GENERALIZED SPRINGER THEORY FOR $D$-MODULES
ON A REDUCTIVE LIE ALGEBRA

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Abstract. Given a reductive group $G$, we give a description of the abelian category of $G$-equivariant $D$-modules on $\mathfrak{g} = \text{Lie}(G)$, which specializes to Lusztig’s generalized Springer correspondence upon restriction to the nilpotent cone. More precisely, the category has an orthogonal decomposition into blocks indexed by cuspidal data $(L,\mathcal{E})$, consisting of a Levi subgroup $L$, and a cuspidal local system $\mathcal{E}$ on a nilpotent $L$-orbit. Each block is equivalent to the category of $D$-modules on the center $z_{p_L}$ of $l$ which are equivariant for the action of the relative Weyl group $W_{G,L}/L$. The proof involves developing a theory of parabolic induction and restriction functors, and studying the corresponding monads acting on categories of cuspidal objects. It is hoped that the same techniques will be fruitful in understanding similar questions in the group, elliptic, mirabolic, quantum, and modular settings.

Main results. In his seminal paper [Lus84], Lusztig proved the Generalized Springer Correspondence, which gives a description of the category of $G$-equivariant perverse sheaves on the nilpotent cone $\mathcal{N}_G \subseteq \mathfrak{g} = \text{Lie}(G)$, for a reductive group $G$:

$$\text{Perv}_G(\mathcal{N}_G) \cong \bigoplus_{(L,\mathcal{E})} \text{Rep}(W_{G,L}).$$

The sum is indexed by cuspidal data: pairs $(L,\mathcal{E})$ of a Levi subgroup $L$ of $G$ and simple cuspidal local system on a nilpotent orbit for $L$, up to simultaneous conjugacy. For each such Levi $L$, $W_{G,L} = W_{G,L}/L$ denotes the corresponding relative Weyl group.

The main result of this paper is that Lusztig’s result extends to a description of the abelian category $\mathcal{M}(\mathfrak{g})^G$ of all $G$-equivariant $D$-modules on $\mathfrak{g}$:

**Theorem A.** There is an equivalence of abelian categories:

$$\mathcal{M}(\mathfrak{g})^G \cong \bigoplus_{(L,\mathcal{E})} \mathcal{M}(\mathfrak{z}(l))^{W_{G,L}},$$

where the sum is indexed by cuspidal data $(L,\mathcal{E})$.

Here $\mathfrak{z}(l)$ denotes the center of the Lie algebra $l$ of a Levi subgroup $L$ which carries an action of the finite group $W_{G,L}$, and $\mathcal{M}(\mathfrak{z}(l))^{W_{G,L}}$ denotes the category of $W_{G,L}$-equivariant $D$-modules on $\mathfrak{z}(l)$, or equivalently, modules for the semidirect product $\mathfrak{z}(l) \rtimes W_{G,L}$. If we restrict to the subcategory of modules with support on the nilpotent cone (which can be identified with the category of equivariant

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1In fact, in the cases when $L$ carries a cuspidal local system, $W_{G,L}$ is a Coxeter group and $\mathfrak{z}(l)$ its reflection representation.
perverse sheaves via the Riemann-Hilbert correspondence), the corresponding blocks restrict to the
category of representations of $W_{G,L}$, which recovers the Generalized Springer Correspondence.

**Example.** In the case $G = GL_n$, it is known that there is a unique cuspidal datum up to conjugacy,
corresponding to a maximal torus of $GL_n$. Thus we have:

$$M(gl_n)^{GL_n} \cong M(A^n)^{S_n} \cong \mathcal{D}_{A^n} \rtimes S_n - \text{mod}$$

**Remark.** The result for $GL_n$ can also be seen by using the functor of quantum Hamiltonian reduc-
tion, which, by the Harish-Chandra homomorphism of Levasseur-Stafford [LS95][LS96], naturally
takes values in $(\mathcal{D}_{A^n})^{S_n} - \text{mod} \cong \mathcal{D}_{A^n} \rtimes S_n - \text{mod}$. In general, quantum Hamiltonian reduction
will only see the Springer block (corresponding to the unique cuspidal datum associated with the
maximal torus).

The proof of Theorem A is based on the idea that the category $M(g)^G$ can be described in
terms of parabolic induction from cuspidal objects in the category $M(l)^L$ associated to a Levi
subgroup $L$ of $G$, where cuspidal objects are precisely those which themselves cannot be obtained
from parabolic induction from a smaller Levi subgroup. The blocks of $M(g)^G$ corresponding to
cuspidal data $(L,E)$, consist precisely of summands of parabolic induction from cuspidal objects
on $M(l)^L$. A priori, parabolic induction and restriction are a pair of adjoint triangulated functors
between the equivariant derived categories $D(g)^G$ and $D(l)^L$. In this paper, we prove:

**Theorem B.** Parabolic induction and restriction restrict to bi-adjoint exact functors between
abelian categories:

$$\text{ind}^G_L : M(l)^L \xleftarrow{\cong} M(g)^G, \quad \text{res}^G_L : M(g)^G \xrightarrow{\cong} M(l)^L$$

In addition, these functors are independent of the choice of parabolic subgroup $P$ containing $L$, and
satisfy a Mackey formula:

$$M_{\text{st}}(\mathfrak{M}) := \text{res}^G_L \text{ind}^G_L (\mathfrak{M}) \cong \bigoplus_{w \in W_L \backslash W_G / W_L} \text{ind}^M_{M \cap w L} \text{res}^w_{M \cap w L} \hat{w} \ast (\mathfrak{M})$$

When the object $\mathfrak{M}$ is cuspidal, the Mackey formula reduces to:

$$L_{\text{st}}(\mathfrak{M}) \cong \bigoplus_{w \in W_{G,L}} w \ast (\mathfrak{M}).$$

The core technical result of this paper states that the above isomorphism defines an isomorphism
of monads acting on the category of cuspidal objects of $M(l)^L$.

There is also a geometric characterization of the blocks corresponding to a fixed Levi $L$. The
singular support of a $G$-equivariant $D_g$-module gives rise to a closed subvariety of the variety
$\text{comm}(g)$ of commuting elements of $g$. We define a locally closed partition of $\text{comm}(g)$, indexed by
conjugacy classes of Levi subgroups $L$ of $G$:

$$\text{comm}(g) = \bigsqcup_{(L)} \text{comm}(g)_{\equiv (L)}.$$ 

In particular, we have a closed subset $\text{comm}(g)_{\equiv (L)}$ (respectively, $\text{comm}(g)_{\equiv (L)}$) given by the union of
$\text{comm}(g)_{\equiv (M)}$ for Levi subgroups $M$ which contain (respectively, properly contain) a conjugate of $L$.

**Theorem C.** Given a non-zero indecomposable object $\mathfrak{M} \in M(g)^G$, the following are equivalent:
The results can be interpreted as follows: the partition of $\text{comm}(g)$ gives rise to a decomposition of the category of $G$-equivariant $D$-modules (which, a priori, is only semiorthogonal) by considering the subcategories of objects with certain singular support with respect to the partition. Theorem $C$ implies that this agrees with the recollement situation given by parabolic induction and restriction functors, which in this case happens to be an orthogonal decomposition.

Remark. The category of character sheaves consists of objects in $M(g)^G$ whose singular support is contained in $g \times N_G$. For such objects, the singular support condition in Theorem $C$ reduces to a support condition with respect to the corresponding (Lusztig) partition of $g$.

Background. The study of equivariant differential equations and perverse sheaves on reductive groups (or Lie algebras) has a rich history. Given a complex reductive group $G$, Harish-Chandra showed that any invariant eigendistribution on (a real form of) $G$ satisfies a certain system of differential equations [HC64]. The Harish-Chandra systems were reinterpreted by Hotta and Kashiwara [HK84] using $D$-module theory, and explained the connection to Springer theory (the geometric construction of representations of the Weyl group), as developed by Springer [Spr78], Kazhdan-Lusztig [KL80], and Borho–MacPherson [BM83]. In his seminal paper [Lus84] Lusztig defined and classified cuspidal local systems and proved the generalized Springer Correspondence, which classifies equivariant perverse sheaves on the unipotent cone of $G$. He then went on to develop the theory of character sheaves, leading to spectacular applications to the representation theory of finite groups of Lie type. These ideas were recast in the $D$-module setting by Ginzburg [Gin89, Gin93] and Mirkovic [Mir04] leading to many simplifications. More recent work of Rider and Russell explore the derived direction of generalized Springer theory [Ach11, Rid13, RR14].

One aspect of the work presented here which is fundamentally different from that of previous authors, is that our results concern all equivariant $D$-modules, not just character sheaves (in particular, not just holonomic $D$-modules). The theory of character sheaves (as reformulated by Ginzburg) is about equivariant $D$-modules on $G$ for which the Harish-Chandra center $Z = (\mathcal{D}_G)^{G \times G}$ acts locally finitely; thus character sheaves are discrete over the space $\text{Spec}(Z)$ of central characters - character sheaves with incompatible central characters don’t “talk” to each other. This becomes even clearer in the Lie algebra setting, where character sheaves can just be defined as the Fourier transforms of orbital $D$-modules (i.e. those with support on finitely many orbits). Many of our results in this paper have well-known analogues in the theory of character (or orbital) sheaves. For example, there are a variety of different proofs of exactness of parabolic induction and restriction (Theorem $B$) in the setting of character and orbital sheaves (see e.g. Lusztig [Lus85], Ginzburg [Gin93], Mirkovic [Mir04], Achar [Ach11]); however, each of these proofs makes essential use of the character sheaf restriction, so they do not lead to a proof of Theorem $B$ (as far as the author is aware, the idea behind our proof is essentially new).

The change in focus from character sheaves to all equivariant $D$-modules is conceptually important as it allows one to do harmonic analysis on the category $M(g)^G$. For example, one might want to try to formulate a Plancherel theorem: express an arbitrary equivariant $D$-module as a direct integral of character sheaves.
The setting of $G$-equivariant $D$-modules on $\mathfrak{g}$ is, in a certain sense, the simplest example in a family of settings:

- **The mirabolic setting** Let $X = \mathfrak{gl}_n \times \mathbb{C}^n$. The group $GL_n$ acts on $X$ by the adjoint action on the first factor, and the standard representation on the second factor. We consider the category of $GL_n$-equivariant $D$-modules on $X$; more generally, we can consider the category of $c$-monodromic $D$-modules for any $c \in \mathbb{C}$. There is a well known relationship between such $D$-modules and modules for the spherical subalgebra of the rational Cherednik algebra (with parameter $c$); these ideas have been the subject of considerable interest, notably in the work of Ginzburg with Bellamy, Etingof, Finkelberg, and Gan [BG15, EG02, FG10, GG06]. Our results in this paper suggest a new approach to this topic via parabolic induction and restriction functors. Work of McGerty and Nevins [MN14] describes a recollement situation associated to a certain stratification of the cotangent bundle $T^*X$; it seems natural to construct this recollement from such induction and restriction functors.

- **The group setting.** Most of the results in this paper carry over to the category of $G$-equivariant $D$-modules on the group $G$. This topic will be addressed in a sequel to this paper. One reason that this setting is more interesting is that this category appears as the value assigned to a circle by a certain topological field theory (TFT) $Z_G$. This TFT is constructed and studied in work of the author with David Ben-Zvi and David Nadler [BZGN17] (based on earlier work of Ben-Zvi and Nadler [BZN09]), where we show that the value of $Z_G$ on an oriented surface $\Sigma$ is the Borel-Moore homology of the Betti moduli stack of $G$-local systems on $\Sigma$. The cohomology (and Hodge theory) of closely related varieties are the subject of conjectures of Hausel, Letellier, and Rodriguez-Villegas [HRV08, HLRV11].

- **The quantum setting** The ring of quantum differential operators, $D^q_G$, is a certain deformation of the ring of functions on $G \times G$. It has been studied extensively by Semenov-Tyan-Shanski, Alekseev, Backelin–Kremnizer, and Jordan [Ale93, BK06, STS94, Jor09]. In the beautiful papers [BZBJ15, BZBJ16] of Ben-Zvi–Brochier–Jordan it was shown that $D^q_G$ (or more precisely its category of equivariant modules $M^q_G$) is obtained as the factorization homology on a genus one curve with coefficients in the category $\text{Rep}^q(G)$ of representations of the quantum group.

- **The elliptic setting.** Let $G_E$ be the moduli of degree 0, semistable $G$-bundles on an elliptic curve $E$, together with a framing at the identity element of $E$; this is a smooth variety with an action of $G$, changing the framing. The $G$-equivariant geometry of $G_E$ is closely related to the geometry of $G$ and $\mathfrak{g}$ (which can be defined in the same way as $G_E$, but replacing $E$ with a nodal or cusp curve). The category of $G$-equivariant $D$-modules on $G_E$ is the home of elliptic character sheaves. The starting point of the study of vector bundles on elliptic curves was the work of Atiyah [Ati57], with subsequent work by Friedman–Morgan–Witten [FMW98], and Baranovsky–Ginzburg (later with Evens) [BG96, BEG03]. More recent work in this direction has been done by Ben-Zvi–Nadler [BZN15], Li–Nadler [LN15], and Fratila [Frat10]. Elliptic character sheaves can be thought of as a model for character sheaves for the loop group $LG$. Thus, the study of generalized Springer theory in this setting is relevant both to local and global (over $E$) versions of geometric Langlands.

Outline of the paper.
In Section 1, we define derived functors of parabolic induction and restriction, and show that they satisfy a general form of the Mackey theorem. In Section 2, we study certain partitions of \( g \) and the variety of commuting elements \( \text{comm}(g) \) indexed by Levi subgroups \( L \) of \( g \). In Section 3, we show that the parabolic induction and restriction functors restrict to exact functors on the level of abelian categories. This is essentially proved by observing that, generically parabolic induction and restriction are controlled by direct and inverse image under an étale map, and the behavior over the non-regular locus is controlled by the key observation that the non-regular locus of \( \text{comm}(l) \) cannot contain the singular support of a non-trivial \( D \)-module. We characterize those objects which are killed by parabolic restriction, by their singular support with respect to the partition of \( \text{comm}(g) \) introduced in Section 2. In Section 4, we show that the monad associated to the induction/restriction adjunction acting on a block of cuspidal objects is described by the action of the relative Weyl group. The monad can be thought of as a relative version of the Borel-Moore homology of the Steinberg variety with its convolution operation. As in other incarnations of Springer theory, we first consider the restriction of the monad to the regular locus and then extend. In particular, this gives a new proof of the classical results of Springer theory. This allows us to deduce Theorem A. The existence of a recollement situation may be already deduced from the theory of induction and restriction; the fact that the recollement is split follows from the semisimplicity of the group algebra of the Weyl group. In Appendix A, we give an overview of some of the category theory required in this paper. In particular, we explain how the Barr-Beck theorem allows one to upgrade certain adjunctions to recollement situations. In Appendix B, we gather some general results on \( D \)-modules.

**Notation.** The following overview of notational conventions may be helpful when reading this paper.

- The following is a summary of the notation for \( D \)-modules (further details can be found in Appendix B). The abelian category of (all) \( D \)-modules on a smooth variety or stack \( X \) is denoted \( M(X) \); the (unbounded) derived category is denoted \( D(X) \). Thus if \( U \) is a smooth algebraic variety with an action of an affine algebraic group \( K \), and \( X = U/K \) is the quotient stack, we write \( M(X) \) or \( M(U)^K \) for the abelian category of \( K \)-equivariant \( D \)-modules on \( U \). The equivariant derived category \( D(X) = D(U)^K \) is a \( t \)-category with heart \( M(X) \) (which is not equivalent to the derived category of \( M(X) \) in general). If \( X \) is not smooth, then \( M(X) \) means the subcategory of \( D \)-modules on some smooth ambient variety or stack with support on \( X \).
- Throughout the paper, \( G \) always refers to a complex reductive group \( P \) and \( Q \) are parabolic subgroups with unipotent radicals \( U \) and \( V \) respectively, and \( L = P/U, M = Q/V \) are the

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In this paper it will be convenient to make use of the categorical language of monads as a convenient tool. However, in our setting, the categories on which the monads act can all be represented as modules for an algebra. In this case, the notion of a monad is very concrete: a (colimit preserving) monad acting on the category \( R - \text{mod} \) for a ring \( R \) is an algebra object in the monoidal category of \( R \)-bimodules. This is the same thing as an \( R \)-ring, i.e., a ring \( A \) together with a (not necessarily central) ring map \( R \to A \). A module for the monad \( A \) in \( R - \text{mod} \) is nothing more than an \( A \)-module.
Levi factors. Frequently, we will consider splittings of $L$ and $M$ as subgroups of $G$, and we usually want them to contain a common maximal torus. Lie algebras are denoted by fraktur letter $\mathfrak{g}, \mathfrak{p}, \mathfrak{q}$, etc. as usual.

- We use the notation $\mathfrak{g}$ to refer to the quotient stack $\mathfrak{g}/G$. Thus $\mathbf{M}(\mathfrak{g})$ means the same thing as $\mathbf{M}(\mathfrak{g})^G$. Similarly, we have $\mathfrak{p}, \mathfrak{l}$ etc. This, and further notation for related stacks (e.g. the Steinberg stack $\mathcal{O}_{\mathfrak{g} G}$) is introduced in Subsection 1.1.
- Functors of parabolic induction and restriction (introduced in Subsection 1.2) will be denoted by $\text{Ind}^G_{P,L}$ and $\text{Res}^G_{P,L}$ in the derived category context, and $\text{ind}^G_{P,L}$ and $\text{res}^G_{P,L}$ in the abelian category context (once we have deduced that they are $t$-exact). The Steinberg functors are the composites of parabolic induction with restriction, and are denoted by $\text{St}$ (in the derived setting) or $\text{st}$ (in the abelian setting).
- Given an element $x \in \mathfrak{g}$, we write $H(x) = H_G(x)$ for the centralizer $C_G(x)$ of the semisimple part of $x$ with respect to the Jordan decomposition. This is a Levi subgroup of $G$.
- We frequently work with the poset $\text{Levi}_G$ of Levi subgroups of $G$ up to conjugacy, ordered by inclusion. Thus $(M) \leq (L)$ means that some conjugate of $M$ is contained in $L$.
- We use a superscript on the left to denote the adjoint action or conjugation. Thus $^g x$ means $\text{Ad}(g)(x)$ (for some $g \in G, x \in \mathfrak{g}$), and if $P$ is a subgroup of $G$, then $^g P$ means $gP g^{-1}$.

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1. Mackey Theory

In this section we will define the functors of parabolic induction and restriction between categories of $D$-modules (or constructible sheaves), and define the Mackey filtration in that setting.

1.1. The adjoint quotient and Steinberg stacks. Throughout this subsection, we will fix a connected reductive algebraic group $G$, and parabolic subgroups $P$ and $Q$ of $G$. We will denote by $U$ and $V$ the unipotent radicals of $P$ and $Q$, and the Levi quotients will be denoted $L = P/U$, $M = Q/V$, respectively. The corresponding Lie algebras will be denoted by lower case fraktur letters as usual; thus the Lie algebras of $G, P, Q, U, V, L, M$ shall be denoted $\mathfrak{g}, \mathfrak{p}, \mathfrak{q}, \mathfrak{u}, \mathfrak{v}, \mathfrak{l}, \mathfrak{m}$ respectively.

Recall that an algebraic group acts on its Lie algebra by the adjoint action. For ease of reading we will denote the adjoint quotient stacks with an underline as follows: $\underline{\mathfrak{g}}/G = \underline{\mathfrak{g}}$, $\underline{\mathfrak{p}}/P = \underline{\mathfrak{p}}$, $\underline{\mathfrak{l}}/L = \underline{\mathfrak{l}}$, etc. We have a diagram of stacks

\[
\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{r} & \mathfrak{p} \\
& & \xrightarrow{s} \mathfrak{l}
\end{array}
\]
Of course, there is an analogous diagram involving \( q \) and \( m \). The fiber product \( q \times_m p \) will be denoted by \( \mathcal{Q} \mathfrak{g} \mathcal{P} \) and referred to as the Steinberg stack. It is equipped with projections

\[
\begin{array}{ccc}
m & \leftarrow & \mathcal{Q} \mathfrak{g} \mathcal{P} \\
\alpha & & \beta \\
\mathcal{P} & \rightarrow & \mathfrak{l}
\end{array}
\]

Explicitly we may write the Steinberg stack as a quotient

\[
\left\{(x, g) \in \mathfrak{g} \times G \mid x \in q \cap g^{-1} p\right\}/(Q \times P),
\]

where the \((q, p) \in Q \times P\) acts by sending \((x, g)\) to \((q x, q g p^{-1})\). In this realization, the morphisms \( \alpha \) and \( \beta \) are given by \( \alpha(x, q) = (x + v) \in m \) and \( \beta(x, q) = (q x + u) \in \mathfrak{l} \). The Steinberg stack is stratified by the (finitely many) orbits of \( Q \times P \) on \( G \) and all of the strata have the same dimension. For each orbit \( w \) in \( Q_\mathcal{P} \), we denote by \( \mathcal{Q} \mathfrak{g} \mathcal{P}_w \) the corresponding strata in \( \mathcal{Q} \mathfrak{g} \mathcal{P} \).

**Lemma 1.1.** Given any lift \( \tilde{w} \in G \) of \( w \), we have an equivalence of stacks

\[
\mathcal{Q} \mathfrak{g} \mathcal{P}_w \simeq (q \cap \tilde{w} p)_{ad}(Q \cap \tilde{w} P).
\]

**Remark 1.2.** The stack \( \mathcal{Q} \mathfrak{g} \mathcal{P} \) is the bundle of Lie algebras associated to the inertia stack of \( Q_\mathcal{P} \); the equidimensionality of the stratification may be seen as a consequence of the orbit stabilizer theorem.

**Remark 1.3.** The morphism \( s : \mathfrak{p} \rightarrow \mathfrak{l} \) is not representable, however it is safe, in the sense of Appendix B.3. Note that \( s \) factorizes as \( \mathfrak{p} \leftarrow \mathfrak{p}/P \rightarrow \mathfrak{t}/P \rightarrow \mathfrak{t}/L = \mathfrak{l} \).

The first morphism is representable, and the second morphism gives rise to a derived equivalence of \( D \)-modules (but the \( t \)-structure is shifted). The benefit of working with such non-representable morphisms is that the shifts that usually appear in the definitions of parabolic induction and restriction are naturally encoded in our definition.

To better understand Diagram 1 and the Steinberg stack, let us consider the following spaces. The \( P \)-flag variety, \( \mathcal{F} \ell \mathcal{P} \approx G/P \), is defined to be the collection of conjugates of \( p \) in \( \mathfrak{g} \) with its natural structure of a projective algebraic variety. The flag variety carries a tautological bundle of Lie algebras,

\[
\tilde{\mathfrak{g}}_P = \{(x, p') \in \mathfrak{g} \times \mathcal{F} \ell \mathcal{P} \mid x \in p'\} \rightarrow \mathcal{F} \ell \mathcal{P}.
\]

The bundle of Lie algebras \( \tilde{\mathfrak{g}}_P \) has a naturally defined ideal \( \tilde{\mathfrak{g}}_U \) (whose fibers are conjugates of \( \mathfrak{u} \)) and corresponding quotient \( \tilde{\mathfrak{g}}_L \). The group \( G \) acts on \( \tilde{\mathfrak{g}}_P \) and \( \tilde{\mathfrak{g}}_L \) and we have natural identifications \( \tilde{\mathfrak{g}}_P/G = \mathfrak{p}/P = \mathfrak{p} \) and \( \tilde{\mathfrak{g}}_L/G = \mathfrak{t}/ad P \).

Now we can reinterpret Diagram 1 as follows:

\[
\begin{array}{ccc}
\mathfrak{g}/G & \leftarrow & \tilde{\mathfrak{g}}_P/G \\
\sigma' & & \sigma'' \\
\mathfrak{t}/l & \rightarrow & \mathfrak{l}/\mathfrak{p}/\mathfrak{t} & \mathfrak{t}/L/G
\end{array}
\]

The morphism \( \sigma' \) is representable and smooth of relative dimension \( \dim(\mathfrak{u}) \), whereas \( \sigma'' \) is smooth of relative dimension \( \dim(\mathfrak{u}) \). Thus the morphism \( s \) is smooth of relative dimension 0. Note that \( \sigma'' \) induces an equivalence on the category of \( D \)-modules, which preserves the \( t \)-structures up to a
shift of \( \dim(u) \). Thus the top row of the diagram induces the same functor on \( D \)-modules up to a shift.

The Steinberg stack may be written in terms of the varieties \( \tilde{\mathfrak{g}}_Q \) and \( \tilde{\mathfrak{g}}_P \) as follows:

\[
Q\mathfrak{g}_P = (\tilde{\mathfrak{g}}_Q \times_{\mathfrak{g}} \tilde{\mathfrak{g}}_P) \backslash G = \{(x, q', p') \in \mathfrak{g} \times F\ell_Q \times F\ell_P \mid x \in p' \cap q'\} / G.
\]

1.2. Parabolic induction and restriction. The functors of parabolic induction and restriction are defined by:

\[
r_*s^! \colon \text{Ind}_{P,L}^G : \text{D}(\mathfrak{l}) \longrightarrow \text{D}(\mathfrak{g}) : \text{Res}_{P,L}^G = s_*r^!
\]

where the notation is as in Diagram 1 above. The morphism \( r \) is proper, and \( s \) is smooth of relative dimension 0, thus \( \text{Ind}_{P,L}^G \) is left adjoint to \( \text{Res}_{P,L}^G \).

**Example 1.4.** Let us make the definitions a little bit more explicit in the case in the case \( P = B \) (a Borel subgroup) and \( H = B/N \) (the canonical Cartan). Parabolic induction of an object \( \mathfrak{M} \in \mathfrak{M}(\mathfrak{h}) \) is just given by

\[
\text{Ind}_{B,H}^G (\mathfrak{M}) = (\tilde{\mathfrak{g}} \to \mathfrak{g})_* (\tilde{\mathfrak{g}} \to h)^! (\mathfrak{M})[-\dim(n)]
\]

On the other hand, given an object \( \mathfrak{N} \in \mathfrak{M}(\mathfrak{g}) \) (i.e. a \( G \)-equivariant \( \mathfrak{D}_g \)-module), parabolic restriction (forgetting about equivariance for now) is given by a derived tensor product:

\[
\text{Res}_{B,H}^G (\mathfrak{N}) \simeq \mathfrak{N} \bigotimes_{\text{Sym}(n) \otimes \text{Sym}(n)} \mathbb{C}[-\dim(n)].
\]

Here, we identify \( \mathfrak{g}^* \) with \( \mathfrak{g} \) using an invariant form, so that \( \mathcal{O}_\mathfrak{g} = \text{Sym}(\mathfrak{g}^*) \) becomes identified with \( \text{Sym}(\mathfrak{g}) \). Then \( \text{Sym}(n) \otimes \text{Sym}(n) \) is a commutative subalgebra of \( \mathfrak{D}_\mathfrak{g} \) (the first factor of \( \text{Sym}(n) \) is really functions on \( n^- \), and the second is constant coefficient differential operators on \( n \)), and so the relative tensor product above is computed by a Koszul complex. Later, we will see that the functor \( \text{Res}_{B,L}^G \) is \( t \)-exact, which due to the shifts, means that the Koszul complex is non-zero only in the middle degree!

We define the functor

\[
\text{St} = \text{Ind}_{P,L}^G : \text{D}(\mathfrak{l}) \to \text{D}(\mathfrak{m})
\]

(we will often drop the subscripts and/or superscripts when the context is clear). By base change, we have \( \text{St} \simeq \alpha_* \beta^! \), where:

Recall that the stack \( Q\mathfrak{g}_P \) has a stratification indexed by double cosets \( Q \backslash G / P \), with strata \( Q\mathfrak{g}_P^{\alpha \beta} \).

Thus the functor \( \text{St} \) has a filtration (in the sense of Definition A.9) indexed by the poset \( Q\backslash G / P \) (we will refer to this filtration as the Mackey filtration).
1.3. The Mackey Formula. For each $g \in G$, consider the conjugate parabolic $gP$ in $G$ and its Levi quotient $gL$. The image of $Q \cap gP$ in $M$ is a parabolic subgroup of $M$ which we will denote $M \cap gP$. The corresponding Levi quotient will be denoted $M \cap gL$. Similarly, the image of $Q \cap gP$ in $gL$ is a parabolic subgroup, denoted $Q \cap gP$; whose Levi subgroup is canonically identified with $M \cap gL$. Analogous notation will be adopted for the corresponding Lie algebras. The conjugation morphism $g(\cdot) : \mathbf{1} \to g\mathbf{1}$ gives rise to an equivalence $g_\# : \text{D}(\mathbf{1}) \to \text{D}(gL)$.

Remark 1.5. If we choose Levi splittings of $L$ in $P$ and $M$ in $Q$ such that $M \cap gL$ contains a maximal torus of $G$, then the notation above may be interpreted literally.

Proposition 1.6 (Mackey Formula). For each lift $\hat{w} \in G$ of $w \in Q \backslash G/P$, there is an equivalence

$$\text{St}_w \simeq \text{Ind}^M_{M \cap wP, M \cap wL} \text{Res}^{gL}_{Q \cap wL, M \cap wL} \hat{w}_\#: \text{D}(\mathbf{1}) \to \text{D}(m).$$

Proof. Recall that $Q \text{Ad}^w P \simeq (q \cap wP)_{/\text{ad}}(Q \cap wP)$. Consider the commutative diagram:

\[\text{(2)}\]

\[
\begin{array}{ccccccc}
\text{St}_w & \xrightarrow{\alpha_w} & \text{Ind}^M_{M \cap wP, M \cap wL} \text{Res}^{gL}_{Q \cap wL, M \cap wL} \hat{w}_\#: \text{D}(\mathbf{1}) & \xrightarrow{\beta_w} & \text{D}(m) \\
\text{m} & \xrightarrow{\text{Ad}} & \text{m} & \xrightarrow{\text{Ad}} & \text{m} & \xrightarrow{\text{Ad}} & \text{m} \\
\text{P} & \xrightarrow{\text{Ad}} & \text{P} & \xrightarrow{\text{Ad}} & \text{P} & \xrightarrow{\text{Ad}} & \text{P} \\
\text{Q} & \xrightarrow{\text{Ad}} & \text{Q} & \xrightarrow{\text{Ad}} & \text{Q} & \xrightarrow{\text{Ad}} & \text{Q} \\
\end{array}
\]

Traversing the correspondence along the top of the diagram (from right to left) gives the functor $\text{St}_w$, whereas traversing the two correspondences along the very bottom of the diagram gives the composite $\text{Ind}^M_{M \cap wP, M \cap wL} \circ \text{Res}^{gL}_{Q \cap wL, M \cap wL} \circ \hat{w}_\#$. The three squares marked $\Diamond$ are all cartesian. The lowest square is not cartesian, however it is still true that the two functors $\text{D}(m \cap wP)^{Q \cap wL} \to \text{D}(m \cap wP)$ obtained by either traversing the top or bottom of this square are naturally isomorphic (the base-change morphism is an isomorphism). Indeed, the stacks only differ by trivial actions of unipotent groups, thus all the $D$-module functors in the square are equivalences, differing at worst by a cohomological shift. The fact that the shift is trivial follows just from the commutativity of the diagram. Thus the entire diagram behaves as if it were a base change diagram as required. 

\[\square\]

1.4. Compatibility under repeated induction or restriction. Suppose $P'$ is a parabolic subgroup of $G$ which contains $P$. Then $L' := P'/U'$ has a parabolic subgroup $L' \cap P$ and the corresponding Levi quotient is canonically isomorphic to $L$ (one can either choose a splitting of $L'$ in $P'$, or just define $L' \cap P$ to be the image of $P$ in $L'$).
Proposition 1.7. The functors of parabolic induction and restriction factor as follows:

\[
\text{Res}^G_{P,L} \simeq \text{Res}^L_{P \cap L',L} \text{Res}^{G'}_{P',L'}
\]
\[
\text{Ind}^G_{P,L} \simeq \text{Ind}^G_{P',L'} \text{Ind}^{L'}_{P \cap L',L'}
\]

Proof. The appropriate diagram is now:

As in the proof of Proposition 1.6, the middle square is cartesian modulo unipotent gerbes. Thus the result follows from base change.

□

2. Geometry of the adjoint quotient and commuting variety

In this section, we will study certain loci of \( g \) and the variety \( \text{comm}(g) \) of commuting elements of \( g \). In the case \( G = GL_n \), these loci record linear algebraic data concerning coincidences amongst the eigenvalues of matrices and simultaneous eigenvalues of commuting matrices.

2.1. The Lusztig partition of \( g \). For each \( x \in g \), let \( H(x) = H_G(x) = C_G(x) \). This is a Levi subgroup of \( G \).

Definition 2.1. For each Levi subgroup \( L \) of \( G \), we write:

\[
\mathfrak{g}(L) = \{ x \in \mathfrak{g} \mid H_G(x) \text{ is conjugate to } L \}
\]

We will write \( \mathfrak{g}^{\diamond} \) for the subset \( \mathfrak{g}(G) \); this is referred to as the cuspidal locus. It consists of elements \( x \) whose semisimple part is central in \( \mathfrak{g} \). In other words, \( \mathfrak{g}^{\diamond} = \mathfrak{z}(\mathfrak{g}) + \mathcal{N}_G \). At the other extreme, \( \mathfrak{g}(H) = \mathfrak{g}_r \) is precisely the subset of regular semisimple elements of \( \mathfrak{g} \).

These subsets define a partition of \( \mathfrak{g} \) in to \( G \)-invariant, locally closed subsets, indexed by conjugacy classes of Levi subgroups. More precisely, there is a continuous map

\[
\mathfrak{g} \to \mathcal{L}_{\text{evi}}^G
\]

\[
x \mapsto (H(x))
\]

where \( \mathcal{L}_{\text{evi}}^G \) is the poset of conjugacy classes of Levi subgroups, and \((L) \preceq (M)\) means that some conjugate of \( L \) is contained in \( M \). We will use the notation \( \mathfrak{g}_{(L)}, \mathfrak{g}_{> (L)}, \) etc. to denote the corresponding subsets of \( \mathfrak{g} \).

Example 2.2. In the case \( G = GL_n \), conjugacy classes of Levi subgroups are in one-to-one correspondence with partitions of \( n \). Given a partition \( p = (p_1, \ldots, p_k) \), we may take \( L_p = GL_{p_1} \times \cdots \times GL_{p_k} \). Then \( \mathfrak{g}_{(L_p)} \) consists of matrices with \( k \) distinct eigenvalues \( \lambda_1, \ldots, \lambda_k \) such that the dimension of the generalized eigenspace corresponding to \( \lambda_i \) is \( p_i \).
Note that the stratification of $\g$ is $G$-invariant, and the question of which subset an element $x \in \g$ belongs to depends only on the semisimple part of $x$ (for example in Example 2.2 for $GL_n$, we saw that the partition depends only on the eigenvalues). Thus we are really just defining a partition $\mathcal{c} = \cup \mathcal{c}_\ell$ of the space $\mathcal{c} = \g//G$ of semisimple conjugacy classes, such that $\mathfrak{g}_\mathcal{c}$ is the preimage of $\mathcal{c}_\ell$ under the characteristic polynomial map $\chi : \g \to \mathcal{c}$. This partition can be described in explicit linear algebraic (or combinatorial) terms as follows.

Let $H$ be a maximal torus of $G$ with Weyl group $W$. There are finitely many Levi subgroups of $G$ which contain $H$: they correspond bijectively with certain root subsystems of the corresponding root system of $(G, H)$ (they are precisely those generated by a subset of simple roots for some choice of polarization of the root system). For a Levi subgroup $L$ containing $H$, consider the linear subspace $\mathfrak{z}(l) \subseteq \mathfrak{h}$. The maximal such subspaces (corresponding to the minimal Levi subgroups) are the root hyperplanes of $\mathfrak{h}$, and more general subspaces of the form $\mathfrak{z}(l)$ are intersections of root hyperplanes. The stratification of $\mathcal{c}$ can be described in terms of the images of these subspaces under the quotient map $\pi : \mathfrak{h} \to \mathcal{c} = \mathfrak{h}/W$. Given an element of $x \in \mathfrak{g}_\mathcal{c}$, we can conjugate so that $x_s \in \mathfrak{z}(l)$; thence $\chi(x) = \chi(x_s) = \pi(x_s)$. Thus we have a commuting diagram:

\[
\begin{array}{ccc}
\mathfrak{h} & \to & \mathcal{c} \\
\downarrow & & \downarrow \\
\mathfrak{z}(l) & \to & \mathfrak{g}_\mathcal{c}
\end{array}
\]

(Note that the left hand square is not cartesian in general: $\pi^{-1}(\mathcal{c}_{\mathcal{c}_\ell})$ consists of the union of $W$-translates of $\mathfrak{z}(l)$.) Similarly, $\mathcal{c}_{\mathcal{c}_\ell}$ is the image under $\pi$ of the union of $\mathfrak{z}(l')$ for Levi subgroups $L'$ which contain $L$. Using these ideas, we observe the following:

**Proposition 2.3.** The closure of $\mathfrak{g}_\mathcal{c}$ is $\mathfrak{g}_{\mathcal{c}_\ell}$.

**Proof.** As the map $\chi : \g \to \mathcal{c}$ is open, we have that the preimage $\chi^{-1}$ commutes with the operation of taking closure. Thus it is sufficient to check that $\mathcal{c}_{\mathcal{c}_\ell}$ is the closure of $\mathcal{c}_\ell$. Moreover, as the map $\pi : \mathfrak{h} \to \mathcal{c}$ is closed, we have that the operation of taking the image under $\pi$ commutes with closure. Thus we are reduced to showing that $\mathfrak{z}(l)$ is the closure of the subvariety give by the complement of the union of proper linear subspaces $\mathfrak{z}(l')$. But the complement of a proper linear subspace in $\mathfrak{z}(l)$ is open and dense, and thus so is a finite intersection of such, which shows the required closure property in $\mathfrak{h}$. $\Box$

2.2. The regular locus in a Levi. Closely related to the Lusztig partition is the notion of regularity.

**Definition 2.4.** Let $L$ be a Levi subgroup of $G$. An element $x \in \mathfrak{l}$ is called regular (or $G$-regular) if $H_G(x) \subseteq L$. The set of regular elements in $\mathfrak{l}$ is denoted $\mathfrak{l}^G_{\text{reg}}$ or just $\mathfrak{l}^\text{reg}$. It’s complement will be denoted $\mathfrak{l}^{\text{nonreg}}$.

The intersection of the $G$-regular locus of $\mathfrak{l}$ with the cuspidal locus of $\mathfrak{l}$ is given by

$$\mathfrak{l}^\text{reg} : = \mathfrak{l}^\text{reg} \cap \mathfrak{c} = \mathfrak{z}(l)^{\text{reg}} + \mathcal{N}_L$$

and consists of elements $x \in \mathfrak{l}$ such that $H_G(x) = L$. By construction, we have that $\mathfrak{g}_\mathcal{c}$ is the $G$-saturation of $\mathfrak{l}^\text{reg}$, and $\mathfrak{g}_{\mathcal{c}_\ell}$ is the $G$-saturation of $\mathfrak{l}^\text{reg}$.
Remark 2.5. If $H$ is a maximal torus of $G$, then $\mathfrak{h}^{\text{nonreg}}$ is the union of root hyperplanes in $\mathfrak{h}$. More generally, for a Levi subgroup $L$, $\mathfrak{z}(L)^{\text{nonreg}}$ is the union of hyperplanes $\mathfrak{z}(\mathfrak{m}_1), \ldots, \mathfrak{z}(\mathfrak{m}_k)$, where $M_1, \ldots, M_k$ are the minimal Levi subgroups of $G$ which contain $L$. However, $\mathfrak{f}^{\text{nonreg}}$ is not cut out by linear equations in $\mathfrak{f}$ in general.

Remark 2.6. The Lusztig stratification (see [Lus95], Section 6) is the refinement of the Lusztig partition indexed by pairs $(L, \mathcal{O})$ of a Levi subgroups $L$ and a nilpotent orbit $\mathcal{O}$ in $\mathfrak{f}$, up to $G$-conjugacy. Given an element $x \in \mathfrak{g}(L)$, we may assume (after conjugating $x$) that $H_L(x) = L$, and thus $x_n$ is a nilpotent element of $\mathfrak{f}$; we say that $x \in \mathfrak{g}(L, \mathcal{O})$ if $x_n \in \mathcal{O}$. As $\mathfrak{g}(L)$ is the $G$-saturation of $\mathfrak{f}^{\text{reg}} = \mathfrak{N}_L + \mathfrak{z}(L)^{\text{reg}}$, we see that $\mathfrak{g}(L, \mathcal{O})$ is the $G$-saturation of $\mathfrak{g} + \mathfrak{z}(L)^{\text{reg}}$.

Example 2.7. Continuing Example 2.2 for $G = GL_n$, note that partitions $q = (q_1, \ldots, q_\ell)$ which refine the partition $p$ index nilpotent orbits $\mathcal{O}_q$ of $L_p$; the corresponding Lusztig stratum $\mathfrak{g}(L_p, \mathcal{O}_q)$ consists of matrices with $k$ distinct eigenvalues with the dimension of generalized eigenspaces according to the partition $p$, and the size of the Jordan blocks prescribed by the refined partition $q$.

2.3. Étale maps and Galois covers. Here we record some results about the restriction of the diagram $\mathfrak{g} \to \mathfrak{p} \to \mathfrak{f}$ to the various loci defined above. These results (or rather their analogues in the group setting) may be found in Lusztig [Lus84], but we include proofs for completeness.

Let us first observe the following preliminary result.

Lemma 2.8. Let $x \in \mathfrak{f}$. The following are equivalent:

1. $x \in \mathfrak{f}^{\text{reg}}$
2. $C_\mathfrak{g}(x_s) \subseteq \mathfrak{f}^{\text{reg}}$
3. $C_\mathfrak{g}(x) \subseteq \mathfrak{f}^{\text{reg}}$

Proof. Recall that $x \in \mathfrak{f}^{\text{reg}}$ means that $C_\mathfrak{g}(x_s) \subseteq L$. It is standard that this is equivalent to (2): the centralizer $C_\mathfrak{g}(x_s)$ is a Levi subgroup, and $C_\mathfrak{g}(x_s)$ is its Lie algebra.

Now suppose $x \in \mathfrak{g}$ and consider $\alpha = \text{ad}(x): \mathfrak{g} \to \mathfrak{g}$. Thus $C_\mathfrak{g}(x) = \ker(\alpha)$ and $C_\mathfrak{g}(x_s) = \ker(\alpha_s)$. Choose an invariant bilinear form, and write $\mathfrak{g} = \mathfrak{g}\mathfrak{l}^{\mathfrak{f}}$. Statement (2) is equivalent to $\ker(\alpha_s|_{\mathfrak{f}^{\text{reg}}}) = 0$, whereas Statement (3) is equivalent to $\ker(\alpha_s|_{\mathfrak{f}^{\text{reg}}}) = 0$. Thus the result follows from a standard fact in linear algebra, which we record in the following lemma.

Lemma 2.9. Suppose $\alpha \in \text{End}(V)$ is a linear transformation of a finite dimensional vector space. Then $\alpha$ is invertible if and only if $\alpha_s$ is invertible.

Given a parabolic $P$ containing $L$ as a Levi factor, we define $\mathfrak{p}^{\text{reg}}$ to be the preimage of $\mathfrak{f}^{\text{reg}}$ under the projection $\mathfrak{p} \to \mathfrak{f}$.

Proposition 2.10. The natural morphism

$$\mathfrak{p}^{\text{reg}} \to \mathfrak{f}^{\text{reg}}$$

is an equivalence.

Proof. The fibers of the morphism $\mathfrak{p} \to \mathfrak{f}$ are the quotient $\mathfrak{f}/\alpha_s\mathfrak{f}$. Thus, we need to show that that $U$ acts simply transitively on $x + \mathfrak{u}$, where $x \in \mathfrak{f}^{\text{reg}}$ (here, we consider $\mathfrak{f}$ to be a subalgebra of
or in other words, the following map is an isomorphism:

\[ U \to x + u \]
\[ g \to \text{Ad}(g)x \]

As \( U \) is unipotent, it suffices to show that the corresponding map

\[ u \to x + u \]
\[ u \to \text{ad}(u)(x). \]

is an isomorphism. If \( \text{ad}(u)(x) = \text{ad}(v)(x) \), then \( u - v \) is an element of \( C_p(x) \cap u \), so must be zero. Thus the map is injective. Any injective morphism between affine spaces of the same dimension must be an isomorphism. □

**Proposition 2.11.** The morphism of stacks

\[ d^\text{reg}_*: \mathcal{L}^\text{reg} \to \mathcal{P}_L^\text{reg} \]

is étale.

**Proof.** A morphism of (derived) stacks is étale if and only if its relative tangent complex is acyclic. Given a point \( x \in \mathcal{P}_L^\text{reg} \), the relative tangent complex at \( x \) is given by the total complex of:

\[
\begin{array}{c}
g \leftarrow \mathcal{I} \\
\downarrow \text{ad}(x) & \downarrow \text{ad}(x) \\
g \leftarrow \mathcal{I}
\end{array}
\]

(The columns are the tangent complex of \( g \) and \( \mathcal{I} \) at the point \( x \), and the horizontal maps is the map of complexes induced by \( d : \mathcal{I} \to g \).) We can write \( g = p \oplus \mathcal{I} \). The acyclicity of the complex is equivalent to the statement that \( \text{ad}(x)|_{\mathcal{I}} \) is invertible; this follows from the fact that \( x \) is regular and Lemma 2.8. □

**Remark 2.12.** In more down-to-earth terminology, Proposition 2.11 states that \( \mathfrak{g}_P \to \mathfrak{g} \) is étale over the regular locus.

**Proposition 2.13.** The map \( d^\text{reg}_*: \mathcal{L}^\text{reg} \cong \mathcal{P}_L^\text{reg} \to \mathcal{G}_L^\text{reg} \) is a \( W_{G,L} \)-Galois cover.

**Proof.** The inclusion of \( \mathcal{P}_L^\text{reg} \) in to \( \mathcal{G}_L^\text{reg} \) induces a morphism of stacks:

\[
\mathcal{L}_{G,L}^\text{reg}/W_{G,L} \cong \mathcal{P}_L^\text{reg}/N_G(L) \to \mathcal{G}_L^\text{reg}/G \cong \mathcal{G}_L^\text{reg}/G.
\]

The proposition is equivalent to the statement that this is an isomorphism. To see this, note that every \( x \in \mathcal{G}_L^\text{reg} \) is conjugate to an element of \( \mathcal{L}_{G,L}^\text{reg} \). Moreover, if \( x \in \mathcal{L}_{G,L}^\text{reg} \), \( C_G(x) = C_{N_G(L)}(x) = C_L(x) \), from which the result follows. □

The results of this subsection can be summarized in the following diagram:\footnote{The idea behind this proof was communicated to me by Dragos Fratila.}

\footnote{Given a parabolic \( P \) with Levi factor \( L \), we define \( p_\partial \) to be the preimage \( l_\partial + u \).}
Proposition 2.14. There is a commutative diagram:

![Diagram](image_url)

where:

- the vertical morphisms are open embeddings;
- the morphisms coming out of the page are closed embeddings; and
- all squares on the right hand side of the diagram, and all squares coming out of the page are cartesian.

2.4. The Lusztig partition of the commuting variety. In this subsection, we generalize the partition of Subsection 2.1 to a partition of the variety \( \text{comm}(g) \) of commuting elements in \( g \). Recall that given an element \( x \in g \), \( H(x) = H_G(x) = C_G(x_s) \) is defined to be the centralizer of the semisimple part of \( x \). Similarly, given \( (x, y) \in \text{comm}(g) \), we define \( H(x, y) = H(x) \cap H(y) \). It is the stabilizer of the pair of commuting semisimple elements \( (x_s, y_s) \).

Given a Levi subgroup \( L \) of \( G \), define

\[
\text{comm}(g)_{(L)} = \{(x, y) \in \text{comm}(g) \mid H(x, y) \text{ is conjugate to } L\},
\]

Note that \( H(x) \) and \( H(y) \) are Levi subgroups of \( G \) which contain a common maximal torus (as \( x_s \) and \( y_s \) commute), thus their intersection is indeed a Levi subgroup of \( G \). As for the partition of \( g \), we write \( \text{comm}(g)_{G} \) for \( \text{comm}(g)_{(G)} \), and (for example) \( \text{comm}(g)_{\xi(L)} \) for the union of \( \text{comm}(g)_{(M)} \), where \( (M) \notin (L) \) in the poset \( \text{Levi}_G \).

Example 2.15. Continuing Examples 2.2 and 2.7, let us consider the case \( G = GL_n \). Given a partition \( p = (p_1, \ldots, p_k) \) with corresponding Levi subgroup \( L_p \), the subspace \( \text{comm}(g_{(L_p)}) \) consists of pairs of commuting matrices \( (x, y) \) with \( k \) distinct simultaneous eigenvalues \( \lambda_1, \ldots, \lambda_k \), such that the dimension of the \( \lambda_i \)-generalized simultaneous eigenspace (i.e. the simultaneous eigenspace for the commuting semisimple matrices \( x_s, y_s \) is \( p_i \).

Remark 2.16. Let us examine how the loci in the commuting variety interact with those in \( g \). We have the following identities:

- \( \text{comm}(g)_{G} = (g_{\otimes} \times g_{\otimes}) \cap \text{comm}(g) = \mathfrak{z}(g) \times \mathfrak{z}(g) \times \text{comm}(\mathcal{N}_G) \).
- \( \text{comm}(g)_{(L)} \cap (g \times g) = (g_{(L)} \times g_{\otimes}) \cap \text{comm}(g) \)

Moreover, there is an inclusion

\[
\text{comm}(g)_{\xi(L)} \subseteq (g_{\xi(L)} \times g_{\xi(L)}) \cap \text{comm}(g)
\]
but it is not an equality in general. For example, take $G = GL_3$, and $L = GL_2$ to be the maximal torus, consisting of diagonal matrices. Then consider the pair $(x, y) \in \text{comm}(gl_3)$, where $x = \text{diag}(0,0,1)$ and $y = \text{diag}(1,0,0)$. Then $H(x) = GL_2 \times GL_1$ and $H(y) = GL_1 \times GL_2$ (thought of as block matrices). Thus both $x$ and $y$ are contained inside $gl_{\text{reg}}(GL_2)$ (which is the complement of the regular semisimple locus). However $H(x) \cap H(y) = GL_1^2$, so $(x, y)$ is not contained in $\text{comm}(gl_2)_{\text{reg}}(GL_2)$. In words, $x$ and $y$ are not regular semisimple matrices (they have repeated eigenvalues), but they are “simultaneously regular semisimple” (they have distinct simultaneous eigenvalues).

Given a Levi subgroup $L$ of $G$, we define

$$\text{comm}(l)_{\text{reg}} = \{(x, y) \in \text{comm}(l) \mid H_G(x) \cap H_G(y) \subseteq L\}.$$

For a parabolic subgroup $P$ containing $L$ as a Levi factor, we define $\text{comm}(p)_{\text{reg}}$ as the preimage of $\text{comm}(l)_{\text{reg}}$ under the canonical map $\text{comm}(p) \to \text{comm}(l)$. Similarly $\text{comm}(l)_{\text{nonreg}}$ is the complement of $\text{comm}(l)_{\text{reg}}$ in $\text{comm}(l)$.

**Remark 2.17.** It is natural to ask whether the partition of $\text{comm}(g)$ considered here refines to a stratification as in the case of $\mathfrak{g}$. In the case of $\mathfrak{g}$, this stratification arises essentially from the stratification of the nilpotent cone in to orbits (this is the theory of Jordan normal form in the $GL_n$ case). Following the analogy between $\mathfrak{g}$ and $\text{comm}(g)$, we are reduced to finding a stratification of $\text{comm}(N_G)$. However, unlike for $N_G$, there are infinitely many $G$-orbits on $\text{comm}(N_G)$, so it is not so clear how to proceed. The variety $\text{comm}(N_G)$ is studied by Premet [Pre03], where it is shown that the irreducible components of $\text{comm}(N_G)$ are in bijection with distinguished nilpotent orbits for $G$ (and moreover the variety is equidimensional).

### 2.5. Generalized Grothendieck-Springer correspondence for commuting stacks.

In this subsection, we present a version of Proposition 2.11 for commuting varieties: the analogue of the Grothendieck-Springer map for commuting varieties is étale over the regular locus.

We have a diagram:

$$
\begin{array}{cccc}
\text{comm}(g) & \xleftarrow{r} & \text{comm}(p) & \xrightarrow{s} & \text{comm}(l) \\
\text{comm}(g)_{\text{reg}}(L) & \xleftarrow{r_{\text{reg}}} & \text{comm}(p)_{\text{reg}} & \xrightarrow{s_{\text{reg}}} & \text{comm}(l)_{\text{reg}}
\end{array}
$$

**Proposition 2.18.** The morphisms $r_{\text{reg}}$ and $s_{\text{reg}}$ are étale morphisms of derived stacks.

Strictly speaking, we will not need this result in the remainder of the paper. However, certain key results (e.g. Lemma 2.19 and Proposition 3.20) are “morally” based on Proposition 2.18 above, although we are able to reduce the proof to Proposition 2.11 (which states that $\mathcal{I}_{\text{reg}} \simeq \mathcal{P}_{\text{reg}} \to \mathfrak{g}$ is étale) using the Fourier transform. Proposition 2.18 may become useful for future work in the group and elliptic setting.

**Sketch of proof of Proposition 2.18.** In fact, we will prove the statement for the natural derived enhancement of the commuting stacks, obtained by taking the derived fiber of the commutator map at zero. This implies the result for ordinary commuting stacks. A morphism of derived stacks of finite presentation is étale if and only if the relative tangent complex is acyclic (see e.g. [TV08], 2.2.2). This may be checked point by point as follows.
Given a point \((x, y) \in \mathsf{comm}(l)\), we have the following diagram of tangent complexes:

\[
\begin{array}{ccc}
& g & p \\
(-ad_y, ad_x) & & s \rightarrow l \\
& g \oplus g & p \oplus p \rightarrow l \oplus l \\
\end{array}
\]

The columns in the diagram above are the tangent complexes of \(\mathsf{comm}(g), \mathsf{comm}(p), \mathsf{comm}(l)\) respectively, and the horizontal maps are morphisms of complexes induced by the maps \(r\) and \(s\). We want to show that if \((x, y)\) is regular, then the horizontal maps are quasi-isomorphisms. Note that we have a splitting \(g = \frak{u} \oplus l \oplus \frak{u}\), which is preserved by the vertical morphisms. Thus it is enough to show that the complex

\[
\begin{array}{ccc}
& u & \\
(-ad_y, ad_x) & & \frak{u} \oplus \frak{u} \\
& \frak{u} \oplus \frak{u} & (a_d) \downarrow \frak{u} \\
\end{array}
\]

together with the equivalent complex for \(\frak{u}\), is acyclic. As \((x, y)\) is regular, it means that the kernel of \(\ker(\text{ad}_L|_\frak{u}) \cap \ker(\text{ad}_L|_\frak{u}) = 0\). The result follows by standard linear algebra. \(\square\)

3. Properties of Induction and Restriction

In this section, we will prove that parabolic restriction and induction, restrict to exact functors on the level of abelian categories, and characterize the kernel of parabolic restriction in terms of singular support.

3.1. Induction and restriction over the regular locus. In this subsection we will keep the conventions of Section 1 (so \(P\) and \(Q\) are parabolic subgroups of \(G\) with Levi quotients \(L, M\) etc.), but additionally fix Levi splittings of \(L\) and \(M\) inside \(P\) and \(Q\). Consider the set \(\mathcal{S}(M, L)\) which is defined to be the set of conjugates \(^gL\) of \(L\) such that \(M \times ^gL\) contains a maximum torus of \(G\) (for convenience, let us assume that \(M \times L\) contains a maximum torus \(H\) of \(G\)). There is a bijection \(M \setminus \mathcal{S}(M, L)/L \approx \mathcal{Q}/G/P\).

According to the results of Subsection 3.1 (see Proposition 2.14), parabolic induction and restriction on the regular loci are just given by pullback and pushforward along the \(\acute{e}tale\) map \(d^{\text{res}} : \frak{g}^{\text{res}} \rightarrow \frak{g}^{\text{res}}(L)\). More precisely, we have

\[
\text{Res}^G_{P,L}(\mathfrak{M})|_{\frak{g}^{\text{res}}} \cong (d^{\text{res}})^{-1}(\mathfrak{M}).
\]

This result has a number of important consequences. First let us give the following key definition:

**Definition 3.1.** An object \(\mathfrak{M} \in \mathbf{D}(\frak{g})\) is called \textit{cuspidal} if \(\text{Res}^G_{P,L}(\mathfrak{M}) \cong 0\) for every proper parabolic subgroup \(P\) of \(G\). The full subcategory of cuspidal objects is denoted \(\mathbf{D}(\frak{g})_{\text{cus}}\).
Proposition 3.2. Suppose $\mathcal{M} \in D(\mathfrak{g})$ and $\text{Res}^G_{P,L}(\mathcal{M}) \simeq 0$. Then $\text{Supp}(\mathcal{M}) \subseteq \mathfrak{g}_{\xi(L)}$. In particular, if $\mathcal{M}$ is cuspidal, then $\text{Supp}(\mathcal{M}) \subseteq \mathfrak{g}_\emptyset$.

Proof. Given $\mathcal{M}$ with $\text{Res}^G_{P,L}(\mathcal{M}) \simeq 0$, we have

$$d^*(\mathcal{M}) \simeq \text{Res}^G_{P,L}(\mathcal{M})|_{\text{reg}} \simeq 0.$$ 

But $d$ is étale with image $\mathfrak{g}_{\xi(L)}$, thus $\mathcal{M}|_{\mathfrak{g}_{\xi(L)}} \simeq 0$. Thus $\mathcal{M}$ is supported in $\mathfrak{g}_{\xi(L)}$. Now, if $\mathcal{M}$ is cuspidal, then $\text{Res}^G_{P,L}(\mathcal{M}) \simeq 0$ for every proper parabolic subgroup $P$. Thus $\mathcal{M}$ is supported in the intersection of $\mathfrak{g}_{\xi(L)}$ for all proper Levi subgroups $L$ of $G$; this intersection is precisely $\mathfrak{g}_\emptyset$. Thus $\mathcal{M}$ is supported in $\mathfrak{g}_\emptyset$ as required. □

The next result explains how the Mackey filtration is naturally split on the regular locus. Recall that we have fixed another Levi subgroup $\mathfrak{m}$ inside a parabolic $Q$ (in addition to $L$ and $P$).

Proposition 3.3. Let $j : m_{\text{reg}} \hookrightarrow m$ denote the inclusion. There is a canonical natural isomorphism:

$$j^! \circ M_Q \text{St}^w_{P,L} \simeq \bigoplus_{\text{we } M_Q \text{St}^w_{P,L}} j^! \circ M_Q \text{St}_{P,L}.$$ 

Proof. We consider the open substack $(Q\mathfrak{sl}_P)^{\text{reg}} := \mathfrak{q}^{\text{reg}} \times_{\mathfrak{g}} \mathfrak{p}$ of $Q\mathfrak{sl}_P$. We denote by $(Q\mathfrak{sl}_P)^{\text{reg}}$ the intersection of $Q\mathfrak{sl}_P^{\text{reg}}$ with $(Q\mathfrak{sl}_P)^{\text{reg}}$. Recall that the functor $M_Q \text{St}_{P,L}$ is given by $\alpha \circ \beta$ where:

$$m \xrightarrow{\alpha} Q\mathfrak{sl}_P \xrightarrow{\beta} 1$$

Thus $j^! \circ M_Q \text{St}_{P,L}$ is given by $\alpha^{\text{reg}} \circ \beta^!$ (where $\alpha^{\text{reg}} : Q\mathfrak{sl}_P^{\text{reg}} \to m_{\text{reg}}$). The result now follows from Lemma 3.4 below. □

Lemma 3.4. Each stratum in the stratification

$$(Q\mathfrak{sl}_P)^{\text{reg}} \simeq \bigcup_{w \in Q \setminus G/P} (Q\mathfrak{sl}_P)^{\text{reg}}.$$ 

is both open and closed (i.e. the stratification is a disjoint union of connected components).

Proof. Consider the opposite parabolic $\overline{Q}$ of $Q$ with respect to the Levi subgroup $\mathfrak{m}$ (with Lie algebra $\mathfrak{a}$ etc.). We have isomorphisms $\mathfrak{q}^{\text{reg}} \simeq m^{\text{reg}} \simeq \mathfrak{a}^{\text{reg}}$, and thus

$$Q\mathfrak{sl}_P^{\text{reg}} \simeq m^{\text{reg}} \times_{\mathfrak{g}} \mathfrak{p} \simeq \mathfrak{a}\mathfrak{sl}_P^{\text{reg}}.$$ 

Note that the bijections

$$Q \setminus G/P \simeq M \setminus S(M,L)/L \simeq \overline{Q} \setminus G/P$$

induce the opposite partial order on $M \setminus S(M,L)/L$. Thus the closure relations amongst the strata $(Q\mathfrak{sl}_P)^{\text{reg}}$ are self opposed. It follows that each stratum is both open and closed, as required. □
3.2. **Singular support and parabolic restriction.** Let us review some of the material from Appendix [B.4](#). Consider the ring $\mathcal{D}_g$ equipped with the filtration by order of a differential operator. Let $\mathfrak{M}$ be a coherent $\mathcal{D}_g$-module. We may choose a good filtration for $\mathfrak{M}$ and take the associated graded module to obtain a finitely generated module $\text{gr} \mathfrak{M}$ for $\mathcal{O}_{T\hat{g}}$, or in other words a coherent sheaf on $g \times g$ (we fix an invariant symmetric bilinear form to identify $T^*g = g \times g^*$ with $g \times g$). The singular support $SS(\mathfrak{M})$ of $\mathfrak{M}$ is defined to be the support of $\text{gr} \mathfrak{M}$ (the module $\text{gr} \mathfrak{M}$ may depend on the choice of filtration, but its support is independent of this choice). The singular support is a conical closed subvariety of $g \hat{} g$ (with respect to the $C^\infty$-action that acts trivially on the first factor and acts on the second with weight 1).

If a coherent $\mathcal{D}_g$-module $\mathfrak{M}$ is $G$-equivariant, we may equip it with an equivariant filtration: a filtration such that each subobject is closed under the $G$-action. The associated graded module $\text{gr} \mathfrak{M}$ will then be (scheme theoretically) supported on the subvariety $\text{comm}(g)$ of commuting elements of $g$. More generally, if $\mathfrak{M}$ is any $G$-equivariant $\mathcal{D}_g$-module (not necessarily coherent), we can write it as a union of coherent submodules. Note that every coherent submodule $\mathfrak{N}$ of a $G$-equivariant module $\mathfrak{M}$ is contained in a $G$-equivariant submodule. (Indeed, take a finite set of generators for $\mathfrak{N}$, find a finite dimensional $G$-invariant subspace $V$ of $\mathfrak{M}$ containing the generators, then consider the submodule of $\mathfrak{M}$ generated by $V$. This will be $G$-equivariant, coherent, and contains $\mathfrak{N}$.) Thus every $G$-equivariant $\mathcal{D}_g$-module is a union of $G$-equivariant coherent submodules. We define the singular support of a $G$-equivariant $\mathcal{D}_g$-module to be the union of the singular supports of all the coherent submodules of $\mathfrak{M}$. Given a complex in $\mathcal{D}(g)$, its singular support is defined as the union of the singular supports of its cohomology modules.

Now we consider the interaction of parabolic restriction with singular support. As usual, let $P$ be a parabolic subgroup of $G$ with Levi factor $L$. Consider the following diagram, and observe that the middle square is cartesian.

![Diagram](#)

**Lemma 3.5.** Given $\mathfrak{M} \in \text{M}(g)$, we have:

$$SS(\text{Res}_{P,L}^G(\mathfrak{M})) \subseteq BA^{-1}SS(\mathfrak{M}).$$

**Proof.** This follows from Proposition [B.7](#) and smooth base change. 

Now suppose that $Q$ is a parabolic subgroup of $L$ with Levi factor $M$ (so we have a chain of Levi subgroups $M \leq L \leq G$). Recall from Subsection [2.4](#) the nonregular locus $\text{comm}(l)^{\text{nonreg}}$ consists of commuting pairs of elements $x, y \in L$ such that $H_G(x, y)$ is not contained in $L$.

5Recall that $G$-equivariance is a condition rather than data in this case, i.e. $\text{M}(g)$ is a full subcategory of $\text{M}(g)$. When we refer to the singular support of an equivariant $D$-module, we mean the singular support of the underlying non-equivariant $D$-module.
Lemma 3.6. If \( \mathcal{M} \) has singular support contained in \( \text{comm}(l)_{\text{nonreg}} \), then \( \mathcal{M} = \text{Res}^G_{Q,M}(\mathcal{M}) \) has singular support contained in \( \text{comm}(m)_{\text{nonreg}} \).

Proof. By Lemma 3.5 the singular support of \( \mathcal{M} \) is contained in \( B_L A^{-1}_L \text{comm}(l)_{\text{nonreg}} \), where

\[
\begin{align*}
I \times I \xrightarrow{A_L} q \times q \xrightarrow{B_L} m \times m
\end{align*}
\]

Given \((x, y)\) in \( \text{comm}(l)_{\text{nonreg}} \), we can conjugate so that \((x, y)\) are contained in \( q \), and \( x_s, y_s \) are contained in \( m \) (in fact, we can ensure that \( x \) and \( y \) are contained in a common Borel inside \( q \), and \( x_s, y_s \) in a common Cartan). Then the image of \((x, y)\) under the projection to \( m \times m \) again has semisimple part \((x_s, y_s)\). Thus \( H_G(B_L(x, y)) = H_G(x, y) \) which is not contained in \( L \) and thus not contained in \( M \), as required. \( \square \)

3.3. Fourier transform and parabolic restriction. The Fourier transform functor for \( D \)-modules defines an involution:

\[
F_g : D(g) \simarrow D(g).
\]

Lemma 3.7 (Lusztig, Mirković [Mir04], 4.2). The functors of induction and restriction commute with the Fourier transform functor for all parabolic subgroups \( P \) with Levi factor \( L 
\]

\[
\begin{align*}
F_g \text{Res}^G_{P,L} &\simeq \text{Res}^G_{P,L} F_g \\
F_g \text{Ind}^G_{P,L} &\simeq \text{Ind}^G_{P,L} F_g
\end{align*}
\]

Sketch of proof. In the case of parabolic restriction, the lemma follows from the observation that the following diagram is cartesian, and taking duals swaps the right hand side with the left hand side:

\[
\begin{array}{ccc}
g & \xleftarrow{\circlearrowleft} & p \\
\downarrow & & \downarrow \\
g/\mu & \xleftarrow{\circlearrowleft} & i
\end{array}
\]

Lemma 3.8. Let \( \mathcal{M} \) be an object of \( D(g) \).

1. If \( \text{Res}^G_{P,L}(\mathcal{M}) \simeq 0 \), then the singular support of \( \mathcal{M} \) and of \( F(\mathcal{M}) \) is contained in \((g_{\xi(L)} \times g_{\xi(L)}) \cap \text{comm}(g)\).
2. If \( \mathcal{M} \) is cuspidal, then the singular support of \( \mathcal{M} \) and of \( F(\mathcal{M}) \) is contained in \( \text{comm}(g)_{\circlearrowleft} \).

Proof. By Proposition 3.2 we have that \( \text{Supp}(\mathcal{M}) \subseteq g_{\xi(L)} \). By Lemma 3.7 we also have that \( \text{Res}^G_{P,L}(F(\mathcal{M})) \simeq 0 \), and thus \( \text{Supp}(F(\mathcal{M})) \subseteq g_{\xi(L)} \). By Lemma 3.4 we see that \( SS(\mathcal{M}) \subseteq g_{\xi(L)} \times g_{\xi(L)} \); the same argument applies to \( F(\mathcal{M}) \), which proves the first claim. If \( \mathcal{M} \) is cuspidal, then by the first part of the lemma, \( SS(\mathcal{M}) \) and \( SS(F(\mathcal{M})) \) are contained in \((g_{\circlearrowleft} \times g_{\circlearrowleft}) \cap \text{comm}(g)\); but this is precisely \( \text{comm}(g)_{\circlearrowleft} \), as required (see Remark 2.16). \( \square \)
Remark 3.9. We will see that the converse to the second part of Lemma 3.8 is true, but the converse to the first part fails: we must replace \((\mathfrak{g}_\mathbb{C}(L) \times \mathfrak{g}_\mathbb{C}(L)) \cap \text{comm}(\mathfrak{g})\) with \(\text{comm}(\mathfrak{g}) \mathfrak{g}_\mathbb{C}(L)\) (see Theorem 3.14).

3.4. \(D\)-modules with nonregular singular support must be zero. The goal of this subsection is to prove the following proposition, which forms the basis for the proof of many of the key results in this paper.

**Proposition 3.10.** Let \(L\) be a proper Levi subgroup of \(G\) and \(\mathfrak{M} \in \mathfrak{M}(\mathfrak{l})\). If \(\text{SS}(\mathfrak{M}) \subseteq \text{comm}(\mathfrak{l})^{\text{nonreg}}\) then \(\mathfrak{M} \simeq 0\).

**Example 3.11.** If \(G = SL_2\), and \(L \simeq H \simeq \mathbb{C}^2\), the maximal torus, then Proposition 3.10 says that if a \(D\)-module \(\mathfrak{M} \in \mathfrak{M}(\mathfrak{h})\) has singular support contained in \(\{0\} \subseteq T^*\mathfrak{h} \simeq \mathfrak{h} \times \mathfrak{h}\), then \(\mathfrak{M} \simeq 0\). This statement is a direct consequence of “Bernstein’s inequality”.

The remainder of the subsection will be devoted to the proof of Proposition 3.10. The idea of the proof is as follows.

- First we prove the proposition in the case when \(\mathfrak{M}\) is cuspidal. In this case the singular support of \(\mathfrak{M}\) is contained in \(\text{comm}(\mathfrak{l})^\odot\) by Lemma 3.8 where the nonregularity constraints are linear. Using a general result in symplectic linear algebra (Lemma 3.11), we deduce that there are no nonempty coisotropic subvarieties of \(\text{comm}(\mathfrak{l})^{\text{nonreg}} \cap \text{comm}(\mathfrak{l})^\odot\). By Gabber’s theorem on the coisotropy of singular support, this proves Proposition 3.10 in the case when \(\mathfrak{M}\) is cuspidal (see Appendix 3.4 for the definition of coisotropic and the statement of Gabber’s theorem).

- Now we reduce the general case to the cuspidal case, as follows. If \(\mathfrak{M}\) in Proposition 3.10 is not zero, we can take a parabolic restriction to a minimal Levi \(M\) so that \(\mathfrak{M} := \text{Res}_{Q,M}^G(\mathfrak{M})\) is nonzero and (necessarily) cuspidal. Then \(\mathfrak{M}\) will itself have nonregular singular support in \(\text{comm}(\mathfrak{m})\) by Lemma 3.14. Thus we can apply the cuspidal case to conclude that \(\mathfrak{M}\) must be zero, contradicting the assumptions.

We begin with the following lemma.

**Lemma 3.12.** There are no non-empty coisotropic subvarieties of \(\mathfrak{l} \times \mathfrak{l}\) contained in \(\text{comm}(\mathfrak{l})^\odot\).

**Proof.** Consider the decomposition \(\mathfrak{l} = \mathfrak{z}(\mathfrak{l}) \oplus \mathfrak{r}',\) where \(\mathfrak{r}' = [\mathfrak{l}, \mathfrak{l}]\). Note that \(\mathfrak{r}^\odot \cap \text{comm}^{\text{nonreg}} = \mathfrak{z}(\mathfrak{l})^{\text{nonreg}} \times \mathcal{N}_L\) which is contained in \(\mathfrak{z}(\mathfrak{l})^{\text{nonreg}} \times \mathfrak{r}'\). In other words, the (non)regularity of an element of \(\mathfrak{l}^\odot \subseteq \mathfrak{l}\) depends only on its projection to \(\mathfrak{z}(\mathfrak{l})\) (which in this case can be thought of as the semisimple part in the Jordan decomposition). Similarly, \(\text{comm}(\mathfrak{l})^\odot \cap \text{comm}(\mathfrak{l})^{\text{nonreg}}\) is contained in \(\mathfrak{z}(\mathfrak{l}) \times \mathfrak{z}(\mathfrak{l})^{\text{nonreg}} \times \mathfrak{r}' \times \mathfrak{r}'\). Thus it is enough to check that there are no coisotropic subvarieties contained in the latter subset.

Given \(x \in \mathfrak{z}(\mathfrak{l})^{\text{nonreg}},\) the centralizer \(H(x) = C_G(x)\) is a Levi subgroup of \(G\) which contains \(L\) as a proper subgroup; given a Levi subgroup \(M\) of \(G\) which contains \(L\), the subset of \(\mathfrak{z}(\mathfrak{l})\) consisting of elements \(y\) with \(H_G(y) = M\) is precisely \(\mathfrak{z}(\mathfrak{m}) \subseteq \mathfrak{z}(\mathfrak{l})\). Let us consider the finitely many Levi subgroups of \(G\) which properly contain \(L\); there are finitely many such subgroups \(M_1, M_2, \ldots, M_k\). Thus \(\mathfrak{z}(\mathfrak{l})^{\text{nonreg}}\) is the union of linear subspaces \(\mathfrak{z}(\mathfrak{m}_i)\) as \(i\) ranges from 1 to \(k\).

Similarly, \((\mathfrak{z}(\mathfrak{l}) \times \mathfrak{z}(\mathfrak{l}))^{\text{nonreg}}\) consists of pairs \((x, y) \in \mathfrak{z}(\mathfrak{l}) \times \mathfrak{z}(\mathfrak{l})\) such that \(H_G(x, y) = C_G(x) \cap C_G(y) \neq L\). This is a union of linear subvarieties \(\mathfrak{z}(\mathfrak{m}_i) \times \mathfrak{z}(\mathfrak{m}_j)\) where \((i, j)\) satisfy the condition:

\[
\mathfrak{z}(\mathfrak{m}_i) + \mathfrak{z}(\mathfrak{m}_j) \neq \mathfrak{z}(\mathfrak{l}).
\]
Thus \((\mathfrak{g}(l) \times \mathfrak{g}(l))_{\text{nonreg}} \times l' \times l'\) is a union of subspaces of the form \(\mathfrak{g}(m_i) \times \mathfrak{g}(m_j) \times l' \times l'\) for \((i, j)\) satisfying condition 4. This is precisely the context of Lemma 3.11 setting \(V = l = \mathfrak{g}(l) \times l'\), and \(K_i = \mathfrak{g}(m_i) \times l'\). Thus we deduce Lemma 3.12 from Lemma 3.11. 

We are now able to prove Proposition 3.10 in the case where \(\mathcal{R}\) is cuspidal, which we record as Lemma 3.13 below.

**Lemma 3.13.** Let \(\mathcal{R}\) be as in Proposition 3.10 and suppose in addition that \(\mathcal{R}\) is cuspidal. Then \(\mathcal{R} \simeq 0\).

**Proof.** By assumption and Lemma 3.8 the singular support of \(\mathcal{R}\) is contained in \(\text{comm}(l)_{\text{nonreg}}\). By Lemma 3.12 this locus does not support any coisotropics. By Corollary B.10 of Gabber’s theorem, we deduce that \(\mathcal{R} \simeq 0\), as required. 

**Proof of Proposition 3.10.** Now suppose \(\mathcal{R}\) is any object of \(\text{D}(\mathfrak{g})\) with singular support contained in \(\text{comm}(l)_{\text{nonreg}}\); we must show that \(\mathcal{R} \simeq 0\). To this end, suppose \(\mathcal{R} \neq 0\), and let \(Q\) be a minimal parabolic subgroup of \(L\) such that \(\mathcal{R} := \text{Res}^G_{L,M}(\mathcal{R})\) is non-zero (where \(M\) is a Levi factor of \(Q\)). Then \(\mathcal{R}\) is a cuspidal object of \(\text{D}(\mathfrak{m})\) by construction, and by Lemma 3.10 it has singular support contained in \(\text{comm}(\mathfrak{m})_{\text{nonreg}}\). Thus \(\mathcal{R}\) in \(\text{D}(\mathfrak{m})\) satisfies the conditions of Proposition 3.10 in the cuspidal case, and thus must be zero by Lemma 3.13. This contradicts the assumptions, and thus \(\mathcal{R} \simeq 0\), as required. 

3.5. **The Kernel of Parabolic Restriction.** The following result contains the core content of Theorem C from the introduction. The proof will be given at the end of this subsection.

**Theorem 3.14.** Given an object \(\mathcal{M} \in \text{D}(\mathfrak{g})\) and a Levi subgroup \(L\) of \(G\), the following are equivalent:

1. \(\text{Res}^G_{L,P} \mathcal{M} \simeq 0\) for any parabolic subgroup containing \(L\) as a Levi factor;
2. \(\text{Res}^G_{L,P} \mathcal{M} \simeq 0\) for some Levi subgroup containing \(L\) as a Levi factor;
3. \(\text{Supp}(\mathcal{M}) \subseteq \text{comm}(\mathfrak{g})_{\xi(L)}\).

Let \(\text{D}(\mathfrak{g})_{\xi(L)}\) denote the subcategory of \(\text{D}(\mathfrak{g})\) consisting of objects \(\mathcal{M}\) with singular support contained in \(\text{comm}(\mathfrak{g})_{\xi(L)}\). Theorem 3.14 implies that it agrees with the kernel of parabolic restriction for any proper parabolic subgroup \(P\) containing \(L\) as a Levi factor.

In particular, we have the following characterization of cuspidal objects (closely related statements can be found in [Mir04], Corollary 5.7 and Theorem 6.7).

**Corollary 3.15.** The following conditions on an object \(\mathcal{M} \in \text{D}(\mathfrak{g})\) are equivalent

1. \(\mathcal{M}\) is cuspidal;
2. \(\text{SS}(\mathcal{M}) \subseteq \text{comm}(\mathfrak{g})_{\varnothing}\);
3. \(\text{Supp}(\mathcal{M}) \subseteq \mathfrak{g}_\varnothing\) and \(\text{Supp}(\mathcal{F}(\mathcal{M})) \subseteq \mathfrak{g}_\varnothing\).

**Remark 3.16.** Note that the analogue of Statement (3) from Corollary 3.15 does not appear in Theorem 3.14. In general, it is not true that having both \(\mathcal{M}\) and \(\mathcal{F}(\mathcal{M})\) supported in \(\mathfrak{g}_\varnothing\) implies \(\text{SS}(\mathcal{M}) \subseteq \text{comm}(\mathfrak{g})_{\xi(H)}\) (which was erroneously implied in an earlier version of this paper). For example, consider the case \(G = SL_2 \times SL_2\), and \(\mathcal{M} = \delta_{(0)} \boxtimes \mathcal{O}_{St_{2}} \in \text{D}(\mathfrak{g})\). Then \(\mathcal{M}\) and its Fourier transform are supported in \(\mathfrak{g}_\varnothing\) (the complement of the regular semisimple locus), however the singular support of \(\mathcal{M}\) is not contained in \(\text{comm}(\mathfrak{g})_{\xi(H)}\). Moreover, this object has a non-zero parabolic restriction to the torus, as expected.
Given a Levi subgroup $L$ of $G$, there are various closed subsets $\operatorname{comm}(g)_J \subseteq \operatorname{comm}(g)$ corresponding to closed subsets $J$ of the poset $\mathcal{L}(G)$. Thus we can define subcategories $\mathbf{D}(g)_J$ consisting of objects with singular support in $\operatorname{comm}(g)_J$. By Theorem 3.14 we can identify these subcategories in terms of the intersection of the kernel of various parabolic restriction functors. In particular:

**Corollary 3.17.** Given a Levi subgroup $L$ of $G$, we have:

1. $\mathbf{D}(g)_<(L)$ consists of objects killed by parabolic restriction to $L$.
2. $\mathbf{D}(g)_=(L)$ consists of objects for which parabolic restriction to $L$ is cuspidal.
3. $\mathbf{D}(g)_>(L)$ consists of objects killed by parabolic restriction to $M$ whenever $M$ does not contain a conjugate of $L$.
4. $\mathbf{D}(g)_\geq(L)$ consists of objects killed by parabolic restriction to $M$ whenever $M$ does not properly contain a conjugate of $L$.

**Remark 3.18.** The notation indicates that $\mathbf{D}(g)_\geq(L)$ is supposed to be generated by parabolic inductions from cuspidals on Levi subgroups which are $\geq (L)$ (i.e. which contain a conjugate of $L$). This will be made precise later; for now, it is easier to define these subcategories “negatively”, as above.

Now we come to the proof of Theorem 3.14, which is equivalent to the following statement.

**Lemma 3.19.** Let $P$ be a parabolic subgroup of $G$ with Levi factor $L$, and $\mathfrak{M} \in \mathbf{M}(g)$. Then $SS(\mathfrak{M}) \subseteq \operatorname{comm}(g)_<(L)$ if and only if $\operatorname{Res}^G_{P,L}(\mathfrak{M}) \simeq 0$.

**Proof of “only if” direction.** Suppose $SS(\mathfrak{M}) \subseteq \operatorname{comm}(g)_<(L)$. It follows from Lemma 3.9 that $SS(\operatorname{Res}^G_{P,L}(\mathfrak{M})) \subseteq \operatorname{comm}(\text{nonreg})$. Thus $\operatorname{Res}^G_{P,L}(\mathfrak{M}) \simeq 0$ by Proposition 3.11 as required.

**Proof of “if” direction.** Suppose $\operatorname{Res}^G_{P,L}(\mathfrak{M}) \simeq 0$. We must show that $SS(\mathfrak{M}) \subseteq \operatorname{comm}(g)_<(L)$. We proceed by induction on the rank of $G$: thus we assume the proposition where $G$ is replaced by any proper Levi subgroup of $G$ (when the rank is zero, the claim is vacuously true). Suppose $(x,y) \in SS(\mathfrak{M})$; we must show that $(x,y) \in \operatorname{comm}(g)_<(L)$, or in other words, that $H(x,y) = H(x) \cap H(y)$ is not contained in any conjugate of $L$. By Lemma 3.18 we know that $x$ and $y$ are both contained in $g_<(L)$ (i.e. $H(x)$ and $H(y)$ are not contained in any conjugate of $L$).

Case 1: $H(x) = G$. In this case $H(x,y) = H(y)$, which is not contained in any conjugate of $L$, as required.

Case 2: $H(x) = L$ is a proper subgroup of $G$. By conjugating $(x,y)$ if necessary, we may assume that $M \subset L$ contains a maximal torus $H$ of $G$. Let $Q$ be the unique parabolic subgroup of $G$ with Levi factor $M$ such that $Q \cap P$ contains a Borel. Note that $x \in \operatorname{Supp}(\mathfrak{M}) \cap g_<(L)$, and thus $\mathfrak{M}' = \operatorname{Res}^G_{Q,M}(\mathfrak{M}) \neq 0$ by Proposition 3.2. Moreover, $\mathfrak{M}'_{\text{res}}$ is given by the restriction $(d^\text{reg})^!(\mathfrak{M})$ along the étale open $d^\text{reg} : [\mathfrak{m}]_{\text{res}} \to [\mathfrak{g}]$ (see Subsection 3.1). Let $SS(\mathfrak{M})_x$ denote the intersection of $SS(\mathfrak{M})$ with the cotangent fiber at $x$. As the singular support is preserved by étale pullbacks, we see that $SS(\mathfrak{M})_x$ (which is necessarily contained in $C_g(x_x) = m$) is identified with $SS(\mathfrak{M}')_x$. On the other hand by Proposition 3.7 (and the premise $\operatorname{Res}^G_{P,L}(\mathfrak{M}) \simeq 0$), we have

$$\operatorname{Res}^M_{M \cap P, M \cap L}(\mathfrak{M}') \simeq \operatorname{Res}^G_{Q \cap P, M \cap L}(\mathfrak{M}) \simeq \operatorname{Res}^L_{Q \cap L, M \cap L} \operatorname{Res}^G_{P,L}(\mathfrak{M}) \simeq 0$$

Thus, by the inductive hypothesis (with $G$ replaced by $M$ and $L$ replaced by $L \cap M$) we see that $SS(\mathfrak{M}') \subseteq \operatorname{comm}(m)_<(L \cap M)$. It follows that $SS(\mathfrak{M})_x = SS(\mathfrak{M}')_x \subseteq m_<(L \cap M)$. Thus we have
that $H_G(x, y) = M \cap H_G(y) = H_M(y)$ is not contained in any conjugate of $L$, or equivalently, $(x, y) \in \text{comm}(g)_{x(L), L}$, as required. \hfill \Box

3.6. Parabolic restriction is $t$-exact. The goal of this subsection is to prove the following result.

**Proposition 3.20.** The functor $\text{Res}_{P,L}^G : D(\mathfrak{g}) \rightarrow D(\mathfrak{l})$ is $t$-exact (it preserves the heart of the $t$-structure).

The idea of the proof is to show that if $\mathfrak{m} \in M(\mathfrak{g})$, the non-zero cohomology objects of the complex $\text{Res}_{P,L}^G(\mathfrak{m})$ have singular support contained in $\text{comm}(l)^{\text{nonreg}}$, and thus must be zero by Proposition 3.10. By induction, we may assume Proposition 3.20 for groups of smaller rank than $G$ (for example, for proper Levi subgroups of $G$). We first observe the following.

**Lemma 3.21.** Suppose $\mathfrak{m} \in M(\mathfrak{g})$ and consider the complex $\mathfrak{n} = \text{Res}_{P,L}^G(\mathfrak{m})$. For any integer $i \neq 0$, the cohomology object $\mathcal{H}^i(\mathfrak{n})$ and its Fourier transform $\mathcal{F}\mathcal{H}^i(\mathfrak{n})$ both have singular support contained in $\text{comm}(l)^{\text{nonreg}}$.

**Proof.** Recall from Subsection 3.4 that the morphism $d^{reg} : \mathfrak{q}_{\leq (L)} \rightarrow \mathfrak{l}^{reg}$, and $\text{Res}_{P,L}^G(\mathfrak{m})|_{\mathfrak{p}^{reg}} \approx (d^{reg})^* (\mathfrak{m})$. As restriction along an étale open is $t$-exact, we have that $\mathcal{H}^i(\mathfrak{n})|_{\mathfrak{p}^{reg}} \approx 0$. The same result for the Fourier transform $\mathcal{F}(\mathcal{H}^i(\mathfrak{n}))$ follows from the fact that Fourier transform is $t$-exact and commutes with parabolic restriction (by Lemma 3.7). Thus the result follows from Lemma 3.14. \hfill \Box

**Remark 3.22.** Lemma 3.21 already proves Proposition 3.20 in the case $G = SL_2$ and $L = H$ is a maximal torus. Then we see that for any integer $i \neq 0$, $\mathcal{H}^i(\mathfrak{n})$ and its Fourier transform must be supported at $0 \in \mathfrak{h}$; it follows that $\mathcal{H}^i(\mathfrak{n}) \approx 0$ and thus $\mathfrak{n} = \text{Res}_{B,H}^G(\mathfrak{m})$ is concentrated in degree $0$ as required.

We next consider the case when the parabolic restriction is cuspidal.

**Lemma 3.23.** Suppose $\mathfrak{m} \in M(\mathfrak{g})$, and $\mathfrak{n} := \text{Res}_{P,L}^G(\mathfrak{m})$ is cuspidal. Then $\mathcal{H}^i(\mathfrak{n}) \approx 0$ for each $i \neq 0$ (i.e., $\mathfrak{n}$ is a complex concentrated in degree $0$).

**Proof.** By Proposition 3.10, it suffices to show that $\mathcal{H}^i(\mathfrak{n}) \subseteq \text{comm}(l)^{\text{nonreg}}$ for every $i \neq 0$. The structure of the proof is similar to the “if” part of the proof of Lemma 3.14.

Suppose that $i \neq 0$ and $(x, y) \in SS(\mathcal{H}^i(\mathfrak{n}))$. By the assumption that $\mathfrak{n}$ is cuspidal, we have that $(x, y) \in \text{comm}(l)^{\text{reg}}$, or in other words $x, y \in l^G$; it follows that $L$ is contained in $H_G(x)$ and $H_G(y)$. We wish to show that $(x, y) \in \text{comm}(l)^{\text{nonreg}}$, or in other words $H_G(x) \cap H_G(y) \neq L$. By Lemma 3.21 we see that $H_G(x) \neq L$ and $H_G(y) \neq L$.

Case 1: $H_G(x) = G$. In this case $H_G(y) = H_G(x, y)$ is not equal to $L$, as required.

Case 2: $H_G(x) : = M$ is a proper Levi subgroup of $G$ (which necessarily contains $L$). Let $Q$ be a parabolic which contains $P$ and has $M$ as a Levi factor, and we set $\mathfrak{m}' = \text{Res}_{Q,M}^G(\mathfrak{m})$. Note that $x \in \mathfrak{m}'^{\text{reg}}$, and thus $x$ is not in the support of $\mathcal{H}^i(\mathfrak{m}')$ by Lemma 3.21.

By the inductive hypothesis $\text{Res}_{M \cap P,L}^M$ is $t$-exact and thus $\mathcal{H}^i(\mathfrak{n}) = \text{Res}_{M \cap P,L}^M \mathcal{H}^i(\mathfrak{m}')$. As $x$ is not in the support of $\mathcal{H}^i(\mathfrak{m}')$, it cannot be in the support of $\mathcal{H}^i(\mathfrak{n}) = \text{Res}_{M \cap P,L}^M \mathcal{H}^i(\mathfrak{m}')$ contradicting the choice of $x$. \hfill \Box
Proof of Proposition 3.20 using Lemma 3.23. Let \( \mathfrak{M} \in \mathbf{M}(\mathfrak{g}) \) and write \( \mathfrak{N} := \text{Res}^G_{P,L}(\mathfrak{M}) \in \mathbf{D}(\mathfrak{g}) \)
Suppose that a cohomology object \( H^i(\mathfrak{N}) \) is non-zero for some \( i \neq 0 \). Then there must be a parabolic subgroup \( P' \) of \( G \) with Levi factor \( L' \) contained in \( L \) such that \( \text{Res}^L_{P',L,L'} H^i(\mathfrak{N}) \) is nonzero and cuspidal. By the inductive hypothesis, \( \text{Res}^L_{P',L,L'} \) is \( t \)-exact, and thus
\[
\text{Res}^L_{P',L,L'} H^i(\mathfrak{N}) = H^i \left( \text{Res}^L_{P',L,L'}(\mathfrak{N}) \right)
\]
As \( \text{Res}^G_{P,L}(\mathfrak{M}) \) is cuspidal, we can apply Lemma 3.23 to deduce that \( \text{Res}^L_{P',L,L'} H^i(\mathfrak{N}) \) is zero, contradicting the assumption.

3.7. Second adjunction. The following result was proved by Drinfeld-Gaitsgory [DG14] (using ideas developed by Braden [Bra03]).

Theorem 3.24. Let \( P^- \) denote the opposite parabolic subgroup of \( P \) with respect to \( L \). Then \( \text{Res}^G_{P^-} \) is left adjoint to \( \text{Ind}^G_{P,L} \).

Remark 3.25. The theorem gives a “cycle of adjoints” of length 4:
\[
\cdots \dashv \text{Ind}^G_{P,L} \dashv \text{Res}^G_{P,L} \dashv \text{Ind}^G_{P',L} \dashv \cdots
\]

Corollary 3.26. The functors of induction and restriction satisfy the following:
1. \( \text{Ind}^G_{P,L} \) is \( t \)-exact.
2. \( \text{Res}^G_{P,L} \) preserves coherent \( D \)-modules.

The corollary implies that there are exact functors:
\[
\text{ind}^G_{P,L} : \mathbf{M}(\mathfrak{g}) \leftarrow \mathbf{M}(\mathfrak{g}) : \text{res}^G_{P,L}
\]
which preserve coherent objects (induction automatically preserves coherent \( D \)-modules as it is a composite of a smooth pullback and proper pushforward).

Remark 3.27. It is possible to prove Corollary 3.26 without using the second adjunction. For example, one can prove the exactness of parabolic induction by a similar technique to Proposition 3.20.

Proof of Corollary 3.26. The functor \( \text{Ind}^G_{P,L} \) has a \( t \)-exact left adjoint (namely \( \text{Res}^G_{P^-} \)) and thus is left \( t \)-exact (i.e., preserves complexes supported in non-negative degrees). But it also has a \( t \)-exact right adjoint (namely, \( \text{Res}^G_{P,L} \)) and thus is also right \( t \)-exact, as required.

The second claim follows from the fact that \( \text{res}^G_{P,L} \) has a right adjoint which preserves (filtered) colimits (in this case, the right adjoint \( \text{ind}^G_{P,L} \) itself admits a right adjoint). We will spell this argument out below.

Let us first note that \( \mathbf{M}(\mathfrak{g}) \) is a full subcategory of \( \mathbf{M}(\mathfrak{g}) \) (warning: this is not true for the equivariant derived category), which is the category of modules for the Noetherian ring \( \mathcal{O}_\mathfrak{g} \). In particular, coherent objects are the same as finitely generated modules, and every object in \( \mathbf{M}(\mathfrak{g}) \) is a union of its coherent subobjects.

Given a coherent object \( \mathfrak{M} \in \mathbf{M}(\mathfrak{g}) \), consider the canonical map:
\[
\mathfrak{N} := \text{res}^G_{P,L}(\mathfrak{M}) \rightarrow \bigcup_{i \in I} \mathfrak{M}_i,
\]
where the union is over the coherent submodules \( N_i \) of \( \mathfrak{M} \). Consider the corresponding map under the (second) adjunction

\[
\mathfrak{M} \xrightarrow{\beta} \text{ind}_{P - L}^G \left( \bigcup_{i \in I} N_i \right) = \bigcup_{i \in I} \text{ind}_{P - L}^G (N_i).
\]

Here we have used the fact that \( \text{ind}_{P - L}^G \) commutes with filtered colimits (in this case, the union).

As \( \mathfrak{M} \) is coherent, and each \( \text{ind}_{P - L}^G N_i \) is coherent, the map \( \beta \) factors through a finite union

\[
\mathfrak{M} \xrightarrow{\beta_0} \bigcup_{i \in I_0} \text{ind}_{P - L}^G (N_i) \to \bigcup_{i \in I} \text{ind}_{P - L}^G (N_i).
\]

It follows that \( \alpha \) must also factor through a finite union; in other words, \( \mathfrak{M} \) is a finite union of coherent submodules so is itself coherent. □

4. Generalized Springer Theory

In this section we apply the theory of parabolic induction and restriction to the give a description of the abelian category \( M_p g q \).

4.1. The recollement situation associated to parabolic induction and restriction. To conclude this section, we will show how our results on parabolic induction and restriction give rise to a recollement situation for the category \( M_p g q \). Later, we will see that the recollement is, in fact, an orthogonal decomposition, so this result is not strictly required for the proof of the main theorems. However, it is helpful for other contexts (e.g. the mirabolic setting) to understand how a collection of induction and restriction functors which satisfy a Mackey formula, give rise to a recollement situation.

We adopt the same notation as in Corollary 3.17 for the abelian category \( M(g) \). Thus, \( M_p g q \) is the subcategory consisting of objects with singular support in \( \text{comm} (g) \), or by Corollary 3.17, those objects which are killed by \( \text{res}^G_{Q,M} \) for Levi subgroups \( M \) which do not contain a conjugate of \( L \). There is a similar definition of \( M_p g q \). Note that these are all Serre subcategories (as they are the kernel of exact functors).

Lemma 4.1. Parabolic induction and restriction restrict to a pair of adjoint functors:

\[
\text{ind}_{P,L}^G : M(\mathfrak{l})_{\text{cusp}} \rightleftarrows M(g)_{\geq (L)} : \text{res}_{P,L}^G
\]

Proof. It is clear from the above remarks that parabolic restriction takes objects of \( M(g)_{\geq (L)} \) to cuspidal objects on \( L \). On the other hand, if \( \mathfrak{M} \) is a cuspidal object of \( M(\mathfrak{l}) \), then it follows from the Mackey formula that \( \text{ind}_{P,L}^G (\mathfrak{M}) \) is in \( M(g)_{\geq (L)} \). Indeed, if \( M \) is a Levi subgroup which doesn’t contain a conjugate of \( L \), and \( Q \) a corresponding parabolic, then the terms in the Mackey formula for \( \text{res}_{Q,M}^G \text{ind}_{P,L}^G (\mathfrak{M}) \) all vanish. □

Note that the kernel of \( \text{res}_{P,L}^G \) restricted to \( M(g)_{\geq (L)} \) is \( M(g)_{> (L)} \). Let us denote by \( M(g)_{\geq (L)} \) the quotient category of \( M(g)_{\geq (L)} \) by \( M(g)_{> (L)} \). According to the results of Appendix A.2, we may deduce:
Proposition 4.2. There is a recollement situation:

\[
\begin{array}{ccc}
M(\mathfrak{g}) >_{(L)} & \xrightarrow{i_{>}(L)} & M(\mathfrak{g}) \geq_{(L)} \\
\downarrow & & \downarrow
\end{array}
\]

In particular:

- Every object \( \mathfrak{M} \in M(\mathfrak{g}) \geq_{(L)} \) fits in to an exact sequence:

\[
\begin{array}{cccccc}
i_{\leq}(L) & \mathfrak{M} & \longrightarrow & \mathfrak{M} & \longrightarrow & i_{>}(L)\ast i_{\leq}(L)\mathfrak{M} & \longrightarrow & 0
\end{array}
\]

- The projection functor \( i_{\leq}(L)\ast i_{\leq}(L)\mathfrak{M} \) is given by:

\[
\text{coker} \left( \text{ind}_{P,L}^{G} \text{res}_{P,L}^{G} \text{ind}_{P,L}^{G} \text{res}_{P,L}^{G} (\mathfrak{M}) \longrightarrow \text{ind}_{P,L}^{G} \text{res}_{P,L}^{G} (\mathfrak{M}) \right)
\]

where the map is given by the difference of the two morphisms coming from the counit of the adjunction.

- The quotient category \( M(\mathfrak{g}) \geq_{(L)} \) can be identified via \( i_{>}(L)\ast \mathfrak{M} \) with the subcategory of \( M(\mathfrak{g}) \geq_{(L)} \) consisting of objects which are parabolically induced from cuspidals on \( L \).

- Any object \( \mathfrak{M} \in M(\mathfrak{g}) \leq_{(L)} \) can be written as an extension of an object in \( M(\mathfrak{g}) \geq_{(L)} \) by a quotient of an object in the essential image of \( \text{ind}_{P,L}^{G} |_{\text{cusp}} \).

It follows that we can express any object in \( M(\mathfrak{g}) \) as an iterated extension of (quotients of) objects which are parabolically induced from cuspidals on various Levis. This can be seen by the following algorithm:

1. First, pick a minimal Levi \( L \) such that \( \mathfrak{M} \in M(\mathfrak{g}) \leq_{(L)} \). Note that \( L \) could be equal to \( G \) (if \( \mathfrak{M} \) is cuspidal) or to the maximal torus \( H \) (if \( \text{res}_{B,H}^{G} (\mathfrak{M}) \) is non-zero).
2. If \( L = G \), stop.
3. Consider the exact sequence in (5).
4. Replace \( \mathfrak{M} \) by \( i_{>}(L)\ast i_{\leq}(L)\mathfrak{M} \), and go back to step one.

Remark 4.3. In Subsection 4.6 we will see that the situation is much nicer: the monad \( L,P \text{st}_{P,L} \) restricted to cuspidal objects, is equivalent to the group monad of \( W_{G,L} \). As explained in Appendix A.3, this means that every object in the essential image of \( M(\mathfrak{g})_{(L)} \) is a direct summand (as opposed to just a quotient) of a parabolic induction from some cuspidal on \( L \).

Remark 4.4. There is an analogous recollement situation in the derived setting too, see [Gun].

4.2. The Springer block. In this subsection, we show how to deduce the “non-generalized” part of the generalized Springer theory presented in this paper (this is sufficient to prove the main results in the case \( G = GL_{n} \)). Fix a Borel subgroup \( B \) containing a maximal torus \( H \) with Weyl group \( W \). To simplify notation we will write \( \text{ind} = \text{ind}_{B,H}^{G}, \text{res} = \text{res}_{B,H}^{G}, \text{st} = \text{res} \circ \text{ind} \), and \( \mathfrak{g}_{L} = B_{L} \mathfrak{g}_{H} \).

Note that the adjunction \( \text{ind}, \text{res} \) fits in to the set-up of Appendix A.2. With that in mind, let \( M(\mathfrak{g})_{(H)} \) denote the Serre subcategory of \( M(\mathfrak{g}) \) consisting of objects killed by the functor \( \text{res} \), and \( M(\mathfrak{g})_{(H)} \) the quotient category of \( M(\mathfrak{g}) \) by \( M(\mathfrak{g})_{(H)} \). As explained in A.2 we can identify
M(\mathfrak{g})_{(H)} with the full subcategory of M(\mathfrak{g}) consisting of quotients of objects in the essential image of ind (we refer to this subcategory as the Springer block). Moreover, by the Barr-Beck Theorem we have the following:

**Lemma 4.5.** The functor res identifies M(\mathfrak{g})_{(H)} with modules for the monad st = res ∘ ind acting on M(\mathfrak{h}).

The Weyl group W acts on M(\mathfrak{h}), and W-equivariant objects are the same thing as modules for the monad W (see Appendix A.3). The following theorem captures the essence of the main results of this paper.

**Theorem 4.6.** There is an equivalence of monads st » W˚.

**Lemma 4.5** then gives the following immediate corollary:

**Corollary 4.7.** There is an equivalence of categories $M(\mathfrak{g})_{(H)} » M(\mathfrak{h}) » D(\mathfrak{h})_{\text{mod}}$.

We recover the classical result of Springer theory from Theorem 4.7 by restricting to the subcategory $M(\mathfrak{g})_{(H)}$ consisting of objects supported on the nilpotent cone.

**Corollary 4.8** ([BM83]). There is an equivalence of categories $M(\mathfrak{g})_{\mathrm{Spr}} » \text{Rep}(W)$.

In particular, the set of irreducible representations of W inject in to the set of simple objects of $M(\mathfrak{g})_{\mathrm{Spr}}$, which are indexed by pairs (O, L) of a nilpotent orbit O and a simple equivariant local system L on O.

**Remark 4.9.** In the case $G = GL_n$ or $PGL_n$, the Springer block is the whole of M(\mathfrak{g}), and thus the main result Theorem 4.6 is implied by Corollary 4.7 in this case. In general (e.g. for $G = SL_2$), there are other blocks that must be taken in to account.

In order to prove Theorem 4.6 we will identify the abelian category of colimit preserving endofunctors of $M(\mathfrak{h}) \simeq \mathcal{O}_\mathfrak{h} \text{ mod}$ with $\mathcal{O}_\mathfrak{h}$-bimodules, or equivalently, as objects in $M(\mathfrak{h} \times \mathfrak{h})$. We refer to an objects of $M(\mathfrak{h} \times \mathfrak{h})$ as integral kernels. Given an integral kernel $\mathfrak{R}$ the corresponding functor is given by

$$\mathfrak{M} \mapsto p_2^* (p_1^*(\mathfrak{M}) \otimes \mathfrak{R})$$

where $p_1, p_2 : \mathfrak{h} \times \mathfrak{h} \rightarrow \mathfrak{h}$ are the projections. Similarly, the composition of endofunctors corresponds to convolution of integral kernels:

$$\mathfrak{R}_1 * \mathfrak{R}_2 := p_{13}^* (p_{12}^*(\mathfrak{R}_1) \otimes p_{23}^*(\mathfrak{R}_2))$$

where $p_{12}, p_{23}$, and $p_{13}$ are the projection maps $\mathfrak{h}^2 \rightarrow \mathfrak{h}^2$. Thus monads acting on $M(\mathfrak{h})$ are the same as algebra objects in the monoidal category $M(\mathfrak{h} \times \mathfrak{h})$, under convolution

Let $\mathfrak{R}_{st}$ denote the integral kernel corresponding to the monad st. The integral kernel corresponding $w_*$ (for $w \in W$) is given by $\mathcal{O}_w := \Gamma_{w*} \mathcal{O}_\mathfrak{h}$ where $\Gamma_w : \mathfrak{h} \rightarrow \mathfrak{h} \times \mathfrak{h}$ is the graph of the $w$-action. In this language, the Mackey formula says that $\mathfrak{R}_{st} \in M(\mathfrak{h} \times \mathfrak{h})$ carries a filtration such

---

More algebraically, we could just think of a monad acting on $M(\mathfrak{h}) = \mathcal{O}_\mathfrak{h} \text{ mod}$ as a $\mathcal{O}_\mathfrak{h}$-ring. However, it will be more convenient to think of it geometrically as an object in $M(\mathfrak{h} \times \mathfrak{h})$. For example, we will want to localize over certain open subsets of $\mathfrak{h} \times \mathfrak{h}$. 

---
that the associated graded object is $\bigoplus_w \mathcal{O}_w$. Theorem 4.6 states that the Mackey filtration on $\mathcal{K}$ is canonically split, and moreover the splitting $\mathcal{K} \simeq \bigoplus_w \mathcal{O}_w$ is an isomorphism of monads.

**Remark 4.10.** The corresponding statement is false in the derived category setting: the Mackey filtration is non-split even in the $SL_2$ case, as shown in [Gun].

**Remark 4.11.** One can check that the kernel $\mathcal{K}_{st}$ is given by $H_0^p(\alpha \times \beta)_{\ast} \omega_{st}$, where $h_{st}$ are the projection maps on the Steinberg stack. Thus the $D$-module $\mathcal{K}_{st}$ over $h \times h$ records the fiberwise Borel-Moore homology (in a certain degree) of $\mathcal{K}_{st}$. The monad structure on the functor $\mathcal{K}_{st}$ reflects the convolution structure on Borel-Moore homology of the fibers. Theorem 4.22 can be thought of as a “relative” version of the statement that the Borel-Moore homology of the nilpotent Steinberg stack (which is the fiber of $\mathcal{K}_{st}$ over $(0,0) \in h \times h$) gives the group algebra of the Weyl group (see Chriss-Ginzburg [CG97]). The proof uses the same basic principle (as do all proofs in Springer theory): the fact that there is an honest Weyl group action over the regular locus, and specialize to the non-regular locus. The difference between our approach and that of other authors, is that the focus is on the entire integral kernel representing the Steinberg functor, rather than the endomorphism algebra of a single object (the Springer sheaf).

4.3. **Proof of Theorem 4.6.** The proof is divided into three elementary lemmas: the first implies that there exists a splitting of the Mackey filtration on $\mathcal{K}_{st}$; the second implies that there is a canonical splitting (as monads) over the regular locus; the third implies that such a splitting extends uniquely.

**Lemma 4.12.** Let $v, w \in W$, $v \neq w$. Then
\[ \text{Ext}^i_{M(h \times h)}(\mathcal{O}_v, \mathcal{O}_w) = 0, \]
for $i = 0, 1$.

**Lemma 4.13.** There is an isomorphism of algebra objects:
\[ \mathcal{K}_{st}|_{h^{reg} \times h^{reg}} \simeq \bigoplus_{w \in W} \mathcal{O}_w|_{h^{reg} \times h^{reg}}. \]

**Lemma 4.14.** We have:
\[ \text{Hom}_{M(h \times h)}(\mathcal{O}_v, \mathcal{O}_w) \simeq \text{Hom}_{M(h^{reg} \times h^{reg})}(\mathcal{O}_v|_{reg}, \mathcal{O}_w|_{reg}). \]

Let us deduce Theorem 4.6 now, leaving the proofs of the various lemmas for the end of this subsection. Lemma 4.12 shows that there must be some splitting of the Mackey filtration, i.e. there exists an isomorphism $\mathcal{K}_{st} \simeq \bigoplus_w \mathcal{O}_w$. Lemma 4.13 defines an isomorphism of monads
\[ \phi^{reg}: \mathcal{K}_{st}|_{h^{reg} \times h^{reg}} \to \bigoplus_w \mathcal{O}_w|_{h^{reg} \times h^{reg}} \]
after restricting to the regular locus. Lemma 4.14 shows that this isomorphism extends to an isomorphism in $M(h \times h)$:
\[ \phi: \mathcal{K}_{st} \to \bigoplus_w \mathcal{O}_w. \]
It remains to show that the monad structures on $\mathfrak{st}$ and on $W_*$ agree. The monad structure on the object $\bigoplus \mathcal{O}_w$ is given by a collection of isomorphisms

$$\tau_{v,w} : \mathcal{O}_w \ast \mathcal{O}_v \rightarrow \mathcal{O}_{wv},$$

for each pair $(v, w) \in W^2$. Similarly, the monad structure on $\mathfrak{st}$ induces a (a priori different) monad structure on $(W_{G,L})_*$ via the natural isomorphism $\phi$; we denote the corresponding structure morphisms by $\tau'_{v,w}$. Lemma 4.13 implies that $\tau = \tau'$ after restricting to the regular locus (we know $\phi$ is an isomorphism of monads there). On the other hand, by Lemma 4.14, if $\tau$ and $\tau'$ agree on the regular locus, then they must agree on all of $\mathfrak{h} \times \mathfrak{h}$. Thus $\phi$ is an isomorphism of monads as required. This completes the proof of Theorem 4.6.

Proof of Lemma 4.12. Let $\mathfrak{h}^{v,w} = \{x \in \mathfrak{h} \mid v(x) = w(x)\}$. We have the cartesian diagram:

$$
\begin{array}{ccc}
\mathfrak{h}^{v,w} & \xrightarrow{\gamma_{w}} & \mathfrak{h} \\
\gamma_{w} \downarrow & & \downarrow \Gamma_{v} \\
\mathfrak{h} & \xrightarrow{\Gamma_{w}} & \mathfrak{h} \times \mathfrak{h}
\end{array}
$$

where $\gamma_{w} = \gamma_{v}$ is the inclusion of $\mathfrak{h}^{v,w}$ into $\mathfrak{h}$. By our assumption $w \neq v$, $\mathfrak{h}^{v,w} \neq \mathfrak{h}$. Let $d$ denote the codimension of $\mathfrak{h}^{v,w}$ in $\mathfrak{h}$. We compute:

$$
R \text{Hom}_D(\mathfrak{h}^{v,w})(\mathcal{O}_v, \mathcal{O}_w) = R \text{Hom}_D(\mathfrak{h}^{v,w})(\Gamma_{w} \ast \mathcal{O}_v, \Gamma_{w} \ast \mathcal{O}_w) \\
= R \text{Hom}_D(\mathfrak{h}^{v,w})(\mathcal{O}_v, \mathcal{O}_w, \Gamma_{w} \ast \mathcal{O}_h) \\
= R \text{Hom}_D(\mathfrak{h}^{v,w})(\mathcal{O}_v, \mathcal{O}_w, \Gamma_{w} \ast \mathcal{O}_h) \\
= R \text{Hom}_D(\mathfrak{h}^{v,w})(\mathcal{O}_v, \mathcal{O}_w, \Gamma_{w} \ast \mathcal{O}_h) \\
= H^{\mathfrak{h}^{v,w}}_{dR}(\mathfrak{h}^{v,w})[d], \mathcal{O}_h^{v,w}[-d]) = H^{\mathfrak{h}^{v,w}}_{dR}(\mathfrak{h}^{v,w}).
$$

As $d > 0$, $\text{Ext}^1(\mathcal{O}_v, \mathcal{O}_w) = H^{1-2d}_{dR}(\mathfrak{h}^{v,w}) = 0$ as required. □

Proof of Lemma 4.13. By Proposition 2.14, the diagram which defines parabolic induction and restriction restricts to the $W_{G,L}$-Galois cover:

$$
\mathfrak{h}^{\text{reg}} \times L \cong \mathfrak{g}^{\text{reg}} \cong \mathfrak{g}^{\text{reg}} \to \mathfrak{g}(L)
$$

In particular, the restriction of the Steinberg stack $(\mathfrak{st}_L)_l$ is equivalent to the action groupoid of $W$ on $\mathfrak{g}^{\text{reg}}$. This directly translates into the claimed result on $\mathfrak{st}_L$. □

The proof of Lemma 4.14 is a straightforward calculation (the result only depends on the fact that $\mathfrak{h}$ and $\mathfrak{h}^{\text{reg}}$ are both connected).

Remark 4.15. Lemma 4.14 does not hold in the derived category (i.e. the $R\text{Hom}$ complexes are not quasi-isomorphic). This is why one cannot deduce the corresponding statement to Theorem 4.22 in the derived setting.
4.4. Orthogonality of parabolic induction from non-conjugate Levi subgroups. The following result is an important step in the orthogonal decomposition of Theorem 4.11.

Proposition 4.16. Suppose $L$ and $M$ are non-conjugate Levi subgroups. Given $\mathfrak{M} \in D^b_{\text{coh}}(\mathfrak{L})_{\text{cusp}}$, and $\mathfrak{N} \in D^b_{\text{coh}}(\mathfrak{M})_{\text{cusp}}$ we have that $\text{Ind}^G_{P,L}(\mathfrak{N})$ and $\text{Ind}^G_{Q,L}(\mathfrak{M})$ are orthogonal, i.e.

$$R\text{Hom}(\text{Ind}^G_{P,L}(\mathfrak{N}), \text{Ind}^G_{Q,L}(\mathfrak{M})) \simeq R\text{Hom}(\text{Ind}^G_{Q,L}(\mathfrak{M}), \text{Ind}^G_{P,L}(\mathfrak{N})) \simeq 0.$$  

Proof. First let assume $L$ is not conjugate to a subgroup of $M$. Let $\mathfrak{N} \in D^b_{\text{coh}}(\mathfrak{L})_{\text{cusp}}$ and $\mathfrak{M} \in D^b_{\text{coh}}(\mathfrak{L})_{\text{cusp}}$. As $M$ is not conjugate to a subgroup of $L$, for any $g \in G$, $M \cap gL$ is a proper subgroup of $gL$. Thus

$$Q_M \text{St}^w_{P,L}(\mathfrak{N}) \simeq \text{Ind}^M_{M \cap gP,M \cap gL} \text{Res}^L_{Q,M \cap gP,M \cap gL} \hat{w}_\ast \mathfrak{M} \simeq 0$$

as $\hat{w}_\ast \mathfrak{M}$ is cuspidal for each $\hat{w}$. By the Proposition 1.10 (the Mackey formula), $Q_M \text{St}^w_{P,L}(\mathfrak{N}) \simeq 0$. Thus

(6) $$R\text{Hom}(\text{Ind}^G_{P,L}(\mathfrak{N}), \text{Ind}^G_{Q,M}(\mathfrak{N})) = R\text{Hom}(\mathfrak{N}, Q_M \text{St}^w_{P,L}(\mathfrak{N})) \simeq 0.$$  

Note that the Verdier duality functor $\mathbb{D}_1$ preserves the category of cuspidal objects in $D^b(\mathfrak{L})$ and the functor $\text{Ind}^G_{P,L}$ intertwines the Verdier duality functors. Applying $\mathbb{D}_1$ with $\mathfrak{N}$ and $\mathfrak{M}$ replaced by $\mathbb{D}_1 \mathfrak{N}$ and $\mathbb{D}_1 \mathfrak{M}$, we obtain:

$$R\text{Hom}(\text{Ind}^G_{P,L}(\mathfrak{N}), \text{Ind}^G_{Q,M}(\mathfrak{N})) = R\text{Hom}(\mathbb{D}_1 \mathfrak{N}, \text{Ind}^G_{Q,M}(\mathfrak{N})) = R\text{Hom}(\text{Ind}^G_{Q,M}(\mathfrak{N}), \mathbb{D}_1 \mathfrak{N}) = 0,$$

Thus $\text{Ind}^G_{P,L}(\mathfrak{N})$ is orthogonal to $\text{Ind}^G_{Q,M}(\mathfrak{M})$ whenever $M$ is not conjugate to a subgroup of $L$. By switching the roles of $M$ and $L$, we obtain that $\text{Ind}^G_{P,L}(\mathfrak{N})$ and $\text{Ind}^G_{Q,M}(\mathfrak{M})$ are also orthogonal whenever $L$ is not conjugate to a subgroup of $M$. Thus $\text{Ind}^G_{P,L}(\mathfrak{N})$ and $\text{Ind}^G_{Q,M}(\mathfrak{M})$ are orthogonal whenever $M$ is not conjugate to $L$. (Note that $M$ is conjugate to $L$ if and only if $M$ is conjugate to a subgroup of $L$ and $L$ is conjugate to a subgroup of $M$).

Remark 4.17. The proposition immediately implies that direct summands of parabolic inductions from cuspidals on non-conjugate Levi subgroups are orthogonal. We will see in the following section that every object of the abelian category $M(Q,L)$ is a direct summand of $\text{ind}^G_{P,L}(\mathfrak{M})$ for some cuspidal object $\mathfrak{M} \in M(L)$. This proves that the subcategories $M(Q,L)$ are orthogonal for non-conjugate Levis, $L$.

In the proof above, we noted that the terms of the Mackey filtration of $M(Q,M)_{\text{St}^w_{P,L}(\mathfrak{N})}$ vanish unless $L$ is conjugate to a subgroup of $M$. Let us observe that if $M$ is conjugate to $L$, then the Mackey formula takes a special form:

Proposition 4.18. Suppose $\mathfrak{N} \in D^b(\mathfrak{L})_{\text{cusp}}$. Then $L \text{St}^w_{P,L}(\mathfrak{N}) \simeq 0$ unless $w \in W_{G,L} = N_G(L)/L$ (which is naturally identified with a subset of $L\backslash S(L,L)/L \sim P\backslash G/P$).

Thus $L \text{St}^w_{P,L}(\mathfrak{N})$ is an iterated extension of $w_\ast(\mathfrak{N})$ where $w$ ranges over $W_{G,L}$ when $\mathfrak{N}$ is cuspidal.
4.5. The category of cuspidal objects. Let $g' = g/\mathfrak{z}(g) \cong [g, g]$; thus $g = \mathfrak{z}(g) \times g'$. Note that the torus $Z^\circ(G)$ acts trivially on $g$, and equivariance for a connected group acting trivially does not affect the abelian category of $D$-modules. Thus equivariance for $G$ is the same thing as equivariance for $G' = G/Z^\circ(G)$. In particular, given an object $\mathcal{M} \in M(\mathfrak{z}(l))$ and $\mathcal{R} \in M(g')$, we have an object $\mathcal{M} \boxtimes \mathcal{R} \in M(g)$.

Lemma 4.19. An object $\mathcal{M} \boxtimes \mathcal{R} \in M(g)$ is cuspidal if and only if $\mathcal{R} \in M(g')$ is cuspidal.

Proof. Suppose $P$ is a parabolic subgroup of $G$ with Levi factor $L$. Set $P' = P/Z^\circ(G)$ and $L' = L/Z^\circ(G)$ (and similarly for the Lie algebras). We have the diagram:

$$
\begin{array}{cccc}
0 & \xrightarrow{\mathfrak{g}} & 0 & \xrightarrow{1} \\
\mathfrak{z}(g) \times g' & \xrightarrow{1} & \mathfrak{z}(g) \times p' & \xrightarrow{1} \mathfrak{z}(g) \times l'
\end{array}
$$

Thus, the functor of parabolic restriction factors as:

$$
\text{res}_{P,L}^{G} \cong \text{res}_{P',L'}^{G'} \boxtimes \text{id}_{M(\mathfrak{z}(l))}.
$$

The lemma follows immediately from this observation.

Definition 4.20. The category of orbital sheaves is defined to be $M(\mathcal{N}_G)$ (which we identify with a full subcategory of $M(g)$). The category of character sheaves is defined to be the essential image of $M(\mathcal{N}_G)$ under Fourier transform.

Let $\mathcal{E} = IC(\mathcal{O}, \mathcal{E})$ be a simple orbital sheaf, where $\mathcal{E}$ is a simple equivariant local system on a nilpotent orbit $\mathcal{O}$. Such a local system $\mathcal{E}$ is called cuspidal if $\mathcal{E}$ is a cuspidal orbital sheaf in $M(g)$, or equivalently $\mathcal{E} = \mathcal{O}_{\mathfrak{z}(l)} \boxtimes \mathcal{E}$ is a cuspidal character sheaf. Given a cuspidal local system $\mathcal{E}$, we define the full subcategory $M(g)(\mathcal{E})$ of $M(g)$ to consist of objects supported on $\mathcal{N}_G \times \mathfrak{z}(g)$ of the form $\mathcal{E} \boxtimes \mathcal{M}$, where $\mathcal{M}$ is any object in $M(\mathfrak{z}(g))$.

Proposition 4.21. The category of cuspidal objects decomposes as an orthogonal sum

$$
M(g)_{\text{cusp}} \cong \bigoplus \mathcal{M}(\mathfrak{z}(l)),
$$

indexed by cuspidal data $(\mathcal{O}, \mathcal{E})$. Moreover, $M(g)(\mathcal{E}) \cong M(\mathfrak{z}(g))$ via $\mathcal{M} \boxtimes \mathcal{E} \mapsto \mathcal{M}$.

Proof. Recall that every cuspidal object of $M(g)$ is supported on $\mathfrak{g}_G = \mathfrak{z}(g) \times \mathcal{N}_G$ (see Lemma 3.8). Also recall that the category $M(\mathcal{N}_G)$ is semisimple\footnote{This follows from the parity vanishing condition in [Lus86] Theorem 24.8.} Thus by Lemma 4.19 any object of $M(g)_{\text{cusp}}$ can be written as a direct sum of objects of the form $\mathcal{E} \boxtimes \mathcal{M}$, where $\mathcal{E}$ is a simple cuspidal object of $M(\mathcal{N}_G)$ and $\mathcal{M}$ is any object of $M(\mathfrak{z}(g))$. As the endomorphisms of $\mathcal{E}$ are one dimensional, we see that the functor $\mathcal{M} \mapsto \mathcal{E} \boxtimes \mathcal{M}$ is fully faithful, as claimed.
4.6. The relative Steinberg and Weyl monads. Let us fix a parabolic subgroup $P$ with Levi factor $L$. As explained in Subsection 4.1, we have a monadic adjunction:

$$\text{ind}_{G, P,L}^{G} : \text{M}^\mathfrak{g}_{\text{cusp}} \leftrightarrow \text{M}^\mathfrak{g}_{(L)} : \text{res}_{G, P,L}^{G}$$

where $\text{M}^\mathfrak{g}_{(L)}$ can be identified with the full subcategory of $\text{M}^\mathfrak{g}$ which is generated under colimits by the essential image of $\text{ind}_{G, P,L}^{G}$ restricted to cuspidal objects.

The corresponding monad is given by

$$L, p^{\text{st}}_{P,L} = \text{res}_{G, P,L}^{G} \text{ind}_{G, P,L}^{G} : \text{M}^\mathfrak{g}_{\text{cusp}} \to \text{M}^\mathfrak{g}_{\text{cusp}}$$

By the Mackey Formula (Proposition 1.6), the functor $L, p^{\text{st}}_{P,L}$ carries a filtration indexed by $P$ for which the associated graded piece corresponding to $w \in P \backslash G / P$ is $\text{ind}_{\text{M}_{\text{cusp}}}^{\text{M}} \text{res}_{\text{M}_{\text{cusp}}}^{\text{M}} \hat{w}_{G,L}$. By Proposition 4.18, the restriction of the monad $L, p^{\text{st}}_{P,L}$ to the subcategory of cuspidal objects has a filtration indexed by $W_{G,L}$ (with its poset structure defined by identification with a subset of $P \backslash G / P$) for which the associated graded functor is $\text{M}^\mathfrak{g}_{G,L}$.

The following key result generalizes Theorem 4.6 in the Springer block case.

**Theorem 4.22.** There is an equivalence of monads $L, p^{\text{st}}_{P,L} \simeq (W_{G,L})_{\hat{w}}$ acting on $\text{M}^\mathfrak{g}_{\text{cusp}}$.

As in the case of the Springer block, the basic idea behind the proof of Theorem 4.22 is to first observe that the statement holds when the functors are restricted to the regular locus, and then extend over the non-regular locus. This can be proved in a similar way to the Springer case, but it will be slightly more convenient to consider one cuspidal block at a time so we can apply the Lemmas in Subsection 4.3 directly. For this to make sense, we must have that the Steinberg monad preserves the decomposition

$$\text{M}^\mathfrak{g}_{\text{cusp}} = \bigoplus_{(\mathcal{E})} \text{M}^\mathfrak{g}_{(\mathcal{E})}$$

This is equivalent to the statement that $W_{G,L}$ acts trivially on the set of cuspidal data $(\mathcal{O}, \mathcal{E})$. This fact is a key feature of the generalized Springer correspondence, which we state below.

**Theorem 4.23** ([Lus84], Theorem 9.2).

1. Every cuspidal datum $(L, \mathcal{E})$ on $L$ is fixed under the action of $W_{G,L}$.
2. There is a summand of $\text{ind}_{G, P,L}^{G}(\mathcal{E})$ which appears with multiplicity one.

**Remark 4.24.** The results of [Lus84] are phrased in terms of the endomorphism algebra $A_{\mathcal{E}}$ of $\text{ind}_{G, P,L}^{G}(\mathcal{E})$. In particular, Lusztig shows in Theorem 9.2 that $A_{\mathcal{E}}$ is isomorphic to the group algebra of $W_{G,L}$. We can view this statement as a combination of three separate statements:

1. The algebra $A_{\mathcal{E}}$ is a twisted group algebra of $W_{G,L, \mathcal{E}}$ (the stabilizer of the cuspidal datum $(\mathcal{O}, \mathcal{E})$ in $W_{G,L}$).
2. The cuspidal datum $(\mathcal{O}, \mathcal{E})$ is fixed by $W_{G, L}$ (so $W_{G,L, \mathcal{E}} = W_{G,L}$).
3. The endomorphism algebra $A_{\mathcal{E}}$ has a module of dimension one (it follows that $A_{\mathcal{E}}$ is equivalent to the non-twisted group algebra).

The first statement above is a consequence of Theorem 4.22 and can be proved directly using the techniques in this paper. The two other statements (which are equivalent to those in Theorem 4.23) require more delicate calculations, encoded in the dimension estimates appearing in the proof of [Lus84], Theorem 9.2; we will not attempt to give a self-contained treatment of these results in the present paper.
Remark 4.25. In fact, Theorem 9.2 in [Lus84] gives more. Let $P_1, \ldots, P_r$ be a complete list of minimal parabolic subgroups of $G$ properly containing $P$, and $L_1, \ldots, L_r$ corresponding Levi factors containing $L$. Then $W_{L_i, L}$ is a subgroup of $W_{G, L}$ of order at most 2 (it acts faithfully on the 1-dimensional vector space $\mathfrak{z}(l)/\mathfrak{z}(l)$ by reflections). Let $s_i$ denote the generator of $W_{G, L}$ which acts by reflections on $\mathfrak{z}(l)/\mathfrak{z}(l)$. In particular, if $L$ is a Levi subgroup of $G$ which admits a cuspidal local system on a nilpotent $L$-orbit, then all the subgroups $W_{L_i, L}$ are non-trivial (and thus of order 2). In this case $W_{G, L}$ is the Weyl group of a certain root system on $\mathfrak{z}(l)/\mathfrak{z}(l)$ (see [Lus76] 5.9). This is a very strong restriction on which Levi subgroups of $G$ admit cuspidals. For example, the Levi subgroup $L_p$ of $SL_n$ consisting of block diagonal matrices corresponding to a partition $n = p_1 + \ldots + p_k$ can only admit a cuspidal when $p_1 = \ldots = p_k$ for some divisor $p_k$ of $n$, in which case $W_{G, L_p} = S_k$ acting by permuting the blocks of a matrix (and there are in fact cuspidals in that case see [Lus84] 10.3).

Remark 4.26. It is a result of Baal-Carter theory that if $O$ is a distinguished nilpotent $L$-orbit, then the $G$-saturation of $O$ intersects $l$ precisely at $O$ (see [Som98]). In particular $O$ is preserved by the action of $W_{G, L}$. The first part of Theorem 4.23 can be thought of as a generalization of this fact, where the set of distinguished orbits is refined by the set of cuspidal data.

Proof of Theorem 4.22 By Theorem 4.23 and the Mackey formula, parabolic induction and restriction restrict to a monadic adjunction for each cuspidal datum $(L, E)$:

$$
\text{ind}_{P, L}^G : M(\mathfrak{l}(E)) \hookrightarrow M(\mathfrak{g}(L, E)) : \text{res}_{P, L}^G
$$

Let $st_\mathcal{E}$ denote the restriction of the Steinberg monad to the block $M(\mathfrak{l}(E))$, which is equivalent to $M(\mathfrak{z}(l))$. Thus we can identify endofunctors of the block with $M(\mathfrak{z}(l) \times \mathfrak{z}(l))$. Just as for Theorem 4.10 the result then follows from the three lemmas in Subsection 4.3 after replacing $\mathfrak{l}$ by $\mathfrak{z}(l)$, and $W$ by $W_{G, L}$.

Corollary 4.27. Every simple cuspidal object $E$ in $M(N_L)$ carries a $W_{G, L}$-equivariant structure. In particular there is a $W_{G, L}$-equivariant equivalence: $M(\mathfrak{l}(E)) \simeq M(\mathfrak{z}(l))$

Proof. According to Theorem 4.22 $E$ is the parabolic restriction of an object in $M(N_G)$ (namely the simple summand of $\text{ind}_{P, L}^G(E)$ appearing with multiplicity one). This carries the structure of a $L, P, L$-module. This is the same as a $W_{G, L}$-equivariant structure by Theorem 4.22. □

4.7. Proof of the main results. In this subsection we will deduce Theorem A and Theorem B from Theorem 4.22. First we note:

Corollary 4.28. For each cuspidal datum $(L, E)$, there is an equivalence of categories:

$$
M(\mathfrak{g}(L, E)) \simeq M(\mathfrak{z}(l))^{W_{G, L}}.
$$

Proof. This is an application of the Barr-Beck theorem (see Appendix A.4). By construction, the functor

$$
\text{res}_{P, L}^G : M(\mathfrak{g}(L, E)) \to M(\mathfrak{g}(E))
$$

is monadic, and by Theorem 4.22 the corresponding monad is given by $(W_{G, L})_*$ acting on $M(\mathfrak{l}(E))$ (which is $W_{G, L}$-equivariantly identified with $M(\mathfrak{z}(l))$ by Corollary 4.27). □

By Proposition A.7 and Remark A.8 we obtain:
Lemma 4.29. For every object \( \mathfrak{M} \in \text{M}(\mathfrak{g})_{(L, \mathcal{E})} \), we have
\[
\mathfrak{M} \cong \left( \text{ind}^G_{P,L} \text{res}^G_L(\mathfrak{M}) \right)^{\text{W}_G,L}.
\]
In particular, \( \mathfrak{M} \) is a direct summand of \( \text{ind}^G_{P,L} \text{res}^G_L(\mathfrak{M}) \).

Thus the category \( \text{M}(\mathfrak{g})_{(L, \mathcal{E})} \) is just the Karoubian completion of the essential image of \( \text{ind}^G_{P,L} \) on \( \text{M}(\mathfrak{g})_{(\mathcal{E})} \) (i.e. the subcategory consisting of direct summands of parabolic induction from \( \text{M}(\mathfrak{g})_{(\mathcal{E})} \)).

Remark 4.30. The corresponding statement is not true for the derived blocks \( \text{D}(\mathfrak{g})_{(L, \mathcal{E})} \): not every such object is a direct summand of a parabolic induction. See [Gun] for further details.

Now we show that parabolic induction of cuspidal objects is independent of the choice of parabolic (later we will drop the cuspidal condition).

Corollary 4.31. Suppose \( P \) and \( P' \) are two parabolic subgroups which contain \( L \) as a Levi factor. Then there is a canonical natural isomorphism \( \text{ind}^G_{P,L} \cong \text{ind}^G_{P',L} \) after restricting to \( \text{M}(\mathfrak{g})_{(\mathcal{E})} \).

Proof. By adjunction, the data of a natural transformation \( \text{ind}^G_{P,L} \rightarrow \text{ind}^G_{P',L} \) is equivalent to that of a natural transformation
\[
\text{Id}_{\text{M}(\mathfrak{g})_{(\mathcal{E})}} \rightarrow L.P\text{st}_{P',L}
\]
By the same argument as in the proof of Theorem 4.22, there is a canonical natural isomorphism \( L.P\text{st}_{P',L} \cong (\text{W}_{G,L})_\ast \). This isomorphism is constructed by first observing that such an isomorphism exists once we restrict to the regular locus (where neither functor depends on the choice of parabolic), and there is a unique extension over the non-regular locus. The inclusion of the identity \( 1 \in \text{W}_{G,L} \) defines the required natural transformation \( (7) \). □

By the uniqueness of adjoints, it follows that parabolic restriction \( \text{res}^G_{P,L} \) is also independent of the choice of parabolic when restricted to \( \text{M}(\mathfrak{g})_{(L)} \) (we may drop this last assumption soon).

Corollary 4.32. The subcategories \( \text{M}(\mathfrak{g})_{(L, \mathcal{E})} \) and \( \text{M}(\mathfrak{g})_{(M, \mathcal{F})} \) are pairwise orthogonal if \( (L, \mathcal{E}) \) and \( (M, \mathcal{F}) \) are non-conjugate cuspidal data.

Proof. By Proposition 4.10 if \( M \) is not conjugate to \( L \) then the essential image of \( \text{ind}^G_{P,L}\text{cusp} \) is orthogonal to the essential image of \( \text{ind}^G_{Q,M}\text{cusp} \). Thus it follows that any direct summand of such objects must also be orthogonal, so by Lemma 4.29 \( \text{M}(\mathfrak{g})_{(L)} \) is orthogonal to \( \text{M}(\mathfrak{g})_{(M)} \). It remains to consider the case when \( M = L \), but \( \mathcal{E} \) is not conjugate to \( \mathcal{F} \). Recall that by the results of Subsection 4.4, \( \text{M}(\mathfrak{g})_{(\mathcal{E})} \) is orthogonal to \( \text{M}(\mathfrak{g})_{(\mathcal{F})} \), and these blocks are preserved by the action of \( \text{W}_{G,L} \) by Theorem 4.23. Applying the Mackey formula as in the proof of Proposition 4.10 gives the result. □

At this point, the proof of Theorems A, B, and C is just a matter of putting together the pieces.

Proof of Theorem A. We have shown that the subcategories, \( \text{M}(\mathfrak{g})_{(L, \mathcal{E})} \) are pairwise orthogonal, and each is equivalent to \( \text{M}((\mathfrak{g})_{W_G,L}) \) (for the relevant \( L \)). By the recollement situation of 4.1 every object of \( \text{M}(\mathfrak{g})_{(L, \mathcal{E})} \) is an iterated extension of objects of \( \text{M}(\mathfrak{g})_{(L, \mathcal{E})} \), we now know that these extensions must all split. We have proved Theorem A
\[
\text{M}(\mathfrak{g}) = \bigoplus_{(L, \mathcal{E})} \text{M}(\mathfrak{g})_{(L, \mathcal{E})} \cong \bigoplus_{(L, \mathcal{E})} \text{M}((\mathfrak{g})_{W_G,L})
\]
Proof of Theorem C. If \( M \) is a non-zero indecomposable object of \( \mathbf{M}(\mathfrak{g}) \), then it must lie in a unique block \( \mathbf{M}(\mathfrak{g})_{(L, E)} \). In that case, it is a direct summand of \( \text{ind}_P^G(\mathfrak{g}) \) for some \( \mathfrak{g} \in \mathbf{M}(\mathfrak{g})_{(L, E)} \) (namely, \( \mathfrak{g} = \text{res}_{P, L}^G(\mathfrak{g}) \)). By definition, the singular support of \( \mathfrak{g} \) is contained in \( \text{comm}(\mathfrak{g})_{> (L)} \), but not contained in \( \text{comm}(\mathfrak{g})_{> (L)} \) (otherwise \( \text{res}_{P, L}^G(\mathfrak{g}) \neq 0 \) and thus \( \mathfrak{g} \neq 0 \)).

It remains to clear up a few points in Theorem B, namely, the dependence on the parabolic and the splitting of the Mackey filtration. These points have been addressed (by Corollary 4.31 and Theorem 4.22) in the case of parabolic induction and restriction on a single cuspidal block. Thus, we must look at the functor of induction and restriction on all blocks at once.

Consider the functor \( \text{res}_{Q,M}^G \) for a fixed parabolic subgroup \( Q \) with Levi factor \( M \). If \( L \) is not conjugate to a subgroup of \( M \), then by the Mackey formula, \( \text{res}_{Q,M}^G \) kills the summand \( \mathbf{M}(\mathfrak{g})_{(L, E)} \).

On the other hand, if \( L \subseteq M \) and \( P \subseteq Q \), then \( (L, E) \) can also be considered as a cuspidal datum for \( M \). The compatibility of Proposition 1.7 implies that the functor \( \text{res}_{Q,M}^G \) restricts to a functor from \( \mathbf{M}(\mathfrak{g})_{(L, E)} \) to \( \mathbf{M}(\mathfrak{m})_{(L, E)} \). Under the equivalence of (8), this corresponds to the forgetful functor:

\[
\Upsilon_{W,G,L}^{W,M,L} : \mathbf{M}(\mathfrak{g})_{W,G,L} \to \mathbf{M}(\mathfrak{g})_{W,M,L}
\]

A similar statement holds for \( \text{ind}_{Q,M}^G \), which corresponds to the induction functor:

\[
\Gamma_{W,G,L}^{W,M,L} : \mathbf{M}(\mathfrak{g})_{W,G,L} \to \mathbf{M}(\mathfrak{g})_{W,M,L}
\]

Putting this all together we have the following commutative diagram (when read from top left to bottom right or from top right to bottom left):

\[
\begin{array}{ccc}
\mathbf{M}(\mathfrak{g}) & \xrightarrow{\text{res}_{Q,M}^G} & \mathbf{M}(\mathfrak{m}) \\
\oplus_{(L, E), L \subseteq G} \mathbf{M}(\mathfrak{g})_{W,G,L} & \xrightarrow{\text{ind}_{Q,M}^G} & \oplus_{(L, E), L \subseteq M} \mathbf{M}(\mathfrak{g})_{W,M,L}
\end{array}
\]

The remaining parts of Theorem B now follow: the Mackey formula must split (as it is equivalent to the Mackey formula corresponding to induction and restriction for finite groups) and the functors parabolic induction and restriction are independent of the choice of parabolic (as they are on each block).

Appendix A. Monads and Recollement situations

We review the Barr-Beck theorem in the context of Grothendieck abelian categories and explain how certain adjunctions give rise to recollement situations. The case when the monad is given by the action of a finite group is of particular interest to us. The proof of the Barr-Beck theorem can be found in [BW85], and the background material on abelian categories in [Pop73]. A modern
treatment of the theory of recollement situations for abelian categories can be found in [FP]. We will only give brief sketches of the proofs here.

Throughout this appendix, we maintain the following:

**Assumptions A.1.** Let $\mathcal{C}$ and $\mathcal{D}$ be Grothendieck abelian categories, $F : \mathcal{D} \to \mathcal{C}$ is an functor with a left adjoint $F^L$. We assume that both $F$ and $F^L$ are exact and preserve direct sums (it follows from the adjoint functor theorem that $F$ also has a right adjoint $F^R$).

**A.1. The Barr-Beck theorem and recollement situations.** Let $T = FF^L$ denote the corresponding monad acting on $\mathcal{C}$. We denote by $\mathcal{C}^T$ the category of $T$-modules (also known as $T$-algebras) in $\mathcal{C}$. Note that for any object $d \in \mathcal{D}$, $F(d)$ is a module for $T$. Thus we have the following diagram:

$\begin{array}{ccc}
\mathcal{D} & \xrightarrow{F} & \mathcal{C} \\
\downarrow{F^L} & & \downarrow{C^T} \\
\mathcal{C}^T & \xleftarrow{\iota} & \\
\end{array}$

**Definition A.2.** A functor $F : \mathcal{D} \to \mathcal{C}$ is called **conservative** if whenever $F(x) \approx 0$ then $x \approx 0$.

**Theorem A.3 (Barr-Beck [BW85]).** The functor $\tilde{F} : \mathcal{D} \to \mathcal{C}^T$ has a fully faithful left adjoint, $J$. If $F$ is conservative, then $\tilde{F}$ and $J$ are inverse equivalences.

There is an explicit formula for the left adjoint, as follows. Given an object $c \in \mathcal{C}^T$, consider the diagram

$$F^L F F^L c \cong F^L c$$

where one of the maps is given by the $T$-module structure on $c$, and the other is given by the counit map of the adjunction. Then $J(c)$ is defined to be the coequalizer of this diagram, or in other words, the cokernel of the map given by the difference of the two maps above. One can check that $J$ is left adjoint to $\tilde{F}$ and that the unit $c \sim \tilde{F} J(c)$, so that $J$ is fully faithful as required.

**A.2. Adjunctions and Recollement situations.** We further study the case when the functor $F$ is not necessarily conservative. Let $\mathcal{K}$ denote the kernel of $F$, i.e. the full subcategory of $\mathcal{D}$ consisting of objects $d$ such that $F(d) \approx 0$. Let $Q$ denote the quotient category $\mathcal{D}/\mathcal{K}$, which is the localization of $\mathcal{D}$ with respect to the multiplicative system of morphisms that are taken to isomorphisms under the functor $F$. The subcategory $\mathcal{K}$ is automatically localizing, and thus the quotient morphism $\iota^* : \mathcal{D} \to Q$ has a fully faithful right adjoint, denoted $i_\circ$. (see e.g. [Pop73]). On the other hand, $F$ descends to a conservative functor on the quotient $Q$, and thus by the Barr-Beck theorem, we can identify $Q$ with the category of $T$-modules, $\mathcal{C}^T$. Using this identification, the bar construction defines a fully faithful left adjoint, which we denote $j_\circ$.

Let $i_\circ : \mathcal{K} \to \mathcal{D}$ denote the embedding. Consider the functor $\iota : \mathcal{D} \to \mathcal{D}$ given by the cokernel of the unit map $Id_{\mathcal{D}} \to i_\circ^* j_\circ$. The essential image of the functor $\iota$ is contained in $\mathcal{K}$, and thus we can write $\iota = i_{a} i^*$ for where $i^* : \mathcal{D} \to \mathcal{K}$ is left adjoint to $i_a$. Similarly, the kernel of the counit map $i_a j^* : Id_{\mathcal{D}}$ is of the form $i_a i_1^*$, where $i_1^*$ is right adjoint to $i_a$.

---

8The usual definition of a conservative functor is a functor $F$ such that if $F(\phi)$ is an isomorphism, then $\phi$ is an isomorphism. This definition is equivalent to the one above, in our context, by considering the the kernel and cokernel of $\phi$. 

These facts are summarized as follows:

**Theorem A.4.** There is a recollement situation:

![Recollement Diagram]

In particular:
- The functor $j^*$ is left adjoint to $j_*$ and right adjoint to $j_!$.
- The functor $i_*$ is left adjoint to $i^!$ and right adjoint to $i^*$.
- The functors $i_!, j_!, j_*$ are fully faithful.
- There are exact sequences of functors
  
  $$0 \to i_* i^! \to \text{Id}_D \to j_* j^! \to j_* j^! \to \text{Id}_D \to i_* i^! \to 0$$

Also note that, by the construction of the left adjoint $j_*$, the essential image of $Q$ in $j_*$ consists of quotients of objects in the essential image of $F^L$.

**Remark A.5.** If the categories $C$ and $D$ are compactly generated, and the functor $F$, in addition, takes compact objects in $D$ to compact objects in $C$, then the right adjoint $F^R$ preserves direct sums. In that case, the recollement situation of Theorem A.4 restricts to one on the level of small categories of compact objects (this is the case for the main example in this paper).

**A.3. Monads and finite group actions.** Suppose additionally that a finite group $W$ acts on $C$. This means that there are functors

$$w_* : C \to C,$$

together with natural isomorphisms $\phi_{w,v} : w_* v_* \simeq (wv)_*$ satisfying the natural cocycle condition. There is an associated monad $W_*$ acting on $C$ given by the formula:

$$W_*(c) = \bigoplus_{w \in W} w_*(c).$$

The category $C^W$ of $W$-equivariant objects, is equivalent to the category of modules for the monad $W_*$.  

**Remark A.6.** The functor $W_*$ also admits the structure of a comonad acting on $C$ (and $W$-equivariant objects in $C$ can also be identified with comodules for this comonad). Considering this monad and comonad structure together, one arrives at the notion of Frobenius monad.

**Proposition A.7.** Suppose there is an isomorphism of monads $T \simeq W_*$. Given an object $c \in C^W$, $W$ acts by automorphisms on the object $F^L(c)$ such that the object $j!(c)$ is given by the coinvariants $F^L(c)_W$ of this action.

**Proof.** The $W$ action comes from the identities:

$$\text{End}(F^L(c)) \simeq \text{Hom}(c, FF^L(c)) \simeq \text{Hom}(c, W_*(c)) \simeq Z[W] \otimes \text{End}(c).$$
The identification of $j_!(c)$ with the coinvariants is by inspection of the formula for $j_!$ as a coequalizer.

**Remark A.8.**

1. Note that the functor of coinvariants for a finite group action is exact (and agrees with the functor of invariants via the norm map). In particular, $j_!(c)$ is a direct summand of $F^L(c)$.

2. If $\mathcal{C}$ and $\mathcal{D}$ are $\mathbb{C}$-linear abelian categories and $c \in \mathcal{C}^T$ is a simple object, then $\text{End}(F^L(c)) \cong \mathbb{C}[W]$.

**A.4. Filtrations.** The following notation will be useful for us later.

**Definition A.9.** A filtration of an object $a$ in a category $\mathcal{C}$, indexed by a poset $(I, \leq)$, is a functor

$$(I, \leq) \to \mathcal{C}/a,$$

$$i \mapsto a_{\leq i} \to a.$$ 

In the cases of interest to us, $I$ will be a finite poset with a maximal element $i_{\text{max}}$, and we demand in addition that $a_{\leq i_{\text{max}}} \to a$ is an isomorphism.

We will apply this definition in two settings: either $\mathcal{C}$ is a Grothendieck abelian category as in the previous subsections, or $\mathcal{C}$ is a triangulated category (or stable $\infty$-category). In the abelian category setting we will ask for the structure maps to be monomorphisms, but in the triangulated/stable setting, we have no such condition.

Let $a_{<i}$ denote the colimit of $a_{\leq j}$ over $j < i$. For any $i \in I$, we set $a_i$ to be the cokernel (or cone) of $a_{<i} \to a_{\leq i}$. Thus, we think of the object $a$ as being built from $a_i$ by a sequence of extensions. The associated graded object is defined to be $\bigoplus_{i \in I} a_i$.

**Appendix B. D-modules**

There is a vast literature on the theory of $D$-modules; a good elementary reference is the book of Hotta–Takeuchi–Tanisaki [HTT08]. The triangulated category of equivariant $D$-modules was defined by Bernstein–Lunts [BL94] (in the context of sheaves) and Beilinson–Drinfeld [BD], and $\infty$-categorical enhancements have been considered in [GR11] [BZN09] (see also the recent book project of Gaitsgory–Rozenblyum [GR17]). Below we outline some of the key properties of the theory that we will need for this paper.

**B.1. On smooth affine varieties.** Suppose $U$ is a smooth affine algebraic variety, and let $\mathfrak{D}_U$ denote the ring of algebraic differential operators on $U$. We will write $\mathbf{M}(U)$ for the category of $\mathfrak{D}_U$-modules. An object of $\mathbf{M}(U)$ is called coherent if it is finitely generated (or equivalently, finitely presented), and we denote the subcategory of such objects by $\mathbf{M}_{\text{coh}}(U)$.

We denote by $\mathbf{D}(U)$, the (unbounded) derived category of $\mathbf{M}(U)$. Really, I will want to consider the corresponding stable $\infty$-category, but for the purposes of this paper we will only need to consider its homotopy category, which is the usual, triangulated, derived category. The compact objects of $\mathbf{D}(U)$ are given by perfect complexes of $\mathfrak{D}_U$-modules (or equivalently, bounded complexes whose cohomology objects are finitely generated $\mathfrak{D}_U$-modules, as $\mathfrak{D}_U$ has finite homological dimension); such complexes are called coherent.

Given a morphism of smooth affine algebraic varieties $f : U \to V$
we have transfer bimodules $\mathcal{D}_{U \rightarrow V}$ and $\mathcal{D}_{V \rightarrow U}$. Derived relative tensor product defines functors:

$$f^\circ : D(V) \rightarrow D(U)$$
$$f_* : D(U) \rightarrow D(V).$$

If $f$ is smooth, then $f^\circ$ is $t$-exact and preserves coherence; if $f$ is a closed embedding, then $f_*$ is $t$-exact and preserves coherent objects.

B.2. **On quotient stacks.** Let $K$ be an affine algebraic group acting on the smooth variety $U$. There is a morphism of algebras

$$\mu^* : U(\mathfrak{k}) \rightarrow \mathcal{D}_U$$

representing the infinitesimal action. Given an object $\mathfrak{M}$ of $\mathcal{M}(U)$ (i.e. a $\mathcal{D}_U$-module), a $K$-equivariant structure on $\mathfrak{M}$ is an action of $K$ on $\mathfrak{M}$ by $\mathcal{D}_U$-module morphisms, such that the corresponding action of $\mathfrak{k}$ agrees with the one coming from $\mu^*$. We write $\mathcal{M}(U)^K$ or $\mathcal{M}(X)$ for the abelian category of $K$-equivariant $\mathcal{D}_U$-modules (where $X = U/K$ is the quotient stack).

Note that if $K$ is connected, then $K$-equivariance is actually a *condition*: the $K$-action (when it exists) is determined by $\mu^*$. Thus $\mathcal{M}(U)^K$ embeds as a full subcategory of $\mathcal{M}(U)$ in that case.

It turns out that the “correct” definition of $D(X) = D(U)^K$ does not agree with the derived category of $\mathcal{M}(X)$ in general. Definitions and properties of the equivariant derived category $D(X)$ (in varying degrees of generality) can be found in [BD, BZN09, GR17]. This is a triangulated category which carries a $t$-structure, whose heart is $\mathcal{M}(X)$. The subcategory $D_{coh}^b(X)$ consists of complexes with finitely many non-zero cohomology groups, each of which are coherent (as $D$-modules on $Y$). The category $D(X)$ is compactly generated and the subcategory $D_{coh}(X)$ of compact objects is contained in $D_{coh}^b(X)$ (these subcategories agree when $X$ is a scheme, but may differ if $X$ is a stack).

B.3. **Functors.** Suppose $U$ and $V$ are smooth varieties, $K$ is an algebraic group acting on $U$, $L$ is an algebraic group acting on $V$, and we fix a homomorphism $\phi : K \rightarrow L$. A morphism $\tilde{f} : U \rightarrow V$ which is equivariant for the group actions (via $\phi$) gives rise to a morphism of quotient stacks $f : U/K \rightarrow V/L$. This morphism is *representable* if $\phi$ is injective, and *safe* if the kernel of $\phi$ is unipotent.

**Remark B.1.** The term *safe* was defined more generally in the paper [DG13] of Drinfeld-Gaitsgory. It is shown in that paper that safe morphisms of stacks give rise to a well-behaved pair of functors on $D$-modules.

The morphism $f$ of quotient stacks is *smooth* if and only if the corresponding morphism $\tilde{f} : U \rightarrow V$ is smooth; in that case, the *relative dimension* of $f$ is given by the relative dimension of $\tilde{f}$ minus the dimension of the kernel of $K \rightarrow L$. The morphism $f$ is called proper if it is representable and $\tilde{f}$ is proper.

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9One approach is to use the theory of stable $\infty$-categories (developed by Lurie in [Lur09, Lur11]), which can be thought of as an enhancement of theory of triangulated categories (the homotopy category of a stable $\infty$-category is a triangulated category). In this enhanced setting, $D(X)$ can be defined as the limit (in some appropriate $\infty$-category of stable $\infty$-categories) of the cosimplicial diagram of categories obtained from the Čech simplicial object associated to the cover $U \rightarrow X$ (this is a homotopical formulation of the notion of descent). In this paper we will not need to consider the $\infty$-categorical enhancements, only their underlying triangulated category.
Example B.2. Let $B$ be a Borel subgroup of a reductive algebraic group, $U$ the unipotent radical of $B$, and $H = B/U$. Then the morphism

$$B_{/\text{ad}} \to H_{/\text{ad}}$$

is safe and smooth of relative dimension 0. Similarly, $U_{/\text{ad}}$ is safe and smooth of relative dimension 0 over a point. On the other hand, the morphism $pt/H \to pt$ is not safe. The morphism $B/B \to G/G$ is proper.

Given a safe morphism of quotient stacks $f : X \to Y$, we have functors:

$$f_* : D(X) \to D(Y),$$

and

$$f^* : D(Y) \to D(X),$$

induced by the corresponding morphisms for $\hat{C}(f) : \hat{C}(U/K) \to \hat{C}(V/L)$ (to make sense of this, one must use the base-change theorem; see Proposition B.4 below). We define the functor $f^!$ as $f^*[\dim(X) - \dim(Y)]$.

Following the convention of [BZN09], we define $D_X(\mathfrak{m})$ by the usual formula for $D$-module duality when $\mathfrak{m} \in D^b_{\text{coh}}(X)$, and extend by continuity to define the functor

$$D_X : D(X) \to D(X)' .$$

Remark B.3. If $\mathfrak{m}, \mathfrak{n} \in D^b_{\text{coh}}(X)$, then

$$R\text{Hom}(\mathfrak{m}, \mathfrak{n}) \simeq R\text{Hom}(D(\mathfrak{m}), D(\mathfrak{n})).$$

This formula fails in general, if we drop the coherence assumption.

Here, we gather all the properties of $D$-module functors that we may need in this paper.

Proposition B.4 ([GR17] [HTT08]).

1. If $f$ is proper, then $f_* \simeq D_Y f_* D_X$ preserves coherence and is right adjoint to $f^!$. We sometimes write $f_!$ instead of $f_*$ in that case.

2. If $f$ is smooth of relative dimension $d$, then $f^!$ preserves coherence and $f_* := f^![-2d]$ is left adjoint to $f_*$. The functor $f^* = f^![-d]$ $t$-exact, and $f^* \simeq D_X f^* D_Y$.

3. If

$$\begin{array}{ccc}
X \times_W V & \overset{j}{\longrightarrow} & V \\
\downarrow f & & \downarrow g \\
X & \underset{f}{\longrightarrow} & W
\end{array}$$

is a cartesian diagram of stacks, then the base change morphism is an isomorphism: $g^! f_* \simeq g_* f^!$.

4. We have the projection formula:

$$f_* (f^! \mathfrak{m} \otimes \mathfrak{n}) \simeq \mathfrak{m} \otimes f_* (\mathfrak{n}).$$

5. The category $D(X)$ carries a symmetric monoidal tensor product

$$\mathfrak{m} \otimes \mathfrak{n} := \Delta^!(\mathfrak{m} \boxtimes \mathfrak{n}) \simeq \mathfrak{m} \otimes_{\mathcal{O}_X} \mathfrak{n}[ - \dim(X) ].$$
(6) We have an internal \(\text{Hom} \): 
\[
\text{Hom}(\mathcal{M}, \mathcal{N}) := D(\mathcal{M}) \otimes \mathcal{N}.
\]
If \(\mathcal{M}\) and \(\mathcal{N}\) are in \(D_{\text{coh}}^{b}(X)\) then 
\[
R\text{Hom}(\mathcal{M}, \mathcal{N}) = p_{\ast}X\text{Hom}(\mathcal{M}, \mathcal{N}).
\]

B.4. Singular support. Given a smooth algebraic variety \(V\), we will equip the ring of differential operators \(D_{V}\) with the order filtration, in which the functions Sym\((V^{\ast})\) have degree 0, and vector fields Sym\((V^{\ast}) \otimes V\) are given degree 1. We can form its Rees algebra \(D_{V}^{\ast}\) which is a graded algebra over \(\mathbb{C}[\hbar]\). It is again generated by Sym\((V)\) and Sym\((V^{\ast})\), with the relation:
\[
[v, w^{\ast}] = hw^{\ast}(v).
\]
We denote the abelian category of graded \(D_{V}^{b}\)-modules by \(M^{b}(V)\), and its derived category \(D^{b}(V)\).

We have the subcategory \(M_{\text{coh}}^{b}(V)\) of finitely generated, graded \(D_{V}^{b}\)-modules, and the subcategory \(D_{\text{coh}}^{b}(V)\) of complexes with finitely many nonzero cohomology groups, each of which are coherent.

Taking the fiber at \(\hbar = 0\), we obtain:
\[
D_{V}^{0} := D_{V}^{b}/hD_{V}^{b} \cong \mathcal{O}_{T^{\ast}V}.
\]
On the other hand, taking the fiber at \(\hbar = 1\), recovers the ring \(D_{V}\).

Given a module \(\mathcal{M} \in M(V)\), a compatible filtration on \(\mathcal{M}\) defines an object \(\mathcal{M}_{\hbar} \in M_{\hbar}(V)\) by taking the Rees construction. A good filtration of \(\mathcal{M}\) is a compatible filtration such that the corresponding Rees module \(\mathcal{M}_{\hbar}\) is coherent. Given such a filtration we may take the fiber \(\mathcal{M}_{0} := \mathcal{M}/h\mathcal{M}\) at \(\hbar = 0\) which is a module for \(D_{V}^{0} = \text{Sym}(V \oplus V^{\ast})\), or in other words, a coherent sheaf on \(V \times V^{\ast}\). The singular support \(SS(\mathcal{M})\) of \(\mathcal{M}\) is defined to be the support of \(\mathcal{M}_{0}\): it is a conic subset of \(V \times V^{\ast}\), which is independent of the choice of filtration.

Given any \(D_{V}\)-module \(\mathcal{M}\), we define the singular support of \(\mathcal{M}\) (as a set), to be the union of the singular support of all its coherent submodules (we only need this definition to make sense of statements of the form “\(SS(\mathcal{M})\) is contained in a certain closed subvariety”). Given a \(D_{V}^{b}\)-module \(\mathcal{M}_{\hbar}\), which specializes to \(\mathcal{M}\) at \(\hbar = 1\), it is not necessarily the case that \(\text{Supp}(\mathcal{M}_{\hbar}) = SS(\mathcal{M})\). However, the following fact (which the author was not able to locate in the literature) gives a containment.

**Proposition B.5.** Suppose \(\mathcal{M}_{\hbar} \in M_{\hbar}(V)\). Then for every coherent submodule \(\mathcal{N}\) of \(\mathcal{M}_{1}\),
\[
SS(\mathcal{N}) \subseteq \text{Supp}(\mathcal{M}_{0}).
\]

**Proof.** One issue is that \(\mathcal{M}_{\hbar}\) may not arise from a filtration (let alone a good filtration) on \(\mathcal{M}_{1}\); in other words, \(\mathcal{M}_{\hbar}\) may have \(h\)-torsion. On the other hand, replacing \(\mathcal{M}_{\hbar}\) by its \(h\)-torsion free quotient, can only make the support of \(\mathcal{M}_{0}\) smaller. Thus we may assume \(\mathcal{M}_{\hbar}\) is torsion free (and thus is obtained from a filtration on \(\mathcal{M}_{1}\) by the Rees construction).

It is enough to check that the singular support of every cyclic submodule \(D_{V}u_{1}\) of \(\mathcal{M}_{1}\) is contained in \(\text{Supp}(\mathcal{M}_{0})\). Let us lift the element \(u_{1} \in \mathcal{M}_{1}\) to \(u_{\hbar} \in \mathcal{M}_{\hbar}\), and consider the cyclic module \(D_{V}^{b}u_{\hbar}\).

This is the Rees construction of the induced (necessarily good) filtration on \(D_{V}u_{1}\). Moreover, the specialization \(D_{V}u_{1}\) to \(\hbar = 0\) is a submodule of \(\mathcal{M}_{0}\) (as everything is flat over \(\hbar\)), and thus has support contained in \(\text{Supp}(\mathcal{M}_{0})\) as required. \(\square\)
Given a morphism of smooth algebraic varieties
\[ f : V \to W, \]
we have functors
\[ f_! : D(V) \to D(W), \]
and
\[ f^! : D(W) \to D(V). \]
These may be defined by transfer bimodules in much the same way as the usual $D$-module functors.

Recall that the morphism $f$ gives rise to the correspondence:
\[ T \leftarrow T \leftarrow \pi^! \to \to \leftarrow T \leftarrow T \leftarrow T^* \]

The following proposition is easily checked (see, for example, [CK15]).

**Proposition B.6.** Given a map $f : V \to W$, as above, and $M, N \in M(V)$, $\pi : M(W)$, we have:
\[ \text{gr } f_! M \cong p_2 \pi f^! \text{gr } M \]
\[ \text{gr } f^! N \cong \pi f p_2 \text{gr } N \]

In particular, combining Proposition B.5 with B.6, we obtain:

**Proposition B.7.** Given a map $f : V \to W$, as above, and $M, N \in M(V)$, $\pi : M(W)$, we have:
\[ SS(f_! M) \subseteq p_2 \pi f^{-1} SS(M) \]
\[ SS(f^! N) \subseteq \pi f p_2^{-1} SS(N) \]

The following result may be checked by noting that the associated graded of a good filtered $D_V$-module $M$ is locally isomorphic to $M$ as an $O_V$-module.

**Lemma B.8 (Kas03, Proposition 2.8).** Given $M \in M(V)$, we have $\pi(S(M)) = \text{Supp}(M)$, where $\pi : T^* V \to V$ is the projection.

**B.5. Coisotropic subvarieties and Gabber’s Theorem.** Recall that a linear subspace $W$ of a symplectic vector space $(V, \omega)$ is called coisotropic if the symplectic orthogonal $W^\perp$ is contained in $W$. A (not necessarily linear) closed subvariety $C$ of $V$ is called coisotropic if the tangent space $T_p C$ at any smooth point $p$ is a linear coisotropic subspace of $V \simeq T_p V$.

Let us recall the following theorem of Gabber [Gab81].

**Theorem B.9 (Gabber [Gab81]).** The singular support of a coherent $D$-module on a vector space $V$ is a coisotropic subvariety of $T^* V = V \times V^*$.

**Corollary B.10.** Suppose $M \in D(V)$, and $X \subseteq T^* V$ is a subset which contains no non-empty coisotropic subvarieties. If $SS(M) \subseteq X$ then $M \cong 0$.

**Proof.** Take any coherent submodule $M'$ of any cohomology object $\mathcal{H}^i(M)$ of the complex $M$. Then its singular support is contained in $X$ and thus is empty; it follows that $M' \cong 0$, and thus the entire complex $M$ is equivalent to zero as required. \[\square\]

---

\[\text{This condition is equivalent to the vanishing ideal of } C \text{ being closed under the Poisson bracket in } O(V) - \text{ see [CG92], Proposition 1.5.1.}\]

\[\text{For an elementary proof in a much more general setting, due to F. Knop, see [Gins] Theorem 1.2.5} \]
The following lemma gives a family of examples of subvarieties \( X \) satisfying the condition of Corollary B.10.

**Lemma B.11.** Suppose \( V \) is a vector space with a non-degenerate symmetric bilinear form, and \( K_1, \ldots, K_n \) are non-degenerate proper linear subspaces. We consider \( V \times V \) as a symplectic vector space using the symmetric form on \( V \). Let \( N \subset V \times V \) denote the union of \( K_i \times K_j \) for pairs \((i, j)\) such that \( K_i + K_j \neq V \). Then there are no non-empty coisotropic subvarieties of \( V \) which are contained in \( N \).

**Proof.** Suppose \( C \) is a coisotropic subvariety of \( V \times V \) which is contained in \( N \), and let \( p \) be a smooth point of \( C \). Then the tangent space \( T_p C \) is a linear subspace of \( V \times V \) contained in \( K_i \times K_j \) for some \((i, j)\) where \( K_i + K_j \neq V \). Thus it is sufficient to prove that \( K_i \times K_j \) does not contain any linear coisotropic subspaces.

As \( K_i + K_j \neq V \), there exists a non-zero element \( v \in (K_i + K_j)\perp = K_i^\perp \cap K_j^\perp \) (where \( \perp \) denotes the orthogonal with respect to the symmetric form on \( V \)). Suppose \( W \) is a subspace of \( K_i \times K_j \). Then \((v, v)\) is orthogonal to \( W \) with respect to the symplectic form, but \((v, v)\) is not contained in \( K_i \times K_j \) and thus not contained in \( W \). Thus \( W \) cannot contain its own symplectic orthogonal, and thus cannot be coisotropic, as required.

**Example B.12.** If \( K \subset V \) is a single proper non-degenerate subspace, then Lemma B.11 states that \( K \times K \) cannot contain a coisotropic subvariety of \( V \times V \).

B.6. **Fourier Transform.** Let \( V \) be a complex vector space with dual space \( V^* \). Fourier transform for \( D\)-modules is given by an isomorphism \( \mathcal{D}_V \simeq \mathcal{D}_{V^*} \) which identifies \( V^* \subset \mathcal{D}_V \) (given by linear functions) with \( V^* \subset \mathcal{D}_{V^*} \) (given by constant coefficient derivations) by \(-1\) times the identity map, and identifies \( V \subset \mathcal{D}_V \) and \( V \subset \mathcal{D}_{V^*} \) by the identity map. This isomorphism gives rise to \( t\)-exact inverse equivalences:

\[
\mathcal{D}(V) \xrightarrow{\mathcal{F}_V} \mathcal{D}(V^*)
\]

**Lemma B.13.** Suppose \( i : A \hookrightarrow V \) is the inclusion of a linear subspace; let \( p : V^* \to A^* \) denote the adjoint to \( i \). We have:

\[
\mathcal{F}_{A^*} = p_* \mathcal{F}_V : \mathcal{D}(V) \to \mathcal{D}(A^*)
\]

\[
\mathcal{F}_V i_* \simeq p^* \mathcal{F}_A : \mathcal{D}(A) \to \mathcal{D}(V^*).
\]

**Proof.** We have that \( i^* \) and \( p_* \) are both given by the formula \( \text{Sym}(A^*) \otimes_{\text{Sym}(V^*)} (-) \); it is readily checked that they agree as functors after identifying the source and target via Fourier transform as required (similarly for \( i_* \) and \( p^* \)).

**Lemma B.14.** Suppose \( \mathfrak{M} \in \mathcal{M}(V) \), \( \text{Supp}(\mathfrak{M}) \) is contained in a closed conical subset \( C \subset V \), and \( \text{Supp}(\mathfrak{F}(\mathfrak{M})) \) is contained in a closed conical subset \( D \subset V^* \). Then \( SS(\mathfrak{M}) \subseteq C \times D \).

**Proof.** By Lemma B.8, \( SS(\mathfrak{M}) \subseteq C \times V^* \). Let \( I_C \subseteq \text{Sym}(V) \) denote the homogeneous (radical) ideal whose corresponding vanishing set is \( C \). Pick a coherent submodule \( \mathfrak{M}' \) of \( \mathfrak{M} \), and a good filtration \( F_0 \subseteq F_1 \subseteq \ldots \) of \( \mathfrak{M}' \). Then as \( \text{Supp}(\mathfrak{F}(\mathfrak{M}')) \) \( \subseteq \text{Supp}(\mathfrak{F}(\mathfrak{M})) \), for every \( m \in F_i \) there exists a positive integer \( N \) such that \( I_C^N m = 0 \); thus \( I_C^N \mathfrak{M}' = 0 \) also, which means that \( SS(\mathfrak{M}) \subseteq V \times C \) as claimed.
Remark B.15. The converse of Lemma [B.14] does not hold in general. For example, take $\mathfrak{M} = F\delta_\alpha$ to be the Fourier transform of a delta-function $D$-module at a non-zero point $\alpha \in V^*$. 

Example B.16. Combining Example [B.12], Corollary [B.10] and Lemma [B.14], we see that if a $\mathcal{O}_V$-module $\mathfrak{M}$ and its Fourier transform are both supported on a non-degenerate closed subspace of $K$ of $V$, then $\mathfrak{M} \simeq 0$.

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