The large-$N_c$ limit of borelized spectral sum rules and the slope of radial Regge trajectories

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Abstract

We put forward a new phenomenological method for calculating the slope of radial trajectories from values of ground states and vacuum condensates. The method is based on a large-$N_c$ extension of borelized spectral sum rules. The approach is applied to the light non-strange vector, axial, and scalar mesons. The extracted values of slopes proved to be approximately universal and are in the interval $1.4 \pm 0.1$ GeV$^2$. As a by-product, the given method leads to prediction of the second radial trajectory with ground state mass lying near 0.6 GeV.

1 Introduction

It is widely believed that confinement in QCD leads to approximately linear radial Regge trajectories (see, e.g., [1,2]). The linearity has a natural explanation within various hadron string models [3]. The most important quantity in this picture is the slope of trajectories. The slope is expected to be nearly universal as arising from flavor-independent non-perturbative gluodynamics which thereby sets a mass scale for light hadrons. In view of absence of analytical description for hadron mass generation, it is interesting to construct phenomenological methods that would allow to estimate the value of slope and check its universality basing on some inputs from QCD. Among the phenomenological approaches to the hadron spectroscopy, the method of spectral sum rules [4] is perhaps the most related with QCD. In many cases, it permits to calculate reliably the masses of ground states on the radial trajectories. We propose an extension of this approach which allows to estimate the slope using essentially the same technic.

The method of QCD sum rules was originally introduced by Shifman, Vainshtein and Zakharov (SVZ) [4] and it turned out to be extremely fruitful in the hadron spectroscopy [5][6]. The idea of this approach is based on
the assumption that a quark-antiquark pair being injected into the strong QCD vacuum does not perturb appreciably the vacuum structure. This allows to parametrize the unknown properties of non-perturbative vacuum by some universal phenomenological characteristics called vacuum condensates. According to this approach, hadrons with different quantum numbers have different masses and other static characteristics because their currents react differently with the vacuum medium. A manifestation of this difference are different coefficients in the Operator Product Expansion (OPE) of correlators of the corresponding quark currents which can be calculated from QCD. Assuming the existence of resonance in some energy range, one is able to calculate its static characteristics via the dispersion relations and the OPE, with higher radial excitations being regarded as a part of perturbative continuum. In the case of the light hadrons, a borelized version of sum rules is usually exploited as the Borel transform effectively singles out the ground state suppressing contributions from the rest of spectrum and simultaneously improving the convergence of OPE [4].

The SVZ sum rules does not allow to calculate the full decay width since hadrons are considered as infinitely narrow states. A typical accuracy of the method is thus of the order of 10 - 20%. On the other hand, the narrow-width approximation for mesons has a solid theoretical basis — the large-$N_c$ (planar) limit in QCD [7, 8]. In this limit, the one-hadron states saturate completely the two-point correlation functions of hadron currents $j$,

\[
\langle j(q)j(-q) \rangle = \sum_n \frac{F^2_n}{q^2 - M^2_n}.
\]  

(1)

The large-$N_c$ scaling of quantities is: $M_n = \mathcal{O}(1)$ for masses, $F^2_n = \langle 0|j|n \rangle^2 = \mathcal{O}(N_c)$ for residues, $\Gamma = \mathcal{O}(1/N_c)$ for decay width. The sum in (1) must contain an infinite number of terms in order to reproduce the logarithmic behavior of the correlator at large $q^2$ following from the asymptotic freedom [8]. Assuming some ansatz for the radial mass spectrum, the expression (1) can be summed up, expanded at large $Q^2 = -q^2$ and compared with the corresponding OPE in QCD. One obtains a set of sum rules — each sum rule represents an equation corresponding to some $k$ in the expansion $1/Q^{2k}$, $k = 0, 1, 2, \ldots$ of both sides in (1). Such planar sum rules were considered many times in the case of the linear Regge ansatz for radial spectrum motivated by the phenomenology [9, 10],

\[
m^2_n = an + m^2_0, \quad n = 0, 1, 2, \ldots,
\]  

(2)

sometimes with certain non-linear corrections to this spectrum.
This planar approach to QCD sum rules has a certain shortcoming: The sum rules corresponding to different $k$ are treated on equal footing while the accuracy of sum rules deteriorates rapidly with increasing $k$ because of a bad convergence. In the given paper, we propose a novel treatment of the planar QCD sum rules. The idea is to apply the Borel transform to the infinite sum in the r.h.s. of (1) and derive an expression for the slope $a$ in (2) in the same way as one finds an expression for a mass of ground state in the classical SVZ sum rules. In other words, we propose to consider borelized planar sum rules and analyze them following a well elaborated technics. We will consider $m_0$ in (2) as the mass of ground state. The value of $m_0$ will be regarded as being known from the old SVZ sum rules or from experimental data. In the first case, the standard calculation of $m_0$ within SVZ sum rules represents just the first step to finding the whole radial spectrum. We construct thus an extension of SVZ sum rules which allows to obtain the radial spectrum using essentially the same number of input parameters. The main output of the analysis is the slope $a$. In essence, we propose a new way for calculating this quantity with the help of a well developed method. We will apply this approach to the light non-strange vector, axial and scalar mesons. In the latter case, our approach leads to a unexpected result related with appearance of the second scalar trajectory beginning with a rather light state having mass near 600 MeV.

The paper is organized as follows. In Section 2, we formulate our approach. Its application to the vector, axial, and scalar channel is considered in Section 3. The conclusion and various remarks are given in Section 4.

2 SVZ sum rules for the slope of radial trajectories

2.1 The vector case

The case of vector $\rho$-mesons is canonical for the SVZ sum rules method [4]. Let us apply this method to the radial spectrum (2) in the large-$N_c$ limit.

The basic theoretical object is the two-point vector correlator $\Pi(Q^2)$ in Euclidean space defined by

$$\Pi_{\mu\nu} = (g_\mu g_\nu - g_{\mu\nu} q^2)\Pi(Q^2),$$

where $\Pi_{\mu\nu}$ represents the T-product of two vector currents interpolating the neutral $\rho^0$-meson,

$$\Pi_{\mu\nu} = i \int d^4x e^{ipx} \langle 0|T\{j\mu(x), j\nu(0)}|0\rangle,$$

2

3
Here \( u \) and \( d \) are quark fields, and \( q \) is the photon spacelike momentum, \( q^2 = -Q^2 \). The OPE for \( \Pi(Q^2) \) reads \(^5\),

\[
\Pi(Q^2) = \frac{1}{8\pi^2} \left( 1 + \frac{\alpha_s}{\pi} \right) \ln \frac{\mu^2}{Q^2} + \frac{\langle m_q \bar{q}q \rangle}{M^4} + \frac{1}{24Q^4} \left( \frac{\alpha_s}{\pi} \left( G_{\mu\nu}^a \right)^2 \right) - \frac{14\pi\alpha_s}{9} (\bar{q}q)^2, 
\]

where \( q \) stands for \( u \) or \( d \) quark, the coefficient in front of the last term is given in the large-\( N_c \) limit (it differs by the common factor \((N_c^2 - 1)/N_c^2\) from the corresponding coefficient in Refs. \(^4\[4, 5]\)), and further \( O(Q^{-8}) \) terms are neglected. Applying the Borel transform,

\[
L_M \Pi(Q^2) = \lim_{Q^2/n \rightarrow \infty} \frac{1}{(n-1)!} \frac{1}{(Q^2)^n} \left( -\frac{d}{dQ^2} \right)^n \Pi(Q^2),
\]

(6)

to the OPE \(^5\) we get

\[
L_M \Pi(Q^2) = \frac{1}{8\pi^2} \left( 1 + \frac{\alpha_s}{\pi} \right) + \frac{\langle m_q \bar{q}q \rangle}{M^4} + \frac{1}{24M^4} \left( \frac{\alpha_s}{\pi} \left( G_{\mu\nu}^a \right)^2 \right) - \frac{7\pi\alpha_s}{9M^6} (\bar{q}q)^2. 
\]

(7)

The vector correlator \( \Pi(Q^2) \) satisfies a dispersion relation with one subtraction,

\[
\Pi(q^2) = \frac{1}{\pi} \int_{4m_q^2}^{\infty} ds \frac{\text{Im} \Pi(s)}{s - q^2 + i\varepsilon} + \Pi(0).
\]

(8)

In the large-\( N_c \) limit, the mesons are infinitely narrow and saturate completely the two-point correlators \(^3\),

\[
\Pi(q^2) = \sum_n \frac{F_n^2}{q^2 - m_n^2 + i\varepsilon}, 
\]

(9)

where the residues are defined by \( \langle 0 | j_\mu | V_n \rangle = F_n m_n \varepsilon_\mu \). The experimental information on the electromagnetic decay constants \( F_n \) of radially excited light vector and axial mesons is poor. The quark-hadron duality requires that \( F_n \) must be constant for the exactly linear spectrum (or decrease at least exponentially with \( n \)) \(^9\). For the sake of simplicity (a minimal number of inputs), we will assume that \( F_n \) represent just a universal constant in the large-\( N_c \) limit, \( F_n = F \). The given assumption will lead to rather reasonable numerical predictions and this is enough for the zero-width approximation. The imaginary part of \( \Pi(Q^2) \) takes then a simple form

\[
\text{Im} \Pi(q^2) = \sum_n \pi F^2 \delta(q^2 - m_n^2). 
\]

(10)
The Borel transform of (8) in this case is [4]:

\[
L_M \Pi(Q^2) = \frac{1}{\pi M^2} \int_0^{\infty} e^{-s/M^2} \text{Im} \ Pi(s) ds = \frac{F^2}{M^2} \sum_n e^{-m_n^2/M^2}, \tag{11}
\]

where we neglected the \( O(m_q^2) \) contribution. Substituting the linear spectrum [2] and summing up we obtain

\[
L_M \Pi(Q^2) = \frac{F^2}{M^2} \frac{e^{-m_0^2/M^2}}{1 - e^{-a/M^2}}. \tag{12}
\]

The first sum rule arises from equating the relations (5) and (12),

\[
\frac{F^2 e^{-m_0^2/M^2}}{1 - e^{-a/M^2}} = \frac{M^2}{8\pi^2} \left[ 1 + \frac{\alpha_s}{\pi} + \frac{8\pi^2}{M^2} \langle m_q \bar{q} q \rangle + \frac{\pi^2}{3M^2} \left\langle \frac{\alpha_s}{\pi} (G_{\mu\nu}^a)^2 \right\rangle - \frac{56}{9} \frac{\pi^3 \alpha_s}{M^6} \langle \bar{q} q \rangle^2 \right]. \tag{13}
\]

Following the prescriptions of SVZ method [4], we can get the second sum rule by taking derivative of Eq. (13) with respect to \( 1/M^2 \). The meson mass appears directly in the fraction \( \frac{d(13)}{d(1/M^2)} \). The given ”combined” sum rule has the form

\[
m_0^2 = M^2 \left( \frac{h_0 - \frac{h_2}{M^2} - \frac{2h_4}{M^4}}{h_0 + \frac{h_1}{M^2} + \frac{h_2}{M^4} + \frac{h_3}{M^6}} - \frac{a}{e^{a/M^2} - 1} \right), \tag{14}
\]

where the condensate terms \( h_i \) are presented in Table 1. In the same table, we display the corresponding terms \( h_i \) for the axial and scalar cases [4, 5].

| Mesons | \( h_0 \) | \( h_1 \) | \( h_2 \) | \( h_3 \) |
|--------|-------------|-------------|-------------|-------------|
| \( \rho \) | \( 1 + \frac{\alpha_s}{\pi} \) | 0 | \( 8\pi^2 \langle m_q \bar{q} q \rangle + \frac{\pi^2}{3} \left\langle \frac{\alpha_s}{\pi} (G_{\mu\nu}^a)^2 \right\rangle \) | \( -\frac{56}{9} \frac{\pi^3 \alpha_s}{M^6} \langle \bar{q} q \rangle^2 \) |
| \( a_1 \) | \( 1 + \frac{\alpha_s}{\pi} \) | \( -8\pi^2 f_{\pi}^2 \) | \( -8\pi^2 \langle m_q \bar{q} q \rangle + \frac{\pi^2}{3} \left\langle \frac{\alpha_s}{\pi} (G_{\mu\nu}^a)^2 \right\rangle \) | \( \frac{88}{9} \pi^3 \alpha_s \langle \bar{q} q \rangle^2 \) |
| \( f_0 \) | \( 1 + \frac{11\alpha_s}{3\pi} \) | 0 | \( 8\pi^2 \langle m_q \bar{q} q \rangle + \frac{\pi^2}{3} \left\langle \frac{\alpha_s}{\pi} (G_{\mu\nu}^a)^2 \right\rangle \) | \( \frac{176}{9} \pi^3 \alpha_s \langle \bar{q} q \rangle^2 \) |

The first term in the r.h.s. of Eq. (14) corresponds to the limit \( s_0 \to \infty \) in the canonical expressions for the meson masses in the SVZ sum rules [5]. The energy cutoff \( s_0 \) is infinite in our case as we take into account an infinite number of radial excitations. The second term is new and reflects contribution of highly excited states. If we knew the slope \( a \) we could find from Eq. (13) the mass of ground state \( m_0 \) making use of the standard stability criterion on the Borel [4]. But we will prefer the opposite procedure: Since \( m_0 \) is usually well known, it is interesting to calculate numerical values of slopes for various radial trajectories.
2.2 The scalar case

The simplest scalar correlator is defined by replacing the vector current in (4) by the scalar one \( j = \bar{q}q \). The OPE for the scalar correlator reads [5]

\[
\Pi_s(q^2) = \frac{3}{8\pi^2} \left(1 + \frac{11\alpha_s}{3\pi}\right) Q^2 \ln \frac{Q^2}{\mu^2} + \frac{3}{Q^2} \langle m_q \bar{q}q \rangle \\
+ \frac{1}{8Q^2} \left(\frac{\alpha_s}{\pi} (G_{\mu\nu}^a)^2\right) - \frac{22\pi\alpha_s}{3Q^4} \langle \bar{q}q \rangle^2, 
\]

(15)

where the coefficient in the last term is written in the large-\( N_c \) limit. We define the spectral representation as

\[
\Pi_s(q^2) = \sum_n G^2 m_n^2 \frac{q^2 - m_n^2 + i\epsilon}{q^2 - m_n^2 + i\epsilon}. 
\]

(16)

Here the constant \( G^2 \) represents a scalar analog of vector residue \( F^2 \). Substituting the linear spectrum (2) and repeating the operations of the previous Section we will arrive at the following "combined" sum rule in the scalar channel,

\[
m_0^4 \left(e^{a/M^2} - 1\right) + 2am_0^2 = L \left(a - m_0^2 + m_0^2 e^{a/M^2}\right) - a^2 e^{a/M^2} + 1 - 1
\]

(17)

where

\[
L \equiv M^2 \frac{2h_0 + \frac{h_3}{M^2}}{h_0 + \frac{h_2}{M^2} - \frac{h_3}{M^4}},
\]

(18)

and the coefficients \( h_i \) are given in Table 1. The relation (17) represents a quadratic equation for the intercept \( m_0^2 \). The corresponding two solutions are

\[
m_0^2 = \frac{a}{1 - e^{a/M^2}} + \frac{L}{2} \pm \frac{\sqrt{L^2 \left(e^{a/M^2} - 1\right)^2 / 4 - a^2 e^{a/M^2}}}{e^{a/M^2} - 1}.
\]

(19)

Below we will discuss both solutions.

3 Numerical fits and predictions

3.1 Input parameters

The numerical values of input parameters \( h_i \) in Table 1 which we will use are displayed in Table 2. Below these numbers are briefly commented.
Table 2: The numerical values of coefficients $h_i$ in our fits.

| Channel | $h_0$ | $h_1$ | $h_2$ | $h_3$ |
|---------|-------|-------|-------|-------|
| $\rho$  | 1     | 0     | 0.032 | −0.030|
| $a_1$   | 1     | −0.674| 0.046 | 0.048 |
| $f_0$   | 1     | 0     | 0.032 | −0.095|

We set $h_0 = 1$ since taking the perturbative threshold $s_0 \to \infty$ (infinite number of radial states) we should formally have $\alpha_s \to 0$ due to the asymptotic freedom. This is tantamount to neglecting the loop corrections to the unit operator in the OPE.

The values of gluon and quark condensates are taken from Ref. [4]:

$$\left\langle \frac{\alpha_s}{\pi} \left( G^g_{\mu\nu} \right)^2 \right\rangle = (330 \text{ MeV})^4,$$

and

$$\left\langle \bar qq \right\rangle = -(250 \text{ MeV})^3.$$

The first value is scale-independent while the second one is taken roughly at the scale $\mu = 1 \text{ GeV}$. From the Gell-Mann–Oakes –Renner relation,

$$m_\pi^2 f_\pi^2 = -(m_u + m_d) \left\langle \bar qq \right\rangle,$$

one gets $m_u + m_d \approx 10.7 \text{ MeV}$. We consider the isospin limit for the masses of current quarks, $m_u = m_d \equiv m_q$. Thus we get a numerical value for another renormalization invariant condensate of dimension four,

$$\left\langle m_q \bar qq \right\rangle = -(95.6 \text{ MeV})^4.$$

All these inputs lead to the values of $h_1$ and $h_2$ in Table 2.

The operator $\alpha_s(\bar qq)$ has a small anomalous dimension. We will regard the corresponding v.e.v. $\alpha_s \left\langle \bar qq \right\rangle$ as a constant. The condensate $h_3$ in the $\rho$-channel is taken from Ref. [4]. All other $h_3$ can be then obtained from a rescaling prescribed by the coefficients in the last column of Table 1. Our numerical results, however, will be only slightly dependent on $h_3$ or independent of it.

As was mentioned above we will regard $m_0$ as the mass of ground state obtained within the classical SVZ sum rules. This reduces the number of input parameters and makes the method more attractive: Fixing the values of condensates in the OPE, one can extract the values of both intercept $m_0$ and slope $a$. The first step (extraction of $m_0$) is nothing but the standard SVZ sum rule method while the second step represents our extension of this method to the case of infinite linear radial spectrum.
3.2 Vector mesons

Our strategy is as follows. Consider the $\rho$-meson. We plot $m_0$ from Eq. (14) as a function of Borel parameter $M^2$ at different values of $a$. A typical plot is presented in Fig. 1. These plots possess the so-called “Borel window” — a stability region near the minimum where $m_0$ is approximately constant. We find the value of $a$ at which $m_0$ coincides with the value of $m_\rho$ extracted in the usual SVZ sum rules [4,5]. The obtained $a$ is our prediction for the slope. The slope of $\rho$-trajectory turns out to be near $a_\rho = 1.52 \pm 0.07 \text{ GeV}^2$. The predicted masses for this value of slope are presented in Table 3 together with a tentative assignment to experimental data [11]. We give an uncertainty in mass related with uncertainty in extraction of $m_\rho$ from the classical SVZ sum rules. The latter uncertainty comes from uncertainty in values of vacuum condensates. In order to avoid double counting of uncertainties we do not use the uncertainties in condensates when calculate $a$. There are of course uncertainties arising from the limits $N_c \to \infty$ and $\alpha_s \to 0$, and from assumed universality of residues. We estimate these uncertainties at the level of 10%.

![Figure 1: The mass of $\rho$-meson at $a = 1.52 \text{ GeV}^2$ as a function of Borel parameter (14).](image)

In the axial-vector case, the stability region exists only at large values of Borel parameter, $M \to \infty$, see Fig. 2. The same situation takes place within the classical SVZ sum rules [5]. Normalizing our $m_0$ to the value $m_{a_1} = 1.15 \pm 0.04 \text{ GeV}$ obtained in Ref. [5], we get $a_{a_1} = 1.30 \pm 0.18 \text{ GeV}^2$. The corresponding mass spectrum is presented in Table 4.

Due to the second term in Eq. (14), however, an alternative possibility appears. This additional contribution results in a local maximum in Fig. 2. One can interpret the region near this extremum as an “emergent” Borel window. The value of $m_0$ in this region is surprisingly close to the mass of...
Table 3: The radial spectrum of $\rho$-mesons for the slope $a = 1.52 \pm 0.07\text{ GeV}^2$. The masses are given in MeV. The first 4 predicted states are tentatively assigned to the resonances $\rho(770)$, $\rho(1450)$, $\rho(1900)$, and $\rho(2270)$ which presumably form the $S$-wave radial trajectory.

| $n$ | $m_\rho$ (th) | $m_\rho$ (exp) |
|-----|---------------|---------------|
| 0   | 770 ± 10      | 775           |
| 1   | 1450 ± 20     | 1465 ± 25     |
| 2   | 1910 ± 40     | 1870–1920     |
| 3   | 2230 ± 50     | 2265 ± 40     |
| 4   | 2580 ± 50     | —             |

Figure 2: The mass of $a_1$ meson at $a = 1.30\text{ GeV}^2$.

axial resonance $a_1(1260)$ which is traditionally interpreted as an axial partner of $\rho(770)$ if the chiral symmetry were not spontaneously broken. Taking this solution as the ground state we obtain an alternative prediction for the tower of radially excited axial states. The corresponding spectrum is also shown in Table 4.

### 3.3 Scalar mesons

The scalar case has two solutions. The first one (with plus sign) exists at any $M^2$ while the second one appears above some positive value of the Borel parameter, see Fig. 3. The first solution corresponds to the value of scalar mass extracted in the classical SVZ sum rules. The stability region lies at $M \to \infty$ as in the axial case. The standard SVZ method gives $m_{f_0} = 1.00 \pm 0.03\text{ GeV}$. Normalizing the first solution to this prediction we obtain $a_{f_0} = 1.38 \pm 0.07\text{ GeV}^2$. If we substitute this value of slope to the second solution we get mass of the "emerged" lighter scalar meson, $m_{f_0} \approx 0.62\text{ GeV}$. Our solution predicts thus two parallel scalar trajectories. The ground state on the first trajectory can be identified with $f_0(980)$ and on
Table 4: The radial spectrum of $a_1$-mesons for the slope $a = 1.30 \pm 0.18$ GeV$^2$. The first 4 predicted states are tentatively assigned to the resonances $a_1(1230)$, $a_1(1640)$, $a_1(1930)$, and $a_1(2270)$ [11].

| $n$ | 0 | 1 | 2 | 3 | 4 |
|-----|---|---|---|---|---|
| $m_{a_1}$ (th) | 1150 ± 40 | 1620 ± 60 | 1980 ± 90 | 2280 ± 120 | 2550 ± 140 |
| $m_{a_1}$ (exp) | 1230 ± 40 | 1647 ± 22 | 1930$^{+30}_{-70}$ | 2270$^{+55}_{-40}$ | — |

The existence of two parallel radial scalar trajectories seems to agree with the experimental data [1]. The masses of predicted radial states and a tentative comparison with the observed scalar mesons for two trajectories are displayed in Tables 5 and 6, correspondingly.

![Figure 3: The mass of scalar meson at $a = 1.38$ GeV$^2$.](image)

### 4 Discussions and conclusions

We have put forward a new extension of borelized SVZ sum rules. Using an example of a simple model with minimum of inputs, we demonstrated how this extension allows to extract the value of slope of linear radial trajectories from static characteristics of QCD vacuum — the vacuum condensates — in the large-$N_c$ limit of QCD. The obtained slopes for the light non-strange vector, axial, and scalar trajectories agree well with the phenomenology. This may justify a posteriori the approximations and assumptions made for a simple demonstration of the method. Our analysis confirms a known hypothesis
Table 5: The radial spectrum of the first $f_0$-trajectory for the slope $a = 1.38 \pm 0.07 \text{GeV}^2$. The first 5 predicted states are tentatively assigned to the resonances $f_0(980)$, $f_0(1500)$, $f_0(2020)$, $f_0(2200)$, and $X(2540)$ [11].

| $n$ | $m_{f_0}$ (th 1) | $m_{f_0}$ (exp 1) |
|-----|------------------|------------------|
| 0   | 1000 ± 30        | 990 ± 20         |
| 1   | 1540 ± 20        | 1504 ± 6         |
| 2   | 1940 ± 40        | 1992 ± 16        |
| 3   | 2270 ± 50        | 2189 ± 13        |
| 4   | 2560 ± 50        | 2539 ± 14        |

Table 6: The radial spectrum of the second $f_0$-trajectory for the slope $a = 1.38 \pm 0.07 \text{GeV}^2$. The first 5 predicted states are tentatively assigned to the resonances $f_0(500)$, $f_0(1370)$, $f_0(1710)$, $f_0(2100)$, and $f_0(2330)$ [11].

| $n$ | $m_{f_0}$ (th 2) | $m_{f_0}$ (exp 2) |
|-----|------------------|------------------|
| 0   | 620              | 400–550          |
| 1   | 1330 ± 30        | 1200–1500        |
| 2   | 1780 ± 40        | 1723$^{+5}_{-5}$ |
| 3   | 2130 ± 50        | 2101 ± 7         |
| 4   | 2430 ± 60        | 2300–2350        |

that the slope of radial trajectories is approximately universal for all light non-strange mesons. We thus obtained an independent estimate for its value, $a = 1.4 \pm 0.1 \text{GeV}^2$. This value is consistent with a typical phenomenological estimate, $a = 1.25 \pm 0.15 \text{GeV}^2$ [1].

In the limit of infinite Borel parameter, our borelized sum rule for the vector and axial channel becomes one of the usual planar sum rules which were considered many times in the past [9]. Indeed, taking the limit $M^2 \to \infty$ in Eq. (14) we get

$$m_0^2 = -h_1 + \frac{a}{2}. \quad (20)$$

The sum rule (20) represents a planar analog of the first Weinberg sum rule. For comparison, we display in Table 7 some fits based on relation (20).

Table 7: The radial spectrum of vector and axial mesons in the limit $M^2 \to \infty$.

| Meson | $h_1$, $\text{GeV}^2$ | $m_0$, $\text{GeV}$ | $a$, $\text{GeV}^2$ | $m_1$ | $m_2$ | $m_3$ | $m_4$ |
|-------|----------------------|---------------------|---------------------|-------|-------|-------|-------|
| $\rho$ | 0                    | 770                 | 1.19                | 1330  | 1720  | 2040  | 2310  |
| $a_1$  | $-0.674$             | 1150                | 1.30                | 1620  | 1980  | 2280  | 2550  |

In the scalar channel, our borelized sum rule has no analog in the planar
sum rules without borelization. The reason is that taking the limit $M^2 \rightarrow \infty$ in Eq. (17) we arrive at identity $0 = 0$. Applying this limit to the solutions (19) we get

$$m_0^2 = \frac{a}{2} \pm \frac{1}{2} \sqrt{\frac{a^2}{3} - 8h_2}.$$  \hspace{1cm} (21)

The relation (21) gives an analytical expression for the masses of two lightest scalar states which we obtained from Fig. 3. It is seen that these masses do not depend on the condensate $h_3$.

The prediction of the second scalar trajectory is a rather surprising feature of our borelized planar sum rules. The ground state on the second radial trajectory turns out to be significantly lighter than on the first trajectory. It looks tempting to identify this state with the elusive $\sigma$ (called also $f_0(500)$) meson [11]. The lightest scalar state in the standard SVZ sum rules lies near 1 GeV [5] and cannot be made significantly lighter within this method [12]. Our extension of the SVZ method leads thus to a new result. It is interesting to check whether a similar result appears in the framework of unborelized planar sum rules. A recent analysis of Ref. [10] gives a positive answer. A light scalar state near 0.5 GeV; however, emerged in Ref. [10] in a different way.

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