Anomalous transport and current fluctuations in a model of diffusing Levy walkers

Abhishek Dhar\textsuperscript{1} and Keiji Saito\textsuperscript{21,2}

\textsuperscript{1}International Centre for Theoretical Sciences, TIFR, Bangalore 560012, India
\textsuperscript{2}Department of Physics, Keio University, Yokohama 223-8522, Japan

Abstract

A Levy walk is a non-Markovian stochastic process in which the elementary steps of the walker consist of motion with constant speed in randomly chosen directions and for a random period of time. The time of flight is chosen from a long-tailed distribution with a finite mean but an infinite variance. Here we consider an open system with boundary injection and removal of particles, at prescribed rates, and study the steady state properties of the system. In particular, we compute density profiles, current and current fluctuations in this system. We also consider the case of a finite density of Levy walkers on the ring geometry. Here we introduce a size dependent cut-off in the time of flight distribution and consider properties of current fluctuations.

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I. INTRODUCTION

The Levy walk model is a well-studied model to describe anomalous diffusion, in particular super-diffusion [1-4]. Among experimental systems the Levy walk model has been used to model the motion of photons in random medium [5] and the diffusion of cold atoms [6]. The migration of various species of animals has also been proposed to follow Levy walk statistics though recent observational data has questioned this [7]. A closely related model is that of Levy flights [1] and fractional diffusion. For the Levy flight model, each elementary step takes the same time but the step length is chosen from a long-tailed distribution. Then, by definition, the second moment of the position of the walker diverges at any time.

Most studies of the Levy walk so far have focused on the motion of a single Levy walker and a description of the propagator which tells us how an initially localized distribution spreads in time. This is known to be non-Gaussian, with a standard distribution \( \sigma \) that grows with time \( t \) as \( \sigma^2 \sim t^\gamma \) with \( 1 < \gamma < 2 \). One can also consider the many-particle situation where we could, for example, be interested in an open system where Levy walkers can enter or leave the system at the boundaries with prescribed rates. For different rates of injection and emission one can have current-carrying steady states. One could also consider a ring geometry with a finite density of walkers and ask questions on relaxation and fluctuations. These questions are much less studied and is the focus of this paper.

The problem of steady state transport in a system of non-interacting Levy walkers arises naturally in the case of anomalous heat transport in one-dimensional systems [8, 9]. A number of recent studies indicate that a good description of anomalous heat conduction in one dimensional systems is obtained by modeling the motion of the heat carriers as Levy random walks instead of simple random walks [10-14]. Some of the indicators of anomalous heat transport include — (i) in steady states the dependence of the heat current \( J \) on system size \( L \) shows the scaling behaviour \( J \sim L^{-1+\alpha} \) with \( \alpha > 0 \), (ii) the temperature profiles across systems in nonequilibrium steady states are found to be nonlinear, even for very small applied temperature differences and, (iii) the spreading of heat pulses in anharmonic chains is super-diffusive. All these features seem to be captured by the Levy walk description [14]. The problem of Levy flights in bounded domains has also been studied in the context of first passage time distributions and hitting probabilities [15, 17] and these are also related to some of the steady state properties of the Levy walk model studied here.
In this paper, we expand on our earlier work in [14] where several exact results on the Levy walk model were presented in the context of steady state anomalous heat transport. We show that the steady state current has a power law dependence on the system-size and is non-locally connected to the density gradient in contrast to normal diffusive transport. We also derive the exact cumulant generating function of current for both the open and the ring geometry. These exact results are consistent with numerical observations that have earlier been found for mechanical models for one-dimensional heat conduction. We believe our analysis will be helpful in discussions of other types of superdiffusive transport such as Levy transport of light in random medium [5].

The plan of the paper is as follows. In Sec. (II) we first consider a single Levy walker on the infinite line. We define the precise model studied here and discuss properties of the Levy propagator and also various moments of the distribution. In Sec. (III) we discuss the setting up of the transport problem in an open system of many non-interacting Levy walkers. We show how steady state properties like density profile and average current in the system can be computed. In Sec. (IV) we discuss current fluctuations and show how the cumulant generating function can be computed exactly. Current fluctuations on a ring geometry are discussed in Sec. (V) and we conclude with a discussion in Sec. (VI).

II. LEVY DIFFUSION ON THE INFINITE LINE

We consider first a Levy walk on the infinite line. Each step of the walk consists in choosing the step length \( x \) and the time for the step \( t \) from the joint distribution \( \eta(x,t) \). Thus \( \eta(x,t)dxdt \) is the probability that a step has length between \( x \) and \( x+dx \) and is of a duration between \( t \) and \( t+dt \). Here we consider the distribution

\[
\eta(x,t) = \frac{1}{2} \left[ \delta(x-vt) + \delta(x+vt) \right] \phi(t) \tag{1}
\]

Our choice of the step distribution corresponds to choosing a time of flight from the distribution \( \phi(t) \) and then moving at speed \( v \) in either direction, with equal probability. We define

\[
\psi(t) = \int_t^\infty dt' \phi(t') \tag{2}
\]

as the probability of choosing a time of flight \( \geq t \) and

\[
\chi(t) = \int_t^\infty d\tau \psi(\tau) \tag{3}
\]
FIG. 1: Plot of typical trajectories of the Levy walk in one-dimension where the flight time distribution is given by Eq. (8) with $\beta = 1.5$.

We would like to find the propagator $P(x, t)$ which gives the probability that the walker is between $x$ and $x + dx$ at time $t$, given that it was at $x = 0$ at time $t = 0$. To this end first let us define $Q(x, t) dx \, dt$ be the probability that the walker has precisely landed in the interval $dx$ during the time interval $dt$. Note that this does not include trajectories which were crossing $x$ at time $t$. We then have the following equations:

$$Q(x, t) = \int_{-\infty}^{\infty} dx' \int_0^t dt' \, Q(x - x', t - t') \, \eta(x', t') + \delta(x) \, \delta(t), \quad (4)$$
\[ P(x, t) = \int_{-\infty}^{\infty} dx' \int_{0}^{t} dt' Q(x-x', t-t') \xi(x', t') , \quad (5) \]

where \( \xi(x, t) = \frac{1}{2} [\delta(x-vt) + \delta(x+vt)] \psi(t) \quad (6) \)

is the probability that during a single step (starting from the origin), the walker is in the region \( x - x + dx \) at time \( t \). Let us define the Fourier-Laplace transform of a function \( f(x, t) \) as:

\[ \tilde{f}(k, s) = \int_{-\infty}^{\infty} dx \int_{0}^{\infty} dt \, e^{ikx} e^{-st} f(x, t) . \]

Then Eqs. (4,5,6) give:

\[ \tilde{Q}(k, s) = \tilde{Q}(k, s) \tilde{\eta}(k, s) + 1 \\
\tilde{P}(k, s) = \tilde{Q}(k, s) \tilde{\xi}(k, s) , \]

where \( \tilde{\eta}(k, s) = [\tilde{\phi}(s-ik) + \tilde{\phi}(s+ik)]/2, \tilde{\xi}(k, s) = [\tilde{\psi}(s-ik) + \tilde{\psi}(s+ik)]/2 \). Hence we get

\[ \tilde{\psi}(s) = \int_{0}^{\infty} dte^{-st}\psi(t) = [1 - \tilde{\phi}(s)]/s. \]

This result also directly follows from noting that the probability \( P(x, t) \) satisfies

\[ P(x, t) = \frac{1}{2} \psi(t)\delta(|x| - vt) + \frac{1}{2} \int_{0}^{t} dt' \phi(t')[P(x-vt', t-t') + P(x+vt', t-t')] . \]

Here we consider Levy walkers with a time-of-flight distribution

\[ \phi(t) = \frac{\beta}{t_{o}(1 + t/t_{o})^{\beta+1}} , \quad 1 < \beta < 2 . \quad (8) \]

which decays like a power law \( \phi(t) \simeq A t^{-\beta-1} \) with \( A = \beta t_{o}^{\beta} \) at large times. For this range of \( \beta \) the mean flight time \( \langle t \rangle = \int_{0}^{\infty} dt \, t \, \phi(t) = t_{o}/(\beta - 1) \) is finite but \( \langle t^2 \rangle = \infty \).

For asympotic properties it is useful to find the form of \( \tilde{P}(k, s) \) for small \( k, s \). The laplace transform \( \tilde{\phi} \) is given by:

\[ \tilde{\phi}(s) = \int_{0}^{\infty} dt e^{-st} \phi(t) = 1 - \langle t \rangle s + b \beta(st_{o})^{\beta} + \cdots , \quad (9) \]

where \( b = \frac{1}{\beta(\beta - 1)} \int_{0}^{\infty} dz e^{-z} z^{1-\beta} = \frac{1}{\beta(\beta - 1)} \Gamma(2 - \beta) . \)
Hence we get:

\[ \tilde{P}(k, s) = \frac{1 - c[(s - ik)^{\beta - 1} + (s + ik)^{\beta - 1}]}{s - c[(s - ik)^{\beta} + (s + ik)^{\beta}]} , \]  

(10)

where \( c = bA/(2\langle t \rangle) \). Taking the inverse Fourier-Laplace transform of this gives us the propagator of the Levy walk on the infinite line. This corresponds to a pulse whose central region is a Levy-stable distribution with a scaling \( x \sim t^{1/\beta} \). This can be seen by expanding Eq. (10) for \( vk/s << 1 \) to get \( \tilde{P}(k, s) = [s - c \cos(\beta \pi/2)(vk)^{\beta - 1}]^{-1} \). The difference with the Levy-stable distribution is that the Levy-walk propagator has ballistic peaks of magnitude \( t^{1-\beta} \) at \( x = \pm vt \) and vanishes outside this. The overall behaviour of the propagator is as follows [2]:

\[
\begin{align*}
P(x, t) &\sim t^{-1/\beta} e^{-ax^2/t^{2/\beta}} \quad |x| \lesssim t^{1/\beta} \\
&\sim t^{x^{\beta-1}} \quad t^{1/\beta} \lesssim |x| < vt \\
&\sim t^{1-\beta} \quad |x| = vt \\
&= 0 \quad |x| > vt .
\end{align*}
\]

(11)

The time evolution of the Levy-walk propagator, obtained from direct simulations of the Levy walk, is shown in Fig. [2]. We also plot the Levy-stable distribution obtained by taking the Fourier transform of \( P(k, t) = e^{-c \cos(\beta \pi/2)|k|^\beta} \).

Various moments of the distribution can be found since

\[
\langle x^n \rangle(t) = \frac{d^n}{d(ik)^n} \int dk e^{ikx} P(x, t) \bigg|_{k=0} ,
\]

while its Laplace transform is given by

\[
\langle x^n \rangle(s) = \frac{d^n}{d(ik)^n} \tilde{P}(k, s) \bigg|_{k=0} .
\]

Using Eq. (10) we get the following leading behaviour

\[
\langle x^2 \rangle(s) \simeq 4(\beta - 1)cv^2s^{\beta-4} ,
\]

\[
\langle x^4 \rangle(s) \simeq 8(\beta - 1)(2 - \beta)(3 - \beta)c^4v^4s^{\beta-6} ,
\]

and this implies (for large \( t \)) [14]

\[
\begin{align*}
\langle x^2 \rangle &\simeq \frac{2 A v^2}{(3 - \beta)(2 - \beta)\beta \langle \tau \rangle} t^\gamma , \quad \gamma = 3 - \beta , \\
\langle x^4 \rangle &\simeq \frac{4 A v^4}{(5 - \beta)(4 - \beta)\beta \langle \tau \rangle} t^{\gamma + 2} .
\end{align*}
\]

(12) (13)
FIG. 2: Plot of the scaled distribution $t^{2/3}P(x,t)$ versus $x/t^{2/3}$ of the Levy walk on the open line for $\beta = 3/2$ at three different times. Also shown is a plot of the Levy-stable distribution (see text). The inset shows a plot of the mean square displacement and the fourth moment and a comparision with the exact asympotic forms (dashed lines) given by Eqs.(12,13). In all plots the time $t_0$ and $v$ are set to one.

We see that for $1 < \beta < 2$ the motion is superdiffusive [3, 4].

Higher dimensions: Our results for the Levy walk model are easy to generalize to higher dimensions. Let us consider a walk where in each step the walker chooses a flight time from
the same distribution \( \phi(t) \) while the joint distribution \( \eta(\bar{x}, t) \) is given by

\[
\eta(\bar{x}, t) = \frac{\delta(|\bar{x}| - \mathbf{v}t)}{[2(\pi)^{d/2}/\Gamma(d/2)] |\bar{x}|} \phi(t)
\] (14)

Corresponding to the one-dimensional equations Eqs. (4,5) for a Levy walk on the open line, we now have the following equations:

\[
Q(x, t) = \int_{-\infty}^{\infty} dx' \int_{0}^{t} dt' Q(x - x', t - t') \eta(x', t') + \delta(x) \delta(t),
\] (15)

\[
P(x, t) = \int_{-\infty}^{\infty} dx' \int_{0}^{t} dt' Q(x - x', t - t') \xi(x', t'),
\] (16)

where \( \xi(x, t) = \frac{1}{2} \frac{\delta(|x| - vt)}{[(2\pi)^{d/2}/\Gamma(d/2)] |x|^{d-1}} \psi(t) \). (17)

Taking the Fourier Laplace transform then gives

\[
\tilde{P}(k, s) = \frac{\tilde{\xi}(k, s)}{1 - \tilde{\eta}(k, s)},
\] (18)

with

\[
\tilde{\eta}(k, s) = \langle \tilde{\phi}(s - iv|k| \cos \theta) \rangle = \frac{\int_{0}^{\pi} d\theta \sin^{d-2} \theta \tilde{\phi}(s - iv|k| \cos \theta)}{\int_{0}^{\pi} d\theta \sin^{d-2} \theta},
\]

\[
\tilde{\xi}(k, s) = \langle \tilde{\psi}(s - iv|k| \cos \theta) \rangle,
\] (19)

and where \( \langle ... \rangle \) denotes an average over the polar angle \( \theta \). Proceeding as for the one-dimensional case we get the analogue of Eq. (10):

\[
\tilde{P}(k, s) = \frac{1 - 2c \langle (s + i|k| \cos \theta)^{\beta-1} \rangle}{s - 2c \langle (s + i|k| \cos \theta)^\beta \rangle}.
\] (20)

III. LEVY DIFFUSION IN A FINITE SYSTEM CONNECTED TO INFINITE RESERVOIRS

To study an open system of Levy walkers we consider a finite system connected to two semi-infinite reservoirs on which the density of walkers is maintained at fixed values. Thus we consider our system to be the finite segment between \((0, L)\) and this is connected on the two sides to reservoirs. The left reservoir consists of the region \( x \leq 0 \) while the right reservoir consists of the region \( x \geq L \). We set \( Q(x, t) = Q_l \) for points on the left reservoir and \( Q(x, t) = Q_r \) for those on the right. In general if we know the distributions \( Q(x, \tau) \) and
\( P(x, \tau) \) for all times \(-\infty < \tau < t\) then the distribution at time \( t \) is given by:

\[
Q(x, t) = \int_{-\infty}^{\infty} dy \frac{1}{2v} Q(y, t - |x - y|/v) \ \phi(|x - y|/v) ,
\]

\[
P(x, t) = \int_{-\infty}^{\infty} dy \frac{1}{2v} Q(y, t - |x - y|/v) \ \psi(|x - y|/v) .
\]

(21)

In the above expressions \( Q(x, t) \) gets contributions from walkers starting from all possible points \( y \) and landing precisely at \( x \) at time \( t \). On the other hand \( P(x, t) \) gets contributions from walkers starting at \( y \) and being either at or passing \( x \) at time \( t \). Since the distribution \( Q(x, t) \) is constrained to take either of the values \( Q_l \) or \( Q_r \) in the reservoirs, the above equation gives, for points on the system

\[
Q(x) = Q_l \frac{1}{2} \psi(x/v) + Q_r \frac{1}{2} \psi((L - x)/v)
\]

\[
P(x) = Q_l \frac{1}{2} \chi(x/v) + Q_r \frac{1}{2} \chi((L - x)/v)
\]

(23)

where we used the definitions of \( \psi \) and \( \chi \) from Eqs. (2) and (3). We see that the equations above can be interpreted by thinking of a system where particles enter from the left boundary with flight times \( t \) at a rate \( Q_l \ \psi(t) \) and from the right boundary at a rate \( Q_r \ \psi(t) \).

A. Steady state density profile

In the steady state we have \( Q(x, t) = Q(x) \) and \( P(x, t) = P(x) \), hence we get:

\[
Q(x) = \int_{0}^{L} dy \frac{1}{2v} \phi(|x - y|/v) \ \psi((L - x)/v) Q(y) + \frac{Q_l}{2} \psi(x/v) + \frac{Q_r}{2} \psi((L - x)/v) ,
\]

\[
P(x) = \int_{0}^{L} dy \frac{1}{2v} \psi(|x - y|/v) \ \phi(|x - y|/v) Q(y) + \frac{Q_l}{2} \chi(x/v) + \frac{Q_r}{2} \chi((L - x)/v) .
\]

(25)

(26)

It can be verified by direct substitution that the solution of Eq. (25) is given by

\[
Q(x) = (Q_l - Q_r) H(x) + Q_r
\]

where \( H(x) \) is the probability that a Levy walker starting at position \( x \) will first hit the left reservoir (i.e. the region \( x < 0 \)) before it hits the right reservoir (i.e. \( x > L \)), and satisfies

\[
H(x) - \int_{0}^{L} dy \frac{1}{2v} \phi(|x - y|/v) \ H(y) = \frac{1}{2} \psi(x/v) .
\]

(27)

(28)
If one considers a Levy flight with distribution \( \rho(z) = [\phi(z/v) + \phi(-z/v)]/(2v) \) of steps \( z \), the probability \( H(x) \) that starting at \( x \), the flight hits first the left bath satisfies exactly Eq. (28). Hence by following the same mathematical steps as in [15] to study equations such as (28), one can show that, in the large \( L \) limit, the solution \( H(x) \) satisfies

\[
\int_0^L dy \, \psi(|x - y|/v) \, \text{Sgn}(x-y) \, H'(y) = 0 ,
\]

with \( H(0) = 1 \) and \( H(L) = 0 \). For a \( \phi(\tau) \) decaying as in (8), the solution of Eq. (29) is [14]

\[
H'(x) = -B[x(L-x)]^{\beta/2-1} .
\]

Integrating this and imposing the boundary conditions \( H(0) = 1 \) and \( H(L) = 0 \), one obtains the constant \( B = \Gamma(\beta)/\Gamma(\beta/2)^2 \, L^{1-\beta} \). From Eq. (27) we then finally get \( Q(x) \). Substituting \( \psi(x/v) = -vd\chi(x/v)/dx \) in (26) we obtain \( P(x) = \chi(0)Q(x) - \int_0^L dx' \chi(|x - x'|/v)\text{Sgn}(x-x')Q'(x')/2 \). Using Eq. (30) we then get in the limit of large \( L \):

\[
P(x) = \chi(0)Q(x) = \langle \tau \rangle Q(x) .
\]

In Fig. (3), we compare numerical results obtained by solving Eqs. (25,26) with the exact results of Eqs. (30,31). The profiles are nonlinear and look similar to those observed for temperature profiles in 1D heat conduction [8, 9].

**B. Steady state current**

To derive an expression for the current operator, we need to write a continuity equation of the form \( \partial P(x,t)/\partial t + \partial J(x,t)/\partial x = 0 \). To this end let us make a change of the integration variables in Eqs. (21,22) from \( y \) to \( t' = t - |x-y|/v \) to get the following forms:

\[
Q(x,t) = \frac{1}{2} \int_{-\infty}^t dt' \left[ Q(x - vt + vt', t') + Q(x + vt - vt', t') \right] \phi(t - t') ,
\]

\[
P(x,t) = \frac{1}{2} \int_{-\infty}^t dt' \left[ Q(x - vt + vt', t') + Q(x + vt - vt', t') \right] \psi(t - t') .
\]

Taking a time-derivative of Eq. (33) we get

\[
\frac{\partial P(x,t)}{\partial t} = Q(x,t)\psi(0)
\]

\[
+ \frac{1}{2} \int_{-\infty}^t dt' \left( -v \frac{\partial}{\partial x} \right) \left[ Q(x - vt + vt', t') - Q(x + vt - vt', t') \right] \psi(t - t')
\]

\[
+ \frac{1}{2} \int_{-\infty}^t dt' \left[ Q(x - vt + vt', t') + Q(x + vt - vt', t') \right] \frac{d}{dt} \psi(t - t') .
\]
FIG. 3: Plot of $Q(x)$ for different system sizes, for the Levy walk with $\beta = 1.5$, $Q_l = 1.0$, $Q_r = 0.5$. The data are obtained by solving Eqs. (25, 26) with discretized space. The distribution of flight times is $\phi(\tau) = \beta/t_o/(1 + \tau/t_o)^{\beta+1}$.

Using $\psi(0) = 1$, $-d\psi/dt = \phi$ and Eq. (32), the first and last terms on the right hand side vanish, and we get the current continuity equation with the following form of the current operator

$$J(x,t) = \frac{1}{2} \int_{-\infty}^{\infty} dy \, Q(x-y, t-|y|/v) \, \text{Sgn}(y) \, \psi(|y|/v) .$$

(34)

This equation is easy to understand physically. The contribution to the integral coming from $y > 0$ corresponds to particles crossing the point $x$ from left to right which started their flight at $x - y$ at time $t - y/v$ (the factor $\psi(y/v)$ comes from the fact that these particles have a flight time longer than $y/v$). Similarly the other part of the integral (from $y < 0$) corresponds to a right-to-left current.

In the steady state, setting $Q(x, t) \equiv Q(x)$ we get the result

$$J(x) = \frac{1}{2} \int_{-\infty}^{\infty} dy \, Q(x-y) \, \text{Sgn}(y) \, \psi(|y|/v) .$$

(35)

Using the values of $Q$ in the reservoirs and the steady state solution given by Eqs. (27, 28) we evaluate the current at $x = 0$ and obtain

$$J = \frac{(Q_l - Q_r)}{2} \left[ \int_0^{\infty} dy \, \psi(y/v) - \int_0^L dy \, H(y) \psi(y/v) \right] .$$

(36)
Since the system is non-interacting, this result can be obtained directly by noting that the current is due to particles which enter from the left and leave to the right (and to the symmetric contribution). We then simply need to know the rate at which the non-interacting particles enter the system on the left side and leave the system into the right reservoir. This is given by

\[ p_l = Q_l \int_{-\infty}^{0} dy \int_{(L-y)/v}^{\infty} d\tau \phi(\tau) \]

\[ + \dfrac{Q_l}{2} \int_{-\infty}^{0} dy \int_{0}^{L} \dfrac{dx}{v} \left[ 1 - H(x) \right] \phi[(x-y)/v] , \]

(37)

with a similar expression for the right to left rate \( p_r \). The net current given by \( J = p_l - p_r \) is easily seen to be identical to Eq. (36).

After a partial integration and using the fact that \( Q(0) = Q_l \) and \( Q(L) = Q_r \), one gets \[ J(x) = -\dfrac{v}{2} \int_{0}^{L} dy \chi(|x-y|/v) Q'(y) . \]

(38)

We note that \( dJ/dx = 0 \) gives Eq. (29) and so the current is independent of \( x \), as expected. Evaluating the current at \( x = 0 \) and using Eq. (30), we get for large \( L \)

\[ J \approx (Q_l - Q_r) \dfrac{A v^\beta \Gamma(\beta) \Gamma(1-\dfrac{\beta}{2})}{2 \beta(\beta - 1) \Gamma(\dfrac{\beta}{2})} L^{\alpha-1}, \quad \alpha = 2 - \beta . \]

(39)

Thus unlike in normal diffusive systems where \( J \sim 1/L \) here we have a anomalous conductivity exponent \( \alpha \neq 0 \). From Eq. (12) we then get the relation \( \alpha = \gamma - 1 \), between the conductivity exponent for transport and the exponent for Levy-walk diffusion. In the context of anomalous heat conduction this relation, between the exponents for the Levy model, was noted in [10], numerically observed in 1D heat conduction models [12, 18] and a derivation based on linear response theory has recently been proposed [19].

In the large \( L \) limit by using Eq. (31) in Eq. (38) we obtain \[ J = -\dfrac{v}{2\langle\tau\rangle} \int_{0}^{L} dy \chi(|x-y|/v)P'(y) . \]

(40)

This is the analogue of the usual diffusion equation \( J = -D\partial P(x)/\partial x \) in the case of diffusive systems and can be interpreted as current being non-locally connected to the density gradient.
IV. CURRENT FLUCTUATIONS IN THE OPEN SYSTEM

Since the particles are independent the current fluctuations can be described by a Poissonian process characterized by the rate at which walkers injected from the left reservoir end up (either after a non stop flight or a non direct flight) into the right reservoir or walkers injected from the right reservoir end up into the left reservoir. In the steady state let us look at the cumulant generating function

$$Z(\lambda) = \langle e^{\lambda Q} \rangle,$$

(41)

where $Q$ is the net total number of particles that have crossed from the left reservoir into the system in the time interval $0 - \tau$. The total current is carried by independent Levy walkers and we can identify contributions from three distinct independent processes, namely —

(1) A particle entered the system at some $s < 0$ from the left reservoir with a flight time $t' < L/v$ and then exits into the left reservoir at some time $t$ in the interval $0 - \tau$. We define a random variable $z_\ell(t', s, \tau)$ which has value 1 whenever this process occurs and is otherwise 0. Similarly $z_r(t', s, \tau)$ is defined for the process when the particle enters from right reservoir at $s < 0$ and exits into left reservoir at $t \in (0, \tau)$.

(2) A particle enters the system from the left reservoir at some time $s \in (0, \tau)$ with a flight time $t'$ and thereafter does not exit back into the left reservoir. We define a variable $y_\ell(t', s, \tau)$ which has value 1 whenever this process occurs and is otherwise zero.

(3) A particle enters the system from the right reservoir at time $s \in (0, \tau)$ with a flight time $t'$ and then exits into the left reservoir some time during the remaining time interval. We define a variable $y_r(t', s, \tau)$ which has value 1 whenever this process occurs and is otherwise zero.

Let us divide the time interval $(0, \tau)$ into small intervals of size $ds$. Hence we can write

$$Q = \sum_{s=-\infty}^{0} \sum_{t'=0}^{L/v} [ z_\ell(t', s, \tau) + z_r(t', s, \tau) ] + \sum_{s=0}^{\tau} \sum_{t'=0}^{\infty} [ y_\ell(t', s, \tau) + y_r(t', s, \tau) ].$$

Since the particles are all independent we can write

$$Z(\lambda) = \prod_{s=-\infty}^{0} \prod_{t'=0}^{L/v} \langle e^{-\lambda z_\ell(t', s, t)} \rangle \langle e^{-\lambda z_r(t', s, t)} \rangle \times \prod_{s=0}^{\tau} \prod_{t'=0}^{\infty} \langle e^{\lambda y_\ell(t', s, t)} \rangle \langle e^{-\lambda y_r(t', s, t)} \rangle.$$

(42)

Let us define $\alpha_\ell(t), \alpha_r(t)$ to be the rates at which particles enter from the left and right reservoirs respectively with flight times $t$. Let $\gamma_{\ell,\ell}(t', t)$ be the rate of escape into the left
reservoir at time \( t \), of particles that entered the system from the left reservoir at time \( t = 0 \) with a flight time \( t' \), while \( \gamma_{r,\ell}(t', t) \) is the rate of escape into the left reservoir at time \( t \) of a particle which entered the system from the right reservoir at time \( t = 0 \) with a flight time \( t' \). Similarly we define \( \gamma_{r,\ell}(t', t) \) and \( \gamma_{\ell,\ell}(t', t) \).

Then we have

\[
\prod_{s=-\infty}^{0} \prod_{t'=0}^{L/v} \langle e^{-\lambda_y(t', s, \tau)} \rangle = \prod_{s=-\infty}^{0} \prod_{t'=0}^{L/v} \{ e^{-\lambda} \alpha_{\ell}(t') ds dt' \} \ D_{\ell,\ell}(t', s, \tau) \\
\quad + \left[ 1 - \alpha_{\ell}(t') ds dt' \ D_{\ell,\ell}(t', s, \tau) \right] \\
= e^{(e^{-\lambda} - 1)} \int_{s=-\infty}^{0} ds \int_{t'=0}^{L/v} dt' \ \alpha_{\ell}(t') \ D_{\ell,\ell}(t', s, \tau), \quad (43)
\]

\[
\prod_{s=-\infty}^{0} \prod_{t'=0}^{L/v} \langle e^{-\lambda_y(t', s, \tau)} \rangle = \prod_{s=-\infty}^{0} \prod_{t'=0}^{L/v} \{ e^{-\lambda} \alpha_{r}(t') ds dt' \} \ D_{r,\ell}(t', s, \tau) \\
\quad + \left[ 1 - \alpha_{r}(t') ds dt' \ D_{r,\ell}(t', s, \tau) \right] \\
= e^{(e^{-\lambda} - 1)} \int_{s=-\infty}^{0} ds \int_{t'=0}^{L/v} dt' \ \alpha_{r}(t') \ D_{r,\ell}(t', s, \tau), \quad (44)
\]

where \( D_{\ell,\ell}(t', s, \tau) \) \((D_{r,\ell})\) is the probability that a particle which entered the system from the left (right) reservoir at time \( s \) with a flight time \( t' \) is emitted into the left reservoir during the time interval \((0, \tau)\). This is given by \( D_{\ell,\ell}(t', s, \tau) = \int_{0}^{\tau} dt S_{\ell}(t', t - s) \gamma_{\ell,\ell}(t', t - s) \) where \( S_{\ell}(t', t) = e^{-\int_{0}^{\tau} ds (\gamma_{r,\ell}(t', s) + \gamma_{r,\ell}(t', s))} \) is the probability of survival up to time \( t \) of a particle entering the system from the left reservoir at time \( t = 0 \) with a flight time \( t' \).

Similarly we have

\[
\prod_{s=0}^{\tau} \prod_{t'=0}^{\infty} \langle e^{-\lambda_y(t', s, \tau)} \rangle = \prod_{s=0}^{\tau} \prod_{t'=0}^{\infty} e^{-\lambda} \alpha_{\ell}(t') ds dt' \ S_{\ell,\ell}(t', \tau - s) \\
\quad + \left[ 1 - \alpha_{\ell}(t') ds dt' \ S_{\ell,\ell}(t', \tau - s) \right] \\
= e^{(e^{-\lambda} - 1)} \int_{s=0}^{\tau} ds \int_{t'=0}^{\infty} dt' \ \alpha_{\ell}(t') \ S_{\ell,\ell}(t', \tau - s), \quad (45)
\]

\[
\prod_{s=0}^{\tau} \prod_{t'=0}^{\infty} \langle e^{-\lambda_y(t', s, \tau)} \rangle = \prod_{s=0}^{\tau} \prod_{t'=0}^{\infty} e^{-\lambda} \alpha_{r}(t') ds dt' \ D_{r,\ell}(t', \tau - s) \\
\quad + \left[ 1 - \alpha_{r}(t') ds dt' \ D_{r,\ell}(t', \tau - s) \right] \\
= e^{(e^{-\lambda} - 1)} \int_{s=0}^{\tau} ds \int_{t'=0}^{\infty} dt' \ \alpha_{r}(t') \ D_{r,\ell}(t', \tau - s), \quad (46)
\]

where \( S_{\ell,\ell}(t', s) \) is the probability that a particle entering the system from the left reservoir at time \( s = 0 \) with a flight time \( t' \) does not exit into the left reservoir in the time interval \( 0 - s \) and \( D_{r,\ell}(t', s) \) is the probability that a particle entering the system from the right reservoir at time \( s = 0 \), with a flight time \( t' \), exits into the left reservoir in the interval \( 0 - s \).
We now note that in Eq. (43), \( \int_0^{L/v} dt' \alpha_\ell(t') D_{\ell,\ell}(t',s,\tau) \) is the average rate at which particles injected into system from left reservoir at time \( s \) are injected back into the left reservoir between times \( 0 - \tau \). Clearly for large \( \tau \) large compared to the residence time of a walker inside the system, this will be \( O(\tau^0) \) and decay as \( e^s \) for large \( s \). Hence we expect, that for large \( \tau \), \( \int_{-\infty}^{0} ds \int_0^{L/v} dt' \gamma_\ell(t') D_{\ell,\ell}(t',s,\tau) \sim O(\tau^0) \) and similarly \( \int_{-\infty}^{0} ds \int_0^{L/v} dt' \gamma_r(t') D_{r,\ell}(t',s,\tau) \sim O(\tau^0) \).

On the other hand in Eq. (45), \( \int_0^{\infty} dt' \alpha_\ell(t') S_{\ell,\ell}(t',\tau-s) \) is the average rate at which particles injected into system at time \( s \) from the left reservoir are not reinjected back into the left reservoir in the time interval \( \tau - s \) while, in Eq. (46), \( \int_0^{\infty} dt' \alpha_r(t') D_{r,\ell}(t',\tau-t) \) is the average rate at which particles injected into system at time \( s \) from the right reservoir are injected into the left reservoir in the time interval \( \tau - s \). Both of these quantities are \( O(\tau^0) \). Hence for large \( \tau \),

\[
\int_{s=0}^{\tau} ds \int_{0}^{\infty} dt' \alpha_\ell(t') S_{\ell,\ell}(t',\tau-s) = p_L \tau,
\]

\[
\int_{s=0}^{\tau} ds \int_{0}^{\infty} dt' \alpha_r(t') D_{r,\ell}(t',\tau-s) = p_R \tau,
\]

where \( p_L \) is the rate at which walkers injected from the left reservoir end up (either after a non stop flight or a non direct flight) into the right reservoir and \( p_R \) is the rate at which walkers injected from the right reservoir end up into the left reservoir.

Hence we get the leading large \( \tau \) behaviour as \( Z(\lambda) \sim e^{\mu(\lambda)\tau} \) where

\[
\mu(\lambda) = p_L(e^\lambda - 1) + p_R(e^{-\lambda} - 1) .
\]

(47)

In the case \( Q_\ell = 1 \) and \( Q_r = 0 \) let \( P \) be the rate at which walkers, which will end up into the right reservoir, are injected from the left reservoir. For general \( Q_\ell \) and \( Q_r \), because the walkers are independent and because they have no preferred direction, one has \( p_L = Q_\ell P \) and \( p_R = Q_r P \). Hence we get for the cumulant generating function of \( Q \)

\[
\mu(\lambda) = J \frac{Q_\ell(e^\lambda - 1) + Q_r(e^{-\lambda} - 1)}{Q_\ell - Q_r} ,
\]

(48)

where \( J = (Q_\ell - Q_r)P \). We can check that \( \mu'(\lambda = 0) \) gives the current \( J \). In Fig. 2, we check the validity of Eq. (48) from direct simulations of the open system. From Eq. (48), for the case \( Q_\ell = 1, Q_r = 0 \) we expect all cumulants \( \langle Q^n \rangle/\tau \) to be independent of \( n \) for sufficiently large \( \tau \), and we check this for different system sizes. The simulations are done by injecting

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FIG. 4: Monte-Carlo results for $\langle Q^2 \rangle_c / \langle Q \rangle$ and $\langle Q^3 \rangle_c / \langle Q \rangle$ (filled circles) as a function of measurement time $\tau$ for several system sizes with the parameters $\beta = 1.5$, $(Q_l, Q_r) = (1, 0)$ for the same model as in Figure 1. The data agree with the results of our theory (48). The inset shows the density profiles as produced from the Monte-Carlo simulations compared to the exact numerical solutions from Eqs. (25,26) [solid black lines].

Particles with flight time $t$ at rates $Q_l \psi(t)$, $Q_r \psi(t)$ from the left and right reservoirs [see discussion after Eqs. (23,24)]. These particles perform Levy walks till they exit from the system.
Not surprisingly we also note that the following fluctuation theorem symmetry relation [20] is satisfied:

\[
\mu(\lambda) = \mu[-\lambda - (\ln Q_l - \ln Q_r)].
\]  

(49)

**Additivity principle:** An interesting observation is that the generating function of the integrated current \(\mu(\lambda)\) [given by Eq. (19) in main text] matches exactly with the formula obtained from the additivity principle (AP) [21] which gives an expression for \(\mu_{AP}(\lambda)\) in terms of the the conductivity \(D\) and equilibrium current fluctuations \(\sigma\) defined respectively as

\[
D(Q) = \lim_{\Delta Q \to 0} L J/\Delta Q,
\]

\[
\sigma(Q) = L \lim_{t \to \infty} \langle Q^2 \rangle / t.
\]

The expression for \(\mu(\lambda)\) from AP is

\[
\mu_{AP}(\lambda) = -\frac{K}{L} \left[ \int_{Q_l}^{Q_r} dQ \frac{D(Q)}{\sqrt{1 + 2 K \sigma(Q)}} \right]^2,
\]

with \(\lambda = \int_{Q_l}^{Q_r} dQ \frac{D(Q)}{\sigma(Q)} \left[ \frac{1}{\sqrt{1 + 2 K \sigma(Q)}} - 1 \right].
\]

(51)

(this is a parametric expression: as \(K\) varies, \(\mu\) and \(\lambda\) vary). From our exact results for \(\mu(\lambda)\) we find \(D = L p\) and \(\sigma = L \mu''(\lambda = 0) = 2 D Q\). Using these in Eq. (51) and after explicitly performing the integrals we find \(\mu_{AP}(\lambda) = \mu(\lambda)\). This result is somewhat surprising since the additivity principle is expected normally to hold for diffusive systems (here \(D\) and \(\sigma\) have a \(L\)-dependence, whereas in usual diffusive systems they don’t). We note that somewhat similar behavior has been observed in deterministic models where anomalous transport satisfies the AP [22].

**V. CURRENT FLUCTUATIONS IN RING GEOMETRY**

In the ring geometry, the system consists of a fixed number \(N\) of particles which perform independent Levy walks on a ring of length \(L\). In the steady state, the density of particles is uniform. As the walkers are independent, the cumulants of the integrated current \(Q\) are related to those of the displacement \(x(t)\) of a single walker on the infinite line (in the steady
state). The number of times that a single walker crosses a fixed point in the ring geometry is approximately $x(t)/L$. Hence in the steady state where the distribution of particle is uniform, the $n$th order of cumulant is given by

$$\langle Q^n \rangle_c \sim \frac{N}{L^n} \langle x(t)^n \rangle_c = \frac{\rho}{L^{n-1}} \cdot \langle x(t)^n \rangle_c$$  \hspace{1cm} (52)

where $\rho$ is the density on the ring. If the walkers perform on the ring the same Levy walks as on the infinite line, the cumulants of $x(t)$ and therefore those of $Q$ grow, as in \cite{12,13}, faster than linearly with time (same exponent but a different prefactor as, on the ring, the walker is in its steady state rather than starting a flight at $t = 0$).

We now introduce the modified distribution $\phi(t)$ with a cut-off time $t_L \sim L^\delta$. This cutoff can be introduced on physical grounds. For example some possible cut-offs that can be argued are —

(a) as for the open geometry, the length of the flights cannot exceed the system size and therefore $\delta = 1$.

(b) Alternatively $t_L$ should be of the order of $t^*$, the typical relaxation time in the system. From the form $P(k,t) = e^{-c \cos(\beta \pi/2) |k| \beta t}$, valid for $vk << 1/t$, we can pull out a relaxation time for the shortest wave number on the ring $k = 2\pi/L$, giving $t^* \sim k^{-\beta}$ and this would give $\delta = \beta$.

(c) Finally from the mean square displacement $\langle x^2 \rangle \sim t^{3-\beta}$ we obtain a relaxation time $t^* = L^{2/(3-\beta)}$ and this gives $\delta = 2/(3 - \beta)$.

With such a cut-off $\tau_L$, the moments of the flight times $\langle t^n \rangle$ would be finite and the motion would be diffusive. On the infinite line, the propagator can then be expanded as

$$\tilde{P}(k, s) = \frac{\sum_{n=1}^{\infty} (-1)^n \langle t^n \rangle [(s - ikv)^n - (s + ikv)^n]}{\sum_{n=1}^{\infty} (-1)^n \langle t^n \rangle [(s - ikv)^n + (s + ikv)^n]} \cdot$$ \hspace{1cm} (53)

The Laplace transforms of various moments can be computed and are given by

$$\langle x^2 \rangle(s) \approx \frac{v^2 \langle t^2 \rangle}{\langle t \rangle^2} \frac{1}{s^2} + \frac{v^2(3\langle t^2 \rangle - 4\langle t \rangle \langle t^3 \rangle)}{6\langle t \rangle^2} \frac{1}{s},$$

$$\langle x^4 \rangle(s) \approx \frac{6v^4 \langle t^4 \rangle^2}{\langle t^2 \rangle^2} \frac{1}{s^4} + \frac{v^4(6\langle t^2 \rangle^3 - 10\langle t \rangle \langle t^2 \rangle \langle t^3 \rangle + \langle t^2 \rangle^2 \langle t^4 \rangle)}{\langle t^3 \rangle^2} \frac{1}{s^2}.$$

One would then get for the first two cumulants of $x$ at large $t$ \cite{14}

$$\frac{\langle x^2 \rangle_c}{v^2 t} \sim \frac{\langle t^2 \rangle}{\langle t \rangle}, \quad \frac{\langle x^4 \rangle_c}{v^4 t} \sim \frac{\langle t^4 \rangle}{\langle t \rangle} - 6 \frac{\langle t^2 \rangle \langle t^3 \rangle}{\langle t \rangle^2} + 3 \frac{\langle t^2 \rangle^3}{\langle t \rangle^3}.$$ \hspace{1cm} (54)
It becomes increasingly cumbersome to evaluate higher moments. For the sixth order cumulant we obtain

$$\frac{\langle x^6 \rangle_c}{v^6 t} = \frac{\langle t^6 \rangle}{\langle t \rangle} - 15 \frac{\langle t^2 \rangle^2}{\langle t \rangle^2} - 15 \frac{\langle t^3 \rangle \langle t^4 \rangle}{\langle t \rangle^3} + 60 \frac{\langle t^2 \rangle \langle t^4 \rangle}{\langle t \rangle^3} + 90 \frac{\langle t^2 \rangle^2 \langle t^3 \rangle^2}{\langle t \rangle^5} - 150 \frac{\langle t^2 \rangle^3 \langle t^3 \rangle}{\langle t \rangle^4} + 45 \frac{\langle t^2 \rangle^5}{\langle t \rangle^5}.$$  

We now use these in Eq. (52) to obtain current fluctuations in the ring geometry. We immediately see that the moments of $Q$ will now grow linearly in time. With the cut-off $t_L \sim L^\delta$, we have $\langle t^n \rangle \sim L^{(2-\beta)\delta}$ and hence we get

$$\frac{\langle Q^2 \rangle_c}{t} \sim L^{(2-\beta)\delta-1}, \quad (55)$$

$$\frac{\langle Q^4 \rangle_c}{t} \sim L^{(4-\beta)\delta-3}, \quad (56)$$

$$\frac{\langle Q^6 \rangle_c}{t} \sim L^{(6-\beta)\delta-5}, \quad (57)$$

and in general we conjecture $\langle Q^{2n} \rangle_c/t \sim L^{(2n-\beta)\delta-(2n-1)}$.

In one-dimensional mechanical models such as hard-point gas and anharmonic chains, heat is mediated by phonons which are weakly scattered. One can then think of these as performing Levy walks and indeed this picture is consistent with simulation data on energy diffusion\[10,13\]. Here we now see that the cut-off time $\tau_L$ also gives a possible explanation for the behavior seen in simulations on the ring of hard-point alternate gas of\[23\], where the cumulants grow linearly in time with different system size dependence. There we find that $\langle Q^2 \rangle_c/t \sim L^{-0.5}$ and $\langle Q^4 \rangle_c/t \sim L^{0.5}$. From this one gets from Eqs.(55) and (56) $\beta \sim 5/3$ and $\delta \sim 3/2$.

which leads through (39) to a value $\alpha = 1/3$ for the anomalous Fourier’s law of the hard-point alternate gas in the open geometry consistent with most of the simulations done so far\[9,23\] for this system. Furthermore, Eq.(57) predicts

$$\frac{\langle Q^4 \rangle_c}{t} \sim L^{1.5}. \quad (58)$$

Also we note that the value $\delta = 3/2$ is consistent to the mechanism (c) discussed above which gives $\delta = 2/(3 - \beta)$, since for this system $\beta = 2 - \alpha = 5/3$. 


VI. DISCUSSION

The Levy walk model is a natural extension of the ordinary random walk model where we now allow the walker to move in randomly chosen directions for long time intervals. Several studies suggest that this is could be a good model to describe the motion of phonons in low-dimensional systems and photons in disordered medium. In this work we have studied steady state transport by non-interacting Levy walkers. We have computed the average current, the density profile and the large deviation function of the integrated current, in the open geometry when the system is connected at its two ends to reservoirs. Current fluctuations on a ring geometry is also studied where we argue that it is natural to study a Levy walk with a system-size cut-off for the flight time distribution. High-dimensional analysis might be relevant for analyzing Levy transport of light through random medium which has been called a Levy glass [5].

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