LIPSCHITZ TENSOR PRODUCT

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ABSTRACT. Inspired by ideas of R. Schatten in his celebrated monograph [22] on a theory of cross-spaces, we introduce the notion of a Lipschitz tensor product $X \boxtimes E$ of a pointed metric space $X$ and a Banach space $E$ as a certain linear subspace of the algebraic dual of $\text{Lip}_0(X, E^*)$. We prove that $X \boxtimes E$ is linearly isomorphic to the linear space of all finite-rank continuous linear operators from $(X^d, \tau_p)$ into $E$, where $X^d$ denotes the space $\text{Lip}_0(X, \mathbb{K})$ and $\tau_p$ is the topology of pointwise convergence of $X^d$. The concept of Lipschitz tensor product of elements of $X^d$ and $E^*$ yields the space $X^d \boxtimes E^*$ as a certain linear subspace of the algebraic dual of $X \boxtimes E$. To ensure the good behavior of a norm on $X \boxtimes E$ with respect to the Lipschitz tensor product of Lipschitz functionals (mappings) and bounded linear functionals (operators), the concept of dualizable (respectively, uniform) Lipschitz cross-norm on $X \boxtimes E$ is defined. We show that the Lipschitz injective norm $\varepsilon$, the Lipschitz projective norm $\pi$ and the Lipschitz $p$-nuclear norm $d_p$ ($1 \leq p \leq \infty$) are uniform dualizable Lipschitz cross-norms on $X \boxtimes E$. In fact, $\varepsilon$ is the least dualizable Lipschitz cross-norm and $\pi$ is the greatest Lipschitz cross-norm on $X \boxtimes E$. Moreover, dualizable Lipschitz cross-norms $\alpha$ on $X \boxtimes E$ are characterized by satisfying the relation $\varepsilon \leq \alpha \leq \pi$. In addition, the Lipschitz injective (projective) norm on $X \boxtimes E$ can be identified with the injective (respectively, projective) tensor norm on the Banach-space tensor product between the Lipschitz-free space over $X$ and $E$. In terms of the space $X^d \boxtimes E^*$, we describe the spaces of Lipschitz compact (finite-rank, approximable) operators from $X$ to $E^*$.

INTRODUCTION

The Lipschitz space $\text{Lip}_0(X, E)$ is the Banach space of all Lipschitz maps $f$ from a pointed metric space $X$ to a Banach space $E$ that vanish at the base point of $X$, under the Lipschitz norm given by

$$\text{Lip}(f) = \sup \left\{ \frac{\|f(x) - f(y)\|}{d(x, y)} : x, y \in X, x \neq y \right\}.$$ 

The elements of $\text{Lip}_0(X, E)$ are referred to as Lipschitz operators. If $\mathbb{K}$ is the field of real or complex numbers, the space $\text{Lip}_0(X, \mathbb{K})$, denoted by $X^d$, is called the Lipschitz dual of $X$. A comprehensive reference for the basic theory of the spaces of Lipschitz functions is the book [23] by N. Weaver.

The use of techniques of the theory of algebraic tensor product of Banach spaces to tackle the problem of the duality for Lipschitz operators from $X$ to $E$ goes back to the seventies with the works [17] [18] of J. A. Johnson. Recently, the second-named author [4] has adopted this approach to describe the duals of spaces of Lipschitz $p$-summing operators from $X$ to $E^*$ for $1 \leq p \leq \infty$. The notion of Lipschitz $p$-summing operators between metric spaces, a nonlinear generalization of $p$-summing operators, was introduced by J. D. Farmer and W. B. Johnson in [10], where a nonlinear version of the Pietsch factorization theorem was established. The article [10] has motivated the study of Lipschitz versions of different types of bounded linear operators in the works [4] [5] [6] [7] [16].

The reading of the paper [4] invites to give a definition for the tensor product of $X$ and $E$. In [17], J. A. Johnson proved that the dual of the closed linear subspace of $\text{Lip}_0(X, E^*)^*$ spanned by the functionals $\varepsilon, \pi$ on $\text{Lip}_0(X, E^*)^*$ with $x \in X$ and $e \in E$, defined by $(\varepsilon, \pi)(f) = (f(x), e)$, is isometrically isomorphic to $\text{Lip}_0(X, E^*)$. It is well known that the dual of the projective tensor product of Banach spaces $E$ and $F$ can be identified with the space of all bounded linear operators from $E$ to $F^*$, so the predual of $\text{Lip}_0(X, E^*)$ provided by Johnson’s result plays the
role of the projective tensor product in the linear theory and this fact suggests to call Lipschitz tensor product of $X$ and $E$ the linear subspace of the algebraic dual of $\text{Lip}_p(X, E^*)$ spanned by the functionals $\delta_x \otimes e$. In [4], this space is called the space of $E$-valued molecules on $X$, a generalization of the Arens–Eells space $\mathcal{E}(X)$ of scalar-valued molecules on $X$ (see [1] [23]).

Our purpose is to develop a theory of the Lipschitz tensor product $X \boxtimes E$ of a pointed metric space $X$ and a Banach space $E$ by following the original ideas of R. Schatten [22] used to construct the algebraic tensor product of two Banach spaces. We are also motivated by the problem of researching the spaces of Lipschitz compact (finite-rank, approximable) operators from $X$ to $E^*$ introduced in [16].

We now describe the contents of this paper. In Section 1, we introduce and study the Lipschitz tensor product $X \boxtimes E$. We show that $\langle X \boxtimes E, \text{Lip}_p(X, E^*) \rangle$ forms a dual pair and identify linearly the space $\text{Lip}_p(X, E^*)$ with a linear subspace of the algebraic dual of $X \boxtimes E$, and the space $X \boxtimes E$ with the space $\mathcal{F}(\langle X^p, \tau_p \rangle; E)$ of all finite-rank continuous linear operators from $(X^p, \tau_p)$ to $E$, where $\tau_p$ denotes the topology of pointwise convergence of $X^p$.

In Section 2, we define the concept of a Lipschitz tensor product of a Lipschitz functional $g \in X^p$ and a bounded linear functional $\phi \in E^*$ as a certain linear functional $g \boxtimes \phi$ on $X \boxtimes E$, and consider the associated Lipschitz tensor product of $X \boxtimes E$, denoted by $X^p \boxtimes E^*$, as the linear subspace of the algebraic dual of $X \boxtimes E$ spanned by the elements $g \boxtimes \phi$. It is shown that the space $X^p \boxtimes E^*$ is linearly isomorphic to the space $\text{Lip}_{p\ell}(X, E^*)$ of all Lipschitz finite-rank operators from $X$ to $E^*$.

In Section 3, we give the notion of a Lipschitz tensor product of a base-point preserving Lipschitz map between pointed metric spaces $h : X \to Y$ and a bounded linear operator between Banach spaces $T : E \to F$ as a certain linear operator $h \boxtimes T$ from $X \boxtimes E$ to $Y \boxtimes F$.

In Section 4, we consider a norm $\alpha$ on $X \boxtimes E$ and obtain a normed space $X \boxtimes_\alpha E$ and its completion $X \boxtimes_\alpha^\ast E$. We are interested in the so-called Lipschitz cross-norms which are those satisfying the condition $\alpha(\delta_x \boxtimes \phi) = d(x, y)\|\phi\|$ for all $x, y \in X$ and $\phi \in E$. Desirable attributes for Lipschitz cross-norms on $X \boxtimes E$ is that they behave well with respect to the formation of Lipschitz tensor products of functionals and operators. In this line, we study dualizable Lipschitz cross-norms and uniform Lipschitz cross-norms on $X \boxtimes E$.

In Section 5, given a dualizable Lipschitz cross-norm $\alpha$ on $X \boxtimes E$, we construct a norm $\alpha'$ on $X^p \boxtimes E^*$ called the associated Lipschitz norm of $\alpha$. The space $X^p \boxtimes E^*$ with the norm $\alpha'$ will be denoted by $X^p \map_{\alpha'} E^*$ and its completion by $X^p \map_{\alpha'}^\ast E^*$.

In Sections 6, 7, 8 and 9, we investigate, respectively, the dual norm $L$ induced on $X \boxtimes E$ by the norm $\text{Lip}_p(X, E^*)$, the Lipschitz injective norm $e$, the Lipschitz projective norm $\pi$ and the Lipschitz $p$-nuclear norm $d_p$ with $p \in [1, \infty]$. All these norms on $X \boxtimes E$ are uniform and dualizable Lipschitz cross-norms. In fact, $e$ is the least dualizable Lipschitz cross-norm and $\pi$ is the greatest Lipschitz cross-norm on $X \boxtimes E$. Furthermore, dualizable Lipschitz cross-norms $\alpha$ on $X \boxtimes E$ are characterized as those which satisfy the relation $\varepsilon \leq \alpha \leq \pi$. We also prove that $L$ agrees with $\pi$ and justify the terminologies “injective” and “projective” for the Lipschitz norms $\varepsilon$ and $\pi$, respectively. We identify $X \boxtimes_\varepsilon E$ and its completion $X \boxtimes_\varepsilon^\ast E$ with $\mathcal{F}(\langle X^p, \tau_p \rangle; E)$ and its closure in the operator norm topology, respectively. We also show that the Lipschitz injective (projective) norm on $X \boxtimes E$ can be identified with the injective (respectively, projective) tensor norm on the Banach-space tensor product between the Lipschitz-free space over $X$ and $E$.

In Section 10, we deal with the space $\text{Lip}_{p\ell}(X, E^*)$ of all Lipschitz finite-rank operators from $X$ to $E^*$ and its closure in the Lipschitz norm topology, the space of Lipschitz approximable operators from $X$ to $E^*$. It is proved that the former space is isometrically isomorphic to $X^p \map_{\varepsilon'} E^*$, and the latter to $X^p \map_{\pi'} E^*$. The approximation property for Banach spaces was introduced by Grothendieck in his famous memory [15]. We show that if $X^p$ has the approximation property, then the space of all Lipschitz compact operators from $X$ to $E^*$ is isometrically isomorphic to $X^p \map_{\pi'} E^*$ for any Banach space $E$. We close the paper giving a new expression of the norm $\pi'$.

**Notation.** Given two metric spaces $(X, d_X)$ and $(Y, d_Y)$, let us recall that a map $f : X \to Y$ is said to be Lipschitz if there is a real constant $C \geq 0$ such that $d_Y(f(x), f(y)) \leq C d_X(x, y)$ for all $x, y \in X$. The least constant $C$ for which
the preceding inequality holds will be denoted by Lip$(f)$, that is,
\[
\text{Lip}(f) = \sup \left\{ \frac{d_y(f(x), f(y))}{d_x(x, y)} : x, y \in X, \ x \neq y \right\}.
\]

A pointed metric space $X$ is a metric space with a base point in $X$ which we always will denote by 0. We will consider a Banach space $E$ over $\mathbb{K}$ as a pointed metric space with the zero vector as the base point. As is customary, $B_E$ and $S_E$ stand for the closed unit ball of $E$ and the unit sphere of $E$, respectively. Given two pointed metric spaces $X$ and $Y$, Lip$_0(X, Y)$ denotes the set of all base-point preserving Lipschitz maps from $X$ to $Y$.

For two linear spaces $E$ and $F$, Lip$(E, F)$ stands for the linear space of all linear operators from $E$ into $F$. In the case that $E$ and $F$ are Banach spaces, we denote by Lip$(E, F)$ the Banach space of all bounded linear operators from $E$ into $F$ with the usual norm, and by $\mathcal{F}(E, F)$ its subspace of finite-rank bounded linear operators. In particular, the algebraic dual $\mathcal{L}(E, \mathbb{K})$ and the topological dual Lip$(E, \mathbb{K})$ are denoted by $E'$ and $E^{**}$, respectively. For each $e \in E$ and $\phi \in E'$, we frequently will write $\langle \phi, e \rangle$ instead of $\phi(e)$.

1. Lipschitz Tensor Products

The Lipschitz tensor product of a pointed metric space $X$ and a Banach space $E$, which we will denote from now on by $X \otimes E$, can be constructed as a space of linear functionals on Lip$_0(X, E^{**})$.

**Definition 1.1.** Let $X$ be a pointed metric space and $E$ a Banach space. For each $x \in X$, let $\delta_{(x,0)} : \text{Lip}_0(X, E^{**}) \to E^{**}$ be the linear map defined by
\[
\delta_{(x,0)}(f) = f(x) \quad (f \in \text{Lip}_0(X, E^{**})).
\]

For each $(x, y) \in X^2$, let $\delta_{(x,y)} : \text{Lip}_0(X, E^{**}) \to E^{**}$ be the linear map defined by
\[
\delta_{(x,y)} = \delta_{(x,0)} - \delta_{(y,0)}.
\]

Let $\Delta(X, E^{**})$ denote the linear subspace of Lip$(\text{Lip}_0(X, E^{**}), E^{**})$ spanned by the set $\{\delta_{(x,y)} : (x, y) \in X^2\}$. For any $\gamma \in \Delta(X, E^{**})$ and $e \in E$, let $\gamma \otimes e : \text{Lip}_0(X, E^{**}) \to \mathbb{K}$ be the linear functional given by
\[
(\gamma \otimes e)(f) = \langle \gamma(f), e \rangle \quad (f \in \text{Lip}_0(X, E^{**})).
\]

In particular, for any $(x, y) \in X^2$ and $e \in E$, let $\delta_{(x,y)} \otimes e$ be the element of Lip$_0(X, E^{**})'$ defined by
\[
(\delta_{(x,y)} \otimes e)(f) = \langle \delta_{(x,y)}(f), e \rangle = \langle \delta_{(x,0)}(f) - \delta_{(y,0)}(f), e \rangle = \langle f(x) - f(y), e \rangle, \quad \forall f \in \text{Lip}_0(X, E^{**}).
\]

The Lipschitz tensor product $X \otimes E$ is defined as the vector subspace of Lip$_0(X, E^{**})'$ spanned by the set
\[
\{\delta_{(x,y)} \otimes e : (x, y) \in X^2, \ e \in E\}.
\]

The following properties of the Lipschitz tensor product can be checked easily.

**Lemma 1.1.** Let $\lambda \in \mathbb{K}$, $(x, y), (x_1, y_1), (x_2, y_2) \in X^2$ and $e, e_1, e_2 \in E$.

(i) $\lambda \left( \delta_{(x,y)} \otimes e \right) = (\lambda \delta_{(x,y)}) \otimes e = \delta_{(x,y)} \otimes (\lambda e)$.

(ii) $\delta_{(x_1, y_1)} + \delta_{(x_2, y_2)} \otimes e = \delta_{(x_1, y_1)} \otimes e + \delta_{(x_2, y_2)} \otimes e$.

(iii) $\delta_{(x,y)} \otimes (e_1 + e_2) = \delta_{(x,y)} \otimes e_1 + \delta_{(x,y)} \otimes e_2$.

(iv) $\delta_{(x,y)} \otimes e = \delta_{(x,y)} \otimes 0 = 0$.

We say that $\delta_{(x,y)} \otimes e$ is an elementary Lipschitz tensor. Note that each element $u$ in $X \otimes E$ is of the form $u = \sum_{i=1}^{n} \lambda_i \delta_{(x_i, y_i)} \otimes e_i$, where $n \in \mathbb{N}$, $\lambda_i \in \mathbb{K}$, $(x_i, y_i) \in X^2$ and $e_i \in E$. This representation of $u$ is not unique. It is worth noting that each element $u$ of $X \otimes E$ can be represented as $u = \sum_{i=1}^{n} \delta_{(x_i, y_i)} \otimes e_i$ since $\lambda(\delta_{(x,y)} \otimes e) = \delta_{(x,y)} \otimes (\lambda e)$. This representation of $u$ admits the following refinement.
Lemma 1.2. Every nonzero Lipschitz tensor \( u \in X \otimes E \) has a representation in the form \( \sum_{i=1}^{m} \delta_{(z_i,0)} \otimes d_i \), where
\[
m = \min \left\{ k \in \mathbb{N} : \exists z_1, \ldots, z_k \in X, d_1, \ldots, d_k \in E \mid u = \sum_{i=1}^{k} \delta_{(z_i,0)} \otimes d_i \right\}
\]
and the points \( z_1, \ldots, z_m \) in \( X \) are distinct from the base point \( 0 \) of \( X \) and pairwise distinct.

Proof. Let \( u \in X \otimes E \) and let \( \sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i \) be a representation of \( u \). We have
\[
u = \sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i = \sum_{i=1}^{n} (\delta_{(x_i,0)} - \delta_{(y_i,0)}) \otimes e_i = \sum_{i=1}^{n} \delta_{(x_i,0)} \otimes e_i + \sum_{i=1}^{n} \delta_{(y_i,0)} \otimes (-e_i) = \sum_{i=1}^{2n} \delta_{(x_i,0)} \otimes e_i + \sum_{i=n+1}^{2n} \delta_{(y_i,0)} \otimes (-e_{i-n}) = \sum_{i=1}^{2n} \delta_{(z_i,0)} \otimes d_i,
\]
where
\[
\delta_{(z_i,0)} \otimes d_i = \begin{cases} \delta_{(x_i,0)} \otimes e_i & \text{if } i = 1, \ldots, n, \\ \delta_{(y_i,0)} \otimes (-e_{i-n}) & \text{if } i = n + 1, \ldots, 2n. \end{cases}
\]

Then, by the well-ordering principle of \( \mathbb{N} \), there exists a smallest natural number \( m \) for which there is a representation of \( u \) in the form \( \sum_{i=1}^{m} \delta_{(z_i,0)} \otimes d_i \). Since \( u \neq 0 \), it is clear that \( z_i \neq 0 \) for some \( i \in \{1, \ldots, m\} \). This implies that \( z_j \neq 0 \) for all \( j \in \{1, \ldots, m\} \). Otherwise, observe that \( \sum_{i=1}^{m} \delta_{(z_i,0)} \otimes d_i \) would be a representation of \( u \) containing \( m - 1 \) terms and this contradicts the definition of \( m \). Moreover, if \( z_j = z_k \) for some \( j, k \in \{1, \ldots, m\} \) with \( j \neq k \), we would have
\[
u = \sum_{i=1}^{m} \delta_{(z_i,0)} \otimes d_i + \left( \delta_{(z_j,0)} \otimes (d_j + d_k) \right),
\]
and this is impossible. Hence the points \( z_i \) are pairwise distinct. \( \square \)

We can concatenate the representations of two elements of \( X \otimes E \) to get a representation of their sum.

Lemma 1.3. Let \( u, v \in X \otimes E \) and let \( \sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i \) and \( \sum_{i=1}^{m} \delta_{(x_i',y_i')} \otimes e_i' \) be representations of \( u \) and \( v \), respectively. Then \( \sum_{i=1}^{n+m} \delta_{(x_i',y_i')} \otimes e_i' \), where
\[
(\alpha_i, \beta_i) = \begin{cases} (x_i, y_i) & \text{if } i = 1, \ldots, n, \\ (x_{i-n}, y_{i-n}) & \text{if } i = n + 1, \ldots, n + m, \end{cases} \quad e_i' = \begin{cases} e_i & \text{if } i = 1, \ldots, n, \\ e_{i-n} & \text{if } i = n + 1, \ldots, n + m, \end{cases}
\]
is a representation of \( u + v \).

We now describe the action of a Lipschitz tensor \( u \in X \otimes E \) on a function \( f \in \text{Lip}_0(X, E^*) \).

Lemma 1.4. Let \( u = \sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i \in X \otimes E \) and \( f \in \text{Lip}_0(X, E^*) \). Then
\[
u(f) = \sum_{i=1}^{n} \langle f(x_i) - f(y_i), e_i \rangle.
\]

Our next aim is to characterize the zero Lipschitz tensor. For it we need the following Lipschitz operators.
Lemma 1.5. Let $g \in X^*$ and $\phi \in E^*$. The map $g \cdot \phi: X \to E^*$, given by $(g \cdot \phi)(x) = g(x)\phi$, belongs to $\text{Lip}_0(X, E^*)$ and $\text{Lip}(g \cdot \phi) = \text{Lip}(g)\|\phi\|$.  

Proof. Clearly, $g \cdot \phi$ is well defined. Let $x, y \in X$. For any $e \in E$, we obtain 
$$||(g \cdot \phi)(x) - (g \cdot \phi)(y), e|| = ||(g(x) - g(y)) \phi, e|| = ||g(x) - g(y)|| ||\phi|| |e||,$$
and so $||(g \cdot \phi)(x) - (g \cdot \phi)(y)|| \leq \text{Lip}(g)\|\phi\| |d(x, y)|$. Then $g \cdot \phi \in \text{Lip}_0(X, E^*)$ and $\text{Lip}(g \cdot \phi) \leq \text{Lip}(g)\|\phi\|$. For the converse inequality, note that 
$$\|g(x) - g(y)||\phi|| = ||(g \cdot \phi)(x) - (g \cdot \phi)(y)|| \leq \text{Lip}(g \cdot \phi)d(x, y)$$
for all $x, y \in X$, and therefore $\text{Lip}(g \cdot \phi) \leq \text{Lip}(g \cdot \phi)$. \hfill $\Box$

Proposition 1.6. If $u = \sum_{i=1}^n \delta_{(x_i,y_i)} \otimes e_i \in X \otimes E$, then the following assertions are equivalent:  

(i) $u = 0$.
(ii) $\sum_{i=1}^n (g(x_i) - g(y_i)) \langle \phi, e_i \rangle = 0$ for every $g \in B_{X^*}$ and $\phi \in B_{E^*}$.
(iii) $\sum_{i=1}^n (g(x_i) - g(y_i)) e_i = 0$ for every $g \in B_{X^*}$.

Proof. (i) implies (ii): If $u = 0$, then $u(f) = 0$ for all $f \in \text{Lip}_0(X, E^*)$. Since $u = \sum_{i=1}^n \delta_{(x_i,y_i)} \otimes e_i$, it follows that 
$$\sum_{i=1}^n (f(x_i) - f(y_i)) \langle \phi, e_i \rangle = 0$$
for all $f \in \text{Lip}_0(X, E^*)$ by Lemma 1.4. For any $g \in B_{X^*}$ and $\phi \in B_{E^*}$, the function $g \cdot \phi$ is in $\text{Lip}_0(X, E^*)$ by Lemma 1.5, and therefore we have
$$\sum_{i=1}^n (g(x_i) - g(y_i)) \langle \phi, e_i \rangle = \sum_{i=1}^n (g(x_i) - g(y_i)) \phi, e_i = \sum_{i=1}^n (g \cdot \phi)(x_i) - (g \cdot \phi)(y_i), e_i = 0.$$

(ii) implies (iii): If (ii) holds, then $\langle \phi, \sum_{i=1}^n (g(x_i) - g(y_i)) e_i \rangle = 0$ for every $g \in B_{X^*}$ and $\phi \in B_{E^*}$. Since $B_{E^*}$ separates the points of $E$, it follows that $\sum_{i=1}^n (g(x_i) - g(y_i)) e_i = 0$ for all $g \in B_{X^*}$.

(iii) implies (i): By Lemma 1.2, we can write $u = \sum_{i=1}^m \delta_{(z_i,0)} \otimes d_i$, where the points $z_i$ in $X$ are pairwise distinct and different from the base point 0. It follows that 
$$\sum_{i=1}^n \delta_{(x_i,y_i)} \otimes e_i + \sum_{i=1}^m \delta_{(z_i,0)} \otimes (d_i) = u - u = 0,$$
and, by using the fact proved above that (i) implies (iii), we have 
$$\sum_{i=1}^n (g(x_i) - g(y_i)) e_i + \sum_{i=1}^m (g(z_i) - g(0)) (-d_i) = 0$$
for all $g \in B_{X^*}$. If (iii) holds, we get that 
$$\sum_{i=1}^m g(z_i) d_i = \sum_{i=1}^n (g(x_i) - g(y_i)) e_i = 0$$
for all $g \in B_{X^*}$. Set 
$$r = \min \left\{ \{d(z_i, z_j): i, j \in \{1, \ldots, m\}, i \neq j\} \cup \{d(z_i, 0): i \in \{1, \ldots, m\}\} \right\}.$$ 
Clearly, $r > 0$. Given $j \in \{1, \ldots, m\}$, define $g_j: X \to \mathbb{R}$ by
$$g_j(x) = \max \left\{ 0, r - d(x, z_j) \right\}.$$ 
It is easy to check that $g_j \in B_{X^*}$, $g_j(z_j) = r$ and $g_j(z) = 0$ for all $i \in \{1, \ldots, m\} \setminus \{j\}$. Hence $0 = \sum_{i=1}^m g_j(z_i) d_i = rd_j$, therefore $d_1 = d_2 = \cdots = d_m = 0$ and thus $u = 0$. \hfill $\Box$

According to Definition 1.1, $X \otimes E$ is a linear subspace of $\text{Lip}_0(X, E^*)'$. Furthermore, we have the next fact.
Theorem 1.7. \((X \otimes E, \text{Lip}_0(X, E^*))\) forms a dual pair, where the bilinear form \((\cdot, \cdot)\) associated to the dual pair is given, for \(u = \sum_{i=1}^n \delta_{(x_i, y_i)} \otimes e_i \in X \otimes E\) and \(f \in \text{Lip}_0(X, E^*)\), by
\[
\langle u, f \rangle = \sum_{i=1}^n (f(x_i) - f(y_i), e_i).
\]

Proof. Note that \(\langle u, f \rangle = u(f)\) by Lemma 1.4. It is plain that \((\cdot, \cdot) : (X \otimes E) \times \text{Lip}_0(X, E^*) \to \mathbb{K}\) is a well-defined bilinear map and that \(\text{Lip}_0(X, E^*)\) separates points of \(X \otimes E\). Moreover, if \(f \in \text{Lip}_0(X, E^*)\) and \(\langle u, f \rangle = 0\) for all \(u \in X \otimes E\), then \(\langle f(x), e \rangle = \langle \delta_{(x, 0)} \otimes e, f \rangle = 0\) for all \(x \in X\) and \(e \in E\). This implies that \(f = 0\) and thus \(X \otimes E\) separates points of \(\text{Lip}_0(X, E^*)\). This completes the proof of the theorem.

Since \((X \otimes E, \text{Lip}_0(X, E^*))\) is a dual pair, \(\text{Lip}_0(X, E^*)\) can be identified with a linear subspace of \((X \otimes E)^*\) as follows.

Corollary 1.8. For every map \(f \in \text{Lip}_0(X, E^*)\), the functional \(\Lambda(f) : X \otimes E \to \mathbb{K}\), given by
\[
\Lambda(f)(u) = \sum_{i=1}^n (f(x_i) - f(y_i), e_i)
\]
for \(u = \sum_{i=1}^n \delta_{(x_i, y_i)} \otimes e_i \in X \otimes E\), is linear. We say that \(\Lambda(f)\) is the linear functional on \(X \otimes E\) associated to \(f\). The map \(f \mapsto \Lambda(f)\) is a linear monomorphism from \(\text{Lip}_0(X, E^*)\) into \((X \otimes E)^*\).

Proof. Let \(f \in \text{Lip}_0(X, E^*)\). By Theorem 1.7, note that \(\Lambda(f)(u) = \langle u, f \rangle\) for all \(u \in X \otimes E\). It is immediate that \(\Lambda(f)\) is a well-defined linear functional on \(X \otimes E\) and that \(f \mapsto \Lambda(f)\) from \(\text{Lip}_0(X, E^*)\) into \((X \otimes E)^*\) is a well-defined linear map. Finally, let \(f \in \text{Lip}_0(X, E^*)\) and assume that \(\Lambda(f) = 0\). Then \(\langle u, f \rangle = 0\) for all \(u \in X \otimes E\). Since \(X \otimes E\) separates points of \(\text{Lip}_0(X, E^*)\), it follows that \(f = 0\) and this proves that the map \(\Lambda\) is one-to-one.

We next show that \((X \otimes E, \text{Lip}_0(X, E^*))\) is linearly isomorphic to the linear space \(\mathcal{F}((X^p, \tau_p); E)\) of all finite-rank linear operators from \(X^p\) into \(E\) which are continuous from the topology of pointwise convergence \(\tau_p\) of \(X^p\) to the norm topology of \(E\).

Definition 1.2. Let \(X\) be a pointed metric space. For \(f \in X^p\), \(x \in X\) and \(\varepsilon \in \mathbb{R}^+\), we put
\[
B(f, x, \varepsilon) = \{g \in X^p : |g(x) - f(x)| < \varepsilon\}.
\]
Let \(S\) be the family of sets \(\{B(f, x, \varepsilon) : f \in X^p, x \in X, \varepsilon \in \mathbb{R}^+\}\). Then the topology of pointwise convergence \(\tau_p\) on \(X^p\) is the topology generated by \(S\).

We can check that \((X^p, \tau_p)\) is a locally convex space. Next we describe its dual space.

Lemma 1.9. Let \(X\) be a pointed metric space. Then \((X^p, \tau_p)^* = \text{lin}(\delta_x : x \in X) \subset (X^p)^*\), where \(\delta_x\) is the functional on \(X^p\) defined by \(\delta_x(g) = g(x)\).

Proof. Define the linear functional \(T : X^p \to \mathbb{K}\) by \(T(g) = \sum_{i=1}^n \lambda_i g(x_i)\) for all \(g \in X^p\), where \(n \in \mathbb{N}\), \(\lambda_1, \ldots, \lambda_n \in \mathbb{K}\) and \(x_1, \ldots, x_n \in X\). Put \(r = 1 + \sum_{i=1}^n |\lambda_i|\) and let \(\varepsilon > 0\) be arbitrary. If \(g \in \bigcap_{i=1}^n B(0, x_i, \varepsilon/r)\), then \(|T(g)| \leq \sum_{i=1}^n |\lambda_i||g(x_i)| < \varepsilon\). This proves that \(T\) is continuous on \(X^p\) when it is equipped with the topology \(\tau_p\). Conversely, we need to show that every element \(S\) in \((X^p, \tau_p)^*\) is of that form. Since \(S\) is continuous in the \(\tau_p\)-topology, there is an open neighborhood \(V\) of 0 such that \(|S(g)| < 1\) for all \(g \in V\). We can suppose that \(V = \bigcap_{i=1}^n B(0, x_i, \varepsilon)\) for suitable \(n \in \mathbb{N}\), \(x_1, \ldots, x_n \in X\) and \(\varepsilon > 0\). Take \(f \in \bigcap_{i=1}^n \ker \delta_{x_i}\). Then \(mf \in V\) for each \(m \in \mathbb{N}\). By the linearity of \(S\), it follows that \(|S(f)| < 1/m\) for all \(m \in \mathbb{N}\) and so \(S(f) = 0\). This shows that \(\bigcap_{i=1}^n \ker \delta_{x_i} \subset \ker S\) and the lemma follows from a known fact of linear algebra.

Theorem 1.10. The map \(J : X \otimes E \to \mathcal{F}((X^p, \tau_p); E)\), given by
\[
J(u)(g) = \sum_{i=1}^n (g(x_i) - g(y_i)) e_i
\]
for $u = \sum_{i=1}^{n} \delta_{(x_i, y_i)} \otimes e_i \in X \otimes E$ and $g \in X^{\#}$, is a linear isomorphism.

**Proof.** Let $u = \sum_{i=1}^{n} \delta_{(x_i, y_i)} \otimes e_i \in X \otimes E$. It is immediate that $J(u) : X^{\#} \to E$ is well defined and linear. Note that $J(u)(X^{\#}) \subset \{e_1, \ldots, e_n\}$ and hence $J(u)$ has finite-dimensional range. In order to prove that $J(u)$ is continuous from $(X^{\#}, \tau_p)$ to $E$, it is sufficient to see that $J(u)$ is continuous at $0$. Let $\varepsilon > 0$. Denote $r = 2(1 + \sum_{i=1}^{n} \|e_i\|)$ and put $z_i = x_i$ for $i = 1, \ldots, n$ and $z_i = y_{i-n}$ for $i = n + 1, \ldots, 2n$. Take $U = r^{2n}B(0, z_i, \varepsilon/r)$. For any $g \in U$, we have

$$
\|J(u)(g)\| \leq \sum_{i=1}^{n} (\|g(x_i)\| + \|g(y_i)\|) \|e_i\| \leq \sum_{i=1}^{n} 2\varepsilon \|e_i\| < \varepsilon,
$$

as required. Moreover, by Proposition 1.6, the map $u \mapsto J(u)$ from $X \otimes E$ to $\mathcal{F}((X^{\#}, \tau_p); E)$ is well defined. We now show that $J$ is linear. If $A \in \mathbb{K}$, then $Au = \sum_{i=1}^{n} \delta_{(x_i, y_i)} \otimes (\lambda e_i)$ and so

$$
J(Au)(g) = \sum_{i=1}^{n} (g(x_i) - g(y_i)) \lambda e_i = \lambda \sum_{i=1}^{n} (g(x_i) - g(y_i)) e_i = \lambda J(u)(g)
$$

for all $g \in X^{\#}$. Now, let $v = \sum_{i=1}^{n} \delta_{(x_i', y_i')} \otimes e_i' \in X \otimes E$ and take the representation $\sum_{i=1}^{n+m} \delta_{(x'_i, y'_i)} \otimes e'_i$ of $u + v$ given in Lemma 1.3. Then we have

$$
J(u + v)(g) = \sum_{i=1}^{n+m} (g(x'_i) - g(y'_i)) e'_i
$$

for all $g \in X^{\#}$. It remains to show that $J$ is bijective. On one hand, assume that $J(u) = 0$. Then $J(u)(g) = \sum_{i=1}^{n} (g(x_i) - g(y_i)) e_i = 0$ for all $g \in X^{\#}$, this implies that $u = 0$ by Proposition 1.6 and so $J$ is one-to-one. On the other hand, if $T \in \mathcal{F}((X^{\#}, \tau_p); E)$, take a basis $\{e_1, \ldots, e_n\}$ of $T(X^{\#})$. For each $g \in X^{\#}$, there are unique $\lambda_1^{(g)}, \ldots, \lambda_n^{(g)} \in \mathbb{K}$ such that $T(g) = \sum_{i=1}^{n} \lambda_i^{(g)} e_i$. For each $i \in \{1, \ldots, n\}$, let $y^i : T(X^{\#}) \to \mathbb{K}$ be given by $y^i(T(g)) = \lambda_i^{(g)}$. The uniqueness of the representation of each element of $T(X^{\#})$ implies that each $y^i$ is linear. Hence $y^i$ is continuous on $T(X^{\#})$ since the linear space $T(X^{\#})$ is finite-dimensional. Then each $T_i = y^i \circ T$ belongs to $(X^{\#, \tau_p})^*$ and $T(g) = \sum_{i=1}^{n} T_i(g)e_i$ for all $g \in X^{\#}$. Since $(X^{\#, \tau_p})^* = \text{lin}(\{\delta_x : x \in X\})$ by Lemma 1.9, for each $i \in \{1, \ldots, n\}$ there are $m(i) \in \mathbb{N}$, $x_1^{(i)}, \ldots, x_{m(i)}^{(i)} \in \mathbb{K}$ and $x_1^{(i)}, \ldots, x_{m(i)}^{(i)} \in X$ such that $T_i = \sum_{j=1}^{m(i)} \lambda_j^{(i)} \delta_{x_j^{(i)}}$. Then, for each $g \in X^{\#}$, we may write

$$
T(g) = \sum_{i=1}^{n} T_i(g)e_i = \sum_{m} g(x_j)u_j = J\left(\sum_{j=1}^{m} \delta_{(x_j, 0)} \otimes u_j\right)(g)
$$

for certain $m \in \mathbb{N}$, $x_1, \ldots, x_m \in X$ and $u_1, \ldots, u_m \in E$. This proves that $J$ is onto.

**Corollary 1.11.** Let $X$ be a pointed metric space. If $E$ is a finite-dimensional Banach space, then $X \otimes E$ is linearly isomorphic to $L((X^{\#}, \tau_p); E)$. In particular, $X \otimes \mathbb{K}$ is linearly isomorphic to $(X^{\#}, \tau_p)^* = \text{lin}(\{\delta_x : x \in X\})$. □
2. Lipschitz tensor product functionals

We introduce the concept of Lipschitz tensor product functional of a Lipschitz functional and a bounded linear functional.

**Definition 2.1.** Let $X$ be a pointed metric space and $E$ a Banach space. Let $g \in X^*$ and $\phi \in E^*$. The map $g \boxtimes \phi : X \boxtimes E \to \mathbb{K}$, given by

$$(g \boxtimes \phi)(u) = \sum_{i=1}^{n} (g(x_i) - g(y_i)) \langle \phi, e_i \rangle$$

for $u = \sum_{i=1}^{n} \delta_{(x_i,y_i)} \boxtimes e_i \in X \boxtimes E$, is called the Lipschitz tensor product functional of $g$ and $\phi$.

By Lemma 1.4, note that

$$(g \boxtimes \phi)(u) = \sum_{i=1}^{n} ((g \cdot \phi)(x_i) - (g \cdot \phi)(y_i), e_i) = u(g \cdot \phi).$$

The following result which follows easily from this formula gathers some properties of these functionals.

**Lemma 2.1.** Let $g \in X^*$ and $\phi \in E^*$. The functional $g \boxtimes \phi : X \boxtimes E \to \mathbb{K}$ is a well-defined linear map satisfying $\lambda (g \boxtimes \phi) = (\lambda g) \boxtimes \phi = g \boxtimes (\lambda \phi)$ for any $\lambda \in \mathbb{K}$. Moreover, $(g_1 + g_2) \boxtimes \phi = g_1 \boxtimes \phi + g_2 \boxtimes \phi$ for all $g_1, g_2 \in X^*$ and $g \boxtimes (\phi_1 + \phi_2) = g \boxtimes \phi_1 + g \boxtimes \phi_2$ for all $\phi_1, \phi_2 \in E^*$.

**Definition 2.2.** Let $X$ be a pointed metric space and $E$ a Banach space. The space $X^* \boxtimes E^*$ is defined as the linear subspace of $(X \boxtimes E)^*$ spanned by the set $\{g \boxtimes \phi : g \in X^*, \phi \in E^*\}$. This space is called the associated Lipschitz tensor product of $X \boxtimes E$.

From the aforementioned formula we also derive easily the following fact.

**Lemma 2.2.** For any $\sum_{j=1}^{m} g_j \boxtimes \phi_j \in X^* \boxtimes E^*$ and $\sum_{i=1}^{n} \delta_{(x_i,y_i)} \boxtimes e_i \in X \boxtimes E$, we have

$$\left( \sum_{j=1}^{m} g_j \boxtimes \phi_j \right) \left( \sum_{i=1}^{n} \delta_{(x_i,y_i)} \boxtimes e_i \right) = \sum_{j=1}^{m} \sum_{i=1}^{n} \delta_{(x_i,y_j)} \boxtimes e_j.$$

Each element $u^*$ in $X^* \boxtimes E^*$ has the form $u^* = \sum_{j=1}^{m} \lambda_j (g_j \boxtimes \phi_j)$, where $m \in \mathbb{N}$, $\lambda_j \in \mathbb{K}$, $g_j \in X^*$ and $\phi_j \in E^*$, but this representation is not unique. Since $\lambda (g \boxtimes \phi) = (\lambda g) \boxtimes \phi = g \boxtimes (\lambda \phi)$, each element of $X^* \boxtimes E^*$ can be expressed as $\sum_{j=1}^{m} g_j \boxtimes \phi_j$. This representation can be improved as follows.

**Lemma 2.3.** Every nonzero element $u^*$ in $X^* \boxtimes E^*$ has a representation $\sum_{j=1}^{m} g_j \boxtimes \phi_j$ such that the functions $g_1, \ldots, g_m$ in $X^*$ are nonzero and the functionals $\phi_1, \ldots, \phi_m$ in $E^*$ are linearly independent.

**Proof.** Let $u^* \in X^* \boxtimes E^*$, $u^* \neq 0$. Since $0 \boxtimes \phi = 0$, we can take a representation for $u^*$, $\sum_{i=1}^{n} h_i \boxtimes \phi_i$, where $h_1, \ldots, h_n$ are nonzero. If the vectors $\phi_1, \ldots, \phi_n$ are linearly independent, we have finished. Otherwise, take $F = \text{lin}(\{\phi_1, \ldots, \phi_n\})$ and choose a subset of $\{\phi_1, \ldots, \phi_n\}$, which is a basis for $F$, $\phi_1, \ldots, \phi_p$ (after reordering) for some $p < n$. For each $i \in \{p+1, \ldots, n\}$ we can express the vector $\phi_i$ as a unique linear combination in the form...
\[
\phi_i = \sum_{k=1}^p \lambda_i^{(k)} \phi_k, \text{ where } \lambda_1^{(1)}, \ldots, \lambda_p^{(1)} \in \mathbb{K}. \] Using Lemma 2.1, we can write
\[
u^* = \sum_{i=1}^p h_i \otimes \phi_i + \sum_{i=p+1}^n h_i \otimes \phi_i \]
\[
= \sum_{i=1}^p h_i \otimes \phi_i + \sum_{i=p+1}^n \left( \sum_{k=1}^p \lambda_i^{(k)} \phi_k \right) \]
\[
= \sum_{i=1}^p h_i \otimes \phi_i + \sum_{k=1}^p \left( \sum_{i=p+1}^n \lambda_i^{(k)} (h_i \otimes \phi_k) \right) \]
\[
= \sum_{i=1}^p h_i \otimes \phi_i + \sum_{k=1}^p \left( \sum_{i=p+1}^n \lambda_i^{(k)} h_i \right) \otimes \phi_k \]
\[
= \sum_{j=1}^m \left( h_j + \sum_{i=p+1}^n \lambda_i^{(j)} h_i \right) \otimes \phi_j. \]
Denote \( g_j = h_j + \sum_{i=p+1}^n \lambda_i^{(j)} h_i \) for each \( j \in \{1, \ldots, p\} \). Since \( \nu^* \neq 0 \), after reordering, we can take \( m \leq p \) for which \( g_j \neq 0 \) for all \( j \leq m \) and \( g_j = 0 \) for all \( j > m + 1 \). Then \( \sum_{j=1}^m g_j \otimes \phi_j \) is a representation of \( \nu^* \) satisfying the required conditions.

Our next aim is to show that the associated Lipschitz tensor product \( X^\# \boxtimes E^* \) is linearly isomorphic to the space of Lipschitz finite-rank operators from \( X \) to \( E^* \). This class of Lipschitz operators appears in [17, 16].

Let us recall that if \( X \) is a set and \( E \) is a vector space, then a map \( f : X \to E \) is said to have finite-dimensional rank if the subspace of \( E \) generated by \( f(X) \), \( \text{lin}(f(X)) \), is finite-dimensional in whose case the rank of \( f \), denoted by \( \text{rank}(f) \), is defined as the dimension of \( \text{lin}(f(X)) \).

For a pointed metric space \( X \) and a Banach space \( E \), we denote by \( \text{Lip}_0(X,E^*) \) the set of all Lipschitz finite-rank operators from \( X \) to \( E^* \). Clearly, \( \text{Lip}_0(X,E^*) \) is a linear subspace of \( \text{Lip}_0(X,E^*) \). For any \( g \in X^\# \) and \( \phi \in E^* \), we consider in Lemma 1.5 the elements \( g \cdot \phi \) of \( \text{Lip}_0(X,E^*) \) defined by \( (g \cdot \phi)(x) = g(x)\phi \) for all \( x \in X \). Note that \( \text{rank}(g \cdot \phi) = 1 \) if \( g \neq 0 \) and \( \phi \neq 0 \). Now we prove that these elements generate linearly the space \( \text{Lip}_0(X,E^*) \).

**Lemma 2.4.** Every element \( f \in \text{Lip}_0(X,E^*) \) has a representation in the form \( f = \sum_{j=1}^m g_j \cdot \phi_j \), where \( m = \text{rank}(f) \), \( g_1, \ldots, g_m \in X^\# \) and \( \phi_1, \ldots, \phi_m \in E^* \).

**Proof.** Suppose that \( \text{lin}(f(X)) \) is \( m \)-dimensional and let \( \{\phi_1, \ldots, \phi_m\} \) be a basis of \( \text{lin}(f(X)) \). Then, for each \( x \in X \), the element \( f(x) \in f(X) \) is expressible in a unique form as \( f(x) = \sum_{j=1}^m \lambda_j^{(x)} \phi_j \) with \( \lambda_1^{(x)}, \ldots, \lambda_m^{(x)} \in \mathbb{K} \). For each \( j \in \{1, \ldots, m\} \), define the linear map \( y^j : \text{lin}(f(X)) \to \mathbb{K} \) by \( y^j(f(x)) = \lambda_j^{(x)} \) for all \( x \in X \). Let \( g_j = y^j \circ f \). Clearly, \( g_j \in X^\# \) and, given \( x \in X \), we have \( f(x) = \sum_{j=1}^m \lambda_j^{(x)} \phi_j = \sum_{j=1}^m g_j(x) \phi_j \). Hence \( f = \sum_{j=1}^m g_j \cdot \phi_j \).

**Theorem 2.5.** The map \( K : X^\# \boxtimes E^* \to \text{Lip}_0(X,E^*) \), defined by
\[
K \left( \sum_{j=1}^m g_j \otimes \phi_j \right) = \sum_{j=1}^m g_j \cdot \phi_j, \]
is a linear isomorphism.

**Proof.** The map \( K \) is well defined by applying Lemma 2.2 and Theorem 1.7. Clearly, \( K \) is linear. Moreover, it is surjective by Lemma 2.4 and injective by Lemma 2.2.
3. Lipschitz tensor product operators

We introduce the concept of Lipschitz tensor product operator of a Lipschitz operator and a bounded linear operator.

**Definition 3.1.** Let $X, Y$ be pointed metric spaces and $E, F$ Banach spaces. Let $h \in \text{Lip}_0(X, Y)$ and $T \in \mathcal{L}(E, F)$. The map $h \otimes T : X \otimes E \to Y \otimes F$, given by

$$(h \otimes T)(u) = \sum_{i=1}^{n} \delta_{(h(x),h(y))} \otimes T(e_i)$$

for $u = \sum_{i=1}^{n} \delta_{(x,y)} \otimes e_i \in X \otimes E$, is called the Lipschitz tensor product operator of $h$ and $T$.

**Lemma 3.1.** Let $h \in \text{Lip}_0(X, Y)$ and $T \in \mathcal{L}(E, F)$. Then $h \otimes T : X \otimes E \to Y \otimes F$ is a well-defined linear operator.

**Proof.** Let $u = \sum_{i=1}^{n} \delta_{(x,y)} \otimes e_i$ and $v = \sum_{i=1}^{m} \delta_{(x',y')} \otimes e'_i$ be in $X \otimes E$. If $u = v$, then Proposition 1.6 says us that $\sum_{i=1}^{n} (g(x_i) - g(y_i)) e_i = \sum_{i=1}^{m} (g(x'_i) - g(y'_i)) e'_i$ for all $g \in B_{X'}$. In particular, this holds for all function in $B_{X'}$ of the form $(f \circ h)/(1 + \lambda h)$ with $\lambda$ varying in $B_{\mathbb{R}}$. It follows that $\sum_{i=1}^{n} (f(h(x_i)) - f(h(y_i))) T(e_i) = \sum_{i=1}^{n} (f(h(x'_i)) - f(h(y'_i))) T(e'_i)$ for all $f \in B_{X'}$, and this implies that $\sum_{i=1}^{n} \delta_{(h(x),h(y))} \otimes T(e_i) = \sum_{i=1}^{m} \delta_{(h(x'),h(y'))} \otimes T(e'_i)$ again by Proposition 1.6. Hence the map $h \otimes T$ is well defined.

We see that $h \otimes T$ is linear. Let $\lambda \in \mathbb{K}$. Then $\lambda u = \sum_{i=1}^{n} \delta_{(x,y)} \otimes (\lambda e_i)$, and Definition 3.1 and Lemma 1.1 give

$$(h \otimes T)(\lambda u) = \sum_{i=1}^{n} \delta_{(h(x),h(y))} \otimes \lambda T(e_i)$$

$$= \sum_{i=1}^{n} \delta_{(h(x),h(y))} \otimes \lambda T(e_i)$$

$$= \lambda \sum_{i=1}^{n} \delta_{(h(x),h(y))} \otimes T(e_i)$$

$$= \lambda(h \otimes T)(u).$$

Take $u + v = \sum_{i=1}^{n+m} \delta_{(x'',y'')} \otimes e''_i$ as in Lemma 1.3. Then we have

$$(h \otimes T)(u + v) = \sum_{i=1}^{n+m} \delta_{(h(x''),h(y''))} \otimes T(e''_i)$$

$$= \sum_{i=1}^{n} \delta_{(h(x''),h(y''))} \otimes T(e''_i) + \sum_{i=n+1}^{n+m} \delta_{(h(x''),h(y''))} \otimes T(e''_i)$$

$$= \sum_{i=1}^{n} \delta_{(h(x),h(y))} \otimes T(e_i) + \sum_{i=n+1}^{n+m} \delta_{(h(x'',h(y''))} \otimes T(e''_{i-n})$$

$$= \sum_{i=1}^{n} \delta_{(h(x),h(y))} \otimes T(e_i) + \sum_{i=1}^{m} \delta_{(h(x'),h(y'))} \otimes T(e'_i)$$

$$= (h \otimes T)(u) + (h \otimes T)(v).$$

□

4. Lipschitz cross-norms

We denote the linear space $X \otimes E$ endowed with a norm $\alpha$ by $X_{\alpha} \otimes E$, and its completion by $X_{\bar{\alpha}} \otimes E$. We are looking for a norm on the linear space $X \otimes E$, and for our purposes it is convenient to work with norms that satisfy the following conditions.
Definition 4.1. Let $X$ be a pointed metric space and $E$ a Banach space. We say that a norm $\alpha$ on $X \otimes E$ is a Lipschitz cross-norm if
\[ \alpha \left( \delta_{(x,y)} \otimes e \right) = d(x, y) \| e \| \]
for all $(x, y) \in X^2$ and $e \in E$.

A Lipschitz cross-norm $\alpha$ on $X \otimes E$ is said to be dualizable if given $g \in X^*$ and $\phi \in E^*$, we have
\[ \left| \sum_{i=1}^{n} (g(x_i) - g(y_i)) \langle \phi, e_i \rangle \right| \leq \text{Lip}(g) \| \phi \| \alpha \left( \sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i \right) \]
for all $\sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i \in X \otimes E$.

A Lipschitz cross-norm $\alpha$ on $X \otimes E$ is called uniform if given $h \in \text{Lip}_0(X, X)$ and $T \in \mathcal{L}(E, E)$, we have
\[ \alpha \left( \sum_{i=1}^{n} \delta_{(h(x_i), y_i)} \otimes T(e_i) \right) \leq \text{Lip}(h) \| T \| \alpha \left( \sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i \right) \]
for all $\sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i \in X \otimes E$.

The dualizable Lipschitz cross-norms on $X \otimes E$ may be characterized by the boundedness of the Lipschitz tensor product functionals.

Proposition 4.1. A Lipschitz cross-norm $\alpha$ on $X \otimes E$ is dualizable if and only if, for each $g \in X^*$ and $\phi \in E^*$, the linear functional $g \otimes \phi : X \otimes E \to \mathbb{K}$ is bounded and $\| g \otimes \phi \| = \text{Lip}(g) \| \phi \|$. 

Proof. Let $\alpha$ be a Lipschitz cross-norm on $X \otimes E$. Given $g \in X^*$ and $\phi \in E^*$, if $\alpha$ is dualizable, we have
\[ \left| \left( g \otimes \phi \right) \left( \sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i \right) \right| = \left| \sum_{i=1}^{n} (g(x_i) - g(y_i)) \langle \phi, e_i \rangle \right| \leq \text{Lip}(g) \| \phi \| \alpha \left( \sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i \right) \]
for all $\sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i \in X \otimes E$. Hence the linear functional $g \otimes \phi$ is bounded on $X \otimes E$ and $\| g \otimes \phi \| \leq \text{Lip}(g) \| \phi \|$. 

The opposite inequality $\text{Lip}(g) \| \phi \| \leq \| g \otimes \phi \|$ is deduced from the fact that
\[ |g(x) - g(y)| |\langle \phi, e \rangle| = \left| g \otimes \phi \left( \delta_{(x,y)} \otimes e \right) \right| \leq \| g \otimes \phi \| \alpha \left( \delta_{(x,y)} \otimes e \right) = \| g \otimes \phi \| |d(x, y)||e|| \]
for all $x, y \in X$ and $e \in E$.

Conversely, if for any $g \in X^*$ and $\phi \in E^*$, the linear functional $g \otimes \phi : X \otimes E \to \mathbb{K}$ is bounded and $\| g \otimes \phi \| = \text{Lip}(g) \| \phi \|$, then
\[ \left| \sum_{i=1}^{n} (g(x_i) - g(y_i)) \langle \phi, e_i \rangle \right| = \left| g \otimes \phi \left( \sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i \right) \right| \]
\[ \leq \| g \otimes \phi \| \alpha \left( \sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i \right) \]
\[ = \text{Lip}(g) \| \phi \| \alpha \left( \sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i \right) \]
for all $\sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i \in X \otimes E$, and so $\alpha$ is dualizable. \(\square\)

Similarly, the boundedness of the Lipschitz tensor product operators characterizes the uniform Lipschitz cross-norms on $X \otimes E$.

Proposition 4.2. A Lipschitz cross-norm $\alpha$ on $X \otimes E$ is uniform if and only if, for each $h \in \text{Lip}_0(X, X)$ and $T \in \mathcal{L}(E, E)$, the linear operator $h \otimes T : X \otimes E \to X \otimes E$ is bounded and $\| h \otimes T \| = \text{Lip}(h) \| T \|$. 

Proof. Let $\alpha$ be a Lipschitz cross-norm on $X \boxtimes E$. If $\alpha$ is uniform, given $h \in \text{Lip}_0(X, X)$ and $T \in \mathcal{L}(E, E)$, we have

$$\alpha\left( (h \boxtimes T) \left( \sum_{i=1}^{n} \delta_{(x,y)} \otimes e_i \right) \right) = \alpha\left( \sum_{i=1}^{n} \delta_{(h(x),h(y))} \otimes T(e_i) \right) \leq \text{Lip}(h) \|T\| \alpha\left( \sum_{i=1}^{n} \delta_{(x,y)} \otimes e_i \right)$$

for all $\sum_{i=1}^{n} \delta_{(x,y)} \otimes e_i \in X \boxtimes E$. It follows that the linear operator $h \boxtimes T$ is bounded on $X \boxtimes_a E$ and $\|h \boxtimes T\| \leq \text{Lip}(h) \|T\|$. For the reverse inequality, notice that

$$d(h(x), h(y)) \|T(e)\| = \alpha\left( \delta_{(h(x),h(y))} \otimes T(e) \right)$$

$$= \alpha\left( (h \boxtimes T) \left( \delta_{(x,y)} \otimes e \right) \right)$$

$$\leq \|h \boxtimes T\| \alpha\left( \delta_{(x,y)} \otimes e \right)$$

$$= \|h \boxtimes T\| d(x,y) \|e\|$$

for all $x, y \in X$ and $e \in E$, and therefore $\text{Lip}(h) \|T\| \leq \|h \boxtimes T\|$.

Conversely, if for each $h \in \text{Lip}_0(X, X)$ and $T \in \mathcal{L}(E, E)$, the linear map $h \boxtimes T : X \boxtimes_a E \to X \boxtimes_a E$ is bounded and $\|h \boxtimes T\| = \text{Lip}(h) \|T\|$, then

$$\alpha\left( \sum_{i=1}^{n} \delta_{(h(x),h(y))} \otimes T(e_i) \right) = \alpha\left( (h \boxtimes T) \left( \sum_{i=1}^{n} \delta_{(x,y)} \otimes e_i \right) \right)$$

$$\leq \|h \boxtimes T\| \alpha\left( \sum_{i=1}^{n} \delta_{(x,y)} \otimes e_i \right)$$

$$= \text{Lip}(h) \|T\| \alpha\left( \sum_{i=1}^{n} \delta_{(x,y)} \otimes e_i \right)$$

for all $\sum_{i=1}^{n} \delta_{(x,y)} \otimes e_i \in X \boxtimes E$, and so $\alpha$ is uniform. \qed

Remark 4.1. A reading of the proofs of the two preceding propositions shows that a Lipschitz cross-norm $\alpha$ on $X \boxtimes E$ is dualizable (uniform) if for each $g \in X^\#$ and $\phi \in E^\ast$, then $g \boxtimes \phi \in (X \boxtimes_a E)^\ast$ and $\|g \boxtimes \phi\| \leq \text{Lip}(g) \|\phi\|$ (respectively, if for each $h \in \text{Lip}_0(X, X)$ and $T \in \mathcal{L}(E, E)$, then $h \boxtimes T \in \mathcal{L}(X \boxtimes_a E, X \boxtimes_a E)$ and $\|h \boxtimes T\| \leq \text{Lip}(h) \|T\|$).

5. Associated Lipschitz cross-norms

We introduce the concept of Lipschitz cross-norm on the associated Lipschitz tensor product of $X \boxtimes E$.

Definition 5.1. Let $X$ be a pointed metric space and $E$ a Banach space. We say that a norm $\beta$ on $X^\# \boxtimes E^\ast$ is a Lipschitz cross-norm if $\beta(g \boxtimes \phi) = \text{Lip}(g) \|\phi\|$ for all $g \in X^\#$ and $\phi \in E^\ast$.

We now construct the following Lipschitz cross-norm on $X^\# \boxtimes E^\ast$.

Proposition 5.1. Let $\alpha$ be a dualizable Lipschitz cross-norm on $X \boxtimes E$. The map $\alpha' : X^\# \boxtimes E^\ast \to \mathbb{R}$, given by

$$\alpha'(u^*) = \sup \left\{ \left( \sum_{j=1}^{m} g_j \boxtimes \phi_j \right) \left( \sum_{i=1}^{n} \delta_{(x,y)} \otimes e_i \right) : \alpha\left( \sum_{i=1}^{n} \delta_{(x,y)} \otimes e_i \right) \leq 1 \right\}$$

for $u^* = \sum_{j=1}^{m} g_j \boxtimes \phi_j \in X^\# \boxtimes E^\ast$, is a Lipschitz cross-norm on $X^\# \boxtimes E^\ast$.

Proof. Let $u^* = \sum_{j=1}^{m} g_j \boxtimes \phi_j \in X^\# \boxtimes E^\ast$. If $\sum_{i=1}^{n} \delta_{(x,y)} \otimes e_i \in X \boxtimes E$, we have

$$\left( \sum_{j=1}^{m} g_j \boxtimes \phi_j \right) \left( \sum_{i=1}^{n} \delta_{(x,y)} \otimes e_i \right) = \sum_{j=1}^{m} (g_j \boxtimes \phi_j) \left( \sum_{i=1}^{n} \delta_{(x,y)} \otimes e_i \right).$$
and since α is dualizable, by applying Proposition 4.1 gives

$$\left\| \sum_{j=1}^{m} g_j \otimes \phi_j \left( \sum_{i=1}^{n} \delta_{(x,y)} \otimes e_i \right) \right\| \leq \sum_{j=1}^{m} \left\| g_j \otimes \phi_j \right\| \left( \sum_{i=1}^{n} \delta_{(x,y)} \otimes e_i \right)$$

$$\leq \sum_{j=1}^{m} \| g_j \otimes \phi_j \| \alpha \left( \sum_{i=1}^{n} \delta_{(x,y)} \otimes e_i \right)$$

$$= \sum_{j=1}^{m} \text{Lip}(g_j) \| \phi_j \| \alpha \left( \sum_{i=1}^{n} \delta_{(x,y)} \otimes e_i \right).$$

Then the supremum on the right side of the equality in the statement exists. It is immediate that α’ is a well-defined map. Given λ ∈ ℱ, it is plain that λα’ = \sum_{j=1}^{m} g_j \otimes (\lambda \phi_j) and therefore

$$\alpha’(\lambda u^*) = \sup \left\{ \left\| \sum_{j=1}^{m} g_j \otimes (\lambda \phi_j) \left( \sum_{i=1}^{n} \delta_{(x,y)} \otimes e_i \right) \right\| : \alpha \left( \sum_{i=1}^{n} \delta_{(x,y)} \otimes e_i \right) \leq 1 \right\}.$$

Clearly, the supremum on the right side above is equal to |λ|α’(u*) and thus α’(λu*) = |λ|α’(u*).

Let v* = \sum_{j=1}^{r} g'_j \otimes \phi'_j ∈ X* ⊗ E*. Then u* + v* = \sum_{j=1}^{m} g_j \otimes \phi_j + \sum_{j=1}^{r} g'_j \otimes \phi'_j. Since

$$\left\| \sum_{j=1}^{m} g_j \otimes \phi_j + \sum_{j=1}^{r} g'_j \otimes \phi'_j \left( \sum_{i=1}^{n} \delta_{(x,y)} \otimes e_i \right) \right\| \leq \left\| \sum_{j=1}^{m} g_j \otimes \phi_j \left( \sum_{i=1}^{n} \delta_{(x,y)} \otimes e_i \right) \right\| + \left\| \sum_{j=1}^{r} g'_j \otimes \phi'_j \left( \sum_{i=1}^{n} \delta_{(x,y)} \otimes e_i \right) \right\|$$

$$\leq (\alpha’(u*) + \alpha’(v*)) \alpha \left( \sum_{i=1}^{n} \delta_{(x,y)} \otimes e_i \right)$$

for all \sum_{i=1}^{n} \delta_{(x,y)} \otimes e_i ∈ X ⊗ E, it follows that α’(u* + v*) ≤ α’(u*) + α’(v*).

Assume u* ≠ 0. By Lemma 2.3 we can take a representation of u*, \sum_{j=1}^{m} g_j \otimes \phi_j, such that g_1, …, g_m in X* are nonzero and \phi_1, …, \phi_m in E* are linearly independent. Then g_1(x) ≠ 0 for some x ∈ X. The linear independence of the \phi_j’s yields that \sum_{j=1}^{m} g_j(x)\phi_j ≠ 0. Now, take e ∈ E for which \sum_{j=1}^{m} g_j(x)\phi_j(e) ≠ 0, that is, \left( \sum_{j=1}^{m} g_j \otimes \phi_j \right) (\delta_{(x,e)} \otimes e) ≠ 0. This implies that \delta_{(x,e)} \otimes e ≠ 0, and so

$$\alpha’(u*) ≥ \frac{\left\| \sum_{j=1}^{m} g_j \otimes \phi_j \right\| \left( \delta_{(x,e)} \otimes e \right)}{\alpha \left( \delta_{(x,e)} \otimes e \right)} > 0.$$

In this way, we have proved that α’ is a norm on X* ⊗ E*.

In order to show that α’ is a Lipschitz cross-norm on X* ⊗ E*, let g ∈ X* and φ ∈ E*. For all x, y ∈ X and e ∈ E, we have

$$|\langle g(x) - g(y), \phi, e \rangle| = \left| \left( g \otimes \phi \right) \left( \delta_{(x,y)} \otimes e \right) \right| \leq \alpha’(g \otimes \phi) \alpha \left( \delta_{(x,y)} \otimes e \right) = \alpha’(g \otimes \phi) d(x, y) \|e\|,$$

and from this we infer that Lip(g) ||φ|| ≤ α’(g ⊗ φ). For the reverse, since α is dualizable, we have

$$\left\| \left( g \otimes \phi \right) \left( \sum_{i=1}^{n} \delta_{(x,y)} \otimes e_i \right) \right\| = \left\| \sum_{i=1}^{n} \langle g(x_i) - g(y_i), \phi, e_i \rangle \right\| ≤ \text{Lip}(g) \|\phi\| \alpha \left( \sum_{i=1}^{n} \delta_{(x,y)} \otimes e_i \right)$$

for any \sum_{i=1}^{n} \delta_{(x,y)} \otimes e_i ∈ X ⊗ E, and this implies that α’(g ⊗ φ) ≤ Lip(g) ||φ||.
Definition 5.2. Let $X$ be a pointed metric space and $E$ be a Banach space. Let $\alpha$ be a dualizable Lipschitz cross-norm on $X \boxtimes E$. The norm $\alpha'$ on $X^{\#} \boxtimes E^{\ast}$ is called the associated Lipschitz norm of $\alpha$. The vector space $X^{\#} \boxtimes E^{\ast}$ with the norm $\alpha'$ will be denoted by $X^{\#} \boxtimes_{\alpha'} E^{\ast}$ and its completion by $X^{\#} \boxtimes_{\alpha} E^{\ast}$.

Remark 5.1. Note that if $\alpha$ is a dualizable Lipschitz cross-norm on $X \boxtimes E$, then $X^{\#} \boxtimes_{\alpha} E^{\ast}$ is a linear subspace of $(X \boxtimes_{\alpha} E)^{\ast}$.

6. The induced Lipschitz dual norm

Definition 6.1. For each $u = \sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i \in X \boxtimes E$, define:

$$L(u) = \sup \left\{ \frac{1}{n} \sum_{i=1}^{n} \langle f(x_i) - f(y_i), e_i \rangle : f \in \operatorname{Lip}_0(X,E^{\ast}), \ |\operatorname{Lip}(f)| \leq 1 \right\}.$$ 

Note that the supremum on the right side above exists and $L(u) \leq \sum_{i=1}^{n} d(x_i,y_i) ||e_i||$ because

$$\left| \sum_{i=1}^{n} \langle f(x_i) - f(y_i), e_i \rangle \right| \leq \sum_{i=1}^{n} \left| \langle f(x_i) - f(y_i), e_i \rangle \right| \leq \sum_{i=1}^{n} ||f(x_i) - f(y_i)|| ||e_i|| \leq \operatorname{Lip}(f) \sum_{i=1}^{n} d(x_i,y_i) ||e_i||$$

for all $f \in \operatorname{Lip}_0(X,E^{\ast})$. Moreover, $L$ defines a map from $X \boxtimes E$ to $\mathbb{R}$ by Lemma 1.4.

Theorem 6.1. The linear space $X \boxtimes E$ is contained in $\operatorname{Lip}_0(X,E^{\ast})$ and $L$ is the dual norm of the norm $\operatorname{Lip}$ of $\operatorname{Lip}_0(X,E^{\ast})$ induced on $X \boxtimes E$. Moreover, $L$ is a Lipschitz cross-norm on $X \boxtimes E$.

Proof. Let $x,y \in X$ and $e \in E$. Since $\delta_{(x,y)} \otimes e$ is a linear map on $\operatorname{Lip}_0(X,E^{\ast})$ and

$$\left| \langle (\delta_{(x,y)} \otimes e)(f) \rangle \right| = ||f(x) - f(y)|| \leq ||f(x) - f(y)|| ||e|| \leq \operatorname{Lip}(f)(d(x,y)||e||)$$

for all $f \in \operatorname{Lip}_0(X,E^{\ast})$, then $\delta_{(x,y)} \otimes e \in \operatorname{Lip}_0(X,E^{\ast})^{\ast}$ and thus $X \boxtimes E \subset \operatorname{Lip}_0(X,E^{\ast})^{\ast}$. For every $u = \sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i \in X \boxtimes E$, we have

$$L(u) = \sup \left\{ ||w(f)|| : f \in \operatorname{Lip}_0(X,E^{\ast}), ||Lip(f)|| \leq 1 \right\}$$

by Lemma 1.4 and therefore $L$ is the dual norm of the norm Lip of $\operatorname{Lip}_0(X,E^{\ast})$ induced on $X \boxtimes E$. Finally, we prove that $L$ is a Lipschitz cross-norm. By above-proved, $\left| \langle (\delta_{(x,y)} \otimes e)(f) \rangle \right| \leq d(x,y)||e||$ for all $f \in \operatorname{Lip}_0(X,E^{\ast})$ with $\operatorname{Lip}(f) \leq 1$, and hence $L(\delta_{(x,y)} \otimes e) \leq d(x,y)||e||$. For the reverse estimate, take $\phi \in E^{\ast}$ with $||\phi|| = 1$ satisfying $|\langle \phi, e \rangle| = ||e||$, and consider the map $f : X \rightarrow E^{\ast}$ given by

$$f(z) = (d(0,x) - d(z,x))\phi \quad (z \in X).$$

An easy verification shows that $f$ is in $\operatorname{Lip}_0(X,E^{\ast})$ with $\operatorname{Lip}(f) \leq 1$ and

$$\left| \langle (\delta_{(x,y)} \otimes e)(f) \rangle \right| = ||(f(x) - f(y), e)\rangle = ||(d(x,y)\phi, e)\rangle = ||\delta_{(x,y)} \langle \langle \phi, e \rangle \rangle = d(x,y)||e||,$$

and therefore $d(x,y)||e|| = ||(\delta_{(x,y)} \otimes e)(f)\rangle \leq L(\delta_{(x,y)} \otimes e)$. □

The following result is essentially known. For completeness we include it here with an alternate proof.

Theorem 6.2. [17, Theorem 4.1] Let $X$ be a pointed metric space and let $E$ be a Banach space. Then $\operatorname{Lip}_0(X,E^{\ast})$ is isometrically isomorphic to $(X \boxtimes_{\alpha} E)^{\ast}$, via the map $\Lambda : \operatorname{Lip}_0(X,E^{\ast}) \rightarrow (X \boxtimes_{\alpha} E)^{\ast}$ given by

$$\Lambda(f)(u) = \sum_{i=1}^{n} \langle f(x_i) - f(y_i), e_i \rangle.$$
for \( f \in \text{Lip}_0(X, E^*) \) and \( u = \sum_{i=1}^{n} \delta_{(x_i, y_i)} \otimes e_i \in X \boxtimes E \). Its inverse \( \Lambda^{-1} : (X \widehat{\boxtimes} E)' \to \text{Lip}_0(X, E^*) \) is defined by

\[
\Lambda^{-1}(\varphi)(x, e) = \langle \varphi, \delta_{(x,0)} \otimes e \rangle
\]

for \( \varphi \in (X \widehat{\boxtimes} E)', \ x \in X \) and \( e \in E \).

**Proof.** By Corollary [1.8] the map \( f \mapsto \Lambda(f) \) is a linear monomorphism from \( \text{Lip}_0(X, E^*) \) into \( (X \boxtimes E)' \). In fact, \( \Lambda(f) \in (X \widehat{\boxtimes} E)' \) and \( \|\Lambda(f)\| \leq \text{Lip}(f) \).

Note that

\[
|\Lambda(f)(u)| = \left| \sum_{i=1}^{n} \langle f(x_i) - f(y_i), e_i \rangle \right| \leq \text{Lip}(f)L(u)
\]

for all \( u = \sum_{i=1}^{n} \delta_{(x_i, y_i)} \otimes e_i \in X \boxtimes E \). Therefore \( \Lambda(f) \) is bounded on \( X \boxtimes E \), hence on \( X \widehat{\boxtimes} E \) by the denseness of the former set in the latter one. In order to see that \( \Lambda \) is a surjective isometry, let \( \varphi \) be an element of \( (X \widehat{\boxtimes} E)' \). Define \( f : X \to E^* \) by

\[
\langle f(x), e \rangle = \varphi(\delta_{(x,0)} \otimes e) \quad (x \in X, \ e \in E).
\]

It is plain that \( f(x) \) is a well-defined bounded linear functional on \( E \) and that \( f \) is well defined. Observe that

\[
|\langle f(x) - f(y), e \rangle| = |\varphi(\delta_{(x,0)} \otimes e)| \leq \|\varphi\|L(\delta_{(x,0)} \otimes e) = \|\varphi\|d(x,y)\|e\|
\]

for all \( e \in E \), and so \( \|f(x) - f(y)\| \leq \|\varphi\|d(x,y) \). Hence \( f \in \text{Lip}_0(X, E^*) \) and \( \text{Lip}(f) \leq \|\varphi\| \). For any \( u = \sum_{i=1}^{n} \delta_{(x_i, y_i)} \otimes e_i \in X \boxtimes E \), we get

\[
\Lambda(f)(u) = \sum_{i=1}^{n} \langle f(x_i) - f(y_i), e_i \rangle = \sum_{i=1}^{n} \varphi(\delta_{(x_i, y_i)} \otimes e_i) = \varphi \left( \sum_{i=1}^{n} \delta_{(x_i, y_i)} \otimes e_i \right) = \varphi(u).
\]

Hence \( \Lambda(f) = \varphi \) on a dense subspace of \( X \widehat{\boxtimes} E \) and, consequently, \( \Lambda(f) = \varphi \). Moreover, \( \text{Lip}(f) \leq \|\varphi\| = \|\Lambda(f)\| \).

This completes the proof of the theorem. \( \square \)

### 7. The Lipschitz injective norm

We introduce the Lipschitz injective norm on \( X \boxtimes E \).

**Definition 7.1.** For each \( u = \sum_{i=1}^{n} \delta_{(x_i, y_i)} \otimes e_i \in X \boxtimes E \), define:

\[
\varepsilon(u) = \sup \left\{ \left| \sum_{i=1}^{n} (g(x_i) - g(y_i)) \langle \phi, e_i \rangle \right| : g \in B_{X^*}, \ \phi \in B_{E^*} \right\}.
\]

Notice that the supremum on the right side in the previous definition exists since

\[
\left| \sum_{i=1}^{n} (g(x_i) - g(y_i)) \langle \phi, e_i \rangle \right| \leq \sum_{i=1}^{n} |g(x_i) - g(y_i)| \|\phi\| \|e_i\|
\]

\[
\leq \sum_{i=1}^{n} \text{Lip}(g) d(x_i, y_i) \|\phi\| \|e_i\|
\]

\[
\leq \sum_{i=1}^{n} d(x_i, y_i) \|e_i\|
\]

for all \( g \in B_{X^*} \) and \( \phi \in B_{E^*} \). Note that

\[
\sum_{i=1}^{n} (g(x_i) - g(y_i)) \langle \phi, e_i \rangle = \left( g \otimes \phi \right) \left( \sum_{i=1}^{n} \delta_{(x_i, y_i)} \otimes e_i \right),
\]

and, consequently, \( \varepsilon \) defines a map from \( X \boxtimes E \) to \( \mathbb{R} \) by Lemma [2.1].

**Theorem 7.1.** \( \varepsilon \) is a uniform and dualizable Lipschitz cross-norm on \( X \boxtimes E \).
Proof. Let \( u = \sum_{i=1}^{n} \delta_{(x_i,y)} \otimes e_i \in X \otimes E \). Suppose that \( \varepsilon(u) = 0 \). Then \( \sum_{i=1}^{n} (g(x_i) - g(y_i)) \langle \phi, e_i \rangle = 0 \) for all \( g \in B_{X^*} \) and \( \phi \in B_{E^*} \), and this happens if and only if \( u = 0 \) by Proposition [1.6]

For any \( \lambda \in \mathbb{K} \), we have \( \lambda u = \sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes (\lambda e_i) \), and therefore

\[
\varepsilon(\lambda u) = \sup \left\{ \left| \sum_{i=1}^{n} (g(x_i) - g(y_i)) \langle \phi, \lambda e_i \rangle \right| : g \in B_{X^*}, \ \phi \in B_{E^*} \right\} = |\lambda| \varepsilon(u).
\]

Let \( v = \sum_{i=1}^{m} \delta_{(x_i',y_i')} \otimes e_i' \in X \otimes E \). Then \( \sum_{i=m+1}^{n+m} \delta_{(x_i',y_i')} \otimes e_i' \), being as in Lemma [1.3] is a representation for \( u + v \).

For any \( g \in B_{X^*} \) and \( \phi \in B_{E^*} \), it holds that

\[
\left| \sum_{i=1}^{n+m} (g(x'_i) - g(y'_i)) \langle \phi, e'_i \rangle \right| \leq \sum_{i=1}^{n+m} \left| (g(x'_i) - g(y'_i)) \langle \phi, e'_i \rangle \right| = \sum_{i=1}^{n} (g(x'_i) - g(y'_i)) \langle \phi, e'_i \rangle + \sum_{i=n+1}^{n+m} (g(x'_i) - g(y'_i)) \langle \phi, e'_i \rangle
\]

\[
= \sum_{i=1}^{n} (g(x'_i) - g(y'_i)) \langle \phi, e'_i \rangle + \sum_{i=n+1}^{n+m} (g(x'_i) - g(y'_i)) \langle \phi, e'_i \rangle
\]

\[
\leq \varepsilon(u) + \varepsilon(v),
\]

and therefore \( \varepsilon(u + v) \leq \varepsilon(u) + \varepsilon(v) \). Hence \( \varepsilon \) is a norm on \( X \otimes E \).

We claim that \( \varepsilon \) is a Lipschitz cross-norm. Take \( \delta_{(x,y)} \otimes e \in X \otimes E \). For any \( g \in B_{X^*} \) and \( \phi \in B_{E^*} \), we have

\[
|g(x) - g(y)| \langle \phi, e \rangle \leq \text{Lip}(g) d(x,y) ||\phi|| ||e|| \leq d(x,y) ||e||,
\]

and so \( \varepsilon(\delta_{(x,y)} \otimes e) \leq d(x,y) ||e|| \). For the converse inequality, we can find \( g_0 \in B_{X^*} \) and \( \phi_0 \in B_{E^*} \) such that \( |g_0(x) - g_0(y)| = d(x,y) \) and \( \langle \phi_0, e \rangle = ||e|| \). For example, \( g_0(z) = d(0,z) - d(x,z) \) for all \( z \in X \). Then

\[
\varepsilon(\delta_{(x,y)} \otimes e) \geq |g(x) - g(y)| \langle \phi, e \rangle = d(x,y) ||e||,
\]

and this proves our claim.

Now take \( g \in X^* \) and \( \phi \in E^* \). By Definition [7.1], we have

\[
\left| \sum_{i=1}^{n} (g(x_i) - g(y_i)) \langle \phi, e_i \rangle \right| \leq \text{Lip}(g) ||\phi|| \varepsilon \left( \sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i \right)
\]

for all \( \sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i \in X \otimes E \). Hence the norm \( \varepsilon \) is dualizable. Then, by Proposition [5.1] \( \varepsilon' \) is a Lipschitz cross-norm on \( X^* \otimes E^* \).

Finally, we prove that the norm \( \varepsilon \) is uniform. Let \( h \in \text{Lip}(X,X) \) and \( T \in \mathcal{L}(E,E) \). Let \( T^* \) denote the adjoint operator of \( T \). We now recall that the Lipschitz adjoint map \( h^* : X^* \to X^* \), given by \( h^*(g) = g \circ h \) for all \( g \in X^* \), is a continuous linear operator and \( \|h^*\| = \text{Lip}(h) \). Indeed, it is clear that \( h^* \) is linear. Let \( g \in X^* \) and \( x, y \in X \). We have

\[
\|h^*(g)(x) - h^*(g)(y)\| = |g(h(x)) - g(h(y))| \leq \text{Lip}(g) d(h(x),h(y)) \leq \text{Lip}(g) \text{Lip}(h) d(x,y),
\]

hence \( h^*(g) \in X^* \) and \( \text{Lip}(h^*(g)) \leq \text{Lip}(g) \text{Lip}(h) \). It follows that \( h^* \) is bounded and \( \|h^*\| \leq \text{Lip}(h) \). Taking the function defined on \( X \) by \( g(z) = d(h(x),0) - d(h(x),z) \) which is in \( B_{X^*} \), we get

\[
d(h(x),h(y)) = |g(h(x)) - g(h(y))| = \|h^*(g)(x) - h^*(g)(y)\| \leq \text{Lip}(h^*(g)) d(x,y),
\]

which gives \( \text{Lip}(h) \leq \text{Lip}(h^*(g)) \leq \|h^*\| \text{Lip}(g) \leq \|h^*\| \) and so \( \|h^*\| = \text{Lip}(h) \).
Given $\sum_{i=1}^{n} \delta_{(x_i, y_i)} \otimes e_i \in X \boxtimes E$, we have
\[
\left| \sum_{i=1}^{n} \left(g(h(x_i)) - g(h(y_i))\right) \langle \phi, T(e_i) \rangle \right| = \left| (h^g(g) \otimes T^*(\phi)) \left(\sum_{i=1}^{n} \delta_{(x_i, y_i)} \otimes e_i \right) \right|
\leq \varepsilon' \left(h^g(g) \otimes T^*(\phi)\right) \varepsilon \left(\sum_{i=1}^{n} \delta_{(x_i, y_i)} \otimes e_i \right)
= \text{Lip}(h^g(g)) \|T^*(\phi)\| \varepsilon \left(\sum_{i=1}^{n} \delta_{(x_i, y_i)} \otimes e_i \right)
\leq \text{Lip}(h) \|T\| \varepsilon \left(\sum_{i=1}^{n} \delta_{(x_i, y_i)} \otimes e_i \right)
\]
for all $g \in B_{X^*}$ and $\phi \in B_{E^*}$, and hence
\[
\varepsilon \left(\sum_{i=1}^{n} \delta_{(h(x_i), h(y_i))} \otimes T(e_i) \right) \leq \text{Lip}(h) \|T\| \varepsilon \left(\sum_{i=1}^{n} \delta_{(x_i, y_i)} \otimes e_i \right),
\]
which proves that the norm $\varepsilon$ is uniform. \hfill \Box

**Theorem 7.2.** $\varepsilon$ is the least dualizable Lipschitz cross-norm on $X \boxtimes E$.

**Proof.** According to Theorem 7.1, $\varepsilon$ is a dualizable Lipschitz cross-norm on $X \boxtimes E$. Let $\alpha$ be a dualizable Lipschitz cross-norm on $X \boxtimes E$ and assume, for contradiction, that
\[
\alpha \left(\sum_{i=1}^{n} \delta_{(x_i, y_i)} \otimes e_i \right) < \varepsilon \left(\sum_{i=1}^{n} \delta_{(x_i, y_i)} \otimes e_i \right)
\]
for some $\sum_{i=1}^{n} \delta_{(x_i, y_i)} \otimes e_i \in X \boxtimes E$. By the definition of $\varepsilon$, there exist $g \in B_{X^*}$ and $\phi \in B_{E^*}$ such that
\[
\alpha \left(\sum_{i=1}^{n} \delta_{(x_i, y_i)} \otimes e_i \right) < \left| \sum_{i=1}^{n} (g(x_i) - g(y_i)) \langle \phi, e_i \rangle \right|.
\]
By Proposition 5.1, $\alpha'$ is a Lipschitz cross-norm on $X^* \boxtimes E^*$, and we have
\[
\left| \sum_{i=1}^{n} (g(x_i) - g(y_i)) \langle \phi, e_i \rangle \right| = \left| (g \otimes \phi) \left(\sum_{i=1}^{n} \delta_{(x_i, y_i)} \otimes e_i \right) \right| \leq \alpha'(g \otimes \phi) \left(\sum_{i=1}^{n} \delta_{(x_i, y_i)} \otimes e_i \right).
\]
Hence $\alpha'(g \otimes \phi) > 1$ and thus $\text{Lip}(g) \|\phi\| < \alpha'(g \otimes \phi)$. This contradicts that $\alpha'$ is a Lipschitz cross-norm. Therefore $\alpha \geq \varepsilon$ and this proves the theorem. \hfill \Box

The completion $X \boxtimes E$ of $X \boxtimes E$ is called the injective Lipschitz tensor product of $X$ and $E$. Next we justify this terminology.

**Theorem 7.3.** Let $X$ be a pointed metric space and let $E$ be a Banach space. Let $X_0 \subset X$ be a subset of $X$ containing 0, and let $E_0$ be a closed linear subspace of $E$. Then $X_0 \boxtimes E_0$ is a linear subspace of $X \boxtimes E$. 
Proof. Let \( u = \sum_{i=1}^{n} \delta_{(x_i, y_i)} \otimes e_i \in X_0 \otimes E_0 \). It is sufficient to prove that \( \varepsilon_{X_0 \otimes E_0}(u) = \varepsilon_{X_0 \otimes E_0}(u) \), where

\[
\varepsilon_{X_0 \otimes E_0}(u) = \sup \left\{ \sum_{i=1}^{n} (g_0(x_i) - g_0(y_i)) \langle \phi_0, e_i \rangle : g_0 \in B_{X^\delta}, \phi_0 \in B_{E_0} \right\},
\]

\[
\varepsilon_{X_0 \otimes E_0}(u) = \sup \left\{ \sum_{i=1}^{n} (g(x_i) - g(y_i)) \langle \phi, e_i \rangle : g \in B_{X^\delta}, \phi \in B_{E_0} \right\}.
\]

By applying the classical Hahn–Banach theorem we can extend each \( \phi_0 \in B_{E_0} \) to a \( \phi \in B_{E_0} \), and by applying the nonlinear Hahn–Banach theorem we can extend each \( g_0 \in B_{X^\delta} \) to \( g \in B_{X^\delta} \), hence we see that \( \varepsilon_{X_0 \otimes E_0}(u) \leq \varepsilon_{X_0 \otimes E_0}(u) \). Conversely, by restricting the functionals \( \phi \in B_{E_0} \) to \( E_0 \) and the Lipschitz functions \( g \in B_{X^\delta} \) to \( X_0 \), we obtain that \( \varepsilon_{X_0 \otimes E_0}(u) \leq \varepsilon_{X_0 \otimes E_0}(u) \). \( \square \)

We can identify \( X_0 \hat{\otimes} E \) with the space of all approximable bounded linear operators of \( (X^\delta, \tau_P) \) to \( E \).

**Proposition 7.4.** The map \( J: X \hat{\otimes} E \to \mathcal{F}((X^\delta, \tau_P); E) \), defined by

\[
J(u)(g) = \sum_{i=1}^{n} (g(x_i) - g(y_i)) e_i
\]

for \( u = \sum_{i=1}^{n} \delta_{(x_i, y_i)} \otimes e_i \in X \hat{\otimes} E \) and \( g \in X^\delta \), is an isometric isomorphism. As a consequence, \( X \hat{\otimes} E \) is isometrically isomorphic to the closure in the operator norm topology of \( \mathcal{F}((X^\delta, \tau_P); E) \).

**Proof.** By Theorem [11,10] \( J \) is a linear bijection. If \( u = \sum_{i=1}^{n} \delta_{(x_i, y_i)} \otimes e_i \in X \hat{\otimes} E \), we have

\[
||J(u)|| = \sup \{ ||J(u)(g)|| : g \in B_{X^\delta} \}
\]

\[
= \sup \left\{ \phi \left( \sum_{i=1}^{n} (g(x_i) - g(y_i)) e_i \right) : g \in B_{X^\delta}, \phi \in B_{E_0} \right\}
\]

\[
= \sup \left\{ \sum_{i=1}^{n} (g(x_i) - g(y_i)) \langle \phi, e_i \rangle : g \in B_{X^\delta}, \phi \in B_{E_0} \right\}
\]

\[
= \varepsilon(u).
\]

The consequence is immediate. \( \square \)

Let \( \mathcal{F}(X) \) be the Lipschitz-free Banach space over a pointed metric space \( X \). Let us recall that \( \mathcal{F}(X) \) is the closed linear subspace of \( (X^\delta)^* \) spanned by the set \( \{ \delta_x : x \in X \} \), where for each \( x \in X \), \( \delta_x \) is the evaluation functional at the point \( x \) defined on \( X^\delta \). Combining Proposition \[7,23\] and Corollary \[11,11\] we can prove that if \( X \) is a pointed metric space, then \( X \hat{\otimes} E \) is isometrically isomorphic to \( \mathcal{F}(X) \); in fact, much more is true. We show below that the space \( X \hat{\otimes} E \) can be identified with the injective Banach-space tensor product \( \mathcal{F}(X) \hat{\otimes} E \). First, let us recall some fundamental properties of the space \( \mathcal{F}(X) \).

**Theorem 7.5.** \[11,23\] pp. 39-41] Let \( X, Y \) be pointed metric spaces, and \( E \) a Banach space.

(i) The dual of \( \mathcal{F}(X) \) is (canonically) isometrically isomorphic to \( X^\delta \), with the duality pairing given by \( \langle g, \delta_x \rangle = g(x) \) for all \( g \in X^\delta \) and \( x \in X \). Moreover, on bounded subsets of \( X^\delta \), the weak* topology coincides with the topology of pointwise convergence.

(ii) The map \( \iota_X: x \mapsto \delta_x \) is an isometric embedding of \( X \) into \( \mathcal{F}(X) \).

(iii) For any Lipschitz map \( T: X \to Y \) with \( T(0) = 0 \), there is a unique linear map \( \hat{T}: \mathcal{F}(X) \to \mathcal{F}(Y) \) such that \( \hat{T} \circ \iota_X = \iota_Y \circ T \). Furthermore, \( ||\hat{T}|| = \text{Lip}(T) \).

(iv) For any Lipschitz map \( T: X \to E \) with \( T(0) = 0 \), there is a unique linear map \( \hat{T}: \mathcal{F}(X) \to E \) such that \( \hat{T} \circ \iota_X = T \). Furthermore, \( ||\hat{T}|| = \text{Lip}(T) \).
It is because of the universal properties above that the space $\mathcal{F}(X)$ is called the Lipschitz-free space over $X$, or simply the free space over $X$. These spaces have been recently used as tools in nonlinear Banach space theory, see [13,19] and the survey [14].

**Proposition 7.6.** The map $I: X \boxtimes E \to \mathcal{F}(X) \otimes_{\pi} E$, defined by

$$I(u) = \sum_{i=1}^{n} (\delta_{x_i} - \delta_{y_i}) \otimes e_i$$

for $u = \sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i \in X \boxtimes E$, is a linear isometry. As a consequence, $X \boxtimes E$ is isometrically isomorphic to $\mathcal{F}(X) \otimes_{\pi} E$.

**Proof.** Let $u = \sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i \in X \boxtimes E$. Since $\mathcal{F}(X)^* \equiv X^d$, note that the norm of $\sum_{i=1}^{n} (\delta_{x_i} - \delta_{y_i}) \otimes e_i$ in $\mathcal{F}(X) \otimes_{\pi} E$ is given by

$$\sup \left\{ \left| \sum_{i=1}^{n} \langle g, \delta_{x_i} - \delta_{y_i} \rangle \langle \phi, e_i \rangle \right| : g \in B_{X^d}, \phi \in B_E \right\}.$$ 

Since $\langle g, \delta_{x_i} - \delta_{y_i} \rangle$ is precisely $g(x_i) - g(y_i)$, Proposition [1.6] shows that $I$ is well defined (and thus linear) and moreover a quick glance at Definition [7,1] shows that $I$ is an isometry.

Recall that the linear span of $\{\delta_{(x,y)}\}_{x,y \in X}$ is dense in $\mathcal{F}(X)$, hence the tensors of the form $\sum_{i=1}^{n} (\delta_{x_i} - \delta_{y_i}) \otimes e_i$, with $x_i, y_i \in X$ and $e_i \in E$, are dense in $\mathcal{F}(X) \otimes_{\pi} E$. This shows that the map $I$ has dense range, and thus $X \boxtimes E$ is isometrically isomorphic to $\mathcal{F}(X) \otimes_{\pi} E$. 

\[ \square \]

8. The Lipschitz projective norm

We introduce the Lipschitz projective norm on $X \boxtimes E$.

**Definition 8.1.** For each $u \in X \boxtimes E$, define:

$$\pi(u) = \inf \left\{ \sum_{i=1}^{n} d(x_i, y_i) \|e_i\| : u = \sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i \right\},$$

the infimum being taken over all representations of $u$.

**Theorem 8.1.** $\pi$ is a uniform and dualizable Lipschitz cross-norm on $X \boxtimes E$ such that $L \leq \pi$.

**Proof.** Let $u \in X \boxtimes E$ and let $\sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i$ be a representation of $u$. Using Definition [6.1], we have seen that $L(u) \leq \sum_{i=1}^{n} d(x_i, y_i) \|e_i\|$. Since this holds for every representation of $u$, it follows that $L(u) \leq \pi(u)$. Suppose that $\pi(u) = 0$. Since $L(u) \leq \pi(u)$ and $L$ is a norm on $X \boxtimes E$, then $u = 0$.

We check that $\pi(\lambda u) = |\lambda| \pi(u)$. If $\lambda \in \mathbb{R}$, then $\lambda u = \sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes (\lambda e_i)$ and so

$$\pi(\lambda u) \leq \sum_{i=1}^{n} d(x_i, y_i) \|\lambda e_i\| = |\lambda| \sum_{i=1}^{n} d(x_i, y_i) \|e_i\|.$$ 

Since the representation of $u$ is arbitrary, this implies that $\pi(\lambda u) \leq |\lambda| \pi(u)$. If $\lambda = 0$, we have $\pi(\lambda u) = 0 = |\lambda| \pi(u)$ since $\pi(u) \geq 0$ for all $u \in X \boxtimes E$. Assume that $\lambda \neq 0$. Similarly, we have $\pi(u) = \pi(\lambda^{-1} (\lambda u)) \leq |\lambda^{-1}| \pi(\lambda u)$, thus $|\lambda| \pi(u) \leq \pi(\lambda u)$ and hence $\pi(\lambda u) = |\lambda| \pi(u)$.

We show that $\pi(u + v) \leq \pi(u) + \pi(v)$ for all $u, v \in X \boxtimes E$. Let $\varepsilon > 0$. Then there are representations $u = \sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i$ and $v = \sum_{i=1}^{m} \delta_{(x_i',y_i')} \otimes e_i'$ such that $\sum_{i=1}^{n} d(x_i, y_i) \|e_i\| < \pi(u) + \varepsilon/2$ and $\sum_{i=1}^{m} d(x_i', y_i') \|e_i'\| < \pi(v) + \varepsilon/2$.

We can concatenate these representations to get a representation $\sum_{i=1}^{n+m} \delta_{(x_i',y_i')} \otimes e_i''$ for $u + v$ as in Lemma [1.3]. By
Since the value of 
therefore we conclude that

We now prove that

By the arbitrariness of

For every

Since the value of 

it follows that

Therefore the Lipschitz cross-norm \( \pi \) is dualizable by Proposition 4.2 and Remark 4.1.

Similarly, by applying Proposition 4.2 and Remark 4.1 we see that the Lipschitz cross-norm \( \pi \) is uniform. Let 

The value of 

is independent of the representation of 

by Lemma 5.1 and therefore we conclude that

\[ \pi \left( h \otimes T \left( \sum_{i=1}^{n} \delta_{(x,y,i)} \otimes e_{i} \right) \right) \leq \text{Lip}(h) \|T\| \pi \left( \sum_{i=1}^{n} \delta_{(x,y,i)} \otimes e_{i} \right). \]

\( \Box \)
The Lipschitz projective norm on $X \boxtimes E$ and the dual norm of the norm Lip of Lip$_p(X, E^*)$ induced on $X \boxtimes E$ coincide as we see next.

**Corollary 8.2.** Let $X$ be a pointed metric space and $E$ a Banach space. Then $\pi = L$ on $X \boxtimes E$.

**Proof.** By Theorem 8.1 $L \leq \pi$. To prove that $L \geq \pi$, suppose by contradiction that $L(u_0) < 1 < \pi(u_0)$ for some $u_0 \in X \boxtimes E$. Denote $B = \{u \in X \boxtimes E : \pi(u) \leq 1\}$. Clearly, $B$ is a closed and convex set in $X \boxtimes E$. Applying the Hahn–Banach separation theorem to $B$ and $[u_0]$ and taking into account Theorem 6.2 there exists some $f \in \text{Lip}_p(X, E^*)$ with $\text{Lip}(f) = 1$ such that $\text{Re}(\Lambda(f))(u_0) > 1 \geq \sup \{\text{Re}(\Lambda(f)(u) : u \in B\}$. Since $L(u_0) \geq |\Lambda(f)(u_0)| \geq \text{Re}(\Lambda(f))(u_0)$, then $L(u_0) > 1$ and this is a contradiction. □

**Theorem 8.3.** $\pi$ is the greatest Lipschitz cross-norm on $X \boxtimes E$.

**Proof.** We have seen in Theorem 8.1 that $\pi$ is a Lipschitz cross-norm on $X \boxtimes E$. Now, let $\alpha$ be a Lipschitz cross-norm on $X \boxtimes E$ and let $u \in X \boxtimes E$. If $\sum_{i=1}^{n} \delta_{(x_i, y_i)} \otimes e_i$ is a representation of $u$, we have

$$\alpha(u) = \sum_{i=1}^{n} \delta_{(x_i, y_i)} \otimes e_i \leq \sum_{i=1}^{n} \alpha(\delta_{(x_i, y_i)} \otimes e_i) = \sum_{i=1}^{n} d(x_i, y_i) \|e_i\|.$$ \hspace{1cm} \text{(1)}

Now the very definition of $\pi$ gives $\alpha(u) \leq \pi(u)$. □

Dualizable Lipschitz cross-norms on $X \boxtimes E$ are characterized by being between the Lipschitz injective and Lipschitz projective norms.

**Proposition 8.4.** A norm $\alpha$ on $X \boxtimes E$ is a dualizable Lipschitz cross-norm if and only if $e \leq \alpha \leq \pi$.

**Proof.** If $\alpha$ is a dualizable Lipschitz cross-norm on $X \boxtimes E$, then $e \leq \alpha \leq \pi$ by Theorems 8.2 and 8.3. Conversely, if $\alpha$ is a norm on $X \boxtimes E$ that lies between $e$ and $\pi$, then $\alpha(\delta_{(x_i, y_i)} \otimes e) = d(x, y) \|e\|$ follows immediately from the fact that $e$ and $\pi$ are Lipschitz cross-norms. Let $g \in X^*$ and $f \in E^*$. Then

$$\left| \sum_{i=1}^{n} (g(x_i) - g(y_i)) \langle \phi, e_i \rangle \right| \leq \text{Lip}(g) \|e\| \left( \sum_{i=1}^{n} \delta_{(x_i, y_i)} \otimes e_i \right) \leq \text{Lip}(g) \|e\| \alpha \left( \sum_{i=1}^{n} \delta_{(x_i, y_i)} \otimes e_i \right)$$

for all $\sum_{i=1}^{n} \delta_{(x_i, y_i)} \otimes e_i \in X \boxtimes E$, and so the Lipschitz cross-norm $\alpha$ is dualizable. □

Let us recall that a completion of a normed space $E$ is a Banach space $\tilde{E}$ that includes a dense linear subspace isometric to $E$. Every normed space has a completion and the completion is unique up to isometric isomorphism. By [9, Lemma 3.10], every element $\tilde{e}$ of the completion $\tilde{E}$ of $E$ can be written as $\tilde{e} = \sum_{n=1}^{\infty} e_n$, where $e_n \in E$ and $\sum_{n=1}^{\infty} \|e_n\| < \infty$. Moreover, $\|\tilde{e}\| = \inf \{\sum_{n=1}^{\infty} \|e_n\| \}$. The infimum is taken over all series in $E$ summing up to $\tilde{e}$. Combining this with Definition 8.1, we obtain the following result.

**Theorem 8.5.** Every element $u \in X \boxtimes E$ admits a representation

$$u = \sum_{i=1}^{\infty} \delta_{(x_i, y_i)} \otimes e_i$$

such that

$$\sum_{i=1}^{\infty} d(x_i, y_i) \|e_i\| < \infty.$$ \hspace{1cm} \text{(2)}

Moreover,

$$\pi(u) = \inf \left\{ \sum_{i=1}^{\infty} d(x_i, y_i) \|e_i\| : u = \sum_{i=1}^{\infty} \delta_{(x_i, y_i)} \otimes e_i, \sum_{i=1}^{\infty} d(x_i, y_i) \|e_i\| < \infty \right\}.$$ \hspace{1cm} \text{(3)}

The dual pairing satisfies the formula

$$\left( \sum_{i=1}^{\infty} \delta_{(x_i, y_i)} \otimes e_i, f \right) = \sum_{i=1}^{\infty} (f(x_i) - f(y_i), e_i)$$

for all $f \in E^*$.
for all \( f \in \text{Lip}_0(X, E^*) \).

Next we consider the boundedness of the linear operator \( h \otimes T : X \otimes E \to Y \otimes F \) for the Lipschitz projective norms.

**Proposition 8.6.** Let \( X, Y \) be pointed metric spaces and \( E, F \) Banach spaces. Let \( h \in \text{Lip}_0(X,Y) \) and \( T \in \mathcal{L}(E,F) \). Then there exists a unique bounded linear operator \( h \otimes_T : X \hat{\otimes} Y \to Y \hat{\otimes} F \) such that \( (h \otimes_T)(u) = (h \otimes T)(u) \) for all \( u \in X \otimes E \). Furthermore, \( \| h \otimes_T \| = \text{Lip}(h) \| T \| \).

**Proof.** Let \( u \in X \otimes E \) and let \( \sum_{i=1}^n \delta_{(x_i,y_i)} \otimes e_i \) be a representation of \( u \). We have

\[
\pi((h \otimes T)(u)) = \pi \left( \sum_{i=1}^n \delta_{(x_i,y_i)} \otimes T(e_i) \right) \leq \sum_{i=1}^n \| h(x_i), h(y_i) \| \| T(e_i) \| \leq \text{Lip}(h) \| T \| \sum_{i=1}^n \| d(x_i, y_i) \| e_i \|.\]

An appeal to Definition \[8.1\] yields \( \pi((h \otimes T)(u)) \leq \text{Lip}(h) \| T \| \pi(u) \). Therefore \( h \otimes T \) is bounded from \( X \otimes E \) to \( Y \otimes F \) and \( \| h \otimes T \| \leq \text{Lip}(h) \| T \| \). Moreover, the converse estimate follows easily from

\[
d(h(x), h(y)) \| T(e) \| = \pi(\delta_{(x,y)} \otimes T(e)) = \pi((h \otimes T)(\delta_{(x,y)} \otimes e))) \leq \| h \otimes T \| \| d(x,y) \| e\|
\]

for all \( x, y \in X \) and \( e \in E \). Thus, we have \( \| h \otimes T \| = \text{Lip}(h) \| T \| \). Finally, it is well known that \( h \otimes T \) has a unique bounded linear extension to an operator \( h \otimes_T : X \hat{\otimes} Y \to Y \hat{\otimes} F \) with \( \| h \otimes_T \| = \text{Lip}(h) \| T \| \).

It turns out that there is a very close relationship between the Lipschitz projective norm and the projective tensor product of Banach spaces. In fact, just as it was the case for the injective norm in Proposition \[7.6\], the Lipschitz projective norm on \( X \otimes E \) can be identified with the projective norm on the tensor product of \( \mathcal{F}(X) \) and \( E \). The authors wish to thank Richard Haydon for suggesting that this might be true.

**Proposition 8.7.** The map \( I : X \otimes \pi \to \mathcal{F}(X) \otimes \pi \), defined by

\[
I(u) = \sum_{i=1}^n \delta_{(x_i,y_i)} \otimes e_i
\]

for \( u = \sum_{i=1}^n \delta_{(x_i,y_i)} \otimes e_i \in X \otimes E \), is a linear isometry. Moreover, \( X \hat{\otimes} E \) is isometrically isomorphic to \( \mathcal{F}(X) \hat{\otimes} E \).

**Proof.** As in the proof of Proposition \[7.6\], Proposition \[1.6\] guarantees that the map \( I \) is well defined (and it is clearly linear). Letting \( u = \sum_{i=1}^n \delta_{(x_i,y_i)} \otimes e_i \in X \otimes E \), note that using the fact that the map \( x \mapsto \delta_x \) is an isometry from \( X \) into \( \mathcal{F}(X) \),

\[
\left\| \sum_{i=1}^n (\delta_{x_i} - \delta_{y_i}) \otimes e_i \right\|_{\mathcal{F}(X) \hat{\otimes} E} \leq \sum_{i=1}^n \left\| \delta_{x_i} - \delta_{y_i} \right\|_{\mathcal{F}(X)} \| e_i \| = \sum_{i=1}^n \| d(x_i, y_i) \| e_i \|.\]

Taking the infimum over all representations of \( u \), we conclude that \( \| I(u) \| \leq \| u \| \). Let \( \eta > 0 \) be given. From Corollary \[8.2\] there exists \( f \in \text{Lip}_0(X, E^*) \) with \( \text{Lip}(f) \leq 1 \) such that \( \langle u, f \rangle > \| u \| - \eta \), where the pairing is the one given in Theorem \[8.5\]. By Theorem \[7.5\] the linear extension \( \hat{f} : \mathcal{F}(X) \to E^* \) of \( f \) has norm at most one. From the properties of the projective tensor product of Banach spaces, the dual of \( \mathcal{F}(X) \hat{\otimes} E \) can be identified with \( \mathcal{L}(\mathcal{F}(X), E^*) \), where the pairing is given by

\[
\left\langle \sum_{i=1}^n y_i \otimes e_i, T \right\rangle = \sum_{i=1}^n \langle T y_i, e_i \rangle \quad (y_i, e_i \in \mathcal{F}(X), e_i \in E, T \in \mathcal{L}(\mathcal{F}(X), E^*)).\]
In particular,
\[ ||I(u)|| \geq \langle I(u), \hat{f} \rangle = \sum_{i=1}^{n} \langle \hat{f}(\delta_{x_i} - \delta_{y_i}), e_i \rangle = \sum_{i=1}^{n} \langle f(x_i) - f(y_i), e_i \rangle = \langle u, f \rangle > ||u|| - \eta.\]

Letting \( \eta \) go to 0, we obtain that \( ||I(u)|| \geq ||u|| \), and thus \( I \) is a linear isometry. Since the sums of the form \( \sum_{i=1}^{n} \delta_{(x_i, y_i)} \otimes e_i \) and \( \sum_{i=1}^{n} (\delta_{x_i} - \delta_{y_i}) \otimes e_i \) are dense respectively in \( X \widehat{\otimes}_{\pi} E \) and \( \mathcal{F}(X) \widehat{\otimes}_{\pi} E \), \( I \) extends to an isometric isomorphism between \( X \widehat{\otimes}_{\pi} E \) and \( \mathcal{F}(X) \widehat{\otimes}_{\pi} E \). \( \square \)

Proposition 8.7 implies in particular that given a pointed metric space \( X \), there is a Banach space \( A \) such that \( X \widehat{\otimes}_{\pi} E \) is isometric to \( A \widehat{\otimes}_{\pi} E \) for every Banach space \( E \). The authors would like to thank Jesús Castillo for pointing out a result in categorical Banach space theory that shows this was to be expected. Without going into all the details, let us outline the argument. First, a theorem of Fuchs [11] Section 6] (a nice presentation can be found in [3] Proposition 5.6), where the reader can also find the definitions of the categorical terms we use below) states the following: if \( \mathcal{F} \), \( \mathcal{G} \) are two covariant Banach functors such that for any Banach spaces \( E \) and \( F \), we have that \( \mathcal{L}(\mathcal{G}(E), F) \) is linearly isometric to \( \mathcal{L}(E, \mathcal{G}(F)) \), then there exists a Banach space \( A \) such that for every Banach space \( E \), \( \mathcal{G}(E) \) is linearly isometric to \( A \widehat{\otimes}_{\pi} E \) and \( \mathcal{G}(F) \) is linearly isometric to \( L(A, F) \). Now consider a fixed pointed metric space \( X \). Note that it induces two covariant Banach functors \( X \widehat{\otimes}_{\pi} (-) \) and \( \mathcal{L}(X, -) \). Arguments closely related to those that led us to prove Corollary 8.2 show that, for any Banach spaces \( E \) and \( F \), we have \( \mathcal{L}(X \widehat{\otimes}_{\pi} E, F) \) is linearly isometric to \( \mathcal{L}(E, \mathcal{L}(X, F)) \), so Fuchs’ result applies.

The space \( X \widehat{\otimes}_{\pi} E \) is called the projective Lipschitz tensor product of \( X \) and \( E \). This term derives from the following result. Before stating it, recall that a Lipschitz map \( f: X \to Z \) is called C-co-Lipschitz if for every \( x \in X \) and \( r > 0 \), \( f(B(x, r)) \supset B(f(x), r/C) \). Moreover, it is called a Lipschitz quotient if it is surjective, Lipschitz and co-Lipschitz.

**Theorem 8.8.** Let \( X, Y \) be pointed metric spaces and \( q: X \to Z \) a Lipschitz quotient that is \( l \)-Lipschitz and \( C \)-co-Lipschitz for every \( C > 1 \). Let \( E \) be a Banach space, \( E_0 \) a closed linear subspace of \( E \) and \( Q: E \to E/E_0 \) the natural quotient map. Then \( q \otimes Q: X \widehat{\otimes}_{\pi} E \to Z \widehat{\otimes}_{\pi} (E/E_0) \) is a quotient operator.

**Proof.** Thanks to Proposition 8.7 and the behavior of the projective tensor norm with respect to quotients [20] Proposition 2.5], it suffices to prove that if \( q: X \to Z \) is such a Lipschitz quotient then the induced map \( \tilde{q}: \mathcal{F}(X) \to \mathcal{F}(Z) \) is a linear quotient operator. Notice that from Theorem 7.8 \( ||\tilde{q}|| = \text{Lip}(q) = 1 \). Now let \( u \in \mathcal{F}(Z) \), and let \( \varepsilon > 0 \). From Proposition 8.7 \( \mathcal{F}(X) \equiv \mathcal{F}(X) \widehat{\otimes}_{\pi} E \equiv X \widehat{\otimes}_{\pi} E \). Thus from Theorem 8.7 there exists a representation \( u = \sum_{j=1}^{\infty} \delta_{(x_j, z_j)} \otimes a_j \) such that \( (1 + \varepsilon)\text{Lip}(a_j) \geq \sum_{j=1}^{\infty} |a_j| d(z_j, z'_j) \). For each \( j \), choose \( x_j, x'_j \in X \) such that \( q(x_j) = z_j \), \( q(x'_j) = z'_j \) and \( d(x_j, x'_j) \leq (1 + \varepsilon) d(z_j, z'_j) \). Setting \( u' = \sum_{j=1}^{\infty} \delta_{(x_j, x'_j)} \otimes a_j \), clearly \( \tilde{q}(u') = u \) (so it follows that \( \tilde{q} \) is surjective) and
\[ \text{Lip}(u') \leq \sum_{j=1}^{\infty} |a_j| d(x_j, x'_j) \leq (1 + \varepsilon) \sum_{j=1}^{\infty} |a_j| d(z_j, z'_j) \leq (1 + \varepsilon)^2 \text{Lip}(u). \]

Since this holds for all \( \varepsilon > 0 \), it follows that \( \text{Lip}(u) = \inf \{ \text{Lip}(u'): \tilde{q}(u') = u \} \). \( \square \)

In a similar manner, the projective norm respects complemented subspaces. We say that a subset \( Z \subset X \) that contains the point 0 is a Lipschitz retract of \( X \), or that it is Lipschitz complemented in \( X \), if there exists a Lipschitz map (called a Lipschitz retraction) \( r: X \to Z \) such that \( r(z) = z \) for all \( z \in Z \).

**Proposition 8.9.** Let \( Z \) be a Lipschitz retract of \( X \), and let \( F \) be a complemented subspace of \( E \). Then \( Z \widehat{\otimes}_{\pi} F \) is complemented in \( X \widehat{\otimes}_{\pi} E \) and the norm on \( Z \widehat{\otimes}_{\pi} F \) induced by the Lipschitz projective norm of \( X \widehat{\otimes}_{\pi} E \) is equivalent to the Lipschitz projective norm on \( Z \widehat{\otimes}_{\pi} F \). If \( Z \) is Lipschitz complemented with a Lipschitz retraction of Lipschitz constant one and \( F \) is complemented by a linear projection of norm one, then \( Z \widehat{\otimes}_{\pi} F \) is a subspace of \( X \widehat{\otimes}_{\pi} E \) and is also complemented by a projection of norm one.
Proof. This follows from the corresponding result for the projective tensor product (see [20, Proposition 2.4]), after noting that a Lipschitz retraction \( r: X \to Z \) (that in particular sends 0 to 0) extends to a linear projection \( \tilde{r}: \mathcal{F}(X) \to \mathcal{F}(Z) \subset \mathcal{F}(X) \) with \( \|\tilde{r}\| = \text{Lip}(r) \).

Calculating the projective norm of an element in a tensor product of Banach spaces is generally difficult, but there is a particular case where the calculation is relatively easy: for any Banach space \( E \), \( \ell_1 \hat{\otimes}_\pi E \) is isometrically isomorphic to \( \ell_1(E) \) (see [20, Example 2.6]). In the nonlinear setting, trees play a role analogous to that of \( \ell_1 \) in the linear theory, so the following result is not surprising.

**Proposition 8.10.** Let \( T = (X,E) \) be a graph with finite vertex set \( X \) and edge set \( E \) which is a tree, that is, it contains no cycles. Consider \( T \) as a pointed metric space, with distance function given by the shortest-path distance and a distinguished fixed point \( 0 \in X \). Let \( G \) be a Banach space. Then \( T \hat{\otimes}_\pi G \) is isometrically isomorphic to \( \ell_1(E;G) \).

Proof. We say that a vertex \( x \in X \) is positive (negative) if it is at an even (respectively, odd) distance from 0 \( \in X \). Note that, since \( T \) is a tree, the endpoints of every edge in \( E \) have different parities. Therefore every edge \( \{x,y\} \) in \( E \) will be written as \( (x,y) \) with \( x \) negative and \( y \) positive.

Consider \( x,y \in X \). Let \( n = d(x,y) \) and \( \{x = z_0, z_1, \ldots, z_n = y\} \) be the unique minimal-length path in \( T \) joining \( x \) and \( y \). Since, for each \( v \in G \),

\[
\|v\| d(x,y) = \sum_{i=1}^{n} \|v\| d(z_i, z_{i-1}),
\]

in order to calculate \( \pi(u) \) for \( u \in T \hat{\otimes}_\pi G \), it suffices to consider only representations involving \( \delta_{(x,y)} \) with \( (x_i, y_i) \in E \). By the triangle inequality, in the representation we can consolidate all terms corresponding to the same edge \( (x_i, y_i) \in E \), so we can consider only representations of the form

\[
u = \sum_{(x,y) \in E} \delta_{(x,y)} \otimes v_{(x,y)}.\]

But, for each \( u \in T \hat{\otimes}_\pi G \), there is only one such representation, something easily seen by induction on the size of the tree. If we define \( J: T \hat{\otimes}_\pi G \to \ell_1(E;G) \) by \( u \mapsto (v_{(x,y)})_{(x,y) \in E} \), \( J \) is then clearly a linear isometry that extends to an isometric isomorphism between \( T \hat{\otimes}_\pi G \) and \( \ell_1(E;G) \).

More generally, in the linear case we have that \( \ell_1(\mu) \hat{\otimes}_\pi E \) is isometrically isomorphic to \( L_1(\mu; E) \) for any measure \( \mu \) (see [20, Example 2.19]). In our nonlinear setting, a possible analogue will be given by a generalization of Proposition 8.10 to a more general class of metric trees. This will depend heavily on the identification of the Lipschitz-free space over such trees carried out in [12]. Before stating the result, let us recall a definition. An \( \mathbb{R} \)-tree is a metric space \( X \) satisfying the following two conditions: (1) For any points \( a \) and \( b \) in \( X \), there exists a unique isometry \( \phi \) of the closed interval \( [0, d(a, b)] \) into \( X \) such that \( \phi(0) = a \) and \( \phi(d(a, b)) = b \); (2) Any one-to-one continuous mapping \( \varphi: [0, 1] \to X \) has the same range as the isometry \( \phi \) associated to the points \( a = \varphi(0) \) and \( b = \varphi(1) \).

**Corollary 8.11.** Let \( X \) be an \( \mathbb{R} \)-tree and \( E \) a Banach space. Then there exists a measure \( \mu \) such that \( X \hat{\otimes}_\pi E \) is isometric to \( L_1(\mu; E) \).

Proof. By [12, Corollary 3.3], there exists a measure \( \mu \) such that \( \mathcal{F}(X) \) is isometrically isomorphic to \( L_1(\mu) \). From Proposition 8.7, \( X \hat{\otimes}_\pi E \) is isometrically isomorphic to \( \mathcal{F}(X) \hat{\otimes}_\pi E \). Finally, from [20, Example 2.19], \( L_1(\mu) \hat{\otimes}_\pi E \) is isometric to \( L_1(\mu; E) \).

9. The Lipschitz \( p \)-nuclear norms

We introduce now the Lipschitz \( p \)-nuclear norms \( d_p \) on \( X \hat{\otimes}_\pi E \) for \( 1 \leq p \leq \infty \). They are Lipschitz versions of the known tensor norms of Chevet [8] and Saphar [21]. Similar versions were introduced in [31] on spaces of \( E \)-valued molecules on \( X \), where they were shown to be in duality with spaces of Lipschitz \( p' \)-summing maps.
Definition 9.1. Let $1 \leq p \leq \infty$. Let $E$ be a Banach space and let $e_1, \ldots, e_n \in E$. Define:

$$
\|(e_1, \ldots, e_n)\|_p = \begin{cases} 
\left( \sum_{i=1}^{n} |e_i|^p \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\
\max_{1 \leq i \leq n} |e_i| & \text{if } p = \infty.
\end{cases}
$$

Let $X$ be a pointed metric space and let $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$. Define:

$$
\left\| (\delta_{(x_1, y_1)}, \ldots, \delta_{(x_n, y_n)}) \right\|_{L^p} = \begin{cases} 
\sup_{g \in B_{E^*}} \left( \sum_{i=1}^{n} |g(x_i) - g(y_i)|^p \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\
\sup_{1 \leq i \leq n} \max |g(x_i) - g(y_i)| & \text{if } p = \infty.
\end{cases}
$$

Let $p'$ be the conjugate index of $p$ defined by $p' = p/(p-1)$ if $p \neq 1$, $p' = \infty$ if $p = 1$, and $p' = 1$ if $p = \infty$. For each $u \in X \boxtimes E$, define:

$$
d_p(u) = \inf \left\{ \left\| (\delta_{(x_1, y_1)}, \ldots, \delta_{(x_n, y_n)}) \right\|_{L^{p'}} \left\| (e_1, \ldots, e_n) \right\|_p : u = \sum_{i=1}^{n} \delta_{(x_i, y_i)} \otimes e_i \right\},
$$

the infimum being taken over all representations of $u$.

Theorem 9.1. For $1 \leq p \leq \infty$, $d_p$ is a uniform and dualizable Lipschitz cross-norm on $X \boxtimes E$.

Proof. Let $u \in X \boxtimes E$ and let $\sum_{i=1}^{n} \delta_{(x_i, y_i)} \otimes e_i$ be a representation of $u$. Clearly, $d_p(u) \geq 0$. Let $\lambda \in \mathbb{K}$. Since $\sum_{i=1}^{n} \delta_{(x_i, y_i)} \otimes (\lambda e_i)$ is a representation of $\lambda u$, Definition 9.1 yields

$$
d_p(\lambda u) \leq \left\| (\delta_{(x_1, y_1)}, \ldots, \delta_{(x_n, y_n)}) \right\|_{L^{p'}} \left\| (\lambda e_1, \ldots, \lambda e_n) \right\|_p = |\lambda| \left\| (\delta_{(x_1, y_1)}, \ldots, \delta_{(x_n, y_n)}) \right\|_{p'} \left\| (e_1, \ldots, e_n) \right\|_p.
$$

If $\lambda = 0$, it follows that $d_p(\lambda u) = 0 = |\lambda| d_p(u)$. For $\lambda \neq 0$, since the preceding inequality holds for every representation of $u$, we deduce that $d_p(\lambda u) \leq |\lambda| d_p(u)$. For the converse estimate, note that $d_p(u) = d_p(\lambda^{-1}(\lambda u)) \leq |\lambda^{-1}| d_p(\lambda u)$ by using the proved inequality, thus $|\lambda| d_p(u) \leq d_p(\lambda u)$ and hence $d_p(\lambda u) = |\lambda| d_p(u)$.

We prove the triangular inequality for $d_p$ as follows. Let $u, v \in X \boxtimes E$ and let $\varepsilon > 0$. If $u = 0$ or $v = 0$, there is nothing to prove. Assume $u \neq 0 \neq v$. We can choose representations for $u$ and $v$ say

$$
u = \sum_{i=1}^{m} \delta_{(x_i', y_i')} \otimes e_i',
$$
such that

$$
\left\| (\delta_{(x_1', y_1')}, \ldots, \delta_{(x_n', y_n')}) \right\|_{L^{p'}} \left\| (e_1', \ldots, e_n') \right\|_p \leq d_p(u) + \varepsilon
$$
and

$$
\left\| (\delta_{(x_1', y_1')}, \ldots, \delta_{(x_n', y_n')}) \right\|_{L^{p'}} \left\| (e_1', \ldots, e_n') \right\|_p \leq d_p(v) + \varepsilon.
$$

Fix $r, s \in \mathbb{R}^+$ arbitrarily and define

$$
\delta_{(x_i', y_i')} = \begin{cases} 
re_i & \text{if } i = 1, \ldots, n, \\
s^{-1} \delta_{(x_{i-n}', y_{i-n}')}, & \text{if } i = n+1, \ldots, n+m
\end{cases},
$$
and

$$
e_i'' = \begin{cases} 
e_i & \text{if } i = 1, \ldots, n, \\
en_i^{m-n} & \text{if } i = n+1, \ldots, n+m.
\end{cases}
$$

It is plain that $u + v = \sum_{i=1}^{n+m} \delta_{(x_i', y_i') \otimes e_i''}$ and therefore we have

$$
d_p(u + v) \leq \left\| (\delta_{(x_1', y_1')}, \ldots, \delta_{(x_{n+m}', y_{n+m}')}} \right\|_{p'} \left\| (e_1'', \ldots, e_{n+m}'') \right\|_p.
$$

We distinguish three cases.
Case 1. $1 < p < \infty$. An easy verification yields
\[
\left( \left\| \delta(x_{1}^\prime, y_{1}^\prime), \ldots, \delta(x_{n+m}^\prime, y_{n+m}^\prime) \right\|_{p}^{\prime} \right)^{r} \leq \left( \left\| \delta(x_{1}, y_{1}), \ldots, \delta(x_{n}, y_{n}) \right\|_{p}^{L_w} \right)^{r} + \left( s \left\| \delta(x_{1}^\prime, y_{1}^\prime), \ldots, \delta(x_{n}^\prime, y_{n}^\prime) \right\|_{p}^{L_w} \right)^{r}
\]
and
\[
\left( \left\| e_{1}^\prime, \ldots, e_{n+m}^\prime \right\|_{p}^{p} = \left( \left\| e_{1}, \ldots, e_{n} \right\|_{p}^{p} + s \left\| e_{1}^\prime, \ldots, e_{m}^\prime \right\|_{p}^{p} \right)^{p}.
\]
Using Young’s inequality, it follows that
\[
d_{p}(u + v) \leq \left\| \delta(x_{1}^\prime, y_{1}^\prime), \ldots, \delta(x_{n+m}^\prime, y_{n+m}^\prime) \right\|_{p}^{\prime} \left\| e_{1}^\prime, \ldots, e_{n+m}^\prime \right\|_{p}^{p} \leq \left( \left\| \delta(x_{1}, y_{1}), \ldots, \delta(x_{n}, y_{n}) \right\|_{p}^{L_w} \right)^{r} + \left( \left\| e_{1}, \ldots, e_{n} \right\|_{p}^{p} \right)^{r}
\]
and
\[
d_{p}(u + v) \leq \left( \left\| \delta(x_{1}^\prime, y_{1}^\prime), \ldots, \delta(x_{n+m}^\prime, y_{n+m}^\prime) \right\|_{p}^{L_w} \right)^{r} + \left( \left\| e_{1}, \ldots, e_{n} \right\|_{p}^{p} \right)^{r}.
\]
Since $r, s$ were arbitrary in $\mathbb{R}^+$, taking
\[
r = (d_{p}(u) + \varepsilon)^{-1/p}, \quad s = (d_{p}(v) + \varepsilon)^{-1/p},
\]
we obtain that $d_{p}(u + v) \leq d_{p}(u) + d_{p}(v) + 2\varepsilon$.

Case 2. $p = 1$. Now we have
\[
d_{1}(u + v) \leq \left\| \delta(x_{1}^\prime, y_{1}^\prime), \ldots, \delta(x_{n+m}^\prime, y_{n+m}^\prime) \right\|_{1}^{\prime} \left\| e_{1}^\prime, \ldots, e_{n+m}^\prime \right\|_{1}
\]
and taking, in particular, $r = \left\| \delta(x_{1}, y_{1}), \ldots, \delta(x_{n}, y_{n}) \right\|_{1}^{L_w}$ and $s = \left\| \delta(x_{1}^\prime, y_{1}^\prime), \ldots, \delta(x_{n}^\prime, y_{n}^\prime) \right\|_{1}^{L_w}$, we infer that $d_{1}(u + v) \leq d_{1}(u) + d_{1}(v) + 2\varepsilon$.

Case 3. $p = \infty$. We have
\[
d_{\infty}(u + v) \leq \left\| \delta(x_{1}^\prime, y_{1}^\prime), \ldots, \delta(x_{n+m}^\prime, y_{n+m}^\prime) \right\|_{\infty}^{\prime} \left\| e_{1}^\prime, \ldots, e_{n+m}^\prime \right\|_{\infty}
\]
and for $r = \left\| e_{1}, \ldots, e_{n} \right\|_{\infty}$ and $s = \left\| e_{1}^\prime, \ldots, e_{m}^\prime \right\|_{\infty}$, we deduce that $d_{\infty}(u + v) \leq d_{\infty}(u) + d_{\infty}(v) + 2\varepsilon$.

In any case, $d_{p}(u + v) \leq d_{p}(u) + d_{p}(v) + 2\varepsilon$ and so $d_{p}(u + v) \leq d_{p}(u) + d_{p}(v)$ by the arbitrariness of $\varepsilon$. Hence $d_{p}$ is a seminorm for $1 \leq p \leq \infty$. Now, we claim that $\varepsilon(u) \leq d_{p}(u) \leq \pi(u)$. Indeed, we have
\[
\left\| \sum_{i=1}^{n} (g(x_{i}) - g(y_{i})) \phi, e_{i} \right\| \leq \sum_{i=1}^{n} \left\| g(x_{i}) - g(y_{i}) \right\| \left\| e_{i} \right\| \leq \left\| \delta(x_{1}, y_{1}), \ldots, \delta(x_{n}, y_{n}) \right\|_{p}^{L_w} \left\| (e_{1}, \ldots, e_{n}) \right\|_{p}
\]
for every $g \in B_{X}$ and $\phi \in B_{F_{p}}$, where we have used Hölder’s inequality in the case $1 < p < \infty$. Therefore $\varepsilon(u) \leq \left\| \delta(x_{1}, y_{1}), \ldots, \delta(x_{n}, y_{n}) \right\|_{p}^{L_w} \left\| (e_{1}, \ldots, e_{n}) \right\|_{p}$, and since it holds for each representation of $u$, we deduce that $\varepsilon(u) \leq d_{p}(u)$. Since $\varepsilon$ is a Lipschitz cross-norm, this implies that $d_{p}$ is a norm and that
\[
\left\| e \right\| d(x, y) = \varepsilon(\delta_{x, y} \otimes e) \leq d_{p}(\delta_{x, y} \otimes e)
\]
for all \( x, y \in X \) and \( e \in E \). Moreover, Definition \[9.1\] gives

\[
d_p(\delta_{(x,y)} \otimes e) \leq \|e\| \sup_{g \in B^q} |g(x) - g(y)| = \|e\| d(x,y).
\]

Hence \( d_p \) is a Lipschitz cross-norm. Then \( d_p \leq \pi \) by Theorem \[8.3\] as we wanted. Now our claim implies that \( d_p \) is dualizable by Proposition \[8.4\].

Finally, to prove that \( d_p \) is uniform, take \( h \in \text{Lip}_0(X, X) \) and \( T \in \mathcal{L}(E, E) \). Let \( u \in X \otimes E \) and pick a representation \( \sum_{i=1}^n \delta_{(x_i,y_i)} \otimes e_i \) for \( u \). We have

\[
\left\| (\delta_{(h(x_1),h(y_1))}, \ldots, \delta_{(h(x_n),h(y_n))}) \right\|_{L^p} \leq \text{Lip}(h) \left\| (\delta_{(x_1,y_1)}, \ldots, \delta_{(x_n,y_n)}) \right\|_{L^p} \|T\| \|(e_1, \ldots, e_n)\|_p.
\]

Taking infimum over all the representations of \( u \), it follows that \( d_p(h \otimes T)(u) \leq \text{Lip}(h) \|T\| d_p(u) \). \( \square \)

Next we show that the Lipschitz 1-nuclear norm \( d_1 \) is justly the Lipschitz projective norm \( \pi \).

**Proposition 9.2.** For every \( u \in X \otimes E \),

\[
d_1(u) = \inf \left\{ \sum_{i=1}^n d(x_i,y_i) \|e_i\| : u = \sum_{i=1}^n \delta_{(x_i,y_i)} \otimes e_i \right\}
\]

taking the infimum over all representations of \( u \).

**Proof.** Let \( u \in X \otimes E \) and let \( \sum_{i=1}^n \delta_{(x_i,y_i)} \otimes e_i \) be a representation of \( u \). We have

\[
\pi(u) \leq \sum_{i=1}^n d(x_i,y_i) \|e_i\|
\]

\[
= \sum_{i=1}^n \left( \sup_{g \in B^q} |g(x_i) - g(y_i)| \right) \|e_i\|
\]

\[
\leq \sum_{i=1}^n \max_{1 \leq i \leq m} \left( \sup_{g \in B^q} |g(x_i) - g(y_i)| \right) \|e_i\|
\]

\[
= \left\| (\delta_{(x_1,y_1)}, \ldots, \delta_{(x_n,y_n)}) \right\|_{L^\infty} \|(e_1, \ldots, e_n)\|_1
\]

and therefore \( \pi(u) \leq d_1(u) \). The converse inequality follows from Theorems \[9.1\] and \[8.3\]. \( \square \)

10. **Lipschitz approximable operators**

The notions of Lipschitz compact operators and Lipschitz approximable operators from \( X \) to \( E \) were introduced in \[16\]. Let us recall that a Lipschitz operator \( f \in \text{Lip}_0(X, E) \) is said to be Lipschitz compact if its Lipschitz image \((f(x) - f(y))/d(x,y) : x, y \in X, x \neq y\) is relatively compact in \( E \) and \( f \) is said to be Lipschitz approximable if it is the limit in the Lipschitz norm Lip of a sequence of Lipschitz finite-rank operators from \( X \) to \( E \).

We show that the spaces of Lipschitz finite-rank operators and Lipschitz approximable operators can be identified as spaces of continuous linear functionals.

**Theorem 10.1.** The map \( K : X^a \mathrel{\otimes}_{\text{op}} E^\ast \rightarrow \text{Lip}_0(X, E^\ast) \), defined by

\[
K \left( \sum_{j=1}^m g_j \otimes \phi_j \right) = \sum_{j=1}^m g_j \cdot \phi_j,
\]

is an isometric isomorphism. As a consequence, the space of all Lipschitz approximable operators from \( X \) to \( E^\ast \) is isometrically isomorphic to \( X^a \mathrel{\otimes}_{\text{op}} E^\ast \).
Proof. By Theorem 2.5, K is a linear bijection. For any $\sum_{j=1}^{m} g_j \otimes \phi_j \in X^{\#} \boxtimes E^{\#}$, we have

$$\pi'\left(\sum_{j=1}^{m} g_j \otimes \phi_j\right) = \sup \left\{ \left(\sum_{j=1}^{m} g_j \otimes \phi_j\right) \left(\sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i\right) : \pi' \left(\sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i\right) \leq 1 \right\}$$

$$= \sup \left\{ \left(\sum_{j=1}^{m} g_j \otimes \phi_j\right) - \left(\sum_{j=1}^{m} g_j \otimes \phi_j\right) \left(\sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i\right) : \pi' \left(\sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i\right) \leq 1 \right\}$$

$$= \left\| \Lambda \left(\sum_{j=1}^{m} g_j \otimes \phi_j\right) \right\|$$

$$= \text{Lip} \left(\sum_{j=1}^{m} g_j \otimes \phi_j\right)$$

by using Proposition 5.1, Lemmas 2.2 and 1.4, Corollary 8.2, and Theorem 6.2. Hence K is an isometry. The consequence follows from a known result of Functional Analysis.

From Theorems 6.2 and 10.1 and Corollary 8.2, we infer the next consequence.

Corollary 10.2. The space $X^{\#} \boxtimes_{\text{c}} E^{\#}$ is isometrically isomorphic to $(X \boxtimes_{\text{c}} E)^{\#}$ if and only if each Lipschitz operator from X to $E^{\#}$ is Lipschitz approximable.

We recall that a Banach space E is said to have the approximation property if given a compact set $K \subset E$ and $\varepsilon > 0$, there is a finite-rank bounded linear operator $T : E \to E$ such that $\|Tx - x\| < \varepsilon$ for every $x \in K$. The approximation property was thoroughly studied by Grothendieck in [15]. In [16, Corollary 2.5], it was shown that $X^{\#}$ has the approximation property if and only if the space of all Lipschitz approximable operators from X to E is the space of all Lipschitz compact operators from X to E. Using this fact and Theorem 10.1, we derive the following result.

Corollary 10.3. Let $X$ be a pointed metric space such that $X^{\#}$ has the approximation property. Then, for any Banach space E, the space of all Lipschitz compact operators from X to $E^{\#}$ is isometrically isomorphic to $X^{\#} \boxtimes_{\text{c}} E^{\#}$.

By Theorem 6.2 and Corollary 10.3, we have the following.

Corollary 10.4. Let $X$ be a pointed metric space such that $X^{\#}$ has the approximation property and let E be a Banach space. Then $X^{\#} \boxtimes_{\text{c}} E^{\#}$ is isometrically isomorphic to $(X \boxtimes_{\text{c}} E)^{\#}$ if and only if each Lipschitz operator from X to $E^{\#}$ is Lipschitz compact.

We close this section with a new formula for the norm $\pi'$. By [16, Lemma 1.1], the closed unit ball of the Lipschitz-free Banach space $\mathcal{F}(X)$ over a pointed metric space $X$ coincides with the closure of the convex balanced hull of the set $\{(\delta_x - \delta_y)/d(x,y) : x, y \in X, x \neq y\}$ in $(X^{\#})^*$, where $\delta_x$ is the evaluation functional at $x$ defined on $X^{\#}$. It is well known that every element in the convex balanced hull of that set is of the form

$$\sum_{i=1}^{n} \lambda_i \frac{\delta_{x_i} - \delta_{x_j}}{d(x_i,y_j)}$$

for some $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, $\sum_{i=1}^{n} |\lambda_i| \leq 1$ and $(x_1, y_1), \ldots, (x_n, y_n) \in X^2$ with $x_i \neq y_j$ for $i \in \{1, \ldots, n\}$.
**Definition 10.1.** For each \( \sum_{j=1}^{m} g_j \otimes \phi_j \in X^g \otimes E^* \), define:

\[
\varepsilon \left( \sum_{j=1}^{m} g_j \otimes \phi_j \right) = \sup \left\{ \sum_{j=1}^{m} \gamma(g_j) \langle \phi_j, e \rangle : \gamma \in B_{F(X)}, \ e \in B_E \right\}.
\]

Note that this supremum exists since, for all \( \gamma \in B_{F(X)} \) and \( e \in B_E \),

\[
\left| \sum_{j=1}^{m} \gamma(g_j) \langle \phi_j, e \rangle \right| \leq \sum_{j=1}^{m} |\gamma||\text{Lip}(g_j)||\rho||e|| \leq \sum_{j=1}^{m} \text{Lip}(g_j)||\phi_j||.
\]

**Theorem 10.5.** The associated Lipschitz norm \( \pi' \) of \( \pi \) on \( X \otimes E \) is \( \varepsilon \) on \( X^g \otimes E^* \).

**Proof.** Let \( \sum_{j=1}^{m} g_j \otimes \phi_j \in X^g \otimes E^* \). We have

\[
\pi' \left( \sum_{j=1}^{m} g_j \otimes \phi_j \right) = \sup \left\{ \left( \sum_{j=1}^{m} g_j \otimes \phi_j \right) \left( \sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i \right) \right\} : \pi \left( \sum_{j=1}^{n} \delta_{(x_i,y_i)} \otimes e_i \right) \leq 1 \}
\]

\[
= \sup \left\{ \left( \sum_{j=1}^{m} g_j \otimes \phi_j \right) \left( \delta_{(x,y)} \otimes e \right) \right\} : \pi \left( \delta_{(x,y)} \otimes e \right) \leq 1 \}
\]

\[
= \varepsilon \left( \sum_{j=1}^{m} g_j \otimes \phi_j \right).
\]

In order to justify these equalities, denote by \( \alpha(\sum_{j=1}^{m} g_j \otimes \phi_j) \) and \( \beta(\sum_{j=1}^{m} g_j \otimes \phi_j) \) the first and the second supremum which appear above. The first equality follows from Proposition 5.1. To see that \( \alpha(\sum_{j=1}^{m} g_j \otimes \phi_j) \leq \beta(\sum_{j=1}^{m} g_j \otimes \phi_j) \), we will use the easy fact that if \( n \in \mathbb{N}, a_1, \ldots, a_n \in \mathbb{R}^+ \) and \( b_1, \ldots, b_n \in \mathbb{R}^+ \), then

\[
\frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} \leq \max \left\{ \frac{a_1}{b_1}, \ldots, \frac{a_n}{b_n} \right\}.
\]

Fix \( \sum_{j=1}^{n} \delta_{(x_i,y_i)} \otimes e_i \in X \otimes E \), nonzero. If \( \sum_{i=1}^{p} \delta_{(x_i',y_i')} \otimes e_i' = \sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i \), we have

\[
\frac{\left( \sum_{j=1}^{m} g_j \otimes \phi_j \right) \left( \sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i \right)}{\sum_{j=1}^{p} d(x_i',y_i') \|e_i'\|} = \left( \frac{\sum_{j=1}^{m} g_j \otimes \phi_j}{\sum_{j=1}^{p} d(x_i',y_i') \|e_i'\|} \right) \frac{\sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i}{\sum_{j=1}^{p} d(x_i',y_i') \|e_i'\|}
\]

\[
\leq \max \left\{ \left( \frac{\sum_{j=1}^{m} g_j \otimes \phi_j}{\sum_{j=1}^{p} d(x_i',y_i') \|e_i'\|} \right) \frac{\sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i}{\sum_{j=1}^{p} d(x_i',y_i') \|e_i'\|} : 1 \leq i \leq p \right\}
\]

\[
\leq \beta \left( \sum_{j=1}^{m} g_j \otimes \phi_j \right).
\]

By the definition of \( \pi \), it follows that

\[
\left\| \sum_{j=1}^{m} g_j \otimes \phi_j \left( \sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i \right) \right\| \leq \beta \left( \sum_{j=1}^{m} g_j \otimes \phi_j \right) \pi \left( \sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i \right).
\]

This ensures that \( \alpha(\sum_{j=1}^{m} g_j \otimes \phi_j) \leq \beta(\sum_{j=1}^{m} g_j \otimes \phi_j) \). The converse inequality is clearly certain.
Now, given any $\delta_{(x,y)} e \in X \boxtimes E$ with $0 < \pi(\delta_{(x,y)} e) \leq 1$, we obtain
\[
\left| \sum_{j=1}^{m} g_j \boxtimes \phi_j \right| (\delta_{(x,y)} e) = \left| \sum_{j=1}^{m} \left( g_j(x) - g_j(y) \right) \phi_j \right| (d(x,y) \epsilon e) = \left| \sum_{j=1}^{m} \left( \delta_x - \delta_y \right) \left( \frac{d(x,y)}{d(x,y)} \right) \phi_j \right| \leq \epsilon \left( \sum_{j=1}^{m} g_j \boxtimes \phi_j \right)
\]
since $(\delta_x - \delta_y)/d(x,y) \in S_{F(X)}$ and $d(x,y)e \in B_E$. Passing to the supremum we arrive at $\beta(\sum_{j=1}^{m} g_j \boxtimes \phi_j) \leq \epsilon(\sum_{j=1}^{m} g_j \boxtimes \phi_j)$.

Finally, we show that $\epsilon(\sum_{j=1}^{m} g_j \boxtimes \phi_j) \leq \pi'(\sum_{j=1}^{m} g_j \boxtimes \phi_j)$. For any $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, $\sum_{i=1}^{n} |\lambda_i| \leq 1$, $(x_i, y_i) \in X^2$ and $x_i \neq y_i$ for each $i \in \{1, \ldots, n\}$, and $e \in B_E$, we have
\[
\left| \sum_{j=1}^{m} \left( \sum_{i=1}^{n} \lambda_i \frac{\delta_{(x_i,y_i)} - \delta_{(y_i,y_i)}}{d(x_i,y_i)} \right) (g_j \boxtimes \phi_j) \right| (e) = \left| \sum_{j=1}^{m} \left( \sum_{i=1}^{n} \lambda_i \frac{g_j(x_i) - g_j(y_i)}{d(x_i,y_i)} \right) (g_j \boxtimes \phi_j) \right| (e)
\]
\[
= \left| \sum_{j=1}^{m} \left( \sum_{i=1}^{n} \frac{\lambda_i}{d(x_i,y_i)} \left( (g_j \boxtimes \phi_j) (\delta_{(x_i,y_i)} \boxtimes e) \right) \right) \right|
\]
\[
= \left| \left( \sum_{j=1}^{m} g_j \boxtimes \phi_j \right) \left( \sum_{i=1}^{n} \delta_{(x_i,y_i)} \boxtimes \frac{\lambda_i e}{d(x_i,y_i)} \right) \right|
\]
\[
\leq \pi' \left( \sum_{j=1}^{m} g_j \boxtimes \phi_j \right)
\]

since
\[
\pi \left( \sum_{i=1}^{n} \delta_{(x_i,y_i)} \boxtimes \frac{\lambda_i e}{d(x_i,y_i)} \right) \leq \sum_{i=1}^{n} \pi \left( \delta_{(x_i,y_i)} \boxtimes \frac{\lambda_i e}{d(x_i,y_i)} \right) = \sum_{i=1}^{n} d(x_i,y_i) \frac{|\lambda_i||e|}{d(x_i,y_i)} = ||e|| \sum_{i=1}^{n} |\lambda_i| \leq 1.
\]

By the density of the elements $\sum_{i=1}^{n} \lambda_i (\delta_{(x_i,y_i)} - \delta_{(y_i,y_i)})/d(x_i,y_i)$ in $B_{F(X)}$, we infer that
\[
\sum_{j=1}^{m} \gamma(g_j) (\phi_j, e) \leq \pi' \left( \sum_{j=1}^{m} g_j \boxtimes \phi_j \right)
\]
for all $\gamma \in B_{F(X)}$ and $e \in B_E$. Taking supremum over all such $\gamma$ and $e$, we conclude that $\epsilon(\sum_{j=1}^{m} g_j \boxtimes \phi_j) \leq \pi'(\sum_{j=1}^{m} g_j \boxtimes \phi_j)$ and this completes the proof.

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