Energy Scattering for Schrödinger Equation with Exponential Nonlinearity in Two Dimensions

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Abstract

When the spatial dimensions \( n = 2 \), the initial data \( u_0 \in H^1 \) and the Hamiltonian \( H(u_0) \leq 1 \), we prove that the scattering operator is well-defined in the whole energy space \( H^1(\mathbb{R}^2) \) for nonlinear Schrödinger equation with exponential nonlinearity \( (e^{\lambda |u|^2} - 1)u \), where \( 0 < \lambda < 4\pi \).

1 Introduction

We consider the Cauchy problem for the following nonlinear Schrödinger equation

\[
iu_t + \Delta u = f(u),
\]

\[
f(u) := (e^{\lambda |u|^2} - 1)u,\]

in two spatial dimensions with initial data \( u_0 \in H^1 \) and \( 0 < \lambda < 4\pi \). Solutions of the above problem satisfy the conservation of mass and Hamiltonian

\[
M(u; t) := \int_{\mathbb{R}^2} |u|^2 dx = M(u_0),
\]

\[
H(u; t) := \int_{\mathbb{R}^2} (|\nabla u|^2 + F(u)) dx = H(u_0),
\]

where

\[
F(u) = \frac{1}{\lambda} (e^{\lambda |u|^2} - \lambda |u|^2 - 1).
\]

Nakamura and Ozawa\[16\] showed the existence and uniqueness of the scattering operator of \( (1.1) \) with \( (1.2) \). Then, Wang\[19\] proved the smoothness of this scattering operator. However, both of these results are based on the assumption of small initial data \( u_0 \). In this paper, we remove this assumption and show that for arbitrary initial data \( u_0 \in H^1(\mathbb{R}^2) \) and \( H(u_0) \leq 1 \), the scattering operator is always well-defined.

Wang et al.\[20\] proved the energy scattering theory of \( (1.1) \) with \( f(u) = (e^{\lambda |u|^2} - 1 - \lambda |u|^2 - \frac{\lambda}{2} |u|^4)u \), where \( \lambda \in \mathbb{R} \) and the spatial dimension \( n = 1 \). Ibrahim et al.\[10\] showed the existence and asymptotic completeness of the wave

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operators for (1.1) with \( f(u) = (e^{\lambda |u|^2} - 1 - \lambda |u|^2)u \) when the spatial dimensions \( n = 2, \lambda = 4\pi \) and \( H(u_0) \leq 1 \). Under the same assumptions as [10], Colliander et al. [5] proved the global well-posedness of (1.1) with (1.2):

**Theorem 1.1.** Assume that \( u_0 \in H^1(\mathbb{R}^2) \), \( H(u_0) \leq 1 \) and \( \lambda = 4\pi \). Then problem (1.1) with (1.2) has a unique global solution \( u \) in the class \( C(\mathbb{R}, H^1(\mathbb{R}^2)) \).

**Remark 1.1.** In fact, by the proof in [5], the global well-posedness of (1.1) with (1.2) is also true for \( 0 < \lambda \leq 4\pi \).

In this paper, we further study the scattering of this problem. Note that \( f(u) = (e^{\lambda |u|^2} - 1)u = \sum_{k=1}^{\infty} \frac{1}{(2k+1)!} u^{2k+2} \). Nakanishi [15] proved the existence of the scattering operators in the whole energy space \( H^1(\mathbb{R}^2) \) for (1.1) with \( f(u) = |u|^p u \) when \( p > 2 \). Then, Killip et al. [12] and Dodson [7] proved the existence of the scattering operators in \( L^2(\mathbb{R}^2) \) for (1.1) with \( f(u) = |u|^2 u \). Inspired by these two works, we use the concentration compactness method, which was introduced by Kenig and Merle in [11], to prove the existence of the scattering operators for (1.1) with (1.2).

For convenience, we write (1.1) and (1.2) together, i.e.

\[
iu + \Delta u = f(u) := (e^{\lambda |u|^2} - 1)u, \quad u(0, x) = u_0,
\]

where \( u_0 \in H^1(\mathbb{R}^2) \) and \( 0 < \lambda \leq 4\pi \). Our main result is:

**Theorem 1.2.** Assume that the initial date \( u_0 \in H^1(\mathbb{R}^2) \), \( H(u_0) \leq 1 \) and \( 0 < \lambda < 4\pi \). Let \( u \) be a global solution of (1.3). Then

\[
\|u\|_{L^4_{t,x}(\mathbb{R} \times \mathbb{R}^2)} < \infty.
\]

In Section 2, Lemma 2.4 will show us that Theorem 1.2 implies the following scattering result:

**Theorem 1.3.** Assume that the initial date \( u_0 \in H^1(\mathbb{R}^2) \), \( H(u_0) \leq 1 \) and \( 0 < \lambda < 4\pi \). Then the solution of (1.3) is scattering.

We will prove Theorem 1.2 by contradiction in Section 5. In Section 2, we give some nonlinear estimates. In Section 3, we prove the stability of solutions. In Section 4, we give a new profile decomposition for \( H^1 \) sequence which will be used to prove concentration compactness.

Now, we introduce some notations:

\[
\langle x \rangle = \sqrt{1 + |x|^2}, \quad \langle \cdot, \cdot \rangle \text{ is the inner product in } L^2(\mathbb{R}^2),
\]

\[
G(u) := \bar{u} f(u) - F(u) = e^{\lambda |u|^2} |u|^2 - \frac{1}{\lambda} (e^{\lambda |u|^2} - 1) = \sum_{k=1}^{\infty} \frac{k\lambda^k |u|^{2k+2}}{(k+1)!},
\]

\[
E = E(u; t) := M(u; t) + H(u; t).
\]

We define

\[
\|u\|_{H^s_q(\mathbb{R}^2)} := \|(I - \Delta)^{s/2} u\|_{L^q(\mathbb{R}^2)}, \quad \|u\|_{\dot{H}^s_q(\mathbb{R}^2)} := \|(-\Delta)^{s/2} u\|_{L^q(\mathbb{R}^2)}.
\]

For Banach space \( X = H^s_q(\mathbb{R}^2), \dot{H}^s_q(\mathbb{R}^2) \) or \( L^q(\mathbb{R}^2) \), we denote

\[
\|u\|_{L^p(\mathbb{R}; X)} := \left( \int_{\mathbb{R}} \|u(t)\|^p_X dt \right)^{1/p},
\]
When $q=r$, we abbreviate $L^q_tL^r_x$ as $L_t^r$. When $q$ or $r$ are infinity, or when the domain $\mathbb{R} \times \mathbb{R}^2$ is replaced by $I \times \mathbb{R}^2$, we make the usual modifications. Specially, we denote

$$S(u) := \|u\|_{L^4_t(L^8_x)}^4.$$ 

If $t_0 \in \mathbb{R}$, we split $S(u) = S_{\leq t_0}(u) + S_{\geq t_0}(u)$, where

$$S_{\leq t_0}(u) := \int_{-\infty}^{t_0} \int_{\mathbb{R}^2} |u(t,x)|^4 \, dx \, dt$$

and

$$S_{\geq t_0}(u) := \int_{t_0}^{+\infty} \int_{\mathbb{R}^2} |u(t,x)|^4 \, dx \, dt.$$ 

For any two Banach spaces $X$ and $Y$, $\| \cdot \|_{X \cap Y} := \max\{\| \cdot \|_X, \| \cdot \|_Y\}$. $C$ denotes positive constant. If $C$ depends upon some parameters, such as $\lambda$, we will indicate this with $C(\lambda)$.

**Remark 1.2.** Note that $0 < \lambda < 4\pi$ in Theorem 1.2, we only need to prove the result for $0 < \lambda < 4(1 - 4\varepsilon)\pi$, $\varepsilon \in (0, 1/8)$. Hence, we always suppose that $0 < \lambda < 4(1 - 4\varepsilon)\pi$ in the context.

Moreover, we always suppose that the initial date $u_0$ of (1.3) satisfies $u_0 \in H^1(\mathbb{R}^2)$ and $H(u_0) \leq 1$.

## 2 Nonlinear Estimates

In order to estimate (1.2), we need the following Trudinger inequality.

**Lemma 2.1.** (H) Let $\lambda \in [0, 4\pi)$. Then for all $u \in H^1(\mathbb{R}^2)$ satisfying $\|\nabla u\|_{L^2(\mathbb{R}^2)} \leq 1$, we have

$$\int_{\mathbb{R}^2} \left( e^{\lambda|u|^2} - 1 \right) \, dx \leq C(\lambda)\|u\|_{L^2(\mathbb{R}^2)}^2.$$ 

Note that for $\forall \alpha \geq 1$,

$$(e^{\lambda|u|^2} - 1)\alpha \leq e^{\lambda\alpha|u|^2} - 1.$$ 

By Lemma [2.4] and Hölder inequality, for $\lambda \in (0, 4\pi)$ and $\forall \beta \geq 0$, we have

$$\int_{\mathbb{R}^2} \left( e^{\lambda|u|^2} - 1 \right) |u|^\beta \, dx \leq \|e^{\lambda|u|^2} - 1\|_{L^{1/(1-\beta)}(\mathbb{R}^2)} \|u\|_{L^{\beta/(\beta-1)}(\mathbb{R}^2)}^\beta \leq\|u\|_{L^2(\mathbb{R}^2)}^2 \|u\|_{H^1(\mathbb{R}^2)}^\beta \leq C(\lambda, \beta)\|u\|_{L^2(\mathbb{R}^2)}^2.$$ (2.1)

and thus

$$\int_{\mathbb{R}^2} \left( e^{\lambda|u|^2} - \lambda|u|^2 - 1 \right) \, dx \leq \lambda \int_{\mathbb{R}^2} \left( e^{\lambda|u|^2} - 1 \right) |u|^2 \, dx \leq C(\lambda)\|u\|_{L^2(\mathbb{R}^2)}^2.$$ (2.2)

**Lemma 2.2.** (Strichartz estimates) For $s = 0$ or $1$,

$$2 \leq r, p < \infty, \quad \frac{1}{\gamma(p)} + \frac{1}{p} = \frac{1}{2}, \quad \frac{1}{p} + \frac{1}{p'} = 1,$$
(the pairs \((\gamma(p), p)\) were called admissible pairs) we have
\[
\|e^{i(t-t_0)\Delta}u(t_0)\|_{L_t^\gamma(x;H^p_x)} \leq C\|u(t_0)\|_{H^r(x;R^2)},
\]
(2.3)
\[
\|f(u)\|_{L_t^\gamma(x;H^p_x)} \leq C\|f(u)\|_{L_t^\gamma(x;H^p_x)},
\]
(2.4)
Lemma 2.3. (Proposition 2.3, [20]) Let \(1 < r < p < \infty\) be fixed indices. Then for any \(q \in [p, \infty)\)
\[
\|u\|_{L_t^q(x;R^2)} \leq C(p, r)q^{1/r'}q^{1/r}\|u\|_{L_t^p(x;R^2)}\|u\|_1^{1-p/q}H_{s/r'}^r(x;R^2).
\]
As is shown in [8] and [15], to obtain the scattering result, it suffices to show that any finite energy solution has a finite global space-time norm. So, if Theorem 1.2 is true, we only need to prove the following theorem.
Lemma 2.4. (Theorem 1.2 implies Theorem 1.3) Let \(u\) be a global solution of (1.3), \(\|u\|_{L_t^4(x;R^2)} < \infty\). Then, for all admissible pairs, we have
\[
\|u\|_{L_t^\gamma(x;H^p_x)} < \infty.
\]
(2.5)
Proof. Defining \(X = L^{2/(1-2\varepsilon)}(I; H^1_{1/\varepsilon}), Y = L^4(I; H^1_x)\), by Strichartz estimates, (2.1) and (2.2),
\[
\|u\|_{X \cap Y} \leq C\|S\|_{H^r(x;R^2)} + C\|\lambda|u|^2\|_{L_t^{1/r}(I;H^1_y)}
+ C\|(|\epsilon|u|^{2} - \lambda|u|^2 - 1)u\|_{L_t^{2/(1+2\varepsilon)}(I;H^1_{1/(1-\varepsilon)})}
\leq C(E) + \lambda C\|u\|_{L_t^4(I;R^2)}\|u\|_{Y}
+ \lambda C\|u\|_{L_t^4(I;R^2)}\|u\|_{X}(\|\epsilon|u|^{2} - \lambda|u|^2 - 1)u\|_{L_t^{2/(1-4\varepsilon)}(I;R^2)}
\leq C(E) + C(E)(\|u\|_{L_t^4(I;R^2)} + \|u\|_{L_t^4(I;R^2)})\|u\|_{X \cap Y}.
\]
(2.6)
Using the same way as in Bourgain [8], one can split \(R\) into finitely many pairwise disjoint intervals
\[
R = \bigcup_{j=1}^J \bar{I_j}, \quad \|u\|_{L_t^4(I_j \times R^2)} \leq \eta, \quad C(E)(\eta^2 + \eta^\varepsilon) \leq 1/2.
\]
(2.7)
By (2.6),
\[
\|u\|_{L_t^{2/(1-2\varepsilon)}(I_j;H^1_{1/\varepsilon}) \cap L_t^4(I_j;H^1_x)} \leq C(E).
\]
(2.8)
As \(\varepsilon \in (0, 1/8)\) can be chosen small arbitrarily, by interpolation,
\[
\|u\|_{L_t^{\gamma}((I_j;H^p_x)} \leq C(E),
\]
(2.9)
for all admissible pairs and \(j = 1, 2, \ldots, J\). The desire result follows. \(\square\)
3 Stability

Lemma 3.1. (Stability) For any \( A > 0 \) and \( \sigma > 0 \), there exists \( \delta > 0 \) with the following property: if \( u : I \times \mathbb{R}^2 \rightarrow \mathbb{C} \) satisfies \( \|u\|_{L^1_t L^2_x(I \times \mathbb{R}^2)} \leq A \) and approximately solves (3.1) in the sense that

\[
\left\| \int_0^t e^{i(t-\tau)\Delta} (iu_t + \Delta u - f(u)) d\tau \right\|_{L^1_t L^2_x(I \times \mathbb{R}^2)} \leq \delta;
\]

and \( v_0 \in H^1(\mathbb{R}^2) \) satisfies \( H(v_0) \leq 1 \) and \( \|e^{i(t-t_0)\Delta}(u(t_0) - v_0)\|_{L^1_t L^2_x(I \times \mathbb{R}^2)} \leq \delta \) for some \( t_0 \in I \), then there exists a solution \( v : I \times \mathbb{R}^2 \rightarrow \mathbb{C} \) to (3.1) with \( v(t_0) = v_0 \) such that \( \|u - v\|_{L^1_t L^2_x(I \times \mathbb{R}^2)} \leq \sigma \).

Proof. Let \( v : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{C} \) be the global solution with the initial data \( v(t_0) = v_0 \). Denote \( u = u + w \) on the interval \( I \), then

\[
iw_t + \Delta w = (f(u + w) - f(u)) - (iu_t + \Delta u - f(u))
\]

and \( \|e^{i(t-t_0)\Delta}w(t_0)\|_{L^1_t L^2_x(I \times \mathbb{R}^2)} \leq \delta \). Let \( X = L^4_t L^2_x(0,t) \cap L^{2/(1-2\varepsilon)}(I \times \mathbb{R}^2) \), by Strichartz estimates, (3.1) and triangle inequality, we have

\[
\|w\|_X \lesssim \delta^{2\varepsilon/(1-2\varepsilon)} + \int_0^t e^{i(t-\tau)\Delta} (f(u + w) - f(u)) d\tau + \delta
\]

\[
\lesssim 2\delta^{2\varepsilon/(1-2\varepsilon)} + \left( \|u\|_{L^4_x L^2_t(0,t)}^2 + \|u + w\|_{L^4_t L^2_x(I \times \mathbb{R}^2)}^2 \right) \|w\|_{L^4_t L^2_x(I \times \mathbb{R}^2)}
\]

\[
+ \left( \|u\|_{L^4_x L^2_t(0,t)}^2 + \|u + w\|_{L^4_t L^2_x(I \times \mathbb{R}^2)}^2 \right) \|w\|_{L^{2/(1-2\varepsilon)} L^{1/\varepsilon}(I \times \mathbb{R}^2)}
\]

\[
\lesssim 2\delta^{2\varepsilon/(1-2\varepsilon)} + (2A^2 + 2A^{8\varepsilon}) \|w\|_X + \|w\|_X^{1+8\varepsilon} + \|w\|_X^3.
\]

When \( A \) and \( \delta = \delta(\sigma) \) both are sufficiently small, standard continuity argument gives \( \|w\|_X \leq \sigma \). When \( A \) is large, we only need to subdivide the time interval \( I \) and then the result follows by an iterate process. \( \square \)

4 Linear Profile Decomposition

In this section, we will give the linear profile decomposition for Schrödinger equation in \( H^1(\mathbb{R}^2) \). First, we give some definitions and lemmas.

Definition 4.1. (Symmetry group, [18]) For any phase \( \theta \in \mathbb{R}/2\pi \mathbb{Z} \), position \( x_0 \in \mathbb{R}^2 \), frequency \( \xi_0 \in \mathbb{R}^2 \), and scaling parameter \( \lambda > 0 \), we define the unitary transformation \( g_{\theta,\xi_0,x_0,\lambda} : L^2_x(\mathbb{R}^2) \rightarrow L^2_x(\mathbb{R}^2) \) by the formula

\[
g_{\theta,\xi_0,x_0,\lambda} f(x) := \frac{1}{\lambda} e^{i\theta} e^{i\xi_0 \cdot \frac{x-x_0}{\lambda}} f\left(\frac{x-x_0}{\lambda}\right).
\]

We let \( G \) be the collection of such transformations; this is a group with identity \( g_{0,0,0,1} \), inverse \( g_{-\theta,-\xi_0,-x_0,-1} = g_{\theta,\xi_0,x_0,\lambda}^{-1} \) and group law

\[
g_{\theta_0,\xi_0_0,x_0_0,\lambda_0} g_{\theta_1,\xi_1_0,x_1_0,\lambda_1} = g_{\theta_0 + \theta_1,\xi_0 + \xi_1,\lambda_0 + \theta_1 / \lambda, \lambda_1 / \lambda}.
\]

We let \( G L^2(\mathbb{R}^2) \) be the modulo space of \( G \)-orbits \( Gf := \{gf : g \in G\} \) of \( L^2(\mathbb{R}^2) \), endowed with the usual quotient topology. If \( u : I \times \mathbb{R}^2 \rightarrow \mathbb{C} \) is a
function, we define $T_{g_0,\xi_0,\lambda_0,\lambda} u : \lambda^2 I \times \mathbb{R}^2 \to \mathbb{C}$ where $\lambda^2 I := \{\lambda^2 t : t \in I\}$ by the formula

$$
(T_{g_0,\xi_0,\lambda_0,\lambda} u)(t,x) := \frac{1}{\lambda} e^{i\theta} e^{ix\xi_0} e^{-it|\xi_0|^2} u\left(\frac{t}{\lambda^2}, \frac{x-x_0-2\xi_0 t}{\lambda}\right),
$$

or equivalently

$$
(T_{g_0,\xi_0,\lambda_0,\lambda} u)(t) = g_0 - i(t|\xi_0|^2, \xi_0, x_0+2\xi_0 t, \lambda(u\left(\frac{t}{\lambda^2}\right))).
$$

If $g \in G$, we can easily prove that $M(T_0 u) = M(u)$ and $S(T_0 u) = S(u)$.

**Definition 4.2.** (Enlarged group, [13]) For any phase $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, position $x_0 \in \mathbb{R}^2$, frequency $\xi_0 \in \mathbb{R}^2$, scaling parameter $\lambda > 0$, and time $t_0$, we define the unitary transformation $g_{\theta,\xi_0,\lambda,x_0,t_0} : L^2_\mathbb{R}(\mathbb{R}^2) \to L^2_\mathbb{R}(\mathbb{R}^2)$ by the formula

$$
g_{\theta,\xi_0,\lambda,x_0,t_0} = g_{\theta,\xi_0,\lambda,x_0}\lambda^{t_0},
$$
or in other words

$$
g_{\theta,\xi_0,\lambda,x_0,t_0} f(x) := \frac{1}{\lambda} e^{i\theta} e^{ix\xi_0} (e^{it_0|\xi_0|^2} f)\left(\frac{x-x_0}{\lambda}\right).
$$

Let $G'$ be the collection of such transformations. We also let $G'$ act on global space-time function $u : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$ by defining

$$
T_{g_{\theta,\xi_0,\lambda,x_0,t_0}} u(t,x) := \frac{1}{\lambda} e^{i\theta} e^{ix\xi_0} e^{-it|\xi_0|^2} (e^{it_0|\xi_0|^2} u)\left(\frac{t}{\lambda^2}, \frac{x-x_0-2\xi_0 t}{\lambda}\right),
$$
or equivalently

$$
(T_{g_{\theta,\xi_0,\lambda,x_0,t_0}} u)(t) = g_{\theta,\xi_0,\lambda,x_0,t_0}(u\left(\frac{t}{\lambda^2}\right)).
$$

**Lemma 4.1.** (Linear profiles for $L^2$ sequence, [13]) Let $u_n$ be a bounded sequence in $L^2_\mathbb{R}(\mathbb{R}^2)$. Then (after passing to a subsequence if necessary) there exists a family $\phi^{(j)}$, $j = 1, 2, \cdots$ of functions in $L^2_\mathbb{R}(\mathbb{R}^2)$ and group elements $g_{n}^{(j)} \in G'$ for $j, n = 1, 2, \cdots$ such that we have the decomposition

$$
u_n = \sum_{j=1}^{l} g_{n}^{(j)} \phi^{(j)} + u_n^{(I)} \quad (4.1)
$$

for all $l = 1, 2, \cdots$; here $u_n^{(I)} \in L^2_\mathbb{R}(\mathbb{R}^2)$ is such that its linear evolution has asymptotically vanishing scattering size:

$$
\lim_{l \to \infty} \lim_{n \to \infty} S(e^{it\Delta} u_n^{(I)}) = 0. \quad (4.2)
$$

Moreover, for any $j \neq j'$,

$$
\frac{\lambda_n^{(j')}}{\lambda_n^{(j)}} + \frac{\lambda_n^{(j)}}{\lambda_n^{(j')}} + \lambda_n^{(j)} \lambda_n^{(j')} |\xi_n^{(j)} - \xi_n^{(j')}|^2 > \frac{|x_n^{(j)} - x_n^{(j')}|^2}{\lambda_n^{(j)} \lambda_n^{(j')}} + \frac{t_n^{(j)} (\lambda_n^{(j)})^2 - t_n^{(j')} (\lambda_n^{(j')}^2}{\lambda_n^{(j)} \lambda_n^{(j')}} \to \infty. \quad (4.3)
$$
Furthermore, for any \( l \geq 1 \) we have the mass decoupling property

\[
\lim_{n \to \infty} \left[ M(u_n) - \sum_{j=1}^{l} M(\phi^{(j)}) - M(w_n^{(l)}) \right] = 0; \tag{4.4}
\]

for any \( j \leq l \), we have

\[
(g_n^{(j)})^{-1} w_n^{(l)} \rightharpoonup 0, \text{ weakly in } L^2_\Delta(\mathbb{R}^2). \tag{4.5}
\]

**Remark 4.1.** If the orthogonal condition \([4, 13]\) holds, then (see \([4, 13]\))

\[
\lim_{n \to \infty} (g_n^{(j)} \phi^{(j)}, g_n^{(j')} \phi^{(j')})_{L^2(\mathbb{R}^2)} = 0, \quad j \neq j',
\]

\[
\lim_{n \to \infty} (g_n^{(j)} \phi^{(j)}, w_n^{(l)})_{L^2(\mathbb{R}^2)} = 0.
\]

Moreover, if \( v^{(j)}, v^{(j')} \in L^4_{t,x}(\mathbb{R} \times \mathbb{R}^2) \), then (see \([2, 13]\)), for any \( 0 < \theta < 1 \)

\[
\lim_{n \to \infty} \| T_{g_n^{(j)}} v^{(j)} (1 - \theta T_{g_n^{(j')}} v^{(j')})^{\theta} \|_{L^4_{t,x}(\mathbb{R} \times \mathbb{R}^2)} = 0; \tag{4.6}
\]

if \( v^{(1)}, \ldots, v^{(l)} \in L^4_{t,x}(\mathbb{R} \times \mathbb{R}^2) \), then (see \([2, 13]\), Lemma 5.5)

\[
\lim_{n \to \infty} S(l \sum_{j=1}^{l} g_n^{(j)} v^{(j)}) \leq \sum_{j=1}^{l} S(v^{(j)}). \tag{4.7}
\]

**Remark 4.2.** As each linear profile \( \phi^{(j)} \) in Lemma \([4, 13]\) is constructed in the sense that

\[
eq \lambda_{n,j} \exp \left\{ \left. \frac{\xi^{(j)}}{2} x \right|_{n \to \infty} \right\} u_n^{(j)}(x) \rightharpoonup \phi^{(j)}
\]

weakly in \( L^2_\Delta(\mathbb{R}^2) \) (see \([13]\), after passing to a subsequence in \( n \), rearrangement, translation, and refining \( \phi^{(j)} \) accordingly, we may assume that the parameters satisfy the following:

1) \( t_n^{(j)} \to \pm \infty \) as \( n \to \infty \), or \( t_n^{(j)} \equiv 0 \) for all \( n,j; \)

2) \( \lambda_{n,j}^{(j)} \to 0 \) or \( \lambda_{n,j}^{(j)} \equiv 1 \) for all \( n,j; \)

3) \( |\xi_n^{(j)}| \to \infty \) as \( n \to \infty \), or \( \xi_n^{(j)} \equiv \xi^{(j)} \) with \( |\xi^{(j)}| < \infty \).

4) When \( \lambda_n^{(j)} \equiv 1, \xi_n^{(j)} \equiv \xi^{(j)} \) and \( |\xi^{(j)}| < \infty \), we can let \( \xi^{(j)} \equiv 0 \).

Our main result in this section is the following lemma:

**Lemma 4.2.** (Linear profiles for \( H^1 \) sequence) Let \( u_n \) be a bounded sequence in \( H^1(\mathbb{R}^2) \). Then up to a subsequence, for any \( J \geq 1 \), there exists a sequence \( \phi_\alpha \) in \( H^1(\mathbb{R}^2) \) and a sequence of group elements \( g_\alpha = g_{\alpha, \lambda_{\alpha}, \xi_{\alpha}} \) in \( G' \)

\[
u \to \infty \}
\]

such that

\[
u = \sum_{\alpha=1}^{J} g_{\alpha} \phi_\alpha + R(n,J). \tag{4.8}
\]

Here, for each \( \alpha, \lambda_{\alpha} \) and \( \xi_{\alpha} \) must satisfy

\[
\lambda_{\alpha} \equiv 1 \text{ and } \xi_{\alpha} \equiv 0, \text{or } \lambda_{\alpha} \to \infty; \tag{4.9}
\]
\( R(n, J) \in H^1(\mathbb{R}^2) \) is such that
\[
\lim_{J \to \infty} \limsup_{n \to \infty} S(e^{it \Delta} R(n, J)) = 0. \tag{4.10}
\]

Moreover, for any \( \alpha \neq \alpha' \), one has the same orthogonal conditions as \( \text{(4.3)} \). For any \( J \geq 1 \), one has the following decoupling properties
\[
\lim_{n \to \infty} \{ \| u_n \|^2_{L^2(\mathbb{R}^2)} - \sum_{\alpha=1}^{J} \| \phi_\alpha \|^2_{L^2(\mathbb{R}^2)} - \| R(n, J) \|^2_{L^2(\mathbb{R}^2)} \} = 0, \tag{4.11}
\]
\[
\lim_{n \to \infty} \{ \| u_n \|^2_{H^1(\mathbb{R}^2)} - \sum_{\alpha=1}^{J} \| g_{n\alpha} \phi_\alpha \|^2_{H^1(\mathbb{R}^2)} - \| R(n, J) \|^2_{H^1(\mathbb{R}^2)} \} = 0, \tag{4.12}
\]
\[
\lim_{J \to \infty} \limsup_{n \to \infty} \{ H(u_n) - \sum_{\alpha=1}^{J} H(g_{n\alpha} \phi_\alpha - H(R(n, J))) \} = 0. \tag{4.13}
\]

Proof. Let
\[
\Box_k = F^{-1} \chi_k F, \quad \chi_k = \begin{cases} 1 & 2^{k-1} < |\xi| \leq 2^k; \\ 0 & \text{else.} \end{cases}
\]
Then, we have
\[
u_n = \sum_{k=-\infty}^{+\infty} \Box_k u_n := \sum_{|k| \leq N} \Box_k u_n + R_N,
\]
and
\[
\| u_n \|^2_{L^2(\mathbb{R}^2)} = \sum_{|k| \leq N} \| \Box_k u_n \|^2_{L^2(\mathbb{R}^2)} + \| R_N \|^2_{L^2(\mathbb{R}^2)},
\]
\[
\| u_n \|^2_{H^1(\mathbb{R}^2)} = \sum_{|k| \leq N} \| \Box_k u_n \|^2_{H^1(\mathbb{R}^2)} + \| R_N \|^2_{H^1(\mathbb{R}^2)},
\]
\[
\lim_{N \to \infty} \limsup_{n \to \infty} \| R_N \|^2_{L^2(\mathbb{R}^2)} = 0.
\]

By Lemma 4.1, after passing to a subsequence if necessary, we can obtain
\[
\Box_k u_n = \sum_{j=1}^{l_k} g_{nk}^{(j)} \phi_k^{(j)} + w_{nk}^{(j)} \tag{4.14}
\]
with the stated properties 1)-4) and \( \text{(1.1)} \)-\( \text{(1.5)} \). Denote
\[
\Lambda_{1,0} = \{(k, j) \mid |k| \leq N, 1 \leq j \leq \ell_k, \lambda_{nk}^{(j)} \equiv 1, \xi_{nk}^{(j)} \equiv 0 \};
\]
\[
\Lambda_{1,\infty} = \{(k, j) \mid |k| \leq N, 1 \leq j \leq \ell_k, \lambda_{nk}^{(j)} \equiv 1, |\xi_{nk}^{(j)}| \to \infty \};
\]
\[
\Lambda_0 = \{(k, j) \mid |k| \leq N, 1 \leq j \leq \ell_k, \lambda_{nk}^{(j)} \to 0 \};
\]
\[
\Lambda_{\infty,0} = \{(k, j) \mid |k| \leq N, 1 \leq j \leq \ell_k, \lambda_{nk}^{(j)} \to \infty, \xi_{nk}^{(j)} \equiv \xi_{nk}^{(j)}, |\xi_{nk}^{(j)}| < 2^{k-1} \};
\]
\[
\Lambda_{\infty,1} = \{(k, j) \mid |k| \leq N, 1 \leq j \leq \ell_k, \lambda_{nk}^{(j)} \to \infty, \xi_{nk}^{(j)} \equiv \xi_{nk}^{(j)}, |\xi_{nk}^{(j)}| \in [2^{k-1}, 2^k) \};
\]
\[
\Lambda_{\infty,\infty} = \{(k, j) \mid |k| \leq N, 1 \leq j \leq \ell_k, \lambda_{nk}^{(j)} \to \infty, |\xi_{nk}^{(j)}| \to \infty \text{ or } \xi_{nk}^{(j)} \equiv \xi_{nk}^{(j)}, |\xi_{nk}^{(j)}| > 2^k \}.
\]
Step 1. We prove that

\[ u_n = \sum_{(k,j) \in \Lambda_{1,0} \cup \Lambda_{\infty,1}} g_{nk}^{(j)} \phi_k^{(j)} + R \]  

(4.15)

with \( g_{nk}^{(j)} \phi_k^{(j)} = \Box_k g_{nk}^{(j)} \phi_k^{(j)} \) and for each fixed \( N \),

\[ \lim_{n \to \infty} \| u_n \|^2_{L^2(\mathbb{R}^2)} - \sum_{(k,j) \in \Lambda_{1,0} \cup \Lambda_{\infty,1}} \| g_k^{(j)} \|^2_{L^2(\mathbb{R}^2)} - \| R \|^2_{L^2(\mathbb{R}^2)} = 0, \]  

(4.16)

\[ \lim_{n \to \infty} \| u_n \|^2_{H^1(\mathbb{R}^2)} - \sum_{(k,j) \in \Lambda_{1,0} \cup \Lambda_{\infty,1}} \| g_k^{(j)} \|^2_{H^1(\mathbb{R}^2)} - \| R \|^2_{H^1(\mathbb{R}^2)} = 0, \]  

(4.17)

\[ \lim_{N \to \infty} \lim_{n \to \infty} \limsup_{n \to \infty} S(e^{it\Delta} R) = 0, \]  

(4.18)

where

\[ R = R_N + R_w, \quad R_w = \sum_{|k| \leq N} w^{(k)}_n. \]

By (4.2) and \( \lim_{N \to \infty} \limsup_{n \to \infty} \| R_N \|^2_{L^2(\mathbb{R}^2)} = 0, \) (4.15) holds obviously. For (4.15), we prove it by induction. For every \( n \), suppose that

\[ \Box_k u_n = g_{nk}^{(1)} \phi_k^{(1)} + w_{nk}^{(1)}. \]  

(4.19)

Case 1. If \( (k,1) \in \Lambda_{1,\infty} \cup \Lambda_{0} \cup \Lambda_{\infty,0} \cup \Lambda_{\infty,\infty} \), we have \( \phi_k^{(1)} = 0 \).

In fact, by (4.20),

\[ \phi_k^{(1)} = (g_{nk}^{(1)})^{-1} \Box_k u_n - (g_{nk}^{(1)})^{-1} w_{nk}^{(1)}. \]

Thus

\[ \| \phi_k^{(1)} \|^2_{L^2(\mathbb{R}^2)} = \langle (g_{nk}^{(1)})^{-1} \Box_k u_n - (g_{nk}^{(1)})^{-1} w_{nk}^{(1)}, \phi_k^{(1)} \rangle \]

\[ = \langle u_n, \Box_k g_{nk}^{(1)} \phi_k^{(1)} \rangle - \langle (g_{nk}^{(1)})^{-1} w_{nk}^{(1)}, \phi_k^{(1)} \rangle. \]  

(4.20)

Using (4.3),

\[ \langle (g_{nk}^{(1)})^{-1} w_{nk}^{(1)}, \phi_k^{(1)} \rangle \to 0 \text{ as } n \to \infty. \]  

(4.21)

By direct calculation,

\[ \Box_k g_{nk}^{(1)} \phi_k^{(1)} = F^{-1} \chi_k(\xi) \lambda_{nk}^{(1)} e^{ix \xi} \xi e^{-ix \lambda_{nk}^{(1)}(\xi + \xi_{nk}^{(1)})} e^{-i \lambda_{nk}^{(1)}(\lambda_{nk}^{(1)}(\xi + \xi_{nk}^{(1)})^2 \xi + \xi_{nk}^{(1)} \phi_k^{(1)}(\lambda_{nk}^{(1)}(\xi + \xi_{nk}^{(1)})))} \]

\[ = \frac{1}{\lambda_{nk}^{(1)}} e^{ix \xi} \xi e^{-ix \lambda_{nk}^{(1)}(\xi + \xi_{nk}^{(1)})} (F^{-1} \chi_k(\xi) \lambda_{nk}^{(1)} e^{-ix \lambda_{nk}^{(1)}(\xi + \xi_{nk}^{(1)})} \cdot \frac{x}{\lambda_{nk}^{(1)}}). \]  

(4.22)

Let \( n \to \infty \), when \( (k,1) \in \Lambda_{1,\infty} \),

\[ \| \Box_k g_{nk}^{(1)} \phi_k^{(1)} \|^2_{L^2(\mathbb{R}^2)} \leq \int_{2^{k-1} \leq |\xi + \xi_{nk}^{(1)}| \leq 2^k} |F \phi_k^{(1)}|^2 d\xi \to 0; \]  

(4.23)
when \((k, 1) \in \Lambda_0\),
\[
\|\Box_k g_{nk}^{(1)} \phi_k^{(1)}\|_{L^2(\mathbb{R}^2)}^2 \leq \int_{\lambda_{nk}^{(1)}2^{k-1} \leq |\xi| + \lambda_{nk}^{(1)}\xi_{nk}^{(1)} \leq \lambda_{nk}^{(1)}2^k} |\mathcal{F}\phi_k^{(1)}|^2 d\xi \to 0; \tag{4.24}
\]
when \((k, 1) \in \Lambda_{\infty, 0}\),
\[
\|\Box_k g_{nk}^{(1)} \phi_k^{(1)}\|_{L^2(\mathbb{R}^2)}^2 \leq \int_{|\xi| \geq \lambda_{nk}^{(1)}(2^{k-1} - |\xi|^{(1)})} |\mathcal{F}\phi_k^{(1)}|^2 d\xi \to 0; \tag{4.25}
\]
when \((k, 1) \in \Lambda_{\infty, \infty}\),
\[
\|\Box_k g_{nk}^{(1)} \phi_k^{(1)}\|_{L^2(\mathbb{R}^2)}^2 \leq \int_{|\xi| \geq \lambda_{nk}^{(1)}(|\xi|_{nk}^{(1)} - 2^k)} |\mathcal{F}\phi_k^{(1)}|^2 d\xi \to 0. \tag{4.26}
\]
By (4.20)–(4.26), \(\|\phi_k^{(1)}\|_{L^2(\mathbb{R}^2)}^2 = 0\) and thus \(\phi_k^{(1)} = 0\).

Case 2. If \((k, 1) \in \Lambda_{1, 0} \cup \Lambda_{\infty, 1}\), we have
\[
\|g_{nk}^{(1)} \phi_k^{(1)} - \Box_k g_{nk}^{(1)} \phi_k^{(1)}\|_{L^2(\mathbb{R}^2)} \to 0 \text{ as } n \to \infty. \tag{4.27}
\]
Let \(\chi_{A_k^{(1)}}\) be the characteristic function of the set \(A_k^{(1)}\) and \(P_{A_k^{(1)}} = \mathcal{F}^{-1} \chi_{A_k^{(1)}} \mathcal{F}\), then
\[
g_{nk}^{(1)} \left(P_{A_k^{(1)}} \phi_k^{(1)} + P_{A_k^{(1)}}(g_{nk}^{(1)})^{-1} w_{nk}^{(1)}\right) = P_{A_k^{(1)}}(g_{nk}^{(1)} \phi_k^{(1)} + w_{nk}^{(1)}) = P_{A_k^{(1)}} \Box_k u_n,
\]
where
\[
P_{A_k^{(1)}} = \mathcal{F}^{-1} \chi_{A_k^{(1)}}(\lambda_{nk}^{(1)}(\xi - \xi_{nk}^{(1)})) \mathcal{F}.
\]
Note that
\[
\langle P_{A_k^{(1)}} \phi_k^{(1)}, P_{A_k^{(1)}}(g_{nk}^{(1)})^{-1} w_{nk}^{(1)} \rangle = \langle P_{A_k^{(1)}} \phi_k^{(1)}, (g_{nk}^{(1)})^{-1} w_{nk}^{(1)} \rangle \to 0 \text{ as } n \to \infty,
\]
we have
\[
\lim_{n \to \infty} \left(\|P_{A_k^{(1)}} \Box_k u_n\|_{L^2(\mathbb{R}^2)}^2 - \|P_{A_k^{(1)}} \phi_k^{(1)}\|_{L^2(\mathbb{R}^2)}^2 - \|P_{A_k^{(1)}}(g_{nk}^{(1)})^{-1} w_{nk}^{(1)}\|_{L^2(\mathbb{R}^2)}^2\right) = 0. \tag{4.28}
\]
When \((k, 1) \in \Lambda_{1, 0}\), we have \(P_{A_k^{(1)}} = P_{A_k^{(1)}}\). Choosing \(A_k^{(1)} = \{\xi \mid |\xi| \leq 2^{k-1} \text{ or } |\xi| > 2^{k}\}\), then by (4.28), \(P_{A_k^{(1)}} \phi_k^{(1)} = 0\), the desired result follows.

When \((k, 1) \in \Lambda_{\infty, 1}\) and \(|\xi_{nk}^{(1)}| \in (2^{k-1}, 2^k)\), we have
\[
\|g_{nk}^{(1)} \phi_k^{(1)} - \Box_k g_{nk}^{(1)} \phi_k^{(1)}\|_{L^2(\mathbb{R}^2)}^2
\leq \int_{|\xi| + \lambda_{nk}^{(1)}\xi_{nk}^{(1)} \leq \lambda_{nk}^{(1)}2^{k-1}} |\mathcal{F}\phi_k^{(1)}|^2 d\xi + \int_{|\xi| + \lambda_{nk}^{(1)}\xi_{nk}^{(1)} > \lambda_{nk}^{(1)}2^k} |\mathcal{F}\phi_k^{(1)}|^2 d\xi
\leq \int_{|\xi| \geq \lambda_{nk}^{(1)}(|\xi_{nk}^{(1)}| - 2^{k-1})} |\mathcal{F}\phi_k^{(1)}|^2 d\xi + \int_{|\xi| \geq \lambda_{nk}^{(1)}(2^k - |\xi|_{nk}^{(1)})} |\mathcal{F}\phi_k^{(1)}|^2 d\xi
\to 0 \text{ as } n \to \infty
\]
When \((k, 1) \in \Lambda_{\infty, 1}\) and \(|\xi_{nk}^{(1)}| = 2^k\), we denote \(\xi = (\xi_1, \xi_2)\) and \(\xi_{nk}^{(1)} = (\xi_{nk1}, \xi_{nk2})\). The line \(\xi_2 = \frac{\xi_{nk1}}{\xi_{nk2}} \xi_1\) (when \(\xi_{nk2} = 0\), we use the line \(\xi_1 = 0\)
instead) separates the frequency space \(L^2(\mathbb{R}^2)\) into two half-planes. We let \(A_k^{(1)}\) to be the half-plane which contains the point \(\zeta_k^{(1)}\), then

\[
P_{A_k^{(1)}} \square_k u_n = \mathcal{F}^{-1} \chi_{A_k^{(1)}}(\lambda_{nk}^{(1)}(\xi - \zeta_k^{(1)})) \chi_k \mathcal{F} u_n = 0.
\]

By (4.29), we have \(P_{A_k^{(1)}} \phi_k^{(1)} = 0\). Note that

\[
\lim_{n \to \infty} \|g_{nk}^{(1)}(1 - P_{A_k^{(1)}})\phi_k^{(1)} - \square_k g_{nk}^{(1)} \phi_k^{(j)}\|_2^2(\mathbb{R}^2) = 0
\]

\[
\lim_{n \to \infty} \|(1 - \chi_{A_k^{(1)}}) - \chi_k(\frac{t}{\lambda_{nk}^{(1)} + \xi_k^{(1)}})\|_2^2(\mathbb{R}^2) \leq \lim_{n \to \infty} \int_{\xi \in \mathbb{R}^2 \setminus \{(1,1), (2k^2, 2k^2)\}} |\mathcal{F} \phi_k^{(1)}|^2 d\xi = 0,
\]

(4.27) holds.

When \((k, 1) \in \Lambda_{\infty, 1}\) and \(|\xi_k^{(1)}| = 2k^2 - 1\), let \(A_k^{(1)}\) to be the half-plane which does NOT contain the point \(\zeta_k^{(1)}\), we can prove (4.27) similarly as above.

By the proof above and absorbing the error into \(w_{nk}^{(1)}\), we can suppose \(g_{nk}^{(1)} \phi_k^{(1)} = \square_k g_{nk}^{(1)} \phi_k^{(2)}\) and \((k, 1) \in \Lambda_{1,0} \cup \Lambda_{\infty, 1}\). Denote \(u_n^{(1)} = u_n - g_{nk}^{(1)} \phi_k^{(1)}\) and suppose

\[
\square_k u_n^{(1)} = g_{nk}^{(2)} \phi_k^{(2)} + w_{nk}^{(2)}.
\]

Repeating the proof above, we can obtain \(g_{nk}^{(2)} \phi_k^{(2)} = \square_k g_{nk}^{(2)} \phi_k^{(2)}\) and \((k, 2) \in \Lambda_{1,0} \cup \Lambda_{\infty, 1}\). By the orthogonal condition (4.3), following the proof in [13], we can obtain (4.15).

By the orthogonal condition (4.3), following the proof in [13], we can obtain that for fix \(k\) and \(j \neq j'\),

\[
\lim_{n \to \infty} \langle g_{nk}^{(j)} \phi_k^{(j)}, g_{nk}^{(j')} \phi_k^{(j')} \rangle_{H^1(\mathbb{R}^2)} = \lim_{n \to \infty} \langle \square_k g_{nk}^{(j)} \phi_k^{(j)}, \square_k g_{nk}^{(j')} \phi_k^{(j')} \rangle_{H^1(\mathbb{R}^2)} = 0,
\]

\[
\lim_{n \to \infty} \langle g_{nk}^{(j)} \phi_k^{(j)}, w_{nk}^{(j')} \rangle_{H^1(\mathbb{R}^2)} = \lim_{n \to \infty} \langle \square_k g_{nk}^{(j)} \phi_k^{(j)}, \square_k w_{nk}^{(j')} \rangle_{H^1(\mathbb{R}^2)} = 0,
\]

(4.16) and (4.17) were proved.

**Step 2.** For arbitrary \((k_1, j_1), (k_2, j_2) \in \Lambda_{1,0} \cup \Lambda_{\infty, 1}\), we define \((k_1, j_1) \sim (k_2, j_2)\) if the orthogonal condition (4.3) is NOT true for any subsequence, that is

\[
\frac{\lambda_{nk_1}^{(j_1)}}{\lambda_{nk_2}^{(j_2)}} + \frac{\lambda_{nk_2}^{(j_2)}}{\lambda_{nk_1}^{(j_1)}} + \frac{\lambda_{jk_1}}{\lambda_{jk_2}} + \gamma_{nk_1} \gamma_{nk_2} = |\zeta_{j_1}|^2 + \frac{|f_{nk_2}^{(2)} - f_{nk_1}^{(2)}|^2}{\lambda_{nk_2}^{(j_2)} \lambda_{nk_1}^{(j_1)}} + \frac{|f_{nk_2}^{(2)} - f_{nk_1}^{(2)}|^2}{\lambda_{nk_2}^{(j_2)} \lambda_{nk_1}^{(j_1)}} < \infty \text{ for } \forall n.
\]

By the definition above, if \((k_1, j_1) \sim (k_2, j_2)\), we have

\[
\lambda_{nk_1}^{(j_1)} \sim \lambda_{nk_2}^{(j_2)}, \lambda_{nk_1}^{(j_1)} \xi_{nk_1} \sim \lambda_{nk_2}^{(j_2)} \xi_{nk_2}, \lambda_{jk_1}^{(j_1)} \sim \lambda_{jk_2}^{(j_2)}, \gamma_{nk_1}^{(j_1)} \sim \gamma_{nk_2}^{(j_2)}.
\]

Note that

\[
g_{\theta, x_0, x_0, \lambda, t_0} f(x) := \frac{1}{\lambda} e^{it_0} e^{i\xi x} \lambda \phi(t_0 \Delta) f\left(\frac{x - x_0}{\lambda}\right),
\]
by Remark 4.2, we can put these two profiles together as one profile. Then, we can denote \((A_{1,0} \cup A_\infty) / \sim = \{1, 2, \ldots, J\}\). \([4.8], [4.12]\) were proved.

Specially, as \(C_c^\infty\) is dense in \(L^2\), we can also suppose \(F\phi_n \in C_c^\infty\) and hence \(\phi_n \in H^1(\mathbb{R}^2)\).

**Step 3.** We prove \([4.13]\) now. By \([1.12]\), we only need to prove that for \(\forall m \in \mathbb{N}, m \geq 2\)

\[
\lim_{J \to \infty} \limsup_{n \to \infty} \left\{ \| u_n \|_{L^{2m}(\mathbb{R}^2)}^2 - \sum_{\alpha=1}^J \| g_{n\alpha} \phi_n \|_{L^{2m}(\mathbb{R}^2)}^2 - \| R(n, J) \|_{L^{2m}(\mathbb{R}^2)}^2 \right\} = 0.
\]

As

\[
\| R(n, J) \|_{L^{2m}(\mathbb{R}^2)} \lesssim \| R(n, J) \|_{L^1(\mathbb{R}^2)}^{1/(m-1)} \| R(n, J) \|_{H^1(\mathbb{R}^2)^{(m-2)/(m-1)}}
\]

and for \(\frac{1}{3} < \theta < \frac{1}{2}\),

\[
\| R(n, J) \|_{L^1(\mathbb{R}^2)} \leq \| e^{it\Delta} R(n, J) \|_{L^\infty L^1(\mathbb{R} \times \mathbb{R}^2)} \lesssim \| (\partial_t)^{\theta} e^{it\Delta} R(n, J) \|_{L^1_{\theta,\tau}(\mathbb{R} \times \mathbb{R}^2)}
\]

\[
= \| e^{it\Delta} R(n, J) \|_{L^1_{t,x}(\mathbb{R} \times \mathbb{R}^2)} \| (\partial_t)^{1/2} e^{it\Delta} R(n, J) \|_{L^2_{\theta,\tau}(\mathbb{R} \times \mathbb{R}^2)}
\]

\[
= \| e^{it\Delta} R(n, J) \|_{L^1_{t,x}(\mathbb{R} \times \mathbb{R}^2)} \| e^{it\Delta} R(n, J) \|_{L^2_{t,x}(\mathbb{R} \times \mathbb{R}^2)}^{20}\n\]

\[
\lesssim \| e^{it\Delta} R(n, J) \|_{L^1_{t,x}(\mathbb{R} \times \mathbb{R}^2)}^{20}
\]

by \([4.10]\), we have

\[
\lim_{J \to \infty} \limsup_{n \to \infty} \| R(n, J) \|_{L^{2m}(\mathbb{R}^2)}^2 = 0
\]

We separate the set \(1 \leq \alpha \leq J\) into two subsets:

\[\Lambda_1 = \{1 \leq \alpha \leq J \mid \lambda_{n\alpha} \equiv 1, \xi_{n\alpha} \equiv 0\}, \quad \Lambda_\infty = \{1 \leq \alpha \leq J \mid \lambda_{n\alpha} \to \infty\}\].

When \(\alpha \in \Lambda_\infty\),

\[
\lim_{n \to \infty} \| g_{n\alpha} \phi_n \|_{L^{2m}(\mathbb{R}^2)}^2 = \lim_{n \to \infty} (\lambda_{n\alpha})^{-1+1/m} \| e^{it_{n\alpha} \Delta} \phi_n \|_{L^{2m}(\mathbb{R}^2)}^2
\]

\[
\leq \lim_{n \to \infty} (\lambda_{n\alpha})^{-1+1/m} \| \phi_n \|_{H^1(\mathbb{R}^2)} = 0.
\]

Hence, in order to prove \((4.30)\), one only need to prove

\[
\lim_{n \to \infty} \left\{ \| \sum_{\alpha \in \Lambda_1} g_{n\alpha} \phi_n \|_{L^{2m}(\mathbb{R}^2)}^2 - \sum_{\alpha \in \Lambda_1} \| g_{n\alpha} \phi_n \|_{L^{2m}(\mathbb{R}^2)}^2 \right\} = 0.
\]

If \(\alpha \in \Lambda_1\) and \(t_{n\alpha} \to \infty\), for a function \(\tilde{\phi}_n \in \dot{H}^{1/2} \cap L^{4/3}\) we have

\[
\| g_{n\alpha} \phi_n \|_{L^4(\mathbb{R}^2)} \leq \| g_{n\alpha} \phi_n - g_{n\alpha} \tilde{\phi}_n \|_{L^4(\mathbb{R}^2)} + \| g_{n\alpha} \tilde{\phi}_n \|_{L^4(\mathbb{R}^2)}
\]

\[
\leq \| \phi_n - \tilde{\phi}_n \|_{\dot{H}^{1/2}(\mathbb{R}^2)} + \| t_{n\alpha} \|^{-1/2} \| \tilde{\phi}_n \|_{L^{4/3}(\mathbb{R}^2)}
\]

By approximating \(\phi_n\) by \(\tilde{\phi}_n \in C_c^\infty\) in \(\dot{H}^{1/2}\) and sending \(n \to \infty\) we have \(\| g_{n\alpha} \phi_n \|_{L^4(\mathbb{R}^2)} \to 0\). Note that \(g_{n\alpha} \phi_n \in H^1\), we obtain \(\| g_{n\alpha} \phi_n \|_{L^{2m}(\mathbb{R}^2)} \to 0\) for \(\forall m \geq 2\).
If \( \alpha \in A_1 \) and \( t_{n_0} \equiv 0 \), we have orthogonal condition \( |x_{n_0} - x_{n_0'}| \to \infty \) for any \( \alpha \neq \alpha' \). Thus

\[
\lim_{n \to \infty} \left\{ \left\| \sum_{\alpha \in \Lambda_1, t_{n_0} \equiv 0} g_{n_0} \phi_{\alpha} \right\|_{L^{2m}(\mathbb{R}^2)}^{2m} - \sum_{\alpha \in \Lambda_1, t_{n_0} \equiv 0} \left\| g_{n_0} \phi_{\alpha} \right\|_{L^{2m}(\mathbb{R}^2)}^{2m} \right\} = \lim_{n \to \infty} \left\{ \left\| \sum_{\alpha \in \Lambda_1, t_{n_0} \equiv 0} \phi_{\alpha} (\cdot - x_{n_0}) \right\|_{L^{2m}(\mathbb{R}^2)}^{2m} - \sum_{\alpha \in \Lambda_1, t_{n_0} \equiv 0} \left\| \phi_{\alpha} (\cdot - x_{n_0}) \right\|_{L^{2m}(\mathbb{R}^2)}^{2m} \right\} = 0.
\]

(1.30) holds and then (1.13) was proved.

\[\square\]

5 The Proof of Theorem 1.2

Let \( u \) be a solution of (1.3), by Strichartz estimate and (2.0),

\[
\|u\|_{L^1_t(L^3_x(\mathbb{R}^2))} \leq C\|u_0\|_{L^2(\mathbb{R}^2)} + C\|u\|_{L^2_t(L^{1/2}(\mathbb{R}^2))} + C(E)\|u\|_{L^2_t(L^{1/2}(\mathbb{R}^2))} \|u\|_{L^2_t(L^{1/2}(\mathbb{R}^2))} \|u\|_{L^2_t(L^{1/2}(\mathbb{R}^2))}
\]

(5.1)

When \( \|u_0\|_{L^2(\mathbb{R}^2)} \ll 1 \), by standard continuity argument, we have

\[
\|u\|_{L^1_t(L^{1/2}(\mathbb{R}^2))} \leq C\|u_0\|_{L^2(\mathbb{R}^2)} < \infty.
\]

(5.2)

Hence, if \( M(u) \ll 1 \), then \( \|u\|_{L^1_t(L^{1/2}(\mathbb{R}^2))} < \infty \). In particular, we have global existence and scattering in both directions.

For any mass \( m \geq 0 \), we define

\[ A(m) := \sup\{S(u) : u \text{ is the global solution of (1.3)}, M(u) \leq m, H(u) \leq 1\}. \]

Then \( A : [0, +\infty) \to [0, +\infty] \) is a monotone increasing function of \( m \). As \( A \) is left-continuous and finite for small \( m \), there must exist a unique critical mass \( m_0 \in (0, +\infty) \) such that \( A(m) \) is finite for all \( m < m_0 \) but infinite for all \( m \geq m_0 \).

To prove Theorem 1.2 one only needs to prove that the critical mass \( m_0 \) is infinite. We will prove that by contradiction.

Proposition 5.1. Suppose that the critical mass \( m_0 \) is finite. Let \( u_n : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C} \) for \( n = 1, 2, \ldots \) be a sequence of solutions and \( t_n \in \mathbb{R} \) be a sequence of times such that \( \lim\sup_{n \to \infty} M(u_n) = m_0 \) and

\[
\lim_{n \to \infty} S_{\geq t_n}(u_n) = \lim_{n \to \infty} S_{\leq t_n}(u_n) = +\infty.
\]

(5.3)

Then there exists a sequence of \( x_n = x_{n,n} \in \mathbb{R}^2 \) such that \( u_n(t_n, x + x_n) \) has a subsequence which converges strongly in \( L^2_t(\mathbb{R}^2) \).

Proof. We can take \( t_n = 0 \) for all \( n \) by translating \( u_n \) in time. Thus,

\[
\lim_{n \to \infty} S_{\geq 0}(u_n) = \lim_{n \to \infty} S_{\leq 0}(u_n) = +\infty.
\]

(5.4)
By Lemma 4.2, up to a subsequence if necessary, we have
\[
u_n(0) = \sum_{\alpha \in \Lambda_1 \cup \Lambda_\infty} g_{n\alpha} \phi_\alpha + R(n, J),
\]
where \(\Lambda_1\) and \(\Lambda_\infty\) were defined by (4.31). Suppose that
\[
g_{n\alpha} = h_{n\alpha} e^{i t_{n\alpha} \Delta}
\]
where \(t_{n\alpha} \in \mathbb{R}\) and \(h_{n\alpha} \in G\). By (4.11),
\[
\sum_{\alpha \in \Lambda_1 \cup \Lambda_\infty} M(\phi_\alpha) \leq \limsup_{n \to \infty} M(u_n(0)) \leq m_0
\]
Hence,
\[
\sup_{\alpha \in \Lambda_1 \cup \Lambda_\infty} M(\phi_\alpha) \leq m_0.
\]
We define the nonlinear profile \(v_\alpha : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}\) as follows:
\[\begin{align*}
\Diamond & \text{ When } \alpha \in \Lambda_1, \\
& \quad \text{ if } t_{n\alpha} \equiv 0, \text{ we define } v_\alpha \text{ to be the global solution of (1.3) with initial data } v_\alpha(0) = \phi_\alpha. \\
& \quad \text{ if } t_{n\alpha} \to +\infty, \text{ we define } v_\alpha \text{ to be the global solution of (1.3) which scatters to } e^{it_{n\alpha} \phi_\alpha} \text{ when } t \to +\infty. \\
& \quad \text{ if } t_{n\alpha} \to -\infty, \text{ we define } v_\alpha \text{ to be the global solution of (1.3) which scatters to } e^{it_{n\alpha} \phi_\alpha} \text{ when } t \to -\infty.
\end{align*}\]
\[\begin{align*}
\Diamond & \text{ When } \alpha \in \Lambda_\infty, \\
& \quad \text{ if } t_{n\alpha} \equiv 0, \text{ we define } v_\alpha \text{ to be the global solution of } iu_t + \Delta u = |u|^2 u \text{ with initial data } v_\alpha(0) = \phi_\alpha. \\
& \quad \text{ if } t_{n\alpha} \to +\infty, \text{ we define } v_\alpha \text{ to be the global solution of } iu_t + \Delta u = |u|^2 u \text{ which scatters to } e^{it_{n\alpha} \phi_\alpha} \text{ when } t \to +\infty. \\
& \quad \text{ if } t_{n\alpha} \to -\infty, \text{ we define } v_\alpha \text{ to be the global solution of } iu_t + \Delta u = |u|^2 u \text{ which scatters to } e^{it_{n\alpha} \phi_\alpha} \text{ when } t \to -\infty.
\end{align*}\]
If we define
\[
\tilde{u}_n = \sum_{\alpha \in \Lambda_1 \cup \Lambda_\infty} T_{h_{n\alpha}}[v_\alpha(\cdot + t_{n\alpha})](t) + e^{i t \Delta} R(n, J)
\]
for \(n, J = 1, 2, \ldots\), then we have the following two lemmas:

**Lemma 5.1.** ([18], Lemma 5.1)
\[
\lim_{n \to \infty} M(\tilde{u}_n(0) - u_n(0)) = 0.
\]

**Lemma 5.2.** If
\[
\lim_{J \to \infty} \limsup_{n \to \infty} \|\tilde{u}_n\|_{L^4_{t,x}(\mathbb{R} \times \mathbb{R}^2)} < \infty, \quad \|v_\alpha\|_{L^4_{t,x}(\mathbb{R} \times \mathbb{R}^2)} < \infty \quad (\forall \alpha),
\]
then
\[
\lim_{J \to \infty} \limsup_{n \to \infty} \left\| \int_0^t e^{i(t-\tau)\Delta} ((i \partial_t + \Delta) \tilde{u}_n - f(\tilde{u}_n))(\tau) d\tau \right\|_X = 0.
\]
where \(X = L^4_{t,x} \cap L^2_{t}((1-2\epsilon) L^2_\epsilon(\mathbb{R} \times \mathbb{R}^2))\).
Proof. Denote
\[ v_{n\alpha} = T_{h_{n\alpha}}[v_{\alpha}(\cdot + t_{n\alpha})]. \]

By the definition of \( \tilde{u}_n \), we have
\[ \tilde{u}_n = \sum_{\alpha \in \Lambda_1 \cup \Lambda_\infty} v_{n\alpha} + e^{it\Delta} R(n, J) \]
and
\[ (i\partial_t + \Delta)\tilde{u}_n = \sum_{\alpha \in \Lambda_1} f(v_{n\alpha}) + \sum_{\alpha \in \Lambda_\infty} |v_{n\alpha}|^2 v_{n\alpha}. \]

Thus, by triangle inequality, it suffices to show that
\[ \lim_{J \to \infty} \lim_{n \to \infty} \int_0^t e^{i(t-\tau)\Delta}(f(\tilde{u}_n - e^{it\Delta} R(n, J)) - f(\tilde{u}_n))(\tau) d\tau \| \chi = 0, \quad (5.8) \]
\[ \lim_{n \to \infty} \int_0^t e^{i(t-\tau)\Delta} \left( f(\sum_{\alpha \in \Lambda_1 \cup \Lambda_\infty} v_{n\alpha}) - \sum_{\alpha \in \Lambda_1 \cup \Lambda_\infty} f(v_{n\alpha}) \right)(\tau) d\tau \| \chi = 0 \quad (5.9) \]
and
\[ \lim_{n \to \infty} \int_0^t e^{i(t-\tau)\Delta} \sum_{\alpha \in \Lambda_\infty} \left( f(v_{n\alpha}) - |v_{n\alpha}|^2 v_{n\alpha} \right)(\tau) d\tau \| \chi = 0 \quad (5.10) \]

Using (4.13) and the same estimates as in (3.3), we have
\[ \| \int_0^t e^{i(t-\tau)\Delta}(f(\tilde{u}_n - e^{it\Delta} R(n, J)) - f(\tilde{u}_n))(\tau) d\tau \| \chi \leq \| \tilde{u}_n \|_{L^4_{t,x}(\mathbb{R} \times \mathbb{R}^2)}^2 \| e^{it\Delta} R(n, J) \|_{L^4_{t,x}(\mathbb{R} \times \mathbb{R}^2)} + \| e^{it\Delta} R(n, J) \|_{L^4_{t,x}(\mathbb{R} \times \mathbb{R}^2)} \| e^{it\Delta} R(n, J) \|_{L^{4/(1-2\epsilon)}_{t,x}(\mathbb{R} \times \mathbb{R}^2)}. \]

By (4.10),
\[ \lim_{J \to \infty} \lim_{n \to \infty} \| e^{it\Delta} R(n, J) \|_{L^4_{t,x}(\mathbb{R} \times \mathbb{R}^2)} = 0. \]

As
\[ \| e^{it\Delta} R(n, J) \|_{L^{2/(1-2\epsilon)}_{t,x}(\mathbb{R} \times \mathbb{R}^2)} \leq \| e^{it\Delta} R(n, J) \|_{L^{2/(1-2\epsilon)}_{t,x}(\mathbb{R} \times \mathbb{R}^2)} \| e^{it\Delta} R(n, J) \|_{L^{2/(1-\epsilon)}_{t,x}(\mathbb{R} \times \mathbb{R}^2)}, \]
and
\[ \| e^{it\Delta} R(n, J) \|_{L^{2/(1-\epsilon)}_{t,x}(\mathbb{R} \times \mathbb{R}^2)} \leq \| R(n, J) \|_{L^2(\mathbb{R}^2)} < \infty, \]
we have
\[ \lim_{J \to \infty} \lim_{n \to \infty} \| e^{it\Delta} R(n, J) \|_{L^{2/(1-2\epsilon)}_{t,x}(\mathbb{R} \times \mathbb{R}^2)} = 0. \]

By (5.13), (5.8) was obtained.
Using Strichartz estimate,
\[
\| \int_0^t e^{i(t-\tau)\Delta} \left( f(v_{n\alpha}) - |v_{n\alpha}|^2 v_{n\alpha} \right)(\tau) \, d\tau \|_X \\
\leq \| \sum_{m=2}^{\infty} \frac{|v_{n\alpha}|^{2m} v_{n\alpha}}{m!} \|_{L_t^1 L_x^6(\mathbb{R} \times \mathbb{R}^2)} \leq \sum_{m=2}^{\infty} \frac{1}{m!} \| v_{n\alpha} \|_{L_t^4 L_x^8(\mathbb{R} \times \mathbb{R}^2)} \| v_{n\alpha} \|_{L_t^2 L_x^\infty(\mathbb{R} \times \mathbb{R}^2)}^{2m} \\
\leq \sum_{m=2}^{\infty} \frac{1}{m!} (\lambda_{n\alpha})^{-2(m-1)} \| v_{n\alpha} \|_{L_t^4 L_x^8(\mathbb{R} \times \mathbb{R}^2)} \| v_{n\alpha} \|_{L_t^2 L_x^\infty(\mathbb{R} \times \mathbb{R}^2)}^{2m}.
\]

By Lemma 2.3,
\[
\| v_{n\alpha} \|_{L_t^{4m} L_x^\infty(\mathbb{R} \times \mathbb{R}^2)} \leq C m^{1/2+1/2m} \| v_{n\alpha} \|_{L_t^4 L_x^8(\mathbb{R} \times \mathbb{R}^2)}^{1/m} \| v_{n\alpha} \|_{L_t^\infty L_x^2(\mathbb{R} \times \mathbb{R}^2)}^{1-1/m}.
\]

Note that \( \| v_{n\alpha} \|_{L_t^4 L_x^8(\mathbb{R} \times \mathbb{R}^2)} < \infty \), \( (5.10) \) was obtained.

To prove Lemma 5.2 we only left to prove (5.9). Note that

\[
\left| f \left( \sum_{\alpha=1}^J z_{\alpha} \right) - \sum_{\alpha=1}^J f(z_{\alpha}) \right| \lesssim \sum_{\alpha \neq \alpha'} |z_{\alpha}| \| e^{\lambda t} |z_{\alpha}|^2 - 1 |,
\]

by (4.13), we have
\[
\| \int_0^t e^{i(t-\tau)\Delta} \left( f \left( \sum_{\alpha \in \Lambda_{1} \cup \Lambda_{\infty}} v_{n\alpha} \right) - \sum_{\alpha \in \Lambda_{1} \cup \Lambda_{\infty}} f(v_{n\alpha}) \right)(\tau) \, d\tau \|_X \\
\lesssim \sum_{\alpha \neq \alpha'} \left( \| v_{n\alpha} \|_{L_t^{1/2} L_x^{\infty}(\mathbb{R} \times \mathbb{R}^2)} \| v_{n\alpha'} \|_{L_t^{1/2} L_x^{\infty}(\mathbb{R} \times \mathbb{R}^2)} \right) \left( \| v_{n\alpha} \|_{L_t^{2} L_x^{4}(\mathbb{R} \times \mathbb{R}^2)} \| v_{n\alpha'} \|_{L_t^{2} L_x^{4}(\mathbb{R} \times \mathbb{R}^2)} \right) \\
+ \sum_{\alpha \neq \alpha'} \left( \| v_{n\alpha} \|_{L_t^{1/2} L_x^{2}(\mathbb{R} \times \mathbb{R}^2)} \| v_{n\alpha'} \|_{L_t^{1/2} L_x^{2}(\mathbb{R} \times \mathbb{R}^2)} \right) \left( \| v_{n\alpha} \|_{L_t^{1/2} L_x^{2}(\mathbb{R} \times \mathbb{R}^2)} \| v_{n\alpha'} \|_{L_t^{1/2} L_x^{2}(\mathbb{R} \times \mathbb{R}^2)} \right) \left( \| v_{n\alpha} \|_{L_t^{2} L_x^{4}(\mathbb{R} \times \mathbb{R}^2)} \| v_{n\alpha'} \|_{L_t^{2} L_x^{4}(\mathbb{R} \times \mathbb{R}^2)} \right) \\
\lesssim \sum_{\alpha \neq \alpha'} \left( \| v_{n\alpha} \|_{L_t^{1/2} L_x^{2}(\mathbb{R} \times \mathbb{R}^2)} \| v_{n\alpha'} \|_{L_t^{1/2} L_x^{2}(\mathbb{R} \times \mathbb{R}^2)} \right) \left( \| v_{n\alpha} \|_{L_t^{2} L_x^{4}(\mathbb{R} \times \mathbb{R}^2)} \| v_{n\alpha'} \|_{L_t^{2} L_x^{4}(\mathbb{R} \times \mathbb{R}^2)} \right) \left( \| v_{n\alpha} \|_{L_t^{2} L_x^{4}(\mathbb{R} \times \mathbb{R}^2)} \| v_{n\alpha'} \|_{L_t^{2} L_x^{4}(\mathbb{R} \times \mathbb{R}^2)} \right)
\]

By (4.8), we immediately obtain (5.9).

By (5.9), suppose
\[
\sup_{\alpha \in \Lambda_{1} \cup \Lambda_{\infty}} M(\phi_{n\alpha}) \leq m_0 - \sigma
\]
for some \( \sigma > 0 \), we will prove that this leads to a contradiction. By the definition of \( A(m) \) and (5.2), we have
\[
A(m) \leq B m \text{ for all } 0 \leq m \leq m_0 - \sigma,
\]
where \( B = B(\sigma) \in (0, +\infty) \). Then \( v_{n\alpha} \) satisfies
\[
M(v_{n\alpha}) = M(\phi_{n\alpha}) \leq m_0 - \sigma,
\]
(5.13)
\[ S(v_0) \leq A(M(\phi_0)) \leq BM(\phi_0). \] (5.14)

By (4.7), (5.5) and (5.14), we have
\[
\lim_{J \to \infty} \limsup_{n \to \infty} S(\tilde{u}_n) \leq Bm_0.
\] (5.15)

Using Lemma 5.1 and Lemma 5.2, we have
\[ M(\tilde{u}_n(0) - u_n(0)) \leq \delta, \quad S(\tilde{u}_n) \leq 2Bm_0, \]
for \( \delta > 0 \) sufficiently small, \( J = J(\delta) \) and \( n = n(J, \delta) \) sufficiently large. By Lemma 3.1, we obtain that
\[ S(u_n) \leq 3Bm_0 \]
which contradicts (5.4). Thus, (5.11) fails for \( \forall \sigma > 0 \), and then
\[
\sup_{\alpha \in \Lambda_1 \cup \Lambda_{\infty}} M(\phi_\alpha) = m_0.
\]
Comparing this with (5.5), we have
\[ u_n(0) = h_n e^{it_n \Delta} \phi + R_n \] (5.16)
with \( t_n \) converging to \( \pm \infty \) or \( t_n \equiv 0 \), \( h_n \in G \), \( M(\phi) = m_0 \) and
\[ M(R_n) \to 0, \quad S(e^{it \Delta} R_n) \to 0 \] as \( n \to \infty \).

Specially, the parameters \( \lambda_n, \xi_n \) of \( h_n \) must satisfies
\[ \lambda_n \equiv 1 \] and \( \xi_n \equiv 0 \), or \( \lambda_n \to \infty \).

Since there is only one profile now, we have
\[ \tilde{u}_n = T_{h_n} v + e^{it \Delta} R_n. \]

When \( \lambda_n \to \infty \), by the scattering of cubic Schrödinger equation (see [12], [7]), we have \( S(v) < \infty \) and \( \lim_{n \to \infty} S(\tilde{u}_n) < \infty \). By Lemma 5.1, Lemma 5.2 and Lemma 3.1, we obtain that for \( n \) sufficiently large, \( S(u_n) < \infty \) which contradicts (5.4).

When \( \lambda_n \equiv 1 \), \( \xi_n \equiv 0 \) and \( t_n \to +\infty \), by Strichartz estimate and monotone convergence we have
\[ \lim_{n \to \infty} S(\tilde{u}_n(0)) = 0. \]

Thus
\[ \lim_{n \to \infty} S(\tilde{u}_n(0)) = \lim_{n \to \infty} S(\tilde{u}_n(0)) = 0. \]

Since \( \lim_{n \to \infty} S(e^{it \Delta} R_n) = 0 \), we can see from (5.16) that
\[ \lim_{n \to \infty} S(\tilde{u}_n(0)) = 0. \]

By Lemma 5.1 (with \( 0 \) as the approximate solution and \( u_n(0) \) as the initial data), we have
\[ \lim_{n \to \infty} S(\tilde{u}_n(0)) = 0. \]
which contradicts one of the estimates in (5.4).

When \( \lambda_n \equiv 1, \xi_n \equiv 0 \) and \( t_n \to -\infty \), the argument is similar and we can obtain a contradiction by using the other half of (5.4).

Now, the only case left is \( \lambda_n \equiv 1, \xi_n \equiv 0 \) and \( t_n \equiv 0 \). In this case, we have

\[
M(u_n(0) - h_n \phi) = M(R_n) \to 0 \quad \text{as} \quad n \to \infty.
\]

Thus \((h_n)^{-1}u_n(0) = e^{\theta_n}u_n(0, x + x_n)\) converges to \(\phi\) in \(L^2_2(\mathbb{R}^2)\). After passing to a subsequence if necessary and refining \(\phi\), the desired result follows.

Let \(\{u_n\}\) be the sequence given in Proposition 5.1 and suppose \(u_n(0, x + x_n)\) converges to \(u_0\) strongly in \(L^2_2(\mathbb{R}^2)\), then \(M(u_0) \leq m_0\). Let \(u\) be the global solution with initial data \(u(0) = u_0\), by Lemma 3.4 we must have

\[S_{\leq 0}(u) = S_{\leq 0}(u) = +\infty.
\]

By the definition of \(m_0\), \(M(u_0) \geq m_0\) and hence \(M(u) = m_0\).

Since \(u\) is locally in \(L^4_t\), for \(\forall t_n \in \mathbb{R}\), we have

\[S_{\geq t_n}(u) = S_{\leq t_n}(u) = +\infty.
\]

Using Proposition 5.4 for \(\{u(t_n)\}\), we have \(u(t_n, x + x(t_n))\) converges in \(L^2_2(\mathbb{R}^2)\). By Ascoli-Arzelà Theorem, that is

**Proposition 5.2.** Suppose that the critical mass \(m_0\) is finite. Then there exists a global solution \(u\) of mass exactly \(m_0\) satisfies that for every \(\eta > 0\) there exists \(0 < C(\eta) < \infty\) such that

\[
\int_{|x-x(t)| \geq C(\eta)} |u(t,x)|^2 \, dx + \int_{|\xi| \leq C(\eta)} |\dot{u}(t,\xi)|^2 \, d\xi \leq \eta \tag{5.17}
\]

for all \(t \in \mathbb{R}\), where the functions \(x, \xi : \mathbb{R} \to \mathbb{R}^2\).

**Proposition 5.3.** The solution described in Proposition 5.2 does not exist.

Once we proved Proposition 5.3, we can say that \(m_0 = \infty\) and thus Theorem 1.2 is true. In order to prove Proposition 5.3 we need the following lemmas.

**Lemma 5.3.** (1.5, Lemma 5.2) Let \(u\) be a global solution of (1.3). Then we have

\[
\iint_{\mathbb{R} \times \mathbb{R}^2} \frac{(t)^2 G(u)}{(t)^3 + |x|^3} \, dxdt \leq C(E). \tag{5.18}
\]

**Lemma 5.4.** (1.5, Lemma 6.2) Let \(u\) be a global solution of (1.3). Let \(B\) be a compact subset of \(\mathbb{R}^2\). Then for any \(R > 0\) and \(T > 0\), we have

\[
\int_{B(R)} |u(T,x)|^2 \, dx \geq \int_B |u(0,x)|^2 \, dx - C(E)T/R, \tag{5.19}
\]

where \(B(R) := \{x \in \mathbb{R}^2 | \exists y \in B \text{ s.t. } |x-y| < R\}\).

The proof of Proposition 5.3. By Lemma 5.4, choosing \(\eta\) sufficiently small,

\[
\int_{|x-x(0)| \leq C(\eta) + R|t|} |u(t,x)|^2 \, dx \geq \int_{|x-x(0)| \leq C(\eta)} |u(0,x)|^2 \, dx - C(E)/R \geq m_0 - \eta - C(E)/R.
\]
By Proposition 5.2
\[ \int_{|x-x(t)| \leq C(\eta)} |u(t, x)|^2 dx \geq m_0 - \eta. \]

For a fixed large number \( R \), we must have \( |x(t) - x(0)| \leq 2C(\eta) + R|t| \). By Lemma 5.3 and Hölder inequality,
\[
\begin{align*}
\infty &> \int_{\mathbb{R} \times \mathbb{R}^2} \frac{\langle t \rangle^2 |u|^4}{\langle t \rangle^3 + |x|^3} \, dx \, dt \\
&\geq \int_{\mathbb{R}} \int_{|x-x(t)| \leq C(\eta)} \frac{\langle t \rangle^2 |u|^4}{\langle t \rangle^3 + |x|^3} \, dx \, dt \\
&\geq \int_{\mathbb{R}} \int_{|x-x(t)| \leq C(\eta)} \frac{|u|^4}{\langle t \rangle + 1} \, dx \, dt \\
&\geq \int_{\mathbb{R}} \int_{|x-x(t)| \leq C(\eta)} \frac{|u|^2}{\langle t \rangle + 1} \, dx \, dt \\
&\geq (m_0 - \eta) \int_{\mathbb{R}} \frac{1}{\langle t \rangle + 1} \, dt = \infty.
\end{align*}
\]

This is a contradiction. Proposition 5.3 was obtained. \( \square \)

References

[1] S. Adachi and K. Tanaka, Trudinger type inequalities in \( \mathbb{R}^N \) and there best exponents, Proc. Amer. Math. Soc. 128(2000), 2051-2057.
[2] H. Bahouri, P. Gérard, High frequency approximation of solutions to critical nonlinear wave equations, Amer. J. Math. 121(1999), 131-175.
[3] J. Bourgain, Global well-posedness of defocusing 3D critical NLS in the radial case, J. Amer. Math. Soc. 12 (1999), 145-171.
[4] P. Begout, A.Vargas, Mass concentration phenomena for the \( L^2 \)-critical nonlinear Schrödinger equation, Transactions of the American Mathematical Society, 359(2007), 5257-5282.
[5] J. Colliander, S. Ibrahim, M. Majdoub and N. Masmoudi, Energy critical NLS in two space dimensions, J. Hype. Diff. Equa. 6(2009), 549-575.
[6] J. Colliander, M. Keel, G.Staffilani, H. Takaoka, and T. Tao, Viriel, Morawetz, and interaction Morawetz inequalities, Notes(2006).
[7] B. Dodson, Global well-posedness and scattering for the defocusing, \( L^2 \)-critical, nonlinear Schrödinger equation when \( d=2 \), arXiv:1006.1375v1 [math.AP].
[8] J.Ginibre and G.Velo, Scattering theory in the energy space for a class of nonlinear Schrödinger equations, J.Math.pures Appl., 64(1985), 363-401.
[9] J.Ginibre and G.Velo, Time decay of finite energy solutions of the Klein-Gordon and Schrödinger equations, Ann. Inst. H. Poincaré Phys. Théor. 43(1985), 399-442.
[10] S. Ibrahim, M. Majdoub, N. Masmoudi and K. Nakanishi, Energy scattering for 2D critical wave equation, arXiv:0806.3150v1 [math.AP].

[11] C. Kenig and F. Merle, Global well-posedness, scattering, and blowup for the energy-critical, focusing, non-linear Schrödinger equation in the radial case, Invent. Math. 166(2006), 645-675.

[12] R. Killip, T. Tao and M. Visan, The cubic nonlinear Schrödinger equation in two dimensions with radial data, arXiv:0707.3188v2 [math.AP].

[13] F. Merle, L. Vega, compactness at blow-up time for $L^2$ solutions of the critical nonlinear Schrödinger equation in 2D, Internat. Math. Res. Not. 8(1998), 399-425.

[14] J. Moser, A sharp form of an inequality by N. Trudinger, Indiana Univ. Math. J.20(1971), 1077-1092.

[15] K. Nakanishi, Energy scattering for nonlinear Klein-Gordon and Schrödinger equations in spatial dimensions 1 and 2, J. Func. Anal. 169(1999), 201-225.

[16] M. Nakamura, T. Ozawa, Nonlinear Schrödinger equations in the Sobolev space of critical order, J. Funct. Anal., 155(1998), 364-380.

[17] S. Shao, Sharp linear and bilinear restriction estimates for the paraboloid in the cylindrically symmetric case, arXiv:0706.3759v3 [math.AP].

[18] T. Tao, M. Visan and X. Zhang, Minimal-mass blowup solutions of the mass-critical NLS, arXiv:0609690v2 [math.AP].

[19] B. Wang, The smoothness of scattering operators for Sinh-Gordon and nonlinear Schrödinger equations, Acta. Math. Sin., 18(2002), 549-564.

[20] B. Wang, C. Hao and H. Hudzik, Energy scattering theory for the nonlinear Schrödinger equations with exponential growth in lower spatial dimensions, J. Diff. Equa., 228(2006), 311-338.