DERIVATIVES FOR THE INTERSECTION LOCAL TIME OF FRACTIONAL BROWNIAN MOTIONS*

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Abstract. Let $B^{H_i} = \{B^{H_i}_t, \, t \geq 0\}, \ i = 1, 2$ be two independent fractional Brownian motions on $\mathbb{R}$ with respective indices $H_i \in (0, 1)$ and $H_1 \leq H_2$. In this paper, we consider their intersection local time $\ell_t(a)$. We show that $\ell_t(a)$ is differentiable in the spatial variable and we introduce the so-called generalized hybrid quadratic covariation $[f(B^{H_1} - B^{H_2}), B^{H_1}]^{(GH)}$. When $H_1 < \frac{1}{2}$, we construct a Banach space $\mathcal{H}$ of measurable functions such that the quadratic covariation exists in $L^2(\Omega)$ for all $f \in \mathcal{H}$, and the Bouleau-Yor type identity

$$[f(B^{H_1} - B^{H_2}), B^{H_1}]^{(GH)} = -\int_{\mathbb{R}} f(a) \ell_t'(a) da$$

holds. When $H_1 \geq \frac{1}{2}$ and $H_2 < \frac{2}{3}$, we show that the quadratic covariation exists in $L^2(\Omega)$ and the above Bouleau-Yor type identity holds for all Hölder functions $f$ of order $\nu > 2H_1 - 1$.

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1. Introduction

In the study of stochastic area integrals for Brownian motion $B$, Rogers-Walsh [27, 28] were led to analyze the following functional

$$A(t, x) = \int_0^t 1_{[0, \infty]}(x - B_s) ds, \quad t \geq 0, \ x \in \mathbb{R}.$$
By using the classical Itô calculus they showed that the process \( \{A(t, B_t), t \geq 0\} \) is not a semi-martingale. In fact, they showed that the process

\[
(1.1) \quad A(t, B_t) - \int_0^t \mathcal{L}(s, B_s)dB_s
\]

has finite non-zero \(4/3\)-variation, where \( \mathcal{L}(t, x) = \int_0^t \delta(B_s - x)ds \) is the local time. By a formal application of Itô’s formula with respect to the Brownian motion and using

\[
\frac{d}{dx}1_{\{x \geq 0\}} = \delta(x), \quad \frac{d^2}{dx^2}1_{\{x \geq 0\}} = \delta'(x)
\]

in the sense of Schwartz’s distribution, Rosen [29] developed a new approach to the study of \( A(t, B_t) \) as follows:

\[
A(t, B_t) - \int_0^t \mathcal{L}(s, B_s)dB_s = t + \frac{1}{2} \int_0^t \int_0^s \delta'(B_s - B_r)drds
\]

for all \( t > r \geq 0 \), and one can consider the process

\[
\alpha'_t(a) := -\int_0^t \int_0^s \delta'(B_s - B_r - a)drds, \quad t \geq 0, a \in \mathbb{R}
\]

which are called the derivatives of self-intersection local time (in short, DSLT) of Brownian motion. By using the idea, Yan et al. [36] deduced the existence of process

\[
\beta'_t(a) := -\int_0^t \int_0^s s^{2H-1}ds \int_0^s \delta'(B_s^H - B_r^H - a)dr, \quad t \geq 0, a \in \mathbb{R},
\]

which are called the DSLT of fractional Brownian motion (fBm) \( B^H \). Moreover, Jung-Markowsky [20, 21] considered some in-depth results for \( \beta'_t(a) \). As an extension it is natural to consider the derivatives of the intersection local time (DILT) of fBms.

A standard fBm with Hurst index \( H \in (0, 1) \) is a centered Gaussian process \( B^H = \{B^H_t, 0 \leq t \leq T\} \) such that \( B^H_0 = 0 \) and

\[
E [B^H_t B^H_s] = \frac{1}{2} [t^{2H} + s^{2H} - |t - s|^{2H}]
\]

for \( t, s \geq 0 \). For \( H = 1/2 \), \( B^H \) coincides with the standard Brownian motion \( B \). Let now \( B^{H_1} = \{B^{H_1}_t, t \geq 0\}, i = 1, 2 \) be two independent fBms with respective indices \( H_i \in (0, 1) \) on \( \mathbb{R} \) and \( H_1 \leq H_2 \). The so-called intersection local time of \( B^{H_1} \) and \( B^{H_2} \), denoted by \( \ell_t(a) \), is formally defined by

\[
\ell_t(a) = \int_0^t s^{2H_2-1}ds \int_0^s \delta(B_r^{H_1} - B_s^{H_2} - a)dr, \quad t \geq 0, a \in \mathbb{R},
\]

and the DILT \( \ell'_t(a) \) is defined by

\[
\ell'_t(a) := -\lim_{\varepsilon \to 0} \int_0^t s^{2H_2-1}ds \int_0^s p_\varepsilon'(B_r^{H_1} - B_s^{H_2} - a)dr,
\]

provided the limit exists in \( L^2(\Omega) \), where

\[
p_\varepsilon(x) = \frac{1}{\sqrt{2\pi \varepsilon}}e^{-\frac{x^2}{2\varepsilon}} \equiv \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi}e^{-\frac{1}{2}\varepsilon\xi^2}d\xi
\]

with \( \varepsilon > 0 \). In fact, it seems interesting to study the DILT when we consider the functional

\[
A^{H_1}(t, x) := \int_0^t 1_{[0, \infty)}(x - B^{H_1}_s)ds = \int_{-\infty}^x \mathcal{L}^{H_1}(t, y)dy, \quad t \geq 0, \ x \in \mathbb{R}
\]
and process $\{A^{H_1}(t, B_t), t \geq 0\}$, where $B$ is a standard Brownian motion, fBm $B^{H_1}$ is independent of $B$ and $\mathcal{L}^{H_1}(x, t) = \int_0^t \delta(B^{H_1}_s - x) ds$ is the local time of fBm $B^{H_1}$. It is important to note (see Table 2 in Geman-Horowitz [14]) the local time $\mathcal{L}^{H_1}(x, t)$ is absolutely continuous in $x$ if $H_1 < \frac{1}{2}$ and it is continuously differentiable in $x$ if $H_1 < \frac{3}{4}$. Thus, it is natural to ask whether the process $\{A^{H_1}(t, B_t), t \geq 0\}$ is a $(\mathcal{F}_t^{H_1})$-semimartingale or not, where $(\mathcal{F}_t^{H_1})$ is the filtration generated by $B$ and $B^{H_1}$. By Itô’s formula we have formally

$$A^{H_1}(t, B_t) = \int_0^t \mathcal{L}^{H_1}(s, B_s) dB_s + \int_0^t 1_{[0, \infty)}(B_s - B_s^{H_1}) ds - \frac{1}{2} \int_0^t \int_0^s \delta'(B_r^{H_1} - B_s) dr ds.$$ 

Therefore, one can consider the process

$$- \int_0^t \int_0^s \delta'(B_r^{H_1} - B_s) dr ds, \quad t \geq 0,$$

and more generally one can consider the process

$$- \int_0^t s^{2H_2-1} ds \int_0^s \delta'(B_r^{H_1} - B_s^{H_2} - a) dr, \quad t \geq 0$$

with $a \in \mathbb{R}$. In this paper we only consider the DILT $\ell'_t(a)$ on $\mathbb{R}^1$ and in Yan-Yu [37] we will consider the higher derivatives derivatives of the intersection local time of fBms on $\mathbb{R}^d$.

This paper is organized as follows. In Section 2 we present some preliminaries for fBm. In Section 3 we prove the existence of the DILT $\ell'_t(0)$ and consider the process

$$X_t := A(t, B^{H_2}) - \int_0^t \mathcal{L}^{H_1}(s, B^{H_2}) dB^{H_1}_s, \quad t \geq 0$$

with $H_1 \leq H_2$, where the integral is the Skorohod integral. In particular, we show that the process $X$ has zero $p$-variation if $p > \frac{2}{2-H_2}$, provided $H_1 < \frac{1}{2}$. In Section 4 we prove the joint Hölder continuity of the DILT $\ell'_t(a)$. As a corollary we have

$$\ell'_t(a) = \frac{d}{da} \ell_t(a),$$

provided either $\min\{H_1, H_2\} < \frac{1}{2}$ or $\max\{H_1, H_2\} < \frac{2}{3}$, and moreover, the occupation formula

$$\int_0^t s^{2H_2-1} ds \int_0^s f'(B_r^{H_1} - B_s^{H_2}) dr = - \int_\mathbb{R} f(a) \ell'_t(a) da$$

holds for any $f \in C^1(\mathbb{R})$ and $t \in [0, T]$. In Section 5 and 6 we study the so-called generalized hybrid quadratic covariation defined as follows.

**Definition 1.1.** Let $0 < H_1, H_2 < 1$ and let $f$ be a Borel function on $\mathbb{R}$. The limit

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^{2H_2}} \int_0^t s^{2H_2-1} ds \int_0^s \{\Delta_\varepsilon f(B_r^{H_1} - B_s^{H_2})\} (B_{r+\varepsilon}^{H_1} - B_r^{H_1}) dr$$

is called the generalized hybrid quadratic covariation (in short, GHQC), provided the limit exists in $L^1(\Omega)$, denoted by $[f(B^{H_1} - B^{H_2})]_t^{GH}$, where

$$\Delta_\varepsilon f(B_r^{H_1} - B_s^{H_2}) := f(B_{r+\varepsilon}^{H_1} - B_s^{H_2}) - f(B_r^{H_1} - B_s^{H_2})$$

for all $\varepsilon > 0$. 

When $0 < H_1 < \frac{1}{2}$, the GHQC is considered in Section 5. By considering the decomposition

$$\frac{1}{\varepsilon^{2H_1}} \int_0^t s^{2H_1 - 1} ds \int_0^s \{ \Delta_{\varepsilon} f(B^{H_1}_r - B^{H_2}_r) \} \left( B^{H_1}_{r+\varepsilon} - B^{H_1}_r \right) dr$$

(1.2) \[ = \frac{1}{\varepsilon^{2H_1}} \int_0^t s^{2H_1 - 1} ds \int_0^s f(B^{H_1}_r - B^{H_2}_r) \left( B^{H_1}_{r+\varepsilon} - B^{H_1}_r \right) dr \]

for all $\varepsilon > 0$ and estimating the two right terms, respectively, we construct a Banach space $\mathcal{H}$ of measurable functions such that the GHQC exists in $L^2(\Omega)$ for all $f \in \mathcal{H}$. In particular, when $f \in C^1(\mathbb{R})$ we have

$$[f(B^{H_1}_r - B^{H_2}_r), B^{H_1}_r]^{(GH)} = \int_0^t s^{2H_1 - 1} ds \int_0^s f'(B^{H_1}_r - B^{H_2}_r) dr$$

for all $0 < H_1 < 1$. Moreover, for all $f \in \mathcal{H}$ and $0 < H_1 < \frac{1}{2}$ we show that the integral

$$\int_{\mathbb{R}} f(a) \ell_t(a) da$$

is well-defined and the following Bouleau-Yor type identity holds:

$$[f(B^{H_1}_r - B^{H_2}_r), B^{H_1}_r]^{(GH)} = - \int_{\mathbb{R}} f(a) \ell_t(a) da \equiv - \int_{\mathbb{R}} f(a) \ell_t(da).$$

When $H_1 \geq \frac{1}{2}$, the GHQC is considered in Section 6. It is clear that the decomposition (1.2) is inefficacy, and we need a new idea for $H_1 > \frac{1}{2}$. In fact, for $f(x) = x$ we have

$$\frac{1}{\varepsilon^{2H_1}} \int_0^t s^{2H_1 - 1} ds \int_0^s E \left[ (B^{H_1}_r - B^{H_2}_r)(B^{H_1}_{r+\varepsilon} - B^{H_1}_r) \right] dr$$

(1.3) \[ = \frac{1}{\varepsilon^{2H_1}} \int_0^t s^{2H_2 - 1} ds \int_0^s E \left[ B^{H_1}(B^{H_1}_{r+\varepsilon} - B^{H_1}_r) \right] dr \]

$$\to \infty,$$

as $\varepsilon \downarrow 0$. Thus, when $\frac{1}{2} \leq H_1 \leq H_2 < \frac{2}{3}$, by estimating integrally the express

$$\frac{1}{\varepsilon^{2H_1}} \int_0^t s^{2H_2 - 1} ds \int_0^s \{ \Delta_{\varepsilon} f(B^{H_1}_r - B^{H_2}_r) \} \left( B^{H_1}_{r+\varepsilon} - B^{H_1}_r \right) dr,$$

and by using the existence of the Young integral

$$\int_{\mathbb{R}} f(a) \ell_t(da),$$

we show that the GHQC exists in $L^2(\Omega)$ and the Bouleau-Yor type identity (1.3) holds for all Hölder functions $f$ of order $\nu > 2H_1 - 1$. In Section 7 we give a necessary and sufficient condition such that the DILT $\ell_t(0)$ is smooth in the sense of Meyer-Watanabe. In Appendix A we give some basic estimates.

2. Fractional Brownian motion

In this section, we briefly recall some basic results of fBm with $0 < H < 1$. For more aspects on the material we refer to E. Alós et al [1], Biagini et al [2], Cheridito-Nualart [4], Decreusefond-Üstünel [6], Gradinaru et al [15], Hu [18], Mishura [22], Nourdin [24], Nualart [25] and references therein. For simplicity we let $C$ stand for a positive constant depending only on the subscripts.
and its value may be different in different appearance, and this assumption is also adaptable to $c$.

As we pointed out before, a zero mean Gaussian process $B^H = \{B^H_t, 0 \leq t \leq T\}$ defined on $(\Omega, \mathcal{F}^H, P)$ is called the fBm with Hurst index $H \in (0, 1)$ if $B^H_0 = 0$ and

$$E \left[ B^H_t B^H_s \right] = \frac{1}{2} \left[ t^{2H} + s^{2H} - |t - s|^{2H} \right]$$

for $t, s \geq 0$. FBm $B^H$ admits the integral representation of the form

$$B^H_t = \int_0^t K_H(t, s) dB^H_s, \quad 0 \leq t \leq T,$$

where $B$ is a standard Brownian motion and the kernel $K_H(t, u)$ satisfies

$$\frac{\partial K_H}{\partial t}(t, s) = \kappa_H \left( H - \frac{1}{2} \right) \left( \frac{s}{t} \right)^{-H} (t - s)^{H - \frac{3}{2}}$$

with a normalizing constant $\kappa_H > 0$. Let $\mathcal{H}$ be the completion of the linear space $\mathcal{E}$ generated by the indicator functions $1_{[0, t], t \in [0, T]}$ with respect to the inner product

$$\langle 1_{[0, t]}, 1_{[0, t]} \rangle_{\mathcal{H}} = \frac{1}{2} \left[ t^{2H} + s^{2H} - |t - s|^{2H} \right].$$

The application $\varphi \in \mathcal{E} \to B^H(\varphi)$ is an isometry from $\mathcal{E}$ to the Gaussian space generated by $B^H$ and it can be extended to $\mathcal{H}$. Denote by $\mathcal{S}$ the set of smooth functionals of the form

$$F = f(B^H(\varphi_1), B^H(\varphi_2), \ldots, B^H(\varphi_n)),$$

where $f \in C^\infty_b(\mathbb{R}^n)$ ($f$ and all its derivatives are bounded) and $\varphi_i \in \mathcal{H}$. The derivative operator $D^H$ (the Malliavin derivative) of a functional $F$ of the form above is defined as

$$D^H F = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(B^H(\varphi_1), B^H(\varphi_2), \ldots, B^H(\varphi_n)) \varphi_j.$$

The derivative operator $D^H$ is then a closable operator from $L^2(\Omega)$ into $L^2(\Omega; \mathcal{H})$. We denote by $\mathbb{D}^{1,2}$ the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,2} := \sqrt{E|F|^2 + E\|D^H F\|_{\mathcal{H}}^2}.$$  

The divergence integral $\delta^H$ is the adjoint of derivative operator $D^H$. That is, we say that a random variable $u$ in $L^2(\Omega; \mathcal{H})$ belongs to the domain of the divergence operator $\delta^H$, denoted by $\text{Dom}(\delta^H)$, if

$$E \left| \langle D^H F, u \rangle_{\mathcal{H}} \right| \leq c \|F\|_{L^2(\Omega)}$$

for every $F \in \mathcal{S}$. In this case $\delta^H(u)$ is defined by the duality relationship

$$E \left[ F \delta^H(u) \right] = E \langle D^H F, u \rangle_{\mathcal{H}}$$

for any $u \in \mathbb{D}^{1,2}$. We have $\mathbb{D}^{1,2} \subset \text{Dom}(\delta^H)$. We will use the notation

$$\delta^H(u) = \int_0^T u_s dB^H_s.$$
to express the Skorohod integral of a process $u$, and the indefinite Skorohod integral is defined as $\int_0^t u_s dB^H_s = \delta^H(u1_{[0,t]})$. We have the following Itô type formula:

$$f(t, B^H_t) = f(0, 0) + \int_0^t \frac{\partial}{\partial x} f(s, B^H_s) dB^H_s$$

$$+ \int_0^t \frac{\partial}{\partial s} f(s, B^H_s) ds + H \int_0^t \frac{\partial^2}{\partial x^2} f(s, B^H_s) s^{2H-1} ds$$

for any $f \in C^{1,2}(\mathbb{R} \times \mathbb{R})$.

Recall that fBm $B^H$ has a local time $\mathcal{L}^H(x, t)$ continuous in $(x, t) \in \mathbb{R} \times [0, \infty)$ which satisfies the occupation formula (see Geman-Horowitz [14])

$$\int_0^t \Phi(B^H_s) ds = \int_\mathbb{R} \Phi(x) \mathcal{L}^H(x, t) dx$$

for every nonnegative bounded function $\Phi$ on $\mathbb{R}$, and such that

$$\mathcal{L}^H(x, t) = \int_0^t \delta(B^H_s - x) ds = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \lambda(s \in [0, t], |B^H_s - x| < \epsilon),$$

where $\lambda$ denotes Lebesgue measure and $\delta(x)$ is the Dirac delta function. Moreover, $\mathcal{L}^H(x, t)$ is absolutely continuous in $x$ if $H < \frac{1}{2}$, it is continuously differentiable in $x$ if $H < \frac{1}{4}$, and its smoothness in the space variable increases when $H$ decreases. For these, see Table 2 in Geman-Horowitz [14].

### 3. Existence of the DILT of fBMs

In this section we will consider the existence of the DILT of fBms. Our main aim is to find the exact result of the existence. Let $B^{H_1} = \{B^H_{t_i}, t \geq 0\}$, $i = 1, 2$ be two independent fractional Brownian motions with respective indices $H_i \in (0, 1)$. The so-called intersection local time, denoted by $\ell_i(a)$, is formally defined by

$$\ell_i(a) = \int_0^t s^{2H_2 - 1} ds \int_0^s \delta(B^H_{r_1} - B^H_{s_2} - a) dr, \quad t \geq 0, \quad a \in \mathbb{R},$$

where $\delta$ denotes the Dirac delta function. Nualart and Ortiz-Latorre [26] has showed the random variable exists in $L^2$ (see also, Chen-Yan [5], Jiang-Wang [19] and Wu-Xiao [33]). We approximate the Dirac delta function by the heat kernel

$$p_\varepsilon(x) = \frac{1}{\sqrt{2\pi\varepsilon}} e^{-x^2/2\varepsilon} = \frac{1}{2\pi} \int_\mathbb{R} e^{i\xi x} e^{-\xi^2/2\varepsilon} d\xi,$$

with $\varepsilon > 0$. Then $\ell_i(a) := \lim_{\varepsilon \to 0} \ell_{\varepsilon, i}(a)$ exists in $L^2(\Omega)$, where

$$\ell_{\varepsilon, i}(a) = \int_0^t s^{2H_2 - 1} ds \int_0^s p_\varepsilon(B^H_{r_1} - B^H_{s_2} - a) dr.$$

Denote

$$\ell'_{\varepsilon, i}(a) = - \int_0^t s^{2H_2 - 1} ds \int_0^s p_\varepsilon'(B^H_{r_1} - B^H_{s_2} - a) dr$$

$$= -\frac{i}{2\pi} \int_0^t s^{2H_2 - 1} ds \int_0^s dr \int_\mathbb{R} \xi e^{i\xi(B^H_{r_1} - B^H_{s_2} - a)} e^{-\xi^2/2\varepsilon} d\xi.$$

We define the process

$$\ell'_i(a) := \lim_{\varepsilon \to 0} \ell'_{\varepsilon, i}(a),$$

for any $a \in \mathbb{R}$, and such that
provided the limits exist in $L^2(\Omega)$, which is called the derivatives of the intersection local time (DILT) of fractional Brownian motions.

**Theorem 3.1.** For every $t > 0$, $\ell'_{\varepsilon,t}(0)$ converges in $L^2(\Omega)$, as $\varepsilon \downarrow 0$ if $0 < H_1 < \frac{1}{3\alpha}$ and $0 < H_2 < \frac{1}{3(1-\alpha)}$ for all $\alpha \in [0,1]$.

**Corollary 3.1.** If $\frac{1}{H_1} + \frac{1}{H_2} > 3$, $\ell'_{\varepsilon,t}(0)$ converges in $L^2(\Omega)$, as $\varepsilon \downarrow 0$ for every $t > 0$. In particular, if either $H_1 \wedge H_2 < \frac{1}{2}$ or $H_1 \vee H_2 < \frac{2}{3}$, $\ell'_{\varepsilon,t}(0)$ converges in $L^2(\Omega)$, as $\varepsilon \downarrow 0$ for every $t > 0$.

To prove the theorem we need some notations and preliminaries. For $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. For simplicity throughout this paper we assume that the notation $F \times G$ means that there are positive constants $c_1$ and $c_2$ such that

$$c_1 G(x) \leq F(x) \leq c_2 G(x)$$

in the common domain of definition for $F$ and $G$. Define the functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}_+$, $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ as follows:

$$f_i(x) := 4x^{2H_i} - (1 + x^{2H_i} - (1 - x)^{2H_i})^2, \quad i = 1, 2,$$

$$g(x, y) := 4\left(x^{2H_1} + y^{2H_2}\right) - 2\left(1 + x^{2H_1} - (1 - x)^{2H_1}\right) \left(1 + y^{2H_2} - (1 - y)^{2H_2}\right).$$

For these functions we have the following estimates which are proved in Appendix A.

**Lemma 3.1.** Let $f_i$ be defined as above and let $0 < H_i < 1$, $i = 1, 2$. Then we have

$$f_i(x) \asymp x^{2H_i} (1 - x)^{2H_i}, \quad i = 1, 2$$

for all $x \in [0,1]$.

**Lemma 3.2.** Let $g$ be defined as above and let $0 \leq H_1, H_2 \leq 1$. Then we have

$$g(x, y) \asymp x^{2H_1} (1 - y)^{2H_2} + y^{2H_2} (1 - x)^{2H_1}$$

for all $x, y \in [0,1]$.

**Proof of Theorem 3.1.** Denote $T = \{0 < r < s < t, \ 0 < r' < s' < t\}$ and

$$T_1 = \{0 < r' < r < s' < s < t\},$$

$$T_2 = \{0 < r' < s' < r < s < t\},$$

$$T_3 = \{0 < r < r' < s' < s < t\}$$

for any $t > 0$, and

$$\lambda_{r,s} := \text{Var}(B_{r}^{H_1} - B_{s}^{H_2}) = r^{2H_1} + s^{2H_2}, \quad \mu := E\left[(B_{r}^{H_1} - B_{s}^{H_2})(B_{r'}^{H_1} - B_{s'}^{H_2})\right]$$

for $s > r > 0$. Then we have

$$E\ell'_{\varepsilon,t}(0) = -\frac{i}{2\pi} \int_{0}^{t} s^{2H_2 - 1} ds \int_{0}^{s} dr \int_{\mathbb{R}} \xi E e^{i\xi (B_{r}^{H_1} - B_{s}^{H_2})} e^{-\xi^2/2} d\xi$$

$$= -\frac{i}{2\pi} \int_{0}^{t} s^{2H_2 - 1} ds \int_{0}^{s} dr \int_{\mathbb{R}} \xi e^{-\frac{1}{2} \lambda_{r,s} \varepsilon} \xi^2 d\xi = 0$$
and

$$E \left[ \ell'_{\varepsilon,t}(0)^2 \right] = \frac{-1}{(2\pi)^2} \int_{\mathbb{T}} (ss')^{2H_2-1} drd\eta d\xi = \int_{\mathbb{R}^2} \xi \eta e^{\frac{1}{2}(\xi^2 + \eta^2)} \cdot E \exp \left( i(B_{r'}^H - B_{s'}^H) + i(B_{r'}^H - B_{s'}^H) \right) d\xi d\eta$$

$$= \frac{-1}{(2\pi)^2} \int_{\mathbb{T}} (ss')^{2H_2-1} drd\eta d\xi \cdot \exp \left( -\frac{1}{\mu} \left[ (\lambda_{r,s} + \varepsilon) \xi^2 + 2\mu \eta + \varepsilon - (\lambda_{r',s'} + \varepsilon) \eta^2 \right] \right) d\xi d\eta$$

$$= \frac{1}{2\pi} \int_{\mathbb{T}} \frac{\mu (ss')^{2H_2-1}}{(\lambda_{r,s}\lambda_{r',s'} - \mu^2)^{3/2}} drd\eta d\xi$$

for all $\varepsilon > 0$, which deduce that $\ell'_{\varepsilon,t}(0) \in L^2(\Omega)$ if and only if

$$\int_{\mathbb{T}} \frac{\mu (ss')^{2H_2-1}}{(\lambda_{r,s}\lambda_{r',s'} - \mu^2)^{3/2}} drd\eta d\xi = 2 \sum_{j=1}^{3} \int_{T_j} \frac{\mu (ss')^{2H_2-1}}{(\lambda_{r,s}\lambda_{r',s'} - \mu^2)^{3/2}} drd\eta d\xi < \infty$$

for all $t \geq 0$.

Let us now estimate the expression

$$\lambda_{r,s}\lambda_{r',s'} - \mu^2$$

for $(r, s, r', s') \in \bigcup_{j=1}^{3} T_j$ in the two cases.

**Case I.** For $(r, s, r', s') \in T_1 \cup T_2$, taking $r' = xr$, $s' = ys$ and $0 \leq x, y \leq 1$, we have

$$\lambda_{r',s'} = r^{2H_1}x^{2H_1} + s^{2H_2}y^{2H_2}$$

and

$$\mu = \frac{1}{2} r^{2H_1} (1 + x^{2H_1} - (1 - x)^{2H_1}) + \frac{1}{2} s^{2H_2} (1 + y^{2H_2} - (1 - y)^{2H_2}).$$

Combining these with Lemma 3.1, Lemma 3.2 and Young’s inequality, we get

$$\lambda_{r,s}\lambda_{r',s'} - \mu^2 = \frac{1}{4} \left\{ r^{4H_1} f_1(x) + r^{2H_1} s^{2H_2} g(x,y) + s^{4H_2} f_2(y) \right\}$$

$$\times r^{4H_1} x^{2H_1} (1 - x)^{2H_1} + s^{4H_2} y^{2H_2} (1 - y)^{2H_1}$$

$$+ r^{2H_1} s^{2H_2} (x^{2H_1} - (1 - y)^{2H_2} + y^{2H_2} (1 - x)^{2H_1})$$

$$\times (r^{2H_1} x^{2H_1} + s^{2H_2} y^{2H_2}) (r^{2H_1} (1 - x)^{2H_1} + s^{2H_2} (1 - y)^{2H_2})$$

$$\times ((r')^{2H_1} + (s')^{2H_2}) ((r - r')^{2H_1} + (s - s')^{2H_2})$$

$$\geq (r')^{2\alpha} H_1 (s')^{2(1 - \alpha) H_2} (r - r')^{2\alpha} H_1 (s - s')^{2(1 - \alpha) H_2}$$

for all $0 \leq \alpha \leq 1$, which deduces

$$\int_{T_1} \frac{\mu (ss')^{2H_2-1}}{(\lambda_{r,s}\lambda_{r',s'} - \mu^2)^{3/2}} drd\eta d\xi$$

$$\leq \int_{T_1} \frac{\mu (ss')^{2H_2-1}}{(r')^{3\alpha} H_1 (s')^{3(1 - \alpha) H_2} (r - r')^{3\alpha} H_1 (s - s')^{3(1 - \alpha) H_2} < \infty$$

if $0 < H_1 < \frac{1}{3\alpha}$ and $0 < H_2 < \frac{1}{3(1 - \alpha)}$. 
Case II. For \((r, s, r', s') \in \mathbb{T}_3\), taking \(r = xr'\), \(s' = ys\) and \(0 \leq x, y \leq 1\), we have
\[
\lambda_{r, s} = (r')^{2H_1}x^{2H_1} + s^{2H_2}, \quad \lambda_{r', s'} = (r')^{2H_1} + s^{2H_2}y^{2H_2}
\]
and
\[
\mu = \frac{1}{2}(r')^{2H_1} (1 + x^{2H_1} - (1 - x)^{2H_1}) + \frac{1}{2}s^{2H_2} (1 + y^{2H_2} - (1 - y)^{2H_2}).
\]
It follows from Lemma 3.1, Lemma 3.2 and Young’s inequality that
\[
\lambda_{r, s} \lambda_{r', s'} - \mu^2 = \frac{1}{4} \left\{ (r')^{4H_1} f_1(x) + (r')^{2H_1} s^{2H_2} g(x, y) + s^{4H_2} f_2(y) \right\}
\]
(3.6)
\[
\geq (r')^{2\alpha H_1} (s')^{2(1-\alpha)H_2} (r' - r)^{2\alpha H_1} (s - s')^{2(1-\alpha)H_2}
\]
for all \(0 \leq \alpha \leq 1\), which deduces
\[
\int_{\mathbb{T}_1} \frac{\mu(ss')(2H_2 - 1)}{(\lambda_{r, s} \lambda_{r', s'} - \mu^2)^{3/2}} dr ds dr' ds' < \infty
\]
if \(0 < H_1 < \frac{1}{3\alpha}\) and \(0 < H_2 < \frac{1}{3(1-\alpha)}\).

Finally, we claim that the sequence \(\{\ell_{\varepsilon,t}(0, \varepsilon > 0)\}\) is of Cauchy in \(L^2(\Omega)\). For any \(\delta, \varepsilon > 0\) we have
\[
E(\|\ell_{\varepsilon,t}(0) - \ell_{\delta,t}(0)\|^2) = \frac{1}{4\pi^2} \int_{\mathbb{T}} dr ds dr' ds' \int_{\mathbb{R}^2} \xi \eta E \exp \left\{ i\xi(B^H_r - B^H_s) + i\eta(B^H_{r'} - B^H_{s'}) \right\}
\]
\[
\cdot \left( e^{-\frac{\xi^2}{2\sigma^2}} - e^{-\frac{\delta^2}{2\sigma^2}} \right) \left( e^{-\frac{\eta^2}{2\sigma^2}} - e^{-\frac{\varepsilon^2}{2\sigma^2}} \right) d\xi d\eta dr ds
\]
\[
= \frac{1}{4\pi^2} \int_{\mathbb{T}} dr ds dr' ds' \int_{\mathbb{R}^2} e^{-\frac{1}{2}(\lambda_{r, s} \xi^2 + 2\mu \xi \eta + \lambda_{r', s'} \eta^2)} \left( 1 - e^{-\frac{|\xi - \delta|^2}{2\sigma^2}} \right)
\]
\[
\cdot \left( 1 - e^{-\frac{|\xi - \varepsilon|^2}{2\sigma^2}} \right) e^{-\frac{\lambda_{r, s} \xi^2 \eta^2}{2\sigma^2}} d\xi d\eta.
\]
Thus, dominated convergence theorem yields
\[
E(\|\ell_{\varepsilon,t}(0) - \ell_{\delta,t}(0)\|^2) \rightarrow 0
\]
as \(\varepsilon \rightarrow 0\) and \(\delta \rightarrow 0\), which leads to \(\{\ell_{\varepsilon,t}(0, \varepsilon > 0)\}\) is a Cauchy sequence in \(L^2(\Omega, \mathcal{F}, P)\). Consequently, \(\lim_{\varepsilon \rightarrow 0} \ell_{\varepsilon,t}(0)\) exists in \(L^2(\Omega, \mathcal{F}, P)\). This completes the proof.

At the end of the section, we throughout assume that \(H_1 \leq H_2\) and consider the integral functional
\[
A(t, x) = \int_0^t 1_{[0, \infty)}(x - B^H_s) ds = \int_{-\infty}^x \mathcal{L}^{H_1}(t, y) dy,
\]
where
\[
\mathcal{L}^{H_1}(t, x) = \int_0^t \delta(B^H_s - x) ds
\]
is the local time of fBm \(B^{H_1}\).

Proposition 3.2. (a) The functional \(A(t, x)\) is jointly continuous in \((t, x)\); (b) For fixed \(x\), \(A(t, x)\) is an increasing Lipschitz continuous function of \(t\); (c) For fixed \(t\), \(A(t, x)\) is an increasing \(C^1\)-function of \(x\) with
\[
\frac{\partial}{\partial x} A(t, x) = \mathcal{L}^{H_1}(t, x).
\]
In particular, if $0 < H_1 < \frac{1}{2}$, $\frac{\partial}{\partial x} A(t, x)$ is absolutely continuous in $x$ for every $t \geq 0$, and if $0 < H_1 < \frac{1}{2}$, $A \in C^{1, 2}(\mathbb{R}_+ \times \mathbb{R})$ almost surely and

$$\frac{\partial^2}{\partial x^2} A(t, x) = \int_0^t \delta'(B_s^{H_1} - x)ds.$$  

Thus, the formal applications of Itô’s formula and the fact in the sense of Schwartz’s distribution

$$\frac{d}{dx} 1_{[0, \infty)}(x) = \delta(x), \quad \frac{d^2}{dx^2} 1_{[0, \infty)}(x) = \delta'(x)$$

yield

$$A(t, B_t^{H_2}) = \int_0^t \mathcal{L}^H_1(s, B_s^{H_2})dB_s^{H_2} + \int_0^t 1_{[0, \infty)}(B_s^{H_2} - B_s^{H_1})ds + H_2 \ell'(0).$$  

Recall that for a process $X = \{X_t; 0 \leq t < \infty\}$ and a fixed $T > 0$, we define the $p$-variation $V_p(X; T)$ of $X$ on $[0, T]$ by

$$V_p(X; T) := \lim_{n \to \infty} \sum_{j=0}^{n-1} |X_{t_{j+1}} - X_{t_j}|^p,$$

in $L^1(\Omega)$, where $\tau_n = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ is a partition of $[0, T]$ such that $|\Delta_n| = \max_j |t_j - t_{j-1}| \to 0$.

**Theorem 3.3.** Given $\alpha \in [0, 1]$. If $0 < 3\alpha H_1 < 1$ and $0 < 3(1 - \alpha)H_2 < 1$, we then have

$$V_p(\ell'_t(0); T) = 0$$

for all $t \in [0, T]$, provided $p > \frac{2}{2 - 2\alpha(1 - \alpha)H_2}$.

The above theorem can be proved by using the estimate (4.11) in the next section. To get a exact result here we simply prove it as follows.

Denote $\mathcal{D}_1 = \{0 < r' < r < s' < s, t_1 < s, s' < t_2\}$, $\mathcal{D}_2 = \{0 < r' < s' < r < s, t_1 < s, s' < t_2\}$ and $\mathcal{D}_3 = \{0 < r < r' < s' < s, t_1 < s, s' < t_2\}$. Now, it is enough to obtain the following estimate:

$$\int_{\mathcal{D}_j} \frac{\mu(ss')^{2H_2-1}}{(\lambda_{r,s}\lambda_{r',s'} - \mu^2)^{3/2}} drdsdr'ds' \leq C_{H_1, H_2, T}(t_2 - t_1)^{2-3(1-\alpha)H_2}$$

for all $t_1, t_2 \in [0, T], t_1 < t_2$. To end this, from the proof of Theorem 3.1 $H_1 \leq H_2$ and the estimates

$$\mu \leq \sqrt{(r^{2H_1} + s^{2H_2})((r')^{2H_1} + (s')^{2H_2})} \leq C_T s^{H_1}(s')^{H_1},$$

$$(r')^{2H_1} + (s')^{2H_2} \geq (r')^{\frac{2}{3}H_1}(s')^{\frac{4}{3}H_2}$$

for all $(r, s, r', s') \in \mathcal{D}_1$, we have

$$\int_{\mathcal{D}_1} \frac{\mu(ss')^{2H_2-1}}{(\lambda_{r,s}\lambda_{r',s'} - \mu^2)^{3/2}} drdsdr'ds'$$

$$\times \int_{\mathcal{D}_1} \frac{\mu(ss')^{2H_2-1}}{s^{2H_2+H_1-1}(s')^{H_1-1}} drdsdr'ds'$$

$$\leq C_T \int_{\mathcal{D}_1} (r')^{H_1}(r - r')^{3\alpha H_1}(s - s')^{3(1-\alpha)H_2}$$

$$\leq C_{H_1, H_2, T}(t_2 - t_1)^{2-3(1-\alpha)H_2}$$
for all $\alpha \in (0,1)$. Similarly, we also have
\[
\int_{D_j} \frac{\mu(ss')^{2H_2-1}}{(\lambda_r, s\lambda_{r', s'} - \mu^2)^{3/2}} dr ds' ds' \leq C_{H_1, H_2, T} (t_2 - t_1)^{2-3(1-\alpha)H_2}, \quad j = 2, 3.
\]

**Corollary 3.2.** Given $T > 0$.

1. If $0 < H_1 < \frac{1}{3}$ and $p > 1$, then we have $V_p(\ell'_t(0); T) = 0$ for all $t \in [0, T]$;
2. If $\frac{1}{3} \leq H_1 < \frac{1}{2}$ and $p > \frac{2}{2-H_2}$, then we have $V_p(\ell'_t(0); T) = 0$ for all $t \in [0, T]$;
3. If $0 < H_2 < \frac{1}{3}$ and $p > \frac{2}{2-H_2}$, then we have $V_p(\ell'_t(0); T) = 0$ for all $t \in [0, T]$;
4. If $\frac{1}{3} \leq H_2 < \frac{2}{3}$ and $p > \frac{4}{4-3H_2}$, then we have $V_p(\ell'_t(0); T) = 0$ for all $t \in [0, T]$.

4. Hölder continuity of the DILT of fBMs

In this section, we consider joint Hölder continuity and the occupation-time formulas for the DILT $\{\ell'_t(a); t \geq 0, a \in \mathbb{R}\}$. Our main object of this section is to explain and prove the following theorem.

**Theorem 4.1.** Let $0 < H_1 \leq H_2 < 1$. If either $H_1 < \frac{1}{2}$ or $H_2 < \frac{2}{3}$, then the processes $\ell'_{x,t}(a)$ converges almost surely, and in $L^p(\Omega)$ for all $p \in (0, \infty)$, as $\varepsilon$ tends to zero. Moreover, the process $\ell'_t(a)$ has a modification which is a.s. jointly Hölder continuous in $(a, t)$ and Hölder continuous in $a$.

In order to prove the theorem we denote $\tilde{D}_{t', t}(u) = \{t < u_n < u_{n-1} < \cdots < u_1 < t'\}$ for $t' > t \geq 0$, and
\[
\tilde{\Lambda}(t, t', n, \gamma) := \int_{\tilde{D}_{t', t}(u) \cup \tilde{D}_{t', t'}(u)} \prod_{j=1}^n (v_j)^{2H_2-1} dv_j du_j \int_{\mathbb{R}^n} e^{-(\sum_{k=1}^n \xi_k^1)^2(u_n)^{2H_1} - (\sum_{k=1}^n \xi_k^2(v_n)^{2H_2}}
\]
\[
\cdot \prod_{j=1}^{n-1} e^{-(\sum_{k=1}^j \xi_k^1)^2(u_j-u_{j+1})^{2H_1} - (\sum_{k=1}^{j-1} \xi_k^2(v_j-v_{j+1})^{2H_2}} \prod_{j=1}^n |\xi_j|^{1+\gamma} d\xi_j
\]
with $\gamma \geq -1$ and $n = 1, 2, \ldots$, where $\xi_1^1, \xi_2^2, \ldots, \xi_n^2$ and $\xi_1^1, \xi_2^1, \ldots, \xi_n^2$ are two arbitrary relabeling of the set $\{\xi_1, \xi_2, \ldots, \xi_n\}$.

**Lemma 4.1.** Let $0 < H_1 \leq H_2 < \frac{2}{3}$ and $n = 1, 2, \ldots$

- If $H_2 \geq \frac{1}{2}$, we have
\[
\tilde{\Lambda}(t, t', n, \gamma) \leq C(t' - t)^{n(1-(\frac{3}{2}+\gamma)H_2)}
\]
for all $0 < t < t' \leq T$ and $\gamma < \frac{2-3H_2}{2H_2}$;

- If $0 < H_2 < \frac{1}{2}$, we have
\[
\tilde{\Lambda}(t, t', n, \gamma) \leq C(t' - t)^{n(1-2\gamma)H_2}
\]
for all $0 < t < t' \leq T$ and $\gamma < \frac{1}{2}$.

The above lemma will be proved in Appendix [A]

**Proof of Theorem 4.1.** By Kolmogorov continuity criterion, it suffices to show
\[
E |\ell'_{t', \varepsilon}(a') - \ell'_{t', \varepsilon}(a)|^n \leq C|(t', \varepsilon', a') - (t, \varepsilon, a)|^{n\lambda}, \quad n = 2, 4, \ldots
\]
for all \( t, t' \in [0, T], a, a' \in \mathbb{R}, \varepsilon, \varepsilon' > 0 \) and some \( \lambda \in (0, 1] \), where \(|\cdot|\) denotes the Euclidean distance in \( \mathbb{R}^3 \). The estimate (4.4) will be done in three parts and denote \( \mathbb{D}_{t,t'} = \{ l \leq r < s \leq l' \} \).

**Step I.** We need to obtain the estimate

\[
E \left| \ell'_{t,\varepsilon}(a) - \ell'_{t',\varepsilon'}(a) \right|^n \leq C|\varepsilon' - \varepsilon|^{n \lambda}, \quad n = 2, 4, \ldots
\]

for all \( \varepsilon, \varepsilon' > 0, a \in \mathbb{R} \) and some \( \lambda \in [0, 1] \). Using (3.2) we have

\[
E \left| \ell'_{t,\varepsilon}(a) - \ell'_{t',\varepsilon'}(a) \right|^n
= \frac{1}{(2\pi)^n} \left| \int_{(D_0,t)^n} \prod_{j=1}^n (s_j)^{2H_2-1} ds_j dr_j \int_{\mathbb{R}^n} E \prod_{j=1}^n \xi_j e^{i\xi_j(B^{H_1}_{r_j} - B^{H_2}_{s_j})} (e^{\varepsilon^2} - e^{-\varepsilon^2}) e^{-i\xi_j a} d\xi_j \right|
\]

\[
\leq C_n |\varepsilon' - \varepsilon|^{n \lambda} \left| \int_{(D_0,0)^n} \prod_{j=1}^n (s_j)^{2H_2-1} ds_j dr_j \int_{\mathbb{R}^n} E \prod_{j=1}^n \xi_j e^{i\xi_j(B^{H_1}_{r_j} - B^{H_2}_{s_j})} \left| \prod_{j=1}^n |\xi_j|^{1+2\lambda} d\xi_j \right| \right|
\]

\[
\equiv C_n |\varepsilon' - \varepsilon|^{n \lambda} \Lambda(0, t, n, 2\lambda)
\]

for all \( n = 2, 4, \ldots \) and some \( \lambda \in [0, 1] \) by the inequality

\[
|e^{-\varepsilon x} - e^{-\varepsilon' x}| \leq C x^\lambda |\varepsilon - \varepsilon'|^\lambda
\]

for all \( x > 0 \) and \( \lambda \in [0, 1] \).

Now, we need to prove \( \Lambda(0, t, n, 2\lambda) < \infty \). Let us first consider the product inside the expectation. This expectation in the integrand will take different forms over different regions of integration, depending on the ordering of \( r_j, s_j \). Fix such an ordering and let \( u_1 > u_2 > \cdots > u_n \) and \( v_1 > v_2 > \cdots > v_n \) be the relabeling of the sets \( \{r_1, r_2, \ldots, r_n\} \) and \( \{s_1, s_2, \ldots, s_n\} \) respectively. Notice that

\[
\text{Var} \left( \sum_{j=1}^n \xi_j B^{H}_{u_j} \right) = \text{Var} \left( \sum_{j=1}^{n-1} \sum_{k=1}^{j} \xi_k (B^{H}_{u_j} - B^{H}_{u_{j+1}}) + \sum_{k=1}^{n} \xi_k B^{H}_{u_n} \right)
\]

\[
\geq \kappa \left( \sum_{j=1}^{n-1} \xi_j (u_j - u_{j+1})^{2H} + \kappa \sum_{k=1}^{n} \xi_k^2 (u_n)^{2H} \right)
\]

for a constant \( \kappa > 0 \) and any fBm \( B^{H} \) with Hurst index \( H \in (0, 1) \) by the local nondeterminism of fBm \( B^{H} \), where \( \xi_1', \xi_2', \ldots, \xi_n' \) is a relabeling (related to \( \{u_1, u_2, \ldots, u_n\} \)) of the set \( \{\xi_1, \xi_2, \ldots, \xi_n\} \).

We get

\[
\text{Var} \left( \sum_{j=1}^n \xi_j (B^{H_1}_{r_j} - B^{H_2}_{s_j}) \right) = \text{Var} \left( \sum_{j=1}^n \xi_j B^{H_1}_{r_j} \right) + \text{Var} \left( \sum_{j=1}^n \xi_j B^{H_2}_{s_j} \right)
\]

\[
\geq \kappa \left( \sum_{j=1}^{n-1} \xi_j (u_j - u_{j+1})^{2H_1} + \kappa \sum_{k=1}^{n} \xi_k^2 (u_n)^{2H_1} \right)
\]

\[
+ \kappa \left( \sum_{j=1}^{n-1} \xi_j (v_j - v_{j+1})^{2H_2} + \kappa \sum_{k=1}^{n} \xi_k^2 (v_n)^{2H_2} \right),
\]
where the set $\xi_1^\prime, \xi_2^\prime, \ldots, \xi_n^\prime$ is a relabeling (related to $\{v_1, v_2, \ldots, v_n\}$) of the set $\{\xi_1, \xi_2, \ldots, \xi_n\}$. It follows from Lemma 10 that

$$\Lambda(0, t, n, 2\lambda) \leq C_n \Lambda(0, t, n, 2\lambda)$$

$$= C_n \int_{\mathbb{R}_0(t)} \prod_{j=1}^n (v_j)^{2H_2-1} dv_j du_j \int_{\mathbb{R}^n} -\frac{1}{2} \kappa(\sum_{k=1}^n (\xi_k^\prime)^2(u_n)^{2H_1} \prod_{j=1}^{n-1} e^{-(\sum_{k=1}^j (\xi_k^\prime)^2(v_j-v_{j+1})^{2H_2}} \prod_{j=1}^n |\xi_j|^{1+2\lambda} d\xi_j$$

$$< \infty$$

for $t \in [0, T]$ and $n = 2, 4, \ldots$, provided $\lambda < \frac{2-3H_2}{4H_2}$ for $H_2 \geq \frac{1}{2}$ and $\lambda < \frac{1}{4}$ for $H_2 < \frac{1}{2}$, which proves the estimate

$$E |\ell(t, \varepsilon)(a) - \ell(t, \varepsilon)(a)|^n \leq C|\varepsilon'|^\lambda |\varepsilon|^n \lambda$$

for $n = 2, 4, \ldots$ and $t \in [0, T]$, if we choose $\lambda$ so that $\lambda < \frac{2-3H_2}{4H_2}$ for $H_2 \geq \frac{1}{2}$ and $\lambda < \frac{1}{4}$ for $H_2 < \frac{1}{2}$.

**Step II.** We obtain the estimate

$$E |\ell(t, \varepsilon)(b) - \ell(t, \varepsilon)(a)|^n \leq C|b-a|^{n\lambda}, \quad n = 2, 4, \ldots$$

for all $\varepsilon > 0$, $a, b \in \mathbb{R}$, $t \geq 0$ and some $\lambda \in [0, 1]$. We have, for all $a, b \in \mathbb{R}$ and $t \in [0, T]$

$$E |\ell(t, \varepsilon)(b) - \ell(t, \varepsilon)(a)|^n$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{D}_0(t)} \prod_{j=1}^n (s_j)^{2H_2-1} ds_j dj \int_{\mathbb{R}^n} E \prod_{j=1}^n \xi_j e^{i\xi_j(B_{s_j}^j - B_{s_j}^j)} (e^{-i\xi_j b} - e^{-i\xi_j a}) e^{-\frac{\xi_j^2}{2}} d\xi_j$$

$$\leq C|b-a|^{n\lambda} \int_{\mathbb{D}_0(t)} \prod_{j=1}^n (s_j)^{2H_2-1} ds_j dj \int_{\mathbb{R}^n} E \prod_{j=1}^n \xi_j e^{i\xi_j(B_{s_j}^j - B_{s_j}^j)} \prod_{j=1}^n |\xi_j|^{1+\lambda} d\xi_j$$

for all $n = 1, 2, \ldots$ and $n \in [0, 1]$ by the inequality

$$e^{-ixb} - e^{-ixa} \leq C|x|^\lambda |b-a|^\lambda$$

for all $x \in \mathbb{R}$ and $\lambda \in [0, 1]$. This shows that the estimate (4.10) holds for $n = 2, 4, \ldots$ and $t \in [0, T]$, if we choose $\lambda$ so that $\lambda < \frac{2-3H_2}{2H_2}$ for $H_2 \geq \frac{1}{2}$ and $\lambda < \frac{1}{2}$ for $H_2 < \frac{1}{2}$.

**Step III.** We obtain the estimate

$$E |\ell(t, \varepsilon)(a) - \ell(t, \varepsilon)(t')|^n \leq C|t' - t|^n \lambda, \quad n = 2, 4, \ldots$$

for all $\varepsilon > 0$, $a \in \mathbb{R}$, $t' > t \geq 0$ and some $\lambda \in [0, 1]$. In order to prove the estimate (4.12) we have, for all $a \in \mathbb{R}, t', t \in [0, T]$ and $t < t'$

$$E |\ell(t', \varepsilon)(a) - \ell(t, \varepsilon)(a)|^n$$

$$= \frac{1}{(2\pi)^n} \int_{[0, t'] \cap \mathbb{D}_0(t')} \prod_{j=1}^n (s_j)^{2H_2-1} ds_j dj \int_{\mathbb{R}^n} E \prod_{j=1}^n \xi_j e^{i\xi_j(B_{s_j}^j - B_{s_j}^j)} e^{-i\xi_j a} e^{-\frac{\xi_j^2}{2}} d\xi_j$$

$$\leq \int_{[t, t']^n} \int_{[0, s_1] \times [0, s_2] \times \cdots \times [0, s_n]} \prod_{j=1}^n (s_j)^{2H_2-1} ds_j dj \int_{\mathbb{R}^n} E \prod_{j=1}^n e^{i\xi_j(B_{s_j}^j - B_{s_j}^j)} \prod_{j=1}^n |\xi_j| d\xi_j$$
for all \( n = 2, 4, \ldots \). It follows from Step I and Lemma 4.1 that
\[
E |\ell'_{t,\epsilon}(a) - \ell'_{t,\epsilon}(a)|^n \leq C \overline{A}(t, t', n, 0) \leq C(t' - t)^{n\beta},
\]
if we choose \( \beta \) so that \( 0 < \beta \leq 1 - \frac{4}{2}H_2 \) for \( H_2 \geq \frac{1}{2} \) and \( 0 < \beta \leq \frac{1}{2}H_2 \) for \( 0 < H_2 < \frac{1}{2} \). Thus, we have obtained the desired estimate (14).

Finally, as two direct consequences of Step II and Step III we see that
\[
E |\ell'_t(b) - \ell'_t(a)|^n \leq C|b - a|^{n\lambda}, \quad n = 2, 4, \ldots
\]
and
\[
E |\ell'_t(a) - \ell'_t(a)|^n \leq C|t' - t|^{n\beta}, \quad n = 2, 4, \ldots
\]
for all \( a, b \in \mathbb{R} \) and \( t', t \geq 0 \) if we choose \( \lambda \) and \( \beta \) so that
\[
\lambda < \begin{cases} \frac{2 - 3H_2}{2H_2}, & H_2 \geq \frac{1}{2}, \\ \frac{1}{2}, & 0 < H_2 < \frac{1}{2} \end{cases}, \quad \beta < \begin{cases} 1 - \frac{3}{2}H_2, & H_2 \geq \frac{1}{2}, \\ \frac{1}{2}H_2, & 0 < H_2 < \frac{1}{2}. \end{cases}
\]
These show that \( \ell'_t(a) \) exists in \( L^p(\Omega) \) for all \( p > 0, t \in [0, T], a \in \mathbb{R} \), and has a modification which is a.s. jointly Hölder continuous in \((a, t)\), and it is also Hölder continuous of order \( \gamma < \min\{\frac{2 - H_2}{2H_2}, 1\} \) when \( \frac{1}{2} \leq H_1 \leq H_2 < 1 \).

**Theorem 4.2.** Let \( 0 < H_1, H_2 < 1 \). The processes \( \ell_{\epsilon, t}(a), \epsilon > 0 \) converges almost surely, and in \( L^p(\Omega) \) for all \( p \in (0, \infty) \), as \( \epsilon \) tends to zero. Moreover, the process \( \ell_t(a) \) has a modification which is a.s. jointly Hölder continuous in \((a, t)\). In particular, \( a \mapsto \ell_t(a) \) is Hölder continuous of order \( \gamma < \min\{\frac{2 - H_2}{2H_2}, 1\} \) when \( \frac{1}{2} \leq H_1 \leq H_2 < 1 \).

**Proof.** In a same way as proof of Theorem 4.1 one can obtain the estimate
\[
E |\ell'_{t,\epsilon}(a') - \ell_{t,\epsilon}(a)|^n \leq C \left( |\epsilon' - \epsilon|^{n\lambda} + |a' - a|^{n\alpha} + |t' - t|^{n\beta} \right), \quad n = 2, 4, \ldots
\]
for all \( t, t' \in [0, T], a, a' \in \mathbb{R}, \epsilon, \epsilon' > 0 \) and some \( \lambda, \beta, \alpha \in (0, 1] \). In fact, we can take
\[
0 < \lambda < \frac{3}{4} \wedge \frac{2 - H_2}{4H_2}, \quad 0 < \alpha < 1 \wedge \frac{2 - H_2}{2H_2}, \quad 0 < \beta < \frac{3H_2}{2} \wedge \frac{2 - H_2}{2},
\]
and in particular we have
\[
0 < \lambda < \frac{2 - H_2}{4H_2}, \quad 0 < \alpha < 1 \wedge \frac{2 - H_2}{2H_2}, \quad 0 < \beta < \frac{2 - H_2}{2}
\]
for all \( H_2 \geq \frac{1}{2} \), and prove the theorem. \( \square \)

**Corollary 4.1.** Let \( 0 < H_1 \leq H_2 < 1 \). If either \( H_1 < \frac{1}{2} \) or \( H_2 < \frac{2}{3} \), we have \( \ell'_t(a) = \frac{d}{da} \ell_t(a) \) a.s. for all \( t \geq 0 \) and \( a \in \mathbb{R} \), i.e. \( \ell_t(a) \) is differentiable in \( a \) for all \( t \geq 0 \) and
\[
\frac{d}{da} \ell_t(a) = \lim_{\epsilon \to 0} \ell'_{t,\epsilon}(a),
\]
amost surely, and in \( L^p(\Omega) \) with \( p > 0 \).

**Proof.** It is clear that \( \ell'_{t,\epsilon}(a) = \frac{d}{da} \ell_{t,\epsilon}(a) \) for any \( \epsilon, t > 0, a \in \mathbb{R} \) and hence
\[
\ell_{t,\epsilon}(a) = \ell_{t,\epsilon}(b) + \int_b^a \ell'_{t,\epsilon}(x)dx
\]
for all \( a, b \in \mathbb{R} \) and \( \epsilon > 0 \). On the other hand, the estimates (4.1) and (4.12) assure us of a locally uniform and hence continuous limits
\[
\ell'_t(a) = \lim_{\epsilon \to 0} \ell'_{t,\epsilon}(a)
\]
The locally uniform convergence (4.13) and (4.14) imply that
\[
\ell_t(a) = \lim_{\varepsilon \to 0} \ell_{t,\varepsilon}(a).
\]
and therefore \( \frac{d}{da} \ell_t(a) = \ell'_t(a) \). This completes the proof. \( \square \)

**Theorem 4.3.** Let \( 0 < H_1 \leq H_2 < 1 \).

1. If either \( H_1 < \frac{1}{2} \) or \( H_2 < \frac{2}{3} \), we then have

\[
\int_0^t s^{2H_2-1} ds \int_0^s f'(B_{t, s}^{H_1} - B_{s}^{H_2}) dr = -\int_{\mathbb{R}} f(a) \ell'_t(a) da
\]
for any \( f \in C^1(\mathbb{R}) \) and \( t \in [0, T] \).

2. If \( f \) is continuous, then

\[
\int_0^t s^{2H_2-1} \int_0^s f(B_{r, s}^{H_1} - B_{s}^{H_2}) dr ds = \int_{\mathbb{R}} f(a) \ell_t(a) da
\]
for all \( t \in [0, T] \).

As an immediate result of the above theorem, we have
\[
\int_{\mathbb{R}} f'(a) \ell_t(a) da = -\int_{\mathbb{R}} f(a) \ell'_t(a) da
\]
for all \( t \in [0, T] \).

**Proof of Theorem 4.3.** By the locally \( L^1 \) convergence in (4.13) and noting that both \( B^{H_1} \) and \( B^{H_2} \) are a.s. bounded on \( [0, t] \), we see that the following manipulations hold:
\[
\int_{\mathbb{R}} f(a) \ell'_t(a) da = \lim_{\varepsilon \to 0} \int_{\mathbb{R}} f(a) \ell'_{t,\varepsilon}(a) da
\]
\[
= \lim_{\varepsilon \to 0} \int_{\mathbb{R}} f(a) da \int_0^t s^{2H_2-1} ds \int_0^s p'_t(B_{t, s}^{H_1} - B_{s}^{H_2} - a) dr
\]
\[
= -\lim_{\varepsilon \to 0} \int_0^t s^{2H_2-1} ds \int_0^s dr \int_{\mathbb{R}} f'(a) p_{\varepsilon}(B_{t, s}^{H_1} - B_{s}^{H_2} - a) da
\]
\[
= -\lim_{\varepsilon \to 0} \int_0^t s^{2H_2-1} ds \int_0^s dr \int_{\mathbb{R}} p_{\varepsilon}(B_{t, s}^{H_1} - B_{s}^{H_2})
\]
\[
= -\int_{\mathbb{R}} f(B_{t, s}^{H_1} - B_{s}^{H_2}) ds
\]
for any \( f \in C^1(\mathbb{R}) \) with compact support, where the negative sign from the integration by parts in the third identity cancels the negative sign of the chain rule in the \( a \) derivative of \( p_{\varepsilon} \). Since both \( B^{H_1} \) and \( B^{H_2} \) are bounded a.s. we have that \( a \to \ell'_t(a) \) has compact support a.s. so that the above manipulations hold for all \( C^1 \)-functions \( f \).

Similarly, one can obtain the identity (4.16), and the theorem follows. \( \square \)
5. The Generalized Hybrid Quadratic Covariation, Case $H_1 < \frac{1}{2}$

In this section we throughout let $H_1 \leq H_2$, and inspired by the occupation formula (4.15), our main aim is to obtain the following Bouleau-Yor type identity
\[(5.1) \quad [f(B^{H_1} - B^{H_2}), B^{H_1}]_{t}^{(GH)} = - \int_{\mathbb{R}} f(a)\ell'_t(a)da, \quad t \geq 0\]
for some suitable Borel functions $f$. Recall that the quadratic covariation $[f(B), B]$ of Brownian motion $B$ can be characterized as
\[(5.2) \quad [f(B), B]_t = - \int_{\mathbb{R}} f(a)\mathcal{L}^B(da, t),\]
where $f$ is locally square integrable and $\mathcal{L}^B(x, t)$ is the local time of Brownian motion. This is called the Bouleau-Yor identity. More works for this can be found in Bouleau-Yor [3], Eisenbaum [8, 9], Errami-Russo [10], Feng–Zhao [11, 12], Follmer et al [13], Gradinaru et al [16], Moret-Nualart [23], Rogers–Walsh [28], Russo–Vallois [30, 31], Yan et al [34, 35] and the references therein.

Denote
\[J_\varepsilon(H_1, H_2, t, f) = \frac{1}{\varepsilon^{2H_1}} \int_0^t s^{2H_2-1} ds \int_0^s \{ f(B_{r+\varepsilon}^{H_1} - B_s^{H_2}) - f(B_r^{H_1} - B_s^{H_2}) \} \left( B_{r+\varepsilon}^{H_1} - B_r^{H_1} \right) dr\]
for any Borel function $f$ and $\varepsilon > 0$. Recall that the GHQC $[f(B^{H_1} - B^{H_2}), B^{H_1}]_t^{(GH)}$ defined by
\[(5.3) \quad [f(B^{H_1} - B^{H_2}), B^{H_1}]_t^{(GH)} := \lim_{\varepsilon \downarrow 0} J_\varepsilon(H_1, H_2, t, f),\]
provided the limit exists in $L^1(\Omega)$.

**Corollary 5.1.** If $f \in C^1(\mathbb{R})$, we then have
\[(5.4) \quad [f(B^{H_1} - B^{H_2}), B^{H_1}]_t^{(GH)} = \int_0^t s^{2H_2-1} ds \int_0^s f'(B_r^{H_1} - B_s^{H_2}) dr\]
for all $t \in [0, T]$. In particular, we have
\[ [B^{H_1} - B^{H_2}, B^{H_1}]_t^{(GH)} = \frac{t^{2H_2+1}}{2H_2+1}. \]

As a direct consequence of the above corollary and occupation formula (4.15), for all $f \in C^1(\mathbb{R})$ and $t \geq 0$ we have
\[(5.5) \quad [f(B^{H_1} - B^{H_2}), B^{H_1}]_t^{(GH)} = - \int_{\mathbb{R}} f(a)\ell'_t(a)da.\]

In order to prove the existence of the GHQC, we decompose $J_\varepsilon(H_1, H_2, t, f)$ as follows:
\[ J_\varepsilon(H_1, H_2, t, f) = \frac{1}{\varepsilon^{2H_1}} \int_0^t s^{2H_2-1} ds \int_0^s f(B_{r+\varepsilon}^{H_1} - B_s^{H_2}) \left( B_{r+\varepsilon}^{H_1} - B_r^{H_1} \right) dr \]
\[ - \frac{1}{\varepsilon^{2H_1}} \int_0^t s^{2H_2-1} ds \int_0^s f(B_r^{H_1} - B_s^{H_2}) \left( B_{r+\varepsilon}^{H_1} - B_r^{H_1} \right) dr \]
\[ = J_\varepsilon^+(H_1, H_2, t, f) - J_\varepsilon^-(H_1, H_2, t, f) \]
for all $\varepsilon > 0$ and $t \in [0, T]$, and consider the set
\[ \mathcal{H} = \{ f : \text{ measurable functions on } \mathbb{R} \text{ such that } \|f\|_\mathcal{H} < \infty \}, \]
Lemma 5.1. Let $\mu \in C_c(\mathbb{R})$, the set of infinitely differentiable functions with compact support.

Recall that the space $H_\infty$ is a Banach space. For the space we have

$$\|f\|_\infty := \int_0^T s^{2H_2 + H_1 - 1} ds \int_0^s dr \int_\mathbb{R} |f(x)|^2 e^{-\frac{r^2}{2\pi^2 \lambda_{r,s}}} dx < \infty.$$  

Then, $\mathcal{H}$ is a Banach space. For the space we have

1. $\mathcal{H} \supset L^2(\mathbb{R}) \supset C^\infty_0(\mathbb{R})$, the set of elementary functions of the form

$$f_\Delta(x) = \sum_i f_{i,1}(x_i,x_{i+1})(x)$$

is dense in $\mathcal{H}$, where $\{x_i, 0 \leq i \leq l\}$ is a finite sequence of real numbers such that $x_i < x_{i+1}$.

2. the space $\mathcal{H}$ contains all polynomial growth functions.

3. the space $\mathcal{H}$ contains all Borel functions $f$ satisfying

$$\int_0^T s^{H_2 + H_1} ds \int_\mathbb{R} |f(x)|^2 e^{-\frac{r^2}{2\pi^2 \lambda_{r,s}}} dx < \infty.$$  

Recall that the space $C^\mu$ of $\mu$-Hölder continuous functions $f : [0, T] \to \mathbb{R}$, equipped with the norm

$$\|f\|_{(\mu)} := \|f\|_\infty + \sup_{x,y \in \mathbb{R}} \frac{|f(x) - f(y)|}{|x - y|^\mu} < \infty,$$

where $\mu \in (0,1)$ and $\|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)|$.

Corollary 5.2. For all $H_1, H_2 \in (0,1)$, we have $\mathcal{H} \supset C^\mu$ for all $\mu \in (0,1]$.

Lemma 5.1. Let $B^H$ be a fBm with Hurst index $0 < H < 1$.

1. For $0 < H < \frac{1}{2}$ we have

$$|E \left[ B^H_t (B^H_t - B^H_s) \right]| \leq (t - s)^{2H},$$

$$|E \left[ B^H_t (B^H_t - B^H_r) \right]| \leq (s - r)^{2H},$$

$$|E \left[ B^H_r (B^H_t - B^H_s) \right]| \leq (t - s)^{2H},$$

$$|E \left[ B^H_s (B^H_t - B^H_r) \right]| \leq (t - s)^{2H}$$

for all $t > s > r > 0$;

2. For $\frac{1}{2} < H < 1$ we have

$$|E \left[ B^H_t (B^H_t - B^H_s) \right]| \leq C_H t^{2H-1}(t - s),$$

$$|E \left[ B^H_t (B^H_t - B^H_r) \right]| \leq C_H t^{2H-1}(s - r),$$

$$|E \left[ B^H_r (B^H_t - B^H_s) \right]| \leq C_H r^{2H-1}(t - s),$$

$$|E \left[ B^H_s (B^H_t - B^H_r) \right]| \leq C_H s^{2H-1}(t - r)$$

for all $t > s > r > 0$.

Lemma 5.2. Let $\lambda_{r,s}$ and $\mu$ be defined in Section 3 and $\rho^2 = \lambda'_{r,s} \lambda_{r,s} - \mu^2$. If $f \in C^\infty(\mathbb{R})$ admit compact support. Then we have

$$|E \left[ f' (B^H_{r_1} - B^H_{s_1}) f' (B^H_{r_2} - B^H_{s_2}) \right]|$$

$$\leq \frac{\sqrt{\lambda'_{r,s} \lambda_{r,s}}}{\rho^2} \left( E \left[ |f(B^H_{r_1} - B^H_{s_1})|^2 \right] E \left[ |f(B^H_{r_2} - B^H_{s_2})|^2 \right] \right)^{1/2}$$

(5.6)
and
\[
|E \left[ f''(B_{r}^{H_1} - B_{s}^{H_2})f(B_{r}^{H_1} - B_{s}^{H_2}) \right]| \leq \frac{3\lambda_{r,s'}}{p^2} \left( E \left[ |f(B_{r}^{H_1} - B_{s}^{H_2})|^2 \right] E \left[ |f(B_{r'}^{H_1} - B_{s'}^{H_2})|^2 \right] \right)^{1/2}
\]
for all \((r, s, r', s') \in \mathbb{T}.

The lemmas above are some elementary calculations and the next lemma will be proved in Appendix [A].

**Lemma 5.3.** Let \(B^H\) be a fBm with Hurst index \(0 < H < 1\). We then have
\[
|E \left[ (B_t^H - B_s^H)(B_{t'}^H - B_{s'}^H) \right]| \leq C_H \frac{(t - s)(t' - s')}{(s - t')^{2-2H}}
\]
for all \(0 < s' < t' < s < t\). In particular, for \(0 < H < \frac{1}{2}\) we have
\[
|E \left[ (B_t^H - B_s^H)(B_{t'}^H - B_{s'}^H) \right]| \leq \frac{(t - s)^{2H}(t' - s')^{2H}}{(s - t')^{2H}}
\]
for all \(0 < s' < t' < s < t\).

**Theorem 5.1.** Let \(0 < H_1 < \frac{1}{2}\) and \(f \in \mathcal{A}\). Then, the GHQC \([f(B_{t}^{H_1} - B_{s}^{H_2}), B_{t}^{H_1}]^{(GH)}\) exists in \(L^2(\Omega)\) and
\[
E \left[ |f(B_{t}^{H_1} - B_{s}^{H_2}), B_{t}^{H_1}]^{(GH)}\right|^2 \leq C_{H_1,H_2,T\|f\|^2_{\mathcal{A}}}
\]
for all \(t \in [0, T]\).

In order to prove the theorem we claim that the following two statements:

- for any \(\varepsilon > 0\), \(t \in [0, 1]\), and \(f \in \mathcal{A}\), \(J_{\varepsilon}^{-}(H_1, H_2, f, t) \in L^2(\Omega)\). That is,
\[
E \left| J_{\varepsilon}^{-}(H_1, H_2, f, t) \right|^2 \leq C_{H_1,H_2,T\|f\|^2_{\mathcal{A}}},
\]
\[
E \left| J_{\varepsilon}^{+}(H_1, H_2, f, t) \right|^2 \leq C_{H_1,H_2\|f\|^2_{\mathcal{A}}},
\]

- \(J_{\varepsilon}^{-}(H_1, H_2, f, t)\) and \(J_{\varepsilon}^{+}(H_1, H_2, f, t)\) are two Cauchy’s sequences in \(L^2(\Omega)\) for all \(t \in [0, 1]\). That is,
\[
E \left| J_{\varepsilon_1}^{-}(H_1, H_2, f, t) - J_{\varepsilon_2}^{-}(H_1, H_2, f, t) \right|^2 \rightarrow 0,
\]
and
\[
E \left| J_{\varepsilon_1}^{+}(H_1, H_2, f, t) - J_{\varepsilon_2}^{+}(H_1, H_2, f, t) \right|^2 \rightarrow 0
\]
for all \(t \in [0, 1]\), as \(\varepsilon_1, \varepsilon_2 \downarrow 0\).

We split the proof of two statements into two parts, and we let \(T = 1\) for simplicity.

**Part I: Proof of the estimates** \(5.11\) and \(5.12\). We have
\[
E|J_{\varepsilon}^{-}(H_1, H_2, f, t)|^2 = \frac{1}{\varepsilon^{4H_1}} \int_0^t \int_0^t (ss')^{2H_2-1} ds'ds
\cdot \int_0^s \int_0^{s'} E \left[ f(B_{r}^{H_1} - B_{s}^{H_2})f(B_{r}^{H_1} - B_{s}^{H_2})(B_{r}^{H_1} - B_{r}^{H_1})(B_{r}^{H_1} - B_{r}^{H_1})(B_{r}^{H_1} - B_{r}^{H_1}) \right] drdr'
\]
for all $\varepsilon > 0$ and $t \geq 0$. Now, let us estimate the expression

$$
\Phi_{\varepsilon_1, \varepsilon_2}(r, s, r', s'; H_1, H_2) := E \left[ f(B_{s}^{H_1} - B_{s}^{H_2}) f(B_{r'}^{H_1} - B_{s'}^{H_2}) (B_{r+\varepsilon}^{H_1} - B_{r'}^{H_1}) (B_{r'+\varepsilon}^{H_1} - B_{s'}^{H_1}) \right]
$$

for all $\varepsilon_1, \varepsilon_2 > 0$ and $s > r > 0, s' > r' > 0$. Thus, it is enough to obtain the estimates (5.11) and (5.12) for all $f \in \mathcal{E}$. By approximating we can assume that $f$ is an infinitely differentiable function with compact support. It follows from the duality relationship (2.1) that

$$
\Phi_{\varepsilon_1, \varepsilon_2}(r, s, r', s'; H_1, H_2)
= E \left[ f(B_{s}^{H_1} - B_{s}^{H_2}) f(B_{r'}^{H_1} - B_{s'}^{H_2}) (B_{r+\varepsilon}^{H_1} - B_{r'}^{H_1}) \right]
\int_{r'}^{r'+\varepsilon_2} dB_t^{H_1}
= E \left[ B_{r}^{H_1} (B_{r'+\varepsilon_2}^{H_1} - B_{r'}^{H_1}) \right] E \left[ f'(B_{s}^{H_1} - B_{s}^{H_2}) f(B_{r'}^{H_1} - B_{s'}^{H_2}) (B_{r+\varepsilon}^{H_1} - B_{r'}^{H_1}) \right]
\int_{r'}^{r'+\varepsilon_2} dB_t^{H_1}
+ E \left[ B_{r'}^{H_1} (B_{r'+\varepsilon_2}^{H_1} - B_{r'}^{H_1}) \right] E \left[ f(B_{s}^{H_1} - B_{s}^{H_2}) f'(B_{r'}^{H_1} - B_{s'}^{H_2}) (B_{r+\varepsilon}^{H_1} - B_{r'}^{H_1}) \right]
\int_{r'}^{r'+\varepsilon_2} dB_t^{H_1}
+ E \left[ (B_{r+\varepsilon}^{H_1} - B_{r'}^{H_1}) (B_{r'+\varepsilon}^{H_2} - B_{r'}^{H_2}) \right] E \left[ f(B_{s}^{H_1} - B_{s}^{H_2}) f(B_{r'}^{H_1} - B_{s'}^{H_2}) \right]
\int_{r'}^{r'+\varepsilon_2} dB_t^{H_1}
$$

(5.15)

$$
= E \left[ B_{r}^{H_1} (B_{r'+\varepsilon_2}^{H_1} - B_{r'}^{H_1}) \right] E \left[ B_{r'}^{H_1} (B_{r+\varepsilon}^{H_1} - B_{r'}^{H_1}) \right] E \left[ f''(B_{s}^{H_1} - B_{s}^{H_2}) f(B_{r'}^{H_1} - B_{s'}^{H_2}) \right]
\int_{r'}^{r'+\varepsilon_2} dB_t^{H_1}
+ E \left[ B_{r'}^{H_1} (B_{r'+\varepsilon_2}^{H_1} - B_{r'}^{H_1}) \right] E \left[ B_{r}^{H_1} (B_{r+\varepsilon}^{H_1} - B_{r'}^{H_1}) \right] E \left[ f'(B_{s}^{H_1} - B_{s}^{H_2}) f'(B_{r'}^{H_1} - B_{s'}^{H_2}) \right]
\int_{r'}^{r'+\varepsilon_2} dB_t^{H_1}
+ E \left[ B_{r}^{H_1} (B_{r'+\varepsilon_2}^{H_1} - B_{r'}^{H_1}) \right] E \left[ B_{r'}^{H_1} (B_{r+\varepsilon}^{H_1} - B_{r'}^{H_1}) \right] E \left[ f(B_{s}^{H_1} - B_{s}^{H_2}) f'(B_{r'}^{H_1} - B_{s'}^{H_2}) \right]
\int_{r'}^{r'+\varepsilon_2} dB_t^{H_1}
+ E \left[ B_{r'}^{H_1} (B_{r'+\varepsilon_2}^{H_1} - B_{r'}^{H_1}) \right] E \left[ B_{r}^{H_1} (B_{r+\varepsilon}^{H_1} - B_{r'}^{H_1}) \right] E \left[ f'(B_{s}^{H_1} - B_{s}^{H_2}) f'(B_{r'}^{H_1} - B_{s'}^{H_2}) \right]
\int_{r'}^{r'+\varepsilon_2} dB_t^{H_1}
+ E \left[ (B_{r+\varepsilon}^{H_1} - B_{r'}^{H_1}) (B_{r'+\varepsilon}^{H_2} - B_{r'}^{H_2}) \right] E \left[ f(B_{s}^{H_1} - B_{s}^{H_2}) f(B_{r'}^{H_1} - B_{s'}^{H_2}) \right]
\int_{r'}^{r'+\varepsilon_2} dB_t^{H_1}
$$

(5.16)

$$
\sum_{j=1}^{5} \Psi_j(r, s, r', s', \varepsilon_1, \varepsilon_2)
$$

for all $s > r > 0, s' > r' > 0$ and $\varepsilon_1, \varepsilon_2 > 0$. In order to end the proof we claim to estimate

$$
\Lambda_j := \frac{1}{\varepsilon^{2H_1}} \left| \int_{(D_{0,t})^2} (ss')^{2H_2-1} \Psi_j (r, s, r', s', \varepsilon, \varepsilon) dr' ds' dr ds \right|, \quad j = 1, 2, 3, 4, 5
$$

(5.16)

for all $\varepsilon > 0$ small enough, where $D_{0,t} = \{0 \leq r < s \leq t\}$

For $j = 5$, from the fact

$$
|E \left[ (B_{r+\varepsilon}^{H_1} - B_{r'}^{H_1}) (B_{r'+\varepsilon}^{H_1} - B_{s'}^{H_1}) \right] | \leq \varepsilon^{2H_1} \leq \frac{\varepsilon^{4H_1}}{(r - r')^{2H_1}}
$$

for $0 < r - r' < \varepsilon$ and the estimate (5.9) we have

$$
\Lambda_5 \leq \frac{1}{\varepsilon^{4H_1}} \int_{0}^{t} \int_{0}^{t} (ss')^{2H_2-1} ds ds' \int_{0}^{s} dr \int_{0}^{s'} dr' \cdot \left( E \left[ |f(B_{s}^{H_1} - B_{s}^{H_2})|^2 \right] + E \left[ |f'(B_{s}^{H_1} - B_{s}^{H_2})|^2 \right] \right) \left| E \left[ (B_{r+\varepsilon}^{H_1} - B_{r'}^{H_1}) (B_{r'+\varepsilon}^{H_1} - B_{s'}^{H_1}) \right] \right|
\leq C_{H_1} \int_{0}^{t} s^{2H_2-1} ds \int_{0}^{s} E \left[ |f(B_{s}^{H_1} - B_{s}^{H_2})|^2 \right] ds' \int_{0}^{s'} \frac{dr'}{(r - r')^{2H_1}}
\leq C_{H_1, H_2} \|f\|_{s}^{2}
$$

for all $0 < \varepsilon \leq 1$ and $t \in [0, 1]$. 
For $j = 1$, from Lemma 5.1 (5.7) and proof of Theorem 3.1 we have

$$A_1 \leq \int_{(D_0,t)^2} (ss')^{2H_2-1}E \left[ f''(B^H_1 - B^H_2) \right] \left[ f'(B^H_1 - B^H_2) \right] |dr' ds' dr ds$$

$$\leq \int_{(D_0,t)^2} (ss')^{2H_2-1} |dr' ds' dr ds \frac{3\lambda_{r,s'} \rho^2}{2} E \left[ |f(B^H_1 - B^H_2)|^2 \right]$$

$$+ \int_{(D_0,t)^2} (ss')^{2H_2-1} |dr' ds' dr ds \frac{3\lambda_{r,s'} \rho^2}{2} E \left[ |f(B^H_1 - B^H_2)|^2 \right]$$

$$\leq C_{H_1,H_2} \int_0^t s^{2H_2-1} ds \int_0^s |dr'| \int_0^s |r - r'|^{2H_1}$$

for all $0 < \varepsilon \leq 1$ and $t \in [0,1]$. Similarly, we can obtain the estimate (5.16) for $j = 2, 3, 4$, and the estimates (5.11) follows.

Similarly one can prove the estimate (5.12) and the first statement follows.

**Part II: Proof of the estimates (5.13) and (5.14).** Without loss of generality one may assume that $\varepsilon_1 > \varepsilon_2$ and for $f \in \mathcal{H}$ we take the sequence $\{f_{\Delta,n}\} \subset \mathcal{E}$ such that $f_{\Delta,n} \to f$ in $\mathcal{H}$. Then we have

$$E|J_{\varepsilon_1}(H_1, H_2, t, f) - J_{\varepsilon_2}(H_1, H_2, t, f)|^2$$

$$\leq 3E|J_{\varepsilon_1}(H_1, H_2, t, f - f_{\Delta,n})|^2 + 3E|J_{\varepsilon_2}(H_1, H_2, t, f - f_{\Delta,n})|^2$$

$$+ 3E|J_{\varepsilon_1}(H_1, H_2, t, f_{\Delta,n}) - J_{\varepsilon_2}(H_1, H_2, t, f_{\Delta,n})|^2$$

$$\leq C_{H_1,H_2} \|f - f_{\Delta,n}\|_{\mathcal{H}}^2 + 3E|J_{\varepsilon_1}(H_1, H_2, t, f_{\Delta,n}) - J_{\varepsilon_2}(H_1, H_2, t, f_{\Delta,n})|^2$$

for all $\varepsilon_1, \varepsilon_2 > 0$ and all $n \geq 1$. Thus, it is enough to obtain the estimates (5.13) and (5.14) for all $f \in \mathcal{E}$. By approximating we can assume that $f$ is an infinitely differentiable function with compact support. It follows from (5.15) that

$$E|J_{\varepsilon_1}(H_1, H_2, t, f) - J_{\varepsilon_2}(H_1, H_2, t, f)|^2$$

$$= \frac{1}{\varepsilon_1 H_1} \int_{D_0,t} (ss')^{2H_2-1} E[f(B^H_1 - B^H_2 f(B^H_1 - B^H_2)]$$

$$\cdot (B^{H_1}_{r+\varepsilon_1} - B^{H_1}_r) (B^{H_2}_{r+\varepsilon_1} - B^{H_2}_r) |dr' ds' dr ds$$

$$- \frac{2}{\varepsilon_1 H_1 \varepsilon_2 H_1} \int_{D_0,t} (ss')^{2H_2-1} E[f(B^H_1 - B^H_2 f(B^H_1 - B^H_2)]$$

$$\cdot (B^{H_1}_{r+\varepsilon_1} - B^{H_1}_r) (B^{H_2}_{r+\varepsilon_2} - B^{H_2}_r) |dr' ds' dr ds$$

$$+ \frac{1}{\varepsilon_2 H_1} \int_{D_0,t} (ss')^{2H_2-1} E[f(B^H_1 - B^H_2 f(B^H_1 - B^H_2)]$$

$$\cdot (B^{H_1}_{r+\varepsilon_2} - B^{H_1}_r) (B^{H_2}_{r+\varepsilon_2} - B^{H_2}_r) |dr' ds' dr ds$$

$$= \frac{1}{\varepsilon_1 H_1 \varepsilon_2 H_1} \int_{D_0,t} \left\{ \varepsilon_1 H_1 \Phi_{\varepsilon_1,\varepsilon_1} - \varepsilon_2 H_1 \Phi_{\varepsilon_1,\varepsilon_2} \right\} (ss')^{2H_2-1} |dr' ds' dr ds$$

$$+ \frac{1}{\varepsilon_1 H_1 \varepsilon_2 H_1} \int_{D_0,t} \left\{ \varepsilon_2 H_1 \Phi_{\varepsilon_2,\varepsilon_2} - \varepsilon_2 H_1 \Phi_{\varepsilon_1,\varepsilon_2} \right\} (ss')^{2H_2-1} |dr' ds' dr ds$$

$$+ \frac{1}{\varepsilon_1 H_1 \varepsilon_2 H_1} \int_{D_0,t} \left\{ \varepsilon_1 H_1 \Phi_{\varepsilon_1,\varepsilon_1} - \varepsilon_2 H_1 \Phi_{\varepsilon_1,\varepsilon_2} \right\} (ss')^{2H_2-1} |dr' ds' dr ds$$

$$+ \frac{1}{\varepsilon_2 H_1 \varepsilon_2 H_1} \int_{D_0,t} \left\{ \varepsilon_2 H_1 \Phi_{\varepsilon_2,\varepsilon_2} - \varepsilon_2 H_1 \Phi_{\varepsilon_1,\varepsilon_2} \right\} (ss')^{2H_2-1} |dr' ds' dr ds$$

$$+ \frac{1}{\varepsilon_2 H_1 \varepsilon_2 H_1} \int_{D_0,t} \left\{ \varepsilon_1 H_1 \Phi_{\varepsilon_1,\varepsilon_1} - \varepsilon_2 H_1 \Phi_{\varepsilon_1,\varepsilon_2} \right\} (ss')^{2H_2-1} |dr' ds' dr ds$$
with $\Phi_{\varepsilon_{i}, \varepsilon_{j}} := \Phi_{\varepsilon_{i}, \varepsilon_{j}}(r, s, r', s'; H_{1}, H_{2})$. Now, in order to end the proof we need to introduce the following convergence:

$$
(5.17)\quad \frac{1}{\varepsilon_{i}^{4H_{1}} \varepsilon_{j}^{2H_{1}}} \int_{D_{0,t}} (ss')^{2H_{2} - 1} \{ \varepsilon_{i}^{2H_{1}} \Phi_{\varepsilon_{i}, \varepsilon_{i}} - \varepsilon_{i}^{2H_{1}} \Phi_{\varepsilon_{1}, \varepsilon_{2}} \} dr'ds'drds \rightarrow 0
$$

with $i, j = 1, 2, i \neq j$, as $\varepsilon_{1}, \varepsilon_{2} \rightarrow 0$. By symmetry, we only need to show that this holds for $i = 1, j = 2$. Denote

$$
A_{0,0}(r, r', \varepsilon, j) := \varepsilon_{j}^{2H_{1}} E \left[ (B_{r_{+}^{\varepsilon}} - B^{-H_{1}}_{r_{-}^{\varepsilon}})(B_{r_{+}^{\varepsilon}} - B^{-H_{1}}_{r_{-}^{\varepsilon}}) \right] - \varepsilon_{j}^{2H_{1}} E \left[ (B_{r_{+}^{\varepsilon}} - B^{-H_{1}}_{r_{-}^{\varepsilon}})(B_{r_{+}^{\varepsilon}} - B^{-H_{1}}_{r_{-}^{\varepsilon}}) \right]
$$

$$
A_{0,2}(r, r', \varepsilon, j) := \varepsilon_{j}^{2H_{1}} E \left[ B_{r_{-}^{\varepsilon}}(B_{r_{-}^{\varepsilon}} - B^{-H_{1}}_{r_{-}^{\varepsilon}}) \right] E \left[ B_{r_{-}^{\varepsilon}}(B_{r_{-}^{\varepsilon}} - B^{-H_{1}}_{r_{-}^{\varepsilon}}) \right] - \varepsilon_{j}^{2H_{1}} E \left[ (B_{r_{+}^{\varepsilon}} - B^{-H_{1}}_{r_{-}^{\varepsilon}})(B_{r_{-}^{\varepsilon}} - B^{-H_{1}}_{r_{-}^{\varepsilon}}) \right] E \left[ B_{r_{-}^{\varepsilon}}(B_{r_{-}^{\varepsilon}} - B^{-H_{1}}_{r_{-}^{\varepsilon}}) \right]
$$

$$
A_{2,0}(r, r', \varepsilon, j) := \varepsilon_{j}^{2H_{1}} E \left[ B_{r_{-}^{\varepsilon}}(B_{r_{-}^{\varepsilon}} - B^{-H_{1}}_{r_{-}^{\varepsilon}}) \right] E \left[ B_{r_{-}^{\varepsilon}}(B_{r_{-}^{\varepsilon}} - B^{-H_{1}}_{r_{-}^{\varepsilon}}) \right] - \varepsilon_{j}^{2H_{1}} E \left[ (B_{r_{+}^{\varepsilon}} - B^{-H_{1}}_{r_{-}^{\varepsilon}})(B_{r_{-}^{\varepsilon}} - B^{-H_{1}}_{r_{-}^{\varepsilon}}) \right] E \left[ B_{r_{-}^{\varepsilon}}(B_{r_{-}^{\varepsilon}} - B^{-H_{1}}_{r_{-}^{\varepsilon}}) \right]
$$

$$
A_{1,1}(r, r', \varepsilon, j) := \varepsilon_{j}^{2H_{1}} E \left[ B_{r_{-}^{\varepsilon}}(B_{r_{-}^{\varepsilon}} - B^{-H_{1}}_{r_{-}^{\varepsilon}}) \right] E \left[ B_{r_{-}^{\varepsilon}}(B_{r_{-}^{\varepsilon}} - B^{-H_{1}}_{r_{-}^{\varepsilon}}) \right] - \varepsilon_{j}^{2H_{1}} E \left[ (B_{r_{+}^{\varepsilon}} - B^{-H_{1}}_{r_{-}^{\varepsilon}})(B_{r_{-}^{\varepsilon}} - B^{-H_{1}}_{r_{-}^{\varepsilon}}) \right] E \left[ B_{r_{-}^{\varepsilon}}(B_{r_{-}^{\varepsilon}} - B^{-H_{1}}_{r_{-}^{\varepsilon}}) \right]
$$

with $j = 1, 2$. It follows that

$$
\varepsilon_{j}^{2H_{1}} \Phi_{\varepsilon_{i}, \varepsilon_{i}} - \varepsilon_{i}^{2H_{1}} \Phi_{\varepsilon_{1}, \varepsilon_{2}} = A_{0,0}(r, r', \varepsilon, j) E \left[ f(B_{r_{+}^{\varepsilon}} - B_{s_{-}^{\varepsilon}}) f(B_{r_{+}^{\varepsilon}} - B_{s_{-}^{\varepsilon}}) \right] + A_{0,2}(r, r', \varepsilon, j) E \left[ f(B_{r_{-}^{\varepsilon}} - B_{s_{-}^{\varepsilon}}) f(B_{r_{-}^{\varepsilon}} - B_{s_{-}^{\varepsilon}}) \right] + A_{2,0}(r, r', \varepsilon, j) E \left[ f'(B_{r_{-}^{\varepsilon}} - B_{s_{-}^{\varepsilon}}) f(B_{r_{-}^{\varepsilon}} - B_{s_{-}^{\varepsilon}}) \right] + A_{1,1}(r, r', \varepsilon, j) E \left[ f'(B_{r_{-}^{\varepsilon}} - B_{s_{-}^{\varepsilon}}) f'(B_{r_{-}^{\varepsilon}} - B_{s_{-}^{\varepsilon}}) \right]
$$

with $i, j = 1, 2$ and $i \neq j$. Now, it is enough to prove

$$
(5.18)\quad \int_{D_{0,t}} A_{k,l}(r, r', \varepsilon_{1}, 2) E \left[ f(k)(B_{r_{+}^{\varepsilon}} - B_{s_{-}^{\varepsilon}}) f(l)(B_{r_{-}^{\varepsilon}} - B_{s_{-}^{\varepsilon}}) \right] (ss')^{2H_{2} - 1} dr'ds'drds \rightarrow 0
$$

for all $k, l \in \{0, 1, 2\}$ and $k + l \in \{0, 2\}$, as $\varepsilon_{1}, \varepsilon_{2} \rightarrow 0$. Without loss of generality one may assume that $\varepsilon_{1} > \varepsilon_{2}$.

For $k = l = 0$ and $0 < |r - r'| < \varepsilon_{2}$ we have

$$
|A_{0,0}(r, r', \varepsilon_{1}, 2)| \leq \varepsilon_{2}^{4H_{1}} \varepsilon_{1}^{2H_{1}} + \varepsilon_{1}^{2H_{1}} \varepsilon_{2}^{2H_{1}} \leq \frac{2\varepsilon_{2}^{2H_{1}} \varepsilon_{1}^{2H_{1}}}{|r - r'|}
$$

by Cauchy’s inequality with $2H_{1} < \gamma \leq 1$. It follows from (A.11) with $\alpha = \frac{\gamma}{2 - 2H_{1}}$ that

$$
\frac{1}{\varepsilon_{1}^{4H_{1}} \varepsilon_{2}^{2H_{1}}} |A_{0,0}(r, r', \varepsilon_{1}, 2)| \leq C_{H_{1}} \left( \frac{1}{|r - r' - \varepsilon_{1}|^{\gamma}} + \frac{1}{|r - r'|^{\gamma}} \right) \varepsilon_{1}^{\gamma - 2H_{1}} \rightarrow 0
$$
for all $s, r > 0$ and $2H_1 < \gamma \leq 1$, as $\varepsilon_1, \varepsilon_2 \to 0$. Consequently, Lebesgue’s dominated convergence theorem deduces the convergence (5.18) with $k = l = 0$ because

$$
\int_{D_{0,t}} \left( \frac{1}{|r - r' - \varepsilon_1|^{2H_1}} + \frac{1}{|r - r'|^{2H_1}} \right) \cdot |E\left[ f(B_{r_1}^{H_1} - B_{s_2}^{H_2}) f(B_{r_1}^{H_1} - B_{s_2}^{H_2}) \right]| \left| (ss')^{2H_2-1} dr' ds' dr ds \right| \leq C_{H_1, H_2} \|f\|^2_{\mathcal{F}}
$$

for all $\varepsilon_1 > 0$.

When $k + l = 2$, by the fact

$$(5.19) \quad b^a - a^\alpha \leq b^{\alpha - \beta} (b - a)^\beta$$

with $b > a > 0$ and $0 < \alpha \leq \beta \leq 1$, we have

$$
|\varepsilon_2^{2H_1} E[B_{r_1}^{H_1}(B_{r_1+\varepsilon_1}^{H_1} - B_{r_1}^{H_1})] - \varepsilon_1^{2H_1} E[B_{r_1}^{H_1}(B_{r_1+\varepsilon_2}^{H_1} - B_{r_1}^{H_1})]| = \frac{1}{2} \varepsilon_2^{2H_1} \left( (r + \varepsilon_1)^{2H_1} - (r + \varepsilon_2)^{2H_1} - r^{2H_1} \right) - \frac{1}{2} \varepsilon_1^{2H_1} \left( (r + \varepsilon_2)^{2H_1} - (r - \varepsilon_1)^{2H_1} \right)
$$

$$
(5.20) \quad = \frac{1}{2} \varepsilon_2^{2H_1} \left( (r + \varepsilon_1)^{2H_1} - (r - 2H_1) \right) - \frac{1}{2} \varepsilon_1^{2H_1} \left( (r + \varepsilon_2)^{2H_1} - (r - 2H_1) \right)
$$

$$
\leq \frac{1}{2} \left( (r + \varepsilon_1)^{2H_1 - \beta} \varepsilon_2^{2H_1} \varepsilon_1^\beta + (r + \varepsilon_2)^{2H_1 - \beta} \varepsilon_1^{2H_1} \varepsilon_2^\beta \right) \leq r^{2H_1 - \beta} \varepsilon_2^{2H_1} \varepsilon_1^\beta
$$

for all $2H_1 < \beta \leq 1$ and $r > 0$. Similarly, by (5.19) we also have

$$
|E \left[ B_{r_1}^{H_1}(B_{r_1+\varepsilon_1}^{H_1} - B_{r_1}^{H_1}) \right]| = \frac{1}{2} \left( (r + \varepsilon)^{2H_1} - (r - \varepsilon)^{2H_1} + |r - r'|^{2H_1} - |r - r' - \varepsilon|^{2H_1} \right)
$$

$$
(5.21) \quad \leq \frac{1}{2} \left( (r')^{2H_1 - \beta} + |r - r'|^{2H_1 - \beta} \right) \varepsilon^\beta
$$

for all $|r - r'| > 0, r', r > 0$ and $2H_1 < \beta \leq 1$. Combining these with Lemma 5.1, we get

$$
\frac{1}{\varepsilon_1^{4H_1} \varepsilon_2^{2H_1}} |A_{0,2}(r, r', \varepsilon_1, 2)| = \frac{1}{\varepsilon_1^{4H_1} \varepsilon_2^{2H_1}} |E[B_{r_1}^{H_1}(B_{r_1+\varepsilon_1}^{H_1} - B_{r_1}^{H_1})]| + \varepsilon_2^{2H_1} E[B_{r_1}^{H_1}(B_{r_1+\varepsilon_1}^{H_1} - B_{r_1}^{H_1})] - \varepsilon_1^{2H_1} E[B_{r_1}^{H_1}(B_{r_1+\varepsilon_2}^{H_1} - B_{r_1}^{H_1})] \to 0 \quad (\varepsilon_1, \varepsilon_2 \to 0),
$$

$$
\frac{1}{\varepsilon_1^{4H_1} \varepsilon_2^{2H_1}} |A_{2,0}(r, r', \varepsilon_1, 2)| = \frac{1}{\varepsilon_1^{4H_1} \varepsilon_2^{2H_1}} \left| E[B_{r_1}^{H_1}(B_{r_1+\varepsilon_1}^{H_1} - B_{r_1}^{H_1})] \right| + \varepsilon_2^{2H_1} E[B_{r_1}^{H_1}(B_{r_1+\varepsilon_1}^{H_1} - B_{r_1}^{H_1})] - \varepsilon_1^{2H_1} E[B_{r_1}^{H_1}(B_{r_1+\varepsilon_2}^{H_1} - B_{r_1}^{H_1})] \to 0 \quad (\varepsilon_1, \varepsilon_2 \to 0)
$$

and

$$
\frac{1}{\varepsilon_1^{4H_1} \varepsilon_2^{2H_1}} |A_{1,1}(r, r', \varepsilon_1, 2)| \leq \frac{1}{\varepsilon_1^{4H_1} \varepsilon_2^{2H_1}} \left| E[B_{r_1}^{H_1}(B_{r_1+\varepsilon_1}^{H_1} - B_{r_1}^{H_1})] \right| + \varepsilon_2^{2H_1} E[B_{r_1}^{H_1}(B_{r_1+\varepsilon_1}^{H_1} - B_{r_1}^{H_1})] - \varepsilon_1^{2H_1} E[B_{r_1}^{H_1}(B_{r_1+\varepsilon_2}^{H_1} - B_{r_1}^{H_1})] + \frac{1}{\varepsilon_1^{4H_1} \varepsilon_2^{2H_1}} \left| E[B_{r_1}^{H_1}(B_{r_1+\varepsilon_1}^{H_1} - B_{r_1}^{H_1})] \right| \to 0 \quad (\varepsilon_1, \varepsilon_2 \to 0)
$$
for all \( r, r' > 0 \), which deduce the convergence (5.18) for \( k + l = 2 \) by Lebesgue’s dominated
convergence theorem because

\[
\frac{1}{\varepsilon_1^{4H_1} \varepsilon_2^{2H_1}} |A_{k,l}(r, r', \varepsilon_1, 2)| \leq 2
\]

and

\[
\int_{D_{0,t}} \left| E[f^{(k)}(B_{s}^{H_1})f^{(l)}(B_{r}^{H_1})]\right| (ss')^{2H_1-1} dr'ds'drds \leq C_{H_1,H_2}\|f\|_{\mathcal{H}}^2
\]

for \( k + l = 2 \) by Lemma 5.1, Lemma 5.2 and proof of Theorem 3.1.

Consequently, the convergence (5.17) holds for \( i = 1, j = 2 \) and (5.13) follows. Similarly one can prove (5.14). Thus, we have established the second statement and Theorem 5.1 follows. \( \square \)

At the end of this section, we obtain the Bouleau-Yor type identity (5.1).

**Lemma 5.4.** Let \( 0 < H_1 < \frac{1}{2} \) and let \( f, f_1, f_2, \ldots \in \mathcal{H} \). If \( f_n \to f \) in \( \mathcal{H} \), as \( n \) tends to infinity, then we have

\[
[f_n(B^{H_1} - B^{H_2}), B^{H_1}]^{(GH)}_t \to [f(B^{H_1} - B^{H_2}), B^{H_1}]^{(GH)}_t
\]

in \( L^2 \) as \( n \to \infty \).

**Proof.** The convergence follows from

\[
E \left| [f_n(B^{H_1} - B^{H_2}), B^{H_1}]^{(GH)}_t - [f(B^{H_1} - B^{H_2}), B^{H_1}]^{(GH)}_t \right|^2 \leq C_{H_1,H_2,T}\|f_n - f\|_{\mathcal{H}}^2 \to 0,
\]

as \( n \) tends to infinity. \( \square \)

**Lemma 5.5.** Let \( 0 < H_1 < \frac{1}{2} \). For any \( f_\Delta = \sum_j f_j 1_{(a_{j-1}, a_j]} \in \mathcal{E} \), we define

\[
\int_{\mathbb{R}} f_\Delta(a)\ell'_t(a)da := \sum_j f_j [\ell_t(a_j) - \ell_t(a_{j-1})].
\]

Then the integral is well-defined and

\[
\int_{\mathbb{R}} f_\Delta(a)\ell'_t(a)da = -[f_\Delta(B^{H_1} - B^{H_2}), B^{H_1}]^{(GH)}_t
\]

almost surely, for all \( t \in [0, T] \).

**Proof.** For the function \( f_\Delta(u) = 1_{(x,y]}(u) \) we define the sequence of smooth functions \( f_n, n = 1, 2, \ldots \) by

\[
f_n(u) = \int_{\mathbb{R}} f_\Delta(u - v)\zeta_n(v)dv = \int_x^y \zeta_n(u - v)dv
\]

for all \( u \in \mathbb{R} \), where \( \zeta_n, n \geq 1 \) are the so-called mollifiers given by

\[
\zeta_n(u) := n\zeta(nu), \quad n = 1, 2, \ldots
\]

and

\[
\zeta(u) := \begin{cases} \frac{1}{u^{(a-1)}} & , u \in (0, 2), \\ 0, \quad \text{otherwise}, \end{cases}
\]
with a normalizing constant $c$ such that $\int_{\mathbb{R}} \zeta(u) du = 1$. Then \( \{f_n\} \subset \mathcal{C}^\infty(\mathbb{R}) \cap \mathcal{H} \) and \( f_n \) converges to \( f \) in \( \mathcal{H} \), as \( n \) tends to infinity. It follows from the occupation formula (4.15) that

\[
[f_n(B^{H_1} - B^{H_2}), B^{H_1}]_{t,LMQ} = \int_0^t s^{2H_2-1} ds \int_0^s f_n'(B^H_s - B^H_s) dr
\]

\[
= -\int_{\mathbb{R}} f_n(a) \ell'(a) da + \int_{\mathbb{R}} (\int_y^\infty \zeta_n(a-u) du) \ell'(a) da
\]

\[
= -\int_{\mathbb{R}} \int_y^\infty \zeta_n(a-u) \ell'(a) da + \int_{\mathbb{R}} \int_y^\infty \zeta_n(a-u) \ell(t) da
\]

\[
= \int_{\mathbb{R}} \ell(t) \zeta_n(a-y) - \zeta_n(a-x) da + \int_{\mathbb{R}} \ell(t) \zeta_n(a-x) da
\]

\[
\to \ell(t)(x) - \ell(t)(y)
\]

almost surely, as \( n \to \infty \), by the continuity of \( a \mapsto \ell_t(a) \). On the other hand, Lemma 5.4 implies that there exists a subsequence \( \{f_{n_k}\} \) such that

\[
[f_{n_k}(B^{H_1} - B^{H_2}), B^{H_1}]_{t,GH} \to [f_\triangle(B^{H_1} - B^{H_2}), B^{H_1}]_{t,GH}
\]

for all \( t \in [0, T] \), almost surely, as \( k \to \infty \), which deduces

\[
[f_\triangle(B^{H_1} - B^{H_2}), B^{H_1}]_{t,GH} = \ell_t(x) - \ell_t(y)
\]

for all \( t \in [0, T] \), almost surely. Thus, the identity

\[
\int_{\mathbb{R}} f_\triangle(a) \ell'_t(a) da = -[f_\triangle(B^{H_1} - B^{H_2}), B^{H_1}]_{t,GH}
\]

follows from the linearity property. This completes the proof. \( \square \)

Thanks to the above lemma we can show that

(5.26) \[
\lim_{n \to \infty} \int_{\mathbb{R}} g_\triangle,n(a) \ell'_t(a) da = \lim_{n \to \infty} \int_{\mathbb{R}} g_\triangle,n(a) \ell'_t(a) da = -[f(B^{H_1} - B^{H_2}), B^{H_1}]_{t,GH}
\]

in \( L^2(\Omega) \) if

\[
\lim_{n \to \infty} f_\triangle,n(a) = \lim_{n \to \infty} g_\triangle,n(a) = f(a)
\]

in \( \mathcal{H} \), where \( \{f_\triangle,n\}, \{g_\triangle,n\} \subset \mathcal{E} \). Thus, by the density of \( \mathcal{E} \) in \( \mathcal{H} \) we can define

\[
\int_{\mathbb{R}} f(a) \ell'_t(a) da := \lim_{n \to \infty} \int_{\mathbb{R}} f_\triangle,n(a) \ell'_t(a) da
\]

for any \( f \in \mathcal{H} \), where \( \{f_\triangle,n\} \subset \mathcal{E} \) and

\[
\lim_{n \to \infty} f_\triangle,n(a) = f(a)
\]

in \( \mathcal{H} \). Thus, we have proved the following theorem.

**Theorem 5.2.** Let \( 0 < H_1 < \frac{1}{2} \) and \( f \in \mathcal{H} \). The integral

\[
\int_{\mathbb{R}} f(a) \ell'_t(a) da
\]

is well-defined for all \( t \in [0, T] \), and the Bouleau-Yor type identity

(5.27) \[
[f(B^{H_1} - B^{H_2}), B^{H_1}]_{t,GH} = -\int_{\mathbb{R}} f(a) \ell'_t(a) da
\]

holds for all \( t \in [0, T] \).
6. The Generalized Hybrid Quadratic Covariation, Case $H_1 \geq \frac{1}{2}$

In this section we consider the GHQC with $\frac{1}{2} \leq H_1 \leq H_2 < \frac{2}{3}$ and obtain a similar Bouleau-Yor type identity. It is important to note that the same method as Section 5 is inapplicability for $H_1 > \frac{1}{2}$. Essentially, for $H_1 > \frac{1}{2}$ we have

$$E \left[ (B_{r+\varepsilon}^{H_1} - B_r^{H_1}) (B_{r+\varepsilon}^{H_2} - B_r^{H_2}) \right] \sim \varepsilon^2 \neq o \left( \varepsilon^{4H_1} \right) \quad (\varepsilon \to 0)$$

for $r > r' + \varepsilon$ and the decomposition

$$J_\varepsilon(H_1, H_2, t, f) = \frac{1}{\varepsilon^{2H_1}} \int_0^t \int_0^s \int_0^t f(s) B_r^{H_1} - B_s^{H_1} (B_{r+\varepsilon}^{H_1} - B_r^{H_1}) dr ds ds$$

$$- \frac{1}{\varepsilon^{2H_1}} \int_0^t \int_0^s \int_0^t f(s) B_r^{H_1} - B_s^{H_2} (B_{r+\varepsilon}^{H_2} - B_r^{H_2}) dr ds ds$$

$$= J_\varepsilon^+(H_1, H_2, t, f) - J_\varepsilon^-(H_1, H_2, t, f)$$

does not bring any information because

$$E J_\varepsilon^+(H_1, H_2, t, f) \to \infty \quad (\varepsilon \to 0),$$

in general. For example, for $f(x) = x$ we have

$$\frac{1}{\varepsilon^{2H_1}} \int_0^t \int_0^s \int_0^t E \left[ (B_r^{H_1} - B_s^{H_2}) (B_{r+\varepsilon}^{H_1} - B_r^{H_1}) \right] dr ds ds$$

$$= \frac{1}{\varepsilon^{2H_1}} \int_0^t \int_0^s \int_0^t E \left[ B_r^{H_1} (B_{r+\varepsilon}^{H_1} - B_r^{H_1}) \right] dr ds ds$$

$$\to \infty,$$

as $\varepsilon \downarrow 0$. Thus, we must estimate $E|J_\varepsilon(H_1, H_2, t, f)|^2$ integrally when $H_1 > \frac{1}{2}$. Moreover, we shall also use the Young integration

$$\int_\mathbb{R} f(a) \ell_t(da),$$

in order to study the existence of the GHQC $[f(B^{H_1} - B^{H_2}), B^{H_1}]^{(GH)}$.

**Lemma 6.1.** Let $H_1 \geq \frac{1}{2}$.

- If $\frac{1}{2} \leq H_2 < \frac{2}{3}$, then for any $f \in C^{(\nu)}$ with $\nu > 2H_1 - 1$, the Young integral

$$\int_\mathbb{R} f(a) \ell_t(da) \equiv \int_\mathbb{R} f(a) \ell_t^*(a) da$$

is well-defined for all $t \geq 0$;

- If $\frac{2}{3} \leq H_2 < 1$, then for $f \in C^{\nu}$ with $\nu > \frac{3H_2 - 2}{2H_2}$, the Young integral

$$\int_\mathbb{R} f(a) \ell_t(da)$$

is well-defined for all $t \geq 0$;

- Let either $\nu > 2H_1 - 1$ with $\frac{1}{2} \leq H_2 < \frac{2}{3}$ or $\nu > \frac{3H_2 - 2}{2H_2}$ with $\frac{2}{3} \leq H_2 < 1$. If $f, f_1, f_2, \ldots \in C^{\nu}$ and $f_n \to f$ in $C^{\nu}$, as $n$ tends to infinity, then we have

$$\int_\mathbb{R} f_n(a) \ell_t(da) \to \int_\mathbb{R} f(a) \ell_t(da)$$

in $L^2$ as $n \to \infty$. 


The above lemma follows from the Hölder continuity of $a \mapsto \ell_t(a)$. For more aspects on Young integration we refer to Dudley-Norvaisa [7] and Young [38]. To estimate $E|J_\varepsilon(H_1, H_2, t, f)|^2$, we denote
\[
\Delta_\varepsilon f(B_{r+\varepsilon}^H - B_s^H) := f(B_{r+\varepsilon}^H - B_s^H) - f(B_r^H - B_s^H)
\]
and
\[
\Upsilon_\varepsilon(H_1, H_2) := E \left[ \Delta_\varepsilon f(B_{r+\varepsilon}^H - B_s^H) \Delta_\varepsilon f(B_{r'+\varepsilon}^H - B_{s'}^H)(B_r^H - B_{r+\varepsilon}^H)(B_{r' - \varepsilon}^H - B_{r+\varepsilon}^H) \right]
\]
for all $\varepsilon > 0$, $s, r, s', r' > 0$ and Borel functions $f \in C^\nu$ with $\nu > 0$. Then we have
\[
E|J_\varepsilon(H_1, H_2, t, f)|^2 = \frac{1}{\varepsilon^4 H_1} \int_0^t \int_0^t \int_0^t \int_0^t (s_s')^{2H_2-1} ds' ds \int_0^t \int_0^t \Upsilon_\varepsilon(H_1, H_2) dr' dr
\]
for all $\varepsilon > 0$ and $t \in [0, T]$. By approximating we can assume that $f$ is an infinitely differentiable function with compact support. It follows from the duality relationship (2.1) and
\[
D_{u_i}^H \Delta_\varepsilon f(B_{r+\varepsilon}^H - B_s^H) = 1_{[0, r+\varepsilon]}(u) f'(B_{r+\varepsilon}^H - B_s^H) - 1_{[0, r]}(u) f'(B_r^H - B_s^H)
\]
that
\[
\Upsilon_\varepsilon(H_1, H_2) = E \left[ \Delta_\varepsilon f(B_{r+\varepsilon}^H - B_s^H) \Delta_\varepsilon f(B_{r'+\varepsilon}^H - B_{s'}^H)(B_r^H - B_{r+\varepsilon}^H)(B_{r' - \varepsilon}^H - B_{r+\varepsilon}^H) \int_{r'}^{r' + \varepsilon} dB_{t_i}^H \right]
\]
for all $s > r > 0$, $s' > r' > 0$ and $\varepsilon > 0$. We have
\[
\Psi_\varepsilon(1; H_1, H_2) = E \left[ (B_{r+\varepsilon}^H - B_{r'}^H)(B_{r'+\varepsilon}^H - B_{r'}^H) \right] E \left[ B_{r+\varepsilon}^H(B_{r'}^H - B_{r'}^H) \right] E \left[ f''(B_{r+\varepsilon}^H - B_{s'}^H) \Delta_\varepsilon f(B_{r'}^H - B_{s'}^H) \right]
\]
\[
+ E \left[ (B_{r'+\varepsilon}^H - B_{r'}^H) \Delta_\varepsilon f(B_{r'}^H - B_{s'}^H) f'(B_{r'}^H - B_{s'}^H)(B_{r'}^H - B_{r'}^H) \right] E \left[ (B_{r'}^H - B_{r'}^H)(B_{r'}^H - B_{r'}^H) \right] E \left[ \Delta_\varepsilon f(B_{r'}^H - B_{s'}^H) \Delta_\varepsilon f(B_{r'}^H - B_{s'}^H) \right]
\]
\[
+ E \left[ (B_{r'}^H - B_{r'}^H)(B_{r'}^H - B_{r'}^H) \right] E \left[ \Delta_\varepsilon f(B_{r'}^H - B_{s'}^H) \Delta_\varepsilon f(B_{r'}^H - B_{s'}^H) \right] E \left[ \Delta_\varepsilon f(B_{r'}^H - B_{s'}^H) \Delta_\varepsilon f(B_{r'}^H - B_{s'}^H) \right]
\]
\[
= \sum_{j=1}^5 \Psi_\varepsilon(j; H_1, H_2)
\]
\[\Psi_\varepsilon(2; H_1, H_2) = E \left[ B_{r'}^{H_1}(B_{r'+\varepsilon}^{H_1} - B_{r'}^{H_1}) \right] E \left[ (B_{r'}^{H_1} - B_r^{H_1})^2 \right]
\cdot E \left[ f''(B_{r'+\varepsilon}^{H_1} - B_{s'}^{H_2})\Delta_{\varepsilon} f(B_{r'}^{H_1} - B_{s'}^{H_2}) \right]
+ E \left[ B_{r'}^{H_1}(B_{r'+\varepsilon}^{H_1} - B_{r'}^{H_1}) \right] E \left[ B_r^{H_1}(B_{r'+\varepsilon}^{H_1} - B_r^{H_1}) \right]
\cdot E \left[ \Delta_{\varepsilon} f''(B_{r'+\varepsilon}^{H_1} - B_{s'}^{H_2})\Delta_{\varepsilon} f(B_{r'}^{H_1} - B_{s'}^{H_2}) \right]
+ E \left[ B_{r'}^{H_1}(B_{r'+\varepsilon}^{H_1} - B_{r'}^{H_1}) \right] E \left[ (B_{r'+\varepsilon}^{H_1} - B_{r'}^{H_1})(B_{r'+\varepsilon}^{H_1} - B_{r'}^{H_1}) \right]
\cdot E \left[ \Delta_{\varepsilon} f'(B_{r'}^{H_1} - B_{s'}^{H_2})\Delta_{\varepsilon} f'(B_{r'}^{H_1} - B_{s'}^{H_2}) \right],\]

for all \(\varepsilon > 0\) and \(s, r > 0\).

**Lemma 6.2.** For all \(\varepsilon \geq 0, s > r > 0, s' > r' > 0\) and \(f \in C_0^\infty(\mathbb{R}) \cap C'\) with \(\nu > 0\) we denote

\[\Theta_\varepsilon(2, \Delta 0) := E[f''(B_{r'+\varepsilon}^{H_1} - B_{s'}^{H_2})\Delta_{\varepsilon} f(B_{r'}^{H_1} - B_{s'}^{H_2})].\]

Then we have

\[|\Theta_\varepsilon(2, \Delta 0)| \leq C_{H_1, H_2, T}\|f\|_{(2)}^2(r^{r'})^{-\frac{3\nu}{2}}(s s'|s - s'|)^{-\frac{3\nu}{2}} \left( \frac{1_{\{r' > r\}}}{|r - r'|^{\frac{3\nu}{2}}} + \frac{1_{\{r' > r\}}}{|r + \varepsilon - r'|^{\frac{3\nu}{2}}} \right) \varepsilon^\nu.\]
Proof. To prove the lemma we let \( \varphi_{\varepsilon_1, \varepsilon_2}(x, y) \) be the density function of

\[
(B_{r+\varepsilon_1} - B_{s_1}^H, B_{s_1}^H - B_{s_1'})
\]

with \( s > r > 0, s' > r' > 0 \) and \( \varepsilon_1, \varepsilon_2 \geq 0 \). That is

\[
\varphi_{\varepsilon_1, \varepsilon_2}(x, y) = \frac{1}{2\pi \rho_{\varepsilon_1, \varepsilon_2}} \exp \left\{ -\frac{1}{2\rho_{\varepsilon_1, \varepsilon_2}^2} \left( \lambda_{r+\varepsilon_2, s'} x^2 - 2\mu_{\varepsilon_1, \varepsilon_2} x y + \lambda_{r+\varepsilon_2, y}^2 \right) \right\},
\]

where \( \lambda_{r+\varepsilon_2, s} = E \left[ (B_{r+\varepsilon_1} - B_{s_1}^H)^2 \right], \mu_{\varepsilon_1, \varepsilon_2} = E \left[ (B_{r+\varepsilon_1} - B_{s_1}^H)(B_{s_1}^H - B_{s_1'}) \right] \) and

\[
\rho_{\varepsilon_1, \varepsilon_2}^2 = \lambda_{r+\varepsilon_2, s} \lambda_{r+\varepsilon_2, s'} - \mu_{\varepsilon_1, \varepsilon_2}^2.
\]

Then we have, by making substitutions

\[
x = \sqrt{\lambda_{r+\varepsilon_2, s}} u, \quad y = \frac{\mu_{\varepsilon_1, \varepsilon_2}}{\sqrt{\lambda_{r+\varepsilon_2, s}}} u + \frac{\rho_{\varepsilon_1, \varepsilon_2}}{\sqrt{\lambda_{r+\varepsilon_2, s}}} v,
\]

\[
E \left[ f''(B_{r+\varepsilon_1} - B_{s_1}^H) f(B_{s_1}^H - B_{s_1'}) \right] = \int_{\mathbb{R}^2} f(x) f(y) \frac{\partial^2}{\partial x^2} \varphi_{\varepsilon_1, \varepsilon_2}(x, y) dx dy
\]

\[
= \int_{\mathbb{R}^2} f(x) f(y) \left\{ \frac{1}{\rho_{\varepsilon_1, \varepsilon_2}} (\lambda_{r+\varepsilon_2, s}' x - y \mu_{\varepsilon_1, \varepsilon_2})^2 - \frac{\lambda_{r+\varepsilon_2, s}^2}{\rho_{\varepsilon_1, \varepsilon_2}^2} \right\} \varphi_{\varepsilon_1, \varepsilon_2}(x, y) dx dy
\]

\[
= \int_{\mathbb{R}^2} f \left( \sqrt{\lambda_{r+\varepsilon_2, s}} u \right) f \left( \frac{\rho_{\varepsilon_1, \varepsilon_2}}{\sqrt{\lambda_{r+\varepsilon_2, s}}} v + \frac{\mu_{\varepsilon_1, \varepsilon_2}}{\sqrt{\lambda_{r+\varepsilon_2, s}}} u \right)
\]

\[
\cdot \left( \frac{u^2}{\lambda_{r+\varepsilon_2, s}} - \frac{2\mu_{\varepsilon_1, \varepsilon_2}}{\rho_{\varepsilon_1, \varepsilon_2}^2} \frac{\mu_{\varepsilon_1, \varepsilon_2}}{\lambda_{r+\varepsilon_2, s}} u \right) \frac{1}{2\pi} e^{-\frac{1}{2}(u^2+v^2)} dudv
\]

for all \( s > r > 0, s' > r' > 0 \) and \( \varepsilon \geq 0 \) and

\[
\Theta_{\varepsilon}(2, \Delta 0) = E \left[ f''(B_{r+\varepsilon_1} - B_{s_1}^H) f(B_{s_1}^H - B_{s_1'}) \right] - E \left[ f''(B_{r+\varepsilon_1} - B_{s_1}^H) f(B_{s_1} - B_{s_1'}) \right]
\]

\[
= \int_{\mathbb{R}^2} f \left( \sqrt{\lambda_{r+\varepsilon_2, s}} u \right) \frac{1}{2\pi} e^{-\frac{1}{2}(u^2+v^2)}
\]

\[
\cdot \left\{ \left( \frac{\rho_{\varepsilon_1, \varepsilon_2}}{\sqrt{\lambda_{r+\varepsilon_2, s}}} v + \frac{\mu_{\varepsilon_1, \varepsilon_2}}{\sqrt{\lambda_{r+\varepsilon_2, s}}} u \right) \left( \frac{u^2}{\lambda_{r+\varepsilon_2, s}} - \frac{2\mu_{\varepsilon_1, \varepsilon_2}}{\rho_{\varepsilon_1, \varepsilon_2}^2} \frac{\mu_{\varepsilon_1, \varepsilon_2}}{\lambda_{r+\varepsilon_2, s}} u \right) - \frac{\lambda_{r+\varepsilon_2, s}^2}{\rho_{\varepsilon_1, \varepsilon_2}^2} \right\} dudv
\]

\[
= \int_{\mathbb{R}^2} f \left( \sqrt{\lambda_{r+\varepsilon_2, s}} u \right) \frac{1}{2\pi} e^{-\frac{1}{2}(u^2+v^2)}
\]

\[
\cdot \left\{ \frac{2uv}{\lambda_{r+\varepsilon_2, s}} \right\} dudv
\]

\[
\equiv \Theta_{\varepsilon}(2, \Delta 0, 1) + \Theta_{\varepsilon}(2, \Delta 0, 2)
\]

for all \( s > r > 0, s' > r' > 0 \) and \( \varepsilon \geq 0 \). We need to estimate \( \Theta_{\varepsilon}(2, \Delta 0, 1) \) and \( \Theta_{\varepsilon}(2, \Delta 0, 2) \). By symmetry we may assume that \( s > s' \).
For all \( r, r' > 0 \) and \( \varepsilon \geq 0 \) we have

\[
|\mu_{\varepsilon, 0} - \mu_{\varepsilon, \varepsilon}| = |E[(B_{r+\varepsilon}^{H_1} - B_{s}^{H_2})(B_{r'}^{H_1} - B_{s}^{H_2})] - E[(B_{r+\varepsilon}^{H_1} - B_{s}^{H_2})(B_{r'}^{H_1} - B_{s}^{H_2})]| \\
= |E[(B_{r+\varepsilon}^{H_1} - B_{s}^{H_2})(B_{r'}^{H_1} - B_{s}^{H_2})]| \\
= |E[B_{r+\varepsilon}^{H_1}(B_{r'}^{H_1} - B_{s}^{H_2})]| \leq C_{H_1} (r + \varepsilon)^{2H_1 - 1} \varepsilon
\]

by Lemma 5.1 and \(|\lambda_{r',s'} - \lambda_{r'+\varepsilon, s'} | = (r')^{2H_1} - (r' + \varepsilon)^{2H_1} | \leq C_{H_1} (r' + \varepsilon)^{2H_1 - 1} \varepsilon\), which imply that

\[
|\rho_{\varepsilon, \varepsilon} - \rho_{\varepsilon, 0}| = |\lambda_{r+\varepsilon, s}\lambda_{r'+\varepsilon, s'} - \mu_{\varepsilon, \varepsilon} - \lambda_{r+\varepsilon, s}\lambda_{r'+\varepsilon, s'} + \mu_{\varepsilon, 0}| \\
\leq \lambda_{r+\varepsilon, s}|\lambda_{r'+\varepsilon, s'} - \lambda_{r'+\varepsilon, s'}| + |\mu_{\varepsilon, \varepsilon} - \mu_{\varepsilon, 0}| \\
\leq C_{H_1}\lambda_{r+\varepsilon, s}(r' + \varepsilon)^{2H_1 - 1} \varepsilon + C_{H_1}(\mu_{\varepsilon, \varepsilon} + \mu_{\varepsilon, 0})(r + \varepsilon)^{2H_1 - 1} \varepsilon \\
\leq C_{H_1}\lambda_{r+\varepsilon, s}(r' + \varepsilon)^{2H_1 - 1} \varepsilon + C_{H_1}\sqrt{\lambda_{r+\varepsilon, s}\lambda_{r'+\varepsilon, s'}(r' + \varepsilon)^{2H_1 - 1} \varepsilon}
\]

for all \( s, s', r, r' > 0 \) and \( \varepsilon \geq 0 \). It follows from the fact (see the proof of Theorem 3.1)

\[
\rho_{\varepsilon, \varepsilon}^2 = \lambda_{r+\varepsilon, s}\lambda_{r'+\varepsilon, s'} - \mu_{\varepsilon, \varepsilon}^2 \times ((r' + \varepsilon)^{2H_1} + (s')^{2H_2})(|r - r'|^{2H_1} + |s - s'|^{2H_2})
\]

for \( s > s', s' > r' + \varepsilon \) and \( s > r + \varepsilon \) that

\[
\frac{1}{\lambda_{r+\varepsilon, s}} \left| \frac{\mu_{\varepsilon, 0}}{\rho_{\varepsilon, 0}} - \frac{\mu_{\varepsilon, \varepsilon}}{\rho_{\varepsilon, \varepsilon}} \right| = \frac{(\lambda_{r+\varepsilon, s})^{-1}}{\rho_{\varepsilon, 0}} |\rho_{\varepsilon, \varepsilon}\mu_{\varepsilon, 0} - \rho_{\varepsilon, \varepsilon}\mu_{\varepsilon, \varepsilon}| \\
\leq \frac{(\lambda_{r+\varepsilon, s})^{-1}}{\rho_{\varepsilon, 0}} |\mu_{\varepsilon, 0} - \mu_{\varepsilon, \varepsilon}| + \frac{(\lambda_{r+\varepsilon, s})^{-1}}{\rho_{\varepsilon, 0}} |\mu_{\varepsilon, \varepsilon} - \mu_{\varepsilon, \varepsilon}| \\
\leq C_{H_1}(r' + \varepsilon)^{2H_1 - 1} \varepsilon \\
+ \frac{C_{H_1}\sqrt{\lambda_{r'+\varepsilon, s'}(r' + \varepsilon)^{2H_1 - 1} \varepsilon}}{((r')^{H_1} + (s')^{H_2})(|r + \varepsilon - r'|^{H_1} + |s - s'|^{H_2})(|r - r'|^{H_1} + |s - s'|^{H_2})} \\
\leq C_{H_1, H_2, T}(r'^{-H_1}(ss')^{-H_2}|s - s'|^{-H_2}r - r'|^{-H_1} \varepsilon)
\]

for \( s > s', s' > r' + \varepsilon, s > r + \varepsilon \geq 0 \). Similarly, we also have

\[
\frac{1}{\lambda_{r+\varepsilon, s}} \left| \frac{\mu_{\varepsilon, 0}}{\rho_{\varepsilon, 0}} - \frac{\mu_{\varepsilon, \varepsilon}}{\rho_{\varepsilon, \varepsilon}} \right| = \frac{(\lambda_{r+\varepsilon, s})^{-1}}{\rho_{\varepsilon, 0}^2} \left| \rho_{\varepsilon, \varepsilon}\mu_{\varepsilon, 0}^2 - \rho_{\varepsilon, \varepsilon}\mu_{\varepsilon, \varepsilon}^2 \right| \\
\leq \frac{(\lambda_{r+\varepsilon, s})^{-1}}{\rho_{\varepsilon, 0}^2} |\mu_{\varepsilon, 0} - \mu_{\varepsilon, \varepsilon}| + \frac{(\lambda_{r+\varepsilon, s})^{-1}}{\rho_{\varepsilon, 0}^2} |\mu_{\varepsilon, \varepsilon} - \mu_{\varepsilon, \varepsilon}| \\
(6.3)
\]

\[
\leq C_{H_1, H_2, T}(r'^{-H_1}(ss')^{-H_2}|s - s'|^{-H_2}r - r'|^{-H_1} \varepsilon) \\
+ C_{H_1, H_2, T}(r'^{-H_1}(ss')^{-H_2}|s - s'|^{-H_2}r - r'|^{-H_1} \varepsilon1_{r > r'}) \\
+ C_{H_1, H_2, T}(r'^{-H_1}(ss')^{-H_2}|s - s'|^{-H_2}r - r'|^{-H_1} \varepsilon1_{r > r'})
\]
and
\[
\left| \frac{\lambda_{r',s'}^\varepsilon - \lambda_{r'+\varepsilon,s'}}{\rho_{s,0}^2} - \frac{\lambda_{r'+\varepsilon,s'}}{\rho_{s,0}^2} \right| \leq \frac{1}{2} \left| \frac{\lambda_{r',s'} - \lambda_{r'+\varepsilon,s'}}{\rho_{s,0}^2} \right| + \frac{\lambda_{r',s'}}{\rho_{s,0}^2 \rho_{s,0}^2} \left| \rho_{s,0}^2 - \rho_{s,0}^2 \right|
\leq \frac{1}{2} \left| \frac{\lambda_{r'} - \lambda_{r'+\varepsilon}}{\rho_{s,0}^2} \right| + \frac{\lambda_{r',s'}}{\rho_{s,0}^2} \left| \rho_{s,0}^2 - \rho_{s,0}^2 \right|
\leq C_{H_1,H_2,T}(r')^{-H_1} (s')^{-H_2} |r + r' - |s - s'|^{-H_2} \varepsilon
\]
\[
+ C_{H_1,H_2,T}(r')^{-\frac{3H_1}{2}} (s')^{-\frac{3H_2}{2}} |s - s'|^{-\frac{3H_2}{2}} |r - r'|^{-\frac{3H_1}{2}} \varepsilon \mathbb{1}_{r > r'}
\]
\[
+ C_{H_1,H_2,T}(r')^{-\frac{3H_1}{2}} (s')^{-\frac{3H_2}{2}} |s - s'|^{-\frac{3H_2}{2}} |r + r' - |r + r'|^{-\frac{3H_1}{2}} \varepsilon \mathbb{1}_{r > r'}
\]
for \(s > s', s' > r + \varepsilon, s > r + \varepsilon\). Consequently, we get
\[
\Theta_\varepsilon(2, \Delta_0, 2) = \int_{\mathbb{R}^2} \left| f(\sqrt{\lambda_{r+\varepsilon,s} u}) \right| \left| f(\sqrt{\lambda_{r+\varepsilon,s} u}) \right| \left| \frac{2uv}{\lambda_{r+\varepsilon,s}} \left( \frac{\mu_{s,0}}{\rho_{s,0}} - \frac{\mu_{s,0}}{\rho_{s,0}} \right) + u^2 + \frac{\mu_{s,0}}{\rho_{s,0}} \left( \frac{\mu_{s,0}}{\rho_{s,0}} - \frac{\mu_{s,0}}{\rho_{s,0}} \right) + \frac{\lambda_{r'+\varepsilon,s'}}{\rho_{s,0}^2} - \frac{\lambda_{r'+\varepsilon,s'}}{\rho_{s,0}^2} \right| \left| \frac{1}{2\pi} e^{-\frac{1}{2}(u^2+v^2)} \right| dudv
\]
\[
\leq C_{H_1,H_2,T} \| f \|_{(r')}^2 \frac{1}{|r - r'|^{\frac{3H_1}{2}}} \frac{1}{|r - r' + \varepsilon|^{\frac{3H_2}{2}}}
\]
and
\[
\Theta_\varepsilon(2, \Delta_0, 1) = \int_{\mathbb{R}^2} \left| f(\sqrt{\lambda_{r+\varepsilon,s} u}) \right| \left| f(\sqrt{\lambda_{r+\varepsilon,s} u}) \right| \left| \frac{2uv}{\lambda_{r+\varepsilon,s}} \left( \frac{\mu_{s,0}}{\rho_{s,0}} - \frac{\mu_{s,0}}{\rho_{s,0}} \right) + u^2 + \frac{\mu_{s,0}}{\rho_{s,0}} \left( \frac{\mu_{s,0}}{\rho_{s,0}} - \frac{\mu_{s,0}}{\rho_{s,0}} \right) + \frac{\lambda_{r'+\varepsilon,s'}}{\rho_{s,0}^2} - \frac{\lambda_{r'+\varepsilon,s'}}{\rho_{s,0}^2} \right| \left| \frac{1}{2\pi} e^{-\frac{1}{2}(u^2+v^2)} \right| dudv
\]
\[
\leq C_{H_1,H_2,T} \| f \|_{(r')}^2 \frac{1}{|r - r'|^{\frac{3H_1}{2}}} \frac{1}{|r + r' - \varepsilon|^{\frac{3H_2}{2}}}
\]
for all \(s' > r + \varepsilon\) and \(s > r + \varepsilon\), which proves
\[
|\Theta_\varepsilon(2, \Delta_0)| \leq |\Theta_\varepsilon(2, \Delta_0, 2)| + |\Theta_\varepsilon(2, \Delta_0, 1)|
\]
\[
\leq C_{H_1,H_2,T} \| f \|_{(r')}^2 \frac{1}{|r - r'|^{\frac{3H_1}{2}}} \frac{1}{|r + r' - \varepsilon|^{\frac{3H_2}{2}}}
\]
\[
for s > s', s' > r + \varepsilon, s > r + \varepsilon. Similarly, we can consider some other cases for
\[
\{s, r, s', r'\} \notin \{s > s', s > r + \varepsilon, s' > r' + \varepsilon\},
\]
and the lemma follows. \(\square\)

**Lemma 6.3.** For all \(\varepsilon \geq 0, s > r > 0, s' > r' > 0\) and \(f \in C_0^\infty(\mathbb{R}) \cap C^\nu\) with \(\nu > 0\) we denote
\[
(6.4) \quad \Theta_\varepsilon(1, \Delta_1) := E[f(B_{r+\varepsilon}^H - B_s^H) \Delta_\varepsilon f(B_{r+\varepsilon}^H - B_s^H)]
\]
Then we have
\[
|\Theta_\varepsilon(1, \Delta_1)| \leq C_{H_1,H_2,T} \| f \|_{(r')}^2 \frac{1}{|r - r'|^{\frac{3H_1}{2}}} \frac{1}{|r + r' - \varepsilon|^{\frac{3H_2}{2}}}
\]
Proof. In a same way as proof of Lemma 6.2 we have

\[ E[f'(B_{r+\varepsilon}^H - B_{s}^{H})']\cdot f'(B_{r+\varepsilon}^H - B_{s}^{H})] = \int_{\mathbb{R}^2} f(x)\, f(y) \frac{\partial^2}{\partial x \partial y} \varphi_{x, \varepsilon}(x, y)\, dx\, dy \]

\[ = \int_{\mathbb{R}^2} f(x)\, f(y) \left\{ \frac{1}{\rho_{x, \varepsilon}^2} (\lambda_{x, \varepsilon} x - y\mu_{x, \varepsilon}) (\lambda_{x, \varepsilon} y - \mu_{x, \varepsilon} x) + \frac{\mu_{x, \varepsilon}}{\rho_{x, \varepsilon}^2} \right\} \varphi_{x, \varepsilon}(x, y)\, dx\, dy \]

\[ = \int_{\mathbb{R}^2} f(\sqrt{\lambda_{x, \varepsilon} s} u) f\left( \frac{\rho_{x, \varepsilon}}{\rho_{x, \varepsilon}^2} v + \frac{\mu_{x, \varepsilon}}{\rho_{x, \varepsilon}^2} u \right) \left( uv + \frac{\mu_{x, \varepsilon}}{\rho_{x, \varepsilon}^2} (1 - v^2) \right) \frac{1}{2\pi} e^{-\frac{1}{2}(u^2 + v^2)}\, du\, dv \]

for all \( s > r > 0 \) and \( s' > r' > 0 \). This deduces

\[ \Theta_{\varepsilon}(1, \Delta 1) = E[f'(B_{r+\varepsilon}^H - B_{s}^{H})']\cdot f'(B_{r+\varepsilon}^H - B_{s}^{H})] - E[f'(B_{r+\varepsilon}^H - B_{s}^{H})']\cdot f'(B_{r'}^{H} - B_{s'}^{H})] \]

\[ = \int_{\mathbb{R}^2} f(\sqrt{\lambda_{x, \varepsilon} s} u) f\left( \frac{\rho_{x, \varepsilon}}{\rho_{x, \varepsilon}^2} v + \frac{\mu_{x, \varepsilon}}{\rho_{x, \varepsilon}^2} u \right) \left( uv + \frac{\mu_{x, \varepsilon}}{\rho_{x, \varepsilon}^2} (1 - v^2) \right) \frac{1}{2\pi} e^{-\frac{1}{2}(u^2 + v^2)}\, du\, dv \]

\[ = \int_{\mathbb{R}^2} f(\sqrt{\lambda_{x, \varepsilon} s} u) f\left( \frac{\rho_{x, \varepsilon}}{\rho_{x, \varepsilon}^2} v + \frac{\mu_{x, \varepsilon}}{\rho_{x, \varepsilon}^2} u \right) \left( uv + \frac{\mu_{x, \varepsilon}}{\rho_{x, \varepsilon}^2} (1 - v^2) \right) \frac{1}{2\pi} e^{-\frac{1}{2}(u^2 + v^2)}\, du\, dv \]

for all \( s > r > 0 \), \( s' > r' > 0 \), and \( \varepsilon > 0 \), and the lemma follows from the proof of Lemma 6.2. \( \square \)

**Lemma 6.4.** Let \( \frac{1}{2} < H_1 \leq H_2 < \frac{2}{3} \) and let \( f \in C' \) with \( \nu \geq 2H_1 - 1 \). Then

\[ E|J_{\varepsilon}(H_1, H_2, t, f)|^2 \leq C_{H_1, H_2, \nu} \| f \|_{(\nu)}^2 \]

for all \( \varepsilon > 0 \) and \( t \in [0, T] \).

Proof. Given \( \varepsilon > 0 \) and \( t \in [0, T] \). We need to estimate

\[ \Gamma_j := \frac{1}{\varepsilon^4 H_1} \int_0^t \int_0^t (ss')^{2H_2 - 1} ds\, ds' \int_0^s \int_0^{s'} |\Psi_{\varepsilon}(j; H_1, H_2)| dr\, dr, \quad j = 1, 2, 3, 4, 5. \]

For \( j = 5 \), by (A,10) we have

\[ E[(B_{r+\varepsilon}^H - B_{s}^{H})(B_{r+\varepsilon}^H - B_{s}^{H})] \leq \frac{C_{H_1, \varepsilon}^2}{(r - r' - \varepsilon)^2 - 2H_1} \]

for \( r > r' + \varepsilon \), and

\[ E[(B_{r+\varepsilon}^H - B_{r'}^{H})(B_{r+\varepsilon}^H - B_{r'}^{H})] \leq \varepsilon^{2H_1} \leq \frac{\varepsilon^2}{(r - r')^2 - 2H_1} \]

for \( r' < r < r' + \varepsilon \), which deduce

\[ \Gamma_5 = \frac{1}{\varepsilon^4 H_1} \int_0^t \int_0^t (ss')^{2H_2 - 1} ds\, ds' \int_0^s \int_0^{s'} |\Psi_{\varepsilon, \varepsilon}(5; H_1, H_2)| dr\, dr \leq C_{H_1, H_2, \nu} \| f \|_{(\nu)}^2 \]

for all \( \varepsilon > 0 \) and \( t \in [0, T] \).

For \( j = 1 \) we have

\[ E[(B_{r+\varepsilon}^H - B_{r'}^{H})(B_{r'+\varepsilon}^H - B_{r'}^{H})] \leq \varepsilon^{2H_1} \]
by Cauchy’s inequality and
\[
|E[f'(B_{r+\varepsilon}^1 - B_{r+\varepsilon}^2)]| = \left| \int_{\mathbb{R}^2} f(x,y) \frac{\partial^2}{\partial x \partial y} \varphi_{\varepsilon}(x,y) dx dy \right|
\]
\[
\leq \int_{\mathbb{R}^2} |f(x,y)| \frac{1}{\rho_{\varepsilon}} (\lambda_{r+\varepsilon}y - \mu_{\varepsilon})(\lambda_{r+\varepsilon}y - \mu_{\varepsilon}) + \frac{\mu_{\varepsilon}}{2\rho_{\varepsilon}^2} \varphi_{\varepsilon}(x,y) dx dy
\]
\[
\leq \|f\|_{L^2}(\mu) \left( \mu_{\varepsilon} + \lambda_{r+\varepsilon} \lambda_{r+\varepsilon} \right) \frac{1}{\rho_{\varepsilon}^2}
\]
for all \( s > r > 0 \) and \( s' > r' > 0 \). It follows from Lemma 5.1, Lemma 6.2 and Lemma 6.3 that
\[
\Gamma_1 = \frac{1}{\varepsilon^4 H_1} \int_0^t \int_0^t (ss')^{2H_2 - 1} dsds' \int_0^s \int_0^{s'} |\Psi_{\varepsilon}(1; H_1, H_2)| dr' dr \leq C_{H_1,H_2,T,\nu} \|f\|_{L^2}^2
\]
for all \( \varepsilon > 0 \). Similarly, we also have
\[
\Gamma_j = \frac{1}{\varepsilon^4 H_1} \int_0^t \int_0^t (ss')^{2H_2 - 1} dsds' \int_0^s \int_0^{s'} |\Psi_{\varepsilon}(j; H_1, H_2)| dr' dr \leq C_{H_1,H_2,T,\nu} \|f\|_{L^2}^2
\]
for all \( \varepsilon > 0 \) and \( j = 2, 3, 4 \), and the lemma follows. \( \square \)

Now, we can obtain our main object of this section.

**Theorem 6.1.** Let \( f \in C^\nu \) with \( \nu \geq 2H_1 - 1 \) and \( \frac{1}{2} < H_1 \leq H_2 < \frac{2}{3} \). Then, the GHQC \( [f(B_{H_1} - B_{H_2}), B_{H_1}^{GH}] \) exists and the Bouleau-Yor type identity
\[
(f(B_{H_1} - B_{H_2}), B_{H_1}^{GH}) = - \int_\mathbb{R} f(x) \ell_t'(x) dx
\]
holds. Moreover, we have
\[
E \left| f(B_{H_1} - B_{H_2}), B_{H_1}^{GH} \right|^2 \leq C_{H_1,H_2,T} \|f\|_{L^2}^2
\]
for all \( t \in [0,T] \).

**Proof.** Given \( f \in C^\nu \). Define the sequence of smooth functions
\[
n(x) = \int_\mathbb{R} f(x - y, \zeta_n(y) dy = \int_0^2 f(x - y, n(y) dy, \quad n = 1, 2, \ldots
\]
for all \( x \in \mathbb{R} \), where the mollifiers \( \zeta_n, n = 1, 2, \ldots \) are given by (5.21). Then \( \{n\} \subset C_0(\mathbb{R}) \cap C^\nu \), \( f_n \) converges to \( f \) in \( C^\nu \) and
\[
J_\varepsilon(H_1, H_2, t, f_n) \longrightarrow - \int_\mathbb{R} f_n(x) \ell_t'(x) dx
\]
in \( L^2 \) by Corollary 5.1 as \( \varepsilon \) tends to 0, for all \( n \geq 1 \).

On the other hand, by Lemma 6.4, we have
\[
E \left| J_\varepsilon(H_1, H_2, t, f) + \int_\mathbb{R} f(x) \ell_t'(x) dx \right|^2 \leq 3E \left| J_\varepsilon(H_1, H_2, t, f) - J_\varepsilon(H_1, H_2, t, f_n) \right|^2
\]
\[
+ 3E \left| J_\varepsilon(H_1, H_2, t, f_n) + \int_\mathbb{R} f_n(x) \ell_t'(x) dx \right|^2 + 3E \left| \int_\mathbb{R} f_n(x) \ell_t'(x) dx - \int_\mathbb{R} f(x) \ell_t'(x) dx \right|^2
\]
\[
\leq 3C_{H_1,H_2,T,\nu} \|f - f_n\|_{L^2}^2 + 3E \left| J_\varepsilon(H_1, H_2, t, f_n) + \int_\mathbb{R} f_n(x) \ell_t'(x) dx \right|^2
\]
\[
+ 3E \left| \int_\mathbb{R} f_n(x) \ell_t'(x) dx - \int_\mathbb{R} f(x) \ell_t'(x) dx \right|^2
\]
for all \( n, \varepsilon > 0 \) and \( t \in [0,T] \). Thus, the theorem follows from Lemma 6.1. \( \square \)
It is possible to extend formula (6.6) to any Hölder function of order \( \nu > 2H_1 - 1 \) by means of a localization argument. In fact, for any \( k \geq 0 \) we may consider the set

\[
\Omega_k = \left\{ \sup_{0 \leq t \leq T} |B_t^H| < k \right\}
\]

and let \( f_k \) be a Hölder function such that

\[
f_k(x) = \begin{cases} 
  f(-k), & \text{if } x < -k, \\
  f(x), & \text{if } -k \leq x \leq k, \\
  f(k), & \text{if } x > k.
\end{cases}
\]

Then \( f_k \in C^\nu \) with \( \nu > 2H_1 - 1 \) for every \( k \geq 0 \). By the above theorem we know that

\[
[f_k(B_t^{H_1} - B_t^{H_2}), B_t^H]^{(GH)} = -\int_{\mathbb{R}} f_k(x) \ell_t'(x) dx
\]
on the set \( \Omega_k \). Letting \( k \) tend to infinity we get the desired formula (6.6) for any Hölder function of order \( \nu > 2H_1 - 1 \).

**Remark 1.** Inspired by Lemma 6.1 we conjecture that the GHQC exists and the identity

\[
(6.9) \quad [f(B_t^{H_1} - B_t^{H_2}), B_t^H]^{(GH)} = -\int_{\mathbb{R}} f(x) \ell_t'(x) dx, \quad t \in [0, T]
\]

holds for all \( \frac{2}{3} \leq H_1, H_2 < 1 \) and \( f \in C^\nu \) with \( \nu > \frac{3H_1 - 2}{2H_2} \).

Finally, from the proof of Theorem 6.1 we can find that (6.6) is also true for \( H_1 = \frac{1}{2} \).

**Theorem 6.2.** Let the Brownian motion \( B \) be independent of \( f\text{Bm} B^H \) with \( H < \frac{2}{3} \) and let \( f \) be a Hölder function of order \( \nu \in (0, 1] \). Then, the GHQC \( [f(B - B^H), B]^{(GH)} \) and the Young integral

\[
\int_{\mathbb{R}} f(x) \ell_t'(x) dx = \int_{\mathbb{R}} f(x) \ell_t'(x) dx
\]

exist, and the Bouleau-Yor type identity

\[
(6.10) \quad [f(B - B^H), B]^{(GH)} = -\int_{\mathbb{R}} f(x) \ell_t'(x) dx,
\]

holds for all \( t \in [0, T] \), where \( \ell_t'(x) \) is the DILT of \( B \) and \( B^H \).

**7. Smoothness of the DILT of fBMs**

In the final section, we consider the smoothness of \( \ell_t'(0) \) for all \( t > 0 \). We start with the recall of chaos expansion (see, for examples, Nualart [25], Watanabe [32], Hu [17] and references therein). Let

\[
L^2(\Omega, \mathbb{P}) = \bigoplus_{n=0}^{\infty} C_n.
\]

be the chaos decomposition of \( L^2(\Omega, \mathbb{P}) \). Namely, for any functional \( F \in L^2(\Omega, \mathbb{P}) \), there are \( F_n \in C_n, \ n = 0, 1, 2, \cdots \) such that

\[
(7.1) \quad F = \sum_{n=0}^{\infty} F_n.
\]
The decomposition (7.1) is called the chaos expansion of $F$. From the orthogonality, we have
\[ E(F^2) = \sum_{n=0}^{\infty} E(|F_n|^2). \]
For $F \in L^2(\Omega, P)$ with the decomposition (7.1) we define the three operators $\Pi(u), \Lambda(u), \Psi_{\Lambda}$, $u \in [0, 1]$ on $L^2(\Omega, P)$ as follows:
\[ \Pi(u)F = \sum_{n=0}^{\infty} u^n F_n, \quad \Lambda(u) = \Pi(\sqrt{u})F, \]
\[ \Psi_{\Lambda}(u) = \frac{d}{du} \|\Lambda(u)\|^2 = \sum_{n=0}^{\infty} n u^{n-1} E|F_n|^2, \]
where $\|F\|^2 = E(F^2)$. Clearly, we have $\Lambda(1) = F$.

Recall that a random variable $F = \sum_{n=0}^{\infty} F_n \in L^2(\Omega, P)$ is smooth in the sense of Meyer-Watanabe [32], provided $\sum_{n=0}^{\infty} n E(|F_n|^2) < \infty$.

Denote by $\mathcal{U}$ the spaces of smooth functionals in the sense of Meyer-Watanabe.

Lemma 7.1 (Hu [17], Nualart [25]). Let $F \in L^2(\Omega, P)$. Then $F \in \mathcal{U}$ if and only if $\Psi_{\Lambda}(1) < \infty$.

For the DILT $\ell_t(0)$, we have the following smoothness theorem.

Theorem 7.1. For any $t > 0$, $\ell_t'(0) \in \mathcal{U}$ if and only if
\[ \int_{\mathbb{T}} \frac{\mu(\lambda_{r,s} \lambda_{r',s'} + 2\mu^2)}{(\lambda_{r,s} \lambda_{r',s'} - \mu^2)^{\frac{5}{2}}} (ss')^{2H_2-1} dsdr'ds' < \infty \]
for all $t \geq 0$.

According to the theorem above, the estimates (3.5) and (3.6), we have

Corollary 7.1. Let either $H_1 \wedge H_2 < \frac{1}{4}$ or $H_1 \vee H_2 < \frac{3}{5}$. Then $\ell_t'(0) \in \mathcal{U}$ for all $t > 0$.

In order to prove Theorem 7.1 we need some preliminaries. Let $H_n(x), x \in \mathbb{R}$ be the Hermite polynomials of degree $n$,
\[ H_n(x) = (-1)^n \frac{1}{n!} x^n e^{-\frac{x^2}{2}}, \quad n = 0, 1, 2, \ldots, \]
and let $X, Y$ be two random variables with joint Gaussian distribution such that $E(X) = E(Y) = 0$ and $E(X^2) = E(Y^2) = 1$. Then for all $n, m \geq 0$ we have (see, for example, Nualart [25])
\[ E[H_n(X)H_m(Y)] = \begin{cases} 0, & m \neq n, \\ \frac{1}{m!}[E(XY)]^n, & m = n. \end{cases} \]

Proof of Theorem 7.1. For $\varepsilon > 0, t \geq 0$ we denote
\[ \Lambda_{\varepsilon}(u) := \Pi(\sqrt{u})\ell'_{\varepsilon, t}(0), \quad \Lambda(u) := \Pi(\sqrt{u})\ell_t'(0). \]

Thus, by Lemma 7.1 it suffices to prove that (7.2) if and only if $\Psi_{\Lambda}(1) < \infty$.

Denote $i^2 = -1$ and
\[ \sigma(r, s, \xi) = \sqrt{\lambda_{r,s} \xi^2} \equiv |\xi| \sqrt{r^{2H_1} + s^{2H_2}} \]
for $\xi \in \mathbb{R}$ and $s > 0$. Recall that

$$e^{tx - \frac{1}{2}t^2} = \sum_{n=0}^{\infty} t^n H_n(x)$$

for all $t > 0$ and $x \in \mathbb{R}$. We see that $\ell_\varepsilon'(0) \in L^2(\Omega)$ by Theorem 3.1 and

$$\ell_{\varepsilon,t}(0) = -\int_0^t \int_0^s p_x^{H_1}(B_r^{H_1} - B_s^{H_2})dr = \frac{i}{2\pi} \sum_{n=0}^{\infty} \int_0^t \int_0^s \int_0^1 e^{ix(B_r^{H_1} - B_s^{H_2}) - \frac{1}{2}x^2} d\xi ds dr$$

for all $t > 0$. Combining this with

$$E\left[H_n\left(\frac{\xi(B_r^{H_1} - B_s^{H_2})}{\sigma(r,s,\xi)}\right) \right] H_n\left(\frac{\eta(B_{r'}^{H_1} - B_{s'}^{H_2})}{\sigma(r',s',\eta)}\right)$$

for all $n \geq 0$, we get

$$\Psi_n(1) = \sum_{n=0}^{\infty} nE(|F_n|^2) = \sum_{n=0}^{\infty} \frac{(\xi\eta)^n}{n!} \left[ E\left(\frac{\xi(B_r^{H_1} - B_s^{H_2})}{\sigma(r,s,\xi)}\right) \right]$$

for all $n \geq 0$, we get

$$\Psi_n(1) = \sum_{n=0}^{\infty} nE(|F_n|^2) = \sum_{n=0}^{\infty} \frac{(\xi\eta)^n}{n!} \left[ E\left(\frac{\xi(B_r^{H_1} - B_s^{H_2})}{\sigma(r,s,\xi)}\right) \right]$$

for all $n \geq 0$.
for all $t \geq 0$. It follows from the fact
\[
\int_{\mathbb{R}} \xi^{2n} \exp \left\{ -\frac{1}{2} \xi^2 (\lambda_{r,s} + \varepsilon) \right\} d\xi = 2 \int_0^{\infty} \xi^{2n} \exp \left\{ -\frac{1}{2} \xi^2 (\lambda_{r,s} + \varepsilon) \right\} d\xi = 2^{n + \frac{1}{2}} \Gamma(n + \frac{1}{2}) (\lambda_{r,s} + \varepsilon)^{-(n + \frac{1}{2})}
\]
for all $\varepsilon > 0$ and $n \geq 1$ that
\[
\Psi_{\lambda_{r,s}}(1) = \sum_{n=1}^{\infty} \frac{(\Gamma(n + \frac{1}{2}))^2 2^{2n+1}}{4\pi^2 (2n - 2)!} \int_{\mathbb{T}} \frac{\mu^{n-1}(ss')^{2H_2-1} dr ds dr' ds'}{(\lambda_{r,s} + \varepsilon)(\lambda_{r',s'} + \varepsilon)}^{n + \frac{1}{2}}
\]
\[
= \sum_{n=1}^{\infty} \frac{(2n - 1)!! 2^{2n+1}}{4\pi (2n - 2)! 2^{2n}} \int_{\mathbb{T}} \frac{\mu^{n-1}(ss')^{2H_2-1} dr ds dr' ds'}{(\lambda_{r,s} + \varepsilon)(\lambda_{r',s'} + \varepsilon)}^{n + \frac{1}{2}}
\]
\[
= \sum_{n=1}^{\infty} \frac{1}{2\pi (2n - 2)!!} (2n - 1) \int_{\mathbb{T}} \frac{\mu^{n-1}(ss')^{2H_2-1} dr ds dr' ds'}{(\lambda_{r,s} + \varepsilon)(\lambda_{r',s'} + \varepsilon)}^{n + \frac{1}{2}}
\]
for all $t \geq 0$. According to the fact
\[
\sum_{n=1}^{\infty} \frac{(2n - 1)!!}{(2n - 2)!!} (2n - 1) x^{2n-1} = x(1 + 2x^2)(1 - x^2)^{-\frac{5}{2}}, \quad x \in (-1, 1)
\]
and the identity
\[
\frac{\mu(\lambda_{r,s}\lambda_{r',s'} + 2\mu^2)}{(\lambda_{r,s}\lambda_{r',s'} - \mu^2)^{\frac{5}{2}}} = \frac{\mu}{\sqrt{\lambda_{r,s}\lambda_{r',s'}}} \left( 1 + 2\frac{\mu^2}{\lambda_{r,s}\lambda_{r',s'}} \right) \left( 1 - \frac{\mu^2}{\lambda_{r,s}\lambda_{r',s'}} \right)^{-\frac{5}{2}} \left( \frac{1}{\lambda_{r,s}\lambda_{r',s'}} \right)
\]
for all $s, r > 0$ and $s \neq r$, we get
\[
\Psi_{\lambda_{r,s}}(1) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{\mu((\lambda_{r,s} + \varepsilon)(\lambda_{r',s'} + \varepsilon) + 2\mu^2)}{(\lambda_{r,s} + \varepsilon)(\lambda_{r',s'} + \varepsilon) - \mu^2)^{\frac{5}{2}}} (ss')^{2H_2-1} dr ds dr' ds'
\]
for all $t \geq 0$. This shows that the limit
\[
\lim_{\varepsilon \to 0} \Psi_{\lambda_{r,s}}(1) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{\mu(\lambda_{r,s}\lambda_{r',s'} + 2\mu^2)}{(\lambda_{r,s}\lambda_{r',s'} - \mu^2)^{\frac{5}{2}}} (ss')^{2H_2-1} dr ds dr' ds'
\]
holds for all $t \geq 0$, and the theorem follows. \(\square\)

**Appendix A. Some Basic Technique Estimates**

In this appendix we give proofs of some lemmas.

**Proof of Lemma 3.1.** For (3.3) we have
\[
f_i(x) = \left(2x^{H_i} + 1 + x^{2H_i} - (1 - x)^{2H_i}\right) \left(2x^{H_i} - 1 - x^{2H_i} + (1 - x)^{2H_i}\right)
\]
\[
= \left(2x^{H_i} + 1 + x^{2H_i} - (1 - x)^{2H_i}\right) \left(2x^{H_i} - 1 - x^{2H_i} + (1 - x)^{2H_i}\right)
\]
\[
= \left\{(1 - x)^{H_i} - (1 - x^{H_i})\right\} \{1 - x\} (1 - x^{H_i})\}
\]
\[
\cdot \left\{(1 + x^{H_i}) - (1 - x^{H_i})\right\} \{1 + x^{H_i} (1 - x)^{H_i}\}.
\]
Clearly, we have
\[
(1 - x)^{H_i} + 1 - x^{H_i} \asymp (1 - x)^{H_i},
\]
\[
(1 + x^{H_i}) - (1 - x)^{H_i} \asymp x^{H_i},
\]
\[
(1 + x^{H_i}) + (1 - x)^{H_i} \asymp 1,
\]
(A.1)
\[
(1 - x)^{H_i} - (1 - x^{H_i}) \leq x^{H_i}(1 - x)^{H_i}
\]
for all \(x \in [0, 1]\). Now, we claim that
\[
(1 - x)^{H_i} - (1 - x^{H_i}) \geq C_{H_i} x^{H_i}(1 - x)^{H_i}.
\]
To end this, we have
(A.2)
\[
(1 + x)^{H_i} \leq 1 + (2^{H_i} - 1)x^{H_i}
\]
with \(0 \leq x \leq 1\). The inequality (A.2) is a calculus exercise, and it is stronger than the well known (Bernoulli) inequality
\[
(1 + x)^{H_i} \leq 1 + H_i x^{H_i} \leq 1 + x^{H_i},
\]
because \(2^{H_i} - 1 \leq H_i\) for all \(0 \leq H_i \leq 1\). It follows that
\[
1 = (1 - x + x)^{H_i} \leq (1 - x)^{H_i} \vee x^{H_i} + (2^{H_i} - 1) \left[(1 - x)^{H_i} \land x^{H_i}\right]
\]
for \(0 \leq x \leq 1\), which deduces
(A.3)
\[
(1 - x)^{H_i} - (1 - x^{H_i}) \geq (2 - 2^{H_i})(1 - x)^{H_i} \land x^{H_i}
\]
\[
\geq (2 - 2^{H_i})(1 - x)^{H_i} x^{H_i}
\]
for all \(x \in [0, 1]\). This gives (3.3).
\(\square\)

**Proof of Lemma 3.2.** We split the proof in two cases.

**Case I.** \(\min\{H_1, H_2\} \leq \frac{1}{2}\). By symmetry we may assume that \(H_1 \leq H_2\). Then \(2H_1 \leq 1\) and
\[
(1 - x)^{2H_1} + x^{2H_1} \geq 1,
\]
which deduces
\[
A(x, y) : = x^{2H_1} + y^{2H_2} + (1 - x)^{2H_1} + (1 - y)^{2H_2} - (1 - x)^{2H_1}(1 - y)^{2H_2} - x^{2H_1}y^{2H_2} - 1
\]
\[
\geq y^{2H_2} + (1 - y)^{2H_2} - (1 - x)^{2H_1}(1 - y)^{2H_2} - x^{2H_1}y^{2H_2} \geq 0.
\]
It follows that
\[
g(x, y) = 4x^{2H_1} + 4y^{2H_2} - 2((1 + x^{2H_1}) - (1 - x)^{2H_1})(1 + y^{2H_2}) - (1 - y)^{2H_2})
\]
\[
= 2 \left[x^{2H_1}(1 - y)^{2H_2} + y^{2H_2}(1 - x)^{2H_1} + A(x)\right]
\]
\[
\geq 2x^{2H_1}(1 - y)^{2H_2} + 2y^{2H_2}(1 - x)^{2H_1}
\]
for all \(x, y \in [0, 1]\). On the other hand, by inequality (A.1), we get
\[
x^{2H_1} + (1 - x)^{2H_1} - 1 \leq x^{2H_1}(1 - x)^{2H_1}
\]
\[
\leq 2^{2H_2}x^{2H_1}(1 - x)^{2H_1} \left(y^{2H_2} + (1 - y)^{2H_2}\right)
\]
\[
\leq 2^{2H_2} \left(y^{2H_2}(1 - x)^{2H_1} + (1 - y)^{2H_2}x^{2H_1}\right)
\]
for all \(x, y \in [0, 1]\). It follows that
\[
A(x, y) = [y^{2H_2} - x^{2H_1}y^{2H_2}] + [(1 - y)^{2H_2} - (1 - x)^{2H_1}(1 - y)^{2H_2}] + [x^{2H_1} + (1 - x)^{2H_1} - 1]
\leq y^{2H_2}(1 - x)^{2H_1} + (1 - y)^{2H_2}x^{2H_1} + x^{2H_1}(1 - x)^{2H_1}
\leq (1 + 2^{2H_2}) [x^{2H_2}(1 - x)^{2H_1} + (1 - x)^{2H_2}x^{2H_1}]
\]
for all \(x \in [0, 1]\), which implies that
\[
g(x, y) = 2 \left[ x^{2H_1}(1 - y)^{2H_2} + y^{2H_2}(1 - x)^{2H_1} + A(x, y) \right]
\leq 2 \left( 2 + 2^{2H_2} \right) [y^{2H_2}(1 - x)^{2H_1} + (1 - y)^{2H_2}x^{2H_1}]
\]
for all \(x \in [0, 1]\). Thus, we have proved the inequalities (3.3) for \(\min\{H_1, H_2\} \leq \frac{1}{2}\).

**Case II.** \(\min\{H_1, H_2\} \geq \frac{1}{2}\). By symmetry we may assume that \(H_1 \leq H_2\). Then \(2H_2 \geq 2H_1 \geq 1\). It follows that
\[
g(x, y) = 4x^{2H_1}(1 - y)^{2H_2} + 4y^{2H_2}(1 - x)^{2H_1}
- 2 \left( 1 - x^{2H_1} - (1 - x)^{2H_1} \right) (1 - y^{2H_2} - (1 - y)^{2H_2})
\leq 4x^{2H_1}(1 - y)^{2H_2} + 4y^{2H_2}(1 - x)^{2H_1}
\]
for all \(x, y \in [0, 1]\). On the other hand, by the inequality (A.3) and the fact
\[
(1 - u)^{H_i} + u^{H_i} \geq 1, \quad i = 1, 2
\]
with \(u \in [0, 1]\) it follows that
\[
1 - u^{2H_i} = (1 - u^{H_i})(1 + u^{H_i})
\leq [(1 - u)^{H_i} - (2 - 2^{H_i})(1 - u)^{H_i}u^{H_i}] (1 + u^{H_i})
\leq (1 - u)^{H_i} (1 - (2 - 2^{H_i})u^{H_i}) (1 + u^{H_i})
= (1 - u)^{H_i} \left[ 1 - u^{H_i} - (2 - 2^{H_i})u^{2H_i} + 2^{H_i}u^{H_i} \right]
\leq (1 - u)^{H_i} \left[ (1 - u)^{H_i} - (2 - 2^{H_i})(1 - u)^{H_i}u^{H_i} - (2 - 2^{H_i})u^{2H_i} + 2^{H_i}u^{H_i} \right]
= (1 - u)^{2H_i} + 2^{H_i}u^{H_i}(1 - u)^{H_i} - (2 - 2^{H_i})u^{H_i}(1 - u)^{H_i}\left[ (1 - u)^{H_i} + u^{H_i} \right]
\leq (1 - u)^{2H_i} + 2(2^{H_i} - 1)u^{H_i}(1 - u)^{H_i}
\]
for all \(u \in [0, 1]\) and \(i = 1, 2\), which yields
\[
\frac{1}{2}g(x, y) = 2x^{2H_1}(1 - y)^{2H_2} + 2y^{2H_2}(1 - x)^{2H_1}
- (1 - x^{2H_1} - (1 - x)^{2H_1}) (1 - x^{2H_2} - (1 - x)^{2H_2})
\geq 2x^{2H_1}(1 - y)^{2H_2} + 2y^{2H_2}(1 - x)^{2H_1}
- 4(2^{H_1} - 1)(2^{H_2} - 1)(1 - x)^{H_1}(1 - y)^{H_2}x^{H_1}y^{H_2}
= 2(2^{H_1} - 1)(2^{H_2} - 1) \left( x^{H_1}(1 - y)^{H_2} - y^{H_2}(1 - x)^{H_1} \right)^2
+ [2 - 2(2^{H_1} - 1)(2^{H_2} - 1)] \left( x^{2H_1}(1 - y)^{2H_2} + y^{2H_2}(1 - x)^{2H_1} \right)
\geq 2 \left[ 1 - (2^{H_1} - 1)(2^{H_2} - 1) \right] \left( x^{2H_1}(1 - y)^{2H_2} + y^{2H_2}(1 - x)^{2H_1} \right)
\]
for all \(x, y \in [0, 1]\). This proves
\[
g(x, y) \geq 4 \left[ 1 - (2^{H_1} - 1)(2^{H_2} - 1) \right] \left( x^{2H_1}(1 - y)^{2H_2} + y^{2H_2}(1 - x)^{2H_1} \right)
\]
Proof of Lemma 4.4. Let first $\frac{1}{2} \leq H_2 < \frac{2}{3}$. Notice that
\[
\int_t^{t'} e^{-x^2u^{2\alpha}} du = \int_0^{t'-t} e^{-x^2(v+t)^{2\alpha}} dv \leq \int_0^{t'-t} e^{-x^2v^{2\alpha}} dv
\]
for all $\alpha \in (0, 1)$ and $x \in \mathbb{R}$ by the fact
\[
\int_0^1 e^{-x^2u^{2\alpha}} du \asymp \frac{1}{1 + |x|^{1/\alpha}}.
\]
By making substitutions $u_j - u_{j+1} = r_j, u_n = r_n$ ($j = 1, 2, \ldots, n - 1$) and $v_j - v_{j+1} = s_j, v_n = s_n$ ($j = 1, 2, \ldots, n - 1$), we get
\[
\Lambda_1(0, t, n, \xi) := \int_{\mathbb{R}_+^{(n)}} \prod_{j=1}^{n-1} e^{-\frac{1}{2} \kappa \left( \sum_{k=1}^{j} \xi_k' \right)^2 (u_j - u_{j+1})^{2H_1}} \cdot e^{-\frac{1}{2} \kappa \left( \sum_{k=1}^{n} \xi_k'' \right)^2 (u_n)^{2H_1}} du_1 \ldots du_n
\]
(A.5)
\[
\leq \prod_{j=1}^{n} \int_0^t e^{-\frac{1}{2} \kappa \left( \sum_{k=1}^{j} \xi_k'' \right)^2 s_j^{2H_1}} ds_j \times \prod_{j=1}^{n} (1 + \left( \sum_{k=1}^{j} \xi_k'' \right)^{1/H_1})^{-1}
\]
and
\[
\Lambda_2(t, t', n, \xi) := \int_{\mathbb{R}_+^{(n)}} \prod_{j=1}^{n-1} e^{-\frac{1}{2} \kappa \left( \sum_{k=1}^{j} \xi_k'' \right)^2 (v_j - v_{j+1})^{2H_2}} \cdot e^{-\frac{1}{2} \kappa \left( \sum_{k=1}^{n} \xi_k'' \right)^2 (v_n)^{2H_2}} dv_1 \ldots dv_n
\]
(A.6)
\[
\leq \int_{[t, t']^n} \prod_{j=1}^{n} \int_0^t e^{-\frac{1}{2} \kappa \left( \sum_{k=1}^{j} \xi_k'' \right)^2 s_j^{2H_2}} ds_j \leq (nT)^{(2H_2-1)n} \int_{[t, t']^n} \prod_{j=1}^{n} e^{-\frac{1}{2} \kappa \left( \sum_{k=1}^{j} \xi_k'' \right)^2 s_j^{2H_2}} ds_j \leq (2T)^{(nH_2-1)n} \int_{[t, t']^n} \prod_{j=1}^{n} (1 + (t' - t) \left( \sum_{k=1}^{j} \xi_k'' \right)^{1/H_2})^{-1}
\]
for all $t \in [0, T]$. On the other hand, by setting $\sum_{k=1}^{j} \xi_k' = x_j$, $j = 1, 2, \ldots, n$ we see that
\[
\prod_{j=1}^{n} |x_j| = \prod_{j=1}^{n} |x_j - x_{j-1}| \leq \prod_{j=1}^{n} (|x_j| + |x_{j-1}|) \leq \prod_{j=1}^{n} (1 + |x_j|)(1 + |x_{j-1}|)
\]
\[
\leq \prod_{j=1}^{n} (1 + |x_j|)^2 \leq 2 \prod_{j=1}^{n} (1 + |x_j|^2)
\]
with $x_0 = 0$. It follows that
\[
\int_{\mathbb{R}^n} \prod_{j=1}^{n} (1 + \left( \sum_{k=1}^{j} \xi_k'' \right)^{1/H_1})^{-2} |x_j|^{1+\gamma} dx_j \leq \int_{\mathbb{R}^n} \prod_{j=1}^{n} (1 + |x_j|^{1/H_1})^{-2} |x_j - x_{j-1}|^{1+\gamma} dx_j
\]
(A.7)
\[
\leq 2 \prod_{j=1}^{n} \int_{\mathbb{R}} (1 + |x_j|^{1/H_1})^{-2} (1 + |x_j|^2)^{1+\gamma} dx_j < \infty,
\]
for all $x, y \in [0, 1]$, and the proof is completed. \qed
provided $\gamma < 2\frac{3H_{\gamma}}{2H_{2}}$. Similarly, we also have
\[
\int_{\mathbb{R}^{n}} \prod_{j=1}^{n} \left(1 + (t' - t)\left|\sum_{k=1}^{j} \xi''_{k} \right|^{1/2}ight)^{2} \left|\xi_{j}\right|^{1+\gamma} d\xi_{j} \\
\leq \int_{\mathbb{R}^{n}} \prod_{j=1}^{n} (1 + (t' - t)\left|x_{j}\right|^{1/2})^{-2} \left|x_{j} - x_{j-1}\right|^{1+\gamma} d\xi_{j} \\
\leq 2 \prod_{j=1}^{n} \int_{\mathbb{R}} (1 + (t' - t)\left|x_{j}\right|^{1/2})^{-2} (1 + \left|x_{j}\right|^{2})^{1+\gamma} d\xi_{j} \\
\leq C(t' - t)^{-n(3+2\gamma)H_{2}},
\]
providing $\gamma < 2\frac{3H_{2}}{2H_{2}}$. Consequently, we have
\[
\tilde{A}(t, t', n, \gamma) = \int_{\mathbb{R}^{n}} \Lambda_{1}(0, t, n, \xi) \Lambda_{2}(t, t', n, \xi) \prod_{j=1}^{n} \left|\xi_{j}\right|^{1+\gamma} d\xi_{j} \\
(A.8) \\
\leq C \left(\int_{\mathbb{R}^{n}} \left[\Lambda_{1}(0, t, n, \xi)\right]^{2} \prod_{j=1}^{n} \left|\xi_{j}\right|^{1+\gamma} d\xi_{j} \int_{\mathbb{R}^{n}} \left[\Lambda_{2}(t, t', n, \xi)\right]^{2} \prod_{j=1}^{n} \left|\xi_{j}\right|^{1+\gamma} d\xi_{j}\right)^{1/2} \\
\leq C(t' - t)^{(1-(\frac{3}{2}+\gamma)H_{2}n)}
\]
for all $\gamma < 2\frac{3H_{2}}{2H_{2}}$ and $0 \leq t < t' < T$, and the estimate (4.2) follows.

Let next $0 < H_{2} < \frac{1}{2}$. Then (A.7) holds and an elementary calculus can show that the estimates
\[
\int_{t}^{t'} u^{2\alpha-1} e^{-x^{2}u^{2\alpha}} du = \int_{0}^{t'-t} (v + t)^{2\alpha-1} e^{-x^{2}(v+t)^{2\alpha}} dv \\
\leq \int_{0}^{t'-t} v^{2\alpha-1} e^{-x^{2}v^{2\alpha}} dv \\
= \frac{1 - e^{-x^{2}(t'-t)^{2\alpha}}}{2\alpha x^{2}} \times \frac{(t'-t)^{2\alpha}}{1 + (t'-t)^{2\alpha}x^{2}}
\]
hold for all $0 < 2\alpha < 1$ and $x \in \mathbb{R}$. Similar to (A.6) and (A.8) one can obtain the following estimates:
\[
\Lambda_{2}(t, t', n, \xi) \leq \int_{[t, t']^{n}} \prod_{j=1}^{n} \left(\sum_{i=j}^{n} s_{i}\right)^{2H_{2}-1} \prod_{j=1}^{n} e^{-\frac{1}{2}\kappa_{0} \left(\sum_{k=1}^{j} \xi''_{k}\right)^{2}s_{j}^{2H_{2}}} ds_{j} \\
\leq \int_{[t, t']^{n}} \prod_{j=1}^{n} \left(\sum_{i=j}^{n} s_{i}\right)^{2H_{2}-1} \prod_{j=1}^{n} e^{-\frac{1}{2}\kappa_{0} \left(\sum_{k=1}^{j} \xi''_{k}\right)^{2}s_{j}^{2H_{2}}} ds_{j} \\
= \prod_{j=1}^{n} \int_{[t, t']^{n}} \left(\sum_{i=j}^{n} s_{i}\right)^{2H_{2}-1} e^{-\frac{1}{2}\kappa_{0} \left(\sum_{k=1}^{j} \xi''_{k}\right)^{2}s_{j}^{2H_{2}}} ds_{j} \\
\leq C(t' - t)^{2nH_{2}} \prod_{j=1}^{n} \left(1 + (t' - t)^{2H_{2}} \left(\sum_{k=1}^{j} \xi''_{k}\right)^{2}\right)^{-1}
\]
for all $0 < t < t' \leq T$. Combining this with (A.9), (A.7) and (A.5) lead to the estimate (4.3). \□

Proof of Lemma 5.3. For $0 < s' < t' < s < t \leq T$ we define the function $x \mapsto G_{s,t}(x)$ on $[s', t']$ by
\[
G_{s,t}(x) = (s - x)^{2H} - (t - x)^{2H}.
\]
Thanks to mean value theorem, we see that there are \( \xi \in (s', t') \) and \( \eta \in (s, t) \) such that

\[
2E \left[ (B_t^H - B_s^H)(B_{t'}^H - B_{s'}^H) \right] = G_{s,t}(t') - G_{s,t}(s)
\]

\[
= 2H(t' - s') \left[ (t - \xi)^{2H-1} - (s - \xi)^{2H-1} \right]
\]

\[
= 2H(2H - 1)(t' - s')(t - s)(\eta - \xi)^{2H-2},
\]

which gives

(A.10)

\[
|E \left[ (B_t^H - B_s^H)(B_{t'}^H - B_{s'}^H) \right]| \leq 2H(2H - 1)\frac{(t' - s')(t - s)}{(s - t')^{2-2H}},
\]

which gives (5.8). In order to prove (5.9), noting that

\[
\left| \frac{E \left[ (B_t^H - B_s^H)(B_{t'}^H - B_{s'}^H) \right]}{(t - s)^H(t' - s')^H} \right| \leq 1,
\]

we see that

\[
\left| E \left[ (B_t^H - B_s^H)(B_{t'}^H - B_{s'}^H) \right] \right| \leq \left( \frac{E \left[ (B_t^H - B_s^H)(B_{t'}^H - B_{s'}^H) \right]}{(t - s)^H(t' - s')^H} \right)^\alpha
\]

for all \( \alpha \in [0, 1] \). Combining this with (A.10) we get

(A.11)

\[
\left| E \left[ (B_t^H - B_s^H)(B_{t'}^H - B_{s'}^H) \right] \right| \leq \frac{(t - s)^{(1-\alpha)H+\alpha}(t' - s')^{(1-\alpha)H+\alpha}}{(s - t')^{\alpha(2-2H)}}.
\]

Since \( 0 < H < \frac{1}{2} \) we can take \( \alpha = H/(1 - H) \) and (5.9) follows. \( \square \)

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