Variational approach for Bose–Einstein condensates in strongly disordered traps

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Abstract
Recently, Nattermann and Pokrovsky (2008 Phys. Rev. Lett. 100 060402) have proposed a scaling approach for studying Bose–Einstein condensates in strongly disordered traps. In this paper we implement their scaling argument in the framework of the variational method for solving the time-dependent Gross–Pitaevskii equation. We consider atomic gases with both short-range s-wave interaction and long-range anisotropic dipolar interaction. The theory is addressed to the regime of strong disorder and weak interactions where the physics is dominated by the collective pinning due to the disorder. The phenomenon of condensate fragmentation in dipolar gases is also analysed.

(Some figures in this article are in colour only in the electronic version)

Introduction
Motivated by the recent progress in ultracold alkali atoms in disordered traps [2–4], Nattermann and Pokrovsky proposed [1] a semiquantitative approach for Bose–Einstein condensates in strong random potentials where standard perturbative methods are expected to fail. Their analysis is based on the evaluation of the mean-field energy due to the fluctuations of the random potential according to a scaling argument introduced by Larkin [5] for studying the effect of defects in flux-line lattices. The scaling approximation requires that the correlation length of the disorder be much shorter than the correlation length of the liquid. In this paper, we implement the method of Nattermann and Pokrovsky in the framework of the time-dependent variational method of Perez-Garcia et al [6]. As a result, we obtain a simple set of equations, which describe the equilibrium and the low-energy dynamics at zero temperature of a Bose–Einstein condensate in a strong random potential. Besides the oscillator and the scattering length, a new quantity enters into the problem, namely the Larkin length \( L \) associated with the collective pinning of the condensate.

The time-dependent variational method represents a very good approximation of the Gross–Pitaevskii equation [7, 8]. Moreover, it can interpolate quite successfully from the low-density regime to the strong coupling Thomas–Fermi gas. The present extension to disordered traps, however, presents some restrictions. Since the fluctuating centre of attraction of the disorder may not coincide with the centre of the harmonic trap, the model is fully consistent only when either the harmonic trap or the disorder is responsible for the localization of the condensate. Moreover, the Gaussian variational ansatz rules out from the outset the solutions describing the multi-fragmented state of the condensate [1]. Despite these limitations, novel quantitative predictions can be obtained with regard to the disorder dominated regime. In this limit the condensate becomes non-superfluid and is characterized by a generalized correlation length \( L \) larger than the size of the cloud.

The paper is organized as follows. In section 1.1 we introduce the general formulation of the problem. We determine the size of the ground state at equilibrium and we calculate the expressions for the frequencies of the low-lying excitations. In section 1.2 we discuss in detail the 3D gas in the presence of a strong delta-correlated disorder. We show that, reducing the atomic scattering length by means of Feshbach resonance techniques, the predictions of the theory for the single connected fragment might be tested experimentally. In section 1.3 low-dimensional gases are considered. Section 1.4 contains some considerations on the

\[ L = \frac{\hbar^2}{m R_0} \]

\[ \xi_{\text{heal}} = \ell \left( \frac{N a}{\ell} \right)^{-\frac{1}{5}} \]

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applicability of the method in the case of a disorder with finite correlation length. In section 2.1 we introduce a long-range anisotropic dipolar interaction and we study the stability of the single connected fragment. The nature of fragmentation in the presence of the dipolar interaction is analysed in section 2.2. The static on-average characteristic properties of the fragments are determined at the level of the Imry–Ma level of approximation [9].

1. Time-dependent variational method with disorder

1.1. Variational equations

We begin considering the Hamiltonian of a gas of trapped bosons interacting through s-wave scattering:

\[ \hat{H} = \int d^3x \left( -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{cap}}(x) \right) + U(x) + \frac{4\pi \hbar^2 a}{m} \bar{\psi} \bar{\psi}, \]

(1)

where \( a \) is the s-wave scattering length and the harmonic trapping potential is \( V_{\text{cap}}(x) = \sum_{i=x,y,z} m \omega_i^2 \bar{R}_i^2/2 \). The disorder potential \( U(x) \) is chosen to be Gaussian distributed characterized by the average values \( \langle U \rangle = 0 \) and \( \langle U(x)U(x') \rangle = \left( \kappa^2/b^3 \right) K_b(x-x') \), where \( b \) denotes the correlation length of the disorder. We will assume that \( K_b(x) \) is a smeared out \( \delta \)-function \( K_b(x) = e^{-x^2/b^2/(2\pi)^{3/2}} \).

The variational calculation follows the same outline as the clean case. Therefore, in what follows, we refer the readers to [6] for the details of the derivation. We consider the normalized variational wavefunction in the case of a 3D fully anisotropic configuration

\[ \psi(x, t) = \frac{N^{1/2}}{\pi^{3/4} \tilde{R}^2(t)} e^{-\sum_{i=x,y,z}^{\frac{1}{2} i} \left( \kappa^2/b^3 \right) K_b^{(i)}(x-x')}, \]

(2)

where we have defined the geometric average \( \tilde{R} = \left[ R_x(t) R_y(t) R_z(t) \right]^{1/3} \) and the variables \( R_x \) and \( R_y \) are time-dependent variational parameters. \( R_z \) is related to the size of the system while the imaginary width \( i B_0 \) of the Gaussian ansatz is necessary to include the fluctuations in the variational procedure. Without this latter the minimum principle would just lead to the condition of static equilibrium.

Physically, the true ground state of the stationary Gross–Pitaevskii equation in the presence of interactions and disorder is expected to deviate strongly from the ground state of the noninteracting system. Nevertheless, for a clean system, it has been shown that the dynamical trial wavefunction (2) is a very good approximation of Gross–Pitaevskii at finite \( N \) [6–8]. Inserting the trial wavefunction in the Lagrangian relative to the Hamiltonian of (1) we find

\[ L = L_0 + \frac{\kappa}{\pi^{3/4}} N \prod_{i=1}^{3} \left[ 2 \bar{R}_i^2(t) + \frac{b^2}{\kappa} \right]^{1/4} \]

(3)

where \( L_0 \) is the well-known contribution in the absence of disorder [6]. The new term represents the on-average contribution of the disorder fluctuations according to the scaling argument of Nattermann and Pokrovsky [1, 5]. It introduces a new relevant length scale, namely the Larkin length \( L = \left( 2^{3/2} \pi^{3/4} \hbar^2/m^2 \kappa^2 \right) \), associated with the pinning energy due to the disorder [6]. The length scale \( L \) was introduced by Larkin [5] in connection with the onset of collective pinning of vortex lines in type-II superconductors. In the case of a trapped gas, the analogy of the (delocalizing) elastic energy relative to the lattice distortion corresponds to the kinetic energy of the atoms. Since the scaling argument requires to have both \( L \) and \( b \) much smaller than the generalized healing length \( \xi_{\text{heal}} \) (see footnote 1), i.e. \( L \ll \xi_{\text{heal}} \) and \( b \ll \xi_{\text{heal}} \), the validity of (3) is in general restricted to the quantum limit \( b \ll L \). The classical limit \( b \gg L \), when many levels occupy the typical well, will be briefly discussed separately.

In what follows it is convenient to rescale the quantities in units of the harmonic oscillator such as \( r_i = R_i/\ell \) and \( (\hbar/\ell) \equiv (b/\ell)(\ell/R_0) \) where \( \ell = \sqrt{\hbar/m\omega} \) is the harmonic oscillator length. The anisotropy of the external trapping is taken into account by setting \( \omega_1 = \lambda_1 \omega \). Without loss of generality we will consider a trap with cylindrical symmetry with anisotropy factors \( \lambda_1 = \lambda_y = 1 \) and \( \lambda_z = \lambda \). In the absence of disorder this implies that the angular momentum along the z-axis is conserved and we can label the modes by the azimuthal angular quantum numbers \( m \).

Next, we derive the Euler–Lagrange equations for the variables \( R_i(t) \) and \( B_i(t) \) relative to the Lagrangian (3). Then, eliminating the \( B_i \) variables, we obtain a closed system of differential equations for \( R_i \). These equations can be viewed as describing the motion of a point-like particle in an effective potential. The position of equilibrium at rest of the particle is determined by the minima of the effective potential. Using rescaled units \( r_{0x} = r_0 \) and \( r_{0z} \), this leads ultimately to the conditions

\[ r_{0x}^4 + y \frac{r_{0x}}{r_{0z}^{1/4}} f_3 \prod_{j=1}^{3} f_j^{3/4} = 1 + \alpha \frac{r_{0z}}{r_{0x}}, \]

(4)

\[ \lambda^2 r_{0x}^2 + y \frac{r_{0x}}{r_{0z}^{3/2}} f_3 \prod_{j=1}^{3} f_j^{3/4} = 1 + \alpha \frac{r_{0z}}{r_{0x}}, \]

(5)

where we have defined the functions \( f_3(r_{0z}) \equiv 2(2+\sqrt{2}) \) and \( g_3(r_{0z}) \equiv (b/\ell)^2 f_3/2 \), and we have introduced the Thomas–Fermi parameter \( \alpha = \sqrt{27/8} (N a/\ell) \) together with the disorder strength parameter \( y = \sqrt{\ell/L} \). In these two latter equations, the lhs describes the confinement due to the trap and to the disorder. On the rhs this effect is counterbalanced by the kinetic energy and the repulsive mean-field interactions. The limit \( L \to \infty \) corresponds to the theory for a clean system [6].

The low-lying excitations of the system can be calculated by means of a harmonic expansion \( r_i(t) = r_{0i} + \delta r_i(t) \) around the equilibrium positions (4) and (5). Making a dynamical

2 For a delta-correlated Gaussian distributed random potential, the Larkin length introduced in [1] coincides essentially with the elastic mean free path \( \ell_{el} \). At very low temperature, this latter is related to the scattering time \( \tau_e \) of a particle of velocity \( v_e = \hbar/k/m \) through the relation \( \tau_e = v_e \tau_e \). For a dilute bosonic gas in a delta-correlated disorder we have \( 1/\tau_e = m k^2/8\pi^2 \hbar^3 \) and thus \( \ell_{el} = \sqrt{8\pi L} \). For a long-range correlated disorder with non-singular correlator \( \langle U(x)U(x') \rangle = (\kappa^2/b^3) K_b(k-x)/b \), the elastic mean free path is momentum dependent and at long wavelength has the form \( \ell_{el} \propto K_b(k) \), where the shape of \( K_b \) depends on \( K_b \).
ansatz for \( \delta r_1(t) \), the calculus of the frequencies for the small oscillations is reduced to the calculation of the eigenvalues of a \( 3 \times 3 \) real symmetric matrix \( A_{ij} \), whose elements are

\[
A_{11} = A_{22} = 1 + \frac{3}{r_0^2} + \frac{2\alpha}{r_0^2} \rho_0 \Xi \nonumber \]
\[
- \frac{3\gamma}{2r_0^2} f_z^{3/2} f_z^{1/4} \left\{ 1 - \frac{5}{3} g_x \right\} \nonumber \]
\[
A_{12} = \frac{\alpha}{r_0^2 \rho_0} - \frac{\gamma}{2r_0^2} f_z^{3/2} f_z^{1/4} \left\{ 1 - g_y \right\} \nonumber \]
\[
A_{13} = \frac{\alpha}{r_0^2 \rho_0} - \frac{\gamma}{2r_0^2} f_z^{3/2} f_z^{1/4} \left\{ 1 - g_z \right\} \nonumber \]
\[
A_{33} = \lambda^2 + \frac{3}{r_0^2} + \frac{2\alpha}{r_0^2} \rho_0 \Xi \nonumber \]
\[
- \frac{3\gamma}{2r_0^2} f_z^{3/2} f_z^{1/4} \left\{ 1 - \frac{5}{3} g_z \right\} . \nonumber \]

We denote the eigenvalues as \( \omega_{a,b,c} \). In the absence of disorder \( \omega_{a,b,c} \) corresponds to the frequencies of the quadrupole modes with quantum numbers \( n = 0 \) and \( l = 2 \), while \( \omega_c \) is the frequency of the monopole mode with \( n = 1, l = 0 \).

It is important to remember that the scaling argument leading to (3) applies under the strong disorder condition \( \ell \gg L \). Since the centre of the attractive domain formed by the disorder may not coincide with the centre of the harmonic trap, the variational ground state is exact only when the system is trapped either by the static fluctuations of the random potential or by the harmonic trap. In principle, one could try to extend the ansatz (2) including an additional variational parameter which describes the centre of the condensate. However, the disorder term in (3) would not bring any new contributions to the equation of motion of the new variable with respect to that of the clean model [6]. This shows that the interplay between the two different confining mechanisms cannot be incorporated in the theory, which is a direct consequence of the on-average nature of the Larkin energy considered in (3). Therefore, the crossover region has to be considered at best as an extrapolation.

Moreover, our ansatz for the wavefunction cannot describe a multi-fragmented state, which is expected to occur in a wide range of parameters [1]3. In this case our solution may occur as a metastable state [1]. Due to these limitations, in the next sections we will focus our analysis mainly on the single fragment non-superfluid regime where the cloud is trapped by the disorder and the physics is dominated by the Larkin length.

### 1.2. Strong disorder with zero correlation length

In the limit of a very short correlated disorder \( b \leq L \) the corrections due to finite correlation length are very small. Therefore, as first approximation, we can consider a delta-correlated disorder, and put \( b = 0, f_z = 1 \) and \( g_1 = 0 \) in (3)–(6). Moreover, in this limit and for low-energy oscillations, the wavelength of the modes is much larger than the distance over which the disorder varies. Therefore, the oscillations can be considered self-averaging and we expect the values of the frequencies to be independent of the specific realization.

At low densities and strong disorder, when both the radial and the axial oscillator lengths of the traps are larger than the Larkin length, the gas is confined mainly by the random potential and the cloud is spherically symmetric. The equilibrium conditions in (4) and (5), which determine the size of the system, can be approximated by \( \gamma \sqrt{\ell_0^2} = 1 + \alpha/\rho_0 \). In the limit \( N \ll (\ell/a)^5/7 \) the interactions can be neglected, the size of the cloud is \( R \approx L \) and we find

\[
\omega_a = \omega_b = c_0 \theta (\ell/L)^2 \nonumber \]
\[
\omega_c = (c_0/2) \omega (\ell/L)^2. \nonumber \]

where the constant \( c_0 \) depends on the normalization of the variational trial wavefunction we have chosen. Nevertheless, although the exact physical value of the constant \( c_0 \) cannot be rigorously determined by our approach, the ratio \( \omega_b/\omega_c \) = 2 of the quadrupole and the monopole does not depend on it. For a larger number of particles \( L \ll Na \ll (\ell/a)^5/7 \ell \) the size of the system is determined by the competition between the disorder and the interaction and we find

\[
\omega_a = \omega_c \propto \frac{\ell^2}{(Na)^{1/6}L^{5/6}} \left( \frac{L}{Na} \right)^{1/6} \nonumber \]
\[
\omega_b \propto \frac{\ell^2}{(Na)^{1/6}L^{5/6}}. \nonumber \]

In this interval the generalized healing length \( \xi \) reaches a point at which the size of the cloud is on the order of \( (Na)^{2/3} L^{1/3} \). Moreover, the two curves exhibit an ‘avoided crossing-like’ feature. For an even larger number of particles the external harmonic potential dominates over the disorder and the frequencies approach the well-known Thomas–Fermi [11]. The crossover between the different regimes of the values of the quadrupole and monopole modes is illustrated in figure 1(a) for some typical experimental parameters.

At \( N \gg (\ell/L)^5/7 (\ell/a) \) the crossover to the Thomas–Fermi regime of the clean theory occurs [1]. Deep into the Thomas–Fermi regime the system is superfluid with \( \xi_{\text{heal}} \ll L \). The critical number of particles \( N_c \) where transition to superfluid occurs can be estimated by equating the Larkin length with the superfluid healing length obtained from equations (4) and (5) and the definition given in footnote 1. This leads to \( N_c \sim \xi^2/aL^3 \) in agreement with the theory of [10, 17]. In the Thomas–Fermi limit the disorder can be treated perturbatively and a small negative linear shift in \( \xi_{\text{heal}}/L \) would be expected [12]. However, in figure 1(a) a positive shift of the frequency of the breathing mode is found. This is not surprising since the present theory has to be considered as exact only in the opposite limit \( \xi_{\text{heal}} \gg L \) where the disorder is dominating.

3 In this paper we neglect systematically the logarithmic corrections found recently in [10].
The frequencies plotted in figure 1(a) refer to a single connected condensate. However, above $N \gtrsim \mathcal{L}/a$ the single connected condensate is a higher energy metastable state, since the ground state is expected to undergo fragmentation [1]. In the fragmented state, we do not expect sharply defined frequencies. More likely, they should be distributed in some interval. Note that the possibility of observing the single connected condensate, in the regime where fragmentation is expected, could arise by means of a sudden decrease of the oscillator frequency. However, it is difficult to make further estimates about the different timescales involved in the experiment. In order to prevent fragmentation, we would like to push the single fragment solution at larger particle numbers. For a given disorder ($\mathcal{L}$) and trap configuration ($\ell$) such that the strong disorder condition $\ell \gg \mathcal{L}$ is satisfied, this can be achieved by maximizing the ratio $\mathcal{L}/a$. Experimentally, this can be obtained by tuning the scattering length close to zero (zero-crossing) by means of a Feshbach resonance as proposed in figure 1(b).

In the case of negative scattering length there is only a metastable state of a finite radius. This state becomes unstable at a critical particle number $N_c$. In figure 2(a), we plot the frequencies of the three modes in the case of an attractive interaction for an isotropic trap with $\omega = 2\pi 25$ Hz, $a = -100 a_0$ and $\mathcal{L} = 200 a$. The divergence signals the instability of the gas when the number of particles reaches the critical value $N_c$.

### 1.3. Anisotropic traps and lower dimensions

We have seen that, at low densities, when the physics is determined only by the balance between the random potential and the kinetic energy, the frequencies of the oscillations do not depend on the external harmonic potential. Therefore, the anisotropy of the trap becomes irrelevant. Increasing the number of particles, the mean-field interaction becomes important and the harmonic trap affects indirectly the results through the Thomas–Fermi parameter $\alpha$. At even larger particle number the system enters the Thomas–Fermi regime where the harmonic trap dominates over the disorder and the anisotropy plays its usual role [11]. If we consider, for
example, an elongated cigar along the axial $z$-direction, with $\omega_{1,2} \gg \omega_z$, this description holds as long as we have $L \ll \ell_{\perp} \ll \ell_{||}$. However, when $\ell_{\perp} \ll L \ll \ell_{||}$, the system can pass through the different regimes in the axial directions while its ground state in the radial direction is frozen to that of the harmonic oscillator of the radial harmonic confinement.

Hereewith, we limit our discussion to the single connected solution, neglecting the fragmented low-dimensional condensate where it occurs [1]. Using the definitions $a_{1D} = \ell_{\perp}^2/a$ and $L_{1D} = (L\ell_{\perp})^{1/3}$ we have that for small $N \ll a_{1D}/L_{1D}$ the axial radius is determined by the equation $\gamma L_{1D}^{3/2} = 1$. This leads to the results

$$\omega_{\perp} = \omega_{\parallel} = 2\omega_{L},$$

$$\omega_{\perp} \propto \omega_{\parallel}(\ell_{\perp}/L)^{2/3}.$$  

At larger $N$ such that $a_{1D}/L_{1D} \ll N \ll a_{1D}/L_{1D}(\ell_{\perp}/L_{1D})^{4/5}$ the axial radius is determined in the leading order by the equation $\gamma L_{1D}^{3/2} = a\ell_{\perp}$ and we have in first approximation

$$\omega_{\perp} = \omega_{\parallel} = 2\omega_{L},$$

$$\omega_{\perp} \propto \omega_{\parallel}(\ell_{\perp}/L)^{3/2}(\ell_{\perp}/N\lambda)^{5/2}. $$

At even larger values the Thomas–Fermi regime is realized in the axial direction while in the radial direction the ground-state wavefunction can still be determined by the free harmonic oscillator. Therefore, we have that the frequency of the breathing mode approaches the value $\omega_{c} = \sqrt{3\omega_{1}}$ characteristic of the so-called 1D mean-field regime [13]. In order to appear, this regime requires the two conditions $N a L/\ell_{\perp} \ll 1$ and $(N a/\sqrt{\lambda} L_{\perp})^{4/5} \gg 1$ [14]. Finally, for $N a L/\ell_{\perp} \gg 1$ the gas enters the full Thomas–Fermi regime and the ‘quadrupole’ mode approaches the strongly anisotropic 3D Thomas–Fermi result [11] $\omega_{c} = \sqrt{5/3} \omega_{1}$ while $\omega_{z} = 2\omega_{L}$ and $\omega_{\perp} = 2\omega_{L}$. The full crossover is shown in figure 2(b) for the lowest axial mode $\varphi_1$ in a $^{87}$Rb condensate confined in a strongly anisotropic cigar-shaped trap with $\omega_2 = 2\pi \times 150$ Hz, $\omega_{1} = 0.01\omega_{2}$, $a = 1500a_{0}$ and $L = 300a_{0}$. For these parameters, $\ell_{z}/L \approx 0.36$ and $\ell_{\perp}/L \approx 3.7$. Moreover, the gas cannot enter the strong interacting Tonks regime since it would require to fulfill the two conditions $\ell_{z}/a \ll \ell_{\perp}/2a_{0}^{2}$ and $N \lambda \ll 1$ simultaneously [14]. In our case, we have $\sqrt{2} \ell_{\perp}/a \approx 11$. Nevertheless, the Tonks gas remains beyond the reach of our mean-field approach.

The treatment of a 2D disc-like geometry trap when $\lambda \rightarrow \infty$ essentially follows the same outline. In this case we define $\omega_{1,2} \equiv \omega_{||} \ll \omega_{\perp} = \omega_{c}$. Similar to the 1D configuration, a non-trivial interplay between disorder and low dimensionality arises only when $\ell_{\perp} \ll L \ll \ell_{||}$. Except that for very large $N$, the ground state in the $z$-directional coincidence with the lowest eigenstate of the harmonic trap, namely $r_{\perp}^{2} = 1/\lambda^{2}$. The 2D analogue of (12) and (14) are respectively $\omega_{\perp} \propto \omega_{||}(\ell_{\perp}/L)$ in the small $N$ limit and $\omega_{\perp} \propto (\ell_{\perp}/L)(\ell_{\perp}/N\lambda)^{3/2}$ in the moderate interacting regime. At larger values of $N$, for sufficiently strong anisotropies, the size of the cloud in the $x, y$ plane is determined by the competition between harmonic trap and interactions while in the $z$-direction the ground state can still be that of the free harmonic oscillator. This is the 2D analogue of the 1D mean-field regime for which $\omega_{c} = 2\omega_{1}$ [15]. Finally, when $N \rightarrow \infty$, the $\lambda \rightarrow \infty$ limit of the 3D Thomas–Fermi is recovered where $\omega_{c} = \sqrt{15/3} \omega_{1}$ [11].

1.4. Finite correlation length

The delta-correlated disorder approximation, discussed so far, is appropriate for a disorder with very short correlation length, i.e. when $b \ll \ell$. In this case, the correlation length of the disorder is generally much shorter than the correlation length of the system, which represents a necessary condition for applying the Larkin scaling argument. The small corrections due to finite $b$ can be taken into account using the full equations (4)–(6). Differently, when $b \gg \ell$, the use of the expression for disorder energy in (3) is somehow questionable. Moreover, also self-averaging is expected to be much less efficient than in the limit delta-correlated disorder. Therefore, we conclude that the description based on (4)–(6) holds as far as the limiting case of a delta-correlated disorder is approached.

Nevertheless, it is interesting to make the following remark. The classical limit $b \gg \ell$ has been recently investigated in [10, 16]. Near one of its typical minima, the random potential can be approximated by a harmonic potential well of depth $U_{0} = \hbar^{2} / 4a_{0}^{2} b^{2}$ and width $b$. The bound state is located very close to the minima of the random potential and the number of levels in a well is large. From $U_{0} \equiv m_{0} \omega_{b}^{2} / 2$ we obtain the oscillator length of the typical fluctuating well as $\ell_{b} = b/(\ell_{b})^{1/2}$ [17, 16]. Interestingly, this length can be obtained from (4) in the limit of a small number of particles where the harmonic trap and the mean-field energy can be neglected. At a larger particle number the size of the fragment is determined by the competition of the mean-field interaction and the disorder energy. This yields $R \sim (N a L_{1D} / b^{2})^{1/5}$, which agrees with the size of the fragments in the multi-domain state found in [17] by different methods. This analysis reveals that the Gaussian ansatz (2) is able to reproduce the ground-state localized state also for disorder with long-range correlated potential.

2. Dipolar gas

2.1. Stability of the gas

The time-dependent variational approach can be extended to the presence of the long-range anisotropic dipolar interaction [18]. If we assume that the dipoles aligned along the $z$-direction, the interaction between two polarized dipoles can be written as

$$V(r) = \mu_{0} \mu^{2} \frac{1}{4 \pi} \sqrt{\frac{\pi}{5}} \frac{Y_{20}(\theta)}{r^{5}},$$

where $\theta$ is the angle between $r$ and the direction along which the dipoles are pointing. The constant $\mu_{0}$ is the magnetic moment of the atoms. The strength of the dipole interaction relative to the short-range potential will be expressed through the dimensionless quantity $\varepsilon \equiv \sqrt{\pi / 5} (\mu_{0} a^{2} m / 4\pi^{2} \hbar^{2} a)$. Alternatively, the strength of the two potentials can be compared by defining a characteristic
length scale associated with the dipole–dipole interaction. From the uncertainty principle, we can define \( a_{\text{dip}} \) as the distance at which the dipole potential energy (15) equals the kinetic energy. This yields

\[
a_{\text{dip}} = \sqrt{\frac{\pi}{\alpha a_0^2 m}} \sqrt{\frac{\epsilon}{E_0}},
\]

where we have chosen the prefactor such that \( \epsilon = a_{\text{dip}}/a \).

In the presence of the disorder and dipole interaction, the equilibrium conditions of (4)–(5) become

\[
r^4_{\text{dip}} + \gamma r^2_{\text{dip}} f_{r_0} \prod_{j=1}^3 f_{r_0}^{1/4} = 1 + \alpha \frac{r_{\text{dip}}}{r_0} \left[ 1 - \epsilon \mathcal{F} \left( \frac{r_{\text{dip}}}{r_0} \right) \right] \]

(17)

\[
\lambda^2 r^4_{\text{dip}} + \gamma^2 r^2_{\text{dip}} f_{r_0} \prod_{j=1}^3 f_{r_0}^{1/4} = 1 + \alpha \frac{r_{\text{dip}}}{r_0} \left[ 1 - \epsilon \mathcal{G} \left( \frac{r_{\text{dip}}}{r_0} \right) \right],
\]

(18)

with [18] \( \mathcal{F}(\xi) = [\sqrt{\frac{5}{\pi}}/6(1 - \xi^2)^{3/2}] - 4\xi^4 - 7\xi^2 + 2 + 9\xi^4 H(\xi), \mathcal{G}(\xi) = [\sqrt{\frac{5}{\pi}}/3(1 - \xi^2)^{3/2}] - 2\xi^4 + 10\xi^2 + 1 - 9\xi^4 H(\xi) \) and \( H(\xi) = \arctanh \sqrt{1 - \xi^2}/\sqrt{1 - \xi^2} \). Analogously to the clean gases, these equations have at most only one stable solution. Here we are interested in the regime of strong disorder and moderate interactions where the gas is confined mainly by the random potential and is constituted by a single connected fragment. In this limit the anisotropy of the harmonic trap does not play any role and, in the absence of the dipolar interaction, the cloud would have spherical symmetry. However, turning on the dipolar potential the fragment tends to become more prolate along the direction of the dipoles orientation. The situation is illustrated in figure 3(a), where we have considered a very short scattering length in order to shift towards large \( N \) the onset of multiple fragmentation which occurs above \( N \geq \lambda L/a \). Moreover, such a small value of the scattering length permits us to access the strong dipolar regime \( a_{\text{dip}}/a \geq 1 \) even for gases made of atoms with small magnetic moments. This regime has been recently realized in experiments with \( ^{52}\text{Cr} \) condensates near a Feshbach resonance [20, 21]. For any fixed \( N \), the fragment collapses at some critical value of \( a_{\text{dip}}/a \). In figure 3(b) we plot this threshold as a function of the number of particles at fixed \( a \) for two different values of the Larkin length. For given \( a \) and \( N \) at shorter Larkin length the condensate is less stable against the attractive dipole interaction. Note that a shorter Larkin length amounts to a stronger centre of attraction due to the fluctuations of the random potential and thus to a smaller volume occupied by the fragment.

2.2. Fragmented state

So far we have not discussed the possibility of a multi-fragmented state since the variational method is based on a single connected ground state. However, in [1] it has been shown that for a large region of the parameters space, strong disorder favours fragmentation. It is therefore interesting to analyse the situation when a dipolar interaction is present as well. In the spirit of [1], we evaluate the Hamiltonian operator on the ground state (2). Assuming cylindrical symmetry and rescaling energies in units of the harmonic trap energy \( \hbar \omega \), the energy per particle of the system can be written as

\[
\frac{\epsilon}{\hbar \omega} = \frac{1}{2r^2} \left[ 1 + \frac{\sigma^2}{2} \right] + \frac{1}{2} \left( 1 + \frac{\lambda^2}{2\sigma^2} \right) - \frac{\sqrt{\xi}}{2r} \sqrt{\frac{\sigma}{2r}} \left[ 1 - \epsilon I_1(\sigma) \right],
\]

in terms of the variational parameters \( r \equiv r_s \) and \( \sigma = r/r_s \). The deformation function \( I_1(\xi) = \sqrt{\frac{5}{\pi}}[1 + 2\xi^2 - 3\xi^4 H(\xi)]/(1 - \xi^2) \) is continuous, monotonic and positive for \( \xi < 1 \) and negative for \( \xi > 1 \).

In the absence of the dipolar interaction and for a strong given disorder potential the total energy \( E(N) = NE \) has a minimum \( E_1(N_i) \) at negative energy as shown in figure 4(a). According to the Imry–Ma criterion [1], at \( N \) larger than \( N_i \) it is energetically favourable to split the gas into fragments of energy \( E_i \). Eventually, at even larger \( N \), the oscillator energy per particle of the fragmented state reaches the energy per
particle from the disorder and the crossover to the Thomas–Fermi regime occurs [1].

The effect of the dipolar interaction on the phenomenon of the fragmentation can be seen as follows. The attractive dipole interaction shrinks the total condensate volume causing an increase of the kinetic energy which tends to delocalize the condensate against the collective pinning induced by the disorder. This is consistent with the numerical results shown in figure 4(a). At fixed disorder, finite values of the dipolar interaction shift upwards the minimum of the energy \( E = EN \) until it becomes positive. Then, strictly speaking, the possibility of having fragmentation is ruled out at least at the level of the Imry–Ma argument. Nevertheless, at sufficiently small dipolar interaction fragmentation can occur. In the presence of anisotropic interactions the Imry–Ma [9] argument does not imply that domains have to be spherical. The typical fragment will be preferentially cigar shaped with the long axis parallel to the dipoles [22]. This means that, while in a single realization of the disorder the various fragments should appear with random shape, they will have on average a characteristic deformation. The dependence on the dipolar interaction of the typical deformation can be estimated evaluating the equilibrium radii \( \tilde{r}_0 \) and \( \tilde{r}_\infty \) in the minimum of the energy curve \( E(N) \). The result obtained is shown in figure 4(b) for two different realizations of the disorder. The curves stop where the Imry–Ma argument for the existence of fragmentation becomes invalid. Note that, for the chosen parameters, this boundary is smaller than the critical dipolar strength at which the cloud would collapse.

Conclusions

We have proposed a hybrid approach, originated from the variational method for the Gross–Pitaevskii theory and the Larkin–Imry–Ma scaling argument of [1], aimed to study zero temperature Bose–Einstein condensates in strongly disordered traps. Similarly as for other disordered systems, the Larkin–Imry–Ma scaling analysis, despite its intrinsic limitations, allows us to make semi-quantitative predictions in situations where no mean-field description seems possible [23]. The theory addresses the problem of the condensate in the limit of strong disorder and moderate interaction. There, the dynamics of the condensate is dominated by a new length scale, namely the Larkin length, associated with the collective pinning of the condensate. In the case of isotropic interaction, we have investigated the stability and the low-lying excitations of the system at different dimensionalities when the ground state consists of a single connected condensate. We have suggested that, for given disorder strength and trapping parameters, this regime could be observed experimentally by tuning the scattering length near to zero using Feshbach resonance techniques [24]. In the case of dipolar interactions, our theory offers in addition the possibility of investigating the effects of the anisotropic interaction on the phenomenon of fragmentation. Predictions about the typical deformation ratio of the fragments can be made. We conclude that the realization of types of disorder with correlation length smaller than all other length scales could allow us to test the theory in the laboratory.

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Figure 4. (a) Energy of a 52Cr condensate in a strong random potential in the absence of the dipolar interaction (red) and for \( a_{dip} = 0.4a \) (blue) as a function of \( N \). The other parameters are \( a = 100a_0 \), \( \lambda = 1 \), \( \omega = 2\pi 50 \) Hz, \( L \sim 30a \) and \( \ell/L \sim 9.5 \). The inset shows the corresponding energy per particle. (b) Typical deformation of the fragments as a function of the relative dipolar strength for two different values of the disorder \( L \sim 50a \) (green), \( L \sim 30a \) (red). The other parameters have been chosen as in figure 4(a). 52Cr atoms, \( a = 100a_0 \), \( \lambda = 1 \), \( \omega = 2\pi 50 \) Hz, upper \( L \sim 50a \), lower \( L \sim 30a \).
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