Some characterizations of quasi Yamabe solitons

Absos Ali Shaikh and Prosenjit Mandal

Abstract. In this article, we have proved some results in connection with the potential vector field having finite global norm in quasi Yamabe soliton. We have derived some criteria for the potential vector field on the non-positive Ricci curvature of the quasi Yamabe soliton. Also, a necessary condition for a compact quasi Yamabe soliton has been formulated. We further showed that if the potential vector field has a finite global norm in a complete non-trivial, non-compact quasi Yamabe soliton with finite volume, then the scalar curvature becomes constant and the soliton reduces to a Yamabe soliton.

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1. Introduction and preliminaries

An \( n \)-dimensional Riemannian manifold \((M, g)\) is said to be a quasi Yamabe soliton [4] if it possesses a smooth vector field \( Y \) satisfying

\[
2(R - \rho)g + 2\beta Y^\flat \otimes Y^\flat = \mathcal{L}_Y g,
\]

where \( R \) denotes the scalar curvature of the metric \( g \), \( Y^\flat \) is the dual 1-form of \( Y \), \( \mathcal{L}_Y g \) represents Lie derivative of the metric tensor \( g \) in the direction of the vector field \( Y \), \( \rho \) is a constant and \( \beta \) is a smooth function on \( M \). In the above Eq. (1) if we put \( \beta = 0 \), then we get the following equation

\[
2(R - \rho)g = \mathcal{L}_Y g,
\]

and in this case \((M, g)\) is called a Yamabe soliton. The Yamabe solitons are the special solutions of Yamabe flow, \( \frac{\partial g}{\partial t} = -Rg \), which was introduced by Hamilton [7,8]. If \( \rho > 0 \), \( \rho = 0 \) or \( \rho < 0 \), then the Yamabe soliton is called shrinking, steady or expanding respectively.
In the local coordinates system the Eq. (1) can also be expressed in the following form
\[ \nabla_i Y_j + \nabla_j Y_i = 2(R - \rho)g_{ij} + 2\beta Y_i Y_j. \] (2)

A Riemannian manifold \((M, g)\) is called a generalized quasi Yamabe gradient soliton [14], if \(Y\) is gradient of some smooth function \(f \in C^\infty(M)\) and in this case (1) takes the form
\[ \nabla_i \nabla_j f = (R - \rho)g_{ij} + \beta \nabla_i f \nabla_j f. \] (3)

In the above Eq. (3) if we suppose \(\beta\) as a constant, then in this case \((M, g)\) is called a quasi Yamabe gradient soliton [9].

We note that the vector field \(\nabla_i f\) is a special torse-forming vector field in the sense of Yano [21], and for \(\beta = \text{constant}\) it is a special concircular vector field [20]. In this case the vector field \(-\beta f \nabla_i f\) is equidistant vector field in the sense of Sinyukov [18] (also see, Mikeš et al. [10]). However, global problems about these vector fields can be found in [11].

The study of quasi Yamabe gradient soliton is a well known research area in differential geometry. In [9], Huang and Li proved that a compact quasi Yamabe gradient soliton is of constant scalar curvature. It has been shown in [19] that the scalar curvature of a non-compact quasi Yamabe gradient soliton may not be constant, and also showed that such a soliton has a warped product structure with warping function as the potential of the soliton. Later Neto and Oliveira [14], have extended these results to the generalized quasi Yamabe gradient solitons. By motivating of the above study and the results in [2,3,5,6,12,15–17,24,25], in the present paper, we have established that in a complete non-trivial non-compact quasi Yamabe soliton having finite volume, if the potential vector field has finite global norm, then \(R\) is constant and the soliton reduces to a Yamabe soliton. Also, we have deduced certain conditions for the potential vector field on quasi Yamabe soliton of non-positive Ricci curvature and an integral condition for compact quasi Yamabe soliton.

Let \(A_m(M)\) be the space of all smooth \(m\)-forms in \(M\), for any \(m, 0 \leq m \leq n\). For \(\eta, \zeta \in A_m(M)\), the local inner product \((\eta, \zeta)\) of \(\eta\) and \(\zeta\) is defined by [13],
\[ (\eta, \zeta) = \eta_{i_1 \ldots i_m} \zeta^{i_1 \ldots i_m}, \]
where \(\eta = \eta_{i_1 \ldots i_m} du^{i_1} \wedge \cdots \wedge du^{i_m}, \zeta = \zeta_{i_1 \ldots i_m} du^{i_1} \wedge \cdots \wedge du^{i_m}\) and \(\zeta^{i_1 \ldots i_m} = g^{i_1 j_1} \cdots g^{i_m j_m} \zeta_{j_1 \ldots j_m}\). For a fixed \(m \geq 0\), the Hodge star operator \(* : A_m(M) \to A_{n-m}(M)\) is defined by
\[ *\eta = \text{sgn}(\mathcal{A}, \mathcal{P}) \eta_{j_1 \ldots j_m} du^{j_1} \wedge \cdots \wedge du^{j_{n-m}}, \]
for \(\eta = \eta_{i_1 \ldots i_m} du^{i_1} \wedge \cdots \wedge du^{i_m} \in A_m(M)\). Here \(j_1 < \cdots < j_{n-m}\) is the rearrangement of the complement of \(i_1 < \cdots < i_m\) in the set \(\{1, \ldots, n\}\) in increasing order and \(\text{sgn}(\mathcal{A}, \mathcal{P})\) is the sign of the permutation \(i_1, \ldots, i_m, j_1, \ldots, j_{n-m}\). If \(M\) is an oriented Riemannian manifold, then the global inner product \(\langle \eta, \zeta \rangle\) of \(\eta\) and \(\zeta\) is defined by
\[ \langle \eta, \zeta \rangle = \int_M \eta \wedge *\zeta. \]
and the global norm of $\eta$ is defined by $\|\eta\|^2 = \langle \eta, \eta \rangle$ and note that $\|\eta\|^2 \leq \infty$. The natural adjoint operator of the exterior derivative $d : A_m(M) \to A_{m+1}(M)$ is called the co-differential operator $\delta : A_m(M) \to A_{m-1}(M)$ and is defined by

$$\delta = (-1)^m *^{-1} d* = (-1)^{n(m+1)+1} * d*.$$ 

For a 1-form $\eta$, we obtain

$$(d\eta)_{ij} = \nabla_i \eta_j - \nabla_j \eta_i$$ 

and

$$(\delta \eta) = -\nabla^i \eta_i,$$

where $\nabla^i = g^{ij} \nabla_j$. For more details on co-differential operator and Hodge operator see [13]. Let $A^0_m(M)$ be the subspace of $A_m(M)$ containing all $m$-forms in $M$ with compact support and the completion of $A^0_m(M)$ with respect to the global inner product be $L^2_m(M)$.

However, throughout the paper we have used the same notation $Y$ both as vector field and its dual 1-form. Hence both the vector field $Y$ and its dual can respectively be expressed locally as $Y = Y^i \partial_i$ and $Y^\flat = Y^i dx^i = g_{ij} Y^j dx^i$, with respect to the Riemannian metric $g$.

2. Main results

Definition 2.1. [23] If $Y^\flat$ belongs to $L^2(M) \cap A_1(M)$, then $Y$ is said to be a vector field with finite global norm.

For a fixed point $z \in M$, $\tau(x)$ is the distance from $z$ to $x$, for each $x \in M$ and $B_\tau$ is the open ball with center at $z$ and radius $\tau > 0$. Then for some constant $K > 0$, the Lipschitz continuous function $\eta_\tau$ [22], satisfies

$$0 \leq \eta_\tau(x) \leq 1 \quad \forall x \in M,$$

$$|d\eta_\tau| \leq \frac{K}{\tau} \quad \text{almost everywhere on } M,$$

$$\eta_\tau(x) = 1 \quad \forall x \in B_\tau,$$

$$\text{supp } \eta_\tau \subset B_{2\tau}.$$ 

Then taking the limit as $\tau \to \infty$, we get $\lim_{\tau \to \infty} \eta_\tau = 1$.

Now, we have

$$g(dY, dY) = \frac{1}{2} \{(\nabla_i Y_j - \nabla_j Y_i)(\nabla^i Y^j - \nabla^j Y^i)\}$$

$$= \frac{1}{2} \{4(\nabla_i Y_j)(\nabla^i Y^j) - 4(R - \rho)g_{ij}(\nabla^i Y^j) - 2\beta Y_j(Y^i Y^j + \nabla^j Y^i)\}$$

$$= \frac{1}{2} \{4(\nabla_i Y_j)(\nabla^i Y^j) - 4(R - \rho)((R - \rho)n + \beta|Y|^2)

-4\beta((R - \rho)|Y|^2 + \beta|Y|^4))$$

$$= 4g(\nabla Y, \nabla Y) - 2n(R - \rho)^2 - 2s. \quad (4)$$
where \( s = \{2\beta(R - \rho) + \beta^2|Y|^2\}|Y|^2 \). We choose \( \beta, R \) and \( \rho \) in such a way that \( s \geq 0 \). Moreover,

\[
g(\delta Y, \delta Y) = (\nabla^i Y_i)(\nabla^j Y_j) \\
= g((R - \rho)n + \beta|Y|^2, (R - \rho)n + \beta|Y|^2) \\
= \left|(R - \rho)n + \beta|Y|^2\right|^2.
\]

(5)

Then from the above two relations (4) and (5), we obtain the following lemma:

**Lemma 2.1.** In a quasi Yamabe soliton the potential vector field \( Y \) satisfies the following relations:

\[
4\left\| \eta_\tau \nabla Y \right\|^2_{g_{2r}} - 2n\left\| \eta_\tau (R - \rho) \right\|^2_{g_{2r}} - 2\left\| \eta_\tau \sqrt{s} \right\|^2 = \left\| \eta_\tau dY \right\|^2_{g_{2r}},
\]

(6)

and

\[
\left\| \eta_\tau ((R - \rho)n + \beta|Y|^2) \right\|^2_{g_{2r}} = \left\| \eta_\tau \delta Y \right\|^2_{g_{2r}}.
\]

(7)

Combining Lemmas 2 and 3 of [23], we have

**Lemma 2.2.** [23] For any smooth 1-form \( Y \) in \( M \), we have

\[
4\langle \eta_\tau, d\eta_\tau \otimes Y, \nabla Y \rangle_{g_{2r}} + \langle \eta_\tau, \nabla^2 Y, \eta_\tau Y \rangle_{g_{2r}} + 2\langle \eta_\tau, \nabla Y, \eta_\tau \nabla Y \rangle_{g_{2r}} = 0.
\]

(8)

\[
\langle \eta_\tau, \nabla^2 Y, \eta_\tau Y \rangle_{g_{2r}} + \langle \eta_\tau, dY, \eta_\tau dY \rangle_{g_{2r}} + 2\langle \eta_\tau, dY, d\eta_\tau \wedge Y \rangle_{g_{2r}} + \langle \eta_\tau, \delta Y, \eta_\tau \delta Y \rangle_{g_{2r}} - 2\langle \eta_\tau, \delta Y, \ast(d\eta_\tau \wedge \ast Y) \rangle_{g_{2r}} = \langle \eta_\tau, Y, \eta_\tau Y \rangle_{g_{2r}},
\]

(9)

where \( (\nabla Y)_{ij} = \nabla_i Y_j \), \( (\nabla^2 Y)_i = \nabla^j \nabla_j Y_i \) and the Ricci transformation on \( A_1(M) \) is \( (\mathcal{R} Y)_i = R^j_i Y_j \).

**Lemma 2.3.** [1] For any smooth \( m \)-form \( Y \) in \( M \), there exists a \( \tau \) independent positive constant \( C \) satisfying

\[
\frac{C}{\tau^2} \left\| Y \right\|^2_{g_{2r}} \geq \left\| d\eta_\tau \otimes Y \right\|^2_{g_{2r}},
\]

\[
\frac{C}{\tau^2} \left\| Y \right\|^2_{g_{2r}} \geq \left\| d\eta_\tau \wedge Y \right\|^2_{g_{2r}},
\]

\[
\frac{C}{\tau^2} \left\| Y \right\|^2_{g_{2r}} \geq \left\| d\eta_\tau \wedge \ast Y \right\|^2_{g_{2r}}.
\]

Using Lemmas 2.1 and 2.3, we have

\[
\left| 2\langle \eta_\tau, dY, d\eta_\tau \wedge Y \rangle_{g_{2r}} \right| \leq 2\left\| \eta_\tau dY \right\|_{g_{2r}} \left\| d\eta_\tau \wedge Y \right\|_{g_{2r}} \leq \frac{1}{4} \left\| \eta_\tau dY \right\|^2_{g_{2r}} + 4\left\| d\eta_\tau \wedge Y \right\|^2_{g_{2r}} \leq \left\| \eta_\tau \nabla Y \right\|^2_{g_{2r}} - \frac{n}{2} \left\| \eta_\tau (R - \rho) \right\|^2_{g_{2r}} - \frac{1}{2} \left\| \eta_\tau \sqrt{s} \right\|^2_{g_{2r}} + \frac{4C}{\tau^2} \left\| Y \right\|^2_{g_{2r}}
\]

(10)
and

\[ |2\langle \eta, \delta Y, *(d\eta \wedge *Y) \rangle_{g_{2r}}| \leq 2\|\eta, \delta Y\|_{g_{2r}} \|d\eta \wedge *Y\|_{g_{2r}} \leq \frac{1}{5}\|\eta, \delta Y\|^2_{g_{2r}} + 5\|d\eta \wedge *Y\|^2_{g_{2r}} \leq \frac{1}{5}\|\eta, ((R - \rho)n + \beta|Y|^2)\|^2_{g_{2r}} + \frac{5C}{\tau^2}\|Y\|^2_{g_{2r}}. \] (11)

Thus using Lemma 2.2, we calculate:

\[ \langle \eta, \mathcal{R}Y, \eta, Y \rangle_{g_{2r}} = -4\langle \eta, d\eta \otimes Y, \nabla Y \rangle_{g_{2r}} - 2\langle \eta, \nabla Y, \eta, \nabla Y \rangle_{g_{2r}} + \langle \eta, dY, \eta, dY \rangle_{g_{2r}} + \langle \eta, \delta Y, \eta, \delta Y \rangle_{g_{2r}} - 2\langle \eta, \delta Y, *(d\eta \wedge *Y) \rangle_{g_{2r}} \geq -\frac{1}{2}\|\eta, \nabla Y\|^2_{g_{2r}} - \frac{8C}{\tau^2}\|Y\|^2_{g_{2r}} - 2\|\eta, \nabla Y\|^2_{g_{2r}} + 4\|\eta, \nabla Y\|^2_{g_{2r}} - 2n\|\eta, (R - \rho)n + \beta|Y|^2\|^2_{g_{2r}} + \frac{n}{2}\|\eta, (R - \rho)n + \beta|Y|^2\|^2_{g_{2r}} - \frac{4C}{\tau^2}\|Y\|^2_{g_{2r}} - \frac{1}{2}\|\eta, \nabla Y\|^2_{g_{2r}} - \frac{3n}{2}\|\eta, (R - \rho)n + \beta|Y|^2\|^2_{g_{2r}} - \frac{3}{2}\|\eta, \nabla Y\|^2_{g_{2r}} + \frac{4}{5}\|\eta, (R - \rho)n + \beta|Y|^2\|^2_{g_{2r}}. \] (12)

Again using Lemma 2.2, we calculate:

\[ \langle \eta, \mathcal{R}Y, \eta, Y \rangle_{g_{2r}} \leq \frac{1}{2}\|\eta, \nabla Y\|^2_{g_{2r}} + \frac{8C}{\tau^2}\|Y\|^2_{g_{2r}} - 2\|\eta, \nabla Y\|^2_{g_{2r}} + 4\|\eta, \nabla Y\|^2_{g_{2r}} - 2n\|\eta, (R - \rho)n + \beta|Y|^2\|^2_{g_{2r}} + \frac{n}{2}\|\eta, (R - \rho)n + \beta|Y|^2\|^2_{g_{2r}} + \frac{4C}{\tau^2}\|Y\|^2_{g_{2r}} + \frac{1}{2}\|\eta, \nabla Y\|^2_{g_{2r}} + \frac{5C}{\tau^2}\|Y\|^2_{g_{2r}} - \frac{5n}{2}\|\eta, (R - \rho)n + \beta|Y|^2\|^2_{g_{2r}} - \frac{5}{2}\|\eta, \nabla Y\|^2_{g_{2r}} + \frac{6}{5}\|\eta, (R - \rho)n + \beta|Y|^2\|^2_{g_{2r}}. \] (13)

If \( Y \) is a vector field of finite global norm, then (12) and (13) reduces to the following inequalities

\[ \frac{1}{2}\|\nabla Y\|^2 - \frac{3n}{2}\|R - \lambda\|^2 - \frac{3}{2}\|\sqrt{s}\|^2 + \frac{4}{5}\| (R - \rho)n + \beta|Y|^2 \|^2 \leq \limsup \tau \to \infty \langle \eta, \mathcal{R}Y, \eta, Y \rangle_{g_{2r}}, \] (14)

and

\[ \frac{7}{2}\|\nabla Y\|^2 - \frac{5n}{2}\|R - \rho\|^2 - \frac{5}{2}\|\sqrt{s}\|^2 + \frac{6}{5}\| (R - \rho)n + \beta|Y|^2 \|^2 \geq \liminf \tau \to \infty \langle \eta, \mathcal{R}Y, \eta, Y \rangle_{g_{2r}}. \] (15)
respectively. Thus from (15), we obtain the following theorem:

**Theorem 2.4.** Let \((M, g, Y, \beta)\) be a quasi Yamabe soliton. If \(Y\) is a vector field of finite global norm, then the following inequality holds:

\[
\liminf_{\tau \to \infty} \langle \eta_\tau \mathcal{R}Y, \eta_\tau Y \rangle_{B_{2\tau}} \leq \frac{7}{2} \| \nabla Y \|^2 + \frac{6}{5} \| (R - \rho) n + \beta |Y|^2 \|^2.
\]

**Theorem 2.5.** Let \((M, g, Y, \beta)\) be an \(n \geq 2\)-dimensional quasi Yamabe soliton with \(\beta(R - \rho) \geq 0\) and Ricci curvature non-positive. If \(Y\) is a vector field of finite global norm, then the following relation holds:

\[
\frac{5}{7} \| \nabla Y \|^2 \leq \int_M \beta^2 |Y|^4.
\]

**Proof.** If the Ricci curvature is non-positive, then

\[
\limsup_{\tau \to \infty} \langle \eta_\tau \mathcal{R}Y, \eta_\tau Y \rangle_{B_{2\tau}} \leq 0.
\]

Hence, (14) yields

\[
0 \geq \frac{1}{2} \| \nabla Y \|^2 - \frac{3n}{2} \| R - \rho \|^2 - \frac{3}{2} \| \sqrt{s} \|^2 + \frac{4}{5} \| (R - \rho) n + \beta |Y|^2 \|^2 \leq 0.
\]

The above inequality together with the definition of global norm implies

\[
0 \geq \frac{1}{2} \| \nabla Y \|^2 - \frac{3n}{2} \int_M (R - \rho)^2 - \frac{3}{2} \int_M \left\{ 2\beta (R - \rho) + \beta^2 |Y|^2 \right\} |Y|^2
\]

\[
+ \frac{4}{5} \int_M \left\{ (R - \rho) n + \beta |Y|^2 \right\}^2
\]

\[
\geq \frac{1}{2} \| \nabla Y \|^2 + \int_M \left\{ - \frac{3n}{2} (R - \rho)^2 - 3\beta (R - \rho)|Y|^2 - \frac{3}{2} \beta^2 |Y|^4
\]

\[
+ \frac{4}{5} (R - \rho)^2 n^2 + \frac{8}{5} (R - \rho) n \beta |Y|^2 + \frac{4}{5} \beta^2 |Y|^4 \right\}
\]

\[
\geq \frac{1}{2} \| \nabla Y \|^2 + \int_M \left\{ (R - \rho)^2 \left( \frac{4n^2}{5} - \frac{3n}{2} \right) + \beta (R - \rho) \left( \frac{8n}{5} - 3 \right) |Y|^2
\]

\[
- \frac{3}{2} \beta^2 |Y|^4 + \frac{4}{5} \beta^2 |Y|^4 \right\}.
\]

Using \(n \geq 2\) and \(\beta(R - \rho) \geq 0\) in the above inequality, we obtain

\[
0 \geq \frac{1}{2} \| \nabla Y \|^2 - \frac{7}{10} \int_M \beta^2 |Y|^4.
\]

This follows the desired result. \(\square\)

**Theorem 2.6.** If \((M, g, Y, \beta)\) is a compact quasi Yamabe soliton, then

\[
\int_M \{(n + 2)|Y|^2 \text{div}(Y) + 2(n - 1)\beta |Y|^4 \} = 0.
\]
Proof. Taking trace of (1) we have
\[ \text{div}(Y) = (R - \rho)n + \beta|Y|^2. \] (18)
Now from (1) and (18) we get
\[ \frac{1}{2} (\mathcal{L}_Y g)(Y, Y) = \frac{1}{n} (\text{div}(Y) - \beta|Y|^2)g(Y, Y) + \beta Y_i Y_j(Y, Y), \] (19)
which gives
\[ ng(\nabla_Y Y, Y) = (\text{div}(Y) - \beta|Y|^2)|Y|^2 + n\beta|Y|^4. \] (20)
Now
\[ \text{div}(|Y|^2 Y) = |Y|^2 \text{div}(Y) + 2g(\nabla_Y Y, Y) \]
\[ = |Y|^2 \text{div}(Y) + \frac{2}{n} \left\{ (\text{div}(Y) - \beta|Y|^2)|Y|^2 + n\beta|Y|^4 \right\} \]
\[ = \frac{n + 2}{n} |Y|^2 \text{div}(Y) + \frac{2(n - 1)}{n} \beta|Y|^4, \] (21)
which follows the proof of the theorem. □

Theorem 2.7. Let \((M, g, Y, \beta)\) be a completely non-compact non-trivial quasi Yamabe soliton whose volume is finite. If \(Y\) is of finite global norm and one of the following conditions
(i) \(\beta \geq 0\) and \(R \geq \rho\),
(ii) \(\beta \leq 0\) and \(R \leq \rho\), holds, then the scalar curvature \(R\) must be constant and \((M, g, Y, \beta)\) becomes a Yamabe soliton.

Proof. For any positive \(\tau\), we obtain
\[ \frac{1}{\tau} \int_{\mathcal{B}_{2\tau}} |Y| dv \leq \left( \int_{\mathcal{B}_{2\tau}} \langle Y, Y \rangle dv \right)^{1/2} \left( \int_{\mathcal{B}_{2\tau}} \left( \frac{1}{\tau} \right)^2 dv \right)^{1/2} \]
\[ \leq \|Y\|_{\mathcal{B}_{2\tau}} \left( \frac{1}{\tau} \right)^{1/2} \left( V(M) \right)^{1/2}, \]
where \(V(M)\) represents the volume of \(M\). Therefore, we get
\[ \liminf_{\tau \to \infty} \frac{1}{\tau} \int_{\mathcal{B}_{2\tau}} |Y| dv = 0. \]
Again, we have
\[ C \frac{1}{\tau} \int_{\mathcal{B}_{2\tau}} |Y| dv \geq \left| \int_{\mathcal{B}_{2\tau}} \eta_\tau \text{div} Y dv \right|. \]
Now with the help of quasi Yamabe soliton equation (1) we get
\[ \int_{\mathcal{B}_{2\tau}} \{ (R - \rho)n + \beta|Y|^2 \} dv = 0, \]
The above relation together with condition either (i) or (ii) gives \(R = \rho\) and \(\beta = 0\). This completes the proof. □
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Declarations

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Absos Ali Shaikh and Prosenjit Mandal
Department of Mathematics
University of Burdwan, Golapbag
Burdwan
West Bengal 713104
India
e-mail: aask2003@yahoo.co.in ; aashaikh@math.buruniv.ac.in

Prosenjit Mandal
e-mail: prosenjitmandal235@gmail.com

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