Invariant Submanifolds of Hyperbolic Sasakian Manifolds and \( \eta \)-Ricci-Bourguignon Solitons

Sudhakar K Chaubey\(^a\), M. Danish Siddiqi\(^b\), D. G. Prakasha\(^c\)

\(^a\) Section of Mathematics, Department of Information Technology, University of Technology and Applied Sciences-Shinas, P.O. Box 77 Postal Code 324, Oman
\(^b\) Department of Mathematics, College of Science, Jazan University, Jazan, Saudi Arabia
\(^c\) Department of Studies in Mathematics, Davangere University, Shivagangothri Campus, Davangere - 577 007, India

Abstract. We set the goal to study the properties of invariant submanifolds of the hyperbolic Sasakian manifolds. It is proven that a three-dimensional submanifold of a hyperbolic Sasakian manifold is totally geodesic if and only if it is invariant. Also, we discuss the properties of \( \eta \)-Ricci-Bourguignon solitons on invariant submanifolds of the hyperbolic Sasakian manifolds. Finally, we construct a non-trivial example of a three-dimensional invariant submanifold of five-dimensional hyperbolic Sasakian manifold and validate some of our results.

1. Introduction

The concept of Ricci-Bourguignon flow as an extension of Ricci flow [22] has been introduced by J. P. Bourguignon [10] based on some unprinted article of Lichnerowicz and a paper of Aubin [2]. Ricci-Bourguignon flow is an intrinsic geometric flow on Riemannian manifolds, whose fixed points are solitons. Therefore, the Ricci-Bourguignon solitons generate self-similar solution to the Ricci-Bourguignon flow [10]:

\[
\frac{\partial g}{\partial t} = -2(Ric - \rho g), \quad g(0) = g_0,
\]

where \( Ric \) is the Ricci curvature tensor, \( R \) is the scalar curvature with respect to the semi-Riemannian metric \( g \) and \( \rho \) is a non-zero real constant. It should be noticed that for special values of the constant \( \rho \) in equation (1), we obtain the following situations for the tensor \( Ric - \rho g \) appearing in equation (1). In particular [10], we have

(i) for \( \rho = \frac{1}{2} \), the Einstein tensor \( Ric - \frac{R}{2} g \) (for Einstein soliton [5]),
(ii) for \( \rho = \frac{1}{n} \), the traceless Ricci tensor \( Ric - \frac{R}{n} g \),
(iii) for \( \rho = \frac{1}{2(n-1)} \), the Schouten tensor \( Ric - \frac{K}{2(n-1)} g \) (for Schouten soliton [10]),
In fact, for sufficiently small $t$, the equation (1) has a unique solution for $\rho < \frac{1}{2(\nu - 1)}$.

On the other hand, quasi-Einstein metrics or Ricci solitons serve as solutions to Ricci flow equation $\frac{\partial g}{\partial t} = -2\text{Ric}$, $g(0) = g_0$ [11, 22]. Aubin [2] has given the notion of Ricci-Bourguignon flow in a complete Riemannian manifold. Recently, De et al. [19] and Danish [30] have studied the properties of Ricci-Bourguignon solitons. A (semi-)Riemannian manifold of dimension $n \geq 3$ is said to be Ricci-Bourguignon soliton [2] if

$$\frac{1}{2}L_V g + \text{Ric} + (\lambda - \rho \lambda)g = 0,$$

where $L_V$ denotes the Lie derivative operator along the vector field $V$ (called soliton or potential vector field), $\rho$ is a non-zero constant and $\lambda$ is a real constant. Similar to the Ricci soliton, a Ricci-Bourguignon soliton $(M, g, V, \lambda, \rho)$ is called expanding if $\lambda > 0$, steady if $\lambda = 0$ and shrinking if $\lambda < 0$.

Perturbing the equation that define (2) Ricci-Bourguignon solitons by multiple of a certain $(0, 2)$-tensor field $\eta \otimes \eta$, we obtain a slightly more general notion, namely $\eta$-Ricci-Bourguignon solitons [30] such as

$$L_V g + 2\text{Ric} + 2(\lambda - \rho \lambda)g + 2\omega \eta \otimes \eta = 0,$$

where $\omega$ is a real constant and $\eta$ is 1-form. Particularly, if we choose $\rho = 0$ in equation (3), then the $\eta$-Ricci Bourguignon soliton reduces to the $\eta$-Ricci soliton (see [6], [7], [12]-[14], [18], [20], [24], [31], [32], [34]). We say that $(M, g, f, \lambda, \rho)$ is a gradient Ricci-Bourguignon soliton if the potential vector field $V$, defined in (2), is the gradient of some smooth function $f$ on $M$. Here, the soliton equation (2) takes the following form as:

$$\text{Hess} f + \text{Ric} + (\lambda - \rho \lambda)g = 0,$$

where $\text{Hess} f$ is the Hessian of $f$.

Motivated by the contact structure, Upadhyay and Dube [33] introduced the notion of an almost hyperbolic contact $(f, g, \eta, \xi)$-structure. A $(2n+1)$-dimensional differentiable manifold of class $C^\infty$ equipped with the structure $(f, g, \eta, \xi)$ is known as an almost hyperbolic contact manifold. Further, it was studied by number of authors [1, 3, 25, 28]. Let $T_p(M)$ denote the tangent space of the almost hyperbolic contact manifold $M$ at point $p$. Then a vector field $v \in T_p(M)$, $v \neq 0$, is said to be time-like (resp., null, space-like, and non-space-like) if it satisfies $g_0(v, v) < 0$ (resp., $= 0$, $> 0$, and $\leq 0$) [17, 27]). If $\{e_1, e_2, \ldots, e_{2n}, e_{2n+1} = \xi\}$ represents a local orthonormal basis of $M$, then the Ricci tensor $\text{Ric}$ and scalar curvature $R$ of an almost hyperbolic contact metric manifold, respectively, are defined as follows:

$$\text{Ric}(X, Y) = \sum_{i=1}^{2n+1} e_i g(\hat{R}(e_i, X)e_i, e_i) = \sum_{i=1}^{2n} e_i g(\hat{R}(e_i, X)e_i, e_i) - g(\hat{R}(\xi, X)e_i, e_i),$$

$$R = \sum_{i=1}^{2n+1} e_i \text{Ric}(e_i, e_i) = \sum_{i=1}^{2n} e_i \text{Ric}(e_i, e_i) - \text{Ric}(\xi, \xi)$$

for all $X, Y \in TM$, where $e_i = g(e_i, \xi)$, $\xi$ is a unit time-like vector field, $\hat{R}$ represents the curvature tensor of $M$ and $TM$ denotes the tangent bundle of $M$.

We structure our work as follows: Section 2 gathers the basic information of hyperbolic Sasakian manifolds whereas in Section 3 we give some basic tools of submanifolds of the hyperbolic Sasakian manifolds. The properties of invariant submanifolds of the hyperbolic Sasakian manifolds are studied in Section 4. Sections 5 and 6 deal with the study of $\eta$-Ricci-Bourguignon solitons on invariant submanifolds of hyperbolic Sasakian manifolds. We give a non-trivial example of an invariant submanifold of hyperbolic Sasakian manifold in Section 7.
2. Hyperbolic Sasakian manifolds

Let $\tilde{M}$ be an $(n = 2m + 1)$-dimensional differentiable manifold of differentiability class $C^\infty$. Then the structure $(\phi, \xi, \eta)$ satisfying
\[
\phi^2 = I + \eta \otimes \xi,
\]
is said to be an almost hyperbolic contact structure [33], where $I$ denotes the identity transformation and $\phi$, $\eta$, $\xi$ and $\otimes$ are the tensor fields of type $(1,1)$, $(0,1)$, $(1,0)$ and tensor product, respectively. The manifold $\tilde{M}$ equipped with the structure $(\phi, \xi, \eta)$ is called an almost hyperbolic contact manifold. From (7), it is noticed that
\[
\phi \xi = 0, \quad \eta(\xi) = -1 \quad \text{and} \quad \text{rank}(\phi) = n - 1.
\]
If the associated semi-Riemannian metric $g$ of $\tilde{M}$ satisfies
\[
g(X, \xi) = \eta(X) \quad \text{and} \quad g(\phi X, \phi Y) + g(X, Y) + \eta(X)\eta(Y) = 0
\]
for all $X, Y \in T\tilde{M}$, then the structure $(\phi, \xi, \eta, g)$ is called an almost hyperbolic contact metric structures and $\tilde{M}$ with the structure $(\phi, \xi, \eta, g)$ is known as an almost hyperbolic contact metric manifold. From (9), it is obvious that $g(\phi X, Y) = -g(X, \phi Y), \forall X, Y \in T\tilde{M}$.

An almost hyperbolic contact metric manifold is said to be an almost hyperbolic Sasakian manifold if the 2-form defined as $\Phi(X, Y) = g(\phi X, Y)$ satisfies $-2d\Phi = d\eta$, which is equivalent to
\[
(\tilde{\nabla}_X \phi)(Y) = g(X, Y)\xi - \eta(Y)X. \quad \text{Then} \quad \tilde{\nabla}_X \xi = -\phi X,
\]
for $X, Y \in T\tilde{M}$. Here $\tilde{\nabla}$ represents the Levi-Civita connection of $\tilde{M}$. It is obvious from (7)-(10) that a hyperbolic Sasakian manifold satisfies the following relations
\[
\tilde{\nabla}(X, Y)\xi = \eta(Y)X - \eta(X)Y,
\]
\[
\eta(\tilde{\nabla}(X, Y)Z) = \eta(X)g(Y, Z) - \eta(Y)g(X, Z),
\]
\[
\tilde{\nabla}(\xi, X)Y = g(X, Y)\xi - \eta(Y)X
\]
for $X, Y, Z \in T\tilde{M}$.

3. Submanifolds of hyperbolic Sasakian manifolds

Let $M$ be a submanifold immersed in a hyperbolic Sasakian manifold $\tilde{M}$. We use the same notation $g$ for the induced metric of $M$. Let $TM$ be a set of all vector fields tangent to $M$, and $T^\perp M$ is a set of all vector fields normal to $M$. Then the Gauss and Weingarten formulas are given by
\[
\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y),
\]
\[
\tilde{\nabla}_X N = -A_N X + \nabla_X^\perp N
\]
for $X, Y \in TM$ and $N \in T^\perp M$, where $\nabla$ and $\nabla^\perp$ are the connections in $TM$ and $T^\perp M$, respectively. The second fundamental form $h$ and the shape operator $A_N$ are connected by the relation
\[
g(A_N X, Y) = g(h(X, Y), N), \quad \forall X, Y \in TM, N \in T^\perp M.
\]

A submanifold $M$ of an $n$-dimensional hyperbolic Sasakian manifold $\tilde{M}$ is said to be invariant if the structure vector field $\xi$ is tangent to $M$ everywhere on $M$ and $\phi X$ is tangent to $M$ for any vector field $X$ tangent to $M$ at every point of $M$, that is, $\phi(TM) \subset TM$ (see [15], [23], [29], [35]).

A submanifold $M$ of a hyperbolic Sasakian manifold $\tilde{M}$ is said to be totally umbilical [15] if
\[
h(X, Y) = g(X, Y)H,
\]
where $H$ is the mean curvature on $M$ for $X, Y \in TM$. Moreover, if $h(X, Y) = 0$ for all $X, Y \in TM$, then $M$ is totally geodesic and if $H = 0$, then $M$ is minimal in $\tilde{M}$.
4. Invariant submanifold of a hyperbolic Sasakian manifold

This section is dedicated to study the properties of invariant submanifolds of hyperbolic Sasakian manifolds.

Lemma 4.1. If $M$ is an invariant submanifold of a hyperbolic Sasakian manifold $\tilde{M}$, then there exists the distributions $D$ and $D^\perp$ on $M$ such that

$$TM = D \oplus D^\perp \oplus \langle \xi \rangle, \quad \phi(D) \subset D^\perp \quad \text{and} \quad \phi(D^\perp) \subset D.$$  

Proof. Since the characteristic vector field $\xi$ is tangent to the invariant submanifold $M$ of $\tilde{M}$, we have $TM = D^1 \oplus \langle \xi \rangle$. Let $X \in D^1$. Then $g(\xi, X) = 0$ and $g(X, \phi X) = 0$. This shows that $\phi X$ is orthogonal to $\xi$ and $X$. If possible, we suppose that $D^1 = D \oplus D^\perp$, where $X \in D \subset D^1$ and $\phi X \in D^\perp \subset D^1$. If $\phi X \in D^\perp$, then from equation (7) we have $\phi(\phi X) = \phi^2 X = X + g(\xi, X)\xi = X \in D$. Again if $X \in D$ and $\phi X = Y \in D^\perp$. Then we can easily show that for $X \in D$, $\phi X \in D^\perp$ and for $Y \in D^\perp$, $\phi Y \in D$. Hence, the statement of proposition is proved. \( \square \)

Lemma 4.2. The second fundamental form $h$ on an invariant submanifold $M$ of a hyperbolic Sasakian manifold $\tilde{M}$ satisfies

$$h(X, \xi) = 0 \quad \text{and} \quad h(X, \phi Y) = h(\phi X, Y) = \phi h(X, Y).$$

Proof. In view of equations (10) and (14), we immediately get the first result. Taking the covariant derivative of $\phi Y$ along the vector field $X$ and making use of equation (14), we get

$$(\nabla_X\phi)(Y) = (\nabla_X\phi)(Y) + h(X, \phi Y) - \phi h(X, Y).$$

This equation along with equation (10), after taking normal part, give the second part of Lemma 4.2. \( \square \)

Lemma 4.3. An invariant submanifold $M$ of a hyperbolic Sasakian manifold $\tilde{M}$ satisfies the following relations:

$$\nabla_X \xi = -\phi X,$$

$$\mathcal{R}(X, Y)\xi = \eta(Y)X - \eta(X)Y,$$

$$\mathcal{R}(\xi, X)Y = g(X, Y)\xi - \eta(Y)X,$$

$$Q\xi = (n - 1)\xi, \quad \text{Ric}(X, \xi) = (n - 1)\eta(X).$$

Proof. The straight forward calculation follows Lemma 4.3. \( \square \)

Lemma 4.4. [26] An invariant submanifold $M$ of a hyperbolic Sasakian manifold $\tilde{M}$ is also hyperbolic Sasakian.

Now, we prove the following:

Theorem 4.5. Every 3-dimensional invariant submanifold $M$ of a hyperbolic Sasakian manifold is totally geodesic.

Proof. Let $M$ be a 3-dimensional invariant submanifold of a hyperbolic Sasakian manifold $\tilde{M}$. Then from Lemma 4.4, it is obvious that $M$ is also a hyperbolic Sasakian manifold. We have from (7)

$$\phi^2 h(X, Y) = h(X, Y) + \eta(h(X, Y))\xi.$$  

From Lemma 4.1, we have

$$TM = D \oplus D^\perp \oplus \langle \xi \rangle.$$  

Let $X_1, Y_1 \in D$. Then from the Lemma 4.2, we have

$$h(X_1, \phi Y_1) = \phi h(X_1, Y_1).$$
Also, Lemma 4.2 tells that
\[ \phi h(X_1, \phi Y_1) = \phi h(X_1, Y_1). \]
Since \( h(X_1, Y_1) \in T^1M \), \( h(X_1, Y_1) \) is orthogonal to \( \xi \in TM \). Thus, the above equation gives
\[ \phi h(X_1, \phi Y_1) = h(X_1, Y_1) = h(\phi X_1, \phi Y_1) = \phi^2 h(X_1, Y_1). \]

Next, we suppose that \( X_2, Y_2 \in D^\perp \) and \( X_2 = \phi X_1, Y_2 = \phi X_2 \). We have
\[ h(X_2, Y_2) = h(\phi X_1, \phi Y_1) = h(X_1, Y_1), \]
where equation (18) is used. As we know that \( h \) is bilinear, for therefore \( X_1, Y_1 \in D \) and \( X_2, Y_2 \in D^\perp \), we get
\[ h(X_1 + X_2 + \xi, Y_1) = h(X_1, Y_1) + h(X_2, Y_1) + h(\xi, Y_1), \]
\[ h(X_1 + X_2 + \xi, -Y_2) = -h(X_1, Y_2) - h(X_2, Y_2) - h(\xi, Y_2) \]
and
\[ h(X_1 + X_2 + \xi, \xi) = h(X_1, \xi) + h(X_2, \xi) + h(\xi, \xi). \]

It is well-known that on a hyperbolic Sasakian manifold, \( h(X, \xi) = 0, \forall X \in TM \). By considering this result together with the last expressions, we have
\[ h(X_1 + X_2 + \xi, Y_1 - Y_2 + \xi) = h(X_2, Y_1) - h(X_1, Y_2). \]

If \( U, W \in TM \), then we can write \( U \) and \( W \) as
\[ U = X_1 + X_2 + \xi, \text{ and } W = Y_1 - Y_2 + \xi. \]

We have
\[ h(U, W) = h(X_1 + X_2 + \xi, Y_1 - Y_2 + \xi) = h(X_2, Y_1) - h(X_1, Y_2), \]
\[ \phi h(U, W) = h(X_2, \phi Y_1) - h(\phi X_1, Y_2) = 0, \]
\[ \phi^2 h(U, W)) = 0 \implies h(U, W) = 0. \]

Then follows the statement. \( \square \)

**Theorem 4.6.** Every totally geodesic submanifold \( M \) of a hyperbolic Sasakian manifold is invariant.

**Proof.** We suppose that the submanifold \( M \) of a hyperbolic Sasakian manifold \( \tilde{M} \) is totally geodesic, that is,
\[ h(X, Y) = 0, \forall X, Y \in TM. \]

We shall prove that the submanifold \( M \) of a hyperbolic Sasakian manifold \( \tilde{M} \) is invariant. Thus, we have to show \( \phi X \notin T^2M \). To prove this, if possible, we suppose that the vector field \( \phi X \) has a component, say \( LX \) along the normal vector field of \( M \). It is obvious that \( A_{1X}Y \in TM, \forall X, Y \in TM \). Let \( Z = A_{1X}Y \neq 0 \). Then
\[ g(Z, Z) = g(A_{1X}Y, Z) = g(h(Y, Z), LX) = 0. \]

Since \( Z \) is a non-null and non-zero vector field of \( TM \) implies \( g(Z, Z) \neq 0 \). Thus our hypothesis that \( \phi X \) has a component along \( T^2M \) is inadmissible. Hence \( \phi X \in TM \) and therefore the submanifold \( M \) of \( \tilde{M} \) is invariant. \( \square \)

In view of Theorem 4.6, we can state the following corollary as:
Corollary 4.7. Every 3-dimensional geodesic submanifold $M$ of a hyperbolic Sasakian manifold $\tilde{M}$ is invariant.

By considering the Theorem 4.5 and Corollary 4.7, we can state:

Theorem 4.8. A 3-dimensional submanifold $M$ of a hyperbolic Sasakian manifold $\tilde{M}$ is totally geodesic if and only if it is invariant.

Next, we have the following results:

Lemma 4.9. Let $M$ be an invariant submanifold of a hyperbolic Sasakian manifold $\tilde{M}$. Then

$$\nabla_Y (\nabla h)(Z, \xi) = -h(Z, \nabla_Y \xi)$$

(20)

for any $Y, Z \in TM$.

Proof. From Lemma 4.2 we turn up

$$\nabla_Y (\nabla h)(Z, \xi) = \nabla_Y h(Z, \xi) - h(\nabla_Y Z, \xi) - h(Z, \nabla_Y \xi),$$

which gives the statement of Lemma 4.9.  

Corollary 4.10. Let $M$ be an invariant submanifold of a hyperbolic Sasakian manifold $\tilde{M}$. Then

$$\nabla_Y (\nabla h)(Z, \xi) = h(Z, \varphi Y)$$

(21)

for any $Y, Z \in TM$.

Proof. By using equations (10) and (20), we get (21).

Corollary 4.11. Let $M$ be an invariant submanifold of a hyperbolic Sasakian manifold $\tilde{M}$. Then the following conditions are equivalent.

1. $M$ is totally geodesic,
2. $h$ is parallel,
3. $(\nabla_Y \nabla h)(\xi, \xi) = 0$,

where $Y$ and $Z$ are arbitrary vector fields on $M$.

Theorem 4.12. Let $M$ be an invariant submanifold of a hyperbolic Sasakian manifold $\tilde{M}$. Then

$$\varphi (A_N Z) = A_{\varphi N} Z = -A_N \varphi Z$$

(22)

for all $Z \in TM$, $N \in TM^\perp$.

Proof. Adopting (9) and (16) for all $Z \in TM$, $N \in TM^\perp$ we turn up

$$g(\varphi (A_N Z), W) = -g(A_N Z, \varphi W) = -g(h(Z, \varphi Y), N)$$

$$= -g(h(\varphi Z, W), N) = -g(A_N \varphi Z, W).$$

Then, we have $\varphi (A_N Z) = -A_N \varphi Z$. Now using Lemma (4.2), we lead to

$$g(A_{\varphi N} Z, W) = g(h(Z, W), \varphi N) = -g(\varphi (h(Z, W)), N) = -g(h(Z, \varphi W), N)$$

$$= -g(A_N Z, \varphi W) = g(\varphi (A_N Z), W).$$

Thus, we have the result.
5. $\eta$-Ricci-Bourguignon solitons on invariant submanifolds

Let us adopt $(g, \xi, \lambda, \omega, \rho)$ as an $\eta$-Ricci-Bourguignon soliton on an invariant submanifold $M$ of a hyperbolic Sasakian manifold $\tilde{M}$. Then we have

$$\mathcal{L}_\xi g(X, Y) + 2\text{Ric}(X, Y) + 2(\lambda + \rho R)g(X, Y) + 2\omega\eta(X)\eta(Y) = 0. \tag{23}$$

From (10) and (14), we turn up

$$-\phi X = \nabla_X \xi = \nabla_X \xi + h(X, \xi). \tag{24}$$

If $M$ is invariant in $\tilde{M}$, then $-\phi X, \xi \in TM$ and therefore equating the tangential and normal parts of (24) we find that

$$\nabla_X \xi = -\phi X \quad \text{and} \quad h(X, \xi) = 0. \tag{25}$$

Again from (10) we lead to

$$(\mathcal{L}_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 0. \tag{26}$$

Adopting (23) and (25) we turn up

$$\text{Ric}(X, Y) = -(\lambda - \rho R)g(X, Y) - \omega\eta(X)\eta(Y), \tag{27}$$

which implies that $M$ is $\eta$-Einstein. Also, from (17) and (25) we get $\eta(X)H = 0$, i.e., $H = 0$, since $\eta(X) \neq 0$ (in general) and therefore $M$ is minimal in $\tilde{M}$. Thus, we turn up the following:

**Theorem 5.1.** If $(g, \xi, \lambda, \omega, \rho)$ is an $\eta$-Ricci-Bourguignon soliton on an invariant submanifold $M$ of a hyperbolic Sasakian manifold $\tilde{M}$. Then we have the following:

1. $M$ is $\eta$-Einstein,
2. $M$ is minimal and
3. $\xi$ is a Killing vector field in $\tilde{M}$.

Now, by choosing the different values of $\rho$ in equation (27), we obtain the following table:

| $\rho$ | Solitons | Expression of Ricci tensor |
|-------|----------|-----------------------------|
| $\frac{1}{\tau}$ | $\eta$-Einstein solitons | $\text{Ric}(X, Y) = -(\lambda - \frac{\tau}{2})g(X, Y) - \omega\eta(X)\eta(Y)$ |
| $\frac{1}{2n+1}$ | $\eta$-Schouten solitons | $\text{Ric}(X, Y) = -(\lambda - \frac{1}{2n+1})g(X, Y) - \omega\eta(X)\eta(Y)$ |
| 0 | $\eta$-Ricci solitons | $\text{Ric}(X, Y) = -\lambda g(X, Y) - \omega\eta(X)\eta(Y)$ |

From table (28) and Theorem 5.1, we state the following corollaries:

**Corollary 5.2.** If $(g, \xi, \lambda, \omega, \rho)$ is an $\eta$-Einstein or $\eta$-Schouten or $\eta$-Ricci solitons on an invariant submanifold $M$ of $\tilde{M}$, then we have

1. $M$ is $\eta$-Einstein,
2. $M$ is minimal and
3. $\xi$ is Killing vector field in $\tilde{M}$.

From Lemma (4.3), we have

$$\text{Ric}(X, \xi) = (n-1)\eta(X),$$

and henceforth

$$\text{Ric}(\xi, \xi) = -(n-1). \tag{29}$$

Also from (27) we turn up

$$\text{Ric}(\xi, \xi) = -(\lambda - \rho R) - \omega. \tag{30}$$

Thus from (29) and (30) we obtain $\lambda = \rho R + [(n-1) - \omega]$. In particular, if we choose $\omega = 0$ then the $\eta$-Ricci-Bourguignon soliton becomes the Ricci-Bourguignon soliton and $\lambda = \rho R + (n-1)$. This leads to the following:
Theorem 5.3. Let \((g, \xi, \lambda, \omega, \rho)\) be an \(\eta\)-Ricci-Bourguignon soliton on an invariant submanifold \(M\) of \(\tilde{M}\). Then \(\lambda + \omega = \rho R + n - 1\). Also, the Ricci-Bourguignon soliton \((g, \xi, \lambda, \rho)\) on \(M\) is steady, expanding or shrinking according as \(\rho R > -(n - 1), \rho R = -(n - 1)\) or \(\rho R < -(n - 1)\), respectively.

Also, in consequence of Theorem 5.3 we can easily obtain the followings corollaries:

Corollary 5.4. An Einstein soliton \((g, \xi, \lambda, \frac{1}{2})\) on an invariant submanifold \(M\) of a hyperbolic Sasakian manifold \(\tilde{M}\) is steady, expanding or shrinking according as \(\frac{8}{2} > -(n - 1), \frac{8}{2} = -(n - 1)\) or \(\frac{8}{2} < -(n - 1)\), respectively.

Corollary 5.5. A Schouten soliton \((g, \xi, \lambda, \frac{1}{2(n-1)})\) on an invariant submanifold \(M\) of a hyperbolic Sasakian manifold \(\tilde{M}\) is steady, expanding or shrinking according as \(\frac{R}{2(n-1)} > -(n - 1), \frac{R}{2(n-1)} = -(n - 1)\) or \(\frac{R}{2(n-1)} < -(n - 1)\), respectively.

Corollary 5.6. A Ricci soliton \((g, \xi, \lambda, 0)\) on an invariant submanifold \(M\) of a hyperbolic Sasakian manifold \(\tilde{M}\) is shrinking, expanding or steady if \(\text{Ric}(\xi, \xi) > 0, < 0\) or \(= 0\), respectively.

6. \(\eta\)-Ricci-Bourguignon solitons with concircular vector field on invariant submanifolds

This section deals with the study of \(\eta\)-Ricci-Bourguignon solitons with concircular vector field on an invariant submanifold \(M\) of a hyperbolic Sasakian manifold \(\tilde{M}\).

In 1939, A. Fialkow [21] has been proposed the theory of concircular vector fields on a Riemannian manifold. A vector field \(v\) on a (semi-)Riemannian manifold \(M\) is said to be a concircular vector field if it satisfies

\[
\nabla_U v = \mu U, \tag{31}
\]

for any \(U \in TM\), where \(\nabla\) denotes the Levi-Civita connection of the metric \(g\) and \(\mu\) is a non-trivial smooth function on \(M\). The concircular vector fields are also known as geodesic fields because their integral curves are geodesics [21]. Recently, Chen [16] studied the properties of Ricci solitons on submanifolds of a Riemannian manifold equipped with a concircular vector field. Particularly, if we choose \(\mu = 1\) in equation (31), then the concircular vector field \(v\) is called concurrent vector field.

For an invariant submanifold, from Lemma (4.1), we can write

\[
v = v^T + v^+, \tag{32}
\]

where \(v \in TM, v^T \in D\) and \(v^+ \in D^+\).

Since \(v\) is a concircular vector field on \(\tilde{M}\) and from (32), we get

\[
\mu U = \tilde{\nabla}_U v^T + \tilde{\nabla}_U v^+ \tag{33}
\]

for any \(U \in D\). Also, from (14) and (15), we turn up

\[
\mu U = \nabla_U v^T + h(U, v^T) - A_{v^+} U + \nabla_U^+ v^+. \tag{34}
\]

By comparing the tangential and normal components of equation (34), we conclude that

\[
h(U, v^T) = -\nabla_U^+ v^+, \quad \nabla_U v^T = \mu U - A_{v^+} U. \tag{35}
\]

Now, we prove the following theorem as:

Theorem 6.1. Let \(M\) be an invariant submanifold of \(\tilde{M}\) admitting an \(\eta\)-Ricci-Bourguignon soliton with a concircular vector field \(v\). Then the Ricci tensor \(\text{Ric}_D\) on the invariant distribution \(D\) is given by

\[
\text{Ric}_D(U, W) = \left\{ \left( \lambda - \frac{\rho R}{2} + \mu \right) g(U, W) + g(h(U, W), v^+) + \omega \eta(U) \eta(W) \right\} \tag{36}
\]

for any \(U, W \in D\).
Proof. By the definition of Lie-derivative, we have
\[(L_{v}g)(U, W) = g(V_{U}v^{\top}, W) + g(U, V_{W}v^{\top}).\]  
(37)

Adopting (35) together with (37), the above equation infers that
\[(L_{v}g)(U, W) = 2\mu g(U, W) - 2g(h(U, W), v^{\perp}).\]  
(38)

Again, since the invariant submanifold \(M\) admits an \(\eta\)-Ricci-Bourguignon soliton, therefore we observe that
\[(L_{v}g)(U, W) + 2Ric_{D}(U, W) + 2(\lambda - \rho R)g(U, W) + 2\omega\eta(U)\eta(W) = 0.\]  
(39)

From equations (38) and (39), we obtain the statement of Theorem 6.1. \(\square\)

In particular, if we take \(\mu = 1\) in equation (31) then we will get concurrent vector field. Let us suppose that \(v\) is a concurrent vector field and \((g, v, \lambda, \omega, \rho)\) is an \(\eta\)-Ricci-Bourguignon soliton in an invariant submanifold \(M\) of a hyperbolic Sasakian manifold. Then by following the similar process of Theorem 6.1, we can state the following:

**Theorem 6.2.** If an invariant submanifold \(M\) of a hyperbolic Sasakian manifold \(\tilde{M}\) admits an \(\eta\)-Ricci-Bourguignon soliton with the concircular vector field \(v\), then the invariant distribution \(D\) of \(M\) is \(\eta\)-Einstein, provided the invariant distribution \(D\) of \(M\) is \(D\)-geodesic.

In view of Theorem 6.1, we can write the following corollaries as:

**Corollary 6.3.** Let \(M\) be an invariant submanifold of \(\tilde{M}\) admitting an \(\eta\)-Ricci-Bourguignon soliton with a concurrent vector field \(v\). Then the Ricci tensor \(\text{Ric}_{D}\) on the invariant distribution \(D\) is given by
\[\text{Ric}_{D}(U, W) = -\{\left(\lambda - \frac{\rho R}{2} + 1\right)g(U, W) + g(h(U, W), v^{\perp}) + \omega\eta(U)\eta(W)\}\]  
for any \(U, W \in D\).

**Corollary 6.4.** Suppose that an invariant submanifold \(M\) of \(\tilde{M}\) admits an \(\eta\)-Ricci-Bourguignon soliton with a concurrent vector field \(v\). If the invariant distribution \(D\) of \(M\) is \(D\)-geodesic, then the invariant distribution \(D\) is \(\eta\)-Einstein.

**Remark 6.5.** It is noticed that if we choose \(D^{\perp}\) distribution and \(D^{\perp}\)-geodesic condition in the above theorems and corollaries, then after following the similar steps we obtain the results as stated in Theorem 6.1, Theorem 6.2 and Corollary 6.3, Corollary 6.4.

Now, with the help of Table (28) and Theorem 6.1, we can easily obtain following corollaries as:

**Corollary 6.6.** If an invariant submanifold \(M\) of a hyperbolic Sasakian manifold \(\tilde{M}\) admits an \(\eta\)-Einstein soliton with a concircular vector field \(v\), then the Ricci tensor \(\text{Ric}_{D}\) on the invariant distribution \(D\) is given by
\[\text{Ric}_{D}(U, W) = -\{\left(\lambda - \frac{R}{2} + \mu\right)g(U, W) + g(h(U, W), v^{\perp}) + \omega\eta(U)\eta(W)\}\]  
for any \(U, W \in D\).

**Corollary 6.7.** Let \(M\) be an invariant submanifold of a hyperbolic Sasakian manifold \(\tilde{M}\) admitting an \(\eta\)-Schouten soliton with a concircular vector field \(v\). Then the Ricci tensor \(\text{Ric}_{D}\) of the invariant distribution \(D\) is given by
\[\text{Ric}_{D}(U, W) = -\{(\lambda - \frac{R}{2(n-1)} + \mu)g(U, W) + g(h(U, W), v^{\perp}) + \omega\eta(U)\eta(W)\}\]  
for any \(U, W \in D\).

**Corollary 6.8.** Let \(M\) be an invariant submanifold of a hyperbolic Sasakian manifold \(\tilde{M}\) admitting an \(\eta\)-Ricci soliton with a concircular vector field \(v\). Then the Ricci tensor \(\text{Ric}_{D}\) on the invariant distribution \(D\) is given by
\[\text{Ric}_{D}(U, W) = -\{(\lambda + \mu)g(U, W) + g(h(U, W), v^{\perp}) + \omega\eta(U)\eta(W)\}\]  
for any \(U, W \in D\).
7. Example

Let $\mathbb{R}^n$ be an $n$-dimensional real number space. Define $M^5 = \{(x_1, x_2, x_3, x_4, x_5) : x_i \in \mathbb{R}, i = 1, 2, ..., 5\}$. Let $\{e_1, e_2, e_3, e_4, e_5\}$ be a set of linearly independent vector fields of $M^5$ given by

$$
e_1 = e^{2x_1} \frac{\partial}{\partial x_1} - 2x_2 \frac{\partial}{\partial x_3}, \quad e_2 = \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_2}, \quad e_3 = \frac{\partial}{\partial x_3}, \quad e_4 = x_4 \frac{\partial}{\partial x_4} - 2x_5 \frac{\partial}{\partial x_3}, \quad e_5 = \frac{\partial}{\partial x_5}.
$$

We define the $(1,1)$-tensor field $\phi$ of $M^5$ as

$$\phi e_1 = e_2, \quad \phi e_2 = e_1, \quad \phi e_3 = 0, \quad \phi e_4 = e_5, \quad \phi e_5 = e_4.$$  

Also, we define the associated metric $g$ of $M^5$ by the following relation.

$$g(e_i, e_j) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -1
\end{pmatrix}.$$  

By the linearity property of $\phi$ and $g$, we can show that the relations

$$\phi^2 e_i = e_i + \eta(e_i) \xi, \quad \eta(\xi) = -1$$

hold for $i = 1, 2, 3, 4, 5$ and $\xi = e_5$. Again, for $\xi = e_5$, $M^5$ satisfies $g(e_i, e_j) = \eta(e_i), g(\phi e_i, e_j) = -g(e_i, \phi e_j)$ and $g(\phi e_i, \phi e_j) = -g(\phi e_i, e_j) - \eta(e_i)\eta(e_j)$, where $i, j = 1, 2, 3, 4, 5$. It can be easily obtained that

$$[e_i, e_j] = \begin{cases}
2e_3, & i = 1, \quad j = 2 \\
2e_3, & i = 4, \quad j = 5 \\
0, & \text{otherwise},
\end{cases}$$

where $i, j = 1, 2, 3, 4, 5$. Koszul’s formula for the Levi-Civita connection $\nabla$ is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y])$$

for all vector fields $X, Y$ and $Z$ on $M^5$. In view of this formula and the above results, we have

$$\nabla_{e_i} e_1 = 0, \quad \nabla_{e_2} e_3 = 0, \quad \nabla_{e_3} e_2 = -e_2, \quad \nabla_{e_4} e_5 = 0, \quad \nabla_{e_5} e_5 = 0,$$

$$\nabla_{e_1} e_1 = -e_3, \quad \nabla_{e_2} e_2 = -e_2, \quad \nabla_{e_3} e_3 = -e_1, \quad \nabla_{e_4} e_4 = 0, \quad \nabla_{e_5} e_5 = 0,$$

$$\nabla_{e_5} e_1 = 0, \quad \nabla_{e_2} e_3 = 0, \quad \nabla_{e_3} e_2 = -e_1, \quad \nabla_{e_4} e_3 = -e_3, \quad \nabla_{e_5} e_4 = 0,$$

$$\nabla_{e_5} e_1 = 0, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_3} e_3 = -e_5, \quad \nabla_{e_4} e_4 = 0, \quad \nabla_{e_5} e_5 = -e_5.$$  

From the above equations, it is obvious that $\nabla_X \xi = -\phi X$ for all $X \in TM^5$ and $\xi = e_3$. Thus, the structure $(\phi, \xi, \eta, g)$ is an almost hyperbolic Sasakian structure and $M^5$ equipped with the structure $(\phi, \xi, \eta, g)$ is an almost hyperbolic Sasakian manifold of dimension 5.

We suppose that $f$ is an isometric immersion from $M^3$ to $M^5$ defined by $f(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2, x_3, 0, 0)$. Let the triplet $(x_1, x_2, x_3)$ be the standard coordinates in $\mathbb{R}^3$. We define $M^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3$ such that $(x_1, x_2, x_3) \neq 0\}$. If we consider the vector fields

$$e_1 = e^{2x_1} \frac{\partial}{\partial x_1} - 2x_2 \frac{\partial}{\partial x_3}, \quad e_2 = \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_2}, \quad e_3 = \frac{\partial}{\partial x_3}$$

on $M^3$, then they form a basis for $M^3$. By considering these vectors, we can easily find the components of Lie bracket for $e_1, e_2$ and $e_3$ as:

$$[e_1, e_2] = 2e_3, \quad [e_1, e_3] = 0, \quad [e_2, e_3] = 0.$$  

(43)
and other components can be obtained by using the skew-symmetric property of Lie bracket. Let the associated metric $g$ of $M^3$ is defined by

$$g(e_i, e_j) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$  

Suppose that the 1-form $\eta$ with respect to the metric $g$ is defined as $\eta(X) = g(X, \xi)$, $\forall \, X \in TM$. The $(1, 1)$-type tensor $\phi$ of $M^3$ is defined by

$$\phi e_1 = e_2, \quad \phi e_2 = e_1, \quad \phi e_3 = 0.$$  

By the linearity of $\phi$ and $g$, we can easily see that for $\xi = e_3$

$$\phi^2 X = X + \eta(X)\xi, \quad \eta(\xi) = -1$$  

and

$$g(\phi X, \phi Y) = -g(X, Y) - \eta(X)\eta(Y), \quad \forall \, X, Y \in TM.$$  

From the Koszul’s formula, we have

$$\nabla_{\xi} e_1 = 0, \quad \nabla_{\xi} e_2 = e_3, \quad \nabla_{\xi} e_3 = -e_2, \quad \nabla_{e_1} e_1 = -e_3, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.$$  

It is obvious from these equations that $\nabla \xi = -\phi X$ for $\xi = e_3$ and $X \in TM$ holds on $M^3$. Thus, $(M^3, g)$ is a 3-dimensional hyperbolic Sasakian manifold. It is obvious that $(M^3, g)$ is a submanifold of $(\tilde{M}^3, \tilde{g})$. To achieve our goal, we have to prove that $M^3$ is an invariant as well as totally geodesic.

If possible, we suppose that $\langle e_1 \rangle = D$ and $\langle e_2 \rangle = D^\perp$, then the tangent space $TM$ of $M^3$ takes the form $TM = D \oplus D^\perp \oplus \langle \xi \rangle$. Let $U \in D$ and $W \in D^\perp$, then we can write $U = a e_1$ and $W = \beta e_2$, where $a$ and $\beta$ are the smooth functions. We have

$$\phi U = \phi(\alpha e_1) = \alpha \phi e_1 = \alpha e_2 \in D^\perp \in TM$$  

and

$$\phi W = \phi(\beta e_2) = \beta \phi e_2 = \beta e_1 \in D \in TM.$$  

Hence, we can say that $M^3 = M$ under consideration is an invariant submanifold of $M^3 = \tilde{M}$.

From equation (14), we have

$$h(e_i, e_j) = \nabla_{e_i} e_j - \nabla_{e_j} e_i.$$  

It is evident from the above results that

$$h(e_i, e_j) = 0, \quad \forall \, i, \, j = 1, 2, 3.$$  

Let $U, \, W \in TM$. Then we can write $U = a_1 e_1 + \beta_1 e_2 + \gamma_1 e_3$ and $W = a_2 e_1 + \beta_2 e_2 + \gamma_2 e_3$, where $a_i, \, \beta_i,$ and $\gamma_i,$ for $i = 1, 2,$ are scalars. We have

$$h(U, W) = h(a_1 e_1 + \beta_1 e_2 + \gamma_1 e_3, a_2 e_1 + \beta_2 e_2 + \gamma_2 e_3) = a_1 a_2 h(e_1, e_1) + a_1 \beta_2 h(e_1, e_2) + a_1 \gamma_2 h(e_1, e_3) + \beta_1 a_2 h(e_2, e_1) + \beta_1 \beta_2 h(e_2, e_2) + \beta_1 \gamma_2 h(e_2, e_3) + \gamma_1 a_2 h(e_3, e_1) + \gamma_1 \beta_2 h(e_3, e_2) + \gamma_1 \gamma_2 h(e_3, e_3).$$  

The last two equations give

$$h(U, W) = 0, \quad \forall \, U, \, W \in TM.$$
This shows that the 3-dimensional hyperbolic Sasakian submanifold $M^3$ of the 5-dimensional hyperbolic Sasakian manifold $M^5$ is totally geodesic. Hence, the statement of Theorem 4.8 is verified.

Also, by using equations (43), (44) and the metric of $M^5$ in $\mathcal{R}(U, Z)W = \nabla_U \nabla_Z W - \nabla_Z \nabla_U W - \nabla_{[UZ]} W$, we find the non-zero components of curvature tensor $\mathcal{R}$ as:

$$\mathcal{R}(e_1, e_2)e_3 = 3e_2, \quad \mathcal{R}(e_1, e_3)e_1 = -e_3, \quad \mathcal{R}(e_2, e_3)e_2 = 3e_1, \quad \mathcal{R}(e_1, e_3)e_3 = -e_1, \quad \mathcal{R}(e_2, e_3)e_3 = -e_2.$$

The other components of the curvature tensor can be obtained by the symmetric properties. The above results together with equations (5) and (6) reveal that

$$\text{Ric}(e_1, e_1) = \text{Ric}(e_3, e_3) = -2, \quad \text{Ric}(e_2, e_2) = 2 \quad \text{and} \quad R = -2.$$

By the straightforward calculations, we can show that the Lemma 4.2, Lemma 4.3, Lemma 4.4, Theorem 4.5 and Corollary 4.7 hold on $M^3$.

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