Set Stabilizability of Quantum Systems

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Abstract

We explore set-stabilizability by constrained controls, and both controllability and stabilizability can be regarded as the special case of set-stabilizability. We not only clarify how to define equilibrium points of Schrödinger Equations, but also establish the necessary and sufficient conditions for stabilizability of quantum systems. Unfortunately, it is revealed that the necessary conditions are quite strict for stabilizability of some concrete quantum systems like nuclear spin systems, and this further justifies the introduction of set-stabilizability notion. It is also exemplified that set-stabilizability can be used for investigating quantum information processing problems including quantum information storage and entangled states generation.

Key words: Quantum Systems, Controllability, Stabilizability

1 Motivation and Introduction

The concepts of controllability and observability are important contribution of control theorists to the science, technology, and engineering domain. With the introduction of these structural concepts, we begin to deeply understand the relationship between the input-output description and state-space description. The concept of controllability was first proposed for linear systems by R. E. Kalman in his remarkable paper\cite{1} in 1960s. Controllability of nonlinear systems was further investigated by H. J. Sussmann and V. Jurdjevic\cite{2} and R. Hermann and A. J. Krener\cite{3} in 1970s.

Quantum control theory has been developed ever since last century\cite{4,5,6,7}. Recently, quantum information and quantum computation is the focus of research\cite{8}. A great progress has been made in the domain of quantum control\cite{9,10}, in which the controllability of quantum systems is a fundamental issue. In 1980s, controllability of quantum systems was first explored by G. M. Huang, T.-J. Tarn and J. W. Clark\cite{5}. The different notations of controllability have been exploited in \cite{11,12,13,14,15,16,17}. Specially, the controllability of quantum open systems has been studied by some researchers\cite{18,19,20,21}.

To manipulate a quantum system, it is not sufficient to just know whether or not the quantum system is controllable. It is necessary to know how to construct permissible controls to steer the quantum system within the the specified time $T_s$ in some applications. We need explore controllability of quantum systems under different permissible control conditions such as bounded controls and time continuous function controls, and exploit the impact of admissible control conditions on the performance indices including transition time.

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Preprint submitted to Automatica
Given a control system, whether quantum or classical, the first and most important question about its various properties is to investigate whether or not it is stable. The most useful and general approach for studying the stability and stabilizability is the theory introduced by Russian mathematician A. M. Lyapunov[22] in the late 19th century. Lyapunov’s pioneering work on stability received little attention outside Russia, although it was translated into French in 1908 (at the instigation of Poincare), and reprinted by Princeton University Press in 1947. The publication of the work of Lure and a book[23] by La Salle and Lefschetz brought Lyapunov’s work to the attention of the larger control engineering community in the early 1960’s. Several quantitative stability concepts like finite-time stability[24], Lipschitz stability[25], partial stability[26] and practical stability[23] had been investigated based on Lyapunov’s great work in 20th century. Set stability of dynamical systems was specifically discussed by Heinen James Albin[27] in 1969. In 2002, S. K. Phooi et al.[28] further proposed the broad-sense Lyapunov function and generalized the notion of stability in the sense of Lyapunov.

From control theory point of view, we not only need to investigate whether a dynamical system is stable or not, but also need to explore whether or not a controlled dynamical system is stabilizable by permissible control. In recent applications like quantum information storage, one of the important questions is whether or not a given state of quantum systems can be stabilizable by permissible controls. To study the stabilizability problem of quantum systems, we will have to exploit how to define the equilibrium points of Schrödinger Equation. Just by investigating stabilizability problem of controlled quantum closed system, we gradually realize that the physical conditions for stabilizability of quantum systems are too strict in some concrete systems like nuclear-soin systems. These observations indicate that we need to weaken a more general framework.

To overcome the aforementioned difficulties, we try to generalize the concepts of both controllability and stabilizability. With quantum control problems in mind, we will propose a new notion of set-stabilizability: given a pair of quantum state sets \( S_0 \) and \( S_1 \), a quantum system is \( S_1 \)-stabilizability from \( S_0 \) within the specified time \( T_s \) under constrained control conditions if, for any initial state in \( S_0 \), on can always find permissible controls to steer the quantum system from an arbitrary initial state in \( S_0 \) to another arbitrary state in the set \( S_1 \) in the finite time \( t_f \) with \( 0 < t_f \leq T_s \), and to further keep the system state stay in the set \( S_1 \) when \( t \geq T_s \).

It is interesting to underline the following observations: (1) When \( S_1 \) is a one-point set, set-stabilizability notation is reduced to stabilizability. (2) When both \( S_0 \) and \( S_1 \) are the state space itself and unconstrained controls are permitted, the concept of set stabilizability is reduced to controllability proposed by Kalman[1].

However, the set-stabilizability is not proposed to generalize the concepts of both controllability and stabilizability just for the sake of generalization, without proper motivations. This work can be regarded as one of explorations made by many researchers who hope to investigate what kind of control goal is achievable for quantum systems by various feedback control[29,30,31,32,33,34,35,36,37,38,39,40,41,42,43]. Stabilization of open quantum systems has been studied by [44,45]. In this research, we exploit what kind of stabilizability can be expected for quantum closed systems by constrained open-loop controls, and we would like to emphasize that set stabilizability is attained by coherent control even when stabilizability itself is not an achievable control goal. It should be also underlined that this research is very different from set value analysis[46] and set dynamics[47].

The rest of this paper are organized as follows. In Sect. II, the concept and properties of set-stabilizability are presented for general dynamical systems, and it is revealed that both controllability and stabilizability are the special case of set stabilizability. The notation of set stabilizability and stabilizability are specifically discussed for quantum closed systems. In Sect. III, the set stabilizability and stabilizability problems of single-qubit systems are explored under different constrained controls. We present the necessary and sufficient conditions for quantum systems and give some further discussions on the strictness of stabilizability in Sect. IV, and it is also exemplified that set-stabilizability notation can be used for exploring entanglement generation of two-qubit systems in this section. The paper concludes with Sect. V.

2 Basic Concept and Basic Lemma

2.1 Set stabilizability and its properties

For the purpose of further discussions, we first recite the axiomatic definition of a dynamical system presented by Kalman in 1960s[1].

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Definition 1. A dynamical system is a mathematical structure defined by the following axioms:

\( (D_1) \) There is given a state space \( \Sigma \) and a set of values of time \( T \) at which the behaviour of the system is defined; \( \Sigma \) is a topological space and \( T \) is an ordered topological space which is a subset of the real numbers.

\( (D_2) \) There is given a topological space \( \Omega \) of functions of time defined on \( T \), which are the admissible inputs to the system.

\( (D_3) \) For any initial time \( t_0 \) in \( T \), any initial state \( x_0 \) in \( \Sigma \), and any input \( u \) in \( \Omega \) defined for \( t \geq t_0 \), the future states of the system are determined by the transition \( \varphi : \Omega \times T \times T \times \Sigma \to \Sigma \), which is written as \( \varphi_u(t_0; t_0, x_0) = x_t \). This function is defined only for \( t \geq t_0 \). Moreover, any \( t_0 \leq t_1 \leq t_2 \) in \( T \), any \( x_0 \) in \( \Sigma \), and any fixed \( u \) in \( \Omega \) defined over \([t_0, t_1] \cap T\), the following relations hold:

\[
\varphi_u(t_0; t_0, x_0) = x_0, \\
\varphi_u(t_2; t_0, x_0) = \varphi_u(t_2; t_1, \varphi_u(t_1; t_0, x_0)).
\]

(1)

In addition, the system must be nonanticipatory, i.e., if \( u, v \in \Omega \) and \( u = v \) on \([t_0, t_1] \cap T\) we have

\[
\varphi_u(t; t_0, x_0) = \varphi_v(t; t_0, x_0).
\]

(3)

\( (D_4) \) Every output of the system is a function \( \psi : T \times \Sigma \to \text{reals} \).

\( (D_5) \) The functions \( \varphi \) and \( \psi \) are continuous, with respect to the topologies defined for \( \Sigma \), \( T \), and \( \Omega \) and the induced product topologies.

In other words, a dynamical system can be described by \( \Xi = \{ \Sigma, T, \Omega, \varphi, \psi \} \). For a dynamical system without output, it can be reduced to \( \Xi = \{ \Sigma, T, \Omega, \varphi \} \).

Subsequently, we will recite the concepts of controllability and stabilizability in the aforementioned abstract framework of dynamical systems.

Definition 2. The dynamical system \( \Xi = \{ \Sigma, T, \Omega, \varphi \} \) is controllable at time \( t_0 \in T \), if for any pair of initial state \( x_0 \) and target state \( x_1 \) in the state space \( \Sigma \), there always exist \( t_1 \in T \) with \( t_0 \leq t_1 < \infty \) and admissible control \( u \in \Omega \) such that \( x_1 = \varphi_u(t_1; t_0, x_0) \).

Definition 3. Let \( x_1 \in \Sigma \), then the state \( x_1 \) of the dynamical system \( \Xi = \{ \Sigma, T, \Omega, \varphi \} \) is stabilizable from \( \Sigma \) at time \( t_0 \in T \), if for any initial state \( x_0 \in \Sigma \), there exist \( t_1 \in T \) with \( t_0 \leq t_1 \leq \infty \) and admissible control \( u \in \Omega \) such that \( \varphi_u(t; t_0, x_0) \to x_1 \) when \( t \geq t_1 \).

Suppose that both \( S_0 \) and \( S_1 \) are subsets of the state space \( \Sigma \), we will introduce a new concept of set stabilizability as follows.

Definition 4. The dynamical system \( \Xi = \{ \Sigma, T, \Omega, \varphi \} \) is \( S_1 \)-stabilizable from \( S_0 \) at time \( t_0 \in T \), if for any initial state \( x_0 \in S_0 \) and another arbitrary target state \( x_1 \in S_1 \), there exist \( t_1 \in T \) with \( t_0 \leq t_1 \leq \infty \) and admissible control \( u \in \Omega \) such that \( x_1 = \varphi_u(t_1; t_0, x_0) \) and \( \varphi_u(t; t_0, x_0) \in S_1 \) when \( t_1 \leq t \in T \).

From the aforementioned definition, we can easily establish the following properties of set stabilizability.

Proposition 1 Suppose that \( S'_0 \subseteq S_0 \), a dynamical system \( \Xi = \{ \Sigma, T, \Omega, \varphi \} \) is \( S_1 \)-stabilizable from \( S'_0 \) at time \( t_0 \in T \) if it is \( S_1 \)-stabilizable from \( S_0 \) at time \( t_0 \in T \).

\( (P_1) \) Let \( S'_0 \subseteq S_0 \), a dynamical system \( \Xi = \{ \Sigma, T, \Omega, \varphi \} \) is \( S_1 \)-stabilizable from \( S'_0 \) at time \( t_0 \in T \) if it is \( S_1 \)-stabilizable from \( S_0 \) at time \( t_0 \in T \).

\( (P_2) \) Let \( \Omega' \subseteq \Omega \), \( \Xi = \{ \Sigma, T, \Omega, \varphi \} \) is stabilizable from \( S_0 \) at time \( t_0 \in T \) if \( \Xi' = \{ \Sigma, \Theta, \Omega', \varphi \} \) is \( S_1 \)-stabilizable from \( S_0 \) at time \( t_0 \in T \).
(P₃) A dynamical system \( \Xi = \{ \Sigma, T, \Omega, \varphi \} \) is controllable at time \( t_0 \in T \) if and only if it is \( \Sigma \)-stabilizable from \( \Sigma \) at time \( t_0 \in T \).

(P₄) Let \( \Sigma_1 \) is a subspace of \( \Sigma \), a dynamical system \( \Xi = \{ \Sigma, T, \Omega, \varphi \} \) is controllable on \( \Sigma_1 \) at time \( t_0 \in T \) if and only if it is \( \Sigma_1 \)-stabilizable from \( \Sigma_1 \) at time \( t_0 \in T \).

(P₅) The state \( x_1 \) of dynamical system \( \Xi = \{ \Sigma, T, \Omega, \varphi \} \) is stabilizable from \( S_0 \) at time \( t_0 \in T \) if it is \( \{ x_1 \} \)-stabilizable from \( S_0 \) at time \( t_0 \in T \).

The proof of the aforementioned properties are quite straightforward, but the properties are of major importance: set stabilizability is regarded as the generalization of both controllability and stabilizability.

Because we need to steer systems within the specified finite time in many applications, we further propose the notation of set stabilizability within the specified time span \( T_s \) as follows.

**Definition 5.** Given \( T_s > 0 \), the dynamical system \( \Xi = \{ \Sigma, T, \Omega, \varphi \} \) is \( S_1 \)-stabilizable within the specified time \( t_s \) from \( S_0 \) at time \( t_0 \in T \), if for any initial state \( x_0 \in S_0 \) and another arbitrary target state \( x_1 \in S_1 \), there exist \( t_1 \in T \) with \( t_1 - t_0 \leq T_s \) and admissible control \( u \in \Omega \) such that \( x_1 = \varphi_a(t_1; t_0, x_0) \) and \( \varphi_a(t; t_0, x_0) \in \mathcal{S}_1 \) when \( t_1 \leq t \in T \).

### 2.2 Equilibrium points of Schrödinger Equation

In this paper, we will study a special subclass of dynamical systems, those which are quantum closed systems, to illustrate why set-stabilizable is useful.

Before investigating stabilizability of quantum closed systems, we will have to discuss how to define an equilibrium point of Schrödinger Equation

\[
\frac{i\hbar}{\partial t} \psi(t) = H \psi(t)
\]

where \( \psi(t) \) is a pure state in Hilbert space. For the purpose of simplicity, we set \( \hbar = 1 \) in the whole paper.

From the mathematical point of view, it seems that zero vector is the sole equilibrium point of Eq. (4). Unfortunately, zero vector is nonsense from the viewpoint of physics. We need to give some further investigation on this issue. To obtain some intuitive pictures about the solution of Eq. (4), let us consider a two-level quantum system given by

\[
i \frac{d}{dt} \psi(t) = \sigma_z |\psi(t)\rangle
\]

where \( \sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1| \). From the physical point of view, both \( |0\rangle \) and \( |1\rangle \) are the equilibrium points of Eq. (5).

From the mathematical point of view, the solution \( \psi(t) \) of Eq. (5) with the initial state \( |\psi(0)\rangle = |0\rangle \) (or \( |\psi(0)\rangle = |1\rangle \)) satisfies \( |\psi(t)\rangle |\psi(t)\rangle = |0\rangle\langle 0| \) (or \( |\psi(t)\rangle |\psi(t)\rangle = |1\rangle\langle 1| \)). In other words, \( |\psi(t)\rangle |\psi(t)\rangle = |\psi(0)\rangle |\psi(0)\rangle \) if \( |\psi(0)\rangle = |i\rangle \) with \( i = 0, 1 \). It is also revealed that \( [\sigma_z, |i\rangle\langle i|] = 0 \) for \( i = 0, 1 \), where \( [ \cdot, \cdot ] \) is Lie bracket which is specified by \( [A, B] = AB - BA \).

Based on aforementioned observations, we have the following definition:

**Definition 6:** \( |\psi_s\rangle \) is called an equilibrium point of Eq. (4) if \( [H, |\psi_s\rangle \langle \psi_s|] = 0 \).

**Remark:** Denote \( \rho(t) = |\psi(t)\rangle \langle \psi(t)| \), then Eq. (4) can be written as \( \frac{d}{dt} \rho(t) = [H, \rho(t)] \). This implies that \( \rho(t) = \rho_s = |\psi_s\rangle \langle \psi_s| \) is the static solution of \( \frac{d}{dt} \rho(t) = [H, \rho(t)] \) if \( |\psi_s\rangle \) is called the equilibrium point of Eq. (4).

**Proposition 2:** \( |\psi_s\rangle \) is an equilibrium point of Eq. (4) if and only if \( |\psi_s\rangle \) is an eigenvector of Hamiltonian \( H \).

### 2.3 Set stabilizability for quantum closed systems

Consider a controlled finite-dimensional quantum system without output

\[
i \frac{d}{dt} \psi(t) = [H_0 + H_c(t)] \psi(t)
\]
where $|\psi(t)|$ is a pure state in Hilbert space, and $H_0$ and $H_c$ are system Hamiltonian and controlled Hamiltonian, respectively.

For the controlled quantum system governed by (6), set stabilizability can be specially defined as follows:

**Definition 4a**: For an arbitrary initial state $|\psi_0\rangle \in S_0$ and another arbitrary target state $|\psi_f\rangle \in S_1$, the quantum system Eq. (6) is $S_1$-stabilizable from $S_0$ at time $t_0$ if there exist a finite time $t_1 \geq t_0$ and permissible control $H_c(t)$ such that the system can be transferred from $|\psi(0)| = |\psi_0\rangle \in S_0$ to $|\psi(t_1)| = |\psi_f\rangle$ and $|\psi(t)| \in S_1$ when $t \geq t_1$.

To overcome decoherence, we need to steer quantum systems within the specified time span $T_s$. In this situation, we can present the notion of set stabilizability within the specified time span $T_s$ by modifying the Definition 4a.

**Definition 5a**: Given $T_s > 0$, $\forall |\psi_0\rangle \in S_0$ and $|\psi_f\rangle \in S_1$, the quantum system Eq. (6) is $S_1$-stabilized within the specified time span $T_s$ from $S_0$ at time $t_0$ if there exist a finite time $t_1$ with $t_1 - t_0 \leq T_s$ and permissible control $H_c(t)$ such that the system can be transferred from $|\psi(0)| = |\psi_0\rangle \in S_0$ to $|\psi(t_1)| = |\psi_f\rangle$ and $|\psi(t)| \in S_1$ when $t \geq t_1$.

3 Set stabilizability for two-level quantum systems

In this section, we will focus on set stabilizability of two-level quantum systems.

### 3.1 Model description and notation

Let $|0\rangle$ and $|1\rangle$ be a basis of Hilbert space of two-level quantum systems. A two-level quantum system is governed by

$$ \frac{d}{dt}|\psi(t)\rangle = i[\omega_0 S_z + u_x(t)S_x + u_y(t)S_y]|\psi(t)\rangle $$

where $S_z = \frac{1}{2}\sigma_z = i(|0\rangle\langle 0| - |1\rangle\langle 1|)$, $S_x = \frac{1}{2}\sigma_x = i(|1\rangle\langle 0| + |0\rangle\langle 1|)$, $S_y = \frac{1}{2}\sigma_y = i(|1\rangle\langle 0| - |0\rangle\langle 1|)$, $u_x(t)$ and $u_y(t)$ are adjustable scale functions, and $|u_x(t)| \leq L_x$ and $|u_y(t)| \leq L_y$. For simplicity, we assume that $L_x = L_y = g_0$ in this paper.

The state space is denoted as $\Sigma = \text{span}\{|0\rangle, |1\rangle\}$ and can be further parameterized as:

$$ \Sigma = \{|\psi\rangle = \cos \frac{\theta}{2}|0\rangle + e^{i\phi} \sin \frac{\theta}{2}|1\rangle : \theta \in [0, \pi], \phi \in [0, 2\pi]\} $$

where $\theta$ and $\phi$ are Bloch parameters.

For the further discussions, we introduce the following notations:

$$ C_{\theta_f} = \{|\psi\rangle = \cos \frac{\theta_f}{2}|0\rangle + e^{i\phi} \sin \frac{\theta_f}{2}|1\rangle : \phi \in [0, 2\pi]\} $$

(9)

and

$$ P_{\theta_f, \phi_f} = \{|\psi_f\rangle = \cos \frac{\theta_f}{2}|0\rangle + e^{i\phi_f} \sin \frac{\theta_f}{2}|1\rangle \} $$

(10)

where $C_{\theta_f}$ represents a circle on a Bloch sphere, $P_{\theta_f, \phi_f}$ is regarded as a point on the circle $\Sigma_{\theta_f}$, $\theta_f$ is a fixed value in $[0, \pi]$ and $\phi_f$ is a fixed value in $[0, 2\pi]$.

To obtain some intuitive pictures of $C_{\theta_f}$ and $P_{\theta_f, \phi_f}$, we plot them in Fig 1.

Two kind of permission control sets are considered: bounded control with bound $g_0$ $\Omega_B(g_0) = \{u_x(t)S_x + u_y(t)S_y : |u_x(t)| \leq g_0, |u_y(t)| \leq g_0\}$ and bounded time-continuous controls with bound $g_0$ $\Omega_{BC}(g_0) = \{u_x(t)S_x + u_y(t)S_y : u_x(t), u_y(t) \in C^0, |u_x(t)| \leq g_0, |u_y(t)| \leq g_0\}$ where $C^0$ is the space of time-continuous functions.
3.2 Stabilizability and circle-set stabilizability for two-level quantum systems

In this subsection, we will first establish the necessary and sufficient conditions for that the given target state is stabilizable by bounded controls in $\Omega_B(g_0)$.

**Theorem 1**: Given $\theta_f \in [0, \pi]$ and $\phi_f \in [0, 2\pi)$, $\forall t_0 > 0$, the controlled qubit system Eq. (7) is $P_{\theta_f, \phi_f}$-stabilizable from $\Sigma$ at $t_0$ by $\Omega_B(g_0)$ if and only if

$$\omega_0 \cdot |\tan \theta_f| \cdot \max\{|\sin \phi_f|, |\cos \phi_f|\} \leq g_0$$

(11)

**Proof**: See in Appendix.

Subsequently, we will further give a theorem for circle-set stabilizability of two-level quantum systems.

**Theorem 2**: For $\forall \theta_f \in [0, \pi]$, $\forall g_0 > 0$ and $\forall t_0 > 0$, the controlled quantum system Eq. (7) is $C_{\theta_f}$-stabilizable from $\Sigma$ by $\Omega_{BC}(g_0)$.

**Proof**: See in Appendix.

**Remark 2**: It should be underlined that $|\psi(t)| = \cos \frac{\theta_f}{2} |0\rangle + e^{i(\phi_f - 2k\pi)} \sin \frac{\theta_f}{2} |1\rangle = |\psi_f\rangle$ when $t = t_f + \frac{2k\pi}{\omega_0}$ where $k \in \mathbb{Z}^+$. This means that for all $|\psi_f\rangle \in \Sigma$, it can be dynamically stored by permissible control in $\Omega_{BC}(g_0)$. We immediately conclude that the controlled two-level quantum system Eq. (7) is $C_{\theta_f}$-stabilizable from $\Sigma$ by $\Omega_B(g_0)$ for all $\theta_f \in [0, \pi]$ since $\Omega_{BC}(g_0)$ is the subset of $\Omega_B(g_0)$.

3.3 Circle-set stabilizability within the specified time for two-level quantum systems

In this subsection, we will establish a theorem to explore whether or not the system (7) is $C_{\theta_f}$-stabilized from $\Sigma$ within the specified time $T_s$ by admissible controls in $\Omega_{BC}(g_0)$.

**Theorem 3**: For any $\theta_f \in [0, \pi]$ and any $t_0 > 0$, the controlled two-level system Eq. (7) is $C_{\theta_f}$-stabilized within the specified time $T_s$ from $\Sigma$ at $t_0$ by $\Omega_{BC}(g_0)$ if

$$\frac{4\pi}{g_0} + \frac{8\pi}{\omega_0} \leq T_s.$$ 

(12)

Furthermore, Eq. (12) can be improved if the permissible controls are chosen from $\Omega_B(g_0)$ instead of $\Omega_{BC}(g_0)$. 
Theorem 4: Let $\theta_f \in [0, \pi]$ and $\phi_f \in [0, 2\pi)$, $\forall t_0 > 0$, the controlled two-level system Eq. (7) is $C_{\theta_f}$-stabilizable within the specified time $T_s$ from $\Sigma$ at $t_0$ by $\Omega_B(g_0)$ if

$$\frac{\pi}{g_0} + \frac{8\pi}{\omega_0} \leq T_s. \quad (13)$$

or

$$\frac{4\pi}{g_0} + \frac{6\pi}{\omega_0} \leq T_s. \quad (14)$$

3.4 Circle-set stabilization of two-level quantum systems with multiple constrains

In this subsection, we first exploit the sufficient conditions for set-stabilizability with both time constraint $T_s$ and energy constraint $E_s$.

For the given $\theta_0$, $\phi_0$, $\theta_f$ and $\phi_f$, we will investigate whether or not there exist permissible controls such that the controlled qubit system Eq. (7) is $C_{\theta_f}$-stabilizable with the following constrained conditions

$$t_f - t_0 \leq T_s \quad (15)$$

and

$$\int_{t_0}^{t_f} [u_x^2(t) + u_y^2(t)] dt \leq E_s, \quad (16)$$

When the unbounded control are permitted, we have the following theorem:

Theorem 5: Given $T_s$ and $E_s$, for $\forall \theta_0$, $\phi_0$, $\theta_f$ and $\phi_f$, there exist unbounded controls such that the controlled qubit system Eq. (7) is $C_{\theta_f}$-stabilizable with time-energy constrains Eqs. (15) and (16) if $T_s \geq \frac{2\pi}{\omega_0}$ and $E_s \geq \omega_0 \cdot \pi$.

Proof: If $T_s \geq \frac{2\pi}{\omega_0}$ and $E_s \geq \omega_0 \cdot \pi$, then there exists at least $k = 2$ such that

$$\frac{\omega_0 \pi \sin^2 \frac{\theta_0 + \theta_f}{2}}{2E_s} + \frac{\phi_f - \phi_0}{2\pi} - \frac{\cos \frac{\theta_0 + \theta_f}{2}}{2} \leq k \quad (17)$$

and

$$k \leq T_s \omega_0 + \phi_f - \phi_0 - \pi \cos \frac{\theta_0 + \theta_f}{2}. \quad (18)$$

and

$$2k\pi - \phi_f + \phi_0 + \pi \cos \frac{\theta_0 + \theta_f}{2} > 0 \quad (19)$$

hold for $\forall \theta_0$, $\phi_0$, $\theta_f$ and $\phi_f$.

Therefore, we establish the sufficient conditions for $C_{\theta_f}$-stabilizable with time-energy constrains.

4 Stabilizability and set stabilizability of quantum systems

The necessary and sufficient conditions can be established for that the state $|\psi_f\rangle$ of quantum system Eq. (6) is stabilizable from $S_0$.

Theorem 6: The state $|\psi_f\rangle$ of quantum system Eq. (6) is stabilizable from $S_0$ if and only if two following conditions are satisfied:

(R1) $|\psi_f\rangle$ is reachable from any state in $S_0$ by a coherent control $H_c(t)$;

(E2) there exists a static control Hamiltonian $H_c$ such that $|\psi_s\rangle$ is an eigenvector of Hamiltonian $H_0 + H_c$. 7
Remark: Unfortunately, the condition (E2) is very difficult to be satisfied. For two-level quantum systems Eq. (6), the condition (E2) is reduced to Eq. (11). To obtain some intuitive pictures of how Eq. (11) is quite strict, we write Eq. (11) as
\[
|\tan \theta_f| \cdot \max\{|\sin \phi_f|, |\cos \phi_f|\} \leq \frac{g_0}{\omega_0}
\] (20)
and plot the range of stabilizable states with ratio \(\frac{g_0}{\omega_0}\) = 0.1, 0.2, 0.5, 1 in Fig 2.

![Fig. 2. Stabilizable state regions with different ratio \(\frac{g_0}{\omega_0}\)](image)

Because \(\frac{g_0}{\omega_0} \leq 10^{-3} \ll 1\) holds for nuclear spin systems[48], we immediately realize from Fig 2 that Eq. (11) is very strict in some experimental quantum systems.

This observation about stabilizability is in remarkable contrast with that about set-stabilizability in Theorem 2: the controlled two-level quantum system Eq. (7) is \(C_{\theta_f}\)-stabilizable from \(\Sigma\) at \(t_0\) by \(\Omega_{BC}(g_0)\) for \(\forall \theta_f \in [0, \pi]\), \(\forall g_0 > 0\) and \(\forall t_0 > 0\).

For two-level quantum system
\[
\frac{d}{dt}|\psi(t)\rangle = i[\omega_0 S_z + u_x(t)S_x]|\psi(t)\rangle
\] (21)
the necessary condition for (E2) is that \(|\psi_f\rangle = \cos \frac{\theta_f}{2}|0\rangle + e^{i\phi_f} \sin \frac{\theta_f}{2}|1\rangle\) with \(\phi_f = 0, \pi\), this is also in remarkable contrast with the observation that the controlled two-level quantum system Eq. (21) is \(C_{\theta_f}\)-stabilizable from \(\Sigma\) at \(t_0\) by bounded control \(|u_x| \leq g_0\) for \(\forall \theta_f \in [0, \pi]\), \(\forall g_0 > 0\) and \(\forall t_0 > 0\).

For two-level quantum system
\[
\frac{d}{dt}|\psi(t)\rangle = i[\omega_0 S_z + u_y(t)S_y]|\psi(t)\rangle
\] (22)
the necessary condition for (E2) is that \(|\psi_f\rangle = \cos \frac{\theta_f}{2}|0\rangle + e^{i\phi_f} \sin \frac{\theta_f}{2}|1\rangle\) with \(\phi_f = \frac{\pi}{2}, \frac{3\pi}{2}\), this is also in remarkable contrast with the observation that the controlled two-level quantum system Eq. (22) is \(C_{\theta_f}\)-stabilizable from \(\Sigma\) at \(t_0\) by bounded control \(|u_y| \leq g_0\) for \(\forall \theta_f \in [0, \pi]\), \(\forall g_0 > 0\) and \(\forall t_0 > 0\).

From the aforementioned discussions, we realize that the necessary conditions are too strict for stabilizability of quantum systems but some kinds of set stabilizability notions are available.

We further present another example to illustrate that sufficient conditions are easier to be fulfilled for set stabilizability.
Consider a controlled two-qubit system governed by the equation
\[ \frac{d}{dt} |\psi(t)\rangle = [H_0 + H_c(t)] |\psi(t)\rangle \]  
(23)
where \( H_0 = -\omega_0 \sigma_x^2 + \omega_0 \sigma_z^2 \) is system Hamiltonian and \( H_c(t) = \sum_{i,j=x,y} u_{ij}(t) \sigma_i^1 \otimes \sigma_j^2 \) are controlled Hamiltonian, respectively.

Let \( \Sigma_s = \text{span}\{|0_12\rangle, |1_10\rangle\} \) be a subspace for two-qubit system, we have
\[ \Sigma_s = \{\cos \frac{\theta}{2} |0_12\rangle + e^{i\phi} \sin \frac{\theta}{2} |1_10\rangle : \theta \in [0, \pi], \phi \in [0, 2\pi]\} \]  
(24)
Denote a maximal entangled state subset as
\[ E_M = \{\frac{\sqrt{2}}{2} (|0_12\rangle + e^{i\phi} |1_10\rangle) : \phi \in [0, 2\pi]\} \]  
(25)
Introducing \( |0^L\rangle = |0_12\rangle \) and \( |1^L\rangle = |1_10\rangle \), we have
\[ \sigma_x^L = |0^L\rangle \langle 0^L| - |1^L\rangle \langle 1^L| = \frac{1}{2} (\sigma_x^2 \otimes I^2 - I^1 \otimes \sigma_z^2) \]  
(26)
and
\[ \sigma_z^L = |1^L\rangle \langle 0^L| + |0^L\rangle \langle 1^L| = \frac{1}{2} (\sigma_x^2 \otimes \sigma_z^2 + \sigma_y^1 \otimes \sigma_z^2) \]  
(27)
From the geometric point of view, the subspace \( \Sigma_s \) can be regarded as a Bloch sphere of the encoded qubit
\[ \Sigma_s = \{\cos \frac{\theta}{2} |0^L\rangle + e^{i\phi} \sin \frac{\theta}{2} |1^L\rangle : \theta \in [0, \pi], \phi \in [0, 2\pi]\} \]  
(29)
and the maximal entangled state subset
\[ E_M = \{\cos \frac{\pi}{4} |0^L\rangle + e^{i\phi} \sin \frac{\pi}{4} |1^L\rangle : \phi \in [0, 2\pi]\} \]  
(30)
can be treated as a circle on the Bloch sphere.

Let \( S_i^L = \frac{1}{2} \sigma_i^L \) with \( i = x, y, z \), \( u_{xx}(t) = u_{yy}(t) = u_x^L(t) \) and \( u_{xy}(t) = -u_{yx}(t) = u_y^L(t) \), then Eq. (23) can be rewritten as
\[ \frac{d}{dt} |\psi(t)\rangle = 4[-\omega_0 S_x^L + u_x^L(t) S_y^L + u_y^L(t) S_x^L] |\psi(t)\rangle \]  
(31)
Let \( \Omega_{BC}^L(g_0) = \{\sum_{i,j=x,y} u_{ij}(t) \sigma_i^1 \otimes \sigma_j^2 : |u_{ij}| \leq g_0\} \) and \( \Omega_B^L(g_0) = \{\sum_{i,j=x,y} u_{ij}(t) \sigma_i^1 \otimes \sigma_j^2 : |u_{ij}| \leq g_0\} \) for \( i, j = x, y \), we have the following Corollaries:

**Corollary 1:** For \( \forall g_0 > 0 \) and \( \forall t_0 > 0 \), the controlled qubit system Eq. (23) is \( E_M \)-stabilizable from \( \Sigma_s \) at time \( t_0 \) by \( \Omega_{BC}^L(g_0) \).

**Corollary 2:** For \( \forall t_0 > 0 \), the controlled qubit system Eq. (23) is \( E_M \)-stabilizable within \( T_s \) from \( \Sigma_s \) at time \( t_0 \) by \( \Omega_{BC}^L(g_0) \) if \( \frac{1}{g_0} + \frac{\pi}{2g_0} \leq T_s \).
Corollary 3: For $\forall t_0 > 0$, the controlled qubit system Eq. (23) is $E_M$-stabilizable within $T_s$ from $\Sigma$ at time $t_0$ by $\Omega_B(g_0)$ if \( \frac{\omega_0}{g_0} + \frac{\pi}{g_0} \leq T_s \) or \( \frac{\omega_0}{g_0} + \frac{3\pi}{2g_0} \leq T_s \).

The aforementioned results suggest that set-stabilizability can be used for studying entangled state generation problem.

5 Conclusions

In summery, we explored set-stabilizability by constrained open-loop controls in this research. Both controllability and stabilizability can be regarded as the special case of set-stabilizability. The necessary and sufficient conditions are also established for stabilizability of quantum closed systems, and it is further revealed that the necessary conditions are too strict for stabilization of some concrete quantum systems like nuclear spin systems. This further justifies the introduction of the set-stabilizability notion.

We also clarify how to define an equilibrium point of Schrödinger Equation from the physical point of view. Strictly speaking, stabilizability problems for quantum systems should be discussed in terms of density operators and master equations. It is exemplified that set-stabilizability can be used for investigating quantum information processing problems including quantum information storage and entangled state generation.

In our opinion, it should be further investigated that what kind of set-stabilizability is achieved for quantum open systems.

Acknowledgements

Partially supported by the National Nature Science Foundation of China under Grant Nos. 61273202 and 61134008.

6 Appendix

6.1 Proof of Theorem 1

(1) First, we need to prove that if Eq. (11) holds, there always exist bounded controls in $\Omega_B(g_0)$ to transit the qubit system from an arbitrary initial state $|\psi(t_0)\rangle = |\psi_0\rangle = \cos \theta_0 |0\rangle + e^{i\phi_0} \sin \theta_0 |1\rangle \in \Sigma$ to another arbitrary target state $|\psi(t_f)\rangle = |\psi_f\rangle = \cos \theta_f |0\rangle + e^{i\phi_f} \sin \theta_f |1\rangle$ with $t_0 < t_f < +\infty$, and $|\psi(t)\rangle\langle\psi(t)| = |\psi_f\rangle\langle\psi_f|$ with $t \geq t_f$.

Choose the permissible controls as follows:

\[
\begin{aligned}
    u_x(t) &= \begin{cases} 
        g \cos[\omega_f(t - t_0) + \varphi_1] & t \in [t_0, t_f) \\
        \omega_0 \cdot \tan \theta_f \cdot \cos \phi_f & t \in [t_f, +\infty) 
    \end{cases} \\
    u_y(t) &= \begin{cases} 
        g \sin[\omega_f(t - t_0) + \varphi_1] & t \in [t_0, t_f) \\
        \omega_0 \cdot \tan \theta_f \cdot \sin \phi_f & t \in [t_f, +\infty) 
    \end{cases}
\end{aligned}
\]

(32)

and

\[
\begin{aligned}
    \varphi_1 &= \phi_0, \\
    g &= \frac{\omega_0 \pi \sin \frac{\theta_f + \phi_f}{2}}{\phi_k^{iap}}, \\
    \omega_f &= \frac{-(2k\pi - \phi_f + \phi_0)\omega_0}{\phi_k^{iap}}
\end{aligned}
\]

(33)
and
\[ t_f = \frac{\phi_{fap}^k}{\omega_0} + t_0 \]  \hspace{1cm} (37)

where
\[ \phi_{fap}^k = 2k_{fap} \pi - \phi_f + \phi_0 + \pi \cos \frac{\theta_0 + \theta_f}{2} \]  \hspace{1cm} (38)

and \( k_{fap} \) is such an integer that
\[ k_{fap} \geq \frac{\phi_f - \phi_0 - \pi \cos \frac{\theta_0 + \theta_f}{2}}{2 \pi} + \frac{\omega_0 \sin \frac{\theta_0 + \theta_f}{2}}{2 \omega_0} \]  \hspace{1cm} (39)

i) It is demonstrated by some calculations that \( |\psi(t_f)\rangle = \cos \frac{\theta_f}{2} |0\rangle + e^{i\phi_f} \sin \frac{\theta_f}{2} |1\rangle \).

ii) When \( t \geq t_f \), the whole system’s Hamiltonian is represented by \( \frac{\omega_0}{\cos \theta_f} H_f \) with \( H_f = [\cos \theta_f \sigma_z + \sin \theta_f \cos \phi_f \sigma_x + \sin \theta_f \cos \phi_f \sigma_y] \). It is easy to check that \( [H_f, |\psi_f\rangle\langle \psi_f|] = 0 \).

iii) Since Eq. (11) holds, \( |\omega_0 \cdot \tan \theta_f \cdot \cos \phi_f| \leq g_0 \) and \( |\omega_0 \cdot \tan \theta_f \cdot \sin \phi_f| \leq g_0 \). Note that \( g \leq g_0 \), we conclude that the permissible controls given by Eqs. (32-33) belongs to \( \Omega_B(g_0) \).

From the aforementioned observations i)-iii), we conclude that the controlled qubit system Eq. (7) is \( P_{\theta_f, \phi_f} \)-stabilizable from \( \Sigma \) at \( t_0 \) by \( \Omega_B(g_0) \) if Eq. (11) holds.

(2) Second, we need to prove that if the controlled qubit system Eq. (7) is \( P_{\theta_f, \phi_f} \)-stabilizable from \( \Sigma \) at \( t_0 \) by \( \Omega_B(g_0) \), then Eq. (11) holds.

If there exist permissible controls in \( \Omega_B(g_0) \) such that Eq. (7) is \( P_{\theta_f, \phi_f} \)-stabilizable from \( \Sigma \), then there exists static Hamiltonian \( H_c = u_x \sigma_x + u_y \sigma_y \in \Omega_B(g_0) \) such that \( H_0 + H_c \) satisfies \( [H_0 + H_c, |\psi_f\rangle\langle \psi_f|] = 0 \). Note that \( [H_0 + H_c, |\psi_f\rangle\langle \psi_f|] = 0 \) if and only if there exists such a scale \( \gamma \in R \) that \( H_0 + H_c = \gamma |\cos \theta_f \sigma_z + \sin \theta_f \cos \phi_f \sigma_x + \sin \theta_f \cos \phi_f \sigma_y| \). Therefore, the following equations should hold simultaneously:
\[ \omega_0 = \gamma \cos \theta_f \]  \hspace{1cm} (40)

and
\[ u_x = \gamma \sin \theta_f \cos \phi_f \]  \hspace{1cm} (41)

and
\[ u_y = \gamma \sin \theta_f \sin \phi_f \]  \hspace{1cm} (42)

Thus \( u_x = \frac{\omega_0}{\cos \theta_f} \sin \theta_f \cos \phi_f \) and \( u_y = \frac{\omega_0}{\cos \theta_f} \sin \theta_f \sin \phi_f \). Recall that \( H_c \in \Omega_B(g_0) \), i.e., \( |u_x| \leq g_0 \) and \( |u_y| \leq g_0 \), we conclude that Eq. (11) holds.

This completes the proof of Theorem 1.

6.2 Proof of Theorem 2

To prove the theorem, it is sufficient to show there always exist bounded time-continuous functional controls in \( \Omega_{BC}(g_0) \) to transfer the qubit system from an arbitrary initial state \( |\psi(t_0)\rangle = |\psi_0\rangle = \cos \frac{\theta_0}{2} |0\rangle + e^{i\phi_0} \sin \frac{\theta_0}{2} |1\rangle \in \Sigma \) to another arbitrary target state \( |\psi(t_f)\rangle = |\psi_f\rangle = \cos \frac{\theta_f}{2} |0\rangle + e^{i\phi_f} \sin \frac{\theta_f}{2} |1\rangle \in \Sigma \) with \( t_0 < t_f < +\infty \), and \( |\psi(t)\rangle \in \mathcal{C}_{\theta_f} \).

We construct the following permissible controls
\[ u_x(t) = g(t) \cos \omega_0(t - t_1) \]  \hspace{1cm} (43)

and
\[ u_y(t) = -g(t) \sin \omega_0(t - t_1) \]  \hspace{1cm} (44)
Therefore, we conclude that an arbitrary initial state to another arbitrary target state.

Case 1: \( \phi \)

\[
\begin{cases}
0 & t \in [t_0, t_1) \\
g[1 - (t_{1+1} - t_{1-1})^{n}] & t \in [t_1, (t_{1+1} - t_{1-1})] \\
g[1 - (2t_{1+1} - t_{1-1})^{n}] & t \in [(t_{1+1} - t_{1-1}), t_f) \\
0 & t \in [t_f, +\infty)
\end{cases}
\] (45)

where

\[
g(t) = \begin{cases}
0 & t \in [t_0, t_1) \\
g[1 - (t_{1+1} - t_{1-1})^{n}] & t \in [t_1, (t_{1+1} - t_{1-1})] \\
g[1 - (2t_{1+1} - t_{1-1})^{n}] & t \in [(t_{1+1} - t_{1-1}), t_f) \\
0 & t \in [t_f, +\infty)
\end{cases}
\]

with

\[
t_1 = \frac{\phi_0 + \frac{3\pi}{\omega_0}}{t_0}
\] (46)

and

\[
k_{dn} = \min\{k \in \mathbb{Z}^+ | k \geq \frac{(n + 1)\omega_0}{ng_0} \frac{4\pi + \theta_f - \theta_0}{2\pi} + \frac{\phi_f}{2\pi} - \frac{1}{4}\}
\] (47)

and

\[
g = \omega_0 \frac{n + 1}{n} \frac{4\pi + \theta_f - \theta_0}{2k_{dn}\pi + \frac{\pi}{2} - \phi_f} \leq g_0
\] (48)

and

\[
t_f = \frac{2k_{dn}\pi + 2\pi - \phi_f + \phi_0}{\omega_0} + t_0
\] (49)

After some calculations, it is demonstrated that \( |\psi(t_1)| = \cos \frac{\theta_0}{2}|0\rangle + i\sin \frac{\theta_0}{2}|1\rangle \) and \( |\psi(t_f)| = \cos \frac{\theta_f}{2}|0\rangle + e^{i\phi_f} \sin \frac{\theta_f}{2}|1\rangle \). Furthermore we have \( |\psi(t)| = \cos \frac{\theta_f}{2}|0\rangle + e^{i(\phi_f - \omega_0(t-t_f))} \sin \frac{\theta_f}{2}|1\rangle \in C_{\theta_f} \), when \( t \geq t_f \).

Because the aforementioned analyses hold for any pair of initial and target states, this implies that one can always construct a local \( n^{th} \)-order function to dynamically modulate amplitude and further steer quantum systems from an arbitrary initial state to another arbitrary target state.

Therefore, we conclude that \( \forall \theta_f \in [0, \pi] \), the controlled qubit system Eq. (7) is \( C_{\theta_f} \)-stabilizable from \( \Sigma \) by \( \Omega_{BC}(g_0) \).

6.3 Proof of Theorem 3

\textbf{Proof:} Notice in the proof of Theorem 2 that one can choose \( t_1 = \frac{\phi_0 - \frac{3\pi}{2}}{2g_0} + t_0 \) and \( t_f = \frac{2k_{dn}\pi - \phi_f + \phi_0}{\omega_0} + t_0 \) if \( \phi_0 - \frac{\pi}{2} \geq 0 \), and observe that one can select

\[
k_{dn} = \min\{k \in \mathbb{Z}^+ | k \geq \frac{(n + 1)\omega_0}{ng_0} \frac{\theta_f - \theta_0}{2\pi} + \frac{\phi_f}{2\pi} - \frac{1}{4}\}
\] (50)

and

\[
g = \omega_0 \frac{n + 1}{n} \frac{\theta_f - \theta_0}{2k_{dn}\pi + \frac{\pi}{2} - \phi_f} \leq g_0
\] (51)

if \( \theta_f - \theta_0 \geq 0 \).

Therefore, we can carry on the analysis to estimate the transition time \( t_f - t_0 \) by considering four different cases:

Case 1: \( \phi_0 - \frac{\pi}{2} < 0 \) and \( \theta_f - \theta_0 < 0 \).

If \( n > \frac{2\omega_0}{g_0} \), then \( \frac{2\omega_0}{g_0} < \frac{1}{4} \). Furthermore, we have from Eq. (47) that \( k_{dn} \leq \frac{2\omega_0}{g_0} + 2 \).

Thus, if we choose \( n > \frac{2\omega_0}{g_0} \), then

\[
t_f - t_0 = \frac{2k_{dn}\pi + 2\pi - \phi_f + \phi_0}{\omega_0} \leq \frac{4\pi}{g_0} + \frac{8\pi}{\omega_0}
\] (52)

Case 2: \( \phi_0 - \frac{\pi}{2} < 0 \) and \( \theta_f - \theta_0 \geq 0 \).
If $n > \frac{2\omega_0}{g_{0}}$, then $\frac{\omega_0}{2g_{0}} < \frac{1}{4}$. Furthermore we obtain from Eq. (50) that $k_{dn} \leq \frac{\omega_0}{2g_{0}} + 2$.

Thus, if we choose $n > \frac{2g_{0}}{\omega_0}$, then

$$t_f - t_0 = \frac{2k_{dn}\pi + 2\pi - \phi_f + \phi_0}{\omega_0} \leq \frac{\pi}{g_{0}} + \frac{8\pi}{\omega_0}$$

(53)

Case 3: $\phi_0 - \frac{\pi}{2} \geq 0$ and $\theta_f - \theta_0 < 0$.

If $n > \frac{8g_{0}}{\omega_0}$, then $\frac{2g_{0}}{\omega_0} < \frac{1}{4}$. Furthermore, we have from Eq. (47) that $k_{dn} \leq \frac{2\omega_0}{g_{0}} + 2$.

Thus, if we choose $n > \frac{8g_{0}}{\omega_0}$, then

$$t_f - t_0 = \frac{2k_{dn}\pi - \phi_f + \phi_0}{\omega_0} \leq \frac{4\pi}{g_{0}} + \frac{6\pi}{\omega_0}.$$  

(54)

Case 4: $\phi_0 - \frac{\pi}{2} \geq 0$ and $\theta_f - \theta_0 \geq 0$.

If $n > \frac{2\omega_0}{g_{0}}$, then $\frac{2\omega_0}{g_{0}} < \frac{1}{4}$. Furthermore, we obtain from Eq. (50) that $k_{dn} \leq \frac{\omega_0}{2g_{0}} + 2$.

Thus, if we choose $n \geq \frac{2g_{0}}{\omega_0}$, then

$$t_f - t_0 = \frac{2k_{dn}\pi - \phi_f + \phi_0}{\omega_0} \leq \frac{\pi}{g_{0}} + \frac{6\pi}{\omega_0}.$$ 

(55)

From Eqs.(52-55), we have

$$t_f - t_0 \leq \frac{4\pi}{g_{0}} + \frac{8\pi}{\omega_0}.$$  

(56)

for any $|\psi_0\rangle \in \Sigma$ and any $|\psi_f\rangle \in C_{\theta_f}$.

Therefore, this completes the proof of Theorem 3.

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