Conic Idempotent Residuated Lattices

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Abstract

We give a structural decomposition of conic idempotent residuated lattices, showing that each of them is an ordinal sum of certain simpler partially ordered structures. This ordinal sum is indexed by a totally ordered residuated lattice, which serves as its skeleton and is both a subalgebra and nuclear image, and we equationally characterize which totally ordered residuated lattices appear as such skeletons. Using the two inverse operations induced by the residuals, we further characterize both congruence and subalgebra generation in conic idempotent residuated lattices. We show that every variety generated by conic idempotent residuated lattices enjoys the congruence extension property. In particular, this holds for semilinear idempotent residuated lattices. Moreover, we provide a detailed analysis of the structure of idempotent residuated chains serving as index sets on two levels: as certain enriched Galois connections and as enhanced monoidal preorders. Using this, we show that although conic idempotent residuated lattices do not enjoy the amalgamation property, the natural class of rigid and conjunctive conic idempotent residuated lattices has the strong amalgamation property, and consequently has surjective epimorphisms. We extend this result to the variety generated by rigid and conjunctive conic idempotent residuated lattices, and establish the amalgamation, strong amalgamation, and epimorphism-surjectivity properties for several important subvarieties. Based on this algebraic work, we obtain local deduction theorems, the deductive interpolation property, and the projective Beth definability property for the corresponding substructural logics.

1. INTRODUCTION

Residuated lattices are prominent ordered algebraic structures that simultaneously generalize lattice-ordered groups, ideal lattices of rings, and relation algebras. In recent years, they have played an especially salient role as algebraic models of substructural logics \cite{23,42}, which comprise a diverse family of deductive systems that includes linear, relevance, many-valued, and intuitionistic logics. Thanks to this connection, residuated lattices and substructural logics together constitute a cohesive whole, and many natural properties have independently interesting manifestations in both order algebraic and logical terms. The structure of residuated lattices is therefore significant to both ordered algebra and logic. In this study, we investigate the structure of conic idempotent residuated lattices \cite{34,10,8,7}, a common generalization of totally ordered idempotent residuated lattices and Heyting algebras. We give a number of decompositions of conic idempotent residuated lattices in terms of simple combinatorial data, and further use these structural descriptions to obtain several results concerning properties at the intersection of

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algebra and logic. In algebraic terms, we establish that every variety generated by conic idempotent residuated lattices has the congruence extension property, and obtain the strong amalgamation property and epimorphism-surjectivity property for several natural varieties generated by conic idempotent residuated lattices. In logical terms, we obtain local deduction theorems, the deductive interpolation property, and the projective Beth definability property for the corresponding deductive systems. By specializing our work to the totally ordered case, we also give novel contributions to the theory of idempotent residuated chains and the logics they algebraize, advancing a line of research portrayed in [43, 30, 23, 9].

Residuated lattices possess an underlying multiplication. This presents two possibilities at opposite extremes: the case where the multiplication is cancellative, and the case where it is idempotent. The cancellative case encompasses lattice-ordered groups [2], and negative cones of lattice-ordered groups [3], (therefore it is also applicable to MV-algebras [11]), and has been studied extensively in these and other contexts. Residuated lattices with idempotent multiplication are at the opposite extreme, and include Heyting algebras, the models of intuitionistic logic, and Sugihara monoids [18], the models of relevance logic with the mingle rule. Idempotent residuated lattices appear indirectly in the study of many varieties of residuated lattices; for example, in ordinal sum representations of MTL-algebras [33], in poset product decompositions of GBL-algebras and their applications [35, 17, 20], and in various representations over Boolean spaces [45, 19]. Furthermore, congruences in finite residuated lattices correspond bijectively to idempotent elements that are negative and central [25], while positive idempotent elements in residuated lattices give rise to localizations via double-division conulei [29]. Idempotent and cancellative residuated lattices are complementary, and often both kinds of algebras play a role in describing the structure of other kinds of residuated lattices.

Due to their importance to the structure of residuated lattices generally, idempotent residuated lattices are the subject of an emerging literature that has been expanding rapidly in recent years. Heyting algebras and their 0-free reducts, often called Brouwerian algebras, are the most extensively studied class idempotent residuated lattices. They include Boolean algebras as a subvariety, and are the subject of a vast body of work. Some general features of commutative idempotent residuated lattices were studied in [44], and commutative idempotent residuated chains have attracted considerable attention, especially in connection to relevance logic (see, e.g., [43, 41, 40, 18]). Although the study of idempotent residuated lattices goes all the way back to George Boole and Boolean algebras, the study of non-commutative idempotent residuated lattices has a shorter history. [23] exhibits uncountably many (not necessarily commutative) totally ordered idempotent residuated lattices, generating uncountably many distinct atoms in the subvariety lattice of residuated lattices. [9] studies idempotent residuated chains from a semigroup-theoretic point of view, while [30] investigates totally ordered and conservative algebras, primarily in the commutative and finite case. Conic idempotent residuated lattices are studied from the perspective of semigroup theory in [10, 8], and [7] investigates some subvarieties of conic idempotent residuated lattices in terms of categorical equivalences with expanded Brouwerian algebras. Quite recently, [36] describes the structure of finite commutative idempotent involutive residuated lattices as unions of Boolean algebras assembled along a distributive lattice grid.
In this paper, we advance the existing literature on idempotent residuated lattices and bring to the surface some aspects that have not been stressed in the previously cited studies. In particular, the aim of the present research is to develop a structural description of conic idempotent residuated lattices that is sufficient to understand amalgamation of these structures. The amalgamation property is highly sensitive to subalgebra generation, and therefore obtaining structural decompositions that capture the subalgebra generation process is indispensable for our aims. Because the existing semigroup-theoretic work on the structure of conic idempotent residuated lattices \([10, 8]\) does not describe the subalgebra generation process, we must supplement this work.

Our key addition to the existing work is our emphasis on the inversion operations \(x \mapsto x^r = x\backslash 1\) and \(x \mapsto x^\ell = x/1\), which were introduced and used in \([23]\). In the totally ordered case, all operations can be defined in terms of the lattice operations and these two inversion operations; in the conic case, a restricted Brouwerian implication is also needed, as we make explicit in Theorem 2.20. Moreover, these are the only operations that contribute to subalgebra generation in the totally ordered case; in the conic case, the lattice operations contribute as well. The control on subalgebra generation that the inversion operations provide is crucial to our study of amalgamation in the sequel. Consideration of these operations also allows us to get a handle on congruence generation, laying the necessary groundwork for our study of the congruence extension property.

This study touches on many aspects of conic idempotent residuated lattices, several of which are of independent interest. Although we believe that this work is best understood as an integrated whole, we have tried to structure our exposition to accommodate readers who are interested in particular aspects of this work. In particular, we expect that the portions of this paper concerning idempotent residuated chains are of special interest to many readers, and where possible have tried to make these sections accessible in isolation from the broader study. As a guide to the reader, we outline our contributions as follows.

First, in Section 2, we lay out background on idempotent residuated lattices, and provide several motivating examples. Placing an emphasis on the inverse operations \(x \mapsto x^r\) and \(x \mapsto x^\ell\) throughout, we provide concrete descriptions of multiplication and division in conic idempotent residuated lattices, and completely describe centralizers of elements in these. We identify in each conic idempotent residuated lattice \(A\) a totally ordered “skeleton,” which comprises both a subalgebra of \(A\) and the image of \(A\) under the nucleus \(x \mapsto \gamma(x) = x^{\ell r} \wedge x^{r \ell}\). We exhibit a transparent decomposition of each conic idempotent residuated lattice \(A\) as an ordinal sum of the partially-ordered sets \(\gamma^{-1}(a)\), where \(a\) ranges over elements of the skeleton of \(A\). For negative \(a\), \(\gamma^{-1}(a)\) is a Brouwerian algebra, and for positive \(a\), \(\gamma^{-1}(a)\) is a prelattice (a bounded lattice possibly missing its bottom element). As a byproduct of this decomposition, we characterize the totally ordered residuated chains appearing as skeletons as those satisfying the identity \(x = x^{\ell r} \wedge x^{r \ell}\), which we call quasi-involutive idempotent residuated chains. We also compare our decomposition results to those of \([10, 8]\).

\(^1\)Recall that a class \(\mathcal{K}\) of algebras has the amalgamation property if whenever \(A, B, C \in \mathcal{K}\) and \(A\) is a subalgebra of \(B\) and \(C\), there exists an algebra \(D \in \mathcal{K}\) so that \(B\) and \(C\) both embed in \(D\) in such a way that the resulting diagram commutes. See [38] for a broad survey of the amalgamation property in algebra.
In Section 3, we use the work of the previous section to describe congruence generation in conic idempotent residuated lattices. In Lemmas 3.1 and 3.2, we give a concrete description of the generation of congruence filters using terms defined via the inversion operations \( x \mapsto x^r \) and \( x \mapsto x^\ell \). As corollaries, we obtain that every variety generated by conic idempotent residuated lattices has the congruence extension property; in particular, this holds for the variety of semilinear idempotent residuated lattices. This is surprising, as the congruence extension property usually holds in varieties that enjoy some degree of commutativity. Anticipating their importance in amalgamation, we also characterize the finitely subdirectly irreducible semiconic idempotent residuated lattices as those for which 1 is join irreducible.

In Section 4, we describe totally ordered idempotent residuated chains, aiming in particular to describe the structure of those appearing as skeletons in the decomposition of Section 2. In Section 4.1, we characterize the \( \{\wedge, \vee, r, \ell, 1\} \)-reducts of idempotent residuated chains as a universal class consisting of certain enriched Galois connections, which we call idempotent Galois connections. We show that idempotent Galois connections are definitionally equivalent to idempotent residuated chains. Then, in Section 4.2, we introduce flow diagrams as a diagrammatic tool for understanding the action of the inversion operations in idempotent residuated chains. Flow diagrams provide a bridge between idempotent Galois connections and enhanced monoidal preorders, introduced in Sections 4.3 and 4.4 as another definitionally equivalent presentation of idempotent residuated chains. Enhanced monoidal preorders improve on the monoidal preorder representations considered in [30] by expanding the latter by information about which elements are positive and negative, as well as with a minimality and maximality condition that is crucial in the infinite setting. In conjunction with idempotent Galois connections and flow diagrams, enhanced monoidal preorders provide a pictorial and combinatorial description of arbitrary idempotent residuated chains. In Section 4.5, we describe subalgebra generation in terms of enhanced monoidal preorders in a transparent way. Then, in Section 4.6, we identify a fundamental problem with amalgamating quasi-involutive idempotent residuated chains, and introduce the class of *-involutive idempotent residuated chains in order to rectify this problem; algebras that are not *-involutive always give rise to V-formations that do not possess an amalgam. In Section 4.7, we completely characterize one-generated *-involutive idempotent residuated chains in terms of their enhanced monoidal preorders and in Section 4.8, characterize arbitrary *-involutive idempotent residuated chains as nested sums of one-generated ones.

In Section 5, we combine the ingredients assembled in the previous section in order to study amalgamation in conic idempotent residuated lattices. We illustrate in Subsection 5.1 that the amalgamation property fails for idempotent residuated chains (via non *-involutive examples). Actually, we prove that there are V-formations of idempotent residuated chains that do not have an amalgam in the whole variety of semilinear idempotent residuated lattices, thus the latter lacks the amalgamation property. Further, in Section 5.3, we show that the amalgamation property fails even for rigid conic idempotent residuated lattices (i.e., those whose skeleton is *-involutive); in fact, the amalgamation property fails even for the variety of rigid semiconic idempotent residuated lattices and its commutative subvariety (Theorem 5.5). The problem causing the failure of amalgamation is that some blocks \( \gamma^1(a) \) may not be lattices. This problem is avoided if and only if the conic
idempotent residuated lattice is conjunctive (i.e., satisfies $\gamma(x \land y) = \gamma(x) \land \gamma(y)$).

In Theorem 5.7 we show that the class of *-involutive idempotent residuated chains has the strong amalgamation property, and in Theorem 5.8 that the class of rigid conjunctive conic idempotent residuated lattices has the strong amalgamation property. Using some new tools on extending the strong amalgamation property, we show that the varieties of *-involutive semilinear idempotent residuated lattices and rigid conjunctive semiconic idempotent residuated lattices both have the strong amalgamation property. In Section 5.7 we further establish the amalgamation and strong amalgamation property for several subvarieties, notably studying this property for the commutative subvarieties. As a consequence of the strong amalgamation property, we also obtain the epimorphism-surjectivity property for a number of the varieties we consider.

Finally, in Section 6 we explain the connection between the algebraic properties we have considered and their logical analogues: local deduction theorems, the deductive interpolation property, and the projective Beth definability property. We show that the deductive systems corresponding to all of the varieties we have considered enjoy a local deduction theorem. For each variety that we have shown to have the amalgamation property, we show that the corresponding logic has the deductive interpolation property; when the variety has the strong amalgamation property, the corresponding logic also has the projective Beth definability property.

2. The structure of conic idempotent residuated lattices

2.1. Residuated lattices. A residuated lattice is an algebra $A = (A, \land, \lor, \cdot, \backslash, /, 1)$, where $(A, \land, \lor)$ is a lattice, $(A, \cdot, 1)$ is a monoid, and $\cdot, \backslash, /$ are binary operations on $A$ such that

$$y \leq x \backslash z \iff x \cdot y \leq z \iff x \leq z / y$$

for all $x, y, z \in A$. In the event that the operation $\cdot$ is commutative, we have $x \backslash y = y / x$ for all $x, y$ and we denote the common value by $x \rightarrow y$.

A residuated lattice $A$ is called conic if every element of $A$ is comparable to the monoid identity. The name conic is due to the fact that conic residuated lattices $A$ are exactly the ones for which $A = A^- \cup A^+$, where $A^- = \{x \in A : x \leq 1\}$ is the negative cone and $A^+ = \{x \in A : x \geq 1\}$ is the positive cone of $A$. We will say that elements of $A^+$ are positive or have positive sign and that elements of $A^-$ are negative or have negative sign; note that the element 1 has both signs, so it has simultaneously the same sign and the opposite sign of any other element. Examples of conic residuated lattices include integral residuated lattices (for which the monoid identity is the top element) and residuated chains (which are totally ordered).

The members of the variety generated by the conic residuated lattices are called semiconic residuated lattices. Note that conic residuated lattices are axiomatized by the positive universal sentence: For all $x, y \leq 1$ or $1 \leq x$. Following the general procedure of obtaining axiomatizations of varieties generated by positive universal classes given in [22], the variety of residuated lattices generated by the conic ones may be axiomatized relative to all residuated lattices by the identities

$$1 = \gamma_1(x \land 1) \lor \gamma_2((x \backslash 1) \land 1),$$

where $\gamma_1$ and $\gamma_2$ range over all iterated conjugates. Recall that the left and right conjugate of $a$ by $x$ are the elements $\lambda_x(a) = x \backslash ax \land 1$ and $\rho_x(a) = xa / x \land 1$; iterated
conjugates are arbitrary compositions of left and right conjugates with no restriction to the conjugating elements used. Iterated conjugates play an important role in describing congruence generation and will be discussed further in Section 3. In the commutative case, conjugates can be omitted and therefore semiconmutative residuated lattices are axiomatized by
\[ 1 = (x \land 1) \lor ((x \setminus 1) \land 1). \]

A residuated lattice \( A \) is called idempotent if it satisfies \( x^2 = x \). It is called integral if it satisfies \( x \leq 1 \). The integral idempotent residuated lattices are exactly the Brouwerian algebras, i.e., residuated lattices that satisfy \( xy = x \land y \). In particular, they are commutative.

\[ \text{Figure 1. Conic and integral.} \]

**Example 2.1.** Odd Sugihara monoids are idempotent residuated lattices that are commutative, semilinear (subdirect products of residuated chains) and 1-involutive \( ((x \to 1) \to 1 = x) \). The totally-ordered ones are of the form \( A = \{ a_i : i \in I \} \cup \{ \cdot \} \cup \{ b_i : i \in I \} \), where \( I \) is a chain, ordered by \( b_i < b_j < 1 < a_m < a_n \), for \( i < j \) and \( m > n \); see Figure 2. Multiplication is commutative and is defined by \( a_i b_j = a_i \) if \( i < j \), \( a_i b_j = b_j \) if \( i > j \), and \( a_i b_i = b_i \). Note that the product of the two elements is always one of them and also that it is the furthest away from 1; when there is a tie (they have the same distance from 1) the product is the meet. This notion of distance from 1 is here captured by means of the index set \( I \), and we will specify the intervals that this distance defines in general in Corollary 4.1. The resulting structure is a residuated lattice and we define \( x^* \) as \( x \to 1 \). We can see that \( x \) and \( x^* \) have opposite sign and that whether \( y \) is closer to 1 than \( x \) is depends on whether \( y \) is in the interval between \( x \) and \( x^* \); to be more precise whether \( y \) is in \( (x, x^*) \) or \( (x^*, x) \) (depending on whether \( x \) is negative or positive) or not. Therefore, multiplication can be defined in terms of * and the order.

**Example 2.2.** Non-commutative analogues of Sugihara chains are given in [23]; see Figure 2 on the right. These chains have the same ordering as Sugihara monoids and they come in many versions. Actually, in [23] it is shown that uncountably many different minimal non-trivial varieties are generated by such chains. By contrast, the subvariety lattice of Sugihara monoids is a countable chain. The variation in these algebras is encoded in a subset \( J \) of the index set \( I \). Multiplication is defined the same way as Sugihara chains for elements with different index, but the products where there is a tie are determined by \( J \):
1. if $i \in J$, then $a_i b_i = a_i$ and $b_i a_i = b_i$ (Left), and
2. if $i \notin J$, then $a_i b_i = b_i$ and $b_i a_i = a_i$ (Right).

It is worth noting that for no index $i$ do we have $a_i b_i = a_i$ and $b_i a_i = a_i$, or $a_i b_i = b_i$ and $b_i a_i = b_i$. We will see in Lemma 2.14(2) that this is not a coincidence.

Also, although these same-index products are the only instances of non-commutativity, these algebras are fundamentally different from Sugihara chains in terms of their subalgebras. In Sugihara chains, for every $i$, the set $\{b_i, 1, a_i\}$ is a subalgebra and every Sugihara chain is a nested sum (to be defined formally in Section 4.8) of copies of this 3-element algebra. However, the above non-commutative algebras have no proper, nontrivial subalgebras; each one of them is generated by any non-identity element. Since subalgebras play an important role in amalgamation, these two types of chains behave very differently.

2.2. Idempotency, conicity, and commutativity. A residuated lattice $A$ is called conservative if $xy \in \{x, y\}$, for all $x, y \in A$. Sugihara chains, as well as their non-commutative variants, are conservative. In the presence of conicity, conservativity is a very strong property, as it implies that the algebra is a chain; see Corollary 2.7. Since we want to study non-linear conic algebras, we will need a more relaxed version of conservativity. As we will see in Corollary 2.6, the correct notion is that of weak conservativity: $A$ is called weakly conservative if $xy \in \{x, y, x \land y, x \lor y\}$, for all $x, y \in A$.

Example 2.3. Note that weakly conservative residuated lattices, even commutative ones, need not be conic; this is in contrast to the conservative case, where they have to be chains. For example, there is a four-element residuated lattice that is weakly conservative, but not conic. Consider the set $A$ consisting of four elements ordered by $\perp < 1, a < \top$; multiplication is idempotent and commutative, 1 is the identity element, $\perp$ is an absorbing element and $\top a = a$; see Figure 3.

The proofs of following properties are straightforward.

Lemma 2.4 ([44, Lemma 2.1]). Let $A$ be an idempotent residuated lattice and $x, y \in A$. 

![Figure 2. Sugihara chains and some non-commutative analogues.](image-url)
Figure 3. Weakly conservative, idempotent and commutative, but not conic.

1. If $x, y \leq 1$, then $xy = x \land y$.
2. If $1 \leq x, y$, then $xy = x \lor y$.
3. For $x \leq 1 \leq y$, if $xy \leq 1$ then $xy = x$ and if $1 \leq xy$ then $xy = y$; likewise for $yx$.

In conic idempotent residuated lattices, conservativity does not always hold, nor does commutativity. However, for each pair of elements, one of the two properties always holds, as we show below. This fact will play an important role in many of our arguments.

**Corollary 2.5.** In conic idempotent residuated lattices, if two elements have the same sign then they commute and if they have opposite sign then their product is conservative.

**Corollary 2.6.** For a residuated lattice $A$ the following are equivalent.
1. $A$ is conic and idempotent.
2. $A$ is conic and weakly conservative.
3. For all $x, y \in A$, we have $xy \in \{x \land y, x \lor y\}$.

**Proof.** That (1) implies (2) follows from Lemma 2.4. Conversely, from (2) we get $x^2 \in \{x, x \lor x, x \land x\}$, so $x^2 = x$. Also, from (2) we obtain that if $x$ and $y$ have the same sign then $xy \in \{x \land y, x \lor y\}$ and if they have different signs then they are comparable, so $x, y \in \{x \land y, x \lor y\}$; hence (2) implies (3). Finally, from (3) we obtain weak conservativity and also for all $x \in A$ we have $x = x1 \in \{x \land 1, x \lor 1\}$, hence $x \leq 1$ or $1 \leq x$. \hfill $\square$

**Corollary 2.7.** [30, Lemma 2.5] All conservative conic residuated lattices are chains.

**Proof.** For any two elements $x$ and $y$, by conservativity we have $xy \in \{x, y\}$. By Corollary 2.4 we also have $xy \in \{x \land y, x \lor y\}$. Therefore, the sets $\{x, y\}$ and $\{x \land y, x \lor y\}$ intersect and this shows that $x$ and $y$ are comparable. \hfill $\square$

We will see that in commutative conic idempotent residuated lattices the term $x^* = x \to 1$ plays an important role, as illustrated by odd Sugihara monoids (see Example 2.1). For the general (non-commutative) case we define two *inverses*. If $A$ is a residuated lattice and $x \in A$, we define $x^\ell := x1$ and $x^\ell := 1/x$; in particular $xx^\ell \leq 1$, $x^\ell x \leq 1$ and $1^\ell = 1^\ell = 1$. Also, as a direct consequence of residuation we have that $(x^\ell, x^\ell)$ forms a Galois connection. We say that $A$ is *cyclic* if $x^\ell = x^\ell$ for all $x \in A$.

**Proposition 2.8.** Let $A$ be a conic idempotent residuated lattice. Then $A$ is cyclic if and only if $A$ is commutative.
Proof. It is trivial that commutativity implies cyclicity. For the converse, note that if \( x \) and \( y \) have the same sign, then they commute by Corollary 2.5. If \( x \) and \( y \) do not have the same sign then we have conservativity: \( \{xy, yx\} \subseteq \{x, y\} \). Assuming, without loss of generality, that \( x \leq 1 < y \), and using Lemma 2.4(3), we have \( xy = x \) iff \( xy \leq 1 \) (by cyclicity) iff \( yx = x \). Therefore, \( xy = yx \). \( \square \)

Example 2.9. We mention that there is a four-element conic (actually, even totally ordered) idempotent residuated lattice that is not commutative/cyclic. It consists of four elements ordered by \( \perp < a < 1 < \top \); multiplication is idempotent, 1 is the monoid identity, \( \perp \) is an absorbing element and \( \top a = \top \) while \( a \top = a \). This is a finite version of the non-commutative residuated chains of Figure 2.

2.3. The role of the inverses. For a residuated lattice \( A \), we define the set
\[
A^t = \{a^t : a \in A\} \cup \{a^r : a \in A\}
\]
of all (left and right) inverses of elements of \( A \).

We say that an element \( x \) of a residuated lattice \( A \) is conical if for all \( y \in A \), \( x \leq y \) or \( y \leq x \); in other words \( A = \uparrow x \cup \downarrow x \). In these terms, a residuated lattice \( A \) is conic iff 1 is conical.

Lemma 2.10. Let \( A \) be a conic idempotent residuated lattice and let \( x \in A \). Then each of \( x^t \) and \( x^r \) are conical elements and both have the same sign, which is opposite to the sign of \( x \). In particular, \( A^t \) is a chain.

Proof. We will prove the claim for \( x^r \); it will follow for \( x^t \) by an analogous argument. First note that if \( x \leq 1 \), then \( 1 \leq x^r = x^r \) and if \( 1 \leq x \) then \( x^r = x \leq 1 \leq 1 = 1 \); hence \( x \) and \( x^r \) always have opposite sign. We will now show that every \( y \in A \) is comparable to \( x^r \).

If \( y \) and \( x^r \) have opposite signs, then they are obviously comparable, so we may assume that they have the same sign. Then by Corollary 2.5, they commute: \( yx^r = x^r y \). Since \( x \) and \( x^r \) always have opposite signs, we have two cases: \( x \leq 1 \leq x^r, y \) and \( x^r, y \leq 1 \leq x \). In both cases \( x \) and \( y \) have opposite signs, so \( xy = x \) or \( xy = y \). If \( xy \leq 1 \), then \( y \leq x \leq 1 = x^r \), so \( y \) and \( x^r \) are comparable. Therefore, we only need to consider the two cases: \( x = xx^r \leq 1 \leq x^r, y = xy \) and \( x^r = xx^r, y \leq 1 \leq x = xy \).

In the first case, we have \( y \vee x^r = yx^r = xyx^r = xx^ry = xy = y \), whence \( x^r \leq y \). In the second case we obtain \( x^r \wedge y = x^r y = xx^ry = xyx^r = xx^r = x^r \), whence \( x^r \leq y \). \( \square \)

Lemma 2.11. Let \( A \) be a residuated lattice and let \( x \) be an idempotent element of \( A \). Then \( x \leq 1 \) iff \( x^r = x \setminus x \) iff \( x^t = x/x \).

Proof. If \( x \leq 1 \), then \( x/x \leq x/1 \) and \( x(x/1) = xx(x/1) \leq x1 = x \) so \( x/1 \leq x/x \). Conversely, if \( x^r = x \setminus x \), then \( x = x1 \leq x(x/x) \leq x(x/1) \leq 1 \). The proof for \( x^t \) is similar. \( \square \)

The following lemma provides multiple equivalent descriptions of the product of an conic idempotent residuated lattice in terms of the position of these conical elements; all of these descriptions will be needed in the various later proofs. Recall the usual interval notation in posets, such as \( \langle a, b \rangle = \{x : a \leq x \leq b\} \); in particular if \( a \not< b \), then \( \langle a, b \rangle \) is empty. We also write \( a || b \) if \( a \) and \( b \) are incomparable. Recall that \( a \leq b^r = b \setminus 1 \) iff \( b a \leq 1 \) iff \( b \leq 1/a = a^t \). Also, using Lemma 2.10, \( b^r < a \) iff \( a \not< b^r \) iff \( b \not< a^t \) iff \( a^t \not< b \).
Lemma 2.12. Let $A$ be a conic idempotent residuated lattice, and let $x, y \in A$.

1. Then

$$xy = \begin{cases} 
x \land y & x, y \leq 1 \\
x \lor y & x, y \geq 1 \\
y & y \leq 1 \leq y^f < x \text{ or } x \leq 1 \leq y \leq x^r \\
\implies x^r < y \leq 1 \leq x \text{ or } x \leq 1 \leq y \leq x^r \\
\implies y \in (x^r, 1] \cup [1, x] \iff y^f \in [x, 1] \cup [1, x) \\
x \leq 1 \leq x^r < y \text{ or } y \leq 1 \leq x \leq y^f \\
\implies y^f < x \leq 1 \leq y \text{ or } y \leq 1 \leq x \leq y^f \\
\implies x \in (y^f, 1] \cup [1, y^f] \iff x^r \in [y, 1] \cup [1, y) 
\end{cases}$$

2. Also

$$xy = \begin{cases} 
x \land y & x, y \leq 1 \text{ and } x \parallel y \\
x \lor y & x, y \geq 1 \text{ and } x \parallel y \\
y & y \leq 1 \leq x \text{ or } x \leq 1 \leq y \leq x^r \\
\implies x^r < y \leq 1 \leq x \text{ or } x \leq 1 \leq y \leq x^r \\
\implies y \in (x^r, 1] \cup [x, x^r] \\
x \leq 1 \leq x^r < y \text{ or } y \leq 1 \leq x \leq x^r \\
\implies y^f < x \leq 1 \leq y \text{ or } y \leq 1 \leq x \leq y^f \\
\implies x \in (y^f, y] \cup [y, y^f] 
\end{cases}$$

3. and

$$xy = \begin{cases} 
x \land y & x \leq 1 \iff x \leq y^f \iff y \leq x^r \\
x \lor y & 1 < xy \iff y^f < x \iff x^r < y 
\end{cases}$$

Proof. By Lemma 2.11 we have that if $x, y \leq 1$, then $xy = x \land y$, and similarly that if $x, y > 1$, then $xy = x \lor y$. In the remaining cases, assume that $x$ and $y$ have different signs. Then $xy \in \{x, y\}$. Observe that if $x \leq 1 \leq y$, then using Lemma 2.11 we have: $xy = x$ iff $xy \leq x$ iff $y \leq x$ but $y \leq x^r$.

On the other hand, under the assumption $x \leq 1 \leq y$ and using the fact that $x^r$ is conical, we have: $xy = y$ iff $xy \neq x$ iff $xy \leq x$ iff $y \leq x^r$ iff $x^r < y$.

In the remaining cases, we have $x \leq 1 \leq x$. So, $xy = y$ iff $x \leq y/y$ iff $x \leq y^f$. Likewise, $xy = x$ iff $xy \leq y$ iff $y^f < x$. Combining these remarks establishes the first conditions of each case in 1.

For the remaining equivalent conditions in 1, we note that $y \leq 1 \leq y^f < x$ is equivalent to the conjunction $y \leq 1$ and $y^f < x$ (since $1 \leq y^f$ follows from $y \leq 1$). Also, using the conicity of inverses, we have the following equivalences: $y^f < x$ iff $x \leq y^f$ iff $y \leq 1$ iff $y \leq x^r$ iff $x^r < y$. Therefore, $y \leq 1 \leq y^f < x$ is equivalent to $x^r < y \leq 1$ (i.e., to $y \in (x^r, 1]$), which in turn is equivalent to $x^r < y \leq 1 \leq x$. The remaining cases follow by analogous reasoning. The proofs of 2 and 3 follow from 1.

Corollary 2.13. For every $x$ in a conic idempotent residuated lattice $A$, we have:

1. $xx^r = x \land x^r$.
2. $x^f x = x^f \land x$.
3. $x^f x \leq xx^f$ and $xx^r \leq x^r x$.
4. If $x$ is conical, then $xy \in \{x, y\}$, for all $y$.

Proof. Properties 1 and 2 follow directly from Lemma 2.12. For property 3 we have $x^f x = x^f x \leq xx^f$, by conservativity. Property 4 follows from Lemma 2.12(2).
2.4. Commutativity and centrality. If $A$ is a residuated lattice and $a \in A$, then as usual the centralizer of $a$ is the set

$$C(a) = \{x \in A : ax = xa\}.$$  

The center of $A$ is

$$Z(A) = \bigcap_{a \in A} C(a) = \{x \in A : (\forall y \in A)(xy = yx)\}.$$  

The elements of the center are called central.

Recall that for every $a$ in a conic idempotent residuated lattice $A$, the elements $a^r$ and $a^l$ are conical, hence comparable, so $\{a^l \land a^r, a^l \lor a^r\} = \{a^r, a^l\}$. We define $a^* := a^l \lor a^r$. If $A$ is commutative, then $a^* = (a \to 1) \lor (a \to 1) = a \to 1$, so this definition extends the one used in the commutative case.

Recall that a left-zero semigroup $S$ is a semigroup that satisfies the identity $xy = x$; a right-zero semigroup satisfies $xy = y$.

**Lemma 2.14.** For every element $a$ in a conic idempotent residuated lattice $A$, $a$ is central iff $a^l = a^r$. Also,

1. If $a$ is central (equivalently, $a^l = a^r$), then $C(a) = A$ and $\{a, a^*\}$ forms a semilattice with multiplication equal to the inherited meet.
2. If $a$ is not central (equivalently, $a^l \neq a^r$), then $C(a) = \{a^*\}$ and $\{a, a^*\}$ forms a left-zero or a right-zero semigroup.

Also, there is no element between the conical elements $a^l$ and $a^r$ and these elements form a covering pair.

**Proof.** By Lemma 2.12(3) we have that for $x \neq y$, $xy = yx$ iff they have the same sign iff $(y \leq x^l \iff y \leq x^r)$. Therefore, $x$ commutes with precisely those elements $y$ for which $(y \leq x^l$ and $y \leq x^r')$ or $(y \not\leq x^r$ and $y \not\leq x^l')$, i.e. with precisely those elements $y$ for which $y \leq x^l \land x^r$ or $x^r \lor x^l < y$; of course if $x = y$, then $xy = yx$ and the value of $y = x$ is included in the above intervals. Therefore, $C(x) = (-\infty, x^l \land x^r] \cup (x^r \lor x^l, \infty) = (x^l \land x^r, x^r \lor x^l)^c$.

1. If $x^l = x^r$, then $C(x) = (-\infty, x^l \land x^r] \cup (x^r \lor x^l, \infty) = A$ and $x$ is central. By Corollary 2.13, $xx^* = x \land x^*$.  
2. Now assume that $x^l \neq x^r$. Then $x^l \land x^r < x^l \lor x^r$, so $x^l \lor x^r \not\in (-\infty, x^l \land x^r] \cup (x^l \lor x^r, \infty) = C(x)$, thus $x$ does not commute with $x^l \lor x^r$; in particular $x$ is not central.

Now, assume that $x$ does not commute with either of $y$ or $z$; we will show that $y = z$. Since two elements of the same sign always commute, it follows that $y$ and $z$ each have different sign than $x$, hence $y$ and $z$ have the same sign. In particular, $y$ and $z$ commute. Since $x$ and $y$ have different signs we have $xy \in \{x, y\}$ and also $yx \in \{x, y\}$, so $\{xy, yx\} \subseteq \{x, y\}$. Also, since $xy \neq yx$ and $x \neq y$ we get $xy = x$ and $yx = y$, or $xy = y$ and $yx = x$; in other words, $\{x, y\}$ is either a left-zero semigroup of a right-zero semigroup. The same holds for $\{x, z\}$. If $\{x, y\}$ and $\{x, z\}$ are both left-zero semigroups, we have

$$y = xy = y(xz) = (yx)z = yz = zy = (zx)y = z(xy) = z(xy) = zx = z.$$  

If $\{x, y\}$ is a left-zero and $\{x, z\}$ is a right-zero semigroup, then

$$z = xz = (xy)z = x(yz) = (xz)y = zy = z(yx) = (zy)x = (yz)x = y(zx) = yx = y.$$
The proof for the remaining two cases is completely symmetric. As a result, the complement of $C(x)$ is equal to $\{x^r \lor x^s\}$.

Finally, to see that $x^r$ and $x^s$ form a covering pair, note that $(x^r \land x^s, x^r \lor x^s)^c = C(x) = \{x^r \lor x^s\}$, whence $(x^r \land x^s, x^r \lor x^s) = \{x^r \lor x^s\}$, so there are no elements strictly in between $x^r \land x^s$ and $x^r \lor x^s$. Since $\{x^r \land x^s, x^r \lor x^s\} = \{x^r, x^s\}$, there are no elements strictly between $x^r$ and $x^s$.

**Corollary 2.15.** If $a$ and $b$ are non-commuting elements in a conic idempotent residuated lattice, then $a^* = b$, $b^* = a$ and $\{a, b\}$ forms a left-zero or a right-zero semigroup.

Recall that $x^* = x^r \lor x^s$, and further define $x^* = x^r \land x^s$.

**Corollary 2.16.** For all $a, b$ in a residuated lattice we have $a \leq b^s$ iff $b \leq a^s$. In other words the pair $(a^s, a^s)$ is a Galois connection. Moreover, in conic idempotent residuated lattices, $(a^s, a^s)$ forms a splitting pair: for all $c, c \leq a^s$ or $a^s \leq c$.

**Proof.** In every residuated lattice we have $a \leq b^s$ iff $a \leq b^r \land b^r$ iff $a \leq b^r$ and $a \leq b^r$ iff $b \leq a^s$ and $b \leq a^s$ iff $b \leq a^s$.

By Lemma 2.14 there is no element between $a^s$ and $a^s$, so since these elements are comparable, every element $b$ is either above their join $a^s$ or below their meet $a^s$.

2.5. **Nuclear images.** Recall that $x^* = x^r \lor x^s$. In the commutative case, $x^* = x^r = x^s = x \rightarrow 1$ and the map $x \mapsto x^{**}$, whose image is $A^{**} = \{a^* : a \in A\}$, plays a prominent role in the study of commutative residuated lattices due to the fact that it is submultiplicative: $x^{**}y^{**} \leq (xy)^{**}$.

The correct generalization in the non-commutative case has a slightly more involved form: we define the map $\gamma$ on $A$ by $\gamma(x) := x^{fr} \land x^{rf}$. We show that $\gamma$ is a closure operator whose image is precisely the set of inverses $A^i = \{a^i : a \in A\} \cup \{a^r : a \in A\}$. Moreover, we prove that $\gamma$ naturally partitions $A$ into blocks, which are linearly ordered, and that $A^i = \gamma[A]$ is a subalgebra of $A$, hence a residuated lattice itself, and it will serve as the skeleton of the decomposition of $A$.

**Remark 2.17.** Note that in the commutative case $\gamma(x) = x^{**} = x^{**}$; however no version of this formula where $^*$ or $^+$ is replaced by $^r$ or $^r$ is correct in the non-commutative case since in general $x^{fr} \land x^{fr} \neq x^{fr}, x^{fr}, x^{fr}, x^{fr}$. Moreover, it follows by Lemma 2.10 that $x \mapsto x^{**}$ is a closure operator, so it is natural to wonder whether it coincides with $\gamma$, but it does not. Indeed, if $b < 1$, $b^{fr} = 1$ and $b^{fr} > 1$, then $b^* = 1$, so $b^{**} = 1$, but as we will show later $b^{fr} = b$, so $\gamma(b) = b^{fr} \land b^{fr} = 1 \land b = b$.

**Lemma 2.18.** If $A$ is a conic idempotent residuated lattice, then:

1. $\gamma$ is a closure operator.
2. $A^i = \{x \in A : x = x^{fr} \land x^{fr}\} = \gamma[A]$.
3. $\gamma(x)^r = x^r$ and $\gamma(x)^r = x^r$, for all $x \in A$.
4. The sets $\gamma^{-1}(a)$, $a \in A^i$, form convex subposets of $A$ with top element $a$.
5. The sets $\gamma^{-1}(a)$, $a \in A^i$, are ordered linearly according to the value of $a$:
   - If $x \in \gamma^{-1}(a)$ and $y \in \gamma^{-1}(b)$, where $a, b$ are distinct elements of $A^i$, then $x \leq y$ iff $a \leq b$.

**Proof.** 1. That $\gamma$ is a closure operator follows from the fact that it is the meet of the two closure operators $x \mapsto x^{fr}$ and $x \mapsto x^{fr}$; see 2.5 for details.
2. We observe that in every residuated lattice we have \( b = a^\ell \), for some \( a \in A \), iff \( b = (b')^\ell \). Indeed, for the non-trivial direction, if \( b = a^\ell = 1/a \), then \( ba \leq 1 \), so \( a \leq b^{-1}b' \); hence \( 1/b' \leq 1/a \); i.e. \( (b')^\ell \leq a^\ell = b \). The reverse inequality, \( b \leq (b')^\ell \), holds always.

Now, we have that \( b \in A^1 \) iff \( b = a^\ell \) or \( b = a^r \), for some \( a \in A \), iff \( b = (b')^r \) or \( b = (b')^\ell \). Since we always have \( b \leq (b')^r \), \( (b')^r \), i.e. \( b \leq (b')^\ell \land (b')^r \), and since \( (b')^\ell \) and \( (b')^r \) are comparable in \( A \), this is equivalent to \( b = (b')^\ell \land (b')^r \).

3. From \( x \leq \gamma(x) \) we obtain \( \gamma(x)^r \leq x^r \). The reverse inequality, \( x^r \leq \gamma(x)^r \), is equivalent to \( \gamma(x)x^r \leq 1 \), which holds since \( (x^r \land x^\ell)x^r \leq (x^r)^r x^r \leq 1 \). The second equality is proved symmetrically.

4. That \( \gamma^{-1}(a) \) is convex follows from the monotonicity of \( \gamma \).

5. The linear ordering of the elements in the sets follows from the fact that the elements in \( A^x \) are conical, by Lemma 2.10.

A nucleus on a residuated lattice \( A \) is a closure operator \( \gamma \) on \( A \) satisfying
\[
\gamma(x) \cdot \gamma(y) \leq \gamma(x \cdot y).
\]

Given a residuated lattice \( A \) and a nucleus \( \gamma \) on \( A \), the nuclear image of \( \gamma \) is the residuated lattice \( A_\gamma = (\gamma[A], \land, \lor, \gamma, \neg, \gamma(1)) \), see [25], where for \( \ast \in \{\lor, \cdot\} \),
\[
x \ast_\gamma y := \gamma(x \ast y).
\]

In the commutative case \( \gamma(x) = x^* = (x \to 1) \to 1 \) is always a nucleus; see [25]. However, the map \( \gamma(x) = x^\ell \land x^r \) is not a nucleus in arbitrary residuated lattices. We will prove that in the conic idempotent case it is. Also, even though both \( x^\ell \) and \( x^r \) are closure operators, neither one of them is a nucleus, thus the importance of considering their meet. We first need the following important technical lemma.

**Lemma 2.19.** If \( x \) is a strictly negative \( \gamma \)-fixed point in a conical idempotent residuated lattice \( A \), then \( x \) is meet irreducible. In particular, for a negative \( \gamma \)-fixed point \( a \in A \), the block \( \gamma^{-1}(a) \) is a sublattice of \( A \).

**Proof.** Toward a contradiction, suppose that \( x = y \land z \) where \( y, z \geq x \). Then \( y \) and \( z \) are incomparable, \( y, z < 1 \), since \( A \) is conic, and \( yz = y \land z = x \). The inverses \( x^\ell, y^\ell, z^\ell, x^r, y^r, z^r \) are positive.

Given that \( x \leq y \) we have \( y^\ell \leq x^\ell \) and \( y^r \leq x^r \). If we had \( y^\ell \geq x^\ell \) and \( y^r \geq x^r \), then we would get \( y^\ell = x^\ell \) and \( y^r = x^r \), which would imply \( \gamma(y) = \gamma(x) \), a contradiction. Given that all of elements of \( A^r \) are comparable, we therefore get that either \( y^\ell < x^\ell \) or \( y^r < x^r \). Likewise we get \( z^\ell < x^\ell \) or \( z^r < x^r \), thus yielding four different cases.

In case \( y^\ell < x^\ell \) and \( z^\ell < x^\ell \), by Lemma 2.12(1) we get \( x^\ell y = x^\ell \) and \( x^\ell z = x^\ell \), so \( x = x^\ell x = x^\ell yz = x^\ell z = x^\ell \), a contradiction. Likewise the case \( y^r < x^r \) and \( z^r < x^r \) leads to a contradiction.

The remaining two cases are symmetric, so we consider only one of them. Assume that \( y^r < x^r \) and \( z^r < x^r \), (so \( x^\ell y = x^\ell \) and \( zx^r = x^r \)) and that we are not already in the above cases, so \( y^r = x^r \) and \( z^r = x^r \). If \( x^r < x^\ell \), then \( z^r < x^r < x^\ell = z^\ell \), contradicting the fact that there is no element strictly between \( z^r \) and \( z^\ell \), as shown in Lemma 2.14. If \( x^r < x^\ell \), then \( y^r < x^r < y^\ell \), contradicting the fact that there is no element strictly between \( y^\ell \) and \( y^r \). For the remaining case of \( x^r = x^\ell \), we have \( x^r y = x^r \) and \( zz^r = x^r \), hence \( x^r = x^r y = zx^r y \). By Lemma 2.14 \( x^r \) commutes with at least one of \( y \) and \( z \). In the first case we get \( zz^r y = zyx^r = xx^r = x \).
and in the second case we get $zx^r y = x^r zy = x^r x = x$. Both cases lead to the contradiction that $x^r = x$. \hfill \qed

A residuated lattice is called \emph{quasi-involutive} if it satisfies $x^{\ell r} \wedge x^{r \ell} = x$. We call a join semilattice $A$ with a greatest element a \emph{prelattice} if $A$ can be made into a lattice by adjoining a new bottom element, i.e., $A^\downarrow := A \cup \{ \bot \}$ is a lattice if the order $\leq$ on $A$ is extended by setting $\bot \leq a$ for all $a \in A$.

**Theorem 2.20.** If $A$ is a conic idempotent residuated lattice, then

1. $\gamma$ is a nucleus ($\gamma(x) = x^{\ell r} \wedge x^{r \ell}$).
2. $A^i$ is the universe of a totally-ordered quasi-involutive subalgebra $A^i$ of $A$.
3. Every block $\gamma^{-1}(a)$ is a prelattice, for all $a \in A^i$. Furthermore
   (a) If $a$ has no lower cover in $A^i$, then $\gamma^{-1}(a)$ is a lattice.
   (b) For negative $a \in A^i$, the block $\gamma^{-1}(a)$ is a Brouwerian lattice and the implication in this Brouwerian lattice is given by $x \Rightarrow y = (x \setminus y) \wedge a = (y/x) \wedge a$.
   (c) If $a \in A^i$ is not central then $\gamma^{-1}(a)$ is trivial.
4. For $x, y \in A$, we have

$$
\begin{cases}
  x \supseteq y = \begin{cases}
    x^r \vee y = \gamma(x)^r \vee y & x \leq y \\
    x^r \wedge y = \gamma(x)^r \wedge y & \gamma(y) < \gamma(x), \text{ or } 1 < \gamma(x) = \gamma(y) \text{ and } x \not\leq y
  
  \end{cases}
\\
  x \Rightarrow a y = \begin{cases}
    x^r \vee y = \gamma(x)^r \vee y & x \leq y \\
    x^r \wedge y = \gamma(x)^r \wedge y & \gamma(y) < \gamma(x), \text{ or } 1 < \gamma(x) = \gamma(y) \text{ and } x \not\leq y
  
  \end{cases}
\\
  y / x = \begin{cases}
    x^r \vee y = \gamma(x)^r \vee y & x \leq y \\
    x^r \wedge y = \gamma(x)^r \wedge y & \gamma(y) < \gamma(x), \text{ or } 1 < \gamma(x) = \gamma(y) \text{ and } x \not\leq y
  
  \end{cases}
\\
  x \Rightarrow a y = \begin{cases}
    x^r \vee y = \gamma(x)^r \vee y & x \leq y \\
    x^r \wedge y = \gamma(x)^r \wedge y & \gamma(y) < \gamma(x), \text{ or } 1 < \gamma(x) = \gamma(y) \text{ and } x \not\leq y
  
  \end{cases}

\end{cases}
$$

\text{Proof. By Lemma 2.18, } \gamma \text{ is a closure operator. To show that it satisfies } \gamma(x) \gamma(y) \leq \gamma(xy), \text{ we consider cases depending on the signs of } x \text{ and } y.

If $1 \leq x, y$, then also $1 \leq \gamma(x), \gamma(y)$. In view of Lemma 2.12(1), we need to verify that $\gamma(x) \vee \gamma(y) \leq \gamma(x \vee y)$, i.e., $\gamma(x), \gamma(y) \leq \gamma(x \vee y)$; this is true by the monotonicity of $\gamma$.

If $x, y \leq 1$, then also $\gamma(x), \gamma(y) \leq \gamma(1) = 1$. In view of Lemma 2.12(1), we need to verify that $\gamma(x) \wedge \gamma(y) \leq \gamma(x \wedge y)$. If $x$ and $y$ are comparable, then this holds because of the monotonicity of $\gamma$. If $x$ and $y$ are incomparable, there cannot be an element of $A^i$ between them. Because all elements in $A^i$ are conical, and because $\gamma(z)$ is defined to be the smallest element of $A^i$ that is above $z$, we have that $\gamma(x) = \gamma(y)$.

By Lemma 2.19, $\gamma(x \wedge y) = \gamma(x) = \gamma(y)$, from which $\gamma(x) \wedge \gamma(y) \leq \gamma(x \wedge y)$ follows.

The two cases, where $x$ and $y$ have opposite sign, are symmetric, so we only consider the case $x \leq 1 \leq y$. We distinguish cases according to the position of $y$ relative to $x^r$.

If $x^r < y$, then $xy = y$ and $\gamma(xy) = \gamma(y)$. So we need to show $\gamma(x) \gamma(y) \leq \gamma(y)$, which follows from the fact that $\gamma(x) \leq \gamma(1) = 1$.

If $y \leq x^r$, then we have $y \leq x^r = x^{\ell r} = x^{r \ell} \setminus 1$, so $x^{\ell r} y \leq 1$. Therefore, we obtain $x^{\ell r} \leq 1/y = y^{r \ell} = 1/y^{\ell r}$, hence $x^{\ell r} y^{r \ell} \leq 1$. We want to show that $\gamma(x) \gamma(y) \leq \gamma(xy)$. By Lemma 2.12(1), $y \leq x^{\ell r}$ implies $xy = x$ and $\gamma(xy) = \gamma(x)$, so we need to show that $\gamma(x) \gamma(y) \leq \gamma(x)$. By Lemma 2.12(1), this is equivalent to $\gamma(x) \gamma(y) \leq 1$; i.e. to $(x^{\ell r} \wedge x^{r \ell})(y^{r \ell} \wedge y^{\ell r}) \leq 1$. The latter follows from the inequalities $(x^{\ell r} \wedge x^{r \ell})(y^{r \ell} \wedge y^{\ell r}) \leq x^{\ell r} y^{r \ell} \leq 1$. 


2. By construction $A_\gamma$ is a subalgebra of $A$ with respect to meet and the division operations. By Lemma 2.14(1), $A^i$ is a chain, hence closed under the lattice operations. It is further closed under multiplication, because of Lemma 2.12(2) and the fact that it is totally ordered. Finally, $\gamma(1) = 1$. That $A^i$ is quasi-involutive follows from the fact that $\gamma$ fixes all of the elements of its image.

3. To see that $\gamma^{-1}(a)$ is a prelattice, note that $\gamma^{-1}(a)$ is closed under $\lor$ since it is a convex subposet of $A$ with top element $a$ by Lemma 2.18(4). Now suppose there exist $x, y \in \gamma^{-1}(a)$ with $x \land y \notin \gamma^{-1}(a)$. Since all inverse elements strictly below $a$ are also strictly below $x$ and $y$ and they are the tops of their $\gamma$-preimage, the element $a$ must have a lower cover $b$ of $a$ in $A^i$ and $x \land y = b$. Since for all pairs of $x, y \in \gamma^{-1}(a)$ with $x \land y \notin \gamma^{-1}(a)$ we have $x \land y = b$, we get that $\gamma^{-1}(a) \cup \{b\}$ is a lattice, whence $\gamma^{-1}(a)$ is a prelattice. Moreover, if $a$ does not have a lower cover in $A^i$, then $\gamma^{-1}(a)$ has to be closed under meets as well, so it is a lattice.

For (b), suppose that $a \in A^i$ is negative; we have $(x \land a)(y \land a) \leq xy \land aa = xy \land a$, for all $x, y \in A$. So, the map $\sigma: x \mapsto x \land a$ is a co-nucleus (see 2.3) with image $\{x \in A : x \leq a\}$, and this co-nuclear image has the structure of an integral, idempotent residuated lattice, i.e., a Brouwerian algebra. By the definition of the operations of the co-nuclear image, the multiplication, which equals the meet, and the join in this Brouwerian algebra are the same as in $A$, while the implication is given by $x \Rightarrow y = (x \land a) \lor a = (y/x) \land a$. Because every strictly negative element of $A^i$ is meet-irreducible by Lemma 2.14(1), the set $\gamma^{-1}(a)$ is closed under $\land$ and hence a filter in $A_\sigma$. As filters in Brouwerian algebras are subalgebras, (b) follows.

For (c), suppose that $a \in A^i$ is not central and let $x \in \gamma^{-1}(a)$. Then $a^t \neq a^r$, and by Lemma 2.18(3) we have $a^t = \gamma(x)^t = x^t$ and $a^r = \gamma(x)^r = x^r$. Thus $x^t \neq x^r$, $x$ is not central, and Lemma 2.14 provides that the only element not commuting with $x$ is $x^t = x^t \lor x^r = a^t \lor a^r = a^*$. Since $x$ and $a$ are not central elements and they both fail to commute with $x^t = a^*$, we have $x = a$ and $\gamma^{-1}(a) = \{a\}$.

4. First note that the two alternatives are equal by Lemma 2.18(3). We will prove only the formula for $x \setminus y$, as the other formula is proved by a symmetric argument.

Let $x \leq y$. Using Corollary 2.13, we get $x(x^r \lor y) = xx^r \lor yy \leq x \lor y \leq y \lor y = y$. To prove that $x^r \lor y$ is the largest element $w$ such that $xw \leq y$, consider $z \leq x^r \lor y$, which is equivalent to $z \leq x^r$ and $z \leq y$ due to the comparability of $x^r$ and $y$; we want to argue that from these it follows that $xz \leq y$. Equivalently, we will establish that $z \leq x^r$ and $xz \leq y$ imply that $z \leq y$. Indeed, from the first assumption we get $xz \leq 1$, so $1 \leq zz$; this implies that $z = 1z \leq xzz = xz$; hence $z \leq y$ follows by transitivity.

Now let $x \leq y$ and assume $\gamma(y) < \gamma(x)$, or $1 < \gamma(x) = \gamma(y)$. First we check that $x(x^r \land y) \leq y$. Since $x(x^r \land y) \leq xx^r \land xy$, it is enough to show that $xx^r \leq y$ or $xy \leq y$. We consider various cases. If $1 \leq x$, then $xy \leq 1y = y$. If $1 \leq x$ (which implies $x^r \leq 1$), then we consider subcases on how $y$ compares with $x^r$. If $y \leq x^r$, then $y \leq x^r \leq 1 \leq x \leq x^t \leq y^t$, so $xy = y$. If $x^t \leq y$, then since $x^t \leq 1 \leq x$, by conservativity of elements with opposite sign we get $xx^t = x^t \leq y$. Next, to prove that $x^r \land y$ is the largest element $w$ such that $xw \leq y$, let $z$ be such that $xz \leq y$. We must show $z \leq x^r \land y$. By the weak conservativity of multiplication, we know that $xz \leq \{x \land z, x, z \land w\}$. Since $x \leq y$, the assumption $xz \leq y$ implies that $xz \notin \{x, x \lor z\}$. Thus by Lemma 2.12(2) there are two possibilities: $xz = z$ and $x \in (z^t, z] \cup [z, z^r)$, or $xz = x \land z$ and $x, z \leq 1$ are incomparable. Suppose that
xz = z and x ∈ ([z]^s, z] ∪ [z, z]^r]. If x ∈ ([z]^r, z], then x ≤ z = xz ≤ y contradicts the assumption that x ≤ y. Thus z ≤ x ≤ z^r, whence z ≤ z^r ≤ x^r. Since z = xz ≤ y by hypothesis, we obtain z ≤ x^r ∧ y as desired. In the remaining subcase, we have xz = x ∧ z and x, z ≤ 1 are incomparable. Lemma 2.18 gives that γ(x) = γ(z) in this case. By hypothesis, we have γ(y) < γ(x) or 1 < γ(x) = γ(y). If γ(y) < γ(x), then by Lemma 2.18 we have y ≤ γ(x) ≤ x, z. But γ(y) < x ∧ z = xz by Lemma 2.19 so this contradicts xz ≤ y. On the other hand, 1 < γ(x) = γ(y) contradicts x ≤ 1.

Finally, suppose that x ≤ y and γ(x) = γ(y) = a ≤ 1. In this event, by 3(b) x ⇒ a y is defined and it is equal to x/\ y ∧ a, so it is in γ^−1(a), since y = 1/\ y ∧ a ≤ x/\ y ∧ a ≤ a. Also, x(x ⇒ a y) = x(x/\ y ∧ a) ≤ x(x/\ y) ≤ y. Now suppose that z ∈ A is such that z ≤ x ⇒ a y. We will show that xz ≤ y. Lemma 2.18 implies that γ(z) ≤ a, so a ≤ γ(z). If a < γ(z), then x ≤ a < z. By weak conservativity, xz is x ∧ z = x or x ∨ z = z. If xz = x ∧ z = x, then xz ≤ y as x ≤ y by assumption. If xz = z = x ∨ z, then y ≤ a < xz and xz ≤ y. In the only remaining case, γ(z) = a = γ(x) = γ(y). Then, since ⇒ a is the residual of ∧ in γ^−1(a), we obtain xz ≤ y iff z ≤ x ⇒ a y.

In light of the importance of the nucleus γ, we provide one more useful characterization of multiplication in conic idempotent residuated lattices.

**Corollary 2.21.** Let A be a conic idempotent residuated lattice, let x, y ∈ A, and set s = γ(x) and t = γ(y). Then:

\[
xy = \begin{cases} 
  x \land y & s, t \leq 1 \\
  x \lor y & s, t > 1 \\
  x & st = s \text{ and } s, t \text{ have opposite sign} \\
  y & st = t \text{ and } s, t \text{ have opposite sign}
\end{cases}
\]

Proof. The result can essentially be read off from Lemma 2.12. If s, t ≤ 1 then x, y ≤ 1 and xy = x \land y, and if s, t > 1 then 1 < x, y and xy = x \lor y. For the rest, s, t have opposite sign so by Lemma 2.13(3) st = s or st = t and by Lemma 2.18 x ≤ s < 1 < y ≤ t or y ≤ t < 1 < x ≤ s. Suppose first that x ≤ s < 1 < y ≤ t. If st = s, then since multiplication is order preserving we have xy ≤ st = s < 1, so xy = x by Lemma 2.4. If st > t, then by Lemma 2.12 we have t^r < s, whence y^r = t^r < x ≤ s < 1 < y ≤ t and xy = y. A similar argument applies if y ≤ t < 1 < x ≤ s. This establishes the first equality. The second equality follows from the fact that if s ≠ t have the same sign, then st = s \land t iff xy = x \land y and st = s ∨ t iff xy = x \lor y.

\[\square\]

2.6. **Decomposition systems.** We now show the inverse of the above decomposition. A **decomposition system** is a structure \((S, \{A_s : s ∈ S\})\), where S is an idempotent residuated chain and, for every s ∈ S, A_s is a prelattice with top element s such that:

1. If s has no lower cover in S, then A_s is a lattice.
2. For negative s ∈ S, the block A_s is a Brouwerian lattice.
3. If s is not central, then A_s is trivial.
Theorem 2.20 shows that if $A$ is a conic idempotent residuated lattice, then $(A^1, \{\gamma^{-1}(s) : s \in A^1\})$ is a decomposition system and furthermore $A^i$ is quasi-involutive. For this reason, we will only have use for such quasi-involutive decomposition systems, even though the constructions below work for all decomposition systems.

Given such a decomposition system $D = (S, \{A_s : s \in S\})$, we consider the ordinal sum $A_D := \bigoplus_{s \in S} A_s$. The order $\leq$ on $A_D$ is the order of the usual ordinal sum of posets. For $s \in S$, we use $s^r$ and $s^l$ as usual. We define, for $x \in A_s$ and $y \in A_t$,

$$x \cdot y = \begin{cases} x \wedge y & s = t \leq 1 \\ x \vee y & s = t > 1 \\ x & st = s \text{ and } s \neq t \\ y & st = t \text{ and } s \neq t \\ \end{cases}$$

$$x \setminus y = \begin{cases} s^r \vee y & \text{if } x \leq y \\ s^r \wedge y & \text{if } t < s, \text{ or } 1 < s = t \text{ and } x \nleq y \\ x \Rightarrow_s y & \text{if } s = t \leq 1 \text{ and } x \nleq y. \\ \end{cases}$$

Note that since $S$ is an idempotent residuated chain, all the results we have proved about conic idempotent residuated lattices apply to $S$ and we will make use of them in the following proof.

**Theorem 2.22.** Given a decomposition system $D$, the algebra $A_D$ is a conic idempotent residuated lattice.

**Proof.** It is clear that $A_D$ is a lattice because if $s \in S$ lacks a lower cover, then $A_s$ is a lattice. Moreover, 1 is comparable to every element of $A_D$.

It is obvious that 1 is a neutral element for $\cdot$, and we claim that $\cdot$ is associative. Let $x, y, z \in A_D$, and suppose that $x \in A_s$, $y \in A_t$, and $z \in A_u$. If $x, y, z \leq 1$ or $x, y, z > 1$, then $x(yz) = (xy)z$ follows from the associativity of $\wedge$ and $\vee$; in particular, this holds if $s = t = u$. If $s, t, u$ are pairwise distinct, it follows from the associativity and conservativity of multiplication in $S$ that $x(yz) = (xy)z$. In the only remaining case, $\{s, t, u\}$ has exactly two distinct elements. Recall that $xy = x$ iff $st = s$, and likewise for the other elements.

If $s = t \neq u$, then in case $su = s$, we have $x(yz) = xy = (xy)z$ and in case $su = u$, we have $(xy)z = z = xz = x(yz)$. A symmetric argument applies in the case where $s \neq t = u$. Finally, if $s = u \neq t$, then in case $st = ts = t$, we have $x(yz) = xy = y = yz = (xy)z$; in case $st = ts = s$, we have $x(yz) = xz = (xy)z$; in the remaining two cases $s$ is not central, $A_u$ is trivial, and hence $x = z = u$. In case $st = s$ and $ts = t$, we have $(xy)z = xz = x = xy = x(yz)$ and in case that $st = t$ and $ts = s$, we have $x(yz) = xz = z = yz = (xy)z$.

We now prove three claims from which the result follows.

**Claim 1.** For all $x, z \in A_D$, $x \leq z$ iff $1 \leq x \setminus z$.

**Proof of Claim 1.** Suppose that $x \in A_s$ and $z \in A_u$. If $x \leq z$, then $x \setminus z = s^r \vee z$ by definition. In the event that $z < 1$, we have that $s \leq u$ and $z \leq u \leq 1$ implies that $1 \leq s^r$. Since if $z \neq 1$ we have $1 \leq z$ by conicity, it follows that $1 \leq s^r \vee z = x \setminus z$. Conversely, if $1 \leq x \setminus z$ then we consider the three cases according to the definition of $\setminus$. Note that it cannot be that $s = u \leq 1$ and $x \nleq z$ since then $1 \leq x \setminus z = x \Rightarrow_s z$, which would yield $x \leq z$ since $\Rightarrow_s$ is residuated. On the other hand, if $1 \leq x \setminus z$
then we have $1 \leq s^r \land z$, whence $1 \leq s^r$ and $1 \leq z$. From $1 \leq s^r$ we infer $x \leq s \leq 1$, so $x \leq z$. In the only remaining possibility we have that $x \leq z$ by assumption, so the claim follows.

**Claim 2.** For all $x, y, z \in A_D$, $xy \leq z$ implies $y \leq x \setminus z$.

**Proof of Claim 2.** Let $x, y, z \in A_D$ with $xy \leq z$, and suppose that $x \in A_s$, $y \in A_t$, and $z \in A_u$. We consider four cases, each of which corresponds to one of the clauses determining the value of $xy$ in the definition of $\cdot$.

For Case 1, suppose $s = t \leq 1$. Here $x, y \leq 1$ and $xy = x \land y$. Since $s \leq 1$ we have $1 \leq s^r$. If $u = s = t$, then $z \leq u \leq 1 \leq s^r$, and $x \land y \leq z$ if and only if $y \leq x \Rightarrow_s z$ (in the event that $x \gtrless z$) or $y \leq s^r \lor z = s^r$ (in the event that $x \leq z$). Thus $y \leq x \setminus z$ in the subcase that $u = s = t$. On the other hand, if $u \neq s = t$, then we have $s = t < u$ as a consequence of $x \land y \leq z$. Then $x \setminus z = s^r \lor z \geq 1 \geq y$.

For Case 2, suppose $s = t > 1$. In this case, $x, y > 1$ and $xy = x \lor y$. We have that $xy \leq z$ gives $x, y \leq z$, so $y \leq z \leq s^r \lor z = x \setminus z$.

For Case 3, suppose $s \neq t$ and $st = s$. In this case, $xy = x \leq z$, so by Claim 1 we have $1 \leq x \setminus z = s^r \lor z$. Since $s^r$ is comparable to all elements in $A_D$, we have that $s^r \lor z \in \{s^r, z\}$ and hence $1 \leq s^r$ or $1 \leq z$. Note that $st \leq s$ iff $t \leq s'$.\(\Vert\) by Lemma 2.11(2), so $y \leq t \leq s^r$. As $x \leq z$ we have $x \setminus z = s^r \lor z$, so $y \leq s^r \leq s^r \lor z = x \setminus z$.

For Case 4, suppose $s \neq t$ and $st = t$. Here $xy = y \leq z$. Note that $st \leq t$ if $s < t$ and $st = t$ iff $t \leq s^r$. Thus $y \leq t \leq s^r$ and $y \leq z$ give that $t \leq s^r \land z$, $s^r \lor z$. Thus $y \leq x \setminus z$ if $x \setminus z \in \{s^r \land z, s^r \lor z\}$. In the only remaining subcase to consider, we have $s = u \leq 1$ and $x \nleq z$. Then $y \leq z$ and $u \neq t$ implies $t < s = u$, so $y \leq x \Rightarrow_s z = x \setminus z$ follows because $x \Rightarrow_s z \in A_s$.

**Claim 3.** For all $x, y, z \in A_D$, $y \leq x \setminus z$ implies $xy \leq z$.

**Proof of Claim 3.** Let $x, y, z \in A_D$ with $y \leq x \setminus z$, and suppose that $x \in A_s$, $y \in A_t$, and $z \in A_u$. We consider three cases, each of which corresponds to one of the clauses determining the value of $x \setminus z$ in the definition of $\setminus$.

For Case 1, suppose $x \leq z$. Here $y \leq x \setminus z = s^r \lor z \in \{s^r, z\}$. If $y \leq z$, then $x \lor y \leq z$ and $xy \leq z$ is immediate. If $y \nleq z$, then $s^r \lor z = s^r$, $y \leq s^r$ and hence $t \leq s^r$. Then by Lemma 2.12 we have $st = s \land t$. If $s \neq t$, then by definition $xy$ is the smaller of $x$ and $y$, so certainly $xy \leq x \leq z$. If $s = t$, then $xy \leq z$. Then $x \nleq z$, possibly $xy = x \lor y$. Note that $xy = x \lor y$ holds only if $s = t > 1$, and then $s^r \leq 1$ and $1 < x \leq z$ contradicts $s^r \lor z = s^r$. Thus this subcase does not occur.

For Case 2, suppose $u < s$, or $1 < s = u$ and $x \nleq z$. In this case, we have $y \leq x \setminus z = s^r \land z$, whence $y \leq s^r$ and $y \leq z$. Observe that $y \leq s^r$ implies $t \leq s$, so $st = s \land t$. If $s \neq t$, then $xy$ is the minimum of $x$ and $y$, so $xy \leq t \leq z$. If $s = t$, then $s = t \leq s^r$ implies $s = t \leq 1$, so $xy = x \lor y \leq y \leq z$.

For Case 3, suppose $s = u \leq 1$ and $x \nleq z$. In this event, $y \leq x \setminus z = x \Rightarrow_s z \in A_s$. It follows that $t \leq s$. If $t < s$, then $xy = x \land y = y \in A_t$, so $xy \leq z$ as $s < t$. If $t = s = u$, then $y \leq x \Rightarrow_s z$ implies $xy = x \land y \leq z$ by the fact that $\Rightarrow_s$ is the residual of $\land$ in $A_s$.

Combining Claims 2 and 3, it follows that $\setminus$ is a residual to $\cdot$. The proof for $/ \cdot$ is similar, and this yields the result. $\square$

We say that a decomposition system $(\mathcal{S}, \{A_s : s \in S\})$ is quasi-involutive if $\mathcal{S}$ is quasi-involutive.
Remark 2.23. Theorem [2.20] shows that every conic idempotent residuated lattice is the ordinal sum of its decomposition system. Since these decompositions systems are quasi-involutive, considering only quasi-involutive decomposition systems is enough. A conic idempotent residuated lattice may be obtained/represented as the ordinal sum of another decomposition system, but this representation is not optimal unless the decomposition system is quasi-involutive. In particular, given an idempotent residuated chain (which can be viewed as a decomposition system with singleton blocks), we can obtain a decomposition of it in terms of its inverse elements, which is more basic, refined, and informative. This study therefore has applications even on idempotent residuated chains.

Remark 2.24. The decomposition systems for commutative conic idempotent residuated lattices are simpler. The two inverses collapse to one, \( x^* = x^t = x^r \), so quasi-involutivity becomes equivalent to involutivity, \( x^{**} = x \). The skeletons are therefore odd Sugihara monoids.

We may also view decomposition systems as (multisorted) partial algebraic structures. In a decomposition system \((S, \{A_s : s \in S\})\):

1. For every negative \( s \in S \), \( A_s \) is a total algebra in the language of Brouwerian algebras.
2. For every \( s \in S \), \( A_s \) is a topped prelattice: a total algebra with respect to join and with respect to a constant operation that produces the top element as an element of \( S \), and a partial operation with respect to meet.
3. \( S \) is a total algebra in the language of residuated lattices, and a partial algebra with respect to an operation \( s \mapsto s^t \), where if \( A_s \) is a proper prelattice (i.e., a prelattice that is not a lattice), then \( s^t \) is a lower cover of \( s \) in \( S \).

A decomposition system \((S_A, \{A_s : s \in S_A\})\) is said to be a subsystem of another decomposition system \((S_B, \{B_s : s \in S_B\})\) if it is a sub-partial algebra in the above sense: The subsystem is closed under all operations when they are defined in the bigger system. Unfolding this definition, \((S_A, \{A_s : s \in S_A\})\) is a subsystem of \((S_B, \{B_s : s \in S_B\})\) if:

1. \( S_A \) is subalgebra of \( S_B \).
2. For every \( s \in S_A \), \( A_s \) is a Brouwerian subalgebra of \( B_s \).
3. For every \( s \in S_A \), \( A_s \) is a topped subprelattice of \( B_s \) (in particular if \( B_s \) is a lattice, then \( A_s \) is a lattice).
4. If \( A_s \) is a proper prelattice (hence also \( B_s \) is a proper prelattice) then \( S_A \) contains the lower cover \( s^t \) of \( s \) in \( S_B \).

**Lemma 2.25.** Let \((S_A, \{A_s : s \in S_A\})\) be the decomposition system corresponding to the algebra \( A \) and \((S_B, \{B_s : s \in S_B\})\) be the decomposition system corresponding to the algebra \( B \). Then \((S_A, \{A_s : s \in S_A\})\) is a subsystem of \((S_B, \{B_s : s \in S_B\})\) iff \( A \) is a subalgebra of \( B \).

**Proof.** Assume that \( A \) and \( B \) are conic idempotent residuated lattices and consider their associated decomposition systems. If \((S_A, \{A_s : s \in S_A\})\) is a subsystem of \((S_B, \{B_s : s \in S_B\})\) then by using the conicity of the elements of the skeleton, it follows that \( A \) is a subalgebra of \( B \) in terms of the lattice operations and multiplication; for closure under meet we use closure under the partial operation \( s \mapsto s^t \). To show that it is also closed under division, first note that \( A \) is closed under Brouwerian implication of \( A_s \) for each \( s \in S_A \). Also, if \( x, y \in A \), then \( x \in A_s \).
for some $s = \gamma(x)$. Since $s$ is the top element of $A_s$, we have $s \in A$. Also, since $s \in S_A$ and $S_A$ is a subalgebra of $S_B$, we also have $s^l, s^r \in A$. Finally, since $s^l, s^r$ are conical we have $s^r \land y, s^r \lor y, s^l \land y, s^l \lor y \in A$. Therefore, $A$ is a subalgebra of $B$. Checking that if $A$ is a subalgebra of $B$ then $(S_A, \{A_s : s \in S_A\})$ is a subsystem of $(S_B, \{B_s : s \in S_B\})$ is equally routine. □

The correspondence between conic idempotent residuated lattices and quasi-involutive decomposition systems can be lifted to a categorical equivalence. Since only the correspondence of embeddings is needed for our application to the amalgamation property (see Section 5), we content ourselves with Lemma 2.25 and we do not prove the categorical equivalence in detail.

2.7. Comparison with the literature. Here we compare our decomposition to the literature. [8] provides a very similar decomposition for conic idempotent residuated lattices, but adopts a more semigroup-theoretic approach and neglects many order-theoretic aspects emphasized in the present study. In particular, [8] does not mention the nucleus $\gamma$, or the fact that the idempotent residuated chains that serve as skeletons are quasi-involutive. The block of an element $a$ (what we call $\gamma^{-1}(a)$) is defined as

$$C_a^A = \begin{cases} \{b \in A^+ : \forall c \in A^-, bc = c \iff ac = c \text{ and } cb = c \iff ca = a\} & \text{if } a \in A^+ \\ \{b \in A^- : \forall c \in A^+, bc = c \iff ac = c \text{ and } cb = c \iff ca = a\} & \text{if } a \in A^- \end{cases}$$

The elements of the chain $A^i$ are obtained as the joins of each $C_a^A$. This definition of $A^i$ is more technical than the one given here, and is hard to comprehend order-theoretically because it obfuscates the natural way in which $A^i$ is a residuated chain (i.e., as a nuclear image of $A$).

Furthermore, in [8] the analysis of the structure is done using the Green’s relation $D$, which is the join of Green’s relations $L = \{(a, b) : Aa = Ab\}$ and $R = \{(a, b) : aA = bA\}$. It is not explained in [8], but $(a, b) \in D$ iff $(ab = a \text{ and } ba = b)$ or $(ab = b \text{ and } ba = a)$, that is iff $\{a, b\}$ forms a left-zero or a right-zero semigroup. The quotient $A/D$ is considered and shown to be a semilattice and some results are phrased in terms of the induced order on this set, thus making it natural for semigroup theorists, but not transparent for order-algebraists. In particular, it is harder to work with equivalence classes than it is to work with appropriate representatives.

More importantly, as mentioned in the introduction, the literature on (non-commutative) idempotent residuated lattices largely ignores the importance of the unary maps $x \mapsto x^l$ and $x \mapsto x^r$, which were introduced and used in [23]; they are used only for the description of subdirectly irreducible algebras, but not in the decomposition. The inversion operations are important because multiplication and the divisions can be defined in terms of the order and the inverses, giving a better handle on the subalgebra generation process. Together with the pictorial, order-theoretic depiction of the decomposition we have given, this insight into subalgebra generation (which we describe in detail in Lemma 4.16) is crucial for our work on amalgamation in Section 5.
3. CONGRUENCE GENERATION IN SEMICONIC IDEMPOTENT RESIDUATED LATTICES

In this section, we prove that the variety of semiconic idempotent residuated lattices has the congruence extension property. This is a necessary component in our proof that the strong amalgamation property lifts from appropriately-chosen classes of conic idempotent residuated to the varieties they generate (Theorem 5.14). It is also of independent interest: While commutative residuated lattices have the congruence extension property, non-commutative residuated lattices typically do not, and it is surprising that the semiconic idempotent residuated lattices enjoy the congruence extension property even in the absence of commutativity. Moreover, the congruence extension property implies the corresponding logic has a local deduction theorem (see Section 6).

Recall, an algebra $B$ has the congruence extension property (or CEP) if for any subalgebra $A$ of $B$, if $\Theta$ is a congruence of $A$, then there exists a congruence $\Psi$ of $B$ such that $\Psi \cap A^2 = \Theta$. A variety is said to have the CEP if each of its members does. To prove that semiconic idempotent residuated lattices have the CEP, we provide a detailed analysis of congruence generation for these algebras. We also use these results to characterize finitely subdirectly irreducible semiconic idempotent residuated lattices, which will be used in lifting the strong amalgamation property from generating classes to varieties in Section 5.6.

3.1. Congruences in idempotent semiconic residuated lattices and the congruence extension property. Recall from [25] that congruences in residuated lattices are in bijective correspondence with congruence filters. Given a residuated lattice $A$, a subset $F$ is called a congruence filter if it is a lattice filter, a submonoid of $A$, and it is closed under conjugation: if $y \in F$ and $x \in A$, then $x \backslash yx, xy/x \in F$. The congruence filter associated to a congruence $\theta$ is $F_\theta = \uparrow[1]_\theta$, and the congruence associated to a filter $F$ is given by: $a \theta_F b$ iff $a \backslash b, b \backslash a \in F$.

The fact that the conjugation closure conditions for congruence filters involve terms which contain parameters often creates complications. These terms are not needed in the commutative case, where congruence filters are filter submonoids. We will show that even though conjugates cannot be omitted in the non-commutative semiconic case, their behavior is relatively tame. In particular, we can obtain closure terms that do not involve parameters: conjugates can be replaced by the term $s(y) = y \wedge y^{\ell\ell} \wedge y^{rr}$, as we show in Lemma 3.1(2).

**Lemma 3.1.** Let $A$ be a semonic idempotent residuated lattice.

1. $A$ satisfies the identities $y \wedge y^{\ell\ell} \wedge y^{rr} \leq xy/x, x \backslash yx$.
2. A subset of $A$ is a congruence filter iff it is a filter, a submonoid, and it is closed under the term $s(y) = y \wedge y^{\ell\ell} \wedge y^{rr}$.
3. If $y \in A^l$, then $y^{\ell\ell} \wedge y^{rr} \leq y$.

**Proof.** 1. It suffices to show that the identity holds for conic idempotent residuated lattices. We will prove that $y \wedge y^{\ell\ell} \wedge y^{rr} \leq xy/x$, or equivalently

$$(y \wedge y^{\ell\ell} \wedge y^{rr})x \leq xy,$$

as the proof of the other inequality is similar.

For $y \neq x^* = x^{\ell} \lor x^{r}$ we have $yx = xy$, by Lemma 2.14(2). So, $(y \wedge y^{\ell\ell} \wedge y^{rr})x \leq yx = xy$. For $y = x^* = x^{\ell} \lor x^{r}$, we distinguish two cases.
If \( x^r < x^\ell \), then \( y = x^\ell \). By Corollary 2.13(1), we have \( x^\ell x = x \land x^\ell \) and since \( x^\ell \) is conic we get \( x x^e \in \{ x, x^e \} \), so \((y \land y^\ell \land y^rr) x \leq xy = x^\ell x = x \land x^\ell \leq xx^e = xy \).

If \( x^\ell < x^r \), then \( y = x^r \). By Corollary 2.13(1,2), we have \( x^r x = x \land x^r \) and \( xx^e = x \land x^r \). Also, from \( x \leq x^r \) we get \( x x^r \leq x^e \). So, using Corollary 2.13(3), we have

\[
(y \land y^\ell \land y^rr)x \leq y^r x = x^r x \leq x^e x \leq xx^e = xy.
\]

2. This follows from (1) and the fact that the terms \( y, y^\ell, y^rr \) all have to be in every congruence filter containing \( y \). Indeed, let \( y \) be in a congruence filter \( F \). We will show that \( y^rr \in F \), and the proof for \( y^\ell \) is similar. For \( z = y \land 1 \) we have \( z \in F \) so \( z^r \land z^r \in F \); also \( z \leq 1 \leq z^r \) and \( \gamma(z) < \gamma(z^r) \) for \( z \neq 1 \). Therefore, By Corollary 2.13 and Theorem 2.20(4), we have \( z \land z^r = z^r \land z = z^r \land z \leq z^rr \leq y^rr \). Therefore, \( y^rr \in F \).

3. Recall that \( y^\ell \) and \( y^r \) are comparable. If \( y^\ell = y^r \), then since \( y \in A^\ell \) we have \( y = y^rr \) or \( y = y^\ell \), so \( y^r = y^\ell = y \). If \( y^r < y^\ell \), then \( y^\ell \not\leq y^r \), so \( yy^\ell \not\leq 1 \), hence \( y \not\leq y^\ell \), thus \( y^\ell < y \). Likewise, if \( y^\ell < y^r \), then \( y^rr < y \). In all three cases, we get \( y^\ell \land y^rr \leq y \).

In the following, we use the abbreviation \( y^{ur} := y^{aru} \), where \( u \in \{ r, \ell \} \) appears \( n \)-many times. We also define the terms

\[
s_1(y) = y \land y^\ell \land y^rr,
\]

\[
s_2(y) = y \land y^{\ell\ell} \land y^{\ell rrr} \land y^{rr\ell} \land y^{rrrr}
\]

and for all \( n \)

\[
s_n(y) := y \land \bigwedge \{ y^{c_1c_2...c_n} : c_1, c_2, ... , c_n \in \{ \ell, r \} \}.\]

We further write \( s(y) \) for \( s_1(y) \), and write \( s^n \) for the \( n \)-fold composition of \( s \) with itself. Also, we define \( t_n(y) := 1 \land s_n(y) \) and \( t(y) := 1 \land s(y) \), and use \( t^n \) for the \( n \)-fold composition of \( t \) with itself. Note that for \( y \leq 1 \) we have \( y^\ell, y^rr \leq 1 \) and by induction \( s_n(y) \leq 1 \), hence for \( y \leq 1 \) we have \( t_n(y) = s_n(y) \). Finally, if \( A \) is a residuated lattice and \( Y \subseteq A \), we write \( Y \land 1 := \{ y \land 1 : y \in Y \} \).

**Lemma 3.2.** Let \( A \) be a semiconic idempotent residuated lattice and \( Y \subseteq A \).

1. \( A \) satisfies the identities \( s^n(y) = s_n(y) \) and \( t^n(y) = t_n(y) \).
2. The congruence filter of \( A \) generated by \( Y \) is given by \( \langle Y \rangle = \uparrow \{ t_n(y) : n \in \mathbb{N}, y \in Y \cup \{ 1 \} \} \).
3. \( Y = \{ y \in Y^F : y \in Y \land 1 \} \), where \( Y^F \) is the \( \land \)-subalgebra generated by \( Y \cup \{ 1 \} \).
4. If \( Y \) is a closed under meets, then \( \langle Y \rangle = \{ y \land s_n(y) : n \in \mathbb{N}, y \in Y \land 1 \} \).
5. If \( Y \) is a congruence filter and \( a \in A \), then \( \langle Y \cup \{ a \} \rangle = \{ y \land s_n(a) : n \in \mathbb{N}, y \in Y \} \).

**Proof.**

1. It suffices to prove the identities for the conic case. Note that the elements \( 1, y^\ell, y^rr \) are conical, so \( t(y) = 1 \land y \land y^\ell \land y^rr = \min \{ 1, y, y^\ell, y^rr \} \). Since \( y^\ell \) and \( y^rr \) are order-preserving, we have \( (1 \land y \land y^\ell \land y^rr)^\ell = 1 \land y^\ell \land y^rr \).


and likewise for \( r^r \). Therefore we obtain
\[
t(t(y)) = 1 \land y \land y^r \land y^r = (1 \land y \land y^r \land y^r)^{r^r} \land (1 \land y \land y^r \land y^r)^{r^r} \\
= 1 \land y \land y^r \land y^r \land y^r \land y^r \land y^{r^r} \land y^{r^r} \\
= 1 \land y \land y^{r^r} \land y^{r^r} \land y^{r^r} \land y^{r^r} \\
= t_2(y),
\]
where we used Lemma 3.1(3) on \( y^r \) and \( y^{r^r} \). Proceeding inductively, we obtain that \( t^n(y) = t_n(y) \). The same argument applies to the terms \( s_n \).

2. By Theorem 3.47(3c) of \([23]\), the congruence filter generated by \( Y \) is the upward closure of products of iterated conjugates of elements from \( Y \land 1 \). By Lemma 3.1(1), each left or right conjugate \((xy/x) \land 1,(x\langle yx \rangle \land 1 \mid y \in Y \land 1 \) by an element \( x \in A \) is above the element \( t(y) = 1 \land y \land y^r \land y^{r^r} \). Iterated conjugates are above iterated applications of the term \( t \). Products of iterated conjugates are above products of iterated applications of \( t \), and these products are equal to meets, as \( t \) takes values in the negative cone. Therefore, the congruence filter generated by \( Y \subseteq A \) is contained in \( \uparrow\{t_n(y_1) \land \cdots \land t_n(y_k) : k, n_1, \ldots, n_k \in \mathbb{N}, y_1, \ldots, y_k \in Y \land 1 \} \).

Conversely, if \( z \in Y \), then \( y = z \land 1 \in Y \land 1 \) is in the congruence filter of \( A \) generated by \( Y \). By Lemma 3.1(2) any congruence filter is closed under \( r^r \) and \( r^r \), so \( y^{r^2} \) and \( y^{r^2} \) are in \( \langle Y \rangle = \langle Y \land 1 \rangle \), for all \( n \in \mathbb{N} \). So, \( t_n(y) \in \langle Y \rangle \) and thus \( \uparrow\{t_n(y_1) \land \cdots \land t_n(y_k) : k, n_1, \ldots, n_k \in \mathbb{N}, y_1, \ldots, y_k \in Y \land 1 \} \subseteq \langle Y \rangle \).

3. Since \( \langle Y \rangle \) is a congruence filter, it contains \( \uparrow\{t_n(y) : n \in \mathbb{N}, y \in Y^F \} \).

Conversely, if \( y_1, \ldots, y_k \in Y \land 1 \) and \( k, n_1, \ldots, n_k \in \mathbb{N} \), then for \( y := y_1 \land \cdots \land y_k \) we have that \( y \in Y^F \), so \( y \in \langle Y \rangle \), and \( y \leq y_1, \ldots, y_k \); so for \( n = \max\{n_1, \ldots, n_k\} \) we have \( t_n(y) \leq t_n(y_1) \land \cdots \land t_n(y_k) \leq t_n(y_1) \land \cdots \land t_n(y_k) \) and hence \( \langle Y \rangle \subseteq \mathbb{N}, y \in Y^F \} \).

4. Since \( Y \land 1 \) is a closed under meet, it is equal to \( Y^F \), and the result follows from (3).

5. This follows from (2) and the fact that if \( Y \) is a congruence filter, then \( t_{n_1}(y_1) \land \cdots \land t_{n_k}(y_k) \in Y \), for all \( k, n_1, \ldots, n_k \in \mathbb{N}, y_1, \ldots, y_k \in Y \); the fact that we can replace \( t_n(a) \) by \( s_n(a) \) is due to the fact that \( y \) ranges over all elements of \( Y \), hence also over 1.

We will not need it, but it can be easily verified that \( y^{r^r} = y \land y^r \) and that \( y^{r^r} \) is equal to \( y^{r^r} \) if \( 1 \leq y \), to \( y \) if \( y \leq 1 \leq y^r \), and to \( y^r \) if \( y \leq 1 \leq y^r \). Also, for all values of \( x \) different from \( y^r \) and \( y^r \), we have \( xy = yx \), so \( y \leq x \). This constitutes another way of establishing Lemma 3.1(1).

The generation of congruence filters with parameter-free terms, given in Lemma 3.2, ensures that the corresponding logical system has a local deduction theorem and that variety has the congruence extension property (see Section 5). This will be useful in the proof of Theorem 5.14.

**Corollary 3.3.** The variety of semiconic idempotent residuated lattices has the congruence extension property.

Note that the generation of congruence filters cannot be simplified by imposing a bound on the value of \( n \) in the terms \( t_n \) in Lemma 3.2(2), as can be seen by the non-commutative chains in Figure 2. This means that we cannot get a deduction theorem (only a local deduction theorem) for the corresponding logical system and that the variety does not have equationally definable principal congruences.
Since the characterization of congruence generation applies to all subvarieties of the variety of idempotent semiconic residuated lattices, we obtain the congruence extension property for every such subvariety. In particular we get the following interesting result:

**Corollary 3.4.** The variety of semilinear idempotent residuated lattices has the congruence extension property.

### 3.2. Finitely subdirectly irreducibles.

On the way to establishing the main result of this paper, Theorem 5.14, we will characterize finitely subdirectly semiconic idempotent residuated lattices. For this, we will need some more control of conjugation (which is captured by the term \(\lambda_x(a) := x \setminus ax \land 1\) and \(\rho_x(a) := xa/x \lor 1\)).

First, we will show that if \(x\) is orthogonal to \(y\) (in the sense that \(x \lor y = 1\)), then every conjugate of \(x\) is orthogonal to \(y\); in other words conjugation stays within each given dimension (orthogonal direction). Recall that iterated conjugates are obtained by repeated applications of the terms \(\lambda_x(a) := x \setminus ax \land 1\) and \(\rho_x(a) := xa/x \lor 1\).

**Lemma 3.5.** Semiconic idempotent residuated lattices satisfy the implications:

1. \(x \lor y = 1 \Rightarrow \lambda_z(x) \lor y = 1\).
2. \(x \lor y = 1 \Rightarrow x \lor \rho_w(y) = 1\).
3. \(x \lor y = 1 \Rightarrow \gamma_1(x) \lor \gamma_2(y) = 1\), for all iterated conjugates \(\gamma_1\) and \(\gamma_2\).
4. \(x \lor y = 1 \Rightarrow t(x) \lor y = 1\).
5. \(x \lor y = 1 \Rightarrow x \lor t(y) = 1\).
6. \(x \lor y = 1 \Rightarrow s_n(x) \lor s_m(y) = 1\) for all \(n, m \in \mathbb{N}\).

**Proof.** 1. Since every semiconic is a subdirect product of conic, and quasiequations are preserved under subalgebras and direct products, it is enough to prove the quasiequations for conic. The quasiequations hold trivially if \(x = 1\) or \(y = 1\), so in the following we assume that \(x \neq 1 \neq y\). If \(x \lor y = 1\) in a conic idempotent residuated lattice \(A\), then \(x\) and \(y\) are in the block of 1. Therefore, \(x' = x^r = 1\), so \(t(x) = x\); \(x\) is central and \(x \leq \lambda_z(x)\); hence \(\lambda_z(x) \lor y = 1\) and \(t(x) \lor y = 1\). This proves 1 and 4.

The proofs of 2 and 5 are similar to that of 1 and 4, and 3 is obtained by repeated applications of the terms \(\lambda_x(a) := x \setminus ax \land 1\) and \(\rho_x(a) := xa/x \lor 1\). The proofs of 1 and 4, and 3 is obtained by repeated applications of the terms \(\lambda_x(a) := x \setminus ax \land 1\) and \(\rho_x(a) := xa/x \lor 1\).

Property 6 follows by iterated application of 4 and 5 together with Lemma 3.2(2) (and by noting that \(t_n(x) = s_n(x)\) when \(x\) is negative). □

We recall the previously-mentioned equational axiomatizations for semiconic (idempotent) residuated lattices, using results from [22].

**Lemma 3.6.**

1. Semiconic residuated lattices are axiomatized by the identities \(\gamma_1(x \land 1) \lor \gamma_2(x^r \land 1) = 1\), where \(\gamma_1, \gamma_2\) range over iterated conjugates.
2. Commutative semiconic residuated lattices are axiomatized by \(xy = yx\) and \((x \land 1) \lor (x^r \land 1) = 1\), as well as by \(xy = yx\), 1 \(\leq x \lor (x \land 1)\) and \(1 \land (x \lor y) = (1 \land x) \lor (1 \land y)\).
3. (Commutative) semiconic idempotent residuated lattices are axiomatized by adding the identity \(x^2 = x\) to the axiomatization given in 1 (respectively, 2).

**Proof.** 1. Conic residuated lattices are axiomatized by the positive universal formula \((1 \leq x\) or \(x \leq 1)\). Following the general procedure of [22] we rewrite the
formula in the equivalent way \((1 \leq x \text{ or } 1 \leq x')\) and finally \((1 = x \land 1 \text{ or } 1 = x' \land 1)\). By the results in [22], semiconic residuated lattices are axiomatized by the equations: \(1 = \gamma_1(x \land 1) \lor \gamma_2(x' \land 1)\), where \(\gamma_1\) and \(\gamma_2\) range over all iterated conjugates.

2. In the commutative case conjugates are redundant, so we obtain the simpler axiomatization \(1 = (x \land 1) \lor (x' \land 1)\). The equivalence of the second axiomatization is straightforward.

3. Obvious. □

We have developed enough understanding of congruence generation to be able to characterize the finitely subdirectly irreducible semiconic idempotent residuated lattices. This characterization will be needed in the proof of Theorem 5.14.

**Lemma 3.7.** A semiconic idempotent residuated lattice is finitely subdirectly irreducible iff \(1 = \gamma(t, f)\). Also, each one of these assumptions imply that the algebra is conic.

**Proof.** Assume that \(A\) is a finitely subdirectly irreducible semiconic idempotent residuated lattice. It follows from the proof of Theorem 9.73(2) of [24] that \(A\) satisfies the conicity positive universal formula \(x \leq 1 \lor 1 \leq x\) if it satisfies the equations of semiconicity of Lemma 3.6(1); hence \(A\) is conic. Now assume further that \(a \lor b = 1\), for some strictly negative \(a, b \in A\). Then \(a\) and \(b\) must be incomparable and they must be in the same block \((\gamma(a) = \gamma(b) = 1)\). In particular, \(a^\ell = a^r = b^r = b^r = 1\) and so \(t_n(a) = a\) and \(t_n(b) = b\), for all \(n\). Hence, by Lemma 3.2(2), the congruence filters generated by \(a\) and \(b\), respectively, are \(\uparrow a\) and \(\uparrow b\); so they intersect at \(\uparrow 1\), since \(a \lor b = 1\). Since \(A\) is finitely subdirectly irreducible, we get \(\uparrow a = \uparrow 1\) or \(\uparrow b = \uparrow 1\); hence \(a = 1\) or \(b = 1\).

Conversely, assume that \(1\) is join irreducible in a semiconic idempotent residuated lattice \(A\). By Lemma 3.6(1), \(A\) satisfies the equation \(1 = (x \land 1) \lor (x' \land 1)\). By join irreducibility, we get \(x \land 1 = 1\) or \(x' \land 1 = 1\), so \(x \leq 1\) or \(1 \leq x\) and \(A\) is conic. Now let \(F \cap G = \uparrow 1\) for some non-trivial congruence filters \(F\) and \(G\). If \(F\) and \(G\) each contains a strictly negative inverse element, say \(i_f\) and \(i_g\), then since \(i_f\) and \(i_g\) must be comparable, we get that \(F\) and \(G\) both contain \(i_f \lor i_g = \max\{i_f, i_g\}\), a contradiction. If \(G\) contains a strictly negative inverse but \(F\) does not, then \(F^{-}\) is fully contained in the block of \(1\) and \(G\) contains the block of \(1\); a contradiction. So, neither \(F\) nor \(G\) can contain a strictly negative inverse and thus \(F^{-}\) and \(G^{-}\) are fully contained in the block of \(1\). Also, \(F^{-}\) and \(G^{-}\) cannot be contained in each other, so there exist strictly negative elements \(a \in F \setminus G\) and \(b \in G \setminus F\). Since \(\uparrow a \subseteq F\) and \(\uparrow b \subseteq G\), we have \(\uparrow a \cap \uparrow b = \uparrow 1\). Therefore, \(a \lor b = 1\), a contradiction. □

**Corollary 3.8.** A semiconic idempotent residuated lattice is finitely subdirectly irreducible iff \(1\) is join irreducible. Also, each one of these assumptions imply that the algebra is conic.

4. **The Structure of (Quasi-Involutive) Idempotent Residuated Chains**

We have shown that each conic idempotent residuated lattice can be decomposed as an ordinal sum based on a quasi-involutive idempotent residuated chain skeleton, where the blocks are arbitrary prelattices, lattices, or Brouwerian algebras. Since the blocks are fairly well understood, we now describe the structure of quasi-involutive idempotent residuated chains. More generally, we study the
structure of all idempotent residuated chains by using three tools: An equivalent formulation of idempotent residuated chains in terms of their \(\{\land, \lor, 1, r, \ell\}\)-reducts (see Section 4.1), a pictorial presentation of these called flow diagrams (see Section 4.2), and another equivalent formulation in terms of certain enhancements of their monoidal preorders (see Sections 4.3 and 4.4).

With an eye on amalgamation, in Section 4.6 we use the tools developed so far to identify a crucial property, rigidity, that is necessary for amalgamating quasi-involution idempotent residuated chains. Rigidity states that the quasi-involution skeleton is *-involutive. Then, in Section 4.7, we study one-generated rigid quasi-involution idempotent residuated chains, and in Section 4.8 we show that all rigid quasi-involution idempotent residuated chains are nested sums of one-generated ones.

As previously mentioned, the decomposition of Section 2 applies to idempotent residuated chains since they are examples of conic idempotent residuated lattices. Combined with the decomposition results given before, the results of this section provide a fine-grained analysis of the structure of idempotent residuated chains. Since residuated chains are the subject of a considerable literature, we expect the results of this section to therefore be of broad interest outside of its applications to the conic case.

Lemma 2.12(2) and Theorem 2.20(4) specialize to to chains as follows:

**Corollary 4.1.** In every idempotent residuated chain we have

\[
xy = \begin{cases}
x & y \in (x^r, x] \text{ or } y \in [x, x^r] \\
y & x \in (y^\ell, y] \text{ or } x \in [y, y^\ell]
\end{cases}
\]

\[
\begin{aligned}
x \land y & \leq 1 \iff x \leq y^\ell \iff y \leq x^r \\
x \lor y & < 1 \iff y^\ell < x \iff x^r < y
\end{aligned}
\]

\[
x \setminus y = \begin{cases}
x^r \lor y & x \leq y \\
x^r \land y & y < x
\end{cases}
\]

\[
y / x = \begin{cases}
x^\ell \lor y & x \leq y \\
x^\ell \land y & y < x
\end{cases}
\]

### 4.1. The reducts of idempotent residuated chains.

In view of the above corollary, subalgebra generation is performed by closure under the unary operations \(r\) and \(\ell\) and closure under 1. Also, every idempotent residuated chain is definitionally-equivalent to a structure over the language \(\{\land, \lor, r, \ell, 1\}\). We will characterize these \(\{\land, \lor, r, \ell, 1\}\)-reducts of idempotent residuated chains. This characterization will be useful in connecting idempotent residuated chains to enhanced monoidal preorders (see Sections 4.3 and 4.4).

By Lemma 2.14 and Corollary 2.16 if \(A\) is an idempotent residuated chain, then the reduct \((A, \land, \lor, r, \ell, 1)\) is a chain such that \((\ell, r)\) forms a Galois connection, \(1^\ell = 1^r = 1\), and for all \(x\) there is no element between \(x^\ell\) and \(x^r\). We call such algebras idempotent Galois connections. Note that \((A, \land, \lor, \ell, r, 1)\) is an idempotent Galois connection iff it satisfies the following definition in terms of positive universal sentences:

1. \((A, \land, \lor)\) is a lattice,
2. \(1^\ell = 1^r = 1\),
3. \(x \leq x^r, x \leq x^\ell, (x \lor y)^r = x^r \land x^\ell, (x \lor y)^\ell = x^r \land x^\ell, x^{r\ell} = x^\ell\) and \(x^{\ell r} = x^r\) (Galois connection on a lattice; see Lemma 7.26 of [13]),
4. \(x \leq y\) or \(y \leq x\) (totally ordered),
5. \(y \leq x^r \land x^\ell\) or \(x^r \lor x^\ell \leq y\) (there is no element between \(x^\ell\) and \(x^r\)).

Note that condition 5 can be written as \((y \leq x^\ast\text{ or } x^\ast \leq y)\), i.e. \((x^\ast, x^\ast)\) forms a splitting pair.
Conversely, given $C = (C, \wedge, \vee, \ell, \cdot, r, 1)$, the algebra $R(C) = (C, \wedge, \vee, \cdot, \ell, /, 1)$ is defined by
\[
xy = \begin{cases} 
  x \wedge y & x \leq y' \ (\Leftrightarrow y \leq x^r) \\
  x \vee y & y' < x \ (\Leftrightarrow x \leq y') 
\end{cases}
\quad
x'y = \begin{cases} 
  x^r \vee y & x \leq y \\
  x^r \wedge y & y < x 
\end{cases}
\quad
y/x = \begin{cases} 
  x^r \vee y & x \leq y \\
  x^r \wedge y & y < x 
\end{cases}
\]

Lemma 4.2. If $C = (C, \wedge, \vee, \ell, r, 1)$ is an idempotent Galois connection, then $R(C)$ is an idempotent residuated chain.

Proof. First we show that multiplication is order-preserving; assume that $x \leq z$. If $z \leq y'$, then also $x \leq y'$. Therefore, $xy = x \wedge y \leq z \wedge y = zy$. If $y' < z$, then $xy \leq x \vee y \leq z \vee y = zy$. Since $C$ is a chain, this implies that multiplication distributes over join and meet.

Note that for $x \leq 1 = 1^e = 1^r$, we have $x1 = x \wedge 1 = x$ and $1x = 1 \wedge x = x$; for $1^e = 1^r = 1 < x$, we have $x1 = x \vee 1 = x$ and $1x = 1 \vee x = x$. Therefore, 1 is the multiplicative unit. Clearly, multiplication is conservative, hence idempotent.

To show that it is associative, we consider $x, y, z$ and show that $(xy)z = x(yz)$ by distinguishing cases.

If $x \leq y'$ and $y' < z$, then since there are no elements between $y'$ and $y''$, we get $x \leq z$. So, $(xy)z = (x \wedge y)z = xz \wedge yz = xz \wedge (y \vee z) = xz$, because $xz \leq z = y \vee z$, and $x(yz) = x(y \wedge z) = xy \wedge xz = (x \wedge y) \wedge xz = xz$, because $x \wedge y \leq x = xx \leq xz$.

If $y' < x$ and $z \leq y''$, then since there are no elements between $y'$ and $y''$, we get $z \leq x$. So, $(xy)z = (x \vee y)z = xz \vee yz = xz \vee (y \wedge z) = xz$, because $y \wedge z \leq z = zz \leq xz$, and $x(yz) = x(y \wedge z) = xy \wedge xz = (x \vee y) \wedge xz = xz$, because $xz \leq xx = x \leq x \vee y$.

If $x \leq y'$ and $z \leq y''$, then $y \leq x^r$ and $y \leq z$, so $(xy)z = (x \wedge y)z = xz \wedge yz = xz \wedge y \wedge z$ and $x(yz) = x(y \wedge z) = xy \wedge xz = x \wedge y \wedge xz$. If $x \leq y'$, then $xz = x \wedge z$ and the two sides are equal to $x \wedge y \wedge z$. If $z \leq x^r$, then $xz = x \vee z$, and the two sides are equal to $x \vee y \vee z$. If $x \leq z$, then $z \leq x^r$ and $xz = x \wedge z$, so one side is equal to $y \wedge z$ and the other to $y \wedge x$. Moreover, we obtain $y \leq z \leq x < y$ and $y \leq x^r < z$, so both sides equal to $y$.

Finally, we will show $xy \leq z$ iff $y \leq x \vee z$, for all $x, y, z \in A$, by considering cases. The equivalence to $x \leq y/z$ is analogous.

Assume $z < x$. Then $xy \leq z$ implies $xy \neq x$, so $y = xy \leq z < x$. Note that if we had $x^r < y$ then $xy = x \vee y = x$, a contradiction; so $y \leq x^r$.

Therefore, $xy \leq z$ iff $z \leq x \vee y = x$. On the other hand, $y \leq x \vee z$ iff $y \leq x^r \wedge z$ iff $y \leq x^r$ and $y \leq z$. So, $xy = x \wedge y = y$, since $y \leq z < x$.

Therefore, $y \leq x \vee z$ iff $x \geq y = y \leq z$. Consequently, $xy \leq z$ iff $y \leq x \vee z$.

Assume $x \leq z$ and $xy = x$. Then $xy \leq z$ is true. By way of contradiction, if $y \leq x \vee z$ were false we would have $x^r \vee z < y$ so $x^r \vee z < y \vee z$; hence $x \leq z < y$ and so $x = xy = x \vee y = y > x$, a contradiction.

Assume $x \leq z$, $xy = y$ and $y \leq z$. Then $xy \leq z$ is true. Also $y \leq x \leq x^r \vee z$, so also $y \leq x \vee z$ is true.

Assume $x \leq z$, $xy = y$ and $z < y$. So $z < xy$; hence $xy \leq z$ is false. Also note that we cannot have $y \leq x^r$, as that would imply the contradiction $y = xy =$.
The condition that for all \( x \) there is no element between \( x^\ell \) and \( x^r \) does not follow from the other conditions, as a 5-element example shows, and it is precisely what guarantees associativity of multiplication.

**Corollary 4.3.** Idempotent residuated chains are definitionally equivalent to idempotent Galois connections.

### 4.2. Flow diagrams

We now provide a visual way to move between the action of the inverse operations \( \ell \), \( r \), and the ordering of the chain. This will eventually lead to the connection with the enhanced monoidal preorders.

It follows from Corollary 2.15 that if \( a, b \) are non-commuting elements of an idempotent residuated chain, then \( a \) and \( b \) have different sign, \( a^\ast = b \), \( b^\ast = a \) and \( \{a, b\} \) forms either a left-zero or a right-zero semigroup. If \( \{a, b\} \) forms a right-zero semigroup and \( a \) is positive (and \( b \) negative), we write \( a R b \). If \( \{a, b\} \) forms a left-zero semigroup and \( a \) is positive (and \( b \) negative), we write \( a L b \). We write \( a C b \) if \( a \) is positive (and \( b \) negative) and \( a^\ell = a^r = b \) (i.e., if \( ab = ba \)). Note that these relations are not symmetric because they encode the signs of the elements. We follow the naming convention of [22] of above and below the identity element 1, for \( a \) and \( b \). Also, recall that \( x \prec y \) means that \( x \) is a lower cover of \( y \).

**Lemma 4.4.** Let \( a, b \) be elements of an idempotent residuated chain \( A \) such that \( a \) is positive and \( b \) is negative.

1. The following are equivalent:
   1. \( a R b \).
   2. \( a^\ell \prec a^r = b \).
   3. \( b^r \prec b^\ell = a \).
   4. \( a^\ell \prec a \) and \( b = a^r \).
   5. \( b^r \prec b \) and \( a = b^\ell \).
   6. \( a = a^r \) is not central and \( b = a^\ell \).
   7. \( b = b^r \) is not central and \( a = b^\ell \).

2. The following are equivalent:
   1. \( a L b \).
   2. \( a^r \prec a^\ell = b \).
   3. \( b^\ell \prec b^r = a \).
   4. \( a^\ell \prec a \) and \( b = a^r \).
   5. \( b^r \prec b \) and \( a = b^\ell \).
   6. \( a = a^r \) is not central and \( b = a^\ell \).
   7. \( b = b^r \) is not central and \( a = b^\ell \).

**Proof.** 1(a-e). Given that \( a \) is positive and \( b \) is negative, \( a R b \) is equivalent to the conjunction of \( ab = b \) and \( ba = a \). By Lemma 2.12(1), \( ab = b \) is equivalent to \( a \leq b^\ell \), which is also equivalent to \( ab \leq 1 \) and \( b \leq a^r \). Also, by Lemma 2.12(1), \( ba = a \) is equivalent to \( b^r < a \), which is also equivalent to \( a \nleq b^r \), \( ba \leq 1 \), \( b \nleq a^\ell \), and \( a^\ell < b \).

Therefore, \( a R b \) is equivalent to \( b^r < a \leq b^\ell \), and since \( b^r \) and \( b^\ell \) form a covering pair (by Lemma 2.14), this is further equivalent to \( b^r \prec a = b^\ell \).

Also, \( a R b \) is equivalent to \( a^\ell < b \leq a^r \), and since \( a^\ell \) and \( a^r \) form a covering pair (by Lemma 2.14), this is further equivalent to \( a^\ell \prec b = a^r \).

All these establish that \( a \), \( b \), and \( c \) are equivalent. Clearly, \( d \) follows from \( b \) and \( c \). Conversely, if \( d \) holds, then in particular \( b^r < a \) and \( b \leq a^r \), which is equivalent to \( a \), by the calculations above. The equivalence of \( e \) is established in similar way. 2(a-e) are established in a similar way.

Now to show that 1(f) is equivalent to 1(a), we note that \( f \) follows from \( b \) and \( c \). Conversely, if \( f \) holds, then \( a \) is not central, so one of 1(a) and 2(a)
hold. It cannot be the case that 2(a) holds because that would imply 2(b), which contradicts 1(f); so 1(a) holds. Likewise we get the equivalence of 1(g) to the rest of 1, as well as the equivalence of 2(f) and 2(g) to the rest of 2. □

The following result is an immediate consequence of Lemma 4.4.

**Corollary 4.5.** Let $A$ be an idempotent residuated chain.

1. If $a$ is a positive non-central element of $A$, then exactly one of the following situations happen.
   - (a) $a^{rr} \prec a^{rf} = aR a = a^* \succ a^r = a^\ell$.
   - (b) $a^{\ell r} \prec a^{rf} = aL a = a^* \succ a^\ell = a^r$.

2. If $b$ is a negative non-central element of $A$, then exactly one of the following situations happen.
   - (a) $b^r = b^* \prec b^r = bL b = b^\ell \succ b^r$.
   - (b) $b^\ell = b^* \prec b^r = b^r L b = b^\ell \succ b^r$.

3. If $x$ is a central element of $A$, then $x^* = x^r = x^\ell = x^\ell$.

Figure 4 demonstrates the two situations and provides an explanation for the notation used.

![Figure 4. The flow diagrams for non-central elements](image)

**4.3. From idempotent residuated chains to enhanced monoidal preorders.**

In this section and the next, we are finally ready to introduce the main tool for amalgamating idempotent residuated chains, enhanced monoidal preorders. These are definitionally equivalent to both idempotent Galois connections and to idempotent residuated chains, and will be key in the arguments in the sequel.

Given an idempotent residuated chain $A$, the *natural order*, also considered in [9], is defined by: $x \leq_n y$ iff $xy = yx = x$.

The *monoidal preorder*, also considered in [30], is defined by: $x \sqsubseteq y$ iff $xy = yx = x$.

We use the convention that Hasse diagrams for preordered sets are similar to ones for ordered sets with the only difference being that mutually-comparable elements are placed on the same level and are connected by horizontal line segments. For a preorder $\sqsubseteq$ we write $x \sqsubseteq y$ if $x \sqsubseteq y$ and $y \sqsubseteq x$; this is a stronger demand than simply asking that $x \subseteq y$ and $y \not\subseteq x$. Observe that distinct $x, y$ are mutually-comparable in $\sqsubseteq$ iff $x L y$ or $y L x$. Likewise, distinct $x, y$ are mutually-incomparable iff $x R y$ or $y R x$.

**Lemma 4.6.** The following hold in idempotent residuated chains.
1. The relation $\leq_n$ is an order, and the relation $\sqsubseteq$ is a preorder.
2. $x \leq_n y$ iff $x \sqsubseteq y$, for all $x, y$.
3. $xy = x$ iff $x \sqsubseteq y$. Also, $xy = y$ iff $y \not\sqsubseteq x$.
4. $x \sqsubseteq y$ iff $y \neq x = xy = yx$.

Proof. For 1, observe that if $xy = x$ and $yz = y$, then $xz = xyz = xy = x$. Thus $\sqsubseteq$ is transitive. Also, if $yx = x$ and $zy = y$, then $zx = zyx = yx = x$, so also $\leq_n$ is transitive. Since $xx = x$, $\sqsubseteq$ is also reflexive, hence it is a preorder. Likewise, $\leq_n$ is reflexive. Finally, if $x \leq_n y$ and $y \leq_n x$, then $x = xy = y$, so $\leq_n$ is an order.

Note that $x \sqsubseteq y$ iff ($x \subseteq y$ and $y \not\sqsubseteq x$) iff ($xy = x$ and $yx = x \neq y$) (due to conservativity) iff $x <_n y$. This gives 2; 3 follows by definition.

Finally, $x \sqsubseteq y$ iff ($x \subseteq y$ and $y \not\sqsubseteq x$) iff ($xy = x$ and $yx = x \neq y$) iff $y \neq x = xy = yx$. This gives 4. \qed

Lemma 4.6 shows that the monoidal preorder completely encodes the multiplication operation and contains more information than the natural order. Still, there are different idempotent residuated structures on the same set that have the same monoidal preorder, as can be seen from Figure 5.

![Figure 5](image-url)

**Figure 5.** Two algebras (sides) with the same monoidal order (middle).

However, as we show in Corollary 4.13 if we further add the information of which elements are positive and which are negative (and impose a maximality and a minimality condition), then we can fully recover the whole residuated lattice structure. We will work toward this direction now, by establishing some useful properties of the monoidal preorder.

**Lemma 4.7.** Let $x, y$ be elements of an idempotent residuated chain.

1. $x \sqsubseteq y$ or $y \sqsubseteq x$ or $x = y$ or $xy \neq yx$.
2. If $x, y \leq 1$, then $x \sqsubseteq y$ iff $x < y$. If $x, y \geq 1$, then $x \sqsubseteq y$ iff $x > y$. 

3. If $x$ and $y$ do not commute, then they are on the same layer: For all $z$ we have $z \sqsubseteq x$ iff $z \sqsubseteq y$, and $x \sqsubseteq z$ iff $y \sqsubseteq z$.

Proof. 1. If $x \not \sqsubseteq y$, $y \not \sqsubseteq x$, $x \neq y$ and $xy = yx$, then by Lemma 3.6(4), we get $x \neq xy = yx \neq y$, which contradicts conservativity.

2. If $x, y \leq 1$, then by Lemma 2.12(1) $xy = x \land y = yx$, so $x \sqsubseteq y$ iff $xy = x$ and $yx \neq y$ iff $x \land y = x$ and $x \lor y \neq y$ iff $x < y$. If $x, y \geq 1$, then by Lemma 2.12(1) $xy = x \lor y = yx$, so $x \sqsubseteq y$ iff $xy = x$ and $yx \neq y$ iff $x \lor y = x$ and $x \lor y \neq y$ iff $y < x$.

3. If $a$ and $b$ do not commute then they have opposite signs, say $a$ is the positive and $b$ is the negative, and either $a \sqsubseteq b$ or $a \sqsupseteq b$. We will show that for all $c \in A$ we have $c \sqsubseteq b$ iff $c \sqsubseteq a$, i.e. that $cb = bc = c \neq b$ iff $ca = ac = c \neq a$. Assume first that $a \sqsupseteq b$ so $ab = b$ and $ba = a$. If $cb = bc = c \neq b$, then $c \neq a$ (because $c$ commutes with $b$ and $a$ does not) so $c$ commutes with $a$ (because $b$ is the unique element that does not commute with $a$) and $cb = cba = ca$, hence $ac = ca = cb = c$. Conversely, if $ca = ac = c \neq a$, then $c \neq b$, so $c$ commutes with $b$ and $bc = bac = ac$, hence $b = cb = ca = c$. The proof for the case where $a \sqsubseteq b$ is similar.

We will show that for all $c \in A$ we have $b \sqsubseteq c$ iff $a \sqsubseteq c$, i.e. that $cb = bc = b \neq c$ iff $ca = ac = a \neq c$. Assume first that $a \sqsubseteq b$ so $ab = a$ and $ba = b$. If $cb = bc = b \neq c$, then $c \neq a$ and $ac = abc = ab = a$. Also, $c$ commutes with $a$ (since $c \neq b$), hence $ac = ca = a$. The converse and also the proof for the case where $a \sqsupseteq b$ are similar. □

Lemma 4.8. Let $A$ be an idempotent residuated chain.

1. For $b \in A^-$, $b^*$ is the smallest element of $A^+$ such that $b \sqsubseteq b^*$.
2. For $a \in A^+$, $a^*$ is the largest element of $A^-$ such that $a^* \sqsubseteq a$.
3. $\sqsubseteq$ is layered: If two distinct elements are not related by $\sqsubseteq$ or $\sqsupseteq$, then they have different signs and they are in the same layer (their $\sqsubseteq$-upsets and $\sqsupseteq$-downsets coincide).

Proof. To verify the first two conditions, we refer to Corollary 4.5 and Figure 4.

1. Assume that $b$ is strictly negative. If $a \sqsubseteq b$ or $a \sqsupseteq b$, for some (necessarily positive) $a$, then $b^*$ is positive and $b^* \prec a$, so $b^*$ is the largest element of $(A^+, \leq)$ such that $b^* < a$ and, by Lemma 3.7(2), $b^*$ is the smallest element of $(A^+, \sqsubseteq)$ such that $a \sqsubseteq b^*$. By Lemma 3.7(3), $b^*$ is the smallest element of $(A^+, \sqsupseteq)$ such that $b \sqsubseteq b^*$. Now, if $b$ is central, then $b^* = b^* = b^*$ is a positive element, $bb^* = b^*b$ and $b^*b = b^*b = b^* \land b = b$; also $b^* \neq b$. Therefore, $b^* \prec b^*$ which is equivalent to $b \sqsubseteq b^*$ by Lemma 4.6(2). To show that $b^*$ is the smallest element of $(A^+, \sqsubseteq)$ with this property, or equivalently that it is the largest element of $(A^+, \leq)$ with this property, let $b \sqsubseteq c$ for some $c \in A^+$, so $cb = bc = b \neq c$. By Lemma 2.12(3) we have $c \leq b^*$.

2. Assume that $a$ is strictly positive. If $a \sqsubseteq b$ or $a \sqsupseteq b$, for some (necessarily negative) $b$, then $a^*$ is negative and $a^* \prec b$, so $a^*$ is the largest element of $(A^-, \leq)$ such that $a^* < b$ and by Lemma 3.7(2) the largest element of $(A^-, \sqsubseteq)$ such that $a^* \sqsubseteq b$. Since $a$ and $b$ do not commute, by Lemma 3.7(3) their $\sqsubseteq$-downsets are equal, so $a^*$ is the largest element of $(A^-, \sqsupseteq)$ such that $a^* \sqsupseteq a$. Now, if $a$ is central so $a^* = a^* = a^*$ is a negative element, $aa^* = a^*a$ and by Lemma 2.13(2) $a^*a = a^*a = a^* \land a = a^* = a^*$; also $a^* \neq a$. Therefore, $a^* \prec a$ which is equivalent to $a^* \sqsubseteq a$ by Lemma 4.6(2). To show that $a^*$ is the largest element of $(A^-, \sqsubseteq)$ with this property, or equivalently (by Lemma 4.7(2)) that it is the largest element of
\((A^-, \leq)\) with this property, let \(c \sqsubseteq a\) for some \(c \in A^-\), so \(ca = ac = c \neq a\). By Lemma 2.12(3) we have \(c \leq a^* = a^*\).

3. To show that \(\sqsubseteq\) is layered, we note that if two distinct elements are not comparable by \(\sqsubseteq\) nor \(\sqsupseteq\), then by Lemma 4.7(1) they cannot not commute, so they have different signs and by Lemma 4.7(3) they are on the same layer. It is clear that \(1^* = 1\). □

Given an idempotent residuated chain \(A\), we associate to it the structure \((A, \sqsubseteq, A^+, A^-, 1, \star)\). Note that by Lemma 4.7(2) the positive cone appears inverted in the monoidal preorder.

We now characterize these structures abstractly; conditions 1 and 2 below are extracted from [9], where they are stipulated for the the natural order, but we formulate them in terms of the the richer monoidal preorder of [30].

\((P, \sqsubseteq, P^+, P^-, 1, \star)\) is an enhanced monoidal preorder, if \((P, \sqsubseteq)\) is a pre-ordered set with sole maximum element \(1\) (\(x \sqsubseteq 1\), for all \(x \neq 1\)), \(P^+\) and \(P^-\) are totally-ordered subsets of \(P\) (i.e., the restriction of \(\sqsubseteq\) to each of \(P^+, P^-\) antisymmetric and total) such that \(P^+ \cup P^- = P\) and \(P^+ \cap P^- = \{1\}\) and \(\star\) is a unary operation on \(P\) such that \(1 \star = 1\) and for all other elements

1. For \(b \in P^-\), \(b^\star\) is the smallest element of \(P^+\) such that \(b \sqsubseteq b^\star\)
2. For \(a \in P^+, a^\star\) is the largest element of \(P^-\) such that \(a^\star \sqsubseteq a\).
3. The preordered is layered: If two distinct elements are not related by \(\sqsubseteq\) nor \(\sqsupseteq\), then they have different signs and their \(\sqsubseteq\)-upsets and downsets coincide.

Lemma 4.9. If \(A\) is an idempotent residuated chain, then \((A, \sqsubseteq, A^+, A^-, 1, \star)\) is an enhanced monoidal preorder.

Proof. By Lemma 4.6 \(\sqsubseteq\) is a preorder and for all \(x \neq 1\), we have \(x1 = x\) and \(1x \neq 1\), so \(x \sqsubseteq 1\). Also, by Lemma 4.7(2), \(A^+\) and \(A^-\) are chains of \(\sqsubseteq\) that intersect at \(\{1\}\) and union to \(A\). The remaining three conditions follow from Lemma 4.8. □

4.4. From enhanced monoidal preorders to idempotent residuated chains.

We now prove that the newly introduced enhanced monoidal preorders are definitionally equivalent to idempotent residuated chains.

First, we extend a previous definition to an arbitrary preorder \(\sqsubseteq\). If the \(\sqsubseteq\)-upsets and \(\sqsupseteq\)-downsets of two elements coincide, we say that they are in the same layer, we write \(x \equiv \star\) for this equivalence relation and define a layer to be an equivalence class of \(\equiv^\star\). (Note that this is exactly how we defined the notion of being in the same layer in Lemma 4.7(3).) We will denote by \(x\) the layer of \(x\).

Note that the layers form a chain with top element \(1\), under the ordering: for all \(x, y\), \(x \equiv \star \leq x \equiv \star\) iff they are in the same layer or \(x \sqsubseteq y\). Clearly, this definition is independent of the representatives (as elements of the same layer have the same \(\sqsubseteq\)-upsets and \(\sqsupseteq\)-downsets), reflexivity and antisymmetry follow from the properties of \(\sqsubseteq\), and antisymmetry follows from Lemma 3.10(2), below.

Lemma 4.10. Let \((P, \sqsubseteq, P^+, P^-, 1, \star)\) be an enhanced monoidal preorder.

1. There are at most two elements in the same layer. In this case, they have different sign.
2. No set of pairwise mutually-comparable elements has more than two elements. In this case, these elements are in the same layer.
3. No set of pairwise mutually-incomparable elements has more than two elements. In this case, they are in the same layer.
4. Each layer contains at most two elements, which are thus either mutually comparable or mutually incomparable.

5. If the layer of an element is a singleton, then the element is comparable with any element of different sign.

Proof. 1. First note that only 1 is in the same layer as 1. Also, note that if \( x, y \in P^- \) and \( x \equiv y \), then in particular \( \{ z \in P^- \mid z \sqsubseteq x \} = \{ z \in P^- \mid z \sqsubseteq y \} \), so since \( P^- \) is a chain we get \( x = y \). Likewise, \( x, y \in P^+ \) and \( x \equiv y \), then \( x = y \). Therefore, if \( x \equiv y \) for distinct \( x, y \), then they have different sign. Also, it is impossible to have three distinct elements in the same layer, as two of them would have the same sign and thus they will be equal.

2. If \( x, y \in P^- \), then by antisymmetry they cannot be mutually comparable unless they are equal; likewise for \( P^+ \). So, elements that are mutually comparable have to have different sign. Thus, no three distinct elements are pairwise mutually comparable. Now let \( x, y \) be distinct mutually comparable and let \( z \sqsubseteq x \). Then by transitivity \( z \sqsubseteq y \). If we also had \( y \sqsubseteq z \), then we would have \( y \sqsubseteq z \), a contradiction. Therefore, \( x \) and \( y \) have the same \( \sqsubseteq \)-downsets. Likewise, they have the same upsets.

Property 3 follows from Property 3 of the definition of enhanced monoidal preorders. Property 4 follows from the others.

5. If two elements are not comparable, then in particular they are not related by \( \sqsubseteq \) nor \( \sqsupseteq \), so by condition 3 in the definition of enhanced monoidal preorders, they are in the same layer. So, if the layer of \( x \) is a singleton and \( y \) is incomparable to \( x \), then \( y = x \), a contradiction. So, \( y \) has to be comparable to \( x \).

We will draw the Hasse diagrams for enhanced monoidal preorders on two columns, following the convention of putting the elements of the chain \( P^+ \) on the left column and the elements of the chain \( P^- \) on the right column. We place 1 on the left.

Figure 6 shows how the two chains of Figure 5 can be distinguished via their enhanced monoidal preorders.

Remark 4.11. Note that in the second algebra/preorder in Figure 6 we have \( b_3^\ast = b_3^\circ = b_5^\circ = a_5 \). This can be seen from Figure 1/2/3 and also by the fact that \( a_5 \) is the \( \leq \)-biggest, i.e. \( \sqsubseteq \)-smallest, positive element such that \( b_3 \sqsubseteq a_5 \). In other words, \( a_5 \) is the \( \sqsubseteq \)-smallest positive element in a layer above \( b_3 \). But it is not in the smallest layer above \( b_3 \), which is \( \{ b_4 \} \), as that does not contain any positive elements. Therefore, it is delicate to argue about the position of \( b^\ast \), for some negative \( b \), as for some negative \( b \) it may not be in the layer above \( b \). The correct way to argue is to consider elements of \( P^+ \) closest to \( b \) from above, and this is how the arguments were structured in the proof of Lemma 1.8.

Figure 7 shows diagrammatically how to move seamlessly between enhanced monoidal preorders and flow diagrams.

Conversely, given an enhanced monoidal preorder \( (P, \sqsubseteq, P^+, P^-, 1, *) \), we define the ordered algebra \( A \) with underlying set \( A = P \), with order given by \( x \leq y \) iff \( (x, y \in P^- \) and \( x \sqsubseteq y \) or \( (x, y \in P^+ \) and \( y \sqsubseteq x \) or \( (x \in P^- \) and \( y \in P^+ \)).

The inverses are given by: \( x^\circ = x^\ast = x^\ast * \), if \( x \) is a \( \sqsubseteq \)-conical element (\( \sqsubseteq \)-comparable to every element) and, for \( a \in P^+ \) and \( b \in P^- \),

1. \( a^\circ = b, a^\ast = a^\ast, b^\ast = b^\ast, b^\circ = a \), if \( a, b \) are mutually comparable and

2. \( a^\circ = a^\ast, a^\ast = b \), \( b^\circ = a, b^\ast = b^\ast \), if \( a, b \) are incomparable.
Observe that $\ell$ and $r$ are well-defined by Lemma 4.10. Multiplication and the divisions are given by:

$$xy = \begin{cases} x \wedge y & x \leq y^\ell \\ x \vee y & y^\ell < x \end{cases}$$

$$x \backslash y = \begin{cases} x^r \vee y & x \leq y \\ x^r \wedge y & y < x \end{cases}$$

$$y/x = \begin{cases} x^r \vee y & x \leq y \\ x^r \wedge y & y < x \end{cases}$$

**Lemma 4.12.** The algebra associated to an enhanced monoidal preorder is an idempotent residuated chain.
Therefore, in all cases

Note that the inverse \( b' \in \{b, b'\} \) of a negative element \( b \) is a positive element.

Also, it is clear from the definition that there is no element \( \leq \)-between \( x' \) and \( x'' \). Finally, since \( 1 \) is \( \leq \)-conical and both positive and negative, the definitions yield \( 1'' = 1' = 1 \). Therefore, \( A' := (A, \wedge, \vee, \cdot, \ast, \top, \bot) \) is a idempotent Galois connection.

By Lemma 4.2, the expansion \( R(A') \) of \( A' \) is an idempotent residuated lattice. Since the multiplication and divisions of \( A \) are defined in the same way as \( R(A') \), we get that \( A = R(A') \).

Corollary 4.13. The defined correspondences between idempotent residuated chains and enhanced monoidal preorders are inverses of each other.

Proof. Let \( A \) be the algebra associated to the enhanced monoidal preorder \( (P, \sqsubseteq, P^+, P^-, 1, \ast) \). Since \( P^- \) and \( P^+ \) are sub-chains of \( P \) and since every element of \( P^- \) is less than every element of \( P^+ \), we get that \( (A, \sqsubseteq) \) is a chain.

First we show that \( x'' < y \leq x \) is equivalent to \( 1 < x \leq y \); in other words that under the assumption \( x > 1 \), the statements \( x'' < y \leq x \) and \( x \leq y \) are equivalent. We have \( x'' < y \leq x \) iff \( xy = x \) in \( A \) iff \( x'' < y \leq x \) iff \( x'' < y \leq x \) iff \( (1 < x \leq y \) or \( 1 \geq x \leq y \sqsubseteq x'') \) iff \( (1 < x \leq y \) or \( 1 \geq x \leq y \sqsubseteq x'') \) iff \( x \leq y \). Therefore, \( A' = P \).

with \( x^* \subseteq x \). So, in that case \( x^r \in y \subseteq \ell_1 \) is equivalent to \( x \subseteq y \subseteq \ell_{p-1} \), by Lemma 4.10(5).

Now we assume \( x \leq 1 \) and prove that \( x \leq y \leq x^r \) is equivalent to \( x \subseteq y \). We have \( x \leq y \leq x^r \) iff \((x \leq y \leq 1 \leq y \leq x^r) \) iff \((x \subseteq y \subseteq \ell_{p-1} \) or \( 1 \geq \ell_{p+1} \geq y \geq x) \) iff \((x \subseteq y \subseteq \ell_{p-1} \) or \( 1 \geq \ell_{p+1} \geq y \geq x) \) if \( x \subseteq y \). We used the equivalence of \( 1 \geq \ell_{p+1} \geq y \geq x \) and \( 1 \geq \ell_{p+1} \geq y \geq x \), which we now prove. Note from the definition of \( x^r \) in \( A \) that if there is \( z \neq x \) in the layer of \( x \) with \( z \) comparable to \( x \), then \( x^r = z \), so in that case \( x^r \subseteq y \subseteq \ell_{p+1} \) is equivalent to \( x \subseteq y \subseteq \ell_{p+1} \) (since \( x^r \), \( x \) have the same \( \succeq \)-upsets and they are comparable). On the other hand, if there is \( z \neq x \) in the layer of \( x \) with \( z \) incomparable to \( x \), then \( x^r = x^* \), which is the smallest element of \( \ell^+ \) with \( x \subseteq x^* \), or equivalently with \( z \subseteq x^* \) since \( x \) and \( z \) are in the same layer. Therefore, in this case \( x^r \subseteq y \subseteq \ell_{p+1} \) is equivalent to \( z \subseteq y \subseteq \ell_{p+1} \) and to \( x \subseteq y \subseteq \ell_{p+1} \), since \( x, z \) are in the same layer and incomparable. Finally, if \( x \) is the only element of its layer, then \( x^r = x^* \), which is the smallest element of \( \ell^+ \) with \( x \subseteq x^* \). So, in that case \( x^r \subseteq y \subseteq \ell_{p+1} \) is equivalent to \( x \subseteq y \subseteq \ell_{p+1} \), by Lemma 4.10(5).

Now let \( A \) be an idempotent residuated chain, \((A, \subseteq, \ell, \Gamma, 1, \star)\) its enhanced monoidal preorder, and let \( B = (B, \wedge, \vee, \cdot, \backslash, /, 1_B) \) be the idempotent residuated chain associated to that. By the definitions it follows that \( B = A \) and \( 1_B = 1 \). Also, \( x \leq_B y \) iff \((x, y \in A^r \) and \( x \leq y \)) or \((x, y \in A^r \) and \( y \leq x \)) or \((x, y \in A^r \) and \( y \leq x \)) or \((x, y \in A^r \) and \( y \leq x \)) iff \((x, y \in A^r \) and \( xy = x \)) or \((x, y \in A^r \) and \( yx = y \)) or \((x, y \in A^r \) and \( yx = y \)) iff \((x, y \in A^r \) and \( x \leq y \)) or \((x, y \in A^r \) and \( y \) commute). Therefore, \( x \) is central in \( A \) and \( x^r = x^* \). Since this is exactly how the inverses of \( x = x \) in \( B \) are defined, we get that the inverses of \( x \) in \( A \) and in \( B \) coincide. If \( x \) is an element of \( B \) in the same layer as another element \( y \), incomparable to \( x \), then \( x \not\subseteq y \) and \( y \not\subseteq x \), so \( xy \neq x \) and \( yx \neq y \). By conservativity, we get \( xy = y \) and \( yx = x \). Then the values of \( x^r \) and \( x^r \) given by Corollary 4.15 coincide with the definition of \( x^r \) and \( x^r \) in \( B \). Likewise, the values of \( x^r \) and \( x^r \) in \( A \) and in \( B \) coincide, if there is a \( y \) incomparable to \( x \) and in the same layer as \( x \). Consequently, the \{\( \wedge, \vee, \ell, r, 1 \}\)-reducts of \( A \) and \( B \) coincide. By Corollary 4.11 we get that \( A = B \).

Therefore, the common notation between idempotent residuated chains and enhanced monoidal preorders (such as \( \star \), \( \subseteq \), etc.) is consistent and we will use it when talking about either one of these structures.

**Remark 4.14.** The motivation for considering the natural order in [30] is the connection to semigroup theory, and the motivation for the monoidal preorder in [30] seems to be the desire to capture the behavior of multiplication. However, our reason for considering the enhanced monoidal preorder is very different. After all, by Lemma 4.12 we already know that multiplication can be captured in the language \{\( \wedge, \vee, \ell, r, 1 \}\}, i.e. by the idempotent Galois connection reduct. Hence subalgebra generation, which is crucial for proving amalgamation, occurs solely via \( \ell \) and \( r \). Unfortunately, this generation takes place remotely: A positive element \( a \) will generate negative elements \( a^\ell \) and \( a^r \), which are both far away from \( a \) in the chain ordering. The benefit of the enhanced monoidal preorder is that the action of \( \ell \) and \( r \) happens locally: \( a^\ell \) and \( a^r \) are adjacent to \( a \) in the enhanced monoidal preorder. Therefore, the reason that we consider enhanced monoidal preorders is that, in that setting, subalgebras are convex subsets. The insight that convex subsets should be
kept intact is key to how to amalgamate such structures in a transparent way (see Section 5).

4.5. Subalgebras. Note that if \( x \) is not central, then \( x \neq x^* \) and its layer is exactly \( \{ x, x^* \} \). Also, if \( x \) is central, then its layer is \( \{ x \} \). For all \( x \), we define \( x^{**} \) to be \( x \), if \( x \) is central, and \( x^* \), if \( x \) is not central. Therefore, \( x^{**} \) is the only other element in the layer of \( x \), if the layer has two elements, and it is equal to \( x \) if the layer has only one element. The next lemma follows directly from Corollary 4.5 and the definition of \( x^{**} \).

**Lemma 4.15.** Let \( A \) be idempotent residuated chain and \( x \in A \). We have \( \{ x^\ell, x^r \} = \{ x^*, x^{**} \} \).

**Lemma 4.16.** Let \( A \) and \( B \) be idempotent residuated chains and let \( P_A \) and \( P_B \) be the corresponding enhanced monoidal preorders. Then \( A \) is a subalgebra of \( B \) iff \( P_A \) is closed under \( \leftrightarrow, \ast \) and \( 1 \).

**Proof.** Recall that subalgebra generation is done via \( \ell, r \) and \( 1 \). Also, we know that \( \{ x^\ell, x^r \} = \{ x^*, x^{**} \} \), for all \( x \), due to the comparability of the two inverses. From Figure 7 we see that for non-central elements, closure under \( \ast \) is the same as closure under same-layer elements. \( \square \)

Figure 8 shows three enhanced monoidal preorders. We call the corresponding idempotent residuated chains \( A \), \( B \) and \( C \). The subalgebras of \( A \) are \( \{ 1 \} \), \( \{ b_4, b_5, a_5, 1 \} = \uparrow b_4 \), \( \{ b_1, a_2, b_2, a_3, 1 \} = \downarrow a_3 \) and \( A \) itself. The subalgebras of \( B \) are \( \{ 1 \} \), \( \{ a_4, b_4, b_5, a_5, 1 \} = \uparrow \{ b_5, b_4, \ldots, b_1, a_2, b_2, a_3, 1 \} \) and \( B \) itself. The subalgebras of \( C \) are only \( \{ 1 \} \), \( \{ b_4, b_5, a_5, 1 \} \) and \( C \) itself. Note that \( \{ b_1, a_2, b_2, b_3, 1 \} \) is not a subalgebra of \( C \), since \( b_2^* = b_3 = a_5 \).

**Figure 8.** The enhanced monoidal preorders of three algebras: \( A \), \( B \) and \( C \).

4.6. Rigidity. In Section 5 we investigate the amalgamation property for idempotent residuated chains. However, we show that amalgamation fails in this class, and unfortunately even if the V-formation consists of quasi-involutive ones; see Section 5.1. There we identify the problem as: \( x^* \) belonging to a subalgebra without \( x \) also belonging to this subalgebra.

We will show below in Lemma 4.18 that, in the context of quasi-involutive idempotent residuated chains, this problem does not occur iff the the residuated chain
is \(*\)-involutive: That is, if it satisfies \(x^{**} = x\). An enhanced monoidal preorder is called \(*\)-involutive if its associated idempotent residuated chain is.

Recall that \(\mathcal{P}\) denotes the layer of \(x\), and that the layers form a chain. Given an enhanced monoidal preorder, we consider the preorder \(\subseteq_{\ast}\) given by \(x \subseteq_{\ast} y\) iff \(x \subseteq y\) or \(x, y\) are in the same layer; we write \(\downarrow_{\ast}\) and \(\uparrow_{\ast}\) for the downward and upward closures of this relation. In the following we will denote by \(\langle x \rangle_{1}\) the subalgebra generated by \(x\) and by \(\langle x \rangle\) the 1-free subalgebra (subalgebra with respect to all the operations except possibly for 1) generated by \(x\). By Corollary 4.14 and Lemma 4.15 we get that \(\langle x \rangle = \langle x^{1} \cdots x^{n} : n \in \mathbb{N}, c_{i} \in \{\ast, \leftrightarrow\} \rangle\) and \(\langle x \rangle_{1} = \langle x \rangle \cup \{1\}\).

**Lemma 4.17.** Let \(\mathbf{A}\) be an idempotent residuated chain.

1. If \(b\) is a negative central element of \(\mathbf{A}\), then \(\langle b \rangle \subseteq \uparrow_{\ast} b\).
2. If \(a\) is a positive central element of \(\mathbf{A}\), then \(\langle a \rangle \subseteq \downarrow_{\ast} a\).
3. If \(b\) is a negative element, \(b < b^{**}\) and \(b < c \leq b^{**}\), then \(c\) is central.
4. If \(x \in \langle x^{\ast} \rangle\) for all \(x \in \mathbf{A}\), then \(\mathbf{A}\) is \(*\)-involutive.
5. Let \(\mathbf{P}\) be a \(*\)-involutive enhanced monoidal preorder.

   (a) For every strictly negative \(b \in \mathbf{P}\), there is a layer of \(\mathbf{P}\) directly above the layer of \(b\) and \(b^{\ast}\) is in that layer.

   (b) Also, for every strictly positive \(a \in \mathbf{P}\), there is a layer of \(\mathbf{P}\) directly below the layer of \(a\) and \(a^{\ast}\) is in that layer.

**Proof.** 1. We prove that \(\uparrow_{\ast} b\) is closed under \(*\) and \(\leftrightarrow\); since it also contains \(b\), the result will follow. Let \(c \in \uparrow_{\ast} b\). Since \(c^{\ast}\) is in the same layer as \(c\), it follows that \(c^{\ast} \in \uparrow_{\ast} b\). We will now show that also \(c^{\ast} \in \uparrow_{\ast} b\). If \(c\) is negative, then \(c \in \uparrow_{\ast} b\) implies \(\overline{b} \leq \overline{c} < \overline{c}\), since \(c \sqsubset c^{*}\); hence \(c^{*} \in \uparrow_{\ast} b\). If \(c\) is positive, then by the definition of \(b^{\ast}\) we have \(b \sqsubseteq b^{\ast} \sqsubseteq_{P_{+}} c\). So \(b \sqsubseteq c\) and by the definition of \(c^{\ast}\) we get \(b \leq c^{\ast}\); hence \(b \sqsubseteq c^{\ast}\).

   The proof of 2 is analogous.

   3. From \(b < b^{**}\) we get that \(\overline{b} < \overline{b^{**}} < \overline{b^{\ast}}\). Assume that \(b < c \leq b^{**}\); hence \(c\) is negative. If \(c\) is not central, then \(\overline{c} \neq c^{\ast}\) and \(c^{\ast}\) is positive. Also, \(\overline{b} < \overline{c} = \overline{c^{\ast}} \leq b^{**} < b^{\ast}\). So, \(b \sqsubseteq c^{\ast} \sqsubseteq_{P_{+}} b^{\ast}\), a contradiction to the definition of \(b^{\ast}\).

   4. We will prove the contrapositive. If \(\mathbf{A}\) is not \(*\)-involutive, then \(x \neq x^{**}\) for some \(x\). By Lemma 2.10 \(x \leq x^{**}\), so \(x < x^{**}\). We will show that this implies \(x \notin \langle x^{\ast} \rangle\). We will do the proof for the case where \(x\) is some negative element \(b\), as the proof for positive elements is analogous. From \(b < b^{**}\), \(\langle b \rangle \subseteq \uparrow_{\ast} b\). By (1) we get that \(\langle b^{**} \rangle \subseteq \uparrow_{\ast} b^{**}\). Also \(\langle b^{**} \rangle = \langle b^{\ast} \rangle\), since \((\ast, \ast)\) is a Galois connection and \(b^{**} = b^{\ast}\), so \(b^{\ast} \subseteq \uparrow_{\ast} b^{**}\). At the same time, from \(b < b^{**}\), we get \(b \notin \uparrow_{\ast} b^{**}\). Therefore, \(b \notin \langle b^{\ast} \rangle\).

   5a. Recall that if \(b\) is negative, then \(\overline{b} < \overline{b^{\ast}}\), since \(b \sqsubseteq b^{\ast}\). If there is a positive \(c\) such that \(\overline{b} < \overline{c} < \overline{b^{\ast}}\), then \(b \sqsubseteq c^{\ast} \sqsubseteq_{P_{-}} b^{\ast}\), a contradiction to the definition of \(b^{\ast}\). If there is a negative \(c\) such that \(\overline{b} < \overline{c} < \overline{b^{\ast}}\), then \(b \sqsubseteq c \sqsubseteq_{P_{-}} b^{\ast}\), so \(b \sqsubseteq_{P_{-}} c \sqsubseteq b^{\ast}\), by the definition of \(b^{**}\); hence \(b < b^{**}\), a contradiction to \(*\)-involutivity. Therefore, there is no level between \(\overline{b}\) and \(\overline{b^{\ast}}\). The proof of 5b is analogous. \(\square\)

**Lemma 4.18.**

For a quasi-involutive idempotent residuated chain \(\mathbf{B}\) the following are equivalent:

1. \(\mathbf{B}\) is \(*\)-involutive
2. For each \(x \in \mathbf{B}\), \(x \in \langle x^{\ast} \rangle_{1}\).
3. For each \(x \in \mathbf{B}\) and each subalgebra \(\mathbf{A}\) of \(\mathbf{B}\), \(x^{\ast} \in \mathbf{A}\) implies \(x \in \mathbf{A}\).
subalgebra of the skeleton but \( x^{\ast} \).

\[ x^{\ast} \in \text{conic idempotent residuated lattices occurs when, for some } x^{\ast} \text{ that are exactly the } \ast \text{-involutive idempotent residuated chains, which by Lemma 4.19(2) are precisely those whose skeleton is } \ast \text{-involutive. Hence it suffices to focus on } \ast \text{-involutive quasi-involutive idempotent residuated chains, which by Lemma 4.19(2) are exactly the } \ast \text{-involutive idempotent residuated chains.}

\]

Lemma 4.19.

1. A conic idempotent residuated lattice is rigid iff its quasi-involutive skeleton is rigid iff its quasi-involutive skeleton is \(*-involutive."

2. If an idempotent residuated chain is \(*-involutive, then it is also quasi-involutive.

Proof. 1. The rigidity equations for a conic idempotent residuated lattice \( A \) stipulate that every element of the quasi-involutive skeleton \( A^i \) is fixed under double star, and this is also what rigidity for its quasi-involutive skeleton \( A^i \) stipulates.

2. For every element \( x \) in a \(*-involutive idempotent residuated chain \( A \), we have that \( x = x^{**} = (x^\ell \lor x^r)^\ell \lor (x^\ell \lor x^r)^r \). Since, \( x^\ell \) and \( x^r \) are comparable, we get that \( x \in \{x^{\ell r}, x^{r\ell}\} \subseteq A^i \). Hence, \( A = A^i \) and \( A \) is quasi-involutive.

As previously mentioned, we will see that one of the main obstacles to amalgamating conic idempotent residuated lattices occurs when, for some \( x \), \( x^{\ast} \) is in some subalgebra of the skeleton but \( x \) is not. The conic idempotent residuated lattices for which this does not occur are exactly the rigid ones, which by Lemma 4.19(1) are precisely those whose skeleton is \(*-involutive. Hence it suffices to focus on \(*-involutive quasi-involutive idempotent residuated chains, which by Lemma 4.19(2) are exactly the \(*-involutive idempotent residuated chains.

4.7. \(*-involutive idempotent residuated chains: the one-generated case."

We will characterize the \(*-involutive idempotent residuated chains as nested sums of one-generated ones. First, we provide a description of the one-generated \(*-involutive idempotent residuated chains.

Note that one of the most obvious consequences of \(*-involutivity is that the only \( x \) with \( x^{\ast} = 1 \) is \( x = 1 \), since then \( x^{**} = 1^{\ast} = 1 = 1 \). In particular this implies that, for non-identity \( x \), we cannot have \( x^{\ell} = 1 \) or \( x^{r} = 1 \), as that would imply that \( x \) is negative, thus that \( x^\ell \) and \( x^r \) are positive, so \( x^{\ast} = x^\ell \land x^r = 1 \). Therefore, in the presence of \(*-involutivity, 1 is isolated: It is not equal to the inverse of a non-identity element. Note that 1 is isolated in an idempotent residuated chain iff in the associated enhanced monoidal preorder, it is not the case that there exists a strictly negative element in a layer directly below the layer of 1. So, if there is a layer directly below 1, it contains a positive element only (or else there is no layer covered by the layer of 1).

Therefore, in this setting, not only do we have \( \langle x \rangle_1 = \langle x \rangle \cup \{1\} \), for every \( x \), but we also have \( \langle x \rangle = \langle x \rangle_1 - \{1\} \). Therefore, we characterize the 1-free subalgebras of the form \( \langle x \rangle \), for \( x \) in a \(*-involutive idempotent residuated chain.
We have already seen examples of non-trivial one-generated \(*\)-involutive idempotent residuated chains in Example 2.2 and Figure 2 and we now also include their bounded variants and their finite variants. We introduce them by means of their enhanced monoidal preorders.

An enhanced monoidal preorder is called a \textit{vertical crown} iff 1 is isolated (in particular its size is not 2) and it has at most two non-identity central elements. Figure 9 shows all the possible shapes that vertical crowns can take.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{vertical_crowns.png}
\caption{Vertical crowns (dashed lines may be independently present or absent)}
\end{figure}

To explain the structures in the picture more precisely, recall the ordinal sum $\bigoplus_{i \in I} P_i$ of (pairwise disjoint) (pre)orders $P_i = (P_i, \leq_{P_i})$ over a chain $(I, \leq_I)$: It is based on their disjoint union $\bigcup_{i \in I} P_i$ with ordering given by $x \leq y$ iff $(x \in P_i, y \in P_j$ and $i <_I j)$ or $(x, y \in P_i$ and $x \leq_{P_i} y)$. We naturally extend the notion of the ordinal sum to \textit{enhanced preorders} (i.e., preorders $(P, \leq)$ enhanced with two subsets $P^+$ and $P^-$) in the natural way: For $P = \bigoplus_{i \in I} P_i$, we define $P^+ := \bigcup_{i \in I} P_i^+$ and $P^- := \bigcup_{i \in I} P_i^-$. Examples of enhanced preorders are:

- $1_L$: a singleton with $(1_L)^+ = 1_L$ and $(1_L)^- = \emptyset$,
- $1_R$: a singleton with $(1_R)^+ = \emptyset$ and $(1_R)^- = 1_R$,
- $2_L$: a doubleton \{a, b\} with $(2_L)^+ = \{a\}$, $(2_L)^- = \{b\}$, $a \leq b$ and $b \leq a$,
- $2_R$: a doubleton \{a, b\} with $(2_R)^+ = \{a\}$, $(2_R)^- = \{b\}$, $a \not\leq b$ and $b \not\leq a$.

A \textit{vertical crown} is the enhanced monoidal preorder expanding (by \* and 1, in the unique and obvious way) an enhanced preorder of one of the following types:

1. $P_{Z, L} = (\bigoplus_{i \in I} P_i) \oplus 1_L$, where $I = \mathbb{Z}$,
2. $P_{N, L} = 1_R \oplus (\bigoplus_{i \in I} P_i) \oplus 1_L$, where $I = \mathbb{N}$,
3. $P_{N^0, L} = (\bigoplus_{i \in I} P_i) \oplus 1_L \oplus 1_L$, where $I = \mathbb{N^0}$,
4. $P_{n, L} = 1_R \oplus (\bigoplus_{i \in I} P_i) \oplus 1_L \oplus 1_L$, where $I = \{1, 2, \ldots, n\}$.

In all cases, $L \subseteq I$ and $P_i = 2_L$ for $i \in L$, while $P_i = 2_R$ for $i \notin L$. In the case of $P_{n, L}$, we allow $n = 0$, hence $I = \emptyset$, which yields the 3-element Sugihara monoid.

Since these enhanced preorders have a unique expansion to enhanced monoidal preorders, where 1 is their top element and \* has the usual definition, we use the same names for the corresponding enhanced monoidal preorders. An idempotent residuated chain whose enhanced monoidal preorder is a vertical crown is called \textit{crownian}. A subset $Q$ of an enhanced monoidal preorder $P$ is called \textit{layer-convex} if $q_1, q_2 \in Q$ and $\overline{q_1} \leq \overline{p} \leq \overline{q_2}$, then $p \in Q$.

Lemma 4.20.
1. For a non-identity element $x$ in a *-involutive enhanced monoidal preorder, 
\langle x \rangle forms a layer-convex subset.

2. A non-trivial idempotent residuated chain is one-generated and *-involutive iff it is crownian.

Proof. 1. Recall that if $x$ is a non-identity element in an idempotent residuated chain, then \( \langle x \rangle = \{x^{c_1 \cdots c_n} : n \in \mathbb{N}, c_i \in \{*, \leftrightarrow\}\}; \) also by definition \((x^{*})^{*} = x\). If we assume *-involutivity, we also have $x^{**} = x$, so the $c_i \in \{*, \leftrightarrow\}$ can be assumed to alternate. Therefore, \( \langle x \rangle = \{a_n : n \in \mathbb{Z}\} \cup \{b_n : n \in \mathbb{Z}\} = \{\ldots, a_{-1}, b_{-1}, a_0, b_0, a_1, b_1, \ldots\} \), where $a_0 = x$, $b_0 = a_0^{\leftrightarrow}$, and for all $n \in \mathbb{N}$,

\[
\begin{align*}
\bullet & \quad a_{n+1} = b_n^{-1} \text{ and } b_n = a_n^{\leftrightarrow}, \\
\bullet & \quad b_{-n+1} = a_n^{\leftrightarrow} \text{ and } a_n = b_{-n}. \\
\end{align*}
\]

In other words we have

\[
\langle x \rangle = \{a_0 \leftrightarrow b_0, a_1 \leftrightarrow b_1, \ldots\}
\]

where $u \leftrightarrow v$ denotes \((u^* = v \text{ and } v^* = u)\) and $u \leftrightarrow v$ denotes \((u^{*+} = v \text{ and } v^{*+} = u)\).

Note that $a_i$ and $b_i$ are in the same layer and, by Lemma 4.17(1), $b_i$ and $a_{i+1}$ are in adjacent layers. Therefore, $\langle x \rangle$ is layer-convex.

2. Recall that every one-generated idempotent residuated chain is of the form \( \langle x \rangle = \langle x \rangle \cup \{1\}; \) since the algebra is non-trivial, we will assume $x \neq 1$ below.

If none of the elements of $\langle x \rangle$ is central, then in particular $y \neq y^{**}$. Note that $*$ always moves to the next layer, and actually in a NW-SE diagonal (the level directly above when applied to a negative element, and the level directly below when applied to a positive element). Moreover, $\leftrightarrow$ moves to the other element of the same layer. It follows that all of these elements are distinct, they form a zigzag pattern (with horizontal and NW-SE diagonal lines), and $b_i = \overline{b_i} < \overline{b_j} = \overline{b_j}$ for all $i < j \in \mathbb{Z}$. Therefore, the enhanced monoidal preorder is $\mathbf{P}_{\mathbb{Z},L}$, for some $L \subseteq \mathbb{Z}$.

Now assume that one of the strictly negative elements of $\langle x \rangle$ is central. By possible renaming, we may assume that this element is $b_0$. Then $a_0 = b_0^{\leftrightarrow} = b_0$. Also, by Lemma 4.17(1), \( \langle b_0 \rangle \subseteq \cup_{i} b_i \), and to be more precise because $a_0 = b_0$ and because of Equation (4.17(2)), we have \( \{\ldots, a_{-1}, b_{-1}, a_0 \} \subseteq \{b_0, a_1, b_1, \ldots\} \). Thus \( \langle x \rangle = \{b_0, a_1, b_1, \ldots\} \). Now, if furthermore no strictly positive element of $\langle x \rangle$ is central, then in particular for every positive element $a \in \langle x \rangle$, the element $a^{**}$ is different from $a$ (and negative). Also, $\overline{a_i} = \overline{b_i} < \overline{b_j} = \overline{b_j}$ for all $i < j \in \mathbb{N}$. Therefore, the enhanced monoidal preorder is $\mathbf{P}_{\mathbb{N},L}$, for some $L \subseteq \mathbb{N}$. On the other hand, if furthermore there is a strictly positive element of $\langle x \rangle = \{b_0, a_1, b_1, \ldots\}$ that is central, say $a_{n+1}$, then $b_{n+1} = a_{n+1}^{**} = a_{n+1}$ and because of Equation (4.17(2)), we have \( \{b_{n+1}, a_{n+2}, b_{n+2}, \ldots\} \subseteq \{\ldots, a_n, b_n, a_{n+1} \} \subseteq \{b_0, a_1, b_1, \ldots, a_n, b_n, a_{n+1} \} \). Thus $\langle x \rangle = \{b_0, a_1, b_1, \ldots, a_n, b_n, a_{n+1} \}$. Therefore, in this case the enhanced monoidal preorder is $\mathbf{P}_{\mathbb{N},L}$, for some $L \subseteq \{1, 2, \ldots, n\}$.

Dually to one of the situations above, if there is a strictly positive element of $\langle x \rangle$ that is central, but no strictly negative element is central, then the enhanced monoidal preorder is $\mathbf{P}_{\mathbb{N}^{\circ},L}$, for some $L \subseteq \mathbb{N}$.

Conversely, all vertical crowns are enhanced monoidal preorders. Also they are generated by any one of their elements.

\[\square\]

4.8. Nested sums and *-involutive idempotent residuated chains. Having described the one-generated subalgebras of *-involutive idempotent residuated

chains, we now show what arbitrary *-involutive idempotent residuated chains look like. To do so, we will make use of the nested sum construction.

Loosely speaking, given two structures $K$ and $L$ over a language with a constant 1, this construction is performed by replacing $1_K$ in $K$ ($= K[1_K]$) by $L$ to obtain their nested sum $K[L]$. We will also denote $K[L]$ by $K ⊞ L$, as this notation allows iterations of this construction: $K_1 \cdots K_n$. We will make use of this construction where the structures are either residuated lattices or enhanced monoidal preorders.

Nested sums of residuated lattices are defined in [23]. Here we will only describe how nested sums specialize to the conic idempotent case. Assume that $A_i$, for $i ∈ I$, are conic residuated lattices, where $I$ is a totally ordered index set. Also, assume that for all $i ∈ I$ except possibly for the top element of $I$, if it exists, we have $a^t \neq 1_{A_i}$, $a^r \neq 1_{A_i}$, $a ∨ b \neq 1$ and $a ∧ b \neq 1$, for all non-identity elements $a, b ∈ A_i$. Their nested sum is an algebra $\bigoplus_{i∈I} A_i$ whose underlying set is the union $\{1\} \cup \bigcup_{i∈I} (A_i \setminus \{1_{A_i}\})$ (we identify all the identity elements). The order and operations extend the operations on the $A_i$‘s, which become subalgebras of the nested sum, by the following additional clauses, where $a_i ∈ A_i$ and $a_j ∈ A_j$ are non-identity elements and $i <_I j$: $a_i < a_j$ iff $a_i < 1_{A_i}$, and $a_j < a_i$ iff $1_{A_i} < a_i$; also,

$$a_i \bullet a_j = a_i \bullet 1_{A_i}$$

and

$$a_j \bullet a_i = 1_{A_i} \bullet a_i,$$

where $\bullet$ ranges over multiplication and the divisions. In particular, focusing on the lattice operations, for $i <_I j$, $A_j$ is nested inside $A_i$ at the location of the identity element. It is shown in [23] that the nested sum is a residuated lattice. It is obvious that conicity, linearity, and idempotency are preserved.

Note that every non-trivial Sugihara monoid is a nested sum of copies of the 3-element Sugihara monoid. In particular, these copies are subalgebras.

We now define nested sums of enhanced monoidal preorders $P_i$, over a chain $I$. We again assume that for all $i ∈ I$ except possibly for the top element of $I$, if it exists, we have $a^* \neq 1_{P_i}$ for every non-identity element $a ∈ P_i$. The nested sum is the enhanced monoidal preorder $\bigoplus_{i∈I} P_i$, whose underlying set is the union $\{1\} \cup \bigcup_{i∈I} P_i \setminus \{1_{P_i}\}$. The operation $*$ is defined as in each $P_i$. For the positive and negative elements, we defined $P^+ := \bigcup_{i∈I} P_i^+$ and $P^- := \bigcup_{i∈I} P_i^-$. The preorder is defined to extend the preorders on the structures $P_i$ by setting 1 to be the largest element, and for non-identity $a_i ∈ P_i$ and $a_j ∈ P_j$, stipulating that $a_i ⊆ a_j$ iff $i < j$. In essence, the preorder is the ordinal sum $\bigoplus_{i∈I} (P_i \setminus \{1_{P_i}\}) \uplus \{1\}$.

Nested sums for integral residuated chains have been considered in the literature. Confusingly, they are often called ordinal sums [11,33], in conflict with the use of the term for (pre)ordered sets. Also, [23] refers to the generalization beyond the integral case as generalized ordinal sums. Here we try to fix the terminological clash caused by the overload of the term ordinal sum by introducing the term nested sums.

The following is an immediate consequence of the work of this section.

**Lemma 4.21.** Nested sums of idempotent residuated chains correspond bijectively to nested sums of the corresponding enhanced monoidal preorders.

**Lemma 4.22.** The *-involutive idempotent residuated chains are exactly the nested sums of crowns, i.e., those whose enhanced monoidal preorders are crowns. Also, one-generated subalgebras intersect only at $\{1\}$ and they are generated by any of their non-identity elements.
Proof. By Lemma 4.21 it suffices to show that an enhanced monoidal preorder is *-involutive iff it is a nested sum of vertical crowns. By Lemma 4.20, for every non-identity element \( x \) in a *-involutive enhanced monoidal preorder, \( \langle x \rangle \) is layer-convex, and together with 1 forms an enhanced monoidal preorder (the one corresponding to the subalgebra \( \langle x \rangle_1 \)). As a result, the enhanced monoidal preorder is an ordinal sum of the various \( \langle x \rangle \), hence the nested sum of the various \( \langle x \rangle_1 \). The converse follows from the fact that vertical crowns are *-involutive and that the nested sum of *-involutive enhanced monoidal preorders remains *-involutive. \( \square \)

5. Amalgamation

We now have all the ingredients for proving the strong amalgamation property. Our approach is to first prove that the class of conic algebras has the amalgamation property, and then extend this result to the semiconic case. Unfortunately, the amalgamation property fails for the whole class of conic idempotent residuated lattices; as we have already mentioned, it even fails for the class of idempotent residuated chains, as we show in Section 5.1. However, the analysis of the structure of conic idempotent residuated lattices—and in particular their decomposition into nested sums and their subalgebra generation—allows us to identify the two obstacles to amalgamation. By stipulating the conditions of being rigid and conjunctive (i.e., all prelattices appearing in the decomposition are lattices; see Section 5.3), we are able to prove the strong amalgamation for this class of conic idempotent residuated lattices and then extend the result to the semiconic variety this class generates.

Let \( \mathcal{K} \) be a class of similar algebras. A V-formation in \( \mathcal{K} \) is an ordered quintuple \((\mathbf{A}, \mathbf{B}, \mathbf{C}, f_\mathbf{B}, f_\mathbf{C})\), where \( \mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K} \) and \( f_\mathbf{B} : \mathbf{A} \to \mathbf{B} \) and \( f_\mathbf{C} : \mathbf{A} \to \mathbf{C} \) are embeddings. Given a V-formation \( \mathcal{V} = (\mathbf{A}, \mathbf{B}, \mathbf{C}, f_\mathbf{B}, f_\mathbf{C}) \) in \( \mathcal{K} \) and a class \( \mathcal{M} \) of algebras in the type of \( \mathcal{K} \), an amalgam of \( \mathcal{V} \) in \( \mathcal{M} \) is an ordered triple \((\mathbf{D}, g_\mathbf{B}, g_\mathbf{C})\), where \( \mathbf{D} \in \mathcal{M} \) and \( g_\mathbf{B} : \mathbf{B} \to \mathbf{D} \) and \( g_\mathbf{C} : \mathbf{C} \to \mathbf{D} \) are embeddings such that \( g_\mathbf{B} \circ f_\mathbf{B} = g_\mathbf{C} \circ f_\mathbf{C} \).

A class \( \mathcal{K} \) of similar algebras is said to have the amalgamation property in \( \mathcal{M} \) if every V-formation in \( \mathcal{K} \) has an amalgam in \( \mathcal{M} \). The class \( \mathcal{K} \) is said to have the amalgamation property if \( \mathcal{K} \) has the amalgamation property in \( \mathcal{K} \).

We say that a V-formation \((\mathbf{A}, \mathbf{B}, \mathbf{C}, f_\mathbf{B}, f_\mathbf{C})\) is reduced if \( \mathbf{A} \) is a subalgebra of each of \( \mathbf{B} \) and \( \mathbf{C} \), and \( f_\mathbf{B} \) and \( f_\mathbf{C} \) are the inclusion maps. Note that for classes closed under isomorphisms, we can assume without loss of generality that all V-formations are reduced. The amalgamation property ensures that the maps \( g_\mathbf{B} \) and \( g_\mathbf{C} \) do not send two different elements of \( \mathbf{A} \) to the same element of \( \mathbf{D} \), but an element of \( \mathbf{B} \) and an element of \( \mathbf{C} \) may be sent to the same element of \( \mathbf{D} \) since the intersection of the images of these maps may be properly larger than the image of \( \mathbf{A} \). In other words, \( \mathbf{B} \) and \( \mathbf{C} \) may be identified with subalgebras of \( \mathbf{D} \) simultaneously, but only if we are allowed to rename their elements that have the same image in \( \mathbf{D} \). Therefore, for classes closed under isomorphisms the amalgamation property can be stated as: If the algebras \( \mathbf{B} \) and \( \mathbf{C} \) have a common subalgebra \( \mathbf{A} \), then there exists algebras \( \mathbf{B}' \) and \( \mathbf{C}' \) isomorphic to \( \mathbf{B} \) and \( \mathbf{C} \), respectively, also having \( \mathbf{A} \) as a common subalgebra, and there exists an algebra \( \mathbf{D} \) that contains \( \mathbf{B}' \) and \( \mathbf{C}' \) as subalgebras.

The situation is much more transparent if a stronger version of the amalgamation holds. A class of similar algebras \( \mathcal{K} \) has the strong amalgamation property iff it has the amalgamation property and amalgams \((\mathbf{D}, g_\mathbf{B}, g_\mathbf{C})\) may be selected so that \((g_\mathbf{B} \circ f_\mathbf{B})[\mathbf{A}] = g_\mathbf{B}[\mathbf{B}] \cap g_\mathbf{C}[\mathbf{C}]\). In this event, we say that \((\mathbf{D}, g_\mathbf{B}, g_\mathbf{C})\) is a strong
amalgam. It is easy to see that the strong amalgamation property has a much easier formulation if the class \( K \) of similar algebras is closed under isomorphisms: If the algebras \( B \) and \( C \) in \( K \) intersect at a common subalgebra \( A \), there exists an algebra \( D \) in \( K \) having \( B \) and \( C \) as subalgebras. Note that no renaming of elements (as in \( B' \) and \( C' \)) is necessary.

5.1. **Failure of amalgamation for idempotent residuated chains.** We consider the idempotent residuated chains \( B \) and \( C \) given in terms of their enhanced monoidal preorder in Figure 10, and their common subalgebra \( A \), where \( A = \{1\} \). We will show that this witnesses the failure of the amalgamation property in conic idempotent residuated lattices even when the V-formation consists of idempotent residuated chains.

![Figure 10](image_url)

**Figure 10.** Failure of amalgamation for conic; \( B \) is on the left, \( C \) is on the middle and the right and \( A = \{1\} \). All algebras are chains and connected; they are not rigid.

If there would be an idempotent residuated chain \( D \) that would serve as an amalgam for this V-formation, then \( B \) and \( C \) could be identified with subalgebras of \( D \), where the intersection of \( B \) and \( C \) could be larger than \( A \). We will argue in the algebra \( D \). Since \( b_2^r = 1 \) in \( B \) and thus also in \( D \), the element \( a_3 \) has to be an upper cover of 1 in \( D \), by Corollary 4.5. Likewise, since \( (b_2')^r = 1 \), the element \( a_3' \) is an upper cover of 1 in \( D \); hence \( a_3 = a_3' \) in \( D \). Therefore, \( b_2 = a_3^r = (a_3')^r = b_2' \) in \( D \), hence \( a_3'^r = (b_2')^r = a_3 = a_3' \) in \( D \), which is a contradiction to the fact that \( C \) is a subalgebra of \( D \) and in \( C \) the elements \( a_2' \) and \( a_4' \) are different.

More generally, if an idempotent residuated chain \( B \) has a subalgebra \( A \) and some element outside of \( A \) (in the above example this was \( b_3 \) whose \( * \) lies inside \( A \) (this was \( b_3^* = b_3 = 1 \)), then a similar failure of amalgamation can be produced. The condition that ensures precisely that this issue does not come up is rigidity, as we proved in Section 4.6. Therefore, we will restrict our attention to rigid conic idempotent residuated lattices.

5.2. **Failure of amalgamation for semilinear idempotent and semiconic idempotent residuated lattices.** It is conceivable that a V-formation of idempotent residuated chains does not have a totally ordered amalgam, but it does have
an amalgam in the variety of semilinear idempotent residuated lattices. We now show that the amalgamation property fails for the variety of semilinear idempotent residuated lattices.

The one sided amalgamation property (1AP) \[21\] is defined in the same way as the amalgamation property, but by removing the demand that \( g_B : B \to D \) is an embedding and only stipulating that it is a homomorphism, while still insisting that \( g_C \) is an embedding; we refer to \( D \) as a 1-amalgam of the associated V-formation. We prove that in the setting of residuated lattices 1-amalgamation and amalgamation are more closely related than in arbitrary algebras.

Lemma 5.1. If \((A, B, C, f_B, f_C)\) is a V-formation of residuated lattices and \(B\) is subdirectly irreducible with monolith \(A\), then every residuated lattice that is a 1-amalgam of the V-formation is actually an amalgam.

Proof. Assume that the residuated lattice \(D\) is a 1-amalgam for the V-formation. By the commutativity of the diagram we obtain that \(g_B\) has to be injective on \(A\).

If \(A\) is the monolith of \(B\), then \(A\) is the smallest non-trivial convex normal subalgebra of \(B\) and any other such must contain \(A\). In particular, since the kernel of \(g_B\) is a convex normal subalgebra of \(B\), it must contain \(A\) or be trivial. If it contains \(A\), then all elements of \(A\) are mapped to 1 under \(g_B\), a contradiction to the injectivity of \(g_B\) on \(A\). Therefore, the kernel of \(g_B\) is trivial, hence \(g_B\) is injective. Consequently, \(D\) is an amalgam of the V-formation. \(\square\)

Theorem 5.2. The 1AP (hence also the AP) fails for the class of idempotent residuated chains. Also, it fails for the class of finitely subdirectly irreducible conic idempotent residuated lattices.

Proof. We consider the idempotent residuated chains \(B\) and \(C\) given in terms of their enhanced monoidal preorder in Figure 11 and their common subalgebra \(A\), where \(A = \{b, a, 1\}\). Assume there is a residuated lattice \(D\) that would serve as a 1-amalgam for this V-formation. Note that by the characterization given in Lemma 5.2, \(A\) is the smallest non-trivial convex normal subalgebra of \(B\) and any other such must contain \(A\). By Lemma 5.1, \(D\) is actually an amalgam of the V-formation and we can identify \(B\) and \(C\) with subalgebras of \(D\), and \(g_B\) and \(g_C\) can be taken to be the inclusion maps. We will argue in the algebra \(D\).

First assume that \(D\) is an idempotent residuated chain. Note that \(b_B^* = a\) and \(b_C^* = a\), so both \(a_B\) and \(a_C\) are a covers of \(a\) in \(D\). Therefore, \(a_B = a_C\), a contradiction on how \(a_B, b_B\) and \(a_C, b_C\) multiply. Thus the 1AP fails for the class of idempotent residuated chains.

By the characterization in Lemma 5.7, the algebras \(A, B\) and \(C\) are finitely subdirectly irreducible conic idempotent residuated lattices. We now assume that \(D\) is also a finitely subdirectly irreducible conic idempotent residuated lattice. Also, note the algebras \(A, B\) and \(C\) are quasi-involutive, since every element in them is the (left or right) inverse of some element. Therefore, the images of these algebras in \(D\) (via \(g_B\) and \(g_C\)) are also quasi-involutive and thus are contained in the quasi-involutive skeleton \(S\) of \(D\). Thus, \(S\) is also an amalgam of the V-formation, which happens to be an idempotent residuated chain. This contradicts the fact established above that this particular V-formation does not have an amalgam among idempotent residuated chains. This establishes that the 1AP fails for the class of finitely subdirectly irreducible conic idempotent residuated lattices. \(\square\)
Corollary 5.3. The amalgamation property fails for the variety of semilinear idempotent residuated lattices. Also, it fails for the variety of semiconic idempotent residuated lattices.

Proof. In [21] it is shown that if a variety $\mathcal{V}$ has the CEP and the class $\mathcal{V}_{FSI}$ of its finitely subdirectly irreducibles is closed under subalgebras, then $\mathcal{V}$ has the AP iff $\mathcal{V}_{FSI}$ has the 1AP. Taking $\mathcal{V}$ to be the variety of semilinear idempotent residuated lattices, Corollary 3.4 shows that $\mathcal{V}$ has the CEP, while Corollary 3.8 shows that $\mathcal{V}_{FSI}$ is exactly the class of idempotent residuated chains, which is clearly closed under subalgebras. Since $\mathcal{V}_{FSI}$ fails the 1AP by Theorem 5.2, we obtain the failure of the AP for $\mathcal{V}$.

Likewise, if we take $\mathcal{V}$ to be the variety of semiconic idempotent residuated lattices, Corollary 3.4 shows that $\mathcal{V}$ has the CEP, while Lemma 3.7 ensures that $\mathcal{V}_{FSI}$ is closed under subalgebras. Since $\mathcal{V}_{FSI}$ fails the 1AP by Theorem 5.2, we obtain the failure of the AP for $\mathcal{V}$.

\[ \square \]

5.3. Failure of amalgamation for rigid conic idempotent residuated lattices. We show that even with the above restriction of rigidity, amalgamation still fails for conic idempotent residuated lattices, even in the commutative case. Let $A$, $B$ and $C$ be the commutative conic idempotent residuated lattices given in Figure 12 where the decomposition systems are based on $\{a^*, b^*, 1, b, a\}$ and $\{a^*, c^*, d^*, 1, d, c, a\}$.

Assume that there is a conic idempotent residuated lattice $D$ that serves as an amalgam, where elements of $B$ and $C$ may be identified. In the amalgam $D$ the element $b$ is an inverse and so is $c$, hence they are conical and thus comparable. If $b < c$ in $D$, then since $b_1 \land b_2 = b$ and since $c$ is conical, we would get $b_1, b_2 \leq c$, hence $a = b_1 \lor b_2 \leq c$, a contradiction. A similar argument with $c_1$ and $c_2$ shows that we cannot have $c < b$ in $D$, so $c = b$ in $D$. A dual argument involving $b'_1, b'_2$ and $d_1, d_2$ shows that in $D$ we must also have $b = d$, implying the contradiction that $c = d$.
Figure 12. Failure of amalgamation for commutative rigid conic idempotent residuated lattices: the algebras A, B, and C.

More generally, the issue is that the positive blocks are not lattices and the meet of elements of the same block is the top element of the next block below. The condition that captures precisely the conic idempotent residuated lattices for which every block is a lattice is being conjunctive. A semiconic idempotent residuated lattice is called conjunctive if it satisfies $\gamma(x \land y) = \gamma(x) \land \gamma(y)$.

**Lemma 5.4.** A conic idempotent residuated lattice is conjunctive iff all of its (positive) blocks are lattices.

**Proof.** Assume that all blocks are lattices. Condition $\gamma(x \land y) = \gamma(x) \land \gamma(y)$ holds if $x$ and $y$ are comparable, so we only need to verify it for the case where $x$ and $y$ are in the same block, i.e. $\gamma(x) = \gamma(y)$. Since all blocks are lattices, we also have $\gamma(x \land y) = \gamma(x) = \gamma(x) \land \gamma(y)$.

Conversely, assume the equation is satisfied. Each block is closed under join, so we only need to check closure under meet. If $x$ and $y$ are in the same block, then $\gamma(x) = \gamma(y)$. Since $\gamma(x \land y) = \gamma(x) \land \gamma(y)$, it follows that $x \land y$ is also in the same block.

Therefore, we will restrict our attention to conjunctive rigid conic idempotent residuated lattices.

5.4. **Failure of amalgamation for rigid semiconic idempotent residuated lattices.** As an aside we mention that amalgamation fails even for the variety of rigid semiconic idempotent residuated lattices, by taking a more special $V$-formation.

**Theorem 5.5.** The amalgamation property fails for the variety of rigid semiconic idempotent residuated lattices, as well as for its commutative subvariety.

**Proof.** By [21], if a variety $\mathcal{V}$ has the CEP and the class $\mathcal{V}_{FSI}$ of its finitely subdirectly irreducibles is closed under subalgebras, then $\mathcal{V}$ has the AP iff $\mathcal{V}_{FSI}$ has the 1AP. Taking $\mathcal{V}$ to be the variety of (commutative) rigid semiconic idempotent
residuated lattices, Corollary 3.4 shows that \( \mathcal{V} \) has the CEP, while Corollary 3.8 shows that \( \mathcal{V}_{FSI} \) is exactly the class of (commutative) rigid conic residuated chains where 1 is join irreducible, which is clearly closed under subalgebras. We will show that \( \mathcal{V}_{FSI} \) fails the 1AP, so we will obtain the failure of the AP for \( \mathcal{V} \).

Let \( A, B \) and \( C \) be the commutative rigid conic idempotent residuated lattices given in Figure 13, where the decomposition systems are based on \( \{a^*, b^*, 1, b, a\} \) and \( \{a^*, c^*, d^*, 1, d, c, a\} \), and note that 1 is join irreducible in them, so they are in \( \mathcal{V}_{FSI} \). Assume that there is a \( D \in \mathcal{V} \) that serves as a 1-amalgam; we view \( C \) as a subalgebra of \( D \). In the amalgam \( D \) the element \( g_B(b) \) is an inverse and so is \( c \), hence they are conical and thus comparable. Since the lattice \( [b, a] \) is isomorphic to \( M_3 \), its image under \( g_B \) is either a single point or isomorphic to \( M_3 \). If \( g_B(b) < c \) in \( D \), then \( g_B(b) < c < a = g_B(a) \), so \( g_B([b, a]) \) cannot be a single point (as \( g_B(b) < g_B(a) \)) nor can it be isomorphic to \( M_3 \), as that would contradict the conicity of \( c \) (for example \( c \leq g_B(b_1), g_B(b_2) \) would imply \( c \leq g_B(b_1) \wedge g_B(b_2) = g_B(b_1 \wedge b_2) = g_B(b) \)). So, by the conicity of \( c \), we get \( c \leq g_B(b) \). A similar argument using the interval \( [1, b] \) shows that \( g_B(b) \leq d \) in \( D \), so \( c \leq d \) in \( D \), a contradiction to the injectivity of \( g_C \).

\[ \square \]

Figure 13. Failure of the 1AP for (commutative) rigid conic idempotent FSI residuated lattices: the algebras \( A, B, \) and \( C \).

5.5. **Amalgamation for conjunctive rigid conic idempotent residuated lattices.** The following fact is well known and can also be established easily using residuated frames [24].

**Lemma 5.6.** The varieties of lattices and of Brouwerian algebras each have the strong amalgamation property.

**Theorem 5.7.** The class of \( * \)-involutive idempotent residuated chains has the strong amalgamation property.
Proof. Let B and C be *-involutory idempotent residuated chains that intersect at a common subalgebra A. Lemma 4.22 entails that A is a union of one-generated subalgebras of B and also a union of one-generated subalgebras of C. In particular, if a non-trivial one-generated subalgebra of A is part of a one-generated subalgebra of B (or of C), then the two one-generated subalgebras are identical.

Therefore, the nested sum decomposition of the corresponding enhanced monoidal preorders afforded by Lemma 4.22 is such that for the corresponding index sets/chains we have that \( I_B \cap I_C = I_A \). We can now amalgamate the index sets, as chains to get \( I_B \cup I_C \), where the ordering is any linear ordering extending the union of the two orderings. The enhanced monoidal preorder of the algebra D is obtained as the nested sum over that index set, where the one-generated subalgebras are exactly the ones of B and the ones from C; if an index is in both \( I_B \) and \( I_C \), then it is in \( I_A \) and the corresponding one-generated subalgebra in A, B and C are identical. □

**Theorem 5.8.** The class of rigid conjunctive conic idempotent residuated lattices has the strong amalgamation property.

Proof. Let B and C be a rigid conjunctive conic idempotent residuated lattices and assume that their intersection is a common subalgebra A. We will produce a rigid conjunctive conic idempotent residuated lattice D that will have B and C as subalgebras. Let \( S_A, S_B \) and \( S_C \) be the residuated chains of quasi-involutive elements of the three algebras. By rigidity, we get that these skeletons are *-involutive, so by Lemma 5.7 we have a *-involutive residuated chain \( S_D \) that has \( S_B \) and \( S_C \) as subalgebras.

Also, we denote the corresponding blocks by \( A_s, \ s \in S_A, B_s, \ s \in S_B, \) and \( C_s, \ s \in S_C, \) which are all lattices by Lemma 5.4. For a negative \( s \in S_D \) we have that \( B_s \) and \( C_s \) are Brouwerian algebras and \( A_s \) is their common subalgebra. By Lemma 5.6 there exists a Brouwerian algebra \( D_s \) that has \( B_s \) and \( C_s \) as subalgebras. Also, for a positive \( s \in S_D \) we have that \( B_s \) and \( C_s \) are lattices and \( A_s \) is their common sublattice. By Lemma 5.6 there exists a lattice \( D_s \) that has \( B_s \) and \( C_s \) as sublattices. Therefore, \((S_D, \{D_s : s \in S_D\})\) is a decomposition system, so by Theorem 2.22 it corresponds to a conic idempotent residuated lattice D, which is conjunctive by Lemma 5.4 and rigid since its skeleton is *-involutive. Also, by Lemma 2.25 this system has \((S_B, \{B_s : s \in S_B\})\) and \((S_C, \{C_s : s \in S_C\})\) as subsystems, so D has B and C as subalgebras. □

### 5.6. Strong amalgamation for rigid conjunctive semiconic idempotent residuated lattices.

In order to lift the amalgamation result from the conic to the semiconic case, we require a variant of the following theorem.

**Theorem 5.9** ([39, Theorem 9]). Let \( S \) be a subclass of a variety \( V \) satisfying the following conditions:

1. \( S \) contains all subdirectly irreducible members of \( V \);
2. \( S \) is closed under isomorphisms and subalgebras;
3. For any algebra \( B \in V \) and subalgebra \( A \) of \( B \), if \( \Theta \) is a congruence of \( A \) and \( A/\Theta \in S \), then there exists a congruence \( \Psi \) of \( B \) such that \( \Psi \cap A^2 = \Theta \) and \( B/\Psi \in S \);
4. \( S \) has the amalgamation property in \( V \).

Then \( V \) has the amalgamation property.
To obtain a version of the above for the strong amalgamation property, we exploit the link between strong amalgamation and epimorphism surjectivity. A class \( K \) of similar algebras is said to have epimorphism surjectivity or the ES property if every epic homomorphism between members of \( K \) is surjective. A subalgebra \( A \) of \( B \in K \) is said to be \( K \)-epic if the inclusion homomorphism \( A \hookrightarrow B \) is an epic homomorphism, i.e., if for any \( C \in K \) and any homomorphisms \( h, k : B \to C \) we have that \( h \upharpoonright_A = k \upharpoonright_A \) implies \( h = k \). Observe that if \( A, B \in K \) and \( h : A \to B \) is an \( K \)-epic homomorphism, then \( h(A) \) is a \( K \)-epic subalgebra of \( B \). Consequently, if \( K \) is closed under taking subalgebras, then \( K \) has the ES property if and only if no \( B \in K \) has a proper \( K \)-epic subalgebra.

The following is an immediate consequence of [32, Corollary 2.5.20]

**Lemma 5.10.** Suppose that \( K \) is a class of similar algebras. If \( K \) has the strong amalgamation property, then \( K \) has the ES property.

We will also make use of the following results.

**Lemma 5.11** ([32, Corollary 2.5.23]). Suppose that \( K \) is a class of similar algebras that is closed under subalgebras and direct products. Then \( K \) has the strong amalgamation property if and only if \( K \) has the amalgamation property and the ES property.

**Lemma 5.12** ([6, Theorem 22]). Let \( V \) be an arithmetical variety such that its class \( V_{FSI} \) of finitely subdirectly irreducible members is a universal class. Then \( V \) has the ES property if and only if \( V_{FSI} \) has the ES property.

The next result puts the previous facts together to obtain a variant of Theorem 5.9 for strong amalgamation.

**Theorem 5.13.** Let \( V \) be an arithmetical variety, and denote by \( V_{FSI} \) its class of finitely subdirectly irreducible members. Suppose that:

1. \( V_{FSI} \) forms a universal class;
2. For any algebra \( B \in V \) and subalgebra \( A \) of \( B \), if \( \Theta \) is a congruence of \( A \) and \( A/\Theta \in V_{FSI} \), then there exists a congruence \( \Psi \) of \( B \) such that \( \Psi \cap A^2 = \Theta \) and \( B/\Psi \in V_{FSI} \);
3. \( V_{FSI} \) has the strong amalgamation property in \( V \).

Then \( V \) has the strong amalgamation property.

**Proof.** By Lemma 5.11 it suffices to show that \( V \) has both the amalgamation property and the ES property.

We show first that \( V \) has the amalgamation property by using Theorem 5.9 for \( S = V_{FSI} \). Because each subdirectly irreducible is finitely subdirectly irreducible, condition (1) of Theorem 5.9 is satisfied. Condition (2) of Theorem 5.9 is satisfied because \( V_{FSI} \) forms a universal class by the hypotheses. Conditions (3) and (4) of Theorem 5.9 are satisfied directly from the assumptions, noting that \( V_{FSI} \) having the strong amalgamation property in \( V \) implies that in particular \( V_{FSI} \) has the amalgamation property in \( V \). It follows that \( V \) has the amalgamation property.

To conclude the proof, we show that \( V \) has the ES property. To do so, we show that \( V_{FSI} \) has the ES property and apply Lemma 5.12. Let \( B \in V_{FSI} \) and suppose that \( A \) is a proper subalgebra of \( B \). As \( V_{FSI} \) is a universal class, we have \( A \in V_{FSI} \). Renaming elements as necessary, let \( C \) be an isomorphic copy of \( B \) with \( B \cap C = A \) and let \( h : B \to C \) be an isomorphism with \( h[A] = A \).
Then $C \in \mathcal{V}_{FSI}$. By assumption, the reduced $V$-formation $(A, B, C)$ has a strong amalgam $D \in \mathcal{V}$. Let $s : D \to \prod_{i \in I} D_i$ be a subdirect representation of $D$ where each $D_i$ is subdirectly irreducible (and hence in $\mathcal{V}_{FSI}$). Because $A$ is a proper subalgebra of $B$, there exists $x \in B \setminus A$. Since $h(A) = A$ we have $h(x) \in C \setminus A$. Since $B \cap C = A$, we have $h(x) \neq x$. Thus $s(h(x)) \neq s(x)$, and consequently there exists $i \in I$ such that $s(h(x))(i) \neq s(x)(i)$. Let $\pi : D \to D_i$ be the canonical projection map, and let $j : B \to D$ be the inclusion map. By construction, the maps $\pi \circ s \circ h, \pi \circ s \circ j : B \to D_i$ are equal on $A$, but $(\pi \circ s \circ h)(x) \neq (\pi \circ s \circ j)(x)$. Thus $A$ is not a $\mathcal{V}_{FSI}$-epic subalgebra of $B$. It follows that $B$ has no proper $\mathcal{V}_{FSI}$-epic subalgebras, whence $\mathcal{V}_{FSI}$ has the ES property as desired.

Theorem 5.14. Let $\mathcal{V}$ be a variety of semicomponent idempotent residuated lattices, and suppose that $\mathcal{V}_{FSI}$ has the strong amalgamation property in $\mathcal{V}$. Then $\mathcal{V}$ has the strong amalgamation property, and hence the ES property.

Proof. We deploy Theorem 5.13. The variety $\mathcal{V}$ is arithmetical by [25, p. 94]. By Lemma 5.7, $\mathcal{V}_{FSI}$ consists precisely of the conic algebras in $\mathcal{V}$ for which 1 is join-irreducible (i.e., satisfying the universal sentence $(\forall x, y)(x \lor y = 1 \Rightarrow x = 1 \lor y = 1)$, so it is a universal class and thus condition (1) of Theorem 5.13 is satisfied. Condition (3) of Theorem 5.13 holds by assumption. For condition (2), suppose that $A$ and $B$ are semicomponent idempotent residuated lattices, that $A$ is a subalgebra of $B$, and that $\Theta$ is a congruence of $A$ such that $A/\Theta$ is finitely subdirectly irreducible, hence also conic by Lemma 3.7.

Let $F$ be the congruence filter of $A$ corresponding to $\Theta$. Since $F_B := \uparrow_B F$ is a filter of $B$ containing 1, it follows from Lemma 3.2(3) that $(F_B) = \uparrow_B t_n(y) : n \in \mathbb{N}, y \in F_B$. Since $t_n$ is monotone this is equal to $\uparrow_B \{t_n(y) : n \in \mathbb{N}, y \in F\}$ and since $F$ is closed under $t_n$ it is equal to $\uparrow_B F = F_B$. Thus $F_B$ is a congruence filter of $B$. Also note that $F_B \cap A = F$, because if $y \in F_B \cap A$, then $y \in A$ and $x \leq y$ for some $x \in F$, so $y \in F$. Therefore, $F_B$ is an element of the poset

$$P = \{G \subseteq B : G \text{ is a congruence filter of } B \text{ and } F = G \cap A\},$$

which is, therefore, non-empty.

By Zorn’s Lemma, $P$ has a maximal element $G$. We will show that $G$ is prime. Toward a contradiction, let $x, y \in B$ such that $x \lor y \in G$, $x \not\in G$, and $y \not\in G$. Since $1 \in G$ and $B$ satisfies distributivity at 1, we may assume without loss of generality that $x$ and $y$ are negative (meeting with 1 if necessary). By the maximality of $G$ in $P$, each of the congruence filters $\langle G \cup \{x\} \rangle$ and $\langle G \cup \{y\} \rangle$ is not in $P$. Thus each of $H_x = \langle G \cup \{x\} \rangle \cap A$ and $H_y = \langle G \cup \{y\} \rangle \cap A$ properly contains $F$, whence there exist elements $a \in H_x \setminus F$ and $b \in H_y \setminus F$. Without loss of generality we may assume that $a$ and $b$ are both negative (considering the result of meeting with 1 if necessary). By Lemma 3.2(4), there exists $g, h \in G^-$ and $m, n \in \mathbb{N}$ such that $g \land s_m(x) \leq a$ and $h \land s_n(x) \leq b$. Now since $x \lor y \in G$, we have that $[1]_G \leq [x]_G \lor [y]_G$, and since we chose $x$ and $y$ to be negative we have $[x]_G \lor [y]_G = [1]_G$. By Lemma 3.5(5), it follows that $[s_m(x)]_G \lor [s_n(y)]_G = [1]_G$, so:

$$[1]_G = [s_m(x)]_G \lor [s_n(y)]_G = [g \land s_m(x)]_G \lor [h \land s_n(y)]_G \leq [a]_G \lor [b]_G.$$
\[ b\hat{f} = [1]_F, \text{ so } a \in F \text{ or } b \in F. \] This is a contradiction to the choice of \( a \) and \( b \), so it follows that \( G \) is prime.

To complete the proof, observe that \( G \) being prime implies that \([1]_G\) is join-irreducible in the quotient \( B/G \), so by Lemma 5.7 we have \( B/G \in V_{FSI} \). □

The following is immediate from combining Theorem 5.14 with Theorem 5.8.

**Theorem 5.15.** The variety of rigid conjunctive semiconic idempotent residuated lattices has the strong amalgamation property, hence the ES property.

5.7. **Subvarieties.** The method of obtaining Theorem 5.15 applies, mutatis mutandis, to a host of subvarieties. We do not catalog these exhaustively, and only discuss a few prominent examples. Most notably, since \( ^\ast \)-involutive idempotent residuated chains are finitely subdirectly irreducible, rigid (by Lemma 4.19), and conjunctive, we immediately obtain strong amalgamation for the corresponding semilinear variety by combining Theorems 5.7 and 5.14.

**Theorem 5.16.** The variety of \( ^\ast \)-involutive semilinear idempotent residuated lattices has the strong amalgamation property, and hence the ES property.

The commutative \( ^\ast \)-involutive idempotent residuated chains are exactly the odd Sugihara monoids, so in the commutative setting Theorems 5.7 and 5.10 are well-known (see e.g., [27]). The fact that odd Sugihara monoid chains admit strong amalgamation allows us to prove the follow result:

**Theorem 5.17.** The variety of commutative conjunctive semiconic idempotent residuated lattices has the strong amalgamation property, hence the ES property.

**Proof.** Observe that every commutative semiconic idempotent residuated lattice is rigid. Moreover, it follows from Corollary 2.21 that a conic idempotent residuated lattice is commutative iff its quasi-involutive skeleton is commutative (i.e., an odd Sugihara monoid). Inspection of the proofs of Theorems 5.7 and Theorem 5.8 shows that commutativity is preserved in taking the amalgams in each case, so the result follows by Theorem 5.14. □

Because distributive lattices do not have the strong amalgamation property [16], we cannot generally apply the same methodology to distributive subvarieties of rigid conjunctive semiconic idempotent residuated lattices. However, we may obtain the amalgamation property:

**Theorem 5.18.** Each of the following subvarieties of rigid conjunctive semiconic idempotent residuated lattices has the amalgamation property.

1. The variety of distributive, rigid, and conjunctive semiconic idempotent residuated lattices.
2. The variety of distributive, commutative, and conjunctive semiconic idempotent residuated lattices.

**Proof.** 1. Let \( A \) be a rigid conjunctive conic idempotent residuated lattice, and let \((S_A, \{A_s : s \in S_A\})\) be its decomposition system. Then \( A \) is distributive iff each \( A_s, s > 1, \) is a distributive lattice with designated top element. Because the variety of distributive lattices with top element has the amalgamation property, by repeating the proof of Theorem 5.8 (and taking amalgams rather than strong amalgams of positive blocks) we obtain that the class of distributive, rigid, and conjunctive conic idempotent residuated lattices has the amalgamation property. Lifting the
amalgamation property to the variety of distributive, rigid, and conjunctive semiconic idempotent residuated lattices is a routine application of Theorem 5.9 using the proof of Theorem 5.14 to establish Condition 3 of Theorem 5.9.

2. This follows immediately from the proofs of 1 and Theorem 5.17. □

Although topped distributive lattices only have the amalgamation property, the trivial subvariety has strong amalgamation thanks to the paucity of V-formations. The recent paper [7] studies the subvariety of commutative semiconic idempotent residuated lattices that satisfy the identity

\[(x \lor 1)^{**} = x \lor 1,\]

called semionic generalized Sugihara monoids. Although it is not discussed in [7], the conic members of this subvariety can be specified by stipulating that strictly positive blocks in the decomposition system are trivial:

**Proposition 5.19.** Let \(A\) be a conic idempotent residuated lattice, and let \((S, \{A_s : s \in S\})\) be its decomposition system. Then \(A\) is a semionic generalized Sugihara monoid if and only if \(S\) is an odd Sugihara monoid and \(A_s = \{s\}\) for each strictly positive \(s \in S\).

**Proof.** Suppose that \(A\) is a semionic generalized Sugihara monoid. The skeleton \(S\) of \(A\) is an odd Sugihara monoid by commutativity. Moreover, if \(s \in S\) with \(s > 1\) and \(a \in A\) with \(a^{**} = s\), then \(a = a \lor 1\) and hence \(s = a^{**} = (a \lor 1)^{**} = a \lor 1 = a\), so \(A_s = \{s\}\).

Conversely, suppose \(S\) is an odd Sugihara monoid and \(A_s = \{s\}\) for every strictly positive \(s \in S\). That \(A\) is commutative follows from the fact that its skeleton is commutative. If \(a \in A\), then either \(a \lor 1 = 1\) or \(a \lor 1 > 1\). If \(a \lor 1 = 1\), then \((a \lor 1)^{**} = 1 = a \lor 1\) is immediate. If \(a \lor 1 > 1\), then \(s = (a \lor 1)^{**}\) being a strictly positive element of \(S\) implies \(A_s = \{s\}\), so \(a \lor 1 = s = (a \lor 1)^{**}\). □

The conic members of the subvariety of central semionic generalized Sugihara monoids, axiomatized relative to semionic generalized Sugihara monoids by

\[1 \leq x^{**} \lor (x^{**} \to x),\]

may further be characterized by stipulating that the only non-trivial block is the block of the identity element:

**Proposition 5.20.** Let \(A\) be a conic idempotent residuated lattice, and let \((S, \{A_s : s \in S\})\) be its decomposition system. Then \(A\) is a central semionic generalized Sugihara monoid if and only if \(S\) is an odd Sugihara monoid and \(A_s = \{s\}\) for each \(s \in S \setminus \{1\}\).

**Proof.** Observe that if \(A\) is a central semionic generalized Sugihara monoid and \(a^{**} = s < 1\), where \(s \in S\), then \(1 \leq a^{**} \lor (a^{**} \to a)\) implies \(1 \leq a^{**} \to a\) by the conicity of \(a^{**}\). Hence \(s = a^{**} = a\), so \(A_s = \{s\}\). Conversely, if \(A_s = \{s\}\) for all \(s < 1\), then an easy calculation shows that \(1 \leq a^{**} \lor (a^{**} \to a)\) for all \(a \in A\). The rest follows from Proposition 5.19. □

By using a categorical equivalence generalizing [28], [7] obtains the strong amalgamation property for the variety of central semionic generalized Sugihara monoids. However, it does not discuss amalgamation of semionic generalized Sugihara monoids.
beyond the central case. Because the conic members in each variety can be described by stipulating that certain blocks are trivial, it follows from the results of this section that both varieties have the strong amalgamation property:

**Corollary 5.21.** Each of the following varieties has the strong amalgamation property, hence the ES property.

1. The variety of semiconic generalized Sugihara monoids.
2. The variety of central semiconic generalized Sugihara monoids.

**Proof.** Both results follow from applying Theorem 5.14 and by noting that in each case the amalgam obtained in the proof of Theorem 5.8 may be taken to preserve trivial blocks. □

6. **Logical applications**

Our study to this point has been entirely algebraic, but our main motivations come from substructural logic. Here we record the logical applications of the work of the previous sections. First, we recall some of the elements of abstract algebraic logic (referring to [5, 15] for more background information). Let \( \mathcal{L} \) be an algebraic language, and write \( \text{Fm}_\mathcal{L} \) for the set of \( \mathcal{L} \)-terms (which we will understand as formulas of a given sentential logic). A **consequence relation** over \( \mathcal{L} \) is a relation \( \vdash \subseteq \mathcal{P}(\text{Fm}_\mathcal{L}) \times \text{Fm}_\mathcal{L} \) from sets of formulas to formulas that satisfies the following four properties, where \( \Gamma \cup \Pi \cup \{ \varphi \} \subseteq \text{Fm}_\mathcal{L} \):

1. If \( \varphi \in \Gamma \), then \( \Gamma \vdash \varphi \).
2. If \( \Gamma \vdash \varphi \) and \( \Gamma \subseteq \Pi \), then \( \Pi \vdash \varphi \).
3. If \( \Gamma \vdash \varphi \) and \( \Pi \vdash \psi \) for every \( \psi \in \Gamma \), then \( \Pi \vdash \varphi \).
4. If \( \Gamma \vdash \varphi \), then \( \sigma[\Gamma] \vdash \sigma(\varphi) \) for every substitution \( \sigma \).

A **deductive system** is a consequence relation that is moreover **finitary**: If \( \Gamma \vdash \varphi \), then there exists a finite subset \( \Gamma' \subseteq \Gamma \) such that \( \Gamma' \vdash \varphi \). A **finitary sentential logic** is a pair \( (\mathcal{L}, \vdash) \) consisting of an algebraic language \( \mathcal{L} \) and a deductive system \( \vdash \) over \( \mathcal{L} \), but informally we will refer to any deductive system as a **logic**. If \( \vdash \) is a logic and \( \varphi \) is a formula with \( \emptyset \vdash \varphi \), then we write \( \vdash \varphi \) and say that \( \varphi \) is a **theorem** of the logic. If \( \varphi \) is a formula and \( \Gamma \) is a set of formulas, then we write \( \text{Var}(\varphi) \) for the set of variables appearing in \( \varphi \) and \( \text{Var}(\Gamma) = \bigcup_{\psi \in \Gamma} \text{Var}(\psi) \).

The deductive systems that we deal with here are especially well-behaved. A deductive system \( \vdash \) is said to be **equivalential** if there exists a set of formulas \( \Delta \) in at most two free variables such that, for variables \( x, y, x_1, \ldots, x_n, y_1, \ldots, y_n, \ldots \):

1. \( \vdash \Delta(x, x) \).
2. \( \{ x \} \cup \Delta(x, y) \vdash y \).
3. \( \bigcup_{i=1}^{n} \Delta(x_i, y_i) \vdash \Delta(r(x_1, \ldots, x_n), r(y_1, \ldots, y_n)) \) for any \( n \)-ary function symbol \( r \) (read here as an \( n \)-ary connective of the logic).

Here we have used the abbreviation that \( \Delta(x, y) = \{ d(x, y) : d \in \Delta \} \). In an equivalential logic, the set of formulas \( \Delta \) intuitively captures being logically equivalent on a syntactic level; this is intended to generalize the behavior of the singleton \( \Delta(x, y) = \{ x \leftrightarrow y \} \) in classical propositional logic, which witnesses that the latter is equivalential.

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2For our purposes, a **substitution** is just an endomorphism of the absolutely free algebra \( \text{Fm}_\mathcal{L} \) over \( \mathcal{L} \).
Being algebraizable is even stronger than being equivalential: Every algebraizable deductive system is equivalential by [5]. We do not give a full account of algebraizability here, but refer the reader to [5,15] for a more detailed discussion. We note, however, that if $K$ is a quasivariety of residuated lattices and $L = \{\land,\lor,\cdot,\vee,\top,\bot,1\}$, then we may naturally define a deductive system $\vdash_K$ by stipulating, for $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_L$: $\Gamma \vdash_K \varphi$ iff there exists a finite subset $\Gamma' \subseteq \Gamma$ such that the quasiequation

$$\land_{\varphi \in \Gamma'} 1 = 1 \land \psi \Rightarrow 1 = 1 \land \varphi$$

is valid in $K$. As proven in [25], residuated lattices give the equivalent algebraic semantics (in the sense of Blok and Pigozzi [5]) of extensions of the 0-free Full Lambek. Loosely speaking, this means that every deductive system corresponding to an extension of the 0-free Full Lambek is equivalent to one of the form $\vdash_K$ for some quasivariety $K$ of residuated lattices. Axiomatic extensions of the 0-free Full Lambek calculus correspond to $\vdash_V$ where $V$ is a variety of residuated lattices. Since varieties of residuated lattices are the algebraic counterpart of substructural logics in the aforementioned sense, the results of the previous section may be interpreted in terms of the corresponding axiomatic extensions of the Full Lambek calculus.

Note that in the fact that these logics are equivalent is witnessed by the set $\Delta = \{x \land y \land \neg y \lor x\}$. We indicate the logical meaning of the this paper’s results presently, but before doing so outline some natural and well-studied properties of deductive systems.

First, we say that a deductive system $\vdash$ has the (deductive) interpolation property if whenever $\Gamma \vdash \varphi$, there exists a set of formulas $\Gamma' \subseteq \text{Fm}_L$ with $\text{Var}(\Gamma') \subseteq \text{Var}(\Gamma) \cap \text{Var}(\varphi)$ and $\Gamma \vdash \Gamma'$, $\Gamma' \vdash \varphi$. Intuitively, the interpolant $\Gamma'$ gives a ‘reason’ for why $\varphi$ follows from $\Gamma$ within the logic $\vdash$.

Next, we say that a deductive system $\vdash$ has a local deduction theorem if there exists a family $\Lambda$ of sets of binary formulas such that, for all $\Gamma \cup \{\varphi,\psi\}$, $\Gamma, \varphi \vdash \psi$ iff there exists $\lambda \in \Lambda$ such that for all $\ell \in \lambda$, $\Gamma \vdash l(\varphi,\psi)$. When $\Lambda$ is a singleton, we say simply that $\vdash$ has a deduction theorem. This is the case for classical propositional logic, where we may take the sole member $\Lambda$ to be $\{x \rightarrow y\}$.

Now suppose that $\vdash$ is an equivalential deductive system, and let $X,Y,Z$ be pairwise disjoint sets of variables with $X \neq \emptyset$. Let $\Gamma \subseteq \text{Fm}_L$ with $\text{Var}(\Gamma) \subseteq X \cup Y \cup Z$. The set of formulas $\Gamma$ is said to implicitly define $Z$ in terms of $X$ via $Y$ if for every $z \in Z$ and every substitution $\sigma$ with $\sigma(x) = x$ for all $x \in X$, we have $\Gamma \cup \sigma[\Gamma] \vdash \Delta(z,\sigma(z))$. When $Y = \emptyset$, we simply say that $\Gamma$ implicitly defines $Z$ in terms of $X$. We also say that $\Gamma$ explicitly defines $Z$ in terms of $X$ via $Y$ if for every $z \in Z$ there exists a formula $t_z$ with $\text{Var}(t_z) \subseteq X$ such that $\Gamma \vdash \Delta(z,t_z)$. When $Y = \emptyset$, we just say that $\Gamma$ explicitly defines $Z$ in terms of $X$.

We say that an equivalential deductive system $\vdash$ has the infinite Beth definability property if for any $\Gamma \subseteq \text{Fm}_L$ with $\text{Var}(\Gamma) \subseteq X \cup Z$, we have that if $\Gamma$ implicitly defines $Z$ in terms of $X$, then $\Gamma$ also explicitly defines $Z$ in terms of $X$. On the other hand, $\vdash$ is said to have the finite Beth definability property if we add that $Z$ is finite in the previous definition. Furthermore, $\vdash$ has the projective Beth definability property if for any $\Gamma \subseteq \text{Fm}_L$ with $\text{Var}(\Gamma) \subseteq X \cup Y \cup Z$, if $\Gamma$ implicitly defines $Z$ in terms of $X$ via $Y$, then $\Gamma$ also explicitly defines $Z$ in terms of $X$ via $Y$. 
The algebraic study of deductive systems is motivated, in part, by correspondences between properties like those just introduced and various associated algebraic properties. ‘Bridge theorems’ announcing connections of this sort may be found throughout the literature. We will use several of these bridge theorems, the first of which may be found in [4]:

**Theorem 6.1.** Let $\vdash$ be an algebraizable deductive system whose equivalent algebraic semantics is the variety $\mathcal{V}$. Then $\mathcal{V}$ has a local deduction theorem iff $\mathcal{V}$ has the congruence extension property.

The presence of a local deduction theorem simplifies a number of other algebra-to-logic links. In particular, we will use the following result proven in [12]:

**Theorem 6.2.** Let $\vdash$ be an algebraizable deductive system with a local deduction theorem, and suppose that the variety $\mathcal{V}$ is its equivalent algebraic semantics. Then $\vdash$ has the deductive interpolation property iff $\mathcal{V}$ has the amalgamation property.

In Section 5, we established that several varieties have the epimorphism-surjectivity property by proving that they have the strong amalgamation property. In fact, the strong amalgamation property entails the strong epimorphism-surjectivity property (see the discussion in [28]): A variety $\mathcal{V}$ has the strong epimorphism-surjectivity property if whenever $B \in \mathcal{V}$, $A$ is a subalgebra of $B$, and $b \in B - A$, then there exists $C \in \mathcal{V}$ and homomorphisms $f, g : B \to C$ such that $f \upharpoonright A = g \upharpoonright A$ but $f(b) \neq g(b)$. The following result from [31] links strong epimorphism-surjectivity to Beth definability:

**Theorem 6.3.** Let $\vdash$ be an algebraizable deductive system and suppose that the variety $\mathcal{V}$ is its equivalent algebraic semantics. Then $\vdash$ has the projective Beth definability property iff $\mathcal{V}$ has the strong epimorphism-surjectivity property.

For brevity, we now assign some names to several of the varieties of this study:

- $S$ denote the variety of semiconic idempotent residuated lattices.
- $T$ denote the variety of semilinear idempotent residuated lattices.
- $R$ denote the variety of rigid, conjunctive semiconic idempotent residuated lattices.
- $T^*$ denote the variety of $*$-involutive semilinear idempotent residuated lattices.
- $\mathcal{DR}$ denotes the variety of distributive, rigid, and conjunctive semiconic idempotent residuated lattices.
- $\mathcal{CR}$ denotes the variety of commutative, rigid, and conjunctive semiconic idempotent residuated lattices.
- $\mathcal{DCR}$ denotes the variety of distributive, commutative, and conjunctive semiconic idempotent residuated lattices.
- $\mathcal{SGSM}$ denotes the variety of semiconic generalized Sugihara monoids.
- $\mathcal{CSGSM}$ denotes the variety of central semiconic generalized Sugihara monoids.

For each of the listed varieties $\mathcal{V}$, $\vdash_\mathcal{V}$ is an axiomatic extension of the 0-free Full Lambek calculus algebraized by $\mathcal{V}$.

The following result is immediate from Corollary 3.4, Lemma 3.2, and Theorem 6.1:

**Theorem 6.4.** Every extension of $\vdash_S$ has a local deduction theorem. In particular, this holds for $\vdash_T$, the logic algebraized by semilinear idempotent residuated lattices.
By applying Theorem 6.2 and the previous theorem, we obtain from Theorems 5.15, 5.16, 5.17, 5.18, and Corollary 5.21 the following result:

**Theorem 6.5.** Each of $\vdash_R, \vdash_{T^*}, \vdash_{DR}, \vdash_{CR}, \vdash_{SGSM},$ and $\vdash_{CSGSM}$ has the deductive interpolation property.

We may apply Theorem 6.3 to all but two of the logics mentioned in the previous theorem, arriving at our final result:

**Theorem 6.6.** Each of $\vdash_R, \vdash_{T^*}, \vdash_{CR}, \vdash_{SGSM},$ and $\vdash_{CSGSM}$ has the projective Beth definability property.

**References**

[1] Agliano, P. and Montagna, F.: Varieties of BL-algebras. I. General properties. J. Pure Appl. Algebra 181, 105–129 (2003).

[2] Anderson, M. and Feil, T.: *Lattice-Order Groups: An Introduction*. Reidel Texts in the Mathematical Sciences, vol. 4. Springer Dordrecht (1988).

[3] Bahls, P., Cole, J., Galatos, N., Jipsen, P., and Tsinakis, C.: Cancellative residuated lattices. Algebra Universalis 50, 83–106 (2003).

[4] Blok, W. and Pigozzi, D.: Local deduction theorems in algebraic logic. In: AndrÉka, H., Monk, J., Ne
temi, I. (eds.), Algebraic Logic, Colloquia Mathematica Societatis János Bolyai 54, pp. 75–109. Budapest, Hungary (1988).

[5] Blok, W. and Pigozzi, D.: Algebraizable logics. Mem. Amer. Math. Soc. 73 (1989).

[6] Campercholi, M.A.: Dominions and primitive positive functions. J. Symbolic Logic 83, 40–54 (2018).

[7] Chen, W.: On semiconic idempotent commutative residuated lattices. Algebra Universalis 81, paper 36 (2020).

[8] Chen, W. and Chen, Y.: Variety generated by conical residuated lattice-ordered idempotent monoids. Semigroup Forum 98, 431–455 (2019).

[9] Chen, W. and Zhao, X.: The structure of idempotent residuated chains. Czechoslovak Math. J. 59, 453–479 (2009).

[10] Chen, W., Zhao, X., and Guo, X.: Conical residuated lattice-ordered monoids. Semigroup Forum 79, 244–278 (2009).

[11] Cignoli, R. L. O., D’Ottaviano, I. M. L., and Mundici, D.: *Algebraic Foundations of Many-Valued Reasoning*, Trends in Logic, Studia Logica Library, vol. 7. Kluwer Academic Publishers, Dordrecht (2000).

[12] Czelakowski, J. and Pigozzi, D.: Amalgamation and interpolation in abstract algebraic logic. In: Caicedo, X., Montenegro, C.H. (eds.), Models, Algebras, and Proofs, Lecture Notes in Pure and Applied Mathematics, vol. 203, pp. 187–265. Marcel Dekker (1999).

[13] Davey, B.A. and Priestley, H.A.: *Introduction to Lattices and Order*, 2nd Edition. Cambridge University Press (2002).

[14] Day, A. and Jejek, J.: The amalgamation property for varieties of lattices. Trans. Amer. Math. Soc. 286, 251–256 (1984).

[15] Font, J.: *Abstract Algebraic Logic: An Introductory Textbook*. College Publications (2016).

[16] Fried, E. and Grätzer, G.: Strong amalgamation in distributive lattices. J. Algebra 128, 446–455 (1990).

[17] Fussner, W.: Poset products as relational models. Studia Logica 110, 95–120 (2021).

[18] Fussner, W., and Galatos, N.: Categories of models of R- mingle. Ann. Pure Appl. Logic 170, 1188–1242 (2019).

[19] Fussner, W. and Ugolini, S.: A topological approach to MTL-algebras. Algebra Universalis 80, paper 38 (2019).

[20] Fussner, W., and Zuluaga Botero, W.: Some modal and temporal translations of generalized basic logic. In: Fahrenberg, U. et al. (eds.), *Proceedings of the 19th International Conference on Relational and Algebraic Methods in Computer Science* (RAMiCS), 2021, LNCS 13027, pp. 176–191.

[21] Fussner, W., and Metcalfe, G.: Transfer theorems for finitely subdirectly irreducible algebras. Manuscript, 2022. Available at [arXiv:2205.00148](https://arxiv.org/abs/2205.00148).
[22] Galatos, N.: Equational bases for joins of residuated-lattice varieties, Studia Logica 76(2) (2004), 227–240.
[23] Galatos, N.: Minimal varieties of residuated lattices. Algebra Universalis 52(2), 215–239 (2005).
[24] Galatos, N. and Jipsen, P.: Residuated frames with applications to decidability. Trans. Amer. Math. Soc. 365, 1219–1249 (2013).
[25] Galatos, N., Jipsen, P., Kowalski, T., and Ono, H.: Residuated Lattices: An Algebraic Glimpse at Substructural Logics. Elsevier (2007).
[26] Galatos, N. and Ono, H.: Algebraization, parametrized local deduction theorem and interpolation for substructural logics over FL. Studia Logica 83, 279–308 (2006).
[27] Galatos, N., and Raftery, J.G.: A category equivalence for odd Sugihara monoids and its applications. J. Pure Appl. Algebra 216, 2177–2192 (2012).
[28] Galatos, N., and Raftery, J.G.: Idempotent residuated structures: Some category equivalences and their applications. Trans. Amer. Math. Soc. 367, 3189–3223 (2014).
[29] Galatos, N. and Jipsen, P.: The structure of generalized BI-algebras and weakening relation algebras. Algebra Universalis 81, 35 (2020).
[30] Gil-Ferez, J., Jipsen, P., and Metcalfe, G.: Structure theorems for idempotent residuated lattices. Algebra Universalis 81, paper 28 (2020).
[31] Hoogland, E.: Algebraic characterizations of various Beth definability properties. Studia Logica 65, 91–112 (2000).
[32] Hoogland, E.: Definability and interpolation: Model-theoretic investigations. Ph.D. thesis, Institute for Logic, Language and Computation, University of Amsterdam (2001).
[33] Horčík, R., Noguera, C., and Petrik, M.: On n-contractive fuzzy logics. Math. Log. Q. 53, 268–288 (2007).
[34] Hsieh, A., and Raftery, J.G.: Semiconic idempotent residuated structures. Algebra Universalis 61, 413–430 (2009).
[35] Jipsen, P. and Montagna, F.: Embedding theorems for classes of GBL-algebras. J. Pure Appl. Algebra 214, 1559–1575 (2010).
[36] Jipsen, P., Tuyt, O, and Valota, D.: The structure of finite commutative idempotent involutive residuated lattices. Algebra Universalis 82, paper 57 (2021).
[37] Jónsson, B.: Universal relational systems. Math. Scand. 4, 193–208 (1956).
[38] Kiss, E.W., Márki, L., Pröhle, P., and Tholen, W.: Categorical algebraic properties: A compendium of amalgamation, congruence extension, epimorphisms, residual smallness, and injectivity. Stud. Sci. Math. Hung. 18, 79–141 (1983).
[39] Metcalfe, G., Montagna, F., and Tsinakis, C.: Amalgamation and interpolation in ordered algebras. J. Algebra 402, 21–82 (2014).
[40] Olson, J.S.: The subvariety lattice for representable idempotent commutative residuated lattices. Algebra Universalis 67, 43–58 (2012).
[41] Olson, J. and Raftery, J.: Positive Sugihara monoids. Algebra Universalis 57, 75–99 (2007).
[42] Paoli, F.: Substructural Logics: A Primer. Springer Dordrecht (2002).
[43] Raftery, J.: Representable idempotent commutative residuated lattices. Trans. Amer. Math. Soc. 359, 4405–4427 (2007).
[44] Stanoský, D.: Commutative idempotent residuated lattices. Czechoslovak Math. J. 57, 191–200 (2007).
[45] Torrens, A.: W-algebras which are Boolean products of members of SR[1] and CW-algebras. Studia Logica 46, 265–274 (1987).