Symmetrizers and antisymmetrizers for the BMW algebra

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Abstract

Let $n \in \mathbb{N}$ and $B_n(r,q)$ be the generic Birman-Murakami-Wenzl algebra with respect to indeterminants $r$ and $q$. It is known that $B_n(r,q)$ has two distinct linear representations generated by two central elements of $B_n(r,q)$ called the symmetrizer and antisymmetrizer of $B_n(r,q)$. These generate for $n \geq 3$ the only one dimensional two sided ideals of $B_n(r,q)$ and generalize the corresponding notion for Hecke algebras of type $A$. The main result Theorem 20 in this paper explicitly determines the coefficients of these elements with respect to the graphical basis of $B_n(r,q)$.

Key words: Birman-Murakami-Wenzl algebra, symmetrizer, antisymmetrizer

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1. Introduction

The sum (signed sum) of all group elements of the symmetric group $S_n$ in the group ring $R\mathcal{S}_n$ are central elements generating precisely the one dimensional (two sided) ideals and hence linear characters of $\mathcal{S}_n$ for any commutative ring $R$. They are called symmetrizer (antisymmetrizer) of $R\mathcal{S}_n$ and play an important role in the representation theory of symmetric groups. This applies in particular to tensor space, where Schur-Weyl duality connects...
representation theory of symmetric groups and general (or special) linear groups. All this generalizes to Hecke algebras of type $A$ and the various quantizations of general (special) linear groups.

The Brauer algebras were defined by Richard Brauer in 1937 \[2\] as centralizing algebras of symplectic and orthogonal groups acting on tensor space. Quantized versions of those were first defined and studied independently by Birman and Wenzl \[1\] and by Murakami \[9\]. They are multi parameter algebras which degenerate in the classical limit to Brauer algebras and are called today Birman-Murakami-Wenzl algebras (BMW-algebras for short). It is known that there are generalisations of symmetrizer and antisymmetrizer for BMW-algebras and these generate the only one dimensional (two sided) ideals of those provided that $n$ is greater than two (see e. g. \[4\]).

Morton and Wassermann \[8\] showed that the BMW-algebra $B_n(r, q)$ with parameters $r$ and $q$ is isomorphic to Kauffman’s tangle algebra \[6\], which is generated by $n$-tangles. Moreover, $B_n(r, q)$ possesses a basis consisting of tangles whose shadows in the plane (that is not distinguishing over- and undercrossings) are precisely the Brauer diagrams producing Brauer’s graphical basis of the Brauer algebra. In this paper we shall determine explicit formulas for the coefficients of the symmetrizer $x = x_n$ and the antisymmetrizer $y = y_n$ with respect to such a basis.

$B_n(r, q)$ comes with a filtration by ideals $I_f (0 \leq f \leq \lfloor \frac{n}{2} \rfloor)$, where $I_f$ is the ideal generated by tangles with at least $2f$ horizontal edges. Already in \[4\] it was shown that the coefficients of $x$ and $y$ differ on tangles with the same number of horizontal edges only by a power of $q$ depending on the number of crossings in the tangle. However, these coefficients of $x$ and $y$ could not be determined in \[4\]. In this paper, we shall not only reprove that fact but also explicitly calculate these coefficients, see Theorem 20. In particular, we verify the conjecture proposed in \[4\], line -11, page 2921.

We remark that several other forms of the symmetrizer and antisymmetrizer (and more generally, a complete set of pairwise orthogonal primitive idempotents) have been given in \[3\], \[5\], \[7\] by using the Jucys-Murphy operators of BMW algebras and/or fusion procedure and/or inductive constructions. However, it seems to be difficult to derive the coefficients of $x_n$ and $y_n$ in the graphical basis of $B_n(r, q)$ from those formulae. Our approach here for determining the coefficients works directly with a special basis of $n$-tangles (introduced in \[4\]) and calculations with those using the relations for these basis elements.
2. Preliminaries

In this section we recall some basic results on Birman-Murakami-Wenzl algebras.

**Definition 1.** [1, 9] The generic Birman-Murakami-Wenzl algebra $B_n = B_n(r, q)$ is the unital associative $Q(r, q)$-algebra generated by the elements $T_i^\pm 1$ and $E_i$ for $1 \leq i \leq n - 1$ subject to the relations:

\[
T_i - T_i^{-1} = (q - q^{-1})(1 - E_i), \quad \text{for } 1 \leq i \leq n - 1, \tag{1}
\]

\[
E_i^2 = \delta E_i, \quad \text{for } 1 \leq i \leq n - 1, \tag{2}
\]

\[
T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad \text{for } 1 \leq i \leq n - 2, \tag{3}
\]

\[
T_i T_j = T_j T_i, \quad \text{for } |i - j| > 1, \tag{4}
\]

\[
E_i E_{i+1} E_i = E_i, \quad E_i E_{i+1} E_{i+1} = E_{i+1}, \quad \text{for } 1 \leq i \leq n - 2, \tag{5}
\]

\[
T_i T_{i+1} E_i = E_{i+1} E_i, \quad T_{i+1} T_i E_{i+1} = E_i E_{i+1}, \quad \text{for } 1 \leq i \leq n - 2, \tag{6}
\]

\[
E_i T_i = T_i E_i = r^{-1} E_i, \quad \text{for } 1 \leq i \leq n - 1, \tag{7}
\]

\[
E_i T_{i+1} E_i = r E_i, \quad E_{i+1} T_i E_{i+1} = r E_{i+1}, \quad \text{for } 1 \leq i \leq n - 2, \tag{8}
\]

where $\delta = 1 + \frac{r - r^{-1}}{q - q^{-1}}$.

The Birman-Murakami-Wenzl algebra is isomorphic to an algebra given in terms of certain diagrams which we will introduce now.

**Definition 2.** [6]

1. A tangle is a knot diagram inside a rectangle consisting of a finite number of vertices in the top and the bottom row of the rectangle (not necessarily the same number) and a finite number of arcs inside the rectangle such that each vertex is connected to another vertex by exactly one arc, arcs either connect two vertices or are closed curves. Two tangles are regularly isotopic if they are related by a sequence of Reidemeister Moves II and III (see [9]) and isotopies fixing the boundary of the rectangle.

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\cdot \quad \cdot \quad \cdot \\
\cdot \quad \cdot \quad \cdot
\end{array}
\end{array} = \\
\begin{array}{c}
\begin{array}{c}
\cdot \quad \cdot \quad \cdot \\
\cdot \quad \cdot \quad \cdot
\end{array}
\end{array}
\end{align*}
\]

2. Kauffman’s tangle algebra is the $Q(r, q)$-algebra generated by all tangles with $n$ vertices in the top row and in the bottom row subject to the following relations which can be applied to a local disk of the tangle:
The following theorem was shown using Kauffman’s invariant of knots.

**Theorem 3.** [8] Kauffman’s tangle algebra is isomorphic to the Birman-Murakami-Wenzl algebra. The isomorphism maps

\[
T_i \mapsto \begin{array}{c|c|c}
\ldots & \times & \ldots \\
\hline
\end{array},
\quad E_i \mapsto \begin{array}{c|c|c}
\ldots & \circ & \ldots \\
\hline
\end{array},
\]

here the crossing and the horizontal strands respectively connect the \(i\)-th and \(i+1\)-st vertices of the top and bottom row.

In view of Theorem 3, each equation in the Birman-Murakami-Wenzl algebra has a tangle analogue. From now on, we will identify both algebras via the isomorphism and denote Kauffman’s tangle algebra as well by \(B_n = B_n(r, q)\).

A **Brauer n-diagram** is a planar graph in a rectangle of a plane, which consist of two rows (top row and bottom row) of \(n\) points, with each point joined to precisely one other point (distinct from itself). We label the points in the top row by \(\{1, 2, \ldots, n\}\) and the points in the bottom row by \(\{1^-, 2^-, \ldots, n^-\}\). Any edge of the form \((i, j)\) or \((i^-, j^-)\), where \(1 \leq i < j \leq n\), will be called a horizontal edge, while other edges (of the form \((i, j^-)\)) will be called vertical edges. One can obtain from each tangle a Brauer diagram by forgetting the orientation of crossings. Conversely, for each Brauer diagram \(d\) one can choose a tangle \(T(d)\) such that two strands cross at least once, by choosing orientation of the crossings.

**Lemma 4** ([8]). For each Brauer \(n\)-diagram \(d\) choose a tangle \(T(d)\) by choosing an orientation for each crossing (such that two strands never cross twice). Then the set

\[
\{T(d) \mid d \text{ Brauer diagram with } n \text{ vertices on the top/bottom row}\}
\]

is a basis for the BMW-algebra \(B_n\).
We fix some notation: If \( f \) is a non-negative number, such that \( 2f \leq n \), then let

\[
\hat{E}_f = E_1E_3\cdots E_{2f-1} = \begin{array}{cccc}
\lor & \lor & \cdots & \lor \\
\lor & \lor & \cdots & \lor \\
\end{array}
\]

A composition \( \mu \vdash n \) of \( n \) is a sequence \( \mu = (\mu_1, \mu_2, \ldots, \mu_k) \) of non-negative integers \( \mu_i \) such that \( \sum_{i=1}^{k} \mu_i = n \). If \( \mu \) is a composition of \( n \) then let \( \mathfrak{S}_\mu = \mathfrak{S}_{\mu_1} \times \mathfrak{S}_{\mu_2} \times \cdots \times \mathfrak{S}_{\mu_k} \) be the corresponding Young subgroup of \( \mathfrak{S}_n \) and let \( \mathcal{D}_\mu \) be the set of distinguished right coset representatives of minimal length, such that \( \mathfrak{S}_n = \mathfrak{S}_\mu \mathcal{D}_\mu \). Throughout, we use the convention that \( \mathfrak{S}_n \) acts on \( \{1, 2, \ldots, n\} \) from the right hand side. In other words,

\[
((a)\sigma)\tau = (a)(\sigma\tau), \quad \forall a \in \{1, 2, \ldots, n\}, \forall \sigma, \tau \in \mathfrak{S}_n.
\]

Let \( \mathcal{H}_f \) be the set of \( d \in \mathfrak{S}_n \) satisfying the following conditions:

- \((1)d < (3)d < \ldots < (2f - 1)d\), and
- \((2i - 1)d < (2i)d\) for \( i = 1, \ldots, f \), and
- \((2f + 1)d < (2f + 2)d < \ldots < (n)d\).

Note that \( \mathcal{H}_f \) is in bijection with the possible positions of precisely \( f \) horizontal edges in the lower part of a Brauer diagram (resp. in the upper part). To \( d \in \mathcal{H}_f \) is associated the configuration of \( f \) horizontal edges such that the vertex \((2i - 1)d\) is connected to the vertex \((2i)d\) for \( i = 1, \ldots, f \). Note that this set is usually denoted by \( \mathcal{D}_f \). We choose \( \mathcal{H}_f \) to avoid confusion with the sets \( \mathcal{D}_\mu \).

If \( w \in \mathfrak{S}_n \) and \( w = s_{i_1}s_{i_2}\cdots s_{i_l} \) is a reduced expression, then let \( \ell(w) := l, T_w := T_{i_1}T_{i_2}\cdots T_{i_l} \). Note that \( \ell(w), T_w \) is independent of the reduced expression of \( w \). Then by translating Lemma 4 we obtain

**Lemma 5** ([4]).

\[
\left\{ T_uT_{\sigma}\hat{E}_fT_d \mid 0 \leq f \leq \left[ \frac{n}{2} \right], u \in \mathcal{H}_f^{-1}, d \in \mathcal{H}_f, \sigma \in \mathfrak{S}_{\{2f+1, \ldots, n\}} \right\}
\]

is a basis of \( B_n \).
3. Symmetrizers and antisymmetrizers

The Birman-Murakami-Wenzl algebra admits two one-dimensional representations $\rho_1$ and $\rho_2$ given by

$$
\begin{align*}
\rho_1(T_i) &= q, & \rho_1(E_i) &= 0 \\
\rho_2(T_i) &= -q^{-1}, & \rho_2(E_i) &= 0
\end{align*}
$$

The symmetrizer (antisymmetrizer) is the quasi-idempotent generating the ideal in $B_n$ corresponding to the representation $\rho_1$ ($\rho_2$), that is the unique (up to scalars) quasi-idempotent $x$ in $B_n$ such that $xE_i = 0$ and $xT_i = qx$ ($xE_i = 0$ and $xT_i = -q^{-1}x$). The goal of this paper will be to determine the (anti-)symmetrizer. Since the $\mathbb{Q}$-linear ring automorphism $B_n \to B_n : T_i \mapsto T_i, E_i \mapsto E_i, r \mapsto r, q \mapsto -q^{-1}$ maps the antisymmetrizer to the symmetrizer, we restrict ourselves to determine the symmetrizer $x \in B_n$.

If $S \subseteq S_n$ is an arbitrary subset of the symmetric group, let $\widehat{S} = \sum_{w \in S} q^{l(w)}T_w$. The following lemma gives further information on the symmetrizer $x$:

**Lemma 6 ([4]).** We have

$$x = \sum_f c_f x_f$$

where $c_f \in \mathbb{Q}(r,q)$ and $x_f = \widehat{\mathcal{H}}_f^{-1}\mathcal{G}_{\{2f+1, \ldots, n\}}\mathcal{G}_{\mu}$.  

**Remark 7.** If $S, Y, Z \subseteq S_n$, $S = Y \cdot Z$ and if each element $w \in S$ can be uniquely written as $w = yz$ with $y \in Y, z \in Z$ and then $l(w) = l(y) + l(z)$, then clearly $\widehat{S} = \widehat{Y} \widehat{Z}$.

In particular, we have

**Lemma 8.**

1. If $\mu \vdash n$, then $\widehat{S}_n = \widehat{\mathcal{G}}_\mu \widehat{D}_\mu$.

2. If $\lambda, \mu \vdash n$ such that $\mathcal{G}_\lambda \subseteq \mathcal{G}_\mu$, then $\widehat{D}_\lambda = \mathcal{G}_\mu \cap \widehat{D}_\lambda$. Hence, if $\mu = (\mu_1, \ldots, \mu_k)$ and $\lambda = (\nu^1, \nu^2, \ldots, \nu^k)$ where $\nu^i$ is a composition of $\mu_i$, then we have $\widehat{D}_\lambda = \iota_1(\widehat{D}_{\nu^1})\iota_2(\widehat{D}_{\nu^2}) \cdots \iota_k(\widehat{D}_{\nu^k})\widehat{D}_\mu$ where $\iota_i$ is the obvious embedding of $\mathcal{G}_{\mu_i}$ into $\mathcal{G}_n$.

**Remark 9.** Applying the antiautomorphism of $B_n$, which maps $T_i$ to $T_i$ and $E_i$ to $E_i$ to Lemma 8 shows that $\widehat{S}_n = \widehat{D}_\mu^{-1}\widehat{\mathcal{G}}_\mu$, etc. All diagram identities which will be shown in the next lemmas produce new identities by applying this antiautomorphism.
One may write the results of Lemma 8 in terms of diagrams with the following convention: If a box filled with $\hat{S}$ occurs in a diagram, we mean the sum over all $w \in S$, the summands being $q^{l(w)}$ times the diagram with the box replaced by $T_w$.

For example, the equation $E_1(1 + qT_1) = (1 + qr^{-1})E_1$ can be depicted as

$$\hat{\mathcal{S}}_2 = \begin{array}{c} \vdots \\ \hat{\mathcal{S}}_2 \\ \vdots \end{array} + q = (1 + qr^{-1})$$

Note that the upper horizontal edge is not involved in these calculations, so we can omit it.

$$\hat{\mathcal{S}}_2 = (1 + qr^{-1})$$

(10)

This equation can be embedded into a larger piece of tangle, such that new tangles arise, the equation holds for the new tangles:

The assertions of Lemma 8 are:

$$\hat{\mathcal{S}}_n = \hat{\mathcal{S}}_\mu$$

(11)

$$\hat{\mathcal{D}}_\lambda = \hat{\mathcal{D}}_{\nu,1} \hat{\mathcal{D}}_{\nu,2} \cdots \hat{\mathcal{D}}_{\nu,k}$$

(12)
if $\mu = (\mu_1, \ldots, \mu_k)$, $\lambda = (\nu^1, \nu^2, \ldots, \nu^k)$ and $\nu^i$ is a composition of $\mu_i$.

To determine the coefficients $c_f$, we need several technical lemmas.

**Lemma 10.** Suppose $n \geq 3$. Then

$$
\hat{S}_n = \hat{S}_n \quad \text{(13)}
$$

**Proof.** For $n = 3$ the lemma can be computed directly:

$$
\hat{S}_3 = (1 + q r^{-1}) \left( q^2 + q + 1 \right)
$$

On the other hand,

$$
\hat{S}_3 = (1 + q r^{-1}) \left( q^2 + q + 1 \right)
$$

This shows the lemma for $n = 3$. For the general case, note that it suffices to shift the horizontal edge on top of $\hat{S}_n$ by one to the left (or right). The result follows from equation (11) for $\lambda = (1^k, 2, 1^{n-k-2})$ and the case $n = 3$.  

**Lemma 11.** For $n \geq 2$, we have

$$
\hat{S}_n \cdot D_{(1,n-1)} = \left( \sum_{i=0}^{n-1} q^{2i} \right) \hat{S}_n + \frac{1 - q^2}{q^{-1}r + 1} \hat{S}_n E_1 D_{(2,n-2)}.
$$

**Proof.** First, note that $D_{(1,n-1)} = \left\{ s_1 s_2 \cdots s_{i-1} \mid i = 1, \ldots, n \right\}$ and $D_{(2,n-2)} = \left\{ s_2 s_3 \cdots s_{i-1} s_1 s_2 \cdots s_{j-1} \mid 1 \leq j < i \leq n \right\}$. We first show that

$$
\hat{S}_n T_1 T_2 \cdots T_{i-1} = q^{i-1} \hat{S}_n + \frac{1 - q^2}{r + q} \hat{S}_n E_1 \sum_{j=1}^{i-1} q^{j-1} T_2 T_3 \cdots T_{i-1} T_1 T_2 \cdots T_{j-1} \quad \text{(14)}
$$
for $i = 1, \ldots, n$. We argue by induction on $i$. Equation (14) is trivial for $i = 1$. Suppose, equation (14) holds for $i$. We have

$$
\tilde{S}_n T_1 T_2 \cdots T_i = (\tilde{S}_n T_1 T_2 \cdots T_{i-1}) T_i
$$

$$
= q^{i-1} \tilde{S}_n T_i + \frac{1 - q^2}{r + q} \tilde{S}_n E_i \sum_{j=1}^{i-1} q^{j-1} T_2 T_3 \cdots T_{i-1} T_1 T_2 \cdots T_{j-1} T_i
$$

Note that $T_i$ commutes with $T_1 T_2 \cdots T_{j-1}$, thus the second part of the right hand side is exactly the sum over $j = 1, \ldots, i - 1$ in equation (14) for $i + 1$ instead of $i$.

To simplify the notations, we shall often use $a^k$ to represents the $k$-tuple $a, a, \ldots, a$. Remark 9 for $\lambda = (1^{i-1}, 2, 1^{n-i-1})$ shows that $\tilde{S}_n = \tilde{D}_\lambda^{-1}(1 + qT_i)$.

It can be verified by direct calculation that

$$
(1 + qT_i) T_i = (1 + qT_i) \cdot \left( q + \frac{1 - q^2}{r + q} E_i \right).
$$

Since $T_x T_y = T_{xy}$ whenever $\ell(xy) = \ell(x) + \ell(y)$, it is easy to see that $(1 + qT_i)$ is a right factor of $\tilde{S}_n$. Therefore, we obtain

$$
q^{i-1} \tilde{S}_n T_i = q^{i-1} \tilde{S}_n \left( q + \frac{1 - q^2}{r + q} E_i \right) = q^i \tilde{S}_n + \frac{1 - q^2}{r + q} q^{i-1} \tilde{S}_n E_i
$$

$q^i \tilde{S}_n$ is the summand $q^{i-1} \tilde{S}_n$ in equation (14) with $i$ replaced by $i+1$. Apply Remark 9 to equation (13) to obtain

$$
\tilde{S}_n E_i = \tilde{S}_n \tilde{S}_n \cdots \tilde{S}_n = \tilde{S}_n E_1 T_2 T_3 \cdots T_i T_1 T_2 \cdots T_{i-1}
$$

Thus, $\frac{1 - q^2}{r + q} q^{i-1} \tilde{S}_n E_i$ is the summand for $j = i$ in equation (14) with $i$ replaced by $i+1$. This shows equation (14). The result follows by summing $q^{i-1}$ times the term in equation (14), since

$$
\sum_{1 \leq j < i \leq n} q^{j-1+i-1} T_2 T_3 \cdots T_{i-1} T_1 T_2 \cdots T_{j-1} = q^{P_{(2,n-2)}}
$$
The diagram version of Lemma 11 is

\[
\widehat{\mathcal{S}}_n^{(1,n-1)} = \left( \sum_{i=0}^{n-1} q^{2i} \right) \widehat{\mathcal{S}}_n + \frac{1 - q^2}{q^{-1}r + 1} \widehat{\mathcal{S}}_n^{(2,n-2)}
\]

(15)

Next, we will show several identities involving \( \mathcal{H}_f \). Note, that \( \mathcal{H}_f = \mathcal{H}_f(n) \) not only depends on \( f \), but also on \( n \). If \( j - i \geq 2f \), we write \( \mathcal{H}_{f\{i+1,\ldots,j\}} \) for the image of \( \mathcal{H}_f(j - i) \) under the embedding \( T_k \mapsto T_{i+k}, E_k \mapsto E_{i+k} \). We will use a similar notation for \( \mathcal{D}_\mu \).

If \( \widehat{\mathcal{H}}_f \) occurs in a diagram, then \( i \) and \( j \) might be obtained from the diagram: \( i \) is the number of vertical edges on the left of the box filled with \( \widehat{\mathcal{H}}_f \) and \( j - i \) is the number of edges going into the box. Thus, in the diagrams we simply write \( \widehat{\mathcal{H}}_f(n) \) or simply \( \widehat{\mathcal{H}}_f \) if \( n \) is clear from the context.

**Lemma 12.** We have

\[
\widehat{\mathcal{H}}_f(n) = \mathcal{H}_f(2f)\mathcal{D}(2f,n-2f)
\]

In diagrams,

\[
\mathcal{H}_f(n) = \mathcal{H}_f(2f)\mathcal{D}(2f,n-2f)
\]

(16)

**Proof.** Since \( \mathcal{D}(2f,n-2f) \) is a set of distinguished right coset representatives of \( \mathcal{S}(2f,n-2f) \) in \( \mathcal{S}_n \) and \( \mathcal{H}_f(2f) \subseteq \mathcal{S}(2f,n-2f) \), each element of \( \mathcal{H}_f(2f)\mathcal{D}(2f,n-2f) \) can be uniquely written as a product of elements of these sets, and lengths add. Thus \( \mathcal{H}_f(2f)\mathcal{D}(2f,n-2f) = \mathcal{H}_f(2f)\mathcal{D}(2f,n-2f) \) and it suffices to show that \( \mathcal{H}_f(2f)\mathcal{D}(2f,n-2f) = \mathcal{H}_f(n) \). Both sets have the same cardinality, namely \( \frac{n!}{2f(n-2f)!} \) and it is easy to see that \( \mathcal{H}_f(2f)\mathcal{D}(2f,n-2f) \subseteq \mathcal{H}_f(n) \) by verifying the defining conditions of \( \mathcal{H}_f \).

\[\Box\]
Lemma 12 shows that $\mathcal{H}_f(n)$ can be deduced from $\mathcal{H}_f(2f)$. The following lemmas deal with the case $n = 2f$.

**Lemma 13.** We have

\[
\hat{\mathcal{H}}_f(2f) = \hat{\mathcal{H}}_{f-1}^{\{3,\ldots,2f\}} \hat{\mathcal{D}}_{(1,2f-2)}^{\{2,\ldots,2f\}}
\]

or in diagrams

\[\begin{tikzpicture}
\draw (0,0) rectangle (1,1); \node at (0.5,0.5) {$\hat{\mathcal{H}}_f(2f)$};
\draw (2,0) rectangle (3,1); \node at (2.5,0.5) {$\hat{\mathcal{H}}_{f-1}$};
\draw (4,0) rectangle (5,1); \node at (4.5,0.5) {$\hat{\mathcal{D}}_{(1,2f-2)}$};
\end{tikzpicture}\]

\[=\]

\[\begin{tikzpicture}
\draw (0,0) rectangle (1,1); \node at (0.5,0.5) {$\hat{\mathcal{D}}_{(3,2f-4)}$};
\draw (2,0) rectangle (3,1); \node at (2.5,0.5) {$\hat{\mathcal{D}}_{(1,2f-4)}$};
\end{tikzpicture}\]

**Proof.** This can be proved with the same arguments used for Lemma 12 by showing that $\hat{\mathcal{H}}_{f-1}^{\{3,\ldots,2f\}} \hat{\mathcal{D}}_{(1,2f-2)}^{\{2,\ldots,2f\}} = \hat{\mathcal{H}}_{f}^{\{1,\ldots,2f\}}$. Note that both sets have cardinality $\frac{(2f)!}{2^f}$. \(\square\)

**Lemma 14.** We have

\[
\hat{\mathcal{H}}_2^{\{1,2,3,4\}} \hat{\mathcal{D}}_{(3,2f-4)}^{\{2,\ldots,2f\}} = \hat{\mathcal{D}}_{(1,2f-4)}^{\{2,\ldots,2f\}}
\]

or

\[\begin{tikzpicture}
\draw (0,0) rectangle (1,1); \node at (0.5,0.5) {$\hat{\mathcal{H}}_2$};
\draw (2,0) rectangle (3,1); \node at (2.5,0.5) {$\hat{\mathcal{D}}_{(3,2f-4)}$};
\end{tikzpicture}\]

\[=\]

\[\begin{tikzpicture}
\draw (0,0) rectangle (1,1); \node at (0.5,0.5) {$\hat{\mathcal{D}}_{(1,2f-4)}$};
\end{tikzpicture}\]

**Proof.** By equation (17) for $f = 2$ (note that $\hat{\mathcal{H}}_1(1) = 1$) and equation (12), both sides are equal to

\[\begin{tikzpicture}
\draw (0,0) rectangle (1,1); \node at (0.5,0.5) {$\hat{\mathcal{D}}_{(1,2)}$};
\draw (2,0) rectangle (3,1); \node at (2.5,0.5) {$\hat{\mathcal{D}}_{(3,2f-4)}$};
\end{tikzpicture}\]

\[\square\]
Recall that we consider the case $n = 2f$. Let $s_i = s_{2i}s_{2i-1}s_{2i+1}s_{2i} = (2i-1, 2i+1)(2i, 2i+2)$. The subgroup of $S_{2f}$ generated by $s_i$ for $i = 1, \ldots, f-1$ is isomorphic to the symmetric group on $f$ letters. We will denote this subgroup by $\mathfrak{S}_f$.

Let $v \in S_f$. Then $v H_f$ is the set of $d \in \mathcal{D}_{(2f)}$ such that $(1)v^{-1}d < (3)v^{-1}d < (5)v^{-1}d < \ldots < (2f-1)v^{-1}d$. Thus we have

$$\mathcal{D}_{(2f)} = S_f H_f$$

and each element $d \in \mathcal{D}_{(2f)}$ can be uniquely written as $d = \mathfrak{S}_f d'$ with $\mathfrak{S}_f, d' \in H_f$. In general, $l(d) = l(v) + l(d')$ is not true.

Let $\mathfrak{S}_{(1,f-1)} = \langle s_2, s_3, \ldots, s_{f-1} \rangle$, which is a Young subgroup in $S_f$, and let $\mathcal{D}_{(1,f-1)} = \{ s_1s_2 \cdots s_k \mid k = 0, \ldots, f-1 \}$ be the corresponding set of coset representatives. Then we have

**Lemma 15.**

$$\mathcal{D}_{(1,f-1)} H_f (2f) = \mathcal{H}_{f-1}^{(3, \ldots, 2f)} \mathcal{D}_{(2,2f-2)}$$

In diagrams,

$$\begin{array}{c}
\mathcal{D}_{(1,f-1)} H_f (2f) \\
\cdots \\
\cdots \\
\end{array} = 
\begin{array}{c}
\mathcal{H}_{f-1} \\
\cdots \\
\cdots \\
\end{array}$$

(19)

**Proof.** Clearly, $\mathcal{H}_{f-1}^{(3, \ldots, 2f)} \mathcal{D}_{(2,2f-2)} = \mathcal{H}_{f-1}^{(3, \ldots, 2f)} \mathcal{D}_{(2,2f-2)}$. Thus it suffices to show that $\mathcal{H}_{f-1}^{(3, \ldots, 2f)} \mathcal{D}_{(2,2f-2)} = \mathcal{D}_{(1,f-1)} H_f$. Both sets have cardinality $\frac{(2f)!}{2f(f-1)!}$. Now, $\mathcal{D}_{(1,f-1)} H_f$ is the set of $d \in \mathcal{D}_{(2f)}$ such that $(3)d < (5)d < \ldots < (2f-1)d$. On the other hand, $\mathcal{H}_{f-1}^{(3, \ldots, 2f)} \mathcal{D}_{(2,2f-2)}$ is a subset of $\mathcal{D}_{(2f)}$ and each element of $\mathcal{H}_{f-1}^{(3, \ldots, 2f)} \mathcal{D}_{(2,2f-2)}$ satisfies the condition $(3)d < (5)d < \ldots < (2f-1)d$. \hfill \square

**Lemma 16.** We have

$$s_1 \mathcal{D}_{(1,f-2)}^{(2, \ldots, f)} H_f (2f) = s_1 \mathcal{H}_2^{(1, \ldots, 4)} \mathcal{H}_{f-2}^{(5, \ldots, 2f)} \mathcal{D}_{(3,2f-4)}^{(2, \ldots, 2f)}$$

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In diagrams,

\[
\begin{array}{c}
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet} \\
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet} \\
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet} \\
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet}
\end{array}
\]

\[
\begin{array}{c}
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet} \\
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet} \\
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet} \\
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet}
\end{array}
\]

\[
\begin{array}{c}
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet} \\
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet} \\
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet} \\
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet}
\end{array}
\]

\[
\begin{array}{c}
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet} \\
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet} \\
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet} \\
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet}
\end{array}
\]

\[
\begin{array}{c}
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet} \\
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet} \\
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet} \\
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet}
\end{array}
\]

\[\hat{\mathcal{D}}_{(1, f-2)}^{(2, \ldots, f)} \mathcal{H}_f(2f) = \hat{\mathcal{S}}_1 \mathcal{H}_2 \hat{\mathcal{D}}_{(3,2f-4)} \mathcal{H}_{f-2} \hat{\mathcal{D}}_{(2f, n-2f)} \mathcal{H}_f(2f) \quad (20)\]

**Proof.** Note that \( \mathcal{D}_{(1,f-1)}^{(1, \ldots, 1)} = \{1\} \cup \mathcal{S}_1 \mathcal{D}_{(1,f-2)}^{(2, \ldots, f)} \). Thus, \( \mathcal{S}_1 \mathcal{D}_{(1,f-2)}^{(2, \ldots, f)} \mathcal{H}_f(2f) \) is the set of \( d \in \mathcal{D}_{(2f)} \), such that \((3)d < (5)d < \ldots < (2f-1)d \) and \((1)d > (3)d \). The elements of \( \mathcal{S}_1 \mathcal{H}_2^{(1, \ldots, 1)} \mathcal{H}_f^{(5, \ldots, n)} \mathcal{D}_{(3,2f-4)}^{(2, \ldots, n)} \) satisfy this condition and both sets have cardinality \( \frac{(2f)!}{(f-1)(2f)!} \).

\[\hat{\mathcal{D}}_{(1, f-1)}^{(1, \ldots, 1)} \mathcal{H}_f(n) \mathcal{D}_{(2f)}^{(2, \ldots, n)} \]

**Proposition 17.** Let \( f \geq 0 \) and \( \mathcal{H}_f = \mathcal{H}_f(n) \). Let \( \hat{\mathcal{E}}_f \) be the bottom part of \( \hat{\mathcal{E}}_f \). Then we have

1. \( \hat{\mathcal{E}}_f \mathcal{D}_{(1, f-1)}^{(1, \ldots, 1)} \mathcal{H}_f(n-2f) = (1 + qr^{-1})^f \hat{\mathcal{E}}_f \mathcal{D}_{(2f, n-2f)}^{(2, \ldots, n)} \mathcal{H}_f(2f) \)

2. \( \hat{\mathcal{E}}_f \mathcal{D}_{(2f, n-2f)}^{(2, \ldots, n)} = a_f \hat{\mathcal{E}}_f \mathcal{H}_f \)

where \( a_0 = 1 \) and \( a_i = a_{i-1} \cdot \sum_{k=0}^{i-1} q^{2k} \) for \( i > 0 \).

**Proof.** For \( f = 0 \), the proposition is trivial.

1. Note that \( \mathcal{D}_{(1,1)} = \mathcal{S}_2 \). We have

\[
\begin{array}{c}
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet} \\
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet} \\
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet} \\
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet}
\end{array}
\]

\[
\begin{array}{c}
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet} \\
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet} \\
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet} \\
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet}
\end{array}
\]

\[
\begin{array}{c}
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet} \\
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet} \\
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet} \\
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet}
\end{array}
\]

\[
\begin{array}{c}
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet} \\
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet} \\
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet} \\
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet}
\end{array}
\]

\[
\begin{array}{c}
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet} \\
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet} \\
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet} \\
\text{bullet} \quad \text{bullet} \quad \text{bullet} \quad \text{bullet}
\end{array}
\]

which shows the first claim.
2. We have

\[ \hat{D}(2f_n-2f) = \hat{D}(2f_n-2f) \hat{D}(2f) \]

and

\[ \hat{H} f(16) = \hat{D}(2f_n-2f) \hat{H} f \]

Thus, without loss of generality we can assume that \( n = 2f \). We show that

\[ \hat{e} f \hat{D}_{(1,f-1)} \hat{H} f = \left( \sum_{k=0}^{f-1} q^{2k} \right) \hat{e} f \hat{H} f \quad \text{and} \]

\[ \hat{e} f \hat{D}_{(2f)} = a f \hat{e} f \hat{H} f \quad \text{(22)} \]

For \( f = 1 \) these equations are trivial. Let \( f = 2 \), then \( \mathcal{D}_{(1,1)} = \mathcal{S}_2 = \{1, s_1\} \) and thus \( \mathcal{D}_{(1,1)} \mathcal{H}_2 = \mathcal{D}_{(2,2)} \). We have

\[ \{T_d \mid d \in \mathcal{H}_2\} = \left\{ \begin{array}{c} \text{\_\_\_\_} \quad \text{\_\_\_\_}\_ \quad \text{\_\_\_\_\_} \end{array} \right\} \]

\[ \{T_d \mid d \in s_1 \mathcal{H}_2\} = \left\{ \begin{array}{c} \text{\_\_\_\_} \quad \text{\_\_\_\_\_} \quad \text{\_\_\_\_\_\_} \end{array} \right\} \]

Then

\[ \hat{e}_2 s_1 \mathcal{H}_2 = q^2 \bigcirc + q^3 \bigcirc + q^4 \bigcirc \]

\[ = q^2 \left( q^2 \bigcirc + q \bigcirc + \bigcirc \right) = q^2 \hat{e}_2 \hat{H}_2 \quad \text{(23)} \]
It follows that equations (21) and (22) hold for $f = 2$. Suppose now that (21) and (22) hold for $f - 1$. Then, $\mathcal{D}_f = \{1\} \cup \mathcal{S}_1 \mathcal{D}_{f-2}$ and thus

\[
\hat{e}_f \mathcal{D}_{f-1} \mathcal{H}_f = \hat{D}_{f-1} \mathcal{H}_f
\]

By (23), this equals

\[
\hat{H}_f \mathcal{H}_f = \hat{D}_{f-1} \mathcal{H}_f + q^2 \hat{H}_f \mathcal{H}_{f-2} + \mathcal{D}_{3,2f-4}
\]

Applying equation (12) to the compositions $\lambda = (1, 2, 2f - 4)$ and
\( \mu = (1, 2f - 2) \) and then equation \([19]\) for \( f - 1 \), we obtain

\[
\hat{e}_f D_{(1,f-1)} H_f = \begin{array}{c}
\vdots \\
\hat{H}_f \\
\vdots 
\end{array} + q^2
\begin{array}{c}
\vdots \\
\hat{D}_{(1,2f-2)} \\
\vdots 
\end{array} + q^2 \sum_{k=0}^{f-2} q^{2k} \begin{array}{c}
\vdots \\
\hat{H}_f \\
\vdots 
\end{array} + q^2 \sum_{k=0}^{f-2} q^{2k} \begin{array}{c}
\vdots \\
\hat{D}_{(1,2f-2)} \\
\vdots 
\end{array}
\]

\[
\text{induction} \quad \hat{H}_f + q^2 \sum_{k=0}^{f-2} q^{2k} \hat{H}_f + q^2 \sum_{k=0}^{f-2} q^{2k} \hat{D}_{(1,2f-2)}
\]

Thus, equation \([21]\) holds for \( f \). Now,

\[
\hat{e}_f D_{(2f)} = \begin{array}{c}
\vdots \\
\hat{D}_{(2f)} \\
\vdots 
\end{array} + \begin{array}{c}
\vdots \\
\hat{D}_{(2f-1)} \\
\vdots 
\end{array} + q^2 \sum_{k=0}^{f-2} q^{2k} \begin{array}{c}
\vdots \\
\hat{H}_{f-1} \\
\vdots 
\end{array} + q^2 \sum_{k=0}^{f-2} q^{2k} \begin{array}{c}
\vdots \\
\hat{E}_{f-1} \\
\vdots 
\end{array}
\]

Then \([22]\) follows from \([19]\) and \([21]\).

Recall that \( x_f = \hat{H}_f^{-1} \mathcal{G}_{(2f+1,\ldots,n)} \hat{E}_f \hat{H}_f \) and let \( y_f = \mathcal{G}_n \hat{E}_f \hat{H}_f \). We have

\[
y_f = \mathcal{G}_n \hat{E}_f \hat{H}_f = D_{(12f,n-2f)}^{-1} \mathcal{G}_{(12f,n-2f)} \hat{E}_f \hat{H}_f = D_{(12f,n-2f)}^{-1} \hat{E}_f \mathcal{G}_{(12f,n-2f)} \hat{H}_f
\]

\[
= D_{(12f,n-2f)}^{-1} \hat{E}_f \mathcal{G}_{(12f,n-2f)} \hat{H}_f
\]

\[
= \frac{1}{1 + qr^{-1})f \alpha_f \hat{H}_f^{-1} \hat{E}_f \mathcal{G}_{(12f,n-2f)} \hat{H}_f = (1 + qr^{-1})f \alpha_f x_f
\]

In order to compute \( y_f E_1 \) and thus \( x_f E_1 \), we decompose \( H_f \) into 5 pairwise disjoint subsets. Note that if \( d \in H_f \) then \((1)d^{-1} = 1\) or \((1)d^{-1} = 2f + 1\).
Let
\[
\mathcal{H}_f^{(a)} = \{ d \in \mathcal{H}_f \mid (1)d^{-1} = 1, (2)d^{-1} = 2 \}
\]
\[
\mathcal{H}_f^{(b)} = \{ d \in \mathcal{H}_f \mid (1)d^{-1} = 1, (2)d^{-1} = 3 \}
\]
\[
\mathcal{H}_f^{(c)} = \{ d \in \mathcal{H}_f \mid (1)d^{-1} = 1, (2)d^{-1} = 2f + 1 \}
\]
\[
\mathcal{H}_f^{(d)} = \{ d \in \mathcal{H}_f \mid (1)d^{-1} = 2f + 1, (2)d^{-1} = 1 \}
\]
\[
\mathcal{H}_f^{(e)} = \{ d \in \mathcal{H}_f \mid (1)d^{-1} = 2f + 1, (2)d^{-1} = 2f + 2 \}
\]

We have

**Lemma 18.** \( \mathcal{H}_f \) is a disjoint union of \( \mathcal{H}_f^{(a)}, \mathcal{H}_f^{(b)}, \mathcal{H}_f^{(c)}, \mathcal{H}_f^{(d)}, \) and \( \mathcal{H}_f^{(e)} \), and

\[
\mathcal{H}_f^{(a)} = \mathcal{H}_{f-1}^{\{3,\ldots,n\}}
\]
\[
\mathcal{H}_f^{(b)} = s_2 \mathcal{H}_{f-2}^{\{5,\ldots,n\}} \mathcal{D}_{(1,1,n-3)}^{\{3,\ldots,n\}}
\]
\[
\mathcal{H}_f^{(c)} = s_2 s_{f-1} \cdots s_2 \mathcal{H}_{f-1}^{\{4,\ldots,n\}} \mathcal{D}_{(1,n-3)}^{\{3,\ldots,n\}}
\]
\[
\mathcal{H}_f^{(d)} = s_2 s_{f-1} \cdots s_2 \mathcal{H}_{f-1}^{\{4,\ldots,n\}} \mathcal{D}_{(1,n-3)}^{\{3,\ldots,n\}}
\]
\[
\mathcal{H}_f^{(e)} = s_2 s_{f-1} \cdots s_1 s_{f+1} \mathcal{H}_f^{\{3,\ldots,n\}}
\]

Furthermore, in each of these products lengths add.

**Proof.** It follows directly from the definition that \( \mathcal{H}_f \) is a disjoint union of \( \mathcal{H}_f^{(a)}, \mathcal{H}_f^{(b)}, \mathcal{H}_f^{(c)}, \mathcal{H}_f^{(d)}, \) and \( \mathcal{H}_f^{(e)} \). For the products on the right hand side of above five equalities, it is easy to check that lengths add. It is also easy to see that the right hand sides are subsets of the left hand sides respectively. By a simple counting, the number of elements of the right hand sides add up to the cardinality of \( \mathcal{H}_f \). This implies the above five equalities. \( \square \)

**Proposition 19.** We have

\[
y_0 E_1 = \hat{\mathcal{S}}_n E_1
\]
\[
y_f E_1 = \frac{(rq^{2n-2f-2} - q^{-1})(1 + qr^{-1})}{q - q^{-1}} \hat{\mathcal{S}}_n E_f \mathcal{H}_{f-1}^{\{3,\ldots,n\}} + q^{2f} \hat{\mathcal{S}}_n E_{f+1} \mathcal{H}_f^{\{3,\ldots,n\}}
\]

for \( 1 \leq f \leq \left[ \frac{n}{2} \right] \). Here, we set \( \hat{\mathcal{S}}_n E_{f+1} \mathcal{H}_f^{\{3,\ldots,n\}} = 0 \), if \( f = \left[ \frac{n}{2} \right] \) i. e. if \( n = 2f \) or \( n = 2f + 1 \).
Proof. The first equality is trivial. Let $f \geq 1$. In order to obtain $y_f E_1 = \widehat{\mathcal{S}}_n E_f \mathcal{H}_f E_1$, we compute $\widehat{\mathcal{S}}_n E_f \mathcal{H}_f^{(\alpha)} E_1$ for each $\alpha \in \{a, b, c, d, e\}$.

First, let $\alpha = a$. By Lemma 18 $\mathcal{H}_f^{(a)} = \mathcal{H}_f^{(3 \ldots n)}$ and thus

$$\widehat{\mathcal{S}}_n E_f \mathcal{H}_f^{(a)} E_1 = \widehat{\mathcal{S}}_n E_f \mathcal{H}_f^{(3 \ldots n)} E_1 = \mathcal{H}_f^{(3 \ldots n)} = \delta \widehat{\mathcal{S}}_n E_f \mathcal{H}_f^{(3 \ldots n)}$$

Note, that (24) holds for all $f \in \{1, \ldots, \left[\frac{n}{2}\right]\}$. If $\alpha = b$ and $f \geq 2$, then by Lemma 18 $\mathcal{H}_f^{(b)} = q T \mathcal{H}_f^{(5 \ldots n)} D_{(1,1,n-4)}$ and thus

$$\widehat{\mathcal{S}}_n E_f \mathcal{H}_f^{(b)} E_1 = q r$$

By Proposition 17 this equals

$$qr(1 + qr^{-1})^{2-f} a_{f-2}$$

$$= qr(1 + qr^{-1}) \frac{a_{f-1}}{a_{f-2}}$$

$$= qr(1 + qr^{-1}) \left( \sum_{k=0}^{f-2} q^{2k} \right) \widehat{\mathcal{S}}_n E_f \mathcal{H}_f^{(3 \ldots n)}$$

Note that $\mathcal{H}_f^{(b)} = \emptyset$ and $\sum_{k=0}^{f-2} q^{2k} = 0$ if $f = 1$. Hence, this equation is also valid for $f = 1$. 

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Consider now the cases $\alpha = c$ and $\alpha = d$. First, let $f \neq \lfloor \frac{n}{2} \rfloor$. Then we have

\[ \hat{\mathcal{S}}_n \hat{E}_f \left( \hat{\mathcal{H}}_f^{(c)} + \hat{\mathcal{H}}_f^{(d)} \right) E_1 \]

\[ = \hat{\mathcal{S}}_n \hat{E}_f \left( q^{2f-1} T_2 T_{2f-1} \cdots T_2 + q^{2f} T_2 T_{2f-1} \cdots T_2 T_1 \right) \hat{\mathcal{H}}_f^{(4, \ldots, n)} \hat{D}_{(1,n-3)}^{(3, \ldots, n)} E_1 \]

\[ = q^{2f-1} + q^{2f} \hat{\mathcal{H}}_{f-1} \]

\[ = q^{2f} (rq^{-1} + 1) \]

By equation (12), we have

\[ \hat{D}_{(1^{2f-2}, n-2f-1)}^{(4, \ldots, n)} \hat{D}_{(1,n-3)}^{(3, \ldots, n)} = \hat{D}_{(1^{2f-1}, n-2f-1)}^{(3, \ldots, n)} = \hat{\mathcal{S}}_{2f-1}^{(3, \ldots, 2f+1)} \hat{D}_{(2f-1,n-2f-1)}^{(3, \ldots, n)} \]
Applying equation (13), we obtain

\[
\hat{S}_n \hat{E}_f \left( \hat{H}_f^{(c)} + \hat{H}_f^{(d)} \right) E_1
\]

\[
= \frac{q^{2f}(rq^{-1} + 1)}{a_{f-1}(1 + qr^{-1})f^{-1}}
\]

By equation (12) and (11),

\[
\hat{S}_{2f-1} \hat{D}_{(2f-1,n-2f-1)} = \hat{D}_{(1^{f-1},n-2f-1)} \hat{D}_{(1^{f-2},n-2f)}
\]

\[
\hat{S}_n = \hat{D}_{(1^{f-1},n-2f)} \hat{S}_{n-2f}
\]

Thus by equation (15), we have

\[
= \sum_{k=0}^{n-2f-1} q^{2k} \hat{S}_{n-2f} + \frac{1 - q^2}{q^{-1}r + 1}.
\]
The first summand equals
\[
\left( \sum_{k=0}^{n-2f-1} q^{2k} \right) \hat{S}_n \hat{E}_f \hat{D}_{(12f-2,n-2f)} = a_{f-1}(1+qr^{-1})f^{-1} \left( \sum_{k=0}^{n-2f-1} q^{2k} \right) \hat{S}_n \hat{E}_f \hat{H}_{f-1}^{(3,\ldots,n)}
\]

By Proposition 17, the second summand is
\[
\frac{1 - q^2}{q^{-1}r + 1} \hat{S}_n \hat{E}_{f+1} \hat{D}_{(2f+1,n-2f)} \hat{D}_{(12f-2,n-2f)} = \frac{(1 - q^2)(1 + qr^{-1})f^{-1}}{q^{-1}r + 1} \hat{S}_n \hat{E}_{f+1} \hat{D}_{(2f+1,n-2f)} \hat{D}_{(12f-2,n-2f)} = \frac{(1 - q^2)(1 + qr^{-1})f^{-1} a_f}{q^{-1}r + 1} \hat{S}_n \hat{E}_{f+1} \hat{H}_f^{(3,\ldots,n)}
\]

Taking this altogether, we get
\[
\hat{S}_n \hat{E}_f \left( \hat{H}_f^{(c)} + \hat{H}_f^{(d)} \right) E_1
\]
\[
= \frac{q^{2f}(r q^{-1} + 1)}{a_{f-1}(1 + qr^{-1})f^{-1}} \left( \sum_{k=0}^{n-2f-1} q^{2k} \right) \hat{S}_n \hat{E}_f \hat{H}_{f-1}^{(3,\ldots,n)} + \frac{(1 - q^2)(1 + qr^{-1})f^{-1} a_f}{q^{-1}r + 1} \hat{S}_n \hat{E}_{f+1} \hat{H}_f^{(3,\ldots,n)}
\]
\[
= q^{2f}(r q^{-1} + 1) \left( \sum_{k=0}^{n-2f-1} q^{2k} \right) \hat{S}_n \hat{E}_f \hat{H}_{f-1}^{(3,\ldots,n)} + q^{2f}(1 - q^2) \left( \sum_{k=0}^{f-1} q^{2k} \right) \hat{S}_n \hat{E}_{f+1} \hat{H}_f^{(3,\ldots,n)}
\]

(26)

If \( f = \left[ \frac{n}{2} \right], \) i.e. \( n = 2f \) or \( n = 2f + 1 \), then the same considerations can be done, except for omitting the second summand. But \( \hat{S}_n \hat{E}_{f+1} \hat{H}_f^{(3,\ldots,n)} = 0 \) by our convention. Thus equation (26) holds for \( f \in \{1, \ldots, \left[ \frac{n}{2} \right] \} \).
Finally, if $\alpha = e$ and $f \neq \left[ \frac{n}{2} \right]$, then

\[
\begin{align*}
\widehat{\mathcal{S}}_n \hat{E}_f \hat{H}_f^{(e)} E_1 &= q^{4f} \widehat{\mathcal{S}}_n \hat{E}_f T_{2f} T_{2f-1} \cdots T_1 \hat{T}_{2f+1} \hat{T}_{2f} \cdots T_2 \hat{H}_f^{(3, \ldots, n)} E_1 \\
&= q^{4f} \widehat{\mathcal{S}}_n \hat{E}_{f+1} \hat{H}_f^{(3, \ldots, n)}
\end{align*}
\]

Again, if $f = \left[ \frac{n}{2} \right]$ then this equation also holds. Adding (24), (25), (26) and (27) yields the proposition.

Theorem 20. We have

\[
\frac{c_f}{c_{f-1}} = - \frac{(q - q^{-1}) q^{2f-2} \sum_{k=0}^{f-1} q^{2k}}{r q^{2n-2f-2} - q^{-1}} = - \frac{q^{4f-3} - q^{2f-3}}{r q^{2n-2f-2} - q^{-1}}
\]

for $f \geq 1$.

Proof. Let $b_0 = 0$ and $b_f = \frac{(1 + qr^{-1})(rq^{2n-2f-2} - q^{-1})}{q - q^{-1}}$ for $f \geq 1$, $d_f = q^{2f}$ and $\mathcal{E}_f = \widehat{\mathcal{S}}_n \hat{E}_{f+1} \hat{H}_f^{(3, \ldots, n)}$. Note that in view of Proposition 17

\[
\mathcal{E}_f = a_{f+1} (1 + qr^{-1})^{f+1} \hat{H}_{f+1}^{-1} \hat{\mathcal{S}}_{(2f+1, \ldots, n)} \hat{E}_{f+1} \hat{H}_f^{(3, \ldots, n)},
\]

and this is a linear combination of basis elements. In particular, the set $\{ \mathcal{E}_f, 0 \leq f \leq \left[ \frac{n}{2} \right] \}$ is linearly independent. We have

\[
0 = \sum_{0 \leq f \leq \left[ \frac{n}{2} \right]} c_f x_f E_1 = \frac{1}{a_f (1 + qr^{-1})^f} \sum_{0 \leq f \leq \left[ \frac{n}{2} \right]} c_f y_f E_1
\]

\[
= \sum_{0 \leq f \leq \left[ \frac{n}{2} \right]} \left( \frac{b_f c_f}{a_f (1 + qr^{-1})^f} \mathcal{E}_{f-1} + \frac{d_f c_f}{a_f (1 + qr^{-1})^f} \mathcal{E}_f \right)
\]

\[
= \sum_{0 \leq f \leq \left[ \frac{n}{2} \right]} \left( \frac{b_f c_f}{a_f (1 + qr^{-1})^f} \mathcal{E}_{f-1} + \frac{d_f c_f}{a_f (1 + qr^{-1})^f} \mathcal{E}_f \right)
\]

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It follows that for $f \geq 1$
\[
\frac{b_f c_f}{a_f (1 + qr^{-1})^f} + \frac{d_f-1 c_f-1}{a_f-1 (1 + qr^{-1})^{f-1}} = 0
\]
and thus
\[
\frac{c_f}{c_{f-1}} = -\frac{d_f-1 a_f (1 + qr^{-1})^f}{a_f-1 (1 + qr^{-1})^{f-1} b_f} = -\frac{d_f-1 (1 + qr^{-1}) \sum_{k=0}^{f-1} q^{2k}}{b_f (1 + qr^{-1})} \frac{q^{2f-2}(1 + qr^{-1})(q - q^{-1}) \sum_{k=0}^{f-1} q^{2k}}{(1 + qr^{-1})(r q^{2n-2f-2} - q^{-1})} = -\frac{q^{2f-2}(q - q^{-1}) \sum_{k=0}^{f-1} q^{2k}}{(r q^{2n-2f-2} - q^{-1})}
\]

Corollary 21. Let $x = \sum_{f=0}^{\lfloor \frac{n}{2} \rfloor} c_f x_f$ be as in Theorem 20. Then we have
\[
x E_i = 0 \\
x T_i = qx
\]
for $i = 1, \ldots, n-1$.

Proof. First note that $x_f = \frac{1}{a_f(1+qr^{-1})^f} \hat{H}_f^{-1} \hat{E}_f \hat{G}_n$ by the same arguments that showed $x_f = \frac{1}{a_f(1+qr^{-1})^f} y_f$. Thus $x = y \hat{E}_f \hat{G}_n$ for some $y \in B_n$. In the proof of Lemma 11 we showed that $\hat{G}_n T_i = q \hat{G}_n + \frac{1-q^2}{r+q} \hat{G}_n E_i$. Thus it suffices to show that $x E_i = 0$ for $i = 1, \ldots, n-1$.

For $i = 1$ this holds by definition of $x$. Note that in the proof of Theorem 20 the bottom horizontal edge of $E_1$ was not involved in our calculations. Thus we actually have
\[
\begin{array}{c}
\vdots \\
x \\
\vdots
\end{array} = 0
\]
But then by Lemma 10
\[
x E_i = \begin{array}{c}
\vdots \\
x \\
\vdots
\end{array} = \begin{array}{c}
\vdots \\
x \\
\vdots
\end{array} = 0
\]
\]
Remark 22. Note that Corollary 21 provides an alternative proof of Lemma 6. In particular, as an application we can now prove a conjecture of [4]. For \( l \in \mathbb{N} \) let \( B_{n,q} := B_n(-q^{2l+1}, q) \) be the specialized BMW-algebra over \( \mathbb{Q}(q) \) and let \( V \) be a \( 2l \)-dimensional vector space over \( \mathbb{Q}(q) \). Then \( B_{n,q} \) acts on tensor space \( V^{\otimes n} \) and satisfies Schur-Weyl duality together with the action of the symplectic quantum group \( \mathcal{U}_q(\mathfrak{sp}_{2l}) \) on tensor space. In [4], a generator of the annihilator in \( B_{n,q} \) of \( V^{\otimes n} \) for \( n > l \) is constructed. It is given as the specialized antisymmetrizer \( y = y_{l+1} \) of \( B_{l+1,q} \), embedded into \( B_{n,q} \). The conjecture proposed in [4, Page 2921, line -11] now states:

**Corollary 23.** The generator \( y = y_{l+1} \) of the annihilator can be rescaled, such that the coefficient of the basis element \( T_uT_\sigma \hat{E}_fT_d \) (see Lemma 6) in \( y \) is \( q^k \cdot (-q)^{-l(u)-l(\sigma)-l(d)} \) for some \( k \in \mathbb{Z} \). In particular, it is up to sign a power of \( q \).

**Proof.** Let \( c_f(r,q) \) be the coefficients \( c_f \) from Lemma 6. Then the antisymmetrizer has the form

\[
y = \sum_{0 \leq f \leq \lfloor n/2 \rfloor} c_f(r, -q^{-1}) y_f
\]

with \( y_f = \sum_{u,\sigma,d} (-q)^{-l(u)-l(u)-l(d)} T_uT_\sigma \hat{E}_fT_d \). We have

\[
\frac{c_f(r, -q^{-1})}{c_{f-1}(r, -q^{-1})} = \frac{q^{3-4f}(1 - q^{2f})}{q + rq^{2+2f-2n}}.
\]

Now specializing \( r \) to \( -q^{2l+1} \) and \( n \) to \( l+1 \) (as in the setting of [4, Theorem 5.4]), we get \( q^{2-4f} \), which is a power of \( q \). This proves the conjecture proposed in [4].

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