Robust mean field social control problems with applications in analysis of opinion dynamics

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1. Introduction

Mean field (MF) games and control have become a heated topic for large-population systems over the past decade in system control, applied mathematics, and economics (Bensoussan et al., 2013; Caines et al., 2017; Gomes & Saude, 2014). It has been found wide applications in many areas such as dynamic production adjustment (Wang & Huang, 2019), vaccination games (Arefin et al., 2019), resource allocation in internet of things (Larranaga et al., 2020), etc. A significant feature of MF systems is the interactive weakly-coupling structure across considerable agents. The interactions between agents are negligible, but the aggregate impact of overall populations to a given agent is significant. The main methodology of MF systems is to model the coupling among agents as the population aggregation effect, also called MF effect. Agents interact with each other through the MF effect. Constrained by the computation complexity and communication capacity, it is difficult and impractical to analyse the large-population system under centralised information structure. Therefore, decentralised strategies, which only use local information, are more practical and attract intensive research attention. With consistent MF approximations, the high-dimensional multi-agent optimisation problem can be transformed into a low-dimensional optimal control problem for a representative agent. Along this line, decentralised solutions may be obtained accordingly and the complexity of the original problem is greatly reduced.

MF game theory was initiated by the parallel works of Huang et al. (2007) and Lasry and Lions (2007). Agents are competitive to minimise their own costs. According to the different dynamics and cost setup, the literature of MF games can be classified into linear-quadratic (LQ) and nonlinear types. The LQ type is commonly adopted because of its analytical tractability (see Bensoussan et al., 2013; Huang, 2010; Huang et al., 2003, 2007; Li & Zhang, 2008; Wang & Zhang, 2012a, 2012b). Especially, Huang et al. (2003, 2007) studied ɛ-Nash equilibrium strategies for LQ MF games with discounted costs based on the Nash certainty equivalence (NCE) approach. The NCE approach was then applied to the cases with long run average costs in Li and Zhang (2008) and with Markov jump parameters in Wang and Zhang (2012a). For other aspects of MF games, the readers are referred to MF games with a major player in Huang (2010); Wang and Zhang (2012b) and oblivious equilibria for large-scale Markov decision processes in Weintraub et al. (2008). The nonlinear type is also of great importance because of its modelling generality; see Lasry and Lions (2007), Huang et al. (2006), Carmona and Delarue (2013) and Weintraub et al. (2008) for a detailed understanding.

Besides the above non-cooperative MF games, the social optimal control problems with MF models have also received much attention. The so-called social optimal control problem refers to that all agents are cooperative to optimise the social cost—the sum of costs of all agents, which was considered as a type of team decision problem in Radner (1962) and Ho (1980). The MF social control problem was first studied in Huang et al. (2012) and the Social Certainty Equivalence methodology was developed. Wang and Zhang (2017) studied the social optimal decision problem of the Markov jump MF model, and gave the asymptotic team-optimal solution. Huang and Nguyen (2019) investigated the MF team problem with a major player by analysing forward-backward stochastic differential equations (FBSDEs). See Arabneydi and Mahajan (2016) and Arabneydi and Aghdam (2020) for the case involving interaction among multiple teams.
All the works described above assume that the dynamic structure of models is accurate. However, the mathematical models we often studied are only approximations of actual problems. Therefore, it is meaningful to study robust MF games and control with unmodelled dynamics. The cases of MF games and social control with common uncertainty were studied in Huang and Huang (2017) and Wang et al. (2021) by introducing a disturbance as the control from a major agent, respectively. Such modelling can accommodate considerable situations by noting that common uncertainty might represent an external factor affecting simultaneously all the agents in the large-population system. Furthermore the partial equivalence between robust MF games and risk-sensitive games was obtained in Moon and Başar (2017). Then the result of robust MF games was applied into opinion dynamics in Bauso et al. (2016) by introducing local disturbance for each agent. Based on the above discussion, the MF social control with local disturbance is also worthy of research and initial investigations were studied in the conference papers (Liang & Wang, 2019; Yu et al., 2018).

Social networks and opinion dynamics have also drawn increasing attention from the systems and control community in recent years (see Anderson & Ye, 2019; Bauso et al., 2016; Chen et al., 2019; Friedkin et al., 2016; Jia et al., 2015). An important feature of social networks is the exchange and discussion of opinions between individuals. The opinion dynamics focus on how individuals’ opinions form and evolve over time through interactions with their peers. It has been observed that numerous phenomena, such as emulation, mimicry and herding behaviours, often occur when multiple social groups interact (Bauso et al., 2016). A successful model of opinion dynamics used in small-scale networks is the agent-based model where each individual is represented by an agent and the individual opinion on a topic is represented by a real value, evolving in time, such as Degroot model (see Chen et al., 2019; Jia et al., 2015). A graph is introduced to describe the interactions among agents and the evolution of opinions over time can be obtained which can reveal whether the phenomena of emulation, mimicry, and herding behaviours have occurred. However, for larger and larger networks, the agent-based model may be more and more inappropriate (Anderson & Ye, 2019). Hence, there is a need to introduce new models or methods to capture the interactions among a large numbers of agents.

It has been observed that the evolution of opinions as a result of the interactions among agents has the flavor of an averaging process (Bauso et al., 2016). This inspires us to use the MF control theory to describe the evolution of large population opinions. We model the interactions among agents as the MF coupling term. Each agent is paired with a cost functional where herd behaviour or crowd-seeking attitudes are rewarding. Hence each agent attempts to steer the individual opinion to the average opinion. These guide us to study the evolution of opinions within the framework of MF control.

In this paper, we study MF LQ social optimal control problems with unmodelled dynamics. Different from the previous works (Bauso et al., 2016; Yu et al., 2018), MF terms are involved not only in the cost functional, but also in the state equation. Due to the presence of state coupling, when perturbing the control or disturbance of an agent, the states of all agents will change. Compared with the situation of no state coupling (Huang et al., 2012), this brings about much difficulty in the control design. In order to design decentralised strategies based on MF approximations, we construct two auxiliary robust control problems by social variational and direct decoupling methods, respectively. Both auxiliary optimal control problems are identical when MF approximations are applied. By solving the auxiliary robust optimal control problem with consistent requirement, we design a set of decentralised strategies. By verifying the convexity-concavity property of the robust social control problem, the decentralised strategies are shown to have the property of asymptotic robust social optimality. Some general conditions are given for existence and uniqueness of the solution to the consistent system.

Besides, we apply the above results into the analysis of opinion dynamics in social networks. Our results can be seen as a generalisation of Bauso et al. (2016), in which the authors studied the integrator opinion dynamics by MF games and obtained the convergence property of the population distribution. In the case of single population, we give a detailed analysis of the opinions evolution and convergence properties in a general situation. The difference between agents’ opinions and the network average opinion is proved to be stochastically bounded. An explicit expression between probability distribution and bound of opinions deviation from average opinion is obtained by eigenvalue analysis, analytic geometry and martingale inequality. The long time behaviour of opinions evolution is also analysed by stochastic stability theory. In the case of multiple populations, we make a discussion for local interactions among populations of agents via graphon theory.

The main contributions of the paper are listed as follows.

- Using variational analysis and direct decoupling methods, we derive two equivalent auxiliary robust optimal control problems. By solving the auxiliary optimal control problem subject to consistent MF approximations, we obtain a set of decentralised strategies.
- By verifying the convexity-concavity property of the social control problem with unmodelled dynamics, it is proved that the decentralised strategies have the property of asymptotic robust social optimality.
- The evolution of agents’ opinions is analysed in a general situation by the MF control approach. An explicit expression between probability distribution and bound of opinions deviation from average opinion is obtained by using eigenvalue analysis and martingale inequality. The long time behaviour of opinions is also analysed by stochastic stability theory. All opinions are shown to reach agreement with the average opinion in a probabilistic sense.
- Local interactions among multiple sub-populations are considered via graphon theory. For the case of heterogeneous agents, decentralised strategies are constructed with the help of infinite dimensional systems theory.

The rest of the paper is organised as follows. In Section 2, we formulate the MFSC problem with unmodelled dynamics. In Section 3, centralised strategies are analysed by the methods of variational analysis and direct decoupling. In Section 4, decentralised strategies are designed by solving the auxiliary problem.
subject to consistent MF approximations. In Section 5, the set of decentralised strategies is proved to have asymptotic social optimality. In Section 6, we apply the results into the analysis of opinion dynamic and show the interaction of individual opinions in the case of single population and multiple sub-populations. In Section 7, the evolution of opinions is simulated. Section 8 concludes the paper.

Notation: Throughout this paper, we denote by $\mathbb{R}^k$ the $k$-dimensional Euclidean space, and $\mathbb{R}^{n \times k}$ the set of all $n \times k$ matrices. We use $\| \cdot \|$ to denote the 2-norm for a vector. For a matrix $M$, $M^T$ denotes its transpose. For a symmetric matrix $M$, $M > 0$ means that $M$ is positive (non-negative) definite. For a symmetric matrix $Q$ and a vector $z$, $\|z\|_Q^2 = z^T Q z$. For a family of $\mathbb{R}^n$-valued random variables $\{x(\lambda), \lambda \geq 0\}$, $\sigma(x(\lambda), \lambda \leq t)$ is the $\sigma$-algebra generated by the collection of random variables. Let $\otimes$ be the Kronecker product. Let $C([0, T], \mathbb{R}^n)$ denote the space of all $\mathbb{R}^n$-valued continuous functions on $[0, T]$, and $C_b([0, \infty), \mathbb{R}^n)$ be the space of all the bounded continuous functions on $[0, \infty)$. Let $L^2(0, T; \mathbb{R})$ be the space of all $\mathbb{R}$-valued measurable functions $\varphi : [0, T] \to \mathbb{R}$ such that $\mathbb{E} \int_0^T \|\varphi(t)\|^2 dt < \infty$. We use $C, C_1$ etc. to denote generic constants, which may vary from place to place.

2. Socially optimal control problems

2.1 Dynamics and costs

Consider a large-population system with $N$ agents. The dynamics of the $i$th agent is given by the following stochastic differential equation (SDE)

$$dx_i(t) = \left[ Ax_i(t) + Bu_i(t) + Gx^{(N)}(t) + f_i(t) \right] dt + \sigma \, dW_i(t), \quad x_i(0) = x_{i0}, \quad t \geq 0, \quad 1 \leq i \leq N,$$

where $x_i(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}^m$ are the state and input of agent $i$, respectively. $\{W_i(t), 1 \leq i \leq N\}$ are a sequence of mutually independent $d$-dimensional Brownian motions defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. $x^{(N)}(t) = (1/N) \sum_{i=1}^N x_i(t)$ is the MF term which reflects the overall effect of the population to each agent. $f_i(t) \in \mathbb{R}^n$ is an unknown local disturbance, which can be considered as the disturbance from the external environment or unmodelled dynamics. The constant matrices $A, B, G$ and $\sigma$ have compatible dimensions. For notational brevity, the time argument of a process is often suppressed when its value at time $t$ is used. Denote $u = (u_1, \ldots, u_N)$ and $f = (f_1, \ldots, f_N)$.

In this paper, we consider the following tracking-type cost functional for agent $i$:

$$J_i(u(\cdot), f(\cdot)) = \mathbb{E} \int_0^T \left[ \|x_i(t) - \Gamma x^{(N)}(t) - \eta_i\|^2_Q + \|u_i(t)\|^2_{R_1} - \|f_i(t)\|^2_{R_2} \right] dt + \mathbb{E} \int_0^T \|x_i(T) - \Gamma_0 x^{(N)}(T) - \eta_0\|^2_H.$$

The goal of agents is to track $\Gamma x^{(N)} + \eta$. The constant matrices or vectors $Q, H, R_1, R_2, \Gamma, \Gamma_0, \eta$ and $\eta_0$ have compatible dimensions; $Q \geq 0, H \geq 0, R_1 > 0$, and $R_2 > 0$. The social cost is defined by

$$j^{(N)}_{soc}(u(\cdot), f(\cdot)) = \sum_{i=1}^N J_i(u(\cdot), f(\cdot)).$$

2.2 Two solutions based on different information patterns

We study social optimal control problems under two types of information sets. First, we define the following admissible control sets for different information patterns:

$$U^c_i = \left\{ u_i \mid u_i(t) \text{ is adapted to } \sigma(x_j(t), W_j(s), s \leq t), \quad 1 \leq j \leq N, \quad \mathbb{E} \int_0^T \|u_i(t)\|^2 dt < \infty \right\},$$

$$U^d_i = \left\{ u_i \mid u_i(t) \text{ is adapted to } \sigma(x_i(s), W_i(s), s \leq t), \quad \mathbb{E} \int_0^T \|u_i(t)\|^2 dt < \infty \right\},$$

where $1 \leq i \leq N$. Each strategy $u_i \in U^c_i$ is generated by the information of all agents, hence strategies in $U^c_i$ are centralised. On the contrary, each strategy $u_i \in U^d_i$ is generated by the information of agent $i$, so strategies in $U^d_i$ is decentralised. Then there are two problems based on the different control sets.

(PA) Find a social solution $u(\cdot)$ with centralised strategies to minimise the social cost $j^{(N)}_{soc}(\cdot, f(\cdot))$ under the worst-case disturbance, i.e. $\inf_{\hat{u} \in U^c_i} \sup_{f \in U^{(N)}_{soc}} j^{(N)}_{soc}(\hat{u}(\cdot), f(\cdot))$.

(PP) Find a social solution $u(\cdot)$ with decentralised strategies to minimise the social cost $j^{(N)}_{soc}(u(\cdot), f(\cdot))$ under the worst-case disturbance, i.e. $\inf_{\hat{u} \in U^d_i} \sup_{f \in U^{(N)}_{soc}} j^{(N)}_{soc}(\hat{u}(\cdot), f(\cdot))$.

We impose the following assumption.

(AI) The initial states $x_i(0), 1 \leq i \leq N$ are independent with $\mathbb{E}x_i(0) = x_0$ and there exists a finite constant $C_0$ independent of $N$ such that $\max_{1 \leq i \leq N} \mathbb{E}\|x_i(0)\|^2 \leq C_0$.

3. Centralised strategies

In this section, we explore centralised strategies for Problem (PA) by variational analysis and direct decoupling methods, and obtain two auxiliary problems (P1) and (P1').

Suppose that $j^{(N)}_{soc}(u(\cdot), f(\cdot))$ is minimised by $\hat{u} = (\hat{u}_1, \ldots, \hat{u}_N)$ under the worst-case disturbance $\hat{f} = (\hat{f}_1, \ldots, \hat{f}_N)$, where $\hat{u}_i \in U^c_i$ and $\hat{f}_i \in U^c_i$. Let $\hat{x}_i$ be the corresponding state under $\hat{u}$ and $\hat{f}$, and denote $\hat{x}^{(N)} = (1/N) \sum_{i=1}^N \hat{x}_i$ and $\hat{x}^{(N)}_i = (1/N) \sum_{j \neq i} \hat{x}_j$. Define $u_{-i} = (u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_N)$; $\hat{u}_{-i}$, $\hat{f}_{-i}$ and $\hat{f}_{-i}$ are defined similarly.

3.1 The variational method

The variational analysis is based on person-by-person optimality, and the benefit is that one can only perturb the control of a representative agent and keep the controls of other agents unchanged. Fix $\hat{u}_{-i}$ and $\hat{f}_{-i}$, and perturb $u_i$ and $f_i$ respectively. Denote $\delta x^{(N)} = (1/N) \sum_{i=1}^N \delta x_i$ and $\delta x^{(N)}_{-i} = (1/N) \sum_{j \neq i} \delta x_j$.
The difficulty in obtaining the evolution of the state of a representative agent is that the dynamics of all agents are coupled by the MF term. The first step is to obtain the variation of the MF term \( \dot{x}^{(N)} \). We only examine the perturbation of \( u_i \); the case of perturbing \( f_i \) is similar. By perturbing \( u_i = \hat{u}_i + \delta u_i \) and fix \( f_i = \hat{f}_i \) in (1), we obtain

\[
\frac{d\delta x_j}{dr} = A\delta x_j + G\delta x^{(N)}, \quad \delta x_j(0) = 0, \quad j \neq i.
\]

Noting for agent \( j, \delta x_j(0) = 0, j \neq i \), this implies \( \delta x_j = \delta x_k \), for any \( j, k \neq i \), which further leads to

\[
\frac{d\delta x_j}{dr} = \left(A + \frac{N-1}{N}\right)G\delta x_j + \frac{G}{N}\delta x_i, \quad \delta x_j(0) = 0.
\]

By solving this equation, we obtain

\[
\delta x_j(t) = \frac{1}{N} \delta x_i(t) + \frac{N-1}{N^2} \int_0^t e^{(A + \frac{N-1}{N}G)(t-\tau)} \delta x_i(\tau) \, d\tau,
\]

which gives

\[
\delta x^{(N)}(t) = \frac{1}{N} \delta x_i(t) + \frac{N-1}{N^2} \int_0^t e^{(A + \frac{N-1}{N}G)(t-\tau)} \delta x_i(\tau) \, d\tau.
\]

Now we turn to derive the variation of the social cost functional. From (2), the cost variation of agent \( i \) is given by

\[
\delta J_i(\delta u_i, \hat{f}_i) = 2\mathbb{E} \int_0^T \left[ (\hat{x}_i - \Gamma \hat{x}^{(N)} - \eta)^T Q(\delta x_i - \Gamma \delta x^{(N)}) + \hat{u}_i^T R_1 \delta u_i \right] dt
\]

\[
+ 2\mathbb{E} \left[ (\hat{x}_i(T) - \Gamma_0 \hat{x}^{(N)}(T) - \eta_0)^T H(\delta x_i(T) - \Gamma_0 \delta x^{(N)}(T)) \right].
\]

For agent \( j, j \neq i \), we have

\[
\delta J_j(\delta u_i, \hat{f}_i) = 2\mathbb{E} \int_0^T \left[ (\hat{x}_j - \Gamma \hat{x}^{(N)} - \eta)^T Q(\delta x_j - \Gamma \delta x^{(N)}) \right] dt
\]

\[
+ 2\mathbb{E} \left[ (\hat{x}_j(T) - \Gamma_0 \hat{x}^{(N)}(T) - \eta_0)^T H(\delta x_j(T) - \Gamma_0 \delta x^{(N)}(T)) \right],
\]

which further implies

\[
\sum_{j \neq i} \delta J_j(\delta u_i, \hat{f}_i) = 2\mathbb{E} \int_0^T \left[ (\hat{x}^{(N)}_i - \frac{N-1}{N} \Gamma \hat{x}^{(N)} - \frac{N-1}{N} \eta)^T Q(\delta x_i - \Gamma \delta x^{(N)}) \right] dt
\]

\[
+ 2\mathbb{E} \left[ (\hat{x}^{(N)}_i(T) - \frac{N-1}{N} \Gamma_0 \hat{x}^{(N)}(T) - \frac{N-1}{N} \eta_0)^T H(\delta x_i(T) - \Gamma_0 \delta x^{(N)}(T)) \right].
\]

Thus, the variation of the social cost functional is given by

\[
\delta J_{soc}^{(N)}(\delta u_i, \hat{f}_i)
\]

\[
= \delta J_i(\delta u_i, \hat{f}_i) + \sum_{j \neq i} \delta J_j(\delta u_i, \hat{f}_i)
\]

\[
= 2\mathbb{E} \int_0^T \left\{ (\hat{x}_i - \Gamma \hat{x}^{(N)} - \eta)^T Q\left[ \left(I - \frac{1}{N} \Gamma \right) \delta x_i \right.ight.
\]

\[
- \frac{N-1}{N^2} \Gamma \int_0^T e^{(A + \frac{N-1}{N}G)(t-\tau)} \delta x_i(\tau) d\tau 
\]

\[
+ \left( \Gamma \hat{x}^{(N)}_i(\tau) - \frac{N-1}{N} \Gamma_0 \hat{x}^{(N)}(\tau) - \frac{N-1}{N} \eta_0 \right)^T 
\]

\[
\times Q \left[ -\Gamma \delta x_i + \left(I - \frac{N-1}{N} \Gamma \right) \int_0^T e^{(A + \frac{N-1}{N}G)(t-\tau)} \delta x_i(\tau) d\tau \right]
\]

\[
\left. + \hat{u}_i^T R_1 \delta u_i \right] dt
\]

\[
+ 2\mathbb{E} \left\{ (\hat{x}_i(T) - \Gamma_0 \hat{x}^{(N)}(T) - \eta_0)^T \right.
\]

\[
\times H \left[ \left(I - \frac{N-1}{N} \Gamma_0 \right) \delta x_i(T) + \left(I - \frac{N-1}{N} \Gamma_0 \right) \int_0^T e^{(A + \frac{N-1}{N}G)(t-\tau)} \delta x_i(\tau) d\tau \right]
\]

\[
\left. + \hat{u}_i^T R_1 \delta u_i \right] \left[ (\hat{x}_i(T) - \Gamma_0 \hat{x}^{(N)}(T) - \eta_0)^T 
\]

\[
+ \frac{N-1}{N^2} \Gamma_0 \int_0^T e^{(A + \frac{N-1}{N}G)(t-\tau)} \delta x_i(\tau) d\tau 
\]

\[
+ \left( \Gamma \hat{x}^{(N)}_i(\tau) - \frac{N-1}{N} \Gamma_0 \hat{x}^{(N)}(\tau) - \frac{N-1}{N} \eta_0 \right)^T 
\]

\[
\times Q \left[ -\Gamma \delta x_i + \left(I - \frac{N-1}{N} \Gamma \right) \int_0^T e^{(A + \frac{N-1}{N}G)(t-\tau)} \delta x_i(\tau) d\tau \right]
\]

\[
\left. + \hat{u}_i^T R_1 \delta u_i \right] \}.
\]

When \( N \) is large enough, we may approximate \( \hat{x}^{(N)} \) and \( \hat{x}^{(N)}_i \) by a deterministic function \( \hat{x}^{(N)} \) and \( \hat{x}^{(N)}_i \). The zero first-order variational condition combined with MF approximations gives

\[
2\mathbb{E} \int_0^T \left[ \hat{x}^T Q \delta x_i - (\hat{x} + \eta)^T Q \delta x_i - (\hat{x} - \hat{x} - \eta)^T Q \Gamma \delta x_i 
\]

\[
+ (\hat{x} - \hat{x} - \eta)^T Q(I - \Gamma) 
\]

\[
+ \int_0^T e^{(A+G)(t-\tau)} \delta x_i d\tau + \hat{x}(T) - \Gamma_0 \hat{x}(T) - \eta_0 \right)^T 
\]

\[
\times H(I - \Gamma_0) e^{(A+G)(T-\tau)} \delta x_i + \hat{u}_i^T R_1 \delta u_i \right] dt
\]

\[
+ 2\mathbb{E} \left[ \hat{x}^T(T) H \delta x_i(T) - (\Gamma_0 \hat{x}(T) + \eta_0)^T H \delta x_i(T) 
\]

\[
- (\hat{x}(T) - \Gamma_0 \hat{x}(T) - \eta_0)^T H \Gamma_0 \delta x_i(T) \right\} = 0.
\]

By using the Gâteaux derivative or variation technique (see, e.g., Firoozi et al., 2020), the above equation is the zero first-order variational condition for the following cost functional:
\[ \tilde{J}(u_i, f_i) = \mathbb{E} \int_0^T \left[ x_i^T Q x_i - 2(\Gamma \tilde{x} + \eta)^T Q x_i \ight. \\
- 2(\tilde{x} - \Gamma \tilde{x} - \eta)^T Q \Gamma x_i + 2v^T G x_i \\
+ u_i^T R_1 u_i - f_i^T R_2 f_i \right] dt \\
+ \mathbb{E} \left[ x_i^T(T) H x_i(T) - 2(\Gamma_0 \tilde{x}(T) + \eta_0)^T H_0 x_i(T) \right. \\
- 2(\tilde{x}(T) - \Gamma_0 \tilde{x}(T) - \eta_0)^T H_0 x_i(T) \right], \]

where

\[ \nu(t) = \int_t^T e^{(A+G)^T(t-s)} (I - \Gamma)^T Q(I - \Gamma) \tilde{x}(\tau) - \eta] d\tau \]

+ \left[ (I - \Gamma) \tilde{x}(\tau) - \eta \right] \left. \right|_0^t \right. \\
\times Q(I - \Gamma) e^{(A+G)^T(t-s)} dt G x_i(t) dt. \]

Denote \( Q_t \triangleq Q_t ^T Q_t + \Gamma^T Q_t \tilde{\eta}_t \triangleq (I - \Gamma^T) Q_t \eta_t, H_t \triangleq \Gamma_0 + \Gamma_0^T H_0 + H_0 \Gamma_0 \eta_0 \triangleq (I - \Gamma_0^T) H_0 \eta_0. \)

Based on the above discussion, we construct the following auxiliary robust optimal control problem.

(P1) Find \( \hat{u}_i \in U^d_i \) to minimise the cost functional \( \hat{J}_i(u_i, f_i) \) under the worst-case disturbance \( \hat{f}_i \in U^d_i \), in which

\[ dx_i = (A \hat{x}_i + B \hat{u}_i + G \hat{x} + f_i) dt + \sigma dW_i, \quad x_i(0) = x_{i0}, \]

\[ \hat{J}_i(u_i, f_i) = \mathbb{E} \int_0^T \left[ x_i^T Q x_i + 2(\Gamma \tilde{x} - \tilde{\eta})^T x_i \ight. \\
+ u_i^T R_1 u_i - f_i^T R_2 f_i \right] dt \\
+ \mathbb{E} \left[ x_i^T(T) H x_i(T) + 2(\Gamma_0 \tilde{x}(T) + \eta_0)^T x_i(T) \right], \]

where \( v \) is determined by

\[ \nu = -(A + G)^T v + (Q - Q \tilde{x} - \tilde{\eta}, \nu(T) = (H - H_0) \tilde{x}(T) - \eta_0. \]

3.2 The direct decoupling method

Now we obtain an exact problem by direct decoupling. Without loss of generality, we assume that the terminal terms of cost functionals are 0. The methodology is that first decouple the MF term in the dynamics and then separate the social cost functional to obtain the terms affected by \( u_i \) and \( f_i \).

**Lemma 3.1:** If the initial Problem (PA) has a centralised saddle-point solution \((\hat{u}, \hat{f})\), then it is necessary that \((\hat{u}, \hat{f})\) is a saddle-point solution for the following problem:

(P1') Find \( \hat{u}_i \in U^d_i \) to minimise the cost functional \( J'_i(u_i, f_i) \) under the worst-case disturbance \( \hat{f}_i \in U^d_i \), in which

\[ dx_i = (A \hat{x}_i + B \hat{u}_i + G \hat{x}(N)) dt + \sigma dW_i, \]

\[ \hat{x}_i(0) = x_{i0}, \]

\[ \hat{J}_i(u_i, f_i) = \mathbb{E} \int_0^T L(\hat{x}_i, \hat{x}_i(N), u_i, f_i) dt, \]

where \( j \neq i \) and

\[ L(\hat{x}_i, \hat{x}_i(N), u_i, f_i) = \left\{ \left( (I - \Gamma) \hat{x}_i - \frac{N - 1}{N^2} \Gamma \right) \left( (I - \Gamma) \hat{x}_i - \frac{N - 1}{N^2} \Gamma \right)^T \right\} \]

\[ \times Q \left[ (I - \Gamma(N)) \hat{x}_i - \frac{N - 1}{N^2} \Gamma \int_0^T e^{(A+\frac{N_i}{N^2}G)(t-s)} \hat{x}_i ds + \eta \right], \]

\[ + (N-1) \left( \frac{N}{N-1} I - \Gamma \right) \left( \Gamma \hat{x}_i - \frac{N - 1}{N^2} \Gamma \right) \int_0^T e^{(A+\frac{N_i}{N^2}G)(t-s)} \hat{x}_i ds \]

\[ + \left\{ \left( \frac{N}{N-1} I - \Gamma \right) \left( \frac{N - 1}{N^2} \Gamma \right) \right\} \]

\[ \times \left[ \frac{NI - (N-1)\Gamma}{N} \hat{x}_i(N) \hat{x}_i(N) \hat{x}_i - \frac{N - 1}{N^2} \eta \right], \]

\[ + \| u_i \|_{R_1}^2 - \| f_i \|_{R_2}^2 \right\}. \]

**Proof:** Note that \( u_i, f_i \in U^d_i \), and \( \hat{u}_i, \hat{f}_i \) have been specified in advance and does not change with \( u_i, f_i \in U^d_i \), where \( \hat{u}_i(N) = \frac{(1/N) \sum_{j \neq i} \hat{u}_j} {\frac{N}{N} \sum_{j \neq i} \hat{f}_j} = \frac{(1/N) \sum_{j \neq i} \hat{u}_j} {\frac{N}{N} \sum_{j \neq i} \hat{f}_j} \). Because the dynamics of all agents are coupled by MF term, even if \( \hat{u}_i \) is specified in advance, the state of the auxiliary problem \( \hat{x}_i, j \neq i \), is also varying along with \( u_i \). The first one which needs to do is that separate the system equation into two parts. The first part evolves with
we have
\[
\begin{align*}
J_i &= \mathbb{E} \int_0^T (Z_i + Z'_i) \, dt, \\
&= \mathbb{E} \int_0^T \left( \left( \frac{N}{N-1} I - \Gamma \right) \phi_i - \Gamma \tilde{x}_i \right) \, dt
\end{align*}
\]

where
\[
Z_i = \left\| \phi_i - \frac{\Gamma}{N} \tilde{x}_i - \Gamma \phi_i \right\|_Q^2
\]

and
\[
Z'_i = \| \phi'_i + \eta \|^Q_2.
\]

For agent \( j, j \neq i \), one can obtain
\[
J_j = \mathbb{E} \int_0^T (Z_j + Z'_j) \, dt,
\]

where
\[
Z_j = \left\| \phi_j - \frac{\Gamma}{N} \tilde{x}_j - \Gamma \phi_j \right\|_Q^2 + 2 \left( \phi_j - \frac{\Gamma}{N} \tilde{x}_j - \Gamma \phi_j - \eta \right)^T Q \left( \phi_j - \frac{\Gamma}{N} \tilde{x}_j - \Gamma \phi_j \right) + \| \phi_j \|^2_{R_1} - \| \phi_j \|^2_{R_2}.
\]

Then by (8)–(13), for the social cost \( J^{(N)}_{soc} \) we have
\[
J^{(N)}_{soc} = I_i + \sum_{j \neq i} J_j = \mathbb{E} \int_0^T \left( (Z_i + Z'_i) + \sum_{j \neq i} (Z_j + Z'_j) \right) \, dt,
\]

where
\[
\sum_{j \neq i} Z_j = (N - 1) \left\| \left( \frac{N}{N-1} I - \Gamma \right) \phi_i - \Gamma \tilde{x}_i \right\|_Q^2 + 2 \left[ \left( \frac{N}{N-1} I - \Gamma \right) \phi_i - \Gamma \tilde{x}_i \right]^T Q \left( \frac{N}{N-1} I - \Gamma \right) \phi_i - \Gamma \tilde{x}_i \right\|_Q^2 - [NI + (N - 1) \Gamma] \phi_i - (N - 1) \eta \right].
\]

From the viewpoint of optimal control, for the social cost functional we can only keep the terms related to \( u_i \) and \( f_i \) and omit the irrelevant terms that are not useful in the optimisation problem. Then we have \( J^{(N)}_{soc} (u_i, f_i) = \mathbb{E} \int_0^T (Z_i + \sum_{j \neq i} Z_j) \, dt \) and \( L(\tilde{x}_i, \tilde{x}^{(N)}_{-i}, u_i, f_i) \) satisfies (7). Thus, optimising \( J^{(N)}_{soc} (u_i, \tilde{u}_{-i}; f_i, \tilde{f}_{-i}) \) is equivalent to minimising \( J^{(N)}_{soc} (u_i, f_i) \) in Problem (P1').

**Remark 3.1:** For large \( N \), it is plausible to approximate \( \hat{x}^{(N)}_{-i} \) in (4)–(7) by a deterministic function \( \hat{x} \). Then a corresponding robust optimal control problem is obtained. Note that Problem (P1') is identical to Problem (P1) when we let \( N \to \infty \) and use MF approximations in Problem (P1'). So Lemma 1 provides a
more precise result. Because $L(x_i, \dot{x}^{(N)}_{-i}, u, f_i)$ only depends on $x_i$, $\dot{x}^{(N)}_{-i}, f_i$, and $u_i$, the structure of $L(x_i, \dot{x}^{(N)}_{-i}, u, f_i)$ has a good feature for replacing $\dot{x}^{(N)}_{-i}$ with a deterministic function $\dot{x}$ and designing decentralised strategies.

4. Decentralised strategies design

In the previous section, we obtained the optimal control Problems (P1) and (P1') for a representative agent. Because there is a MF term $\dot{x}^{(N)}$ in the dynamics (4) and $\dot{x}^{(N)}_{-i}$ in the cost functional (7), the solutions of (P1') are centralised. In order to obtain a decentralised strategy, we now solve Problem (P1) by the maximum principle.

Lemma 4.1: For Problem (P1), suppose that (A1) holds, and there exists a symmetric matrix $S$ satisfying the following differential Riccati equation (DRE)

$$
\dot{S}(t) + S(t)A + A^T S(t) + S(t)R_2^{-1} S(t) + Q = 0, \quad S(T) = H.
$$

Then we have the following results.

(i) The cost functional $J_i(u_i, f_i)$ is strictly concave in $f_i$ for all $u_i \in U_i^I$.

(ii) The saddle-point solution (the optimal control and the worst-case disturbance) is given by

$$
\begin{align*}
\dot{u}_i(t) &= -R_1^{-1} B^T [P(t)\dot{x}_i(t) + s(t)], \\
\dot{f}_i(t) &= R_2^{-1} [P(t)\dot{x}_i(t) + s(t)],
\end{align*}
$$

where $P$ is the solution of the following DRE

$$
P(t) + A^T P(t) + P(t)A + P(t)(R_2^{-1} - BR_1^{-1} B^T)P(t) + Q = 0
$$

with the terminal condition $P(T) = H$ and $s$ satisfies the following differential equation

$$
\dot{s}(t) = -\bar{A}^T s(t) + (Q \Gamma - PG)\bar{x}(t) + \bar{\eta} - G^T v(t)
$$

with the terminal condition $s(T) = -H_{\bar{\gamma}_0}\bar{x}(T) - \bar{\eta}_0$, where $\bar{A} \triangleq A + R_2^{-1} P - BR_1^{-1} B^T P$.

Proof: (i) Since (14) has a solution $S$, the following problem admits a unique solution (see Başar & Olsder, 1998, Chapter 6.5):

$$
\min_{\dot{f}_i} \left\{ \int_0^T \left[ -y_i(t)^T Q y_i(t) + f_i(t)^T R_2 f_i(t) \right] dt - y_i^T(T) H y_i(T) \right\}
$$

subject to

$$
dy_i(t) = [Ay_i(t) + f_i(t)] dt, \quad y_i(0) = 0.
$$

This gives the strict convexity of $-J_i(u_i, f_i)$ in $f_i$, which further implies that (P1) is strictly convex in $u_i$.

(ii) It is easy to verify that (P1) is strictly convex in $u_i$. From the proof of Theorem A.4 in Başar and Olsder (1998), (P1) has a saddle-point strategy. By the maximum principle, the following FBSDE has a unique adapted solution $(\dot{x}_i, p_i, q_i)$:

$$
\begin{align*}
&d\dot{x}_i = \left[ \bar{A}\dot{x}_i + G\dot{x} + (R_2^{-1} - BR_1^{-1} B^T) p_i \right] dt + \sigma \, dW_i, \\
&dp_i = (-A^T p_i - Q\dot{x}_i - G^T v + Q_T \bar{x} + \bar{\eta}) dt + q_i \, dW_i, \\
x_i(0) = x_{i0}, \quad p_i(T) = H\dot{x}_i(T) - H_{\gamma_0}\bar{x}(T) - \eta_0.
\end{align*}
$$

and the saddle-point strategy is given by $u_i = -R_1^{-1} B^T p_i, f_i = R_2^{-1} p_i$. By Ma and Yong (1999, Chapter 5, Theorem 3.7), we obtain that the DRE (16) admits a solution and $p_i = P\dot{x}_i + s$, where $s$ satisfies (17). Thus the saddle point solution is given by (15).

After the strategies in (15) are applied, the dynamics of agent $i$ can be written as

$$
\begin{align*}
\dot{\bar{x}}_i(t) &= \left[ \bar{A}\bar{x}_i(t) + (R_2^{-1} - BR_1^{-1} B^T) s(t) + G^{(N)}(I) \right] dt \\
&\quad + \frac{1}{N} \sigma \sum_{j=1}^N dW_j(t), \\
\bar{x}_i(0) &= x_{i0}, \quad i = 1, \ldots, N.
\end{align*}
$$

Then the dynamics of $\dot{x}^{(N)}(t)$ is obtained as follows:

$$
\begin{align*}
\dot{x}^{(N)}(t) &= [\bar{A} + G]\bar{x}(t) + (R_2^{-1} - BR_1^{-1} B^T) s(t) \\
&\quad + \frac{1}{N} \sigma \sum_{j=1}^N dW_j(t), \\
\bar{x}(0) &= \frac{1}{N} \sum_{i=1}^N x_i(0).
\end{align*}
$$

When $N \to \infty$, $(1/N) \sum_{j=1}^N \sigma \, dW_j$ vanishes due to the law of large numbers. As an approximation to $\dot{x}^{(N)}$, we obtain the aggregate effect $\bar{x}$ which is a deterministic function satisfying

$$
\frac{d\bar{x}(t)}{dt} = (\bar{A} + G)\bar{x}(t) + (R_2^{-1} - BR_1^{-1} B^T) s(t), \quad \bar{x}(0) = \bar{x}_0.
$$

Now we obtain the following system of consistent equations

$$
\begin{align*}
\frac{d\bar{x}}{dt} &= (\bar{A} + G)\bar{x} + (R_2^{-1} - BR_1^{-1} B^T) s, \quad \bar{x}(0) = \bar{x}_0, \\
\frac{dv}{dt} &= -(A + G)^T v + (Q_\Gamma - Q)\bar{x} + \bar{\eta}, \\
\dot{v}(t) &= (H - H_{\gamma_0})\bar{x}(T) - \eta_0, \\
\frac{ds}{dt} &= -G^T v - \bar{A}^T s + (Q_\Gamma - PG)\bar{x} + \bar{\eta}, \\
\dot{s}(t) &= -H_{\gamma_0}\bar{x}(T) - \eta_0.
\end{align*}
$$

For further analysis, we impose the following assumptions:

(A2) For the matrix $R_2, R_2 > (r^*)^2 I$, where $r^* = \inf \{r > 0 : S \text{ solves (14)} \}$ in which $R_2 = r^2 I$.

(A3) There exists a solution $(\bar{x}, v, s)$ in $C([0, T], \mathbb{R}^{1n})$ for the equation system (21).

Denote

$$
M_{11} = \bar{A} + G, \quad M_{12} = \left[ 0 \quad R_2^{-1} - BR_1^{-1} B^T \right],
$$

$$
M_{21} = (A + G)^T, \quad M_{22} = (H - H_{\gamma_0})^T.
$$
Then (21) can be rewritten as

\[
\begin{bmatrix}
\dot{x}_0 \\
\dot{v}
\end{bmatrix} =
\begin{bmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{bmatrix}
\begin{bmatrix}
x \\
v
\end{bmatrix}
+ \begin{bmatrix}
0 \\
\bar{\eta}
\end{bmatrix}
\begin{bmatrix}
\dot{x}(0) \\
v(T)
\end{bmatrix}
+ \begin{bmatrix}
\bar{x}_0 \\
\bar{v}
\end{bmatrix}.
\]

(22)

We now give a sufficient condition that ensures (A3).

**Proposition 4.1:** If the following DRE

\[
K(t) = M_{21} + M_{22}K(t) - K(t)M_{11} - K(t)M_{12}K(t)
\]

with the terminal condition \(K(T) = [(H - H_{\Gamma_0})^T, -H_{\Gamma_0}^T]^T\)

admits a solution \(K \in C([0, T], \mathbb{R}^{2n \times n})\), then (A3) holds.

**Proof:** Denote \(z = [v^T, s^T]^T\). Let \(z = Kx + \alpha\). By (22), we have

\[
\dot{z}(t) = [\dot{K}(t) + K(t)M_{11} + K(t)M_{12}K(t)]\dot{x}
+ \dot{\alpha}(t) + K(t)M_{12}\alpha(t)
= [M_{21} + M_{22}K(t)]\dot{x}(t) + M_{22}\alpha(t) + [\bar{\eta}^T, \bar{\eta}^T]^T.
\]

Thus, we obtain the DRE (23) and \(\alpha\) satisfies the following differential equation

\[
\dot{\alpha}(t) = [M_{22} - K(t)M_{12}]\alpha(t) + [\bar{\eta}^T, \bar{\eta}^T]^T
\]

with the terminal condition \(\alpha(T) = [-\bar{\eta}_0^T, \bar{\eta}_0^T]^T\). By (23) and (24), we have

\[
\dot{x}(t) = [M_{11} + M_{12}K(t)]\dot{x} + M_{12}\alpha(t), \quad \dot{x}(0) = \bar{x}_0.
\]

Since \(K \in C([0, T], \mathbb{R}^{2n \times n})\) and \(\alpha \in C([0, T], \mathbb{R}^n)\), then \(\dot{x} \in C([0, T], \mathbb{R}^n)\). By \(z = Kx + \alpha\), we have \((v, s) \in C([0, T], \mathbb{R}^n)\).

By the local Lipschitz property of (23) and the proof of Proposition 1, if the DRE (23) admits a solution \(K \in C([0, T], \mathbb{R}^{2n \times n})\), then we can solve the equations system (21) and obtain a unique solution \((\dot{x}, v, s)\) in \(C([0, T], \mathbb{R}^{2n})\). We further have the following result for the general condition of existence and uniqueness of the solution to (21). See Huang and Zhou (2020) for more related discussions. Let the fundamental solution matrix of (22) be given by the matrix ODE

\[
\frac{\partial}{\partial t} \Phi(t, \tau) = 
\begin{bmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{bmatrix} \Phi(t, \tau), \quad \Phi(t, t) = I,
\]

where

\[
\Phi(t, \tau) = 
\begin{bmatrix}
\Phi_{11}(t, \tau) & \Phi_{12}(t, \tau) \\
\Phi_{21}(t, \tau) & \Phi_{22}(t, \tau)
\end{bmatrix}.
\]

Denote

\[
Z_1 = \Phi_{22}(T, 0) - \begin{bmatrix}
H - H_{\Gamma_0} \\
- H_{\Gamma_0}
\end{bmatrix} \Phi_{12}(T, 0),
\]

\[
Z_2 = \left(\Phi_{21}(T, 0) - \begin{bmatrix}
H - H_{\Gamma_0} \\
- H_{\Gamma_0}
\end{bmatrix} \Phi_{11}(T, 0)\right)\bar{x}_0
+ \int_0^T \left(\Phi_{22}(T, \tau) - \begin{bmatrix}
H - H_{\Gamma_0} \\
- H_{\Gamma_0}
\end{bmatrix} \Phi_{12}(T, \tau)\right) \left[\begin{array}{c}
\bar{\eta} \\
\bar{\eta}_0
\end{array}\right] d\tau.
\]

\[
\text{Proposition 4.2:} \ (i) \ The \ consistent \ system \ (21) \ has \ a \ solution \ if \ and \ only \ if \ there \ exists \ a \ z(0) \ such \ that \ Z_1z(0) + Z_2 = 0. \ (ii) \ If \ det(Z_1) \neq 0, \ the \ solution \ of \ (21) \ is \ unique.
\]

**Proof:** Let \(\bar{\eta} = [\bar{\eta}^T, \bar{\eta}_0^T]^T\). By (22) we have

\[
\begin{bmatrix}
\dot{x}(T) \\
\bar{z}(T)
\end{bmatrix} =
\begin{bmatrix}
\Phi_{11}(T, 0) & \Phi_{12}(T, 0) \\
\Phi_{21}(T, 0) & \Phi_{22}(T, 0)
\end{bmatrix}
\begin{bmatrix}
\bar{x}_0 \\
\bar{z}(0)
\end{bmatrix}
+ \int_0^T \left(\Phi_{22}(T, \tau) - \begin{bmatrix}
H - H_{\Gamma_0} \\
- H_{\Gamma_0}
\end{bmatrix} \Phi_{12}(T, \tau)\right) \left[\begin{array}{c}
\bar{\eta} \\
\bar{\eta}_0
\end{array}\right] d\tau.
\]

Note that \(z(0)\) is to be determined. The above equation is equivalent to

\[
\Phi_{21}(T, 0)\bar{x}_0 + \Phi_{22}(T, 0)\bar{z}(0) + \int_0^T \Phi_{22}(T, \tau)\bar{\eta} d\tau
= \begin{bmatrix}
H - H_{\Gamma_0} \\
- H_{\Gamma_0}
\end{bmatrix} \Phi_{11}(T, 0)\bar{x}_0 + \begin{bmatrix}
H - H_{\Gamma_0} \\
- H_{\Gamma_0}
\end{bmatrix} \Phi_{12}(T, 0)\bar{z}(0)
+ \begin{bmatrix}
H - H_{\Gamma_0} \\
- H_{\Gamma_0}
\end{bmatrix} \int_0^T \Phi_{22}(T, \tau)\bar{\eta} d\tau - \begin{bmatrix}
\bar{\eta}_0 \\
\bar{\eta}_0
\end{bmatrix}.
\]

Therefore (21) has a solution if and only if there exists a \(z(0)\) such that \(Z_1z(0) + Z_2 = 0\). Furthermore, Equation (22) exists a unique solution if \(det(Z_1) \neq 0\).}

5. **Asymptotic social optimality**

In this section, we analyse the asymptotic performance of the decentralised strategies (15), where \(P\) and \(s\) are given by (16) and (17). We first provide some preliminary results.

**Lemma 5.1:** Assume that (A1)–(A3) hold. Under the decentralised strategies (15), the following holds:

\[
\max_{0 \leq t \leq T} \mathbb{E}\|\tilde{x}^{(N)}(t) - \tilde{x}(t)\|^2 dt = O \left(\frac{1}{N}\right).
\]

**Proof:** See Appendix 1.

**Lemma 5.2:** Assume that (A1)–(A3) hold. For any \(N\), there exists \(C_1\) independent of \(N\) such that

\[
\sum_{i=1}^{N} \mathbb{E} \int_0^T (\|\tilde{x}_i(t)\|^2 + \|\tilde{u}_i(t)\|^2 + \|\tilde{v}_i(t)\|^2) dt < NC_1.
\]

**Proof:** See Appendix 1.

**Lemma 5.3:** Under (A1)–(A3), the following equation holds:

\[
v(t) = P(t)\tilde{x}(t) + s(t), \quad v(T) = H\tilde{x}(T) - H_{\Gamma_0}\tilde{x}(T) - \bar{\eta}_0.
\]
**Proof:** Let \( \phi(t) = \nu(t) - P(t)\bar{x}(t) - s(t) \). By (21) and some elementary computations, we obtain
\[
d\phi(t) = -A^T \phi(t) \, dt, \quad \phi(T) = 0.
\]
The solution of the above differential equation is \( \phi(t) \equiv 0 \). This completes the proof. \[\blacksquare\]

Denote \( J_{soc}^{(N)}(\hat{u}, \hat{f}) \) as the social cost under the decentralised strategies (15). Let \( x = (x_1^T, \ldots, x_N^T)^T, u = (u_1, \ldots, u_N)^T, f = (f_1^T, \ldots, f_N^T)^T, 1 \triangleq (1, \ldots, 1), W \triangleq (W_1^T, \ldots, W_N^T)^T, A \triangleq \text{diag}(A_1, \ldots, A_N), B \triangleq \text{diag}(B_1, \ldots, B_N), \sigma \triangleq \text{diag}(\sigma_1, \ldots, \sigma_N), \) \( R_1 \triangleq \text{diag}(R_{11}, \ldots, R_{1N}), R_2 \triangleq \text{diag}(R_{21}, \ldots, R_{2N}), Q \triangleq \text{diag}(Q_1, \ldots, Q_N), x_0 \triangleq (x_{10}^T, \ldots, x_{N0}^T)^T, \) and \( H \triangleq \text{diag}(H_1, \ldots, H_N) \). Using the above notions, we can write Problem (PA) as follows:
\[
J_{soc}^{(N)}(u, f) = \mathbb{E} \int_0^T \left[ x^T(t) \dot{Q} x(t) + u^T(t) R_1 u(t) - f^T(t) R_2 f(t) \right] dt + \mathbb{E} \left[ x^T(T) \dot{H} x(T) - 2 \hat{\eta}_0 x(T) \right]
\]
subject to
\[
dx(t) = [\dot{\hat{x}}(t) + Bu(t) + f(t)] dt + \dot{\sigma} dW(t), \quad x(0) = x_0,
\]
where \( \dot{\hat{x}} = \dot{\hat{x}} \), \( \dot{\sigma} = \dot{\sigma} \), \( \dot{\hat{\eta}} = \dot{\hat{\eta}} \), and \( \dot{\eta}_0 = \dot{\eta}_0 \).

Assume that
(A4) The following DRE
\[
\dot{S}(t) + \hat{A}^T S(t) + S(t) \hat{A} - \hat{Q} + S(t) R_2^{-1} S(t) = 0, \quad S(T) = \hat{H}
\]
holds a solution \( S \in C([0, T], \mathbb{R}^{nN \times nN}) \).

**Theorem 5.1:** Suppose (A1)–(A4) hold. The set of decentralised strategies \( \hat{u} = (\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_N) \) and \( \hat{f} = (\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_N) \) given by (15) has asymptotic robust social optimality, i.e.
\[
\frac{1}{N} J_{soc}^{(N)}(\hat{u}, \hat{f}) - \frac{1}{N} \inf_{u \in U} \sup_{f \in F} J_{soc}^{(N)}(u, f) \leq O \left( \frac{1}{\sqrt{T}} \right).
\]

**Proof:** See Appendix 2. \[\blacksquare\]

### 6. Applications in opinion dynamics

In this section, we apply the above results into analysing opinion dynamics in social networks. Consider a population of scalar agents, where each agent is characterised by an opinion \( x_i \). In opinion dynamics, we can control the variable \( u_i \) as the variation rate of the opinion of agent \( i \). A zealot attempts to change agents’ opinions in a way that is proportional to his advertisement efforts \( f_i \), \( i = 1, \ldots, N \). For the dynamics (1), by choosing \( A = a, G = -a, \) and \( B = 1 \), we can write the opinion dynamics of a social network in the following form
\[
dx_i(t) = \left[ ax_i(t) - ax_i^{(N)}(t) + u_i(t) + f_i(t) \right] dt + \sigma dW_i(t), \quad t \geq 0, \quad 1 \leq i \leq N.
\]

The corresponding cost functional of agent \( i \) is given by
\[
J_i(u(\cdot), f(\cdot)) = \mathbb{E} \int_0^T \left[ \|x_i(t) - x_i^{(N)}(t)\|^2 + r_1 \|u_i(t)\|^2 + r_2 \|f_i(t)\|^2 \right] dt + h \mathbb{E} \|x_i(T) - x_i^{(N)}(T)\|^2.
\]

Note that the above cost is in a LQ tracking-type form. Minimising such cost implies that all agents are willing to mimic the average population behaviour as happens in herd behaviours or crowd-seeking attitudes. The social cost functional is given by
\[
J_{soc}^{(N)}(u(\cdot), f(\cdot)) = \sum_{i=1}^N J_i(u(\cdot), f(\cdot)).
\]

The above system is given by
\[
\dot{\hat{u}}_i(t) = -\frac{1}{r_1} \left[ p(t) \dot{\hat{x}}_i(t) + s(t) \right], \quad \dot{\hat{f}}_i(t) = \frac{1}{r_2} \left[ p(t) \dot{\hat{x}}_i(t) + s(t) \right], \quad t \geq 0,
\]

where \( p \) satisfies the following DRE
\[
\dot{p}(t) + \frac{r_1 - r_2}{r_1 r_2} p^2(t) + 2 a p(t) + 1 = 0, \quad p(T) = h,
\]
and \( s \) is determined by
\[
\begin{align*}
\dot{s} &= \left[ a + \left( \frac{1}{r_2} - \frac{1}{r_1} \right) p \right] s + (1 + p a) \dot{x}, \\
\dot{x} &= \left( \frac{1}{r_2} - \frac{1}{r_1} \right) p \dot{x} + \left( \frac{1}{r_2} - \frac{1}{r_1} \right) s, \quad \dot{x}(0) = \dot{x}_0.
\end{align*}
\]

By Proposition 4.1, we further have the following result.

**Proposition 6.1:** For the Equations (28) and (29), we have \( s = -p \dot{x} \), where \( p \) is the solution of the DRE (28).

**Proof:** From Proposition 4.1, there exists a \( 2 \times 1 \) matrix \( K \) satisfying
\[
\begin{bmatrix}
0 \\
1 + p a
\end{bmatrix} + \begin{bmatrix}
0 \\
-a - \frac{r_1 - r_2}{r_1 r_2}
\end{bmatrix} \begin{bmatrix}
k_1 \\
k_2
\end{bmatrix} = \begin{bmatrix}
k_1 \\
k_2
\end{bmatrix} \frac{r_1 - r_2}{r_1 r_2} p
\]
\[
+ \begin{bmatrix}
k_1 \\
k_2
\end{bmatrix} \begin{bmatrix}
\alpha_1(T) \\
\alpha_2(T)
\end{bmatrix} = \begin{bmatrix}
0 \\
-h
\end{bmatrix}
\]

and a \( 2 \times 1 \) matrix \( \alpha = [\alpha_1 \ \alpha_2]^T \) satisfying
\[
\begin{align*}
[\alpha_1] &= \begin{bmatrix}
0 \\
-\frac{r_1 - r_2}{r_1 r_2}
\end{bmatrix} k_1 \\
0 - a - \frac{r_1 - r_2}{r_1 r_2} (p + k_2)
\end{bmatrix} \begin{bmatrix}
\alpha_1(T) \\
\alpha_2(T)
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix},
\end{align*}
\]

By (28) and (30), we obtain \( k_1 = 0 \) and \( k_2 = -p \). This further implies that (31) admits a unique solution \( \alpha(t) \equiv 0 \) in \( C([0, T], \mathbb{R}^2) \). Thus we obtain \( s = -p \dot{x} \). \[\blacksquare\]
Remark 6.1: Compared (29) with (4.30) of Bauso et al. (2016), the optimal control and worst-case disturbance (27) is the same as that of Bauso et al. (2016) if we choose \( r_2 = \gamma^2 \). This shows that the team solution we studied in this paper coincides with the game solution of Bauso et al. (2016) for the system (25)–(26).

Therefore, we may extend the (unified) solution to analyse the local interactions between multiple populations with the help of graphon theory.

Suppose that the population state average \( x^{(N)} \) is accessed, we obtain \( s = -p x^{(N)} \) by replacing \( \tilde{x} \) with \( x^{(N)} \). This is reasonable because by Lemma 5.1, \( x^{(N)} \) and \( \tilde{x} \) are close enough when the number of agents is large enough. Then the corresponding control law and disturbance are given by

\[
\begin{aligned}
&u_i(t) = -\frac{p(t)}{r_1}[x_i(t) + x^{(N)}(t)], \\
&f_i(t) = \frac{p(t)}{r_2}[x_i(t) + x^{(N)}(t)].
\end{aligned}
\]

Substituting \( u_i \) and \( f_i \) into the opinion dynamics (25), the closed-loop dynamics of \( x_i \) can be written as

\[
dx_i = \left[ a + \left( \frac{1}{r_2} - \frac{1}{r_1} \right) p \right] (x_i - x^{(N)}) \, dt + \sigma \, dW_i.
\]

Furthermore, the dynamics of MF term \( x^{(N)} \) satisfies

\[
dx^{(N)}(t) = \frac{1}{N} \sum_{i=1}^{N} \sigma \, dW_i(t).
\]

It can be seen that \( \mathbb{E} x^{(N)}(t) = \tilde{x}_0 \), since \( \mathbb{E} x_i(0) = \tilde{x}_0 \). By (33) and (34) we obtain

\[
d(x_i(t) - x^{(N)}(t)) = \left\{ a + \frac{r_1 - r_2}{r_1 r_2} p \right\} (x_i(t) - x^{(N)}(t)) \, dt + \frac{N - 1}{N} \sigma \, dW_i(t) - \frac{1}{N} \sum_{j \neq i} \sigma \, dW_j(t).
\]

Now we construct a microscopic model for the social network. To this end, let us collect all opinions into an opinion vector \( X(t) = [x_1(t), \ldots, x_N(t)]^T \). We have

\[
dX = \left\{ a + \left( \frac{1}{r_2} - \frac{1}{r_1} \right) p \right\} (X - X^{(N)}) \, dt + \zeta \, dW,
\]

where \( X^{(N)} \) is an \( N \times 1 \) vector whose terms all are \( x^{(N)} \), \( W = [W_1, \ldots, W_N]^T \) is a \( N \)-dimensional Brownian motion and \( \zeta = \text{diag}[\sigma, \sigma, \ldots, \sigma] \). For later analysis, let us introduce the Laplacian and averaging matrices. Denote

\[
M = \begin{bmatrix}
1 & 1 & \ldots & 1 \\
\frac{1}{N} & \frac{1}{N} & \ldots & \frac{1}{N} \\
\frac{1}{N} & \frac{1}{N} & \ldots & \frac{1}{N} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{N} & \frac{1}{N} & \ldots & 1
\end{bmatrix},
\]

where \( L \) is called the Laplacian matrix of a fully connected network and \( M \) is called the averaging matrix. Note that \( L = I - M, L^T = L \) and \( MX = x^{(N)} \mathbf{1} = X^{(N)} \).

Now, our aim is to analyse the deviation of the opinions from their average. For the error vector, we can write the expression below which relates \( e(t) \) to \( X(t) \):

\[
e(t) = X(t) - x^{(N)}(t) \mathbf{1} = (I - M)X(t).
\]

Note that \( Me(t) = 0 \). By (35) and (36) we obtain

\[
de(t) = \left[ a + \left( \frac{1}{r_2} - \frac{1}{r_1} \right) p \right] Le(t) \, dt + L \zeta \, dW(t).
\]

The following result establishes that the error process \( \{e(t), \ t \geq 0\} \) is stochastically bounded, which implies that all opinions reach agreement with the average opinion in a probabilistic sense.

**Theorem 6.1:** For the system (25)–(26), suppose that \( a + (\frac{1}{r_2} - \frac{1}{r_1})p(t) \leq 0, \ t \in [0, T] \) holds. Under the control law and disturbance (32), for each \( \pi \in (0, 1] \) there exists an \( \varepsilon(\pi) > 0 \) such that

\[
\mathbb{P}(\sup_{0 \leq t \leq T} ||e(t)|| \leq \varepsilon(\pi)) > 1 - \pi,
\]

where \( \varepsilon(\pi) = \max \left\{ \sqrt{\frac{2a}{\pi}}, \sqrt{\frac{2a}{\pi} \sqrt{\mathbb{E}[e^2(t)]}} \right\} \) with

\[
\kappa = \frac{\sqrt{(N^2 - N)^2}}{\sqrt{-2a - 2(\frac{1}{r_2} - \frac{1}{r_1})p_{\min}}} \quad \text{and} \quad p_{\min} = \min_{0 \leq t \leq T} p(t).
\]

**Proof:** See Appendix 3.

**Remark 6.2:** Note that when \( a = 0 \), the dynamics (25) can be degenerated into the model of Bauso et al. (2016). Theorem 2 can be seen as a generalisation of Theorem 4.3 of Bauso et al. (2016) and it additionally gives an explicit expression between probability distribution and bound of opinions deviation from average opinion. To be specific, we further show the relationship between bound of opinions deviation \( \varepsilon \) and the probability \( \pi \).

### 6.1 Long time behaviour of opinions evolution

Now we consider the long time behaviour of opinions evolution in the social network. Note that the system (25)–(26) is controllable and observable. For (28), we introduce the following algebraic Riccati equation (ARE):

\[
r_1 r_2 p^2 + 2ap + 1 = 0.
\]

Suppose that the above ARE has a solution in \( \mathbb{R} \) such that \( a + (\frac{1}{r_2} - \frac{1}{r_1})p < 0 \). Let the decentralised control law with the
Theorem 6.2: Let \( s \in C_p([0, \infty), \mathbb{R}) \) be determined by the following differential equations:

\[
\begin{align*}
\dot{x}(t) &= -\left( a + \frac{r_1 - r_2}{r_1 r_2} p \right) x(t) + \left( 1 + pa \right) \bar{x}(t), \\
\dot{x}(t) &= \frac{r_1 - r_2}{r_1 r_2} p \bar{x}(t) + \frac{r_1 - r_2}{r_1 r_2} x(t), \quad \bar{x}(0) = \bar{x}_0.
\end{align*}
\]

By direct computations, we have the following proposition.

**Proposition 6.2**: For (39), we have \( s = -p \bar{x} \), where \( p \) satisfies the ARE (38).

**Proof**: Suppose \( s = k \bar{x} \). By (39) we obtain

\[
\frac{r_1 - r_2}{r_1 r_2} k \bar{x} + \frac{r_1 - r_2}{r_1 r_2} p \bar{x} = -\left( a + \frac{r_1 - r_2}{r_1 r_2} p \right) \bar{x} + (1 + pa) \bar{x}.
\]

By the ARE (38), we further obtain \( k = -p \) as the solution to the Equation (40).

As the discussion above, we have \( s = -p x^{(N)} \) by replacing \( \bar{x} \) with \( x^{(N)} \). The control law with the disturbance are given by

\[
\begin{align*}
 u_i &= -\frac{p}{r_1} (x_i - x^{(N)}), \\
f_i &= \frac{p}{r_2} (x_i - x^{(N)}).
\end{align*}
\]

Let us collect all opinions into an opinion vector \( X(t) = [x_1(t), \ldots, x_N(t)]^T \). Under the above control law (41), we obtain the dynamics of the opinion vector satisfying

\[
dX(t) = \left[ a + \frac{1}{r_1 - \frac{1}{r_1}} p \right] (X - X^{(N)}) dt + \zeta dW(t).
\]

Define the error vector as \( e(t) = X(t) - x^{(N)} 1 = (I - M)X(t) \). By (42), the dynamics of \( e(t) \) satisfies

\[
de(t) = \left[ a + \frac{1}{r_2 - \frac{1}{r_2}} p \right] Le(t) dt + L\zeta dW(t).
\]

Then we have the counterpart of Theorem 6.1.

**Theorem 6.2**: For the system (25)–(26), assume that (38) has a solution such that \( a + (\frac{1}{r_2 - \frac{1}{r_2}} p) \leq 0 \). Under the control law and disturbance (41), for each \( \pi \in (0, 1] \) there exists an \( \varepsilon(\pi) > 0 \) such that

\[
P(\sup_{0 \leq t < \infty} \| e(t) \| \leq \varepsilon(\pi)) > 1 - \pi,
\]

where \( \varepsilon(\pi) = \max\left\{ \sqrt{\frac{2\pi}{\kappa}}, \sqrt{\frac{E[e(t) e(0)]}{\pi}} \right\} \) with

\[
\kappa = \frac{\sqrt{(N^2 - N)\sigma^2}}{\sqrt{-2[a + (\frac{1}{r_2 - \frac{1}{r_2}} p)]}}.
\]

**Proof**: Since this proof is similar to that of Theorem 6.1, we focus on different parts. By (43) and \( M e(t) = 0 \), we obtain

\[
de(t) = \left[ a + \frac{1}{r_2 - \frac{1}{r_2}} p \right] (L + M)e(t) dt + L\zeta dW(t).
\]

Similarly, letting \( A \equiv [a + (\frac{1}{r_2 - \frac{1}{r_2}} p)](L + M) \), rewrite the dynamics for the error vector as \( de(t) = Ae(t) dt + L\zeta dW(t) \). The eigenvalues of \( A \) are \( \lambda_1 = \lambda_2 = \cdots = \lambda_N = a + (\frac{1}{r_2 - \frac{1}{r_2}} p) \leq 0 \). This implies that \( A \) is non-positive definite. Define the Lyapunov function \( V(e) = \frac{1}{2} e^T e \). Applying the infinitesimal generator (A3) to the Lyapunov function \( V(e) \), we have

\[
\mathcal{L}V(e) = e(t)^T A e(t) + \frac{1}{2} N^2 \sigma^2 \left[ \frac{N - 1}{N^2} + \left( 1 - \frac{1}{N} \right)^2 \right].
\]

Next we show that there exists a finite positive scalar \( \kappa \) and \( N_c \in \mathbb{N} \) such that \( \mathcal{L}V(e) \leq 0 \) for all \( e(t) \notin N_c \). Let \( \bar{A} = -A/\alpha \). Then \( \bar{A} \) is positive definite. By analytic geometry theory, \( x \in \mathbb{R}^N : x^T A x = 1 \) represents a ball in \( N \)-dimensional Euclidean space \( \mathbb{R}^N \) with the radius being \( 1/\sqrt{\lambda} \), where \( \lambda \) is the eigenvalue of \( A \). Hence the radius of the ball is

\[
r = \sqrt{(N^2 - N)\sigma^2 / -2[a + (\frac{1}{r_2 - \frac{1}{r_2}} p)]}.
\]

Letting \( \kappa = r \), we have \( \mathcal{L}V(e) \leq 0 \) for all \( e(t) \notin N_c \). This implies that \( V(e(t)) \) is a non-negative supermartingale when \( e(t) \) is not in \( N_c \). By Lemma 6, we obtain

\[
\varepsilon(\pi) = \max\left\{ \sqrt{\frac{2\pi}{\kappa}}, \sqrt{\frac{2Ee(\varepsilon(0))}{\pi}} \right\}.
\]

By a similar argument in the proof of Theorem 2, the theorem follows.

**6.2 Local interactions among multiple sub-populations**

In the above we considered the case of a single homogeneous population. Now we extend the results to the case of multiple heterogeneous sub-populations by virtue of graphon MF games (see Caines & Huang, 2019; Gao & Caines, 2020; Gao et al., 2020).

Let \( G = (V, E, M) \) be a weighted graph of order \( N \) with the set of nodes \( V = \{v_1, v_2, \ldots, v_N\} \), set of edges \( E \subset V \times V \), and a weighted adjacency matrix \( M = [m_{ij}] \). Each node in the graph represents a cluster hence we have \( N_c \) clusters. Let \( C_q \) denote the set of agents in the \( q \)th cluster. Then \( \sum_{i=1}^{N_c} |C_q| = N \). The opinion dynamics of agent \( i \) in the cluster \( C_q \) is given by

\[
dx_i(t) = [a_q x_i(t) - a_q \bar{z}_q(t) + u_i(t) + f_i(t)] dt + \sigma_q dW_i(t).
\]

Compared with the dynamics of single population (25), the system average opinion \( x^{(N)} \) is replaced by the local average opinion \( z_q \equiv \frac{1}{N_c} \sum_{i=1}^{N_c} m_{ij} [1] \sum_{j \in C_q} x_j \). Similarly replacing \( x^{(N)} \) by \( z_q \) in (26), the cost functional of agent \( i \) is given by

\[
J_i(u(\cdot), f(\cdot)) = \mathbb{E} \int_0^T \left[ \| x_i(t) - z_q(t) \|^2 + r_i^u \| u_i(t) \|^2 - r_i^f \| f_i(t) \|^2 \right] dt + h_q \mathbb{E} \| x_i(T) - z_i(T) \|^2.
\]

We first consider the problem of infinite nodal population on finite networks. By taking the local population limit (i.e. \( |C_q| \rightarrow \infty \)).
for all $q \in V$, the opinion dynamics of the generic agent $\alpha$ in the cluster $C_q$ is then given by
\[
dx{\alpha}(t) = [a_q x_\alpha(t) - a_q \bar{z}_q(t) + u_\alpha(t) + f_\alpha(t)] \, dt + \sigma_q \, dW_\alpha(t),
\]
and the corresponding cost functional is given by
\[
J_\alpha(u(\cdot), f(\cdot)) = \mathbb{E} \int_0^T \left[ \|x_\alpha(t) - \bar{z}_q(t)\|^2 + r_1^q \|u_\alpha(t)\|^2 - r_2^q \|f_\alpha(t)\|^2 \right] \, dt + h_q \mathbb{E}\|x_\alpha(T) - z_\alpha(T)\|^2
\]
where
\[
\bar{z}_q(t) = \frac{1}{N_c} \sum_{l=1}^{N_c} m_q \bar{x}_l(t), \quad \alpha \in C_q,
\]
\[
\bar{x}_l(t) = \lim_{|C_l| \to \infty} \frac{1}{|C_l|} \sum_{j \in C_l} x_j(t).
\]
The dynamics of local MF $\bar{x}_l$ for cluster $C_l$ is given by
\[
\bar{x}_l(t) = a_l \bar{x}_l(t) - a_l \bar{z}_l(t) + \bar{u}_l(t) + \bar{f}_l(t),
\]
where $\bar{u}_l = \lim_{|C_l| \to \infty} \frac{1}{|C_l|} \sum_{j \in C_l} u_j$ and $\bar{f}_l = \lim_{|C_l| \to \infty} \frac{1}{|C_l|} \sum_{j \in C_l} f_j$. Based on (44), the network weighted MF for cluster $C_q$ is then given by
\[
\hat{z}_q(t) = \frac{1}{N_c} \sum_{l=1}^{N_c} m_q [a_l \bar{x}_l(t) - a_l \bar{z}_l(t) + \bar{u}_l(t) + \bar{f}_l(t)].
\]

**Remark 6.3:** For the above network with $N_c$ sub-populations, every sub-population generates a local MF $\bar{z}_q$ during local interaction among multiple sub-populations according to the predefined graph $G$, and the objective of agents in a cluster $C_q$ is now to tracking the local MF $\bar{z}_q$. This reflects a typical crowd-seeking behaviour based on local interactions.

As stated in Remark 6.1, the team solution coincides with its game solution of the system (25)–(26). Therefore we may apply the above (unified) solution for a representative agent $\alpha$ in cluster $C_q$, and then obtain
\[
u_\alpha(t) = -\frac{1}{r_1^q} [p_q(t)x_\alpha(t) + s_q(t)],
\]
\[
f_\alpha(t) = \frac{1}{r_2^q} [p_q(t)x_\alpha(t) + s_q(t)],
\]
where $\alpha \in C_q$ and $p_q$ satisfies the following DRE
\[
\dot{p}_q(t) + \frac{r_1^q - r_2^q}{r_2^q r_2^q} p_q^2(t) + 2a_q p_q(t) + 1 = 0, \quad p_q(T) = h_q,
\]
and $s_q$ is determined by
\[
\dot{s}_q(t) = \left[ a_q + \frac{r_1^q - r_2^q}{r_1^q r_2^q} p_q(t) \right] s_q(t) + \hat{z}_q(t)
\]
\[
+ \frac{1}{N_c} \sum_{l=1}^{N_c} m_q [a_l p_l(t) \bar{x}_l(t)]
\]
with the terminal condition $s_q(T) = -\frac{1}{N_c} \sum_{l=1}^{N_c} m_q h_l \bar{x}_l(T)$. If every agent adopts the above strategy, then the dynamics of $\hat{z}_q$.
is given by
\[
\tilde{z}_q = \frac{1}{N_c} \sum_{i=1}^{N_c} m_{q|l} \left[ (a_i + \frac{r_1^2 - r_2^2}{r_1^2 r_2^2} p_i) \tilde{x}_i - a_i \tilde{z}_1 + \frac{r_1^2 - r_2^2}{r_1^2 r_2^2} s_i \right]
\]
with the initial conditions \(\tilde{z}_q(0) = \frac{1}{N_c} \sum_{i=1}^{N_c} m_{q|l} \tilde{x}_i(0)\), \(q \in V\). Assume that the initial states of cluster \(C_q\) are independent with the mathematical expectation \(\tilde{x}_q(0)\), \(q = 1, \ldots, N_c\). Let \(s = (s_1, s_2, \ldots, s_{N_c})^T\), \(\tilde{z} = (\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_{N_c})^T\), \(\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{N_c})\) and \(\tilde{x}_0 = (\tilde{x}_{10}, \tilde{x}_{20}, \ldots, \tilde{x}_{N_c0})^T\), then \((\tilde{x}, \tilde{z}, s)\) is given by
\[
\begin{align*}
\dot{\tilde{x}}(t) &= [A + RP(t)]\tilde{x}(t) - A\tilde{z}(t) + RS(t), \\
\dot{\tilde{z}}(t) &= \frac{1}{N_c} M [A + RP(t)]\tilde{x}(t) - \frac{1}{N_c} M\tilde{A}\tilde{z}(t) + \frac{1}{N_c} MRS(t), \\
\dot{s}(t) &= \frac{1}{N_c} MAP(t)\tilde{x}(t) + \tilde{z}(t) - [A + RP(t)]s(t),
\end{align*}
\]
(46)
with \(\tilde{x}(0) = \tilde{x}_0\), \(\tilde{z}(0) = \frac{1}{N_c} M\tilde{x}_0\), \(s(T) = -\frac{1}{N_c} M\tilde{H}\tilde{x}(T)\), where
\[
A \triangleq \text{diag}(a_1, a_2, \ldots, a_{N_c}), \quad R \triangleq \text{diag}(\frac{r_1^2 - r_2^2}{r_1^2 r_2^2}, \ldots, \frac{r_{N_c}^2 - r_{N_c}^2}{r_{N_c}^2 r_{N_c}^2}),
\]
\[
P \triangleq \text{diag}(p_1, p_2, \ldots, p_{N_c}), \quad H \triangleq \text{diag}(h_1, h_2, \ldots, h_{N_c}).
\]

We now consider the infinite sub-populations case by graphon theory. Graphons are represented by symmetric Lebesgue measurable functions \(W : [0,1]^2 \rightarrow \mathbb{R}\). Thus, they may be interpreted as operators \(W : L^2(0,1;\mathbb{R}) \rightarrow L^2(0,1;\mathbb{R})\) or as weighted graphs on the vertex set \([0,1]\) (Lovász, 2012). Assume that the mathematical expectation of initial states of infinite sub-populations is \(\tilde{x}_{0} \in L^2(0,1;\mathbb{R})\), which is taken as a mean-square integrable function on \([0,1]\). See Gao and Caines (2020); Gao et al. (2020) for more details. Taking \(N_c \rightarrow \infty\), it follows by (46) that
\[
\begin{align*}
\dot{\tilde{x}}(t) &= [A + RP(t)]\tilde{x}(t) - A\tilde{z}(t) + RS(t), \\
\dot{\tilde{z}}(t) &= M [A + RP(t)]\tilde{x}(t) - MA\tilde{z}(t) + MRS(t), \\
\dot{s}(t) &= MAP\tilde{x}(t) + \tilde{z}(t) - [A + RP(t)]s(t),
\end{align*}
\]
(47)
where \(\tilde{x}(0) = \tilde{x}_0 \in L^2(0,1;\mathbb{R}), S = -M\tilde{H}\tilde{x}(T) \in L^2(0,1;\mathbb{R}),\)
\(\tilde{z}(0) = M\tilde{x}_0 \in L^2(0,1;\mathbb{R}), A, R, P, H\) are operators from \(L^2(0,1;\mathbb{R})\) to \(L^2(0,1;\mathbb{R})\), and \(M\) is the corresponding graphon. Therefore, the optimal control strategy and the worst-case disturbance for a generic agent \(\alpha\) in cluster \(C_\theta\) with \(\theta \in [0,1]\) is given by
\[
\begin{align*}
u_\alpha(t) &= -\frac{1}{r_0} [p_\theta(t) x_\alpha(t) + S_\theta(t)], \\
\nu_\alpha(t) &= \frac{1}{r_0} [p_\theta(t) x_\alpha(t) + S_\theta(t)],
\end{align*}
\]
where \(\alpha \in C_\theta, \theta \in [0,1], (S_\theta(t))_{\theta \in [0,1]} \in (0,1,1)\) satisfies (47) and \(p_\theta\) satisfies the DRE (45).
7. Simulation

To better demonstrate the efficacy of the proposed decentralised strategies, two numerical examples are carried out in this section. The first example shows the convergence of individuals’ opinions to the average opinion in a homogeneous social network. The second example shows the opinion evolution in the case of multiple sub-populations.

Example 7.1: In this example, we choose $\alpha = 0.5, \sigma = 1, r_1 = 1, r_2 = 1.5$ to simulate the opinion evolution in a homogeneous social network with 20 agents. The initial opinions are taken independently from a uniform distribution on $[0, 100]$. By computation, the solution of ARE (38) is $p = 3.79$, and $\alpha + (\frac{1}{r_2} - \frac{1}{r_1})p = -0.76$ satisfying the assumption of Theorem 6.2.

From Figure 1, it can be seen that the opinions reach consensus roughly despite being affected by the model uncertainty.
All opinions evolve within a certain range around average opinion. From Figure 2, it is shown that $\bar{x}$ is close to $\bar{x}^{(N)}$. This shows the rationality of MF approximations by which a deterministic function $\bar{x}$ is substituted for the MF term $x^{(N)}$.

Example 7.2: In this example we consider the case of multiple sub-populations with heterogeneous agents. The parameters are listed in Table 1. The adjacency matrices are given as

$$M_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0.25 & 0.5 & 0.25 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}.$$  

Figures 3 and 4 show that the opinions converge to the weighted average opinion under the adjacency matrices $M_1$ and $M_2$, respectively, which generalises the results from the single population case to the multiple sub-populations case.

8. Conclusions

In this paper, we have considered the social optimality of LQ MF control with unmodelled dynamics. The socially optimal problem is analysed by variational and direct decoupling methods, which leads to two equivalent auxiliary robust optimal control problems. Decentralised strategies are designed by consistent MF approximations, and it is proved that the decentralised strategies have asymptotic social optimality. Applying the MF control approach, the evolution of opinions are analysed over finite and infinite horizons, respectively. It is shown that the opinions converge to the average opinion in a probabilistic sense. For the future work, an interesting problem is to consider robust MF social control problems with major players.

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Appendix 1. Proofs of Lemmas 5.1 and 5.2

Proof of Lemma 5.1: From (19) and (20), we have

\[
E[\tilde{x}^{(N)}(t) - \tilde{x}(t)]^2 \, dt = e^{(G+A)t} E[\tilde{x}^{(N)}(0) - \tilde{x}(0)]^2 \\
+ \frac{1}{N} \sum_{i=1}^{N} \int_{0}^{t} e^{(G+A)(t-s)} \sigma \, dW_i(s),
\]

By (A1), one can obtain

\[
E[\tilde{x}^{(N)}(t) - \tilde{x}(t)]^2 = \frac{2}{N} E[e^{(A+G)t}] \left\{ \max_{i \leq t \leq N} E[\tilde{x}_0]^2 + \int_{0}^{t} \text{tr}[\sigma^T e^{-(A+G+\tilde{A}^T+G^T+\sigma^T)]} \, dt \right\}.
\]

This completes the proof.

Proof of Lemma 5.2: By Proposition 4.1, we obtain that \( \tilde{x} \) and \( s \in C([0, T], \mathbb{R}^N) \) and

\[
\max_{0 \leq t \leq T} E[\tilde{x}^{(N)}(t) - \tilde{x}(t)]^2 = O \left( \frac{1}{N} \right),
\]

which further gives

\[
E \int_{0}^{T} \|\tilde{x}^{(N)}(t)\|^2 \, dt < \infty.
\]

Denote \( g = (R_2^{-1} - BR_1^{-1}B^T)s + G\tilde{x}(N) \). Then we have \( E \int_{0}^{T} \|g(t)\|^2 \, dt < \infty \) and

\[
\tilde{x}_i(t) = e^{\tilde{A}t} \tilde{x}_i(0) + \int_{0}^{t} e^{\tilde{A}(t-s)} g(s) \, ds + \int_{0}^{t} e^{\tilde{A}(t-s)} \sigma \, dW_i(s).
\]

Thus there exists \( C_2 \) independent of \( N \) such that

\[
E \int_{0}^{T} \|\tilde{x}(t)\|^2 \, dt \leq C_2.
\]

This with (15) completes the proof.

Appendix 2. Proof of Theorem 5.1

Proof: Let \( \tilde{x}_i = x_i - \tilde{x}_i, \tilde{u}_i = u_i - \tilde{u}_i, \tilde{f}_i = f_i - \tilde{f}_i \) and \( \tilde{x}^{(N)} = \frac{1}{N} \sum_{i=1}^{N} \tilde{x}_i. \)

Then by (1) and (18), we have

\[
d\tilde{x}_i = (A \tilde{x}_i + G\tilde{x}(N) + B\tilde{u}_i + \tilde{f}_i) \, dt, \quad \tilde{x}_i(0) = 0. \tag{A1}
\]

It follows from (3) that \( J_{oc}^{(N)}(u, \tilde{f}) = \sum_{i=1}^{N} (I_i(\tilde{u}_i, \tilde{f}_i) + I_i(\tilde{u}, \tilde{f})) \), where

\[
I_i(\tilde{u}_i, \tilde{f}_i) = \int_{0}^{T} \left[ \|\tilde{x}_i - \Gamma \tilde{x}^{(N)}\|^2 + \|\tilde{u}_i\|^2 + \|\tilde{f}_i\|^2 \right] \, dt + E[\|\tilde{x}_i(T) - \Gamma \tilde{x}^{(N)}(T)\|^2],
\]

\[
l_i = 2E \int_{0}^{T} \left[ (\tilde{x}_i - \Gamma \tilde{x}^{(N)} - \eta)^T Q(\tilde{x}_i - \Gamma \tilde{x}^{(N)}) + \tilde{u}_i^T R_1 \tilde{u}_i - \tilde{f}_i^T R_2 \tilde{f}_i \right] \, dt + 2E \left[ (\tilde{x}(T) - \Gamma \tilde{x}^{(N)}(T) - \eta_0)^T H \tilde{x}(T) - \Gamma \tilde{x}^{(N)}(T) \right].
\]

From this we have

\[
J_{oc}^{(N)}(u, \tilde{f}) = \sum_{i=1}^{N} I_i(\tilde{u}_i, \tilde{f}_i)
\]

\[
= \int_{0}^{T} \left( \tilde{x}^T Q \tilde{x} + \tilde{u}^T R_1 \tilde{u} - \tilde{f}^T R_2 \tilde{f} \right) \, dt + E[\tilde{x}^T(T) \hat{H}(T)],
\]

which implies

\[
\inf_{u, \tilde{f}, \tilde{u}, \tilde{f}} \sup_{u, \tilde{f}} \int J_{oc}^{(N)}(u, \tilde{f}) \geq \inf_{u, \tilde{f}, \tilde{u}, \tilde{f}} \sup_{u, \tilde{f}} \int \left( \tilde{x}^T Q \tilde{x} + \tilde{u}^T R_1 \tilde{u} - \tilde{f}^T R_2 \tilde{f} \right) \, dt + E[\tilde{x}^T(T) \hat{H}(T)] \geq 0. \tag{A2}
\]

By (A4), Problem (PA) is concave in \( f \) and

\[
\sup_{f \in L^2} \inf_{u, \tilde{f}} J_{oc}^{(N)}(u, \tilde{f})
\]

\[
= \inf_{u, \tilde{f}, \tilde{u}, \tilde{f}} \sup_{u, \tilde{f}} \int J_{oc}^{(N)}(u, \tilde{f}) \geq 0.
\]
\[
\leq E \int_0^T (\dot{x}^T Q \dot{x} - R_1 \dot{x}_1) \, dt + E[\dot{x}^T (T) \dot{H}_x(T)] \leq 0.
\]

By Mou and Yong (2006) with (A2), this gives \( \inf_{u(t) \in U} \sup_{p \in P} I^{\infty}(u, \dot{f}) = 0 \). By direct computation, we obtain
\[
\sum_{i=1}^N I_i = 2E \int_0^T \left[ \sum_{i=1}^N \dot{x}_i^T \left[ Q(\dot{x}_i - (I - \Gamma) \dot{x} - \eta) - \Gamma^T Q(I - \Gamma) \dot{x} - \eta \right] \right. \\
+ \sum_{i=1}^N \dot{R}_i \dot{u}_i - \sum_{i=1}^N \dot{R}_i \dot{y}_i \left. \right] \, dt \\
+ \sum_{i=1}^N 2E \int_0^T (\dot{x}^{(N)} - \bar{x})^T Q_1 \dot{x}_i \, dt \\
+ 2 \sum_{i=1}^N E \left[ \dot{x}_i^T (T) \left[ H \dot{x}_i(T) - \Gamma_0 \dot{x}_i(T) - \eta_0 \right] \right. \\
- \Gamma_0^T H((I - \Gamma_0) \dot{x}_i(T) - \eta_0) + (\dot{x}^{(N)}(T) - \bar{x}(T))^T H \Gamma_0 \dot{x}_i(T).)
\]

Define the Lyapunov function \( V(\epsilon) = \frac{1}{2} \epsilon^T \epsilon \). Applying this infinitesimal generator (A3) to \( V(\epsilon) \), we have
\[
\mathcal{L}V(\epsilon) = \epsilon^T A \epsilon + \frac{1}{2} \epsilon^T \epsilon \left[ \frac{N - 1}{N^2} + \left( 1 - \frac{1}{N} \right)^2 \right].
\]

In the above equation, the second term is a positive real number and denoted as \( \alpha \). Now we show that there exists a finite positive scalar \( \alpha \) and a neighbourhood of zero with size \( \alpha \), denoted by \( N = \{ x \in \mathbb{R}^N : V(x) \leq \alpha \} \), such that \( \mathcal{L}V(\epsilon) \leq 0 \) for all \( \epsilon \notin N \). Let \( \tilde{A} = -\alpha \tilde{A} \). Thus \( \tilde{A} \) is positive definite.

The theorem follows.

**Appendix 3. Proof of Theorem 6.1**

In order to prove Theorem 6.1, we need a lemma.

**Lemma A.1:** Suppose there exists a twice differentiable function \( V \) and a number \( \kappa > 0 \) such that for \( V(\epsilon(t)) > \kappa \), \( \mathcal{L}V(\epsilon(t)) \leq 0 \), where \( \mathcal{L} \) is the infinitesimal generator of the process \( \{ \epsilon(t), t \geq 0 \} \). Let \( \tau_1 = \tau_1(\kappa) \) be the first entry time of \( \{ \epsilon(t) \in \mathbb{R}^N \mid V(\epsilon) \leq \kappa \} \). Then for each \( \pi > 0 \) there exists a \( \epsilon(\pi) \) such that
\[
\mathbb{P} \left( \sup_{t \leq \tau_1} V(\epsilon(t)) \leq \epsilon(\pi) \right) > 1 - \pi,
\]
where \( \epsilon(\pi) = E[V(\epsilon(0))/\pi] \).

**Proof:** The main proof is based on Theorems 3 and 18 of Thuygenes (1997). We choose \( N > \kappa \), and stop the process when it leaves the region \( \{ \epsilon(t) \in \mathbb{R}^N \mid \kappa < V(\epsilon) < N \} \). Then \( V \) applied to the stopped process is a supermartingale because \( \mathcal{L}V(\epsilon(t)) \geq 0 \). When letting \( N \to \infty \), we have that the process stops for \( V(\epsilon(t)) = \kappa \) w.p. 1. By Theorem 18 of Thuygenes (1997), we obtain
\[
\mathbb{P} \left( \sup_{t \leq \tau_1} V(t) > \kappa \right) > 1 - \frac{E[V(\epsilon(0))/\pi]}{\epsilon}.
\]

Letting \( \epsilon = E[V(\epsilon(0))/\pi] \), we complete the proof.