The relativistic Brownian motion: interdisciplinary applications

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Abstract. Relativistic Brownian motion theory will be applied to the study of analogies between physical and economic systems, emphasizing limiting cases in which Gaussian distributions are no longer valid. The characteristic temperatures of the particles will be associated with the concept of variance, and this will allow us to choose whether the pertinent distribution is classical or relativistic, while working specific situations. The properties of particles can be interpreted as economic variables, in order to study the behavior of markets in terms of Lévy financial processes, since markets behave as stochastic systems. As far as we know, the application of the Jüttner distribution to the study of economic systems is a new idea.

1. Introduction
Brownian motion corresponds to a random displacement of matter in a thermal bath, produced by collisions between particles. In non-relativistic physics, a Gaussian distribution is used to describe the velocities of particles. In other contexts, the use of classical functions to describe fluctuations is based on the Central Limit Theorem (CLT). In relativistic systems, significant deviations are observed with respect to Gaussian distributions. A Lévy process, used a lot in economics and finance, corresponds to a continuous-time stochastic process that starts at $t = 0$ and admits a CLT. The most common Lévy processes are the Wiener process and the Poisson process. As we can see, in physics and in economics and finance the use of stochastic processes is very common, and it is important to mention that in a stochastic process exists time dependence and randomness[1].

Econophysics is a study area which uses the mathematical methods of physics to develop important knowledge in economics and finance. Specially, models that use classical distribution functions have been developed in this area. For example, the Boltzmann-Gibbs distribution of money[2]. In the core of the curious and very interesting comparison between physics and econophysics models relies the validity of conservation laws. E.g. in a collision there is conservation of momentum and energy; as well as in a money exchange, there is money conservation. What we are trying to show in this paper, is that relativistic Brownian motion, can also be used to develop models in econophysics, so it deserves a deeper look. Using a Jüttner distribution, important results are obtained when compared to its Maxwell-Boltzmann distribution counterparts[3].
2. Comparison of the Mean Square Displacement for the Non-relativistic and the Relativistic Cases

For a massive particle, of mass $m$, immersed in a relativistic thermal bath, a relativistic distribution function, like the Jüttner distribution, is appropriate to describe the Brownian motion. This is because the particles can’t move at velocities higher than the speed of light $c$ [3].

The mean square displacement for the non-relativistic case at temperature $T$ and time $t$ is given by

$$\langle [r(t)]^2 \rangle = \frac{6KT}{\mu} t. \quad (1)$$

In Eq. 1 $\mu$ is the friction coefficient and $K$ is the Boltzmann constant. In the relativistic case the mean square displacement is now given by

$$\langle [r(t)]^2 \rangle = \frac{2}{\mu} \left[ 3KT + mc^2 \frac{K_1 \left( \frac{1}{z} \right)}{K_2 \left( \frac{1}{z} \right)} \right] t, \quad (2)$$

where $z = KT/mc^2$ and $K_i$ represents the modified Bessel function of type $i$.

Fig. 1 shows a stochastic process compatible with a Brownian motion. The model consists of random successive steps, moving right or left at each step. The value of the parameter $z$ was 0.1 and 2000 steps were used. The most important observation of this graphic is that for the Jüttner’s mean square displacement the fluctuations are smaller than their Maxwell-Boltzmann’s counterpart. Exporting this observation to financial systems a conclusion is that Jüttnerian stochastic processes are less uncertain than the Gaussian ones. Markets don’t fluctuate infinitely, as is implied by Maxwell-Boltzmann’s statistics. As shown in Fig. 2, for large values of $z$ the characteristic speed of the particles of the thermal bath in the non-relativistic case tends to infinity, while the relativistic value tends to be constant.

3. Final Remarks

This study was made for a non-degenerate relativistic single component gas, where the Brownian particles are immersed. The models in physics help to understand the financial markets because they also behave in a random character. A very interesting interaction between relativistic

![Figure 1](image.png)

**Figure 1.** This graphic shows the behavior of the mean square displacement in a one dimensional random walk process in a relativistic (blue) and in the non-relativistic (red) cases. The Gaussian random walk shows larger fluctuations compared to the Jüttnerian case, for the same thermal bath temperatures. In the case of price fluctuations this behavior possesses a potential interest.
Figure 2. This graphic shows the behavior of the characteristic velocity of the particles of the thermal bath. One can see that for a larger value of \( z \) the non-relativistic speed grows to infinity, while the relativistic speed tends to a constant value.

Physics and economics and finance is a fact which will help us to understand in a better way the modelling of money and wealth.

References

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