CMB anisotropies from primordial inhomogeneous magnetic fields

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Primordial inhomogeneous magnetic fields of the right strength can leave a signature on the CMB temperature anisotropy and polarization. Potentially observable contributions to polarization B-modes are generated by vorticity and gravitational waves sourced by the magnetic anisotropic stress. We compute the corresponding CMB transfer functions in detail including the effect of neutrinos. The shear rapidly causes the neutrino anisotropic stress to cancel the stress from the magnetic field, suppressing the production of gravitational waves and vorticity on super-horizon scales after neutrino decoupling. A significant large scale signal from tensor modes can only be produced before neutrino decoupling, and the actual amplitude is somewhat uncertain. Plausible values suggest primordial nearly scale invariant fields with $B_\lambda \sim 10^{-10} \text{G}$ today may be observable from their large scale tensor anisotropy. They can be distinguished from primordial gravitational waves by their non-Gaussianity. Vector mode vorticity sources B-mode power on much smaller scales with a power spectrum somewhat similar to that expected from weak lensing, suggesting amplitudes $B_\lambda \sim 10^{-9} \text{G}$ may be observable on small scales for a spectral index $n \sim -2.9$. In the appendix we review the covariant equations for computing the vector and tensor CMB power spectra that we implement numerically.

I. INTRODUCTION

Magnetic fields are ubiquitous in the universe, with $\sim 10^{-6} \text{G}$ coherent fields observed on galactic and cluster scales. However their origin is not well understood (for a review see Ref. [1]). Tiny seed fields $\lesssim 10^{-20} \text{G}$ may have been amplified by a dynamo mechanism to give the much larger fields we now see, though to what extent this process can work in practice is not yet clear [2]. Alternatively initial fields of strength $\sim 10^{-9} \text{G}$ can give rise to galactic fields of the observed values without a functioning dynamo mechanism. Such fields have potentially interesting observational signatures on the CMB, and if present would provide powerful constraints on models of the early universe. The absence of such signatures may also serve as a consistency check on models of galaxy evolution that would be observationally incompatible with initial fields this large.

A primordial field of $\sim 10^{-9} \text{G}$ can leave a signature in the B-mode (curl-like) CMB polarization. Since the scalar (density) perturbations do not produce B-modes at linear order, the B-modes are a much cleaner signal of additional physics than very small fractional changes to the temperature or E-mode polarization. However B-modes are produced at second order through weak lensing [3, 4], and are also generated by primordial gravitational waves (tensor modes) [5]. Other possible sources include topological defects [6, 7]. The focus of many future CMB observations will be on observing the B-modes, so it is useful to assess in detail the various possible components and how they can be distinguished from each other.

Primordial fields with a blue spectrum compatible with nucleosynthesis are far too weak on cosmological scales to leave an interesting signature [8]. In this paper we consider in detail the CMB signal expected from $\sim 10^{-9} \text{G}$ primordial fields with a nearly scale invariant spectrum. Such fields are not well motivated by current theoretical models, which mostly give much smaller amplitudes or a much bluer spectrum [1, 9, 10, 11]. Observation of primordial fields at this level would therefore be a powerful way to rule out many models. However Ref. [12] present one model in which observably interesting CMB signatures may be produced. Previous semi-analytical work has investigated the CMB signal from both tensor [13, 14] and vector modes [15, 16] sourced by magnetic fields. Here we give a more detailed numerical analysis of the full linearized equations. We include the effect of neutrinos as they change the way that magnetic fields source gravitational waves and vorticity on super-horizon scales. Our final CMB power spectra include the contribution to the B-mode signal from both the tensor and vector modes. We do not consider helical modes [14, 17] which can be detected via their parity-violating correlations, nor the effects of Faraday rotation [18, 19] (which can be identified by the frequency dependence). We also assume that reionization is relatively sharp and unmodified by energy injection into the IGM from decay of the small scale magnetic field [20]. For a discussion of constraints on homogeneous magnetic fields see e.g. Ref. [21] and references therein.

II. COVARIANT EQUATIONS

We consider linear perturbations in a flat FRW universe evolving according to general relativity with a cosmological constant and cold dark matter, and approximate the neutrinos as massless. Perturbations can be described covariantly in terms of a 3+1 decomposition with respect to some choice of observer velocity $u_a$ (we use natural units, and the signature where $u_a u^a = 1$),

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The stress-energy tensor can be decomposed with respect to \( u_a \) as
\[
T_{ab} = \rho u_a u_b - p h_{ab} + 2 u_a q_b + \pi_{ab} \tag{1}
\]
where \( \rho \) is the energy density, \( p \) is the pressure, \( q_a \) is the heat flux and \( \pi_{ab} \equiv T_{(ab)} \) is the anisotropic stress. Angle brackets around indices denote the projected (orthogonal to \( u_a \)) symmetric trace-tree part (PSTF). The tensor
\[
h_{ij} \equiv g_{ij} - u_i u_j,
\]
where \( g_{ij} \) is the metric tensor, projects into the instantaneous rest space orthogonal to \( u_a \). It defines a spatial derivative \( D_a \equiv h_{ab} \nabla_b \) orthogonal to \( u_a \) where \( \nabla_a \) is the covariant derivative. Spatial derivatives can be used to quantify perturbations to background quantities, for example the pressure perturbation can be described covariantly in terms of \( D_a p \).

Conservation of total stress-energy \( \nabla^a T_{ab} = 0 \) implies an evolution equation for the total heat flux \( q_a \)
\[
\dot{q}_a + \frac{4}{3} \Theta q_a + (\rho + p) A_a - D_a p + D^b \pi_{ab} = 0 \tag{3}
\]
where \( \dot{q}_a \equiv u^b \nabla_b q_a \), \( \Theta \equiv \nabla^a u_a \) is three times the Hubble expansion, and \( A_a \equiv u_b \nabla^b u_a \) is the acceleration. The evolution equation for the heat flux \( q_a^i \equiv (\rho^i + p^i) v_a^i \) of each matter component present is of the form
\[
\dot{q}_a^i + \frac{4}{3} \Theta q_a^i + (\rho^i + p^i) A_a - D_a p^i + D^b \pi_{ab}^i = L_a^i \tag{4}
\]
where \( L_a^i \) is an interaction force term. Conservation of total stress energy \( q_a = \sum q_a^i \) implies that \( \sum L_a^i = 0 \).

For magnetic fields the components of the stress-energy tensor are given by
\[
\rho^B = 3p^B = -\frac{1}{2}(E^2 + B^2) \tag{5}
\]
\[
\pi_{ab}^B = -E_a (E_b) - B_a (B_b) \tag{6}
\]
\[
q_{ab}^B = -(E \times B)_a \tag{7}
\]
where \( E_a \) and \( B_a \) are the electric and magnetic field projected vectors. We take \( B^2 \) and \( E^2 \) to be first order, and to this order \( E_a \) and \( B_a \) are frame invariant. For most of its evolution the universe is a good conductor so we may take \( E_a = 0 \) in all linear frames: the magnetic fields are frozen in, and in this approximation almost all the complications of MHD disappear. The linearized Bianchi identity for the electromagnetic field tensor implies
\[
\dot{B}_a + \frac{2}{3} \Theta B_a = 0 \tag{8}
\]
so the magnetic field simply redshifts as \( 1/S^2 \) where \( S \) is the scale factor, and hence \( \pi^B_{ab} \propto \rho^B \propto 1/S^4 \). More general equations can be found in e.g. Ref. \[24\].

The Poynting vector heat flux is zero \( (q_a^B = 0) \) in all linear frames because we have set \( E_a = 0 \). Since \( A_a \) is first order, on linearizing we have the Lorentz force \( L_a^B \) given by the evolution equation \[4\]
\[
D^a \pi_{ab}^B - D_a p^B = L_a^B. \tag{9}
\]
This is consistent with the usual \( \text{curl } B \times B \) expression. The opposite force acts on the baryons to ensure total momentum conservation, which with the Thomson scattering terms \[24\] gives the baryon velocity evolution equation:
\[
\dot{v}_a + \frac{1}{3} \Theta v_a + A_a - \frac{D_a p^b}{\rho^b} = -\frac{\rho^b}{\rho^0} \left[ n_e \sigma_T \left( \frac{4}{3} v_a - I_a \right) + \frac{D^b \pi_{ab}^B - D_a p^B}{\rho^b} \right] \tag{10}
\]
where \( n_e \) is the electron number density, \( \sigma_T \) is the Thomson scattering cross-section, \( \rho^b \) is the photon energy density, and we neglect baryon pressure terms of the form \( \rho^b / \rho^0 \ll 1 \). Thus magnetic fields source baryon vorticity, as well as providing extra density and pressure perturbations, and anisotropic stresses.

We define the vorticity vector \( \Omega_a \equiv \text{curl } u_a \) where for a general tensor
\[
\text{curl } X_{a_1 \ldots a_l} = \eta_{b c d (a_1} u^b D^c X^d_{a_2 \ldots a_l)} \tag{11}
\]
and round brackets denote symmetrization. It is transverse \( D^a \Omega_a = 0 \). Remaining quantities we shall need are the ‘electric’ \( E_{ab} \) and ‘magnetic’ \( H_{ab} \) parts of the Weyl tensor \( C_{abcd} \)
\[
E_{ab} \equiv C_{abcd} u^c u^d \quad H_{ab} \equiv \frac{1}{2} \eta_{abcd} C_{e c d} u^e u^f \tag{12}
\]
(which are frame invariant) and the shear \( \sigma_{ab} \equiv D_{(a} u_{b)} \). The Einstein equation and the Bianchi identity give the constraint equations
\[
D^a \sigma_{ab} - \frac{1}{2} \text{curl } \Omega_b - \frac{3}{2} D_b \Theta - \kappa q_b = 0 \tag{13a}
\]
\[
D^a E_{ab} - \kappa \left( \frac{\Theta}{3} q_b + \frac{1}{3} D_b \rho + \frac{1}{2} D^a \pi_{ab} \right) = 0 \tag{13b}
\]
\[
D^a H_{ab} - \frac{1}{2} \kappa [(\rho + p) \Omega_b + \text{curl } q_b] = 0 \tag{13c}
\]
\[
H_{ab} - \text{curl } \sigma_{ab} - \frac{1}{2} D_{(a} \Omega_{b)} = 0, \tag{13d}
\]

Footnote:
\footnote{Note that unlike many other authors we use natural rather than Gaussian units. Due to the signature choice \(-E^2 \geq 0\).}
and the evolution equations

\[
\dot{\Omega}_a + \frac{2}{3} \Theta \Omega_a = \text{curl} A_a,
\]

\[
\dot{\sigma}_{ab} + \frac{2}{3} \Theta \sigma_{ab} = -E_{ab} - \frac{\kappa}{2} \pi_{ab},
\]

\[
E_{ab} + \Theta E_{ab} = \text{curl} H_{ab} + \frac{\kappa}{2} \left[ \pi_{ab} - (\rho + p) \sigma_{ab} + \frac{\Theta}{3} \pi_{ab} \right],
\]

\[
\dot{H}_{ab} + \Theta H_{ab} = -\text{curl} E_{ab} - \frac{\kappa}{2} \text{curl} \pi_{ab}. \tag{14}
\]

Here \(\kappa \equiv 8\pi G\).

The distribution functions for the various species can be expanded into multipole moments. For example the photon multipole tensors \(I_{A_l} \equiv I_{(a_1 \ldots a_l)}\) are defined as moments of the distribution of the photon intensity \(I(e)\) per solid angle as:

\[
I_{A_l} \equiv \int d\Omega_e \, I(e) \, e_{(A_l)}, \tag{15}
\]

where the direction vector \(e_a\) is normalized to \(e^a e_a = -1\) and \(e_{(A_l)} = e_{(a_1 \ldots a_l)}\) are irreducible PSTF tensors. The \(e_{(A_l)}\) are orthogonal:

\[
\frac{1}{4\pi} \int d\Omega_e \, e_{(A_l)} e^{(B_m)} = \delta_{m}^{(l)} \frac{(-2)^l(l!)^2}{(2l+1)!} h_{(a_1 \ldots a_l)}^{b_1 \ldots b_l} \tag{16}
\]

The \(I_{A_l}\) multipole tensors have \(2l + 1\) degrees of freedom, \(I_{ab} = \pi_{ab}\) is the anisotropic stress, \(I_a = \rho_\gamma^p\) is the heat flux and \(I = \rho_\gamma\). The temperature anisotropy can then be expanded as:

\[
\frac{\Delta T(e)}{T} = \sum_l \frac{(2l+1)!}{4(-2)^l(l!)^2} \frac{I_{A_l} e_{(A_l)}}{\rho_\gamma} = \sum_l \sum_{m=-l}^l a_{lm} Y_{lm}(e), \tag{17}
\]

where the latter expansion in terms of spherical harmonics \(Y_{lm}\) is the non-covariant version of the expansion in \(I_{A_l}\). The CMB power spectrum is defined in terms of the variance of the spherical harmonic components \(a_{lm}\) by:

\[
C_l \equiv \langle |a_{lm}|^2 \rangle = \frac{\pi}{4} \frac{(2l)!}{(-2)^l(l!)^2} (I_{A_l} I_{A_l}^*) / \rho_\gamma^2. \tag{18}
\]

Analogous results for the polarization are given in Ref. \[27\], where \(E_{A_l}\) is a gradient-like multipole of the polarization tensor and \(B_{A_l}\) is a curl-like multipole.

The covariant equations can be expanded in terms of scalar, vector and tensor harmonics. The details and definitions are given in the appendix. In the following sections we analyse in detail the tensor and vector equations, where each quantity is a component of a harmonic expansion, and \(k\) labels are suppressed. Scalar modes can source temperature and E-polarization CMB signals, however since we are mostly interested in the B-polarization signal, which is not sourced by scalar modes, we do not discuss scalar modes here. A partial analysis of scalar modes is given in Ref. \[28\].

### III. Tensors

Expanding in \(m = 2\) tensor harmonics \[A12\] (and suppressing \(m\) indices), the constraint equations \[13\] imply that the Weyl tensor variable \(H\) is related to the shear by \(H = \sigma\). The evolution equations \[14\] then give

\[
k^2(E' + HE) - k^3\sigma + \frac{\kappa}{2} S^2(\rho + p) k\sigma = \frac{\kappa}{2} S^2 (\Pi' + H\Pi) \tag{19}
\]

where primes denote derivatives with respect to conformal time \(\eta\) and \(H \equiv 3\Theta/3\) is the conformal Hubble parameter. The Weyl tensor variable \(E\) and the shear \(\sigma\) define the new variable

\[
H_T \equiv -2E - \frac{\sigma'}{k} \tag{20}
\]

to correspond to the metric perturbation variable of non-covariant approaches. It satisfies \(H_T'' = -k\sigma\), and the above equations combine to give the well known evolution equation

\[
H_T'' + 2H_H_T' + k^2 H_T = k S^2 \Pi. \tag{21}
\]

Magnetic fields provide a component of the anisotropic stress \(\Pi\) and hence source gravitational waves, and we quantify the magnetic field source by the dimensionless ratio \(B_0 \equiv \Pi_B / \rho_\gamma\). The covariant tensor equations are discussed in more detail in Ref. \[20\].

Equations for the evolution of the tensor multipoles are obtained from the appendix \((m = 2\) in Eqs. \[A13\] and \[A14\]). We use a series expansion in conformal time \(\eta\) to identify the regular primordial modes in the early radiation dominated era. Defining \(\omega \equiv \Omega_m H_0 / \sqrt{\Omega_R}\), where \(\Omega_R = \Omega_\gamma + \Omega_\nu\), and \(H_0\) and \(\Omega_\nu\) are the Hubble parameter and densities (in units of the critical density) today, the Friedmann equation gives

\[
S = \frac{\Omega_m H_0^2}{\omega^2} \left( \omega \eta + \frac{1}{4} \omega^2 \eta^2 + \mathcal{O}(\eta^5) \right). \tag{22}
\]

Defining the ratios \(R_\nu \equiv \Omega_\nu / \Omega_R\), \(R_\gamma \equiv \Omega_\gamma / \Omega_R\) and keeping lowest order terms the regular solution (with zero initial anisotropies for \(l > 2\)) is

\[
H_T = H_T^{(0)} \left( 1 - \frac{5}{24} \frac{(k^2)^2}{4 R_\nu + 15} + \frac{15}{28} \frac{R_\gamma B_0 (k^2)^2}{4 R_\nu + 15} \right),
\]

\[
\sigma = \frac{5 H_T^{(0)} k \eta}{4 R_\nu + 15} - \frac{15 R_\gamma B_0 k^2}{14 (4 R_\nu + 15)} \tag{23}
\]

\[
\pi_\nu = -\frac{R_\gamma B_0}{R_\nu} \left( 1 - \frac{15}{14} \frac{(k^2)^2}{4 R_\nu + 15} \right) + \frac{4}{3} \frac{(k^2)^2 H_T^{(0)}}{4 R_\nu + 15}
\]

where \(H_T^{(0)}\) is the initial value (after neutrino decoupling) and \(\pi_\nu \equiv \Pi_\nu / \rho_\nu\). The \(B_0 \neq 0\) mode (with \(H_T^{(0)} = 0\) has compensating anisotropic stresses: the sum of the
magnetic and neutrino terms gives the total source term

\[ \kappa S^2 \Pi = \frac{45}{14} R_\gamma k^2 B_0 + 15 \left( 1 - \frac{(45 - 2R_\nu)\omega \eta}{2R_\nu + 15} \right) + \mathcal{O}(\eta^2) \]

rather than the \( S^2 \rho_\gamma \propto 1/\eta^2 \) result expected without collisionless radiation. For \( k \ll H \) there is therefore no sourcing of gravitational waves during radiation domination, so \( H_T \propto (k\eta)^2 \) if it was zero initially. Collisionless fluids suppress generation of gravitational waves on super-horizon scales.

Before neutrino decoupling there is no neutrino anisotropic stress so the magnetic field source is not compensated. Taking \( \eta_{\text{in}} \) as the magnetic field production time (at which we take \( H_T = \sigma = 0 \)), the solution for \( k\eta \ll 1 \) is approximately\(^3\)

\[ H_T \approx 3R_\gamma B_0 \left( \ln(\eta/\eta_{\text{in}}) + \frac{\eta_{\text{in}}}{\eta} - 1 \right) \]

\[ \sigma \approx -\frac{3R_\gamma B_0}{k\eta} \left( 1 - \frac{\eta_{\text{in}}}{\eta} \right). \]

After neutrino decoupling this mode must convert into a combination of the above regular modes, which can then be used to compute the observable signature. As the neutrino coupling is switched off the neutrino anisotropic stress becomes important, and with no scattering evolves as

\[ \pi_\nu = -\frac{k}{3} J_3 + \frac{8}{15} k\sigma. \]

Since the octopole \( J_3 \) and \( \pi_\nu \) will be zero before neutrino decoupling, for modes well outside the horizon we have \( \pi_\nu \sim -B_0/\eta \) just after decoupling (we assume \( \eta \gg \eta_{\text{in}} \)), and hence the neutrino anisotropic stress grows logarithmically \( \pi_\nu \sim -B_0 \ln(\eta/\eta_{\text{in}}^*) \). It therefore reaches the constant value \( \pi_\nu \sim -B_0 \) of the regular solution in about one e-folding. At this point \( H_T \) ceases to grow logarithmically because the magnetic anisotropic stress source is now cancelled by the neutrinos, and \( H_T \) gives the amplitude of the usual regular solution \( H_T^{(0)} \).

Thus after neutrino decoupling we expect a combination of the usual passive primordial tensor mode and the regular compensated sourced mode. As we show explicitly below, the compensated mode can be neglected compared to the small scale vector mode contribution. The passive tensor mode has

\[ H_T^{(0)} \approx 3R_\gamma B_0 \ln(\eta_{\nu}^*/\eta_{\text{in}}) \]

where \( \eta_{\nu}^* \) is the time of neutrino decoupling (assuming magnetic field generation is during radiation domination at \( \eta \sim \eta_{\text{in}} \) and before neutrino decoupling). Since the transverse traceless part of the metric tensor

\[ P_h = \sum_{k \pm} 2H_T Q_k \]

this corresponds to a primordial power spectrum for \( h \) of

\[ P_h \approx |6R_\gamma \ln(\eta_{\nu}^*/\eta_{\text{in}})|^2 P_B. \]

Power spectra are defined in Eq. \( A26 \) and \( R_\gamma \sim 0.6 \).

Thus the tensor covariance part of the magnetic field signal is very similar to that expected from primordial gravitational waves, and can be computed trivially by using the above tensor power spectrum in the numerical codes CAMB \( 29 \) or CMBFAST \( 30 \). However unlike primordial gravitational waves the spectral index of \( P_h \) here is expected to be at least slightly blue, and the signal should be non-Gaussian because \( B_0 \) is quadratic in the magnetic field. Allowing for subtracting the B-mode lensing signal, levels of \( P_h \sim 10^{-15} \) may be observable\(^4\) \( 31 \). This corresponds to \( B_3 \sim 10^{-10} G \) for \( \eta_{\text{in}}/\eta_{\nu}^* \sim 10^{-6} \) and a close to scale invariant spectrum. To distinguish this from primordial gravitational waves from inflation one would need to detect the non-Gaussianity or small scale power from the vector modes.

The compensation mechanism in principle works with any collisionless relativistic fluid, even when it only makes up a fraction \( R_\nu \rightarrow 0 \) of the energy density \( (R_\nu \) can be interpreted as any collisionless component). However if there is collisionless relativistic matter only from a time \( \eta_{\nu} \) after magnetic field production, partial compensation will be effective at a time \( \eta \sim \eta_{\nu} e^{R_\nu/Hc} \). For small fractions \( R_\nu \) this is a very large time, so the mechanism is inefficient for components that are only a small fraction of the density allowed by the nucleosynthesis bound. Neutrinos themselves can only suppress gravitational wave production after neutrino decoupling at \( z \sim 10^3 \).

IV. VECTORS

Unlike tensors modes, vectors are not in general frame invariant. It is therefore convenient to choose the frame \( u_a \) to simplify the analysis. At linear order one can always write \( u_a = u^\perp_a + v_a \), where \( u^\perp_a \) is hypersurface orthogonal and \( v_a \) is first order, so \( \text{curl } u_a = \text{curl } v_a \). For a zero order scalar quantity \( X \) it follows that

\[ D_a X = D_a^\perp X - v_a \dot{X}. \]

For vector modes \( (D_a^\perp X)^{(1)} = 0 \), and it is convenient to choose the frame \( u_a \) to be hypersurface orthogonal so that \( \text{curl } u_a = 0 \) and hence \( (D_a X)^{(1)} = 0 \), where the bar denotes evaluation in the zero vorticity frame. From its

\[^4\] Observalbe in the sense that in the null hypothesis that there are no non-lensing B-modes, the residual noise level would be consistent with this level. Actual subtraction of CMB lensing in the presence of B-modes from magnetic field sourced vector modes may be extremely difficult, but one should still be able to detect violations of the null hypothesis that there are none.
As in the tensor case the anisotropic stresses compensate on super-horizon scales, so they source negligible shear \( \bar{\sigma} \) on these scales. However, unlike the tensor case, any non-zero \( \bar{\sigma} \) present on super-horizon scales at neutrino decoupling decays away rapidly and has no observable effect, so the evolution prior to neutrino decoupling is irrelevant. The observable signature comes from the vorticity sourced on sub-horizon scales by the magnetic anisotropic stress on its own.

In the approximation that recombination is sharp at \( \eta = \eta_* \), the photon multipoles are given approximately from the integral solution (32) by

\[
\frac{I_l(\eta_0)}{4} \approx \left[ (\bar{v} + \bar{\sigma}) \Psi_l + \frac{\zeta}{4} \frac{d\Psi_l}{d\chi} \right]_{\eta_*} + \int_{\eta_*}^{\eta_0} d\eta \bar{v} \bar{\sigma} \Psi_l, \tag{39}
\]

where \( \Psi_l \equiv l j_l(\chi)/\chi, j_l(\chi) \) is a spherical Bessel function and \( \chi \equiv k(\eta_0 - \eta) \). Thus the Newtonian gauge vorticity \( \bar{\sigma} \) has last scattering and integrated Sachs-Wolfe (ISW) contributions to the temperature anisotropy. In the absence of neutrino anisotropic stress \( \bar{\sigma} \sim -3 R B_0 (k \eta) \) during radiation domination, as in the tensor case. On super-horizon scales this is large, and would give a large scale contribution from \( \bar{\sigma} \) orders of magnitude larger than the small super-horizon Doppler contribution from \( \bar{v} \). Previous work [14, 34] has neglected the contributions from \( \bar{\sigma} \) around last scattering, an approximation that is good on small scales. However on super-horizon scales when \( \bar{\sigma} \) is large, the diverging \( \bar{\sigma} \) contributions are far from negligible. Previous power spectra are approximately the right shape on large scales, but for the wrong reason: the large scale anisotropies are small because of neutrino compensation suppressing the source term for \( \bar{\sigma} \), not because they are insensitive to \( \bar{\sigma} \). Similar comments apply to the polarization power spectra.

There is one caveat to the above. The decay of \( \bar{\sigma} \) from neutrino decoupling to last scattering amounts to a decay factor of about \((z_*^+ / z_*)^2 \sim 10^{12}\). However on arbitrarily large scales the \( 1/(k n \bar{\sigma}) \) evolution of \( \bar{\sigma} \) before neutrino decoupling can be larger than this, so on the very largest scales there can still be a contribution to \( \bar{\sigma} \) at last scattering. In the asymptotic limit the dipole \( I_1 \) appears singular, though the quadrupole \( I_2 \) is finite. The constraint \( n > -3 \) ensures that the total power from large scale modes is not singular if the individual modes are not. Here we neglect this effect, effectively assuming the power spectrum cuts off on sufficiently large scales that are otherwise unobservable. It is unclear whether there is a serious infrared problem with a nearby scale invariant spectrum or not. We simply compute the CMB transfer function from a given anisotropy stress \( B_0 \), and defer the issue of what the spectrum actually is, how it could be generated, and its actual asymptotic behaviour.

V. NUMERICAL RESULTS

In this paper the focus is on calculation of accurate CMB transfer functions from a given initial distribution.
As a convenient ansatz for computing sample $C_l$ power spectra we assume a Gaussian primordial magnetic field distribution, with power spectrum $P_B \propto k^{n-3}$ (the definition of $n$ is conventional).

One can define a smoothed magnetic field $B_\lambda$ using a Gaussian smoothing of width $\lambda$ (we choose $\lambda = 1$ Mpc) and express the power spectrum in terms of $B_\lambda$ as in Refs. [13, 14]. In harmonic space the anisotropic stress is given by a convolution of the underlying magnetic fields, so the power spectrum for the anisotropic stress at a given $k$ feeds the power from across the $P_B$ spectrum. For vectors and tensors the resultant power spectrum is given approximately by [14]

$$P_{B_0} \approx \frac{4}{2n+3} \left[ \frac{(2\pi)^n B_\lambda^2}{2\Gamma(\frac{n+3}{2})\rho_G} \right]^2 \times \left\{ \frac{k_D}{k_\lambda} \right\}^{2n+3} \left( \frac{k}{k_\lambda} \right)^3 + \frac{n}{n+3} \left( \frac{k}{k_\lambda} \right)^{2n+6}$$  \hspace{1cm} (40)

for $-3 < n$. The scale $k_D$ comes from a small scale damping cut-off [13, 14], and does not affect the power spectrum significantly for nearly scale invariant power spectra with $n \sim -3$. The spectrum is singular at $n = -3$, which comes from the singular build up of super-horizon power for a scale invariant $B$ spectrum with no large scale cut-off. Since $B_0 \equiv \Pi/\rho_G$ is quadratic in $B$, the spectrum of $B_0$ will be non-Gaussian, so the power spectrum only contains a subset of the available information, though it is useful to assess the detectability amplitude.

As a sample example we take $B_\lambda = 3 \times 10^{-9}$ G and $n = -2.9$ (as in Ref. [13]), which implies

$$P_{B_0} \approx 1.16 \times 10^{-13} \left( \frac{k}{k_\lambda} \right)^{0.2}.$$  \hspace{1cm} (41)

Since data will only constrain $P_{B_0}$ directly, we take this equation to be exact for our numerical results so they may easily be related to different power spectra for $P_{B_0}$ (which may or may not come from the assumed power law spectrum of $B_\lambda$ fluctuations). The power spectrum $P_{B_0}$ scales as $B_\lambda^4$ as do the CMB power spectra, so large changes in overall amplitude can be obtained from relatively small changes in the primordial field: the value of $B_\lambda$ has to be quite finely tuned to give a CMB signature that is neither totally dominant nor totally negligible.

Numerical results from vector modes with $B_\lambda = 3 \times 10^{-9}$ G are shown in Fig. 1 in comparison with the spectra expected from primordial curvature perturbations and possible primordial gravitational waves. For this $B_\lambda$ the effect on the temperature power spectrum is negligible: only if $B_\lambda \gtrsim 8 \times 10^{-9}$ G could there be a significant contribution to the power at $l \gtrsim 2000$, perhaps contributing some of the power observed on these scales [32].

The contributions to the most easily distinguished B-modes are shown in Fig. 2 including the tensor contribution. It is clear that the compensated tensor mode has a
negligible observational signature and can be neglected. The exact amplitude of the large scale tensor signal from gravitational waves sourced before neutrino decoupling is uncertain because we do not know the time (or mechanism) of field generation, nor have we modelled neutrino decoupling in detail.

Our vector mode results are in broad agreement with the semi-analytical results of [15]. The main qualitative difference is that our $T E$ cross-correlation changes sign in the damping region. The quantitative results differ somewhat across the spectrum due to our more detailed full analysis of the damping, recombination, inclusion of neutrinos and modelling of reionization (we have also used a slightly different primordial power spectrum). The results in Ref. [11] for the large scale vector and tensor spectra are too large by a factor$^5$ of $(2\pi)^3$ (giving constraints on $B_\lambda$ too small by a factor of $(2\pi)^{3/4} \sim 4$). There was another normalization error in [37] but corrected in [15]. Refs. [12, 14] provide analytical solutions valid for tensor modes that are super-horizon at decoupling, which give $C_l$ spectra qualitatively valid for $l \lesssim 100$. However this approximation was also used for $l < 500$ in Ref. [14], and so their result is qualitatively incorrect at $l \gtrsim 100$. Their tensor polarization results also suffer qualitative problems because the peak in the visibility was neglected. All previous analyses of sourced tensor modes have neglected neutrino compensation, giving results somewhat larger (but the result is still, in any case, somewhat uncertain).

VI. CONCLUSIONS

Nearly scale invariant primordial magnetic fields can give a potentially observable CMB signal if they happen to be $\gtrsim 10^{-10}$G. The observational B-mode signature comes from tensor modes, giving a non-Gaussian spectrum otherwise essentially identical to that expected from primordial gravitational waves, and vector modes giving power on small scales.

Any possible future detection of primordial gravitational waves from large scale B-modes should be carefully distinguished from that produced by magnetic fields. Any primordial signal is expected to be Gaussian, so Gaussianity tests can be used to distinguish them (methods for robustly isolating the B-mode component on sections of the sky are given in Ref. [37], and can be used to construct a set of cut-sky modes that should be Gaussian if they are due to inflation). Low frequency observations may also be able to detect Faraday rotation [19], which would be a clear signal of magnetic fields. Small scale B-mode observations from magnetic fields will need to be carefully distinguished from the weak lensing signal. Regular primordial vector modes, though theoretically unmotivated, can also in principle give a significant small and large scale B-mode signal [33]. They may be distinguished by their sharper fall in power on very small scales due to the lack of sources. Topological defects [6] can be identified by the lack of non-Gaussian tensor mode power on large scales. Note that throughout we have been assuming idealized observations; in practice foregrounds and systematics may well pose very serious problems (see e.g. Ref. [38]).

Our analysis is significantly more detailed than previous work, in that we have numerically solved the full set of linearized equations. There is a qualitatively important mechanism of a neutrino anisotropic stress compensation on super-horizon scales that was previously neglected. Computing the full transfer functions is rather straightforward, and we encourage future workers in this area to at least compare their semi-analytic results with the numerical answer to ensure that important physical effects have not be accidentally overlooked. Our modified version of CAMB [29] for efficiently computing vector mode power spectra is publicly available, and may also be useful for computing anisotropies from other sources, for example topological defects or second order effects.

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5 An inconsistency between their definition Eq. 2.17 and the equation for the $C_l$, Eq. 5.1

6 [http://camb.info](http://camb.info)
APPENDIX A: MULTIPLE EQUATIONS, HARMONIC EXPANSION AND \( C_l \)

In this appendix we review in a streamlined fashion the multipole equations, solutions, and equations for the \( C_l \) needed for numerical calculation \cite{21, 26, 27}. The definitions used here are precisely those used in the CAMB \cite{29} numerical implementation. Equivalent results using the total angular momentum method are given in Ref. \cite{32}.

The photon multipole evolution is governed by the geodesic equation and Thomson scattering, giving \cite{27}

\[
\dot{I}_{Ai} + 4/3 \Theta I_Ai + D^D(b_{Ai}) - \frac{l}{2l + 1} D_{(a_i} I_{Ai_{-1})} + 4/3 I A_{a_1} \delta_{l1} - \frac{8}{15} I \sigma_{a_1 a_2} \delta_{l2} = -n_e \sigma_T \left( I_{Ai} - I_{0i} - \frac{4}{3} I a_{a_1} \delta_{l1} - \frac{8}{15} \zeta_{a_1 a_2} \delta_{l2} \right) \quad \text{(A1)}
\]

where \( I_{Ai} \) is taken to be zero for \( l < 0 \) and

\[
\zeta_{ab} \equiv \frac{3}{4} I_{ab} + \frac{9}{2} \zeta_{ab}
\]

is a source from the anisotropic stress and E-polarization. The equation for the density perturbation \( D_a I \) is obtained by taking the spatial derivative of the above equation for \( l = 0 \). The corresponding evolution equations for the polarization multipole tensors are \cite{27}

\[
\begin{align*}
\dot{E}_{Ai} + 4/3 \Theta E_{Ai} + (l + 3)(l - 1) \frac{1}{(l + 1)^2} D^D E_{bAi} - \frac{l}{2l + 1} D_{(a_i} E_{Ai_{-1})} + 2 \frac{1}{l + 1} \text{curl} B_{Ai} &= -n_e \sigma_T (E_{Ai} - \frac{2}{15} \zeta_{a_1 a_2} \delta_{l2}) \\
\dot{B}_{Ai} + 4/3 \Theta B_{Ai} + (l + 3)(l - 1) \frac{1}{(l + 1)^2} D^D B_{bAi} - \frac{l}{2l + 1} D_{(a_i} B_{Ai_{-1})} + 2 \frac{1}{l + 1} \text{curl} E_{Ai} &= 0.
\end{align*}
\]

For numerical solution we expand the covariant equations into scalar, vector and tensor harmonics. The resulting equations for the modes at each wavenumber can be studied easily and also integrated numerically.

1. Scalar, vector, tensor decomposition

It is useful to do a decomposition into \( m \)-type tensors, scalar \((m = 0)\), vector \((m = 1)\) and 2-tensor \((m = 2)\) modes. They describe respectively density perturbations, vorticity and gravitational waves. In general a rank-\(\ell\) PSTF tensor \( X_{Ai} \) can be written as a sum of \( m \)-type tensors

\[
X_{Ai} = \sum_{m=0}^{l} X_{Ai}^{(m)}.
\]

Each \( X_{Ai}^{(m)} \) can be written in terms of \( l - m \) derivatives of a transverse tensor

\[
X_{Ai}^{(m)} = D_{(A_{i}m} \Sigma_{A_{m})}.
\]

where \( D_{Ai} \equiv D_{a_1} D_{a_2} \ldots D_{a_l} \) and \( \Sigma_{A_{m}} \) is first order, PSTF and transverse \( D^m \Sigma_{A_{m} = 0} \). The ‘scalar’ component is \( X_{Ai}^{(0)} \), the ‘vector’ component is \( X_{Ai}^{(1)} \), etc. Since General Relativity gives no sources for \( X_{Am} \) with \( m > 2 \) usually only scalars, vectors and \( (2-)\)-tensors are considered. At linear order they evolve independently.

2. Harmonic expansion

For numerical work we perform a harmonic expansion in terms of zero order eigenfunctions of the Laplacian \( Q_{Am}^m \),

\[
D^2 Q_{Am}^m = \frac{k^2}{S^2} Q_{Am}^m.
\]

For \( m > 0 \) there are eigenfunctions with positive and negative parity, which we can write explicitly as \( Q_{Am}^{m\pm} \) when required. Since

\[
D^2 (\text{curl} Q_{Am}^m) = \text{curl} (D^2 Q_{Am}^m) = \frac{k^2}{S^2} \text{curl} Q_{Am}^m
\]

they are related by the curl operation. Using the result

\[
\text{curl} \text{curl} Q_{Am}^m = \frac{k^2}{S^2} Q_{Am}^m
\]

we can choose to normalize the \( \pm \) harmonics the same way so that

\[
\text{curl} Q_{Am}^{m\pm} = \frac{k}{S} Q_{Am}^{m\mp}.
\]
A rank-$l$ PSTF tensor of either parity may be constructed from $Q_{A_m}^{m\pm}$ as

$$Q_{A_l}^m = \left(\frac{S}{k}\right)^{l-m} D_{(A_l-m)} Q_{A_m}^m$$

(A10)

and an $X_{A_l}^{(m)}$ component of $X_{A_l}$ may be expanded in terms of these tensors. They satisfy

$$D^2 Q_{A_l}^m = \frac{k^2}{S^2} Q_{A_l}^m$$

$$D^{a\pm} Q_{A_{l-1},a_l}^m = \frac{k}{S} (l^2 - m^2) Q_{A_l}^m$$

$$\text{curl} Q_{A_l}^m = \frac{m k}{l S} Q_{A_l}^m$$

where $l \geq m$.

Dimensionless harmonic coefficients are defined by

$$\sigma_{ab}^{(m)} = \sum_{k,\pm} \frac{k}{S} \sigma^{(m)\pm} Q_{ab}^{m\pm}$$

$$H_{ab}^{(m)} = \sum_{k,\pm} \frac{k^2}{S^2} H^{(m)\pm} Q_{ab}^{m\pm}$$

$$Q_a^{(m)} = \sum_{k,\pm} q_a^{(m)\pm} Q_a^{m\pm}$$

$$E_a^{(m)} = \sum_{k,\pm} \frac{k^2}{S^2} E^{(m)\pm} Q_a^{m\pm}$$

$$A_{ab}^{(m)} = \sum_{k,\pm} \frac{k}{S} A^{(m)\pm} Q_{ab}^{m\pm}$$

$$\Omega_{ab}^{(m)} = \sum_{k,\pm} \frac{k}{S} \Omega_a^{(m)\pm} Q_{ab}^{m\pm}$$

$$\Omega_a^{(m)} = \sum_{k,\pm} \frac{k}{S} \Omega_a^{(m)\pm} Q_{a}^{m\pm}$$

$$\Psi_{ap}^{(m)} = \sum_{k,\pm} \frac{k}{S} \delta X^{(m)\pm} Q_{ap}^{m\pm}$$

(A12)

where the $k$ dependence of the harmonic coefficients is suppressed. We also often suppress $m$ and $\pm$ indices for clarity. The other multipoles are expanded in analogy with $I_{A_l}$. The heat flux for each fluid component is given by $q_i = (p_i + p) v_i$, where $v_i$ is the velocity, and the total heat flux is given by $q = \sum_i q_i$. We write the baryon velocity simply as $v$, and define a constant $B_{0}^{(m)} = \Omega^{(m)}_{B}/\rho_{\gamma}$ to quantify the magnetic field anisotropic stress source. In the frame in which $\Omega_a = 0$ gradients are purely scalar $\delta X^{(1)} = 0$.

3. Harmonic multipole equations

Expanded into harmonics, the photon multipole equations become

$$I_l' + \frac{k}{2l + 1} \left[\frac{(l+1)^2 - m^2}{l+1} I_{l+1} - I_{l-1}\right] = -S n_{e} \sigma_{T} \left(I_l - \delta_{l0} I_0 - \frac{4}{3} \delta_{l1} \nu - \frac{2}{15} \zeta \delta_{l2}\right)$$

$$+ \frac{8}{15} k \sigma \delta_{l2} - 4 h' \delta_{l0} - \frac{4}{3} k A \delta_{l1}$$

(A13)

where $l \geq m$, $I_0 = \delta p_{\gamma}/\rho_{\gamma}$, $I_l = 0$ for $l < m$, and $m$ superscripts are implicit. The scalar source is $h' = (\delta S/S)'$. The equation for the neutrino multipoles (after neutrino decoupling) is the same but without the Thomson scattering terms (for massive neutrinos see Ref. [32]). The polarization multipole equations become

4. Integral solutions

Solutions to the Boltzmann hierarchies can be found in terms of line of sight integrals. The $I_l$ hierarchy has homogeneous solutions (i.e. solutions to Eq. [A13] with RHS set to zero) given by derivatives of

$$\Psi_{l}^{m}(k\eta) = \frac{l!}{(l-m)!} j_l(k\eta)$$

(A15)

where $j_l(x)$ is a spherical Bessel function. These can be used to construct Green’s function solutions to the full equations. For the polarization the result is less obvious, though solutions can easily be verified once found. For
vector modes \((m = 1)\) the solutions are \[33\]

\[
I_l(\eta_0) = 4 \int_0^\eta d\eta e^{-\tau} \left[ S_n e_{\sigma_T} \breve{\Psi}_l^1(\chi) + \left( k \sigma + S_n e_{\sigma T} \frac{\zeta}{4} \right) \frac{d\Psi_l^1(\chi)}{d\chi} \right]
\]

(A16)

\[
E_l^\pm(\eta_0) = \frac{l(l-1)}{l+1} \int d\eta S_n e_{\sigma_T} e^{-\tau} \left[ \frac{1}{\chi} \frac{dj_l(\chi)}{d\chi} + \frac{j_l(\chi)}{\chi^2} \right] \zeta^\pm
\]

(A17)

\[
B_l^\pm(\eta_0) = -\frac{l(l-1)}{l+1} \int d\eta S_n e_{\sigma_T} e^{-\tau} \left[ \frac{d^2j_l(\chi)}{d\chi^2} + \frac{4}{\chi} \frac{dj_l(\chi)}{d\chi} - \left( 1 - \frac{2}{\chi^2} \right) j_l(\chi) \right] \zeta^\mp
\]

(A18)

where \(\chi \equiv k(\eta_0 - \eta)\). For tensors \((m = 2)\) the solutions are \[27\]

\[
I_l(\eta_0) = 4 \int_0^\eta d\eta e^{-\tau} \left[ k \sigma + S_n e_{\sigma T} \frac{\zeta}{4} \right] \Psi_l^2(\chi)
\]

(A19)

\[
E_l^\pm(\eta_0) = \frac{l(l-1)}{(l+1)(l+2)} \int d\eta S_n e_{\sigma_T} e^{-\tau} \left[ \frac{d^2j_l(\chi)}{d\chi^2} + \frac{4}{\chi} \frac{dj_l(\chi)}{d\chi} - \left( 1 - \frac{2}{\chi^2} \right) j_l(\chi) \right] \zeta^\pm
\]

(A20)

\[
B_l^\pm(\eta_0) = -\frac{2l(l-1)}{(l+1)(l+2)} \int d\eta S_n e_{\sigma_T} e^{-\tau} \left[ \frac{d^2j_l(\chi)}{d\chi^2} + \frac{2}{\chi} \frac{dj_l(\chi)}{d\chi} \right] \zeta^\mp.
\]

(A21)

Here \(\tau\) is the optical depth from \(\eta\) to \(\eta_0\), \(\tau' = -S_n e_{\sigma T}\).

5. Power spectra

Using the harmonic expansion of \(I_{A_l}\) in Eq. \([18]\) the contribution to the \(C_l\) from type-\(m\) tensors becomes

\[
C_l^{TT(m)} = \frac{\pi}{4} \frac{(2l)!}{(-2)^l(l!)^2} \sum_{k,k',\pm} \langle I_{A_l}^{\pm}(l,k)Q_{A_{k'}}^\pmQ^A_{k'} \rangle.
\]

(A22)

The multipoles \(I_l\) can be related to some primordial variable \(X_{A_m} = \sum_k (X^+ Q_{A_m}^m + X^- Q_{A_m}^{-m})\) via a transfer function \(T_{lX}^m\) defined by \(I_l = T_{lX}^m X\). Statistical isotropy and orthogonality of the harmonics implies that

\[
\langle X_{k}^{\pm} X_{k'}^{\pm} \rangle = f_X(k) \delta_{kk'}
\]

(A23)

where \(\delta_{kk'} Y_k = Y_{k'}\) and \(f_X(k)\) is some function of the eigenvalue \(k\). The normalization of the \(Q_{A_k}^m\) is given by

\[
\int dV Q_{A_k}^m Q_{A_k}^{mA_l} = \int dV \left( \frac{-S}{k} \right)^{l-m} Q_{A_k}^m D^{A_l-m} Q_{A_l}^m = \frac{(-2)^l(-1)^m(l+m)!(l-m)!}{(2l)!} N
\]

(A24)

where we have integrated by parts repeatedly, then repeatedly applied the identity for the divergence \([A11]\). The normalization is \(N \equiv \int dV Q_{A_k}^m Q_{A_k}^{m}\). By statistical isotropy \(C_l = (1/V) \int dV C_l\) and hence

\[
C_l^{TT(m)} = \frac{\pi}{4} \frac{(l+m)!}{(-2)^m(l!)^2} \sum_{k,\pm} N \frac{V}{\int |T_{lX}^m(k)|^2 f_X(k)}
\]

(A25)

We choose to define a power spectrum \(P_X(k)\) so that the real space isotropic variance is given by

\[
\langle |X_{A_m} X_{A_m}^*| \rangle = \sum_{k,\pm} \frac{|N|}{V} f_X(k) = \int d\ln k P_X(k)
\]

(A26)

so the CMB power spectrum becomes

\[
C_l^{TT(m)} = \frac{\pi}{4} \frac{(l+m)!}{2^m(l!)^2} \int d\ln k P_X(k) |T_{lX}^m(k)|^2.
\]

(A27)

Note that we have not had to choose a specific representation of \(Q_{A_m}\) or \(\sum_k\).
The polarization $C_l$ are obtained similarly and in general we have

$$C_l^{JK(m)} = \frac{\pi}{4} \left[ \frac{(l+1)(l+2)}{l(l-1)} \right]^{1/2} \frac{(2l)!}{(-2)^{l!(l!)}^2} \frac{\langle J_{A_l} K_{A_j} \rangle}{\rho_p^2}$$

$$= \frac{\pi}{4} \left[ \frac{(l+1)(l+2)}{l(l-1)} \right]^{1/2} \frac{(l+m)!}{2^n n^l n^m} \int d\ln k P_X(k) J_l^X K_l^X$$

where $JK$ is $TT$ ($p=0$), $EE$ or $BB$ ($p=2$) or $TE$ ($p=1$). Our conventions for the polarization are consistent with CMBFAST [5] and CAMB [29]. We have assumed a parity symmetric ensemble, so $C_l^{TB} = C_l^{BT} = 0$.

For tensors we use $H_T$ where $h_{ij} = \sum_{k,l} 2 H_T Q_{ij}^k$ and $h_{ij}$ is the transverse traceless part of the metric tensor. This introduces an additional factor of 1/4 into the result for the $C_l$ in terms of $P_h$ and $T_T^{h}$.

The numerical factors in the hierarchy and $C_l$ equations depend on the choice of normalization for the $\ell$ and $k$ expansions. Neither $e_{(A_j)}$ or $Q_{A_i}$ are normalized, so there are compensating numerical factors in the expression for the $C_l$. If desired one can do normalized expansions, corresponding to an $\ell$-dependent re-scaling of the $I_l$ and other harmonic coefficients, giving expressions in more manifest agreement with Ref. [22].

[1] D. Grasso and H. R. Rubinstein, Phys. Rept. 348, 163 (2001), astro-ph/0009061.
[2] A. Brandenburg and K. Subramanian (2004), astro-ph/0405052.
[3] M. Zaldarriaga and U. Seljak, Phys. Rev. D58, 023003 (1998), astro-ph/9803150.
[4] W. Hu, Phys. Rev. D62, 043007 (2000), astro-ph/0001303.
[5] U. Seljak and M. Zaldarriaga, Phys. Rev. Lett. 78, 2054 (1997), astro-ph/9609169.
[6] U. Seljak, U.-L. Pen, and N. Turok, Phys. Rev. Lett. 79, 1615 (1997), astro-ph/9704231.
[7] L. Pogosian, S. H. H. Tye, I. Wasserman, and M. Wyman, Phys. Rev. D68, 023506 (2003), hep-th/0304188.
[8] C. Caprini and R. Durrer, Phys. Rev. D65, 023517 (2002), astro-ph/0106244.
[9] R. Durrer and C. Caprini, JCAP 0311, 010 (2003), astro-ph/0305059.
[10] K. Enqvist, A. Jokinen, and A. Mazumdar (2004), hep-ph/0404269.
[11] R. Durrer and J. Yokoyama, Phys. Rev. D69, 043507 (2004), astro-ph/0310824.
[12] R. Durrer, P. G. Ferreira, and T. Kahniashvili, Phys. Rev. D61, 043001 (2000), astro-ph/9911940.
[13] A. Mack, T. Kahniashvili, and A. Kosowsky, Phys. Rev. D65, 123004 (2002), astro-ph/0105504.
[14] K. Subramanian, T. R. Seshadri, and J. D. Barrow, Mon. Not. Roy. Astron. Soc. 344, L31 (2003), astro-ph/0303014.
[15] L. Pogosian, T. Vachaspati, and S. Winitzki, New Astron. Rev. 47, 859 (2003), astro-ph/0210639.
[16] C. Caprini, R. Durrer, and T. Kahniashvili, Phys. Rev. D69, 063006 (2004), astro-ph/0304556.
[17] C. Scoccola, D. Harari, and S. Mollerach (2004), astro-ph/0405396.
[18] A. Kosowsky and A. Loeb, Astrophys. J. 469, 1 (1996), astro-ph/9601055.
[19] S. K. Sethi and K. Subramanian (2004), astro-ph/0405413.
[20] G. Chen, P. Mukherjee, T. Kahniashvili, B. Ratna, and Y. Wang (2004), astro-ph/0403695.
[21] G. F. R. Ellis, D. R. Matravers, and R. Treciokas, Ann. Phys. (N.Y.) 150, 455 (1983).
[22] A. Challinor and A. Lasenby, Astrophys. J. 513, 1 (1999), astro-ph/9804301.
[23] C. G. Tsagas and J. D. Barrow, Class. Quant. Grav. 15, 3523 (1998), gr-qc/9803032.
[24] A. Challinor, Class. Quant. Grav. 17, 871 (2000), astro-ph/9906474.
[25] A. Challinor, Phys. Rev. D62, 043004 (2000), astro-ph/9911481.
[26] S. Koh and C. H. Lee, Phys. Rev. D62, 083509 (2000), astro-ph/0006357.
[27] A. Lewis, A. Challinor, and A. Lasenby, Astrophys. J. 538, 473 (2000), astro-ph/9911177.
[28] R. Durrer and C. Caprini, JCAP 0311, 010 (2003), astro-ph/0305059.
[29] K. Enqvist, A. Jokinen, and A. Mazumdar (2004), hep-ph/0404269.
[30] M. S. Turner and L. M. Widrow, Phys. Rept. D37, 2743 (1988).
[31] R. Durrer and J. Yokoyama, Phys. Rev. D69, 043507 (2004), astro-ph/0310824.
[32] R. Durrer, P. G. Ferreira, and T. Kahniashvili, Phys. Rev. D61, 043001 (2000), astro-ph/9911940.
[33] A. Mack, T. Kahniashvili, and A. Kosowsky, Phys. Rev. D65, 123004 (2002), astro-ph/0105504.
[34] K. Subramanian, T. R. Seshadri, and J. D. Barrow, Mon. Not. Roy. Astron. Soc. 344, L31 (2003), astro-ph/0303014.
[35] L. Pogosian, T. Vachaspati, and S. Winitzki, New Astron. Rev. 47, 859 (2003), astro-ph/0210639.
[36] C. Caprini, R. Durrer, and T. Kahniashvili, Phys. Rev. D69, 063006 (2004), astro-ph/0304556.
[37] C. Scoccola, D. Harari, and S. Mollerach (2004), astro-ph/0405396.
[38] A. Kosowsky and A. Loeb, Astrophys. J. 469, 1 (1996), astro-ph/9601055.
[39] A. Lewis and A. Challinor, Phys. Rev. D66, 023531 (2002), astro-ph/020357.