Abstract This paper investigates well posedness of utility maximization problems for financial markets where stock returns depend on a hidden Gaussian mean reverting drift process. Since that process is potentially unbounded, well posedness cannot be guaranteed for utility functions which are not bounded from above. For power utility with relative risk aversion smaller than that of log-utility this leads to restrictions on the choice of model parameters such as the investment horizon and parameters controlling the variance of the asset price and drift processes. We derive sufficient conditions to the model parameters leading to bounded maximum expected utility of terminal wealth for models with full and partial information.

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1 Introduction

In this paper we investigate utility maximization problems for a financial market where asset prices follow a diffusion process with an unobservable Gaussian mean reverting drift modelled by an Ornstein-Uhlenbeck process. It is motivated by our papers [18,19] where we examine in detail the maximization of expected power utility of terminal wealth which is treated as a stochastic optimal control problem under partial information. A special feature of these papers is that for the construction of optimal portfolio strategies investors estimate the unknown drift not only from observed asset prices. They also incorporate external sources of information such as news, company reports, ratings or their own intuitive views on the future asset performance. These outside sources of information are called expert opinions.
In the present paper, we focus on the well posedness of the above stochastic control problem, which is often overlooked or taken for granted in the literature. For Gaussian drift processes which are potentially unbounded, well posedness in general cannot be guaranteed for utility functions which are not bounded from above. This is the case for log-utility and power utility with relative risk aversion smaller than that of log-utility. For log-utility well posedness can be shown quite easily and holds without restriction to the model parameters. However, the case of power utility is much more demanding and leads to restrictions on the choice of model parameters such as the investment horizon, the risk aversion parameter of the power utility function, parameters controlling the variance of the asset price and drift processes, and the filter process describing the conditional covariance of the Kalman filter.

**Literature review.** The above phenomenon was already observed in Kim and Omberg [20] for a financial market with an observable drift modeled by an Ornstein-Uhlenbeck process. They coined the terminology nirvana strategies. Such strategies generate in finite time a terminal wealth with a distribution leading to infinite expected utility. Note that this is a property of the distribution of terminal wealth and realizations of terminal wealth need not to be infinite. The same holds for the generating strategies which might be even suboptimal. That phenomenon was also observed in Korn and Kraft [22, Sec. 3] who coined it “I-unstability”, in Angoshtari [1][2] and Lee and Papanicolaou [24] who studied power utility maximization problems and their well posedness for financial market models with cointegrated asset price processes and in Battauz et al. [3] for markets with defaultable assets. For the case of partial information Colaneri et al. [8] provides some results for markets with a single risky asset ($d = 1$). Kim and Omberg [20] also pointed out that financial market models allowing investors to attain nirvana do not properly reflect reality. Thus, there are not only mathematical reasons to exclude combinations of model parameters allowing for attaining nirvana, i.e., not ensuring well-posed optimization problems. This problem is addressed in the present paper and we derive sufficient conditions to the model parameters leading to bounded maximum expected utility of terminal wealth for portfolio selection problems under full and partial information.

Portfolio selection problems for market models with partial information on the drift have been intensively studied in the last years. We refer to Lakner [23] and Brendle [6] for models with Gaussian drift and to Rieder and Bäuerle [25], Sass and Haussmann [28] for models in which the drift is described by a continuous-time hidden Markov chain. A generalization of these approaches and further references can be found in Björk et al. [4].

Utility maximization problems for investors with logarithmic preferences in market models with non-observable Gaussian drift process and discrete-time expert opinions are addressed in a series of papers [16][17][29][30][31] of the present authors and of Sass and Westphal. The case of continuous-time expert opinions and power utility maximization is treated in a series of papers by Davis and Lleo, see [9][10][11]. For models with drift processes described by continuous-time hidden Markov chains and power utility maximization we refer to Frey et al. [14][15]. Finally, the computation of optimal strategies using dynamic programming methods for the power utility maximization problems addressed in this paper can be found in our papers [18][19].

**Our contribution.** The paper addresses well posedness of power utility maximization problems under partial information on the not directly observable drift of risky assets. It derives sufficient conditions to the model parameters ensuring bounded objective functions,
and under which the dynamic programming approach can be applied for their solution. Such conditions are often taken for granted or overlooked and restrict the choice of model parameters for investors which are less risk averse than the log-utility investors.

To the best of our knowledge, our results for the case of multi-asset markets and partial information are new to the literature. They extend known results for the corresponding optimization problems under full information to the case of partial information. A first main result is Theorem 3.3 providing an upper bound for the expected terminal wealth expressed in terms of the solution to some matrix Riccati differential equation and involving the current value of the non-observable drift. That result allows to deduce sufficient conditions to the model parameters ensuring the well posedness of the utility maximization problem under full information in Corollary 3.4. The respective conditions for the case of partial information follow from the projection of the above upper bound on the investor filtration of the partially informed investor and lead to our second main result given in Theorem 3.7. It allows to derive well posedness conditions for the problem under partial information Corollary 3.8.

The derived results appear to be helpful for the analysis of portfolio selection problems under partial information in general, and not only to the specific situation where investors draw information for estimating unobservable drifts from return observations which are combined with additional information from expert opinions, as in this work and our papers [18,19].

We provide numerical results to illustrate the theoretical findings for a market model with a single risky asset. Here, the sufficient conditions for well posedness become quite explicit. This allows an insightful visualization of the set of feasible model parameters.

**Paper organization.** In Section 2 we introduce the financial market model with partial information on the drift and formulate the portfolio optimization problem. The well posedness of that problem is studied in Section 3. We derive sufficient conditions to the model parameters ensuring the well posedness of the utility maximization problem under full as well as partial information. These conditions become quite explicit for market models with a single risky asset which are considered in Subsection 3.4. Section 4 illustrates the theoretical findings by results of some numerical experiments and visualizes the derived restrictions on the model parameters. The appendix collects proofs which are removed from the main text.

**Notation.** Throughout this paper, we use the notation $I_d$ for the identity matrix in $\mathbb{R}^{d \times d}$, $0_d$ denotes the null vector in $\mathbb{R}^d$, $0_{d \times m}$ the null matrix in $\mathbb{R}^{d \times m}$. For a symmetric and positive-semidefinite matrix $A \in \mathbb{R}^{d \times d}$ we call a symmetric and positive-semidefinite matrix $B \in \mathbb{R}^{d \times d}$ the square root of $A$ if $B^2 = A$. The square root is unique and will be denoted by $A^{1/2}$. For a generic process $X$ we denote by $\mathcal{G}^X$ the filtration generated by $X$.

## 2 Financial Market and Optimization Problem

### 2.1 Price Dynamics

Our financial market model comprises one risk-free and multiple risky assets. The setting is based on Gabih et al. [16,17] and Sass et al. [29,31,30] and also used in our papers [18,19]. For a fixed date $T > 0$ representing the investment horizon, we work on a filtered prob-
ability space \((\Omega, \mathcal{G}, \mathbb{G}, \mathbb{F})\), with filtration \(\mathbb{G} = (\mathcal{G}_t)_{t \in [0,T]}\) satisfying the usual conditions. All processes are assumed to be \(\mathbb{G}\)-adapted.

We consider in our market model discounted asset prices with the risk-free asset as numéraire. Then the risk-free asset has a constant price \(S^0 = 1\). Further, there are \(d\) risky securities whose excess log-return or risk premium process \(R = (R^1, \ldots, R^d)\) is defined by the SDE

\[
dR_t = \mu_t \, dt + \sigma_R \, dW^R_t,
\]

for a given \(d_1\)-dimensional \(\mathbb{G}\)-adapted Brownian motion \(W^R\) with \(d_1 \geq d\). The volatility matrix \(\sigma_R \in \mathbb{R}^{d \times d_1}\) is assumed to be constant over time such that \(\Sigma_R := \sigma_R \Sigma_R^\top\) is positive definite. In the remainder of this paper we will call \(R\) simply returns. In this setting the discounted process \(S = (S^1, \ldots, S^d)\) of the risky securities reads as

\[
dS_t = \text{diag}(S_t) \, dR_t, \quad S_0 = s_0,
\]

with some fixed initial value \(s_0 = (s_0^1, \ldots, s_0^d)\). Note that for the solution to the above SDE it holds

\[
\log S_t^i - \log s_0^i = \int_0^t \mu^i_j \, ds + \frac{d_1}{2} \sum_{j=1}^{d_1} \left( \sigma^i_j W^R_{t}^{R_j} - \frac{1}{2} (\sigma^i_j)^2 t \right) = R^i_t - \frac{1}{2} \sum_{j=1}^{d_1} (\sigma^i_j)^2 t, \quad i = 1, \ldots, d.
\]

So we have the equality \(\mathbb{G}^R = \mathbb{G}^\log S = \mathbb{G}^S\).

The dynamics of the drift process \(\mu = (\mu_t)_{t \in [0,T]}\) in (2.1) are given by the stochastic differential equation (SDE) defining an Ornstein–Uhlenbeck process

\[
d\mu_t = \kappa(\overline{\mu} - \mu_t) \, dt + \sigma_\mu \, dW^\mu_t,
\]

where \(\kappa \in \mathbb{R}^{d \times d}, \sigma_\mu \in \mathbb{R}^{d \times d_2}\) and \(\overline{\mu} \in \mathbb{R}^{d}\) are constants. We require that all eigenvalues of \(\kappa\) have a positive real part (that is, \(-\kappa\) is a stable matrix) and that \(\Sigma_\mu := \sigma_\mu \Sigma_\mu^\top\) is positive definite. Further, \(W^\mu\) is a \(d_2\)-dimensional Brownian motion such that \(d_2 \geq d\). For the sake of simplification and shorter notation we assume that the Wiener processes \(W^R\) and \(W^\mu\) driving the return and drift process, respectively, are independent. For the general case we refer to Brendle [6], Colaneri et al. [8] and Fouque et al. [13]. Here, \(\overline{\mu}\) is the mean-reversion level, \(\kappa\) the mean-reversion speed and \(\sigma_\mu\) describes the volatility of \(\mu\). The initial value \(\mu_0\) is assumed to be a normally distributed random variable independent of \(W^\mu\) and \(W^R\) with mean \(\overline{\mu}_0 \in \mathbb{R}^d\) and covariance matrix \(\overline{\sigma}_0 \in \mathbb{R}^{d \times d}\) assumed to be symmetric and positive semi-definite.

2.2 Partial Information

Our mathematical market model reflects the fact that investors in real financial markets do not have full access to market information. They can instead observe the historical data of the return process \(R\) but they neither, but they neither observe the factor process \(\mu\) nor the Brownian motion \(W^R\). Further, investors know the model parameters such as \(\sigma_R, \kappa, \overline{\mu}, \sigma_\mu\) and the distribution \(\mathcal{N}(\overline{\mu}_0, \overline{\sigma}_0)\) of the initial value \(\mu_0\).

Information about the drift \(\mu\) can be drawn from observing the asset prices from which the returns \(R\) can be derived. However, it is well-known that estimating the drift with a
reasonable degree of precision based only on historical asset prices is known to be almost impossible. This is nicely described in Rogers [27, Chapter 4.2]. Here, the author analyzes that problem for a model in which the drift is even constant. Reliable estimate require extremely long time series of data which are usually not available. Furthermore, the assumption of a constant drift over longer periods of time is rather unrealistic. Drifts tend to fluctuate randomly over time and drift effects are often overlaid by volatility.

For these reasons, portfolio managers and traders also rely on external sources of information such as news, company reports, ratings and benchmark values. Further, they increasingly turn to data outside of the traditional sources that companies and financial markets provide. Examples are social media posts, internet search, satellite imagery, sentiment indices, pandemic data, product review trends and are often related to Big Data analytics.

In the literature, these external sources of information are referred to as expert opinions or more generally as alternative data, see Chen and Wong [7], Davis and Lleo [11]. We use the first term here. After appropriate mathematical modeling, they are included as additional noisy observations in the drift estimation and the construction of optimal portfolio strategies. This approach goes back to Black and Litterman [5] and their celebrated Black-Litterman model, which is an extension of the classic one-period Markowitz model.

A first modeling approach considers expert opinions as noisy signals about the current state of the drift arriving at discrete time points forming an increasing sequence $\{T_k\}_{k \in \mathbb{N}}$ with values in $[0, T]$ and $\mathbb{I} \subseteq \mathbb{N}_0$. The literature distinguishes between a given finite number of deterministic time points as in [16,18,29,31] or random time points which are the jump times of a Poisson process with some given intensity as in [17,31]. The signals or “expert views” at time $T_k$ are modelled by $\mathbb{R}^d$-valued Gaussian random vectors $Z_k = (Z_k^1, \ldots, Z_k^d)^\top$ with

$$Z_k = \mu_{T_k} + \Gamma^{\frac{1}{2}} \epsilon_k, \quad k \in \mathbb{N}_0,$$

where the matrix $\Gamma \in \mathbb{R}^{d \times d}$ is symmetric and positive definite. Further, $(\epsilon_k)_{k \in \mathbb{N}_0}$ is a sequence of independent standard normally distributed random vectors, i.e., $\epsilon_k \sim \mathcal{N}(0, I_d)$. It is also independent of both the Brownian motions $W^R, W^\mu$ and the initial value $\mu_0$ of the drift process. Thus given $\mu_{T_k}$ the expert opinion $Z_k$ is $\mathcal{N}(\mu_{T_k}, \Gamma)$-distributed. So, $Z_k$ can be considered as an unbiased estimate of the unknown state of the drift at time $T_k$. The matrix $\Gamma$ is a measure of the expert’s reliability. Its diagonal entries $\Gamma_{ii}$ are just the variances of the expert’s estimate of the drift component $\mu^i$ at time $T_k$. The larger $\Gamma_{ii}$ the less reliable is the expert’s estimate.

Expert opinions can also take the form of relative views, which are estimates of the difference in drift between two stocks rather than an absolute view of the drift of a single stock. We refer for this extension to Schöttle et al. [32] where the authors show how to switch between these two models for expert opinions by means of a pick matrix.

In a second modeling approach the expert opinions do not arrive at discrete time points but continuously over time as in the BLCT model of Davis and Lleo [9,10,11]. This is motivated by the results of Sass et al. who show in [31] for periodically arriving and in [30] for randomly arriving expert opinions that for increasing arrival intensity $\lambda$ and an expert’s variance $\Gamma$ growing linearly in $\lambda$ asymptotically for $\lambda \to \infty$ the information drawn from these expert opinions is essentially the same as the information one gets from observing yet another diffusion process. This diffusion process can then be interpreted as an expert who
gives a continuous-time estimation about the current state of the drift. Let this estimate be given by the diffusion process

\[ dJ_t = \mu dt + \sigma dW_t^J \]

where \( W_t^J \) is a \( d_J \)-dimensional Brownian motion independent of \( W_t^R \) and \( W_t^H \) and such that with \( d_J \geq d \). The volatility matrix \( \sigma_t \in \mathbb{R}^{d \times d_J} \) is assumed to be constant over time such that the matrix \( \Sigma_t := \sigma_t \sigma_t^T \) is positive definite. In [18] we show that based on this model and on the diffusion approximations provided in [31,30] one can find efficient approximative solutions to utility maximization problems for partially informed investors observing high-frequency discrete-time expert opinions.

2.3 Investor Filtration

In view of the different levels of information on the financial markets, we consider different types of investors. Mathematically, the information available to an investor can be described by the investor filtration \( \mathbb{F}^H_t = (\mathcal{F}^H_t)_{t \in [0,T]} \). Here, \( H \) denotes the information regime for which we consider the cases \( H = R,Z,J,F \) and the investor with filtration \( \mathbb{F}^H_t = (\mathcal{F}^H_t)_{t \in [0,T]} \) is called the \( H \)-investor. The \( R \)-investor only observes the return process \( R \), the \( Z \)-investor combines observations of returns with the discrete-time expert opinions \( Z_k \) arriving at time \( T_k \), while the \( J \)-investor observes the return process together with the continuous-time expert \( J \). Finally, the \( F \) investor is fully informed and can directly observe the drift process \( \mu \) and, of course, the return process. In a market with stochastic drift, this case is not realistic, but we use it as a benchmark.

The \( \sigma \)-algebras \( \mathcal{F}^H_t \) representing the \( H \)-investor’s information at time \( t \in [0,T] \) are defined at initial time \( t = 0 \) by \( \mathcal{F}^F_0 = \sigma \{ \mu_0 \} \) for the fully informed investor, and by \( \mathcal{F}^H_0 = \mathcal{F}^I_0 \subset \mathcal{F}^F_0 \) for \( H = R,Z,J \), i.e., for the partially informed investors. Here, \( \mathcal{F}^I_0 \) denotes the \( \sigma \)-algebra representing prior information about the initial drift \( \mu_0 \), e.g., from observing returns or expert opinions in the past, before the trading period \( [0,T] \). Note that all partially informed investors \( (H = R,J,Z) \) start at \( t = 0 \) with the same initial information given by \( \mathcal{F}^I_0 \). For \( t \in (0,T] \) we define

\[
\begin{align*}
\mathcal{F}^R_t &= \sigma(R_s, s \leq t) \lor \mathcal{F}^I_0, \\
\mathcal{F}^Z_t &= \sigma(R_s, s \leq t, (T_k,Z_k), T_k \leq t) \lor \mathcal{F}^I_0, \\
\mathcal{F}^J_t &= \sigma(R_s,J_s, s \leq t) \lor \mathcal{F}^I_0, \\
\mathcal{F}^F_t &= \sigma(R_s, \mu_s, s \leq t).
\end{align*}
\]

It is assumed that the above \( \sigma \)-algebras \( \mathcal{F}^H_t \) are augmented by the null sets of \( \mathbb{P} \). Further, we assume that the conditional distribution of the initial value drift \( \mu_0 \) given \( \mathcal{F}^I_0 \) is the normal distribution \( \mathcal{N}\left(m_0,q_0\right) \) with mean \( m_0 \in \mathbb{R}^d \) and covariance matrix \( q_0 \in \mathbb{R}^{d \times d} \) assumed to be symmetric and positive semi-definite. For more details and examples we refer to [18] Sec. 2.3.

2.4 Drift Estimates and Filtering

The investors’ trading decisions are based on their knowledge of the drift process \( \mu \). The fully informed \( F \)-investor observes the drift directly, the partially informed \( H \)-investor for
The estimator’s accuracy can be described by the conditional covariance matrix
\[ Q^H_t := \mathbb{E}[(\mu_t - M^H_t)(\mu_t - M^H_t)^\top] | \mathcal{F}^H_t. \]

In our market model the signal \( \mu \), the observations and the initial value of the filter are jointly Gaussian. Therefore, we are in the setting of the Kalman filter, and the conditional distribution of the drift at time \( t \) is Gaussian, which is completely characterized by the conditional mean \( M^H_t \) and the conditional covariance \( Q^H_t \). For the associated filter equations describing the dynamics of the filter processes \( M^H \) and \( Q^H \) we refer to [18, 29, 31] for the case of expert opinions arriving at fixed arrival times and to [17, 19, 31] for random arrival times which are the jump times of a Poisson process. While \( M^H \) solves a SDE driven by the return process with random jumps at the expert’s arrival dates, the conditional covariance \( Q^H \) is governed by a Riccati ODE between the arrival dates and exhibits jumps at the arrival dates. The jump sizes are a deterministic function of \( Q^H \) before the jump. Thus, for fixed arrival times, \( Q^H \) is deterministic and can be computed offline already in advance whereas for random arrival times, \( Q^H \) is only piecewise deterministic. Then it has to be computed online and to be included into the state of associate control problems.

We finally note that since the conditional distribution of \( \mu_t \) given \( \mathcal{F}^H_t \) is the Gaussian distribution \( \mathcal{N}(M^H_t, Q^H_t) \) and the two filter processes are Markov processes it holds for all \( \mathcal{F}^H_t \)-measurable random variables \( Y \) with \( t \leq s \leq T \), that there exists some measurable function \( h \) such that
\[ \mathbb{E}[Y | \mathcal{F}^H_t] = h(M^H_t, Q^H_t). \tag{2.3} \]

### 2.5 Portfolio and Optimization Problem

The self-financing trading of an investor can be described by the initial capital \( x_0 > 0 \) and the \( \mathbb{F}^H \)-adapted trading strategy \( \pi = (\pi_t)_{t \in [0, T]} \), with \( \pi_t \in \mathbb{R}^d \). The \( i \)-th component \( \pi^i_t \) represents the proportion of the current portfolio wealth invested in the \( i \)-th stock at time \( t \). The assumption that \( \pi \) is \( \mathbb{F}^H \)-adapted reflects that investment decisions have to be based only on information available to the \( H \)-investor. These are observations of assets returns for \( H = R \), returns combined with expert opinions for \( H = Z, J \), or returns combined with the drift process for \( H = F \). Following the strategy \( \pi \) the investor generates a wealth process \( (X^\pi_t)_{t \in [0, T]} \), whose dynamics reads as
\[ \frac{dX^\pi_t}{X^\pi_t} = \pi^\top_t dR_t + \pi^\top_t \mu_t \, dt + \pi^\top_t \sigma_t \, dW^R_t, \quad X^\pi_0 = x_0. \tag{2.4} \]

We denote by
\[ \mathcal{A}^H = \left\{ \pi = (\pi_t)_{t} : \pi_t \in \mathbb{R}^d, \pi \text{ is } \mathbb{F}^H \text{-adapted, } X^\pi > 0, \mathbb{E} \left[ \int_0^T \| \pi_t \|^2 \, dt \right] < \infty \right\} \tag{2.5} \]
the class of admissible trading strategies. The investor aims to maximize expected utility of terminal wealth using a utility function \( U : \mathbb{R}_+ \rightarrow \mathbb{R} \) which models the risk aversion of the investor. Here, we use the power utility function

\[
U_\theta(x) := \frac{x^\theta}{\theta}, \quad \theta \in (-\infty, 0) \cup (0, 1).
\]

As limiting case for \( \theta \to 0 \) the family of power utility function contains the logarithmic utility \( U_0(x) := \ln x \), since we have \( U_\theta(x) - \frac{1}{\theta} = \frac{x^{\theta-1}}{\theta} \to \log x \). The optimization problem thus reads as

\[
V^H_0 := \sup_{\pi \in \mathcal{A}^H} D^H_0(\pi) \quad \text{with} \quad D^H_0(\pi) = \mathbb{E} \left[ U_\theta(X^\pi_T) \mid \mathcal{F}_0^H \right], \quad \pi \in \mathcal{A}^H, \quad (2.6)
\]

where we call \( D^H_0(\pi) \) reward or performance of the strategy \( \pi \) and \( V^H_0 \) the value of the problem to given model parameters, in particular to given initial capital \( x_0 \). For \( H \neq F \) this is an optimization problem under partial information since we have required that the strategy \( \pi \) is adapted to the investor filtration \( \mathbb{F}^H \). However, the drift coefficient of the wealth equation (2.4) contains the non-observable drift \( \mu \) and is therefore not \( \mathbb{F}^H \)-adapted. For \( x_0 > 0 \) the solution of the SDE (2.4) is strictly positive. This ensures that the terminal wealth \( X^\pi_T \) is in the domain of logarithmic and power utility.

For problems of the above type, in the literature as outlined in Sec.1, dynamic programming is a powerful solution method which is frequently applied. The key idea is to embed the optimization problem (2.6) into a family of problems in which the initial date is moved from \( t = 0 \) to an arbitrary time point \( t \in [0, T] \), and the initial value of the wealth process \( X^\pi_t = x_0 \), as well as those of other state processes included in the analysis, are replaced by the respective values of the states at time \( t \). Then one ties all these problems together and derives a partial differential equation known as the Hamilton-Jacobi-Bellman (HJB) equation.

We introduce for a fixed strategy \( \pi \in \mathcal{A}^H \) the notation \( \mathcal{F}^{H,X}_t = \mathcal{F}_t^H \vee \sigma \{ X^\pi_t \} \) and note that \( \mathcal{F}^{H,X}_0 = \mathcal{F}_0^H \) since \( X^\pi_0 = x_0 \) is the given and fixed initial capital. Then the optimization problems of the above mentioned family are indexed by time \( t \in [0, T] \) and read

\[
V^H_t := \sup_{\pi \in \mathcal{A}^H} D^H_t(\pi) \quad \text{with} \quad D^H_t(\pi) = \mathbb{E} \left[ U_\theta(X^\pi_T) \mid \mathcal{F}_t^{H,X} \right], \quad \pi \in \mathcal{A}^H. \quad (2.7)
\]

In view of the result (2.3) for the Kalman filter and exploiting the Markov property of the wealth process, it holds for all \( \mathcal{F}_s^{\mu,X} \)-measurable random variables \( U \) with \( t \leq s \leq T \) that there exists some measurable function \( h \) such that

\[
\mathbb{E}[U \mid \mathcal{F}_s^{H,X}] = \mathbb{E}[U \mid Y^H_s] = h(Y^H_s). \quad (2.8)
\]

Here, \( Y^H \) denotes a state process which is given by the triple \( Y^H = (X, M^H, Q^H) \) taking values in the state space \( \mathcal{Y}^H = (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^{d \times d} \) for the information regimes with partial information \( (H = R, J, Z) \). The regime with full information \( (H = F) \) can be incorporated by choosing the state process as the pair \( Y^F = (X, \mu) \) taking values in the state space \( \mathcal{Y}^F = (0, \infty) \times \mathbb{R}^d \). By setting in (2.8) \( U = U_\theta(X^\pi_T) \) and \( s = T \), the conditional expectation in (2.7) defining the reward \( D^H_t(\pi) \) at time \( t \) can be expressed as

\[
D^H_t(\pi) = \mathbb{E} \left[ U_\theta(X^\pi_T) \mid Y^H_t \right] = D^H(t, Y^H_t, \pi).
\]
The function
\[ D^H(t, y; \pi) = \mathbb{E} \left[ U_\theta(X_t^\pi) \mid Y_t^H = y \right], \quad \pi \in \mathcal{A}^H, \] (2.9)
is called reward function and performance criterion for the strategy \( \pi \), and
\[ V^H(t, y) = \sup_{\pi \in \mathcal{A}^H} D^H(t, y; \pi) \] (2.10)
value function for the family of optimization problems (2.7).

Note that the conditional variance process \( Q^H \) for the information regimes \( H = R, J \) and for the regime \( H = Z \) with fixed information dates is deterministic. Thus it can be computed offline and removed from the state process \( Y^H \), as in [18]. For the regime \( H = Z \) with random information dates, however, \( Q^H \) is a stochastic process and must be included in \( Y^H \), see [19].

3 Well Posedness of the Optimization Problem

3.1 Well Posedness

Solving the utility maximization problem (2.6) for the various information regimes \( H = R, Z, J, F \) requires conditions under which the optimization problem is well posed. Under these conditions the maximum expected utility of terminal wealth cannot explode in finite time as it is the case for so-called nirvana strategies described in Kim and Omberg [20] and Angoshtari [1]. Such strategies generate in finite time a terminal wealth with a distribution leading to infinite expected utility although the realizations of terminal wealth may be finite.

We start by describing the model of the financial market via the parameter
\[ \rho := \{T, \theta, d, \sigma_R, \sigma_\mu, \kappa, \bar{\mu}, \bar{x}_0, \bar{q}_0, m_0, q_0, m_0, q_0\} \]
taking values in a suitable chosen set of parameter values \( \mathcal{P} \). For emphasizing the dependence on the parameter \( \rho \) we rewrite (2.9) and (2.10) for \( t \in [0, T] \) as
\[ D^H_\rho(t, y; \pi) = \mathbb{E} \left[ U_\theta(X_t^\pi) \mid Y_t^H = y, \rho \right], \quad \pi \in \mathcal{A}^H, \] (3.1)
\[ V^H_\rho(t, y) = \sup_{\pi \in \mathcal{A}^H} D^H_\rho(t, y; \pi). \]

For a given parameter \( \rho \) we want to study if the performance criterion of the optimization problem (3.1) is well-defined in the following sense.

**Definition 3.1** For a given financial market with parameter \( \rho \in \mathcal{P} \) we say that the utility maximization problem (3.1) for the \( H \)-investor is well-posed, if for every fixed \( (t, y) \in [0, T] \times \mathcal{Y}^H \) there exists a constant \( C^H_\rho = C^H_\rho(\rho, t, y) < \infty \) depending on \( \rho, t, y \) such that
\[ V^H_\rho(t, y) \leq C^H_\rho. \]

The set
\[ \mathcal{P}^H = \{ \rho \in \mathcal{P} : \text{problem (3.1) is well-posed} \} \subset \mathcal{P} \]
is called set of feasible parameters of the financial market model for which (3.1) is well-posed.
3.2 Log-Utility and Power Utility with $\theta < 0$

For power utility with parameter $\theta < 0$ it holds $U_\theta(x) < 0$. Hence, in that case we can choose $C_0^H = 0$ and the optimization problem is well-posed for all model parameters $p \in \mathcal{P}$ with $\theta < 0$. For log-utility ($\theta = 0$) the utility function is no longer bounded from above but it is shown in [18, Subsec. 4.1] and [30, Sec. 4] that the value function $V^H_p(t,y)$ is bounded from above by some positive constant $C^H_V = C^H_Y(\rho,t,y)$ for any selection of the model parameters in $\rho \in \mathcal{P}$. Hence it holds $\{\rho \in \mathcal{P} : \theta \leq 0\} \subset \mathcal{P}^H$. More challenging is the case of power utility with positive parameter $\theta \in (0,1)$ which is also not bounded from above. That case is investigated in the remainder of this section. We note that this approach can also be applied to log-utility leading to an alternative proof of well posedness for the maximization of expected log-utility, for details we refer to Kondakji [21, Sec. 4.2].

3.3 Power Utility with $\theta \in (0,1)$

For the study of well posedness it will be convenient to extend the concept of the fully informed $F$-investor who has access to observations of the return and drift process to an (artificial) investor who observes also the sequence of discrete-time expert opinions $(T_k, Z_k)$ and the continuous-time expert opinion process $J$, as well as the Wiener processes $W^R, W^\mu, W^J$. That investor is called $G$-investor and defined by the investor filtration $\mathbb{F}^G = \mathbb{G}$ which is the underlying filtration to which all stochastic process of the financial market model are adapted. Comparing the $F$- and $G$-investor the additional information from the observation of expert opinions and the driving Wiener processes $W^R, W^\mu, W^J$ will not lead to superior performance of the $G$-investor in the considered utility maximization problem, since the distribution of the wealth process $X^\pi$ is fully determined by the return process $R$ and the drift process $\mu$. The latter is known to the $G$-investor and does not need to be estimated. Thus, the associated state process $Y^G$ can be chosen as the pair $(X^\pi, \mu)$. However, for the $G$-investor we have the inclusion $\mathbb{F}^H \subset \mathbb{F}^G$ for $H = R, Z, J, F$. Note that for the $F$-investor we only have $\mathbb{F}^R \subset \mathbb{F}^F$ but in general $\mathbb{F}^Z, \mathbb{F}^J \subset \mathbb{F}^F$. Analogous to the other investors we define for $H = G$ the set of admissible strategies $\mathcal{A}^G$, the performance of a strategy $D^G_t$, the value $V^G_t$, the reward function $D^G_p(t,y,\pi)$ and the value function $V^G_p(t,y)$ as in (2.5) and (2.7) through (2.10), respectively.

Next we want to derive estimates of the value $V^H_p$ of the $H$-investor in terms of the value $V^G_t$ of the $G$-investor. Let us fix a strategy $\pi \in \mathcal{A}^H \subset \mathcal{A}^G$, then tower property of the conditional expectation with $\mathcal{F}^{H,X}_t \subset \mathcal{F}^{G,X}_t$ implies

$$D^H_t(\pi) = \mathbb{E}[U_\theta(X^\pi_t) | \mathcal{F}^{H,X}_t] = \mathbb{E}[\mathbb{E}[U_\theta(X^\pi_t) | \mathcal{F}^{G,X}_t] | \mathcal{F}^{H,X}_t] = \mathbb{E}[D^G_t(\pi) | \mathcal{F}^{H,X}_t].$$

Using relations (2.8) through (2.10) yields for all information regimes $H$

$$D^H_t(\pi) = D^H_p(t,Y^H_t,\pi) = \mathbb{E}[D^G_p(t,Y^G_t,\pi) | \mathcal{F}^{H,X}_t] = \mathbb{E}[D^G_p(t,Y^G_t,\pi) | Y^H_t].$$

Taking supremum over all admissible strategies in $\mathcal{A}^H$ it follows for all fixed $Y^H_t = y \in \mathcal{Y}^H$

$$V^H_p(t,y) = \sup_{\pi \in \mathcal{A}^H} \mathbb{E}[D^H_p(t,y,\pi)] = \sup_{\pi \in \mathcal{A}^H} \mathbb{E}[D^G_p(t,Y^G_t,\pi) | Y^H_t = y].$$
Using \( \mathcal{A}^H \subset \mathcal{A}^G \) and properties of the supremum we find the estimates

\[
V^H_\rho(t, y) \leq \mathbb{E}\left[ \sup_{\pi \in \mathcal{A}^H} D_\rho^G(t, Y^G_t, \pi) | Y^H_t = y \right] \\
\leq \mathbb{E}\left[ \sup_{\pi \in \mathcal{A}^G} D_\rho^G(t, Y^G_t, \pi) | Y^H_t = y \right] = \mathbb{E}\left[ V^G_\rho(t, Y^G_t) | Y^H_t = y \right].
\] (3.2)

In the sequel we will derive conditions under which \( V^G_\rho(t, y) \) with \( y = (x, m) \) is bounded for any fixed \( t \in [0, T] \) and \( X^x_t = x, \mu_t = m \). Then estimate (3.2) will allow us to derive conditions for the boundedness of \( V^H(t, y) \) for the other information regimes \( H \). We will need the following lemma where we denote by \( \mu^t,m \) the drift at time \( u \in [t, T] \) starting at time \( \sigma_t = [0, T] \) from \( m \in \mathbb{R}^d \). The proof is given in Appendix A.

**Lemma 3.2.** Let \( \gamma \in \mathbb{R} \setminus \{0\}, t \in [0, T] \), \( z > 0 \), and the stochastic process \( (\Psi_{s, z, m}^s)_{s \in [t, T]} \) be defined by

\[
\Psi_{s, z, m} := z \exp\left\{ \gamma \int_t^s (\mu^t,m)^\top \Sigma^{-1} \mu^t,m \ du \right\}
\] (3.3)

and the function \( d : [0, T] \times \mathbb{R}^d \to \mathbb{R} \) be defined by \( d(t, m) := \mathbb{E}[\Psi_{t, 1, m}] \), for \( t \in [0, T] \) and \( m \in \mathbb{R}^d \). Then it holds

\[
d(t, m) := \exp\left\{ m^\top A_\gamma(t)m + B_\gamma^\top(t)m + C_\gamma(t) \right\}.
\] (3.4)

Here \( A_\gamma(t) \), \( B_\gamma(t) \) and \( C_\gamma(t) \) are functions in \( t \in [0, T] \) taking values in \( \mathbb{R}^{d \times d} \), \( \mathbb{R}^d \) and \( \mathbb{R} \), respectively, satisfying the following system of ODEs

\[
\frac{dA_\gamma(t)}{dt} = -2A_\gamma(t)\Sigma \mu A_\gamma(t) + \kappa^\top A_\gamma(t) + A_\gamma(t) \kappa - \gamma \Sigma^{-1}, \quad A_\gamma(T) = 0_{d \times d},
\] (3.5)

\[
\frac{dB_\gamma(t)}{dt} = -2A_\gamma(t)\kappa \mu + \left[ \kappa^\top - 2A_\gamma(t)\Sigma \mu \right] B_\gamma(t), \quad B_\gamma(T) = 0_d,
\] (3.6)

\[
\frac{dC_\gamma(t)}{dt} = -\frac{1}{2} B_\gamma^\top(t)\Sigma \mu B_\gamma(t) - B_\gamma^\top(t) \kappa \mu - tr\{\Sigma \mu A_\gamma(t)\}, \quad C_\gamma(T) = 0.
\] (3.7)

Note that equation (3.5) is a Riccati equation for the symmetric matrix-valued function \( A_\gamma \), while equation (3.6) is a system of \( d \) linear differential equations whose solution \( B_\gamma \) is obtained given \( A_\gamma \). Finally, given \( A_\gamma \) and \( B_\gamma \) the scalar function \( C_\gamma \) is obtained by integrating the right hand side of (3.7).

**Theorem 3.3** For a model parameter \( \rho \) with \( \theta \in (0, 1) \) the value function of the G-investor satisfies for \( y = (x, m) \in \mathcal{Y}^G = (0, \infty) \times \mathbb{R}^d \)

\[
V^G_\rho(t, y) \leq \frac{x^\theta}{\theta} d^{1-\theta}(t, m),
\] (3.8)

where the function \( d : [0, T] \times \mathbb{R}^d \to \mathbb{R} \) is given by (3.4) for \( \gamma = \frac{\theta}{2(1-\theta)^2} \).

**Proof.** The proof is given in Appendix B.
The last theorem together with the fact that for $\theta \leq 0$ the problem is well posed (see the reasoning at the beginning of this section) allows to give the following characterization of the $G$-investor’s set $\mathcal{P}^G$ of feasible model parameters by the inclusion $\overline{\mathcal{P}}^G \subset \mathcal{P}^G$ where

$$\mathcal{P}^G = \{ \rho \in \mathcal{P} : \theta \in (0, 1) \text{ and } d(t, m) \text{ given in (3.4) is bounded for every fixed } (t, m) \in [0, T] \times \mathbb{R}^d \} \cup \{ \rho \in \mathcal{P} : \theta \leq 0 \}.$$  

We are now in a position to characterize the set of feasible model parameters $\mathcal{P}^H$ for $H = R, Z, J, F$ by combining the estimate (3.8) for $V^G_\rho(t, y)$ from Theorem 3.3 with (3.2) stating that $V^H_\rho(t, y) \leq \mathbb{E}[V^G_\rho(t, Y^G_t)|Y^H_t = y]$. Recall, that for the partially informed investors ($H = R, Z, J$) the state process is $Y^G_t = (X^{\pi, M^H, Q^H})$. For the $G$- and $F$-investor it is $Y^{G/F}_t = (X^{\pi, \mu})$. Thus, substituting the estimate (3.8) for $V^G_\rho$ into (3.2) yields for the partially informed investors for $y = (x, m, q)$

$$V^H_\rho(t, y) \leq \frac{\lambda}{\theta} \mathbb{E}\left[d^{1-\theta}(t, \mu_t)|M^H_{t} = m, Q^H_{t} = q\right], \quad H = R, Z, J;$$  

and for the fully informed investor for $y = (x, m)$

$$V^F_\rho(t, y) \leq \frac{\lambda}{\theta} \mathbb{E}\left[d^{1-\theta}(t, \mu_t)|\mu_t = m\right].$$  

**Well posedness for full information ($H = F$).** For the $F$-Investor the drift is known and from inequality (3.11) it follows for $y = (x, m) \in \mathcal{Y}^F = (0, \infty) \times \mathbb{R}^d$

$$V^F_\rho(t, y) \leq \frac{\lambda}{\theta} d^{1-\theta}(t, m),$$

which implies that the inclusion given in (3.9) for $\mathcal{P}^G$ also holds for the set of feasible model parameters $\mathcal{P}^F$ for the $F$-investor.

The restrictions in (3.9) to the feasible model parameters $\rho$ for $\theta \in (0, 1)$ are given implicitly via the boundedness of $d(t, m)$ where $d$ is given in (3.4). They can be further analyzed by studying conditions for non-explosive solutions of Riccati equation (3.5) for the matrix-valued function $A_\gamma$ on the investment horizon $[0, T]$. The boundedness of the solution to (3.5) carries over to the boundedness of the solution to the linear differential equation (3.6) for $B_\gamma$ and also to $C_\gamma$ which is obtained by integrating the right hand side of (3.7). Thus we obtain

**Corollary 3.4 (Sufficient condition for well posedness, full information)**

The utility maximization problem (3.1) for the fully informed $F$-investor is well-posed for all parameters $\rho \in \overline{\mathcal{P}}^F \subset \mathcal{P}^F$ where

$$\mathcal{P}^F = \left\{ \rho \in \mathcal{P} : \theta \in (0, 1), A_\gamma \text{ is bounded on } [0, T] \text{ for } \gamma = \frac{\theta}{2(1-\theta)^2} \right\} \cup \{ \rho \in \mathcal{P} : \theta \leq 0 \},$$

and $A_\gamma$ is the solution to Riccati equation (3.5).
We observe that the sufficient condition derived in Corollary 3.4 does not depend on the initial values of the state process \((x_0, m_0)\) but only on \(T\), and the constant parameters \(\theta, \sigma_R, \sigma_{\mu}, \kappa\). Note that, if the solution \(A_T\) to the terminal value problem for the Riccati equation (3.5) does not explode on \([0, T]\), i.e., it is bounded, then it also does not explode on \([t, T]\) for any \(t \in [0, T]\). Thus, for the well posedness of the \(F\)-investor’s optimization problem (3.1) it is sufficient that \(A_T(t)\) is bounded for \(t = 0\). Below we will see that for utility maximization problems under partial information we need stronger conditions.

It is well known, that in general closed-form solutions of Riccati differential equations are available only for the one-dimensional case (\(d = 1\)). More details about this special case can be found below in Subsec. 3.4.

**Well posedness for partial information** (\(\mathbf{H} = R, Z, J\)). For the partially informed investors the random variable \(d(t, \mu_t)\) in (3.10) is no longer \(\mathcal{F}_t^H X\)-measurable and we have to compute the conditional expectation using the conditional distribution of the drift value \(\mu_t\) given \(\mathcal{F}_t^H X\). We recall that we are in the setting of the Kalman filter. Thus, the conditional distribution of \(\mu_t\) is Gaussian and completely characterized by the conditional mean \(M_t^H\) and the conditional covariance \(Q_t^H\).

The result is given below in Theorem 3.7 for which we need the following lemma. The proofs can be found in Appendix C and D respectively.

**Lemma 3.5** Let \(U, \Sigma\) be symmetric \(d \times d\) matrices, \(\Sigma\) positive semidefinite with an associated decomposition \(\Sigma = PP^\top\) with a \(d \times d\)-matrix \(P\). Further, let the eigenvalues of \(\Sigma U\) be denoted by \(\lambda_1, \ldots, \lambda_d\).

1. The eigenvalues \(\lambda_1, \ldots, \lambda_d\) of \(\Sigma U\) are also the eigenvalues of \(P^\top UP\). They are all real.

2. Let \(K = I_d - 2\Sigma U, a \in \mathbb{R}^d, Y \sim \mathcal{N}(0_d, \Sigma)\) be a \(d\)-dimensional zero-mean Gaussian random vector with covariance matrix \(\Sigma\), and \(V\) the real-valued random variable defined by the quadratic form \(V = Y^\top UY + a^\top Y\).

   If \(\lambda_i < \frac{1}{2}\) for all \(i = 1, \ldots, d\), then it holds for the exponential moment of \(V\)

   \[
   \mathbb{E}[e^V] = \mathbb{E}\left[\exp(Y^\top UY + a^\top Y)\right] = (\det(K))^{-1/2}\exp\left\{\frac{1}{2}a^\top K^{-1}\Sigma a\right\}. \tag{3.12}
   
   
3. For the terms of the right hand side of (3.12) it holds

   \[
   \det(K) = \prod_{j=1}^d (1 - 2\lambda_j) \quad \text{and} \quad a^\top K^{-1}\Sigma a = \sum_{j=1}^d c_j^2 (1 - 2\lambda_j)^{-1}, \tag{3.13}
   
   where \(c_1, \ldots, c_d\) are the entries of the vector \(c = D^\top P^\top a\), and \(D\) is the orthogonal \(d \times d\)-matrix diagonalizing the symmetric matrix \(P^\top UP\) such that it holds \(P^\top UP = \Lambda D\top\) with \(\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_d)\).

**Remark 3.6** The expressions in (3.13) are helpful for the actual computation of the expectation in (3.12). For large dimensions \(d\) and a covariance matrix \(\Sigma\) of low-rank \(r \ll d\) the computational efficiency can be improved by working with a low-rank decomposition \(\Sigma = P_r P_r^\top\) with a \(d \times r\)-matrix \(P_r\), and an eigenvalue decomposition of the \(r \times r\)-matrix \(P_r^\top U P_r = D_r \Lambda_r D_r^\top\) with an orthogonal \(r \times r\)-matrix \(D_r\) and the diagonal matrix \(\Lambda_r\) obtained form \(\Lambda\) by removing \(d - r\) zero eigenvalues on the diagonal.
Theorem 3.7 Let for the information regimes $H = R, Z, J$ with partial information and $t \in [0, T]$ denote the conditional mean and covariance describing the Gaussian conditional distribution of the drift $\mu$, given $\mathcal{F}^t_H$ by $M^t_H = m \in \mathbb{R}^d$ and $Q^t_H = q \in \mathbb{R}^d \times \mathbb{R}^d$. Further, let the solution $A_\gamma$ to the Riccati equation (3.5) for $\gamma = \frac{\theta}{2(1-\theta)}$ be bounded on $[0, T]$, and assume that all eigenvalues of $K = K(t) = I_d - 2(1-\theta)qA_\gamma(t)$ are positive. Then it holds for all $x > 0$ and $y = (x, m, q)$

$$V^H_\theta(t, y) \leq \frac{\theta}{\theta} C^H_\theta \quad \text{with} \quad C^H_\theta = d^{1-\theta}(t, m)(\det(K))^{-1/2}\exp\left\{\frac{1}{2}a^T K^{-1} qa\right\},$$

where $a = a(t) = (1-\theta)2A_\gamma(t)m + B_\gamma(t)$ and $d(t, m)$ is given in (3.4).

We observe that the upper bound for $V^H_\theta(t, y)$ given in Theorem 3.7 is finite if $d(t, m)$ is finite and all eigenvalues of $K = I_d - 2(1-\theta)qA_\gamma(t)$ are positive. As in the full information case, the first condition is satisfied if the Riccati equation (3.5) for $A_\gamma$ is bounded on $[0, T]$. Recall, this depends only on the choice of the constant model parameters $\theta, \sigma_R, \sigma_o, \kappa$ but not on $M^t_H = m$, and it implies boundedness of $A_\gamma$ on $[t, T]$ for all $t \in [0, T]$. However, the second condition is for $\theta \in (0, 1)$ an additional restriction for the partially informed case and states that the conditional covariance $Q^H = q$ is not “too large” such that all eigenvalues of $K$ are positive. We have to require that the conditional covariance $Q^H$ process starting at $t = 0$ with $Q^H_0 = q_0$ is such that the eigenvalues of $K = K(t) = I_d - 2(1-\theta)Q^H_\gamma(t)$ remain positive on the entire time interval $[0, T]$. Note that for the regimes $H = R, J$ and for $H = Z$ with discrete-time expert opinions at fixed arrival times, $Q^Z$ a deterministic function and fully specified by its initial value $q_0$. However, for the regime $H = Z$ with discrete-time expert opinions at random arrival times, $Q^Z$ is a stochastic process, it depends on the random timing of expert’s views and is not only specified by its initial value $q_0$.

For $\theta \in (0, 1)$ it is known that if the solution $A_\gamma$ exists on $[0, T]$, it is symmetric and positive semidefinite, see Roduner [26] Theorem 1.2. Further, the conditional covariance $Q^H$ is also symmetric and positive semidefinite. However, the product $Q^H A_\gamma$ of the two symmetric matrices is generally no longer symmetric, and the properties of such matrices may no longer apply. But it is known that $Q^H A_\gamma$ has the same eigenvalues as $D = SQ^H_S$ with $S$ denoting the unique symmetric and positive semidefinite square root of $A_\gamma$, that is $A_\gamma = SS$. Since $D$ is symmetric and positive semidefinite its eigenvalues and therefore the eigenvalues of $Q^H A_\gamma$ are nonnegative. Finally, let $\lambda = \lambda(Q^H A_\gamma) \geq 0$ be an arbitrary eigenvalue of $Q^H A_\gamma$. Then $1 - 2(1-\theta)\lambda$ is an eigenvalue of $K = I_d - 2(1-\theta)Q^H A_\gamma$. Thus, the condition that the eigenvalues of $K$ are positive implies that all eigenvalues of $Q^H A_\gamma(t)$ are required to be strictly smaller than $\frac{1}{2(1-\theta)}$ for all $t \in [0, T]$. Let $\lambda_{\max}(G)$ denote the largest eigenvalue of a generic matrix $G$ with real and nonnegative eigenvalues, then this condition can be stated as

$$\lambda_{\max}(Q^H A_\gamma(t)) < \frac{1}{2(1-\theta)}, \quad \text{for all } t \in [0, T]. \quad (3.14)$$

Summarizing, from the above theorem we deduce the following sufficient condition for well posedness.

Corollary 3.8 (Sufficient condition for well posedness, partial information)

The utility maximization problem (3.1) for the partially informed $H$-investor, $H = R, J, Z,$
is well-posed for all parameters $\rho \in \mathcal{P}^H \subset \mathcal{P}^H$ where

$$\mathcal{P}^H = \left\{ \rho \in \mathcal{P} : \theta \in (0, 1), A_\gamma \text{ is bounded on } [0, T] \text{ for } \gamma = \frac{\theta}{2(1-\theta)^2}, \right. \left. \text{ and } \lambda_{\max}(Q^H_\gamma A_\gamma(t)) < \frac{1}{2(1-\theta)}, \text{ for all } t \in [0, T] \right\} \cup \{\rho \in \mathcal{P} : \theta \leq 0\}.$$ 

### 3.4 Market Models With a Single Risky Asset

The above conditions for the well posedness given in terms of the boundedness of $A_\gamma(t)$ on $[0, T]$, and condition (3.14) to the eigenvalues of $Q^H_\gamma A_\gamma(t)$ are quite abstract and its verification requires that the solution of the Riccati ODE (3.5) is bounded on $[0, T]$. While in the multi-dimensional case Riccati ODEs in general can be solved only numerically these equations enjoy a closed-form solution in the one-dimensional case. This allows to give more explicit characterizations of the set of feasible parameters for market models with a single risky asset only. The following lemma gives explicit conditions to the model parameters under which (3.5) has a bounded solution on $[0, T]$. For the proof we refer to Kondakji [21, Lemma A.1.3, A.2.2 and A.2.3]

**Lemma 3.9** Let $d = 1$, $\theta \in (0, 1), \gamma = \frac{\theta}{2(1-\theta)}$, and

$$\Delta_\gamma = 4\kappa^2 \left( 1 - 2\gamma \left( \frac{\sigma_\mu}{\kappa \sigma_R} \right)^2 \right) \quad \text{and} \quad \delta_\gamma := \frac{1}{2} \sqrt{|\Delta_\gamma|}. \quad (3.15)$$

Then it holds for the Riccati differential equation (3.5) on $[0, T]$

1. For $\Delta_\gamma \geq 0$ there is a bounded solution for all $T > 0$.
2. For $\Delta_\gamma < 0$ a bounded solution exists only if $T < T^E_\gamma$ with the explosion time

$$T^E_\gamma := \frac{1}{8\gamma} \left( \frac{\pi}{2} + \arctan \left( \frac{\kappa}{\delta_\gamma} \right) \right). \quad (3.16)$$

The above lemma allows to give more explicit sufficient conditions for well posedness given for the general multi-dimensional case in Corollary 3.4 and 3.8. They can be formulated in terms of the parameters $\kappa, \sigma_\mu, \sigma_R$ describing the variance of the asset price and drift process, the investment horizon $T$, the parameter $\theta$ of the utility function and the conditional covariance process $Q^H$. Analyzing the inequality $T < T^E_\gamma$ and (3.16) we obtain

**Corollary 3.10 (Sufficient condition for well posedness, single risky asset)**

Let $d = 1, \gamma = \frac{\theta}{2(1-\theta)}$, and $\Delta_\gamma, T^E_\gamma$ as given in (3.15) and (3.16), respectively.

1. The utility maximization problem (3.1) for the fully informed $F$-investor is well-posed for all parameters $\rho \in \mathcal{P}^F \subset \mathcal{P}^F$ where

$$\mathcal{P}^F = \left\{ \rho \in \mathcal{P} : \theta \in (0, 1), \kappa, \sigma_\mu, \sigma_R \text{ such that either } \Delta_\gamma \geq 0, \text{ or } \Delta_\gamma < 0 \text{ and } T < T^E_\gamma \right\} \cup \{\rho \in \mathcal{P} : \theta \leq 0\}.$$
The utility maximization problem (3.1) for the partially informed $H$-investor, $H = R, J, Z$, is well-posed for all parameters $\rho \in \mathcal{D}^H \subset \mathcal{P}^H$ where

$$\mathcal{D}^H = \{ \rho \in \mathcal{P} : \theta \in (0, 1), \kappa, \sigma_\mu, \sigma_R \text{ such that either } \Delta_\gamma \geq 0, \text{ or } \Delta_\gamma < 0 \text{ and } T < T_\gamma^E, \} \cup \{ \rho \in \mathcal{P} : \theta \leq 0 \}.$$

(3.17)

### 4 Numerical Results

In this section we illustrate the theoretical findings of the previous sections by results of some numerical experiments. They are based on a stock market model where the unobservable drift $(\mu_t)_{t \in [0, T]}$ follows an Ornstein-Uhlenbeck process as given in (2.2) whereas the volatility is known and constant. For simplicity, we assume that there is only one risky asset in the market, i.e. $d = 1$. If not stated otherwise, our numerical experiments are based on model parameters as given in Table 4.1.

| Drift mean reversion level $\mu_0$ | Investment horizon $T$ | 1 year |
|---|---|---|
| mean reversion speed $\kappa$ | Power utility parameter $\theta$ | 0.3 |
| volatility $\sigma_\mu$ | Volatility of stock $\sigma_R$ | 0.25 |
| mean of $\mu_0$ $m_0$ | Initial estimate $m_0 = m_0$ | 0 |
| variance of $\mu_0$ $\sigma_\mu_0$ $\frac{\sigma_\mu_0^2}{2\kappa} = 0.16$ | $q_0 = q_0$ | 0.16 |

Table 4.1 Model parameters for the numerical experiments

In Section 3 we have specified sufficient conditions to the model parameters for which the optimization problem is well-posed. For market models with a single risky asset these conditions are given Corollary 3.10. In Figure 4.1 we visualize the subset $\mathcal{D}^F$ of the set of feasible parameters $\mathcal{D}^F$ for which well posedness for the utility maximization problem of the fully informed investor can be guaranteed. In particular, we show the dependence of $\mathcal{D}^F$ on the investment horizon $T$, the power utility parameter $\theta$, the volatility $\sigma_\mu$ of the drift and the volatility $\sigma_R$ of the stock price.

The two top panels show the subset $\mathcal{D}^F$ depending on $\theta, T$ and $\sigma_R$. It can be seen that for negative $\theta$, i.e. for investors which are more risk averse than the log-utility investor, the optimization problem is always well-posed. Moreover, the top left panel shows that for the selected parameters the optimization problem is well-posed for all $T > 0$ if the parameter $\theta$ does not exceed the critical value $\theta^E \approx 0.36$. For $\theta > \theta^E$, i.e. for investors with sufficiently small risk-aversion, the optimization problem is no longer well-posed for all investment horizons $T$, but only up to a critical investment horizon $T^E = T^E(\theta)$ depending on $\theta$ and given in (3.16). The larger $\theta$, the smaller is that critical investment horizon $T(\theta)$. For the limiting case $\theta \rightarrow 1$ it holds $T^E(\theta) \rightarrow 0$. The top right panel shows for an investment horizon fixed to $T = 1$ the subset $\mathcal{D}^F$ depending on $\theta$ and the volatility $\sigma_R$ of stock price. It can be seen that larger values for the stock volatility allow to choose larger values of $\theta$.

The two panels in the middle illustrate the influence of the drift volatility $\sigma_\mu$ on the subset $\mathcal{D}^F$. The left panel shows that the optimization problem is well-posed for all $T > 0$ as long as the volatility $\sigma_\mu$ of the drift does not exceed the critical value $\sigma_\mu^E \approx 1.15$. For
volatilities $\sigma_\mu > \sigma_\mu^E$ the optimization problem is well-posed only for investment horizons $T$ smaller than the critical horizon $T^E = T^E(\sigma_\mu)$ that depends on $\sigma_\mu$ and is given in (3.16).

In the right panel we investigate for fixed investment horizon $T = 1$ the dependence of $\mathcal{P}_F$ on the drift volatility $\sigma_\mu$ and the power utility parameter $\theta$. While for $\theta < 0$ there are no further restrictions on the parameters, this is no longer true for $\theta \in (0, 1)$. The larger the volatility $\sigma_\mu$, the smaller one has to choose $\theta$.

The bottom two panels illustrate the influence of stock volatility $\sigma_R$ on the subset $\mathcal{P}_F$. In contrast to the volatility $\sigma_\mu$ of the drift, smaller values of $\sigma_R$ imply that the optimization problem is well-posed only for smaller $T$ as it can be seen in the bottom left panel. If $\sigma_R$ does not exceed the critical value $\sigma_R^E \approx 0.22$, then the optimization problem is well-posed only up to a critical investment horizon $T^E = T^E(\sigma_R)$ that depends on $\sigma_R$. The larger $\sigma_R$, the larger the horizon can be set. However, for $\sigma_R$ exceeding the critical value $\sigma_R^E$ the control problem is well defined for any horizon time $T > 0$. The bottom right panel shows the dependence of $\mathcal{P}_F$ on the two volatilities $\sigma_R$ and $\sigma_\mu$. Note that the two regions are separated by a straight line as it can be deduced from (3.15) and (3.16).

Finally, we consider the case of a partial informed investor. Then an additional condition on the covariance process $Q^H$ of the filter has to imposed to ensure well posedness. We refer to Corollary 3.8 for the multi-dimensional case and to Corollary 3.10 and (3.17) for the special case of markets with a single risky asset considered here. The sufficient condition requires that $Q^H < 1/(2(1 - \theta)A_\gamma(t))$ which is satisfied for the model parameters from Table 4.1. First, it is known that $A_\gamma$ is decreasing on $[0, T]$. Second, proper-
ties of the conditional covariance process (see [18,29,30]) imply $Q^H_t \leq Q^R_t$. Further, for $q_0 > Q^R_\infty = \lim_{t\to\infty} Q^R_t$ we have that $Q^R$ is decreasing. According to [16, Prop. 4.6] it holds $Q^R_\infty = \sigma_R \sqrt{\sigma^2_R \kappa^2 + \sigma^2_\mu} - \kappa \sigma^2_R = 0.125$ and thus $Q^R_\infty < q_0 = 0.16$ which yields that we have $Q^H_t \leq q_0 \leq 1.25$ since $\frac{1}{2(1-\theta)A_\gamma(t)} \geq \frac{1}{2(1-\theta)A_\gamma(0)} = 0.964$. This shows that the problem is well-posed.

**Conclusion**

The paper derived sufficient conditions for the well posedness of power utility maximization problems under full and partial information on the not directly observable drift of risky assets. For power utility with relative risk aversion smaller than that of log-utility these conditions ensure the absence of nirvana strategies as well as bounded value functions arising in the solution with dynamic programming techniques. They lead to restrictions on the choice of the model parameters such as the investment horizon and the risk aversion parameter of the power utility function, parameters controlling the variance of the asset price and drift processes. For the fully informed investor well posedness does not depend on the choice of the parameters $x_0, \bar{m}_0, \underline{m}_0$ defining the initial values of the state process $Y^F = (X^\pi, \mu)$. However, and somewhat surprisingly, for the partially informed investors ($H = R, Z, J$), well posedness is only guaranteed if one component of the state process $Y^H = (X^\pi, M^H, Q^H)$, namely the filter process $Q^H$ of conditional covariance is “sufficiently small” on the entire time interval $[0, T]$. For the regimes $H = R, J$ and for $H = Z$ with discrete-time expert opinions at fixed arrival times, $Q^Z$ is a deterministic function and is therefore fully specified by its initial value $q_0$. But for the regime $H = Z$ with discrete-time expert opinions at random arrival times, $Q^Z$ is a stochastic process that depends on the random arrival times and therefore not only specified by its initial value $q_0$.

The well posedness conditions are related to non-explosive solutions of certain terminal value problems for matrix Riccati differential equations on the time interval $[0, T]$. They become more explicit for financial markets with a single risky asset. For that case the paper provides numerical results and visualizes the set of feasible model parameters. For the actual solution of the analyzed portfolio optimization problems we refer to our papers [18,19].

**A Proof of Lemma 3.2**

**Proof.** Consider first the function $g \in C^{1,2}$ defined as follows

$$g : [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} ; \quad g(t, z, m) := \mathbb{E} \left[ \Psi_t^{\mu, z, m} \right].$$  \hspace{1cm} (A.1)

Then it holds $g(T, z, m) = \mathbb{E} \left[ \xi_T^{z, m} \right] = \mathbb{E} [z] = z$.

The dynamics of the drift $\mu$ and the process $\Psi$ for $s \in [t, T]$ read as

$$\begin{pmatrix} \frac{d\mu^l_t}{d\Psi_t^{z, m}} \\ \frac{d\Psi_t^{z, m}}{d\psi_t^{z, m}} \end{pmatrix} = \begin{pmatrix} \kappa (\mu_t - \mu^l_t) \\ \gamma \mu_t^{l, m} (\mu_t^{l, m})^\top \Sigma^{-1} \mu_t^{l, m} \end{pmatrix} ds + \begin{pmatrix} \sigma_\mu \\ 0_{1 \times d_\mu} \end{pmatrix} dW^\mu_t + \begin{pmatrix} \mu_t^{l, m} \\ \mu_t^{l, m} \end{pmatrix} d\tilde{W}^\mu_t ; \quad \begin{pmatrix} \mu_t^{l, m} \\ \Psi_t^{z, m} \end{pmatrix} = \begin{pmatrix} m \\ \psi_t^{z, m} \end{pmatrix}.$$

The drift and the diffusion coefficients of the last equation satisfy the Lipschitz- and linear growth conditions. Moreover, the Feynman-Kac-Formula for the expectation from \(\{A.1\} \) leads to the fol-
lowing partial differential equation for $g$

\[
0 = \frac{\partial}{\partial t} g(t, z, m) + \nabla_m^\top g(t, z, m) \kappa(\mu - m) + \frac{1}{2} \text{tr}\{ \nabla_{mm} g(t, z, m) \Sigma_\mu \} \\
+ \gamma z m^\top \Sigma_R^{-1} m \frac{\partial g(t, z, m)}{\partial z},
\]

(A.2)

with $g(T, z, m) = z$ as terminal condition and $\nabla_m$ and $\nabla_{mm}$ denoting gradient and Hessian, respectively. The above terminal value problem can be solved with the following separation ansatz

\[g(t, z, m) = z d(t, m), \quad d(T, m) = 1.\] (A.3)

At time $t$ we have $\mu_t^1, m = m$ and $\Psi_t^{1, m} = 1$ so that we obtain $\mathbb{E}[\Psi_T^{1, m}] = g(t, 1, m) = d(t, m)$ which is the function $d$ defined in Lemma (3.2). Plugging (A.3) into (A.2) leads to the following linear parabolic PDE for $d$

\[
0 = \frac{\partial}{\partial t} d(t, m) + \nabla_m^\top d(t, m) \kappa(\mu - m) + \frac{1}{2} \text{tr}\{ \nabla_{mm} d(t, m) \Sigma_\mu \} + \gamma m^\top \Sigma_R^{-1} m d(t, m),
\]

with terminal value $d(T, m) = 1$. For solving the above PDE the ansatz

\[d(t, m) = \exp\{ m^\top A_\gamma(t) m + B_\gamma^\top(t) m + C_\gamma(t) \} \]

leads to the system of ODEs for $A_\gamma, B_\gamma$ and $C_\gamma$, which are given in (3.5), (3.6) and (3.7). \qed

**B Proof of Theorem 3.3**

**Proof.** In the proof we follow an approach presented in Angoshtari [1, Theorem 1.8]. Without loss of generality we give the proof for $t = 0$ and show that it holds $V_t^G(0, y) \leq \frac{\varphi}{\pi} d^{1-\theta}(0, m)$ for all $y = (x, m) \in \mathcal{Y}^G = (0, \infty) \times \mathbb{R}^d$.

Let $(\xi_t)_{t \in [0, T]}$ be a stochastic process satisfying the SDE

\[d\xi_t = -\xi_t \mu_t^\top \Sigma_R^{-1} \sigma_R dW_t^R, \quad \xi_0 = 1, \quad \mu_0 = m,\] (B.1)

with the solution

\[\xi_t = \exp\left\{ -\frac{1}{2} \int_0^t \| \mu_s^\top \Sigma_R^{-1} \sigma_R \|^2 ds - \int_0^t \mu_s^\top \Sigma_R^{-1} \sigma_R dW_s^R \right\}.
\]

For $t_0 \in [0, T]$ we denote by $\mu_{t_0, m}$ the solution to the SDE (2.2) for the drift process $\mu$ starting at time $t_0$ with initial value $m$, by $X_t^{\pi_0, x, m}$ the solution to the wealth equation (2.4) with initial values $(x, m)$ and by $\xi_{t_0, z, m}$ the solution of (B.1) at time $t$ with initial values $(z, m)$. Applying Itô’s-formula it holds

\[d(X_t^{\pi_0, x, m}, \xi_t^{0,1,m}) = X_t^{\pi_0, x, m} d\xi_t^{0,1,m} + \xi_t^{0,1,m} dX_t^{\pi_0, x, m} + d\langle X_t^{\pi_0, x, m}, \xi_t^{0,1,m} \rangle_t
\]

\[= \xi_t^{0,1,m} X_t^{\pi_0, x, m} [\pi_t^\top \sigma_R - \mu_t^0 m^\top \Sigma_R^{-1} \sigma_R] dW_t^R.
\]

Moreover, Fatou’s Lemma implies that the non-negative process $\langle X_t^\gamma \xi_t \rangle$ is a supermartingal, and as a consequence it holds

\[x - \mathbb{E}[X_T^{\pi_0, x, m}, \xi_T^{0,1,m}] \geq 0,\] (B.2)
From the other hand let \( f : \mathbb{R}^+ \to \mathbb{R} \) be the associated Legendre-Fenchel transformation of the utility function \( U_\theta(x) \) defined for every \( w > 0 \) by
\[
f(w) := \sup_{x \in \mathbb{R}^+} \{ U_\theta(x) - xw \} = \frac{1 - \theta}{\theta} w^{-\frac{\theta}{\gamma}}. \tag{B.3}
\]

Since \( \xi_T^{0,1,m} > 0 \), it holds for every \( w > 0 \)
\[
f(\xi_T^{0,1,m} w) = \sup_{x \in \mathbb{R}^+} \{ U_\theta(x) - x\xi_T^{0,1,m} w \} \geq U_\theta(x_T^{0,0,x,m}) - X_T^{0,1,m} \xi_T^{0,1,m} w. \tag{B.4}
\]

Now for \( w > 0 \) and \( y = (x,m) \in \mathcal{Y}^G = (0,\infty) \times \mathbb{R}^d \) inequality (B.2) implies that
\[
D^G_\rho(0,y;\pi) = \mathbb{E}[U_\theta(X_T^{0,0,x,m})] \leq \mathbb{E}[U_\theta(X_T^{0,0,x,m})] + w(x - \mathbb{E}[X_T^{0,0,x,m} \xi_T^{0,1,m} w])
\]
\[
= \mathbb{E}[U_\theta(X_T^{0,0,x,m}) - X_T^{0,0,x,m} \xi_T^{0,1,m} w + xw]
\]
\[
\leq \mathbb{E}[f(\xi_T^{0,1,m} w)] + xw,
\]
where the last inequality follows from (B.4). For the term \( f(\xi_T^{0,1,m} w) \) we now apply (B.3) to obtain
\[
D^G_\rho(0,y;\pi) \leq \frac{1 - \theta}{\theta} w^{-\frac{\theta}{\gamma}} \mathbb{E}\left[ (\xi_T^{0,1,m})^{-\frac{\theta}{\gamma}} \right] + xw.
\]

Since the last inequality holds for every admissible strategy \( \pi \in \mathcal{A}^G \) and for every \( w > 0 \), we can take the supremum over all strategies \( \pi \in \mathcal{A}^G \) on the left-hand side and the infimum over all \( w > 0 \) in the right-hand side to obtain
\[
V^G_\rho(0,y) = \sup_{\pi \in \mathcal{A}^G} D^G_\rho(0,y;\pi) \leq \frac{\gamma^\theta}{\theta^\theta} \left( \mathbb{E}\left[ (\xi_T^{0,1,m})^{-\frac{\theta}{\gamma}} \right] \right)^{1-\theta}. \tag{B.5}
\]

The problem is now reduced to investigate if the expectation in the r.h.s. of (B.5) is bounded. It holds
\[
(\xi_T^{0,1,m})^{-\frac{\theta}{\gamma}} = \exp\left\{ \frac{\theta}{1-\theta} \left( \frac{1}{2} \int_0^T \left[ \left( \mu_s^{0,m} \right)^{\top} \Sigma_R^{-1} \sigma_R \right]^2 \; ds + \int_0^T \left( \mu_s^{0,m} \right)^{\top} \Sigma_R^{-1} \sigma_R \; dW_s^R \right) \right\}
\]
\[
= \Lambda_T \cdot \Psi_T^{0,1,m},
\]
where \( \Psi_T^{0,1,m} \) is given in (3.3) with \( \gamma = \frac{\theta}{\Sigma(1-\theta)^2} \). The term \( \Lambda_T \) is given by
\[
\Lambda_T = \exp\left\{ \int_0^T \frac{\theta}{1-\theta} \left( \mu_s^{0,m} \right)^{\top} \Sigma_R^{-1} \sigma_R \; dW_s^R - \frac{1}{2} \int_0^T \left[ \frac{\theta}{1-\theta} \left( \mu_s^{0,m} \right)^{\top} \Sigma_R^{-1} \sigma_R \right]^2 \; ds \right\}.
\]

We now introduce a new probability measure \( \mathbb{P}^* \) given by \( d\mathbb{P}^* = \Lambda_T \) so that the expectation from (B.5) can be expressed as
\[
\mathbb{E}\left[ (\xi_T^{0,1,m})^{-\frac{\theta}{\gamma}} \right] = \mathbb{E}\left[ \Lambda_T \cdot \Psi_T^{0,1,m} \right] = \mathbb{E}^*\left[ \Psi_T^{0,1,m} \right] = d(0,m).
\]

This expectation can be expressed according to (3.4) in Lemma 3.2 and its proof given in Appendix A where \( \mathbb{E}^* \) denotes the expectation under the new probability measure. \( \square \)
C Proof of Lemma 3.5

Proof. First claim. Note that the product of the symmetric matrices $\Sigma$ and $U$ needs not to be symmetric, the latter would immediately imply real eigenvalues.

Since $\Sigma$ is positive semidefinite there exists a $d \times d$-matrix $P$ such that $\Sigma = PP^\top$. It is well-known that for $d \times d$ matrices $A, B$ it holds that $AB$ and $BA$ have the same eigenvalues. Setting $A = P$ and $B = P^\top U$ it follows that $\Sigma U = PP^\top U$ and $P^\top UP$ have the same eigenvalues. They are real since $P^\top UP$ is symmetric.

Second claim. The decomposition $\Sigma = PP^\top$ allows the representation $Y = PZ$ with an $d$-dimensional standard normally distributed random vector $Z = N(0_d, I_d)$, since the mean of $PZ$ is $\mathbb{E}[PZ] = 0_d$ and its covariance matrix is $\mathbb{E}[PZZ^\top P^\top] = P I_d P^\top = \Sigma$. Then, we have $Y^\top UY + a^\top Y = Z^\top P^\top UPZ + a^\top PZ$, so that it holds

$$
\mathbb{E}[e^Y] = \mathbb{E}[\exp (Y^\top UY + a^\top Y)] = \mathbb{E}[\exp (Z^\top P^\top UPZ + a^\top PZ)]
$$

$$
= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp(z^\top P^\top UPz + a^\top Pz - \frac{1}{2} z^\top z)dz. \quad (C.1)
$$

The eigenvalues of $\Sigma U$ are real and satisfy by assumption $\lambda_i < \frac{1}{2}, i = 1, \ldots, n$. This implies that the eigenvalues of $K = I_d - 2\Sigma U$ which are given by $1 - 2\lambda_i$ are real and positive, and $K$ is invertible. We define the $d \times d$ matrix $G = I_d - 2P^\top UP$. Recall that the matrices $\Sigma U$ and $P^\top UP$ have the same eigenvalues. Thus $G$ is also invertible since its eigenvalues are given by $1 - 2\lambda_i > 0$. Further, it holds

$$
\det(G) = \det(K) = \prod_{i=1}^d (1 - 2\lambda_i). \quad (C.2)
$$

Rearranging terms in the integral of Equation (C.1) yields

$$
z^\top P^\top UPz + a^\top Pz - \frac{1}{2} z^\top z = -\frac{1}{2} (z^\top Gz - 2z^\top P^\top a) = \frac{1}{2} a^\top PG^{-1}P^\top a - \frac{1}{2} (z - b)^\top G(z - b),
$$

with $b = G^{-1}P^\top a$. Using that $h(z) = (2\pi)^{-d/2}(\det(G^{-1}))^{-1/2} \exp\left\{-\frac{1}{2} (z - b)^\top G(z - b)\right\}$ is the probability density function of the non-degenerate Gaussian distribution $N(b, G^{-1})$ with the normalization $\int_{\mathbb{R}} h(z)dz = 1$, it follows from (C.1) and (C.2)

$$
\mathbb{E}[e^Y] = (\det(G))^{-1/2} \exp\left\{\frac{1}{2} a^\top PG^{-1}P^\top a\right\} = (\det(K))^{-1/2} \exp\left\{\frac{1}{2} a^\top K^{-1} \Sigma a\right\}.
$$

For the last equality we have applied the fact that $PG^{-1}P^\top = K^{-1}\Sigma$. The proof of this equality is based on the fact that for $d \times d$-matrices $A$ and $B$ which are such that $C = I_d - AB$ is invertible, it holds

$$(I_d - AB)^{-1} = I_d + A(I_d - BA)^{-1}B.$$

This can easily seen by verifying the defining property of an inverse matrix, i.e., $CC^{-1} = C^{-1}C = I_d$. Setting $A = P^\top U$ and $B = P$ we obtain the identity $PG^{-1}P^\top = K^{-1}\Sigma$ from

$$
G^{-1} = (I_d - 2P^\top UP)^{-1} = I_d + 2P^\top U(I_d + 2PP^\top U)^{-1}P = I_d + 2P^\top UK^{-1}P
$$

and finally

$$
PG^{-1}P^\top = P(I_d + 2P^\top UK^{-1}P)P^\top = PP^\top + 2PP^\top UK^{-1}PP^\top = \Sigma + 2\Sigma UK^{-1}\Sigma = (I_d + 2\Sigma UK^{-1})\Sigma = (K + 2\Sigma U)K^{-1}\Sigma = I_dK^{-1}\Sigma. \quad (C.3)
$$
Third claim. The first identity in (3.13) was already proven in (C.2). For the second identity we use $D^\top UP = DAD^\top$ and that the matrix $D$ is orthogonal, i.e., $DD^\top = D^\top D = I_d$. Then, according to (C.3) it holds

$$K^{-1}\Sigma = PG^{-1}P^\top = P(I_d - 2P^\top UP)^{-1}P^\top = P(DD^\top - 2DD^\top P^\top UPDD^\top)^{-1}P^\top = P(D(I_d - 2\Lambda)D^\top)^{-1}P^\top = PD(I_d - 2\Lambda)^{-1}D^\top P^\top.$$

Using $c = D^\top P^\top a$ we obtain

$$a^\top K^{-1}\Sigma a = a^\top PD(I_d - 2\Lambda)^{-1}D^\top P^\top a = c^\top (I_d - 2\Lambda)^{-1}c = \sum_{j=1}^d c_j^2 (1 - 2\lambda_j)^{-1}. \quad \Box$$

D Proof of Theorem 3.7

Proof. We recall inequality (3.10) stating $V^H_\rho (t,y) \leq \frac{\sigma^2}{\bar{\rho}} \mathbb{E} [d^{1-\theta} (t,\mu_t) | M^H_t = m, Q^H_t = q]$, for $H = R, Z, J$, and $y = (x,m,q)$. For the $H$-investors the conditional distribution of $\mu_t$ given $M^H_t = m, Q^H_t = q$ is the Gaussian distribution $\mathcal{N}(m, q)$. Thus we can deduce for the conditional expectation

$$\mathbb{E}[d^{1-\theta} (t,\mu_t) | M^H_t = m, Q^H_t = q] = \mathbb{E}[d^{1-\theta} (t,m + q^{1/2}\epsilon)]$$

with a random variable $\epsilon \sim \mathcal{N}(0,I_d)$ independent of $\mathcal{F}^H_t$. Substituting into (3.10) and using representation (3.4) we deduce

$$V^H_\rho (t,y) \leq \mathbb{E} [d^{1-\theta} (t,\mu_t) | M^H_t = m, Q^H_t = q] = \mathbb{E} \left[ \exp \left\{ (1-\theta) \left( (m + q^{1/2}\epsilon)^\top A\gamma(t) (m + q^{1/2}\epsilon) + B\gamma(t)(m + q^{1/2}\epsilon) + C\gamma(t) \right) \right\} \right].$$

To simplify the notation we write in the following $A, B, C$ instead of $(1-\theta)A\gamma(t)$, $(1-\theta)B\gamma(t)$, $(1-\theta)C\gamma(t)$, respectively. Rearranging terms yields

$$V^H_\rho (t,y) \leq \mathbb{E} \left[ \exp \left\{ m^\top Am + B^\top m + C \right\} \exp \left\{ (q^{1/2}\epsilon)^\top Aq^{1/2}\epsilon + (2m^\top A + B^\top)q^{1/2}\epsilon \right\} \right] = d^{1-\theta} (t,m) \mathbb{E} \left[ \exp \left\{ (q^{1/2}\epsilon)^\top Aq^{1/2}\epsilon + (2Am + B)q^{1/2}\epsilon \right\} \right] = d^{1-\theta} (t,m) \mathbb{E} \left[ \exp \left\{ Y^\top Ay + a^\top Y \right\} \right], \quad (D.1)$$

where $Y = q^{1/2}\epsilon \sim \mathcal{N}(0_d, \Sigma)$ is a zero-mean Gaussian random vector with covariance matrix $\Sigma = q$ and $a = 2Am + B = (1-\theta)(2A\gamma(t)m + B\gamma(t))$. Applying Lemma 3.5 with $U = A = (1-\theta)A\gamma(t)$ and $K = I_d - 2qA = I_d - 2(1-\theta)qA\gamma(t)$ yields

$$\mathbb{E} \left[ \exp \left\{ Y^\top Ay + a^\top Y \right\} \right] = (\det(K))^{-1/2} \exp \left\{ a^\top K^{-1}qa \right\},$$

and substituting this expression into (D.1) proves the claim. \quad \Box

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