Distribution of continuous-variable entanglement by separable Gaussian states

Ladislav Mišta, Jr. 1,2 and Natalia Korolkova 2

1Department of Optics, Palacký University, 17. listopadu 50, 772 07 Olomouc, Czech Republic
2School of Physics and Astronomy, University of St. Andrews, North Haugh, St. Andrews, KY16 9SS, UK

(Dated: April 25, 2008)

Entangling two systems at distant locations using a separable mediating ancilla is a counterintuitive phenomenon proposed for qubits by T. Cubitt et al. [Phys. Rev. Lett. 91, 037902 (2003)]. We show that such entanglement distribution is possible with Gaussian states, using a certain three-mode fully separable mixed Gaussian state and linear optics elements readily available in experiments. Two modes of the state become entangled by sequentially mixing them on two beam splitters, while the third one remains separable in all stages of the protocol.

Quantum entanglement is a striking property of composite quantum systems that lies at the heart of the fundamental quantum information protocols such as quantum teleportation 1 or quantum cryptography 2. The typical scenario involves two parties in distant laboratories, Alice holding the quantum system \(a\) and Bob in possession of the system \(b\). They need to establish a quantum channel between their remote location in the form of the shared entangled state, which cannot be prepared merely by local operations on systems \(a\) and \(b\) and classical communication between Alice and Bob 2. Having only separable systems at hand and not having a possibility to meet each other in one place, Alice and Bob can entangle their distant quantum systems only by employing another ancillary quantum system \(c\), which first couples with the system \(a\), then is send over to the remote location where it interacts with \(b\). This is a required global operation that facilitates entanglement between \(a\) and \(b\). Remarkably, \(a\) can be entangled with \(b\) by sending the ancilla \(c\) that becomes never entangled with the subsystem \((ab)\) 4. In 4 the counterintuitive effect of entanglement distribution by separable ancilla was studied in the context of finite-dimensional systems.

In this paper we show how to turn this idea into a practical concept. We consider infinite-dimensional quantum systems, e.g., light modes. We propose a feasible three-step protocol where two distant separable modes \(A\) and \(B\) become entangled after interacting stepwise with the third mode \(C\). At any stage of the protocol, the mode \(C\) is separable from the subsystem \((AB)\). Our scheme relies entirely on Gaussian states and the challenging nonlinear controlled-NOT gates of the previous idea 4 are replaced by simple beam splitters. Therefore, the protocol can be implemented with Gaussian states and operations that are currently available in the laboratory. Moreover, the proposed protocol allows a more simple deterministic distribution of entanglement than the previous qubit protocol that requires an additional operation on systems \(b\) and \(c\) on Bob’s side 4.

Our protocol is schematically depicted in Fig. 1. The aim of the protocol is to entangle mode \(A\) in Alice’s laboratory with separable mode \(B\) in Bob’s distant laboratory by sending a separable mediating ancillary mode \(C\) from Alice to Bob. For pure quantum states this is not possible 4. Therefore, Alice and Bob have to construct by local operations and classical communication (LOCC) a suitable mixed fully separable Gaussian state of three modes \(A\), \(B\) and \(C\). The engineering of such a state is the aim of step 1 and represents the most challenging part of the problem. Alice and Bob start with three pure single-mode Gaussian states. They prepare modes \(A\) and \(B\) in the same momentum-squeezed vacuum states and rotate them clockwise and anticlockwise, respectively, by the same suitable angle. The ancillary mode \(C\) is initially on Alice’s side and is in a vacuum state. Alice and Bob then displace locally the three modes by random correlated displacements \(D_A\), \(D_B\) and \(D_C\) with Gaussian distribution characterized by a correlation matrix \(Q(x)\) specified below. This procedure generates the desired three-mode mixed Gaussian state with separability properties tunable by changing the parameter \(x\).

The actual entanglement distribution commence in step 2. By mixing modes \(A\) and \(C\) on a balanced beam splitter \(BS_{AC}\) Alice entangles mode \(A\) with the pair of modes \((BC)\) while mode \(B\) is separable from \((AC)\) and mode \(C\) is separable from \((AB)\). Next, she sends mode \(C\) to Bob. In step 3 Bob mixes modes \(B\) and \(C\) on a
balanced beam splitter $BS_{BC}$ finally entangling $A$ and $B$, while $C$ still remains separable from $(AB)$.

The modes $A$, $B$, and $C$ are described by three pairs of canonically conjugate quadrature operators $x_j$, $p_j$, $j = A, B, C$. The operators satisfy the canonical commutation rules that can be compactly expressed as $[\xi_j, \xi_k] = -i\Omega_{jk}$, where $\xi = (x_A, p_A, x_B, p_B, x_C, p_C)^T$ is the vector of quadratures and $\Omega = \otimes_{i=1}^{3} J$ is the symplectic matrix, where $J = -i\sigma_y \sigma_y$ denotes the $y$ Pauli matrix). Quantum states of three-mode system can be represented in phase space by the Wigner function $W$ of six real variables separated state.

vector of first moments $\bar{x}$ associated by adding a vacuum mode $\bar{x} = (\bar{x}_A, \bar{x}_B, \bar{x}_C)^T$, where $\bar{x} = (\bar{x}_A, \bar{x}_B, \bar{x}_C)^T$, and a noise term in the form of a positive semidefinite matrix $\rho_x$, $\rho_x \geq \rho$ with elements $\gamma_{jk} = \text{Tr} \left( \rho \left( \frac{1}{2} \left( \xi_j - \xi_k \right) \xi_k - \xi_j \right) \right)$, $j, k = 1, \ldots, 6$, where $(A, B) = AB + BA$.

Preparation of the three-mode fully separable state. We start with the three-mode Gaussian state, which CM is composed from the CM of an entangled state

\[
\gamma_{AB} = \begin{pmatrix}
  e^{2d} a & 0 & -e^{-2d} c & 0 \\
  0 & e^{-2d} a & 0 & e^{-2d} c \\
  -e^{-2d} c & 0 & e^{2d} a & 0 \\
  0 & -e^{2d} c & 0 & e^{2d} a 
\end{pmatrix}
\]  

(1)

and a noise term in the form of a nonnegative multiple of a positive semidefinite matrix $P \equiv q_1 q_1^T + q_2 q_2^T$:

\[
\gamma_1(x) = \gamma_{AB} \oplus \mathbb{1}_C + x(q_1 q_1^T + q_2 q_2^T),
\]  

(2)

where $x \geq 0$. The parameters involved in the CM $\gamma_{AB}$ are given by $a = \cosh(2r)$, $c = \sinh(2r)$ and we assume $d \geq r > 0$. This is a two-mode squeezed vacuum state with the squeezing parameter $r$ with modes $A$ and $B$ squeezed, in addition, by local squeezing operations $S_A = S_B = \text{diag}(e^d, e^{-d})$. Our design of the noise term in Eq. (2) is inspired by the method used to construct various three-mode entangled Gaussian states and $q_1, q_2$ read

\[
q_1 = (0, \sin \phi, 0, -\sin \phi, \sqrt{2}, \sqrt{2})^T,
\]

\[
q_2 = (\cos \phi, 0, \cos \phi, 0, \sqrt{2}, \sqrt{2})^T,
\]

\[
\tan \phi = e^{-2r} \sinh(2d) + \sqrt{1 + e^{-4r} \sinh^2(2d)}
\]

with $\sin \phi, \cos \phi > 0$. This additional noise is chosen such that for sufficiently large $x$ the CM $\gamma_1(x)$ describes a fully separable state.

The state described by CM $\gamma_{AB}$ can be naturally prepared by mixing on a balanced beam splitter $U_{AB}$ modes $A$ and $B$, each in a pure momentum-squeezed vacuum state with the variances of the position quadratures $\langle (\Delta x_A)^2 \rangle = e^{2(d-r)}$ and $\langle (\Delta x_B)^2 \rangle = e^{2(d+r)}$ respectively. The entire three-mode state then can be created by adding a vacuum mode $C$ with CM $\mathbb{1}_C$ to the CM $\gamma_{AB}$ and performing local random correlated displacements of modes $A$, $B$ and $C$ distributed with Gaussian distribution with correlation matrix $xP$. Making use of the criterion of full separability for three-mode Gaussian states one then finds that for all $x \geq x_{\text{sep}}$, where

\[
x_{\text{sep}} = \frac{2\sinh(2r)}{\delta},
\]  

(4)

where $\delta = e^{2d} \sin^2 \phi + e^{-2d} \cos^2 \phi$, the CM $\gamma_1(x)$ describes a fully separable state. However, although the state is fully separable the way of its preparation described above is not suitable for our purposes. Namely, it is not prepared by LOCC but instead requires Alice and Bob to meet to implement the beam splitting operation $U_{AB}$ on their modes $A$ and $B$.

Still the CM $\gamma_1(x)$ corresponds to a fully separable state and therefore there exists a recipe how to create this state by LOCC. The recipe is based on the three-mode separability criterion according to which a three-mode Gaussian state with CM $\gamma_1(x)$ is fully separable iff there exist single-mode CMs $\gamma_A$, $\gamma_B$, $\gamma_C$ such that

\[
Q(x) \equiv \gamma_1(x) - \gamma_A \oplus \gamma_B \oplus \gamma_C \geq 0.
\]  

(5)

Interestingly, such single-mode CMs can be indeed found for $x \geq x_{\text{sep}}$ in the form

\[
\gamma_{AB} = \left( \begin{array}{ccc}
\alpha + \beta & \mp \tau & \alpha - \beta \\
\mp \tau & \alpha & \pm \tau \\
\alpha - \beta & \pm \tau & \alpha
\end{array} \right),
\]  

(6)

where

\[
\alpha = \frac{e^{-2r}}{2\delta} \left[ e^{4r} + \cosh(4d) - \sinh(4d) \cos(2\phi) \right],
\]

\[
\beta = \frac{e^{-2r}}{2\delta} \left[ \{ e^{4r} - \cosh(4d) \} \cos(2\phi) + \sinh(4d) \right],
\]

\[
\tau = \frac{\sin(2r)}{\delta} \sin(2\phi),
\]  

(7)

and the parameters satisfy the purity condition $\alpha^2 + \beta^2 + \tau^2 + 1 = 1$. The CM $\gamma_C$ represents a vacuum state. The CM $\gamma_A (\gamma_B)$ corresponds to the pure momentum-squeezed vacuum state with squeezing parameter $s = \frac{1}{2} \ln (\alpha + \sqrt{\alpha^2 - 1})$ rotated clockwise (anticlockwise) by the phase $\theta = \text{arctan} \left( \frac{\sqrt{\alpha^2 - 1} - \beta}{\sqrt{\alpha^2 - 1} + \beta} \right)$.

It remains to show that the matrix $Q(x)$ is positive semidefinite for $x \geq x_{\text{sep}}$. It is sufficient to show that for $x = x_{\text{sep}}$ since if $Q(x_{\text{sep}}) \geq 0$, then $Q(x) = Q(x_{\text{sep}}) \oplus (x - x_{\text{sep}})P \geq 0$ for all $x \geq x_{\text{sep}}$ because $(x - x_{\text{sep}})P$ is also positive semidefinite. To obtain the eigenvalues of the matrix $Q(x_{\text{sep}})$, we will calculate the eigenvalues of the matrix $U_{AB}Q(x_{\text{sep}})U_{AB}^T$, which possesses the same eigenvalues. They read explicitly as $\lambda_1, \lambda_2, \lambda_3 = 0$, $\lambda_4 = 9x_{\text{sep}}$, and $\lambda_5 = (e^{4d} \sin^2 \phi + e^{-4d} \cos^2 \phi) x_{\text{sep}}$. All of the eigenvalues are nonnegative and therefore the matrix $Q(x)$ for $x \geq x_{\text{sep}}$ is indeed positive semidefinite.

Creation of the fully separable state with CM $\gamma_1(x)$, where $x \geq x_{\text{sep}}$, is now straightforward. Initially, Alice prepares in her laboratory mode $A$ in a pure single-mode
squeezed state with CM $\gamma_A$ and the ancillary mode $C$ in the vacuum state. Similarly, Bob prepares the mode $B$ in a pure single-mode squeezed state with CM $\gamma_B$.

In the next step, Alice and Bob displace locally their modes by random correlated displacements distributed according to the Gaussian distribution with correlation matrix $Q(x)$. As a result, they prepare by LOCC a three-mode fully separable Gaussian state with CM $\mathcal{G}$. For the sake of simplicity here and in what follows we do not write explicitly the dependence of CMs on the parameter $x$ and we implicitly assume that $x \geq x_{\text{sep}}$.

**Entanglement distribution.** In step 2 Alice superimposes modes $A$ and $C$ of a fully separable state described by the CM $\gamma_1$ on a balanced beam splitter $U_{AC}$ [7] that transforms the CM as

$$\gamma_2 = U_{AC}\gamma_1 U_{AC}^T.$$  

Apparently, the CM is separable with respect to partition $B - (AC)$. More interestingly, mode $C$ can remain separable from the subsystem $(AB)$ if we choose the parameters $d, r$ and $x$ properly. To prove this, we apply to CM [3] the separability criterion based on the symplectic invariants [10]. The criterion utilizes the matrix $\gamma_2^{T_2} = A_C \gamma_2 A_C$, where $A_C = \text{diag}(1,1,1,1,1,-1)$, that describes CM $\gamma_2$ after partial transposition with respect to the mode $C$ [3,8]. The matrix $\gamma_2^{T_2}$ has three symplectic invariants denoted $I_1, I_2$ and $I_3 = \det(\gamma_2)$ that can be obtained as coefficients of the characteristic polynomial of the matrix $\Omega\gamma_2^{T_2}$, i.e.

$$\det(\Omega\gamma_2^{T_2} - yI) = y^6 + I_1 y^4 + I_2 y^2 + I_3.\] According to the criterion mode $C$ is separable from modes $(AB)$ iff

$$\Sigma \equiv I_1 - I_2 + I_3 - 1 \geq 0$$  

holds [14]. Calculating the invariants using the equations above we arrive at the following simple expression:

$$\Sigma = x(uv + v),$$  

where $u$ and $v$ are complex functions of parameters $d$ and $r$ given elsewhere [15]. In this paper we are interested in demonstrating the possibility of entangling $A$ and $B$, while keeping $C$ separable. So we show that for particular values of parameters $d$ and $r$ we get $\Sigma > 0$ and hence CM $\gamma_2$ is separable for $x$ larger than a certain threshold value $x_{\text{th}}$. Eq. (10) determines a parabola in the $(x, \Sigma)$ plane that intersects the $x$-axis in the origin. Taking $e^{2(d-r)} = 3/2$ and $e^{2(d+r)} = 2$ (which corresponds to $e^{-2x} \approx 0.6387$ and $\theta \approx 5.73^\circ$) we get $u > 0$ and $v < 0$ and the parabola is oriented upwards. Making use of Eqs. (11) and (13), the threshold value $x_{\text{th}} = -v/u \approx 1.04 > x_{\text{sep}} \approx 0.2043$, and hence for $x > x_{\text{th}}$ we have $\Sigma > 0$ and the CM (3) is separable with respect to the partition $C - (AB)$.

The protocol is finalized in the step 3. After receiving mode $C$ from Alice, Bob superimposes this mode with his mode $B$ on another balanced beam splitter $BS_{BC}$ [7]. The CM of the resulting state reads

$$\gamma_3 = U_{BC}\gamma_2 U_{BC}^T.$$  

Remarkably, $\gamma_3$ exhibits entanglement between modes $A$ and $B$ whereas mode $C$ remains separable from $(AB)$.

To verify entanglement between modes $A$ and $B$ we express the two-mode CM $\gamma_{3,AB}$ of the reduced state of modes $A$ and $B$ in the block form

$$\gamma_{3,AB} = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix},$$  

with the submatrices $A, B,$ and $C$ of the form:

$$C = \begin{pmatrix} c_+ + (g_1 h_1 - \frac{1}{\sqrt{2}}) x & -(h_0 + \frac{1}{\sqrt{2}}) x \\ (h_1 - \frac{1}{\sqrt{2}}) x & c_- - (g_0 h_0 + \frac{1}{\sqrt{2}}) x \end{pmatrix},$$  

$$A = \begin{pmatrix} a_+ + (g_1^2 + 1)x & (g_0 + g_1)x \\ (g_0 - g_1)x & a_- + (g_0^2 + 1)x \end{pmatrix},$$  

$$B = \begin{pmatrix} b_+ + (h_1^2 + \frac{1}{2}) x & h_0 - h_1 x \\ h_0 + h_1 x & b_- + (h_0^2 + \frac{1}{2}) x \end{pmatrix},$$  

where $a_\pm = (e^{\pm 2d} a + 1)/2$, $b_\pm = (e^{\pm 2d} a + \sqrt{2} c) + 1)/4$; $c_\pm = (e^{\pm 2d} (a + \sqrt{2} c) - 1)/2\sqrt{2}$, $g_1 = 1 + \sin(\phi + j \theta)/\sqrt{2}$, $h_1 = \sqrt{2} - (1/2) \sin(\phi + j \theta)/\sqrt{2}$, $\nu = 0$. The entanglement of CM $\gamma_{3,AB}$ can be proved if we calculate the so called symplectic eigenvalues [10] of the matrix $\gamma_{3,AB}^{T_{3,AB}} = A_{2,2}(\gamma_{3,AB})$, $A_{2,2}$, where $A_{2,2} = \text{diag}(1,1,1,1,-1)$ [17]. The matrix has two symplectic eigenvalues $\nu, \nu'$ that can be computed from the eigenvalues of the matrix $\Omega_{3,AB}$, where $\Omega_{3,AB} = \Sigma_x^{1/2} J$ and are equal to $(\pm i \nu, \pm i \nu')$ [17]. The mode $A$ is entangled with the mode $B$ if $\nu < 1$ or $\nu' < 1$. The lower symplectic eigenvalue of the matrix $\gamma_{3,AB}^{T_{3,AB}}$ can be expressed as [17]

$$\nu = \sqrt{(\kappa - \sqrt{\kappa^2 - 4\det(\gamma_{3,AB})})/2},$$  

where $\kappa = \det(A) + \det(B) - 2\det(C)$ and $A, B, C$ are defined in Eq. (13). By the same token as in step 2, we revert to the case $e^{2(d-r)} = 3/2$, $e^{2(d+r)} = 2$. Taking $x = 1.041 > x_{\text{th}}$, we obtain an exact expression for $\nu$ in terms of square roots that approximately equals to $\nu \approx 0.9571 < 1$, which is a clear evidence of the entanglement between modes $A$ and $B$. Note, that by measuring a quadrature $(x_C + p_C)/\sqrt{2}$ on mode $C$ the eigenvalue $\nu$ can be further reduced (i.e. entanglement can be increased [18]) to $\nu_m = 0.9421$.

Finally, we have to show that the ancillary mode $C$ is separable from the two-mode subsystem $(AB)$. We use again the simplectic invariants criterion of [10]. Analogous to step 2, we calculate the characteristic polynomial of the matrix $\Omega_{3,AB}^{T_{3,AB}}$ and find three symplectic invariants $J_1$, $J_2$ and $J_3 = \det(\gamma_3)$, which have to obey the condition $\Sigma = J_1 - J_2 + J_3 - 1 = x(uv + z) \geq 0$ (c.f. [9,10]). The parameters $w$ and $z$ are again complex functions of the parameters $d, r$ [19]. Assuming $e^{2(d-r)} = 3/2$, $e^{2(d+r)} = 2$ and $x = 1.041$, we obtain $\Sigma \approx 0.3957 > 0$. Thus the mode $C$ is separable from $(AB)$. 

FIG. 2: Performance of our protocol in dependence of the variances $\langle (\Delta x_A)^2 \rangle = e^{2(d-r)}$ and $\langle (\Delta x_B)^2 \rangle = e^{2(d+r)}$. Gray-scale region: all assumptions of the protocol are satisfied and $\nu < 1$. White region to the right of the slash: either $\nu \geq 1$ or $C$ is entangled with $(AB)$ in some stage of the protocol. The contour lines display the values of symplectic eigenvalue $\nu$.

In an experiment, verification of entanglement of modes $A$ and $B$ in step 3 can be done by measuring the entire CM $\gamma_{AB}$ similarly as in [19] and applying Simon’s [12] or Duan’s [13] separability criterion. The separability of mode $C$ from the pair of modes $(AB)$ in steps 2 and 3 can be proved by measuring the three-mode CMs $\gamma_2$ and $\gamma_3$. For Gaussian states one then can use the positive partial transposition criterion [10, 11] that is sufficient for separability of these $1 \times 2$-mode systems [8].

As a numerical evidence of the robustness of the protocol, we obtained eigenvalue $\nu = 0.9787$ for $\gamma_1 + 2 \times 10^{-2} \mathbb{I}$ corresponding to the initial CM perturbed by a weak isotropic noise. Numerical analysis also verifies the performance of the protocol for a broad range of variances $\langle (\Delta x_{A,B})^2 \rangle = e^{2(d+r)}$ depicted by a gray region in Fig. 2.

In conclusion, we have demonstrated the possibility to distribute entanglement without sending entanglement in infinite-dimensional systems. Remarkably, one can entangle two distant modes by a separable mode using experimentally feasible Gaussian states and operations involving single-mode squeezed states, correlated displacements and beam splitters, dispensing with the CNOT gates of the qubit case. The distributed entanglement is distillable [20] and therefore can be used for quantum communication. In contrast with two qubits, two light modes can be entangled deterministically even without any additional operation on Bob’s system and ancilla beyond step 3 (cf. [4]). Furthermore, we have elaborated the procedure to design three-mode Gaussian states with desired separability and noise properties. Together with the possibility to distribute distillable continuous-variable entanglement without sending it through the channel, it prepares the ground for better understanding and engineering of optical quantum networks, continuous-variable cryptography and other entanglement-based communication protocols using light modes and/or atomic ensembles. The support of the EU project COVAQIAL (FP6-511004) under STREP and the Czech Ministry of Education (Grant Nos. MSM 6198959213 and LC06007) is acknowledged.

[1] C. H. Bennett et al., Phys. Rev. Lett. 70, 1895 (1993).
[2] A. Ekert, Phys. Rev. Lett. 67, 661 (1991).
[3] R. F. Werner, Phys. Rev. A 40, 4277 (1989).
[4] T. S. Cubitt et al., Phys. Rev. Lett. 91, 037902 (2003).
[5] E. P. Wigner, Phys. Rev. 40, 749 (1932).
[6] G. Giedke et al., Phys. Rev. A 64, 052303 (2001).
[7] The $6 \times 6$ matrix $U_{ij}$ of a BS$_{ij}$ between the modes $i$ and $j$ transforms $x$ quadratures as $x'_i = (x_i + x_j)/\sqrt{2}$, $x'_j = (x_i - x_j)/\sqrt{2}$ and $p$ quadratures in the same way.
[8] R. F. Werner and M. M. Wolf, Phys. Rev. Lett. 86, 3658 (2001).
[9] A. Serafini, Phys. Rev. Lett. 96, 110402 (2006).
[10] A. Peres, Phys. Rev. Lett. 77, 1413 (1996).
[11] M. Horodecki et al., Phys. Lett. A 223, 1 (1996).
[12] R. Simon, Phys. Rev. Lett. 84, 2726 (2000).
[13] L.-M. Duan et al., Phys. Rev. Lett. 84, 2722 (2000).
[14] Strictly speaking, $\Sigma > 0$ suffices for separability of CM $\gamma_2$ as there are entangled CMs giving $\Sigma = 0$ [9].
[15] L. Mišta, Jr. and N. Korolkova, (in preparation) (2007).
[16] J. Williamson, Am. J. Math. 58, 141 (1936).
[17] G. Vidal and R. F. Werner, Phys. Rev. A 65, 032314 (2002).
[18] G. Adesso et al., Phys. Rev. A 70, 022318 (2004).
[19] J. DiGuglielmo et al., Phys. Rev. A 76, 012323 (2007).
[20] G. Giedke et al., Quant. Inf. Comp. 1, 79 (2001).