Abstract. The purpose of this article is to study whether a given tropical curve $\Gamma$ in $\mathbb{R}^n$ can be realized as the tropicalization of an algebraic curve whose non-archimedean skeleton is faithfully represented by $\Gamma$. We give an affirmative answer to this question for a large class of tropical curves that includes all trivalent tropical curves, but also many tropical curves of higher valence. We then deduce that for every metric graph $G$ with rational edge lengths, there exists a smooth algebraic curve in a toric variety whose analytification has skeleton $G$, and the corresponding tropicalization is faithful. Our approach is based on a combination of the theory of toric schemes over discrete valuation rings and logarithmically smooth deformation theory, expanding on a framework introduced by Nishinou and Siebert.

1. Introduction

1.1. Let $N$ be a finitely generated free abelian group, and write $M = \text{Hom}(N, \mathbb{Z})$ for its dual. We denote by $N_\mathbb{R}$ the vector space $N \otimes \mathbb{R}$. Let $K$ be a non-archimedean field, let $T$ be the split algebraic torus $\text{Spec} K[M]$, and let $T^{\text{an}}$ be the non-archimedean analytic space associated to $T$ in the sense of [Ber90]. Following [Gub07], [Gub13], and [EKL06], one can define a continuous tropicalization map $\text{trop} : T^{\text{an}} \longrightarrow N_\mathbb{R}$.

A point $x \in T^{\text{an}}$ corresponds to a multiplicative seminorm $| \cdot |_x$ on $K[M]$ extending the norm on $K$ and its image $\text{trop}(x) \in N_\mathbb{R}$ is defined by the equation
$$\langle m, \text{trop}(x) \rangle = -\log |\chi^m|_x$$
for all $m \in M$, where $\chi^m$ denotes the character in $K[M]$ associated to $m$.

Given a subvariety $Y$ of $T$, its associated tropical variety $\text{Trop}(Y)$ is the image of the closed analytic subspace $Y^{\text{an}}$ of $T^{\text{an}}$ onto $N_\mathbb{R}$ under the tropicalization map; that is,
$$\text{Trop}(Y) = \text{trop}(Y^{\text{an}}).$$

By the Bieri-Groves Theorem [BG84, Theorem A] and [EKL06, Theorem 2.2.3] the set $\text{Trop}(Y)$ can be endowed with the structure of a finite rational polyhedral complex in $N_\mathbb{R}$. Moreover, if $Y$ is a curve, then $\text{Trop}(Y)$ can be canonically endowed with the structure of a metric graph using the lattice length on $N_\mathbb{R}$ with respect to $N$. For the basics of tropical geometry we refer the reader to [Gub13] and [MS14].

1.2. Let us now assume that $K$ is not trivially valued, and denote by $R$ its valuation ring. Let $C$ be a smooth, complete, and connected curve over $K$. While the underlying topological space of the non-archimedean analytic curve $C^{\text{an}}$ is an infinite graph, Berkovich shows in [Ber90, Section 4.3] that it has the homotopy type of a finite graph. More precisely, he associates to every semistable $R$-model $C$ of $C$, i.e. a flat and proper $R$-scheme $C$ with
generic fiber C and nodal special fiber $C_s$, a subset $\Sigma_C$ of $C^{an}$, called a skeleton, and shows that it is a deformation retract of $C^{an}$. Note that a semistable model $C$ of $C$ always exists, possibly after a finite separable base change, by the semistable reduction theorem [DM69].

As an abstract graph, the skeleton $\Sigma_C$ can be described explicitly as the dual graph of the special fiber $C_s$ of $C$. Recall that a dual graph of a curve $C_s$ is the finite graph containing a vertex for every irreducible component of $C_s$ and an edge between two (not necessarily distinct) vertices whenever the corresponding components meet in a node. Moreover, there is a metric on $\Sigma_C$ which is defined as follows: the étale local equation of a node is of the form $xy = r$ for some $r$ in the maximal ideal of $R$, and the length of the corresponding edge of $\Sigma_C$ is defined to be $\text{val}(r)$; we then endow $\Sigma_C$ with the shortest path metric.

Let $K^{\text{alg}}$ denote a fixed algebraic closure of $K$. Given a finite set $V$ of $K^{\text{alg}}$-points of $C$, we can associate an enlarged skeleton $\Sigma_{C,V}$ by adding to $\Sigma_C$, for every $v \in V$, an unbounded edge emanating from the vertex corresponding to the component of $C_s$ which contains the specialization of $v$. Again, there is an embedding of this enlarged skeleton as a deformation retract $\Sigma_{C,V}$ of $C^{an}$, together with a metric on $\Sigma_{C,V}$, extending the one of $\Sigma_C$, such that all the new rays have infinite length. Therefore, the metric graph $\Sigma_{C,V}$ is finite if and only if $V = \emptyset$, and in this case, $\Sigma_{C,V} = \Sigma_C$. We refer the reader to [BPR13, Section 3] for the details of this construction.

1.3. Given an embedding of $C$ into a toric variety $X$ with big torus $T$, we say that the corresponding tropicalization is faithful with respect to a skeleton $\Sigma_{C,V}$ if trop induces an isometric homeomorphism from $\Sigma_{C,V}$ onto its image in $\text{Trop}(C \cap T)$. If, in addition, the skeleton $\Sigma_{C,V}$ surjects onto $\text{Trop}(C \cap T)$, we say that the tropicalization is totally faithful with respect to $\Sigma_{C,V}$. Examples in [BPR11, Section 2.5] show that many tropicalizations are not faithful; so it is natural to ask the following question:

**Question 1.** Given a smooth, complete, and connected curve $C$ over $K$ and a skeleton $\Sigma_{C,V}$ of $C^{an}$, can we find an embedding of $C$ into a toric variety $X$ such that the corresponding tropicalization is faithful with respect to $\Sigma_{C,V}$?

In [BPR11, Section 6] the authors give a positive answer to this question in the case when $K$ is algebraically closed and $V$ is empty. For higher-dimensional varieties the same question has recently been considered in [GRW14]. In general, however, the methods of [BPR11] do not give us a totally faithful tropicalization.

In this article we study the following related question:

**Question 2** (Faithful realizability). Given a tropical curve $\Gamma \subseteq N_R$, can we find a smooth, complete and connected curve $C$ over $K$, a skeleton $\Sigma_{C,V}$ of $C^{an}$, and an embedding of $C$ into a toric variety $X$ with dense torus $T$ such that

$$\text{Trop}(C \cap T) = \Gamma$$

and the tropicalization is faithful with respect to $\Sigma_{C,V}$? Moreover, can we do so in such a way that the tropicalization is totally faithful with respect to $\Sigma_{C,V}$?

This question is a refinement of the classical question of realizability of tropical curves that asks whether there is an algebraic curve $C$ that can be realized as a subvariety of a toric variety $X$ with dense torus $T$ such that $\text{Trop}(C \cap T) = \Gamma$, where $\Gamma \subseteq N_R$ is a tropical curve.
1.4. We give a positive answer to both parts of Question 2, working over a finite extension of \( \mathbb{C}((t)) \), when \( \Gamma \subseteq \mathbb{N}_\mathbb{R} \) is a tropical curve with rational edge lengths and fulfills the following three conditions:

(A) \( \Gamma \) is non-superabundant.
(B) \( \Gamma \) is a smooth tropical curve.
(C) Every subgraph \( G \) of \( \Gamma \) contains a vertex that is adjacent to at most two other vertices of \( G \).

The notion of non-superabundancy for tropical curves, which will be defined at the beginning of Section 4, is a natural genericity condition, originally introduced in [Mik05, Definition 2.22], roughly saying that the deformation space of \( \Gamma \) as an embedded metric graph in \( \mathbb{N}_\mathbb{R} \) has the expected dimension; in our situation we adopt the point of view of [Kat12, Section 1]. Note that condition (C) is always satisfied by trivalent tropical curves. Moreover, we also obtain that the tropicalization map induces a surjection of \( \Sigma_{\mathcal{C},V} \) onto \( \Gamma \). More precisely, we prove the following theorem:

**Theorem 1.1.** Let \( \Gamma \) be tropical curve with rational edge lengths that fulfills the conditions (A), (B) and (C). Then there exists

- a finite extension \( K \) of \( \mathbb{C}((t)) \) with valuation ring \( R \),
- a toric scheme \( \mathfrak{X} \) over \( R \) with big torus \( T \),
- a complete smooth curve \( C \) over \( K \),
- a semistable \( R \)-model \( \mathcal{C} \) of \( C \) together with an embedding of \( \mathcal{C} \) into \( \mathfrak{X} \), and
- a finite set \( V \subseteq C(K^{\text{alg}}) \) of marked points,

such that

\[ \text{Trop}(C \cap T) = \Gamma \]

and the tropicalization is totally faithful with respect to \( \Sigma_{\mathcal{C},V} \).

Now we can combine Theorem 1.1 with a result of Cartwright–Dudzik–Manjunath–Yao [CDMY14], which states the following: Given a metric graph \( G \) with rational edge lengths, there is a tropical curve \( \Gamma \) in \( \mathbb{R}^n \) fulfilling the above conditions, such that \( G \) is the skeleton of \( \Gamma \). We will recall their result and adjust it to our needs in Theorem 5.2. This leads to the following theorem:

**Theorem 1.2.** For every metric graph \( G \) with rational edge lengths there exists

- a tropical curve \( \Gamma \) in \( \mathbb{R}^n \), where
  \[ n = \max \left\{ 3, \max \{ \deg v - 1 | v \in E(G) \} \right\}, \]
  - a finite extension \( K \) of \( \mathbb{C}((t)) \) with valuation ring \( R \),
  - a toric scheme \( \mathfrak{X} \) over \( R \) with big torus \( T = \mathbb{G}_m^n \),
  - a complete smooth curve \( C \) over \( K \),
  - a semistable \( R \)-model \( \mathcal{C} \) of \( C \) together with an embedding of \( \mathcal{C} \) into \( \mathfrak{X} \), and
  - a finite set \( V \subseteq C(K^{\text{alg}}) \) of marked points,

such that

\[ \text{Trop}(C \cap T) = \Gamma, \]

the skeleton \( \Sigma_{\mathcal{C}} \) of \( C^{\text{can}} \) is equal to \( G \), and the tropicalization is totally faithful with respect to \( \Sigma_{\mathcal{C},V} \).
In particular, if $G$ has no univalent vertices then the skeleton $\Sigma_C$ in Theorem 1.2 is the minimal skeleton of $C_{\text{an}}$, since each proper subgraph of $G$ has homotopy type different than the homotopy type of $C_{\text{an}}$.

Note that the methods of [CDMY14] allow us to consider more general graphs $G$ that may have some infinite rays. Our proof carries over to this situation, giving us the same data as in Theorem 1.2 together with an additional subset $V' \subseteq V$ of marked points such that $G$ is now equal to the skeleton $\Sigma_{C,V'}$.

1.5. Our proof of Theorem 1.1 generalizes the methods developed in [NS06] and extended in [Nis14]. Let us now outline the steps that go into this proof: Let $\Gamma$ be a tropical curve in $N_{\mathbb{R}}$ with rational edge lengths fulfilling the requirements (A), (B) and (C). Choose a finite extension $K = \mathbb{C}((t^{1/3}))$ of $\mathbb{C}((t))$ such that all vertices of $\Gamma$ have coordinates in the value group of $K$. Define $\Delta$ to be the fan in $N_{\mathbb{R}} \times \mathbb{R}_{\geq 0}$ obtained by putting a copy of $\Gamma$ in the height one part $N_{\mathbb{R}} \times \{1\}$ of $N_{\mathbb{R}} \times \mathbb{R}_{\geq 0}$ and taking cones over all edges and vertices of $\Gamma$. The fan $\Delta$ defines a toric scheme $X = X_\Delta$ over $R = \mathbb{C}[[t^{1/3}]]$.

(I) In Section 2 we define a suitable nodal curve $C_0$ in the special fiber $X_s$ of $X$, whose dual graph is the skeleton of $\Gamma$.

(II) The special fiber $X_s$ is log smooth over the standard log point $O_0 = (\text{Spec} \mathbb{C}, \mathbb{N})$; and the curve $C_0$, endowed with the log structure induced from $X_s$, is log smooth over $O_0$. These observations allow us to apply log smooth deformation theory in Section 3 and Section 4 in order to show that we can lift the nodal curve $C_0$ to a proper, flat semistable curve $C \subseteq X$ over $R$ with special fiber $C_0$. The generic fiber $C$ of $C$ is a smooth complete curve in the generic fiber $X$ of $X$, which is a $T$-toric variety over $K$. We set $V = C \cap (X - T)$.

(III) Finally, in Section 5, we verify that this construction satisfies the properties asserted in Theorem 1.1: Lemma 5.1 shows that in this case $\text{Trop}(C \cap T) = \Gamma$, and the results of [BPR11, Section 6] allow us to show that $\text{trop}|_{\Sigma_{C,V}} : \Sigma_{C,V} \to \text{Trop}(C)$ is totally faithful.

The crucial technical insight of this paper is contained in Section 4, where we give a combinatorial interpretation to the homomorphism of cohomology groups on $C_0$ controlling the logarithmic deformation theory used in Step (II) of our proof. In particular, we deduce that if $\Gamma$ is non-superabundant, i.e. if it satisfies condition (A), this homomorphism is surjective and the deformations are unobstructed.

The conditions (B) and (C) we impose on our tropical curve $\Gamma$ are only used in Step (I) of our proof. The proof would work equally well with any other condition on $\Gamma$ that allows us to construct a suitable nodal curve $C_0$ in $X_s$. Note that the classical condition of trivality used extensively in the literature implies condition (C), therefore Theorem 5.1.7 holds in particular for all smooth non-superabundant trivalent tropical curves.

1.6. The question of realizability of tropical curves has initially been studied by Mikhalkin [Mik05] in order to count algebraic curves in toric surfaces. Nishinou and Siebert [NS06] generalize his results to loopless (hence non-superabundant) trivalent tropical curves in higher dimension, and Tyomkin [Tyo12] further generalizes this to non-superabundant trivalent tropical curves of higher genus. See [Gro11, Chapter 4] for an expository account of the methods of [NS06]. Special conditions ensuring the realizability of some superabundant tropical curves are studied by Speyer in [Spe05], and by Katz in [Kat12]. In the upcoming
Nishinou extends the approach of [NS06] to more general trivalent tropical curves (see also [Nis09, Section 3]). Our Theorem 1.1 in particular gives realizations for a wide class of tropical curves, including tropical curves of higher valences and higher genera.

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2. Construction of the special fiber

We start by recalling the definition of a smooth tropical curve, which is the one-dimensional version of [MZ13, Definition 1.14]. For a planar tropical curve this also coincides with [Mik05, Definition 2.18].

Definition 2.1. A tropical curve \( \Gamma \) in \( \mathbb{N}_R \) is said to be smooth, if for every point \( x \in \Gamma \) there exists a basis \( \{b_1, \ldots, b_n\} \) of \( \mathbb{N} \) and a positive integer \( 1 \leq d \leq n \) such that the rays and segments of \( \Gamma \) adjacent to \( x \) are in the directions of \( \{b_1, \ldots, b_d, -\sum_{i=1}^{d} b_i\} \).

This means that, since the balancing condition is automatically satisfied, a smooth tropical curve is a union of finitely many segments and rays in \( \mathbb{N}_R \) departing from points in \( \mathbb{N}Q \) and satisfying the condition in 2.1.

Given a smooth tropical curve \( \Gamma \), its skeleton is the metric graph \( G \subseteq \Gamma \) defined as follows:

- Its vertices are the points of \( \Gamma \) having at least three adjacent segments or rays,
- its edges are the (bounded) line segments in \( \Gamma \), and
- the length of the edge between the two vertices \( v_1 \) and \( v_2 \) is the biggest positive real number \( l \) such that \( v_2 - v_1 = lv \) for some \( v \in \mathbb{N} \).

That is, if we view the tropical curve \( \Gamma \) as a graph with unbounded edges, then \( G \) is the graph obtained from \( \Gamma \) by removing all unbounded edges. If \( v \) is a vertex of \( \Gamma \), the bounded valence of \( v \), i.e. the number of bounded edges of \( \Gamma \) departing from \( v \), is equal to the valence of \( v \) in \( G \) and smaller than or equal to the valence of \( v \) in \( \Gamma \).

Let \( \Gamma \) be a tropical curve in \( \mathbb{N}_R \). Let \( K = \mathbb{C}((t^{1/\ell})) \) be a finite extension of \( \mathbb{C}((t)) \) such that all the vertices of the underlying graph of \( \Gamma \) lie in \( N_{v(K^x)} = N \otimes \mathbb{Z} v(K^x) \), where \( v(K^x) \) is the value group of \( K \), and let \( R = \mathbb{C}[[t^{1/\ell}]] \) be the valuation ring of \( K \). Define \( \Delta \) to be
the fan in $N_R \times \mathbb{R}_{\geq 0}$ obtained by putting a copy of $\Gamma$ into the height one part $N_R \times \{1\}$ of $N_R \times \mathbb{R}_{\geq 0}$ and taking cones over all edges and vertices of $\Gamma$. Then $\Delta$ is $v(K^\times)$-admissible in the sense of [Gub13, Definition 7.5] and we can set $\mathfrak{X} = \mathfrak{X}_\Delta$ to be the toric scheme over $R$ defined by $\Delta$.

**Remark 2.2.** We could compactify $\mathfrak{X}$ by completing the cone $\Delta$, but prefer not to do so, since we only want to keep the toric strata that are relevant to our construction.

Using the correspondence of [Gub13, 7.9] we can give an explicit description of the toric scheme $\mathfrak{X}$. Its generic fiber $\mathfrak{X}_\eta$ is the toric variety over $K$ associated to the fan $\Delta_0 = \Delta \cap (N_R \times \{0\})$, which is the recession cone of $\Gamma$. Its special fiber $\mathfrak{X}_s$ is reduced by [Gub13, Lemma 7.10], since the valuation $v$ is discrete and the vertices of $\Gamma$ are in $N_{v(K^\times)}$. The irreducible components of $\mathfrak{X}_s$ are toric varieties over the residue field $\mathbb{C}$, and they correspond bijectively to the vertices of $\Gamma$. Whenever two vertices $v$ and $w$ are connected by an edge $e$, the two components $X_v$ and $X_w$ are glued along the boundary divisor corresponding to $e$.

If $\Gamma$ is smooth and $v$ is a vertex of $\Gamma$ with $\text{val}_\Gamma(v) = d + 1$, from the explicit description of $\Gamma$ around $v$ of 2.1 we deduce that the corresponding component $X_v$ of $\mathfrak{X}_s$ is isomorphic to $P^d \times \mathbb{G}_m^{n-d}$, where $P_d := \mathbb{P}^d \setminus \{\text{orbits of codimension 2 or higher}\}$, and the boundary divisors of $P_d$ are all isomorphic to $\mathbb{G}_m^{d-1}$.

**Example 2.3.** Consider the admissible cone $\Delta \subseteq \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$ obtained as described above by placing at height 1 the following tropical curve $\Gamma \subseteq \mathbb{R}^2$:

![Tropical Curve](image)

We obtain a toric scheme $\mathfrak{X}$ over $\mathbb{C}[[t]]$ whose generic fiber $X$ is the toric variety associated to the recession cone $\Delta_0 = \Delta \cap (\mathbb{R}^2 \times \{0\})$ of $\Gamma$, which is the following fan in $\mathbb{R}^2$:

![Fan](image)

Therefore $X$ is the toric surface obtained by blowing up $\mathbb{P}^2_{\mathbb{C}[[t]]}$ in three points. The special fiber $\mathfrak{X}_s$ of $\mathfrak{X}$ consists of six copies of $\mathbb{P}^2_{\mathbb{C}}$ without their closed torus invariant points, glued over the one-dimensional toric strata as indicated by their moment polytopes below:
In order to construct a suitable curve in the special fiber of $\mathfrak{X}$, we need to study more carefully the condition (C). The next lemma provides a condition which is equivalent to (C) but easier to work with in practice.

**Lemma 2.4.** Let $G$ be a graph. Then $G$ satisfies condition (C) if and only if it satisfies the following condition:

$(C')$ There exists an ordering $\{v_1, v_2, \ldots\}$ of the vertices of $G$ such that for every $i$ the vertex $v_i$ is adjacent to at most two of the vertices $\{v_j\}_{j<i}$.

**Proof.** If $G$ satisfies (C) we inductively construct an ordering of the vertices of $G$ by picking at the $i$-th step a vertex $v_i$ which has bounded valence at most two in the subgraph of $G$ generated by the vertices which haven’t been picked yet. Then the ordering we obtain by reversing the one we just defined satisfies $(C')$. Conversely, assume by contradiction that $G$ satisfies $(C')$ but not (C), and let $G'$ be a nonempty subgraph of $G$ such that all the vertices of $G'$ have bounded valence at least three. Let $\{v_i\}_i$ be an ordering of the vertices of $G$ as in $(C')$ and let $i_0$ be the biggest index such that $v_{i_0} \in G'$. But $v_{i_0}$ has bounded valence at least three in $G'$ and all other vertices of $G'$ are smaller than $v_{i_0}$, contradicting our hypothesis that the ordering $\{v_i\}_i$ satisfies $(C')$. □

The following proposition is the main result of this section.

**Proposition 2.5.** Suppose that $\Gamma$ is a tropical curve satisfying the conditions (B) and (C). Then there is a complete and connected curve $C_0 \subseteq \mathfrak{X}_s$ fulfilling the following properties:

(i) For each vertex $v$ of $\Gamma$ denote by $C_v$ the intersection of $C_0$ with the irreducible component $X_v$ of $\mathfrak{X}_s$, then $C_v \cong \mathbb{P}^1$.

(ii) Each $C_v$ intersects every toric boundary stratum of $\mathfrak{X}_v$ transversally.

(iii) If $v$ and $w$ are two vertices of $\Gamma$ connected by an edge, then the components $C_v$ and $C_w$ intersect in a node.

**Proof.** Let $\{v_i\}_i$ be an ordering of the vertices of $\Gamma$ as in condition $(C')$. For every $i$, we inductively define a smooth rational curve $C_{v_i}$ in $P_d \times \{1\} \subseteq X_{v_i} \cong P_d \times G_{m-d}$ subject to the following condition: if $v_j$ is a vertex of $\Gamma$ adjacent to $C_{v_i}$, for $j < i$, the two curves $C_{v_i}$ and $C_{v_j}$ intersect in a point of $X_{v_i} \cap X_{v_j}$. Such a curve $C_{v_i}$ exists since the condition that we are imposing is the passage through at most two given points. Moreover, we can choose the curves $C_{v_i}$ meeting transversally each boundary stratum of $P_d \times \{1\}$, since a generic line in $\mathbb{P}^d$ intersects any coordinate hyperplane transversally away from the strata of codimension two or higher. Finally, let $C_0$ be the union of the curves $C_{v_i}$. □

**Example 2.6.** Consider the tropical curve $\Gamma$ and the associated toric scheme $\mathfrak{X}$ as in Example 2.3. Then the curve $C_0$ in $\mathfrak{X}_s$ constructed in Proposition 2.5 is a loop consisting of six copies of $\mathbb{P}^1_C$, and it can be visualized using tropical lines in the moment polytopes as follows:
Remark 2.7. The condition (C) is satisfied by every trivalent tropical curve. Indeed, if $\Gamma$ is a trivalent tropical curve, then $\Gamma$ has at least one unbounded edge. To see this, let $A$ be the minimal affine subspace of $N_\mathbb{R}$ containing $\Gamma$. Since $\Gamma$ is trivalent, $A$ is not a point. Now, the balancing condition implies that every half plane $H^+$ of $A$ intersects $\Gamma$, or otherwise $\Gamma$ would be contained in a hyperplane of $A$ parallel to the one defining $H^+$, and so $\Gamma$ contains an unbounded edge $e$. Therefore, the vertex of $\Gamma$ adjacent to $e$ has bounded valence at least two, hence we can remove it as a first step for showing that condition (C’) is satisfied. By removing one vertex, and all the adjacent edges, one new vertex of valence at most two appears. We can repeat the argument until we remove all vertices of $\Gamma$, thus showing that $\Gamma$ satisfies condition (C’), and therefore also condition (C). Moreover, all tropical curves of genus at most three satisfy condition (C), since the smallest graph with all vertices of bounded valence at least three is a tetrahedron.

Remark 2.8. Note that the curve $C_0$ constructed in Proposition 2.5 is a nodal curve, and its dual graph is equal to the graph underlying the skeleton of $\Gamma$. We remind the reader that the data of an $n$-dimensional toric scheme over $R$ is essentially equivalent to the notion of a toric degeneration, a toric morphism from a complex toric variety of dimension $n + 1$ to $\mathbb{A}^1_C$ as in [NS06, Section 3], and the embedding $C_0 \subseteq \mathcal{X}_s$ is a pre-log curve in the sense of [NS06, Definition 4.3].

3. Log smooth deformation theory

In this section, we explain under which conditions we can use log smooth deformation theory to lift the nodal curve $C_0$ to a semistable curve $\mathcal{C}$ over $R$ in $\mathcal{X}$. Our approach is a generalization of the methods developed in [NS06].

We use logarithmic geometry in the sense of [Kat89], a theoretical framework that makes it possible to treat certain singularities, such as toric or normal crossings singularities, as if they were smooth. For the basics of this theory we refer the reader to [Kat89] and [Gro11, Chapter 3].

In our setting, we endow the scheme $O = \text{Spec } R$ with the divisorial log structure defined by its special fiber. Its generic fiber is then $\text{Spec } K$ with the trivial log structure, while its special fiber is the standard log point $(\text{Spec } \mathbb{C}, N)$, which we denote by $O_0$. We endow a toric scheme $\mathcal{X}$ with the divisorial log structure defined by its toric boundary; then $\mathcal{X}$ is log smooth over $O$. If $Y \to X$ is a morphism of log schemes, we denote by $\Theta_{Y/X}$ the log tangent sheaf of $Y$ over $X$. By [Oda88, Proposition 3.1], there is then a natural isomorphism $\Theta_{X/O} \cong \mathcal{O}_X \otimes N$.

Finally, we endow the nodal curve $C_0 \subseteq \mathcal{X}_s$ with the log structure inherited from the log structure of $\mathcal{X}_s$. Then $C_0$ is log smooth over the standard log point $O_0$. Note that the log
structure of $C_0$ not only encodes information about the nodal points, but also marks the points of the intersection of $C_0$ with those toric boundary divisors of $X_0$ which lie in only one component of $X_0$. In the situation of Section 2, this means that $C_0$ has one marked point for each unbounded edge of $\Gamma$.

We can now develop the log smooth deformation theory we need for our proof. Let us denote $R_k$ the ring $\mathbb{C}[\![t^{1/k}]\!]/(t^{k+1})$ and endow $O_k = \text{Spec} R_k$ with the log structure induced by $\mathbb{N} \to R_k : a \mapsto t^a$. Note that we have natural closed immersions $O_k' \hookrightarrow O_k$ for $0 \leq k' \leq k$.

**Definition 3.1.** Let $C_0$ be a log smooth curve over $O_0$. A $k$-th order deformation of $C_0$ is a log smooth morphism $C_k \to O_k$ whose base change to $O_0$ is $C_0 \to O_0$.

Suppose we are given a $(k-1)$-st order deformation $C_{k-1} \to O_{k-1}$ of $C_0 \to O_0$. By [Gro11, Proposition 3.40] there is an element $\text{ob}(C_{k-1}/O_{k-1})$ in $H^2(C_0, \Theta_{C_0/O_0})$ such that $C_{k-1} \to O_{k-1}$ lifts to a $k$-th order deformation $C_k \to O_k$ if and only if $\text{ob}(C_{k-1}/O_{k-1}) = 0$. Since $C_0$ is of dimension one, we have $H^2(C_0, \Theta_{C_0/O_0}) = 0$ and therefore such a lift always exists. Moreover, the set of such lifts is a torsor over $H^1(C_0, \Theta_{C_0/O_0})$. What is more complicated is lifting the log smooth curve together with its embedding into $\mathfrak{x}$.

Let $f = f_0 : C_0 \hookrightarrow \mathfrak{x}_s$ be a strict closed immersion of $C_0$ into the special fiber $\mathfrak{x}_s$ of a toric scheme $\mathfrak{x}$, and consider the commutative diagram

$$
\begin{array}{ccc}
C_0 & \xrightarrow{f_0} & \mathfrak{x} \\
\downarrow & & \downarrow \\
O_0 & \longrightarrow & O
\end{array}
$$

**Definition 3.2.** A $k$-th lift of $f_0$ is a commutative diagram

$$
\begin{array}{ccc}
C_k & \xrightarrow{f_k} & \mathfrak{x} \\
\downarrow & & \downarrow \\
O_k & \longrightarrow & O
\end{array}
$$

where $f_k : C_k \to \mathfrak{x}$ is log smooth and $C_k \to O_k$ is a $k$-th lift of $C_0 \to O_0$.

The following proposition is the main result of this section; it expands on the argument in [Gro11, Theorem 3.41]. We refer the reader to [NS06, Lemma 7.2 and Proposition 7.3] for the original results.

**Proposition 3.3.** Let $f_{k-1}$ be a $(k-1)$-st lift of $f_0$ and suppose that the canonical homomorphism

$$
H^1(C_0, \Theta_{C_0/O_0}) \longrightarrow H^1(C_0, f_0^*\Theta_{\mathfrak{x}/O})
$$

(3.1)

is surjective. Then there exists a $k$-th lift of $f_0$ that extends $f_{k-1}$.

In [NS06] the authors assume that $C_0$ is rational. In this case $H^1(C_0, f_0^*\Theta_{\mathfrak{x}/O}) = 0$ and the homomorphism (3.1) is always surjective.

**Proof of Proposition 3.3.** Assume that we are given a $(k-1)$-st lift $f_{k-1}$ and let $C_k \to O_k$ be a lift of $C_{k-1} \to O_{k-1}$.

Suppose that (3.1) is surjective. Choose an affine open cover $(U_i)$ of $C_0$ and let $U_{ij} = U_i \cap U_j$. For every $i$, let $U_i^k$ and $U_i^{k-1}$ be log smooth thickenings of $U_i$ over $O_k$ and $O_{k-1}$ respectively.
Since the $U_i$ are affine, log smooth thickenings exist and are unique by [Gro11, Proposition 3.38]; in particular we can assume that $U_i^k$ is a lifting of $U_i^{k-1}$ to a log smooth scheme over $O_k$. We have gluing morphisms
\[
\theta_{ij}^k : U_{ij}^k \longrightarrow U_{ji}^k
\]
and
\[
\theta_{ij}^{k-1} : U_{ij}^{k-1} \longrightarrow U_{ji}^{k-1}
\]
that fulfill $\theta_{ij}^{k-1} = \theta_{ij}^k|_{U_{ij}^{k-1}}$ for all $i$ and $j$. Moreover, we have lifts $f_i^{k-1} : U_i^{k-1} \rightarrow \mathfrak{X}$ that satisfy the compatibility condition $f_i^{k-1} = f_i^k \circ \theta_{ij}^{k-1}$ on $U_{ij}^{k-1}$.

Since $\mathfrak{X}$ is log smooth over $O$, we can find lifts $f_i^k : U_i^k \rightarrow \mathfrak{X}$ of the $f_{ij}^{k-1}$ to the thickening $U_i^k$ of $U_i^{k-1}$ for every $i$. By [Kat89, Proposition 3.9] the set of such lifts $f_i^k$ on $U_i^k$ forms a torsor over $H^0(U_i, f^*\Theta_{\mathfrak{X}/O}|_{U_i})$.

Now compare the two liftings $f_i^k$ and $f_j^k \circ \theta_{ij}^{k-1}$ on $U_{ij}^k$. Note that they both lift $f_i^{k-1} = f_j^{k-1} \circ \theta_{ij}^{k-1}$ on $U_{ij}^{k-1}$ and therefore differ by a section $\psi_{ij} \in H^0(U_{ij}, f^*\Theta_{\mathfrak{X}/O}|_{U_{ij}})$, i.e. we have
\[
f_i^k = f_j^k \circ \theta_{ij}^{k-1} + t^\wedge \psi_{ij}.
\]
The $\psi_{ij}$ define a 2-cocycle of $f^*\Theta_{\mathfrak{X}/O}$ on $C_0$, since $H^2(f^*\Theta_{\mathfrak{X}/O}) = 0$.

Since (3.1) is surjective, there is a 2-cocycle $\phi_{ij}^k$ for $\Theta_{C_0/O}$ such that $f \circ \phi_{ij}^k = \psi_{ij}$ for all $i$ and $j$. We can now replace the lift $C_k$ of $C_{k-1}$ by the lift $\tilde{C}_k$ of $C_{k-1}$ that is given by the gluing maps $\tilde{\theta}_{ij}^k = \theta_{ij}^k + t^\wedge \phi_{ij}^k$. But then we have
\[
f_i^k = f_j^k \circ \theta_{ij}^k + t^\wedge \psi_{ij}\]
\[
= f_j^k \circ (\tilde{\theta}_{ij}^k - t^\wedge \phi_{ij}^k) + t^\wedge \psi_{ij}\]
\[
= f_j^k \circ \tilde{\theta}_{ij}^k - t^\wedge \phi_{ij}^k + t^\wedge \psi_{ij} = f_j^k \circ \tilde{\theta}_{ij}^k
\]
and therefore we can glue the local lifts $f_i^k$ to a global lift $f_k : \tilde{C}_k \rightarrow \mathfrak{X}$. \hfill \Box

In our setting, we can deduce the following result:

**Proposition 3.4.** Let $\Gamma$, $\mathfrak{X}$, and $C_0$ be as in Section 2, and assume that the homomorphism (3.1) is surjective. Then there exists a semistable curve $C$ over $R$ with smooth generic fiber $C$ and special fiber $C_0$, together with a closed immersion $C \hookrightarrow \mathfrak{X}$ extending $C_0 \hookrightarrow \mathfrak{X}$.

**Proof.** Taking the direct limit of all $C_k$, we obtain a formal scheme $\mathcal{C}$ over $O$, with a closed immersion into the $t$-adic formal completion $\tilde{\mathfrak{X}}$ of $\mathfrak{X}$. Since $C_0$ is complete, all $C_k$ are proper over $O_k$ and therefore $\mathcal{C}$ is proper over $O$. By Grothendieck’s existence theorem [Gro61, Théorème 5.1.4], $\mathcal{C}$ is then the $t$-adic formal completion of a closed subscheme $\mathfrak{C}$ of $\tilde{\mathfrak{X}}$, proper over $O$. Note that by construction the special fiber of $\mathcal{C}$ is equal to $C_0$.

We want to show that with the log structure induced from $\mathfrak{X}$ the $R$-scheme $\mathcal{C}$ is log smooth over $O$. Since this can be checked in an étale neighborhood in $\mathfrak{C}$ of a node $p$ of $C_0$, without loss of generality we can assume that $C_0 = \text{Spec} \left( \mathbb{C}[x, y]/(xy) \right)$. For $e > 0$, set $\mathcal{C}_e = \text{Spec} \left( R[x, y]/(xy - t^e/\ell^e) \right)$ with log smooth logarithmic structure as in [Gro11, Example 3.26]. By the description of log smooth curves of [Kat00, Proposition 1.1], there exists some $e > 0$ such that the special fiber of $\mathcal{C}_e$ is $C_0$. Therefore, for every $k > 0$ the restriction $\mathcal{C}_e \times_O O_k$ is the unique log smooth lifting of $C_0 \rightarrow O_0$ to $O_k$. This implies that $\mathcal{C} = \mathcal{C}_e$, so $\mathcal{C}$ is log smooth over $O$.\hfill \Box
Then the generic fiber $C$ of $\mathcal{C}$ is log smooth over $K$. Therefore $C$ has only toric singularities, hence it is smooth, since it is one-dimensional. Since $\mathcal{C}$ is proper over $O$, the curve $C$ is complete. □

**Remark 3.5.** Assume we are in the situation of Proposition 3.4. Then the log structure of $C$, which is the one induced by the log structure of $X$, contains information not only about the nodes of $C_0$ but also about finitely many sections of $C \to O$, which are disjoint since $C$ is log smooth. In the special fiber, these sections cut out precisely the marked points of $C_0$, which correspond to the unbounded edges of $\Gamma$. On the other hand, in the generic fiber they cut out a finite set $V \subseteq C(K^{\text{alg}})$ of marked points of $C$, which is precisely the intersection of $C$ with the toric boundary of $X$. Therefore, this construction naturally gives rise to a generalized skeleton $\Sigma_{C,V}$ of $C^{\text{an}}$.

### 4. The abundancy map in cohomology

Let $\Gamma \subseteq N_\mathbb{R}$ be a tropical curve with skeleton $G$, and denote by $E_G$ the set of edges of $G$. Choose a direction of every edge $e \in E_G$ and denote by $\vec{e} \in N_\mathbb{R}$ the minimal non-zero integral vector in this direction.

We denote by $H_1(\Gamma, \mathbb{Z})$ the first simplicial homology group of $\Gamma$. An element of $H_1(\Gamma, \mathbb{Z})$ is a formal sum $\sum_{e \in E_G} a_e[e]$, with integer coefficients, forming a cycle in $\Gamma$. The next definition is taken from [Kat12, Section 1].

**Definition 4.1.** A tropical curve $\Gamma$ is said to be non-superabundant, if the abundancy map $\Phi_\Gamma : \mathbb{R}^{E_G} \to \text{Hom} \left( H_1(\Gamma, \mathbb{Z}), N_\mathbb{R} \right)$

$$(\ell_e) \mapsto \left( \sum_{e \in E_G} a_e[e] \mapsto \sum_{e \in E_G} \ell_e a_e \vec{e} \right),$$

is surjective.

The map $\Phi_\Gamma$ can be interpreted as follows: to each set of edge-lengths, it associates the function sending a cycle on $\Gamma$ to the total displacement traveled when going around the edge directions with the prescribed lengths.

Therefore, the intersection of the kernel of $\Phi_\Gamma$ with $\mathbb{R}^{E_G}_{>0}$ can be seen as the moduli space of metric graphs embedded in $\mathbb{R}^n$ which have the same combinatorial type and edge-directions as the skeleton of $\Gamma$. Whenever $\Gamma$ is trivalent, then for $\Gamma$ to be non-superabundant means that this moduli space has the expected dimension.

Note that, while the map $\Phi_\Gamma$ depends on the choice of a direction of the edges of $\Gamma$, whether $\Gamma$ is non-superabundant or not is independent of such a choice.

Now let $\Gamma$ be a tropical curve, let $X$ be the toric scheme as constructed in Section 2, and let $C_0$ be a nodal curve in $X_s$ fulfilling the conclusion of Proposition 2.5. The crucial result of this section is the following proposition, which gives a cohomological interpretation of the abundancy map.

**Proposition 4.2.** There are a homomorphism $\delta : \mathbb{C}^{E_G} \to H^1(C_0, \Theta_{C_0/O_0})$, and an isomorphism $H^1(C_0, f^*\Theta_{X/O}) \cong \text{Hom} \left( H_1(\Gamma), N_\mathbb{C} \right)$ such that the induced homomorphism $\mathbb{C}^{E_G} \to H^1(C_0, \Theta_{C_0/O_0}) \to H^1(C_0, f^*\Theta_{X/O}) \cong \text{Hom} \left( H_1(\Gamma), N_\mathbb{C} \right)$ is equal to $\Phi_\Gamma \otimes \mathbb{C}$.
Proof. Note that there are compatible one-to-one correspondences between the vertices \( v \) of \( \Gamma \) and the components \( C_v \) of \( C_0 \) as well as between the edges \( e \) of \( G \) and the nodes \( p(e) \) of \( C_0 \). We have two normalization exact sequences (written horizontally) below, where the signs are chosen in accordance with the orientation on the edges of \( G \):

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \Theta_{C_0/O_0} & \rightarrow & \prod_v (\Theta_{C_0/O_0})|_{C_v} & \rightarrow & \prod_e \mathbb{C}_{p(e)} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{O}_{C_0} \otimes N & \rightarrow & \prod_v (\mathcal{O}_{C_0/O_0} \otimes N)|_{C_v} & \rightarrow & \prod_e N_{C_{p(e)}} & \rightarrow & 0 
\end{array}
\]

The first and the second vertical maps are given by composing the natural maps \( \Theta_{C_0/O_0} \rightarrow f^* \Theta_{X/O} \) with the natural isomorphism \( f^* \Theta_{X/O} \cong \mathcal{O}_{C_0} \otimes N \) of [Oda88, Proposition 3.1]. These maps induce the third vertical map. Taking the long exact cohomology sequences of these two short exact sequences, we obtain the following commutative square:

\[
\begin{array}{ccccccccc}
H^1(C_0, \mathcal{O}_{C_0/O_0}) & \rightarrow & H^1(C_0, \mathcal{O}_{C_0} \otimes N) & \rightarrow & \prod_e \mathbb{C} & \rightarrow & \prod_e N_{C} & \rightarrow & 0 \\
\delta & \downarrow & \delta & & \downarrow & & \downarrow & & \\
\prod_e \mathbb{C} & \rightarrow & \prod_e N_{C} & & & & & & 
\end{array}
\]

Now we need the following lemma:

**Lemma 4.3.** There is an isomorphism

\[
\alpha : H^1(C_0, \mathcal{O}_{C_0} \otimes N) \cong \text{Hom}(H_1(\Gamma), N_{C}).
\]

such that the composition \( \alpha \circ \delta : \prod_e N_{C} \rightarrow \text{Hom}(H_1(\Gamma), N_{C}) \) is given by sending a family \( (u_e)_e \) of elements \( u_e \in N_{C} \) to the homomorphism

\[
\sum_e a_e[e] \mapsto \sum_e a_e u_e
\]

in \( \text{Hom}(H_1(\Gamma), N_{C}) \).

**Proof.** Consider the normalization short exact sequence

\[
0 \rightarrow \mathcal{O}_{C_0} \otimes N \rightarrow \prod_v \mathcal{O}_{C_v} \otimes N \rightarrow \prod_e N_{C_{p(e)}} \rightarrow 0.
\]

The associated long exact cohomology sequence is

\[
0 \rightarrow H^0(C_0, \mathcal{O}_{C_0} \otimes N) \rightarrow \prod_v H^0(C_v, \mathcal{O}_{C_v} \otimes N) \rightarrow \prod_e N_{C_{p(e)}} \rightarrow H^1(C_0, \mathcal{O}_{C_0} \otimes N) \rightarrow 0,
\]

since by the rationality of \( C_v \) we have \( H^1(C_v, \mathcal{O}_{C_v}) = 0 \) and therefore \( H^1(C_v, \mathcal{O}_{C_v} \otimes N) = 0 \) for all components \( C_v \) of \( C_0 \).

The first three terms of the above sequence are \( \text{Hom}(\cdot, N_{C}) \) of the simplicial chain complex

\[
\mathbb{Z}^{\# E_G} \rightarrow \mathbb{Z}^{V_G} \rightarrow \mathbb{Z} \rightarrow 0.
\]

defining \( H_1(\Gamma) \). Therefore we obtain an isomorphism \( H^1(\mathcal{O}_{C} \otimes N) \cong H^1(\Gamma, N_{C}) \), and the latter is isomorphic to \( \text{Hom}(H_1(\Gamma), N_{C}) \) by the universal coefficient theorem for \( \Gamma \). In this case the homomorphism \( \prod_e N_{C} \rightarrow \text{Hom}(H_1(\Gamma), N_{C}) \) is given by sending a family \( (u_e)_e \) of
elements $u_e \in N_C$ to the homomorphism

$$
\sum_e a_e[e] \mapsto \sum_e a_e u_e
$$

in $\text{Hom} (H_1(\Gamma), N_C)$. \hfill \Box

Let us now finish the proof of Proposition 4.2. By Lemma 4.3, it is enough to show that the homomorphism

$$
\prod_e C \to \prod_e N_C
$$

is given by sending $(l)_e$ to the the family $(l e)_e$ in $\prod_e N_C$. But this follows from the following lemma applied to every component of the special fiber $\mathcal{X}_0$ of $\mathcal{X}$ separately, since for every component $C_i$ of $C_0$, the tropicalization $\text{Trop}(C)$ is the local cone around $v$ in $\Gamma$. \hfill \Box

**Lemma 4.4.** Let $X = X(\Delta)$ be a toric variety corresponding to a one-dimensional fan $\Delta$ in $N_R$. Suppose $C \subseteq X$ is a curve and $\text{Trop}(C)$ contains a cone $\rho \in \Sigma$ with multiplicity 1. Then $C$ meets the divisor $D$ corresponding to $\rho$ transversely at a smooth point $x \in C$, and $(\Theta_{C/C})_x$ is the one-dimensional subspace of $(\Theta_{X/C})_x = N_C$ spanned by $\rho$.

**Proof.** Note that by [Gro11, Example 3.32] we have a canonical isomorphism $\Theta_{X/C} \cong \mathcal{O}_X \otimes N$. If we fix a basis $e_1, \ldots, e_n$ of $N$, this induces coordinates $x_1, \ldots, x_n$ on the torus $T = \text{Spec} \mathbb{C}[M]$, and $\Theta_{X/C}$ is generated by the global sections

$$
x_1 \frac{\partial}{\partial x_1}, \ldots, x_n \frac{\partial}{\partial x_n}.
$$

The section $x_i \cdot \partial/\partial x_i$ maps to $1 \otimes e_i$ in $\mathcal{O}_X \otimes \mathbb{Z}^n$, under the natural identification of $\Theta_{X/C}$ with $\mathcal{O}_X \otimes N$.

After a change of coordinates, we may assume that $\rho$ is spanned by $e_1$. The $T$-invariant affine open $U_\rho$ is $\text{Spec} \mathbb{C}[x_1, x_2^\pm, \ldots, x_n^\pm]$, with the divisor $D$ being the locus where $x_1 = 0$.

The tropical multiplicity of $\rho$ in $C$ is naturally identified with the intersection number $D \cdot C$ [KP11, Lemma 2.3], so $D$ meets $C$ with total multiplicity 1. By the positivity of intersection multiplicities [Ful98, Lemma 7.1(a)] and criterion for intersection multiplicity 1 [Ful98, Proposition 7.2], it follows that $D$ meets $C$ transversely at a single point $x$.

It remains to show that $(\Theta_{C/C})_x$ is spanned by the section $x_1 \cdot \partial/\partial x_1$. Since $\text{Trop}(C)$ contains the ray spanned by $e_1$ with multiplicity 1, the initial degeneration (as in [BPR11, Section 2.1], also see Section 5 below) of $C$ with respect to the weight vector $e_1$ is a translate of the one-parameter subgroup corresponding to $e_1$. Say this translate is $(t, a_2, \ldots, a_n)$, for some $a_2, \ldots, a_n$ in $\mathbb{C}^*$. The ideal defining this initial degeneration is generated by

$$
\{(x_2 - a_2), \ldots, (x_n - a_n)\}.
$$

Therefore, we have functions $f_2, \ldots, f_n$ in the ideal $I_C$ such that the leading term of $t_i$ with respect to the weight vector $e_1$ is $x_i - a_i$. In other words, there is a function $g_i \in \mathbb{C}[x_1, x_2^\pm, \ldots, x_n^\pm]$ such that

$$
f_i = x_i - a_i + x_1 g_i
$$

for $2 \leq i \leq n$.

Note that the differential form

$$
d f_i = dx_i + x_1 d g_i + dx_1 g_i
$$
vanishes on $(\Theta_{C/C})_x$. In particular, $(\Theta_{C/C})_x$ is contained in the subspace of $(\Theta_{X/C})_x$ where $df_i$ vanishes.

Note that $\langle z_j \cdot \partial/\partial z_j, x_1dg_i + dx_1g_i \rangle$ vanishes at $x = (0, a_2, \ldots, a_n)$ for all $j$, and $\langle z_j \cdot \partial/\partial z_j, dx_i \rangle$ is nonvanishing if and only if $i = j$. This shows that $(\Theta_{C/C})_x$ is contained in the subspace of $(\Theta_{X/C})_x$ spanned by
\[
\left\{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_i}, \ldots, \frac{\partial}{\partial x_n} \right\},
\]
for $2 \leq i \leq n$. Intersecting these conditions shows that $(\Theta_{C/C})_x$ is the subspace spanned by $x_1 \cdot \partial/\partial x_1$, as required. \hfill \Box

**Remark 4.5.** In particular, if $\Gamma$ is trivalent, then the homomorphism (3.1) is precisely $\Phi_1 \otimes \mathbb{C}$. Indeed, in this case, for every vertex $v$, we have $\Theta_{C_0/O_0|C_v} \cong \mathcal{O}_{\mathbb{P}^1}(-1)$, so $H^i(C_0, \Theta_{C_0/O_0|C_v}) = 0$ for $i = 0, 1$, and therefore the homomorphism $\delta$ in Proposition 4.2 is an isomorphism. This is the case considered in [Nis14].

5. **Proofs of Theorems 1.1 and 1.2**

Let $\Gamma \subseteq \mathbb{N}$ be a tropical curve, let $\mathfrak{X}$ be the toric $R$-scheme defined as in Section 2, and denote by $X$ the generic fiber of $\mathfrak{X}$.

**Lemma 5.1.** Let $C$ be a flat $R$-curve in $\mathfrak{X}$ and denote by $C \subseteq X$ its generic fiber. If the special fiber $C_s$ is proper over $\mathbb{C}$ and intersects every torus orbit contained in the special fiber of $\mathfrak{X}$, then $\text{Trop}(C \cap T) = \Gamma$.

**Proof.** By [Gab13, Proposition 11.12] the properness of $C_s$ implies that $\text{Trop}(C) \subseteq \Gamma$, and therefore the vertices of $\text{Trop}(C)$ are a subset of $\Gamma$. By Tevelev’s Lemma [Gab13, Lemma 11.6] $\text{Trop}(C)$ intersects the relative interior of every face in $\Gamma$. In particular, the vertices of $\text{Trop}(C)$ coincide with the vertices of $\Gamma$. Again, since $\text{Trop}(C)$ intersects the relative interior of every face of $\Gamma$, all one-dimensional faces of $\Gamma$ already have to be contained in $\text{Trop}(C)$, and therefore $\text{Trop}(C) = \Gamma$. \hfill \Box

In the proof of Theorem 1.1 we use the initial degeneration $\text{in}_P(C)$ of $C$ along an open face $P$ of $\text{Trop}(C)$. In our situation $\text{in}_P(C)$ can be defined as the $\mathbb{C}$-scheme
\[
\text{in}_P(C) = (C_s \cap \mathfrak{X}_P) \times T_P,
\]
where $\mathfrak{X}_P$ is the torus orbit in $\mathfrak{X}_s$ corresponding to $P$, and $T_P$ is the subgroup of the reduction of the torus $T = \text{Spec } R[M]$ that stabilizes $\mathfrak{X}_P$. By [HK12, Lemma 3.6] this coincides with the usual definition of the initial degeneration of $C$ at a point $p \in P$, as for example in [BPR11, Section 2.1], since $C$ is proper over $O = \text{Spec } R$ and the multiplication map $T \times_O C \rightarrow \mathfrak{X}$ is flat by the same argument as in [Hac08, Lemma 2.7] and surjective because $C$ meets each torus orbit of $\mathfrak{X}$.

**Proof of Theorem 1.1.** Let $\Gamma$ be a tropical curve with rational edge lengths fulfilling the conditions (A), (B) and (C), let $K$ and $\mathfrak{X}$ be be defined as in Section 2, and $C_0$ be a curve in $\mathfrak{X}_s$ as constructed in Proposition 2.5. Since $\Gamma$ is non-superabundant, Proposition 4.2 and Proposition 3.4 imply that there is a semistable curve $C$ over $R$ with smooth generic fiber $C$ and special fiber $C_0$, together with a closed immersion $C \hookrightarrow \mathfrak{X}$. By Lemma 5.1 we have $\text{Trop}(C \cap T) = \Gamma$. By construction of $C_0$, all the initial degenerations of $C$ along
open faces of $\Gamma$ are smooth and irreducible, and so the tropical multiplicities are all one. Therefore, [BPR11, Corollary 6.11] implies that the tropicalization is faithful with respect to the skeleton $\Sigma_{C,V}$ of $C^an$, where $V = C \setminus (C \cap T)$ is the set of marked points as described in Remark 3.5.

Theorem 1.2 follows from Theorem 1.1 and the following result, which is a slight extension of a theorem of Cartwright–Dudzik–Manjunath–Yao.

**Theorem 5.2 ([CDMY14]).** Let $G$ be a metric graph with rational edge lengths. Set

$$n = \max \left\{ 3, \max \{ \deg v - 1 \mid v \in E_G \} \right\}$$

and let $N$ be a free abelian group of rank $n$. Then there exists a tropical curve $\Gamma \subseteq N_R$ with rational edge lengths, whose skeleton is isomorphic to $G$, and which satisfies conditions (A), (B) and (C) from the introduction.

**Proof.** Fix an integral basis $v_1, \ldots, v_n$ of $N_R$. Let $\Gamma$ be a tropical curve as given by [CDMY14, Theorem 1.1]. As noted in [CDMY14, Remark 2.8], the only part that is not included there is the non-superabundance of $\Gamma$. Let $G'$ be the minimal finite graph underlying the skeleton of $\Gamma$ and consider a spanning tree $T$ of $G'$. Then the set of $\epsilon \in E_{G' \setminus T}$ parametrizes a basis \{c_\epsilon\} of $H_1(\Gamma, \mathbb{Z})$, and $c_\epsilon$ is the only element of the basis which contains $\epsilon$.

In order to show the surjectivity of the abundancy map it is enough to show that for all $\epsilon \in E_{G' \setminus T}$ and $1 \leq i \leq n$ the homomorphisms

$$f_{\epsilon,i} : H_1(\Gamma) \to N_R$$

$$c_\epsilon' \mapsto \begin{cases} v_i & \text{if } \epsilon' = \epsilon \\ 0 & \text{if } \epsilon' \neq \epsilon. \end{cases}$$

are in the image of $\Phi_T$. By the construction of [CDMY14], every $\epsilon$ will contain an edge $e$ of $G'$ such that $\overleftarrow{e}$ is parallel to $v_i$. Since $c_\epsilon$ is the only element of the basis which contains $\epsilon$, if we take the vector $\ell \in \mathbb{R}^{\#E_{G'}}$ with value $|v_i|$ in the $\epsilon$-th entry and 0 otherwise, we obtain $\Phi(\ell) = f_{\epsilon,i}$. \hfill $\square$

**Remark 5.3.** It is possible to extend Theorem 1.1, and therefore Theorem 1.2, to the equicharacteristic $p > 0$ case. Let $\Gamma$ be a tropical curve fulfilling conditions (A), (B) and (C) from the introduction. Then for all but finitely many prime numbers $p$ there exists a finite field extension $K$ of $\mathbb{F}_p((t))$ such that we can construct a suitable curve $C_0$ in the special fiber of $\mathfrak{X}$ using the method of Section 2. The results of Section 3 remain valid over any discrete valuation ring of the form $k[[t]]$, where $k$ is an arbitrary field, so in particular they hold over $R$. Observe that the abundancy map $\Phi_T$ defined in 4.1 is the base change $\Phi_{T,Z} \otimes \mathbb{R}$ of the map $\Phi_T : \mathbb{Z}^{E_G} \to \text{Hom}(H_1(\Gamma, \mathbb{Z}), N)$ defined by

$$(\ell_\epsilon) \mapsto \left( \sum_{e \in E_G} a_e[\epsilon] \mapsto \sum_{e \in E_G} \ell_\epsilon a_e \overleftarrow{e} \right)$$

as in Definition 4.1. For a field $L$ we denote by $\Phi_{T,L}$ the base change of $\Phi_{T,Z}$ to $L$. Given two fields $L$ and $L'$ of the same characteristic, the surjectivity of $\Phi_{T,L}$ is equivalent to the surjectivity of $\Phi_{T,L'}$, and in particular $\Gamma$ is non-superabundant if and only if $\Phi_{T,Q}$ is surjective. In this case the map $\Phi_{T,p}$ is surjective for all but finitely many prime numbers $p$. Therefore, if $\Gamma$ is non-superabundant, log deformations of $C_0$ can be constructed for $p$ big enough and
it follows that, given a tropical curve $\Gamma$ satisfying the hypotheses of Theorem 1.1, for all but finitely many prime numbers $p$ we can find a finite field extension $K$ of $\mathbb{F}_p((t))$ such that the conclusion of Theorem 1.1 holds over $K$.

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