Bell inequalities and distillability in \(N\)-quantum-bit systems

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The relation between Bell inequalities with two two-outcome measurements per site and distillability is analyzed in systems of an arbitrary number of quantum bits. We observe that the violation of any of these inequalities by a quantum state implies that pure-state entanglement can be distilled from it. The corresponding distillation protocol may require that some of the parties join into several groups. We show that there exists a link between the amount of the Bell inequality violation and the size of the groups they have to form for distillation. Thus, a strong violation is always sufficient for full \(N\)-partite distillability. This result also allows for a security proof of multi-partite quantum key distribution (QKD) protocols.

I. INTRODUCTION

The statistical correlations between the outcomes of experiments performed by different observers in a composed quantum system can in general not be reproduced by local variable models (LV) (in the sense of Einstein-Podolsky-Rosen [1]). This impossibility is shown by proving that quantum correlations violate some constrains, known as Bell inequalities [2], that any LV theory satisfies. Thus, a quantum state \(\rho\) in a composite systems of \(N\) parties, \(\mathcal{C}^{d_1} \otimes \mathcal{C}^{d_2} \otimes \cdots \otimes \mathcal{C}^{d_N}\), where \(d_i\) is the dimension of the Hilbert space associated to party \(i\) \((i = 1, \ldots, N)\), doesn’t admit a LV description when it violates a Bell inequality. It is not difficult to see that separable states,

\[
\rho = \sum_j p_j |\psi_j^1\rangle \langle \psi_j^1| \otimes |\psi_j^2\rangle \langle \psi_j^2| \otimes \cdots \otimes |\psi_j^N\rangle \langle \psi_j^N|,
\]

(1)

i.e. those states that can be written as a mixture of product pure states, do not violate any Bell inequality [3]. States that are not separable are called entangled. Entanglement is then a necessary condition for the violation of a Bell inequality.

The understanding and interpretation of quantum correlations has notably changed in the last years. Entanglement has turned out to be a practical resource, since it is the key ingredient for most of the recent quantum information applications, such as teleportation [4] and quantum key distribution [5]. In all these new information processing protocols, some results that cannot be achieved in Classical Information Theory become possible by using entangled states. These processes do not have classical analog because they are based on entanglement, which is an intrinsic quantum feature. Nevertheless, it is not clear whether all entangled states are useful for quantum information tasks.

In systems of two parties, \(\mathcal{C}^{d_1} \otimes \mathcal{C}^{d_2}\), the most representative entangled state is

\[
|\Psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |i\rangle \otimes |i\rangle,
\]

(2)

where \(d = \min(d_1, d_2)\) and \(\{|i\rangle\}\) are orthonormal bases in the two local systems. The state (3) is the maximally entangled state of two \(d\)-dimensional systems, often called qubits. A state \(\rho\) is definitely useful for quantum information applications when out of possibly many copies of it, the parties are able to distill some amount of maximally entangled states using only local operations and classical communication (LOCC). If this is the case, the state \(\rho\) is said to be distillable [6]. This condition is equivalent to see if some pure-state entanglement can be extracted from the original state, since all entangled pure states are distillable [6]. It is known that there are mixed states, called bound entangled, that are not distillable in spite of being entangled [6]. It is an open question whether this type of states are useful for quantum information. For systems of more parties the picture is more complicated, and it is not even known what the fundamental types of pure-state entanglement are [6]. However, as we will see, one can extend the notion of distillability to the multi-partite scenario: a quantum state shared by \(N\) parties is \(N\)-party distillable when out of many copies of it all the parties can extract by LOCC pure-state \(N\)-party entanglement, i.e. a pure state that is bipartite entangled with respect to any splitting of the parties into two groups.

Distillability and the violation of Bell inequalities are two manifestations of entanglement. The first is related to the usefulness of a state for quantum information processing. On the other hand, Bell inequality violation demonstrates the inadequacy of classical LV models. Is it possible to relate these two concepts? This is the main motivation for the present work: to search for a connection between Bell inequality violation and distillability. We consider systems of \(N\) quantum bits, or qubits, and the complete set of Bell correlation inequalities with two two-outcome measurement per site (see below). We demonstrate that there exists a link between their violation and state-distillability.

The structure of the article is the following. In the next section we describe more precisely our \(N\)-qubit scenario. We introduce the family of Bell inequalities we consider and we extend the concept of distillability to
these systems. In Section III we establish a first result: if a $N$-qubit state violates a Bell inequality of this family, it is at least bipartite distillable. This result is used as a basis for the main result of the paper, described in Section IV, the amount of violation is connected to the degree of distillability of the state. In particular, if the violation of an $N$-qubit inequality exceeds the bound $2^{N-2}/2$, the state is fully distillable. These results are valid for the whole family of inequalities that we study. We move then to consider a subset of these inequalities that detects truly $N$-qubit entanglement, and we prove that their violation is also sufficient for $N$-party distillability (section V). Finally, in Section VI we discuss the connection of these results with the security of multi-party protocols, that we have chosen not to include in the main text in order to enhance its readability and to underscore the physical strength of the results.

II. N-QUBIT SYSTEMS

In this article we deal with states shared by an arbitrary number of observers, $N$, such that the dimension of each local Hilbert space is two (qubits). Let us describe here more precisely the type of Bell inequalities and distillability protocols that we consider in these systems.

A. Bell inequalities

A complete set of Bell correlation inequalities for $N$-qubit systems was found by Werner and Wolf, and independently Zukowski and Brukner, in [14,15]. Every local observer, $i = 1, \ldots, N$, can measure two observables, $O^i_1 = O_i$ and $O^i_2 = O'_i$, of two outcomes labelled by $\pm 1$. Thus, after many rounds of measurements, all the parties collect a list of experimental numbers, and they can construct the corresponding list of correlated expectation values, $E(j_1, j_2, \ldots, j_N) = \langle O^1_{j_1} \otimes O^2_{j_2} \otimes \cdots \otimes O^N_{j_N} \rangle$, where $j_i = 1, 2$. The general expression for the Werner-Wolf-Zukowski-Brukner (WWZB) inequalities is given by a linear combination of the correlation expectation values,

$$I_N(\vec{c}) = \sum_{j_1, \ldots, j_N} c(j_1, \ldots, j_N) E(j_1, \ldots, j_N) \leq 1,$$

where the conditions for the coefficients $c$ can be found in [14,15]. This set is complete in the following sense. If none of these inequalities is violated, there exists a LV model for the list of data $E(j_1, \ldots, j_N)$. If any of these inequalities is violated, the observed correlations do not admit a LV description. Thus, this family of inequalities can be thought of as the generalization of the CHSH inequality [16,17].

$$I_2 = \frac{1}{2} (E(1, 1) - E(1, 2) + E(2, 1) + E(2, 2)) \leq 1 \quad (4)$$

to an arbitrary number of subsystems. Indeed it reduces to the CHSH inequality when $N = 2$.

Now consider a system composed of $N$-qubits and quantum observables corresponding to von Neumann measurements $\sigma(i) \equiv \vec{n} \cdot \vec{\sigma}$, where $\vec{n}$ is a normalized three-dimensional real vector and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. Thus, any observable is defined by a real unit vector $\vec{n}$. It is known that for any Bell inequality with the corresponding local measuring apparatus, one can define the so-called Bell operator $B = B(\vec{c}, \{\vec{n}, \vec{n}'\})$ such that $I_N(\vec{c}) = \text{tr}(\rho B)$. Then, $\rho$ violates the corresponding Bell inequality if

$$\text{tr}(\rho B) > 1. \quad (5)$$

The spectral decomposition of all these Bell operators is known [18], and implies that the maximal violation is always obtained for Greenberger-Horne-Zeilinger (GHZ) states of $N$ qubits $|\text{GHZ}_N\rangle = (|0, \ldots, 0\rangle + |1, \ldots, 1\rangle)/\sqrt{2}$. Indeed, one can find $(i)$ a local computational basis $\{ |0\rangle, |1\rangle \}$ for each qubit $j$, $(ii)$ $2^{N-1}$ non-negative numbers $b_k$ and $(iii)$ $2^{N-1}$ parameters $\theta_k$ such that

$$B_N = \sum_{k=0}^{2^{N-1}-1} b_k (Q^+_k - Q^-_k) \quad (6)$$

where $Q^+_k$ are the projectors on the generalized GHZ-states $|\theta^+_k\rangle$ defined as

$$|\theta^+_k\rangle = \frac{1}{\sqrt{2}} (e^{i\theta_k}|k\rangle \pm |\bar{k}\rangle). \quad (7)$$

In these expressions we label the product states in the computational basis by $|k\rangle$ with $k \in \{0, 1, \ldots, 2^N - 1\}$, where the correspondence is given by the binary expansion, i.e. $|0\rangle = |00\ldots0\rangle$, $|1\rangle = |00\ldots1\rangle$, and so on until $|2^N-1\rangle = |11\ldots1\rangle$. We also define $|\bar{k}\rangle = |2^N-1-k\rangle$; written as tensor product, $|\bar{k}\rangle$ is obtained from $|k\rangle$ by exchanging all the zeros and ones. The set of the $|\theta^+_k\rangle$ with $k \in \{0, 1, \ldots, 2^{N-1} - 1\}$ and $\sigma = \pm$ is a basis of eigenstates of $B_N$, that we shall call the theta basis. The values of the coefficients $b_k$ and $\theta_k$ depend on the specific Bell’s inequality and the chosen measurements $\{\vec{n}, \vec{n}'\}$.

An important member of this family of inequalities is the Mermin-Belinskii-Klyshko (MBK) inequality [19,20]. Given a set of measurements $\{\vec{n}, \vec{n}'\}$, the $N$-qubit Bell operator for these inequalities is defined recursively as

$$M_N = \frac{1}{2} (\sigma(\vec{n}) + \sigma(\vec{n}') \otimes M_{N-1} + (\sigma(\vec{n}) - \sigma(\vec{n}') \otimes M_{N-1}^\dagger, \quad (8)$$

where $\sigma(\vec{n}) \equiv \vec{n} \cdot \vec{\sigma}$.
where \( M'_n \) is obtained from \( M_n \) interchanging \( \hat{n}_i \) and \( \hat{n}'_i \), and \( M_1 = \sigma(\hat{n}_1) \). The maximal quantum violation of the set of inequalities (3) is obtained for the MBK one (10), with some particular choice of measurements, and it is equal to \( 2^{(N-1)/2} \), i.e., quantum violations of WWZB inequalities are in the range \((1, 2^{(N-1)/2})\).

B. Distillability

The concept of state distillability is very often related to the usefulness of a state for quantum information tasks. In the bipartite case, a state \( \rho \) is distillable when out of possibly many copies of it the two parties can extract a two-qubit maximally entangled state, or singlet,

\[
|\Phi\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle),
\]

by LOCC. It is not completely evident how to extend this definition to a multi-partite scenario. In this work we will use the following generalization: a quantum state shared by \( N \) parties is \( N \)-partite distillable when it is possible to distill states like (3) between any pairs of parties using LOCC. This is equivalent to demand that any \( N \)-partite entangled state, in particular an \( N \)-qubit GHZ state, can be obtained by LOCC. Indeed, once all the parties are connected by singlets, one of them can prepare locally any of these states and send it to the rest by teleportation. On the other hand if the parties share an \( N \)-partite pure entangled state, there exists local projections such that \( N - 2 \) qubits project the remaining two parties into a bipartite entangled pure state (19), which is always distillable to a state like (3).

In the multipartite case the situation is subtler than in the bipartite one. Consider a state \( \rho \) which is not \( N \)-partite distillable. It may happen that if some of the parties join into several groups (or establish quantum channels between them), the state becomes distillable (see 21). The original state is now shared by \( L < N \) parties, and it is \( L \)-partite distillable. However it is important to stress that some of the parties can extract singlets without using any global quantum operation between them. They already had some distillable entanglement that was hidden and that can be extracted by joining some of the parties. We have two extreme cases: (i) the parties can perform all the operations locally, and then the state is \( N \)-partite distillable as was defined above, or (ii) they have to join into two groups, and then the state is said to be bipartite distillable (21). All the other cases are between these two possibilities and, of course, there also non-distillable states. According to this classification, we will estimate the degree of distillability in an \( N \)-qubit state by means of the minimal size of the groups the parties have to create in order to distill pure-state entanglement between them.

III. VIOLATION OF BELL INEQUALITIES IMPLIES BIPARTITE DISTILLABILITY

One of the present authors has proven some time ago (21) that for a specific Bell operator \( B_N \), namely the MBK operator with some given settings, all the states that violate the corresponding Bell inequality are bipartite distillable, although the partition may not be such that one of the parties is a single qubit (22). Refs. (21,22) are the only two studies of the link between violation of Bell and distillability in \( N \)-qubit systems before this one (23). As a first step, we provide the generalization of the result of Ref. (21) for an arbitrary inequality of the WWZB family.

Theorem 1 Consider an \( N \)-qubit state \( \rho \). If there exists a Bell operator \( B_N \) in the WWZB family such that the corresponding inequality is violated, that is such that \( \text{Tr}(\rho B_N) > 1 \), then \( \rho \) is bipartite distillable.

Proof: The proof goes along two steps.

First step. If \( \rho \) is such that \( \text{Tr}(\rho B_N) > 1 \) for a Bell operator with two measurement per qubit, then it has been proven in Ref. (14) that there exists at least one partition of the \( N \) qubits into two disjoint groups, \( A \) and \( A' \), such that the partial transpose (24) of \( \rho \), \( \rho^{\perp A} \), has a negative eigenvalue (the state is said to be NPT). This condition is necessary for distillability (19), but it is conjectured not to be sufficient (22), except for \( 4^2 \otimes 4^d \) systems. Now, from the state \( \rho \), we form the state

\[
\rho_D = \sum_{k=0}^{2^{N-1}-1} \sum_{\sigma=\pm} \lambda_k^\sigma Q_k^\sigma \text{, with } \lambda_k^\sigma = \text{Tr}(\rho Q_k^\sigma)
\]

by keeping only the terms that are diagonal in the theta basis associated to \( B_N \). Note that we do not claim that there is an LOCC operation such that \( \rho \rightarrow \rho_D \). By construction, \( \text{Tr}(\rho B_N) = \text{Tr}(\rho_D B_N) \): \( \rho_D \) violates the same inequality as \( \rho \). As a consequence of the result in Ref. (14), we know that there exists at least one bipartite splitting \( A \sim A' \) of the qubits such that \( \rho_D^{\perp A} \) has a negative eigenvalue. As stressed several times, this is not in general a guarantee that \( \rho_D \) is distillable. However, it turns out that here NPT is a sufficient condition for distillability.

To see this, we write the matrix \( \rho_D \) in the product basis. This gives

\[
\rho_D = \sum_{k=0}^{2^{N-1}-1} \left( \frac{\lambda_k^+ + \lambda_k^-}{2} (|k\rangle\langle k| + |\bar{k}\rangle\langle \bar{k}|) + \frac{\lambda_k^+ - \lambda_k^-}{2} (e^{i\theta_k}|k\rangle\langle \bar{k}| + h.c.) \right).
\]

This means that there are non-zero elements only in the two main diagonals of the matrix. It is easy to get convinced that the partial transposition with respect to any partition \( A \sim A' \) will preserve this structure (fig. 1). As an example, take \( N = 5 \) qubits, \( A = \{2,3\} \). Then

\[
|00111\rangle (11000) \xrightarrow{T_A} |01011\rangle (10100) : \quad (12)
\]
an element of the anti-diagonal is sent onto another element of the anti-diagonal. Of course, the elements of the main diagonal remain unchanged.

This being the structure of $\rho_D^{T_A}$, the negativity of some eigenvalue can be studied by looking at each $2 \times 2$ bloc

$$M(k) = \begin{pmatrix} (\rho_D^{T_A})_{kk} & (\rho_D^{T_A})_{k\bar{k}} \\ (\rho_D^{T_A})_{\bar{k}k} & (\rho_D^{T_A})_{\bar{k}\bar{k}} \end{pmatrix},$$

for $k = 0, ..., 2^{N-1} - 1$. Since $\rho_D$ is NPT for some bipartite splitting, there exist at least two values of $k$, say $K$ and $K'$, and a bipartite splitting $A - A'$ such that: (i) $|K'\rangle\langle K'|$ is sent onto $|K\rangle\langle K'|$ by the partial transposition $T_A$, and (ii) the determinant of the $2 \times 2$ block $M(K)$ is negative:

$$(\lambda_K^+ + \lambda_K^-)^2 - (\lambda_K^+ - \lambda_K^-)^2 < 0. \quad (14)$$

Now it is not difficult to see that the $N$-qubit state $\rho_D$ is bipartite distillable. According to the partition $A - A'$, the state can be locally projected into the subspace $\mathcal{H}(K, K')$ spanned by $|K\rangle$, $|\bar{K}\rangle$, $|K'\rangle$ and $|\bar{K}'\rangle$ [24].

This subspace is isomorphic to $\mathbb{C}^2 \otimes \mathbb{C}^2$, and one can relabel $|K\rangle = |00\rangle$, $|\bar{K}\rangle = |11\rangle$, $|K'\rangle = |01\rangle$ and $|\bar{K}'\rangle = |10\rangle$. The projected two-qubit state satisfies NPT, then it is distillable [27].

![FIG. 1. Schematic representation of the effect of the partial transposition on the matrix $\rho_D$, written in the product basis.](image)

This is the end of the first step of the proof. The only problem left is that the transformation $\rho \rightarrow \rho_D$ may well be impossible with LOCC [26]. We have not a final proof for this impossibility. Note however that the theta basis in which we should diagonalize $\rho$ is defined by $2^{N-1}$ highly non-local parameters, the phases $\theta_k$, whose value is determined by the details of the observable $B_N$ [10]. We cannot get easily rid of the $\theta_k$ by using the freedom left in the construction, that is by a redefinition of local phases $|0\rangle_j \rightarrow e^{i\theta_j} |0\rangle_j$ and $|1\rangle_j \rightarrow e^{i\theta_j} |1\rangle_j$, because there are only $2N$ such phases. As we said in the main text, instead of looking for an hypothetical LOCC protocol leading from $\rho$ to $\rho_D$ defined in [10], we take another approach.

**Second step.** In the previous step, we have identified the subspace in which to project the state, locally for the partition $A - A'$. It is the subspace $\mathcal{H}(K, K')$ spanned by $|K\rangle$, $|\bar{K}\rangle$, $|K'\rangle$ and $|\bar{K}'\rangle$, or alternatively, by $|\theta_K\rangle$, $|\theta_{K'}\rangle$, $|\theta_{K}\rangle$ and $|\theta_{K'}\rangle$. We begin by applying a local phase redefinition $U$ that erases the phases $\theta_K$ and $\theta_{K'}$: thus $U|\theta_{K,K'}\rangle = |\psi_{K,K'}^\pm\rangle$ where the $\psi$’s are the GHZ states without phases

$$|\psi_{K,K'}^\pm\rangle = \frac{1}{\sqrt{2}} (|k\rangle \pm |\bar{k}\rangle).$$

This $U$ of course does not erase all the other phases $\theta_k$, but this is not a problem.

Dür and Cirac [20] have shown that any $N$-qubit state $\rho$ can be brought by LOCC to a state diagonal in a GHZ-basis like [13],

$$\rho_D' = \sum_{k=0}^{2^{N-1}-1} \sum_{\sigma=\pm} \mu_k^\sigma P_k^\sigma, \quad \text{with } \mu_k^\sigma \equiv \text{Tr}(\rho P_k^\sigma) \quad (16)$$

where $P_k^\sigma$ is the projector on the GHZ state $|\psi_{K,K'}^{\pm\sigma}\rangle$. All the diagonal terms, $\langle \psi_{K,K'}^{\pm\sigma}\rangle|\psi_{K,K'}^{\pm\sigma}\rangle$, are kept unchanged, while the rest of terms go to zero. Thus in our case we can bring $U\rho U^\dagger$ onto

$$\rho_D' = \sum_{\sigma=\pm} \lambda_K^\sigma P_K^\sigma + \lambda_{K'}^\sigma P_{K'}^\sigma + \sum_{k \neq K,K'} \sum_{\sigma=\pm} \mu_k^\sigma P_k^\sigma, \quad (17)$$

just using local operation on each sub-system. Note that for $K$ and $K'$ the $\lambda_k^\sigma$ are indeed the same that appear in the construction [10] of $\rho_D$. Obviously, when written in the product basis, $\rho_D'$ has exactly the same structure as $\rho_D$, that is, non-zero elements only in the two main diagonals. Contrary to what happened for $\rho_D$, we do not know if $\rho_D'$ violates the original Bell’s inequality. However this is not important here, we simply have to apply the same procedure that we followed for $\rho_D$: take the partition $A - A'$ that brings $|K'\rangle\langle K'|$ onto $|K\rangle\langle K|$, thus by construction the determinant of $M(K')$ built from $\rho_D^{T_A}$ is the same as the determinant of $M(K)$ built from $\rho_D^{T_A}$. Thus $A - A'$ can locally project $\rho_D'$ into $\mathcal{H}(K, K')$, and the resulting two-qubit state will satisfy NPT and will thus be distillable. This concludes the proof. $\square$

In summary, the way of distilling a singlet from a state $\rho$ that violates a Bell inequality is: (a) determine on the paper the $2 \times 2$ subspace $\mathcal{H}(K, K')$ in which to project and the corresponding partition; (b) erase locally the phases $\theta_K$ and $\theta_{K'}$ and apply the Dür-Cirac protocol, and finally (c) project onto $\mathcal{H}(K, K')$. In the proof, we used three known results: the spectral decomposition for Bell operators with two measurements per qubit [10], the fact that any $\rho$ that violates one of these Bell inequalities is NPT for at least one partition [10], and the depolarization protocol of [20]. The new insight is provided by the peculiar structure of the matrices $\rho_D$ and $\rho_D'$ that makes NPT a sufficient condition for distillability.

**IV. THE AMOUNT OF VIOLATION AND THE DEGREE OF DISTILLABILITY**
A. Main result

In this section we prove the main result of the article: there exists a link between the amount of Bell violation and the degree of state distillability. We have just shown that if a state violates any of the WWZB inequalities (3), then it is bipartite distillable. As it has been mentioned above, the range of quantum violations, \((1, 2^{(N-1)/2}]\), is quite broad, specially for a large number of qubits. This suggests that a finer classification of state distillability properties can be done depending on the amount of Bell violation. This is the scope of this section. Let us start by proving the following

**Lemma 1:** Consider an \(N\)-qubit state \(\rho_N\) that violates an inequality of (3), with Bell operator \(B_N\), by an amount \(\beta_N\), i.e.

\[
\text{tr}(\rho_N B_N) = \beta_N > 1.
\]

Then, it is possible to obtain by LOCC a new state \(\rho_{N-1}\) of \(N-1\) qubits violating another inequality of (3), with Bell operator \(B_{N-1}\), by an amount \(\beta_{N-1} \geq \beta_N/\sqrt{2}\).

**Proof:** It was shown in (14) that the CHSH is the elementary inequality for the whole set (3). This means that for \(N\) qubits any of these inequalities can be written as

\[
B_N = \frac{1}{2} [\sigma(n_1) + \sigma(n_2)] B_{N-1}^+ + \sigma(n_1) B_{N-1}^- - \sigma(n_2) B_{N-1}^+,
\]

where \(B_{N-1}^\pm\) are WWZB Bell operators of \(N-1\) qubits —of course, the special relation that \(B_N\) is obtained from \(B_{N-1}^+\) by interchanging \(\hat{n}_1\) and \(\hat{n}_2\) holds only for the MBK inequality. Using local unitary operations, the \(N\)th qubit can put \(\hat{n}_N\) and \(\hat{n}_N'\) in the \(xy\) plane, their bisectrix being the \(x\) axis. Denote by \(2\delta\) the angle between the two vectors, \(0 \leq \delta \leq \pi/2\). Then, we have that the state \(\rho_N\) satisfies

\[
\text{tr}(\rho_N B_N) = \cos \delta \text{tr}(\rho_N \sigma_x \otimes B_{N-1}^+) + \sin \delta \text{tr}(\rho_N \sigma_y \otimes B_{N-1}^-) = \beta_N.
\]

Suppose now that \(\text{tr}(\rho_N \sigma_x \otimes B_{N-1}^+) \geq \text{tr}(\rho_N \sigma_y \otimes B_{N-1}^-)\) (of course a similar demonstration is possible for the other case). Then it follows from (20) that

\[
\text{tr}(\rho_N \sigma_x \otimes B_{N-1}^+) \geq \beta_N \cos \delta + \beta_N \sin \delta \geq \beta_N/\sqrt{2}.\]

If we use the spectral decomposition \(\sigma_x = |+\rangle\langle +| - |\rangle\langle |\rangle\), and we denote by \(\tilde{B}_{N-1}^+ \equiv -B_{N-1}^+\), which is a new Bell operator of \(N-1\) qubits, we have

\[
\text{tr}(|\rangle\langle +| \rho_N |+\rangle B_{N-1}^+) + \text{tr}(|\rangle\langle || \rho_N |\rangle B_{N-1}^-) \geq \beta_N \sqrt{2}
\]

Define the normalized states \(\rho_{\pm} \equiv (\pm \rho_N |\pm\rangle)/p_{\pm}\) of \(N-1\) qubits, where \(p_{\pm} \equiv \text{tr}(|\rangle\langle +| \rho_N |\pm\rangle)/p_{\pm}\). The physical meaning of these states is the following: if the \(N\) qubits start with state \(\rho_N\) and party \(N\) measures \(\sigma_x\), the rest of the qubits are projected into \(\rho_{\pm}\) with probability \(p_{\pm}\). Again without loss of generality, consider the case in which \(\text{tr}(\rho_+ B_{N-1}^+) \geq \text{tr}(\rho_- B_{N-1}^-)\). Then from (22) it is easy to see that

\[
\text{tr}(\rho_+ B_{N-1}^+) = \beta_{N-1} \geq \frac{\beta_N}{\sqrt{2}}.
\]

Thus, starting from \(\rho_N\) that has a Bell violation equal to \(\beta_N\), any qubit can locally project with some probability the other \(N-1\) qubits into a new state \(\rho_{N-1}\) that violates a new inequality by an amount of at least \(\beta_N/\sqrt{2}\).

We note that \(\beta_{N-1} \geq \beta_N/\sqrt{2}\) can always be obtained with non-zero probability. In the case where \(p_+\) (or \(p_-\)) is zero, \(\rho_N\) is a product state containing a \(\sigma_x\) eigenstate on the relevant tensor factor. However, since \(\langle +| \sigma_y + |\rangle\langle +|\rangle\) is zero, Eq. (20) tells us that in this situation \(\beta_N = \beta_{N-1}\) right from the beginning, such that no measurement is required at all.

Of course, it is likely that some of the inequalities used in the derivation of this lemma are not tight. However they cannot be improved if we do not have more information about the specific state or Bell operator. We can now combine this lemma with the result shown in Section I for proving the following

**Theorem 2:** Consider an \(N\) qubit state \(\rho_N\) violating one of the WWZB inequalities by an amount \(\beta\) such that

\[
1 < 2^{\frac{\beta}{\sqrt{2}} < \beta < 2^{\frac{\beta+1}{\sqrt{2}}}.
\]

Then pure-state entanglement can be distilled if the parties can join into groups of at most \(p-1\) qubits.

**Proof:** Using the lemma seen above, any qubit can perform a local projection such that the amount of Bell violation is decreased by a factor \(\sqrt{2}\). In the worst case, after \(N-p\) of these local projections, the rest of \(p\) qubits share a state \(\rho_p\) having a Bell violation of \(1 < \beta_p \leq \sqrt{2}\). A new local projection is not possible since it might imply that the resulting state is not entangled. At this point, and since \(\rho_p\) is still non-local, we can use the result of Section I: the state \(\rho_p\) is bipartite distillable. Thus, these \(p\) qubits can distill pure-state entanglement between them if they can join into groups of at most \(p-1\) parties.

This theorem gives the searched link between Bell violation and the degree of state distillability. As in any distillation scenario we have at our disposal many copies of the state \(\rho_N\). Thus, the parties can use the different copies for connecting all of them. The amount of Bell violation bounds the size of the groups they have to form in any of these distillation processes. It gives an estimation of the \(L\)-partite distillability of the state (\(2 \leq L \leq N\)), or in other words, the number of quantum channels to be established between the parties for distillation (see figure 2). If we focus now on \(N\)-partite distillability, we have
Corollary 1: If an $N$-qubit state violates any of the WWZB inequalities by an amount $\beta_N > 2^{(N-2)/2}$, then it is $N$-partite distillable.

Proof: It follows easily from the previous theorem. All the parties but 1 and $l$, with $l = 2, \ldots, n$, perform the local projection. Then the two qubits $l$ and 1 end, with some probability, with a state $\rho_{1l}$ violating the CHSH inequality. This two-qubit state is entangled and then it is distillable [27]. In this way, the first party shares singlets with all the others, so the initial state is $N$-party distillable. \(\Box\)

**FIG. 2.** The figure shows the case of a seven-qubit state having a Bell violation of $4\sqrt{2}$. After local projections by four parties, the other three are left with a state violating a WWZB inequality. Then, if two of them share a quantum channel, they can distill pure-state entanglement. For the blue partition, for instance, parties 2, 3, 4 and 6 perform the local projection in such a way that 1, 5 and 7 end with a three-qubit state violating a WWZB inequality. Now, they can distill entanglement if (at most) two of them share a quantum channel, say parties 1 and 7. The parties run this protocol for all the different groups as in the figure, and at the end party 7 can prepare any $N$-qubit pure state locally and use teleportation for sending the corresponding qubit to the rest of parties. Thus, any $N$-qubit state can be prepared between all the parties using several copies of the initial state, provided that quantum channels are established between some of them.

B. Some comments

Another possible manifestation of entanglement is the negativity of the partial transposition [24]. If a state $\rho_N$ has a non-positive partial transposition (the state is NPT) with respect to some bipartite splitting of the parties, then it is not separable for this splitting. It is also known that if the state is distillable, then it is NPT [3]. The relation between Bell violation and partial transposition in $N$-qubit systems was analyzed in [28] for the MBK case. Since distillability is sufficient for NPT, our results are a generalization of the ones in [28] to the whole set of WWZB inequalities. Indeed, defining as $\beta_{\text{max}}$ the maximal Bell violation for an $N$-qubit state $\rho$, if $\beta_{\text{max}} > 2^{(N-p)/2}$, every subset containing $p$ parties has at least one NPT partition. In a similar way as in [29], one can consider a partition of the $N$ qubits into $p$ nonempty and disjoint subsets $\alpha_1, \ldots, \alpha_p$ and the collection $\mathcal{P}$ of all unions of these sets with the empty set. The set $\mathcal{P}$ has $2^p$ elements. Then, if $\forall \alpha \in \mathcal{P}$ we have $\rho^\alpha > 0$, then $\beta_{\text{max}} \leq 2^{(N-p)/2}$. In particular, if $\beta_{\text{max}} > 2^{(N-2)/2}$, all the partitions are NPT (as it should be, since the state is fully distillable).

As it has been already mentioned, it may happen that for a particular situation the bounds presented here are not good. However, they cannot be improved: a better estimation of the distillability properties of an $N$-qubit state is not possible if we only know the amount of Bell violation. The clearest example is a $N$-qubit state of the form $|\text{GHZ}_{N-1}\rangle|0\rangle$. This state violates the MBK inequality for $N$ qubits by a quite large amount, $2^{(N-2)/2}$, even if it is clearly not $N$-partite distillable — even worse, one of the qubits is not entangled at all. Similar examples for GHZ states of $N-p+1$ qubits in $N$-qubit systems show that the bounds given in Theorem 1 are indeed tight. In the next Section, we show that additional knowledge about the meaning of a given inequality leads to an improvement of these bounds.

C. Amount of violation and weight of the GHZ state

Some further insight on the meaning of the main result above can be gained by noting that a state must have a large overlap with the GHZ state in order to violate an inequality of the WWZB family by a large amount. This is not astonishing, and can be quantified.

Given the $N$-qubit state $\rho_N$, one can always redefine local bases (or apply local unitary operations) in order to maximize

$$ r = \langle \text{GHZ}_N | \rho_N | \text{GHZ}_N \rangle. \quad (25) $$

It can be proven (see Appendix A) that for all Bell operator $B$ in the WWZB family, normalized so that the LV limit is at 1, it holds

$$ \text{tr}(\rho_N B) \leq \beta(r) = 2^{\frac{N-1}{2}} \sqrt{r^2 + \frac{(1-r)^2}{2^{N-1} - 1}}. \quad (26) $$

A necessary condition to detect $N$-qubit entanglement is therefore $\beta(r) > 2^{(N-2)/2}$. Let us see what this condition implies when the number of qubits is varied.

For two qubits, the condition $\beta(r) > 1$ is fulfilled for $r > 1/2$; but $r$ is the weight of a Bell state in $\rho$, and it is known that $r > 1/2$ is a sufficient condition for the state to be entangled. Consequently, for two qubits this bound
simply says that if a state violates a Bell inequality, then it is entangled (and thence distillable).

For three qubits, the condition $\beta(r) > \sqrt{2}$ is fulfilled for $r > (1 + \sqrt{3})/4 \sim 0.683$. In Appendix B, we show that this is sufficient but not necessary for full distillability, by giving an explicit protocol. As expected, the violation of a Bell inequality by a large amount is a sufficient, but not a necessary condition for full distillability. This is in perfect analogy with the two-qubit case, where there are distillable (that is, entangled) states that do not violate any Bell inequality. This is also in agreement with the facts that states of the form $\cos \alpha |000\rangle + \sin \alpha |111\rangle$, for an arbitrary number of qubits, never give a large violation (if any) when $\alpha$ is small enough \cite{31}; however, such states are distillable to $\{|GHZ_N\rangle\}$ by filtering and classical communication.

In the limit of a large number of qubits, we have $\beta(r) \sim 2^{(N-1)/2} r$. This means that for a violation $\beta_N > 2^{(N-p)/2}$, $p \geq 2$, as in Theorem 1, the overlap $r$ with the GHZ state must be larger than $2^{(3-p)/2}$. In particular, for the violation implying $N$-qubit entanglement ($p = 2$) one must have $r > 1/\sqrt{2} \sim 0.71$.

V. N-PARTY ENTANGLEMENT AND N-PARTY DISTILLABILITY

In this Section we focus more specifically on full distillability. We have proven in Corollary 1 that if an $N$-qubit state violates any of the WWZB inequalities \cite{24} by an amount $\beta_N > 2^{(N-2)/2}$, then it is $N$-partite distillable. This is a very general result, and because of this generality it cannot be improved: only $\beta_N > 2^{(N-2)/2}$ guarantees truly $N$-qubit entanglement for the MBK inequalities \cite{23}, then a fortiori for the whole set of WWZB inequalities; and we have just proven that the $N$-qubit entanglement detected by this criterion is fully distillable.

Recently, other inequalities have been constructed whose violation guarantees $N$-qubit entanglement \cite{30,31}. However, the amount of violation is smaller. Specifically, for these inequalities $\beta_N = 2^{(N-2)/2}$ is the maximum amount of violation allowed by QM, and the criterion for $N$-qubit entanglement reads $\beta_N > 2^{(N-3)/2}$. Thus, the general criterion of Corollary 1 is not fulfilled. It seems, however, reasonable to conjecture that the $N$-qubit entanglement detected by these specific inequalities is also fully distillable. We are going to prove that this is indeed the case. But first, we must introduce the inequalities under consideration.

A. Uffink’s inequality

In Ref. \cite{31}, Uffink discussed a quadratic inequality that detects $N$-qubit entanglement. The experimental data for it are the same as for the WWZB family: each party can perform two von Neumann measurements, $\{\hat{n}_i, \hat{n}_i'\}$. The settings are chosen in order to maximize

$$U_N(\rho) = \sqrt{\text{tr}(\rho M_N^2) + \text{tr}(\rho M_N')^2},$$

where $M_N$ and $M_N'$ are the MBK operators defined above. The LV limit is $U_{LV} = \sqrt{2}$, since in LV, as well as for quantum product states, the average value of both $M_N$ and $M_N'$ can reach 1. The QM bound is found to be $U_N(\rho) = 2^{N-1}$, and if

$$U_N(\rho) > 2^{N-2}$$

then $\rho$ exhibits $N$-qubit entanglement. Of course, since the LV limit is set to $\sqrt{2}$ instead of being set to 1, this corresponds to a violation $\beta_N > 2^{N-2}$. At first sight, Uffink’s inequality looks fundamentally different from the WWZB set of inequalities, since these are linear constraints while Uffink’s parameter $U_N(\rho)$ involve squaring correlation coefficients. However, using the basic optimization $\sqrt{x^2 + y^2} = \max(x \cos \gamma + \sin \gamma y)$, one can rewrite \cite{28} as $U_N(\rho) = \max_\gamma \text{tr}(\rho U_{N,\gamma})$ where we have defined the linear operator

$$U_{N,\gamma} \equiv \cos \gamma M_N + \sin \gamma M_N'.$$

Thus, Uffink’s quadratic inequality turns out to be a compact way of writing a set of linear inequalities, which are satisfied if the inequalities belonging to the WWZB family are satisfied. In particular, there exists $\gamma$ such that $U_N(\rho) = \text{tr}(\rho U_{N,\gamma})$. In a geometrical picture, Uffink’s parameter $U_N(\rho)$ defines a circle in the plane given by $x = \langle M_N \rangle = \text{tr}(\rho M_N)$ and $y = \langle M_N' \rangle = \text{tr}(\rho M_N')$. The $U_{N,\gamma}$ define all the tangents to this circle: if a point lies outside the circle, it also lies beyond some tangent to the circle. In summary, Uffink’s result reads: if there exist some settings and an angle $\gamma$ such that $\text{tr}(\rho U_{N,\gamma}) > 2^{(N-2)/2}$, then the state $\rho$ has $N$-qubit entanglement.

Note that the “generalized Svetlichny inequalities” discussed in Ref. \cite{30}, that also detect $N$-partite entanglement, are included in the discussion of the Uffink’s inequality. In fact, the operators defining these last inequalities are $S_N = U_{N,0} = M_N$ for $N$ even and $S_N = U_{N,\pi/4}$ for $N$ odd; and the condition for $N$-partite entanglement is exactly $\text{tr}(\rho S_N) > 2^{(N-2)/2}$.

Before showing that \cite{28} implies full distillability, we want to motivate our interest in Uffink’s inequality by showing that this inequality is indeed stronger than the MBK inequality. On the one hand, it is evident from \cite{27} that $\text{tr}(\rho M_N) > 2^{(N-2)/2}$ implies $U_N(\rho) > 2^{(N-2)/2}$, even for the same settings. On the other hand, we can exhibit states for which $U_N(\rho) > 2^{(N-2)/2}$ but $\text{tr}(\rho M_N) \leq 2^{(N-2)/2}$. For instance, consider the three-qubit state $\cos \alpha |000\rangle + \sin \alpha |W\rangle$, $\alpha \in [0, \pi/2]$, where $|W\rangle = (|110\rangle + |101\rangle + |011\rangle)/\sqrt{3}$. We found numerically that the MBK inequality is violated for $\alpha > \pi/63$, while the Uffink’s inequality is violated for $\alpha > \pi/8$. It is not astonishing that Uffink’s inequality is stronger than
those previously reported, because it allows optimization not only on the settings \( \{ n_i, \hat{n}_i' \} \), but also on a non-trivial parameter \( \gamma \).

In conclusion, we have an inequality whose absolute violation is weaker than the MBK, but which is a better detector of \( N \)-partite entanglement. We turn now to the proof that the violation of this inequality also implies full distillability.

### B. Distillability from Uffink’s inequality

To show full distillability from the violation of Uffink’s inequality, we begin by applying similar techniques as above. Indeed, the two MBK operators appearing in (20) can be written as

\[
M_N = \frac{1}{2} \left( (\sigma(\hat{n}_N) + \sigma(\hat{n}_N')) \otimes M_{N-1} \right)
\]

and similarly for \( M'_{N} \). Now, consider that we start with an \( N \)-qubit state \( \rho_N \) satisfying \( \text{tr}(\rho_N U_{N, \gamma}) > 2^{(N-2)/2} \) for some \( \gamma \). If qubit \( N \) applies the same measurement as in the previous section, the other \( N-1 \) qubits can be projected with some probability into a state of \( N-1 \) qubits violating an inequality \( U_{N-1, \gamma'} \). Thus, the different parties apply this projection until the point where three of them end with a three-qubit state satisfying \( \text{tr}(\rho_3 U_{3, \gamma}) > \sqrt{2} \).

So we reduce the proof that the violation of the Uffink inequality implies full distillability to the simplest case, the one involving three qubits. Now, using the techniques introduced in [13], it can be shown (Appendix C) that a state satisfying \( \text{tr}(\rho_3 U_{3, \gamma}) > \sqrt{2} \) is necessarily such that

\[
\langle \text{GHZ} | \rho_3 | \text{GHZ} \rangle \geq 0.628.
\]

In this case, the distillation protocol of Appendix B can be applied: a three-qubit state violating the Uffink inequality is three-party distillable. Going back, this means that if \( \text{tr}(\rho_N U_{N, \gamma}) > 2^{(N-2)/2} \), GHZ states of three qubits can be distilled between any group of three parties, which means that \( \rho_N \) is \( N \)-partite distillable. This concludes the proof.

It is interesting to note that in some cases fully distillability can be associated to a small Bell violation. Indeed consider a systems of three qubits and the Svetlichny inequality \( S_3 = U_{3, \beta} \). The value attained by LV models is exactly the bound above which we have 3-party entanglement, i.e. \( \sqrt{2} \). Thus, in this case an infinitesimal Bell violation \( \text{tr}(\rho_3 S_3) = \sqrt{2} + \epsilon \) is sufficient for full distillability [13].

### VI. BELL INEQUALITIES AND THE SECURITY OF QKD PROTOCOLS

The existence of a link between Bell inequality violation and the security of QKD protocols was first noticed in [33]. There it was shown that the violation of the CHSH inequality is a necessary and sufficient condition for the security of the BB84 protocol [12], under the assumption of individual attacks and using privacy amplification [15].

This connection was later extended [36] to the following multi-partite QKD schemes: a sender encodes a key into \( N-1 \) qubits shared between \( N-1 \) observers in such a way that all of them must cooperate in order to retrieve the key. The quantum version of this protocol, also known as secret sharing, uses the correlations in a GHZ state of \( N \) qubits (see [37] for details). The main result of [36] was to show that the mutual information between the sender and the authorized partners (all the receivers) is greater than the one between the sender and the unauthorized partners (the eavesdropper and the dishonest receiver) if and only if the authorized partners can violate the MBK inequality by an amount greater than \( 2^{(N-2)/2} \). It has to be emphasized that this security criterion is based on the very plausible fact that the difference between the mutual information for honest and dishonest parties should allow for some kind of classical privacy amplification protocol. However the existence of this protocol is at the present unknown.

The region for security coincides with the Bell violation sufficient for \( N \)-party distillability. Thus, if the parties share a state with the sufficient amount of Bell violation, they can use the quantum distillation protocol shown here for distilling a GHZ state and then run the quantum secret sharing protocol [15]. In this way Eve is disentangled from the honest parties, and the protocol is secure. Note that this is a quantum privacy amplification protocol, while, as we have just stressed, the existence of a classical protocol for these schemes in this security region remains an open question.

Thus the results of the present paper prove that (i) the ”plausible” security criterion put forward in [36] is definitely a security criterion, at least for quantum privacy amplification; and (ii) the criterion is extended to the violation of any of the WWZB inequalities by \( \beta_N > 2^{(N-2)/2} \), and to all the inequalities allowing for \( N \)-partite distillability like Uffink’s. Further investigation in this direction is still required, but our results strengthen the interpretation of Bell inequalities as security indicators for quantum cryptography schemes.

### VII. DISCUSSION

Bell inequalities are usually presented as a way for testing Quantum Mechanics vs LV theories. Then, its importance is normally related to the insight they give in our interpretation of the quantum world. Recently, due to the new quantum information applications, the current understanding of quantum correlations has significantly changed. Nowadays it is important to detect when the
correlations in a quantum state are useful for quantum information tasks, i.e. they are distillable. In this work we have shown that Bell inequalities can also be useful for this role. Indeed, since present knowledge on multi-particle entanglement is far from being exhaustive, they provide a powerful tool for understanding distillability properties of N-qubit states.

What does a multi-particle Bell violation exactly mean from the point of view of non-locality? It is clear that Bell violation implies some form of non-locality if we want to describe the correlations within classical probability theory (i.e. within a hidden variable model) how? However it has been stressed recently that even larger violations of an N-party Bell inequality may be not sufficient for truly N-party non-locality. As an example, consider a three-qubit state saturating the maximal violation of the MBK inequality \(M_3\). An hybrid non-local model where two of the parties have non-local correlations and the third is separated can reach the same value. This leads to the search for inequalities that detect full N-party non-locality. It turns out that these inequalities are equal to the usual MBK one for the case of an even number of particles, and to the so-called Svetlichny inequality (that is, with the notation of this article, \(U_{N,\pi/4}\)) for odd \(N\) (see [30] for details). From our results it follows that in both cases, full non-locality implies full distillability.

Full distillability seems to be a quite strong entanglement criterion for multi-particle states. Indeed the set of local operations assisted with classical communication becomes less powerful when the number of parties increases. From the point of view of Bell inequalities, a quite large violation is needed for N-party distillability. Taking into account that the range of Bell violations is quite broad for large \(N\), it is likely that to demand N-party distillability is much more stringent than to demand Bell violation. It is interesting to analyze the result given in [22] in view of the ideas presented in this article. In [23] Duré constructs a multi-qubit state violating the MBK inequality that has all local partial transpositions positive, which means that the parties acting alone cannot distill entanglement at all. His result fits quite well into our picture: a strong violation is never reached by this state, since this would imply \(N\)-partite distillability.

In conclusion, in this paper we have shown the existence of a link between Bell violation and state distillability in N-qubit systems: one can estimate the degree of distillability of an N-qubit state from the amount of its Bell violation (or non-locality). In this way Bell inequalities provide information about the usefulness of the state for quantum information applications. Indeed, a strong Bell violation is sufficient for the security of multi-particle QKD protocols.

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**APPENDIX A**

The spectral decomposition of any Bell operator in the WWZB family is of the form (3)\(^{1}\). The \(b_k\) are non-negative and we suppose \(b_0\) to be the largest eigenvalue, since we can always relabel the local bases in such a way that the largest eigenvalue corresponds to \(Q_0^+\). Moreover by a local unitary transformation we absorb the phase \(\theta_0\), so \(Q_0^+ = P_0^+\) is the projector onto \(|\text{GHZ}_N\rangle\). If the Bell operator is normalized so that the LV is set to 1, then the eigenvalues satisfy the constraint given in Eq. (25) of \([14]\):

\[
2^{N-1-1} \sum_{k=0}^{N-1-1} b_k^2 \leq 2^{N-1} \tag{32}
\]

with equality for the MBK operators. For each N-qubit state \(\rho_N\), we have

\[
\text{tr}(\rho_N B_N) = \sum_{k=0}^{2^{N-1}-1} b_k (\lambda_k^+ - \lambda_k^-) \tag{33}
\]

with \(\lambda_k^\pm\) given in (3). We suppose that \(\lambda_0^+ = r = \langle \text{GHZ}_N | \rho_N | \text{GHZ}_N \rangle\) is the maximum of the \(\lambda_k^\pm\). By keeping only the positive terms, \(\lambda_k \equiv \lambda_k^+\), and normalizing the probabilities so that

\[
\sum_{k=1}^{2^{N-1}-1} \lambda_k = 1 - r. \tag{34}
\]

we obtain the upper bound

\[
\text{tr}(\rho_N B_N) \leq \beta(r) = \sup \left( b_0 r + \sum_{k=1}^{2^{N-1}-1} b_k \lambda_k \right) \tag{35}
\]

where the supremum is taken over the \(b_k\) compatible with (3) and the \(\lambda_k\) satisfying (3). Using Lagrange multipliers, it turns out that for fixed \(b_0\) the extremum is reached when

\[
\lambda_k = \frac{1 - r}{2^{N-1}-1}, \quad b_k = \sqrt{\frac{2^{N-1} - b_0^2}{2^{N-1} - 1}}, \tag{36}
\]

for all \(k = 1, \ldots, 2^{N-1}-1\). Writing \(b_0 = 2^{N-1} \cos \eta\), we obtain for \(\beta(r)\) the expression

\[
\beta(r) = \max_{\eta} \left( r \cos \eta + (1 - r) \frac{\sin \eta}{\sqrt{2^{N-1}-1}} \right) 2^{N-1}. \tag{37}
\]

Using \(\max_{\eta} \left( \cos \eta x + \sin \eta y \right) = \sqrt{x^2 + y^2}\) we get [26].
Just one comment to point out a common mistake when dealing with these Bell operators. We have said that the bound $\text{tr}(\rho_N B) \leq \beta(r)$ is rough in general. One might expect however that the bound is exact for the MBK operator and the state

$$\rho_N(r) = r P_0^+ + \frac{1-r}{2^{N-1}-1} \sum_{k=1}^{2^{N-1}-1} P_k^+.$$  \hspace{1cm} (38)

This guess would be correct if \(\rho_3\), with equality, were the unique constraint on the eigenvalues of the MBK operator; but this is not the case \(\rho_3\). Therefore it may happen that the set of $b_k$ that optimize $\beta(r)$ is not a set of possible eigenvalues. This remark already applies to the case of three qubits: as discussed in the text, $\beta(r) = \sqrt{2}$ for $r = r_3 \approx 0.683$. However numerically one can verify that $\rho_3(r_3)$ does not violate the Mermin inequality; the family of states $\rho_3(r)$ starts to violate the Mermin inequality only at $r \sim 0.687$.

**APPENDIX B**

Here we present a simple protocol for full distillability for three qubits. There is no claim of efficiency, even less of optimality, for such a protocol. Let us consider the basis of GHZ states \(|15\rangle\) for three qubits. As usual, we shall write $|\psi_0^+\rangle = |\text{GHZ}_3\rangle$. One can locally depolarize any state $\rho$ onto a state which is diagonal in the GHZ basis keeping the diagonal terms as in \(|16\rangle\). This is the first step of the distillation protocol:

$$\rho \rightarrow \rho_D = \sum_{k=0}^{2^{N-1}-1} \sum_{\sigma=\pm} \mu_k^\sigma P_k^\sigma$$  \hspace{1cm} (39)

With further local operations, we can arrange (just for definiteness) that $\mu_0^+$ is the maximum of the $\mu_k^\sigma$.

In the second step, one of the three parties, say Charlie, measures his own qubit in the $\sigma_x$ basis and communicates the result $s_x^C = \pm 1$ to the other two parties Alice and Bob. This way, Alice and Bob share several copies of each of the two two-qubit conditional states $\rho_{AB}(s_x^C = \pm 1)$. The idea now is rather trivial: if at least one of these states is entangled, then Alice and Bob can distill a singlet out of many copies of it. Without loss of generality, we can concentrate on $\rho_{AB} = \rho_{AB}(s_x^C = +1)$. In the computational basis, this state reads

$$\rho_{AB} = \begin{pmatrix} a & c \\ \frac{1}{2} - a & \frac{1}{2} - c \\ c & d & a \end{pmatrix}$$  \hspace{1cm} (40)

with

$$a = \frac{1}{2} \sum_{\sigma} (\mu_0^\sigma + \mu_3^\sigma),$$

$$b = \frac{1}{2} \sum_{\sigma} (\mu_0^\sigma - \mu_3^\sigma),$$

$$c = \frac{1}{2} \sum_{\sigma} (\mu_0^\sigma + \mu_3^\sigma),$$

$$d = \frac{1}{2} \sum_{\sigma} (\mu_1^\sigma + \mu_2^\sigma).$$  \hspace{1cm} (41)

A necessary and sufficient condition for distillability is that $\langle \rho_{AB} \rangle_{T,A}$ has at least a negative eigenvalue. In the present case, due the form of the matrix, this is a very simple condition to write. Without loss of generality, we can suppose that the negative eigenvalue is in the block

$$M = \begin{pmatrix} \frac{1}{2} - a & c \\ c & \frac{1}{2} - a \end{pmatrix}.$$  \hspace{1cm} (42)

This block must have a positive eigenvalue since the trace is non-negative, so the necessary and sufficient condition for distillability is simply $\det M < 0$. After some algebra, defining $p^+ = \mu_0^+ + \mu_3^+$, this condition reads explicitly

$$p^+ + p^- - 2p^+p^- > \frac{1}{2}.$$  \hspace{1cm} (43)

We don’t need to study the domain of validity of this condition in very sharp detail. Actually, for our purpose we simply have to notice that if $\mu_0^+ > 1/2$ then \(43\) is satisfied. But $\mu_0^+ = \langle \text{GHZ}_3 | \rho | \text{GHZ}_3 \rangle$, thus whenever

$$\langle \text{GHZ}_3 | \rho | \text{GHZ}_3 \rangle > \frac{1}{2}.$$  \hspace{1cm} (44)

Alice and Bob can distill a singlet if they collaborate with Charlie. But \(44\) is actually symmetric in the roles of the three parties; then if \(44\) holds, any pair of parties can distill a singlet. This is sufficient for full distillability.

**APPENDIX C**

We present here the detailed proof of the fact that the violation of the three-qubit Uffink inequality implies full distillability. This proof uses tools from the spectral decomposition of WWZB Bell operators. We begin by applying the results of Ref. \(4\) to the three-qubit operators \(M_3\) and \(U_{3,\gamma}\).

**A. Spectral decomposition of \(M_3\)**

The settings \{$\hat{n}_i, \hat{n}'_i\}$ that define the Bell operator are supposed to lie in the \((x, y)\) plane for each qubit $i = 1, 2, 3$ — this can always be achieved by local unitary operations on the state. Defining

$$\hat{n}_i = \cos \alpha_i \hat{x} + \sin \alpha_i \hat{y}$$  \hspace{1cm} (45)

and a similar definition for $\hat{n}'_i$, the settings are parametrized by the angles $\alpha = \{\alpha_i, \alpha'_i\}$. We shall also define $\delta_i = \alpha_i - \alpha'_i$. 

10
For all three qubits, let $|0\rangle$ and $|1\rangle$ be the two eigenvectors of $\sigma_z$. To write the operators in a compact way, for all $k \in \{0,1,2,3\}$, let $\hat{\sigma}(k)$ be the operator acting as the Pauli matrices in the subspace spanned by $|+\rangle = |02_k3_k\rangle$ and $|-\rangle = |12_k3_k\rangle$, where $(k_2,k_3)$ is the binary expression of $k$ and $\hat{k}_j = 1 - k_j$ for $j = 2,3$. It follows from (46) that

$$M_3 = \bigoplus_k b_k \hat{n}_k \cdot \hat{\sigma}(k)$$

and similarly for $M'_3$. The expressions (46) and (47) imply in particular the fact, not stressed explicitly in (44), that $M_3$ and $M'_3$ have the same eigenvalues. The way of obtaining the eigenvalues $b_k$ and the parameters $\theta_k$ from the settings $\alpha$ has been discussed in [16]: one has

$$b_k e^{i\theta_k} = f_k(\alpha) = e^{i(\alpha_1 + \beta_2 + \beta_3)} + e^{i(\alpha_1 + \beta_2 + \beta_3)} + e^{i(\alpha_1 + \beta_2 + \beta_3)} - e^{i(\alpha_1' + \beta_2' + \beta_3')}$$

with $\beta_j = k_j \alpha_j$. The explicit form of $b_k$ is not very elegant, but will be needed in what follows:

$$b_k = \left[ \left( \prod_i \cos \delta_i \right)^2 + \left( \prod_i k_i \sin \delta_i + \sum_i k_i \sin \delta_i \right)^2 \right]^{1/2}$$

where $i = 1,2,3$ and $k_1 = 1$. 

B. The Uffink operator $U_{3,\gamma}$

The Uffink operator $U_{3,\gamma}$ reads

$$U_{3,\gamma} = \cos \gamma M_3 + \sin \gamma M'_3 = \bigoplus_k u_k \hat{m}_k \cdot \hat{\sigma}(k)$$

where $\hat{m}_k$ is the unit vector along the direction $\cos \gamma \hat{n}_k + \sin \gamma \hat{n}_k'$ and where the eigenvalues are given by

$$u_k = b_k \sqrt{1 + \sin 2\gamma (\hat{n}_k \cdot \hat{n}_k')}.$$ 

Note that $(\hat{n}_k \cdot \hat{n}_k') = \cos(\theta_k - \theta'_k)$. Now, it follows from (52) that

$$m_k(\alpha) = f_k(\alpha) f_k(\alpha')^* = |b_k|^2 e^{i(\theta_k - \theta'_k)}$$

with $\alpha'$ meaning exchanging the settings $\alpha_3 \leftrightarrow \alpha'_3$. By comparison with (53) we find

$$u_k^2 = |m_k(\alpha)|^2 + \sin 2\gamma \Re(m_k(\alpha))$$

$$= b_k^2 + \sin 2\gamma \prod_i \cos \delta_i.$$ 

Here we can apply the same estimate as in Appendix B. Note that for the Uffink operator the bound (52) does not hold in general, since the LV limit is not set to 1. But we have just to replace that constraint by

$$\sum_k u_k^2 = 4 \left( 1 + \sin 2\gamma \cos \delta_1 \cos \delta_2 \cos \delta_3 \right)$$

since for the MBK operators $M_3$ it holds $\sum_k b_k^2 = 4$. Apart from that, the calculation with the Lagrange multipliers is the same. Without loss of generality, we can take $u_0$ to be the highest eigenvalue of $U_{3,\gamma}$, and with local unitary operations we can choose $|GHZ_3\rangle = (|000\rangle + |111\rangle)\sqrt{2}$ as the associated eigenvector. Writing $\hat{\delta} = \{\delta_1, \delta_2, \delta_3\}$, we obtain

$$\beta_\gamma(r) = \sup_{\hat{\delta}} \left[ u_0(\hat{\delta}) r + \bar{u}(\hat{\delta})(1 - r) \right]$$

with $r = \langle GHZ_3 | \rho | GHZ_3 \rangle$ and

$$\bar{u}(\hat{\delta}) = \left( \sum_k u_k^2 - u_0^2 \right) \frac{1}{3}.$$ 

The optimization over $\hat{\delta}$ can be done numerically. One finds there are some settings $\hat{\delta}$ and some values $\gamma$ for which one can get $\beta_\gamma(r) > \sqrt{2}$ only if $r \geq r_U \sim 0.628$. In conclusion: a necessary condition to violate the Uffink inequality for three qubits is that

$$\langle GHZ_3 | \rho | GHZ_3 \rangle \geq 0.628.$$ 

Thus in particular (14) is fulfilled, and we can distill singlets between any two parties using the protocol described in Appendix C.

Finally, we want to mention that in this case

$$\rho_3(r) = r P^+_0 + \frac{1 - r}{3} \sum_{k \neq 0} P^+_k$$

gives $U_3(\rho_3(r)) > \sqrt{2}$ for $r \geq 0.628$. Comparing with the remark that concludes Appendix A, we see that the same state gives $\text{tr}(\rho_3(r) M_3) > \sqrt{2}$ only for $r > 0.687$. This is another manifestation of the strength of the Uffink inequality.

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