Comparing the constructions of Goldberg, Fuller, Caspar, Klug and Coxeter and a general approach to local symmetry-preserving operations

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Supplementary material

A first step towards general local operations preserving orientation-preserving symmetries but not necessarily orientation-reversing automorphisms

Although the emphasis of this paper is on the work of Goldberg, Fuller, Caspar and Klug and on operations preserving all symmetries, we want to make a first step – details are postponed to a later paper – towards operations that preserve orientation-preserving automorphisms but not necessarily orientation-reversing automorphisms. In the case of Goldberg, these operations could be described by two subdivisions – one of a black and one of a white triangle. In general, such a description is not possible, and it is necessary to describe how a quadrangle containing a white and a black triangle that share an edge must be subdivided. See Figure 3 and Figure 4 from the main text for an illustration of the following definition.

Definition 1 Let $T$ be a periodic 3-connected tiling of the Euclidean plane with chamber system $C_T$, and let $v_2, v_0, v_1, v'_0$ be points in the Euclidean plane so that:

(i) $v_2$ is the center of a rotation $\rho_{v_2}$ by 60 degrees in counterclockwise direction that is a symmetry of the tiling.

(ii) $v'_0 = \rho_{v_2}(v_0)$

(iii) $v_1$ is the midpoint between $v'_0$ and $v_0$ and the center of a rotation $\rho_{v_1}$ by 180 degrees that is a symmetry of the tiling (so also $v'_0 = \rho_{v_1}(v_0)$).

Let $Q$ be a simple cycle in $C_T$ through $v_2, v_0, v_1, v'_0$ (in this order). For $\{x, y\} \in \{\{v_2, v_0\}, \{v_2, v'_0\}, \{v_1, v_0\}, \{v_1, v'_0\}\}$ let $P_{x,y}$ denote the path on $Q$ from $x$ to $y$ not containing any other vertex of $\{v_2, v_0, v_1, v'_0\}$.

If $P_{v_2,v'_0} = \rho_{v_2}(P_{v_2,v_0})$ and $P_{v_1,v'_0} = \rho_{v_1}(P_{v_1,v_0})$, then we call the quadrangle $Q$ with corner points $v_2, v_0, v_1, v'_0$ labelled with the names $v_2, v_0, v_1, v'_0$ and subdivided into chambers as given by $C_T$, a local operation that preserves orientation-preserving symmetries, lopsp operation for short.

Let $P$ be a polyhedron or tiling given as a chamber system. We call a pair of (black and white) chambers sharing a 1-vertex and a 2-vertex a double chamber. Each chamber is contained in exactly one double chamber. The result of applying a lopsp operation $O$ to a polyhedron or tiling $P$ is given by subdividing each double chamber of $P$ with $O$ by identifying $v_2$ with the 2-vertex of the double chamber, $v_0$ with the 0-vertex of the white chamber, $v_1$ with the 1-vertex of the double chamber, and $v'_0$ with the 0-vertex of the black chamber. Note that this operation is purely combinatorial and that the angles of $O$ are not preserved.

As already mentioned in the section about Goldberg’s approach where the chiral case is discussed, also here there are various ways to draw the boundaries of the quadrangle $Q$ that lead to equivalent operations, that is, operations that when applied to a polyhedron give the same result. We will postpone working out the details to a later paper.

It is tempting to conjecture that for each lopsp operation there is a way to choose the quadrangle so that the subdivision of the double chamber can be expressed by two subdivisions of the chambers – one for the black chambers and one for the white chambers.
Figure S1: A lopsp operation that cannot be described by separate subdivisions of black and white chambers.

Figure S2: The lopsp operation propellor and the periodic tiling from which it originates.

Figure S1 gives an example of an operation where this is not the case. In order to avoid misunderstandings, we mention the fact already here, but a proof has to be postponed as first the necessary prerequisites (e.g. results about different ways to choose the edges of the quadrangles that lead to equivalent operations) have to be formally introduced and proven.

Again, it must be shown that operations used in the literature are covered by this definition. The mirror image plays a special role as an operation as it does not change the combinatorial structure of a polyhedron. It is neither an lsp operation nor a lopsp operation. On the level of chamber systems it can be described as changing the orientation of the chambers, so black ones become white and the other way around. We will show that the well known operations propellor, snub and whirl can be obtained from tilings. Other operations (e.g. gyro) can be obtained by a combination of lsp operations and these lopsp operations. These operations are basic to the field. People who are not familiar with these operations (or at least their names) can take their representations as lopsp operations as the definition. In Figure S2, Figure S3, and Figure S4 the quadrangles $v_2, v_0, v_1, v'_0$ are shown next to the tiling for better visibility. All chiral Caspar-Klug operations can be represented this way.
Although we could not find publications where decorations of double chambers are used to represent lopsop operations like propellor, snub or whirl, it may be considered common knowledge among people who work with chamber systems that such a representation is possible. Again, Definition 1 establishes the inverse direction and gives a criterion for which decompositions of double chambers define an operation.