Fiducial inference then and now

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December 4, 2021

Dedicated to the memory of Mervyn Stone, 1932–2020

Abstract

We conduct a review of the fiducial approach to statistical inference, following its journey from its initiation by R. A. Fisher, through various problems and criticisms, on to its general neglect, and then to its more recent resurgence. Emphasis is laid on the functional model formulation, which helps clarify the very limited conditions under which fiducial inference can be conducted in an unambiguous and self-consistent way.

Key words: conditioning inconsistency; functional model; marginalization consistency; partitionability; pivot; structural model

1 Introduction

According to Zabell (1992) “the fiducial argument stands as Fisher’s one great failure”, a sentiment that has been echoed by others. Fisher never constructed a fully-fledged theory of fiducial inference, but developed his ideas by means of examples and ad hoc responses to increasingly complex problems or challenges raised by others. Few other statisticians have taken the fiducial argument seriously, and after some sporadic activity (mostly critical) in the decades following Fisher’s introduction of the idea in 1930, it almost completely disappeared from the scene. The trio of Encyclopedia articles Edwards (1983); Buehler (1983); Stone (1983) is a useful resource for the state of the enterprise up to 1982. In recent times, however, there has been a resurgence of interest in the fiducial programme, as evidenced by works such as Hannig (2009) and Martin and Liu (2016), and the success of the series of annual Bayesian, Fiducial & Frequentist (BFF) conferences, since 2014.

In this article I give a personal review of the main contributions, positive and negative, to fiducial inference, in both earlier and later periods. In §2 I describe the original argument, centered on inference for a correlation coefficient. Section 3 introduces an extension to more complex problems, based on the idea of a pivotal function. Such a function arises naturally when the problem possesses properties of invariance under a group of transformation, as described in §4. A variant of this is Fraser’s structural model described in §4.2, while a further extension is the functional model of §5 which forms a basis for the rest of the article. In §6 we show that, under certain conditions, two different routes to marginalizing a fiducial distribution give the same answer.

In §7 we start to see some problems with the fiducial argument. In particular, when attempting to condition in a fiducial distribution, we again have two possible routes, but they generally yield different answers.

To this point we have only considered simple models, essentially those where the dimensions of the parameter and the data are the same. Non-simple models require additional conditioning, as described in §8. However this requires an additional property, partitionability, in the absence of which there is no well-defined fiducial distribution.

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Section 11 considers cases in which the fiducial argument fails to yield a distribution for the parameter, but only a distribution for a set containing the parameter. This is linked to the Dempster-Shafer theory of belief functions, and has been a focus of recent work. In this case too a partitionability property is required for well-defined inference.

Some concluding thoughts are gathered in §11

2 Fisher’s original fiducial argument

The fiducial argument was introduced by Fisher (1930) by means of the following example.

We have $n$ observations from a bivariate normal distribution. Let random variable $R$ be the sample correlation, and let parameter variable $\Phi$ be the population correlation. Then the sampling distribution of $R$ depends only on ($n$ and) the value $\phi$ of $\Phi$. The form of this distribution (Fisher 1915) is not expressible by means of simple functions; however the rest of Fisher’s argument does not involve the specific form of this distribution. Indeed, letting $F(R; \Phi)$ be the cumulative distribution function of $R$ when $\Phi = \phi$, the general argument applies to any problem satisfying the following sufficient (but not entirely necessary) regularity conditions:

(i). The range of each of $R$ and $\Phi$ is an open interval in the real line

(ii). $F(r; \phi)$ is a continuous function of each of its arguments

(iii). For fixed $\phi$, $F(r; \phi)$ strictly increases as $r$ increases, taking all values in $(0, 1)$

(iv). For fixed $r$, $F(r; \phi)$ strictly decreases as $\phi$ increases, taking all values in $(0, 1)$

It follows from the probability integral transformation (Angus 1994) that, for all $\phi$, the distribution of $F(R; \phi)$, given $\Phi = \phi$, is uniform on $[0, 1]$. That is, $E = F(R; \Phi) \sim U[0, 1]$, independently of $\Phi$: $E \perp \Phi$.

Since, for any $\gamma \in [0, 1]$, when $\Phi = \phi$

$$Pr\{F(R; \phi) \leq \gamma\} = \gamma,$$

a level-$\gamma$ confidence set for $\Phi$ is, for observed $R = r$, $I(r; \gamma) := \{\phi : F(r; \phi) \leq \gamma\}$. In fact, because $F(r; \phi)$ is a decreasing function of $\phi$, this is an upper confidence interval: $I(r; \gamma) = \{\phi(\gamma), \infty\}$, where $F(r; \phi(\gamma)) = \gamma$.

Fisher now takes this argument further. He regards the uniform distribution for $E = F(R; \Phi)$, and so 11, as remaining valid, even after observing data with $R = r$. Equivalently, he takes $E \perp R$ (compare the sampling property $E \perp \Phi$). Thus he assumes, for any $r$,

$$F(r, \Phi) \sim U[0, 1].$$

This argument has now assigned to the parameter $\Phi$ the status of a random variable. Indeed, after observing $R = r$, 12, in conjunction with 111 and 114 implies

$$Pr(\Phi \leq \phi) = Pr\{F(r; \Phi) \geq F(r; \phi)\} = 1 - F(r; \phi).$$

On account of 114 this yields a full “fiducial distribution function” for $\Phi$: under differentiability, the associated “fiducial density” of $\Phi$ is $-\frac{\partial}{\partial \phi} F(r; \phi)$. In particular, $Pr\{\Phi \in I(r; \gamma)\} = Pr\{\Phi \geq \phi(\gamma)\} = F(r, \phi(\gamma)) = \gamma$, so transforming a confidence statement for $\Phi$ into a probability statement for $\Phi$—an interpretation of a confidence interval that is typically castigated as showing a gross misunderstanding of its nature.

Fisher, noting that 113 does not follow from standard probability arguments, termed it a “fiducial probability”. However, he appeared to believe that (subject to some caveats—see §3.1 below) it can still be interpreted as a regular probability. Savage (1961) memorably described the fiducial argument as “a bold attempt to make the Bayesian omelet without breaking the Bayesian eggs”. Lindley (1958) considered the general case 113114 of Fisher’s construction, and showed that the fiducial distribution does not arise as a Bayesian posterior, for any prior, except in the special case of a location model with uniform prior on the location parameter.
3 Pivotal inference

A more general fiducial construction relies on the existence of a pivot \((\text{Barnard 1980})\).

Let \(X\) have distribution governed by parameter \(\Theta\). A pivot \(E\) is a function of \(X\) and \(\Theta\) with known distribution \(P_{0}\), not depending on the value \(\theta\) of \(\Theta\): \(E \perp \Theta\), with \(E \sim P_{0}\).

**Example 3.1** In \(\S 2\) \(E = F(R; \Phi)\) is a pivot, with distribution \(U[0, 1]\). We have seen how this can be used to supply a fiducial distribution.

**Example 3.2** For a sample of size \(n\) from the normal distribution with mean \(M\) and variance \(\Sigma^{2}\), having sample mean \(\bar{X}\) and sample variance \(S^{2}\), \(E_{1} = (\bar{X} - M)/\Sigma\) is a pivot, with the normal distribution \(N(0, 1/n); E_{2} = S/\Sigma\) is a pivot, with distribution \(\chi_{n-1}^{2}/(n - 1)\); and \(E_{3} = \sqrt{n}E_{1}/E_{2} = (\bar{X} - M)/(S/\sqrt{n})\) is a pivot, with the Student distribution \(t_{n-1}\).

Given a suitable pivot \(E = f(X, \Theta)\), a fiducial distribution is obtained by regarding the distribution \(P_{0}\) of \(E\) as still relevant, even after observing the data. Equivalently, instead of \(E \perp \Theta\), we regard \(E \perp X\). Thus after observing \(X = x\), we suppose \(f(x, \Theta) \sim P_{0}\). When \(\Theta\) (or a desired function \(\Psi\) of \(\Theta\)) can be expressed as a function of \(f(x, \Theta)\)—the case of invertibility—this delivers a fiducial distribution for \(\Theta\) (or \(\Psi\)).

In Example 3.2, using \(E_{3}\) we have \((\bar{X} - M)/(s/\sqrt{n}) \sim t_{n-1}\), which can be solved as \(M = \bar{X} - (s/\sqrt{n})E_{3}\), yielding fiducial distribution \(M \sim \bar{X} + (s/\sqrt{n})E_{n-1}\). Similarly using \(E_{2}\) we obtain a fiducial distribution for \(\Sigma\): \(\Sigma = s/E_{2} \sim s/\sqrt{\chi_{n-1}^{2}/(n - 1)}\). The pivot \(E_{1}\), by itself, is not invertible, and does not yield a fiducial distribution. However, the bivariate pivot \((E_{1}, E_{2})\) is invertible, and yields a joint fiducial distribution for \((M, \Sigma)\), represented by

\[
M = \bar{X} - sE_{1}/E_{2} \quad \text{(4)}
\]
\[
\Sigma = s/E_{2} \quad \text{(5)}
\]

where still \(E_{1} \sim N(0, 1/n), E_{2} \sim \sqrt{\chi_{n-1}^{2}/(n - 1)}\) (these moreover retaining their sampling distribution independence). In particular, the induced marginals for \(M\) and \(\Sigma\) agree with those above based directly on \(E_{3}\) and \(E_{2}\), as above. However, \(M\) and \(\Sigma\) are not independent. Learning \(\Sigma = \sigma\) is equivalent to learning \(E_{2} = s/\sigma\). This does not change the \(N(0, 1/n)\) distribution of \(E_{1}\), and we now have \(M = \bar{X} - \sigma E_{1}\). So we have conditional fiducial distribution

\[
M \mid (\Sigma = \sigma) \sim N(\bar{X}, \sigma^{2}/n). \quad \text{(6)}
\]

### 3.1 Validity

There will typically be many available pivotal functions. For instance, in Example 3.2 we could retain just the first \(n/2\) (say) observations, and use the sample mean and variance computed from these. In his early writings, Fisher insisted that, to make use of all the available information, a fiducial distribution should be based on the minimal sufficient statistic.

In addition, Fisher indicated that a fiducial distribution should be regarded as yielding an appropriate inference only if the following vaguely stated conditions are satisfied:

(i). there is no available prior information about the unknown parameter;

(ii). ("principle of irrelevance"): the data are uninformative about the pivot.

While (i) obviously precludes having a Bayesian prior distribution, its intended scope is much wider (and much vaguer). As for (ii) Hacking (1965) (see also Harris and Harding (1984)) attempted to rigorize it as requiring that the likelihood function based on any data, re-expressed as function of the data and the pivot, be the same (up to proportionality) for any data. He then showed that, in the univariate case, this holds if and only if (possibly after transformation) we have a location model, in which case, as observed by Lindley (1958) the fiducial distribution agrees with the formal posterior based on an improper uniform prior.
4 Group-structured models

Let observable \( X \) take values in a space \( \mathcal{X} \), and identifiable parameter \( \Theta \) take values in \( \mathcal{T} \). We suppose \( \mathcal{T} \) can be identified with a group \( G \) of transformations acting on \( \mathcal{X} \). We denote the image of \( x \in \mathcal{X} \) under \( g \in G \) by \( g \circ x \), and further suppose the group action is exact, so that, given \( x_0, x_1 \in \mathcal{X} \) there is at most one \( g \in G \) such that \( g \circ x_1 = x_2 \) (this condition can be relaxed: see Bondar (1972)).

We shall investigate cases in which the family \( \mathcal{P} = \{ P_\theta : \theta \in \mathcal{T} \} \) of distributions over \( \mathcal{X} \) is equivariant under the action of \( G \): that is, if \( X \sim P_\theta \), then \( g \circ X \sim P_{g\theta} \) (where \( g\theta \) is the group product).

4.1 Simple group model

In the simplest case, \( G \) acts transitively (as well as exactly) on \( \mathcal{X} \), so that, for any \( x_1, x_2 \in \mathcal{X} \) there exists exactly one \( g \in G \) such that \( x_2 = g \circ x_1 \). Fix some \( x_0 \in \mathcal{X} \), and henceforth identify any \( x \in \mathcal{X} \) with the unique \( g \in G \) such that \( x = g \circ x_0 \). We can thus take \( \mathcal{X} = G \), with \( g \circ x \) becoming the group product \( gx \). Let \( P_0 = P_{\iota} \), with \( \iota \) the identity element of \( G \). Define \( E := \Theta^{-1}X \). Then, by equivariance, conditional on \( \Theta = \theta \), \( E \sim P_0 \). Hence \( E \) is a pivot, and could be used to construct a fiducial distribution: after observing \( X = x \), take \( \Theta^{-1}X \sim P_0 \). It can be shown that this construction satisfies Hacking’s version of the principle of irrelevance. Moreover [Fraser 1961], assuming \( G \) is locally compact, the resulting fiducial distribution is identical to a Bayesian posterior distribution, based on the (typically improper) right-invariant distribution (right Haar measure) for \( \Theta \) over \( G \).

Example 4.1 In Example 8.2 we can consider both \( (\bar{X}, S) \) and \( (M, \Sigma) \) as elements of the location-scale group, with multiplication \((a,b)(A,B) = (a+bA,bB)\), and identity \( \iota = (0,1) \). Under \( P_0 \), \( \bar{X} \sim N(0,n^{-1}) \) and \( S \sim \chi^2_{n-1}/(n-1) \), independently; then \( (\mu,\sigma)(\bar{X}, S) = (\mu + \sigma \bar{X}, \sigma S) \sim (N(\mu, \sigma^2/n), \sigma \sqrt{\chi^2_{n-1}/(n-1)}) \) (independently), which is \( P_{(\mu,\sigma)} \). Thus we have equivariance.

We have pivot
\[
E = (E_1, E_2) = (M, \Sigma)^{-1}(\bar{X}, S) = \begin{pmatrix} \bar{X} - M \cr \Sigma S \end{pmatrix}.
\]

We recover the same joint fiducial distribution represented by (4) and (5). Moreover, this is the same as the posterior distribution based on the right-invariant prior, having density element \( d\mu d\sigma/\sigma \).

Example 4.2 Let \( G \) be the group of lower triangular matrices with positive diagonal. An observable random \( 2 \times 2 \) matrix \( S \) has the Wishart distribution \( W(\nu; \Sigma) \), where \( \nu \geq 2 \) and \( \Sigma \) is positive definite. Then \( S \) is almost surely non-singular. We can alternatively represent \( S \) by the unique \( L \in G \) such that \( S = LL^T \), and similarly \( \Sigma \) by \( \Lambda \in G \) with \( \Sigma = \Lambda \Lambda^T \). We write the implied distribution of \( L \), depending on \( \Lambda \), as \( L \sim \mathcal{L}(\nu; \Lambda) \). It is then easy to see that, for fixed \( A \in G \), \( AL \sim \mathcal{L}(\nu; \Lambda A) \), so that the problem is equivariant under \( G \). It follows that a pivot is \( E = \Lambda^{-1}L \), with distribution \( \mathcal{L}(\nu; I) \)—under which Mauldon (1955) the non-zero entries of \( E \) are independent, with \( E_{11} \sim \sqrt{\chi^2_{\nu}} \), \( E_{22} \sim \sqrt{\chi^2_{\nu-1}} \), and \( E_{21} \sim N(0,1) \). So the fiducial distribution of \( \Sigma = \Lambda \Lambda^T \), given data \( S = s = ll^T \), is that of \( (E^T E)^{-1}I \) when \( E \sim \mathcal{L}(\nu; I) \). In this case the right-invariant prior density element, under the action of \( G \), can be expressed in terms of the entries of \( \Sigma \) as
\[
(d\sigma_{11}/\sigma_{11}) d\sigma_{12} d\sigma_{22}, \tag{7}
\]
and the fiducial distribution of \( \Sigma \) agrees with its posterior, based on (7) as prior.
4.2 Structural models

Fraser (1961) Fraser (1968) has a somewhat different take on group-structured models, which takes the group structure as part of the specification of the problem. He posits, as part of the very set-up, a nominated group $G$ acting on $X$, and an “error variable” $E$, with known distribution $P_0$ over $X$. Both $E$ and the parameter $\Theta$, which takes values in $G$, are regarded as having independent existence. The observable $X$ is then defined by $X = \Theta \circ E$. There is thus additional algebraic structure, over and above the implied parametric family of distributions for $X$ given $\Theta = \theta$. This extended structure is termed a “structural model”.

Since the implied distributional model is equivariant under $G$, we can now construct a fiducial distribution as in §4.1 (see also §8.2 below for the non-transitive case). Fraser terms this a “structural distribution”.

Note that, as demonstrated in §7.1 below, distinct structural models can correspond to the same distributional model, and yield different structural distributions for its parameter. Since a structural model is considered to comprise more than just its induced distributional model—including, in particular, specification of the group $G$ as a key ingredient—this is not regarded as an inconsistency.

5 Functional models

Dawid and Stone (1982) propose the functional model, a generalization of the structural model. We have arbitrary sample space $X$ and parameter space $\Theta$. We again consider an “error variable” $E$, taking values in a space $E$ that now may be different from $X$. The observable $X$ is defined, algebraically, as $X = f(\Theta, E)$, where $f : T \times E \to X$ is a specified function, and $E$ has a known distribution $P_0$ over $E$, independently of the value of $\Theta$. For simplicity we denote the function simply by $X = \Theta \circ E$. Then the distribution $P_0$ of $X$ given $\Theta = \theta$ is that of $\theta \circ E$ where $E \sim P_0$.

The structural approach, which is a special case of the functional approach, identifies $\theta \in T = G$ with the function (an element of $G$) $e \mapsto \theta \circ e$ on $X$. In the functional approach, by contrast, it is more helpful to consider $e \in E$ as the function $\theta \mapsto \theta \circ e$, mapping $T$ into $X$. (Note that, as a function, $e$ is written to the right of its argument $\theta$.)

5.1 Simple functional model (SFM)

In the simplest case, for any $x \in X$, $e \in E$, there exists exactly one $\theta$ such that $x = \theta \circ e$: we write $\theta = x \circ e^{-1}$, since this determines the inverse function $e^{-1} : X \to T$. In this case the fiducial distribution, for data $x$, is obtained from $\Theta = x \circ E^{-1}$, with $E \sim P_0$.

In the special case that $E$ can be expressed as a function of $(X, \Theta)$, it serves as a pivot. The model is then termed pivotal, and the fiducial distribution agrees with that constructed as in §6.

5.1.1 Monotonic functional model

When $X = T = \mathbb{R}$ and each $e$ acts as a strictly monotonic function of $\theta$, the fiducial distribution is fully determined by the distributional model for $X$ given $\Theta$: in particular, the finer details of the functional model do not enter. Thus when $e$ is a decreasing function, the fiducial probability $Pr(\Theta \leq \theta) = P_0(x \circ E^{-1} \leq \theta) = P_0(x \geq \theta \circ E) = Pr_0(X \leq x)$. One can show that, under regularity conditions parallel to those in §2 and by a similar argument, a 1-sided fiducial interval is also a confidence interval.

Example 5.1 Let $X = T = \mathbb{R}$, $E = (\mathbb{R}^+)^3$. The function $x = \theta \circ e$ is given by $x = (\theta e_1 + e_3)/e_2$, which is strictly increasing in $\theta$. This model is not pivotal, but we can solve for $\theta$: $\theta = x \circ e^{-1} = (xe_2 - e_3)/e_1$. So the fiducial distribution of $\Theta$, for data $X = x$, is that of $(xe_2 - E_3)/E_1$, with $E$ having its initially assigned distribution $P_0$. 

5
As a special case, suppose that, under \( P_0 \), \( E_1 \sim \sqrt{\chi^2_{n-1}} \), \( E_2 \sim \sqrt{\chi^2_{n-2}} \), and \( E_3 \sim N(0, 1) \), all independently. Define \( R = X/\sqrt{1 + X^2} \), \( \Phi := \Theta/\sqrt{1 + \Theta^2} \). It then turns out (see Example 6.2 below) that (compare §2):

(i). the sampling distribution of \( R \) is that of a sample correlation coefficient, based on \( n \) independent observations from a bivariate normal distribution with population correlation coefficient \( \Phi \)

(ii). the fiducial distribution of \( \Phi \) agrees with Fisher’s fiducial distribution, based on \( R \). This follows from [i] as a consequence of the monotonic structure of this model.

\[ \Box \]

6. Marginalization Consistency

In a SFM \( X = \Theta \circ E \), with \( E \sim P_0 \), let \( W = w(X) \) be a function of \( X \). Suppose that \( w = w(\theta \circ e) \) can be expressed as a function of \( \omega \) and \( e \), where \( \omega = \omega(\theta) \) is some function of \( \theta \): we write this function as \( w = \omega \circ e \). With \( \Omega := \omega(\Theta) \) we thus have a new model \( W = \Omega \circ E \): in particular, the sampling distribution of \( W \) depends only on the value of \( \Omega \). We require that this model itself be a SFM, so that, given \((w, e)\), we can solve \( w = \omega \circ e \) for \( \omega \), which solution we write as \( \omega = \omega(\theta \circ e^{-1}) \) (though the function \( e^{-1} \), now acting on \( \omega \), has a different meaning here than in \( \theta = x \circ e^{-1} \)). We term the model \( W = \Omega \circ E \) a reduction of \( X = \Theta \circ E \).

Given data \( X = x \), we have two different routes to computing the fiducial distribution of \( \Omega = \omega(\Theta) \):

(i). Obtain the fiducial distribution of \( \Theta \) based on data \( X = x \), using the full SFM \( X = \Theta \circ E \); then marginalize this to get the implied distribution of \( \Omega = \omega(\Theta) \).

(ii). Start from the reduced SFM \( W = \Omega \circ E \), and obtain the associated fiducial distribution of \( \Omega \), based on the reduced data \( W = w(x) \).

To see that these give the same result we argue as follows. Route [i] represents \( \Theta = x \circ E^{-1} \), and so produces the distribution of \( \omega(x \circ E^{-1}) \). Route [ii] represents \( \Omega = w(x) \circ E^{-1} \). In both cases \( E \sim P_0 \). Now if \( x = \theta \circ e \) then \( \theta = x \circ e^{-1} \). Also \( w(x) = \omega(\theta \circ e) \), so \( \omega(\theta) = w(x) \circ e^{-1} \). Hence \( \omega(x \circ e^{-1}) = w(x) \circ e^{-1} \) whence \( \omega(x \circ E^{-1}) = w(x) \circ E^{-1} \), showing that both routes yield the same representation, and hence the same fiducial distribution, for \( \Omega \) (in particular, the marginal fiducial distribution of \( \Omega \) in the route [ii] analysis must depend on the data \( x \) only through \( w = w(x) \)).

**Example 6.1** Example [4.1] can be regarded as a SFM (in fact a structural model): \((\bar{X}, S) = (M, \Sigma) \circ (E_1, E_2)\), where \( \circ \) is group product in the location-scale group. That is,

\[
\bar{X} = M + \Sigma E_1 \tag{8}
\]

\[
S = \Sigma E_2 \tag{9}
\]

with \( E_1 \sim N(0, 1/n) \), \( E_2 \sim \sqrt{\chi^2_{n-1}/(n-1)} \), independently.

Define \( W = \bar{X}/S, \Omega = M/\Sigma \). Then

\[
W = \frac{\Omega + E_1}{E_2}. \tag{10}
\]

This is itself a SFM \( W = \Omega \circ F \) (though not structural), so is a reduction of the original SFM.

\[ ^{1} \text{Appendix A2 of Dawid and Stone (1982)} \text{ characterises such a reduction in terms of group actions. In particular, if the initial SFM is structural, with } T \text{ a group } G \text{ of transformations of } X = \mathfrak{E}, \text{ a reduction is obtained by taking } W \text{ and } \Omega \text{ as maximal invariants under a subgroup } K \text{ of } G, \text{ acting on } X \text{ and } T \text{ respectively.} \]
Dempster (1963) noted that (as indeed follows from the above) the sampling distribution of \(W\) depends only on \(\Omega\), so that a fiducial distribution for \(\Omega\) can be constructed from these univariate sampling distributions using Fisher’s approach of inverting the distribution function. By monotonicity, the route (ii) analysis of the reduced model \(W = \Omega * F\) will also deliver this fiducial distribution. Dempster (1963) further showed that this agrees with the distribution of \(\Omega = M/\Sigma\) obtained by marginalising the joint fiducial distribution of \((M, \Sigma)\) represented by (4) and (5)—as also arises from the route (i) analysis of the initial SFM (8)–(9). Here we see marginal consistency in action.

Example 6.2 Example 4.2 can be regarded as the (structural) SFM \(L = \Lambda E\) (all lower triangular matrices), with \(E_{11} \sim \sqrt{\chi^2_\nu}, E_{22} \sim \sqrt{\chi^2_{\nu-1}}, \) and \(E_{21} \sim N(0,1)\), independently. Thus

\[
L_{11} = \Lambda_{11} E_{11} \\
L_{12} = \Lambda_{12} E_{11} + \Lambda_{22} E_{22} \\
L_{22} = \Lambda_{22} E_{22}.
\]

Defining \(X = L_{12}/L_{11}, \Theta = \Lambda_{12}/\Lambda_{11}\), we obtain a reduction \(X = \Theta * E\), given by the SFM \(X = (\Theta E_{11} + E_{12})/E_{22}\). Note that (with minor notational changes) this is identical with the special case considered in Example 5.1.

From Example 4.2 we have \(S = LL^T\), i.e.

\[
\begin{pmatrix}
S_{11} & S_{12} \\
S_{12} & S_{22}
\end{pmatrix}
= \begin{pmatrix}
L_{11}^2 & L_{11} L_{12} \\
L_{11} L_{12} & L_{12}^2 + L_{22}^2
\end{pmatrix},
\]

with a similar expression for \(\Sigma\) in terms of \(\Lambda\). The sample correlation based on \(S\) is

\[
R := \frac{S_{12}}{\sqrt{S_{11} S_{22}}} = \frac{X}{\sqrt{1 + X^2}},
\]

and similarly the population correlation is \(\Phi := \Theta/\sqrt{1 + \Theta^2}\). The former identity explains the distribution of \(R\) asserted in (i) of Example 4.2. There we deduced that the fiducial distribution of \(\Phi\), based on \(R\), agrees with that derived by Fisher. By marginalization consistency, this must also be true for the marginal distribution of \(\Phi = \Sigma_{12}/\sqrt{\Sigma_{11} \Sigma_{22}}\) formed from the full fiducial distribution of \(\Sigma\) given \(S\), based on the lower-triangular structural model.

6.1 Marginalization paradox

Although the marginalization consistency property seems to speak in favour of fiducial inference, at least in some problems, it becomes a problem for Bayesian inference with improper priors.

We know that the full fiducial distribution, used in the route (i) analysis, is also the Bayesian posterior, based on the right-invariant prior distribution. Thus the output of the route (i) analysis is the marginal distribution of \(\Omega\) in this Bayesian posterior. By marginalization consistency, the output of route (ii) analysis—which depends on the data \(X\) only through \(W\)—must then likewise agree with this marginal distribution. It therefore seems reasonable to believe that the marginal Bayesian distribution of \(\Omega\), depending as it does only on \(W\), could arise as a Bayesian posterior based on the likelihood from the reduced model for \(W\) (depending only on \(\Omega\)). But by the result of Lindley (1958) (see §2), if—as in both the above examples—the reduced model has univariate \(W\) and \(\Omega\) but is not equivalent to a location model, this can not be the case. We then have an example of a marginalization paradox (Dawid et al. 1973) in improper Bayesian inference.

7 Some difficulties

7.1 Choice of group

Mauldon (1955) pointed out a problem with Example 4.2 if we simply interchange the order in which we consider the variables (equivalent to now using equivariance under the upper triangular,
rather than lower triangular, group), the analysis proceeds essentially as before, but we obtain a different fiducial distribution. This can most easily be seen by noting that the right-invariant prior \( p(\Sigma) \) is altered on interchanging the suffices 1 and 2, leading to a different posterior, hence fiducial, distribution for \( \Sigma \).

One possible escape from this bind is not to allow the use of just any group \( G \) under which the statistical model happens to be equivariant, but to specify an appropriate group as part of the very structure of the problem—this thus requiring an additional ingredient in the model, over and above its purely distributional properties. This tallies with the position adopted in Fraser’s structural modelling—see § 4.2.

Although use of the upper triangular group produces a different fiducial distribution for \( \Sigma \) than that based on the lower triangular group, nevertheless, by an argument parallel to that of Example 6.2, the implied distribution for \( \Phi \) again agrees with Fisher’s, and thus is the same in both cases.

\[ \Box \]

7.2 Marginalization inconsistency

Example 7.1 Consider the \( n \)-variate SFM \( X = \Theta \circ E \) given by \( X_i = \Theta_i + E_i \) (\( i = 1, \ldots, n \)), with \( E_i \sim N(0, 1) \), all independently. On observing \( X = x \), the fiducial distribution has \( \Theta_i \sim N(x_i, 1) \), independently.

Let \( W = \sum_{i=1}^n X_i^2 \), \( \Omega = \sum_{i=1}^n \Theta_i^2 \). The marginal fiducial distribution of \( \Omega \), given data \( x \), depends only on \( w = \sum_{i=1}^n x_i^2 \); it is non-central \( \chi^2 \) with non-centrality parameter \( w \): \( \Omega \sim \chi^2_n(w) \).

Also, the sampling distribution of \( W \), when \( \Theta = \theta \), depends only on \( \omega = \sum_{i=1}^n \theta_i^2 \); it is \( \chi^2_n(\omega) \).

Nevertheless, \( W = \sum_{i=1}^n (\Theta_i + E_i)^2 \) can not be expressed as a function of \( \Omega \) and \( E \), so we do not have a reduction of the initial model. We note that \( W - \Theta \) has sampling expectation \( n \), but fiducial expectation \( -n \), which suggests a serious inadequacy in the marginalized fiducial distribution.

We can attempt to derive a “route (ii)”-type fiducial distribution of \( \Omega \), by Fisherian inversion of the distribution function of \( W \) given \( \Omega \). In this case condition (iii) of § 2 does not hold, and we obtain only an incomplete distribution—which obviously can not agree with the complete marginal fiducial distribution obtained from route (i) analysis, so we do not have marginalization consistency. In fact, with \( n = 50 \), we get 95% central fiducial interval \((109, 196)\) by marginalizing the full fiducial distribution to \( \Omega \), compared to \((21, 89)\) based on the distribution of \( W \) given \( \Omega \).

This example indicates that sensible marginalization of a joint fiducial distribution may not be possible when not based on a reduction of a functional model. Wilkinson (1977) embraces inconsistencies such as in this example by his noncoherence principle, which allows the overall joint fiducial to coexist with the “marginal” based on the reduced data—which is not the actual marginal. But then fiducial distributions do not satisfy the axioms of probability theory.

Other examples of marginalization inconsistency, evidenced by incompatibilities between fiducial and confidence statements, are the Behren-Fisher problem, looking at the difference between the means of two normal distributions with different, unknown, variances, and the Fieller-Creasy problem, looking at the ratio of two normal means, with known variances (Wallace 1980).

7.3 Conditional consistency?

Example 7.2 Consider again Example 3.2, and suppose we want to construct a conditional fiducial distribution for \( M \), given \( \Sigma = \sigma \). Again we have two possible routes to do this:

(i). Condition the joint fiducial distribution on \( \Sigma = \sigma \), leading to (6).

(ii). Note than, when \( \Sigma = \sigma \) is fixed, the sampling model now has

\[
\begin{align*}
X &= M + \sigma E_1 \\
S &= \sigma E_2
\end{align*}
\]
On observing \((\mathbf{r}, s)\) we learn \(E_2 = s/\sigma\), so should condition on this. The same reasoning that led to (9) again applies, so yielding the same answer. We have “conditional consistency”.

\[\Box\]

**Example 7.3** Introduce \(W\) and \(\Omega\) as in Example 6.1 related by the reduced SFM (10). We have seen that the marginal fiducial distribution for \(\Omega\) is the same under the two routes of computation. What about the conditional fiducial distribution of \(\Sigma\), given \(\Omega = \omega\)?

We can reexpress the full model as

\[\begin{align*}
W &= \frac{\Omega + E_1}{E_2} \\
S &= \frac{\Sigma E_2}{E_2}.
\end{align*}\]  

Again we can identify two routes to construct a conditional distribution for \(\Sigma\), given \(\Omega = \omega\).

(i). Form the joint fiducial distribution of \((\Omega, \Sigma)\), and condition this on \(\Omega = \omega\).

Given data \((x, s)\) (with \(x/s = w\)) the joint fiducial distribution is represented by:

\[\begin{align*}
\Omega &= wE_2 - E_1 \\
\Sigma &= s/E_2.
\end{align*}\]  

So conditioning on \(\Omega = \omega\) is equivalent to conditioning on

\[wE_2 - E_1 = \omega.\]  

We should therefore condition \(E_2\) on this, and then invert (14), so obtaining \(\Sigma = s/E_2\), where \(E_2\) has its distribution conditioned on (15).

(ii). Alternatively we can argue as follows, using (13). We have observed \(W = w\); since we are assuming \(\Omega = \omega\), we have thus learned

\[\frac{\omega + E_1}{E_2} = w.\]  

The conditioning of \(E_2\) should therefore be on (16).

Dempster (1963) showed that we get different answers, depending on whether we condition on (15) or on (16). So here we have conditioning inconsistency—and it is not clear how we should resolve it. \(\Box\)

Note that the logical information expressed by (15) and (16) is the same in both cases. How then can it matter which we condition on? The point is that, when we condition, it is not only the logical content of the condition that matters, but which partition of the space it is embedded in. This is the point of the “Borel-Kolmogorov paradox”, which concerns conditioning on an event of probability 0. But the paradox can arise even when we have positive probabilities.

A parable may help.

**Example 7.4** Suppose Mr Smith tells you: “I have two children, who are not twins.” At this point you regard each of them as equally likely to be a boy (B) or a girl (G), independently. He then says: “One of them is a boy”. Given this information, what is the probability he has two boys?

**Argument 1** Initially you assessed 4 equally likely cases: BB, BG, GB, GG. The new information rules out GG, leaving 3 cases, just one of which is BB. The conditional probability is thus 1/3.
Argument 2 You might consider that, if he had 2 boys, he would have said “They are both boys”. The fact that he did not then implies a conditional probability of 0.

Moral: When conditioning on information, we must take account of what other information might have been obtained. Otherwise put, we must specify the question (explicit or implicit) that the received information answers. Was it the question “Do you have a boy?”, or the question “How many boys do you have?”?

□

In Example 7.3 the question relevant to (15) is “What is the value of \(wE_2 - E_1\)? (answer: \(\omega\)). The question relevant to (16) is “What is the value of \((\omega + E_1)/E_2\)? (answer: \(w\)). Correspondingly the elements of the partition relevant to (15) are of the form \(wE_2 - E_1 = \omega'\), for varying \(\omega'\), while those relevant to (16) are of the form \((\omega + E_1)/E_2 = w'\), for varying \(w'\). Only when \(w' = w\) and \(\omega' = \omega\) do the answers even contain equivalent logical information. Even then, as the partitions differ, so do the conditional distributions.

In both Example 7.2 and Example 7.3 we wished to condition on a parameter-function that itself figures in a reduced functional model. However, only in Example 7.2 is this model pivotal. When this is the case, but not more generally, we will obtain the same partition, and hence the same result, by following each of the two routes, (i) and (ii).

8 Non-simple models

8.1 Ancillary information

In many cases there is no simple sufficient statistic. Fisher (1956) towards the end of the book, suggested—as usual by means of examples—an alternative approach. Suppose we can identify a statistic \(S\) that is ancillary, i.e. has the same distribution under any \(P_i\); and a further statistic \(T\) such that, together, \((S,T)\) are equivalent to the full data \(X\) (or, more generally, are jointly sufficient). Given data \((S,T) = (s,t)\), we can first restrict attention to the conditional distribution of \(T\), given \(S = s\); and then try to identify a pivotal function of \((T,\Theta)\) in this conditional distribution. Finally we invert this pivot to obtain a fiducial distribution.

Example 8.1 Let \(X = (X_1, \ldots, X_n)\) arise as a random sample from a general location model, with sampling density of the form

\[
f(x | \theta) = g(x - \theta).
\]

Typically there is no simple sufficient statistic. However, it seems natural to base inference on a location statistic, such as the sample mean \(\bar{X}\), or (for \(n\) odd) the sample median, \(\tilde{X}\). But since the sampling distributions of the pivots \(E_1 = \bar{X} - \Theta\) and \(E_2 = \tilde{X} - \Theta\) are typically very different, we seem to have a problem of choice.

This can be resolved as follows. Let \(T = \bar{X}\), and \(S = (X_i - \bar{X} : i = 1, \ldots, n)\). Then \((S,T)\) are together equivalent to \(X\), \(S\) is ancillary, and \(E = T - \Theta\) is a pivot, both unconditionally and conditionally on \(S = s\). Letting \(P^s\) denote the distribution of \(E\) given \(S = s\) (the same for all \(\theta\)), a fiducial distribution can be obtained by regarding \(t - \Theta\) as having distribution \(P^s\). It is easy to show that, if we had instead used \(T = \tilde{X}\) (or any other location statistic), the identical fiducial distribution would have been obtained.

The above seemingly well-specified procedure becomes less so when we take into account the results of Basu (1959) that there is, typically, a plethora of incommensurate choices of an ancillary to condition on.

\[\Box\]

8.2 Non-transitive group models

We now consider the general case of a group-structured model, as introduced in §81, where we do not assume transitivity: given \(x_1, x_2 \in X\), there may be no \(g\) such that \(x_2 = gx_1\). When there is such a \(g\) we write \(x_1 \approx x_2\). It is easily checked that \(\approx\) is an equivalence relation on \(X\): the equivalence classes under the action of \(G\) are termed the orbits of \(G\) in \(X\). Let \(S = s(X)\) label
orbits. It is then readily seen that $S$ is ancillary; this is the group ancillary, and, unlike general ancillaries, is essentially unique. We choose some arbitrary representative point $x_s$ in the orbit labelled by $S = s$. For any $x \in \mathcal{X}$, there is a unique $g \in G$ such that $X = g \circ x_s$; we denote this by $t(x)$. Let $T = t(X)$. Then $\Theta^{-1}T$ is a pivot, even conditional on $S = s$. We can thus apply the construction of § 8.1 to obtain a fiducial distribution. Again, this will coincide with the Bayesian posterior distribution, based on the right-invariant prior—which may be easier to compute.

**Example 8.2** In Example 8.1, the problem is equivariant under the location group, and $S$ and $T$ satisfy the above requirements. Since the right-invariant prior has density element $d\theta$, the fiducial distribution, which could be daunting to compute directly, must have density element proportional to $\prod_{i=1}^n g(x_i - \theta) d\theta$. 

As noted in § 8.1, a given problem may be equivariant under more than one group, and these may induce different fiducial distributions. This problem is defined away when we start with a structural model, which includes specification of the relevant group $G$. Then the above recipe yields a unique fiducial distribution.

### 8.3 Non-simple functional models

In a general functional model $X = \Theta \circ E$, we term $x \in \mathcal{X}$ and $e \in \mathcal{E}$ compatible when $x = \theta \circ e$ for some $\theta \in \mathcal{T}$. We first assume invertibility: that such $\theta$ is unique (compare the group-theoretic concept of exactness), and write $\theta = x \circ e^{-1}$, noting that in this case $e^{-1}$ is a partial function, operating only on $x$'s that are compatible with $e$.

Let $\mathcal{E}_x = \{e : x$ and $e$ are compatible$\}$. On observing $X = x$, we learn the logical information $E \in \mathcal{E}_x$, but no other logical information about $E$.

We should thus aim to adjust the distribution of $E$ to account for this new information, yielding a revised distribution $E \sim P_x$, say—where $P_x$ is confined to $\mathcal{E}_x$. Then $\Theta = x \circ E^{-1}$ is well-defined, and a fiducial distribution can be formed by assigning to $E$ the distribution $P_x$ over $\mathcal{E}_x$.

But how might we compute $P_x$? As seen in Example 7.3 conditioning on the logical information $E \in \mathcal{E}_x$ is only well-defined when $\mathcal{E}_x$ is embedded in a suitable partition. We have to consider what other information we might have obtained, in other circumstances. Such information would be of the form $E \in \mathcal{E}_y$, as $y$ varies in $\mathcal{X}$. Conditioning would thus be justified when the $\{\mathcal{E}_y : y \in \mathcal{X}\}$ form a partition, which will be the case when, for $x, y \in \mathcal{X}$, $\mathcal{E}_x$ and $\mathcal{E}_y$ are either identical or disjoint—in which case we term the FM partitionable. Equivalently, there exist, essentially unique, functions $a(\cdot)$ on $\mathcal{X}$, $u(\cdot)$ on $\mathcal{E}$, such that $x \in \mathcal{X}$ and $e \in \mathcal{E}$ are compatible just when $a(x) = u(e)$. Since the observable $X = \Theta \circ E$ is necessarily compatible with the error variable $E$, $a(X) = u(E)$, and so is ancillary—the functional ancillary. The fiducial distribution of $\Theta$, for data $X = x$, is now that of $x \circ E^{-1}$, with $E \sim P$ conditioned on $u(E) = a(x)$.

However, a non-partitionable FM does not support unambiguous fiducial inference.

### 8.4 Examples

We do not have a general necessary and sufficient condition for a functional model to be partitionable. This will however hold when the model is structural, or a reduction of a structural model.

**Example 8.3 Location-scale model**

Let $\mathcal{E} = \mathcal{X} = \mathbb{R}^3$, $\Theta = (M, \Sigma) \in \mathcal{T} = \mathbb{R} \times \mathbb{R}^+$. The structural model $X = \Theta \circ E$ is given by $X_i = M + \Sigma E_i, \ i \in \{1, \ldots, n\}$. The functional ancillary can be taken as $a(x) = ((x_i - \overline{x})/s_x : i = 1, \ldots, n)$, where $s^2_x = \sum_{i=1}^n (x_i - \overline{x})^2/(n-1)$; and $u(e) = a(e)$. The fiducial distribution is represented by $\Theta = (\overline{x} - s_x \overline{E}/s_{\Sigma}, s_x/s_{\Sigma})$, where the initial distribution of $(\overline{E}, s_{\Sigma})$ is conditioned on $(E_i - \overline{E})/s_{\Sigma} = (x_i - \overline{x})/s_x, \ i = 1, \ldots, n$. It can alternatively be derived as the Bayesian posterior distribution based on the right-invariant prior, having density element $d\mu d\sigma/\sigma$. □
Example 8.4 Reduced structural model
We have observable \( W \in \mathcal{W} = \{ w \in \mathbb{R}^n : s_w = 1 \} \), parameter \( \Omega \in \mathbb{R} \), error variable \( E \in \mathcal{E} = \mathbb{R}^n \). The functional model \( W = \Omega \circ E \) is given by \( W_i = (\Omega + E_i)/s_E, i = 1, \ldots, n \). This is a reduction of the structural model of Example 8.3, induced by \( W = X/s_X, \Omega = M/\Sigma \). It is partitionable, with \( u(e) = ((e_i - \bar{e})/s_e : i = 1, \ldots, n), a(w) = ((w_i - \bar{w}) : i = 1, \ldots, n) \). The fiducial distribution is represented by \( \Omega = ws_E - E \), with the distribution of \( E \) conditioned on \( (E_i - \bar{E})/s_E = w_i - \bar{w}, i = 1, \ldots, n \). It is not a Bayesian posterior based on the likelihood in the reduced model, though it does agree with the marginal for \( \Omega \) in the full fiducial distribution of Example 8.3 (which is a Bayesian posterior).

Example 8.5 Non-partitionable model
Let \( \mathcal{X} = \mathcal{E} = \mathbb{R}, \mathcal{T} = \mathbb{R}^+ \). Consider the functional model \( X = \Theta + E \). Then \( \mathcal{E}_x = (-\infty, x) \). Conditioning on \( E \in \mathcal{E}_x \) appears, prima facie, straightforward: just truncate the initial distribution of \( E \) to \( (-\infty, x) \). However, as the \( \{ \mathcal{E}_x : x \in \mathbb{R} \} \) do not form a partition, it is arguable whether this is appropriate.

9 Non-invertible models
Consider a functional model \( X = \Theta \circ E, E \sim P_0 \). Now we drop the invertibility requirement, so that \( \tau_{x,e} := \{ \theta : x = \theta \circ e \} \) may be a set with more than one element.

9.1 Simple non-invertible functional model
We first suppose the model simple, so that any \( x \in \mathcal{X} \) and \( e \in \mathcal{E} \) are compatible: equivalently, \( \tau_{x,e} \) is never empty.

On observing \( X = x \), no new logical information is obtained about \( E \). The usual fiducial argument now implies that we can still regard \( E \sim P_0 \). But even were we to know the realised value \( e \) of \( E \), we could only infer \( \Theta \in \tau_{x,e} \). In the absence of knowledge of \( e \), the fiducial argument represents our knowledge of \( \Theta \) by \( \Theta \in \mathcal{T}_x \), where \( \mathcal{T}_x := \tau_{x,E} \), with \( E \sim P_0 \), is a random subset of \( \mathcal{T} \).

This kind of partial probabilistic knowledge, based on random sets, lies at the heart of the Dempster-Shafer theory of inference [Dempster 2008]. Using it, we can go on to define the belief and plausibility functions for \( \Theta \), after observing \( X = x \):

\[
\text{Bel}_x(\Theta \in A) = P_0(\mathcal{T}_x \subseteq A)
\]
\[
\text{Pl}_x(\Theta \in A) = P_0(\mathcal{T}_x \cap A \neq \emptyset).
\]

9.2 Recent variations
Fiducial theory was largely ignored for many decades. However recent years have seen a resurgence of interest, much of it related to non-invertibility. [Hannig (2009)] carries through an analysis similar to that of §9.1 but, in order to finish with a probability distribution for \( \Theta \), adds a further step, in which, given the compatible set \( \mathcal{T}_x \), a single value in \( \mathcal{T}_x \) is selected at random, from some specified conditional distribution. There is of course sensitivity to this specification, and there does not seem to be any principled way to resolve this. The theory of inferential models [Martin and Liu 2016] uses a different auxiliary construction, which effectively replaces \( \text{Bel}_x \) by a new belief function \( \text{Bel}^*_x \), bounded above by \( \text{Bel}_x \). Again there is a choice of the extra specification. In both approaches, some guidance on this may be found by aiming towards compliance with frequentist (e.g., confidence) properties.
9.3 General non-invertible functional model

We now generalize by allowing $\tau_{x,e} = \emptyset$, equivalent to $x$ and $e$ being incompatible.

On now observing $X = x$, we obtain new logical information about $E$, namely

$$E \in E_x := \{ e : \tau_{x,e} \neq \emptyset \}.$$  

Only when $e \in E_x$ could we have made the observation $X = x$ (for some $\theta \in \Theta$). In order to support fiducial inference, the initial distribution $P_0$ of $E$ must be adjusted, somehow, to a new distribution, $P^x$, supported on $E_x$. But how?

Again, things are reasonably straightforward if the model is partitionable, i.e., for all $x, x' \in X$, $E_x$ and $E_{x'}$ are either identical or disjoint. This will hold if and only if there exist functions $a(\cdot)$ on $X$ and $u(\cdot)$ on $E$, such that $e \in E_x$ exactly when $u(e) = a(x)$. Then learning $X = x$ is equivalent to learning $u(E) = a(x)$, and conditioning on this information is unproblematic: letting $(P_a)$ be the family of conditional distribution of $E$ given $u(E) = a$ (well-defined under partitionability), we take $E \sim P_a(x)$ (and so Bel$_x$, Pl$_x$). However, when the model is not partitionable it is not clear how (or indeed whether) to construct $P^x$. Hannig (2009) suggests ways of identifying a suitable function on which to condition, but there typically remains a multiplicity of apparently reasonable choices.

Example 9.1 Consider the following functional model (a variation on Example 4 of Hannig (2009)):

$$X_1 = \Theta_1/E_1$$  \hspace{1cm} (17)

$$X_2 = (\Theta_1 + \Theta_2)/E_2$$  \hspace{1cm} (18)

$$X_3 = (\Theta_1 + 2\Theta_2)/E_3.$$  \hspace{1cm} (19)

Fixing data $X = x$ and solving (17) and (18), we obtain

$$\Theta_1 = x_1E_1$$ \hspace{1cm} (20)

$$\Theta_2 = x_2E_2 - x_1E_1,$$ \hspace{1cm} (21)

and then inserting these in (19) we get the compatibility condition, $E \in E_x$, expressed as

$$\frac{2x_2E_2 - x_1E_1}{E_3} = x_3.$$ \hspace{1cm} (22)

A similar analysis that starts by solving (18) and (19) yields

$$\frac{2x_2E_2 - x_3E_3}{E_1} = x_1,$$ \hspace{1cm} (23)

Both (22) and (23) are (necessarily) equivalent to each other, and to

$$x_1E_1 - 2x_2E_2 + x_3E_3 = 0.$$ \hspace{1cm} (24)

The partitions generated by the variables on the left-hand sides of (22), (23) and (24) are all different, so that conditioning (20) and (21) on them will give different answers. And indeed, since the model is non-partitionable, there is no correct answer as to how (or whether) we should condition. To see non-partitionability directly, note that, as expressed by (24), each $E_x$ is a plane in $\mathbb{R}^3$. When not identical, any two such planes must intersect in a line, so can not be disjoint. Hannig (2009) notes that, in examples such as this, it matters how we condition on the information $E \in E_x$, and makes some ad hoc recommendations. But in view of non-partitionability, it could be argued that no conditioning of any kind is justifiable, and that the model simply does not support fiducial inference. \hfill $\Box$
Example 9.2 Let $X = \mathcal{E} = [0, 1]^n$, $T = [0, 1]$. Under $P_0$, $(E_i : i = 1, \ldots, n)$ are independently uniform over $[0, 1]$. The functional model is given by $X_i = \mathbb{1}(E_i \leq \Theta)$, $i = 1, \ldots, n$. Then, when $\Theta = \theta$, the $X_i$ are $n$ independent Bernoulli($\theta$) variables. This functional model is the basis of Example 6 of [Hannig (2009)]

We see that $e$ and $x$ are compatible ($\tau_{x,e} \neq \emptyset$) when $x_i = 1, x_j = 0$ if and only if $e_i < e_j$. Thus on observing $X = x$, we learn

$$E \in \mathcal{E}_x := \{e : e_i < e_j \text{ just when } x_i = 1, x_j = 0\},$$

and then $\Theta$ lies in the random interval between two order statistics:

$$T_E := [E(r), E(r+1)) \quad (r = \sum x_i).$$

We have to confine the distribution of $E$ to $\mathcal{E}_x$, but how? Noting that $P_0(\mathcal{E}_x) > 0$, an obvious approach is simply to truncate $P_0$ to the set $\mathcal{E}_x$. We may then note that (25) is a condition on the way in which the $(E_i)$ are ordered, which is independent of their order-statistic, which is what determines $T_E$. So the distribution of the random interval $T_E$ will be unaffected by the truncation to (25)—allowing us to use its unconditional distribution, based on the order statistics of a random sample from the uniform distribution on $[0, 1]$. This is the approach of [Hannig (2009)]

Nevertheless, this model is not partitionable, as may be seen by noting that, when $x = 1$ (the vector with all $x_i = 1$), we have $\mathcal{E}_1 = \mathcal{E}$—which is not disjoint from or identical with any other $\mathcal{E}_x$.

So, in the light of examples such as Example 7.4 and in the absence of a clear question that is answered by the information $E \in \mathcal{E}_x$, it is debatable whether the above argument is appropriate. Once again there seems to be no fully justifiable fiducial inference available.

\[\square\]

10 Concluding comments

While the fiducial argument has some prima facie appeal, all attempts to formulate a fully coherent theory of fiducial inference have fallen foul of inconsistency and counter-examples. The investigations of [Dawid and Stone (1982)] based on functional models, were an attempt to see just how far the theory could be taken before crashing onto the rocks—but crash it eventually did. Many of the difficulties are associated with the need to specify, unambiguously, a relevant partition for performing probabilistic conditioning. In some cases, as in Example 7.3 two equally natural routes to take account of new or assumed information lead to different partitions and hence conflicting fiducial distributions—a parallel inconsistency in predictive inference was exhibited by [Dawid and Wang (1993)]. In other cases, as in §9.3 there is no natural embedding of the information obtained within any partition whatsoever, rendering fiducial inference undefined.

Nevertheless, even though methods derived from fiducial theory may have limited validity from a fully principled theoretical standpoint, that is not to deny that they may prove useful for other purposes—for example, for constructing exact or approximate confidence regions. But—simple pivotal cases apart—there is no guarantee that this will be the case, so that further investigations are required in individual cases.

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