THE RADIAL GLASSEY CONJECTURE WITH MINIMAL REGULARITY

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Abstract. In this paper, we give a proof of the radial Glassey conjecture with minimal regularity assumption. In the process, we prove a weighted fractional chain rule, which is of independent interest. We also show well-posedness for 3-D quadratic semi-linear wave equations with radial data in the almost scale-critical Sobolev space, which improves the earlier result of Klainerman and Machedon.

1. Introduction

Let \( n \geq 2, p > 1, (a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\} \), \( \Box = \partial_t^2 - \Delta \), and consider the following small-amplitude nonlinear wave equations, with \((t, x) \in \mathbb{R}^+ \times \mathbb{R}^n\),

\[
\Box u = a|\partial_t u|^p + b|\nabla u|^p, \ u(0, x) = u_0(x) = u_0 \in H^s_{\text{rad}}, \ \partial_t u(0, x) = u_1(x) \in H^{s-1}_{\text{rad}}.
\]

Here \( H^s_{\text{rad}} \) stands for the space of spherically symmetric functions lying in the usual Sobolev space \( H^s \). For this problem with data small enough in certain sense, we would like to investigate the long time existence of solutions with sharp lower bound of the lifespan, and the minimal regularity assumption, measured by \( s \), on the initial data.

For general compactly supported smooth data, it was conjectured that the equation (1.1) admit a global small solution if and only if

\[
p > p_c(n) \equiv 1 + \frac{2}{n-1},
\]

which is referred to as the Glassey conjecture in literatures. It is known to be true for dimension two and three, as well as the high dimensional radial case, see [7, 19, 20] and references therein for the history, as well as the analogs for asymptotically flat manifolds and exterior domains.

The Cauchy problem (1.1) has been investigated in our previous work, Hidano-Wang-Yokoyama [7], where it was shown that \( s = 2 \) is sufficient for existence. Moreover, the smallness of the initial data was measured in certain “multiplicative form”, which strongly suggests that, the minimal regularity for the problem is given by

\[
s_o \equiv \max \left( \frac{3}{2}, s_c \right).
\]

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Here $s_c = \frac{3}{2} + 1 - \frac{4}{p-1}$ is the regularity for the problem to be scale invariant in the homogeneous Sobolev space $\dot{H}^s$, which is well known to be a lower bound for the problem to be locally well-posed (see, e.g., Fang-Wang [2] and references therein). Notice that for $p > 1$, 
$$s_c > 3/2 \iff p > p_c .$$

In this paper, we prove that $s > s_c$ is sufficient for the radial Glassey conjecture to hold, which gives an affirmative answer for the natural regularity problem raised in [7]. In the process, a weighted fractional chain rule, see Theorem 2.5, plays a key role in the proof, which itself should be of independent interest.

Our first main result concerns the global existence for $p > p_c(n) = 1 + 2/(n-1)$, for small radial data in $H^s$ with $s > s_c$.

**Theorem 1.1.** Let $n \geq 2$ and $p \in (p_c(n), 1 + 2/(n-2))$ (when $n = 2$, it is understood to be $p \in (3, \infty)$). For any $s_1, s$, with
\begin{equation}
\max\left( \frac{3}{2}, \frac{n+2}{2} - \frac{n}{p-1} \right) < s_1 < s_c < s < 2 ,
\end{equation}
there exists $\varepsilon > 0$, such that the Cauchy problem (1.1) admits a global radial solution $u \in CH^s \cap C^1 H^{s-1}$ for any radial data $(u_0, u_1) \in H^s \times H^{s-1}$ with $\|u(0)\|_{H^{s-1}} \leq \varepsilon$.

For the remaining case $1 < p \leq p_c(n)$, when $(a, b) = (1, 0)$, it is known that there is an upper bound of the lifespan $T_{\varepsilon} \leq \left\{ \begin{array}{ll}
\exp(C\varepsilon^{-(p-1)}), & p = p_c , \\
C\varepsilon^{1/(s_c - 3/2)} = c\varepsilon^{(2(p-1)/n-1)/(p-1)} , & 1 < p < p_c ,
\end{array} \right.$
for generic, compactly supported, smooth data of size $\varepsilon$, see Zhou [21] and references therein. Our second main result proves the existence part, with sharp lower bound of the lifespan, for the radial initial data with low regularity.

**Theorem 1.2.** Let $n \geq 2$ and $1 < p \leq p_c$. For any $s \in (3/2, 2)$, there exist $c, \varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$, the Cauchy problem (1.1) admits a solution in $[0, T_{\varepsilon}] \times \mathbb{R}^n$, where
\begin{equation}
T_{\varepsilon} = \left\{ \begin{array}{ll}
\exp(C\varepsilon^{-(p-1)}) , & p = p_c , \\
c\varepsilon^{(2(p-1)/n-1)/(p-1)} , & 1 < p < p_c ,
\end{array} \right.
\end{equation}
for any radial data $(u_0, u_1) \in H^s \times H^{s-1}$ with
$$\varepsilon^2 = \left\{ \begin{array}{ll}
\|\partial u(0)\|_{H^{s-1}}^2 + \|\partial u(0)\|_{H^{s-1}} \|\partial u(0)\|_{H^{s-2}} , & p = p_c , \\
\|\partial u(0)\|_{H^{s-1}} \|\partial u(0)\|_{H^{s-2}} , & p < p_c .
\end{array} \right.$$
Remark 1.3. As is clear from the proof of Theorem 1.2, the statement remains true for \( \varepsilon_0 = \infty \) when \( 1 < p < p_c \).

As we have mentioned, the regularity assumption \( s > s_c \) in Theorem 1.1 is minimal, in the sense that such a result is not valid for \( s < s_c \). A natural further problem is to ask whether we can go down further to the critical threshold, \( s = s_c \). It turns out that it is true when the spatial dimension is two.

By exploiting generalized Strichartz estimates of Smith-Sogge-Wang [17], we are able to obtain the following:

**Theorem 1.3** (Global solutions). Let \( n = 2 \) and \( p > 5 \). Then there exists a small constant \( \varepsilon_0 > 0 \), such that the Cauchy problem \( (1.1) \) has a unique global solution satisfying \( u \in C([0, \infty); H^{s_c}) \cap C^1([0, \infty); H^{s_c-1}) \) and \( \partial u \in L_t^{p-1}L_x^\infty \), whenever the initial data \((u_0, u_1)\) belongs to \( H_{s_c} \times H_{s_c-1} \) with

\[
(\phi, \psi) \in H_{s_c} \times H_{s_c-1}, \quad \varepsilon \leq \varepsilon_0.
\]

In addition, there exists a constant \( C > 0 \), such that the solution satisfies

\[
\|\partial u\|_{L_t^p H^{s_c-1} \cap L_t^{p-1}L_x^\infty} \leq C\varepsilon, \quad \|\partial u\|_{L_t^p L_x^\infty} \leq C\|(\phi, \psi)\|_{H_{s_c} \times L_x^2}
\]

Moreover, when \( p > 3 \) and the initial data are radial, the same results remain valid for the radial solutions.

**Remark 1.4.** As is known, if the problem is locally well-posed in \( H^s \), then \( s \geq \max(s_c, (n+5)/4) \) (see, e.g., Lindblad [12], Fang-Wang [2]). The restriction \( p > 5 \) is a natural condition, as \( s_c > (n+5)/4 \) if and only if \( p > 1 + 4/(n-1) \).

Our paper is organized as follows. In the next section, we collect various basic estimates to be used, including trace estimates, local energy estimates, Strichartz estimates and fractional chain rule. In particular, we prove a weighted fractional chain rule, Theorem 2.5. Then in Section 3, we give the proof of the radial Glassey conjecture with \( s > \max(s_c, 3/2) \), Theorems 1.1-1.2. In the final section, we prove the radial Glassey conjecture with critical regularity, for dimension two.

2. Preliminaries

In this section, we collect various basic estimates to be used. All of these estimates are well known, except a novel weighted fractional chain rule, Theorem 2.5.

2.1. Trace estimates: spatial decay. At first, let us record the trace estimates, which will provide spatial decay for functions, see, e.g., (1.3), (1.7) in Fang-Wang [4] and references therein.

**Lemma 2.1** (Trace estimates). Let \( n \geq 2 \) and \( 1/2 < s < n/2 \). Then we have

\[
(2.1) \quad \|r^{n/2-s}f\|_{L_t^\infty H_x^{s-1/2}} \lesssim \|f\|_{H^s}, \quad \|r^{(n-1)/2}f\|_{L_t^\infty L_x^2} \lesssim \|f\|_{B^{1/2}_{p,1}},
\]

for any \( f \in C_0^\infty(\mathbb{R}^n) \). Here \( B_{p,q}^s \) is the homogeneous Besov space and \( H^s_x \) is the Sobolev space on the unit sphere. In particular, when \( u \) is spatially radial, then \( |\partial u| \lesssim |\partial u(t, r)| \) and so

\[
(2.2) \quad \|r^{n/2-s}u\|_{L_t^\infty L_x^\infty} \lesssim \|\partial u\|_{L_t^\infty H^s}, \quad \|r^{(n-1)/2}\partial u\|_{L_t^\infty L_x^2} \lesssim \|\partial u\|_{L_t^\infty H^{s-1/2}},
\]
2.2. Space-time estimates. We will need to exploit the following space-time estimates for solutions to the linear wave equations: local energy estimates, as well as the generalized Strichartz estimates.

At first, we record the required local energy estimates (which are also known as KSS type estimates) for the operator $\Box$.

Lemma 2.2. Let $n \geq 2$. Then for any $\delta_1, \delta_2 > 0$ we have
\begin{equation}
\|u\|_{L^\infty_t L^2_x} \lesssim \|\partial u(0)\|_{L^2_x} + \|r^{(1/2)-\delta_1}(r)^{\delta_1} + \delta_2 \Box u\|_{L^2_t L^2_x},
\end{equation}
where $\partial u = (\partial_t u, \nabla u)$ is the space-time gradient, and
\begin{equation}
\|u\|_{L^1_t L^\infty_x} \equiv \|r^{-(1/2) + \delta_1}(r)^{-\delta_1 - \delta_2} \Box u\|_{L^\infty_t L^2_x} = \|\partial u\|_{L^\infty_t L^2_x}.
\end{equation}
In particular, for any $T > 0$ and $\delta_1 \in (0, 1/2]$, we have
\begin{equation}
\ln (2 + T) \lesssim \|\partial u(0)\|_{L^2_x} + (\ln (2 + T))^{1/2} \|r^{(1/2) - \delta_1}(r)^{\delta_1} \Box u\|_{L^2_t L^2_x}
\end{equation}
and
\begin{equation}
T^{-\delta_1} \|r^{-(1/2) + \delta_1} \Box u\|_{L^2_t L^2_x} \lesssim \|\partial u\|_{L^\infty_t L^2_x} + \|r^{(1/2) - \delta_2} \Box u\|_{L^2_t L^2_x},
\end{equation}
where $L^q_t L^r_x = L^q([0,T])$. The estimates were formulated and proved in [7, Lemma 3.2] for $n \geq 3$, by multiplier method, see also [14, 19]. We remark that $n = 2$ is also admissible, as (2.3) is a special case of the localized energy estimates in Metcalfe-Tataru [15, Theorem 1]. It is essentially known from Keel-Smith-Sogge [10] that (2.4)-(2.5) are implied by (2.3), see e.g. [9, Section 7.2] and [20, Section 3.4]. Local energy estimates have rich history and we refer [15, 13] for more exhaustive history of such estimates.

For the problem with dimension two and critical regularity, we will also use the following generalized Strichartz estimates of Smith-Sogge-Wang [17], with the previous radial estimates in Fang-Wang [3]. See also [9] for the high dimensional analogs. For Strichartz estimates, see [3] and references therein.

Lemma 2.3 (Generalized Strichartz estimates). Let $n = 2$, $q \in (2,4)$, and $s = 1 - 1/q$. Then we have the following inequality
\begin{equation}
\|\partial u\|_{L^q([0,\infty); L^\infty_t L^2_x(\mathbb{R}^2))} \lesssim \|\partial u(0)\|_{H^s_x} + \|\Box u\|_{L^1_t H^s_x}.
\end{equation}
In addition, we have the classical Strichartz estimates for $q \in (4, \infty)$,
\begin{equation}
\|\partial u\|_{L^q([0,\infty); L^r_t \mathbb{R}^2)} \lesssim \|\partial u(0)\|_{H^s_x} + \|\Box u\|_{L^1_t H^s_x}.
\end{equation}
In particular, when $u$ is (spatially) radial, we have
\begin{equation}
\|\partial u\|_{L^q([0,\infty); L^r_t \mathbb{R}^2)} \lesssim \|\partial u(0)\|_{H^s_x} + \|\Box u\|_{L^1_t H^s_x}, \quad q > 2.
\end{equation}

2.3. Fractional chain rule. For the problem with dimension two, we would need to use the following fractional chain rule, see Taylor [18, Chapter 2 Proposition 5.1] and references therein.

Lemma 2.4 (Fractional chain rule). Assume $F: \mathbb{R}^k \to \mathbb{R}^l$ is a $C^1$ map, satisfying $F(0) = 0$ and
\begin{equation}
|F'(\tau v + (1 - \tau)w)| \leq \mu(\tau)|G(v) + G(w)|,
\end{equation}
where $\mu : [0,1] \to [0,\infty)$ is a non-decreasing map with $\mu(0) = 0$ and the assumptions on $G$ are as in Lemma 2.3.
Lemma 2.7. Let $w \in L^1([0,1])$. Then for $s \in (0,1)$ and $q,q_1 \in (1,\infty)$, $q_2 \in (1,\infty)$, with $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, we have
\begin{equation}
\|D^s F(u)\|_{L_q^\infty} \lesssim \|G(u)\|_{L_{q_2}} \|D^s u\|_{L_{q_1}},
\end{equation}
where $D = \sqrt{-\Delta}$.

2.4. Weighted fractional chain rule. To apply the local energy estimates for the nonlinear problem, we are naturally required to introduce the weighted fractional chain rule.

For the weight functions, we recall the Muckenhoupt $A_p$ class, which by definition,
$$w \in A_1 \iff \mathcal{M}w(x) \leq Cw(x), \text{a.e. } x \in \mathbb{R}^n,$$
$$w \in A_p \ (1 < p < \infty) \iff \left( \int_Q w(x)dx \right)^{\frac{1}{p}} \left( \int_Q w^{1-p'}(x)dx \right)^{\frac{1}{p'}-1} \leq C|Q|^p, \forall \text{ cubes } Q,$$
with $\mathcal{M}w(x) = \sup_{r>0} r^{-n} \int_{B_r(x)} w(y)dy$ denotes the Hardy-Littlewood Maximal function.

Theorem 2.5 (Weighted fractional chain rule). Let $s \in (0,1)$, $q,q_1,q_2 \in (1,\infty)$ with $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Under the same assumption on $F$ as in Theorem 2.4, if $(w_1 w_2)^q \in A_q$, $w_1^{q_1} \in A_{q_1}$, $w_2^{q_2} \in A_{q_2}$, then
\begin{equation}
\|w_1 w_2 D^s F(u)\|_{L_q^\infty} \lesssim \|w_1 D^s u\|_{L_{q_1}} \|w_2 G(u)\|_{L_{q_2}}.
\end{equation}
In addition, when $q_2 = \infty$ and $q \in (1,\infty)$, if $w_1^q, (w_1 w_2)^q \in A_q$ and $w_2^{-1} \in A_1$, we have
\begin{equation}
\|w_1 w_2 D^s F(u)\|_{L_q^\infty} \lesssim \|w_1 D^s u\|_{L_q} \|w_2 G(u)\|_{L_\infty}.
\end{equation}

The proof follows essentially the same lines as in Taylor [18, Chapter 2 Proposition 5.1], with additional care of the weighted estimates for Calderón-Zygmund, Littlewood-Paley operators and Maximal functions. We postpone the proof to the end of this section, see subsection 2.5.

As a direct corollary, with $w_2 = w_1^{-2} = w^{-1}$ and the fact that
$$w \in A_1 \Rightarrow w \in A_2 \Rightarrow w^{-1} \in A_2,$$
we obtain the following:

Corollary 2.6. Let $w \in A_1$. Under the same assumption on $F(u)$ in Theorem 2.4, we have
\begin{equation}
\|w^{-1/2} D^s F(u)\|_{L_2^\infty} \lesssim \|u^{1/2} D^s u\|_{L_2} \|w^{-1} G(u)\|_{L_2^\infty}.
\end{equation}

The actual weight function we will choose is $w(x) = r^{-1+2\delta_1}(r^{-2\delta_1-2\delta_2}.$

Lemma 2.7. Let $w(x) = r^{-1+2\delta_1}(r^{-2\delta_1-2\delta_2},$ with $0 \leq 1 - 2\delta_1 \leq 1 + 2\delta_2 < n$. Then $w \in A_1(\mathbb{R}^n)$.

Proof. Though proof of this lemma is rather elementary, for the sake of completeness we provide a proof. It amounts to proving that for any $r > 0$, and almost every $x \in \mathbb{R}^n$,
\begin{equation}
\int_{B_r(x)} |y|^{-1+2\delta_1} \langle y \rangle^{-2\delta_1-2\delta_2} dy \leq C_r^n |x|^{-1+2\delta_1} \langle x \rangle^{-2\delta_1-2\delta_2}.
\end{equation}
We deal with two cases separately. First, if $|x| \leq 1$, then as $\delta_1 + \delta_2 \geq 0$, we have

$$|y|^{-1+2\delta_1} \langle y \rangle^{-2\delta_1 - 2\delta_2} \leq |y|^{-1+2\delta_1} \in A_1$$

provided that $1 - 2\delta_1 \in [0, n)$ (recall that $|x|^a \in A_1$ iff $a \in (-n, 0]$, Grafakos [6, Example 7.1.7, page 506]). Then

$$r^{-n} \int_{B_r(x)} |y|^{-1+2\delta_1} \langle y \rangle^{-2\delta_1 - 2\delta_2} dy \leq C |x|^{-1+2\delta_1} \leq C |x|^{-1+2\delta_1} \langle x \rangle^{-2\delta_1 - 2\delta_2}, \forall r > 0. \tag{2.15}$$

For the case $|x| \geq 1$, recall that $|x|^{-1-2\delta_2} \in A_1$ if $1 + 2\delta_2 \in [0, n)$, and so

$$r^{-n} \int_{B_r(x)} |y|^{-1-2\delta_2} dy \leq C |x|^{-1-2\delta_2}, \forall r > 0. \tag{2.16}$$

Then, if $r < |x|/2$, we have $|y| \geq |x| - r \geq |x|/2$ and so $|y| \simeq \langle y \rangle \simeq |x|$ for $y \in B_r(x)$. So

$$r^{-n} \int_{B_r(x)} |y|^{-1-2\delta_2} dy \leq C r^{-n} \int_{B_r(x)} |x|^{-1-2\delta_2} dy \leq C |x|^{-1+2\delta_1} \langle x \rangle^{-2\delta_1 - 2\delta_2}. \tag{2.17}$$

Else, if $r \geq |x|/2$, by (2.16),

$$\int_{B_r(x)} |y|^{-1+2\delta_1} \langle y \rangle^{-2\delta_1 - 2\delta_2} dy \leq \int_{B_r} |y|^{-1+2\delta_1} + \int_{B_r(x) \setminus B_r} |y|^{-1-2\delta_2} dy \leq C + C r^n |x|^{-1-2\delta_2} \leq C r^n |x|^{-1-2\delta_2}. \tag{2.18}$$

This proves the Lemma. \qed

2.5. Proof of Theorem 2.5. At first, it is known (see, e.g., Muscalu-Schlag [16, Theorem 7.21, page 191]) that if $T$ is a strong Calderón-Zygmund operator, then

$$\|T(f)\|_{L^p(wdx)} \leq C \|f\|_{L^p(wdx)}, w \in A_p, p \in (1, \infty). \tag{2.19}$$

Based on this fact, it is easy to adapt the argument in [16, Section 8.2] to conclude the weighted Littlewood-Paley square-function estimate

$$\|w S_j f\|_{L^p} \simeq \|wf\|_{L^p}, w^p \in A_p, f \in L^p(wdx), p \in (1, \infty) \tag{2.20}$$

where $S_j = \phi_j \ast$ is the standard Littlewood-Paley operator, with $\phi_j(x) = 2^{jn} \phi(2^j x)$, supp $\hat{\phi} \subset \{||\xi|| \leq 2^{-2}, 2^2\}$.

By repeating essentially the same argument as in the proof of Taylor [18, (5.6), page 112], we can obtain

$$|S_j D^s F(u)(x)| \lesssim 2^{js} \sum_{k \in \mathbb{Z}} \min(1, 2^{-j}) (M(S_k u)(x), M(H)(x) + M(HS_k u)(x)), \tag{2.21}$$

where $H(x) \equiv G(u(x))$. 





By (2.18) and (2.19), we know that for \((w_1)^q,(w_1w_2)^q \in A_q\) with \(q \in (1,\infty)\),
\[
\|w_1w_2D^sF(u)\|_{L^q} \lesssim \|w_1w_2S_jD^sF(u)\|_{L^q}\|_{L^q}\|_{L^q}\|_{L^q}
\]
\[
\lesssim \|w_1w_22^{ks}\min(1,2^{(k-s)})(M(S_ku)M(H) + M(HS_ku))\|_{L^q}\|_{L^q}\|_{L^q}
\]
\[
\lesssim \|w_1w_22^{ks}\min(2^{(j-k)s},2^{(k-j)(1-s)})(M(S_ku)M(H) + M(HS_ku))\|_{L^q}\|_{L^q}\|_{L^q}
\]
\[
\lesssim \|w_1w_22^{ks}(M(S_ku)M(H) + M(HS_ku))\|_{L^q}\|_{L^q}\|_{L^q}.
\]
where we used Young’s inequality with the assumption \(s \in (0,1)\) in the last inequality.

By applying Minkowski’s and Hölder’s inequalities to the last expression we have
\[
\|w_1w_2D^sF(u)\|_{L^q} \lesssim \|w_1w_22^{ks}M(S_ku)MH\|_{L^q}\|_{L^q}\|_{L^q} + \|w_1w_22^{ks}M(HS_ku)\|_{L^q}\|_{L^q}\|_{L^q}
\]
\[
\lesssim \|w_2M(H)\|_{L^q\|_{L^q}}\|_{L^q}\|_{L^q}\|_{L^q} + \|w_1w_22^{ks}HS_ku\|_{L^q\|_{L^q}}\|_{L^q\|_{L^q}},
\]
for any \(q_1,q_2 \in (1,\infty]\) with \(\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}\). The last term in the above we used the weighted vector valued inequality for the Hardy-Littlewood inequality which is due to Andersen-John [1, Theorem 3.1]

\[
(2.20) \quad \int \|Mf_j\|_{L^q}^{2}wdx \leq C \int \|f_j\|_{L^q}^{2}wdx, \; p,q \in (1,\infty), \; w \in A_q.
\]

Using (2.20) again and Hölder’s inequality,
\[
\|w_1w_2D^sF(u)\|_{L^q} \lesssim \|w_2M(H)\|_{L^q\|_{L^q}}\|_{L^q}\|_{L^q} + \|w_12^{ks}S_ku\|_{L^q\|_{L^q}}\|_{L^q\|_{L^q}}.
\]

Then, by (2.18) and its variant
\[
\|w_12^{ks}S_ku\|_{L^q\|_{L^q}} = \|w_1\tilde{S}kD^su\|_{L^q\|_{L^q}} \lesssim \|w_1D^su\|_{L^q\|_{L^q}}, \; w_1 \in A_{q_1}, \; q_1 \in (1,\infty),
\]
we get that

\[
(2.21) \quad \|w_1w_2D^sF(u)\|_{L^q} \lesssim \|w_2M(H)\|_{L^q\|_{L^q}} + \|w_1D^su\|_{L^q\|_{L^q}}.
\]

If \(q_2 < \infty\), (2.11) follows directly from (2.21), by applying (2.20) for the term involving \(M(H)\). To handle the remaining case \(q_2 = \infty\), we observe that for \(w \in A_1\),

\[
(2.22) \quad \|w^{-1}M(H)\|_{L^\infty} \lesssim \|w^{-1}H\|_{L^\infty}.
\]

This is trivial, since by the definition of \(A_1\), we know that for a.e. \(x \in \mathbb{R}^n\),
\[
H(x) \leq w(x)\|w^{-1}H\|_{L^\infty} \Rightarrow M(H)(x) \leq M(w)(x)\|w^{-1}H\|_{L^\infty} \leq Cw(x)\|w^{-1}H\|_{L^\infty},
\]
which gives us (2.22). Together with (2.21), we get (2.12) and this completes the proof.

3. Radial Glassey conjecture

In this section, we give the proof of Theorems 1.1-1.2, based on Lemmas 2.1-2.2 and Theorem 2.5.
3.1. Global existence. As usual, we prove the existence of solutions for (1.1) through iteration. Observe that \( s_c \in (3/2, 2) \) for \( p \in (1 + 2/(n - 1), 1 + 2/(n - 2)) \). For fixed \( s \in (s_c, 2) \) and \((u_0, u_1) \in H^s_{\text{rad}} \times H^{s-1}_{\text{rad}} \), we define the iteration map
\[
\Phi[u] := H[u_0, u_1] + I[N[u]],
\]
where \( H[\phi, \psi] \) is the solution map of the linear homogeneous Cauchy problem with data \((\phi, \psi)\), \( I[F] \) is the solution map of the linear inhomogeneous Cauchy problem \( \square u = F \) with vanishing data, and the nonlinear term
\[
N[u] := a|\partial u|^p + b|\nabla x u|^p.
\]
Notice that \( \Phi \) preserves radial property.

For any \( d \in (0, 1) \), we have
\[
D^d \Phi[u] = H[D^d u_0, D^d u_1] + I[D^d N[u]].
\]
Let \( w(x) = r^{1 - 2d_1}(r)^{-2d_1 - 2d_2} \) with \( d_1, d_2 > 0 \). Applying Lemma 2.2 for \( D^d \Phi[u] \), we get
\[
\|D^d \Phi[u]\|_{L^p} \leq \|u^{1/2} D^d \Phi[u]\|_{L^2_{t,x}} + \|\partial D^d \Phi[u]\|_{L^p_{t,x}} (3.2)
\]
\[
\leq \|\partial D^d u(0)\|_{L^2} + \|w^{-1/2} D^d N[u]\|_{L^2_{t,x}}.
\]
If \( d_1 \in (0, 1/2) \) and \( d_2 \in (0, (n - 1)/2) \), then we can apply Lemma 2.7 and Corollary 2.6 to conclude
\[
\|D^d \Phi[u]\|_{L^p} \leq \|\partial D^d u(0)\|_{L^2} + \|w^{1/2} D^d \partial u\|_{L^2_{t,x}} \|w^{-1} \partial u\|_{L^2_{t,x}} d, d \in (0, 1).
\]
Now, for any given \( s, \) satisfying (1.3), there exist \( d_1 \in (0, 1/2) \) and \( d_2 \in (0, (n - 1)/2) \), such that
\[
s = s_c + \frac{2d_1}{p - 1} = \frac{n}{2} + 1 - \frac{1 - 2d_1}{p - 1},
\]
\[
s_1 = s_c - \frac{2d_2}{p - 1} = \frac{n}{2} + 1 - \frac{1 + 2d_2}{p - 1}.
\]
Thus by (2.2), we know that for radial \( u \),
\[
\|r^{1 - 2\delta_1} \partial u\|_{L^p_{t,x}} \leq \|D^s \partial u\|_{L^p_{t,x}} \|D^{s-1} \partial u\|_{L^p_{t,x}} \|D^{s-1} \partial u\|_{L^p_{t,x}} \|D^{s-1} \partial u\|_{L^p_{t,x}},
\]
and so
\[
\|w^{-1} \partial u\|_{L^p_{t,x}} \leq \|(r^{1 - 2\delta_1})^{(r)^{2d_1 + 2d_2} 1/(p - 1)} \partial u\|_{L^p_{t,x}} \leq \|D^{s-1} \partial u\|_{L^p_{t,x}} \|D^{s-1} \partial u\|_{L^p_{t,x}} \|D^{s-1} \partial u\|_{L^p_{t,x}} \|D^{s-1} \partial u\|_{L^p_{t,x}}.
\]
In conclusion, we arrive at, for any \( d \in (0, 1) \),
\[
\|D^d \Phi[u]\|_{L^p} \leq C_d \|\partial u(0)\|_{H^d} + C_d \|w^{1/2} D^d \partial u\|_{L^2_{t,x}} \|\partial u\|_{L^p_{t,x}} \|D^{s-1} \partial u\|_{L^p_{t,x}} \|D^{s-1} \partial u\|_{L^p_{t,x}} \|D^{s-1} \partial u\|_{L^p_{t,x}}.
\]
Similarly, as
\[
\|N[u] - N[v]\| \leq \|(\partial u)^{p-1} + |\partial v|^{p-1})\|\partial (u - v)\|,
\]
we have
\[
\|w^{1/2} \Phi[u] - \Phi[v]\|_{L^2_{t,x}} + \|\partial \Phi[u] - \Phi[v]\|_{L^p_{t,x}} \leq C_1 \|w^{1/2} \partial (u - v)\|_{L^2_{t,x}} \|\partial u\|_{L^p_{t,x}} \|\partial v\|_{L^p_{t,x}} \|D^{s-1} \partial u\|_{L^p_{t,x}} \|D^{s-1} \partial v\|_{L^p_{t,x}}.
\]
Based on (3.3) with \( d = s_1 - 1 \), \( s - 1 \) and (3.4), it is standard to conclude the unique global solvability for the equation (1.1), in the space
\[
\{ u \in CH^s_{\text{rad}} \cap C^1 H^{s-1}_{\text{rad}}, \|\partial u\|_{L^p_{t,x}(H^{s-1} \cap H^{s-1})} \leq (3 \max(C_1, C_{s_1-1}, C_{s-1}))^{-1/(p-1)} \}.\]
for any initial data with
\[ \|\partial u(0)\|_{H^{s-1}} + \|\partial u(0)\|_{H^{s-1}} \leq 2(3 \max(C_1, C_{s_1-1}, C_{s-1}))^{-p/(p-1)}. \]

This completes the proof of Theorem 1.1.

3.2. Critical case. For \( n \geq 2 \) and \( p = p_c \), we have \( s_c = 3/2, 1/(p - 1) = (n - 1)/2 \) and we follow the same vein to give the proof. For any \( s \in (3/2, 2) \), there exists \( \delta \in (0, 1/2) \) such that
\[ s = s_c + \frac{2\delta}{p-1} = n/2 + 1 - \frac{2\delta}{p-1}. \]

By (2.2), we have, for radial \( u \),
\[ \|r^{1-2\delta} \partial u\|_{L^{s_c}_p} \lesssim \|\partial u\|_{L^p_\infty}; \|r^{1-2\delta} \partial^2 u\|_{L^{s_c}_p} \lesssim \|\partial u\|_{L^p_\infty}\|\partial u\|_{L^p_\infty}, \]
and so
\[ \|(r^{-1-2\delta})^{1/(p-1)} u\|_{L^{s_c}_p} \lesssim \|\partial u\|_{L^p_\infty}\|\partial u\|_{L^p_\infty}. \]

As \( \delta \in (0, 1/2) \), with \( w(x) = r^{-1-2\delta} (r)^{-2\delta} \), we can apply Lemma 2.2 (2.4) for \( D^d\Phi[u], \) Corollary 2.6 and Lemma 2.7 to get that, for \( T \geq 2 \) and \( d \in (0, 1) \),
\[ \|\Phi[u]\|_{KSS_d} = (\ln T)^{-1/2} \|u^1/2D^d\Phi[u]\|_{L^2_{t,x}} + \|\partial D^d\Phi[u]\|_{L^\infty_{t,x}} \leq \|\partial D^d u(0)\|_{L^2_{t,x}} + (\ln T)^{1/2} \|w^{-1/2} D^d N[u]\|_{L^\infty_{t,x}} \leq \|\partial D^d u(0)\|_{L^2_{t,x}} + (\ln T)^{1/2} \|u\|_{KSS_d} \|r^{-1-2\delta} (r^2)^{1/(p-1)} u\|_{L^p_{t,x}}. \]

Thus, for any \( s \in (3/2, 2) \), with the choice \( d = s - 1, 2 - s \), there exist \( 0 < \varepsilon_0, c \ll 1 \), such that the Cauchy problem (1.1) admits a solution up to \( T_c = \exp(c \varepsilon_0^{-(p-1)}) \), provided that \( \partial u(0) \in H^{s-1} \) with
\[ \|\partial u(0)\|_{H^{s-1}} \lesssim \|\partial u(0)\|_{H^{s-1}} \|\partial u(0)\|_{H^{2-s}} = \varepsilon^2 \lesssim \varepsilon_0^2. \]

3.3. Subcritical case. In the same spirit, for \( 1 < p < p_c \), let \( \delta \in (0, 1/2) \) be such that
\[ 1 - 2\delta = \frac{n-1}{2} \frac{1}{p-1}. \]

For \( \delta \in (0, 1/2) \), with \( w(x) = r^{-1+2\delta} \in A_1 \), we can apply Lemma 2.2 (2.5), Corollary 2.6 to get that, for any \( T > 0 \) and \( d \in (0, 1) \),
\[ \|\Phi[u]\|_d \leq T^{-\delta} \|r^{-1+\delta} D^d\Phi[u]\|_{L^2_{t,x}} + \|\partial D^d\Phi[u]\|_{L^\infty_{t,x}} \leq \|\partial D^d u(0)\|_{L^2_{t,x}} + T^{\delta} \|r^{-1+2\delta} D^d N[u]\|_{L^\infty_{t,x}} \leq \|\partial D^d u(0)\|_{L^2_{t,x}} + T^{2\delta} \|u\|_d \|\partial u\|_{L^p_{t,x}} \|u\|_{L^p_{t,x}} \|\partial u\|_{L^p_{t,x}}. \]

Combining with (2.2), for any \( s \in (3/2, 2) \), there exist \( c > 0 \), such that we can prove existence up to
\[ T_c = c \varepsilon_0^{2(p-1)/(p-1)} \]
provided that \( \partial u(0) \in H^{s-1} \) with \( \|\partial u(0)\|_{H^{2-s}} \|\partial u(0)\|_{H^{s-1}} = \varepsilon^2. \)
4. Radial Glassey conjecture with critical regularity: dimension two

In this section, for spatial dimension two, we use the generalized Strichartz estimates to prove Theorem 1.3, concerning the radial Glassey conjecture with critical regularity.

At first, when $p > 5$, recall $s_\ast = 2 - 1/(p - 1) \in (1, 2)$, we apply (2.7) of Lemma 2.3 to get

$$
\|(\Phi[u], \partial_t \Phi[u])\|_{L^\infty_t(\dot{H}^{s_c} \times \dot{H}^{s_c - 1})} + \|\partial_t \Phi[u]\|_{L^{p-1}_t L^{\infty}_x} \\
\lesssim \|(\phi, \psi)\|_{\dot{H}^{s_c} \times \dot{H}^{s_c - 1}} + \|N[u]\|_{L^1_t \dot{H}^{s_c - 1}} \\
\lesssim \varepsilon + \|\partial u\|_{L^{p-1}_t L^{\infty}_x}^{p-1} \|\partial u\|_{L^\infty_t L^{s_c}_x},
$$

where in the last inequality, we have applied Lemma 2.4 with $F(v) = |v|^p$, $G(v) = F'(v)$ and $v = \partial u$. Similarly, we have

$$
\|(\Phi[u], \partial_t \Phi[u])\|_{L^\infty_t(\dot{H}^1 \times L^2)} \lesssim \|(\phi, \psi)\|_{\dot{H}^1 \times L^2} + \|N[u]\|_{L^1_t L^2_x} \\
\lesssim \|(\phi, \psi)\|_{\dot{H}^1 \times L^2} + \|\partial u\|_{L^{p-1}_t L^{\infty}_x}^{p-1} \|\partial u\|_{L^\infty_t L^{s_c}_x}.
$$

Moreover, we have

$$
\|(\Phi[u] - \Phi[v], \partial_t (\Phi[u] - \Phi[v]))\|_{L^\infty_t(\dot{H}^1 \times L^2)} \\
\lesssim \|N[u] - N[v]\|_{L^1_t L^2_x} \\
\lesssim \|\partial u\|_{L^{p-1}_t L^{\infty}_x} + \|\partial \Phi[u]\|_{L^{p-1}_t L^{\infty}_x} \|\partial u - \partial v\|_{L^\infty_t L^{s_c}_x}.
$$

With these three estimates, it is easy to prove the existence and uniqueness for (1.1) in $C_t^{\dot{H}^{s_c}} \cap C_t^{\dot{H}^{s_c - 1}}$ with $\partial u \in L_r^{p-1} L^{\infty}_x$, when $(\phi, \psi) \in \dot{H}^{s_c} \times \dot{H}^{s_c - 1}$ and $\varepsilon$ is small enough.

When $p > 3$, we see that if the initial data are radial and we use (2.8) of Lemma 2.3 instead of (2.7), then the same proof applies. This completes the proof of Theorem 1.3.

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