SINGULAR (LIPSCHITZ) HOMOLOGY AND HOMOLOGY OF INTEGRAL CURRENTS

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Abstract. We compare the homology groups $H^IC_n(X)$ of the chain complex of integral currents with compact support of a metric space $X$ with the singular Lipschitz homology $H^L_n(X)$ and with ordinary singular homology. If $X$ satisfies certain cone inequalities all these homology theories coincide. On the other hand, for the Hawaiian Earring the homology of integral currents differs from the singular Lipschitz homology and it differs also from the classical singular homology $H_n(X)$.

1. Introduction and results

Building on the theory of metric currents (see [AK00]), Stefan Wenger introduced the homology of integral metric currents with compact support ([Wen07]) in complete quasiconvex metric spaces that admit certain cone inequalities. As the axioms of Eilenberg–Steenrod for a homology theory are satisfied, this homology is isomorphic to the singular (Lipschitz) homology on finite CW-complexes.

Here, we compare this homology of metric currents with the singular homology and the singular Lipschitz homology. If locally cycles can be filled by a diameter-controlled chain, all three theories are identical. In the special case of the Hawaiian Earring we show that the later homology theories do not coincide with the first one.

Definition 1.1. A subset $S$ of a metric space $X$ admits a strong $\gamma$-Lipschitz contraction, $\gamma > 0$, if there is a map $\phi : [0,1] \times S \to X$ and $x_0 \in X$ such that $\phi(1,\cdot) = \text{id}_S$, $\phi(0,s) = x_0$, $\forall s \in S$, and

$$d(\phi(t,s),\phi(t',s')) \leq \gamma \text{diam}(S)|t-t'| + \gamma d(s,s').$$

The space $X$ admits locally strong Lipschitz contractions if for all $x \in X$ there is $r_x > 0$ and $\gamma_x > 0$ such that every subset $U \subset B_{r_x}(x)$ admits a strong $\gamma_x$-Lipschitz contraction.

Remark 1.2. What we call a strong $\gamma$-Lipschitz contraction is also called simply a $\gamma$-Lipschitz contraction (e.g. in [Wen07]). However we want to emphasize that the contracting map has this diameter bound for the first entry.

For example complete CAT($\kappa$)-spaces admit locally strong $\gamma$-Lipschitz contractions; we come back to this in subsection 3.1.

We denote the group of $k$-dimensional integral currents in $X$ with compact support by $I^IC_k(X)$. We let $C^LC_k(X)$ be the group of singular Lipschitz $k$-chains, and we let $C_k(X)$ be the group of singular $k$-chains, both with coefficients in $\mathbb{Z}$. For $c \in C^LC_k(X)$ the current induced by $c$ is denoted by $[c]$. The boundary of a current $T$ is $\partial T$; the boundary of a chain $c$ is $bc$. For a (Lipschitz) chain $c$, we denote by $sd^m(c)$ the $m$-th barycentric subdivision of $c$. For $A \subset X$, $B_r(A) := \{y \in X; d(A,y) < r\}$

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is the open \( r \)-neighbourhood of \( A \). The support of the current \( T \) is denoted by \( \text{spt}(T) \), and we write \( \text{im}(c) \) for the union of the images of the simplices of a chain. Definitions of these concepts can be found in section 2.

We write \( H_k^{LC}(X) \) for the homology of the chain complex of integral currents with compact support of a metric space \( X \), \( H_k^S(X) \) for the singular Lipschitz homology and \( H_n(X) \) for ordinary singular homology.

**Theorem 1.3.** Let \( X \) be a complete metric space. For \( T \in \mathbf{I}_0(X) \) there exists \( c \in \mathbf{C}_0^L(X) \) with \( T = [c] \).

Suppose the complete metric space \( X \) admits locally strong Lipschitz contractions, let \( \epsilon > 0 \) and \( n \in \mathbb{N} \). Given \( T \in \mathbf{I}_n^c(X) \) with \( \partial T = [c] \) for \( c \in \mathbf{C}_{n-1}^L(X) \) and \( bc = 0 \), there exist \( N \in \mathbb{N} \), \( T_1, \ldots, T_N \in \mathbf{I}_n^c(X) \), \( c_1, \ldots, c_N \in \mathbf{C}_n^L(X) \) and \( V_1, \ldots, V_N \in \mathbf{I}_n^c(X) \) such that

\[
\begin{align*}
\text{i)} & \quad \sum T_i = T \text{ and } \partial V_i = T_i - [c_i] \\
\text{ii)} & \quad \exists m \in \mathbb{N} \cup \{0\} : b(\sum c_i) = sdm(c). \quad \text{iii)} \quad \text{spt}(V_i) \cup \text{im}(c_i) \subset B_\epsilon(\text{spt}(T) \cup \text{im}(c)) \text{ and } \text{diam}(\text{spt}(V_i)) < \epsilon.
\end{align*}
\]

Note that i) implies that \( T \) and \( \sum [c_i] \) are homologous.

**Corollary 1.4.** If \( X \) and the closed subset \( A \subset X \) admit locally strongly Lipschitz contractions then the homology of integral Lipschitz chains is isomorphic to the homology of integral currents with compact support:

\[
H_k^S(X, A) \cong H_k^{LC}(X, A).
\]

The isomorphism is induced by the map \([\ ] : \mathbf{C}_k^L(X) \rightarrow \mathbf{I}_k^c(X)\).

**Theorem 1.5.** Let \( X \) be a metric space. Then \( \mathbf{C}_0^L(X) = \mathbf{C}_0(X) \).

Suppose that the metric space \( X \) admits locally strong Lipschitz contractions, let \( \epsilon > 0 \) and \( n \in \mathbb{N} \). Given \( c \in \mathbf{C}_n(X) \) with \( bc \in \mathbf{C}_{n-1}^L(X) \), there exist \( N \in \mathbb{N} \), \( c_1, \ldots, c_N \in \mathbf{C}_n(X) \) and \( c_1^L, \ldots, c_N^L \in \mathbf{C}_n^L(X) \) and \( e_1, \ldots, e_N \in \mathbf{C}_{n-1}(X) \) such that

\[
\begin{align*}
\text{i)} & \quad \exists m \in \mathbb{N} \cup \{0\} : \sum c_i = sdm(c) \text{ and } b\overline{c_i} = c_i - c_i^L \\
\text{ii)} & \quad \text{im}(e_i) \subset B_\epsilon(\text{im}(c)) \text{ and } \text{diam}('\text{im}(e_i)) < \epsilon.
\end{align*}
\]

**Corollary 1.6.** If \( X \) and the subset \( A \subset X \) admit locally strongly Lipschitz contractions then the homology of integral Lipschitz chains is isomorphic to the singular homology:

\[
H_k^S(X, A) \cong H_k(X, A).
\]

The isomorphism is induced by the inclusion \( \mathbf{C}_k^L(X) \subset \mathbf{C}_k(X) \).

The above theorems are proved in a more general setting where we make assumptions on local \( k \)-cycles for \( 0 \leq k \leq n \), see section 3.

The Hawaiian Earring \( \mathbb{H} \subset \mathbb{R}^2 \) is given by the countable union of the circles

\[
L_n = \{ x \in \mathbb{R}^2 : \|x - (1/n, 0)\| = 1/n \}
\]

with radius \( 1/n \) and centre \( (1/n, 0) \), \( n \in \mathbb{N} \). As metric on \( \mathbb{H} \) we set

\[
d(x, y) := \begin{cases} 
\|x - y\|, & \text{if } \exists n \in \mathbb{N} : x, y \in L_n; \\
\|x\| + \|y\|, & \text{otherwise}
\end{cases}
\]

(we could as well take the length metric of \( \mathbb{H} \)). Note that any neighbourhood of \((0, 0)\) in \( \mathbb{H} \) contains all but finitely many of the \( L_n \). Thus \( \mathbb{H} \) is not locally contractible, and in particular not a CW-complex.

We show in section 4 that the maximal divisible subgroup of \( H_1^{LC}(\mathbb{H}) \) is trivial whereas the maximal divisible subgroups of \( H_1^L(\mathbb{H}) \) and \( H_1(\mathbb{H}) \) are non-trivial. This implies that

**Theorem 1.7.** \( H_1^{LC}(\mathbb{H}) \) is isomorphic neither to \( H_1^L(\mathbb{H}) \) nor to \( H_1(\mathbb{H}) \).
Let $M(X)$ be the space of multi-linear functional on $D^k(X)$ satisfying the following properties:

i) If $\pi_1$ converges point-wise to $\pi_i$ as $j \to \infty$ and if $\sup_j \text{Lip}(\pi_1) < \infty$ then $T(f, \pi_1^1, \ldots, \pi_k^1) = T(f, \pi_1, \ldots, \pi_k)$.

ii) If $\{x \in X; f(x) \neq 0\}$ is contained in the union $\bigcup_{i=1}^k B_i$ of Borel sets $B_i$ and if $\pi_i$ is constant on $B_i$ then $T(f, \pi_1, \ldots, \pi_k) = 0$.

iii) There exists a finite Borel measure $\mu$ on $X$ such that

\begin{equation}
|T(f, \pi_1, \ldots, \pi_k)| \leq \prod_{i=1}^k \text{Lip}(\pi_i) \int_X |f| d\mu
\end{equation}

for all $(f, \pi_1, \ldots, \pi_k) \in D^k(X)$.

Let $M_k(X)$ be the set of $k$-dimensional currents.

The minimal Borel measure $\mu$ with (2.1) is called mass of $T$ and written as $\|T\|$; set $M(T) := \|T\|(X)$.

The support $\text{spt}(T)$ of $T$ is the closed set given by

$\text{spt}(T) := \{x \in X; \|T\|(B_r(x)) > 0, \forall r > 0\}$.

Remark 2.2. As is done in [AK00, Wen07], we will henceforth assume that the cardinality of $X$ is an Ulam number. Then for $T \in M_k(X)$ one has $\|T\|(X \setminus \text{spt}(T)) = 0$ (see [AK00]).

Let $\chi_C$ be the characteristic function of a set $C$. For a Borel set $A \subset X$ and $T \in M_k(X)$ we define the restriction of $T$ to $A$ as

$T[A](f, \pi_1, \ldots, \pi_k) := T(\chi_A f, \pi_1, \ldots, \pi_k)$,

which is in $M_k(X)$ (the above expression is well-defined since $T$ extends to a functional whose first argument lies in $L^\infty(X, ||T||)$, see [AK00]).

The boundary of $T \in M_k(X)$ is the functional

$\partial T(f, \pi_1, \ldots, \pi_{k-1}) := T(1, f, \pi_1, \ldots, \pi_{k-1})$.

We say that $T$ is normal if $\partial T \in M_{k-1}(X)$; the space of normal currents is denoted by $N_k(X)$. From the locality property for currents (condition ii) we get $\partial(\partial T) = 0$.

For a Lipschitz map $g : X \to Y$ to a complete metric space $(Y, d)$ and a current $T \in M_k(X)$ we define the push-forward of $T$ by $g$ to be $g_T(f, \pi_1, \ldots, \pi_k) := T(f \circ g, \pi_1 \circ g, \ldots, \pi_k \circ g)$ for $(f, \pi_1, \ldots, \pi_k) \in D^k(Y)$. Then, $\partial(g_T) = g_{\partial T}$.

Definition 2.3 ([Wen07, 2.3]). A current $T \in M_k(X)$ with $k \geq 1$ is said to be rectifiable if
holds:

i) \(||T|||\) is concentrated on a countably \(\mathcal{H}^k\)-rectifiable set and

ii) \(||T|||\) vanishes on \(\mathcal{H}^k\)-negligible sets.

The current \(T\) is called integer rectifiable if, in addition, the following property holds:

iii) For any Lipschitz map \(\phi : X \to \mathbb{R}^k\) and any open set \(U \subset X\) there exists \(\theta \in L^1(\mathbb{R}^k, \mathbb{Z})\) such that

\[
\phi_\#(T[U](f, \pi_1, \ldots, \pi_k)) = \int_{\mathbb{R}^k} \theta f \det(\frac{\partial \pi_i}{\partial s_j}) d\mathcal{L}^k(s)
\]

for all \((f, \pi_1, \ldots, \pi_k) \in D^k(X)\).

A 0-dimensional (integer) rectifiable current is a \(T \in M_0(X)\) of the form

\[
T(f) = \sum_{i=1}^{\infty} \theta_i f(x_i),
\]

\(f\) Lipschitz and bounded, for suitable \(\theta_i \in \mathbb{R}\) (\(\theta_i \in \mathbb{Z}\) respectively, note that in this case the sum has to be finite) and \(x_i \in X\). We write \(T = \sum \theta_i [x_i]\).

The space of integral currents, i.e. integer rectifiable and normal currents (equivalently defined as integer rectifiable currents with integer rectifiable boundary, see [AK00, Theorem 8.6]), is denoted by \(I_k\) and

\[
I_k^\#(X) := \{T \in I_k(X); \text{spt}(T)\text{ is compact}\}
\]

is the space of integral currents with compact support. So, \(\partial|I_k(X); I_{k-1}(X), \partial|I_k^\#(X) \to I_{k-1}^\#(X)\) and \(g\#T \in I_k^\#(Y)\) for \(T \in I_k(X)\) and \(g\) Lipschitz.

Let \(A \subset X\) be a closed subset, we define (see [Wen07, p. 159])

\[
Z^I_k(X, A) := \{T \in I_k(X); \partial T \in I_{k-1}(A)\}
\]

\[
B^I_k(X, A) := \{R + \partial S; R \in I_k(A), S \in I_{k+1}(X)\}.
\]

The homology of integral currents with compact support is

\[
H^I_k(X, A) := Z^I_k(X, A)/B^I_k(X, A).
\]

If \(A = \emptyset\) we write \(H^I_k(X)\). For \(T \in N_k(X)\) and a Lipschitz function \(d : X \to \mathbb{R}\) we set

\[
\langle T, d, r+ \rangle := \partial(T[\{d \leq r\}) - (\partial T)[\{d \leq r\}
\]

\[
= (\partial T)[\{d > r\}) - \partial(T[\{d > r\}).
\]

If \(T \in I_k^\#(X)\) then, for almost all \(r \in \mathbb{R}\), \(\langle T, d, r+ \rangle \in I_{k-1}^\#(X)\). This is called the slice of \(T\) by \(d\) at \(r\) and we denote it by \(\langle T, d, r \rangle\). Note that \(\langle \partial T, d, r+ \rangle = -\partial \langle T, d, r \rangle\) and that

\[
\text{spt}(\langle T, d, r \rangle) \subset d^{-1}\{r\} \cap \text{spt}(T).
\]

2.2. Singular (Lipschitz) chains. Let

\[
\Delta^k := \{(s_0, \ldots, s_k) \in \mathbb{R}^{k+1}; \sum_j s_j = 1 \text{ and } 0 \leq s_j, \forall j \} \subset \mathbb{R}^{k+1}
\]

be the standard simplex. Sometimes it is more convenient to consider \(\Delta^k\) as a subset of \(\mathbb{R}^k\); for this we choose an isometry \(\phi : \mathbb{R}^k \to \{s \in \mathbb{R}^{k+1}; \sum_j s_j = 1\}\) and define \(\Delta^k = \phi^{-1}(\Delta^k)\). Let \(X'\) be a metric space (not necessarily complete).
Definition 2.4. A singular $k$-simplex $c$ is a continuous map $c : \Delta^k \to X'$.

A singular $k$-chain $c$ over an abelian group $G$ is a finite formal sum $c = \sum_{i=1}^{m} n_i c_i$, where $n_i \in G$ and $c_i$ are singular simplices.

A singular Lipschitz $k$-chain is a singular $k$-chain $c = \sum_{i=1}^{m} n_i c_i$ where all $c_i$ are Lipschitz maps.

We use only $G = \mathbb{Z}$ and speak of integral (Lipschitz) $k$-chains. We denote by $C_k(X')$ the free abelian group of integral $k$-chains and by $C_k^L(X')$ the subgroup of integral Lipschitz $k$-chains.

Let $[e_0, e_1, \ldots, e_k]$ be the vertices of the standard $k$-simplex for $k \geq 1$, then the boundary of a singular (Lipschitz) $k$-simplex $c$ is the (Lipschitz) $(k-1)$-chain

$$bc := \sum_{j=0}^{k} (-1)^j c_{[e_{j_0} \ldots e_{j_k}]},$$

where $e_j$ means that $e_j$ is omitted. By setting $bc = \sum_{i=1}^{m} n_i b c_i$ for a $k$-chain $c = \sum_{i=1}^{m} n_i c_i$ and defining $bc = 0$ for 0-chains, we get homomorphisms $b : C_k^L(X') \to C_{k-1}^L(X')$ such that $b(bc) = 0$. For a subset $A \subset X'$ let

$$Z_k^L(X', A) := \{ c \in C_k^L(X'); \; bc \in C_{k-1}^L(A) \}$$

and

$$B_k^L(X', A) := \{ c + bc; \; c \in C_k^L(A), \; \partial c \in C_{k+1}^L(X') \}$$

If $A = \emptyset$ we write $H_k^L(X')$. There is a natural comparison homomorphism $H_k^L(X, A) \to H_k(X, A)$ which is induced by the inclusions $C_k^L(S) \to C_k^L(S)$ for subsets $S \subset X$.

For the complete metric space $X$, any integral Lipschitz $k$-chain $c = \sum_{i=1}^{m} n_i c_i$ induces an integral current $[c]$ with compact support defined by

$$[c](f, \pi_1, \ldots, \pi_k) := \sum_{i=1}^{m} n_i \int_{\Delta^k \subset \mathbb{R}^k} f \circ \hat{c}_i \det \left( \frac{\partial (\pi_j \circ \hat{c}_i)}{\partial x_l} \right) d\mathcal{L}^k(s),$$

where $\hat{c}_i = c_i \circ \phi$. The maps $[.] : C_k^L(X) \to \Gamma_k^L(X)$ are homomorphisms. By Stokes’ theorem we get $[bc] = \partial [c]$, so $[.]$ is a chain map for these complexes. We also refer to the induced maps between the homologies of these complexes as comparison maps.

For $c = \sum_{i=1}^{m} n_i c_i \in C_k(X')$ with $n_i \neq 0$ we define

$$\text{im}(c) := \bigcup_{i=1}^{m} \text{im}(c_i)$$

(this is well-defined since $C_k(X')$ is free over the $k$-simplices, so the representation of $c$ as sum of simplices is unique).

The push-forward of $c \in C_k(X')$ to the metric space $Y$ by the continuous map $g : X' \to Y$ is the chain $g \# c := \sum_{i=1}^{m} n_i (g \circ c_i) \in C_k(Y)$. Again, $b(g \# c) = g \# (bc)$. If $c \in C_k^L(X')$ and $g : X' \to Y$ is Lipschitz then $g \# c \in C_k^L(Y)$; if $X$ and $Y$ are complete and $g : X \to Y$ is Lipschitz, $[g \# c] = g \# [c]$ for all $c \in C_k^L(X)$.

In order to get smaller simplices (i.e. simplices with smaller diameter) we use barycentric subdivision. This standard construction can be found in [Hat02] or [Mun84]. We only list the facts that we need later. Let $m$ be given and $sd^m(c)$ denote the singular (Lipschitz) $k$-chain resulting from the $m$-th barycentric subdivision of the singular (Lipschitz) $k$-chain $c$, then:
i) For $k \geq 0$ there is a homomorphism $D_{m,X} : C^{(L)}_k(X') \to C^{(L)}_{k+1}(X')$ such that for each $k$-chain $c$

$$b(D_{m,X}(c)) + D_{m,X}(bc) = sd^m(c) - c. \quad (2.3)$$

Furthermore, $D_{m,X}$ is natural: for $f : X' \to Y$ Lipschitz, $f \# \circ D_{m,X} = D_{m,Y} \circ f \#$.

ii) Applying iterated barycentric subdivision, we can get arbitrary small diameter of the image of the resulting simplices.

iii) For $c \in C^L_k(X')$ holds $[sd^m(c)] = [c]$.

iv) $b(sd^m(c)) = sd^m(b(c))$.

2.3. Cone inequalities. Here we give the definitions of miscellaneous cone inequalities. These inequalities are used in Theorem 3.2 and Theorem 3.4 which generalize Theorem 1.3 and 1.5. We recall the construction of $[0,1] \times T$ for a normal current $T$ (from [Wen07] which is a modified version of the one in [AK00]) and something similar for chains. This allows us to show in Proposition 3.1 that spaces which admit locally strong Lipschitz contractions also admit these cone inequalities.

Definition 2.5. Let $k \geq 1$ and let

$$\mathcal{F} := \{ F : \mathbb{R} \to \mathbb{R} ; F \text{ is continuous and non-decreasing with } F(0) = 0 \}.$$

- $X$ admits a local cone inequality for $\mathbf{I}^{k}(X)$ if for every $x \in X$ there exists $r_x > 0$ and $F_x \in \mathcal{F}$ such that for every $T \in \mathbf{I}^{k}(X)$ with $\partial T = 0$ and $\text{spt}(T) \subset B_{r_x}(x)$ there exists a $\bar{T} \in \mathbf{I}^{k+1}(X)$ satisfying $\partial \bar{T} = T$ and $\text{diam}(\text{spt}(\bar{T})) \leq F_x(\text{diam}(\text{spt}(T)))$.

- $X$ admits a local cone inequality for $C^{(L)}_k(X)$ if for every $x \in X$ there exists $r_x > 0$ and $F_x \in \mathcal{F}$ such that for every $c \in C^{(L)}_k(X)$ with $bc = 0$ and $\text{im}(c) \subset B_{r_x}(x)$ there exists a $\bar{c} \in C^{(L)}_{k+1}(X)$ satisfying $b\bar{c} = c$ and $\text{diam}(\text{im}(\bar{c})) \leq F_x(\text{diam}(\text{im}(c)))$.

- $X$ admits a local cone inequality for $C^{(L)}_0(X)$ if for every $x \in X$ there exists $r_x > 0$ and $F_x \in \mathcal{F}$ such that for every $c = \sum n_i c_i \in C^{(L)}_0(X)$ with $\sum n_i = 0$ and $\text{im}(c) \subset B_{r_x}(x)$ there exists an $\bar{c} \in C^{(L)}_1(X)$ satisfying $b\bar{c} = c$ and $\text{diam}(\text{im}(\bar{c})) \leq F_x(\text{diam}(\text{im}(c)))$.

Note that our definition of cone type inequalities for currents is different from the one used in [Wen07]: there one has the condition that locally there exists a filling with controlled mass whereas we consider only compactly supported currents and our condition is that locally there exists a filling with controlled diameter.

The following lemma is from [Mun84, Lemma 30.6] (therein it is stated for the singular theory in topological spaces, see below).

Lemma 2.6. There exists, for each metric space $X$ and each non-negative integer $k$, a homomorphism

$$K_X : C^{(L)}_k(X) \to C^{(L)}_{k+1}([0,1] \times X)$$

having the following property: If $c \in C^{(L)}_k(X)$ is a singular simplex, then

$$bK_X(c) + K_X(bc) = j_{\#}(c) - i_{\#}(c). \quad (2.4)$$

Here the map $i : X \to [0,1] \times X$ carries $x$ to $(0,x)$; and the map $j : X \to [0,1] \times X$ carries $x$ to $(1,x)$.

We only adumbrate the proof (carried out in [Mun84] for the continuous case) to indicate that this holds for Lipschitz chains too: One wants to look at $[0,1] \times c$ as a chain in $C^{(L)}_{k+1}([0,1] \times X)$, where $c$ is a singular simplex in $C^{(L)}_k(X)$. To do this, one
first gives a decomposition of $[0, 1] \times \Delta^k$ into a $(k + 1)$-chain consisting of (regular) simplices in $\mathbb{R}^{k+2}$. Then one carries this decomposition over to $[0, 1] \times X$ for every simplex $c \in C^{(L)}_k(X)$ in an intuitive way, producing a chain $\tilde{c}$ in $C^{(L)}_{k+1}([0, 1] \times \text{im}(c))$. Clearly, this construction respects the Lipschitz continuity.

We can also define $[0, 1] \times T$ for a normal current $T$ (see [Wen07, p. 146]): Given a function $f : [0, 1] \times X \to \mathbb{R}$ we set $f_t(x) := f(t, x)$, so $f_t$ is a function from $X$ to $\mathbb{R}$. For $T \in N_k(X)$ we define the normal $k$-current $[t] \times T$ on $[0, 1] \times X$ by $([t] \times T)(f, \pi_1, \ldots, \pi_k) := T(f_t, \pi_1, \ldots, \pi_k)$.

**Definition 2.7** ([Wen07], Definition 3.1). For a normal current $T \in N_k(X)$ the functional $[0, 1] \times T$ on $D^{k+1}([0, 1] \times X)$ is given by

$$
(0, 1] \times T)(f, \pi_1, \ldots, \pi_k) := \sum_{i=1}^{k+1} (-1)^{i+1} \int_0^1 T(f_t, \partial \pi_i / \partial t, \pi_1, \ldots, \pi_{i-1}, \pi_i, \pi_{i+1}, \ldots, \pi_{k+1})dt
$$

for $(f, \pi_1, \ldots, \pi_{k+1}) \in D^{k+1}([0, 1] \times X)$.

**Proposition 2.8** ([Wen07], Theorem 3.2). For $T \in N_k(X)$, $k \geq 1$, with bounded support the functional $[0, 1] \times T$ is a $(k+1)$-dimensional normal current on $[0, 1] \times X$ with boundary $\partial([0, 1] \times T) = [1] \times T - [0] \times T - [0, 1] \times \partial T$. Moreover, if $T \in I_k(X)$ then $[0, 1] \times T \in I_{k+1}([0, 1] \times X)$.

3. **Proof of Theorem 1.3, Theorem 1.5 and their corollaries**

3.1. **Lipschitz contractions and cone inequalities.** Examples of spaces that admit locally strong Lipschitz contractions are Banach spaces or CAT$(\kappa)$-spaces for $\kappa \in \mathbb{R}$. This is discussed on pp. 146 and 147 in [Wen07]; as well from there follows the last statement of the following proposition.

**Proposition 3.1.** Let $X$ be a metric space that admits locally strong Lipschitz contractions. Then $X$ admits local cone inequalities for $C^j_L(X)$ and $C^j_S(X)$ for all $j \geq 0$.

If in addition $X$ is complete then it also admits local cone inequalities for $I^j_k(X)$, $k \geq 1$.

**Proof.** Denote by $x_0^0$ the constant $n$-simplex with image $x_0 \in X$. Note that $bx_0^0 = 0$ if $n$ is odd or zero and $bx_0^n = x_0^{n-1}$ if $n$ is even.

Let $x \in X$ with $r_x > 0$ and $\gamma_x > 0$ as in the definition of locally strong Lipschitz contractions. Note that for a $\gamma_x$-contraction $\phi$ of a set $S$ we have

$$
\text{diam}(\phi([0, 1] \times S)) \leq 2\gamma_x \text{diam}(S).
$$

Now we use the homomorphism $K := K_X : C^L_k(X) \to C^L_{k+1}([0, 1] \times X)$ from Lemma 2.6. Let $c \in C^L_k(X)$ with $\text{im}(c) \subset B_{r_x}(x)$. Let $\phi : [0, 1] \times \text{im}(c) \to X$ be a $\gamma_x$-contraction. The push-forward of $K(c)$ by $\phi$ is clearly in $C^L_{k+1}(X)$. If $bc = 0$ we have $K(bc) = 0$. In this case, for $c = \sum n_i c_i$, we have by (2.4)

$$
b(\phi \# K(c)) = c - \sum n_i x_0^k.
$$

Since $bc = 0$ we have, for $k$ even and positive, $\sum n_i = 0$; i.e. $\tilde{c} := \phi \# K(c)$ is a filling of $c$ in this case. If $k$ is odd, filling of $c$ is given by

$$
\tilde{c} := \phi \# K(c) + \sum n_i x_0^{k+1}
$$

Note that, if $c$ is a Lipschitz chain, so is $\tilde{c}$. If $k = 0$, $c = \sum n_i x_i$ with $\sum n_i = 0$ a Lipschitz filling is given by the 1-chain $\tilde{c} := \sum n_i \phi(t, x_i)$. Furthermore,

$$
\text{diam}(\text{im}(\tilde{c})) = \text{diam}(\text{im}(\phi \# K(c))) \leq 2\gamma_x \text{diam}(\text{im}(c)).
$$
Now let $X$ be complete. If $T \in \mathbf{I}_t(X)$ has $\text{spt}(T) \subset B_{r_t}(x)$ and $\partial T = 0$, we get a filling $T := \phi_{\#}([0,1] \times T)$ in $\mathbf{I}_t(X)$ (by Proposition 2.8) with $\text{spt}(\bar{T}) \subset \text{im}(\phi)$, i.e. $\text{diam}(\text{spt}(\bar{T})) \leq \gamma \text{diam}(\text{spt}(T))$. If in addition $T \in \mathbf{I}_k(X)$ then $[0,1] \times T \in \mathbf{I}_{k+1}([0,1] \times X)$ and therefore $\text{spt}(\bar{T})$ is compact.

Concluding we see that if $X$ admits locally strong Lipschitz contractions then $X$ admits local cone inequalities for $\mathbf{I}_k(X)$, $k \geq 1$, and $C^{(j)}_k(X)$, $j \geq 0$, with $F_{s_1}(t) = 2\gamma s t$.

\[\Box\]

3.2. \textbf{Proof of Theorem 1.3.} Theorem 1.3 follows by Proposition 3.1 from the more general fact stated below.

\textbf{Theorem 3.2.} Let $X$ be a complete metric space. Then for $T \in \mathbf{I}_0(X)$ there exists $c \in C_0(X)$ with $T = [c]$.

Suppose the complete metric space $X$ admits local cone inequalities for $\mathbf{I}_1(X)$ and $C^j_k(X)$ for $j = 1, \ldots, n$ and $k = 0, \ldots, n - 1$ and let $\epsilon > 0$. Given $T \in \mathbf{I}_0(X)$ with $\partial T = [c]$ for $c \in C^L_n(X)$ and $bc = 0$, there exist $N \in \mathbb{N}$, $T_1, \ldots, T_N \in \mathbf{I}_N(X)$, $c_1, \ldots, c_N \in \mathbf{C}_N(X)$ and $V_1, \ldots, V_N \in \mathbf{I}_{N+1}(X)$ such that

\begin{align*}
&i) \sum T_i = T \text{ and } \partial V_i = T_i - [c] \\
&ii) \exists m \in \mathbb{N}_0: b(\sum c_i) = sd^m(c) \\
&iii) \text{spt}(V_i) \cup \text{im}(c_i) \subset B_s(\text{spt}(T) \cup \text{im}(c)) \text{ and } \text{diam}(\text{spt}(V_i)) < \epsilon.
\end{align*}

\textbf{Remark 3.3.} By a part of a chain $c = \sum_{i=1}^m n_i c_i$ we mean a chain $c' = \sum_{i=1}^m n^i c_i$ such that $n_i = n^i$ or $n^i = 0$. Let $c \in C^{(L)}_n(X)$, $n \geq 1$ and $U \subset X$. For $\epsilon > 0$ there exist a chain $\hat{c} \in C^{(L)}_n(B_\epsilon(U))$ that is a part of $sd^m(c)$ for some $m \in \mathbb{N}_0$ and such that $\text{im}(sd^m(c) - \hat{c}) \subset X \setminus U$. To see this, it is enough to consider a singular $n$-simplex $c$. Let $m \in \mathbb{N}$ be such that $sd^m(c) = \sum m_i c_i$ with $\text{diam}(c_i) < \epsilon$ for all $i$. Set $\hat{c} := \sum m_i c_i' \in C^{(L)}_n(B_\epsilon(U))$, where $m_i = m_i'$ if $\text{im}(c_i') \cap U \neq \emptyset$ and $m_i = 0$ otherwise.

This fact will be used in the proof to construct a simplicial boundary close to a slice of a simplicial cycle with a filling of the difference: Let $bc = 0$ and $U = \{d \leq r\}$ for a Lipschitz function $d : X \to \mathbb{R}$ and $r \in \mathbb{R}$. Then, since $|c|/U = |c| - |\hat{c}|(X \setminus U)$, we have $|c|/U = |bc| - \partial|\hat{c}|((X \setminus U))$. Thus, $|c|/(X \setminus U)$ has boundary $\partial|\hat{c}|((X \setminus U)) = |bc| - (|c|, d, r+)

\textbf{Proof of Theorem 3.2.} We argue inductively on the dimension of the current; for the induction step we use another induction on the number of balls needed to cover $\text{spt}(T) \cup \text{im}(c)$. Let $n = 0$. The decomposition (2.2) gives the desired equality for integral currents, i.e. $T = \sum_{i=1}^m n_i [q_i]$ and $c := \sum_{i=1}^m n_i q_i$.

Now let $n > 0$. For $x \in X$, let $s_x > 0$ denote the minimum of all radii in the (finitely many) cone inequalities for $x$ and $F_x$ the maximum function of all those $x$-dimension functions for $x$. We can assume that $F_x(r) \geq r$. Let $0 < R_x < r_x$ be such that

\[2(R_x + F_x(2R_x + 2F_x(2R_x))) < \epsilon.\]

Cover $\text{spt}(T) \cup \text{im}(c)$ by balls of radius $R_x$ and centers in $\text{spt}(T) \cup \text{im}(c)$. We get a finite subcover with centers $x_1, \ldots, x_M$; set $R_i := R_{x_i}$ and $F_i := F_{x_i}$. We show that there are $V_i, c_i$ and $T_i, i = 1, \ldots, M, \text{ with properties } i) \text{ and } ii)$ of Theorem 3.2 that fulfill moreover

\[\text{spt}(V_i) \cup \text{im}(c_i) \subset B_{s_i}(x_i);\]

this clearly implies iii).

If $M = 1$ the cone inequality for $C^L_{n-1}(X)$ gives a filling $c_1 \in C^L_n(X)$ of $c$ (note that for $n = 1$, $c = \sum n_i q_i$ necessarily $\sum n_i = T(1,1) = 0$). We have now
im(c_1) ⊂ B_{R_1+F_1(2R_1)}(x_1). Set T_1 := T; now the cone inequality for I^n(X) gives a filling V_1 ∈ I^n_{1+1}(X) of T − [c_1] with support in B_{\frac{R}{2}}(x_1).

If M > 1 let 0 < R < R_1 be such that spt(T) ∪ im(c) ⊂ B_R(x_M) ∪ \bigcup_{i=1}^{M-1} B_{R_i}(x_i). Set \alpha := \frac{R_3-R}{R_1} and choose r \in (R+\alpha, R+3\alpha); note that \|T\| = \|T\| - \|\alpha\| < r + \alpha < R_M.

We slice T by \{d(x) := d(x_M, x) at r. We can assume that \langle T, d, r \rangle ∈ I^n_{n-1}(X) (and hence \langle \partial T, d, r \rangle ∈ I^n_{n-2}(X) for n > 1) and that \|\partial T\|((d^{-1}(r)) = 0 for n = 1.

For n = 1 set directly S := T\{d > r\}; this is an integer rectifiable 1-current with compact support. Since \partial S = \langle \partial T\{d > r\} − \langle T, d, r \rangle \in I^0(X) we have S ∈ I^0(X). As above we see that \partial S = \sum n_i [q_i] with \sum n_i = 0. By induction there are c_1, \ldots, c_{M-1}, T_1, \ldots, T_{M-1} and V_1, \ldots, V_{M-1} with b\sum c_i = \sum n_i [q_i], property i) of Theorem 3.2 for S instead of T and with (3.2). Now, T_M := T − spt(T_M) ⊂ B_{R_1}(x_M) and boundary \partial(T_M) = \partial T − \partial S =: \sum u'_j [p_j]; again, \sum u'_j = 0. So there is a c_M ∈ C^L_1(B_{R_1+F_1(2R_1)}(x_M)) and a filling V_M of T_M − [c_M] with spt(V_M) sup \im(c_M) ⊂ B_{R_1+F_1(2R_1+F_1(2R_1))}(x_M) ⊂ B_{\frac{R}{2}}(x_M), proving the case n = 1.

If n > 1, choose c' := \alpha/2. We can apply remark 3.3 for c, c' and the slice by d at r to get \tilde{c} ∈ C^L_{n-1}(B_{R_1+c'}(x_M)) and m_1 ∈ \N_0 with \im(sdm^{m_1}(c) − \tilde{c}) ⊂ X \setminus B_r(x_M).

Now, T' := \langle T, d, r \rangle − \tilde{c}\{d > r\} has by (3.1)
\partial T' = −\langle \partial T, d, r \rangle − \partial(\tilde{c}\{d > r\}) = −[\tilde{c}],
in particular, T' ∈ I^n_{n-1}(X). By induction assumption for T' and c' and c := −\tilde{b} there exist T'_1, \ldots, T'_K ∈ I^n_{n-1}(X), V'_1, \ldots, V'_K ∈ I^n_n(X), c'_1, \ldots, c'_K ∈ C^L_{n-1}(X) and m_2 ∈ \N_0 with
\partial \sum V'_i = T' − \sum [c'_i], \quad sdm^{m_2}(−\tilde{b}) = b\sum c'_i
and
\im(\sum c'_i) \cup spt(\sum V'_i) ⊂ B_{c'}(spt(T') ∪ im(b\tilde{c})) ⊂ B_{R_1}(x_M) \setminus B_R(x_M).

So we have an (a priori only) integer rectifiable n-current with compact support
S := T\{d > r\} + \sum V'_i
with spt(S) ⊂ \bigcup_{i=1}^{M-1} B_{R_i}(x_i). The boundary is
\partial S = \partial T\{d > r\} + T' − \sum c'_i
= \langle \partial T\{d > r\}, − \langle T, d, r \rangle + \langle T, d, r \rangle − [\tilde{c}]\{d > r\} − \sum [c'_i]
= [\tilde{c}]\{d > r\} − [\tilde{c}]\{d > r\} − \sum [c'_i]
= [sd^{m_1}(c) − \tilde{c} − \sum c'_i] ∈ I^n_{n-1}(X),
so S is integral. By construction, \im(sdm^{m_1}(c) − \tilde{c}) and \im(\sum c'_i) are subsets of \bigcup_{i=1}^{M-1} B_{R_i}(x_i) and for \tilde{c} := sdm^{m_2}(sd^{m_1}(c) − \tilde{c}) − \sum c'_i we have b(\tilde{c}) = sdm^{m_1+m_2}(bc) = 0.

By induction for S there are T_1, \ldots, T_{M-1}, c_1, \ldots, c_{M-1}, V_1, \ldots, V_{M-1} and m_3 with properties i) and ii) of Theorem 3.2 and (3.2). Set T_M := T − S, then
\partial T_M = \partial T − \partial S = [\tilde{c}] − [\tilde{c}]
= [sd^{m_1+m_2+m_3}(c) − sdm^{m_2}(sd^{m_1}(c) − \tilde{c}) − \sum c'_i] .
With c'' := sdm^{m_3}(\tilde{c}) + sdm^{m_3}(\sum c'_i), we have \im(c'') ⊂ B_{R_1}(x_M) and b(c'') = sdm^{m_3}(sd^{m_2}(bc) + b\sum c'_i) = 0. So by the cone inequalities there exist fillings c_M ∈
$C_n^L(X)$ of $c''$ and $V_M \in \mathbb{I}_{n+1}^L(X)$ of $T - S - [c_M]$ with image and support in $B^L_x(x_M)$.

Finally,

$$b\left(\sum_{i=1}^M c_i\right) = sd^m_3(\tilde{c}) + c''$$

$$= sd^{m_1 + m_2 + m_3}(c) - sd^{m_2 + m_3}(\tilde{c}) - sd^m_3\left(\sum c'_i\right)$$

$$+ sd^{m_2 + m_3}(\tilde{c}) + sd^m_3\left(\sum c'_i\right)$$

$$= sd^m_3(c)$$

for $m = m_1 + m_2 + m_3$.

3.3. Proof of Theorem 1.5. The following implies Theorem 1.5 by Proposition 3.1.

**Theorem 3.4.** Let $X$ be a metric space. Then $C_0(X) = C_0^L(X)$.

Suppose the metric space $X$ admits local cone inequalities for $C_k^L(X)$ and $C_k^L(X)$ for $k = 0, \ldots, n$ and $j = 0, \ldots, n - 1$ and let $\epsilon > 0$. Given $c \in C_n(X)$ with $bc \in C_{n-1}^L(X)$ there exist $N \in \mathbb{N}$, $m \in \mathbb{N}_0$, $c_1, \ldots, c_N \in C_n(X)$, $c_1^L, \ldots, c_N^L \in C_{n}^L(X)$ and $c_i, \ldots, c_{N} \in C_{n+1}(X)$ such that

1. $\sum c_i = sd^m(c)$ and $b\tilde{c}_i = c_i - c'_i$

2. $\text{im}(\tilde{c}_i) \subset B_\alpha(\text{im}(c))$ and $\text{diam}(\text{im}(\tilde{c}_i)) < \epsilon$.

The proof of this theorem is essentially the same as the proof of Theorem 3.2, when slicing is exchanged by subdivision.

**Proof of Theorem 3.4.** Let $n = 0$. Clearly, $C_0(X) = C_0^L(X)$. Let now $n > 0$ and assume that the theorem holds for $n - 1$. For $x \in X$, let $r_x$ denote the minimum of all radii in the (finitely many) cone inequalities for $x$ and $F_x$ the maximum function of all those diameter functions for $x$ and assume that $F_x(r) \geq r$. Let $0 < R_x < r_x$ be such that

$$2(R_x + F_x(2R_x + 2F_x(2R_x))) < \epsilon.$$ 

Cover $\text{im}(c)$ by balls with center in $\text{im}(c)$ and of radius $R_x$ and choose a finite covering: let the centers be $x_1, \ldots, x_m$, denote $R_0 := R_x$ and $F_i := F_{x_i}$. We show by induction that there are $c_i, c_i^L$ and $\tilde{c}_i, i = 1, \ldots, M$, with property i) of Theorem 3.4 and the property

$$\text{im}(\tilde{c}_i) \subset B^L_\alpha(x_i).$$

If $M = 1$ the claim follows directly from the cone inequalities. If $M > 1$ let $0 < R < R_M$ such that $\text{im}(c) \subset B_R(x_M) \cup \bigcup_{i=1}^{M-1} B_{R_i}(x_i)$. Set $\alpha := \frac{R_M - R}{4}$ and choose $r \in (R + \alpha, R + 3\alpha)$; note that $R < r - \alpha < r + \alpha < R_M$. Now let $m_1 \geq 0$ be such that each simplex of $sd^{m_1}(c)$ has image with diameter less than $\alpha/2$.

Let $c^+$ be the part of $sd^{m_1}(c)$ consisting of all simplices whose image has non-empty intersection with $X \setminus B_{R_\alpha}(x_M)$ and let $c^- := sd^{m_1}(c) - c^+$. Let $\tilde{c}$ denote the part of the boundary of $c^+$ that is not Lipschitz (note that is also the negative of the non-Lipschitzian part of $bc^-$) and set $\tilde{c}^L := bc^+ - \tilde{c}$.

Now, $b\tilde{c} = b(bc^+ - \tilde{c}^L) = -b\tilde{c}_i^L \in C_{n-2}^L(X)$, and $\text{im}(\tilde{c}) \subset \bigcup_{i=1}^{M-1} B_{R_i}(x_i)$. By induction for $\tilde{c}$ with $c' := \alpha/2$, there are $c'_1, \ldots, c'_N \in C_{n-1}(X)$, $c'_1^L, \ldots, c'_N^L \in C_{n-1}^L(X)$, $c'_1, \ldots, c'_N \in C_n(X)$ and $m_2 \in \mathbb{N}_0$ such that

$$\sum c'_i = sd^{m_2}(\tilde{c}), b\tilde{c}'_i = c'_i - c'_i^L$$

and $\text{im}(\sum c'_i) \subset B_{c'}(\text{im}(\tilde{c})) \subset B_{r+\alpha}(x_M) - B_{r-\alpha}(x_M)$.

Set

$$z := sd^{m_2}(c^+) - \sum c'_i \in C_n(X).$$
Then, \( \text{im}(z) \subset \bigcup_{i=1}^{M-1} B_{R_i}(x_i) \) and

\[
\begin{align*}
bz &= sd^{m_2}(c^L + \tilde{c}) + \sum c^L_i - \sum c^L_i \\
&= sd^{m_2}(c^L) + \sum c^L_i \in C^{L}_{n-1}(X).
\end{align*}
\]

By induction for \( z \) there are \( c_1, \ldots, c_{M-1} \in C_n(X), c^L_1, \ldots, c^L_{M-1} \in C^{L}_{n}(X), \)
\( \tilde{c}_1, \ldots, \tilde{c}_{M-1} \in C_{n+1}(X) \) and \( m_0 \in N_0 \) so that
\[
\sum c_i = sd^{m_3}(z),
\]
\[
b\tilde{c}_i = c_i - c^L_i \text{ and } \text{im}(\tilde{c}_i) \subset B^{L}_2(x_i).
\]

Now, let \( m := m_1 + m_2 + m_3 \) and

\[
c_M := sd^{m}(c) - sd^{m_3}(z)
\]
\[
= sd^{m}(c) - sd^{m_2+m_3}(c^+) + sd^{m_3}(\sum c^L_i)
\]
\[
= sd^{m_2+m_3}(c^-) + sd^{m_3}(\sum c^L_i),
\]

In particular, \( \text{im}(c_M) \subset B_{R_M}(x_M) \). Denote by \( c^L_M \) the Lipschitzian part of
the boundary of \( c^- \) and \( c^L_M := bc^- - c^L_M \); recall that \( c^L_M = \tilde{c} \) and that \( \sum c^L_i = sd^{m_3}(\tilde{c}) \).

Then
\[
bc_M = sd^{m_2+m_3}(c^L_M + c^L_M) + sd^{m_3}(\sum c^L_i - \sum c^L_i)
\]
\[
= sd^{m_2+m_3}(c^L_M) - sd^{m_3}(\sum c^L_i),
\]
i.e. \( bc_M \in C^{L}_{n-1}(X) \). The cone inequalities give now fillings \( c^L_M \in C^{L}_{n}(X) \) of \( bc_M \) and \( c_M \in C_{n+1}(X) \) of \( c_M = c^L_M + c^L_M \) such that (3.3) is satisfied. Finally,
\[
\sum_{i=1}^{M} c_i = sd^{m_3}(z) + sd^{m}(c) - sd^{m_3}(z) = sd^{m}(c).
\]

3.4. Proof of Corollary 1.4 and 1.6. Assuming Theorem 1.3 and 1.5, the proofs of Corollary 1.4 and 1.6 are the same up to minor changes. Therefore we only give the proof of Corollary 1.4.

Proof of Corollary 1.4. We show that the chain homomorphism \([ ]: C^{L}_{n}(X) \rightarrow \Gamma^{L}_{n}(X) \)
induces an isomorphism \([ ]: H^{L}_{n}(X, A) \rightarrow H^{IC}_{n}(X, A) \) for all \( n \geq 0 \). Note that \([ ] \)
sends \( Z^{L}_{n}(X, A) \) to \( Z^{IC}_{n}(X, A) \) and \( B^{L}_{n}(X, A) \) to \( B^{IC}_{n}(X, A) \), so \([ ] \) induces a homomorphism from \( H^{L}_{n}(X, A) \) to \( H^{IC}_{n}(X, A) \).

Let \( T \in \mathcal{Z}^{IC}_{n}(X, A) \), as \( \partial T \in \mathcal{I}^{n-1}(A) \) and \( \partial(\partial T) = [0] \) there exists \( c \in C^{L}_{n-1}(A) \) with \( bc = sd^{m}(0) = 0 \) and \( V \in \mathcal{I}^{n}(A) \) with \( \partial V = \partial T - [c] \). Now, \( T - V \) has boundary \( \partial(T - V) = [c], \) i.e. there exists \( \tilde{c} \in C^{L}_{n}(X) \) with \( b\tilde{c} = sd^{m}(c) \) (thus, \( \tilde{c} \in \mathcal{Z}^{L}_{n}(X, A) \)) and there exists \( \tilde{V} \in \mathcal{I}^{n+1}(A) \) with \( \partial \tilde{V} = (T - V) - [\tilde{c}]; \) hence \( V + \partial \tilde{V} \in \mathcal{B}^{IC}_{n}(X, A) \) and
\[
[c] + V + \partial \tilde{V} = T,
\]
showing the surjectivity of the homomorphism.

Let \( c, \tilde{c} \in Z^{L}_{n}(X, A) \) with \( [c] + B^{IC}_{n}(X, A) = [\tilde{c}] + B^{IC}_{n}(X, A) \), i.e. there exists \( R + \partial S \in B^{IC}_{n}(X, A): \)
\[
[c] + R + \partial S = [\tilde{c}].
\]
Then \( \partial R = [b(\tilde{c} - c)] \) and \( b(c - \tilde{c}) \in C^{L}_{n-1}(A) \), so there exists \( c_{1} \in C^{L}_{n}(A) \), \( V \in \mathcal{I}^{n+1}(A) \) with \( \partial V = R - [c_{1}] \) and \( bc_{1} = sd^{m_{1}}(b(c - \tilde{c})) \). Now, by (2.3) we get
\[
b(D_{m_{1}, X}(\tilde{c} - c)) + D_{m_{1}, X}(b(c - \tilde{c})) = sd^{m_{1}}(\tilde{c} - c) - \tilde{c} + c.
Set \( c_2 := D_{m_1, X}(\bar{c} - c) \in C^L_{n+1}(X) \); note that by naturality \( D_{m_1, X}(b(\bar{c} - c)) = D_{m_1, A}(b(\bar{c} - c)) =: c_3 \in C^L_n(A) \).

On the other hand,
\[
\partial(S + V) = [\bar{c} - c] - [c_1] = [sd^{m_1}(\bar{c} - c) - c_1].
\]

Set \( c_4 := sd^{m_1}(\bar{c} - c) - c_1 \in C^L_n(X) \); then \( bc_4 = 0 \). So there is a filling \( c_5 \in C^L_{n+1}(X) \)
with \( bc_5 = sd^{m_2}(c_4) \). Thus, \( c_6 := D_X(c_4) \in C^L_{n+1}(X) \) has
\[
bc_6 = sd^{m_2}(c_4) - c_4 = bc_3 - sd^{m_1}(\bar{c} - c) + c_1.
\]

Together,
\[
c + c_1 - c_3 + b(c_5 - c_2 - c_6) = \bar{c}
\]
with \( c_1 - c_3 \in C^L_n(A) \) and \( c_5 - c_2 - c_6 \in C^L_{n+1}(X) \), therefore the homomorphism is injective.

\[\square\]

4. Proof of Theorem 1.7

The goal of this section is to show that the maximal divisible subgroup of \( H^1_{1C}(\mathbb{H}) \) is trivial. In subsection 4.2 we will show that the maximal divisible subgroups of \( H^1_1(\mathbb{H}) \) and \( H_1(\mathbb{H}) \) are non-trivial. This implies that \( H^1_{1C}(\mathbb{H}) \) is not isomorphic to either one of these groups.

Recall that the Hawaiian Earring \( \mathbb{H} \subset \mathbb{R}^2 \) is the countable union of the circles
\[
L_n = \{ x \in \mathbb{R}^2; \| x - (1/n, 0) \| = 1/n \},
\]
with metric given by
\[
d(x, y) := \begin{cases} \| x - y \|, & \text{if } \exists n \in \mathbb{N} : x, y \in L_n, \\ \| x \| + \| y \|, & \text{otherwise}. \end{cases}
\]

4.1. Metric Currents and the Hawaiian Earring. In the proof we use the fact that the first homology group of the complex of integral currents on \( S^1 \) is isomorphic to \( \mathbb{Z} \). This follows from Corollary 1.4 and 1.6 since \( H_1(S^1) \cong \mathbb{Z} \).

By definition, we have \( I_1(X) = 0 \) for any \( \mathcal{H}^2 \) null set \( X \). It follows that \( H^1_{1C}(\mathbb{H}) \) and \( H^1_{1C}(L_n) \) are simply the kernels of the maps \( \partial : I_1(\mathbb{H}) \to I_0(\mathbb{H}) \) and \( \partial : I_1(L_n) \to I_0(L_n) \) respectively. Thus, showing that an element of \( H^1_{1C}(\mathbb{H}) \) is zero is equivalent to showing that the integral current representing it is zero.

Proposition 4.1. The maximal divisible subgroup of \( H^1_{1C}(\mathbb{H}) \) is trivial.

Proof. Let \( T \in I_1(\mathbb{H}) \) be an element of the maximal divisible subgroup of \( H^1_{1C}(\mathbb{H}) \). Let \( n \in \mathbb{N} \). We write \( p_n : \mathbb{H} \to L_n \) for the map which sends \( x \in \mathbb{H} \) to \( (0, 0) \) if \( x \notin L_n \) and to itself otherwise. We denote the inclusion \( L_n \to \mathbb{H} \) by \( i_n \). The above remark shows that \( H^1_{1C}(L_n) \cong \mathbb{Z} \). We claim that \( H^1_{1C}(p_n)(T) = (p_n)_{\#}(T) = 0 \). To see this, let \( k \in \mathbb{Z} \) be an arbitrary integer. By assumption there exists an element \( T' \in H^1_{1C}(\mathbb{H}) \) such that \( T = k \cdot T' \). It follows that \( (p_n)_{\#}(T) = k \cdot (p_n)_{\#}(T') \in \mathbb{Z} \), i.e. every integer divides \( (p_n)_{\#}(T) \), which shows that \( (p_n)_{\#}(T) = 0 \). We find in particular that \( (i_n p_n)_{\#}(T) = 0 \).

On the other hand, we have \( (i_n p_n)_{\#}(T) = T|L_n \). This follows since both currents have support \( L_n \) and since for any function \( f : \mathbb{H} \to \mathbb{R} \), the restriction of \( f \) to \( L_n \) agrees with the restriction of \( f \circ i_n \circ p_n \) to \( L_n \). These two facts imply the desired equality (by [Lan08, Lemma 3.2 and Theorem 4.4]).

Together this implies that
\[
\| T \|(L_n) = M(T|L_n) = M((i_n p_n)_{\#}(T)) = M(0) = 0,
\]
and therefore that
\[ M(T) = \|T\|((H)) \leq \sum_{n=1}^{\infty} \|T\|((L_n)) = 0, \]
which follows by countable subadditivity of \( \|T\| \). We find that \( T = 0 \), i.e. that the maximal divisible subgroup of \( H_1^L(\mathbb{H}) \) is indeed trivial. \( \square \)

4.2. The Maximal Divisible Subgroup of \( H_1^L(\mathbb{H}) \).

4.2.1. Overview. In [EK00] it is shown that the maximal divisible subgroup of \( H_1(\mathbb{H}) \) is non-trivial. In order to construct a nontrivial element of the maximal divisible subgroup of \( H_1^L(\mathbb{H}) \) we follow the construction given in the proof of Theorem 4.14 in [Eda92]. Since we do not have a concise description of the Lipschitz maps \( \sigma \) to an explicit construction of certain Lipschitz maps \( \sigma_n : [0, 2\lambda(n)] \to \mathbb{H}, n \in \mathbb{N} \) (where \( \lambda(n) \leq 1 \) is a real number). The \( \sigma_n \) have the following properties:

i) For \( n \geq 2 \), \( [\sigma_{n-1}] = n \cdot [\sigma_n] \) in \( H_1^L(\mathbb{H}) \) and

ii) The element \( [\sigma_1] \in H_1^L(\mathbb{H}) \) maps to a nonzero element under the homomorphism \( H_1^L(\mathbb{H}) \to H_1(\mathbb{H}) \).

The idea behind the construction is the following. We first choose a sequence of maps \( \sigma_n : [0, \lambda(n)] \to \mathbb{H}, n \in \mathbb{N} \) which represent commutators of certain standard loops in \( \pi_1(\mathbb{H}) \). We construct the maps \( \sigma_n \) in such a way that for \( n \geq 2 \) the equation
\[ \sigma_{n-1} = c_{n-1} \cdot \sigma_n \cdot \cdots \cdot \sigma_n^{\text{n times}} \]
holds. This could be depicted as follows:

\[ \begin{array}{c}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\vdots
\end{array} \]

Condition i) is satisfied since \( c_n \) is Lipschitz homotopic to a constant map. The commutators \( c_n \) are inserted to ensure that the element \( [\sigma_1] \in H_1^L(\mathbb{H}) \) does not vanish, which follows from the stronger fact that its image under the comparison map \( H_1^L(\mathbb{H}) \to H_1(\mathbb{H}) \) is a non-zero element. This is equivalent to proving that the corresponding element \( [\sigma_1]_{\pi_1} \in \pi_1(\mathbb{H}) \) does not lie in the commutator subgroup. We prove this in Proposition 4.6 by reducing the problem to a question about commutator subgroups of free groups.

4.2.2. Preliminaries. Let \( S \) be the set of finite sequences of (non-zero) natural numbers, i.e. of maps \( \{1, \ldots, n\} \to \mathbb{N} \) for \( n \in \mathbb{N} \). We write \( s = (s_1, \ldots, s_n) \) for the sequence with \( s(i) = s_i \). We call \( n \) the length of \( s \) and denote it by \( \ell(s) \).

For \( k, m \in \mathbb{N} \) let \( k \cdot m \) denote the sequence \( s \in S \) of length \( k \) with \( s(i) = m \), \( 1 \leq i \leq k \). The concatenation of sequences \( + : S \times S \to S \) is given by the map which sends \((s, s') \in S \times S \) to the sequence \( s'' \) given by \( s''(i) = s(i) \), \( 1 \leq i \leq \ell(s) \) and \( s''(j) = s'(j - \ell(s)) \) for \( \ell(s) + 1 \leq j \leq \ell(s) + \ell(s') \).

Let \( B = \{s \in S ; s(i) \leq i \text{ and } \ell(s) > 0\} \). We write \( S_n \) for the subset of \( S \) consisting of all sequences of length \( n \) and \( B_n \) for \( B \cap S_n \). There is a linear order relation \( \preceq \) on \( S \) such that for \( s, t \in S, s \preceq t \) if and only if one of the following holds:
where the sum is taken over any enumeration (and for $s$

\[ \tau(n) = \frac{1}{2^n n!} \]

and for $s \in B$ let

\[ \tau(s) = \sum_{t < s} \lambda(\ell(t)), \]

where the sum is taken over any enumeration $(t_n)_{n \in \mathbb{N}}$ of \{t \in B; t \prec s\} if this set is infinite. Note that the sum does not depend on the chosen enumeration and that the second part of ii) follows immediately from the first and equation (4.1).

The ordered set $B$ can be embedded in the unit interval $[0, 1]$. For $n \in \mathbb{N}$ let

\[ \lambda(n) = \frac{1}{2^n n!} \]

holds, which shows that the series is absolutely convergent. Using a suitable enumeration we find that for $s \preceq s'$ the useful formula

\[ \tau(s') - \tau(s) = \sum_{s \preceq t \prec s'} \lambda(\ell(t)) \]

holds.

The following lemma summarizes the basic properties of the map $\tau : B \to [0, 1]$.

**Lemma 4.2.** Let $n, m \in \mathbb{N}$, $s, s' \in B_n$, and let $s'' \in S$ be such that $s + s'$ and $s' + s''$ belong to $B$.

i) The map $\tau$ is strictly order preserving.

ii) If $s \preceq t \preceq s + (1)$, then $t \in \{s, s + (1)\}$. Moreover $\tau(s + (1)) - \tau(s) = \lambda(n)$.

iii) Let $1 \leq m \leq n$. Then $s + (m) \preceq t \prec s + (m + 1)$ if and only if $t(i) = s(i)$ for $1 \leq i \leq n$ and $t(n + 1) = m$.

iv) For $1 \leq m \leq n$, $\tau(s + (m + 1)) - \tau(s + (m)) = 2\lambda(n + 1)$.

v) The equality $\tau(s + s'') - \tau(s' + s'') = \tau(s) - \tau(s')$ holds.

**Proof.** Assume $s \preceq s'$. Then we have by definition

\[ \tau(s') = \sum_{t < s} \lambda(\ell(t)) = \sum_{t < s} \lambda(\ell(t)) + \sum_{s \preceq t \prec s} \lambda(\ell(t)) \]

\[ \geq \sum_{t < s} \lambda(\ell(t)) + \lambda(\ell(s)) + \sum_{t \preceq s} \lambda(\ell(t)) = \tau(s) \]

since $\lambda(\ell(s)) > 0$. This shows that $\tau$ is indeed strictly order preserving.

Note that the second part of ii) follows immediately from the first and equation (4.1). Now let $s \preceq t \preceq s + (1)$ and let $1 \leq j \leq n$. Then $s(j) < t(j)$ would imply that $s + (1) \prec t$, but this contradicts the hypothesis $t \preceq s + (1)$. Conversely, $s(j) > t(j)$ contradicts $s \preceq t$, so we find that $\ell(t) \geq n$ and $t(j) = s(j)$ for $1 \leq j \leq n$. If $\ell(t) = n$ we have $t = s$, so we can assume that $\ell(t) > n$ and hence that $t(n + 1) \geq 1$. If $t(n + 1)$ were greater than 1 we would have $s + (1) \prec t$, so we can conclude that $t(n + 1) = 1$. But $\ell(t) > n + 1$ then implies $s + (1) \prec t$ (by case ii) of the definition of $\prec$ which is again a contradiction. So we finally find that $t = s + (1)$, which establishes the second claim.

To see iii) we have to consider two cases. First note that if $t = s + (m)$, the conclusion holds trivially, i.e. we can assume that $s + (m) \prec t$. Then by definition we have to consider the two cases
a) there exists \( k \leq \min(n+1, \ell(t)) \) such that \( s + \langle m \rangle(k) \neq t(k) \) and \( s + \langle m \rangle(j) < t(j) \) for the minimal \( j \) with \( s + \langle m \rangle(j) \neq t(j) \), or

b) \( n + 1 \leq \ell(t) \) and \( s + \langle m \rangle(j) = t(j) \) for all \( 1 \leq j \leq n + 1 \).

In case b) we are obviously finished, so we can assume that we are in the situation of case a). If \( j \leq n \) we have \( s + \langle m + 1 \rangle \prec t \) which contradicts our assumption. It follows that \( s(i) = t(i) \) for \( 1 \leq i \leq n \) and that \( m < t(n + 1) \). But this shows that \( s + \langle m + 1 \rangle \preceq t \). Since we also have \( t \prec s + \langle m + 1 \rangle \) we find that case a) cannot occur under the assumptions of iii).

The proof of iv) follows directly from iii) and equation (4.1). We have

\[
\tau(s + \langle m + 1 \rangle) - \tau(s + \langle m \rangle) = \sum_{t < s + \langle m + 1 \rangle} \lambda(\ell(t)) - \sum_{t < s + \langle m \rangle} \lambda(\ell(t))
\]

\[
= \sum_{s + \langle m \rangle \preceq t < s + \langle m + 1 \rangle} \lambda(\ell(t))
\]

\[
= \lambda(\ell(s + \langle m \rangle)) + \sum_{k > n + 1} \left( \sum_{t \in C_k} \lambda(k) \right)
\]

where \( C_k = \{ t \in B_k; t(j) = s(j) \text{ for } 1 \leq j \leq n \text{ and } t(n + 1) = m \} \). Since the cardinality of \( C_k = \frac{k!}{(n+1)!} \) we find that

\[
\tau(s + \langle m + 1 \rangle) - \tau(s + \langle m \rangle) = \lambda(\ell(s + \langle m \rangle)) + \sum_{k > n + 1} \frac{k!}{(n+1)!} \lambda(k)
\]

\[
= \lambda(n + 1) + \frac{1}{(n+1)!} \sum_{k > n + 1} \frac{k!}{2^k k!}
\]

\[
= \lambda(n + 1) + \frac{1}{2^{n+1} (n+1)!} \sum_{k=1}^{\infty} \frac{1}{2^k} = 2\lambda(n + 1).
\]

It remains to show v). There are natural numbers \( m_1, \ldots, m_d \) such that \( s'' = \langle m_1 \rangle + \ldots + \langle m_d \rangle \). By induction we can reduce the problem to the case \( s'' = \langle m \rangle \).

Using iv) we can further reduce this to the case \( m = 1 \). Indeed, for \( m > 1 \) we have

\[
\tau(s + \langle m \rangle) = \tau(s + \langle m \rangle) - \tau(s + \langle m - 1 \rangle) + \tau(s + \langle m - 1 \rangle)
\]

\[
= \tau(s + \langle m - 1 \rangle) + 2\lambda(n + 1) = \ldots = \tau(s + \langle 1 \rangle) + 2(m - 1)\lambda(n + 1)
\]

which shows that

\[
\tau(s + \langle m \rangle) - \tau(s' + \langle m \rangle) = \tau(s + \langle 1 \rangle) - \tau(s' + \langle 1 \rangle).
\]

With ii) and equation (4.1) we find that \( \tau(s + \langle 1 \rangle) = \tau(s) + \lambda(n + 1) \) and therefore that the equality

\[
\tau(s + s'') - \tau(s' + s'') = \tau(s + \langle 1 \rangle) - \tau(s' + \langle 1 \rangle) = \tau(s) - \tau(s')
\]

holds, as claimed. \( \square \)

**Lemma 4.3.** The set

\[
I' = \bigcup_{s \in B} [\tau(s), \tau(s + \langle 1 \rangle)] \subseteq [0, 1]
\]

is dense in \([0, 1]\).

**Proof.** Let \( x \in [0, 1] \) and assume that for all \( s \in B, x \notin [\tau(s), \tau(s + \langle 1 \rangle)] \). We construct a sequence \((s_i)_{i \in \mathbb{N}} \) with \( \tau(s_i) < x \) as follows. Let \( s_1 = \langle 1 \rangle \) (which trivially satisfies \( \tau(s_1) = 0 < x \)), and if \( s_k \) is defined for all \( k < i \), let \( s_i \) be the maximal element of \( \{ s \in B_i; \tau(s) < x \} \). This set is not empty since it contains \( s_{i-1} + \langle 1 \rangle \).

Indeed, by induction hypothesis we have \( \tau(s_{i-1}) < x \) and \( x \notin I' \) implies in particular
\(x \notin [\tau(s_{i-1}), \tau(s_{i-1} + (1))],\) so \(\tau(s_{i-1} + (1)) < x.\) Moreover, for any \(i \in \mathbb{N}\) there exists \(m \in \mathbb{N}\) such that
\[
s_{i+1} = s_i + (m).
\]
This follows since \(s_i < s_i + (1) \leq s_{i+1}\) and therefore one of the following must hold:
a) there is a \(k \leq i\) with \(s_i(k) \neq s_{i+1}(k)\) and \(s_i(j) < s_{i+1}(j)\) for the minimal \(j\) with
\(s_i(j) \neq s_{i+1}(j),\) or
b) the sequence \(s_{i+1}\) extends \(s_i,\) i.e. \(s_{i+1} = s_i + \langle m \rangle.\)
But case a) would contradict the maximality of \(s_i: s' = \langle s_{i+1}(1), \ldots, s_{i+1}(i) \rangle\) is an
element of \(B_i,\) the equation
\[
\tau(s') \leq \tau(s' + \langle s_{i+1}(i) + 1 \rangle) = \tau(s_{i+1}) < x
\]
holds, and \(s' \prec s_i\) since \(s_i(j) < s_{i+1}(j) = s'(j).\)
This implies in particular that \((\tau(s_i))_{i \in \mathbb{N}}\) is a strictly increasing sequence and
that \(\sup \{\tau(s_i); i \in \mathbb{N}\} = \lim_{i \to \infty} \tau(s_i) \leq x.\) We will show that this is an equality
and therefore that \(x\) lies in the closure of \(I'.\)
First note that the set
\[
C = \{i \in \mathbb{N}; s_i(i) < i\}
\]
is either empty or infinite. If \(C\) were finite and non-empty, it would contain a
maximal element \(n \in \mathbb{N}.\) We write \(s'\) for the sequence \(\langle s_n(1), s_n(2), \ldots, s_n(n - 1) \rangle\)
and let \(m = s_n(n).\) Since \(n \in \mathbb{C}\) we have \(m < n.\) By maximality of \(n\) and with equation (4.2) it
follows that for \(i > n\) we must have \(s_i = s_n + (n + 1, n + 2, \ldots, i).\)
Now let \(t \prec s' + (m + 1).\) Then we have either \(t \prec s_n\) or \(t(j) = s'(j)\) for \(1 \leq j \leq n - 1\)
and \(t(n) = m\) (by Lemma 4.2 iii)). But in the latter case we must have \(t \preceq s_{i(t)}\)
since, for any \(i > n, s_i\) is the maximal element of \(B_i\) satisfying the given condition.
So in either case there is an \(i \in \mathbb{N}\) with \(t < s_i.\) It follows that
\[
\tau(s' + (m + 1)) = \sum_{t \prec s' + (m + 1)} \lambda(\ell(t)) \leq \sup \left\{ \sum_{t \preceq t'} \lambda(\ell(t')); t \prec s' + (m + 1) \right\}
\]
\[
\leq \sup \left\{ \sum_{t \preceq s_k} \lambda(\ell(t')); k \in \mathbb{N} \right\} = \lim_{i \to \infty} \tau(s_i) \leq x
\]
and hence, because \(x \in I',\) that \(\tau(s' + (m + 1)) < x.\) But this contradicts the
maximality of \(s_n\) in \(\{s \in B_n; \tau(s) < x\},\) so \(C\) must be infinite.
We first consider the case \(C = \varnothing.\) By construction we must have \(s_i = \langle 1, 2, \ldots, i \rangle\)
and it follows that
\[
\tau(s_i) \geq \sum_{k=1}^{i-1} \left( \sum_{t \in B_k} \lambda(k) \right) = \sum_{k=1}^{i-1} \frac{1}{2^k} = 1 - \frac{1}{2^{i-1}},
\]
which shows that \(\lim_{i \to \infty} \tau(s_i) = 1.\) But this implies that \(x = 1\) and that \(x\) is the
limit of \((\tau(s_i))_{i \in \mathbb{N}}.\)
It remains to show that the same holds if \(C\) is infinite. We prove this by contra-
diction, so assume now that \(\lim_{i \to \infty} \tau(s_i) < x.\) Choose \(n \in \mathbb{N}\) such that \(2\lambda(n) < \varepsilon,\)
where \(\varepsilon = x - \lim_{i \to \infty} \tau(s_i).\) Since the set \(C\) is infinite it follows that there exists
\(N > n\) such that \(s_N \in C.\) But now Lemma 4.2 iv) implies that we have
\[
\tau(\langle s_N(1), s_N(2), \ldots, s_N(N - 1) \rangle + \langle s_N(N) + 1 \rangle) = \tau(s_N) + 2\lambda(N) < \tau(s_N) + \varepsilon \leq x,
\]
which contradicts the maximality of \(s_N\) in \(\{s \in B_N; \tau(s) < x\}.\) So we have indeed
found a sequence \((\tau(s_i))_{i \in \mathbb{N}}\) in \(I'\) with \(\lim_{i \to \infty} \tau(s_i) = x.\)
□
s \prec s' \in B$, the intersection of the two intervals $[\tau(s), \tau(s + \langle 1 \rangle)] \cap [\tau(s'), \tau(s' + \langle 1 \rangle)]$ is either empty or the singleton $\{\tau(s')\}$. So if we have a 1-Lipschitz function
\[ \varphi_s : [\tau(s), \tau(s + \langle 1 \rangle)] \to \mathbb{H} \]
for each $s \in B$ with $\varphi_s(\tau(s)) = \varphi_s(\tau(s + \langle 1 \rangle)) = x_0$, there is a unique 1-Lipschitz function $\varphi : [0, 1] \to X$ with
\[ \varphi|[\tau(s), \tau(s + \langle 1 \rangle)] = \varphi_s. \]
We use this construction to define $\sigma_1 : [0, 1] \to \mathbb{H}$.

The following construction is useful for the computation of certain finite sums in $H^1_\mathcal{L}(X)$. Given two Lipschitz maps $\sigma : [0, a] \to X$ and $\sigma' : [0, b] \to X$, $a, b \in \mathbb{R}$ with $\sigma(a) = \sigma'(0)$, their concatenation $\sigma \cdot \sigma' : [0, a + b] \to X$ is given by
\[ \sigma \cdot \sigma'(t) = \begin{cases} \sigma(t) & \text{if } t \leq a \\ \sigma'(t - a) & \text{if } a \leq t \leq b \end{cases}. \]
If $\sigma$ and $\sigma'$ represent elements of $H^1_\mathcal{L}(X)$, so does $\sigma \cdot \sigma'$ and moreover the equation
\[ [\sigma \cdot \sigma'] = [\sigma] + [\sigma'] \]
holds. This follows as in the continuous case.

4.3. Construction of the $\sigma_n$. We write $\varphi_n$ for the Lipschitz function $[0, 1] \to \mathbb{H}$ which traverses the $n$-th loop $L_n$ of $\mathbb{H}$ with constant speed. Explicitly, for $t \in [0, 1]$ the equation
\[ \varphi_n(t) = \left( \frac{-\cos(2\pi t)}{n} + \frac{1}{n}, \frac{\sin(2\pi t)}{n} \right) \]
holds. The Lipschitz constant of $\varphi_n$ is bounded by $\mu_n = 2\pi/n$. Choose a sequence $(n_k)_{k \in \mathbb{N}}$ such that
\[ 4\mu_{n_k} \leq \lambda(k) \]
and $n_{k+1} > n_k + 1$. We write
\[ e_k = \varphi_{n_k} \cdot \varphi_{n_k + 1} \cdot \varphi_{n_k}^{-1} \cdot \varphi_{n_k + 1}^{-1} \circ \psi : [0, \lambda(k)] \to \mathbb{H} \]
for the composition of (a representative of) the commutator of $\varphi_{n_k}$ and $\varphi_{n_k + 1}$ with the reparametrization $\psi : [0, \lambda(k)] \to [0, 4]$ which sends $t$ to $4t/\lambda(k)$. By choice of the sequence $(n_k)_{k \in \mathbb{N}}$ we find that $e_k$ is a 1-Lipschitz function. For $s \in B$ and $t \in [\tau(s), \tau(s + \langle 1 \rangle)]$ let $(\sigma_1)_s(t) = e_{\ell(s)}(t - \tau(s))$ and note that $(\sigma_1)_s(\tau(s)) = (\sigma_1)_s(\tau(s + \langle 1 \rangle)) = (0, 0)$. By the comment succeeding Lemma 4.3 there is a unique 1-Lipschitz function $\sigma_1 : [0, 1] \to \mathbb{H}$ such that
\[ \sigma_1|[\tau(s), \tau(s + \langle 1 \rangle)](t) = e_{\ell(s)}(t - \tau(s)). \]
holds. For $n > 1$ Let $\sigma_n : [0, 2\lambda(n)] \to \mathbb{H}$ be the function which sends $t \in [0, 2\lambda(n)]$ to
\[ \sigma_n(t) = \sigma_1|[\tau(n(\langle 1 \rangle), \tau((n - 1)(\langle 1 \rangle) + (\langle 2 \rangle))(t + \tau(n \cdot (\langle 1 \rangle))). \]

Proposition 4.4. For $n > 1$ the equation
\[ \sigma_{n-1} = e_{n-1} \cdot \sigma_n \cdot \cdots \cdot \sigma_n \]
holds, and consequently $[\sigma_{n-1}] = n \cdot [\sigma_n]$ in $H^1_\mathcal{L}(\mathbb{H})$. 


Proof. Let \(a = \tau((n-1) \cdot \langle 1 \rangle), t \in [0, 2\lambda(n-1)]\) and let \(x = t + a\). If \(t \leq \lambda(n-1)\) or equivalently \(x \in \tau((n-1) \cdot \langle 1 \rangle), \tau(n \cdot \langle 1 \rangle))\) we find that
\[
\sigma_{n-1}(t) = \sigma_1(x) = \sigma_1[\tau((n-1) \cdot \langle 1 \rangle), \tau(n \cdot \langle 1 \rangle)](x) = c_{n-1}(x-a) = c_{n-1}(t) = c_{n-1} \cdot \sigma_n \cdot \ldots \cdot \sigma_n(t)
\]
holds. We can therefore restrict our attention to those points \(t \in [\lambda(n-1), 2\lambda(n-1)]\) with the property that \(t + a\) lies in \(I'\), so we can assume that there is a sequence \(s_0 \in B\) such that \(x = t + a\) lies in the interior of \(\tau(s_0)\). But \(t \geq \lambda(n-1)\) implies that \(x \geq \tau(n \cdot \langle 1 \rangle)\), and it follows that \(n \cdot \langle 1 \rangle \leq s_0\) since \(s_0\) is maximal with respect to \(\tau(s_0) \leq x\) (see Lemma 4.2 ii)). On the other hand, we have
\[
\tau(s_0) < x \leq \tau((n-1) \cdot \langle 1 \rangle) + 2\lambda(n-1) = \tau((n-2) \cdot \langle 1 \rangle + 2\lambda(n-1))
\]
and therefore \((n-1) \cdot \langle 1 \rangle \leq s_0 < (n-2) \cdot \langle 1 \rangle + 2\lambda(n-1)). By Lemma 4.2 iii) it follows that \(s_0 = (n-1) \cdot \langle 1 \rangle + s'\) for some sequence \(s' \in S\). But \(n \cdot \langle 1 \rangle \leq (n-1) \cdot \langle 1 \rangle + s'\) implies that \(s'\) is of the form \(\langle m \rangle + s\) for some \(m \in \mathbb{N}\) and a (possibly empty) sequence \(s \in S\). Hence \(s_0 = (n-1) \cdot \langle 1 \rangle + \langle m \rangle + s\) and we find that
\[
\sigma_{n-1}(t) = \sigma_1(x) = c_{n+\ell(s)} \left( x - \tau((n-1) \cdot \langle 1 \rangle + \langle m \rangle + s) \right)
\]
holds, where the crucial step is an application of Lemma 4.2 v) and the last equality follows from Lemma 4.2 ii) and iii). On the other hand, since we have
\[
\tau((n-1) \cdot \langle 1 \rangle + \langle m \rangle) \leq x \leq \tau((n-1) \cdot \langle 1 \rangle + \langle m + 1 \rangle)
\]
it follows that \(\lambda(n-1) + (m-1) \cdot 2\lambda(n) \leq t \leq \lambda(n-1) + m \cdot 2\lambda(n)\), again from Lemma 4.2 ii) and iii). So in this case we have
\[
c_{n-1} \cdot \sigma_n \cdot \ldots \cdot \sigma_n(t) = \sigma_n \cdot \ldots \cdot \sigma_n(t - \lambda(n-1))
\]
and we find that the claimed equality holds for a dense subset. The second statement follows since we have in general \([\sigma \cdot \sigma'] = [\sigma] + [\sigma']\). This implies in particular
that the commutator \([c_{n-1}]\) vanishes (since \(H^n_+ (\mathbb{H})\) is abelian) and therefore that \([\sigma_{n-1}] = n \cdot [\sigma_n]. \)

\[ \square \]

**Corollary 4.5.** The element \([\sigma_1]\) lies in the maximal divisible subgroup of \(H^n_+ (\mathbb{H})\).

**Proof.** The set

\[ D = \{m \cdot [\sigma_n]; m \in \mathbb{Z}, n \in \mathbb{N} \} \]

is a divisible subgroup of \(H^n_+ (\mathbb{H})\). To see this, let \(m, m' \in \mathbb{Z}, n, n' \in \mathbb{N} \) with \(n \leq n'\). Then \([\sigma_n] = \frac{n!}{m!} [\sigma_{n'}]\) by the above proposition. It follows that

\[ m \cdot [\sigma_n] - m' \cdot [\sigma_{n'}] = \left( m \cdot \frac{n!}{m!} - m' \right) [\sigma_{n'}], \]

so \(D\) is indeed a subgroup. But it is also divisible, for if \(l \in \mathbb{N}\) we have

\[ m \cdot [\sigma_n] = m \cdot \frac{(l \cdot n)!}{n!} [\sigma_{l \cdot n}] = l \cdot \left( m \cdot \frac{(l \cdot n - 1)!}{(n - 1)!} \right) [\sigma_{l \cdot n}] \]

and \(m \cdot \frac{(l \cdot n)!}{(n - 1)!} [\sigma_{l \cdot n}] \in D. \)

\[ \square \]

It remains to show that \([\sigma_1] \neq 0. We will show that the image \([\sigma_1]_S \in H_1 (\mathbb{H})\) of \([\sigma_1]\) under the comparison map \(H^n_+ (\mathbb{H}) \to H_1 (\mathbb{H})\) is non-zero. Using the isomorphism

\[ H_1 (\mathbb{H}) \cong \pi_1 (\mathbb{H}) / [\pi_1 (\mathbb{H}), \pi_1 (\mathbb{H})] \]

we find that this is equivalent to showing that the loop \(\sigma_1\) does not represent an element of the commutator subgroup of \(\pi_1 (\mathbb{H}).\)

Let \(p_k : \mathbb{H} \to L_{n_k} \cup L_{n_k+1}\) be the map which is the identity on \(L_{n_k} \cup L_{n_k+1}\) and which sends an element \(x \in \mathbb{H} \setminus L_{n_k} \cup L_{n_k+1}\) to \((0, 0).\) This is a (Lipschitz) continuous map and therefore induces a group homomorphism \(\pi_1 (p_k) : \pi_1 (\mathbb{H}) \to \pi_1 (L_{n_k} \cup L_{n_k+1}).\) Since \(L_{n_k} \cup L_{n_k+1}\) is homeomorphic to \(S^1 \cup S^1, \pi_1 (L_{n_k} \cup L_{n_k+1})\) is isomorphic to the free group on two generators \((a, b).\) Equation (4.4) implies that \(\pi_1 (p_k) ([\varphi_{n_k}]_{\pi_1}) = [p_k \circ \varphi_{n_k}]_{\pi_1} = a\) and \(\pi_1 (p_k) ([\varphi_{n_k+1}]_{\pi_1}) = [p_k \circ \varphi_{n_k+1}]_{\pi_1} = b\) under this isomorphism. We find in particular that

\[ (4.8) \]

\[ \pi_1 (p_k) ([c_k]) = aba^{-1}b^{-1}. \]

**Proposition 4.6.** The element \([\sigma_1]_{\pi_1} \in \pi_1 (\mathbb{H})\) does not lie in the commutator subgroup \([\pi_1 (\mathbb{H}), \pi_1 (\mathbb{H})].\)

**Proof.** From now on we identify \(\pi_1 (L_{n_k} \cup L_{n_k+1})\) with \((a, b).\) We will first show that \(\pi_1 (p_k)\) maps \([\sigma_1]_{\pi_1}\) to \((aba^{-1}b^{-1})^{k!}\). By Lemma 4.3 the map \(p_k \circ \sigma_1\) is uniquely determined by its restriction to the intervals \([\tau (s), \tau (s + 1)]\). By equation (4.5) it follows that

\[ p_k \circ [\sigma_1]_{[\tau (s), \tau (s + 1)]} (t) = p_k \circ c_{\ell (s)} (t - \tau (s)) = \begin{cases} c_k (t - \tau (s)) & \text{if } \ell (s) = k \\ (0, 0) & \text{else} \end{cases} . \]

Since there are precisely \(k!\) sequences \(s \in B\) with \(\ell (s) = k\) we find that \(p_k \circ \sigma_1\) is homotopic (rel. endpoints) to

\[ \underbrace{c_k \circ \cdots \circ c_k}_{k! \text{ times}} \]

and together with equation (4.8) that \(\pi_1 (p_k) ([\sigma_1]_{\pi_1}) = (aba^{-1}b^{-1})^{k!}\).

Assume now that \([\sigma_1]_{\pi_1}\) does lie in the commutator subgroup, i.e. that \([\sigma_1]_{\pi_1} = [x_1, y_1] \cdots [x_n, y_n]\) for some \(x_i, y_i \in \pi_1 (\mathbb{H})\). Since \(\pi_1 (p_k)\) is a group homomorphism, it follows that for all \(k \in \mathbb{N}\) the element \((aba^{-1}b^{-1})^{k!}\) of \((a, b)\) can be written as a product of \(n\) elementary commutators. On the other hand, in [Cul81, example 2.6] it was shown that \((aba^{-1}b^{-1})^{k!}\) cannot be written as a product of less than \(|k!/2| + 1\) elementary commutators. Since \(k \in \mathbb{N}\) was arbitrary this is clearly a contradiction. \[ \square \]
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