Enhanced algorithms for solving the spectral discretization of the
vorticity–velocity–pressure formulation of the Navier–Stokes problem

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Abstract

The objective of the article is to improve the algorithms for the resolution of the
spectral discretization of the vorticity–velocity–pressure formulation of the
Navier–Stokes problem in two and three domains. Two algorithms are proposed. The
first one is based on the Uzawa method. In the second one we consider a modified
global resolution. The two algorithms are implemented and their results are
compared.

Keywords: Navier–Stokes problem; Spectral method; Uzawa method; Global
resolution

1 Introduction

The Navier–Stokes system models the flow of a fluid, for example, the movements of air
in the atmosphere, ocean currents, the flow of water in a pipeline and many other fluid-
flow phenomena. The change of one of the parameters associated with the Navier–Stokes
equations (the domain, the boundary conditions, the data, the variational formulation, the
approximation method, …) gives us a new research problem. In the reference article [1],
the authors handle the Navier–Stokes equations with no standard boundary conditions
on the velocity and the pressure in a convex domain.

In this work, Ω is assumed to be a bounded simply connected domain of \(\mathbb{R}^d\) (\(d = 2, 3\))
and \(\partial \Omega\) is its continuous Lipschitz boundary. We focus on the following Navier–Stokes
system:

\[
\begin{aligned}
\nabla \cdot \mathbf{u} & = 0 \quad \text{in } \Omega, \\
\nabla \mathbf{u} : \nabla \mathbf{v} + \nabla \mathbf{P} & = \mathbf{f} \quad \text{in } \Omega, \\
\mathbf{u} \cdot \mathbf{n} & = 0 \quad \text{on } \partial \Omega, \\
\eta (\nabla \times \mathbf{u}) & = 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

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where \( \varphi \) is the velocity, \( P \) the pressure, \( f \) is the density of forces, \( \varphi \) is the fluid viscosity and \( n \) is an outer unit vector normal to \( \partial \Omega \).

System (1) can be written as (see [2, 3]):

\[
\begin{aligned}
\nu (\text{curl } \mu) + (\mu \times \varphi) + \nabla p &= f \quad \text{in } \Omega, \\
\text{div } \varphi &= 0 \quad \text{in } \Omega, \\
\mu &= \text{curl } \varphi \quad \text{in } \Omega, \\
\varphi \cdot n &= 0 \quad \text{on } \partial \Omega, \\
\eta(\mu) &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]  

where

- \( \mu = \text{curl } \varphi \) is the vorticity verifying \( \varphi \cdot \nabla \varphi = \mu \times \varphi + \frac{1}{2} \text{grad} |\varphi|^2 \),
- \( p = P + \frac{1}{2} |\varphi|^2 \) is the dynamic pressure,
- \( \eta \) is an operator such that \( \eta(\text{curl } \varphi) \) is the normal (resp. the tangential) boundary component of \( \text{curl } \varphi \) when \( d = 2 \) (resp. \( d = 3 \)).

The above problem has been solved in several previous works using finite-element discretization, see [2–6], while the extension to the discretization by the spectral method has been handled in [7, 8] for solving stationary and nonstationary Navier–Stokes equations. In a previous work [8], we studied (2) using the spectral discretization method. The resulting linear system has been solved using the GMRES iterative method (see [9]) since the matrix of the resulting linear system is not symmetric.

To improve the resolution of this problem, we propose in this work two new methods in order to reduce the number of iterations and execution time. The first method consists in applying the Uzawa algorithm (see [10, 11]) after reducing the unknowns of the system by eliminating the vorticity. The second method consists of modifying the iterative algorithm used to linearize the resolution of the Navier–Stokes problem in order to obtain a linear system with a symmetric positive-definite matrix. Then, the conjugate gradient method is used.

The paper is organized as follows:

- The continuous and discrete weak formulation are described in Sect. 2.
- Sect. 3 provides the details of the obtained linear system and its implementation using the two proposed algorithms.
- In Sect. 4, we present and compare the two algorithms for the resolution of the discrete problem based on some numerical tests.

### 2 Continuous and the discrete weak formulations

The weak formulation of problem (2) can be written as: find \((\mu, \varphi, p) \in H_0(\text{curl}, \Omega) \times H_0(\text{div}, \Omega) \times L^2_0(\Omega)\) solution of

\[
\begin{aligned}
a(\mu, \varphi; v) + T(\mu, \varphi; v) + b(v, p) &= (f, v), \quad \forall v \in H_0(\text{div}, \Omega), \\
b(\varphi, q) &= 0, \quad \forall q \in L^2_0(\Omega), \\
c(\mu, \varphi; \psi) &= 0, \quad \forall \psi \in H_0(\text{curl}, \Omega),
\end{aligned}
\]  

where
We note that meshes (see [14, Sect. 2]), we construct our discrete spectral spaces as follows.

In the following, we use the Gauss–Lobatto quadrature formula to construct the discrete space of the vorticity \( \mathcal{Y}_N \), the discrete space of the pressure \( \mathcal{M}_N \), and the discrete space of the vorticity \( \mathcal{Y}_N \).

\[
\mathcal{X}_N = H_0(\text{div}, \Omega) \cap \begin{cases} \mathcal{P}_{N,N-1}(\Omega) \times \mathcal{P}_{N-1,N}(\Omega) & \text{if } d = 2, \\ \mathcal{P}_{N-1,N,N-1}(\Omega) \times \mathcal{P}_{N,N-1,N-1}(\Omega) \times \mathcal{P}_{N-1,N-1,N}(\Omega) & \text{if } d = 3, \end{cases}
\]

\[
\mathcal{Y}_N = \begin{cases} H_0^1(\Omega) \cap \mathcal{P}(\Omega) & \text{if } d = 2, \\ H_0(\text{curl}, \Omega) \cap (\mathcal{P}_{N-1,N,N}(\Omega) \times \mathcal{P}_{N,N-1,N}(\Omega) \times \mathcal{P}_{N,N,N-1}(\Omega)) & \text{if } d = 3, \end{cases}
\]

\[
\mathcal{M}_N = L^2_0(\Omega) \cap \mathcal{P}_{N-1}(\Omega).
\]

In the following, we use the Gauss–Lobatto quadrature formula to construct the discrete problem

\[
\int_{-1}^{1} \varphi(x) \, dx = \sum_{j=0}^{N} \varphi(\xi_j) \rho_j, \quad \forall \varphi \in \mathcal{P}_{2N-1}(-1,1),
\]
where $\xi_j, 0 \leq j \leq N$, are the roots of the polynomial $(1 - x^2)\chi'_N$, $\chi_N$ is the Legendre polynomial of degree $N$ defined on $[-1, 1]$, and $\rho_j, 0 \leq j \leq N$, is the associated set of the positive weights.

The following inequality
\[
\|\varphi_N\|_{L^2(-1,1)}^2 \leq \sum_{j=0}^N \varphi_N^2(\xi_j) \rho_j \leq 3\|\varphi_N\|_{L^2(-1,1)}^2 \quad \forall \varphi_N \in \mathbb{P}_N(-1,1)
\]

enables us to show that the continuous and discrete norms are equivalent (see [15]).

The discrete scalar product is defined such that for the continuous functions $\varphi$ and $\psi$ on $\Omega$, we have:
\[
(\varphi, \psi)_N = \begin{cases} 
\sum_{i=0}^N \sum_{l=0}^N \varphi(\xi_i, \xi_l) \psi(\xi_i, \xi_l) \rho_i \rho_l & \text{if } d = 2, \\
\sum_{i=0}^N \sum_{l=0}^N \varphi(\xi_i, \xi_l, \xi_l') \psi(\xi_i, \xi_l, \xi_l') \rho_i \rho_l \rho_l' & \text{if } d = 3.
\end{cases}
\]

Then, using the Galerkin method and the numerical integration based on the Gauss–Lobatto quadrature formula (5), we deduce for a continuous data $f$ on $\Omega$ the following discrete variational formulation:

Find $(\mu_N, \varphi_N, p_N)$ in $\mathbb{Y}_N \times \mathbb{X}_N \times \mathbb{M}_N$ such that
\[
a_N(\mu_N, \varphi_N; \nu_N) + T_N(\mu_N, \varphi_N; \nu_N) + b_N(\varphi_N, p_N) = (f, \nu_N)_N, \quad \forall \nu_N \in \mathbb{X}_N, \\
b_N(\varphi_N, q_N) = 0, \quad \forall q_N \in \mathbb{M}_N, \\
c_N(\mu_N, \varphi_N, \psi_N) = 0 \quad \forall \psi_N \in \mathbb{Y}_N. \tag{6}
\]

The bilinear forms $a_N(\cdot, \cdot, \cdot)$, $b_N(\cdot, \cdot)$, and $c_N(\cdot, \cdot)$ are continuous and are formulated as follows:
\[
a_N(\mu_N, \varphi_N; \nu_N) = \langle \text{curl} \mu_N, \nu_N \rangle_N, \quad b_N(\varphi_N, q_N) = -\langle \text{div} \varphi_N, q_N \rangle_N, \\
c_N(\mu_N, \varphi_N; \psi_N) = (\mu_N, \psi_N)_N - (\varphi_N, \text{curl} \psi_N)_N.
\]

However, the discrete trilinear form $T_N(\cdot, \cdot, \cdot)$ is defined as:
\[
T_N(\mu_N, \varphi_N; \nu_N) = (\mu_N \times \varphi_N, \nu_N)_N.
\]

Using Brouwer’s fixed-point theorem ([10], Chap. IV, Cor. 1.1), problem (6) has a unique solution, since the discrete bilinear form $b_N(\cdot, \cdot)$ coincides with the continuous one on $\mathbb{X}_N \times \mathbb{M}_N$ due to the exactness of the quadrature formula and verifies the inf-sup condition (see [8], Lem. 3.9)
\[
\sup_{\nu_N \in \mathbb{X}_N} \frac{b_N(\nu_N, p_N)}{\|\nu_N\|_{H(\text{div}, \Omega)}} \geq \alpha \|p_N\|_{L^2(\Omega)}, \quad \forall p_N \in \mathbb{M}_N. \tag{7}
\]

In the case of $d = 2$ the error estimate between the continuous and discrete problems is given by (see [7] for the proof)
\[
\|\mu - \mu_N\|_{H(\text{curl}, \Omega)} + \|\varphi - \varphi_N\|_{H(\text{div}, \Omega)} + \left|\log(N)\right|^{-\frac{1}{2}} \|p - p_N\|_{L^2(\Omega)} \leq c(N^{-\frac{1}{2}}(\|\mu\|_{H^{1}(\Omega)} + \|\varphi\|_{H^1(\Omega)}^2 + \|p\|_{H^1(\Omega)}^2) + N^{-\frac{\gamma}{2}} \|f\|_{H^1(\Omega)}^2). \tag{8}
\]
where \( c \) is a positive constant independent of \( N \), \((\mu_N, \phi_N, p_N)\) in \( \mathbb{Y}_N \times \mathbb{X}_N \times \mathbb{M}_N \) and \((\mu, \varphi, p) \in H^{s+1}(\Omega) \times H^s(\Omega) \times H^s(\Omega), \ s > 1 \).

In the case \( d = 3 \), the error estimate is difficult to prove and remains an open problem.

### 3 Numerical algorithms

We present in this section two numerical algorithms to improve the numerical resolution of discrete problem (6). In the following we consider \( \Omega = [-1, 1]^2 \).

#### 3.1 The iterative algorithm

To linearize the Navier–Stokes problem, we consider the following iterative algorithm.

- **Step 1**: Solve the following linear Stokes problem:
  
  \[
  \forall \nu_N \in \mathbb{X}_N, \quad a_N(\mu^0_N, \phi^0_N; \nu_N) + b_N(\nu_N, p^0_N) = (f, \nu_N)_N, \\
  \forall q_N \in \mathbb{M}_N, \quad b_N(\phi^0_N, q_N) = 0, \\
  \forall \psi_N \in \mathbb{Y}_N, \quad c_N(\mu^0_N, \phi^0_N; \psi_N) = 0.
  \]

  (9)

- **Step 2**: Solve the following problem, by assuming that the \( i-1 \) iterative solution \((\mu^{i-1}_N, \phi^{i-1}_N; p^{i-1}_N)\) is known.
  
  \[
  a_N(\mu^i_N, \phi^i_N; \nu_N) + T_N(\mu^{i-1}_N, \phi^{i-1}_N; \nu_N) + b_N(\nu_N, p^i_N) = (f, \nu_N)_N + T_N(\mu^{i-1}_N, \phi^{i-1}_N; \nu_N), \\
  \forall \nu_N \in \mathbb{X}_N, \\
  b_N(\phi^i_N, q_N) = 0, \quad \forall q_N \in \mathbb{M}_N, \\
  c_N(\mu^i_N, \phi^i_N; \theta_N) = 0 \quad \forall \theta_N \in \mathbb{Y}_N.
  \]

  (10)

We stop the iterations when the following inequality is satisfied

\[
\left( \| \mu^i_N - \mu^{i-1}_N \|^2_{H(\text{curl}; \Omega)} + \| \phi^i_N - \phi^{i-1}_N \|^2_{H(\text{div}; \Omega)} \right)^{\frac{1}{2}} \leq \epsilon,
\]

where \( \epsilon \) a positive small real number.

#### 3.2 The linear matrix system

Consider the Lagrange polynomials of degree \( \leq 1 \), \( \kappa_j \in \mathbb{P}_N([-1,1]) \), such that for, \( 0 \leq j, k \leq N \), \( \kappa_j(\xi_k) = \delta_{jk} \) (the Kronecker symbol).

Since the discrete velocity function does not have the same degree in the different directions, we consider \( f^j \in 0, \ldots, N \) the fixed integer and we define the following polynomial of degree \( N - 1 \)

\[
\kappa_f^j(x) = \kappa_j(x) \frac{\xi_j - \xi_f}{x - \xi_f}, \quad j \in f^j,
\]
where \( J^j \) is the set \( \{0, \ldots, N\} \setminus \{j^j\} \). Then, we write the unknowns \((\mu^i_N; \varphi^i_N; p^i_N)\) in two dimensions as:

\[
\begin{align*}
\mu^i_N(x, y) &= \sum_{j=1}^{N-1} \sum_{k=1}^{N-1} \mu^i_{jk} \zeta_j(x) \kappa_k(y), \\
\varphi^i_N(x, y) &= \sum_{j=1}^{N-1} \sum_{k=1}^{N-1} \varphi^i_{jk} \zeta_j(x) \kappa_k(y), \\
p^i_N(x, y) &= \sum_{j \in J^j, k \in J(k, j) \neq (0,0)} \rho^i_{jk} \zeta_j(x) \kappa_k(y).
\end{align*}
\]

We consider the real pressure

\[ p^i_N(x, y) = \tilde{p}^i_N(x, y) - \frac{1}{4} (\tilde{P}_N, 1)_N \]

since the pseudopressure \( \tilde{p} \) is not in \( L^2_0(\Omega) \).

We denote by \((\mu^i_N; \varphi^i_N; p^i_N)\) the components of the solution \((W, U, P)\) on grid \((\xi_j, \eta_k)\) \(0 \leq j, k \leq N\). Problem (10) can be written as the following linear system:

\[
\begin{cases}
MW - A^T UI = 0, \\
AW + (DU + DU) - B^T P = F + N, \\
BU = 0,
\end{cases}
\]

where

- \( A^T \) and \( B^T \) are, respectively, the transposed matrix of \( A \) and \( B \).
- The matrix \( A \) is defined as:

\[
A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}.
\]

Since \( \text{curl}(\mu^i_N) = (\partial_y \mu^i_N, -\partial_x \mu^i_N) \), the coefficient of the matrices \( A_1 \) and \( A_2 \) correspond, respectively, to the terms \((\partial_y \mu^i_N, \varphi^i_N)\) and \((\partial_x \mu^i_N, \varphi^i_N)\). In two dimensions, the matrices \( A_1 \) and \( A_2 \) are square, their dimension is \((N-1)^2\).

- The matrix \( B = [B_1, B_2] \) where \( B_1 \) and \( B_2 \) are two matrices, the coefficients of which are deduced, respectively, from the terms \((\partial_x \varphi^i_N, p_N)\) and \((\partial_y \varphi^i_N, p_N)\). We note that the matrices \( B_1 \) and \( B_2 \) are not square, their dimension is \(N^2 - 1\) lines and \(2N(N-1)\) columns.
- \( M \) is a diagonal matrix, the coefficients of which are found from the values of \((\mu^i_N; \mu^i_N)\). Its dimension is \(N(N-1)^2\).
- The matrices \( D \) and \( N \) are, respectively, written as \( D = (D_1, D_2) \) and \( N_1 = (N_1, N_2) \) and are, respectively, calculated from the terms \( T_N(\mu^i_N^{-1}, \varphi^i_N; v_N) \) and \( T_N(\mu^i_N^{-1}, \varphi^i_N^{-1}; v_N) \).

### 3.3 Algorithm 1

To solve the linear system (11), we propose to remove the vorticity from the first equation and replace it in the second one.
This modification permits us to use the Uzawa algorithm (see [10, 11]) with only the velocity and the pressure as unknowns.

\[
\begin{align*}
W &= (M^{-1}A^T)U, \\
(AM^{-1}A^T + 2D)U - B^TP &= F + N, \\
BU &= 0.
\end{align*}
\tag{12}
\]

Let \( \tilde{A} = AM^{-1}A^T + 2D \). The second and third equations of (12) can be decoupled as

\[
U = \tilde{A}^{-1}((F + N) - B^TP),
\tag{13}
\]

and

\[
(B\tilde{A}^{-1}B^T)P = \tilde{A}^{-1}(F + N).
\tag{14}
\]

Since \( B\tilde{A}^{-1}B^T \) is symmetric and positive defined, we use the conjugate gradient method to solve system (14) at each iteration. Then, we deduce the velocity \( U \) and vorticity \( W \), respectively, from (13) and the first equation of (12).

The Uzawa algorithm used is written as:

**Uzawa algorithm**

- Let \( P_0 = 0 \).
- Initialization step:

\[
U_0 = \tilde{A}^{-1}((F + N) - B^TP_0), \quad W_0 = (M^{-1}A^T)U_0.
\]

- Iterations: \( n \geq 0 \)

  From \( U_n \) and \( P_n \):

\[
G_n = -BU_n, \\
V_n = \tilde{A}^{-1}B^T G_n, \\
\rho_n = \frac{\|G_n\|^2}{(B^T G_n, V_n)}, \\
P_{n+1} = P_n - \rho_n G_n, \\
U_{n+1} = U_n + \rho_n V_n, \\
W_{n+1} = M^{-1}A^T U_{n+1}.
\]

**3.4 Algorithm 2**

We propose a second algorithm to solve problem (11). We begin first by inverting the matrix \( G \) given by

\[
G = \begin{pmatrix}
M & -A^T & 0 \\
-A & 2D & B^T \\
0 & B & 0
\end{pmatrix}.
\]
and then find $U$, $W$ and $P$.

Since $G$ is symmetric and positive defined, we use the preconditioned gradient conjugate method.

4 Numerical results

4.1 Comparison of two algorithms

We consider the Bercovier–Engelman continuous solution of problem (2) (see [16]) given by

$$
\begin{align*}
\mu(x,y) &= 6((1-x^2)(1-4x^2)(1-y^2)^3 + (1-y^2)(1-4y^2)(1-x^2)^3), \\
\varphi_x(x,y) &= -6(1-y^2)^2y(1-x^2)^3, \\
\varphi_y(x,y) &= 6(1-x^2)^2x(1-y^2)^3, \\
p(x,y) &= xy.
\end{align*}
$$

The aim of this first numerical test is to study the accuracy of the two methods. We consider the viscosity $\nu = 5 \times 10^{-2}$ and the tolerance $\epsilon = 10^{-10}$.

In Fig. 1 we present a comparison of the convergence order of the two presented algorithms based on the error curves on the solution $(\mu, \varphi, p)$ when $N$ varies between 5 and 20. Figure 1(a) and (b) correspond, respectively, to the resolution using Algorithms 1 and 2. This shows clearly that the obtained error is better when using Algorithm 2.

$\kappa(\tilde{A})$ and $\kappa(G)$ are, respectively, the condition number of the matrix $\tilde{A}$ and that of the matrix $G$.

Tables 1 and 2 show the number of convergence iterations and the corresponding average error for the unknowns (vorticity, velocity and pressure). While comparing the two results, it is clear once again that the second method has a better accuracy and converges within a few iterations.

![Figure 1](image_url) Convergence for the solution defined in (15)

Table 1 Iterations and errors for Algorithm 1

| $N$ | 5  | 7  | 15 | 20 |
|-----|----|----|----|----|
| Iterations | 45 | 192 | 345 | 677 |
| Error | 2.0012 | 0.5672 | 0.0032 | $10^{-3}$ |
Table 2 Iterations and errors for Algorithm 2

| N   | 5    | 7    | 15   | 20   |
|-----|------|------|------|------|
| Iterations | 7    | 15   | 21   | 28   |
| Error     | 3.1411 | 0.0012 | 10^{-15} | 10^{-25} |

Table 3 The condition number for the matrices of the two algorithms

| N   | 5    | 7    | 12   | 20   |
|-----|------|------|------|------|
| \( \kappa(\tilde{A}) \) | 50.21 | 501.36 | 1.708 \times 10^5 | 5.708 \times 10^7 |
| \( \kappa(G) \) | 111.12 | 27.08 | 67.05 | 91.38 |

Figure 2 The discrete solution \((\mu_N, \phi_{Nx}, \phi_{Ny}, p_N)\)

We conclude that:

- Although Algorithm 1 based on the Uzawa method is easier to implement and requires much less memory space, it converges after a large number of iterations, which is due to the fact that the conditioning \( \kappa(\tilde{A}) \) of the matrix \( A \) is high if we compare it with the conditioning \( \kappa(G) \) of the matrix \( G \), see Table 3 (hundreds of iterations for an error of \( 10^{-5} \) order, see Fig. 1 and Table 1).
- The convergence order of Algorithm 2 is much better than the order produced by Algorithm 1 (22 iterations for a relative error approximatively equal to \( 10^{-25} \), see Fig. 1 and Table 2).

4.2 Solution using Algorithm 2

Consider the numerical resolution of the problem (6) with \( f = (0, x^2 y) \) using Algorithm 2. Figure 2 shows, from top to bottom and left to right, the obtained values of the vorticity \( \mu_N \), the two components of the velocity \( (\phi_{Nx}, \phi_{Ny}) \) and the pressure \( p_N \), for \( N = 30 \).
In Fig. 3 we present the vector field of the velocity corresponding to the data $f = (0, 0)$, and a boundary condition $g$ given by (see [8]),

$$
g(-1, y) = -(1 - y^2), \quad g(1, y) = (1 - y^2), \quad g(x, \pm 1) = 0, \quad (16)$$

where $N = 40$. We note that the vorticity $\mu_N$ and pressure $p_N$ are null, since we handle a Poiseuille linear flow.

5 Conclusion and future work

In this work, we show the efficiency of the global resolution compared to the Uzawa algorithm adapted to the resolution of the discrete problem issued from the spectral discretization of the vorticity, velocity and pressure formulation of the Navier–Stokes problem. We achieved a good convergence with the global resolution through the transformation of the matrix to a symmetric and positive defined one. As future work, we are looking at applying the global method to the discretization by the spectral element methods for handling more complex domains.

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Competing interests

The authors declare that they have no competing interests.

Authors’ contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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