UNIVERSAL HODGE BUNDLE AND MUMFORD ISOMORPHISMS ON THE ABELIAN INDUCTIVE LIMIT OF TEICHMÜLLER SPACES

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Abstract

Let \( \mathcal{M}_g \) denote the moduli space of compact Riemann surfaces of genus \( g \). Mumford had proved, for each fixed genus \( g \), that there are isomorphisms asserting that certain higher \( DET \) bundles over \( \mathcal{M}_g \) are certain fixed (genus-independent) tensor powers of the Hodge line bundle on \( \mathcal{M}_g \). We obtain a coherent, genus-independent description of the Mumford isomorphisms over certain infinite-dimensional “universal” parameter spaces of compact Riemann surfaces (with or without marked points). We work with an inductive limit of Teichmüller spaces comprising complex structures on a certain “solenoidal Riemann surface”, \( H_{\infty,\text{ab}} \), which appears as the inverse limit of an inverse system of surfaces of different genera connected by abelian covering maps. We construct the universal Hodge and higher \( DET \) line bundles on this direct limit of Teichmuller spaces (in the sense of Shafarevich). The main result shows how such \( DET \) line bundles on the direct limit carry coherently-glued Quillen metrics and are related by the appropriate Mumford isomorphisms.

Our work can be viewed as a contribution to a non-perturbative formulation of the Polyakov measure structure in a genus-independent fashion.

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**INTRODUCTION:**

The chief purpose of the present paper is to obtain a genus-independent description of the Mumford isomorphisms over the moduli spaces of Riemann surfaces (with or without marked points). Mumford proved that, for each fixed genus $g$, there are isomorphisms asserting that certain higher $DET$ bundles over the moduli space $M_g$ are certain fixed (genus-independent) tensor powers of the Hodge line bundle on $M_g$. (Hodge itself is the first of these $DET$ bundles, and generates the Picard group of $M_g$.)

In order to establish a coherent genus-independent description of the Mumford isomorphisms, we work with an inductive limit of Teichmüller spaces corresponding to a projective family of surfaces of different genera connected by abelian covering maps. Since a covering space between surfaces induces (by a contravariant functor) an immersion of the Teichmüller space of the surface of lower genus into the Teichmüller space of the covering surface, we can think of line bundles living on the direct limit of these Teichmüller spaces (in the sense of [Sha]). The main result shows how such “$DET$” line bundles on the direct limit carry coherently-glued Quillen metrics and are related by the appropriate Mumford isomorphisms; these isomorphisms actually become unitary isometries with respect to the Quillen hermitian structures.

All this is, of course, closely related to the Polyakov bosonic string theory, – and our work can be viewed as an attempt at a non-perturbative formulation of the Polyakov measure structure in a genus-independent mode. Regarding the Polyakov measure (which is a measure on each moduli space $M_g$ and describable via the Mumford isomorphisms), and its relation to our work, we shall have more to say in Section 5c.

**Acknowledgements:** We would sincerely like to thank E. Arbarello, E. Looijenga and M.S. Narasimhan for their interest and many helpful discussions, and D. Prasad for pointing out to us the paper of Moore [M]. We specially thank Professor Looijenga for generously supplying us, (in early January 1994), with a proof of a strong form of the discreteness of the Picard group of mod-
ulti spaces of surfaces with marked points and level-structure. That result is needed critically in our construction, and we have inserted the proof he sent us as an Appendix to this work. Later on he also pointed out to us that the discreteness result can be found in a recent preprint of R. Hain (see References).

The first two authors would like to thank RIMS/Kyoto University, TIFR Bombay, and ICTP Trieste, (in chronological order), for creating very pleasant opportunities for us to get together and collaborate on this project.

Section 1. BASIC BACKGROUND:

1a. $T_g$ and $M_g$: Let $X$ be a compact oriented $C^\infty$ surface of genus $g \geq 2$. Let $T(X) = T_g$ denote the corresponding Teichmüller space and $M_g$ denote the moduli space of all complex structures on $X$. $M_g$ is a complex analytic V-manifold of dimension $3g - 3$ that is known to be a quasi-projective variety. $T_g$ itself can be constructed as the orbifold universal covering of $M_g$ in the sense of Thurston; conversely, $M_g$ can be recovered from $T_g$ as the quotient by the action of the mapping class group (="Teichmüller modular group"), $MC_g$, acting biholomorphically and properly discontinuously on the contractible, Stein, bounded, $(3g - 3)$-dimensional domain, $T_g$. In the sequel, unless otherwise stated, we will assume $g \geq 3$.

Remark: For simplicity, in the main body of the paper we deal with the situation of closed Riemann surfaces without punctures and prove our main theorems in this context. In Section 6 we shall show how to extend our main results also to the case of moduli of Riemann surface with some fixed number of marked points.

1b. Bundles on moduli: Our first critical need is to study and compare the Picard groups of algebraic/holomorphic line bundles over $M_g$. We introduce three closely related groups of bundles:

\begin{equation}
\text{(1.1) } \operatorname{Pic}_{alg}(M_g) = \text{the group of algebraic line bundles over } M_g.
\end{equation}
Pic\(_{hol}(\mathcal{M}_g) = H^1(\mathcal{M}_g, \mathcal{O}^*) = \text{the group of holomorphic line bundles over } \mathcal{M}_g

(1.3) \quad Pic_{fun}(\mathcal{M}_g) = \text{the Picard group of the moduli functor.}

An element of \( Pic_{fun}(\mathcal{M}_g) \) consists of the prescription of a holomorphic line bundle \( L_F \) on the base space \( S \) for every complex analytic family of Riemann surfaces of genus \( g \) (\( C^\infty \) locally trivial Kodaira-Spencer family) \( F = (\gamma : V \to S) \) over any complex analytic base \( S \). Moreover, for every commutative diagram of families \( F_1 \) and \( F_2 \) having the morphism \( \alpha \) from the base \( S_1 \) to \( S_2 \), there must be assigned a corresponding isomorphism between the line bundle \( L_{F_1} \) and the pullback via \( \alpha \) of the bundle \( L_{F_2} \). For compositions of such pullbacks these isomorphisms between the prescribed bundles must satisfy the obvious compatibility condition. See [Mum], [HM], [AC]. [Note: Mumford has considered this Picard group of the moduli functor also over the Deligne-Mumford compactification of \( \mathcal{M}_g \).]

There is, of course, a natural map of \( Pic_{alg}(\mathcal{M}_g) \) to \( Pic_{hol}(\mathcal{M}_g) \) by stripping off the algebraic structure and remembering only the holomorphic structure. Also it is clear that any actual line bundle over \( \mathcal{M}_g \) produces a corresponding element of \( Pic_{fun}(\mathcal{M}_g) \) simply by pulling back to the arbitrary base \( S \) using the canonical classifying map for the given family \( F \) over \( S \). In fact, Mumford has shown that \( Pic_{alg}(\mathcal{M}_g) \) is a subgroup of finite index in \( Pic_{fun}(\mathcal{M}_g) \), and that the latter is torsion-free. Therefore \( Pic_{alg}(\mathcal{M}_g) \) is also torsion free. See [Mum] and [AC].

Now, it is known that the cohomology \( H^1(\mathcal{M}_g, \mathcal{O}) \) vanishes. [In fact, by Kodaira-Spencer theory of infinitesimal deformations of structure, we know that the above cohomology represents the tangent space to \( Pic_{hol}(\mathcal{M}_g) \). Since by Mumford and Harer's results (see below) we know that Pic is discrete, this cohomology must vanish.] It therefore follows that the Chern class homomorphism \( c_1 : Pic_{hol}(\mathcal{M}_g) \to H^2(\mathcal{M}_g, \mathbb{Z}) \) is an injection. Since Harer
([Har]) has shown that the $H^2(MC_g,\mathbb{Z})$ is $\mathbb{Z}$, it follows that $Pic_{hol}(\mathcal{M}_g)$ is also torsion free, and in fact therefore $Pic_{hol}(\mathcal{M}_g)$ must be isomorphic to $\mathbb{Z}$ (for $g \geq 5$).

The torsion-free property guarantees that we stand to lose nothing by tensoring with $\mathbb{Q}$; namely, the maps:

\begin{align*}
(1.4) \quad Pic_{alg}(\mathcal{M}_g) & \to Pic_{alg}(\mathcal{M}_g) \otimes \mathbb{Q} \\
Pic_{hol}(\mathcal{M}_g) & \to Pic_{hol}(\mathcal{M}_g) \otimes \mathbb{Q}
\end{align*}

are injective. (All tensor products are over $\mathbb{Z}$.) We can thus think of the original Picard groups respectively as discrete lattices embedded inside these $\mathbb{Q}$-vector spaces. These two rational Picard groups above are of fundamental interest for us. We wish to show that they are naturally isomorphic:

**Lemma 1.1:** The structure-stripping map (that forgets the algebraic nature of the transition functions for bundles in $Pic_{alg}(\mathcal{M}_g)$), induces a canonical isomorphism $Pic_{alg}(\mathcal{M}_g) \otimes \mathbb{Q} \to Pic_{hol}(\mathcal{M}_g) \otimes \mathbb{Q}$.

**Proof:** The Chern class homomorphism provides us with two maps:

\begin{align*}
(1.5) \quad c_a : Pic_{alg}(\mathcal{M}_g) \otimes \mathbb{Q} & \to H^2(\mathcal{M}_g, \mathbb{Q}) \\
(1.6) \quad c_h : Pic_{hol}(\mathcal{M}_g) \otimes \mathbb{Q} & \to H^2(\mathcal{M}_g, \mathbb{Q})
\end{align*}

which, by naturality, give a commutative triangle when combined with the structure-stripping map of the Lemma. The Lemma will therefore be proved if we show that each of the two Chern class maps above are themselves isomorphisms.

Since, as we mentioned above, $H^1(\mathcal{M}_g, \mathcal{O}) = \{0\}$, we deduce that both $c_a$ and $c_h$ are injections. To prove that they are surjective it clearly suffices to find some algebraic line bundle over $\mathcal{M}_g$ with a non-vanishing Chern class (because, by Harer’s result we know that $H^2(\mathcal{M}_g, \mathbb{Q}) = \mathbb{Q}$.)

Towards this aim, and since it is of basic importance to us, we introduce the Hodge line bundle, $\lambda$, as a member of $Pic_{fun}(\mathcal{M}_g)$. This is easy. Consider a family of genus $g$ Riemann surfaces, $F = (\gamma : E \to S)$ over $S$. Then the
“Hodge bundle” on the parameter space $S$ is defined to be $\det(R^1\gamma_*\mathcal{O} - R^0\gamma_*\mathcal{O})$. Here the $R^i$ denote the usual direct image and higher derived functors (see, for example, [H]). One thus obtains an element of $\text{Pic}_{\text{fun}}(\mathcal{M}_g)$ by definition. It is a known fact that $\text{Pic}_{\text{fun}}(\mathcal{M}_g)$ is generated by $\lambda$; moreover, for $g \geq 3$, $\text{Pic}_{\text{fun}}(\mathcal{M}_g) = \mathbb{Z}$. (See Theorem 1 [AC].)

Now recall Mumford’s assertion that $\text{Pic}_{\text{alg}}(\mathcal{M}_g)$ is a subgroup of finite index within $\text{Pic}_{\text{fun}}(\mathcal{M}_g) = H^2(MC_g, \mathbb{Z})$ ([Mum]). Therefore some integer multiple (i.e., tensor power) of $\lambda$ must lie in $\text{Pic}_{\text{alg}}(\mathcal{M}_g)$ and have non-zero Chern class. We are therefore through. $\square$

Remark: By Mumford, some tensor power of each element of $\text{Pic}_{\text{fun}}(\mathcal{M}_g)$ is an honest algebraic line bundle on moduli. As a corollary of our Lemma we see that some power every holomorphic line bundle over moduli space carries an algebraic structure.

Discussion of the Hodge line bundle: Let $\mathcal{M}_g^\circ$ denote the open set of moduli space comprising Riemann surfaces without nontrivial automorphisms. Let us utilise the universal genus $g$ family over $T_g$, and restrict the family over $T_g^\circ$ (automorphism-free points of Teichmüller space). Clearly then, by quotienting out the fix-point free action of $MC_g$ on this open subset of $T_g$, we obtain over $\mathcal{M}_g^\circ$ a Hodge line bundle whose fiber over the Riemann surface $X$ is $\lambda^g H^0(X, K_X)^*$. (We let $K_X$ denote the canonical line bundle of $X$.) What we see from the proof is that although the Hodge line bundle may not exist on the whole of $\mathcal{M}_g$ as an actual (algebraic or holomorphic) line bundle, some integer power of the Hodge bundle over $\mathcal{M}_g^\circ$ necessarily lives as an actual algebraic line bundle over the full moduli space. Therefore, when we go to the rational Picard groups over $\mathcal{M}_g$, the Hodge itself exists as an element of $\text{Pic}_{\text{alg}}(\mathcal{M}_g) \otimes \mathbb{Q} = \text{Pic}_{\text{hol}}(\mathcal{M}_g) \otimes \mathbb{Q}$. This is of crucial importance for our work.

Our chief object in making the above careful considerations is to be able to think of the Picard group over moduli as a group of $MC_g$-invariant line bundles on the Teichmüller space. Indeed, there is a 1-1 correspondence between the Picard group $\text{Pic}_{\text{hol}}(\mathcal{M}_g)$ on $\mathcal{M}_g$ and the set of pairs of the
form \((L, \psi)\), where \(L\) is a holomorphic line bundle on \(T_g\) and \(\psi\) is a lift of the action \(MC_g\) (the mapping class group) on \(T_g\) to the the total space of \(L\). The pull-back of a bundle on \(\mathcal{M}_g\) to \(T_g\) is naturally equipped with a lift of the action of \(MC_g\). Conversely, a line-bundle on \(T_g\) equipped with a lift of the action of \(MC_g\) descends to a line bundle on \(\mathcal{M}_g\). Define

\[
\text{Pic}(T_g) := \{(L, \psi) \mid L \text{ and } \psi \text{ as above}\}
\]

We will refer to the elements of \(\text{Pic}(T_g)\) as modular invariant line-bundles. Two elements \((L, \psi)\) and \((L', \psi')\) are called isomorphic if there is an isomorphism \(h : L \to L'\) which is equivariant with respect to the action of \(MC_g\). In other words, \(\psi' \circ h = h \circ \psi\).

\(\text{Pic}(T_g)\) has a natural structure of an abelian group, and, from the above remarks, is isomorphic as a group to \(\text{Pic}_{\text{hol}}(\mathcal{M}_g)\).

Now we introduce hermitian structure on elements of \(\text{Pic}(T_g)\). Let \(\text{Pich}(T_g)\) be the set of triplets of the form \((L, \psi, h)\), where \((L, \psi) \in \text{Pic}(T_g)\) and \(h\) is a hermitian metric on \(L\) which is invariant under the action \(MC_g\) given by \(\psi\). Note that a modular invariant metric \(h\), as above, induces a modular invariant metric on any power of \(L\). Hence \(\text{Pich}(T_g)\) has a structure of an abelian group.

**1c. DET bundles for families of d-bar operators:** Given, as before, any Kodaira-Spencer family \(\mathcal{F} = (\gamma : V \to S)\), of compact Riemann surfaces of genus \(g\), and a holomorphic vector bundle \(E\) over the total space \(V\), we can consider the base \(S\) as parametrizing a family of elliptic d-bar operators, as is standard. The operator corresponding to \(s \in S\) acts along the fiber Riemann surface \(X_s = \gamma^{-1}(s)\):

\[
\tilde{\partial}_s : C^\infty(\gamma^{-1}(s), E) \to C^\infty(\gamma^{-1}(s), E \otimes \Omega^{0,1}_{X_s})
\]

One defines the associated vector space of one dimension given by:

\[
\text{DET}(\tilde{\partial}_s) = (\bigwedge^\text{max} \ker \tilde{\partial}_s)^* \otimes (\bigwedge^\text{max} \text{coker} \tilde{\partial}_s)
\]
and it is known that these complex lines fit together naturally over the base
space $S$ giving rise to a holomorphic line bundle over $S$ called $\text{DET}(\bar{\partial})$. In
fact, this entire construction is natural with respect to morphisms of families
and pullbacks of vector bundles.

We could have followed the above construction through for the universal
genus $g$ family over $\mathcal{T}_g$, with the vector bundle $E$ being, for example, the
trivial line bundle over the universal curve, or the vertical (relative) tangent
bundle, or any of its higher tensor powers. It is clear that setting $E$ to
be the trivial line bundle over $V$ for any family $F = (\gamma : V \to S)$, the
above prescription for $\text{DET}$ provides merely another description of the Hodge
line bundle over the base $S$. Indeed, the fiber of $\text{DET} = \text{Hodge}$ over $s$ is
$\lambda^g \cdot H^0(X_s, K_{X_s})^*$, and the naturality of the $\text{DET}$ construction mentioned
above shows that we thus get a member of $\text{Pic}_{\text{fun}}(\mathcal{M}_g)$.

By the same token, setting over any family $F$ the vector bundle $E$ to be
the $m^{th}$ tensor power of the vertical tangent bundle along the fibers, we get
by the $\text{DET}$ construction a well-defined member

$$\lambda_m \in \text{Pic}_{\text{fun}}(\mathcal{M}_g), \quad m = 0, 1, 2, 3, \ldots$$

Clearly, $\lambda_0$ is the Hodge bundle in the Picard group of the moduli functor. By
“Teichmüller’s lemma” one notes that $\lambda_1$ represents the canonical bundle over
the moduli space, the fiber at any Riemann surface $X$ being the determinant
line of the space of holomorphic quadratic differentials thereon.

1d. Mumford isomorphisms: By applying the Grothendieck-Riemann-
Roch theorem it was proved ([Mum]) that as elements of $\text{Pic}_{\text{fun}}(\mathcal{M}_g)$ one
has the following isomorphisms:

$$\lambda_m = (6m^2 + 6m + 1) \text{ tensor power of } \text{Hodge}(\lambda_0)$$

It is well-known, (Satake compactification combined with Hartogs theo-
rem), that there are no non-constant holomorphic functions on $\mathcal{M}_g$ ($g \geq 3$).
Therefore the choice of an isomorphism of $\lambda_m$ with $\lambda_0^{6(m^2 + 6m + 1)}$ is unique
up to a nonzero scalar. We would like to put canonical Hermitian metrics
on these DET bundles so that this essentially unique isomorphism actually becomes an unitary isometry. This is the theory of the:

1e. Quillen metrics on DET bundles: If we prescribe a conformal Riemannian metric on the fiber Riemann surface $X_s$, and simultaneously a hermitian fiber metric on the vector bundle $E_s$, then clearly this will induce a natural $L^2$ pairing on the one dimensional space $DET(\bar{\partial}_s)$ described in (1.9). Even if one takes a smoothly varying family of conformal Riemannian metrics on the fibers of the family, and a smooth hermitian metric on the vector bundle $E$ over $V$, these $L^2$ norms on the DET-lines may fail to fit together smoothly (basically because the dimensions of the kernel or cokernel for $\bar{\partial}_s$ can jump as $s$ varies over $S$). However, Quillen, and later Bismut-Freed and other authors, have described a “Quillen modification” of the $L^2$ pairing which always produces a smooth Hermitian metric on $DET$ over $S$, and has functorial properties. [Actually, in the cases of our interest the Riemann-Roch theorem shows that the dimensions of the kernel and cokernel spaces remain constant over moduli – so that the $L^2$ metric is itself smooth. Nevertheless the Quillen metrics will be crucially used by us because of certain functorial properties, and curvature properties, that they enjoy.]

In fact, using the metrics assigned on the Riemann surfaces (the fibers of $\gamma$), and the metric on $E$, one gets $L^2$ structure on the spaces of $C^\infty$ sections that constitute the domain and target for our d-bar operators. Hence $\bar{\partial}_s$ is provided with an adjoint operator $\bar{\partial}_s^*$, and one can therefore construct the positive (Laplacian) elliptic operator as the composition:

$$\Delta_s = \bar{\partial}_s^* \circ \bar{\partial}_s,$$

These Laplacians have a well-defined (zeta-function regularized) determinant, and one sets:

(1.12) Quillen norm on fiber of $DET$ above $s = (L^2$ norm on that fiber)$\det(\Delta_s)^{1/2}$

*This turns out to be a smooth metric on the line bundle $DET$. See [Q], [BF].*
In the situation of our interest, the vector bundle $E$ is the vertical tangent (or cotangent) line bundle along the fibers of $\gamma$, or its powers, so that the assignment of a metric on the Riemann surfaces already suffices to induce a hermitian metric on $E$. Hence one gets a Quillen norm on the various $DET$ bundles $\lambda_m (\in Pic_{fun}(\mathcal{M}_g))$ for every choice of a smooth family of conformal metrics on the Riemann surfaces. The Mumford isomorphisms (over any base $S$) become isometric isomorphisms with respect to the Quillen metrics.

The curvature form (i.e., first Chern form) on the base $S$ of the Quillen DET bundles has a particularly elegant expression:

$$c_1(DET, \text{Quillen metric}) = -\int_{V\mid S} (Ch(E)\text{Todd}(T_{vert}))$$

where the integration represents integration of differential forms along the fibers of the family $\gamma : V \to S$.

We now come to one of our main tools in this paper. By utilising the uniformisation theorem (with moduli parameters), the universal family of Riemann surfaces over $\mathcal{T}_g$, and hence any holomorphic family $F$ as above, has a smoothly varying family of Riemannian metrics on the fibers given by the constant curvature -1 Poincare metrics. The Quillen metrics arising on the $DET$ bundles $\lambda_m$ from the Poincare metrics on $X_s$ has the following beautiful property for its curvature:

$$c_1(\lambda_m, \text{Quillen}) = (12\pi^2)^{-1}(6m^2 + 6m + 1)\omega_{WP}, \; m = 0, 1, 2, ..$$

where $\omega_{WP}$ denotes the (1,1) Kähler form on $\mathcal{T}_g$ for the classical Weil-Petersson metric of $\mathcal{T}_g$. (We remind the reader that the cotangent space to the Teichmüller space at $X$ can be canonically identified with the vector space of holomorphic quadratic differentials on $X$, and the WP Hermitian pairing is obtained as

$$\langle \phi, \psi \rangle_{WP} = \int_X \bar{\phi} \psi (Poin)^{-1}$$

(Here $(Poin)$ is the area form on $X$ induced by the Poincare metric.) That the curvature formula (1.13) takes the special form (1.14) for the Poincare
family of metrics has been shown by Zograf and Takhtadzhyan [ZT]. Indeed, (1.13) specialised to $E = T_{\text{vert}}^\otimes m$ becomes simply $-(6m^2 + 6m + 1)/12$ times $\int_V |s_1(T_{\text{vert}})|^2$. This last integral represents, for the Poincare-metrics family, $\pi^{-2}$ times the Weil-Petersson symplectic form. See also [BF], [BK], [Bos].

Applying the above machinery, we will investigate in this paper the behaviour of the Mumford isomorphisms in the situation of a covering map between surfaces of different genera. Since the Weil-Petersson form is natural with respect to coverings, the above facts will be very useful.

Section 2. COVERINGS AND DET BUNDLES:  
2a. Induced morphisms on moduli: Over the genus $g$ closed oriented surface $X$ let:

$$\pi : \tilde{X} \longrightarrow X$$

be any unramified covering space, where $\tilde{X}$ is a surface of genus $\tilde{g}$.

If $X$ is equipped with a complex structure then there is exactly one complex structure on $\tilde{X}$ such that the map $\pi$ is a holomorphic map. In other words, a complex structure on $X$ induces a complex structure on $\tilde{X}$. One may tend to think that this correspondence induces a morphism from $\mathcal{M}_g$ to $\mathcal{M}_{\tilde{g}}$. But a closer analysis shows that this is not in general the case. In fact, the space $\mathcal{M}_g$ can be naturally identified with $\text{Comp}/\text{Diff}^+$, where $\text{Comp}$ is the space of all complex (=conformal) structures on $X$ and $\text{Diff}^+$ is the group of all orientation-preserving diffeomorphisms of $X$. But arbitrary element of $\text{Diff}^+$ can not be lifted to a diffeomorphism of $\tilde{X}$ – and this is the genesis of why there is in general no induced map of $\mathcal{M}_g$ to $\mathcal{M}_{\tilde{g}}$.

Since the Teichmüller space $\mathcal{T}_g$ is canonically identified with the quotient $\text{Comp}/\text{Diff}_0$, (where $\text{Diff}_0 \subset \text{Diff}^+$ is the sub-group consisting of all those diffeomorphisms which are homotopic to the identity map), we see however that there is in fact a map induced by the covering between the relevant Teichmüller spaces. Indeed, using the homotopy lifting property, a diffeomorphism homotopic to identity lifts to a diffeomorphism of $\tilde{X}$; moreover, the lifted diffeomorphism is also homotopic to the identity. Therefore, any
covering $\pi$ induces a natural morphism between the complex manifolds $T_g$ and $\tilde{T}_g$ obtained by “pullback” of complex structure: $\text{Teich}(\pi) = \pi_T : T_g \to \tilde{T}_g$.

It is well-known that the induced map is a holomorphic injective immersion that preserves the Teichmüller distance.

However, for our work we need to compare certain line bundles at an appropriate moduli-space (or finite covering thereof) level, where we can be sure that the space of line bundles is discrete. Therefore it is imperative that we find whether the covering $\pi$ induces a map at the level of some finite coverings of moduli space. To investigate this – and give an affirmative answer in certain topologically restricted situations – we introduce the finite coverings of moduli space called:

*Moduli of Riemann surfaces with level structure:* Fix any integer $k$ at least two. Let $\mathcal{M}_g[k]$ denote the moduli with level $k$. In other words, $\mathcal{M}_g[k]$ is the parameter space of pairs of the form $(M, \rho)$, where $M$ is a Riemann surface of genus $g$, and $\rho$ is a basis of the $\mathbb{Z}/k\mathbb{Z}$ module $H^1(M, \mathbb{Z}/k\mathbb{Z})$.

Now, Teichmüller space parametrises pairs $(M, \rho)$ where $\rho$ is a presentation of the fundamental group of the Riemann surface $M$. Hence it is clear that $\mathcal{M}_g[k]$ is a quotient of the Teichmüller space by a normal subgroup of the Teichmüller modular group. $\mathcal{M}_g[k]$ is a finite branched cover over $\mathcal{M}_g$, $\psi : \mathcal{M}_g[k] \to \mathcal{M}_g$. The covering group is $Sp(2g, \mathbb{Z}/k\mathbb{Z})$.

Now suppose that the covering space $\pi$ is a cyclic covering: namely, $\pi : \tilde{X} \to X$ be a Galois cover, with the Galois group being $\mathbb{Z}/l\mathbb{Z}$, where $l$ is a prime number. Of course, the number of sheets $l$ is the ratio of the Euler characteristics, i.e., $(\tilde{g} - 1)/(g - 1)$. Fix $k$ any positive integer that is a multiple of the sheet number $l$. Under these special circumstances we show:

**Proposition 2.1.** The covering $\pi$ induces natural maps

\[(i) \quad \pi_T : T_g \to \tilde{T}_g \]

\[(ii) \quad \pi_k : \mathcal{M}_g[k] \to \mathcal{M}_{\tilde{g}} \]
Moreover, if \( q : \mathcal{M}_g[k] \longrightarrow \mathcal{M}_g[l] \) is the projection induced by the homomorphism \( \mathbb{Z}/k \rightarrow \mathbb{Z}/l \) then the following diagram commutes

\[
\begin{array}{ccc}
\mathcal{M}_g[k] & \xrightarrow{id} & \mathcal{M}_g[k] \\
\downarrow q & & \downarrow \pi_k \\
\mathcal{M}_g[l] & \xrightarrow{\pi_l} & \mathcal{M}_{\tilde{g}}
\end{array}
\]

**Proof:** Part (i) we have already explained above (without any topological restrictions on the covering).

In order to prove that \( \pi \) induces a map from \( \mathcal{M}_g[k] \) to \( \mathcal{M}_{\tilde{g}} \) note that \( \mathcal{M}_g[k] \) can be identified with \( \text{Comp}/D(k) \), where \( D(k) \subset \text{Diff}^+ \) denotes the subgroup consisting of all those diffeomorphism whose induced homomorphism on \( H_1(X, \mathbb{Z}/k\mathbb{Z}) \) is identity. So we need to check that any element \( f \in D(k) \) lifts to a diffeomorphism of \( \tilde{X} \). From the given condition, \( \pi \) is an unramified cyclic cover of order \( l \). Now the set of unramified cyclic covers of order \( l \) of \( X \) is parametrized by \( PH^1(X, \mathbb{Z}/l) \). (This denotes the projective space of the first cohomology vector space over the field \( \mathbb{Z}/l \).) To see this, note that any such cover \( h : Y \rightarrow X \) induces a homomorphism

\[
h_* : H_1(Y, \mathbb{Z}/l) \longrightarrow H_1(X, \mathbb{Z}/l)
\]

with the quotient being isomorphic to \( \mathbb{Z}/l \). So the quotient map \( H_1(X, \mathbb{Z}/l) \rightarrow H_1(X, \mathbb{Z}/l)/\text{Im}(h_*) \) gives an element \( \bar{h} \in PH^1(X, \mathbb{Z}/l) \). Conversely, given an element of the \( PH^1 \), there is a homomorphism \( h_* \), and consequently a covering of \( X \) determined by the subgroup of the fundamental group corresponding to \( \text{Im}(h_*) \).

Clearly the covering

\[
f \circ h : Y \longrightarrow X
\]

corresponds to \( f^*\bar{h} \in PH^1(X, \mathbb{Z}/l) \). But \( f^* \) on \( H^1(X, \mathbb{Z}/k) \) is the identity, and hence \( f^*\bar{h} = \bar{h} \). We are through. \( \square \)

**Remark:** It is clear that the map \( \pi_T \), which exists for arbitrary covering \( \pi \), will not descend in general to \( \mathcal{M}_g \) because of the non-existence of lifts for
arbitrary diffeomorphisms. That is why we needed to use cyclic coverings and fix a level structure on the genus $g$ moduli. On the other hand it is worth noting that the map $\pi_k$ does not respect level structure on the target side. Namely, even when the covering $\pi$ is cyclic of order $l$ as above, there is no induced map from $\mathcal{M}_g[k]$ to $\mathcal{M}_{\bar{g}}[k]$ (even for $k = l$). That can be proved by constructing an example: in fact one can write down a diffeomorphism on $X$ which acts trivially on homology mod $l$ but such that each lifting to $\tilde{X}$ acts nontrivially on the homology mod $l$ up on $\tilde{X}$.

We also asked ourselves whether the inductive limit space construction, corresponding to the projective system of coverings between surfaces, can be carried through at the Torelli spaces level. Again it turns out that an arbitrary diffeomorphism that acts trivially on the homology of $X$ may not lift to such a diffeomorphism on $\tilde{X}$.

It therefore appears to be in the very nature of things that we are forced to do our direct limit construction of spaces and bundles at the Teichmüller level rather than at some intermediate moduli level.

2b. Comparison of Hodge bundles: We are now in a position to compare the two candidate Hodge bundles that we get over $\mathcal{M}_g[k]$; one by the $\pi_k$ pullback of Hodge from $\mathcal{M}_{\bar{g}}$, and the other being its ‘own’ Hodge arising from pulling back Hodge on $\mathcal{M}_g$.

Notations: Let $\lambda = \lambda_0$ denote, as before, the Hodge bundle on $\mathcal{M}_g$ (a member of $\text{Pic}_{\text{fun}}(\mathcal{M}_g)$, as explained), and let $\tilde{\lambda}$ denote the Hodge line bundle over $\mathcal{M}_{\bar{g}}$. Also let $\psi: \mathcal{M}_g[k] \to \mathcal{M}_g$ denote the natural projection from moduli with level-structure to moduli. Let $\omega = \omega_{WP}$ and $\tilde{\omega}$ represent the Weil-Petersson forms (i.e., the Kähler forms corresponding to the WP hermitian metrics) on $\mathcal{M}_g$ and $\mathcal{M}_{\bar{g}}$, respectively.

The naturality of Weil-Petersson forms under coverings is manifest in the following basic Lemma:

**Lemma 2.2.** The 2-forms $l(\psi^*\omega)$ and $(\pi_k)^*\tilde{\omega}$ on $\mathcal{M}_g[k]$ coincide.

**Proof.** This is actually a straightforward computation. Recall that the cotangent space to the moduli space is given by the space of quadratic differ-
entials. Now at any point $\alpha := (M, \rho) \in \mathcal{M}_g[k]$, the morphism of cotangent spaces induced by the map $\pi_k$ of Proposition 2.1(ii) is a map:

$$(d\pi_k)^* : T^*_{\pi_k(\alpha)} \mathcal{M}_g \rightarrow T^*_\alpha \mathcal{M}_g[k]$$

The action on any quadratic differential $\phi$ on the Riemann surface $\pi_k(\alpha)$, (i.e., $\phi \in H^0(\pi_k(\alpha), K^2)$), is given by:

$$(2.2) \quad (d\pi_k)^* \phi = 1/l( \sum_{f \in Deck} f^* \phi)$$

Here $Deck$ denotes, of course, the group of deck transformations for the covering $\pi$. Now recall that a covering map $\pi$ induces a local isometry between the respective Poincare metrics, and that $l$ copies of $X$ will fit together to constitute $\tilde{X}$. The lemma therefore follows by applying formula (2.2) to two quadratic differentials, and pairing them by Weil-Petersson pairing as per definition (1.15).

The above Lemma, combined with the statements about the curvature form of the Hodge bundles being $(12\pi^2)^{-1} \omega_{WP}$, show that the curvature forms (with respect to the Quillen metrics) of the two line bundles $\lambda^l$ and $(\pi_k)^* \tilde{\lambda}$ coincide. Do the bundles themselves coincide? Yes:

**Theorem 2.3.** The two line-bundles $(\psi)^* \lambda^l$ and $(\pi_k)^* \tilde{\lambda}$, on $\mathcal{M}_g[k]$, are isomorphic. If $g \geq 3$, then such an isomorphism:

$$(F : (\psi)^* \lambda^l \rightarrow (\pi_k)^* \tilde{\lambda})$$

is uniquely specified up to the choice of a nonzero scaling constant. The same assertions hold for the higher $DET$ bundles.

**Proof.** Of course, the basic principle of the proof is that “curvature of the bundle determines the bundle uniquely” over the moduli space $\mathcal{M}_g[k]$. That would follow automatically if we know that the Picard group of $\mathcal{M}_g[k]$ is discrete. This is in fact known by a recent preprint of R. Hain, and E. Looijenga has provided us with an independent proof of (a somewhat stronger) fact. See the Appendix to this paper.
Without using the Hain-Looijenga proof we can prove what is necessary for our purposes by using a certain cohomology vanishing theorem for $l$-adic symplectic groups to be found in work of Moore [M]:

Firstly note that, in view of the commutative diagram in Proposition 2.1, it is enough to prove things for the case $k = l$. The Quillen metrics on $\lambda$ and $\tilde{\lambda}$ induce hermitian metrics on the two bundles under scrutiny: $\xi := (\psi)^*\lambda^l$ and $\xi' := (\pi_1)^*\tilde{\lambda}$. We know their curvatures coincide. So the line-bundle $\xi^* \otimes \xi'$ with the induced metric is flat. In other words, after holonomy consideration, $\xi^* \otimes \xi'$ is given by a unitary character $\rho$ of the fundamental group $\Gamma' := \pi_1(M_g[l])$.

Let us temporarily denote by $\Gamma := MC_g$ the mapping class group for genus $g$. Also let $G := Sp(2g, \mathbb{Z}/l)$, be the symplectic group defined over the field $\mathbb{Z}/l$. We have an exact sequence

$$0 \rightarrow \Gamma' \rightarrow \Gamma \rightarrow G \rightarrow 0$$

which induces the following long exact sequence of cohomologies

$$\rightarrow Hom(\Gamma, U(1)) \rightarrow Hom(\Gamma', U(1)) \rightarrow H^2(G, U(1)) \rightarrow$$

Note that all the actions of $U(1)$ are taken to be trivial.

Now we claim that in order to prove the theorem it is enough to show that $H^2(G, U(1)) = 0$. This is because, in that case the above exact sequence of cohomologies would imply that $\rho$ is the pull-back of a character on $\Gamma$. In other words, $\xi^* \otimes \xi'$ would be shown to be a pull-back of a flat hermitian line-bundle on $M_g$. Now, the rational Chern classes of a flat line bundle vanish, and a line bundle on $M_g$ with vanishing 1-st Chern class is actually a trivial bundle (Theorem 2, [AC]). Thus if $H^2(G, U(1)) = 0$, then the triviality of the line-bundle $\xi^* \otimes \xi'$ is established.

So all we have to show is the vanishing of $H^2(G, U(1))$. That follows from the literature on $l$-adic linear groups. Indeed, from Chapter III of [M] we get that $\pi_1(G) = 0$. But then the Lemma (1.1) of [M] implies that $H^2(G, U(1)) = 0$. 

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The uniqueness assertion follows since for $g \geq 3$, $\mathcal{M}_g[k]$ does not admit any nonconstant global holomorphic function. 

2c. Lifting the identifications to Teichmüller space: Since the principal level structure keeps changing as we deal with different covering spaces, we must perform pullback these bundles to Teichmüller space and use our results above to produce a canonical identification of the bundles also at the Teichmüller level.

Notations: Let $p : \mathcal{T}_g \rightarrow \mathcal{M}_g$ denote the quotient projection from Teichmüller to moduli. Similarly, set $\tilde{\phi} : \mathcal{T}_g \rightarrow \mathcal{M}_g$. There is the natural map $q : \mathcal{T}_g \rightarrow \mathcal{M}_g[k]$ to the intermediate moduli space with level-structure, and we have $\psi \circ q = p$. (Recall, $\psi$ denotes the finite covering from $\mathcal{M}_g[k]$ onto $\mathcal{M}_g$.) Furthermore, by functoriality of the operation of pullback of complex structure, one has $\pi_k \circ q = \tilde{\phi} \circ \pi_T$.

Remark: Since Teichmüller spaces are contractible Stein domains, any two holomorphic line bundles on $\mathcal{T}_g$ are isomorphic; but, since $\mathcal{T}_g$ admits plenty of nowhere zero non-constant holomorphic functions, a priori there is no natural isomorphism. Our problem in manufacturing $DET$ bundles over the inductive limit of Teichmüller spaces is to find natural connecting bundle maps.

Theorem 2.4 The isomorphism $F$ of Theorem 2.3 induces an isomorphism of $\lambda^l$ and $\tilde{\lambda}$ pulled back to $\mathcal{T}_g$:

$$\tilde{F} : p^*\lambda^l \rightarrow (\pi_T)^*\tilde{\lambda}$$

$\tilde{F}$ does not depend on the choice of the level $k$.

Proof: To see how this map $\tilde{F}$ depends upon $k$ replace $k$ by $k'$, both being multiples of the sheet number $l$. We may assume $k'$ itself is a multiple of $k$. (If $k'$ is not a multiple of $k$ we can go to the L.C.M. of the two and proceed with the following argument.) One has the following commutative diagram

$$
\begin{array}{ccc}
\mathcal{T}_g & \xrightarrow{id} & \mathcal{T}_g \\
\downarrow & & \downarrow \scriptstyle{f} \\
\mathcal{M}_g[k'] & \xrightarrow{h} & \mathcal{M}_g[k]
\end{array}
$$
where \( id \) denotes the identity map and \( h \) is the natural projection. Clearly, \( \pi_{k'} = \pi_k \circ h \). Therefore \( \bar{F} \) does not depend upon the choice of \( k \).

So we are able to pick up a unique isomorphism between \( p^* \lambda^l \) and \((\pi_T)^* \bar{\lambda} \), ambiguous only up to a non-zero scaling constant. 

In the next section we will examine some functorial properties of this chosen isomorphism.

**Section 3. FROM CYCLIC TO ABELIAN COVERS:**

We aim to generalise the Theorem 2.4 above to the situation where the covering \( \pi \) admits a factorisation into a finite succession of abelian Galois coverings.

**3a. Factorisations into cyclic coverings:** Let \( f : Z \to X \) be an abelian Galois covering of a surface of genus \( g \) by a surface of genus \( \gamma \), having \( N \) sheets. Suppose that \( f \) allows decompositions in two ways as a product of cyclic coverings of prime order. One therefore has a commutative diagram of covering maps:

\[
\begin{array}{ccc}
Z & \xrightarrow{id} & Z \\
\downarrow \rho_1 & & \downarrow \rho_2 \\
Y_1 & & Y_2 \\
\downarrow \pi_1 & & \downarrow \pi_2 \\
X & \xrightarrow{id} & X
\end{array}
\] (3.1)

Here \( \pi_i \circ \rho_i = f, \ i = 1, 2 \), and \( \pi_i \) and \( \rho_i \) are cyclic Galois covers of prime orders \( l_i \) and \( m_i \) respectively. Let \( \alpha_1, \alpha_2 \) be the genera of \( Y_1, Y_2 \) respectively. For the above coverings we have the following commutative diagram of maps between Teichmüller spaces

\[
\begin{array}{ccc}
\mathcal{T}_g & \xrightarrow{id} & \mathcal{T}_g \\
\downarrow \pi^T_1 & & \downarrow \pi^T_2 \\
\mathcal{T}_{\alpha_1} & & \mathcal{T}_{\alpha_2} \\
\downarrow \rho^T_1 & & \downarrow \rho^T_2 \\
\mathcal{T}_\gamma & \xrightarrow{id} & \mathcal{T}_\gamma
\end{array}
\]
Now, \( N = l_1 m_1 = l_2 m_2 \); set \( \pi^T := \pi^T_1 \circ \rho^T_1 = \pi^T_2 \circ \rho^T_2 \).

Following the prescription described in the previous section, corresponding to each of the two different decomposition of the covering \( Z \to X \), we will get two isomorphisms of the bundles over \( T_g \):

\[
F_1, \ F_2 : \ p^* \lambda^N \to \pi^{T*} \xi
\]

where we have denoted by \( \lambda \) the Hodge bundle on \( T_g \) and by \( \xi \) the Hodge bundle on \( T_\gamma \). For our final construction we require the crucial Proposition below:

**Proposition 3.1.** The two isomorphisms \( F_1 \) and \( F_2 \) are constant (nonzero) multiples of each other.

**Proof:** As in Proposition 2.1, we get the induced maps:

\[
\pi_{1,N} \text{ (resp. } \pi_{2,N} : \ M_g[N] \to M_{\alpha_1} \text{ (resp. } M_{\alpha_2})
\]

For \( i = 1, 2 \), let \( q_i \) be the projection of \( M_{\alpha_i}[m_i] \) onto \( M_{\alpha_i} \).

The technique of the proof is to go over to bundles on the fiber product of moduli spaces with level structures and show that isomorphisms are unique (up to scalar) there. Then the result desired will follow.

The fiber product \( \mathcal{P} := M_g[N] \times_{(\mathcal{M}_{\alpha_1} \times \mathcal{M}_{\alpha_2})} (M_{\alpha_1}[m_1] \times M_{\alpha_2}[m_2]) \) fits into the following commutative diagram

![Diagram](image)

Again let \( f_i : \ M_{\alpha_i}[m_i] \to M_{\gamma} \) be the morphisms induced on moduli by the relevant covering spaces, \( (i = 1, 2) \). Also let \( pr_i \) denote the projection of \( M_{\alpha_1}[m_1] \times M_{\alpha_2}[m_2] \) to the \( i \)-th factor.

In the above notation we have:

\[
f_1 \circ pr_1 \circ p_2 = f_2 \circ pr_2 \circ p_2.
\]
From Theorem 2.3 and diagram (3.3) it now follows that the following two line bundles over \( \mathcal{P} \) are isomorphic: \((f_1 \circ pr_1 \circ p_2)^*\xi \) is isomorphic to \((\psi \circ p_1)^*\lambda^N\). (Here \( \psi \), as in previous sections, denotes the projection of \( \mathcal{M}_g[N] \) to \( \mathcal{M}_g \).

Now, the Teichmüller space \( T_g \) has a canonical mapping to the fiber product \( \mathcal{P} \). Clearly the isomorphism \( F_i \) under concern \((i = 1, 2)\), is the pullback of an isomorphism between \((f_i \circ pr_1 \circ p_2)^*\xi \) and \((\psi \circ p_1)^*\lambda^N\) over \( \mathcal{P} \). The desired result will follow if we show that two such isomorphisms over \( \mathcal{P} \) can only differ by a multiplicative scalar.

But that last result is true for any space on which the only global holomorphic functions are the constants. That is known to be true for the moduli space \( \mathcal{M}_g[N] \). But the fiber product sits over this by a finite covering map \( p_1 \). Therefore the result is true for the space \( \mathcal{P} \) as well. (Indeed, if \( \mathcal{P} \) has a nontrivial function \( f \), then by averaging the powers of \( f \) over the fibers of \( p_1 \) one would obtain nontrivial functions on \( \mathcal{M}_g[N] \). This argument is well-known.) The Proposition is proved.

**3b. Canonical isomorphism from abelian covering:** The above proposition holds, and the same proof goes through, if the covering map \( f \) is factored by more than one intermediate covering. In other words, for a situation as below

\[
\begin{array}{ccccccc}
Z & \rightarrow & Y_1^1 & \rightarrow & Y_1^2 & \cdots & \rightarrow & X \\
\downarrow^{id} & & \downarrow & & \downarrow & & \downarrow^{id} \\
Z & \rightarrow & Y_2^1 & \rightarrow & Y_2^2 & \cdots & \rightarrow & X
\end{array}
\]

But the structure theorem for finite abelian group says that any finite abelian group is a direct sum of cyclic abelian groups whose orders are prime powers. Moreover any cyclic group of order prime power \( l^n \) fits as the left-end term of an exact sequence of abelian groups where each quotient group is \( \mathbb{Z}/l \). The upshot is that an abelian covering factorises into a sequence of cyclic covers of prime order.

Utilising Proposition 3.1 and the above remarks we therefore obtain the sought-for generalisation of Theorem 2.4:
Theorem 3.2: Let $\pi : Y \rightarrow X$ be a covering space, of order $N$; let $g$ and $\tilde{g}$ be the genera of $X$ and $Y$, respectively. Assume that $\pi$ can be factored into a sequence of abelian Galois covers. Then there is a canonical isomorphism (uniquely determined up to a nonzero-constant):

$$\tilde{F} = \rho : p^*\lambda^N \rightarrow \pi^*_T\tilde{\lambda}$$

Here $\pi_T : T_g \rightarrow T_{\tilde{g}}$ is the map induced by $\pi$ at the Teichmüller space level, and $\lambda$ and $\tilde{\lambda}$ denote respectively the Hodge bundles on $T_g$ and $T_{\tilde{g}}$.

A similar statement holds using the $m^{th}$ DET bundles instead of the Hodge bundle. In fact, $p^*(\lambda^\otimes N_m)$ and the pullback $\pi^*_T\tilde{\lambda}_m$ are also related by a canonical isomorphism over $T_g$.

3c. Functorial properties: We will require the following compatibility property for these chosen isomorphisms.

Let

$$Z \xrightarrow{f_1} Y \xrightarrow{f_2} X$$

be such that both $f_1$ and $f_2$ are covers, of orders $n_1$ and $n_2$ respectively. Denote the genera of $Y$ and $Z$ by $g_1$ and $g_2$ respectively. Let $f_{1T} : T_{g_1} \rightarrow T_{g_2}$, $f_{2T} : T_{g} \rightarrow T_{g_1}$ and $(f_2 \circ f_1)_T : T_{g} \rightarrow T_{g_2}$ be the induced maps of Teichmüller spaces. If $\lambda_1$ and $\lambda_2$ denote the Hodge bundles on $T_{g_1}$ and $T_{g_2}$, respectively, and $\lambda$ the Hodge on $T_{g}$, then we have a commutative diagram:

$$\begin{array}{ccc}
\lambda^{n_1n_2} & \xrightarrow{id} & \lambda^{n_1n_2} \\
\downarrow f_2^{-} & & \downarrow (f_2 \circ f_1)^{-} \\
f_{2T}^*\lambda_1^{n_1} & \xrightarrow{f_1^{-}} & (f_2 \circ f_1)_T^*\lambda_2
\end{array}$$

(3.4)

In the diagram above the homomorphism of Hodge bundles induced by any given covering has been denoted by using the superscript "−" over the notation for that covering.

Section 4. HERMITIAN BUNDLE ISOMORPHISMS:
Recall the group of modular invariant line bundles on Teichmüller space, $Pich(T_g)$, carrying hermitian structure, that was introduced in Section 1b. We consider its “rational version” by tensoring (over $\mathbb{Z}$) with $\mathbb{Q}$:

\begin{equation}
Pich(T_g)_{\mathbb{Q}} := Pich(T_g)\otimes_{\mathbb{Z}}\mathbb{Q}
\end{equation}

It is to be noted that hermitian metrics, curvature forms (as members of cohomology of the base with $\mathbb{Q}$-coefficients), etc. make perfect sense for elements of rational Picard groups just as for usual line bundles.

Lemma 1.1 now places things in perspective. We see from that result that $Pich(T_g)_{\mathbb{Q}}$ is canonically isomorphic to $Pic_{alg}(M_g)\otimes\mathbb{Q}$ as well as $Pic_{hol}(M_g)\otimes\mathbb{Q}$.

We have seen in Section 1 that the Hodge bundle, and the higher DET bundles, admit their Quillen metric, such that the curvature of the holomorphic Hermitian connection is a multiple of the Weil-Petersson form. Therefore, these bundles $(\lambda_m, Quillen)$ for $m = 0, 1, 2, ..$ constitute some interesting and canonical elements of $Pich(T_g)_{\mathbb{Q}}$.

As before, suppose that $\pi : Y \to X$ is a covering that allows factorisation into abelian covers, as in the set up of Theorem 3.2. Let the degree of the covering be $N$.

First of all we note that the isomorphism $\rho$ (Theorem 3.2) induces a modular invariant structure on $\pi_\tau^*\tilde{\lambda}$. Note that though $\rho$ is defined only up to a non-zero constant, since multiplication by scalars commutes with the modular invariance structure on $\lambda$, the modular invariant structure obtained on $\pi_\tau^*\tilde{\lambda}$ does not depend upon the exact isomorphism chosen. Thus we can put a restriction on the isomorphism $\rho$, of Theorem 3.2, associated to the covering $\pi$. We demand that it be unitary with respect to the Quillen metric on $(\psi)^*\lambda^N$ and the pull-back, by $\pi_k$, of the Quillen metric on $\tilde{\lambda}$. In that case $\tilde{F} = \rho$ is determined up to a scalar of norm one.

We have therefore proved the following Proposition that is fundamental to our construction of Universal DET bundles over inductive limits of Teichmüller spaces:
Proposition 4.1: Let \( \pi \) be a covering with \( N \) sheets as above. Then \( \lambda \) and \( \pi^*((1/N)\lambda) \) coincide as elements of \( \text{Pich}(\mathcal{T}_g)_Q \). Here \( (1/N)\lambda \) is considered as an element of \( \text{Pich}(\mathcal{T}_g)_Q \). By the same token, \( \lambda_m \) and the pullback via \( \pi_T \) of the \( (1/N) \) times the \( m \)th DET bundle over \( \mathcal{T}_g \) coincide.

Remark: In the diagram 3.4 (Section 3c), if all the bundles are equipped with respective Quillen metric then the diagram becomes a commutative diagram of Hermitian bundles. In other words, all the morphisms in 3.4 are unitary.

More generally, let \( (L, h) \in \text{Pich}(\mathcal{T}_g) \). Denote the pulled-back line bundle by \( L' := \pi^*_T L \in \text{Pic}(\mathcal{T}_g) \). Because of modular invariance, notice that \( (L, h) \) becomes a hermitian line bundle on \( \mathcal{M}_g \). Consider the hermitian line bundle \( \pi^*_k(L, h) \) on \( \mathcal{M}_g[k] \), where \( \pi_k \) is the induced map of Proposition 2.1. The line-bundle \( \psi^*L' \), (where \( \psi : \mathcal{M}_g[k] \rightarrow \mathcal{M}_g \) is the natural projection), is canonically isomorphic to \( \pi^*_kL \). Now, using this isomorphism, and averaging the metric \( \pi^*_k h \) over the fibers of \( \psi \), a hermitian metric on the bundle \( L' \) over \( \mathcal{M}_g \) is obtained. In other words \( L' \) on \( \mathcal{T}_g \) gets a modular invariant hermitian structure. Thus we have a natural mapping:

\[
\pi^Q_T : \text{Pich}(\mathcal{T}_g)_Q \longrightarrow \text{Pich}(\mathcal{T}_g)_Q
\]

Proposition 4.2 The natural homomorphism (4.2) enjoys the following properties:

(1). The image of \( \pi^Q_T \) on the Hodge bundle with Quillen metric coincides with \( N \) times the Hodge with Quillen metric over \( \mathcal{M}_g \), precisely as stated in Proposition 4.1. The obvious corresponding assertion holds also for the higher DET bundles.

(2). For a pair of coverings

\[
\begin{align*}
Z & \xrightarrow{f_1} Y & \xrightarrow{f_2} X
\end{align*}
\]

as in the set up of 3.4, let \( \pi^Q_1 \) be the homomorphism given by (4.2) for the covering \( f_1 \). Similarly, \( \pi^Q_2 \) and \( \pi^Q \) be the homomorphisms for \( f_2 \) and \( f_2 \circ f_1 \).
respectively. Then
\[ \pi_2^Q \circ \pi_1^Q = \pi^Q \]

Section 5. THE MAIN RESULTS:

5a. Abelian lamination and its complex structures: Consider the inverse system of isomorphism classes of finite covers of \( X \), which allow factorization into a sequence of abelian coverings. The mappings in this directed system are those that commute with the projections onto \( X \). One constructs thereby an interesting Riemann surface lamination (“solenoidal Riemann surface”, see [S], [NS]), by taking the inverse limit of this projective system of surfaces. This inverse limit, which we denote by \( H_{\infty,ab} \), is a compact space that is fibered over \( X \) with a Cantor set as fiber. Each path component (“leaf”) of this Riemann surface lamination is a covering space over the original base surface \( X \), the covering group being the commutator subgroup of the fundamental group of \( X \).

Remark: The general “universal hyperbolic lamination” of this type, where all finite coverings over \( X \) are allowed, has been studied in [NS]. That inverse limit space was denoted \( H_\infty \), and its Teichmüller space was shown to possess a convergent Weil-Petersson pairing.

Let this projective system of covering mappings be indexed by some set \( I \), and for \( i \in I \), let \( p_i : X_i \to X \) be the covering and \( \text{Teich}(p_i) = q_i : \mathcal{T}_g \to \mathcal{T}_{g_i} \) be the induced injective immersion between the Teichmüller spaces. Thus, corresponding to the inverse system of surfaces we have a direct system of Teichmüller spaces.

Let \( g_i \) denote the genus of \( X_i \). The direct limit space

\[ (5.1) \quad \mathcal{T}_\infty := \lim_{\to} \mathcal{T}_{g_i} \]

of these Teichmüller spaces constitute the space of “TLC” [transversely locally constant] complex structures on the Riemann surface lamination \( H_{\infty,ab}^n \).

In fact, the lamination has a Teichmüller space parametrising complex structures on it. This space is precisely the completion in the Teichmüller
metric of the space $T_\infty$. It is the Teichmüller space of the abelian laminaction, $T(H_\infty, ab)$ and it is Bers-embeddable and possesses the structure of an infinite dimensional separable complex manifold. Notice that the various finite dimensional Teichmüller spaces for Riemann surfaces of genus $g_i$ are all contained faithfully inside this infinite dimensional “universal” Teichmüller space, from its very construction.

Remark: We emphasize that the above construction is stable with respect to change of the initial base surface $X$. In fact, starting from two distinct genera $X_1$ and $X_2$, the inverse limit solenoidal surface, and the corresponding direct limit of Teichmüller spaces, will be isomorphic whenever one finds a common covering surface $Y$ covering both the bases by a (product of) abelian covers. That is easily done provided the genera of $X_1$ and $X_2$ are both at least two. The families become cofinal from $Y$ onwards. Hence the limiting objects are isomorphic.

We are interested in studying the space of complex structures on this laminated surface, and constructing the canonical $DET$ hermitian line bundle over this Teichmüller space, in order to obtain the Mumford isomorphisms between the relevant $DET$ bundles over this inductive limit space.

5b. Line bundles on ind spaces: A line bundle on the inductive limit of an inductive system of varieties or spaces, is, by definition ([Sha]), a collection of line bundles on each stratum (i.e., each member of the inductive system of spaces) together with compatible bundle maps. The compatibility condition for the bundle maps is the obvious one relating to their behaviour with respect to compositions, and guarantees that the bundles over the strata themselves fit into an inductive system. A bundle with hermitian metric is a collection with hermitian metrics such that the connecting bundle maps are unitary. Such a direct system of line bundles can clearly be thought of as an element of the inverse limit of the Picard varieties of the stratifying spaces. See [KNR] [Sha].

A “rational” hermitian line bundle over the inductive limit is thus clearly an element of the inverse limit of the $Pic \otimes \mathbb{Q}$ ‘s, namely of the space
The following two theorems were our chief aim.

**Theorem 5.1: Universal DET line bundles:** There exist canonical elements of the inverse limit $\lim \leftarrow Pich(T_{g_i})_Q$, namely hermitian line bundles on the ind space $T_\infty$, representing the Hodge and higher DET bundles with respective Quillen metrics:

$$\Lambda_m \in \lim \leftarrow Pich(T_{g_i})_Q, \ m = 0, 1, 2, ..$$

The pullback of $\Lambda_m$ to each of the stratifying Teichmüller spaces $T_{g_i}$ is $(n_i)^{-1}$ times the corresponding Hodge or higher DET bundle with Quillen metric over $T_{g_i}$.

**Proof:** The foundational work is already done in Theorem 3.2 and Proposition 4.1 above. In fact, let $\lambda_{0,i}$ represent the Hodge bundle with Quillen metric in $Pich(T_{g_i})_Q$. Then Proposition 4.1 implies that, for $i \in I$ taking the element

$$(1/n_i)\lambda_{0,i} \in Pich(T_{g_i})_Q$$

provides us a compatible family of hermitian line bundles (in the rational Pic) over the stratifying Teichmüller spaces – as required in the definition of line bundles over ind spaces. The connecting family of bundle maps is determined (up to a scalar) by Theorem 3.2.

The property that the connecting unitary bundle maps for the above collection are compatible, follows from Theorem 3.2 and the commutative diagram (3.4) of Section 3. Notice that prescribing a base point in $T_{g_i}$ and a vector of unit norm over it, fixes uniquely all the scaling factor ambiguities in the choices of the connecting bundle maps. We have therefore constructed the universal Hodge, $\Lambda_0$, over $T_\infty$.

Naturally, the above analysis can be repeated verbatim for the higher d-bar families, and one thus obtains elements $\Lambda_m$ for each positive integer $m$. Again the pullback of $\Lambda_m$ to any of the stratifying $T_{g_i}$ produces $(n_i)^{-1}$ times the $m$-th DET bundle with Quillen metric living over that space. $\square$
Theorem 5.2: Universal Mumford isomorphisms: Over the direct limit Teichmüller space $T_\infty$ we have

$$\Lambda_m = (6m^2 + 6m + 1)\Lambda_0$$

as an equality of hermitian line bundles.

Proof: Follows directly from the genus-by-genus isomorphisms of (1.11), and our universal line bundle construction of this paper. $\square$

5c. Polyakov measure on $M_g$ and our bundles: The quantum theory of the closed bosonic string is the theory of a sum over random surfaces (”world-sheets”) swept out by strings propagating in Euclidean spacetime $\mathbb{R}^d$. Owing to the conformal invariance property enjoyed by the Polyakov action, that summation finally reduces to integrating out a certain measure - called the Polyakov measure - on the moduli space of conformally distinct surfaces, namely on the parameter spaces of Riemann surfaces. It is well-known ([P], [Alv], [BM]) that the original sum over the infinite-dimensional space of random world-sheets (of genus $g$), reduces to an integral over the finite-dimensional moduli space, $M_g$, of Riemann surfaces of genus $g$ precisely when the background spacetime has dimension $d = 26$. In the physics literature, this is described by saying that the “conformal anomaly” for the path-integral vanishes when $d = 26$.

That magic dimension 26, at which the conformal anomaly cancels, can also be explained as the dimension in which the “holomorphic anomaly” cancels. The vanishing of the holomorphic anomaly at $d = 26$ can be interpreted as the statement of Mumford’s isomorphism (1.11) above for the case $m = 1$. In fact, that result shows, in combination with our interpretation of the DET bundles $\lambda$ and $\lambda_1$ (see Section 1c), that the holomorphic line bundle $K \otimes \lambda^{-m}$ over $M_g$ is the trivial bundle when $m = d/2 = 13$. Here $K = \lambda_1$ denotes the canonical line bundle, and $\lambda$ the Hodge bundle, over $M_g$.

The Polyakov volume form on $M_g$ can now be given the following simple interpretation. Fixing a volume form (up to scale) on a space amounts to fixing a fiber metric (up to scale) on the canonical line bundle over that space.

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But the Hodge bundle $\lambda$ has its natural Hodge metric (arising from the $L^2$ pairing of holomorphic 1-forms on Riemann surfaces). Therefore we may transport the corresponding metric on $\lambda^{13}$ to $K$ by Mumford’s isomorphism, (as we know the choice of this isomorphism is unique up to scalar) – thereby obtaining a volume form on $\mathcal{M}_g$. [BK] showed that this is none other than the Polyakov volume. Therefore, the presence of Mumford isomorphisms over the moduli space of genus $g$ Riemann surfaces describes the Polyakov measure structure thereon.

Above we have succeeded in fitting together the Hodge and higher DET bundles over the ind space $T_\infty$, together with the relating Mumford isomorphisms. We thus have from our results a structure on $T_\infty$ that qualifies as a genus-independent, universal, version of the Polyakov structure.

Remark: Since the genus is considered the perturbation parameter in the above formulation of the standard perturbative bosonic Polyakov string theory, we may consider our work as a contribution towards a non-perturbative formulation of that theory.

Remark: In [NS] we had shown the existence of a convergent Weil-Petersson pairing for the Teichmüller space of the universal lamination $H_\infty$. Now, Polyakov volume can be written as a multiple by a certain combination of Selberg zeta functions of the Weil-Petersson volume. Using therefore Theorems 5.1 and 5.2 in conjunction with the Weil-Petersson on $T(H_\infty, ab)$, we have an interpretation of that combination of Selberg zetas as an object fitted together over each stratifying Teichmüller space.

It should be noted in this context that the line bundles over $T_\infty$ that we have constructed do not appear as restrictions to the strata from any rational line bundle over the completed space $T(H_\infty, ab)$.

Section 6: EXTENSION TO MODULI WITH MARKED POINTS:
6a. Moduli spaces with $r$ distinguished points: As always, denote by $X$ a compact oriented $C^\infty$ surface of genus $g$, and

$$S = \{p_1, \ldots, p_r\}$$
a finite ordered subset of \( r \) distinct points on \( X \).

Let \( \mathcal{M}_g^r \) stands for the moduli space of \( r \)-pointed compact Riemann surfaces of genus \( g \). By definition, it is the space of \( \text{Diff}^+(X, \{p_1, \ldots, p_r\}) \)-orbits of conformal structures on \( X \). The corresponding Teichmüller space is denoted by \( \mathcal{T}^r_g \).

Fix any integer \( k \geq 2 \). Let \( \mathcal{M}_g^r[k] \) be the moduli space with level \( k \). In other words, \( \mathcal{M}_g^r[k] \) is the moduli of triplets of the form \((M, \Gamma, \rho)\), where \( M \) is a Riemann surface of genus \( g \), \( \Gamma \) is an ordered subset of cardinality \( r \) and \( \rho \) is a basis of the \( \mathbb{Z}/k\mathbb{Z} \) module \( H_1(M - \Gamma, \mathbb{Z}/k\mathbb{Z}) \). \( \mathcal{M}_g^r[k] \) is a complex manifold of dimension \( 3g - 3 + r \) covering the orbifold \( \mathcal{M}_g^r \).

**Generators for Pic:** Of course, the Hodge line bundle can be introduced as before as an element of \( \text{Pic}_{\text{fun}}(\mathcal{M}_g^r) \). Moreover, for each \( i \in \{1, \ldots, r\} \), there is a line bundle \( L_i \) on \( \mathcal{M}_g^r \), whose fiber \( L_{i,x} \), over \( x \in \mathcal{M}_g^r \), is \( K_{p_i} \) (recall \( p_i \) is the \( i \)-th element of \( S \)). Here \( K_x \) denotes the fiber of the canonical bundle at \( x \) on any Riemann surface. The Picard group of of the moduli functor for \( \mathcal{M}_g^r \), \( \text{Pic}_{\text{fun}}(\mathcal{M}_g^r) \), is known to be generated by the Hodge bundle together with these \( \{L_i\} \). See [AC].

**6b. Construction of morphisms between moduli:** Let

\[
\pi : \tilde{X} \longrightarrow X
\]

be a Galois cover, with cyclic Galois group being \( \mathbb{Z}/l\mathbb{Z} \), where \( l \) is a prime number. *We assume that \( \pi \) is ramified exactly over \( S \).* Since the Galois cover is of prime order, it is easy to see that the map \( \pi \) is necessarily totally ramified over each point of \( S \). Let the genus of \( \tilde{X} \) be \( \tilde{g} \).

Using the same techniques as for Proposition 2.1, we can show the following:

**Proposition 6.1:** Let the level number \( k \) be fixed at any multiple of the number of sheets \( l \). Then the covering \( \pi \) induces natural maps

\[
(i)\pi_T : \mathcal{T}_g^r \longrightarrow \mathcal{T}_{\tilde{g}}^r
\]
Moreover if \( q : \mathcal{M}_g^r[k] \to \mathcal{M}_g^r[l] \) be the projection induced by the homomorphism \( \mathbb{Z}/k \to \mathbb{Z}/l \) then the diagram corresponding to the one shewn in Proposition 2.1 again commutes.

Now let \( \pi : Z \to X \) be an abelian Galois covering which is totally ramified over the subset \( S \), and nowhere else. Exactly the same results as proved in Sections 3 and 4 above go through with essentially no extra trouble. The set of points of ramification at any covering stage remains of cardinality \( r \), and one needs to keep track of this subset all along.

Indeed, if we identify, as in Section 4, the hermitian bundles on \( \mathcal{M}_g^r \) with modular invariant bundles on \( T_g^r \), then associated to the covering \( \pi \) there is a morphism which is precisely the analog of morphism (4.2):

\[
\pi_T^{r, Q} : \text{Pich}(T_g^r)_Q \to \text{Pich}(\mathcal{T}_g^r)_Q
\]

(6.1) exists because, as we have seen for the Hodge bundles, the bundles \( L_i \) are also well-behaved under pull-back by \( \pi \) (again the \( N \)-th power of \( L_i \) identifies with \( \tilde{L}_i \)). As we mentioned above, Hodge together with these \( r \) bundles generate \( \text{Pic}_{\text{fun}}(\mathcal{M}_g^r) \). Therefore (6.1) is defined and well-behaved. This homomorphism satisfies the analogue of Proposition 4.2.

6c. **Line bundles on the inductive limit** \( T^r_\infty \): Consider the inverse system of isomorphism classes of finite covers of \( X \), totally ramified over the finite subset \( S \), which allow factorization into a sequence of abelian coverings. The mappings in this directed system are those that commute with the projections onto \( X \). As in the case without punctures, one constructs thereby an interesting Riemann surface lamination by taking the inverse limit of this projective system of surfaces. This inverse limit, which we denote by \( H^r_{\infty, \text{ab}} \), is a compact space that is fibered over \( X \) with a Cantor set as fiber.

Let this projective system of covering mappings be indexed by the set \( I \), and for \( i \in I \), let \( p_i : X_i \to X \) be the covering and let \( g_i : \mathcal{T}_g^r \to \mathcal{T}_{g_i}^r \) be the induced map between Teichmüller spaces. Here \( g_i \) denotes the genus of \( X_i \).
and \( n_i \) is the order of the covering. As before, this constitutes an inductive system of Teichmüller spaces, and we have the direct limit

\[
T^r_\infty := \lim_{\to} T^r_{g_i}
\]

of these Teichmüller spaces constituting the space of “TLC” [transversely locally constant] complex structures on the Riemann surface lamination \( H^r_{\infty,ab} \).

The completion in the Teichmüller metric of the space \( T^r_\infty \) is the Teichmüller space of the \( r \)-pointed abelian lamination; it is denoted \( T(H^r_{\infty,ab}) \). \( T^r_\infty \) itself comprises the space of “TLC” complex structures on the Riemann surface lamination \( H^r_{\infty,ab} \). Notice that the various finite dimensional Teichmüller spaces for Riemann surfaces of genus \( g_i \) with \( r \) distinguished points are all contained faithfully inside this infinite dimensional “universal” Teichmüller space.

The main result, Theorem 5.1 above, as well as its proof, goes through and has the corresponding statement for each of the DET bundles. It should be noted that the Pic-generator bundles \( L_i, i = 1, \ldots, r \), also fit together coherently over the ind space \( T^r_\infty \) (as members of its rational Pic).

**Theorem 6.2: Universal DET line bundles:** There exist canonical elements of the inverse limit \( \lim_{\to} \text{Pic}(T^r_{g_i})_\mathbb{Q} \), namely hermitian line bundles on the ind space \( T^r_\infty \), representing the Hodge and higher DET bundles with respective Quillen metrics.

The pullback of \( \Lambda_m \) to each of the stratifying Teichmüller spaces \( T^r_{g_i} \) is \( (n_i)^{-1} \) times the corresponding Hodge or higher DET bundle with Quillen metric.
Appendix: Discreteness of the Picard variety of moduli spaces with marked points and level structure

Professor Eduard Looijenga (Utrecht) has sent us the following argument for:

**Theorem A:** The Picard group of the moduli space of pointed Riemann surfaces with a principal level structure is discrete. In fact, the stronger assertion that the corresponding mapping class group has trivial first cohomology (with \( \mathbb{Z} \) coefficients) holds.

**Remark:** As mentioned before, see also [Hain]. The above result is clearly of independent interest and reproves our basic Theorem 2.3, namely that “curvature determines the bundle” on such moduli spaces.

**Proof:** Let us be given a compact connected oriented surface \( S \) of genus \( g \).

Write \( V_g \) for \( H^1(S) \) and denote by \( \Gamma_g \) resp. \( Sp_g \) the mapping class group of \( S \) resp. the group of integral symplectic transformations of \( V_g \). There is a natural surjection \( \Gamma_g \to Sp_g \) whose kernel \( T_g \) is known as the Torelli group of \( S \). In a series of papers Dennis Johnson proved that if \( g \geq 3 \) (an assumption in force from now on), \( T_g \) is finitely generated and that \( H^1(T_g) = \text{Hom}(T_g, \mathbb{Z}) \) can be naturally identified with the quotient of \( \wedge^3V_g \) by \( \omega \wedge V_g \), where \( \omega \in \wedge^2V_g \) corresponds to the intersection product. (This is precisely the primitive cohomology in degree 3 of the Jacobian \( H_1(S; \mathbb{R}/\mathbb{Z}) \) of \( S \), but that fact will not play a role here.) This isomorphism is \( Sp_g \)-equivariant and so there are no nonzero elements in \( H^1(T_g) \) invariant under \( Sp_g \). This is also true for any subgroup of \( Sp_g \) of finite index; in particular it is so for the principal level \( n \) congruence subgroup \( Sp_g[n] \). If we define the subgroup \( \Gamma_g[n] \) of \( \Gamma_g \) by the exact sequence

\[
0 \to T_g \to \Gamma_g[n] \to Sp_g[n] \to 0,
\]

then what we want is the vanishing of \( H^1(\Gamma_g[n]) \). The spectral sequence associated to this exact sequence

\[
E_2^{p,q} = H^p(Sp_g[n]; H^q(T_g)) \Rightarrow H^{p+q}(\Gamma_g[n])
\]


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gives in degree 1 the short exact sequence

\[ 0 \to H^1(\text{Sp}_g[n]) \to H^1(\Gamma_g[n]) \to H^1(T_g)^{\text{Sp}_g[n]} \to \]

We noted already that the term on the right is trivial. So it remains to see that \( H^1(\text{Sp}_g[n]) \cong H^1(\text{Sp}(2g, \mathbb{Z})[n]) \) is trivial. A. Borel has shown that in this range the real cohomology of \( \text{Sp}(2g, \mathbb{Z})[n] \) can be represented by translation-invariant forms on the corresponding Siegel space. It is therefore trivial, too. (The result we are alluding to is in Theorem 7.5 in his paper [Bo], although sharper results have been obtained by him since. The constants involved in applying Theorem 7.5 can be noted from [KN] (reference [16] of [Bo]). In the case at hand, both constants “\( c \)” and “\( m \)” that are involved are at least 1, and hence our argument is valid, provided \( g \) is at least 3.)

In order to obtain the corresponding result for the moduli space of \( k \)-pointed genus \( g \) curves we fix some notation: let \( z_1, z_2, z_3, \ldots \) be a sequence of distinct points of \( S \) and put \( V_g^k := H^1(S - \{z_1, \ldots, z_k\}) \). Notice that the restriction maps give inclusions \( V_g = V_g^0 = V_g^1 \subset V_g^2 \subset \cdots \). Denote the group of automorphisms of \( V_g^k \) that leave \( V_g \subset V_g^k \) invariant, act symplectically on \( V_g \) and are the identity on \( V_g^k/V_g \) by \( \text{Sp}_g^k \) and let \( \Gamma_g^k \) stand for the connected component group of the group \( \text{Diff}^+(S, \{z_1, \ldots, z_k\}) \) of orientation preserving diffeomorphisms of \( S \) that leave \( z_1, \ldots, z_k \) pointwise fixed. It is known that the natural homomorphism \( \Gamma_g^k \to \text{Sp}_g^k \) is surjective.

We prove with induction on \( k \) that \( H^1(\Gamma_g^k[n]) \) is trivial. So assume \( k \geq 1 \) and the claim proved for \( k - 1 \). We have an exact sequence

\[ 1 \to \pi_1(S - \{z_1, \ldots, z_{k-1}\}, z_k) \to \Gamma_g^k[n] \to \Gamma_g^{k-1}[n] \to 1. \]

Its associated spectral sequence for cohomology gives the short exact sequence

\[ 0 \to H^1(\Gamma_g^{k-1}[n]) \to H^1(\Gamma_g^k[n]) \to (V_g^{k-1})^{\text{Sp}_g^{k-1}} \to \]

The left-hand term is trivial by assumption and it is easy to see that the right-hand term is trivial, too. This gives the vanishing of the middle term.
Let $\mathcal{M}_g^k$ stands for the moduli space of $k$-pointed compact Riemann surfaces of genus $g$ (Section 6a). and let $\mathcal{M}_g^k[n]$ mean the moduli with level $n$. We see from above that for $g \geq 3$, $\mathcal{M}_g^k[n]$ has vanishing first Betti number and has therefore a discrete Picard group, as desired. □

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