A Mathematical Model of Protectant and Curative Fungicide Application and its stability analysis

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Abstract. In this paper we introduce a mathematical model for fungicide application with effect of protectant and curatives factor. We show the value of the Basic Reproduction Number (\(R_0\)) of the fungal disease, which is computed from the largest eigen value of the next generation matrix of the model. The result show that in the region where \(R_0\) greater than one there is only one single stable endemic equilibrium. However, in region where \(R_0\) less than one some parameters affect the number of possible equilibria. Some numerical simulation are also given to illustrate our analytical results.

1. Introduction

Recently Castle et al. (2012) have developed a mathematical model of fungicide application. They argued that most models of fungicides application have been done by simply modifying the parameters of the underlying epidemiological models, for example by reducing the infection rates and/or increasing the host recovery rates. Many work have been done to investigate the effect of the application of fungicide for several underlying mathematical but the dynamics of the fungicides themselves are not well explored. Castle et al. were pioneering this work by investigating the dynamics of fungicide application to infected plant by considering the curative factor effect of the fungicide and showed how the timing of initial infection relative to host population growth affects invasion and persistence in a chemically controlled system. It followed by the work of Anggriani et al. (2015) to consider the optimal control of the fungicide application.

Since Castle et al (2012) have argued that generally the application of fungicide can be modeled by one or more of the following assumptions that “there is complete coverage by the fungicide for either the entire host population or a fixed subset of the host population, that a generic, multipurpose, fungicide has been applied to the hosts (i.e. they have not readily separated out the effects of different fungicide types such as protectants, curatives or eradicants); or that the fungicides are permanent (i.e. that the chemicals do not decay or that their effects do not change over time)”, then we develop a mathematical model to combine the effect of curative and protective function of the fungicide in this paper. Specifically the paper discusses a model of fungicide application in a fungal disease epidemic, where explicitly take fungicide dynamics into the model together with the curative and protective effect of the fungicide. We show the value of the Basic Reproduction Number (\(R_0\)) where it is computed from the largest eigen value of the next generation matrix. In the region where \(R_0\) greater than one there is only one single stable endemic equilibrium. However, in region where \(R_0\) less
than one some parameters affect the number of possible equilibria (Chavez et al 2002). Some numerical simulation are also given to illustrate our results.

2. Mathematical Model

Following Castle et al (2012) in this model we assume that:

1. The fungal disease spread through one population;
2. There is a simple density dependent growth of the host up to a carrying capacity;
3. Infected host tissue consumes resources but does not contribute to new host growth;
4. The hosts cannot be re-infected after they cease to be infectious;
5. A protectant fungicide affects susceptible hosts, reducing their capacity to become infected;
6. All susceptible hosts are protected;
7. The population is divided into two classes, susceptible ($\hat{S}$), infected ($\hat{I}$) and protected ($\hat{P}$).

| Table 1. Factor that influence the model |
|-----------------------------|
| Symbol | Explanation |
| $\rho$ | All the factor that influence effectiveness of fungicide ($0 \leq \rho \leq 1$) |
| $t$ | Time ($\hat{p}$) |
| $\beta$ | Growth rate of host ($\hat{g}$) |
| $\mu$ | Removal rate ($\hat{u}$) |
| $\pi$ | Fungicide application rate ($\hat{p}$) |
| $\delta$ | Fungicide decay rate ($\hat{d}$) |
| $\alpha$ | Curatives rate of fungicide ($\hat{a}$) |
| $\epsilon$ | Fungicide effectiveness |
| $k$ | Disease free carrying capacity |

Table 1 shows the symbols used in the subsequent model. Using the symbols in Table 1 and from the assumption, we can build a transition diagram.

![Transition Diagram]

**Figure 1** Scheme of susceptible ($\hat{S}$), infected ($\hat{I}$) and protected ($\hat{P}$) population (modified from Castle)

Based on the assumption and by referring to the work of Angriani et al 2015, the differential equations model for the protective and curative explicit model is given by:

\[
\begin{align*}
\dot{S} &= (\kappa - \mu S - \beta S I - \omega) - \omega S P - \mu S \\
\dot{I} &= \beta S I - \mu I \\
\dot{P} &= \omega S P + \alpha I P - \mu P
\end{align*}
\]
\[
\frac{d\hat{S}}{dt} = \hat{\beta}(\hat{S} + \hat{P}) \left(1 - \frac{\hat{S} + \hat{P} + \hat{I}}{k}\right) - \pi \hat{S} - \delta \hat{P} - \frac{\hat{\beta}\hat{I}}{k} \tag{1}
\]
\[
\frac{d\hat{P}}{dt} = \hat{\beta}\hat{S} - \delta \hat{P} - \frac{e\hat{\beta}\hat{I}}{k} + \alpha I,
\]
\[
\frac{d\hat{I}}{dt} = \frac{\hat{\beta}I(\hat{S} + e\hat{P})}{k} - \mu \hat{I} - \alpha I.
\]

After the reduction and normalization by changing the compartment from density to proportion (substituting \( S_i = \hat{S} k^{-1}, I_i = \hat{I} k^{-1}, P_i = \hat{P} k \) and \( \theta = \hat{\theta} k^{-1}, \pi = \hat{\pi} k^{-1}, \delta = \hat{\delta} k^{-1}, \mu = \hat{\mu} k^{-1}, \alpha = \hat{\alpha} k^{-1}, t = \hat{t} k \)) we get the non-dimensional system:

\[
\frac{dS}{dt} = \theta(S_i + P_i)(1 - (S_i + P_i + I_i)) - \pi S_i + \delta P_i - S_i I_i,
\]
\[
\frac{dI}{dt} = I_i(S_i + eP_i) - \mu I_i + \alpha I_i,
\]
\[
\frac{dP}{dt} = \pi S_i - \delta P_i - eP_i I_i + \alpha I_i.
\]

Anggraini et al. (2015) showed the optimal control of the curvature intervention but did not present the stability analysis of the equilibrium points. Here we present the stability analysis of the resulting equilibrium points.

### 3. Mathematical Analysis

The non-endemic fixed point of the equation (2) is

\[
(S_{\infty}, P_{\infty}, I_{\infty}) = \left(\frac{\delta}{\delta + \pi}, \frac{\pi}{\delta + \pi}, 0\right)
\]

and the endemic fixed point is

\[
S_i = \mu a - e P_i,
\]
\[
I_i = 2\delta P_i e + 2\delta e P_i \mu + 2\delta e P_i a + \pi e P_i + \delta P_i + \theta \mu (1 - \mu) + \alpha \theta(1 - 2P_i) + \theta P_i (1 - P_i)
\]
\[
- \varepsilon \theta^2 P_i^2 - \varepsilon e P_i - \alpha^2 \theta - \pi \mu - \pi a - 2\delta \mu a - 2\delta P_i \mu / (\varepsilon \theta P_i + \theta \mu + \mu + \alpha - \varepsilon e P_i - e P_i)
\]

By substituting \( S_i \) and \( P_i \) before to the equation (2) the third line, then \( P_i \) is:

\[
(\varepsilon \theta - 2\delta \varepsilon^2 + \varepsilon^2 \theta) P_i^2 + \\
(-3\varepsilon^2 \theta a + 2\varepsilon \theta \mu + 4\varepsilon \theta \mu a - 2\varepsilon^4 \theta \mu + \delta \theta \mu - \theta \mu a) + 2\varepsilon \theta \mu - \alpha \theta - \delta \theta a - 2\delta \mu \varepsilon + 2\varepsilon \mu a + 3\varepsilon \mu a)
\]
\[
(-2\varepsilon \theta \mu a + 3\varepsilon \theta a - 2\varepsilon \theta \mu a - 2\varepsilon \theta a - 2\varepsilon \theta \mu a + 2\varepsilon \theta \mu a + 2\varepsilon \theta \mu a)
\]
\[
(\varepsilon \theta a + \varepsilon \theta \mu + \alpha \theta \mu - \varepsilon \theta \mu + 2\varepsilon \theta \mu a - 2\varepsilon \theta \mu a - 2\varepsilon \theta \mu a - 2\varepsilon \theta \mu a - 2\varepsilon \theta \mu a - 2\varepsilon \theta \mu a - 2\varepsilon \theta \mu a - 2\varepsilon \theta \mu a)
\]
\[
= 0
\]
By using Routh Hurwitz criteria, we can conclude that the polynomial has the real positive root provided that condition
\[ 0 < \varepsilon < \frac{1}{4}, \quad 0 < \mu < \frac{3\varepsilon\alpha}{2}, \quad \pi\mu + \alpha + \pi\alpha < \delta\mu + 2\varepsilon\alpha + \delta\alpha + 2\pi\mu + 2\pi\alpha \]
is satisfied, with the Basic Reproduction Ratio given by
\[ R_{EPC} = \frac{1}{\mu} \left( \frac{\delta + \varepsilon}{\delta + \pi} - \alpha \right) = R_0 - \frac{\alpha}{\mu} \]

**Stability Analysis**

The jacobian matrix for the equation (2) is

\[
MJ_\varepsilon = \begin{bmatrix}
\partial(1 - S_i - P_i - I_i) - \partial(S_i + P_i) - \pi - I_i & -\partial(S_i + P_i) - S_i & \partial(1 - S_i - P_i - I_i) - \partial(S_i + P_i) + \delta \\
I_i & S_i + \varepsilon P_i - \mu - \alpha & I_i \\
\pi & -\varepsilon P_i + \alpha & -\delta - I_i \\
\end{bmatrix}
\]

The stability testing at non-endemic fixed point s given as follows.

**Theorem 1**

The non-endemic fixed point \((S_{i0}, P_{i0}, I_{i0})\) locally asymptotically stable if \(R_{EPC} < 1\) and not stable if \(R_{EPC} > 1\).

**Proof:**

Following (Diekmann dan Heesterbeek, 2000) substitution of the non-endemic fixed point into the jacobian matrix results in:

\[
J_\varepsilon(E_i) = \begin{bmatrix}
\partial \left(1 - \frac{\delta}{\delta + \pi} - P_i\right) - \partial \left(\frac{\delta}{\delta + \pi} + P_i\right) - \pi - \partial \left(\frac{\delta}{\delta + \pi} + P_i\right) - \partial \left(1 - \frac{\delta}{\delta + \pi} - P_i\right) - \partial \left(\frac{\delta}{\delta + \pi} + P_i\right) + \delta \\
0 & \frac{\delta}{\delta + \pi} + \varepsilon P_i - \mu - \alpha & 0 \\
\pi & -\varepsilon P_i + \alpha & -\delta \\
\end{bmatrix}
\]

From the jacobian matrix, we can get three eigen values:

\[
\lambda_1 = \frac{\delta + \varepsilon P_i \delta + \varepsilon P_i \pi - \mu \delta - \mu \pi - \delta \alpha - \alpha \pi}{\delta + \pi} = \mu(R_{EPC} - 1) < 0,
\]

\[
\lambda_2 = -(\delta + \pi) < 0,
\]

\[
\lambda_3 = -\frac{\partial(-\pi + 2P_i + 2\delta P_i + \delta)}{\delta + \pi} = -\frac{\pi \partial + \delta \partial}{\delta + \pi} < 0
\]

It is clear that if \(R_{EPC} < 1\), all of the eigen values are negative, and hence the equilibrium point of \((S_{i0}, P_{i0}, I_{i0})\) is locally asymptotically stable.
4. Numerical Simulation

In this simulation suppose that growth rate of host \( \vartheta = 0.5 \), the removal rate \( \mu = 0.2 \). The initial condition to the class of susceptible, infected, and protected respectively are \((0.7; 0.3; 0)\). And the other parameters divided into two condition, without fungicide application has a data set \( \pi = \delta = \varepsilon = \alpha = 0 \) and with fungicide has a data set \( \pi = 0.1; \varepsilon = 0.1; \alpha = 0.05, \delta = 0.001 \). Matlab solution for the system is given in figure 2 to 4.

![Figure 2](image1.png)

**Figure 2.** The graph of the influence of the fungicide to the susceptible host \((S_t)\)

In Figure 2, when the fungicide is applied, the susceptible population will decrease towards zero.

![Figure 3](image2.png)

**Figure 3.** The graph of the influence of the fungicide to the infected population \((I_t)\)
In Figure 3, when the fungicide is applied, the infected population will decrease towards zero.

![Graph showing decrease in infected population](image)

**Figure 4.** The graph of the influence of the fungicide to the protected population ($P_t$).

In Figure 4, when the fungicide is applied, the protected population increase toward one.

5. **Conclusion**

We have discussed a model of the protective and curative of fungicide. It is shown that the model has two equilibrium points, i.e non–endemic and endemic points. For the protective and curative explicit model the non-endemic fixed point is stable at $R_{ec} < 1$. The result of the simulation shown that fungicide application are effective to decrease the infection of the population.

6. **Acknowledgment**

This work was supported by the Academic Leadership Grant (ALG) 2015 from the Universitas Padjadjaran.
7. References

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