Quantum dynamics of the harmonic oscillator on a cylinder

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Abstract

Evolution of coherent states is considered for a particle confined to a cylinder moving in a harmonic oscillator potential. Because of the discontinuous changes as time goes by of the phase representing the position of a particle on a parallel (circle) the trajectory pattern of quantum averages specifying coordinates on a cylinder is very complex and in some aspects resembles chaotic one.
Despite the fact that the helical motion of a charged particle in a uniform magnetic field is described in many textbooks the theory of quantization on a cylinder is far from complete. For example, the coherent states for the quantum mechanics on a cylinder were not, to the best of our knowledge, discussed in the literature. We point out that coherent states for a charged particle in a magnetic field are obtained by reduction of the dynamics to the transversal motion [1]. However, it is clear that in general such reduction cannot be applied. An example is a particle on a cylinder with radial magnetic field analyzed in [2]. In this work we study the dynamics of the harmonic oscillator on a cylinder. More precisely, we discuss the time development of the wave packets when the initial condition is a coherent state. Since the coherent states for a cylinder are not stable but do not spread as in the case of the motion in a plane, the dynamics is nontrivial. In particular, the geometry of jump points related to discontinuity of the phase representing the position of a particle on a parallel is similar to the Poincaré section of chaotic trajectories of nonlinear dynamical systems.

We first summarize the basic facts about the quantum mechanics on a cylinder

\[
\begin{align*}
x_1 &= \cos \varphi, \\
x_2 &= \sin \varphi, \\
x_3 &= l,
\end{align*}
\]

(1)

where \( \varphi \in [0, 2\pi) \) specifies the position of a particle on a parallel and \( l \in (-\infty, \infty) \) is the coordinate of a particle on a meridian (generator). On taking into account the topological equivalence of the infinite circular cylinder (1) with the product of the circle and real line \( S^1 \times \mathbb{R} \), we arrive at the following algebra adequate for the study of the motion on a cylinder:

\[
\begin{align*}
[\hat{J}, U] &= U, & [\hat{J}, U^\dagger] &= -U^\dagger, & [\hat{l}, \hat{p}_l] &= iI, \\
[\hat{J}, \hat{l}] &= [\hat{J}, \hat{p}_l] = [U, \hat{l}] = [U, \hat{p}_l] = [U^\dagger, \hat{l}] = [U^\dagger, \hat{p}_l] &= 0,
\end{align*}
\]

(2)

where \( \hat{J} \) is the angular momentum operator, \( U = e^{i\hat{\varphi}} \) is the unitary operator representing the position of a quantum particle on a (unit) circle \( \mathbb{S}^1 \), \( \hat{l} \) is the position operator on a generator of a cylinder, \( \hat{p}_l \) is the corresponding momentum, and we set \( \hbar = 1 \). The operators act in the Hilbert space \( L^2(S^1 \times \mathbb{R}) \) for the quantum mechanics on a cylinder specified by the scalar
product
\[ \langle f | g \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \int_{-\infty}^{\infty} dl \, f^*(\varphi, l) g(\varphi, l). \] (3)

In the following we restrict to the case of integer eigenvalues of the operator \( \hat{J} \), then the functions \( f(\varphi, l) \) are periodic functions of \( \varphi \) with period \( 2\pi \). The operators act in the representation (3) as follows

\[ \begin{align*}
\hat{J} f(\varphi, l) &= -i \frac{\partial}{\partial \varphi} f(\varphi, l), \quad U f(\varphi, l) = e^{i\varphi} f(\varphi, l), \\
\hat{p}_l f(\varphi, l) &= -i \frac{\partial}{\partial l} f(\varphi, l), \quad \hat{l} f(\varphi, l) = l f(\varphi, l).
\end{align*} \] (4)

Consider now the coherent states for a particle on a cylinder. The topological equivalence of a cylinder and the product \( S^1 \times \mathbb{R} \) indicates that the coherent states are common eigenvectors of the commuting operators: \( X = e^{-\hat{J} + \frac{1}{2}U} \) defining via an eigenvalue equation the coherent states for the quantum mechanics on a circle \[3\] and the annihilation operator \( a = \frac{1}{\sqrt{2}}(\hat{l} + i\hat{p}) \) used for the definition of the standard coherent states for a particle on a line, that is

\[ \begin{align*}
X f_{\xi,z}(\varphi, l) &= \xi f_{\xi,z}(\varphi, l), \\
a f_{\xi,z}(\varphi, l) &= z f_{\xi,z}(\varphi, l),
\end{align*} \] (5a, 5b)

where \( \xi = e^{-J + i\alpha} \), where \( J \) is the classical orbital momentum and \( \alpha \) is the classical angle parametrizing the classical phase space for a particle on a circle \[3\] and \( z = \frac{1}{\sqrt{2}}(q + ip) \).

Clearly, the coherent state \( f_{\xi,z}(\varphi, l) \) is simply the product of the coherent states for the quantum mechanics on a circle and a real line in the coordinate representation. Hence, using the formulae for a particle on a circle \[4\] and well-known relations for the standard coherent states we find

\[ f_{\xi,z}(\varphi, l) = \pi^{-1/4} \theta_3 \left( \frac{1}{2\pi} (\varphi - \alpha - iJ) \left| \frac{i}{2\pi} \right| \exp[-\frac{1}{2}(l - q)^2 + ip(l - \frac{i}{2}q)] \right), \] (6)

where \( \theta_3(v|\tau) \) is the Jacobi theta function \[5\].

We now discuss the harmonic oscillator on a cylinder. We begin by recalling that the Hamiltonian for a classical particle with unit mass confined to a surface of a cylinder (1) moving in a harmonic oscillator potential is given by

\[ H = \frac{p_l^2}{2} + \frac{J^2}{2} + \frac{\omega^2}{2} l^2, \] (7)
where \( p_t = \dot{l} \) is the momentum corresponding to the motion in the meridian and \( J = \dot{\varphi} \) is the conserved angular momentum. We point out that the potential \( \frac{\omega^2 l^2}{2} \) differs only by a constant from \( \frac{\omega^2}{2} (x_1^2 + x_2^2 + x_3^2) \). Of course, besides the circular motion in the \( x_1 \) and \( x_2 \) plane and an equilibrium point, the solution describes the superposition of the uniform circular motion

\[
\varphi = \varphi_0 + J t, \tag{8}
\]

with the angular velocity \( \omega_J = J \), and the harmonic oscillations along the meridian with frequency \( \omega \)

\[
l = l_0 \cos \omega t + \frac{p_{l0}}{\omega} \sin \omega t, \quad p_l = p_{l0} \cos \omega t - \omega l_0 \sin \omega t. \tag{9}
\]

In the case of commensurable \( \omega \) and \( \omega_J \) the motion is periodic and the trajectory is the closed curve on the cylinder with the upper and lower bound

\[
l = \pm \sqrt{l_0^2 + \left( \frac{p_{l0}}{\omega} \right)^2},
\]

respectively, otherwise the motion is quasiperiodic and the trajectory densely fills the strip \( |l| \leq \sqrt{l_0^2 + \left( \frac{p_{l0}}{\omega} \right)^2} \), \( 0 \leq \varphi < 2\pi \).

In quantum mechanics the dynamics of the harmonic oscillator on a cylinder defined by the classical Hamiltonian (7) is described by the Schrödinger equation

\[
i \frac{\partial f(\varphi, l; t)}{\partial t} = \hat{H} f(\varphi, l; t), \quad f(\varphi, l; 0) = f_0(\varphi, l), \tag{10}
\]

where the quantum Hamiltonian is

\[
\hat{H} = \frac{\hat{p}_l^2}{2} + \frac{\hat{j}^2}{2} + \omega^2 \hat{l}^2. \tag{11}
\]

As is well known the Hamiltonian \( \hat{H} \) can be written in the form

\[
\hat{H} = \frac{\hat{j}^2}{2} + \omega \left( N_\omega + \frac{1}{2} \right), \tag{12}
\]

where \( N_\omega = a_\omega^\dagger a_\omega \) is the number operator expressed by means of the Bose annihilation operators

\[
a_\omega = \sqrt{\frac{\omega}{2}} \left( \hat{l} + \frac{i}{\omega} \hat{p}_l \right). \tag{13}
\]

The coherent states of the harmonic oscillator with the Hamiltonian (11) can be immediately obtained by the formal generalization of coherent states defined by (5) relying on replacement of (5.b) with

\[
a_\omega f_{\xi, z; \omega}(\varphi, l) = z f_{\xi, z; \omega}(\varphi, l), \tag{14}
\]
where \( z = \sqrt{\frac{\pi}{2}} (q + \frac{i}{\omega}p) \). Hence, using the coordinate representation of the standard coherent states of the harmonic oscillator we find that the coherent states \( f_{\xi,z;\omega}(\varphi, l) \) are given by

\[
f_{\xi,z;\omega}(\varphi, l) = \left(\frac{\omega}{\pi}\right)^{\frac{3}{4}} \theta_3 \left(\frac{1}{2\pi}(\varphi - \alpha - iJ)\right) \exp \left[ -\frac{\omega}{2} (l - q)^2 + ip(l - \frac{q}{2}) \right].
\]

Suppose now that the initial state \( f_0(\varphi, l) \) in (10) is the coherent state (15). On taking into account that \( \hat{J} \) and \( \hat{N}_\omega \) commute, making use of the formula for the free evolution of the coherent states on a circle [4] and well-known relations describing stability of the standard coherent states with respect to the Hamiltonian of the harmonic oscillator, we arrive at the following solution to the Schrödinger equation (10):

\[
f_{\xi,z;\omega}(\varphi, l; t) = e^{-it\hat{J}^2/2} e^{-it\omega(N_\omega + \frac{1}{2})} f_{\xi,z;\omega}(\varphi, l)
= \left(\frac{\omega}{\pi}\right)^{\frac{1}{4}} \theta_3 \left(\frac{1}{2\pi}(\varphi - \alpha - iJ)\right) e^{-it\omega/2} \exp \left[ -\frac{\omega}{2} t^2 \right.
- \frac{1}{2} \omega^2 \left( q^2 \cos \omega t - \frac{i}{2\omega} p^2 e^{-i\omega t} \sin \omega t \right. - \left. \frac{1}{2} i qpe^{-2i\omega t} + \omega \left( q + \frac{i}{\omega} p \right) e^{-i\omega t} \right].
\]

Since the Jacobi theta function \( \theta_3(v, \tau) \) is a periodic function of \( \tau \) with period \( T = 2 \), so in view of (16) the frequency of circular motion is equal to \( \frac{1}{2} \), therefore the function \( f_{\xi,z;\omega}(\varphi, l; t) \) is periodic function of time for rational \( \omega \) and quasiperiodic for irrational \( \omega \). Furthermore, from (16) it follows that the probability density for the coordinates in the normalized coherent state is

\[
p_{\xi,z}(\varphi, l; t) = \frac{|f_{\xi,z;\omega}(\varphi, l; t)|^2}{\|f_{\xi,z;\omega}\|^2}
= \sqrt{\frac{\omega}{\pi}} \left| \theta_3 \left(\frac{1}{2\pi}(\varphi - \alpha - iJ)\right) \frac{1}{2\pi}(i - t) \right|^2 \exp \left[ -\omega \left( l - q \cos \omega t - \frac{p}{\omega} \sin \omega t \right)^2 \right].
\]

As with (16) the probability density is periodic (quasiperiodic) for rational (irrational) \( \omega \). We point out that the coherent states for a particle on a circle are not stable with respect to the free evolution generated by the Hamiltonian \( \hat{J}^2/2 \). However, in opposition to the case of the standard coherent states for a particle on a real line, the wave packets referring to the angular part of the function (16) do not spread but oscillate during the course of time. As a result of oscillations of the corresponding probability density i.e. the angular part of (17), an interesting phenomenon occurs which can be regarded as quantum jumps on a circle described in [4]. We recall that the probability density has at \( t = t_* = (2k + 1)\pi \), where \( k \) is integer, two identical maxima. Therefore at \( t = t_* \) a particle can be detected with equal
FIG. 1. The front-view of quantum probability density (17), where \( \omega = 1, \alpha = 0.75\pi, J = 1, q = -0.7, \) and \( p = 0.2. \) For \( t = \pi \) the two maxima are identical, so a particle can be detected with equal probability on two different points on a cylinder.

(maximal) probability at two different points on a circle. As a consequence of such behavior of the probability density there is a discontinuity in the angle that on the quantum level is identified with \( \text{Arg}(U(t)) \), where \( U(t) = e^{itJ^2/2}Ue^{-itJ^2/2} \), related to the jumps of the phase by \( \pi \). Such discontinuity takes place in the discussed case of integer eigenvalues of \( \hat{J} \) only for \( J \) integer. Now, in the light of the observations concerning a circle it is clear that in the discussed case of the wave packets (16) we deal with quantum jumps on a cylinder. The jumps occur at \( t = t_* = (2k + 1)\pi \), where \( k \) is integer, when the probability density (17) has two identical maxima (see Fig. 1), and they are related to the discontinuity in the angle \( \text{Arg}(U(t))_{\xi,z} \), where \( \langle U(t)\rangle_{\xi,z} \) is the expectation value of the operator \( U(t) \) in the normalized
coherent state $f_{\xi,z;\omega}$ given by

\[
\langle U(t) \rangle_{\xi,z} = e^{-1/4} e^{i\alpha} \frac{\theta_2\left(\frac{t}{\pi} - \frac{1}{2}\right) |\frac{1}{2}|}{\theta_3\left(\frac{1}{4} + \frac{1}{\pi}\right)}.
\]  

(18)

Such discontinuity takes place only for $J$ integer and is connected with the jumps of phase by $\pi$ corresponding to the transformation $(x_1, x_2) \rightarrow (-x_1, -x_2)$ of the coordinates of a particle on a cylinder. We study the quantum dynamics of the harmonic oscillator on a cylinder by means of expectation values of operators representing the angle and position on a meridian (generator). More precisely, we set in (1)

\[
\varphi = \text{Arg}\langle U(t) \rangle_{\xi,z}, \quad l = \langle \hat{l}(t) \rangle_{\xi,z} = q \cos \omega t + \frac{p}{\omega} \sin \omega t,
\]

(19)

where $\langle U(t) \rangle_{\xi,z}$ is given by (18), and $\hat{l}(t) = e^{i\omega(N\omega + \frac{1}{2})} e^{-i\omega(N\omega + \frac{1}{2})}$. We point out that $\theta_2(v|\tau)$ is a periodic function of $v$ with period $T = 2$, so $\varphi$ given by (18) and (19) is periodic with period $4\pi$ and thus the frequency of circular motion is $1/2$. We conclude that the trajectory on a cylinder given by (1) and (19) is periodic for rational $\omega$ and quasiperiodic for irrational $\omega$. Interestingly, as follows from computer simulations the jump points concentrate around the generator corresponding to the angle $\varphi = \alpha - \frac{\pi}{2}$, where $\alpha$ is the parameter labelling the coherent state referring to the classical angle. Since both the classical and quantum dynamics of the harmonic oscillator on a cylinder given by (8), (9) and (19), respectively is quasiperiodic for irrational frequency $\omega$, no wonder that the set of jump points turns out to be very complicated for irrational $\omega$ that is illustrated in Fig. 2. We point out that the geometry of jump points presented in Fig. 2 resembles the Poincaré section of chaotic trajectories of nonlinear dynamical systems. The difference of the pattern referring to the quasiperiodic motion with irrational $\omega$ and periodic motion with rational $\omega$ is also remarkable.

Concluding, a quantum particle on a cylinder in a coherent state moving in a harmonic oscillator potential shows exotic behaviour which can be interpreted as pseudo-stochastic quantum jumps. It seems that besides the quantum mechanics of constrained systems, the results concerning the dynamics of the harmonic oscillator on a cylinder would be also of importance for the theory of quantization of a quasiperiodic motion. Finally, the provided example of nontrivial dynamics with discontinuous trajectories would be also of interest in the theory of dynamical systems.
FIG. 2. Top: fragment of the surface of the cylinder with jump points, where \( \omega = \frac{1}{2}(1 + \sqrt{5}) \approx 1.61803 \), that is \( \omega \) is the golden ratio — the most irrational number of all irrational numbers. The remaining parameters are the same as in Fig. 1. Bottom: the jump points in the case with rational \( \omega = 1.62 \) approximating the golden ratio, that is the frequency for the top figure. Despite the fact that the relative error of the approximation of the golden ratio by \( \omega = 1.62 \) is of order 0.1%, the difference of the pattern from the top and bottom figure referring to the classical quasiperiodic and periodic motion, respectively is noticeable.

ACKNOWLEDGEMENTS

This work was supported by the grant N202 205738 from the National Science Centre.

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