An observation: cut-off of the weight $w$ does not increase the $A_{p_1,p_2}$-“norm” of $w$

Alexander Reznikov, Vasiiliy Vasyunin, Alexander Volberg

Abstract
We consider weights $w$ and their cut-offs: $w_a(t) = w(t)$ if $w(t) \leq a$ and $w_a(t) = a$ if $w(t) \geq a$. We consider a generalized $A_p$-“norm” and prove that the “norm” of $w_a$ is not greater then the “norm” of $w$. Our proof in the case $w \in A_2$ is especially simple.

1 Introduction
Put $I$ be a cube in $\mathbb{R}^n$ and $p_1 > p_2$. For every summable function $\varphi$ and any subset $J \subset I$ we denote

$$\langle \varphi \rangle_J = \frac{1}{|J|} \int_J \varphi(t) dt,$$

where $|J|$ is Lebesgue measure of $J$. For simplicity, when we take an average over the whole cube $I$, we’ll drop the subindex and write $\langle \varphi \rangle$.

Take a nonnegative function $w$. By Hölder’s inequality we have

$$\langle w^{p_1} \rangle_J^{1/p_1} \langle w^{p_2} \rangle_J^{1/p_2} \geq 1 \quad \forall J \subset [0,1].$$

(1)

We would like to consider an upper bound of the left-hand side. Precisely, we define

$$[w]_{p_1,p_2} = \sup \left( \langle w^{p_1} \rangle_J^{1/p_1} \langle w^{p_2} \rangle_J^{1/p_2} \right),$$

where the supremum is taken over all cubes $J$, $J \subset I$. If $[w]_{p_1,p_2} < \infty$ then we say that $w \in A_{p_1,p_2}$. Note that if $p_1 = 1$, $p_2 = 1 - p' = -\frac{1}{p-1}$ then we get a famous class $A_p$. In this case instead of $[.,1,1-p']$ we write $[..]_p$.

Notice that the reverse Hölder inequality is also included as a particular case of these classes $A_{p_1,p_2}$.

We shall assume that $p_i \neq 0, \pm\infty$, but it is clear that everything remains true in the limit cases. In the case $p = 0$ the expression $\langle w^{p} \rangle_J^{1/p}$ has to be replaced by $\exp(\langle \log w \rangle_J)$. It has to be replaced by $\sup_J w$ in the case $p = +\infty$ and by $\inf_J w$ in the case $p = -\infty$.

We also point out the $A_2$-case: when $p_1 = 1$ and $p_2 = -1$ we have

$$[w]_2 = \sup_{J \subset I} \left( \langle w \rangle_J \langle w^{-1} \rangle_J \right)$$
Observe that function $w$ can be unbounded or not separated from zero. However, for some problems it is convenient to consider only bounded, separated from zero weights.

For a given $a$, $a > 0$, we define

$$w_a(t) = \begin{cases} w(t), & w(t) \leq a \\ a, & w(t) > a. \end{cases}$$

It was well known that the following inequality is true:

$$[w_a]_p \leq c \cdot [w]_p$$

with a constant $c$. The main purpose of this text is to delete this constant and write 1 instead.

## 2 Main results

We are going to prove the following general theorem.

**Theorem 2.1.** Let $p_1 > p_2$; let $w$ be a nonnegative function, defined on $I \subset \mathbb{R}^d$. Take

$$w_a(t) = \begin{cases} w(t), & w(t) \leq a \\ a, & w(t) > a. \end{cases}$$

Then for every cube $J$, $J \subset I$, the following is true:

$$\langle w^{p_1}_a \rangle^\frac{1}{p_1} J \langle w^{p_2}_a \rangle^\frac{1}{p_2} J - \langle w^{p_1}\rangle^\frac{1}{p_1} J \langle w^{p_2}\rangle^\frac{1}{p_2} J \leq 0. \quad (2)$$

Consequently,

$$[w_a]_{p_1,p_2} \leq [w]_{p_1,p_2}.$$

This theorem gives an answer to a similar question, when we cut from below. Precisely,

**Corollary 2.2.** Denote

$$w^a(t) = \begin{cases} w(t), & w(t) \geq a \\ a, & w(t) < a. \end{cases}$$

Then the following inequality holds:

$$[w^a]_{p_1,p_2} \leq [w]_{p_1,p_2}.$$

This corollary is an immediate consequence of the theorem, since instead of $a, w, p_1, p_2$ we can consider $\frac{1}{a}, \frac{1}{w}, -p_2, -p_1$. 


Corollary 2.3. Take a function $w \in A_{p_1,p_2}$. For every integer $n$, $n \geq 1$, denote

$$
\phi_n(t) = \begin{cases} 
n, & w(t) > n \\
w(t), & \frac{1}{n} < w(t) \leq n \\
\frac{1}{n}, & w(t) \leq \frac{1}{n} \end{cases}.
$$

Then

$$
[w]_{p_1,p_2} \leq [\phi_n]_{p_1,p_2} \quad \text{(3)}
$$

$$
\lim_{n \to \infty} [\phi_n]_{p_1,p_2} = [w]_{p_1,p_2}. \quad \text{(4)}
$$

We give an independent proof of (2) in one leading particular cases of the class $A_2$.

The possibility to approximate a function in the class $A_p$ by bounded functions from the same class and with the control of their $A_p$ constants (and we have the best possible control here) can be used in various places. First of all, [1] shows how this can be used to show that the set $\{p : w \in A_p\}$ is open. Secondly, working with Bellman function proofs of various sharp reverse Hölder inequalities or sharp John–Nirenberg type inequalities (see e.g. [3], [4]), one needs an approximation of a weight $w$ in $A_p$ (and, more generally, $w \in A_{p_1,p_2}$) by the weights bounded from above and from below and of at most the same $A_p$ constant. We show how this can be easily achieved by just a standard “cut-off” procedure on weights. Seems like this has not been observed in the literature, even though it amounts to a very simple remark.

3 History of the question

As far as we know, the known result for $w_a$, $w^n$ and $\phi_n$ is the following inequality:

$$
[w_a]_{p_1,p_2} \leq 2[w]_{p_1,p_2}.
$$

In this work we erase the constant 2. We should cite the work [1], where the different approach is described. Authors consider weights

$$
\frac{s + w(t)}{s^2 + sw(t) + 1},
$$

which are bounded and which also satisfy (3) and (4) as $s \to +0$. However, we think that our approach is more natural if one wants to get a bounded weight separated from zero.

4 Proof of the Corollary 2.3

Inequality (3) follows from the main theorem 2.1 and from the corollary 2.2. Thus, we need to prove (4). By the monotone convergence theorem we have

$$
\langle \phi_{n_k}^p \rangle \to \langle w^p \rangle.
$$
therefore, for every $J \subset I$, the following is true:

$$\langle \varphi_n \rangle_{J}^{1/p} \langle \varphi_n \rangle_{J}^{-1/p^2} \to \langle w^{p_1} \rangle_{J}^{1/p} \langle w^{p_2} \rangle_{J}^{-1/p^2}, \quad n \to \infty.$$  

Therefore,

$$\langle w^{p_1} \rangle_{J}^{1/p} \langle w^{p_2} \rangle_{J}^{-1/p^2} = \lim \left( \langle \varphi_n \rangle_{J}^{1/p} \langle \varphi_n \rangle_{J}^{-1/p^2} \right) = \lim \inf \left( \langle \varphi_n \rangle_{J}^{1/p} \langle \varphi_n \rangle_{J}^{-1/p^2} \right) \quad (5)$$

$$\leq \lim \inf [\varphi_n]_{p_1,p_2} \leq \lim \sup [\varphi_n]_{p_1,p_2} \leq [w]_{p_1,p_2}. \quad (6)$$

Passing to the supremum over $J$ in the right-hand side, we get

$$[w]_{p_1,p_2} \leq \lim \inf [\varphi_n]_{p_1,p_2} \leq \lim \sup [\varphi_n]_{p_1,p_2} \leq [w]_{p_1,p_2},$$

which finishes the proof.

In next three sections we prove the main Theorem 2.1.

5 The case of $A_2$-weights

We separate this case since here everything is in some sense linear, and we can prove everything without taking derivatives. In this case $p_1 = 1$, $p_2 = -1$. Fix a cube $J \subset I$ and denote

$$J_1 = \{ t \in J : w(t) \leq a \}, \quad J_2 = \{ t \in J : w(t) > a \},$$

$$x_i = \langle w \rangle_{J_i}, \quad y_i = \langle \frac{1}{w} \rangle_{J_i}, \quad \alpha_i = \frac{|J_i|}{|J|}.$$  

Then

$$\langle w \rangle_{J} \langle w^{-1} \rangle_{J} - \langle w_a \rangle_{J} \langle w_a^{-1} \rangle_{J}$$

$$= (\alpha_1 x_1 + \alpha_2 x_2)(\alpha_1 y_1 + \alpha_2 y_2) - (\alpha_1 x_1 + \alpha_2 a)(\alpha_1 y_1 + \alpha_2 a^{-1})$$

$$= \alpha_1 \alpha_2 (x_1 y_2 + x_2 y_1 - y_1 a - x_1 a^{-1}) + \alpha_2^2 (x_2 y_2 - 1).$$

The expression in the second parentheses is positive and therefore it is sufficient to check that the expression in the first parentheses is positive as well.

$$x_1 y_2 + x_2 y_1 - y_1 a - x_1 a^{-1} = x_1 (y_2 - a^{-1}) + y_1 (x_2 - a)$$

$$= \langle x_1 (w^{-1} - a^{-1}) + y_1 (w - a) \rangle_{J_2} = \langle \frac{w - a}{w a} (way_1 - x_1) \rangle_{J_2}.$$

Since $y_1 \geq \frac{1}{a}$ and $x_1 \leq a$, we have $way_1 - x_1 \geq w - a$, which finishes the proof.

6 Proof of the general case

In this section we present a fully general proof.
We keep all notation from the preceding section with a natural modification. Fix a cube $J \subset I$ and put

$$J_1 = \{ t \in J : w(t) \leq a \}, \quad J_2 = \{ t \in J : w(t) > a \},$$

$$x_i = \langle w^{p_1} \rangle_{J_i}, \quad y_i = \langle w^{p_2} \rangle_{J_i}, \quad \alpha_i = \frac{|J_i|}{|J|}.$$ Then we want to prove

$$\langle w^{p_1} \rangle_{\frac{1}{J}} \langle w^{p_2} \rangle_{\frac{1}{J}} - \langle w^{p_1} \rangle_{\frac{1}{a} J} \langle w^{p_2} \rangle_{\frac{1}{a} J} = (\alpha_1 x_1 + \alpha_2 x_2)^{\frac{1}{p_1}} (\alpha_1 y_1 + \alpha_2 y_2)^{\frac{1}{p_2}} - (\alpha_1 x_1 + \alpha_2 a^{p_1})^{\frac{1}{p_1}} (\alpha_1 y_1 + \alpha_2 a^{p_2})^{\frac{1}{p_2}} \geq 0. \tag{7}$$

By Hölder’s inequality, we get $x_1^{\frac{1}{p_1}} \geq y_1^{\frac{1}{p_2}}$. Therefore, if we denote $y_2^{\frac{1}{p_2}}$ by $u$, then $x_2^{\frac{1}{p_1}} = su$ for a number $s \geq 1$ and expression (7) we need to estimate can be written as the following function of $s$ and $u$:

$$\varphi(s, u) = (\alpha_1 x_1 + \alpha_2 s^{p_1} u^{p_1})^{\frac{1}{p_1}} (\alpha_1 y_1 + \alpha_2 u^{p_2})^{\frac{1}{p_2}} - (\alpha_1 x_1 + \alpha_2 a^{p_1})^{\frac{1}{p_1}} (\alpha_1 y_1 + \alpha_2 a^{p_2})^{\frac{1}{p_2}}.$$ Since

$$\frac{\partial \varphi}{\partial s} = \alpha_2 s^{p_1-1} u^{p_1} (\alpha_1 x_1 + \alpha_2 s^{p_1} u^{p_1})^{\frac{1}{p_1}-1} \geq 0,$$

the function $\varphi$ is increasing in $s$ and therefore $\varphi(s, u) \geq \varphi(1, u)$, i.e., it has the minimal value when $w(t)$ is equal to $u$ on $J_2$ identically.

Now we have $u = w(t)|_{J_2} > a$ and since $\varphi(1, a) = 0$, the desired inequality will be proved after checking that $\frac{\partial \varphi}{\partial u}(1, u) \geq 0$.

$$\frac{\partial \varphi}{\partial u}(1, u)
= \alpha_2 u^{-1} (\alpha_1 x_1 + \alpha_2 u^{p_1})^{\frac{1}{p_1}-1} (\alpha_1 y_1 + \alpha_2 u^{p_2})^{\frac{1}{p_2}-1} \times u^{p_1} (\alpha_1 y_1 + \alpha_2 u^{p_2}) - u^{p_2} (\alpha_1 x_1 + \alpha_2 u^{p_1})]
= \alpha_1 \alpha_2 u^{-1} (\alpha_1 x_1 + \alpha_2 u^{p_1})^{\frac{1}{p_1}-1} (\alpha_1 y_1 + \alpha_2 u^{p_2})^{\frac{1}{p_2}-1} [u^{p_1} y_1 - u^{p_2} x_1]$$

and we are done because $u^{p_1} y_1 - u^{p_2} x_1 \geq 0$. Indeed, since $u \geq w(t)$ and $p_1 \geq p_2$, we have $u^{p_1-p_2} \geq w(t)^{p_1-p_2}$, whence $u^{p_1} u^{p_2} \geq u^{p_2} w^{p_1}$. Therefore,

$$u^{p_1} y_1 - u^{p_2} x_1 = \langle u^{p_1} w^{p_2} - u^{p_2} w^{p_1} \rangle_{J_1} \geq 0,$$

what completes the proof.

**References**

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