UNIFORM APPROXIMATIONS
BY FOURIER SUMS ON CLASSES
OF CONVOLUTIONS OF PERIODIC FUNCTIONS

ANATOLY S. SERDYUK
Institute of Mathematics NAS of Ukraine
Tereschenkivska st. 3, 01601 Kyiv, Ukraine
E-mail: serdyuk@imath.kiev.ua

TETIANA A. STEPANYUK
Johann Radon Institute for Computational and Applied Mathematics (RICAM)
Austrian Academy of Sciences, Altenbergerstrasse 69 4040, Linz, Austria;
Institute of Mathematics of NAS of Ukraine,
3, Tereshchenkivska st., 01601, Kyiv-4, Ukraine
E-mail: tania.stepanyuk@ukr.net

Summary
We establish asymptotic estimates for exact upper bounds of uniform approximations by
Fourier sums on the classes of 2π–periodic functions, which are represented by convolutions of functions
\( \varphi(\varphi \pm 1) \) from unit ball of the space \( L_1 \) with fixed kernels \( \Psi_{\beta} \) of the
form
\[
\Psi_{\beta}(t) = \sum_{k=1}^{\infty} \psi(k) \cos \left( k t - \frac{\beta \pi}{2} \right), \quad \sum_{k=1}^{\infty} k \psi(k) < \infty, \psi(k) \geq 0, \beta \in \mathbb{R}.
\]

1. Introduction. Let \( L_1 \) be the space of 2π–periodic functions \( f \) summable on \([0, 2\pi)\), in
which the norm is given by the formula \( \|f\|_1 = \frac{2\pi}{0} |f(t)|\,dt \); \( L_\infty \) be the space of measurable
and essentially bounded 2π–periodic functions \( f \) with the norm \( \|f\|_\infty = \text{ess sup} |f(t)| \); \( C \)
be the space of continuous 2π–periodic functions \( f \), in which the norm is specified by the equality
\( \|f\|_C = \max_t |f(t)| \).

Let \( \psi(k) \) be an arbitrary fixed sequence of real, nonnegative numbers and let \( \beta \) be a

2000 Mathematics Subject Classification: Primary 41A36.
Key words and phrases: Fourier sums, classes of convolutions of periodic functions, asymptotic equality
The paper is in final form and no version of it will be published elsewhere.
A.S. SERDYUK, T.A. STEPANYUK

fixed real number.

We set

\[ B_0^1 := \{ \varphi : \|\varphi\|_1 \leq 1, \varphi \perp 1 \}. \]  

(1)

Further let \( C_{\beta,1}^\psi \) be the set of all functions \( f \), which are represented for all \( x \) as convolutions of the form

\[ f(x) = \frac{a_0}{2} + \frac{1}{\pi} \int_{-\pi}^\pi \varphi(t)\Psi_\beta(x-t)dt, \quad a_0 \in \mathbb{R}, \quad \varphi \in B_0^1, \]  

(2)

where \( \Psi_\beta \) is a fixed kernel of the form

\[ \Psi_\beta(t) = \sum_{k=1}^{\infty} \psi(k) \cos(kt - \frac{\beta\pi}{2}), \quad \psi(k) \geq 0, \quad \beta \in \mathbb{R}, \]  

(3)

and the following condition holds:

\[ \sum_{k=1}^{\infty} \psi(k) < \infty. \]  

(4)

Condition (4) provides an embedding \( C_{\beta,1}^\psi \subset C \).

For \( \psi(k) = e^{-\alpha k^r}, \alpha, r > 0 \), the kernels \( \Psi_\beta(t) \) of the form (3) are called generalized Poisson kernels \( P_{\alpha,r,\beta}(t) = \sum_{k=1}^{\infty} e^{-\alpha k^r} \cos(kt - \frac{\beta\pi}{2}) \) and the classes of functions \( f \), generated by these kernels are the classes of generalized Poisson integrals \( C_{\beta,1}^{\alpha,r} \).

For the classes \( C_{\beta,1}^\psi \) we consider the quantities

\[ \mathcal{E}_n(C_{\beta,1}^\psi) = \sup_{f \in C_{\beta,1}^\psi} \|f(\cdot) - S_{n-1}(f;\cdot)\|_C, \]  

(5)

where \( S_{n-1}(f;\cdot) \) are the partial Fourier sums of order \( n-1 \) for a function \( f \).

Approximations by Fourier sums on other classes of differentiable functions in uniform metric were investigated in works [1]–[10].

In this paper we consider Kolmogorov–Nikolsky problem about finding of asymptotic equalities of the quantity (5) as \( n \to \infty \).

2. Main result. The following statement holds.

THEOREM 2.1. Let \( \sum_{k=1}^{\infty} k\psi(k) < \infty, \psi(k) \geq 0, k = 1, 2, ... \) and \( \beta \in \mathbb{R} \). Then as \( n \to \infty \) the following asymptotic equality holds

\[ \mathcal{E}_n(C_{\beta,1}^\psi) = \frac{1}{\pi} \sum_{k=n}^{\infty} \psi(k) + \frac{O(1)}{n} \sum_{k=1}^{\infty} k\psi(k + n), \]  

(6)

where \( O(1) \) is a quantity uniformly bounded in all parameters.

Proof. According to (2) and (5) we have that

\[ \mathcal{E}_n(C_{\beta,1}^\psi) = \frac{1}{\pi} \sup_{\varphi \in B_0^1} \left\| \int_{-\pi}^\pi \varphi(t)\Psi_{\beta,n}(x-t)dt \right\|_C, \]  

(7)
where

\[ \Psi_{\beta,n}(t) := \sum_{k=n}^{\infty} \psi(k) \cos \left( kt - \frac{\beta \pi}{2} \right), \quad \beta \in \mathbb{R}. \tag{8} \]

Taking into account the invariance of the sets \( B_{1}^{0} \) under shifts of the argument, from (7) we conclude that

\[ E_{n}(\mathcal{C}_{\beta,1}^{\psi}) = \frac{1}{\pi} \sup_{\varphi \in B_{1}^{-\pi}} \int_{-\pi}^{\pi} \varphi(t) \Psi_{\beta,n}(t) dt. \tag{9} \]

On the basis of the duality relation (see, e.g., [2, Chapter 1, Section 1.4]) we obtain

\[ \sup_{\varphi \in B_{1}^{-\pi}} \int_{-\pi}^{\pi} \Psi_{\beta,n}(t) \varphi(t) dt = \inf_{\lambda \in \mathbb{R}} \| \Psi_{\beta,n}(t) - \lambda \|_{C}, \tag{10} \]

We represent the function \( \Psi_{\beta,n}(t) \), which is defined by formula (8), in the form

\[ \Psi_{\beta,n}(t) = g_{\psi,n}(t) \cos \left( nt - \frac{\beta \pi}{2} \right) + h_{\psi,n}(t) \sin \left( nt - \frac{\beta \pi}{2} \right), \tag{11} \]

where

\[ g_{\psi,n}(t) := \sum_{k=0}^{\infty} \psi(k + n) \cos kt, \tag{12} \]

\[ h_{\psi,n}(t) := -\sum_{k=0}^{\infty} \psi(k + n) \sin kt. \tag{13} \]

It is obvious that

\[ \inf_{\lambda \in \mathbb{R}} \| \Psi_{\beta,n}(t) - \lambda \|_{C} \leq \| \Psi_{\beta,n} \|_{C} \tag{14} \]

and

\[ \| \frac{1}{2} \left[ \Psi_{\beta,n} \left( t + \frac{\pi}{n} \right) - \Psi_{\beta,n}(t) \right] \|_{C} \leq \inf_{\lambda \in \mathbb{R}} \| \Psi_{\beta,n}(t) - \lambda \|_{C}. \tag{15} \]

In view of representation (11) and applying mean value theorem, we obtain that

\[ \frac{1}{2} \left[ \Psi_{\beta,n} \left( t + \frac{\pi}{n} \right) - \Psi_{\beta,n}(t) \right]_{C} = \frac{1}{2} \left[ 2\Psi_{\beta,n}(t) + g_{\psi,n} \left( t + \frac{\pi}{n} \right) \cos \left( nt - \frac{\beta \pi}{2} \right) + h_{\psi,n} \left( t + \frac{\pi}{n} \right) \sin \left( nt - \frac{\beta \pi}{2} \right) \right. \]

\[ \left. - \left( g_{\psi,n}(t) \cos \left( nt - \frac{\beta \pi}{2} \right) + h_{\psi,n}(t) \sin \left( nt - \frac{\beta \pi}{2} \right) \right) \right]_{C} \]

\[ = \| \Psi_{\beta,n} \|_{C} + \mathcal{O}(1) \left( \| g_{\psi,n} \left( t + \frac{\pi}{n} \right) - g_{\psi,n}(t) \|_{C} + \| h_{\psi,n} \left( t + \frac{\pi}{n} \right) - h_{\psi,n}(t) \|_{C} \right) \]

\[ = \| \Psi_{\beta,n} \|_{C} + \mathcal{O}(1) \left( \frac{1}{n} \| g'_{\psi,n} \|_{C} + \frac{1}{n} \| h'_{\psi,n} \|_{C} \right) \]

\[ = \| \Psi_{\beta,n} \|_{C} + \mathcal{O}(1) \sum_{k=1}^{\infty} k\psi(k + n). \tag{16} \]
So, formulas (9), (10) and (14)–(16) imply

\[ E_n(C^\psi_{,1})C = \frac{1}{\pi}||\Psi_{\beta,n}||C + \frac{O(1)}{n} \sum_{k=1}^{\infty} k\psi(k + n). \]  

(17)

The kernel \( \Psi_{\beta,n} \) can be written in the form

\[ \Psi_{\beta,n}(t) = \sqrt{g^2_{\psi,n}(t) + h^2_{\psi,n}(t)} \times \]

\[ \times \left( \frac{g_{\psi,n}(t)}{\sqrt{g^2_{\psi,n}(t) + h^2_{\psi,n}(t)}} \cos \left( nt - \frac{\beta \pi}{2} \right) + \frac{h_{\psi,n}(t)}{\sqrt{g^2_{\psi,n}(t) + h^2_{\psi,n}(t)}} \sin \left( nt - \frac{\beta \pi}{2} \right) \right) \]

\[ = \sqrt{g^2_{\psi,n}(t) + h^2_{\psi,n}(t)} \cos \left( nt - \frac{\beta \pi}{2} - \arg(g_{\psi,n}(t) + ih_{\psi,n}(t)) \right). \]  

(18)

Let

\[ t_0 := \frac{1}{n} \left( \frac{\beta \pi}{2} + \arg(g_{\psi,n}(t) + ih_{\psi,n}(t)) \right). \]  

(19)

Then

\[ ||\Psi_{\beta,n}||C \geq \Psi_{\beta,n}(t_0) = \sqrt{g^2_{\psi,n}(t_0) + h^2_{\psi,n}(t_0)} \geq |g_{\psi,n}(t_0)|. \]  

(20)

Using mean value theorem we have that

\[ |g_{\psi,n}(t_0)| = g_{\psi,n}(0) + |g_{\psi,n}(t_0) - g_{\psi,n}(0)| = g_{\psi,n}(0) + \frac{O(1)}{n} ||g'_{\psi,n}||C \]

\[ = \sum_{k=n}^{\infty} \psi(k) + \frac{O(1)}{n} \sum_{k=1}^{\infty} k\psi(k + n). \]  

(21)

On the other hand it is clear that

\[ ||\Psi_{\beta,n}||C \leq \sum_{k=n}^{\infty} \psi(k). \]  

(22)

Thus,

\[ E_n(C^\psi_{,1})C = \frac{1}{\pi} \sum_{k=n}^{\infty} \psi(k) + \frac{O(1)}{n} \sum_{k=1}^{\infty} k\psi(k + n). \]  

(23)

Theorem 2.1 is proved.

Corollary 2.2. Let the sequence \( \psi(k) \), which generates the classes \( C^\psi_{,1} \), satisfies the condition \( D_0 \), i.e., \( \psi(k) > 0 \) and

\[ \lim_{k \to 0} \frac{\psi(k + 1)}{\psi(k)} = 0. \]

Then, the following asymptotic equality holds as \( n \to \infty \)

\[ E_n(C^\psi_{,1})C = \frac{1}{\pi} \psi(n) + \frac{O(1)}{n} \sum_{k=n+1}^{\infty} k\psi(k), \]  

(24)

where \( O(1) \) is a quantity uniformly bounded in all parameters.
Proof. Indeed, let \( \psi \in D_0 \), then the right hand of \( 5 \) can be written in the form
\[
E_n(C_\psi C, 1) C = \frac{1}{\pi} \psi(n) + O(1) \left( \sum_{k=n+1}^{\infty} \psi(k) + \frac{1}{n} \sum_{k=0}^{\infty} k \psi(k + n) \right)
\]
\[
= \frac{1}{\pi} \psi(n) + \frac{O(1)}{n} \sum_{k=1}^{\infty} (k + n) \psi(k + n)
\]
\[
= \frac{1}{\pi} \psi(n) + \frac{O(1)}{n} \sum_{k=n+1}^{\infty} k \psi(k).
\]

Corollary 2.2 is proved.  

Asymptotic equality \( 24 \) with written in another form reminder was obtained earlier in \([4]\) and \([5]\).

Corollary 2.3. Let \( \psi(k) = e^{-\alpha k r}, r > 1, \alpha > 0 \) and \( \beta \in \mathbb{R} \). Then as \( n \to \infty \) the following asymptotic equality holds
\[
E_n(C_{\alpha,r,1}) C = e^{-\alpha n r} \left( \frac{1}{\pi} + O(1) e^{-\alpha n r^{-1}} \left( 1 + \frac{1}{\alpha r(n+1)^{r-1}} \right) \right),
\]
where \( O(1) \) is a quantity uniformly bounded in \( n \) and \( \beta \).

Proof. Formula \( 24 \) implies that as \( n \to \infty \)
\[
E_n(C_{\alpha,r,1}) C = \frac{1}{\pi} e^{-\alpha r} + \frac{O(1)}{n} \sum_{k=n+1}^{\infty} k e^{-\alpha k r}.
\]

It is easy to see that
\[
\frac{1}{n} \sum_{k=n+1}^{\infty} k e^{-\alpha k r} < \frac{1}{n} \left( (n+1) e^{-\alpha (n+1)^r} + \int_{n+1}^{\infty} t e^{-\alpha t r} dt \right).
\]

Integrating by parts, we get
\[
\int_{n+1}^{\infty} t e^{-\alpha t r} dt = \int_{n+1}^{\infty} t^2 \frac{1}{\alpha t r} \left( -e^{-\alpha t r} \right)' dt \leq \frac{1}{\alpha r(n+1)^r} \int_{n+1}^{\infty} t^2 \left( -e^{-\alpha t r} \right)' dt
\]
\[
= \frac{1}{\alpha r(n+1)^r} \left( (n+1)^2 e^{-\alpha (n+1)^r} + 2 \int_{n+1}^{\infty} t e^{-\alpha t r} dt \right).
\]

From the last equality we have
\[
\left( 1 - \frac{2}{\alpha r(n+1)^r} \right) \int_{n+1}^{\infty} t e^{-\alpha t r} dt \leq \frac{(n+1)^2 e^{-\alpha (n+1)^r}}{\alpha r(n+1)^r},
\]
which is equivalent to
\[
\int_{n+1}^{\infty} t e^{-\alpha t r} dt \leq \frac{e^{-\alpha (n+1)^r}}{\alpha r(n+1)^{r-2}} \frac{\alpha r(n+1)^r}{\alpha r(n+1)^r - 2} = \frac{e^{-\alpha (n+1)^r}}{\alpha r(n+1)^{r-2}} \left( 1 + \frac{2}{\alpha r(n+1)^r - 2} \right).
\]
Relations (27) and (30) yield that
\[ \frac{1}{n} \sum_{k=n+1}^{\infty} ke^{-\alpha k^r} = O \left( e^{-\alpha(n+1)^r} + \frac{e^{-\alpha(n+1)^r}}{\alpha r(n+1)^{r-2}} \left( 1 + \frac{2}{\alpha r(n+1)^{r-2}} \right) \right). \] (31)

Combining (26) and (31) we obtain
\[ E_n(C_{\alpha,r})_{\beta,1} = e^{-\alpha n^r} + O \left( e^{-\alpha(n+1)^r} + \frac{e^{-\alpha(n+1)^r}}{\alpha r(n+1)^{r-2}} \left( 1 + \frac{2}{\alpha r(n+1)^{r-2}} \right) \right). \] (33)

Corollary 2.3 is proved.

Formula (25) was obtained in [4] and [5].

For classes \( C_{\alpha,1} \beta,1 \), generated by classes of Poisson kernels
\[ P_{\alpha,1,\beta}(t) = \sum_{k=1}^{\infty} e^{-\alpha k} \cos \left( k \frac{t}{\beta} - \frac{\beta \pi}{2} \right), \quad \alpha > 0, \beta \in \mathbb{R}, \] (32)
the following statement holds.

Corollary 2.4. Let \( \alpha > 0 \) and \( \beta \in \mathbb{R} \). Then the following asymptotic equality holds as \( n \to \infty \)
\[ E_n(C_{\alpha,r})_{\beta,1} = e^{-\alpha n^r} \left( \frac{1}{\pi} \frac{1}{1 - e^{-\alpha}} + \frac{O(1)}{n} \frac{e^{-\alpha}}{(1 - e^{-\alpha})^2} \right), \] (33)
where \( O(1) \) is a quantity uniformly bounded in all parameters.

Proof. Denote \( q = e^{-\alpha} \). Then, from Theorem 2.1 it follows
\[ E_n(C_{\alpha,r})_{\beta,1} = \frac{1}{\pi} \sum_{k=n}^{\infty} q^k + \frac{O(1)}{n} \sum_{k=n}^{\infty} kq^{k+n} \]
\[ = \frac{1}{\pi} \frac{q^n}{1 - q} + \frac{O(1)}{n} \left( \sum_{k=n}^{\infty} kq^k - n \sum_{k=n}^{\infty} q^k \right) \]
\[ = \frac{1}{\pi} \frac{q^n}{1 - q} + \frac{O(1)}{n} \left( \frac{nq^n(1 - q) + q^{n+1}}{(1 - q)^2} - \frac{nq^n}{1 - q} \right) \]
\[ = \frac{1}{\pi} \frac{q^n}{1 - q} + \frac{O(1)}{n} \frac{q^{n+1}}{(1 - q)^2}, \] (34)
where we have used that
\[ \sum_{k=n}^{\infty} kq^k = \frac{nq^n(1 - q) + q^{n+1}}{(1 - q)^2}. \]

Corollary 2.4 is proved.

The asymptotic equality (33) was proved in [4] and [5].

Corollary 2.5. Let \( \psi(k) = e^{-\alpha k^r}, \ 0 < r < 1, \ \alpha > 0, \ \beta \in \mathbb{R}, \) Then as \( n \to \infty \) the following asymptotic equality holds
\[ E_n(C_{\alpha,r})_{\beta,1} = \frac{e^{-\alpha n^r}}{\pi \alpha r} n^{1-r} \left( 1 + O(1) \left( \frac{1}{n^r} + \frac{1}{n^{1-r}} \right) \right), \] (35)
where \( O(1) \) is a quantity uniformly bounded in \( n \) and \( \beta \).

Proof. Theorem 2.1 allows to write

\[
E_n(C^{\psi,1}_\beta) = \frac{1}{\pi} \sum_{k=0}^{\infty} e^{-\alpha k r} + \frac{O(1)}{n} \sum_{k=0}^{\infty} k e^{-\alpha (k+n) r}.
\]  (36)

Formulas (90) and (91) of the work [7] imply that

\[
\sum_{k=0}^{\infty} e^{-\alpha (k+n) r} = e^{-\alpha n r} \left( 1 + O\left( \frac{1}{n^r} + \frac{1}{n^{1-r}} \right) \right).
\]  (37)

From formulas (94) and (97) of the work [7] it follows that

\[
\frac{1}{n} \sum_{k=1}^{\infty} k e^{-\alpha (k+n) r} = O(1) \frac{1}{n} e^{-\alpha n r} (n^{2-2r} + n) = O(1) e^{-\alpha n r} n^{1-r} \left( \frac{1}{n^r} + \frac{1}{n^{1-r}} \right).
\]  (38)

Combining (36)–(38) we obtain (24). Corollary 2.5 is proved. ■

Asymptotic equality (24) was proved in [7].

By \( \mathfrak{M} \) we denote the set of all convex (downward) continuous functions \( \psi(t), t \geq 1 \), such that \( \lim_{t \to \infty} \psi(t) = 0 \).

Assume that the sequence \( \psi(k), k \in \mathbb{N} \), specifying the class \( C^{\psi,1}_\beta \) is the restriction of the functions \( \psi(t) \) from \( \mathfrak{M} \) to the set of natural numbers.

We also consider the following characteristics of functions \( \psi \in \mathfrak{M} \):

\[
\alpha(t) := \frac{\psi(t)}{t|\psi'(t)|}
\]

and

\[
\lambda(t) := \frac{\psi(t)}{|\psi'(t)|}.
\]

Theorem 2.6. Let \( \psi \in \mathfrak{M}, \alpha(t) \downarrow 0, \lambda(t) \to \infty, \lambda'(t) \to 0 \) as \( t \to \infty \) and \( \beta \in \mathbb{R} \). Then as \( n \to \infty \) the following asymptotic equality holds

\[
E_n(C^{\psi,1}_\beta) = \psi(n) \lambda(n) \left( \frac{1}{\pi} + O\left( \frac{1}{\lambda(n)} + \alpha(n) + \varepsilon_n \right) \right),
\]  (39)

where \( \varepsilon_n := \sup_{t \geq n} |\lambda'(t)| \) and \( O(1) \) is a quantity uniformly bounded in \( n \) and \( \beta \).

Proof. From Theorem 2.1 we have that the following asymptotic equality holds as \( n \to \infty \)

\[
E_n(C^{\psi,1}_\beta) = \frac{1}{\pi} \sum_{k=0}^{\infty} \psi(k) + \frac{O(1)}{n} \sum_{k=0}^{\infty} k \psi(k + n).
\]  (40)

Notice that

\[
\sum_{k=0}^{\infty} k \psi(k + n) = \sum_{k=0}^{\infty} k \psi(k) - n \sum_{k=0}^{\infty} \psi(k) \psi(n) + \int_{n}^{\infty} t \psi(t) dt - n \int_{n}^{\infty} \psi(t) dt.
\]  (41)
Let $\lambda'(t) \to 0$ as $t \to \infty$. Then integrating by parts, we get

$$I_1 := \int_n^\infty \psi(t) dt = \int_n^\infty -\psi'(t)\lambda(t) dt = \psi(n)\lambda(n) + \int_n^\infty \psi(t)\lambda'(t) dt$$

$$= \psi(n)\lambda(n) + \lambda'(\theta_1)I_1,$$

where $\theta_1$ is a some point from the interval $[n, \infty)$.

Then

$$I_1 (1 - \lambda'(\theta_1)) = \psi(n)\lambda(n)$$

and

$$I_1 = \psi(n)\lambda(n) \left(1 + \frac{\lambda'(\theta)}{1 - \lambda'(\theta)}\right) = \psi(n)\lambda(n) \left(1 + \mathcal{O}(1) \varepsilon_n\right).$$

Again integrating by parts

$$I_2 := \int_n^\infty t\psi(t) dt = \int_n^\infty \left[\frac{t^2}{t\psi'(t)}(-\psi'(t)) dt\right] = \alpha(\theta_2) \int_n^\infty \frac{t^2}{t\psi'(t)} dt$$

$$= \alpha(\theta_2) \left(n^2\psi(n) + 2 \int_n^\infty t\psi(t)\right),$$

where $\theta_2$ is a some point from the interval $[n, \infty)$.

Assume that $\alpha(t)$ monotonically decreases. Then

$$I_2 \leq \alpha(n)n^2\psi(n) + 2\alpha(n)I_2,$$

which is equivalent to

$$I_2 (1 - 2\alpha(n)) \leq \alpha(n)n^2\psi(n) = \psi(n)n\frac{\psi(n)}{|\psi'(n)|}$$

and

$$\frac{1}{n}I_2 \leq \psi(n)\frac{\psi(n)}{|\psi'(n)|} \frac{1}{1 - 2\alpha(n)}.$$

Hence, if $\alpha(t) \downarrow 0$, then

$$\frac{1}{n}I_2 \leq \psi(n)\frac{\psi(n)}{|\psi'(n)|} \left(1 + \frac{2\alpha(n)}{1 - 2\alpha(n)}\right)$$

(42)

and

$$I_1 = \psi(n)\frac{\psi(n)}{|\psi'(n)|} + \psi(n)\frac{\psi(n)}{|\psi'(n)|} \mathcal{O}(\varepsilon_n).$$

(43)
Combining (42) and (43), we obtain
\[
\frac{1}{n} \sum_{k=0}^{\infty} k \psi(k + n) \leq \psi(n) \frac{\psi(n)}{|\psi'(n)|} \left( 1 + \frac{2\alpha(n)}{1 - 2\alpha(n)} \right) \]
\[- \left( \psi(n) \frac{\psi(n)}{|\psi'(n)|} + \psi(n) \frac{\psi(n)}{|\psi'(n)|} \mathcal{O}(\varepsilon_n) \right) \]
\[= \psi(n) + \psi(n) \frac{\psi(n)}{|\psi'(n)|} \left( \frac{2\alpha(n)}{1 - 2\alpha(n)} + \mathcal{O}(\varepsilon_n) \right) \]
\[= \psi(n) \frac{\psi(n)}{|\psi'(n)|} \mathcal{O} \left( \frac{1}{\lambda(n)} + \alpha(n) + \varepsilon_n \right). \tag{44} \]

Moreover, taking into account (43),
\[
\sum_{k=n}^{\infty} \psi(k) = I_1 + \mathcal{O}(1) \psi(n) = \psi(n) \frac{\psi(n)}{|\psi'(n)|} \left( 1 + \mathcal{O}(1) \left( \varepsilon_n + \frac{1}{\lambda(n)} \right) \right). \tag{45} \]

Formulas (40), (44) and (45) imply (39). Theorem 2.6 is proved.

**Corollary 2.7.** Let \( \psi(k) = (k + 2)^{-\ln \ln(k + 2)}, \beta \in \mathbb{R} \) and \( k \in \mathbb{N} \). Then as \( n \to \infty \) the following asymptotic equality holds
\[
\mathcal{E}_n(C_{\beta,1}^\psi) C^\psi = \frac{1}{\pi} \frac{\psi(n)}{\ln \ln(n + 2)} + \mathcal{O}(\psi(n)). \tag{46} \]

**Proof.** If \( \psi(k) = e^{-\ln(k + 2) \ln \ln(k + 2)} \), then
\[
\psi'(t) = -e^{-\ln(t + 2) \ln \ln(t + 2)} \left( \frac{\ln \ln(t + 2)}{t + 2} + \frac{\ln(t + 2)}{(t + 2) \ln(t + 2)} \right) \]
\[= -e^{-\ln(t + 2) \ln \ln(t + 2)} \frac{\ln \ln(t + 2) + 1}{t + 2}. \]

Doing elementary calculations, we get
\[
\lambda(t) = \frac{t + 2}{\ln \ln(t + 2) + 1} = \frac{t}{\ln \ln(t + 2)} + \mathcal{O}(1), \tag{47} \]
\[
\alpha(t) = \frac{t + 2}{t} \frac{1}{\ln \ln(t + 2) + 1} \tag{48} \]
and
\[
\lambda'(t) = \frac{\ln \ln(t + 2) + 1 - \frac{1}{\ln(t + 2)}}{(\ln \ln(t + 2) + 1)^2} \leq \frac{1}{\ln \ln(t + 2)}. \tag{49} \]
Substituting (47)–(49) in (39) we obtain (46). Corollary 2.7 is proved.

**Corollary 2.8.** Let \( \psi(k) = e^{-\ln^2 k}, \beta \in \mathbb{R} \) and \( k \in \mathbb{N} \). Then as \( n \to \infty \) the following asymptotic equality holds
\[
\mathcal{E}_n(C_{\beta,1}^\psi) C^\psi = \frac{1}{2\pi} \frac{\psi(n)n}{\ln n} + \mathcal{O}(\psi(n)). \tag{50} \]

**Proof.** It is easy to see
\[
\psi'(t) = -2 \frac{1}{t} t^{-\ln^2 t} \ln t. \tag{51} \]
A.S. SERDYUK, T.A. STEPANYUK

Formula (51) yields
\[ \lambda(t) = \frac{t}{2 \ln t}, \quad \alpha(t) = \frac{1}{2 \ln t} \] (52)

and
\[ \lambda'(t) = \frac{\ln t - 1}{2(\ln t)^2} \leq \frac{1}{2 \ln t}. \] (53)

Formulas (51)–(53) and (39) imply (50). Corollary 2.8 is proved.

Corollary 2.9. Let \( \psi(k) = e^{-\frac{k}{k+1}}, \beta \in \mathbb{R} \) and \( k \in \mathbb{N} \). Then as \( n \to \infty \) the following asymptotic equality holds
\[ E_n(C^\psi_{\beta,1})_C = \psi(n) \ln(n+1) \left( \frac{1}{\pi} + \mathcal{O}\left(\frac{1}{\ln(n+1)}\right) \right). \] (54)

Proof. Doing elementary calculations, we get
\[ \psi'(t) = -e^{-\frac{t}{t+1}} \frac{\ln(t+1) - 1}{ln^2(t+1)}, \]
\[ \lambda(t) = \frac{\ln^2(t+1)}{\ln(t+1) - 1}, \quad \alpha(t) = \frac{\ln^2(t+1)}{t\ln(t+1) - t} = \mathcal{O}\left(\frac{1}{t \ln(t+1)}\right) \] (55)

and
\[ \lambda'(t) = \frac{1}{t+1} - \frac{1}{t+1(\ln(t+1) - 1)^2} \leq \frac{1}{t+1}. \] (56)

Formulas (55), (56) and (59) imply (54). Corollary 2.9 is proved.

Acknowledgements Second author is supported by the Austrian Science Fund FWF project F5506-N26 (part of the Special Research Program (SFB) “Quasi-Monte Carlo Methods: Theory and Applications”) and partially is supported by grant of NASU for groups of young scientists (project No16-10/2018)

References

[1] A. Kolmogoroff, Zur Grössenordnung des Restgliedes Fourierschen Reihen differenzierbarer Funktionen, (in German) Ann. Math. (2), 36:2 (1935) 521–526.
[2] N.P. Korneichuk, Exact Constants in Approximation Theory, Vol. 38, Cambridge Univ. Press, Cambridge, New York 1990.
[3] S.M. Nikolskii, Approximation of functions in the mean by trigonometrical polynomials, (in Russian) Izv. Akad. Nauk SSSR, Ser. Mat. 10 (1946) 207-256.
[4] A.S. Serdyuk, Approximation of classes of analytic functions by Fourier sums in uniform metric, Ukr. Math. J. 57:8 (2005) 1275–1296.
[5] A.S. Serdyuk, Approximation of classes of analytic functions by Fourier sums in the metric of the space \( L_p \), Ukr. Math. J. 57:10 (2005) 1395–1408.
[6] A.S. Serdyuk, T.A. Stepanyuk, Order estimates for the best approximations and approximations by Fourier sums of the classes of convolutions of periodic functions of low smoothness in the integral metric, Ukr. Math. J. 65:12 (2015) 1862–1882.
[7] A.S. Serdyuk, T.A. Stepanyuk, Uniform approximations by Fourier sums on classes of generalized Poisson integrals, Analysis Mathematica 45 (1) (2019), 201–236.
[8] A.I. Stepanets, Methods of Approximation Theory. VSP: Leiden, Boston 2005.

[9] S.B. Stechkin, An estimate of the remainder term of Fourier series for differentiable functions, (in Russian) Tr. Mat. Inst. Steklova 145 (1980) 126–151.

[10] S.A. Telyakovskii, Approximation of differentiable functions by partial sums of their Fourier series, Mathematical Notes 4:3 (1968) 668–673.