Growing Graceful and Harmonious Trees

Edinah K. Gnang ∗, Isaac Wass †

May 15, 2019

Abstract

We describe symbolic constructions for listing and enumerating induced edge label sequences of graphs. Our constructions settles in the affirmative a conjecture of R. Whitty [W08] on the existence of determinantal constructions which list/enumerate gracefully labeled trees. The constructions is easily adapted to the purpose of enumerating harmoniously labeled trees. We conclude the paper with descriptions of graph labeling algorithms.

1 Introduction

The Kotzig-Ringel [R64, Gal05] conjecture, better known as the Graceful Labeling Conjecture (GLC), asserts that every tree admits at least one graceful labeling and the Graham-Sloane [GS80] conjecture, also known as the Harmonious Labeling Conjecture (HLC), asserts that every tree admits at least one harmonious labeling. Both the graceful and the harmonious labelings assign integers to vertices of a graph to create a bijection between vertices and induced edge labels. In the context of the GLC, induced edge labels correspond to absolute differences of integers assigned to the vertices spanning each edge. In the context of the HLC, induced edge labels correspond to residue classes ( modulo the number of vertices ) of sums of integers assigned to the vertices spanning each edge. For notional convenience, we consider a functional reformulation of both problems. A rooted tree on \( n > 0 \) vertices is naturally associated with a discrete function

\[
f \in ([0, n) \cap \mathbb{Z})^{[0, n) \cap \mathbb{Z}} \text{ subject to } \left| f^{(n-1)} ([0, n) \cap \mathbb{Z}) \right| = 1 \tag{1}
\]

where

\[
\forall i \in [0, n) \cap \mathbb{Z}, \quad f^{(0)}(i) := i, \quad \text{and} \quad \forall k \geq 0, \quad f^{(k+1)}(i) = f^{(k)}(f(i)) = f \left( f^{(k)}(i) \right).
\]

An arbitrary \( f \in ([0, n) \cap \mathbb{Z})^{[0, n) \cap \mathbb{Z}} \) is associated with a functional directed graph \( G_f = (V, E) \) where

\[
V := [0, n) \cap \mathbb{Z} \quad \text{and} \quad E := \{(i, f(i)) : i \in [0, n) \cap \mathbb{Z}\}.
\]

We characterize induced edge labelings of a functional directed graph \( G_f \) associated with \( f \in ([0, n) \cap \mathbb{Z})^{[0, n) \cap \mathbb{Z}} \) as follows:

- Induced subtractive edge labels given by \( \{|f(i) - i| : i \in [0, n) \cap \mathbb{Z}\} \) determine whether or not \( G_f \) is gracefully labeled.
- Induced additive edge labels given by \( \{f(i) + i \mod n : i \in [0, n) \cap \mathbb{Z}\} \) determine whether or not \( G_f \) is harmoniously labeled.

Moreover a given functional directed graph \( G_f \) associated with \( f \in ([0, n) \cap \mathbb{Z})^{[0, n) \cap \mathbb{Z}} \) is said to be

∗Department of Applied Mathematics and Statistics, Johns Hopkins, egnang1@jhu.edu
†Department of Mathematics, Iowa State University icwass@iastate.edu
Proof function

Proposition 0a

rected graphs.

by performing the change of variable 

Consequently


given function 

then the corresponding functional directed graph

Proposition 0b

harmonious if there exist

graceful if there exist



The following two propositions respectively express the permutation reformulation of grace and harmony of functional directed graphs.

Proposition 0a: (Permutation formulation of the GLC) A functional directed graph \( G_f \) associated with an arbitrary function

\[
f \in ([0, n] \cap \mathbb{Z})^{[0, n] \cap \mathbb{Z}}\] is graceful iff

\[
\exists \sigma \in S_n/\text{Aut}(G_f) \ \text{and} \ \gamma \in S_n \ \text{such that} \ \forall \ i \in [0, n] \cap \mathbb{Z}, \ \ f(i) \in \sigma^{-1}(i) \pm \gamma(i)
\]

Proof: On the one hand, the proof of necessity, follows from the fact that \( G_f \) being graceful implies that

\[
\exists \sigma \in S_n/\text{Aut}(G_f), \ \text{such that} \ \{|\sigma f(j) - \sigma(j)| : j \in [0, n] \cap \mathbb{Z}\} = [0, n] \cap \mathbb{Z},
\]

by performing the change of variable \( j = \sigma^{-1}(i) \) we have

\[
\{|\sigma f \sigma^{-1}(i) - \sigma \sigma^{-1}(i)| : i \in [0, n] \cap \mathbb{Z}\} = [0, n] \cap \mathbb{Z}.
\]

Consequently

\[
|\sigma f \sigma^{-1}(i) - i| = \gamma(i), \ \forall \ i \in [0, n] \cap \mathbb{Z}
\]

for some permutation \( \gamma \in S_n \). Note that \( \gamma(i) \) is the weight of the outgoing edge leaving vertex \( i \) in the gracefully labeled graph \( G_{\sigma f \sigma^{-1}(i)} \).

\[
\Rightarrow \sigma f \sigma^{-1}(i) = i \pm \gamma(i), \ \forall \ i \in [0, n] \cap \mathbb{Z},
\]

\[
\Rightarrow f \sigma^{-1}(i) = \sigma^{-1}(i) \pm \gamma(i), \ \forall \ i \in [0, n] \cap \mathbb{Z},
\]

a change of variable similar to the previous one yields the desired result

\[
f(i) \in \sigma^{-1}(i) \pm \gamma(i), \ \forall \ i \in [0, n] \cap \mathbb{Z}.
\]

On the other hand the proof of sufficiency follows from the fact that if a given function \( f \in ([0, n] \cap \mathbb{Z})^{[0, n] \cap \mathbb{Z}} \) is such that

\[
\forall \ i \in [0, n] \cap \mathbb{Z}, \ \ f(i) \in \sigma^{-1}(i) \pm \gamma(i)
\]

then the corresponding functional directed graph \( G_f \) is isomorphic to the gracefully labeled functional directed graph \( G_{\sigma^{-1} f \sigma} \) and thereby completes the proof.

Proposition 0b: (Permutation reformulation of the HLC) An arbitrary functional directed graph \( G_f \) associated with

\[
f \in (\mathbb{Z}/n\mathbb{Z})^{\mathbb{Z}/n\mathbb{Z}}\] is harmonious iff

\[
\exists \sigma \in S_n/\text{Aut}(G_f) \ \text{and} \ \gamma \in S_n \ \text{such that} \ \forall \ i \in \mathbb{Z}/n\mathbb{Z}, \ \ f(i) \equiv \sigma^{-1}((n-1)\sigma(i) + \gamma(i))
\]

Proof: The proof of necessity, follows from the fact that \( G_f \) being harmonious implies that

\[
\exists \sigma \in S_n/\text{Aut}(G_f), \ \text{s.t.} \ \{\sigma f(j) + \sigma(j) : j \in \mathbb{Z}/n\mathbb{Z}\} = \mathbb{Z}/n\mathbb{Z}
\]
by performing the change of variable $j = \sigma^{-1}(i)$ we have
\[ \{ \sigma f \sigma^{-1}(i) + \sigma \sigma^{-1}(i) : i \in \mathbb{Z}/n\mathbb{Z} \} = \mathbb{Z}/n\mathbb{Z}, \]
Consequently
\[ \sigma f \sigma^{-1}(i) + i \equiv \gamma(i), \quad \forall i \in \mathbb{Z}/n\mathbb{Z} \]
for some permutation $\gamma \in S_n$. Note that $\gamma(i)$ is the weight of the outgoing edge leaving vertex $i$ in the harmoniously labeled functional directed graph $G_{\sigma f \sigma^{-1}(i)}$.

On the other hand the proof of sufficiency follows from the fact that if an arbitrarily given function $f \in (\mathbb{Z}/n\mathbb{Z})^2$ is subject to the condition
\[ f(i) \equiv \sigma^{-1}((n-1)\sigma(i) + \gamma \sigma(i)), \quad \forall i \in \mathbb{Z}/n\mathbb{Z} \]
then $G_f$ is isomorphic to the harmoniously labeled functional directed graph $G_{\sigma f \sigma^{-1}}$ and thereby completes the proof.

Let $\text{GrL}(G_f)$ denote the set of distinct graceful relabelings of the functional directed graph $G_f$ while $\text{HaL}(G_f)$ denote the set of distinct harmonious relabelings of $G_f$. *Induced edge label sequence* of $G_f$ refers to the non-decreasing sequence of induced edge labels. For instance the function in Figure 1
\[ f : [0,6) \cap \mathbb{Z} \to [0,6) \cap \mathbb{Z} \]
defined by
\[ f(0) = 0, f(1) = 0, f(2) = 0, f(3) = 0, f(4) = 3, f(5) = 3, \]
is a *functional spanning subtree* of the complete graph ( or functional tree for short ) on 6 vertices. The attractive fixed point condition from Eq. (1) is met since $f^2([0,6) \cap \mathbb{Z}) = \{0\}$. The edge set of $G_f$ is $\{(0,0), (1,0), (2,0), (3,0), (4,3), (5,3)\}$, the corresponding induced subtractive edge label sequence is equal to the corresponding induced additive edge label sequence and given by
\[ (0,1,1,2,2,3). \]
The GLC and HLC are easily verified for the families of functional stars ( which include identically constant functions ). This is seen from the fact that the constant zero function
\[ f : [0,n) \cap \mathbb{Z} \to [0,n) \cap \mathbb{Z} \]
such that
\[ f(i) = 0, \quad \forall i \in [0, n) \cap \mathbb{Z} \]
is simultaneously gracefully and harmoniously labeled. In particular
\[ |\text{GrL}(G_f)| = 2 \quad \text{and} \quad |\text{HaL}(G_f)| = n. \]

In particular the number of distinct subtractive edge label sequences of the functional star on \( n \) vertices is \( \left\lfloor \frac{n}{2} \right\rfloor + (n - 2 \left\lfloor \frac{n}{2} \right\rfloor) \).

Our main results are symbolic constructions based upon Gaussian elimination for listing/enumerating induced edge label sequences of graphs. Our constructions settles in the affirmative a conjecture of R. Whitty [W08] on the existence of determinantal constructions which list/enumerate gracefully labeled trees. The constructions is easily adapted to the purpose of listing/enumerating harmoniously labeled trees. We conclude the paper with descriptions of graph labeling algorithms.

This article is accompanied by an extensive SageMath[18] graceful graph package which implements the symbolic constructions described here. The package is made available at the link: [https://github.com/gnang/Graceful-Graphs-Package](https://github.com/gnang/Graceful-Graphs-Package)

Acknowledgement: The first author would like to thank Noga Alon for introducing him to the subject. We are grateful to Harry Crane, Mark Daniel Ward, Yuval Filmus, Andrei Gabrielov, Edward R. Scheinerman and Jeanine Gnang for insightful discussions and suggestions.

This research was partially supported by NSF-DMS grants 1603823, 1604458, and 1604773, “Collaborative Research: Rocky Mountain - Great Plains Graduate Research Workshops in Combinatorics” and the NSA grant H98230-18-1-0017.

2 Counting gracefully labeled functional digraphs.

There are \( n! \) gracefully labeled undirected graphs on \( n \) vertices with \( n \) edges. For many of these gracefully labeled undirected graphs there exist no edge orientation which result in a functional directed graph. The probability that an undirected graph on \( n \) vertices with \( n \) edges chosen uniformly at random, happens to be gracefully labeled is equal to
\[
\frac{n!}{n \left( \begin{array}{c} n \\ \frac{n}{2} \end{array} \right) \left( \begin{array}{c} n \frac{n}{2} \end{array} \right)} = \frac{(n - 1)!}{\left( \begin{array}{c} n / 2 \\ n - 1 \end{array} \right)}.
\]

The enumeration of gracefully labeled functional directed graph is harder. The number of gracefully labeled functional directed graph on \( n \) vertices is trivially upper bounded \( n! 2^{n-1} \). The permutation reformulation in Prop. 0a. permits an improvement to the trivial upper bound while simultaneously providing a non-trivial lower bound. Recall that a functional directed graph \( G_f \) associated with an arbitrary \( f \in ([0, n) \cap \mathbb{Z})^{([0, n) \cap \mathbb{Z}]} \), is graceful if there exists fixed permutations \( \sigma \in S_n/\text{Aut}(G_f) \) and \( \gamma \) such that
\[
f(i) \in \sigma (\sigma^{-1}(i) \pm \gamma \sigma^{-1}(i)), \quad \forall i \in [0, n) \cap \mathbb{Z}, \]
\[
\implies \sigma^{-1} f \sigma (i) \in i \pm \gamma (i), \quad \forall i \in [0, n) \cap \mathbb{Z}.
\]
Consequently, \( G_f \) is gracefully labeled when \( \sigma \) lies in the coset \( \text{id} \cdot \text{Aut}(G_f) \). Assume that \( f(0) = 0 \), to ensure that the gracefully labeled functional directed graph \( G_f \) has no isolated vertices. The permutation \( \gamma \) must be chosen such that
\[
\forall 0 < i < n, \quad |\{i \pm \gamma (i)\} \cap ([0, n) \cap \mathbb{Z})| > 0 \Rightarrow \begin{cases} 
\gamma (i) \leq i \\
or \\
\forall i \in [0, n) \cap \mathbb{Z}.
\end{cases}
\]

(2)
Alternatively, the functional directed graph $G_f$ having no isolated vertices is gracefully labeled iff
\[ f(i) = i + s(i) \gamma(i), \quad \forall i \in [0, n) \cap \mathbb{Z}, \]
where $s \in \{-1, 1\}^{[0,n)\cap\mathbb{Z}}$. The following proposition therefore follows from the permutation criterion (2).

Proposition 1: The number of permutations of the labels in $[0, n) \cap \mathbb{Z}$ subject to the restriction (2) for $n > 2$ is given by
\[ |\{ \gamma \in S_n \text{ such that } \gamma(0) = 0 \text{ and } \forall i \in [0, n) \cap \mathbb{Z}, \text{ either } \gamma(i) < n - i \text{ or } \gamma(i) \leq i \}| = \left( \left\lfloor \frac{n-1}{2} \right\rfloor \right) \cdot \left( \left\lceil \frac{n-1}{2} \right\rceil \right) \cdot \frac{(n-1)!}{(n-1)!} \cdot (n-1)! \cdot \frac{(n-1)!}{(n-1)!} \cdot \frac{(n-1)!}{(n-1)!}.
\]
The corresponding sequence appears in the OEIS database as \([A010551]\).

Proof: The proof follows by observing that for such a permutation $\gamma$ there is a single choice for the pre-image of $(n-1)$ and this choice is determined by
\[ \gamma(n-1) = n-1 \Rightarrow s(n-1) = -1. \]
Following this choice there are two possible choices for the pre-image of $(n-2)$ as determined by
\[ \gamma(n-2) = n-2 \Rightarrow s(n-2) = 1 \quad \text{or} \quad \gamma(n-2) = n-2 \Rightarrow s(n-2) = -1. \]
(3)

Following the first two choices, there will be three remaining choices for the pre-image of $(n-3)$. The possible choices for the pre-image of $(n-3)$ (not accounting for the pre-image choices already made for $(n-1)$ and $(n-2)$) are determined by
\[ \gamma(n-3) = n-3 \Rightarrow s(n-3) = 1 \quad \text{or} \quad \gamma(n-3) = n-3 \Rightarrow s(n-3) = -1 \quad \text{or} \quad \gamma(n-3) = n-3 \Rightarrow s(n-3) = -1. \]
Similarly following these three choices there will be four remaining choices for the pre-image of $(n-4)$. All the possible choices (not accounting for the pre-images choices already made for $(n-1), (n-2)$ and $(n-3)$) are determined by
\[ \gamma(n-4) = n-4 \Rightarrow s(n-4) = 1 \quad \text{or} \quad \gamma(n-4) = n-4 \Rightarrow s(n-4) = -1 \quad \text{or} \quad \gamma(n-4) = n-4 \Rightarrow s(n-4) = -1. \]
yields the following lower and upper bounds for \( \tau \)

\[
\begin{align*}
(\gamma (n-3) = (n-4) \Rightarrow s(n-3) = -1) \\
\text{or} \\
(\gamma (n-2) = (n-4) \Rightarrow s(n-2) = -1)
\end{align*}
\]

The argument proceeds similarly all the way up to the choices for the pre-images of \( \left\lceil \frac{n-1}{2} \right\rceil \). These possible pre-image assignments account for the factorial factor \( \left\lceil \frac{n-1}{2} \right\rceil ! \). Note that for each one of these pre-image choices, the output of the sign function \( s \) is uniquely determined. Finally, the remaining factorial factor arises from taking all possible permutations of the remaining integers thus completing the proof. \( \square \)

Let \( \tau_n \) denote the number of gracefully labeled functional directed graphs on \( n \) vertices having no isolated vertices. Proposition 1 yields the following lower and upper bounds for \( \tau_n \) given by

\[
\left( \left\lfloor \frac{n-1}{2} \right\rfloor! \right) \left( \left\lfloor \frac{n-1}{2} \right\rfloor! \right) \leq \tau_n \leq \left( \left\lfloor \frac{n-1}{2} \right\rfloor! \right) \left( \left\lfloor \frac{n-1}{2} \right\rfloor! \right) n^{2 \left\lceil \frac{n+1}{2} \right\rceil}.
\]

The extra factor of 2 in the lower bound accounts for the complementary graceful labeling. The extra factor of \( n \) in the upper bound accounts for alternative possible placements of the unique loop edge. Incidentally the argument used to prove Proposition 1 describes an optimal algorithm for constructing the set of signed permutations noted \( \mathbb{SP}_n \) (used to construct gracefully labeled functional directed graphs having no isolated vertices) defined by

\[
\mathbb{SP}_n := \left\{ g \text{ s.t. } \forall i \in [0,n) \cap \mathbb{Z}, 0 \leq (i + g(i)) = (i + s(i) \gamma(i)) < n, \text{ for some } s \in \{-1,1\}^{[0,n) \cap \mathbb{Z}} \text{ and } \gamma \in S_n \gamma(0) = 0 \right\}.
\]

An ensuing determinental sieve follows from the directed Matrix Tree Theorem \( [Z85] \). The edge variables associated with distinct gracefully labeled functional directed graphs having no isolated vertices make up distinct monomial terms of the polynomial identity below

\[
\sum_{g \in \mathbb{SP}_n} A[0,0] \det \left( \begin{array}{c}
\sum_{0 \leq i < n} A[i,i + g(i)] I[:,i] I[i + g(i),:] 1_{n \times 1} \\
|\{|f(i) - i| : i \in [0,n) \cap \mathbb{Z}\}| = n
\end{array} \right) = \prod_{0 \leq i < n} A[i,f(i)]
\]

We used in the expression above the colon notation. Recall that in the colon notation \( A[1:,1:] \) refers to the \((n-1) \times (n-1)\) matrix obtained by deleting row 0 and column 0 of the matrix. Similarly \( I[:,i] \) denotes the \( i \)-th column of the identity matrix and \( A \) denotes a symbolically weighted \( n \times n \) adjacency matrix for the directed complete graph on \( n \) vertices which includes loop edges, with entries given by

\[
A[i,j] = a_{ij}, \quad \forall 0 \leq i, j < n.
\]

Note that Whitty shows in \( [W08] \)

\[
A[0,0] \det \{ (\mathbf{Y} - A)[1:,1:] \} = \sum_{f^{(n-1)}([0,n) \cap \mathbb{Z}) = \{0\}} \text{sgn}(|f - \text{id}|) \prod_{0 \leq i < n} A[\min(i,f(i)),\max(i,f(i))]
\]

\[
\text{sgn}(|f - \text{id}|) \prod_{0 \leq i < n} A[\min(i,f(i)),\max(i,f(i))]
\]

\[
\sum_{f^{(n-1)}([0,n) \cap \mathbb{Z}) = \{0\}} |\{|f(i) - i| : i \in [0,n) \cap \mathbb{Z}\}| = n
\]
where
\[
\forall 0 \leq i, j < n, \quad A[i, j] = A[\min(j - (n - 1) + i - 1), \max(j - (n - 1) + i - 1, i)],
\]
\[
\forall 0 \leq i, j < n, \quad Y[i, j] = A[\min(i, (n - 1) - j + i + 1), \max(i, (n - 1) - j + i + 1)].
\]

Whitty also conjectures in [W08] the existence of similar determinental constructions whose monomials are free of the signing \(\text{sgn}(|f - \text{id}|)\) which results in undesirable cancelations when the value 1 is assigned to every variable.

### 3 Generatingfunctionology of induced edge labelings

Motivated by Whitty’s conjecture, we derive here various generating functions constructions whose coefficient enumerate special functional directed graphs which have a given induced edge label sequence. Our first construction is directly obtained from the listing of functional directed graphs.

**Proposition 2**: Let \(A\) denote a symbolically weighted \(n \times n\) adjacency matrix for the directed complete graph on \(n\) vertices which includes loop edges, then
\[
\det\{\text{diag}(A \cdot 1_{n \times 1})\} = \sum_{f \in (\{0, n\} \cap \mathbb{Z})^{\{0, n\}}: \forall \omega} \prod_{0 \leq i < n} A[i, f(i)].
\]

**Proof**: The proof immediately follows from the following identity
\[
\det\{\text{diag}(A \cdot 1_{n \times 1})\} = \prod_{0 \leq i < n} \left( \sum_{0 \leq j < n} A[i, j] \right) = \sum_{f \in (\{0, n\} \cap \mathbb{Z})^{\{0, n\}}: \forall \omega} \prod_{0 \leq i < n} A[i, f(i)].
\]
The only terms which contribute to the sum are terms where exactly one entry of each row \(A\) is selected. We devise from the factored form in an efficient algorithm for expressing a generating function whose coefficients enumerate the number of distinct functional directed graphs which have the same given induced edge label sequence.

**Corollary 3**: Let \(X\) and \(Y\) denote a symbolically weighted \(n \times n\) adjacency matrix for the directed complete graph on \(n\) vertices which includes loop edges, with entries given by
\[
X[i, j] = x^{(n+1)(j-i)}, \quad Y[i, j] = x^{(\omega^{(i+j)})}, \quad \forall 0 \leq i, j < n, \ \omega_n = e^{\frac{2\pi i}{n}} \sqrt{-1}
\]
then the polynomial \(\det\{\text{diag}(X \cdot 1_{n \times 1})\}\) and \(\det\{\text{diag}(Y \cdot 1_{n \times 1})\}\) in the variable \(x\) yield the generating function whose coefficients enumerate the number of distinct functional directed graphs on \(n\) vertices which have the same given induced subtractive and additive edge label sequence respectively.

**Proof**: It follows from Proposition 2 that
\[
\det\{\text{diag}(X \cdot 1_{n \times 1})\} = \prod_{0 \leq i < n} \left( \sum_{0 \leq j < n} x^{(n+1)(j-i)} \right) = \left( \sum_{f \in (\{0, n\} \cap \mathbb{Z})^{\{0, n\}}: \forall \omega} \prod_{0 \leq i < n} x^{(n+1)(f(i)-i)} \right).
\]
\[
\det\{\text{diag}(Y \cdot 1_{n \times 1})\} = \prod_{0 \leq i < n} \left( \sum_{0 \leq j < n} x^{\omega^{(i+j)}} \right) = \left( \sum_{f \in (\{0, n\} \cap \mathbb{Z})^{\{0, n\}}: \forall \omega} \prod_{0 \leq i < n} x^{\omega^{(i+f(i))}} \right).
\]
Consequently, the coefficient of the term \( \prod_{0 < j \leq n} x^{(n+1)n-j} b_{n-j} \) or \( \prod_{0 < j \leq n} x^{(\omega_n)n-j} b_{n-j} \) enumerates the number of distinct functional directed graphs whose induced edge label sequence is such that \( b_{n-j} \) of its \( n \) edges are labeled \( (n-j) \).

Note that for the monomial \( \prod_{0 < j \leq n} x^{(n+1)n-j} b_{n-j} \) to have non-vanishing coefficient in the generating functions \( \text{det} \{\text{diag} (X \cdot 1_{n \times 1})\} \) and \( \text{det} \{\text{diag} (Y \cdot 1_{n \times 1})\} \) it is necessary that

\[
n = \left( \sum_{0 < j \leq n} b_{n-j} \right) \quad \text{and} \quad 0 \leq b_{n-j} \leq n, \quad \forall 0 < j \leq n.
\]

In particular to upper bound the number of terms with non-vanishing coefficient in the generating function \( \text{det} \{\text{diag} (X \cdot 1_{n \times 1})\} \) it suffices to count the number of non negative integer solutions to the constraints

\[
n = \sum_{0 < j \leq n} b_{n-j} \quad \text{subject to} \quad \left\{ \begin{array}{ll}
0 \leq b_{n-j} & \leq 2j \quad \text{if} \quad 0 < j < \left\lceil \frac{n}{2} \right\rceil \\
b_{n-j} & \geq 0 \quad \text{otherwise}
\end{array} \right.,
\]

Recall that there are \( \binom{2n-1}{n} \) non negative integer solutions to the constraints

\[
n = \left( \sum_{0 < j \leq n} b_{n-j} \right), \quad \text{such that} \quad b_{n-j} \geq 0 \forall 0 < j \leq n.
\]

We can account for the upper and lower bound conditions \( 0 \leq b_{n-j} \leq 2j \) for all \( 0 < j \leq \left\lceil \frac{n}{2} \right\rceil \) using inclusion exclusion as follows

\[
\left( \binom{2n-1}{n} \right) - \sum_{\phi \neq J \subseteq \{0, \ldots, \left\lceil \frac{n}{2} \right\rceil -1\}} (-1)^{|J|-1} \left| \bigcap_{j \in J} \{b_{n-j} > 2j\} \right|.
\]

This yields the asymptotic upper bound of \( O \left( \frac{4^n}{\sqrt{n}} \right) \) for the number of non vanishing terms in both generating functions.

The generating functions are thus efficiently determined via Lagrange interpolation using up to \( O \left( \frac{4^n}{\sqrt{n}} \right) \) evaluations of the variable \( x \).

The term of lowest degree in the generating function \( \text{det} \{\text{diag} (X \cdot 1_{n \times 1})\} \) corresponds to the identity function noted \( \text{id} \) specified by

\[
\text{id} (i) = i \quad \forall 0 \leq i < n,
\]

and the corresponding term is \( x^{(n+1)n} \). On the other hand, the terms of largest degree in the generating function \( \text{det} \{\text{diag} (X \cdot 1_{n \times 1})\} \) is

\[
\prod_{\left\lceil \frac{n-1}{2} \right\rceil \leq i < n} x^{2(n+1)i} \quad \text{if} \quad (n-1) \equiv 1 \text{ mod } 2
\]

or

\[
x^{(n+1)\left\lceil \frac{n-1}{2} \right\rceil} \prod_{\left\lceil \frac{n-1}{2} \right\rceil \leq i < n} x^{2(n+1)i} \quad \text{if} \quad n-1 \equiv 0 \text{ mod } 2
\]

When \( n \) is even the corresponding functional directed graph is associated with the function

\[
f \in ([0, n) \cap \mathbb{Z})^{[0,n) \cap \mathbb{Z}} \quad \text{such that} \quad f (i) = \begin{cases} n-1 & \text{if } 0 \leq i \leq \left\lceil \frac{n-1}{2} \right\rceil \\ 0 & \text{if } \left\lceil \frac{n-1}{2} \right\rceil \leq i < n \end{cases}
\]
When \( n \) is odd there are two such functional directed graphs respectively associated with the functions by

\[
f, g \in ([0, n) \cap \mathbb{Z})^{[0,n) \cap \mathbb{Z}}
\]

such that

\[
f(i) = \begin{cases} 
  n - 1 & \text{if } 0 \leq i < \frac{n-1}{2} \\
  0 & \text{if } \frac{n-1}{2} \leq i < n 
\end{cases}
\]

and

\[
g(i) = \begin{cases} 
  n - 1 & \text{if } 0 \leq i \leq \frac{n-1}{2} \\
  0 & \text{if } \frac{n-1}{2} < i < n
\end{cases}
\]

We summarize these observations by stating that for the monomial \( \prod_{0 \leq j \leq n} x^{(n+1)^{n-j} b_{n-j}} \) to have non-vanishing coefficient in the generating function \( \det \{ \text{diag}(X \cdot 1_{n \times 1}) \} \) it is necessary that

\[
n = \sum_{0 \leq j \leq n} b_{n-j},
\]

\[
0 \leq b_{n-j} \leq \min \{ 2j, n \}, \quad \forall 0 < j \leq n,
\]

and

\[
n \leq \sum_{0 < j \leq n} (n+1)^{n-j} b_{n-j} \leq \begin{cases} 
  2 \left( \sum_{\frac{n-1}{2} \leq i < n} (n+1)^i \right) & \text{if } (n-1) \equiv 1 \mod 2 \\
  (n+1)^{\frac{n-1}{2}} + 2 \left( \sum_{\frac{n-1}{2} \leq i < n} (n+1)^i \right) & \text{if } n - 1 \equiv 0 \mod 2
\end{cases}
\]

Similarly, the term of lowest degree in the generating function \( \det \{ \text{diag}(Y \cdot 1_{n \times 1}) \} \) is also \( x^{(n+1)^{n-j} b_{n-j}} \) associated with the additive induced edge label sequence of the function

\[
f(i) = \begin{cases} 
  0 & \text{if } i = 0 \\
  n - i & \text{otherwise} 
\end{cases}, \forall 0 \leq i < n.
\]

The term of highest degree in \( \det \{ \text{diag}(Y \cdot 1_{n \times 1}) \} \) is \( x^{n \omega_n^{(n-1)}} \) associated with the additive induced edge label sequence of

\[
f(i) = n - 1 - i, \forall 0 \leq i < n.
\]

We summarize these observations by stating that for the monomial \( \prod_{0 < j \leq n} x^{(\omega_n)^{n-j} b_{n-j}} \) to have non-vanishing coefficient in the generating function \( \det \{ \text{diag}(Y \cdot 1_{n \times 1}) \} \) it is necessary that

\[
n = \sum_{0 < j \leq n} b_{n-j},
\]

\[
0 \leq b_{n-j} \leq n, \quad \forall 0 \leq j \leq n,
\]

and

\[
n \leq \sum_{0 < j \leq n} (n+1)^{n-j} b_{n-j} \leq n (n+1)^{(n-1)}
\]

The next proposition refines the construction to obtain instead the generating function whose coefficients enumerate the number of distinct functional trees which have the same given induced edge label sequence. We will use here a variant of Gantmacher’s notation where for

\[
0 \leq i_0 < \cdots < i_t < \cdots < i_{k-1} < n
\]

and

\[
0 \leq j_0 < \cdots < j_t < \cdots < j_{k-1} < n
\]
the matrix
\[
M = \begin{bmatrix}
  f_0, \ldots, f_{i}, \ldots, f_{k-1} \\
  j_0, \ldots, j_{i}, \ldots, j_{k-1}
\end{bmatrix}
\]
denotes the \( k \times k \) sub-matrix formed by retaining only the rows and columns of an \( n \times n \) matrix \( M \) indexed by \( \{i_t\}_{0 \leq t < n} \) and \( \{j_t\}_{0 \leq t < n} \) respectively.

**Proposition 4** : Let \( X \) denote a symbolically weighted \( n \times n \) adjacency matrix for the directed complete graph on \( n \) vertices which includes loop edges, with entries given by
\[
X[i, j] = x^{(n^i - i)}, \quad Y[i, j] = x^{(\omega_n^{(i+j)})}, \quad \forall 0 \leq i, j < n,
\]
then the polynomials
\[
\begin{align*}
\sum_{0 \leq i < n} X[i, i] \det \left\{ \left( \text{diag}(X \cdot 1_{n \times 1}) - X \right) \right\} \left[ \begin{array}{c}
0, \ldots, i-1, i+1, \ldots, n-1 \\
0, \ldots, i-1, i+1, \ldots, n-1
\end{array} \right], \\
\sum_{0 \leq i < n} Y[i, i] \det \left\{ \left( \text{diag}(Y \cdot 1_{n \times 1}) - Y \right) \right\} \left[ \begin{array}{c}
0, \ldots, i-1, i+1, \ldots, n-1 \\
0, \ldots, i-1, i+1, \ldots, n-1
\end{array} \right],
\end{align*}
\]
in the variable \( x \) corresponds respectively to the generating function whose coefficients enumerate the number of distinct functional trees on \( n \) vertices having a given induced subtractive and additive edge label sequence respectively.

**Proof** : Recall Tutte’s Directed Matrix Tree Theorem (DMTT), which asserts that
\[
\sum_{|f^{(n-1)}(\{0,n\}) \cap \mathbb{Z}| = 1} \prod_{0 \leq i < n} A[i, f(i)] = \sum_{0 \leq i < n} A[i, i] \det \left\{ \left( \text{diag}(A \cdot 1_{n \times 1}) - A \right) \right\} \left[ \begin{array}{c}
0, \ldots, i-1, i+1, \ldots, n-1 \\
0, \ldots, i-1, i+1, \ldots, n-1
\end{array} \right],
\]
we refer the reader to the elegant combinatorial proof of the DMTT in Zeilberger [Z85]. The desired generating function is thus obtained by substituting the matrix \( A \) by the symbolic matrices \( X \) and \( Y \) respectively from which it follows that
\[
\begin{align*}
\sum_{|f^{(n-1)}(\{0,n\}) \cap \mathbb{Z}| = 1} \prod_{0 \leq i < n} x^{n^{f(i)-i}} &= x \sum_{0 \leq i < n} \det \left\{ \left( \text{diag}(X \cdot 1_{n \times 1}) - X \right) \right\} \left[ \begin{array}{c}
0, \ldots, i-1, i+1, \ldots, n-1 \\
0, \ldots, i-1, i+1, \ldots, n-1
\end{array} \right], \\
\sum_{|f^{(n-1)}(\{0,n\}) \cap \mathbb{Z}| = 1} \prod_{0 \leq i < n} x^{\omega_n^{(i+1)}} &= x \sum_{0 \leq i < n} x^{(\omega_n^{i+1})} \det \left\{ \left( \text{diag}(Y \cdot 1_{n \times 1}) - Y \right) \right\} \left[ \begin{array}{c}
0, \ldots, i-1, i+1, \ldots, n-1 \\
0, \ldots, i-1, i+1, \ldots, n-1
\end{array} \right].
\end{align*}
\]
thus completing the proof □.

### 4 Some combinatorial variants of Immanants.

The generating function constructions described in the previous section motivate the following variants of the permanent, determinant and conjecturally of immanant polynomials associated with a \( n \times n \) matrix \( A \) and a function \( f \in (\{0, n\} \cap \mathbb{Z})^{[0, n]} \cap \mathbb{Z} \)
\[
\text{mPer}_f (A) = \sum_{\sigma \in S_n / Aut_f} \prod_{0 \leq i < n} A[i, \sigma f^{-1}(i)],
\]
\[
\text{mDet}_f (A) = \sum_{\sigma \in S_n / \text{Aut}_f} \text{sgn} (\sigma f^{-1}) \prod_{0 \leq i < n} A [i, \sigma f^{-1} (i)],
\]
where
\[
\text{sgn} (\sigma f^{-1}) = \text{sgn} \left( \prod_{0 \leq i < j < n} \left( (\sigma f^{-1} (j) - \sigma f^{-1} (i)) \right) \right)
\]

Conjecturally modified immanants are of the general form
\[
\text{mImm}_{\lambda, f} (A) = \sum_{\sigma \in S_n / \text{Aut}_f} \chi (\sigma f^{-1}) \prod_{0 \leq i < n} A [i, \sigma f^{-1} (i)],
\]
where \( \chi \) appropriately generalizes the notion of characters. Let \( C_n \) denote an arbitrary choice of one of the largest subset of \( S_n \) whose elements are pairwise non-isomorphic as functional directed graphs. The relationship between the proposed modified permanent/determinant and their classical counterparts is expressed by
\[
\text{Per} (A) = \sum_{f \in C_n} \text{mPer}_f (A) \quad \text{and} \quad \text{det} (A) = \sum_{f \in C_n} \text{mDet}_f (A). \tag{6}
\]
For instance in the case \( n = 3 \) a representative choice for \( C_3 \) is given by
\[
C_3 = \{ [(0,0), (1,1), (2,2)], [(0,1), (1,0), (2,2)], [(0,1), (1,2), (2,0)] \}.
\]
Similarly, let \( T_n \) denote an arbitrary choice of one of the largest subset of functional trees whose elements are pairwise non-isomorphic as functional directed graphs then we have
\[
\sum_{f \in T_n} \text{mPer}_f (A) = \sum_{f \in T_n} \text{mPer}_f (A) = \prod_{|f^{(n-1)}(\{0, n\} \cap \mathbb{Z})| = 1} A [i, f (i)].
\]
\[
\sum_{f \in T_n} \text{mDet}_f (A) = \sum_{f \in T_n} \text{mDet}_f (A) = \prod_{|f^{(n-1)}(\{0, n\} \cap \mathbb{Z})| = 1} \text{sgn} (f) \prod_{0 \leq i < n} A [i, f (i)].
\]
For instance in the case \( n = 3 \) a representative choice for \( T_3 \) is given by
\[
T_3 = \{ [(0,0), (1,2), (2,0)], [(0,0), (1,0), (2,0)] \}.
\]
The Tutte’s DMTT therefore relates the modified permanent to the modified determinant as follows
\[
\sum_{f \in T_n} \text{mPer}_f (A) = \sum_{0 \leq i < n} A [i, i] \sum_{f \in C_n} \text{mDet}_f \left\{ (\text{diag} (A \cdot 1_{n \times 1}) - A) \right\} \prod_{0 \leq i < n} A [i, i, n - 1]. \tag{7}
\]
Let \( f \in (\{0, n\} \cap \mathbb{Z})^{0 \leq n} \) subject to \( |f^{(n-1)}(\{0, n\} \cap \mathbb{Z})| = 1 \) and let \( A \) denote a symbolically weighted \( n \times n \) adjacency matrix for the directed complete graph on \( n \) vertices allowing for loop edges with entries given by
\[
\forall 0 \leq i, j < n, \quad A [i, j] = a_{ij}.
\]
It follows from the identity
\[
A [i, i] \det \left\{ (\text{diag} (A \cdot 1_{n \times 1}) - A) \right\} = \sum_{f^{(n-1)}(\{0, \ldots, n-1\}) = \{i\}} \prod_{0 \leq j < n} A [j, f (j)],
\]
\[
A [i, i] \det \left\{ (\text{diag} (A \cdot 1_{n \times 1}) - A) \right\} = \sum_{f^{(n-1)}(\{0, \ldots, n-1\}) = \{i\}} \prod_{0 \leq j < n} A [j, f (j)],
\]
that the polynomial $m_{\text{Per}}(A)$ may be alternatively derived by a variant of Gaussian elimination procedure. The summands of $m_{\text{Per}}(A)$ are obtained via row operations. Each row linear combination step is followed by setting to zero some of the terms which appear in the numerators of intermediary rational functions. More precisely we set to zero any term in the numerator which expresses a product of edge variables associated graphs which are not sub-isomorphic $G_f$. Formally speaking we reduce modulo the monomial ideal generated by edge monomials associated with graph which are not sub-isomorphic to $G_f$. Note that by symmetry it follows that

$$\sum_{|f^{(n-1)}((0,n)\cap\mathbb{Z})|=1} \frac{\text{Aut}_f |m_{\text{Per}}(A)|}{n!} = \sum_{0 \leq i < n} A[i,i] \det \left\{ (\text{diag}(A1_{n \times 1}) - A) \begin{bmatrix} 0, \ldots, i-1, i+1, \ldots, n-1 \\ 0, \ldots, i-1, i+1, \ldots, n-1 \end{bmatrix} \right\} \quad (8)$$

5. Node splitting and edge contraction labelings

In the previous section we described how obtain the graceful labelings of a given functional tree via a variant of the Gaussian elimination procedure. In this section we describe a different algorithm for obtaining graceful labelings of an arbitrary input functional directed graph. The node splitting procedure devises from an input gracefully labeled undirected graph $G$ new gracefully labeled undirected graphs on $n+1$ vertices and $n$ non-loop edges as follows:

**Step 1**: Delete an arbitrary but fixed edge subset of $G$.

**Step 2**: Split the vertex labeled 0 into the new edge $(n, 0)$.

**Step 3**: Output all the gracefully labeled graphs obtained by placing new edges spanning vertex pairs which recover missing edge labels.

The edge contraction procedure takes as input a gracefully labeled graph on $n$ vertices and devises gracefully labeled graphs on $(n-1)$ vertices as follows:

**Step 1**: Delete a fixed edge subset of $G$ which include every edges adjacent to the vertex labeled $(n-1)$.

**Step 2**: Discard the vertex labeled $(n-1)$.

**Step 3**: Output all the gracefully labeled graphs obtained by placing new edges spanning vertex pairs which recover missing edge labels.

Each graph obtained from the input graph $G$ by the edge contraction procedure is a direct parent $G$. More generally, a graph $H$ is a parent (not necessarily a direct parent) of $G$ if the graph $H$ can be obtained from $G$ by a sequence of edge contraction procedures. By the same token $G$ is a obtained from $H$ by a sequence of node splitting procedures. $G$ is called a descendent of $H$. For example all gracefully labeled graphs on three vertices or more are descendent of the gracefully labeled graph on two vertices.

5.1 The contraction/splitting labeling procedure

We co-opt the edge contraction and node splitting constructions to devise a graceful relabelings procedure. It is easy to see that an input graph $G_f$ associated with $f \in ((0,n) \cap \mathbb{Z})^{(0,n)\cap\mathbb{Z}}$ admits a graceful labeling if and only if there exist a solution
to the system of algebraic equations in the vertex variables \( \{x_0, \ldots, x_{n-1}\} \) given by

\[
-1 = \sum_{0 \leq i \neq f(i) < n} \left( \frac{x_{f(i)}}{x_i} \right)^{-j} + \left( \frac{x_{f(i)}}{x_i} \right)^j \quad 0 < j \leq 2n-1.
\] 

(9)

admits a solution. Note that each constraint in Eq. (9) is invariant to the change of variable

\[ x_k \leftrightarrow x_{f(k)} \]

for every choice of \( k \in [0, n) \cap \mathbb{Z} \). Consequently, the system in Eq. (9) is associated with the underlying undirected graph of \( G_f \) noted \( \tilde{G}_f \).

\[
\text{Aut} \tilde{G}_f := \left\{ \sigma \in S_n \mathbin{\text{ s.t. }} \left( \sum_{0 \leq i \neq f(i) < n} \left( \frac{x_{\sigma f^{-1}(i)} x_j}{x_j} \right) \right)^{-1} + \left( \frac{x_{\sigma f^{-1}(i)}}{x_j} \right) = \left( \sum_{0 \leq i \neq f(i) < n} \left( \frac{x_{f(i)}}{x_i} \right)^{-1} + \left( \frac{x_{f(i)}}{x_i} \right) \right) \right\}.
\]

The undirected edge \( \{(i, f(i)), (f(i), i)\} \) and \( \{(j, f(j)), (f(j), j)\} \) for \( 0 < i < j < n \) are said to lie in a common edge orbit of \( \tilde{G}_f \) induced by the action of \( \text{Aut} \tilde{G}_f \) on the vertex set if

\[
\exists \sigma \in \text{Aut} \tilde{G}_f, \text{ s.t. } \left( \frac{x_{\sigma f^{-1}(i)}}{x_j} \right)^{-1} + \left( \frac{x_{\sigma f^{-1}(i)}}{x_j} \right) = \left( \frac{x_{f(i)}}{x_j} \right)^{-1} + \left( \frac{x_{f(i)}}{x_j} \right).
\]

Similarly, two vertices \( x_i \) and \( x_j \) for \( 0 \leq i < j < n \) lie in a common vertex orbit of \( \tilde{G}_f \) induced by the action of \( \text{Aut} \tilde{G}_f \) if

\[
\exists \sigma \in \text{Aut} \tilde{G}_f, \text{ s.t. } x_{\sigma(i)} = x_j.
\]

We describe edge contraction contraction/splitting labeling procedure which finds all graceful relabeling of \( \tilde{G}_f \). The contraction/splitting labeling algorithm proceeds via two complementary subroutines. The first subroutine is an edge contraction routine. It determines the set of possible parent graphs which result from sequences of edge contractions. The edge contraction subroutine terminates once it reaches a star tree. A single iteration of the edge contraction subroutine on an input graph \( \tilde{G}_f \) on \( n \) vertices proceeds as follows:

**Step 1**: If none of the edges of \( \tilde{G}_f \) spans specially marked blue and red vertices, then select a non-isolated edge representative of some arbitrarily chosen edge orbit of \( \tilde{G}_f \). The procedure separately selects a different non-isolated edge representative \( \tilde{G}_f \) per iteration. Otherwise if \( \tilde{G}_f \) already has a special selected edge which spans specially marked blue and red vertices then the procedure skips Step 2 and moves onto Step 3.

**Step 2**: Arbitrarily mark blue and red the vertices spanned by the edge selected in Step 1.

**Step 3**: Remove from \( \tilde{G}_f \) every unselected edges incident to the red vertex.

**Step 4**: Contract the special selected edge into the specially marked blue vertex. If the specially marked red vertex was a leaf node, the iteration outputs a set of candidate parent graphs resulting from all the possible ways of selecting a new specially marked red vertex among the vertices adjacent to the specially marked blue vertex. Only one red vertex is chosen per orbit of vertices adjacent to the blue vertex. Otherwise, if the selected edge is not a leaf edge then the iteration proceeds to Step 5.
Step 5: Output a set of candidate parent graphs resulting from all the ways of replacing edges removed in Step 3 with new edges incident to the specially marked blue vertex. If $n$ is odd, then at most one of the new edges can span the specially marked blue vertex and a vertex previously adjacent to the contracted red vertex. In which case this particular vertex is to be assigned the label $(\frac{n-1}{2})$. Otherwise, if $n$ is even then none of the new edges can be incident to a vertex which was previously adjacent to the contracted red vertex.

Repeatedly applying the edge contraction routine eventually leads to a terminating parent star tree. The second subroutine is a constrained node splitting routine. The constrained node splitting routine recovers possible vertex relabellings via sequences of node splittings. As input, the second subroutine takes a gracefully labeled star tree. The input star tree corresponds to one of the terminating star trees obtained by the edge contraction subroutine. The constrained node splitting routine prunes candidate parent graphs in order to reverse the steps of the edge contractions subroutine, thereby uncovering graceful labelings when they exist.

A given input graph admits no graceful labeling if every sequence of node splittings avoids the input graph. The contraction/splitting labeling procedure is a special purpose elimination procedure. The simplest family of functional directed graph (having no isolated vertices) which admit no graceful labeling belongs to the family of graphs defined for $n \geq 5$ by

$$f : [0, n) \cap \mathbb{Z} \to [0, n) \cap \mathbb{Z}$$

such that

$$f(i) = \begin{cases} 
  i + 1 \mod 3 & \text{if } i \in \{0, 1, 2\} \\
  3 & \text{otherwise}
\end{cases}.$$

The case $n = 5$ yields the smallest $f \in ([0, 5) \cap \mathbb{Z})^{[0, n)} \cap \mathbb{Z} \setminus S_5$ which admits no graceful labeling. Note that the presence of a cycle in a functional directed graph $G_f$ which has no isolated vertex does not necessarily imply that $G_f$ is ungraceful. This is illustrated by the functional directed graph prescribed by

$$f : [0, 7) \cap \mathbb{Z} \to [0, 7) \cap \mathbb{Z}$$

such that

$$f(i) = \begin{cases} 
  i + 1 \mod 3 & \text{if } i \in \{0, 1, 2\} \\
  3 & \text{if } i = 3 \\
  i - 1 & \text{otherwise}
\end{cases},$$

admits a graceful labeling.

Theorem 5: The contraction/splitting procedure identifies every graceful relabeling of a given input functional graph $G_f$ associated with an arbitrary $f \in ([0, n) \cap \mathbb{Z})^{[0, n)} \cap \mathbb{Z}$.

Proof: We prove the claim by establishing the validity of the reduction. It suffices to show that the we can derive from every solutions to the original system of $2n - 1$ equations in Eq. 18 solutions to a family of smaller systems of $2(n - 1) - 1$ equations in $n - 1$ variables. Each smaller system is associated with some choice of vertices $k, l$ where $k \neq l$ and $\{(k, l), (l, k)\} \subset \tilde{G}_f$ in conjunction with a subset $S \subset [0, n) \cap \mathbb{Z}$ such that $|S| = |f^{-1}(\{k\})|$. The smaller system of equations is thus specified by constraints of the form

$$\left\{ 
-1 = \sum_{u \in S} \left( \frac{x_l}{x_u} \right)^{-j} + \left( \frac{x_l}{x_u} \right)^j + \sum_{i \in [0, n) \cap \mathbb{Z} \setminus \{(k) \cup f^{-1}(\{k\})\}} \left( \frac{x_{f(i)}}{x_i} \right)^{-j} + \left( \frac{x_{f(i)}}{x_i} \right)^j \right\}_{0 < j \leq 2(n - 1) - 1},$$

14
Note that every solution to the original system of equation
\[
-1 = \sum_{0 \leq i \neq f(i) < n} \left( \frac{x_{f(i)}}{x_i} \right)^{-j} + \left( \frac{x_{f(i)}}{x_i} \right)^{j}
\]
determines a graceful relabeling of \( G_f \). If the system admits at least one solution then \( G_f \) is graceful. In which case, every graceful relabeling of \( G_f \), is such that the vertex labeled 0 must be adjacent to the vertex labeled \((n-1)\). The smaller system therefore express the system of equation associated with a direct parent graph of \( G_f \). Consequently solutions to system derived from every valid choice of adjacent vertices \( k, l \) and subset \( S \) determine graceful relabelings of candidate direct parent graph of \( G_f \). It therefore follows that graceful re-labeling of \( G_f \) and its siblings are obtained by performing an iteration of the node splitting procedure to each candidate parent. In conclusion \( G_f \) admits no graceful labeling if the node splitting procedure avoids \( G_f \) when starting from its candidate parent graphs. Thus completing the proof.

Note that the procedure operates on the underlying undirected graphs and intermediary steps may very well result in undirected graphs for which there is no edge orientation that renders the graph functional. It is safe to disregard such candidate parent graphs.

6 The method of creative stabilizing.

6.1 Functional paths and functional stars.

In this section we investigate the extent to which symmetries reveal properties of induced edge labelings. Bounds on \(|\text{GrL}(G_f)|\) can be expressed in terms of the size of the automorphism group of \( G_f \) associated with \( f \in ([0, n) \cap \mathbb{Z})^n \mathbb{Z} \) as follows
\[
0 \leq |\text{GrL}(G_f)| \leq \left| \frac{s_n}{\mathcal{I}_n} \right| / |\text{Aut} G_f|,
\]
where \( s_n/\mathcal{I}_n \) denotes the set of representatives of the complementarity labeling equivalence class
\[
\{ \sigma, (n-1) - \sigma \}.
\]
We consider the two extreme examples of graceful functional trees for which \(|\frac{s_n/\mathcal{I}_n}{\text{Aut} G_f}|\) is respectively smallest and largest possible.
\[
f, g : [0, n) \cap \mathbb{Z} \rightarrow [0, n) \cap \mathbb{Z}, \quad \forall 0 < i < n, \quad f(i) = 0 \text{ and } g(i) = \begin{cases} 0 & \text{if } i = 0 \\ i - 1 & \text{otherwise} \end{cases}.
\]
For \( n \geq 3 \), we formally define the set of functional stars to be the set of functional directed graphs whose edge set make up distinct terms of the multivariate polynomial
\[
\sum_{0 \leq i < n} a[i, i] \det \left( \text{diag} \left( A_{1 \times 1} \right) - A \right) \left[ \begin{array}{c} 0, \ldots, i - 1, i + 1, \ldots, n - 1 \\ 0, \ldots, i - 1, i + 1, \ldots, n - 1 \end{array} \right]
\]
\[
\text{mod} \left\{ A \left[ i_{\varnothing}(0), i_{\varnothing}(1) \right] A \left[ i_{\varnothing}(1), i_{\varnothing}(2) \right] A \left[ i_{\varnothing}(2), i_{\varnothing}(3) \right] A \left[ i_{\varnothing}(3), i_{\varnothing}(3) \right] \right\} \left\{ A \left[ i_{\varnothing}(0), i_{\varnothing}(1) \right] A \left[ i_{\varnothing}(1), i_{\varnothing}(2) \right] A \left[ i_{\varnothing}(2), i_{\varnothing}(2) \right] \right\} \left\{ A \left[ i_{\varnothing}(3), i_{\varnothing}(3) \right] \right\} = 4 \quad \forall g \in S_4
\]
where $A$ denotes a symbolically weighted $n \times n$ adjacency matrix for the directed complete graph on $n$ vertices which includes loop edges, with entries given by

$$A[i,j] = a_{ij}, \quad \forall 0 \leq i, j < n.$$ 

In the case $n = 4$ the multivariate polynomial whose terms list all functional stars is given by

$$\sum_{0 \leq i < 4} A[i,i] \det \left\{ (\text{diag}(A1_{4 \times 1}) - A) \left[ \begin{array}{c} 0, \ldots, i - 1, i + 1, \ldots, 3 \\ 0, \ldots, i - 1, i + 1, \ldots, 3 \end{array} \right] \right\} \mod \left\{ A[i_0(i_0), i_1(i_1)] A[i_1(i_2), i_2(i_2)] A[i_2(i_3), i_3(i_3)] \right\} = \left\{ \begin{array}{c} 0 \leq i, i_1, i_2, i_3 \end{array} \right\} = 4,$$

Similarly, for $n \geq 3$, we formally define the set of functional paths to be the set of functional directed graphs whose edges set make up distinct terms of the multivariate polynomial

$$\sum_{0 \leq i \leq n} A[i,i] \det \left\{ (\text{diag}(A1_{n \times 1}) - A) \left[ \begin{array}{c} 0, \ldots, i - 1, i + 1, \ldots, n - 1 \\ 0, \ldots, i - 1, i + 1, \ldots, n - 1 \end{array} \right] \right\} \mod \left\{ A[i_0(i_0), i_1(i_1)] A[i_1(i_2), i_2(i_2)] A[i_2(i_3), i_3(i_3)] \right\} = \left\{ \begin{array}{c} 0 \leq i, i_1, i_2, i_3 \end{array} \right\} = 4.$$ 

In the case $n = 4$ the multivariate polynomial whose terms describes the set of all functional path is

$$\sum_{0 \leq i < 4} A[i,i] \det \left\{ (\text{diag}(A1_{4 \times 1}) - A) \left[ \begin{array}{c} 0, \ldots, i - 1, i + 1, \ldots, 3 \\ 0, \ldots, i - 1, i + 1, \ldots, 3 \end{array} \right] \right\} \mod \left\{ A[i_0(i_0), i_1(i_1)] A[i_1(i_2), i_2(i_2)] A[i_2(i_3), i_3(i_3)] \right\} = \left\{ \begin{array}{c} 0 \leq i, i_1, i_2, i_3 \end{array} \right\} = 4.$$ 

Note that if $G_f$ is a functional star then depending on the location of the loop-edge

$$|\text{Aut}G_f| \in ((n - 1)!, (n - 2)!).$$
Similarly, if $G_g$ is a functional path then depending on the location of the loop-edge

$$|\text{Aut}G_g| \in \{1, 2\}.$$  

It is easy to see that the set of induced edge label sequences is the same across all functional paths and similarly the set of induced edge label sequences is also the same across all functional stars. Furthermore for every $n \geq 3$ functional stars and functional paths respectively minimize and maximize the cardinality of induced edge label sequences. In particular for

$$f, g \in (\{0, n\} \cap \mathbb{Z})^{[0, n) \cap \mathbb{Z}},$$

such that

$$\forall 0 \leq i < n, \quad f(i) = 0 \quad \text{and} \quad g(i) = \begin{cases} 0 & \text{if } i = 0 \\ i - 1 & \text{otherwise} \end{cases},$$

we have

$$|\text{Aut}G_f| = (n - 1)! \quad \text{and} \quad 1 = |\text{Aut}G_g|.$$  

The following proposition determines the set of induced edge label sequences of a functional stars.

**Proposition 6**: Let $X, Y$ denote a symbolically weighted $n \times n$ adjacency matrix for the directed complete graph on $n$ vertices which includes loop edges, with entries given by

$$X[i, j] = x^{n|j - i|}, \quad Y[i, j] = x^{(\omega_n^i + i)}$$

then

$$m_{\text{Per}}(X) = \begin{cases} \prod_{0 \leq i < n} x^{n|\frac{n-1}{2} - i|} & \text{if } n \equiv 1 \mod 2 \\ 2 \prod_{0 \leq i < n} x^{n|\frac{n}{2} - i|} & \text{if } n \equiv 0 \mod 2 \end{cases}$$

and

$$m_{\text{Per}}(Y) = n x^{\sum_{0 \leq i < n} \omega_n^i}$$

e numerate the number of distinct vertex relabelings of a given functional star which have the same induced edge label sequence.

**Proof**: Since all functional trees have the same set of induced edge label sequences, without loss of generality it suffices to consider the $n$ constant functions

$$f \in ([0, n) \cap \mathbb{Z})^{[0, n) \cap \mathbb{Z}} \quad \text{subject to} \quad \forall 0 < i < n, \quad f(i) = k.$$  

We observe that an induced edge label sequence of $G_f$ is completely determined by the label of the vertex of highest degree. The factor 2 follows from the invariance of the induced edge labeling label sequence to the operation of replacing each vertex label $i$ with the new label $(n - 1) - i$ for all $0 \leq i < n$. In particular, if the entries of the symbolically weighted adjacency matrix are given by

$$A[i, j] = a_{ij} x^{n|j - i|}, \quad \forall 0 \leq i, j < n,$$

then for any constant function

$$f \in ([0, n) \cap \mathbb{Z})^{[0, n) \cap \mathbb{Z}} \quad \text{subject to} \quad \forall 0 \leq i < n, \quad f(i) = k.$$
we have

$$\text{mPer}_f (A) = \begin{cases} \left( \prod_{0 \leq i < n} a_{i, \frac{a_{i-1,i} + 1}{2}} x^{\frac{n-1}{2}} \right) + \sum_{0 \leq t < \left\lfloor \frac{n}{2} \right\rfloor} \left( \prod_{0 \leq i < n} a_{i,t} \prod_{0 \leq i < n} a_{n-1-i,n-1-t} \right) \prod_{0 \leq i < n} x^{n-1-i} & \text{if } n \equiv 1 \mod 2 \\ \sum_{0 \leq t < \left\lfloor \frac{n}{2} \right\rfloor} \left( \prod_{0 \leq i < n} a_{i,t} \prod_{0 \leq i < n} a_{n-1-i,n-1-t} \right) \prod_{0 \leq i < n} x^{n-1-i} & \text{if } n \equiv 0 \mod 2 \end{cases}. $$

Consequently, assigning 1 to each variable in the set \( \{a_{ij}\}_{0 \leq i,j < n} \), yields the desired result thus completing the proof for subtractive edge label setting. In the setting of the additive edge label sequence we have

$$\text{mPer}_f (Y) = \sum_{0 \leq t < n} \prod_{0 \leq i < n} x^{(\omega_{i}+i)} = n x^{\left( \sum_{0 \leq k < n} \omega_{k} \right)}. $$

In particular, if the entries of the adjacency matrix are given by

$$B [i, j] = a_{ij} x^{(n-1-i)}, \quad \forall 0 \leq i, j < n,$$

then for any constant function

$$f \in ([0, n) \cap \mathbb{Z})^{[0, n) \cap \mathbb{Z}} \text{ subject to } \forall 0 \leq i < n, \quad f (i) = k,$$

we have

$$\text{mPer}_f (B) = \sum_{0 \leq t < n} \prod_{0 \leq i < n} a_{i,t} x^{(\omega_{i}+i)} = x^{\left( \sum_{0 \leq k < n} \omega_{k} \right)} \sum_{0 \leq t < n} \prod_{0 \leq i < n} a_{i,t}$$

thus completing the proof for the additive edge label case.□

6.2 Graceful sub-orbits.

We now describe the conjugation algorithm which enables us to focus on the graceful sub-orbit of functional directed graphs. The conjugation algorithm resembles the Buchberger’s algorithm in its emphasis on orderings, monomial ideals as well as the fact the algorithm consists in reducing multivariate polynomials modulo an increasing number of algebraic relations. For any multivariate polynomial \( Q \), \( \text{TermSet}(Q) \) denote the set

$$\{ t : \text{ is a nonzero term in the canonical expansion of } Q \}. $$

For example, for

$$Q (x, y, z) = (x + y + z)^2 + (2x - y - 2z) x \implies \text{TermSet} (Q) = \{ 3x^2, 2yz, xy, z^2, y^2 \}. $$

Let \( A \) denote a symbolically weighted \( n \times n \) adjacency matrix for the directed complete graph on \( n \) vertices which includes loop edges with entries given by

$$A [i, j] = a_{ij} x^{j-i}, \quad \forall 0 \leq i, j < n.$$

Assuming the elements of \( S_n \) to be lexicographically ordered, the conjugation algorithm is expressed as a recurrence relation for set of monomials. The initial set corresponds to monomials in the polynomial construction which lists gracefully labeled functional directed graphs as follows

$$\text{Lm}_0 = \text{Lm}_{\text{lex(id)}} = \text{TermSet} \left( \det \{ \text{diag} (A \cdot 1_{n \times 1}) \} \mod \{ x_j^2 \}_{0 \leq j < n} \right),$$

18
and the recurrence relation is defined for any \( \sigma \in S_n \setminus \{ \text{id} \} \) by

\[
L_{\text{lex}}(\sigma) = L_{\text{lex}}(\sigma -1) \cup \text{TermSet} \left( \sum_{0 \leq i < n} A[i, i] \prod_{0 \leq i < n} A[i, f(i)] \right).
\]

Consequently, the set

\[
L_{n!-1} = L_{\text{lex}(n-1 - \text{id})} = \bigcup_{\sigma \in S_n} \text{TermSet} \left( \sum_{0 \leq i < n} A[i, i] \det \left( \left( \text{diag}(A1_{n \times 1}) - A \right) \begin{bmatrix} 0, \ldots, i-1, i+1, \ldots, n-1 \\ 0, \ldots, i-1, i+1, \ldots, n-1 \end{bmatrix} \right) \right) \mod \{ x_j^2 \}_{0 \leq j < n}.
\]

and the recurrence prescribed for any \( \sigma \in S_n \setminus \{ \text{id} \} \) by

\[
L_{\text{lex}}(\sigma) = L_{\text{lex}}(\sigma -1) \cup \text{TermSet} \left( \sum_{0 \leq i < n} A[i, i] \prod_{0 \leq i < n} A[i, f(i)] \right).
\]

Consequently, the set

\[
L_{n!-1} = L_{\text{lex}(n-1 - \text{id})} = \bigcup_{\sigma \in S_n} \text{TermSet} \left( \sum_{0 \leq i < n} A[i, i] \prod_{0 \leq i < n} A[i, \sigma f^{-1}(i)] \right)
\]

list all graceful trees. The GLC is thus equivalent to the assertion that

\[
L_{n!-1} = \text{TermSet} \left( \sum_{0 \leq i < n} A[i, i] \det \left( \left( \text{diag}(A1_{n \times 1}) - A \right) \begin{bmatrix} 0, \ldots, i-1, i+1, \ldots, n-1 \\ 0, \ldots, i-1, i+1, \ldots, n-1 \end{bmatrix} \right) \right).
\]

By construction, \( L_{n!-1} \) is invariant under conjugation in the sense that

\[
\forall \sigma \in S_n, \ L_{n!-1} = \text{TermSet} \left( \sum_{0 \leq i < n} A[i, i] \prod_{0 \leq i < n} A[i, \sigma f^{-1}(i)] \right).
\]

The following theorem describes a construction for listing gracefully labeled functional directed trees rooted at 0.

**Theorem 7**: Let \( A \) denote a symbolically weighted adjacency matrix then
is more directly achieved by exploiting the fact that every vertex has outgoing degree equal to one as follows

\[ \sum_{g \in SP_n, 0 \leq i < n} \prod_{0 \leq i < n} A[i, i + g(i)] = \]

\[ \sum_{f \in \{0, n\} \cap \mathbb{Z}} \prod_{0 \leq i < n} \left( A[n - 1 - f(i), i - f(i)]^{1 - k_i} A[i - f(i), n - 1 - f(i)]^{k_i} \right) \]  

(10)

**Proof:** Let \( A \) denote the adjacency matrix of a symbolically weighted complete graph on \( n \) vertices (allowing for loop edges) whose entries are specified by

\[ A[i, j] = a_{ij} x_{j-i}, \quad \forall 0 \leq i, j < n, \]

we have

\[ \text{det} \left( \text{diag} \left( A \cdot 1_{n \times 1} \right) \right) \mod \{ x_j^2 \}_{0 \leq j < n} = \sum_{f \in \{0, n\} \cap \mathbb{Z}} \prod_{0 \leq i < n} A[i, f(i)] \]

\[ |\{ f(i) - j : i \in [0, n] \cap \mathbb{Z} \} | = n \]

The expression is simplified by removing the absolute value in the indexing by considering the following variant of the symbolic edge weights

\[ A[i, j] = a_{ij} x_j, \quad \forall 0 \leq i, j < n, \]

where

\[ \frac{1}{2} \prod_{0 \leq i < n} \sum_{0 \leq j \leq i} A[i - j, n - 1 - j] + A[n - 1 - j, i - j] \mod \{ x_k^2 \}_{0 \leq k < n} = \]

\[ \sum_{f \in \{0, n\} \cap \mathbb{Z}} \prod_{0 \leq i < n} A[i, f(i)] \cdot \]

\[ |\{ f(i) - j : i \in [0, n] \cap \mathbb{Z} \} | = n \]

The expression above list the set of all gracefully labeled functional directed graphs. Note that reducing the polynomial above modulo the algebraic relations

\[ \{ x_k^2 \equiv 0 : k \in [0, n) \cap \mathbb{Z} \}, \]

is more directly achieved by exploiting the fact that every vertex has outgoing degree equal to one as follows

\[ \frac{1}{2} \sum_{f \in \{0, n\} \cap \mathbb{Z}} \prod_{0 \leq i < n} A[n - 1 - f(i), i - f(i)]^{1 - k_i} A[i - f(i), n - 1 - f(i)]^{k_i} \]

\[ \forall j \in [0, n) \cap \mathbb{Z} \setminus \{ n - 1 \}, f(j) \leq j \]

\[ \prod_{0 \leq j < n} x_j = \prod_{0 \leq j < n} x_j^{f(j)} \sum_{0 \leq j < n} x_j^{1 - k_j} \]

\[ k_j \in \{0, 1\} \]

The desired result by restricting the listing to gracefully labeled graphs rooted at 0 given by

\[ \sum_{g \in SP_n, 0 \leq i < n} \prod_{0 \leq i < n} A[i, i + g(i)] = \]
\[
\sum_{f^{n-1}([0,n) \cap \mathbb{Z}) = \{n-1\}} \prod_{0 \leq i < n-1} \left( A[i, i - f(i) \mod n]^{1 - k_i} A[i - f(i) \mod n, n - 1 - f(i)]^{k_i} \right). \\
\forall j \in [0,n) \cap \mathbb{Z}\setminus\{n-1\}, f(j) > j \\
\prod_{0 \leq j < n} x_j = \prod_{0 \leq j < n} x_j^{k_j} \prod_{0 \leq j < n} x_j^{1 - k_j} \\
k_j \in \{0, 1\}
\]

Thereby completing the proof.

As a corollary of theorem 7 it follows that

\[
\left( \sum_{g \in \text{SP}_n} \sum_{\sigma \in S_n/\text{Aut}(G_{id+p})} \prod_{0 \leq i < n} A[i, \sigma^{-1}(\sigma(i) + g\sigma(i))] \prod_{0 \leq i < n} A[i, f(i)] \right) = \sum_{f^{n-1}([0,n) \cap \mathbb{Z}) = \{0\}} \prod_{0 \leq i < n} A[i, f(i)]
\]

such that

\[
\sigma^{-1}(\sigma(j) + g\sigma(j)) \leq j, \forall j \in [0,n) \cap \mathbb{Z}
\]

where

\[
c_f = |\{\theta \in S_n/\text{Aut}(G_i) \text{ s.t. } \{[\theta f^{-1}(i) \mod n] : 0 < i < n\} = n - 1\}| \cdot |\{\theta \in S_n/\text{Aut}(G_i) \text{ s.t. } \theta f^{-1}(i) \leq i, 0 < i < n\}|.
\]

The GLC is therefore equivalent to the assertion that \(c_f > 0\) for all functions functional trees rooted at 0 where

\[
f(i) < i, \forall i \in [0,n) \cap \mathbb{Z}\setminus\{0\}.
\]

We conclude the paper by proposing a stronger form of the GLC.

**Conjecture 8**: Let \(A\) denote a symbolically weighted \(n \times n\) adjacency matrix for the directed complete graph on \(n\) vertices (which allows for loop edges), with entries given by

\[
A[i, j] = a_{ij}, \quad \forall \ 0 \leq i, j < n,
\]

then

\[
\left( \prod_{0 \leq i < n} A[i, f(i)] \right) \in \text{TermSet} \left\{ \sum_{0 \leq i < n} A[i, i] \det \left( \text{diag}(A_{1 \times 1}) - A \right) \left[ \begin{array}{c} 0, \ldots, i - 1, i + 1, \ldots, n - 1 \\ 0, \ldots, i - 1, i + 1, \ldots, n - 1 \end{array} \right] \right\}
\]

\[
\text{mod} \left( \begin{array}{c} A[i_{g(0)}, i_{g(1)}] A[i_{g(1)}, i_{g(2)}] A[i_{g(2)}, i_{g(3)}] A[i_{g(3)}, i_{g(0)}] \equiv 0 \\ A[i_{g(0)}, i_{g(1)}] A[i_{g(1)}, i_{g(2)}] A[i_{g(2)}, i_{g(3)}] A[i_{g(3)}, i_{g(2)}] \equiv 0 \end{array} \right) \left\{ \begin{array}{c} \{i_0, i_1, i_2, i_3\} = 4 \\ g \in S_4 \end{array} \right\}
\]

21
\[0 = \sum_{\sigma \in (\mathfrak{S}_n / \mathcal{I}_n) / \text{Aut} G_f} \sum_{g \in T_n} \prod_{\gamma \in (\mathfrak{S}_n / \mathcal{I}_n) / \text{Aut} G_g} \left( \prod_{0 \leq i < n} 2^n |\sigma_f^{-1}(i) - i| - \prod_{0 \leq j < n} 2^n |\gamma_g^{-1}(j) - j| \right).\]

In other words every induced edge label sequence of a functional star is common to every functional tree.

References

[AlS03] R.E.L. Aldred, J. Siran, M. Siran, A note on the number of graceful labelings of paths, Discrete Mathematics 261 (2003), 27-30
[Ad06] M. Adamaszek, Efficient enumeration of graceful permutations 2006, arXiv:math/0608513
[A010551] OEIS Foundation Inc. (2016), The On-Line Encyclopedia of Integer Sequences, http://oeis.org/A010551
[Gal05] J.A. Gallian, A dynamic survey of Graph Labeling, Electronic J. Comb. DS6 (2000), vol 6.
[GS80] R. L. Graham and N. J. A. Sloane, On additive bases and harmonious graphs, SIAM J. Alg. Discrete Meth., 1 (1980) 382-404.
[R64] G. Ringel, Problem 25, in Theory of Graphs and its Applications, Proc. Symposium Smolenice 1963, Prague (1964) 162.
[S18] W.A. Stein et al., Sage Mathematics Software (Version 8.3), The Sage Development Team, (2018) , http://www.sagemath.org.
[W08] Robin W. Whitty, Rook polynomials on two-dimensional surfaces and graceful labelings of graphs, In Discrete Mathematics, Volume 308, Issues 5-6, 2008, Pages 674-683
[Z85] Doron Zeilberger, A combinatorial approach to matrix algebra, Discrete Mathematics, Volume 56, Issue 1, 1985, Pages 61-72,