Minimum energy problems with external fields on locally compact spaces

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In memory of Makoto Ohtsuka (1922–2007)

Abstract. The paper deals with minimum energy problems in the presence of external fields on a locally compact space $X$ with respect to a function kernel $\kappa$ satisfying the energy and consistency principles. For quite a general (not necessarily lower semicontinuous) external field $f$, we establish sufficient and/or necessary conditions for the existence of $\lambda_{A,f}$ minimizing the Gauss functional

$$\int \kappa(x,y) d(\mu \otimes \mu)(x,y) + 2 \int f \, d\mu$$

over all positive Radon measures $\mu$ with $\mu(X) = 1$, concentrated on quite a general (not necessarily closed or bounded) $A \subset X$, thereby giving an answer to a question raised by M. Ohtsuka (J. Sci. Hiroshima Univ., 1961). Such results are specified for the Riesz kernels $|x - y|^{\alpha - n}$, $0 < \alpha < n$, on $\mathbb{R}^n$, $n \geq 2$, and are illustrated by some examples. Furthermore, we provide various alternative characterizations of the minimizer $\lambda_{A,f}$, and as a by-product we analyze the strong and vague continuity of $\lambda_{A,f}$ under the exhaustion of $A$ by compact $K \subset A$. The results obtained hold true and are new for many interesting kernels in classical and modern potential theory.

1. Statement of the problem. Main results

C.F. Gauss investigated the variational problem of minimizing the Newtonian energy evaluated in the presence of an external field, nowadays called the Gauss functional (or, in constructive function theory, the weighted energy), over positive charges $\phi ds$ on the boundary surface of a bounded domain in $\mathbb{R}^3$ (see [17]).

A far-reaching generalization of the original Gauss variational problem, employing vector-valued Radon measures $\mu = (\mu_i)_{i \in I}$ on a locally compact space $X$ as charges and replacing the Newtonian kernel by a function kernel $\kappa$ on $X$, has grown into an eminent branch of modern potential theory, initiated in the fundamental work by M. Ohtsuka [26]. See e.g. the author’s papers [27]–[30], where vector-valued Radon measures $\mu$ were even allowed to be infinite dimensional. Regarding the analytic, constructive, and numerical analysis of the Gauss variational problem for scalar Borel measures on $\mathbb{R}^n$, $n \geq 2$, with respect to the logarithmic or Riesz kernels, see the monographs [2, 25] and numerous references therein, as well as [10], [18]–[20].

Throughout the present paper, $X$ denotes a locally compact (Hausdorff) space, $\mathcal{M}$ the linear space of all (real-valued scalar Radon) measures $\mu$ on $X$ equipped with the vague (= weak*) topology of pointwise convergence on the class $C_0(X)$ of all continuous functions $\varphi : X \to \mathbb{R}$ of compact support, and $\mathcal{M}^+$ the cone of all positive $\mu \in \mathcal{M}$, where $\mu \in \mathcal{M}$ is positive if and only if $\mu(\varphi) \geq 0$ for all positive $\varphi \in C_0(X)$.

A kernel $\kappa$ on $X$ is meant to be a symmetric function from $\Phi(X \times X)$, where $\Phi(Y)$ consists of all lower semicontinuous (l.s.c.) functions $g : Y \to (-\infty, \infty]$ such

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that $g \geq 0$ unless the topological space $Y$ is compact. Then the energy

$$I(\mu) := \int \kappa(x, y) \, d(\mu \otimes \mu)(x, y), \quad \mu \in \mathcal{M},$$

with respect to the kernel $\kappa$ is well defined (as a finite number or $+\infty$) on all of $\mathcal{M}^+$, and represents there a vaguely l.s.c. function (see Section 2 below for more details). We denote by $\mathcal{E}^+$ the set of all $\mu \in \mathcal{M}^+$ with $I(\mu) < \infty$.

For any $A \subset X$, denote by $\mathcal{E}^+(A)$ the class of all $\mu \in \mathcal{E}^+$ concentrated on $A$ [31, Section V.5.7], and by $\mathcal{E}^+(A)$ its subclass consisting of all $\mu$ with $\mu(X) = 1$.

1.1. **Statement of the problem.** Fix a universally measurable function $f : X \to [-\infty, \infty]$, to be treated as an external field acting on charges (measures) on $X$. Given $A \subset X$, let $\mathcal{E}^+_f(A)$ stand for the class of all $\mu \in \mathcal{E}^+(A)$ such that $f$ is $\mu$-integrable [1] (Chapter IV, Sections 3, 4). Then the $f$-weighted energy (= the Gauss functional)

$$I_f(\mu) := I(\mu) + 2 \int f \, d\mu$$

is well defined and finite for all $\mu \in \mathcal{E}^+_f(A)$, and one can introduce the extremal value $^{1}$

$$w_f(A) := \inf_{\mu \in \mathcal{E}^+_f(A)} I_f(\mu) \in [-\infty, \infty].$$

(See Lemmas 3.2, 3.3 and Corollary 3.4 for necessary and/or sufficient conditions for this to hold.) Then $\mathcal{E}^+_f(A) \neq \emptyset$, and hence the following problem makes sense.

**Problem 1.1. Does there exist $\lambda = \lambda_{A,f} \in \mathcal{E}^+_f(A)$ with**

$$I_f(\lambda_{A,f}) = w_f(A)?$$

Problem 1.1 is often referred to as the inner Gauss variational problem [28]. We call its solutions $\lambda_{A,f}$ (if they exist) the inner $f$-weighted equilibrium measures of $A$.

Assume for a moment that $A = K \subset X$ is compact, and that $f \in \Phi(X)$. Then the solutions $\lambda_{K,f}$ do exist, which follows easily from the vague compactness of the class of admissible measures, and the vague lower semicontinuity of the $f$-weighted energy $I_f(\mu)$ on $\mathcal{M}^+$. However, this fails to hold if either $A$ is noncompact or $f \not\in \Phi(X)$, and then Problem 1.1 becomes "rather difficult" (Ohtsuka [29, Section 2.2]).

In the remainder of the present section, we shall always assume that

$$-\infty < w_f(A) < \infty.$$  

(1.3)

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In the remainder of this subsection as well as throughout Section 1.2 we impose on $\kappa$ the following permanent requirement:

(H) The kernel $\kappa$ is perfect, or equivalently it satisfies the energy and consistency principles (see [13], cf. Section 2.2 below).

Then all (signed) measures $\mu \in \mathcal{M}$ with $I(\mu) < \infty$ form a pre-Hilbert space $\mathcal{E}$ with the inner product $\langle \mu, \nu \rangle := \int \kappa(x, y) \, d(\mu \otimes \nu)(x, y)$ and the energy norm $\|\mu\| := \sqrt{I(\mu)}$. Moreover, the cone $\mathcal{E}^+$ then becomes complete in the strong topology, determined by this norm, and the strong topology on $\mathcal{E}^+$ is finer than the (induced) vague topology.

These facts made it possible to develop the theory of inner capacitary measures as well as that of inner balayage, the latter however additionally requiring the (perfect) kernel $\kappa$ to satisfy the domination principle (B. Fuglede [13] and N. Zorii [33]–[36].

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1As usual, the infimum over the empty set is interpreted as $+\infty$. We also agree that $1/(+\infty) = 0$ and $1/0 = +\infty$.

2In general, such $\lambda_{K,f}$ are not unique (unless of course the kernel $\kappa$ satisfies the energy principle).
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cf. Sections 2.4, 2.5 below). Along with the advantages of using perfect kernels, these two theories are crucial to the analysis of Problem 1.1 performed in the present work.

1.2. Main results. Maintaining the interaction between the strong and the vague topologies on the (strongly complete) cone \( E^+ \) as the main tool in the present study, we obtain sufficient and/or necessary conditions for the solvability of Problem 1.1 for noncompact (and even nonclosed) sets \( A \subset X \) and for quite general (not necessarily lower semicontinuous) external fields \( f \), thereby giving an answer to the above-quoted question by Ohtsuka [26, Section 2.2] (see Theorems 1.2, 1.5, and Corollaries 1.3, 1.8 below). Furthermore, we establish various alternative characterizations of the solution \( \lambda_{A,f} \) to Problem 1.1 (see Theorems 1.2, 1.5, and as a by-product we prove its strong and vague continuity under the exhaustion of \( A \) by compact \( K \subset A \) (Theorem 1.4), thereby justifying the term "inner \( f \)-weighted equilibrium measure".

For this purpose, we impose on \( f \) the following permanent requirement:

\[(H_2)\] The external field \( f \) is representable in the form\(^3\)

\[ f = \psi + U^\vartheta, \quad \text{where } \psi \in \Phi(X) \text{ and } \vartheta \in \mathcal{E}, \tag{1.4} \]

\[ U^\vartheta(\cdot) := \int \kappa(\cdot, y) \, d\vartheta(y) \text{ being the potential with respect to the kernel } \kappa. \]

(As shown in Lemma 3.3 below, assumption (1.3) is then equivalent to \( w_f(A) < \infty \), which in turn holds if and only if \( \text{cap}_\kappa(\{x \in A : \psi(x) < \infty\}) > 0 \). Here and in the sequel, \( \text{cap}_\kappa(\cdot) \) denotes the inner capacity of a set with respect to the kernel \( \kappa \).)

Along with (H1) and (H2), some/all of the following three hypotheses on \( \kappa, A, \) and \( f \) will often be required:

\[(H_3)\] The class \( E^+(A) \) is closed in the strong topology on \( E^+ \). (This in particular occurs if \( A \subset X \) is quasiclosed (quasicompact), that is, if \( A \) can be approximated in outer capacity by closed (compact) sets [14]; see Section 2.3 below.)

\[(H_4)\] The (perfect) kernel \( \kappa \) satisfies the first and the second maximum principles (for definitions see Section 2.2).

\[(H'_2)\] The following particular case of (1.4) takes place:

\[ f = -U^\zeta, \quad \text{where } \zeta \in \mathcal{E}^+ \text{ and } \zeta(X) \leq 1. \tag{1.5} \]

**Theorem 1.2.** Assume that (H1)–(H3) are fulfilled, and that

\[ \text{cap}_\kappa(A) < \infty. \tag{1.6} \]

Then the solution \( \lambda = \lambda_{A,f} \) to Problem 1.1 does exist.\(^4\)\(^5\) Furthermore, \( \lambda_{A,f} \) is uniquely determined within \( \mathcal{E}^+_f(A) \) by either of the two characteristic properties\(^6\)

\[ U^\lambda \geq c_{A,f} \text{ n.e. on } A, \]

\[ U^\lambda \leq c_{A,f} \text{ \lambda-a.e. on } X, \]

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\(^3\) In the recent works on Problem 1.1 mainly dealing with the logarithmic or Riesz kernels on \( \mathbb{R}^n \) (see [2, 10, 25] and references therein), the external field \( f \) is always required to be of the class \( \Phi(\mathbb{R}^n) \), whereas the presence of an alternative/additional source of energy, generated by a signed charge \( \vartheta \in \mathcal{E} \), cf. (1.4) or (1.5), agrees well with the original electrostatic nature of the problem.

\(^4\) The solution to Problem 1.1 is unique (if it exists), which follows easily from the convexity of the class of admissible measures and the energy principle (see [27, Lemma 6], cf. Section 3 below).

\(^5\) \( \lambda_{A,f} \) would not necessarily exist if (H3) were omitted from the hypotheses (cf. footnote 19 pertaining to the unweighted case \( f = 0 \) and the Newtonian kernel \( |x - y|^{2-n} \) on \( \mathbb{R}^n, n \geq 3 \)).

\(^6\) For more details about such characteristic inequalities for \( \lambda_{A,f} \), see the author’s earlier paper [27] (cf. Theorems 3.5 and 3.6 below). As seen from there, the latter part of Theorem 1.2 actually holds true under much more general assumptions than stated.
where $U^\lambda_f := U^\lambda + f$ denotes the $f$-weighted potential of $\lambda$, and

$$c_{A,f} := \int U^\lambda_f d\lambda = w_f(A) - \int f d\lambda \in (-\infty, \infty)$$ \hspace{1cm} (1.7)

is said to be the inner $f$-weighted equilibrium constant for the set $A$.

Here and in the sequel, the abbreviation n.e. (nearly everywhere) means, as usual, that the set of all $x \in A$ where the inequality fails is of inner capacity zero.

**Corollary 1.3.** Under (permanent) requirements $(H_1)$ and $(H_2)$, Problem 1.1 is (uniquely) solvable for any quasicompact $A \subset X$.

**Proof.** For quasicompact $A$, (1.6) necessarily holds, the capacity of a compact set with respect to a strictly positive definite kernel being finite. Since $(H_3)$ is fulfilled as well (see Theorem 2.13 below), the corollary follows directly from Theorem 1.2. \hfill \Box

Given $A \subset X$, denote by $\mathcal{C}_A$ the upward directed set of all compact subsets $K$ of $A$, where $K_1 \subseteq K_2$ if and only if $K_1 \subset K_2$. If a net $(x_K)_{K \in \mathcal{C}_A} \subset Y$ converges to $x_0 \in Y$, $Y$ being a topological space, then we shall indicate this fact by writing $x_K \to x_0$ in $Y$ as $K \uparrow A$.

**Theorem 1.4.** Under (permanent) requirements $(H_1)$ and $(H_2)$, assume that the solution $\lambda_{A,f}$ to Problem 1.1 exists. Then

$$\lambda_{K,f} \to \lambda_{A,f} \text{ strongly and vaguely in } \mathcal{E}^+ \text{ as } K \uparrow A,$$

(1.8) with $\lambda_{K,f}$ being the solution to Problem 1.1 with $A := K$ (such solutions $\lambda_{K,f}$ do exist for all $K \in \mathcal{C}_A$ large enough). If moreover $f = U^\delta$ with $\delta \in \mathcal{E}$, then also

$$\lim_{K \uparrow A} c_{K,f} = c_{A,f},$$

(1.9) $c_{A,f}$ and $c_{K,f}$ being introduced by formula (1.7) applied to $A$ and $K$, respectively.

**Theorem 1.5.** Let $(H_1)$, $(H_3)$, $(H_4)$, and $(H_2')$ be fulfilled, and let $\zeta^A$ denote the inner balayage of $\zeta$ onto $A$, the measure $\zeta$ appearing in $(H_2')$. If moreover\footnote{See Theorems 1.2, 1.5, 1.10 and Corollaries 1.3, 1.8 for sufficient conditions for this to occur.}

$$\text{either } \text{cap}_*(A) < \infty, \text{ or } \zeta^A(X) = 1,$$

(1.10) then, and only then, the solution $\lambda_{A,f}$ to Problem 1.1 does exist, and it has the form\footnote{Under hypotheses $(H_1)$, $(H_3)$, and $(H_4)$, assumption $\zeta^A(X) = 1$ is fulfilled, for instance, if $\zeta \in \mathcal{E}^+$ is a measure of unit total mass concentrated on $A$, i.e. $\zeta \in \mathcal{E}^+(A)$. Yet another possibility, pertaining to the $\alpha$-Riesz kernels $|x-y|^{\alpha-n}$ on $\mathbb{R}^n$, $n > 2$, where $0 < \alpha < n$ and $\alpha \leq 2$, requires $\zeta \in \mathcal{E}^+$ to be an arbitrary measure on $\mathbb{R}^n$ with $\zeta(\mathbb{R}^n) = 1$, and $A$ to be not inner $\alpha$-thin at infinity, which according to [22, 32] means that}

$$\lambda_{A,f} = \begin{cases} \zeta^A + \eta_{A,f} & \text{if } \text{cap}_*(A) < \infty, \\ \zeta^A & \text{otherwise,} \end{cases}$$

(1.11)

where $q \in (1, \infty)$ and $A_k := A \cap \{x \in \mathbb{R}^n : q^k \leq |x| < q^{k+1}\}$. For the latter, see [32, Corollary 5.3].

**Representation** (1.11) is particularly useful in applications. For instance, for the $\alpha$-Riesz kernel $|x-y|^{\alpha-n}$ on $\mathbb{R}^n$, $n \geq 2$, of order $0 < \alpha < n$, $\alpha < n$, it enables us to give an answer to Open question 2.1 raised by Ohtsuka in [26, Section 2.12]. In view of [31, Theorem 8.5] and [32, Theorem 4.2, Corollary 5.4], such answer is in the positive if $\alpha < 2$, and it is in the negative otherwise.
where $\gamma_A$ denotes the inner capacitary measure on $A$, while

$$\eta_{A,f} := \frac{1 - \zeta^A(X)}{\text{cap}_A} \in [0, \infty).$$  \hspace{1cm} (1.12)$$

Furthermore, the above $\lambda_{A,f}$ can alternatively be characterized by means of any one of the following three assertions:

(i) $\lambda_{A,f}$ is the unique measure of minimum energy norm in the class

$$\Lambda_{A,f} := \{ \mu \in \mathcal{E}^+ : U^\mu_f \geq \eta_{A,f} \text{ n.e. on } A \},$$

$\eta_{A,f}$ being introduced by formula (1.12). That is, $\lambda_{A,f} \in \Lambda_{A,f}$ and

$$\|\lambda_{A,f}\| = \min_{\mu \in \Lambda_{A,f}} \|\mu\|.$$  \hspace{1cm} (1.13)

(ii) $\lambda_{A,f}$ is the unique measure of minimum potential in the class $\Lambda_{A,f}$, introduced by means of (1.13). That is, $\lambda_{A,f} \in \Lambda_{A,f}$ and

$$U^{\lambda_{A,f}} = \min_{\mu \in \Lambda_{A,f}} U^\mu \text{ on } X.$$  \hspace{1cm} (1.14)

(iii) $\lambda_{A,f}$ is the only measure in $E^+(A)$ having the property

$$U^{\lambda_{A,f}} = \eta_{A,f} \text{ n.e. on } A.$$  \hspace{1cm} (1.15)

In addition, the inner $f$-weighted equilibrium constant $c_{A,f}$, introduced by (1.7), admits an alternative representation

$$c_{A,f} = \eta_{A,f},$$  \hspace{1cm} (1.16)

and hence (1.9) can be specified as follows:

$$c_{K,f} \downarrow c_{A,f} \text{ in } \mathbb{R} \text{ as } K \uparrow A.$$  \hspace{1cm} (1.17)

**Corollary 1.6.** Under the assumptions of Theorem 1.5, if moreover $X$ is $\sigma$-compact\footnote{A locally compact space is said to be $\sigma$-compact if it is representable as a countable union of compact sets \cite[Section I.9, Definition 5]{footnote}.} then $\lambda_{A,f}$ is a measure of minimum total mass in the class $\Lambda_{A,f}$, i.e.

$$\lambda_{A,f}(X) = \min_{\mu \in \Lambda_{A,f}} \mu(X) \ (= 1).$$  \hspace{1cm} (1.18)

**Remark 1.7.** However, extremal property (1.16) cannot serve as an alternative characterization of the inner $f$-weighted equilibrium measure $\lambda_{A,f}$, for it does not determine $\lambda_{A,f}$ uniquely within $\Lambda_{A,f}$. Indeed, consider the $\alpha$-Riesz kernel $|x - y|^{\alpha-n}$ of order $\alpha \leq 2, \alpha < n$, on $\mathbb{R}^n$, $n \geq 2$, a proper, closed subset $A$ of $\mathbb{R}^n$ that is not $\alpha$-thin at infinity (take, for instance, $A := \{ |x| \geq 1 \}$), and let $f$ be given by (1.5) with $\zeta \in \mathcal{E}^+(\mathbb{R}^n \setminus A)$. Applying \cite[Corollary 5.3]{footnote} we get

$$\zeta^A(\mathbb{R}^n) = \zeta(\mathbb{R}^n) = 1.$$  \hspace{1cm} (1.19)

Noting that then $\eta_{A,f} = 0$, and hence $\zeta, \zeta^A \in \Lambda_{A,f}$ (cf. (2.11)), we conclude from (1.17) that there are actually infinitely many measures of minimum total mass in $\Lambda_{A,f}$, for so is every measure of the form $a\zeta + b\zeta^A$, where $a, b \in [0, 1]$ and $a + b = 1.$
Let hypotheses \((H_1), (H_3), (H_4),\) and \((H'_2)\) be fulfilled; and let \(\text{cap}_x(A) = \infty\), for if not, the solution \(\lambda_{A,f}\) to Problem \(1.1\) does exist by Theorem \(1.2\) or \(1.5\). Then the following corollary to Theorem \(1.5\) holds, where \(\zeta\) is the measure appearing in \(1.5\).

**Corollary 1.8.** The minimizer \(\lambda_{A,f}\) does not exist if \(\zeta(X) < 1\), and it exists if \(\zeta\) is a measure of unit total mass concentrated on \(A\). In the latter case, \(\lambda_{A,f} = \zeta\).

**Remark 1.9.** All the above-quoted results hold true and are new for the Green kernels associated with the Laplacian on Greenian sets in \(\mathbb{R}^n\), \(n \geq 2\) (thus in particular for the Newtonian kernel \(|x - y|^{2-n}\) on \(\mathbb{R}^n\), \(n \geq 3\)), as well as for the \(\alpha\)-Riesz kernels \(|x - y|^{\alpha-n}\) and the associated \(\alpha\)-Green kernels on \(\mathbb{R}^n\), where \(0 < \alpha < 2 \leq n\). Furthermore, Theorems \(1.2, 1.4\) and Corollary \(1.3\) also hold true and are new for the \(\alpha\)-Riesz kernels of order \(2 < \alpha < n\) as well as for the logarithmic kernel \(\log |x - y|\) on a closed disc in \(\mathbb{R}^2\) of radius \(1 < \infty\), and for the Deny kernels on \(\mathbb{R}^n\), defined with the aid of Fourier transformation (see condition \((A)\) in \([7, \text{Section 1}]\), cf. \([23, \text{Section VI.1.2}]\)). For more details, see Example \(2.11\) below.

### 1.3. Applications to the Riesz kernels

Let \(X := \mathbb{R}^n\), \(n \geq 2\), and let \(\kappa(x, y)\) be the \(\alpha\)-Riesz kernel \(|x - y|^{\alpha-n}\) of order \(0 < \alpha < n\). Then the above-quoted results on the (un)solvability of Problem \(1.1\) can be specified as follows.

**Theorem 1.10.** Assume that an external field \(f\) is of form \((1.4)\), that \(A \subset \mathbb{R}^n\) is quasiclosed (or, more generally, that \(\mathcal{E}^+(A)\) is strongly closed), and that \(w_f(A) < \infty\).

(a) If moreover \(\text{cap}_x(A) < \infty\), then the solution \(\lambda_{A,f}\) to Problem \(1.1\) does exist. In the remaining case \(\text{cap}_x(A) = \infty\), assume \(\alpha \leq 2\), and consider \(f\) of form \((1.5)\). Then the following \((b)-(e)\) hold true, where \(\zeta\) is the measure appearing in \((1.5)\).

(b) \(\lambda_{A,f}\) exists if and only if \(\zeta^A(\mathbb{R}^n) = 1\), and in the affirmative case \(\lambda_{A,f} = \zeta^A\).

(c) \(\lambda_{A,f}\) does not exist if \(\zeta(\mathbb{R}^n) < 1\), and it exists if \(\zeta\) is a measure of unit total mass concentrated on \(A\). In the latter case, \(\lambda_{A,f} = \zeta\).

(d) If \(A\) is not inner \(\alpha\)-thin at infinity, then \(\lambda_{A,f}\) exists if and only if \(\zeta(\mathbb{R}^n) = 1\), and in the affirmative case \(\lambda_{A,f} = \zeta^A\).

(e) Assume that \(\overline{A} := \text{Cl}_{\mathbb{R}^n}A\) is \(\alpha\)-thin at infinity, and that \(D := \mathbb{R}^n \setminus \overline{A}\) is connected unless \(\alpha < 2\). Then \(\lambda_{A,f}\) does not exist whenever \(\zeta(D) > 0\).

The above results on the solvability of Problem \(1.1\) for the \(\alpha\)-Riesz kernels, included in the present work rather for illustration purposes, can actually be much strengthened, which we plan to pursue in a subsequent paper.

**Example 1.11.** On \(\mathbb{R}^3\), consider the kernel \(1/|x - y|\) and the rotation bodies

\[
F_i := \{x \in \mathbb{R}^3 : 0 \leq x_1 < \infty, \ x_2^2 + x_3^2 \leq g_i^2(x_1)\}, \ i = 1, 2, 3,
\]

where

\[
\begin{align*}
g_1(x_1) & := x_1^s \quad \text{with } s \in [0, \infty), \\
g_2(x_1) & := \exp(-x_1^s) \quad \text{with } s \in (0, 1], \\
g_3(x_1) & := \exp(-x_1^s) \quad \text{with } s \in (1, \infty).
\end{align*}
\]

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12The latter assertion would fail in general if \(\zeta\) were not required to be concentrated on \(A\). See Theorem \(1.10\)(e); for illustration, see the last claim in Example \(1.11\) (pertaining to the set \(F_2\)).

13If \(A\) is not inner \(\alpha\)-thin at infinity, then the requirement \(\text{cap}_x(A) = \infty\) does hold automatically, cf. \([32, \text{Section 2}]\), and hence can be dropped.

14Actually, we require \(D\) to be connected (when \(\alpha = 2\)) only in order to simplify the formulation. Compare with \([31]\) (Lemma 7.1 and Theorems 7.2, 8.7).
As can be derived from estimates in [23, Section V.1, Example], $F_1$ is not 2-thin at infinity, $F_2$ is 2-thin at infinity, though has infinite Newtonian capacity, whereas $F_3$ is of finite Newtonian capacity. Therefore, by Theorem 1.10, $\lambda_{F_3, f}$ exists for any $f$ of form (1.4), see (a). Let now $f$ be of form (1.5) with the measure $\zeta$ involved. Then $\lambda_{F_1, f}$ exists if and only if $\zeta(\mathbb{R}^n) = 1$, see (d); whereas $\lambda_{F_2, f}$ exists if and only if both $\zeta(\mathbb{R}^n) = 1$ and $S(\zeta) \subset F_2$ hold true, see (c) and (e). (Here $S(\zeta)$ is the support of $\zeta$.)

**Figure 1.** The set $F_1$ in Example 1.11 with $\varrho_1(x_1) = 1/x_1$.

**Figure 2.** The set $F_2$ in Example 1.11 with $\varrho_2(x_1) = \exp(-x_1)$.

The rest of the paper is organized as follows. For the convenience of the reader, in Section 2 we review some basic facts of the theory of potentials on locally compact spaces, the main emphasis being placed on the theory of inner capacitary measures and that of inner balayage. Section 3 provides preliminary results on Problem 1.1 and Section 4 gives proofs to Theorems 1.2, 1.4, 1.5, 1.10 and Corollaries 1.6, 1.8.
2. ON THE THEORY OF POTENTIALS ON LOCALLY COMPACT SPACES

Throughout the rest of the paper, we shall use the notations and conventions introduced in Section 1.

For the theory of measures and integration on a locally compact (Hausdorff) space \( X \), we refer to N. Bourbaki \([4]\). A comprehensive application of this theory to the theory of potentials on \( X \) with respect to function kernels satisfying the principle of consistency, is presented in the pioneering paper by Fuglede \([13]\).

Lemma 2.1 (\([11]\) Section IV.1, Proposition 4). For any l.s.c. function \( g : X \to [0, \infty] \), the mapping \( \mu \mapsto \int g \, d\mu \) is vaguely l.s.c. on \( \mathcal{M}^+ \), the integral here being understood as an upper integral.

For an arbitrary set \( A \subset X \), denote by \( \mathcal{M}^+(A) \) the cone of all \( \mu \in \mathcal{M}^+ \) concentrated on \( A \), which means that \( A^c := X \setminus A \) is locally \( \mu \)-negligible, or equivalently that \( A \) is \( \mu \)-measurable and \( \mu = \mu|_A \), \( \mu|_A := 1_A \cdot \mu \) being the trace of \( \mu \) to \( A \) \([12]\) Section IV.14.7]. (Note that for \( \mu \in \mathcal{M}^+(A) \), the indicator function \( 1_A \) of \( A \) is locally \( \mu \)-integrable.) The total mass of \( \mu \in \mathcal{M}^+(A) \) is \( \mu(X) = \mu_+(A), \mu_-(A) \) and \( \mu^0(A) \) denoting the inner and the outer measure of \( A \), respectively. If moreover \( A \) is closed, or if \( A^c \) is contained in a countable union of sets \( Q_j \) with \( \mu^+(Q_j) < \infty \) then for any \( \mu \in \mathcal{M}^+(A) \), \( A^c \) is \( \mu \)-negligible, that is, \( \mu^0(A^c) = 0 \). In particular, if \( A \) is closed, \( \mathcal{M}^+(A) \) consists of all \( \mu \in \mathcal{M}^+ \) with support \( S(\mu) \subset A \), cf. \([4]\) Section IV.2.2].

Lemma 2.2 (cf. \([13]\) Lemma 1.2.2]). For any l.s.c. function \( g : X \to [0, \infty] \), any measure \( \mu \in \mathcal{M}^+ \), and any \( \mu \)-measurable set \( A \subset X \),

\[
\int g \, d\mu|_A = \lim_{K \uparrow A} \int g \, d\mu|_K.
\]

A kernel on \( X \) is thought of as a symmetric function \( \kappa \in \Phi(X \times X) \) (Section 1); thus, either \( \kappa(x, y) \) is \( \geq 0 \) for all \( (x, y) \in X \times X \), or the space \( X \) is compact.

Remark 2.3. The case of compact \( X \) can mostly be reduced to that of \( \kappa \geq 0 \) by replacing the kernel \( \kappa \) by \( \kappa' := \kappa + q \geq 0 \), where \( q \in [0, \infty) \), which is always possible since a lower semicontinuous function on a compact space is lower bounded.

Remark 2.4. For similar reasons, Lemmas 2.1 and 2.2 actually hold true for any \( g \in \Phi(X) \) — even if \( g \not\geq 0 \). Indeed, then \( X \) must be compact, and hence \( g \) can be replaced by \( g' := g + q \geq 0 \), where \( q \in [0, \infty) \). For Lemma 2.1 use the vague continuity of the mapping \( \mu \mapsto \mu(X) \) on \( \mathcal{M}^+ \), the space \( X \) being compact, while for Lemma 2.2 use the fact that for any \( \mu \in \mathcal{M}^+ \) and any \( \mu \)-measurable set \( A \subset X \),

\[
\lim_{\kappa \uparrow A} \mu|_K(X) = \mu|_A(X) < \infty.
\]

Given (signed) \( \mu, \nu \in \mathcal{M} \), define the potential and the mutual energy by

\[
U^\mu(x) := \int \kappa(x, y) \, d\mu(y), \quad x \in X,
\]

\[
I(\mu, \nu) := \int \kappa(x, y) \, d(\mu \otimes \nu)(x, y),
\]

respectively, provided the right-hand side is well defined as a finite number or \( \pm \infty \) (for more details see \([13]\) Section 2.1]). For \( \mu = \nu \), \( I(\mu, \nu) \) defines the energy \( I(\mu) := I(\mu, \mu) \) of \( \mu \in \mathcal{M} \). In particular, if \( \mu, \nu \geq 0 \), then \( U^\mu(x) \), resp. \( I(\mu, \nu) \), is always well

\[\text{[15]}\]If the latter holds, \( A^c \) is said to be \( \mu \)-\( \sigma \)-finite \([12]\) Section IV.7.3]. This in particular occurs if the measure \( \mu \) is bounded (that is, with \( \mu(X) < \infty \)), or if the locally compact space \( X \) is \( \sigma \)-compact.
defined and represents a l.s.c. function of \((x, \mu) \in X \times \mathcal{M}^+\), resp. of \((\mu, \nu) \in \mathcal{M}^+ \times \mathcal{M}^+\) (the principle of descent \cite[Lemma 2.2.1]{13}, cf. Lemma 2.1 and Remark 2.4). Thus
\[
U_\mu \in \Phi(X) \text{ for all } \mu \in \mathcal{M}^+.
\]

**Lemma 2.5.** For any \(A \subset X\) and any \(\mu \in \mathcal{M}^+(A)\),
\[
I(\mu) = \lim_{K \uparrow A} I(\mu|_K).
\]

**Proof.** This is seen from Lemma 2.2 (cf. Remark 2.4) in view of the fact that for any \(\mu\)-measurable set \(Q \subset X\), \((\mu \otimes \mu)|_Q = \mu|_Q \otimes \mu|_Q\), cf. \cite[Lemma 1.2.5]{13}. □

In what follows, we always assume a kernel \(\kappa\) to satisfy the energy principle (or equivalently, to be strictly positive definite), which means that \(I(\mu) \geq 0\) for all (signed) \(\mu \in \mathcal{M}\) whenever \(I(\mu)\) is well defined, and moreover \(I(\mu) = 0\) only for zero measure. Then all \(\mu \in \mathcal{M}\) with \(I(\mu) < \infty\) form a pre-Hilbert space \(\mathcal{E}\) with the inner product \(\langle \mu, \nu \rangle := I(\mu, \nu)\) and the energy norm \(\|\mu\| := \sqrt{I(\mu)}\) \cite[Section 3.1]{13}. The topology on \(\mathcal{E}\) determined by this norm is said to be strong.

### 2.1 Capacities of a set

For any \(A \subset X\), denote
\[
\mathcal{M}^+(A) := \{\mu \in \mathcal{M}^+(A) : \mu(X) = 1\},
\]
\[
\mathcal{E}^+(A) := \mathcal{E} \cap \mathcal{M}^+(A), \quad \mathcal{E}^+(A) := \mathcal{E} \cap \mathcal{M}^+(A),
\]
\[
w(A) := \inf_{\mu \in \mathcal{E}^+(A)} I(\mu) \in [0, \infty].
\]

The value
\[
\operatorname{cap}_s(A) := 1/w(A) \in [0, \infty]
\]
is said to be the (Wiener) inner capacity of the set \(A\) (with respect to the kernel \(\kappa\)).

It is seen from Lemma 2.5 that \(\operatorname{cap}_s(A)\) would be the same if the admissible measures in (2.2) were required to be of compact support. This in turn yields
\[
\operatorname{cap}(K) \uparrow \operatorname{cap}_s(A) \text{ as } K \uparrow A.
\]

**Lemma 2.6 (cf. \cite[Lemma 2.3.1]{13}).** For any \(A \subset X\),
\[
\operatorname{cap}_s(A) = 0 \Leftrightarrow \mathcal{E}^+(A) = \{0\} \Leftrightarrow \mathcal{E}^+(K) = \{0\} \text{ for all } K \in \mathcal{E}_A.
\]

**Lemma 2.7.** Given \(\mu \in \mathcal{E}^+\), let \(A \subset X\) be a \(\mu\)-measurable and \(\mu\)-\(\sigma\)-finite set with \(\operatorname{cap}_s(A) = 0\). Then \(A\) is \(\mu\)-negligible.

**Proof.** As \(A\) is \(\mu\)-measurable and \(\mu\)-\(\sigma\)-finite, \(\mu^*(A) = 0\) will follow once we show that \(\mu(K) = 0\) for every compact \(K \subset A\), which however is obvious by Lemma 2.6. □

A proposition \(\mathcal{P}\) involving a variable point \(x \in X\) is said to hold nearly everywhere (n.e.) on \(A \subset X\) if \(\operatorname{cap}_s(E) = 0\), \(E\) being the set of all \(x \in A\) where \(\mathcal{P}(x)\) fails. Replacing here \(\operatorname{cap}_s(E)\) by \(\operatorname{cap}^*(E)\) leads to the concept of quasi-everywhere (q.e.).

The potential \(U^\mu\) of any \(\mu \in \mathcal{E}\) is (well defined and) finite quasi-everywhere on \(X\) \cite[Corollary to Lemma 3.2.3]{13}. Furthermore, for any two given \(\mu, \nu \in \mathcal{E}\),
\[
\mu = \nu \iff U^\mu = U^\nu \text{ n.e. on } X \iff U^\mu = U^\nu \text{ q.e. on } X,
\]
which follows from \cite[Lemma 3.2.1]{13} by making use of the energy principle.

In the study of inner potential theoretical concepts, the following strengthened version of countable subadditivity for inner capacity is particularly useful.

\footnote{We write \(\operatorname{cap}(A)\) in place of \(\operatorname{cap}_s(A)\) if \(A\) is capacitable, that is, if \(\operatorname{cap}_s(A) = \operatorname{cap}^*(A)\), where \(\operatorname{cap}^*(A)\) is the outer capacity of \(A\), defined as \(\inf \operatorname{cap}_s(D)\), \(D\) ranging over all open sets containing \(A\). This occurs, for instance, if \(A\) is compact \cite[Lemma 2.3.4]{13} or open.}
Lemma 2.8. For arbitrary $A \subset X$ and universally measurable $U_j \subset X$, $j \in \mathbb{N}$,
\[ \text{cap}_s \left( \bigcup_{j \in \mathbb{N}} A \cap U_j \right) \leq \sum_{j \in \mathbb{N}} \text{cap}_s(A \cap U_j). \]

Proof. Noting that a strictly positive definite kernel is pseudo-positive, cf. [13, p. 150],
we derive the lemma from [13] (see Lemma 2.3.5 and the remark following it). For the
Newtonian kernel $|x-y|^{2-n}$ on $\mathbb{R}^n$, $n \geq 3$, this goes back to H. Cartan [6, p. 253]. \qed

2.2. Potential-theoretic principles. The following two maximum principles are
often used throughout the paper. A kernel $\kappa$ is said to satisfy Frostman’s maximum
principle (= the first maximum principle) if for any $\mu \in \mathcal{E}^+$ with $U^\mu \leq 1$ on $S(\mu)$,
the same inequality holds everywhere on $X$; and it is said to satisfy the domination
principle (= the second maximum principle) if for any $\mu, \nu \in \mathcal{E}^+$ with $U^\mu \leq U^\nu \mu$-a.e.,
the same inequality is fulfilled on all of $X$.

For classical kernels, the principle of positivity of mass goes back to J. Deny [8]. In
the generality stated below, it was established in [23] (Theorem 2.1 and Remark 2.1).

Theorem 2.9. Assume that a locally compact space $X$ is $\sigma$-compact, and that a
(strictly positive definite) kernel $\kappa$ satisfies the first and second maximum principles.
For any $\mu, \nu \in \mathcal{E}^+$ with $U^\mu \leq U^\nu$ $\mu$-a.e. on $X$, we have $\mu(X) \leq \nu(X)$.
In the case where $X$ is compact, this remains valid with the second maximum principle dropped.

Unless explicitly stated otherwise, from now on a (strictly positive definite) kernel
$\kappa$ will always be assumed to satisfy the consistency principle, or equivalently to be
perfect, which means that the cone $\mathcal{E}^+$ is complete in the induced strong topology, and
moreover that the strong topology on $\mathcal{E}^+$ is finer than the (induced) vague topology
on $\mathcal{E}^+$ [13, Section 3.3]. Thus, for a perfect kernel $\kappa$, any strong Cauchy net $(\mu_s) \subset \mathcal{E}^+$
converges both strongly and vaguely to the same unique limit $\mu_0 \in \mathcal{E}^+$, the strong
and the vague topologies on $\mathcal{E}^+$ being Hausdorff.\footnote{Since the space $\mathcal{M}$ equipped with the vague topology does not necessarily satisfy the first axiom
of countability, the vague convergence cannot in general be described in terms of sequences.
We follow Moore and Smith’s theory of convergence [24], based on the concept of nets. However, if a
locally compact space $X$ is second-countable, then the space $\mathcal{M}$ is first-countable [36, Lemma 4.4],
and the use of nets in $\mathcal{M}$ may often be avoided.}

Remark 2.10. As seen from a well-known counterexample by Cartan [5], the
whole pre-Hilbert space $\mathcal{E}$ is in general strongly incomplete, and this is the case even
for the Newtonian kernel $|x-y|^{2-n}$ on $\mathbb{R}^n$, $n \geq 3$, despite the fact that the Newtonian
kernel is perfect (cf. also [23, Theorems 1.18, 1.19]).

Example 2.11. The $\alpha$-Riesz kernel $|x-y|^{\alpha-n}$ of order $\alpha \in (0,2]$, $\alpha < n$, on
$\mathbb{R}^n$, $n \geq 2$ (thus in particular the Newtonian kernel $|x-y|^{2-n}$ on $\mathbb{R}^n$, $n \geq 3$), is
perfect, and it satisfies the first and second maximum principles [23, Theorems 1.10, 1.15, 1.18, 1.27, 1.29].
The same holds true for the associated $\alpha$-Green kernel on an arbitrary open subset of $\mathbb{R}^n$, $n \geq 2$
[10, Theorems 4.6, 4.9, 4.11]. The (2-)Green kernel on a planar Greenian set is likewise strictly positive definite
[9, Section I.XIII.7] and perfect [11], and it fulfills the first and the second maximum principles (see [11, Theorem 5.1.11] or [9, Section I.V.10]). The restriction of the logarithmic kernel
$-\log |x-y|$ to a closed disc in $\mathbb{R}^2$ of radius $< 1$ satisfies the energy principle as well
as Frostman’s maximum principle [23, Theorems 1.6, 1.16], and hence it is perfect
[13, Theorem 3.4.2]. (However, the domination principle then fails in general; it does
hold only in a weaker sense where $\mu, \nu$ involved in the above-quoted definition meet
the cone at the concept of a certain be chosen to be strongly bounded, hereditary [15, Definition 5.2] and vaguely compact [13, Lemma 2.5.1]. Since potentials \( U \) now [15, Corollary 6.2] with is vaguely closed according to [4, Section III.2, Proposition 6], whence we only need to show that

\[
\mathcal{E}'(A) \subset \mathcal{E}^+(\overline{A}),
\]

for \( \mathcal{E}^+(\overline{A}) \) is strongly closed (see Theorem 2.13 below). Here \( \overline{A} := \text{Cl}_X A \).

**Definition 2.12** (Fuglede [14]). A set \( A \subset X \) is said to be *quasiclosed* if

\[
\inf \{ \text{cap}'(A \triangle F) : F \text{ closed, } F \subset X \} = 0,
\]

\( \triangle \) being the symmetric difference. Replacing here "closed" by "compact", we arrive at the concept of a *quasicompact* set.

**Theorem 2.13.** If \( A \subset X \) is quasiclosed (or in particular quasicompact), then the cone \( \mathcal{E}^+(A) \) is strongly closed, and hence

\[
\mathcal{E}'(A) = \mathcal{E}^+(A).
\]

**Proof.** Given a net \( (\mu_s) \subset \mathcal{E}^+(A) \) converging strongly (hence vaguely) to \( \mu_0 \in \mathcal{E}^+ \), we only need to show that \( \mu_0 \) is concentrated on \( A \). For closed \( A \), the cone \( \mathcal{M}^+(A) \) is vaguely closed according to [1, Section III.2, Proposition 6], whence \( \mu_0 \in \mathcal{M}^+(A) \).

For quasiclosed \( A \), note that for every \( q \in (0, \infty) \), \( \mathcal{E}^+_q := \{ \mu \in \mathcal{E}^+ : \|\mu\| \leq q \} \) is hereditary [15, Definition 5.2] and vaguely compact [13, Lemma 2.5.1]. Since \( (\mu_s) \) can certainly be chosen to be strongly bounded, \( (\mu_s) \subset \mathcal{E}^+_q \) for some \( q \in (0, \infty) \). Applying now [15, Corollary 6.2] with \( J := \mathcal{E}^+_q \) and \( H := A \), we infer that \( \tilde{J} := \mathcal{E}^+_q \cap \mathcal{M}^+(A) \) is vaguely compact, and consequently, \( (\mu_s) \subset (\tilde{J}) \) has a vague limit point \( \nu_0 \in \tilde{J} \). The vague topology being Hausdorff, \( \nu_0 = \mu_0 \), whence \( \mu_0 \in \tilde{J} \subset \mathcal{M}^+(A) \) as desired.

---

### 2.4. Inner capacitary measures.

Assume for a moment that a set \( A = K \subset X \) is compact, and \( w(K) < \infty \); then the infimum in (2.2) is an actual minimum, the energy being vaguely l.s.c. on \( \mathcal{M}^+ \) (the principle of descent), whereas the class \( \mathcal{M}^+(K) \) being vaguely compact (cf. [4, Section III.1, Corollary 3]). But if \( A \subset X \) is noncompact, the class \( \mathcal{M}^+(A) \) is no longer vaguely compact, and the infimum in (2.2) is in general not attained among the admissible measures — not even for perfect kernels.

To overcome this inconvenience, Fuglede [13, Theorem 4.1] has formulated a dual minimum energy problem, which leads to the same concept of inner capacity, but it is already solvable. Under the assumptions of the present study, where the kernel \( \kappa \) is assumed throughout to be perfect, Fuglede’s result can be specified as follows.

**Theorem 2.14** ([13, Theorem 6.1]). For any \( A \subset X \),

\[
\inf_{\nu \in \Gamma_A} \|\nu\|^2 = \text{cap}_s (A),
\]

where

\[
\Gamma_A := \{ \nu \in \mathcal{E}^+ : U^\nu \geq 1 \text{ n.e. on } A \}.
\]

---

\( ^{18} \)However, these kernels do not satisfy any of the two maximum principles, since the \( \alpha \)-Riesz potentials \( U^\mu \), where \( 2 < \alpha < n \) and \( \mu > 0 \), are superharmonic on \( \mathbb{R}^n \) (see [23, Theorems 1.4, 1.5]).
If \( \text{cap}_s(A) < \infty \), the infimum in (2.6) is an actual minimum with the unique minimizer \( \gamma_A \in \Gamma_A \), called the inner capacitary measure of \( A \) and having the properties

\[
\gamma_A(X) = \| \gamma_A \|^2 = \text{cap}_s(A),
\]

\[
U^{\gamma_A} \geq 1 \text{ n.e. on } A,
\]

\[
U^{\gamma_A} \leq 1 \text{ on } S(\gamma_A),
\]

\[
U^{\gamma_A} = 1 \text{ } \gamma_A\text{-a.e. on } X.
\]

If moreover Frostman’s maximum principle is fulfilled, then also

\[
U^{\gamma_A} = 1 \text{ n.e. on } A.
\]

The following convergence theorem is often useful in applications.

**Theorem 2.15** (33, Theorem 8.1). For any \( A \subset X \) with \( \text{cap}_s(A) < \infty \),

\[
\gamma_K \to \gamma_A \text{ strongly and vaguely in } E^+ \text{ as } K \uparrow A.
\]

If moreover the first and the second maximum principles both hold, then also

\[
U^{\gamma_K} \uparrow U^{\gamma_A} \text{ pointwise on } X \text{ as } K \uparrow A.
\]

In general, \( \gamma_A \) is not concentrated on the set \( A \) itself. However, \( \gamma_A \in E^+(A) \) would necessarily hold if \( E^+(A) \) were assumed to be strongly closed. (Indeed, then \( \gamma_A \in E'(A) = E^+(A) \), the former relation being implied by Theorem 2.15.)

**Theorem 2.16** (cf. 33, Theorem 7.2]). Given \( A \subset X \) with \( \text{cap}_s(A) < \infty \), assume that the class \( E^+(A) \) is strongly closed (or in particular that the set \( A \) is quasiclosed). Then necessarily \( \gamma_A \in E^+(A) \). If moreover Frostman’s maximum principle is fulfilled, then \( \gamma_A \) is uniquely characterized within \( E^+(A) \) by property (2.9) 20.

For other alternative characterizations of the inner capacitary measure \( \gamma_A \) which hold true even for arbitrary \( A \), we refer to 35, Theorems 9.1–9.3].

2.5. **Inner balayage.** Throughout this subsection, a (perfect) kernel \( \kappa \) is additionally required to satisfy the domination principle.

Assume for a moment that \( A = K \subset X \) is compact. Then one can easily prove by generalizing the classical Gauss variational method (see [6], cf. also 23, Section IV.5.23]) that for any given \( \mu \in E^+ \), there exists \( \mu^K \in E^+(K) \) uniquely determined within \( E^+(K) \) by the equality \( U^{\mu^K} = U^{\mu} \) n.e. on \( K \), the uniqueness being obvious from (2.4) applied to the space \( X := K \) and the kernel \( \kappa' := \kappa|_{K \times K} \). This \( \mu^K \) is said to be the balayage of \( \mu \in E^+ \) onto (compact) \( K \).

If now \( A \subset X \) is arbitrary, there is in general no \( \nu \in E^+(A) \) having the property \( U^\nu = U^\mu \) n.e. on \( A \). Nevertheless, a substantial theory of inner balayage (sweeping), generalizing Cartan’s theory [6] of Newtonian balayage to a suitable kernel \( \kappa \) on a locally compact space \( X \), was developed (see 33–36], and this was performed by means of several alternative approaches described in Theorem 2.17 below.

For any \( \mu \in E^+ \) and any \( A \subset X \), denote

\[
\Gamma_{A,\mu} := \{ \nu \in E^+: U^\nu \geq U^\mu \text{ n.e. on } A \}.
\]

19 For instance, if \( A := B_r \) is an open ball \( \{ |x| < r \} \), \( r > 0 \), in \( \mathbb{R}^n \), \( n \geq 3 \), and if \( \kappa(x,y) \) is the Newtonian kernel \( |x - y|^{2-n} \), then \( \gamma_{B_r} \) is the positive measure of total mass \( r^{2-n} \) uniformly distributed over the sphere \( \{ |x| = r \} \). Thus \( S(\gamma_{B_r}) \cap B_r = \emptyset \). (Compare with Theorem 2.16.)

20 We emphasize that, if the requirement of the strong closedness of the class \( E^+(A) \) is dropped, then there is in general no \( \nu \in E^+(A) \) with \( U^\nu = 1 \) n.e. on \( A \) (see 35, footnote 3).
Theorem 2.17 ([35, Theorem 3.1]). There exists precisely the same unique measure $\mu^A \in \Gamma_{A,\mu} \cap \mathcal{E}'(A)$, called the inner balayage of $\mu \in \mathcal{E}^+$ to $A \subset X$, and solving each of the following three extremal problems:

$$\left\| \mu^A \right\| = \min_{\nu \in \Gamma_{A,\mu}} \left\| \nu \right\|,$$

$$U^{\mu^A} = \min_{\nu \in \Gamma_{A,\mu}} U^\nu \text{ on } X,$$

$$\left\| \mu - \mu^A \right\| = \min_{\nu \in \mathcal{E}'(A)} \left\| \mu - \nu \right\|.$$ \hfill (2.10)

This $\mu^A$ is said to be the inner balayage of $\mu$ to $A$, and it has the properties

$$U^{\mu^A} = U^\mu \text{ n.e. on } A,$$ \hfill (2.11)

$$U^{\mu^A} = U^\mu \text{ } \mu^A \text{-a.e.},$$ \hfill (2.12)

$$U^{\mu^A} \leq U^\mu \text{ on } X.$$  

Moreover, (2.11) characterizes the inner balayage $\mu^A$ uniquely within $\mathcal{E}'(A)$.

Thus, according to (2.10), the inner balayage $\mu^A$ can in particular be characterized as the (unique) orthogonal projection of $\mu \in \mathcal{E}^+$ in the pre-Hilbert space $\mathcal{E}$ onto the (strongly complete, convex) cone $\mathcal{E}'(A)$, cf. [12, Theorem 1.12.3].

Corollary 2.18. For arbitrary $A$ with $\text{cap}_s(A) < \infty$,

$$\gamma_A = (\gamma_A)^A. $$ \hfill (2.13)

Proof. Since $\gamma_A \in \mathcal{E}'(A)$ (cf. Section 2.4), whereas the orthogonal projection of any $\nu \in \mathcal{E}'(A)$ onto $\mathcal{E}'(A)$ is certainly the same $\nu$, (2.10) applied to $\gamma_A$ gives (2.13). \hfill $\square$

In general, $\mu^A \notin \mathcal{E}^+(A)$. (Indeed, for $A := B_r \subset \mathbb{R}^n$, $\kappa(x,y) := |x-y|^{2-n}$, and $\mu := \gamma_{B_r}$, where $n \geq 3$ and $r \in (0, \infty)$, this is obvious from Corollary 2.18 and footnote 19.) Nonetheless, the following assertion holds true.

Corollary 2.19. If $\mathcal{E}^+(A)$ is strongly closed (or in particular if $A$ is quasiclosed), then the inner balayage $\mu^A$ can alternatively be characterized as the orthogonal projection of $\mu$ in the pre-Hilbert space $\mathcal{E}$ onto the cone $\mathcal{E}^+(A)$; i.e. $\mu^A \in \mathcal{E}^+(A)$ and

$$\left\| \mu - \mu^A \right\| = \min_{\nu \in \mathcal{E}^+(A)} \left\| \mu - \nu \right\|.$$  

The same $\mu^A$ is uniquely determined within $\mathcal{E}^+(A)$ by property (2.11).

Proof. Since then $\mathcal{E}'(A) = \mathcal{E}^+(A)$, this is obvious from Theorem 2.17. \hfill $\square$

The following Theorem 2.20 justifies the term "inner balayage". (It is worth emphasizing here that, although a kernel $\kappa$ was assumed in [33] to be positive, this restriction on $\kappa$ is unnecessary for the validity of the results from [33] quoted below.)

Theorem 2.20 ([33, Theorem 4.8]). For any $\mu \in \mathcal{E}^+$ and any $A \subset X$,

$$\mu^K \to \mu^A \text{ strongly and vaguely in } \mathcal{E}^+ \text{ as } K \uparrow A,$$

$$U^{\mu^K} \uparrow U^{\mu^A} \text{ pointwise on } X \text{ as } K \uparrow A.$$  

We complete this brief review of the theory of inner balayage with the following useful properties of inner swept measures (see [33, Propositions 7.2, 7.4], cf. [34]).
**Proposition 2.21** (Balayage "with a rest"). If \( A \subset Q \), then for any \( \mu \in \mathcal{E}^+ \),
\[
\mu^A = (\mu^Q)^A.
\]

**Proposition 2.22.** If Frostman’s maximum principle is fulfilled, then
\[
\mu^+(X) \leq \mu(X) \quad \text{for any} \quad \mu \in \mathcal{E}^+ \quad \text{and} \quad A \subset X.
\]

(2.14)

3. Preliminary results on Problem 1.1

Keeping the (permanent) assumption of the energy principle, in the present section we do not require a kernel \( \kappa \) to be perfect.

Fix an external field \( f \) admitting the representation\(^{21}\)
\[
f = f_1 - f_2, \quad \text{where} \quad f_1, f_2 \in \Phi(X) \quad \text{and} \quad \int f_2 \, d\mu < \infty \quad \text{for all} \quad \mu \in \mathcal{E}^+.
\]

(3.1)

Observing with the aid of Lemma\(^{22}\)[2.6] that then necessarily \( f_2 < \infty \) n.e. on \( X \), we infer that such \( f \) is well defined and takes values in \((-\infty, \infty] \) n.e. on \( X \). Also note that \( f \) is universally measurable, i.e. \( \nu \)-measurable for all \( \nu \in \mathcal{M}^+ \).

For any \( \mu \in \mathcal{E}^+ \), define the \( f \)-weighted potential \( U^f_\mu \) by means of the formula
\[
U^f_\mu := U^\mu + f;
\]

then \( U^f_\mu \) is well defined and takes values in \((-\infty, \infty] \) n.e. on \( X \). This follows from the fact that the same holds true for both \( U^\mu \) and \( f \), by making use of the countable subadditivity of inner capacity on universally measurable sets (Lemma\(^{23}\)2.8).

Let \( \mathcal{E}^+_f \) stand for the (convex) class of all \( \mu \in \mathcal{E}^+ \) such that the function \( f \) (or, equivalently, \( f_1 \)) is \( \mu \)-integrable. Then for every \( \mu \in \mathcal{E}^+_f \), the \( f \)-weighted energy \( I_f(\mu) \), introduced by (1.1), is finite and, by Lebesgue–Fubini’s theorem\(^{24}\)[4, Section V.8, Theorem 1], is representable in the form
\[
I_f(\mu) = \int (U^\mu_\mu + f) \, d\mu \in (-\infty, \infty).
\]

For arbitrary \( A \subset X \), denote
\[
\mathcal{E}^+_f(A) := \mathcal{E}^+_f \cap \mathcal{M}^+(A), \quad \mathcal{E}^+_\mu(A) := \mathcal{E}^+_\mu \cap \mathcal{M}^+(A),
\]

and let \( w_f(A) \) stand for the infimum of \( I_f(\mu) \), where \( \mu \) ranges over \( \mathcal{E}^+_f(A) \), see (1.2). If this infimum is finite, see (1.3)\(^{25}\), one can consider the inner Gauss variational problem on the existence of \( \lambda = \lambda_{A,f} \in \mathcal{E}^+_f(A) \) with \( I_f(\lambda_{A,f}) = w_f(A) \), see Problem\(^{26}\) 1.1.

The present section contains preliminary results on Problem\(^{27}\) 1.1 some of them being established in a much more general form in the author’s earlier paper\(^{27}\).

To begin with, we note that a solution \( \lambda_{A,f} \) to Problem\(^{28}\) 1.1 is unique (if it exists), which follows easily from the convexity of the class \( \mathcal{E}^+_f(A) \) and the energy principle, by use of the parallelogram identity in the pre-Hilbert space \( \mathcal{E} \) (cf.\(^{27}\) [27, Lemma 6]).

\(\quad\)

\(^{21}\)Regarding the terminology used here, we follow N.S. Landkof [23], p. 264,

\(^{22}\)It is worth noting that, if moreover \( \kappa \geq 0 \) and \( \text{cap}_+(A) < \infty \), or if \( A \) is compact, then actually \( \mu^A(X) = \int U^\mu d\gamma_A \). See [33] (Eq. (7.3) and Proposition 7.6), cf. [34].

\(^{23}\)In particular, (3.1) necessarily holds for \( f \) of form (1.4). In fact, in view of (2.1) one can take \( f_1 := \psi + U^{\sigma^+} \) and \( f_2 := U^{\sigma^-} \), where \( \sigma^+ \) and \( \sigma^- \) denote the positive and negative parts of \( \sigma \in \mathcal{E} \) in the Hahn–Jordan decomposition. The required inequality \( \int U^{\sigma^-} \, d\mu < \infty \) is indeed fulfilled for all \( \mu \in \mathcal{E}^+ \) since, by virtue of the positive definiteness of the kernel, \( \mathcal{E} = \mathcal{E}^+ - \mathcal{E}^+ \) (see [13, Section 3.1]).

\(^{24}\)It follows immediately from (1.3) that \( \text{cap}_+(A) > 0 \) (for if not, \( \mathcal{E}^+(A) = \emptyset \) by Lemma 2.6). Actually, even a stronger assertion then necessarily holds — see Lemma 3.2.
Lemma 3.1 (cf. [27, Lemma 4]). Given $\kappa$, $A$, and $f$,

$$w_f(K) \downarrow w_f(A) \text{ as } K \uparrow A. \quad (3.2)$$

In fact, (3.2) can be derived from the relation

$$I_f(\mu) = \lim_{K \uparrow A} I_f(\mu|_K) \text{ for all } \mu \in \mathcal{E}_f^+(A), \quad (3.3)$$

which in turn follows by applying Lemma 2.2 to each of the $\mu$-integrable functions $\kappa$, $f_1$, and $f_2$ (cf. Remark 2.4), $f_1$ and $f_2$ appearing in (3.1). Formula (3.3) also implies that the extremal value $w_f(A)$ would be the same if the admissible measures $\mu$ in (1.2) were required to be of compact support $S(\mu) \subset A$ (cf. [27, Remark 2]).

Lemma 3.2 (cf. [27, Lemma 5]). $w_f(A) < \infty$ holds if and only if

$$\text{cap}^+(\{x \in A : f_1(x) < \infty\}) > 0. \quad (3.4)$$

Lemma 3.3. Assume an external field $f$ is of form (1.4), that is, $f = \psi + U^\vartheta$, where $\psi \in \Phi(X)$ and $\vartheta \in \mathcal{E}$. Then (1.3) is equivalent to $w_f(A) < \infty$, and hence to

$$\text{cap}^+(\{x \in A : \psi(x) < \infty\}) > 0. \quad (3.5)$$

Proof. The first part of the claim will follow once we show that

$$w_f(A) > -\infty.$$ 

For any given $\vartheta \in \mathcal{E}$,

$$I_{U^\vartheta}(\mu) = \|\mu\|^2 + 2 \int U^\vartheta \, d\mu = \|\mu + \vartheta\|^2 - \|\vartheta\|^2 \quad \text{for all } \mu \in \mathcal{E}^+, \quad (3.6)$$

and hence, by the energy principle,

$$w_{U^\vartheta}(A) \geq -\|\vartheta\|^2 > -\infty.$$ 

It thus remains to prove that for any given $\psi \in \Phi(X)$,

$$\inf_{\mu \in \mathcal{E}_f^+(A)} \int \psi \, d\mu > -\infty.$$ 

This however is obvious if $\psi \geq 0$, while the remaining case of compact $X$ is treated as described in Remark 2.4, by use of the equality $\mu(X) = 1$ for all $\mu \in \mathcal{E}_f^+(A)$.25

In view of Lemma 3.2, the latter part of the claim is reduced to the equivalence of (3.4) and (3.5), which however is obvious from Lemma 2.8 and the fact that the potential of a measure of finite energy is finite q.e. on $X$ (cf. footnote 23). \hfill \Box

Corollary 3.4. Assume that $f = U^\vartheta$, where $\vartheta \in \mathcal{E}$. Then

$$\mathcal{E}_f^+(A) = \mathcal{E}^+(A). \quad (3.7)$$

Furthermore, assumption (1.3) is fulfilled if and only if $\text{cap}^+(A) > 0$.

Proof. Noting from (3.6) that $I_f(\mu)$ is finite for all $\mu \in \mathcal{E}^+$, we obtain (3.7), while the latter part of the claim is an immediate consequence of Lemma 3.3 with $\psi = 0$. \hfill \Box

---

25The above proof also implies the following observation, to be useful in the sequel: if an external field $f$ is of form (1.4), then $I_f(\mu) > -\infty$ for all $\mu \in \mathcal{E}$. 

3.1. **Characteristic properties of** $\lambda_{A,f}$. In Theorems 3.5 and 3.6, assume again an external field $f$ to be of form (3.1). Also assume (1.3) to be fulfilled (see Lemmas 3.2 and Corollary 3.4 for necessary and/or sufficient conditions for this to hold).

**Theorem 3.5** (cf. [27, Theorems 1, 2]). If the solution $\lambda = \lambda_{A,f}$ to Problem 1.1 exists, then its $f$-weighted potential $U_{f}^{\lambda}$ has the properties

\begin{equation}
U_{f}^{\lambda} \geq c_{A,f} \text{ n.e. on } A, \tag{3.8}
\end{equation}

\begin{equation}
U_{f}^{\lambda} = c_{A,f} \text{ } \lambda\text{-a.e. on } X, \tag{3.9}
\end{equation}

where

\begin{equation}
c_{A,f} := \int U_{f}^{\lambda} \, d\lambda = w_{f}(A) - \int f \, d\lambda \in (-\infty, \infty) \tag{3.10}
\end{equation}

is said to be the inner $f$-weighted equilibrium constant for $A$.

If moreover $f \in \Phi(X)$, then also

\begin{equation}
U_{f}^{\lambda} \leq c_{A,f} \text{ on } S(\lambda),
\end{equation}

and hence

\begin{equation}
U_{f}^{\lambda} = c_{A,f} \text{ n.e. on } S(\lambda) \cap A.
\end{equation}

Relations (3.8) and (3.10), resp. (3.9) and (3.10), characterize the minimizer $\lambda_{A,f}$ uniquely within $\mathcal{E}_{f}^{+}(A)$. In more detail, the following theorem holds true.

**Theorem 3.6** (cf. [27, Proposition 1]). For $\mu \in \mathcal{E}_{f}^{+}(A)$ to be the (unique) solution $\lambda_{A,f}$ to Problem 1.1 it is necessary and sufficient that either of the following two characteristic inequalities be fulfilled:

\begin{equation}
U_{f}^{\mu} \geq \int U_{f}^{\lambda} \, d\mu =: c_{1} \text{ n.e. on } A, \tag{3.11}
\end{equation}

\begin{equation}
U_{f}^{\mu} \leq w_{f}(A) - \int f \, d\mu =: c_{2} \text{ } \mu\text{-a.e. on } X. \tag{3.12}
\end{equation}

If (3.11) or (3.12) holds true, then equality actually prevails in (3.12), and moreover

\begin{equation}
c_{1} = c_{2} = c_{A,f},
\end{equation}

the inner $f$-weighted equilibrium constant $c_{A,f}$ being introduced by formula (3.10).

4. **Proofs of the main results**

Throughout this section,

\begin{equation}
w_{f}(A) < \infty, \tag{4.1}
\end{equation}

and so the class $\mathcal{E}_{f}^{+}(A)$ of admissible measures in Problem 1.1 is nonempty.

Assume for a moment that $A = K \subset X$ is compact, and that $f \in \Phi(X)$. Then Problem 1.1 is solvable for an arbitrary (not necessarily perfect) kernel $\kappa$ on $X$, which follows from the vague compactness of the class $\mathcal{M}^{+}(K)$ [4, Section III.1, Corollary 3] and the vague lower semicontinuity of the $f$-weighted energy $I_{f}(\mu)$ on $\mathcal{M}^{+}$, the latter being obvious from Lemma 2.1 applied to each of $\kappa \in \Phi(X \times X)$ and $f \in \Phi(X)$ (cf. also Remark 2.4). However, this fails to hold if either of the above two requirements is omitted, and moreover Problem 1.1 then becomes in general unsolvable.

To analyze Problem 1.1 for quite a general (not necessarily of the class $\Phi(X)$) external field $f$ and for quite a general (not necessarily closed) set $A \subset X$, from now on we assume that the kernel $\kappa$ is perfect, and that $f$ is of form (1.4), i.e.

\begin{equation}
f = \psi + U^{\vartheta}, \text{ where } \psi \in \Phi(X) \text{ and } \vartheta \in \mathcal{E}
\end{equation}
What is clear so far is that to some converge to than the vague topology on (strong Cauchy. The cone which yields (4.4) when combined with (4.2).

\[ \mu \rightarrow \xi_{A,f} \text{ strongly and vaguely.} \]

Another consequence of (4.3) is that every \( \xi_{A,f} \in \mathcal{E}^+ \) such that, for all \( (\mu_s)_{s \in \mathbb{S}} \subset \mathcal{M}_f(A) \),

\[
\lim_{s \in \mathbb{S}} I_f(\mu_s) = w_f(A).
\]

The (nonempty) set of all those \( (\mu_s)_{s \in \mathbb{S}} \) is denoted by \( \mathbb{M}_f(A) \).

**Lemma 4.1.** There is the unique \( \xi_{A,f} \in \mathcal{E}^+ \) such that, for all \( (\mu_s)_{s \in \mathbb{S}} \subset \mathbb{M}_f(A) \),

\[
\mu_s \rightarrow \xi_{A,f} \text{ strongly and vaguely.}
\]

This \( \xi_{A,f} \) is said to be the extremal measure (in Problem 1.1).

**Proof.** We shall first show that for any \( (\mu_s)_{s \in \mathbb{S}} \) and \( (\nu_t)_{t \in \mathbb{T}} \) from \( \mathbb{M}_f(A) \),

\[
\lim_{(s,t) \in \mathbb{S} \times \mathbb{T}} \| \mu_s - \nu_t \| = 0,
\]

\[ S \times T \text{ being the directed product of the directed sets } S \text{ and } T \text{ (see e.g. } \text{[21] p. 68}). \]

In fact, due to the convexity of the class \( \mathcal{E}^+_f(A) \), for any \( (s, t) \in S \times T \) we have

\[
4w_f(A) \leq 4I_f\left(\frac{\mu_s + \nu_t}{2}\right) = \| \mu_s + \nu_t \|^2 + 4 \int f(\mu_s + \nu_t).
\]

Applying the parallelogram identity in the pre-Hilbert space \( \mathcal{E} \) therefore gives

\[
0 \leq \| \mu_s - \nu_t \|^2 \leq -4w_f(A) + 2I_f(\mu_s) + 2I_f(\nu_t),
\]

which yields (4.4) when combined with (4.2).

Taking the two nets in (4.4) to be equal, we see that every \( \xi_{A,f} \) is strong Cauchy. The cone \( \mathcal{E}^+ \) being strongly complete, \( (\nu_t)_{t \in \mathbb{T}} \) must converge strongly to some \( \xi_{A,f} \in \mathcal{E}^+ \). The same unique \( \xi_{A,f} \) also serves as the strong limit of any other \( (\mu_s)_{s \in \mathbb{S}} \subset \mathbb{M}_f(A) \), which is obvious from (4.4). The strong topology on \( \mathcal{E}^+ \) being finer than the vague topology on \( \mathcal{E}^+ \) by virtue of the perfectness of the kernel, \( (\mu_s)_{s \in \mathbb{S}} \subset \mathbb{M}_f(A) \) converge to \( \xi_{A,f} \) also vaguely.

**Remark 4.2.** In general, the extremal measure \( \xi_{A,f} \) is not concentrated on \( A \). What is clear so far is that

\[
\xi_{A,f} \in \mathcal{E}'(A) \subset \mathcal{E}^+(\overline{A}),
\]

the former relation being clear from (4.3), and the latter from (2.5).

**Remark 4.3.** Another consequence of (4.3) is that

\[
\xi_{A,f}(X) \leq 1,
\]

the map \( \mu \mapsto \mu(X) \) being vaguely l.s.c. on \( \mathcal{M}^+ \) (Lemma 2.4 with \( g = 1 \)). Equality necessarily prevails in (4.6) if \( A = K \) is compact, cf. [21] Section III.1, Corollary 3].

If the unweighted case \( f = 0 \) takes place, we shall drop the index \( f \), and shall simply write \( \lambda_A, \mathcal{M}(A) \), and \( \xi_{A} \) in place of \( \lambda_{A,f}, \mathcal{M}_f(A) \), and \( \xi_{A,f} \), respectively.
Lemma 4.4. For the extremal measure $\xi := \xi_{A,f}$, we have
\[-\infty < I_f(\xi_{A,f}) \leq w_f(A) < \infty. \tag{4.7}\]

Proof. The first inequality in (4.7) is fulfilled by footnote 25 and the last by (permanent) assumption (4.1). In view of (4.2), it is therefore enough to show that
\[I_f(\xi) \leq \lim_{s \in S} I_f(\mu_s), \tag{4.8}\]

\[(\mu_s)_{s \in S} \in M_f(A)\] being fixed.

Assume first that $f \in \Phi(X)$. Since $\mu_s \to \xi$ both strongly and vaguely, Lemma 2.1 applied to $f$ gives (4.8), for (cf. Remark 2.4)
\[I_f(\xi) = \|\xi\|^2 + 2\int f \, d\xi \leq \lim_{s \in S} \left(\|\mu_s\|^2 + 2\int f \, d\mu_s\right) = \lim_{s \in S} I_f(\mu_s).\]

Otherwise, $f = \psi + U^\vartheta$, where $\psi \in \Phi(X)$ and $\vartheta \in \mathcal{E}$. Similarly as in the preceding paragraph, (4.8) will follow once we verify the inequality
\[\int (\psi + U^\vartheta) \, d\xi \leq \lim_{s \in S} \int (\psi + U^\vartheta) \, d\mu_s,\]

which, again by Lemma 2.1 applied to $\psi$, is reduced to the equality
\[\langle \vartheta, \xi \rangle = \lim_{s \in S} \langle \vartheta, \mu_s \rangle. \tag{4.9}\]

Applying the Cauchy–Schwarz (Bunyakovski) inequality to the (signed) measures $\vartheta$ and $\xi - \mu_s$, $s \in S$, elements of the pre-Hilbert space $\mathcal{E}$, we get
\[|\langle \vartheta, \xi - \mu_s \rangle| \leq \|\vartheta\| \cdot \|\xi - \mu_s\|,
\]

which by the strong convergence of $(\mu_s)_{s \in S}$ to $\xi$ proves (4.9), whence the lemma. \qed

Lemma 4.5. The following two assertions are equivalent:

(i) There exists the solution $\lambda = \lambda_{A,f}$ to Problem \ref{problem1.1}.

(ii) The extremal measure $\xi = \xi_{A,f}$ belongs to the class $\hat{\mathcal{E}}^+(A)$\footnote{Compare with (4.5) and (4.6)}.

If either of (i) or (ii) is fulfilled, then actually
\[\xi_{A,f} = \lambda_{A,f}. \tag{4.10}\]

Proof. Assume (i) holds true. The trivial sequence $(\lambda)$ being obviously minimizing:
\[(\lambda) \in M_f(A),\]

it must converge strongly (and vaguely) to the extremal measure $\xi$, see Lemma 4.1. The strong topology on $\mathcal{E}$ being Hausdorff, this yields (4.10), whence (ii).

Assuming now that (ii) holds, we note from (4.7) that $\xi \in \hat{\mathcal{E}}_f^+(A)$, whence
\[I_f(\xi) \geq w_f(A).
\]

The opposite being valid again by (4.7), this establishes (i) with $\lambda := \xi$. \qed

Lemma 4.6. For each $K \in \mathcal{C}_A$ large enough, there exists the (unique) solution $\lambda_{K,f}$ to Problem \ref{problem1.1} with $A := K$, and moreover
\[\lambda_{K,f} \to \xi_{A,f} \text{ strongly and vaguely as } K \uparrow A. \tag{4.11}\]
Proof. For each $K \in \mathcal{C}_A$ large enough ($K \geq K_0$), we have $w_f(K) < \infty$, cf. (4.1) and (3.2). Therefore, by Lemma 4.1 with $A := K$, there is the unique $\xi_{K,f} \in \mathcal{E}^+$ such that every $(\mu_s)_{s \in S} \in \mathbb{M}_f(K)$ converges to $\xi_{K,f}$ strongly and vaguely. Noting that $\xi_{K,f} \in \mathcal{E}^+(K)$ (see Remarks 4.2 and 4.3), we now derive from Lemma 4.5 that $\xi_{K,f}$ serves as the (unique) solution $\lambda_{K,f}$ to Problem 1.1 with $A := K$, which establishes the first part of the lemma. But $(\lambda_{K,f})_{K \geq K_0} \subset \mathcal{E}^+(A)$, and moreover, by Lemma 3.1

$$\lim_{K \uparrow A} I_f(\lambda_{K,f}) = \lim_{K \uparrow A} w_f(K) = w_f(A),$$

which shows that, actually,

$$(\lambda_{K,f})_{K \geq K_0} \in \mathbb{M}_f(A).$$

Applying Lemma 4.1 once again, we obtain (4.11), whence the whole lemma. □

4.2. Proof of Theorem 1.2. Let the assumptions of the theorem be fulfilled. Due to $(H_3)$, $\mathcal{E}'(A) = \mathcal{E}^+(A)$, which substituted into (4.5) gives

$$\xi \in \mathcal{E}^+(A),$$

(4.12)

$\xi = \xi_{A,f}$ being the extremal measure in Problem 1.1 (cf. Lemma 4.1).

We aim to show that in the case $\text{cap}_A(A) < \infty$, $\xi$ serves as the solution $\lambda_{A,f}$ to Problem 1.1. By Lemma 4.5, this will follow once we verify that $\xi \in \mathcal{E}^+(A)$, which in view of (4.12) is equivalent to the assertion

$$\xi(X) = 1.$$ (4.13)

Fix a minimizing net $(\mu_s)_{s \in S} \in \mathbb{M}_f(A)$. Taking a subnet (if necessary) and changing notation, we can certainly assume $(\mu_s)_{s \in S}$ to be strongly bounded:

$$\sup_{s \in S} \|\mu_s\| < \infty.$$ (4.14)

Since $\mu_s \rightharpoonup \xi$ vaguely, cf. (4.3), applying [11] Section IV.4, Corollary 3] gives

$$\int 1_K d\xi \geq \limsup_{s \in S} \int 1_K d\mu_s$$

(4.15)

for every compact $K \subset X$, the indicator function $1_K$ being bounded, of compact support, and upper semicontinuous on $X$. On the other hand,

$$\xi(X) = \lim_{K \uparrow X} \xi(K) = \lim_{K \uparrow X} \int 1_K d\xi,$$

which together with (4.6) and (4.15) results in

$$1 \geq \xi(X) \geq \limsup_{(s,K) \in S \times \mathcal{C}_X} \int 1_K d\mu_s = 1 - \liminf_{(s,K) \in S \times \mathcal{C}_X} \int 1_{A \setminus K} d\mu_s,$$

the equality being implied by the fact that every $\mu_s$ is a positive measure of unit total mass concentrated on $A$. (Here $\mathcal{C}_X$ is the upward directed set of all compact subsets $K$ of $X$.) The proof of (4.13) is thus reduced to that of

$$\liminf_{(s,K) \in S \times \mathcal{C}_X} \int 1_{A \setminus K} d\mu_s = 0.$$ (4.16)

By Theorem 2.14 applied to $A \setminus K$, $K \in \mathcal{C}_X$ being arbitrarily chosen, there exists the (unique) inner capacitary measure $\gamma_{A \setminus K}$, minimizing the energy $\|\mu\|^2$ over the (convex) set $\Gamma_{A,K} \subset \mathcal{E}^+$. For any $K' \in \mathcal{C}_X$ such that $K \subset K'$, we have $\Gamma_{A,K} \subset \Gamma_{A,K'}$, and applying [13] Lemma 4.1] with $\mathcal{H} := \mathcal{E}$ and $\Gamma := \Gamma_{A,K'}$ gives

$$\|\gamma_{A \setminus K} - \gamma_{A \setminus K'}\|^2 \leq \|\gamma_{A \setminus K}\|^2 - \|\gamma_{A \setminus K'}\|^2.$$ (4.17)
Since \( |A|_K|^2 = \operatorname{cap}_s (A \setminus K) \) by (2.7), \( |A|_K|^2 \) decreases as \( K \) ranges through \( E_X \), which together with (4.17) implies that the net \((|A|_K)|_{K \in E_X} \subset E^+\) is Cauchy in the strong topology on \( E^+ \). Noting that \((|A|_K)|_{K \in E_X}\) converges vaguely to zero, \(^{27}\) we get
\[
|A|_K \to 0 \text{ strongly in } E^+ \text{ as } K \uparrow X, \tag{4.18}
\]
the kernel \( \kappa \) being perfect.

But, by (2.8) applied to \( A \setminus K \),
\[
U^{A \setminus K} \geq 1_{A \setminus K} \text{ n.e. on } A \setminus K, \tag{4.19}
\]
hence \( \mu_s \)-a.e. for all \( s \in S \), the latter being derived from Lemma 2.7 due to the fact that \( A \setminus K \) along with \( A \) is \( \mu_s \)-measurable, whereas \( \mu_s \in E^+ \) is bounded. Integrating (4.19) with respect to \( \mu_s \) we therefore obtain, by the Cauchy–Schwarz inequality,
\[
\int 1_{A \setminus K} \, d\mu_s \leq \int U^{A \setminus K} \, d\mu_s \leq \| |A|_K \| \cdot \| \mu_s \| \text{ for all } K \in E_X \text{ and } s \in S,
\]
which combined with (4.14) and (4.18) results in (4.16).

Thus, under the assumptions made, the solution \( \lambda_{A,f} \) to Problem 1.1 does indeed exist. The remaining part of the theorem, uniquely characterizing \( \lambda_{A,f} \) within \( \hat{E}^+_f (A) \), follows immediately by applying Theorem 3.6.

4.3. Proof of Theorem 1.4. Under (permanent) hypotheses \((H_1)\) and \((H_2)\), assume that the solution \( \lambda_{A,f} \) to Problem 1.1 exists; then by Lemma 4.1 it must coincide with the extremal measure \( \xi_{A,f} \), determined by Lemma 4.1. On the other hand, according to Lemma 4.6 for every \( K \in E_A \) large enough \((K \geq K_0)\), there is the solution \( \lambda_{K,f} \) to Problem 1.1 with \( A := K \); and moreover the net \((\lambda_{K,f})|_{K \geq K_0}\) converges strongly and vaguely to \( \xi_{A,f} \), see (4.11). Substituting \( \lambda_{A,f} = \xi_{A,f} \) into (4.11) we get (1.8).

Assume now that \( f = U^\vartheta \), where \( \vartheta \in E \). By Theorem 3.5, the inner \( f \)-weighted equilibrium constant \( \kappa_{A,f} \) can be written in the form
\[
\kappa_{A,f} = \int U^{\lambda_{A,f}} \, d\lambda_{A,f} = \| \lambda_{A,f} \|^2 + \int U^{\vartheta} \, d\lambda_{A,f} = \| \lambda_{A,f} \|^2 + \langle \vartheta, \lambda_{A,f} \rangle,
\]
and likewise
\[
\kappa_{K,f} = \| \lambda_{K,f} \|^2 + \langle \vartheta, \lambda_{K,f} \rangle \quad (K \geq K_0).
\]
Since \( \lambda_{K,f} \to \lambda_{A,f} \) strongly in \( E^+ \) as \( K \uparrow A \), see (1.8), applying the Cauchy–Schwarz inequality to \( \vartheta \) and \( \lambda_{K,f} - \lambda_{A,f} \) gives \( \langle \vartheta, \lambda_{K,f} \rangle \to \langle \vartheta, \lambda_{A,f} \rangle \) (as \( K \uparrow A \)), whence (1.9).

4.4. Proof of Theorem 1.5. Let the assumptions of the theorem be fulfilled, and let \( \zeta \) be the measure appearing in \((H_3)\). The proof is divided into four steps.

Step 1. Assume first that \( \operatorname{cap}_s (A) < \infty \). Since \( E^+_X (A) \) is strongly closed according to \((H_3)\), \( \zeta^A \) and \( \gamma_A \), the inner balayage of \( \zeta \) to \( A \) and the inner capacitary measure of \( A \), respectively, are both concentrated on \( A \), i.e.
\[
\zeta^A, \gamma_A \in E^+_X (A) \tag{4.20}
\]
(see Corollary 2.19 and Theorem 2.16). We aim to show that
\[
\omega := \zeta^A + \eta_{A,f} \gamma_A, \tag{4.21}
\]
the constant \( \eta_{A,f} \), being introduced by the equality in (1.12), serves as the (unique) solution \( \lambda_{A,f} \) to Problem 1.1. This will provide an alternative proof of the solvability of Problem 1.1 (compare with Theorem 1.2 and its proof, given in Section 4.2).

---

^{27}\text{Indeed, for any given } \varphi \in C_0 (X), \text{ there exists a relatively compact open set } G \subset X \text{ such that } \varphi(x) = 0 \text{ for all } x \not\in \overline{G}. \text{ Hence, } \gamma_{A \setminus K} (\varphi) = 0 \text{ for all } K \in E_X \text{ with } K \supset \overline{G}, \text{ and the claim follows.}
Indeed, applying (2.14) gives
\[ 0 \leq \zeta^A(X) \leq \zeta(X) \leq 1, \quad (4.22) \]
the last inequality being valid by virtue of (1.5). Thus \( \eta_{A,f} \in [0, \infty) \), and combining (1.12), (2.7), (4.20), and (4.21) shows that, actually, \( \omega \in \bar{E}^+(A) \). Hence, by (3.7), \( \omega \in \bar{E}^+_f(A) \).

According to Theorem 3.6, \( \omega = \lambda_{A,f} \) will therefore follow once we verify the inequality
\[ U_\omega \geq \eta_{A,f} \omega(X) = \eta_{A,f}. \quad (4.23) \]

Substituted into (4.24), this gives (4.23). Thus the solution \( \lambda_{A,f} \) to Problem 1.1 does exist, and moreover \( \lambda_{A,f} = \omega \) and \( \eta_{A,f} = c_{A,f} \), \( c_{A,f} \) being the inner \( f \)-weighted equilibrium constant. This establishes (1.14) as well as the representation
\[ \lambda_{A,f} = \zeta^A + \eta_{A,f} \gamma_A. \quad (4.25) \]

As \( (\gamma_A)^A = \gamma_A \) (Corollary 2.18), identity (4.25) can be rewritten in the form
\[ \lambda_{A,f} = (\zeta + \eta_{A,f} \gamma_A)^A, \quad (4.26) \]
the inner balayage being additive on positive measures of finite energy. By virtue of Theorem 2.17 applied to \( \mu := \zeta + \eta_{A,f} \gamma_A \in E^+ \), \( \lambda_{A,f} \) can therefore be characterized as the unique measure of minimum energy norm, resp. of minimum potential, within the class of all \( \nu \in E^+ \) having the property
\[ U_\nu \geq U_\zeta + \eta_{A,f} \gamma_A \quad \text{n.e. on } A. \]

Noting from (2.9) with the aid of Lemma 2.8 that this inequality is equivalent to
\[ U_\nu \geq \eta_{A,f} \quad \text{n.e. on } A, \]
we arrive at assertion (i), resp. (ii), of the theorem.

Finally, by making use of Corollary 2.19 we derive from (4.26) that \( \lambda_{A,f} \) is the only measure in \( E^+ (A) \) having the property
\[ U^{\lambda_{A,f}} = U_\zeta + \eta_{A,f} \gamma_A \quad \text{n.e. on } A, \]
or equivalently (cf. Lemma 2.8)
\[ U_{\lambda_{A,f}} = \eta_{A,f} \quad \text{n.e. on } A. \]

This establishes assertion (iii).

Step 2. Let now \( \text{cap}_*(A) = \infty \). By virtue of assumption (1.10), then necessarily \( \zeta^A(X) = 1 \). The class \( E^+ (A) \) being strongly closed according to \((H_3)\), we actually have \( \zeta^A \in E^+_f(A) \) (Corollary 2.19), whence (Corollary 3.4)
\[ \zeta^A \in \bar{E}^+_f(A). \quad (4.27) \]

Our aim is to show that \( \zeta^A \) serves as the (unique) solution to Problem 1.1, i.e.
\[ \zeta^A = \lambda_{A,f}. \quad (4.28) \]
By (2.12) with \( \mu := \zeta \), \( U^\zeta = U^{\zeta^A} - U^\zeta = 0 \) \( \zeta^A \)-a.e., whence
\[
\int U^\zeta \, d\zeta^A = 0,
\]
which combined with (2.11) applied to \( \mu := \zeta \) gives
\[
U^\zeta = \int U^\zeta \, d\zeta^A \text{ n.e. on } A.
\]
By Theorem 3.6, this together with (4.27) implies (4.28) as well as \( c_{A,f} = 0 \). Furthermore, noting from (1.12) that \( \eta_{A,f} \) also equals \( 0 \), we obtain (1.14).

We finally observe that, due to the equalities \( \zeta^A = \lambda_{A,f} \) and \( \eta_{A,f} = 0 \) thereby established, assertions (i)-(iii), providing alternative characterizations of \( \lambda_{A,f} \), can be derived directly from Theorem 2.17 with \( \mu := \zeta \).

Step 3. The aim of this step is to verify that, under hypotheses \((H_1), (H_3), (H_4), (H_\lambda^0)\), assumption (1.10) is not only sufficient, but also necessary for the existence of the solution \( \lambda_{A,f} \). Assume to the contrary that \( \lambda_{A,f} \) exists, but (1.10) does not hold; in view of (1.5) and (2.14), then necessarily
\[
\mathrm{cap}_*(A) = \infty \quad \text{and} \quad \zeta^A(X) < 1.
\]
According to Lemma 4.6, for each \( K \in \mathfrak{C}_A \) large enough \( (K \geq K_0) \), there is the solution \( \lambda_{K,f} \) to Problem 1.1 with \( A := K \); and moreover the net \( (\lambda_{K,f})_{K \geq K_0} \) converges strongly and vaguely to the extremal measure \( \xi_{A,f} \), determined by Lemma 4.1. We assert that, due to the former relation in (4.29),
\[
\zeta_{A,f} = \xi_{A,f}.
\]
Note that the capacity of any compact set is finite, the kernel \( \kappa \) being strongly positive definite by assumption. Therefore, by (1.12) and (4.25) applied to \( K \),
\[
\lambda_{K,f} = \zeta^K + \tilde{\eta}_{K,f} \xi_K \quad \text{for all } K \geq K_0,
\]
where \( \lambda_K := \gamma_K / \mathrm{cap}(K) \) is the (unique) solution to problem (2.2) with \( A := K \), and
\[
\tilde{\eta}_{K,f} := 1 - \zeta^K(X).
\]
But the net \( (\tilde{\eta}_{K,f})_{K \geq K_0} \subset \mathbb{R} \) is bounded since, by (4.22) with \( A := K \),
\[
0 \leq \zeta^K(X) \leq \zeta(X) \leq 1 \quad \text{for all } K \geq K_0.
\]
Furthermore, by Theorem 2.20
\[
\zeta^K \to \zeta^A \quad \text{strongly and vaguely in } \mathcal{E}^+ \text{ as } K \uparrow A.
\]
Thus, if we show that
\[
\lambda_K \to 0 \quad \text{strongly in } \mathcal{E}^+ \text{ as } K \uparrow A,
\]
identity (4.30) will follow from (4.31) by passing to the limit as \( K \uparrow A \), and making use of the triangle inequality in the pre-Hilbert space \( \mathcal{E} \).

It is seen from (2.3) that the net \( (\lambda_K)_{K \geq K_0} \) is minimizing in Problem 1.1 with \( f = 0 \), i.e. \( (\lambda_K)_{K \geq K_0} \in \mathbb{M}(A) \). Applying Lemmas 4.1 and 4.6 we therefore conclude that there exists the unique extremal measure \( \xi_A \) in Problem 1.1 with \( f = 0 \), and moreover \( \lambda_K \to \xi_A \) strongly in \( \mathcal{E}^+ \) as \( K \uparrow A \). This yields
\[
\|\xi_A\|^2 = \lim_{K \uparrow A} \|\lambda_K\|^2 = \lim_{K \uparrow A} w(K) = 0,
\]
the last equality being valid due to the assumption \( \mathrm{cap}_*(A) = \infty \). By virtue of the energy principle, \( \xi_A = 0 \), which proves (4.32), whence (4.30).
Since $\lambda_{A,f}$ exists by assumption, applying Lemma 4.5 therefore gives

$$\lambda_{A,f} = \xi_{A,f} = \zeta^A,$$

which however is impossible, for $\zeta^A(X) < 1$ by (4.29). Contradiction.

**Step 4.** To complete the proof of the theorem, it remains to establish (1.15). Applying (1.12) and (1.14) to each $K \in \mathcal{C}_A$ large enough ($K \geq K_0$), we get

$$c_{K,f} = \frac{1 - \zeta^K(X)}{\text{cap}(K)}.$$

In view of (1.9), (1.15) will therefore follow once we show that the net decreases, or equivalently, that the net $(\zeta^K(X))_{K \in \mathcal{C}_A}$ increases, cf. (2.3). But for any $K, K' \in \mathcal{C}_A$ such that $K' \geq K$, we have $\zeta^K = (\zeta^{K'})_{K}$ by Proposition 2.21, hence

$$\zeta^K(X) \leq \zeta^{K'}(X)$$

by Proposition 2.22 whence the claim.

### 4.5. Proof of Corollary 1.6

As pointed out in Theorem 1.3, $\lambda_{A,f} \in \Lambda_{A,f}$, the class $\Lambda_{A,f}$ being introduced by (1.13). We thus only need to show that in the case of $\sigma$-compact $X$,

$$\lambda_{A,f}(X) \leq \mu(X),$$

(4.33)

$\mu \in \Lambda_{A,f}$ being arbitrarily given. But according to Theorem 1.5(ii), then necessarily

$$U^{\lambda_{A,f}} \leq U^\mu \quad \text{everywhere on } X,$$

and (4.33) follows by making use of the principle of positivity of mass (Theorem 2.9).

### 4.6. Proof of Corollary 1.8

Let hypotheses $(H_1), (H_3), (H_4)$, and $(H_5')$ be fulfilled, and let $\text{cap}_{\ast}(A) = \infty$. To verify the first part of the corollary, assume moreover that $\zeta(X) < 1$. In view of (2.14), then

$$\zeta^A(X) \leq \zeta(X) < 1,$$

which implies, by use of Theorem 1.5, that Problem 1.1 indeed has no solution.

It remains to consider the case of $\zeta$ concentrated on $A$. Then the orthogonal projection of $\zeta$ onto the (strongly closed by $(H_3)$, hence strongly complete) cone $\mathcal{E}^+(A)$ is certainly the same $\zeta$, which implies, by virtue of Corollary 2.19 that

$$\zeta^A = \zeta.$$

Applying Theorem 1.5 once again, we therefore conclude that the solution $\lambda_{A,f}$ to Problem 1.1 exists if and only if $\zeta(X) = \zeta^A(X) = 1$, and in the affirmative case

$$\lambda_{A,f} = \zeta^A = \zeta,$$

cf. the latter formula in representation (1.11). This completes the whole proof.

### 4.7. Proof of Theorem 1.10

Consider the $\alpha$-Riesz kernel of order $\alpha \in (0, n)$ on $\mathbb{R}^n, n \geq 2$, a set $A \subset \mathbb{R}^n$ with strongly closed $\mathcal{E}^+(A)$, and an external field $f$ of form (1.4) such that $w_f(A) < \infty$, or equivalently with $\text{cap}_{\ast}(\{x \in A : \psi(x) < \infty\}) > 0$. If moreover $\text{cap}_{\ast}(A) < \infty$, then Problem 1.1 is solvable according to Theorem 1.2, the $\alpha$-Riesz kernel of arbitrary order $\alpha$ being perfect. This establishes assertion (a).

If now $\text{cap}_{\ast}(A) = \infty$, assume that $\alpha < 2$; and let $f$ be of form (1.5) — then $w_f(A) < \infty$ is equivalent to $\text{cap}_{\ast}(A) > 0$. Furthermore, then the first and second maximum principles hold true, hence Theorem 1.5 and Corollary 1.8 can be utilized. By a direct application of Theorem 1.5 we thus see that $\lambda_{A,f}$ exists if and only if $\zeta^A(\mathbb{R}^n) = 1$, and in the affirmative case $\lambda_{A,f} = \zeta^A$, cf. (1.11). This validates (b).
Assertion (c) is obtained as a direct application of Corollary 1.8, or alternatively, it can be derived from (b) with the aid of arguments similar to those in Section 4.6. The "only if" part of (d) follows directly from (c). For the "if" part, let $A$ be not inner $\alpha$-thin at infinity, and let $\zeta(R^n) = 1$. Applying [32, Corollary 5.3] gives

$$\zeta^A(R^n) = \zeta(R^n) = 1,$$

which implies, by making use of (b), that $\lambda_{A,f}$ does exist, and moreover $\lambda_{A,f} = \zeta^A$.

To verify (e), assume that $A$ is $\alpha$-thin at infinity. In the Newtonian case $\alpha = 2$, also assume for the sake of simplicity that $D := (A)^c$ is connected. We aim to show that in the case $\zeta(D) > 0$, $\lambda_{A,f}$ does not exist. As seen from (b), this will follow once we prove that $\zeta^A(R^n) < 1$, which in turn is reduced to proving

$$\zeta|_D^A(R^n) < \zeta|_D(R^n). \quad (4.34)$$

Due to the $\alpha$-thinness of $A$ at infinity, there exists the $\alpha$-Riesz equilibrium measure $\gamma$ of $A$, treated in an extended sense where $I(\gamma)$ as well as $\gamma(R^n)$ may be infinite (for details, see [23, Section V.1.1], cf. [31, Section 5] and [32, Sections 1.3, 2.1]). Applying [31, Theorem 8.7], we therefore get

$$(\zeta|_D)\gamma(A(R^n)) \leq (\zeta|_D)^\gamma(A(R^n)),$$

whence (4.34), for, in consequence of Propositions 2.21 and 2.22.

$$((\zeta|_D)^\gamma(A(R^n)) \leq ((\zeta|_D)^\gamma)(R^n).$$

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