Frölicher-smooth geometries, quantum jet bundles 
and BRST symmetry

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Abstract

We attempt a clarification of geometric aspects of quantum field theory by using the
notion of smoothness introduced by Frölicher and exploited by several authors in the
study of functional bundles. A discussion of momentum and position representations in
curved spacetime, in terms of generalized semi-densities, leads to a definition of quantum
configuration bundle which is suitable for a treatment of that kind. A consistent approach
to Lagrangian field theories, vertical infinitesimal symmetries and related currents is then
developed, and applied to a formulation of BRST symmetry in a gauge theory of the
Yang-Mills type.

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Introduction and summary

Our understanding of quantum field theory (QFT) may benefit from clarifications regarding the differential geometric notions underlying it. Usually one considers a quantised version of a classical field, which is a section of a finite-dimensional bundle over spacetime, by seeing it as “operator-valued”. Apparently, the standard formalism of differential geometry seamlessly carries over to this infinite-dimensional setting, but a closer look reveals complications.

One should also bear in mind that QFT is presently better understood and effective in a very specialized context, where one essentially works in Minkowski spacetime with a chosen observer, and background parallel transport of internal configurations is available. These background structures allow various simplifications and identifications; in particular, the spatial Fourier transform yields a full correspondence between position and momentum representations. A possible covariant formulation on a curved fixed background, on the other hand, besides giving us some opportunity of studying gravitational effects in particle physics, constrains our setting to be meaningful at a higher level, and also allows us to distinguish where exactly the various needed assumptions enter the picture.

A useful ingredient in that quest is the notion of smoothness introduced by Frölicher [23], or F-smoothness, which has been studied and exploited by several authors [24 30 9 10 11 12 16] and provides a convenient approach to infinite-dimensional bundles. The differential geometric notions of tangent, vertical and jet functors, bundle connections, the Frölicher-Nijenhuis algebra of tangent-valued forms, all can be introduced in the context of functional and distributional bundles over a finite dimensional base manifold by a fairly direct and general procedure, without heavy involvement in infinite-dimensional topology. In particular, the notion of a smooth connection of a functional bundle has been applied in the context of the “covariant quantization” approach to Quantum Mechanics [13, 26, 23, 29, 31, 31, 32, 38, 40, 41], while the geometry of distributional bundles is used in an approach, proposed by this author, to the physics of interacting quantum particles [17, 21].

In the literature one finds works about BRST cohomology and related issues which treat the matter in a much more ample and general way than this one [6, 1, 7, 2, 3]. Admittedly we do not aim at such generality, but rather at a close examination of some essential notions in the concrete context of Yang-Mills theories. Thus we try to avoid intricacies such as infinite jets or Fréchet space topology, while at the same time we try not to overlook certain complications which, particularly in the physical literature, are often dealt with in a rather informal way.

F-smoothness enters the present paper in two distinct ways: the geometry of quantum state bundles, which was introduced in previous papers and is reviewed here, and the geometry of those bundles over spacetime whose sections are the quantum fields. The relation between these two complementary aspects is examined in [11] and it is argued that their equivalence is, in general, only partial and dependent on a suitable linearization and the choice of a spacetime synchronization; but we remark that a distinguished time is actually needed in all approaches to QFT.

In [2] we lay down the basics of F-smooth geometry for quantum fields. The quantum configuration bundle is obtained from the classical bundle via fiber tensor product by a certain infinite dimensional $\mathbb{Z}_2$-graded algebra which was defined in [11]. Jet bundles and connections can be introduced by a straightforward adaptation of the general procedure already used in other contexts, while the notion of fiber partial derivative and dual bundles require special care. We then apply those ideas to a jet-bundle approach to Lagrangian field theory, drawing the guidelines from the well-developed classical theory [25, 29, 31, 32, 33, 38, 39]. There one deals with arbitrary exterior forms on jet bundles of the configuration bundle, which can be
actually introduced in the said quantum context but present us with some complications. In order to deal with vertical infinitesimal symmetries, however, the restricted notion of basic form (or “totally horizontal” form [35]) is sufficient. The symmetry naturally acts on such forms by a “pseudo-Lie” derivation which raises the order of the form. The result of applying this to a Lagrangian density splits as a horizontal differential plus (essentially) the Euler-Lagrange operator, whence we derive a special version of the Noether theorem and the notions of current and charge associated with the symmetry.

Finally we apply these ideas to a formulation of BRST symmetry in a gauge theory of the Yang-Mills type. The classical geometric background and the issue of gauge freedom are discussed in a fairly general setting, though we provisionally disregard the distinction between the right-handed and left-handed sectors and symmetry breaking. Then we introduce the vertical vector field generating the symmetry, and calculate the corresponding current. A comparison of the equivalent first-order and second-order versions of the ghost Lagrangian yields some insight about Noether’s theorem in this context.

1 Quantum bundles and quantum fields

1.1 Finitely-generated multi-particle algebra

Let \( Z \rightarrow X \) be a finite-dimensional vector bundle. Denote by \( \mathcal{Z}^1 \) the vector space of all sections \( z : X \rightarrow Z \) which vanish outside some finite subset \( X_2 \subset X \),

\[
\mathcal{Z}^1 = \left\{ |z\rangle = \sum_{x \in X} |z(x)\rangle \right\}, \quad X \supset X_2 \text{ finite}
\]

(we denote \( z \) and \( z(x) \in Z_x \) as “ket” objects when it is convenient). Note that the term “section” is used here in a broad sense, and \( X \) could actually be an arbitrary set; in practice, we’ll deal with smooth bundles.

This \( \mathcal{Z}^1 \) is our template for the space of states of one particle of some type. We define the associated “\( n \)-particle state” space \( \mathcal{Z}^n \) to be either the symmetrised tensor product \( \vee^n \mathcal{Z}^1 \) (bosons) or the anti-symmetrised tensor product \( \wedge^n \mathcal{Z}^1 \) (fermions). The “multi-particle state” space is \( \mathcal{Z} = \bigoplus_{n=0}^{\infty} \mathcal{Z}^n \) (constituted by finite sums with arbitrarily many terms). If \( y \in \mathcal{Z}^m, \ z \in \mathcal{Z}^n \), then we define \( y \odot z \in \mathcal{Z}^{m+n} \) to be either \( y \vee z \) or \( y \wedge z \). By abuse of language we call this the “exterior product” of \( y \) and \( z \), and extend it to any elements in \( \mathcal{Z} \) by linearity.

Let \( Z^* \rightarrow X \) be the dual vector bundle. The dual space of \( \mathcal{Z}^1 \) is the vector space of all sections \( X \rightarrow Z^* \), which we can formally write as infinite sums \( \zeta = \sum_{x \in X} \langle \zeta(x) \rangle \). Actually \( \langle \zeta, z \rangle = \sum_{x \in X} \langle \zeta(x), z(x) \rangle \), a finite sum. For our purposes we may as well work with its subspace \( \mathcal{Z}^1 \) constituted of all such sections which vanish outside some finite subset. Exterior product (with the same parity as that of the associated \( \mathcal{Z}^1 \)) yields now spaces \( \mathcal{Z}^n \) and the “dual multi-particle space” \( \mathcal{Z}^* \equiv \bigoplus_{n=0}^{\infty} \mathcal{Z}^n \).

Next we introduce an “interior product” \( \lambda \mid \psi \), where \( \psi \in \mathcal{Z} \) and \( \lambda \in \mathcal{Z}^* \). For fermions, this is the usual interior product \( i\lambda \langle \psi \rangle \) of exterior algebra. For bosons it can be defined similarly, as tensor contraction with appropriate symmetrization and normalization, so that the rule \( \langle \zeta \odot \lambda \rangle \mid \psi = \lambda \mid \langle \zeta \mid \psi \rangle \) holds for all \( \zeta \in \mathcal{Z}^1, \lambda \in \mathcal{Z}^* \).

A general theory of quantum particles has several particle types. Correspondingly, one considers several multi-particle state spaces (or “sectors”) \( \mathcal{Z}', \mathcal{Z}'', \mathcal{Z}''' \) etc. The total state space is now defined to be

\[
\mathcal{V} := \mathcal{Z}' \otimes \mathcal{Z}'' \otimes \mathcal{Z}''' \otimes \cdots = \bigoplus_{n=0}^{\infty} \mathcal{V}^n
\]
where $\mathcal{V}^n$, constituted of all elements of tensor rank $n$, is the space of all states of $n$ particles of any type. We observe that if $\mathcal{X}$ and $\mathcal{Y}$ are any two vector spaces, then their antisymmetric tensor algebras fulfill the isomorphisms

$$\Lambda^p(\mathcal{X} \oplus \mathcal{Y}) \cong \bigoplus_{h=0}^p (\Lambda^{p-h} \mathcal{X}) \otimes (\Lambda^h \mathcal{Y}) \ , \quad (\Lambda \mathcal{X}) \otimes (\Lambda \mathcal{Y}) \cong \Lambda (\mathcal{X} \oplus \mathcal{Y}) .$$

Hence all fermionic sectors can be described by a unique overall antisymmetrised tensor algebra. A similar observation holds true for the bosonic sectors, while we regard mutual ordering of fermionic and bosonic sectors as inessential. Similarly one constructs a “dual” space $\mathcal{V}^* := \mathcal{Z}^* \otimes \mathcal{Z}^\ast \otimes \mathcal{Z}^{\ast \ast} \ldots = \bigoplus_{n=0}^\infty \mathcal{V}^{*n}$.

If we now let the grade $[\phi]$ of a monomial element (a “decomposable tensor”) $\phi \in \mathcal{V}$ to be the parity of the number of fermion factors it contains, then we can see $\mathcal{V}$ as a “super-algebra” (a $\mathbb{Z}_2$-graded algebra), products being performed in the appropriate tensor factors. Similarly, the interior product is defined by performing interior products in the appropriate tensor factors.

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$$\psi \circ \phi = (-1)^{[\phi][\psi]} \phi \circ \psi \ , \quad (\zeta \circ \xi) | \psi = \xi | (\zeta | \psi) ,$$

$$z | (\phi \circ \psi) = (z | \phi) \circ \psi + (-1)^{[z][\phi]} \phi \circ (z | \psi) , \quad \phi, \psi \in \mathcal{V}, \ \zeta, \xi \in \mathcal{V}^{*1} ,$$

valid whenever each of the involved factors has a definite grade. A linear map $X : \mathcal{V} \to \mathcal{V}$ is called a super-derivation (or anti-derivation) of grade $[X]$ if $[X \psi] = [X] + [\psi]$ and if it fulfills the graded Leibnitz rule

$$X(\phi \circ \psi) = (X \phi) \circ \psi + (-1)^{[X][\phi]} \phi \circ X \psi .$$

The absorption operator associated with $\zeta \in \mathcal{V}^{*1}$ and the emission operator associated with $z \in \mathcal{V}^1$ are the linear maps $\mathcal{V} \to \mathcal{V}$ respectively defined as

$$a[\zeta] \phi \equiv \zeta | \phi , \quad a^* [z] \phi \equiv z \circ \phi , \quad \phi \in \mathcal{V} .$$

Similarly we have operators $a[z], a^* [\zeta] : \mathcal{V}^* \to \mathcal{V}^*$. We obtain $\lambda | a[\zeta] \psi = (a^* [\zeta] \lambda) | \psi$ for all $\lambda \in \mathcal{V}^*$, namely $a[\zeta]$ and $a^* [\zeta]$ are mutually transposed endomorphisms.

By finite compositions and finite linear combinations, absorption and emission operators generate a vector subspace $\mathcal{Q} \subset \text{End} \mathcal{V}$. This turns out to be a $\mathbb{Z}_2$-graded algebra (the algebra product being the composition of endomorphisms) by letting the grades of $a[\zeta]$ and $a^* [z]$ be $[\zeta]$ and $[z]$ , respectively. The super-bracket of $X, Y \in \mathcal{Q}$ is then defined by

$$\{[X, Y] \} := XY - (-1)^{[X][Y]} YX$$

whenever both $X$ and $Y$ have definite grade, and extended by linearity. In particular, for all $y, z \in \mathcal{V}^1$ and $\zeta, \xi \in \mathcal{V}^{*1}$ we get

$$\{[a[\zeta], a[\xi]]\} = \{[a^*[z], a^* [\zeta]]\} = 0 \ , \quad \{[a[\zeta], a^* [z]]\} = (\zeta, z) \ 1 .$$

Remark. If $X$ and $Y$ are derivations then $\{[X, Y]\}$ turns out to be a derivation of grade $[X] + [Y]$. Furthermore, the map $\text{ad}_X : Y \mapsto \text{ad}_X Y \equiv \{[X, Y]\}$ turns out to be a derivation of grade $[\text{ad}_X] = [X]$ in $\mathcal{Q}$.

1 This expression, as well as others which follow, involves decomposable elements; extension by linearity must be understood.
On the other hand we also have, on the same underlying vector space, a graded algebra \( \mathcal{O} \equiv \bigoplus_{n=0}^{\infty} \mathcal{O}^n \) where \( \mathcal{O}^0 \equiv \mathbb{C} \) and \( \mathcal{O}^n \) is the space spanned by compositions of \( n \) emission and absorption operators. Now we observe that as a vector space \( \mathcal{O} \) is naturally isomorphic to \( \mathcal{V} \otimes \mathcal{V}^* \cong \bigoplus_{n=0}^{\infty} (\mathcal{V}^n \oplus \mathcal{V}^1) \). The identification is obtained by using the above super-commutation rules in order to move all absorption operators to the right of any emission operators (normal order). On the other hand, if we replace the last rule by letting \( a[z] \) and \( a^*[z] \) super-commute, then we get an isomorphism of \( \mathbb{N} \)-graded algebras. The situation is quite similar to that of a Clifford algebra, where the scalar product determines a new algebra structure on the exterior algebra of a vector space; the new product is not “super-commutative”.

In most practical cases \( Z \rightarrow X \) is a complex bundle with a Hermitian structure in its fibers. This yields an isomorphism \( \# : Z^* \rightarrow \mathcal{Z} : \zeta \mapsto \zeta^\# \) over \( X \), where \( \mathcal{Z} \rightarrow X \) is the conjugate bundle; the inverse of \( \# \) is denoted as \( b : \bar{z} \mapsto \bar{z}^\# \). The induced anti-isomorphism \( \mathcal{Z}^{1} \rightarrow \mathcal{Z}^{*1} \) is traditionally denoted as \( |z| \mapsto \langle z \rangle \). Extending it to an anti-isomorphism \( \mathcal{Z} \leftrightarrow \mathcal{Z}^* \) is straightforward.

We also note that whenever a sector corresponding to a complex bundle \( Z \) is considered, then the theory also includes the sector corresponding to \( \bar{Z} \). Accordingly, for \( \zeta \in \mathcal{V}^{*1} \) we have the “anti-particle” emission operator \( a^1[\zeta] \equiv a^*[\zeta^\#] \).

Remark. In the case of Dirac spinors one uses two different Hermitian structures: the (intrinsic) Dirac conjugation is denoted as \( \bar{\gamma} \). Accordingly, for \( \zeta \in \mathcal{V}^{*1} \) we have the “anti-particle” emission operator \( a^1[\zeta] \equiv a^*[\bar{\zeta}^\#] \).

1.2 Quantum states as generalised semi-densities

Let \( Z \rightarrow X \) be a finite-dimensional complex vector bundle, \( \dim_X X = m \). Assume that \( X \) is orientable, and choose a positive semi-vector bundle\(^2\) \( \mathcal{V} \equiv \mathcal{V} \equiv (\wedge^m TX)^+ \). A section \( X \rightarrow \mathcal{V}^{-1/2} \otimes Z \) is called a \( Z \)-valued semi-density. The vector space of all such sections which are smooth and have compact support is denoted as \( \mathcal{D}_c(X, Z) \). The dual space of \( \mathcal{D}_c(X, Z^*) \) in the standard test map topology \(^{[36]} \) is indicated as \( \mathcal{D}(X, Z) \) and called the space of \( Z \)-valued generalised semi-densities. In particular, a sufficiently regular ordinary section \( \theta : X \rightarrow \mathcal{V}^{-1/2} \otimes Z \) is in \( \mathcal{D}(X, Z) \) via the rule \( (\theta, \sigma) := \int_X \langle \theta(x), \sigma(x) \rangle, \sigma \in \mathcal{D}_c(X, Z^*) \).

Semi-densities have a special status among all kinds of generalised sections because of the natural inclusion \( \mathcal{D}_c(X, Z) \subset \mathcal{D}(X, Z) \). Furthermore, if a fibered Hermitian structure of \( Z \rightarrow X \) is assigned then one has the space \( \mathcal{L}^2(X, Z) \) of all ordinary semi-densities \( \theta \) such that \( \langle \theta^\dagger, \theta \rangle < \infty \). The quotient \( \mathcal{H}(X, Z) \equiv \mathcal{L}^2(X, Z)/\mathcal{O} \) is then a Hilbert space (here \( \mathcal{O} \subset \mathcal{L}^2(X, Z) \) denotes the subspace of all almost-everywhere vanishing sections), and we get a so-called rigged Hilbert space \(^{[4]} \).

\( \mathcal{D}_c(X, Z) \subset \mathcal{H}(X, Z) \subset \mathcal{D}(X, Z) \).

Elements in \( \mathcal{D}(X, Z) \setminus \mathcal{H}(X, Z) \) can then be identified with the (non-normalizable) generalised states of the common terminology.

Let \( \delta[x] \) be the Dirac density on \( X \) with support \( \{x\}, x \in X \). A generalised semi-density is said to be of Dirac type if it is of the form \( \delta[x] \otimes v \in \mathcal{D}(X, Z) \) with \( v : X \rightarrow \mathcal{V}^{1/2} \otimes Z \). We define \( \mathcal{D}(X, Z) \) to be the space of all finite linear combinations of Dirac-type semi-densities. An important result in the theory of distributions \(^{[36]} \) implies that \( \mathcal{D}(X, Z) \) is dense in \( \mathcal{D}(X, Z) \), namely any generalised semi-density can be approximated with arbitrary
1.2 Quantum states as generalised semi-densities

precision (in the sense of the topology of distributional spaces) by a finite linear combination of Dirac-type semi-densities.

An isomorphism \( Z_1 \leftrightarrow D_1 \equiv D(X, Z) \) is determined by the assignment of a volume form \( \eta : X \to \mathcal{V}^{-1} \). If \( (b_\alpha) \) is a frame of \( Z \to X \) (for notational simplicity we assume its domain to be the whole \( X \) ), then this isomorphism is characterised by the correspondence

\[
|x\rangle \otimes b_\alpha(x) \leftrightarrow B_{x\alpha} \equiv \delta[x] \otimes \eta^{-1/2} \otimes b_\alpha(x).
\]

The set \( \{B_{x\alpha}\} \subset D_1 \) is called a generalised basis. Accordingly we introduce a handy “generalised index” notation. We write \( B x_\alpha \equiv \delta[x] \otimes \eta^{-1/2} \otimes b^{\alpha}(x) \), where \( (b^{\alpha}) \) is the dual classical frame. Though contraction of any two distributions is not defined in the ordinary sense, a straightforward extension of the discrete-space operation yields

\[
\langle B^{x_\alpha'}, B_{x\alpha} \rangle = \delta^{x'}_x \delta^{\alpha'}_\alpha,
\]

where \( \delta^{x'}_x \) is the generalised function usually indicated as \( \delta(x' - x) \). This is consistent with “index summation” in a generalised sense: if \( z \in D_0(X, Z) \) and \( \zeta \in D_0(X, Z^*) \) are test semi-densities, then we write

\[
z^{x_\alpha} \equiv z^{\alpha}(x) \equiv \langle B^{x_\alpha}, z \rangle,
\]

\[
\zeta_{x\alpha} \equiv \zeta^{\alpha}(x) \equiv \langle \zeta, B_{x\alpha} \rangle,
\]

\[
\langle \zeta, z \rangle \equiv \zeta_{x\alpha'} z^{x_\alpha} \langle B^{x_\alpha'}, B_{x\alpha} \rangle \equiv \int_X \zeta^{\alpha}(x) z^{\alpha}(x) \eta(x),
\]

namely we interpret index summation with respect to the continuous variable \( x \) as integration, provided by the chosen volume form. This formalism can be extended to the contraction of two generalised semi-densities whenever it makes sense.

We extend the above constructions to multi-particle spaces. The identification \( Z_1 \leftrightarrow D_1 \) extends to \( Z^n \leftrightarrow D^n \equiv \otimes^n D_1 \). Then \( Z^n \) turns out to be dense in \( Z^n \), defined to be either the symmetrised or the antisymmetrised subspace of \( D(X^n, \otimes^n Z) \), \( X^n \equiv X \times \cdots \times X \). Next we set \( Z \equiv \bigoplus_{n=0}^\infty Z^n \), and assemble several particle types into one total state space \( \mathcal{V} := Z' \otimes Z'' \otimes \cdots \). An analogous construction yields the “dual” space \( \mathcal{V}^* \). Furthermore, using the spaces \( D_0 \) of test semi-densities we obtain subspaces \( \mathcal{V}_0 \subset \mathcal{V} \) and \( \mathcal{V}_0^* \subset \mathcal{V}^* \). All these spaces are naturally \( \mathbb{Z}_2 \)-graded.

Also the constructions related to the operator algebra (§1) can be extended to the present setting. If \( \zeta \in \mathcal{V}^{x_1} \) and \( z \in \mathcal{V}^1 \) then the absorption operator \( a[\zeta] \) and the emission operator \( a^*[z] \) are well-defined linear maps \( \mathcal{V}_0 \to \mathcal{V} \). The vector space \( \mathcal{O}^1 \) of all sums of the kind \( a[\zeta] + a^*[z] \) has the subspace \( \mathcal{O}^1 \) of all finite linear combinations of absorption and emission operators associated with Dirac-type semi-densities. In particular we write \( a^{x_\alpha} \equiv a[B^{x_\alpha}], \ a^{x_\alpha}_\alpha \equiv a^{x_\alpha}[B_{x\alpha}] \), and obtain super-commutation rules

\[
\{a^{x_\alpha}, a^{x'_{\alpha'}}\} = \{a^{x_\alpha}_\alpha, a^{x'_{\alpha'}}_{\alpha'}\} = 0,
\]

\[
\{a^{x_\alpha}, a^{x'_{\alpha'}}\}_\alpha = \delta^{x'}_{x} \delta^{\alpha'}_{\alpha},
\]

where the latter is to be understood in a generalised sense: for \( \zeta \in \mathcal{V}^{x_1}, z \in \mathcal{V}^1 \), we write

\[
\{a[\zeta], a^*[z]\} = \{\zeta_{x\alpha} a^{x_\alpha}, z^{x'_{\alpha'}} a^{x'_{\alpha'}}_{\alpha'}\} = \zeta_{x\alpha} z^{x'_{\alpha'}} \{a^{x_\alpha}, a^{x'_{\alpha'}}_{\alpha'}\} = \langle \zeta, z \rangle.
\]

Next we denote as \( \mathcal{O}^n \), \( n \in \mathbb{N} \), the vector space spanned by all compositions of normally ordered \( n \) emission and absorption operators. A product \( \mathcal{O}^n \times \mathcal{O}^p \to \mathcal{O}^{n+p} \) can be defined as composition together with normal reordering, obtained by imposing the modified rule

\[
\{a^{x_\alpha}, a^*_{x'_{\alpha'}}\} = 0.
\]
Setting $\mathcal{O}^0 \equiv \mathbb{C}$ we obtain a graded algebra $\mathcal{O} \equiv \bigoplus_{n=0}^{\infty} \mathcal{O}^n$ of linear maps $\mathcal{V}_o \to \mathcal{V}$ (we remark that here, differently from normal order is needed for obtaining an algebra of such maps). Moreover, $\mathcal{O}$ turns out to be a $\mathbb{Z}_2$-graded algebra by letting the grades of $a[\zeta]$ and $a'[z]$ be $[\zeta]$ and $[z]$, and we have the isomorphism $\mathcal{O} \cong \mathcal{V} \otimes \mathcal{V}^*$. Let $Z : \mathbb{R} \to \mathcal{O}$ be a local curve such that $\lim_{\lambda \to 0} Z(\lambda) \chi \in \mathcal{V}$ exists in the sense of distributions for all $\chi \in \mathcal{V}_o$. Then $\lim_{\lambda \to 0} Z(\lambda)$ is a well-defined linear map $\mathcal{V}_o \to \mathcal{V}$ which belongs, in general, to an extended space $\mathcal{O}^* \supset \mathcal{O}$. In this paper we won’t be concerned with the issue of defining a topology on $\mathcal{O}^*$, as the notion of F-smoothness will suffice for our purposes.

1.3 Frölicher-smoothness

Let $\mathcal{M}$ be any set. A family $\mathcal{C}_\mathcal{M}$ of curves $\mathbb{R} \to \mathcal{M}$ determines a family $\mathcal{FC}_\mathcal{M}$ of maps $\mathcal{M} \to \mathbb{R}$ by the rule

$$f \in \mathcal{FC}_\mathcal{M} \iff f \circ c \in C^\infty(\mathbb{R}) \forall c \in \mathcal{C}_\mathcal{M}.$$ 

Conversely, a set $\mathcal{FC}_\mathcal{M}$ of functions $\mathcal{M} \to \mathbb{R}$ determines a set $\mathcal{CF}_\mathcal{M}$ of curves in $\mathcal{M}$ by

$$c \in \mathcal{CF}_\mathcal{M} \iff f \circ c \in C^\infty(\mathbb{R}) \forall f \in \mathcal{FC}_\mathcal{M}.$$ 

An F-smooth structure on $\mathcal{M}$ is defined to be a couple $(\mathcal{C}_\mathcal{M}, \mathcal{FC}_\mathcal{M})$ such that $\mathcal{C}_\mathcal{M}$ and $\mathcal{FC}_\mathcal{M}$ determine each other, namely $\mathcal{FC}_\mathcal{M} = \mathcal{FC}_\mathcal{M}$ and $\mathcal{CF}_\mathcal{M} = \mathcal{C}_\mathcal{M}$. Note that either any set $\mathcal{C}_0$ of curves in $\mathcal{M}$, or any set $\mathcal{FC}_0$ of functions on $\mathcal{M}$, generate an F-smooth structure either by $\mathcal{FC}_\mathcal{M} := \mathcal{FC}_0$ or by $\mathcal{C}_\mathcal{M} := \mathcal{FC}_0$. If $(\mathcal{N}, \mathcal{C}_\mathcal{N}, \mathcal{FC}_\mathcal{N})$ is another F-smooth structure, then a map $\Phi : \mathcal{M} \to \mathcal{N}$ is called F-smooth if $\Phi \circ c \in \mathcal{C}_\mathcal{N}$ for all $c \in \mathcal{C}_\mathcal{M}$, or equivalently if $f \circ \Phi \in \mathcal{FC}_\mathcal{N}$ for all $f \in \mathcal{FC}_\mathcal{M}$.

It can be proved [5] that a function $f : \mathcal{M} \to \mathbb{R}$ on a classical manifold $M$ is smooth (in the standard sense) if and only if the composition $f \circ c$ is a smooth function of one variable for any smooth curve $c : \mathbb{R} \to \mathcal{M}$. Thus one has a unified notion of smoothness based on smooth curves, including classical manifolds as well as functional and distributional spaces. This notion of smoothness, which was introduced by Frölicher [23], behaves naturally with regard to inclusions and cartesian products, so it yields a convenient general setting for dealing with functional spaces and functional bundles [24, 31, 8, 30, 16].

1.4 Frölicher-smooth quantum state bundles

F-smoothness in a distributional space is defined quite naturally: a curve $c$ is called F-smooth if $t \to \langle c(t), u \rangle$ is smooth for any test element $u$. Accordingly, distributional bundles are defined to be F-smooth vector bundles, over a classical base manifold, whose fibers are distributional spaces. The geometry of these bundles, and in particular the connections on them, have been studied in a previous paper [16].

Bundles of generalised semi-densities over particle momenta (“quantum bundles”) are a special case, which has been applied to a partly original approach to quantum particle physics [17]. For a given particle type, the underlying “classical” (i.e. finite-dimensional) geometric structure is that of a 2-fibered bundle $Z \to P_m \to M$ (the top fibers describing the ‘internal degrees of freedom’ of the considered particle type), where $(M, g)$ is Einstein’s spacetime and $P_m \subset P \cong T^\ast M$ is the sub-bundle over $M$ of future shells for the particle’s mass $m$ (4-momentum bundle). At each $x \in M$ we perform the construction presented in [11, 12] with the generic manifold $X$ now replaced by $(P_m)_x$. We get spaces $\mathcal{E}_x^1$ and $\mathcal{E}_x^1 \equiv \mathcal{D}((P_m)_x, Z_x)$.
The fibered sets $\mathcal{Z}^1 := \bigsqcup_{x \in M} \mathcal{Z}^1_x$ and $\mathcal{Z}^1 := \bigsqcup_{x \in M} \mathcal{Z}^1_x$ turn out to have a natural F-smooth vector-bundle structure over $M$. Now the multi-particle state bundles

$$\mathcal{Z} := \bigoplus_{n=0}^{\infty} \mathcal{Z}^n \hookrightarrow M,$$

$$\mathcal{Z} := \bigoplus_{n=0}^{\infty} \mathcal{Z}^n \hookrightarrow M,$$

turn out to be F-smooth vector bundles. The same is true for the sub-bundles of $\mathcal{Z}$ constructed from $\mathcal{D}(P_m, \mathcal{Z})$, whose fibers are constituted of finite linear combinations of Dirac-type semi-densities. A natural isomorphism $\mathcal{Z}^1 \leftrightarrow \mathcal{D}(P_m, \mathcal{Z})$ is determined by the mass-shell Leray form. Considering more particle types, one eventually gets the total quantum bundle

$$\mathcal{V} := \mathcal{Z}^1 \otimes \mathcal{Z}^n \otimes \mathcal{Z}^m \otimes \cdots \otimes \bigoplus_{n=0}^{\infty} \mathcal{V}^n \hookrightarrow M.$$

Note that the quantum bundles for particle types of different mass are constructed over different mass-shell bundles.

In order to proceed we must now consider an orthogonal splitting $T^*M \equiv \mathcal{P} = \mathcal{P}_\parallel \oplus \mathcal{P}_\perp$ into "timelike" and "spacelike" $g$-orthogonal subbundles over $M$, which can be seen as associated to the choice of an observer and is needed in order to build a theory of quantum particles and their interactions. Indeed, while we may wish we had an observer-independent theory, we must accept that some kind of an observer and its proper time are always used in particle physics at the present state of the art. In §1.5 we’ll discuss the viability of a somewhat weaker requirement.

Let $\eta_\perp$ be the volume form, associated with the metric, on the fibers of $\mathcal{P}_\perp \hookrightarrow M$. The orthogonal projection $\mathcal{P} \rightarrow \mathcal{P}_\parallel$ yields a distinguished diffeomorphism $\mathcal{P}_m \leftrightarrow \mathcal{P}_\perp$ for each $m$. The pull-back of $\eta_\perp$ is then a volume form on the fibers of $\mathcal{P}_m$, which is denoted for simplicity by the same symbol. The Leray form can be then written as $\omega_m(p) = (2p_0)^{-1} \eta_\perp(p), p \in \mathcal{P}_m$, where $p_0 \equiv \mathcal{E}_m(p) = (m^2 + p_0^2)^{1/2}$.

It will be convenient to use the “spatial part” $p_\perp$ of the 4-momentum $p$ as a label, that is a generalised index for quantum states. For each $p \in \mathcal{P}_m$ let $\delta_m[p]$ be the Dirac density with support $\{p\}$ on the same fiber of $\mathcal{P}_m$, and $\delta(y_\perp - p_\perp)$ the generalised function characterised by $\delta_m[p](y) = \delta(y_\perp - p_\perp) \, d^3y$ in terms of linear coordinates $(y_\perp) \equiv (y_0, y_1, y_2, y_3) \equiv (y_0, y_\perp)$ in the fibers of $\mathcal{P}$. Now consider the section $\mathcal{P}_m \rightarrow \mathcal{D}(P_m, \mathcal{Z}) : p \mapsto X_p$ defined as follows; for each $p \in \mathcal{P}_m$ we can regard $X_p$ as a generalised function of the variable $y_\perp$, with the expression

$$X_p(y) := l^{-3/2} \delta(y_\perp - p_\perp) \sqrt{d^3y}.$$

Here $l$ is a constant length needed in order to get an unscaled (“conformally invariant”) semi-density (compare with the usual “box quantization” argument). Eventually, we get the distinguished isomorphism $\mathcal{Z}^1 \leftrightarrow \mathcal{D}(P_m, \mathcal{Z})$ which is determined by the correspondence $|z\rangle \leftrightarrow X_p \otimes z, z \in \mathcal{Z}_p$.

The dual frame of $\{\mathcal{B}_p\}$ is $\{\mathcal{B}^{\alpha}_p\}$ where $\mathcal{B}^{\alpha}_p = X^p \otimes b^{\alpha}$. Here, $\{b^{\alpha}\}$ is the classical dual frame of $\{b_\alpha\}; X^p$, the dual of $X_p$, is actually the same semi-density. We obtain $\{\mathcal{B}^{\alpha}_p, \mathcal{B}_q\beta\} = l^{-3} \delta(p_\perp - q_\perp) \delta_\beta^\alpha$, an unscaled relation.

Let $h : \mathcal{P}_m \rightarrow \mathcal{Z}^* \otimes \mathcal{Z}^*$ be the tensor describing the Hermitian structure of $\mathcal{Z}$. Then $(b^{\alpha})# = h^{\alpha\beta} b_\beta : \mathcal{P}_m \rightarrow \mathcal{Z}$ (we follow the common convention of indicating “conjugate” indices by a dot). We extend this Hermitian structure to the quantum bundle by setting $$(\mathcal{B}^{\alpha}_p)^# := X_p \otimes (b^{\alpha})# \equiv h^{\alpha\beta} X_p \otimes b_\beta \in \mathcal{D}(P_m, \mathcal{Z}).$$

---

3 This is a distinguished volume form on the shells, usually denoted as $\delta(p^2 - m^2)$. 

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1.4 Frölicher-smooth quantum state bundles

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Recalling the generalised index summation convention, and using the spatial momentum as a generalised index, we can see this Hermitian structure as associated with the tensor having the generalised components \( H^{\hat{\rho} \hat{\alpha}, \rho \alpha} = H^{\hat{\rho} \rho} h^{\alpha \alpha} \) where \( H^{\hat{\rho} \rho} = \delta(p_\rho - p_\nu) \).

### 1.5 Detectors and quantum configuration space

We consider a timelike submanifold \( T \subset M \) and call it a detector. A momentum space formalism for particle interactions, in terms of generalised semi-densities, can be exhibited as a sort of a complicated ‘clock’ carried by the detector.

A generalised frame of free one-particle states along \( T \) can be obtained as follows. At some arbitrarily fixed event \( x_0 \in T \subset M \) we choose a classical frame \( (b_\alpha) \) of the bundle \( Z \sim (P_m)_{x_0} \), and note that the family of generalised semi-densities \( \{B_{p\alpha}(x_0)\} \) is a generalised frame of \( Z_{x_0} \sim (P_m)_{x_0} \). Then we transport \( B_{p\alpha} \) along \( T \) by means of Fermi transport \([19, 21]\) for the spacetime and spinor factors \([3] \) and, for the remaining factors, by means of parallel transport relatively to a background connection of \( Z \) which will have to be assumed (see also \([3.1]\)). We write

\[
B_{p\alpha} : T \to \mathcal{D}(P_m, Z)_T : t \mapsto B_{p\alpha}(t) = X_{p(t)} \otimes b_\alpha,
\]

where \( p : T \to P_m : t \mapsto p(t) \) is Fermi-transported and \( t = 0 \) at \( x_0 \). This yields a trivialization

\[
\mathcal{D}(P_m, Z)_T \cong T \times \mathcal{D}(P_m, Z)_{x_0},
\]

which can be seen as determined by a suitable connection called the free-particle connection. Eventually, the above arguments can be naturally extended to multi-particle bundles and states. When several particle types are considered, we get a trivialization \( \mathcal{V}_T \cong T \times Q \) of the total quantum state bundle, where \( Q \equiv \mathcal{V}_{x_0} \) can be seen as the “quantum configuration space”. The quantum interaction, an added term that modifies the free-field connection, can be constructed by assembling the classical interaction with a distinguished quantum ingredient. By construction, the free-particle transport preserves particle type and number, while the interaction doesn’t \([17, 21]\).

Let \( (TM)_T \to T \) denote the restriction of the tangent bundle of \( M \) to base \( T \). We have the splitting \( (TM)_T = (TM)_T^\perp \oplus (TM)_T^\parallel \) into “timelike” and “spacelike” \( g \)-orthogonal sub-bundles. Exponentiation determines, for each \( t \in T \), a diffeomorphism from a neighbourhood of 0 in \( (TM)_t^\perp \) to a spacelike submanifold \( M_t \subset M \), and so a 3-dimensional foliation of a neighbourhood \( N \equiv \bigcup_{t \in T} M_t \subset M \) of \( T \).

A tempered generalised semi-density on \( (P_m)_t \) yields, via Fourier transform, a generalised semi-density on \( (TM)_t^\parallel \). A suitable restriction then yields, via exponentiation, a generalised semi-density on \( M_t \) (remember \([36]\) that a distribution can be restricted to an open set). This correspondence can be extended to \( Z \)-valued semi-densities by means of background linear connections of the various “internal” bundles. We may view these auxiliary connections not as classical gauge fields but rather as a “mean field” background structure, analogous to the gravitational background, whereas quantum gauge fields are of a different nature (more about that later). Eventually, the trivialisation \( \mathcal{V}_T \cong T \times Q \) can be extended as \( \mathcal{V}_N \cong N \times Q \).

Note how this essentially amounts to a natural extension of the generalised quantum frames \( (B_{p,\alpha}) \) over \( T \) to generalised frames over \( N \). In flat spacetime, and for an inertial detector, we essentially get the usual correspondence between momentum-space and position-space representation.

---

\[ \text{Fermi and parallel transport coincide if the detector is inertial.} \]
Accordingly, we also get the operator algebra
\[ \mathcal{O} \cong \mathcal{Q} \otimes \mathcal{Q}^* \cong \mathcal{V}_{x_0} \otimes \mathcal{V}_{x_0}^*, \]
where, as previously discussed, the identification is determined via normal ordering.

### 1.6 Quantum Fields

Taking the fiber Hermitian structure of \( Z \to P_m \) into account, any \( \zeta \in \mathcal{D}(P_m, Z^*) \) yields an emission operator \( a^\dagger [\zeta] \equiv a^\dagger [\zeta^\#] : \psi \mapsto \zeta^\# \circ \psi \) and an absorption operator \( a[\zeta] : \psi \mapsto \zeta \circ \psi \). We write
\[ a^\alpha(p_\perp) \equiv a^\alpha := a^\alpha[B^{0\alpha}], \quad a^{\dagger \alpha}(p_\perp) \equiv a^{\dagger \alpha} := a^\dagger[B^{0\alpha}^\#], \]
thus seeing \( a^\alpha \) and \( a^{\dagger \alpha} \) as generalised functions of momentum. Consistently with the generalised index notation we also write \( a[\zeta] = \zeta_{\rho \alpha} a^{\rho \alpha}, \ a^{\dagger}[\zeta] = \zeta_{\rho \alpha} a^{\dagger \rho \alpha} \), and eventually
\[ a[.] = a^\alpha B^\alpha, \quad a^{\dagger}[.] = a^{\dagger \rho \alpha} B^{\rho \alpha}. \]

We now consider a notion of quantum field on curved spacetime based on the choice of a detector and the related constructions presented in \( \S 1.5 \). If \( (M, g) \) is Minkowski spacetime and \( T \) is inertial (a straight line) then we have the orthogonal decomposition \( M = T \times X \), namely \( M_t \equiv X \) (a Euclidean space) \( \forall t \in T \), and define the free quantum field, in the considered sector, to be \( \phi(x) \equiv \phi^{(+)}(x) + \phi^{(-)}(x) \) with
\[
\phi^{(+)}(x) := \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{\sqrt{2p_0}} e^{-i\langle p, x \rangle} a^{\alpha}(p_\perp) b^{\alpha}(p_\perp),
\]
\[
\phi^{(-)}(x) := \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{\sqrt{2p_0}} e^{i\langle p, x \rangle} a^{\dagger \alpha}(p_\perp) b^{\alpha}(p_\perp),
\]
where \( p_0 \equiv (m^2 + p^2)^{1/2} \), \( x \equiv (t, x_\perp) \), \( \langle p, x \rangle = p_0 t + \langle p_\perp, x_\perp \rangle \). Thus \( \phi^{(+)}(x) \) and \( \phi^{(-)}(x) \) are, respectively, a Fourier transform of \( a[.] \) and a Fourier anti-transform of \( a^{\dagger}[.] \) related to the Leray form of the mass shell.\(^5\) In curved spacetime, all this still makes sense if the neighbourhood \( N \) of \( T \) coincides with \( M \) (namely we have a global time \( \times \) space decomposition of \( M \)). Alternatively, it could be viewed as a kind of linearization by replacing \( M_t \equiv X \) with \( (TM)_{x_0} \) in the whole construction.

**Remark.** \( a[.] \) and \( a^{\dagger}[.] \) can be seen as generalised maps \( T \times (P_\perp \times Z^*)_{x_0} \to \mathcal{O} \), or equivalently, because of linearity, as morphisms \( T \times P_\perp \to Z_T \otimes \mathcal{O} \) over \( T \) and distributions on \( P_\perp \to T \).

Our next step will be expressing the above \( \phi \) as a local section of some vector bundle over \( M \), in order to recover the usual approach in which one “quantizes” a classical field by rendering it “operator-valued”. The basic issue here is that \( Z \), in general, is a vector bundle over \( P_m \). In many practical cases one actually has a “semi-trivial” bundle \( P_m \times_M Z \), where \( Z \to M \) is the true “internal” vector bundle. The notable exceptions are given by spin bundles. For fermions, these are sub-bundles of semi-trivial bundles anyway \( \S 3.2 \). For gauge bosons, the same holds true when a gauge has been chosen \( \S 3.1 \). So \( b^\beta(p_\perp) = K^\beta_\beta(p_\perp) b^\alpha_\alpha \) and \( b^\beta(p_\perp) = K^\beta_\beta(p_\perp) b^\alpha_\alpha \), where \( (b^\alpha_\alpha) \) is a new frame, independent of momentum. The Hermitian metric yields an identification \( Z \cong Z^* \), allowing us to write \( b^\alpha_\alpha = b^\alpha_\alpha \) and \( \phi(x) = \phi^\alpha(x) b^\alpha_\alpha(t) \), with
\[
\phi^\alpha(x) := \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{\sqrt{2p_0}} K^\alpha_\beta(p_\perp) \left( e^{-i\langle p, x \rangle} a^\beta(p_\perp) + e^{i\langle p, x \rangle} a^{\dagger \beta}(p_\perp) \right).
\]
\(^5\) As indicated by the factor \( (2p_0)^{-1/2} \), which could be absorbed either in \( \{B_{\rho \alpha}\} \) or in the interaction.
Eventually we can identify $\phi(x)$ with an element in the fiber at $x$ of a vector bundle over $M$ by parallel transport of $b'_a$ along the spacelike geodesic from $t$ to $x$ (relatively to the gauge-fixing connection). Its components $\phi^\alpha(x)$ are valued in a fixed algebra of linear operators on the space $\mathcal{Q}$ of quantum states.

Remark. The meaning of the above integral should be specified. The integrand can be seen as a generalised map $\Phi_x : P \rightarrow \mathcal{O}$, the target being the fixed vector space of linear maps $\mathcal{Q}_\circ \rightarrow \mathcal{Q}$ introduced in §1.2 ($\mathcal{Q}_\circ \subset \mathcal{Q}$ is the subspace of all test elements). Then the requirement

$$\langle \zeta, (\int d^3p \Phi_x(p_\perp))\chi \rangle = \int d^3p \langle \zeta, \Phi_x(p_\perp)\chi \rangle,$$

$\zeta, \chi \in \mathcal{Q}_\circ,$

characterises the integral as belonging to the extended operator space $\mathcal{O}^\ast$ (§1.2). Similar observations can be made in other situations where integrals of $\mathcal{O}$-valued maps occur (§2.2, §2.4).

Henceforth, for notational simplicity, we’ll indicate as $\mathcal{O}$ the above said extended operator space. We can now see the quantum field as a generalised section (in the distributional sense) $M \rightarrow \mathcal{O} \otimes_M \mathbb{Z}$. The set of all quantum fields of a theory can be described as a generalised section $M \rightarrow \mathcal{E} \equiv \mathcal{O} \otimes \mathcal{E}$, where $\mathcal{E} \rightarrow M$ is the classical “configuration bundle” (the finite-dimensional vector bundle whose sections are the classical fields). In other terms, the “quantum bundle” $\mathcal{E} \rightarrow M$ replaces the classical configuration bundle. Note that all tensor products and contractions, via multiplication in $\mathcal{O}$, again generate $\mathcal{O}$-valued fields. In particular, a generalised map $M \rightarrow \mathcal{O}$ may be called a quantum scalar. Finally, point-wise multiplication of field components is super-commutative by virtue of the modified rule (1).

The above $\phi$ is an essentially unique, well-defined object, fulfilling the Klein-Gordon equation and determined by the underlying classical geometry. We call it a free field. We’ll be interested, more generally, in “interpolating fields”, namely generalised sections $M \rightarrow \mathcal{E}$ which are solutions of a differential equation containing interactions among various “sectors” in $\mathcal{E}$.

Remark. Admittedly, the above scheme relies on the choice of a detector and, in general, only makes sense as a local linearisation. However, some kind of an observer and the preferred time related to it are always needed in quantum physics, and we are lead to conclude that the equivalence between position and momentum representations may be only local and partial in non-flat spacetime. We might argue that the particle/field complementarity issue is still unresolved: the two views can’t be both fundamental and on the same footing.

2 F-smooth geometry and Lagrangian field theory

2.1 F-smooth geometry for quantum fields

The notion of F-smoothness enters our approach to quantum fields essentially in two ways, the first being the construction of bundles of states over momenta (§1.3). Then, in order to deal with quantum fields as sketched in §1.3, we must extend the fundamental differential geometric notions for classical bundles to the case of a quantum bundle $\mathcal{E} \rightarrow M$.

We start from the F-smooth structure of $\mathcal{O}$ characterised by a suitable set $\mathcal{C}_0$, defined to be the family of all curves $C : \mathbb{R} \rightarrow \mathcal{O}$ such that, for any test element $\chi \in \mathcal{Q}_\circ \subset \mathcal{Q}$, the curve $C \chi : \mathbb{R} \rightarrow \mathcal{Q} : s \mapsto C(s)\chi$ is F-smooth. In particular, if $\alpha : \mathbb{R} \rightarrow \mathcal{Q}^\ast$ and $c : \mathbb{R} \rightarrow \mathcal{Q}^\ast$

---

6 We remark that $\phi$ is a combination of particle absorption and anti-particle emission, while the conjugate field $\bar{\phi}$ is a combination of anti-particle absorption and particle emission.
are F-smooth curves, then the curves \( s \mapsto a[a(s)] \) and \( s \mapsto a'[c(s)] \) are F-smooth. A curve \( \hat{C} : \mathbb{R} \to \mathcal{O} \otimes E \) is said to be F-smooth if

\[
\langle \sigma, \hat{C} \rangle : \mathbb{R} \to \mathcal{O} : s \mapsto \langle \sigma, \hat{C}(s) \rangle
\]

is F-smooth for any classical smooth section \( \sigma : M \to E^* \).

Now \( \varphi : E \equiv \mathcal{O} \otimes E \to M \) turns out to be an F-smooth vector bundle: if \( X \subset M \) is an open subset and \( (x,y) : E_X \to X \times Y \) is a smooth linear trivialisation, then one has an F-smooth local trivialisation \( E_X \to X \times Y \), still denoted as \( (x,y) \), where \( Y \equiv \mathcal{O} \otimes Y \). In particular, a linear fibered chart \( (x^a, y^i) \) on \( E \) can be seen as a linear fibered chart on \( E \), the fiber “coordinates” \( y^i \) being now \( \mathcal{O} \)-valued.

To the reader which is familiar with the geometry of jet bundles, parts of this section will look, at first sight, just as reminders; this is exactly one of the virtues of the F-smoothness approach. However we note that certain points do require some care.

**Tangent, vertical and jet spaces**

It can be shown [16] that if \( c \) is an F-smooth curve valued into a distributional space then there exists a unique curve \( \partial c \), valued into the same space, such that \( \langle c, u \rangle' = \langle \partial c, u \rangle \) for any test element \( u \) (here the prime denotes the ordinary derivative of functions \( \mathbb{R} \to \mathbb{R} \)). Hence, if \( C : \mathbb{R} \to \mathcal{O} \) is an F-smooth curve then the rule \( \partial C(s)_\chi := \partial(C(s)\chi) \forall \chi \in \mathcal{Q}_c \) defines a unique curve \( \partial C : \mathbb{R} \to \mathcal{O} \); moreover, the tangent prolongation \( TC : \mathbb{R} \times \mathbb{R} \to T\mathcal{O} \equiv \mathcal{O} \otimes \mathcal{O} \) is well defined as \( TC(s,\tau) := \langle (C(s)), \tau \partial C(s) \rangle \).

If \( \hat{C} : \mathbb{R} \to E \equiv \mathcal{O} \otimes E \) is F-smooth then \( X^a \circ \hat{C} : \mathbb{R} \to \mathbb{R} \) and \( y^i \circ \hat{C} : \mathbb{R} \to \mathcal{O} \). Two such curves \( \hat{C} \) and \( \hat{C}_1 \) are said to be first-order equivalent at (say) 0 \( \in \mathbb{R} \) if \( \hat{C}_1(0) = \hat{C}(0) \equiv y \in E \) and their “components” have the same derivatives at 0, namely \( \partial(x^a \circ \hat{C})(0) = \partial(x^a \circ \hat{C}_1)(0) \) and \( \partial(y^i \circ \hat{C})(0) = \partial(y^i \circ \hat{C}_1)(0) \) in any chart. The equivalence class \( T_y \mathcal{E} \) is called the tangent space of \( \mathcal{E} \) at \( y \), and naturally turns out to be a vector space. Each local linear trivialization \( (x,y) : E_X \to X \times Y \) determines a local F-smooth trivialization

\[
T(x,y) \equiv \langle Tx, Ty \rangle : TE \equiv \bigsqcup_{y \in \mathcal{E}} T_y \mathcal{E} \to TX \times TY ,
\]

where \( Y \equiv \mathcal{O} \otimes Y \), \( T \mathcal{Y} \equiv \mathcal{Y} \times \mathcal{Y} \). Hence, the fibered set \( \pi_\mathcal{E} : T\mathcal{E} \to \mathcal{E} \) turns out to be an F-smooth bundle. We have another F-smooth bundle with the same total space, namely

\[
T_{\varphi} : T\mathcal{E} \to TM : \partial \hat{C} \to \partial(\varphi \circ \hat{C}) .
\]

Moreover we have the vertical subbundle \( \nabla \mathcal{E} := \text{Ker} T_{\varphi} \subset T\mathcal{E} \), the natural identification \( \nabla \mathcal{E} = \mathcal{E} \times_M \mathcal{E} \), and the exact sequence over \( \mathcal{E} \)

\[
0 \to \nabla \mathcal{E} \to T\mathcal{E} \to \mathcal{E} \times TM \to 0 .
\]

The subbundle of \( T^*M \otimes_\mathcal{E} T\mathcal{E} \) which projects over the identity of \( TM \) is called the first jet bundle and denoted by \( J\mathcal{E} \to \mathcal{E} \). This is an affine bundle over \( \mathcal{E} \), with ‘derived’ vector bundle \( T^*M \otimes_\mathcal{E} \nabla \mathcal{E} \). The restriction of \( T^*x \otimes T(x,y) \) is a local bundle trivialization

\[
J(x,y) : J\mathcal{E}_X \to J(X \times \mathcal{Y}) \cong \mathcal{Y} \times (T^*X \otimes \mathcal{Y}) .
\]

If \( (x^a, y^i) \) is a linear fibered coordinate chart of \( \mathcal{E} \), then the naturally induced charts on \( T\mathcal{E} \) and \( J\mathcal{E} \) are denoted by \( (x^a, y^i; \dot{x}^a, \dot{y}^i) \) and \( (x^a, y^i, y_a) \), respectively. Also higher jet spaces can be defined similarly to the finite-dimensional case, and have the same basic formal properties.
As the classical bundle $E \rightarrow M$ is a vector bundle, $JE \rightarrow M$ turns out to be a vector bundle, too (while $JE \rightarrow E$ is affine). Then it’s not difficult to exhibit a distinguished isomorphism $JE \cong O \otimes JE$. Similarly, $VE \cong O \otimes VE$ and $TE \cong O \otimes TE$, where the latter tensor product is over $TM$.

Prolongations

If $f : E \rightarrow O$ is F-smooth then its tangent prolongation $Tf \equiv (f, df) : TE \rightarrow TO \equiv O \times O$ is characterised by the requirement that the rule $\langle df, \partial \hat{C} \rangle = \partial(f \circ \hat{C})$ holds for any F-smooth curve $\hat{C} : \mathbb{R} \rightarrow E$. Then $df : TE \rightarrow O$ turns out to be a linear morphism (closely related to the usual notion of functional derivative).

Setting $v \equiv \partial \hat{C}(0) \in TE$ we also write $v.f \equiv \partial(f \circ \hat{C})(0)$; more generally, if $v : E \rightarrow TE$ is F-smooth then we obtain the F-smooth function $v.f \equiv \langle df, v \rangle$, called the derivative of $f$ along $v$. It’s not difficult to see that this operation fulfills the ordinary Leibnitz rule

$$v.(fg) = (v.f)g + f.v.g = \langle df, v \rangle g + f \langle dg, v \rangle.$$  

We then write $d(fg) = dfg + fdg$ with the understanding that it is applied to $v$ by applying $df$ and $dg$ to $v$ without changing the order of any factors.

If $\Phi : E \rightarrow E'$ is a morphism of quantum bundles then we define $T\Phi : TE \rightarrow TE'$ to be characterised by $T\Phi \circ \partial \hat{C} = \partial(\Phi \circ \hat{C})$, and $V\Phi : VE \rightarrow VE'$ as its vertical restriction.

If $\phi : M \rightarrow E$ is an F-smooth section, then $T\phi : TM \rightarrow TE$ projects over the identity of $TM$, so that it can be viewed as a section $j_\phi : M \rightarrow JE$ (the first jet prolongation of $\phi$). The jet prolongation morphism $J\Phi : JE \rightarrow JE'$ is defined by the requirement $j(\Phi \circ \phi) = J\Phi \circ j_\phi$ for any $\phi$. We then get the tangent, vertical and jet functors, which can be iterated essentially as in finite-dimensional geometry. In particular, we indicate the $k$-th (holonomic) jet prolongation of $E \rightarrow M$ as $J_k E \rightarrow J_{k-1} E \rightarrow \cdots \rightarrow E \rightarrow M$.

The induced fiber coordinates $y^i_A$ on jet bundles, where $A$ is a base manifold multi-index, are defined as usual by $y^i_A \circ j_k \phi = \partial_A \phi^i$ for all sections $\phi : M \rightarrow E$, $k = |A|$ being the multi-index’ length.

Straightforward extensions of the above constructions yield bundles $VJ_k E$ and $J_k V E$, which turn out to be naturally isomorphic to each other.

Partial derivatives

Let $(x^a, y^i)$ be a linear fibered coordinate chart of $E \rightarrow M$. The fiber coordinates $(y^i)$ can be seen as elements of the dual frame of a local frame $(b_i)$, and as $O$-valued fiber coordinates on $E$. If $f : E \rightarrow O$ then the partial derivatives $\frac{\partial}{\partial y^i} f$ are well defined, since the coordinate curves $\mathbb{R} \rightarrow M$ corresponding to the base coordinates $(x^a)$ can be seen as $E$-valued via the trivialisation associated with the chart itself.

Instead, the coordinate curves corresponding to the fiber coordinates $(y^i)$ can’t be naturally seen as $E$-valued, so that we have no obvious meaning for $\partial_i f \equiv \partial f / \partial y^i$, and wonder whether we can apply the known results and coordinate formulas of classical Lagrangian field theory on jet bundles. Now observe that a vertical vector field $v : E \rightarrow VE$ can be written as $v^i \partial_i$, where $\partial_i \equiv \frac{\partial}{\partial y^i}$ denotes the classical vectors tangent to the fiber coordinate curves and the components $v^i$ are $O$-valued. Moreover $\langle df, v \rangle = v.f$ has a well-defined meaning, as previously discussed. In particular, if $\lambda \in O$ then we write $\langle df, \lambda \partial_i \rangle \equiv \partial_i f \lambda \equiv \partial_i f (\lambda)$, namely we regard $\partial_i f$ as a function on $E$ valued into linear maps $O \rightarrow O$. Thus $\langle df, v \rangle = \partial_i f v^i$ gets
2.1 F-smooth geometry for quantum fields

a well-defined meaning, though the right-hand side must not be intended as ordinary multiplication. We now write \( df = \partial_a f \, dx^a + \partial_i f \, dy^i \), so that if \( v = v^a \partial_a + v^i \partial_i \): \( \mathcal{E} \to T\mathcal{E} \) is an arbitrary vector field then \( \langle df, v \rangle = \partial_a f \, v^a + \partial_i f \, v^i \).

In practical calculations one may wish to be able to write down an explicit expression for \( \partial_i f \). This can be easily done when \( f \) is a polynomial in the linear fiber coordinates \( y^i \), as it is indeed true in many cases of interest (§3.3). One possibility would be marking in some way the “insertion point” for the \( \mathcal{O} \)-argument in each monomial in \( \partial_i f \). But recalling (§1.3) that the product of field components at any given spacetime point is defined in such a way to be super-commutative, and observing that the fiber coordinates can be (and are) chosen in such a way that each one belongs to a sector of definite \( \mathbb{Z}_2 \)-grade, we can just define \( \frac{\partial f}{\partial y^i} \equiv \partial_i f \) by

\[
\frac{\partial f}{\partial y^i} \lambda = \langle df, \lambda \partial_i \rangle \quad \forall \lambda \in \mathcal{O} \text{ such that } [\lambda] = [y^i].
\]

In this way, explicit expressions of partial derivatives are the same as in the classical situation, possibly up to a sign in each monomial.

The above discussion naturally extends to partial derivation with respect to the induced fiber coordinates \( y^i \) on jet bundles. As a consequence, several basic differential geometric operations on the “quantum bundle” \( \mathcal{E} \to M \) turn out to have, formally, the same expressions as their classical counterparts on the “classical bundle” \( E \to M \), though some extra care will be needed in the ordering of factors.

Connections

We denote by \( \vartheta : J\mathcal{E} \times_{\mathcal{E}} T\mathcal{E} \to V\mathcal{E} \) the complementary morphism, over \( \mathcal{E} \), of the inclusion \( J\mathcal{E} \to T^*M \otimes_{\mathcal{E}} T\mathcal{E} \). Its coordinate expression is \( \vartheta^i = dy^i - y^l_i \, dx^l \). The basic idea associated with \( \vartheta \) is that a point \( \xi \in J_e\mathcal{E}, e \in \mathcal{E} \), determines a linear projection \( T_e\mathcal{E} \to V_e\mathcal{E} \) and hence a splitting of \( T_e\mathcal{E} \) into the direct sum \( H_\xi\mathcal{E} \oplus V_\xi\mathcal{E} \) of a “\( \xi \)-horizontal” subspace and the vertical subspace.

A connection on \( \phi : \mathcal{E} \to M \) is defined to be an F-smooth section \( \Gamma : \mathcal{E} \to J\mathcal{E} \). As in the finite-dimensional case, a connection can be assigned by choosing any one of various equivalent structures. First, \( \Gamma \) can be viewed as a linear morphism \( \mathcal{E} \times_M T^*M \to T\mathcal{E} \) over \( \mathcal{E} \) such that \( (\pi_\mathcal{E}, T\varphi) \circ \Gamma \) turns out to be the identity of \( \mathcal{E} \times_M T^*M \). The image \( H_\varphi\mathcal{E} := \Gamma(\mathcal{E} \times_M T^*M) \) is a vector subbundle of \( T\mathcal{E} \); the restriction of \( \Gamma \circ (\pi_\mathcal{E}, T\varphi) \) is the identity of \( H_\varphi\mathcal{E} \). If \( v : M \to T^*M \) is a smooth vector field, then \( \Gamma[v] : \mathcal{E} \to T\mathcal{E} \) is an F-smooth vector field, called its horizontal lift. Moreover we have the complementary morphism

\[
\Omega_\varphi := \vartheta \circ \Gamma \equiv 1 - \Gamma : T\mathcal{E} \to V\mathcal{E} \equiv \mathcal{E} \times_M \mathcal{E},
\]

so that the map \( (\Gamma \circ (\pi_\mathcal{E}, T\varphi), \Omega_\varphi) \) determines the decomposition \( T\mathcal{E} = H_\varphi\mathcal{E} \oplus_E V\mathcal{E} \).

The covariant derivative of an F-smooth section \( \phi : M \to \mathcal{E} \) is defined to be the linear morphism over \( M \)

\[
\nabla \phi \equiv \nabla[\Gamma] \phi := pr_2 \circ \Omega_\varphi \circ T\phi : T^*M \to \mathcal{E}.
\]

If \( v : M \to T^*M \) is a vector field then we write \( \nabla_v \phi \equiv \nabla \phi \circ v \).

We say that \( \Gamma \) is a linear connection if it is a linear morphism \( \mathcal{E} \to J\mathcal{E} \) over \( M \). Then its curvature tensor can be defined as a section \( R_\varphi : M \to \Lambda^2 T^*M \otimes_M \text{End}(\mathcal{E}) \), formally characterised as in the finite-dimensional case.

Any linear morphism of finite-dimensional vector bundles naturally yields a linear morphism of the corresponding quantum bundles (via tensor product by \( \mathcal{O} \)). In particular, a
linear connection $\Gamma : E \to J E$ of the classical configuration bundle yields a linear quantum connection $\mathcal{E} \to J \mathcal{E}$, which is indicated by the same symbol.

**Dual quantum bundles**

The cotangent space is one notion of classical geometry which does not readily extends to the context of F-smooth geometry. Such extension can be done in the present special context, still without being involved with duality in a general sense.

We define the dual bundle of $\mathcal{E} \equiv \mathcal{O} \otimes E \to M$ to be simply $\mathcal{E}^* \equiv \mathcal{O} \otimes E^* \to M$, so avoiding functional duals. An element in $\mathcal{E}^*_x$, $x \in M$, can be seen as a linear map $\mathcal{E}_x \to \mathcal{O}$. Moreover we define the vertical dual to be $V^* \mathcal{E} : = \mathcal{E} \times_M E^*$. If $f : \mathcal{E} \to \mathcal{O}$ then its fiber differential $df : \mathcal{E} \to V^* \mathcal{E}$ is well-defined as the restriction of $df$ to $V \mathcal{E}$.

A straightforward extension of the above procedure yields the bundles $V^* J_k \mathcal{E}$, $k \in \mathbb{N}$.

In order to introduce the cotangent bundle $T^* \mathcal{E} \to \mathcal{E}$ we first note that any $\lambda \in \mathcal{O} \otimes T^* M$ can be viewed as an F-smooth linear morphism $T^* \mathcal{E} \to \mathcal{O}$ over $\mathcal{E}$ via the rule $v \mapsto \langle \lambda, T\varphi(v) \rangle$. The image of this inclusion is a vector bundle $H^* \mathcal{E} \to \mathcal{E}$. We define $T^* \mathcal{E} \to \mathcal{E}$ to be be the smallest vector bundle, containing $H^* \mathcal{E}$ as a sub-bundle, whose fibers are constituted by F-smooth linear maps $T^* \mathcal{E} \to \mathcal{O}$ whose restrictions to $V \mathcal{E}$ are in $V^* \mathcal{E}$. As in the classical case we have the exact sequence over $0 \to H^* \mathcal{E} \to T^* \mathcal{E} \to V^* \mathcal{E} \to 0$, which splits over $J \mathcal{E}$.

### 2.2 Basic forms and Lagrangian field theory

In classical Lagrangian field theory \cite{Saunders2000, Golubitsky1988, Ratiu1997, Abraham1988, Marsden1974, Marsden1974a} one deals with arbitrary exterior forms on jet bundles of the configuration bundle. Furthermore one deals with the Lie bracket of arbitrary vector fields and the Frölicher-Nijenhuis bracket of vector-valued forms. All such notions and operations can be introduced in the above described quantum context, though some complications do arise. For our present purposes, however, we’ll only need the restricted notions of vertical dual and of basic (or totally horizontal) form.

A “basic” $q$-form of order $k$ is defined to be an F-smooth morphism $\alpha : J_k \mathcal{E} \to \mathcal{O} \otimes \wedge^q T^* M$ over $M$, $q,k \in \{0\} \cup \mathbb{N}$. A basic 0-form, in particular, is a map $f : J_k \mathcal{E} \to \mathcal{O}$. It’s easy to see that there exists a unique 1-form of order $k+1$, indicated as $d_h f : J_{k+1} \mathcal{E} \to \mathcal{O} \otimes T^* M$ and called the horizontal differential of $f$, with the property that for any F-smooth section $\phi : M \to \mathcal{E}$ one has $d_h f \circ J_{k+1} \phi = d(f \circ J_k \phi)$. We write its coordinate expression as $d_a f \, dx^a$, with

$$d_a f \equiv \partial_a f + \sum_{0 \leq |A| \leq k} \partial^A_a f y^i_A + \cdots + \partial^A f y^i_A \equiv \partial_a f + \cdots + \partial^A f y^i_A \equiv \partial_a f + \cdots + \partial^A f y^i_A.$$  

where $\partial^A_a f \equiv \partial f / \partial y^i_A$. Note that $d_a f$ can be seen as the coordinate expression of the holonomic restriction of $J f$, the jet functor applied to $f$ seen as a morphism over $M$. Similarly we indicate by $d_h f$ the expression of the holonomic restriction of $J^{\cdots} f$, the jet functor iterated $|A|$ times.

We extend $d_h$ to act on arbitrary basic forms\footnote{For a general definition of horizontal and vertical differentials in the finite-dimensional context see e.g. Saunders \cite{Saunders2000}.} by requiring it to be an anti-derivation of degree 1 vanishing on closed classical basic forms. If $\alpha = \alpha_{a_1 \ldots a_q} dx^{a_1} \wedge \cdots \wedge dx^{a_q}$, then we get the coordinate expression $d_h \alpha = d_h \alpha_{a_1 \ldots a_q} dx^{a_1} \wedge \cdots \wedge dx^{a_q}$. It’s then immediate to check that the horizontal differential is nilpotent, i.e. $d_h^2 = 0$. 


2.3 Vertical infinitesimal symmetries and currents

Let $m = \dim M$. A $k$-order Lagrangian density is an $m$-form $\mathcal{L} : J_k \mathcal{E} \rightarrow \mathcal{O} \otimes \wedge^m T^*M$. We write its coordinate expression as $\ell d^m x$, with $\ell : J_k \mathcal{E} \rightarrow \mathcal{O}$. The related field equation for fields $\phi : M \rightarrow \mathcal{E}$ is $\mathcal{F}_i := \sum_{0 \leq |\alpha| \leq k} (-1)^{|\alpha|} d_\alpha \partial_i \ell \equiv \partial_i \ell - d_a \partial_i \ell + d_{ab} \partial_i \ell - \ldots$

are the components of the Euler-Lagrange operator $\mathcal{F}[\mathcal{L}] : J_{2k} \mathcal{E} \rightarrow \wedge^m T^*M \otimes \mathcal{E} V^* \mathcal{E}$. As in the classical case, the field equation can be derived from the requirement that the action integral \(\int \mathcal{L} \circ J_k \phi\) be stationary on any compact set for any variation of the field $\phi$ which is fixed on the set’s boundary (in other terms, the functional derivative of the action vanishes whenever it is applied to an “increment” with compact support).

A straightforward extension of the notion of fiber differential introduced in \(\text{[2.1]}\) yields the fiber derivative of a basic form, which for the Lagrangian density reads

\[
\bar{\partial} \mathcal{L} = d^m x \otimes \bar{\partial} \ell : J_{2k} \mathcal{E} \rightarrow \wedge^m T^*M \otimes \mathcal{E} V^* \mathcal{E}.
\]

Introducing the shorthand $P^A_i \equiv \partial_i^A \ell$ we have

\[
\mathcal{F}[\mathcal{L}] = d^m x \otimes \left( \sum_{0 \leq |\alpha| \leq k} (-1)^{|\alpha|} d_\alpha P^A_i \right) \, dy^i, \quad \bar{\partial} \mathcal{L} = d^m x \otimes \left( \sum_{0 \leq |\alpha| \leq k} P^A_i \, dy^i \right).
\]

2.3 Vertical infinitesimal symmetries and currents

We consider a morphism $v : J \mathcal{E} \rightarrow V \mathcal{E}$ over $\mathcal{E}$, written in coordinates as $v^i : J \mathcal{E} \rightarrow \mathcal{O}$. Its $k$-th holonomic prolongation

\[
v^{(k)} = \sum_{0 \leq |\alpha| \leq k} d_\alpha v^i \, \partial_i^A : J_{k+1} \mathcal{E} \rightarrow V J_k \mathcal{E}
\]

can be introduced as the restriction of the $k$-jet prolongation $J_k v : J_k J \mathcal{E} \rightarrow J_k V \mathcal{E}$ to the holonomic subbundle $J_{k+1} \mathcal{E} \subset J_k \mathcal{E}$, taking the natural isomorphism $J_k V \mathcal{E} \cong V J_k \mathcal{E}$ into account.

We now define a new operation $\delta[v]$ acting on basic forms $\alpha : J_k \mathcal{E} \rightarrow \mathcal{O} \otimes \wedge^q T^*M$ as

\[
\delta[v] \alpha := \bar{\partial} \alpha | (v^{(k)}) : J_{k+1} \mathcal{E} \rightarrow \mathcal{O} \otimes \wedge^q T^*M.
\]

This is, in a sense, a generalization of the standard Lie derivative; but note that it raises the order of the form acted on. The extension to non-vertical vector fields and non-basic forms is possible but more intricate, and won’t be needed in this paper.

Remark. Setting

\[
\Lambda^k_0 := \text{Sections}(J_k \mathcal{E} \rightarrow \mathcal{O} \otimes \wedge^q T^*M), \quad k, q \in \{0\} \cup \mathbb{N}, \ 0 \leq q \leq \dim M,
\]

where $J_0 \mathcal{E} \equiv \mathcal{E}$, then $\forall q$ and for any given $v : J \mathcal{E} \rightarrow T \mathcal{E}$ we obtain the (infinite) sequence

\[
\Lambda^q_0 \xrightarrow{\delta[v]} \Lambda^q_1 \xrightarrow{\delta[v]} \ldots \xrightarrow{\delta[v]} \Lambda^q_k \xrightarrow{\delta[v]} \ldots
\]

By direct calculations it is not difficult to check:

**Proposition 2.1** The operations $d_H$ and $\delta[v]$ commute.
In the case of the Lagrangian density we obtain the coordinate expression
\[ \delta [v] \mathcal{L} = \left( \sum_{0 \leq |\alpha| \leq k} P_i^\alpha \partial_\alpha v^i \right) d^m x . \]

The above horizontal form has a straightforward physical interpretation. Actually it’s not difficult to see that if \( \phi : M \rightarrow \mathcal{E} \) and we make the replacement \( \delta \phi \rightarrow \delta \phi + \epsilon v \circ j_k \phi \), then the variation of the action functional \( I[\phi] \equiv \int_L \circ j_k \phi \), at first order in \( \epsilon \in \mathbb{R}^+ \), is just \( \epsilon \int \delta [v] \mathcal{L} \circ j_k \phi \). Accordingly, we say that \( v \) is a symmetry of the field theory under consideration if \( \delta [v] \mathcal{L} = 0 \).

Observing that there is a natural inclusion \( V^* \mathcal{E} \hookrightarrow \mathcal{V}^* J_k \mathcal{E} \) (complementary to the fibering \( VJ_k \mathcal{E} \rightarrow V \mathcal{E} \)), we can compare \( d \mathcal{L} \) and \( F[\mathcal{L}] \). Contracting their difference with \( v_{(k)} \) we get the new 2\( k \)-order density
\[ \delta [v] \mathcal{L} - F[v] = \sum_{1 \leq |\alpha| \leq k} \left( P_i^\alpha \partial_\alpha v^i - (-1)^{|\alpha|} \partial_\alpha P_i^\alpha v^i \right) d^m x . \]

**Theorem 2.1** \( \delta [v] \mathcal{L} - F[v] \) is an exact horizontal differential.

**Proof:** Consider \( P = dx_a \otimes \mathcal{P}^a : J_{2k-1} \mathcal{E} \rightarrow \wedge^{m-1} T^* M \otimes V^* J_{k-1} \mathcal{E} \), where \( dx_a \equiv \partial_a |d^m x \) and
\[ \mathcal{P}^a = \sum_{0 \leq |\alpha| \leq k-1} (-1)^{|\alpha|} \partial_\alpha P_i^{A+a} d y^a_b . \]

Then one may check that actually \( d_H (P) v_{(k-1)} = \delta [v] \mathcal{L} - F[v] \).

An analogous of \( P \), or an exterior form related to it, is called “momentum” in classical Lagrangian field theories, where it is related to the notion of **Poincaré-Cartan form** [29, 32]. We’ll be only involved with orders one and two, respectively yielding
\[ \mathcal{P}^a | v = P_i^a v^i \] and \[ \mathcal{P}^a | v_{(1)} = (P_i^a - \partial_b P_i^{ab}) v^i + P_i^{ab} \partial_b v^i . \]

**Definition 2.1** A horizontal form \( \mathcal{J} : J_1 \mathcal{E} \rightarrow \mathcal{O} \otimes \wedge^{m-1} T^* M \) is called a conserved current if \( d_H \mathcal{J} \circ j_{k+1} \phi \) vanishes for any critical section \( \phi : M \rightarrow \mathcal{E} \) (that is, for any section fulfilling the field equation \( F_i \circ j_k \phi = 0 \)).

Using theorem 2.1 we then immediately prove the following version of **Noether’s theorem:**

**Theorem 2.2** Let \( \mathcal{J} : J \mathcal{E} \rightarrow V \mathcal{E} \) be a morphism over \( \mathcal{E} \) and \( \mathcal{N} : J_k \mathcal{E} \rightarrow \mathcal{O} \otimes \wedge^{m-1} T^* M \) a basic form such that \( \delta [v] \mathcal{L} = d_H \mathcal{N} \). Then \( \mathcal{J} [v] := P \circ v_{(k-1)} - \mathcal{N} \) is a conserved current.

**2.4 Symmetry and charge**

In relation to the setting discussed in [11, 6] we now consider a spacetime synchronization, and use coordinates adapted to it: synchronicity submanifolds are characterized by \( x^0 = \text{constant} \). We set \( \pi_i \equiv P_i^0 \equiv \partial^0_i \ell \). In a Hamiltonian setting, which we are not discussing in detail here, this plays the role of the “conjugate momentum” associated with \( y^i \). Field components and canonical momenta fulfill the **equal-time super-commutation rules**
\[ \{ \phi^i (x), \pi_j (x') \} = i \delta^i_j \delta (x - x') \sqrt{g} (x) , \]
\[ \{ \phi^i (x), \phi^j (x') \} = \{ \pi_i (x), \pi_j (x') \} = 0 . \]
where \( x \equiv (t, x) \), \( x' \equiv (t, x') \), and we used the shorthand \( \pi_i(x) \equiv \pi_i \circ j \phi(x) \). Indeed, these rules can be directly checked to hold true for free fields, and their validity for critical sections can be inferred by arguments based on the form of the dynamics. Note that the product of field components valued at different spacetime points is defined in terms of the non-supercommutative product in \( O \) (\ref{eq:brst}), and related formulas must be intended in a generalized distributional sense.

If \( J = J^0 \, dx \circ j \phi : J \mathcal{E} \rightarrow \mathcal{O} \otimes \wedge^3 T^* M \) is a conserved current then

\[
J^0 = \pi_i \, \nu^i : J \mathcal{E} \rightarrow \mathcal{O}
\]

is called the associated charge density. For any critical section \( \phi : M \rightarrow \mathcal{E} \) we define the related charge \( Q \in \mathcal{O} \) as

\[
Q := \int_{M_t} J \circ j \phi = \int_{M_t} (J^0 \circ j \phi) \, dx_0 \equiv \int_{M_t} \pi_i(x) \, \nu^i(x) \, d^3 x ,
\]

where \( M_t \) is any synchronicity manifold. This is independent of \( t \) because \( \phi \) is critical, and certainly finite if \( \phi \) has compact spatial support. Now we suppose \( [\mathcal{Q}] = [J^0] = 0 \), so that

\[
\{Q, \phi^i(x)\} = [Q, \phi^i(x)] = \int d^3 x' \left[ \pi_j(x') \, \nu^j(x'), \phi^i(x) \right] ,
\]

where \( \nu^j(x') = \nu^j \circ j \phi(x') \) and the like. Moreover we suppose \( \{\phi^i(x), \nu^j(\phi(x'))\} = 0 \). Then

\[
\{Q, \phi^i(x)\} = \pm \int d^3 x' \left[ \pi_j(x'), \phi^i(x) \right] \nu^j(x') = -i \int d^3 x' \, \delta_j^i \, \delta(x - x') \, \sqrt{|g|}(x) \, \nu^j(x') = -i \nu^i(\phi)(x) \, \sqrt{|g|}(x) .
\]

Since \( \nu^i(\phi) \equiv \delta[v] \phi^i \), we can rewrite the above result as

\[
\delta[v] \phi^i = i \left[ Q, \phi^i \right] |g|^{-1/2} .
\]

One also says that the infinitesimal symmetry \( v \) is generated by the corresponding charge \( Q \).

We recall that the functional derivative \( DF[\phi] \) of a “field functional” \( \phi \mapsto F[\phi] \in \mathcal{O} \) is the linear map acting on fields \( \theta : M \rightarrow \mathcal{E} \) as \( DF[\phi][\theta] := \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (F[\phi + \epsilon \theta] - F[\phi]) \). Then we set

\[
\delta[v] F[\phi] := DF[\phi][v \circ j \phi] ,
\]

which provides a natural extension of \( \delta[v] \) since, when \( F[\phi] = F \circ j_k \phi \) with \( F : J_k \mathcal{E} \rightarrow \mathcal{O} \), then indeed we have \( (\delta[v] F) \circ j_k \phi = \delta[v] F[\phi] \). The relation \( \delta[v] F = i \left[ Q, F \right] |g|^{-1/2} \) can be easily seen to hold at least when \( F \) is a fiber polynomial.

3 Gauge theory and BRST symmetry

3.1 Remarks about gauge fixing

If the “matter field” of a classical theory is a section of a vector bundle \( E \rightarrow M \), then the classical “gauge field” is a linear connection of that bundle, whose true physical meaning is encoded in the curvature tensor. Hence, different connections may yield the same physical field. Locally, connections can be described as tensor fields by choosing a gauge, namely
a local “flat” connection \( \gamma_0 \): an arbitrary linear connection \( \gamma \) is then characterised by the difference

\[
\alpha \equiv \gamma - \gamma_0 : M \to T^* M \otimes \text{End } E \equiv T^* M \otimes M \text{End } E^* .
\]

The fibers of the vector bundle \( \text{End } E \to M \) are constituted by all linear endomorphisms of the respective fibers of \( E \), and are naturally Lie algebras via the ordinary commutator. In fact, this is the Lie algebra bundle of the group bundle \( \text{Aut } E \to M \) of all fibered automorphisms of \( E \). Moreover \( E \) is usually endowed with some fibered geometric structure, which selects the ("internal" symmetry) Lie-group subbundle \( G \to M \) of all automorphisms preserving it; the fibers of \( G \) are isomorphic Lie groups, though distinguished isomorphisms among them don’t exist in general.\(^8\) A section \( M \to G \) is called a (local) gauge transformation. The Lie algebra bundle of \( G \) is a sub-bundle \( \mathcal{L} \subset \text{End } E \). If we restrict ourselves to consider connections which make the fiber geometric structure covariantly constant, then the difference of any two such connections is \( \mathcal{L} \)-valued.

A linear connection can be seen as a section \( M \to \Gamma \), where \( \Gamma \subset \mathcal{J}E \otimes M E^* \to M \) is the affine sub-bundle projecting over the identity \( 1_E \). Its “derived” vector bundle (the bundle of “differences of linear connections”) is \( D\Gamma = T^* M \otimes_M \text{End } E \). A fiber symmetry determines an affine sub-bundle \( \Gamma_G \subset \Gamma \), and \( D\Gamma_G = T^* M \otimes_M \mathcal{L} \).

Gauge freedom can be viewed in terms of momenta by observing that one usually tries to describe a radiative electromagnetic field as a tensor field of the form \( F = k \wedge b \), with \( k, b : M \to P \) such that \( k^\# \) is a geodesic null vector field and \( g^\#(k, b) = 0 \). Though in curved spacetime there is no guarantee that we can find a closed such \( F \), a natural viewpoint shift suggest that this algebraic type is suitable for describing photons. Then the couple \( (k, b) \) constitutes a redundant description.

Recalling the notion of a quantum field discussed in [16] and [21] we realize that all fields must be sections of some vector bundle. This is achieved by fixing a gauge, but the resulting theory has now too many “degrees of freedom”. In order to view this aspect in terms of the momentum formulation, we treat a gauge field as a section \( \alpha : P \to P \otimes \mathcal{L}, \ P \equiv T^* M \). A close examination at point interactions in terms of 2-spinors [22] shows that the replacement \( \alpha(p) \to \rho(p) + \alpha(p) \), with \( \chi : P \to \mathcal{L} \); actually does not affect scattering matrix calculations. Hence the physical meaning of the gauge field is encoded in its equivalence class, \( \alpha \) and \( \alpha' \) being equivalent if their difference is of the kind \( p \otimes \chi \). The equivalence class of \( \alpha \) also uniquely determines the “curvature-like” tensor

\[
\rho[\alpha] := i p \wedge \alpha + \alpha \bar{\alpha},
\]

where the notation \( \alpha \bar{\alpha} \beta \) stands for exterior product of \( \mathcal{L} \)-valued forms together with composition (if \( \alpha \) and \( \beta \) are \( \mathcal{L} \)-valued 1-forms then \( (\alpha \bar{\alpha} \beta)_{ab} = c^i_{jk} \alpha^i \beta^j \beta^k \) where \( c^i_{jk} \equiv [l_j, l_k]^i \) are the “structure constants” in the chosen special frame \( (l_k) \) of \( \mathcal{L} \)). The Lagrangian density for the field \( \alpha \), written in terms of \( \rho[\alpha] \), contains all the needed self-interaction terms.

In scattering matrix calculations, gauge freedom is exploited by a suitable gauge fixing, namely by inserting, in the gauge particle propagator, terms that won’t affect the final result. On turn, these derive from an added term in the Lagrangian which is not gauge-invariant, namely does not “pass to the quotient” when we deal with the above said equivalence classes, though it is a natural geometric object when \( \alpha \) is seen as a tensor field.

---

\(^8\) Usually people prefer to deal with a fixed group, by exploiting the notion of principal bundle.
3.2 Geometric background for gauge field theory

In previous papers [14, 15, 18, 20], a fairly general geometric background for gauge field theories, encompassing at least all the sectors needed in the standard model, was shown to arise naturally from a few elementary blocks (“minimal geometric data”). While having that in mind, for a preliminary discussion we can conveniently use a simplified setting in which we provisionally disregard the differences between the right-handed and the left-handed sectors, and issues related to symmetry breaking.

Let $W \to M$ be the bundle of Dirac spinors, and $m$ the considered fermion’s mass. The “semi-trivial” bundle $P_m \times M W \to P_m$ has the distinguished decomposition $W^+ \oplus F^-$, with $W_p^\pm := \ker(m \pm \gamma_p)$, where $\gamma$ denotes the Dirac map. We call $W^+$ and $W^-$ the electron and positron bundles, respectively [15, 18]. The sub-bundles $W^\pm$ are mutually orthogonal in the Hermitian metric associated with Dirac conjugation; this has signature $(++--)$, and the sign of its restriction to $W^\pm$ is the same as the label. We have a distinguished transformation expressing a Dirac frame $(u_A(p); v_B(p))$, $A, B = 1, 2$, which is adapted to the decomposition $W_p = W_p^+ \oplus W_p^-$, in terms of a frame independent of $p$, e.g. the Dirac frame $(\zeta_\alpha)$ associated with the observer $(\alpha = 1, 2, 3, 4)$. This transformation is inserted in the integral defining the components of the fermion field (§ 1.6).

Let now $F \to M$ be a complex vector bundle, endowed with a fibered positive Hermitian metric, describing the internal fermion structure besides spin, and $\mathcal{L} \subset \text{End} F$ the associated Lie algebra bundle over $M$ (§ 3.1). Our classical configuration bundle has then a fermion sector $Y \equiv W \otimes M F$ and the gauge field sector $P \otimes M \mathcal{L}$. In order to deal with gauge symmetry we’ll have to include also some “ghost sectors”, whose classical configuration bundles are either $\mathcal{L}$ or $\mathcal{L}^\ast$. Besides the Dirac frame, we deal with a frame $(f_i)$ of $F$ and a frame $(t_i)$ of $\mathcal{L}$. Whenever no confusion arises, a dual frame is denoted by the same symbol and shifted index position. We also write $t_i = t_i^j f_j \otimes f^i$ and $t^i = t^i_j f_i \otimes f^j$, so that the natural inclusion $\iota : \mathcal{L} \to \text{End} F$ has the local expression $t = t^i \otimes t^j f_i \otimes f^j$. We also have a Hermitian structure on $\text{End} F$, given by $(X, Y) \mapsto \text{Tr}(X^\dagger Y)$; its restriction to $\mathcal{L}$, constituted by all traceless anti-Hermitian endomorphisms, is negative-definite, thus $t_i^\dagger = -t_i$ if $(t_i)$ is orthonormal as we’ll assume. The following table gives, for each sector, the notation used for the related field components, its grade (or parity), and the classical configuration bundles:

| field            | grade | components | cl. config. bundle |
|------------------|-------|------------|-------------------|
| fermion          | 1     | $\psi^\alpha$ | $F \otimes W$ |
| gauge boson      | 0     | $A^I_a$     | $T^*M \otimes \mathcal{L}$ |
| ghost            | 1     | $\omega_I$  | $\mathcal{L}$  |
| anti-ghost       | 1     | $\omega_I$  | $\mathcal{L}^\ast$ |
| Nakanishi-Lautrup| 0     | $n^I$       | $\mathcal{L}$  |

Note that the ghost, anti-ghost and Nakanishi-Lautrup (NL) fields are all mutually independent, the natural isomorphism $\mathcal{L} \leftrightarrow \mathcal{L}^\ast$ not withstanding.

3.3 The Lagrangian

We consider a first-order Lagrangian density $\mathcal{L} = \ell \, d^4 x : \mathcal{J} \mathcal{E} \to \mathcal{O} \otimes \wedge^m T^* M$, with

$$\ell = \ell_\psi + \ell_A + \ell_{\text{ghost}} : \mathcal{J} \mathcal{E} \to \mathcal{O}$$
sum of fermion, gauge and ghost sectors terms defined as
\[ \ell_\psi = \left( \frac{1}{2} \left( \bar{\psi}_{\alpha i} \nabla \psi^{\alpha i} - \nabla \bar{\psi}_{\alpha i} \psi^{\alpha i} \right) - m \bar{\psi}_{\alpha i} \psi^{\alpha i} \right) \sqrt{|g|} , \]
\[ \ell_A = -\frac{1}{4} g^{ab} g^{cd} F_{ac} F_{bd} \sqrt{|g|} \equiv -\frac{1}{4} g^{ab} g^{cd} F_{acj} F_{bdj} \sqrt{|g|} , \]
\[ \ell_{\text{ghost}} = g^{ab} \omega_{1,a} \nabla_b \omega^l \sqrt{|g|} + n_i \left( f^l + \frac{1}{2} \xi n^l \right) \sqrt{|g|} , \quad \xi \in \mathbb{R} , \]
where \( g \) is the spacetime metric, \(|g| \equiv |\det g|\), \( \bar{\psi} : M \to \mathcal{O} \otimes F^* \otimes W^* \) is the Dirac adjoint of \( \psi \), and
\[ \nabla \bar{\psi}_{\alpha i} = g^{ab} \gamma_b^{\alpha \beta} \left( \psi^{\beta j} - A_{aj}^i \psi^{\beta j} - \Gamma_{IJ}^{i} \gamma_{\beta} \right) , \]
\[ \nabla \psi^{\alpha i} = g^{ab} \left( \bar{\psi}_{\alpha i} + A_{ai}^j \bar{\psi}_{\beta j} + \Gamma_{IJ}^{j} \gamma_{\beta} \right) \gamma^a \beta , \]
\[ \Gamma_{IJ}^{i} = \frac{1}{2} \Gamma^{\alpha \beta \gamma} \left( \gamma_j \gamma_k - \gamma_k \gamma_j \right) , \]
\[ F_{acj} = F_{acj}^i t^i_j = A_{[a,c]}^i + c_{jH}^j A_a^i A_c^j = A_{[a,c]}^i + [A_a , A_c]^j , \]
\[ \nabla_b \omega^l \equiv \omega^l_b + c_{jH}^j \omega^l W_a^j A_b^i , \quad f^l \equiv (\ast dA)^l = \frac{1}{\sqrt{|g|}} d_a (g^{ab} \sqrt{|g|}) A_b . \]

Here \( \Gamma : W \to JW \) denotes the spin connection, describing the interaction between spin and gravitation. As it is customary, for notation simplicity we indicate field components and the related fiber coordinates by the same symbols.

While here we are not directly involved with the field equations for all the fields, we note that for the NL field we get \( n^l = -\frac{1}{2} f^l \). The NL-term in \( \ell_{\text{ghost}} \) then becomes \(-\frac{1}{2} \xi \sqrt{|g|} f^i \), which is recognized as the “gauge fixing term” usually added to the gauge field Lagrangian.

### 3.4 BRST symmetry

We consider the morphism \( v : J\mathcal{E} \to V\mathcal{E} \) which has the coordinate expression
\[ v = v^{\alpha i} \frac{\partial}{\partial \psi^{\alpha i}} + v^{\alpha i} \frac{\partial}{\partial \bar{\psi}_{\alpha i}} + v^l_a \frac{\partial}{\partial A^i_a} + v^l \frac{\partial}{\partial \omega^l} + v_l \frac{\partial}{\partial \alpha} , \]
\[ v^{\alpha i} = \theta \lambda^{ij} \psi^{\alpha j} , \quad v^{\alpha i} = \theta \lambda^{ij} \psi^{\alpha j} , \quad v^l_a = \theta \nabla_a \omega^l , \quad v^l = \frac{1}{2} \theta c_{jH} \omega^j \omega^H , \quad v_l = \theta n_l , \]
where \( \theta \in \mathcal{O} \) is such that \( |\theta| = 1 \), so that the components of \( v \) have the same parities as their respective sectors.

If \( \Phi : J_k \mathcal{E} \to \wedge^q T^* M \) is a basic form then \((\ref{2.3}) \delta[v] \Phi : J_{k+1} \mathcal{E} \to \wedge^q T^* M \). We then define the BRST transformation \( s \) by the rule
\[ \theta s \Phi = \delta[v] \Phi . \]

In particular \( s \) acts on functions \( \mathcal{E} \to \mathcal{O} \) \((k = q = 0)\), and for fiber coordinates we obtain essentially the usual formulas by which it is usually introduced as a “transformation of the fields” \( \ref{24} \):
\[ s\psi^{\alpha i} = \lambda^{ij} \psi^{\alpha j} , \quad s\bar{\psi}_{\alpha i} = \lambda^{ij} \psi^{\alpha j} , \quad sA_a^i = \nabla_a \omega^l + c_{jH}^j \omega^j A_a^i , \]
\[ s\omega^l = \frac{1}{2} c_{jH} \omega^j \omega^H , \quad s\alpha = n_l , \quad sn_l = 0 . \]
3.4 BRST symmetry

It’s not difficult then to check that s is nilpotent, \( s^2 = 0 \). Consequently, for each exterior degree \( 0 \leq q \leq 4 \) we have the exact sequence \( \ldots \to S^q \to \Lambda^q \to \Lambda^{q+1} \to S \to \ldots \) (see [2.3]).

With regard to practical calculations we note that proposition [2.1] implies that \( \delta[v] \) and s commute with the derivation of fields relatively to base coordinates. We also note that the actions of \( \delta[v] \) and s on fiber polynomials are, respectively, a derivation and an anti-derivation, and that they respectively preserve and change grade.

We now discuss \( \delta[v]\mathcal{L} \), using the observation that \( \mathcal{L}_{\text{ghost}} \) is s-exact up to a horizontal differential.

**Proposition 3.1** We have \( \delta[v]\mathcal{L}_{\text{ghost}} = \theta d_H\mathcal{N} \) where

\[
\mathcal{N} \equiv \left< n, \ast \nabla \omega \right> = g^{ab} \sqrt{|g|} \, n_I \, \nabla_b \omega^I \, dx_a .
\]

**Proof:** We have \( s f^i = \frac{1}{\sqrt{|g|}} d_a (g^{ab} \sqrt{|g|} \, \nabla_b \omega^i) \), whence

\[
s \left( \varpi_I \left( f^I + \frac{1}{2} \xi n^I \right) \sqrt{|g|} \right) = n_I \left( f^I + \frac{1}{2} \xi n^I \right) \sqrt{|g|} - \varpi_I \left( g^{ab} \sqrt{|g|} \, \nabla_b \omega^i \right) =
\]

\[
= n_I \left( f^I + \frac{1}{2} \xi n^I \right) \sqrt{|g|} - d_a \left( \varpi_I \left( g^{ab} \sqrt{|g|} \, \nabla_b \omega^i \right) \right).
\]

namely \( \mathcal{L}_{\text{ghost}} = sK + d_H(\varpi, \ast \nabla \omega) \) with \( K \equiv \varpi_I \left( f^I + \frac{1}{2} \xi n^I \right) \sqrt{|g|} \, d^I x \). Hence

\[
\delta[v]\mathcal{L}_{\text{ghost}} = \theta s^2 K + \theta s d_H(\varpi, \ast \nabla \omega) = 0 + \theta d_H(\varpi, \ast \nabla \omega) ,
\]

where we used \( \delta[v]d_H = d_H\delta[v] \) (proposition [2.1]) and \( s\nabla_b \omega^i = s^2 A^i_b = 0 \). 

The above proposition implies \( \delta[v]\mathcal{L} = \delta[v](\mathcal{L}_\psi + \mathcal{L}_A) + \theta d_H\mathcal{N} \). Now \( \mathcal{L}_\psi + \mathcal{L}_A \) is the standard Lagrangian density of electrodynamics, dependent on spinor and e.m. fields (independent of ghosts). The action of \( \delta[v] \) on these is exactly that of an infinitesimal gauge transformation, represented by \( \theta \omega \). Hence \( \mathcal{L}_\psi + \mathcal{L}_A = \delta[v]-\text{closed} \), and finally we get \( \delta[v]\mathcal{L} = \theta d_H\mathcal{N} \).

Recalling theorem [2.2] we now get a current which we express as \( \theta J^a dx_a \), obtaining

\[
J^a = \left( -i \, \langle \bar{\psi} \gamma^\alpha \omega \psi \rangle + \langle F^{ab} \, \nabla_b \omega \rangle + g^{ab} (n_I \, \nabla_b \omega^I - \frac{1}{2} \varpi_{I,b} \, c_{JH} \omega^J \omega^H) \right) \sqrt{|g|} .
\]

We write \( J = J_{(\psi,A)} + J_{\text{ghost}} \) where \( J_{(\psi,A)} \) is constituted by the first two terms above; then \( \theta J_{(\psi,A)} \) is the current related to an infinitesimal gauge transformation \( \theta \omega \) of the matter and gauge fields. Actually, a calculation shows that \( d_H J_{(\psi,A)} = 0 \).

**Remark.** The symmetry \( \omega \to e^{i \alpha} \omega \), \( \omega \to e^{-i \alpha} \omega \), \( \alpha \in \mathbb{R} \), corresponds to the infinitesimal symmetry \( v_{FP} = \omega^I \frac{\partial}{\partial \phi^I} - \omega^I \frac{\partial}{\partial \bar{\phi}^I} \). We have \( \delta[v_{FP}]\mathcal{L} = 0 \) and get the *Faddeev-Popov current*

\[
J_{FP} = \int_{FP} dx_a = g^{ab} \left( \varpi_{I,b} \omega^I + \varpi_I \nabla_b \omega^I \right) \sqrt{|g|} \, dx_a ,
\]

which fulfills

\[
\int J_{FP} = g^{ab} n_I,b \omega^I \sqrt{|g|} \, dx_a + J_{\text{ghost}} .
\]

Moreover we find \( [v_{FP}, v] = v \).

With regard to the charge associated with an infinitesimal symmetry ([2.4]), the case of the BRST symmetry is peculiar because \( v \) depends on the choice of \( \theta \), which we get rid of by setting \( \delta[v] \equiv \theta s \) and defining the charge \( Q \) in such a way that the charge in the general sense is actually \( \theta Q \). The argument of [2.4] applies because \( \delta Q = 0 \), and it can be checked that the equal-time identity \( \left( \left< \phi^i(x) , \varphi^i(x') \right> \right) = 0 \) holds in each sector. Hence

\[
\theta s \phi^i = i \left[ \theta Q , \phi^i \right] |g|^{-1/2} = i \theta \left[ Q , \phi^i \right] |g|^{-1/2}
\]

\[
\Rightarrow s \phi^i = i \left[ Q , \phi^i \right] |g|^{-1/2} \quad (i \text{ is a generic index}).
\]
Finally, we remark that the ghost Lagrangian is often presented as an equivalent 2nd order density $-\varpi_i \, d_a (g^{ab} \nabla_b \omega^i \sqrt{|g|})$, rather than the 1st order density $g^{ab} \varpi_{i,a} \nabla_b \omega^i \sqrt{|g|}$. Actually this amounts to considering a new Lagrangian $L' = L - d_M M$ where

$$M \equiv \langle \varpi, * \nabla \omega \rangle = g^{ab} \varpi_i \nabla_b \omega^i \sqrt{|g|} \, dx_a.$$  

We immediately see that $\delta[v]M = \theta N$, so that

$$\delta[v]L' = \delta[v]L - \delta[v]d_M M = \theta d_M N - d_M \delta[v]M = 0.$$  

Hence (theorem 2.2) we obtain the 2-nd order current

$$P^a \big|_{v(1)} = (P^a_i - d_b P^{ab}_i) v^i + P^{ab}_i d_b v^i,$$

which, by a straightforward calculation, can be shown to coincide with $\theta J$.

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