Cesàro Convergent Sequences in the Mackey Topology

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Abstract. A Banach space $X$ is said to have property $(\mu^s)$ if every weak*-null sequence in $X^*$ admits a subsequence, such that all of its subsequences are Cesàro convergent to 0 with respect to the Mackey topology. This is stronger than the so-called property (K) of Kwapień. We prove that property $(\mu^s)$ holds for every subspace of a Banach space which is strongly generated by an operator with Banach–Saks adjoint (e.g., a strongly super weakly compactly generated space). The stability of property $(\mu^s)$ under $\ell^p$-sums is discussed. For a family $\mathcal{A}$ of relatively weakly compact subsets of $X$, we consider the weaker property $(\mu^s_{\mathcal{A}})$ which only requires uniform convergence on the elements of $\mathcal{A}$, and we give some applications to Banach lattices and Lebesgue–Bochner spaces. We show that every Banach lattice with order continuous norm and weak unit has property $(\mu^s_{\mathcal{A}})$ for the family of all $L$-weakly compact sets. This sharpens a result of de Pagter, Dodds, and Sukochev. On the other hand, we prove that $L^1(\nu, X)$ (for a finite measure $\nu$) has property $(\mu^s_{\mathcal{A}})$ for the family of all $\delta S$-sets whenever $X$ is a subspace of a strongly super weakly compactly generated space.

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1. Introduction

A subset $C$ of a Banach space $X$ is said to be Banach–Saks if every sequence $(x_n)_n$ in $C$ admits a Cesàro convergent subsequence $(x_{n_j})_j$, i.e., the sequence of arithmetic means $(\frac{1}{k} \sum_{j=1}^{k} x_{n_j})_k$ is convergent (in the norm topology) to some element of $X$. A Banach space $X$ is said to have the Banach–Saks property if its closed unit ball $B_X$ is a Banach–Saks set. An operator $T : Y \to$
$X$ between Banach spaces is said to be Banach–Saks if so is $T(B_Y)$. Every Banach–Saks set is relatively weakly compact (see, e.g., [28, Proposition 2.3]), and so, every space having the Banach–Saks property is reflexive [31], and every Banach–Saks operator is weakly compact. The converse statements are not true in general [2]. Every super-reflexive space (like $L^p(\nu)$ for a non-negative measure $\nu$ and $1 < p < \infty$) has the Banach–Saks property (see, e.g., [12, p. 124]). For any non-negative measure $\nu$, the space $L^1(\nu)$ enjoys the weak Banach–Saks property; that is, every weakly compact subset of $L^1(\nu)$ is Banach–Saks, a result due to Szlenk [40] (cf. [12, p. 112]). At this point, it is convenient to recall the Erdős–Magidor theorem [15] (cf. [28, Corollary 2.6] and [33, Theorem 2.1]) which implies, in particular, that every sequence in a Banach–Saks set admits a subsequence, such that all of its subsequences are Cesàro convergent to the same limit:

**Theorem 1.1** (Erdős–Magidor). Every bounded sequence $(x_n)_n$ in a Banach space $X$ admits a subsequence $(x_{n_j})_j$, such that

(i) either all subsequences of $(x_{n_j})_j$ are Cesàro convergent (to the same limit);

(ii) or no subsequence of $(x_{n_j})_j$ is Cesàro convergent.

As we will see, for a finite measure $\nu$, the weak Banach–Saks property of $L^1(\nu)$ yields a somehow similar property for its dual $L^1(\nu)^* = L^\infty(\nu)$ by considering the $w^*$-topology and the Mackey topology, namely: every $w^*$-null sequence in $L^\infty(\nu)$ admits a subsequence such that all of its subsequences are Cesàro convergent to 0 with respect to $\mu(L^\infty(\nu), L^1(\nu))$. Recall that, for an arbitrary Banach space $X$, the Mackey topology $\mu(X^*, X)$ is the (locally convex) topology on $X^*$ of uniform convergence on all weakly compact subsets of $X$. Therefore, for any finite measure $\nu$, the space $L^1(\nu)$ satisfies the following property which is the main object of study of this paper:

**Definition 1.2.** A Banach space $X$ is said to have property ($\mu^*$) if every $w^*$-null sequence in $X^*$ admits a subsequence, such that all of its subsequences are Cesàro convergent to 0 with respect to $\mu(X^*, X)$.

The paper is organized as follows. In the preliminary section (Sect. 2), we point out that property ($\mu^*$) is stronger than the so-called property (K) invented by Kwapień in connection with some results of Kalton and Pełczyński [24]:

**Definition 1.3.** A Banach space $X$ is said to have property (K) if every $w^*$-null sequence in $X^*$ admits a convex block subsequence which converges to 0 with respect to $\mu(X^*, X)$.

Property (K) (and some variants) have been also studied by Frankiewicz and Plebanek [19], Figiel, Johnson and Pełczyński [17], de Pagter, Dodds and Sukochev [10], and Avilés and the author [1]. In Sect. 2, we also give some basic examples of Banach spaces having property ($\mu^*$). For a reflexive space $X$, ($\mu^*$) is equivalent to the Banach–Saks property of $X^*$ (Proposition 2.2). In particular, any super-reflexive space has ($\mu^*$). For a $C(L)$ space (where $L$ is
a compact Hausdorff topological space), \((\mu^s)\) is equivalent to the fact that \(C(L)\) is Grothendieck (Proposition 2.4). Therefore, for instance, \(\ell_\infty\) has property \((\mu^s)\).

In Sect. 3, we discuss the role of “strong generation” in the study of property \((\mu^s)\). To be more precise, we need some terminology:

**Definition 1.4.** Let \(X\) be a Banach space, and let \(H\) and \(G\) be two families of subsets of \(X\). We say that \(H\) is **strongly generated** by \(G\) if, for every \(H \in H\) and every \(\varepsilon > 0\), there is \(G \in G\), such that \(H \subseteq G + \varepsilon B_X\). If, in addition, \(G = \{nG_0 : n \in \mathbb{N}\}\) for some \(G_0 \subseteq X\), we simply say that \(H\) is **strongly generated** by \(G_0\).

We will be mainly interested in the case \(H = \text{wk}(X)\), the family of all weakly compact subsets of the Banach space \(X\).

**Definition 1.5.** Let \(X\) be a Banach space and let \(G\) be a family of subsets of \(X\). We say that \(X\) is **strongly generated by** \(G\) if \(\text{wk}(X)\) is strongly generated by \(G\). If in addition \(G = \{nT(B_Y) : n \in \mathbb{N}\}\) for some operator \(T : Y \to X\) from a Banach space \(Y\), we say that \(Y\) **strongly generates** \(X\) or that \(T\) strongly **generates** \(X\).

Banach spaces which are strongly generated by a reflexive space (i.e., SWCG spaces) or by a super-reflexive space have been widely studied (see, e.g., [16,25,29] and the references therein). All SWCG spaces and their subspaces have property (K), see [1, Corollary 2.3]. We show that property \((\mu^s)\) is enjoyed by every subspace of a Banach space which is strongly generated by an operator with Banach–Saks adjoint (Theorem 3.1). This assumption is satisfied by the so-called strongly super weakly compactly generated spaces \((S^2\text{WCG})\) studied recently by Raja [35] and Cheng et al. [8]. In particular, any Banach space which is strongly generated by a super-reflexive space (e.g., \(L^1(\nu)\) for a finite measure \(\nu\)) has property \((\mu^s)\).

We prove that an SWCG space \(X\) has property \((\mu^s)\) if (and only if) every \(w^*\)-null sequence in \(X^*\) admits a subsequence which is Cesàro convergent to 0 with respect to \(\mu(X^*,X)\). We do not know whether such equivalence holds for arbitrary Banach spaces. The case of SWCG spaces is generalized to Banach spaces which are strongly generated by less than \(p\) weakly compact sets (Theorem 3.7). Recall that \(p\) is the least cardinality of a family \(\mathcal{M}\) of infinite subsets of \(\mathbb{N}\), such that:

- \(\bigcap \mathcal{N}\) is infinite for every finite subfamily \(\mathcal{N} \subseteq \mathcal{M}\).
- There is no infinite set \(A \subseteq \mathbb{N}\), such that \(A \setminus M\) is finite for all \(M \in \mathcal{M}\).

In general, \(\omega_1 \leq p \leq \aleph_0\). Under CH cardinality less than \(p\) just means countable, but in other models, there are uncountable sets of cardinality less than \(p\) (see, e.g., [5] for more information).

In Sect. 4, we study the stability of property \((\mu^s)\) under \(\ell^p\)-sums for \(1 \leq p \leq \infty\). Pełczyński showed that the \(\ell^1\)-sum of \(c\) copies of \(L^1[0,1]\) fails property (K), see [17, Example 4.1] (cf. [19]). In particular, this implies that property \((\mu^s)\) is not preserved by arbitrary \(\ell^1\)-sums. We prove that \((\mu^s)\) is preserved by \(\ell^1\)-sums of less than \(p\) summands (Theorem 4.2), as well as by
arbitrary $\ell^p$-sums whenever $1 < p < \infty$ (Theorem 4.4). On the other hand, in Example 4.6, we point out the existence of a sequence of finite-dimensional spaces whose $\ell^\infty$-sum fails property (K), which answers a question left open in [1, Problem 2.19].

In Sect. 5, we consider a natural weakening of properties ($\mu^*$) and (K) to deal with certain families of relatively weakly compact sets. This idea is applied to some Banach lattices and Lebesgue–Bochner spaces.

**Definition 1.6.** Let $X$ be a Banach space and let $\mathcal{A}$ be a family of subsets of $X$. We say that

(i) a sequence $(x^*_j)_j$ in $X^*$ is Cesàro convergent to 0 uniformly on each element of $\mathcal{A}$ if

$$\lim_{k \to \infty} \sup_{x \in \mathcal{A}} \left| \frac{1}{k} \sum_{j=1}^{k} x^*_j(x) \right| = 0 \quad \text{for every } A \in \mathcal{A};$$

(ii) $X$ has property ($\mu^*_\mathcal{A}$) if every $w^*$-null sequence in $X^*$ admits a subsequence, such that all of its subsequences are Cesàro convergent to 0 uniformly on each element of $\mathcal{A}$;

(iii) $X$ has property (K$\mathcal{A}$) if every $w^*$-null sequence in $X^*$ admits a convex block subsequence which converges to 0 uniformly on each element of $\mathcal{A}$.

For instance, the so-called property ($k$) of Figiel, Johnson and Pelczynski [17] coincides with property (K$\mathcal{A}$) when $\mathcal{A}$ is the family

$$\{ T(C) : T : L^1([0,1]) \to X \text{ operator}, C \in \text{wk}(L^1([0,1])) \},$$

see [10, Lemma 8.1]. Every weakly sequentially complete Banach lattice with weak unit has property ($k$), see [17, Proposition 4.5(b)]. This can also be obtained as a consequence of a result of de Pagter, Dodds, and Sukochev (see [10, Theorem 5.3]) stating that every Banach lattice $X$ with order continuous norm and weak unit has property (K$\mathcal{A}$) when $\mathcal{A}$ is the family of all order bounded subsets of $X$. We sharpen those results by proving that, in fact, such Banach lattices have property ($\mu^*_\mathcal{A}$) when $\mathcal{A}$ is the family described in (1.1) or the family of all $L$-weakly compact sets, respectively (Corollary 5.2 and Theorem 5.1).

Finally, we focus on the Lebesgue–Bochner space $L^1(\nu, X)$, where $\nu$ is a finite measure and $X$ is a Banach space. It is known that if $X$ contains a subspace isomorphic to $c_0$, then $L^1([0,1], X)$ contains a complemented subspace isomorphic to $c_0$, see [14]. When applied to the space $\ell^\infty$, this shows that properties ($\mu^*$) and (K) do not pass from $X$ to $L^1([0,1], X)$ in general (cf. [17, Remark 6.5]). In fact, in Theorem 5.6, we prove that $L^1([0,1], X)$ fails property (K$\mathcal{A}$), for the family $\mathcal{A}$ of all $\delta\mathcal{S}$-sets of $L^1([0,1], X)$, whenever $X$ contains a subspace isomorphic to $c_0$.

**Definition 1.7.** A set $K \subseteq L^1(\nu, X)$ is said to be a $\delta\mathcal{S}$-set if it is uniformly integrable and, for every $\delta > 0$, there exists a weakly compact set $W \subseteq X$, such that $\nu(f^{-1}(W)) \geq 1 - \delta$ for every $f \in K$. 

The collection of all $\delta S$-sets of $L^1(\nu, X)$ will be denoted by $\delta S(\nu, X)$ or simply $\delta S$ if no confusion arises. These sets play an important role when studying weak compactness in Lebesgue–Bochner spaces. Any $\delta S$-set of $L^1(\nu, X)$ is relatively weakly compact, while the converse is not true in general. For more information on these sets, see [37] and the references therein.

Concerning positive results, we show that $L^1(\nu, X)$ has property $(\mu_s^\delta S)$ whenever $X$ is a subspace of a $S^2WCG$ space (Theorem 5.8). In general, the assumption that $X^*$ has the Banach–Saks property is not enough to ensure that $L^1(\nu, X)$ has property $(\mu_s^\delta S)$ (Example 5.9).

2. Notation and Preliminaries

The symbol $|S|$ stands for the cardinality of a set $S$. All our vector spaces are real. Given a sequence $(f_n)_n$ in a vector space, a convex block subsequence of $(f_n)_n$ is a sequence $(g_k)_k$ of the form:

$$g_k = \sum_{n \in I_k} a_n f_n,$$

where $(I_k)_k$ is a sequence of finite subsets of $\mathbb{N}$ with $\max(I_k) < \min(I_{k+1})$ and $(a_n)_n$ is a sequence of non-negative real numbers, such that $\sum_{n \in I_k} a_n = 1$ for all $k \in \mathbb{N}$. An operator is a continuous linear map between Banach spaces. By a subspace of a Banach space, we mean a closed linear subspace. Given a Banach space $X$, its norm is denoted by either $\|\cdot\|_X$ or simply $\|\cdot\|$, and we write $B_X = \{x \in X : \|x\| \leq 1\}$. The topological dual of $X$ is denoted by $X^*$ and the adjoint of an operator $T$ is denoted by $T^*$. The evaluation of $x^* \in X^*$ at $x \in X$ is denoted by either $x^*(x)$ or $\langle x^*, x \rangle$. The weak (resp. weak*) topology on $X$ (resp. $X^*$) is denoted by $w$ (resp. $w^*$).

Lemma 2.1. Let $X$ be a Banach space and let $A$ be a family of subsets of $X$. If $X$ has property $(\mu_s^A)$, then it also has property $(K_A)$.

Proof. Bear in mind that if $(u_n)_n$ is a sequence in a topological vector space which is Cesàro convergent to 0, then it admits a convex block subsequence converging to 0. Indeed, define

$$v_k := \frac{1}{2^k} \sum_{n=1}^{2^k} u_n \quad \text{and} \quad w_k := \frac{1}{2^{k-1}} \sum_{n=2^{k-1}+1}^{2^k} u_n$$

for every $k \in \mathbb{N}$. Then, $(w_k)_k$ is a convex block subsequence of $(u_n)_n$ converging to 0, because $(v_k)_k$ converges to 0 and $v_k = \frac{1}{2}(v_{k-1} + w_k)$ for all $k \geq 2$.

In particular, property $(\mu^s)$ implies property $(K)$. The converse is not true in general, see Remark 2.3 below.

Proposition 2.2. Let $X$ be a Banach space. The following statements are equivalent:

(i) $X^*$ has the Banach–Saks property.
(ii) \( X \) is reflexive and has property \((\mu^s)\).

(iii) \( X \) contains no subspace isomorphic to \( \ell^1 \) and has property \((\mu^s)\).

**Proof.** (i) \( \Rightarrow \) (ii): Clearly, the Banach–Saks property of \( X^* \) implies that \( X \) has property \((\mu^s)\). On the other hand, as we mentioned in the introduction, every space with the Banach–Saks property is reflexive.

The implication (ii)\( \Rightarrow \) (iii) is obvious, while (iii) \( \Rightarrow \) (ii) follows from Lemma 2.1 and the fact that any Banach space with property (K) without subspaces isomorphic to \( \ell^1 \) is reflexive, see [1, Theorem 2.1].

Finally, (ii) \( \Rightarrow \) (i) follows from the fact that, if \( X \) is reflexive, then \( \mu(X^*, X) \) agrees with the norm topology of \( X^* \) and \( B_{X^*} \) is \( w^* \)-sequentially compact. \( \square \)

**Remark 2.3.** Typical examples of Banach spaces having property (K) are SWCG spaces and Grothendieck spaces (i.e., spaces for which every \( w^* \)-convergent sequence in the dual is weakly convergent). In particular, reflexive spaces have property (K). However, there are reflexive spaces \( Y \) which fail the Banach–Saks property (see [2]); by Proposition 2.2, the (reflexive) space \( X = Y^* \) fails property \((\mu^s)\).

**Proposition 2.4.** Let \( L \) be a compact Hausdorff topological space. The following statements are equivalent:

(i) \( C(L) \) is Grothendieck.

(ii) \( C(L) \) has property \((\mu^s)\).

**Proof.** \( C(L)^* \) is isomorphic (in fact, order isometric) to the \( L^1 \)-space of a non-negative measure, so it has the weak Banach–Saks property. The implication (i) \( \Rightarrow \) (ii) follows at once from this.

(ii) \( \Rightarrow \) (i): Apply Lemma 2.1 and the fact that \( C(L) \) is Grothendieck if (and only if) it has property (K), see [1, Corollary 2.5]. \( \square \)

A subspace \( Y \) of a Banach space \( X \) is said to be \( w^* \)-extensible in \( X \) if every \( w^* \)-null sequence in \( Y^* \) admits a subsequence which can be extended to a \( w^* \)-null sequence in \( X^* \). It is easy to check that: (i) any complemented subspace is \( w^* \)-extensible; and (ii) if \( B_{X^*} \) is \( w^* \)-sequentially compact, then any subspace is \( w^* \)-extensible in \( X \) (see [41, Theorem 2.1]). The following is straightforward:

**Remark 2.5.** Let \( X \) be a Banach space and let \( Y \subseteq X \) be a subspace which is \( w^* \)-extensible in \( X \). If \( X \) has property \((\mu^s_A)\), then \( Y \) has property \((\mu^s_A)\), as well.

In general, the statement of Remark 2.5 is not true for arbitrary subspaces. For instance, \( \ell^\infty \) has property \((\mu^s)\) (by Proposition 2.4 and the fact that \( \ell^\infty \) is Grothendieck), but \( c_0 \) does not (since it fails (K), see, e.g., [24, Proposition C]).

**3. Strong Generation and Property \((\mu^s)\)**

Our first result in this section provides a sufficient condition for property \((\mu^s)\).
**Theorem 3.1.** Let $X$ be a Banach space which is strongly generated by an operator with Banach–Saks adjoint. Then, every subspace of $X$ has property $(\mu^*)$.

The proof of Theorem 3.1 requires two lemmas which will be used again later.

**Lemma 3.2.** Let $X$ be a Banach space. Let $\mathcal{H}$ and $\mathcal{G}$ be two families of subsets of $X$, such that $\mathcal{H}$ is strongly generated by $\mathcal{G}$.

(i) If $(x_n^*)$ is a bounded sequence in $X^*$ converging to 0 uniformly on each element of $\mathcal{G}$, then it also converges to 0 uniformly on each element of $\mathcal{H}$.

(ii) If $X$ has property $(\mu^*_G)$, then it also has property $(\mu^*_H)$.

**Proof.** (ii) is an immediate consequence of (i) applied to the corresponding sequences of Cesàro means. For the proof of (i), let $c > 0$ be a constant, such that $\|x_n^*\|_{X^*} \leq c$ for all $n \in \mathbb{N}$. Fix $H \in \mathcal{H}$ and take any $\varepsilon > 0$. Pick $G \in \mathcal{G}$, such that $H \subseteq G + \varepsilon B_X$, and choose $n_0 \in \mathbb{N}$, such that $\sup_{x \in G} |x_n^*(x)| \leq \varepsilon$ for all $n \geq n_0$. Then, $\sup_{x \in H} |x_n^*(x)| \leq (1 + c)\varepsilon$ for all $n \geq n_0$. 

**Lemma 3.3.** Let $X$ and $Y$ be Banach spaces and let $T : Y \to X$ be an operator such that $T^*$ is Banach–Saks. Then, $X$ has property $(\mu^*_{(T(B_Y))})$.

**Proof.** Let $(x_n^*)$ be a $w^*$-null sequence in $X^*$. Since $T^* : X^* \to Y^*$ is Banach–Saks and $w^*$-$w^*$-continuous, there is a subsequence of $(x_n^*)$, not relabeled, such that for every further subsequence $(x_{n_k}^*)$, we have that $(T^*(x_{n_k}^*))$ is Cesàro convergent to 0 in norm; that is:

$$\lim_{N \to \infty} \sup_{y \in B_Y} \left\| \frac{1}{N} \sum_{k=1}^{N} x_{n_k}^*, T(y) \right\| = \lim_{N \to \infty} \left\| \frac{1}{N} \sum_{k=1}^{N} T^*(x_{n_k}^*) \right\|_{Y^*} = 0.$$  

This shows that $X$ has property $(\mu^*_{(T(B_Y))})$. 

**Proof of Theorem 3.1.** Since any Banach–Saks operator is weakly compact, the space $X$ is SWCG. In particular, $X$ is weakly compactly generated, and so, $B_{X^*}$ is $w^*$-sequentially compact (see, e.g., [12, p. 228, Theorem 4]). Therefore, every subspace is $w^*$-extensible in $X$ and so it suffices to prove that $X$ has property $(\mu^*)$ (see Remark 2.5 and the paragraph preceding it). To this end, let $Y$ be a Banach space and let $T : Y \to X$ be an operator, such that $T^*$ is Banach–Saks and $wk(X)$ is strongly generated by $T(B_Y)$. By Lemmas 3.3 and 3.2(ii), $X$ has property $(\mu^*)$. 

An operator between Banach spaces $T : Y \to X$ is said to be super weakly compact if the ultrapower $T^U : Y^U \to X^U$ is weakly compact for every free ultrafilter $U$ on $\mathbb{N}$. This is equivalent to being uniformly convexifying in the sense of Beauzamy [3] (see, e.g., [21, Theorem 5.1]). A Banach space $X$ is said to be strongly super weakly compactly generated ($S^2$WCG) if it is strongly generated by a super weakly compact operator (see [35]). In general:

- strongly generated by a super-reflexive space \( \Rightarrow \) $S^2$WCG \( \Rightarrow \) strongly generated by an operator with Banach–Saks adjoint.
The first implication is clear. The second one holds, because an operator is super weakly compact if and only if its adjoint is super weakly compact (see [3, Proposition II.4]) and any super weakly compact operator is Banach–Saks (see [4, Théorème 3]). An example of a $S^2$WCG space which is not strongly generated by a super-reflexive space can be found in [35, Example 3.10]. The converse of the second implication above is neither true in general, even for reflexive spaces. Indeed, there are spaces with the Banach–Saks property which are not super-reflexive (see [31]), while every $S^2$WCG reflexive space is super-reflexive (cf. [35, Theorem 1.9]).

**Corollary 3.4.** Every subspace of a $S^2$WCG Banach space has property $(\mu^*)$.

**Corollary 3.5.** Let $\nu$ be a finite measure. Then, every subspace of $L^1(\nu)$ has property $(\mu^*)$.

**Remark 3.6.** The previous results are formulated in terms of “subspaces”, because the corresponding classes of Banach spaces are not hereditary. Indeed, Mercourakis and Stamati (see [29, Theorem 3.9(ii)]) showed that there exist subspaces of $L^1[0, 1]$ which are not SWCG.

The Erdős–Magidor Theorem 1.1 is also valid for Fréchet spaces and, under certain set theoretic assumptions, for other classes of locally convex spaces, see [33]. Along this way, we have the following:

**Theorem 3.7.** Let $X$ be a Banach space which is strongly generated by a family $\mathcal{G} \subseteq wk(X)$ with $|\mathcal{G}| < p$ (e.g., a SWCG space). Then, $X$ has property $(\mu^*)$ if (and only if) every $w^*$-null sequence in $X^*$ admits a subsequence which is Cesàro convergent to 0 with respect to $\mu(X^*, X)$.

To deal with the proof of Theorem 3.7, we need two lemmas. The first one is a standard diagonalization argument.

**Lemma 3.8.** Let $\{S_\alpha\}_{\alpha<\gamma}$ be a collection of families of infinite subsets of $\mathbb{N}$, where $\gamma$ is an ordinal with $\gamma < p$. Suppose that, for each $\alpha < \gamma$, we have the following:

(i) if $B \subseteq \mathbb{N}$ is infinite and $B \setminus A$ is finite for some $A \in S_\alpha$, then $B \in S_\alpha$;
(ii) every infinite subset of $\mathbb{N}$ contains an element of $S_\alpha$.

Then, every infinite subset of $\mathbb{N}$ contains an element of $\bigcap_{\alpha<\gamma} S_\alpha$.

**Proof.** Fix an infinite set $B \subseteq \mathbb{N}$. We will first construct by transfinite induction a collection $\{B_\alpha : \alpha < \gamma\}$ of subsets of $B$ with the following properties:

$(P_\alpha)$ $B_\alpha \in S_\alpha$ for all $\alpha < \gamma$;
$(Q_{\alpha, \beta})$ $B_\beta \setminus B_\alpha$ is finite whenever $\alpha \leq \beta < \gamma$.

For $\alpha = 0$, we just use (ii) to select any subset $B_0$ of $B$ belonging to $S_0$. Suppose now that $1 \leq \gamma' < \gamma$ and that we have already constructed a collection $\{B_\alpha : \alpha < \gamma'\}$ of subsets of $B$, such that $(P_\alpha)$ and $(Q_{\alpha, \beta})$ hold whenever $\alpha \leq \beta < \gamma'$. In particular, for any finite set $I \subseteq \gamma'$, the intersection $\bigcap_{\alpha \in I} B_\alpha$ is infinite. Since $\gamma' < p$, there is an infinite set $B_{\gamma'} \subseteq B$, such that $B_{\gamma'} \setminus B_\alpha$
is finite for all $\alpha < \gamma'$. Property (i) implies that $B_{\gamma'} \in \bigcap_{\alpha<\gamma'} \mathcal{S}_\alpha$. Now property (ii) implies that, by passing to a further subset of $B_{\gamma'}$, if necessary, we can assume that $(P_{\gamma'})$ holds. Clearly, $(Q_{\alpha,\beta})$ also holds for every $\alpha \leq \beta \leq \gamma'$. This finishes the inductive construction.

Since $\gamma < \mathfrak{p}$ and for any finite set $I \subseteq \gamma$ the intersection $\bigcap_{\alpha \in I} B_\alpha$ is infinite, there is an infinite set $C \subseteq B$, such that $C \setminus B_\alpha$ is finite for every $\alpha < \gamma$. From (i), it follows that $C \in \bigcap_{\alpha < \gamma} S_\alpha$. \hfill $\square$

The second lemma will also be used in Sect. 4.

**Lemma 3.9.** Let $\{E_i\}_{i \in I}$ be a family of topological vector spaces with $|I| < \mathfrak{p}$ and let $E := \prod_{i \in I} E_i$ be equipped with the product topology. For each $i \in I$, we denote by $\rho_i : E \to E_i$ the $i$th coordinate projection. Let $(u_n)_n$ be a sequence in $E$ satisfying the following condition:

\((\star)\) for every infinite set $A \subseteq \mathbb{N}$ and every $i \in I$ there is an infinite set $B \subseteq A$, such that the subsequence $(\rho_i(u_n))_{n \in C}$ is Cesàro convergent to 0 in $E_i$ for every infinite set $C \subseteq B$.

Then, there is a subsequence of $(u_n)_n$, such that all of its subsequences are Cesàro convergent to 0 in $E$.

**Proof.** We will apply Lemma 3.8. For each $i \in I$, let $\mathcal{S}_i$ be the family of all infinite sets $A \subseteq \mathbb{N}$, such that for every infinite set $C \subseteq A$, the corresponding subsequence $(\rho_i(u_n))_{n \in C}$ is Cesàro convergent to 0 in $E_i$. It suffices to check that conditions (i) and (ii) of Lemma 3.8 hold for this choice.

Indeed, (ii) follows immediately from (\(\star\)). On the other hand, fix $i \in I$, $A \in \mathcal{S}_i$ and an infinite set $B \subseteq \mathbb{N}$, such that $B \setminus A$ is finite. To check that $B \in \mathcal{S}_i$, take any strictly increasing sequence $(n_k)_k$ in $B$. There is $k_0 \in \mathbb{N}$, such that $n_k \in A$ for all $k > k_0$, and hence, $(\rho_i(u_{n_k}))_{k > k_0}$ is Cesàro convergent to 0 in $E_i$ and so is $(\rho_i(u_{n_k}))_k$. This shows that $B \in \mathcal{S}_i$. Thus, condition (i) of Lemma 3.8 is also satisfied. \hfill $\square$

**Proof of Theorem 3.7.** To prove that $X$ has property $(\mu^*)$, it suffices to check that it has property $(\mu^*_G)$ (Lemma 3.2(ii)). For each $G \in \mathcal{G}$, let $R_G : X^* \to C(G)$ be the operator given by $R_G(x^*) := x^*|_G$ (the restriction of $x^*$ to $G$).

Let $(x_n^*)_n$ be a $w^*$-null sequence in $X^*$. Fix an infinite set $A \subseteq \mathbb{N}$ and $G \in \mathcal{G}$. Since $(R_G(x_n^*))_{n \in A}$ is bounded, we can apply the Erdös–Magidor Theorem 1.1 to find an infinite set $B \subseteq A$, such that either all subsequences of $(R_G(x_n^*))_{n \in B}$ are Cesàro convergent (to the same limit), or no subsequence of $(R_G(x_n^*))_{n \in B}$ is Cesàro convergent. The assumption on $X$ excludes the second possibility and ensures that all subsequences of $(R_G(x_n^*))_{n \in B}$ are Cesàro convergent to 0.

We can now apply Lemma 3.9 to the family of Banach spaces $\{C(G)\}_{G \in \mathcal{G}}$ and the sequence $(u_n)_n$ in $\prod_{G \in \mathcal{G}} C(G)$ defined by $u_n := (R_G(x_n^*))_{G \in \mathcal{G}}$. Therefore, there is a subsequence of $(x_n^*)_n$, such that all of its subsequences are Cesàro convergent to 0 uniformly on each $G \in \mathcal{G}$. This proves that $X$ has property $(\mu^*_G)$.

$\square$
4. $\ell^p$-Sums

The $\ell^p$-sum (1 ≤ p ≤ $\infty$) of a family of Banach spaces $\{X_i\}_{i \in I}$ is denoted by

$$\left( \bigoplus_{i \in I} X_i \right)_{\ell^p}.$$ 

When $p \neq \infty$, we identify the dual of $(\bigoplus_{i \in I} X_i)_{\ell^p}$ with $(\bigoplus_{i \in I} X_i^*)_{\ell^q}$, where $q$ is the conjugate exponent of $p$, i.e., $1/p + 1/q = 1$, and for each $j \in I$, we denote by

$$\pi_j : \left( \bigoplus_{i \in I} X_i \right)_{\ell^p} \to X_j \quad \text{and} \quad \rho_j : \left( \bigoplus_{i \in I} X_i^* \right)_{\ell^q} \to X_j^*$$

the $j$th coordinate projections.

**Lemma 4.1.** Let $\{X_i\}_{i \in I}$ be a family of Banach spaces and $X := \left( \bigoplus_{i \in I} X_i \right)_{\ell^1}$. Let $(x_n^*)_n$ be a bounded sequence in $X^*$, such that for every $i \in I$ the sequence $(\rho_i(x_n^*))_n$ is $\mu(X_i^*, X_i)$-null. Then, $(x_n^*)_n$ is $\mu(X^*, X)$-null.

**Proof.** It is known that $wk(X)$ is strongly generated by the family $G$ consisting of all weakly compact subsets of $X$ of the form:

$$\bigcap_{i \in I} \pi_i^{-1}(W_i) \cap \bigcap_{i \in I \setminus J} \pi_i^{-1}(\{0\}),$$

where $J \subseteq I$ is finite and $W_i \in wk(X_i)$ for every $i \in I$ (see, e.g., [23, Lemma 7.2(ii)]). Clearly, $(x_n^*)_n$ converges to 0 uniformly on each element of $G$. From Lemma 3.2(i), it follows that $(x_n^*)_n$ converges to 0 with respect to $\mu(X^*, X)$.

**Theorem 4.2.** Let $\{X_i\}_{i \in I}$ be a family of Banach spaces having property $(\mu^*)$. If $|I| < p$, then $(\bigoplus_{i \in I} X_i)_{\ell^1}$ has property $(\mu^*)$.

**Proof.** Write $X := \left( \bigoplus_{i \in I} X_i \right)_{\ell^1}$. Let $(x_n^*)_n$ be a $w^*$-null sequence in $X^*$. Then, $(\rho_i(x_n^*))_n$ is $w^*$-null in $X_i^*$ for all $i \in I$. Since each $X_i$ has property $(\mu^*)$, we can apply Lemma 3.9 to the family of locally convex spaces $\{(X_i^*, \mu(X_i^*, X_i))\}_{i \in I}$ and the sequence $(u_n)_n$ in $\prod_{i \in I} X_i^*$ defined by $u_n := (\rho_i(x_n^*))_{i \in I}$. Thereafter, there is a subsequence of $(x_n^*)_n$, not relabeled, such that for every further subsequence $(x_{n_k}^*)_k$ and every $i \in I$, the sequence $(\rho_i(x_{n_k}^*))_k$ is Cesàro convergent to 0 with respect to $\mu(X_i^*, X_i)$. Now, apply Lemma 4.1 to the sequence of arithmetic means of any such subsequence $(x_{n_k}^*)_k$ to conclude that $(x_{n_k}^*)_k$ is Cesàro convergent to 0 with respect to $\mu(X^*, X)$.

The following lemma isolates an argument used in the proof of Theorem 2.15 in [1], which says that property (K) is preserved by arbitrary $\ell^p$-sums for $1 < p < \infty$. We will use it to prove the same statement for property $(\mu^*)$ (Theorem 4.4 below).

**Lemma 4.3.** Let $\{X_i\}_{i \in I}$ be a family of Banach spaces, let $1 < p < \infty$ and write $X := \left( \bigoplus_{i \in I} X_i \right)_{\ell^p}$. Let $(y_j^*)_j$ be a sequence in $X^*$, such that:

(i) for every $i \in I$, the sequence $(\rho_i(y_j^*))_j$ is $\mu(X_i^*, X_i)$-null;
(ii) there is a norm convergent sequence \((\tilde{\xi}_j)_j\) in \(\ell^q(I)\), such that
\[
\tilde{\xi}_j \geq \xi_j := (\|\rho_i(y_j^*)\|_{X_i^*})_{i \in I}
\]
pointwise in \(\ell^q(I)\) for all \(j \in \mathbb{N}\).

Then, \((y_j^*)_j\) is \(\mu(X^*, X)\)-null.

**Proof.** Fix any weakly compact set \(L \subseteq B_X\) and \(\varepsilon > 0\). Since \((\tilde{\xi}_j)_j\) is norm convergent in \(\ell^q(I)\), there exist a finite set \(I_0 \subseteq I\) and \(j_0 \in \mathbb{N}\), such that
\[
\sup_{j > j_0} \left( \sum_{i \in I_0} \psi_i(\tilde{\xi}_j)^q \right)^{1/q} \leq \varepsilon,
\]
where \(\psi_i \in \ell^q(I)^*\) denotes the \(i\)th coordinate functional. Bearing in mind that
\[
\|\rho_i(y_j^*)\|_{X_i^*} = \psi_i(\xi_j) \leq \psi_i(\tilde{\xi}_j)
\]
for every \(i \in I\) and \(j \in \mathbb{N}\), from (4.1), we get the following:
\[
\sup_{j > j_0} \left( \sum_{i \in I_0} \|\rho_i(y_j^*)\|_{X_i^*}^q \right)^{1/q} \leq \varepsilon.
\]
The previous inequality and Hölder’s one imply that
\[
\sup_{j > j_0} \sum_{i \in I_0} |b_i| \cdot \|\rho_i(y_j^*)\|_{X_i^*} \leq \varepsilon \quad \text{for every } (b_i)_{i \in I} \in B_{\ell^p(I)}.
\]

For each \(i \in I\), the sequence \((\rho_i(y_j^*)_j)_j\) converges to 0 uniformly on the weakly compact set \(\pi_i(L) \subseteq X_i\). Thus, we can find \(j_1 > j_0\), such that
\[
|\langle \rho_i(y_j^*), \pi_i(x) \rangle| \leq \frac{\varepsilon}{|I_0|} \quad \text{for every } j > j_1, i \in I_0 \text{ and } x \in L.
\]

Therefore, for every \(j > j_1\) and \(x \in L \subseteq B_X\), we have the following:
\[
|\langle y_j^*, x \rangle| \leq \sum_{i \in I} |\langle \rho_i(y_j^*), \pi_i(x) \rangle| \overset{(4.3)}{\leq} \varepsilon + \sum_{i \in I_0} |\langle \rho_i(y_j^*), \pi_i(x) \rangle| \leq \varepsilon + \sum_{i \in I_0} \|\pi_i(x)\|_{X_i^*} \cdot \|\rho_i(y_j^*)\|_{X_i^*} \overset{(4.2)}{\leq} 2\varepsilon.
\]
This shows that \((y_j^*)_j\) is \(\mu(X^*, X)\)-null. \(\square\)

**Theorem 4.4.** Let \(\{X_i\}_{i \in I}\) be a family of Banach spaces having property \((\mu^*)\) and let \(1 < p < \infty\). Then, \((\bigoplus_{i \in I} X_i)_{\ell^p}\) has property \((\mu^*)\).

**Proof.** Write \(X := (\bigoplus_{i \in I} X_i)_{\ell^p}\). Let \((x_n^*)_n\) be a \(w^*\)-null sequence in \(X^*\). Define the following:
\[
v_n := (\|\rho_i(x_n^*)\|_{X_i^*})_{i \in I} \in \ell^q(I) \quad \text{for all } n \in \mathbb{N},
\]
so that \(v_n\|_{\ell^q(I)} = \|x_n^*\|_{X^*}\). Since \((v_n)_n\) is bounded and \(\ell^q(I)\) has the Banach–Saks property, there exist a subsequence of \((v_n)_n\), not relabeled, such that all subsequences of \((v_n)_n\) are Cesàro convergent in norm (to the same limit).
On the other hand, since every element of $X^* = (\bigoplus_{i \in I} X_i^*)_{\ell^q}$ is countably supported, we can assume without loss of generality that $I$ is countable. Then, as in the proof of Theorem 4.2, we can find a subsequence of $(x^*_n)_n$, not relabeled, such that for every further subsequence $(x^*_{n_k})_k$ and every $i \in I$, the sequence $(\rho_i(x^*_{n_k}))_k$ is Cesàro convergent to 0 with respect to $\mu(X^*_i, X_i)$.

We claim that any subsequence $(x^*_n)_n$ is Cesàro convergent to 0 with respect to $\mu(X_*, X)$. Indeed, define $y^*_j := \frac{1}{j} \sum_{k=1}^{j} x^*_n$ for all $j \in \mathbb{N}$. We will show that $(y^*_j)_j$ is $\mu(X_*, X)$-null by checking that it satisfies conditions (i) and (ii) of Lemma 4.3. Obviously, (i) holds. On the other hand, for each $i \in I$ and $j \in \mathbb{N}$, we have the following:

$$\|\rho_i(y^*_j)\|_{X_i^*} \leq \frac{1}{j} \sum_{k=1}^{j} \|\rho_i(x^*_n)\|_{X_i^*} = \psi_i \left( \frac{1}{j} \sum_{k=1}^{j} v_{n_k} \right),$$

where $\psi_i \in \ell^q(I)^*$ denotes the $i$th coordinate functional. Hence, condition (ii) of Lemma 4.3 holds by taking $\tilde{\xi}_j := \frac{1}{j} \sum_{k=1}^{j} v_{n_k}$ for all $j \in \mathbb{N}$. □

**Remark 4.5.** The previous result provides examples of separable Banach spaces having property $(\mu^*)$ which do not embed isomorphically into any SWCG space, like $\ell^p(\ell^1)$ and $\ell^p(L^1[0,1])$ for $1 < p < \infty$ (see [25, Corollary 2.29]).

The following example answers in the negative a question left open in [1, Problem 2.19]. It also shows that property $(\mu^*)$ is not preserved by countable $\ell^\infty$-sums.

**Example 4.6.** There is a sequence $(X_n)_n$ of finite-dimensional Banach spaces, such that $(\bigoplus_{n \in \mathbb{N}} X_n)_{\ell^\infty}$ fails property (K).

**Proof.** Johnson [22] proved the existence of a sequence $(X_n)_n$ of finite-dimensional Banach spaces, such that, for every separable Banach space $X$, its dual $X^*$ is isomorphic to a complemented subspace of $(\bigoplus_{n \in \mathbb{N}} X_n)_{\ell^\infty}$.

Bearing in mind that property (K) is inherited by complemented subspaces, the fact that the space $(\bigoplus_{n \in \mathbb{N}} X_n)_{\ell^\infty}$ fails property (K) follows from the existence of separable Banach spaces whose dual fails property (K), like $C[0,1]$ and the predual of the James tree space $JT$. Indeed, $C[0,1]^*$ is isomorphic to the $\ell^1$-sum of $\epsilon$ many copies of $L^1[0,1]$ (see, e.g., [38, p. 242, Remark 5]), and so, it fails property (K), according to Pelczyński’s example mentioned in the introduction. On the other hand, $JT$ is not reflexive and contains no subspace isomorphic to $\ell^1$, and hence, $JT$ fails property (K) (see [1, Theorem 2.1]). □

5. Applications

5.1. Banach Lattices

Given a Banach lattice $X$, we write

$$L_w(X) := \{A \subseteq X : A \text{ is } L\text{-weakly compact}\}.$$
Recall that a bounded set $A \subseteq X$ is said to be $L$-weakly compact if every disjoint sequence contained in $\bigcup_{x \in A} [-|x|, |x|]$ (the solid hull of $A$) is norm null. Every $L$-weakly compact set is relatively weakly compact, while the converse does not hold in general. $L$-weak compactness and relative weak compactness are equivalent for subsets of the $L^1$-space of a non-negative measure. More generally, if $X$ has order continuous norm, then a set $A \subseteq X$ is $L$-weakly compact if and only if it is approximately order bounded, i.e., for every $\varepsilon > 0$, there is $x \in X^+$, such that $A \subseteq [-x, x] + \varepsilon B_X$. For more information on $L$-weakly compact sets, see [30, §3.6].

**Theorem 5.1.** Let $X$ be a Banach lattice with order continuous norm and weak unit. Then, $X$ has property $(\mu^*_Lw(X))$.

**Proof.** Such a Banach lattice is order isometric to a Köthe function space over a finite measure space (see, e.g., [27, Theorem 1.b.14]). Therefore, we can assume that $X$ is a Köthe function space over a finite measure space, say $(\Omega, \Sigma, \mu)$. Let $i : L^\infty(\mu) \to X$ be the inclusion operator. Since $X$ has order continuous norm, $i^*(X^*) \subseteq L^1(\mu)$, and so, $i^* : X^* \to L^1(\mu)$ is $w^*-w$-continuous (see, e.g., [27, p. 29]).

The order continuity of the norm also ensures that $L_w(X)$ is strongly generated by $i(B_{L^\infty(\mu)})$ (see, e.g., [32, Lemma 2.37(iii)]). Therefore, to prove that $X$ has property $(\mu^*_Lw(X))$, we only have to check that $X$ has property $(\mu^*_{i(B_{L^\infty(\mu)})})$ (by Lemma 3.2(ii)). To this end, let $(x^*_n)_n$ be a $w^*$-null sequence in $X^*$. Then, $(i^*(x^*_n))_n$ is weakly null in $L^1(\mu)$, which has the weak Banach–Saks property. Hence, there is a subsequence of $(x^*_n)_n$, not relabeled, such that any further subsequence $(i^*(x^*_n))_k$ is Cesàro convergent to 0 in the norm of $L^1(\mu)$. In particular:

$$\lim_{N \to \infty} \sup_{f \in B_{L^\infty(\mu)}} \left\| \frac{1}{N} \sum_{k=1}^N x^*_n ; i(f) \right\|_{L^1(\mu)} = \lim_{N \to \infty} \left\| \frac{1}{N} \sum_{k=1}^N i^*(x^*_n) \right\|_{L^1(\mu)} = 0.$$

This shows that $X$ has property $(\mu^*_{i(B_{L^\infty(\mu)})})$. \hfill \square

If $X$ is a weakly sequentially complete Banach lattice, then $L_w(X) \supseteq Q(X) := \{ T(C) : T : L^1[0, 1] \to X \text{ operator, } C \in \text{wk}(L^1[0, 1]) \}$. Indeed, in this case, $X$ has order continuous norm (see, e.g., [30, Theorems 2.4.2 and 2.5.6]) and any operator $T : L^1[0, 1] \to X$ is regular (see, e.g., [30, Theorem 1.5.11]), so it maps approximately order bounded (i.e., relatively weakly compact) subsets of $L^1[0, 1]$ to approximately order bounded (i.e., $L$-weakly compact) subsets of $X$. Thus, we get the following:

**Corollary 5.2.** Let $X$ be a weakly sequentially complete Banach lattice with weak unit. Then, $X$ has property $(\mu^*_Q(X))$.

A Banach lattice is said to have the positive Schur property (PSP) if every weakly null sequence of positive vectors is norm null. Obviously, this property is satisfied by the $L^1$-space of any non-negative measure. The PSP implies weak sequential completeness (see, e.g., [30, Theorem 2.5.6]) and so
the order continuity of the norm. On the other hand, it is known that the PSP is equivalent to saying that every relatively weakly compact set is \( L \)-weakly compact (see, e.g., [20, Theorem 3.14]). Therefore, Theorem 5.1 implies that every Banach lattice with the PSP and weak unit has property \( (\mu_s) \). In fact, such a Banach lattice is \( S^2 \text{WCG} \), as we show in Corollary 5.4 below. The key is the following lemma (cf. [18, Corollary 5.6]):

**Lemma 5.3.** Let \( X \) be a Köthe function space with order continuous norm over a finite measure space \( (\Omega, \Sigma, \nu) \). Then, the inclusion operator \( i : L^\infty(\nu) \to X \) is super weakly compact.

**Proof.** Since \( X \) has order continuous norm, every order interval of \( X \) is weakly compact (see, e.g., [30, Theorem 2.4.2]). Hence, \( i \) is weakly compact and so is \( i^* : X^* \to L^1(\nu) \) (see the proof of Theorem 5.1). Every weakly compact operator taking values in the \( L^1 \)-space of a finite measure is super weakly compact, see, e.g., [3, p. 123, Remarque] (cf. [18, Proposition 5.5]). Therefore, \( i^* \) is super weakly compact and the same holds for \( i \) (see [3, Proposition II.4]). \( \square \)

**Corollary 5.4.** Let \( X \) be a Banach lattice with the positive Schur property and weak unit. Then, \( X \) is \( S^2 \text{WCG} \).

**Proof.** As in Theorem 5.1, we can assume that \( X \) is a Köthe function space over a finite measure space \( (\Omega, \Sigma, \nu) \). Then, \( X \) is strongly generated by the inclusion operator \( i : L^\infty(\nu) \to X \) (bear in mind that every weakly compact subset of \( X \) is \( L \)-weakly compact). On the other hand, \( i \) is super weakly compact by Lemma 5.3. \( \square \)

The previous corollary is an improvement of [6, Proposition 5.6], where it was shown that such Banach lattices are SWCG.

**Remark 5.5.** The conclusion of Theorem 5.1 and Corollaries 5.2 and 5.4 can fail in the absence of weak unit. For instance, let \( X \) be the \( \ell^1 \)-sum of \( c \) many copies of \( L^1[0,1] \). Then, \( X \) has the PSP and fails property \( (\mu^s_{Q(X)}) \), since it does not have property \( (k) \) (see [17, Example 4.1]).

### 5.2. Lebesgue–Bochner Spaces

The first result of this subsection is based on the proof of Emmanuele’s result [14] on complemented copies of \( c_0 \) in Lebesgue–Bochner spaces (cf. [7, Theorem 4.3.2]).

**Theorem 5.6.** Let \( X \) be a Banach space containing a subspace isomorphic to \( c_0 \). Then, \( L^1([0,1],X) \) fails property \( (K_0S) \).

For the proof of Theorem 5.6, we need a lemma.

**Lemma 5.7.** Let \( X \) be a Banach space containing a \( c_0 \)-sequence \( (x_n)_n \). Let \( (h_n)_n \) be a sequence of \( \{-1,1\} \)-valued measurable functions on \([0,1]\) and let \( (I_k)_k \) be a sequence of finite subsets of \( \mathbb{N} \) with \( \max(I_k) < \min(I_{k+1}) \) for all \( k \in \mathbb{N} \). Then:

\[
\left\{ \sum_{n \in I_k} h_n(\cdot)x_n \mid k \in \mathbb{N} \right\}
\]
is a $\delta S$-set of $L^1([0, 1], X)$.

**Proof.** It suffices to show that the set

$$C := \left\{ \sum_{n \in I_k} a_n x_n : k \in \mathbb{N}, (a_n)_{n \in I_k} \in \{-1, 1\}^{I_k} \right\}$$

is relatively weakly compact in $X$. To this end, let $(y_m)_m$ be a sequence in $C$. For each $m \in \mathbb{N}$, we write the following:

$$y_m = \sum_{n \in I_{k_m}} a_{n, m} x_n$$

for some $k_m \in \mathbb{N}$ and $a_{n, m} \in \{-1, 1\}$. By passing to a subsequence, not relabeled we can assume that one of the following alternatives holds:

- There is $k \in \mathbb{N}$ such that $k_m = k$ for all $m \in \mathbb{N}$. In this case, $(y_m)_m$ is a bounded sequence in a finite-dimensional subspace of $X$ and, therefore, it admits a norm convergent subsequence.
- $k_m < k_{m+1}$ for all $m \in \mathbb{N}$. In this case, since $(x_n)_n$ is a $c_0$-sequence, the same holds for $(y_m)_m$, and so, it is weakly null.

Thus, $C$ is relatively weakly compact, as required. □

**Proof of Theorem 5.6.** We denote by $(e_n)_n$ and $(e^*_n)_n$ the usual bases of $c_0$ and $\ell^1$, respectively. Let $(x_n)_n$ be a $c_0$-sequence in $X$ and let $(r_n)_n$ be the sequence of Rademacher functions on $[0, 1]$. Then, $(r_n(\cdot)x_n)_n$ is a $c_0$-sequence in $L^1([0, 1], X)$ which spans a complemented subspace

$$Z := \text{span}\{r_n(\cdot)x_n : n \in \mathbb{N}\} \subseteq L^1([0, 1], X),$$

see, e.g., the proof of Theorem 4.3.2 in [7]. Let $P : L^1([0, 1], X) \to Z$ be a projection and let $T : Z \to c_0$ be the isomorphism satisfying $T(r_n(\cdot)x_n) = e_n$ for all $n \in \mathbb{N}$. Consider the operator $S := T \circ P : L^1([0, 1], X) \to c_0$. Note that $(S^*(e^*_n))_n$ is a $w^*$-null sequence in $L^1([0, 1], X^*)$.

**Claim.** $(S^*(e^*_n))_n$ does not admit convex block subsequences converging to 0 uniformly on each $\delta S$-set. Indeed, let $(g_k)_k$ be any convex block subsequence of $(S^*(e^*_n))_n$. Write $g_k = \sum_{n \in I_k} a_n S^*(e^*_n)$, where $(I_k)_k$ is a sequence of finite subsets of $\mathbb{N}$ with $\max(I_k) < \min(I_{k+1})$ and $a_n \geq 0$ satisfy $\sum_{n \in I_k} a_n = 1$. Then, for each $k \in \mathbb{N}$, we have the following:

$$\left\langle g_k, \sum_{n \in I_k} r_n(\cdot)x_n \right\rangle = \left\langle \sum_{n \in I_k} a_n e^*_n, \sum_{n \in I_k} S(r_n(\cdot)x_n) \right\rangle$$

$$= \left\langle \sum_{n \in I_k} a_n e^*_n, \sum_{n \in I_k} e_n \right\rangle = \sum_{n \in I_k} a_n = 1.$$

Hence, $(g_k)_k$ does not converge to 0 uniformly on the $\delta S$-set:

$$\left\{ \sum_{n \in I_k} r_n(\cdot)x_n : k \in \mathbb{N} \right\}$$

(Lemma 5.7). This proves that $L^1([0, 1], X)$ fails property $(K_{\delta S})$. □
From now on, \((\Omega, \Sigma, \nu)\) is a finite measure space. Given a Banach space \(X\), the identity operator \(i : L^2(\nu, X) \to L^1(\nu, X)\) strongly generates \(L^1(\nu, X)\). Indeed, this can be checked as in the case of real-valued functions, bearing in mind that any weakly compact subset of \(L^1(\nu, X)\) is uniformly integrable (see, e.g., [13, p. 104, Theorem 4]). On the other hand, \(L^2(\nu, X)\) is super-reflexive whenever \(X\) is super-reflexive, see [9] (cf. [11, Ch. IV, Corollary 4.5]). In particular, \(L^1(\nu, X)\) is strongly generated by a super-reflexive space (and so, it has property \((\mu^s)\), by Corollary 3.4) whenever \(X\) is super-reflexive.

**Theorem 5.8.** Let \(X\) be a \(S^2\) WCG Banach space and let \(Z \subseteq X\) be a subspace. Then, \(L^1(\nu, Z)\) has property \((\mu^s_{\delta\Sigma(\nu, Z)})\).

**Proof.** The space \(L^1(\nu, X)\) is weakly compactly generated because \(X\) is (see, e.g., [13, p. 252, Corollary 11]). Therefore, as in the proof of Theorem 3.1, it suffices to check that \(L^1(\nu, X)\) has property \((\mu^s_{\delta\Sigma(\nu, X)})\).

Let \(Y\) be a Banach space which strongly generates \(X\) through a super weakly compact operator \(T : Y \to X\). We can assume that \(Y\) is reflexive and \(T\) is injective, according to Theorem 4.5 and Proposition 4.6 in [34]. Then, \(\delta\Sigma(\nu, X)\) is strongly generated by the set:

\[
H := \{ h \in L^1(\nu, X) : h(\omega) \in T(B_Y) \text{ for } \mu\text{-a.e. } \omega \in \Omega \}
\]

(see [36, proof of Theorem 2.7]). Let \(\bar{T} : L^2(\nu, Y) \to L^2(\nu, X)\) be the “composition” operator defined by the formula:

\[
\bar{T}(f) := T \circ f \text{ for all } f \in L^2(\nu, Y).
\]

Let \(i : L^2(\nu, X) \to L^1(\nu, X)\) be the identity operator and define \(S := i \circ \bar{T}\). Since \(Y\) is reflexive and \(T\) is injective, we have \(H \subseteq S(\rho B_{L^2(\nu, Y)})\) for \(\rho := (\nu(\Omega))^{1/2}\) (see [26, proof of Theorem 1]), and hence, \(\delta\Sigma(\nu, X)\) is strongly generated by \(S(\rho B_{L^2(\nu, Y)})\).

Since \(T\) is super weakly compact, so is \(\bar{T}\) (see [3, p. 126, Corollaire]), hence \(S\) is super weakly compact as well. Therefore, \(S^*\) is super weakly compact (see [3, Proposition II.4]) and so \(S^*\) is Banach–Saks (see [4, Théorème 3]). From Lemmas 3.3 and 3.2(ii), we conclude that \(L^1(\nu, X)\) has property \((\mu^s_{\delta\Sigma(\nu, X)})\).

Our final example shows that, in the statement of Theorem 5.8, the \(S^2\) WCG property cannot be replaced by the Banach–Saks property of the dual. It is known (and not difficult to check) that if \(X\) is a reflexive Banach space, then every relatively weakly compact subset of \(L^1(\nu, X)\) is a \(\delta\Sigma\)-set, and so, in this particular setting properties \((\mu^s)\) and \((\mu^s_{\delta\Sigma})\) are equivalent.

**Example 5.9.** There exists a Banach space \(X\) such that \(X^*\) has the Banach–Saks property, but \(L^1([0, 1], X)\) fails property \((\mu^s)\).

**Proof.** Schachermayer [39] constructed an example of a Banach space \(E\) having the Banach–Saks property, such that \(L^2([0, 1], E)\) does not have it. The failure of the property is witnessed by a uniformly bounded weakly null sequence \((f_n)_n\) in \(L^2([0, 1], E)\) (see [39, proof of Proposition 3]).
Set $X := E^*$ and let $i : L^2([0,1], X) \to L^1([0,1], E)$ be the identity operator. Since $E$ is reflexive, the same holds for $L^2([0,1], E)$ and we have

$$L^2([0,1], E)^* = L^2([0,1], X) \quad \text{and} \quad L^1([0,1], X)^* = L^\infty([0,1], E),$$

see, e.g., [13, IV.1]. Moreover, $(f_n)_n$ is a $w^*$-null sequence in $L^1([0,1], X)^*$. No subsequence $(f_{n_k})_k$ is Cesàro convergent to 0 in the norm of $L^2([0,1], E)$, and so, it cannot be Cesàro convergent to 0 uniformly on the weakly compact set $i(B_{L^2([0,1], X)})$. This shows that $L^1([0,1], X)$ fails property $(\mu^s)$. □

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