RIGIDITY AND DERIVED ISOMORPHISM PROBLEM FOR ENVELOPING ALGEBRAS

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Abstract. We prove that there are no injective homomorphisms between enveloping algebras of non-isomorphic semi-simple Lie algebras of the same dimension. We also describe the center of reduction modulo large prime \( p \) of the enveloping algebra of an algebraic Lie algebra \( \mathfrak{g} \) with the property that the center of its enveloping algebra is a polynomial algebra and \( \text{Sym}(\mathfrak{g}) \) has no non-trivial semi-invariants with respect to the adjoint \( \mathfrak{g} \)-action, in particular showing that it is generated by the Harish-Chandra part and \( p \)-center, and is a complete intersection ring. As an application, we solve the derived isomorphism problem of enveloping algebras for a class of Lie algebras.

1. Introduction

The isomorphism problem for enveloping algebras of Lie algebras is a basic open question in ring theory. Recall that it asks whether a \( \mathbb{C} \)-algebra isomorphism between enveloping algebras of Lie algebras implies an isomorphism of the underlying Lie algebras. For the background and the detailed discussion of this well-known problem we refer the reader to the survey article by Usefi [U].

In analogy with the derived isomorphism problem for rings of differential operators for smooth affine varieties, in [T] we considered the following natural generalization.

**Conjecture 1.** Let \( \mathfrak{g}, \mathfrak{g}' \) be finite dimensional Lie algebras over \( \mathbb{C} \). If the derived categories of bounded complexes of \( \mathfrak{g} \)-modules and \( \mathfrak{g}' \)-modules are equivalent, then \( \mathfrak{g} \cong \mathfrak{g}' \).

In our approach to the above conjecture we follow the well-known blueprint of “dequantization” by reducing to the modulo large prime \( p \) technique, which allows a translation of questions about various quantizations to the ones about Poisson algebras. This is largely motivated by the proof of equivalence between the Dixmier and the Jacobian conjectures given by Belov-Kanel and Kontsevich [BK]. Using this approach, we proved in [T] that certain class of Lie algebras that includes Frobenius Lie algebras (see Remark 1.2 below) are derived invariants of their enveloping algebras, also a finite subgroup of automorphisms of \( \mathfrak{Ug} \) is isomorphic to a subgroup of \( \text{Aut}(\mathfrak{g}) \) for a semi-simple \( \mathfrak{g} \).

Recall that given an associative flat \( \mathbb{Z} \)-algebra \( R \) and a prime number \( p \), then the center \( Z(R/pR) \) of its reduction modulo \( p \) acquires the natural Poisson bracket, defined as follows. Given central elements \( a, b \in Z(R/pR) \), let \( z, w \in R \) be their lifts respectively. Then the Poisson bracket \( \{a, b\} \) is defined to be

\[
\frac{1}{p}[z, w] \mod p \in Z(R/pR).
\]

In particular, given a a Lie algebra \( \mathfrak{g} \) over \( S \subset \mathbb{C} \)–a finitely generated ring such that \( \mathfrak{g} \) is a finite free \( S \)-module, then for a prime \( p > 0 \), the center \( Z(\mathfrak{Ug}_p) = Z(\mathfrak{Ug}/p\mathfrak{Ug}) \) of the enveloping algebra of \( \mathfrak{g}_p = \mathfrak{g}/p\mathfrak{g} \) is equipped with the natural \( S/pS \)-linear Poisson bracket defined as above. In this context, the importance of understanding \( Z(\mathfrak{Ug}_p) \) (besides its fundamental role in representation theory of \( \mathfrak{g}_p \)) in regards with the derived isomorphism problem above lies in its derived invariance:

1After completing this manuscript, we learned that the full solution of the isomorphism problem of enveloping algebras has been announced very recently in [CPNW].
if $U_\mathfrak{g}$ and $U_\mathfrak{g}'$ are derived equivalent, then $Z(U_\mathfrak{g}_p) \cong Z(U_\mathfrak{g}'_p)$ as Poisson $S/pS$-algebras. This easily follows from the derived invariance of the Hochschild cohomology and the Gershenhaber bracket (see [1] Lemma 4). Now recall that if $I \subset Z(U_\mathfrak{g}_p)$ is a Poisson ideal, then the Poisson bracket induces Lie bracket on $I/I^2$. In particular, if $m$ is a Poisson ideal such that $Z(U_\mathfrak{g}_p)/m = S/pS$, then $m/m^2$ is a finite $S/pS$-Lie algebra. Therefore, the collection of isomorphisms classes of $S/pS$-Lie algebras $m/m^2$, as $m$ ranges over Poisson ideals of $Z(U_\mathfrak{g}_p)$ such that $Z(U_\mathfrak{g}_p)/m = S/pS$, is a derived invariant of $U_\mathfrak{g}$. The significance of this derived invariant of $\mathfrak{g}$ is highlighted by the fact that given Poisson ideal $m \subset Z(U_\mathfrak{g}_p)$ as above, as the key computation by Kac and Radul shows (see Lemma 1.1 below), there is a canonical Lie algebra homomorphism $\mathfrak{g}_p \to m/m^2$ (when $\mathfrak{g}$ is an algebraic Lie algebra). There is one distinguished such maximal Poisson ideal—the augmentation ideal $m(\mathfrak{g}_p) = Z(U_\mathfrak{g}_p) \cap \mathfrak{g}_p U_\mathfrak{g}_p$.

Now we recall a key computation of the Poisson bracket for restricted Lie algebras due to Kac and Radul [KR]. Let $R$ be a commutative reduced ring of characteristic $p > 0$. Let $\mathfrak{g}$ be a restricted Lie algebra over $R$ with the restricted structure map $x \to x^{[p]}$, $x \in \mathfrak{g}$. As usual, $Z_p(\mathfrak{g})$ denotes the $p$-center of $U_\mathfrak{g}$: the central $R$-subalgebra of the enveloping algebra $U_\mathfrak{g}$ generated by elements of the form $x^p - x^{[p]}$, $x \in \mathfrak{g}$. It is well-known that the map $x \to x^p - x^{[p]}$ induces homomorphism of $R$-algebras

$$i : \text{Sym}(\mathfrak{g}) \to Z_p(\mathfrak{g}),$$

where $Z_p(\mathfrak{g})$ is viewed as an $R$-algebra via the Frobenius map $F : R \to R$. The homomorphism $i$ is an isomorphism when $R$ is perfect. Recall also that the Lie algebra bracket on $\mathfrak{g}$ defines the Kirillov-Kostant Poisson bracket on the symmetric algebra $\text{Sym}(\mathfrak{g})$.

The following is the above mentioned key result from [KR].

**Lemma 1.1.** Let $S$ be a finitely generated integral domain over $\mathbb{Z}$. Let $\mathfrak{g}$ be an algebraic Lie algebra over $S$. Then $Z_p(\mathfrak{g}_p)$ is a Poisson subalgebra of $Z(U_\mathfrak{g}_p)$, moreover the induced Poisson bracket coincides with the negative of the Kirillov-Kostant bracket:

$$\{a^p - a^{[p]}, b^p - b^{[p]} \} = -([a, b]^p - [a, b]^{[p]}), \quad a \in \mathfrak{g}_p, b \in \mathfrak{g}_p.$$

Given an algebraic Lie algebra $\mathfrak{g}$ over $S \subset \mathbb{C}$--a finitely generated ring, by $Z_{HC}(\mathfrak{g}_p)$ (the Harish-Chandra part of the center) we denote the image of $Z(U_\mathfrak{g})$ in $Z(U_\mathfrak{g}_p)$. Towards the goal of understand $Z(U_\mathfrak{g}_p)$, it is a natural and important problem to establish whether $Z(U_\mathfrak{g}_p)$ is generated over its $p$-center $Z_p(\mathfrak{g}_p)$ by $Z_{HC}(\mathfrak{g}_p)$ (see the discussion in [KR] ). Although this is not always the case (see Remark 1.1), we show that it does hold for a class of Lie algebras satisfying the following assumption.

**Assumption 1.** Let $S \subset \mathbb{C}$ be a finitely generated ring, and $\mathfrak{g}$ be an algebraic Lie algebra over $S$. Let $\text{Sym}(\mathfrak{g})^p = \mathcal{O}$ be generated by $f_1, \cdots, f_m$ over $S$. We have the quotient map

$$\pi : \mathfrak{g}^* = \text{Spec} \text{Sym}(\mathfrak{g}) \to \text{Spec} \mathcal{O} = Y.$$

Denote by $Y_m$ the smooth locus of $Y$. Let $U = \{ x \in \mathfrak{g}^*, \pi(x) \in Y_m, d\pi_x \text{ is onto} \}$. Assume that $\text{Sym} \mathfrak{g}$ has no nontrivial $\mathfrak{g}$-semi-invariants, and $\mathfrak{g}^* \setminus \pi^{-1}(Y_m)$ has codimension $\geq 3$ in $\mathfrak{g}^*$.

Then we have the following.

**Theorem 1.1.** Let a Lie algebra $\mathfrak{g}$ be as in Assumption 1. Let $g_i \in Z(U_\mathfrak{g})$ be the symmetrization of $f_i$. Then for all $p \gg 0$, we have

$$\text{Sym}(\mathfrak{g}_p)^p = \text{Sym}(\mathfrak{g}_p)^p[f_1, \cdots, f_m], \quad Z(U_\mathfrak{g}_p) = Z_p(\mathfrak{g}_p)[g_1, \cdots, g_m].$$

Given a base change $S \to \mathbb{k}$ to an algebraically closed field of characteristic $p$ and a Poisson maximal ideal $m$ in $Z(U_\mathfrak{g}_k)$ (under the Poisson bracket induced from the Poisson bracket on $Z(U_\mathfrak{g}_p)$,) the Lie
algebra $m/m^2$ is isomorphic to a quotient of $\mathfrak{g}_k/Z(\mathfrak{g}_k) \oplus k^m$, where $k^m$ is viewed as an abelian Lie algebra.

In view of Theorem 1.1 it is tempting to make the following

**Conjecture 2.** Let $g$ be an algebraic Lie algebra over a finitely generated ring $S \subset \mathbb{C}$, such that $\text{Sym} \mathfrak{g}$ has no nontrivial $g$-semi-invariants Then for all large enough primes $p \gg 0$, $Z(Ug_p)$ is generated over $Z_p$ by the image of $Z(\mathfrak{l}g)$.

The following is a significant strengthening of Theorem 1.1 for Lie algebras $g$ with additional assumption that $(\text{Sym} \mathfrak{g})^0$ is a polynomial algebra.

**Theorem 1.2.** Let $g$ be a Lie algebra of a connected algebraic group $G$ over a finitely generated ring $S \subset \mathbb{C}$, such that $\text{Sym} \mathfrak{g}$ has no nontrivial $g$-semi-invariants, and $\text{Sym}(g)^0 = S[f_1, \cdots, f_n]$ is a polynomial algebra. Let $g_i \in Z(\mathfrak{l}g)$ be the symmetrization of $f_i$. Then for all primes $p \gg 0$, $Z(Ug_p)$ is a free $Z_p(g_p)$-module with a basis $\{ g^a = g_1^a \cdots g_n^a, 0 \leq \alpha_i < p \}$. Moreover given a base change $S \to k$ to an algebraically closed field $k$ of characteristic $p$, then $Z(Ug_k) = Z(Ug_p) \otimes_{S/pS} k$ is a complete intersection and

$$Z(Ug_k) \cong Z_p(g_k) \otimes_{Z_p(g_k) \cap Z_{HC}(g_k)} Z_{HC}(g_k), \quad Z_{HC}(g_k) = (\mathfrak{l}g)^{G_k}.$$ 

Given a Poisson maximal ideal $m$ in $Z(Ug_k)$ (under the Poisson bracket induced from the Poisson bracket on $Z(Ug_p)$) the Lie algebra $m/m^2$ is isomorphic to $g_k/Z(g_k) \oplus V$, where $V$ is a trivial Lie algebra spanned by images of $g_i - a_i, a_i \in k$, where $g_i - a_i \in m$.

The proof of Theorems 1.1, 1.2 crucially relies on the first Kac-Weisfeler conjecture, which was established in [MSTT] for $p \gg 0$.

**Remark 1.1.** It is easy to see that assumption about non-existence of nontrivial $G$-semi-invariants of $\text{Sym}(g)$ is essential. Let $g$ be the following 3-dimensional Lie algebra $g = \mathbb{C}x \oplus \mathbb{C}y \oplus \mathbb{C}z$ with the bracket $[z, x] = x, [z, y] = y, [x, y] = 0$. Then it is easy to see that $Z(Ug_k)$ is not generated by $Z_p$ and $Z_{HC}$, as $Z_{HC} = k$, while $xy^{p-1} \in Z(Ug_k) \setminus Z_p$.

It was proved in [AP] that given a finite subgroup $W$ of automorphisms of the enveloping algebra of a semi-simple Lie algebra $g$, such that $(\mathfrak{l}g)^W$ is isomorphic to an enveloping algebra of a Lie algebra $g'$, then $W$ must be trivial and $g' = g$. On the other hand, Caldero [C] showed that given semi-simple Lie algebras $g, g'$ and finite subgroups of automorphisms of corresponding enveloping algebras $W \subset \text{Aut}(\mathfrak{l}g), W' \subset \text{Aut}(\mathfrak{l}g')$ such that $(\mathfrak{l}g)^W \cong (\mathfrak{l}g')^{W'}$, then $g \cong g'$.

Recall that the index of a Lie algebra $g$ is defined as the smallest $\dim g^\chi$ over $\chi \in g^*$.

We have the following rigidity result about homomorphisms between enveloping algebras.

**Theorem 1.3.** Let $g$ be a semi-simple Lie algebra over $\mathbb{C}$, $g'$ be a Lie algebra satisfying Assumption 7 and $\dim g = \dim g'$. Let $f : \mathfrak{l}g \to \mathfrak{l}g'$ be a $\mathbb{C}$-algebra homomorphism such that, either $f$ is injective, or $f|_{Z(\mathfrak{l}g)}$ is injective and $\text{index}(g) \leq \text{index}(g')$. Then $g \cong g'$ and $f(Z(\mathfrak{l}g)) \subset Z(\mathfrak{l}g')$.

As an application of Theorem 1.2 to the derived equivalence problem we have the following.

**Theorem 1.4.** Let $g, g'$ be algebraic Lie algebras satisfying assumptions in Theorem 1.2. If $\mathfrak{l}g$ is derived equivalent to $\mathfrak{l}g'$, then $g/Z(g) \cong g'/Z(g')$.

We recall the following simple result that illustrates usefulness of ”dequantizing” $\mathfrak{l}g$ to $\text{Sym} \mathfrak{g}_k$ in regards with the isomorphism problem for enveloping algebras.
**Remark 1.2.** Theorems 1.2 and 1.4 were proved in [1] under much more restrictive assumptions on $g$: there we assumed that $(f_1, \ldots, f_n)$ is a regular sequence, $\text{Sym}(g)/(f_1, \ldots, f_n)$ is a normal domain and the coadjoint action of $G$ on $\text{Spec}(\text{Sym}(g)/(f_1, \ldots, f_n))$ has an open orbit. In particular, no nilpotent Lie algebras can satisfy these conditions. On the other hand, Theorems 1.2 and 1.4 can be applied to many nilpotent algebras, as the class of Lie algebras with the property that $\text{Sym}(g)^p$ is a polynomial algebra is large (see [O]).

2. THE FIRST KAC-WIESEFLER CONJECTURE

In this section we recall some results associated with the first Kac-Weisfeiler conjecture that are used in proof of our main results. Recall that the first Kac-Weisfeiler conjecture asserts that for a $p$-restricted Lie algebra $g$ over $k$, the maximal possible dimension of an irreducible $g$-module is $p^{\frac{1}{2}(\dim(g) - \text{index}(g))}$. Equivalently the rank of $\mathfrak{u}g$ over its center equals $p^{\dim(g) - \text{index}(g)}$.

Let $g$ be a Lie algebra of an algebraic group $G$ defined over a finitely generated ring $S \subset \mathbb{C}$. Then for all $p \gg 0$ and a base change $S \to k$ to an algebraically closed field of characteristic $p$, the first Kac-Weisfeiler conjecture was established for $g_k$ in [MSTT]. Namely we have proved the following. As usual, $D(g)$ denotes the skew field of fractions of $\mathfrak{u}g$. Similarly, by $C(g)$ we will denote the field of fractions of $\text{Sym}(g)$.

**Theorem 2.1 (MSTT, Theorems 3.8 and 3.9).** Let $g$ be an algebraic Lie algebra over a finitely generated ring $S \subset \mathbb{C}$. Then for all $p \gg 0$ and a base change $S \to k$ to an algebraically closed field of characteristic $p$, the fraction field of $Z(\mathfrak{u}g_k)$ is generated by the image of $Z(D(g))$ and the fraction field of $\text{Frac}(Z_p(g_k))$.

Also, the following equality of degrees of field extensions holds:

$$[\text{Frac}(Z(\mathfrak{u}g_k)) : \text{Frac}(Z_p(g_k))] = p^{\text{index}(g)} = [C(g_k)^g_k : C(g_k)^p]$$

**Remark 2.1.** The equality

$$[C(g_k)^g_k : C(g_k)^p] = p^{\text{index}(g)}$$

follows from the fact that on the one hand (as proved in [MSTT, Theorem 3.8, 3.9])

$$[\text{Frac}(\text{gr} Z(\mathfrak{u}g_k)) : C(g_k)^p] \geq p^{\text{index}(g)},$$

and on the other hand it was proved in [PS] that

$$[C(g_k)^g_k : C(g_k)^p] \geq p^{\dim(g) - \text{index}(g)}.$$

We also need to recall the following simple result from commutative algebra (for a proof see [MSTT, Lemma 3.11].)

**Lemma 2.1.** Let $S \subset \mathbb{C}$ be a finitely generated ring. Let $A$ be a finitely generated commutative algebra over $S$ such that $A_C$ is a domain. Let $B \subset A$ be a finitely generated $S$-subalgebra. Then for all $p \gg 0$ and a base change $S \to k$ to an algebraically closed field $k$ of characteristic $p$ the rank of $A_k$ over $B_k A_k^p$ is $p^{\dim(A) - \dim(B)}$. 

3. The proofs

We crucially rely on the following result of Panyushev-Yakimova [PY, Remark 1.3] (see also [PY, Proposition 1.2], [JS Proposition 5.2]).

Proposition 3.1. Let \( \mathfrak{g} \) be a Lie algebra over \( \mathbb{C} \) such that \( \text{Sym}(\mathfrak{g}) \) has no proper \( \mathfrak{g} \)-semi-invariants. Let \( \mathcal{O} = \text{Sym}(\mathfrak{g})^g \) be finitely generated. Put \( Y = \text{Spec} \mathcal{O} \). We have the quotient map \( \pi : \mathfrak{g}^* = \text{Spec} \text{Sym}(\mathfrak{g}) \rightarrow \text{Spec} \mathcal{O} = Y \). Denote by \( Y_{\text{sm}} \) the smooth locus of \( Y \). Let \( U = \{ x \in \mathfrak{g}^*, \pi(x) \in Y_{\text{sm}}, d\pi_x \text{ is onto} \} \). Then \( \mathfrak{g}^* \setminus U \) has codimension \( \geq 2 \) in \( \mathfrak{g}^* \).

Proof of Theorems. We start by proving Theorem 1.1. Since \( \mathfrak{g} \) has no nontrivial semi-invariants in \( \text{Sym}(\mathfrak{g}) \), it follows easily that

\[
(\text{Frac}(\text{Sym}(\mathfrak{g}))^g) = \text{Frac}((\text{Sym}(\mathfrak{g}))^g).
\]

Put \( \mathcal{O} = \mathbb{k}[f_1, \cdots, f_m] \) and \( n = \text{index}(\mathfrak{g}) \). In particular \( n = \dim(\mathcal{O}) \). Also put

\[
A = \text{Sym}(\mathfrak{g}_k), \quad B = \text{Sym}(\mathfrak{g}_k)^g \mathcal{O}, \quad B' = (\text{Sym}(\mathfrak{g}_k)^g \mathcal{O} = (\text{Sym}(\mathfrak{g}_k))^{p\mathfrak{k}}.
\]

Clearly \( B \subset B' \). Let \( K = \text{Frac}(A) \) and \( K_0 = \text{Frac}(B) \).

We first claim that

\[
[K_0 : K^p] = p^n.
\]

Indeed, this follows at once since the degree of \( K \) over \( K_0 \) is \( p^{\dim(\mathfrak{g}) - n} \) by Lemma 2.1 and

\[
[K : K^p] = p^{\dim(\mathfrak{g})}.
\]

Next we argue that \( B \) a normal domain. Indeed, it follows from Theorem 3.1 that for all \( p \gg 0 \) and a base change \( S \rightarrow \mathbb{k} \), the complement of

\[
U_k = \{ x \in \mathfrak{g}_k^*, \pi(x) \in Y_{\text{sm}}, d\pi_x \text{ is onto} \}.
\]
in \( \mathfrak{g}_k^* \) has codimension \( \geq 2 \). We will argue that for a prime ideal \( I \) in \( U_k \), the ring \( B_I \) is regular ring, while for a prime in \( \pi^{-1}(Y_{\text{sm}}), B_I \) is a Cohen-Macaulay ring. Let \( I \) be a prime ideal in \( A \) such that \( I \in \pi^{-1}(Y_{\text{sm}}) \). Put

\[
I' = I \cap A^p, \quad I'' = I \cap \mathcal{O}.
\]

Also put \( C = \mathcal{O}_{I''} \subset A_{I'} \cap \mathcal{O}_{I''} \). We claim that the multiplication map \( \phi \) induces an isomorphism

\[
\phi : (A^p)_{I'} \otimes_C \mathcal{O}_{I''} \cong B_I.
\]

Indeed \( \phi \) is a surjective map from a free \( A^p \)-module of rank \( p^n \) (as \( \mathcal{O}_{I''} \) is a free \( C \)-module of rank \( p^n \) since \( I'' \) belongs to the smooth locus of \( Y \)) onto a \( A^p \)-module of rank \( p^n \). Thus \( \phi \) must be an isomorphism. In particular, it follows that \( B_I \) is a Cohen-Macaulay ring. Now if in addition \( I \in U_k \), then it follows easily that \( A_{I'} \otimes_C \mathcal{O}_{I''} \) is a regular ring. Hence \( B \) is regular in codimension 1 and Cohen-Macaulay in codimension 2. Therefore it is a normal domain by Serre’s criterion of normality.

It follows that \( B' \) and \( B \) have the same field of fractions as they are both extensions of degree \( p^n \) of \( \text{Frac}(Z_p) \) by Theorem 2.1 and \( B \subset B' \). Thus normality of \( B \) yields that \( B = B' \). Hence

\[
(\text{Sym}(\mathfrak{g}_k))^{p\mathfrak{k}} = (\text{Sym}(\mathfrak{g}_k))^p[f_1, \cdots, f_m], \quad Z(\mathfrak{U} \mathfrak{g}_k) = Z_p[g_1, \cdots, g_m].
\]

Now we proceed to prove Theorem 1.2. We first show that \( B \) is a free \( A^p \)-module with basis \( \{ f^\alpha = \prod_{i=1}^m f_i^{\alpha_i}, \alpha_i < p \} \). Indeed, since \( [K_0 : K^p] = p^n \), it follows that \( \{ f^\alpha, \alpha_i < p \} \) are linearly independent over \( K^p \). In particular they form a basis of \( B \) over \( A^p \). Hence, \( Z(\mathfrak{U} \mathfrak{g}_k) \) is a free module over \( Z_p \) with a basis \( \{ g^\alpha, \alpha_i < p \} \) as desired. Thus we obtain that

\[
\text{gr } Z(\mathfrak{U} \mathfrak{g}_k) = B = (\text{Sym}(\mathfrak{g}_k))^{p\mathfrak{k}}.
\]
We have the natural surjective homomorphism (that restricts to the Frobenius homomorphism on \(A\) and sends \(x_i\) to \(f_i^p\))

\[
A[x_1, \ldots, x_n]/(x_1^p - f_1, \ldots, x_n^p - f_n) \to B,
\]

which must be an isomorphism since both rings are free \(A^p\)-modules of rank \(p^n\). Since \((x_1^p - f_1, \ldots, x_n^p - f_n)\) is a regular sequence in \(A[x_1, \ldots, x_n]\), it follows that \(B\) is a complete intersection. Therefore \(Z(\mathcal{U}\mathfrak{g}_k)\) is also a complete intersection.

Our next goal is to show that \(\text{Sym}(\mathfrak{g}_k)^{G_k} = \mathbb{k}[f_1, \ldots, f_n]\), which implies \(\mathcal{U}\mathfrak{g}_k^{G_k} = Z_{HC}(\mathfrak{g}_k)\). Let \(x \in \text{Sym}(\mathfrak{g}_k)^{G_k} \subset B\). As \(B\) is a free \(\text{Sym}(\mathfrak{g}_k)^p\) module with basis \(\{f^\alpha = \prod_{i=1}^n f_i^{\alpha_i}, \alpha_i < p\}\), we may write

\[
x = \sum_{\alpha} x_\alpha f^\alpha, \quad x_\alpha \in (\text{Sym}(\mathfrak{g}_k)^p, \alpha_i < p).
\]

Then

\[
x_\alpha \in (\text{Sym}(\mathfrak{g}_k)^{G_k}).
\]

Replacing \(x\) by \(\frac{1}{\alpha}x_\alpha\) and continuing in this manner, we obtain that

\[
x \in \bigcap_n (\text{Sym}(\mathfrak{g}_k))^{p^n} \mathcal{O} = \mathcal{O}.
\]

We also get that

\[
Z_p(\mathfrak{g}_k) \cap Z_{HC}(\mathfrak{g}_k) = Z_p(\mathfrak{g}_k)^{G_k}.
\]

So

\[
gr(Z_p(\mathfrak{g}_k) \cap Z_{HC}(\mathfrak{g}_k)) = \mathbb{k}[f_1^p, \ldots, f_n^p].
\]

Hence \(Z_{HC}(\mathfrak{g}_k)\) is a free \(Z_p(\mathfrak{g}_k) \cap Z_{HC}(\mathfrak{g}_k)\)-module of rank \(p^n\). Therefore the natural ring homomorphism

\[
Z_p(\mathfrak{g}_k) \otimes_{Z_p(\mathfrak{g}_k) \cap Z_{HC}(\mathfrak{g}_k)} Z_{HC}(\mathfrak{g}_k) \to Z(\mathcal{U}\mathfrak{g}_k)
\]

is a surjective homomorphism of free \(Z_p(\mathfrak{g}_k)\)-modules of rank \(p^n\). Thus it is a Poisson algebra isomorphism.

Let \(m \subset Z(\mathcal{U}\mathfrak{g}_k)\) be a Poisson maximal ideal. Put

\[
m' = m \cap Z_p(\mathfrak{g}_k), \quad m'' = m \cap Z_{HC}(\mathfrak{g}_k), m_1 = m' \cap m''.
\]

Then the Poisson algebra isomorphism above implies the following Lie algebra isomorphism

\[
m/m^2 \cong m'/m'^2 \times m_1/m_1^2 m''/m''^2.
\]

On the other hand, since the Poisson bracket vanishes on \(Z_{HC}\), we conclude that \(m/m^2\) is a trivial central extension of the image of \(m'/m'^2 \cong \mathfrak{g}_k\) and the kernel of the natural homomorphism \(m'/m'^2 \to m/m^2\) is central. On the other hand, if \(g \in Z(m'/m'^2)\), then \(g = f^p\) with \(f \in m_1\), hence the image of \(g\) in \(m/m^2\) is 0. Thus \(m/m^2\) is a trivial central extension of \(\mathfrak{g}_k/Z(\mathfrak{g}_k)\).

\[\square\]

**Proof of Theorem 1.4.** We may assume that Lie algebras \(\mathfrak{g}, \mathfrak{g}'\) and the corresponding derived isomorphism are defined over a finitely generated ring \(S \subset \mathbb{C}\). Let

\[
Z(\mathcal{U}\mathfrak{g}) = S[f_1, \ldots, f_n] \quad \text{and} \quad Z(\mathcal{U}\mathfrak{g}') = S[f'_1, \ldots, f'_n].
\]

Thus we have an \(S\)-algebra isomorphism \(S[f_1, \ldots, f_n] \cong S[f'_1, \ldots, f'_n]\) and a Poisson algebra isomorphism \(Z(\mathcal{U}\mathfrak{g}_k) \cong Z(\mathcal{U}\mathfrak{g}'_k)\). Moreover these isomorphisms are compatible with reduction modulo \(p\) maps so that the following diagram commutes:
\[ S[f_1, \cdots, f_n] \longrightarrow S[f'_1, \cdots, f'_n] \]
\[
\downarrow \\
Z(\mathfrak{U}_k) \longrightarrow Z(\mathfrak{U}'_k)
\]

Let \( m \) be a maximal Poisson ideals in \( Z(\mathfrak{U}_k) \), and let \( m' \subset Z(\mathfrak{U}'_k) \) be the corresponding maximal Poisson ideal under the above isomorphisms. Hence we get an isomorphism of Lie algebras \( m'/m^{2} \cong m'/m^{2} \). On the other hand, by Theorem I.2 m/m' (respectively \( m'/m^{2} \)) is isomorphic to a trivial central extension of \( \mathfrak{g}/Z(\mathfrak{g}) \) (resp. \( \mathfrak{g}'_k/Z(\mathfrak{g}'_k) \)). Since we have the compatible isomorphism \( k[g_1, \cdots, g_n] \cong k[g'_1, \cdots, g'_n] \) we get that
\[ \mathfrak{g}/Z(\mathfrak{g}) \oplus V \cong \mathfrak{g}'_k/Z(\mathfrak{g}'_k) \oplus V' \]
with \( V \cong V' \) being abelian \( k \)-Lie algebra. Thus \( \mathfrak{g}/Z(\mathfrak{g}) \cong \mathfrak{g}'_k/Z(\mathfrak{g}'_k) \). Hence \( \mathfrak{g}/Z(\mathfrak{g}) \cong \mathfrak{g}'/Z(\mathfrak{g}') \).

Proof of Theorem 1.3. Put \( Z(\mathfrak{U}_g') = S[g'_1, \cdots, g'_n] \).

Let \( \phi : \mathfrak{U}_g \rightarrow \mathfrak{U}_g' \) be a \( \mathbb{C} \)-algebra embedding with \( \mathfrak{g} \) semi-simple, \( \mathfrak{g}' \) as above and \( \dim \mathfrak{g} = \dim \mathfrak{g}' \). It is well-known that the maximal Krull dimension of a commutative subalgebra of \( \mathfrak{U}_g \) is at most \( \frac{1}{2} \dim(\mathfrak{g} + \text{index} (\mathfrak{g}' ) ) \), on the other hand \( \mathfrak{U}_g \) contains a commutative subalgebra of dimension \( \frac{1}{2} \dim(\mathfrak{g} + \text{index}(\mathfrak{g}) ) \) as proved by Rybnikov [R]. So we may conclude that \( \text{rank}(\mathfrak{g}) \leq \text{index}(\mathfrak{g}') \). Thus for the remainder of the proof we may (and will) assume that we have a \( \mathbb{C} \)-algebra homomorphism \( \phi : \mathfrak{U}_g \rightarrow \mathfrak{U}_g' \), such that \( \text{rank}(\mathfrak{g}) \leq \text{index}(\mathfrak{g}') \) and \( \phi |_{Z(\mathfrak{U}_g)} \) is injective.

We may also assume that \( \phi, \mathfrak{g}, \mathfrak{g}' \) are defined over \( S \)-a large enough finitely generated subring of \( \mathbb{C} \). Denote by \( \phi_k : \mathfrak{U}_k \rightarrow \mathfrak{U}'_k \) the base change of \( \phi \) to an algebraically closed field of characteristic \( p \gg 0 \). The crucial step of the proof of Theorem 1.3 will be showing that
\[ \phi_k(Z(\mathfrak{U}_k)) \subset Z(\mathfrak{U}'_k). \]

This part of the proof will mimic the corresponding part of the proof equivalence between the Jacobian and the Dixmier conjectures [BK Proposition 2], except we have to work a bit harder as \( \mathfrak{U}_k, \mathfrak{U}'_k \) are not Azumaya algebras.

First we argue that for all \( p \gg 0 \), there exists \( \delta _p \in Z_{HC}(\mathfrak{g}_k) \) that vanishes on the non-Azumaya locus of \( Z(\mathfrak{U}_k) \) and \( \phi_k(\delta _p) \neq 0 \). For this purpose we recall some well-known facts about the singular locus of Spec \( Z(\mathfrak{U}_k) \) following [La].

Let \( \Psi : (\mathfrak{U}_k)^G_k \rightarrow \text{Sym}(\mathfrak{h})^W \) be the usual isomorphism, \( W \) is the Weyl group and \( \mathfrak{h} \) is a Cartan subalgebra, \( \Sigma \) is the set of roots and \( h_\alpha \in \mathfrak{h} \), \( \alpha \in \Sigma \) are the corresponding elements. Put
\[ \delta _a = \Psi^{-1}(\Pi _{\alpha \in \Sigma } (h_\alpha - a)), a \in \mathbb{F}_p; \quad \delta _p = \prod _{a \in \mathbb{F}_p} \delta _a. \]

Now recall that \( \delta _p \) vanishes on the singular locus of Spec \( Z(\mathfrak{U}_k) \) [La Theorem 1], and since the smooth and Azumaya loci of \( Z(\mathfrak{U}_k) \) coincide, \( \delta _p \) vanishes on the compliment of the Azumaya locus. Denote by \( \Delta _1, \cdots, \Delta _t \in (\mathfrak{U}_k)^G_k \) images under \( \Psi^{-1} \) of all elementary symmetric functions on \( h_\alpha, \alpha \in \Sigma \) (viewed as elements of \( \text{Sym}(\mathfrak{h})^W \)). Then each \( \delta _a \) belongs to the \( \mathbb{F}_p \)-span of images of \( \Delta _1, \cdots, \Delta _t \) in \( \mathfrak{U}_k \). Since \( \phi_k(\Delta _1), \cdots, \phi_k(\Delta _t) \) are linearly independent for \( p \gg 0 \). Thus \( \phi_k(\delta _a) \neq 0 \) for all \( a \in \mathbb{F}_p \) and \( p \gg 0 \). Hence \( \phi_k(\delta _p) \neq 0 \).

Recall that by the first Kac-Weisfeiler conjecture, the PI-degree of \( \mathfrak{U}_k \) (respectively \( \mathfrak{U}'_k \)) is \( \frac{1}{2} \dim(\mathfrak{g}) \) (resp. \( \frac{1}{2} \dim(\mathfrak{g}') \)). Since index(\( \mathfrak{g} \)) \( \leq \text{index}(\mathfrak{g}') \). It follows from that the PI-degree of \( \mathfrak{U}'_k \) is at most the PI-degree of \( \mathfrak{U}'_k \). Let \( z \in Z(\mathfrak{U}_k) \) be such that \( \phi_k(z) \notin Z(\mathfrak{U}'_k) \).
Put $S_1 = (\mathfrak{Ug}_k)_{\delta_p}$ and denote by $S_2$ a localization of $\mathfrak{Ug}'_k$ so that $\phi_k(\delta_p)$ is invertible in $S_2$, and $S_2$ is an Azumaya algebra.

So we have a homomorphism of Azumaya algebras $\phi_k : S_1 \rightarrow S_2$ and $z \in Z(S_1)$ such that $\phi_k(z) \notin Z(S_2)$ and $P\text{-}\text{degree}(S_2) \leq P\text{-}\text{degree}(S_1)$. Let $V$ be a simple $S_2$-module on which $\phi_k(z)$ does not act like a scalar. To construct such a module suffices to take a simple module afforded by a character $\chi : Z(S_2) \rightarrow \mathfrak{k}$ such that $\phi_k(z)$ has a nonzero image in $$(S_2/Z(S_2))_\chi = S_2/Z(S_2) \otimes_{Z(S_2)} Z(S_2)/\text{Ker}(\chi).$$

Then $V$ viewed as an $S_1$-module must be simple (forcing $P\text{-}\text{deg}(S_1) = P\text{-}\text{deg}(S_2)$) on which $z$ acts as a non-scalar, a contradiction.

Thus we have a $\mathfrak{k}$-Poisson algebra homomorphism $\tilde{\phi}_k : Z_p(\mathfrak{g}_k) \rightarrow Z(\mathfrak{Ug}'_k)$. Recall that we have distinguished Poisson maximal ideals—the augmentation ideals $m = \mathfrak{g}_k \mathfrak{Ug}_k \cap Z_p(\mathfrak{g}_k)$, $m' = \mathfrak{g}'_k \mathfrak{Ug}'_k \cap Z(\mathfrak{Ug}'_k)$.

Thus $\mathfrak{g}_k \cong m/m^2$ with the isomorphism given by $x \mapsto x^p - x^{[p]}$, $x \in \mathfrak{g}_k$. Denote by $\mathfrak{g}_k^{(1)} \subset m$ the image of $\mathfrak{g}_k$ under the map $x \mapsto x^p - x^{[p]}$, $x \in \mathfrak{g}_k$. Since $\mathfrak{g}_k$ is perfect, it follows that the $\phi_k(m) \subset \{Z(\mathfrak{Ug}'_k), Z(\mathfrak{Ug}'_k)\}$. On the other hand, $\{Z(\mathfrak{Ug}'_k), Z(\mathfrak{Ug}'_k)\}$ is contained in the ideal generated by $[\mathfrak{g}'_k, \mathfrak{g}'_k, \mathfrak{g}'_1, \ldots, \mathfrak{g}'_n]$. The latter is clearly contained in $m'$. Thus we have a homomorphism of Lie algebras $\tilde{\phi}_k : m/m^2 \rightarrow m'/m'^2$.

Let $\bar{I}$ denote the kernel of $\tilde{\phi}_k$. Thus $[\bar{I}, \bar{I}] = \bar{I}$ since $\mathfrak{g}_k$ is semi-simple. Let $I \subset \mathfrak{g}_k^{(1)}$ be the lift of $\bar{I}$. Thus $\phi_k(I) \subset m^2$. Let $\phi_k(I) \subset m^n$ for $n > 1$. Since $\phi_k$ is a Poisson homomorphism, we get that $\phi_k(I) = \phi_k([I, I]) \subset \{m^n, m^m\} \subset m^{2n-1}$.

Hence $\phi_k(I) = 0$. So, we have an ideal $I' \subset \mathfrak{g}_k$, so that $\phi_k(g^p - g^{[p]}) = 0$ for all $g \in I'$. Hence $\phi_k(I') = 0$, so $I = 0$. As by Theorem 1.2 the Lie algebra $m'/m'^2$ is a quotient of $\mathfrak{g}'_k \oplus \mathfrak{k}^m$ and $\dim \mathfrak{g}' = \dim \mathfrak{g}'$, we may conclude that $\mathfrak{g}_k \cong \mathfrak{g}'_k$. Hence $\mathfrak{g} \cong \mathfrak{g}'$.

\begin{flushright} $\square$ \end{flushright}

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