Mutating Epidemic Processes Over Time-Varying Networks in Discrete-Time

Sebin Gracy, Philip. E. Paré, Henrik Sandberg and Karl Henrik Johansson

Abstract

This paper studies epidemic processes over discrete-time periodic time-varying networks. Our objective is to find necessary and sufficient conditions for asymptotic convergence to the disease-free equilibrium (DFE). We provide, in terms of the joint spectral radius of a set of matrices, a sufficient condition for global asymptotic stability (GAS) of the DFE. Subsequently, we provide, in terms of the spectral radius of the product of matrices over an interval of size $p$, a necessary and sufficient condition for GAS of the DFE.

I. INTRODUCTION

The spread of infectious diseases in large populations often leads to non-trivial consequences, and therefore has been an active area of interest across multiple communities, since Bernoulli’s seminal paper [1]. A natural question that arises in this context is: under what conditions will the spread of a disease stop? Towards this end, various models have been studied in the literature; prominent among them being susceptible-infected-recovered (SIR), susceptible-exposed-infected-recovered (SEIR) and susceptible-infected-susceptible (SIS) models– the first of which was developed by Kermack and McKendrick in [2]. The present paper deals with SIS models.

In an SIS model, an agent is either in the susceptible to infection state; that is, the agent is healthy yet it might become infected depending on whether or not it is exposed to the disease, or it is in the infected state. A healthy agent could become infected, with some infection rate $\beta$, as a consequence of its neighbors being infected, whereas an infected agent could be cured, with a healing rate $\delta$, returning it to the susceptible state [3]–[5].

SIS models have been studied for a long time, as evidenced by the numerous manuscripts in the literature; for the continuous-time case, see for instance [6]–[8], whereas for the discrete-time case, see for instance [3], [9]–[12]. In this paper, we are interested in the latter. In this context, for time-invariant graphs, Wang et al., in [9], propose a homogeneous (i.e., each agent having the same infection rate) virus model, and provide an epidemic threshold for the model in terms of the maximum eigenvalue – proportional to the ratio of the infection and healing rates– of the matrix representing the graph structure. Peng et al., in [10] provide similar results, but, unlike [9], account for directed and weighted graphs as well. Ahn and Hassibi, in [3], study the DFE and the non-disease free equilibrium (NDFE) of several models, and provide existence, stability and uniqueness conditions for the NDFE. Paré et al.,
in [11], [12] provide a necessary and sufficient condition for GAS of the DFE. However, the models in [3], [9–
[12] cannot represent settings where the agents are mobile, thereby imposing a time-varying topology, for instance
real-world social networks. This motivates one to focus on SIS models with *time-varying* topology.

Unlike the time-invariant case, the literature regarding SIS models with time-varying topology is relatively thin; for continuous-time setting, see [4], [5], [13], [14], whereas for discrete-time setting, see [15], [16]. Building upon
the model in [9], the authors in [15] study a discrete-time model, and provide a sufficient condition for *local*
exponential stability of the DFE. Bokharaie *et al.*, in [16], by improving upon the model in [9], provide a sufficient
condition for a) *local* exponential stability of the DFE and b) *local* exponential *instability* of the DFE. Mason *et al.*, in [13] showed that the same (sufficient) conditions in [16] imply GAS of the DFE for a continuous-time switched system. Paré *et al.*, in [4], provide a sufficient condition for global exponential stability of DFE, however the main results therein depend on the literature concerning slowly time-varying LTI continuous-time systems, and, therefore, cannot be applied directly to our setup. To the best of our knowledge, for discrete-time time-varying SIS models represented by time-varying graphs, the following question remains open: what are necessary and sufficient
conditions for the DFE to be GAS. The present paper aims to settle this gap in the time-periodic sense.

Our first main contribution is to provide, in terms of the joint spectral radius, a sufficient condition for GAS of
DFE (see Theorem 1). The second main contribution is to provide, in terms of the spectral radius of the product
of matrices over an interval of size $p$, a necessary and sufficient condition for GAS of the DFE (see Theorem 2).

A. Paper Outline

The rest of this paper is organized as follows: we conclude the present section by listing the notations that are
used in the sequel. We formally state the problem being investigated in Section II and gather some key background
material in Section III. The main contributions, namely Theorems 1 and 2 are presented in Section IV. Simulations
are presented in Section V. Finally, we summarize the paper, and present some interesting, albeit related, open
problems in Section VI.

Notation

For any positive integer $n$, we have $[n] = \{1, \ldots, n\}$. Given a matrix $A \in \mathbb{R}^{n \times n}$, $a_{ij}$ denotes the entry
corresponding to the $i^{th}$ row and $j^{th}$ column. Given a matrix $A \in \mathbb{R}^{n \times n}$, the spectral radius is $\rho(A)$. A diagonal
matrix is denoted as $\text{diag}(\cdot)$. Given a vector $x \in \mathbb{R}^n$, its transpose is denoted as $x^\top$. $\mathbb{Z}_{\geq 0}$ denotes the set of
non-negative integers. The Euclidean norm is denoted by $\|\cdot\|$. Given a sequence of matrices $A_{k+p}$, $A_{k+p-1}$, $\ldots$, $A_{k+1}$, $A_k$, their product $A_{k+p+1:k}$ is defined as $A_{k+p+1:k} = A_{k+p} \cdot A_{k+p-1} \cdots A_{k+1} \cdot A_k$. Given a matrix $A$, $A \prec 0$ (resp. $A \preceq 0$) denotes that $A$ is negative definite (resp. negative semidefinite). Given a matrix $A$, $A \succ 0$ (resp. $A \succeq 0$) denotes that $A$ is positive definite (resp. positive semidefinite).

II. PROBLEM FORMULATION

Suppose that there is a disease propagating through a time-varying network of $n$ agents, where *time-varying*
is to be understood in the following sense: the set of agents remains fixed, however the interconnections among
them could possibly be time-varying. As a consequence of the possibly time-varying nature of the interconnections among agents, the healing rate and infection rate of each agent might also be time-dependent, that is, mutating. Thus, the continuous-time dynamics of each agent can be represented as follows:

\[ \dot{x}_i(t) = (1 - x_i)\beta_i(t) \sum_{j=1}^{n} a_{ij}(t)x_j - \delta_i(t)x_i(t), \]

where \( i \) represents the \( i^{th} \) agent, \( x_i \) is the infection level, and for every \( t \in \mathbb{R} \), \( \beta_i(t) > 0 \) (resp. \( \delta_i(t) > 0 \)) denotes the infection (resp. healing) rate. The edge weight between any two agents \( i \) and \( j \) at time \( t \), is denoted by \( a_{ij}(t) > 0 \).

The discrete-time version of (1), obtained by applying Euler’s method [17], is the following:

\[ x_i(k + 1) = x_i(k) + h\left((1 - x_i(k))\beta_i(k) \sum_{j=1}^{n} a_{ij}(k)x_j(k) - \delta_i(k)x_i(k)\right), \]

where \( h \) is the sampling parameter.

The spread of diseases in a network can be modeled using a graph: the nodes representing the agents, and the edges representing the interaction among them. More formally, let \( G_k = (V, E_k) \) represent such a network, where \( V = 1, 2, \ldots, n \) is the vertex set, and \( E_k = \{(x_i, x_j) \mid \beta_i(k)a_{ij}(k) \neq 0\} \) is the edge set.

In matrix form, (2) can be rewritten as:

\[ x(k + 1) = x(k) + h((I - X(k))B(k)A(k) - D(k))x(k) \]

where \( X(k) = \text{diag}(x(k)), B(k) = \text{diag}(\beta_i(k)), D(k) = \text{diag}(\delta_i(k)), \) and \( A(k) = [a_{ij}(k)] \), for every \( i, j \in [n] \).

Let us define \( \bar{B}(k) := B(k)A(k), \) with its entries being denoted as \( \bar{\beta}_{ij}(k) \). Then (3) can be rewritten as:

\[ x(k + 1) = x(k) + h((I - X(k))\bar{B}(k) - D(k))x(k). \]

The DFE is defined as the state where \( x_i(k) = 0 \) for all \( i \in [n] \), which, from (4), implies that \( x_i(\kappa) = 0 \) for all \( \kappa \geq k \), for all \( i \in [n] \). We are interested in ensuring that, independent of the initial condition of an agent, i.e., healthy or sick, the system should eventually reach the DFE state. Against this backdrop, we formally state our objective as follows:

- For the system with dynamics as given in (4), what are the necessary and sufficient conditions such that the DFE is unique and GAS?

In the sequel, we restrict ourselves to periodic systems with periodicity \( p \), where \( p \in \mathbb{Z}_+ \). That is, \( B(k + p) = B(k), A(k + p) = A(k), \) and \( D(k + p) = D(k) \) for all \( k \geq 0 \).

The following assumptions are required for our model to be well-defined:

**Assumption 1:** For all \( i \in [n] \), we have \( x_i(0) \in [0, 1] \).

Intuitively, one can think of \( x_i \) as an approximation of the probability of agent \( i \) being infected, and \( 1 - x_i \) represents an approximation of the probability of agent \( i \) being healthy. Therefore, one can quite naturally assume that the initial values of each agent would lie in the interval \([0, 1]\).

**Assumption 2:** We have \( h\delta_i(k) \geq 0 \) and \( \bar{\beta}_{ij}(k) \geq 0 \) for every \( i, j \in [n], k = 0, 1, \ldots, p - 1 \).

**Assumption 3:** For every \( i, j \in [n], h\delta_i(k) \leq 1 \) and \( h \sum_j \bar{\beta}_{ij}(k) \leq 1 \), where \( k = 0, 1, \ldots, p - 1 \).
Lemma 1: For the system in (4), under the conditions of Assumptions 1-3, \( x_i(k) \in [0, 1] \) for all \( i \in [n] \) and \( k \geq 0 \).

The proof is quite similar to that of Lemma 1 in [11], and is, therefore, omitted.

Lemma 1 ensures that, with respect to the system in (4), the set \([0, 1]^n\) is positively invariant, i.e., once a trajectory of (4) enters the set \([0, 1]^n\), it never leaves it.

The following assumption ensures nontrivial virus spread.

Assumption 4: We have \( h \neq 0 \) and, for \( k = 0, 1, \ldots, p - 1 \), there exists \( i \neq j \) such that \( \bar{\beta}_{ij}(k) > 0 \).

Assumption 5 implies that the adjacency matrix \( \bar{B}(k) \), where \( k = 0, 1, \ldots, p - 1 \), is irreducible, i.e., \( \bar{B}(k) \) cannot be permuted to a block upper triangular matrix.

III. Preliminaries

In this section, we recall some of the results that will be required in the sequel. We first familiarize ourselves with the notions of asymptotic stability in discrete-time time-varying systems.

Consider an autonomous system, described as follows:

\[
x(k + 1) = A(k)x(k),
\]

where \( A : \mathbb{Z}_+ \to \mathbb{R}^{n \times n} \) is a uniformly bounded discrete-time matrix-valued function. The state transition matrix, \( \Phi \), for system (5) is defined as:

\[
\Phi_{k:k_0} = A(k - 1)A(k - 2) \ldots A(k_0 + 1)A(k_0).
\]

We say that an equilibrium of (5) is asymptotically stable if it is stable and attractive. An equilibrium is said to be globally asymptotically stable (GAS) if in addition to being asymptotically stable the system converges to that equilibrium for any initial condition. The next proposition recalls a sufficient condition for DFE of (4) to be GAS.

Proposition 1: [18, Section 5.9 Thm. 27] The DFE of system (4) is globally uniformly asymptotically stable if there is a function \( V : \mathbb{Z}_+ \times \mathbb{R}^n \to \mathbb{R} \) such that i) \( V \) is positive definite, decrescent, and radially unbounded, and ii) \( -\Delta V \) is positive definite.

Note that Proposition 1 gives a sufficient condition for global uniform asymptotic stability (GUAS) of an equilibrium point. Since we are interested in GAS, which, in comparison to GUAS, is a weaker notion, the same conditions as in Proposition 1 can be used for our analysis.

The following lemmas are needed for proving the main results in the next section. More precisely, Lemmas 2 and 3 are needed for sufficiency, while Prop. 2 is useful for proving necessity.

Lemma 2: [19, Prop. 1] Suppose that \( M \) is an irreducible nonnegative matrix such that \( \rho(M) < 1 \). Then there exists a diagonal matrix \( P > 0 \) such that \( M^T PM - P < 0 \).

Lemma 3: [11] Lemma 3 Suppose that \( M \) is an irreducible nonnegative matrix such that \( \rho(M) = 1 \). Then there exists a diagonal matrix \( P > 0 \) such that \( M^T PM - P \preceq 0 \).

Proposition 2: [18, Section 5.9 Theorem 42] Consider the autonomous system

\[
x(k + 1) = f(x(k)).
\]
Define $A = \left[ \frac{\partial f}{\partial x} \right]_{x=0}$. If $A$ has at least one eigenvalue with magnitude greater than one, then $x = 0$ is an unstable equilibrium of (6).

IV. MAIN RESULT

In this section, we first present a sufficient, but not necessary, condition for GAS of the DFE. Subsequently, we present a sufficient and necessary condition for GAS of the DFE.

A classic approach towards addressing stability issues in time-varying networks is to take recourse to the notion of joint spectral radius – first introduced by Rota and Strang in [20] – of a set of matrices; see for instance [13], [16]. In the sequel, we investigate the link between the joint spectral radius of an appropriately-defined set of matrices and GAS of the DFE.

Let $\mathcal{R} = \{ R(0), R(1), \ldots, R(p-1) \}$ denote a set of $p$ matrices $R(k)$, where $k = 0, 1, \ldots, p - 1$. As was defined in [16], the joint spectral radius of $\mathcal{R}$, denoted by $\rho(\mathcal{R})$, is:

$$
\rho(\mathcal{R}) = \lim_{p \to \infty} \sup_{\mathcal{R}(k) \in \mathcal{R}, k = 0, 1, \ldots, p - 1} \| R(p-1)R(p-2) \cdots R(0) \|^\frac{1}{p} \tag{7}
$$

In words, $\rho(\mathcal{R})$ is the largest eigenvalue of the product of $p$ matrices in $\mathcal{R}$ amongst all products of $p$ matrices in $\mathcal{R}$.

We define the following:

$$
M(k) := I - hD(k) + h\bar{B}(k)
$$

$$
\hat{M}(k) := I + h((I - X(k))\bar{B}(k) - D(k))
$$

$$
M_{k+p,k} := M(k + p - 1)M(k + p - 2) \cdots M(k).
$$

With the understanding of joint spectral radius in hand, the following result gives a sufficient condition for GAS of the DFE.

**Theorem 1:** Suppose that the system (4) is $p$-periodic and that Assumptions [11][5] hold. If $\rho(M) \leq 1$, then the DFE is globally asymptotically stable.

The proof technique closely mirrors that of an analogous result in [11] (see Theorem 1), and also relies on some of the material in [21] (see Section 3.A).

**Proof:** We use the cyclic reformulation of a linear periodic system; see [22] Section 6.3. Specifically, define

$$
\hat{M} = \begin{bmatrix}
0 & 0 & \ldots & 0 & M(p-1) \\
M(0) & 0 & \ldots & 0 & 0 \\
0 & M(1) & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & M(p-2) & 0 \\
\end{bmatrix}.
$$

Note that

$$
\hat{M}^p = \begin{bmatrix}
M_{p,0} & 0 & \ldots & 0 \\
0 & M_{p+1,1} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & M_{2p-1,p-1} \\
\end{bmatrix}.
$$

(8)
Since $\tilde{M}^p$ is a block diagonal matrix, the eigenvalues of $\tilde{M}^p$ are the eigenvalues of $M_{p:0}, M_{p+1:1}, \ldots, M_{2p-1:p-1}$. By assumption, $\rho(M) \leq 1$. Hence, from (7), it follows that $\rho(M_{p:0}) \leq 1$, $\rho(M_{p+1:1}) \leq 1$, \ldots, $\rho(M_{2p-1:p-1}) \leq 1$, and therefore, $\rho(\tilde{M}^p) \leq 1$. Since the eigenvalues of $\tilde{M}$ are the $p^{th}$-roots of eigenvalues of $\tilde{M}^p$, it follows that $\rho(\tilde{M}) \leq 1$. From hereon we split the proof in two parts, which are detailed as follows: Case 1: Suppose that $\rho(\tilde{M}) < 1$. By Assumptions 2 and 3, $M(k)$ is nonnegative and irreducible, it follows that $\tilde{M}$ is also nonnegative and irreducible. Therefore, from Lemma 2 there exists a diagonal matrix $Q_1 \succ 0$ such that $\tilde{M}^\top Q_1 \tilde{M} - Q_1 < 0$. Let the diagonal blocks of $Q_1$ be denoted by $[Q_1]_k \in \mathbb{R}^{N \times N}$, $k = 1, 2, \ldots, p$. By defining $P_1(k) = [Q_1]_{k+1}$, $k = 0, 1, \ldots, p - 1$, it is immediate that $M(k)^\top P_1(k + 1)M(k) - P_1(k) < 0$ for $k = 0, 1, \ldots, p - 1$. Consider the following Lyapunov function $V_1(x, k) = x^\top P_1(k)x$. Since $Q_1 \succ 0$ and diagonal, each of the blocks along its diagonal must be positive definite. This implies that, for $k = 0, 1, \ldots, p - 1$ and for $x \neq 0$, $x^\top P_1(k)x > 0$, and hence $V_1(x, k) > 0$.

Define $\Delta V_1(x, k) = V_1(x(k + 1)) - V_1(x(k))$. For $x \neq 0$, and for $k = 0, 1, \ldots, p - 1$, one obtains the following:

$$\Delta V_1(x, k) = x^\top \tilde{M}^\top(k)P_1(k + 1)\tilde{M}(k)x - x^\top P_1(k)x$$

$$= x^\top (M^\top(k)P_1(k + 1)M(k) - P_1(k))x - 2hx^\top \hat{B}^\top(k)X(k)P_1(k + 1)M(k)x(k)$$

$$+ h^2x^\top \hat{B}^\top(k)X(k)P_1(k + 1)X(k)\hat{B}(k)x$$

$$< h^2x^\top \hat{B}^\top(k)X(k)P_1(k + 1)X(k)\hat{B}(k)x - 2hx^\top \hat{B}^\top(k)X(k)P_1(k + 1)M(k)x$$

$$= h^2x^\top \hat{B}^\top(k)X(k)P_1(k + 1)X(k)\hat{B}(k)x - 2h^2x^\top \hat{B}^\top(k)X(k)P_1(k + 1)\hat{B}(k)x$$

$$- 2hx^\top \hat{B}^\top(k)X(k)P_1(k + 1)(I - hD(k))x$$

$$\leq h^2x^\top \hat{B}^\top(k)X(k)P_1(k + 1)X(k)\hat{B}(k)x - 2x^\top \hat{B}^\top(k)X(k)P_1(k + 1)\hat{B}(k)x$$

$$\leq -h^2x^\top \hat{B}^\top(k)X(k)P_1(k + 1)(I - X(k))\hat{B}(k)x$$

$$\leq 0$$

where (10) is due to $M^\top(k)P_1(k + 1)M(k) - P_1(k) < 0$ for $k = 0, 1, \ldots, p - 1$, while inequality (11) is a consequence of Assumptions 2 and 3. Finally, inequality (12) follows from Lemma 1 and the non-negativity of $\hat{B}(k)$ and $P_1(k)$.

Thanks to our assumption of $p$-periodicity, $M(k + p) = M(k)$ for every $k \in \mathbb{Z}_{\geq 0}$. Hence, over every successive interval of size $p$, the matrix $\tilde{M}$ remains the same. This implies that $P_1(k + p) = P_1(k)$ for every $k \in \mathbb{Z}_{\geq 0}$.

Hence, we can use the same Lyapunov function over every successive interval of size $p$. Thus, repeating the same analysis as in over the interval $[0, 1, \ldots, p - 1]$ over every successive interval of size $p$ yields: $V_1(k, x) > 0$ and $\Delta V_1(k, x) \leq 0$ for every $k \in \mathbb{Z}_{\geq 0}$ and for all $x \in [0, 1]^n$. Moreover, it can be immediately seen that $V_1(k, x)$ is radially unbounded, since $V_1(k, x) = \left\| P_1(k)^{\frac{1}{2}}x \right\|^2$. Therefore, from Prop. 1, the system converges asymptotically to the DFE for this case, for all $x(0) \in [0, 1]^n$.

Case 2: Suppose that $\rho(\tilde{M}) = 1$. Since $\tilde{M}$ is also irreducible and nonnegative, then, from Lemma 3 there exists a positive diagonal matrix $Q_2$ such that $\tilde{M}^\top Q_2 \tilde{M} - Q_2 \preceq 0$. By defining, for $k = 0, 1, \ldots, p - 1$, $P_2(k) = [Q_2]_{k+1}$ it is immediate that $M(k)^\top P_2(k + 1)M(k) - P_2(k) \preceq 0$, where $k = 0, 1, \ldots, p - 1$. 


Consider the following Lyapunov function $V_2(k, x) = x^TP_2(k)x$. Observe that, by analogous reasoning as in Case 1, for $k = 0, 1, \ldots, p - 1$ and for $x \neq 0$, $V_2(k, x) > 0$. Define $\Delta V_2(k, x) = V_2(x(k + 1)) - V_2(x(k))$. For $x \neq 0$, from [4], one obtains:

$$
\Delta V_2(k, x) = x^T\tilde{M}^T(k)P_2(k + 1)\tilde{M}(k)x(k) - x^TP_2(k)x
$$

$$
= x^T(M(k)P_2(k + 1)M(k) - P_2(k))x - 2hx^T\tilde{B}^T(k)X(k)P_2(k + 1)M(k)x(k)
$$

$$
+ h^2x^T\tilde{B}^T(k)X(k)P_2(k + 1)X(k)\tilde{B}(k)x
$$

$$
\leq h^2x^T\tilde{B}^T(k)X(k)P_2(k + 1)X(k)\tilde{B}(k)x - 2hx^T\tilde{B}^T(k)X(k)P_2(k + 1)M(k)x
$$

$$
= h^2x^T\tilde{B}^T(k)X(k)P_2(k + 1)X(k)\tilde{B}(k)x - h^2x^T\tilde{B}^T(k)X(k)P_2(k + 1)(I - hD(k))x
$$

$$
- h^2x^T\tilde{B}^T(k)X(k)P_2(k + 1)\tilde{B}(k)x
$$

$$
\leq -h^2x^T\tilde{B}^T(k)X(k)P_2(k + 1)(I - X(k))\tilde{B}(k)x - h^2x^T\tilde{B}^T(k)X(k)P_2(k + 1)M(k)x
$$

$$
\leq -h^2x^T\tilde{B}^T(k)X(k)P_2(k + 1)(I - X(k))\tilde{B}(k)x - h^2x^T\tilde{B}^T(k)X(k)P_2(k + 1)M(k)x
$$

$$
\leq 0.
$$

It can be immediately seen that if $x = 0$, then for $k = 0, 1, \ldots, p - 1$, $\Delta V_2(k, x) = 0$. For every $k \in \mathbb{Z}_{\geq 0}$, by Assumptions 3 and 5, $\tilde{B}(k)$ (and hence $M(k)$) is nonzero and nonnegative, whereas, from Lemma 3, $P_2(k)$ is a positive diagonal matrix. Hence, if, for $k = 0, 1, \ldots, p - 1$, $-h^2x^T\tilde{B}^T(k)X(k)P_2(k + 1)M(k)x = 0$ then $x = 0$.

For reasons, analogous to those outlined in Case 1, the aforesaid analysis can be repeated over every successive interval of size $p$, which yields $V_2(k, x) > 0$ and $\Delta V_2(k, x) \leq 0$ for every $k \in \mathbb{Z}_{\geq 0}$. Moreover, it can be immediately seen that $V_2(k, x)$ is radially unbounded, since $V_2(k, x) = \| P_2(k)^{\frac{1}{2}}x \|^2$. Therefore, from Prop. 1, the DFE is globally asymptotically stable. □

While Theorem 1 concerns periodic systems, the result in [16] Theorem 2.2] is not restricted to periodic systems. However, the following should be noted: First, Theorem 1 gives a sufficient condition for GAS of the DFE, whereas [16] Theorem 2.2] provides only local results. Second, our requirements are less stringent in the sense that we ask for $\rho(M) \leq 1$, where $\rho(M)$ is defined as in [7] with $M(k) = I - hD(k) + h\tilde{B}(k)$, while [16] asks for $\rho(\hat{M}) \leq 1$, where $\rho(\hat{M})$ is defined as in [7] with $\hat{M}(k) = I + h((I - X(k))\tilde{B}(k) - D(k))$. Third, Theorem 1 accounts for the (asymptotic) stability of the equilibrium point $x = 0$ for the case of $\rho(M) = 1$ as well, unlike [16] Theorem 2.2] wherein nothing can be concluded when $\rho(\hat{M}) = 1$. Moreover, the proof technique is also different. On the other hand, the result in Theorem 1 deals with (global, and therefore, local) asymptotic stability of $x = 0$, while the result in [16] Theorem 2.2] gives (local, but not global) exponential stability of $x = 0$.

We will now turn our attention to finding a necessary and sufficient condition for GAS of the DFE. As a first step, the following is immediate from the proof of Theorem 1.
**Corollary 1:** Suppose that the system (4) is \( p \)-periodic and that Assumptions 1–5 hold. If \( \rho(M_{k+p};k) \leq 1 \), where \( k = 0, 1, \ldots p - 1 \), then the DFE is globally asymptotically stable. \[\blacksquare\]

It is natural to ask whether the condition in Corollary 1 is necessary as well. The following proposition addresses this question.

**Proposition 3:** Suppose that the system (4) is \( p \)-periodic. The healthy state is asymptotically stable only if \( \rho(M_{k+p};k) \leq 1 \) for \( k = 0, 1, \ldots p - 1 \). \[\blacksquare\]

**Proof:** Consider the system (4). By way of contraposition, assume there exists \( k' \in [p] \) such that \( \rho(M_{k'+p};k') > 1 \). Let \( z(k) \) be a vector formed by concatenating the states \( x(k) \) over a period of length \( p \). Using (??), define

\[
\tilde{M} = \begin{bmatrix}
0 & 0 & \ldots & 0 & \hat{M}(p-1) \\
\hat{M}(0) & 0 & \ldots & 0 & 0 \\
0 & \hat{M}(1) & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & \hat{M}(p-2) & 0
\end{bmatrix}.
\]

Note that, since \( \hat{M}(k+p) = \hat{M}(k) \) for all \( k \in \mathbb{Z}_{\geq 0} \) because of the assumption of \( p \)-periodicity, \( \tilde{M} \) is a time-invariant matrix. Hence, we can rewrite system (4) as

\[
z(k+1) = \tilde{M}z(k).
\]

Linearizing (13) around the DFE, i.e., \( x = 0 \), yields the following:

\[
z(k+1) = \tilde{M}z(k),
\]

where \( \tilde{M} \) is defined in (8). By the assumption that there exists \( k' \in [p] \) such that \( \rho(M_{k'+p};k') > 1 \), we have that \( \rho(\tilde{M}^p) > 1 \), which implies \( \rho(\tilde{M}) > 1 \). Therefore, from Proposition 2, \( x = 0 \) is an unstable equilibrium of system (13). Thus, since the system in (13) is equivalent to (4), the DFE is an unstable equilibrium of system (4) as well. \[\blacksquare\]

Combining Corollary 1 and Proposition 3 readily yields the following:

**Theorem 2:** Suppose that the system (4) is \( p \)-periodic and that Assumptions 1–5 hold. The DFE of system (4) is globally asymptotically stable if and only if \( \rho(M_{k+p};k) \leq 1 \), for \( k = 0, 1, \ldots, p - 1 \). \[\blacksquare\]

A natural question that arises at this point is the connection between Theorem 2 and an analogous result in [11, Theorem 2]. The following remark addresses it:

**Remark 1:** If system (4) is time-invariant, or equivalently, \( p = 1 \), then the condition in Theorem 2 coincides with the condition in [11, Theorem 2]. To see this, consider the following argument: since \( p = 1 \), for every \( k \in \mathbb{Z}_{\geq 0} \), \( M_{k+p;k} = M \). Therefore \( \rho(M_{k+p};k) = \rho(M) \). Hence \( \rho(M_{k+p};k) \leq 1 \) if and only if \( \rho(M) \leq 1 \), which is the same as the condition in [11, Theorem 2]. \[\blacksquare\]
Fig. 1: Initial condition for simulations. The first node is infected while the other two are in DFE, following the coloring scheme in (15).

V. SIMULATIONS

In this section we provide a set of simulations that illustrate the main results and some unproven behavior. Blue ($b$) represents healthy and red ($r$) represents infected. The coloring of each node $i$ at time $k$ follows

$$x_i r + (1 - x_i)b.$$ \hspace{1cm} (15)

For simplicity we limit ourselves to a system with three nodes and three different adjacency matrices:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$A_2 = A_1^T,$$ and $$A_3 = A_1 + A_2.$$ We set $h = 0.05$, and use the initial condition in Figure 1 (however the results are independent of initial condition). For the first two simulations we use homogeneous virus spread.

For the first simulation we set $\beta^1 = \beta^2 = \beta^3 = 1$ and $\delta^1 = 1$, $\delta^2 = 0.1$, and $\delta^3 = 3$. With these values $\rho(I - h\delta^1 I + h\beta^1 A_1) = 1$, $\rho(I - h\delta^2 I + h\beta^2 A_2) = 1.045$, $\rho(I - h\delta^3 I + h\beta^3 A_3) = 0.95$, and $\rho(M_{k+3:k}) = 0.9927$, for all $k \in \{0, 1, 2\}$. Consistent with the results the system converges to the DFE; see Figure 2.

For the second simulation the parameters are the same as the first simulation except $\delta^3 = 1/3$. This system has $\rho(I - h\delta^3 I + h\beta^3 A_3) = 1.08333$ and $\rho(M_{k+3:k}) = 1.321$, for all $k \in \{0, 1, 2\}$. The system converges to a limit cycle with three states:

$$\begin{bmatrix} 0.628 & 0.628 & 0.628 \\ 0.637 & 0.637 & 0.637 \\ 0.649 & 0.649 & 0.649 \end{bmatrix}.$$
Fig. 2: Final condition for simulation with $\delta^3 = 3$. All nodes are in DFE, depicted by blue.

Fig. 3: Final condition for simulation with $\delta^3 = 1/3$. All nodes become infected, depicted by redish purple following (15).

Since they are close to each other in value, we plot only one in Figure 3.

A question, not addressed within the scope of our main results, is the following: given that there are two networks, one of which is in the healthy state whereas the other is in the infected state, how intense can the interaction be between the healthy agents and the infected ones before they too get infected? Some simulation results seeking to address this question are presented in Figure 4. The two subpopulations have the adjacency matrices given by $A_3$, for all $k \geq 0$, and every third time step the link connecting the two subpopulations is set to some value $\epsilon$. The healing rate for the healthy population is set to 10 and the other is 0.1. The simulations show that, for a small edge weight connecting the two populations, the healthy population remains uninfected. However, as the edge weight gets larger the healthy population slowly becomes infected.
This paper dealt with the problem of finding necessary and sufficient conditions for asymptotic convergence to the DFE in discrete-time periodic time-varying networks. Firstly, in terms of the joint spectral radius, a sufficient condition for GAS of the DFE was provided. Secondly, in terms of the spectral radius of the product of matrices over an interval of size $p$, a necessary and sufficient condition for GAS of the DFE was provided.

Note that the present paper while accounting for time-varying networks relies on the assumption that at each time instant the underlying graph be strongly connected. A possible future line of investigation would be to weaken this assumption, and account for arbitrary time-varying graphs as well. Second, we have restricted our attention to periodic time-varying systems. Hence, yet another line of future research could be to remove the periodicity assumption. Finally, for continuous-time setting, it has been shown that the switched SIS model admits a limit cycle if the condition in Theorem 2 is violated, see [13, Theorem 6.4]. While our simulations suggest that, for the discrete-time setting, violating the condition in Theorem 2 could lead to the existence of a limit cycle, this conjecture remains open, and is the focus of an ongoing investigation.

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