The string topology BV algebra, Hochschild cohomology and the Goldman bracket on surfaces

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Abstract

In 1999 Chas and Sullivan [2] discovered that the homology $\mathbb{H}_*(X)$ of the space of free loops on a closed oriented smooth manifold $X$ has a rich algebraic structure called string topology. They proved that $\mathbb{H}_*(X)$ is naturally a Batalin-Vilkovisky (BV) algebra. There are several conjectures connecting the string topology BV algebra with algebraic structures on the Hochschild cohomology of algebras related to the manifold $X$, but none of them has been verified for manifolds of dimension $n > 1$.

In this work we study string topology in the case when $X$ is aspherical (i.e. its homotopy groups $\pi_i(X)$ vanish for $i > 1$). In this case the Hochschild cohomology Gerstenhaber algebra $HH^*(A)$ of the group algebra $A$ of the fundamental group of $X$ has a BV structure. Our main result is a theorem establishing a natural isomorphism between the Hochschild cohomology BV algebra $HH^*(A)$ and the string topology BV algebra $\mathbb{H}_*(X)$. In particular, for a closed oriented surface $X$ of hyperbolic type we obtain a complete description of the BV algebra operations on $\mathbb{H}_*(X)$ and $HH^*(A)$ in terms of the Goldman bracket [11] of loops on $X$. The only manifolds for which the BV algebra structure on $\mathbb{H}_*(X)$ was known before were spheres [15] and complex Stiefel manifolds [16].

Our proof is based on a combination of topological and algebraic constructions allowing us to compute and compare multiplications and BV operators on both $\mathbb{H}_*(X)$ and $HH^*(A)$. 
1 Introduction

For any topological space $X$ the set $G = \pi_1(X, x_0)$ of homotopy classes of based loops (continuous maps from the circle $S^1$ to $X$ taking a fixed point $s_0$ in $S^1$ to the base point $x_0$ in $X$) forms a group. Forgetting the base points, we obtain the set $LX$ of free loops, i.e. of all continuous maps $\phi : S^1 \to X$. Homotopy classes of free loops are indexed by conjugacy classes of the group $G$. There is a priori no multiplication or other algebraic structure on this set. However, surprisingly, if $X$ is an oriented two-dimensional manifold, the vector space $L$ spanned by homotopy classes of free loops on $X$ has a natural Lie algebra structure. For two transversal loops $\phi_1$ and $\phi_2$ their Goldman bracket is defined by

$$[\phi_1, \phi_2] = \sum_P s(P) \phi_1 \ast_P \phi_2,$$

where the sum is taken over all intersections $P$ of $\phi_1$ and $\phi_2$, the sign $s(P) \in \{\pm 1\}$ is determined by the orientations ($s(P) = 1$ if the area form of the surface evaluated on the two tangent vectors is positive and $s(P) = -1$ if it is negative) and the composition $\phi_1 \ast_P \phi_2$ is the product of $\phi_1$ and $\phi_2$ as elements of $\pi_1(X, P)$.

\begin{center}
\includegraphics[width=0.5\textwidth]{goldman_bracket.png}
\end{center}

Figure 1: The Goldman bracket of two loops on the torus

This Lie algebra structure was found by William Goldman in [11] in his study of the symplectic structure on the moduli space $\text{Hom}(G, K)/K$ of representations of $G$ into a compact Lie group $K$.

The Goldman Lie algebra was the starting point of the foundational work by Chas and Sullivan [2] on string topology. Whereas Goldman considered
only homotopy classes—or connected components of the space of free loops $LX$ of a manifold $X$, Chas and Sullivan studied algebraic structures on the the total homology of $LX$. They called this space $\mathbb{H}_*(X) = H_*(LX)$ the “loop homology” of $X$ and discovered that it has a natural multiplication, a Lie-type bracket, and also a Batalin-Vilkovisky (BV) structure (see Section 2.1).

String topology is a rapidly developing area of mathematical research that connects classical algebraic topology with more recent developments in mathematics influenced by theoretical physics and, in particular, string theory and mirror symmetry (see e.g. [5]).

The operations on the loop homology algebra of a manifold are very difficult to compute. The complete BV algebra structure on $\mathbb{H}_*(X)$ has so far only been computed explicitly for spheres [15] and complex Stiefel manifolds [16] with coefficients in an arbitrary ground field and for real projective spaces [21] with coefficients in $\mathbb{F}_2$.

There are, however, several conjectures relating the loop homology BV algebra with a more computable algebraic object, the Hochschild cohomology of algebras related to $X$.

Hochschild cohomology $HH^*(A)$ of algebras was introduced by Hochschild in 1945 as a tool for studying homological properties of algebras. Recently it found important applications in other areas of mathematics and also in theoretical physics. In 1963 Gerstenhaber [8] found that, in addition to the natural cup product, $HH^*(A)$ has a Lie-type bracket. This bracket and the cup product satisfy a compatibility relation and make $HH^*(A)$ into a so-called Gerstenhaber algebra (see Section 2.1). Recently T. Tradler [17] showed that under certain assumptions the Gerstenhaber structure on $HH^*(A)$ extends to a structure of a Batalin-Vilkovisky algebra. In particular, this is the case when $A$ is the algebra of singular cochains of a simply connected closed oriented manifold (see [15]). In [4] Cohen and Jones established that for a simply connected manifold $X$ there is an isomorphism

$$F: \mathbb{H}_*(X) \to HH^*(C^*(X))$$

between the loop homology of $X$ and the Hochschild cohomology space of the algebra $C^*(X)$ of singular cochains of $X$ and proved that $F$ takes the loop product in $\mathbb{H}_*(X)$ to the cup product on Hochschild cohomology. Since both $\mathbb{H}_*(X)$ and $HH^*(C^*(X))$ possess BV algebra structure, it is conjectured that the Cohen-Jones identification $F$ is an isomorphism of BV algebras.
However, to the best of our knowledge, the question whether $F$ respects the Gerstenhaber bracket or the BV operators remains open.

In this paper we algebraically compute the string topology BV algebra for a large class of manifolds. Namely, we show that for an aspherical smooth closed oriented manifold $X$ of dimension $n$, its loop homology BV algebra $\mathbb{H}_*(X)$ is isomorphic to the Hochschild cohomology BV algebra $HH^*(A)$, where $A$ is the group algebra $A = k[\pi_1(X)]$ of the fundamental group of $X$ with the BV operator given by the dual to the Connes operator $\kappa$ on the Hochschild homology $HH_*^*(A)$.

We first construct a vector space isomorphism between $\mathbb{H}_*(X)$ and $HH_*(A)$ which sends the Chas-Sullivan BV operator $\Delta$ to the Connes operator $\kappa$. Then we construct a Poincaré duality isomorphism $\tau: HH_*(A) \to HH^{n-*}(A)$ and show that $\tau$ takes $\kappa$ to a BV operator for the Gerstenhaber algebra on $HH^{n-*}(A)$. We prove that the resulting vector space isomorphism $\xi: \mathbb{H}_*(X) \to HH^{n-*}(A)$ is an isomorphism of associative algebras and therefore gives an isomorphism of BV algebras.

When $X$ is a closed oriented surface of genus $g \geq 2$ we obtain a complete description of the BV algebra operations on $\mathbb{H}_*(X)$ and $HH^*(A)$ in terms of the Goldman bracket $[\ ]$ of loops on $X$.

We hope that this result and our methods will provide an insight for proving the Cohen-Jones conjecture that $\mathbb{H}_*(X) \cong HH^*(C^*(X))$ for aspherical and other manifolds.

String topology on aspherical manifolds was also the subject of the recent work [1] by Abbaspour, Cohen and Gruher. They described the loop homology product in terms of a new operation on the direct sum of group homologies of modules corresponding to cosets of the fundamental group $G$ (not explicitly in terms of Hochschild cohomology). However, they did not consider the Gerstenhaber or the BV algebra structures.

This work originated as a project to compute the Hochschild cohomology $HH^*(A)$ of the group algebra $A$ of the fundamental group of a closed oriented hyperbolic surface $X$ and express the Gerstenhaber structure on $HH^*(A)$ in terms of the Goldman bracket $[\ ]$.

This problem was motivated by the following result by Crawley-Boevey, Etingof and Ginzburg [6] about quiver algebras.

Let $P$ be the preprojective algebra of a hyperbolic (i.e. non-Dynkin and non-affine) quiver $Q$. The space $L = P/[P,P]$ has a natural Lie algebra structure given by the so-called necklace bracket. Let $V_0$ be the vector space
with basis given by the vertices of $Q$. In [6] it shown that
\[ HH^0(P) = V_0, \quad HH^1(P) = (L/V_0) \oplus k, \quad HH^2(P) = L, \]
and Gerstenhaber algebra operations on $HH^*(A)$ can be expressed in terms of the necklace bracket.

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2 Background and definitions

2.1 Gerstenhaber and BV algebras

Here we recall the definitions of Gerstenhaber and BV algebras (see [9]).

Definition 2.1. A Gerstenhaber algebra is a commutative graded algebra $B^* = \bigoplus_{i \geq 0} B^i$ equipped with a linear map
\[ [\cdot, \cdot] : B^* \otimes B^* \to B^{*-1} \]
of degree $-1$ such that it defines a super Lie algebra structure on the shifted space $B^{*-1}$ and, for $a \in B^i$, the operator $[a, \cdot]$ is a degree $i - 1$ derivation of the product on $B^*$. 

Definition 2.2. A Batalin-Vilkovisky (BV) algebra is a commutative graded algebra $B^*$ with an operator $\Delta : B^* \to B^{*+1}$ (called the BV operator) such that $\Delta \circ \Delta = 0$ and the operation
\[ [a, b] = \Delta(ab) - \Delta(a)b - (-1)^i a\Delta(b) \]
(where $a \in B^i$) defines a Gerstenhaber bracket on $B^*$. 

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2.2 Chas-Sullivan String Topology

Let $X$ be a closed oriented manifold of dimension $n$ and let $LX$ be its space of free loops (maps $\gamma : S^1 \to X$).

**Definition 2.3.** The loop homology $\mathbb{H}_*(X)$ of $X$ is the homology of its loop space, $\mathbb{H}_*(X) = H_*(LX)$. Chas and Sullivan [2] proved that $\mathbb{H}_*(X)$ forms a BV algebra (see below). We will call it the string topology BV algebra.

The BV algebra operations on $\mathbb{H}_*(X)$ are as defined as follows.

The loop product

$$\mathbb{H}_i(X) \otimes \mathbb{H}_j(X) \to \mathbb{H}_{i+j-n}(X) \quad (2)$$

is given by the combination of the intersection product on $X$,

$$\cap : H_i(X) \otimes H_j(X) \to H_{i+j-n}(X), \quad (3)$$

with the composition of loops $LX \times_X LX \to LX$. Here the projection $LX \to X$ is given by the map $ev : \gamma \mapsto \gamma(s_0)$ where $s_0 \in S^1$ is the base point of the circle.

The BV operator

$$\Delta : \mathbb{H}_* \to \mathbb{H}_{*-1} \quad (4)$$

is determined by the natural $S^1$ action $S^1 \times LS \to LS$ induced by the action of $S^1$ on itself. Note that the operator $\Delta$ on $\mathbb{H}_*(LX)$ is defined even if $X$ is not a manifold.

2.3 Hochschild homology and cohomology

We recall here the definitions of Hochschild homology and cohomology (see e.g. [20].)

**Definition 2.4.** Let $M$ be a bimodule over an algebra $A$. The Hochschild homology $HH_*$ and cohomology $HH^*$ of $A$ with coefficients in $M$ are given by

$$HH_i(A, M) = \text{Tor}_i(A, M) \quad \text{and} \quad HH^i(A, M) = \text{Ext}^i(A, M), \quad (5)$$

where the functors Tor, and Ext are taken in the category of $A$-bimodules.
When $M = A$, the graded spaces $HH_*(A) = HH_*(A,A)$ and $HH^*(A) = HH^*(A,A)$ have several natural algebraic structures. Hochschild homology has a natural operator
\[
\kappa : HH_i(A) \to HH_{i+1}(A)
\] (6)
called the Connes operator. It comes from the cyclic structure on the Hochschild complex of $A$ (see [13]).

Hochschild cohomology has two product structures, a cup product
\[
\cup : HH^i(A) \otimes HH^j(A) \to HH^{i+j}(A)
\] (7)
(which is associative and graded commutative), and a Gerstenhaber Lie-type bracket [8, 13]
\[
[,] : HH^i(A) \otimes HH^j(A) \to HH^{i+j-1}(A).
\] (8)
Together these two structures make $HH^*$ into a Gerstenhaber algebra.

Van den Bergh [19] showed that for certain algebras $A$, there exists a non-negative $n$ and a Poincaré duality isomorphism between Hochschild homology and cohomology
\[
\tau : HH_i \to HH^{n-i}.
\] (9)
In a recent preprint [10], Ginzburg proved further that this isomorphism takes the Connes operator $\kappa$ on $HH_*$ to a BV operator $\lambda = \tau \circ \kappa \circ \tau^{-1}$ on $HH^*$ compatible with the Gerstenhaber algebra structure.

3 Statement of results

3.1 String topology BV algebra of aspherical manifolds

The following theorem is the main result of this paper.

**Theorem 3.1.** Let $X$ be a closed connected oriented aspherical manifold of dimension $n$ and let $A$ be the group algebra of the fundamental group $G = \pi_1(X)$. There exists an isomorphism of BV algebras
\[
\xi : \mathbb{H}_*(X) \to HH^{n-*}(A)
\]
where $\mathbb{H}_*(X) = H_*(LX)$ is the Chas-Sullivan loop homology BV algebra and $HH^*(A)$ is the Hochschild cohomology algebra equipped with the BV operator $\lambda$.

The proof of this theorem is given in the next section.
3.2 Hochshild cohomology and Goldman bracket on surfaces

Theorem 3.1 allows us to compute the Hochshild cohomology BV algebra $HH^*(A)$ of the group algebra of the fundamental group and the string topology BV algebra $\mathbb{H}^*(X)$ for a hyperbolic surface $X$ in terms of the Goldman Lie bracket (1) on $X$.

Theorem 3.2. Let $X$ be a compact oriented surface of genus $g > 1$. Let $A = k[G]$ be the group algebra of its fundamental group $G$ and let $L = H_0(LX)$ be the space generated by homotopy classes of free loops equipped with the Goldman Lie bracket.

(i) The Hochshild cohomology graded space of $A$ is naturally isomorphic to the loop homology space, namely

$$HH^i(A) \cong H_{2-i}(LX)$$

for $0 \leq i \leq 2$, and $HH^i(A) = 0$ for $i > 2$. (10)

(ii) Under the above identification, the cup product on the algebra $HH^*(A)$ coincides with the Chas-Sullivan loop homology product (of degree $-2$) on $\mathbb{H}^*(X)$.

(iii) Under the identification (10) the Gerstenhaber bracket on $HH^*(A)$ becomes the Gerstenhaber bracket on the string topology $\mathbb{H}^*(X)$.

(iv) The Gerstenhaber algebra $HH^*(A)$ has a Batalin-Vilkovisky structure given by the operator $\lambda$ of degree $-1$ that corresponds to the Chas-Sullivan string topology BV operator $\Delta$ on $\mathbb{H}^*(X)$ (of degree $+1$).

(v) The non-trivial Hochshild cohomology groups of $A$ are given by

$$HH^0(A) = k, \quad HH^1(A) = H_1(X, k) \oplus L/k\gamma_0, \quad HH^2(A) = L,$$

where $\gamma_0 \in L$ is the class of the trivial loop.

(vi) Non-trivial cup products on $HH^*(A)$ exist only on $HH^1(A)$ and can be expressed in terms of the Goldman bracket on $X$ as follows

$$(\alpha, \gamma) \cdot (\alpha', \gamma') = \langle \alpha, \alpha' \rangle \gamma_0 + \langle \alpha', \gamma \rangle \gamma + \langle \alpha, \gamma' \rangle \gamma' + [\gamma, \gamma'],$$

where $\langle \cdot, \cdot \rangle$ is the intersection pairing on $H_1(X, k)$ and $[\cdot, \cdot]$ is the Goldman bracket on $L$.

(vii) The BV operator $\lambda$ is equal to $0$ on $HH^1(A)$ and is induced by the projection $L \to L/k\gamma_0$ on $HH^2(A) = L$. 
Proof. Parts (i)-(iv) follow from Theorem 3.1.

To show (v) and (vii), we see that $HH^*(A)$ as a space decomposes into the sum over conjugacy classes $C$ of group cohomology

$$HH^*(A) = \bigoplus_C H^*(G, kC)$$

where $kC$ is the vector space spanned by the elements of $C$ on which $G$ acts by conjugation. We can compute $H^*(G, kC)$ using the fact that for any $g \in G, g \neq 1$, its centralizer $Z(g) \cong \mathbb{Z}$. Since conjugacy classes of $G$ correspond to homotopy classes of free loops, this gives the desired result. Finally, to prove (vi) we use the formula $[\gamma, \gamma'] = \Delta(\gamma) \cdot \Delta(\gamma')$ which expresses the Goldman bracket of $\gamma, \gamma' \in L \cong \mathbb{H}_0(X)$ in terms of string topology operations. \qed

4 Proof of Theorem 3.1

4.1 Notation and conventions

We will be using the following notation and conventions.

- $k$ is a field of characteristic zero.
- $X$ is a closed oriented smooth aspherical manifold of dimension $n$, a base point $x_0$ and the fundamental group $G = \pi_1(X, x_0)$.
- $A = k[G]$ is the group algebra of $G$.
- The notation $_AY$ means that $Y$ is a left $A$-module. By $Y_A$ we denote a right $A$-module and by $_AY_A$ an $A-A$ bimodule structure on $Y$. Note that since $A = k[G]$ we have $A \cong A^{op}$ and therefore each left $A$-module is canonically a right $A$-module. In particular, this implies that $A \otimes A$-modules can be viewed as $A-A$ bimodules.
- By $Y_*$ or $Y^*$ we denote a graded vector space or a chain complex, and $Y_{n-*}, Y_{*-1}$ etc. denote the same thing with shifted grading.
- By a map $f : X_* \to Y_*$ or $f : X_* \to Y_{n-*}$ etc., we always mean a homomorphism of complexes or graded spaces.
• Tensor product $\otimes$ is taken over $k$ by default, $\otimes_A$ means the tensor product over $A$, and $\otimes_{A \otimes A}$ denotes the tensor product in the category of $A - A$ bimodules.

• By $C_*(X)$ we denote the singular chains of a topological space $X$. For a cell decomposition $T$ of $X$, we denote by $C_*(T)$ the corresponding chain complex.

4.2 Construction of resolutions

In this section we construct several resolutions of the $A$-module $k$ and of the bimodule $AAA$.

Let $\tilde{X}$ be the universal covering space of $X$. Since $X$ is aspherical, $\tilde{X}$ is contractible. We define $\tilde{R}_* = C_*(\tilde{X}_i)$, the $i$-dimensional singular chains on $\tilde{X}$.

**Proposition 4.1.** $\tilde{R}_*$ is a projective $A$-module resolution of $AAAk$.

**Proof.** Since $G$ acts freely on $\tilde{X}$, $G$ also acts freely on $C_*(\tilde{X})$ and so $\tilde{R}_*$ is a complex of projective (indeed of free) $A$-modules. Since $H_*(\tilde{X})$ is equal to $k$ concentrated in degree zero, $\tilde{R}$ is a projective resolution of $AAAk$.

We will also use $A$-module resolutions of $k$ using cellular chains. Let $T_* = T_0 \cup T_1 \cup \ldots \cup T_n$ be a cellular decomposition of $X$ (where $T_1$ are the 1-simplices, etc.) By the homotopy lifting property of covering spaces, $T$ can be lifted to a cell decomposition $\tilde{T}$ of $\tilde{X}$.

**Proposition 4.2.** $C_*(\tilde{T})$ is a projective $A$-module resolution of $k$.

**Proof.** This is proven analogously to 4.1.

We will work with two particular cell decompositions.

**Definition 4.1.** Let $T$ be a triangulation of $X$ (one exists because $X$ is a smooth manifold), and let $T'$ be the dual cellular decomposition.

We lift these decompositions to decompositions $\tilde{T}$ and $\tilde{T}'$ respectively of $\tilde{X}$. We denote the resolutions $R_* = C_*(\tilde{X})$ and $R'_* = C_*(\tilde{X}')$.

We perform the same constructions on the fiber product $Q = \tilde{X} \times_X \tilde{X}$ of the universal covering of $X$ with itself.
Definition 4.2. Let $T^b$ be the simplicial complex given by the barycentric subdivision of $T$. We lift this decomposition using the covering homotopy property to get a cellular decomposition $T^b_Q$ of $Q$. We define

$$W_\ast = C_\ast(T^b_Q) \quad \text{and} \quad \tilde{W}_\ast = C_\ast(Q).$$

The $G$-action on $\tilde{S}$ defines a $G \times G$ action on $Q$. This makes $Q$ an $A$-bimodule.

Lemma 4.3. The complexes $W_\ast$ and $\tilde{W}_\ast$ are projective $A$-bimodule resolutions of $AA$.

Proof. The modules $W_\ast$ and $\tilde{W}_\ast$ have free $G \times G$ action and are therefore projective. All connected components of $Q$ are contractible, so $H_\ast(Q)$ is concentrated in degree 0. It is a well-known result that the connected components of $Q$ are indexed by elements of $G$ and that $H_0(Q) = A$ with canonical $A$-biaction. This proves the proposition. \qed

Remark 1. We will later consider every point $\tilde{x} \in \tilde{X}$ as a point $x \in X$ and homotopy class of paths $\epsilon$ from the base point $x_0$ to $x$. Analogously, every point $q \in Q$ can be considered as a pair of homotopy classes $\epsilon_1$ and $\epsilon_2$ from $x_0$ to $x$. The connected component of $Q$ containing $q$ is then indexed by the path product $\epsilon_2^{-1}\epsilon_1$.

4.3 Construction of the isomorphism $\rho$

We will define a vector space isomorphism $\rho : HH_\ast \to HH_\ast$ which takes the BV operator $\Delta$ on string homology to the Connes operator $\kappa$ on $HH_\ast$.

Let $A^N$ be the $A = k[G]$-module whose underlying space is $A$ and with $G$-action defined by conjugation, $g.a = gag^{-1}$ for $g \in G$ and $a \in A$.

We will use the following standard fact about Hochschild (co)homology of group algebras.

Proposition 4.4. For a group algebra $A = k[G]$, its Hochschild homology and cohomology is isomorphic to the group homology and cohomology of $G$ with coefficients in $N$:

$$HH_\ast(A) = H_\ast(G, N) \quad \text{and} \quad HH^\ast(A) = H^\ast(G, N).$$
A proof can be found in e.g. [13] or [20].

Let $G$ be any (discrete) group and let $X = K(G, 1) = BG$ (i.e. $X$ is a connected topological space with $\pi_1(X) = G$, $\pi_i(X) = 0$ for $i \geq 2$).

Even if $X$ is not a manifold, the operator $\Delta : H_* (LX) \to H_{*+1}(LX)$ is well-defined.

We will use the following known result.

**Theorem 4.5** ([13] Corollary 7.3.13). There is an isomorphism of vector spaces,

$$\rho : H_*(LX) \to HH_*(k[G])$$

which takes the operator $\Delta$ to the Connes operator $\kappa$ on $HH_*$.

Loday constructs this isomorphism in terms of the geometric realization of the cyclic bar construction and its covering of the geometric realization of the regular bar construction.

In terms of the resolution $\tilde{R}_*$, the map $\rho$ can be computed as follows.

Let $\sigma \in C_i(LX)$ be an $i$-simplex, $\sigma : \Delta^i \to LX$. Composition with the map $ev_{s_0} : LX \to X$ gives us a simplex $\sigma_0 : \Delta^i \to X$. By the homotopy covering theorem, we can (not canonically) choose a lifting of the map $\sigma_0$ to a map $\tilde{\sigma}_0 : \Delta^i \to \tilde{X}$. Let $p \in \Delta^i$ be a point in the simplex. The point gives a loop $\gamma = \sigma(p)$ and a point $\tilde{m} = \tilde{\sigma}(p) \in \tilde{X}$. The point $\tilde{m}$ corresponds to a homotopy class $\epsilon$ of paths from the base point $m_0$ to the point $m = \sigma_0(p)$. The loop $\gamma$ represents a homotopy class $g_m$ of loops with base point $m$. Then the conjugate $\epsilon^{-1}g_m\epsilon$ is a homotopy class of loops based at $m_0$ and gives an element of the fundamental group $g \in G$. Note that the element $g$ is independent of choice of $p \in \Delta$.

**Lemma 4.6.** The element $g \otimes \tilde{\sigma} \in N \otimes G \tilde{R}$ is independent of the choice of the lifting $\tilde{\sigma}$ and therefore well-defined.

**Proof.** Let $\tilde{\sigma}'$ be a different lifting, and let $g'$ be the element of $G$ which we get by the above construction. There is an element $h \in G$ such that $\tilde{\sigma}' = h\tilde{\sigma}$ and this makes $g = h^{-1}gh$ (the action of $h^{-1}$ on $g \in N$). The actions of $h$ and $h^{-1}$ get canceled after taking tensor product over $A$, so $g \otimes_G \tilde{\sigma} = g' \otimes_G \tilde{\sigma}'$. \hfill $\square$

We define $\rho_0 : C_*(LX) \to N \otimes_G \tilde{R}$ by $\rho_0(\sigma) = g \otimes \tilde{\sigma}$.

**Lemma 4.7.** The map $\rho : H_* \to HH_*$ induced by $\rho_0$ on homology coincides with the isomorphism given in [13, Corollary 7.3.13].
Proof. The geometric realization $|B.G|$ of the bar construction of $G$ is a $K(G,1)$ space and therefore homotopic to $X$. Any homotopy equivalence $\psi : |B.G| \to X$ gives a quasiisomorphism of complexes which identifies our construction with Loday’s on the level of homology.

4.4 Intersection product

Now we will construct a non-commutative analogue of the intersection product

$$\mu : R_i \otimes R'_j \to W_{i+j-n}.$$  \hspace{1cm} (12)

Let $\sigma, \sigma'$ be simplices in $T_i, T'_j$ respectively and let $\sigma_0$ and $\sigma'_0$ be their images in $T$ and $T'$. If $\sigma_0$ and $\sigma'_0$ do not intersect, we set $\mu(\sigma \otimes \sigma') = 0$. Otherwise, from the definition of the dual complex, we know that $\sigma_0$ and $\sigma'_0$ intersect in exactly one cell of the barycentric subdivision, $\sigma^b_0 \in T^b_{ij-n}$. For every point $p$ of $\sigma^b_0$, the cell $\sigma$ gives one point in its $\tilde{X}$ fiber and $\sigma'$ gives another. This naturally gives us a lifting of $\sigma^b_0$ to a cell $\sigma^b$ of $\tilde{X} \times_X \tilde{X}$. We have $\sigma^b \in W^b$ and we define $\mu(\sigma \otimes \sigma') = \sigma^b$. We extend $\mu$ to all of $R \otimes R'$ by linearity.

Lemma 4.8. The map $\mu$ respects the $A$-biaction, i.e. for any $a, b \in A$

$$\mu(a\sigma_i \otimes b\sigma_j) = (a \otimes b)(\mu(\sigma \otimes \sigma')).$$

Proof. This can be verified directly from the definition of $\mu$. \hspace{1cm} \Box

Using the map $\mu$, we will define a product

$$\beta_0 : C \otimes C' \to A_A \otimes_A W_{i+j-n}$$

such that the induced map $\beta : HH_i \otimes HH_j \to HH_{i+j-n}$ makes the following diagram commute:

$$
\begin{array}{c}
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\begin{array}{c}
H_i \otimes H_j \xrightarrow{\rho \otimes \rho} HH_i \otimes HH_j \xrightarrow{\beta} H_{i+j-n} \end{array} \\
\end{array} \\
\end{array}
\end{array}
$$

\hspace{1cm} (13)
Lemma 4.9. Let $g \otimes G \sigma \in C_i$ and $g' \otimes G \sigma' \in C'_j$. The equation

$$\beta_0(g \otimes G \sigma \otimes g' \otimes G \sigma') = g(\mu[\sigma \otimes \sigma']^{-1}g' \otimes A)(\mu(\sigma \otimes \sigma'))$$ (14)

gives a well-defined map $\beta_0 : C \otimes C' \rightarrow A \otimes_A W_{i+j-n}$ where $[\mu(\sigma \otimes \sigma')] \in G$ is the element of $G$ corresponding to the connected component of $Q = \tilde{S} \times S \tilde{S}$ which contains the cell $\mu(\sigma \otimes \sigma')$.

Proof. This can be verified by a direct computation. \qed

Lemma 4.10. The map $\rho$ takes the product $\cdot$ on string topology to the operation $\beta$ on $HH_\ast$.

Proof. Let $\gamma \in H_i, \gamma' \in H_j$. Let $c$ be a representative of $\rho(\gamma)$ in the complex $C_i$ and let $c'$ be a representative of $\rho(\gamma')$ in $C'_j$.

It is possible to choose a representative $\gamma_0 \in C_i(LX)$ of $\gamma$ such that $\rho_0(\gamma_0) = c_0$ and similarly a representative $\gamma'_0 \in C'_j(LX)$ of $\gamma'$ such that $\rho_0(\gamma'_0) = c'_0$.

Since all intersections between cells of $T$ and cells of $T'$ are transversal, we can explicitly construct a representative of $\gamma \cdot \gamma'$ by multiplying together all pairs of loops which map $s_0 \in S^1$ to the same point in $X$. We apply this operation to $\gamma_0$ and $\gamma'_0$ to get a new chain of loops, $\gamma''_0$. From our construction of $\beta_0$, we see that $\rho_{X_0}(\gamma''_0) = \beta_0(\gamma \otimes \gamma')$. It follows that the two products, $\cdot$ and $\beta$, coincide on homology. \qed

4.5 Hochschild Poincaré Duality

The pairing

$$\pi_0 = \alpha \circ \mu : R_i \otimes R'_{n-i} \rightarrow A$$

has two adjoint maps

$$i : R_i \rightarrow \text{Hom}_A(R'_{n-i}, A) \quad \text{and} \quad i' : R'_{n-i} \rightarrow \text{Hom}_A(R_i, A).$$

Lemma 4.11. The maps $i : R_i \rightarrow \text{Hom}_A(R'_{n-i}, A)$ and $i' : R'_{n-i} \rightarrow \text{Hom}_A(R_i, A)$ are isomorphism of $A$-modules.

Proof. $R$ is a free $A$-module. Therefore $\text{Hom}(R, A)$ has a basis (as a vector space) of maps $f_\sigma$ for $\sigma \in T_i$ with $f_\sigma(\sigma) = 1$ and $f_\sigma(\sigma_1) = 0$ for a cell $\sigma_1 \neq g\sigma$ for some $g \in G$. We see that for a cell $\sigma' \in T'_{n-i}$ the map $i' : R'_i \rightarrow \text{Hom}_A(R_{n-i}, A)$ takes $\sigma'$ to $f_\sigma$ where $\sigma$ is the unique cell which intersects $\sigma$ in
the space $\tilde{X}$. This means that $\iota'$ bijects a (vector space) basis of $R_{n-i}'$ with a basis of $\text{Hom}_A(R_i, A)$ and is therefore an isomorphism. A similar argument shows that $\iota$ is an isomorphism as well.

**Lemma 4.12.** There exists an isomorphism of complexes

$$\tau_0 : N \otimes_A R_* \to \text{Hom}_A(R_{n-*}', N).$$

*Proof.* We define $\tau_0$ as a composition,

$$\tau_0 = m \circ c \circ (id \otimes \iota) : N \otimes_A R_* \to \text{Hom}_A(R_{n-*}', N),$$

where $c$ is the canonical map

$$c : N \otimes_A \text{Hom}(R_{n-*}', N) \to \text{Hom}_A(R_{n-*}', N \otimes A)$$

and $c : \text{Hom}_A(R_{n-*}', N \otimes A) \to \text{Hom}_A(R_{n-*}', N)$ is the map obtained from the action $A \otimes N \to N$.

It follows from a standard algebraic fact that when $R$ is a free $A$-module of finite rank and $\iota$ is an isomorphism, the map $\tau_0$ is an isomorphism. (This is analogous to the fact that for finite-dimensional vector spaces $V, W$ there is an isomorphism $V^* \otimes W \to \text{Hom}(V, W)$.)

This induces an isomorphism $\tau : HH_* \to HH^{n-*}$ on homology.

Let

$$\lambda = \tau \circ \kappa \circ \tau^{-1} : HH^* \to HH^{*^{-1}}$$

be the operator on $HH^*$ induced by the Connes operator $\kappa$ on $HH_*$ by $\tau$.

**Lemma 4.13.** The map $\lambda$ is a BV operator compatible with the standard Gerstenhaber algebra structure on $HH^*$.

*Proof.* This follows from [10, Theorem 3.3.2] (see also [6, Sec. 6.5]). Ginzburg constructs a Poincaré duality isomorphism between $HH^i$ and $HH_{n-i}$ of an algebra which has two dual resolutions analogously to our construction of $\tau$. He shows using the formalism of non-commutative differential geometry that this isomorphism sends the Connes operator to a BV operator on $HH^*$. 

We have shown that the isomorphism $\rho$ takes the BV operator $\Delta$ on loop homology to the Connes operator $\kappa$ on $HH_*$ and that the isomorphism $\tau$ takes $\kappa$ to a BV operator $\lambda$ on $HH^*$. Let us define

$$\xi = \tau \circ \rho : \mathbb{H}_* \to HH^{n-*}.$$ 

It follows that $\xi$ is an isomorphism which takes $\Delta$ to $\lambda$. 

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Lemma 4.14. The composition

$$\xi = \tau \circ \rho : \mathbb{H}_* \to HH^{n-*}$$

is an associative algebra isomorphism.

Proof. The result follows from the following lemma.

Lemma 4.15. The map $$\tau : HH_* \to HH^{n-*}$$ takes the operation $$\beta$$ to cup product on Hochschild cohomology.

Proof. We first define cup product in terms of the complexes we have used. Let $$\mathcal{A}Tot$$ be the total complex of $$R \otimes R'$$ with the diagonal $$A$$-action.

Lemma 4.16. The complex $$\mathcal{A}Tot$$ is an $$A$$-module resolution of $$k$$.

Proof. This is a known result, see e.g. [20].

We have a map $$\cup_0 : C^i \otimes C'^j \to \text{Hom}_A(Tot_{i+j}, N)$$ given by the multiplication on $$N$$ (which coincides with $$A$$ as a vector space). This means that the homology of the chain complex $$\text{Hom}_A(Tot_* , N)$$ is the Hochschild cohomology, $$HH^*(A)$$.

Lemma 4.17. After passing to cohomology, the map $$\cup_0$$ becomes the Hochschild cup product

$$\cup : HH^i \otimes HH^j \to HH^{i+j}.$$ 

Proof. This is a standard fact from homological algebra, see e.g. [13].

Using $$\cup_0$$ to compute the cup product, we show that this product is the same as the product obtained from $$\beta_0$$.

We define a map of complexes $$\beta'_0 : \text{Hom}_A(Tot_* , N) \to A \otimes_{A \otimes A} W_{n-*}$$ such that the following diagram commutes after passing to homology.

$$\begin{array}{ccc}
C_i \otimes C'_j & \xrightarrow{\beta_0} & A \otimes_{A \otimes A} W_{i+j-n} \\
\downarrow \tau_0 \otimes \tau_0 & & \downarrow \beta'_0 \\
C'^{m-i} \otimes C^{m-j} & \xrightarrow{\cup_0} & \text{Hom}_A(Tot_{2m-i-j} , N).
\end{array}$$

(15)
Because the Poincaré duality map $\tau$ is obtained from an augmentation of $\mu$, we see that $\beta_0'$ is a quasiisomorphism and that on the level of homology, the corresponding map $\beta' : HH^* \to HH_{n-*}$ is the inverse of the isomorphism $\tau$.

This proves that the following diagram commutes.

$$
\begin{array}{ccc}
HH_i \otimes HH_j & \xrightarrow{\beta} & HH_{i+j-n} \\
\tau \otimes \tau & \downarrow & \tau \\
HH^{n-i} \otimes HH^{n-j} & \cup & HH^{2n-i-j}
\end{array}
$$

Combining this with our previous result, we obtain the commutative diagram

$$
\begin{array}{ccc}
H_i \otimes H_j & \xrightarrow{\bullet} & H_{i+j-n} \\
\rho \otimes \rho & \downarrow & \rho \\
HH_i \otimes HH_j & \xrightarrow{\beta} & HH_{i+j-n} \\
\tau \otimes \tau & \downarrow & \tau \\
HH^{n-i} \otimes HH^{n-j} & \cup & HH^{2n-i-j}.
\end{array}
$$

Thus $\xi = \tau \circ \rho : H \to HH_{n-*}$ is an isomorphism of associative algebras. We have also shown that it takes the BV operator $\Delta$ to $\lambda$, the dual of the Connes operator $\kappa$. 
A BV algebra is defined by its product and its BV operator, so we have shown that the BV algebra structures on $\mathbb{H}_{n-*}$ and on $HH_*$ are isomorphic. In particular, this implies that the Lie-like bracket on string topology is mapped to the Gerstenhaber bracket on Hochschild cohomology. This concludes the proof of Theorem 3.1. Q.E.D.

5 Concluding remarks

1. In this paper, we constructed an isomorphism of BV algebra structures between the Hochschild cohomology and loop homology for aspherical oriented closed manifolds. The algebra $\mathbb{H}_*(X)$ has additional algebraic structure (of a 2-dimensional positive-boundary TQFT, see [3]) and it should be possible to compute it algebraically for aspherical manifolds.

2. I hope that the methods of this work might be useful for proving that the isomorphisms of algebras $\mathbb{H}_{-*n} \cong HH^*(C(M), C(M))$ and $\mathbb{H}_{-*n} \cong HH_*(C_*(\Omega(M)), C_*(\Omega(M)))$ for simply connected manifolds also preserve BV structures.

3. I plan to extend the results of this paper for aspherical orbifolds. String topology operations for orbifolds have been recently introduced in [14].

4. It is known that the equivariant homology $H_{*}^{st}(LX)$ of $LX$ is related to the cyclic homology $HC_*(A)$ of the algebra $A = C^*(X)$ of singular cochains of $X$ (see [12]). It should be possible to show that for an aspherical manifold $X$ with $\pi_1(X) = G$, structures on the cyclic cohomology $HC^*(k[G])$ agree with the string topology operations on $H_{*}^{st}(LX)$.

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