Sharpened Uncertainty Principle

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Abstract

For any finite group $G$, any finite $G$-set $X$ and any field $F$, we consider the vector space $\mathbb{F}^X$ of all functions from $X$ to $\mathbb{F}$. When the group algebra $\mathbb{F}G$ is semisimple and splitting, we find a specific basis $\hat{X}$ of $\mathbb{F}^X$, construct the Fourier transform: $\mathbb{F}^X \rightarrow \mathbb{F}^\hat{X}$, $f \mapsto \hat{f}$, and define the rank support $\text{rk-supp}(\hat{f})$; we prove that $\text{rk-supp}(\hat{f}) = \dim \mathbb{F}G f$, where $\mathbb{F}G f$ is the submodule of the permutation module $FX$ generated by the element $f = \sum_{x \in X} f(x) x$. Next, we extend a sharpened uncertainty principle for abelian finite groups by Feng, Hollmann, and Xiang [9] to the following extensive framework: for any field $\mathbb{F}$, any transitive $G$-set $X$ and $0 \neq f \in \mathbb{F}^X$ we prove that

$$|\text{supp}(f)| \cdot \dim \mathbb{F}G f \geq |X| + |\text{supp}(f)| - |X_{\text{supp}(f)}|,$$

where $\text{supp}(f)$ is the support of $f$, and $X_{\text{supp}(f)}$ is a block of $X$ associated with the subset $\text{supp}(f)$ such that $\text{supp}(f)$ is a disjoint union of some translations of the block. Then many (sharpened or classical) versions of finite-dimensional uncertainty principle are derived as corollaries.

Key words: Finite group; group action; support of function; Fourier transform; uncertainty principle.

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1 Introduction

Assume that $\mathbb{F}$ is a field, $\mathbb{C}$ denotes the complex field. For any set $X$, the cardinality of $X$ is denoted by $|X|$. By $\mathbb{F}^X$ we denote the $\mathbb{F}$-vector space of all functions from $X$ to $\mathbb{F}$. For any function $f \in \mathbb{F}^X$, the support of $f$ is defined as the subset $\text{supp}(f) = \{x \mid x \in X, f(x) \neq 0\}$.

Throughout the paper, $G$ is a finite group. By $H \leq G$ we mean that $H$ is a subgroup of $G$. Let $FG$ be the group algebra, i.e., the $\mathbb{F}$-vector space with basis $G$ and equipped with multiplication induced by the multiplication of the group $G$. By $\text{Irr}(G)$ we denote the set of all absolutely irreducible characters of $G$ (maybe valued in an extension of $\mathbb{F}$); by $n_\psi$ we denote the degree of $\psi \in \text{Irr}(G)$.
As said in [18], “uncertainty principle” means a kind of theorems which assert that the supports of a non-zero function and its Fourier transform cannot both be very small.

Donoho-Stark uncertainty principle ([4], or [17]) says that, if $G$ is abelian and $\mathbb{F} = \mathbb{C}$ (hence $\hat{G} := \text{Irr}(G)$ is the dual group), $0 \neq f \in \mathbb{C}^G$ and $\hat{f} \in \mathbb{C}^\hat{G}$, $\hat{f}(\psi) = \sum_{x \in G} f(x)\psi(x)$ for $\psi \in \hat{G}$, is the Fourier transform of $f$, then

$$|\text{supp}(f)| \cdot |\text{supp}(\hat{f})| \geq |G|;$$

and the equality holds if and only if $f = c\chi I_{\gamma H}$, where $c \in \mathbb{C}$, $\chi \in \text{Irr}(G)$, $H \leq G$, $\gamma \in G$ and $I_{\gamma H}$ is the indicator function of the coset $\gamma H$, i.e., for $\alpha \in G$, $I_{\gamma H}(\alpha) = \begin{cases} 1, & \alpha \in \gamma H; \\ 0, & \alpha \notin \gamma H. \end{cases}$

Turn to the general case ($G$ is not necessarily abelian), the Fourier transform and Fourier inversion for the group algebra $\mathbb{C}G$ were described in [14, §6.2]. For $\psi \in \text{Irr}(G)$, choose a representation $\rho^\psi : G \to \text{GL}_{n_\psi}(\mathbb{C})$ (the general linear group) affording $\psi$. For $\alpha \in G$, $\rho^\psi(\alpha) = (\rho^\psi_{ij}(\alpha))_{n_\psi \times n_\psi}^{}$ is a matrix of degree $n_\psi$. Let $\hat{G}$ be the set of all the entry functions $\rho^\psi_{ij} \in \mathbb{C}^G$ for all $\psi \in \text{Irr}(G)$. Then $\hat{G}$ is an orthogonal (but not normal) basis of the function space $\mathbb{C}^G$. For $f \in \mathbb{C}^G$, the Fourier transform $\hat{f} \in \mathbb{C}^\hat{G}$ is: $\hat{f}(\rho^\psi_{ij}) = \sum_{\alpha \in G} f(\alpha)\rho^\psi_{ij}(\alpha)$, $\forall \rho^\psi_{ij} \in \hat{G}$. However, the size $|\text{supp}(\hat{f})|$ varies upon the choices of the representations $\rho^\psi$’s.

Meshulam [12] defined the rank-support of $\hat{f}$ as:

$$\text{rk-supp}(\hat{f}) = \sum_{\psi \in \text{Irr}(G)} n_\psi \cdot \text{rank}(\hat{f}(\rho^\psi)), \quad \text{where } \hat{f}(\rho^\psi) = (\hat{f}(\rho^\psi_{ij}))_{n_\psi \times n_\psi},$$

and proved that ([12], or [18]): for $0 \neq f \in \mathbb{C}^G$,

$$|\text{supp}(f)| \cdot \text{rk-supp}(\hat{f}) \geq |G|;$$

the condition for the equality is also discussed similarly to the abelian case.

Tao [16], Goldstein, Guralnick and Isaacs [10] showed another type of stronger uncertainty principle: if $|G| = p$ is a prime (hence $G$ is cyclic), $0 \neq f \in \mathbb{C}^G$ and $\hat{f} \in \mathbb{C}^\hat{G}$ is the Fourier transform of $f$, then

$$|\text{supp}(f)| + |\text{supp}(\hat{f})| \geq p + 1.$$  

A key for the proof is a result by Chebotarev in 1926 on the minors of the Fourier matrix $(\omega^{ij})_{0 \leq i, j \leq p-1}$, where $\omega$ is a primitive $p$’th root of unity, cf. [3, 15].

The studies mentioned above consider the complex functions. Wigderson and Wigderson [18] exhibited an interesting unified way to do the studies by using various norms of linear operators.
For abelian $G$ and any filed $F$ with characteristic char $F$ satisfying that
\[ \text{char } F = 0 \quad \text{or} \quad \gcd (\text{char } F, |G|) = 1, \quad (1.5) \]
the group algebra $FG$ is semisimple and Fourier transform still works well (by extending $F$ to a splitting field) and Eq.(1.1) still holds. Any function $f \in F^G$ is identified with an element $f = \sum_{\alpha \in G} f(\alpha) \alpha \in FG$. Let $\dim FGf$ denote the dimension of the submodule $FGf$ of $FG$ generated by $f$. It can be observed (cf. Remark 2.11(2) below) that
\[ |\text{supp}(\hat{f})| = \dim FGf. \quad (1.6) \]
With $\dim FGf$ instead of $\text{rk}\supp (\hat{f})$, Goldstein, Guralnick and Isaacs [10] proved that: for any (not necessarily abelian) finite group $G$ and any field $F$ (the condition Eq.(1.5) is no longer necessary), and $0 \neq f \in F^G$,
\[ |\supp (f)| \cdot \dim FGf \geq |G|. \]
In fact, they worked in an extensive framework: for any transitive $G$-set $X$ (cf. [1, §3], or Remark 2.1(1) below, for a definition) and $0 \neq f \in FX$, they consider the submodule $FGf$ generated by $f$ of the permutation $FG$-module $FX$ (cf Eq.(2.3) below for a definition of permutation modules), and proved that
\[ |\supp (f)| \cdot \dim FGf \geq |X|. \quad (1.7) \]
And they got the conditions that the equality in Eq.(1.7) holds.

In coding theory finite field $F$ is considered as the alphabet; a word with entries indexed by a finite group $G$ is just a function $f \in F^G$, and $|\supp (f)|$ is called the Hamming weight of the word $f$. People usually hope that a code word $f$ has big Hamming weight and big dim $FGf$; cf. [11]. From this point of view, [2] and [5] worked with uncertainty principle to explore good codes.

Recently, Feng, Hollmann, and Xiang [9] extended the so-called shifting technique for cyclic codes (i.e. $G$ is cyclic) to abelian codes ($G$ is abelian), and obtained the shift bound for abelian codes. With the shifting technique, they proved a sharpened version of uncertainty principle. Assume that $G$ is abelian and $F$ is any field with the condition Eq.(1.5), and $0 \neq f \in F^G$. They define the stabilizer in $G$ of the subset $\supp (f)$ of $G$ as:
\[ G_{\supp (f)} = \{ \alpha \mid \alpha \in G, \alpha \cdot \supp (f) = \supp (f) \}; \]
then $G_{\supp (f)} \leq G$ and $\supp (f)$ is a disjoint union of some cosets of the subgroup $G_{\supp (f)}$ (note that our notation is somewhat different from [9]). And they proved the following sharpened uncertainty principle:
\[ |\supp (f)| \cdot |\supp (\hat{f})| \geq |G| + |\supp (f)| - |G_{\supp (f)}|. \quad (1.8) \]
From Eq.(1.8), Donoho-Stark uncertainty principle Eq.(1.1) (and the condition for the equality in Eq.(1.1)) follows easily.
In this paper we extend the sharpened uncertainty principle to any transitive $G$-set $X$ of any (not necessarily abelian) finite group $G$ and any field $\mathbb{F}$ (without the condition Eq.(1.5)).

Let $G$ be any finite group, $\mathbb{F}$ be any field with the condition Eq.(1.5), and $X$ be any finite $G$-set with a left $G$-action. After a sketch about function spaces and permutation modules, we construct the dual set $\hat{X}$ which is a specific basis of $\mathbb{F}^X$, and describe Fourier transform: $\mathbb{F}^X \rightarrow \mathbb{F}^\hat{X}$, $f \mapsto \hat{f}$. When $X = G$ is the left regular set, it turns out the usual Fourier transform for $G$ described in [14, §6.2] (as mentioned above). Then, for $f \in \mathbb{F}^X$, we explain how to define the rank-support $\text{rk-supp}(\hat{f})$ for the Fourier transform $\hat{f} \in \mathbb{F}^\hat{X}$. In the extensive framework, we extend Eq.(1.6) to the following (Lemma 2.10 below):

$$\text{rk-supp}(\hat{f}) = \dim \mathbb{F}Gf,$$

where $\mathbb{F}Gf$ is the $\mathbb{F}G$-submodule generated by $f$ of the permutation module $\mathbb{F}X$. All of these are done in Section 2.

Therefore, we can extend the sharpened uncertainty principle with $\dim \mathbb{F}Gf$ instead of $\text{rk-supp}(\hat{f})$ (and the condition Eq.(1.5) on $\mathbb{F}$ can be removed). Once it is done, the sharpened or not sharpened versions of uncertainty principle with $\text{rk-supp}(\hat{f})$ (hence the condition Eq.(1.5) on $\mathbb{F}$ should be kept), or with regular $G$-set $X = G$, are then straightforward corollaries.

In Section 3, $X$ is assumed to be a transitive $G$-set, and $0 \neq f \in \mathbb{F}^X$. By a natural surjection $G \rightarrow X$, $S := \text{supp}(f)$ is lifted to $\mathcal{S} \subseteq G$, and we get the right stabilizer $G_{\mathcal{S}}$ of $\mathcal{S}$, which is a subgroup of $G$. Reducing $G_{\mathcal{S}}$ to $X$, we get the block $X_S$ of $X$ (cf. Remark 3.4 below for a definition of blocks) such that $S$ is a disjoint union of some translations of the block $X_S$. Then we prove the following sharpened uncertainty principle (Theorem 3.9 below):

$$|\text{supp}(f)| \cdot \dim \mathbb{F}Gf \geq |X| + |\text{supp}(f)| - |X_{\text{supp}(f)}|.
(1.9)$$

As a consequence, we get again Eq.(1.7) and the condition that the equality in Eq.(1.7) holds (Corollary 3.10 below). However, the question remains open (Question 3.12): when the equality in Eq.(1.9) holds? And, as pointed out above, many corollaries, with $\text{rk-supp}(\hat{f})$ or with regular set $X = G$, are derived. For example, for $X = G$ to be the left regular $G$-set, the rank-support version of Eq.(1.9) is the following sharpened uncertainty principle (Theorem 3.20 below):

$$|\text{supp}(f)| \cdot \text{rk-supp}(\hat{f}) \geq |G| + |\text{supp}(f)| - |G_{\text{supp}(f)}|.$$

Section 4 is the conclusion of the paper.

2 Fourier transforms for finite group actions

In this paper, $G$ is a finite group of order $|G| = n$ with operation written as multiplication, the identity element is denoted by $1_G$, or 1 for short. And $\mathbb{F}$
is any field; by $F^X$ we denote the multiplicative group of all units (non-zero elements) of $F$.

### 2.1 Function spaces and permutation modules

By $FG$ we denote the group algebra, i.e., $FG$ is the $F$-vector space with basis $G$, and with the multiplication induced by the multiplication of $G$. The following is a natural linear isomorphism:

$$F^G \rightarrow FG, \quad g \mapsto \sum_{\alpha \in G} g(\alpha)\alpha. \quad (2.1)$$

And, for $g, h \in F^G$ we have the convolution $g \ast h \in F^G$ as follows:

$$(g \ast h)(\alpha) = \sum_{\beta \in G} g(\beta)h(\beta^{-1}\alpha), \quad \forall \alpha \in G. \quad (2.2)$$

Then it is easy to check that Eq.(2.1) is an $F$-algebra isomorphism. In this way, we identify any function $g \in F^G$ with the element $g = \sum_{\alpha \in G} g(\alpha)\alpha \in FG$.

**Remark 2.1.** (1) A $G$-set $X$ is a set with a $G$-action on the set, i.e., there is a map $G \times X \rightarrow X$, $(\alpha, x) \mapsto \alpha x$, satisfying that: (i) $(\alpha \beta)x = \alpha(\beta x)$, $\forall \alpha, \beta \in G$, $\forall x \in X$; (ii) $1_G x = x$, $\forall x \in X$. A $G$-set $X$ is said to be transitive if for any $x, y \in X$ there is an $\alpha \in G$ such that $\alpha x = y$. Note that if we take $X = G$ and, for $(\alpha, x) \in G \times G$, set $\alpha x$ to be the multiplication in $G$, it is clearly a transitive $G$-set; the $X = G$ is called the left regular $G$-set.

(2) An $FG$-module $V$ (or, called a $G$-space over $F$) is an $F$-vector space $V$ with a $G$-action on the space (equivalently, a representation of $G$ on the space), i.e. a map $G \times X \rightarrow V$, $(\alpha, v) \mapsto \alpha v$, satisfying that: (i) $(\alpha \beta)v = \alpha(\beta v)$, $\forall \alpha, \beta \in G$, $\forall v \in V$; (ii) $1_G v = v$, $\forall v \in V$; (iii) $\alpha : V \rightarrow V$, $v \mapsto \alpha v$, is a linear transformation, $\forall \alpha \in G$.

In the following, $X$ is always a finite $G$-set with cardinality $|X| = m$. Let $FX$ be the $F$-vector space with basis $X$. And the group $G$ acts on the vector space $FX$ in a natural way:

$$\alpha \left( \sum_{x \in X} f(x)x \right) = \sum_{x \in X} f(x)\alpha x \left( = \sum_{y \in X} f(\alpha^{-1}y)y \right), \quad \forall \alpha \in G, \sum_{x \in X} f(x)x \in FX. \quad (2.3)$$

So $FX$ is an $FG$-module, called the permutation module of the $G$-set $X$.

On the other hand, $G$ acts on the function space $F^X$ also in a natural way: for $\alpha \in G$ and $f \in F^X$, the $\alpha f \in F^X$ is defined as (compare it with Eq.(2.3)):

$$(\alpha f)(x) = f(\alpha^{-1}x), \quad \forall x \in X. \quad (2.4)$$
So \( F^X \) is also an \( FG \)-module. Similarly to Eq.\((2.1)\), we have the following natural \( FG \)-module isomorphism:

\[
F^X \rightarrow FX, \quad f \mapsto \sum_{x \in X} f(x)x.
\] (2.5)

Also, we identify any function \( f \in F^X \) with the element \( f = \sum_{x \in X} f(x)x \in FX \).

Note that, extending the \( G \)-action of Eq.\((2.4)\), for \( g = \sum_{\alpha \in G} g(\alpha)\alpha \in FG \) and \( f \in F^G \) we have \((\sum_{\alpha \in G} g(\alpha)\alpha)f = \sum_{\alpha \in G} g(\alpha)(\alpha f)\); i.e.,

\[
((\sum_{\alpha \in G} g(\alpha)\alpha)f)(x) = \sum_{\alpha \in G} g(\alpha)f(\alpha^{-1}x), \quad \forall x \in X.
\]

Viewing \( g \in F^G \) as in Eq.\((2.1)\), the right hand side of the above equation is just the so-called convolution \( g * f \in F^X \) as follows:

\[
(g * f)(x) = \sum_{\alpha \in G} g(\alpha)f(\alpha^{-1}x), \quad \forall g \in F^G, f \in F^X, x \in X.
\]

Let \( f \in F^X \). The support of \( f \) is the following subset of \( X \):

\[
\text{supp}(f) = \{ x \mid x \in X, f(x) \neq 0 \}.
\] (2.6)

For \( \alpha \in G \), since \( (\alpha f)(x) = f(\alpha^{-1}x) \), \( \forall x \in X \), we see that

\[
\text{supp}(\alpha f) = \alpha \cdot \text{supp}(f), \quad \forall \alpha \in G, f \in F^X,
\] (2.7)

where \( \alpha \cdot \text{supp}(f) = \{ \alpha y \mid y \in \text{supp}(f) \} \). A subset \( S \subseteq X \) is said to be \( G \)-stable if \( \alpha S = S \), \( \forall \alpha \in G \).

**Remark 2.2.** We say that \( f \in F^X \) is a \( G \)-linear function if for any \( \alpha \in G \) the functions \( \alpha f \) and \( f \) are linearly dependent (hence \( \text{supp}(f) \) must be \( G \)-stable, cf, Eq.\((2.7)\)). Assume that \( 0 \neq f \in F^X \) is \( G \)-linear, i.e., for any \( \alpha \in G \) there is an element \( c_\alpha \in F^x \) such that \( \alpha f = c_\alpha f \); then for any \( \alpha, \beta \in G \) we have

\[
c_{\alpha \beta} f = (\alpha \beta) f = \alpha(\beta f) = \alpha(c_\beta f) = c_\beta(\alpha f) = c_\beta c_\alpha f = c_\alpha c_\beta f.
\]

So \( c_{\alpha \beta} = c_\alpha c_\beta \) (since \( f \neq 0 \)), and the map \( \eta : G \rightarrow F^x, \alpha \mapsto c_\alpha \), is a homomorphism. Note that any homomorphism \( \eta : G \rightarrow F^x \) is an irreducible representation of \( G \) over \( F \) of degree 1 (but maybe not absolutely irreducible because \( F \) is any field); and all such representations form a group, denoted by \( \text{Hom}(G, F^x) \), with respect to the function multiplication. Thus, for \( \eta \in \text{Hom}(G, F^x) \), we have the inverse \( \eta^{-1} \in \text{Hom}(G, F^x) \), \( \eta^{-1}(\alpha) = \eta(\alpha^{-1}) = \eta(\alpha^{-1}), \forall \alpha \in G \).

We conclude that: a non-zero function \( f \in F^X \) is \( G \)-linear if and only if there is an \( \eta \in \text{Hom}(G, F^x) \) such that \( \alpha f = \eta(\alpha)f, \forall \alpha \in G \).

If \( X = G \) is the left regular \( G \)-set, then for a non-zero \( G \)-linear function \( f \in F^G \), i.e., \( \alpha f = \eta(\alpha)f \) for an \( \eta \in \text{Hom}(G, F^x) \), we have

\[
f(\alpha) = f(\alpha 1_G) = \alpha^{-1} f(1_G) = \eta^{-1}(\alpha)f(1_G), \quad \forall \alpha \in G;
\]
i.e., \( f = c \eta^{-1} \) where \( c = f(1_G) \in F^x \). In conclusion, \( 0 \neq f \in F^G \) is a \( G \)-linear function if and only if \( f = c \eta \) with \( c \in F^x \) and \( \eta \in \text{Hom}(G, F^x) \).
2.2 Fourier transform for group actions

To describe Fourier transformations, in the rest of this section we assume that the field $\mathbb{F}$ satisfies the condition Eq.(1.5), so that $\mathbb{F} G$ is semisimple. And further, we should extend $\mathbb{F}$ to a splitting field $\mathbb{E}$ for $G$.

Let $\omega$ be a primitive exp$(G)$'th root of unity, where exp$(G)$ is the exponent of $G$ (i.e. the least common multiple of the orders of the elements of $G$). Let $\mathbb{E} = \mathbb{F}(\omega)$ be the extension field over $\mathbb{F}$ by $\omega$. Then $\mathbb{E}$ is a splitting field for $G$, see [14, Theorem 24] for char$\mathbb{F} = 0$, and [14, Proposition 43] for char$\mathbb{F} \neq 0$; and any $\mathbb{E}$-irreducible character $\psi$ is absolutely irreducible.

By $\text{Irr}(G)$ we denote the set of all absolutely irreducible characters of $G$. For any $\psi \in \text{Irr}(G)$, by $n_\psi$ we denote the degree of $\psi$. There is a representation (homomorphism) $\rho^\psi : G \to \text{GL}_{n_\psi}(\mathbb{E})$, affording the character $\psi$, where $\text{GL}_{n_\psi}(\mathbb{E})$ denotes the group of all invertible matrices over $\mathbb{E}$ of degree $n_\psi$. That is, we get $n_\psi^2$ functions $\rho^\psi_{ij} \in \mathbb{E}^G$, $1 \leq i, j \leq n_\psi$, such that, for any $\alpha \in G$, we have the matrix $\rho^\psi(\alpha) = (\rho^\psi_{ij}(\alpha))_{1 \leq i, j \leq n_\psi}$ with trace $\text{Tr}(\rho^\psi(\alpha)) = \psi(\alpha)$. Let

$$\hat{G} = \{ \rho^\psi_i | \psi \in \text{Irr}(G), i, j = 1, \ldots, n_\psi \}. \quad (2.8)$$

For $\rho^\psi_{ij}, \rho^\psi_{k\ell} \in \hat{G}$ and $\alpha, \beta \in G$, by the corollaries of Schur's Lemma ([14, §2.2 Corollaries 2 and 3]) we have the following formula:

$$(\rho^\psi_{ij} \ast \rho^\psi_{k\ell})(\alpha) = \sum_{\beta \in G} \rho^\psi_{ij}(\beta^{-1})\rho^\psi_{k\ell}(\beta\alpha) = \begin{cases} \frac{1}{n_\psi} \rho^\psi_{ij}(\alpha), & \psi = \varphi \text{ and } j = k; \\ 0, & \text{otherwise.} \end{cases} \quad (2.9)$$

Then $\hat{G}$ is a linearly independent (orthogonal in fact) subset of $\mathbb{E}^G$, and

$$|\hat{G}| = \sum_{\psi \in \text{Irr}(G)} n_\psi^2 = n = |G|; \quad (2.10)$$

see [14, Propositions 4, 5 and their corollaries]. So $\hat{G}$ is a basis of $\mathbb{E}^G$.

**Definition 2.3.** We call $\hat{G}$ in Eq.(2.8) by the *dual basis* of $G$.

By the classical decomposition of $\mathbb{E} G$-modules ([14, §2.6 Theorem 8]), $\mathbb{E}^X$ is decomposed into a direct sum $\mathbb{E}^X = \bigoplus_{\psi \in \text{Irr}(G)} V^\psi$, and

$$V^\psi = W^\psi_1 \oplus \cdots \oplus W^\psi_{m_\psi}, \quad \psi \in \text{Irr}(G), \quad (2.11)$$

with each $W^\psi_i$ affording the character $\psi$. Thus, for $\alpha \in G$, the action of $\alpha$ on $W^\psi_i$ induces a linear transformation of $W^\psi_i$ with matrix $\rho^\psi(\alpha) = (\rho^\psi_{ij}(\alpha))_{1 \leq i, j \leq n_\psi}$. In other words, each $W^\psi_i$ has an $\mathbb{E}$-basis $\lambda^\psi_{i1}, \ldots, \lambda^\psi_{im_\psi}$ such that,

$$\alpha \lambda^\psi_{ij} = \sum_{k=1}^{m_\psi} \rho^\psi_{jk}(\alpha)\lambda^\psi_{ik}, \quad \psi \in \text{Irr}(G), \ i = 1, \ldots, n_\psi, \ j = 1, \ldots, m_\psi.$$
In matrix version,
\[
\begin{pmatrix}
\alpha_{11} \psi & \cdots & \alpha_{1n}\psi \\
\cdots & \cdots & \cdots \\
\alpha_{m1} \psi & \cdots & \alpha_{mn}\psi
\end{pmatrix}
= \begin{pmatrix}
\lambda_{11} \psi & \cdots & \lambda_{1n}\psi \\
\cdots & \cdots & \cdots \\
\lambda_{m1} \psi & \cdots & \lambda_{mn}\psi
\end{pmatrix} \begin{pmatrix}
\rho_{11}(\alpha) & \cdots & \rho_{1n}\psi(\alpha) \\
\cdots & \cdots & \cdots \\
\rho_{m1}(\alpha) & \cdots & \rho_{mn}\psi(\alpha)
\end{pmatrix};
\]
or shortly, with \( \lambda^\psi = (\lambda^\psi_{ij})_{1 \leq i \leq m, 1 \leq j \leq n} \) and \( \alpha \lambda^\psi = (\alpha \lambda^\psi_{ij})_{1 \leq i \leq m, 1 \leq j \leq n} \),

\[
\alpha \lambda^\psi = \lambda^\psi \cdot \rho^\psi(\alpha), \quad \forall \alpha \in G, \forall \psi \in \text{Irr}(G). \quad (2.12)
\]

**Definition 2.4.** Let \( \hat{X} = \{ \lambda^\psi_{ij} \mid \psi \in \text{Irr}(G), 1 \leq i \leq m, 1 \leq j \leq n \} \) as above. Then \( \hat{X} \) is a basis of \( \mathbb{E}^\hat{X} \). We say that \( \hat{X} \) is a **G-dual set** of the G-set \( X \).

By \( \mathbb{E} \hat{X} \) we denote the vector space with basis \( \hat{X} \). As usual, any function \( h \in \mathbb{E} \hat{X} \) is linearly extended to a linear function on the space \( \mathbb{E} \hat{X} \). Then, for \( h \in \mathbb{E} \hat{X} \) and \( \alpha \in G \), by Eq.\((2.12)\) we have:

\[
ah(\lambda^\psi) = h(\alpha^{-1} \lambda^\psi) = h(\lambda^\psi \cdot \rho^\psi(\alpha^{-1})) = h(\lambda^\psi) \cdot \rho^\psi(\alpha^{-1}). \quad (2.13)
\]

**Lemma 2.5.** \( \mathbb{E}^\hat{X} \) is a G-space with G-action on the vector space as follows (it is just the entry-wise version of Eq.\((2.13)\)):

\[
ah(\lambda^\psi_{ij}) = \sum_{k=1}^{n} h(\lambda^\psi_{ik}) \rho^\psi_{kj}(\alpha^{-1}), \quad \forall \alpha \in G, \ h \in \mathbb{E} \hat{X}, \ \forall \lambda^\psi_{ij} \in \hat{X}.
\]

**Proof.** It is easy to check that for \( \alpha \in G \) the map \( h \mapsto ah \) is a linear transformation of \( \mathbb{E} \hat{X} \), i.e., \( \alpha(h_1 + h_2) = \alpha h_1 + \alpha h_2, \ \forall h_1, h_2 \in \mathbb{E} \hat{X} \); and \( \alpha(ch) = c(\alpha h) \ \forall h \in \mathbb{E} \hat{X}, \forall c \in \mathbb{E} \). For \( \alpha, \beta \in G, h \in \mathbb{E} \hat{X} \) and \( \lambda^\psi = (\lambda^\psi_{ij})_{m \times n} \),

\[
(\alpha \beta)h(\lambda^\psi) = h(\lambda^\psi) \rho^\psi((\alpha \beta)^{-1}) = (h(\lambda^\psi) \rho^\psi((\beta^{-1})) \rho^\psi((\alpha)^{-1}) = \alpha(\beta h(\lambda^\psi));
\]
so \( (\alpha \beta)h = \alpha(\beta h) \). By Remark 2.1(2), \( \mathbb{E} \hat{X} \) is a G-space. \( \square \)

Now we can define the Fourier transformation for the G-set \( X \).

**Definition 2.6.** The map: \( \mathbb{E}^X \rightarrow \mathbb{E} \hat{X}, \ f \mapsto \hat{f} \), where

\[
\hat{f}(\lambda^\psi_{ij}) = \sum_{x \in X} f(x) \lambda^\psi_{ij}(x), \quad \psi \in \text{Irr}(G), \ i = 1, \cdots, m, \ j = 1, \cdots, n;
\]
is called the **Fourier transformation** for the G-set \( X \), and \( \hat{f} \) is said to be the Fourier transform of \( f \).

**Lemma 2.7.** The Fourier transformation \( \mathbb{E}^X \rightarrow \mathbb{E} \hat{X}, \ f \mapsto \hat{f}, \) is a G-space (\( \mathbb{E} \text{G-module} \) isomorphism.)
Proof. We list $\text{Irr}(G)$, $X$ and index $\widehat{X}$ in lexicographical order:

$$\text{Irr}(G) = \{ \psi_1, \cdots, \psi_r \}, \quad X = \{ x_1, \cdots, x_m \},$$

$$\widehat{X} = \{ \lambda_{11}^{\psi_1}, \cdots, \lambda_{ij}^{\psi_1}, \cdots, \lambda_{m_r n_r}^{\psi_1} \}, \quad \lambda_{ij}^{\psi_1} = \lambda_{ij}^{\psi_k}, \quad m_r = m_{\psi_k}, \quad n_r = n_{\psi_k}.$$  \hfill (2.14)

Then

$$\begin{pmatrix}
\hat{f}(\lambda_{ij}^{11}) \\
\vdots \\
\hat{f}(\lambda_{m_r n_r}^{1}) 
\end{pmatrix} =
\begin{pmatrix}
\lambda_{11}^1(x_{1}) & \cdots & \lambda_{11}^1(x_{m}) \\
\vdots & \ddots & \vdots \\
\lambda_{m_r n_r}^1(x_{1}) & \cdots & \lambda_{m_r n_r}^1(x_{m}) 
\end{pmatrix}
\begin{pmatrix}
f(x_{1}) \\
\vdots \\
f(x_{m}) 
\end{pmatrix}. \hfill (2.15)$$

Thus $f \mapsto \hat{f}$ is a linear map. Since $\lambda_{11}^{\psi_1}, \cdots, \lambda_{ij}^{\psi_1}, \cdots, \lambda_{m_r n_r}^{\psi_1}$ are a basis of $E^X$ (hence $m_1 n_1 + \cdots + m_r n_r = m$), the rows of the $m \times m$ matrix $(\lambda_{ij}^{\psi_1}(x_t))_{m \times m}$ in Eq.(2.15) are linearly independent, i.e., the $m \times m$ matrix is invertible. So $f \mapsto \hat{f}$ is a linear isomorphism. For $\alpha \in G$, $f \in E^X$ and $\lambda^{\psi} = (\lambda_{ij}^{\psi})_{m_r \times n_r}$ as in Eq.(2.12), we have

$$\hat{f}(\alpha^{\psi}) = \sum_{x \in X} (\alpha f)(x) \lambda^{\psi}(x) = \sum_{x \in X} f(\alpha^{-1} x) \lambda^{\psi}(x) = \sum_{y \in X} f(y) \lambda^{\psi}(\alpha y)$$

$$= \sum_{y \in X} f(y) (\alpha^{-1} \lambda^{\psi})(y) = \sum_{y \in X} f(y) \cdot \lambda^{\psi}(y) \cdot \rho^{\psi}(\alpha^{-1}) \quad \text{(by Eq.(2.12))}$$

$$= \sum_{y \in X} (f(y) \lambda^{\psi}(y)) \rho^{\psi}(\alpha^{-1}) = \hat{f}(\lambda^{\psi}) \rho^{\psi}(\alpha^{-1}) = (\alpha \hat{f})(\lambda^{\psi});$$

where the last equality is by Eq.(2.13). Thus $\hat{f} = \alpha \hat{f}$, for all $\alpha \in G$, for all $f \in E^X$. The Fourier transformation $f \mapsto \hat{f}$ is a $G$-space isomorphism. \hfill \square

Remark 2.8. (1) If $F = \mathbb{C}$ (hence $E = \mathbb{C}$) is the complex field, then there is a $G$-dual set $\widehat{X}$ having better properties (in particular, orthogonality) such that the Fourier inversion can be defined in a classical way, cf. [8].

(2) If $X = G$ is the left regular $G$-set, then we take $\widehat{X} = \widehat{G}$ as usual; the dual basis $\widehat{G}$ possesses orthogonality, see Eq.(2.9); and many things can be done better too, cf. [7].

(3) Assume that $G$ is abelian. If $X = G$ is the regular set, then $\widehat{X} = \widehat{G}$ is the dual group of $G$ (and the choice of $\widehat{X}$ is unique up to rescaling), every thing is classical. If $X$ is a transitive $G$-set, it is reduced to the regular set of a quotient group of $G$. Finally, if $X$ is not transitive, $X$ is partitioned into orbits, and it is reduced to each orbit which is transitive; cf. [6].

2.3 The rank-support of the Fourier transform $\hat{f}$

For $f \in E^X$, the support $\text{supp}(f) = \{ x \mid x \in X, \ f(x) \neq 0 \}$ is defined in Eq.(2.6). In coding theory, the cardinality $|\text{supp}(f)|$ is just the Hamming weight of $f$. 9
Since $\mathbb{F}^X \subseteq \mathbb{E}^X$, we can consider the Fourier transform $\hat{f} \in \mathbb{E}^{\hat{X}}$. Because of a technical reason, e.g., see [18, §3.2.2], we define

$$|\supp(\hat{f})| := \sum_{\psi \in \text{Irr}(G)} n_\psi \cdot |\supp(\hat{f}(\lambda^\psi))|,$$

where $\hat{f}(\lambda^\psi) = (\hat{f}(\lambda^\psi)_{ij})_{m_\psi \times n_\psi}$, cf. Eq.(2.12). However, the $G$-dual set $\hat{X}$ is not unique, it depends on

(C1) the choices of the dual basis $\hat{G}$ of $G$; and on

(C2) the choices of the decompositions Eq.(2.11).

Following [18, Definition 3.6], we define

$$|\min\text{-supp}(\hat{f})| = \min_{\hat{X}} |\supp(\hat{f})|,$$

where the minimum is over the choices of the $G$-dual set $\hat{X}$. On the other hand, following [12] (see Eq.(1.2)), we define the rank support as follows:

$$\text{rk-support}(\hat{f}) = \sum_{\psi \in \text{Irr}(G)} n_\psi \cdot \text{rank}(\hat{f}(\lambda^\psi)).$$

Note that $\text{rk-support}(\hat{f})$ is an integer, not a set.

Any change of the above first choice (C1) implies that there is a $P \in \text{GL}_{n_\psi}(\mathbb{E})$ such that the matrix $\hat{f}(\lambda^\psi)$ is changed to $\hat{f}(\lambda^\psi) \cdot P$. A change of the second choice (C2) implies that there is a $Q \in \text{GL}_{m_\psi}(\mathbb{E})$ such that the matrix $\hat{f}(\lambda^\psi) \cdot P$ is changed to $Q \cdot \hat{f}(\lambda^\psi) \cdot P$. There are $P, Q$ such that

$$Q \cdot \hat{f}(\lambda^\psi) \cdot P = \begin{pmatrix} 1 & & \\ & 1 & \\ & & \ddots \end{pmatrix}_{m_\psi \times n_\psi}$$

is a diagonal (maybe not square) matrix with 1 or 0 on the diagonal, and the number of 1 on the diagonal equals $\text{rank}(\hat{f}(\lambda^\psi))$. We summarize the above discussion into the following lemma.

**Lemma 2.9.** Let $\hat{X}$ be defined as in Definition 2.4. For any $f \in \mathbb{F}^X$ we have that $\text{rk-support}(\hat{f}) = |\min\text{-supp}(\hat{f})|$.

Note that, if $X = G$ is the left regular set, then we take $\hat{X} = \hat{G}$ as usual (cf. Remark 2.8(2)), hence there is only one choice, i.e., (C1); as a result, the matrix $\hat{f}(\rho^\psi)$ can only be changed to $\hat{f}(\rho^\psi) \cdot P$; and we can only get an inequality $\text{rk-support}(\hat{f}) \leq |\min\text{-supp}(\hat{f})|$; see [18, Lemma 3.8].

We will investigate the sharpened uncertainty principle with $|\supp(f)|$ and $\text{rk-support}(\hat{f})$. 

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Lemma 2.10. For any \( f \in \mathbb{F}^X \), by \( \dim \mathbb{E}G\hat{f} \) we denote the \( \mathbb{E} \)-dimension of the \( \mathbb{E} \)-submodule \( \mathbb{E}G\hat{f} \) of \( \mathbb{E}^X \) generated by \( \hat{f} \); by \( \dim \mathbb{F}Gf \) we denote the \( \mathbb{F} \)-dimension of the \( \mathbb{F} \)-submodule \( \mathbb{F}Gf \) of \( \mathbb{F}^X \) generated by \( f \). Then

\[
\text{rk-supp}(\hat{f}) = \dim \mathbb{E}G\hat{f} = \dim \mathbb{F}Gf.
\]

Proof. Let \( M_{m \times n}(\mathbb{E}) \) (\( M_n(\mathbb{E}) \) resp.) be the \( \mathbb{E} \)-space of all \( m \times n \) \((n \times n \text{ resp.})\) matrices over \( \mathbb{E} \). Keep the notation in Eq.(2.14). The representations \( \rho^k = \rho^{\alpha_k}, \ k = 1, \cdots, r \) induces an \( \mathbb{E} \)-algebra isomorphism (see [14, Proposition 10]):

\[
\rho : \mathbb{E}G \stackrel{\cong}{\to} M_{m_i}(\mathbb{E}) \times \cdots \times M_{n_r}(\mathbb{E}), \quad \sum_{\alpha \in G} g(\alpha)\alpha \mapsto \left( \sum_{\alpha \in G} g(\alpha)\rho^1(\alpha), \cdots, \sum_{\alpha \in G} g(\alpha)\rho^r(\alpha) \right). \tag{2.18}
\]

Since \( \hat{X} \) is a basis of \( \mathbb{E}^X \), denoting \( \lambda^k = \lambda^{\alpha_k} \) as in Eq.(2.12), we have the following linear isomorphism:

\[
\xi : \mathbb{E}^X \stackrel{\cong}{\to} M_{m_1 \times n_1}(\mathbb{E}) \times \cdots \times M_{m_r \times n_r}(\mathbb{E}), \quad h \mapsto (h(\lambda^1), \cdots, h(\lambda^r)). \tag{2.19}
\]

For \( \alpha \in G \) and \( A = (A^1, \cdots, A_r) \) with \( A^k \in M_{m_k \times n_k}(\mathbb{E}) \), define \( \alpha \circ A \) as follows:

\[
\alpha \circ A = (A^1\rho^1(\alpha^{-1}), \cdots, A_r\rho^r(\alpha^{-1})). \tag{2.20}
\]

It is easy to check that the right hand side of Eq.(2.19) is a \( \mathbb{G} \)-space, we check the condition (i) for \( \mathbb{G} \)-spaces in Remark 2.1(2) as follows: for \( \alpha, \beta \in G, \)

\[
(\alpha\beta) \circ A = (\cdots, A^k\rho^k((\alpha\beta)^{-1}), \cdots) = (\cdots, A^k\rho^k(\beta^{-1}\alpha^{-1}), \cdots) = (\cdots, A^k\rho^k(\beta^{-1})\rho^k(\alpha^{-1}), \cdots) = (\cdots, (A^k\rho^k(\beta^{-1}))(\alpha^{-1}), \cdots) = \alpha \circ \cdots, A^k\rho^k(\beta^{-1}), \cdots = \alpha \circ (\cdots, A^k\rho^k(\beta^{-1}), \cdots).
\]

Computing it entry by entry similarly to the above, for \( \alpha \in G \) and \( h \in \mathbb{E}^\hat{X} \), by Lemma 2.5 (i.e., Eq.(2.13)), we have

\[
\xi(ah) = (\cdots, ah(\lambda^k), \cdots) = (\cdots, h(\lambda^k)\rho^k(\alpha^{-1}), \cdots) = \alpha \circ \xi(h).
\]

Thus \( \xi \) in Eq.(2.19) is an \( \mathbb{E}G \)-module isomorphism. Then, by the isomorphisms Eq.(2.18), Eq.(2.19) and the definition Eq.(2.20),

\[
\xi(\mathbb{E}G\hat{f}) = \mathbb{E}G \circ \xi(\hat{f}) = \hat{f}(\lambda^1)M_{m_1}(\mathbb{E}) \times \cdots \times \hat{f}(\lambda^r)M_{n_r}(\mathbb{E}).
\]

Hence

\[
\dim \mathbb{E}G\hat{f} = \dim (\mathbb{E}G \circ \xi(\hat{f})) = \sum_{k=1}^r \dim (\hat{f}(\lambda^k)M_{n_k}(\mathbb{E})) = \sum_{k=1}^r n_k \cdot \text{rank}(\hat{f}(\lambda^k)).
\]

By definition Eq.(2.17), \( \dim \mathbb{E}G\hat{f} = \text{rk-supp}(\hat{f}) \).

By Lemma 2.7, \( \dim \mathbb{E}G\hat{f} = \dim \mathbb{E}Gf \).

Finally, the \( \mathbb{E} \)-space \( \mathbb{E}Gf \) is obtained from the \( \mathbb{F} \)-space \( \mathbb{F}Gf \) by extending coefficient field; so \( \dim \mathbb{E}Gf = \dim \mathbb{F}Gf \). \( \square \)
Remark 2.11. (1) By Lemma 2.10, we can consider the uncertainty principle on the transitive $G$-set $X$ for any field $F$ with $\dim FGf$ instead of $\text{rk-su}pp(\hat{f})$ (hence the condition Eq. (1.5) on $F$ is no longer necessary). When such an uncertainty principle is proved, then an uncertainty principle on $\text{rk-su}pp(\hat{f})$ (with the condition Eq. (1.5) on $F$), or on the regular set $X = G$, follows directly.

(2) If $G$ is abelian, we can consider only the regular $G$-set $X = G$ (see Remark 2.8(3)), then everything seems easier:

- $n_\psi = 1$ and $\rho^\psi = \psi$ for each $\psi \in \text{Irr}(G)$, $\hat{G} = \text{Irr}(G)$ is the dual group.
- $\text{rk-su}pp(\hat{f}) = |\text{supp}(\hat{f})|$; since $\hat{f}(\lambda^\psi) = \hat{f}(\psi)$ is a $1 \times 1$ matrix in Eq. (2.17).
- Eq. (2.19) and Eq. (2.20) are reduced to
  \[ \xi : \hat{f} = \text{EG} \rightarrow E \times \cdots \times E, \quad h \mapsto (h(\psi_1), \ldots, h(\psi_n)); \]
  and, for $\alpha \in G$ and $(a_1, \ldots, a_n) \in E \times \cdots \times E$,
  \[ \alpha \circ (a_1, \ldots, a_n) = (a_1\psi_1(\alpha^{-1}), \ldots, a_n\psi_n(\alpha^{-1})). \]
- Hence, $\xi(\text{EG}\hat{f}) = \text{EG} \circ \xi(\hat{f}) = \hat{f}(\psi_1)E \times \cdots \times \hat{f}(\psi_n)E$.
- By the above equality, Lemma 2.10 looks obvious.

3 Sharpened uncertainty principle

In this section we assume that $G$ is any finite group of order $|G| = n$ with operation written as multiplication, $F$ is any field, and $X$ is a transitive $G$-set (or, we say that $G$ acts transitively on the set $X$, cf. Remark 2.1(1)).

3.1 Preparations

For any subsets $\mathcal{A} \subseteq G$ and $Y \subseteq X$, we denote $\mathcal{A}^{-1} = \{\alpha^{-1} | \alpha \in \mathcal{A}\}$, and $\mathcal{A}Y = \{\alpha y | \alpha \in \mathcal{A}, y \in Y\}$. We also write $\mathcal{A}Y = \mathcal{A}y$ if $Y = \{y\}$; or $\mathcal{A}Y = \alpha Y$ if $\mathcal{A} = \{\alpha\}$. Note that for any subsets $\mathcal{A}, \mathcal{B} \subseteq G$ and $Y \subseteq X$ the following hold:

\[
(\mathcal{A} \mathcal{B})Y = \mathcal{A}(\mathcal{B}Y), \quad (\mathcal{A} \cup \mathcal{B})Y = (\mathcal{A}Y) \cup (\mathcal{B}Y).
\]

However, the distribution for intersection “$(\mathcal{A} \cap \mathcal{B})Y = \mathcal{A}Y \cap \mathcal{B}Y$” no longer holds in general (e.g., see the notice after Corollary 3.3 below). We denote the difference set by $\mathcal{A} - \mathcal{B} = \{\alpha | \alpha \in \mathcal{A} \text{ but } \alpha \notin \mathcal{B}\}$. In particular, $X - Y$ denotes the complement of $Y$ in $X$.

Given any $x_0 \in X$. Since the $G$-set $X$ is transitive, we have a natural surjective map

\[
\zeta_{x_0} : G \rightarrow X, \quad \zeta_{x_0}(\alpha) = \alpha x_0, \quad \forall \alpha \in G.
\]
For any \( Y \subseteq X \), we denote \( \zeta^{-1}_{x_0}(Y) = \{ \alpha | \alpha \in G, \zeta_{x_0}(\alpha) \in Y \} \); i.e., \( \zeta^{-1}_{x_0}(Y) \) is the full inverse image in \( G \) of \( Y \). Let
\[
G_{x_0} = \{ \alpha | \alpha \in G, \alpha x_0 = x_0 \},
\]
which is a subgroup of \( G \), because: for \( \alpha, \beta \in G_{x_0} \), \( (\alpha \beta)x_0 = \alpha(\beta x_0) = \alpha x_0 = x_0 \), i.e., \( \alpha \beta \in G_{x_0} \). We call the subgroup \( G_{x_0} \) by the stabilizer of \( x_0 \) in \( G \). As usual, \( G/G_{x_0} = \{ \gamma G_{x_0} | \gamma \in G \} \) denotes the set of all left cosets of the subgroup \( G_{x_0} \). For any \( x \in X \), it is obviously that \( \zeta^{-1}_{x_0}(x) = \gamma_x G_{x_0} \) is a left coset of \( G_{x_0} \), where \( \zeta_{x_0}(\gamma_x) = \gamma_x x_0 = x \). Then \( G \) acts by left translation (multiplication) on the set \( G/G_{x_0} \), and we have a bijection:
\[
G/G_{x_0} \to X, \quad \gamma G_{x_0} \mapsto \gamma x_0,
\]
which is a \( G \)-equivalence of \( G \)-sets (cf. [1, §3 Proposition 4]).

**Remark 3.1.** From the equivalence Eq.(3.3), in the following we assume that
\[
|G_{x_0}| = k, \quad |X| = m; \quad \text{hence} \quad n = |G| = km.
\]
By the equivalence, for any \( A \subseteq G \),
\[
A \subseteq \zeta^{-1}_{x_0}(\zeta_{x_0}(A)) = \zeta^{-1}_{x_0}(A x_0),
\]
then \( |A| \leq k \cdot |A x_0| \).

**Lemma 3.2.** Let notation be as above. For any subset \( A \subseteq G \) the following three statements are equivalent to each other.

1. \( \zeta^{-1}_{x_0}(\zeta_{x_0}(A)) = \zeta^{-1}_{x_0}(A x_0) \).  
   
2. \( A \) is a disjoint union of some left cosets of \( G_{x_0} \), i.e., \( \mathcal{A} G_{x_0} = A \).

3. \( |A| = k \cdot |A x_0| \).

**Proof.** (1) \( \Rightarrow \) (2). By the equivalence Eq.(3.3), for any \( x \in X \), \( \zeta^{-1}_{x_0}(x) = \gamma_x G_{x_0} \) where \( \zeta_{x_0}(\gamma_x) = \gamma_x x_0 = x \). Thus, from (1) we get \( \mathcal{A} = \bigcup_{x \in \mathcal{A} x_0} \gamma_x G_{x_0} \).

(2) \( \Rightarrow \) (3). If \( \mathcal{A} = \bigcup_{i=1}^{k} \gamma_i G_{x_0} \) is a disjoint union, then, by the equivalence Eq.(3.3), \( \mathcal{A} x_0 = \{ \gamma_1 x_0, \ldots, \gamma_k x_0 \} \) and \( \gamma_1 x_0, \ldots, \gamma_k x_0 \) are distinct in \( X \). Thus, \( |\mathcal{A}| = k \cdot |A x_0| \).

(3) \( \Rightarrow \) (1). We have seen that \( \mathcal{A} \subseteq \zeta^{-1}_{x_0}(\mathcal{A} x_0) \). Since \( \zeta^{-1}_{x_0}(x) = \gamma_x G_{x_0} \) for \( x \in X \), by (3) we have \( |\zeta^{-1}_{x_0}(\mathcal{A} x_0)| = k \cdot |A x_0| = |\mathcal{A}| \). Hence (1) holds. \( \square \)

If one of the three (hence all) statements in Lemma 3.2 holds, then we say that \( \mathcal{A} \) is an \( x_0 \)-closed subset of \( G \).

**Corollary 3.3.** Let \( \mathcal{A}, \mathcal{B} \) be subsets of \( G \). The following three hold.

1. If \( \mathcal{A} \) is \( x_0 \)-closed, then so is \( \mathcal{B} \mathcal{A} \).

2. If both \( \mathcal{A} \) and \( \mathcal{B} \) are \( x_0 \)-closed, then so are \( \mathcal{A} \cup \mathcal{B} \) and \( \mathcal{A} \cap \mathcal{B} \).

3. If both \( \mathcal{A} \) and \( \mathcal{B} \) are \( x_0 \)-closed, then
\[
(\mathcal{A} \cap \mathcal{B}) x_0 = (\mathcal{A} x_0) \cap (\mathcal{B} x_0).
\]
Proof. (1). \( \mathcal{B} \mathcal{A} G_{x_0} = \mathcal{B} \mathcal{A} \). By Lemma 3.2(2), \( \mathcal{B} \mathcal{A} \) is \( x_0 \)-closed.

(2). If \( \gamma \in \mathcal{A} \cap \mathcal{B} \), then \( \gamma G_{x_0} \subseteq \mathcal{A} \) and \( \gamma G_{x_0} \subseteq \mathcal{B} \), hence \( \gamma G_{x_0} \subseteq \mathcal{A} \cap \mathcal{B} \). By Lemma 3.2(2), \( \mathcal{A} \cap \mathcal{B} \) is \( x_0 \)-closed.

(3). Assume that \( \mathcal{A} \) and \( \mathcal{B} \) are disjoint unions of left cosets of \( G_{x_0} \) as follows:

\[
\mathcal{A} = \alpha_1 G_{x_0} \cup \cdots \cup \alpha_i G_{x_0} \cup \gamma_1 G_{x_0} \cup \cdots \cup \gamma_j G_{x_0},
\]

\[
\mathcal{B} = \beta_1 G_{x_0} \cup \cdots \cup \beta^r G_{x_0} \cup \gamma_1 G_{x_0} \cup \cdots \cup \gamma_j G_{x_0},
\]

such that \( \mathcal{A} \cap \mathcal{B} = \gamma_1 G_{x_0} \cup \cdots \cup \gamma_j G_{x_0} \). Then

\[
(\mathcal{A} \cap \mathcal{B}) x_0 = \{ \gamma_1 x_0, \cdots, \gamma_j x_0 \} = (\mathcal{A} x_0) \cap (\mathcal{B} x_0).
\]

We are done. \( \square \)

Note that the condition “both \( \mathcal{A} \) and \( \mathcal{B} \) are \( x_0 \)-closed” of Corollary 3.3(3) is necessary. For example, if we partition \( G_{x_0} \) into two subsets \( \mathcal{A} \) and \( \mathcal{B} \), then \( (\mathcal{A} \cap \mathcal{B}) x_0 = 0 \neq \{ x_0 \} = (\mathcal{A} x_0) \cap (\mathcal{B} x_0) \).

Remark 3.4. (1) Assume that \( B \subseteq X \). If \( \{ \alpha B \mid \alpha \in G \} \) form a partition of the set \( X \), i.e., for any \( \alpha \in G \), either \( B \cap (\alpha B) = B \) or \( B \cap (\alpha B) = \emptyset \), then we say that \( B \) is a block of the transitive \( G \)-set \( X \). In that case, there are \( \gamma_1, \cdots, \gamma_r \in G \) such that \( X = (\gamma_1 B) \cup \cdots \cup (\gamma_r B) \) is a disjoint union.

The single point \( \{ x \} \ (x \in X) \) and \( X \) are called the trivial blocks.

(2) Let \( x_0 \in B \subseteq X \) (so \( G_{x_0} \subseteq \zeta^{-1}_{x_0}(B) \)). It is known that \( B \) is a block if and only if \( H := \zeta^{-1}_{x_0}(B) \) is a subgroup of \( G \). If it is the case, then \( B \) is \( H \)-stable, hence an \( H \)-set, and the partition (i.e., the disjoint union) \( X = (\gamma_1 B) \cup \cdots \cup (\gamma_r B) \) corresponds the disjoint union \( G = (\gamma_1 H) \cup \cdots \cup (\gamma_r H) \) of the left cosets of the subgroup \( H = \zeta^{-1}_{x_0}(B) \). See [1, §3 Proposition 9].

For any subset \( \mathcal{J} \subseteq G \), following [9] (\( G \) is abelian in [9], but maybe not abelian here) we denote

\[
G_{\mathcal{J}} = \{ \alpha \mid \alpha \in G, \mathcal{J} \alpha = \mathcal{J} \}.
\]

\( G_{\mathcal{J}} \) is a subgroup of \( G \), because: for \( \alpha, \beta \in G_{\mathcal{J}} \), \( \mathcal{J} (\alpha \beta) = (\mathcal{J} \alpha) \beta = \mathcal{J} \beta = \mathcal{J} \) (3.4).

We call \( G_{\mathcal{J}} \) the right stabilizer of \( \mathcal{J} \) in \( G \).

If \( \mathcal{J} \) is an \( x_0 \)-closed subset of \( G \), then \( \mathcal{J} G_{x_0} = \mathcal{J} \), see Lemma 3.2(2); so

\[
G_{x_0} \leq G_{\mathcal{J}} \leq G.
\]

Lemma 3.5. Let \( S \subseteq X \), \( \mathcal{J} = \zeta^{-1}_{x_0}(S) \subseteq G \) and \( G_{\mathcal{J}} \) be defined in Eq.(3.4). Denote

\[
X_S = \zeta_{x_0}(G_{\mathcal{J}}) = G_{\mathcal{J}} x_0.
\]

Then both \( \mathcal{J} \) and \( G_{\mathcal{J}} \) are \( x_0 \)-closed, and \( X_S \) is a block of the transitive \( G \)-set \( X \). (Definition: We call \( X_S \) the block associated with the subset \( S \) of \( X \).)
Proof. By Lemma 3.2(1), $\mathcal{I}$ is $x_0$-closed, hence Eq. (3.5) holds, so that $x_0 \in X_S$ and $G_{\mathcal{I}}G_{x_0} = G_{\mathcal{I}}$, where the later equality implies that $G_{\mathcal{I}}$ is $x_0$-closed too (see Lemma 3.2(2)). Hence

$$G_{\mathcal{I}} = \zeta_{x_0}^{-1}(\zeta_{x_0}(G_{\mathcal{I}})) = \zeta_{x_0}^{-1}(X_S).$$

(3.7)

By Remark 3.4(2), $X_S$ is a block.

Lemma 3.6. Let $S \subseteq X$ and $\mathcal{I} = \zeta_{x_0}^{-1}(S) \subseteq G$ as above. Set $\mathcal{I}' = G - \mathcal{I}$. The following hold.

(1) $|\mathcal{I}| = k \cdot |S|$ and $|G_{\mathcal{I}}| = k \cdot |X_S|$, where $k = |x_0|$ as in Remark 3.1.

(2) There are $\gamma_1, \ldots, \gamma_\ell \in G$ such that $S = (\gamma_1 X_S) \cup \cdots \cup (\gamma_\ell X_S)$ is a disjoint union. In particular, $|X_S| \cdot |S|; $ and, $|S| = |X_S|$ if and only if $S = \gamma X_S$ is a block.

(3) $X_S = \bigcap_{\alpha \in \mathcal{I}} \alpha^{-1}S$.

(4) If $\alpha' \in \mathcal{I}'$ then $\alpha'^{-1} S \cap X_S = \emptyset$.

Proof. (1). Since both $\mathcal{I}$ and $G_{\mathcal{I}}$ are $x_0$-closed, by Lemma 3.2(3), (1) holds.

(2). For $\gamma \in \mathcal{I}$, by the definition Eq.(3.4) of $G_{\mathcal{I}}$, the left coset $\gamma G_{\mathcal{I}} \subseteq \mathcal{I}$.

In this way, we can fined $\gamma_1, \ldots, \gamma_\ell \in G$ such that

$$\mathcal{I} = (\gamma_1 G_{\mathcal{I}}) \cup \cdots \cup (\gamma_\ell G_{\mathcal{I}})$$

is a disjoint union. Then

$$S = \mathcal{I} x_0 = ((\gamma_1 G_{\mathcal{I}}) \cup \cdots \cup (\gamma_\ell G_{\mathcal{I}}))x_0 = (\gamma_1 G_{\mathcal{I}}x_0) \cup \cdots \cup (\gamma_\ell G_{\mathcal{I}}x_0);$$

i.e., $S = (\gamma_1 X_S) \cup \cdots \cup (\gamma_\ell X_S)$. And $(\gamma_1 X_S) \cap (\gamma_j X_S) = (\gamma_1 G_{\mathcal{I}} x_0) \cap (\gamma_j G_{\mathcal{I}} x_0)$; by Corollary 3.3, $\gamma_i G_{\mathcal{I}}$ is $x_0$-closed too, and

$$(\gamma_i X_S) \cap (\gamma_j X_S) = ((\gamma_i G_{\mathcal{I}}) \cap (\gamma_j G_{\mathcal{I}}))x_0 = \emptyset x_0 = \emptyset, \quad \forall 1 \leq i \neq j \leq \ell.$$

(3). By definition, $X_S = G_{\mathcal{I}} x_0$. For $\alpha \in G$, $\alpha x_0 \in G_{\mathcal{I}} x_0$ if and only if $\alpha \in G_{\mathcal{I}}$ (because $G_{\mathcal{I}}$ is $x_0$-closed); if and only if $\beta \alpha \in \mathcal{I}$, if and only if $\alpha \in \beta^{-1}\mathcal{I}$, $\forall \beta \in \mathcal{I}$; if and only if $\alpha x_0 \in \beta^{-1} S$ (because $\beta^{-1}\mathcal{I}$ is $x_0$-closed), $\forall \beta \in \mathcal{I}$; if and only if $\alpha x_0 \in \bigcap_{\beta \in \mathcal{I}} \beta^{-1} S$.

(4). Suppose that there is a $\beta \in \mathcal{I}$ such that (recall that both $\alpha^{-1}\mathcal{I}$ and $G_{\mathcal{I}}$ are $x_0$-closed):

$$\alpha^{-1} \beta x_0 \in \alpha^{-1} S \cap X_S = (\alpha^{-1}\mathcal{I} x_0) \cap (G_{\mathcal{I}} x_0) = (\alpha^{-1}\mathcal{I} \cap G_{\mathcal{I}}) x_0;$$

so $\alpha^{-1} \beta \in G_{\mathcal{I}}$, i.e., $\mathcal{I} \alpha^{-1} \beta = \mathcal{I}$. Then there exists a $\beta_1 \in \mathcal{I}$ such that $\beta_1 \alpha^{-1} \beta = \beta$, hence $\alpha' = \beta_1 \in \mathcal{I}$; which contradicts that $\alpha' \in \mathcal{I}'$.

Lemma 3.7. Let $\emptyset \neq S \subseteq X$, and $S' = X - S$. Let $\mathcal{I} \subseteq G$. The following two statements are equivalent to each other:

(1) $\mathcal{I} S \neq X$.

(2) There exists an $x \in X$ such that $\mathcal{I}^{-1} x \subseteq S'$.
Proof. (1) $\Rightarrow$ (2). If \( X \supseteq \mathcal{A}S = \bigcup_{\alpha \in \mathcal{A}} \alpha S \), then there is an \( x \in X - \bigcup_{\alpha \in \mathcal{A}} \alpha S \). By De Morgan’s law, \( x \in \bigcap_{\alpha \in \mathcal{A}} (X - \alpha S) = \bigcap_{\alpha \in \mathcal{A}} \alpha S' \); i.e., \( \alpha^{-1}x \in S' \), \( \forall \alpha \in \mathcal{A} \), and (2) holds.

(2) $\Rightarrow$ (1). It is proved by reversing the above argument. \[ \Box \]

**Corollary 3.8.** Let \( \emptyset \neq S \subseteq X \) and \( S' = X - S \). Denote \( \mathcal{I} = \zeta_x^{-1}(S) \), \( \mathcal{I}' = \zeta_x^{-1}(S') \) and \( X_S = G_{\mathcal{I}x_S} \). Then
\[
X = (\mathcal{I}'^{-1}S) \cup X_S, \quad (\mathcal{I}'^{-1}S) \cap X_S = \emptyset.
\]

**Proof.** By the assumption, we have \((k = |G_{x_S}|)\) as in Remark 3.1:
\[
G = \mathcal{I} \cup \mathcal{I}', \quad \mathcal{I} \cap \mathcal{I}' = \emptyset; \quad |\mathcal{I}| = k \cdot |S|, \quad |\mathcal{I}'| = k \cdot |S'|
\]
For any \( x \in X \), by Remark 3.1 we have \( |\mathcal{I}'^{-1}x| \geq |\mathcal{I}'|/k > k \cdot |S'|/k = |S'| \). So Lemma 3.7(2) cannot be satisfied, hence Lemma 3.7(1) does not hold; i.e.,
\[
X = \mathcal{I}S = (\mathcal{I}'^{-1} \cup \{\alpha\})S = (\mathcal{I}'^{-1}S) \cup (\alpha S).
\] (3.8)
Then \( X - \mathcal{I}'^{-1}S \subseteq \alpha S, \forall \alpha \in \mathcal{I}' - 1 \). By Lemma 3.6(3),
\[
X - \mathcal{I}'^{-1}S \subseteq \bigcap_{\alpha \in \mathcal{I}'} \alpha^{-1}S = X_S.
\]
That is, \( X = \mathcal{I}'^{-1}S \cup X_S \). By Lemma 3.6(4), \((\mathcal{I}'^{-1}S) \cap X_S = \emptyset\). \[ \Box \]

### 3.2 Main results

Recall that the \( FG \)-module \( F^G \) is isomorphic in a natural way to the left regular \( FG \)-module \( FG \), cf. Eq.(2.1); in this way any function \( g \in F^G \) is identified with
\[
g = \sum_{\alpha \in G} g(\alpha)\alpha \in FG.
\]
And, the \( FG \)-module \( F^X \) is isomorphic in a natural way to the permutation \( FG \)-module \( F^X \), cf. Eq.(2.5); in this way any function \( f \in F^X \) is identified with the element \( f = \sum_{\alpha \in X} f(x) x \in FX \).

Then the surjective map Eq.(3.1) induces a surjective linear map (denoted by \( \zeta_x \)) again
\[
\zeta_x : FG \to FX, \quad \zeta_x \left( \sum_{\alpha \in G} g(\alpha)\alpha \right) = \sum_{\alpha \in G} g(\alpha)\alpha x_0, \quad \forall g \in F^G,
\] (3.9)
which is a surjective \( FG \)-module homomorphism, because: for \( \beta \in G, g \in F^G \),
\[
\zeta_x \left( \sum_{\alpha \in G} g(\alpha)\alpha \right) = \zeta_x \left( \sum_{\alpha \in G} g(\alpha)\beta \alpha \right) = \sum_{\alpha \in G} g(\alpha)\beta \alpha x_0 = \beta \zeta_x \left( \sum_{\alpha \in G} g(\alpha)\alpha \right).
\]

**Theorem 3.9.** Let \( F \) be any field, \( G \) be any finite group and \( X \) be any transitive \( G \)-set. Let \( 0 \neq f \in F^X \) and \( FGf \) be the \( FG \)-submodule of \( FX \) generated by \( f \). Then
\[
|\text{supp}(f)| \cdot \dim FGf \geq |X| + |\text{supp}(f)| - |X_{\text{supp}(f)}|.
\] (3.10)
Proof. Let $S = \text{supp}(f)$ and $S' = X - S$. Note that $S \neq \emptyset$. If $S' = \emptyset$, then $S = X$, $X_S = X$ and $\dim \mathbb{F}Gf \geq 1$, so Eq.(3.10) holds obviously. In the following we assume that $S' \neq \emptyset$.

Fix an $x_0 \in S$, and set $\mathcal{I} = \zeta_{x_0}^{-1}(S)$, $\mathcal{I}' = \zeta_{x_0}^{-1}(S')$. Take $\alpha_1 \in \mathcal{I}'^{-1}$. If $\alpha_1 S \nsubseteq \mathcal{I}'^{-1}S$, take $\alpha_2 \in \mathcal{I}'^{-1}$ such that $\alpha_2 S \nsubseteq \alpha_1 S$. Iteratively in this way, we get $\mathcal{I} = \{\alpha_1, \cdots, \alpha_{i-1}\} \subseteq \mathcal{I}'^{-1}$, $i \geq 2$, such that

$$\mathcal{I}S = \mathcal{I}'^{-1}S, \quad \text{and} \quad \alpha_i S \nsubseteq \alpha_1 S \cup \cdots \cup \alpha_{i-1} S, \quad i = 2, \cdots, t - 1.$$  

(3.11)

Take $\alpha_i \in \mathcal{I}^{-1}$. By Eq.(3.8) and its argument, we get that

$$\alpha_i S \notin \alpha_1 S \cup \cdots \cup \alpha_{i-1} S, \quad \alpha_1 S \cup \cdots \cup \alpha_{i-1} S \cup \alpha_i S = X.$$  

(3.12)

Note that $\text{supp}(\alpha_1 f) = \alpha_1 S$, $1 \leq i \leq t$, cf. Eq.(2.7). For $1 \leq i \leq t$, the support of $\alpha_i f$ is not contained in the union of the supports of $\alpha_{i-1} f, \cdots, \alpha_1 f$, see Eq.(3.11) and Eq.(3.12); we see that $\alpha_i f$ can not be a linear combination of $\alpha_{i-1} f, \cdots, \alpha_1 f$. Thus the following $t$ elements of $\mathbb{F}Gf$:

$$\alpha_1 f, \alpha_2 f, \cdots, \alpha_t f,$$

(3.13)

are linearly independent, hence

$$t \leq \dim \mathbb{F}Gf.$$  

(3.14)

By Corollary 3.8,

$$|X| = |\mathcal{I}'^{-1}S| + |X_S| = |\mathcal{I}S| + |X_S|.$$  

(3.15)

But

$$|\mathcal{I}S| \leq \sum_{i=1}^{t-1} |\alpha_i S| = (t - 1)|S| = t|S| - |S|.$$  

(3.16)

Combining Eq.(3.16), Eq.(3.15) and Eq.(3.14), we obtain that

$$|X| \leq t|S| - |S| + |X_S| \leq |\text{supp}(f)| \cdot \dim \mathbb{F}Gf - |\text{supp}(f)| + |X_{\text{supp}(S)}|,$$

we are done. \hfill \Box

As a consequence, we obtain again the result Eq.(1.7) in [10] as follows.

Corollary 3.10. Let $\mathbb{F}$, $G$, $X$, $f$ and $\mathbb{F}Gf$ be as above in Theorem 3.9. Then

$$|\text{supp}(f)| \cdot \dim \mathbb{F}Gf \geq |X|.$$  

(3.17)

Let $S = \text{supp}(f)$, $x_0 \in S$, $\mathcal{I} = \zeta_{x_0}^{-1}(S)$ and $f^\mathcal{I} = f \circ \zeta_{x_0} \in \mathbb{F}^\mathcal{I}$, i.e., $f^\mathcal{I} (\beta) = f(\beta x_0)$, $\forall \beta \in \mathcal{I}$. The following three are equivalent to each other:

1) The equality in Eq.(3.17) holds.

2) $S$ is a block of $X$, hence $\mathcal{I}$ is a subgroup of $G$ and $S$ is an $\mathcal{I}$-set, and there is an $\eta \in \text{Hom}(\mathcal{I}, \mathbb{F}^\times)$ such that $\beta f = \eta(\beta)f$, $\forall \beta \in \mathcal{I}$, where $\text{Hom}(\mathcal{I}, \mathbb{F}^\times)$ is defined in Remark 2.2.

3) $\mathcal{I}$ is a subgroup of $G$, $f^\mathcal{I} = c \eta$ for a $c \in \mathbb{F}^\times$ and an $\eta \in \text{Hom}(\mathcal{I}, \mathbb{F}^\times)$. 

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Proof. From Lemma 3.6(2) we have the following inequality

$$|\text{supp}(f)| \geq |X_{\text{supp}(f)}|. \quad (3.18)$$

By Theorem 3.9, Eq.(3.17) holds. The equality in Eq.(3.17) holds (i.e., (1) holds) if and only if the following (1') holds:

(1') All the equalities in Eq.(3.17) hold.

Keep the notation in the proof of Theorem 3.9. Since $x_0 \in S$, we have that $G_{x_0} \subseteq \mathcal{I}$, hence $G_{x_0} \subseteq G_{\mathcal{I}} \subseteq \mathcal{I}$. And, the equality in Eq.(3.18) holds if and only if $S = X_S$ is a block (Lemma 3.6(2)), if and only if $G = G_\mathcal{I}$ (Lemma 3.5).

(1) $\Rightarrow$ (2). (1') holds, hence $S = X_S$, $\mathcal{I} = G_\mathcal{I} \subseteq G$ and $S$ is an $\mathcal{I}$-set. The equality in Eq.(3.16) implies that the union of $X$ in Eq.(3.12) is disjoint. Turning back to $G$, we get a disjoint union of left cosets of the subgroup $\mathcal{I}$:

$$G = \alpha_1 \mathcal{I} \cup \cdots \cup \alpha_{t-1} \mathcal{I} \cup \alpha_t \mathcal{I}; \quad (3.19)$$

i.e., $\alpha_1, \cdots, \alpha_t$ form a system of the representatives of all left cosets of $\mathcal{I}$. We set $\alpha_1 = 1_G$. The equality in Eq.(3.14) implies that the elements of Eq.(3.13):

$$f = \alpha_1 f, \alpha_2 f, \cdots, \alpha_t f, \quad (3.20)$$

form a basis of $\mathbb{F} G f$. The following set of elements

$$\{ \gamma f \mid \gamma \in G \} = \{ \alpha_i \beta f \mid 1 \leq i \leq t, \beta \in \mathcal{I} \}$$

linearly generate $\mathbb{F} G f$. Consider $i = 1$. Then $\beta f, \beta \in \mathcal{I}$, is a linear combination of $f, \alpha_2 f, \cdots, \alpha_t f$. Because the supports of $f, \alpha_2 f, \cdots, \alpha_t f$ are disjoint each other and $\text{supp}(\beta f) = \beta S = S$, there is $\gamma_1 f \in \mathbb{F}^X$ such that $\beta f = c_\beta f$. Thus $f$ is an $\mathcal{I}$-linear function on $S$ and, by Remark 2.2, (2) holds.

(2) $\Rightarrow$ (1). $S$ is a block, hence $\mathcal{I}$ is a subgroup, the equality in Eq.(3.18) holds. And the union of $X$ in Eq.(3.12) and the union Eq.(3.19) are both disjoint, so that the equality in Eq.(3.16) holds. Translating from $\mathcal{I}$ to $\alpha_i \mathcal{I}$ (cf. Eq.(3.19)), for any $\beta \in \mathcal{I}$, by (2) we have $\alpha_i \beta f = \eta(\beta)(\alpha_i f), i = 1, \cdots, t$. Thus, Eq.(3.20) are a basis of $\mathbb{F} G f$, hence the equality in Eq.(3.14) holds. In a word, (1') holds.

(2) $\Leftrightarrow$ (3). $\mathcal{I}$ is a subgroup of $G$ if and only if $S$ is a block of $X$ (cf. Remark 3.4(2)). Because $\text{supp}(f) = S$, for $\beta \in \mathcal{I}$ and $c_\beta \in \mathbb{F}^X$, $\beta f = c_\beta f$ if and only if $\beta f_\mathcal{I} = c_\beta f_\mathcal{I}$. By Remark 2.2, (2) is equivalent to (3). \]

**Remark 3.11.** If the equality in Eq.(3.17) holds, then, by the (1') in the proof of Corollary 3.10, the equality in Eq.(3.10) holds too. The following example shows that the inverse is incorrect.

**Example.** Take $G = S_3$ to be the symmetric group of degree 3, and $X = \{x_1, x_2, x_3\}$ to be the set with the canonical $S_3$-action. Let $f \in \mathbb{F}^X$ be as

$$f(x_1) = 1, \quad f(x_2) = -1, \quad f(x_3) = 0.$$
Then $S = \text{supp}(f) = \{x_1, x_2\}$, and (take $x_0 = x_1$ in Eq.\((3.1)\)), $G_{x_1} = \{(1), (23)\}$, $\mathscr{S} = \zeta_{x_1}^{-1}(S) = \{(1), (23), (12), (123)\}$, $G_{\mathscr{S}} = \{(1), (23)\} = G_{x_1}$, and $X_S = \{x_1\}$ is a trivial block. And, $(12)f(x_1) = f((12)^{-1}x_1) = f(x_2) = -1$. Computing in this way, we get the table of function values as follows:

|   | $f$ | $(12)f$ | $(13)f$ | $(123)f$ | $(23)f$ | $(132)f$ |
|---|-----|---------|---------|----------|---------|----------|
| $x_1$ | 1   | -1      | 0       | 0        | 1       | -1       |
| $x_2$ | -1  | 1       | -1      | 1        | 0       | 0        |
| $x_3$ | 0   | 0       | 1       | -1       | -1      | 1        |

From the table we have: $(12)f = -f$, $(13)f = -(13)f$, $(123)f = -(23)f$; $(23)f = f - (13)f$. So $f$, $(13)f$ are a basis of $FGf$, and $\dim FGf = 2$. Hence

$$|\text{supp}(f)| \cdot \dim FGf = 2 \cdot 2 = 4 = 3 + 2 - 1 = |X| + |\text{supp}(f)| - |\text{supp}(f)|.$$  

Thus, the equality in Eq.\((3.10)\) holds, but the equality in Eq.\((3.17)\) does not hold.

The following is an interesting question which has been posted by Feng, Hollmann, and Xiang \cite{9} for abelian case.

**Question 3.12.** Under what conditions the equality in Eq.\((3.10)\) holds?

### 3.3 Corollaries

When we take $X = G$ to be the left regular $G$-set, the next two corollaries follow at once.

**Theorem 3.13.** Let $G$ be any finite group and $F$ be any field. Let $0 \neq f \in F^G$ and $FGf$ be the left ideal of $F^G$ generated by $f$. Then (where $G_{\text{supp}(f)}$ is defined in Eq.\((3.4)\))

$$|\text{supp}(f)| \cdot \dim FGf \geq |G| + |\text{supp}(f)| - |G_{\text{supp}(f)}|. \tag{3.21}$$

**Proof.** It follows from Theorem 3.9 straightforwardly.

The next one follows from Corollary 3.10 (note that Corollary 3.10 (2) and (3) are the same for the regular $G$-set).

**Corollary 3.14.** Let $G$, $F$, $f$ and $FGf$ be as above in Theorem 3.13. Then

$$|\text{supp}(f)| \cdot \dim FGf \geq |G|. \tag{3.22}$$

And the following two are equivalent to each other:

1. The equality in Eq.\((3.22)\) holds.
2. $\text{supp}(f) = \gamma H$, $\gamma \in G$, is a left coset of a subgroup $H$ of $G$, and there is a homomorphism $\eta : H \rightarrow F^\times$ and an element $c \in F^\times$ such that $f|_{\text{supp}(f)} = c \cdot \gamma \eta$, i.e., $f(\gamma \beta) = c \eta(\beta)$, $\forall \beta \in H$.  

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In Corollary 3.10, taking $X = G$ to be the left regular $G$-set, we obtain Eq.(3.22). Note that (1) is the same as the Corollary 3.10(1). When we identify the regular $G$-set $G$ with the group $G$ through the $\zeta_{x_0}$ in Eq.(3.1), in fact, we take $x_0 = 1_G$. Here it is possible that $1_G \notin S = \text{supp}(f)$. We can take $\gamma^{-1} \in G$ such that $1_G \in \gamma^{-1} S$. Hence $1_G \in \text{supp}(\gamma^{-1} f)$ (cf. Eq.(2.7)), and Corollary 3.10(3) states that $H := \gamma^{-1} S$ is subgroup and $\gamma^{-1} f = c \eta$ for an element $c \in \mathbb{F}^\times$ and an $\eta \in \text{Hom}(H, \mathbb{F}^\times)$. In other words, $S = \text{supp}(f) = \gamma H$ and $f|_S = c \cdot \gamma \eta$. Therefore, the equivalence of (1) and (2) follows from the equivalence of Corollary 3.10 (1) and (3).

**Remark 3.15.** If $G$ is abelian, and Eq.(1.5) satisfied, and $\mathbb{F}$ contains a primitive exp($G$)'th root of unity (hence $\hat{f}$ is defined and $\text{supp}(\hat{f}) = \text{rk-supp}(\hat{f}) = \dim \mathbb{F} G f$), then for any subgroup $H$ of $G$ and any homomorphism $\eta : H \rightarrow \mathbb{F}^\times$ (i.e., any $\eta \in \hat{H}$), there is a homomorphism $\chi : G \rightarrow \mathbb{F}^\times$ (i.e., a $\chi \in \hat{G}$) such that the restriction $\chi|_H = \eta$ (cf. [13, Ch.6 Proposition 1]). So the statement (2) of Corollary 3.14 can be rewritten as $f = c' \chi I_{\gamma H}$, where $c' = c \chi(\gamma)^{-1}$ and $I_{\gamma H}(\alpha) = \begin{cases} 1, & \alpha \in \gamma H; \\ 0, & \alpha \notin \gamma H. \end{cases}$ It is just the classical result stated after Eq.(1.1).

If $G$ is non-abelian, however, then for a homomorphism $\eta : H \rightarrow \mathbb{F}^\times$, there is in general no homomorphism $\chi : G \rightarrow \mathbb{F}^\times$ such that $\chi|_H = \eta$. For example, if $H$ is contained in the derivative subgroup (commutator subgroup) of $G$, then for any homomorphism $\chi : G \rightarrow \mathbb{F}^\times$, $\chi(x) = 1, \forall x \in H$; thus $\chi|_H \not\equiv \eta$ provided $\eta : H \rightarrow \mathbb{F}^\times$ is not a trivial homomorphism.

**Remark 3.16.** It is easy to inflate the example in Remark 3.11 to an example on the group $G = S_3$ such that, in the inflated example, the equality in Eq.(3.21) holds but the equality in Eq.(3.22) does not. The following is the regular set version of Question 3.12.

**Question 3.17.** Under what conditions the equality in Eq.(3.21) holds?

However, we think, Question 3.12 would be done provided the above question could be solved.

As shown in Remark 2.11(1), the following sharpened uncertainty principle for finite group actions follows from Theorem 3.9 and Corollary 3.10 at once.

**Theorem 3.18.** Let $G$ be any finite group and $X$ be any transitive $G$-set. Let $\mathbb{F}$ be any field satisfying the condition Eq.(1.5), $0 \neq f \in \mathbb{F}^X$ and $\hat{f}$ be the Fourier transform of $f$. Then (where $\text{rk-supp}(\hat{f})$ is defined in Eq.(2.17)):

$$|\text{supp}(f)| \cdot \text{rk-supp}(\hat{f}) \geq |X| + |\text{supp}(f)| - |\text{supp}(\hat{f})|.$$  \hspace{1cm} (3.23)

**Corollary 3.19.** Let $G$, $X$, $\mathbb{F}$, $f \in \mathbb{F}^X$ and $\hat{f}$ be as the same as above. Then

$$|\text{supp}(f)| \cdot \text{rk-supp}(\hat{f}) \geq |X|.$$  \hspace{1cm} (3.24)

And the equality holds if and only if Corollary 3.10(2) holds.
As for Theorem 3.13 and Corollary 3.14, taking $X = G$ to be the left regular $G$-set, we get the following at once.

**Theorem 3.20.** Let $G$ be any finite group and $\mathbb{F}$ be any field satisfying the condition Eq. (1.5). Let $0 \neq f \in \mathbb{F}^G$ and $\hat{f}$ be the Fourier transform of $f$. Then

$$|\text{supp}(f)| \cdot \text{rk-supp}(\hat{f}) \geq |G| + |\text{supp}(f)| - |G_{\text{supp}(f)}|. \quad (3.25)$$

**Corollary 3.21.** Let $G, \mathbb{F}, f \in \mathbb{F}^G$ and $\hat{f}$ be as the same as above. Then

$$|\text{supp}(f)| \cdot \text{rk-supp}(\hat{f}) \geq |G|. \quad (3.26)$$

And the equality holds if and only if Corollary 3.14(2) holds.

### 4 Conclusion

Feng, Hollmann, and Xiang [9] exhibited a sharpened uncertainty principle for any abelian finite group $G$ and $0 \neq f \in \mathbb{F}^G$ (where $\mathbb{F}$ is a field such that $\mathbb{F}G$ is semisimple and $\hat{f}$ is the Fourier transform of $f$ in a splitting field):

$$|\text{supp}(f)| \cdot |\text{supp}(\hat{f})| \geq |G| + |\text{supp}(f)| - |G_{\text{supp}(f)}|. \quad (4.1)$$

In this paper we extend it to an extensive framework: any transitive $G$-set $X$ where $G$ is any finite group, any non-zero function $f \in \mathbb{F}^X$ where $\mathbb{F}$ is any field.

For the purpose, first, by assuming that $\mathbb{F}G$ is semisimple, we construct the $G$-dual set $\hat{X}$ of the $G$-set $X$, and extended the classical Fourier transform to the $G$-actions, so that the Fourier transform $\hat{f} \in \mathbb{F}^X$ of $f$ is constructed.

Second, we extended the quantity $|\text{supp}(\hat{f})|$ for abelian finite groups to the so-called rank support $\text{rk-supp}(\hat{f})$ for group actions. Moreover, we found that $\text{rk-supp}(\hat{f}) = \dim \mathbb{F}Gf$, where $\dim \mathbb{F}Gf$ is the $\mathbb{F}$-dimension of the submodule $\mathbb{F}Gf$ of the permutation module $\mathbb{F}X$ generated by the element $f = \sum_{x \in X} f(x)x$. Therefore, we investigated the sharpened uncertainty principle with $\dim \mathbb{F}Gf$ instead of $\text{rk-supp}(\hat{f})$, and we proved that:

$$|\text{supp}(f)| \cdot \dim \mathbb{F}Gf \geq |X| + |\text{supp}(f)| - |X_{\text{supp}(f)}|. \quad (4.2)$$

A benefit of the replacement of $\text{rk-supp}(\hat{f})$ by $\dim \mathbb{F}Gf$ is that $\mathbb{F}$ can be any field (without the condition “$\mathbb{F}G$ is semisimple”).

The “shifting technique” in [9] works well in the dual $\hat{G}$ of abelian $G$ to prove Eq.(4.1); but it doesn’t work in $\hat{X}$ of transitive $G$-set $X$. An important benefit of the replacement of $\text{rk-supp}(\hat{f})$ by $\dim \mathbb{F}Gf$ is that we can using the “translating technique” in the $G$-set $X$ to prove our sharpened uncertainty principle Eq.(4.2). And then many (sharpened or classical) versions of finite-dimensional uncertainty principle are derived as corollaries.
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