Maslov’s concept of phase transition from Bose-Einstein to Fermi-Dirac distribution.
Results of interdisciplinary workshop in MSU.

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1 Introduction

At the end of 2017, an interdisciplinary scientific seminar was organized at Moscow University, devoted to the study and development of a new scientific concept created by V.P. Maslov, allowing you to take a fresh look at the statistics of Bose-Einstein and Fermi-Dirac ideal gases. This new point of view allows us to interpret the indicated statistics as particular cases of statistical properties in number theory, on the one hand, and to indicate the limits of phase transitions from Bose to Fermi distributions.

Such a sort of phase transitions have already been observed in experiments on the decay of helium $He_4$, which forms a Bose gas, into helium $He_3$ and a fermion. The concept built by Maslov allows practical calculations of thermodynamic potentials of Bose and Fermi gases in the region of phase transitions.

Scientists from Moscow State University and other Moscow scientific centers take part in the seminar: academician V.P. Maslov, academician A.T. Fomenko, corresponding member RAS V.E. Nazaikinsky, the head of the department of general topology, Professor Yu.V. Sadovnichy, Professor of the RUDN A.Yu. Savin, Professor A.G. Kushner, and many of their students and employees. The result of the seminar was the development of a mathematical model for the calculation of thermodynamic potentials and their comparisons for Bose and Fermi gases.
2 Formulation of the problem

The Bose–Einstein formulas for the equilibrium occupation numbers in a Bose gas of noninteracting particles are well known in physics and gives the analogue in the number theory. Physicists claim that these formulas, which follow from the indistinguishability of particles, are only true in the quantum case, while in the classical case one should use the Boltzmann distribution.

However, V.P. Maslov put forward the idea that the same indistinguishability argument can be used to justify the Bose–Einstein distribution in the classical case, which, in conjunction with the notion of fractional number of degrees of freedom, permitted him in particular to show how the van der Waals equation emerges for a classical gas of noninteracting particles.

He gave the first rigorous estimates for the probabilities of large deviations from the Bose–Einstein distribution for large energies.

The Bose–Einstein distribution is closely related to the theory of partitions and number theory; for example, the case of two degrees of freedom corresponds to the partitio numerorum problem in number theory.

Let

\[ \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \lambda_n \geq \cdots \]

be a nondecreasing sequence of positive integers tending to \( +\infty \). Given a positive integer \( M \), consider the Diophantine equation

\[ \sum_{k=1}^{\infty} \lambda_k n_k M \]

with unknowns \( n_k \in \mathbb{Z}_+, k = 1, 2, \ldots \).

The Diophantine equation can be viewed as an (integer-valued) model of a physical system of Bose particles (the number of particles is not fixed) in which, for each \( k \), \( n_k \) particles sit in the \( k \)th eigenstate (with energy \( \lambda_k \)), the total energy of the system being equal to \( M \). The numbers \( n_k \) are known as the occupation numbers.

Note that the problem can be stated in a different way. For each \( j \in \mathbb{N} \), let

\[ N_j = \sum_{k: \lambda_k = j} n_k \]

be the total number of particles at the energy level \( j \). Then the Diophantine equation becomes

\[ \sum_{j=1}^{\infty} jN_j = M, \quad \sum_{j} N_j = N. \quad (1) \]

It is a classical problem of partitioning of a number.

This problem can be formulated as: How many ways can a positive integer \( M \) be represented as a sum of \( N \leq M \) positive integers,

\[ M = a_1 + a_2 + \cdots + a_N? \]
This problem statement is equivalent to: What is the number of sets of integers $N_i \geq 0$ such that equalities (1) are satisfied? It is obvious that any set of integers $N_i = 0$ satisfying system (1) corresponds to one partition of $M$.

We must note that in the book by Landau and Lifshitz and in quantum thermodynamics, where bosons and fermions are considered, two equations are used: one of them contains the number $N$ of particles, and the other contains the energy $E$. The number of particles can be regarded as an integer, but the energy $E$ is not an integer (it even has a dimension).

The fundamental number theory problem of decomposing a number $M$ into $N$ terms in the space of integers splits into two cases: repeated terms are admissible in the first case and not admissible in the second case.

It is well known that the first case (with repeated terms) corresponds to the Bose–Einstein distribution, and the second case (without repeated terms) corresponds to the Fermi–Dirac distribution. This fact indicates that analytic number theory must be related to statistical physics. Here, we solve the analytic continuation problem in the case of a transition from Bose statistics to Fermi statistics.

We let $N_i$ denote the number of particles at the $i$th energy level. In the case of Gentile statistics, there can be at most $K$ particles at each energy level. According to the generally accepted definition, we have $K=1$ for the Fermi–Dirac distribution and $K=\infty$ for the Bose–Einstein distribution. But it obviously follows that $N_i \leq N$ for the Bose system, and hence $K \leq N$ for the Bose system. It hence follows that the maximum $K$ is equal to $N$ and not infinity. This physical conclusion substantially changes the formulas.

The mass $m$ and the Planck constant $\hbar$ were introduced as parameters in Landau and Lifshitz book. In number theory, we can assume that all these constants and the volume $V$ are equal to unity. To compare number theory formulas and statistical distributions, we introduce the notation for some quantities that are equal to unity in number theory.

Let us introduce a general parameter $\Phi$ that is consistent with the notation in the book by Landau and Lifshitz:

$$\Phi = \lambda^{2(\gamma+1)}VT^{\gamma+1} = V(\lambda^2T)^{\gamma+1},$$

where $V$ is the volume, $T$ is the temperature, and $\lambda = \sqrt{2\pi m}/2\pi\hbar$.

Then in the case $N = K$, we obtained crucial new equations for the energy $E$ and the number $N$ of particles:

$$E = \Phi T(\gamma + 1) \left( Li_{2+\gamma}(a) - \frac{1}{(N+1)^{\gamma+1}} Li_{2+\gamma}(a^{N+1}) \right),$$

$$N = \Phi \left( Li_{1+\gamma}(a) - \frac{1}{(N+1)^{\gamma}} Li_{1+\gamma}(a^{N+1}) \right).$$
where $\gamma = D/2 - 1$, $D$ is the number of degrees of freedom, and $a$ is the activity. Here $Li_s(z)$ is the polylogarithm function:

$$Li_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$$

### 3 Results

In the case of Gentile statistics, the self-consistent equations have the form

$$N = \Phi T(\gamma + 1) \left( Li_{2+\gamma}(a) - \frac{1}{(N^\alpha + 1)^{\gamma+1}} Li_{2+\gamma}(a^{N^\alpha+1}) \right),$$

$$M = \Phi \left( Li_{1+\gamma}(a) - \frac{1}{(N^\alpha + 1)^{\gamma}} Li_{1+\gamma}(a^{N^\alpha+1}) \right).$$

The value $\alpha$ varies from 0 to 1. For $\alpha = 1$, the Bose distribution occurs, while for $\alpha = 0$, the Fermi distribution occurs. When $\alpha = 0.5$, the Fermi distribution for the main term of the asymptotics occurs. In the interval between $\alpha = 0$ and $\alpha = 0.5$, there is a concentration of curves depicted in Figures 3 and 4.

![Fig. 3. Dependence of $M$ on $a$ obtained numerically. The solid curves correspond to $W = 1000, \gamma = 0, \alpha = 0.9, 0.8, 0.7, 0.6, 0.5, 0.4, 0.3, 0.2, 0.1, 0.01$ (from right to left). The dotted line is $N = -1/log(a)$.](image)

![Fig. 4. Dependence of $N$ on $a$ obtained numerically corresponding to the plot of $N(a)$ turned over. The solid curves correspond to $W = 1000, \gamma = 0, \alpha = 0.9, 0.8, 0.7, 0.6, 0.5, 0.4, 0.3, 0.2, 0.1, 0.01$ (from right to left). The dotted line is $N = -1/log(a)$.](image)
So it was shown that, depending on the hidden parameter, purely quantum problems behave like classical ones. It is shown that the Bose-Einstein and the Fermi-Dirac distributions, which until now were regarded as dealing with quantum particles, describe, for the appropriate values of the hidden parameter, the macroscopic thermodynamics of classical molecules.

We stress that the transition of a boson into a fermion occurs not at the point \( a = 0 \) but at a nonzero \( a \). We let \( a_0 \) denote a value of the activity \( a \) at which the number of particles in the Bose-Einstein distribution is \( N = 0 \). At this point, there are already no bosons. This is the instant of transition when the boson has already disappeared, splitting into two fermions, and one of the fermions must somehow be lost. What happens with it? At what distance from the nucleus shell does it move? This can be calculated by determining the energy value required for boson decay. This energy is very small although the fermion separated in a macroscopic volume \( V \).

When one of the fermions leaves this volume, this means that it disappears. Namely, this means that the fermion separates not in the nucleus shell but in a greater volume. In this situation, there is a jump of spin in a macroscopic volume. This conclusion agrees well with experiments, in particular, described by Bell.

We note that the fermions do not interchange, which means that the fermions are numbered. This energy of the spin jump as the specific energy can be calculated.

The jump of the compressibility factor from the Fermi system into the Bose system is determined by the formula

\[
\Delta Z(a) = Z|_{Fermi} - Z|_{Bose} = \frac{Li_{2+\gamma}(-a)}{Li_{1+\gamma}(-a)} - \frac{Li_{2+\gamma}(a)}{Li_{1+\gamma}(a)}.
\]

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