Proof systems: from nestings to sequents and back

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Abstract. In this work, we explore proof theoretical connections between sequent, nested and labelled calculi. In particular, we show a general algorithm for transforming a class of nested systems into sequent calculus systems, passing through linear nested systems. Moreover, we show a semantical characterisation of intuitionistic, multi-modal and non-normal modal logics for all these systems, via a case-by-case translation between labelled nested to labelled sequent systems.

1 Introduction

The quest of finding good proof systems for different logics has been the main research topic for proof theorists since Gentzen’s seminal work [10]. The definition of good is, of course, subjective. While it is widely accepted that a proposed calculus has to be sound and complete w.r.t. a given semantics, other aspects such as analyticity, simplicity, and efficiency are often taken into account for considering a calculus “adequate”.

One of the best known formalisms for proposing analytic proof systems is Gentzen’s sequent calculus. While its simplicity makes it an ideal tool for proving meta-logical properties, sequent calculus is not expressive enough for constructing analytic calculi for many logics of interest. The case of modal logic is particularly problematic, since sequent systems for such logics are usually not modular, and they mostly lack relevant properties such as separate left and right introduction rules for the modalities. These problems are often connected to the fact that the modal rules in such calculi usually introduce more than one connective at a time, e.g. as in the rule k for modal logic K:

\[
\frac{B_1, \ldots, B_n \vdash A}{\Box B_1, \ldots, \Box B_n \vdash \Box A} \quad k
\]

One way of solving this problem is by considering extensions of the sequent framework that are expressive enough for capturing these modalities using separate left and right introduction rules. This is possible e.g. in labelled sequents [22] or in nested sequents [2]. In the labelled sequent framework, usually the semantical characterisation is explicitly added to sequents. In the nested framework in contrast, a single sequent is replaced with a tree of sequents, where successors of a sequent (nestings) are interpreted under a given modality. The nesting rules of these calculi govern the transfer of formulae between the different sequents, and they are local, in the sense that it is sufficient to transfer only one formula at a time. As an example, the labelled and nested versions for the necessity right rule (\(\Box R\)) are

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where $y$ is a fresh variable in the $\square_R$ rule. Reading bottom up, while the labelled system creates a new variable $y$ related to $x$ via a relation $R$ and changes the label of $A$ to $y$, in $\square_R$ a new nesting is created, and $A$ is moved there. It seems clear that nestings and semantical structures are somehow related. Indeed, a direct translation between proofs in labelled and nested systems for some normal modal logics is presented in [11], while in [7] it is shown how to relate nestings with Kripke structures for intuitionistic logic. In this work, we show how to smoothly generalise this relationship to multi-modalities, where intuitionistic logic and normal modal logics are particular cases.

Since nested systems have being also proposed for other modalities, such as the non-normal ones [4], an interesting question is whether this semantical interpretation can be generalised to other systems as well. In [20] a labelled approach was used for setting the grounds for proof theory of some non-normal modal systems based on neighbourhood semantics. In parallel, we have proposed [16] modular systems based on nestings for several non-normal modal logics. We will relate these two approaches for the logics $M$ and $E$, hence clarifying the nesting-semantics relationship for such logics.

While nested sequents allow for modular proposal of proof systems, it comes with a price: the obvious proof search procedure is of suboptimal complexity since it constructs potentially exponentially large nested sequents [2]. In this work, we show that a class of nested systems can be transformed into sequent systems via a linearisation procedure, where sequent rules can be seen as nested macro-rules. In this way, we do not only recover simplicity but also efficiency by defining an optimal proof search procedure, in the sense that it matches the computational complexity of the validity problem in the logic.

Finally, by relating nested and sequent systems, we are able to extend the semantical interpretation also to the sequent case, hence closing the relationship between systems and shedding light on the semantical interpretation of several sequent based systems.

**Organisation and contributions.** Sec. 2 presents the basic notation for sequent systems; Sec. 3 shows sufficient conditions for a nested system to be linearised, so that to be presented as linear a nested system; Sec 4 shows how to transform linear nested systems into sequent systems; Sec 5 presents the basic notation for labelled systems; Sec. 6, 7 and 8 show the results under the particular views of intuitionistic, multi-modal and non-normal logics; Sec. 9 concludes the paper.

## 2 Sequent systems

Contemporary proof theory started with Gentzen’s work [10], and it has had a continuous development with the proposal of several proof systems for many logics.

**Definition 1.** A sequent is an expression of the form $\Gamma \vdash \Delta$ where $\Gamma$ (the antecedent) and $\Delta$ (the succedent) are finite multisets of formulae. A sequent calculus (SC) consists of a set of rules, of the form

$$
\begin{array}{c}
S_1 \quad \cdots \quad S_n \\
\hline
S \quad r
\end{array}
$$
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\[ \Gamma, A \rightarrow B, \Delta \vdash A, \Gamma, B \rightarrow \Delta \vdash \Delta \rightarrow L \]
\[ \Gamma, A \rightarrow B, \Delta \vdash A \rightarrow B, \Delta \vdash \Gamma, A \rightarrow B, \Delta \rightarrow \Delta \rightarrow R \]
\[ \Gamma, A, B \vdash \Delta \vdash \Gamma, A, \Gamma, B, \Delta \rightarrow \Delta \rightarrow R \]
\[ \Gamma, A, B \rightarrow \Delta \vdash A \rightarrow B, \Delta \vdash \Gamma, A \rightarrow B, \Delta \vdash \Delta \rightarrow L \]
\[ \Gamma, A \rightarrow B, \Delta \vdash A \rightarrow B, \Delta \vdash \Gamma, A \rightarrow B, \Delta \rightarrow \Delta \rightarrow R \]
\[ \Gamma, \bot \vdash \Delta \vdash \Gamma, \bot \vdash \Delta \rightarrow \Delta \rightarrow R \]
\[ \Gamma, P \vdash \Delta \vdash \Gamma, P \vdash \Delta \rightarrow \Delta \rightarrow R \]

Fig. 1: Multi-conclusion intuitionistic calculus SC\text{mLJ}. P is atomic.

where the sequent \( S \) is the conclusion inferred from the premise sequents \( S_1, \ldots, S_n \) in the rule \( r \). If the set of premises is empty, then \( r \) is an axiom. where the formula in the conclusion sequent in which a rule is applied is the principal formula, and its sub-formulae in the premises are the auxiliary formulae.

A derivation is a finite directed tree with nodes labelled by sequents and a single root, axioms at the top nodes, and where each node is connected with the (immediate) successor nodes (if any) according to the inference rules. The height of a derivation is the greatest number of successive applications of rules in it, where an axiom has height 0.

As an example, Fig. 1 presents SC\text{mLJ} \([17]\), a multiple conclusion sequent system for propositional intuitionistic logic. The rules are exactly the same as in classical logic, except for the implication rules. While the left rule copies the implication in the left premise, the right implication forces all formulae in the succedent of the conclusion sequent to be weakened (when viewed bottom-up). This guarantees that, on applying the \((\rightarrow R)\) rule on \( A \rightarrow B \), the formula \( B \) should be proved assuming only the pre-existent antecedent context extended with the formula \( A \), creating an interdependency between \( A \) and \( B \).

3 Nested systems

Nested systems \([2,21]\) are extensions of the sequent framework where a single sequent is replaced with a tree of sequents.

Definition 2. A nested sequent is defined inductively as follows:

(i) if \( \Gamma \vdash A \) is a sequent, then it is a nested sequent;
(ii) if \( \Gamma \vdash A \) is a sequent and \( G_1, \ldots, G_n \) are nested sequents, then \( \Gamma \vdash A, [G_1], \ldots, [G_n] \) is a nested sequent.

A nested system (NS) consists of a set of inference rules acting on nested sequents. An auxiliary structure is either an auxiliary formula or a nesting created by the principal formula of a nested rule.

For readability, we will denote by \( \Gamma, \Delta \) sequent contexts and by \( \Lambda \) multisets of nestings. In this way, every nested sequent has the shape \( \Gamma \vdash A, \Lambda \) where elements of \( \Lambda \) have the shape \([\Gamma', \Delta', \Lambda']\) and so on. We will denote by \( T \) an arbitrary nested sequent.

Nodes in a nesting tree are called positions \([12]\). This terminology is transferred to formulae, so that we also refer to the position of a formula in a nesting. We will
We will denote by focusing based on (also a proof strategy) that allows for reconciling the added extra superior expressiveness and modularity of nested sequents over ordinary sequents with the computational behaviour of the standard sequent framework.

Normal forms to proofs, where application of rules can be re-organised so to follow a determinate shape.

Fig. 2: Nested system NS\textsubscript{mLJ}.

**Definition 3.** Let \( A, B \) be formulae occurring in nested sequents \( \Gamma, \Gamma' \) respectively. Then \( A \preceq B \) iff \( \mathbf{id}_\Gamma(A) \leq \mathbf{id}_\Gamma(B) \).

Rules in nested systems will be represented using holed contexts.

**Definition 4.** A nested-holed context is a nested sequent that contains a hole of the form \( \{ \} \) in place of nestings. We represent such a context as \( \Gamma(\{ \} \). Given a holed context and a nested sequent \( \Gamma' \), we write \( \Gamma(\{ \}) \) to stand for the nested sequent where the hole \( \{ \} \) has been replaced by \( \{ \} \), assuming that the hole is removed if \( \Gamma' \) is empty and if \( \Gamma \) is empty then \( \Gamma(\{ \}) = \Gamma' \). The depth of \( \Gamma(\{ \} \), denoted by \( \mathbf{dp}(\Gamma(\{ \}) \), is the number of nodes on a branch of the nesting tree of \( \Gamma(\{ \} \) of maximal length.

For example, \( \{ \Gamma' \vdash A, \{ \} \} \) \( \{ \Gamma' \vdash A' \} \) \( \{ \Gamma' \vdash A' \} \) while \( \{ \{ \} \} \) \( \{ \Gamma' \vdash A' \} \) \( \{ \Gamma' \vdash A' \} \). Fig. 2 presents the NS\textsubscript{mLJ} [7], a nested system for mLJ.

**Normal forms in NS** While adding a tree structure to sequents enhances the expressiveness of the nesting framework when compared with the sequent one, the price to pay is that the obvious proof search procedure is of suboptimal complexity, since it constructs potentially exponentially large nested sequents [2]. Hence the quest for proposing proof search strategies for taming the proof search space (see e.g. [3] for proof strategies based on focusing). In what follows, we will propose a normalisation procedure (hence also a proof strategy) that allows for reconciling the added extra superior expressiveness and modularity of nested sequents over ordinary sequents with the computational behaviour of the standard sequent framework.

Permutability of rules is often used in sequent systems in order to establish a notion of normal forms to proofs, where application of rules can be re-organised so to follow a determinate shape.
Definition 5. Let $S$ be a nested sequent in a NS. We say that a rule $r_2$ permutes down $r_1$ ($r_2 \downarrow r_1$) if, for every derivation in which $r_1$ operates on $S$ and $r_2$ operates on one or more of $r_1$’s premises (but not on auxiliary structures of $r_1$), there exists another derivation of $S$ in which $r_2$ operates on $S$ and $r_1$ operates on zero or more of $r_2$’s premises (but not on auxiliary structures of $r_2$). If $r_2 \downarrow r_1$ and $r_1 \downarrow r_2$ we will say that $r_1, r_2$ are permutable. If all pair of rules in NS permute we say that NS is fully permutable.

In the case of nested systems, permutability alone is often not enough for proposing effective proof strategies, since rules can be applied deep inside a nesting and also among nestings. In what follows, we will restrict the rules so to allow only exchange of formulae in inner nestings.

Definition 6. Let $r$ be a rule in NS with principal formula $A$ and auxiliary formulae $A_1, \ldots, A_k$. We say that $r$ is inter-nested if $A \prec A_i$ or $A \parallel A_i$ for some $i \in \{1, \ldots, k\}$. NS is n-directed if, for every inter-nested rule $r$, $A \prec A_i$, $\forall i \in \{1, \ldots, k\}$. NS is shallow n-directed if, for each $i \in \{1, \ldots, k\}$, there exists a $j_i \in \mathbb{N}$ such that $\text{id}_\gamma(A_i) = \text{id}_\gamma(A)_{j_i}$.

Observe that the nested rules for the modalities $5$ and $b$ are not n-directed (see [3]). It is interesting to note that there is no cut-free sequent calculus for $S5$ either.

Lemma 7. Let NS be a n-directed nested system and let $\Lambda_i$ be nestings in NS. If $\vdash \Lambda_1, \ldots, \Lambda_n$ is derivable then $\vdash \Lambda_i$ is derivable for some $i \in \{1, \ldots, n\}$.

Proof. Let $\pi$ be a proof of $\vdash \Lambda_1, \Lambda_2, \ldots, \Lambda_n$. Since NS is n-directed, any instance of a rule $r$ acting in $\Lambda_i$ is such that the auxiliary formulae remain in $\Lambda_i$. This means that they cannot be principal in any rule $s$ acting in $\Lambda_j$, $i \neq j$. Hence $\pi$ can be re-ordered so that there is no interleaving of rules applied to different nestings.

We observe that the theorem is valid in the other direction if weakening on nestings is available (e.g. with a weakening absorbing initial axiom).

That is, in n-directed systems the comma on the right context is intuitionistic. Moreover, if the n-directness is shallow, a depth first normalization procedure can be defined, which induces a depth first proof strategy.

Theorem 8. Let NS be a fully permutable, shallow n-directed nested system. Then any proof $\pi$ of a nested sequent $\Gamma$ can be re-organised so that to satisfy the following normalisation procedure:

- local phase: apply, in any order, rules that do not move formulae between nestings;
- nesting phase: apply, in any order, all possible rules creating nestings;
- lift phase: apply inter-nested rules;
- deep phase: start the process again deep inside a sub-nesting;
- Axioms are applied eagerly.

Proof. Let $\pi$ be a proof of $\Gamma \vdash A, \Lambda$ in NS. Since NS is fully permutable, $\pi$ can be re-organised so to apply first all local rules acting in $\Gamma, A$ then to create nestings via rules in the nesting phase. At this point, leaves in the (open) derivation will have the form $\Gamma' \vdash A', \Lambda'$ with $\Lambda \subseteq A'$. Since NS is shallow n-directed, rules in the nesting phase
cannot add formulae to the sequent contexts and rules in the lift phase have principal formulae in \( \Gamma', \Delta' \) and auxiliary formulae in \( \Lambda' \). Hence no more rules in \( \pi \) are applied over formulae in the sequent contexts and the proof must continue in the nestings with no interleaving of rules applied to different nestings (Lemma\[7\]).

We will show in Sec. [6,7,8] representative examples of systems falling into the class of fully permutable, shallow n-directed nested systems. A quick remark is that fully permutability can be substituted by a weaker condition: local rules permute down the nesting ones.

4 Linear nested systems

In [14,16] we proposed the concept of end-active, blocked form linear nested sequents [13]. In this section, we will show how such systems can both: be generated from shallow n-directed nested systems; and recover sequent systems.

**Definition 9.** The set \( LNS \) of linear nested sequents is given recursively by:

1. if \( \Gamma \vdash \Delta \) is a sequent then \( \Gamma \vdash \Delta \in LNS \)
2. if \( \Gamma \vdash \Delta \) is a sequent and \( G \in LNS \) then \( \Gamma \vdash \Delta/G \in LNS \).

A linear nested system (LNS) consists of a set of inference rules acting on linear nested sequents. We call each sequent in a linear nested sequent a component.

An application of a linear nested sequent rule is end-active if the rightmost components of the premises are active and the only active components (in premise and conclusion) are the two rightmost ones.

In other words, a linear nested sequent is a finite list of sequents. Since this data structure matches a path in a nested tree, Lemma [7] immediately entails that proofs in n-directed nested systems can be linearised. This linearisation process can also be extended to nested rules. We will illustrate this formalisation for the shallow case.

Let NS be shallow n-directed. Observe that the depth first procedure assures that nested rules can be restricted so to be applied in the two deepest sub-nestings of a nesting. This implies that the nested sequents in conclusion and premises in any rule have the form \( T[\Gamma \vdash \Delta, \{T_1\}, \ldots, \{T_n\}] \) with \( dp(\{T_n\}) \leq 1 \) (see Def. [4]). After the local phase, rules do not alter the sequent contexts \( \Gamma, \Delta \). Thus, after the lift phase, a sequent \( \Gamma \vdash \Delta, \Lambda_1, \ldots, \Lambda_n \) will be provable iff \( \Lambda_i \) is provable, for some \( i \in \{1, \ldots, n\} \). Hence the nesting phase can be restricted to the creation of a single nesting and nested contexts of the form \( T[\Gamma \vdash \Delta, \{T'\}] \) present on inference rules can be written as the linear nested sequent \( G/G \vdash \Delta/G \vdash \Delta' \), where \( G \) carries the position of the sequent \( \Gamma \vdash \Delta' \) in \( T' \).

Fig. [5] presents the end-active system \( LNS_{mLJ} \), which is the linearisation of the system \( NS_{mLJ} \).

**Recovering sequent systems** Observe that end-activeness in linear nested systems alone is not enough for faithfully translating the depth first normalisation strategy since the local phase can be interleaved with the lift and deep phases. For ensuring such
5 Labelled proof systems

While it is widely accepted that nested systems carry the Kripke structure on nestings for intuitionistic and normal modal logics, it is not clear what is the relationship between nestings and semantics for other systems. For example, in [14] we presented a LNS for linear logic (LL), but the interpretation of nestings for this case is still an open problem.

In this work we will relate labelled nested systems [11] with labelled systems [22]. While the results for intuitionistic and modal logics are not new [7,19], we present the first semantical interpretation for nestings in non-normal modal logics. In this section we shall recall some of the terminology for labelled systems.

Labelled nested systems Let SV a countable infinite set of state variables (denoted by x, y, z, ...), disjoint from the set of propositional variables. A labelled formula has the form x : A where x ∈ SV and A is a formula. If Γ = {A₁, ..., Aₙ} is a multiset of formulae, then x : Γ denotes the multiset {x : A₁, ..., x : Aₙ} of labelled formulae. A (possibly empty) set of relation terms (i.e. terms of the form xRy, where x, y ∈ SV) is called a relation set. For a relation set R, the frame Fr(R) defined by R is given by
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\((\mathcal{R}, \mathcal{R})\) where \(|\mathcal{R}| = \{x | xRy \in \mathcal{R} \text{ or } yRx \in \mathcal{R} \text{ for some } y \in \mathcal{SV}\}\). We say that a relation set \(\mathcal{R}\) is \textit{treelike} if the frame defined by \(\mathcal{R}\) is a tree or \(\mathcal{R}\) is empty. A treelike relation set \(\mathcal{R}\) is called \textit{linelike} if each node in \(\mathcal{R}\) has at most one child.

\textbf{Definition 10.} A labelled nested sequent \(\text{LbNS}\) is a labelled sequent \(\mathcal{R}, X \vdash Y\) where

1. \(\mathcal{R}\) is treelike;
2. if \(\mathcal{R} = \emptyset\) then \(X\) has the form \(x:A_1, \ldots, x:A_n\) and \(Y\) has the form \(x:B_1, \ldots, x:B_m\) for some \(x \in \mathcal{SV}\);
3. if \(\mathcal{R} \neq \emptyset\) then every state variable \(y\) that occurs in either \(X\) or \(Y\) also occurs in \(\mathcal{R}\).

A labelled nested sequent calculus is a labelled calculus whose initial sequents and inference rules are constructed from \(\text{LbNS}\).

As in \cite{[11]} labelled nested systems can be automatically generated from nested systems.

\textbf{Definition 11.} Given \(\Gamma \vdash \Delta\) and \(\Gamma' \vdash \Delta'\) sequents, we define \((\Gamma \vdash \Delta) \otimes (\Gamma' \vdash \Delta')\) to be \(\Gamma, \Gamma' \vdash \Delta, \Delta'\). For a state variable \(x\), define the mapping \(T\) from \(\text{NS}\) to \(\text{LbLNS}\) as follows

\[
T(x: \Gamma \vdash \Delta) = xR_{x_1, \ldots, x_R_{x_n}}, (x: \Gamma \vdash x: \Delta) \otimes T(x_{1}, \ldots, x_{n})
\]

with all state variables pairwise distinct.

For the sake of readability, when the state variable is not important, we will suppress the subscript, writing \(T\) instead of \(T_x\). We will shortly illustrate the procedure of constructing labelled nestings using the mapping \(T\). Consider the following application of the rule \(\rightarrow_R\) of Fig. \[2\]

\[
\begin{align*}
\frac{\Gamma \vdash A, A, [A + B]}{\Gamma \vdash A \rightarrow B, A} & \rightarrow_R^n \\
\end{align*}
\]

Applying \(T\) to the conclusion we obtain \(\mathcal{R}, X \vdash Y, X : A \rightarrow B\) where the variable \(x\) label formulae in two components of the \(\text{NS}\), and \(X, Y\) are multisets of labelled formulae. Applying \(T\) to the premise we obtain \(\mathcal{R}, xRy, x : A \vdash Y, y : B\) where \(y\) is a fresh variable (i.e. different from \(x\) and not occurring in \(X, Y\)). We thus obtain an application of the \(\text{LbLNS}\) rule

\[
\begin{align*}
\frac{\mathcal{R}, xRy, x : A \vdash Y, y : B}{\mathcal{R}, xRy, x : A \rightarrow B} & \rightarrow_R^n
\end{align*}
\]

Some rules of the labelled nested system \(\text{LbNS}_{\text{mlj}}\) are depicted in Fig. \[5\]

The following result follows readily by transforming derivations bottom-up \cite{[11]}.

\textbf{Theorem 12.} The mapping \(T\) preserves open derivations, that is, there is a 1-1 correspondence between derivations in a nested sequent system \(\text{NS}\) and in its labelled translation \(\text{LbNS}\).
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6 Intuitionistic logic

In this section we will give a tour on various proof systems for intuitionistic logic, relating them by applying the results presented in the last sections.

**Theorem 13.** Weakening is height-preserving admissible in NS_{mlJ}. Moreover, all introduction rules in NS_{mlJ} are invertible where both the height of the derivation and the minimal level of the active components of rule applications are preserved. Finally, NS_{mlJ} is fully permutable.

**Proof.** The proofs of weakening-admissibility and invertibility are by induction on the depth of the derivation, distinguishing cases according to the last applied rule. Permutability of rules is proven by a case-by-case analysis and it uses the invertibility and weakening results. □
Remark 14. Observe that permutability also holds for the \textit{lift} rule. In fact the case
\[
\frac{A, \Gamma \vdash A, \Lambda, [A, B \vdash C]}{A, \Gamma \vdash A, \Lambda, [B \vdash C]} \quad \text{lif}\]
\[
\frac{A, \Gamma \vdash A, \Lambda, B \rightarrow C}{\rightarrow_R}
\]
is not considered for permutation since \([B \vdash C]\) is an auxiliary structure of the principal formula \(B \rightarrow C\) (see Def. 5).

Since \(\text{NS}_{\text{mLJ}}\) is shallow n-directed, the depth first strategy holds with the following classification of rules: local phase: conjunction, disjunction and implication left rules; nesting phase: implication right rule; lift phase: lift rule.

The results in the previous sections entail the following.

Theorem 15. Systems \(\text{NS}_{\text{mLJ}}, \text{LNS}_{\text{mLJ}}, \text{mLJ} \) and \(\text{LbNS}_{\text{mLJ}}\) are equivalent.

Observe that the proof uses syntactical arguments only, differently from e.g. [7,13].

For establishing a comparison between labels in \(\text{G3I}\) and \(\text{LbNS}_{\text{mLJ}}\), first observe that applications of rule Trans in \(\text{G3I}\) can be restricted to the leaves (i.e. just before an instance of the initial axiom). Also, since weakening is admissible in \(\text{G3I}\) and the monotonicity property: \(x \not\vdash A \land x \leq y \vdash y \not\vdash A\) is derivable in \(\text{G3I}\) (Lemma 4.1 in [6]), the next result follows.

Lemma 16. The following rules are derivable in \(\text{G3I}\) up to weakening.
\[
\frac{\mathcal{R}_x, x: A \rightarrow B + x: A, Y \quad \mathcal{R}_x, x: B + Y \quad \rightarrow_L}{\mathcal{R}_x, x: A \rightarrow B + Y} \quad \text{init'}
\]
Moreover, the rule
\[
\frac{\mathcal{R}_x \leq y, x: A + Y \quad \mathcal{R}_x \leq y, x: A + Y \quad \text{lift'}}{\mathcal{R}_x \leq y, x: A + Y}
\]
is admissible in \(\text{G3I}\).

Proof. For the derivable rules, just note that
\[
\frac{\mathcal{R}_x, x \leq x, x: A \rightarrow B + y, x: A \quad \mathcal{R}_x, x \leq x, x: B + y \quad \rightarrow_L}{\mathcal{R}_x, x: A \rightarrow B + y} \quad \text{Ref}
\]
\[
\frac{\mathcal{R}_x \leq x, x: P + y, x: P \quad \text{init'}\quad \mathcal{R}_x \leq x, x: P + y, x: P \quad \text{Ref}}{\mathcal{R}_x \leq x, x: P + y, x: P}
\]

Using an argument similar to the one in [11], it is easy to see that, in the presence of the primed rules shown above, the relational rules are admissible. Moreover, labels are preserved.

Theorem 17. \(\text{G3I}\) is label-preserving equivalent to \(\text{LbNS}_{\text{mLJ}}\). That is, nestings in \(\text{NS}_{\text{mLJ}}\) and \(\text{LNS}_{\text{mLJ}}\) correspond to worlds in the Kripke structure where the sequent is valid and this is the semantical interpretation of the linear nested system for intuitionistic logic [7].
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\[ K (A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B) \]
\[ \frac{A \text{ nec}}{\Box A} \quad D \vdash \bot \quad T \Box A \rightarrow A \quad 4 \Box A \rightarrow \Box \Box A \]

**Fig. 7:** Modal logic \( K \) contains the propositional tautologies, modus ponens, \( K \) and \( \text{nec} \).

**Wrapping up** Observe that, in labelled line systems, the relation \( R \) relates two components in a sequence, hence \( \leq \) is the transitive closure of \( R \) in any nested path. Since derivability is the same in \( \text{LNS}_{\text{mLJ}} \) and \( \text{LbLNS}_{\text{mLJ}} \) (Thm. 12), this means that a proof in \( \text{LNS}_{\text{mLJ}} \) corresponds to a successful trace in a deep-first proof strategy in \( G3I \), which, by its turn, corresponds to a path in the Kripke semantics of intuitionistic logic.

Moreover, more than internalising the semantics, end-active linear nested systems actually show that we may consider only the upper most words in the Kripke semantics.

Finally, since \( \text{mLJ} \) derivations are equivalent to blocked end-active \( \text{LNS}_{\text{mLJ}} \) derivations, the semantical analysis for \( \text{LNS}_{\text{mLJ}} \) also hold for \( \text{mLJ} \).

In what follows, we will show how this tour on different proof systems can be smoothly extended to (multi-) normal modalities and to non-normal modalities, using propositional classical logic as the base logic.

7 Multi-modal logics

In [15] we presented sequent calculi and linear nested systems for multimodal logics with a simple interaction between modalities, called simply dependent multimodal logics [5]. The language for these logics contains indexed modalities \( \Box_i \) for indices \( i \) from an index set \( N \subseteq \mathbb{N} \) of natural numbers. The axioms are given by extensions of the axioms of modal logic \( K \) for every modality \( \Box_i \) together with interaction axioms of the form \( \Box_i A \rightarrow \Box_j A \).

In this paper we will consider a subset of these logics such that, for each index, the underline logic is an extension of \( K \) with axioms from the set \{D, T, 4\} (see Fig. 7).

**Definition 18.** A simply dependent multimodal logical system is given by a triple \((N, \preceq, F)\), where \( N \) is a finite set of natural numbers, \((N, \preceq)\) is a partial order and \( F \) is a mapping from \( N \) to \( 2^{\{D, T, 4\}} \). Moreover, \((N, \preceq, F)\) is transitive-closed, that is, for every \( i, j \in N, j \preceq i \) if \( K4 \subseteq F(j) \) then \( K4 \subseteq F(i) \).

The logic described by \((N, \preceq, F)\) has modalities \( \Box_i \) for every \( i \in N \) with the axioms and rules of classical propositional logic together with rules and axioms for the modality \( i \) given by the necessitation rule and the \( K \) axiom for \( \Box_i \) as well as the axioms \( F(i) \), and interaction axioms \( \Box_i A \rightarrow \Box_j A \) for every \( i, j \in I \) with \( j \preceq i \), understood as zero-premise rules.

**Remark 19.** Observe that \( K \) and its \{D, T, 4\}-extensions are trivial cases of simply dependent multimodal logics where the index set \( N \) is a singleton. This means that all the results stated in what follows hold for, e.g., \( S4 \) and \( KD \). Also note that the modal axioms \( D \) and \( T \) propagate upwards, so there is no need for reflexive or serial closure.

The following definition extends the concept of frames to simply dependent multimodal logic. The notions of valuations, model and truth in a world of the model are defined as usual (see [11]).
A transitivity axiom 4 classification of rules: local phase the set of indices admissible; all introduction rules in are height-preserving invertible; any pair of rules NS is permutable.

Definition 20. Let $(N, \prec, F)$ be a description for a simply dependent multimodal logic $A (N, \prec, F)$-frame is a tuple $(W, (R_i)_{i \in N})$ consisting of a set $W$ of worlds and an accessibility relation $R_i$ for every index $i \in N$, such that for all $i, j \in N$:

- If the logic $F(i)$ contains KD, then $R_i$ is serial.
- If the logic $F(i)$ contains KT, then $R_i$ is reflexive.
- If the logic $F(i)$ contains K4, then $R_i$ is transitive.
- If $j \preceq i$, then $R_j \subseteq R_i$.

Since here we only consider simply dependent multimodal logics where the different component logics are extensions of K with axioms from $\{D, T, 4\}$, and since the interaction axioms are of a particularly simple shape, standard results e.g. from Sahlqvist theory immediately yield soundness and completeness of the description $(N, \prec, F)$ w.r.t. the logic of the class of $(N, \prec, F)$-frames.

7.1 Indexed nested systems

Nestings can be extended to multi-modalities by adding indexes. An indexed nested sequent [8] is a nested sequent where each sequent node is marked with an index taken from $N$, and it is denoted by $\Gamma \vdash [\subseteq A, [\subseteq A]^
\Delta, \Pi, \Lambda]$. We will denote by $\Upsilon(j)$ the upset of the index $j$, i.e., the set $\{i \in N : j \prec i\}$ and extend this notation to the sets $\uparrow^Ax(j) := \{i \in N : j \prec i, Ax \not\subseteq F(i)\}$ and $\downarrow^Ax(j) := \{i \in N : j \preceq i, Ax \not\subseteq F(i)\}$ where $Ax$ is any of the axioms $D, T, 4$. Thus e.g. the set $\uparrow^Ax(j)$ is the set of indices $i$ with $j \prec i$ such that $K4 \not\subseteq F(i)$, i.e., the logic $F(i)$ does not derive the transitivity axiom 4.

$\mathcal{N}_{(N, \prec, F)}$ (see Fig. 8) is the nested sequent system for the $(N, \prec, F)$ description for a simply dependent multimodal logic. Next result follows the same lines as in Sec. 3.

Theorem 21. $\mathcal{N}_{(N, \prec, F)}$ has the following properties: weakening is height-preserving admissible; all introduction rules in are height-preserving invertible; any pair of rules is permutable.

$\mathcal{N}_{(N, \prec, F)}$ is shallow $n$-directed, hence the depth first strategy holds with the following classification of rules: local phase: propositional and $t_i$ rules; nesting phase: $d_i$ and $\square_R$ rules; lift phase: $\square_{ij_L}$ and $4i_j_L$ rules. This allows the definition of a linear version
Lemma 23. The rules $\mathsf{TL}(d_{ij})$, $\mathsf{TL}(f_{ij})$, $\mathsf{TL}(t_{ij})$, $\mathsf{TL}(4_{ij})$ are derivable in $\mathsf{G3MM}$. 

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of nested systems for the simply dependent multimodal logic given by the description $(N, \leq, F)$, system $\mathsf{LNS}_{(N, \leq, F)}$ presented in Fig. 9. Observe that rules $d_{ij}$ and $\sqcap_{ij}$ are not invertible in $\mathsf{LNS}_{(N, \leq, F)}$. Using blocked forms, the linear nested system emulates the rules of the sequent system $\mathsf{SC}_{(N, \leq, F)}$ in Fig. 10, as illustrated next for the rule $d_{ij}$:

Finally, Def. 11 of Sec. 5 can be extended to the multi-modal case in a trivial way, resulting in the labelled nested system $\mathsf{LbNS}_{(N, \leq, F)}$ (Fig. 11).

Theorem 22. Systems $\mathsf{NS}_{(N, \leq, F)}$, $\mathsf{LNS}_{(N, \leq, F)}$, $\mathsf{SC}_{(N, \leq, F)}$ and $\mathsf{LbNS}_{(N, \leq, F)}$ are equivalent.

By straightforwardly extending the geometric rule scheme presented in 19 for normal modalities to the multi-modal case, we can propose $\mathsf{G3MM}$, a sound and complete labelled sequent system for $(N, \leq, F)$. Figs. 12a and 12b present the modal and relational rules of $\mathsf{G3MM}$ (see also 9).

The next results follow the same lines as the ones in Sec 6.
We now move our attention to non-normal modal logics, i.e., modal logics that are not extensions of K. This means that labels in non-normal modal systems tend the results in [11] for the case of multi-modality. Although our approach is general enough for considering

\[
\frac{\mathcal{R}, xR_y, x : \square A, y : A \vdash Y}{\mathcal{R}, xR'_y, x : \square A \vdash Y} \quad \frac{\mathcal{R}, xR_y, x : \square A, y : A \vdash Y}{\mathcal{R}, xR'_y, x : \square A \vdash Y} \quad \frac{\mathcal{R}, xR_y, x, y : A \vdash Y}{\mathcal{R}, xR'_y, x, y : A \vdash Y} \quad \frac{\mathcal{R}, xR_y, x : \square A, x : A \vdash Y}{\mathcal{R}, xR'_y, x : \square A \vdash Y} \quad \frac{\mathcal{R}, xR_y, x, y : A \vdash Y}{\mathcal{R}, xR'_y, x, y : A \vdash Y} \quad \frac{\mathcal{R}, xR_y, x : \square A \vdash Y}{\mathcal{R}, xR'_y, x : \square A \vdash Y}
\]

Fig. 11: Modal rules for labelled indexed nested system \( \text{LbNS}_{(N, \leq, F)} \).

(a) Modal rules. (b) Multi-modal relational rules. \( y \) is fresh in \( \text{Ser} \).

Fig. 12: Some rules of the labelled system \( \text{G3MM} \).

**Proof.** Suppose \( i \leq j \). \( \mathcal{TL} (\ominus_i f) \) is derivable as

\[
\frac{\mathcal{R}, xR_y, x : \square A, y : A \vdash Y}{\mathcal{R}, xR'_y, x : \square A \vdash Y} \quad \frac{\mathcal{R}, xR_y, x : \square A, y : A \vdash Y}{\mathcal{R}, xR'_y, x : \square A \vdash Y} \quad \frac{\mathcal{R}, xR_y, x, y : A \vdash Y}{\mathcal{R}, xR'_y, x, y : A \vdash Y} \quad \frac{\mathcal{R}, xR_y, x : \square A, x : A \vdash Y}{\mathcal{R}, xR'_y, x : \square A \vdash Y} \quad \frac{\mathcal{R}, xR_y, x, y : A \vdash Y}{\mathcal{R}, xR'_y, x, y : A \vdash Y} \quad \frac{\mathcal{R}, xR_y, x : \square A \vdash Y}{\mathcal{R}, xR'_y, x : \square A \vdash Y}
\]

Derivability of \( \mathcal{TL} (d_j), \mathcal{TL} (t_i), \mathcal{TL} (4_i) \) are also straightforward. For example, if \( j \leq i \) and \( KD \subseteq F(i) \),

\[
\frac{\mathcal{R}, xR_y, x : \square A, y : A \vdash Y}{\mathcal{R}, xR'_y, x : \square A \vdash Y} \quad \frac{\mathcal{R}, xR_y, x : \square A, y : A \vdash Y}{\mathcal{R}, xR'_y, x : \square A \vdash Y} \quad \frac{\mathcal{R}, xR_y, x, y : A \vdash Y}{\mathcal{R}, xR'_y, x, y : A \vdash Y} \quad \frac{\mathcal{R}, xR_y, x : \square A, x : A \vdash Y}{\mathcal{R}, xR'_y, x : \square A \vdash Y} \quad \frac{\mathcal{R}, xR_y, x, y : A \vdash Y}{\mathcal{R}, xR'_y, x, y : A \vdash Y} \quad \frac{\mathcal{R}, xR_y, x : \square A \vdash Y}{\mathcal{R}, xR'_y, x : \square A \vdash Y}
\]

\( \square \)

**Theorem 24.** \( \text{G3MM} \) is label-preserving equivalent to \( \text{LbNS}_{(N, \leq, F)} \).

**Proof.** That every provable sequent in \( \text{LbNS}_{(N, \leq, F)} \) is provable in \( \text{G3MM} \) is a direct consequence of Lemma [23]. For the other direction, observe that rule relational rules can be restricted so to be applied just before a \( \square_i f \) rule.

This means that labels in \( \text{NS}_{(N, \leq, F)} \) represent worlds in a \( (N, \leq, F) \)-frame, and this extends the results in [11] for the case of multi-modality.

**8 Non-normal modal systems**

We now move our attention to non-normal modal logics, i.e., modal logics that are not extensions of K. In this work, we will consider the classical modal logic \( \mathcal{E} \) and the monotone modal logic \( \mathcal{M} \). Although our approach is general enough for considering
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\[ \Gamma, \Delta, \Lambda, [\Sigma, A + B] \vdash \Delta, \Lambda, [\Sigma, A + B] \]

Fig. 13: Modal rules for systems NS\(_E\) and NS\(_M\).

\[ \Gamma, \Delta, \Lambda, [\Sigma, A + B] \vdash \Delta, \Lambda, [\Sigma, A + B] \]

Fig. 14: Modal rules for systems LNS\(_E\) and LNS\(_M\).

nested, linear nested and sequent systems for several extensions of such logics (such as the classical cube or the modal tesseract – see (16)), there are no satisfactory labelled sequent calculi in the literature for such extensions.

For constructing nested calculi for these logics, the sequent rules should be decomposed into their different components. However, there are two complications compared to the case of normal modal logics: the need for (1) a mechanism for capturing the fact that exactly one boxed formula is introduced on the left hand side; and (2) a way of handling multiple premises of rules. The first problem is solved by introducing the indexed nesting [·] to capture a state where a sequent rule has been partly processed; the second problem is solved by making the nesting operator [·] binary, which permits the storage of more information about the premises. Fig. 13 presents a unified nested system for logics NS\(_E\) and NS\(_M\).

NS\(_E\) and NS\(_M\) are shallow n-directed, fully permutable, with invertible rules and, since propositional rules cannot be applied inside the indexed nestings, the modal rules naturally occur in blocks. Hence the nested rules can be restricted to the linear version (16) (Fig. 14), which in turn correspond to macro-rules equivalent to the sequent rules in Fig. 15 for SC\(_E\) and SC\(_M\).

Finally, using the labelling method in Section 5 the rules in Fig. 13 correspond to the rules in Fig. 16 where xNy and xNe(y1, y2) are relation terms.

The semantical interpretation of non-normal modalities \(E, M\) can be given via neighbourhood semantics, that smoothly extends the concept of Kripke frames in the sense that accessibility relations are substituted by a family of neighbourhoods.

Definition 25. A neighbourhood frame is a pair \(F = (W, N)\) consisting of a set \(W\) of worlds and a neighbourhood function \(N : W \rightarrow \mathcal{P}(\mathcal{P}(W))\). A neighbourhood model is a pair \(M = (F, \mathcal{V})\), where \(\mathcal{V}\) is a valuation. We will drop the model symbol when it is clear from the context.

The truth description for the box modality in the neighbourhood framework is

\[ w \vdash \square A \iff \exists X \in N(w).[(X \not\ni \forall A) \land (A \ni X)] \] (1)
An instance of the blocked derivation where $X$ weakened. The same with $Y$.

Observe that, in $\pi$, the label $y_2$ will no longer be active, hence the formula $y_2: A$ can be weakened. The same with $y_1$ in $\pi_2$. Hence $\pi_1/\pi_2$ corresponds to $\pi'_1/\pi'_2$ and the “only if” part holds. The “if” part uses similar proof theoretical arguments as in the intuitionistic or (multi) modal case, observing that applications of the forcing rules can be restricted so to be applied immediately after the modal rules.

Fig. 15: Modal sequent rules for non-normal modal logics SC$_E$ and SC$_M$.

\[
\frac{A \vdash B}{\top, \Box A \vdash \Box B, \bot} \quad \frac{A \vdash B}{\top, \Box A \vdash \Box B, \bot}
\]

Fig. 16: Modal rules for LbNS$_E$ and LbNS$_M$ with $y_1, y_2$ fresh in $\Box^L_R$.

where $X \models^\forall A$ is $\forall x \in X, x \vdash A$ and $A \models X$ is $\forall y, [(y \vdash A) \rightarrow y \in X]$. The rules for $\models^\forall$ and $\models$ are obtained using the geometric rule approach [20] and are depicted in Fig. 17a.

If the neighbourhood frame is monotonic (i.e. $\forall X \subseteq W$, if $X \in N(w)$ and $X \subseteq Y \subseteq W$ then $Y \in N(w)$), it is easy to see [20] that (1) is equivalent to

$$w \vdash \Box A \text{ iff } \exists X \in N(w). X \models^\forall A.$$  \hspace{1cm} (2)

This yields the set of labelled rules presented in Fig. 17b where the rules are adapted from [20] by collapsing invertible proof steps. Intuitively, while the box left rules create a fresh neighbourhood to $x$, the box right rules create a fresh world in this new neighbourhood and move the formula to it.

**Theorem 26.** G3E (resp. G3M) is label-preserving equivalent to LbNS$_E$ (resp. LbNS$_M$).

**Proof.** For sake of readability, we will only show in sequents the principal and auxiliary formulæ on the application of rules. Let $\pi$ be a normal proof of $\mathcal{N}, X \vdash Y$ in LbNS$_E$. An instance of the blocked derivation

\[
\frac{\pi_1}{\mathcal{N}, x\mathcal{N}y_1, y_1: A, y_2: B \vdash y_2, y_1: B} \quad \frac{\mathcal{N}, x\mathcal{N}y_2, y_2: B \vdash y_1, y_2: B}{\mathcal{N}, x\mathcal{N}e(y_1, y_2), x \vdash B, y_1: B}
\]

is transformed into the labelled derivation

\[
\frac{\pi'_1}{y_1: A, X \vdash Y, y_1: B} \quad \frac{\pi'_2}{y_2: B, X \vdash Y, y_2: A} \quad \frac{y_2 \in a, A \vdash a, X \vdash Y, y_2 \in a}{\text{init}} \quad \frac{y_2 \in a, A \vdash a, X \vdash Y, x \vdash a}{\Box^L_R} \quad \frac{X, x \vdash B, A \vdash a, X \vdash Y, x \vdash B}{a \in N(x), a \models^\forall A, A \models X \vdash Y, x \vdash B}
\]

Observe that, in $\pi_1$, the label $y_2$ will no longer be active, hence the formula $y_2: A$ can be weakened. The same with $y_1$ in $\pi_2$. Hence $\pi_1/\pi_2$ corresponds to $\pi'_1/\pi'_2$ and the "only if" part holds. The "if" part uses similar proof theoretical arguments as in the intuitionistic or (multi) modal case, observing that applications of the forcing rules can be restricted so to be applied immediately after the modal rules. \qed
This indicates that the nesting approach is more efficient when compared to labelled systems. Also, it is curious to note that this “two-step” interpretation makes the nested system external, in the sense that nestings cannot be interpreted inside the syntax of the logic. In fact, it makes use of the ( ) modality [4].

9 Conclusion and future work

In this work we gave sufficient conditions for transforming a nested system into a sequent calculus system, passing through linear nested systems. Moreover, we showed a semantical characterisation of intuitionistic, multi-modal and non-normal modal systems, via a case-by-case translation between labelled nested to labelled sequent systems. In this way, we closed the cycle of syntax/semantical characterisation for a class of logical systems.

While some of the presented results are expected (or even not new as the semantical interpretation of nestings in intuitionistic or in single-normal modal logics), our approach is, as far as we know, the first done entirely using proof theoretical arguments. Indeed, the soundness and completeness results are left to the tail case of labelled systems, that carry within the syntax the semantic information explicitly. Moreover, we formally established a relationship between sequent and nested systems for a class of logics. Of course this link is not possible for logics that do not have cut-free sequent systems, such as S5. This suggests that there should be a relationship between the fact that the nesting rules for such logic are not n-directed and the impossibility of the proposal of an adequate sequent calculus system.

Finally, this work can be extended in a number of ways, but perhaps the more natural direction is to complete the syntactical/semantical analysis for the classical cube [16]. This is specially interesting since MNC = K, that is, we should be able to smoothly collapse the neighbourhood approach into the relational one. We observe that nestings play an important role in these transformations, since it enables to modularly build proof systems.
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