Abstract. Let $G$ be a reductive group over a finite field $F = \mathbb{F}_q$. Fix a non-trivial additive character $\psi : F \to \mathbb{Q}_\ell^\times$. In [3] we introduced certain $\gamma$-functions $\gamma_{G,\rho,\psi}$ on the set $\text{Irr}(G)$ of irreducible representations of the finite group $G = G(F)$.

As usual every function $\gamma_{G,\rho,\psi}$ on $\text{Irr}(G)$ gives rise to an $\text{Ad}G$-equivariant function $\Phi_{G,\rho,\psi}$ on $G$. The purpose of this paper is to construct an irreducible perverse sheaf $\Phi_{G,\rho,\psi}$ on $G$ such that the function $\Phi_{G,\rho,\psi}$ is obtained conjecturally by taking traces of Frobenius morphism in the stalks of $\Phi_{G,\rho,\psi}$. In order to do this we need to assume that $\rho$ satisfies certain technical condition (we call $\rho$ good if that condition is satisfied). We prove this conjecture for $G = \text{GL}(n)$ and for $G$ of semi-simple rank one.

We also prove the above conjecture assuming that certain cohomology vanishing holds (we show that this is the case for $G$ of semisimple rank 1). Assuming this vanishing we show that if both $\rho_1$ and $\rho_2$ are good then $\Phi_{G,\rho_1,\psi} \star \Phi_{G,\rho_2,\psi} \cong \Phi_{G,\rho_1 \oplus \rho_2,\psi}$. We also compute the convolution of $\Phi_{G,\rho,\psi}$ with the majority of Lusztig’s character sheaves.

We conjecture that the functor of convolution with $\Phi_{G,\rho,\psi}$ is exact in the perverse $t$-structure.

1. Introduction

1.1. Some notations. In what follows we denote by $F = \mathbb{F}_q$ the finite field with $q$ elements, $\overline{F}$ – its algebraic closure. Choose a prime number $\ell$ which is prime to $q$. Let also $\psi : F \to \mathbb{Q}_\ell^\times$ denote a non-trivial additive character of $F$ with values in $\mathbb{Q}_\ell^\times$. We shall denote algebraic varieties over $F$ by boldface letters (e.g. $G$, $X$ etc). The corresponding ordinary letters (e.g. $G$, $T$ etc) will denote the corresponding sets of $F$-points.

For a finite group $G$ we denote by $\text{Irr}(G)$ the set of isomorphism classes of irreducible representations of $G$.

In what follows we choose a square root $q^{1/2}$ of $q$.

For an algebraic variety $X$ over $\overline{F}$ we shall denote by $\mathcal{D}(X)$ the bounded derived category of $\ell$-adic sheaves on $X$.

If $X$ is defined over $F$ we let $\text{Fr} : X \to X$ denote the geometric Frobenius morphism. We say that an object $\mathcal{F} \in \mathcal{D}(X)$ is endowed with a Weil structure if we are given an isomorphism $\text{Fr}^* \mathcal{F} \cong \mathcal{F}$. To any Weil sheaf on $X$ we associate a function $\chi(\mathcal{F})$ on $X = X(F)$ in the following way. Let $x \in X$ and let $\mathcal{F}_x$ denote the fiber of $\mathcal{F}$ at $x$. This is a complex of $\ell$-adic vector spaces. Since $x$ is fixed by $\text{Fr}$, the Weil structure
on \( F \) gives rise to an automorphism of \( F_x \) which (by abuse of language) we shall also denote by \( \text{Fr} \). Thus we set
\[
\chi(F)(x) = \sum_i (-1)^i \text{Tr}(\text{Fr}, H^i(F_x))
\] (1.1)

Let \( F \) be a Weil sheaf. For a half integer \( n \) we denote by \( F(n) \) the Tate twist of \( F \) (corresponding to the chosen \( q^{1/2} \)). Thus \( \chi(F(n)) = \chi(F)q^{-n} \).

For an algebraic group \( G \) we define two convolution functors \((F, G) \mapsto F \ast G \) and \((F, G) \mapsto F * G \) going from \( D(G) \times D(G) \) to \( D(G) \) in the following way. Let \( \text{m} : G \times G \to G \) denote the multiplication map. Then
\[
F \ast G = \text{m}_!(F \boxtimes G) \quad \text{and} \quad F * G = \text{m}_*(F \boxtimes G).
\] (1.2)

1.2. \( \gamma \)-functions for \( \text{GL}(n) \). Let \((\pi, V)\) be an irreducible representation of \( G = \text{GL}(n, F) \). Consider the operator
\[
\sum_{g \in G} \psi(\text{tr}(g))\pi(g)(-1)^nq^{-n^2/2} \in \text{End}_G V.
\] (1.3)

By Schur’s lemma this operator takes the form \( \gamma_\psi(\pi) \cdot \text{Id}_V \) where \( \gamma_\psi(\pi) \in \mathbb{Q}_l \).

The number \( \gamma_\psi(\pi) \) is called the gamma-function of the representation \( \pi \). One can ”explicitly” compute \( \gamma_\psi(\pi) \) in the following way.

Let \( W \cong S_n \) denote the Weyl group of \( \text{GL}(n) \). Following Deligne and Lusztig (cf. [4]) we can associate to every \( w \) a maximal torus \( T_w \subset \text{GL}(n) \) (defined uniquely up to \( G = \text{GL}(n, F) \)-conjugacy).

Fix \( w \in W \). For a character \( \theta : T_w \to \mathbb{Q}_l^\times \) we set
\[
\gamma_{\psi,w}(\theta) = (-1)^nq^{-n/2} \sum_{t \in T_w} \psi(\text{tr}(t))\theta(t) \in \mathbb{Q}_l.
\] (1.4)

Example. Assume that \( w \in S_n \) is a cycle of length \( n \). Then \( T_w \cong E^\times \) where \( E \) is the (unique up to isomorphism) extension of \( F \) of degree \( n \). In this case \( \gamma_w(\theta) = (-q^{1/2})^{n-1}\gamma_{E,\psi}(\theta) \) for any character \( \theta \) of \( E^\times \), where by \( \gamma_{E,\psi}(\theta) \) we denote the \( \gamma \)-function defined as in (1.3) for the group \( \text{GL}(1, E) \cong E^\times \).

Recall that in [4] Deligne and Lusztig have associated to \( \theta \) a virtual representation \( R_{\theta,w} \) of \( G \) and they have proved that every \( \pi \in \text{Irr}(G) \) is an irreducible constituent of some (in general non-unique) \( R_{\theta,w} \).

The following result is proven in Section 3.

**Theorem 1.3.** Assume that an irreducible representation \((\pi, V)\) appears in \( R_{\theta,w} \) for some \( w \) and \( \theta \) as above. Then
\[
\gamma_{\psi}(\pi) = \gamma_{w,\psi}(\theta)
\] (1.5)

In particular, \( \gamma_{\psi}(\pi) = \gamma_{\psi}(\pi') \) if \( \pi \) and \( \pi' \) appear in the same virtual representation \( R_{\theta,w} \).
1.4. The case of arbitrary group. Let now $G$ be any connected split reductive group over $F$, $G = G(F)$. Let $T$ be the Cartan group of $G$. Let also $T^\vee$ denote the dual torus to $T$ over $\mathbb{Q}_l$. The Weyl group $W$ acts naturally on $T^\vee$.

Assume that we are given an $n$-dimensional representation $\rho : T^\vee \to \text{GL}(n, \mathbb{Q}_l)$ of $T^\vee$ such that for every $w \in W$ the composition $\rho \circ w$ is isomorphic to $\rho$. In other words, $\rho$ is given by a collection $\lambda_1, \ldots, \lambda_n$ of characters of $T^\vee$ which is invariant under the action of $W$.

Let $T^\vee_\rho = G^{m,F}_{n,\mathbb{Q}_l}$. Then we get a natural map $p^\vee_\rho : T^\vee \to T^\vee_\rho$ sending every $t$ to $(\lambda_1(t), \ldots, \lambda_n(t))$.

Let now $T_\rho \simeq G^{m,F}_n$ denote the dual torus to $T^\vee_\rho$ over $F$ and let $p_\rho : T_\rho \to T$ denote the map which is dual to $p^\vee_\rho$. Explicitly one has

\[ p_\rho(x_1, \ldots, x_n) = \lambda_1(x_1) \ldots \lambda_n(x_n). \] (1.6)

Let $W_\rho \simeq S_n$ denote the Weyl group of $\text{GL}(n)$.

Let now $\pi$ be an irreducible representation of $G$. Assume that $\pi$ appears in some $R_{\theta,w}$ for some $\theta : T_w \to \mathbb{Q}_l^\times$. Let $w'$ be any lift of $w$ to $W_\rho$. Then $p_\rho$ induces an $F$-rational map $p_{\rho,w'} : T_{\rho,w'} \to T_w$, hence a homomorphism $p_{\rho,w'} : T_{\rho,w'} \to T_w$. Let $\theta' = p^*_{\rho,w'}(\theta)$. Define

\[ \gamma_{G,\rho,\psi}(\pi) := \gamma_{\psi}(\pi'), \] (1.7)

where $\pi'$ is any irreducible representation of $G_\rho$ which appears in $R_{\theta',w'}$. By Theorem 1.3 one has

\[ \gamma_{G,\rho,\psi}(\pi) = \gamma_{w',\psi}(\theta'). \] (1.8)

Lemma 1.5. The definition of $\gamma_{G,\rho,\psi}(\pi)$ does not depend on the choice of $w'$.

Proof. Let $w''$ be another lift of $w$ to $W_\rho$ and let $\theta''$ be the corresponding character of $T_{w''}$. Then it follows from 1.3 that

\[ \langle R_{\theta',w'}, R_{\theta'',w''} \rangle \neq 0. \] (1.9)

Therefore, our lemma follows from Theorem 1.3. \qed

Sometimes we shall write $\gamma_{w,\rho,\psi}(\theta)$ instead of $\gamma_{G,\rho,\psi}(\pi)$. Let now $\Phi_{G,\rho,\psi}$ denote the unique central function on $G$ such that for every irreducible representation $(\pi, V)$ of $G$ one has

\[ \sum_{g \in G} \Phi_{G,\rho,\psi}(g) \pi(g) = \gamma_{G,\rho,\psi}(\pi) \cdot \text{Id}_V. \] (1.10)

We would like to compute this function explicitly using geometry. More precisely, we are going to do the following.

We say that the representation $\rho$ is good if there exists a character $\sigma : G \to \mathbb{G}_m$ such that for every weight $\lambda$ of $\rho$ as above one has $\langle \lambda, \sigma \rangle > 0$. For any good representation $\rho$ we are going to construct an irreducible perverse sheaf $\Phi_{G,\rho,\psi}$ on $G$ endowed with a Weil structure.
Remark. The condition of being “good” is not very restrictive: if one starts with arbitrary $G$ and $\rho$ one can always make $\rho$ good by passing to $G' = G \times \mathbb{G}_m$ and taking $\rho' = \rho \otimes \text{St}$ where St denotes the standard one-dimensional representation of $\mathbb{G}_m$.

One of our main results is

**Theorem 1.6.** Assume that the semi-simple rank of $G$ is $\leq 1$ or that $G = \text{GL}(n)$. Then

$$\chi(\Phi_{G,\rho,\psi}) = \Phi_{G,\rho,\psi}. \quad (1.11)$$

When $\rho$ is sufficiently generic (i.e. when the cocharacters $\lambda_1, ..., \lambda_n$ span a lattice of rank equal to $\dim T$) the sheaf $\Phi_{G,\rho,\psi}$ as above is explicitly constructed on the set $G_r$ of regular elements in $G$ and on the whole of $G$ it is obtained by means of the Goresky-MacPherson extension.

When the semi-simple rank of $G$ is $\leq 1$ we can also show that the functor of convolution with $\Phi_{G,\rho,\psi}$ enjoys some nice properties. In Theorem 1.7 we compute the convolution of $\Phi_{G,\rho,\psi}$ with Lusztig’s character sheaves. In particular, we show that $\ast$ and $\ast$ convolutions in this case coincide. Also, we prove the following result (assuming again that the semi-simple rank of $G$ is $\leq 1$):

**Theorem 1.7.** Assume that $\rho_1$ and $\rho_2$ are good with respect to the same character $\sigma$ of $G$. Then

$$\Phi_{G,\rho_1 \oplus \rho_2,\psi} \simeq \Phi_{G,\rho_1,\psi} \ast \Phi_{G,\rho_2,\psi} \simeq \Phi_{G,\rho_1,\psi} \ast \Phi_{G,\rho_2,\psi} \quad (1.12)$$

We conjecture that the above theorems hold for general $G$ but we don’t know how to prove this (however, in Section 6 we deduce these results from certain conjectural cohomology vanishing). We also believe in the following

**Conjecture 1.8.** The functors $\mathcal{F} \mapsto \mathcal{F} \ast \Phi_{G,\rho,\psi}$ and $\mathcal{F} \mapsto \mathcal{F} \ast \Phi_{G,\rho,\psi}$ are exact in the perverse $t$-structure.

For example, when $G = \text{GL}(n)$ and $\rho$ is the standard representation this conjecture follows from the corresponding property of the Fourier-Deligne transform.

1.9. Acknowledgements. We are grateful to D. Gaitsgory and R. Bezrukavnikov, N. Katz and F. Loeser for very helpful discussions on the subject.

2. Induction and Restriction Functors

The purpose of this section is to collect some facts about Lusztig’s induction and restriction functors which will be used later.

2.1. **Restriction.** Let $P$ be a parabolic subgroup of $G$ and let $M$ be the corresponding Levi factor. Let $i_P : P \to G$ and $a_P : P \to M$ be the natural maps. Following Lusztig we define the restriction functor $\text{Res}_M^G : \mathcal{D}(G) \to \mathcal{D}(M)$ by setting

$$\text{Res}_M^G(\mathcal{F}) = (a_P)_! i^* \mathcal{F} \quad (2.1)$$
2.2. The space $\tilde{G}$. Let $\tilde{G}$ denote the variety of all pairs $(B, g)$, where

- $B$ is a Borel subgroup of $G$,
- $g \in B$.

One has natural maps $\alpha : \tilde{G} \to T$ and $\pi : \tilde{G} \to G$ defined as follows. First of all, we set $\pi(B, g) = g$. Now, in order to define $\alpha$, let us recall that for any Borel subgroup $B$ of $G$ one has canonical identification $\mu_B : B/U_B \simeq T$, where $U_B$ denotes the unipotent radical of $B$ (in fact, this is how the abstract Cartan group $T$ is defined). Now we set $\alpha(B, g) = \mu_B(g)$.

2.3. Induction. Let now $P = B$. In this case $M$ is the Cartan group $T$. We define the induction functor $\text{Ind}_T^G : D(T) \to D(G)$ setting

$$\text{Ind}_T^G(F) = \pi_! \alpha^*[d](\frac{d}{2})$$

(2.2)

where $\pi$ and $\alpha$ are as above and $d = \dim G - \dim T$.

We set $\text{Spr} = \text{Ind}_T^G(\delta_e)$ where $\delta_e$ denotes the $\delta$-function sheaf at the unit element of $G$. It is known that $\text{Spr}$ is a perverse sheaf supported on the set $N$ of unipotent elements of $G$. Moreover, $\text{Spr}$ is endowed with a natural $W$-action (see for example [4]).

2.4. A reformulation. For a subgroup $H$ of $G$ let $D^H(G)$ denote the derived category of $\ell$-adic sheaves on $G$ which are equivariant with respect to the adjoint action. Then the functor $\text{Ind}_T^G$ can be rewritten as follows. Following [1] let us define the averaging functor $\text{Av}_{G/B} : D^B(G) \to D^G(G)$. Let $\delta : G \times B \to B$ and $\eta : G \times B \to G \times B$ be the natural maps (in the definition of $G \times B$ the group $B$ acts by translations on $G$ and via the adjoint action on $B$). Let also $m_B : G \times B \to G$ be the map sending a pair $(g, b)$ to $gbg^{-1}$ (note that $m_B$ is proper). Let $\mathcal{F} \in D^B(B)$. Then there exists canonical $\hat{G} \in D(G \times B)$ such that $\eta^* \hat{G} = \delta^* \mathcal{F}$. We define

$$\text{Av}_{G/B}(\mathcal{F}) = (m_B)_! \hat{G}.$$  

Abusing the notations we shall denote the composition of $\text{Av}_{G/B}$ with the forgetful functor going from $D^G(G)$ to $D(G)$ by the same symbol. Given any $\mathcal{F} \in D(T)$ its inverse image $\mathcal{F}_B$ to $B$ with respect to the natural map $B \to T$ can be naturally regarded as an object of $D^B(G)$. Then it is easy to see that

$$\text{Ind}_T^G(\mathcal{F}) = \text{Av}_{G/B}(\mathcal{F}_B)[d](\frac{d}{2}).$$

(2.3)

We define now the functors $\tilde{\text{Ind}}_T^G : D(T) \to D(G)$ and $\tilde{\text{Res}}_T^G : D(G) \to D(T)$ by setting

$$\tilde{\text{Ind}}_T^G(\mathcal{F}) = \text{Ind}_T^G \mathcal{F} \otimes H^*(T, \mathbb{Q}_l) \otimes (\mathbb{Q}_l[1](-\frac{1}{2}))^{\otimes 2\dim G} =$$

$$\text{Ind}_T^G \mathcal{F} \otimes (H_\ast^*(T, \mathbb{Q}_l))^{\vee}[2d](d)$$

(2.4)

(2.5)
where \((H^*_c(T, \mathbb{Q}_l))^\vee\) denotes the graded dual to \(H^*_c(B)\) and

\[
\widehat{\text{Res}}_T^G \mathcal{G} = \text{Res}^G_T \mathcal{G} \otimes H^*_c(G, \mathbb{Q}_l) \quad (2.6)
\]

The following facts about the induction and restriction functors are essentially due to Lusztig [13], Theorem 4.4 and Ginzburg [6], Theorem 6.2. However, since these results are stated in \textit{loc. cit.} are stated only for character sheaves, we are going to sketch the proofs.

**Theorem 2.5.** 1. The functor \(\text{Ind}^G_T\) maps perverse sheaves to perverse sheaves.

2. Let \(\mathcal{G}\) be a perverse sheaf on \(G\) which is equivariant with respect to the adjoint action. Let also \(\mathcal{F}\) be any perverse sheaf on \(T\). Then

\[
R\text{Hom}(\widehat{\text{Res}}_T^G \mathcal{G}, \mathcal{F}) = R\text{Hom}(\mathcal{G}, \widehat{\text{Ind}}_T^G \mathcal{F}) \quad (2.7)
\]

Moreover, for any \(\mathcal{F}, \mathcal{G}\) as above the following diagram is commutative.

\[
\begin{array}{ccc}
R\text{Hom}(\widehat{\text{Res}}_T^G \mathcal{G}, \mathcal{F}) & \longrightarrow & R\text{Hom}(\mathcal{G}, \widehat{\text{Ind}}_T^G \mathcal{F}) \\
\downarrow & & \downarrow \\
R\text{Hom}(\widehat{\text{Res}}_T^G \text{Fr}^* \mathcal{G}, \text{Fr}^* \mathcal{F}) & \longrightarrow & R\text{Hom}(\text{Fr}^* \mathcal{G}, \widehat{\text{Ind}}_T^G \text{Fr}^* \mathcal{F})
\end{array} \quad (2.8)
\]

(note that to write vertical arrows one needs to use the natural isomorphisms \(\widehat{\text{Ind}}_T^G \text{Fr}^* \mathcal{F} \simeq \text{Fr}^* \widehat{\text{Ind}}_T^G \mathcal{F}\) and \(\widehat{\text{Res}}_T^G \text{Fr}^* \mathcal{G} \simeq \text{Fr}^* \widehat{\text{Res}}_T^G \mathcal{G}\)).

3. Let \(\mathcal{F}\) be an irreducible perverse sheaf on \(T\). Assume that the support of \(\mathcal{F}\) is a \(W\)-invariant subtorus in \(T\). Then for every \(w \in W\) one has a canonical isomorphism

\[
\text{Ind}_T^G(\mathcal{F}) \simeq \text{Ind}_T^G(w^* \mathcal{F}). \quad (2.9)
\]

**Remark.** One can show that Theorem 2.5(3) holds for any perverse sheaf \(\mathcal{F}\) on \(T\). However, in this case the argument is a little more complicated and we are not going to present it since we don’t need it.

**Proof.** The first assertion of Theorem 2.5 is standard (cf. [13], Section 4.3). Also the second assertion follows from standard adjointness properties of inverse and direct images. Let us prove the third assertion.

Let \(T' \subset T\) denote the support of \(\mathcal{F}\). Then we can find two reductive groups \(G_1\), \(G_2\) and a surjective homomorphism \(\kappa : G_1 \times G_2 \to G\) such that

1) the kernel of \(\kappa\) is a central subtorus in \(G_1 \times G_2\).

2) the preimage of \(T'\) under \(\kappa\) (with respect to some embedding of \(T\) into \(G\)) is equal to a maximal torus \(T_1\) in \(G_1\).

Let \(T_2\) be a (split) maximal torus in \(G_2\). We have the natural map \(\kappa_T : T_1 \times T_2 \to T\) with connected kernel. It is easy to see that it is enough to construct the isomorphism \((2.9)\) for \(\kappa_T^* \mathcal{F}[\dim \ker \kappa_T]\).
Let $W_i$ denote the Weyl group of $G_i$. Then $W = W_1 \times W_2$. On the other hand
\[ \text{Ind}_{G_1 \times G_2}^{T_1 \times T_2}(\kappa^*_T \mathcal{F}[\dim \ker \kappa_T]) = \text{Ind}_{G_1}^{T_1}(\kappa^*_T \mathcal{F}[\dim \ker \kappa_T]) \boxtimes \text{Spr}_{G_2}. \]
Since the second multiple is endowed with a natural action of $W_2$, it is enough to construct an action of $W_1$ on the first multiple. However, it follows from the fact that the map $\pi : \tilde{G} \to G$ is small that $\text{Ind}_{G_1}^{T_1}(\kappa^*_T \mathcal{F}[\dim \ker \kappa_T])$ is equal to the Goresky-MacPherson extension of its restriction to the set of regular semi-simple elements in $G_1$ where the construction of the $G_1$-equivariant structure is standard. \hfill \Box

2.6. Composition. Let $P, M$ be as above and let $W_M \subset W$ be the Weyl group of $M$ (note that the embedding of $W_M$ into $W$ depends on $P$).

**Theorem 2.7.** Let $\mathcal{F}$ be an irreducible perverse $W$-equivariant sheaf on $T$ whose support $\text{supp} \mathcal{F}$ is a $W$-stable subtorus in $T$. Then
\[ \text{Res}_M^G \text{Ind}_T^G(\mathcal{F}) \simeq \text{Ind}_M^W(\text{Ind}_T^M(\mathcal{F})) \quad (2.10) \]
and this isomorphism commutes with the natural actions of $W$ on both sides.

**Remark.** For character sheaves this result is proved in [13].

**Proof.** First of all consider $\text{Res}_M^G \text{Ind}_T^G(\mathcal{F})$ where $\mathcal{F}$ is an arbitrary object of $D(T)$. We claim that it is glued from the complexes $\text{Ind}_T^M(w^*\mathcal{F})$ where $W$ runs over the representatives of the cosets $W/W_M$ of minimal length. Indeed, the sheaf $\text{Res}_M^G \text{Ind}_T^G(\mathcal{F})$ can be computed in the following way.

Consider the product $Z = P \times \tilde{G}$. We let $\delta : Z \to T$ and $\sigma : Z \to M$ be the natural maps. Then $\text{Res}_M^G \text{Ind}_T^G(\mathcal{F}) = \sigma_! \delta^* \mathcal{F}[\delta]\left(\frac{d}{2}\right)$. On the other hand to every $w$ as above there corresponds a locally closed stratum $Z_w$ of $Z$ (consisting of pairs $(B, x \in B \cap P)$ where $B$ and $P$ are in position $w \text{mod} W_M$). We denote by $\delta_w$ and $\sigma_w$ the restrictions of $\delta$ and $\sigma$ to $Z_w$. Then looking at the Cousin complex associated with the stratification $Z_w$ we see that $\text{Res}_M^G \text{Ind}_T^G(\mathcal{F})$ is glued from the complexes $(\sigma_w)_! \delta_w^* \mathcal{F}[\delta]\left(\frac{d}{2}\right)$.

Each $Z_w$ has a natural map $\mu_w$ to $\tilde{M}$. Namely, for every Borel subgroup $B$ of $G$ the image of $P \cap B$ in $M = P/UP$ is a Borel subgroup of $M$ and we set $\mu_w(B, x) = (B \cap P \text{ mod } U_P, x \text{ mod } U_P)$. It is easy to see that $\mu_w$ is a locally trivial fibration with fiber isomorphic to $A^{\dim U_P - l(w)}$. Also the composition of $\mu_w$ with the natural map $\alpha_M : \tilde{M} \to T$ is equal to $w \circ \delta$. This implies that $(\sigma_w)_! \delta_w^* \mathcal{F}[\delta]\left(\frac{d}{2}\right) \simeq \text{Ind}_T^M(w^*\mathcal{F})$. Hence $\text{Res}_M^G \text{Ind}_T^G(\mathcal{F})$ is glued from the complexes $\text{Ind}_T^M(w^*\mathcal{F})$.

Assume now that $\mathcal{F}$ satisfies the conditions of the theorem. Then $\mathcal{F}$ acts naturally on $\text{Res}_M^G \text{Ind}_T^G(\mathcal{F})$ and it is easy to see that it permutes the various subquotients $\text{Ind}_T^M(w^*\mathcal{F})$ from which $\text{Res}_M^G \text{Ind}_T^G(\mathcal{F})$ is glued by the above argument. Hence $\text{Res}_M^G \text{Ind}_T^G(\mathcal{F})$ is isomorphic to $\text{Ind}_W^W \text{Ind}_T^G$.

Assume now that we are given two perverse sheaves $\mathcal{F}_1$ and $\mathcal{F}_2$ on $T$ satisfying the conditions of Theorem [2.6]. Let also $\mathcal{G}$ be a direct summand of $\text{Ind}_T^G(\mathcal{F}_2)$. Then it follows from Theorem [2.6] that $\text{Res}_T^G(\mathcal{G})$ has a natural $W$-equivariant structure and
hence the same is true for $\widetilde{\text{Res}}^G_T(G)$. Therefore we have a natural action of $W$ on $\text{RHom}(\mathcal{F}_1, \widetilde{\text{Res}}^G_T(G))$. On the other hand, since $W$ acts naturally on $\text{Ind}^G_T \mathcal{F}_1$ and on $H^*_c(T, \mathbb{Q})$; hence it also acts on $\widetilde{\text{Ind}}^G_T \mathcal{F}_1$. Therefore $W$ also acts on $\text{RHom}(\widetilde{\text{Ind}}^G_T \mathcal{F}_1, G)$.

The proof of the following lemma is left to the reader.

**Lemma 2.8.** The isomorphism (2.7) commutes with the above $W$-actions.

Let $U$ be the unipotent radical of a Borel subgroup $B$ of $G$. Let $r_U : G \to G/U$ denote the natural map.

**Proposition 2.9.** Let $\mathcal{G}$ be a perverse sheaf on $G$ which is equivariant with respect to the adjoint action. Assume that $(r_U)_!* \mathcal{G}$ is concentrated on $T \subset G/U$. Then for every $\mathcal{F} \in D(T)$ we have

$$\mathcal{G} \ast \text{Ind}^G_T(\mathcal{F}) \simeq \text{Ind}^G_T(\text{Res}^G_T \mathcal{G} \ast \mathcal{F}) \quad (2.11)$$

(in the above formula the restriction functor is taken with respect to the chosen Borel subgroup $B$).

The same result holds if we require that $(r_U)_!* \mathcal{G}$ vanishes outside of $T$ and replace $\ast$-convolution by $\ast$-convolution.

*Proof.* We are going to prove the statement about $\ast$-convolution. The proof for $\ast$-convolution is analogous.

It is easy to see that for any $\mathcal{H} \in D^B(G)$ we have the natural isomorphism

$$\mathcal{G} \ast \text{Av}_{G/B}(\mathcal{H})[d](\frac{d}{2}) \simeq \text{Av}_{G/B}(\mathcal{G} \ast \mathcal{H})[d](\frac{d}{2}) \quad (2.12)$$

Let us apply this to $\mathcal{H} = \mathcal{F}_B$. Then the right hand side of (2.12) is equal to the right hand side of (2.11). On the other hand, the assumption that $r_U^* \mathcal{G}$ vanishes outside of $T = B/U$ together with $U$-equivariance of $\mathcal{F}_B$ imply that

$$\mathcal{G} \ast \mathcal{F}_B = (\text{Res}^G_T(\mathcal{G}) \ast \mathcal{F})_B \quad (2.13)$$

which finishes the proof.

\[
\square
\]

3. **Character sheaves and Deligne-Lusztig representations**

3.1. **Maximal tori.** Let us recall the classification of conjugacy classes of $\mathbb{F}_q$-rational maximal tori in $G$. Recall that we assume that $G$ is split.

Let $T$ denote the abstract Cartan group of $G$ (with its canonical $F$-rational structure). Given $w \in W$ we can construct a new Frobenius morphism $\text{Fr}_w : T \to T$ sending every $t \in T$ to $w(\text{Fr}(t))$. In this way we get a new $F$-rational structure on $T$. We will denote the resulting torus by $T_w$. It is clear that if $w$ and $w'$ belong to the same conjugacy class then $T_w$ and $T_{w'}$ are isomorphic.

The following result is proved in [4]:

\[
\square
\]
Theorem 3.2. For every \( w \in W \) there exists an embedding of \( T_w \) in \( G \) and in this way we get a bijection between \( G \)-conjugacy classes of \( F \)-rational maximal tori in \( G \) and conjugacy classes in \( W \).

3.3. Characters of tori. Let \( T \) be any algebraic torus over \( F \) and let \( \theta \) be a character of \( T \). One can associate to \( \theta \) an \( l \)-adic local system \( L_\theta \) on \( T \) in the following way.

Let \( \alpha : T \to T \) be the morphism given by \( \alpha(t) = \text{Fr}(t)t^{-1} \). Then \( \alpha \) is a Galois étale covering with Galois group equal to \( T \). Hence \( T \) acts on the sheaf \( \alpha^*\mathbb{Q}_l \). We set \( L_\theta \) to be the part of \( \alpha^*\mathbb{Q}_l \) on which \( T \) acts by means of \( \theta \).

Let \( w', w'' \in W \) and let \( \theta' \) (resp. \( \theta'' \)) be a character of \( T_{w'} \) (resp. of \( T_{w''} \)). We say that \( \theta' \) and \( \theta'' \) are geometrically conjugate if there exists \( w \in W \) such that \( w^*\mathcal{L}_{\theta'} \simeq \mathcal{L}_{\theta''} \) (note that as varieties over \( \overline{F} \) both \( T_{w'} \) and \( T_{w''} \) are identified with \( T \)). This notion is introduced in [4] in a slightly different language.

3.4. Deligne-Lusztig representations. Let \( w \in W \) and let \( \theta : T_w \to \mathbb{Q}_l^\times \) be a character. In [4] Deligne and Lusztig constructed a virtual representation \( R_{\theta,w} \) of \( G \).

We are going to need the following facts about \( R_{\theta,w} \). Let \( K_G \) denote the Grothendieck group of representations of \( G \). We have a natural pairing \( \langle \,, \rangle : K_G \otimes K_G \to \mathbb{Z} \) such that if \( \pi_1, \pi_2 \in \text{Irr}(G) \) then

\[
\langle \pi_1, \pi_2 \rangle = \begin{cases} 
1 & \text{if } \pi_1 \text{ is isomorphic to } \pi_2 \\
0 & \text{otherwise}
\end{cases} \quad (3.1)
\]

1) For every \( \pi \in \text{Irr}(G) \) there exists \( w \in W \) and \( \theta : T_w \to \mathbb{Q}_l^\times \) such that
\[
\langle \pi, R_{\theta,w} \rangle \neq 0 \quad (3.2)
\]

2) One has
\[
\langle R_{\theta,w}, R_{\theta',w'} \rangle \neq 0 \quad (3.3)
\]

if and only if \( \theta \) and \( \theta' \) are geometrically conjugate.

3.5. Character sheaves. Let \( G \) be an arbitrary reductive algebraic group over \( F \).
Let us recall Lusztig's definition of (some of) the character sheaves.

Let \( \mathcal{L} \) be a tame local system on \( T \). We define \( \mathcal{K}_\mathcal{L} = \text{Ind}_T^G \mathcal{L} \). One knows (cf. [13], [10]) that the sheaf \( \mathcal{K}_\mathcal{L} \) is perverse.

3.6. The Weil structure. Assume now, that for some \( w \in W \) there exists an isomorphism \( \mathcal{L} \simeq \text{Fr}_w^*(\mathcal{L}) \) and let us fix it. It was observed by G. Lusztig in [13] that fixing such an isomorphism endows \( \mathcal{K}_\mathcal{L} \) canonically with a Weil structure (this follows immediately from Theorem 3.2).

Let now \( \theta : T_w \to \mathbb{Q}_l^\times \) be any character. The following result is due to G. Lusztig.

Theorem 3.7.
\[
\chi(\mathcal{K}_\mathcal{L}_\theta) = q^{-\frac{d}{2}} \text{ch}(R_{\theta,w}) \quad (3.4)
\]
4. \(\gamma\)-SHEAVES ON SPLIT TORI

4.1. Let \(T\) be a split torus over \(F\) and let
\[
\rho = \lambda_1 \oplus \ldots \oplus \lambda_n
\]
be a good representation of \(T^\vee\). Recall that this means that there exists a character \(\sigma : T \to \mathbb{G}_m\) such that \(\langle \sigma, \lambda_i \rangle > 0\) for every \(i\).

Each \(\lambda_i\) can be considered as a cocharacter of \(T\). Let \(T_\rho = \mathbb{G}_m^n\). Define the map \(p_\rho : T_\rho \to T\) by setting
\[
p_\rho(t_1, \ldots, t_n) = \lambda_1(t_1) \ldots \lambda_n(t_n).
\]

We have
\[
\text{tr}_\rho : T_\rho \to \mathbb{A}^1 \text{ given by } \text{tr}_\rho(x_1, \ldots, x_n) = x_1 + \ldots + x_n.
\]

Consider the complex \(\Phi_{T,\rho,\psi} := (p_\rho)_! \text{tr}_\rho^* \mathcal{L}_\psi[n](\frac{n}{2})\) (on \(T\)).

**Theorem 4.2.** Assume that \(\lambda_i\) is non-trivial for every \(i = 1, \ldots, n\). Then

1. The complex \(\Phi_{T,\rho,\psi}\) is perverse.
2. \(\text{supp} \Phi_{T,\rho,\psi} = p_\rho(T_\rho)\)
3. Assume in addition that \(\rho\) is good. Then the natural map
\[
\Phi_{T,\rho,\psi} := (p_\rho)_! \text{tr}_\rho^* \mathcal{L}_\psi[n](\frac{n}{2}) \rightarrow (p_\rho)_* \text{tr}_\rho^* \mathcal{L}_\psi[n](\frac{n}{2})
\]
is an isomorphism and \(\Phi_{T,\rho,\psi}\) is an irreducible perverse sheaf on \(T\).

**Proof.** Point (1) of Theorem 4.2 follows from the following result.

**Proposition 4.3.** Let \(\lambda : \mathbb{G}_m \to T\) be a non-trivial character and define \(\Phi_{\lambda} = \lambda_*(\mathcal{L}_{\psi})[1](\frac{1}{2})\). Then the functors \(C_{\lambda,*}, C_{\lambda,*} : \mathcal{D}(T) \to \mathcal{D}(T)\) sending every \(A \in \mathcal{D}(T)\) to \(\Phi_{\lambda} \ast A\) and to \(\Phi_{\lambda} \ast A\) respectively map perverse complexes to perverse ones.

To see that Proposition 4.3 implies Theorem 4.2(1) it is enough to note that the complex \(\Phi_{\lambda}\) is perverse for every \(\lambda\) and that
\[
\Phi_{T,\rho,\psi} \simeq \Phi_{\lambda_1} \ast \ldots \ast \Phi_{\lambda_n}
\]

**Proof of Proposition 4.3.** First of all there exists another torus \(T'_\lambda\) together with an isogeny \(q_\lambda : T'_\lambda \to T\) and an injective cocharacter \(\lambda' : \mathbb{G}_m \to T\) such that \(\lambda = q_\lambda \circ \lambda'\). Let \(S_{\lambda} = T'_\lambda / \mathbb{G}_m\). Set also \(T_{\lambda} = T'_\lambda \times \mathbb{A}^1\).

We have the natural map \(s_{\lambda} : T_{\lambda} \to S_{\lambda}\) which endows \(T_{\lambda}\) with a structure of a line bundle over \(S_{\lambda}\). Moreover, the dual vector bundle can be naturally identified with \(s_{\lambda,-1} : T'_{\lambda,-1} \to S_{\lambda,-1} = S_{\lambda}\). Thus one can consider the Fourier-Deligne transform functor \(F_{\lambda} : \mathcal{D}(T_{\lambda}) \to \mathcal{D}(T'_{\lambda})\) (cf. e.g. [1]).

Let \(j_{\lambda} : T'_{\lambda} \to T_{\lambda}\) denote the natural embedding.

The following lemma is straightforward from the definitions.
**Lemma 4.4.** There exists a natural isomorphism of functors
\[ j_\lambda^* \circ F_{\lambda^{-1}} \circ (j_{\lambda^{-1}})_! \circ q_\lambda^* \simeq q_\lambda^* \circ C_{\lambda,*} \] (4.7)
and
\[ j_\lambda^* \circ F_{\lambda^{-1}} \circ (j_{\lambda^{-1}})_* \circ q_\lambda^* \simeq q_\lambda^* \circ C_{\lambda,*} \] (4.8)
(note that we can take \( T'_\lambda = T_{\lambda^{-1}}' \)).

To see that Lemma 4.4 implies Proposition 4.3 it is enough to note that a complex \( A \in D(T) \) is perverse if and only if \( q_\lambda^*(A) \) is perverse and that the functors \((j_{\lambda^{-1}})_!\) and \( F_{\lambda^{-1}} \) map perverse complexes to perverse ones (for the former this follows from the fact that \( j_\lambda \) is an affine open embedding and for the latter cf. [9]).

**4.5. Proof of Theorem 4.2(2).** We are going to use induction on \( n \). For \( n = 1 \) the statement is obvious. Assume that we know the result for \( n - 1 \). Let \( \rho' = \{\lambda_1, \ldots, \lambda_{n-1}\} \). Then we know that \( \Phi T_{\rho', \psi}! \) is supported on the image of \( p_{\rho} \). We have \( \Phi T_{\rho', \psi}! = \Phi T_{\rho' \psi}! \times \Phi_{\lambda_n} \). Thus our statement follows from Lemma 4.4 and from the following result.

**Proposition 4.6.** Let \( X \) be a scheme over \( \mathbb{F} \), \( \pi : L \to X \) - a line bundle. Denote by \( L^\vee \) the dual line bundle and by \( j : \overset{\circ}{L} \to L \) the embedding of the complement to the zero section (thus we have the natural isomorphism \( \overset{\circ}{L} \simeq L^\vee \)). Let \( F_\psi : D(L) \to D(L^\vee) \) denote the Fourier-Deligne transform corresponding to the additive character \( \psi \). Then for every \( F \in D(L) \) we have
\[ \text{supp}(F_\psi(j!F)) = \mathbb{G}_m \cdot \text{supp}(F) \] (4.9)
(\( \text{here bar denotes the Zariski closure} \)).

**Proof.** The statement is immediately reduced to the case when \( X \) is a point. Thus we have the embedding \( j : \mathbb{G}_m \to \mathbb{A}^1 \) and a complex \( F \in D(\mathbb{G}_m) \). We have to show that \( F_\psi(j!F) \) is non-zero at the generic point of \( \mathbb{A}^1 \) provided that \( F \neq 0 \).

Assume the contrary. Then the Grothendieck-Ogg-Schafarevich formula for the Euler-Poincaré characteristic (cf. for example formula 2.3.1 in [8]) implies that
1) \( F \) is locally constant.
2) For any \( t \in \mathbb{F}^* \) the complex \( F \otimes t^* \mathcal{L}_\psi|_{\mathbb{G}_m} \) is tame at infinity.

Clearly this is possible only if \( F = 0 \). \( \square \)

Let us now pass to the proof of Theorem 4.2(3). Let \( H \) denote the kernel of \( p_{\rho} \) and let \( \delta_H \) be the constant sheaf on \( H \) shifted by \( \dim H \) (regarded as a perverse sheaf on \( T_{\rho'} \)).

We must show that the natural map
\[ \Phi T_{\rho', \psi} = \Phi (p_{\rho})_! \phi_{\rho}^* \mathcal{L}_\psi|_{\mathbb{G}_m} \left( \frac{n}{2} \right) \to \Phi (p_{\rho})_* \phi_{\rho}^* \mathcal{L}_\psi|_{\mathbb{G}_m} \left( \frac{n}{2} \right) \]
is an isomorphism and that $\Phi_{T,\rho,\psi}$ is an irreducible perverse sheaf. Taking inverse image to $T_\rho$ we see that this is equivalent to the following two statements:

1) The natural map $\text{tr}_\rho^* \mathcal{L}_\psi[n](\frac{n}{2}) \ast \delta_H \to \text{tr}_\rho^* \mathcal{L}_\psi[n](\frac{n}{2}) \ast \delta_H$ is an isomorphism.

2) The perverse sheaf $\text{tr}_\rho^* \mathcal{L}_\psi[n](\frac{n}{2}) \ast \delta_H[\dim H]$ is irreducible as an $H$-equivariant perverse sheaf (i.e. it has no $H$-equivariant subsheaves).

Let $j : T_\rho \to \mathbb{G}_m^n \to \mathbb{A}^n$ be the natural embedding. Let also $F_{\mathbb{A}^n}$ denote the Fourier-Deligne transform on $\mathbb{A}^n$. Then arguing as above it is easy to see that

$$\text{tr}_\rho^* \mathcal{L}_\psi[n](\frac{n}{2}) \ast \delta_H = j^* F_{\mathbb{A}^n}(j_! \delta_H)$$

(4.10)

and

$$\text{tr}_\rho^* \mathcal{L}_\psi[n](\frac{n}{2}) \ast \delta_H = j^* F_{\mathbb{A}^n}(j_! \delta_H)$$

(4.11)

Since $F_{\mathbb{A}^n}$ is an auto-equivalence which maps $H$-equivariant perverse sheaves to $H$-equivariant perverse sheaves we see that in order to prove 1 and 2 above it is enough to show that $\mathcal{L}_\psi \otimes L$ is an irreducible perverse sheaf (i.e. it has no $H$-equivariant subsheaves).

For this it is enough to show that $H$ is closed in $\mathbb{A}^n$. But $H$ is a closed subset of $H_\sigma = \ker \sigma$. Hence it is enough to show that $H_\sigma$ is closed in $\mathbb{A}^n$. Let $a_i = \langle \lambda_i, \sigma \rangle$, $i = 1, \ldots, n$. Then

$$H_\sigma = \{(t_1, \ldots, t_n) \in \mathbb{G}_m^n | t_1^{a_1} \ldots t_n^{a_n} = 1\}$$

(4.12)

Since $\rho$ is good it follows that $a_i > 0$ for every $i$. This together with (4.12) clearly implies that $H_\sigma$ is closed in $\mathbb{A}^n$. \hfill \Box

4.7. **Tame local systems.** Recall that an $\ell$-adic local system $\mathcal{L}$ on $T$ is called tame if there exists a finite homomorphism $\pi : T' \to T$ of some other algebraic torus $T'$ to $T$ such that $\pi^* \mathcal{L}$ is trivial.

**Theorem 4.8.** Let $\rho = \oplus \lambda_j$ be as above such that all $\lambda_j$ are non-trivial. Then for every tame local system $\mathcal{L}$ on $T$ one has

$$\dim H^i_c(\Phi_{T,\rho,\psi} \otimes \mathcal{L}) = \begin{cases} 0 & \text{if } i \neq 0, \\ 1 & \text{if } i = 0. \end{cases}$$

(4.13)

The same is true for $H^i(\Phi_{T,\rho,\psi} \otimes \mathcal{L})$.

We set $H_{\rho,\mathcal{L},\psi,!} := H^0_c(\Phi_{T,\rho,\psi} \otimes \mathcal{L}^{-1})$ and $H_{\rho,\mathcal{L},\psi,*} := H^0(\Phi_{T,\rho,\psi} \otimes \mathcal{L}^{-1})$.

**Proof.** Let us prove the statement of Theorem 4.8 for $H^i_c(\Phi_{T,\rho,\psi} \otimes \mathcal{L})$. The proof for $H^i(\Phi_{T,\rho,\psi} \otimes \mathcal{L})$ is analogous.

First of all, we may assume without loss of generality that $T = T_\rho = \mathbb{G}_m^n$, and $\rho : \mathbb{G}_m^n \to \text{GL}(n)$ is the standard embedding. Indeed, the definition of $\Phi_{T,\rho,\psi}$ and the projection formula imply that the statements of Theorem 4.8 hold for $\mathcal{L}$ if and only if they hold for $p_\rho^* \mathcal{L}$.

On the other hand since $\mathcal{L}$ is a one-dimensional tame local system on $\mathbb{G}_m^n$ it follows that there exist tame local systems $\mathcal{L}_1, \ldots, \mathcal{L}_n$ on $\mathbb{G}_m$ such that $p_\rho^* \mathcal{L} \simeq \mathcal{L}_1 \boxtimes \cdots \boxtimes \mathcal{L}_n$. \hfill 12
Since in our case $\Phi_{T,\rho,\psi} = \mathcal{L}_\psi \boxtimes \cdots \boxtimes \mathcal{L}_\psi [n](\frac{n}{2})$, Theorem 4.8(1) follows from the following well-known lemma.

**Lemma 4.9.** Let $\mathcal{L}$ be a one-dimensional local system on $\mathbb{G}_m$. Then $H^i_c(\mathcal{L} \otimes \mathcal{L}_\psi) = 0$ if $i \neq 1$ and $\dim H^1_c(\mathcal{L} \otimes \mathcal{L}_\psi) = 1$.

**Corollary 4.10.** Assume that $\rho$ is good. Then we have
\begin{equation}
\Phi_{T,\rho,\psi} \star \mathcal{L} = H_{\rho,\mathcal{L}_\psi} \otimes \mathcal{L} \tag{4.14}
\end{equation}
and
\begin{equation}
\Phi_{T,\rho,\psi} \star \mathcal{L} = H_{\rho,\mathcal{L}_\psi,*} \otimes \mathcal{L} \tag{4.15}
\end{equation}

5. The basic example

Now we assume again that $G = \text{GL}(n)$. Set $\Phi_{G,\rho,\psi} = \text{tr} \ast \mathcal{L}_\psi [n]^2(\frac{n}{2})$. Let also $\Phi_{T,\rho,\psi} = \text{tr} \ast \mathcal{L}_\psi [n](\frac{n}{2})$ where $\text{tr}_T : T \to \mathbb{A}^1$ is the restriction of the trace morphism to the group $T$ of diagonal matrices in $\text{GL}(n)$.

**Theorem 5.1.**

1. One has an isomorphism of functors
\begin{equation}
\Phi_{G,\rho,\psi} \star \text{Ind}^G_T(\mathcal{F}) \cong \text{Ind}^T_{G}(\Phi_{T,\rho,\psi} \star \mathcal{F}) \tag{5.1}
\end{equation}

2. Let $\mathcal{L}$ be a tame local system on $T$ endowed with an isomorphism $\text{Fr}^\ast \mathcal{L} \cong \mathcal{L}$ for some $w \in W$. In this case both sides of (5.1) are endowed with a natural Weil structure. Then the isomorphism of (5.1) is an isomorphism of Weil sheaves.

**Proof.** Let $\mathfrak{g}$ denote the Lie algebra of $G$, i.e. the algebra of $n \times n$-matrices. We have the natural embedding $j_G : G \hookrightarrow \mathfrak{g}$. Let us identify $\mathfrak{g}$ with its dual space by means of the form $(x, y) \mapsto \text{tr}(xy)$. Let $\mathbf{F}_{\mathfrak{g}} : \mathcal{D}(\mathfrak{g}) \to \mathcal{D}(\mathfrak{g})$ denote the corresponding Fourier transform functor. Then for every $\mathcal{G} \in \mathcal{D}(G)$ we have
\begin{equation}
\Phi_{G,\rho,\psi} \star \mathcal{G} \cong (j_G)^\ast \mathbf{F}_{\mathfrak{g}}((j_G)_! \mathcal{G}') \tag{5.2}
\end{equation}
where $\mathcal{G}'$ denotes the inverse image of $\mathcal{G}$ with respect to inversion map $g \mapsto g^{-1}$. Similarly for any $\mathcal{F} \in \mathcal{D}(T)$ we have
\begin{equation}
\Phi_{T,\rho,\psi} \star \mathcal{F} \cong (j_T)^\ast \mathbf{F}_t((j_T)_! \mathcal{F}') \tag{5.3}
\end{equation}

Let $\widetilde{\mathfrak{g}}$ be the space of all pairs $(\mathfrak{b}, x)$ where $\mathfrak{b}$ is a Borel subalgebra in $\mathfrak{g}$ and $x \in \mathfrak{b}$. We have the natural open embedding $j_{\mathfrak{G}} : \widetilde{G} \hookrightarrow \widetilde{\mathfrak{g}}$.

Let $\mathfrak{t}$ be the Cartan algebra of $G$ (the Lie algebra of $T$) and let $j_T : T \hookrightarrow \mathfrak{t}$ be the natural embedding. Then as in Section 2.3 we can define the induction functor $\text{Ind}^\mathfrak{t}_T : \mathcal{D}(\mathfrak{t}) \to \mathcal{D}(\mathfrak{g})$. It follows immediately from the definitions that for every $\mathcal{F} \in \mathcal{D}(T)$ we have the natural isomorphisms
\begin{equation}
(j_{\mathfrak{G}}) : \text{Ind}^\mathfrak{G}_T(\mathcal{F}) \cong \text{Ind}^\mathfrak{t}((j_T)_! \mathcal{F})
\end{equation}
and

\[ \text{Ind}_T^G F^i \simeq (\text{Ind}_T^G F)^i. \]

Hence the first statement of Theorem 5.1 follows from the following

**Lemma 5.2.** There is a natural isomorphism of functors

\[ F_g \circ \text{Ind}_T^g \simeq \text{Ind}_t^g \circ F_t. \quad (5.4) \]

**Proof.** Let \( \pi : \bar{g} \to g \) and \( \alpha : \bar{g} \to t \) be the natural maps. Then for every \( F \in D(t) \) we have

\[ \text{Ind}_t^g(F) = \pi_{!}\alpha_{*}F[\dim g - \dim t]\left(\frac{\dim g - \dim t}{2}\right). \]

Let \( X \) denote the flag variety of \( G \). Then \( \bar{g} \) can be regarded naturally as a vector subbundle of the trivial vector bundle \( X \times g \) over \( X \). Let \( \eta : X \times g \to g \) be the natural projection and let also \( F_{X \times g} \) denote the Fourier transform in the fiber of the vector bundle \( X \times g \). It is known that in this situation the functor \( \eta_{!} \) commutes with Fourier transform, i.e. there is a natural isomorphism of functors

\[ \eta_{!} \circ F_{X \times g} \simeq F_{g} \circ \eta_{!}. \]

Hence the lemma follows from the following observation: for every \( F \in D(t) \) we have

\[ F_{X \times g}(\alpha_{*}F) \simeq \alpha_{*}F_{t}(F). \]

The second statement of Theorem 5.1 is proved using similar considerations and we leave it to the reader. \( \square \)

**Corollary 5.3.** Assume that an irreducible representation \((\pi, V)\) appears in some Deligne-Lusztig representation \( R_{\theta, w} \). Then

\[ \gamma_{\psi}(\pi) = \gamma_{w, \psi}(\theta) \]

where the notations are as in Section 1.2. In particular, \( \gamma(\pi) = \gamma(\pi') \) if \( \pi \) and \( \pi' \) appear in the same virtual representation \( R_{\theta, w} \).

This follows immediately from Theorem 5.1 and from Theorem 3.7.

6. \( \gamma \)-sheaves: the main results

6.1. The perverse sheaf \( \Phi_{G, \rho, \psi} \). In what follows we assume that \( \rho \) is good and \( W \)-equivariant. We want to define a \( W \)-equivariant structure on \( \Phi_{T, \rho, \psi} \). Recall that we denote \( T_{\rho} = G_{m}^{n} \). Then the group \( W_{\rho} := S_{n} \) acts naturally on \( T_{\rho} \).

Choose \( w \in W \). We need to define an isomorphism \( \iota_{w} : w^{*}(\Phi_{T, \rho, \psi}) \sim \Phi_{T, \rho, \psi} \). Let (as above) \( w' \) be any lift of \( \rho(w) \) to \( W_{\rho} \). Then one has

\[ p_{\rho}(w'(t)) = w(p_{\rho}(t)). \quad (6.1) \]

The sheaf \( \text{tr}_{\rho}^{*}L_{\psi} \) is obviously \( W_{\rho} \)-equivariant. This, together with (6.1), gives rise to an isomorphism \( \iota_{w}' : w^{*}(A_{\rho}) \sim A_{\rho} \). We now define

\[ \iota_{w} := (-1)^{l(w') - l(w)}\iota_{w}' \]

(6.2)
Proposition 6.2. 1. The isomorphism \( \iota_w \) does not depend on the choice of \( w' \).
2. The assignment \( w \mapsto \iota_w \) defines a \( W \)-equivariant structure on the sheaf \( \Phi_{T,\rho,\psi} \).

Proof. Clearly the second statement of Proposition 6.2 follows from the first one. So, we just have to show that \( \iota_w \) does not depend on the choice of \( w' \). For this it is enough to show the following.

- Let \( s \in W_\rho \) be a simple reflection. Assume that \( p_\rho \circ s = p_\rho \) (i.e. \( s \) is a lift of the unit element \( e \in W \) to \( W_\rho \)). Then \( \iota_s \) is equal to multiplication by \(-1\).

For this it is enough to prove the following lemma.

Lemma 6.3. Consider the torus \( \mathbb{G}_m^2 \) with coordinates \( x \) and \( y \). Let \( \mathfrak{m} : \mathbb{G}_m^2 \to \mathbb{G}_m \) be the multiplication map and let \( f : \mathbb{G}_m^2 \to \mathbb{A}^1 \) be given by \( f(x,y) = x + y \). Then the involution \( s \) interchanging \( x \) and \( y \) acts on \( \mathfrak{m}_!(f^*L_\psi) \) by means of multiplication by \(-1\).

Proof. The quotient of \( \mathbb{G}_m^2 \) by the action of \( \mathbb{Z}_2 \) coming from \( s \) is isomorphic to \( \mathbb{A}^1 \times \mathbb{G}_m \) with coordinates \( z = x + y \) and \( t = xy \). Let \( q : \mathbb{G}_m^2 \to \mathbb{G}_m^2 / \mathbb{Z}_2 \) be the natural map. Then it is enough to show that the direct image under \( \mathfrak{m} \) of \( q! (f^*L_\psi)^{\mathbb{Z}_2} \) vanishes. But the latter sheaf is isomorphic \( L_\psi \boxtimes \mathbb{T}_l \) on \( \mathbb{G}_m^2 / \mathbb{Z}_2 \simeq \mathbb{A}^1 \times \mathbb{G}_m \) and the required assertion follows from the fact that \( H^*_c(\mathbb{A}^1,L_\psi) = 0 \).

Proposition 6.4. 1. The sheaf \( \Phi_{G,\rho,\psi} \) is non-zero and irreducible.
2. Assume that \( \text{im} p_\rho \cap T_{rs} \neq \emptyset \) where \( T_{rs} \) denotes the set of regular semi-simple elements in \( T \). Then \( \Phi_{G,\rho,\psi} \) is equal to the Goresky-MacPherson extension of its restriction to the set of regular semi-simple elements in \( G \). Moreover, the restriction of \( \Phi_{G,\rho,\psi} \) to the set \( G_r \) of regular elements in \( G \) is equal to \( s^*(q!\Phi_{T,\rho,\psi})^W_{[\dim G - \dim T]}(\dim G - \dim T) \) where \( s : G_r \to T/W \) and \( q : T \to T/W \) are the natural maps.

Proof. Let us first prove 2. The fact that \( \Phi_{G,\rho,\psi} \) is equal to the Goresky-MacPherson extension of its restriction to the set of regular semisimple elements in \( G \) follows from smallness of the morphism \( \pi : \widetilde{G} \to G \) (indeed, the fact that \( \pi \) is small implies that \( \text{Ind}_T^G \Phi_{T,\rho,\psi} \) is equal to the Goresky-MacPherson extension of its restriction to the set of regular semisimple elements in \( G \) (since in this case \( \Phi_{T,\rho,\psi} \) is equal to the Goresky-MacPherson extension of its restriction to the set of regular semisimple elements in \( T \) and hence the same is true for any of its direct summands).
Let us show that
\[
\Phi_{G,\rho,\psi}|_{G_r} = s^*(q_! \Phi_{T,\rho,\psi})^W [\dim G - \dim T](\frac{\dim G - \dim T}{2}).
\]
Both sides are equal to the Goresky-MacPherson extensions of their restrictions to the set \(G_{rs}\) of regular semi-simple elements. Hence it is enough to establish the above isomorphism on \(G_{rs}\) where it is obvious.

Let us prove 1. Arguing as in the proof of Theorem 2.5 we can assume that one of the following holds:

(i) \(\text{im} p_\rho = T\).

(ii) \(\text{im} p_\rho\) lies in \(Z(G)\) (the center of \(G\)).

Consider case (i). Then to show that \(\Phi_{G,\rho,\psi}\) is irreducible it is enough to show that \(\Phi_{G,\rho,\psi}|_{G_r}\) is irreducible (this follows from 2). On the other hand since
\[
\Phi_{G,\rho,\psi}|_{G_r} = s^*(q_! \Phi_{T,\rho,\psi})^W [\dim G - \dim T](\frac{\dim G - \dim T}{2}).
\]
and since \(s\) has connected fibers it is enough to show that \((q_! \Phi_{T,\rho,\psi})^W\) is irreducible which follows immediately from the irreducibility of \(\Phi_{T,\rho,\psi}\).

Let us now consider case (ii). In this case we claim that \(\Phi_{G,\rho,\psi} = \Phi_{T,\rho,\psi}\) (this means that both sheaves are supported on \(Z(G)\) and are equal there). Indeed, the sheaf \(\text{Ind}_T^G \Phi_{T,\rho,\psi}\) is supported on \(Z(G) \cdot N\) and \(\text{Ind}_T^G \Phi_{T,\rho,\psi} = \Phi_{T,\rho,\psi} \boxtimes \text{Spr}\). Moreover, it follows from (6.2) that the action of \(W\) on \(\text{Ind}_T^G \Phi_{T,\rho,\psi}\) comes from the second multiple and there it is equal to the standard action of \(W\) on \(\text{Spr}\) twisted by the sign character. It is well-known that the sheaf \(\text{Hom}_W(\text{sign}, \text{Spr})\) is equal to \(\delta_e\) where \(\delta_e\) denotes that \(\delta\)-function sheaf at the unit element of \(G\). Hence we have
\[
\Phi_{G,\rho,\psi} = (\text{Ind}_T^G \Phi_{T,\rho,\psi})^W = \Phi_{T,\rho,\psi} \boxtimes (\text{Spr} \otimes \text{sign})^W = \Phi_{T,\rho,\psi} \boxtimes \delta_e = \Phi_{T,\rho,\psi}.
\]

Let us now discuss the relation between the sheaf \(\Phi_{G,\rho,\psi}\) and the function \(\Phi_{G,\rho,\psi}\).

Conjecture 6.5. Let \(P \subset G\) be a parabolic subgroup, \(U \subset P\) – its unipotent radical, \(M = P/U\) – the corresponding Levi group. Let \(q_P : G \rightarrow G/U\) and \(i_P : M \rightarrow G/U\) be the natural morphisms. Assume that \(\rho\) is good. Then \((q_P)_! \Phi_{G,\rho,\psi}\) vanishes outside of \(M\).

Theorem 6.6. We have
\[
\text{Res}_M^G \Phi_{G,\rho,\psi} = \Phi_{M,\rho,\psi}.
\]
Assume now that Conjecture 6.5 holds. Then

1. \[(q_P)_! \Phi_{G,\rho,\psi} \simeq (i_P)_! \Phi_{M,\rho,\psi}\] (6.4)
2. Assume that $\rho_1$ and $\rho_2$ are good with respect to the same character $\sigma$ of $G$. Then

$$\Phi_{G, \rho_1 \oplus \rho_2, \psi} \simeq \Phi_{G, \rho_1, \psi} \ast \Phi_{G, \rho_2, \psi} \simeq \Phi_{G, \rho_1, \psi} \ast \Phi_{G, \rho_2, \psi}$$

(6.5)

3. Let $\mathcal{L}$ be a tame local system on $T$. Then

$$\Phi_{G, \rho, \psi} \ast \mathcal{K}_\mathcal{L} \simeq H_{\rho, \mathcal{L}, \psi} \otimes \mathcal{K}_\mathcal{L}$$

(6.6)

and

$$\Phi_{G, \rho, \psi} \ast \mathcal{K}_\mathcal{L} \simeq H_{\rho, \mathcal{L}, \psi, \ast} \otimes \mathcal{K}_\mathcal{L}.$$  

(6.7)

If $\mathcal{L}$ is endowed with an isomorphism $\text{Fr}^*_w \mathcal{L} \simeq \mathcal{L}$ then these isomorphisms commute with the Weil structures on both sides (note that due to $W$-equivariance of $\Phi_{T, \rho, \psi}$ every isomorphism $\text{Fr}^*_w \mathcal{L} \simeq \mathcal{L}$ endows the spaces $H_{\rho, \mathcal{L}, \psi} \otimes \mathcal{L}^{-1}$ and $H_{\rho, \mathcal{L}, \psi, \ast} = H^0(\Phi_{T, \rho, \psi} \otimes \mathcal{L}^{-1})$ with a Frobenius action).

**Corollary 6.7.** Assume Conjecture 6.5 holds. Then

$$\chi(\Phi_{G, \rho, \psi}) = \Phi_{G, \rho, \psi}.$$  

**Proof.** Let $w \in W$, $\theta : T_w \to \mathbb{O}_l^\times$ and let $\mathcal{L} = \mathcal{L}_\theta$. It follows from Theorem 6.6(3) and Theorem 3.7 that in order to prove Corollary 6.7 it is enough to show that the scalar by which Frobenius acts on $H_{\rho, \mathcal{L}, \psi} \otimes \mathcal{L}^{-1}$ is equal to $\gamma_{w, \rho, \psi}(\theta^{-1})$. It follows from the definitions that it is enough to do it in the case when $G = \text{GL}(n)$ and $\rho$ is the standard representation where it is obvious. □

**Conjecture 6.8.** The functors $F \mapsto F \ast \Phi_{G, \rho, \psi}$ and $F \mapsto F \ast \Phi_{G, \rho, \psi}$ from $D(G)$ to itself are exact with respect to the perverse $t$-structure.

**Theorem 6.9.** Conjecture 6.5 holds for $G$ of semi-simple rank $\leq 1$. In particular, Corollary 6.7 holds for $G$ of semi-simple rank $\leq 1$.

**Proof.** Clearly we can assume that the semi-simple rank of $G$ is equal to 1 (otherwise $G$ is a torus and in this case there is nothing to prove). Also, without loss of generality we may assume that $\text{im} \ p_\rho \cap T_r \neq \emptyset$ (otherwise the support of $\Phi_{G, \rho, \psi}$ lies in the center of $G$ and again there is nothing to prove).

In this case the Weyl group $W$ is isomorphic to $\mathbb{Z}_2$. We denote by $\sigma$ the only non-trivial element in $W$. Let also $T' = T/T \cap [G, G]$ and let $\pi' : T \to T'$ be the natural map. Thus $W \simeq \mathbb{Z}_2$ acts in the fibers of $\pi'$. Hence we get a natural map $\pi : T/W \to T'$.

Let $B$ be a Borel subgroup of $G$ with unipotent radical $U$. Let $g \in G$ such that $g \not\in B$. Then $gu$ is a regular element of $G$ for every $u \in U$. Moreover, the map $s : G_r \to T/W$ identifies $gU$ with one of the fibers of $\pi$. Hence it is enough to show that $\pi'_* \Phi_{T, \rho, \psi} = \pi_* \Phi_{T, \rho, \psi} = 0$. For this it enough to prove that $(\pi'_* \Phi_{T, \rho, \psi})^W = (\pi_* \Phi_{T, \rho, \psi})^W = 0.$
First of all, it follows from Theorem 1.2 (applied to the torus $T'$) that $\pi;\Phi_{T,\rho,\psi} = \pi_\ast\Phi_{T,\rho,\psi}$ and that the sheaf $\Phi' := \pi_\ast\Phi_{T,\rho,\psi}$ is irreducible. Hence $\sigma \in W$ acts on $\Phi'$ by means of multiplication by a scalar. Since $\sigma^2 = 1$ it follows that this scalar must be $\pm 1$.

We claim that the above scalar is equal to $-1$. Since $H^0(\Phi') = H^0(\Phi_{T,\rho,\psi}) \neq 0$, it is enough to check that $\sigma$ acts on $H^0(\Phi_{T,\rho,\psi})$ by means of multiplication by $-1$. Let $\sigma'$ be a lift of $\sigma$ to $S_n$ and let $\iota' : \sigma'\Phi_{T,\rho,\psi} \cong \Phi_{T,\rho,\psi}$ be the corresponding isomorphism (we are using here the notations introduced before Proposition 6.2). Then it follows from Lemma 6.3 that $\iota'$ induces multiplication by $(-1)^{l(\sigma')}$. Hence $\iota_{\sigma} = (-1)^{l(\sigma') - l(\sigma)}\iota' = (-1)^{l(\sigma') - l(\sigma)}$ acts on $H^0(\Phi_{T,\rho,\psi})$ by means of multiplication by $-1$. \hfill $\square$

6.10. **Proof of Theorem 6.6.**

6.11. Let us show that $\text{Res}_M^W \Phi_{G,\rho,\psi} = \Phi_{M,\rho,\psi}$. By Theorem 2.7 we have

$$\text{Res}_M^W \text{Ind}_T^G(\Phi_{T,\rho,\psi}) = \text{Ind}_{W_M}^W \text{Ind}_T^M(\Phi_{T,\rho,\psi}).$$

By Frobenius reciprocity

$$\left(\text{Ind}_{W_M}^W \text{Ind}_T^M(\Phi_{T,\rho,\psi})\right)^W = \left(\text{Ind}_T^M(\Phi_{T,\rho,\psi})\right)^{W_M} = \Phi_{M,\rho,\psi}.$$ 

Hence

$$\text{Res}_M^G \Phi_{G,\rho,\psi} = \left(\text{Res}_M^G \text{Ind}_T^G(\Phi_{T,\rho,\psi})\right)^W = \Phi_{M,\rho,\psi}.$$ 

This clearly implies Theorem 6.6(1) if we assume that Conjecture 6.5 holds.

The fact that Conjecture 6.5 implies Theorem 6.6(3) is an immediate consequence of Proposition 2.9. Hence we just need to prove Theorem 6.6(2). We will do that for $\ast$-convolution. The proof for $\ast$-convolution is analogous.

Consider first $\Phi_{G,\rho_1,\psi} \ast \text{Ind}_T^G(\Phi_{T,\rho_2,\psi})$. Then Conjecture 6.5 and Proposition 2.9 imply that it is isomorphic to

$$\text{Ind}_T^G(\text{Res}_T^G \Phi_{G,\rho_1,\psi}) \ast \Phi_{T,\rho_2,\psi} \cong \text{Ind}_T^G(\Phi_{T,\rho_1,\psi} \ast \Phi_{T,\rho_2,\psi}) \cong \text{Ind}_T^G(\Phi_{T,\rho_1 \oplus \rho_2,\psi}).$$

Our assertion is obtained by taking $W$-invariants on both sides.

We conclude with the following

**Theorem 6.12.** Let $\Phi_{G,\rho,\psi} = \chi(\Phi_{G,\rho,\psi})$. Then

1. For every $w \in W$ and $\theta : T_w \to \overline{Q}_l^\times$ the trace of $\Phi_{G,\rho,\psi}$ in $R_{\theta,w}$ is equal to that of $\Phi_{G,\rho,\psi}$.
2. If $G = \text{GL}(n)$ then

$$\Phi_{G,\rho,\psi} = \Phi_{G,\rho,\psi} \quad (6.8)$$

**Proof.** Let us prove 1. First of all we know that $\text{Tr}(\Phi_{G,\rho,\psi}, R_{\theta,w}) = \gamma_{w,\rho,\psi} \cdot \dim R_{\theta,w}$. By 4(Theorem 7.1) we have $\dim R_{\theta,w} = q^{-\frac{d}{2}} \frac{\#G}{\#T_w}$. Hence we need to show that

$$\text{Tr}(\Phi_{G,\rho,\psi}, R_{\theta,w}) = \gamma_{w,\rho,\psi} \cdot q^{-\frac{d}{2}} \frac{\#G}{\#T_w}. \quad (6.9)$$
Let $\mathcal{L}$ be the local system on $T$ corresponding to $\theta, w$. Thus $\mathcal{L}$ is endowed with the natural isomorphism $\text{Fr}_w^* \mathcal{L} \simeq \mathcal{L}$. We know that the character of $R_{\theta, w}$ is equal to $\chi(\mathcal{K}_L)q^{d/2}$. Also the inverse image of $\mathcal{K}_L$ under the map $g \mapsto g^{-1}$ is equal to $\mathcal{K}_{L^{-1}} = \mathbb{D}\mathcal{K}_L$. Hence it follows that $\text{Tr}(\Phi'_{G, \rho, \psi}, R_{\theta, w})$ is equal to

$$q^{d/2} \sum (-1)^i \text{Tr}(\text{Fr}, H^i_c(\Phi G, \rho, \psi \otimes \mathcal{K}_L)).$$

Recall that for any two complexes $A$ and $B$ on a scheme $X$ we have

$$(\text{Ext}^*(A, B))^\vee = H^*_c(A \otimes \mathbb{D}B).$$

Hence it follows from (2.7) that we have a canonical isomorphism

$$H^*_c(\Phi G, \rho, \psi \otimes \mathbb{D}\text{Ind}_T^G \mathcal{L}) \simeq H^*_c(\text{Res}\Phi G, \rho, \psi \otimes \mathcal{L}^{-1}).$$

Changing $\mathcal{L}$ to $\mathcal{L}^{-1}$ and taking into account the canonical isomorphism $\mathbb{D}\text{Ind}_T^G \mathcal{L}^{-1} \simeq \text{Ind}_T^G \mathcal{L}$ we see that we have a canonical isomorphism

$$H^*_c(\Phi G, \rho, \psi \otimes \text{Ind}_T^G \mathcal{L})[-2d](-d) \otimes H^*_c(T, \overline{Q}_l) \simeq H^*_c(\text{Res}\Phi G, \rho, \psi \otimes \mathcal{L}) \otimes H^*_c(G, \overline{Q}_l).$$

In other words, since we know that $\text{Res}_T^G \Phi G = \Phi T, \rho, \psi$, we get the isomorphism

$$H^*_c(\Phi G, \rho, \psi \otimes \text{Ind}_T^G \mathcal{L})[-2d](-d) \otimes H^*_c(T, \overline{Q}_l) \simeq H^*_c(\Phi T, \rho, \psi \otimes \mathcal{L}) \otimes H^*_c(G, \overline{Q}_l).$$

(6.10)

It is easy to see that this isomorphism commutes with the action of Frobenius on both sides where the actions of Frobenius on $H^*_c(\Phi G, \rho, \psi \otimes \text{Ind}_T^G \mathcal{L}$ and $H^*_c(G, \overline{Q}_l)$ are standard and the actions on $H^*_c(T, \overline{Q}_l)$ and $H^*_c(\Phi T, \rho, \psi \otimes \mathcal{L})$ are via $\text{Fr}_w$. Thus (6.9) follows from (6.10) by taking traces of Frobenius.

Let us prove the second assertion. It is known that if $G = \text{GL}(n)$ then the virtual Deligne-Lusztig representations $R_{\theta, w}$ generate over $\mathbb{Q}$ the Grothendieck group of the category of finite-dimensional representations of $G$ (cf. for example the introduction to [12]). Hence (6.8) follows from Theorem 6.12(1).

\begin{thebibliography}{99}

1. A. Beilinson, J. Bernstein and P. Deligne, \textit{Faisceaux pervers}, in: Analysis and topology on singular spaces, I (Luminy, 1981), 5–171, Astérisque, 100 1982.
2. W. Borho and R. MacPherson, \textit{Partial resolutions of nilpotent varieties}, in: Analysis and topology on singular spaces, II, III (Luminy, 1981), Astérisque, 101-102 (1983), 23–74.
3. A. Braverman and D. Kazhdan, $\gamma$-functions of representations and lifting, to appear in GAFA.
4. P. Deligne and G. Lusztig, \textit{Representations of reductive groups over a finite field}, Jour. of Algebra.
5. R. Godement and H. Jacquet, \textit{Zeta-functions of simple algebras}, Lecture Notes in Mathematics, Vol. 260. Springer-Verlag, Berlin-New York, 1972.
6. V. Ginzburg, \textit{Induction and restriction of character sheaves}, Advances in Soviet mathematics 16 (I. M. Gelfand seminar), I, 1993, 149–168.
7. O. Gabber and F. Loeser, \textit{Faisceaux pervers l-adiques sur un tore}, Duke Math. J. 83 (1996), no. 3, 501–606.

\end{thebibliography}
[8] N. Katz, *Gauss sums, Kloosterman sums, and monodromy groups*, Annals of Mathematics Studies, 116 Princeton University Press, Princeton, 1988.
[9] N. Katz and G. Laumon, *Transformation de Fourier et majoration de sommes exponentielles*, Inst. Hautes Études Sci. Publ. Math. 62 (1985), 361–418
[10] G. Laumon, *Faisceaux caractères (d’après Lusztig)* (French), Séminaire Bourbaki, Vol. 1988/89. Astérisque 177-178, (1989), 231–260.
[11] G. Laumon, *Transformation de Fourier, constantes d’équations fonctionnelles et conjecture de Weil*. Inst. Hautes Études Sci. Publ. Math. 65 (1987), 131–210.
[12] G. Lusztig, *Characters of reductive groups over a finite field*, Annals of Mathematics Studies, 107. Princeton University Press, Princeton, NJ, 1984.
[13] G. Lusztig, *Character sheaves I*, Adv. in Math. 56 (1985), no. 3, 193–237.
[14] I. Mirkovic and K. Vilonen *Characteristic varieties of character sheaves*, Invent. Math. 93 (1988), no. 2, 405–418.
[15] E. B. Vinberg, *On reductive algebraic semi-groups*, in: Lie groups and Lie algebras: E. B. Dynkin’s Seminar, 145–182, Amer. Math. Soc. Transl. Ser. 2, 169, Amer. Math. Soc., Providence, RI, 1995.