Once more about the 52 four-dimensional parallelotopes

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Abstract

There are several works [De29] (and [St73]), [En92], [Con97] and [Va03] enumerating four-dimensional parallelotopes. In this work we give a new enumeration showing that any four-dimensional parallelotope is either a zonotope or the Minkowski sum of a zonotope with the regular 24-cell $\{3, 4, 3\}$. Each zonotopal parallelotope is the Minkowski sum of segments whose generating vectors form a unimodular system. There are exactly 17 four-dimensional unimodular systems. Hence, there are 17 four-dimensional zonotopal parallelotopes. Other 35 four-dimensional parallelotopes are: the regular 24-cell $\{3, 4, 3\}$ and 34 sums of the regular parallelotope with non-zero zonotopal parallelotopes. For the nontrivial enumerating of the 34 sums we use a theorem describing necessary and sufficient conditions when the Minkowski sum of a parallelotope with a segment is a parallelotope.

1 Introduction

A parallelotope is a convex polytope which fills the space facet to facet by its translation copies without intersecting by inner points. Such a filling by parallelotopes is a tiling. The centers of the tiles form a lattice. A parallelotope of dimension $n$ is called primitive if exactly $n + 1$ adjacent parallelotopes meet in each vertex of its lattice tiling. Voronoi defines an L-type of a parallelotope, which is (in modern terms) the isomorphism class of the face lattice of the parallelotope. A special kind of a parallelotope is the Voronoi polytope of a lattice. The Voronoi polytope at a lattice point $v$ is the set of points which are at least as close to $v$ as to any other lattice point. Voronoi conjectured that every parallelotope of a class of parallelotopes of the same L-type is affinely equivalent to a Voronoi polytope (of course, of the same L-type), and he proved this conjecture for primitive parallelotopes.

In four-dimensional space, Voronoi [Vo08] determined all the 3 types of primitive parallelotopes. Using projection along a zone of parallel edges, Delaunay [De29] find 51 types of four-dimensional parallelotopes. The missed 52nd type was discovered by Shtogrin [St73].

Engel verified by computer the result of Delaunay corrected by Shtogrin. In Table 1 of [En92] he gives an informative and useful list of parallelotopes of the 52 types, each of maximal symmetry. Besides, in Fig.1 he gives a partial order between the 52 parallelotopes. This order consists of two disjoint components. These two partial orders correspond to partial orders between zonotopes: the first one amongst zonotopes, which are themselves
parallelotopes, and the second one amongst zonotopes, the Minkowski sum of which with the 24-cell gives a parallelotope. Conway in the chapter "Afterthoughts: Feeling the form of a Four-dimensional lattice" of [Con97] proposes conorms in order to characterize shapes (i.e. types) of parallelotopes. (In fact, Conway enumerates shapes of four-dimensional Voronoi polytopes.) The 17 types of parallelotopes which are zonotopes are parameterized by 16 subgraphs of the complete graph $K_5$ and by the complete bipartite graph $K_{3,3}$. The remaining 35 types of parallelotopes are characterized by shapes of positions minimal conorms in $4 \times 4$ matrices of all conorms. Vallentin [Va03] repeated the above Conway’s computations. He compute in details the 35 nonequivalent conorms.

In [Ry98] Ryshkov asserts that a four-dimensional parallelotope is either the Minkowski sum of segments, or the Minkowski sum of the 24-cell with a set of segments. The segments of the set are parallel to the edges of the 24-cell. But no proof of these assertions was published.

A zonotope is the Minkowski sum of segments. McMullen [McM75] proved that an $n$-dimensional zonotope is a parallelotope if and only if the directing vectors of its segments form an $n$-dimensional unimodular system. It is well known (see, for example, [DG99]) that there are two maximal four-dimensional unimodular systems. These are the graphic system of 10 vectors representing the regular matroid of the complete graph $K_5$ and the cographic system of 9 vectors representing the cographic matroid of the complete bipartite graph $K_{3,3}$. Each subsystem of the maximal cographic unimodular system $K_{3,3}^*$ is graphic and represents a subgraph of $K_5$. Hence, each four-dimensional zonotopal parallelotope is generated either by one of 16 graphic unimodular subsystems of $K_5$ or by the cographic system $K_{3,3}^*$. Note that the graph $K_4$ in the table on page 87 of [Con97] generates a three-dimensional unimodular system. But the rank 4 subgraph $C_{221} + 1$ of $K_5$ is missed in the Conway’s table. (An explanation of the notation $C_{221} + 1$ is given below in Section 3.) Note that in [Va03], zonotopes are described by their facet vectors. The facet vectors of zonotope $Z(M)$, related to a matroid $M$, form a representation of the dual matroid $M^*$. Hence, in contrast to us, Vallentin considers cographic matroids of corresponding graphs.

Remaining 35 parallelotopes are Minkowski sums of zonotopes related to unimodular systems with the regular 24-cell $\{3,4,3\}$, and the 24-cell itself. Note that zonotopes with the same unimodular system can give distinct sums with the 24-cell.

In this paper we give a list of the 52 types of four-dimensional parallelotopes, where for each type we explicitly give the corresponding unimodular system. The case of four-dimensional parallelotopes is a very instructive example of constructions of a large class of $n$-dimensional parallelotopes.

2 Parallelotopes of non-zero width

Venkov introduced in [Ve59] a notion of a polytope of non-zero width in direction of a $k$-dimensional subspace $X^k$ as a polytope whose intersection with any affine $k$-space parallel to $X^k$ is either $k$-dimensional or empty. He studied parallelotopes of non-zero width. The
most interesting are parallelotopes of non-zero width in direction of a line (or in direction of a vector spanning this line).

It is not difficult to see that if a parallelotope $P$ has a non-zero width in direction of a line $l$, then the line $l$ is parallel to some edges of $P$. A set of mutually parallel edges of $P$ is called an edge zone of $P$. Following to Delaunay [De29], Engel called an edge zone closed if each two-dimensional face of $P$ has either two or none of edges of this zone. Otherwise, the zone is called open.

For each edge zone $E$, there is a vector $z(E)$ with integer coordinates which is parallel to the edges of $E$. Voronoi called the vector $z(E)$ a characteristic of any edge of the edge zone $E$. So, any edge $e \in E$ has the form $e = \rho_e z(E)$. The positive number $\rho_e$ is called by Voronoi regulator of the edge $e$.

In [Gr03], the following proposition is proved.

**Proposition 1** For a parallelotope $P$ and a vector $z$, the following assertions are equivalent:

(i) $P$ has a closed edge zone parallel to $z$;
(ii) $P$ has a non-zero width in direction of $z$;
(iii) $P$ is the Minkowski sum of a segment $S(z)$ of the line spanned by $z$ and a parallelotope $P'$ of zero width in direction $z$, i.e., $P = P' + S(z)$.

We say that a parallelotope $P$ is of zero width in direction $z$ if $P$ is not of non-zero width in this direction. The length of the segment $S(z)$ in item (iii) is equal to the length of the shortest edge of the closed zone parallel to $z$. It implies that this zone is open in the parallelotope $P'$. Note that $S(z) = \lambda(z - z)$ for some real $\lambda$, where $z - z = \{x = \alpha z : -1 \leq \alpha \leq 1\}$ is the Minkowski sum of $z$ and $-z$.

If the parallelotope $P'$ in the sum $P = P' + S(z)$ is also of non-zero width in another direction $z_1$, then, by Proposition 1(iii), $P' = P'' + S(z_1)$. Of course, the Minkowski sum is associative and distributive. Hence, if a parallelotope $P$ has a non-zero width in several directions $z \in Q$, then $P = P_0 + \sum_{z \in Q} S(z)$, where $P_0$ is a parallelotope, which has no direction of non-zero width. The sum $Z(Q) = \sum_{z \in Q} S(z)$ is a zonotope. If the original parallelotope $P$ is a zonotope, then $P_0$ is a point and $P = Z(Q)$.

Recall that a set of vectors $U$ is a unimodular system if every vector $u \in U$ has an integer representation in any basic subset $B \subseteq U$ (details see in [DG99]). We say that the set of vectors $Q$ spans $U$ if there are scalars $\beta_z$, $z \in Q$, such that $U = \{\beta_z z : z \in Q\}$. It is proved in [Gr03], that the set of vectors $Q(P)$ of all directions of non-zero width of a parallelotope $P$ spans a unimodular system $U(P)$. McMullen [McM75] proved that a zonotope $Z(Q) = \sum_{z \in Q} S(z)$ is a parallelotope if and only if $Q$ spans a unimodular system.

We obtain the following result.

**Theorem 1** Any parallelotope is:

(i) either a zonotope,
(ii) or a parallelotope of zero width in any direction,
(iii) or the Minkowski sum of a zonotope with a parallelotope of zero width in any direction.
The Proposition 1 shows that, sometimes, one can add a segment to a parallelotope $P_0$ in order to obtain another parallelotope. If the parallelotope $P_0$ has zero width in direction $z$, then obtained parallelotope $P_0 + S(z)$ has another L-type than the original one. In [Gr03] necessary and sufficient conditions are given, when the sum of a parallelotope with a segment is a parallelotope. For to formulate these conditions, we introduce some new notions.

Venkov [Ve54] proved that a polytope $P$ is a parallelotope if and only if $P$ itself and all its facets are centrally symmetric and the projection of $P$ along any $(n - 2)$-dimensional face is either parallelogram, or a centrally symmetric hexagon. The four or six facets, which are projected into edges of a parallelogram or of a hexagon form a 2-belt or a 3-belt, respectively.

A facet $F$ of a parallelotope $P$ is defined by a facet vector $p$, such that the facet $F$ lies in the affine hyperplane $\{x \in \mathbb{R}^n : p^T x = \frac{1}{2} p^T t_i\}$. Here $t$ is the lattice vector connecting the center of $P = P(0)$ with the center of the parallelotope $P(t)$ adjacent to $P$ by the facet $F$. Let $I$ be the set of indices of all pairs of opposite facets of $P$. Then we have:

$$P(0) = \{x \in \mathbb{R}^n : -\frac{1}{2} p_i^T t_i \leq p_i^T x \leq \frac{1}{2} p_i^T t_i, \ i \in I\}. \quad (1)$$

If $P = P(0)$ is a Voronoi polytope, then the facet vectors $p_i$ are parallel to the lattice vectors $t_i$, and we can set $p_i = t_i$. We use the same names 2- and 3-belts for the two and three facet vectors, defining facets of a 2- and a 3-belt, respectively.

It is proved in [Gr03] the following:

**Proposition 2** For a parallelotope $P$ and a vector $z$, the following assertion are equivalent:

(i) the Minkowski sum $P + S(z)$ is a parallelotope;

(ii) vector $z$ is orthogonal to at least one facet vector of each 3-belt of $P$.

3 The Voronoi polytope of the lattice $D_n$

We apply Proposition 2 to the Voronoi polytope $P_V(D_n)$ of the root lattice $D_n$. The facet vectors of the Voronoi polytope $P_V(D_n)$ are $n(n - 1)$ roots of the root system $D_n$. We take the roots in the usual form $e_i \pm e_j$, $1 \leq i < j \leq n$. Here $\{e_i : i \in N\}$, $N = \{1, 2, ..., n\}$, is an orthonormal basis of $\mathbb{R}^n$. According to (1), we have:

$$P_V(D_n) = \{x \in \mathbb{R}^n : -1 \leq x_i \pm x_j \leq 1, \ 1 \leq i < j \leq n\}.$$ 

The vertices of $P_V(D_n)$ are of the following two forms (cf. [CS91]):

$$\pm v_i = \pm e_i, \ i \in N,$$

$$\text{and } v(S) = \frac{1}{2}(e(S) - e(S)), S \subseteq N,$$

where $S = N - S$ and $e(T) = \sum_{i \in T} e_i$ for any $T \subseteq N$. The $2^n$ vertices of the set $\{v(S) : S \subseteq N\}$ are vertices of a unit cube with its center in origin. The $2n$ vertices $\pm v_i, \ i \in N$, form $2n$ pyramids having the $2n$ facets of the unit cube as bases. Hence, the vertex $v_i$ is adjacent to a vertex $v(S)$ only if $S \ni i$. Similarly, the vertex $-v_i = -e_i$ is adjacent to a vertex $v(S)$.
only if \( i \notin S \). The edge \( v(S) - v_i = \frac{1}{2}e(T) - e(T), T = S - \{i\} \), connects these vertices. The vertex \( v(S) \) is adjacent to a vertex \( v(S') \) only if \( v(S) - v(S') = \pm e_i \) for some \( i \in N \).

So, edges of \( P_V(D_n) \) are of the following form

\[
e(S) - e(\overline{S}), S \subseteq N, \text{ or } e_i, i \in N.
\]

Up to sign, there are \( 2^{n-1} + n \) directions of edges, i.e., edge zones.

Each facet of \( P_V(D_n) \) is an \((n - 1)\)-dimensional bipyramid with an \((n - 2)\)-dimensional cube as its base. Hence, each 2-face of \( P_V(D_n) \) is a triangle. This implies that all edge zones of \( P_V(D_n) \) are open.

**Proposition 3** The following assertions are equivalent:

(i) \( P_V(D_n) + S(z) \) is a parallelotope;

(ii) \( z \) is parallel to an edge of \( P_V(D_n) \).

**Proof.** (i)\( \Leftrightarrow \) (ii). By Proposition 2, we have to show that the set of vectors \( z \), each of which is orthogonal to at least one vector \( p \) of each 3-belt of \( P_V(D_n) \) coincides with the set of edges of \( P_V(D_n) \).

The 3-belts of \( P_V(D_n) \) are of the following two types:

\[
(a) \ e_i - e_j, e_j - e_k, e_i - e_k; \ (b) \ e_i + e_j, e_j + e_k, e_i - e_k.
\]

We find all vectors \( z = \sum_{i=1}^{n} z_i e_i \) such that \( z^T p = 0 \) for at least one facet vector \( p \) of each belt. The vector \( z \) cannot have 3 mutually non-equal coordinates. In fact, if there are three such coordinates \( z_i, z_j, z_k \), then \( z \) is not orthogonal to any facet vector of the belt \((e_i - e_j, e_j - e_k, e_i - e_k)\) of type (a). Therefore, the vector \( z \) should be of the form \( z = z'(S) := z_1 e(S) + z_2 e(\overline{S}), S \subseteq N \). Since at least one pair of each triple of indices \( \{i, j, k\} \) lies either in \( S \), or in \( \overline{S} \), the vector \( z'(S) \) is orthogonal to at least one vector of each belt of type (a).

For the vector \( z'(S) \) to be orthogonal to at least one vector of each belt of type (b), it should be either \( z_1 = z_2 \), or \( z_1 = 0 \) and \( |S| = 1 \), or \( z_2 = 0 \) and \( |S| = 1 \), or \( z_1 + z_2 = 0 \). The last condition is necessary for the vector \( z'(S) \) with \( z_1 \neq z_2 \) to be orthogonal to at least one vector of the belt type (b), such that either \( i \in S, k \in \overline{S} \), or \( k \in S, i \in \overline{S} \).

So, we obtain that, up to a multiple, the vector \( z \) has the form

\[
either z = z(S) = e(S) - e(\overline{S}), S \subseteq N, \text{ or } z = e_i, i \in N.
\]

Comparing these vectors with edges (2) of \( P_V(D_n) \), we obtain that all vectors are directed along edges of \( P_V(D_n) \). \( \square \)

4 Four-dimensional unimodular systems and zonotopal parallelotopes

Let \( \Sigma_n \) be an \( n \)-dimensional simplex. It has \( \frac{1}{2}n(n + 1) \) edges. The \( \frac{1}{2}n(n + 1) \) vectors which are parallel to edges of \( \Sigma_n \) and have the same length as corresponding edges, form a maximal
The unimodular system $A_n$. It represents the graphic matroid of the complete graph $K_{n+1}$ which is the one-dimensional skeleton of the simplex $\Sigma_n$. Recall that a set of vectors corresponding to edges of a graph $G$ represents a graphic (cyclic) matroid of the graph $G$ if the sum of vectors (taken in suitable direction) along any cycle of $G$ is zero vector. Changing in this definition cycle by cocycle (cut), we obtain a representation of the cographic matroid of $G$. (See any book on Matroid Theory, for example, [Aig79].)

We identify the vectors of $A_n$ with the corresponding edges of $\Sigma_n$ and $K_{n+1}$. Take the $n$ vectors incident to a vertex $v \in \Sigma_n$ as a basis of $A_n$ and denote them $e_i$, $1 \leq i \leq n$. Suppose that these vectors are directed from the vertex $v$. Then the other $(\frac{n}{2}n(n-1))$ vectors of $A_n$ are $e_i - e_j$, $1 \leq i < j \leq n$.

The Minkowski sum of all vectors of $A_n$ is an $n$-dimensional zonotope which is called permutohedron. It is a primitive parallelotope. Voronoi called its L-type as the principal type.

Unimodular $n$-dimensional subsystems of $A_n$ are related to subgraphs of rank $n$ of $K_{n+1}$. Recall that rank of a graph is the number of its vertices minus the number of its components.

If a graph $G$ is planar, then there exists its dual planar graph $G^+$ edges of which are in one-to-one correspondence with edges of $G$. The graphic matroid of $G$ is isomorphic to the cographic matroid of $G^+$. Both these matroids are represented by a common unimodular system.

A deletion of an element from the cographic matroid of a graph $G$ provides the contraction of the corresponding edge of $G$. It means that the end vertices of the contracted edge are identified and the obtained loop is deleted. For the graph $K_{3,3}$, the contraction of an edge gives a planar graph on 5 vertices. This graph is the subgraph of $K_5$ obtained by deletion from $K_5$ of two non-adjacent edges. It is denoted by $K_5 - 2 \times 1$. The dual $(K_5 - 2 \times 1)^+$ is isomorphic to $K_5 - 2 \times 1$.

Here and below instead of the sum $1 + 1 + ... + 1$ of $k$ ones of [Con97] (denoting $k$ non-adjacent edges), we write $k \times 1$. We use here and below the following Conway’s notations: $C_{ijk...}$ denotes the graph consisting of more than two chains each containing $i, j, k, ...$ edges and all chains connect the same two vertices. But note that $C_k$ is a cycle with $k$ edges, i.e. $C_k = C_{ij}$ with $i + j = k$. $G + k \times 1$ denotes a graph $G$ with $k$ pendant edges. For the matroid of the graph $G + k \times 1$ it is not important, whether the $k$ edges are connected to $G$ or not, or the $k$ edges form a tree or they are disconnected. It is important, that the subgraph induced by these $k$ edges contains no cycle.

In dimension 4, there are two maximal unimodular systems:

1) $A_4$, representing the graphic matroid of the complete graph $K_5$, and
2) the unimodular system, representing the cographic matroid $K_{3,3}^*$ of the complete bipartite graph $K_{3,3}$.

There are 16 subgraphs of rank 4 in $K_5$. They are drawn on p.87 of [Con97]. But the graph $K_4$ on this picture has rank 3. It should be changed by the subgraph $C_{221} + 1$ missed in [Con97]. A correct picture of these graphs is given on p.55 of [Va03], but the graph $C_{221} + 1$ is denoted there as $K_4$.

Since proper cographic submatroids of $K_{3,3}^*$ are isomorphic to graphic ones, in dimension 4, there is only one cographic unimodular system $K_{3,3}^*$, which is not isomorphic to graphic
one. Hence, besides the mentioned above 16 graphic four-dimensional unimodular systems, there is the 17th cographic four-dimensional unimodular system $K_{3,3}^*$. This implies, there are exactly 17 four-dimensional zonotopal parallelotopes. Amongst them only permutohedron is primitive.

Note that all edge zones of a zonotope are closed. All edges of an edge zone have the same length. A deletion of a vector from the unimodular system of a zonotope relates to contraction of the corresponding edge zone.

The correspondence of zonotopal parallelotopes from [De29] with subgraphs of $K_5$ is given in Table 1. In this table, $N_D$ denotes the number given to a parallelotope in [De29] (we call it Delaunay number), and $m$ is the number of segments in the Minkowski sum of the corresponding zonotope. According to [Con97], 2 and 3 denote the subgraphs of $K_5$, which are connected chains of two and three edges, respectively.

Table 1. Four-dimensional zonotopal parallelotopes $Z(G)$

| $N_D$ | 1 | 4 | 19 | 5 | 6 | 7 | 8 | 9 | 10 |
|-------|---|---|----|---|---|---|---|---|----|
| $m$   | 10| 9 | 9  | 8 | 8 | 7 | 7 | 7 | 7 |
| $G$   | $K_5$ | $K_5 - 1$ | $K_{3,3}^*$ | $K_5 - 2 \times 1$ | $K_5 - 2$ | $K_5 - 1 - 2$ | $K_4 + 1$ | $C_{221}$ | $K_5 - 3$ |

| $N_D$ | 11 | 12 | 13 | 16 | 14 | 15 | 17 | 18 |
|-------|----|----|----|----|----|----|----|----|
| $m$   | 6  | 6  | 6  | 6  | 5  | 5  | 5  | 4  |
| $G$   | $C_{222}$ | $C_{321}$ | $C_{221} + 1$ | $C_3 + C_3$ | $C_4 + 1$ | $C_5$ | $C_3 + 2 \times 1$ | $4 \times 1$ |

5 Unextendible unimodular subsystems of $D_4$

The four-dimensional Voronoi polytope $P_V(D_4)$ is the self-dual regular four-dimensional polytope called the 24-cell. Coxeter [Cox48] denotes it as $\{3, 4, 3\}$. The edges of $P_V(D_4)$ are given in (2) where $N = \{1, 2, 3, 4\}$. In this case, up to sign, we have 12 vectors, and these 12 vectors up to the multiple $\sqrt{2}$ are 12 vectors of the root system $D_4$. For convenience, we use below for edges of $P_V(D_4)$ the usual form of the root system $D_4 = \{e_i \pm e_j : 1 \leq i < j \leq 4\}$. Besides, we denote the vector $e_i \pm e_j$ by the symbol $ij^\pm$, where the signs agree.

Note that $D_4$ consists of the following three 4-sets of mutually orthogonal vectors: $\{ij^\pm, kl^\pm\}$, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Call such a 4-set quadruple. Each quadruple relates to one of the three partitions of the 4-set $\{1, 2, 3, 4\}$ into pairs. Call a three mutually orthogonal roots of $D_4$ by a triple. Each triple $t$ is contained in a uniquely determined by $t$ quadruple $q_t \supset t$.

Consider the following two maximal unimodular systems contained in the root system $D_4$: the graphic system $A_4 - e$, where $e$ is any vector of $A_4$, and the cographic system $K_{3,3}^*$. The system $A_4 - e$ represents the graphic matroid of the graph $K_5 - 1$, and the system $K_{3,3}^*$ represents the cographic matroid of the complete bipartite graph $K_{3,3}$. There are many ways to choose in $D_4$ vectors forming the unimodular systems $A_4 - e$ and $K_{3,3}^*$. We choose the vectors as follows. The 6 vectors $ij^\pm, 1 \leq i < j \leq 4$, form the graphic system $A_3$ representing the graphic matroid of the complete graph $K_4$. If the vertices of $K_4$ are denoted by the numbers 1,2,3,4, then the vector $ij^- = e_i - e_j$ represents the edge $(ij)$ connecting the
vertices $i$ and $j$. Now suppose that the vertex 5 of $K_5 - 1$ is not connected with the vertex
1 of its subgraph $K_4$. Then we can relate the vectors $12^+, 13^+, 14^+$ to the edges (25), (35), (45), respectively. It is easy to verify that the 9 vectors $ij^-$, $1 \leq i < j \leq 4$, $1i^+$, $2 \leq i \leq 4$, form the unimodular system $A_4 - e$. This unimodular system consists of the following three triples of mutually orthogonal vectors: $(ij^-, ij^+, kl^-), ij = 12, 13, 14$, corresponding to three partitions of the 4-set $\{i, j, k, l\}$ into pairs.

The graph $K_5 - 1$ is planar. It has three vertices of degree 4 and two vertices of degree
3. The 9 edges of this graph are partitioned into the following two orbits (1) and (2) of the
automorphism group of $K_5 - 1$:

(1) 3 edges with both end vertices of degree 4; they are represented by the roots $23^-$, $24^-$, $34^-$;
(2) 6 edges with end vertices of degree 3 and 4; they are represented by the roots $12^\pm$, $13^\pm$, $14^\pm$.

If we delete in the graph $K_5 - 1$ the edge (24), we obtain the planar graph $G_5 := K_5 - 2 \times 1$. The vertices 1, 2, 4, 5 of the graph $G_5$ have degree 3 and form a 4-cycle with edges (12), (25), (45), (14). The vertex 3 has degree 4 and it is adjacent to the four vertices of the 4-cycle
by edges (13), (23), (35), (34). Now, we consider the dual graph $G_5^*$ which is isomorphic to original one. Let the vertices of the 4-cycle of the $G_5^*$ be $a, b, c, d$ in this order along the 4-cycle. Then edges of this 4-cycle, corresponding to the edges (13), (23), (35), (34) of $G_5$, have the following pairs of end vertices, respectively: $(ab), (bc), (cd), (ad)$. The vertex $v$ of degree 4 is adjacent to the vertices $a, b, c, d$ by edges, corresponding to the edges (14), (12), (25), (45) of $G_5$, respectively. The four last edges form a 4-cocycle of $G_5^+$. The 8 vectors $12^-, 13^-, 14^-, 23^-, 34^-, 12^+, 13^+, 14^+$ related to the edges of $G_5$ and $G_5^+$ represent the graphic matroid of $G_5$ and the cographic matroid of $G_5^+$. The cographic matroid of $G_5^+$ is a submatroid of $K_{3,3}^*$.

The graph $G_5^+$ is obtained from $K_{3,3}$ by the contraction of one of its edges. The operation opposite to the contraction of an edge of the graph $K_{3,3}$ is the splitting of the vertex $v$
of degree 4 in $G_5^+$ into two adjacent vertices $v'$ and $v''$. This splitting is such that the vertex $v'$ is adjacent to the vertices $a$ and $c$, and the vertex $v''$ is adjacent to the vertices $b$ and $d$. We obtain the complete bipartite graph $K_{3,3}$ with mutually non-adjacent vertices $b, d, v'$ of one part and mutually non-adjacent vertices $a, c, v''$ of other part. The vectors related to edges $(av'), (cw')$ and $(dv'), (dv'')$ are $14^-, 12^+ and 12^-, 14^+$, respectively. Since $12^+ - 14^- = 14^+ - 12^- = 24^+$, we should relate the vector $24^+$ to the edge $(v'v'')$.

We see that this representation of the cographic matroid $K_{3,3}^*$ is obtained from the representation of the graphic matroid of the graph $K_5 - 1$ by changing the vector $24^-$ into the vector $24^+$. Call a unimodular subsystem $U \subseteq D_4$ unextendible if the set $U \cup \{r\}$ is not unimodular for all $r \in D_4 - U$.

**Proposition 4** Up to isomorphism, the root system $D_4$ contains exactly the following 3 unextendible unimodular subsystems: a quadruple, $A_4 - e$, and $K_{3,3}^*$.

**Proof.** Let $q = \{ij^-, ij^+, kl^-, kl^+\} \subseteq D_4$ be a quadruple. Obviously it is a unimodular system. It is a basis of the space $\mathbb{R}^4$. Any root $r \in D_4 - q$ has a unique non-integer
Hence, each quadruple of $D_4$ is not unimodular for all $r \in D_4 - q$. Hence, each quadruple of $D_4$ is an unextendible unimodular subsystem of $D_4$.

This implies that any other unimodular subsystem of $D_4$ does not contain a quadruple. Consider a set $U = t_1 \cup t_2 \cup t_3$ of three mutually disjoint triples. Obviously, $U$ is a maximal subset of $D_4$ not containing a quadruple. (Note, there are $4^3 = 64$ such sets.) We show that $U$ is a unimodular system isomorphic either to $A_4 - e$, or to $K^*_3$. Obviously, these systems are unextendible.

Each triple $t \subset U$ is complemented by a unique root $r_t \in D_4$ up to the quadruple $q_t$. Label the set $U$ by the triad $(r_1, r_2, r_3)$ of these complementing roots. For example, the following set $U = \{ij^-, ij^+, kl^- : ij = 12, 13, 14\}$ is labeled by the triad $(34^+, 24^+, 23^+)$. The automorphism group of the root system $D_4$ consists of the following operations: permutations of indices, changing a pair of indices $(ij)$ by the complementing pair $(kl)$, reversing the sign of a unit vector $e_i \rightarrow -e_i$. The collection of all triads is partitioned into two orbits of the automorphism group. One orbit consists of triads with even number of minus signs. Another orbit contains triads with odd number of minus signs. The sets $U$ labeled by triads of these two orbits are isomorphic to unimodular systems $A_4 - e$ and $K^*_3$, respectively.

Denote the zonotopes related to a unimodular system $U$ by $Z(U)$. Since the deletion of the vector $24^-$ from $A_4 - e$ and the vector $24^+$ from $K^*_3$ provides the same unimodular system $U(K_5 - 2 \times 1)$, the contraction of the edge zones of $Z(A_4 - e)$ and $Z(K^*_3)$ corresponding to $24^-$ and $24^+$, respectively, provides zonotopes, which are both isomorphic to $Z(U(K_5 - 2 \times 1))$. \hfill\Box

## 6 Sums of $P_V(D_4)$ with zonotopes

The sums $P_V(D_4) + Z(A_4 - e)$ and $P_V(D_4) + Z(K^*_3)$ are primitive parallelotopes. Their projections along a closed edge zone into three-dimensional space are drawn in Figs.II and III of [De29]; see also [De38]. Their Delaunay numbers are $N_D = 2$ and $N_D = 3$. In Figs.II and III, the closed edge zones of these parallelotopes are denoted by numbers $1, 2, 3, \ldots, 8, 9$. The number 9 corresponds to the edge zone, along which the parallelotope is projected. Any non-primitive parallelotope is obtained from these two ones by contracting some closed edge zones. Some edges are denoted by $0i$, $1 \leq i \leq 8$. This means that after the contraction of the edge zone with number $i$ the edge with number $0i$ is contracted to an edge of $P_V(D_4)$ which is denoted by 0.

Figs.II and III show that the contraction of the edge zone 4 in $P_V(D_4) + Z(A_4 - e)$ and the edge zone 6 in $P_V(D_4) + Z(K^*_3)$ gives the same parallelotope. Comparing the numbers of edge zones of this parallelotope in Figs.II and III, we obtain their correspondence, shown in Table below. Besides, in this Table the vectors of the unimodular systems representing $A_4 - e$ and $K^*_3$ are given.

| Fig. | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | − |
|------|---|---|---|---|---|---|---|---|---|---|
|      |   |   |   |   |   |   |   |   |   |   |
| Fig. II | 2 | 1 | 5 | − | 7 | 8 | 3 | 4 | 9 | 6 |
| roots      | 23−| 14−| 12−| 24−| 34−| 13−| 12+| 13+| 14+| 24+ |
We see that the edge zones 1, 4 and 5 of Fig.1 contains edges of the same orbit (1) of the automorphism group of \( K_5 - 1 \). Hence the contraction in \( P_V(D_4) + Z(A_4 - e) \) of any edge zone of this orbit gives isomorphic parallelotopes. The parallelotope with contracted edge zone 1 is the parallelotope of \[\text{[De29]}\] with the Delaunay number \( N_D = 21 \). This parallelotope is \( P_V(D_4) + Z(U(K_5 - 2 \times 1)) \).

Note that for zonotopes \( Z(U) \), which are parallelotopes, it is not important whether the summing vectors are orthogonal or not. A parallelepiped and a cube have the same L-type. But an orthogonality of summing vectors in \( Z(U) \) affects heavenly onto the L-type of the sum \( P_V(D_4) + Z(U) \). This implied by the fact that any two orthogonal roots do not belong to a 2-face of \( P_V(D_4) \). Hence the sum \( P_V(D_4) + S(r) + s(r') \), where \( r \) and \( r' \) are orthogonal, obtains a new 2-face spanned by \( r \) and \( r' \) contrary to the case, when \( r \) and \( r' \) are not orthogonal.

Facets of \( P_V(D_4) \) are octahedra whose edges are roots of \( D_4 \). Each octahedron has four pairs of opposite parallel triangle faces and six pairs of parallel edges, which are parallel to six distinct roots. This six roots are partitioned into three pairs of orthogonal roots. Each triangle contains one representative root from these three pairs. Each of the six roots belongs to four triangles.

For a facet \( F \) of \( P_V(D_4) \), let \( R(F) \) be the set of the six roots parallel to edges of \( F \). In this case, when edges are parallel to roots, the facet vectors are given by (2). The facet vector \( e_k, 1 \leq k \leq 4 \), defines a facet \( F \), such that \( R(F) = \{ij \pm : i, j \neq k\} \). This is the root system \( D_3 \), which is isomorphic to \( A_3 \). The facet vector of another type \( \frac{1}{2} \sum \epsilon_i \epsilon_j \), where \( \epsilon_i \in \{\pm 1\} \), defines a facet \( F \), such that \( R(F) = \{ij - \epsilon_i \epsilon_j : 1 \leq i < j \leq 4\} \).

Note that, for each pair \( (r, r') \) of orthogonal roots, there is a facet \( F \) of \( P_V(D_4) \), such that \( r, r' \in R(F) \).

Let \( U = U(G) \) be a unimodular system of roots representing a subgraph \( G \subseteq K_5 - 1 \) or \( U = K_{3,3} \). Let \( \pi(U) \) be the set of maximal pairs of orthogonal roots in \( U \). A pair \( (r, r') \subseteq U \) of orthogonal roots in \( U \) is called maximal if there is no root in \( U - \{r, r'\} \) which is orthogonal to both the roots \( r, r' \). Let \( \tau(U) \) be the set of triples of mutually orthogonal roots in \( U \). Recall that each triple \( t \in \tau(U) \) uniquely determines the fourth root \( r_t \in D_4 \), such that \( r_t \) is orthogonal to all roots of \( t \).

Recall that \( P_V(D_4) \) has 24 facets (it is a 24-cell) and all its 16 belts are 3-belts.

**Proposition 5** Let \( Z(U) = \sum_{r \in U} S(r) \). Then it holds:

(i) the sum \( P_V(D_4) + Z(U) \) has \( |\pi(U)| \) 2-belts and \( 16 + 3|\tau(U)| \) 3-belts;

(ii) the sum \( P_V(D_4) + Z(U) \) has \( 24 + 2|\tau(U)| \) facets;

(iii) the sum \( P_V(D_4) + Z(U) + S(r) \) is a parallelotope for all \( r \in D_4 \) such that \( r \neq r_t, t \in \tau(U) \).

**Proof.** Let \( F \) be a facet of \( P_V(D_4) \) and \( r \in R(F) \). The facet \( F \) is transformed to the facet \( F + S(r) \) in the sum \( P_V(D_4) + S(r) \). (Here \( S(r) \) is a segment of the line parallel to the root \( r \).) The facet \( F + S(r) \) has also 4 pairs of parallel faces. But the four triangles of \( F \) containing an edge parallel to \( r \) are transformed into trapezoids with two edges parallel to \( r \).
Now consider the sum \( P(r, r') := P_V(D_4) + S(r) + S(r'), r, r' \in R(F) \). If \( r \) and \( r' \) are not orthogonal, then the facet \( F + S(r) + S(r') \) has also 4 pairs of parallel faces. The two parallel faces which contains edges parallel to both the roots \( r \) and \( r' \) are transformed into pentagons. The two pairs of faces having edges parallel only one of these two roots are transformed into two pairs of trapezoids. One pair of parallel faces is not changed.

If \( r \) and \( r' \) are orthogonal, then each face of \( F \) has only one edge parallel to one of these roots. Hence, each face is transformed into a trapezoid. There are two opposite vertices of \( F \) which are not incident to the edges parallel to these roots. These vertices are transformed into new square faces \( Q \) of \( F + S(r) + S(r') \). So, this facet has now 10 faces.

This shows that the L-type of the sum \( P_V(D_4) + S(r) + S(r') \) depends on whether \( r \) and \( r' \) are orthogonal or not. Besides, if \( r \) and \( r' \) are orthogonal, then there are two pairs of opposite facets which are transformed into polyhedra with 10 faces. These four facets form a new 2-belt. So, each pair of orthogonal roots in \( U \) generates a 2-belt.

Let \( t = (r, r', r'') \in \tau(U) \). Consider the sum \( P_t := P(r, r') + S(r'') \). Let \( F \) be a facet such that \( r, r' \in R(F) \). Then \( r'' \not\in R(F) \). Let \( Q \) be the quadrangle face of \( F + S(r) + S(r') \). Then \( Q + S(r'') \) is a cube. It is a new facet of \( P_t \). This facet, its opposite and the facets of the 2-belt of \( P(r, r') \) form a new 3-belt \( B \). The cube \( Q + S(r'') \) has 3 pairs of opposite faces, and each pair of its faces generates a 3-belt. Hence, for each \( t \in \tau(U) \), the parallelotope \( P_t \) has additionally three belts. We proved (i) and (ii).

For \( t \in \tau(U) \), the root \( r_t \) is orthogonal to the new facet \( Q + S(r'') \). So, \( r_t \) is the facet vector of this facet. Obviously, \( r_t \) is not orthogonal to any facet vector of the new 3-belt \( B \).

By Proposition 2, \( P_t + S(r_t) \) is not a parallelotope. Since the facet vector of the transformed facet \( F + Z(U) \) is the same as the facet vector of \( F \), we obtain (iii). \( \square \)

Table 2 shows the 35 unimodular systems \( U \) of roots sums of which with \( P_V(D_4) \) are parallelotopes. Maximal pairs and triples of mutually orthogonal roots are distinguished by parentheses. Besides of used above denotations of graphs, \( H_k \) denotes the skeleton of a \( k \)-dimensional parallelepiped. As in Table 1, \( N_D \) denotes the Delaunay number of the parallelotope \( P_V(D_4) + Z(U) \) and \( m \) is the number of roots in \( U \). The parallelotope, missed by Delaunay and found by Shtogrin, is denoted by \( \text{St} \). The fifth column gives dimension of the added zonotope \( Z(U) \). Note that \( \dim Z(U) \) is equal to the rank of the graph \( G \), which is represented by the unimodular system \( U \).

For to compare Tables 2 and 1, we add the last column. In this column \( N^0_D \) denotes the Delaunay number \( N_D \) of the corresponding zonotopal parallelotope \( Z(U) \) if \( \dim U = 4 \); \( N^0_D = a_i, \ 1 \leq i \leq 5 \) denotes a 3-dimensional zonotopal parallelotope if \( \dim U = 3 \); and \( N^0_D = \alpha, \beta_1, \beta_2 \) denotes a 2-dimensional zonotopal parallelotope if \( \dim U = 2 \). Note that

- \( a_1 \) denotes a permutohedron = a truncated octahedron;
- \( a_2 \) denotes an elongated dodecahedron;
- \( a_3 \) denotes a prism with a hexagonal base;
- \( a_4 \) denotes a rhombic dodecahedron;
- \( a_5 \) denotes a parallelopiped with mutually orthogonal edges;
- \( a_5' \) denotes a parallelopiped with a pair of parallel rectangle facets;
- \( a_5'' \) denotes a parallelopiped without rectangle facets;
\(\alpha\) denotes a centrally symmetric hexagon;
\(\beta_1\) denotes a rectangle;
\(\beta_2\) denotes a parallelogram without orthogonal edges.
Table 2. Zonotopes $Z(U)$ such that $P_V(D_4) + Z(U)$ is a parallelootope

| $N_D$ | $m$ | roots of the unimodular system $U$ | graph | dim$U$ | $N_D^m$ |
|-------|-----|-----------------------------------|-------|--------|--------|
| 2     | 9   | $(12^-, 12^+, 34^-), (13^-, 13^+, 24^-), (14^-, 14^+, 23^-)$ | $K_5$ | 4      | 1      |
| 3     | 9   | $(12^-, 12^+, 34^-), (13^-, 13^+, 24^-), (14^-, 14^+, 23^-)$ | $K_5^2$ | 4      | 19     |
| 20    | 8   | $(12^-, 12^+, 34^-), (13^-, 13^+, 24^-), (14^+, 23^-)$ | $K_5 - 2$ | 4      | 6      |
| 21    | 8   | $(12^-, 12^+, 34^-), (13^-, 13^+, 24^-), (14^-, 23^-)$ | $K_5 - 2 \times 1$ | 4 | 5     |
| 22    | 7   | $(12^+, 34^-), (13^-, 13^+, 24^-), (14^-, 14^+$ | $K_5 - 3$ | 4 | 10    |
| 23    | 7   | $34^-, (13^-, 13^+, 24^-), (14^-, 14^+, 23^-)$ | $C_{221}$ | 4 | 9     |
| 24    | 7   | $(12^-, 12^+, 34^-), (13^-, 13^+, 24^-)$, $14^+$ | $K_5 - 1 - 2$ | 4 | 7      |
| 25    | 7   | $(12^+, 34^-), (13^-, 13^+, 24^-), (14^+, 23^-)$ | $K_4 + 1$ | 4 | 8      |
| 26    | 7   | $(12^-, 12^+, 34^-), (13^-, 13^+, 24^-), (14^-, 14^+$ | $K_5 - 1 - 2$ | 4 | 7      |
| 27    | 6   | $(12^-, 12^+, 34^-), (13^-, 13^+)$, $14^+$ | $C_{321}$ | 4 | 12    |
| 28    | 6   | $(12^+, 34^-), (13^-, 13^+, 24^-)$, $14^+$ | $C_{221} + 1$ | 4 | 13    |
| 29    | 6   | $(13^-, 13^+, 24^-), (14^-, 14^+, 23^-)$ | $C_{222}$ | 4 | 11    |
| 30    | 6   | $(12^+, 34^-), (13^-, 24^-), (14^-, 14^+$ | $C_{221} + 1$ | 4 | 13    |
| 31    | 6   | $(12^-, 12^+), (13^-, 13^+)$, $14^+$ | $C_{222}$ | 4 | 11    |
| 32    | 6   | $(12^+, 34^-), (13^- 24^-), (14^-, 14^+$ | $C_3 + C_3$ | 4 | 16    |
| 33    | 6   | $(12^+, 34^-), (13^-, 24^-)$, $14^+$ | $K_4$ | 3 | $a_1$ |
| 34    | 5   | $12^+, (13^-, 13^+, 24^-)$, $14^+$ | $C_3 + 2 \times 1$ | 4 | 17    |
| 35    | 5   | $(12^-, 12^+, 34^-)$, $13^-, 14^+$ | $C_5$ | 4 | 15    |
| 36    | 5   | $(13^-, 13^+, 24^-)$, $14^-, 14^+$ | $C_4 + 1$ | 4 | 14    |
| 37    | 5   | $(12^-, 12^+), (13^- 13^+)$, $14^+$ | $C_4 + 1$ | 4 | 14    |
| 38    | 5   | $(12^+, 34^-), (13^-, 13^+)$, $14^+$ | $C_3 + 2 \times 1$ | 4 | 17    |
| 39    | 5   | $(12^+, 34^-), (13^-, 24^-)$, $14^+$ | $C_{221}$ | 3 | $a_2$ |
| 40    | 4   | $(13^-, 13^+, 24^-)$, $14^+$ | $4 \times 1 = H_4$ | 4 | 18    |
| 41    | 4   | $12^+, (13^-, 13^+)$, $14^+$ | $4 \times 1 = H_4$ | 4 | 18    |
| 42    | 4   | $(12^+, 34^-)$, $13^-, 14^+$ | $C_3 + 1$ | 3 | $a_3$ |
| 43    | 4   | $(13^-, 24^-)$, $14^-, 14^+$ | $4 \times 1 = H_4$ | 4 | 18    |
| 44    | 4   | $(13^-, 13^+)$, $(14^-, 14^+$ | $C_4$ | 3 | $a_4$ |
| 45    | 3   | $(14^-, 14^+, 23^-)$ | $3 \times 1 = H_3$ | 3 | $a_5$ |
| 46    | 3   | $(13^-, 13^+)$, $14^-$ | $3 \times 1 = H_3$ | 3 | $a_5'$ |
| 47    | 3   | $12^+, 13^+, 14^+$ | $3 \times 1 = H_3$ | 3 | $a_5''$ |
| St    | 3   | $34^-, 13^+, 14^+$ | $C_3$ | 2 | $\alpha$ |
| 48    | 2   | $(14^-, 14^+$ | $2 \times 1 = H_2$ | 2 | $\beta_1$ |
| 49    | 2   | $13^+, 14^+$ | $2 \times 1 = H_2$ | 2 | $\beta_2$ |
| 50    | 1   | $14^+$ | $1 = H_1$ | 1 |       |
| 51    | 0   | | $24 - \text{cell}$ | |       |
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