About a class of exact string backgrounds

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Abstract

We investigate a class of string backgrounds which have one conserved chiral null current on the world sheet. In the target space they have a null Killing vector and unbroken supersymmetries. This class also known as chiral null model is a generalization of the gravitational wave and fundamental string background and is exact in the $\alpha'$ expansion. The reduction to 4 dimensions yields a stationary IWP solution which couples to 7 gauge fields (one gravi-photon and 6 matter gauge fields) and 4 scalars. Special cases are the Taub-NUT geometry and rotating black holes. These solutions possess a T-self-dual point where the black hole becomes massless. Discussing the S-duality we show that the Taub-NUT geometry allows an S-self-dual point and that the electric black hole corresponds to a magnetic black hole or an H-monopole. We could identify the massless black hole as $N_L = 0$ and confirm the H-monopole as an $N_L = 1$ string states.

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1 Introduction

String theory is restrictive concerning allowed space time geometries. The requirement of 2-dimensional (2D) conformal invariance on the string world sheet fixes allowed backgrounds. There are mainly two different ways in finding consistent string backgrounds. One way is to start with a known 2D conformal field theory and to try to give them a reasonable space time interpretation. A lot of space time geometries and the corresponding conformal field theories are known. However, some classes of backgrounds seem to be not reachable by this approach, e.g. it was not yet possible to find a reasonable 4-dimensional (4D) black hole or nonstatic cosmological solution. Fortunately, there is another way as well. It is known that the conformal points of the world sheet theory correspond to stationary point of a string effective action which has directly a space time interpretation. The drawback of this starting point is that the effective action is known only perturbatively in the string tension $\alpha'$. On the other side there are some special classes of models where the lowest order is already the complete solution, i.e. which do not receive $\alpha'$ corrections. These types of backgrounds are especially interesting if one wants to address questions about singularities, horizons or the general structure of the space time. In the last time it was possible to find many different such solutions including different 4D black holes. For a recent review about known exact string solution see [1]. Although they can represent completely different space time geometries from the string point of view they are often not so different. Either one can relate them simply by a dimensional reduction which is nothing else as a different embedding of the 4D space time into the 10D target space. Obviously this does not change the 2D world sheet theory. Or they can be related by string symmetries like T– or S–duality. Whereas the T–duality is a symmetry which is valid order by order in the $\alpha'$ expansion the situation is not so obvious for the S–duality, where the different orders in the string loop expansion get mixed. This symmetry seems to be of non-perturbative nature and is up to now not proved. But let us assume that this conjecture is right. Thus, a string travelling in such related backgrounds is unable to distinguish between them and these solutions have to be identified or in the functional sense one has to sum only once. It is therefore reasonable to ask what backgrounds can be summarized to equivalent classes and what are the essential (string) properties of one class. This is the basic motivation of our investigation.

In investigating possible string backgrounds a reasonable assumption is the existence of at least one Killing vector. On the world sheet this means the existence of a conserved current. This current becomes chiral if the metric and the antisymmetric
tensor are related in a certain way. Many exact string backgrounds possess at least one chiral current, for the WZW model there are even all currents chiral. If we further assume that the corresponding Killing vector is null we arrive at the chiral null model \[2\]. It is a generalization of the gravitational wave background and fundamental strings \[3, 4, 5, 6\] and it belongs to the group of solutions which are already exact in the lowest order in \(\alpha'\). Furthermore, it has been shown that this model has unbroken supersymmetries \[7\] and thus it is a reasonable bosonic part of a ten dimensional (10D) superstring background. The aim of this paper is to investigate the different solutions belonging to this class.

Since this model gets no \(\alpha'\) corrections it is enough to consider the lowest order of the effective action only

\[
S_{10} = \int d^{10}X \sqrt{\hat{G}} e^{-2\hat{\phi}} \left[ R + 4(\partial \hat{\phi})^2 - \frac{1}{12} \hat{H}^2 \right] +
\]

\[\text{(higher genus terms)} + \text{(non-pert. terms)} \]

(1)

where: \(X^M = \{v, x^1, ..., x^8, u\}\), \(\hat{H} = d\hat{B}\) and \(\hat{B}\) is the 10D antisymmetric tensor. Of course apart from the \(\alpha'\) expansion there is an expansion in the genus of the world sheet (string loops) and with respect to this expansion we do not know much. This expansion goes with the string coupling ‘constant’ \((\sim g_s = e^{2\hat{\phi}})\) and the non-perturbative terms are typical of order \(\mathcal{O}(\exp^{-1/2g_s^2})\). We will neglect these corrections for our further consideration. The starting point is the 10D model and we are going to compactify this theory on a torus without making any simplifications. It has already been shown that after a dimensional reduction one gets electric black hole\[4\] and Taub-NUT spaces. This has been done for constant internal space \[4\] and for the case of an additional modulus field \[4\]. In a previous paper we have already discussed the dimensional reduction of the most general case \[10\]. Here, we are going to discuss mainly the 4D solutions, the charges, the T-self-dual point, the S-duality including the S-self-dual point.

The paper is organized as follows. In the next section we start with the gravitational wave and fundamental string background. The chiral null model is a natural generalization of both. In section 3 we are going to regard this model as the bosonic part of a 10D superstring background and will perform the dimensional reduction to 4 dimensions. As result we find a stationary IWP metric. In the fourth section we argue

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\*Strictly speaking as it has been shown in \[8\] the resulting 4D objects are not black but white holes which are attracting at long distances and repulsive for short distances. But for convenience we will remain at the name black hole.
that special cases of this metric are the Taub-NUT geometry and rotating black holes. Also we investigate the T-self-dual point. Finally, in section 5 we discuss the S-duality and construct the magnetically charged solutions. At the end we try to identify special classes of our solutions as elementary string or solitonic excitations.

2 Gravitational waves and fundamental strings

Let us start with two types of string backgrounds which have been widely discussed in the literature, namely the gravitational wave background (see e.g., in [3]) and the fundamental string background [5]. After a short introduction to these models we show that the chiral null model [2] is the natural generalization of both models.

If we are talking about gravitational wave background we mean the pp-waves, which are plane fronted waves with parallel rays. This background is defined by an covariantly constant null Killing vector

$$D_M k_N = 0, \quad k_M k^M = 0.$$  \hspace{1cm} (2)

As solution in general relativity (i.e. for $\phi = B = 0$) this background is Ricci flat $(R_{MN} = 0)$ and the metric is given by

$$ds^2 = 2dudv - K(x)du^2 - dx^I dx^I, \quad \partial^2 K(x) = 0$$

where the harmonic function $K$ in our case depends only on the transversal coordinates $x^I$; $u$ and $v$ are light cone coordinates. Of course, this represents a possible string background too, namely for a vanishing dilaton and antisymmetric tensor. For non-vanishing antisymmetric tensor we can generalize this wave background to

$$ds^2 = 2du[ dv - \frac{1}{2}K(x)du + \omega_I(x)dx^I ] - dx^I dx^I,$$

$$B = 2du \wedge [ dv + \omega_I(x)dx^I ], \quad \hat{\phi} = 0.$$  \hspace{1cm} (4)

Again the functions $K(x)$ and $\omega_I(x)$ should depend on the transversal coordinates $x^I$ only. The equation of motion for these functions are discussed below. In principle, one could allow a further dependence on $u$ (see in [3]), but throughout this paper we assume that all fields are independent of $v$ and $u$. The covariant constant vector is given by

$$k^M = (k^v = 1, 0).$$  \hspace{1cm} (5)

For the special case where $\partial_I [\omega_J] = constant$ this model can be interpreted as a non-semisimple WZW model [2] (see also [1]).
Secondly, we are interested in the fundamental string, i.e. a macroscopic string in the target space. This background has two Killing vectors lying in the “world sheet” of the this macroscopic string

\[ D_{(M k_N)} = D_{(M l_N)} = 0 \]  

and the metric, antisymmetric tensor and dilaton are given by

\[ ds^2 = 2F(x)du[du + \omega_I dx^I] - dx^I dx^I \]

\[ B = 2F(x)du \wedge [dv + \omega_I dx^I] \quad , \quad e^{2\hat{\phi}} = F(x) . \]

For the two Killing vectors one finds

\[ k^M = (k^v = 1, \vec{0}) \quad , \quad l^M = (\vec{0}, l^u = 1) . \]

For special choices of the $F$ and vanishing $\omega_I$ there exists again a CFT interpretation as $G/H$ gauged WZW model, where $H$ is a nilpotent subgroup of $G$ [4].

Both models can be naturally unified to

\[ ds^2 = 2F(x) du[du - \frac{1}{4}K(x)du + \omega_I(x)dx^I] - dx^I dx^I \]

\[ \hat{B} = 2F(x) du \wedge [dv + \omega_I(x)dx^I] \quad , \quad e^{2\hat{\phi}} = F(x) . \]

This is the chiral null model which has been discussed by Horowitz and Tseytlin [2]. For $F = 1$ we get the gravitational wave background (4) (also known as $K$-model) and for $K = 0$ this model corresponds to the fundamental string (7) ($F$-model). If one inserts these fields in the 2D world sheet Lagrangian one finds

\[ L = 4F(x) \partial u [\partial v - \frac{1}{4}K(x)\partial u + \omega_I(x)\partial x^I] - \partial x^I \partial x^I + \alpha' R^{(2)} \hat{\phi}(x) . \]

Let us recall some properties of this model. First we see that this model has a chiral symmetry on the world sheet: $v \rightarrow v + h(z)$, where $z$ is the complex world sheet coordinate. This translational invariance of $v$ has in the target space the consequence that the background admit a Killing vector: $k_M$ [3] and since there is no kinetic term for $v$ this is a null Killing vector. Furthermore, since we assume that there is no dependence in $u$ (generally this is possible) $l_M$ [8] is a Killing vector too. In the case of gravitational waves ($F = 1$) $k_M$ becomes even covariantly constant. In addition, the chiral symmetry is crucial for the exactness of this model, which means that all higher
\(\alpha'\) corrections in the renormalization group \(\beta\) functions vanish. To see this one has to integrate out \(u\) and \(v\) and find that for the renormalization only tadpole diagrams are relevant. The chiral structure of the Lagrangian makes it impossible to construct other (non-tadpole) divergent diagrams (for details see especially the second Ref. in [2]). Thus, the conformal invariance conditions are given by the lowest order in \(\alpha'\) and, if we drop a linear dilaton part, we have the equations

\[-\partial^2 K(x) = -\partial^2 F^{-1}(x) = 0 \quad , \quad -\partial^J F_{IJ} = 0 \quad \text{and} \quad e^{2\hat{\phi}} \sim F(x) . \quad (11)\]

\((F_{IJ} = \partial [I \omega_J])\). These are the equation of motion for the fields in (11). Investigating the T-duality we find that only the (harmonic) scalar functions \(K\) and \(F^{-1}\) are mixing.

If we take an arbitrary direction in the \((u, v)\) plane, e.g. by transforming \(v = \hat{v} + c u\), then dualizing \(u\) and finally reverse the \(v\) shift we find

\[F' = (2c - K)^{-1} \quad , \quad K' = 2c - F^{-1} \quad , \quad \omega'_I = \omega_I \quad , \quad e^{-2\hat{\phi'}} = F'^{-1} . \quad (12)\]

Using the fact that \(K\) and \(F^{-1} \sim e^{-2\hat{\phi}}\) are harmonic functions we see that the duality transformation changes only the parameter of the solution. The model is explicitly self-dual, i.e. the functions \(F\) and \(K\) remain unchanged if

\[K + F^{-1} = 2c . \quad (13)\]

Fixing the asymptotic values to \(K|_\infty = F|_\infty = 1\), we see that the model is explicitly self-dual only for \(c = 1\), i.e. not for arbitrary directions in the \((u, v)\) plane. Obviously, the wave background (4) with \(F = 1\) is dual to the fundamental string (7) with \(K = 0\) and only if both scalar fields are nontrivial we can have self-duality.

Finally, let us note that embedded in N=1, D=10 supergravity this model admits unbroken supersymmetries [7], [3], [4], [12]. On the other side one has to note that for the exactness of this model the supersymmetry is not crucial. Instead, as it has been shown in [2] the pure bosonic background is already exact in all orders in the \(\alpha'\) expansion.

3 The IWP solution in 4 dimensions

Of course, it is possible to regard the model (11) directly in 4 dimensions. But instead, the unbroken supersymmetries suggest to consider this model as the bosonic part of a 10D superstring background. Then we can ask, what does the corresponding 4D theory look like? Following this question we start with the 10D effective action, perform the dimensional reduction and discuss the 4D fields. In the next section we will relate the results to Taub-Nut and rotating black hole solutions.
Since for our model (10) there are no higher $\alpha'$ corrections the complete effective action (up to non-perturbative and higher genus contributions) is given by (1). Let us now start with the dimensional reduction. Since the reduction procedure preserves the supersymmetry also the 4D background has unbroken supersymmetries (corresponding to N=4). Assuming that the theory does not depend on 6 coordinates and that the internal space is compact we can integrate over internal coordinates and get the 4D theory. This is more or less standard in string compactification (see e.g. in [13]). On the other side if one admits a dependence on the internal coordinates, one can make a Fourier expansion in the internal coordinates. After reduction one gets then states with masses corresponding to the inverse compactification scales (see e.g. [14]). In this philosophy we are in the massless sector.

Now, we have to embed the 4D space time into the 10D target space. We choose $v$ as time coordinate and three of the transversal coordinates as spatial part, i.e. $x^M = (v, x^i|x^r, u)$ and get for the metric and antisymmetric tensor (9)

$$
\hat{G}^{MN} = \begin{pmatrix}
0 & 0 & F \\
0 & -\delta_{ij} & 0 & F \omega_i \\
0 & 0 & -\delta_{mn} & F \omega_m \\
F & F \omega_i & F \omega_m & -F K
\end{pmatrix}, \quad \hat{B}^{MN} = \begin{pmatrix}
0 & 0 & 0 & F \\
0 & 0 & 0 & F \omega_i \\
0 & 0 & 0 & F \omega_m \\
-F & -F \omega_i & -F \omega_m & 0
\end{pmatrix}
$$

(14)

where the last line corresponds to the $u$ coordinate. It seems to be a little disturbing that the former light cone coordinate $u$ becomes now part of the internal space. But as long as we make sure that the internal space remains compact and the 4D space time metric has the right signature there is no reason against this. Following now the standard procedure for dimensional reduction (see e.g. [13, 15]) we write the 10-Bein as

$$
e^\hat{N}_M = \begin{pmatrix}
0 \\
e^\mu \overset{\mu}{A}^r \overset{\hat{r}}{E}^s \\
0 \\
E^\hat{s}
\end{pmatrix}.
$$

(15)

The 4D space-time metric is given by $g_{\mu\nu} = e^{\hat{\mu}} e^{\hat{\nu}} \eta_{i\hat{j}}$ and the internal metric is $G_{rs} = \hat{E}^r \hat{E}^s \delta_{\hat{r}\hat{s}} (r, s = 4, \ldots, 9)$. This form of the 10-Bein has the advantage that the determinant of the metric and thus the volume measure factorizes and we can absorb the internal part in the dilaton

$$
\sqrt{|\hat{G}^{MN}|} e^{-2\phi} = |e^N_M| e^{-2\phi} = \sqrt{|g_{\mu\nu}|} \sqrt{|G_{rs}|} e^{-2\phi} = \sqrt{|g_{\mu\nu}|} e^{-2\phi}
$$

(16)
with the 4D dilaton $\phi$ defined by
\[ \sqrt{|G_{rs}|} e^{-2\phi} = e^{-2\phi}. \] (17)

Using the 10-Bein $\hat{G}$ we can write the 10D metric as
\[ \hat{G}_{MN} = \left( \begin{array}{cc} g_{\mu\nu} + A_\mu^r A_\nu^r & A_\mu^r \\ A_\mu^s & G_{rs} \end{array} \right). \] (18)

Comparing (18) with (14) we find for the internal metric and the gauge field part
\[ G_{rs} = \left( \begin{array}{cc} -\delta_{mn} & F \omega_m \\ F \omega_n & -FK \end{array} \right), \quad A_\mu = (0 \mid A_\mu) \] (19)
with $A_\mu = F \{1, \omega_i\}$ and inserting this metric in (17) yields for the 4D dilaton
\[ e^{-2\phi} = e^{-2\phi} \sqrt{FK - F^2 |\omega_m|^2} = \sqrt{F^{-1}K - |\omega_m|^2} \] (20)
where $|\omega_m|^2 = \omega_1^2 + \omega_2^2 + \ldots$. In the following we are assuming that
\[ F^{-1}K > |\omega_m|^2. \] (21)

This condition means that all eigenvalues of the internal metric are negative, i.e. there is no timelike compactified coordinate. Therefore, this condition ensures that the internal space remains compact, i.e. even $u$ represents a spacial direction. In discussing our results we will return to this point once more (after eq. (36)) but here let us proceed in deriving the 4D fields. For the metric we get finally
\[ ds^2 = \frac{1}{F^{-1}K - |\omega_m|^2} (dt + \omega_i dx^i)^2 - dx^i dx^i = e^{4\phi} (dt + \omega_i dx^i)^2 - dx^i dx^i. \] (22)

For the 4D antisymmetric tensor we are using the definition of $\hat{B}_{\mu\nu}$ and find that it vanishes in our case
\[ B_{\mu\nu} = \hat{B}_{\mu\nu} - A_\mu^r \hat{B}_{rs} A_\nu^s = 0 \] (23)
(where $\hat{B}_{\mu\nu}$ are the 10D components of (4)). In principle there is one further term which, however, vanishes too, because the off-diagonal terms coming from the metric
\[ b \text{The eigenvalues of } G_{rs} \text{ are: } \{-1, -1, -1, -1, -\frac{1}{2} \left(1 + FK \pm \sqrt{(1 - FK)^2 + 4F^2 |\omega_m|^2}\right)\}. \]
and antisymmetric tensor are equal. But nevertheless the 4D torsion is non-vanishing and is proportional to the Chern-Simons term of the gauge field \( A_\mu = F\{1, \omega^i\} \)

\[
H_{\mu\nu\rho} = -A_{[\mu}^r \partial_{\nu} A_{\rho]} = -e^{4\phi} F^{-2} A_{[\mu} \partial_{\nu} A_{\rho]} .
\]  

(24)

Using the equation of motion for \( \omega^i \) we can write the torsion on shell in a different form. From (11) we get

\[
\partial^i \partial_i \omega_j = 0 .
\]

(25)

Immediately, we find the two solutions

\[
\partial_i \omega_j = \epsilon_{ijk} \partial^k a(x) \quad \text{or} \quad \partial_i \omega_j = \text{const.}
\]

(26)

In the second case we have an uniform magnetic field and under certain assumptions the target space is parallelizable and the model corresponds to a product of a non-semisimple WZW model and a free spatial direction. The corresponding 4-D space time is not asymptotically flat. Let us ignore this case here (see [16]). The first case defines a further harmonic scalar field \( a \). Inserting this expression into (24) we find for the torsion

\[
(\det g)^{-\frac{1}{2}} \epsilon^{\lambda\mu\nu} H_{\mu\nu\rho} = -e^{2\phi} \partial^\lambda a
\]

(27)

Therefore, \( a \) is the axion field which determines the torsion \( H = e^{4\phi} a \), in the Einstein frame.

Next, we have to discuss the gauge fields. Gauge fields appear in the Kaluza–Klein procedure as non-diagonal components of the metric and the antisymmetric tensor. The gauge fields coming from the metric are in principle given by (19). But there is a subtlety. Investigating the gauge transformation one realizes that the basic gauge fields have an upper internal index. The reason is, that gauge transformations are generated by local translations in the internal coordinates which have an upper index. Rising the index we get

\[
A_\mu^r = A_{\mu s} G^{sr} = -\frac{1}{2} e^{-2\sigma} (F \omega^r + l^r) A_\mu \quad , \quad \omega^r = (\omega^m, 0) \quad , \quad l^r = (\vec{0}, 1) \quad (28)
\]

(\( \omega^m = \omega_m \), \( l^r \) is the second Killing vector of the 10D theory). On the other side, the gauge fields coming from the antisymmetric tensor have a lower internal index. The corresponding gauge transformation is part of the antisymmetric tensor gauge symmetry. But here, to get the right gauge field coupling in the effective action we have to add an additional term [13] and obtain

\[
B_{r\mu} = \hat{B}_{r\mu} - \hat{B}_{rs} A_\mu^s = \frac{1}{2} e^{-2\sigma} (F \omega^r - K F l^r) A_\mu .
\]

(29)
Finally, in the scalar field sector we have the dilaton, the axion and two modulus fields. The 4D dilaton is given by (20) and the axion is a not yet specified harmonic function. That there are at least two moduli fields becomes already obvious if we look at the eigenvalues of the internal metric (see footnote on page 8). If we diagonalize the internal metric we find that the internal space factorizes in a trivial 4D space (with vanishing antisymmetric tensor) and a non-trivial 2D space with the antisymmetric tensor and metric given by

\[
G_{mn} = \begin{pmatrix}
-\lambda_1 & 0 \\
0 & -\lambda_2
\end{pmatrix}, \quad B_{mn} = \begin{pmatrix}
0 & F|\omega_m| \\
-F|\omega_m| & 0
\end{pmatrix}
\]

where \(\lambda_{1/2}\) are the eigenvalues of the internal metric given in the footnote on page 8. We see that this 2D space is determined by two scalar functions

\[
e^\sigma = \sqrt{G_{mn}} = \sqrt{FK - F^2|\omega_m|^2}, \quad \eta = \sqrt{B_{mn}} = F|\omega_m|
\]

which we can combine to a complex scalar field

\[
T = \eta + ie^\sigma.
\]

This complex scalar function \(T\) parameterizes the non-trivial 2-torus. Note, that the T-duality transformation given in (12) with the self-dual point defined in (13) is not taken with respect to this \(T\) modulus field. The crucial point in our duality transformation was that we first shifted the time \(v\), then performed the duality transformation and finally inverted the time shift, i.e. we have “given the isometry direction \(u\) a piece of time”.

Summarizing our results the general 4-D solution is given by

\[
ds^2 = e^{4\phi}(dt + \omega_i dx^i)^2 - dx^i dx^i, \quad e^{-2\phi} = \sqrt{KF^{-1} - |\omega_m|^2}
\]

\[
H^{\mu\nu\rho} = -e^{2\phi} (\det g)^{-\frac{1}{2}} \epsilon^{\mu\nu\rho\lambda} \partial_\lambda a, \quad e^{2\sigma} = e^{\lambda(\hat{\phi} - \phi)} = K F - F^2 |\omega_m|^2
\]

with \(\omega_m = \omega^m\). The metric is the string version of the Israel-Wilson-Perjes (IWP) metric. In Einstein-Maxwell gravity this metric was constructed as a general stationary solution [17]. Basically, this metric has a non-flat spacial part, but Tod [18] found that the class of stationary metrics admitting a Killing spinor is given by the IWP metrics with flat 3D space. This class can then be embedded into supergravity. The static limit \((\omega_i = 0)\) is given by the Majumdar–Papapetrou black hole. Well-known examples
for IWP metrics are Taub-NUT and rotating black hole space times. The gauge fields can be combined as following

\[
\begin{pmatrix}
A^{(+)}_\mu^r \\
A^{(-)}_\mu^r
\end{pmatrix}
= \frac{1}{\sqrt{2}} \begin{pmatrix}
A_\mu^r + B_{r\mu} \\
A_\mu^r - B_{r\mu}
\end{pmatrix} = -\frac{1}{\sqrt{2}} e^{-2\sigma} \begin{pmatrix}
\frac{1}{2}(KF + 1) l^r \\
F \omega^r - \frac{1}{2}(KF - 1) l^r
\end{pmatrix} A_\mu .
\]

(34)

with \(A_\mu = F \{1, \omega_i\}\). Thus, after the dimensional reduction we have at all three scalars (dilaton, moduli field and axion) and 7 vectors. Because we have started with a supersymmetric theory in 10 dimension we expect to arrive in a N=4 theory in 4 dimensions. Indeed, the vector fields decompose into 6 matter vectors (\(A^{(-)}_\mu^r\)) and one gravi-photon (\(A^{(+)}_\mu^r\)) [15].

All these fields are completely determined by eight harmonic functions

\[-\partial^2 K(x) = -\partial^2 F^{-1}(x) = -\partial^2 \omega_m(x) = -\partial^2 a(x) = 0\]

(35)

where the derivatives acts only on the three spacial coordinates \(\vec{x}\). Note, that the vector \(\omega^i\) is fixed by (26) and \(m\) is an internal index so that all functions are space time scalars.

Before we further investigate the general solution let us discuss the relation to other known solutions. First, if we take the internal space flat, i.e. \(\omega_m = 0\) and \(FK = 1\). We arrive at the IWP solution discussed in [9]. In this case all matter gauge fields vanish (\(A^{(-)}_\mu^r = 0\)) and in addition we have no modulus field (\(\sigma = 0\)). Second, Horowitz and Tseytlin [2] have discussed the case \(\omega_m = 0\) but \(FK \neq 1\), i.e. they have added a nontrivial modulus field and one further (matter) gauge field. One example is the Kaluza–Klein (KK) solution. Setting \(\omega_i = \omega_m = 0\) and \(F = 1\) in the 10D theory the antisymmetric tensor and dilaton vanish and we arrive at the Einstein-Hilbert action. The solution is just the gravitational wave background [3] and after the reduction we have no torsion and only one gauge field coming from (28). Inserting a harmonic function for \(K\) this solution in the Einstein frame is given by the known extremely charged KK-black hole [19]

\[
\begin{aligned}
ds^2 &= e^{2\phi} dt^2 - e^{-2\phi} dx^i dx^i , \quad e^{-4\phi} = e^{2\sigma} = 1 + \frac{2m}{r} \\
H_{\mu\nu\rho} &= 0 , \quad A_0 = -\frac{1}{2} \left(1 + \frac{2m}{r}\right)^{-1} .
\end{aligned}
\]

(36)

Let us now return once more to the condition (21). This condition was crucial for getting an euclidean internal space. If this condition is not fulfilled the internal
metric has a positive eigenvalue (see footnote on page 8) corresponding to one time-like internal direction and therefore a non-compact internal space. On the other side for the fundamental string solution we have $K = 0$, and thus, this condition is not valid. This becomes clear if one remembers how to construct the fundamental string solution \[20\]. One can start with a (uncharged) black string as a direct product of the Schwarzschild metric and a flat direction. After a Lorentz boost in the flat direction one can dualize this direction and get a dilaton and an $H$-charge. Finally, after performing the extremal limit one gets the fundamental string winding around the flat direction. The coordinates $u$ and $v$ are there light cone coordinates, which are both non compact. There are two possibilities to avoid this problem. First is to take at least one $\omega_m$ imaginary corresponding to a Wick rotation in the internal coordinate with the wrong eigenvalue. This was done in \[6\] to get a black hole solution from the fundamental string in 10 dimensions. Another way is to give the fundamental string first a non-zero linear momentum and reduce it then. In this case $K \neq 0$ and can be normalized to $K = 1$ \[2, 12\]. In both cases one can find a region with right signature.

Before we turn to special examples we want to discuss the theory in the neighborhood of the singular points of the scalar functions $F$, $K$, $a$, $\omega^r$ (35). At these points our metric and dilaton as well as the other 4D fields are singular. Although our solution is exact in the $\alpha'$ expansion we should expect that at these points higher genus (string loop) contributions and non-perturbative terms become important. But this not the case. If we remember that the string coupling constant $g_s$ is defined by the dilaton we find near the singularities

$$g_s^2 = e^{4\phi} = \frac{1}{KF^{-1} - |\omega|^2} \to 0,$$

i.e. the string coupling constant vanishes and therefore the higher genus contributions ($\sim g_s$) and the non-perturbative terms ($\sim \exp\{-\frac{2}{g_s}\}$) are expected to be under control. Thus our model does not break down there but seems to behave even better, it becomes asymptotically free. This feature has been discussed for the wave or fundamental string background in \[21\].

4 Relation to Taub-NUT and rotating black hole solutions

In the last section we have performed the dimensional reduction of our model and have discussed the field content in 4 dimensions. The 4D solution (33) is determined by eight
harmonic functions $K(x)$, $F^{-1}(x)$, $a(x)$, $\omega^r(x)$. In this section we are going to discuss special examples for these functions and we will find different space time geometries.

The general solution for the harmonic functions is given by the real or imaginary part of
\[ \sim c_0 + \sum_{k=1}^{N} \frac{c_k}{r_k} \]
where $r_k^2 = (x - x_k)^2 + (y - y_k)^2 + (z - z_k)^2$ and the parameter $(c_0, c_k, x_k, y_k, z_k)$ are complex. Now we have to choose these complex parameter in a proper manner. Calculating the real and imaginary part one finds that the physical distinguishable situations are determined by the different singularity structures. That means, without loss in generality we can assume $c_0$ and $c_k$ as real. Furthermore, to make the situation more clear we assume that we have only one center, i.e. no sum over $k$ ($x_k \to x_0$). In this case we can absorb the real part of $x_0$ by shifting the spatial coordinate, and we set $x_0 = i(\gamma, \beta, \alpha)$ ($\alpha, \beta, \gamma$ are now real parameter). Thus, we write the harmonic functions as
\[ \sim c_1 + \frac{c_2}{\sqrt{(x - i\gamma)^2 + (y - i\beta)^2 + (z - i\alpha)^2}} \]
We have now to distinguish that all, two, one or none of the imaginary parts are zero. Understanding this situation the generalization to multi-center solutions is straightforward.

1. Taub-NUT: $\alpha = \beta = \gamma = 0$. Here, the imaginary parts vanish and the theory is completely spherical symmetric. We can write the harmonic functions as
\[ K = 1 + \frac{2m}{r} \quad , \quad F^{-1} = 1 + \frac{2\tilde{m}}{r} \quad , \quad \omega_m = \frac{2\tilde{q}^m}{r} \quad , \quad a = \frac{2n}{r} \]
where $r^2 = x^2 + y^2 + z^2$ and $\tilde{\omega}$ is defined by the axion $a$ ($\partial_i [\omega_j] = \epsilon_{ijk} \partial_k a$). In this case the harmonic functions $K$ and $F^{-1}$ can always be written as $K = 2c - bF^{-1}$. The constants $c$ and $b$ can be adjusted by a proper choice of the $u$ and $v$ coordinates in 10 dimensions (translation in $v$ and scaling of $u$) and one finds that only $b = \pm 1$ ($m = \pm \tilde{m}$) are non-trivial. We will often distinguish between these two cases. The self-dual case (B3) is given by $b = 1$ or $m = -\tilde{m}$ (to get the right asymptotic behavior we set $c = 1$). The other case ($m = \tilde{m}$) is crucial for a flat internal space. Inserting these fields and a solution for $\omega_i$ in (B3) yields for the metric and dilaton
\[ ds^2 = e^{4\phi} (dt + 2n \cos \theta d\phi)^2 - \left( dr^2 + r^2 d\Omega^2 \right) \quad , \quad e^{4\phi} = \frac{1}{(1 + \frac{2\tilde{m}}{r})(1 + \frac{2n}{r})} \]
with

\[ r_\pm = m + \tilde{m} \pm \sqrt{(m - \tilde{m})^2 + 4|q_m|^2} = \begin{cases} 
\pm 2\sqrt{\tilde{m}^2 + |q|^2}, & m = -\tilde{m} \\
2(m \pm |q|), & m = \tilde{m} 
\end{cases} \quad (42) \]

This is an extremely charged Taub-NUT solution. If we make our internal space flat \((\omega_m = 0, FK = 1 \text{ or } r_+ = r_-, m = \tilde{m})\) this solution again coincides with solution found in \([3, 22]\). This metric has a wire singularity, which makes it impossible to invert the metric along the axes \(\theta = 0, \pi\). To see this one can consider slides with constant \(t\) and \(r\) for which \(ds^2\) does not vanish at the north and south pole (as one would expect for a smooth space time). Though this singularity can be removed by choosing different times at the north and south hemisphere \((t \rightarrow t \pm 4\pi\phi)\). But for consistency at the overlapping region one has to require that the time is periodical \(t \sim t + 8\pi n\) (see e.g. in \([23]\)). Also, this metric is asymptotically not flat. Instead, one can show that for constant radius this metric has an \(S_3\) geometry.

Furthermore, if \(r_- < 0\) \((m\tilde{m} < |q_m|^2)\) the metric contains a pole at \(r = r_-\). An example is the self-dual case \([13]\) where \(r_- = -r_+ \quad (m = -\tilde{m})\). What happens at this point? If we consider the corresponding moduli field \(e^{2\sigma}\) \([11]\)

\[ e^{2\sigma} = \lambda_1\lambda_2 = \frac{r + r_+}{r + 2\tilde{m}} \frac{r - r_-}{r + 2\tilde{m}} \rightarrow \begin{cases} 
1, & \text{for } r \rightarrow \infty \\
\frac{m\tilde{m} - |q_m|^2}{m^2}, & \text{for } r \rightarrow 0 
\end{cases} \quad (43) \]

where \(\lambda_{1/2}\) are the eigenvalues of the internal metric which define the radii of the non-trivial torus. We see that for negative \(r_-\) one radius has a zero and becomes negative for \(r < -r_-\), and thus, this coordinate becomes time like. Also at this point the space time metric changes its signature and becomes euclidean. It has been shown in \([24]\) that at this point the space time has a true singularity. Interestingly, the 10D theory is here completely smooth. In some sense the singularity is compensated by the vanishing volume of the internal space \((\det G_{rs} = 0)\). As it was shown by Kallosh and Linde \([8]\) this singularity becomes smooth and non-singular on the quantum level. On the other side for positive \(r_-\) the compactification radii are bounded for all \(r\), and thus, the internal space remains “invisible” (as long as we choose the compactification scale sufficiently small).

2. Rotating black holes: \(\alpha \neq 0, \beta = \gamma = 0\). Here the theory is axial-symmetric and the singularities of the scalar functions are in the plane \(z = 0\) on the circle \(x^2 + y^2 = \alpha^2\).
We write

\[ K = \text{Re} \left( 1 + \frac{2m}{\sqrt{x^2 + y^2 + (z - i\alpha)^2}} \right), \quad F^{-1} = \text{Re} \left( 1 + \frac{2\tilde{m}}{\sqrt{x^2 + y^2 + (z - i\alpha)^2}} \right) \]

\[ \omega^r = \text{Re} \left( \frac{2y}{\sqrt{x^2 + y^2 + (z - i\alpha)^2}} \right), \quad a = \text{Im} \left( \frac{2n}{\sqrt{x^2 + y^2 + (z - i\alpha)^2}} \right). \]

As we will see below this choice corresponds to the inclusion of non-vanishing angular momentum which is proportional to \( \alpha \). In order to get back the static solution \([\omega = 0]\) for vanishing angular momentum \((\alpha = 0)\) we have to take the imaginary part for the axion field and the real part for the other functions. So, taking these functions and transforming the solution into spheroidal coordinates \((x + iy = \sqrt{r^2 + \alpha^2 \sin^2 \theta} \exp \{\pm i\phi\}; \quad z = r \cos \theta)\) we find

\[ \omega_\phi = \frac{2n\alpha \sin^2 \theta}{R} \quad \text{with} \quad R = \frac{r^2 + \alpha^2 \cos^2 \theta}{r} \]

and for the metric and the dilaton

\[ ds^2 = e^{4\phi} \left( dt + \omega_\phi d\phi \right)^2 - d\vec{x}^2, \quad e^{4\phi} = \frac{1}{(1 + \frac{r}{R})(1 + \frac{r}{R})} \]

\[ d\vec{x}^2 = \frac{r^2 + \alpha^2 \cos^2 \theta}{r^2 + \alpha^2} dr^2 + (r^2 + \alpha^2 \cos^2 \theta)d\theta^2 + (r^2 + \alpha^2) \sin^2 \theta d\phi^2 \]

where \( R = 0 \) defines the ring singularity. Here, as for the Taub-NUT type solution \((41)\) the only influence of the additional gauge fields and the moduli field is a splitting in the dilaton field \((r_+ \neq r_-)\). Again for a flat internal space this solution coincides with \([9]\).

In difference to the Taub-NUT case this solution is asymptotically flat and we can define conserved charges. First from the asymptotic behaviour of metric in the Einstein frame we get the mass \((\text{from } g^E_{00})\) and the angular momentum \((\text{from } g^E_{00})\)

\[ M = \frac{1}{4} (r_+ + r_-) = \frac{1}{2} (m + \tilde{m}) \quad , \quad J = n \alpha \]

and from the gauge fields we can read off the electric charges \((\tilde{A}_0 \propto -\frac{\tilde{Q}}{r})\) and find

\[ \begin{pmatrix} Q^{(+)r}_e \\ Q^{(-)r}_e \end{pmatrix} = \sqrt{2} \begin{pmatrix} \frac{1}{2} (m + \tilde{m}) l^r \\ -q^r - \frac{1}{2} (m - \tilde{m}) l^r \end{pmatrix} \quad , \quad q^r = (q^m, 0). \]

\(^4\)In the classification of time independent metrics static means, that one can diagonalize the metric. In contrast to stationary metrics, which are also time independent, but the non-diagonal part corresponds, e.g., to a non-vanishing angular momentum or Taub-NUT charge.
Now we can express the constants $r_\pm$ in the solutions by these charges and obtain

$$r_\pm = \sqrt{2} \left( |\vec{Q}(+)\rangle \pm |\vec{Q}(-)\rangle \right)$$

(49)

Investigating the dual field strength tensor we see that there are no magnetic charges but magnetic moments \( \tilde{F}_{\theta r} \propto 2 \mu \cos \theta \)

$$\mu^{(-)r} = 0 \quad \mu^{(+)} = \sqrt{2} \alpha l^r .$$

(50)

Having the charges we can ask which of the states saturate a Bogomol’nyi bound. Following the procedure of Sen [25] we get

$$M^2 = \frac{1}{2} e^{-2\phi_0} \vec{Q}_e \cdot (LM_0L + L) \cdot \vec{Q}_e = \frac{|\vec{Q}(+)\rangle|^2}{2}$$

(51)

(the index “0” means asymptotical values and \((LM_0L + L)\) is in our case just the projector on the (+) states). Thus, as expected for a supersymmetric solution the Bogomol’nyi bound is saturated. But only the graviphoton sector \(A^{(+)}_{\mu} r\) enter this bound and the matter gauge fields sector \(A^{(-)}_{\mu} r\) are not restricted. Therefore, the graviphoton has to be a lowest state.

An interesting case in this context is the self-dual configuration \([13]\) where \(\tilde{m} = -m\). Here the black hole states become massless \((M = 0)\) and carry only the matter gauge field charges \(\vec{Q}(-)\) \((\vec{Q}(+) = 0)\). This is consistent with the statement that self-dual configurations are points of enhanced symmetry and often correlated with additional massless states. But note that our duality transformation is not taken with respect to the internal space only but a mixing of the time and the internal coordinate \(u\). The self-dual point with respect to the \(T\) modulus field (e.g. \(T = i\)) does not correspond to massless black holes.

Let us end this section with a discussion of the singularity. As it was pointed out in [3] the solution with flat internal space exhibits a naked singularity. Since the causal structure and the singularities are not affected by a non-flat internal space (as long as \(r_- > 0\)) the metric \([40]\) has still a naked singularity. This can also be seen by a direct comparison with a rotating black hole solution. Starting with a 4-D Kerr solution and using the \(O(d,d+p)\) technique Sen [26] has constructed a general black hole solution including 28 U(1) gauge fields and nontrivial moduli. One can show that both solutions coincide in a special limit\(^d\). On the other side, Sen’s solution has two horizons, which

\(^d\)We have to set \(m = \tilde{m} = \sqrt{n^2 + |q|^2}\) in our solution and take in Sen’s solution the limit \(m^{Sen} \to 0\), \(\beta \to \infty\), but \(m \sinh \beta \cosh \alpha = 2M\) remains fix \((|q_m| = M^{Sen} \tanh \alpha^{Sen}, q^{Sen} = \alpha \cosh \alpha^{Sen})\). Note, that in Sen’s solution all parameters are correlated to each other. But in our solution, e.g., \(n\) is not restricted. The reason is that Sen has generated his solution by a \(O(d, d)\) transformation, which does not yield the most general rotating black hole solution.
vanish in this limit and the singularity becomes naked. But what is the interpretation of this singularity? There has been a lot of discussion that the ring singularity of a Kerr black hole can be regarded as closed elementary string. E.g. Nishino [27] showed some indication that the Kerr solution with a naked singularity is nothing else as the gravitational field generated by a closed string. Furthermore, this interpretation could yield for the entropy the correct density of elementary string states [28]. Let us discuss here only one feature which supports this idea, namely the gyromagnetic ratios. They can be defined by $\vec{\mu} = \frac{1}{2} g J \vec{Q}$ and for our solution we obtain

$$g^{(-)} = 0, \quad g^{(+)} = 2$$

i.e. the matter gauge fields have a zero gyromagnetic ratio and the graviphoton has $g = 2$. As pointed out by Sen [26] these values coincides with the corresponding values for elementary string states. Using a generalization of the results of Russo and Susskind [29] Sen found for $g$

$$g^{(\pm)} = 2 \frac{S^{(\pm)}}{S^{(-)} + S^{(+)}}$$

where $S^{(\pm)}$ is the contribution to the $z$ component to the angular momentum. Since in our case the fields in the right moving sector ($+$ component) saturate the Bogomolnyi bound they have to be in the lowest state. Therefore the angular momentum coming from this part can be neglected with respect to left moving part and one finds the values (52). In addition the value $g = 2$ for the Bogomolnyi states is just the natural value of $g$ if one wants to interpret these states as elementary particles [30]. Of course, that there is no horizon is disturbing but as it was shown in [31] even timelike singularities can be completely non-singular when probed with quantum particles. In analogy to the hydrogen atom this means that a classical singular theory becomes nonsingular quantum mechanically. By the way, singularities play a useful role - they can enable a stable ground state [32].

3. The cases: $\alpha \neq 0$, $\beta \neq 0$, $\gamma = 0$ or $\alpha \neq 0$, $\beta \neq 0$, $\gamma \neq 0$. Let us summarize both cases here. The first one corresponds to two singular points along the line $y = z = 0$ at $x = \pm \sqrt{\alpha^2 + \beta^2}$. In the second case the harmonic functions have no poles at all. But nevertheless they have zeros and thus the metric has singularities. This is interesting because in this case we have no weak coupling region (dilaton $\phi \rightarrow -\infty$ in (20)) but only a strong coupling region (corresponding to the zeros). Since our theory is not well defined there let us ignore this case further. Perhaps it is worthwhile to look on these case a little bit more carefully, but we want to leave this question for future investigations.
The S–duality and H–monopoles

In this section we are going to discuss the strong-weak coupling (S-) duality. This symmetry is a string version of the old Montonen–Olive conjecture that the theory is invariant under replacing the electric with the magnetic charges and simultaneously the inversion of the coupling constant. This symmetry has not yet been proved, but nevertheless there are a lot of good reason to belief that this is really a symmetry of the theory (for a nice short review see [33]). The reason for the difficulties is that one knows the theory only perturbatively (i.e. for weak couplings) and this symmetry has a non-perturbative nature. It mixes the different orders of the perturbative expansion, e.g. the weak with the strong coupling region.

The starting point is to transform the model into the Einstein frame. This frame is defined by a conformal rescaling of the metric in order to decouple the dilaton from the curvature in (1). Then one goes on-shell with the torsion, i.e. one replaces the torsion by the axion $a$. Thus we write

$$G_{\mu\nu} \rightarrow G^E_{\mu\nu} = e^{-2\phi} G_{\mu\nu} , \quad H_{\mu\nu\lambda} = -\sqrt{\det G^E} \epsilon_{\mu\nu\lambda\sigma} \partial^\sigma a$$ (54)

where $G^E_{\mu\nu}$ is now the Einstein metric. As a next step one combines the dilaton and the axion to one complex scalar field

$$\lambda = a + ie^{-2\phi}.$$ (55)

Now, after these transformations in the 4D theory (coming from the dimensional reduction of (1)) one finds that the equations of motion are invariant under [34], [35]

$$\lambda \rightarrow a \frac{\lambda + b}{c \lambda + d} , \quad \text{with} : \quad ad - bc = 1 , \quad a, b, c, d \in \mathbb{R}$$ (56)

$$F^{(m)}_{\mu\nu} \rightarrow (c \text{Re}\lambda + d) F^{(m)}_{\mu\nu} + c \text{Im}\lambda (ML)_{mn} \tilde{F}_{\mu\nu}^{(n)}$$

where

$$\tilde{F}_{\mu\nu}^{(m)} = \frac{1}{2} (\det G^E)^{-\frac{1}{2}} G_{\mu\nu}^{\prime} G_{\mu\nu}^{\prime} \epsilon_{\mu\nu}^{\prime} \lambda \phi F^{(m)}_{\lambda\phi} .$$ (57)

The index $m$ numerates all gauge fields in (28), (29) and the exact form of the matrices $M$ and $L$ can be found in [35]. All other fields, the (Einstein) metric and the modulus field remains unchanged. The S-duality is therefore an $SL(2, \mathbb{R})$ transformation in the complex scalar $\lambda$ and we can generate new solutions which have in general both electric and magnetic charges. Let us consider the special case where the asymptotic values of
the dilaton and the axion remains unchanged \( (\phi|_0 = a|_0 = 0) \), i.e.

\[
\begin{pmatrix}
    a & b \\
    c & d \\
\end{pmatrix} = \begin{pmatrix}
    \cos \gamma & \sin \gamma \\
    -\sin \gamma & \cos \gamma \\
\end{pmatrix} \in SO(2)
\]  

(58)

which means for the field strength

\[
F^{(m)}_{\mu\nu} \rightarrow (-\sin \gamma a + \cos \gamma) F^{(m)}_{\mu\nu} - \sin \gamma e^{-2\phi} (ML)_{mn} \tilde{F}^{(n)}_{\mu\nu}.
\]  

(59)

From the asymptotic values we can get the new charges \( (F_0r \propto \frac{Q_e}{r^2} \) and \( \tilde{F}_{0r} \propto \frac{Q_m}{r^2} \).

For the electric charges only the first part is relevant whereas the magnetic charges are determined by the second part. Inserting the asymptotic value for \( (ML)^{(\pm)} \propto \pm 1 \) we find

\[
\vec{Q}_e^{(\pm)} = \cos \gamma \vec{Q}_e^{(\pm)\text{old}}, \quad \vec{Q}_m^{(\pm)} = \mp \sin \gamma \vec{Q}_e^{(\pm)\text{old}}
\]  

(60)

or the new electric and magnetic charges are correlated by

\[
\vec{Q}_m^{(\pm)} = \mp \tan \gamma \vec{Q}_e^{(\pm)}.
\]  

(61)

The mass and the angular momentum in (47) as defined via the Einstein metric \( G^E_{\mu\nu} \) remain fix. Again, we can express the parameter \( r_\pm \) of our solution by the new charges

\[
r_\pm = \sqrt{2} \left( \sqrt{|\vec{Q}_e^{(+)|^2} + |\vec{Q}_m^{(+)|2} \right) \pm \sqrt{|\vec{Q}_e^{(-)|2} + |\vec{Q}_m^{(-)|2}} \)  

(62)

The case \( \gamma = \frac{\pi}{2} \) or \( \lambda \rightarrow -\frac{1}{\lambda} \). This transformation together with the trivial discrete transformation \( a \rightarrow a + 1 \) generate the discrete subgroup \( SL(2, \mathbb{Z}) \). This subgroup is expected to be a symmetry of the whole quantum theory (by instanton effects the \( SL(2, \mathbb{R}) \) breaks down to this subgroup [34]). In this case our pure electrically charged solution becomes a pure magnetically charged. The transformed axion and dilaton are given by

\[
a' = -\frac{a}{a^2 + e^{-4\phi}}, \quad e^{-2\phi'} = \frac{e^{-2\phi}}{a^2 + e^{-4\phi}}
\]  

(63)

where \( a \) and \( \phi \) are the old axion and dilaton. We see that for vanishing axion the dilaton changes the sign which corresponds to the change from the weak to the strong coupling or vice versa. Although the metric in the Einstein frame does not change during the S-duality the metric in string frame does (due to the new dilaton field)

\[
ds'^2 = e^{2\phi'} ds^2_E = (1 + a^2 e^{4\phi}) \left[ (dt + \omega_i dx^i)^2 - e^{-4\phi} dx^2 \right]
\]  

(64)
where $\phi$ and $\omega_i$ are defined in (11) or (16). Let us discuss special cases.

1. **S-self-duality.** Obviously, there is the special case if

$$a^2 + e^{-4\phi} = 1.$$  \hspace{1cm} (65)

In this case the dilaton remains unchanged and the axion changes only the sign and because the effective action depends only quadratically on the torsion this does not change the equations of motion. Only the gauge fields get a nontrivial transformation. Let us call this case S-self-dual. Inserting the dilaton and axion for the Taub-NUT case (40), (41) we obtain the restriction

$$r_- + r_+ = 0 \quad , \quad r_+ r_- + 4n^2 = 0 \quad \leftrightarrow \quad m = -\tilde{m} \quad , \quad n^2 = \tilde{m}^2 + |q|^2$$  \hspace{1cm} (66)

i.e. the theory is S-self-dual iff it is T-self-dual (coming from (13)) and the axion charge $n$ is correlated with the electric charges ($\tilde{m}$ and $q_r$). Looking on the rotating black hole configuration (14) we find that asymptotically $a^2 \sim r^{-4}$ but $e^{-4\phi} \sim r^{-2}$ and therefore the rotating black hole does not allow the S-self-dual configuration. By the way, the different asymptotic behaviour of the axion is the reason why the black hole configuration is asymptotically flat but the Taub-NUT solution not. To clarify this case a little bit more let us consider the gauge fields. In the rotating black hole case we have seen that our solution has only electric charges. In the Taub-NUT case we have no asymptotically flat region and we cannot define charges as usual. But looking on the asymptotic behavior of the gauge fields we can define an electric and magnetic charge analogue. To simplify let us assume that $\omega_r = q_r = 0$ and consider the gauge field $A^m$ (28)

$$A^m = -\frac{1}{2} \frac{1}{1 - 2a r} \{1, \omega_i\} l^m \quad \text{with:} \quad \omega_i dx^i = 2n \cos \theta d\phi.$$  \hspace{1cm} (67)

If we now look on the asymptotic behavior of the corresponding field strength we find

$$F^m_{0r} \propto \tilde{m} \frac{l^m}{r^2} \quad , \quad \tilde{F}^m_{0r} \propto \frac{n}{r^2} l^m.$$  \hspace{1cm} (68)

Thus $\tilde{m}$ and $n$ play the role of an electric and magnetic charge and the Taub-NUT solution carries both. Furthermore in self-dual case (66) both charges coincide up to a possible sign. If we add the other electric charges, i.e. $q_r \neq 0$ the magnetic charge does not change but the electric charge gets additional terms. The condition (66) then simply means that the length of the electric and magnetic charge vector is equal. Finally for the complex scalar function $\lambda$ we find

$$|\lambda|^2 = |a + ie^{-2\phi}|^2 = (a^2 + e^{-4\phi})^2 = 1.$$  \hspace{1cm} (69)
i.e. for the self-dual case $\lambda$ is a unit vector. In this case we do not change from weak to strong region or vice versa. Instead, we remain in one region (the change of the dilaton is compensated by the axion).

2. Vanishing axion ($a = 0$). In this case $\omega_i$ vanish as well ($\partial_ia = \epsilon_{ijk}\partial_j\omega_k$) and therewith the angular momentum or Taub-NUT parameter. So, both solutions the rotating black hole and the Taub-NUT coincides. This case is interesting since here the weak coupling region is directly transformed into a strong coupling region ($\phi \to -\phi$) and therefore it is just the opposite to the former case.

Setting the axion to zero we get an extreme magnetic black hole coupling to 7 gauge fields

$$ds^2 = dt^2 - (1 + \frac{r_+}{r})(1 + \frac{r_-}{r})d\vec{x}^2, \quad e^{2\phi'} = \sqrt{(1 + \frac{r_+}{r})(1 + \frac{r_-}{r})}$$

where $r_\pm$ are given by (62) and for the magnetic charges one obtains

$$Q_m^{(+)} = -\frac{1}{\sqrt{2}}(m + \tilde{m}) \rho, \quad Q_m^{(-)} = -\sqrt{2}(q^r + \frac{1}{2}(m - \tilde{m}) \rho).$$

For a flat internal space $r_+ = r_-$ ($q^r = 0$ and $m = \tilde{m}$) and one gets the standard extreme magnetic black hole coupling to one gauge field [36]. For the T-self-dual case ($m = -\tilde{m}$) again the black hole becomes massless and neutral for the gravi-photon ($Q_m^{(+)} = 0$).

Another limit yields the H-monopole solution. Here we have to set $r_- = 0$, i.e. $m\tilde{m} = |q|^2$. In this case both magnetic charges in (62) compensate each other $|Q_m^{(+)}| = |Q_m^{(-)}|$ (note that for the $SL(2,\mathbb{Z})$ transformations we have only magnetic charges). Therefore an H-monopole is nothing else as an extreme magnetic black hole with a balance between the magnetic charges coming from the matter and from the graviton sector. As long as $q^r,\tilde{m} \neq 0$ this monopole couples to more gauge fields. To get the standard H-monopole [37] coupling only to one gauge field we set

$$m = q^r = 0 \quad (K = 1)$$

and find

$$ds^2 = dt^2 - (1 + \frac{r_+}{r})d\vec{x}^2, \quad e^{2\phi'} = \sqrt{(1 + \frac{r_+}{r})}$$

$$e^{-2\sigma} = 1 + \frac{r_+}{r}, \quad F_{ij} = \frac{1}{2}\epsilon_{ijk}\partial_k(1 + \frac{r_+}{r})$$

where the gauge field is the S-dual of (28). The gauge field coming from the antisymmetric tensor (29) is trivial in this case. This monopole solution can be obtained by a dimensional reduction of the neutral fivebrane solution [38]. In this case we have no gauge fields in the higher dimensional theory and the magnetic charge comes from the
axion charge (torsion). The interpretation of this solution is that the fivebrane in 10
dimension wraps around five of the six compactified dimensions. This H-monopole is
a solution of the Kaluza–Klein (KK) theory as well. Namely, instead of setting $m = 0$
we can set $\tilde{m} = 0$. This corresponds to $F = 1$ and thus a vanishing 10D dilaton ($\hat{\phi}$, see
(1)). Also, the 10D antisymmetric tensor (14) is constant ($\omega_i = \omega_m = 1$) and therefore
the torsion vanish too. So, the 10D action is given only by the Einstein-Hilbert term.
The corresponding electric charged solution (36) is the S-dual to this case. Both solu-
tions confirm the statement that for a diagonal internal metric the most general static
KK black hole can have at most one electric and one magnetic gauge field [39].

Interestingly all these magnetic solutions are not yet exact in the $\alpha'$ expansion.
One can, however, promote these solutions to exact ones by embedding the 10D spin
connections into a non-abelian gauge group. For the extreme magnetic black hole this
has been done in [14] and for the 5-brane solution see e.g. in [38]. This embedding is
necessary to remove anomaly related higher order terms in $\alpha'$. On the other side the
electric charged solutions are already exact without this embedding. The situation is
not very clear. If we start from the exact magnetic solution including the non-abelian
gauge field, there is no electric analogue. In general it is impossible to embed the spin
connection in a non-abelian gauge field. Only for $F = K = 1$ the holonomy group is
compact (see [2]). We have to postpone this question until the S-duality transformation
of non-abelian gauge fields is better understood.

Despite this shortcoming it is remarkable that the S-duality relates solutions which
are exact in the $\alpha'$ expansion (after this embedding). The point is that this relation
between two exact backgrounds is independent of the “starting dimension”. As long
as we start with a 10D theory and reduce it down to 4 dimensions we get a theory with
$N = 4$ and one expects that there are no quantum corrections and the S-duality maps
in fact exact solutions to exact solutions (see e.g. [35]). But this relation is also valid
for a 6D theory reduced to 4 dimensions yielding $N = 2$. There, one would expect
corrections and the S-duality mixes the orders, i.e. in general a solution in the lowest
order is not expected to transform again in a solution of the lowest order. In other
words, one would expect that corrections of the higher genus or even non-perturbative
terms contribute in the transformed theory to the lowest order (i.e. lowest genus). But
for both types of solution one cannot see such corrections. Even more, above we have
seen that our electric solution is near the singularities asymptotically free. This is a
hint that higher genus contribution and non-perturbative corrections are there under
control. On the other side the magnetic solutions are there in the strong coupling
region, i.e. one could expect a strong influence of higher genus and non-perturbative
contributions. But no such corrections can be seen. Although one knows that the
string fivebrane is in the strong coupling region effectively a SU(2) WZW model (R × S^3 throat limit) and therefore an exact conformal field theory, this is not the case for the H-monopoles. To get this monopole solution one assumes that the fields of the fivebrane solution depend only on three coordinates and thus one destroys the S_3 symmetry. This procedure here suggests that also the H-monopole remains exact in the strong coupling region (as long as S-duality is real a symmetry). Furthermore, if we go the other direction, since the H-monopole solution is an exact model away from the singularity we can speculate that for our original model (1), even away from the weak coupling region, further corrections (no higher genus nor non-perturbative) are negligible.

**Identification of the string states.** Finally, let us try to identify a subset of our solutions in the elementary string spectrum. As usual let us consider the mass formula for string excitations in the Neveu-Schwarz (NS) sector which is given by

\[ M^2 = \frac{1}{2} \left( |\vec{Q}_e^{(+)}|^2 + 2(N_R - \frac{1}{2}) \right) = \frac{1}{2} \left( |\vec{Q}_e^{(-)}|^2 + 2(N_L - 1) \right) \]

where \(\vec{Q}_e^{(\pm)}\) is the electric charge vector (internal momentum contributions) which form an even selfdual lattice and \(N_{R/L}\) are the oscillator contributions. We know that the right moving sector of our solution saturates the Bogomol’nyi bound (51) which means

\[ N_R = \frac{1}{2} . \]

This is the lowest possible value required by GSO projection. Then using (18) we find for the left moving sector

\[ N_L - 1 = \frac{1}{2} \left( |\vec{Q}_e^{(+)}|^2 - |\vec{Q}_e^{(-)}|^2 \right) = \bar{m} \bar{m} - |q|^2 . \]

Of course, because the charges enter the formulare as internal momenta this equation describes first of all electrically charged solutions, i.e. in our case electric black holes. Thus they are interpreted as elementary string states. On the other side the magnetic solutions appearing as the S-dual of the electric solutions are then identified as soliton excitations of the theory. Let us now discuss special cases of our solution.

1. **T-self-dual case.** Here we have \(m = -\bar{m}\) and the black hole solutions (electric or magnetic) become massless. Inserting this into the mass formula yields \(N_L - 1 = -(\bar{m}^2 + |q|^2)\). Therefore, our massless black holes can be identified as string excitation with \(N_L = 0\). In principle the same is true for the S-self-dual case. But there the charges are not well defined (the Taub-NUT geometry is asymptotically not flat). Note, that
for this case it is crucial to have at least two gauge fields. This is maybe the reason why Duff and Rahmfeld [41] were unable to identify a state with $N_L = 0$. Sen [35] could identify such a state as BPS monopoles, i.e. a monopole solution where the 10D theory contains a non-abelian gauge field.

2. H-monopoles, Kaluza-Klein case. Our generalization of the H-monopoles are defined by $m\tilde{m} = |q|^2$ and therefore we get $N_L = 1$. This coincides with the results for the standard H-monopole with only one gauge field ($|q| = 0$) [41], [35]. As we have already seen, this solution is nothing else as the extreme electric [36] or extreme magnetic [73] KK black hole.

3. Extreme dilaton black holes, rotating black holes, ... All other cases have $N_L > 1$. Special examples with only one gauge field and flat internal space ($|q| = 0, m = \tilde{m}$) are, e.g., the rotating dilaton black hole solution (with a naked singularity) or the extreme dilaton black holes.

6 Conclusion

In this paper we have started with a general model which allows a null Killing vector and which has unbroken spacetime supersymmetries. This model also known as chiral null model, is the generalization of the gravitational wave and fundamental string background. As discussed in section two this model possesses a chiral symmetry on the world sheet and is exact in the $\alpha'$ expansion. In addition, there are points of explicit T-self-duality. Regarding this model as the bosonic part of a $D = 10, N = 1$ superstring background we have reduced this model to 4 dimensions. As result we got a stationary IWP solution [73] which couples to 7 gauge fields (one gravi-photon and 6 matter gauge fields) and 4 scalars (dilaton, axion and two moduli). This solution is completely determined by ten harmonic functions which can have different kind of singularities yielding different geometries. Assuming that we have a point like singularity we get the Taub-NUT geometry [41] whereas a ring singularity yields an electric Kerr black hole [46]. In addition to these both cases the harmonic function can be non-singular or can have singularities in two points. Whether these cases have a reasonable space time interpretation remains unclear. As expected, in the T-self-dual point the black hole becomes massless and the charge correlated to the gravi-photon vanishes. Furthermore, we showed that the gravi-photon state saturate a Bogomol’nyi bound. Unfortunately, this black hole solution has a naked singularity as long as the angular momentum does not vanish. On the other side, there are arguments to interpret these states with the

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*See [34], as long as $FK \neq 1$ we have always at least two gauge fields.*
naked ring singularity as elementary closed strings.

In the last section we have investigated the S-duality for this solution. After some general remarks we have discussed the $SL(2,\mathbb{Z})$ transformation $\lambda \to -\lambda^{-1}$. There are two special cases. One is the S-self-dual limit where the dilaton and the axion remains unchanged the S-duality transformed not the weak into the strong coupling region. This case is only possible for the Taub-NUT geometry. Secondly, we have shown that for vanishing axion the magnetic solution is an extreme magnetic black hole (70) or an H-monopole (73). Both solutions couple to more gauge fields and in the limit of only one gauge field we obtained the standard expressions. It turned out that an H-monopole is nothing else as an extreme magnetic black hole with a balance between the magnetic charges coming from the matter and from the graviton sector. Interestingly, both solutions are not exact in the $\alpha'$ expansion. There are anomaly related $\alpha'$ corrections, but one can promote these solutions to exact ones by embedding the torsionfull 10D spin connections into a non-abelian gauge group. Such an embedding can only be understood in the framework of heterotic string model with a non-abelian gauge field in 10 dimensions. On the other hand the electric solutions are exact already without this embedding and can be regarded as a pure bosonic string model. Maybe a better understanding of the S-duality for non-abelian gauge fields can yield a better understanding of this phenomena.

In the last subsection we have tried to identify our solutions as elementary string or soliton excitations of the theory. We found, that the T-self-dual solution corresponds to states with $N_R = \frac{1}{2}$, $N_L = 0$, the H-monopole class to $N_R = \frac{1}{2}$, $N_L = 1$ and all other solutions including the extreme dilaton and rotating black holes to $N_R = \frac{1}{2}$, $N_L > 0$.

In our consideration it remains an open problem why the massless black holes or string excitations are at rest and does not move with the speed of light.

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