ORDERED GROUP INVARIANTS FOR ONE-DIMENSIONAL SPACES

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ABSTRACT. We show that the Bruschlinsky group with the winding order is a homeomorphism invariant for a class of one-dimensional inverse limit spaces. In particular we show that if a presentation of an inverse limit space satisfies the Simplicity Condition, then the Bruschlinsky group with the winding order of the inverse limit space is a dimension group and is a quotient of the dimension group with the standard order of the adjacency matrices associated with the presentation.

1. INTRODUCTION

Ordered groups have been useful invariants for the classification of many different categories. A class of ordered groups, dimension groups, was used in the study of $C^*$-algebras to classify AF-algebras ([8]), and Giordano, Herman, Putnam and Skau ([3, 4]) defined (simple) dimension groups in terms of dynamical concepts to give complete information about the orbit structure of zero-dimen sional minimal dynamical systems. Swanson and Volkmer ([11]) showed that the dimension group of a primitive matrix is a complete invariant for weak equivalence, which is called $C^*$-equivalence by Bratteli, Jørgensen, Kim, and Roush ([5]). And Barge, Jacklitch, and Vago ([2]) showed that, for certain class of one-dimensional inverse limit spaces, two spaces are homeomorphic if and only if their associated substitutions are weak equivalent, and if two inverse limit spaces are homeomorphic and the square of their connection maps are orientation preserving, then the dimension groups of the adjacency matrices associated to the substitutions are order isomorphic.

A recent development ([2, 3, 4, 7, 8, 11]) is the refinement of $\tilde{\mathbb{H}}^1(X)$ as a topological invariant for certain one-dimensional spaces $X$, by making this group an ordered group. Here $\tilde{\mathbb{H}}^1(X)$ is the direct limit of first cohomology groups on graphs approximating the space $X$. There is a natural order on the first cohomology of a graph (a coset is positive if it contains a nonnegative function), and the standard order on $\tilde{\mathbb{H}}^1(X)$ is the direct limit order derived from the natural graph orders (see definition 3.7). Except for parts of [4] and [7], the ordered cohomology results have involved the standard order.

A second order on $\tilde{\mathbb{H}}^1(X)$, the winding order, is geometrically natural as its positive elements are the homotopy classes of continuous orientation preserving maps from $X$ to $S^1$. Boyle and Handelman ([4]) defined the winding order for suspension spaces of zero-dimensional dynamical systems, and showed that in some (but not all) cases it agrees with the standard order. Forrest ([6]) defined the winding order for the first Čech cohomology groups of directed graphs (thus taking

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the step of removing dynamics), and used this to show that whenever two one-dimensional inverse limit spaces are pro-homotopy equivalent, then their first Čech cohomology groups with the standard order are order isomorphic.

In this paper, we extend the definition of the winding order to a large class of one-dimensional spaces, “compact branched matchbox manifolds”. We show that for a compact connected orientable branched matchbox manifold with an inverse limit presentation satisfying the Simplicity Condition, the Bruschlinsky group with the winding order is a simple dimension group, and the winding order equals the standard order. This is a natural extension of the relations between zero-dimensional minimal systems and simple dimension groups in Giordano, Herman, Putnam and Skau ([8, 9]) to an appropriate class of one-dimensional spaces. As a corollary we obtain an independent proof of some results of Forrest and Barge, Jacklitch and Vago ([3, 7]) computing dimension group invariants for the oriented generalized one-dimensional solenoids of Williams ([16, 17, 18]).

The outline of the paper is as follows. In section 2, using works of Aarts and Oversteegen ([1]), Mardešić and Segal ([12]) and Rogers ([14]), we define compact connected orientable branched matchbox manifolds, and show they all have presentations by orientation preserving maps of finite directed nondegenerate graphs. In section 3, we show that the Bruschlinsky group with the winding order of a compact connected orientable branched matchbox manifold with the Simplicity Condition is order isomorphic to the direct limit of the graph groups with the standard order defined from the presentation (and therefore the winding and standard orders agree). And in section 4, we recall the axioms for one-dimensional generalized solenoids and calculate the Bruschlinsky groups with the winding order of an example in which the Bruschlinsky group is not given by the obvious direct limit of presenting matrices.

2. Branched matchbox manifolds and ordered groups

Aarts and Oversteegen ([1]) defined a matchbox manifold to be a separable metric space $Y$ such that each point $y \in Y$ has a neighborhood which is homeomorphic to $S_y \times I_y$ where $S_y$ is a zero-dimensional space and $I_y$ is an open interval. For a topological embedding $h: S_y \times I_y \to Y$, they called $h(S_y \times I_y)$ a matchbox neighborhood of $y \in Y$. A matchbox manifold $Y$ is called orientable if each arc component $C_{\alpha}$, $\alpha \in A$, of $Y$ has a parameterized immersed arc $p_{\alpha}: \mathbb{R} \to C_{\alpha}$ such that each point $y \in Y$ has a matchbox neighborhood $h(S_y \times I_y)$ with the following property: for each $\alpha \in A$ and each $t \in \mathbb{R}$ with $p_{\alpha}(t) \in h(S_y \times I_y)$ there exists an open interval $I$ containing $t$ such that $pr_2 \circ h^{-1} \circ p_{\alpha}$ is increasing on $I$ where $pr_2$ is the canonical projection from $S_{p_{\alpha}(t)} \times I_{p_{\alpha}(t)}$ to $I_{p_{\alpha}(t)}$.

**Theorem 2.1 ([1]).** For a one-dimensional space $Y$, the following are equivalent:

1. $Y$ is an orientable matchbox manifold.
2. $Y$ is the phase space of a flow without rest point.
3. There exists a cross section $K$ with return time map $r_K$ such that $Y$ is the standard suspension of $(K, r_K)$.

**Branched matchbox manifold.** We define a branched matchbox to be a topological space homeomorphic to $U = ((S_1 \times (-1,0]) \cup (S_2 \times [0,1]))/\sim$ such that $S_1$ and $S_2$ are zero dimensional separable metrizable spaces and there is a (closed) equivalence relation $\approx$ on $S_1 \cup S_2$ such that
(1) For every \( s_1 \in S_1 \) (\( \sigma_2 \in S_2 \), respectively) there exists at least one \( s_2 \in S_2 \) (\( \sigma_1 \in S_1 \), respectively) such that \( s_1 \approx s_2 \) (\( \sigma_1 \approx \sigma_2 \), respectively),

(2) \((S_1 \cup S_2)/\approx\) is a zero dimensional metrizable space with the quotient topology and

(3) \((s_1,i) \sim (s_2,j)\) if and only if either \( s_1 \approx s_2 \) and \( i = j = 0 \) or \( s_1 = s_2 \) and \( i = j \).

**Remark 2.2.** In this paper, we will always be concerned with the case that \( S_1 \) and \( S_2 \) are compact.

For \( s_1 \in S_1 \) and \( s_2 \in S_2 \) such that \( s_1 \approx s_2 \), the set

\[
\left( (\{ s_1 \} \times (-1,0)) \cup (\{ s_2 \} \times [0,1]) \right) / \sim
\]

is called a **match**.

A **branched matchbox manifold** is a separable metrizable space \( Y \) together with a collection of maps called **charts** such that

1. a chart is a homeomorphism \( h: V \rightarrow U \) where \( V \) is an open set in \( X \) and \( U \) is a branched matchbox,
2. every point in \( Y \) is in the domain of some chart,
3. for charts \( h_1: V_1 \rightarrow U_1 \) and \( h_2: V_2 \rightarrow U_2 \) the change of coordinates map \( h_2 \circ h_1^{-1}: h_1(V_1 \cap V_2) \rightarrow h_2(V_1 \cap V_2) \) is continuous.

Every branched matchbox \( U \) has the direction given by the second coordinate with a continuous projection \( p_U: U \rightarrow (-1,1) \) defined by \( [(z,j)] \mapsto j \). Following the approach of Aarts and Oversteegen ([3], §3), we call a branched matchbox manifold \( Y \) **orientable** if it can be covered by branched matchboxes with directions agreeing on overlaps, i.e., there are oriented branched matchboxes \( U_i \) with projections \( p_i: U_i \rightarrow (-1,1) \), open sets \( V_i \) covering \( Y \), and homeomorphisms \( h_i: V_i \rightarrow U_i \) such that for every \( i,j \) and every locally one-to-one curve \( \gamma: [0,1] \rightarrow V_i \cap V_j \), \( p_i \circ h_i \circ \gamma \) is increasing if and only if \( p_j \circ h_j \circ \gamma \) is increasing. The particular collection of charts, maximal with respect to this change of coordinate property, is called an **orientation** of the branched matchbox manifold \( Y \).

**Ordered group.** A **preordered group** is a pair \((G, G_+)\) where \( G \) is an Abelian group and the **positive cone** \( G_+ \) is a submonoid of \( G \) which generates \( G \). We write \( g_1 \leq g_2 \) if \( g_2 - g_1 \in G_+ \) for \( g_1, g_2 \in G \). If \((G, G_+)\) satisfies the additional condition that \( G_+ \cap -G_+ = \{0\} \), then \((G, G_+)\) is called an **ordered group**.

An **order unit** in a preordered group is an element \( u \in G_+ \) such that for every \( g \in G \) there exists a positive integer \( n = n(g) \) such that \( g \leq nu \). A preordered group \((G, G_+)\) is **unperforated** if for every \( g \in G \) and positive integer \( n \), \( ng \in G_+ \) implies \( g \in G_+ \). We say that an ordered group \((G, G_+)\) satisfies the **Riesz Interpolation property** if given \( g_1, g_2, h_1, h_2 \in G \) with \( g_i \leq h_j \) (\( i,j = 1,2 \)), then there is a \( k \in G \) such that \( g_i \leq k \leq h_j \).

**Bruschlinys group with the winding order.** For a compact metric space \( Y \), let \( C(Y, S^1) \) be the set of continuous functions from \( Y \) to \( S^1 \), and

\[
R(Y) = \{ \phi \in C(Y, S^1) \mid \phi(y) = \exp(2\pi ig(y)) \text{ for some } g \in C(Y, \mathbb{R}) \}.
\]

Then \( R(Y) \) is the subgroup of functions homotopic to a constant map in \( C(Y, S^1) \). The **Bruschinsky group** of \( Y \) ([3], §4.3) is given by

\[
\text{Br}(Y) = C(Y, S^1)/R(Y).
\]
It is well-known that $\check{H}^1(Y)$, the first Čech cohomology group of $Y$, is isomorphic to the Bruschlinsky group of $Y$ ([4 4.7]).

Now suppose that $Y$ is an oriented compact branched matchbox manifold. Let $C_{\oplus}(Y, S^1)$ be the set of $\phi \in C(Y, S^1)$ such that there exists a map $\psi \in R(Y)$ such that $\phi \cdot \psi$ is non-orientation-reversing, i.e., for every orientation preserving parameterized curve $\gamma: \mathbb{R} \to Y$, $(\phi \cdot \psi)(\gamma)(t)$ does not move in the clockwise direction as $t \in \mathbb{R}$ increases.

Define $\mathrm{Br}_{\oplus}(Y) = \{ \phi | \phi \in C_{\oplus}(Y, S^1) \}$. Then $(\mathrm{Br}(Y), \mathrm{Br}_{\oplus}(Y))$ is a preordered group. We call this preorder the winding order ([4 §4]).

**Remark 2.3 ([4 4.7]).** It is possible that the Bruschlinsky group with the winding order of compact orientable space is not an ordered group.

**Observation 2.4.** Homeomorphic orientable compact metric spaces have order-isomorphic Bruschlinsky groups with the winding order.

**Proposition 2.5 ([10]).** The Bruschlinsky group of a compact branched matchbox manifold is a torsion-free group.

Recall that a continuum is a compact connected metric space.

**Lemma 2.6 ([10]).** Let $Y$ be a continuum, $\phi \in C(Y, S^1)$, and $p_n: S^1 \to S^1$ defined by $z \mapsto z^n$ for every positive integer $n$. Then $n \cdot [\phi] = [p_n \circ \phi]$.

**Proposition 2.7.** The Bruschlinsky group with the winding order of a compact connected oriented branched matchbox manifold $Y$ is unperforated.

**Proof.** Suppose that $\phi \in C(Y, S^1)$ and $n \in \mathbb{Z}_+$ such that $n \cdot [\phi] = [p_n \circ \phi] \in \mathrm{Br}_{\oplus}(Y)$. Then there exists a map $\psi \in R(Y)$ given by $y \mapsto \exp(2\pi ig(y))$ with $g \in C(Y, \mathbb{R})$ such that $(p_n \circ \phi) \cdot \psi$ is non-orientation reversing.

Define $\psi: Y \to S^1$ by $y \mapsto \exp(2\pi i \cdot \frac{1}{n}g(y))$. Then we have $\tilde{\psi} \in R(Y)$ and $(p_n \circ \phi) \cdot \psi = p_n \circ (\phi \cdot \tilde{\psi})$. For every orientation preserving parameterized curve $\gamma: \mathbb{R} \to Y$,

\[(p_n \circ \phi) \cdot \psi \circ \gamma(t) = p_n \circ (\phi \cdot \tilde{\psi}) \circ \gamma(t) = p_n \circ \left( (\phi \cdot \tilde{\psi}) \circ \gamma(t) \right)\]

does not move clockwise on $S^1$ as $t \in \mathbb{R}$ increases. So $\phi \cdot \tilde{\psi}$ is non-orientation-reversing as $n$ is a positive integer. Therefore $\phi \in C_{\oplus}(Y, S^1)$, and $(\mathrm{Br}(Y), \mathrm{Br}_{\oplus}(Y))$ is unperforated.

**Remark 2.8.** If $Y$ is a compact connected orientable matchbox manifold, then the above Propositions 2.3 and 2.7 follow from Propositions 4.5 and 3.4 of [4] and Theorem 2.3.

**One dimensional continua.** In [14], Rogers introduced the following notations for one-dimensional continua.

Suppose that $X_1$ and $X_2$ are graphs and that $\mathcal{V}_i$ and $\mathcal{E}_i$ are the vertex set and the edge set of $X_i$, respectively, $i = 1, 2$. A continuous onto map $f: X_2 \to X_1$ is called simplicial relative to $(\mathcal{V}_1, \mathcal{V}_2)$ if $f(\mathcal{V}_2) \subseteq \mathcal{V}_1$ and for every edge $e_2 \in \mathcal{E}_2$ there is an edge $e_1 \in \mathcal{E}_1$ such that $f|_{e_2 \setminus \mathcal{V}_2}$ is a homeomorphism onto $e_1 \setminus \mathcal{V}_1$ or a constant map. The map $f: X_2 \to X_1$ is simplicial if it is simplicial relative to some vertex
sets of $X_1$ and $X_2$. And $f$ is called \textit{light} if the preimage of each point is totally disconnected.

An inverse limit sequence $\{X_k, f_k\}$ on graphs is called \textit{light simplicial} if each $f_k$ is light simplicial, and is called \textit{light uniformly simplicial} if each $X_k$ is a graph with a vertex set $V_k$ and each map $f_k : X_k \to X_{k-1}$ is light simplicial relative to $(V_{k-1}, V_k)$.

\textbf{Theorem 2.9} \cite{13, 14}. Suppose that $\mathcal{X}$ is a one-dimensional continuum.

(1) $\mathcal{X}$ is homeomorphic to an inverse limit of a light simplicial sequence $\{X_k, f_k\}$ on graphs, and

(2) $\mathcal{X}$ is homeomorphic to a light uniformly simplicial inverse limit on graphs if and only if there exists a map $\pi : \mathcal{X} \to [0, 1]$ such that $\pi^{-1}(\{0, 1\})$ is totally disconnected and $\pi|_e$ is a homeomorphism for every $e$ which is a closure of component of $\mathcal{X}\backslash \pi^{-1}(\{0, 1\})$.

Suppose that $\{X_k, f_k\}$ is a light simplicial sequence on graphs. Let

$$\mathcal{X} = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{} \{ (x_0, x_1, x_2, \ldots) \in \prod_{0}^{\infty} X_k \mid f_{k+1}(x_{k+1}) = x_k \}.$$ For a one-dimensional continuum $Y$, we call the sequence $\{X_k, f_k\}$ a \textbf{presentation} of $Y$ if $\mathcal{X}$ is homeomorphic to $Y$.

\textbf{Notation 2.10}. Suppose that $G$ is a directed graph. We consider a directed edge $e$ of $G$ as the image of a local homeomorphism from $[0, 1]$ to $e$ such that $e(0)$ is the initial point of $e$ and $e(1)$ is the terminal point. Then we can represent each point $x \in e$ as $e(t)$ (possibly $e(0) = e(1)$).

Recall that a continuous map $p : [0, 1] \to G$, a directed graph, is \textit{orientation preserving} if $e^{-1} \circ p : I \to [0, 1]$ is increasing for every interval $I \subset [0, 1]$ such that $p(I)$ is a subset of a directed edge $e$. A continuous map $f : G_1 \to G_2$ between two directed graphs is \textit{orientation preserving} if, for every orientation preserving map $p : [0, 1] \to G_1$, $f \circ p : [0, 1] \to G_2$ is orientation preserving \cite{14}. A directed graph is called \textit{nondegenerate} if every vertex has at least one incoming edge and at least one outgoing edge.

Suppose that $Y$ is a compact connected oriented branched matchbox manifold. Since $Y$ is a one-dimensional continuum, there is a light simplicial presentation $\{X_k, f_k\}$ of $Y$ by Theorem 2.9. The following proposition shows that the orientation of $Y$ decides the directions of edges in each coordinate space $X_k$ so that every connection map $f_k : X_k \to X_{k-1}$ is orientation preserving.

\textbf{Proposition 2.11}. Suppose that $Y$ is a compact connected oriented branched matchbox manifold. Then $Y$ has a light simplicial presentation by orientation preserving maps of directed nondegenerate graphs.

\textbf{Proof}. Suppose that $\{h_U : V \to U\}$ is an orientation of $Y$ where $U$ is a branched matchbox with the projections $p_U : U \to (-1, 1)$. Let $\{X_k, f_k\}$ be a light uniformly simplicial presentation of $Y$ given by Theorem 2.9 and $\pi_k : Y \to X_k$ the canonical projection to the $k$th coordinate space. If $e$ is an edge of $X_k$ with $\pi_k^{-1}(e \backslash V_k) \cap h_U^{-1}(U) \neq \emptyset$, then give the direction to the set $(e \backslash V_k) \cap (\pi_k \circ h_U^{-1}(U))$ so that, for every curve $\gamma : [0, 1] \to \pi_k^{-1}(e \backslash V_k) \cap h_U^{-1}(U)$, $p_U \circ h_U \circ \gamma$ is increasing if and only if $e^{-1} \circ \pi_k \circ \gamma$ is increasing. Since $\{h_U\}$ is an orientation of $Y$, we can extend this
direction on \((e \setminus V) \cap \pi_k \circ h_U^{-1}(U)\) to \(e\), and each edge \(X_k\) has a direction induced by the orientation of \(Y\).

Suppose that \(x = (x_0, x_1, \ldots)\) is a point in \(Y\) such that \(x_k \in X_k\) is a vertex and that \(U\) is a branched matchbox such that the domain of \(h_U\) contains \(x\). Then there is a match \(M \subset U\) containing \(h_U(x)\) such that \(p_U|_M \circ h_U(x) = t\) for some \(t \in (-1, 1)\). Since \(\pi_k \circ h_U^{-1} \circ (p_U|_M)^{-1}((-1, t))\) and \(\pi_k \circ h_U^{-1} \circ (p_U|_M)^{-1}((t, 1))\) are nonempty sets in \(X_k\), there exist an edge \(e\) such that \((\pi_k \circ h_U^{-1} \circ (p_U|_M)^{-1}((-1, t))\) \(\cap e = \emptyset\), which is incoming to \(x_k\), and an edge \(e^+\) such that \((\pi_k \circ h_U^{-1} \circ (p_U|_M)^{-1}((t, 1))) \cap e+ \neq \emptyset\), which is outgoing from \(x_k\). Therefore \(X_k\) is nondegenerate.

Suppose that \(e_k \in E_k\) and \(e_{k-1} \in E_{k-1}\) are two edges such that \(e_{k-1} = f_k(e_k)\), and that \(h_k : V \to U\) is a chart such that \(W = \pi_k \circ h_U^{-1}(U) \cap (e_k \setminus V_k) \neq \emptyset\). Then we have \(f_k(W) \subset \pi_{k-1} \circ h_U^{-1}(U) \cap (e_{k-1} \setminus V_{k-1})\), and for every curve \(\gamma : [0, 1] \to h_U^{-1}(U) \cap \pi_k^{-1}(e_k \setminus V_k)\), \(e_{k-1} \circ \pi_k \circ \gamma\) is increasing if and only if \(U \circ h \circ \gamma\) is increasing.

Let \(\gamma : [a, b] \to h_U^{-1}(U) \cap \pi_k^{-1}(e_k \setminus V_k)\) be given by \(\pi_k \circ \gamma(t) = e_k(t)\). Then we have \(\pi_k \circ \gamma(t) = f_k \circ e_k(t)\), and \(e_{k-1} \circ \pi_k \circ \gamma(t) = e_{k-1} \circ f_k \circ e_k(t)\) is increasing as \(t\) is increasing. Therefore \(f_k : X_k \to X_{k-1}\) is orientation preserving.

**Corollary 2.12.** Suppose that \(Y\) is a compact connected orientable branched matchbox manifold. Then there is a continuous map \(\pi : Y \to S^1\) such that \(\pi^{-1}(1)\) is totally disconnected and \(\pi|_\ell\) is an orientation preserving homeomorphism for every \(\ell\) which is an arc component of \(Y \setminus \pi^{-1}(1)\).

**Proof.** Define \(\pi : Y \to S^1\) by \(x = (x_0, x_1, \ldots) \mapsto \exp(2\pi it)\) where \(t \in [0, 1]\) is given by \(x_0 = e(t) \in e \in E_0\). Then \(\pi\) is well-defined and \(\pi^{-1}(1) = \{x \in Y \mid x_0 \in V_0\}\) is a zero dimensional set. Since \(\ell\), an arc component of \(Y \setminus \pi^{-1}(1)\), is given by \(\ell = (e_0 \setminus V_0, e_1 \setminus V_1, \ldots)\) where \(e_i \in E_i\), \(\pi : \ell \to S^1\) given by \(x = (e_0(t), e_1(t), \ldots) \mapsto \exp(2\pi it)\) is an orientation preserving homeomorphism.

We have the following proposition from theorem [2.9].

**Proposition 2.13.** Every compact connected orientable branched matchbox manifold has a lift uniformly simplicial presentation.

**Standing Assumption 2.14.** From now on, a graph means a finite directed nondegenerate graph.

### 3. Orientable One Dimensional Inverse Limit Spaces

In this section we suppose that \(\overline{X}\) is a compact connected oriented branched matchbox manifold with a presentation \(\{X_k, f_k\}\) such that each \(X_k\) is a graph with a fixed vertex set \(V_k\) and each map \(f_k : X_k \to X_{k-1}\) is an orientation preserving map such that \(f_k(V_k) \subset V_{k-1}\) and \(f_k|_{X_k \setminus V_k}\) is locally one-to-one. Let \(E_k\) be the set of directed edges in \(X_k\) defined by \(V_k\), \(C(E_k, \mathbb{Z})\) the set of integer-valued functions on \(E_k\), and \(C_+(E_k, \mathbb{Z})\) the subset of \(C(E_k, \mathbb{Z})\) with range in the nonnegative integers \(\mathbb{Z}_+\). For each vertex \(p_i\) of \(X_k\), define the vertex function \(v_i \in C(E_k, \mathbb{Z})\) such that for every edge \(e \in E_k\)

\[
v_i(e) = \begin{cases} 
1 & \text{if } e \text{ is an edge from } p_i \text{ to other vertex point}, \\
-1 & \text{if } e \text{ is an edge from other vertex point to } p_i, \\
0 & \text{if } p_i \text{ is the initial and terminal point of } e, \text{ or } p_i \notin e.
\end{cases}
\]
Denote $V_k$ as the set of integral combinations of $\{v_i\} \subset C(\mathcal{E}_k, \mathbb{Z})$, and call an element of $V_k$ a vertex coboundary. Define

$$G^k = C(\mathcal{E}_k, \mathbb{Z})/V_k \text{ and } G^k_+ = C_+(\mathcal{E}_k, \mathbb{Z})/V_k.$$ 

Then $(G^k, G^k_+, 1)$ is a unital preordered group.

**Notation 3.1.** By a path in a graph $X$ we mean a finite sequence $e^{(1)}_1 \cdots e^{(n)}_n$ of edges such that, for $1 \leq i < n$, $s(i) = \pm 1$ represents the direction of $e_i$ and the terminal vertex of $e^{(i)}_i$ is the initial vertex of $e^{(i+1)}_{i+1}$. We write $e^s \in \varphi$ if $\varphi$ is a path and $e$ is an edge such that $e^s$ is a factor of $\varphi$. A cycle is a path $e^{(1)}_1 \cdots e^{(n)}_n$ such that the terminal vertex of $e^{(n)}_n$ is the initial vertex of $e^{(1)}_1$.

We say that a function $g$ in $C(\mathcal{E}_k, \mathbb{Z})$ is zero (nonnegative, respectively) on cycles if the sum of $g(e)$ over the edges $e$ of every cycle in $X_k$ is zero (nonnegative, respectively).

**Lemma 3.2** ([1], §3). Suppose that $g$ is an element of $C(\mathcal{E}_k, \mathbb{Z})$. Then

1. $g$ is an element of $V_k$ if and only if $g$ is zero on cycles in $X_k$, and
2. $[g]$ is an element of $C_+(\mathcal{E}_k, \mathbb{Z})/V_k = G^k_+$ if and only if $g$ is nonnegative on cycles.

Given $g \in C(\mathcal{E}_k, \mathbb{Z})$, define a continuous map

$$\phi_g : X_k \to S^1 \text{ by } x \mapsto \exp(2\pi itg(e)) \text{ for } x = e(t), \ t \in [0, 1].$$

Then $\phi_g$ is well-defined as every vertex point maps to $1 \in S^1$, and $\phi_g$ is an element of $C(X_k, S^1)$.

**Lemma 3.3.** Suppose that $g$ is an element of $C(\mathcal{E}_k, \mathbb{Z})$. Then $g$ is an element of $V_k$ if and only if $\phi_g$ is homotopic to a constant function $1$ in $C(X_k, S^1)$.

**Proof.** Suppose that $g$ is an element of $V_k$. For each vertex function $v_i$ defined at the vertex $p_i$ of $X_k$, define a map $h_{sv_i} : X_k \to S^1$ for $0 \leq s \leq 1$ by

$$h_{sv_i}(e(t)) = \begin{cases} 
  e^{2\pi is} & \text{if } e \text{ is an edge from } p_i \text{ to another vertex point}, \\
  e^{-2\pi is} & \text{if } e \text{ is an edge from another vertex point to } p_i, \\
  e^{2\pi is} & \text{if } p_i \text{ is the initial and terminal point of } e, \\
  1 & \text{otherwise}.
\end{cases}$$

Then $s \mapsto h_{sv_i}$, $0 \leq s \leq 1$, is a homotopy between $\phi_{v_i}$ and $1$.

Now suppose that $\phi_g$ and $1$ are homotopic on $X_k$. Since the winding number of the restriction of $\phi_g$ on every cycle in $X_k$ is a homotopy invariant and $\sum_{e \in \ell} g(e)$ is the winding number for every cycle $\ell$ in $X_k$, we have that $g$ is zero on every cycle, and $g$ is an element of $V_k$ by Lemma 3.2.

Therefore we have a well-defined map

$$\iota_k : G^k \to \text{Br}(X_k) \text{ given by } [g] \mapsto [\phi_g].$$

**Proposition 3.4.** Let $\iota_k$ be defined as above. Then $\iota_k$ is an isomorphism of preordered groups $(G^k, G^k_+, 1)$ and $(\text{Br}(X_k), \text{Br}_+(X_k))$. 
Proof. Since \( \phi_{g+h} = \phi_g \cdot \phi_h \), \( \iota_k \) is a group homomorphism. By Lemma 3.3, \( \phi_g \) is homotopic to a constant function 1 if and only if \( g \) is a vertex coboundary. So we have \( \iota_k : G^k \rightarrow Br(X_k) \) is injective.

To obtain an inverse of \( \iota_k \), suppose that \( \phi \) belongs to \( C(X_k, S^1) \). Then we can choose a map \( \rho : V_k \rightarrow \mathbb{R} \) where \( V_k \) is the vertex set of \( X_k \) such that \( \phi(p) = \phi(2\pi i \rho(p)) \) for every vertex \( p \) of \( X_k \). Define \( S_\rho \in C(X_k, S^1) \) by

\[
e(t) \mapsto \exp \left( 2\pi i (1-t)\rho(e(0)) + t\rho(e(1)) \right), \quad 0 \leq t \leq 1.
\]

Then \( S_\rho \) is homotopic to the constant map 1 by \( H_u = S_{u\rho} \) for \( 0 \leq u \leq 1 \), \( \phi \) is homotopic to \( \phi/S_\rho \), and for every vertex \( p \) of \( X_k \), \( (\phi/S_\rho)(p) = 1 \in S^1 \).

For each edge \( e \in E_k \), let \( r_\phi(e) \) be the number of times the loop \( (\phi/S_\rho)(x) \) winds around \( S^1 \) as \( x = e(t) \) moves on \( e \). Since \( (\phi/S_\rho)(p) = 1 \in S^1 \) for every vertex \( p \) of \( X_k \), \( r_\phi(e) \) is well-defined for each edge \( e \). Then \( r_\phi : e \mapsto r_\phi(e) \) is an element of \( C(E_k, \mathbb{Z}) \), and \( \phi_\phi \) wraps around \( S^1 \) the same number of times as \( \phi/S_\rho \). Therefore \( \phi_\phi \) is homotopic to \( \phi/S_\rho \), and \( [\phi] \mapsto [r_\phi] \) gives the desired inverse to \( \iota_k \).

Clearly if \( g \in C(E_k, \mathbb{Z}_+) \), then \( \iota_k(g) \) is a positive element in the winding order. Conversely if \( [\phi_g] \in Br(X_k) \) is a positive in the winding order, then there exists a map \( \psi \in R(X_k) \) such that \( \phi_g \cdot \psi \) is non-orientation-reversing. It follows that \( g \) has to be nonnegative on cycles, and we have \( [g] \in G^k_+ \) by Lemma 3.2. Therefore \( \iota_k \) is an isomorphism of preordered groups.

Since \( f_{k+1} : X_{k+1} \rightarrow X_k \) is an orientation preserving map, if \( e \) is an edge in \( E_{k+1} \), then \( f_{k+1}(e) \) is a path \( e_1 \cdots e_n \) in \( X_k \). Hence \( f_{k+1} \) induces a map

\[
f_{k+1}^* : C(E_k, \mathbb{Z}) \rightarrow C(E_{k+1}, \mathbb{Z}) \text{ defined by } g \mapsto g \circ f_{k+1}
\]

where \( (g \circ f_{k+1})(e) = \sum_{i=1}^n g(e_i) \) such that \( f_{k+1}(e) = e_1 \cdots e_n \) in \( E_{k+1} \). And \( f_{k+1} \) induces another map

\[
\hat{f}_{k+1}^* : C(X_k, S^1) \rightarrow C(X_{k+1}, S^1) \text{ defined by } \phi \mapsto \phi \circ f_{k+1}.
\]

Lemma 3.5. Let \( f_{k+1}^* \) and \( \hat{f}_{k+1}^* \) be given as above. Then there are well-defined homomorphisms from \( G^k \) to \( G^{k+1} \) and from \( Br(X_k) \) to \( Br(X_{k+1}) \) defined by \( f_{k+1}^* \) and \( \hat{f}_{k+1}^* \), respectively.

Proof. For every \( v \in V_k \) and every cycle \( \ell \) in \( X_{k+1} \), \( f_{k+1}(\ell) \) is a cycle in \( X_k \) and \( f_{k+1}^*(v)(\ell) = v(f_{k+1}(\ell)) = 0 \) by Lemma 3.2. Therefore \( f_{k+1}^*(v) \) is an element of \( V_{k+1} \), and the map \( G^k \rightarrow G^{k+1} \) given by \( [g] \mapsto [f_{k+1}^*(g)] \) is a well-defined homomorphism. That \( f_{k+1}^* \) induces a homomorphism follows from the definition of the Bruschinsky group.

Let us denote these well-defined homomorphisms as \( f_{k+1}^* \) and \( \hat{f}_{k+1}^* \), respectively, if they do not give any confusion.

Proposition 3.6. Let \( \iota_k : G^k \rightarrow Br(X_k) \), \( f_{k+1}^* \) and \( \hat{f}_{k+1}^* \) be given as above. Then we have \( \iota_k \circ f_{k+1}^* = \hat{f}_{k+1}^* \circ \iota_k \) and that \( f_{k+1}^* \) and \( \hat{f}_{k+1}^* \) are order preserving homomorphisms.

Proof. It is not difficult to check, for every \( [g] \in G^k \),

\[
(\iota_{k+1} \circ f_{k+1}^*) ([g]) = (\hat{f}_{k+1}^* \circ \iota_k) ([g]),
\]

Hence \( \iota_k \circ f_{k+1}^* = \hat{f}_{k+1}^* \circ \iota_k \).
The standard isomorphism \( \lim \) which is the direct limit order \( C \) ment in Recal that \( G \) and that \( \iota \) and we have \( \lim \iota k+1 = \hat{f}_{k+1} \iota k \).

To show that \( \hat{f}_{k+1} \) is order preserving, suppose \( [\phi] \in \Br(X_k) \). Then there exists a \( \psi \in R(X_k) \) such that \( \phi \cdot \psi \) is non-orientation-reversing. Since \( f_{k+1}^* \psi = \psi \circ f_{k+1} \) is an element of \( R(X_{k+1}) \) by lemma 3.3 and \( f_{k+1}: X_{k+1} \to X_k \) is orientation preserving, for every orientation preserving parameterized curve \( \gamma: \mathbb{R} \to X_{k+1}, f_{k+1} \gamma \) is an orientation preserving parameterized curve in \( X_k \), and

\[
((\phi \circ f_{k+1}) \cdot (\psi \circ f_{k+1})) (\gamma(t)) = ((\phi \cdot \psi) \circ f_{k+1}) (\gamma(t)) = (\phi \cdot \psi) \circ (f_{k+1} \circ \gamma) (t)
\]
does not move in the clockwise direction as \( t \in \mathbb{R} \) increases. Therefore \( [\phi \circ f_{k+1}] = \hat{f}_{k+1}^* [\phi] \) is an element of \( \Br(X_{k+1}) \), and \( \hat{f}_{k+1} \) is an order preserving homomorphism. Since \( \iota k \) is an order preserving isomorphism by proposition 3.4, \( f_{k+1}^* \iota k = \iota_{k+1}^{-1} \circ \hat{f}_{k+1} \iota k \) is also order preserving.

Then \( \{G^k, f_{k+1}^*\} \) and \( \{\Br(X_k), \hat{f}_{k+1}^*\} \) are directed systems. Let \( \lim \to G^k \) and \( \lim \Br(X_k) \) be the direct limits of \( \{G^k, f_{k+1}^*\} \) and \( \{\Br(X_k), \hat{f}_{k+1}^*\} \), respectively.

**Definition 3.7.** Recall that \( C_+(E_k, \mathbb{Z}) \) is the subset of \( C(E_k, \mathbb{Z}) \) with range in \( \mathbb{Z}_+ \), and that \( G^k_+ \) is given by \( C_+(E_k, \mathbb{Z})/V_k \). Since \( f_{k+1}^*: C(E_k, \mathbb{Z}) \to C(E_{k+1}, \mathbb{Z}) \) defined by \( g \mapsto g \circ f_{k+1} \) is an order preserving homomorphism by proposition 3.6, \( \lim G^k_+ = \lim G^k \) is well-defined. This set, as a positive set, defines the order which is the direct limit order or the standard order on \( \lim \to G^k \).

The **standard isomorphism** \( \lim \to G^k \to \Br(X) \). Suppose \( X = \lim X_k \) and that \( \pi_k: X \to X_k \) is the projection map to the \( k \)th coordinate space. If \( \phi \) is an element in \( C(X_k, S^1) \), then \( \phi \) induces an element \( \phi \circ \pi_k \in C(X, S^1) \). We will use the isomorphism \( \iota_k: G^k \to \Br(X_k) \) and the natural map \( \Br(X_k) \to \Br(X) \) defined by \( \phi \mapsto [\phi \circ \pi_k] \) to make an isomorphism \( \iota_k: \lim G^k \to \Br((X, S^1)). \)

Let \( 1_{X_k}: X_k \to S^1 \) and \( 1_X: X \to S^1 \) be given by \( x_k \mapsto 1 \in S^1 \) and \( x \mapsto 1 \) for all \( x_k \in X_k \) and \( x \in X \), respectively. Suppose that \( \phi \) is an element of \( C(X, S^1) \) such that \( \phi \) is homotopic to \( \chi_k \) by \( H: X_k \times [0, 1] \to S^1 \). Then \( \phi \circ \pi_k \) is homotopic to \( 1_X = 1_{X_k} \circ \pi_k \) by the map \( \overline{H}: X \times [0, 1] \to S^1 \) given by \( \overline{H}(x,t) = H(\pi_k(x),t) \). Thus there is a well-defined map

\[
\pi_k^*: \Br(X_k) \to \Br(X) \text{ given by } [\phi] \mapsto [\phi \circ \pi_k].
\]

Since \( (\phi_1 \cdot \phi_2) \circ \pi_k = (\phi_1 \circ \pi_k) \cdot (\phi_2 \circ \pi_k) \) for all \( \phi_1, \phi_2 \in C(X_k, S^1) \), \( \pi_k^* \) is a homomorphism. That \( f_{k+1} \circ \pi_k = \pi_k: X \to X_k \) implies the following lemma.

**Lemma 3.8.** Let \( \pi_k^* \) and \( \hat{f}_{k+1}^* \) be defined as above. Then for all \( k, \pi_{k+1}^* \hat{f}_{k+1} = \pi_k^* \).

Let \( \varphi_k^*: \Br(X_k) \to \Br(X_k) \) be the natural map for each \( k \). If \( \varphi_k([\phi]) = \varphi_1([\psi]) \) for \( [\phi] \in \Br(X_k) \) and \( [\psi] \in \Br(X_l) \), then there is a positive integer \( m \geq k, l \) such that \( \hat{f}_{k+1}^* \iota_{m+1} \cdots \hat{f}_l^{*1} [\psi] = \hat{f}_{m+1}^* \iota_{m+1} \cdots \hat{f}_l^{*1} [\psi] \). Hence

\[
\pi_k([\phi]) = \pi_{m+1}^* \hat{f}_{m+1}^* \cdots \hat{f}_l^{*1} [\psi] = \pi_{m+1}^* \hat{f}_{m+1}^* \cdots \hat{f}_l^{*1} [\psi] = \pi_l([\psi]),
\]
and there is a well-defined group homomorphism

\[
\pi^*: \lim \Br(X_k) \to \Br(X) \text{ given by } \varphi_k^*([\phi]) \mapsto \pi_k^*([\phi]) = [\phi \circ \pi_k].
\]
Lemma 3.9. Suppose that $\xi$ is an element of $C(\overline{X}, S^1)$. Then there exist $\xi' \in C(\overline{X}, S^1)$ and $k \geq 0$ such that $\xi$ is homotopic to $\xi'$ and $\xi'(x) = \xi'(y)$ if $x_k = y_k$.

Proof. Define a metric $d$ on $\overline{X}$ by $d(x, y) = \sum_{k=0}^{\infty} \frac{1}{2^k}d_k(x_k, y_k)$ where $x = (x_0, x_1, \ldots)$, $y = (y_0, y_1, \ldots) \in \overline{X}$ and $d_k$ is a metric on $X_k$ compatible with its topology. Since $\overline{X}$ is a compact Hausdorff space, every element in $C(\overline{X}, S^1)$ is uniformly continuous. So, for given $\xi$ and $\epsilon > 0$, there exists a nonnegative integer $k$ such that for $x, y \in \overline{X}$, $x_k = y_k$ implies $d(\xi(x), \xi(y)) < \epsilon$.

For $x = (x_0, \ldots, x_k, \ldots) \in \overline{X}$, let's denote $x^k = \{y \in \overline{X} \mid y_k = x_k\}$. Then for all $a, b \in x^k$, $d(\xi(a), \xi(b)) < \epsilon$ and we can choose a point $\tilde{x} \in S^1$ such that $\tilde{x}$ is the center of the smallest interval containing $\xi(x^k)$ in $S^1$. Define $\xi': \overline{X} \to S^1$ by $\xi'(x) = \tilde{x}$. Then it is clear that $\xi' \in C(\overline{X}, S^1)$ and $\xi'(x) = \xi'(y)$ if $x_k = y_k$. Since $d(\xi(x), \xi'(x)) < \epsilon$ for all $x \in \overline{X}$, $\xi$ is homotopic to $\xi'$.

Proposition 3.10. Let $\pi^*$ be defined as above. Then $\pi^*$ is a group isomorphism.

Proof. To show that $\pi^*$ is surjective, suppose $\xi \in C(\overline{X}, S^1)$ and that $\xi'$ and $k$ are given in Lemma 3.9. Define $\phi_k: X_k \to S^1$ by $x_k \to \xi'(x)$ for $x = (x_0, \ldots, x_k, \ldots) \in \overline{X}$. Then $\phi_k$ is well-defined, and it is trivial that $\phi_k \circ \pi_k = \xi'$. Therefore $\xi \in C(\overline{X}, S^1)$ is homotopic to $\phi_k \circ \pi_k$, and $\pi^*: \lim Br(X_k) \to Br(\overline{X})$ is surjective.

Suppose $\xi_1, \xi_2 \in C(\overline{X}, S^1)$ and that $\xi_1$ is homotopic to $\xi_2$. Then by the surjectivity of $\pi^*$, there exist nonnegative integers $k \leq l$ and $\phi \in C(X_k, S^1)$, $\psi \in C(X_l, S^1)$ such that $\xi_1$ is homotopic to $\phi \circ \pi_k$ and $\xi_2$ is homotopic to $\psi \circ \pi_l$. Since $\phi \circ \pi_k = \phi \circ f_{k+1} \circ \cdots \circ f_l \circ \pi_l$, we have

$$\varphi_k^1([\psi]) = \varphi_k^1([\phi \circ f_{k+1} \circ \cdots \circ f_l]) = \varphi_k^1 \circ \tilde{f}_l \circ \cdots \circ \tilde{f}_{k+1} ([\phi]) = \varphi_k ([\phi]).$$

Hence $\pi^*$ is injective.

Therefore the isomorphisms $\iota_k: G^k \to Br(X_k)$ and $\pi^*: \lim Br(X_k) \to Br(\overline{X})$ induce an isomorphism $\iota: \lim G^k \to Br(\overline{X})$.

Order isomorphism. Assume now that the presentation $\{X_k, f_k\}$ satisfies the following Simplicity Condition:

for each $k \geq 1$ there exists $\kappa(k) \geq k$ such that for every $l \geq \kappa(k)$ and $e \in E_l$

$$f_{k+1} \circ \cdots \circ f_l(e) = X_k$$

where $E_l$ is the edge set of $X_l$. Then the winding order on Br($X_k$) and Br($\overline{X}$) is an order.

Theorem 3.11. Suppose that the presentation $\{X_k, f_k\}$ satisfies the above Simplicity Condition. Then $\iota: \left(\lim G^k, \lim G_+^k\right) \to \left(\text{Br}(\overline{X}), \text{Br}_{\oplus}(\overline{X})\right)$ is an isomorphism of ordered groups.

Proof. (Trivial case). Suppose that all but finitely many $X_k$ has a unique edge, i.e., $X_k$ is homeomorphic to a circle $S^1$ with a unique vertex by the standing assumption 2.14 and that the connection map $f_k: X_k \to X_{k-1}$ is the identity map if $X_k = X_{k-1} = S^1$. Then it is obvious that

$$\left(\lim G^k, \lim G_+^k\right) \cong \left(\text{Br}(\overline{X}), \text{Br}_{\oplus}(\overline{X})\right) = \langle \mathbb{Z}, \mathbb{Z}_+ \rangle,$$

and $\iota$ is an isomorphism.
(Nontrivial case). We have that \( \iota \) is a group isomorphism, and clearly \( \iota(\lim G^k_+ \subseteq \Br_T(\mathbf{X}) \). It remains to show that \( \iota \) maps \( \lim G^k_+ \) onto \( \Br_T(\mathbf{X}) \). So we assume that 

\[ [\phi] \text{ is an element of } \Br_T(\mathbf{X}). \]

Then there is an \([h] \) in \( G^k \) for some \( k \geq 0 \) such that 

\[ [\phi] = [\phi_h \circ \pi_k], \]

and we need to show \([h] \in G^k_+ \).

That \([\phi] \) is an element of \( \Br_T(\mathbf{X}) \) implies that there is a map \( \gamma \in R(\mathbf{X}) \) such that 

\[ (\phi_h \circ \pi_k) \cdot \gamma \text{ is non-orientation-reversing.} \]

Since \( \gamma \) is an element of \( R(\mathbf{X}) \), there is a continuous map \( g: \mathbf{X} \rightarrow \mathbb{R} \) such that \( \gamma(x) = \exp(2\pi i g(x)) \). For \( y = (y_0, \ldots, y_k, \ldots) \in \mathbf{X} \), if \( y_k = e(t) \) for \( e \in E_k \) and \( t \in [0, 1] \), then \( \phi_h \circ \pi_k \cdot \gamma \) is defined by \( y \mapsto \exp(2\pi i (h(e) + g(y))) \).

Suppose that \( (\phi_h \circ \pi_k) \cdot \gamma \) is a constant map to \( S^1 \). Then we have 

\[ [(\phi_h \circ \pi_k) \cdot \gamma] = [\phi_h \circ \pi_k] \cdot [\gamma] = [\phi_h \circ \pi_k] = [1] \in \Br(\mathbf{X}) \]

as \( \gamma \) is homotopic to the identity element in \( \Br(\mathbf{X}) \). Hence the equivalence class of \( h \) is the identity element in \( \lim G^k_+ \), for \( \iota: \lim G^k \rightarrow \Br(\mathbf{X}) \) is an isomorphism.

Next suppose that \( (\phi_h \circ \pi_k) \cdot \gamma \) is not constant on \( S^1 \). Then there are nonnegative integer \( m \), a small interval \( I \) contained in some edge \( e' \) of \( X_{k+1} \), and \( \epsilon > 0 \) such that if \( \Gamma \) is any orientation preserving curve in \( \mathbf{X} \) and \( \pi_{k+1}(\Gamma|_{[a,b]}) = I \), then 

\[ \text{length} \{ (\phi_h \circ \pi_k) \cdot \gamma \} \circ \Gamma|_{[a,b]} > \epsilon. \]

Given an arbitrary constant \( L \), by the simplicity condition we can choose a sufficiently large integer \( M \) such that \( e' \) is covered under \( f_k^{k+1} \circ \cdots \circ f_k^{k+M} \) at least \( L \) times by every edge in \( E_{k+1} \).

Define \( H = f_k^{k+1} \circ \cdots \circ f_k^{k+M} h = h \circ f_k^{k+1} \circ \cdots \circ f_k^{k+M} \in C(E_{k+1}, \mathbb{Z}) \). Then by Lemma 3.8, \( \phi_H \circ \pi_{k+1} \in C(\mathbf{X}, S^1) \) is homotopic to \( \phi_h \circ \pi_k \). For \( x = (x_0, \ldots, x_{k+1}, \ldots) \in \mathbf{X} \), as \( x_{k+1} \) moves forward through a directed edge \( e \) of \( E_{k+1} \), its image under \( \phi_H \circ \pi_{k+1} \) moves \( \sum h(e) \cdot n_e(\hat{e}) \) times around \( S^1 \) where \( n_e(\hat{e}) \) is the number of times \( e \) covers \( \hat{e} \in E_k \) under the map \( f_k^{k+1} \circ \cdots \circ f_k^{k+M} \).

**Lemma 3.12.** For every edge \( e \in E_{k+1} \), \( H(e) \geq 2\pi L - 2 \max |g| \).

**Proof of Lemma.** Regard \( e \) as a curve \( e(t), 0 \leq t < 1 \), and pick a curve \( \Gamma: [0, 1] \rightarrow \mathbf{X} \) such that \( \pi_{k+1} \circ \Gamma(t) = e(t) \). As \( t \) increases from 0 to 1, the point 

\[ ((\phi_h \circ \pi_k) \cdot \gamma) \circ \Gamma(t) = (\phi_h \circ \pi_k \circ \Gamma(t)) \cdot (\gamma \circ \Gamma(t)) \]

moves counterclockwise on \( S^1 \) from \( e^{2\pi i g(\Gamma(0))} \) to \( e^{2\pi i (g(\Gamma(1)) + H(e))} \), covering an arclength \( A \) in the plane such that 

\[ A \leq \frac{1}{2\pi} (H(e) + 2 \max |g|). \]

Because \( \phi_h \circ \pi_k \circ \Gamma = \phi_h \circ f_k^{k+1} \circ \cdots \circ f_k^{k+M} \circ \pi_{k+1} \circ \Gamma \), as \( t \) runs from 0 to 1 the curve \( f_k^{k+1} \circ \cdots \circ f_k^{k+M} \circ \pi_{k+1} \circ \Gamma(t) \) wraps around \( e' \) at least \( L \) times, and therefore \( A \geq L \epsilon \). Consequently \( 2\pi L \epsilon - 2 \max |g| \leq H(e) \) as required.

Since we can choose \( M \) to make \( L \) as large as we wish, we can make the choice to guarantee \( H(e) > 0 \) for every edge. Therefore \([H] = [h] \) is an element of \( G^k_+ \).

**Dimension group.** Let \( M \) be an \( r \times s \) nonnegative integer matrix. Then the matrix \( M \) determines a homomorphism \( \mathbb{Z}^s \rightarrow \mathbb{Z}^r \) by the ordinary matrix multiplication. The simplicial order on \( \mathbb{Z}^r \) is the usual ordering \( \mathbb{Z}^r_+ = \{ (n_1, \ldots, n_r) \mid n_i \geq 0 \} \). Then the corresponding homomorphism \( M: \mathbb{Z}^s \rightarrow \mathbb{Z}^r \) is positive with respect to the simplicial order, that is, \( a \geq 0 \) implies \( M(a) \geq 0 \).
Definition 3.13 ([4], §2]). Let \( M_i \) be an \( r(i) \times r(i-1) \) nonnegative integer matrix. For a system of ordered groups and positive maps 
\[
\mathbb{Z}^{r(0)} \xrightarrow{M_1} \mathbb{Z}^{r(1)} \xrightarrow{M_2} \cdots ,
\]
the set theoretic direct limit \( \lim \rightarrow \left( \mathbb{Z}^{r(i)}, M_i \right) \) is an ordered group under the usual limit addition operation with the positive cone \( \lim \rightarrow \left( \mathbb{Z}^{r(i)}_+, M_i \right) \) where \( M_{i\infty} \) is the induced map from \( \mathbb{Z}^{r(i-1)}_+ \) to the direct limit \( \lim \rightarrow \left( \mathbb{Z}^{r(i)}, M_i \right) \).

An ordered group \((G,G_+)\) is called a dimension group if it is order isomorphic to the limit of a system of simplicially ordered groups with positive maps.

Let \((G,G_+)\) be a dimension group. A subgroup \( H \) of \( G \) is called an order ideal if \( H \) is an ordered group with the positive cone \( H_+ = H \cap G_+ \) and \( 0 \leq a \leq b \in H \) implies \( a \in H \). The dimension group \((G,G_+)\) is called simple if it has no proper order ideal.

In a simple dimension group \((G,G_+)\) with an element \( g \in G \), if neither \( g \) nor \(-g\) lies in \( G_+ \), then \( g \) is called an infinitesimal element. If \( u \) is an order unit and \( g \) is an infinitesimal element of \( G \), then \( g + u \) is also an order unit.

It is well known that a dimension group defined as above by matrices \( M_i \) is simple if for every \( i \) there exists \( j \) such that all entries of the matrix \( M_jM_{j-1}\cdots M_{i+1}M_i \) are strictly positive.

Suppose that \( \{X_k,f_k\} \) is a presentation of an (orientable) branched matchbox manifold with the edge set \( E_k \) of \( X_k \). Then for each edge \( e_i \in E_k \), \( f_k(e_i) \) is a path \( e_{i,1} \cdots e_{i,j(i)} \) in \( X_{k-1} \) such that \( s(j) = \pm 1 \) denotes the direction and the terminal point of \( e_{i,j(i)} \) is the initial point of \( e_{i,j+1}(i) \) for \( 1 \leq j < j(i) \). Therefore we can define an induced map \( \hat{f}_k : E_k \to E_{k-1} \) by
\[
\hat{f}_k : e_i \mapsto e_{i,1} \cdots e_{i,j(i)}.
\]

Definition 3.14. Suppose that \( X_k \) has \( n_k \) edges for all \( k \geq 0 \). Then the adjacency matrix \( M_k \) of \( (f_k,E_k,E_{k-1}) \) is an \( n_k \times n_{k-1} \) matrix such that for any edges \( e_i \in E_k \) and \( e_j \in E_{k-1} \), \( M_k(i,j) \) is the number of times \( \hat{f}_k(e_i) \) covers \( e_j \) ignoring the direction of the covering.

Lemma 3.15 ([4], §3]). A countable ordered group is a dimension group if and only if it is unperforated and has the Riesz Interpolation Property.

Proposition 3.16. Suppose that \( \{X_k,f_k\} \) is a presentation of a compact connected orientable branched matchbox manifold with the adjacency matrices \( M_k \). Then
\[
(1) \quad \left( \lim \rightarrow C(E_k,\mathbb{Z}), \lim \rightarrow C_+(E_k,\mathbb{Z}) \right) \cong \left( \lim \rightarrow \left( \mathbb{Z}^{n_k}, M_k \right), \lim \rightarrow \left( \mathbb{Z}^{n_k}_+, M_k \right) \right).
\]
If the presentation satisfies the Simplicity Condition, then
\[
(2) \quad \left( \lim \rightarrow C(E_k,\mathbb{Z}), \lim \rightarrow C_+(E_k,\mathbb{Z}) \right) \text{ and } \left( \lim \rightarrow G^k, \lim \rightarrow G^k_+ \right) \text{ are simple dimension groups.}
\]

Proof. (1) For each \( g \in C(E_{k-1},\mathbb{Z}) \) and \( f_k^* : C(E_{k-1},\mathbb{Z}) \to C(E_k,\mathbb{Z}) \) given by \( g \mapsto g \circ f_k \), if we represent \( g \) as \( (g(e_1),\ldots,g(e_{n_k-1})) \in \mathbb{Z}^{n_k-1} \), then \( C(E_{k-1},\mathbb{Z}) \) is isomorphic to \( \mathbb{Z}^{n_k-1} \) and \( f_k^*(g) = g \circ f_k \) is given by \( M_k \cdot (g(e_1),\ldots,g(e_{n_k-1}))^t \).
Hence we have \( \lim C(\mathcal{E}_{k-1}, \mathbb{Z}) \cong \lim (\mathbb{Z}^{n_k}, M_k) \). Since \( C_+(\mathcal{E}_{k-1}, \mathbb{Z}) \) is the set of elements in \( C(\mathcal{E}_{k-1}, \mathbb{Z}) \) with range in \( \mathbb{Z}_+ \), \( C(\mathcal{E}_{k-1}, \mathbb{Z}) \) is simplicially ordered, and so is \( \lim C(\mathcal{E}_k, \mathbb{Z}) \). Therefore \( \left( \lim \lim C(\mathcal{E}_k, \mathbb{Z}), \lim C_+(\mathcal{E}_k, \mathbb{Z}) \right) \) is order isomorphic to \( \left( \lim \lim (\mathbb{Z}^{n_k}, M_k), \lim (\mathbb{Z}_+^{n_k}, M_k) \right) \).

(2) Suppose that \( H \) is a proper order ideal of \( \left( \lim \lim C(\mathcal{E}_k, \mathbb{Z}), \lim C_+(\mathcal{E}_k, \mathbb{Z}) \right) \) and that \( b \in H_+ \). Then there exist a nonnegative integer \( k \) and \( h \in C_+(\mathcal{E}_k, \mathbb{Z}) \) such that \( b = [h] \in \lim C(\mathcal{E}_k, \mathbb{Z}) \). By the Simplicity Condition, there is a nonnegative integer \( \kappa(k) \geq k \) such that, for every \( l \geq \kappa(k) \) and every edge \( e \in \mathcal{E}_i \), \( f_k^{l+1} \circ \cdots \circ f_i(e) = X_k \). If \( a \in \lim C_+(\mathcal{E}_k, \mathbb{Z}) \), then we can choose a positive integer \( l \geq \kappa(k) \) and \( g \in C_+(\mathcal{E}_l, \mathbb{Z}) \) such that \( a = [g] \). Let \( n = \max g(e) \). Then \( n \cdot b = [n \cdot f_k^* \circ \cdots \circ f_k^{l+1} \circ h] \in H_+ \) and \( n \cdot f_k^* \circ \cdots \circ f_k^{l+1} \circ h - g \in C_+(\mathcal{E}_l, \mathbb{Z}) \).

Therefore all elements of \( \lim \lim C(\mathcal{E}_k, \mathbb{Z}), \lim C_+(\mathcal{E}_k, \mathbb{Z}) \) is a simple dimension group.

The group \( \left( \lim \lim G^k, \lim G^k_+ \right) \cong (\operatorname{Br}(\mathcal{X}), \operatorname{Br}_0(\mathcal{X})) \) is an unperforated ordered group by proposition \( \ref{14} \) and its positive set is the image of the positive set of \( \lim C(\mathcal{E}_k, \mathbb{Z}) \) under the quotient map \( \chi: \lim C(\mathcal{E}_k, \mathbb{Z}) \to \lim G^k \). We claim that with this quotient order, \( \left( \lim G^k, \lim G^k_+ \right) \) satisfies the Riesz Interpolation Property (and therefore by Lemma \( \ref{13} \) is a dimension group.) (We learned this argument from unpublished remarks of David Handelman. The general line of argument is also implicit in remarks on pp. 58 and 66 of \( \ref{11} \).)

Let \( V = \ker \chi \). Note that if \( V \) contains a nonzero positive element \( u \), then for every \( g \in \lim \lim C_+(\mathcal{E}_k, \mathbb{Z}) \) we have for some integer \( n \) that \( 0 \leq g \leq nu \), and therefore \( 0 \leq \chi(g) \leq 0 \), which contradicts the image of \( \chi \) being a nontrivial ordered group. Therefore all elements of \( V \) are infinitesimals.

To show the Riesz Interpolation Property, suppose that \( [a_1], [a_2], [b_1], [b_2] \in \lim G^k \) satisfy \( [a_i] < [b_j] \) \( (i, j = 1, 2) \). Let \( a_i \) and \( b_j \in \lim C_+(\mathcal{E}_k, \mathbb{Z}) \) be preimages of \( [a_i] \) and \( [b_j] \), respectively. Since \( -[a_i] + [b_j] \) is a nonzero positive element of \( \lim G^k \), there exists a \( v_{i,j} \in V \) such that \( -a_i + v_{i,j} + b_j \) is a nonzero positive element of \( \lim C_+(\mathcal{E}_k, \mathbb{Z}) \). Because \( v_{i,j} \) is an infinitesimal element, it follows that \( -a_i + b_j \) is a nonzero positive element of \( \lim C(\mathcal{E}_k, \mathbb{Z}) \), and \( a_i < b_j \) for \( i, j = 1, 2 \). Hence by the Riesz Interpolation Property for \( \lim C(\mathcal{E}_k, \mathbb{Z}) \) there exists an element \( c \in \lim C(\mathcal{E}_k, \mathbb{Z}) \) such that \( a_i \leq c \leq b_j \). Then by definition of the quotient order we have \( [a_i] \leq [c] \leq [b_j] \) for all \( i, j \) as required. Therefore \( \left( \lim G^k, \lim G^k_+ \right) \) is a dimension group by lemma \( \ref{13} \).

Suppose that \( (G, G_+) \) is a proper order ideal of \( \left( \lim G^k, \lim G^k_+ \right) \). Then it is not difficult to see that \( (H, H_+) = (\chi^{-1}(G), \chi^{-1}(G_+)) \) is a proper order ideal of \( \left( \lim C(\mathcal{E}_k, \mathbb{Z}), \lim C_+(\mathcal{E}_k, \mathbb{Z}) \right) \) which is a simple dimension group. Therefore \( \left( \lim G^k, \lim G^k_+ \right) \) is a simple dimension group. \( \square \)
If each graph $X_k$ is a wedge of circle, then $V_k = \{0\}$ as each edge in $X_k$ is a cycle. So we have the following corollary

**Corollary 3.17.** Suppose that the presentation $\{X_k, f_k\}$ satisfies the Simplicity Condition and that each graph $X_k$ is a wedge of circles. Then $\left(\lim_{\to \infty} G_k, \lim_{\to \infty} G_k\right)$ is order isomorphic to $\left(\lim_{\to \infty} (\mathbb{Z}^n, M_k), \lim_{\to \infty} (\mathbb{Z}^n, M_k)\right)$.

The following corollary follows from Observation 2.4 and Theorem 3.11.

**Corollary 3.18.** Suppose that $(X_i, f_i)$ is a compact connected orientable branched matchbox manifold with the Simplicity Condition for $i = 1, 2$. If $X_1$ is homeomorphic to $X_2$, then $\lim_{\to \infty} G^i_1$ is order isomorphic to $\lim_{\to \infty} G^i_2$.

**Remark 3.19.**
1. The dimension group of adjacency matrices is not a homeomorphism invariant. See example 4.4.
2. The isomorphism in corollary 3.18 need not respect distinguished order unit ($[\mathbb{Z}, \mathbb{Z}]$).

4. **One-dimensional generalized solenoid**

One interesting class of branched matchbox manifolds is one-dimensional branched solenoids, including one-dimensional generalized solenoids of Williams ([16], [17], [18]). Let $X$ be a directed graph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$, and $f : X \to X$ a continuous map. We define some axioms which might be satisfied by $(X, f)$ ([18]).

**Axiom 0.** *(Indecomposability)* $(X, f)$ is indecomposable.

**Axiom 1.** *(Nonwandering)* All points of $X$ are nonwandering under $f$.

**Axiom 2.** *(Flattening)* There is $k \geq 1$ such that for all $x \in X$ there is an open neighborhood $U$ of $x$ such that $f^k(U)$ is homeomorphic to $(-\epsilon, \epsilon)$.

**Axiom 3.** *(Expansion)* There are a metric $d$ compatible with the topology and positive constants $C$ and $\lambda$ with $\lambda > 1$ such that for all $n > 0$ and all points $x, y$ on a common edge of $X$, if $f^n$ maps the interval $[x, y]$ into an edge, then $d(f^n x, f^n y) \geq C \lambda^n d(x, y)$.

**Axiom 4.** *(Nonfolding)* $f^n|_{X-V}$ is locally one-to-one for every positive integer $n$.

**Axiom 5.** *(Markov)* $f(V) \subseteq \mathcal{V}$.

Let $\overline{X}$ be the inverse limit space

$$\overline{X} = X \leftarrow \mathcal{L} X \leftarrow \mathcal{L} \cdots = \left\{ (x_0, x_1, x_2, \ldots) \in \prod_0^\infty X \mid f(x_{n+1}) = x_n \right\},$$

and $f : \overline{X} \to \overline{X}$ the induced homeomorphism defined by

$$(x_0, x_1, x_2, \ldots) \mapsto (f(x_0), f(x_1), f(x_2), \ldots) = (f(x_0), x_0, x_1, \ldots).$$

Let $Y$ be a topological space and $g : Y \to Y$ a homeomorphism. We call $Y$ a **1-dimensional generalized solenoid** or **1-solenoid** and $g$ a **solenoid map** if there exist a directed graph $X$ and a continuous map $f : X \to X$ such that $(X, f)$ satisfies all six Axioms and $(\overline{X}, f)$ is topologically conjugate to $(Y, g)$. If $(X, f)$ satisfies all Axioms except possibly the Flattening Axiom, then we call $Y$ a **branched solenoid**. If we can choose the direction of each edge in $X$ so that the connection map $f : X \to X$ is orientation preserving, then we call $(X, f)$ an **orientable presentation**, and $Y$ an **orientable** (branched) solenoid. If $(Y, g)$ is a...
branched solenoid with a presentation \( (X, f) \), then there exist an \( n \times n \) adjacency matrix \( M_{X,f} \) where \( n \) is the cardinal number of the set of edges in \( X \). If \( X \) is a wedge of circles and \( f \) leaves the unique branch point of \( X \) fixed, then we say \( (X, f) \) an elementary presentation.

We have the following proposition from theorem \([3.11]\) and corollary \([3.17]\).

**Proposition 4.1.** Suppose that \( (X, f) \) is an orientable branched solenoid with an adjacency matrix \( M \). Then \( \iota: \left( \lim_{\to} (\mathbb{Z}^n, M), \lim_{\to} (\mathbb{Z}^n_+, M) \right) \to (\text{Br}(X), \text{Br}_\oplus(X)) \) is an epimorphism of ordered groups. If \( (X, f) \) is an elementary presentation, then \( \iota \) is an isomorphism.

**Remark 4.2.** We need the elementary presentation condition for the injectivity of \( \iota \). See example \([4.4]\).

**Example 4.3** ([15, §2] and [11, §7.5]). Let \( X \) be the unit circle on the complex plane. Suppose that 1 and \(-1\) are the vertices of \( X \), and that the upper half circle \( e_1 \) and the lower half circle \( e_2 \) with counterclockwise direction are the edges of \( X \). Define \( f: X \to X \) by \( f: z \mapsto z^2 \). The \( \hat{f}: \mathcal{E}_X \to \mathcal{E}_X^* \) is given by \( \hat{f}: e_1 \mapsto e_1 e_2, \quad e_2 \mapsto e_1 e_2 \), and the adjacency matrix is

\[
M_{X,f} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.
\]

Therefore we have

\[
(\text{Br}(X), \text{Br}_\oplus(X)) = (\mathbb{Z}[1/2], \mathbb{Z}[1/2] \cap \mathbb{R}_+) .
\]

We give Figure 1 to represent the presentation \( (X, f) \) with the wrapping rule \( \hat{f} \).

![Figure 1](image1.png)

**Figure 1.** \( (X, f) \) with the wrapping rule \( \hat{f} \).

Similarly, if \( (Y, g) \) is given by Figure 2, then \( (Y, g) \) does not satisfy the flattening axiom and \( (Y, \tilde{g}) \) is a branched solenoid. The wrapping rule \( \tilde{g}: \mathcal{E}_Y \to \mathcal{E}_Y^* \) is given by \( a \mapsto ab, \quad b \mapsto a \) and the adjacency matrix is

\[
M_{Y,g} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Thus \( \text{Br}(Y) = \mathbb{Z} \oplus \mathbb{Z} \) and \( \text{Br}_\oplus(Y) = \left\{ v \in \mathbb{Z} \oplus \mathbb{Z} \mid v \cdot \left( \frac{1+\sqrt{5}}{2}, 1 \right) > 0 \right\} \cup \{0\} \).

The following example shows that the dimension group of adjacency matrices induced by a presentation is not a homeomorphism invariant.
Example 4.4 ([18, 4.8 and 5.1]). Let $X$ be a wedge of two circles $a, b$ with a unique vertex $p$, and $f: X \to X$ defined by $a \mapsto aab$ and $b \mapsto ab$. So $(X, f)$ is given by Figure 3. Suppose that $Y$ is given by Figure 4 and that the wrapping rule $\tilde{g}: \mathcal{E}_Y \to \mathcal{E}_Y$ is given by

\[\alpha \mapsto \gamma\alpha\beta, \quad \beta \mapsto \gamma, \quad \gamma \mapsto \beta\gamma\alpha\beta.\]

Then it is shown in [18, 4.8] that $(X, f)$ is topologically conjugate to $(\overline{X}, \overline{f})$. And their adjacency matrices are given by the following matrices

\[M_{(X, f)} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad M_{(Y, g)} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}.\]

Since the determinants of $M_{(X, f)}$ and $M_{(Y, g)}$ are 1 and $-1$, respectively, $M_{(X, f)}$ and $M_{(Y, g)}$ are invertible over $\mathbb{Z}$. Hence the dimension group of $M_{(X, f)}$ is $\mathbb{Z}^2$ and that of $M_{(Y, g)}$ is $\mathbb{Z}^3$. Therefore the dimension group of $M_{(X, f)}$ is not isomorphic to the dimension group of $M_{(Y, g)}$.

Since $(X, f)$ is elementary presented, the dimension group of $M_{(X, f)}$ is order isomorphic to the Bruschlinsky group of $(\overline{X}, \overline{f})$. And the Bruschlinsky group of
\((\mathcal{Y}, \mathcal{G})\) is given by the dimension group of \(\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\). Hence we have \(\text{Br}(\mathcal{X}) \cong \text{Br}(\mathcal{Y}) \cong \mathbb{Z} \oplus \mathbb{Z}\) with

\[
\text{Br}_\mathcal{G}(\mathcal{X}) \cong \text{Br}_\mathcal{G}(\mathcal{Y}) \cong \left\{ v \in \mathbb{Z} \oplus \mathbb{Z} \mid v \cdot \left( \frac{1 + \sqrt{5}}{2}, 1 \right) > 0 \right\} \cup \{0\}.
\]

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