The Electroweak $\pi^+ - \pi^0$ Mass Difference
and
Weak Matrix Elements in the $1/N_c$ Expansion

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Abstract

The $\pi^+ - \pi^0$ mass difference generated by the electroweak interactions of the Standard Model is expressed, to lowest order in the chiral expansion and to leading order in $\alpha$, in terms of an integral involving the same correlation function $\Pi_{L,R}(Q^2)$ which governs the well known electromagnetic $\pi^+ - \pi^0$ mass difference. We calculate this contribution within the framework of QCD in the limit of a large number of colours $N_c$. We show how this calculation, which requires non–trivial contributions from next–to–leading terms in the $1/N_c$ expansion, provides an excellent theoretical laboratory for studying issues of long– and short–distance matching in calculations of weak matrix elements of four–quark operators. The electroweak $\pi^+ - \pi^0$ mass difference turns out to be a physical observable which is under good analytical control and which should, therefore, be an excellent testground of numerical evaluations in lattice–QCD and of model calculations in general.
1 Introduction

In the chiral limit where the light quark masses are set to zero, the low–energy effective
Lagrangian of QCD in the presence of electromagnetic interactions has, to lowest order in
\( \alpha = e^2/4\pi \approx 1/137.03 \ldots \), an interaction term without derivatives of order \( \mathcal{O}(p^0) \) in the chiral
expansion

\[
\mathcal{L}_{\text{eff}} = \cdots + e^2 C \text{tr} \left( Q_R U Q_L U^\dagger \right) .
\]  

(1)

Here, \( U \) is the matrix field which collects the octet of pseudoscalar Goldstone fields and
\( Q_R = Q_L = \text{diag}[2/3, -1/3, -1/3] \) the right– and left–handed charges associated with the
electromagnetic couplings of the light quarks. Upon expanding \( U \) in powers of the pseu-
doscalar fields there appear quadratic terms like

\[
\mathcal{L}_{\text{eff}} = \cdots - 2e^2 C \frac{1}{f^2} (\pi^+\pi^- + K^+K^-) + \cdots ,
\]  

(2)

showing that, in the presence of electromagnetic interactions, the charged pion and kaon fields
become massive in agreement with the old current algebra result obtained by Dashen [2],

\[
(m^2_{K^+} - m^2_{K^0})|_{\text{EM}} = (m^2_{\pi^+} - m^2_{\pi^0})|_{\text{EM}} = 2e^2 C + \cdots .
\]  

(3)

In fact, the main contribution to the physical \( \pi^+ - \pi^0 \) mass difference \( i \)s of electromagnetic
origin because quark masses do not contribute significantly to the \( \pi^+ - \pi^0 \) mass splitting [3].

The effective term in eq. (1) results from the integration of a virtual photon in the presence
of the strong interactions in the chiral limit. It is well known [4–7] that the corresponding
coupling constant \( C \) is given by an integral of a correlation function \( \Pi_{LR}(Q^2) \) which is the
invariant amplitude of the two–point function

\[
\Pi_{\mu\nu}^{LR}(q) = 2i \int d^4x e^{iq\cdot x} \langle 0 | T \left( L^\mu(x) R^{\nu}(0) \right) | 0 \rangle ,
\]  

(4)

with currents

\[
L^\mu = \bar{d}(x)\gamma^\mu \frac{1}{2}(1 - \gamma_5) u(x) \quad \text{and} \quad R^\mu = \bar{d}(x)\gamma^\mu \frac{1}{2}(1 + \gamma_5) u(x) .
\]  

(5)

In the chiral limit

\[
\Pi_{\mu\nu}^{LR}(Q^2) = (q^\mu q^\nu - g^{\mu\nu} q^2) \Pi_{LR}(Q^2) ,
\]  

(6)

and

\[
C = \frac{-1}{8\pi^2} \frac{3}{4} \int_0^\infty dQ^2 Q^2 \Pi_{LR}(Q^2) ,
\]  

(7)

with \( Q^2 \) the euclidean momentum squared of the virtual photon. The function \( \Pi_{LR}(Q^2) \) is,
for all values of \( Q^2 \), an order parameter of spontaneous chiral symmetry breakdown (S\( \chi \)SB)
and thus, so is the low–energy constant \( C \). This means that the correlator in eq. (4) has a
smooth behaviour at short distances, so that the integral in eq. (7) converges in the ultraviolet
region:

\[
\lim_{Q^2 \to \infty} Q^4 \Pi_{LR}(Q^2) \to 0 .
\]  

(8)
This behaviour also entails the two Weinberg sum rules \[8\] in the chiral limit. It has furthermore been shown \[9, 10\] that
\[ -Q^2 \Pi_{LR}(Q^2) \geq 0 \quad \text{for} \quad 0 \leq Q^2 \leq \infty, \] (9)
which in particular ensures the positivity of the integral in eq. (8) and thus the stability of the QCD vacuum with respect to small perturbations induced by electromagnetic interactions.

In this letter we shall first show that the same correlation function \( \Pi_{LR}(Q^2) \) governs the \( \pi^+ - \pi^0 \) mass difference in the presence of the electroweak interactions of the Standard Model. This observation and the calculation which follows thereof are of course of phenomenological interest \textit{per se}; but their real relevance lies in the fact that, as we shall show below, they provide an excellent theoretical laboratory to study issues of long– and short–distance matching in calculations of weak matrix elements of four–quark operators. We also think that this example, which is under good analytical control, provides an excellent testground for numerical evaluations in lattice–QCD and for model calculations in general.

2 The Electroweak \( \pi^+ - \pi^0 \) Mass Difference

In the presence of the electromagnetic and weak neutral currents of the Electroweak Model, the QCD lagrangian \( \mathcal{L}_{\text{QCD}} \) becomes
\[ \mathcal{L}_{\text{QCD}} \rightarrow \mathcal{L}_{\text{QCD}} + \bar{q}_R \gamma^\mu r_\mu q_R + \bar{q}_L \gamma^\mu l_\mu q_L, \] (10)
where
\[ l_\mu = e Q_L [A_\mu(x) - \tan \theta_W Z_\mu(x)] + \frac{e}{2 \sin \theta_W \cos \theta_W} T_3 Z_\mu(x), \] (11)
\[ r_\mu = e Q_R [A_\mu(x) - \tan \theta_W Z_\mu(x)], \] (12)
and
\[ Q_L = Q_R = \frac{1}{3} \text{diag}(2, -1, -1), \quad T_3 = \text{diag}(1, -1, -1). \] (13)

There are then two possible terms of order \( \mathcal{O}(p^0) \) in the low–energy effective lagrangian generated by the virtual \( Z \) integration \[1\]:
\[ + e^2 \tan^2 \theta_W C_Z \text{tr}(Q_R U Q_L U^\dagger) \quad \text{and} \quad - e^2 \frac{1}{2 \cos^2 \theta_W} C_Z \text{tr}(Q_R U T_3 U^\dagger), \] (14)
where
\[ C_Z = \frac{1}{8\pi^2} \frac{3}{4} \int_0^\infty dQ^2 \frac{Q^2}{Q^2 + M_Z^2} Q^2 \Pi_{LR}(Q^2). \] (15)
These couplings give rise to physical mass terms
\[ + e^2 \tan^2 \theta_W C_Z \text{tr}(Q_R U Q_L U^\dagger) = -2e^2 \frac{\sin^2 \theta_W}{\cos^2 \theta_W} C_Z \frac{1}{f_\pi^2} \left( \pi^+ \pi^- + K^+ K^- \right) + \cdots, \] (16)
and
\[ - e^2 \frac{1}{2 \cos^2 \theta_W} C_Z \text{tr}(Q_R U T_3 U^\dagger) = 2e^2 \frac{1}{\cos^2 \theta_W} C_Z \frac{1}{f_\pi^2} \left( \pi^+ \pi^- + K^+ K^- \right) + \cdots. \] (17)
\[ 1\]The charged \( W \)–field only couples to left currents and therefore cannot contribute to an \( \mathcal{O}(p^0) \) self–energy.
Collecting together these weak neutral current contributions with the electromagnetic contribution in eq. (3), we arrive at the quite remarkable result that, to lowest order in $e^2$ and in the chiral limit,

$$ (m_{K^+}^2 - m_{K^0}^2)_{\text{SM}} = (m_{\pi^+}^2 - m_{\pi^0}^2)_{\text{SM}} = \frac{2e^2 C_{\text{SM}}}{f^2_\pi}, \tag{18} $$

with

$$ C_{\text{SM}} = \frac{1}{8\pi^2} \frac{3}{4} \int_0^\infty dQ^2 \left( 1 - \frac{Q^2}{Q^2 + M_Z^2} \right) \left( -Q^2 \Pi_{LR}(Q^2) \right). \tag{19} $$

In the limit where $M_Z \to \infty$ we recover the usual expression of Low et al. [4], while there is no induced mass difference in the limit $M_Z \to 0$ where the $SU(2) \times U(1)$ gauge symmetry is unbroken. This can readily be seen by rewriting the external fields in eq. (11) in terms of the usual $W^3_\mu$ and $B_\mu$ gauge fields:

$$ l_\mu = \frac{e}{\cos \theta_W} \left[ Q_L - \frac{1}{2} T_3 \right] B_\mu(x) + \frac{e}{2 \sin \theta_W} T_3 W^3_\mu(x), \tag{20} $$

$$ r_\mu = \frac{e}{\cos \theta_W} Q_R B_\mu(x). \tag{21} $$

Neither the $W^3_\mu$ field, which has a pure left–handed coupling to quarks, nor the $B_\mu$ field, which has a left component proportional to the unit matrix, can generate an effective $O(p^0)$ interaction of Goldstone fields. When the $SU(2) \times U(1)$ gauge symmetry is broken down to $U(1)_{\text{EM}}$, eqs. (18) and (19) show that the net effect of the massive physical $Z$ is to reduce just a little bit the $\pi^+$ mass induced by electromagnetism alone.

It is possible to make a phenomenological evaluation of the integral in eq. (19) using the fact that the function $\Pi_{LR}(Q^2)$ obeys an unsubtracted dispersion relation

$$ \Pi_{LR}(Q^2) = \int_0^\infty dt \frac{1}{t + Q^2} \left[ \frac{1}{\pi} \text{Im}\Pi_V(t) - \frac{1}{\pi} \text{Im}\Pi_A(t) \right], \tag{22} $$

with $\frac{1}{\pi} \text{Im}\Pi_V(t)$ and $\frac{1}{\pi} \text{Im}\Pi_A(t)$ the physical vector and axial–vector spectral functions which are measured in hadronic $e^+e^-$ annihilations and in hadronic $\tau$ decays. The physical spectral functions, however, are not quite the “chiral limit” spectral functions which should be inserted in eq. (19). It is possible to do the appropriate corrections, as discussed e.g. in ref. [1], but for the purposes of this letter we wish to follow a more theoretical procedure. We propose to evaluate the integral in eq. (19) within the framework of QCD in the limit of a large number of colours $N_c$, [12–14], which we shall denote by QCD($\infty$) for short.

The spectral function associated with $\Pi_{LR}(Q^2)$ in QCD($\infty$) consists of the difference of an infinite number of narrow vector states and an infinite number of narrow axial–vector states, together with the Goldstone pion pole:

$$ \frac{1}{\pi} \text{Im}\Pi_{LR}(t) = \sum_V f^2_V M^2_V \delta(t - M^2_V) - \sum_A f^2_A M^2_A \delta(t - M^2_A) - f^2_\pi \delta(t). \tag{23} $$

Since $\Pi_{LR}(Q^2)$ obeys the dispersion relation (22), we find that

$$ -Q^2 \Pi_{LR}(Q^2) = f^2_\pi + \sum_A f^2_A M^2_A \frac{Q^2}{M^2_A + Q^2} - \sum_V f^2_V M^2_V \frac{Q^2}{M^2_V + Q^2}. \tag{24} $$
Furthermore, the two Weinberg sum rules that follow from eq. (8) constrain the couplings and masses of the narrow states as follows:

\[ \sum_V f_V^2 M_V^2 - \sum_A f_A^2 M_A^2 = f_\pi^2 \quad \text{and} \quad \sum_V f_V^2 M_V^2 - \sum_A f_A^2 M_A^2 = 0 , \]  

ensuring the convergence of the integral in eq. (19) in QCD(∞), with the result

\[ C_{SM} = -\frac{3}{32\pi^2} \left( \sum_A \frac{M_Z^2}{M_Z^2 - M_A^2} f_A^2 M_A^4 \log \frac{M_Z^2}{M_A^2} - \sum_V \frac{M_Z^2}{M_Z^2 - M_V^2} f_V^2 M_V^4 \log \frac{M_Z^2}{M_V^2} \right) . \]

The original evaluation by Low et al. [4] of the integral in eq. (7) was made by considering only the phenomenological contributions from the pion pole and the lowest vector and axial–vector states in the narrow width approximation. This approximation, which reproduces very well the experimental determination of the physical \( \pi^+ - \pi^0 \) mass difference, can nowadays be viewed as the approximation to QCD(∞) which consists in restricting the part of the hadronic spectrum which is responsible for \( S \chi SB \) in the channels with the \( J^P \) quantum numbers \( 1^- \) and \( 1^+ \) to their lowest energy states. The rest of the infinite number of narrow states are then treated as dual to their corresponding QCD(∞) perturbative continuum, and therefore do not contribute to order parameters of \( S \chi SB \). An explicit formulation in terms of an effective Lagrangian of this lowest meson dominance (LMD) approximation to QCD(∞), with inclusion of the \( 0^- \) and \( 0^+ \) channels as well, has been recently discussed in ref. [5]. In the LMD approximation to QCD(∞), the integral in eq. (19) gives the result

\[ C_{SM} \simeq \frac{3}{48\pi^2} M_V^2 \left[ \frac{M_A^2}{M_A^2 - M_V^2} \log \frac{M_A^2}{M_V^2} - \frac{M_A^2}{M_A^2 - M_V^2} \left( \log \frac{M_Z^2}{M_V^2} - \frac{M_A^2}{M_A^2 - M_V^2} \log \frac{M_A^2}{M_V^2} \right) \right] , \]  

where we have expanded in powers of \( 1/M_Z^2 \) and retained only the leading term. The first term in the r.h.s. of eq. (27) is the contribution from the virtual photon integration. The \( Z \)–induced contribution consists of a “large” \( \log M_Z^2 \) term and a “constant” term. Overall, it represents a correction of \( 0.097\% \) [4] to the lowest order electromagnetic contribution. The effect, as expected, is very small but nevertheless larger than the present experimental accuracy in the determination of the \( \pi^+ - \pi^0 \) mass difference. Although this observation may bear some interest by itself, we shall not pursue it any further here. We rather dedicate the rest of this note to several theoretically interesting aspects of this calculation.

### 3 \( \Pi_{LR}(Q^2) \) and the Operator Product Expansion

As recently discussed in ref. [4], besides the two Weinberg sum rules in eqs. (25), there are further constraints between masses and couplings of resonances on the one hand and the local order parameters which govern the operator product expansion (OPE) of the \( \Pi_{LR}(Q^2) \) function on the other. In particular, it has been shown [16] that in the large–\( N_c \) limit, the leading \( d = 6 \) order parameter is given by the expression

\[ \lim_{Q^2 \to \infty} Q^6 \Pi_{LR}(Q^2) = -4\pi^2 \left( \frac{\alpha_s}{\pi} + O(\alpha_s^2) \right) \langle \bar{\psi}\psi \rangle^2 = \sum_V f_V^6 M_V^6 - \sum_A f_A^6 M_A^6 , \]

\(^2\)For the numerical evaluation, we have taken \( M_V \) and \( M_A \) as the \( \rho(770) \) and the \( a_1(1260) \) masses, respectively. Our normalization convention corresponds to \( f_\pi = 92.4 \text{ MeV} \).
where \( \langle \bar{\psi} \psi \rangle \) is the usual single flavour quark bilinear condensate (\( \psi = u, d, \) or \( s \)) in the chiral limit, and the Wilson coefficient is the result of a lowest order calculation \(^{17}\) in powers of \( \alpha_s \). We shall show that it is precisely this \( d = 6 \) term which controls the log \( M_Z^2 \) contribution to the \( Z \)-induced part of the integral \( C_Z \) in eq. (13). Indeed, following a Wilsonian approach, we can split this integral into a low–energy region \( 0 \leq Q^2 \leq \mu^2 \ll M_Z^2 \) where the \( Z \) field is integrated out, \( i.e. \) where we approximate \( 1/(Q^2 + M_Z^2) \) by \( 1/M_Z^2 \), and a high–energy region \( \mu^2 \leq Q^2 \leq \infty \) where the \( Z \)-propagator is fully kept but the \( \Pi_{LR}(Q^2) \) function is approximated by the leading \( d = 6 \) term in eq (28). These approximations are equivalent to neglecting higher order corrections in \( 1/M_Z^2 \) and in \( 1/\mu^2 \). We then obtain

\[
C_Z = \left[ \begin{array}{c}
\frac{3}{32\pi^2} \frac{1}{M_Z^2} \left[ \sum_A f_A^2 M_A^6 \log \frac{M_A^2}{\mu^2} - \sum_V f_V^2 M_V^6 \log \frac{M_V^2}{\mu^2} \right] \\
-\frac{3}{32\pi^2} \frac{1}{M_Z^2} \left[ \sum_A f_A^2 M_A^6 - \sum_V f_V^2 M_V^6 \right] \log \frac{M_Z^2}{\mu^2},
\end{array} \right. \tag{29}
\]

where the first line is the result from the low–energy region and the second line the one from the high–energy region. We observe that the separation scale \( \mu^2 \) cancels in the sum of the two terms; in other words, there is an exact matching between the long–distance contribution and the short–distance contribution. This cancellation also occurs in the LMD approximation to QCD(\( \infty \)) and the result coincides then with the one given in eq. (27). The reason why we can exhibit this exact matching is of course due to the fact that the function \( \Pi_{LR}(Q^2) \) is an order parameter for all values of \( Q^2 \) and therefore has duality properties under the transformation \( Q^2 \rightarrow 1/Q^2 \) which, in the large–\( N_c \) limit, we have been able to work out explicitly.

It is also interesting to look at eq. (29) from the point of view of effective field theory. The full integral in eq. (13) can be split as follows

\[
C_Z = \lim_{\mu \to \infty} \{ C_Z(M_Z \to \infty, \mu) + \delta C_Z(M_Z, \mu) \}, \tag{30}
\]

where \( \delta C_Z(M_Z, \mu) \equiv C_Z - C_Z(M_Z \to \infty, \mu) \). Each of the two terms in the sum is now UV–divergent and this is why an ultraviolet cutoff \( \mu \) is needed. The first term in eq. (30) corresponds to the result from the effective low–energy theory, where the \( Z \) field has been integrated out and it is precisely given by the first line in the r.h.s. of eq. (29). The second term \( \delta C_Z(M_Z, \mu) \) corresponds to the matching condition in the effective field theory language and the result is precisely the one given in the second line of eq. (29). Formally one can set \( \mu = M_Z \) (the effective field theory scale in this case) in these expressions since they are actually \( \mu \) independent. Then \( \delta C_Z(M_Z, M_Z) = 0 \) and \( C_Z = C_Z(M_Z \to \infty, M_Z) \).

We are now in a position to discuss how the above calculation proceeds in the framework which is usually adopted to tackle weak matrix element calculations. \(^3\)

### 4 Four–Quark Operators

The relevant term in the Lagrangian of the Standard Model which is responsible for the \( Z \)-induced contribution to the \( \pi^+ - \pi^0 \) mass difference is the neutral current interaction term

\[
\mathcal{L}_{NC} = \frac{e}{2 \sin \theta_W \cos \theta_W} \left[ \bar{q}_L \gamma^\mu T_3 q_L - 2 \sin^2 \theta_W \bar{q}_L \gamma^\mu Q_L q_L - 2 \sin^2 \theta_W \bar{q}_R \gamma^\mu Q_R q_R \right] Z_\mu. \tag{31}
\]

\(^3\)See e.g. the lectures of A. Buras in ref. \(^{18}\) and references therein.
When looking for the induced effective Lagrangian of order $O(p^0)$ which contributes to Goldstone boson masses, it is sufficient to consider left–right operators. In the absence of the strong interactions, the effective four–quark Hamiltonian which emerges after integrating out the $Z$ field is

$$-\mathcal{H}_{\text{eff}} = \frac{-1}{M_Z^2 \cos^2 \theta_W} \left[ \sin^2 \theta_W Q_{LR} - \frac{1}{2} (\bar{q}_L \gamma_\mu T_3 q_L) (\bar{q}_R \gamma^\mu Q_{LR}) \right]$$

$$= \frac{e^2}{M_Z^2} Q_{LR} - \frac{e^2}{M_Z^2 \cos^2 \theta_W} \frac{1}{6} \left( \sum_q q_\mu q \right) (\bar{q}_R \gamma^\mu Q_{LR}) , \quad (32)$$

where

$$Q_{LR} \equiv (\bar{q}_L \gamma_\mu Q_L q_L) (\bar{q}_R \gamma^\mu Q_{LR} q_R) , \quad (33)$$

and summation over quark colour indices within brackets is understood. In fact, to $O(p^0)$, only the first term proportional to the four–quark operator $Q_{LR}$ can contribute. In the presence of the strong interactions, the evolution of $Q_{LR}$ from the scale $M_Z^2$ down to a scale $\mu^2$ can be calculated in the usual way, provided this $\mu^2$ is still large enough for a perturbative QCD (pQCD) evaluation to be valid. In the leading logarithmic approximation in pQCD, and to leading non–trivial order in the $1/N_c$ expansion, the relevant mixing in this evolution which we need to retain is simply given by

$$Q_{LR}(M_Z^2) = Q_{LR}(\mu^2) - 3\frac{\alpha_s}{\pi} \frac{1}{2} \log \frac{M_Z^2}{\mu^2} D_{RL}(\mu^2) + \cdots , \quad (34)$$

where $D_{RL}$ denotes the four–quark density–density operator

$$D_{RL} := \sum_{q,q'} e_q e_{q'} (\bar{q}_L' q_R) (\bar{q}_R q_L') , \quad (35)$$

with $e_q$ and $e_{q'}$ the quark charges in units of the electric charge. This can be seen as follows: in the $MS$ renormalization scheme, the full evolution of the Wilson coefficients $c_Q$ and $c_D$ of the operators $Q_{LR}$ and $D_{LR}$ at the one loop level is governed by the equations (subleading contributions in the $1/N_c$ expansion have been neglected)

$$\mu^2 \frac{d}{d\mu^2} \left( \begin{array}{c} c_Q \\ c_D \end{array} \right) = \frac{1}{4} \frac{\alpha_s N_c}{\pi} \left( \begin{array}{cc} \cdots & \cdots \\ \frac{6}{N_c} & -3 + \cdots \end{array} \right) \left( \begin{array}{c} c_Q \\ c_D \end{array} \right) , \quad (36)$$

with boundary conditions: $c_Q(M_Z) = 1$ and $c_D(M_Z) = \frac{3 \alpha_s}{2 \pi}$. The result in eq. (34) follows when taking $c_D(M_Z) = 0$, which is appropriate when keeping the one-loop leading log only, and from the off–diagonal term in the (transposed) anomalous dimension matrix.

We are then confronted with a typical problem of bosonization of four–quark operators. The bosonization of $D_{RL}$ is only needed to leading order in the $1/N_c$ expansion. To that order and to order $O(p^0)$ in the chiral expansion it can be readily obtained from the bosonization of the factorized density currents, with the result

$$D_{RL} = \sum_{q,q'} e_q e_{q'} (\bar{q}_L' q_R) (\bar{q}_R q_L') \rightarrow 2B \frac{f_2^2}{4} \times 2B \frac{f_2^2}{4} \text{tr} \left( U Q_L U^\dagger Q_R \right) , \quad (37)$$

\[\text{See e.g. the lectures in ref. } [19] \text{ and references therein.}\]
where $B$ is the low energy constant which describes the bilinear quark condensate in the chiral limit,

$$B = -\langle \bar{\psi} \psi \rangle_{f^2 \pi^2}. \quad (38)$$

We find that the overall contribution of the term proportional to the $D_{RL}(\mu^2)$ four–quark operator, which we denote $C_{Z|D_{RL}}$, is given by the expression

$$C_{Z|D_{RL}} = -\frac{1}{M_Z^2} \frac{3}{4} \frac{\alpha_s}{\pi} \langle \bar{\psi} \psi \rangle^2 \frac{1}{2} \log \frac{M_Z^2}{\mu^2}, \quad (39)$$

and it is exactly the same result as the one coming from the $d = 6$ term of the OPE in the previous calculation, the second line in eq. (29), once the duality constraint in eq. (28) is taken into account, i.e. precisely the same contribution as required by the matching condition in the effective field theory analysis above. Equation (39) does not change in an $\overline{\text{MS}}$ renormalization scheme when restricted to the one-loop leading log.

The problem is then reduced to the bosonization of the operator $Q_{LR}(\mu^2)$. We are confronted here with a typical calculation of a hadronic matrix element of a four–quark operator, in this case the matrix element $\langle \pi^+ | Q_{LR}(\mu^2) | \pi^+ \rangle$. The factorized component of the operator $Q_{LR}$, which is leading in $1/N_c$, cannot contribute to the $\mathcal{O}(p^0)$ term of the low–energy effective lagrangian. The contribution we want from this matrix element is therefore the next–to–leading one in the $1/N_c$ expansion and it requires the evaluation of non–factorizable four–quark matrix elements which is a priori a highly non–trivial task. Yet, we know that there is a straightforward integral representation of the observable we want to compute in terms of a two–point function, and we have succeeded in doing the calculation that way within the $1/N_c$–expansion framework. This seems to indicate that there should be a way to do the calculation of the matrix element $\langle \pi^+ | Q_{LR}(\mu^2) | \pi^+ \rangle$ as well. Indeed, we shall next show that, for this particular matrix element, our present knowledge of analytic results in non–perturbative QCD allows us to do the calculation exactly to next–to–leading order in the $1/N_c$ expansion. The deep reason behind it is that we know from the discussions in the previous sections that only two–point functions can appear in the dynamics of this matrix element, and therefore it is enough to have a formulation of the low–energy effective lagrangian of QCD($\infty$) compatible with the OPE constraints at short distances for two–point functions. These are precisely the constraints which have been recently discussed in ref. [16].

The calculation proceeds along much the same lines as first suggested in papers by Bardeen, Buras and Gérard [20, 21] sometime ago 5, except that we shall go beyond loops generated by Goldstone particle interactions alone in order to achieve a correct matching with the logarithmic scale dependence of the short–distance contribution in eq. (33). We have evaluated the matrix element $\langle \pi^+ | Q_{LR}(\mu^2) | \pi^+ \rangle$ within the framework of an effective Lagrangian which is a straightforward generalization to an arbitrary number of massive $J^P = 1^-$ and $J^P = 1^+$ mesonic states of the effective Lagrangian corresponding to the LMD approximation to QCD($\infty$) recently discussed in ref. [15]. Keeping only the terms that are relevant for the evaluation of the $\mathcal{O}(p^0)$ contributions, we have

$$\mathcal{L}_{\text{eff}} = \frac{f^2}{4} \text{tr} D_\mu U^+ D^\mu U + L_{10} \text{tr} U^+ F^{\mu \nu}_{R} U F_{L, \mu \nu}$$

5See also refs. [22–24] and references therein for more recent work.
\[-\frac{1}{4} \sum_V \text{tr} \left( V_{\mu \nu} V^{\mu \nu} - 2M_V^2 V_{\mu \nu} \right) - \frac{1}{4} \sum_A \text{tr} \left( A_{\mu \nu} A^{\mu \nu} - 2M_A^2 A_{\mu \nu} \right) \]
\[-\frac{1}{2\sqrt{2}} \sum_V f_V \text{tr} V_{\mu \nu} f_+^{\mu \nu} - \frac{1}{2\sqrt{2}} \sum_A f_A \text{tr} A_{\mu \nu} f_+^{\mu \nu} + \cdots, \tag{40} \]
with the same notation as in ref. \[3\], \( L_{10} \) being one of the \( \mathcal{O}(p^4) \) constants of the Gasser-Leutwyler Lagrangian \[23\].

The bosonized expressions of the currents \((\bar{q}_L \gamma^\mu Q_Lq_L)\) and \((\bar{q}_R \gamma^\mu Q_Rq_R)\) are obtained from the resulting effective action \( \Gamma_{\text{eff}} \) at tree level,

\[(\bar{q}_L \gamma^\mu Q_Lq_L) \rightarrow (Q_L)_{ij} \frac{\delta \Gamma_{\text{eff}}}{\delta r_{ij}^\mu (x)} \quad \text{and} \quad (\bar{q}_R \gamma^\mu Q_Rq_R) \rightarrow (Q_R)_{ij} \frac{\delta \Gamma_{\text{eff}}}{\delta r_{ij}^\mu (x)} , \tag{41} \]

where \( i, j = u, d, s \) are flavour indices. The contributions from a loop of virtual pions, vector and axial–vector resonances are computed by applying a finite ultraviolet cut-off \( \mu^2 \) to the integral over the corresponding (euclidean) virtual loop momentum. One technical point, as already discussed in ref \[27\] (see eqs. (7.24) to (7.26) of this reference), requires however attention. It is the fact that to next–to–leading order in the \( 1/N_c \) expansion, and besides the loops generated by the bosonized factorized currents that we have just discussed, there are also contact terms generated by the second variation of the effective action. In our case these terms are:

\[Q_{LR}^{\text{contact}} = (Q_L)_{ij} (Q_R)_{kl} \frac{\delta^2 \Gamma_{\text{eff}}}{\delta l_{ij}^\mu (x) \delta r_{kl}^\mu (x)} = \left\{-2f_\pi^2 \text{tr} \left( Q_RUQ_LU^\dagger \right) \delta(0) - 6L_{10} \text{tr} \left( Q_RUQ_LU^\dagger \right) \Box \delta(0) + \cdots \right\} , \tag{42} \]

where the \( \delta \)–function contributions are to be interpreted within the same cut-off regularization:

\[\delta(0) \rightarrow \pi \int_0^{\mu^2} dQ^2 Q^2 = \frac{\pi}{2} \mu^4 \quad \text{and} \quad \Box \delta(0) \rightarrow \pi \int_0^{\mu^2} dQ^2 Q^4 = \frac{\pi}{3} \mu^6 . \tag{43} \]

Adding the two types of contributions, the final result reads

\[\langle \pi^+ | Q_{LR}(\mu^2) | \pi^+ \rangle = \frac{1}{16\pi^2} \left\{ \frac{3}{2} \mu^4 + \frac{L_{10}}{f_\pi^2} \mu^6 + \frac{3}{2} \int_0^{\mu^2} dQ^2 Q^6 \left[ \sum_V \frac{f_V^2}{Q^2 + M_V^2} - \sum_A \frac{f_A^2}{Q^2 + M_A^2} \right] \right\} . \tag{44} \]

The contribution of the Goldstone bosons alone corresponds to the two terms in the first line of the r.h.s. of eq. \[14\]. They display a typical polynomial dependence with respect to the cut-off \( \mu \), which can hardly provide a reasonably good matching with the logarithmic scale dependence coming from the short–distance contributions in \( C_Z|_{D_{RL}} \). In fact, in an \( \overline{MS} \) regularization scheme, as commonly chosen for the evaluation of the short–distance Wilson coefficients, these power divergences will automatically disappear. Simply adding higher resonances does not by itself solve the problem of matching the long– and the short–distances either; however, when the information coming from the short–distance properties of the correlator in eq. \[14\], \( i.e. \) the two Weinberg sum rules \[23\] and the sum rule \[10, 26\]

\[-4L_{10} = \sum_V f_V^2 - \sum_A f_A^2 , \tag{45} \]

\[\overline{\text{MS}} \]
is taken into account, the result of eq. (44) can indeed be recast into a form which reproduces the first line of eq. (29). Notice that in an \( \overline{MS} \) regularization scheme, the integral in eq. (44) should be understood in \( n = 4 - \epsilon \) dimensions and therefore multiplied by \( \mu^{4-n} \) with \( \mu \) the \( \overline{MS} \) regularization scale. The result, when combined with eq. (39) finally yields eq. (29) once again. We insist on the fact that, regardless of the regularization one chooses, the calculation we have done of the matrix element \( \langle \pi^+ | Q_{LR}(\mu^2) | \pi^+ \rangle \) is an exact calculation to next–to–leading order in the \( 1/N_c \) expansion and to \( \mathcal{O}(p^0) \) in the chiral expansion.

5 Comments and Outlook

The main purpose of the above analysis has been to show that the calculation of the electroweak contribution to the \( \pi^+ - \pi^0 \) mass difference in the chiral limit is an interesting theoretical laboratory for testing issues connected with the matching of long and short distances in the evaluation of hadronic weak matrix elements in general.

On the one hand, upon identifying the relevant low–energy constant in terms of the appropriate QCD correlator, we have been able to obtain, in the large–\( N_c \) limit, an exact result in terms of resonance parameters with masses and couplings constrained by the short–distance properties of this correlator.

On the other hand, due to the presence of the large scale set by \( M_Z \), we were able to proceed as done usually in the study of weak non–leptonic processes. In this respect, we encounter a typical situation: the effective Hamiltonian at tree level is given by a current–current four quark operator \( Q_{LR} \), which, upon taking into account short–distance pQCD corrections, mixes with the density–density operator \( D_{RL} \). The matrix element of the latter to \( \mathcal{O}(p^0) \), which is only needed at the leading order in the large–\( N_c \) limit, is easily computed. In the same limit, \( \langle \pi^+ | Q_{LR}(\mu^2) | \pi^+ \rangle \) vanishes, and next–to–leading, non–factorizable, contributions have to be included. A suitable matching of the long–distance contribution with the logarithmic scale dependence coming from the short–distance contribution cannot be achieved, in this case, by evaluating the relevant hadronic matrix elements in terms of an effective theory incorporating only the Goldstone bosons. When the contribution of vector and axial–vector states satisfying the appropriate short–distance constraints are included, we find an exact matching.

We think that the observations which follow from this example point towards promising perspectives for a systematic determination of hadronic weak matrix elements within the framework of QCD(\( \infty \)). The required steps are the following: first one needs an identification of the low–energy constants of the \( \Delta S = 1 \) and \( \Delta S = 2 \) effective chiral Lagrangian in terms of integrals of QCD Green’s functions, \( i.e. \) the equivalent of our eq. (19) above. Next one should proceed to the study of the constraints which the OPE imposes on these Green’s functions in the large–\( N_c \) limit, \( i.e. \) the equivalent of the Weinberg sum rules in eqs. (25) and the sum rules in eqs. (28) and (45). The final step is the construction of an effective Lagrangian, along the lines shown in ref. [15], which incorporates these constraints. This program is at present under study.

In the meantime, it would be interesting to see a calculation done using numerical simulations of lattice QCD of the matrix element \( \langle \pi^+ | Q_{LR}(\mu^2) | \pi^+ \rangle \), so as to be able to appreciate the
performance of the lattice approach in a simple case which we understand well analytically.

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