Binary matroids and local complementation

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Abstract

We introduce a binary matroid $M(IAS(G))$ associated with a looped simple graph $G$. $M(IAS(G))$ classifies $G$ up to local equivalence, and determines the delta-matroid and isotropic system associated with $G$. Moreover, a parametrized form of its Tutte polynomial yields the interlace polynomials of $G$.

Keywords. delta-matroid, graph, interlace polynomial, isotropic system, local complementation, matroid, multimatroid, pivot, Tutte polynomial

Mathematics Subject Classification. 05B35, 05C31

1 Introduction

A graph $G = (V(G), E(G))$ consists of a finite vertex-set $V(G)$ and a finite edge-set $E(G)$. Each edge is incident on one or two vertices; an edge incident on only one vertex is a loop. The two vertices incident on a non-loop edge are neighbors, and the open neighborhood of a vertex $v$ is $N(v) = \{\text{neighbors of } v\}$. A graph in which different edges can be distinguished by their vertex-incidences is a looped simple graph, and a simple graph is a looped simple graph with no loop.

In this paper we are concerned with properties of looped simple graphs motivated by two sets of ideas. The first set of ideas is the theory of the principal pivot transform (PPT) over $GF(2)$. PPT over arbitrary fields was introduced more than 50 years ago by Tucker [38]; see also the survey of Tsatsomeros [37]. According to Geelen [24], PPT transformations applied to the mod-2 adjacency matrices of looped simple graphs are generated by two kinds of elementary PPT operations, non-simple local complementations with respect to looped vertices and edge pivots with respect to edges connecting unlooped vertices. The second set of ideas is the theory of 4-regular graphs and their Euler circuits, initiated more than 40 years ago by Kotzig [27]. Kotzig proved that all the Euler circuits of a 4-regular graph are obtained from any one using $\kappa$-transformations. If a 4-regular graph is directed in such a way that every vertex has indegree 2 and
outdegree 2, then Kotzig [27], Pevzner [29] and Ukkonen [39] showed that all of
the graph’s directed Euler circuits are obtained from any one through certain
combinations of \( \kappa \)-transformations called transpositions by Arratia, Bollobás
and Sorkin [2, 3, 4]. Bouchet [8] and Rosenstiehl and Read [30] introduced a
simple graph associated with any Euler circuit of a connected 4-regular graph,
the alternance graph or interlacement graph; an equivalent link relation matrix
was defined by Cohn and Lempel [20] in the context of the theory of permuta-
tions. These authors showed that the effects of \( \kappa \)-transformations and transpo-
sitions on interlacement graphs are given by simple local complementations and
edge pivots, respectively.

In the late 1980s, Bouchet introduced two new kinds of combinatorial struc-
tures associated with these two theories. On the one hand are the delta-matroids
[9], some of which are associated with looped simple graphs. The funda-
mental operation of delta-matroid theory is a way of changing one delta-matroid into
another, called twisting. Two looped simple graphs are related through PPT
operations if and only if their associated delta-matroids are related through
twisting. On the other hand are the isotropic systems [10, 12], all of which are
associated with fundamental graphs. Two isotropic systems are strongly iso-
morphic if and only if they share fundamental graphs. Moreover, two simple
graphs are related through simple local complementations if and only if they
are fundamental graphs of strongly isomorphic isotropic systems. Properties
of isotropic systems were featured in the proof of Bouchet’s famous “forbidden
minors” characterization of circle graphs [14].

The purpose of this paper is to introduce a binary matroid \( M(IAS(G)) \) con-
structed in a natural way from the adjacency matrix of a looped simple graph
\( G \); we call it the isotropic matroid of \( G \), in honor of Bouchet’s isotropic systems.
This matroid directly determines both the delta-matroid and the isotropic sys-
tem associated with \( G \). Moreover, it is not difficult to characterize the effects
of edge pivots and both kinds of local complementations on isotropic matroids,
so we have a single matroid invariant that serves to classify \( G \) under all the
operations mentioned above.

The paper is set up as follows. First, we recall some basic facts about binary
matroids in Section 2. \( M(IAS(G)) \) is defined in Section 3, where we also discuss
a certain kind of matroid isomorphism between isotropic matroids. The heart
of the paper is Sections 4 and 5, where we show that \( M(IAS(G)) \) determines
\( G \) up to equivalence under various kinds of complementations and pivots. In
Section 6 we show that the delta-matroid and isotropic system associated with \( G \)
are determined by \( M(IAS(G)) \), and in Section 7 we discuss some fundamental
properties of isotropic matroids. In the last section we show that \( M(IAS(G)) \)
has another interesting property: appropriately parametrized Tutte polynomials
of this matroid yield the interlace polynomials introduced by Arratia, Bollobás
and Sorkin [2, 3, 4], and also the modified versions subsequently defined by
Aigner and van der Holst [11], Courcelle [21] and the author [34].

The ideas in this paper came to mind after the resemblance between the
matrices appearing in Aigner and van der Holst’s discussion of interlace poly-
nomials [11] and our nonsymmetric approach to interlacement in 4-regular graphs
was pointed out to us by R. Brijder. We are grateful to him for years of informative correspondence regarding delta-matroids, isotropic systems, PPT and related combinatorial notions.

2 Standard representations of binary matroids

We do not review general results and terminology of graph theory and matroid theory here; instead we refer the reader to standard texts in the field, [25, 28, 40, 41] for instance. All the matroids we consider in this paper are binary:

**Definition 1** Let $S$ be a finite set. A binary matroid $M$ on $S$ is represented by a matrix with entries in $\text{GF}(2)$, whose columns are indexed by the elements of $S$. A subset of $S$ is dependent in $M$ if and only if the corresponding columns of the matrix are linearly dependent.

The binary matroid represented by a matrix is not changed if one row is added to another, or the rows are permuted, or a row of zeroes is adjoined or removed. Also, permuting the columns of a matrix will yield a new matrix that represents an isomorphic binary matroid. Familiar results of elementary linear algebra tell us that consequently, every binary matroid has a representation of the following type:

**Definition 2** Let $I$ be an $r \times r$ identity matrix. A standard representation of a rank-$r$ binary matroid $M$ is a matrix of the form $(I \ | \ A)$ that represents $M$.

If $A$ is a matrix with entries in $\text{GF}(2)$ then $M(IA)$ denotes the matroid with standard representation $(I \ | \ A)$.

Recall that if $B$ is a basis of a matroid $M$, and $x$ is an element of $M$ not included in $B$, then the fundamental circuit of $x$ with respect to $B$ is

$$C(x, B) = \{x\} \cup \{b \in B \mid B \Delta \{b, x\} \text{ is a basis of } M\},$$

where $\Delta$ denotes the symmetric difference. $C(x, B)$ is the unique circuit contained in $B \cup \{x\}$.

A peculiar property of binary matroids is that the fundamental circuits with respect to any one basis contain enough information to determine a binary matroid. The same is not true for general matroids; for instance a matroid on $\{1, 2, 3, 4\}$ with basis $\{1, 2\}$ and fundamental circuits $\{1, 2, 3\}$ and $\{1, 2, 4\}$ might be either $U_{2,4}$ or the circuit matroid of a triangle with one doubled edge. ($U_{2,4}$ is not binary, of course.) Notice that in essence, a standard representation $(I \ | \ A)$ is this kind of description: the matroid elements corresponding to the columns of $I$ constitute a basis $B$, and for each element $x \notin B$, the fundamental circuit $C(x, B)$ includes $x$ together with the elements of $B$ corresponding to nonzero entries of the $x$ column of $A$.

The only part of this section that does not appear in the textbooks mentioned above is the following simple theorem, which tells us how the various standard representations of a binary matroid are related to each other.
Theorem 3 Let $A_1$ and $A_2$ be $r \times (n-r)$ matrices with entries in $GF(2)$. Then $M(I \cup A_1) \cong M(I \cup A_2)$ if and only if $(I \mid A_2)$ can be obtained from $(I \mid A_1)$ using the following three types of operations on matrices of the form $(I \mid A)$:

(a) Permute the columns of $A$.
(b) Permute the columns of $I$ and the rows of $(I \mid A)$, using the same permutation.
(c) Suppose the $jk$ entry of $A$ is $a_{jk} = 1$. Then toggle (reverse) $a_{bc}$ whenever $b \neq j$, $c \neq k$, $a_{jc} = 1$ and $a_{bk} = 1$.

Proof. As noted above, a standard presentation of a rank-$r$ binary matroid $M$ on an $n$-element set $S$ is obtained as follows. First choose a basis $B$, and index its elements as $s_1, \ldots, s_r$. Then index the remaining elements of $S$ as $s_{r+1}, \ldots, s_n$. Finally, let $A$ be the $r \times (n-r)$ matrix whose $jk$ entry is 1 if and only if $s_j$ is an element of the fundamental circuit $C(s_{r+k}, B)$. Operations of types (a) and (b) correspond to re-indexings of $S - B$ and $B$, respectively.

Suppose now that $a_{jk} = 1$, and let $A'$ be the matrix obtained from $A$ by an operation of type (c). Another way to describe $A'$ is this: $(I \mid A')$ is obtained from $(I \mid A)$ by first interchanging the $j$th and $(r+k)$th columns, and then adding the $j$th row of the resulting matrix to every other row in which the original $(r+k)$th column has a nonzero entry. That is, the matrix $(I \mid A')$ is simply the standard representation corresponding to the basis $B \Delta \{s_j, s_{r+k}\}$, with the elements other than $s_j$ and $s_{r+k}$ indexed as they were before.

The theorem follows, because basis exchanges $B \leftrightarrow B \Delta \{b, x\}$ eventually construct every basis of $M$ from any one. \qedsymbol

We refer to an operation of type (c) as a basis exchange involving the $j$th column of $I$ and the $k$th column of $A$. (It would also be natural to call it a pivot, but this term already has other meanings.)

3 M(IAS(G)) and compatible isomorphisms

If $G$ is a looped simple graph then $A(G)$ denotes the adjacency matrix of $G$, i.e., the $|V(G)| \times |V(G)|$ matrix with entries in $GF(2)$ given by: a diagonal entry is 1 if and only if the corresponding vertex is looped, and an off-diagonal entry is 1 if and only if the corresponding vertices are adjacent. $AS(G)$ denotes the matrix $(A(G) \mid I + A(G))$. (S is for “sum.”)

As in Section 2, $M(IAS(G))$ is the binary matroid represented by the $|V(G)| \times (3|V(G)|)$ matrix

$$IAS(G) = (I \mid A(G) \mid I + A(G)).$$

We denote the ground set of this matroid $W(G)$. If $v \in V(G)$ then there are three columns of $IAS(G)$ corresponding to $v$: one in $I$, one in $A(G)$, and one in $I + A(G)$. For notational convenience, and to indicate the connection with our work on interlace polynomials \cite{34}, we use $v_\phi$ to denote the column of $I$ corresponding to $v$, $v_\chi$ to denote the column of $A(G)$ corresponding to $v$, and
to denote the column of $I + A(G)$ corresponding to $v$. We refer to the resulting partition of $W(G)$ into 3-element subsets as canonical.

It is convenient to adopt notation to describe matroid isomorphisms that are compatible with these canonical partitions. Let $S_3$ denote the group of permutations of the three symbols $\phi$, $\chi$ and $\psi$. We use standard notation in $S_3$; for instance 1 is the identity, $(\phi\chi)$ is a transposition, and $(\phi\chi)(\chi\psi) = (\psi\phi\chi)$ is a 3-cycle.

Suppose $G_1$ and $G_2$ are looped simple graphs, and there is an isomorphism $\beta : M(\text{IAS}(G_1)) \to M(\text{IAS}(G_2))$ that is compatible with the canonical partitions. Then the isomorphism consists of two parts. First, there is an induced bijection $V(G_1) \to V(G_2)$; in general we will denote this bijection $\beta$ too, though up to isomorphism we may always presume that $V(G_1) = V(G_2)$ and the induced bijection is the identity map. Second, there is a function $f_\beta : V(G_1) \to S_3$ such that $\beta(v) = \beta(v_{f_\beta(v)}) \forall v \in V(G_1) \forall u \in \{\phi, \chi, \psi\}$. In this situation we say that $\beta$ is a compatible isomorphism determined by $f_\beta$.

Here are two obvious properties of compatible isomorphisms.

**Lemma 4** If $\beta_1 : M(\text{IAS}(G_1)) \to M(\text{IAS}(G_2))$ and $\beta_2 : M(\text{IAS}(G_2)) \to M(\text{IAS}(G_3))$ are compatible isomorphisms, then $\beta_2 \circ \beta_1 : M(\text{IAS}(G_1)) \to M(\text{IAS}(G_3))$ is also a compatible isomorphism, and it is determined by the map $f : V(G_1) \to S_3$ given by $f(v) = f_{\beta_2}(\beta_1(v)) \cdot f_{\beta_1}(v)$.

**Lemma 5** If $\beta : M(\text{IAS}(G_1)) \to M(\text{IAS}(G_2))$ is a compatible isomorphism then so is $\beta^{-1} : M(\text{IAS}(G_2)) \to M(\text{IAS}(G_1))$, and $\beta^{-1}$ is determined by the map $f_{\beta^{-1}} : V(G_2) \to S_3$ given by $f_{\beta^{-1}}(v) = f_{\beta}(\beta^{-1}(v))^{-1}$.

The next property is not quite so obvious.

**Lemma 6** Suppose $\beta : M(\text{IAS}(G_1)) \to M(\text{IAS}(G_2))$ is a compatible isomorphism, determined by the map $f : V(G_1) \to S_3$ with $f(v) = 1 \forall v \in V(G_1)$. Then $G_1$ and $G_2$ are isomorphic.

**Proof.** Up to isomorphism, we may as well presume that $V(G_1) = V(G_2)$ and the bijection $V(G_1) \to V(G_2)$ induced by $\beta$ is the identity map. Then $M(\text{IAS}(G_1))$ and $M(\text{IAS}(G_2))$ are isomorphic matroids on the ground set $W(G_1) = W(G_2)$. As $f(v) \equiv 1$, the isomorphism $\beta$ is the identity map of this ground set.

The identity map preserves the basis $\Phi = \{v_\phi \mid v \in V(G_1)\}$. The identity map is a matroid isomorphism, so it must also preserve fundamental circuits with respect to $\Phi$. Recall the discussion of Section 2: the column of $\text{IAS}(G_i)$ corresponding to $x \notin \Phi$ is determined by the fundamental circuit of $x$ with respect to $\Phi$ in $\text{M}(\text{IAS}(G_i))$. It follows that the matrices $\text{AS}(G_1)$ and $\text{AS}(G_2)$ are identical.

The following consequence will be useful.

**Corollary 7** Suppose $\beta_1 : M(\text{IAS}(G_1)) \to M(\text{IAS}(G_2))$ and $\beta_2 : M(\text{IAS}(G_1)) \to M(\text{IAS}(G_3))$ are compatible isomorphisms, and the associated functions $f_{\beta_1}, f_{\beta_2} : V(G_1) \to S_3$ are the same. Then $G_2$ and $G_3$ are isomorphic.

**Proof.** Apply the lemmas to the composition of $\beta_2$ and $\beta_1^{-1}$. ■
4 Complements and pivots

In this section we prove that the matroid $M(IAS(G))$ classifies $G$ under several different kinds of operations.

4.1 Loop complementation

Suppose $G_1$ is a looped simple graph, $v \in V(G_1)$, and $G_2$ is the graph obtained from $G_1$ by complementing (reversing) the loop status of $v$. Clearly then $IAS(G_2)$ is the matrix obtained from $IAS(G_1)$ by interchanging the $v_\chi$ and $v_\psi$ columns. This interchange is an example of an operation of type (a), so Theorem 3 tells us that there is a compatible isomorphism $M(IAS(G_1)) \rightarrow M(IAS(G_2))$ determined by the map $f : V(G_1) \rightarrow S_3$ given by

$$
 f(w) = \begin{cases} 
 \text{the transposition } (\chi \psi), & \text{if } w = v \\
 1, & \text{if } w \neq v
\end{cases}
$$

The converse also holds:

**Theorem 8** Let $G_1$ and $G_2$ be looped simple graphs, and suppose $v \in V(G_1)$. Then these two conditions are equivalent:

1. Up to isomorphism, $G_2$ is the graph obtained from $G_1$ by complementing the loop status of $v$.
2. There is a compatible isomorphism $\beta : M(IAS(G_1)) \rightarrow M(IAS(G_2))$ such that $f_\beta(v) = (\chi \psi)$ and $f_\beta(w) = 1 \forall w \neq v$.

**Proof.** We have already discussed the implication $1 \Rightarrow 2$. The converse follows from Corollary 7. $\blacksquare$

4.2 Local complementation

Two different versions of local complementation appear in the literature. Simple local complementation was introduced by Bouchet [8] and Rosenstiehl and Read [30], as part of the theory of interlacement in 4-regular graphs. This operation does not involve the creation of loops, so it is the version seen most often in graph theory, where the theory of simple graphs predominates. Non-simple local complementation is part of the theory of the principal pivot transform (PPT) over $GF(2)$. The general theory of PPT was introduced by Tucker [38]; see also the survey of Tsatsomeros [37]. The special significance of non-simple local complementation in PPT over $GF(2)$ was discussed by Geelen [24].

Later (and independently) non-simple local complementation was introduced by Arratia, Bollobás and Sorkin as part of the theory of the two-variable interlace polynomial [4]. We should emphasize that simple local complementations are usually applied only to simple graphs in the first set of references, and non-simple local complementations are usually applied only with respect to looped vertices in the second set of references. Our definitions are not so restrictive.
Definition 9 If $G$ is a looped simple graph and $v \in V(G)$ then the simple local complement of $G$ with respect to $v$ is the graph $G^v_s$ obtained from $G$ by complementing all adjacencies between distinct elements of the open neighborhood $N(v)$. The non-simple local complement of $G$ with respect to $v$ is the graph $G^v_{ns}$ obtained from $G^v_s$ by complementing the loop status of each element of $N(v)$.

Observe that replacing $A(G)$ by $A(G^v_{ns})$ has precisely the same effect on the matrix $IAS(G)$ as a type (c) operation from Theorem 3. As discussed in Section 2, this operation is equivalent to a basis exchange involving $v_\phi$ and either $v_\chi$ (if $v$ is looped) or $v_\psi$ (if $v$ is unlooped). We deduce the following.

Theorem 10 Let $G_1$ and $G_2$ be looped simple graphs, and suppose $v \in V(G_1)$. Then these two conditions are equivalent:

1. $G_2$ is isomorphic to $(G_1)^{v}_{ns}$.
2. There is a compatible isomorphism $\beta : M(IAS(G_1)) \to M(IAS(G_2))$ such that $f_\beta(w) = 1 \forall w \neq v$ and
   $$f_\beta(v) = \begin{cases} 
   (\phi \chi), & \text{if } v \text{ is looped in } G_1 \\
   (\phi), & \text{if } v \text{ is not looped in } G_1
   \end{cases}$$

Proof. As already noted, $1 \Rightarrow 2$ because $IAS((G_1)^{v}_{ns})$ is the same as the matrix associated with the standard presentation of $M(IAS(G_1))$ obtained from $IAS(G_1)$ by a basis exchange involving $v_\phi$ and either $v_\chi$ or $v_\psi$. The converse follows from Corollary 7.  

4.3 Pivots

Here is a well-known definition. The reader who is encountering it for the first time should take a moment to verify that the two indicated triple local complements are indeed the same.

Definition 11 If $v$ and $w$ are neighbors in $G$ then the edge pivot $G^{vw}$ is the triple simple local complement:

$$G^{vw} = (((G^v_s)_w)_s)_n = (((G^w_s)_v)_w)_n.$$  

Note that we do not restrict edge pivots to edges with unlooped vertices.

Corollary 12 Suppose $v \neq w$ are neighbors in $G$, and let $f : V(G) \to S_3$ be the function with $f(x) = 1$ for $w \neq x \neq v$,

$$f(v) = \begin{cases} 
   (\phi \chi), & \text{if } v \text{ is unlooped} \\
   (\phi \psi), & \text{if } v \text{ is looped}
   \end{cases}$$  

and $f(w) = \begin{cases} 
   (\phi \chi), & \text{if } w \text{ is unlooped} \\
   (\phi), & \text{if } w \text{ is looped}
   \end{cases}$.

Then $f$ determines a compatible isomorphism $M(IAS(G)) \to M(IAS(G^{vw}))$. 

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Proof. The reader can easily check that the definition

\[ G^{vw} = ((G^v)^w)_s \]

is equivalent to saying this: \( G^{vw} \) is obtained from \( ((G^v)^w)_s \), by complementing the loop status of \( v \). (The reason is that \( v \) is the only vertex whose loop status is complemented an odd number of times in \( ((G^v)^w)_s \).) It follows that there is a compatible isomorphism \( M(IAS(G)) \rightarrow M(IAS(G^{vw})) \) obtained by composing four compatible isomorphisms, three from Theorem 10 and one from Theorem 8. According to Lemma 4 this compatible isomorphism is determined by the function \( f: V(G) \rightarrow S_3 \) such that \( f(x) = 1 \) for \( w \neq x \neq v \), \( f(w) = (\phi \psi) \) if \( w \) is unlooped in \( G^w \), \( f(w) = (\phi \chi) \) if \( w \) is looped in \( G^w \), and

\[
f(v) = \begin{cases} 
(\chi \psi) \cdot (\phi \chi) \cdot 1 \cdot (\phi \psi), & \text{if } v \text{ is unlooped} \\
(\chi \psi) \cdot (\phi \psi) \cdot 1 \cdot (\phi \chi), & \text{if } v \text{ is looped}
\end{cases}
\]

As in Theorems 8 and 10 Corollary 12 implies that the converse of Corollary 12 is also valid. That is, if the given function \( f \) determines a compatible isomorphism \( M(IAS(G_1)) \rightarrow M(IAS(G_2)) \) then \( G_2 \cong G_1^{vw} \).

Notice that the function \( f \) of Corollary 12 is the combination of two separate functions, one \( \equiv 1 \) except at \( v \) and the other \( \equiv 1 \) except at \( w \). According to Theorem 3 these two separate functions do not come from two separate compatible isomorphisms, though. This fact is reflected in the proof, where the compatible isomorphism \( M(IAS(G)) \rightarrow M(IAS(G^{vw})) \) is described as a composition of four simpler compatible isomorphisms, not two.

There is a different way to describe the compatible isomorphism of Corollary 12 using only two basis exchanges. According to Theorem 4 if \( v \) and \( w \) are neighbors then there is a basis exchange involving \( v_\psi \) and either \( w_\chi \) or \( w_\psi \), as each of these columns has a 1 in the \( v \) row. The matrix resulting from part (c) of Theorem 4 is not of the form \( (I \mid A \mid I + A) \) for a symmetric matrix \( A \), so there is no natural way to interpret such a basis exchange as a graph operation. However, the reader can easily check that if this basis exchange is followed by one involving \( w_\phi \) and either \( v_\chi \) or \( v_\psi \), then the result is of the form \( (I \mid A \mid I + A) \) for a symmetric matrix \( A \). Moreover, \( A \) closely resembles the adjacency matrix of \( G^{vw} \); the positions of \( v \) and \( w \) have been interchanged, though, and depending on the \( \chi, \psi \) choices the loop statuses of \( v \) and \( w \) may also have changed. The fact that the compatible isomorphism \( M(IAS(G)) \rightarrow M(IAS(G^{vw})) \) may be described in this different way, involving a transposition of adjacency information regarding \( v \) and \( w \), is a reflection of the fact that there is a different way to define the pivot. See Section 3 of [3], where Arratia, Bollobás and Sorkin give this different definition, and show that it is related to Definition 13 by applying a “label swap” exchanging the names of \( v \) and \( w \).
4.4 Classifying graphs using compatible isomorphisms

The simplest classification theorem resulting from the above discussion is this immediate consequence of Theorem 8.

**Theorem 13** Let $G_1$ and $G_2$ be looped simple graphs. Then these two conditions are equivalent:

1. Up to isomorphism, $G_2$ can be obtained from $G_1$ by complementing the loop status of some vertices.
2. There is a compatible isomorphism $\beta : M(IAS(G_1)) \to M(IAS(G_2))$ such that $f_\beta(v) \in \{1, (\chi \psi)\}$ for every loop $v \in V(G_1)$.

Other classification theorems have similar statements, but take a little more work to prove.

**Theorem 14** Let $G_1$ and $G_2$ be looped simple graphs. Then these two conditions are equivalent:

1. Up to isomorphism, $G_2$ can be obtained from $G_1$ using edge pivots.
2. There is a compatible isomorphism $\beta : M(IAS(G_1)) \to M(IAS(G_2))$ such that $f_\beta(v) \in \{1, (\phi \psi)\}$ for every looped $v \in V(G_1)$ and $f_\beta(v) \in \{1, (\phi \chi)\}$ for every unlooped $v \in V(G_1)$.

**Proof.** If $G'$ can be obtained from $G$ using edge pivots, simply apply Corollary 12 repeatedly.

For the converse, suppose condition 2 holds, and there are $k$ vertices with $f_\beta(v) \neq 1$. If $k = 0$ then Corollary 17 implies that $G_1 \cong G_2$.

If $k > 0$ then let $v_0 \in V(G)$ have $f_\beta(v_0) \neq 1$, and let $\Phi = \{v_0 \mid v \in V(G_1)\}$. Then $\Phi$ is a basis of $M(IAS(G_1))$, so $\beta(\Phi)$ is a basis of $M(IAS(G_2))$. Consequently,

$$\{\beta(v_0)\} \cup \{v_0 \mid f_\beta(v_0) = 1\}$$

cannot be dependent in $M(IAS(G_2))$, because it is a subset of $\beta(\Phi)$. It follows that the column of $IAS(G_2)$ corresponding to $\beta(v_0)$ must have at least one nonzero entry in a row that corresponds to a vertex $v \neq v_0$ with $f_\beta(v) \neq 1$. Then $v$ is a neighbor of $v_0$, and Corollary 12 implies that there is a compatible isomorphism $\beta' : M(IAS(G_2)) \to M(IAS((G_2)^{v_0}))$ determined by the function $f_{\beta'} : V(G_2) \to S_3$ with $f_{\beta'}(w) = 1 \forall w \notin \{v, v_0\}, f_{\beta'}(v_0) = f_\beta(v_0)$ and $f_{\beta'}(v) = f_\beta(v)$. The composition $\beta' \circ \beta$ is a compatible isomorphism $M(IAS(G_1)) \to M(IAS((G_2)^{v_0}))$, which satisfies condition 2; as there are only $k - 2$ vertices outside $(f_{\beta' \circ \beta})^{-1}(1)$, induction assures us that up to isomorphism, $(G_2)^{v_0}$ may be obtained from $G_1$ using edge pivots.

When restricted to simple graphs, Theorem 14 yields the following.

**Corollary 15** Let $G_1$ and $G_2$ be simple graphs. Then these two conditions are equivalent:
1. Up to isomorphism, $G_2$ can be obtained from $G_1$ using edge pivots.

2. There is a compatible isomorphism $\beta : M(IAS(G_1)) \rightarrow M(IAS(G_2))$ such that $f_\beta(v) \in \{1, (\phi \chi)\} \forall v \in V(G_1)$.

Another consequence of Theorem 14 is this:

**Corollary 16** Let $G_1$ and $G_2$ be looped simple graphs. Then these three conditions are equivalent:

1. Up to isomorphism, $G_2$ can be obtained from $G_1$ using two kinds of operations: non-simple local complementations with respect to looped vertices, and edge pivots with respect to non-looped vertices.

2. Up to isomorphism, $G_2$ can be obtained from $G_1$ using PPT operations.

3. There is a compatible isomorphism $\beta : M(IAS(G_1)) \rightarrow M(IAS(G_2))$ such that $f_\beta(v) \in \{1, (\phi \chi)\} \forall v \in V(G_1)$.

**Proof.** The equivalence 1$\Leftrightarrow$2 is due to Geelen [24], and the implication 1$\Rightarrow$3 follows from Theorems 10 and 14.

Suppose condition 3 holds and there are $k$ vertices with $f_\beta(v) \neq 1$. If $k = 0$ then Corollary 7 implies that $G_1$ and $G_2$ are isomorphic. If $G_1$ has a looped vertex $v_0$ with $f_\beta(v_0) \neq 1$ then there is a compatible isomorphism $\beta' : M(IAS((G_1))_{v_0}) \rightarrow M(IAS(G_2))$ that satisfies condition 3, and for which only $k - 1$ vertices have $f_{\beta'}(v) \neq 1$. Induction then tells us that condition 1 holds. If there is no such $v_0$, then Theorem 14 applies.

Here is our final classification theorem involving compatible isomorphisms.

**Theorem 17** Let $G_1$ and $G_2$ be looped simple graphs. Then these two conditions are equivalent:

1. Up to isomorphism, $G_2$ can be obtained from $G_1$ using local complementations and loop complementations.

2. There is a compatible isomorphism $\beta : M(IAS(G_1)) \rightarrow M(IAS(G_2))$.

**Proof.** The implication 1$\Rightarrow$2 follows from Theorems 8 and 10.

For the converse, suppose condition 2 holds and there are $k$ vertices with $f_\beta(v) \neq 1$. Up to isomorphism, we may presume that $V(G_1) = V(G_2)$ and the bijection induced by $\beta$ is the identity map.

If $k = 0$ then Corollary 7 tells us that $G_1 = G_2$.

The argument proceeds by induction on $k \geq 1$. If there is any vertex with $f_\beta(v_0) = (\chi \psi)$ then the graph $G_2'$ obtained from $G_2$ by complementing the loop status of $v_0$ has the property that there is a compatible isomorphism $\beta' : M(IAS(G_1)) \rightarrow M(IAS(G_2'))$ such that $f_{\beta'}(v) = f_\beta(v) \forall v \neq v_0$, and $f_{\beta'}(v_0) = 1$. The inductive hypothesis tells us that up to isomorphism, $G_2'$ can be obtained from $G_1$ using local complementations and loop complementations. Of course we can then obtain $G_2$ from $G_2'$ by loop complementation, so condition 1 holds.
If there is a looped vertex $v_0$ with $f_\beta(v_0) = (\phi\chi)$ or an unlooped vertex $v_0$ with $f_\beta(v_0) = (\phi\psi)$, then a similar argument applies, with $G'_2 = (G_2)^{v_0}_{\alpha}$. If there is a looped vertex $v_0$ with $f_\beta(v_0) = (\phi\psi)$ or an unlooped vertex $v_0$ with $f_\beta(v_0) = (\phi\chi)$, then the same argument used in the proof of Theorem 14 tells us that there is a vertex $v$ that neighbors $v_0$ in $G_2$ and has $f_\beta(v) \neq 1$. Then the graph $G'_2 = (G_2)^{v_0}_{\alpha}$ has the property that there is a compatible isomorphism $\beta' : M(IAS(G_1)) \to M(IAS(G'_2))$ such that $f_{\beta'}(w) = f_\beta(w) \forall w \notin \{v_0, v\}$, and $f_{\beta'}(v_0) = 1$. The inductive hypothesis tells us that up to isomorphism, $G'_2$ can be obtained from $G_1$ using local complementations and loop complementations; of course we can then obtain $G_2$ from $G'_2$ using local complementations.

Finally, if there is a vertex $v_0$ with $f_\beta(v_0)$ a 3-cycle then the graph $G'_2$ obtained from $G_2$ by complementing the loop status of $v_0$ has the property that there is a compatible isomorphism $\beta' : M(IAS(G_1)) \to M(IAS(G'_2))$ such that $f_{\beta'}(v) = f_\beta(v) \forall v \neq v_0$, and $f_{\beta'}(v_0) = (\chi\psi) : f_\beta(v_0)$ is a transposition. Consequently one of the preceding arguments applies to $G'_2$. ■

**Corollary 18** Let $G_1$ and $G_2$ be simple graphs. Then $G_2$ can be obtained from $G_1$ (up to isomorphism) using simple local complementations if and only if there is a compatible isomorphism $M(IAS(G_1)) \cong M(IAS(G_2))$.

### 4.5 $M(IA(G))$

If $G$ is a looped simple graph then we call the binary matroid $M(IA(G))$ the *restricted* isotropic matroid of $G$; it is represented by the $|V(G)| \times 2|V(G)|$ matrix $IA(G) = (I \mid A(G))$. This use of the term *restricted* is consistent with Bouchet’s use of the term for isotropic systems \[16\]. (The connection between isotropic matroids and isotropic systems is discussed in Section 6.)

Suppose $G_1$ and $G_2$ are looped simple graphs, and there is a compatible isomorphism $\beta : M(IAS(G_1)) \to M(IAS(G_2))$ with $f_\beta(v) \in \{1, (\phi\chi)\} \forall v \in V(G_1)$. Then $\beta(v_{\phi}) = \beta(v_{\psi}) \forall v \in V(G_1)$, so $\beta$ restricts to an isomorphism between the submatroids $M(IA(G_1))$ and $M(IA(G_2))$. Moreover, this restriction of $\beta$ is compatible with the natural partitions of these matroids into pairs, and the restriction determines $\beta$.

**Corollary 15** implies that a simple graph is classified up to pivot equivalence by compatible isomorphisms of $M(IA(G))$. Similarly, Corollary 16 implies that a looped simple graph is classified up to PPT equivalence by compatible isomorphisms of $M(IA(G))$.

A compatible isomorphism $\beta : M(IAS(G_1)) \to M(IAS(G_2))$ with $f_\beta(v) \in \{1, (\phi\psi)\} \forall v$ can be analyzed by first applying loop complementation to all vertices in $G_1$ and $G_2$, and then analyzing the corresponding compatible isomorphism $\beta' : M(IAS(G'_1)) \to M(IAS(G'_2))$, which has $f_{\beta'}(v) \in \{1, (\phi\chi)\} \forall v \in V(G'_1)$. Compatible isomorphisms with $f_\beta(v) \in \{1, (\chi\psi)\} \forall v$ are less interesting, as Theorem 8 tells us that they can be realized using loop complementation.
5 Non-compatible isomorphisms

The discussion of Section 4 relies on the fact that transpositions of the symbols $\phi, \chi, \psi$ describe the effects on isotropic matroids of local complementations, loop complementations, and edge pivots. If $G_1$ and $G_2$ are not related by these graph operations then it might seem possible for $M(\text{IAS}(G_1))$ and $M(\text{IAS}(G_2))$ to be isomorphic, so long as there is no isomorphism compatible with the canonical partitions. In fact, however, this is impossible:

**Theorem 19** Let $G_1$ and $G_2$ be looped simple graphs. If there is an isomorphism between $M(\text{IAS}(G_1))$ and $M(\text{IAS}(G_2))$, then there is a compatible isomorphism between them.

Combining Theorem 19 with Theorem 17 and Corollary 18, we deduce the following.

**Corollary 20** Let $G_1$ and $G_2$ be looped simple graphs. Then $M(\text{IAS}(G_1))$ is isomorphic to $M(\text{IAS}(G_2))$ if and only if up to isomorphism, $G_2$ can be obtained from $G_1$ using local complementations and loop complementations.

**Corollary 21** Let $G_1$ and $G_2$ be simple graphs. Then $M(\text{IAS}(G_1))$ is isomorphic to $M(\text{IAS}(G_2))$ if and only if up to isomorphism, $G_2$ can be obtained from $G_1$ using simple local complementations.

5.1 Triangulations of isotropic matroids

We prove Theorem 19 by carefully analyzing the image of the canonical partition of $W(G_1)$ under a non-compatible isomorphism $M(\text{IAS}(G_1)) \rightarrow M(\text{IAS}(G_2))$. This image satisfies the following.

**Definition 22** Let $G$ be a looped simple graph. A partition $P$ of $W(G)$ into three-element cells is a triangulation if each cell contains either a 3-element circuit of $M(\text{IAS}(G))$ or a loop and a pair of non-loop parallels.

The canonical partition of $W(G)$ is a triangulation, of course. The simplest non-canonical triangulations of $W(G)$ are obtained from the canonical partition by interchanging parallel elements of $M(\text{IAS}(G))$. It is not difficult to see that all parallels in $M(\text{IAS}(G))$ are associated with pendant or twin vertices.

If $v$ is an unlooped degree-one vertex pendant on $w$ then the $v_\chi$ and $w_\phi$ columns of $\text{IAS}(G)$ are the same, so $v_\chi$ and $w_\phi$ are parallel elements of $M(\text{IAS}(G))$; consequently interchanging $v_\chi$ and $w_\phi$ transforms the canonical partition into a non-canonical triangulation of $W(G)$. Similarly, if $v$ is a looped degree-one vertex pendant on $w$ then a non-canonical triangulation of $W(G)$ is obtained from the canonical partition by interchanging $v_\phi$ and $w_\psi$.

If $v$ and $w$ are unlooped, nonadjacent twin vertices – i.e., $N(v) = N(w)$ – then the $v_\chi$ and $w_\chi$ columns of $\text{IAS}(G)$ are the same, so $v_\chi$ and $w_\chi$ are parallel in $M(\text{IAS}(G))$ and there is a non-canonical triangulation of $W(G)$ that differs.
from the canonical one in that $v_\chi$ and $w_\chi$ are interchanged. If $v$ is looped, then the interchange involves $v_\psi$ instead of $v_\chi$; if $w$ is looped, the interchange involves $w_\psi$ instead of $w_\chi$. Similarly, if $v$ and $w$ are unlooped, adjacent twins – i.e., $N(v) - \{w\} = N(w) - \{v\}$ – then $v_\psi$ and $w_\psi$ are parallel in $M(IAS(G))$, and a non-canonical triangulation of $W(G)$ may be obtained from the canonical partition by interchanging $v_\psi$ and $w_\psi$. If $v$ or $w$ is looped then the parallel pair includes $v_\chi$ or $w_\chi$ instead.

Other non-canonical triangulations can be a little more complicated. Suppose $u, v, w, x$ are unlooped vertices in $G$ with $N(v) = \{u, w\}$, $N(w) = \{v, x\}$ and $N(u) - \{v\} = N(x) - \{w\}$. We say $u, v, w, x$ constitute a matched 4-path. A non-canonical triangulation of $W(G)$ may be obtained from the canonical partition by replacing the canonical cells corresponding to $u, v, w, x$ with these four cells: $\{u_\phi, v_\chi, w_\phi, x_\phi\}$, $\{v_\phi, w_\chi, x_\phi\}$, $\{u_\phi, v_\psi, x_\chi\}$ and $\{u_\chi, w_\psi, x_\psi\}$. We refer to this replacement as bending the 4-path.

Suppose $G'$ is obtained from $G$ using local complementations and loop complementations, and $u, v, w, x$ is a matched 4-path in $G$. Then we say $u, v, w, x$ is a matched 4-set in $G'$. The terminology reflects the fact that the subgraph of $G'$ induced by a matched 4-set need not be a path. For instance, a matched 4-path $u, v, w, x$ in $G$ yields a 4-cycle in $G^{uw}$; see Figure 1. The discussion of Section 4 tells us that there is an isomorphism $\beta : M(IAS(G)) \to M(IAS(G'))$ that is compatible with the canonical partitions, so a triangulation $P$ of $W(G)$ induces a triangulation $\beta(P)$ of $W(G')$. In particular, if $P$ is a non-canonical triangulation of $W(G)$ in which $u, v, w, x$ is bent, then we say that $u, v, w, x$ is a bent 4-set in $\beta(P)$.

**Proposition 23** Let $G$ be a looped simple graph, and suppose $P$ is a non-canonical triangulation of $W(G)$ obtained from the canonical partition either by bending a matched 4-set in $G$ or by interchanging two parallel elements of $M(IAS(G))$. Then there is a matroid automorphism $\alpha : M(IAS(G)) \to M(IAS(G))$ such that $\alpha(P)$ is the canonical partition.

**Proof.** If $x$ and $y$ are parallel elements of a matroid, then the transposition $(xy)$ is a matroid automorphism.

Suppose $u, v, w, x$ is a matched 4-path in $G$, and $P$ is obtained from the canonical partition by bending the 4-path. Let $\alpha : W(G) \to W(G)$ be the
permutation
\[ \alpha = (u_\phi x_\phi)(u_\chi v_\phi)(u_\psi w_\chi)(v_\chi x_\psi)(v_\psi w_\psi)(w_\phi x_\chi). \]

We claim that \( \alpha \) defines an automorphism of the matroid \( M(IAS(G)) \). As a first step in verifying the claim, recall that \( \Phi = \{t_\phi \mid t \in V(G)\} \) is a basis of \( M(IAS(G)) \). The image \( \alpha(\Phi) = \{u_\chi, x_\chi\} \cup \{t_\phi \mid v \neq t \neq w\} \) is clearly a spanning set of \( M(IAS(G)) \), as \( v_\phi \) and \( w_\phi \) are both sums of elements of \( \alpha(\Phi) \), so \( \alpha(\Phi) \) is also a basis of \( M(IAS(G)) \). To verify the claim, it is enough to verify that \( \alpha \) preserves fundamental circuits, i.e., \( \alpha(C(z, \Phi)) = C(\alpha(z), \alpha(\Phi)) \) \( \forall z \in W(G) - \Phi \).

If \( y \in V(G) - \{u, v, w, x\} \) is unlooped and \( z = y_\chi \), or \( y \) is looped and \( z = y_\phi \), then \( \alpha(z) = z \) and \( C(z, \Phi) = \{z\} \cup \{t_\phi \mid t \in N(y)\} \). As \( u, v, w, x \) is a matched 4-path, \( u \in N(y) \) if and only if \( x \in N(y) \), and \( v, w \notin N(y) \). It follows that \( \alpha(C(z, \Phi)) = C(z, \Phi) \) and \( C(\alpha(z), \alpha(\Phi)) = C(z, \Phi) \). If \( y \) is unlooped and \( z = y_\phi \), or \( y \) is looped and \( z = y_\chi \), then \( C(z, \Phi) = \{z, y_\phi\} \cup \{t_\phi \mid t \in N(y)\} \), and again \( \alpha(C(z, \Phi)) \) and \( C(\alpha(z), \alpha(\Phi)) \) both coincide with \( C(z, \Phi) \).

The eight remaining elements \( z \in W(G) - \Phi \) are the \( \chi \) and \( \psi \) elements corresponding to \( u, v, w \) and \( x \). It is a simple matter to verify the equalities \( \alpha(C(z, \Phi)) = C(\alpha(z), \alpha(\Phi)) \) individually. This information is displayed in the table below.

| \( z \) | \( C(z, \Phi) \) | \( \alpha(z) \) | \( \alpha(C(z, \Phi)) = C(\alpha(z), \alpha(\Phi)) \) |
|---|---|---|---|
| \( u_\chi \) | \( \{t_\phi \mid t \in N(u)\} \cup \{u_\chi\} \) | \( v_\phi \) | \( \{t_\phi \mid v \neq t \notin N(u)\} \cup \{u_\chi, v_\phi\} \) |
| \( u_\phi \) | \( \{t_\phi \mid t \in N(u)\} \cup \{u_\phi, u_\psi\} \) | \( w_\chi \) | \( \{t_\phi \mid v \neq t \notin N(u)\} \cup \{w_\chi, x_\phi, u_\chi\} \) |
| \( v_\chi \) | \( \{v_\phi, u_\chi, w_\phi\} \) | \( x_\psi \) | \( \{x_\phi, x_\chi, x_\psi\} \) |
| \( v_\phi \) | \( \{u_\phi, v_\phi, v_\psi, w_\phi\} \) | \( w_\phi \) | \( \{u_\chi, w_\psi, x_\phi, x_\chi\} \) |
| \( w_\chi \) | \( \{v_\phi, w_\phi, v_\psi, x_\phi\} \) | \( u_\phi \) | \( \{u_\phi, u_\chi, u_\psi\} \) |
| \( w_\phi \) | \( \{v_\phi, w_\phi, v_\psi, x_\phi\} \) | \( v_\phi \) | \( \{u_\phi, u_\chi, v_\chi\} \) |
| \( x_\chi \) | \( \{t_\phi \mid t \in N(x)\} \cup \{x_\chi\} \) | \( w_\phi \) | \( \{t_\phi \mid w \neq t \notin N(x)\} \cup \{x_\phi, w_\phi\} \) |
| \( x_\psi \) | \( \{t_\phi \mid t \in N(x)\} \cup \{x_\phi, x_\psi\} \) | \( v_\chi \) | \( \{t_\phi \mid w \neq t \notin N(x)\} \cup \{u_\phi, v_\chi, x_\chi\} \) |

As \( \alpha(P) \) is the canonical partition, \( \alpha \) satisfies the proposition.

Suppose now that \( u, v, w, x \) is a matched 4-set in \( G \), and \( P \) is obtained from the canonical partition by bending the 4-set. Then there is a looped simple graph \( G' \) obtained from \( G \) by some sequence of local complementations and loop complementations, such that the resulting compatible automorphism \( \beta : M(IAS(G)) \to M(IAS(G')) \) has the property that \( u, v, w, x \) is a bent 4-path in \( \beta(P) \). We have just verified that there is a matroid automorphism \( \alpha : M(IAS(G')) \to M(IAS(G')) \) under which the image of \( \beta(P) \) is the canonical partition. Then \( \beta^{-1}\alpha\beta \) is an automorphism of \( M(IAS(G)) \) that satisfies the proposition. ■

5.2 Theorem [19]

Most of our proof of Theorem [19] is devoted to showing that every non-canonical triangulation of an isotropic matroid is built from the two particular types of
triangulations discussed in Proposition 23.

Lemma 24 Let $G$ be a looped simple graph, and let $P$ be a non-canonical triangulation of $W(G)$. Suppose no non-canonical cell of $P$ contains two elements of $W(G)$ that correspond to the same vertex of $G$. Then there is a sequence $\Sigma$ of local complementations and loop complementations such that the graph $G'$ obtained by applying $\Sigma$ to $G$ has an unlooped degree-2 vertex $w$ with $\beta_3^{-1}(\{w, v_\phi, v_\psi\}) \in P$. Here $N_G'(w) = \{v, x\}$ and $\beta_3 : M(IAS(G)) \to M(IAS(G'))$ is the compatible isomorphism induced by $\Sigma$.

Proof. To reduce the number of cases that must be considered, we perform loop complementations to remove all loops in $G$.

Let $v$ be a vertex of $G$ such that $\{v_\phi, v_\chi, v_\psi\} \notin P$. Then $P$ contains a cell $\{v_\phi, a_\gamma, b_\delta\}$ with $a \neq b \neq v \neq a$ and $\gamma, \delta \in \{\phi, \chi, \psi\}$. This cell is either a circuit of $M(IAS(G))$ or the union of two disjoint circuits, so the corresponding columns of $IAS(G)$ must sum to 0.

If $\gamma = \phi$ then the $b_\delta$ column of $IAS(G)$ must have nonzero entries in the $a$ and $v$ columns, and not in any other columns; necessarily then $\delta = \chi$ and $N(b) = \{a, v\}$. Similarly, if $\delta = \phi$ then $\gamma = \chi$ and $N(a) = \{b, v\}$.

Suppose now that $\gamma = \psi$. The $a$ entry of the $v_\phi$ column of $IAS(G)$ is 0, and the $a$ entry of the $v_\psi$ column is 1, so the $a$ entry of the $b_\delta$ column must be 1. It follows that $a$ and $b$ are neighbors in $G$, so the $b$ entry of the $a_\psi$ column of $IAS(G)$ is 1. Then the $b$ entry of the $b_\delta$ column must also be 1, so $\delta = \psi$. The $v$ entry of the $v_\psi$ column of $IAS(G)$ is 1, so precisely one of $a, b$ is a neighbor of $v$; say $a \in N(v)$ and $b \notin N(v)$. All in all, we have $\gamma = \delta = \psi$, $v \in N(a)$ and $N(b) = (N(a) \cup \{a\}) - \{b, v\}$. It follows that in $G'_v$, $a$ is an unlooped degree-2 vertex whose only neighbors are $b$ and $v$. Theorems 8 and 11 tell us that there is a compatible isomorphism $\beta : M(IAS(G)) \to M(IAS(G'_v))$ whose associated map $f_\beta : V(G) \to \Sigma_3$ has $f_\beta(v) = 1, f_\beta(a) = (\chi \psi)$ and $f_\beta(b) = (\phi \psi)$, so $\beta(\{v_\phi, a_\gamma, b_\delta\}) = \{v_\phi, a_\chi, b_\delta\}$.

Finally, suppose $\gamma = \chi$. The $a$ entry of the $v_\phi$ column of $IAS(G)$ is 0, and the $a$ entry of the $a_\chi$ column is 0, so the $a$ entry of the $b_\delta$ column must be 0. It follows that $a$ and $b$ are not neighbors in $G$, so the $b$ entry of the $a_\chi$ column of $IAS(G)$ is 0. Then the $b$ entry of the $b_\delta$ column must also be 0, so $\delta = \chi$. The $v$ entry of the $v_\phi$ column of $IAS(G)$ is 1, so precisely one of $a, b$ is a neighbor of $v$; say $a \in N(v)$ and $b \notin N(v)$. All in all, we have $\gamma = \delta = \chi$, $v \in N(a)$ and $N(b) = N(a) - \{v\}$. If $N(b)$ is empty then the only columns of $IAS(G)$ with nonzero entries in the $b$ row are the $b_\phi$ and $b_\psi$ columns, so $b_\phi$ and $b_\psi$ must appear together in a cell of $P$; this cell must be non-canonical as it does not contain $b_\chi$, so it violates the hypothesis that no non-canonical cell of $P$ contains two elements of $W(G)$ corresponding to the same vertex of $G$. By contradiction, then, $N(b)$ is not empty. If $y \in N(b)$ then Theorems 8 and 11 tell us that there is a compatible isomorphism $\beta : M(IAS(G)) \to M(IAS((G'_x)_y))$ whose associated map $f_\beta : V(G) \to \Sigma_3$ has $f_\beta(v) = (\chi \psi)$, $f_\beta(a) = (\phi \psi)(\chi \psi) = (\phi \psi \chi)$ and $f_\beta(b) = (\chi \psi)^2 = 1$, so $\beta(\{v_\phi, a_\gamma, b_\delta\}) = \{v_\phi, a_\psi, b_\chi\}$. ■
Lemma 25 Let $G$ be a looped simple graph, and let $P$ be a non-canonical triangulation of $W(G)$. Suppose no non-canonical cell of $P$ contains two elements of $W(G)$ that correspond to the same vertex of $G$. Then either there is a bent 4-set in $P$, or there is a bent 4-set in a non-canonical triangulation $P'$ obtained from $P$ by interchanging two parallel elements of $M(IAS(G))$.

**Proof.** By Lemma 24 after local complementations and loop complementations we may presume that $G$ has no looped vertex, and that $P$ includes a cell $\{w_\chi, v_\phi, x_\phi\}$. Then $P$ also includes a cell $\{w_\phi, y_\gamma, z_\delta\}$ with $w \neq y \neq z \neq w$. As the $w$ entry of the $w_\phi$ column of $IAS(G)$ is 1, either the $y_\gamma$ or the $z_\delta$ column also has its $w$ entry equal to 1; say it is the $z_\delta$ column. Then $\delta \in \{x, y\}$ and $z \in N(w)$, so $z \in \{v, x\}$; say $z = v$. Notice that if $y = x$ then $\gamma \neq \phi$, as $x_\phi$ appears in the cell $\{w_\chi, v_\phi, x_\phi\}$; but then every element of $\{w_\phi, y_\gamma, z_\delta\}$ corresponds to a column of $IAS(G)$ whose $w$ entry is 1, an impossibility as $\{w_\phi, y_\gamma, z_\delta\}$ is a circuit or a disjoint union of circuits in $M(IAS(G))$. Consequently $y \neq x$.

Summing up: $P$ contains the cells $\{w_\chi, v_\phi, x_\phi\}$ and $\{w_\phi, y_\gamma, z_\delta\}$ with $N(w) = \{v, x\}$ and $y \notin \{v, w, x\}$.

Case 1: If $\gamma = \phi$ then since $\{w_\phi, y_\phi, z_\delta\}$ is a cell of $P$, it must be that $\delta = \chi$ and $N(v) = \{w, y\}$. Among the elements of $W(G)$ not included in $\{w_\phi, y_\phi, v_\chi\}$ or $\{w_\chi, v_\phi, x_\phi\}$, only four correspond to columns of $IAS(G)$ with nonzero $w$ entries, namely $v_\psi$, $w_\psi$, $x_\chi$ and $x_\phi$; two of these must appear in each of two cells of $P$. Similarly, among the elements of $W(G)$ not included in $\{w_\phi, y_\phi, v_\chi\}$ or $\{w_\chi, v_\phi, x_\phi\}$, only four correspond to columns of $IAS(G)$ with nonzero $v$ entries, namely $v_\psi$, $w_\psi$, $y_\chi$ and $y_\phi$; two of these must appear in each of two cells of $P$. By hypothesis, $x_\chi$ and $x_\phi$ do not appear in the same cell of $P$; nor do $y_\chi$ and $y_\phi$. Consequently $P$ has two cells of the form

$$\{\text{one of } x_\chi, x_\phi\} \cup \{\text{one of } y_\chi, y_\phi\} \cup \{\text{one of } v_\psi, w_\psi\}.$$  

As $N(v) \cup N(w) = \{v, w, x, y\}$ and the sum of the columns of $IAS(G)$ corresponding to a cell of $P$ must be 0, it follows that $N(x) - \{v, w, x, y\} = N(y) - \{v, w, x, y\}$.

If $x$ and $y$ are not adjacent in $G$ then among the elements $x_\chi$, $x_\phi$, $y_\chi$, $y_\phi$, $v_\psi$ and $w_\psi$, the only ones that correspond to columns of $IAS(G)$ with nonzero $x$ entries are $x_\phi$ and $w_\psi$, so they must appear in the same cell. Similarly, $y_\phi$ and $v_\psi$ must appear in the same cell. Consequently $\{x_\chi, w_\psi, y_\chi\}$ and $\{x_\phi, v_\psi, y_\phi\}$ are cells of $P$, so $y, v, w, x$ is a bent 4-path in $P$.

On the other hand, if $x$ and $y$ are adjacent in $G$ then among the elements $x_\chi$, $x_\phi$, $y_\chi$, $y_\phi$, $v_\psi$ and $w_\psi$, the only ones that correspond to columns of $IAS(G)$ with $x$ entries equal to 0 are $x_\chi$ and $v_\psi$; these cannot appear in the same cell of $P$. Similarly, the only ones that correspond to columns of $IAS(G)$ with $y$ entries equal to 0 are $y_\chi$ and $w_\psi$; and these cannot appear in the same cell. Consequently $\{x_\chi, w_\psi, y_\phi\}$ and $\{x_\phi, y_\chi, v_\psi\}$ are cells of $P$. In this situation $y, v, w$ and $x$ are the vertices of a 4-cycle of $G$, in this order, with $v$ and $w$ of degree two and $N(x) - \{w, y\} = N(y) - \{v, x\}$. Then $y, w, v, x$ is a matched 4-path in $G^{vw}$. Corollary 12 tells us that there is a compatible isomorphism $\beta : M(IAS(G) \to M(IAS(G^{vw})))$ whose associated map $f_\beta : V(G) \to S_3$ has

16
formulation, we may presume that \( tw_2, \) and \( \psi \) of \( N \) is an entire connected component of \( N \). Consequently \( y \neq 0 \) if and only if \( y \) is a nonzero \( \phi \) and \( \gamma \) is a cell of \( P \). Consequently \( y \) does not contain \( y \) and \( y \) must appear in the same cell of \( P \). This cell does not contain \( y \), so it is not a canonical cell; but no such cell exists, by hypothesis. Consequently \( N(y) \) is a cell of \( P \).

If \( x \neq u \in N(y) \) then \( u \notin \{x, v\} = N(w) \), so Theorems 8 and 10 tell us that there is a compatible isomorphism \( \beta : M(IAS(G)) \to M(IAS(G_w^u)) \) whose associated map \( f_\beta : V(G) \to S_3 \) has \( f_\beta(y) = (\psi, \chi) \), \( f_\beta(z) = (\chi, \psi) \) for \( z \in N(y) \), and \( f_\beta(z) = (\psi, \chi, \psi) \) for \( z \notin N(y) \). As \( w \notin N(y) \), it follows that \( \beta(P) \) is a triangulation of \( W(G_w^u) \) that contains the columns \( \{w_\phi, y_\phi, v_\phi\} \) and \( \{w_\psi, x_\psi, \phi\} \). That is, Case 1 holds in \( G_w^u \).

It remains to consider the possibility that \( N(y) = \{x\} \). Then the only columns of \( IAS(G) \) with nonzero \( y \) entries are those corresponding to \( y_\psi, y_\psi, x_\psi \) and \( x_\psi \), so there must be two cells of \( P \) each of which contains one of \( y_\psi, y_\psi \) and one of \( x, x_\psi \). Also, the fact that \( \{w_\phi, y_\phi, v_\phi\} \) is a cell of \( P \) implies that \( N(v) = \{w, x\} \); hence the only columns of \( IAS(G) \) with nonzero \( v \) entries are those corresponding to \( v_\psi, v_\psi, w_\psi, x_\psi \). As \( \{w_\phi, y_\phi, v_\phi\} \) is a cell of \( P \) there must be two cells of \( P \) each of which contains one of \( v_\psi, w_\psi \) and one of \( x_\psi, x_\psi \). Consequently \( P \) has two cells of the form

\[
\text{one of } \{x, x_\psi\} \cup \text{one of } \{y_\phi, y_\psi\} \cup \text{one of } \{v_\psi, w_\psi\}.
\]

The columns of \( IAS(G) \) corresponding to \( v_\psi \) and \( w_\psi \) both have nonzero \( x \) entries, so \( x_\psi \) and \( y_\psi \) cannot appear in the same cell. Consequently these two cells are

\[
\{x_\psi, y_\psi\} \cup \text{one of } \{v_\psi, w_\psi\} \cup \{x_\psi, y_\psi\} \cup \text{one of } \{v_\psi, w_\psi\}.
\]

It follows that \( N(x) = \{v, w, y\} \) and the subgraph of \( G \) induced by \( \{v, w, x, y\} \) is an entire connected component of \( G \). See Figure 2.

Notice that \( N(v) = \{w, x\} \) and \( N(w) = \{v, x\} \), so \( v \) and \( w \) are adjacent twins, and \( v_\psi \) and \( w_\psi \) are parallel in \( M(IAS(G)) \). Interchanging \( v_\psi \) and \( w_\psi \) if necessary, we may presume that \( \{x_\psi, y_\psi, v_\psi\} \) and \( \{x_\psi, y_\psi, w_\psi\} \) are both cells of \( P \). Theorems 8 and 10 tell us that there is a compatible isomorphism
\( \beta : M(IAS(G)) \to M(IAS(G^w)) \) whose associated map \( f_\beta : V(G) \to S_3 \)
has \( f_\beta(w) = (\phi \psi), \ f_\beta(v) = f_\beta(x) = (\chi \psi) \) and \( f_\beta(z) = 1 \)
for \( z \notin \{v, w, x\} \).
Consequently \( \beta(P) \) contains \( \beta([w_\chi, v_\phi, x_\phi]) = \{w_\chi, v_\phi, x_\phi\}, \ \beta([w_\chi, y_\psi, v_\psi]) = \{w_\chi, y_\psi, v_\psi\} \) and \( \beta([x_\phi, y_\psi, w_\psi]) = \{x_\phi, y_\psi, w_\psi\} \).
It follows that \( v, w, x, y \) is a bent 4-path in \( \beta(P) \).

**Definition 26** If \( G \) is a looped simple graph and \( P \) is a triangulation of \( W(G) \) then the index of \( P \) is \( \|P\| = |\{\text{non-canonical cells of } P\}| \).

**Proposition 27** Let \( P \) be a non-canonical triangulation of \( W(G) \). Then there are an integer \( k \in \{1, ..., \|P\|\} \), a sequence \( G = H_0, ..., H_k \) of graphs and a sequence \( P = P_0, ..., P_k \) of triangulations such that:

1. If \( 1 \leq i \leq k \) then \( H_i \) is obtained from \( H_{i-1} \) through some (possibly empty) sequence of local complementations and loop complementations.
2. If \( 1 \leq i < k \) then \( P_i \) is a non-canonical triangulation of \( W(H_i) \).
3. \( P_k \) is the canonical partition of \( W(H_k) \).
4. If \( 1 \leq i \leq k \) then \( \|P_i\| \in \{\|P_{i-1}\|, \|P_{i-1}\| - 1, \|P_{i-1}\| - 2, \|P_{i-1}\| - 4\} \).
5. If \( \|P_i\| \in \{\|P_{i-1}\|, \|P_{i-1}\| - 1, \|P_{i-1}\| - 2\} \) then \( P_i \) is obtained from \( P_{i-1} \) by interchanging two parallel elements of \( M(IAS(H_{i-1})) \).
6. If \( \|P_i\| = \|P_{i-1}\| \), then \( i < k \) and \( \|P_{i+1}\| = \|P_i\| - 4 \).
7. If \( \|P_i\| = \|P_{i-1}\| - 4 \), then \( P_i \) is obtained from \( P_{i-1} \) by replacing the cells corresponding to a bent 4-set with the four corresponding canonical cells.

**Proof.** Suppose \( v \) is a vertex of \( G \) such that \( \{v_\phi, v_\chi, v_\psi\} \notin P \) and two of \( v_\phi, v_\chi, v_\psi \) appear together in a single cell of \( P \). Then the third element of this cell is parallel to the third of \( v_\phi, v_\chi, v_\psi \). Interchanging these two parallels transforms this cell into the canonical cell corresponding to \( v \), and may also transform another cell of \( P \) into a canonical cell, so the resulting triangulation \( P' \) has \( \|P'\| \in \{\|P\| - 1, \|P\| - 2\} \). If there is no such vertex \( v \), then Lemma 25 applies.

Propositions 23 and 27 tell us that if \( P \) is a non-canonical triangulation of \( M(IAS(G)) \), then there is an automorphism \( \alpha_P \) of \( M(IAS(G)) \) such that \( \alpha_P(P) \) is the canonical partition. Theorem 19 follows, for if \( \gamma : M(IAS(G_1)) \to \)
\(M(IAS(G_2))\) is a non-compatible isomorphism and \(P\) is the image of the canonical partition of \(W(G_1)\) under \(\gamma\), then \(\alpha_P \circ \gamma : M(IAS(G_1)) \to M(IAS(G_2))\) is a compatible isomorphism.

6 Delta-matroids and isotropic systems

The results of this paper show that the theory of binary matroids contains “conceptual imbeddings” of the theories of graphic delta-matroids and isotropic systems, two interesting and useful theories studied by Bouchet in the 1980s and 1990s. Bouchet later introduced a third theory, involving multimatroids, to unify these two. Using terminology of [15], we can summarize our “conceptual imbeddings” by saying two things. First, if \(G\) is a looped simple graph then \(M(IAS(G))\) is a binary matroid that shelters the 3-matroid associated with an isotropic system with fundamental graph \(G\), and the submatroid \(M(IA(G))\) shelters the 2-matroid associated with a delta-matroid with fundamental graph \(G\). (“Sheltering” is a way of containing; that’s why we use the term “imbedding.”) Second, matroidal properties of \(M(IAS(G))\) provide new explanations of the properties of graphic delta-matroids and isotropic systems; that’s what makes the imbeddings “conceptual.” For instance, compatible isomorphisms of isotropic matroids provide a new explanation of the significance of isotropic systems, using the fact that certain kinds of basis exchanges correspond to local complementations. Compatible isomorphisms also provide a new way to conceptualize the work of Brijder and Hoogeboom [17] on the connection between \(S_3\) and certain operations on delta-matroids.

**Definition 28** [11] If \(G\) is a looped simple graph, then the delta-matroid associated to \(G\) is

\[
D(G) = \{ S \subseteq V(G) \mid \text{the submatrix } A(G)[S] \text{ is nonsingular over GF}(2) \}.
\]

Here \(A(G)[S]\) denotes the principal submatrix of \(A(G)\) obtained by removing all rows and columns corresponding to vertices \(v \notin S\). Observe that

\[
D(G) = \{ S \subseteq V(G) \mid \{ s_x \mid s \in S \} \cup \{ v_\psi \mid v \notin S \} \text{ is a basis of } M(IAS(G)) \},
\]

so the matroid \(M(IAS(G))\) determines \(D(G)\). (The index \(\psi\) does not appear in this description of \(D(G)\), so the submatroid \(M(IA(G))\) actually contains enough information to determine \(D(G)\).) Moreover, if \(G_1\) and \(G_2\) are looped simple graphs and there is a compatible isomorphism \(\beta : M(IAS(G_1)) \to M(IAS(G_2))\) with \(f_\beta(v)(\psi) = \psi \forall v \in V(G_1)\), then the set \(X = \{ v \in V(G_1) \mid f_\beta(v) \neq 1 \}\) has the property that

\[
D(G_2) = \{ S \Delta X \mid S \in D(G_1) \}.
\]

Consequently, the significance of symmetric difference (also called “twisting”) for the theory of graphic delta-matroids follows from the results of Section 4, regarding compatible isomorphisms of \(M(IAS(G))\) and \(M(IA(G))\).

It takes a little more work to see how \(M(IAS(G))\) determines an isotropic system.
Definition 29 If $G$ is a looped simple graph then the sub-transversals of $W(G)$ are the elements of $S(W(G)) = \{ S \subseteq W(G) \mid |S \cap \{v_\phi, v_\chi, v_\psi\}| \leq 1 \ \forall v \in V(G)\}$.

The following is easily proven.

Proposition 30 $S(W(G))$ is a $GF(2)$-vector space with addition $S \oplus T$ defined as follows.

1. If $v \in V(G)$ and $S \cap \{v_\phi, v_\chi, v_\psi\} = \emptyset$, then $(S \oplus T) \cap \{v_\phi, v_\chi, v_\psi\} = T \cap \{v_\phi, v_\chi, v_\psi\}$.
2. If $v \in V(G)$ and $T \cap \{v_\phi, v_\chi, v_\psi\} = \emptyset$, then $(S \oplus T) \cap \{v_\phi, v_\chi, v_\psi\} = S \cap \{v_\phi, v_\chi, v_\psi\}$.
3. If $v \in V(G)$ and $S \cap \{v_\phi, v_\chi, v_\psi\} = T \cap \{v_\phi, v_\chi, v_\psi\}$, then $(S \oplus T) \cap \{v_\phi, v_\chi, v_\psi\} = \emptyset$.
4. If $v \in V(G)$ and $S \cap \{v_\phi, v_\chi, v_\psi\}$ and $T \cap \{v_\phi, v_\chi, v_\psi\}$ are nonempty and distinct, then the only element of $(S \oplus T) \cap \{v_\phi, v_\chi, v_\psi\}$ is the one element of $\{v_\phi, v_\chi, v_\psi\}$ that is not included in either $S$ or $T$.

Recall that the power set $P(W(G))$ is an algebra over $GF(2)$, with symmetric difference used for addition and intersection used for multiplication. Let $\sigma : P(W(G)) \rightarrow S(W(G))$ be the $GF(2)$-linear map with $\sigma(\{x\}) = \{x\} \ \forall x \in W(G)$. Then for each $v \in V(G)$, $\{\emptyset, \sigma(\{v_\phi\}), \sigma(\{v_\chi\}), \sigma(\{v_\psi\})\}$ is a subspace of $S(W(G))$; and $S(W(G))$ is the direct product of these subspaces.

Recall also that the cycle space $Z(M(IAS(G)))$ is the $GF(2)$-subspace of $P(W(G))$ consisting of the subsets of $W(G)$ corresponding to sets of columns of $IAS(G)$ that sum to 0.

Definition 31 Let $G$ be a looped simple graph. A transverse cycle of $G$ is an element of $L(G) = S(W(G)) \cap Z(M(IAS(G)))$.

Lemma 32 Suppose $S \in S(W(G))$, and let $S_\phi = \{ v \in V(G) \mid v_\phi \in S \}$, $S_\chi = \{ \text{looped } v \in V(G) \mid v_\chi \in S \}$, $S_\psi = \{ \text{looped } v \in V(G) \mid v_\psi \in S \}$, $S_\chi = \{ \text{unlooped } v \in V(G) \mid v_\chi \in S \}$, and $S_\psi = \{ \text{unlooped } v \in V(G) \mid v_\psi \in S \}$. Then $S \in L(G)$ if and only if the following conditions are met:

1. For every $v \in S_\chi \cup S_\psi$, $|N(v) \cap (S - S_\phi)|$ is even.
2. For every $v \in S_\phi \cup S_\chi \cup S_\psi$, $|N(v) \cap (S - S_\phi)|$ is odd.
3. For every $v \in V(G)$ with $\{v_\phi, v_\chi, v_\psi\} \cap S = \emptyset$, $|N(v) \cap (S - S_\phi)|$ is even.

Proof. $S$ is a transverse cycle of $G$ if and only if the sum of the columns of $IAS(G)$ included in $S$ is 0. That is, $S \in L(G)$ if and only if for every $v \in V(G)$, $S$ contains an even number of columns of $IAS(G)$ with 1s in the $v$ row. ■
Proposition 33 Let \( \Phi(G) = \{ v_\varphi \mid v \in V(G) \} \), and let \( \Psi(G) = \{ v_\psi \mid v \in V(G) \text{ is looped} \} \cup \{ v_\chi \mid v \in V(G) \text{ is unlooped} \} \). Then

\[
\mathcal{L}(G) = \{ \sigma(X \cdot \Psi(G)) \oplus \sigma(N(X) \cdot \Phi(G)) \mid X \subseteq V(G) \}.
\]

Proof. This follows immediately from Lemma 32 with \( X = S - S_\varphi \). □

As \( \sigma\Phi(G) \) and \( \sigma\Psi(G) \) are disjoint elements of \( \mathcal{S}(W(G)) \), and each is of size \( |V(G)| \), they satisfy Bouchet’s definition of supplementary vectors [12]. It follows from Proposition 33 that \( \mathcal{L}(G) \) is an isotropic system with fundamental graph \( G \). The basic theorem of isotropic systems – that two simple graphs are locally equivalent if and only if they are fundamental graphs of strongly isomorphic isotropic systems – now follows immediately from Corollary 18.

It is worth taking a moment to observe that even though \( \mathcal{L}(G) \) includes only the transverse cycles of \( G \), it contains enough information to determine \( G \), and hence also \( M(IAS(G)) \). The reason is simple: For each \( v \in V(G) \), \( \mathcal{L}(G) \) contains precisely one transverse cycle \( \zeta_v \subset \{ v_\chi, v_\psi \} \cup \{ w_\varphi \mid v \neq w \in V(G) \} \).

The open neighborhood of \( v \) is \( N(v) = \{ w \mid w_\varphi \in \zeta_v \} \), and \( v \) is looped if and only if \( v_\psi \in \zeta_v \).

Before proceeding, we take another moment to expand on the following comment of Bouchet [15]:

The theory of isotropic systems can be considered as an extension of the theory of binary matroids, whereas delta-matroids extend arbitrary matroids. However delta-matroids do not generalize isotropic systems.

Jaeger showed that every binary matroid can be represented by some symmetric \( GF(2) \)-matrix, or equivalently, by the adjacency matrix of some looped simple graph [26]. (This result is also discussed in [18].) It follows that every binary matroid can be extended to some isotropic matroid. As the theory of isotropic systems is equivalent to the theory of isotropic matroids, this confirms the first part of Bouchet’s comment. On the other hand, all isotropic matroids are binary so the theory of isotropic systems can also be considered to be a subset of the theory of binary matroids, rather than an extension.

The second sentence of Bouchet’s comment seems questionable. If \( G \) is a looped simple graph then \( G \) is completely determined by \( D(G) \): a vertex \( v \) is looped if and only \( \{ v \} \in D(G) \), two looped vertices \( v \) and \( w \) are adjacent if and only \( \{ v, w \} \notin D(G) \), and otherwise two vertices \( v \) and \( w \) are adjacent if and only \( \{ v, w \} \in D(G) \). Consequently, \( D(G) \) also determines the isotropic systems with fundamental graph \( G \), up to strong isomorphism. All isotropic systems have fundamental graphs, and there are non-graphic delta-matroids, so it would certainly seem that in a sense, delta-matroids do generalize isotropic systems.
7 Some properties of isotropic matroids

In this section we mention several basic properties of isotropic matroids. One basic property was noted above: every binary matroid is a submatroid of some isotropic matroid. A second basic property relates the connected components of a graph to the components of its isotropic matroid.

**Theorem 34** If $G$ is a looped simple graph with connected components $G_1, ..., G_c$ then

$$M(\text{IAS}(G)) = \bigoplus_{i=1}^{c} M(\text{IAS}(G_i)).$$

Moreover, $M(\text{IAS}(G_i))$ is a connected matroid unless $|V(G_i)| = 1$, in which case $M(\text{IAS}(G_i))$ has two components, a loop and a pair of parallel non-loops.

**Proof.** Notice that if $S$ is a set of columns of $\text{IAS}(G)$ corresponding to vertices from one connected component $G_i$, then every nonzero entry of an element of $S$ occurs in a row corresponding to a vertex of $G_i$. Consequently if $C$ is a set of columns of $\text{IAS}(G)$ whose sum is 0, then the subsets $C_i = \{x \in C \mid x$ corresponds to a vertex of $G_i\}$ sum to 0 individually. If $C$ corresponds to a circuit of $M(\text{IAS}(G))$ then the minimality of $C$ implies that only one of these $C_i$ can be nonempty, so $C = C_i$. Thus every circuit of $M(\text{IAS}(G))$ is contained in some submatroid $M(\text{IAS}(G_i))$, so $M(\text{IAS}(G))$ is the direct sum of these submatroids.

If $v$ is an isolated vertex of $G$ then one of $v_\chi, v_\psi$ corresponds to a column of zeroes in $\text{IAS}(G)$, and hence to a loop in $M(\text{IAS}(G))$. The column of $\text{IAS}(G)$ corresponding to the other of $v_\chi, v_\psi$ equals the column corresponding to $v_\phi$. As this column is nonzero, the corresponding elements are parallel non-loops.

It remains to prove that if $|V(G_i)| > 1$, then $M(\text{IAS}(G_i))$ is a connected matroid. If $v \in V(G_i)$ then the columns of $\text{IAS}(G)$ corresponding to $v_\phi, v_\chi, v_\psi$ are nonzero, and sum to 0; hence $\{v_\phi, v_\chi, v_\psi\}$ is a circuit of $M(\text{IAS}(G_i))$. Let $\Phi$ denote the basis $\{w_\phi \mid w \in V(G_i)\}$ of $M(\text{IAS}(G_i))$. If $v$ neighbors $w$ in $G_i$ then $w_\phi$ and $v_\chi$ are both elements of the fundamental circuit $C(v_\chi, \Phi)$, so $\{v_\phi, v_\chi, v_\psi\}$ and $\{w_\phi, w_\chi, w_\psi\}$ are contained in the same component of $M(\text{IAS}(G_i))$. As this holds for all neighbors and $G_i$ is connected, we conclude that all elements of $M(\text{IAS}(G_i))$ lie in the same component.

**Corollary 35** $M(\text{IAS}(G))$ is a connected matroid if and only if $|V(G)| \geq 2$ and $G$ is a connected graph.

7.1 Minors

Given the discussion of Section 6, it is no surprise that some properties of isotropic matroids are suggested by properties of delta-matroids and isotropic systems. For instance, local complementation and vertex deletion are connected to matroid minor operations in much the same way as they are connected to the minor operations of isotropic systems [10, Section 8]. Establishing these
connections is somewhat easier here, though, because the arguments require
only elementary linear algebra.

**Theorem 36** Let $v$ be a vertex of a looped simple graph $G$. Then

$$M(IAS(G - v)) = (M(IAS(G))/v_\phi) - v_\chi - v_\psi.$$  

**Proof.** If $r$ is the rank function of $M(IAS(G))$ then the rank function $r'$
of $(M(IAS(G))/v_\phi) - v_\chi - v_\psi$ is defined by

$$r'(S) = r(S \cup \{v_\phi\}) - 1 \forall S \subseteq W(G) - \{v_\phi, v_\chi, v_\psi\}.$$  

As the only nonzero entry of the $v_\phi$ column of $M(IAS(G))$ is a 1 in the $v$ row,
it is a simple matter to use elementary column operations to verify that $r'(S)$
is the $GF(2)$-rank of the submatrix of $M(IAS(G))$ that involves all rows other
than the $v$ row, and all columns corresponding to elements of $S$. That is, $r'(S)$
is the same as the rank of $S$ in $M(IAS(G - v))$. □

Combining Theorem 36 with Theorem 10 and Corollary 12, we deduce the following.

**Corollary 37** Let $v$ be a vertex of a looped simple graph $G$. Then

$$M(IAS(G_{ns})^w - v)) = \begin{cases} (M(IAS(G))/v_\psi) - v_\phi - v_\chi, & \text{if } v \text{ is not looped in } G \\ (M(IAS(G))/v_\chi) - v_\phi - v_\psi, & \text{if } v \text{ is looped in } G \end{cases}.$$  

**Corollary 38** Let $v$ be a vertex of a looped simple graph $G$, and let $w$ be a neighbor of $v$. Then

$$M(IAS(G^{vw} - v)) \cong \begin{cases} (M(IAS(G))/v_\chi) - v_\phi - v_\psi, & \text{if } v \text{ is not looped in } G \\ (M(IAS(G))/v_\psi) - v_\phi - v_\chi, & \text{if } v \text{ is looped in } G \end{cases}.$$  

Note that $= \text{ appears in Corollary 37}$ because the compatible isomorphism
$\beta : M(IAS(G)) \to M(IAS(G_{ns}))$ of Theorem 10 has $f_\beta(x) = 1 \forall x \neq v$. 
In Corollary 38 we write $\cong$ instead because the compatible isomorphism $\beta : M(IAS(G)) \to M(IAS(G^{vw}))$ of Corollary 12 has $f_\beta(w) \neq 1$.

**Theorem 39** Let $G$ be a looped simple graph, and let $M$ be a binary matroid. Then these two statements are equivalent.

1. $M$ is isomorphic to the isotropic matroid of a graph obtained from $G$ through some sequence of local complementations, loop complementations and vertex deletions.

2. $M$ is isomorphic to a minor of $M(IAS(G))$ obtained by removing some cells of the canonical partition, each cell removed by contracting one element and deleting the other two.
Proof. Recall that if \( v \) is an isolated vertex of \( G \) and \( \{v_1, v_2, v_3\} \) is the corresponding cell of the canonical partition then \( \{v_1, v_2, v_3\} \) contains two components of \( M(\text{IAS}(G)) \), a singleton component containing a loop and a two-element component containing a pair of parallel non-loops. It follows that the result of removing these three elements by deletion and contraction is the same no matter which elements are deleted and which are contracted. According to Theorem 36 then,

\[
(M(\text{IAS}(G))/v_1) - v_2 - v_3 = M(\text{IAS}(G-v))
\]

no matter how the elements of the cell are ordered.

Using the preceding observation for isolated vertices and Theorem 36 Corollary 37 and Corollary 38 for non-isolated vertices, we deduce the equivalence asserted in the statement from Theorem 17.

Corollary 40 \( M(\text{IAS}(G)) \) is a regular matroid if and only if \( G \) has no connected component with more than two vertices.

Proof. If every connected component of \( G \) has one or two vertices, then \( M(\text{IAS}(G)) \) is a direct sum of submatroids of size three or six. The smallest binary matroids that are not regular have seven elements, so \( M(\text{IAS}(G)) \) is a direct sum of regular matroids.

On the other hand, if \( G \) has a connected component with three or more vertices then a sequence of vertex deletions can be applied to \( G \) to yield a three-vertex graph \( H \) isomorphic to a looped version of either the complete graph \( K_3 \) or the path \( P_3 \). For any such \( H, \text{IAS}(H) \) is a \( 3 \times 9 \) matrix with a submatrix whose columns can be permuted to yield

\[
\begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1
\end{pmatrix}.
\]

Consequently, the Fano matroid is a submatroid of \( M(\text{IAS}(H)) \). As \( M(\text{IAS}(H)) \) is a minor of \( M(\text{IAS}(G)) \), it follows that \( M(\text{IAS}(G)) \) is not regular.

7.2 The triangle property and strong maps

Recall Definition 31 a subtransversal of \( W(G) \) is a subset that contains no more than one element from each cell of the canonical partition. The ranks of subtransversals in \( M(\text{IAS}(G)) \) are connected to each other through the triangle property, which is part of Bouchet’s theory of isotropic systems [10, Section 9].

Theorem 41 Suppose \( r \) is the rank function of \( M(\text{IAS}(G)) \), \( S \) is a subtransversal of \( W(G) \) with \( |S| = |V(G)| - 1 \), and \( v \) is the vertex of \( G \) with \( v_\phi, v_\chi, v_\psi \notin S \). Let \( S_\phi = S \cup \{v_\phi\}, S_\chi = S \cup \{v_\chi\} \) and \( S_\psi = S \cup \{v_\psi\} \). Then one of \( S_\phi, S_\chi, S_\psi \) has rank \( r(S) \) in \( M(\text{IAS}(G)) \), and the other two have rank \( r(S) + 1 \).
Proof. Complementing the loop status of \( v \) has the effect of interchanging \( v_\phi \) and \( v_\psi \), and this interchange does not affect the statement of the theorem, so we may suppose without loss of generality that \( v \) is looped. Order the other vertices of \( G \) as \( v_1, \ldots, v_{n-1} \) in such a way that for some \( p \in \{1, \ldots, n\} \), \( v_\phi \in S \) if and only if \( i < p \). Then there is a symmetric \((n-1-p) \times (n-1-p)\) matrix \( B \) such that

\[
| B | = r(S_\phi) = r \begin{pmatrix} I & * & 0 \\ 0 & B & 0 \\ 0 & \rho & 1 \end{pmatrix}, \quad | S_\phi | = r \begin{pmatrix} I & * & * \\ 0 & B & \kappa \\ 0 & \rho & 1 \end{pmatrix}, \quad \text{and } r(S_\psi) = r \begin{pmatrix} I & * & * \\ 0 & B & \kappa \\ 0 & \rho & 0 \end{pmatrix}.
\]

Here \( r \) denotes the rank function of \( M(IAS(G)) \) and also matrix rank over \( GF(2) \); \( I \) is the \((p-1) \times (p-1)\) identity matrix; \( \rho \) is the row vector whose nonzero entries occur in columns such that \( p \leq i \leq n-1 \) and \( v_i \) neighbors \( v \); \( \kappa \) is the transpose of \( \rho \); and \( * \) indicates submatrices that do not contribute to the rank. Using elementary column operations, we deduce that

\[
r(S_\phi) = p + r(B), \quad r(S_\phi) = p - 1 + r \begin{pmatrix} B & \kappa \\ \rho & 1 \end{pmatrix}, \quad \text{and } r(S_\psi) = p - 1 + r \begin{pmatrix} B & \kappa \\ \rho & 0 \end{pmatrix}.
\]

A result mentioned by Balister, Bollobás, Cutler, and Pebody [5, Lemma 2] implies that two of the ranks \( r(S_\phi), r(S_\chi), r(S_\psi) \) are the same, and the other is one less. As each of these ranks is \( r(S) \) or \( r(S) + 1 \), the theorem follows. \( \blacksquare \)

Corollary 42 Let \( S \) be a subtransversal of \( W(G) \), and let \( v \) be a vertex of \( G \) with \( v_\phi, v_\chi, v_\psi \notin S \). Then the closure of \( S \) includes at most one of \( v_\phi, v_\chi, v_\psi \).

Proof. If \( |S| = |V(G)| - 1 \), Theorem 41 tells us that two of \( S_\phi, S_\chi, S_\psi \) have rank \( r(S) + 1 \); hence the corresponding two of \( v_\phi, v_\chi, v_\psi \) are not included in the closure of \( S \). If \( |S| < |V(G)| - 1 \), let \( S' \) be any subtransversal obtained from \( S \) by adjoining one element from each canonical cell not represented in \( S \), other than the cell corresponding to \( v \). Then the closure of \( S' \) contains the closure of \( S \) and two of \( v_\phi, v_\chi, v_\psi \) are not included in the closure of \( S' \). \( \blacksquare \)

Corollary 43 Let \( S \) and \( T \) be disjoint transversals of \( W(G) \), i.e., \( S \cap T = \emptyset \) and \( |S \cup \{v_\phi, v_\chi, v_\psi\}| = 1 = |T \cup \{v_\phi, v_\chi, v_\psi\}| \forall v \in V(G) \). For each \( v \in V(G) \) let \( v_S \) and \( v_T \) be the elements of \( S \cap \{v_\phi, v_\chi, v_\psi\} \) and \( T \cap \{v_\phi, v_\chi, v_\psi\} \), respectively. Then the function \( v_S \mapsto v_T \) defines a strong map from \( M(IAS(G))|S \) to \( (M(IAS(G))|T)^* \).

Proof. For \( A \subseteq V(G) \) let \( A_S = \{a_S \mid a \in A\} \) and \( A_T = \{a_T \mid a \in A\} \). The assertion that \( v_S \mapsto v_T \) defines a strong map is equivalent to this claim: if \( v \notin A \) and the closure of \( A_S \) in \( M(IAS(G))|S \) includes \( v_S \), then the closure of \( A_T \) in \( (M(IAS(G))|T)^* \) includes \( v_T \).

Suppose instead that the closure of \( A_S \) in \( M(IAS(G))|S \) includes \( v_S \), and the closure of \( A_T \) in \( (M(IAS(G))|T)^* \) does not include \( v_T \). A fundamental property of matroid duality is that the closure of \( A_T \) in \( (M(IAS(G))|T)^* \) does not include \( v_T \) if and only if the closure of \( T - A_T - \{v_T\} \) in \( M(IAS(G))|T \) does include
\(v_T\). It follows that the closure of \(U = A_S \cup (T - A_T - \{v_T\})\) in \(M(IAS(G))\) includes both \(v_S\) and \(v_T\). \(U\) is a subtransversal, though, so Corollary 42 tells us that its closure cannot include both \(v_S\) and \(v_T\). By contradiction, then, the claim must hold. ■

Corollary 43 may seem to be a merely technical result, but it generalizes one of the most famous situations in matroid theory. If \(H\) and \(K\) are dual graphs in the plane then they give rise to disjoint transversals \(S\) and \(T\) of \(W(G)\), where \(G\) is an interlacement graph of the medial graph shared by \(H\) and \(K\). In this case the strong map \(v_S \mapsto v_T\) is the familiar isomorphism between the bond matroid of \(H\) and the cycle matroid of \(K\). We refer to [36] for more details of the significance of isotropic matroids in the theory of 4-regular graphs.

8 Interlace polynomials and Tutte polynomials

Motivated by problems that arise in the study of DNA sequencing, Arratia, Bollobás and Sorkin introduced a one-variable graph polynomial, the \textit{vertex-nullity interlace polynomial}, in [2]. In subsequent work [3, 4] they observed that this one-variable polynomial may be obtained from the Tutte-Martin polynomial of isotropic systems studied by Bouchet [13, 16], introduced an extended two-variable version of the interlace polynomial, and observed that the interlace polynomials are given by formulas that involve the nullities of matrices over the two-element field, \(GF(2)\). Inspired by these ideas, Aigner and van der Holst [1], Courcelle [21] and the author [33, 34] introduced several different variations on the interlace polynomial theme.

All these references share the underlying presumption that although the interlace and Tutte-Martin polynomials are connected to other graph polynomials in some ways, they are in a general sense separate invariants. In this section we point out that in fact, the interlace polynomials of graphs can be derived from parametrized Tutte polynomials of isotropic matroids.

One way to define the \textit{Tutte polynomial} of \(M(IAS(G))\) is a polynomial in the variables \(s\) and \(z\), given by the subset expansion

\[
t(M(IAS(G))) = \sum_{T \subseteq W(G)} s^{r^G(W(G)) - r^G(T)} z^{|T| - r^G(T)}.
\]

Here \(r^G\) denotes the rank function of \(M(IAS(G))\). We do not give a general account of this famous invariant of graphs and matroids here; thorough introductions may be found in [6, 19, 22, 25].

Tutte polynomials of graphs and matroids are remarkable both for the amount of structural information they contain and for the range of applications in which they appear. Some applications (electrical circuits, knot theory, network reliability, and statistical mechanics, for instance) involve graphs or networks whose vertices or edges have special attributes of some kind – impedances and resistances in circuits, crossing types in knot diagrams, probabilities of failure and successful operation in reliability, bond strengths in statistical mechanics. A
natural way to think of these attributes is to allow each element to carry two parameters, \( a \) and \( b \) say, with \( a \) contributing to the terms of the Tutte polynomial corresponding to subsets that include the given element, and \( b \) contributing to the terms of the Tutte polynomial corresponding to subsets that do not. Zaslavsky \([42]\) calls the resulting polynomial

\[
\sum_{T \subseteq W(G)} \left( \prod_{t \in T} a(t) \right) \left( \prod_{w \notin T} b(w) \right) s^{r^G(W(G)) - r^G(T)} z^{|T| - r^G(T)}
\]

the parametrized rank polynomial of \( M(IAS(G)) \); we denote it \( \tau(M(IAS(G))) \).

We do not give a general account of the theory of parametrized Tutte polynomials here; the interested reader is referred to the literature, for instance \([7, 23, 31, 32, 42]\). However it is worth taking a moment to observe that parametrized polynomials are very flexible, and the same information can be formulated in many ways. For instance if \( s \) and the parameter values \( b(w) \) are all invertible then formula (1) is equivalent to

\[
s^{r^G(W(G))} \cdot \left( \prod_{w \in W(G)} b(w) \right) \cdot \sum_{T \subseteq W(G)} \left( \prod_{t \in T} \left( \frac{a(t)}{b(t)s} \right) \right) (sz)^{|T| - r^G(T)},
\]

which expresses \( \tau(M(IAS(G))) \) as the product of a prefactor and a sum that is essentially a parametrized rank polynomial with only \( a \) parameters and one variable, \( sz \). We prefer formula (1), though, because we do not want to assume invertibility of the \( b \) parameters.

Suppose that the various parameter values \( a(w) \) and \( b(w) \) are independent indeterminates, and let \( P \) denote the ring of polynomials with integer coefficients in the 2+6 \( |V(G)| \) independent indeterminates \( \{s, z\} \cup \{a(w), b(w) \mid w \in W(G)\} \).

Let \( J \) be the ideal of \( P \) generated by the set of 4 \( |V(G)| \) products \( \{a(vw) a(v_x), a(v_w) a(v_x), a(v_x) a(v_w), b(v_x) b(v_w) \mid v \in V(G)\} \), and let \( \pi : P \to P/J \) be the natural map onto the quotient. Then the only summands of (1) that make nonzero contributions to \( \pi\tau(M(IAS(G))) \) correspond to transversals of the canonical partition of \( W(G) \), i.e., subsets \( T \subseteq W(G) \) with the property that \( |T \cap \{v_w, v_x, v_y\}| = 1 \forall v \in V(G) \). We denote the collection of all such transversals \( T(W(G)) \). Each \( T \in T(W(G)) \) has \( |T| = |V(G)| = r^G(W(G)) \), so \( s \) and \( z \) have the same exponent in the corresponding term of (1):

\[
\pi\tau(M(IAS(G))) = \pi \left( \sum_{T \in T(W(G))} \left( \prod_{t \in T} a(t) \right) \left( \prod_{w \notin T} b(w) \right) (sz)^{|V(G)| - r^G(T)} \right).
\]

Observe that \( \pi \) is injective when restricted to the additive subgroup \( A \) of \( P \) generated by products

\[
\left( \prod_{t \in T} a(t) \right) \left( \prod_{w \notin T} b(w) \right) (sz)^k
\]
where $k \geq 0$ and $T \in \mathcal{T}(W(G))$. Consequently there is a well-defined isomorphism of abelian groups $\pi^{-1} : \pi(A) \to A$, and we have

$$
\pi^{-1} \tau(M(\text{IAS}(G))) = \sum_{T \in \mathcal{T}(W(G))} \left( \prod_{t \in T} a(t) \right) \left( \prod_{w \notin T} b(w) \right) (sz)^{|V(G)|-r^G(T)}. \quad (2)
$$

Note that $\pi^{-1} \tau(M(\text{IAS}(G)))$, the image of the parametrized Tutte polynomial $\tau(M(\text{IAS}(G)))$ under the mappings $\pi$ and $\pi^{-1}$, might also be described as the section of $\tau(M(\text{IAS}(G)))$ corresponding to $\mathcal{T}(W(G))$. Either way, formula (2) describes an element of $P$, where $s$, $z$ and the various parameter values $a(w)$, $b(w)$ are all independent indeterminates.

Arratia, Bollobás and Sorkin [4] define the two-variable interlace polynomial $q(G)$ by the formula

$$
q(G) = \sum_{S \subseteq V(G)} (x-1)^{r(A(G)[S])} (y-1)^{|S|-r(A(G)[S])} = \sum_{S \subseteq V(G)} \left( \frac{y-1}{x-1} \right)^{|S|-r(A(G)[S])} (x-1)^{|S|}.
$$

Here $r(A(G)[S])$ denotes the $GF(2)$-rank of the principal submatrix of $A(G)$ involving rows and columns corresponding to vertices from $S$.

Let $\mathcal{T}_0(W(G)) = \{ T \in \mathcal{T}(W(G)) \mid v \notin T \forall v \in V(G) \}$, and for $T \in \mathcal{T}(W(G))$ let $S(T) = \{ v \in V(G) \mid v \notin T \}$. Then $T \mapsto S(T)$ defines a bijection from $\mathcal{T}_0(W(G))$ onto the power-set of $V(G)$. As $r^G(T)$ is the $GF(2)$-rank of the matrix

$$(\text{columns } v_\phi \text{ with } v \notin S(T) \mid \text{columns } v_\chi \text{ with } v \in S(T))$$

and the columns $v_\phi$ are columns of the identity matrix,

$$r^G(T) = |V(G)| - |S(T)| + r(A(G)[S(T)]).$$

It follows that $q(G)$ may be obtained from $\pi^{-1} \tau(M(\text{IAS}(G)))$ by setting $a(v_\phi) \equiv 1$, $a(v_\chi) \equiv x-1$, $a(v_\psi) \equiv 0$, $b(v_\phi) \equiv 1$, $b(v_\chi) \equiv 1$, $b(v_\psi) \equiv 1$, $s = y-1$ and $z = 1/(x-1)$. These assignments are not unique; for instance the values of $s$ and $z$ may be replaced by $s = (y-1)/u$ and $z = u/(x-1)$ for any invertible $u$.

The reader familiar with the Tutte-Martin polynomials of isotropic systems studied by Bouchet [13, 14] and the interlace polynomials introduced by Aigner and van der Holst [11], Courcelle [21], and the author [33, 34] will have no trouble showing that appropriate values for $s$, $z$ and the $a$ and $b$ parameters yield all of these polynomials from the parametrized rank polynomial $\tau(M(\text{IAS}(G)))$.

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