Research Article

Biharmonic Hypersurfaces in Pseudo-Riemannian Space Forms with at Most Two Distinct Principal Curvatures

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In this paper, we show that biharmonic hypersurfaces with at most two distinct principal curvatures in pseudo-Riemannian space forms, is properly cited.

1. Introduction

Let \( N^{s+1}_r(c) \) be a \((n+1)\)-dimensional pseudo-Riemannian space form with index \( 1 \leq s \leq n+1 \) and constant sectional curvature \( c \). According to \( c = 0, c > 0, \) and \( c < 0 \), \( N^{s+1}_r(c) \) is isometric to pseudo-Euclidean space \( \mathbb{E}^{s+1}_r \), de Sitter space \( \mathbb{S}^{s+1}_r \), and anti-de Sitter space \( \mathbb{H}^{s+1}_r \).

Suppose that \( \phi : M^n_r \to N^{s+1}_r(c), r = s - 1, \) or \( s \) be an isometric immersion of a pseudo-Riemannian hypersurface \( M^n_r \) into \( N^{s+1}_r(c) \). The hypersurface \( M^n_r \) is said to be biharmonic if its bitension field \( \tau_2(\phi) \) vanishes identically, i.e.,

\[
\tau_2(\phi) := \text{trace} \left( \nabla^2 \phi \otimes \phi - \nabla \phi \otimes \nabla \phi \right) \tau(\phi) - \text{trace} \left( \mathcal{R}(\phi, \tau(\phi)) \right) \nabla \phi = 0,
\]

where \( \tau(\phi) = \text{div}(\phi), \mathcal{R}, \nabla, \) and \( \nabla \) are the curvature tensor of \( N^{s+1}_r(c) \), the induced connection by \( \phi \) on the bundle \( \phi^{-1} TN^{s+1}_r(c) \), and the connection of \( M^n_r \), respectively. If the mean curvature of the hypersurface \( M^n_r \) is zero, then we call \( M^n_r \) as minimal. It is generally known that minimal hypersurfaces are biharmonic ones. Conversely, the natural question is whether any biharmonic hypersurface is minimal.

For biharmonic hypersurfaces in pseudo-Euclidean spaces, there is a conjecture in [2] that every biharmonic hypersurface of pseudo-Euclidean space \( \mathbb{E}^{s+1}_r \) is minimal. Up to now, this conjecture has been examined for many biharmonic hypersurfaces, such as \( M^2_r \) of \( \mathbb{E}^{3}_r \) (cf. [2, 3]), \( M^3_1 \) of \( \mathbb{E}^{4}_1 \) (cf. [1]), \( M^3_2 \) of \( \mathbb{E}^{4}_2 \) (cf. [10]), and \( M^2_2 \) in \( \mathbb{E}^{4+1} \) with at most three distinct principal curvatures and diagonalizable shape operator (cf. [5, 7]).

When the ambient space is de Sitter space \( \mathbb{S}^{s+1}_r \), there are also some papers that studied the above problem. Sasahara in [11] considered biharmonic hypersurfaces \( M^2_r \) of \( \mathbb{S}^{3}_r \) and proved that it must be minimal when \( r = 0 \), but may not when \( r = 1 \). Investigators studied biharmonic hypersurfaces with at most two distinct principal curvatures in \( \mathbb{S}^{s+1}_r \) whose shape operator is diagonalizable in [6, 8] and showed that such hypersurface \( M^2_{s-1} \) is minimal, but the hypersurface \( M^n_r \) may not. Naturally, there is a question as to whether any biharmonic hypersurface \( M^2_{s-1} \) in de Sitter space \( \mathbb{S}^{s+1}_r \) is minimal.

The situation is quite different when the ambient space is anti-de Sitter space \( \mathbb{H}^{s+1}_r \). For biharmonic hypersurface \( M^2_r \) of \( \mathbb{H}^{3}_r \), it must be minimal when \( r = 1 \) and may not when \( r = 0 \) (cf. [11]). For biharmonic hypersurface \( M^n_r \) with at most two distinct principal curvatures in \( \mathbb{H}^{s+1}_r \) whose shape operator is diagonalizable, it is minimal when \( r = s \)
and may not when \( r = s - 1 \) (cf. [6, 8]). A natural question is whether any biharmonic hypersurface \( M^s_r \) in anti-de Sitter space \( \mathbb{H}^{2k}_s \) is minimal.

In this paper, we study biharmonic hypersurfaces with at most two distinct principal curvatures in pseudo-Riemannian space forms \( N^{n+1}_s \), without the restriction that the shape operator is diagonalizable. We proved such biharmonic hypersurfaces have constant mean curvature. Furthermore, we find that such biharmonic hypersurfaces \( M^s_{s+1} \) in even-dimensional pseudo-Euclidean space \( \mathbb{E}^{2k}_s \), \( M^s_{s-1} \) in even-dimensional de Sitter space \( \mathbb{S}^{2k}_s \), and \( M^s_{2k-1} \) in even-dimensional anti-de Sitter space \( \mathbb{H}^{2k}_s \) are minimal.

### 2. Preliminaries

#### 2.1. Notions and Formulas of Hypersurfaces in \( N^{n+1}_s \)

Let \( N^{n+1}_s \) be a pseudo-Riemannian space form with index \( s \) and constant sectional curvature \( c \). A nonzero vector \( X \) in \( N^{n+1}_s \) is called time-like, space-like, or light-like, according to whether \( \langle X, X \rangle \) is negative, positive, or zero, respectively.

Let \( M^s_r \) be the nondegenerate hypersurface in \( N^{n+1}_s \). \( \xi \) denotes a unit normal vector field to \( M^s_r \), then \( e = \langle \xi, \xi \rangle = \pm 1 \). Denote by \( \nabla \) and \( \bar{\nabla} \) the Levi-Civita connections of \( M^s_r \) and \( N^{n+1}_s \), respectively. For any vector fields \( X, Y \) tangent to \( M^s_r \), the Gauss formula and Weingarten formula are given by

\[
\begin{align*}
\bar{\nabla}_X Y &= \nabla_X Y + h(X, Y) \xi, \\
\bar{\nabla}_X \xi &= -A(X),
\end{align*}
\]  

(2)

where \( h \) is the scalar-valued second fundamental form and \( A \) is the shape operator of \( M^s_r \) associated to \( \xi \). The mean curvature vector field \( H \) can be expressed as \( \bar{H} = H \xi \), with mean curvature \( \bar{H} = (1/n)\text{tr}A \). For any vector fields \( X, Y, Z \) tangent to \( M^s_r \), the Codazzi and Gauss equations are (cf. [9])

\[
(\nabla_X A) Y = (\nabla_Y A) X,
\]  

(3)

\[
R(X, Y)Z = c(\langle Y, Z \rangle X - \langle X, Z \rangle Y) + e \langle A(Y), Z \rangle A(X) - e \langle A(X), Z \rangle A(Y),
\]  

(4)

here \( R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \).

A hypersurface \( M^s_r \) of \( N^{n+1}_s \) is biharmonic if and only if its mean curvature \( H \) satisfies the following two equations (cf. [4]):

\[
A(\nabla H) = -\frac{n}{2}eH(\nabla H),
\]  

(5)

\[
\Delta H + e\text{tr}A^2 = ncH,
\]  

(6)

where

\[
\Delta H = -\sum_{i=1}^{n} (e_i e_i H - \nabla_{e_i} e_i H),
\]  

(7)

here \( \{e_1, e_2, \ldots, e_n\} \) is a local orthonormal frame of \( T_x M^s_r \).

#### 2.2. The Shape Operator of \( M^s_r \) in \( N^{n+1}_s \)

According to [9] (exercise 18, pp. 260-261), the tangent space \( T_x M^s_r \) at \( x \in M^s_r \) can be expressed as a direct sum of subspaces \( V_k, 1 \leq k \leq m \), that are mutually orthogonal and invariant under the shape operator \( A \), and each \( A|_{V_k} \) (the restriction of \( A \) on \( V_k \)) has form (a) or (b) as follows.

(a) \( A|_{V_k} \) has the form

\[
A|_{V_k} = \begin{pmatrix}
\lambda_k & 1 & \ldots & \ldots & \lambda_k \\
1 & \lambda_k & \ldots & \ldots & 1 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
1 & \ldots & \ldots & \lambda_k & 1
\end{pmatrix},
\]  

(8)

with respect to a basis \( \mathfrak{B}_k = \{u_k, \ldots, u_{k+d} \} \) of \( V_k \). The inner products of the basis elements in \( \mathfrak{B}_k \) are all zero except

\[
\langle u_k, u_{k+d} \rangle = e_k = \pm 1, \quad b + d = \alpha_k + 1.
\]  

(b) \( A|_{V_k} \) has the form

\[
A|_{V_k} = \begin{pmatrix}
\gamma_k & \tau_k & \ldots & 0 & 0 & \ldots & 0 \\
-\tau_k & \gamma_k & \ldots & 0 & 0 & \ldots & 0 \\
0 & 1 & \gamma_k & \ldots & 0 & \ldots & 0 \\
0 & 0 & 1 & \gamma_k & \ldots & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & \ldots & 1 & \gamma_k & \ldots & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 1 & \gamma_k & \ldots & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 & \gamma_k & \ldots & 0 \\
\end{pmatrix},
\]  

(10)

with respect to a basis \( \mathfrak{B}_k = \{\bar{u}_1, \ldots, \bar{u}_{k+d} \} \) of \( V_k \). The inner products of the basis elements in \( \mathfrak{B}_k \) are all zero except

\[
\langle \bar{u}_{k+d}, \bar{u}_j \rangle = 1 = \langle \bar{v}_{k+d}, \bar{v}_j \rangle, \quad b + d = \beta_k + 1.
\]  

We denote by \( t \) the number of terms \( A|_{V_k} \) having form (a). We adjust the order of \( V_k \), \( 1 \leq k \leq m \), such that \( A|_{V_k} \) have
form (a) for $1 \leq k \leq t$ and $A_{|V_t}$ have form (b) for $t + 1 \leq k \leq m$. Denote $A_{i} = A_{|V_{i}}$, $1 \leq i \leq t$, and $A_{j} = A_{|V_{j}}$, $t + 1 \leq j \leq m$. Collecting all the vectors in $\mathfrak{B}_{1}, \ldots, \mathfrak{B}_{t}, \mathfrak{B}_{t+1}, \ldots, \mathfrak{B}_{m}$ in order, we get a basis $\mathfrak{B}$ of $T_{x}M^{n}$. With respect to this basis $\mathfrak{B}$, the shape operator $A$ of the hypersurface $M^{n}$ in $N^{n+1}$ can be expressed as an almost diagonal matrix:

$$A = \text{diag}\{A_{1}, \ldots, A_{t}, \widetilde{A}_{t+1}, \ldots, \widetilde{A}_{m}\},$$

and the inner products of the elements in $\mathfrak{B}$ are all zero except

$$\langle u_{ij}, u_{ij} \rangle = \epsilon_{i} = \pm 1, a + b = \alpha_{t} + 1, 1 \leq i \leq t,$$

$$\langle \bar{u}_{ij}, \bar{u}_{ij} \rangle = 1 = \left(\langle \bar{v}_{ij}, \bar{v}_{ij} \rangle\right), c + d = \beta_{j} + 1, t + 1 \leq j \leq m,$$

where

$$\alpha_{t} + \alpha_{t+1} + \cdots + \alpha_{t} + \left(2(\beta_{t+1} + \beta_{t+2} + \cdots + \beta_{m}) = n.\right)$$

Observe the forms (a) and (b); we see that $A_{i}$, $1 \leq i \leq t$, has only a simple eigenvalue $\lambda_{i}$ and $A_{j}$, $t + 1 \leq j \leq m$, has eigenvalues $\gamma_{j} + \tau_{j} \sqrt{-1}$, $\gamma_{j} - \tau_{j} \sqrt{-1}$. It follows from the form of the shape operator $A$ that $M^{n}_{p}$ has principal curvatures

$$\lambda_{1}, \ldots, \lambda_{t}, \gamma_{t+1} \pm \tau_{t+1} \sqrt{-1}, \ldots, \gamma_{m} \pm \tau_{m} \sqrt{-1}.\right.$$  

So, under the assumption that $M^{n}_{p}$ has at most two distinct principal curvatures, the shape operator $A$ has the following two possible forms:

(I) $t = m$, i.e., $A = \text{diag}\{A_{1}, A_{2}, \ldots, A_{m}\}$, and there are at most two distinct values among $\{\lambda_{1}, \lambda_{2}, \ldots, \lambda_{m}\}$

(II) $t = 0$, i.e., $A = \text{diag}\{\widetilde{A}_{1}, \widetilde{A}_{2}, \ldots, \widetilde{A}_{m}\}$ and $\gamma_{1} = \gamma_{2} = \cdots = \gamma_{m} = \gamma = \gamma_{m} = \tau_{1} = \tau_{2} = \cdots = \tau_{m} = \tau, \tau \neq 0$

For the form (I), we have

$$A(u_{ij}) = \lambda_{i}u_{ij} + u_{ij}, A(u_{ij}) = \lambda_{i}u_{ij}, 1 \leq i \leq m, 1 \leq a \leq \alpha_{t} - 1.\right)$$

And for the form (II), we have

$$A(\bar{u}_{ij}) = \nu_{ij} - \tau\bar{v}_{ij} + \bar{u}_{ij}, A(\bar{u}_{ij}) = \nu_{ij} - \tau\bar{v}_{ij},$$

$$A(\bar{u}_{ij}) = \tau\bar{u}_{ij} + \nu\bar{v}_{ij} + \bar{v}_{ij}, A(\bar{u}_{ij}) = \tau\bar{u}_{ij} + \nu\bar{v}_{ij},$$

with $t + 1 \leq j \leq m, 1 \leq b \leq \beta_{j}$.

3. Theorems

Theorem 1. Let $N^{n+1}(c)$ be a $(n + 1)$-dimensional pseudo-Riemannian space form with index $s$ and constant sectional curvature $c$ and $M^{n}$ be a nondegenerate biharmonic hypersurface of $N^{n+1}(c)$ with at most two distinct principal curvatures, then $M^{n}$ has constant mean curvature.

Proof. From Section 2, the shape operator $A$ has the form (I) or (II). If $A$ has the form (II), then its eigenvalues are not real and $-(n/2)eH$ is not an eigenvalue. It follows from (5) that $\nabla H = 0$, which tells us $H$ is a constant.

For the form (I), if we assume that $H$ is not a constant, then (5) implies that $-(n/2)eH$ is an eigenvalue of the shape operator $A$. When $\lambda_{1} = \cdots = \lambda_{m}$, then $trA = -(n/2)eH$. On the other hand, $trA = neH$. These two expressions imply $H = 0$, a contradiction.

So, in the following, we need only to discuss the situation where there are two distinct values among $\{\lambda_{1}, \ldots, \lambda_{m}\}$. Expression (5) also informs us that $\nabla H$ is an eigenvector of $A$ with corresponding eigenvalue $-(n/2)eH$. In view of (16), $\nabla H$ is one of the directions $u_{ij}, 1 \leq i \leq m$. Without loss of generality, we suppose $\nabla H$ is in the direction of $u_{ij}$: it may be a light-like vector or not. We will follow different processes to lead contradictions for these two cases.

First of all, we give a lot of equations deduced from compatibility and symmetry of the connection, as well as the Codazzi equation.

Observe the inner products of the elements in basis $\mathfrak{B}$ given in Section 2, we can express

$$\nabla H = \sum_{i=1}^{m} \sum_{a=1}^{n} \epsilon_{i}\mu_{i\alpha_{t+1}}(H)u_{ij}.$$  

Since $\nabla H$ is in the direction of $u_{ij}$, the above equation implies that

$$u_{ij}(H) \neq 0, u_{ij}(H) = 0, \ i_{a} \neq 1.\right)$$

Let $\nabla u_{ij} = \sum_{k=1}^{m} \sum_{d=1}^{n} f_{i\alpha_{t+1}}(H)u_{ij}$. Applying compatibility condition to calculate

$$\nabla u_{ij} = \nabla u_{ij}, \nabla u_{ij} = \nabla u_{ij}, \nabla u_{ij} = \nabla u_{ij},$$

we conclude

$$\Gamma_{i\alpha_{t+1}}^{d} = 0,\right)$$

$$\gamma_{i\alpha_{t+1}}^{d} = -\gamma_{i\alpha_{t+1}}^{d}, \gamma_{i\alpha_{t+1}}^{d} = -\gamma_{i\alpha_{t+1}}^{d},\right)$$

for $D \in \{k_{1}, 1 \leq k \leq m, 1 \leq e \leq \alpha_{t}\}, 1 \leq i, j \leq m, 1 \leq a, b \leq \alpha_{t}$, and $1 \leq d \leq \alpha_{t}$.

From the expression ($\nabla u_{ij} u_{ij} - \nabla u_{ij} u_{ij}) = [u_{ij}, u_{ij}]$, $B, C \in \{k_{1}, 1 \leq k \leq m, 1 \leq d \leq \alpha_{t}\}$, and (19), we easily get

$$\Gamma_{i\alpha_{t+1}}^{d} = \Gamma_{i\alpha_{t+1}}^{d}, B, C \neq 1.\right)$$

We state that in the proof, if not otherwise specified, then for $i_{a}$ in $u_{ij}$ and the connection’s coefficients, the ranges of $i$
and \(a\) are as follows: \(1 \leq i \leq m\) and \(1 \leq a \leq \alpha_i\). For the equations about the connection’s coefficients, when \(a = \alpha_i\) (or \(a = 1\)), then the terms about \(i_{a+1}\) (or \(i_{a-1}\)) disappear. And when \(b = \beta_j\) (or \(b = 1\)), then the terms about \(j_{b+1}\) (or \(j_{b-1}\)) and \(j_{b+1}\) (or \(j_{b-1}\)) disappear.

It follows from the Codazzi equation (3) that for any vector fields \(X, Y, Z\) tangent to \(M^n\),

\[
\langle (\nabla_X Y) Z \rangle = \langle (\nabla_Y X) Z \rangle. \tag{24}
\]

Start with this equation, we can get a series of equations about the coefficients of connection.

(i) For \((X, Y, Z) = (u_{j_1}, u_{j_2}, u_{j_3})\) in (24), then combining (19), we obtain

\[
\Gamma^{11}_{i_{1}k_{11}} = \Gamma^{11}_{i_{1}k_{11}}. \tag{25}
\]

Applying (21), (22), and (23), we get from the above equation that

\[
\Gamma^{11}_{i_{1}k_{11}} = \cdots = \Gamma^{11}_{i_{1}k_{11}} = \Gamma^{11}_{i_{1}k_{11}} = 0. \tag{26}
\]

Note that if \(\alpha_i = 1\), then (26) tells us nothing.

(ii) For \((X, Y, Z) = (u_{j_1}, u_{j_1}, u_{j_2})\), \(2 \leq i \leq m\), in (24), then combining (19), we have

\[
\left( \lambda_i + \frac{n}{2} eH \right) \Gamma^{11}_{i_{1}k_{11}} + \Gamma^{11}_{i_{1}k_{11}} = \Gamma^{11}_{i_{1}k_{11}}, \quad 2 \leq i \leq m, \tag{27}
\]

which together with (23) implies that if \(\lambda_i = -(n/2)eH\), then

\[
\Gamma^{11}_{i_{1}k_{11}} = 0, \tag{28}
\]

and if \(\lambda_i \neq -(n/2)eH\), then

\[
\Gamma^{11}_{i_{1}k_{11}} = 0. \tag{29}
\]

(iii) For \((X, Y, Z) = (u_{j_1}, u_{j_2}, u_{j_3})\), \(2 \leq i \leq m\), in (24), then

\[
-\Gamma^{11}_{i_{1}k_{11}} = \left( \lambda_i + \frac{n}{2} eH \right) \Gamma^{11}_{i_{1}k_{11}} + \Gamma^{11}_{i_{1}k_{11}} - \Gamma^{11}_{i_{1}k_{11}}, \tag{30}
\]

which together with (22), (23), (28), and (29) implies that if \(\lambda_i \neq -(n/2)eH\), then

\[
\Gamma^{11}_{i_{1}k_{11}} = 0, \quad 2 \leq i \leq m. \tag{31}
\]

(iv) For \((X, Y, Z) = (u_{j_1}, u_{j_1}, u_{j_2})\), \(2 \leq i \leq m\), in (24), then combining (23), we know

\[
\Gamma^{11}_{i_{1}k_{11}} = \Gamma^{11}_{i_{1}k_{11}}. \tag{32}
\]

From (22), (23), and the above relation, we conclude that

\[
\Gamma^{11}_{i_{1}k_{11}} = 0, \quad 2 \leq i \leq m, \tag{33}
\]

\[
\Gamma^{11}_{i_{1}k_{11}} = \Gamma^{11}_{i_{1}k_{11}}, \quad 2 \leq i \leq m, \quad 1 \leq a, b \leq \alpha_i - 1. \tag{34}
\]

(vi) For \((X, Y, Z) = (u_{j_1}, u_{j_1}, u_{j_2})\), \(2 \leq i \leq m\), in (24), then

\[
\left( \lambda_i + \frac{n}{2} eH - \lambda_j \right) \Gamma^{11}_{i_{1}k_{11}} - \Gamma^{11}_{i_{1}k_{11}} = u_{j_3}(\lambda_i) + \Gamma^{11}_{i_{1}k_{11}} - \Gamma^{11}_{i_{1}k_{11}}, \tag{35}
\]

which together with (22) and (34) gives that

\[
\Gamma^{11}_{i_{1}k_{11}} = \cdots = \Gamma^{11}_{i_{1}k_{11}} = 0, \quad 2 \leq i \leq m, \tag{36}
\]

\[
u_{j_3}(\lambda_i) = \left( \lambda_i + \frac{n}{2} eH - \lambda_j \right) \Gamma^{11}_{i_{1}k_{11}}, \quad 2 \leq i \leq m. \tag{37}
\]

(vii) For \((X, Y, Z) = (u_{j_1}, u_{j_1}, u_{j_2})\), \(2 \leq i, j \leq m\), in (24), then combining (23), we have

\[
\left( \lambda_i - \lambda_j \right) \Gamma^{11}_{i_{1}k_{11}} + \Gamma^{11}_{i_{1}k_{11}} = \Gamma^{11}_{i_{1}k_{11}}, \tag{38}
\]

which together with (23) tells us that when \(\lambda_i \neq \lambda_j\), then

\[
\Gamma^{11}_{i_{1}k_{11}} = 0, \quad 2 \leq i, j \leq m, \tag{39}
\]

\[
\Gamma^{11}_{i_{1}k_{11}} = \Gamma^{11}_{i_{1}k_{11}}. \tag{40}
\]
and when \( \lambda_i = \lambda_j \), then
\[
\Gamma^{i}_{i, j_{\omega_1}, 1} = 0, \tag{43}
\]
\[
\Gamma^{i}_{i, j_{\omega_1}} = \Gamma^{i}_{i, j_{\omega_1}, 1}, \quad \Gamma^{i}_{j_{\omega_1}} = \Gamma^{i}_{j_{\omega_1}, 1}, \tag{44}
\]
with \( 2 \leq i, j \leq m, 1 \leq a \leq \alpha_i, 1 \leq b \leq \alpha_j - 1 \). (43) and (44) give that if \( a > b \), then
\[
\Gamma^{i}_{j_{\omega_1}} = 0. \tag{45}
\]

(viii) For \((X, Y, Z) = (u_{i_1}, u_{i_2}, u_{i_2+1})\) with \(2 \leq i, j \leq m\), and \(i \neq j\) in (24), then
\[
\left( -\frac{n}{2} eH - \lambda_i \right) \Gamma^{i}_{i_{a_{1}}, i_{a_{1}+1}} = \left( \lambda_i - \lambda_j \right) \Gamma^{i}_{i_{a_{1}}, i_{a_{1}+1}} + \Gamma^{i}_{i_{a_{1}}, i_{a_{1}+1}}, \tag{46}
\]

(ix) For \((X, Y, Z) = (u_{i_1}, u_{i_2}, u_{i_2+1})\), \(a \neq d\), in (24), with \(\lambda_j = -(n/2)eH\) and \(\lambda_j \neq -(n/2)eH\), then
\[
\left( -\frac{n}{2} eH - \lambda_i \right) \Gamma^{i}_{i_{a_{1}}, i_{a_{1}+1}} + \Gamma^{i}_{i_{a_{1}}, i_{a_{1}+1}} - \Gamma^{i}_{i_{a_{1}}, i_{a_{1}+1}} = \Gamma^{i}_{i_{a_{1}}, i_{a_{1}+1}} - \Gamma^{i}_{i_{a_{1}}}, \tag{47}
\]

Let
\[
\begin{align*}
A_{i, j_{\omega_1}} &= \Gamma^{i}_{i_{a_{1}}}, + \Gamma^{i}_{i_{a_{1}}} + \cdots + \Gamma^{i}_{i_{a_{1}+1}}, \\
A_{i, j_{\omega_1}} &= \Gamma^{i}_{i_{a_{1}}}, + \Gamma^{i}_{i_{a_{1}}} + \cdots + \Gamma^{i}_{i_{a_{1}+1}}, \\
\vdots \\
A_{i, j_{\omega_1}} &= \Gamma^{i}_{i_{a_{1}}}, + \Gamma^{i}_{i_{a_{1}}} + \cdots + \Gamma^{i}_{i_{a_{1}+1}},
\end{align*} \tag{48}
\]
with \(1 \leq i, j \leq m, 1 \leq b \leq \alpha_i\), and \(\lambda_j = -(n/2)eH, \lambda_j \neq -(n/2)eH\). Since there
\[
\left( -\frac{n}{2} eH - \lambda_i \right) A_{i, j_{\omega_1}} + A_{i, j_{\omega_1}} - A_{i, j_{\omega_1}} = 0, \quad 1 \leq b \leq \alpha_i, 1 \leq e \leq \alpha_i - 2, \tag{49}
\]
\[
\left( -\frac{n}{2} eH - \lambda_i \right) A_{i, j_{\omega_1}} + A_{i, j_{\omega_1}} = 0, \quad 1 \leq b \leq \alpha_i,
\]
which implies that
\[
A_{i, j_{\omega_1}} = 0, \quad 1 \leq b \leq \alpha_i, 1 \leq e \leq \alpha_i - 1, \tag{50}
\]
with \(\lambda_j = -(n/2)eH, \lambda_j \neq -(n/2)eH\), and \(1 \leq i, j \leq m\).

(x) For \((X, Y, Z) = (u_{i_1}, u_{i_2}, u_{i_2+1})\) in (24), with \(\lambda_j = -(n/2)eH\) and \(\lambda_j \neq -(n/2)eH\), then
\[
\left( -\frac{n}{2} eH - \lambda_i \right) \Gamma^{i}_{i_{a_{1}}, i_{a_{1}+1}} + \Gamma^{i}_{i_{a_{1}}, i_{a_{1}+1}} - \Gamma^{i}_{i_{a_{1}}, i_{a_{1}+1}} = \Gamma^{i}_{i_{a_{1}}, i_{a_{1}+1}} - \Gamma^{i}_{i_{a_{1}}}, \tag{51}
\]
which combining (50) gives that
\[
\left( -\frac{n}{2} eH - \lambda_i \right) \sum_{a=1}^{n} \Gamma^{i}_{i_{a_{1}}, i_{a_{1}+1}} + \sum_{a=1}^{n} \Gamma^{i}_{i_{a_{1}}, i_{a_{1}+1}} = a_i u_{j_{\omega_1}} (\lambda_i). \tag{52}
\]

(xi) For \((X, Y, Z) = (u_{i_1}, u_{i_2}, u_{i_2+1})\) in (24), with \(\lambda_j = -(n/2)eH, \lambda_j \neq -(n/2)eH\), and \(i \neq k\), then
\[
\left( -\frac{n}{2} eH - \lambda_k \right) \Gamma^{i}_{i_{a_{1}}, i_{a_{1}+1}} + \Gamma^{i}_{i_{a_{1}}, i_{a_{1}+1}} - \Gamma^{i}_{i_{a_{1}}, i_{a_{1}+1}} = \Gamma^{i}_{i_{a_{1}}, i_{a_{1}+1}} - \Gamma^{i}_{i_{a_{1}}}, \tag{53}
\]
\[
\begin{align*}
& \left( -\frac{n}{2} eH - \lambda_k \right) B_{i, k, d_{\omega_1}} + B_{i, k, d_{\omega_1}} - B_{i, k, d_{\omega_1}+1} = 0, \\
& \left( -\frac{n}{2} eH - \lambda_k \right) B_{i, k, d_{\omega_1}} + B_{i, k, d_{\omega_1}} = 0,
\end{align*} \tag{55}
\]
\[
\begin{align*}
& \lambda_j = -(n/2)eH, \lambda_j \neq -(n/2)eH, \lambda_j \neq -(n/2)eH, \\
& B_{i, k, d_{\omega_1}} = 0, \quad 1 \leq b \leq \alpha_i, 0 \leq e \leq \alpha_i - 1. \tag{56}
\end{align*}
\]

Now, we treat the two cases that \(\text{VH}\) is not light-like or light-like separately and get contradictions.

**Case 1.** \(\text{VH}\) is not light-like.
As \(\text{VH}\) is in the direction of \(u_{i_{a_{1}}}\), \(\text{VH}\) is not light-like means that \(\alpha_i = 1\). Observe equation (38); we find if \(\lambda_j = -(n/2)eH\), \(\text{VH}\) is light-like, which contradicts with (19). So, we conclude that \(\lambda_j \neq -(n/2)eH\), for \(2 \leq i \leq m\). Since there are two distinct values among \(\{\lambda_1, \ldots, \lambda_m\}\) and \(\text{tr} A = n e H\), we have \(\lambda_2 = \cdots = \lambda_m = 3 n e H / 2 (n - 1)\). It follows from (34) and (38) that
\[
\Gamma^{i}_{i_{a_{1}}, i_{a_{1}+1}} = -\frac{3 u_{i_{a_{1}}} (H)}{(n + 2) e H} \quad 2 \leq i \leq m, 1 \leq a \leq \alpha_i. \tag{57}
\]

Denote \(W = \Gamma^{i}_{i_{a_{1}}, i_{a_{1}+1}}\), we have the following lemmas.
Lemma 2. We have

\[ u_t^i(W) + W^2 = \frac{3\varepsilon_i n^2 H^2}{4(n-1)} - \varepsilon c_i. \]  

(58)

Proof. Since \( \lambda_i = 3eH/2(n-1) \) with \( 2 \leq i \leq m \), it follows from (46) that

\[ \left( -\frac{n}{2} eH - \lambda_i \right) r^i_{i,1} = r^j_{i,1}, \quad 2 \leq i, j \leq m, \ i \neq j. \]  

(59)

Combining (43), we have

\[ C_1 r^i_{i,1} = 0, \]  

(60)

with \( 2 \leq i, j \leq m, i \neq j, \) and \( 2 \leq a \leq \alpha_i - 1 \).

Using Gauss equation for \( \{ R(u_i, u_{i,1}) u_{i,1}, u_i \} \), with \( 2 \leq i \leq m \), combining (29), (31), (35), (40), (42), (43) and (60), we have

\[ u_i^i \left( r^i_{i,1} \right)^2 = -\frac{1}{2} \sum_{2 \leq j,m \neq i} \left( \frac{1}{r^i_{i,1}} \right)^2 + \frac{3\varepsilon_i n^2 H^2}{4(n-1)} - \varepsilon c_i. \]  

(61)

From (22) and (23), we know \( r^i_{i,1} = \varepsilon \varepsilon_i r^i_{i,1} \), \( r^i_{i,1} = \varepsilon \varepsilon_i r^i_{i,1} \), with \( 2 \leq i, j \leq m \). So, we can rewrite the above equation as

\[ u_i^i \left( r^i_{i,1} \right)^2 = -\frac{1}{2} \sum_{2 \leq j,m \neq i} \left( \frac{1}{r^i_{i,1}} \right)^2 + \frac{3\varepsilon_i n^2 H^2}{4(n-1)} - \varepsilon c_i, \quad 2 \leq i \leq m. \]  

(62)

For \( 2 \leq i, j \leq m \) and \( i \neq j \), if \( \alpha_j < \alpha_i \), then we have \( \Gamma^i_{j,1} = \Gamma^i_{j,1} = 0 \) from (22), (43), and (44). And if \( \alpha_j = \alpha_i \), from (56), we know

\[ \Gamma^i_{j,1} + \Gamma^i_{j,1} + \cdots + \Gamma^i_{j,1} = 0, \]  

(63)

which together with (44), tells us that

\[ \Gamma^i_{j,1} = 0. \]  

(64)

So, for \( 2 \leq i, j \leq m \) and \( i \neq j \), if \( \alpha_j < \alpha_i \), then

\[ \Gamma^i_{j,1} = 0. \]  

(65)

For \( 2 \leq i, j \leq m \) and \( i \neq j \), if \( \alpha_j > \alpha_i \), combining (65), we get from (46) that

\[ \begin{align*}
&\left( -\frac{n(n+2)eH}{2(n-1)} \right) r^i_{i,1} = -r^i_{i,1}, \\
&\left( -\frac{n(n+2)eH}{2(n-1)} \right) r^i_{i,1} = -r^i_{i,1}, \\
&\vdots \\
&\left( -\frac{n(n+2)eH}{2(n-1)} \right) r^i_{i,1} = -r^i_{i,1}.
\end{align*} \]  

(66)

If \( \alpha_j > \alpha_i + 1 \), then (43) and (44) give us that

\[ r^i_{i,1} = \cdots = r^i_{i,1} = 0. \]  

(67)

As (67), it follows from (66) that for \( 2 \leq i, j \leq m \) and \( i \neq j \), if \( \alpha_j > \alpha_i + 1 \), then

\[ \Gamma^i_{j,1} = \cdots = \Gamma^i_{j,1} = 0. \]  

(68)

For \( 2 \leq i, j \leq m \) and \( i \neq j \), if \( \alpha_j = \alpha_i + 1 \), then combining (22) and (23), we get from (44) and (66) that

\[ \Gamma^i_{j,1} = \Gamma^i_{j,1} = \varepsilon \varepsilon_i r^i_{j,1}, \]  

(69)

\[ \Gamma^i_{j,1} = \alpha_i r^i_{j,1} = \varepsilon \varepsilon_i n(n+2)eH \]  

(69)

Therefore, from (59), (65), (67), (68), and (69), (62) can be simplified to

\[ \begin{align*}
&\frac{n(n+2)eH}{2(n-1)} \Gamma^i_{j,1} = \varepsilon \varepsilon_i \Gamma^i_{j,1}, \\
&\Gamma^i_{j,1} = \varepsilon \varepsilon_i n(n+2)eH \]  

(70)

Choosing \( \{ \alpha_1, \alpha_2, \ldots, \alpha_{h_i} \} \) such that

\[ \alpha_i = \alpha_i = \cdots = \alpha_i, \quad \alpha_i + 1 = \alpha_i + 1, \quad 1 \leq i \leq h, \]  

(71)

and \( \alpha_i - 1, \alpha_i + 1 \notin \{ \alpha_2, \ldots, \alpha_m \} \).
Taking sum in (70) for $i = k_1, k_2, \ldots, k_h$, with $1 \leq k \leq h$, we have

$$
\begin{align*}
\begin{pmatrix}
-\tilde{a}_1 + 1\right)Q_1 &= \xi_1 U, \\
\tilde{a}_1 Q_1 - (\tilde{a}_2 + 1)Q_2 &= \xi_2 U, \\
\tilde{a}_2 Q_2 - (\tilde{a}_3 + 1)Q_3 &= \xi_3 U, \\
\vdots
\end{pmatrix}
\begin{pmatrix}
\tilde{a}_{k-2} Q_{k-2} - (\tilde{a}_{k-1} + 1)Q_{k-1} &= \xi_{k-1} U, \\
\tilde{a}_{k-1} Q_{k-1} &= \xi_k U,
\end{pmatrix}
\end{align*}
$$

(72)

where

$$
\begin{align*}
U &= -u_{i_1}(W) - W^2 + 3\varepsilon_1 n^2 H^2 \\
Q_k &= n(n + 1) \sum_{1 \leq q \leq n} \sum_{k=1}^{1 \leq q \leq n} \varepsilon_{q,k} \left( \Gamma_{i_1 i_1}^{q,k} \right)^2, \quad \rho = k_1, \quad \sigma = \left( k + 1 \right) q.
\end{align*}
$$

(73)

From (72), we can get $Q_k = 0$, $1 \leq k \leq h - 1$, and $U = 0$, i.e., (58) holds.

Lemma 3. We have

$$
\sum_{i=2}^{m} \sum_{a=1}^{a_i} I_{i_1 i_1}^{a_i} u_{i_1} (H) + u_{i_1} u_{i_1} (H) - \varepsilon_1 e_i \left( \frac{n + 8}{4(n - 1)} \right) + n c e_i H = 0.
$$

(74)

Proof. For $1 \leq k \leq m$, if $\alpha_k$ is an even number, we put

$$
\begin{align*}
e_k &= \frac{u_{k_1} - u_{k_2}}{\sqrt{2}}, \quad k_2 = \frac{u_{k_1} + u_{k_2}}{\sqrt{2}},
\end{align*}
$$

(75)

with $1 \leq a \leq \alpha_k/2$. And if $\alpha_k$ is an odd number, we take

$$
\begin{align*}
e_k &= \frac{u_{k_1} - u_{k_2}}{\sqrt{2}}, \quad k_2 = \frac{u_{k_1} + u_{k_2}}{\sqrt{2}}, \quad 1 \leq a < \frac{\alpha_k + 1}{2},
\end{align*}
$$

(76)

and $e_{k_2} = e_{k_2}$. We easily find $E_k = \{e_k, e_{k_2}, \ldots, e_{k_h}\}$ is an orthonormal basis of $V_k$ with $1 \leq k \leq m$. Thus, $E = \{e_k, 1 \leq k \leq m, 1 \leq a \leq \alpha_k\}$ is an orthonormal basis of $T_\beta M^n$.

From (19), (75), and (76), we have when $\alpha_k$ is an even number,

$$
\begin{align*}
\nabla_{\alpha_k} e_k (H) &= \frac{1}{2} \left( I_{k,k}^{i_1} + I_{k_1 k_2}^{i_1} + I_{k_2 k_3}^{i_1} + I_{k_3 k_4}^{i_1} + I_{k_4 k}^{i_1} \right) u_{i_1} (H), \\
\nabla_{\alpha_k} e_{k_2} (H) &= \frac{1}{2} \left( I_{k,k}^{i_1} + I_{k_1 k_2}^{i_1} + I_{k_2 k_3}^{i_1} + I_{k_3 k_4}^{i_1} + I_{k_4 k}^{i_1} \right) u_{i_1} (H),
\end{align*}
$$

(77)

with $1 \leq a \leq \alpha_k/2$, and when $\alpha_k$ is an odd number,

$$
\begin{align*}
\nabla_{\alpha_k} e_k (H) &= \frac{1}{2} \left( I_{k,k}^{i_1} + I_{k_1 k_2}^{i_1} + I_{k_2 k_3}^{i_1} + I_{k_3 k_4}^{i_1} + I_{k_4 k}^{i_1} \right) u_{i_1} (H), \\
\nabla_{\alpha_k} e_{k_2} (H) &= \frac{1}{2} \left( I_{k,k}^{i_1} + I_{k_1 k_2}^{i_1} + I_{k_2 k_3}^{i_1} + I_{k_3 k_4}^{i_1} + I_{k_4 k}^{i_1} \right) u_{i_1} (H),
\end{align*}
$$

(78)

with $1 \leq a < (\alpha_k + 1)/2$.

Note that $e_{i_1} = u_{i_1}$; it follows from (19) that

$$
e_{i_1} (H) \neq 0, \quad e_{i_1} (H) = 0, \quad 2 \leq i \leq m, 1 \leq a \leq \alpha_i.
$$

(79)

By calculating, we get

$$
\text{tr} A^2 = \frac{(n + 8)n^2 H^2}{4(n - 1)}.
$$

(80)

Since the above, we obtain from (6) that

$$
\sum_{i=2}^{m} \sum_{a=1}^{a_i} I_{i_1 i_1}^{a_i} u_{i_1} (H) + u_{i_1} u_{i_1} (H) - \varepsilon_1 e_i \left( \frac{n + 8}{4(n - 1)} \right) + n c e_i H = 0.
$$

(81)

Now, we continue the proof of Theorem 1 for Case 1. Combining (57) and (58), we get

$$
\begin{align*}
&u_{i_1} u_{i_1} (H) - \frac{(n + 2)(n + 5)H}{9} W^2 - \frac{(n + 2)n^2 \varepsilon_i H^3}{4(n - 1)} \\
&+ \frac{1}{3} (n + 2) c e_i H.
\end{align*}
$$

(82)
Substitute the above equation into (74), considering \(a_2 + \cdots + a_m = n(a_1 = 1)\) and (57), we have

\[
\left\{ \frac{2(n-4)(n+2)}{9} W^2 + \frac{(n+2+\varepsilon(n+8))n^2 \varepsilon_1 H^2}{4(n-1)} - \frac{2}{3}(2n+1)\varepsilon_1 \right\} H = 0. 
\]

(83)

Act on (83) with \(u_{i_1}\), and using (57) and (58), we obtain

\[
\left\{ \frac{2(n-4)(n+2)}{9} W^2 + \frac{(n+2+\varepsilon(n+8))n^2 \varepsilon_1 H^2}{4(n-1)} - \frac{2}{3}(2n+1)\varepsilon_1 \right\} u_{i_1} \cdot (H) - 2(n+2)
\]

\[
\cdot \left\{ \frac{2(n-4)}{9} W^2 + \frac{(n-10-\varepsilon(n+8))n^2 \varepsilon_1 H^2}{12(n-1)} + \frac{2(n-4)}{9} \varepsilon_1 \right\} H W = 0.
\]

(84)

As (83) and \(W \neq 0\), the above equation implies that

\[
\left\{ \frac{2(n-4)}{9} W^2 + \frac{(n-10-\varepsilon(n+8))n^2 \varepsilon_1 H^2}{12(n-1)} + \frac{2(n-4)}{9} \varepsilon_1 \right\} H = 0,
\]

(85)

which together with (83) gives that

\[
-\frac{2}{9}(n-1)(n+5)\varepsilon_1 H = 0,
\]

(86)

which implies that \(H\) is a constant, a contradiction.

**Case 2.** \(\nabla H = 0\). Notice that \(\nabla H = 0\) is light-like means \(a_i \geq 2\). Since \(\text{tr}A = n\varepsilon H\), we can suppose \(\lambda_1 = \cdots = \lambda_p = -(n/2)\varepsilon H\) and \(\lambda_{p+1} = \cdots = \lambda_m = (2 + a_1 + \cdots + a_p)n\varepsilon H/(2 + a_{p+1} + \cdots + a_m)\). It follows from (19) and (38) that

\[
I_{i_1 j_1 k_1} = 0, \quad p + 1 \leq i \leq m. 
\]

(87)

Calculate \(\langle R(u_{i_2}, u_{i_1})u_{j_1}, u_{k_1}, \cdots \rangle\) and \(\langle R(u_{j_1}, u_{j_1})u_{i_1}, \cdots \rangle\) with \(p + 1 \leq i \leq m\) by Gauss equation, and letting \(i\) and \(a\) for sum with \(p + 1 \leq i \leq m\) and \(1 \leq a \leq a_i\), combining (21), (22), (23) (26), (28), (29), (35), (42), (45), and (87), we get

\[
\sum_{i=p+1}^{m} \sum_{a_p}^{a} \sum_{d=3}^{a_i-1} I_{i_1 j_1 k_1} + \sum_{k=2}^{p} \sum_{k=2}^{a_k} I_{i_1 j_1 k_1} = 0,
\]

(88)

\[
\sum_{i=p+1}^{m} \sum_{a_p}^{a} \sum_{d=3}^{a_i-1} I_{i_1 j_1 k_1} + \sum_{k=2}^{p} \sum_{k=2}^{a_k} I_{i_1 j_1 k_1} = 0,
\]

(89)

Since (50) and (56), we find

\[
\sum_{i=p+1}^{m} \sum_{a_p}^{a} \sum_{d=3}^{a_i-1} I_{i_1 j_1 k_1} + \sum_{k=2}^{p} \sum_{k=2}^{a_k} I_{i_1 j_1 k_1} = 0.
\]

(90)

So, (88) and (89) can be simplified to

\[
\sum_{i=p+1}^{m} \sum_{a_p}^{a} \sum_{d=3}^{a_i-1} I_{i_1 j_1 k_1} + \sum_{k=2}^{p} \sum_{k=2}^{a_k} I_{i_1 j_1 k_1} = 0,
\]

(91)

\[
\sum_{i=p+1}^{m} \sum_{a_p}^{a} \sum_{d=3}^{a_i-1} I_{i_1 j_1 k_1} + \sum_{k=2}^{p} \sum_{k=2}^{a_k} I_{i_1 j_1 k_1} = 0.
\]

(92)

with \(1 \leq k \leq p\) and \(k_d \neq 1\).

Multiply both sides of (92) with \(-n/2\varepsilon H - \lambda_m\), we get

\[
\sum_{i=p+1}^{m} \sum_{a_p}^{a} \sum_{d=3}^{a_i-1} \left( -\frac{n}{2} \varepsilon H - \lambda_m \right) I_{i_1 j_1} = 0,
\]

(93)

Combining (87) and (93), it gives us

\[
\sum_{i=p+1}^{m} \sum_{a_p}^{a} \sum_{d=3}^{a_i-1} \left( -\frac{n}{2} \varepsilon H - \lambda_m \right) I_{i_1 j_1} = 0,
\]

(94)

\[
\sum_{i=p+1}^{m} \sum_{a_p}^{a} \sum_{d=3}^{a_i-1} \left( -\frac{n}{2} \varepsilon H - \lambda_m \right) I_{i_1 j_1} = 0.
\]

(95)
Since (91), we get from (95) that
\[
\left( -\frac{n}{2} eH - \lambda_m \right) \left( c - \frac{1}{2} nH \lambda_m \right) = 0, \tag{96}
\]
which combining \( \lambda_m \neq -(n/2)eH \) and \( \lambda_m = (2 + \alpha_1 + \cdots + \alpha_m) neH/2(\alpha_{p+1} + \cdots + \alpha_m) \), tells us that \( c - ((2 + \alpha_1 + \cdots + \alpha_m)n^2eH^2/4(\alpha_{p+1} + \cdots + \alpha_m)) = 0 \).

So, \( H \) is a constant, a contradiction.

In view of the two cases, we complete the proof of Theorem 1.

**Remark 4.** With the assumption that the shape operator is diagonalizable, the result of Theorem 1 was proved in [6, 8, 12] when the hypersurface \( M^n \) has at most three distinct principal curvatures.

Applying Theorem 1, we can get the following theorem.

**Theorem 5.** Let \( M^n \) be a nondegenerate biharmonic hypersurface of pseudo-Riemannian space form \( N^{n+1}_s \), with \( ce \leq 0 \). Suppose that \( M^n \) has at most two distinct principal curvatures, which are all real numbers, then \( M^n \) is minimal.

**Proof.** We know from Theorem 1 that \( H \) is a constant; it follows from (6) that
\[
H trA^2 = nceH. \tag{97}
\]

When \( ce \leq 0 \), it is easy to see from (97) that \( H = 0 \). When \( ce \leq 0 \), then (97) implies \( H = 0 \), or \( trA^2 = 0 \). Since \( trA^2 \) is equal to the sum of the squares of all principal curvatures and \( trA^2 \) is equal to the sum of all principal curvatures, so \( trA^2 = 0 \) tells us \( trA^2 = 0 \). Combining \( trA = neH \), we have \( H = 0 \), i.e., \( M^n \) is minimal.

For odd-dimensional hypersurfaces, under the assumption that hypersurfaces have at most two distinct principal curvatures, the principal curvatures are all real. So, from Theorem 5, the following theorems are true.

**Theorem 6.** Let \( M^{2k-1}_x \) be a nondegenerate biharmonic hypersurface of pseudo-Euclidean space \( \mathbb{E}^{2k} \). Suppose that \( M^{2k-1}_x \) has at most two distinct principal curvatures, then \( M^{2k-1}_x \) is minimal.

**Theorem 7.** Let \( M^{2k-1}_x \) be a nondegenerate biharmonic hypersurface of de Sitter space \( S^{2k}(c) \). Suppose that \( M^{2k-1}_x \) has at most two distinct principal curvatures, then \( M^{2k-1}_x \) is minimal.

**Theorem 8.** Let \( M^{2k-1}_x \) be a nondegenerate biharmonic hypersurface of anti-de Sitter space \( \mathbb{H}^{2k}(c) \). Suppose that \( M^{2k-1}_x \) has at most two distinct principal curvatures, then \( M^{2k-1}_x \) is minimal.

**Remark 9.** Under the assumption that the shape operator is diagonalizable, the results of Theorems 6, 7, and 8 were proven not only for odd-dimensional hypersurfaces but also for even-dimensional hypersurfaces (cf. [6, 8]).

**Data Availability**

The data used to support the findings of this study are included within the article.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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