Realizing topological corner states in two-dimensional Su-Schrieffer-Heeger model with next-nearest neighbor couplings

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Abstract. In the present work, we consider a two-dimensional Su-Schrieffer-Heeger model with alternating positive tunneling couplings between the neighboring sites. We show that introducing next-nearest neighbor couplings results in the emergence of topological corner states for some values of the corresponding tunneling coupling. Our work resolves the contradiction that arose when such corner states absent in the original two-dimensional Su-Schrieffer-Heeger model were observed experimentally in its photonic realization based on the array of cylindrical microwave resonators.

1. Introduction
Photonic topological states are actively studied currently due to their robustness against geometrical imperfections of the structure which opens promising opportunities for realizing highly efficient nanophotonic devices [1]. Even greater flexibility in the design of topological structures is enabled by the so-called higher-order topological states which can exist in the form of zero-dimensional excitations localized at the corners of a two-dimensional system. Recently, such types of topological states have been realized in photonic systems [2, 3, 4].

A paradigmatic example of one-dimensional topological model featuring zero-energy topological edge states is the Su-Schrieffer-Heeger model which is the one-dimensional array of sites connected by alternating tunneling couplings between the nearest neighbors of magnitude \(J\) and \(K\) [5]. However, its direct generalization to the two-dimensional case with positive tunneling couplings \(J > 0, K > 0\) does not give rise to the localized corner states due to the absence of band gap at zero energy [2]. On the other hand, zero-dimensional corner states have been recently observed in the electromagnetic realization of a two-dimensional Su-Schrieffer-Heeger model in the form of a two-dimensional array of cylindrical microwave resonators [6].

To resolve this contradiction, we consider a two-dimensional Su-Schrieffer-Heeger model with alternating tunneling couplings \(J > 0, K > 0\) between the nearest neighbors and additional couplings \(M > 0\) between the sites in the second coordination sphere, figure 1. Such tunneling couplings, as well as tunneling couplings between even more distant neighbors, should inevitably arise in photonic systems with long-range electromagnetic interactions.
Figure 1. Geometry of the considered tight-binding model. Positive tunneling couplings \( J, K \) and \( M \) are shown by solid blue, solid green, and dashed red lines, respectively.

2. Tight-binding model

The energy spectrum of the system and associated eigenmodes are found from the eigenvalue problem having the form of a linear system of tight-binding equations

\[
-\mathcal{J}(\beta_{m-1,n} + \beta_{m,n-1}) - \mathcal{K}(\beta_{m+1,n} + \beta_{m,n+1}) - \mathcal{M}(\beta_{m+1,n+1}) = \varepsilon \beta_{m,n},
\]

\[
-\mathcal{J}(\beta_{m+1,n} + \beta_{m,n+1}) - \mathcal{K}(\beta_{m-1,n} + \beta_{m,n-1}) - \mathcal{M}(\beta_{m-1,n-1}) = \varepsilon \beta_{m,n},
\]

\[
-\mathcal{J}(\beta_{m+1,n} + \beta_{m,n+1}) - \mathcal{K}(\beta_{m-1,n} + \beta_{m,n-1}) - \mathcal{M}(\beta_{m-1,n+1}) = \varepsilon \beta_{m,n},
\]

where \( \varepsilon \) is the eigenmode energy and \( \beta_{m,n} \) are wave function coefficients characterizing probability amplitude at the given site \((m,n)\).

Next, we study the system Eq. (1) numerically for fixed values of couplings \( J = 1, K = 3 \) and different next-nearest neighbors couplings \( M \) for the system of the size \( 15 \times 15 \) sites, figure 2. At \( M = 0 \), there is no band gap in the energy spectrum, as seen in figure 2a, and zero-dimensional corner states are absent. However, eigenmode maps demonstrate the presence of one-dimensional edge states (figure 2c) along with bulk excitations (figure 2b,d). At \( M = 3 \), a band gap opens in the energy spectrum and a single spectrally isolated state emerges, marked with a red dot in figure 2e. As seen from the corresponding eigenmode in figure 2h, this state indeed corresponds to a localized zero-dimensional state.

3. Realization with electric circuit

To further support theoretical observations, the proposed model can be realized experimentally in the form of a resonant electric circuit \([7, 8]\). One can show that the obtained linear system of tight-binding equations (1) can be mapped to Kirchhoff’s rules describing currents in the electric circuit consisting of capacitors \( C_J, C_K \) and \( C_M \) which describe tunneling couplings between nearby sites, and grounding inductors \( L \) placed at every site of the circuit, figure 3a. Indeed, for a single site first Kirchhoffs rule reads

\[
-\sigma C_J(\varphi_{m-1,n} + \varphi_{m,n-1}) - \sigma C_M(\varphi_{m+1,n+1}) - \sigma C_K(\varphi_{m+1,n} + \varphi_{m,n+1}) = (-2\sigma C_J - 2\sigma C_K - \sigma C_L - \sigma C_M)\varphi_{m,n},
\]
Figure 2. (a): Spectrum of the eigenvalues $\varepsilon_j$ for the model of size $15 \times 15$ sites with tunneling couplings $J = 1$, $K = 3$, $M = 0$. (b)-(d): Eigenmode profiles for different energies corresponding to bulk excitations (b,d) and edge states (c). (e): The same as (a), but for $J = 1$, $K = 3$, $M = 3$. A band gap opening and the emergence of the spectrally isolated state inside the band gap are observed. (f)-(h): Eigenmode profiles corresponding to the bulk (f), one-dimensional edge (g) and zero-dimensional corner (h) states.

The eigenvalues $\varepsilon_j$ are related to the values of circuit elements as

$$J = 1, \quad K = \frac{C_K}{C_J}, \quad M = \frac{C_M}{C_J},$$

and the eigenvalue $\varepsilon$ is related to the mode frequency as

$$\varepsilon = \frac{\omega_0^2}{\omega^2} \left( 2 + 2 \frac{C_K}{C_J} + \frac{C_M}{C_J} \right), \quad \omega_0 = \frac{1}{\sqrt{L C_J}}.$$

The proposed circuit was numerically simulated with the help of Multisim software package. The values of circuit elements in the simulation are $L = 22 \mu\text{H}$, $C_J = 1 \mu\text{F}$, $C_K = 3.3 \mu\text{F}$, and $C_M = 3.3 \mu\text{F}$. Realistic models of components available in the software libraries were used in the simulation in order to take into account the effects of Ohmic losses and dispersion.

In order to study the resonance spectrum of the model, we consider frequency-dependent total impedances between the sites $(1, 1)$ and $(5, 5)$ and the ground, figure 3b. For the site $(5, 5)$ marked with a red dot in figure 3a, a single resonant peak is observed at the frequency $10.3$ kHz that indicates the presence of a corner state.

To study resonances in the circuit in detail, we simulate the measurement of impedance between all nodes of the circuit and the ground at characteristic frequencies of corresponding eigenmodes of the tight-binding model and plot the results in the form of maps, see figure 3d-h. A good agreement between the numerical calculations for the tight-binding model and the simulation results is observed, and characteristic states are clearly distinguishable even in the small-size $5 \times 5$ circuit.
Figure 3. (a): Topolectrical circuit geometry. Capacitors $C_J$, $C_K$ and $C_M$ corresponding to tunneling couplings $J$, $K$ and $M$ are shown with blue, green and red lines, respectively. (b): Numerically simulated impedance spectra for the sites $(1, 1)$ and $(5, 5)$. (c)-(e): Eigenmode profiles for the bulk (c), edge (d) and corner (e) states in the tight-binding model with $M = 3$ and the size $5 \times 5$ sites. (f)-(h): Impedance maps for the electrical circuit of the size $5 \times 5$ sites corresponding to the same modes as (c)-(e) and simulated for the excitation of circuit sites at the given frequency.

4. Topological properties
Topological states in the considered structure are protected by $C_4$ crystalline symmetry and therefore the relevant topological invariants can be extracted directly from the eigenvalue of
rotation operator by the angle $\pi/2$ [9]. Rotation by $\pi/2$ for a four-site unit cell is defined by the operator:

$$ R_4 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{pmatrix}, $$

(5)

which has the eigenvalues $e^{2\pi i (p-1)/4}$ for $p = 1, 2, 3, 4$.

The topological invariants are determined from the behavior of wave function under $R_4$ and $R_2^4$ transformations as follows:

$$ \chi^{(4)} = \begin{pmatrix}
\#X_1^{(2)} - \#I_1^{(2)} \\
\#M_1^{(4)} - \#I_1^{(4)} \\
\#M_2^{(4)} - \#I_2^{(4)}
\end{pmatrix}, $$

(6)

where upper indices (2) and (4) correspond to the rotation by the angles $\pi$ and $\pi/2$, respectively, lower indices denote the respective eigenvalues of the rotation operator, and $M, X, \Gamma$ are the high-symmetry points in the first Brillouin zone which have coordinates $(\pi, \pi), (\pi, 0), (0, 0)$, respectively. Symbol $\#$ in front of $X, M$ and $\Gamma$ means the number of energy bands below the (in-gap) Fermi level with symmetry operator eigenvalues equal to the lower indices. Taking a rectangular unit cell such that $J$ are the intra-cell links, we find out that the vector $\chi^{(4)}$ is equal to

$$ \chi^{(4)} = \begin{pmatrix}
-1 \\
-1 \\
0
\end{pmatrix}, $$

(7)

in the topological phase. This indicates the presence of corner states with associated corner charge $Q_{\text{corner}} = \frac{1}{4}$ and the polarization $P = (\frac{1}{2}, \frac{1}{2})$ [9] for the respective choice of the unit cell.

5. Conclusion

In the present paper we have shown that the introduction of long-range tunneling couplings leads to the emergence of topological edge states even in the simplest case when only the closest among next-nearest neighbors are connected by additional couplings. In particular, we demonstrate that the two-dimensional Su-Schrieffer-Heeger model can possess symmetry-protected topological corner states even in the case of purely positive tunneling couplings when the second coordination sphere is taken into account.

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References

[1] Ozawa T et al 2019 Rev. Mod. Phys. 91 015006
[2] Mittal S et al 2019 Nat. Photon. 13 692-696
[3] El Hassan A et al 2019 Nat. Photon. 13 697-700
[4] Li M et al 2020 Nat. Photon. 14 89-94
[5] Su W P, Schrieffer J R, and Heeger A J 1979 Phys. Rev. Lett. 42 1698
[6] Xie B-Y et al 2019 Phys. Rev. Lett. 122 233903
[7] Imhof S et al 2018 Nat. Phys. 14 925-929
[8] Olekhn N A et al 2020 Nat. Commun. 11 1436
[9] Benalcazar W A, Bernevig B A, and Hughes T L 2019 Phys. Rev. B 99 245151