On cosmography and flat $\Lambda$CDM tensions at high redshift

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ABSTRACT

Recently constructed high-redshift Hubble diagrams of supernovae, quasars and gamma-ray bursts show 4\,$\sigma$ deviations from flat $\Lambda$CDM, where the discrepancies are observed in cosmographic expansions. In this note, we show that existing approaches to high-redshift cosmography can lead to “tensions” that are simply an artifact of the expansions, even if the underlying model is fully consistent with flat $\Lambda$CDM. Thus, in this note we revisit the 4\,$\sigma$ tensions to ascertain if results have been impacted by the cosmography. In the process, we find that the quasar data, which has been calibrated by external Type Ia supernovae, is surprisingly inconsistent with both supernovae and gamma-ray bursts to the point that combining the datasets is questionable. Results based on cosmography only appear to mask more significant tensions.

1. Introduction

To first approximation the Universe is well described by the $\Lambda$CDM cosmological model built on the assumptions of a cosmological constant and cold dark matter. Recently, this harmony has been disturbed by conflicting determinations of the Hubble constant $H_0$. Indeed, the disagreement between local measurements (Riess et al. 2019; Wang et al. 2019; Freedman et al. 2019) and determinations based on the Cosmic Microwave Background by the Planck Collaboration is now in a 4-6\,$\sigma$ window (Verde et al. 2019). In the wake of Hubble tension/problem/crisis, other tensions have emerged (KiDS+VIKING-450+DES-Y1 2019), which point to potential differences in matter density $\Omega_m$. Since $\Omega_m$ is suppressed by factors of redshift $z$ relative to $H_0$ in the Hubble parameter $H(z)$, these tensions are redshift dependent and understandably less well established: the statistical significance is about 2–3\,$\sigma$.

Against this backdrop claims of 4\,$\sigma$ deviations (Risaliti & Lusso 2019; Lusso et al. 2019) from flat $\Lambda$CDM based on a high-redshift Hubble diagram of Type Ia supernovae (SNe), quasars (QSOs) and gamma-ray bursts (GRBs) are eye-catching. Recently, great strides have been made towards the use of QSOs as standard candles (Risaliti & Lusso 2015) and Risaliti & Lusso have succeeded in producing a high quality catalogue of 1598 QSOs in the redshift range 0.04 < $z$ < 5.1 (Risaliti & Lusso 2019). When combined with Pantheon Type Ia SNe (Scolnic et al. 2018) and an existing GRB compilation (Demianski et al. 2017a; Demianski et al. 2017b), this results in 2800 odd spectroscopically confirmed objects in the redshift range 0.01 < $z$ < 6.67. Since the Pantheon dataset is consistent with flat $\Lambda$CDM, for the 4\,$\sigma$ tension to be believable, it must be attributable to either the high-redshift QSO or GRB data.

But before jumping to conclusions, there may be another interesting explanation. At first sight the log polynomial cosmographic expansion employed in Risaliti & Lusso 2019 and later Lusso et al. 2019 is intriguing, since in the idealised limit, i. e.

\begin{footnotesize}
\item[1] This precludes two GRBs at $z$ = 8.1 and $z$ = 9.3, which were dropped by Lusso et al. 2019 precisely for problems related to cosmography.
\end{footnotesize}

small error bars and good quality data, these expansions are guaranteed to fail to recover flat $\Lambda$CDM over such a wide redshift range, e. g. 0 < $z$ < 7. As we show in section 2, tensions that are an artifact of cosmography can appear at low redshift. Intuitively, this is easily understood: conforming to high-redshift data may come at the price of distorting the low-redshift fit away from flat $\Lambda$CDM, even if the underlying model is flat $\Lambda$CDM. So a priori, it is not clear if the 4\,$\sigma$ tension is real or an artifact of cosmography. Moreover, the fact that tension appears at low redshift (Risaliti & Lusso 2019; Lusso et al. 2019) when high-redshift data is added is already suggestive that the cosmographic expansion is impacting results.

However, in the end it turns out that the core tension comes from the data. That being said, fitting a log polynomial cosmographic expansion to the data fails to capture the full extent of the tension. In contrast to Risaliti & Lusso 2019 and Lusso et al. 2019, where a circuitous route was followed that ultimately culminated in flat $\Lambda$CDM tension, here we do the obvious and fit each dataset independently to flat $\Lambda$CDM. In the process, we observe that the tension between the QSO data and flat $\Lambda$CDM based on canonical values greatly exceeds 4\,$\sigma$ and conservatively may stand closer to a 20\,$\sigma$ discrepancy. Cosmography simply dilutes this to a more palatable 4\,$\sigma$.

Given the success of $\Lambda$CDM, such a large deviation only raises suspicions of the QSO dataset. Since the luminosity distance $d_L(z)$ data we are using was calibrated using external SNe data (Risaliti & Lusso 2019; Lusso et al. 2019), but recent studies treating the free parameters as nuisance parameters show no tension with flat $\Lambda$CDM (Melia 2019; Khadka & Ratra 2019), it appears that the tension is an unintended consequence of the calibration process.

2. Limitations of high-redshift cosmography

Even in its traditional form as a Taylor expansion around $z$ = 0 (Visser 2004), great care should be taken with cosmography. For example, it is well documented that the expansion cannot be trusted beyond $z$ > 1 and instead one should use the improved
expansion parameter (Catton & Visser 2007)

\[
y = \frac{z}{1 + z}
\]  

(1)

Switching between \( z \) and \( y \) overcomes the problem with convergence, since higher powers of \( y \), i.e. \( y^n \), clearly converge. This still leaves the thorny problem concerning where best to truncate the cosmographic expansion. Related problems have been teased out elsewhere (Capozziello et al. 2008, Capozziello et al. 2011;Busse et al. 2015;Zhang et al. 2017). In this section, we focus on log polynomial approach to cosmography adopted in Risaliti & Lusso 2019 and later Lusso et al. 2019, but our analysis serves more generally as a cautionary tale for applications of cosmographic expansions to high-redshift Hubble diagrams. We will later also briefly touch upon traditional cosmography.

The basic idea (Risaliti & Lusso 2019) is to expand the luminosity distance in powers of \( \log_{10}(1 + z) \):

\[
dl(z) = \frac{c \ln(10)}{H_0} \left[ \log_{10}(1 + z) + a_2 \log_{10}^2(1 + z) + a_3 \log_{10}^3(1 + z) + a_4 \log_{10}^4(1 + z) + \ldots \right]
\]  

(2)

where \( H_0, a_2, a_3 \) and \( a_4 \) are free parameters. The leading term in this expansion is fixed by requiring \( H(z = 0) = H_0 \) and this can be confirmed through the following identity for the Hubble parameter \( H(z) \):

\[
H(z) = \left[ \frac{d}{dz} \left( \frac{dL(z)}{c(1 + z)} \right) \right]^{-1}
\]  

(3)

Since flat \( \Lambda \)CDM has only two parameters \( (H_0, \Omega_m) \), to identify the remaining parameters \( a_i \) in terms of the standard model, one expands the above luminosity distance and the flat \( \Lambda \)CDM luminosity distance around \( z = 0 \). Making a comparison term by term, one identifies the following relations (Lusso et al. 2019):

\[
a_2 = \ln(10) \left( \frac{3}{2} + \frac{3}{4} \Omega_m \right), \quad a_3 = \ln^2(10) \left( \frac{9}{8} \Omega_m^2 - \frac{7}{6} \right), \quad a_4 = \ln^3(10) \left( \frac{-135}{64} \Omega_m^3 + \frac{9}{2} \Omega_m^2 - \frac{47}{16} \Omega_m + \frac{5}{8} \right)
\]  

(4)

(5)

(6)

It is worth stressing here that we have performed a Taylor expansion around \( z = 0 \) to identify the parameters \( a_i \) in terms of the single parameter \( \Omega_m \) in flat \( \Lambda \)CDM. For precisely this reason, one may worry that the above relations are only valid at low redshift. However, in the appendix we show that these relations are valid out beyond \( z \approx 4 \) in the sense that the discrepancy with \( \Lambda \)CDM is still less than 1% for canonical values \( (H_0, \Omega_m) = (70, 0.3) \), which we employ throughout this work. For later convenience we record the \( a_i \) based on canonical values:

\[
a_2 = 2.9358, \quad a_3 = 3.54123, \quad a_4 = 1.12066.
\]  

(7)

Note, throughout this work \( H_0 \) is a free parameter.

2.1. Mock data

Before proceeding further, let us consider a simple warm-up exercise to test the polynomial expansion (2). Consider the following seven redshifts \( z_i \in \{0, 1, 2, 3, 4, 5, 6\} \) and their corresponding luminosity distances \( d_L(z_i) \) for a flat \( \Lambda \)CDM model with canonical values. Now fit this data to the fourth order polynomial. Doing so, one will discover that the best-fit values of \( H_0, a_2, a_3 \) and \( a_4 \) are respectively, 5.6\( \sigma \), 6.6\( \sigma \), 10.4\( \sigma \) and 10.3\( \sigma \) removed from their expected values (7). Obviously, since the underlying model is \( \Lambda \)CDM, the tension here is simply an artifact of the cosmographic expansion. In essence, there are simply too few parameters for such a wide range of redshifts.

![Fig. 1. Compilation of SNe, QSOs and GRBs from Lusso et al. 2019](Image)

Of course the reader is justified to protest that we have not considered data with error bars so our above example is simply fanciful. To this end, let us consider a second exercise. Here, we work with the original data of Lusso et al. 2019 comprising SNe, QSOs and GRBs, reproduced in Fig. 1. Relative to the distance moduli of the QSOs, the SNe and GRBs are displaced by +19.36 and +0.54, respectively. The first figure here is easy enough to understand as it is more or less the canonical value for the absolute magnitude \( M \) of Type Ia SNe. We will see later than these shifts lead to best-fit values of \( H_0 \) that are within 1\( \sigma \) of the canonical value \( H_0 = 70 \text{ km s}^{-1} \text{Mpc}^{-1} \).

Now, let us mock up some data. For each triplet \( (z_i, \mu(z_i), \Delta \mu(z_i)) \), where \( \mu(z_i) \) is the distance modulus and \( \Delta \mu(z_i) \) denotes the error in the distance modulus of the original data, we replace the second entry with a new value \( \tilde{\mu}(z_i) \) that is picked from a normal distribution around its flat \( \Lambda \)CDM value \( \mu_{\Lambda \text{CDM}}(z_i) \) with standard deviation given by the error \( \Delta \mu(z_i) \), i.e.

\[
\tilde{\mu}(z_i) \sim N(\mu_{\Lambda \text{CDM}}(z_i), \Delta \mu(z_i)).
\]  

The final result is shown in Fig. 2. Compared to the real data, the mock data more closely follows flat \( \Lambda \)CDM, in line with expectations.

With the mock data in hand, the first thing to do is to confirm that the data is consistent with the underlying flat \( \Lambda \)CDM from where it was generated. In Fig. 3 we show the canonical values versus the best-fit values from Markov Chain Monte Carlo (MCMC) using the Python package emcee (Foreman-Mackey et al. 2013) and confirm that the returned values are within 1\( \sigma \) of the original values. This allows us to quan-

3. We thank Elisabeta Lusso and Guido Risaliti for explaining and sharing their luminosity distance data.
4. We ignore off-diagonal terms in the covariance matrix and just focus on the statistical errors.
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Fig. 2. Mock distance modulus data based on flat $\Lambda$CDM with canonical values.

Fig. 3. Best-fit values of flat $\Lambda$CDM to the mock data.

tify the degree of “noise” we have added in mocking up the data. Clearly, the data is consistent with flat $\Lambda$CDM.

That being said, fits of the fourth order log polynomial (2) make this agreement less clear. Again using MCMC we identify the best-fit values, this time for the parameters $H_0, a_2, a_3$ and $a_4$. The result appears in Fig. 4. Here we see a noticeable difference in that the values of the parameters $H_0, a_2, a_3$ are displaced by 2$\sigma$ when compared to their canonical values (7). $a_4$ performs marginally better, but is still outside of 1$\sigma$.

Since we have replaced a two-parameter model with a four-parameter cosmographic expansion, one would expect any tensions to decrease as the number of parameters is increased. Instead, here we have found the opposite. As argued earlier, this behaviour is not entirely unexpected as depending on the dataset, fitting the fourth order polynomial along the entire range of redshift can lead to “tensions” that are simply an artifact of the cosmography. This is clearly the case here with the mock data, but whether the cosmographic expansion leads to tension or not depends on the data. As the data improves and more closely follows flat $\Lambda$CDM, these tensions become more pronounced.

Fig. 4. Best-fit values of the fourth order polynomial to the mock data.

2.2. Real GRB data

Previously we argued that tensions that are simply an artifact of cosmography can arise when one fits too few parameters over too wide a range of redshifts. So far we have yet to provide a concrete example based on real data, so here we make amends.

In previous studies of GRBs using traditional cosmography with the improved parameter $y$ (1) deviations from flat $\Lambda$CDM have been recorded with statistical significance 2–3$\sigma$ (Demianski et al. 2012). We will now show that such deviations may be down to the cosmographic expansion by studying the GRB dataset with the two GRBs at $z = 8.1$ and $z = 9.3$ reinstated.

Fig. 5. Best-fit values of flat $\Lambda$CDM to the GRB dataset.

Let us begin by confirming that the GRB dataset on its own is consistent with flat $\Lambda$CDM. This provides us with an opportunity to test the $+0.54$ shift in the distance moduli, which
is the outcome of the calibration with respect to Type Ia SNe (Lusso et al. 2019). The best-fit values are illustrated in Fig. 5. Unsurprisingly, since we have few data points the errors are large, but nevertheless we see from that $H_0$ is within 1 $\sigma$ of $H_0 = 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$. We also recognise that the best-fit value of $\Omega_m$ is just under 1 $\sigma$ removed from the canonical value. Overall, there is no hint of any deviation from flat $\Lambda$CDM.

Now, let us switch our attention to the cosmographic expansion, but here we focus on traditional cosmography, which is the setting for earlier papers (Demianski et al. 2012). Following Zhang et al. 2017 for example, we consider a power series expansion in $y$:

$$d_L(y) = \frac{c}{H_0} \left( y + C_1 y^2 + C_2 y^3 + C_3 y^4 + C_4 y^5 \right),$$

where we record the first two terms in the expansion in terms of the deceleration $q_0$ and jerk parameter $j_0$ and refer the reader to Zhang et al. 2017 for the missing expressions:

$$C_1 = \frac{1}{2} (3 - q_0), \quad C_2 = \frac{1}{6} \left( 11 - 5 q_0 + 3 q_0^2 - j_0 \right).$$

The canonical values for $\Lambda$CDM are $(q_0, j_0) = (-0.55, 1)$, so that the canonical flat $\Lambda$CDM values of these parameters are

$$C_1 = 1.775, \quad C_2 = 2.27652.$$  \hspace{1cm} (10)

Using least squares fitting, the best-fit values of the parameters we record in Table 1, where we have inferred 1 $\sigma$ confidence intervals from the returned covariance matrix and in order to save space we have only quoted a single decimal space. This first thing to check is that the luminosity distance is indeed positive over the entire redshift range and this turns out to be the case. Next, from the best-fit values we can see that although $H_0$ is within 1 $\sigma$ of the canonical value, the discrepancies in $q_0$ and $j_0$ are respectively 2 $\sigma$ and 2.4 $\sigma$. While these deviations are consistent with those reported in Demianski et al. 2012 and elsewhere, since we have already checked that the data is consistent with flat $\Lambda$CDM, we confirm that these tensions are an artifact of the cosmographic expansion and not the data.

| $H_0$ | $C_1$ | $C_2$ | $C_3$ | $C_4$ |
|-------|-------|-------|-------|-------|
| 51.2^{+26.4}_{-26.4} | -9.0^{+5.3}_{-5.3} | 72.2^{+28.6}_{-28.6} | -160.5^{+64.6}_{-64.6} | 121.3^{+50.0}_{-50.0} |

Table 1. Best-fit of traditional cosmography in the y parameter to the full GRB dataset using least squares fitting.

Note, the situation with the real data closely mirrors the mock data example: consistency at the 1 $\sigma$ level becomes a slight tension above 2 $\sigma$. Thus, whether one works with the mock high-redshift data or real high-redshift data, we have seen in certain cases that tensions can be exacerbated by cosmographic expansions. This brings us to the pressing question at the heart of this study: to what extent are the 4 $\sigma$ tensions reported in Risaliti & Lusso 2019 and Lusso et al. 2019 attributable to cosmography?

3. Is the tension real?

To ascertain if the tension is real or not, i.e. due to the data or cosmography, it is simplest to return to the original luminosity data (Risaliti & Lusso 2019, Lusso et al. 2019), see Fig. 1, and perform independent fits of flat $\Lambda$CDM separately to SNe, QSO and GRB. On the basis of Fig. 5, we have already established that the GRB data, even with the two highest redshift GRBs, is consistent with flat $\Lambda$CDM.

Let us turn our focus to the Pantheon Type Ia SNe dataset (Scolnic et al. 2018), where one has 1048 SNe in a range 0.01 < $z$ < 2.26. Whether one employs least-square fitting or MCMC, it should come as no surprise that one recovers the $\Omega_m$ value from the fourth entry in Table 8 of Scolnic et al. 2018, thereby underscoring the fact that we have used the full Pantheon covariance matrix including both statistical and systematic uncertainties. More importantly, the best-fit value of $\Omega_m$ is consistent with the canonical flat $\Lambda$CDM value. The +19.36 shift in the distance moduli results in a best-fit value of $H_0 = 69.661^{+0.345}_{-0.345} \text{ km s}^{-1} \text{ Mpc}^{-1}$, which is also within 1 $\sigma$ of the canonical value $H_0 = 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$. Therefore, as promised, the Pantheon dataset is consistent with flat $\Lambda$CDM.

Next we turn our attention to the QSO dataset, where the best-fit MCMC values are presented in Fig. 6. The first thing that strikes one is that in contrast to the SNe dataset, here the preferred $\Omega_m$ value of $\Omega_m = 0.9994^{+0.0003}_{-0.0003}$ saturates the upper bound. Based on the MCMC 1 $\sigma$ confidence interval or the complementary least squares 1 $\sigma$

| $H_0$ | $\Omega_m$ |
|-------|-----------|
| 69.698^{+0.345}_{-0.345} | 0.298^{+0.022}_{-0.022} |
| 73.946^{+0.652}_{-0.652} | 1.003 |
| 68.628^{+0.980}_{-0.980} | 0.460^{+0.251}_{-0.251} |

Table 2. Best-fit values of flat $\Lambda$CDM to respective datasets based on least squares curve fitting.

With 162 data points least-squares fitting is well suited to this task. In contrast, there is a large degeneracy in the parameters and we were unable to get the corresponding MCMC to converge. Combining the GRB dataset with SNe, we have checked that this degeneracy is broken and MCMC recovers the least squares result.

5 With 162 data points least-squares fitting is well suited to this task.

6 However, see Ó Colgáin 2019 for a 2 $\sigma$ discrepancy with flat $\Lambda$CDM at low redshift.
confidence interval (Table 2), this means that the best-fit value is respectively $699 \sigma$ or $21 \sigma$ removed from the canonical value $\Omega_m = 0.3$. While the first figure may be taken with a pinch of salt, the more conservative second figure still constitutes a sizable deviation from flat ΛCDM, importantly one that greatly exceeds $4 \sigma$. However, the problems don’t stop there. The best-fit value of $H_0$ is either $28 \sigma$ or $6 \sigma$ removed from the canonical value $H_0 = 70 \text{km s}^{-1} \text{Mpc}^{-1}$. Even taking the more conservative least squares value at face value, this suggests that the QSO data is completely inconsistent with SNe data: it is questionable to even combine these datasets as they stand. Given that the authors of Risaliti & Lusso 2019 and Lusso et al. 2019 deliberately set out to calibrate the QSOs using Type Ia SNe, this is a rather unexpected outcome. However, it is clear from Fig. 4 of Lusso et al. 2019 that regardless of the value of $\Omega_m$ in the range $0.1 - 0.9$ that there is a $4 \sigma$ discrepancy, so our findings here appear reasonable.

To get a different perspective, it is a useful exercise to bin both the SNe and QSO data and compare. As is standard practice, we consider weighted means in each bin (Pinho et al. 2018):

$$\bar{s}_i = \frac{\sum s_k^i (\sigma^i_k)^{-2}}{\sum (\sigma^i_k)^{-2}}, \quad \sigma^i_k = \frac{1}{\sqrt{\sum s_k^i (\sigma^i_k)^{-2}}}$$  \hspace{1cm} (11)

where $s_k^i \equiv s(z_k)$ denotes the data value at each point $z_k$ with error $\sigma^i_k$ and $N_i$ is the number of data points in each bin $i$. $\bar{s}_i$ and $\sigma^i_k$ denote the new value and error for each bin. Note, in the binning process we adopt the weighted average value of $z_i$ for a given bin. The result of the binning procedure is illustrated in Fig. 7, where we have considered bins of length $\Delta z = 0.1$. From the plot it is clear that beyond $z \approx 0.4$, the binned QSO data is noticeably lower than the binned SNe data, which clearly conforms well to flat ΛCDM with canonical values. The lower binned values correspond to a larger matter density $\Omega_m$, in line with our results above.

![](image)

**Fig. 7.** Comparison between the SNe and QSO binned data.

We close this section by combining the two datasets that are cosmographic to see if there is any hint of tension due to the cosmographic model. In the end the fourth order log polynomial performs well and the best-fit values we present in Table 3. Evidently, the best-fit values are within $1 \sigma$ of their expected values (7) and there is no hint of tension. Given our concerns in the last section, we attribute the absence of tension here to the fact that the high-redshift GRB data is particularly sparse and it carries very little weight relative to the SNe data. We note that if one drops the systematic uncertainties the discrepancy from the expected values increases, but all values are still within $1 \sigma$.

| $H_0$ | $a_2$ | $a_3$ | $a_4$ |
|-------|-------|-------|-------|
| $69.85^{+0.61}_{-0.59}$ | $3.04^{+0.26}_{-0.26}$ | $3.31^{+0.49}_{-1.52}$ | $-0.13^{+2.46}_{-2.36}$ |

**Table 3.** Best-fit of the fourth order log polynomial to the combined SNe+GRB dataset using MCMC.

### 4. Discussion

This letter has two essential take-home messages. The first is that it is hard to make sense of cosmography as a sensible tool at high redshift. This is primarily down to the fact that a large number of parameters is required to recover flat ΛCDM in the limit where data quality is good, i.e. small error bars. With too few parameters, one runs the risk that even if the data is consistent with flat ΛCDM that the data will “stretch” or “distort” the parameters away from their preferred ΛCDM values. We have provided various examples based on mock and real data where this is the case and any “tension” is simply an artifact of cosmography. This makes cosmography at high redshift particularly confusing.

Secondly, as we have seen, tensions based on cosmographic expansions may poorly reflect glaring inconsistencies in the data. To that end, we have shown that the QSO dataset is inconsistent with the SNe and GRB datasets at leading order through a $6 \sigma$ discrepancy in $H_0$ and is inconsistent to a $21 \sigma$ level when matter density $\Omega_m$ is taken into account. The findings of Risaliti & Lusso 2019 and Lusso et al. 2019 based on cosmographic expansions simply obfuscate the matter. Given the existence of a number of recent complementary studies where no tension with flat ΛCDM is reported (Melia 2019, Khadka & Ratra 2019), it is most likely that the $4 \sigma$ tensions reported in Risaliti & Lusso 2019 and Lusso et al. 2019 are an artifact of the calibration process. These teething problems aside, in the big picture QSOs clearly have a bright future as standard candles and will surely open up new doors.

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### Appendix

In the appendix we collect some results connected to comments in the literature that are easily substantiated.

#### Validity of log polynomial expansion

Here we comment on the validity of the relations (4) to (6), which were derived in an expansion around $z = 0$. It is pertinent to ask at what $z$ the fourth order log polynomial (2) starts to show discrepancy with the corresponding flat ΛCDM luminosity distance with the same canonical values ($H_0, \Omega_m) = (70, 0.3)$. Defining

$$\Delta d_L(z) = \frac{d_L^{\text{QSO}}(z) - d_L^{\text{ΛCDM}}(z)}{d_L^{\text{ΛCDM}}(z)}$$  \hspace{1cm} (12)

the result is illustrated in Fig. 2. As can be seen from the plot, the relations (4) to (6) are robust out to $z \approx 4.3$, where we start to witness a $1\%$ discrepancy.

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Fig. 8. % difference between the fourth order log polynomial and flat $\Lambda$CDM based on canonical values.

Although the discrepancy is not so great, the key point is that if there is enough data, especially data consistent with $\Lambda$CDM, in the $6 < z < 7$ range, fitting this will come at the cost of introducing distortions elsewhere.

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