \subset O_F^\times,$$

hence

$$|G_\beta(F)| = \frac{\left|(O_K/p_K^e)^\times : (O_F/p_F)^\times\right|}{(O_F : p_F)^\times} = (O_F^\times : N_{K/F}(O_K^\times)) \cdot \frac{q^e - q^{(e-1)}}{q - 1} \cdot \frac{1 - q^{-f}}{1 - q^{-1}}.$$

2.3 Construction of supercuspidal representations

We will keep the notations of the preceding subsection. The purpose of this subsection is to prove the following theorem:

**Theorem 2.3.1** If $l' = \left\lfloor \frac{r}{2} \right\rfloor \geq 2(e - 1)$, then the compactly induced representation $\pi_{\beta, \theta} = \text{ind}_{G(F)}^{G(O_F)} \delta_{\beta, \theta}$ is an irreducible supercuspidal representation of $G(F) = SL_n(F)$ such that

1) the multiplicity of $\delta_{\beta, \theta}$ in $\pi_{\beta, \theta}|_{G(O_F)}$ is one,

2) $\delta_{\beta, \theta}$ is the unique irreducible unitary constituent of $\pi_{\beta, \theta}|_{G(O_F)}$ which factors through the canonical morphism $G(O_F) \to G(O_F/p_F)$,

3) with respect to the Haar measure on $G(F)$ such that the volume of $G(O_F)$ is one, the formal degree of $\pi_{\beta, \theta}$ is equal to

$$\dim \delta_{\beta, \theta} = \frac{q^{r-n(n-1)/2}}{1 - q^{-f}} \cdot \frac{1}{(O_F^\times : N_{K/F}(O_K^\times))} \cdot \prod_{k=1}^n (1 - k^{-1}).$$

The rest of this subsection is devoted to the proof.

We have the Cartan decomposition

$$G(F) = \bigsqcup_{m \in \mathbb{M}} G(O_F)\varpi_F^m G(O_F)$$

(2.4)

where

$$\mathbb{M} = \left\{ m = (m_1, m_2, \cdots, m_n) \in \mathbb{Z}^n \mid m_1 \geq m_2 \geq \cdots \geq m_n, \right. \left. m_1 + m_2 + \cdots + m_n = 0 \right\}$$

and

$$\varpi_F^m = \begin{bmatrix} \varpi_F^{m_1} \\ \vdots \\ \varpi_F^{m_n} \end{bmatrix}$$

for $m = (m_1, \cdots, m_n) \in \mathbb{M}$. 

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For an integer $1 \leq i < n$, let

$$U_i = \left\{ \begin{bmatrix} 1_i & B \\ 0 & 1_{n-i} \end{bmatrix} \mid B \in M_{i,n-i} \right\}$$

be the unipotent part of the parabolic subgroup

$$P_i = \left\{ \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \in G \mid A \in GL_i, D \in GL_{n-1} \right\}.$$

Put $U_i(p_F^a) = U_i(O_F) \cap K_a(O_F)$ for a positive integer $a$.

**Proposition 2.3.2** If $K/F$ is unramified or $r \geq 4$, then $\text{ind}_{G(O_F)}^{G(F)} \delta_{\beta,\theta}$ is an admissible representation of $G(F)$.

**Proof** We will prove that the dimension of the space of the $K_a(O_F)$-fixed vectors is finite for any integer $a > 0$. The Cartan decomposition (2.4) gives

$$G(F) = \bigcup_{s \in S} K_a(O_F)sG(O_F)$$

with

$$S = \{ k\varpi_F^m \mid k \in K_a(O_F) \setminus G(O_F), m \in \mathbb{M} \}.$$

Then we have

$$\text{ind}_{G(O_F)}^{G(F)} \delta_{\beta,\theta} = \bigoplus_{s \in S} \text{ind}_{K_a(O_F) \cap sG(O_F) s^{-1}}^{K_a(O_F) \cap sG(O_F) s^{-1}} \delta_{\beta,\theta}$$

with $\delta_{\beta,\theta}(h) = \delta_{\beta,\theta}(s^{-1}hs) \ (h \in K_a(O_F) \cap sG(O_F)s^{-1})$. The Frobenius reciprocity gives

$$\text{Hom}_{K_a(O_F)}(1, \text{ind}_{G(O_F)}^{G(F)} \delta_{\beta,\theta}) = \bigoplus_{s \in S} \text{Hom}_{G_a(O_D) \cap sG(O_F)}(1, \delta_{\beta,\theta}).$$

Here $1$ is the one-dimensional trivial representation of $K_a(O_F)$. If

$$\text{Hom}_{K_a(O_F)}(1, \text{ind}_{G(O_F)}^{G(F)} \delta_{\beta,\theta}) \neq 0$$

then there exists a

$$s = k\varpi_F^m \in S \ (k \in G(O_F), m = (m_1, \cdots, m_n) \in \mathbb{M})$$

such that $\text{Hom}_{s^{-1}K_a(O_F) \cap sG(O_F)}(1, \delta_{\beta,\theta}) \neq 0$. If

$$\text{Max}\{m_i - m_{i+1} \mid 1 \leq i < n\} = m_i - m_{i+1} \geq a$$

then $\varpi_F^m U_i(O_F) \varpi_F^{-m} \subset K_a(O_F)$. So we have $U_i(O_F) \subset s^{-1}K_a(O_F)s \cap G(O_F)$ so that

$$\text{Hom}_{U_i(p')} (1, \delta_{\beta,\theta}) \supset \text{Hom}_{s^{-1}K_a(O_F) \cap sG(O_F)}(1, \delta_{\beta,\theta}) \neq 0$$

where $U_i(p') = U_i(O_F) \cap G(p')$. Then the decomposition (2.1) implies that there exists a $g \in G(O_F)$ such that $\chi_{\text{Ad}(g)}(h) = 1$ for all $h \in U_i(p')$, that is

$$\tau\left(\varpi_F^{-p'} \text{tr}(g^\beta g^{-1} \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix})\right) = 0$$
for all \( B \in M_{i,n-1}(O_F) \). This means
\[
g \beta g^{-1} \equiv \begin{bmatrix} A & \ast \\ 0 & D \end{bmatrix} \pmod{p'}
\]
with \( A \in M_i(O_F), D \in M_{n-1}(O_F) \), that is
\[
\chi_\beta(t) \equiv \det(t_{1i} - A) \cdot \det(t_{1n-1} - D) \pmod{p'}.
\]
If \( K/F \) is unramified, this is contradict against 2) of Proposition 2.2.2. If \( K/F \) is ramified, then \( r \geq 4 \) and \( l' \geq 2 \) and a contradiction to 3) of the proposition. So we have
\[
\max\{m_i - m_{i+1} \mid 1 \leq i < n\} < a.
\]
This implies that the number of \( s \in S \) such that \( \text{Hom}^s_{K_a}(O_F) \cap G(O_F) = 0 \) is finite, and then
\[
\dim \text{Hom}_{K_a}(O_F) \left( 1, \text{ind}_{G}^{G(O_F)} \delta_{\beta, \theta} \right) < \infty.
\]

\[\blacksquare\]

**Proposition 2.3.3** If \( l' = \left\lfloor \frac{r}{2} \right\rfloor \geq 2(e - 1) \), then

1) \( \dim \text{C} \text{Hom}_{G(O_F)}^{G(O_F)} \left( \delta_{\beta, \theta}, \text{ind}_{G}^{G(O_F)} \delta_{\beta, \theta} \right) = 1, \)

2) if \( \delta \) is an irreducible representation of \( G(O_F) \) which factors through the canonical surjection \( G(O_F) \to G(O_F/p') \) such that
\[
\text{Hom}_{G(O_F)}^{G(O_F)} \left( \delta, \text{ind}_{G}^{G(O_F)} \delta_{\beta, \theta} \right) \neq 0,
\]
then \( \delta = \delta_{\beta, \theta} \).

[Proof] Cartan decomposition gives
\[
\text{ind}_{G}^{G(O_F)} \delta_{\beta, \theta} = \bigoplus_{m \in M} \text{ind}_{G(O_F)\cap \varpi^m G(O_F)\varpi^{-m}}^{G(O_F)} \delta_{\beta, \theta}^m.
\]
Then Frobenius reciprocity gives
\[
\text{Hom}_{G(O_F)}^{G(O_F)} \left( \delta, \text{ind}_{G}^{G(O_F)} \delta_{\beta, \theta} \right) = \bigoplus_{m \in M} \text{Hom}_{\varpi^m G(O_F)\varpi^{-m} G(O_F)}^{\varpi^{-m} G(O_F)} \left( \delta_{\varpi^{-m}, \delta_{\beta, \theta}} \right).
\]
Now take a \( m = (m_1, \cdots, m_n) \in M \) such that
\[
\text{Hom}_{\varpi^m G(O_F)\varpi^{-m} G(O_F)}^{\varpi^{-m} G(O_F)} \left( \delta_{\varpi^{-m}, \delta_{\beta, \theta}} \right) \neq 0.
\]
Suppose
\[
\max\{m_k - m_{k+1} \mid 1 \leq k < n\} = m_i - m_{i+1} \geq a
\]
with a integer \( 0 < a \leq l' \). Then
\[
U_i(O_F) = \varpi^{-m} U_i(O_F) \varpi^m \cap U_i(O_F) \subset \varpi^{-m} G(O_F) \varpi^m \cap G(O_F)
\]
and we have

$$\text{Hom}_{U_i(p^{r-a})}(\delta^{w_m}, \delta_{\beta, \theta}) \supset \text{Hom}_{\varpi^{-m} G(O_F) \varpi^m G(O_F)}(\delta^{w_m}, \delta_{\beta, \theta}) \neq 0.$$  

Since $\omega^m U_i(p^{r-a}) \varpi^{-m} \subset U_i(p^r) \subset \text{Ker} \delta$, we have

$$\text{Hom}_{U_i(p^{r-a})}(1, \delta_{\beta, \theta}) \neq 0.$$  

Here 1 is the trivial one-dimensional representation of $U_i(p^{r-a})$. Since $r-a \geq l$ and hence $U_i(p^{r-a}) \subset U_i(p^l)$, there exists a $g \in G(O_F)$ such that $\psi_{\text{Ad}(g)\beta}(h) = 1$ for all $h \in U_i(p^{r-a})$, due to the decomposition (2.1). This means

$$g \beta g^{-1} \equiv \begin{bmatrix} A & * \\ 0 & D \end{bmatrix} \pmod{p^a}$$  

with some $A \in M_i(O_F)$ and $D \in M_{n-i}(O_F)$. Then

$$\chi_\beta(t) \equiv \det(t1_i - A) \cdot \det(t1_{n-i} - D) \pmod{p^a}. \quad (2.5)$$

If $a \geq 2$, this decomposition of the characteristic polynomial contradicts to Proposition 2.2.2. So $a = 1$. Then 2) of Proposition 2.2.2 implies that $i = \deg \det(1_i - A)$ is a multiple of $f$. So we have $m_1 - m_2 < e$. Note that we have

$$\omega^m K_{l+1-m_2}(O_F) \omega^{-m} \subset K_l(O_F).$$

So if $\delta$ corresponds, as explained in subsection 2.1 to an adjoint $G(O_F/p_F')$-orbit $\Omega' \subset g(O_F/p_F')$ of $\chi(\text{mod } p_F') (\gamma \in g(O_F))$, then there exist $g, h \in G(O_F)$ such that

$$\chi_{\text{Ad}(g)\beta}(x) = \chi_{\text{Ad}(h)\gamma}(\omega^m x \omega^{-m})$$

for all $x \in K_{l+1-m_2}(O_F)$. This means

$$g \beta g^{-1} - \omega^{-m} h g^{-1} \omega^{-m} \in M_n(p^{1-(m_1-m_2)}).$$

Since $2(m_1 - m_2) \leq 2(e - 1) \leq l'$, we have

$$\omega^m g \beta g^{-1} \omega^{-m} \in M_n(p^{l'-2(m_1-m_2)}) \subset M_n(O_F).$$

Then there exists a $g' \in GL_n(O_F)$ such that $\omega^m g \beta g^{-1} \omega^{-m} = g' g^{-1}$ and hence $g'^{-1} \omega^m g \in K$ due Proposition 2.2.3. On the other hand we have

$$N_{K/F}(g'^{-1} \omega^m g) = \det(g'^{-1} \omega^m g) \in O_F^\times$$

so that $g'^{-1} \omega^m g \in O_K \subset GL_n(O_F)$. Hence $m = (0, \cdots, 0)$. Now we have proved

$$\text{Hom}_{G(O_F)}(\delta, \text{ind}_{G(O_F)^{\text{red}}}^{G(F)}\delta_{\beta, \theta}) = \text{Hom}_{G(O_F)}(\delta, \delta_{\beta, \theta})$$

which implies the two statements of the proposition. \[\Box\]

The admissible representation $\pi_{\beta, \theta} = \text{ind}_{G(O_F)^{\text{red}}}^{G(F)}\delta_{\beta, \theta}$ of $G(F)$ is irreducible. In fact, if there exists a $G(F)$-subspace $0 \not= W \not= \text{ind}_{G(O_F)^{\text{red}}}^{G(F)}\delta_{\beta, \theta}$, we have

$$0 \not= \text{Hom}_{G(F)}(W, \text{ind}_{G(O_F)^{\text{red}}}^{G(F)}\delta_{\beta, \theta}) \subset \text{Hom}_{G(O_F)}(W, \text{ind}_{G(O_F)^{\text{red}}}^{G(F)}\delta_{\beta, \theta})$$

$$= \text{Hom}_{G(O_F)}(W, \delta)$$

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by Frobenius reciprocity. Hence $\delta \mapsto W|_{G(O_F)}$. On the other hand, we have

$$0 \neq \text{Hom}_{G(F)}\left(\text{ind}^G_{G(O_F)}\delta_{\beta,\theta}, \text{ind}^G_{G(O_F)}\delta_{\beta,\theta} \right)/W) = \text{Hom}_{G(O_F)}\left(\delta_{\beta,\theta}, \text{ind}^G_{G(O_F)}\delta_{\beta,\theta} \right)/W,$$

hence $\delta \mapsto \left(\text{ind}^G_{G(O_F)}\delta_{\beta,\theta} \right)/W$. Now $\text{ind}^G_{G(O_F)}\delta_{\beta,\theta}$ is semi-simple $G(F)$-module, we have

$$\dim_\mathbb{C} \text{Hom}_{G(O_F)}(\delta_{\beta,\theta}, \text{ind}^G_{G(O_F)}\delta_{\beta,\theta}) \geq 2$$

which contradicts to the first statement of Proposition 2.3.3.

Then $\pi_{\beta,\theta}$ is a supercuspidal representation of $G(F)$ whose formal degree with respect to the Haar measure $d_{G(F)}(x)$ of $G(F)$ such that $\int_{G(O_F)} d_{G(F)}(x) = 1$ is equal to $\dim \delta_{\beta,\theta}$. We have completed the proof of Theorem 2.3.1.

3 Kaleta’s $L$-parameter

3.1 Local Langlands correspondence of elliptic tori

Let $K_+/F$ be a finite extension, $K/K_+$ a quadratic extension with a non-trivial element $\tau$ of $\text{Gal}(K/K_+)$. Let us denote by $L$ an arbitrary Galois extension over $F$ containing $K$ for which let us denote by

$$\text{Emb}_F(K, L) = \{\sigma|_K \mid \sigma \in \text{Gal}(L/F)\}$$

the set of the embeddings over $F$ of $K$ into $L$.

Let us denote by $V$ the $\overline{F}$-algebra of the functions $v$ on $\text{Emb}_F(K, \overline{F})$ with values in $\overline{F}$. The action of $\sigma \in \text{Gal}(\overline{F}/F)$ on $v \in V$ is defined by $v^\sigma(\gamma) = v(\gamma^{-1})^\sigma$.

Then fixed point subspace $\mathcal{V}^\text{Gal}(\overline{F}/L) = \mathcal{V}(L)$ is the set of the functions on $\text{Emb}_F(K, L)$ with values in $L$, and $\mathcal{V}^\text{Gal}(\overline{F}/F) = \mathcal{V}(F)$ is identified with $K$ via $v \mapsto v(1_K)$.

The action of $\sigma \in \text{Gal}(\overline{F}/F)$ on $g \in SL_{\overline{F}}(V)$ is defined by $v \cdot g^\sigma = (v^\sigma^{-1} \cdot g)^\sigma$. Then the fixed point subgroup $SL_{\overline{F}}(\mathcal{V})^{\text{Gal}(\overline{F}/F)}$ is identified with $SL_F(K)$ via $g \mapsto g|_{G_m}$.

Let $S = \text{Res}_{K/F}G_m$ which is identified with the multiplicative group $\mathbb{C}^\times$. Then $S(F)$ is identified with the multiplicative group $\mathbb{C}^\times$.

The group $X(S)$ of the characters over $\overline{F}$ of $S$ is a free $\mathbb{Z}$-module with $\mathbb{Z}$-basis $\{b_s\}_{s \in \text{Emb}_F(K, \overline{F})}$ where $b_s(s) = s(\delta)$ for $s \in S$. The dual torus $S^\circ = X(S) \otimes_{\mathbb{Z}} \mathbb{C}^\times$ is identified with the group of the functions $s$ on $\text{Emb}_F(K, \overline{F})$ with values in $\mathbb{C}^\times$. The action of $\sigma \in W_F \subset \text{Gal}(\overline{F}/F)$ on $S$ induces the action on $X(S)$ such that $\delta_\sigma = b_\sigma$, and hence the action on $s \in S^\circ$ is defined by $s^\sigma(\gamma) = s(\gamma^{-1})$.

Since we have a bijection $\rho \mapsto \rho|_{K}$ of $W_K \setminus W_F$ onto $\text{Emb}_F(K, \overline{F})$, the $\overline{F}$-algebra $V$ (resp. the torus $S$, $S^\circ$) is identified with the set of the left $W_K$-invariant functions on $W_F$ with values in $\overline{F}$ (resp. $\mathbb{C}^\times$, $\mathbb{C}^\times$).

The local Langlands correspondence for the torus $S$ is the isomorphism

$$H^1(W_F, S) \rightarrow \text{Hom}(W_K, \mathbb{C}^\times) \quad (3.1)$$
given by \([\alpha] \mapsto [\rho \mapsto \alpha(\rho)(1_K)]\). The inverse mapping is defined as follows. Let

\[ l : \text{Emb}_F(K, \overline{F}) \to W_F \]

be a section of the restriction mapping \(W_F \to \text{Emb}_F(K, \overline{F})\), that is \(l(\gamma)|_K = \gamma\) for all \(\gamma \in \text{Emb}_F(K, \overline{F})\) and \(l(1_K) = 1\), and put

\[ J(\gamma, \sigma) = l(\gamma)\sigma l(\gamma\sigma)^{-1} \in W_K \text{ for } \gamma \in \text{Emb}_F(K, \overline{F}), \sigma \in W_F. \]

Take a \(\psi \in \text{Hom}(W_K, \mathbb{C}^\times)\) and define \(\alpha \in Z^1(W_F, \hat{S})\) by

\[ \alpha(\sigma)(\rho) = \alpha(\sigma\rho^{-1})(1) \cdot \alpha(\rho^{-1})(1) \text{ with } \alpha(\sigma)(1) = \psi(J(1_K, \sigma^{-1})) \]

for all \(\sigma, \rho \in W_F\). Then \(\psi \mapsto [\alpha]\) is the inverse mapping of the isomorphism \eqref{eq:isom}.

If we restrict the isomorphism \eqref{eq:isom} to continuous group homomorphisms, we have an isomorphism

\[ H^1_{\text{cont}}(W_F, S) \cong \text{Hom}_{\text{cont}}(K^\times, \mathbb{C}^\times) \]

via \eqref{eq:isom} combined with the isomorphism of the local class field theory

\[ \delta_K : K^\times \cong W_K/[W_K, W_K]. \]

Let \(T\) be a subtorus of \(S\) which is identified with the multiplicative subgroup of \(V^\times\) consisting of the functions \(s\) on \(\text{Emb}_F(K, F)\) to \(F^\times\) such that

\[ \prod_{\gamma \in \text{Emb}_F(K, F)} s(\gamma) = 1. \]

In other words \(T\) is a maximal torus of \(SL_{\overline{F}}(V)\) by identifying \(s \in T\) with \([v \mapsto v \cdot s] \in SL_{\overline{F}}(V)\). The fixed point subgroup \(T^{\text{Gal}(\overline{F}/F)} = T(F)\) is identified with

\[ U_{K/F} = \{ \varepsilon \in O_K^\times \mid N_{K/F}(\varepsilon) = 1 \} \text{ by } s \mapsto s(1_K). \]

The restriction mapping gives a canonical surjection

\[ \text{Hom}_{\text{cont}}(K^\times, \mathbb{C}^\times) \to \text{Hom}_{\text{cont}}(U_{K/F}, \mathbb{C}^\times). \]

The restriction from \(S\) to \(T\) gives a surjection \(X(S) \to X(T)\) whose kernel is the subgroup of \(X(S)\) generated by \(\sum_{\gamma \in \text{Emb}_F(K, \overline{F})} b_{\gamma} \). Then the dual torus is

\[ T^\sim = X(T) \otimes \mathbb{C}^\times = S^\sim/\mathbb{C}^\times \]

where \(\mathbb{C}^\times \subset S^\sim\) is the subgroup of the \(\mathbb{C}^\times\)-valued constant function on \(\text{Emb}_F(K, \overline{F})\) or on \(W_F\).

The exact sequence

\[ 1 \to \mathbb{C}^\times \to S^\sim \to T^\sim \to 1 \]

induces the exact sequence

\[ H^1_{\text{cont}}(W_F, \mathbb{C}^\times) \to H^1_{\text{cont}}(W_F, S^\sim) \to H^1_{\text{cont}}(W_F, T) \to H^2_{\text{cont}}(W_F, \mathbb{C}^\times). \]
Since we have $H^2_{\text{cont}}(W_F, \mathbb{C}^\times) = \{1\}$ by [12], the canonical surjection $S^\sim \to T^\sim$ induces a canonical surjection

$$H^1_{\text{cont}}(W_F, S^\sim) \to H^1_{\text{cont}}(W_F, T^\sim). \quad (3.4)$$

Then we have

**Proposition 3.1.1** There exists an isomorphism

$$H^1_{\text{cont}}(W_F, T^\sim) \to \text{Hom}_{\text{cont}}(U_{K/F}, \mathbb{C}^\times) \quad (3.5)$$

such that the following diagram is commutative:

$$\begin{array}{ccc}
H^1_{\text{cont}}(W_F, S^\sim) & \to & \text{Hom}_{\text{cont}}(K^\times, \mathbb{C}^\times) \\
\downarrow \quad (3.3) & & \downarrow \\
H^1_{\text{cont}}(W_F, T^\sim) & \to & \text{Hom}_{\text{cont}}(U_{K/F}, \mathbb{C}^\times)
\end{array}$$

**[Proof]** See [21] for the arguments with general tori. A direct proof for our specific setting is as follows.

It is enough to show that an $\alpha \in Z^1_{\text{cont}}(W_F, S^\sim)$ mapped to the trivial element in $H^1(W_F, T^\sim)$ by (3.4) if and only if $\alpha$ is mapped to the trivial element of $\text{Hom}_{\text{cont}}(U_{K/F}, \mathbb{C}^\times)$ by the composite of (3.2) and (3.3). Let $c \in \text{Hom}_{\text{cont}}(K^\times, \mathbb{C}^\times)$ be the image of $\alpha$ by (3.2,3), that is, $c(x) = \alpha(\rho)(1_K)$ with $\rho \equiv \delta_K(x) (\text{mod } [K, K])$. By the commutative diagram of the local class field theory

$$\begin{array}{ccc}
W_K & \to & K^\times \\
\downarrow \quad \sigma \mapsto [W_K, W_K] & \downarrow \\
W_F/ [W_F, W_F] & \to & F^\times
\end{array}$$

c is trivial on $U_{K/F}$ if and only if $c$ is extended to $\tilde{c} \in \text{Hom}_{\text{cont}}(F^\times, \mathbb{C}^\times)$. If so, put $\tilde{s}(\sigma) = \tilde{c}(\sigma)/\alpha(\sigma)(1_K)$ for $\sigma \in W_F$. Then the cocycle condition of $\alpha$ implies that $\tilde{s}$ is left $W_F$-invariant, and hence $s \in S^\sim$, and that $s(\sigma) \equiv s^{\sigma^{-1}} (\text{mod } \mathbb{C}^\times)$ for all $\sigma \in W_F$. Conversely if $\alpha(\sigma) \equiv s^{\sigma^{-1}} (\text{mod } \mathbb{C}^\times)$ for some $s \in S^\sim$, then $\tilde{c} = \alpha(\sigma)(1_K) (\sigma \in W_F)$ gives an extension of $c$. ■

Put $L^T = W_F \ltimes T^\sim$. Then a cohomology class $[\alpha] \in H^1_{\text{cont}}(W_F, T^\sim)$ defines a continuous group homomorphism

$$\tilde{\alpha} : W_F \to L^T \quad (\sigma \mapsto (\sigma, \alpha(\sigma))) \quad (3.6)$$

and $[\alpha] \mapsto \tilde{\alpha}$ induces a well-defined bijection

$$H^1_{\text{cont}}(W_F, T^\sim) \to \text{Hom}^*_{\text{cont}}(W_F, L^T)/"T^{-}\text{conjugate}"$$

where $\text{Hom}^*_{\text{cont}}(W_F, L^T)$ denotes the set of the continuous group homomorphisms $\psi$ of $W_F$ to $L^T$ such that $W_F \xrightarrow{\psi} L^T \xrightarrow{\text{proj.}} W_F$ is the identity map.

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3.2 \( \chi \)-datum

In this subsection, let us assume that \( K/F \) is a Galois extension and put \( \Gamma = \text{Gal}(K/F) \). For a \( \gamma \in \Gamma \) of order two (if any), let us denote by \( K_\gamma \) the intermediate subfield of \( K/F \) such that \( \text{Gal}(K/K_\gamma) = \langle \gamma \rangle \).

Let \( S^- \) be the maximal torus of \( GL_n(\mathbb{C}) \) consisting of the non-singular diagonal matrices. Then \( T^- = S^-/\mathbb{C}^\times \) is the maximal torus of the complex projective general linear group \( PGL_n(\mathbb{C}) = GL_n(\mathbb{C})/\mathbb{C}^\times \).

We have an isomorphism \( T^- \rightarrow T^- \) given by

\[
s \mapsto \text{diag}(s(\gamma_1), \ldots, s(\gamma_n))
\]

where \( \text{Emb}_F(K, \overline{F}) = \{ \gamma_i \}_{1 \leq i \leq n} \). The action of \( W_F \) on \( T^- \) induces the action on \( T^- \) which factors through \( \Gamma \).

The Weyl group \( W(S) = N_{GL_n(\mathbb{C})}(S)/S^- \) on \( S^- \) is identified with a subgroup of the permutation group \( S_n \). Then any \( w \in W(S) \) is represented by the permutation matrix \( [w] \in GL_n(\mathbb{C}) \) corresponding to \( w \in W(S) \subset S_n \). Put \( \tilde{w} = [w] (\text{mod } \mathbb{C}^\times) \in N_{PGL_n(\mathbb{C})}(T^-) \).

For any \( \gamma \in \text{Emb}_F(K, \overline{F}) \), let us denote by \( a_\gamma \) an element of \( X(S^-) \) such that \( a_\gamma(s) = s(\gamma) \) for all \( s \in T^- \). Put \( a_i = a_{\gamma_i} \in X(S^-) \) (\( 1 \leq i \leq n \)). Then

\[
\Phi(S^-) = \Phi(S^-) = \{ a_i \cdot a_j^{-1} \mid 1 \leq i, j \neq n, i \neq j \}
\]

is the set of the roots of \( GL_n(\mathbb{C}) \) with respect to \( S^- = S^- \) with the simple roots \( \Delta = \{ a_i = a_i \cdot a_{i+1}^{-1} \mid 1 \leq i < n \} \).

Let \( \{ X_\alpha, X_{-\alpha}, H_\alpha \} \) be the standard triple associate with a simple root \( \alpha \in \Delta \). Then \( s_\alpha \in W(S) \) is represented by

\[
n(s_\alpha) = \exp(X_\alpha) \cdot \exp(-X_{-\alpha}) \cdot \exp(X_\alpha) \in N_{GL_n(\mathbb{C})}(S^-)
\]

and \( W(S^-) \) is generated by \( S = \{ s_\alpha \}_{\alpha \in \Delta} \). For any \( w \in W(T^-) \), let \( w = s_1 s_2 \cdots s_r \) (\( s_i \in S \)) be a reduced presentation and put

\[
n(w) = n(s_1)n(s_2) \cdots n(s_r) \in N_{GL_n(\mathbb{C})}(T^-).
\]

Then

\[
r(w) = [w]^{-1}n(w) = \prod_{\alpha \leq 0, \alpha^{-1} > 0 } \varepsilon(\alpha) \in T^-\]

where \( \prod_{\alpha \leq 0, \alpha^{-1} > 0 } \) is the product over negative roots \( \alpha \in \Phi(T^-) \) such that \( \alpha^{w^{-1}} \) is positive, and \( \varepsilon(\alpha) \in S^- \) with \( \alpha = a_\alpha a_\gamma^{-1} \in \Phi(T^-) \) is the diagonal matrix whose diagonal elements are 1 except \( j \)-the one is \(-1\).

The action of \( \sigma \in W_F \) on \( X(S^-) \) induced from the action on \( S^- \) is such that \( a_\gamma^\sigma = a_{\gamma_\sigma} \) for all \( \gamma \in \text{Emb}_F(K, \overline{F}) \), and it determines an element \( w(\sigma) \in W(S^-) \).

Then \( \chi \) shows that the 2-cocycle \( t \in \mathbb{Z}^2(W_F, S^-) \) defined by

\[
t(\sigma, \sigma') = n(w(\sigma \sigma'))^{-1}n(w(\sigma)) \cdot n(w(\sigma')) \quad (\sigma, \sigma' \in W_F)
\]

is split by \( r_p : W_F \rightarrow S^- \) defined by \( \chi \)-data as follows.
For any \(\lambda \in \Phi(S)\), put
\[
\Gamma_\lambda = \{\sigma \in \Gamma \mid \lambda^\sigma = \lambda\}, \quad \Gamma_{\pm \lambda} = \{\sigma \in \Gamma \mid \lambda^\sigma = \pm \lambda\}
\]
and put \(F_\lambda = L^\Gamma_\lambda\), \(F_{\pm \lambda} = L^\Gamma_{\pm \lambda}\). Then \((F_\lambda : F_{\pm \lambda}) = 1, 2\) and \(\lambda\) is called symmetric if \((F_\lambda : F_{\pm \lambda}) = 2\).

The Galois group \(\Gamma\) acts on \(\Phi(S)\) and
\[
\Phi(S)/\Gamma = \{a_{\lambda,K}a_{\gamma}^{-1} \mid 1 \neq \gamma \in \Gamma\}.
\]
If \(\lambda = a_{\lambda,K}a_{\gamma}^{-1}\), then \(\lambda\) is symmetric if and only if \(\gamma^2 = 1\). In this case \(F_\lambda = K\) and \(F_{\pm \lambda} = K_\gamma\), and choose a continuous character \(\chi_\gamma : F_\lambda^\times \to \mathbb{C}^\times\) such that \(\chi_{\lambda/K_\gamma} : K_\gamma^\times \to \{\pm 1\}\) is the character of the quadratic extension \(K/K_\gamma\).

These characters are parts of a system of \(\chi\)-data \(\chi_\lambda : F_\lambda \to \mathbb{C}^\times\) \((\lambda \in \Phi(T))\) such that
\[
1) \quad \chi_{-\lambda} = \chi_\lambda^{-1}\quad \text{and} \quad \chi_{\lambda^\sigma} = \chi_\lambda(x^\sigma)\quad \text{for all} \quad \sigma \in \Gamma,
\]
\[
2) \quad \chi_\lambda = 1\quad \text{if} \quad \lambda\quad \text{is not symmetric}.
\]

With this \(\chi\)-data and the gauge
\[
p : \Phi(S) \to \{\pm 1\} \text{ s.t. } p(\lambda) = \begin{cases} 1 & : \lambda > 0, \\ -1 & : \lambda < 0, \end{cases}
\]
the mechanism of [13] gives a \(r_p : W_F \to S^{-}\) such that
\[
t(\sigma, \sigma') = r_p(\sigma)^\sigma r_p(\sigma^\sigma)^{-1} r_p(\sigma') \quad \text{for all} \quad \sigma, \sigma' \in W_F
\]
and
\[
r_p(\sigma) = \prod_{\gamma \in \Gamma \text{ s.t. } 0 < \lambda \in \{a_{\lambda,K}a_{\gamma}^{-1}\} \Gamma} \chi_\lambda(x)^\lambda
\]
if \(\delta = (1, x) \in W_{K/F} = \Gamma \rtimes K_\gamma\), where \(\{\alpha\}_\Gamma\) is the \(\Gamma\)-orbit of \(\alpha \in \Phi(T)\) and \(\bar{\lambda}\) is the co-root of \(\lambda\). Then we have a group homomorphism
\[
L^S = W_F \times S^{-} \to GL_n(\mathbb{C}) \quad ((\sigma, s) \mapsto n(w(\sigma))r_p(\sigma)^{-1}s).
\]
(3.7)
If we put \(r(\sigma) = r(w(\sigma))\) for \(\sigma \in W_F\), we have
\[
t(\sigma, \sigma') = r(\sigma)^\sigma r(\sigma^\sigma)^{-1} r(\sigma') \quad (\sigma, \sigma' \in W_F).
\]
Now \(\chi_p(\sigma) = r(\sigma) \cdot r_p(\sigma)^{-1}\) \((\sigma \in W_F)\) define an element of \(Z^1(W_F, S)\) and the group homomorphism (3.7) is
\[
L^S = W_F \times S^{-} \to GL_n(\mathbb{C}) \quad ((\sigma, s) \mapsto [w(\sigma)]\chi_p(\sigma) \cdot s).
\]
(3.8)
Because the actions are trivial on the center \(\mathbb{C}^\times\) of \(GL_n(\mathbb{C})\), the group homomorphism (3.7) or (3.8) can be lifted to the group homomorphism
\[
L^T = W_F \times T^{-} \to PGL_n(\mathbb{C}) \quad ((\sigma, s) \mapsto \hat{w}(\sigma)\bar{\chi}(\sigma) \cdot s)
\]
(3.9)
where \( \tilde{\chi}_p(\sigma) = \chi_p(\sigma) \mod C^\times \in PGL_n(C) \). Let \( c \in \text{Hom}_{\text{cont}}(U_{K/F}, C^\times) \) be the character corresponding to the cohomology class \([\tilde{\chi}_p] \in H^1_{\text{cont}}(W_F, T)\) by the local Langlands correspondence of torus (3.5). If we put

\[
\tilde{c}(x) = \chi_p(1, x)(1) = \prod_{1 \neq \gamma \in \Gamma} \chi_{a_{1_K}^{-1}}(x) \quad (x \in K^\times),
\]

then \( c = \tilde{c}|_{U_{K/F}} \). The following proposition will be used in the next two sections.

**Proposition 3.2.1** We can choose the \( \chi \)-data \( \{ \chi_{\lambda} \}_{\lambda \in \Phi(T)} \) so that \( c(x) = 1 \) for all \( x \in U_{K/F} \cap (1 + p^2_K) \).

**Proof** Take a \( \gamma \in \Gamma \) of order two. Clearly \( (1 + p^2_K) \cap K^\times = 1 + p^2_K \). If \( K/K^\gamma \) is ramified, then \( (1 + p^2_K) \cap K^\gamma = 1 + p^2_K \). So we can assume that \( \chi_{a_{1_K}^{-1}} \) is trivial on \( 1 + p^2_K \).

Then \( c(x) = 1 \) for all \( x \in U_{K/F} \cap (1 + p^2_K) \). \( \blacksquare \)

### 3.3 Explicit value of \( c((-1)^{n-1}) \)

From now on, we will assume that \( K/F \) is a tamely ramified Galois extension and put \( \Gamma = \text{Gal}(K/F) \).

We will prove the following proposition:

**Proposition 3.3.1** According to the parity of \( n = (K : F) \), we have

\[
c((-1)^{n-1}) = \begin{cases} 
1 & : n=\text{odd}, \\
(-1)^{3f-1} & : n=\text{even}.
\end{cases}
\]

In order to prove the proposition, we need the structure of the set of the order two elements of \( \Gamma = \text{Gal}(K/F) \). Put

\[
\text{Gal}(K/F) = \langle \delta, \rho \rangle \quad (3.11)
\]

where \( \text{Gal}(K/K_0) = \langle \delta \rangle \) with the maximal unramified subextension \( K_0/F \) of \( K/F \) and \( \rho|_{K_0} \in \text{Gal}(K_0/F) \) is the inverse of the Frobenius automorphism. There exists a prime element \( \sigma_K \) of \( K \) such that \( \sigma_K \in K_0 \). Then \( \sigma \mapsto \sigma_K^{-1} \mod p_K \) is an injective group homomorphism of \( \text{Gal}(K/K_0) \) into \( K^\times \), and hence \( \delta \mapsto q^{-1} - 1 \). Put \( \rho^f = \delta^m \) with \( 0 \leq m < e \). We have a relation \( \delta^{-1} \delta^m = \delta^m \) due to Iwasawa [10] and hence

\[
\delta^m = \delta^{-1} \delta^m \delta = \delta^{em}
\]

that is \( m(q - 1) \equiv 0 \mod e \). So we have

\[
\rho^f(a^{(a^{-1})}) = 1 \quad (3.12)
\]

Since \( f \) divides \( \text{ord}(\rho) \), we have

\[
\text{ord}(\rho) = f \cdot \frac{e}{\text{GCD}(e, m)}.
\]

The structure of the elements of order two in \( \text{Gal}(K/F) \) plays an important role in our arguments, and we have
Proposition 3.3.2 Assume that $|\text{Gal}(K/F)|$ is even. Then

$$H = \{\gamma \in \text{Gal}(K/F) \mid \gamma^2 = 1\} \subset Z(\text{Gal}(K/F))$$

and

$$H = \begin{cases} 
[1, \delta^f] & : f = \text{odd or } e = \text{even,} \\
[1, \rho^f \delta^e \gamma] & : e = \text{odd, } m = \text{even} \\
[1, \rho^f \delta^m \gamma] & : e = \text{odd, } m = \text{odd} \\
[1, \delta^f, \rho^f \delta^e \gamma, \rho^f \delta^m \gamma] & : f = \text{even, } e = \text{even, } m = \text{even.}
\end{cases}$$

For $\gamma \in \text{Gal}(K/F)$ of order two, the quadratic extension $K/K_\gamma$ is ramified if and only if $\gamma \in \text{Gal}(K/K_0)$.

[Proof] Take a $1 \neq \gamma \in \text{Gal}(K/F)$ such that $\gamma^2 = 1$.

If $\gamma \in \text{Gal}(K/K_0)$, then $e$ is even and $\gamma = \delta^f$ is the unique element of order 2 of the normal subgroup $\text{Gal}(K/K_0)$. So $\gamma \in Z(\text{Gal}(K/F))$. In this case $K_0 \subset K_\gamma$ and $K/K_\gamma$ is ramified extension.

Assume that $\gamma \notin \text{Gal}(K/K_0)$. Then $\gamma|_{K_0} \in \text{Gal}(K_0/F)$ is of order two (hence $f = 2$ is even), and $\gamma = \rho^f \delta^m$ with $0 < a < e$. Then $K/K_\gamma$ is unramified extension, because if it was not the case we have $f(K_\gamma/F) = f(K/F)$ and hence $K_0 \subset K_\gamma$ which means

$$\gamma \in \text{Gal}(K/K_\gamma) \subset \text{Gal}(K/K_0),$$

contradicting to the assumption $\gamma \notin \text{Gal}(K/K_0)$. Then $f(K_\gamma/F) = f'$ and $e(K_\gamma/F) = e$, and hence $e|q^{f'} = 1$. So we have

$$1 = \gamma^2 = \rho^f \rho^{-f} \delta^m \rho^f \delta^m = \delta^{m+aq + a} = \delta^{2a+m},$$

hence $2a \equiv -m (\text{mod } e)$. Then $a \equiv \frac{m}{2}$ or $\frac{e - m}{2} (\text{mod } e)$ if $e$ is even (hence $m$ is even), and

$$a \pmod{e} = \begin{cases} 
\frac{m}{2} & : \text{if } m \text{ is even,} \\
\frac{e - m}{2} & : \text{if } m \text{ is odd}
\end{cases}$$

if $e$ is odd. We have $e|q^{f'} - 1$ hence

$$\delta \gamma = \rho^{f'} \delta^{aq + a} = \rho^{f'} \delta^{1 + a} = \gamma \delta.$$  

Now we have

$$\rho^{f'(q-1)} = 1.$$  

(3.13)

In fact $\text{Gal}(K_\gamma/F) = \langle \delta', \rho' \rangle$ with $\delta' = \delta|_{K_\gamma}$, $\rho' = \rho|_{K_\gamma}$. Then $(\rho')^{f'(q-1)} = 1$, that is

$$\rho^{f'(q-1)} \in \text{Gal}(K/K_\gamma) = \langle \gamma \rangle.$$  

If $\rho^{f'(q-1)} \neq 1$, then $\rho^{f'(q-1)} = \gamma = \rho^{f'} \delta^m$, therefore

$$\rho^{f'q} = \rho^{f'} \delta^m = \delta^{m+a} \in \text{Gal}(K/K_0)$$

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and hence \( f \) divides \( f'q \), contradicting to the assumption that \( q \) is odd. Now we have

\[
\gamma \rho = \rho f' + \delta a = \rho \gamma \cdot \delta (q-1).
\]

For \( a = -\frac{m}{2} \) or \( a = \frac{e-m}{2} \), we have \( a(q-1) \equiv 0 \pmod{\text{e}} \) if and only if

\[
\frac{q-1}{2} \equiv 0 \pmod{\gcd(e,m)}
\]

which is equivalent to \( \rho f' = \rho f'(q-1) = 1 \). Then (3.13) implies \( \gamma \rho = \rho \gamma \).

Then we have \( \gamma \) is an element of the center of \( \text{Gal}(K/F) \).

**Proof of Proposition 3.3.1**  Assume the \( n = (K:F) \) is even. Then

\[
c(-1) = \prod_{1 \neq \gamma \in \Gamma} (-1, K/K_{\gamma})
\]

by (3.10). Here

\[
(-1, K/K_{\gamma}) = \begin{cases} 
1 & : K/K_{\gamma} \text{ is unramified}, \\
(-1)^{q^f - 1} & : K/K_{\gamma} \text{ is ramified}
\end{cases}
\]

for the quadratic extension \( K/K_{\gamma} \) with \( 1 \neq \gamma \in \Gamma \) such that \( \gamma^2 = 1 \). The extension \( K/K_{\gamma} \) is ramified if and only if \( \gamma \in \text{Gal}(K/K_0) = \langle \delta \rangle \), the list of Proposition 3.3.2 shows that

\[
c(-1) = \begin{cases} 
1 & : e \text{=odd}, \\
(-1)^{q^f - 1} & : e \text{=even}.
\end{cases}
\]

Note that

\[
\frac{q^f - 1}{2} = \frac{q-1}{2} (1 + q + q^2 + \cdots + q^{f-1}) \equiv \frac{q-1}{2} f \pmod{2}.
\]

This complete the proof.

### 3.4 \( L \)-parameters associated with characters of tame elliptic tori

By local Langlands correspondence of tori described in Proposition 3.1.1, the continuous character \( \theta \) of \( U_{K/F} \) which parametrizes the irreducible representation \( \delta_{\beta, \theta} \) of \( SL_n(O_F) \) determines, by choosing an extension of \( \theta \) to \( K^\times \), the cohomology class \([\alpha] \in H^1_{\text{cont}}(W_F, S)\) which is mapped onto \([\alpha] \in H^1_{\text{cont}}(W_F, T)\) by (3.7), which is independent of the choice of the extension to \( K^\times \) of \( \theta \). Then we have a group homomorphisms

\[
\varphi_1 : W_F \xrightarrow{\tilde{\alpha}_1} L_S \xrightarrow{\tilde{\mu}} GL_n(\mathbb{C})
\]

and

\[
\varphi : W_F \xrightarrow{\tilde{\alpha}} L_T \xrightarrow{\tilde{\mu}} PGL_n(\mathbb{C}).
\]
The construction of $\varphi$ shows that $\varphi(\sigma) = \varphi_1(\sigma) \, (\bmod \mathbb{C}^\times)$ ($\sigma \in W_F$). The definition of (3.8) shows that

$$
\text{tr} \varphi_1(\sigma) = \sum_{\gamma \in \text{Emb}_F(K_{\mathcal{F}}), \gamma \sigma = \gamma} \chi_p(\gamma) \cdot \alpha(\sigma)(\gamma)
$$

$$
= \sum_{\gamma \in W_K \setminus W_F, \gamma \gamma^{-1} \in W_K} \chi_p(\gamma) \cdot \alpha(\gamma)
$$

$$
= \sum_{\gamma \in W_K \setminus W_F, \gamma \gamma^{-1} \in W_K} \psi_c \cdot \psi_\theta(\gamma^{-1})
$$

for $\sigma \in W_F$. Here $\psi_c$ and $\psi_\theta$ are respectively the elements of $\text{Hom}_{\text{cont}}(W_K, \mathbb{C})$ corresponding to $\tilde{c}$ defined by (3.10) and an extension $\tilde{\theta} \in \text{Hom}_{\text{cont}}(K^\times, \mathbb{C}^\times)$ of $\theta$ by the isomorphism of the local class field theory

$$
W_K \rightarrow W_K/\left[W_K, W_K\right]\stackrel{\sim}{\rightarrow} K^\times.
$$

This shows that $\varphi_1$ is the induced representation of $W_F$ from the character $\psi_c \cdot \psi_\theta$ of $W_K$. So $\varphi_1$ factors through the canonical surjection

$$
W_F \rightarrow W_{K/F} = W_F/\left[W_K, W_K\right]
$$

and, if we put $\vartheta = c \cdot \theta$ and $\tilde{\vartheta}(x) = \tilde{c} \cdot \tilde{\theta}$, the extension of $\vartheta$, then we have

$$
\text{tr} \varphi_1(\sigma, x) = \begin{cases} 
0 & : \sigma \neq 1, \\
\sum_{\gamma \in \text{Gal}(K/F)} \tilde{\vartheta}(\gamma) & : \sigma = 1
\end{cases}
$$

(3.16)

for $(\sigma, x) \in W_{K/F} = \text{Gal}(K/F) \ltimes_{\alpha_{K/F}} K^\times$ with the fundamental class $[\alpha_{K/F}] \in H^2(\text{Gal}(K/F), K^\times)$.

The representation space $V_{\vartheta}$ of the induced representation $\text{Ind}^{W_{K/F}}_{K/F} \tilde{\vartheta}$ is the complex vector space of the $\mathbb{C}$-valued function $v$ on $\text{Gal}(K/F)$ with the action of $(\sigma, x) \in W_{K/F}$

$$
(x \cdot v)(\gamma) = \tilde{\vartheta}(\gamma) \cdot v(\gamma), \quad (\sigma \cdot v)(\gamma) = \tilde{\vartheta}(\alpha_{K/F}(\sigma, \sigma^{-1} \gamma)) \cdot v(\sigma^{-1} \gamma).
$$

(3.17)

A $\mathbb{C}$-basis $\{v_\rho\}_{\rho \in \text{Gal}(K/F)}$ of $V_{\vartheta}$ is defined by

$$
v_\rho(\gamma) = \begin{cases} 
1 & : \gamma = \rho, \\
0 & : \gamma \neq \rho
\end{cases}
$$

Then

$$
x \cdot \rho = \tilde{\vartheta}(x^\rho) \cdot v_\rho, \quad \sigma \cdot \rho = \tilde{\vartheta}(\alpha_{K/F}(\sigma, \rho)) \cdot v_{\sigma \rho}
$$

for $(\sigma, x) \in W_{K/F}$. The following proposition will be used to analyze $\text{Ind}^{W_{K/F}}_{K/F} \tilde{\vartheta}$ in detail.

**Proposition 3.4.1** Assume $l \geq 2$, then for an integer $k \geq 2$

$$
\left\{ \sigma \in \text{Gal}(K/F) \mid \tilde{\vartheta}(x_\sigma) = \tilde{\vartheta}(x) \quad \text{for} \quad \forall x \in 1 + \mathfrak{p}_K^k \right\} = \begin{cases} 
\text{Gal}(K/F) & : k > e(r - 1), \\
\text{Gal}(K/K_0) & : k = e(r - 1), \\
\{1\} & : k < e(r - 1).
\end{cases}
$$

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[Proof] Note that \( \vartheta(x) = \theta(x) \) for all \( x \in U_{K/F} \cap (1 + p_K^e) \) (by Proposition 3.2.1) and \( \vartheta(x) = 1 \) for all \( x \in U_{K/F} \cap (1 + p_K) \). Take an integer \( k \) such that \( 0 \leq k \leq eI \), and hence \( 2 \leq eI \leq er - k \). Then, for any \( x \in O_K \), we have

\[
(1 + \varpi_F \varpi_K^{-k} x)^{1 - \tau} \equiv 1 + \varpi_F (\varpi_K^{-k} x - \varpi_K^{-k} x^\tau) \pmod{p_K^e}
\]
since \( 2(er - k) \geq er \). Hence, for \( \alpha = 1 + \varpi_F \varpi_K^{-k} x \in 1 + p_K^e \) (\( x \in O_K \)), we have

\[
\tilde{\vartheta}(\alpha) = \psi \left( T_{K/F} \left( (\varpi_K^{-k} x - \varpi_K^{-k} x^\tau) \beta \right) \right) = \psi \left( 2T_{K/F}(\varpi_K^{-k} x^\tau) \right).
\]  

(3.18)

Because \( K/F \) is tamely ramified, we have

\[
V_t(K/F) = \{ \sigma \in \text{Gal}(K/F) \mid \text{ord}_K(x^\sigma - x) \geq t + 1 \forall x \in O_K \}
\]

\[
= \begin{cases} 
\text{Gal}(K/F) & : t < 0, \\
\text{Gal}(K/K_0) & : 0 \leq t < 1, \\
\{1\} & : 1 \leq t.
\end{cases}
\]  

(3.19)

Take a \( \sigma \in \text{Gal}(K/F) \). Then, by (3.18), we have

\[
\tilde{\vartheta}(\alpha^\sigma) = \psi \left( 2T_{K/F}(\varpi_K^{-k} x^\sigma) \beta \right) = \psi \left( 2T_{K/F}(\varpi_K^{-k} x^\sigma) \right).
\]

So the statement \( \tilde{\vartheta}(\alpha^\sigma) = \tilde{\vartheta}(\alpha) \) for all \( \alpha = 1 + p_K^e \) is equivalent to the statement \( \varpi_K^{-k} (\beta^\sigma - \beta) \in \mathcal{D}(K/F)^{-1} = p_K^{1-e} \), or to the statement

\[
\text{ord}_K(x^\sigma - x) \geq k + e - 1 \forall x \in O_K
\]

since \( O_K = O_F[\beta] \), which is equivalent to \( \sigma \in V_{k-e} \). Then (3.19) completes the proof. \( \blacksquare \)

**Proposition 3.4.2** The induced representation \( \text{Ind}_{K^K_x/F} \tilde{\vartheta} \) is irreducible.

[Proof] Take a \( 0 \neq T \in \text{End}_{W_{K/F}}(V_\theta) \). Since

\[
Tv_\rho = T(\rho \cdot v_1) = \rho \cdot Tv_1
\]

for all \( \rho \in \text{Gal}(K/F) \), we have \( Tv_1 \neq 0 \). If \( (Tv_1)(\gamma) \neq 0 \) for a \( \gamma \in \text{Gal}(K/F) \), then we have

\[
\tilde{\vartheta}(x^\gamma) \cdot (Tv_1)(\gamma) = (x \cdot Tv_1)(\gamma) = T(x \cdot v_1)(\gamma)
\]

\[
= (T(\tilde{\vartheta}(x) \cdot v_1))(\gamma) = \tilde{\vartheta}(x) \cdot (Tv_1)(\gamma),
\]

and hence \( \tilde{\vartheta}(x^\gamma) = \tilde{\vartheta}(x) \) for all \( x \in K^{\times} \). Then \( \gamma = 1 \) by Proposition 3.4.1. This means \( Tv_1 = c \cdot v_1 \) with a \( c \in \mathbb{C}^{\times} \). Then

\[
Tv_\rho = \rho \cdot (Tv_1) = c \cdot v_\rho
\]

for all \( \rho \in \text{Gal}(K/F) \), and hence \( T \) is a homothety. \( \blacksquare \)

**Remark 3.4.3** The proof of Proposition 3.4.2 shows that the induced representation \( \text{Ind}_{K^K_x/F} \tilde{\vartheta} \) is irreducible if \( \tilde{\vartheta} \) is a character of \( K^{\times} \) such that \( \tilde{\vartheta}(x^\sigma) = \tilde{\vartheta}(x) \) for all \( x \in K^{\times} \) with \( \sigma \in \text{Gal}(K/F) \) implies \( \sigma = 1 \).
4 Formal degree conjecture

In this section, we will assume that $K/F$ is a tamely ramified Galois extension such that the degree $(K:F) = n$ is prime to $p$, and will keep the notations of preceding sections.

4.1 $\gamma$-factor of adjoint representation

The admissible representation of the Weil-Deligne group $W_F \times SL_2(\mathbb{C})$ to $PGL_n(\mathbb{C})$ corresponding to the triple $(\varphi, PGL_n(\mathbb{C}), 0)$ as presented in the appendix A.6 is

$$W_F \times SL_2(\mathbb{C}) \xrightarrow{\text{projection}} W_F \xrightarrow{\varphi} PGL_n(\mathbb{C}) \quad (4.1)$$

which is denoted also by $\varphi$. We will also denote by $\varphi_1$ the representation

$$W_F \times SL_2(\mathbb{C}) \xrightarrow{\text{projection}} W_F \xrightarrow{\varphi_1} GL_n(\mathbb{C}).$$

Then, as a complex linear representation of $W_F \times SL_2(\mathbb{C})$, the adjoint representation of $PGL_n(\mathbb{C})$ on its Lie algebra composed with the $\varphi$ is equivalent to the adjoint representation of $GL_n(\mathbb{C})$ on

$$\widehat{\mathfrak{g}} = \mathfrak{sl}_n(\mathbb{C}) = \{ X \in M_n(\mathbb{C}) | \text{tr}(X) = 0 \}$$

composed with the $\varphi_1$.

The purpose of this subsection is to determine the $\gamma$-factor $\gamma(\varphi, \text{Ad}, \psi, d(x), s)$, as explained in the appendix A.6.

Let us use the notation of (3.11)

$$\Gal(K/F) = \langle \delta, \rho \rangle,$$

that is, $\Gal(K/K_0) = \langle \delta \rangle$ with the maximal unramified subextension $K_0/F$ of $K/F$ and $\rho|_{K_0} \in \Gal(K_0/F)$ is the inverse of the Frobenius automorphism. By the canonical surjection

$$W_F \to W_F/[W_K, W_K] = W_{K/F} = \Gal(K/F) \times \alpha_{K/F} K^\times \subset \Gal(K^{ab}/F),$$

$I_F = \Gal(F^{alb}/F^{ur}) \subset W_F$ is mapped onto

$$\Gal(K/K_0)^{\times} \alpha_{K/F} O_K^\times = \Gal(K^{ab}/F^{ur}).$

Put $v_1 = v_{i-1\delta^{-1}…} \in V_{\vartheta} \ (1 \leq i \leq e, 1 \leq j \leq f)$, and identify $GL_n(\mathbb{C})$ with $GL_n(\mathbb{C})$ by the $\mathbb{C}$-basis $\{ v_{ij} \}_{1 \leq i \leq e, 1 \leq j \leq f}$ of $V_{\vartheta} = \text{Ind}_{K^{\times}}^{W_K} \vartheta$. Then the action (3.17) gives

$$\varphi_1(\delta) = \begin{bmatrix} J_1 & \cdots & \cdots & J_f \end{bmatrix} \in GL_n(\mathbb{C}) \quad (4.2)$$

with

$$J_j = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 1 & \ddots & \ddots & \ddots & \cdots \\ \cdots & \cdots & \ddots & \ddots & \cdots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{1j} & \cdots & \cdots & \cdots & a_{ej} \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

and

$$a_{ij} = \begin{cases} a & \text{if } i = j \\ b & \text{if } i = j+1 \\ c & \text{if } i = j+2 \\ \vdots & \vdots \\ g & \text{if } i = j+g \end{cases}$$

for $1 \leq i, j \leq f$.
\[(a_{ij} = \theta(\alpha_{K/F}(\delta, \delta^{-1}\rho^{-1}))).\] Since the action of \(O_K^\times\) on \(V_\theta\) is diagonal, the space \(\tilde{g}^{tr}\) of the \(\text{Ad} \circ \varphi(I_F)\)-fixed vectors is
\[
\begin{bmatrix}
  a_{1e} & a_{21e} & \cdots & a_{f1e}
\end{bmatrix}
\begin{bmatrix}
  a_i \in \mathbb{C}, \\
  a_1 + a_2 + \cdots + a_f = 0
\end{bmatrix}.
\]
A \(\mathbb{C}\)-basis of it is given by
\[
X_1 = \begin{bmatrix}
P & 0_e & \cdots & 0_e
\end{bmatrix}, \quad X_2 = \begin{bmatrix}
P & 0_e & \cdots & 0_e
\end{bmatrix}, \cdots, X_{f-1} = \begin{bmatrix}
P & 0_e & \cdots & 0_e
\end{bmatrix}
\]
with \(P = \begin{bmatrix} 1_e & -1_e \end{bmatrix}\). Since \(\rho^l\rho^{-1} = \delta^l\) with \(0 < l < e\) and \(ql \equiv 1 (\text{mod } e)\), we have
\[
\rho^l\rho^{-1} = \begin{cases}
\delta^{l(i-1)} & : 1 \leq j < f, \\
\delta^{l(i-1)+m} & : j = f.
\end{cases}
\]
Put
\[
l(i-1) \equiv i' - 1 \pmod{e}, \quad l(i-1) + m \equiv i'' - 1 \pmod{e} \quad (1 \leq i', i'' \leq e)
\]
for \(0 \leq i < e\) and let \([l]_e, [l, m]_e \in \text{GL}_e(\mathbb{Z})\) be the permutation matrix associated respectively with the element
\[
\begin{pmatrix}
1 & 1' & 2' & \cdots & e'
\end{pmatrix}, \quad \begin{pmatrix}
1 & 1'' & 2'' & \cdots & e''
\end{pmatrix}
\]
of the symmetric group of degree \(e\). Then we have
\[
\varphi_1(\rho) = \begin{bmatrix}
0 & 0 & \cdots & 0 & I_f \\
I_1 & 0 & \cdots & 0 & 0 \\
\vdots & & & & \\
I_{f-1} & 0 & \cdots & 0 & 0
\end{bmatrix}
\]
with
\[
I_j = P_j \begin{bmatrix}
b_{1j} & b_{2j} & \cdots & b_{e,j}
\end{bmatrix}, \quad P_j = \begin{cases}
[l]_e & : 1 \leq j < f, \\
[l, m]_e & : j = f
\end{cases}
\]
\((b_{ij} = \theta(\alpha_{K/F}(\rho, \delta^{-1}\rho^{-1}))).\) So the representation matrix of \(\text{Ad} \circ \varphi(\widetilde{F}_r)\) on \(\tilde{g}^{tr}\) with respect to the basis \((4.3)\) is
\[
\begin{bmatrix}
-1 & 1 & 0 & \cdots & 0 \\
-1 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
-1 & 0 & 0 & 1 \\
-1 & 0 & \cdots & 0 & 0
\end{bmatrix}
\]
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Hence we have

\[ L(\varphi, \text{Ad}, s) = \det \left( 1 - q^{-s} \text{Ad} \circ \varphi(\tilde{\text{Fr}})|_{\mathfrak{g}_{\text{Ad} \circ \varphi}(F)} \right)^{-1} = \left( 1 + q^{-s} + q^{-2s} + \cdots + q^{-(f-1)s} \right)^{-1} \]

and

\[ \frac{L(\varphi, \text{Ad}, 1)}{L(\varphi, \text{Ad}, 0)} = f \cdot \frac{1 - q^{-1}}{1 - q^{-f}}. \quad (4.4) \]

Let us denote by \( K^{(k)} = K_{\pi_K, k} \) (\( k = 1, 2, \cdots \)) the field of \( \pi_K \)-th division points of Lubin-Tate theory over \( K \). Then we have an isomorphism

\[ \delta_K : 1 + p^k_K \tilde{\to} \text{Gal}(K^{ab}/K^{(k)}K^{ur}). \]

Because the character \( \tilde{\vartheta} : K^\times \to \mathbb{C}^\times \) comes from a character of

\[ G_{\beta}(O_F/p^f) \subset (O_K/p_K^{\text{cr}})^\times, \]

\( \varphi \) is trivial on \( \text{Gal}(K^{ab}/K^{(er)}K^{ur}) \). Note that \( K^{(er)}K^{ur} = K^{(cr)}F^{ur} \) is a finite extension of \( F^{ur} \). If we use the upper numbering

\[ V^s = V_t(K^{(cr)}F^{ur}/F^{ur}) \]

of the higher ramification group, where \( t \mapsto s \) is the inverse of Hasse function whose graph is

\[ \begin{array}{c|c|c|c|c}
    s & -1 & 1 & 2 & 3 \\
    \hline
    t & q^f - 1 & q^{2f} - 1 & q^{3f} - 1 \\
\end{array} \]

then \( \delta_K \) induces the isomorphism

\[ (1 + p^k_K)/(1 + p_K^{cr}) \tilde{\to} \text{Gal}(K^{(er)}K^{ur}/K^{(k)}K^{ur}) = V^s \]

for \( k - 1 < s \leq k \) (\( k = 1, 2, \cdots \)), and hence, for \( V_t = V_t(K^{(cr)}F^{ur}/F^{ur}) \), we have

\[ |V_t| = \begin{cases} 
    e \cdot q^{nrf} (1 - q^{-f}) & : t = 0, \\
    q^{nr-fk} & : q^{f(k-1)} - 1 < t \leq q^f k - 1.
\end{cases} \]
The explicit actions (4.12) and (3.17) combined with Proposition 3.4.1 shows that the space $\hat{\mathcal{g}}^{V_i}$ of the $\text{Ad} \circ \varphi(V_i)$-fixed vectors in $\hat{\mathcal{g}}$ is

$$\begin{align*}
\begin{cases}
\begin{bmatrix}
1 \\
\vdots \\
1
\end{bmatrix}, & a_i \in \mathbb{C}, \\
a_1 + a_2 + \cdots + a_f = 0
\end{cases}
\end{align*}$$

if $t = 0$,

$$\begin{align*}
\begin{cases}
\begin{bmatrix}
a_1 \\
a_2 \\
\vdots \\
a_n
\end{bmatrix}, & a_i \in \mathbb{C}, \\
a_1 + a_2 + \cdots + a_n = 0
\end{cases}
\end{align*}$$

if $0 < t \leq q^{f(e(r-1)-1)} - 1$, 

$$\begin{align*}
\begin{cases}
\begin{bmatrix}
A_1 \\
A_2 \\
\vdots \\
A_f
\end{bmatrix}, & A_i \in M_n(\mathbb{C}), \\
\text{tr}(A_1 + A_2 + \cdots + A_n) = 0
\end{cases}
\end{align*}$$

if $q^{f(e(r-1)-1)} - 1 < t \leq q^{f(e(r-1)-1)} - 1$ and $\hat{\mathcal{g}}$ if $q^{f(e(r-1)-1)} - 1 < t \leq q^{f(e(r-1)-1)} - 1$. So we have

$$\dim_{\mathbb{C}} \hat{\mathcal{g}}^{V_i} = \begin{cases}
f - 1 : t = 0, \\
n - 1 : 0 < t \leq q^{f(e(r-1)-1)} - 1, \\
f^{e^2 - 1} : q^{f(e(r-1)-1)} - 1 < t \leq q^{f(e(r-1)-1)} - 1, \\
n^2 - 1 : q^{f(e(r-1)-1)} - 1 < t \leq q^{f(e(r-1)-1)} - 1
\end{cases}$$

Hence we have

$$\sum_{t=0}^{\infty} (V_0 : V_t)^{-1} \dim_{\mathbb{C}} (\hat{\mathcal{g}}/\hat{\mathcal{g}}^{V_i})$$

$$= n^2 - f + (n^2 - n) \cdot \frac{1}{e} \cdot \{e(r-1) - 1\} + \frac{1}{e}$$

$$= rn(n-1).$$

Combined with (4.4), we have

$$\gamma(\varphi, \text{Ad}, \psi, d(x), 0) = w(\text{Ad} \circ \varphi) \cdot q^{r(n-1)/2} \cdot f \cdot \frac{1 - q^{-1}}{1 - q^{-f}}$$

(4.5)

where $d(x)$ is the Haar measure on $F$ such that the volume of $O_F$ is one. The explicit value of the root number $w(\text{Ad} \circ \varphi)$ will be given in the subsection 5.1.

### 4.2 $\gamma$-factor of principal parameter

Let $\text{Sym}_{n-1}$ be the symmetric tensor representation of $SL_2(\mathbb{C})$ on the space of the complex coefficient homogeneous polynomials of $X, Y$ of degree $n - 1$, which gives the group homomorphism

$$\text{Sym}_{n-1} : SL_2(\mathbb{C}) \rightarrow GL_n(\mathbb{C})$$

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with respect to the \( \mathbb{C} \)-basis \( \left\{ v_k = \frac{1}{(k-1)!} X^{n-k} Y^{k-1} \right\} \). Then

\[
d \text{Sym}_{n-1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = N_0 = \begin{bmatrix} 0 & 1 & 1 & \cdots & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \cdots & 0 \\ \end{bmatrix}
\]

is the nilpotent element in \( \mathfrak{pgl}_n(\mathbb{C}) = \mathfrak{sl}_n(\mathbb{C}) \) associated with the standard \( \acute{\text{e}} \text{pinglage} \) of the standard root system of \( \mathfrak{sl}_n(\mathbb{C}) \). Then \( \varphi_0 : W_F \times SL_2(\mathbb{C}) \xrightarrow{\text{proj}} SL_2(\mathbb{C}) \xrightarrow{\text{Sym}_{n-1}} GL_n(\mathbb{C}) \xrightarrow{\text{canonical}} PGL_n(\mathbb{C}) \) (4.6) is a representation of Weil-Deligne group with the associated triplet \( (\varphi_0, PGL_n(\mathbb{C}), N_0) \) such that \( \varphi_0 |_{I_F} \) is trivial and

\[
\rho_0(\widetilde{Fr}) = \begin{bmatrix} q^{-(n-1)/2} \\ q^{-(n-3)/2} \\ \ddots \\ q^{(n-3)/2} \\ q^{(n-1)/2} \end{bmatrix} \in PGL_n(\mathbb{C}).
\]

Let \( \text{Ad} : PGL_n(\mathbb{C}) \to GL_\mathfrak{g} \) be the adjoint representation of \( PGL_n(\mathbb{C}) \) on \( \mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}) \). Then

\[
\left\{ N_0^k \mid k = 1, 2, \cdots, n-1 \right\}
\]

is the \( \mathbb{C} \)-basis of \( \mathfrak{g}_{N_0} = \{ X \in \mathfrak{g} \mid [X, N_0] = 0 \} \). The representation matrix of \( \text{Ad} \circ \rho_0(\widetilde{Fr}) \) on \( \mathfrak{g}_{N_0} \) with respect to the \( \mathbb{C} \)-basis (4.7) is

\[
\begin{bmatrix} q^{-1} & q^{-2} & \cdots & q^{-(n-1)} \\ q^{-2} & \cdots & q^{-(n-2)} \\ \vdots & \ddots & \ddots & \vdots \\ q^{-(n-1)} & \cdots & q^{-1} & q^{-2} \\ \end{bmatrix}
\]

so that we have

\[
L(\varphi_0, \text{Ad}, s) = \prod_{k=1}^{n-1} \left( 1 - q^{-(s+k)} \right)^{-1}.
\]

On the other hand [8, p.448] shows

\[
\varepsilon(\varphi_0, \text{Ad}, \psi, d(x), 0) = q^{n(n-1)/2}
\]

where \( d(x) \) is the Haar measure on \( F \) such that the volume of \( O_F \) is one. Since the symmetric tensor representation \( \text{Sym}_{n-1} \) is self-dual, we have

\[
\gamma(\varphi_0, \text{Ad}, \psi, d(x), 0) = q^{n(n-1)/2} \cdot \frac{1 - q^{-1}}{1 - q^{-n}}.
\]

(4.8)
4.3 Verification of formal degree conjecture

Let $\mathcal{A}_\varphi$ be the centralizer of $\text{Im } \varphi$ in $PGL_n(\mathbb{C})$. Let us denote by $\mathcal{A}_\tilde{\varphi}$ the set of the group homomorphism $\lambda : W_{K/F} \to \mathbb{C}^\times$ whose restriction to $K^\times$ is a character $x \mapsto \tilde{\varphi}(x^{\sigma-1})$ with some $\tau \in \text{Gal}(K/F)$ which is uniquely determined by $\lambda$ due to Proposition 3.3.1. Let us call it associated with $\lambda$. If $\tau \in \text{Gal}(K/F)$ is associated with $\lambda \in \mathcal{A}_\tilde{\varphi}$, then we have

$$\tilde{\varphi}(x^{\sigma(\tau-1)}) = \tilde{\varphi}(x^{\tau-1})$$

(4.9)

for all $x \in K^\times$ and $\sigma, \tau \in \text{Gal}(K/F)$, because

$$\lambda(x^\sigma) = \lambda((\sigma,1)^{-1}(1,x)(\sigma,1)) = \lambda(x).$$

This implies that $\mathcal{A}_\tilde{\varphi}$ is in fact a subgroup of the character group of $W_{K/F}$.

Take a $T(\bmod \mathbb{C}^\times) \in \mathcal{A}_\varphi$ with $T \in GL_\mathbb{C}(V_0) = GL_n(\mathbb{C})$. Then we have a character

$$\lambda : W_{K/F} \to \mathbb{C}^\times$$

such that $gT = \lambda(g)Tg$ for all $g \in W_{K/F}$. If $(Tv_1)(\tau) \neq 0$ with $\tau \in \text{Gal}(K/F)$ then $x \cdot T(v_1) = \lambda(x)T(x \cdot v_1)$ for $x \in K^\times$ implies $\tilde{\varphi}(x^\tau) = \lambda(x)\tilde{\varphi}(x)$ for all $x \in K^\times$. So $(Tv_1)(\tau') \neq 0$ for some $\tau' \in \text{Gal}(K/F)$ implies that $\tilde{\varphi}(x^\tau) = \tilde{\varphi}(x^{\tau'})$ for all $x \in K^\times$ and hence $\tau' = \tau$, due to Proposition 3.4.1. Hence we have $Tv_1 = c \cdot v_\tau$ with $c \in \mathbb{C}^\times$. Then we have

$$Tv_\sigma = c \cdot \lambda(\sigma)^{-1} \cdot v_\sigma = c \cdot \lambda(\sigma)^{-1} \tilde{\varphi}(\alpha_{K/F}(\sigma,\tau)) \cdot v_\sigma$$

for all $\sigma \in \text{Gal}(K/F)$. We have

**Proposition 4.3.1** $\mathcal{T} \mapsto \lambda$ gives a group isomorphism of $A_\varphi$ onto $A_\tilde{\varphi}$.

**Proof** It is clear that $\mathcal{T} \mapsto \lambda$ is injective group homomorphism, because $\text{Ind}_{K^\times}^{W_{K/F}} \tilde{\varphi}$ is irreducible. Take any $\lambda \in \mathcal{A}_\tilde{\varphi}$ and the $\tau \in \text{Gal}(K/F)$ associated with it. Define a $T \in GL_\mathbb{C}(V)$ by

$$Tv_\sigma = \lambda(\sigma)^{-1} \tilde{\varphi}(\alpha_{K/F}(\sigma,\tau)) \cdot v_\sigma$$

for all $\sigma \in \text{Gal}(K/F)$. Then we have $gT = \lambda(g) \cdot Tg$ for all $g \in W_{K/F}$. □

We have in fact

**Proposition 4.3.2** $A_\tilde{\varphi}$ is equal to the group of the character $\lambda$ of $W_{K/F}$ which is trivial on $K^\times$. In particular

$$|A_\tilde{\varphi}| = (O_K : N_{K/F}(O_K^\times)) \cdot f.$$  

(4.10)

**Proof** Let $\lambda : W_{K/F} \to \mathbb{C}^\times$ be a group homomorphism such that $\lambda(x) = \tilde{\varphi}(x^{\tau-1})$ for all $x \in K^\times$ with some $\tau \in \text{Gal}(K/F)$. Because of (4.9), We have

$$\tilde{\varphi}(x^{\tau-1})^n = \prod_{\sigma \in \text{Gal}(K/F)} \tilde{\varphi}(x^{\sigma(\tau-1)}) = 1$$

for all $x \in K^\times$. Since $n$ is prime to $p$, the mapping $x \mapsto x^n$ is surjective group homomorphism of $1 + p_K$ onto $1 + p_K$. Hence $1 + p_K \subset \text{Ker}(\lambda|_{O_K^\times})$, that is
\( \tilde{\vartheta}(x^\tau) = \tilde{\vartheta}(x) \) for all \( x \in 1 + \mathfrak{p}_K \). Then \( \tau = 1 \) by Proposition 3.4.1. So \( \lambda \) is trivial on \( K^\times \). Now we have

\[ |A_{\tilde{\vartheta}}| = (K_1 : F) = (K_1 : K_0) \cdot f \]

where \( K_1 \) is the maximal abelian subextension of \( K/F \). On the other hand we have \( N_{K/F}(O_K^\times) = N_{K_1/F}(O_{K_1}^\times) \), and hence we have

\[ (O_K^\times : N_{K/F}(O_K^\times)) = e(K_1/F) = (K_1 : K_0) \]

because \( K_0 \) is the maximal unramified subextension of \( K/F \). So \( \lambda \) is trivial on \( K \times \). Now we have

\[ |A_{\tilde{\vartheta}}| = (K_1 : F) = (K_1 : K_0) \cdot f \]

where \( K_1 \) is the maximal abelian subextension of \( K/F \). On the other hand we have \( N_{K/F}(O_K^\times) = N_{K_1/F}(O_{K_1}^\times) \), and hence we have

\[ (O_K^\times : N_{K/F}(O_K^\times)) = e(K_1/F) = (K_1 : K_0) \]

because \( K_0 \) is the maximal unramified subextension of \( K/F \). ■

Let \( d_{G(F)}(g) \) be the Haar measure on \( G(F) = SL_n(F) \) with respect to which the volume of \( G(O_F) = SL_n(O_F) \) is one. Then the Euler-Poincaré measure \( \mu_{G(F)} \) on \( G(F) \) is

\[ d\mu_{G(F)}(g) = (-1)^{n-1} q^{n(n-1)/2} \prod_{k=1}^{n-1} (1 - q^{-k}) \cdot d_{G(F)}(g) \]

(see [14, 3.4, Théorème 7]). Then, by Theorem 2.3.1 the formal degree of the supercuspidal representation \( \pi_{\beta, \theta} = \text{ind}_{G(O_F)}^{G(F)} \delta_{\beta, \theta} \) with respect to the absolute value of the Euler-Poincaré measure on \( G(F) \) is

\[ q^{(r-1)n(n-1)/2} \frac{1 - q^{-n}}{1 - q^{-1}} (O_F^\times : N_{K/F}(O_K^\times)) \]

Now the formulae (4.5), (4.8) and (4.10) give the following

**Theorem 4.3.3** The formal degree of the supercuspidal representation \( \pi_{\beta, \theta} = \text{ind}_{G(O_F)}^{G(F)} \delta_{\beta, \theta} \) with respect to the absolute value of the Euler-Poincaré measure on \( G(F) \) is

\[ \frac{1}{|A_{\varphi}|} \cdot \frac{\gamma(\varphi, \text{Ad}, \psi, d(x), 0)}{\gamma(\varphi_0, \text{Ad}, \psi, d(x), 0)} \]

where \( d(x) \) is the Haar measure on \( F \) such that the volume of \( O_F \) is one.

Since \( A_{\varphi} \) is a finite abelian group, all the irreducible representation of \( A_{\varphi} \) is one-dimensional. So Theorem 13.3 says that the formal degree conjecture is valid if we consider (1.1) as the Arthur-Langlands parameter of the supercuspidal representation \( \pi_{\beta, \theta} \) and (1.4) as the principal parameter of \( G(F) = SL_n(F) \).

**5 Root number conjecture**

In this section, we will assume that \( K/F \) is a tamely ramified Galois extension such that the degree \( (K : F) = n \) is prime to \( p \), and will keep the notations of preceding sections. Put \( \Gamma = \text{Gal}(K/F) \).
5.1 Root number of adjoint representation

We will identify the representations of $W_{K/F}$ with the representations of $W_F$ which factor through the canonical surjection 

$$W_F \to W_F/[W_K.W_K] = W_{K/F}.$$ 

We will also regard a representation of $\Gamma$ as the representation of $W_{K/F}$ via the projection $W_{K/F} \to \Gamma$. Then we have

Proposition 5.1.1

$$\text{Ad} \circ \varphi = \bigoplus_{1 \neq \pi \in \Gamma} \pi^{\dim \pi} \oplus \bigoplus_{1 \neq \gamma \in \Gamma} \text{Ind}_{K^\times}^{W_{K/F}} \tilde{\vartheta}_\gamma$$

where $\Gamma^\sim$ is the set of the equivalence classes of the irreducible complex representations of $\Gamma$, and $\tilde{\vartheta}_\gamma(x) = \tilde{\vartheta}(x^{1-\gamma})$ $(x \in K^\times)$ for $1 \neq \gamma \in \Gamma$.

[Proof] The adjoint action of $\varphi$ on $gl_n(C) = C \oplus sl_n(C)$ gives $(\text{Ad} \circ \varphi)$ on $\hat{g}$, giving the trivial representation of $W_F = \varphi_1 \otimes \varphi_1^*$. So the character of $\text{Ad} \circ \varphi$ is

$$\chi_{\text{Ad} \circ \varphi}(g) = \chi_{\varphi_1}(g) \cdot \chi_{\varphi_1}(g) = \begin{cases} -1 & : \sigma \neq 1, \\ n - 1 + \sum_{\tau, \gamma \in \Gamma, \tau \neq \gamma} \tilde{\vartheta}(x^{\tau - \gamma}) & : \sigma = 1. \end{cases}$$

for $g = (\sigma, x) \in W_{K/F} = \Gamma \ltimes_{\alpha_{K/F}} K^\times$. Since

$$\sum_{\tau, \gamma \in \Gamma, \tau \neq \gamma} \tilde{\vartheta}(x^{\tau - \gamma}) = \sum_{1 \neq \gamma \in \Gamma} \sum_{\tau \in \Gamma} \tilde{\vartheta}(x^{\tau(\gamma - 1)}) ,$$

we have the required decomposition of $\text{Ad} \circ \varphi$. ■

In order to calculate the $\varepsilon$-factor of $\text{Ad} \circ \varphi$, we will fix the additive character

$$\psi_F : F \xrightarrow{T_{F/Q_p}} Q_p \xrightarrow{\text{canonical}} Q_p/Z_p \xrightarrow{\exp(2\pi \sqrt{-1} \cdot \cdot \cdot)} C^\times$$

of $F$ so that

$$\{ x \in F \mid \varphi_F(xO_F) = 1 \} = \mathcal{D}(F/Q_p)^{-1} = p_F^{-d(F)} ,$$

and the Haar measure $d_F(x)$ on $F$ such that

$$\int_{O_F} d_F(x) = q^{-d(F)} .$$

Then $\psi = \psi_F \circ T_{L/F}$ for any finite extension $L/F$. For the sake of simplicity, put

$$\varepsilon(\ast, \psi_F) = \varepsilon(\ast, \psi_F, d_F(x), 0) .$$

On the other hand the additive character $\psi$ of $F$ is such that

$$\{ x \in F \mid \psi(xO_F) = 1 \} = O_F$$

and the Haar measure $d(x)$ on $F$ is such that

$$\int_{O_F} d(x) = 1 .$$

Then we have

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Theorem 5.1.2 Assume that $r \geq 3$. Then we have

$$\varrho(\varphi, \text{Ad}, d(x), 0) = w(\varphi, \text{Ad}) \cdot q^{n(n-1)/2}$$

with

$$w(\varphi, \text{Ad}) = \vartheta((-1)^{n-1}) \times \begin{cases} (-1)^{\frac{q^{f/2}}{2}} & : e=\text{even}, \\ 1 & : e=\text{odd}. \end{cases}$$

[Proof] Note that

$$\bigoplus_{\pi \neq \pi \in \Gamma} \pi^{\dim \pi} = \text{Ind}_K^F 1_K - 1_F$$

and

$$\text{Ind}_{K^\times}^{W_{K/F}} \tilde{\vartheta}_\gamma = \text{Ind}_{K^\times}^{W_{K/F}} \tilde{\vartheta}_{\gamma-1}$$

for $1 \neq \gamma \in \Gamma$. So, if we choose a subset $S \subset \Gamma$ such that

$$\{ \gamma \in \Gamma \mid \gamma^2 \neq 1 \} = \{ \gamma, \gamma^{-1} \mid \gamma \in S \}$$

and $S \cap S - 1 = \emptyset$, then we have

$$\varrho(\text{Ad} \circ \varphi, \psi) = \lambda(K/F, \psi_F)^n \times \varrho(1_K, \psi_K) \times \varrho(1_F, \psi_F)^{-1}$$

$$\times \prod_{\gamma \in S} \varrho(\tilde{\vartheta}_\gamma, \psi_F) \times \varrho(\tilde{\vartheta}_{\gamma-1}, \psi_F) \times \prod_{1 \neq \gamma \in \Gamma, \gamma^2 = 1} \varrho(\tilde{\vartheta}_\gamma, \psi_F). \quad (5.1)$$

By (A.1), we have

$$\varrho(1_K, \psi_K) \cdot \varrho(1_F, \psi_F)^{-1} = q^{(d(K)/2) \cdot q^{-d(F)/2}}$$

$$= q^{(n-1)d(F)/2 + (n-f)/2}, \quad (5.2)$$

since $K/F$ is tamely ramified and hence $D(K/F) = p_K^{-1}$. For the Gauss sum, we have

$$G_{\psi_{K_0}} \left( \frac{\ast}{K_0} , \varpi_0^{-(d(K_0)+1)} \right)^2 = \left( \frac{-1}{K_0} \right) = (-1)^{(q' - 1)/2}.$$

Then, by Proposition A.3.5 we have

$$\lambda(K/F, \psi_F) = \begin{cases} (-1)^{\frac{q^{f/2}}{2}} & : e=\text{even}, \\ 1 & : e=\text{odd}. \end{cases} \quad (5.3)$$

Note that if $e$ is even, since $e$ divides $q^f - 1$, we have

$$\frac{q^f - 1}{2} \cdot \frac{e}{2} \equiv \frac{q^f - 1}{2} \pmod{2} \equiv \frac{q - 1}{2} \cdot f \pmod{2}.$$

For any $1 \neq \gamma \in \Gamma$, we have

$$n(\tilde{\vartheta}) = \text{Min} \{ 0 < k \in \mathbb{Z} \mid \tilde{\vartheta}_\gamma(1 + p_K) = 1 \} = \begin{cases} e(r - 1) + 1 : \gamma \not\in \text{Gal}(K/K_0), \\ e(r - 1) : \gamma \in \text{Gal}(K/K_0). \end{cases}$$

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by Proposition 3.4.1. So we have
\[
\prod_{\gamma \in S} \varepsilon(\varrho_\gamma, \psi_K) \cdot \varepsilon(\varrho_\gamma^{-1}, \psi_K) \times \prod_{1 \neq \gamma \in \Gamma, \gamma^2 = 1} \varepsilon(\varrho_\gamma, \psi_K)
\]
\[
= q_K^{(e-1)(d(K)+e(r-1))/2} \times q_K^{(n-e)(d(K)+e(r-1)+1)/2}
\]
\[
\times \prod_{\gamma \in S} \left\{ G_{\psi_K}(\varrho_\gamma^{-1}, -w_K^{-(d(K)+n(\varrho_\gamma))}) \cdot \varrho_\gamma(w_K)^{d(K)+n(\varrho_\gamma)} \right\}
\]
\[
\times \prod_{1 \neq \gamma \in \Gamma, \gamma^2 = 1} G_{\psi_K}(\varrho_\gamma^{-1}, -w_K^{-(d(K)+n(\varrho_\gamma))}) \cdot \varrho_\gamma(w_K)^{d(K)+n(\varrho_\gamma)}.
\] (5.4)

For any $1 \neq \gamma \in \Gamma$ such that $\gamma^2 = 1$, put $K = K_\gamma(\xi)$ where $\xi \in K$ such that $\xi^2 \in K_\gamma$. Then 7 Th.3 shows that
\[
G_{\psi_K}(\varrho_\gamma^{-1}, w_K^{-(d(K)+n(\varrho_\gamma))}) \cdot \varrho_\gamma(w_K)^{d(K)+n(\varrho_\gamma)} = \varrho_\gamma(\xi) = \varrho(-1).
\]

On the other hand we have
\[
\sharp\{1 \neq \gamma \in \Gamma \mid \gamma^2 = 1\} = \sharp\{H \subset \Gamma : \text{subgroup s.t. } |H| = 2\}
\]
\[
\equiv 1 \pmod{2} : n \text{ is even,}
\]
\[
\equiv 0 \pmod{2} : n \text{ is odd.}
\]

Then we have
\[
\prod_{1 \neq \gamma \in \Gamma, \gamma^2 = 1} G_{\psi_K}(\varrho_\gamma^{-1}, w_K^{-(d(K)+n(\varrho_\gamma))}) = \varrho(-1)^{n-1} = \varrho((-1)^{n-1})
\] (5.5)

Combining (5.1), (5.2), (5.3), (5.4) and (5.5), we have
\[
\varepsilon(\text{Ad} \circ \varphi, \psi_F) = q^{d(F)(n^2-1)/2+rn(n-1)/2} \cdot \varrho((-1)^{n-1})
\]
\[
\times \begin{cases} (-1)^{2n-1-f} : e \text{ is even,} \\ 1 : e \text{ is odd.} \end{cases}
\]

Since $\psi_F(x) = \psi(w_F^{d(F)x})$ and $d_F(x) = q^{-d(F)/2} \cdot d(x)$, we have the required formula of $\varepsilon(\text{Ad} \circ \varphi, \psi, d_F(0))$ by Proposition A.3.2

5.2 Verification of root number conjecture

Let $D$ be the maximal torus of $SL_n$ consisting of the diagonal matrices. The group $X^\vee(D)$ of the one-parameter subgroup of $D$ is identified with
\[
\mathbb{Z}^n_{\text{tr}=0} = \{(m_1, \ldots, m_n) \in \mathbb{Z}^n \mid m_1 + \cdots + m_n = 0\}
\]
by \( m \mapsto u_m \) where
\[
u_m(t) = \begin{bmatrix} t^{m_1} \\ \vdots \\ t^{m_n} \end{bmatrix}
\]
or we will denote by \( u_m = \sum_{i=1}^{n} m_i u_i \). Then the set of the co-roots of \( SL_n \) with respect to \( D \) is
\[
\Phi^\vee(D) = \{ u_i - u_j \in X^\vee(D) \mid 1 \leq i, j \leq n, i \neq j \}.
\]
Now we have
\[
2 \cdot \rho = \sum_{1 \leq i < j \leq n} (u_i - u_j) = \sum_{i=1}^{n} (n + 1 - 2i) u_i.
\]
So the special central element is
\[
\epsilon = 2 \cdot \rho(-1) = (-1)^{n-1} 1_n \in SL_n(F).
\]
Since
\[
\pi_{\beta, \theta} = \text{Ind}^{G(F)}_{G(O_F)} \delta_{\beta, \theta}, \quad \delta_{\beta, \theta} = \text{Ind}^{G(O_F/p_F)}_{G(O_F/p_F)} \sigma_{\beta, \theta},
\]
by recalling the construction of \( \sigma_{\beta, \theta} \), we have
\[
\pi_{\beta, \theta}(\epsilon) = \delta_{\beta, \theta}(\epsilon) = \sigma_{\beta, \theta}(\epsilon) = \theta((-1)^{n-1}).
\]
On the other hand, since \( \vartheta = c \cdot \theta \), we have
\[
\vartheta(\varpi, \text{Ad}) = \theta((-1)^{n-1})
\]
by Proposition 3.3.1 and Theorem 5.1.2. So we have proved the following theorem

**Theorem 5.2.1** \( \vartheta(\varpi, \text{Ad}) = \pi_{\beta, \theta}(\epsilon) \).

This theorem says that the root number conjecture is valid if we consider \( (4.1) \) as the Arthur-Langlands parameter of the supercuspidal representation \( \pi_{\beta, \theta} \).

**A Local factors**

Fix an algebraic closure \( F^{\text{alg}} \) of \( F \) in which we will take every algebraic extensions of \( F \). Put
\[
\nu_F(x) = (F(x) : F)^{-\text{ord}_F(N_F(x)/F(x))} \text{ for } \forall x \in F^{\text{alg}}
\]
and
\[
O_K = \{ x \in F^{\text{alg}} \mid \nu_F(x) \geq 0 \}, \quad p_K = \{ x \in F^{\text{alg}} \mid \nu(F)(x) > 0 \}.
\]
Then \( K = O_K/p_K \) is an algebraic extension of \( F = O_F/p_F \). If \( K/F \) is a finite extension, fix a generator \( \varpi_K \in O_K \) of \( p_K \).
A.1 Weil group

Let $F^\text{ur}$ be the maximal unramified extension of $F$ and $F^\text{ur} \in \text{Gal}(F^\text{ur}/F)$ the inverse of the Frobenius automorphism of $F^\text{ur}$ over $F$. The Weil group $W_F$ of $F$ is

$$W_F = \{ \sigma \in \text{Gal}(F^\text{alg}/F) \mid \sigma|_{F^\text{ur}} \in \langle Fr \rangle \}$$

The group $W_F$ is a locally compact group with respect to the topology such that $I_F = \text{Gal}(F^\text{alg}/F^\text{ur})$ is an open compact subgroup of $W_F$.

Let $F^\text{ab}$ be the maximal abelian extension of $F$ in $F^\text{alg}$. Then $[W_F, W_F] = \text{Gal}(F^\text{alg}/F^\text{ab})$ and

$$W_F/[W_F, W_F] \xrightarrow{\text{res.}} \{ \sigma \in \text{Gal}(F^\text{alg}/F) \mid \sigma|_{F^\text{ur}} \in \langle Fr \rangle \}.$$ 

So, by the local class field theory, there exists a topological group isomorphism

$$\delta_F : F^\times \to W_F/[W_F, W_F]$$

such that $\delta_F(\pi)|_{F^\text{ur}} = Fr$. Fix a $\tilde{F}r \in \text{Gal}(F^\text{alg}/F)$ such that $\tilde{F}r|_{F^\text{ab}} = \delta_F(\pi)$. Then

$$W_F = \langle \tilde{F}r \rangle \ltimes \text{Gal}(F^\text{alg}/F^\text{ur}).$$

Let $K/F$ be a finite extension in $F^\text{alg}$. Then $K^\text{ur} = K \cdot F^\text{ur}$ and

$$W_K = \{ \sigma \in \text{Gal}(F^\text{alg}/K) \mid \sigma|_{F^\text{ur}} \in \langle Fr \rangle \} = \{ \sigma \in W_F \mid \sigma|_K = 1 \},$$

where $f = (K : F)$, is a closed subgroup of $W_F$. If further $K/F$ is a Galois extension, then $[W_K, W_K] \triangleleft W_F$ and

$$W_{K/F} = W_F/[W_K, W_K] = \{ \sigma \in \text{Gal}(K^\text{alg}/F) \mid \sigma|_{F^\text{ur}} \in \langle Fr \rangle \}$$

is called the relative Weil group of $K/F$. Then we have an exact sequence

$$1 \to K^\times \to W_{K/F} \xrightarrow{\text{res.}} \text{Gal}(K/F) \to 1$$

which is the group extension associated with the fundamental class

$$[\alpha_{K/F}] \in H^2(\text{Gal}(K/F), K^\times),$$

that is, we can identify $W_{K/F} = \text{Gal}(K/F) \times K^\times$ with the group operation

$$\left( (\sigma, x) \cdot (\tau, y) = (\sigma \tau, \alpha_{K/F}(\sigma, \tau) \cdot xy) \right).$$

Let $K_0 = K \cap F^\text{ur}$ be the maximal unramified subextension of $K/F$. Then the fundamental class can be chosen so that $\alpha_{K/F}(\sigma, \tau) \in O_K^\times$ for all $\sigma, \tau \in \text{Gal}(K/K_0)$, and the image $I_{K/F}$ of $I_F = \text{Gal}(F^\text{alg}/F^\text{ur}) \subset W_F$ under the canonical surjection $W_F \to W_{K/F}$ is identified with $\text{Gal}(K/K_0) \times O_K^\times$. 

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A.2 Artin conductor of representations of Weil group

Let \((\Phi, V)\) be a finite dimensional continuous complex representation of the Weil group \(W_F\). Since \(I_F \cap \text{Ker}(\Phi)\) is an open subgroup of \(I_F = \text{Gal}(F^{\text{alg}}/F^\text{ur})\), there exists a finite Galois extension \(K/F^\text{ur}\) such that \(\text{Gal}(F^{\text{alg}}/K) \subset \text{Ker}(\Phi)\). Let

\[ V_k = V_k(K/F^\text{ur}) = \{ \sigma \in \text{Gal}(K/F^\text{ur}) \mid x^\sigma \equiv x \pmod{p_K^{k+1}} \text{ for } \forall x \in O_K \} \]

be the \(k\)-th ramification group of \(K/F^\text{ur}\) put

\[ \tilde{V}_k = \left[ \text{Gal}(F^{\text{alg}}/F^\text{ur}) \xrightarrow{\text{res.}} \text{Gal}(K/F^\text{ur}) \right]^{-1} V_k \]

for \(k = 0, 1, 2, 3, \cdots\). So \(\tilde{V}_0 = I_F\). The Artin conductor \(a(\Phi) = a(V)\) is defined by

\[ a(\Phi) = a(V) = \sum_{k=0}^{\infty} \dim_{\mathbb{C}}(V/V^{\Phi(\tilde{V}_k)}) \cdot |V_0/V_k|^{-1} \]

where

\[ V^{\Phi(\tilde{V}_k)} = \{ v \in V \mid \Phi(\tilde{V}_k)v = v \} \quad (k = 0, 1, 2, 3, \cdots). \]

A.3 \(\varepsilon\)-factor of representations of Weil group

Fix a continuous unitary character \(\psi : F \to \mathbb{C}^\times\) of the additive group \(F\) and a Haar measure \(d(x)\) of \(F\).

Langlands and Deligne \([2]\) show that, for every finite dimensional continuous complex representation \((\Phi, V)\) of \(W_F\), there exists a complex constant \(\varepsilon(\Phi, \psi, d(x)) = \varepsilon(V, \psi, d(x))\) which satisfies the following relations:

1) an exact sequence

\[ 1 \to V' \to V \to V'' \to 1 \]

implies

\[ \varepsilon(V, \psi, d(x)) = \varepsilon(V', \psi, d(x)) \cdot \varepsilon(V'', \psi, d(x)), \]

2) for a positive real number \(r\)

\[ \varepsilon(\Phi, \psi, r \cdot d(x)) = r^{\dim \Phi} \cdot \varepsilon(\Phi, \psi, d(x)), \]

3) for any finite extension \(K/F\) and a finite dimensional continuous complex representation \(\phi\) of \(W_K\), we have

\[ \varepsilon \left( \text{Ind}_{W_K}^{W_F} \phi, \psi, d(x) \right) = \varepsilon \left( \phi, \psi \circ T_{K/F}, d_K(x) \right) \cdot \lambda(K/F, \psi)^{\dim \phi} \]

where \(d_K(x)\) is a Haar measure of \(K\) and

\[ \lambda(K/F, \psi) = \lambda(K/F, \psi, d(x), d_K(x)) = \frac{\varepsilon \left( \text{Ind}_{W_K}^{W_F} 1_K, \psi, d(x) \right)}{\varepsilon \left( 1_K, \psi \circ T_{K/F}, d_K(x) \right)}, \]

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4) if $\dim \Phi = 1$, then $\Phi$ factors through $W_F/\left[W_F \cdot W_F\right]$ and put

$$\chi : F^\times \rightarrow W_F/\left[W_F \cdot W_F\right] \xrightarrow{\delta} \mathbb{C} \times.$$

Then we have

$$\varepsilon(\Phi, \psi, d(x)) = \varepsilon(\chi, \psi, d(x))$$

where the right hand side is the $\varepsilon$-factor of Tate [19].

If the Haar measure $d(x)$ of $F$ is normalized so that the Fourier transform

$$\hat{\varphi}(y) = \int_F \varphi(x) \cdot \psi(-xy) d(x)$$

has inverse transform

$$\varphi(x) = \int_F \hat{\varphi}(y) \cdot \psi(xy) d(y),$$

in other words

$$\int_{O_F} d(x) = q^{-n(\psi)/2} \text{ with } \{ x \in F \mid \psi(xO_F) = 1 \} = p_F^{-n(\psi)},$$

then the explicit value of the $\varepsilon$-factor $\varepsilon(\chi, \psi, d(x))$ is

1) if $\chi|_{O_F^\times} = 1$, then

$$\varepsilon(\chi, \psi, d(x)) = \chi(\varpi)^n(\psi) \cdot q^{n(\psi)/2},$$

(A.1)

2) if $\chi|_{O_F^\times} \neq 1$, then

$$\varepsilon(\chi, \psi, d(x)) = G_\psi(\chi^{-1}, -\varpi^{-(n(\psi)+n)} \cdot \chi(\varpi)^n(\psi) + n \cdot q^{-(n(\psi)+n)/2} \text{ (A.2)}$$

where $f(\chi) = \text{Min}\{0 < n \in \mathbb{Z} \mid \chi(1 + p_F^n) = 1\}$ and

$$G_\psi(\chi^{-1}, -\varpi^{-(n(\psi)+f(\chi))}) = q^{-n/2} \sum_{t \in (O_F/p_F^{f(\chi)})^\times} \chi(t)^{-1} \psi \left( -\varpi^{-(n(\psi)+f(\chi))} t \right)$$

is the Gauss sum.

**Remark A.3.1** The definition of the Gauss sum is normalized so that

$$\left| G_\psi(\chi^{-1}, -\varpi^{-(n(\psi)+f(\chi))}) \right| = 1.$$

We have

**Proposition A.3.2** 1) Put $\psi_a(x) = \psi(ax)$ for $a \in F^\times$. Then

$$\varepsilon(\Phi, \psi_a, d(x)) = \det \Phi(a) \cdot |a|_F^{-\dim \Phi} \cdot \varepsilon(\Phi, \psi, d(x))$$

where

$$\det \Phi : F^\times \rightarrow W_F/\left[W_F \cdot W_F\right] \xrightarrow{\det \phi} \mathbb{C} \times.$$
2) For any \( s \in \mathbb{C} \)

\[
\varepsilon(\Phi, \psi, d(x), s) = \varepsilon(\Phi \otimes | \cdot |_F^s, \psi, d(x)) \\
= \varepsilon(\Phi, \psi, d(x)) \cdot q^{-s(n(\psi) \cdot \dim \Phi + \alpha(\Phi))}.
\]

**Proposition A.3.3** If \( n(\psi) = 0 \) and the Haar measure \( d(x) \) is normalized so that

\[
\int_{O_F} d(x) = 1,
\]

then

\[
\varepsilon(\Phi, \psi, d(x)) = w(\Phi) \cdot q^{\alpha(\Phi)/2} = w(V) \cdot q^{\alpha(V)/2}
\]

with \( w(\Phi) \in \mathbb{C} \) of absolute value one.

**Proposition A.3.4** For finite extensions \( F \subset K \subset L \), we have

\[
\lambda(L/F, \psi) = \lambda(L/K, \psi \circ \delta_{K/F}) \cdot \lambda(K/F, \psi)^{(L:K)}.
\]

When \( K/F \) is a finite tamely ramified Galois extension, the maximal unramified subextension \( K_0 = K \cap F^{ur} \) is a cyclic extension of \( F \) and \( K/K_0 \) is also cyclic extension. So, by means of Proposition A.3.4, we can give the explicit value of \( \lambda(K/F, \psi) \).

Let \( \psi_F : F \to \mathbb{C}^\times \) be a continuous unitary character such that

\[
\{ x \in F \mid \psi_F(xOF) = 1 \} = \mathcal{D}(F/Q_p)^{-1} = p_F^{-d(F)}
\]

and the Haar measure \( d_F(x) \) on \( F \) is normalized so that

\[
\int_{O_F} d_F(x) = q^{-d(F)}.
\]

Let \( K/F \) be a tamely ramified finite Galois extension, and put \( \psi_K = \psi_F \circ \delta_{K/F} \).

Put

\[
e = e(K/F) = (K : K_0), \quad f = f(K/F) = (K_0 : F)
\]

where \( K_0 = K \cap F^{ur} \) is the maximal unramified subextension of \( K/F \). Let

\[
\left( \frac{e}{K_0} \right) = \begin{cases} 
1 & : e \equiv \text{square (mod } p_{K_0}), \\
-1 & : \text{otherwise}
\end{cases}
\quad (e \in \mathcal{O}_{K_0}^\times)
\]

be the Legendre symbol of \( K_0 \). Then we have

**Proposition A.3.5**

\[
\lambda(K/F, \psi_F) = \left\{ \begin{array}{ll}
(-1)^{\frac{e-1}{2} + \frac{e(e+2)}{8}} \cdot G_{\psi_K_0} \left( \left( \frac{*}{K_0} \right), \varpi_0^{-(d(K_0)+1)} \right) : & e = \text{even}, \\
(-1)^{(e-1)d(F)} : & e = \text{odd}
\end{array} \right.
\]

where \( \varpi_0 \) is a prime element of \( K_0 \) such that \( \varpi_0 \in N_{K/F_0}(K^\times) \).
A.4 γ-factors of admissible representations of Weil group

Definition A.4.1 The pair $(\Phi, V)$ is called an admissible representation of $W_F$ if

1) $V$ is a finite dimensional complex vector space and $\Phi$ is a group homomorphism of $W_F$ to $GL_C(V)$,

2) $\text{Ker}(\Phi)$ is an open subgroup of $W_F$,

3) $\Phi(\tilde{\text{Fr}}) \in GL_C(V)$ is semisimple.

Let $(\Phi, V)$ be an admissible representation of $W_F$. Since $I_F = \text{Gal}(F_{\text{alg}}/F_{\text{ur}})$ is a normal subgroup of $W_F$, $\Phi(\tilde{\text{Fr}}) \in GL_C(V)$ keeps $V_{I_F} = \{ v \in V \mid \Phi(\sigma)v = v \forall \sigma \in I_F \}$ stable. Then the $L$-factor of $(\Phi, V)$ is defined by

$$L(\Phi, s) = L(V, s) = \det \left( 1 - q^{-s} \cdot \Phi(\tilde{\text{Fr}})|_{V_{I_F}} \right)^{-1}.$$ 

Since $\Phi : W_F \to GL_C(V)$ is continuous group homomorphism, we have the ε-factor $\varepsilon(\Phi, \psi, d(x), s)$ of $\Phi$. Then the γ-factor of $(\Phi, V)$ is defined by

$$\gamma(\Phi, \psi, d(x), s) = \gamma(V, \psi, d(x), s) = \varepsilon(\Phi, \psi, d(x), s) \cdot \frac{L(\Phi, 1 - s)}{L(\Phi, s)}$$

where $\Phi^\sim$ is the dual representation of $\Phi$.

A.5 Symmetric tensor representation of $SL_2(\mathbb{C})$

The complex special linear group $SL_2(\mathbb{C})$ acts on the polynomial ring $\mathbb{C}[X, Y]$ of two variables $X, Y$ by 

$$g \cdot \varphi(X, Y) = \varphi((X, Y)g) \quad (g \in SL_2(\mathbb{C}), \varphi(X, Y) \in \mathbb{C}[X, Y]).$$

Let 

$$\mathcal{P}_n = \langle X^n, XY^{n-1}, \ldots, XY, Y^n \rangle_{\mathbb{C}}$$

be the subspace of $\mathbb{C}[X, Y]$ consisting of the homogeneous polynomials of degree $n$. The action of $SL_2(\mathbb{C})$ on $\mathcal{P}_n$ defines the symmetric tensor representation $\text{Sym}_n$ of degree $n + 1$. The complex vector space $\mathcal{P}_n$ has a non-degenerate bilinear form defined by

$$\langle \varphi, \psi \rangle = \varphi \left( -\frac{\partial}{\partial Y}, \frac{\partial}{\partial X} \right) \psi(X, Y) \bigg|_{(X, Y) = (0, 0)} \in \mathbb{C}$$

for $\varphi, \psi \in \mathcal{P}_n$. This bilinear form is $SL_2(\mathbb{C})$-invariant 

$$\langle \text{Sym}_n(g)\varphi, \text{Sym}_n(g)\psi \rangle = \langle \varphi, \psi \rangle \quad (g \in SL_2(\mathbb{C}), \varphi, \psi \in \mathcal{P}_n)$$

and

$$\langle \psi, \varphi \rangle = (-1)^n \langle \varphi, \psi \rangle \quad (\varphi, \psi \in \mathcal{P}_n).$$

So we have group homomorphisms 

$$\text{Sym}_n : SL_2(\mathbb{C}) \to SO(\mathcal{P}_n) \text{ if } n \text{ is even}$$

and 

$$\text{Sym}_n : SL_2(\mathbb{C}) \to Sp(\mathcal{P}_n) \text{ if } n \text{ is odd}.$$
A.6 Admissible representations of Weil-Deligne group

Fix a complex Lie group $G$ such that the connected component $G^0$ is a reductive complex algebraic linear group. Then the $G^0$-conjugacy class of the group homomorphisms

$$\varphi : W_F \times SL_2(\mathbb{C}) \to G$$

such that

1) $I_F \cap \ker(\varphi)$ is an open subgroup of $I_F$,

2) $\varphi(\tilde{F}r) \in G$ is semi-simple,

3) $\varphi|_{SL_2(\mathbb{C})} : SL_2(\mathbb{C}) \to G^0$ is a morphism of complex linear algebraic group

corresponds bijectively the equivalence classes of the triples $(\rho, G, N)$ where $N \in \operatorname{Lie}(G)$ is a nilpotent element and

$$\rho : W_F \to G$$

is a group homomorphism such that

1) $\rho|_{I_F} : I_F \to G$ is continuous,

2) $\rho(\tilde{F}r) \in G$ is semi-simple,

3) $\rho(\sigma)N = |\sigma|_F \cdot N$ for $\forall \sigma \in W_F$ where

$$| \cdot |_F : W_F \xrightarrow{\text{can}} W_F/[W_F, W_F] \xrightarrow{\text{l.c.f.t.}} F^\times \xrightarrow{q^{-\ord}_{F}(\cdot)} Q^\times$$

by the relations

$$\rho|_{I_F} = \varphi|_{I_F}, \quad \rho(\tilde{F}r) = \varphi(\tilde{F}r) \cdot \varphi \left( \begin{smallmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{smallmatrix} \right), \quad N = d_{\varphi} \left( \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} \right)$$

(see [8, Prop.2.2]). Here two triples $(\rho, G, N)$ and $(\rho', G', N')$ is equivalent if there exists a $g \in G$ such that $\rho' = g \rho g^{-1}$ and $N' = \operatorname{Ad}(g)N$.

The couple $(\varphi, G)$ or the triple $(\rho, G, N)$ is called an admissible representation of the Weil-Deligne group.

Let $(r, V)$ be a continuous finite dimensional complex representation of $G$ which is algebraic on $G^0$. Then the $L$-factor associated with $(\varphi, G)$ and $(r, V)$ is defined by

$$L(\varphi, r, s) = \det \left( 1 - q^{-s} r \circ \rho(\tilde{F}r)|_{V_N} \right)^{-1},$$

where $V_N = \{ v \in V \mid dr(N)v = 0 \}$ and

$$V_N^{I_F} = \{ v \in V_N \mid r \circ \rho(\sigma)v = v \ \forall \sigma \in I_F \}.$$ 

The $\varepsilon$-actor is defined by

$$\varepsilon(\varphi, r, \psi, d(x), s) = \varepsilon(r \circ \rho, \psi, d(x), s) \cdot \det \left( -q^{-s} r \circ \rho(\tilde{F}r)|_{V_N^{I_F}} \right)$$

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where \( \varepsilon(r \circ \rho, \psi, d(x), s) \) is the \( \varepsilon \)-factor of the representation \((r \circ \rho, V)\) of \( W_F \) defined in the subsection A.4. Finally the \( \gamma \)-factor is defined by

\[
\gamma(\varphi, r, \psi, d(x), s) = \varepsilon(\varphi, r, \psi, d(x), s) \cdot \frac{L(\varphi, r^\vee, 1 - s)}{L(\varphi, r, s)}
\]

where \( r^\vee \) is the dual representation of \( r \).

Let \( \text{Sym}_n \) be the symmetric tensor representation of \( SL_2(\mathbb{C}) \) of degree \( n + 1 \). Then the \( W_F \times SL_2(\mathbb{C}) \)-module \( V \) has a decomposition

\[
V = \bigoplus_{n=0}^{\infty} V_n \otimes \text{Sym}_n
\]

where \( V_n \) is a \( W_F \)-module. Then we have

\[
V_n^{IF} = \bigoplus_{n=0}^{\infty} V_n^{IF} \otimes \text{Sym}_{n,N}
\]

where \( \text{Sym}_{n,N} \) is the highest part of \( \text{Sym}_n \). Since \( r \circ \rho(\widetilde{F}) \) act on \( V_n \otimes \text{Sym}_{n,N} \) by \( q^{-n/2} r \circ \varphi(\widetilde{F}) \), we have

\[
L(\varphi, r, s) = \prod_{n=0}^{\infty} \det \left( 1 - q^{-(s+n/2)} r \circ \varphi(\widetilde{F})|_{V_n^{IF}} \right)^{-1}.
\]

If the Haar measure \( d(x) \) on the additive group \( F \) and the additive character \( \psi : F \to \mathbb{C}^\times \) are normalized so that \( \int_{O_F} d(x) = 1 \) and

\[
\{ x \in F \mid \psi(xO_F) = 1 \} = O_F,
\]

then we have

\[
\varepsilon(\varphi, r, \psi, d(x), s) = w(\varphi, r) \cdot q^{a(\varphi, r)(1/2 - s)}
\]

where

\[
w(\varphi, r) = \prod_{n=0}^{\infty} w(V_n)^{n+1} \cdot \prod_{n=1}^{\infty} \det \left( -\varphi(\widetilde{F})|_{V_n^{IF}} \right)^n
\]

and

\[
a(\varphi, r) = \sum_{n=0}^{\infty} (n + 1) a(V_n) + \sum_{n=1}^{\infty} n \cdot \dim V_n^{IF}.
\]

If \( \varphi|_{SL_2(\mathbb{C})} = 1 \), then \( V_n = 0 \) for all \( n > 0 \) and we have

\[
w(\varphi, r) = w(r \circ \varphi) = w(r \circ \rho), \quad a(\varphi, r) = a(r \circ \varphi) = a(r \circ \rho).
\]

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