Comparative analysis for three models of hysteretic water retention capacity using data on silty soil

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Abstract. The most important of the hydrophysical properties is the water-retention capacity of the soil. This property is formulated as a relationship between the volumetric moisture of the soil and the capillary pressure of soil moisture. Three hysteresis models of soil water-retention capacity are presented. The parameters were identified by point approximation of experimental data (from an authoritative literary source) on the main (boundary) drying and wetting branches of the soil «2003 Silt of Nave-Zaar» using the «SoilHysteresis-v.1.0» computer program developed by the authors. Applying the condition of equality of the values of the exponential parameter for the branches of draining and moistening eliminates the undesirable (methodical) «pump effect». A comparative analysis of hysteresis models based on the identification of significant differences between the errors of the point approximation of the experimental data by the Williams-Kloot test. The advantages of models in which an additional additive parameter is used are revealed. The use of physically justified models of hysteresis of the water-retention capacity of the soil has prospects in the development and application of resource-saving technologies in precision reclamation agriculture.

1. Introduction

For the substantiation of the decisions on the design and operation of hydraulic structures [1-6], on the development of precision farming technologies and the use of agricultural chemicals, reliable information on the hydrological and agrochemical conditions of the land use territory is needed [7, 8]. Such information includes data on the climate, the depth of the aquifers of the parent rock, as well as on the agrochemical composition and hydrophysical properties of the soil [9, 10]. The most important of the hydrophysical properties is the water-retention capacity of the soil. This property is formulated as a relationship between the volumetric water content of the soil \( \theta \) [cm
\(^{-3}\)] and capillary pressure of soil moisture \( \psi \) [cm H
\(_2\)O] [11, 12]. Along with this formulation, a dependency is used to functionally represent the water-retention capacity of the soil. \( S_\psi (\psi) \), where \( S_\psi = (\theta - \theta_\varepsilon) / (\theta_s - \theta_\varepsilon) \) – effective moisture saturation of soil; \( \theta_s \) [cm
\(^{-3}\)] – volumetric water content at full water saturation of the...
soil; \( \theta_i \) [cm\(^3\)/cm\(^3\)] – volumetric water content corresponding to the minimum specific volume of moisture as a fluid in the soil. For the water-retention capacity of the soil, a hysteresis phenomenon is characteristic, due to which the branches of drying and wetting the dependences \( \theta(\psi) \) (or \( S_e(\psi) \)) do not coincide. Direct measurements of indicators characterizing the hydrophysical properties of the soil are very laborious. This is especially true for measuring scanning branches of hysteresis. At the same time, it is precisely the data on the scanning branches filling the hysteresis loop and describing the sequence of soil moisture conditions when the soil is moistened that are of great practical interest in accurate reclamation farming. Such data make it possible to calculate the precision rates for irrigation by the difference between the soil moisture value corresponding to the effective minimum soil moisture capacity and the value of the pre-irrigation soil moisture. Note that due to the lack of such data, the irrigation rates are calculated according to the «traditional» formula according to the difference between the moisture value corresponding to the lowest field capacity on the main drying branch of the water-retention capacity and the value of the pre-irrigation soil moisture. Therefore, the «traditional» norm is overstated. This circumstance is well known to land reclamation practitioners. However, it is not possible to solve the problem in the absence of data on the scanning wetting branches. Moreover, it is not known in advance which scanning wetting branches will be required in the upcoming growing season. In this case, the use of a mathematical model of hysteresis of the water-retention capacity of the soil is uncontested [13].

The purpose of the study was to conduct a comparative analysis of three hysteresis models of soil water-retention capacity. A search is made for the solution of the following tasks:

- description of the water-retention capacity of the soil, taking into account hysteresis in the form of mathematical models;
- identification of model parameters by point approximation of data on the main (boundary) branches of the soil «2003 Silt of Nave-Yaar» hysteresis loop from an authoritative literary source [14];
- comparison and selection of the best physically sound model;
- study of the influence of the condition of equality of values of the exponential parameter of models on the prevention of undesirable artificial (methodological) «pump effect».

2. Materials and methods

Most studies are the development of two well-known models of hysteresis of the water-retention capacity of the soil [15-19]. The first of them is the model of Scott et al. [20], the second is the model of Cool and Parker [21]. The first model is based on the function \( S_e(\psi) \), proposed by Haverkamp et al. [22]; the second model is based on the function \( S_e(\psi) \), proposed by Van Genuchten [23]. In this study, advanced versions of functions are considered as the basis for modeling hysteresis of soil water-retention capacity. \( S_e(\psi) \) of Kosugi [24] and Haverkamp et. al [22]. In advanced versions of these functions, the authors of this article use an additional additive parameter \( \psi_e \) [25]. For the drying branches of \( S_e(\psi) \) this parameter has the meaning of «bubbling pressure», and for the wetting branches of \( S_e(\psi) \) parameter \( \psi_e \) makes sense of «pressure of water entry» into the soil during the displacement of air trapped in dead ends, at the stage of complete saturation of the soil with water.

For hysteresis models with improved functions of Kosugi and Haverkamp et al, the notation used here are Hys-SKT and Hys-SHT, respectively. In case of \( \psi_e = 0 \) these model have the designation Hys-SKT\(_0\) and Hys-SHT\(_0\), respectively. For the model of Cool and Parker with the Van Genuchten function \( S_e(\psi) \) designation Hys-KPVG is used (Table 1). In a study conducted by the authors, three hysteresis models are compared, which are based on various functions \( S_e(\psi) \). As a basis for comparison, the accuracy of the point approximation of the data on the main (boundary) branches of drying and wetting of the hysteresis of the water-retention capacity of the soil «2003 Silt of Nave-Yaar» is used [14].
Table 1 uses the following notations: $\text{erfc}(x) = 1 - \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2} dt$ – complementary error function; in function #1 $\alpha$ (cm $H_2O^{-1}$), $n$ and $m$ – empirical parameters ($m=1-1/n$, $n>1$); in functions #2 and #3 $n=0$, $\alpha = -1/(\psi_{e,w})$ (cm $H_2O^{-1}$), $\psi_e$ [cm $H_2O$], $\psi_0$ [cm $H_2O$] $< \psi_e$ – interpreted parameters (for drying branches $\psi_e = \psi_{e,d} < 0$, for wetting branches $\psi_e = \psi_{e,w} > 0$); under $\psi_e = 0$ functions #2 and #3 accordingly reduced to the original model by Kosugi [24], as well as the original model by Haverkamp et al. [22].

### Table 1. Functions $S_e(\psi)$ used in the hysteresis models.

| Hysteresis model          | Model denomination | Model # | $S_e(\psi)$ description       |
|---------------------------|--------------------|---------|-------------------------------|
| Kool and Parker           | Hys-KPVG           | 1       | $S_e = \begin{cases} \frac{1}{1-m}(-\alpha \psi)^m \psi > 0; \\ 1, \quad \psi \leq 0. \end{cases}$ |
| Scott et al. with enhanced S_e(\psi) by Kosugi  | Hys-SKT            | 2       | $S_e = \begin{cases} \frac{1}{2} \text{erfc} \left( \frac{\sqrt{n}}{4} \ln \left(-\alpha (\psi - \psi_p)\right) \right), \psi < \psi_p; \\ 1, \quad \psi \geq \psi_p. \end{cases}$ |
| Scott et al. with enhanced S_e(\psi) by Haverkamp et al. | Hys-SHT            | 3       | $S_e = \begin{cases} \left(1 - \alpha (\psi - \psi_p)\right)^{-1}, \quad \psi < \psi_p; \\ 1, \quad \psi \geq \psi_p. \end{cases}$ |

In the all hysteresis models, turning points are calculated using the algorithm by Scott et al. [20], which is presented in Table 2. Indexes «d» and «w» correspond to the branches of drying and wetting.

### Table 2. Scanning drying and wetting branches of $\theta(\psi)$ by Scott et al.

| Scanning drying branches, starting from the i-th point on the wetting branch. | Scanning wetting branches, starting from the j-th point on the drying branch. |
|-----------------------------------------------------------------------------|-------------------------------------------------------------------------|
| $\theta = \theta_0 + \left(\theta_S - \theta_0 \right)S_{e,d}$;             | $\theta = \theta_0^* + \left(\theta_S - \theta_0^* \right)S_{e,w}$;        |
| $\theta_S = \theta_S$, $\psi_{e,w} \leq \psi_i$, $\psi < \psi_{e,d}$;     | $\theta_j = \theta_j^*$, $\psi_j < \psi_{e,d}$, $\psi_j \leq \psi_{e,w}$; |
| $\theta_0 = \theta_0$, $\psi_{e,w} \leq \psi_i$, $\psi_{e,d} < \psi_{e,w}$;| $\theta_j = \theta_0 - \theta_{S,e,w} \left(\psi_j \right)^n \frac{1}{1-S_{e,w} \left(\psi_j \right)}$,      |
| $\theta = \theta_0$, $\psi_{e,w} \leq \psi_i$, $\psi_{e,d} < \psi_{e,w}$; | $\psi_j < \psi_{e,d}$, $\psi_j \leq \psi_{e,w}$;                      |
| $\psi_{e,d} \leq \psi_i$, $\psi_{e,w} \leq \psi_{e,w}$;                   | $\theta_j = \theta_0$, $\psi_{e,d} \leq \psi_i$, $\psi_{e,w} \leq \psi_{e,w}$; |

### 3. Results and discussions

Table 3 shows the parameters of the compared models, identified by point approximation of experimental data on the main (boundary) branches of drying and wetting of the water-retention capacity of the studied soil using the «SoilHysteresis-v.1.0» computer program developed by the authors [26]. Using the parameters from Table 3, a computational experiment was carried out to study the effect of the condition of equal values of the exponential parameter $n$ for the branches of drying and wetting of hysteresis on the prevention of an undesirable artificial (methodical) «pump effect». This effect consists in the fact that upon oscillation of the quantity $\psi$ in a fixed range, values of $S_e$ can...
reach physically absurd values, and the scanning branch can cross the main branch of the hysteresis loop.

### Table 3. Parameters of the models.

| Models | $n_d \neq n_w$ | $n_d = n_w$ |
|--------|----------------|-------------|
|        | $\theta_r$ | $\theta_s$ | $\psi_{e,d}$ | $\psi_{0,d}$ | $\alpha_d$ | $\psi_{e,w}$ | $\psi_{0,w}$ | $\alpha_w$ | $n_d$ | $n_w$ |
| #1    | 0.3825     | 0.5770     | -13.46     | 0.0743      | -3.93     | 0.2545     | 1.872      | 1.575      |
| #2    | 0.3832     | 0.5770     | -6.22      | -23.11      | 0.05      | -12.49     | 0.0797     | 0.941      | 0.832      |
| #3    | 0.3821     | 0.5770     | -6.56      | -23.20      | 0.0601    | 0.00       | -12.43     | 0.0805     | 0.948      | 0.8434     |
| #2 $\psi_e=0$ | 0.4004 | 0.5770     | -21.96     | 0.0455      | -10.14    | 0.0986     | 1.484      | 1.031      |
| #3 $\psi_e=0$ | 0.3895     | 0.5770     | -24.45     | 0.0409      | -12.12    | 0.0825     | 1.296      | 0.916      |

Table 4 shows the root mean square errors (RMSE) of the point approximation of the data on the main (boundary) branches. Table 5 shows the results of identifying differences between the errors of the compared three hysteresis models with respect to the point approximation of experimental data on the main branches of the hysteresis of the water-retention capacity of the soil for two variants of the computational experiment ($n_d \neq n_w$ and $n_d=n_w$). Table 6 shows the results of identifying differences between the errors of the two versions of the computational experiment. ($n_d \neq n_w$ and $n_d=n_w$) with respect to the point approximation of the experimental data on the water-retention capacity of the soil for each model. The Williams-Kloot test was used to identify significant differences [27, 28].

### Table 4. Errors of the point approximation of data on the main (boundary) branches of hysteresis.

| Hysteresis branches | $n_d \neq n_w$ | $n_d = n_w$ |
|---------------------|----------------|-------------|
|                     | #1 | #2 | #3 | #3 $\psi_e=0$ | #3 $\psi_e=0$ | #1 | #2 | #3 | #3 $\psi_e=0$ | #3 $\psi_e=0$ |
| Main (boundary): identification by 35 points | 0.0047 | 0.0018 | 0.0019 | 0.0044 | 0.0058 | 0.0070 | 0.0023 | 0.0023 | 0.0073 | 0.0074 |

### Table 5. Reliability of differences in errors of the point approximation of data on the main (boundary) branches of hysteresis.

| $y=(y_1+y_2)/2=\lambda(y_1-y_2)$, where $y$ - experimental data |
|---------------------|----------------|----------------|
| $n_d \neq n_w$ | $y_1$: HYS-VG, | $y_1$: HYS-KT, |
| $y_2$: HYS-KT    | $y_1$: HYS-VG, | $y_1$: HYS-KT, |
| $y_2$: HYS-HT    | $y_1$: HYS-HT |
| $\lambda$ | $\lambda_{0.95}$ | $\lambda_{0.975}$ | $\lambda$ | $\lambda_{0.95}$ | $\lambda_{0.975}$ | $\lambda$ | $\lambda_{0.95}$ | $\lambda_{0.975}$ |
| -0.415 | 0.118 | 0.142 | -0.420 | 0.128 | 0.154 | 0.530 | 0.843 | 1.014 |
| $y_2$ more accurate than $y_1$ | $y_1$: HYS-VG, | $y_1$: HYS-KT, |
| $y_2$: HYS-KT, | $y_1$: HYS-HT, |
| $\lambda_{0.95}$ | $\lambda_{0.975}$ | $\lambda$ | $\lambda_{0.95}$ | $\lambda_{0.975}$ | $\lambda$ | $\lambda_{0.95}$ | $\lambda_{0.975}$ |
Table 6. Comparison of the errors in the point approximation of experimental data on the water-retention capacity of the soil for two variants of a computational experiment with parameters $n_d\neq n_w$ and $n_d=n_w$.

| Hysteresis loops | Models |
|------------------|--------|
|                  | Hys-KPVG | Hys-SKT | Hys-SHT | Hys-SKT$_0$ | Hys-SHT$_0$ |
| Main identification by the 35 points | $\lambda_{0.95}$ | $\lambda_{0.95}$ | $\lambda_{0.95}$ | $\lambda_{0.95}$ | $\lambda_{0.95}$ |
| $y_1$ more accurate than $y_2$ | $0.621$ | $0.283$ | $0.340$ | $0.384$ | $0.426$ | $0.376$ | $0.453$ | $0.266$ | $0.398$ | $0.313$ | $0.376$ | $0.975$ |
| $y_2$ more accurate than $y_1$ | $0.519$ | $0.340$ | $0.384$ | $0.426$ | $0.376$ | $0.453$ | $0.266$ | $0.398$ | $0.313$ | $0.376$ | $0.975$ |

In the formulas, $y_1$ and $y_2$ are the approximation of experimental data by the models $Hys-KPVG$, $Hys-SKT$, $Hys-SHT$, $Hys-SKT_0$, and $Hys-SHT_0$, respectively; $y_1$ and $y_2$ are the experimental data values. The errors are calculated by the formula $y - (y_1 + y_2)/2 = \lambda (y_1 - y_2)$, where $y$ is the experimental data.
For condition \( n_d \neq n_w \) point approximation errors are lower due to a larger number of variable parameters. However, this condition is characterized by a «pump effect». It is proposed to use the condition \( n_d = n_w \) to eliminate this effect.

**Figure 1a.** The presence of the «pump effect» of the model Hys-SKT when \( n_d \neq n_w \).

**Figure 1b.** Elimination of the «pump effect» of the model Hys-SKT when \( n_d = n_w \).

As an example, illustrating the presence of the «pump effect» and its elimination, Figure 1a, b shows the results of a computational experiment with a model Hys-SKT. The arrows in Figure 1a, b depict scenarios of variation of the quantity \( \psi \). Figure 1a shows that in the range of low humidity, the scanning drying branch crosses the main wetting branch of the hysteresis of the water-retention capacity of the soil when \( n_d \neq n_w \). Using parameters, identified in terms of \( n_d = n_w \), leads to the prevention of the marked intersection of the main branch of wetting and, therefore, to the elimination of the unwanted «pump effect», which confirms Figure 1b.

Figure 2 a, b, c, d shows the results of two variants of a computational experiment with models Hys-SKT and Hys-SKT\(_0\) by the point approximation of the main branches of the hysteresis of the water-retention capacity of the soil.

**Figure 2a.** Point approximation of data on the main (boundary) branches using the model Hys-SKT (\( n_d \neq n_w \)).

**Figure 2b.** Point approximation of data on the main (boundary) branches using the model Hys-SKT (\( n_d = n_w \)).
4. Conclusion
From the analysis of the research results, the following conclusions can be drawn:

1. Using the Hys-SKT and Hys-SHT models (in both versions \( n_d \neq n_w \) and \( n_d = n_w \)) for point approximation of data on the main (boundary) branches of drying and wetting of the water-retention capacity of the soil is more preferable in comparison with the model Hys-KPVG. For the studied soil, the exponential parameter \( n \) in the models Hys-SKT and Hys-SHT turned out to be less than unity, which is fundamentally impossible in relation to a similar parameter of the model Hys-KPVG. This circumstance explains the fact that the error of the point approximation using the model Hys-KPVG more than three times the error of the other two models.

2. Using an optional additive parameter \( \psi_e \) significantly contributes to a significant reduction in errors. Usage of models Hys-SKT and Hys-SHT preferable than Hys-SKT\(_0\) and Hys-SHT\(_0\), respectively.

3. Unwanted artificial «pump effect» is eliminated when the condition for the equality of the values of the exponential parameter \( n \) for the branches of drying and wetting.

4. Usage of models Hys-SKT and Hys-SHT will be of great practical importance for the design and operation of hydraulic structures (in particular - reclamation systems), as well as for the development of precision farming technologies.

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