A Bourgain type bilinear estimate for a class of water-wave models

Qifan Li

September, 2010

We consider the general form of the equation of water-wave models on torus $T$

$$\partial_t u + \sum_{k=1}^{N} b_k \partial_x^{2k+1} u + Q(u, \partial_x u, \cdots \partial_x^{2N+1} u) = 0$$

(1)

where $Q$ denotes nonlinear term of the equation and $b_k$s are real constants. This equation was first introduced and studied by [3] on the real line. The symbol of the linear partial differential operator $L = \sum_{k=1}^{N} b_k \partial_x^{2k+1}$ is

$$m(\xi) = \sum_{k=1}^{N} b_k (2\pi i \xi)^{2k+1} = P(\xi)i$$

where $P(\xi) = \sum_{k=1}^{N} c_k \xi^{2k+1}$ and $c_k = (-1)^k (2\pi)^{2k+1} b_k$. We assume $c_k \leq 0$ for all the $k \geq 1$. In particular, generalized Kawahara equations [2] and fifth-order KdV equations [4] are the special cases satisfying this condition. We introduce the Bourgain space-time space $X^{s,b}$ with the norm

$$\|u\|_{X^{s,b}}^2 = \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} (1 + |\lambda + P(n)|^{2b} (1 + |n|^{2s}) |\hat{u}(n, \lambda)|^2) d\lambda.$$

(2)

We are going to establish a bilinear estimate which extends the proposition 7.15 in [1] to the higher order derivatives.

Lemma 0.1. Let functions $u, v : T \times [0, T] \to \mathbb{R}$, we have the bilinear estimate

$$\|uv\|_{L^2_t L^1_x} \lesssim \|u\|_{X^{s,b}} \|v\|_{X^{s,b}}.$$

(3)

Proof. Let

$$A_m = \left( \int_{\mathbb{R}} (1 + |\xi|)^{\frac{N+1}{2s+1}} |\hat{u}_m(\xi)|^2 d\xi \right)^{1/2}$$

$$B_m = \left( \int_{\mathbb{R}} (1 + |\xi|)^{\frac{N+1}{2s+1}} |\hat{v}_m(\xi)|^2 d\xi \right)^{1/2}$$

and

$$u(x, t) = \sum_{m \in \mathbb{Z}} e^{2\pi i (mx - P(m)t)} u_m(t), \quad v(x, t) = \sum_{m \in \mathbb{Z}} e^{2\pi i (mx - P(m)t)} v_m(t).$$

(4)
Let quadratic polynomial $Q(m, l) = P(m + l) - P(m) - P(l)$. We have
\[
(u\bar{v})(x, t) = \sum_{l \in \mathbb{Z}} e^{2\pi i (x-P(l)t)} \sum_{m \in \mathbb{Z}} e^{2\pi i Q(m, l)t} (u_m \overline{v_{m+t}})(t)
\]
and
\[
\|u\bar{v}\|_{L^2_x L^1_t}^2 = \sum_{l \in \mathbb{Z}} \int \left( \sum_{m \in \mathbb{Z}} e^{2\pi i Q(m, l)t} (u_m \overline{v_{m+t}})(t) \right)^2 dt.
\]
For a integer $j > 0$, define Paley-Littlewood operator
\[
\Delta_j f(t) = (1_{2^{-j-1} \leq |\xi| \leq 2^j} \hat{f}(\xi))^{\vee}(t)
\]
where $1_{2^{-j-1} \leq |\xi| \leq 2^j}$ denotes the characteristic function on the set $[2^{-j-1}, 2^j] \cup [-2^j, -2^{-j-1}]$, and set
\[
\Delta_0 f(t) = (1_{|\xi| \leq 1} \hat{f}(\xi))^{\vee}(t).
\]
We have the Paley-Littlewood decomposition for $u_m$ and $v_m$
\[
u_m = \sum_{p \geq 0} \Delta_p u_m, \quad \nu_m = \sum_{q \geq 0} \Delta_q v_m.
\]
We assert that \((5)\) can be estimated by
\[
\|u\bar{v}\|_{L^2_x L^1_t}^2 \lesssim \sum_{l \in \mathbb{Z}} \left( \sum_{q \geq p} \left( \sum_{m \in \mathbb{Z}} e^{2\pi i Q(m, l)t} \|\Delta_p u_m \Delta_q v_{m+l}(t)\|_{L^2_t} \right)^2 \right)
\]
\[
+ \sum_{l \in \mathbb{Z}} \left( \sum_{q \geq p} \left( \sum_{m \in \mathbb{Z}} e^{2\pi i Q(m, l)t} \|\Delta_p v_m \Delta_q u_{m+l}(t)\|_{L^2_t} \right)^2 \right).
\]
Since it is easy to see that
\[
\|u\bar{v}\|_{L^2_x L^1_t}^2 \lesssim \sum_{l \in \mathbb{Z}} \left( \sum_{q \geq p} \sum_{m \in \mathbb{Z}} e^{2\pi i Q(m, l)t} \|\Delta_p u_m \Delta_q v_{m+l}(t)\|_{L^2_t} \right)^2
\]
For the summation over $q \leq p$ inside the brackets, take $m' = m + l$ and $l' = -l$ we have $Q(m' + l', -l') = P(m') - P(m' + l') - P(-l') = -Q(m', l')$. By this reason, we can write
\[
\sum_{l \in \mathbb{Z}} \left( \sum_{q \leq p} \left( \sum_{m \in \mathbb{Z}} e^{2\pi i Q(m, l)t} \|\Delta_p u_m \Delta_q v_{m+l}(t)\|_{L^2_t} \right)^2 \right) = \sum_{l \in \mathbb{Z}} \left( \sum_{q \leq p} \left( \sum_{m' \in \mathbb{Z}} e^{2\pi i Q(m', l')t} \|\Delta_p u_{m'+l'} \Delta_q v_{m'}(t)\|_{L^2_t} \right)^2 \right)
\]
\[
+ \sum_{l' \in \mathbb{Z}} \left( \sum_{q \leq p} \left( \sum_{m' \in \mathbb{Z}} e^{2\pi i Q(m', l')t} \|\Delta_p u_{m'+l} \Delta_q v_{m'}(t)\|_{L^2_t} \right)^2 \right).
\]
This proves the estimate \((6)\).
We will need the pointwise estimate
\[
\sum_m |\Delta_i u_m|^2 \leq 2^j \left( \sum_m \|\Delta_i u_m\|_{L^2_t}^2 \right).
\] (8)

To prove (8), we shall use Jensen inequality and Plancherel theorem.

\[
\sum_m |\Delta_i u_m|^2 = \sum_{m} 2^{2j} \left\{ \frac{1}{2^j} \int_{2^{j-1} \leq |\xi| \leq 2^j} \hat{u}_m(\xi) e^{2\pi i \xi t} \right\}^2 
\leq \sum_m 2^j \int_{2^{j-1} \leq |\xi| \leq 2^j} |\hat{u}_m(\xi)|^2 d\xi = 2^j \left( \sum_m \|\Delta_i u_m\|_{L^2_t}^2 \right).
\]

We distinguish three cases

(i) \(|l|^{2N+1} \leq 2^q\)  \hspace{1cm} (ii) \(|l|^{2N} \leq 2^q < |l|^{2N+1}\)  \hspace{1cm} (iii) \(2^q < |l|^{2N}\).

Contribution of (i). By (8), we have the estimate

\[
\left\| \sum_m e^{2\pi i Q(m,l)t} (\Delta_p u_m \Delta_q v_{m+l})(t) \right\|_{L^2_t} \leq \left( \sum_m |\Delta_p u_m|^2 \right)^{1/2} \left( \sum_m |\Delta_q v_{m+l}|^2 \right)^{1/2} \leq 2^{p/2} \left( \sum_m \|\Delta_p u_m\|_{L^2_t}^2 \right)^{1/2} \left( \sum_m \|\Delta_q v_{m+l}\|_{L^2_t}^2 \right)^{1/2}.
\]

By Plancherel theorem we get

\[
\left\| \sum_m e^{2\pi i Q(m,l)t} (\Delta_p u_m \Delta_q v_{m+l})(t) \right\|_{L^2_t} \lesssim \sum_m 2^{-p \frac{N}{N+2}} \left( \sum_m \int_{2^{j-1} \leq |\xi| \leq 2^j} (1 + |\xi|)^{N+1} |\hat{u}_m(\xi)|^2 d\xi \right)^{1/2} \times \left( \sum_m \int_{2^{j-1} \leq |\xi| \leq 2^j} (1 + |\xi|)^{N+1} |\hat{v}_{m+l}(\xi)|^2 d\xi \right)^{1/2}.
\] (9)

Estimate (9) implies

\[
\sum_{q \geq p} \left\| \sum_m e^{2\pi i Q(m,l)t} (\Delta_p u_m \Delta_q v_{m+l})(t) \right\|_{L^2_t} \lesssim \sum_{2p \geq |l|^{2N+1}} 2^{-p \frac{N}{N+2}} \left( \sum_m \int_{2^{j-1} \leq |\xi| \leq 2^j} (1 + |\xi|)^{N+1} |\hat{u}_m(\xi)|^2 d\xi \right)^{1/2} \times \left( \sum_m B_{m}^2 \right)^{1/2} \lesssim |l|^{-1/4} \left( \sum_{2p \geq |l|^{2N+1}} 2^{-p \frac{N}{N+2}} \sum_m \int_{2^{j-1} \leq |\xi| \leq 2^j} (1 + |\xi|)^{N+1} |\hat{u}_m(\xi)|^2 d\xi \right)^{1/2} \times \left( \sum_m B_{m}^2 \right)^{1/2}.
\]
The last step is followed by Hölder inequality. For \( j \geq 0 \), let positive integer \( |l| = 2^j \) and \( p' = p - j(2N + 1) \), we can obtain the estimate

\[
\sum_{l \in \mathbb{Z}} \left( \sum_{q \geq p} \left\| \sum_{m \in \mathbb{Z}} e^{2\pi i Q(m, l)} (\Delta_p u_m \Delta_q v_{m+l})(t) \right\|_{L^2_t} \right)^2 \lesssim \sum_{l \in \mathbb{Z}} \sum_{2^p \geq |l| \leq 2^{p+1}} \sum_{m} \int_{2^{p-1} \leq |\xi| \leq 2^p} (1 + |\xi|)^{\frac{N+1}{p'+1}} \left| \hat{u}_m(\xi) \right|^2 d\xi \\
\times |l|^{1/2} 2^{p' \frac{N}{p' + 2}} \sum_{m} B_m^2 \\
\approx \sum_{j \geq 0} \sum_{p' \geq 0} \sum_{m} \int_{2^{p'+(2N+1)-1} \leq |\xi| \leq 2^{p'+(2N+1)}} (1 + |\xi|)^{\frac{N+1}{p'+1}} \\
\times \left| \hat{u}_m(\xi) \right|^2 d\xi 2^{-p' \frac{N}{p' + 2}} \sum_{m} B_m^2 \\
= \sum_{p' \geq 0} 2^{-p' \frac{N}{p' + 2}} \sum_{m} A_m^2 \sum_{m} B_m^2 \lesssim \sum_{m} A_m^2 \sum_{m} B_m^2.
\]

Similarly, we have

\[
\sum_{l \in \mathbb{Z}} \left( \sum_{q \geq p} \left\| \sum_{m \in \mathbb{Z}} e^{2\pi i Q(m, l)} (\Delta_p u_m \Delta_q v_{m+l})(t) \right\|_{L^2_t} \right)^2 \lesssim \sum_{m} A_m^2 \sum_{m} B_m^2.
\]

**Contribution of (ii).** Consider the quantity

\[
\left\| \sum_{m \in \mathbb{Z}} e^{2\pi i Q(m, l)} (\Delta_p u_m \Delta_q v_{m+l})(t) \right\|_{L^2_t}.
\]

We first assume \( l \geq 0 \). For a large positive \( K > 0 \), by Plancherel theorem, we write

\[
\left\| \sum_{|m| \leq K} e^{2\pi i Q(m, l)} (\Delta_p u_m \Delta_q v_{m+l})(t) \right\|_{L^2_t} = \left\| \sum_{|m| \leq K} (\Delta_p u_m * \Delta_q v_{m+l})(\xi - Q(m, l)) \right\|_{L^2_{\xi}}.
\]

Since \( \text{supp} (\Delta_p u_m * \Delta_q v_{m+l})(\xi) \subset [-2^{q+1}, 2^{q+1}] \), splitting the summation into \( 62^q/|l|^{2N} \) summations over arithmetic progressions of increment \( 62^q/|l|^{2N} \), say \( M_s \) for \( s = 1, \cdots, 62^q/|l|^{2N} \). if \( m_1, m_2 \in M_s \) then

\[
m_1 = n_1 62^q/|l|^{2N} + d, \quad m_2 = n_2 62^q/|l|^{2N} + d.
\]

where \( d, n_1 \) and \( n_2 \) denote be integers and \( n_1 > n_2, d < 62^q/|l|^{2N} \). We write

\[
(m+l)^k - m^k = \sum_{\alpha + \beta = k, \ \beta \geq 1} a_{\alpha, \beta} m^\alpha l^\beta
\]

and obviously \( a_{\alpha, \beta} \geq 0 \) for all indices \( \alpha, \beta \). In this case we have

\[
|Q(m_1, l) - Q(m_2, l)| = \sum_{k=1}^{2N+1} \sum_{\alpha + \beta = k, \ \beta \geq 1} c_k a_{\alpha, \beta} (m_1^\alpha - m_2^\alpha) l^\beta \geq (m_1 - m_2)^{2N} \geq \max\{62^q, l^{2N}\}.
\]

\[\text{4}\]
By a orthogonality consideration and (10) we write
\[ \left\| \sum_{m \in \mathcal{M}_s} (\hat{\Delta}_p u_m \ast \hat{\Delta}_q v_{m+l})(\xi - Q(m, l)) \right\|_{L^2_t}^2 = \sum_{m \in \mathcal{M}_s} \left\| (\hat{\Delta}_p u_m \ast \hat{\Delta}_q v_{m+l})(\xi - Q(m, l)) \right\|_{L^2_t}^2 = \sum_{m \in \mathcal{M}_s} \left\| \hat{\Delta}_p u_m \ast \hat{\Delta}_q v_{m+l} \right\|^2_{L^2_t} \] (12)

From Young inequality, (12) and (8) we get
\[ \sum_{|m| \leq K} \left\| e^{2\pi i Q(m, l)t} (\hat{\Delta}_p u_m \hat{\Delta}_q v_{m+l})(t) \right\|_{L^2_t}^2 \leq 6 \frac{2^q}{|t|^{2N}} \sum_{s=1}^{62^q/|t|^{2N}} \left\| e^{2\pi i Q(m, l)t} (\hat{\Delta}_p u_m \hat{\Delta}_q v_{m+l})(t) \right\|_{L^2_t}^2 \]
\[ = 6 \frac{2^q}{|t|^{2N}} \sum_{s=1}^{62^q/|t|^{2N}} \sum_{m \in \mathcal{M}_s} \left\| (\hat{\Delta}_p u_m \hat{\Delta}_q v_{m+l})(t) \right\|_{L^2_t}^2 \]
\[ \leq 6 \frac{2^q}{|t|^{2N}} 2^p \sum_m \| \Delta_p u_m \|^2_{L^2_t} \| \Delta_p v_m \|^2_{L^2_t} \]
\[ \leq 6 |t|^{-2N} 2^{Np/(2N+1)} 2^{2Nq/(2N+1)} \sum_m \int_{2^{p-1} \leq |\xi| \leq 2^p} (1 + |\xi|)^{N+1} \times |\hat{u}_m(\xi)|^2 d\xi \int_{2^{q-1} \leq |\xi| \leq 2^q} (1 + |\xi|)^{N+1} |\hat{v}_{m+l}(\xi)|^2 d\xi. \]

Let $K \to \infty$, the estimate holds for the whole integer set. The same estimate can be obtained if we interchange the role of $u_m$ and $v_m$. As for the case $l < 0$, by (7) we can reduce this case to $l \geq 0$.

Therefore, we have
\[ \sum_{l \in \mathbb{Z}} \left( \sum_{q \geq p, \ 2^q < |t|^{2N+1}} \left\| \sum_{m \in \mathbb{Z}} e^{2\pi i Q(m, l)t} (\hat{\Delta}_p u_m \hat{\Delta}_q v_{m+l})(t) \right\|_{L^2_t}^2 \right) \leq \sum_m \sum_l A_m^2 B_{m+l} \]
\[ = \sum_m A_m^2 \sum_l B_{m+l} \]

Contribution of (iii). We have known from (11) that the orthogonality seems more natural in this case. The arguments are similar as the case of (ii) and even simpler.

The result is sharp. For example if we take
\[ u_K(x, t) = \sum_{|n| \leq K} \int_{|\lambda| \leq K^{2N+1}} e^{2\pi i (nx + \lambda t)} d\lambda. \]

Then $\|u_K\|_{L^2_t L^\infty_x} \approx K^{N+1}$ and
\[ \|u_K\|_{L^2_t L^4_x} \approx K^{3/4}(K^{2N+1})^{3/4} \approx K^{3(N+1)/2}. \]
On the other hand,

\[ \|u_K\|_{\chi^{N+1/2}} = \left( \sum_{|n| \leq K} \left( \int_{|\lambda| \leq K^{2N+1}} (1 + |\lambda + P(n)|)^{\frac{N+1}{2N+1}} |\hat{u}_K(n, \lambda)|^2 d\lambda \right)^{\frac{1}{2}} \approx K^{3(N+1)/2} \approx \|u_K\|_{L^4_t L^4_x}. \]

References

[1] J. Bourgain, Fourier restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, Parts II, Geometric Funct. Anal. 3(3) (1993) 209-262.

[2] Anjan Biswas, Solitary wave solution for the generalized Kawahara equation. Applied Mathematics Letters Volume 22, Issue 2, February 2009, Pages 208-210

[3] Zoran Grujic, Henrik Kalisch, Gevrey regularity for a class of water-wave models. Nonlinear Analysis: Theory, Methods and Applications Volume 71, Issues 3-4, August 2009, 1160-1170

[4] Netra Khanal, Ramjee Sharma, Jiahong Wu and Juan-Ming Yuan, A dual-Petrov-Galerkin method for extended fifth-order Korteweg-de Vries type equations. Discrete and Continuous Dynamical Systems-Suppl. (2009).