Weak Essentially Undecidable Theories of Concatenation II

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Abstract

We show that we can interpret concatenation theories in arithmetical theories without coding sequences.

1 Introduction

A computably enumerable first-order theory is called essentially undecidable if any consistent extension, in the same language, is undecidable (there is no algorithm for deciding whether an arbitrary sentence is a theorem). A computably enumerable first-order theory is called essentially incomplete if any recursively axiomatizable consistent extension is incomplete. Since a decidable consistent theory can be extended to a decidable complete consistent theory (see Chapter 1 of Tarski et al. [12]), a theory is essentially undecidable if and only if it is essentially incomplete. Two theories that are known to be essentially undecidable are Robinson arithmetic $Q$ and the related theory $R$ (see Figure 1.1 for the axioms of $R$ and $Q$). The essential undecidability of $R$ and $Q$ is proved in Chapter 2 of [12]. In Chapter 1 of [12], Tarski introduces interpretability as an indirect way of showing that first-order theories are essentially undecidable. The method is indirect because it reduces the problem of essential undecidability of a theory $T$ to the problem of essentially undecidability of a theory $S$ which is known to be essentially undecidable. Interpretability between theories is a reflexive and transitive relation and thus induces a degree structure on the class of computably enumerable essentially undecidable first-order theories.

In [10], we introduce two theories of concatenation $WD$, $D$ and show that they are respectively mutually interpretable with $R$ and $Q$ (see Figure 1.2 for the axioms of $WD$ and $D$). The language of $WD$ and $D$ is $\{0, 1, \circ, \preceq\}$ where $0$ and $1$ are constant symbols, $\circ$ is a binary function symbol and $\preceq$ is a binary relation symbol. The intended model of $WD$ and $D$ is the free semigroup generated by two letters extended with the prefix relation. Extending finitely generated free semigroups with the prefix relation allows us to introduce $\Sigma_1$-formulas which are expressive enough to encode computations by Turing machines (see Kristiansen &
The Axioms of \( \mathbb{R} \)

\[
\begin{align*}
R_1 & \quad \pi + \overline{m} = \pi + m \\
R_2 & \quad \pi \times \overline{m} = \pi \times m \\
R_3 & \quad \pi \neq \overline{m} \quad \text{if } n \neq m \\
R_4 & \quad \forall x \ [ x \leq \pi \rightarrow \bigvee_{k \leq n} x = \overline{k}] \\
R_5 & \quad \forall x \ [ x \leq \pi \lor \pi \leq x]
\end{align*}
\]

The Axioms of \( \mathbb{Q} \)

\[
\begin{align*}
Q_1 & \quad \forall xy \ [ x \neq y \rightarrow Sx \neq Sy ] \\
Q_2 & \quad \forall x \ [ Sx \neq 0 ] \\
Q_3 & \quad \forall x \ [ x = 0 \lor \exists y \ [ x = Sy ] ] \\
Q_4 & \quad \forall x \ [ x + 0 = x ] \\
Q_5 & \quad \forall x \ [ x + Sy = S(x + y) ] \\
Q_6 & \quad \forall x \ [ x \times 0 = 0 ] \\
Q_7 & \quad \forall xy \ [ x \times Sy = x \times y + x ]
\end{align*}
\]

Figure 1.1: Non-logical axioms of the first-order theories \( \mathbb{R}, \mathbb{Q} \). The axioms of \( \mathbb{R} \) are given by axiom schemes where \( n, m, k \) are natural numbers and \( \overline{n}, \overline{m}, \overline{k} \) are their canonical names.

The Axioms of \( WD \)

\[
\begin{align*}
WD_1 & \quad \pi \overline{\beta} = \overline{\alpha \beta} \\
WD_2 & \quad \pi \neq \overline{\beta} \quad \text{if } \alpha \neq \beta \\
WD_3 & \quad \forall x \ [ x \leq \pi \leftrightarrow \bigvee_{\gamma \in \text{Pref}(\alpha)} x = \overline{\gamma}]
\end{align*}
\]

The Axioms of \( D \)

\[
\begin{align*}
D_1 & \quad \forall xyz \ [ (xy)z = x(yz) ] \\
D_2 & \quad \forall xy \ [ x \neq y \rightarrow ( x0 \neq y0 \land x1 \neq y1 ) ] \\
D_3 & \quad \forall xy \ [ x0 \neq y1 ] \\
D_4 & \quad \forall x \ [ x \leq 0 \leftrightarrow x = 0 ] \\
D_5 & \quad \forall x \ [ x \leq 1 \leftrightarrow x = 1 ] \\
D_6 & \quad \forall xy \ [ x \leq y0 \leftrightarrow ( x = y0 \lor x \leq y ) ] \\
D_7 & \quad \forall xy \ [ x \leq y1 \leftrightarrow ( x = y1 \lor x \leq y ) ]
\end{align*}
\]

Figure 1.2: Non-logical axioms of the first-order theories \( WD, D \). The axioms of \( WD \) are given by axiom schemes where \( \alpha, \beta, \gamma \) are nonempty binary strings and \( \overline{\alpha}, \overline{\beta}, \overline{\gamma} \) are their canonical names. \( \text{Pref}(\alpha) \) is the set of all nonempty prefixes of \( \alpha \).

Murwanashyaka [6]). \( \Sigma_1 \)-formulas are formulas on negation normal form where universal quantifiers occur bounded, i.e., they are of the form \( \forall x \leq t \). Axioms \( D_4 - D_7 \) are essential for coding sequences in \( D \) since they allow us to work with \( \Sigma_0 \)-formulas, formulas where all quantifiers are of the form \( \exists \leq t, \forall x \leq t \). In [10], we show that \( \mathbb{Q} \) is interpretable in \( D \) by using especially axioms \( D_4 - D_7 \) to restrict the universe of \( D \) to a domain \( K \) on which the analogue of \( Q_3 \) holds, that is, the sentence \( Q'_3 \equiv \forall x \ [ x = 0 \lor x = 1 \lor \exists y \leq x \ [ x = y0 \lor x = y1 ] ] \).

To improve readability, we use juxtaposition instead of the binary function symbol \( \circ \) of the formal language. Due to the existential quantifier in \( Q'_3 \), we need to ensure that \( \Sigma_{10} \)-formulas are absolute for \( K \).

Since \( D \) and \( \mathbb{Q} \) are mutually interpretable, we can identify differences between
these two theories by investigating the interpretability degrees of the theories we obtain by weakening axioms $D_4 - D_7$, $Q_3$ which are essential for coding sequences in $D$ and $Q$. In addition to $D$ and $WD$, we introduce in $[10]$ two theories $ID$, $ID^*$ (called $C$, $BT$, respectively, in $[10]$) and prove that their interpretability degrees are strictly between the degrees of $WD$ and $D$. But we are not able to determine in $[10]$ whether $ID$ and $ID^*$ are mutually interpretable. We obtain $ID$ and $ID^*$ from $D$ by replacing axioms $D_4 - D_7$ with respectively the axiom schemas

$$ID_4 \equiv \forall x \left[ x \leq \alpha \leftrightarrow \bigvee_{\gamma \in \text{Pref}(\alpha)} x = \gamma \right], \quad ID^*_4 \equiv \forall x \left[ x \subseteq \alpha \rightarrow \bigvee_{\gamma \in \text{Sub}(\alpha)} x = \gamma \right]$$

where $\alpha$ is a nonempty binary string, $\alpha$ is canonical variable-free term that represents $\alpha$, $\text{Pref}(\alpha)$ denotes the set of all nonempty prefixes of $\alpha$, $\text{Sub}(\alpha)$ denotes the set of all nonempty substrings of $\alpha$ and $x \subseteq y$ is shorthand for

$$x = y \lor \exists uv \left[ y = ux \lor y = xv \lor y = uxv \right].$$

In the standard model, $x \subseteq y$ holds if and only if $x \in \text{Sub}(y)$. It is easy to interpret $ID$ in $ID^*$ while it is less obvious whether $ID^*$ is interpretable in $ID$ since the axiom schema $ID^*_4$ puts strong constraints on the concatenation operator while any model of $D_1 - D_3$ can always be extended to a model of $ID$. In Section 3 we show that $ID$ and $ID^*$ are mutually interpretable.

Given mutually interpretability of $ID$ and $ID^*$, a natural question is whether the arithmetical analogues of $ID$ and $ID^*$ are also mutually interpretable. We let $IQ$ and $IQ^*$ be the theories we obtain from $Q$ by replacing axiom $Q_3$ with respectively the axiom schemas

$$IQ_3 \equiv \forall x \left[ x \leq \pi \leftrightarrow \bigvee_{k \leq n} x = \kappa \right], \quad IQ^*_3 \equiv \forall x \left[ x \leq_1 \pi \rightarrow \bigvee_{k \leq n} x = \kappa \right]$$

where $n$ is a natural number, $\pi$ is a canonical variable-free term that represents $n$, $\leq$ is a fresh binary relation symbol that is realized as the less than or equal relation in the standard model and $x \leq_1 y \equiv \exists z \left[ z + x = y \right]$. In Section 4 we show that $IQ$ and $IQ^*$ are mutually interpretable.

We try to identify differences between concatenation theories and arithmetical theories by investigating the comparability of $ID$ and $IQ$ with respect to interpretability. In Section 5 we show that $IQ$ is expressive enough to interpret the theory $ID$ we obtain by extending $ID$ with the axioms

$$\forall xy \left[ x \neq y \rightarrow (0x \neq 0y \land 1x \neq 1y) \right], \forall xy \left[ 0x \neq 1y \right].$$

Since $IQ$ does not have enough resources for coding general sequences, the interpretation we give shows that we can think of concatenation theories as naturally contained in arithmetical theories. In Section 5.2 we show that the idea behind the interpretation of $ID$ in $IQ$ allows us to give a very simple interpretation of $WD$ in $R$. In Section 6 we show that our interpretation of $ID$ in $IQ$ extends in a natural way to an interpretation in $Q$ of Grzegorczyk’s theory of concatenation $TC$ $[11]$ (see Figure 1.3 for the axioms of $TC$). We can think of $D$ as a fragment of $TC$ since $TC$ proves all the axioms of $D$ when we let $x \leq y \equiv x = y \lor \exists z \left[ y = xz \right]$. The intended model of $TC$ is a finitely
The Axioms of TC

\[\begin{align*}
TC_1 & \forall xyz \ [ x(yz) = (xy)z ] \\
TC_2 & \forall x y z w \ [ (xy = zw) \rightarrow ( (x = z \land y = w ) \lor \\
& \exists u \ [ ( z = xu \land uw = y ) \lor ( x = zu \land uy = w ) ] ) ] \\
TC_3 & \forall xy \ [ xy \neq 0 ] \\
TC_4 & \forall xy \ [ xy \neq 1 ] \\
TC_5 & 0 \neq 1
\end{align*}\]

Figure 1.3: Non-logical axioms of the first-order theory TC.

generated free semigroup with at least two generators. We have not been able
to determine whether IQ is interpretable in ID and whether ID is interpretable
in ID.

We summarize our results in the following theorem. We let \( S \leq T \) mean that
\( S \) is interpretable in \( T \). We let \( S < T \) mean \( S \leq T \land T \not\leq S \). We let \( S \cong T \) mean \( S \leq T \land T \leq S \) and we let \( \overline{ID} \) denote the theory we obtain from \( ID \) by
replacing \( ID_4 \) with \( ID_4^* \).

Theorem 1.

\[ R \cong WD < ID \cong ID^* \leq \overline{ID} \cong \overline{ID}^* \leq IQ \cong IQ^* < Q \cong D . \]

It is not difficult to see that the two strict inequalities \( WD < ID, IQ < Q \)
hold. If \( ID \) were interpretable in \( WD \), then \( ID_1 - ID_3 \) would be interpretable in a
finite subtheory of \( WD \). Since any model of \( ID_1 - ID_3 \) is infinite while any finite
subtheory of \( WD \) has a finite model, \( ID \) is not interpretable in \( WD \). Similarly,
if \( Q \) were interpretable in \( IQ \), it would be interpretable in a finite subtheory of
\( IQ \). But, any finite subtheory of \( IQ \) is interpretable in the first-order theory of
the field of real numbers \((R, 0, 1, +, \times)\), which was shown to be decidable by
Tarski [T]. Since \( Q \) is essentially undecidable, it is not interpretable in \( IQ \).

2 Preliminaries

In this section, we clarify a number of notions that we only glossed over in the
previous section.

2.1 Notation and Terminology

We consider the structures

\[ D^- = (\{0, 1\}^+, 0, 1, \_ ) \quad \text{and} \quad D = (\{0, 1\}^+, 0, 1, \_ , \_D) \]

where \( \{0, 1\}^+ \) is the set of all finite non-empty strings over the alphabet \( \{0, 1\} \),
the binary operator \( \_ \) concatenates elements of \( \{0, 1\}^+ \) and \( \_D \) denotes the
prefix relation, i.e., \( x \_D y \) if and only if \( y = x \) or there exists \( z \in \{0, 1\}^+ \)
such that \( y = x \_D z \). The structure \( D^- \) is thus the free semigroup with two
generators. We call elements of \( \{0, 1\}^+ \) bit strings. The structures \( D^- \) and
D are first-order structures over the languages $L_{\mathcal{BT}}^{-} = \{0,\ 1, \circ\}$ and $L_{\mathcal{BT}} = \{0,\ 1, \circ, \mathcal{Z}\}$, respectively.

The language of first-order arithmetic is $L_{\mathcal{NT}} = \{0, S, +, \times\}$ and we denote by \((\mathbb{N}, 0, S, +, \times)\) the standard first-order structure. In first-order number theory, each natural number \(n\) is associated with a numeral \(\overline{n}\) by recursion: \(\overline{0} \equiv 0\) and \(\overline{n+1} \equiv S\overline{n}\). Each non-empty bit string \(\alpha \in \{0,1\}^+\) is associated by recursion with a unique $L_{\mathcal{BT}}$-term \(\overline{\alpha}\), called a biteral, as follows: \(\overline{0} \equiv 0, \overline{1} \equiv 1, \overline{\alpha 0} \equiv (\overline{\alpha} \circ 0)\) and \(\overline{\alpha 1} \equiv (\overline{\alpha} \circ 1)\). The biterals are important if we, for example, want to show that certain sets are definable since we then need to talk about elements of \(\{0,1\}^+\) in the formal theory.

A class is a formula with at least one free variable. Given a class \(\mathcal{I}\) with \(n\) free variables, we write \((\mathcal{I} \phi)\) for \(\phi\) in \(\mathcal{I}(x_1,\ldots,x_n)\). If \(\mathcal{I}\) has two free variables, we also write \(\mathcal{I}xy\) for \(\mathcal{I}(x,y)\). We let \((\exists x_1,\ldots,x_n) \in \mathcal{I} \phi\) and \((\forall x_1,\ldots,x_n) \in \mathcal{I} \phi\) be shorthand for the formulas \(\exists x_1,\ldots,x_n [ \mathcal{I}(x_1,\ldots,x_n) \land \phi]\) and \(\forall x_1,\ldots,x_n [ \mathcal{I}(x_1,\ldots,x_n) \rightarrow \phi]\), respectively. We let \(\{x_1,\ldots,x_n\} \in \mathcal{I} : \psi\) be shorthand for \(\mathcal{I}(x_1,\ldots,x_n) \land \psi\).

### 2. Translations and Interpretations

We recall the method of relative interpretability introduced by Alfred Tarski [12] for showing that first-order theories are essentially undecidable. We restrict ourselves to many-dimensional parameter-free one-piece relative interpretations. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be computable first-order languages. A relative translation \(\tau\) from $\mathcal{L}_1$ to $\mathcal{L}_2$ is a computable map given by:

1. An $\mathcal{L}_2$-formula $\delta(x_1,\ldots,x_m)$ with exactly $m$ free variables. The formula $\delta(x_1,\ldots,x_m)$ is called a domain.

2. For each $n$-ary relation symbol $R$ of $\mathcal{L}_1$, an $\mathcal{L}_2$-formula $\psi_R(\vec{x_1},\ldots,\vec{x_n})$ with exactly $mn$ free variables. The equality symbol $=$ is treated as a binary relation symbol.

3. For each $n$-ary function symbol $f$ of $\mathcal{L}_1$, an $\mathcal{L}_2$-formula $\psi_f(\vec{x_1},\ldots,\vec{x_n},\vec{y})$ with exactly $m(n+1)$ free variables.

4. For each constant symbol $c$ of $\mathcal{L}_1$, an $\mathcal{L}_2$-formula $\psi_c(\vec{y})$ with exactly $m$ free variables.

We extend $\tau$ to a translation of atomic $\mathcal{L}_1$-formulas by mapping an $\mathcal{L}_1$-term $t$ to an $\mathcal{L}_2$-formula $(t)_{\tau,\vec{w}}$ with free variables $\vec{w}$ that denote the value of $t$:

5. For each $n$-ary relation symbol $R$ of $\mathcal{L}_1$

\[
(R(t_1,\ldots,t_n))_{\tau} \equiv \exists \vec{\bar{u}}_1\ldots\vec{\bar{u}}_n [ \bigwedge_{i=1}^{n} \delta(\vec{\bar{u}}_i) \land \bigwedge_{j=1}^{n} (t_j)_{\tau,\vec{w}_j} \land \psi_R(\vec{\bar{u}}_1\ldots\vec{\bar{u}}_n) ]
\]

where $\vec{\bar{u}}_1\ldots\vec{\bar{u}}_n$ are distinct variable symbols that do not occur in $t_1,\ldots,t_n$ and

a) for each variable symbol $x$ of $\mathcal{L}_1$, $(x)_{\tau,\vec{w}} \equiv \bigwedge_{i=1}^{m} w_i = x_i$
b) for each constant symbol \( c \) of \( \mathcal{L}_1 \), \((c)^\tau,\vec{w} \equiv \psi_c(\vec{w})\)

c) for each \( n \)-ary function symbol \( f \) of \( \mathcal{L}_1 \)

\[
(f(t_1, \ldots, t_n))^{\tau,\vec{w}} \equiv \exists \vec{w}_1 \ldots \vec{w}_n \left[ \bigwedge_{i=1}^{n} \delta(\vec{w}_i) \land \bigwedge_{j=1}^{n} (t_j)^{\tau,\vec{w}_j} \land \psi_f(\vec{w}_1 \ldots \vec{w}_n, \vec{w}) \right]
\]

where \( \vec{w}_1 \ldots \vec{w}_n \) are distinct variable symbols that do not occur in \( \bigwedge_{j=1}^{n} (t_j)^{\tau,\vec{w}_j} \).

We extend \( \tau \) to a translation of all \( \mathcal{L}_1 \)-formulas as follows:

6. \((\neg \phi)^\tau \equiv \neg \phi^\tau\)

7. \((\phi \oplus \psi)^\tau \equiv \phi^\tau \oplus \psi^\tau \) for \( \oplus \in \{\land, \lor, \rightarrow, \leftrightarrow\} \)

8. \((\exists x \phi)^\tau \equiv \exists \vec{x} \left[ \delta(\vec{x}) \land \phi^\tau \right] \)

9. \((\forall x \phi)^\tau \equiv \forall \vec{x} \left[ \delta(\vec{x}) \rightarrow \phi^\tau \right] \).

Let \( S \) be an \( \mathcal{L}_1 \)-theory and let \( T \) be an \( \mathcal{L}_2 \)-theory. We say that \( S \) is (relatively) interpretable in \( T \) if there exists a relative translation \( \tau \) such that

- \( T \vdash \exists \vec{x} \delta(\vec{x}) \)

- For each function symbol \( f \) of \( \mathcal{L}_1 \)

\[
T \vdash \bigwedge_{i=1}^{n} \delta(\vec{x}_i) \rightarrow \exists \vec{y} \left[ \delta(\vec{y}) \land \psi_f(\vec{x}_1, \ldots, \vec{x}_n, \vec{y}) \right].
\]

- For each constant symbol \( c \) of \( \mathcal{L}_1 \)

\[
T \vdash \exists \vec{y} \left[ \delta(\vec{y}) \land \psi_c(\vec{y}) \right].
\]

- \( T \) proves \( \phi^\tau \) for each non-logical axiom \( \phi \) of \( S \). If equality is not translated as equality, then \( T \) must prove the translation of each equality axiom.

If \( S \) is relatively interpretable in \( T \) and \( T \) is relatively interpretable in \( S \), we say that \( S \) and \( T \) are mutually interpretable.

The following proposition summarizes important properties of relative interpretability (see Tarski et al. [12] for the details).

**Proposition 2.** Let \( S, T \) and \( U \) be computably enumerable first-order theories.

1. If \( S \) is interpretable in \( T \) and \( T \) is consistent, then \( S \) is consistent.

2. If \( S \) is interpretable in \( T \) and \( T \) is interpretable in \( U \), then \( S \) is interpretable in \( U \).

3. If \( S \) is interpretable in \( T \) and \( S \) is essentially undecidable, then \( T \) is essentially undecidable.
3. MUTUAL INTERPRETABILITY OF ID AND ID∗

The Axioms of ID

\begin{align*}
ID_1 & \quad \forall xyz \ [ \ (xy)z = x(yz) ] \\
ID_2 & \quad \forall xy \ [ \ x \neq y \rightarrow (x0 \neq y0 \land x1 \neq y1 ) ] \\
ID_3 & \quad \forall xy \ [ \ x0 \neq y1 ] \\
ID_4 & \quad \forall x \ [ \ x \leq \overline{0} \leftrightarrow \bigvee_{\gamma \in \text{Pref}(\alpha)} x = \overline{\gamma} ]
\end{align*}

The Axioms of ID∗

\begin{align*}
ID_4^*, \ ID_1^*, \ ID_2^*, \ ID_3^*
\end{align*}

\begin{align*}
ID_4^* & \quad \forall x \ [ \ x \subseteq_s \overline{0} \rightarrow \bigvee_{\gamma \in \text{Sub}(\alpha)} x = \overline{\gamma} ]
\end{align*}

Figure 1.4: Non-logical axioms of the first-order theories ID and ID∗. ID4 and ID∗4 are axiom schemas where α is a nonempty binary string, Pref(α) is the set of all nonempty prefixes of α and Sub(α) is the set of all nonempty substrings of α. Furthermore, \( x \subseteq_s y \equiv x = y \lor \exists uv \ [ y = ux \lor y = xv \lor y = uxv ] \).

3 Mutual Interpretability of ID and ID∗

In this section, we show that ID and ID∗ are mutually interpretable (see Figure 1.4 for the axioms of ID and ID∗). It is easy to see that ID is interpretable in ID∗. We therefore need to focus on the more difficult task of proving that ID∗ is interpretable in ID. It is more difficult to interpret ID∗ in ID because the axiom schema ID∗4 puts strong constraints on the concatenation operator while it is always possible to extend any model of ID1, ID2, ID3 to a model of ID. For example, we can have models of ID where there exist infinitely many pairs \( x, y \) such that \( xy = \alpha \) for each nonempty string \( \alpha \). Indeed, consider the model where the universe is the Cartesian product \( \prod_{i<\omega} \{0,1\}^* \), concatenation is componentwise and each binary string \( \beta \) is mapped to the constant sequence \( (\beta)_{i<\omega} \).

To interpret ID∗ in ID, we need to use the axiom schema ID4 in an essential way to define a function \( * \) that provably in ID satisfies the translation of each axiom of ID∗. The idea is to observe that since we have the right cancellation law in the weak form of ID2, if we had an axiom schema for the suffix relation, denoted \textless_{\text{suffix}}\), analogous to ID4, we could try to define \( * \) by requiring that \( x * y = xy \) only if \( y \leq_{\text{suffix}} xy \). If \( xy \) is a variable-free term and \( y \leq_{\text{suffix}} xy \), then the axiom schema for the suffix relation gives us a finite number of possibilities for the value of \( y \). If we also knew that 0 and 1 were atoms/ indecomposable, we would be able to use ID2 and ID3 to determine that \( x \) and \( y \) are also variable-free terms. To make this idea work, we need to ensure that \( * \) is associative. Our solution is to show that extending ID with an axiom schema for \textless_{\text{suffix}}\) and the axiom \( \forall xy \ [ y \leq_{\text{suffix}} xy \] does not change the interpretability degree.

This section is organized as follows: In Section 3.1, we show that we can extend ID to a theory ID(2) with the same interpretability degree where 0 and 1 are atoms. In Section 3.2, we show that we can extend ID(2) to a theory ID(3) with the same interpretability degree where we have an axiom schema for the suffix relation \textless_{\text{suffix}}\) analogous to the axiom schema ID4. In Section 3.3, we extended...
$\text{ID}^{(3)}$ to a theory $\text{ID}^{(4)}$ with the same interpretability degree and where we have an axiom schema for the substring relation, analogues to $\text{ID}_4$. In Section 3.3, we use the axiom schema for the substring relation to extend $\text{ID}^{(4)}$ to a theory $\text{ID}^{(5)}$ with the same interpretability degree and where the suffix relation $\preceq_{\text{suffix}}$ satisfies additional properties. Finally, in Section 3.5, we show that $\text{ID}^*$ is interpretable in $\text{ID}^{(5)}$.

### 3.1 Atoms

It will prove useful later to know that 0 and 1 are atoms. So, let $\text{ID}^{(2)}$ be $\text{ID}$ extended with the axioms

\[
\text{AT0} \equiv \forall xy \ [ xy \neq 0 ], \quad \text{AT1} \equiv \forall xy \ [ xy \neq 1 ].
\]

**Lemma 3.** $\text{ID}$ and $\text{ID}^{(2)}$ are mutually interpretable.

**Proof.** Since $\text{ID}^{(2)}$ is an extension of $\text{ID}$, it suffices to show that $\text{ID}^{(2)}$ is interpretable in $\text{ID}$. Since the axioms of $\text{ID}$ are universal sentences, it suffices to relativize quantification to a domain $K$ on which the sentences AT0, AT1 hold. We obtain $K$ by successively restricting the universe to subclasses with nice properties.

Let

\[ K_1 = \{ x : \ x = 0 \lor x = 1 \lor \exists y \ [ x = y \ 0 \lor x = y 1 ] \}. \]

Clearly, $0, 1 \in K_1$. Since concatenation is associative, $K_1$ is closed under concatenation.

Let

\[ K_2 = \{ y \in K_1 : \forall x \in K_1 \ [ \bigwedge_{a \in \{0,1\}} xy \neq a ] \}. \]

We show that $0, 1 \in K_2$. Let $a, b \in \{0, 1\}$ and let $x \in K_1$. We need to show $xb \neq a$. Assume for the sake of a contradiction that $xb = a$. Since $x \in K_1$, let $c \in \{0, 1\}$ be such that $x = c$ or $x = uc$ for some $u$. Let $d \in \{0, 1\} \setminus \{c\}$. Then, $xb = a$ implies $ddxb = dda$. By $\text{ID}^3$ and $\text{ID}^2$, $ddx = dd$, which contradicts $\text{ID}_3$. Thus, $0, 1 \in K_2$.

We now show that $K_2$ is closed under the maps $x \mapsto x0$, $x \mapsto x1$. Let $y \in K_2$ and let $b \in \{0, 1\}$. We need to show that $yb \in K_2$. Since $y, b \in K_2 \subseteq K_1$ and $K_1$ is closed under concatenation, $yb \in K_1$. Now, let $a \in \{0, 1\}$ and let $x \in K_1$. We need to show $xyb \neq a$. Assume for the sake of a contradiction that $xyb = a$. Then, $axyb = aa$. By $\text{ID}_3$ and $\text{ID}_2$, we have $axy = a$, which contradicts $y \in K_2$ since $ax \in K_1$ as $a, x \in K_1$ and $K_1$ is closed under concatenation. Thus, $K_2$ is closed under the maps $x \mapsto x0$, $x \mapsto x1$.

The class $K_2$ is not a domain since it may not be closed under concatenation. We obtain $K$ by restricting $K_2$ to a subclass that contains 0 and 1 and is closed under concatenation. Let

\[ K = \{ w \in K_2 : \forall z \in K_2 \ [ zw \in K_2 ] \}. \]

We have $0, 1 \in K$ since $K_2$ contains 0, 1 and is closed under the maps $x \mapsto x0$, $x \mapsto x1$. We now show that $K$ is closed under concatenation. Let $w_0, w_1 \in K$.
3. MUTUAL INTERPRETABILITY OF ID AND ID*

We need to show that $w_0 w_1 \in K$. Since $w_0 \in K \subseteq K_2$ and $w_1 \in K$, we have $w_0 w_1 \in K_2$. Now, let $z \in K_2$. We need to show that $zw_0 w_1 \in K_2$. We do not worry about parentheses since ID tells us that concatenation is associative. Since $w_0 \in K$, we have $zw_0 \in K_2$. Since $w_1 \in K$, we have $zw_0 w_1 \in K_2$. Hence, $w_0 w_1 \in K$. Thus, $K$ is closed under concatenation.

3.2 Suffix Relation

In this section, we show that we can extend $\text{ID}^{(2)}$ to a theory where we have an axiom schema for the suffix relation, analogues to $\text{ID}^{4}$, without changing the interpretability degree. We extend the language of $\text{ID}^{(2)}$ with a fresh binary relation symbol $\preceq_{\text{suff}}$. Given a nonempty binary string $\alpha$, let $\text{Suff}(\alpha)$ denote the set of all nonempty suffixes of $\alpha$: $\gamma \in \text{Suff}(\alpha)$ if and only if $\alpha = \gamma$ or $\exists \delta \in \{0, 1\}^+ [ \alpha = \delta \gamma \land \gamma \in \{0, 1\}^+]$. Let $\text{ID}^{(3)}$ be $\text{ID}^{(2)}$ extended with the following axiom schema

$$\forall x [ x \preceq_{\text{suff}} \overline{\alpha} \leftrightarrow \bigvee_{\gamma \in \text{Suff}(\alpha)} x = \overline{\gamma} ] .$$

Lemma 4. $\text{ID}$ and $\text{ID}^{(3)}$ are mutually interpretable.

Proof. Since $\text{ID}^{(3)}$ is an extension of $\text{ID}$, it suffices by Lemma 3 to show that the suffix relation is definable in $\text{ID}^{(2)}$. We translate the suffix relation as follows: $x \preceq_{\text{suff}} y$ if and only if

1. $y = x \lor \exists u [ y = ux ]$  
2. $\forall u \preceq x [ u = 0 \lor u = 1 \lor \exists v \preceq u [ u = v0 \lor u = v1 ]]$  
3. $\preceq$ is reflexive and transitive on the class $I_x = \{ z : z \preceq x \}, \ x \in I_x$ and $\forall z \in I_x \forall w \preceq z [ w \in I_x ]$.

Given a nonempty binary string $\alpha$, we need to show that

$$\text{ID}^{(2)} \vdash \forall x [ x \preceq_{\text{suff}} \overline{\alpha} \leftrightarrow \bigvee_{\gamma \in \text{Suff}(\alpha)} x = \overline{\gamma} ] .$$

($\Rightarrow$) We show that

$$\text{ID}^{(2)} \vdash \forall x \big[ \left( \bigvee_{\gamma \in \text{Suff}(\alpha)} x = \overline{\gamma} \right) \rightarrow x \preceq_{\text{suff}} \overline{\alpha} \big] .$$

Let $\gamma \in \text{Suff}(\alpha)$. We need to show that $\overline{\gamma} \preceq_{\text{suff}} \overline{\alpha}$ holds. That is, we need to show that $\overline{\gamma}$ and $\overline{\alpha}$ satisfy (1)-(3). It is easy to prove by induction on the length of binary strings that

$$\text{ID} \vdash \overline{\delta} \overline{\zeta} = \overline{\delta \zeta} \text{ for all } \delta, \zeta \in \{0, 1\}^+ .$$

By (8), $\overline{\alpha} = \overline{\gamma}$ or $\overline{\alpha} = \overline{\delta} \overline{\gamma}$ where $\delta$ is a prefix of $\alpha$. Hence, (1) holds. By (8) and the axiom schema $\text{ID}_2$ for the prefix relation, $\overline{\gamma}$ satisfies (2)-(3). Thus, $\overline{\gamma} \preceq_{\text{suff}} \overline{\alpha}$ holds.
We need to show that
\[ \text{ID}^{(2)} \vdash \forall x \ [ x \preceq_{\text{suff}} \pi \to \bigvee_{\gamma \in \text{Suff}(\alpha)} x = \gamma ] . \]  

(\Rightarrow) We prove (**) by induction on the length of \( \alpha \). Assume \( \alpha \in \{0, 1\} \) and \( x \preceq_{\text{suff}} \pi \) holds. By (1), \( x = \pi \) or there exist \( u \) such that \( \pi = ux \). By AT0 and AT1, we have \( x = \pi \). Thus, (**) holds when \( \alpha \in \{0, 1\} \).

We consider the inductive case. Assume \( \alpha = \beta a \) where \( a \in \{0, 1\} \), \( \beta \in \{0, 1\}^+ \) and
\[ \text{ID}^{(2)} \vdash \forall x \ [ x \preceq_{\text{suff}} \beta \to \bigvee_{\gamma \in \text{Suff}(\beta)} x = \gamma ] . \]  

(***)

By definition, \( \pi = \beta \pi = \beta a \pi \). Assume \( x \preceq_{\text{suff}} \pi \) holds. By (1), \( x = \pi \) or there exist \( u \) such that \( \pi = ux \). If \( x = \pi \), we are done. So, assume \( \pi = ux \). By (3), we have \( x \preceq \pi \). Thus, \( x = \pi \) where \( \gamma \in \text{Suff}(\alpha) \).

Assume (ii) holds. Then, \( \beta \pi = \pi = ux = uw \). By ID3, we have \( \pi = c \). By ID2, we have \( \beta = uw \). Furthermore

- \( \forall u \preceq w \ [ u = 0 \lor u = 1 \lor \exists v \preceq u \ [ u = v0 \lor u = v1 ] \] since \( u \preceq w \) and \( w \preceq x \) implies \( u \preceq x \) by (3)

- since \( w \preceq x \) and (3) holds, \( \preceq \) is reflexive and transitive on the class \( I_w = \{ z : z \preceq w \} \), \( w \in I_w \) and \( \forall z \in I_w \forall w \preceq z \ [ w \in I_w ] \).

Thus, \( x \preceq_{\text{suff}} \beta \) holds. By (**), \( w = \delta \) where \( \delta \) is a suffix of \( \beta \). Then, \( x = \alpha \beta = \beta \pi = \beta a \) and \( \beta a \) is a suffix of \( \alpha \). Thus, \( \alpha \) satisfies (**).

Thus, by induction, (**) holds for all nonempty binary strings \( \alpha \).

3.3 Substring Relation

In this section, we show that we can extend ID\(^{(3)}\) to a theory where we have an axiom schema for the substring relation, analogues to ID\(^{(4)}\), without changing the interpretability degree. We extend the language of ID\(^{(3)}\) with a fresh binary relation symbol \( \preceq_{\text{sub}} \). Given a nonempty binary string \( \alpha \), let \( \text{Sub}(\alpha) \) denote the set of all nonempty substrings of \( \alpha \): \( \beta \in \text{Sub}(\alpha) \) if and only if \( \alpha = \beta \) or there exist \( \gamma, \delta \in \{0, 1\}^+ \) such that \( \beta \in \{0, 1\}^+ \) and \( \alpha = \gamma \beta \lor \alpha = \beta \delta \lor \alpha = \gamma \beta \delta \). Let ID\(^{(4)}\) be ID\(^{(3)}\) extended with the following axiom schema

\[ \forall x \ [ x \preceq_{\text{sub}} \pi \leftrightarrow \bigvee_{\gamma \in \text{Sub}(\alpha)} x = \gamma ] . \]

Lemma 5. ID and ID\(^{(4)}\) are mutually interpretable.
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Proof. By Lemma 4, it suffices to show that the substring relation is definable in ID(3). We translate the substring relation as follows

\[ x \preceq_{\text{sub}} y \equiv x \preceq y \vee x \preceq_{\text{suff}} y \vee \exists u \preceq y [ x \preceq_{\text{suff}} u ] . \]

By the axiom schema for the prefix relation and the axiom schema for the suffix relation, it is easy to see that ID(3) proves \( \forall x [ x \preceq_{\text{sub}} \alpha \leftrightarrow \bigvee_{\gamma \in \text{Sub}(\alpha)} x = \gamma ] \) for each nonempty binary string \( \alpha \).

3.4 Suffix Relation II

We are finally ready to equip the suffix relation with two very important properties. Let ID(5) be ID(4) extended with the following axioms

\[
\forall x \left[ \bigwedge_{a \in \{0,1\}} a \preceq_{\text{suff}} xa \right], \quad \forall xy \left[ x \preceq_{\text{suff}} y \rightarrow \bigwedge_{a \in \{0,1\}} xa \preceq_{\text{suff}} ya \right].
\]

To show that ID(5) and ID are mutually interpretable, we need the following lemma. Recall that a class is a formula with at least one free variable and that if \( I \) is a class with one free variable we occasionally write \( x \in I \) for \( I(x) \).

Lemma 6. There exists a class \( J \) with the following properties:

1. \( \text{ID}^{(4)} \vdash t \in J \) for each variable-free term \( t \)
2. \( \text{ID}^{(4)} \vdash \forall x \forall z \in J \left[ \bigwedge_{a \in \{0,1\}} ( z = xa \rightarrow a \preceq_{\text{suff}} z ) \right] \)
3. \( \text{ID}^{(4)} \vdash \forall xy \forall z \in J \left[ \bigwedge_{a \in \{0,1\}} ( ( z = ya \wedge x \preceq_{\text{suff}} y ) \rightarrow ( xa \preceq_{\text{suff}} z ) ) \right] \)
4. \( \text{ID}^{(4)} \vdash \forall z \in J \left[ z = 0 \vee z = 1 \vee \exists u \preceq_{\text{sub}} z [ z = u0 \vee z = u1 ] \right] \)
5. \( \text{ID}^{(4)} \vdash \forall z \in J \forall u \left[ u \preceq_{\text{sub}} z \rightarrow u \in J \right] \).

Proof. We define \( J \) as follows: \( u \in J \) if and only if

1. \( u \preceq_{\text{sub}} u \)
2. \( \forall w \preceq_{\text{sub}} u \left[ w \preceq_{\text{sub}} w \right] \)
3. \( \forall w \preceq_{\text{sub}} u \forall v0 \preceq_{\text{sub}} w \forall v1 \preceq_{\text{sub}} v0 \left[ v1 \preceq_{\text{sub}} w \right] \)
4. \( \forall w \preceq_{\text{sub}} u \forall x \left[ w = x0 \rightarrow 0 \preceq_{\text{suff}} w \right] \)
5. \( \forall w \preceq_{\text{sub}} u \forall x \left[ w = x1 \rightarrow 1 \preceq_{\text{suff}} w \right] \)
6. \( \forall w \preceq_{\text{sub}} u \forall xy \left[ ( w = y0 \wedge x \preceq_{\text{suff}} y ) \rightarrow x0 \preceq_{\text{sub}} w \right] \)
7. \( \forall w \preceq_{\text{sub}} u \forall xy \left[ ( w = y1 \wedge x \preceq_{\text{suff}} y ) \rightarrow x1 \preceq_{\text{sub}} w \right] \).
It follows straight from the definition that \( J \) satisfies clauses (2)-(4). By the axiom schema for the substring relation, the axiom schema for the suffix relation, \( \text{AT}\text{O}, \text{AT}\text{I}, \text{ID}\text{2} \) and \( \text{ID}\text{3}, \) \( J \) satisfies Clause (1). It remains to show that \( J \) also satisfies Clause (5). That is, we need to show that \( u' \) satisf"

We show that \( u' \) satisfies (2)-(iii) and (A)-(E). Consider one of these clauses. It is of the form \( \forall w \leq_{\text{sub}} u' \phi(w) \). We need to show that \( \forall w \leq_{\text{sub}} u' \phi(w) \) holds. Since \( u \in J \), we know that \( \forall w \leq_{\text{sub}} u \phi(w) \) holds. Let \( w \leq_{\text{sub}} u' \). We need to show that \( \phi(w) \) holds. Since \( \forall w \leq_{\text{sub}} u \phi(w) \) holds, it suffices to show that \( w \leq_{\text{sub}} u \) holds. By assumption

\[
 w \leq_{\text{sub}} u' \leq_{\text{sub}} u .
\]

Since \( u \) satisfies (i)
\[
 w \leq_{\text{sub}} u' \leq_{\text{sub}} u .
\]

Then, \( w \leq_{\text{sub}} u \) since \( u \) satisfies (iii). Hence, \( \forall w \leq_{\text{sub}} u \phi(w) \) holds. Thus, \( u' \)

Since \( u' \) satisfies (i)-(iii) and (A)-(E), \( u' \in J \). Thus, \( J \) is downward closed under \( \leq_{\text{sub}} \).

**Lemma 7.** \( \text{ID} \) and \( \text{ID}^{(5)} \) are mutually interpretable.

**Proof.** By Lemma 5, it suffices to show that \( \text{ID}^{(5)} \) is interpretable in \( \text{ID}^{(4)} \). Let \( J \) be the class given by Lemma 6. To interpret \( \text{ID}^{(5)} \) in \( \text{ID}^{(4)} \) it suffices to translate the suffix relation as follows

\[
x \leq_{\text{sub}} y \equiv ( y \in J \land x \leq_{\text{sub}} y ) \lor ( y \notin J \land x = x ).
\]

We need show that the translation of each instance of the axiom schema for the suffix relation is a theorem of \( \text{ID}^{(4)} \). Let \( \alpha \) be a nonempty binary string. We need to show that

\[
 \forall x [ x \leq_{\text{sub}} \gamma \leftrightarrow \bigvee_{\gamma \in \text{Suff}(\alpha)} x = \gamma ] (A)
\]

holds. By Clause (1) of Lemma 6, \( \gamma \in J \). Hence, by the definition of \( \leq_{\text{sub}} \), (A) holds if and only if

\[
 \forall x [ x \leq_{\text{sub}} \gamma \leftrightarrow \bigvee_{\gamma \in \text{Suff}(\alpha)} x = \gamma ] (B)
\]

holds. Observe that (B) is an instance of the axiom schema for the suffix relation. Thus, the translation of each instance of the axiom schema for the suffix relation is a theorem of \( \text{ID}^{(4)} \).

We need to show that the translation of the axiom

\[
 \forall x [ \bigwedge_{a \in \{0,1\}} a \leq_{\text{sub}} xa ] (C)
\]
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is a theorem of ID(4). Let x be arbitrary and let a ∈ {0, 1}. We need to show that \( x \succeq_{s_{a}} x \) holds. Assume \( x \in J \). Then, \( x \succeq_{s_{a}} x \) holds if and only if \( x \succeq_{s} x \) holds. By Clause (2) of Lemma 6, \( x \succeq_{s_{a}} x \) holds. Hence, \( x \succeq_{s_{a}} x \) holds when \( x \in J \). Assume now \( x \notin J \). Then, \( x \succeq_{s_{a}} x \) holds by the second disjunct in the definition of \( \succeq_{s} \). Thus, the translation of (C) is a theorem of ID(4).

We need to show that the translation of the axiom

\[
\forall xy \left[ x \succeq_{s} y \rightarrow \bigwedge_{a \in \{0,1\}} xa \succeq_{s} ya \right]
\]

is a theorem of ID(4). Let a ∈ {0, 1} and assume \( x \succeq_{s} y \) holds. We need to show that \( xa \succeq_{s} ya \) holds. Assume first \( ya \notin J \). Then, \( xa \succeq_{s} ya \) holds by the second disjunct in the definition of \( \succeq_{s} \). Assume next \( ya \in J \). Then, by Clause (4) of Lemma 6, \( ya \in \{0,1\} \) or there exist \( u \succeq_{s} ya \) and \( b \in \{0,1\} \) such that \( ya = ub \). By AT0, AT1 and ID3, we have \( ya = ua \) where \( u \succeq_{s} ya \). By ID2, we have \( y = u \). Hence, \( y \succeq_{s} ya \). By Clause (5) of Lemma 6 \( y \in J \). Thus, since \( x \succeq_{s} y \) holds and \( y \in J \), we have \( x \succeq_{s} y \) by the definition of \( \succeq_{s} \). Then, by Clause (3) of Lemma 6, \( xa \succeq_{s} ya \) holds. Thus, the translation of (D) is a theorem of ID(4).

3.5 Interpretation of ID* in ID

We are finally ready to show that ID* and ID are mutually interpretable.

**Theorem 8.** The theories ID, ID* are mutually interpretable.

**Proof.** To interpret ID in ID*, it suffices to translate \( \preceq \) as follows

\[ x \preceq y \equiv y = x \lor \exists z \left[ y = xz \right]. \]

Given a nonempty binary string α, we have

\[
\forall x \left[ x \preceq x \right] \quad \leftrightarrow \quad \forall x \left[ x = x \lor \exists z \left[ x = xz \right] \right]
\]

This shows that the translation of each instance of the axiom schema ID4 is a theorem of ID*. Thus, ID is interpretable in ID*.

Next, we show that ID* is interpretable in ID. By Lemma 7, it suffices to show that ID* is interpretable in ID(5). Since the axioms of ID* are universal sentences or sentences where existential quantifiers occur in the antecedent (instances of ID1), to interpret ID* in ID(5) it suffices to relativize quantification to a suitable domain K.
We start by defining an auxiliary class \( K_1 \) (this is why we extended ID\(^{(4)} \) to ID\(^{(5)} \)). Let

\[
K_1 = \{ u : \forall x \left[ u \preceq_{\text{suff}} xu \right] \}.
\]

By the axiom \( \forall x \left[ \bigwedge_{a \in \{0,1\}} a \preceq_{\text{suff}} xa \right] \), we have \( 0,1 \in J \). We show that \( K_1 \) is closed under the maps \( u \mapsto u0, \ u \mapsto u1 \). Let \( b \in \{0,1\} \) and let \( u \in K_1 \). We need to show that \( ub \in K_1 \). That is, we need to show that \( ub \preceq_{\text{suff}} xub \) for all \( x \). Since \( u \in K_1 \), we know that

\[
\forall x \left[ u \preceq_{\text{suff}} xu \right]
\]

(\( * \)) holds. Then, by (\( * \)) and the axiom

\[
\forall xy \left[ x \preceq_{\text{suff}} y \rightarrow \bigwedge_{a \in \{0,1\}} xa \preceq_{\text{suff}} ya \right]
\]

we have

\[
\forall x \left[ ub \preceq_{\text{suff}} xub \right].
\]

Hence, \( ub \in K_1 \). Thus, \( K_1 \) is closed under the maps \( u \mapsto u0, \ u \mapsto u1 \).

The class \( K_1 \) is not a domain since it may not be closed under concatenation. We let

\[
K = \{ u \in K_1 : \forall v \in K_1 \left[ vu \in K_1 \right] \}.
\]

Since \( K_1 \) contains \( 0 \) and \( 1 \) and is closed under the maps \( x \mapsto x0, \ x \mapsto x1 \), we have \( 0,1 \in K \). We show that \( K \) is closed under concatenation. Let \( u0, u1 \in K \). We need to show that \( u0u1 \in K \). We start by showing that \( u0u1 \in K_1 \). We have \( u0 \in K \subseteq K_1 \). Hence, \( u0u1 \in K_1 \) since \( u1 \in K_1 \). Now, we need to show that \( \forall v \in K_1 \left[ vu0u1 \in K_1 \right] \). We do not need to worry about parentheses since ID\(_1 \) tells us that concatenation is associative. Let \( v \in K_1 \). We need to show that \( vu0u1 \in K_1 \). Since \( u0 \in K \), we have \( vu0 \in K_1 \). Since \( u1 \in K \), we have \( vu0u1 \in K_1 \). Hence, \( u0u1 \in K \). Thus, \( K \) is closed under concatenation and therefore satisfies the domain conditions.

Since the axioms ID\(_1 \), ID\(_2 \), ID\(_3 \) are universal sentences, their restrictions to \( K \) are theorems of ID\(^{(5)} \). It remains to show that the restriction to \( K \) of each instance of

\[
\text{ID}^*_4 \equiv \forall x \left[ x \subseteq_{\text{sub}} \overline{\alpha} \rightarrow \bigvee_{\gamma \in \text{Sub}(\alpha)} x = \overline{\gamma} \right]
\]

is a theorem of ID\(^{(5)} \). It suffices to show that for each nonempty binary string \( \alpha \)

\[
\forall x,y \in K \left[ xy = \overline{\alpha} \rightarrow \bigvee_{\beta,\gamma \in \text{Sub}(\alpha)} \left( x = \overline{\gamma} \land y = \overline{\beta} \right) \right]. \quad (**)
\]

So, let \( x,y \in K \) and assume \( xy = \overline{\alpha} \). Since \( y \in K \subseteq K_1 \), we know that \( y \preceq_{\text{suff}} xy = \overline{\alpha} \). By the axiom schema for the suffix relation, \( y = \overline{\beta} \) where \( \beta \) is a nonempty suffix of \( \alpha \). So, \( x\overline{\beta} = \overline{\alpha} \). By ID\(_1 \), ID\(_2 \), ID\(_3 \), AT0, AT1, we have that \( x = \overline{\gamma} \) where \( \gamma \) is a nonempty prefix of \( \alpha \) such that \( \alpha = \gamma\beta \). Thus, (\( ** \)) holds for all nonempty binary strings \( \alpha \). Thus, the translation of each instance of \( \text{ID}^*_4 \) is a theorem of ID\(^{(5)} \). \( \square \)
3.6 The Theories \( \overline{\mathcal{ID}}, \overline{\mathcal{ID}}^* \)

The axioms \( \mathcal{ID}_1, \mathcal{ID}_2, \mathcal{ID}_3 \) describe a right cancellative semigroup. It is also natural to consider semigroups that are also left cancellative, for example \((\{0,1\}^+, 0, 1, \_\_\) \). Let \( \overline{\mathcal{ID}} \) and \( \overline{\mathcal{ID}}^* \) be \( \mathcal{ID} \) and \( \mathcal{ID}^* \), respectively, extended with the axioms

\[
\forall xy \ [ x \neq y \rightarrow (0x \neq 0y \land 1x \neq 1y)], \ \forall xy \ [ 0x \neq 1y].
\]

It is not difficult to see that \( \mathcal{T}C \) proves each axiom of \( \mathcal{ID}^* \). It is easily seen that our interpretation of \( \mathcal{ID}^* \) in \( \mathcal{ID} \) is also an interpretation of \( \overline{\mathcal{ID}}^* \) in \( \overline{\mathcal{ID}} \). Thus, \( \overline{\mathcal{ID}} \) and \( \overline{\mathcal{ID}}^* \) are mutually interpretable. We have not been able to determine whether \( \mathcal{ID} \) is interpretable in \( \mathcal{ID}^* \).

**Theorem 9.** \( \overline{\mathcal{ID}} \) and \( \overline{\mathcal{ID}}^* \) are mutually interpretable.

**Open Problem 10.** Is \( \overline{\mathcal{ID}} \) interpretable in \( \mathcal{ID}^* \)?

4 Mutual Interpretablility of \( \mathcal{IQ} \) and \( \mathcal{IQ}^* \)

In this section, we show that \( \mathcal{IQ} \) and \( \mathcal{IQ}^* \) are also mutually interpretable. Recall that \( \mathcal{IQ}^* \) is the theory we obtain from \( \mathcal{IQ} \) by removing \( \leq \) from the language and replacing the axiom schema \( \mathcal{IQ}_3 \) with the axiom schema

\[
\mathcal{IQ}^*_3 \equiv \forall x \ [ x \leq_1 \overline{n} \rightarrow \bigvee_{k \leq n} x = \overline{k}]
\]

where \( x \leq_1 y \equiv \exists z \ [ z + x = y] \).

**Theorem 11.** \( \mathcal{IQ} \) and \( \mathcal{IQ}^* \) are mutually interpretable.

The proof strategy is similar to the one we used to interpret \( \mathcal{ID}^* \) in \( \mathcal{ID} \). Since we obtain an interpretation of \( \mathcal{IQ} \) in \( \mathcal{IQ}^* \) by translating \( \leq \) as \( \leq_1 \), we just need to focus on proving that \( \mathcal{IQ}^* \) is interpretable in \( \mathcal{IQ} \). The proof is structured as follows: In Section 4.1, we extend \( \mathcal{IQ} \) to a theory \( \mathcal{IQ}^+ \) which proves that for each inductive class there exists an inductive subclass that is closed under addition and multiplication. A class is inductive if it contains 0 and is closed under the successor function. In Section 4.2, we extend \( \mathcal{IQ}^+ \) to a theory \( \mathcal{IQ}^{++} \) with the same interpretability degree as \( \mathcal{IQ}^+ \) and where the ordering relation \( \leq \) satisfies additional properties. In Section 4.3, we show that \( \mathcal{IQ}^* \) is interpretable in \( \mathcal{IQ}^+ \). Finally, in Section 7, we show that \( \mathcal{IQ} \) is mutually interpretable with a theory \( \mathcal{IQ}^{(2)} \) that is an extension of \( \mathcal{IQ}^+ \).

4.1 Closure under Addition and Multiplication

A class \( X \) is called inductive if \( 0 \in X \) and \( \forall x \in X \ [ Sx \in X] \). A class \( X \) is called a cut if it is inductive and \( \forall x \in X \forall y \ [ y \leq_1 x \rightarrow y \in X] \). Let \( \mathcal{IQ}^+ \) and \( \mathcal{Q}^+ \) be respectively \( \mathcal{IQ} \) and \( \mathcal{Q} \) extended with the following axioms

- Associativity of addition \( \forall xyz \ [ (x + y) + z = x + (y + z)] \)
- Left distributive law \( \forall xyz \ [ x(y + z) = xy + xz] \)


- Associativity of multiplication \( \forall xyz \[ (xy)z = x(yz) \] \).

Lemma V.5.10 of Hajek & Pudlak [4] says that \( Q^+ \) proves that any inductive class has a subclass that is a cut and is closed under + and \( \times \). The proof of that lemma shows that \( IQ^+ \) proves that any inductive class has an inductive subclass that is closed under + and \( \times \) (see also Section 7).

**Lemma 12.** Let \( X \) be an inductive class. Then, \( IQ^+ \) proves that there exists an inductive subclass \( Y \) that is closed under + and \( \times \).

### 4.2 Ordering Relation

Let \( IQ^{++} \) be \( IQ^+ \) extended with the following axioms

\[
\forall x \[ 0 \leq x \] \quad \forall xy [ x \leq y \rightarrow Sx \leq Sy ]
\]

Using the ideas of Section 3.4, we prove the following lemma.

**Lemma 13.** \( IQ^+ \) and \( IQ^{++} \) are mutually interpretable.

**Proof.** Since \( IQ^{++} \) is an extension of \( IQ^+ \), it suffices to show that \( IQ^{++} \) is interpretable in \( IQ^+ \). Furthermore, it suffices to show that we can translate \( \leq \) in such a way that \( IQ^+ \) proves the translation of each instance of \( IQ_3 \) and the translation of \( \forall x \[ 0 \leq x \] \) and \( \forall xy [ x \leq y \rightarrow Sx \leq Sy ] \).

Let \( u \in G \) if and only if

1. \( u \leq u \)
2. \( \forall w \leq u \[ w \leq w \] \)
3. \( \forall w \leq u \forall v_0 \leq w \forall v_1 \leq v_0 \[ v_1 \leq w \] \)
4. \( \forall w \leq u \[ w = 0 \lor \exists v \leq w \[ w = Sv \] \)
5. \( \forall w \leq u \forall x \[ w = Sx \rightarrow 0 \leq w \] \)
6. \( \forall w \leq u \forall xy \[ ( w = Sy \land x \leq y ) \rightarrow Sx \leq w \] \).

It can be verified that \( IQ \) proves that \( t \in G \) for each variable-free term \( t \) and that \( G \) is downward closed under \( \leq \).

We translate \( \leq \) as follows

\[ x \leq^↓ y \equiv ( y \in G \land x \leq y ) \lor ( y \notin G \land x = x ) \].

Since \( t \in G \) for each variable-free term \( t \), the translation of each instance of the axiom schema \( IQ_3 \) is a theorem of \( IQ^+ \).

We show that \( IQ^+ \) proves the translation of \( \forall x \[ 0 \leq x \] \). Choose an arbitrary \( x \). If \( x \notin G \), then \( 0 \leq^↓ x \) holds by the second disjunction in the definition of \( \leq^↓ \).

Otherwise, \( x \in G \). We need to show that \( 0 \leq x \) holds. If \( x = 0 \), then \( 0 \leq x \) holds by \( IQ_3 \). Otherwise, by (A), there exists \( v \leq x \) such that \( x = Sv \). Then, by (B), \( 0 \leq x \) holds. Thus, \( IQ^+ \models \forall x \[ 0 \leq^↓ x \] \).
5. INTERPRETABILITY OF $\mathbb{ID}$ IN $\mathbb{IQ}$

We show that $\mathbb{IQ}^+$ proves the translation of $\forall x y \ [ x \leq y \rightarrow Sx \leq Sy ]$. Assume $x \leq y$ holds. If $Sy \notin G$, then $Sx \leq^* Sy$ holds by the second disjunct in the definition of $\leq^*$. Otherwise, $Sy \in G$. We need to show that $Sx \leq Sy$ holds. By $Q_2$, $Sy \neq 0$. Hence, by (A), there exists $v \leq Sy$ such that $Sy = Sv$. By $Q_1$, $y = v$. Hence, $y \leq Sy$. Since $G$ is downward closed under $\leq$, we have $y \in G$. Then, by (C), $Sx \leq Sy$ holds. Thus, $\mathbb{IQ}^+ \vdash \forall x y \ [ x \leq^* y \rightarrow Sx \leq^* Sy ]$. □

4.3 Interpretation of $\mathbb{IQ}^*$ in $\mathbb{IQ}^+$

Lemma 14. $\mathbb{IQ}^*$ is interpretable in $\mathbb{IQ}^+$.

Proof. By Lemma [13], it suffices to show that $\mathbb{IQ}^*$ is interpretable in $\mathbb{IQ}^{++}$. We interpret $\mathbb{IQ}^*$ in $\mathbb{IQ}^{++}$ by simply restricting the universe of $\mathbb{IQ}^{++}$ to an inductive subclass $K$ that is closed under $+, \times$ and which is such that $\mathbb{IQ}^{++}$ proves that $\forall x, u \in K \ [ u \leq x + u ]$.

Let

$$K_1 = \{ u : \forall x \ [ u \leq x + u ] \}$$

We have $0 \in K_1$ by the axiom $\forall x \ [ 0 \leq x ]$ and $Q_4$. We show that $K_1$ is closed under $S$. Let $u \in K_1$. We need to show that $Su \in K_1$. That is, we need to show that $Su \leq x + Su$. Since $u \in K_1$, we have $u \leq x + u$. Then, $Su \leq S(x + u)$ by the axiom $\forall xy \ [ x \leq y \rightarrow Sx \leq Sy ]$. By $Q_5$, we have

$$Su \leq S(x + u) = x + Su .$$

Hence, $Su \in K_1$. Thus, $K_1$ contains $0$ and is closed under $S$. By Lemma [12], there exists an inductive subclass $K$ of $K_1$ that is closed under $+$ and $\times$.

We interpret $\mathbb{IQ}^*$ in $\mathbb{IQ}^{++}$ by relativizing quantification to $K$. The translation of each one of the axioms $Q_1 - Q_2$, $Q_4 - Q_7$ is a theorem of $\mathbb{IQ}^{++}$ since universal sentences are absolute for $K$. It remains to show that each instance of $\mathbb{IQ}_3^*$ is a theorem of $\mathbb{IQ}^{++}$. Choose a natural number $n$. We need to show that

$$\mathbb{IQ}^{++} \vdash \forall x, y \in K \ [ x + y = \overline{\pi} \rightarrow \bigvee_{k \leq n} y = \overline{k} ] .$$

Assume $x, y \in K$ and $x + y = \overline{\pi}$. Since $y \in K \subseteq K_1$, we have $y \leq \overline{\pi}$. By the axiom schema $\mathbb{IQ}_3$, there exists $k \leq n$ such that $y = \overline{k}$. Thus, $\mathbb{IQ}^{++}$ proves the translation of each instance of $\mathbb{IQ}_3^*$.

5 Interpreting $\mathbb{ID}$ in $\mathbb{IQ}$

In this section, we show that $\mathbb{ID}$ is interpretable in $\mathbb{IQ}$ (see Figure 1.5 for the axioms of $\mathbb{ID}$ and $\mathbb{IQ}$). The most intuitive way to interpret concatenation theories in arithmetical theories is to construct a formula $\phi_\circ(x, y, z)$ that given $x$ and $y$ defines an object that encodes a computation of $x \circ y$. Unfortunately, $\mathbb{IQ}$ does not have the resources necessary to prove that we can find a domain $I$ on which $\phi_\circ(x, y, z)$ defines a function that satisfies $\mathbb{ID}_1$, $\mathbb{ID}_2$, $\mathbb{ID}_3$, $\mathbb{ID}_5$, $\mathbb{ID}_6$. To prove correctness of recursive definition in Robinson Arithmetic $\mathbb{Q}$, we rely on the axioms $Q_3 \equiv \forall x \ [ x = 0 \lor \exists y \ [ x = Sy ] ]$. The axiom schema $\mathbb{IQ}_3 \equiv \forall x \ [ x \leq \overline{\pi} \leftrightarrow \bigvee_{k \leq n} x = \overline{k} ]$ can only allow us to verify that $\phi_\circ(x, y, z)$
The Axioms of ID

ID₁ \( \forall xyz \ [(xy)z = x(yz)]\)
ID₂ \( \forall xy \ [x \neq y \rightarrow (x0 \neq y0 \land x1 \neq y1)]\)
ID₃ \( \forall xy \ [x0 \neq y1]\)
ID₄ \( \forall x \ [x \leq \pi \leftrightarrow \bigvee_{x \in \text{Pref} \alpha} x = \pi]\)
ID₅ \( \forall xy \ [x \neq y \rightarrow (0x \neq 0y \land 1x \neq 1y)]\)
ID₆ \( \forall xy \ [0x \neq 1y]\)

The Axioms of IQ

Q₁ \( \forall xy \ [x \neq y \rightarrow Sx \neq Sy]\)
Q₂ \( \forall x \ [Sx \neq 0]\)
Q₄ \( \forall x \ [x + 0 = x]\)
Q₅ \( \forall xy \ [x + Sy = S(x + y)]\)
Q₆ \( \forall x \ [x \times 0 = 0]\)
Q₇ \( \forall xy \ [x \times Sy = x \times y + x]\)
IQ₃ ∀x [ x \leq \pi \leftrightarrow \bigvee_{k \leq n} x = k ]

Figure 1.5: Non-logical axioms of the first-order theories ID and IQ.

gives a correct value \( z \) when \( x \) and \( y \) represent variable-free terms. Thus, to interpret ID in IQ, we need a conception of strings as numbers that allows us to translate concatenation without coding sequences. The translation needs to also be simple enough that we can prove its correctness in IQ. In Lemma 4 of [2], Ganea explains how we can translate concatenation as a \( \Delta_0 \)-formula in strong theories such as Peano Arithmetic PA and I\( \Delta_0 \).

Although we show that ID is interpretable in IQ, we have not been able to determine whether the converse holds.

Open Problem 15. Is IQ interpretable in ID?

As mentioned, the main result of this section is the following theorem.

Theorem 16. ID is interpretable in IQ.

The proof of the theorem is structured as follows: In Section 5.1, we explain how we intend to interpret ID in IQ. In Section 5.2, we use this idea to give a simple interpretation of WD in R. In Section 5.3, we show that we can interpret ID in an extension of IQ which we denote IQ(2). Finally, in Section 7, we show that IQ and IQ(2) are mutually interpretable.

5.1 Strings as Matrices

The idea is to think of strings as \( 2 \times 2 \) matrices and to translate concatenation as matrix multiplication. Let us first see how we can use this idea to give a 4-dimensional interpretation of \((\{0, 1\}^{\ast}, \varepsilon, 0, 1, \preceq)\) in \((N, 0, 1, +, \times)\), where \( \varepsilon \)
denotes the empty string and \( \{0,1\}^* = \{0,1\}^+ \cup \{\varepsilon\} \). Let

\[
\varepsilon^\tau := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad 0^\tau := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad 1^\tau := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
\]

Let \( \text{SL}_2(\mathbb{N}) \) denote the monoid generated by \( 0^\tau \) and \( 1^\tau \) under matrix multiplication. The monoid \( \text{SL}_2(\mathbb{N}) \) is a substructure of the special linear group \( \text{SL}_2(\mathbb{Z}) \) of \( 2 \times 2 \) matrices with integer coefficients and determinant 1. Let \( \times \) denote matrix multiplication. Then, \( \{0,1\}^*, \varepsilon, 0, 1, \sim \) is isomorphic to \( (\text{SL}_2(\mathbb{N}), \varepsilon^\tau, 0^\tau, 1^\tau, \times) \). Since \( \text{SL}_2(\mathbb{N}) \) is the set of \( 2 \times 2 \) matrices with natural number coefficients and determinant 1, the isomorphism defines a 4-dimensional interpretation of \( (\{0,1\}^*, \varepsilon, 0, 1, \sim) \) in \( (\mathbb{N}, 0, 1, +, \times) \). The idea is to specify an interpretation of \( \mathcal{T}_0^D \) in \( \mathcal{I}_Q \) by building on this interpretation of \( (\{0,1\}^*, \varepsilon, 0, 1, \sim) \) in \( (\mathbb{N}, 0, 1, +, \times) \). But we need to be careful since the axioms \( \mathcal{I}_Q_1 - \mathcal{I}_Q_2, \mathcal{I}_Q_4 - \mathcal{I}_Q_7 \) have many models.

In Lemma 11 of [10], we use this idea of associating strings with matrices to prove that \( \mathcal{I}_D_1 - \mathcal{I}_D_3 \) has a decidable model. We prove this result by giving a 4-dimensional interpretation of \( \mathcal{I}_D_1 - \mathcal{I}_D_3 \) in the first-order theory of the real closed field \( (\mathbb{R}, 0, 1, +, \times, \leq) \), which is decidable (see Tarski [11]). At the time, we were investigating whether it is possible to remove some of the axioms of \( \mathcal{D} \) and obtain a theory that is essentially undecidable. The possibility of interpreting \( \mathcal{T}_0^D \) in \( \mathcal{I}_Q \) resulted from a careful investigation of the algebraic properties of \( (\mathbb{R}, 0, 1, +, \times, \leq) \) which we need to interpret \( \mathcal{I}_D_1 - \mathcal{I}_D_3 \). Properties (I)-(VIII) in Figure 1.6 are sufficient to interpret \( \mathcal{T}_0^D \) in \( \mathcal{I}_Q \). Extending \( \mathcal{I}_Q \) with (I)-(VIII) allows us to reason about natural numbers in the standard way. In the rest of the paper, we use the Roman numerals (I)-(VIII) to refer exclusively to axioms (I)-(VIII) in Figure 1.6.

The 4-dimensional interpretation of \( (\{0,1\}^*, \varepsilon, 0, 1, \sim) \) in \( (\mathbb{N}, 0, 1, +, \times) \) we described is a many-to-one reduction that maps existential sentences to existential sentences. This means that unsolvability of equations over \( (\{0,1\}^*, \varepsilon, 0, 1, \sim) \) implies unsolvability of equations over \( (\mathbb{N}, 0, 1, +, \times) \). The idea of associating \( (\{0,1\}^*, \varepsilon, 0, 1, \sim) \) with \( \text{SL}_2(\mathbb{N}) \) dates back to A.A. Markov [9]. According to Lothaire [7] (see p. 387), in the 1950s, A. A. Markov hoped that Hilbert’s 10th Problem could be solved by proving unsolvability of word equation, that is, equations over finitely generated free semigroups. In 1970, Yuri Matiyasevich proved that Hilbert’s 10th Problem is undecidable using a completely different method (see for example Davis [1]). In 1977, Makanin [8] proved that the existential theory of a finitely generated free semigroup is decidable.

### 5.2 Interpretation of WD in R

In this section, we show that the isomorphism between \( (\{0,1\}^*, \varepsilon, 0, 1, \sim) \) and \( \text{SL}_2(\mathbb{N}) \) defines a very simple interpretation of \( \mathcal{T}_0^D \) in \( \mathcal{I}_Q \).

**Lemma 17.** Let \( \tau \) be the 4-dimensional translation of \( \{0,1,\circ\} \) in \( \{0,S,+,\times\} \) defined as follows

- 0 and \( \circ \) are translated as \( \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \), respectively
- ○ is translated as matrix multiplication

- the domain \( J \) is the set of all \( 2 \times 2 \) matrices \( \begin{pmatrix} x & y \\ z & w \end{pmatrix} \) where \( x \neq 0 \).

Then, \( \tau \) extends to a translation of \( \{0, 1, \circ, \leq\} \) in \( \{0, S, +, \times, \leq\} \) that defines a 4-dimensional interpretation of \( \text{WD} \) in \( \mathbb{R} \).

Proof. By the axiom schemas \( \text{R}_1 \equiv \pi + m = n + m, \text{R}_2 \equiv \pi \times m = n \times m \), \( \mathbb{R} \) proves the translation of each instance of \( \text{WD}_1 \equiv \pi \beta = \alpha \beta \). By the axiom schema \( \text{R}_3 \), \( \mathbb{R} \) proves the translation of each instance of \( \text{WD}_2 \). It remains to give a translation of \( \leq \) that provably satisfies the axiom schema \( \text{WD}_3 \equiv \forall x \left[ x \leq \alpha \leftrightarrow \bigvee_{\gamma \in \text{Pref}(\alpha)} x = \gamma \right] \).

This is where we use the axiom schema \( \text{IQ}_3 \equiv \forall x \left[ x \leq \pi \leftrightarrow \bigvee_{k \leq n} x = k \right] \), which is a theorem of \( \mathbb{R} \).

Let

\[
K = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} : \begin{pmatrix} x & y \\ z & w \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \land xw = 1 + yz \right\}.
\]

Let

\[
A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}.
\]

Let \( A \preceq B \) if and only if \( A, B \in K \) and there exists a largest element \( m(B) \in \{b_1, b_2, b_3, b_4\} \) with respect to \( \leq \) such that

1. \( A = B \) or
2. there exists \( C \in K \) such that \( a_i, c_i \leq m(B) \) for all \( 1 \leq i \leq 4 \) and \( AC = B \).

Let \( \text{SL}_2(\mathbb{N})^+ \) denote \( \text{SL}_2(\mathbb{N}) \) minus the identity matrix. Assume \( B \) is the translation of a variable-free \( \text{L}_{\text{BT}} \)-term. Then, \( B \in \text{SL}_2(\mathbb{N})^+ \). The bound in (2) tells that \( A, C \in \text{SL}_2(\mathbb{N})^+ \). It is straightforward to verify that if \( A, B, C \in \text{SL}_2(\mathbb{N})^+ \) are such that \( AC = B \), then a bound such as the one in (2) holds. It is then clear that (1)-(2) capture what it means for a finite string to be a prefix of another string. Thus, \( \mathbb{R} \) proves the translation of each instance of \( \text{WD}_3 \).

5.3 Interpretation of \( \overline{\text{ID}} \) in \( \text{IQ}^{(2)} \)

Let \( \text{IQ}^{(2)} \) be \( \text{IQ} \) extended with axioms (I)-(VIII) in Figure 1.6. We can reason in \( \text{IQ}^{(2)} \) about natural numbers in the standard way and will therefore occasionally not refer explicitly to the axioms of \( \text{IQ}^{(2)} \) we use. In this section, we show that \( \overline{\text{ID}} \) is interpretable in \( \text{IQ}^{(2)} \).

We start by making a few simple observations:

- Axiom (IV) tells us that addition is commutative. Hence, by \( \text{Q}_4 \), \( 0 \) is an additive identity. That is, \( \text{IQ}^{(2)} \vdash \forall x \left[ 0 + x = x \land x + 0 = x \right] \).
5. **INTERPRETABILITY OF \( \mathbb{T} \) IN \( \mathbb{IQ} \)**

(I) Left distributivity \( \forall xyz \ [ x(y + z) = xy + xz ] \)

(II) Associativity of + \( \forall xyz \ [ (x + y) + z = x + (y + z) ] \)

(III) Associativity of \( \times \) \( \forall xyz \ [ (xy)z = x(yz) ] \)

(IV) Commutativity of + \( \forall xy \ [ x + y = y + x ] \)

(V) Commutativity of \( \times \) \( \forall xy \ [ xy = yx ] \)

(VI) Right cancellation \( \forall xyz \ [ x + z = y + z \rightarrow x = y ] \)

(VII) Nonnegative Elements \( \forall xy \ [ x + y = 0 \rightarrow (x = 0 \land y = 0) ] \)

(VIII) No Zero Divisors \( \forall xy \ [ xy = 0 \rightarrow (x = 0 \lor y = 0) ] \).

Figure 1.6: Algebraic properties we need in order to interpret \( \mathbb{T} \) in \( \mathbb{IQ} \).

- Recall that 1 = S0. By \( Q_7 \) and \( Q_6 \)
  \[ x1 = x0 + x = 0 + x = x . \]
  Since axiom (V) tells us that multiplication is commutative, 1 is a multiplicative identity. That is, \( \mathbb{IQ}^{(2)} \vdash \forall x \ [ 1x = x \land x1 = x ] \).

- Axiom (VI) tells us that addition is right-cancellative. Since addition is commutative, it is also left-cancellative. That is
  \( \mathbb{IQ}^{(2)} \vdash \forall xyz \ [ z + x = z + y \rightarrow x = y ] \).

- By \( Q_6 \) and (V), \( \mathbb{IQ}^{(2)} \vdash \forall x \ [ x0 = 0 \land 0x = 0 ] \).

**Lemma 18.** Let \( \tau \) be the 4-dimensional translation of \( \{0, 1, \circ\} \) in \( \{0, S, +, \times\} \) defined as follows

- 0 and 1 are translated as \( \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} , \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \), respectively
- \( \circ \) is translated as matrix multiplication
- the domain \( J \) is the set of all \( 2 \times 2 \) matrices \( \begin{pmatrix} x & y \\ z & w \end{pmatrix} \) where \( x \neq 0 \).

Then, \( \tau \) extends to a translation of \( \{0, 1, \circ, \leq\} \) in \( \{0, S, +, \times, \leq\} \) that defines a 4-dimensional interpretation of \( \mathbb{T} \) in \( \mathbb{IQ}^{(2)} \).

**Proof.** We verify that \( J \) satisfies the domain condition. It is clear that 0\( ^\tau \), 1\( ^\tau \) \( \in \) \( J \). It remains to verify that \( J \) is closed under matrix multiplication. Let

\[ A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} , \quad B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} , \quad AB = \begin{pmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{pmatrix} \]
where \( a_1, b_1 \neq 0 \). We need to show that \( a_1b_1 + a_2b_3 \neq 0 \). Axiom (VIII) tells us that models of IQ\(^{(2)}\) do not have zero divisors. Hence, \( a_1b_1 \neq 0 \). Axiom (VII) tells us that 0 is the only element with an additive inverse. Hence, \( a_1b_1 + a_2b_3 \neq 0 \), which implies \( AB \in J \). Thus, \( J \) is closed under matrix multiplication.

It is straightforward to verify that (I)-(V) suffice to prove that matrix multiplication is associative. Thus, IQ\(^{(2)}\) proves the translation of ID\(_1\).

Next, we show that the translation of ID\(_2\) and ID\(_5\) are theorems of IQ\(^{(2)}\). We need to show that

\[
(1) \forall A, B \in J \ [ \ ( A0^\tau = B0^\tau \lor 0^\tau A = 0^\tau B ) \to A = B ]
\]

\[
(2) \forall A, B \in J \ [ \ ( A1^\tau = B1^\tau \lor 1^\tau A = 1^\tau B ) \to A = B ].
\]

We verify (1). First, we show that \( \forall A, B \in J \ [ \ A0^\tau = B0^\tau \to A = B ] \). Assume \( x, a \neq 0 \) and

\[
\begin{pmatrix} x+y & y \\ z+w & w \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} 0^\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix} 0^\tau = \begin{pmatrix} a+b & b \\ c+d & d \end{pmatrix}.
\]

We need to show that \( x = a \) and \( z = c \). We have

\[
x + b = x + y = a + b \land z + d = z + w = c + d.
\]

Since addition is right-cancellative, \( x = a \) and \( z = c \). Thus, for all \( A, B \in J \), if \( A0^\tau = B0^\tau \), then \( A = B \).

We show that \( \forall A, B \in J \ [ \ 0^\tau A = 0^\tau B \to A = B ] \). Assume \( x, a \neq 0 \) and

\[
\begin{pmatrix} x & y \\ x+z & y+w \end{pmatrix} = 0^\tau \begin{pmatrix} x & y \\ z & w \end{pmatrix} = 0^\tau \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ a+c & b+d \end{pmatrix}.
\]

We need to show that \( z = c \) and \( w = d \). We have

\[
a + z = x + z = a + c \land b + w = y + w = b + d.
\]

Since addition is left-cancellative, \( z = c \) and \( w = d \). Thus, for all \( A, B \in J \), if \( 0^\tau A = 0^\tau B \), then \( A = B \). Hence, (1) holds. By similar reasoning, (2) holds. Thus, IQ\(^{(2)}\) proves the translation of ID\(_2\) and ID\(_5\).

We show that the translation of ID\(_3\) is a theorem of IQ\(^{(2)}\). We need to show that \( \forall A, B \in J \ [ \ A0^\tau \neq B1^\tau ] \). Assume for the sake of a contradiction \( x, a \neq 0 \) and

\[
\begin{pmatrix} x+y & y \\ z+w & w \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} 0^\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix} 1^\tau = \begin{pmatrix} a & a+b \\ c & c+d \end{pmatrix}.
\]

Then

\[
a = x + y = x + a + b
\]

where we have omitted parentheses since addition is associative. Since 0 is an additive identity and addition is commutative, \( 0 + a = x + b + a \). Since addition is right-cancellative, \( 0 = x + b \). Since 0 is the only element with an
The Axioms of $TC^\varepsilon$

1. $\forall x \ [ \varepsilon x = x \land x \varepsilon = x ]$
2. $\forall xyz \ [ x(yz) = (xy)z ]$
3. $\forall xyzw \ [ xy = zw \rightarrow \exists u \ [ ( z = xu \land uw = y ) \lor ( x = zu \land uy = w ) ] ]$
4. $0 \neq \varepsilon$
5. $\forall xy \ [ xy = 0 \rightarrow ( x = \varepsilon \lor y = \varepsilon ) ]$
6. $1 \neq \varepsilon$
7. $\forall xy \ [ xy = 1 \rightarrow ( x = \varepsilon \lor y = \varepsilon ) ]$
8. $0 \neq 1$

Figure 1.7: Non-logical axioms of the first-order theory $TC^\varepsilon$.

additive inverse, $x = 0$, which contradicts the assumption that $x \neq 0$. Thus, $IQ^{(2)}$ proves the translation of $ID_3$.

We show that the translation of $ID_6$ is a theorem of $IQ^{(2)}$. We need to show that $\forall A,B \in J \ [ 0^\tau A \neq 1^\tau B ]$. Assume for the sake of a contradiction $x,a \neq 0$ and

$$
\begin{pmatrix}
  x \\
  x+z \\
  y+w
\end{pmatrix}
\begin{pmatrix}
  y \\
  y+w
\end{pmatrix}
= 0^\tau
\begin{pmatrix}
  x \\
  z \\
  w
\end{pmatrix}
= 1^\tau
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}
= \begin{pmatrix}
  a+c & b+d \\
  c & d
\end{pmatrix}
.
$$

Then, $x = a + c = a + x + z$. Hence, $0 = a + z$. Since $0$ is the only element with an additive inverse, $a = 0$, which contradicts the assumption that $a \neq 0$. Thus, $IQ^{(2)}$ proves the translation of $ID_6$.

Finally, we translate $\preceq$ as in the proof of Lemma 17. □

6 Interpretation of $TC$ in $Q$

In this section, we show that our interpretation of $ID$ in $IQ$ extends in a natural way to an interpretation of $TC$ in $Q$. Instead of interpreting $TC$, we interpret the variant $TC^\varepsilon$ where we extend the language of $TC$ with a constant symbol $\varepsilon$ for the identity element. See Figure 1.7 for the axioms of $TC^\varepsilon$. We choose to work with $TC^\varepsilon$ because the identity matrix is naturally present in our interpretation of $ID$ in $IQ$ and because we get a more compact form of the editor axiom ($TC_2$ and $TC_3$). The interpretation we give can be turned into an interpretation of $TC$ by simply removing the identity matrix from the domain (see Appendix A of Visser [13] for mutual interpretability of $TC$ and $TC^\varepsilon$).

Recall that $x \preceq y \equiv \exists r \ [ r + x = y ]$. Let $x \prec y \equiv \exists r \ [ r \neq 0 \land r + x = y ]$.

Let $Q^{(2)}$ be $Q$ extended with axioms (I)-(VI) in Figure 1.6 and the trichotomy law

$$
\forall xy \ [ x \prec y \lor x = y \lor y \prec x ]
.$$

We make a few simple observations:

- Axiom (VII) $\forall xy \ [ x + y = 0 \rightarrow ( x = 0 \land y = 0 ) ]$ is a theorem of $Q^{(2)}$.

Indeed, assume $x + y = 0$. If $y = 0$, then $x = 0$ by $Q_4$. Thus, it suffices
to show that \( y = 0 \). Assume for the sake of contradiction that \( y \neq 0 \). Then, by \( Q_3 \), there exists \( v \) such that \( y = Sv \). By \( Q_5 \)
\[
0 = x + y = x + Sv = S(x + v)
\]
which contradicts \( Q_2 \). Thus, \( x + y = 0 \) implies \( x = y = 0 \).

- Axiom (VIII) \( \forall xy \ [ xy = 0 \to ( x = 0 \ \vee \ y = 0 ) ] \) is a theorem of \( Q^{(2)} \). Indeed, assume \( xy = 0 \) and \( y \neq 0 \). By \( Q_3 \), there exists \( v \) such that \( y = Sv \).

\[
0 = xy = xv + x
\]
which implies \( x = 0 \). Thus, \( xy = 0 \) implies \( x = 0 \ \vee \ y = 0 \).

- \( Q^{(2)} \) proves that 1 is the only element with a multiplicative inverse. Indeed, assume \( xy = 1 \). By commutativity of multiplication, \( Q_2 \) and \( Q_5 \), we have \( x, y \neq 0 \). Hence, by \( Q_3 \), there exist \( u, v \) such that \( x = Su \) and \( y = Sv \). By commutativity of multiplication, \( Q_7 \) and \( Q_5 \)
\[
1 = xy = xv + x = S(xv + u) \land 1 = yx = yu + y = S(yu + v) .
\]
By \( Q_1 \)
\[
0 = xv + u \land 0 = yu + v
\]
which implies \( u = v = 0 \). Hence, \( x = y = 1 \). Thus, \( xy = 1 \) implies \( x = y = 1 \).

This section as structured as follows: In Section 6.1, we show that if we modify our interpretation of \( \mathbb{T} \) in \( IQ^{(2)} \) by choosing as the domain the class \( K \) of all \( 2 \times 2 \) matrices with determinant one, we obtain an interpretation in \( Q^{(2)} \) of the theory we obtain from \( TC^\varepsilon \) by replacing the editor axiom \( TC^\varepsilon_2 \) with the axioms \( D_2, D_3, TD_5, TD_6, \forall x \ [ x = \varepsilon \ \vee \ \exists y \ [ x = y\varepsilon \ \vee \ x = y\varepsilon_1 ] \). In Section 6.2, we extend our interpretation of \( \mathbb{T} \) in \( IQ^{(2)} \) to an interpretation of \( TC^\varepsilon \) in \( Q^{(2)} \) by restricting \( K \) to a subclass on which the editor axiom holds. Finally, in Section 8, we show that we can interpret \( Q^{(2)} \) in \( Q \) by restricting the universe of \( Q \) to a suitable subclass.

### 6.1 Atoms and Predecessors

Let \( K \) denote the class all \( 2 \times 2 \) matrices with determinant 1. That is
\[
K = \{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} : xw = 1 + yz \} .
\]

It is not difficult to verify that \( Q^{(2)} \) proves that \( \det(AB) = 1 \) if \( \det(A) = \det(B) = 1 \). We thus have the following lemma.

**Lemma 19.** \( Q^{(2)} \) proves that \( K \) is closed under \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) and matrix multiplication.

Let us say that \( A \in K \) is an atom in \( K \) if for all \( B, C \in K \), \( A = BC \) implies that one of \( B \) and \( C \) is the identity matrix. The proof of the following lemma is straightforward.
Lemma 20. \( Q^{(2)} \) proves that \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \) and \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) are atoms in \( K \).

In [10], we introduce a theory BTQ and show that \( D \) interprets \( Q \) by showing that it interprets BTQ. We obtain BTQ from ID by replacing \( ID_4 \) with the axiom \( \forall x (x = 0 \lor x = 1 \lor \exists y (x = y0 \lor x = y1)) \). The next lemma shows that if we modify the translation in Lemma 18 by choosing as the domain the class of all elements in \( K \) distinct from the identity matrix, we obtain an interpretation of BTQ in \( Q^{(2)} \).

Lemma 21. Let \( A \in K \). Then, \( Q^{(2)} \) proves that \( A \) is the identity matrix or that there exist \( B, C \in K \) such that \( A = BC \) and \( C \) is one of \( \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \).

Proof. Let \( K \ni A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). By the trichotomy law, we have the following cases

- (1a) \( a = b \land c <_1 d \), (1b) \( a = b \land c = d \), (1c) \( a = b \land d <_1 c \)
- (2a) \( a <_1 b \land c = d \), (2b) \( b <_1 a \land c = d \)
- (3a) \( b <_1 a \land c <_1 d \), (3b) \( a <_1 b \land d <_1 c \), (3c) \( b <_1 a \land d <_1 c \), (3d) \( a <_1 b \land c <_1 d \).

We consider Case (1a). Since \( a = b \land c <_1 d \), let \( d = r + c \) where \( r \neq 0 \). Since \( ad = 1 + bc \) as \( A \in K \), we have

\[ ar + ac = a(r + c) = ad = 1 + bc = 1 + ac \]

Since addition is right-cancellative, \( ar = 1 \), which implies \( a = r = 1 \). Thus

\[ A = \begin{pmatrix} 1 & 1 \\ c & 1 + c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} . \]

We consider Case (1b). Since \( a = b \land c = d \) and \( A \in K \), we have

\[ ad = 1 + bc = 1 + ad . \]

Since 0 is an additive identity and addition is right-cancellative, \( 0 = 1 \) which contradicts \( Q_2 \).

We consider Case (1c). Since \( a = b \land d <_1 c \), let \( c = s + d \) where \( s \neq 0 \). We have

\[ ad = 1 + bc = 1 + a(s + d) = 1 + as + ad . \]

Hence, \( 0 = 1 + as \). Since addition is commutative, \( 0 = S(as + 0) \) by \( Q_5 \), which contradicts \( Q_2 \).

We consider Case (2a). Since \( a <_1 b \land c = d \), let \( b = r + a \) where \( r \neq 0 \). We have

\[ ad = 1 + bc = 1 + (r + a)d = 1 + rd + ad . \]
Hence, \(0 = 1 + rd\) which contradicts \(Q_2\).

We consider Case (2b). Since \(b <_1 a \land c = d\), let \(a = s + b\) where \(s \neq 0\). We have

\[
sd + bd = (s + b)d = ad = 1 + bc = 1 + bd.
\]

Hence, \(sd = 1\) which implies \(s = d = 1\). Thus

\[
A = \begin{pmatrix} 1 + b & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.
\]

We consider Case (3a). Since \(b <_1 a \land c <_1 d\), there exist \(r, s \neq 0\) such that \(a = r + b\) and \(d = s + c\). Since \(ad = 1 + bc\), we have

\[
rs + rc + bs + bc = (r + b)(s + c) = ad = 1 + bc
\]

which implies

\[
rs + rc + bs = 1.
\]

Since \(r, s \neq 0\), we conclude that \(r = s = 1\) and \(b = c = 0\). Thus, \(A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\).

We consider Case (3b). Since \(a <_1 b \land d <_1 c\), there exist \(p, q \neq 0\) such that \(b = p + a\) and \(c = q + d\). Since \(ad = 1 + bc\), we have

\[
ad = 1 + bc = 1 + (p + a)(q + d) = 1 + pq + pd + aq + ad.
\]

Hence, \(0 = 1 + pq + pd + aq\) which contradicts \(Q_2\).

We consider Case (3c). Since \(b <_1 a \land d <_1 c\),

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = E \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{where} \quad E = \begin{pmatrix} a - b & b \\ c - d & d \end{pmatrix}.
\]

Since addition is right-cancellative in \(Q^{(2)}\), we write \(a - b\) and \(c - d\) for the unique elements \(r, s\) such that \(a = r + b\) and \(d = s + d\). We need to show that \(E \in K\). That is, we need to show that \(\det(E) = 1\). First, observe that

\[
(x - y)z = xy - yz \quad \text{and} \quad (1 + xz) - yz = 1 + (xz - yz)
\]

since

\[
(x - y)z + yz = ((x - y) + y)z = xz \quad \text{and} \quad 1 + (xz - yz) + yz = 1 + xz.
\]

Since \(\det(A) = 1\), we have

\[
(a - b)d = ad - bd = (1 + bc) - bd = 1 + b(c - d)\quad \text{.}
\]

Thus, \(E \in K\).

We consider Case (3d). Since \(a <_1 b \land c <_1 d\)

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = G \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{where} \quad G = \begin{pmatrix} a & b - a \\ c & d - c \end{pmatrix}
\]

We need to show that \(G \in K\). That is, we need to show that \(\det(G) = 1\). Since \(\det(A) = 1\), we have

\[
a(d - c) = ad - ac = 1 + bc - ac = 1 + (b - a)c\quad \text{.}
\]

Thus, \(G \in K\).
6. INTERPRETATION OF TC IN Q

6.2 Interpretation of TCε

We are finally ready to extend our interpretation of TRD in IQ(2) to an interpretation of TCε in Q(2). All we need to do is to restrict the class K to a subclass on which the editor axiom holds.

**Theorem 22.** There exists a class I such that the 4-dimensional translation of \( \{ \varepsilon, 0, 1, \circ \} \) in \( \{ 0, 1, \text{S}, +, \times \} \) defined by

- \( \varepsilon \) and 1 are translated as \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \), respectively
- \( \circ \) is translated as matrix multiplication
- the domain is I

defines a 4-dimensional interpretation of TCε in Q(2).

**Proof.** Let

\[ K = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} : xw = 1 + yz \right\}. \]

Lemma 18 and Lemma 20 tell us that the restriction of axioms TCε1 – TCε2, TCε3 – TCε4 to K are theorems of Q(2). Since TCε1 – TCε2, TCε3 – TCε4 are universal sentences, to interpret TCε in Q(2), it suffices to restrict the class K to a subclass I on which the editor axiom TCε2 holds. We need the following three properties that are given by Lemma 18 and Lemma 21:

**DJ** \( \forall A, B \in K \left[ A \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \neq B \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right] \)

**RC** \( \forall A, B \in K \left[ A \neq B \rightarrow \left( A \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \neq B \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \wedge A \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \neq B \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right] \)

**PD** \( \forall A \in K \left[ A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vee \exists B \in K \left[ A = B \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \vee A = B \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right] \right] \)

Let

\[ H = \{ W \in K : \forall XZ \forall Y \in K \left[ XY = ZW \rightarrow \exists U \in K \left[ \begin{array}{c} Z = XU \wedge UW = Y \vee (X = ZU \wedge UY = W) \end{array} \right] \right] \}. \]

It follows from DJ, RC, PD and associativity of matrix multiplication that \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) are elements of H. We show that H is closed under matrix multiplication. So, assume \( W_0, W_1 \in H \). We need to show that \( W_0W_1 \in H \). First, we observe that \( W_0W_1 \in K \) since K is closed under matrix
multiplication and \( H \subseteq K \). Now, let \( X,Y,Z \) be such that \( XY = ZW_0W_1 \) and \( Y \in K \). Since \( W_1 \in H \), we have the following two cases for some \( U_1 \in K \)

\[
(1) \quad X = ZW_0U_1 \land U_1Y = W_1 , \\
(2) \quad ZW_0 = XU_1 \land U_1W_1 = Y .
\]

We consider (1). Since \( K \) is closed under matrix multiplication and \( H \subseteq K \), we have

\[
X = ZW_0U_1 \land W_0U_1Y = W_0W_1 \land W_0U_1 \in K .
\]

We consider (2). Since \( W_0 \in H \) and \( U_1 \in K \), we have one of the following two cases for some \( U_0 \in K \)

\[
(2a) \quad Z = XU_0 \land U_0W_0 = U_1 \land U_1W_1 = Y ; \\
(2b) \quad X = ZU_0 \land U_0U_1 = W_0 \land U_1W_1 = Y .
\]

In case of (2a), we have

\[
Z = XU_0 \land U_0W_0W_1 = U_1W_1 = Y \land U_0 \in K .
\]

In case of (2b), we have

\[
X = ZU_0 \land W_0W_1 = U_0U_1W_1 = U_0W_1 \land U_0 \in K .
\]

By (\( \ast \)), (\( \ast \ast \)) and (\( \ast \ast \ast \)), we have \( W_0W_1 \in H \). Thus, \( H \) is closed under matrix multiplication.

We are finally ready to specify the class \( I \). Let

\[
I = \{ A \in H : \forall B [ B \preceq_K A \rightarrow B \in H ] \}
\]

where

\[
B \preceq_K A \equiv \exists C \in K [ A = BC ] .
\]

It follows from Lemma [20] that \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} , \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) are elements of \( I \).

To show that \( I \) defines a model of \( TC^c \), it suffices to show that \( I \) is closed under matrix multiplication and downward closed under \( \preceq_K \), where the latter ensures that the editor axiom holds restricted to \( I \).

We show that \( I \) is closed under matrix multiplication. Assume \( A_0,A_1 \in I \). We need to show that \( A_0A_1 \in I \). So, assume \( BC = A_0A_1 \) where \( C \in K \). We need to show that \( B \in H \). Since \( A_1 \in I \subseteq H \) and \( C \in K \), we have one of the following cases for some \( U \in K \)

\[
(i) \quad A_0 = BU \land UA_1 = C , \\
(ii) \quad B = A_0U \land UC = A_1 .
\]

In case of (i) we have \( B \preceq_K A_0 \) which implies \( B \in H \) since \( A_0 \in I \). In case of (ii) we have \( U \preceq_K A_1 \) which implies \( U \in H \) since \( A_1 \in I \). Since \( H \) is closed under matrix multiplication and \( A_0 \in I \subseteq H \), we have \( B = A_0U \in H \). Hence, \( A_0A_1 \in I \). Thus, \( I \) is closed under matrix multiplication.

We show that \( I \) is downward closed under \( \preceq_K \). So, assume \( B \preceq_K A \) where \( A \in I \). We need to show that \( B \in I \). That is, we need to show that \( B \in H \) and \( \forall D \preceq_K B \left[ D \in H \right] \). Since \( A \in I \) and \( B \preceq_K A \), it follows from the definition of \( I \) that \( B \in H \). Assume now \( B = DC \) where \( C \in K \). We need to show that \( D \in H \). Since \( B \preceq_K A \), there exists \( E \in K \) such that \( A = BE \). Hence, \( DCE = BE = A \). Since \( C,E \in K \) and \( K \) is closed under matrix multiplication, \( CE \in K \). Hence, \( D \preceq_K A \). Thus, \( I \) is downward closed under \( \preceq_K \).
7. COMMUTATIVE SEMIRINGS I

We complete our proof of interpretability of $\mathcal{ID}$ in $\mathcal{IQ}$ by showing that $\mathcal{IQ}$ and $\mathcal{IQ}^{(2)}$ are mutually interpretable.

**Theorem 23.** $\mathcal{IQ}$ and $\mathcal{IQ}^{(2)}$ are mutually interpretable.

**Proof.** Since $\mathcal{IQ}^{(2)}$ is an extension of $\mathcal{IQ}$, we only need to show that $\mathcal{IQ}^{(2)}$ is interpretable in $\mathcal{IQ}$. Our strategy is to first restrict the universe of $\mathcal{IQ}$ to an inductive class $N_2$ which is such that each of the axioms (I)-(VIII) in Figure 1.6 holds on $N_2$ when we restrict quantification to $N_2$ and treat addition and multiplication as partial functions. Recall that a class $X$ is inductive if $0 \in X$ and $\forall x \in X [ Sx \in X ]$. Now, since the axioms of $\mathcal{IQ}^{(2)}$ are all universal sentences, to interpret $\mathcal{IQ}^{(2)}$ in $\mathcal{IQ}$, it suffices to relativize quantification to a subclass $N$ of $N_2$ that is closed under $0, S, +, \times$.

We start by restricting the universe of $\mathcal{IQ}$ to a subclass $N_0$ where 0 is the only element with an additive inverse, and addition is associative and right-cancellative. Let $u \in N_0$ if and only if

1. $0 + u = u$
2. $\forall x [ x + u = 0 \rightarrow ( x = 0 \land u = 0 ) ]$
3. $\forall x [ Sx + u = S(x + u) ]$
4. $\forall xy [ (x + y) + u = x + (y + u) ]$
5. $\forall xy [ x + u = y + u \rightarrow x = y ]$
6. $0u = 0$.

We verify that $0 \in N_0$. We need to show that 0 satisfies (1)-(6). By $Q_4 \equiv \forall x [ x + 0 = x ]$, 0 satisfies (1)-(5). By $Q_6 \equiv \forall x [ x0 = 0 ]$, 0 satisfies (6). Thus, $0 \in N_0$.

We verify that $N_0$ is closed under $S$. Let $u \in N_0$. We need to show that $Su \in N_0$. That is, we need to show that $Su$ satisfies (1)-(6). We have

$0 + Su = S(0 + u) = Su = Su + 0$

where the first equality holds by $Q_5 \equiv \forall xy [ x + Sy = S(x + y) ]$, the second equality holds since $u$ satisfies (1) and the last equality holds by $Q_4$. Thus, $Su$ satisfies (1).

By $Q_5$ and $Q_2 \equiv \forall x [ Sx \neq 0 ]$

$Sx + Su = S(x + u) \neq 0$.

Thus, $Su$ satisfies (2).

We have

$Sx + Su = S(Sx + u) = SS(x + u) = S(x + Su)$.
where the first equality holds by $Q_5$, the second equality holds since $u$ satisfies (3) and the last equality holds by $Q_5$. Thus, $Su$ satisfies (3).

We have

$$(x + y) + Su = S((x + y) + u) = S(x + (y + u)) = x + (y + Su)$$

where the first equality holds by $Q_5$, the second equality holds since $u$ satisfies (4), and the last equality holds by $Q_5$. Thus, $Su$ satisfies (4).

We have

$$S(x + u) = x + Su = y + Su = S(y + u) \Rightarrow x + u = y + u \Rightarrow x + y$$

where the first implication follows from $Q_4 \equiv \forall xy \ [ Sx = Sy \rightarrow x = y ]$, and the last implication follows from the assumption that $u$ satisfies (5). Thus, $Su$ satisfies (5).

By $Q_7 \equiv \forall xy \ [ x \times Sy = x \times y + x ]$

$$0 \times Su = 0u + 0 = 0 + 0 = 0$$

where the second equality follows from the assumption that $u$ satisfies (6) and the last equality holds by $Q_4$. Thus, $Su$ satisfies (6).

Since $Su$ satisfies (1)-(6), $Su \in N_0$. Thus, $N_0$ is closed under $S$. Since $N_0$ contains 0 and is closed under $S$, the class $N_0$ is inductive.

We restrict $N_0$ to a subclass $N_1$ where addition is commutative, the left distributive law holds, and there are no zero divisors. Let $u \in N_1$ if and only if $u \in N_0$ and

$$(7) \ \forall x \in N_0 \ [ x + u = u + x ]$$

$$(8) \ \forall x \in N_0 \forall y \ [ x(y + u) = xy + xu ]$$

$$(9) \ \forall x \in N_0 \ [ xu = 0 \rightarrow ( x = 0 \lor u = 0 ) ]$$

$$(10) \ \forall x \in N_0 \ [ Sx \times u = xu + u ] .$$

Verify that $N_1$ contains 0. We need to show that 0 \in N_0 and that 0 satisfies (7)-(10). Since $N_0$ is inductive, 0 \in N_0. By $Q_4$ and (1), 0 satisfies (7). By $Q_4$ and $Q_6$, 0 satisfies (8). It is obvious that 0 satisfies (9). By $Q_6$ and $Q_4$, 0 satisfies (10). Since 0 is an element of $N_0$ and satisfies (7)-(10), 0 \in N_1.

We verify that $N_1$ is closed under $S$. Let $u \in N_1$. We need to show that $Su \in N_0$ and that $Su$ satisfies (7)-(10). Since $N_0$ is inductive and $u \in N_1 \subseteq N_0$, $Su \in N_0$. We verify that $Su$ satisfies (7). Let $x \in N_0$. Then

$$x + Su = S(x + u) = S(u + x) = Su + x$$

where the first equality holds by $Q_5$, the second equality holds since $u$ satisfies (7), and the last equality holds since $x$ satisfies (3). Thus, $Su$ satisfies (7).
7. COMMUTATIVE SEMIRINGS I

We verify that $S_u$ satisfies (8). Let $x \in N_0$. We have
\[
x(y + Su) = x \times S(y + u) \quad (Q_5)
= x(y + u) + x \quad (Q_7)
= (xy + xu) + x \quad (u \text{ satisfies (8)} )
= xy + (xu + x) \quad (x \text{ satisfies (4)} )
= xy + (x \times Su) \quad (Q_7).
\]
Thus, $S_u$ satisfies (8).

We verify that $S_u$ satisfies (9). Let $x \in N_0$ and assume $x \times Su = 0$. By $Q_7$, $xu + x = 0$. Since $x$ satisfies (2), $x = 0$. Thus, $S_u$ satisfies (9).

Finally, we verify that $S_u$ satisfies (10). Let $x \in N_0$. We have
\[
Sx \times Su = (Sx \times u) + Sx \quad (Q_7)
= (xu + u) + Sx \quad (u \text{ satisfies (10)} )
= xu + (u + Sx) \quad (Sx \in N_0 \text{ satisfies (4)} )
= xu + S(u + x) \quad (Q_5)
= xu + S(x + u) \quad (x \in N_0 \text{ and } u \text{ satisfies (7)} )
= xu + (x + Su) \quad (Q_5)
= (xu + x) + Su \quad (Su \in N_0 \text{ satisfies (4)} )
= (x \times Su) + Su \quad (Q_7).
\]
Thus, $S_u$ satisfies (10).

Since $S_u$ is an element of $N_0$ and satisfies (7)-(10), $S_u \in N_1$. Thus, $N_1$ is closed under $S$. Since $N_1$ contains 0 and is closed under $S$, the class $N_1$ is inductive.

We restrict $N_1$ to a subclass $N_2$ where multiplication is associative and commutative. Let $u \in N_2$ if and only if $u \in N_1$ and
\[
(11) \forall x, y \in N_1 \ [ (xy)u = x(yu) ]
(12) \forall x \in N_1 \ [ xu = ux ].
\]
We verify that $0 \in N_2$. We need to show that $0 \in N_1$ and that $0$ satisfies (11)-(12). Since $N_1$ is inductive, $0 \in N_1$. By $Q_6$, 0 satisfies (11). By $Q_6$ and (6), 0 satisfies (12). Thus, $0 \in N_2$.

We verify that $N_2$ is closed under $S$. Let $u \in N_2$. We need to show that $Su \in N_1$ and that $Su$ satisfies (11)-(12). Since $N_2 \subseteq N_1$ and $N_1$ is inductive, $Su \in N_1$. We verify that $Su$ satisfies (11). Let $x, y \in N_1$. We have
\[
(xy) \times Su = (xy)u + xy \quad (Q_7)
= x(yu) + xy \quad (u \text{ satisfies (11)} )
= x(yu + y) \quad (x \in N_0 \text{ and } y \in N_1 \text{ satisfies (8)} )
= x(y \times Su) \quad (Q_7).
\]
Thus, $Su$ satisfies (11).

We verify that $Su$ satisfies (12). Let $x \in N_1$. We have
\[
x \times Su = xu + x \quad (Q_7)
= ux + x \quad (u \text{ satisfies (12)} )
= Su \times x \quad (u \in N_0 \text{ and } x \in N_1 \text{ satisfies (10)} ) .
\]
Thus, $S_u$ satisfies (12).

Since $S_u$ is an element of $N_1$ and satisfies (11)-(12), $S_u \in N_2$. Thus, $N_2$ is closed under $S$. Since $N_2$ contains $0$ and is closed under $S$, the class $N_2$ is inductive.

We are almost done. All that remains is to restrict $N_2$ to an inductive class that is closed under addition and multiplication. We start by ensuring closure under addition. Let

$$N_3 = \{ u \in N_2 : \forall x \in N_2 \ [ x + u \in N_2 ] \} .$$

By $Q_4$, $0 \in N_3$. We show that that $N_3$ is closed under $S$. Let $u \in N_3$. Since $N_2$ is inductive and $u \in N_3 \subseteq N_2$, $S_u \in N_2$. By $Q_5$, given $x \in N_2$, we have $x + S_u = S(x + u)$. Since $u \in N_3$, $x + u \in N_2$. Since $N_2$ is inductive, $S(x + u) \in N_2$. Hence, $S_u \in N_3$. Thus, $N_3$ is closed under $S$.

We verify that $N_3$ is closed under $\times$. Let $u, v \in N_3$. We need to show that $u + v \in N_2$ and $\forall x \in N_2 [ x + (u + v) \in N_2 ]$. Since $u \in N_2$ and $v \in N_3$, $u + v \in N_2$. Since $v \in N_3$, $(x + u) + v \in N_2$. Since $v \in N_2 \subseteq N_0$ satisfies (4), $(x + u) + v = x + (u + v)$. Hence, $u + v \in N_3$ Thus, $N_3$ is closed under $\times$.

Let

$$N = \{ u \in N_3 : \forall x \in N_3 [ xu \in N_3 ] \} .$$

We show that $N$ is an inductive class that is closed under $+$ and $\times$. We show that $0 \in N$. Since $N_3$ is inductive, $0 \in N_3$. Let $x \in N_3$. By $Q_6$, $x0 = 0 \in N_3$. Thus, $0 \in N$.

We show that $N$ is closed under $S$. Let $u \in N$. We need to show that $S_u \in N$. Since $u \in N_3$ and $N_3$ is inductive, $S_u \in N_3$. Let $x \in N_3$. By $Q_7$, $x \times S_u = xu + x$. Since $x \in N$, $xu \in N$. Since $N_3$ is closed under addition, $xu + x \in N_3$. Hence, $S_u \in N$. Thus, $N$ is closed under $S$.

We show that $N$ is closed under $\times$. Let $u, v \in N \subseteq N_3$. Since $N_3$ is closed under addition, $u + v \in N_3$. Let $x \in N_3$. Since $u, v \in N$, $xu, xv \in N_3$. Since $N_3$ is closed under addition, $xu + xv \in N_3$. Since $x \in N_3 \subseteq N_0$ and $v \in N_3 \subseteq N_1$, $xu + xv = x(u + v)$. Hence, $u + v \in N$. Thus, $N$ is closed under $\times$.

We show that $N$ is closed under $\times$. Let $u, v \in N \subseteq N_3$. Since $u \in N_3$ and $v \in N$, $uv \in N_3$. Let $x \in N_3$. Since $u \in N$, $xu \in N_3$. Since $v \in N$, $(xu)v \in N_3$. Since $x, u \in N_3 \subseteq N_1$ and $v \in N_3 \subseteq N_2$, $x(uv) \in N_3$. Hence, $uv \in N$. Thus, $N$ is closed under $\times$.

Since $N$ satisfies the domain conditions and all the axioms of $IQ(2)$ hold restricted to $N$ as they are universal sentences, $IQ(2)$ is interpretable in $IQ$. Since $IQ(2)$ is an extension of $IQ$, it follows that $IQ$ and $IQ(2)$ are mutually interpretable.

8 Commutative Semirings II

It is clear that $Q(2)$ is interpretable in $Q$ since each axiom of $Q(2)$ is provable in $I\Delta_0$, which is $Q$ extended with an induction schema for $\Sigma_0$-formulas, and
Given a sentence $\phi$ and a class $M$, let $\phi^M$ denote the sentence we obtain by restricting quantification to $M$.

**Theorem 24.** There exists a class $M$ such that $Q \vdash \phi^M$ for each axiom $\phi$ of $Q^{(2)}$.

**Proof.** Let $N$ be the class in the proof of Theorem 23. Let 

$$u \leq_N v \equiv \exists r \in N \{ u + r = v \}.$$

We restrict $N$ to an inductive subclass $M_0$ that is downward closed under $\leq_N$. Let 

$$M_0 = \{ u \in N : \forall v \leq_N u [ v \in N ] \land \forall x,y \leq_N u [ x \leq_N y \lor y \leq_N x ] \}.$$

We show that $0 \in M_0$. Assume $v + r = 0$. If $r = 0$, then $v = 0$ by $Q_1$. If $v \neq 0$, then by $Q_3$ there exists $t$ such $r = St$. Then, by $Q_3$, $0 = v + r = S(v + t)$ which contradicts $Q_2$. Thus, since $0 \in N$ and $0 + 0 = 0$, we have $0 \in M_0$.

We show that $M_0$ is closed under $S$. Let $u \in M_0$. We need to show that $Su \in M_0$. Since $u \in M_0 \subseteq N$ and $N$ is inductive, $Su \in N$. We show that $\forall v \leq_N Su [ v \in N ]$. Assume $r \in N$ and $v + r = Su$. We need to show that $v \in N$. If $v = 0$, then $v \in N$ since $N$ is an inductive class. Otherwise, by $Q_3$, there exists $w$ such that $Sw = v$. By Clause (3) in the proof of Theorem 23 

$$Su = v + r = Sw + r = S(w + r).$$

By $Q_1$, $w + r = u$. Hence, $w \leq_N u$. Since $u \in M_0$, we have $w \in N$. Since $N$ is an inductive class, $v = Sw \in N$. Thus, $\forall v \leq_N Su [ v \in N ]$.

We show that $\forall x,y \leq_N Su [ x \leq_N y \lor y \leq_N x ]$. Assume $x,y \leq_N Su$. By what we have just shown, $x,y \in N$. If $x = Su$ or $y = Su$, then $x$ and $y$ are comparable with respect to $\leq_N$ since $x,y \leq_N Su$. Otherwise, by $Q_4$ 

$$Su = x + r \land Su = y + t \quad \text{where } r,t \in N \setminus \{0\}.$$

Since $x,y,r,t \in N$, we have 

$$Su = x + r = r + x \land Su = y + t = t + y$$

by Clause (7) in the proof of Theorem 23. By $Q_3$, there exist $r_0,t_0$ such that $r = Sr_0$ and $t = St_0$. Hence 

$$Su = Sr_0 + x = S(r_0 + x) \land Su = St_0 + y = S(t_0 + y)$$

by Clause (3) in the proof of Theorem 23. By $Q_4$, $u = r_0 + x$ and $u = t_0 + y$. Hence, $r_0,t_0 \leq_N u$ which implies $r_0,t_0 \in N$ since $u \in M_0$. Then 

$$u = r_0 + x = x + r_0 \land u = t_0 + y = y + t_0$$
by Clause (7) in the proof of Theorem 23. Hence, $x,y \leq_N u$ which implies that $x$ and $y$ are comparable with respect to $\leq_N$ since $u \in M_0$. Thus, we have $\forall r,y \leq_N Su [ x \leq_N y \lor y \leq_N x ]$. It then follows that $Su \in M_0$.

Since $\leq_N$ is transitive, $M_0$ is downward closed under $\leq_N$. Indeed, assume $w \leq_N v$ and $v \leq_N u$. Then, there exist $r,t \in N$ such that $v = w + r$ and $u = v + t$. Hence, $u = (w + r) + t$. Since $t \in N \subseteq N_0$, we have

$$ u = (w + r) + t = w + (r + t) $$

by Clause (4) in the proof of Theorem 23. Since $r,t \in N$ and $N$ is closed under addition, $r + t \in N$. Hence, $w \leq_N u$. Thus, $\leq_N$ is transitive.

We restrict $M_0$ to a subclass $M_1$ that is closed under addition. Let

$$ M_1 = \{ u \in M_0 : \forall x \in M_0 [ x + u \in M_0 ] \} . $$

The class $M_1$ is shown to be closed under $0,S,+,$ and $\times$ just as in the proof of Theorem 23. We show that $M_1$ is downward closed under $\leq_N$. Assume $u \in M_1$ and $u = v + r$ where $r \in N$. We need to show that $v \in M_1$. So, let $x \in M_0$. We need to show that $x + v \in M_0$. We have

$$ M_0 \ni x + u = x + (v + r) = (x + v) + r $$

by Clause (4) in the proof of Theorem 23. Then

$$ x + v \leq_N x + u \in M_0 . $$

Since $M_0$ is downward closed under $\leq_N$, we have $x + v \in M_0$. Hence, $v \in M_1$. Thus, $M_1$ is downward closed under $\leq_N$.

Finally, we restrict $M_1$ to a domain $M$. Let

$$ M = \{ u \in M_1 : \forall x \in M_1 [ xu \in M_1 ] \} . $$

The class $M$ is shown to be closed under $0,S,+,$ and $\times$ just as in the proof of Theorem 23. We show that $M_1$ is downward closed under $\leq_N$. Assume $u \in M$ and $u = v + r$ where $r \in N$. We need to show that $v \in M$. So, let $x \in M_1$. We need to show that $xv \in M_1$. We have

$$ M_1 \ni xu = x(v + r) = xv + xr $$

by Clause (8) in the proof of Theorem 23. Since $v \leq_N u$, $u \in M \subseteq M_1$ and $M_1$ is downward closed under $\leq_N$, we have $v \in M_1$. Then, by Clause (7) in the proof of Theorem 23.

$$ u = v + r = r + v . $$

Hence, $r \leq_N u$ which implies $r \in M_1$. Since $x,r \in M_1$ and $M_1$ is closed under $\times$, we have $xr \in M_1 \subseteq N$. Then, $xu = xv + xr$ implies $xv \leq_N xu$. Since $xu \in M_1$ and $M_1$ is downward closed under $\leq_N$, we have $xv \in M_1$. Hence, $v \in M$. Thus, $M$ is downward closed under $\leq_N$.

Axioms (I)-(VIII) in Figure 1.6 and the axioms of $Q$ that are universal sentences hold on $M$ when we restrict quantification to $M$ since they hold on $N$ when we restrict quantification to $N$. We show that $Q_3 \equiv \forall x \{ x = 0 \lor \exists y \{ x = Sy \}$
holds on $M$. Assume $x \in M \setminus \{0\}$. By $Q_3$, there exists $y$ such that $x = S y$. We need to show that $y \in M$. By $Q_4$ and $Q_5$, we have

$$x = S y = S y + 0 = S(y + 0) = y + S 0 .$$

Since $M$ is an inductive class, $S 0 \in M \subseteq N$. Hence, $y \leq_N x$. Since $M$ is downward closed under $\leq_N$, we have $y \in M$. Thus, $Q_3$ holds restricted to $M$.

Finally, we show the trichotomy law $\forall x y \left[ x <_1 y \lor x = y \lor y <_1 x \right]$ holds restricted to $M$. Recall that $x <_1 y \equiv \exists r \left[ r \neq 0 \land r + x = y \right]$. Let $x, y \in M$.

Since $M$ is closed under addition and addition on $M$ is commutative

$$y + x = x + y \in M .$$

Then, $x, y \leq_N x + y$. Since $M \subseteq M_0$, we have

$$x \leq_N y \lor y \leq_N x .$$

Assume $y = x + r$ where $r \in N$. By Clause (7) in the proof of Theorem 23, $y = x + r = r + x$. Hence, $r \leq_N y$ which implies $r \in M$. Similarly, if $x = y + t$ where $t \in N$, then $t \in M$. Hence, since $x \leq_N y \lor y \leq_N x$ holds

$$\exists r, t \in M \left[ y = x + r \lor x = y + t \right] .$$

Thus, the trichotomy law holds restricted to $M$. \hfill $\Box$
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