Matrix Product States of Three Families of One-Dimensional Interacting Particle Systems

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Abstract. The steady states of three families of one-dimensional non-equilibrium models with open boundaries, first proposed in [22], are studied using a matrix product formalism. It is shown that their associated quadratic algebras have two-dimensional representations, provided that the transition rates lie on specific manifolds of parameters. Exact expressions for the correlation functions of each model have also been obtained. We have also studied the steady state properties of one of these models, first introduced in [23], with more details. By introducing a canonical ensemble we calculate the canonical partition function of this model exactly. Using the Yang-Lee theory of phase transitions we spot a second-order phase transition from a power-law to a jammed phase. The density profile of particles in each phase has also been studied. A simple generalization of this model in which both the left and the right boundaries are open has also been introduced. It is shown that double shock structures may evolve in the system under certain conditions.

PACS numbers: 02.50.Ey, 02.50.Ga, 05.20.-y, 05.70.Fh, 05.70.Ln

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1. Introduction

The occurrence of non-equilibrium phase transitions is one of the most interesting properties of one-dimensional out-of-equilibrium models \[1, 2\]. In open boundary systems, changing the boundary parameters may result in special kinds of non-equilibrium phase transitions which are called the boundary induced phase transitions. The existence of shocks is another interesting property of these lattice models. A shock is defined as a discontinuity between a high-density and a low-density region in the density profile of particles on the lattice. The shocks have been studied both at the macroscopic level using the hydrodynamic equations and also at the microscopic level by solving the master equation of spatially discrete models. For example, at the macroscopic scale the shocks may appear in the solutions of the Burgers equation with particles conversation \[3\]. At the microscopic level the Burgers equation can be described by the Asymmetric Simple Exclusion Process (ASEP) \[4\]. The ASEP contains one class of particles which can be injected and extracted from the boundaries of a one-dimensional chain while hopping in the bulk with asymmetric rates. This model has been studied widely during the last decade. It is known that by changing the injection and extraction rates, the ASEP undergoes several boundary induced phase transitions \[5\]. The shocks may also appear in the density profile of the particles with the ASEP dynamics for certain values of the reaction rates. The same phenomenon might happen when the fixed number of such particles hop on a ring in the presence of a slow particle called an impurity \[6\]–\[9\].

Recently, it has been shown that among one-dimensional reaction-diffusion systems with open boundaries there are three families of models with a common property: The time evolution equation of a Bernoulli shock measure developed by the Hamiltonian of each of these models is quite similar to the time evolution equation of a single-particle random walk on a lattice with reflecting boundaries \[22\]. This actually takes place provided that some constraints on the reaction rates of these systems are fulfilled. These models are the Asymmetric Simple Exclusion Process (ASEP), the Branching-Coalescing Random Walk (BCRW) and the Asymmetric Kawasaki-Glauber process (AKGP). It is known that in an infinite system, and under some constraints, single shock or even consecutive multiple shocks may evolve in the ASEP \[10\]–\[15\]. In the latter case the time evolution equation for the multiple shock measure generated by the Hamiltonian of the system is similar to the time evolution equation of \(n\) random walkers. In the steady state the ASEP’s probability distribution function can be written in terms of superposition of shocks. From the Matrix Product Formalism (MPF) point of view, in which the steady state weights are written in terms of the expectation values of product of non-commuting operators \[15\], \(n\) consecutive shocks correspond to an \((n + 1)\)-dimensional representation of the ASEP’s quadratic algebra \[16, 17\].

The question that might arise is that whether or not the steady state of the BCRW and that of the AKGP can also be represented by finite dimensional representation of their quadratic algebras since their steady states can be written in terms of interactions of single shock. We will deal with this question in the first part of this paper. We will show that as far as a single shock structure is concerned the steady state of the BCRW and that of the AKGP can be obtained from two-dimensional representations of their quadratic algebras.

In \[23\] we have studied a branching-coalescing model with open boundary in which the particles diffuse, coagulate and decoagulate on a one-dimensional chain. The particles
are also allowed to enter and leave the chain only from the left boundary. This model had already been studied with reflecting boundaries in \[18, 19\]. We have shown that the steady state of this open boundary model can be obtained using the MPF provided that the reaction rates satisfy a constraint. In this case its quadratic algebra has a two-dimensional representation, while in the case of reflecting boundary conditions the quadratic algebra of the model has a four-dimensional representation. Recent investigations show that some of the classical concepts and theories in the equilibrium statistical mechanics, such as the Yang-Lee theory of equilibrium phase transitions \[20\], can be successfully applied to the out-of-equilibrium systems. There are several examples from the driven diffusive systems to the directed percolation models which confirm the applicability of this theory to the non-equilibrium systems \[23\]-\[30\]. We have also studied the phase transition of our model using the Yang-Lee theory by studying the roots of its grand canonical partition function \[23\]. These zeros lie on a circle and intersect the positive real axis at an angle \[\frac{\pi}{2}\]. According to the Yang-Lee theory this reveals a first-order phase transition \[21\]. As we will see this model is an special case of the BCRW; therefore, we expect to see the shock profiles in its steady state. In order to see the shock profiles we will confine the number of particles by working in a canonical ensemble. The sum over the steady state weights with fixed number of particles is defined as the canonical partition function of this model. The canonical partition function can be calculated exactly using the MPF. It turns out that the zeros of this function in the complex plan of one of the parameters lie on a curve which intersects the positive real axis at an angle \[\frac{\pi}{4}\]. At the same time their density vanishes near the critical point. It indicates a second-order phase transition. In the canonical ensemble one can also calculate the density profile of particles on the chain again using the MPF. Our calculations show that in one phase the density profile of particle has an exponential behavior while in the other phase, as we expect, it is a shock.

Another question that might arise is that under what conditions multiple shocks may evolve in this system. We will show that in our model if we let the particles to enter and leave the chain from the both ends of the chain, one can see the double shock structures given that the injection and extraction rates satisfy some constraints. In this case the time evolution equation of the shock measure is similar to that of two random walkers. This phenomenon has already been observed in ASEP \[22, 31, 32\]. This paper is organized as follows: In the second section we will briefly review the basic concepts of the MPF. In the third section we will show that the quadratic algebra of the BCRW, as a generalized form of our branching-coalescing model, has a two-dimensional representation provided that the same constraints introduced in \[22\] are fulfilled. In this section the two-dimensional representation of the AKGP will also be introduced. In the forth section we will switch to our model proposed in \[23\] and find its canonical partition function using the MPF. The numerical estimates of the canonical partition function zeros, as a function of a complex reaction rate, will be obtained. The line of zeros and also their density will be calculated exactly. Finally, we will calculate the density profile of particles in each phase. The fifth section is devoted to the study of double shock structures in a general branching-coalescing model with open boundaries. In the last section we will summarize our results.
2. The matrix product formalism

In this section we review the basic ideas of the Matrix Product Formalism (MPF) first introduced in [5]. We will consider only one-dimensional stochastic systems with nearest neighbors interactions and open boundaries. Systems with long range interactions and open boundaries have also been studied in related literatures [33]. Let us define \( P(C; t) \) as the probability distribution of any configuration \( C \) of a Markovian interacting particle system at the time \( t \). The time evolution of \( P(C; t) \) can be written as a Schrödinger equation in imaginary time

\[
\frac{d}{dt}P(C; t) = HP(C; t)
\]

in which \( H \) is a stochastic Hamiltonian whose matrix elements are the transition rates between different configurations. For the one-dimensional systems of length \( L \) which consist of one species of particles the Hamiltonian \( H \) has the following general form

\[
H = \sum_{k=1}^{L-1} h_{k,k+1} + h_1 + h_L
\]

in which

\[
h_{k,k+1} = I^{\otimes (k-1)} \otimes h \otimes I^{\otimes (L-k-1)}; h_1 = h^{(l)} \otimes I^{\otimes (L-1)}; h_L = I^{\otimes (L-1)} \otimes h^{(r)}
\]

where \( I \) is a \( 2 \times 2 \) identity matrix, \( h \) is a \( 4 \times 4 \) matrix for the bulk interactions and \( h^{(l)} \) \((h^{(r)}) \) is a \( 2 \times 2 \) matrix for particle input and output from the left (right) boundary. In a basis \((00, 01, 10, 11)\) the bulk Hamiltonian along with the boundary Hamiltonians are given by

\[
h = \begin{pmatrix}
\cdot & w_{12} & w_{13} & w_{14} \\
\cdot & w_{21} & w_{23} & w_{24} \\
w_{31} & w_{32} & \cdot & w_{34} \\
w_{41} & w_{42} & w_{43} & \cdot
\end{pmatrix}, \quad h^{(l)} = \begin{pmatrix}
-\alpha & \gamma \\
\alpha & -\gamma
\end{pmatrix}, \quad h^{(r)} = \begin{pmatrix}
-\delta & \beta \\
\delta & -\beta
\end{pmatrix}.
\]

Requiring the conservation of probability, the diagonal terms of matrices should be the negative of the sum of transition rates in each column. In the most general form the interaction rates are

- Diffusion to the left and right \( w_{32}, w_{23} \)
- Coalescence to the left and right \( w_{34}, w_{24} \)
- Branching to the left and right \( w_{43}, w_{42} \)
- Death to the left and right \( w_{13}, w_{12} \)
- Birth to the left and right \( w_{31}, w_{21} \)
- Pair Annihilation and Creation \( w_{14}, w_{41} \)
- Injection and Extraction at the first site \( \alpha, \gamma \)
- Injection and Extraction at the last site \( \delta, \beta \)

In the stationary state we have \( HP^*(C) = 0 \). According to the MPF the stationary probability distribution \( P^*(C) \) is assumed to be of the form

\[
P^*(C) = P^*(\tau_1, \cdots, \tau_L) = \frac{1}{Z_L} \langle W | \prod_{i=1}^{L} (\tau_i D + (1 - \tau_i) E) | V \rangle
\]
where $\tau_i = 0$ if the site $i$ is empty and $\tau_i = 1$ if it is occupied by a particle. The function $Z_L$ is a normalization factor and can be obtained easily using the normalization condition $\sum_C P^*(C) = 1$. The operators $D$ and $E$ stand for the presence of a particle and a vacancy at each site. These operators besides the vectors $\langle W \rangle$ and $|V\rangle$ should satisfy the following algebra

$$h \left( \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \right) = \left( \begin{pmatrix} E \\ D \end{pmatrix} \right) \otimes \left( \begin{pmatrix} E \\ D \end{pmatrix} \right) - \left( \begin{pmatrix} E \\ D \end{pmatrix} \right) \otimes \left( \begin{pmatrix} E \\ D \end{pmatrix} \right),$$

$$(W) h^{(l)} \left( \begin{pmatrix} E \\ D \end{pmatrix} \right) = -(W) \left( \begin{pmatrix} E \\ D \end{pmatrix} \right),$$

$$h^{(r)} \left( \begin{pmatrix} E \\ D \end{pmatrix} \right) |V\rangle = \left( \begin{pmatrix} E \\ D \end{pmatrix} \right) |V\rangle.$$

The operators $\bar{E}$ and $\bar{D}$ are auxiliary operators and do not enter in the calculation of physical quantities. Using (3) and (6) the quadratic algebra associated with the most general reaction-diffusion model can be obtained. By defining $C := D + E$ one can easily see that the mean values of the physical quantities can be written in terms of the non-commuting operators $C$ and $E$ and the vectors $|V\rangle$ and $\langle W \rangle$. For example the mean density of particles at site $i$ has the following form:

$$\langle \rho_i \rangle = \frac{\sum_C \tau_i P^*(C)}{\sum_C P^*(C)} = \frac{\langle W | C^{i-1} (C - E) C^{L-i} | V \rangle}{\langle C^L \rangle},$$

Similarly, any $n$-point density correlation function can be written as

$$\langle \rho_{i_1} \rho_{i_2} \cdots \rho_{i_n} \rangle = \frac{\sum_C \tau_{i_1} \tau_{i_2} \cdots \tau_{i_n} P^*(C)}{\sum_C P^*(C)} = \frac{\langle W | C^{i_1-1} (C - E) C^{i_2-i_1-1} \cdots (C - E) C^{L-i_n} | V \rangle}{\langle C^L \rangle}. $$

In order to calculate (7), (8), or any other quantity, there are two possibilities: one might find a finite or an infinite-dimensional representation for the quadratic algebra of the system or equivalently, one may use the commutation relation between the operators to calculate these quantities rigorously. In the next section we will show that finite-dimensional representation exists for three families of reaction-diffusion systems, given that some constraints are satisfied.

3. Representation of the quadratic algebra

In our recent paper [22] exact travelling wave solutions are obtained for three families of one-dimensional reaction-diffusion models with open boundaries. These models are the Asymmetric Simple Exclusion Process (ASEP), the Branching-Coalescing Random Walk (BCRW) and the Asymmetric Kawasaki-Glauber Process (AKGP). It has been shown that for these models the stationary measure can be written as a linear combination of Bernoulli shock measures defined on a lattice of length $L$ as

$$|m\rangle = \left( \frac{1 - \rho_1}{\rho_1} \right)^{\otimes m} \otimes \left( \frac{1 - \rho_2}{\rho_2} \right)^{\otimes L-m}, \quad 0 \leq m \leq L$$

provided that some constraints on the reaction rates are fulfilled. In [23] $\rho_1$ and $\rho_2$ are the densities of particles at the left and the right domains of the shock position.
The time evolution of generated by the Hamiltonian of the above-mentioned models is given by

$$-H|m\rangle = d_1|m-1\rangle + d_2|m+1\rangle - (d_1 + d_2)|m\rangle \quad 0 < m < L$$

which is a simple single-particle random walk equation for the position of the shock \(m\) with hopping rate \(d_1\) to the left and \(d_2\) to the right. In this section we show that under the same constraints the stationary states of these models can be studied using the MPF and their associated quadratic algebras have two-dimensional representations.

For the ASEP the non-vanishing rates in (4) are \(w_{32}, w_{23}, \alpha, \beta, \gamma\) and \(\delta\). It is known that the quadratic algebra of the ASEP has an \(n\)-dimensional representation \((n \geq 2)\), provided that the parameters of the model satisfy some constraints [16, 17]. The \((n+1)\)-dimensional representations of the quadratic algebra describe the stationary linear combination of shock measures with \(n\) consecutive shocks. The ASEP has already been studied widely in literatures; therefore, will not be considered here (see [2] and references therein).

For the BCRW the non-vanishing parameters in (4) are \(w_{34}, w_{24}, w_{42}, w_{43}, w_{32}, w_{23}, \alpha, \beta\) and \(\gamma\). As we will see, the steady state of this model can be described by two-dimensional representation of its quadratic algebra given that

$$\rho_1 = \rho_2 = \frac{1 - \rho}{\rho} \alpha + (1 - \rho)w_{32} - \frac{1 - \rho}{\rho}w_{34}$$

in which \(\rho_1 = \rho\) is the density of particles on the left-hand side of the shock. The density of particles on the right-hand side of the shock \(\rho_2\) is zero.

For the AKGP, the non-vanishing parameters in (4) are \(w_{12}, w_{13}, w_{42}, w_{43}, w_{32}, \alpha\) and \(\beta\); however, there is no specific constraint on these parameters in order to have the mentioned property. If the particles are allowed only to enter the system from the first site with the rate \(\alpha\) and leave it from the last site of the chain with the rate \(\beta\), then we will have \(\rho_1 = 1\) and \(\rho_2 = 0\).

We have found that the quadratic algebras of the BCRW and the AKGP have two-dimensional representations given that the constraints [11] for the BCRW are fulfilled. For \(c_{11} \neq c_{22}\) we find

$$C = \begin{pmatrix} c_{11} & 0 \\ 0 & c_{22} \end{pmatrix}, \quad E = \begin{pmatrix} e_{11} & e_{12} \\ 0 & e_{22} \end{pmatrix}$$

$$|V\rangle = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad \langle W| = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

and for \(c = c_{11} = c_{22}\) we find

$$C = \begin{pmatrix} \tilde{c} & \tilde{c}_{12} \\ 0 & \tilde{c} \end{pmatrix}, \quad E = \begin{pmatrix} \tilde{e}_{11} & \tilde{e}_{12} \\ 0 & \tilde{e}_{22} \end{pmatrix}$$

$$|V\rangle = \begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{pmatrix}, \quad \langle W| = \begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{pmatrix}$$

The matrix elements are functions of the non-vanishing parameters of each model. The explicate form of the matrices \(C\), \(E\), \(\tilde{C}\) and \(\tilde{E}\) and the vectors \(|V\rangle\) and \(\langle W|\) for each model is given in Appendix. Using the representation [12] for \(c_{11} \neq c_{22}\) and the
definition of the density profile of particles \( \tilde{c} \) one finds in the thermodynamic limit \( (L \to \infty) \)

\[
\langle \rho_i \rangle = \begin{cases} 
(1 - \frac{c_{11}}{c_{11}}) - \frac{v_2c_{12}}{v_1c_{11}} e^{\frac{-L}{c_{11}}}, & c_{11} > c_{22} \\
(1 - \frac{c_{22}}{c_{22}}) - \frac{v_1c_{12}}{v_2c_{11}} e^{\frac{-L}{c_{22}}}, & c_{11} < c_{22}.
\end{cases}
\] (14)

in which we have defined the characteristic length \( \xi = |\ln \frac{c_{11}}{c_{22}}|^{-1} \). Also using (13) and (14) for \( \tilde{c} = c_{11} = c_{22} \)

\[
\langle \rho_i \rangle = (1 - \frac{\tilde{c}_{11}}{\tilde{c}}) + \left( \frac{\tilde{c}_{11}}{\tilde{c}} - \frac{\tilde{c}_{22}}{\tilde{c}} \right) x, \quad 0 \leq x \leq 1
\] (15)

where \( x = \frac{1}{\tilde{c}} \). As can be seen the density profile of particles has an exponential behavior for \( c_{11} \neq c_{22} \) while it is linear on the coexistence line \( c_{11} = c_{22} \). For \( c_{11} > c_{22} \) the density of particles near the left boundary is \( \rho_1 = 1 - \frac{c_{11}}{c_{22}} \) while for \( c_{11} < c_{22} \) the density of particles near the right boundary is \( \rho_2 = 1 - \frac{c_{22}}{c_{11}} \).

Having the two-dimensional representations (12) and (13) any higher order correlation functions can easily be calculated using (8). For instance, in the thermodynamic limit the two-point correlation function is obtained to be

\[
\langle \rho_i \rho_j \rangle = \begin{cases} 
\frac{(d_{11})^2}{c_{11}} + \frac{\nu_2d_{11}d_{12}}{v_1c_{11}^2} e^{\frac{-L}{c_{11}}}, & c_{11} > c_{22} \\
\frac{(d_{22})^2}{c_{22}} + \frac{\nu_1d_{11}d_{12}}{w_2c_{11}^2} e^{\frac{-L}{c_{22}}}, & c_{11} < c_{22}.
\end{cases}
\] (16)

for \( c_{11} \neq c_{22} \) and

\[
\langle \rho_i \rho_j \rangle = \left( \frac{d_{11}}{c} \right)^2 + \frac{\tilde{d}_{11}(\tilde{d}_{22} - \tilde{d}_{11})}{\tilde{c}^2} \left( \frac{i}{L} \right) + \frac{\tilde{d}_{22}(\tilde{d}_{22} - \tilde{d}_{11})}{\tilde{c}^2} \left( \frac{i}{L} \right)
\] (17)

for \( \tilde{c} = c_{11} = c_{22} \). In (16) \( d_{ij} \)'s are defined as \( D_{ij} = (C - E)_{ij} \equiv d_{ij} \) and in (17) \( \tilde{d}_{ij} \)'s are the same elements for the case \( c_{11} = c_{22} \).

4. The Yang-Lee zeros of the canonical partition function

In this section we will focus on some of the steady state properties of our branching-coalescing model first studied in [23]. We will first calculate the canonical partition function of our model with the following non-zero rates:

\[
w_{32} = q, w_{23} = \frac{1}{q}, w_{34} = q, w_{43} = \Delta q, w_{24} = \frac{1}{q}, w_{42} = \frac{\Delta}{q}, \alpha, \gamma.
\] (18)

It can easily be verified that this model is an special case of the BCRW. In this model we have \( \beta = 0 \) and the constraints (11) also give \( \rho = \frac{\Delta}{1 + \Delta} \) and \( \alpha = (\frac{1}{q} - q + \gamma) \Delta \).

By introducing the canonical ensemble we restrict ourself to the case where the total number of particles on the chain \( M \) is fixed. As we mentioned, it has been shown that the classical Yang-Lee theory of the equilibrium phase transitions can successfully be applied to the driven diffusive systems to predict their non-equilibrium phase transitions. By finding the roots of the canonical partition function of our model,
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we will investigate its phase transitions. In order to calculate the canonical partition function we will use the quadratic algebra of the model

\[ [C, \bar{C}] = [E, \bar{E}] = 0 \]

\[ \bar{E}C - EC = (q + q\Delta + q^{-1})EC - q(1 + \Delta)E^2 - q^{-1}C^2 \quad (19) \]

\[ CE - C\bar{E} = (q^{-1} + q^{-1}\Delta + q)CE - q^{-1}(1 + \Delta)E^2 - qC^2 \]

\[ \langle W|((\alpha + \gamma)E + \bar{E} - \gamma C) = \langle W|\bar{C} = 0, \, \bar{E}|V\rangle = \bar{C}|V\rangle = 0 \]

and its representation for \( q^2 \neq 1 + \Delta \)

\[ C = \begin{pmatrix} 1 + \Delta & 0 \\ 0 & q^2 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & \lambda \\ 0 & q^2 \end{pmatrix}, \quad \bar{E} = \begin{pmatrix} \frac{q^2 - 1}{q} & -\frac{\Delta \lambda}{q} \\ q & 0 \end{pmatrix} \quad (20) \]

\[ \bar{C} = 0, \quad |V\rangle = \begin{pmatrix} \frac{\lambda}{q^2 - 1} \\ 1 \end{pmatrix}, \quad |W\rangle = \begin{pmatrix} -\frac{q^2 \alpha \lambda - 1}{(\gamma(1 + \Delta) - q\Delta)} & 1 \end{pmatrix}. \]

For convenience we will use our new notation in which \( \gamma \) (instead of \( \beta \) in [23]) is the extraction rate of particles from the left boundary. It can easily be seen from (20) that the operators \( C \) and \( E \) have the following properties

\[ E(C - E) = (C - E), \quad (C - E)^i = \Delta^{i-1}(C - E) \quad (21) \]

We define the canonical partition function \( Z_{L,M} \) as the sum over the steady states weights with fixed number of particles \( M \). Using (19)-(21) and the approach used in [8, 9] we obtain

\[ Z_{L,M} = q^{2\alpha(1 - \frac{\Delta}{q^2 - 1})\Delta^M} \sum_{i=0}^{L-M} q^{2i} C_{L-i-1}^{M-1} \quad (22) \]

where \( C_i^j = \binom{j}{i} \frac{\beta}{\gamma(j - \beta)} \) is the binomial coefficient. Let us now examine the zeros of the canonical partition function \( Z_{L,M} \) as a function of \( q \) for fixed \( L, M, \alpha, \gamma \) and \( \Delta \). Apart from the zeros generated by the factor behind the sum in (22), the numerical estimates of the roots in the complex-\( q \) plane are plotted in Figure (1) for \( L = 500 \) and \( M = 300 \) (grey dots). As can be seen from Figure (1) the roots accumulate towards a critical point on the positive real-\( q \) axis which will be calculated shortly. In order to calculate the line of zeros analytically we need the thermodynamic behavior of the canonical partition function (22). This can be done using the steepest decent method. We find

\[ Z_{L,M} \sim \begin{cases} \frac{q^{2L} \Delta^M}{(q^{2L} - 1)^M}, & \rho < 1 - q^{-2} \\
\frac{\rho C_M^M \Delta^M}{1 + q^2(\rho - 1)}, & \rho > 1 - q^{-2} \end{cases} \quad (23) \]

in which \( \rho = \frac{M}{L} \). The critical point is obviously \( q_c = \frac{1}{\sqrt{1 - \rho}} \). Defining the extensive part of the free energy as

\[ g = \lim_{L,M \to \infty} \frac{1}{L} \ln Z_{L,M} \quad (24) \]

one can calculate the line of zeros from

\[ \text{Re } g_1 = \text{Re } g_2 \quad (25) \]
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Figure 1. Numerical estimates (grey dots) for the roots of \(\sum_{i=0}^{L-M} q^{2i}C_{L-i-1}^{M-1} \) besides the line of zeros (dark line) given by (26) for \(L = 500\) and \(M = 300\) in the complex-q plane. The exact critical point is \(q_c = \frac{1}{\sqrt{1-\rho}} = 1.581\).

In which \(g_1\) and \(g_2\) are the free energy functions on the left and the right-hand side of the critical point [21]. After some calculations using (23)-(25) we obtain the following equation for the line of zeros in the thermodynamic limit:

\[
\frac{x^2 + y^2}{(x^2 - y^2 - 1)^2 + (2xy)^2} = \frac{(1-\rho)^{\rho-1}}{\rho^\rho}
\]

in which we have defined \(x := \text{Re}(q)\) and \(y := \text{Im}(q)\). This function is plotted in Figure (1) for \(L = 500\), \(M = 300\) and \(\rho = \frac{M}{L} = 0.6\) (dark line). It is seen that the line of zeros lays on the numerical estimates of the zeros. The function (26) intersects the positive real-q axis at the critical point \(x_c = \frac{1}{\sqrt{1-\rho}}\). We can also predict the order of transition by finding the density of roots near the critical point. For \(0 < y \ll 1\) and \((x - \frac{1}{\sqrt{1-\rho}}) \ll 1\) we find that the line of zeros is actually a straight line \(y \sim x - \frac{1}{\sqrt{1-\rho}}\) with the slope \(\frac{\pi}{4}\). The density of zeros on this line can be computed from

\[
\mu(s) = \frac{1}{2\pi} \left\lvert \frac{\partial}{\partial s} \text{Im}(g_1 - g_2) \right\rvert
\]

where \(s\) is the distance from the transition point [21]. It turns out that the density of zeros as a function of the distance in the vertical direction is proportional to \(y\) so in \(y \to 0\) limit the density of zeros vanishes. This expresses a second-order phase transition at the critical point.

In what follows we will calculate the density profile of particles on the chain in each phase using the canonical ensemble. In the canonical ensemble, the density of particles at site \(i\) can be written as

\[
\langle \rho_i \rangle = \sum_C \frac{\tau_i P^*(C)}{\sum_C P^*(C)}
\]

Note that the sums are over the configurations with fixed number of particles \(M\). By using (19)-(21) and after some calculations we obtain

\[
\langle \rho_i \rangle = \frac{\sum_{k=0}^{L-M} q^{2k}C_{L-k-2}^{M-2} + C_{L-2}^{M-1}q^{2(L-i)}\theta(M < i)}{\sum_{k=0}^{L-M} q^{2k}C_{L-k-1}^{M-1}}
\]

in which \(\theta(\cdots)\) is the ordinary Heaviside function. In Figure (2) we have plotted (29) for \(L = 500\), \(M = 300\) and two values of \(q\). With this choice of \(L\) and \(M\) the critical
Figure 2. Plot of the equation (29) for $L = 500$, $M = 300$ and two values of $q$ as it is shown. Below the transition point ($q_c = 1.581$) the density profile has an exponential behavior while it is an error function above the critical point.

point is $q_c = \frac{1}{\sqrt{1 - \rho}} = 1.581$. As can be seen for $q = 1.5 < q_c$, the density profile of particles is constant through the bulk of the chain equal to $\rho = \frac{M}{L} = 0.6$ and drops to zeros only near the right boundary. For $q = 2.5 > q_c$, the density profile of particles is a shock. The high-density region ($\rho_{\text{High}} = 1 - q^{-2} = 0.84$) extending from $i = 1$ to $i = \frac{\rho}{q^{-2}}L$ is separated by a sharp interface from a low-density region ($\rho_{\text{Low}} = 0$). Using the steepest decent method one can easily show that in the power-law phase ($q < q_c$) the density profile of particles $\langle \rho_i \rangle$ has the following analytical form

$$\langle \rho_i \rangle = \rho (1 - e^{-L^{-i+1}/\xi}) \quad (30)$$

in which $\xi$ is the characteristic correlation length equal to $\xi = |\ln(q^2(1 - \rho))|^{-1}$.

5. Generalization: Double shock structures

In our non-conserving driven diffusive model [23] only the left boundary was assumed to be open so that the particles could enter and leave the system from the first site of the chain. In this case the time evolution of the single shock measure was equivalent to that of a random walker on a one-dimensional lattice with reflecting boundaries. In this section we will introduce a simple generalization of this model in which the particles are allowed to enter and leave the system from both ends of the chain. Our main objective will be to investigate the possibility of the existence of double shock structures and the way that they evolve in time. Defining a double shock measure, we will see that on special manifold of parameters its time evolution is equivalent to that of two random walkers on a one-dimensional lattice with reflecting boundaries. Let us assume that apart from the interactions given by (18), the following reactions take place at the right boundary

$$\text{injection at the last site with rate } \delta$$

$$\text{extraction at the last site with rate } \beta.$$ \quad (31)

On a chain of length $L$ we define an uncorrelated double shock measure as

$$|m, n\rangle = \left(\frac{1 - \rho_1}{\rho_1}\right)^{\otimes m} \otimes \left(\frac{1 - \rho_2}{\rho_2}\right)^{\otimes n - m - 1} \otimes \left(\frac{1 - \rho_1}{\rho_1}\right)^{\otimes L - n + 1} \quad (32)$$
Figure 3. Density profile of a double shock measure

in which \( m + 1 \leq n \), \( 0 \leq m \leq L \) and \( 1 \leq n \leq L + 1 \). We have sketched the density profile of a sample double shock measure in Figure (3). It can easily be verified that the time evolution of the double shock measure (32) is equivalent to that of two random walkers provided that

\[
\rho_1 = \frac{\Delta}{1 + \Delta}, \quad \rho_2 = 0, \quad \alpha = (q^{-1} - q + \gamma)\Delta, \quad \delta = (q - q^{-1} + \beta)\Delta. \tag{33}
\]

Under these conditions the time evolution equation of (32) generated by the Hamiltonian of our generalized model will be given by the following equations

\[
-H|\ell_{m,n}\rangle = q|m-1,n\rangle + q^{-1}(1 + \Delta)|m+1,n\rangle + q^{-1}|m,n+1\rangle + (q(1 + \Delta)|m,n-1\rangle - (2 + \Delta)(q^{-1} + q)|m,n\rangle,
\]

for \( m \neq 0, n \neq L + 1, n \neq m + 1 \)

\[
-H|\ell_{0,n}\rangle = \alpha(\frac{1}{\Delta})|0,n+1\rangle + q^{-1}|0,n-1\rangle + q(1 + \Delta)|0,n\rangle - (1 + \Delta)|0,n\rangle,
\]

for \( n \neq 1, L + 1 \)

\[
-H|\ell_{m,L+1}\rangle = q^{-1}(1 + \Delta)|m+1,L\rangle + q^{-1}|m-1,L+1\rangle + \delta(\frac{1+\Delta}{\Delta})|m,L\rangle - \delta(\frac{1+\Delta}{\Delta})|m,L+1\rangle,
\]

for \( m \neq 0, L \)

\[
-H|\ell_{0,L+1}\rangle = \alpha(\frac{1+\Delta}{\Delta})|1,L+1\rangle + \delta(\frac{1+\Delta}{\Delta})|0,L\rangle - \alpha(\frac{1+\Delta}{\Delta})|0,L\rangle
\]

\[-H|\ell_{m,n}\rangle = 0 \text{ for } n = m + 1.
\]

The last equation indicates that the model has a trivial steady states in which the density of particles at all sites is equal to \( \rho_1 = \frac{\Delta}{1 + \Delta} \). This is associated with a one-dimensional representation of the quadratic algebra of this model. As can be seen from (34) the shock fronts move like two biased random walkers with different hopping rates. The shocks are also reflected from the boundaries. In the most general form it can easily be seen that for an open boundary model with the following non-vanishing rates

\[
w_{23}, w_{32}, w_{24}, w_{42}, w_{34}, w_{43}, \alpha, \beta, \gamma, \delta \tag{35}
\]
the above-mentioned approach can be used to show that the double shock structures of kind (32) exist if the reaction rates satisfy the following constraints:

\[ \rho_1 = \frac{w_{42} + w_{43}}{w_{42} + w_{43} + w_{24} + w_{34}} \quad \rho_2 = 0 \]

\[ w_{32} = \frac{w_{24} + w_{34}}{w_{12} + w_{43}} \quad w_{23} = \frac{w_{43} (w_{24} + w_{34})}{w_{42} + w_{43}} \]

\[ \alpha = \frac{w_{42} + w_{43}}{w_{24} + w_{34}} (w_{24} - \frac{2w_{42}(w_{24} + w_{34})}{w_{42} + w_{43}} + \frac{(w_{34} + w_{43})(w_{34} + w_{42})}{w_{24} + w_{34} + w_{42} + w_{43}} + \gamma) \]

\[ \delta = \frac{w_{42} + w_{43}}{w_{24} + w_{34}} (-w_{24} + \frac{2w_{42}(w_{24} + w_{34})}{w_{42} + w_{43}} - \frac{(w_{24} + w_{34})(w_{34} + w_{42})}{w_{24} + w_{34} + w_{42} + w_{43}} + \beta) \].

In this case the time evolution equations for \( |m, n\rangle \) given by (32) will be quite similar to (34). One can readily check that for our generalized model, defined by (18) and (31), the conditions (36) reduce to (33). The steady state of the system can be obtained from the superposition of double shock measures [34]; however, the system has a trivial steady state in which the particles occupy the sites of the chain with the uniform probability \( \rho_1 \) and as before this is associated with a one-dimensional representation of its quadratic algebra.

6. Concluding remarks

Our calculations in this paper can be divided into three parts. In the first part of the paper we have shown that using the MPF the steady state of the ASEP, BCRW and AKGP can be expressed by two-dimensional representations of their quadratic algebras provided that some constraints hold on the reaction rates. These constraints introduce specific manifold of parameters. On these manifolds the stationary measures are given by superpositions of Bernoulli shock measures. Having the explicit form of the two-dimensional representations several physical quantities, such as correlation functions, can easily be calculated.

The BCRW is a generalized form of the model that we had studied in [23]. The second part of our calculations is devoted to the detailed study of the phase transition and also the shock formation in this model as an special case of the BCRW. We have introduced a canonical ensemble in which the number of particles on the chain is fixed. Using the properties of the quadratic algebra the canonical partition function of the model is calculated exactly. The thermodynamic behavior of this function reveals a phase transition which depends on \( q \) and \( \rho \) i.e. one of the reaction rates and the density of particles on the chain. In order to study this phase transition we have used the classical Yang-Lee theory. Our calculations show that the Yang-Lee zeros of the canonical partition function as a function of the complex variable \( q \) lie on a curve which intersect the positive real-\( q \) axis at an angle \( \frac{\pi}{4} \) at the critical point \( q_c = \frac{1}{\sqrt{1-\rho}} \).

According to the Yang-Lee theory of phase transitions, this is reminiscent of a second-order phase transition. The nature of each phase can be studied by calculating the density profile of the particles in the steady state. For \( q < q_c \) the density of particles is constant everywhere on the chain except near the right boundary where it drops to zero exponentially with the correlation length \( \xi = |q^2(1-\rho)|^{-1} \) while it has a shock structure for \( q > q_c \).

In the third part of our calculations we have shown that double shock structures can evolve in a general branching-coalescing model with open boundaries. In this case the boundary parameters lie on specific manifolds of parameters determined by [36]. A
detailed study of the double shock properties will be given elsewhere [34].

The exact solution of the BCRW without any constraints on its reaction rates still remains an open question. The exact steady state of the model defined by the non-vanishing rates [35] under the conditions [36] from the MPF point of view is under consideration [34].

Appendix

It turns out that the representation of the quadratic algebra of the BCRW in the case $c_{11} \neq c_{22}$ (see [12]) is

$$
C = \begin{pmatrix}
\frac{w_{43}}{1-\rho} & 0 \\
0 & (1-\rho)w_{32} + \rho w_{34}
\end{pmatrix},
E = \begin{pmatrix}
\frac{w_{23}}{1-\rho} & \lambda \\
0 & (1-\rho)w_{32} + \rho w_{34}
\end{pmatrix},
$$

$$
\bar{E} = \begin{pmatrix}
-w_{43}(w_{23} - w_{32}(1-\rho) - w_{34}\rho) & -\frac{\lambda w_{23}}{1-\rho} \\
0 & 0
\end{pmatrix},
\bar{C} = 0
$$

$$
|V\rangle = \begin{pmatrix}
\frac{\lambda(1+\frac{\beta}{w_{43}})}{\beta + w_{32}(1-\rho) + w_{34}\rho - w_{23}} \\
1
\end{pmatrix},
|W\rangle = \begin{pmatrix}
\frac{\rho(\omega_{32}(\rho - 1) - w_{34}\rho)}{\lambda(\alpha + w_{32}(1-\rho) + w_{34}\rho)}, 1
\end{pmatrix}.
$$

and in the case $c_{11} = c_{22}$ (see [13]) it is

$$
C = \begin{pmatrix}
w_{32}(1-\rho) + w_{34}\rho & \lambda \\
0 & w_{32}(1-\rho) + w_{34}\rho
\end{pmatrix},
$$

$$
E = \begin{pmatrix}
(1-\rho)(w_{32}(1-\rho) + w_{34}\rho) & \rho(w_{32}(1-\rho) + w_{34}\rho) \\
0 & \rho(w_{32}(1-\rho) + w_{34}\rho)
\end{pmatrix},
$$

$$
\bar{E} = \begin{pmatrix}
\rho^2(w_{32}(1-\rho) + w_{34}\rho)^2 & \eta \\
0 & 0
\end{pmatrix},
\bar{C} = 0,
\bar{C} = 0
$$

$$
|V\rangle = \begin{pmatrix}
v_1 \\
1
\end{pmatrix},
|W\rangle = \begin{pmatrix}
w_1, 1
\end{pmatrix}.
$$

in which

$$
v_1 = -\frac{\eta^2 + w_{32}(\eta + \lambda\beta)^2 - (w_{32} - w_{43})(\eta + \lambda\beta)^2}{\rho^2(w_{32}(1-\rho) + w_{34}\rho)^2} \eta + \lambda\beta(w_{32}(1-\rho) + w_{34}\rho),
$$

$$
w_1 = \frac{\alpha w_{32} - w_{34}\rho}{\alpha(1-\rho) + \rho(w_{32}(1-\rho) + w_{34}\rho)}
$$

and $\lambda$ and $\eta$ are arbitrary parameters. The representation of the quadratic algebra for the AKGP is also found to be

$$
C = \begin{pmatrix}
w_{43} & 0 \\
0 & w_{13}
\end{pmatrix},
E = \begin{pmatrix}
0 & \lambda \\
0 & w_{13}
\end{pmatrix},
\bar{E} = \begin{pmatrix}
0 & \lambda(w_{13} - w_{43}) \\
0 & 0
\end{pmatrix},
\bar{C} = 0
$$

$$
|V\rangle = \begin{pmatrix}
\frac{\lambda(w_{13} - w_{43})}{w_{43} \beta w_{43}} \\
1
\end{pmatrix},
|W\rangle = \begin{pmatrix}
\frac{-\alpha w_{43}}{\lambda(w_{13} - w_{43})}, 1
\end{pmatrix}.
$$

and

$$
C = \begin{pmatrix}
w_{13} & \lambda \\
0 & w_{13}
\end{pmatrix},
E = \begin{pmatrix}
0 & \lambda \\
0 & w_{13}
\end{pmatrix},
\bar{E} = \begin{pmatrix}
0 & -\lambda w_{13} \\
0 & 0
\end{pmatrix},
\bar{C} = 0
$$

$$
|V\rangle = \begin{pmatrix}
\lambda \\
1
\end{pmatrix},
|W\rangle = \begin{pmatrix}
\frac{-\alpha w_{13}}{\lambda w_{13} - \alpha}, 1
\end{pmatrix}.
$$

for the case $c_{11} \neq c_{22}$ and $c_{11} = c_{22}$ respectively.
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