Holographic $\mathcal{N} = 1$ supersymmetric membrane flows

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Abstract

The M-theory lift of $\mathcal{N} = 2$ $SU(3) \times U(1)_R$-invariant RG flow via a combinatorical use of the four-dimensional flow and 11-dimensional Einstein–Maxwell equations was found previously. By taking the three internal coordinates differently and preserving only the $SU(3)$ symmetry from the $\mathbb{CP}^2$ space, we find a new 11-dimensional solution of the $\mathcal{N} = 1$ $SU(3)$-invariant RG flow interpolating from the $\mathcal{N} = 8$ $SO(8)$-invariant UV fixed point to the $\mathcal{N} = 2$ $SU(3) \times U(1)_R$-invariant IR fixed point in four dimensions. We describe how the corresponding three-dimensional $\mathcal{N} = 1$ superconformal Chern–Simons matter theory deforms. By replacing the above $\mathbb{CP}^2$ space with the Einstein–Kahler twofold, we also find out a new 11-dimensional solution of the $\mathcal{N} = 1$ $SU(2) \times U(1)$-invariant RG flow connecting the above two fixed points in four dimensions.

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1. Introduction

The three-dimensional $\mathcal{N} = 6$ $U(N) \times U(N)$ Chern–Simons matter theory [1] with level $k$ describes the low energy limit of $N$ membranes at $\mathbb{C}^4/\mathbb{Z}_k$ singularity. The $\mathcal{N} = 8$ supersymmetry is preserved for $k = 1, 2$. The matter contents and the superpotential of this theory are exactly same as the ones in the theory for D3-branes at the conifold in four dimensions [2]. The three-dimensional membrane theory is related to the four-dimensional $\mathcal{N} = 8$ gauged supergravity theory [3] via AdS/CFT correspondence [4]. The holographic $\mathcal{N} = 2$ $SU(3) \times U(1)_R$-invariant renormalization group (RG) flow connecting the $\mathcal{N} = 8$ $SO(8)$ ultraviolet (UV) point to the $\mathcal{N} = 2$ $SU(3) \times U(1)_R$ infrared (IR) point has been studied in [5–7] long ago while the $\mathcal{N} = 1$ $G_2$-invariant RG flow from the $\mathcal{N} = 8$ $SO(8)$ UV point to the $\mathcal{N} = 1$ $G_2$ IR point has been described in [6, 8]. The former has the $SU(3) \times U(1)_R$-symmetry and the latter has the $G_2$-symmetry, around the IR region. The 11-dimensional M-theory lifts of these two RG flows have been found in [8, 9] by solving the Einstein–Maxwell equations explicitly in 11 dimensions.
The mass-deformed $U(2) \times U(2)$ Chern–Simons matter theory with $k = 1, 2$ preserving the $\mathcal{N} = 2 SU(3) \times U(1)_R$ symmetry has been found in [10, 11] while the mass deformation for this theory preserving the $\mathcal{N} = 1 G_2$ symmetry has been found in [12]. The non-supersymmetric RG flow equations preserving two $SO(7)^\pm$ symmetries have been studied in [13]. The holographic $\mathcal{N} = 1 SU(3)$-invariant RG flow equations connecting $\mathcal{N} = 1 G_2$ point to $\mathcal{N} = 2 SU(3) \times U(1)_R$ point in four dimensions have been studied in [14]. Moreover, the other holographic supersymmetric RG flows have been found and further developments on the four-dimensional gauged supergravity (see also [15, 16]) have been made in [17, 18]. The spin-2 Kaluza–Klein modes around a warped product of $AdS_4$ and a seven-ellipsoid having the $SU(2) \times SU(2) \times U(1)_R$ symmetry for the 11-dimensional lift of $SU(3) \times U(1)_R$-invariant solution in four-dimensional supergravity is described in [20] (see also [21]). The 11-dimensional description preserving the $\mathcal{N} = 2 SU(2) \times U(1) \times U(1)_R$ symmetry is found in [22] and the smaller $\mathcal{N} = 2 U(1) \times U(1) \times U(1)_R$ symmetry flow is discussed in [23]. Further study on [8] is done in [24] recently.

When the 11-dimensional supergravity theory is reduced to four-dimensional $\mathcal{N} = 8$ gauged supergravity, the four-dimensional spacetime is warped by a warp factor $\Delta$ which depends on both four-dimensional coordinates and seven-dimensional internal coordinates. The seven-dimensional internal metric of deformed seven-sphere is obtained from the AdS$_4$ supergravity fields [25, 26]. As they vary, the geometric structure of a round seven-sphere changes. We have the following 11-dimensional metric:

$$ds^2_{11} = \Delta^{-1}(dr^2 + e^{2A(r)} \eta_{\mu\nu} dx^\mu dx^\nu) + ds^2_7,$$

where the three-dimensional metric is $\eta_{\mu\nu} = (−, +, +)$, the radial coordinate is transverse to the domain wall and the scale factor $A(r)$ behaves linearly in $r$ at UV and IR regions.

The AdS$_4$ supergravity fields $(\rho, \chi)$ in four-dimensional gauged supergravity as well as the scale function $A$ satisfy the supersymmetric $SU(3) \times U(1)_R$-invariant RG flow equations [5]

$$\frac{d\rho}{dr} = \frac{1}{8L\rho^4}[(\cosh 2\chi + 1) + \rho^8(\cosh 2\chi - 3)],$$

$$\frac{d\chi}{dr} = \frac{1}{2L\rho^2}(\rho^8 - 3) \sinh 2\chi,$$

$$\frac{dA}{dr} = \frac{1}{4L\rho^2}[3(\cosh 2\chi + 1) - \rho^8(\cosh 2\chi - 3)].$$

There exist two critical points: the $\mathcal{N} = 8 SO(8)$ critical point at which $(\rho, \chi) = (1, 0)$ and the $\mathcal{N} = 2 SU(3) \times U(1)_R$ critical point at which $(\rho, \chi) = (3, \frac{1}{2} \cosh^{-1} 2)$ [27]. The $L$ is the radius of the round seven-sphere $S^7$. We focus on the possible 11-dimensional lift of the RG flows around the $\mathcal{N} = 2 SU(3) \times U(1)_R$ critical point in this paper and those around the $\mathcal{N} = 1 G_2$ critical point have been recently described in [24].

For the specific 7-dimensional internal metric $ds^2_7$, how one can determine the solution for 11-dimensional Einstein–Maxwell equations? For a given 11-dimensional metric and 4-form field strengths, the 11-dimensional bosonic field equations are given by [28]

$$R_M^{\ N} = \frac{1}{4} F_{MPQR} F^{NMPQR} - \frac{1}{72} \delta_M^N F_{PQRS} F^{PQRS},$$

$$\nabla_M F^{MNPQ} = -\frac{1}{72} \epsilon^{MNPQRSTUVWXY} F_{RSTU} F_{VWXY},$$

where the covariant derivative is given by $\nabla_M = \partial_M + E^{-1}(\partial_M E)$ with the elfbein determinant $E = \sqrt{−g_{11}}$. The 11-dimensional epsilon tensor with lower indices is purely numerical in our convention.
In order to construct the various 11-dimensional M-theory lifts of a supersymmetric RG flow (1.2), we impose the nontrivial AdS$_4$ radial coordinate dependence of AdS$_4$ supergravity fields subject to the four-dimensional RG flow equations. Then the geometric parameters in the seven-dimensional metric at certain critical point are controlled by the RG flow equations so that they can be extrapolated from the critical points. After that, an appropriate ansatz for the 4-form field strengths is made. Finally, the 11-dimensional Einstein–Maxwell bosonic equations [28, 29] can be checked in order to complete the M-theory uplift.

At the $\mathcal{N} = 8$ SO(8) critical point where $(\rho, \chi) = (1, 0)$, the seven-dimensional internal space is round seven-sphere $S^7$. Either one can introduce the global coordinates as $\mathbb{CP}^1$ appropriate for the base round 6-sphere $S^6$ or those as the Hopf fibration on $\mathbb{CP}^3$ where there are two $U(1)$ symmetries. Each global coordinates share the common $\mathbb{CP}^2$ space. For the former compactification, we would like to find out the new 11-dimensional solutions by taking the other various Einstein–Kahler twofolds including the above $\mathbb{CP}^2$ space, inside of the six-dimensional manifold. For the latter compactification, the new 11-dimensional solutions were found in [9, 20, 22, 23] where the 3-forms are fixed by the unbroken $U(1)$ symmetry which is nothing but the $U(1)_R$ symmetry.

In this paper, we find out a new exact solution of the $\mathcal{N} = 1$ SU(3)-invariant flow (connecting from the $\mathcal{N} = 8$ SO(8) UV fixed point to $\mathcal{N} = 2$ SU(3) $\times$ U(1)$_R$ IR fixed point) to the 11-dimensional Einstein–Maxwell–Euler equations. This can be described in the three-dimensional $\mathcal{N} = 1$ Chern–Simons matter theory by introducing two mass terms for two adjoint $\mathcal{N} = 1$ superfields. If the two masses are equal to each other, then the previously known $\mathcal{N} = 2$ SU(3) $\times$ U(1)$_R$-invariant flow [10] arises. If both masses are nonzero but not necessarily equal, then the $\mathcal{N} = 1$ SU(3)-invariant flow occurs. We claim that we have found the 11-dimensional uplift of the generic $\mathcal{N} = 1$ flow with unequal mass parameters found by [14].

There exist various four-dimensional Einstein–Kahler twofolds which live in the five-dimensional Sasaki–Einstein space. By replacing the above $\mathbb{CP}^2$ space with $\mathbb{CP}^1 \times \mathbb{CP}^1$ space, Einstein–Kahler twofold and other Einstein–Kahler twofold, respectively, we find out new 11-dimensional solutions of $\mathcal{N} = 1$ SU(2) $\times$ SU(2)-, SU(2) $\times$ U(1)- and U(1) $\times$ U(1)-invariant RG flows connecting the above two fixed points. The corresponding U(1) bundles are also replaced. We will present the middle one which can be generalized to the last and leads to the first for a particular limit. One can take the last one because this includes the first two cases but there is no non-Abelian symmetry group.

In section 2, by changing only three coordinates among seven internal coordinates characterized by previous parametrization [9], one only keeps the SU(3) symmetry inside of the SU(3) $\times$ U(1)$_R$ symmetry. Then, the 11-dimensional metric can be written in terms of these new coordinates and it contains the Fubini–Study metric on $\mathbb{CP}^2$ space [30]. The Ricci tensor can be expressed as a linear combination of the Ricci tensor for the SU(3) $\times$ U(1)$_R$-invariant flow and similarly the 4-forms are also given by a linear combination of 4-forms for the SU(3) $\times$ U(1)$_R$-invariant flow. Then, we find out a new solution for the 11-dimensional Einstein–Maxwell–Euler equations corresponding to the 11-dimensional lift of the $\mathcal{N} = 1$ SU(3)-invariant RG flow connecting from the $\mathcal{N} = 8$ SO(8) UV fixed point to the $\mathcal{N} = 2$ SU(3) $\times$ U(1)$_R$ IR fixed point. In the 11-dimensional point of view, both the metric and 4-forms preserve only the SU(3) symmetry inside of the SU(3) $\times$ U(1)$_R$ symmetry. The possible deformation in the gauge dual, $\mathcal{N} = 1$ Chern–Simons matter theory, is discussed.

In section 3, by considering the Einstein–Kahler twofold and U(1) bundle living in the five-dimensional Sasaki–Einstein manifold $Y^{p,q}$, one only keeps the SU(2) $\times$ U(1) symmetry inside of the SU(2) $\times$ U(1) $\times$ U(1)$_R$ symmetry. We find out a new solution for the 11-dimensional lift of the $\mathcal{N} = 1$ SU(2) $\times$ U(1)-invariant RG flow.
In section 4, we summarize the results of this paper and present some future directions.
In the appendices, we present the detailed expressions for the Ricci tensor, 4-form field strengths and the Maxwell equation.

The new 11-dimensional solutions of $\mathcal{N} = 1 \ SU(2) \times SU(2)$- and $U(1) \times U(1)$-invariant RG flows connecting the above two fixed points in four dimensions can be done similarly.

2. An $\mathcal{N} = 1 \ SU(3)$-invariant supersymmetric flow

- The 11-dimensional metric along the flow.
  Let us introduce the three orthogonal $\mathbb{R}^8$ vectors [30]
  \begin{align}
  U &= (u^1, u^2, u^3, u^4, u^5, u^6, 0, 0) \sin \theta \sin \theta_6, \\
  V_1 &= (0, 0, 0, 0, 0, 1, 0, 1) \cos \theta, \quad (2.1) \\
  V_2 &= (0, 0, 0, 0, 0, 0, 1) \sin \theta \cos \theta_6.
  \end{align}
  The sum of these vectors is restricted on the round seven-sphere $S^7$. That is, $\sum_{i=1}^{8} (X^i)^2 = 1$ with $X = U + V_1 + V_2$. Note that $\sum_{i=1}^{8} (X^i)^2 + (X^8)^2 = \sin^2 \theta$ and the ellipsoidal deformation arises along the remaining $V_1(X^8)$-direction. By looking at the inside of $S^7$, the unit round five-sphere $S^5$ with $\mathbb{C}P^2$-base can be described by the following six variables with the constraint $\sum_{i=1}^{6} (u^i)^2 = 1$, or five angular variables $\theta_i$ ($i = 1, \ldots, 5$):
  \begin{align}
  u^1 + i u^2 &= \sin \theta_1 \cos \left(\frac{\theta_2}{2}\right) e^{\frac{i}{2} (\theta_3 + \theta_4)} e^{i \theta_5} \equiv z^1, \\
  u^3 + i u^4 &= \sin \theta_1 \sin \left(\frac{\theta_2}{2}\right) e^{-\frac{i}{2} (\theta_3 + \theta_4)} e^{i \theta_5} \equiv z^2, \\
  u^5 + i u^6 &= \cos \theta_1 e^{i \theta_5} \equiv z^3.
  \end{align}
  The isometry of five-sphere $S^5$ is given by $SU(3) \times U(1)$ where $SU(3)$ acts on three complex coordinates $z^i$ and the $U(1)$ acts on each $z^i$ as the phase rotations. The five-dimensional metric $(du)^2$ from (2.2) can be rewritten as $(du)^2 = \text{d}s_{S^5}^2 + (u, J \ du)^2$ where $\text{d}s_{S^5}^2$ denotes the Fubini–Study metric on $\mathbb{C}P^2$-base of $S^5$ characterized by four angular variables $\theta_i$ ($i = 1, \ldots, 4$):
  \begin{align}
  \text{d}s_{S^5}^2 &= \text{d}\theta_1^2 + \frac{1}{2} \sin^2 \theta_1 (\sigma_1^2 + \sigma_2^2 + \cos^2 \theta_1 \sigma_3^2), \quad (2.3)
  \end{align}
  and $(u, J \ du) \equiv u^I J_{ij} u^j$ is the Hopf fiber on it and is given by
  \begin{align}
  (u, J \ du) &= \text{d}\theta_5 + \frac{1}{2} \sin^2 \theta_1 \sigma_3. \quad (2.4)
  \end{align}
  The $J$ is the standard Kahler form: $J_{12} = J_{34} = J_{56} = J_{78} = 1$. The one-forms appearing in (2.3) are given by $\sigma_1 = \cos \theta_4 \text{d}\theta_2 + \sin \theta_4 \sin \theta_3 \text{d}\theta_1$, $\sigma_2 = \sin \theta_4 \text{d}\theta_2 - \sin \theta_3 \cos \theta_4 \text{d}\theta_3$ and $\sigma_3 = \text{d}\theta_4 + \cos \theta_2 \text{d}\theta_3$. By extending to the seven-dimensional metric with the differentials $\text{d}\theta$ and $\text{d}\theta_6$, one sees the standard metric for the round seven-sphere $S^7$:
  \begin{align}
  (dX)^2 = (dX, dX) &= \text{d}\theta^2 + \sin^2 \theta \text{d}\Omega_6^2, \\
  \text{d}\Omega_6^2 &= \text{d}\theta_6^2 + \sin^2 \theta_6 [\text{d}s_{S^5}^2 + (u, J \ du)^2]. \quad (2.5)
  \end{align}
  Inside of $S^7$, the six-dimensional metric is nothing but the metric for the unit round six-sphere $S^6$. So far, the background geometry provides the $\mathcal{N} = 8$ maximally symmetric $\text{AdS}_4 \times S^7$ solution [29] characterized by (1.1) for $\rho = 1$ and $\chi = 0$ corresponding to the $SO(8)$-invariant UV fixed point. Then the Ricci tensor in (1.3) has the form of
  \begin{align}
  R_{\mu}^\nu &= \frac{6}{L^2} \text{diag}(-2, -2, -2, -2, 1, 1, 1, 1, 1, 1, 1, 1), \quad (2.6)
  \end{align}
where \( L \) is the radius of \( S^7 \), twice the AdS\(_4\) radius and the only nonzero 4-form field strength satisfying (1.3) with (2.6) is given by \( F_{1234} = \frac{13}{16} \), the so-called Freund–Rubin parametrization [31].

Now we turn on the AdS\(_4\) supergravity scalar fields \((\rho, \chi)\) starting from the above \( SO(8)\)-invariant UV fixed point \((\rho, \chi) = (1, 0)\). They develop a nontrivial profile as a function of \( r \) (1.2) becoming more significantly different from \((\rho, \chi) = (1, 0)\) as one goes to the \( SU(3) \times U(1)_R \) IR fixed point. Let us consider the deformation from the \( \rho \)-supergravity field first. The deformation matrix \( Q \) is given by [9]

\[
Q = \text{diag}(\rho^{-2}, \rho^{-2}, \rho^{-2}, \rho^{-2}, \rho^{-2}, \rho^{-6}, \rho^6).
\]

The quadratic form \( \xi^2 = (X, Q X) \equiv X^A Q_{AB} X^B \) from (2.1) and (2.7) can be calculated to be

\[
\xi^2 = \rho^6 + (\rho^{-2} - \rho^6) \sin^2 \theta \sin^2 \theta_0.
\]

One sees that the metric on the deformed \( \mathbb{R}^8 \) by ellipsoidal squashing via the \( \rho \)-field can be recombined by

\[
\begin{align*}
(dX, Q^{-1} dX) &= \rho^2 (dU)^2 + \frac{1}{\rho^6} \left[ (dV_1)^2 + (dV_2)^2 \right] \\
&= \rho^2 \sin^2 \theta \sin^2 \theta_0 \left[ d_{FS(2)}^2 + (u, J \, du)^2 \right] \\
&\quad + \rho^{-4} \xi^2 \left[ \frac{(- \cos \theta \sin \theta_0 \, d\theta - \sin \theta \cos \theta_0 \, d\theta_0)^2}{(1 - \sin^2 \theta \sin^2 \theta_0)} \right] \\
&\quad + \rho^{-6} \left[ \frac{(\cos \theta_0 \, d\theta_0 - \sin \theta \cos \theta \sin \theta_0 \, d\theta)^2}{(1 - \sin^2 \theta \sin^2 \theta_0)} \right].
\end{align*}
\]

The Fubini–Study metric on \( \mathbb{CP}^2 \) space and its Hopf fiber are given by (2.3) and (2.4), respectively. According to (2.8), the \( \rho^2 \)-terms from \((dU)^2\) in (2.9) are decomposed into two parts, the first two terms of (2.9) and some terms containing \( \xi^2 \), and moreover the \( \rho^{-6} \)-terms from \((dV_1)^2\) and \((dV_2)^2\) are distributed into the remaining term in \( \xi^2 \) and the last term of (2.9). At \( \rho = 1 \) (no deformation), this leads to the standard metric [30] for round seven-sphere \( S^7 \) characterized by (2.5), by recollecting \( d\theta_2 \)- and \( d\theta_0^2 \)-terms. Although the full \( SO(8) \) isometry is broken in (2.9), the \( SU(3) \times U(1) \) isometry is still preserved.

The \( U(1) \) Hopf fiber on the six-dimensional manifold can be written as

\[
(X, J \, dX) \equiv X^A J_{AB} X^B = (U, J \, dU) + (V_1, J \, dV_1) + (V_2, J \, dV_1)
\]

\[
= (u, J \, du) \sin^2 \theta \sin^2 \theta_0 + [\cos \theta_0 \, d\theta - \sin \theta \cos \theta \sin \theta_0 \, d\theta_0],
\]

(10.2)

where the square of last two terms is proportional to the last term of (2.9). By recalling that the Fubini–Study metric on \( \mathbb{CP}^1 \) space can be written in terms of the Fubini–Study metric on \( \mathbb{CP}^2 \) space, \( d_{FS(2)}^2 \), that is given by (2.3) and other pieces, the metric on the six-dimensional manifold from (2.5) and (2.10) is rewritten as

\[
\begin{align*}
(dX, dX) - (X, J \, dX)^2 &= \left[ \frac{(- \cos \theta \sin \theta_0 \, d\theta - \sin \theta \cos \theta_0 \, d\theta_0)^2}{(1 - \sin^2 \theta \sin^2 \theta_0)} \right] \\
&\quad + \sin^2 \theta \sin^2 \theta_0 \left[ d_{FS(2)}^2 + (1 - \sin^2 \theta \sin^2 \theta_0) \right] \\
&\quad \times \left( (u, J \, du) - \frac{(\cos \theta_0 \, d\theta_0 - \sin \theta \cos \theta \sin \theta_0 \, d\theta)^2}{(1 - \sin^2 \theta \sin^2 \theta_0)} \right)^2.
\end{align*}
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In the first term on the right-hand side (RHS), we intentionally recollected $\rho^{-4} \xi^2$-terms in (2.9) which will play the role of one of the frame basis and then the other parts of (2.11) (that will be the other frame basis) are automatically determined once we identify the Fubini–Study metric on CP$^2$ space.

Let us next consider the deformation from the $\chi$-supergravity field. Then the warped $SU(3)$-invariant seven-dimensional metric, by adding (2.9) to the stretched fiber characterized by the $\chi$-field, leads to [30]

$$\mathrm{d} s^2 = \sqrt{\Delta} L^2 \left[ (\mathrm{d} X, Q^{-1} \mathrm{~d} X) + \frac{\sinh^2 \chi}{\xi^2} (X, J \mathrm{~d} X)^2 \right]. \quad (2.12)$$

where the warp factor is given by [9]

$$\Delta = (\xi \cosh \chi)^{-4}. \quad (2.13)$$

Let us use the identity $\cosh^2 \chi - \sin^2 \chi = 1$ in the second term and recollect the first term and the second term with the coefficient $-\frac{1}{\xi^2}$ of (2.12).

By keeping the $\rho^{-4} \xi^2$-term in (2.9) and the $\mathrm{d}s^2_{\text{FS}(2)}$ part as independent five-orthonormal frames and substituting (2.9) and (2.10) into (2.12), one obtains, with (2.4),

$$\mathrm{d} s^2 = \sqrt{\Delta} L^2 \left( \frac{\xi^2}{\rho^2} \left[ -\cos \theta \sin \theta_0 \mathrm{~d} \theta - \sin \theta \cos \theta_0 \mathrm{~d} \theta_0 \right]^2 \right) + \rho^2 \sin^2 \theta \sin^2 \theta_0 \mathrm{d} s^2_{\text{FS}(2)} + \frac{\omega^2}{\xi^2} \left[ \cos \theta_0 \mathrm{~d} \theta - \sin \theta \cos \theta \sin \theta_0 \mathrm{~d} \theta_0 \right] + \mathrm{d} \theta_0^2 + \frac{\sin^2 \theta \sin^2 \theta_0 (u, J \mathrm{~d} u)^2}, \quad (2.14)$$

where the remaining one-form is completely fixed as follows:

$$\omega = \sin \theta \sin \theta_0 \sqrt{1 - \sin^2 \theta \sin^2 \theta_0} \times \left[ \frac{1}{\rho^2} \left[ \cos \theta_0 \mathrm{~d} \theta - \sin \theta \cos \theta \sin \theta_0 \mathrm{~d} \theta_0 \right] + \rho^4 (u, J \mathrm{~d} u) \right]. \quad (2.15)$$

Note that for no deformation ($\rho = 1$), the square of this one-form (2.15) becomes the last term in (2.11). Moreover, the last term in (2.12) without deformation ($\rho = 1$ and $\chi = 0$) leads to the square of (2.10) where $\frac{\cos^2 \chi}{\xi^2}$ becomes 1.

By combining the four-dimensional and seven-dimensional metric (2.14) together with (1.1), one arrives at the following set of frames for the 11-dimensional metric with 11-dimensional coordinates $z^M = (x^1, x^2, x^3, r; \theta, \theta_0, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$:

$$e^1 = \Delta^{-\frac{1}{4}} \rho \sin \theta \sin \theta_0 \mathrm{~d} \theta_0, \quad e^2 = \Delta^{-\frac{1}{4}} \rho \sin \theta \cos \theta_0 \mathrm{~d} \theta_0, \quad e^3 = \Delta^{-\frac{1}{4}} \rho \sin \theta_0 \mathrm{~d} \theta, \quad e^4 = \Delta^{-\frac{1}{4}} \rho \sin \theta \cos \theta_0 \mathrm{~d} \theta_0,$$

$$e^5 = L \Delta^{\frac{1}{4}} \rho \sin \theta \sin \theta_0 \mathrm{~d} \theta_0, \quad e^6 = L \Delta^{\frac{1}{4}} \rho \sin \theta \cos \theta_0 \mathrm{~d} \theta_0, \quad e^7 = L \Delta^{\frac{1}{4}} \rho \sin \theta \cos \theta_0 \mathrm{~d} \theta_0,$$

$$e^8 = L \Delta^{\frac{1}{4}} \rho \sin \theta_0 \mathrm{~d} \theta_0, \quad e^9 = L \Delta^{\frac{1}{4}} \rho \sin \theta \cos \theta_0 \mathrm{~d} \theta_0,$$

$$e^{10} = L \Delta^{\frac{1}{4}} \rho \sin \theta \cos \theta_0 \mathrm{~d} \theta_0, \quad e^{11} = L \Delta^{\frac{1}{4}} \rho \sin \theta \cos \theta_0 \mathrm{~d} \theta_0.$$
\( e^{10} = L \Delta^1 \xi^{-1} \sin \theta \sin \theta_6 \sqrt{1 - \sin^2 \theta \sin^2 \theta_6} \left[ -\frac{1}{\rho^2} \frac{(\cos \theta_6 d\theta - \sin \theta \cos \theta \sin \theta_6 d\theta_6)}{(1 - \sin^2 \theta \sin^2 \theta_6)} \right] \\
+ \rho^4 \left( d\theta_5 + \frac{1}{2} \sin^2 \theta_1 \sigma_3 \right) \] \\
\( e^{11} = L \Delta^1 \xi^{-1} \cosh \chi \left[ \cos \theta_6 d\theta - \sin \theta \cos \theta \sin \theta_6 d\theta_6 \\
+ \sin^2 \theta \sin^2 \theta_6 \left( d\theta_5 + \frac{1}{2} \sin^2 \theta_1 \sigma_3 \right) \right], \quad (2.16) \)

where the quadratic form \( \xi^2 \) is given by (2.8), the warp factor \( \Delta \) is given by (2.13) and the fiber on \( \text{CP}^3 \) space (2.4) is written in terms of the angular variables \( \theta_i \) (\( i = 1, \ldots, 5 \)). The AdS4 supergravity fields \( (\rho, \chi) \) and scale function \( A \) have nontrivial \( r \)-dependence via (1.2). The \( \xi^2 \) and \( \Delta \) depend on the radial coordinate as well as the internal coordinates \( (\theta, \theta_6) \). The one-forms \( \sigma_i \) are the same as previous expressions and are related to the angular coordinates \( (\theta_2, \theta_3, \theta_4) \) as before. There are no \( U(1) \) shifts in the coordinates \( (\theta, \theta_6) \) except \( \theta_5 \). The three coordinates [9] where the base six-dimensional manifold is \( \text{CP}^3 \) space are related to those in this paper or in [30] as follows:

\[
\mu = \cos^{-1} (\sin \theta \sin \theta_6), \\
\phi = \theta_5 - \cos^{-1} \left( \frac{\cos \theta}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}} \right), \\
\psi = \cos^{-1} \left( \frac{\cos \theta}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}} \right). \quad (2.17)
\]

Through the transformation (2.17), one can easily see that the above 11-dimensional metric (2.16) becomes exactly the one in [9]. In order to use the Ricci tensor and 4-form field strengths for the \( SU(3) \)-invariant flow, the partial differentiations between these coordinates are needed and some of them are

\[
\frac{\partial \mu}{\partial \theta} = -\frac{\cos \theta \sin \theta_6}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}}, \quad \frac{\partial \mu}{\partial \theta_6} = \frac{\sin \theta \cos \theta_6}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}}, \quad \frac{\partial \mu}{\partial \theta_5} = \frac{\sin \theta \cos \theta_6}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}}, \quad \frac{\partial \mu}{\partial \theta_4} = \frac{-\sin \theta \cos \theta_6}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}}, \quad \frac{\partial \mu}{\partial \theta_3} = \frac{-\sin \theta \cos \theta_6}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}}, \quad \frac{\partial \mu}{\partial \theta_2} = \frac{-\sin \theta \cos \theta_6}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}}, \\
\frac{\partial \phi}{\partial \theta} = -\frac{\cos \theta_6}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}}, \quad \frac{\partial \phi}{\partial \theta_6} = \frac{\sin \theta \cos \theta_6}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}}, \quad \frac{\partial \phi}{\partial \theta_5} = \frac{\sin \theta \cos \theta_6}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}}, \quad \frac{\partial \phi}{\partial \theta_4} = \frac{-\sin \theta \cos \theta_6}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}}, \quad \frac{\partial \phi}{\partial \theta_3} = \frac{-\sin \theta \cos \theta_6}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}}, \quad \frac{\partial \phi}{\partial \theta_2} = \frac{-\sin \theta \cos \theta_6}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}}, \\
\frac{\partial \psi}{\partial \theta} = \frac{\cos \theta_6}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}}, \quad \frac{\partial \psi}{\partial \theta_6} = \frac{\sin \theta \cos \theta_6}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}}, \quad \frac{\partial \psi}{\partial \theta_5} = \frac{\sin \theta \cos \theta_6}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}}, \quad \frac{\partial \psi}{\partial \theta_4} = \frac{-\sin \theta \cos \theta_6}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}}, \quad \frac{\partial \psi}{\partial \theta_3} = \frac{-\sin \theta \cos \theta_6}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}}, \quad \frac{\partial \psi}{\partial \theta_2} = \frac{-\sin \theta \cos \theta_6}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}}. \quad (2.18)
\]

Moreover, the partial differentiations of new variables \( (\theta, \theta_5, \theta_6) \) with respect to old variables \( (\mu, \phi, \psi) \) from (2.17) can also be obtained.

- **The Ricci tensor and the 4-form field strengths along the flow.** Since the metric (2.16) is related to the metric given in [9] via the change of variables (2.17), one can use the \( SU(3) \times U(1)_R \)-invariant solution summarized in appendix A and find out the new solution which is invariant under the \( SU(3) \) symmetry. The Ricci tensor can be obtained from (2.16) directly or can be determined from the one preserving \( SU(3) \times U(1)_R \) by using the transformation on the coordinates between the two coordinate systems (2.17). That is, the \( SU(3) \)-invariant Ricci tensor is given by

\[
\tilde{R}^N_M = \left( \frac{\partial \tilde{z}^P}{\partial \tilde{z}^M} \right) \left( \frac{\partial \tilde{z}^N}{\partial \tilde{z}^Q} \right) R^Q_P, \quad (2.19)
\]
where the 11-dimensional coordinates are given by
\[
\begin{align*}
\tilde{z}^M &= (x^1, x^2, x^3, r; \theta, \theta_1, \theta_2, \theta_3, \theta_6), \\
z^M &= (x^1, x^2, x^3, r; \mu, \theta_1, \theta_2, \theta_3, \phi, \psi).
\end{align*}
\]
Only three of them are distinct and the new variables \((\theta, \theta_5, \theta_6)\) correspond to old variables \((\mu, \phi, \psi)\). The Ricci tensor \(R^Q_P\) for the \(SU(3) \times U(1)_r\)-invariant flow is explicitly presented in appendix A (A.1) where the flow equations (1.2) are imposed. The transformation rule between the two coordinate systems can be obtained from (2.17). The Ricci tensor \(\tilde{R}^N_M\) for the \(SU(3)\)-invariant flow is explicitly given in appendix B (B.1) and there exist off-diagonal components \((4, 5), (4, 11), (5, 4), (5, 11), (5, 10), (8, 5), (8, 10), (8, 11), (9, 5), (9, 10), (9, 11), (10, 5), (10, 11)\) and \((11, 5)\) and \((11, 10)\). At the IR critical point, the components of the Ricci tensor \((4, 5), (4, 11), (5, 4)\) and \((11, 4)\) vanish.

For the 4-form field strengths, one has
\[
\tilde{F}^*_MNPQ = \frac{\partial z^R}{\partial z^M} \frac{\partial z^S}{\partial z^N} \frac{\partial z^T}{\partial z^P} \frac{\partial z^U}{\partial z^Q} F_{RSTU},
\]
with (2.20), (2.17) and (2.18). These transformed 4-forms are given in appendix B (B.2) in terms of those (A.3) in the \(SU(3) \times U(1)_r\)-invariant flow and the transformed 4-forms with upper indices are described via (B.3) in terms of (A.4). The 4-forms with upper indices \(\tilde{F}^MNPQ\) can be obtained from \(F_{RSTU}\) by using the 11-dimensional inverse metric \((2.16)\) or by multiplying the transformation matrices obtained from (2.17) into the 4-forms with upper indices \(F^MNPQ\) (A.4) of the \(SU(3) \times U(1)_r\)-invariant flow as done in (2.21). The 4-form \(\tilde{F}_{1234}\) is a new object for the \(SU(3)\)-invariant flow. At the IR critical point in four dimensions, the following 4-forms also vanish: \(\tilde{F}_{1235} = \tilde{F}_{123411} = \tilde{F}_{45mn} = \tilde{F}_{45mpnp} = 0\). For the \(SU(3) \times U(1)_r\)-invariant flow, the 4-forms \(F_{1235, 1}F_{12345, mn} F_{mn,p}(m, n, p = 6, \ldots, 11)\) become zero at the IR critical point in four dimensions. Once we suppose that the four-dimensional metric has the domain wall factor \((1.1)\) which breaks the four-dimensional conformal invariance, the mixed 4-forms occur along the whole RG flow.

Note that the internal 3-form corresponding to the above 4-forms (2.21) (the internal part of \(\tilde{F}^{(4)}\) is given by \(d\tilde{C}^{(3)} + d\tilde{C}^{(3)*}\)) has the following expression:
\[
\tilde{C}^{(3)} = \frac{L^3 \tanh \chi}{4[\rho^8 + (1 - \rho^8)(1 - |W|^2)]} (3Z^1 dZ^2 \wedge dZ^3 \wedge dW - \rho^8 W dZ^1 \wedge dZ^2 \wedge dZ^3),
\]
where the rectangular coordinates with (2.1) and (2.2) are given by
\[
Z^i \equiv z^i \sin \theta \sin \theta_6 \quad (i = 1, 2, 3), \quad W \equiv X^7 + iX^8 = \cos \theta + i\sin \theta \cos \theta_6.
\]
Then it is easy to see that the 3-form (2.22) has an explicit \(SU(3)\) symmetry because the 3 of \(SU(3)\) in \(Z\) occurs as \(\epsilon_{ijk} Z^i \wedge dZ^j \wedge dZ^k\) or \(\epsilon_{ijk} dZ^i \wedge dZ^j \wedge dZ^k\) with the \(SU(3)\)-invariant epsilon tensor \(\epsilon_{ijk}\). The \(X^7\) and \(X^8\) are \(SU(3)\) singlets. Since the 3-form along the \((123)\)-directions has \((\theta, \theta_6)\) dependence as well as \(r\)-dependence, it is \(SU(3)\)-singlet.

**Checking the Einstein equation.** One checks the Einstein equation using the solution for the \(SU(3) \times U(1)_r\)-invariant flow. In [9], it was shown that the Ricci tensor \(R^Q_P\), the 4-forms with lower indices (A.3) and the 4-forms with upper indices (A.4) satisfy the field equation (1.3). Now one can use the property of the \(SU(3) \times U(1)_r\)-invariant flow. One replaces the Ricci tensor \(R^Q_P\) in terms of the quadratic 4-forms: \(F^2\) or \(F_{MNPQ}F^{MNPQR}\). After that one finds that the transformed-Ricci tensor \(\tilde{R}^N_M\) can be written in terms of the quadratic 4-forms for the \(SU(3) \times U(1)_r\)-invariant flow from (2.19). Let us return to the
right-hand side of Einstein equation. Using (2.21) and \( \tilde{F}^{MNPQ} \), one can express the RHS in terms of quadratic 4-forms \( F^5 \) or \( F_{MNPQ} F^{NPQR} \) for the \( SU(3) \times U(1)_R \)-invariant flow. One can make the difference between the left-hand side (LHS) and the RHS of Einstein equation and see whether this becomes zero or not. At first sight, some of the components written in terms of quadratic 4-forms in the \( SU(3) \times U(1)_R \)-invariant flow are not exactly vanishing. They contain the terms

- \( F_{MNPQ} F^{NPQR} \), \( M = 4, 5, N = 10, 11 \), \( F_{6PQR} F^{NPQR} \), \( N = 4, 5, 9, 10, 11 \), \( F_{1PQR} F^{NPQR} \), \( N = 4, 5, 6, 8, 9, 10, 11 \), \( F_{8PQR} F^{NPQR} \), \( N = 4, 5, 6, 7, 9 \), (2.23) \( F_{5PQR} F^{NPQR} \), \( N = 4, 5, 6 \), \( F_{MPQR} F^{NPQR} \), \( M = 10, 11 \), \( N = 4, 5 \).

However, after plugging the explicit solution \( F_{MNPR} \) (A.3) and \( F^{MNPR} \) (A.4) for the \( SU(3) \times U(1)_R \)-invariant flow into (2.23), all of these (2.23) are identically vanishing. Recall that these quadratic 4-forms above correspond to the off-diagonal terms of Einstein equation for the \( SU(3) \times U(1)_R \)-invariant flow which vanish identically except the \((4, 5), (5, 4), (8, 10), (8, 11), (9, 10), (9, 11), (10, 11), (11, 10)\)-components from the \( R^N_M \) (A.1). Of course, the above extra piece (2.23) does not possess these nonzero off-diagonal terms. Therefore, we have shown that the solution given by the Ricci tensor \( \tilde{R}^{N}_M \) (2.19), the 4-forms \( \tilde{F}_{MNPQ} \) (2.21) and the 4-forms \( \tilde{F}^{MNQP} \) indeed satisfies the 11-dimensional Einstein equation (1.3).

**Checking the Maxwell equation.** Let us introduce the notation \( \frac{1}{2} \tilde{E} \tilde{\nabla}_M \tilde{F}^{MNPQ} \equiv (\tilde{N} \tilde{P} \tilde{Q}) \) for simplicity and present all the nonzero components for the LHS of Maxwell equations in terms of the 4-forms in the \( SU(3) \times U(1)_R \)-invariant flow, using the property of an 11-dimensional solution. According to the transformation rules, one can express the transformed covariant derivative and transformed 4-forms in terms of those for the \( SU(3) \times U(1)_R \)-invariant flow together with \((\theta, \theta_0)\)-dependent coefficient functions. In other words, by using the Maxwell equation for the \( SU(3) \times U(1)_R \)-invariant flow, one can replace the LHS of the Maxwell equation with the quadratic 4-forms \( F_{RSTU} F_{VWXY} \). The nonzero components are explicitly given in appendix (B.4). The nonzero components of the Maxwell equation are characterized by the indices (123), \((\tilde{n} \tilde{p})\), \((\tilde{m} \tilde{n} \tilde{p})\) and \((\tilde{n} \tilde{m} \tilde{n} \tilde{p})\), with the number of components 1, 8 by choosing two out of six, 8 by choosing two out of six and 13 by choosing three out of six, respectively. Other remaining components of the Maxwell equation become identically zero. Therefore, there exist 30 nonzero components of the Maxwell equation. The (123)-component above consists of 12 terms coming from the quadratic 4-forms \( F_{RSTU} F_{VWXY} \). For the other components, the right-hand side contains a single term, three terms or four terms in quadratic 4-forms.

Similarly, let us return to the RHS of the Maxwell equation. Using (2.21) and the 11-dimensional metric (2.16), one can express this in terms of quadratic 4-forms \( F_{RSTU} F_{VWXY} \) for the \( SU(3) \times U(1)_R \)-invariant flow. We also transform the 11-dimensional determinant according to the transformation rules appropriately. It turns out that the difference between the LHS obtained from the previous paragraph and the RHS in this paragraph of the Maxwell equation becomes zero. Therefore, we have shown that the solution (2.21) for a given 11-dimensional metric (2.16) indeed satisfies the Maxwell equation (1.3).

Now we have shown that the solution (2.21) together with the appendices A and B consists of an exact solution to 11-dimensional supergravity by bosonic field equations (1.3) as long as the deformation parameters \((\rho, \chi)\) of the 7-dimensional internal space and the domain wall amplitude \( A \) develop in the AdS\(_4\) radial direction along the \( N = 1 \) \( SU(3) \)-invariant RG flow (1.2) connecting from the \( N = 8 \) SO(8) UV fixed point to the \( N = 2 \) \( SU(3) \times U(1)_R \) IR fixed point in four dimensions. Compared with the previous
solution for the $SU(3) \times U(1)_R$-invariant flow [9], they share the common $\mathbb{CP}^2$ space inside the seven-dimensional internal space but three remaining coordinates are different from each other. This provides the symmetry breaking of $SU(3) \times U(1)_R$ into its subgroup $SU(3)$ along the whole RG flow in 11 dimensions.

- The mass deformation in dual gauge theory. The mass deformation of BL theory [32] has the fermion mass term in the Lagrangian [33]. For the $\mathcal{N} = 1$ supersymmetry, the bosonic mass terms consist of two independent terms when the 4-form $F_4$ vanishes in four-dimensional gauged supergravity [18]. One writes down the mass-deformed superpotential in the $\mathcal{N} = 1$ superfield notation

$$\Delta W = \frac{1}{2} m_7 \text{Tr} \Phi_7^2 + \frac{1}{2} m_8 \text{Tr} \Phi_8^2. \quad (2.24)$$

The original $\mathcal{N} = 1$ superpotential $W$ has quartic terms in $\Phi_I$ and comes from the D-term and F-term of the $\mathcal{N} = 2$ action [11, 14] and the superpotential $W$ has terms not having $(\Phi_7, \Phi_8)$, linear terms in $\Phi_7$, linear terms in $\Phi_8$ and terms that depend on $\Phi_7$ and $\Phi_8$. When we integrate out $(\Phi_7, \Phi_8)$ in the deformed $W + \Delta W$ with (2.24) at a low energy scale, we obtain the quartic terms coming from the original $W$ which do not contain mass parameters and two kinds of sextic terms with two independent parameters which depend on $(m_7, m_8)$ by solving the equations of motion for $(\Phi_7, \Phi_8)$ in $W + \Delta W$. This should flow to the superconformal field theory in the IR and the gravity dual shows that it will flow to the $\mathcal{N} = 2$ $SU(3) \times U(1)_R$-invariant fixed point. In the IR, when the two parameters are equal to each other in the resulting superpotential $\hat{W}$, it will flow to the $\mathcal{N} = 2$ $SU(3) \times U(1)_R$-invariant fixed point. At the IR critical point, the supersymmetry should be enhanced from $\mathcal{N} = 1$ to $\mathcal{N} = 2$ and there exists the $U(1)_R$ symmetry.

Thus, we have found the $\mathcal{N} = 1$ superconformal Chern–Simons matter theory with the global $SU(3)$ symmetry and expect that the $SU(3)$-invariant $U(N) \times U(N)$ Chern–Simons matter theory for $N > 2$ with $k = 1, 2$ is dual to the background of this paper with $N$ unit of flux. Namely, we have described the 11-dimensional uplift of the generic $\mathcal{N} = 1$ flow with unequal mass parameters found by the authors of [14]³.

³ Since the four-dimensional supersymmetric flow in [6, 14] involves four supergravity fields rather than two which we considered in this paper, one might ask what is the exact meaning of 11-dimensional uplift of [14]? In the context of four-dimensional $\mathcal{N} = 8$ gauged supergravity, the $SU(3)$-invariant sector can be realized by four real supergravity fields. The 11-dimensional metric with common $SU(3)$-invariance is constructed in [30] where the seven-dimensional internal metric is more complicated than the one in (2.12) due to the presence of four fields denoted by $(a, b, c, d)$ rather than two by $(a, c)$. The definition for $(a, b, c, d)$ [30] is given in terms of the fields in [27]. Along the constraints $b = \frac{3}{4} \rho^{-4}$ and $d = c = \cosh \gamma$, the supersymmetric flow characterized by four fields becomes the one in (1.2). Therefore, we have considered the four-dimensional superpotential on a restricted two-dimensional slice (rather than four) of the scalar manifold and the differential equations for the fields are the gradient flow equations of this superpotential.

What happens for the 11-dimensional point of view? Since the 11-dimensional metric is given in terms of four fields, the Ricci tensor can be obtained straightforwardly. This becomes the Ricci tensor (A.1) along the above constraints. What about 4-forms? One expects that the internal 4-forms look like (2.22) but the dependence on the four fields arises in various ways. For the 3-form with membrane indices, one should generalize the four-dimensional superpotential in the $SU(3)$ along the whole RG flow in 11 dimensions. Along the constraints, this 4-forms should be the same as the one in (2.22). Our work in this paper is the first task to complete the 11-dimensional uplift of the supersymmetric flow involving the four supergravity fields which is still an open problem.

In this sense, we have found the 11-dimensional solution for the four-dimensional supersymmetric flow presented in [6, 14] along the constraints and the corresponding $SU(3)$-invariant RG flow to the IR point of 3-dimensional dual gauge theory is the curve $(m_1 \neq m_2)$ connecting the $SO(8)$ point to the $SU(3) \times U(1)_R$ point in figure 1 of [14]. Recall that the $SU(3) \times U(1)_R$-invariant flow is the straight line $m_1 = m_2$ (with the horizontal axis $m_1$ and the vertical axis $m_2$) and its 11-dimensional uplift is found in [9]. The coordinates $(m_1, m_2)$ of [14] correspond to the previous $(m_7, m_8)$ in (2.24). Of course, at the IR (or UV) critical point, our solution is an exact 11-dimensional uplift of [14] because the values of four supergravity fields at the IR (or UV) critical point are located at the above constraints.
3. An $\mathcal{N} = 1$ $SU(2) \times U(1)$-invariant supersymmetric flow

Let us consider the five-dimensional Sasaki–Einstein space $Y^{p,q}$ used in [34] and it consists of the Einstein–Kahler twofold and the $U(1)$ bundle. Let us replace the $\mathbb{CP}^2$ metric (2.3) and the fiber (2.4) with the Einstein–Kahler twofold and the $U(1)$ bundle of $Y^{p,q}$ space, respectively. Then the following set of frames for the 11-dimensional metric can be written as

$$e^1 = \Delta^{-\frac{1}{2}} e^\rho \, dx^1, \quad e^2 = \Delta^{-\frac{1}{2}} e^\theta \, dx^2, \quad e^3 = \Delta^{-\frac{1}{2}} e^\phi \, dx^3, \quad e^4 = \Delta^{-\frac{1}{2}} dr,$$

$$e^5 = L \Delta^{\frac{4}{5}} \frac{\xi}{\rho^2} \left[ -\cos \theta \sin \theta_6 d\theta - \sin \theta \cos \theta_6 d\theta_6 \right],$$

$$e^6 = L \Delta^{\frac{1}{2}} \rho \sin \theta \sin \theta_6 \sqrt{1 - y^2} \frac{1}{6} d\theta_1,$$

$$e^7 = L \Delta^{\frac{1}{2}} \rho \sin \theta \sin \theta_6 \sqrt{1 - y^2} \sin \theta_1 d\phi_1,$$

$$e^8 = L \Delta^{\frac{1}{2}} \rho \sin \theta \sin \theta_6 \frac{1}{\sqrt{wq}} dy,$$

$$e^9 = L \Delta^{\frac{1}{2}} \rho \sin \theta \sin \theta_6 \frac{1}{6} \sqrt{wq} (d\beta + \cos \theta_1 d\phi_1),$$

$$e^{10} = L \Delta^{\frac{1}{2}} \xi^{-1} \sin \theta \sin \theta_6 \sqrt{1 - y^2} \sin^2 \theta_6 \left[ -\frac{1}{\rho^2} \frac{1}{(1 - \sin^2 \theta \sin^2 \theta_6)} \right],$$

$$+ \rho^2 \frac{1}{5} \left[ (d\theta_5 - \cos \theta_1 d\phi_1) + y (d\beta + \cos \theta_1 d\phi_1) \right],$$

$$e^{11} = L \Delta^{\frac{1}{2}} \xi^{-1} \cos \chi \left[ \cos \theta_6 d\theta - \sin \theta \cos \theta_6 d\theta_6 \right]$$

$$+ \sin^2 \theta \sin^2 \theta_6 \frac{1}{5} \left[ (d\theta_5 - \cos \theta_1 d\phi_1) + y (d\beta + \cos \theta_1 d\phi_1) \right],$$

where the $y$-dependent functions with some parameter $a$ are given by

$$w = \frac{2(a - y^2)}{1 - y^2}, \quad q = \frac{a - 3y^2 + 2y^3}{a - y^2}, \quad a \equiv \frac{1}{2} - \frac{(p^2 - 3q^2)}{4p^3} \sqrt{4p^2 - 3q^2}.$$

The quadratic form $\xi^2$ is given by (2.8), the warp factor $\Delta$ is given by (2.13) and the fiber on the Einstein–Kahler twofold is written in terms of the angular variables $\theta_5, \theta_1, \phi_1, y$ and $\beta$. See the frame $e^{10}$ and the frame $e^{11}$. There are $U(1)$ symmetries in $\beta$ and $\theta_5$. The AdS$_4$ supergravity fields $(\rho, \chi)$ and scale function $A$ have nontrivial $r$-dependence via (1.2) as before. The three coordinates [22] are related to those in this paper: $(\mu, \alpha)$ correspond to $(\theta, \theta_6)$ of (2.17) and $\psi = \theta_5$. That is, $\cos \mu = \sin \theta \sin \theta_6$ and $\cos \alpha = \frac{\cos \theta_6}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}}$.

Through the transformation, one can easily see that the above 11-dimensional metric (3.1) becomes exactly the one in [22]. In order to obtain the Ricci tensor and 4-form field strengths for the $SU(2) \times U(1)$-invariant flow, one needs the partial differentiations between these coordinates.

- The Ricci tensor and the 4-form field strengths along the flow. The Ricci tensor can be obtained from (3.1) directly or can be determined from the one preserving $SU(2) \times U(1) \times U(1)$ by using the transformation on the coordinates between the two coordinate systems. That is, the $SU(2) \times U(1)$-invariant Ricci tensor is given by $\tilde{R}^N_M (2.19)$ where the 11-dimensional coordinates are given by $\tilde{z}^M$ in (2.20) and $z = (x^1, x^2, x^3, r; \mu, \theta, \phi_1, y, \beta, \psi, \alpha)$. Only two of them are distinct and the Ricci tensor $R^Q_P$ for $SU(2) \times U(1) \times U(1)$-invariant flow is explicitly presented in
appendix C (C.1). The Ricci tensor (2.19) for $SU(2) \times U(1)$-invariant flow is explicitly given in appendix D (D.1) and there exist off-diagonal components $(4, 5), (4, 11), (5, 4), (5, 10), (5, 11), (7, 5), (7, 10), (7, 11), (9, 5), (9, 10), (9, 11), (10, 5), (10, 11) (11, 4), (11, 5)$ and $(11, 10)$. At the IR critical point in four dimensions, the components $(4, 5), (4, 11), (5, 4)$ and $(11, 4)$ vanish.

For the 4-form field strengths, one can use (2.21). These transformed 4-forms are given, in terms of those (C.2) in the $SU(2) \times U(1) \times U(1)_R$-invariant flow, in appendix D (D.2) and the transformed 4-forms with upper indices, in terms of (C.3), are described by (D.3). At the IR critical point in four dimensions, the following 4-forms also vanish: $\tilde{F}_{1235} = \tilde{F}_{12311} = \tilde{F}_{45mn} = \tilde{F}_{4nmp} = 0$. For the $SU(2) \times U(1) \times U(1)_R$-invariant flow, the $4$-forms $F_{1235}, F_{4nmp}$ and $F_{45mn}$ ($m, n, p = 6, \ldots, 11$) become zero at the IR critical point.

- Checking the Einstein equation. In [22], it is known that the solution by the Ricci tensor (C.1), the 4-forms with lower indices (C.2) and the 4-forms with upper indices (C.3) satisfies the field equation (1.3). The Ricci tensor can be written in terms of the quadratic 4-forms. This implies that the transformed-Ricci tensor can be written in terms of the quadratic 4-forms for the $SU(2) \times U(1) \times U(1)_R$-invariant flow. One can express the RHS of the Einstein equation in terms of quadratic 4-forms for the $SU(2) \times U(1) \times U(1)_R$-invariant flow from relation (2.21).

One can make the difference between the LHS and the RHS of the Einstein equation and see whether this difference is zero or not. Some of the components written in terms of quadratic 4-forms in the $SU(2) \times U(1) \times U(1)_R$-invariant flow are not exactly vanishing. They contain the terms

\[
F_{MPQR}F^{NPQR}, \quad M = 4, 5, N = 10, 11, \quad F_{6PQR}F^{NPQR}, \quad N = 4, 5, 7, 8, 9, 10, 11,
\]

\[
F_{7PQR}F^{NPQR}, \quad N = 4, 5, 6, 8, 9, \quad F_{8PQR}F^{NPQR}, \quad N = 4, 5, 9, 10, 11, \quad F_{9PQR}F^{NPQR}, \quad N = 4, 5, 8, \quad F_{MPQR}F^{NPQR}, \quad M = 10, 11, \quad N = 4, 5.
\]

After plugging the explicit solution for the lower 4-forms (C.2) and the upper 4-forms (C.3) of the $SU(2) \times U(1) \times U(1)_R$-invariant flow, all of these (3.2) are vanishing identically. Recall that these quadratic 4-forms correspond to the off-diagonal terms of the Einstein equation for the $SU(2) \times U(1) \times U(1)_R$-invariant flow which vanish identically except the $(4, 5)$-, $(5, 4)$-, $(7, 10)$-, $(7, 11)$-, $(9, 10)$-, $(9, 11)$-, $(10, 11)$- and $(11, 10)$-components from the Ricci tensor (C.1). The extra piece (3.2) does not possess these nonzero off-diagonal terms. Therefore, we have shown that the solution characterized by (2.19) and (2.21) for the $SU(2) \times U(1)$-invariant flow indeed satisfies the 11-dimensional Einstein equation.

- Checking the Maxwell equation. By using the Maxwell equation for the $SU(2) \times U(1) \times U(1)_R$-invariant flow, one can replace the LHS of the Maxwell equation with the quadratic 4-forms. Eventually, the nonzero components are explicitly given in the appendix (D.4). They are characterized by the following indices $(123), (4\hat{m}\tilde{n}\tilde{p}), (5\hat{n}\tilde{p}), (\tilde{m}\tilde{n}\tilde{p})$, where $\hat{m}, \tilde{n}, \tilde{p} = 6, \ldots, 11$, with the number of components 1, 9 by choosing two out of six, 9 by choosing two out of six and 14 by choosing three out of six, respectively. Other remaining components of the Maxwell equation become identically zero. There exist 33 nonzero components of the Maxwell equation. The RHS of the Maxwell equation for the $(123)$-component above consists of 12 terms coming from the quadratic 4-forms. For the other components, the RHS contains a single term, two terms or three terms in quadratic 4-forms. Similarly, one can express the RHS in terms of quadratic 4-forms for the $SU(2) \times U(1) \times U(1)_R$-invariant flow via (2.21). The difference between the LHS and the RHS of the Maxwell equation becomes zero. Therefore, we have shown that the
solution by the 4-forms (2.21) with 11-dimensional metric (3.1) and the 4-forms with upper indices indeed satisfies the Maxwell equation.

Therefore, we have shown that the solution by the Ricci tensor (D.1), the 4-forms with lower indices (D.2) and the 4-forms with upper indices (D.3) consists of an exact solution to 11-dimensional supergravity by bosonic field equations (1.3) and the deformation parameters \((\rho, \chi)\) and the domain wall amplitude \(A\) develop in the AdS\(_4\) radial direction along the \(\mathcal{N} = 1 \ SU(2) \times U(1)\)-invariant RG flow. Compared with the previous solution [22] for the \(SU(2) \times U(1) \times U(1)_{R}\)-invariant flow, they share the common Einstein–Kahler twofold inside the seven-dimensional internal space but two remaining coordinates are different from each other. This provides the symmetry breaking of \(SU(2) \times U(1) \times U(1)_{R}\) into its subgroup \(SU(2) \times U(1)\) along the whole RG flow.

4. Conclusions and outlook

We have found out four new 11-dimensional solutions of the \(\mathcal{N} = 1 \ SU(3)\)-invariant flow with the Ricci tensor (B.1), the 4-forms (B.2) and the 4-forms (B.3), the \(\mathcal{N} = 1 \ SU(2) \times U(1)\)-invariant flow with the Ricci tensor (D.1), the 4-forms (D.2) and the 4-forms (D.3). In four-dimensional sense, they have common RG flows given by (1.2). Inside of the seven-dimensional internal metric, they have their own Einstein–Kahler twofold and its \(U(1)\) bundle living in the five-dimensional Sasaki–Einstein space \((S^5, Y^{P,q})\) and share other two coordinates characterized by \((\theta, \theta_{\theta})\). Although the 4-forms are obtained from (2.19), one can determine them by constructing the corresponding 3-forms in frame basis directly described in [20, 22, 23].

- One can analyze the similar RG flow descriptions around the \(\mathcal{N} = 1 \ G_2\) critical point. According to the branching rules of \(G_2\) into its subgroups, one expects that the 11-dimensional solution should preserve \(SU(3)\) or \(SU(2) \times SU(2)\) symmetries. It would be interesting to find out the correct 4-forms for a given 11-dimensional metric. Furthermore, the most general five-dimensional Sasaki–Einstein can be considered and the global symmetries become the smaller \(SU(2) \times U(1)\) symmetry or the \(U(1) \times U(1)\) symmetry.

- It is an open problem to find the 11-dimensional lifts of holographic \(\mathcal{N} = 1\) supersymmetric RG flows [14] connecting from the \(\mathcal{N} = 1 \ G_2\) critical point to the \(\mathcal{N} = 2 \ SU(3) \times U(1)_{R}\) critical point. One can think of \(SU(3)-, \ SU(2) \times SU(2)-, \ SU(2) \times U(1)-\) and \(U(1) \times U(1)\)-invariant flows with the global coordinates for \(S^7\) appropriate for the base round 6-sphere \(S^6\) or those as the Hopf fibration on \(CP^3\) space. Along the supersymmetric \(\mathcal{N} = 1\) RG flows, one expects that there exist nontrivial 4-form field strengths and the work of [26] will be useful to obtain these 4-forms explicitly.

- It is an open problem to consider the case where there exist four supergravity fields preserving the \(SU(3)\) symmetry. In a particular limit, one has the 11-dimensional lift [9] of the \(\mathcal{N} = 2 \ SU(3) \times U(1)_{R}\)-invariant flow and for the other limit, one obtains the 11-dimensional lift [8] of the \(\mathcal{N} = 1 \ G_2\)-invariant flow. The decoding of the 4-forms written as the \(SU(3)\)-singlet vacuum expectation values, the covariant derivative in round 7-sphere \(S^7\) and the Killing vectors in [26] will also be useful.

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Appendix A. The SU(3) \times U(1)_R-invariant flow

In this appendix, we summarize the Ricci tensor and the 4-form field strengths for the SU(3) \times U(1)_R-invariant flow [9].

A.1. The Ricci tensor

The nonzero Ricci tensor in the coordinate basis from the 11-dimensional metric [9], after imposing the flow equations (1,2), can be written as follows:

\[ R^1_1 = -\frac{1}{3L^2 u^4 v (c_\mu^2 + u^2 s_\mu^2)^3} \left[ 2 \left( 2 u^8 v^2 (v^2 - 1) s_\mu^4 + 2 v^2 (v^2 + 3) c_\mu^4 + u^6 \left[ -2 (5 + c_\mu^2) ight. \right. \\
+ v^2 (-11 + c_\mu^2) + 4 v^6 c_\mu^2 v^2 (5 - 8 c_\mu^2) + v^4 (3 + 9 c_\mu^2) + 4 v^4 s_\mu^2 c_\mu^2 \\
+ u^4 \left( 6 s_\mu^2 v^2 + v^2 (5 - 8 c_\mu^2 + 5 c_\mu) + v^4 s_\mu^2 \right) \right] \\
= R_1^1 = R_3^3 = -2 R_6^6 = -2 R_7^7 = -2 R_8^8 = -2 R_9^9. \]

\[ R^2_1 = -\frac{1}{6L^2 u^4 v (c_\mu^2 + u^2 s_\mu^2)^3} \left[ -4 u^4 v^2 (2 u^4 - 21 u^2 + 27) c_\mu^4 + 4 u^8 v^2 (2 u^4 - 5 v^2 + 3) s_\mu^4 \\
+ 2 u^2 v^2 \left[ -48 + 60 c_\mu^2 + v^2 (15 - 43 c_\mu^2) + 4 v^4 s_\mu^2 \right] c_\mu^2 \\
- 2 u^6 \left[ 24 c_\mu^2 - 4 v^4 (7 + 4 c_\mu^2) + v^4 (17 + 5 c_\mu^2) - 4 v^6 c_\mu^2 \right] s_\mu^2 \\
- u^4 v^2 \left[ 33 - 48 c_\mu^2 + 27 c_\mu^2 + 4 v^4 \left[ 2 + 4 c_\mu^2 - 7 c_\mu^2 + v^2 (1 + c_\mu^2) \right] \right], \]

\[ R^3_1 = -\frac{1}{2L^3 (c_\mu^2 + u^2 s_\mu^2)^3} u^4 v (v^2 - 3) \\
+ u^4 \left[ -5 - 11 c_\mu^2 + 14 v^2 c_\mu^2 + u^2 (-11 + 9 c_\mu^2 + 2 c_\mu^2 \left[ u^2 - v^2 (u^2 - 7) \right] \right) s_\mu^2, \]

\[ R^4_1 = -\frac{1}{2L u^4 v (c_\mu^2 + u^2 s_\mu^2)^3} (-2 c_\mu^2 (v^2 - 3)) \\
+ u^4 \left[ -5 - 11 c_\mu^2 + 14 v^2 c_\mu^2 + u^2 \left[ u^2 - v^2 (u^2 - 7) \right] \right) s_\mu^2, \]

\[ R^5_1 = -\frac{1}{6L^2 u^4 v (c_\mu^2 + u^2 s_\mu^2)^3} \left[ 4 u^4 v^2 (v^2 - 1) s_\mu^4 + 4 v^2 (v^2 + 3) c_\mu^4 \\
+ u^4 \left[ 6 - 6 c_\mu^2 + v^2 (19 - 16 c_\mu^2 + c_\mu^2) + 5 v^4 (-1 + c_\mu^2) \right] \\
+ 6 u^4 \left[ 9 c_\mu^2 + v^2 (-5 - 5 c_\mu^2) + 2 v^4 s_\mu^2 \right] c_\mu^2 \\
+ 4 u^6 \left[ -1 - v^2 + v^4 + (-5 + v^2 + v^4) c_\mu^2 \right] s_\mu^2, \]

\[ R^8_1 = \frac{1}{4L^2 u^4 v (c_\mu^2 + u^2 s_\mu^2)^3} \left[ c_\mu^2 c_\mu^2 s_\mu^2 u^2 (v^2 - 1) \left( -8 s_\mu^4 u^8 - 24 c_\mu^2 v^2 \right) \\
- 12 s_\mu^4 u^4 (-3 + c_\mu^2 + 2 c_\mu^2 v^2) + 8 c_\mu^2 u^2 (-3 s_\mu^2 - 5 s_\mu^2 v^2 + 2 c_\mu^2 v^4) \\
+ u^4 \left[ 4 (-1 + 5 c_\mu^2) s_\mu^2 + (28 c_\mu^2 - 5 (3 + c_\mu^2)) v^2 + 8 (5 + c_\mu^2) s_\mu^2 v^4 \right]. \]
\[ R_{8}^{11} = - \frac{1}{L^{2}u^\frac{9}{2}(c_\mu^2 + u^2s_\mu^2)^\frac{5}{2}} \left[ 2c_\mu c_\mu^2 s_\mu^2 u \frac{9}{2} (v^2 - 1) (-s_\mu^4 u^6 + 2c_\mu^4 u^4 \\
- s_\mu^4 u^2 (-3 + (1 + 2c_\mu) v^2) + c_\mu^2 u^2 (3s_\mu^2 + (-1 + 2c_\mu) v^2 + 6s_\mu^2 v^4)) \right] \]

\[ R_{9}^{10} = - \frac{1}{4L^{2}u^\frac{9}{2}(c_\mu^2 + u^2s_\mu^2)^\frac{5}{2}} \left[ c_\mu c_\mu^2 u \frac{9}{2} (v^2 - 1) (-8s_\mu^4 u^8 - 24c_\mu^4 u^2 \\
- 12s_\mu^4 u^6 (-3 + c_\mu + 2c_\mu^2 v^2) + 8c_\mu^2 u^2 (-3s_\mu^2 - 5s_\mu^2 v^2 + 2c_\mu^2 v^4) \\
+ u^4 [4(-1 + 5c_\mu) s_\mu^2 + (28c_\mu - 5(3 + c_\mu) v) v^2 + 8(5 + c_\mu) s_\mu^2 v^2)] \right] \]

\[ R_{9}^{11} = \frac{2c_\mu c_\mu^2 s_\mu^2 u \frac{9}{2} (v^2 - 1)}{L^{2}u^\frac{9}{2}(c_\mu^2 + u^2s_\mu^2)^\frac{5}{2}} \left[ -s_\mu^4 u^6 + 2c_\mu^4 u^4 \\
- s_\mu^4 u^2 (-3 + (1 + 2c_\mu) v^2) + c_\mu^2 u^2 (3s_\mu^2 + (-1 + 2c_\mu) v^2 + 6s_\mu^2 v^4) \right] \]

\[ R_{10}^{10} = \left[ -1 + 2v(-3 + 4u^2) (-v^2 - 1 - v(-3 + 4v(v + \sqrt{v^2 - 1}))) \right] \left[ 48L^{2}u^\frac{9}{2}(c_\mu^2 + u^2s_\mu^2)^\frac{5}{2} (v^2 + \sqrt{v^2 - 1})^6 \\
32c_\mu^4 u^2 (v^2 + 3) + 32s_\mu^4 u^4 (v^2 - 1)(6c_\mu^2 + s_\mu^2 v^4) \\
+ 32c_\mu^2 u^2 [12c_\mu^2 + (15 + c_\mu) v^2 + 3s_\mu^2 v^4] - 4c_\mu^4 u^4 [36s_\mu^2 + 6(-1 + c_\mu) v^2 \\
+ (28c_\mu - 3(39 + 5c_\mu) v) v^4 + 4(7 + 14c_\mu + 3c_\mu^2) v^6] \\
- u^6 [24(1 - 5c_\mu) s_\mu^2 + 32c_\mu^2 (1 - 7c_\mu) v^2 \\
+ (-182 + 237c_\mu + 134c_\mu + 3c_\mu^2) v^4 \\
+ 16(1 + 3c_\mu) s_\mu^2 v^6] - 8s_\mu^2 u^8 [36c_\mu^2 (-3 + c_\mu)] \\
- v^2 (-1 - 848c_\mu + c_\mu + 3v^2 (5 + 16c_\mu + 3c_\mu^2 + s_\mu^2 v^2))] \right] \]

\[ R_{10}^{11} = \frac{4c_\mu c_\mu^2 u \frac{9}{2} (v^2 - 1)}{L^{2}u^\frac{9}{2}(c_\mu^2 + u^2s_\mu^2)^\frac{5}{2}} \left[ -s_\mu^4 u^6 + 2c_\mu^4 u^4 - s_\mu^4 u^2 (-3 + (1 + 2c_\mu) v^2) \\
+ c_\mu^2 u^2 (3s_\mu^2 + (-1 + 2c_\mu) v^2 + 6s_\mu^2 v^4) \right] \]

\[ R_{11}^{10} = \left[ -1 + 2v(-3 + 4u^2) (-v^2 - 1 + v(-3 + 4v(v + \sqrt{v^2 - 1}))) \right] \left[ 4L^{2}u^\frac{9}{2}(c_\mu^2 + u^2s_\mu^2)^\frac{5}{2} (v + \sqrt{v^2 - 1})^4 \\
u^2 (v^2 - 1) (-16c_\mu^4 s_\mu^4 u^8 + 16c_\mu^2 (3s_\mu^2 + (-4 + c_\mu) v^2) + 4s_\mu^4 u^6 (9 + 6c_\mu - 3c_\mu^2) \\
+ (1 - 14c_\mu - c_\mu^2) v^4) + 16c_\mu^2 u^2 (-2 + 5c_\mu) s_\mu^2 v^4 + (4 - 3c_\mu) s_\mu^2 v^2 + 2c_\mu u^4 \\
u^4 (45 + 6c_\mu) s_\mu^2 + (53 - 40c_\mu) c_\mu + 3c_\mu v^2 + 32(2 + c_\mu) s_\mu^2 v^4)] \right] \]

\[ R_{11}^{11} = \left[ -1 + 2v(-3 + 4u^2) (-v^2 - 1 + v(-3 + 4v(v + \sqrt{v^2 - 1}))) \right] \left[ 48L^{2}u^\frac{9}{2}(c_\mu^2 + u^2s_\mu^2)^\frac{5}{2} (v + \sqrt{v^2 - 1})^6 \\
32c_\mu^2 u^2 (v^2 + 3) + 32s_\mu^4 u^4 (v^2 - 1)(-6c_\mu^2 + s_\mu^2 v^4) \\
+ 32c_\mu^2 u^2 [6c_\mu^2 + (6 - 8c_\mu) v^2 + 3s_\mu^2 v^4] - 4c_\mu^4 u^4 [-36s_\mu^2 + 48s_\mu^2 v^2 \\
+ (-39 + 124c_\mu + 3c_\mu^2) v^4 - 4(11 + 10c_\mu + 3c_\mu^2) v^6] \\
+ 8s_\mu^2 u^8 [36c_\mu^2 (-3 + c_\mu) + 8(7 + 12c_\mu - c_\mu^2) v^2 \\
- 12(1 + 5c_\mu) v^4 + 3s_\mu^2 v^6] + u^6 [24(1 - 5c_\mu) s_\mu^2 v^2 + 32c_\mu^2 v^4] \\
+ u^8 [(36c_\mu^2 + 3) v^2 + 32c_\mu^2 v^4] \right] \]
where we introduce
\[ u \equiv \rho^4, \quad v \equiv \cosh \chi. \] (A.2)

There are additional nonzero Ricci tensor components \((R_8^{10}, R_8^{11}, R_9^{10} \text{ and } R_9^{11})\) that depend on the internal coordinates \(\theta_1\) or \(\theta_2\), compared to the Ricci tensor in the frame basis \([9, 22]\).

One also obtains (A.1) directly from the Ricci tensor in the frame basis with the help of vielbeins. At the IR fixed point \((u = \sqrt{3} \text{ and } v = \sqrt{\frac{3}{2}})\) in four dimensions, the off-diagonal components \(R_8^3\) and \(R_9^4\) vanish. We use a simplified notation for the trigonometric function as in \(s_\mu \equiv \sin \mu\) and so on.

### A.2. The 4-form field strengths

The nonzero 4-form field strengths satisfying (1.3) for a given Ricci tensor (A.1) and an 11-dimensional metric presented in [9] are summarized as follows:

\[
F_{1234} = \left[\frac{3e^{3A}}{L^4} \cosh^2 \chi \right] \left[ c_2^2 (-5 + \cosh 2\chi) + 2\rho^8 (-2 + c_{2\mu} + \rho^8 s_\mu^2 \sinh^2 \chi) \right],
\]

\[
F_{1235} = \left[\frac{3e^{3A}}{2\rho^2} \right] \left[ 1 + \cosh 2\chi + \rho^8 (-3 + \cosh 2\chi) \right] s_{2\mu},
\]

\[
F_{4567} = \left[\frac{3L^2}{2\rho^2} \right] \left[ c_2^2 c_{0\mu+3\nu+4\psi} s_{0\mu} \right] = -2s_{0\mu} c_{0\mu} c_{0\nu} s_{0\nu} F_{4568},
\]

\[
F_{4568} = \left[\frac{3L^2}{2\rho^2} \right] \left[ c_2^2 s_{0\mu+3\nu+4\psi} s_{0\mu} s_{0\nu} \right] = 2s_{0\mu} c_{0\mu} c_{0\nu} c_{0\nu} F_{4578},
\]

\[
F_{4678} = \left[\frac{3L^2}{16c_2^2 + \rho^8 s_\mu^2} \right] \left[ 2c_2^2 (-2 + \cosh 2\chi) + \rho^8 (-5 + c_{2\mu} \cosh 2\chi + 2\rho^8 s_\mu^2) \right]
\]

\[
\times s_{0\mu} c_{0\mu} c_{0\nu} s_{0\nu+3\psi+4\phi} s_{0\nu} = c_{0\mu} F_{4679} = \frac{1}{2} s_{0\mu} c_{0\mu} F_{46710} = -s_{0\mu} c_{0\mu} F_{46910},
\]

\[
F_{46711} = \left[\frac{3L^2}{4\rho^2 (c_2^2 + \rho^8 s_\mu^2)^2} \right] \times \left[ 6c_2^2 + \rho^8 (1 - 7c_{2\mu} + 2 \cosh 2\chi + \rho^8 (-9 + 5c_{2\mu} + 2 \cosh 2\chi + 2\rho^8 s_\mu^2)) \right]
\]

\[
\times s_{0\mu} c_{0\mu} s_{0\nu} c_{0\nu+3\psi+4\phi} s_{0\nu} = -2s_{0\mu} c_{0\mu} c_{0\nu} c_{0\nu} F_{46911},
\]

\[
F_{4689} = -\left[\frac{3L^2}{16(c_2^2 + \rho^8 s_\mu^2)^2} \right] \times \left[ 2c_2^2 (-2 + \cosh 2\chi) + \rho^8 (-5 + 9c_{2\mu} + 2 \cosh 2\chi + 2\rho^8 s_\mu^2) \right]
\]

\[
\times s_{0\mu} c_{0\mu} c_{0\nu} s_{0\nu+3\psi+4\phi} s_{0\nu} = \frac{1}{2} s_{0\mu} c_{0\mu} F_{46910} = c_{0\mu} F_{468910} = s_{0\mu} s_{0\nu} F_{46910},
\]

\[
F_{46811} = -\left[\frac{3L^2}{4\rho^2 (c_2^2 + \rho^8 s_\mu^2)^2} \right] \times \left[ 6c_2^2 + \rho^8 (1 - 7c_{2\mu} + 2 \cosh 2\chi + \rho^8 (-9 + 5c_{2\mu} + 2 \cosh 2\chi + 2\rho^8 s_\mu^2)) \right]
\]
\[
F_{5678} = \left[ \frac{3L^3 \tanh \chi}{8(c^2_\mu + \rho^8 s^8_\mu)} \right] \left[ 6c^4_\mu + \rho^8 c^2_\mu (9 - 7c_{2\mu}) + 10 \rho^{16} s^4_\mu - 2 \rho^{24} s^4_\mu \right] \\
\times s^1_0 s^1_0 s^1_\mu c_\mu c_{\theta_0 + 3\phi + 4\psi} = c_{\theta_0} F_{5679} = \frac{1}{2} s^1_\mu c^1_0 F_{5679} = s^1_\mu c^1_0 F_{5809}.
\]

\[
F_{5671} = \left[ \frac{3L^3 \tanh \chi}{8(c^2_\mu + \rho^8 s^8_\mu)} \right] \\
\times \{ 4c^4_\mu (2 + c_{2\mu}) + \rho^8 c^2_\mu (13 - 6c_{2\mu} - 3c_{4\mu}) + 12 \rho^{16} (2 + c_{2\mu}) s^4_\mu - 4 \rho^{24} s^4_\mu c_{2\mu} \}
\times s^1_0 c^1_0 s^1_\mu c_{\theta_0 + 3\phi + 4\psi} = -2s^1_\mu c^1_0 F_{5809}.
\]

\[
F_{5689} = -\left[ \frac{3L^3 \tanh \chi}{8(c^2_\mu + \rho^8 s^8_\mu)} \right] \left[ 6c^4_\mu + \rho^8 c^2_\mu (9 - 7c_{2\mu}) + 10 \rho^{16} s^4_\mu - 2 \rho^{24} s^4_\mu \right] \\
\times s^1_0 s^1_0 c_\mu c_{\theta_0 + 3\phi + 4\psi} = \frac{1}{2} s^1_\mu F_{5680} = s^1_\mu c^1_0 F_{5780} = s^1_\mu c^1_0 F_{5790}.
\]

\[
F_{5681} = \left[ \frac{3L^3 \tanh \chi}{4(c^2_\mu + \rho^8 s^8_\mu)} \right] \left[ -2c^4_\mu (2 + c_{2\mu}) + \rho^8 (-5 + 2c_{2\mu} + c_{4\mu} + 2 \rho^8 c_{2\mu} s^4_\mu) \right] \\
\times s^1_0 s^2_0 c^2_\mu c_{\theta_0 + 3\phi + 4\psi}.
\]

\[
F_{5781} = \left[ \frac{3L^3 \tanh \chi}{16(c^2_\mu + \rho^8 s^8_\mu)} \right] s^1_0 s^2_0 c^1_\mu c^1_0 c^1_\mu c_{\theta_0 + 3\phi + 4\psi} \left[ -4c^4_\mu (2 + c_{2\mu}) \\
+ \rho^8 c^2_\mu (-13 + 6c_{2\mu} + 3c_{4\mu}) - 12 \rho^{16} (2 + c_{2\mu}) s^4_\mu + 4 \rho^{24} c_{2\mu} s^4_\mu \right]
\]

\[
\times s^1_\mu F_{5791} = s^1_\mu c^1_0 F_{5801}.
\]

\[
F_{6781} = -\left[ \frac{3L^3 (3 + \rho^8) \tanh \chi}{4(c^2_\mu + \rho^8 s^8_\mu)} \right] c^3_\mu s^3_\mu c_{\theta_0 + 3\phi + 4\psi} s^1_0 c^1_0 = c_{\theta_0} F_{6791} = \frac{1}{2} s^1_\mu c^1_0 F_{6791}.
\]

\[
F_{6801} = s^1_\mu c^1_0 F_{5801} = \frac{1}{2} s^1_\mu F_{5801}.
\]

\[
F_{6891} = -\left[ \frac{3L^3 (3 + \rho^8) \tanh \chi}{4(c^2_\mu + \rho^8 s^8_\mu)} \right] c^3_\mu s^3_\mu c_{\theta_0 + 3\phi + 4\psi} s^1_0 c^1_0 = \frac{1}{2} s^1_\mu F_{6801}.
\]

\[
\text{(A.3)}
\]

One sees that the 4-forms (A.3) have the dependence on \((\theta_4 + 3\phi + 4\psi)\). According to the shifts \(\phi \to \frac{1}{2} \chi\) and \(\psi \to \psi - \chi\), corresponding to the \(U(1)_e\) charge, it is evident that 4-forms preserve this \(U(1)_e\) charge. Note that \(\theta_4\) is one of the Euler angles on \(S^3\) in (2.2). At the IR fixed point in four dimensions, the components \(F_{1235}, F_{3589}\) and \(F_{5890}\) vanish. By using the 4-form field strengths in the frame basis [9, 20] and vielbeins, one also obtains (A.3).

The 4-form field strengths with upper indices can be obtained from those with lower indices (A.3) by multiplying the 11-dimensional inverse metric and they are given as follows:

\[
F^{1234} = -\left[ \frac{3e^{-3A} \rho^4}{L \cosh \frac{\omega}{2} \chi \left( c^2_\mu + \rho^8 s^8_\mu \right)^{\frac{1}{2}}} \right] \left[ c^2_\mu (-5 + \cosh 2 \chi) + 2 \rho^8 (-2 + c_{2\mu} + \rho^8 s^2_\mu \sinh^2 \chi) \right],
\]

\[
F^{1235} = -\left[ \frac{3e^{-3A} \rho^4 \sech \frac{\omega}{2} \chi}{2L^2 \left( c^2_\mu + \rho^8 s^8_\mu \right)^{\frac{1}{2}}} \right] \left[ 1 + \cosh 2 \chi + \rho^8 (-3 + \cosh 2 \chi) \right] s_{2\mu}.
\]
\[
F^{4567} = \frac{6(-3 + \rho^8) \tanh \chi}{L^3 \rho^2 \sinh \frac{\chi}{2} (c_\mu^2 + \rho^8 s_\mu^2)^2} \left[ c_{i_1} c_{i_2} + 4 \rho^8 s_\mu^2 \right] s_{i_1}^{-1} s_{i_2}^{-1} c_{i_2}^{-2} = \frac{1}{2} s_{i_1} c_{i_2} s_{i_2} F^{4589}.
\]

\[
F^{4568} = \frac{6(-3 + \rho^8) \tanh \chi}{L^3 \rho^2 \sinh \frac{\chi}{2} (c_\mu^2 + \rho^8 s_\mu^2)^2} \left[ s_{i_1} c_{i_2} + 4 \rho^8 s_\mu^2 \right] s_{i_1}^{-1} s_{i_2}^{-1} c_{i_2}^{-2} = -c_{i_2} F^{4569}.
\]

\[
F^{4671} = \frac{3}{2L^4 \rho^2 \cosh \frac{\chi}{2} (c_\mu^2 + \rho^8 s_\mu^2)^2} \left[ -s_{i_1} c_{i_2} \tanh \chi (1 + \cosh 2 \chi + \rho^8 (-3 + \cosh 2 \chi)) + 2(\rho^8 - 3) \sinh 2 \chi \left( c_\mu^2 + \rho^8 s_\mu^2 \right) s_{i_1}^{-1} s_{i_2}^{-1} c_{i_2}^{-3} s_{i_3} c_{i_3} + 4 \rho^8 \sinh 2 \chi = \frac{1}{2} s_{i_1} c_{i_2} s_{i_2} F^{4891}.
\]

\[
F^{4681} = \frac{3}{2L^4 \rho^2 \cosh \frac{\chi}{2} (c_\mu^2 + \rho^8 s_\mu^2)^2} \left[ -s_{i_1} c_{i_2} \tanh \chi (1 + \cosh 2 \chi + \rho^8 (-3 + \cosh 2 \chi)) + 2(\rho^8 - 3) \sinh 2 \chi \left( c_\mu^2 + \rho^8 s_\mu^2 \right) s_{i_1}^{-1} s_{i_2}^{-1} c_{i_2}^{-3} s_{i_3} c_{i_3} + 4 \rho^8 \sinh 2 \chi = \frac{1}{2} s_{i_1} c_{i_2} s_{i_2} F^{4891}.
\]

\[
F^{4682} = \frac{3}{2L^4 \rho^2 \cosh \frac{\chi}{2} (c_\mu^2 + \rho^8 s_\mu^2)^2} \left[ -s_{i_1} c_{i_2} \tanh \chi (1 + \cosh 2 \chi + \rho^8 (-3 + \cosh 2 \chi)) + 2(\rho^8 - 3) \sinh 2 \chi \left( c_\mu^2 + \rho^8 s_\mu^2 \right) s_{i_1}^{-1} s_{i_2}^{-1} c_{i_2}^{-3} s_{i_3} c_{i_3} + 4 \rho^8 \sinh 2 \chi = \frac{1}{2} s_{i_1} c_{i_2} s_{i_2} F^{4891}.
\]

\[
F^{5671} = \frac{6(-3 + \rho^8) \sinh \chi}{L^5 \rho^2 \left( c_\mu^2 + \rho^8 s_\mu^2 \right)^2} \left[ 3c_\mu s_{i_1} \sinh 2 \chi + 4 \rho^8 (2 - c_{i_2} c_{i_3}) s_{i_1}^{-1} c_{i_2}^{-1} \sinh 2 \chi + \rho^8 (2 - 3c_{i_2} c_{i_3}) s_{i_1}^{-1} c_{i_2}^{-1} \sinh 2 \chi = -c_{i_2} F^{4691}.
\]

\[
F^{5671} = \frac{6(-3 + \rho^8) \sinh \chi}{L^5 \rho^2 \left( c_\mu^2 + \rho^8 s_\mu^2 \right)^2} \left[ 3c_\mu s_{i_1} \sinh 2 \chi + 4 \rho^8 (2 - c_{i_2} c_{i_3}) s_{i_1}^{-1} c_{i_2}^{-1} \sinh 2 \chi + \rho^8 (2 - 3c_{i_2} c_{i_3}) s_{i_1}^{-1} c_{i_2}^{-1} \sinh 2 \chi = -c_{i_2} F^{4691}.
\]

\[
F^{5671} = \frac{6(-3 + \rho^8) \sinh \chi}{L^5 \rho^2 \left( c_\mu^2 + \rho^8 s_\mu^2 \right)^2} \left[ 3c_\mu s_{i_1} \sinh 2 \chi + 4 \rho^8 (2 - c_{i_2} c_{i_3}) s_{i_1}^{-1} c_{i_2}^{-1} \sinh 2 \chi + \rho^8 (2 - 3c_{i_2} c_{i_3}) s_{i_1}^{-1} c_{i_2}^{-1} \sinh 2 \chi = -c_{i_2} F^{4691}.
\]
In this appendix, we describe the Ricci tensor and the 4-form field strengths for the invariant flow that can be written in terms of corresponding Ricci tensor and 4-form field strengths for the IR fixed point in four dimensions, the components of the 4-form field strengths vanish as \( F^{1235}, F^{4\mu\nu\rho} \) and \( F^{8\mu\nu\rho\sigma} \) vanish as before. It turns out that after computing the RHS of (1.3) using both (A.3) and (A.4), the dependence on the combination \((\theta_2 + 3\phi + 4\psi)\) disappears completely. This coincides with the fact that the Ricci tensor (A.1) does not depend on these variables.

### Appendix B. The SU(3)-invariant flow

In this appendix, we describe the Ricci tensor and the 4-form field strengths for the SU(3)-invariant flow that can be written in terms of corresponding Ricci tensor and 4-form field strengths respectively for the SU(3) \( \times U(1)_R \)-invariant flow.

#### B.1. The Ricci tensor

The Ricci tensor in the coordinate basis from the 11-dimensional metric (2.16), after imposing the flow equations (1.2), can be written in terms of those (A.1) in appendix A as follows:

\[
\begin{align*}
\tilde{R}_1 &= R_1, \\
\tilde{R}_2 &= R_2, \\
\tilde{R}_3 &= R_3, \\
\tilde{R}_4 &= R_4, \\
\tilde{R}_4^5 &= -\left[\frac{\cos \theta \sin \theta_2}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_2}}\right] R_4^5, \\
\tilde{R}_4^{11} &= -\left[\frac{\csc \theta \cos \theta_2}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_2}}\right] R_4^5.
\end{align*}
\]

One also obtains (A.4) from the 4-form field strengths in the frame basis [9, 20] and vielbeins. At the IR fixed point in four dimensions, the components \( F^{1235}, F^{4\mu\nu\rho} \) and \( F^{8\mu\nu\rho\sigma} \) vanish as before. It turns out that after computing the RHS of (1.3) using both (A.3) and (A.4), the dependence on the combination \((\theta_2 + 3\phi + 4\psi)\) disappears completely. This coincides with the fact that the Ricci tensor (A.1) does not depend on these variables.
There are extra nonzero off-diagonal Ricci tensor components \( \tilde{R}^5 \), \( \tilde{R}^{10} \), \( \tilde{R}^{11} \), \( \tilde{R}^6 \), \( \tilde{R}^7 \), \( \tilde{R}^8 \) and \( \tilde{R}^{11} \) for the SU(3)\(_e\) and \( \tilde{R}^5 \) for the SU(3)\(_g\) invariance. Also the 11-dimensional metric (2.16) generates (B.1) directly. At the IR fixed point in four dimensions, the components \( \tilde{R}^5, \tilde{R}^{11} \) and \( \tilde{R}^{11} \) vanish.

\[ \tilde{R}^5 = \left[ \frac{1}{1 + \cot^2 \theta \sec^2 \theta} \right] R^5 - \left[ \cos^2 \theta \frac{\theta}{1 - \sin^2 \theta \sin^2 \theta} \right] (R_{10}^{11} - R_{11}^{11}), \]

\[ \tilde{R}^{10} = - \left[ \frac{\cos \theta}{1 - \sin^2 \theta \sin^2 \theta} \right] (R_{10}^{10} + R_{11}^{11} - R_{11}^{10} + R_{11}^{11}), \]

\[ \tilde{R}^{11} = \left[ \frac{2 \cot \theta \sin 2 \theta}{3 + \cos 2 \theta + 2 \cos 2 \theta \sin^2 \theta} \right] (R^5 + R_{10}^{11} - R_{11}^{11}), \]

\[ \tilde{R}^6 = R^6, \quad \tilde{R}^7 = R^7, \]

\[ \tilde{R}^8 = [\cos \theta_0] R^8, \quad \tilde{R}^{10} = R_{10}^{10}, \quad \tilde{R}^{11} = [\cos \theta_0] R_{10}^{11}, \quad \tilde{R}^{10} = R_{10}^{10} + R_{10}^{11}, \quad \tilde{R}^{11} = [\cos \theta \sin \theta] R_{10}^{11}, \]

\[ \tilde{R}^4 = - \left[ \frac{\cos \theta \cos \theta_0}{\sqrt{1 - \sin^2 \theta \sin^2 \theta}} \right] R^5, \]

\[ \tilde{R}^5 = \frac{3 + \cos 2 \theta + 2 \cos 2 \theta \sin^2 \theta}{1 - \sin^2 \theta \sin^2 \theta} (R^5 - R_{10}^{11} - R_{11}^{11}), \]

\[ \tilde{R}^{11} = \left[ \frac{\cos \theta \sin \theta \sin \theta_0}{1 - \sin^2 \theta \sin^2 \theta_0} \right] (R_{10}^{10} - R_{11}^{10} - R_{11}^{11}), \]

\[ \tilde{R}^{11} = \left[ \frac{1}{1 + \tan^2 \theta \cos^2 \theta} \right] R^5 - \left[ \cos^2 \theta \sin^2 \theta \frac{\theta}{1 - \sin^2 \theta \sin^2 \theta} \right] (R_{10}^{11} - R_{11}^{11}). \]  

(B.1)

B.2. The 4-form field strengths

The 4-form field strengths satisfying (1.3) for a given Ricci tensor (B.1) and an 11-dimensional metric (2.16), in terms of those (A.3) for the SU(3)\(_g\) invariance, are summarized as follows:

\[ \tilde{F}_{1234} = F_{1234}, \quad \tilde{F}_{1235} = \left[ \frac{\cos \theta \sin \theta_0}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_0}} \right] F_{1235}, \]

\[ \tilde{F}_{12311} = \left[ \frac{\cos \theta \sin \theta_0}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_0}} \right] F_{1235}, \]

\[ \tilde{F}_{45mn} = \left[ \frac{\cos \theta_0}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_0}} \right] F_{45mn} = \left[ \frac{\cos \theta_0}{1 - \sin^2 \theta \sin^2 \theta_0} \right] (F_{4mn10} - F_{4mn11}) \]

\( (m, n = 6, \ldots, 10), \)

\[ \tilde{F}_{4mnp} = \left[ \frac{\sin \theta \cos \theta_0}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_0}} \right] F_{45mn} + \left[ \frac{\sin \theta \cos \theta \sin \theta_0}{1 - \sin^2 \theta \sin^2 \theta_0} \right] (F_{4mn10} - F_{4mn11}) \]

\( (m, n = 6, \ldots, 10), \)
\[
F_{5\mu
\rho
\nu\lambda} = -\frac{\cos \theta \sin \theta_0}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_0}} F_{5\mu
\rho
\nu\lambda}
\]

\[
F_{5\mu
\nu\lambda\rho} = \frac{\cos \theta_0}{1 - \sin^2 \theta \sin^2 \theta_0} F_{5\mu
\nu\lambda\rho}
\]

\[
(m, n, p = 6, \ldots, 10), \quad (B.2)
\]

\[
\bar{F}_{5\mu
\nu\lambda\rho} = -\left[ \frac{5 \sin \theta + 4 \cos 2\theta_0 \sin^3 \theta + \sin 3\theta}{8(1 - \sin^2 \theta \sin^2 \theta_0)^2} \right] (F_{5\mu
\nu\lambda\rho} - F_{5\mu
\nu\lambda\rho}^\ast) \quad (m, n = 6, \ldots, 9),
\]

\[
\bar{F}_{m\nu\lambda\rho} = \frac{\sin \theta \cos \theta_0}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_0}} F_{m\nu\lambda\rho}
\]

\[
F_{m\nu\lambda\rho} = \frac{\sin \theta_0 \cos \theta \sin \theta_0}{1 - \sin^2 \theta \sin^2 \theta_0} F_{m\nu\lambda\rho}
\]

\[
(m, n, p = 6, \ldots, 10).
\]

The \( \bar{F}_{1231} \) is new, compared to the \( SU(3) \times U(1)_g \)-invariant flow. At the IR fixed point in four dimensions, the components \( \bar{F}_{1235}, \bar{F}_{1231}, \bar{F}_{4\mu\nu\lambda\rho} \) and \( \bar{F}_{5\mu\nu\lambda\rho} \) vanish.

The 4-form field strengths with upper indices can be obtained from those with lower indices (B.2) by multiplying the 11-dimensional inverse metric (2.16) and they are given as follows:

\[
\bar{F}^{1234}_{1234}, \quad \bar{F}^{1235}_{1235} = -\left[ \frac{\cos \theta \sin \theta_0}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_0}} \right] F^{1235}
\]

\[
\bar{F}^{1231}_{1231} = -\left[ \frac{\csc \theta \cos \theta_0}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_0}} \right] F^{1231}
\]

\[
\bar{F}^{45mn}_{45mn} = [\cos \theta_0] F^{4mn11} - \left[ \frac{\cos \theta \sin \theta_0}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_0}} \right] F^{45mn} \quad (m, n = 6, \ldots, 9),
\]

\[
\bar{F}^{45m0}_{45m0} = [\cos \theta_0] F^{4m011} - \left[ \frac{\cos \theta \sin \theta_0}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_0}} \right] (F^{45m0} + F^{45m0}) \quad (m = 6, \ldots, 9),
\]

\[
\bar{F}^{45m11}_{45m11} = [\cos \theta_0 \sqrt{1 - \sin^2 \theta \sin^2 \theta_0}] F^{45m11} \quad (m = 6, \ldots, 10),
\]

\[
\bar{F}^{4mn11}_{4mn11} = F^{4mn11} + F^{4mn11} \quad (m, n = 6, \ldots, 9),
\]

\[
\bar{F}^{4mn11}_{4mn11} = -[\sin \theta_0 \cot \theta] F^{4mn11} - \left[ \frac{\csc \theta \cos \theta_0}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_0}} \right] F^{45mn} \quad (m, n = 6, \ldots, 10),
\]

\[
\bar{F}^{5mn10}_{5mn10} = -[\cos \theta_0] F^{5mn10} - \left[ \frac{\cos \theta \sin \theta_0}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_0}} \right] (F^{5mn10} + F^{5mn10}) \quad (m = 6, \ldots, 9),
\]

\[
\bar{F}^{5mn11}_{5mn11} = [\csc \theta \sqrt{1 - \sin^2 \theta \sin^2 \theta_0}] F^{5mn11} \quad (m, n = 6, \ldots, 10),
\]

\[
\bar{F}^{mn1011}_{mn1011} = -[\cot \theta \sin \theta_0] F^{mn1011} + \left[ \frac{\csc \theta \cos \theta_0}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_0}} \right] (F^{5mn10} + F^{5mn11}) \quad (m, n, p = 6, \ldots, 9), \quad (B.3)
\]

There exists a new 4-form \( \bar{F}_{1231} \), compared to the 4-form field strengths (A.4) for the \( SU(3) \times U(1)_g \)-invariant flow. At the IR fixed point in four dimensions, the components \( \bar{F}_{1235}, \bar{F}_{1231}, \bar{F}_{4\mu\nu\lambda\rho} \) and \( \bar{F}_{5\mu\nu\lambda\rho} \) vanish. By substituting the Ricci tensor (B.1) and 4-form field strengths (B.2) and (B.3) into the Einstein equation in (1.3), one can check that the LHS coincides exactly with the RHS. In doing this, one uses the fact that the solution characterized by (A.1), (A.3) and (A.4) for the \( SU(3) \times U(1)_g \)-invariant flow really satisfies the Einstein equation. Therefore one concludes that the Einstein equation for the \( SU(3) \)-invariant flow is satisfied.
B.3. The left-hand side of the Maxwell equation

By introducing the notation
\[ \frac{1}{2} E_M \tilde{F}^{MNPQ} = (NPQ), \]
let us present all the nonzero components of left-hand side of Maxwell equations, in terms of 4-forms (A.3), as follows:

\begin{align*}
(123) &= -F_{489\,11} F_{567\,10} + F_{489\,10} F_{567\,11} + F_{479\,11} F_{568\,10} - F_{479\,10} F_{568\,11} \\
&\quad + F_{468\,11} F_{579\,10} - F_{468\,10} F_{579\,11} - F_{467\,11} F_{589\,10} + F_{467\,10} F_{589\,11} \\
&\quad - F_{589\,11} F_{679\,10} + F_{589\,10} F_{679\,11} + F_{4589\,11} F_{679\,10} - F_{4589\,10} F_{679\,11}.
\end{align*}

\begin{align*}
(467) &= F_{1235} F_{489\,10\,11}, & (468) &= -F_{1235} F_{79\,10\,11}, & (469) &= F_{1235} F_{78\,10\,11}, \\
(479) &= -F_{1235} F_{68\,10\,11}, & (4710) &= F_{1235} F_{68\,11\,11}, & (489) &= F_{1235} F_{67\,10\,11}, \\
(4810) &= -F_{1235} F_{67\,9\,11}, & (4910) &= F_{1235} F_{67\,8\,11},
\end{align*}

\begin{align*}
(567) &= [\cos \theta]_b (F_{1235} F_{489\,10\,11} - F_{1234} F_{589\,10\,11}) + \left[ \frac{\cos \theta \sin \theta_b}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_b}} \right] F_{1234} F_{689\,10\,11},
\end{align*}

\begin{align*}
(568) &= -[\cos \theta]_b (F_{1235} F_{479\,10\,11} - F_{1234} F_{579\,10\,11}) - \left[ \frac{\cos \theta \sin \theta_b}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_b}} \right] F_{1234} F_{679\,10\,11},
\end{align*}

\begin{align*}
(569) &= [\cos \theta]_b (F_{1235} F_{478\,10\,11} - F_{1234} F_{578\,10\,11}) + \left[ \frac{\cos \theta \sin \theta_b}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_b}} \right] F_{1234} F_{678\,10\,11},
\end{align*}

\begin{align*}
(579) &= -[\cos \theta]_b (F_{1235} F_{468\,10\,11} - F_{1234} F_{568\,10\,11}) - \left[ \frac{\cos \theta \sin \theta_b}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_b}} \right] F_{1234} F_{668\,10\,11},
\end{align*}

\begin{align*}
(5710) &= [\cos \theta]_b (F_{1235} F_{468\,10\,11} - F_{1234} F_{568\,10\,11}) + \left[ \frac{\cos \theta \sin \theta_b}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_b}} \right] F_{1234} F_{668\,11\,11},
\end{align*}

\begin{align*}
(589) &= [\cos \theta]_b (F_{1235} F_{467\,10\,11} - F_{1234} F_{567\,10\,11}) + \left[ \frac{\cos \theta \sin \theta_b}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_b}} \right] F_{1234} F_{667\,10\,11},
\end{align*}

\begin{align*}
(5810) &= -[\cos \theta]_b (F_{1235} F_{467\,10\,11} - F_{1234} F_{567\,10\,11}) - \left[ \frac{\cos \theta \sin \theta_b}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_b}} \right] F_{1234} F_{667\,11\,11},
\end{align*}

\begin{align*}
(5910) &= [\cos \theta]_b (F_{1235} F_{467\,10\,11} - F_{1234} F_{567\,10\,11}) + \left[ \frac{\cos \theta \sin \theta_b}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_b}} \right] F_{1234} F_{667\,10\,11},
\end{align*}

\begin{align*}
(6711) &= -[\cot \theta \sin \theta_b] (F_{1235} F_{489\,10} + F_{1234} F_{589\,10}) + \left[ \frac{\csc \theta \cos \theta_b}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_b}} \right] F_{1234} F_{689\,10\,11},
\end{align*}

\begin{align*}
(6810) &= F_{1235} (-F_{479\,10} + F_{479\,11}) + F_{1234} (F_{579\,10} - F_{579\,11}),
\end{align*}

\begin{align*}
(6811) &= [\cot \theta \sin \theta_b] (F_{1235} F_{479\,10\,11} - F_{1234} F_{579\,10\,11}) - \left[ \frac{\csc \theta \cos \theta_b}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_b}} \right] F_{1234} F_{679\,10\,11},
\end{align*}

\begin{align*}
(6910) &= F_{1235} (F_{478\,10\,11} - F_{478\,11\,11}) + F_{1234} (-F_{578\,10\,11} + F_{578\,11\,11}),
\end{align*}

\begin{align*}
(6911) &= [\cot \theta \sin \theta_b] (-F_{1235} F_{478\,10\,11} + F_{1234} F_{578\,10\,11}) + \left[ \frac{\csc \theta \cos \theta_b}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_b}} \right] F_{1234} F_{678\,10\,11},
\end{align*}

\begin{align*}
(7910) &= F_{1235} (-F_{689\,10\,11} + F_{689\,11\,11}) + F_{1234} (F_{668\,10\,11} - F_{668\,11\,11}),
\end{align*}
\begin{equation}
(79 \, 11) = [\cot \theta \sin \theta_6 (F_{1235}F_{468.10} - F_{1234}F_{568.10}) - \csc \theta \cos \theta_6 \left( \sqrt{1 - \sin^2 \theta \sin^2 \theta_6} \right) F_{1234}F_{68.10} \, 11].
\end{equation}

\begin{equation}
(7 \, 10 \, 11) = [\cot \theta \sin \theta_6 (-F_{1235}F_{4689} + F_{1234}F_{5689}) + \csc \theta \cos \theta_6 \left( \sqrt{1 - \sin^2 \theta \sin^2 \theta_6} \right) F_{1234}F_{689.11}]
\end{equation}

\begin{equation}
(89 \, 10) = F_{1235}(F_{467.10} - F_{467.11}) + F_{1234}(-F_{567.10} + F_{567.11}),
\end{equation}

\begin{equation}
(89 \, 11) = [\cot \theta \sin \theta_6 (-F_{1235}F_{467.10} + F_{1234}F_{567.10}) + \csc \theta \cos \theta_6 \left( \sqrt{1 - \sin^2 \theta \sin^2 \theta_6} \right) F_{1234}F_{67.10} \, 11].
\end{equation}

\begin{equation}
(8 \, 10 \, 11) = [\cot \theta \sin \theta_6 (F_{1235}F_{4679} - F_{1234}F_{5679}) - \csc \theta \cos \theta_6 \left( \sqrt{1 - \sin^2 \theta \sin^2 \theta_6} \right) F_{1234}F_{679.11}]
\end{equation}

\begin{equation}
(9 \, 10 \, 11) = [\cot \theta \sin \theta_6 (-F_{1235}F_{4678} + F_{1234}F_{5678}) + \csc \theta \cos \theta_6 \left( \sqrt{1 - \sin^2 \theta \sin^2 \theta_6} \right) F_{1234}F_{678.11}].
\end{equation}

(B.4)

One can easily check that the RHS of the Maxwell equation (1.3) with (A.3) and (2.16) is exactly coincident with the above LHS of the Maxwell equation (B.4) as we explained in section 2.

**Appendix C. The SU(2) × U(1) × U(1)_{R}-invariant flow**

In this appendix, we summarize the Ricci tensor and the 4-form field strengths for the $SU(2) \times U(1) \times U(1)_{R}$-invariant flow [22].

**C.1. The Ricci tensor**

The nonzero Ricci tensor in the coordinate basis from the 11-dimensional metric [22], after imposing the flow equations (1.2), can be written as follows:

\begin{equation}
R^1_{\mu} = - \frac{1}{L^2 u^2 v^2 (c_{\mu}^2 + u^2 s_{\mu}^2)^\frac{3}{2}} \left[ 24 u^6 v^2 (v^2 - 1) s_{\mu}^4 + 2 v^2 (v^2 + 3) c_{\mu}^4 
\right.
\end{equation}

\begin{equation}
+ u^6 \left[ -2(-5 + c_{2\mu}) + v^2 (-11 + c_{2\mu}) + 4 v^4 c_{2\mu}^2 \right] s_{\mu}^2 
\end{equation}

\begin{equation}
+ u^6 \left[ 12 c_{\mu}^2 + v^2 (9 - 13 c_{2\mu}) + 4 v^4 s_{\mu}^2 \right] c_{\mu}^2 
\end{equation}

\begin{equation}
+ u^4 \left[ 6 c_{\mu}^2 + v^2 (5 - 8 c_{2\mu} + 5 c_{4\mu}) + v^4 c_{2\mu}^2 \right] 
\right].
\end{equation}

\begin{equation}
= R_{2.2}^2 = R_{3.3}^3 = -2 R_{6.6}^6 = -2 R_{7.7} = -2 R_{8.8} = -2 R_{9.9}.
\end{equation}

\begin{equation}
R^4_{\mu} = \frac{6}{L^2 u^5 v^4 (c_{\mu}^2 + u^2 s_{\mu}^2)^\frac{7}{2}} \left[ -4 v^6 (2 v^4 - 21 v^2 + 27) c_{\mu}^4 
\right.
\end{equation}

\begin{equation}
- 4 u^8 v^2 (2 v^4 - 5 v^2 + 3) s_{\mu}^4 
\end{equation}

\begin{equation}
+ 2 u^2 v^2 \left[-48 + 60 c_{2\mu} + v^2 (15 - 43 c_{2\mu}) + 4 v^4 s_{\mu}^2 \right] c_{\mu}^2 
\end{equation}

\begin{equation}
- 2 u^6 \left[ 24 c_{\mu}^2 - 4 v^2 (7 + 4 c_{2\mu}) + v^4 (17 + 5 c_{2\mu}) - 4 u^4 c_{2\mu}^2 \right] s_{\mu}^2 
\end{equation}

\begin{equation}
- u^8 v^2 \left[ 33 - 48 c_{2\mu} + 27 c_{4\mu} + 4 v^2 \left[ 2 + 4 c_{2\mu} - 7 c_{4\mu} + v^2 (-1 + c_{4\mu}) \right] \right].
\end{equation}

\begin{equation}
R^5_{\mu} = \frac{18}{L^4 (c_{\mu}^2 + u^2 s_{\mu}^2)^\frac{7}{2}} u^2 v^2 \left[ -2 c_{\mu}^2 (v^2 - 3) 
\right.
\end{equation}

\begin{equation}
+ u^2 \left[ -5 - 11 c_{2\mu} + 14 v^2 c_{\mu}^2 + u^2 (-11 + 9 c_{2\mu} + 2 c_{\mu}^2 \left[ u^2 - v^2 (u^2 - 7) \right]) \right] s_{\mu}. 
\end{equation}
\[ R_5^4 = \frac{18}{L^2 u^4 (c_{\mu}^2 + u^2 s_{\mu}^2)^2} \left( -2c_{\mu}^2 (v^2 - 3) + u^2 \left[ -5 - 11c_{2\mu} + 14v^2 c_{\mu}^2 + u^2 \left( -11 + 9c_{2\mu} + 2s_{\mu}^2 \left( u^2 - v^2 (u^2 - 7) \right) \right) \right] \right) s_{2\mu}, \]

\[ R_5^5 = \frac{6}{L^2 u^4 (c_{\mu}^2 + u^2 s_{\mu}^2)^2} \left[ 4u^4 v^2 (v^2 - 1)s_{\mu}^4 + 4u^2 (v^2 + 3)c_{\mu}^4 \right.
\[ + u^4 \left[ 6 - 6c_{2\mu} + v^2 (19 - 16c_{2\mu} + c_{4\mu}) + 5v^4 (-1 + c_{4\mu}) \right]
\[ + 4u^2 \left[ 6c_{\mu}^2 + v^2 (3 - 5c_{2\mu}) + 2v^4 s_{\mu}^2 \right] c_{\mu}^2 \n\[ + 4u^6 \left[ -1 - v^2 + v^4 + (-7 + 5v^2 + v^4)c_{2\mu} \right] s_{\mu}^2 \left. \right]\right], \]

\[ R_7^{10} = \frac{36(y - 1) c_{\mu} c_{2\mu} u^4 (v^2 - 1)}{L^2 v^2 (c_{\mu}^2 + u^2 s_{\mu}^2)^2} \left[ 12c_{\mu}^4 v^2 + u^2 (s_{2\mu}^2 (3 + 5v^2) + 2s_{\mu}^2 u^2 (4 - 2c_{2\mu}) \n\[ + (3 - 9c_{2\mu}) v^2 + 2s_{\mu}^2 (4v^4 + u^2 (-3 + 3v^2))) \right] \right], \]

\[ R_7^{11} = \frac{48(y - 1) c_{\mu} c_{2\mu} u^4 (v^2 - 1)}{L^2 v^2 (c_{\mu}^2 + u^2 s_{\mu}^2)^2} \left[ -s_{\mu}^4 u^6 + 2c_{\mu}^4 u^4 - s_{\mu}^2 v^2 (-3 + (1 + 2c_{2\mu}) v^2) \n\[ + c_{\mu}^2 u^2 (3s_{\mu}^2 + (-1 + 2c_{2\mu}) v^2 + 6s_{\mu}^2 u^4) \right] \right], \]

\[ R_9^{10} = \frac{36 v^2 c_{\mu}^4 u^4 (v^2 - 1)}{L^2 v^2 (c_{\mu}^2 + u^2 s_{\mu}^2)^2} \left[ 12c_{\mu}^4 v^2 + u^2 (s_{2\mu}^2 (3 + 5v^2) + 2s_{\mu}^2 u^2 (4 - 2c_{2\mu}) \n\[ + (3 - 9c_{2\mu}) v^2 + 2s_{\mu}^2 (4v^4 + u^2 (-3 + 3v^2))) \right] \right], \]

\[ R_9^{11} = \frac{48 v^2 c_{\mu}^2 u^4 (v^2 - 1)}{L^2 v^2 (c_{\mu}^2 + u^2 s_{\mu}^2)^2} \left[ -s_{\mu}^4 u^6 + 2c_{\mu}^4 u^4 - s_{\mu}^2 v^2 (-3 + (1 + 2c_{2\mu}) v^2) \n\[ + c_{\mu}^2 u^2 (3s_{\mu}^2 + (-1 + 2c_{2\mu}) v^2 + 6s_{\mu}^2 u^4) \right] \right], \]

\[ R_{10}^{10} = \frac{3}{L^2 u^4 (c_{\mu}^2 + u^2 s_{\mu}^2)^2} \left[ -1 + 2v (v^2 + 4v^2) (-\sqrt{v^2 - 1} + v (-3 + 4v (v + \sqrt{v^2 - 1})) \right] \]

\[ 8s_{\mu}^6 u^4 (v^2 - 1) + 8s_{\mu}^4 u^4 (v^2 + 3) + 8c_{\mu}^4 u^2 v^2 \left( -12c_{\mu}^2 + 15 + c_{2\mu} v^2 + 3s_{\mu}^2 v^4 \right) \]

\[ + 8s_{\mu}^2 u^6 (18c_{\mu}^2 - 7 + 13c_{2\mu} v^2 - 6s_{\mu}^2 u^4 + 3c_{\mu}^2 u^6) \]

\[ + s_{\mu}^4 u^6 (48c_{\mu}^2 + 2 + c_{2\mu}) + 224c_{\mu}^4 v^2 \]

\[ + (5 - 12c_{2\mu} + 57c_{4\mu} v^4 + 8 (-2 + c_{2\mu} + 3c_{4\mu}) v^6) \]

\[ + c_{\mu}^2 u^4 (-36s_{2\mu} + 12s_{\mu}^2 + 8v^8 + (81 - 76c_{2\mu} + 3c_{4\mu}) u^4 + 16s_{\mu}^2 u^4) \].

\[ R_{10}^{11} = \frac{48 v^2 c_{\mu}^2 u^4 (v^2 - 1)}{L^2 v^2 (c_{\mu}^2 + u^2 s_{\mu}^2)^2} \left[ -s_{\mu}^4 u^6 + 2c_{\mu}^4 u^4 - s_{\mu}^2 v^2 (-3 + (1 + 2c_{2\mu}) v^2) \n\[ + c_{\mu}^2 u^2 (3s_{\mu}^2 + (-1 + 2c_{2\mu}) v^2 + 6s_{\mu}^2 u^4) \right] \right], \]

\[ R_{11}^{11} = \frac{108 v^2 c_{\mu}^4 u^4 (v^2 - 1)}{L^2 v^2 (c_{\mu}^2 + u^2 s_{\mu}^2)^2} \left[ -4s_{\mu}^4 u^6 v^2 + 4c_{\mu}^4 (-3 + 5v^2) + 2s_{\mu}^2 u^4 (10c_{\mu}^2 + (1 - 11c_{2\mu}) v^2 \n\[ - 4s_{\mu}^2 v^4) + u^2 (2(c_{2\mu} + c_{4\mu}) + s_{2\mu} v^2 (7 + 2v^2)) \right] \right].\]
\[ R_{11}^{11} = \frac{3}{L^2 u^2 v^2} \left( c_2^2 + u^2 s_2^2 \right) \left( v + \sqrt{v^2 - 1} \right)^6 \]

\[ 8s_4^6 u^3 v \left( v^2 - 1 \right) + 8c_4^6 v^4 \left( v^2 + 3 \right) + 8c_4^4 u^2 v^2 \left( 6c_2^2 + \left( 8 - 8c_2u \right) v^2 + 3s_2^2 v^4 \right) \]

\[ + 8s_4^6 u^2 \left( -18c_2^2 + 4 \left( 2 + 5c_2u_2 \right) v^2 - 12c_2u_2 v^4 + 3c_2^2 v^6 \right) \]

\[ + s_4^2 u^2 \left( -48c_2^2 \left( -2 + c_2u \right) - 8c_2^2 \left( 19 + c_2u \right) v^2 \right) \]

\[ + \left( 47 + 20c_2u_2 + 45c_4u_2 \right) v^4 + 8 \left( 4 + c_2u - 3c_4u \right) v^6 \]

\[ + c_4^2 u^4 \left( 36s_2^2 - 48s_2^2 u^2 + 40c_2u_2 + 9c_4u_2 \right) v^4 + 16s_2^2 v^6 \right), \]

(C.1)

where we use a simplified notation (A.2). There are additional nonzero Ricci tensor components \( R_{10}^{10}, R_{11}^{11}, R_{00}^{00} \) and \( R_{11}^{11} \) that depend on the internal coordinates \( \theta_i \) or \( y \), as compared to the Ricci tensor in the frame basis [22]. One also obtains (C.1) directly from the Ricci tensor in the frame basis with the help of the vielbeins. At the IR fixed point in four dimensions, the off-diagonal components \( R_{21}^{21} \) and \( R_{22}^{22} \) vanish.

### C.2. The 4-form field strengths

The nonzero 4-form field strengths satisfying (1.3) for a given Ricci tensor (C.1) and an 11-dimensional metric presented in [22] are summarized as follows:

\[ F_{1234} = \left[ \frac{3e^{1A}}{L/\rho^2} \right] \left[ c_2^2 \left( -5 + \cosh 2 \chi \right) + 2\rho^2 \left( -2 + c_2u + \rho^2 s_2 \sinh^2 \chi \right) \right] \]

\[ F_{1235} = \left[ \frac{3e^{1A}}{2\rho^2} \right] \left[ 1 + \cosh 2 \chi + \rho^2 \left( -3 + \cosh 2 \chi \right) \right] s_2u. \]

\[ F_{4567} = \left[ \frac{L^2 (\rho^3 - 3) \tanh \chi}{6\rho^2} \right] \frac{1}{2} \left[ 1 - y \right] \left[ \frac{wq c_0 c_2^2 s_{u+\psi}}{c_0} F_{4569} = \frac{wq}{6} s_{u+1} c_0 F_{4578} \right. \]

\[ F_{4568} = \left[ \frac{L^2 (\rho^3 - 3) \tanh \chi}{\rho^2} \right] \frac{3}{2} \left[ 1 - y \right] \left[ \frac{wq c_2^2 c_{u+\psi}}{c_0} F_{4579} = \frac{6}{wq} s_{u+1} F_{4578} \right. \]

\[ F_{4678} = \left[ \frac{L^2 (\rho^3 - 3) \tanh \chi \sinh 2 \chi}{8 \left[ (c_2^2 + \rho^2 s_2^2) \right]^2} \right] \left[ 2c_2^2 \left( -2 + \cosh 2 \chi \right) \right] \]

\[ + \rho^2 \left( -5 + c_2u + 2c_2^2 \cosh 2 \chi + 2\rho^2 s_2 \right) \left( 1 - y \right) \frac{1}{\sqrt{2} wq} \left[ c_0 c_2^2 s_{u+\psi} \right] \]

\[ = \frac{1}{y} c_0 F_{4689} = \left( 1 - y \right) c_0 F_{46810} = \frac{6}{wq} s_{u+1} c_0 F_{47910}, \]

\[ F_{4679} = \left[ \frac{L^2 \rho^3 \tanh \chi}{24 \left( c_2^2 + \rho^2 s_2^2 \right)^2} \right] \left[ 2c_2^2 \left( -2 + \cosh 2 \chi \right) + \rho^2 \left( -5 + c_2u + c_2^2 \cosh 2 \chi + 2\rho^2 s_2^2 \right) \right] \]

\[ \times \left[ \frac{1}{6} \left( 1 - y \right) wq c_0 c_2^2 s_{u+\psi} \right] F_{46710} = c_0 F_{46910} = \frac{1}{6} \left( 1 - y \right) wq c_0 s_{u+1} F_{4789}, \]

\[ = \frac{1}{6} wq c_0 s_{u+1} F_{47810}. \]
One sees that the 4-forms (C → F) preserve this U(1) charge, it is evident that 4-forms preserve this U(1) charge. At the IR fixed point in four dimensions, the components

\[ F_{4671} = \frac{L^2 \rho^6 \tanh \chi}{4 \rho^3 (c_\mu^2 + \rho^8 s_\mu^2)^2} \times \left[ -3c_\mu^2 + \rho^8 (c_\mu^2 - (4 + \cosh 2\chi)s_\mu^2) - \rho^8 (-4 + \cosh 2\chi)s_\mu^2 \right] \times \frac{1}{6} (1 - y) \underbrace{wq c_{\theta h} s_{2\mu} c_{\mu}s_{\psi}}_{c_{\theta h} F_{46911}} = c_{\theta h} F_{46911} = \frac{1}{6} wq c_{\theta h} s_{\theta h}^{-1} F_{47811}. \]

\[ F_{4681} = \frac{L^2 \tanh \chi}{6 \rho^3 (c_\mu^2 + \rho^8 s_\mu^2)^2} \times \left[ -3c_\mu^2 + \rho^8 (c_\mu^2 - (4 + \cosh 2\chi)s_\mu^2) - \rho^8 (-4 + \cosh 2\chi)s_\mu^2 \right] \times \frac{6(1 - y)}{wq} s_{2\mu} c_{\mu}s_{\psi} = -\frac{6}{wq} s_{\theta h}^{-1} F_{47911}. \]

\[ F_{5678} = -\frac{L^3 \tanh \chi}{4 \rho^2 (c_\mu^2 + \rho^8 s_\mu^2)^3} \times \left[ 6c_\mu^4 + \rho^8 c_\mu^2 (9 - 7c_\mu^2) + 10^2 \rho^4 c_\mu^4 - 2\rho^4 c_\mu^4 \right] \times \frac{\sqrt{2}}{3} (1 - y)^{\frac{1}{2}} c_{\mu}s_{\theta h} \underbrace{c_{\theta h} s_{\psi}}_{c_{\theta h} F_{5689}} = y^{-1} (1 - y) c_{\theta h} F_{56910} = \frac{1}{6} wq s_{\theta h}^{-1} c_{\theta h} F_{57810}. \]

\[ F_{5679} = -\frac{L^3 \tanh \chi}{36 \rho^2 (c_\mu^2 + \rho^8 s_\mu^2)^3} \times \left[ 6c_\mu^4 + \rho^8 c_\mu^2 (9 - 7c_\mu^2) + 10^2 \rho^4 c_\mu^4 - 2\rho^4 c_\mu^4 \right] \times \frac{\sqrt{3}}{2} (1 - y) \underbrace{wq s_{\theta h} c_{\mu}s_{\psi}}_{c_{\theta h} F_{56911}} = \frac{1}{6} wq c_{\theta h} s_{\theta h}^{-1} F_{57811}. \]

\[ F_{5681} = -\frac{L^3 \tanh \chi}{(c_\mu^2 + \rho^8 s_\mu^2)^2} \times \left[ c_\mu^2 + 3\rho^8 + \rho^4 c_\mu^2 \right] \times \sqrt{\frac{3(1 - y)}{2wq}} \underbrace{c_{\mu}s_{\theta h}^2 c_{\mu}s_{\psi}}_{c_{\theta h} F_{56891}} = -\frac{6}{wq} s_{\theta h}^{-1} F_{57911}. \]

\[ F_{5681} = \frac{L^3(\rho^8 + 3) \tanh \chi}{(c_\mu^2 + \rho^8 s_\mu^2)^2} \times \left[ \frac{(1 - y)^{\frac{1}{2}}}{\sqrt{6wq}} c_{\mu} c_{\mu} s_{\theta h} c_{\mu}s_{\psi} = \left(1 - y\right) \frac{1}{y} \frac{1}{c_{\theta h} F_{56911}} = (1 - y) c_{\theta h} F_{68011} \right] = \frac{6 (1 - y)}{wq} c_{\theta h} s_{\theta h}^{-1} F_{57911}. \]

\[ F_{5671} = -\frac{L^3(\rho^8 + 3) \tanh \chi}{18 (c_\mu^2 + \rho^8 s_\mu^2)^2} \times \sqrt{\frac{3}{2}(1 - y) wq \left(1 - y\right) c_{\mu} c_{\mu} s_{\theta h} c_{\mu}s_{\psi} = F_{57011}} = \frac{1}{6} c_{\theta h} F_{6911} = \frac{1}{6} c_{\theta h} s_{\theta h}^{-1} wq F_{78911} = \frac{1}{6} c_{\theta h} s_{\theta h}^{-1} wq F_{781011}. \]
\( F_{1234} \) \( F_{4\mu np} \) and \( F_{35\mu np} \) vanish. By using the 4-form field strengths in the frame basis \([22]\) and vielbeins, one also obtains (C.2).

The 4-form field strengths with upper indices can be obtained from those with lower indices \((C.2)\) by multiplying the 11-dimensional inverse metric \([22]\) and they are given as follows:

\[
F_{1234} = \left[ \frac{3e^{-2\chi} \rho^4}{L \cosh^2 \chi \left( c_{\mu}^2 + \rho^8 s_{\mu}^2 \right)^2} \right] \left[ c_{\mu}^2 (-5 + \cosh 2\chi) + 2\rho^8 (-2 + c_{2\mu} + \rho^8 s_{\mu}^2 \sinh^2 \chi) \right],
\]

\[
F_{1235} = \left[ \frac{3e^{-2\chi} \rho^2 \tanh \chi}{2L^2 \left( c_{\mu}^2 + \rho^8 s_{\mu}^2 \right)^2} \right] \left[ 1 + \cosh 2\chi + \rho^4 (-3 + \cosh 2\chi) \right] s_{2\mu},
\]

\[
F_{4568} = \left[ \frac{6(-3 + \rho^8) \tanh \chi}{L^4 \rho^4 \tanh^2 \chi \left( c_{\mu}^2 + \rho^8 s_{\mu}^2 \right)^2} \right] \sqrt{\frac{3wq}{2(1-y)}} \mu c_{\mu} c_{-2} = -\frac{1}{6wq s_{\omega}} F_{4579},
\]

\[
= \frac{1}{6y} wq s_{\omega} F_{4579}^{10} = \frac{1}{6(1-y)} s_{\omega} c^{-1}_{\omega} F_{4590}^{10},
\]

\[
F_{4569} = \left[ \frac{18(-3 + \rho^8) \tanh \chi}{L^4 \rho^4 \tanh^2 \chi \left( c_{\mu}^2 + \rho^8 s_{\mu}^2 \right)^2} \right] \sqrt{\frac{6wq}{(1-y)wq}} s_{\omega} c_{\mu} c_{-2} = -\frac{6}{wq} s_{\omega} c^{-1}_{\omega} F_{4580},
\]

\[
= \frac{6}{wq} s_{\omega} F_{4587}^{10} = \frac{1}{wq} s_{\omega} c^{-1}_{\omega} F_{45810}^{10},
\]

\[
F_{46810} = \left[ \frac{9}{L^4 \rho^4 \cosh^2 \chi \left( c_{\mu}^2 + \rho^8 s_{\mu}^2 \right)^2} \right] \left[ - \sinh 2\chi + \rho^8 (\sinh 2\chi (1 - 3s_{\mu}^2 c_{-2}^2))
\right.

\[\left. + 2(-3 + \cosh 2\chi) \tanh \chi + \rho^8 \sinh 2\chi (3s_{\mu}^2 c_{-2}^2) \right] \sqrt{\frac{3wq}{2(1-y)}} s_{\omega} c_{\mu} c_{-2} = \frac{1}{6wq s_{\omega}} F_{4790}^{10},
\]

\[
F_{46811} = \left[ \frac{3 \tanh \chi}{L^4 \rho^4 \left( c_{\mu}^2 + \rho^8 s_{\mu}^2 \right)^2} \right] \left[ 3c_{\mu} s_{\omega} c_{\mu}^{-1} \sinh 2\chi + \rho^8 s_{\omega} c_{\mu}^{-1} \sinh 2\chi (2 - 3c_{2\mu})
\right.

\[\left. + \rho^{16} (-7 + \cosh 2\chi) \tanh \chi \right] \sqrt{\frac{3wq}{2(1-y)}} s_{\omega} c_{\mu} c_{-2}
\left. = -\frac{1}{6wq s_{\omega}} F_{4791}^{10} = \frac{1}{6y} wq s_{\omega} F_{4710}^{11} = \frac{wq}{6(1-y)} s_{\omega} c^{-1}_{\omega} F_{4910}^{11},
\]

\[
F_{46910} = \left[ \frac{27}{L^4 \rho^4 \cosh^2 \chi \left( c_{\mu}^2 + \rho^8 s_{\mu}^2 \right)^2} \right] \left[ \sinh 2\chi + \rho^8 (\sinh 2\chi (1 - 3s_{\mu}^2 c_{-2}^2))
\right.

\[\left. + 2(-3 + \cosh 2\chi) \tanh \chi + \rho^8 \sinh 2\chi (3s_{\mu}^2 c_{-2}^2) \right] \sqrt{\frac{6}{(1-y)wq}} c_{\omega} s_{\mu} c_{\mu} c_{-2}^{-1}
\]
\[
F^{469\text{11}} = \frac{6}{wq} \left( \frac{L^4 \rho_1^2}{\left(c_{\mu}^2 + \rho^2 s_{\mu}^2\right)^2} \right) \left[ 3c_{\mu}^2 s_{\mu}^{-1} \cosh \chi \sinh \chi + \rho^8 \left(-3c_{\mu}^2 s_{\mu}^{-1} + s_{\mu}^{-1} c_{\mu}^{-1}\right) \sinh 2\chi + \rho^{16} \left(-3 + \sinh^2 \chi\right)s_{\mu} c_{\mu}^{-1} \tanh \chi \right] \sqrt{\frac{6}{(1-y)wq}} c_{\mu + \psi} \\
= -\frac{1}{y} F^{469\text{11}} = \frac{6}{wq} \left( \frac{\rho_1^2}{s_{\mu} c_{\mu}^{-1} f^{48\text{11}}} = \frac{6}{wq} \left( \frac{\rho_1^2}{s_{\mu} c_{\mu}^{-1} f^{48\text{11}}} = \frac{6}{wq} \left( \frac{s_{\mu} c_{\mu}^{-1} f^{48\text{11}}} \right) \right) \right),
\]

\[
F^{568\text{10}} = \frac{6}{wq} \left[ \frac{18 \cosh^2 \chi \sinh^2 \chi}{L^5 \rho_1^3 \left(c_{\mu}^2 + \rho^2 s_{\mu}^2\right)^2} \right] \left[ 3c_{\mu}^2 s_{\mu}^{-1} \cosh^2 \chi + \rho^8 \left(-2 + c_{\mu}^{-2}\right) \sech^2 \chi \left(\cosh^2 \chi + \sinh^2 \chi\right) s_{\mu} c_{\mu}^{-2} \right] \sqrt{\frac{3wq}{2(1-y)}} c_{\mu - \psi} \\
= -\frac{wq}{6y} wq s_{\mu} F^{57\text{11}} = \frac{wq}{6y} s_{\mu} F^{57\text{11}} = \frac{6}{wq} \left( \frac{s_{\mu} c_{\mu}^{-1} f^{59\text{11}}} \right),
\]

\[
F^{569\text{10}} = \frac{6}{wq} \left( \frac{L^6 \rho_1^4}{\left(c_{\mu}^2 + \rho^2 s_{\mu}^2\right)^2} \right) \left[ -3 \sech^2 \chi + \rho^8 (\cosh^2 \chi - 3 \sech^2 \chi) s_{\mu} c_{\mu}^{-2} \right] \sqrt{\frac{6}{(1-y)wq}} c_{\mu + \psi} \\
= \frac{6}{wq} \left( \frac{s_{\mu} c_{\mu}^{-1} f^{57\text{11}}} \right) = \frac{6}{wq} \left( \frac{s_{\mu} c_{\mu}^{-1} f^{57\text{11}}} \right),
\]

\[
F^{569\text{11}} = \frac{18 \rho_1^2 \sinh \chi \cosh^2 \chi}{L^5 \rho_1^3 \left(c_{\mu}^2 + \rho^2 s_{\mu}^2\right)^2} \left[ -2 \cosh^2 \chi + 3 \sech^2 \chi + \rho^8 (2 \cosh^2 \chi - (-2 + c_{\mu}^{-2}) \sech^2 \chi s_{\mu} c_{\mu}^{-2}) \right] \sqrt{\frac{6}{(1-y)wq}} c_{\mu + \psi} \\
= \frac{6}{wq} \left( \frac{s_{\mu} c_{\mu}^{-1} f^{58\text{11}}} \right) = \frac{6}{wq} \left( \frac{s_{\mu} c_{\mu}^{-1} f^{58\text{11}}} \right),
\]

\[
F^{68\text{10}} = \frac{18(\rho^8 + 3) \sinh \chi \sech^2 \chi (c_{\mu}^2 + \rho^2 s_{\mu}^2)^2}{L^5 \rho_1^3} \sqrt{\frac{3wq}{2(1-y)}} c_{\mu - \psi}^3 c_{\mu + \psi} \\
= -\frac{1}{6y} wq s_{\mu} F^{79\text{11}},
\]
\[ F^{9011} = -\left[ \frac{54(\rho^8 + 3)}{L^5 \rho^7} - \sinh \chi \text{sech}^2 \chi \left( \left( c_{\mu}^2 + \rho^8 s_{\mu}^2 \right)^{\frac{1}{2}} \right) \right] \left( \frac{6}{wq} s_{\mu}^{-1} c_{\rho}^{-3} s_{\mu+\rho} \right) \] 
\[ = \frac{6}{wq} s_{\omega}^{-1} F^{9011} = \frac{6}{wq} c_{\omega}^{-1} s_{\omega} F^{9011}. \]  
(C.3)

One also obtains (C.3) from the 4-form field strengths in the frame basis [22] and vielbeins. At the IR fixed point in four dimensions, the components \( F^{1235}, F^{4455 \mu} \) and \( F^{455 \mu} \) vanish. It turns out that after computing the right-hand side of (1.3) using both (C.2) and (C.3), the dependence on the combination \( (\alpha + \psi) \) disappears completely. This coincides with the fact that the Ricci tensor (C.1) does not depend on these variables.

**Appendix D. SU(2) \times U(1)-invariant flow**

In this appendix, we describe the Ricci tensor and the 4-form field strengths for the \( SU(2) \times U(1)-invariant \) flow that can be written in terms of the corresponding Ricci tensor and the 4-form field strengths respectively for the \( SU(2) \times U(1) \times U(1)_K \)-invariant flow.

**D.1. The Ricci tensor**

The Ricci tensor in the coordinate basis from the 11-dimensional metric (3.1), after imposing the flow equations (1.2), can be written in terms of those (C.1) in appendix E as follows:

\[ \tilde{R}_1^1 = R_1^1, \quad \tilde{R}_2^2 = R_2^2, \quad \tilde{R}_3^3 = R_3^3, \quad \tilde{R}_4^4 = R_4^4, \]

\[ \tilde{R}_5^5 = -\cos \theta \sin \theta \left[ R_5^5 - \frac{\cos^2 \theta}{\sqrt{1 - \sin^2 \theta \sin^2 \theta}} \right], \quad \tilde{R}_5^11 = -\csc \theta \cos \theta \left[ \frac{\cos \theta \sin \theta \cos \theta}{\sqrt{1 - \sin^2 \theta \sin^2 \theta}} \right] R_5^5, \]

\[ \tilde{R}_6^6 = R_6^6, \quad \tilde{R}_7^7 = \cos \theta \left[ \frac{\cos \theta}{1 + \tan^2 \theta \cos^2 \theta} \right] R_7^7, \]

\[ \tilde{R}_8^8 = R_8^8, \quad \tilde{R}_9^9 = \cos \theta \left[ \frac{\cos \theta}{1 + \tan^2 \theta \cos^2 \theta} \right] R_9^9, \]

\[ \tilde{R}_{10}^{10} = \cos \theta \left[ \frac{\cos \theta}{1 + \tan^2 \theta \cos^2 \theta} \right] R_{10}^{10}, \quad \tilde{R}_{11}^{11} = -\cos \theta \left[ \frac{\cos \theta}{1 + \tan^2 \theta \cos^2 \theta} \right] R_{11}^{11}. \]  

There are extra nonzero off-diagonal Ricci tensor components \( \tilde{R}_4^{11}, \tilde{R}_5^{10}, \tilde{R}_5^{11}, \tilde{R}_6^{10}, \tilde{R}_5^{11}, \tilde{R}_7^{10}, \tilde{R}_5^{11}, \tilde{R}_10^{11}, \tilde{R}_11^{10} \) for the \( SU(2) \times U(1)-invariant \) flow.
The 4-form field strengths

The 4-form field strengths satisfying (1.3) for a given Ricci tensor (D.1) and the 11-dimensional metric (3.1), in terms of those (C.2) for the $SU(2) \times U(1) \times U(1)_R$-invariant flow, are summarized as follows:

\[
\tilde{F}_{1234} = F_{1234}, \quad \tilde{F}_{1235} = - \left[ \frac{\cos \theta \sin \theta}{\sqrt{1 - \sin^2 \theta \sin^2 \theta}} \right] F_{1235},
\]

\[
\tilde{F}_{12311} = - \left[ \frac{\cos \theta \sin \theta}{\sqrt{1 - \sin^2 \theta \sin^2 \theta}} \right] F_{1235},
\]

\[
\tilde{F}_{45mn} = - \left[ \frac{\cos \theta \sin \theta}{\sqrt{1 - \sin^2 \theta \sin^2 \theta}} \right] F_{45mn} + \left[ \frac{\cos \theta_6}{1 - \sin^2 \theta \sin^2 \theta_6} \right] F_{4mn11},
\]

\[
(m, n, p) = (6, 7), (6, 8), (6, 9), (7, 8), (7, 9),
\]

\[
\tilde{F}_{46mn} = F_{46mn}, \quad (m, n) = (7, 8), (7, 9), (7, 10), (8, 9), (8, 10), (9, 10),
\]

\[
\tilde{F}_{4mn11} = - \left[ \frac{\sin \theta \cos \theta_6}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}} \right] F_{45mn} - \left[ \frac{\sin \theta \cos \theta_6}{1 - \sin^2 \theta \sin^2 \theta_6} \right] F_{4mn11},
\]

\[
(m, n) = (6, 7), (6, 8), (6, 9), (7, 8), (7, 9),
\]

\[
\tilde{F}_{47mn} = F_{47mn}, \quad (m, n) = (8, 9), (8, 10), (9, 10),
\]

\[
\tilde{F}_{56mn} = - \left[ \frac{\cos \theta_6 \sin \theta}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}} \right] F_{56mn} - \left[ \frac{\cos \theta_6}{1 - \sin^2 \theta \sin^2 \theta_6} \right] F_{6mn11},
\]

\[
(m, n) = (7, 8), (7, 9), (7, 10), (8, 9), (8, 10), (9, 10),
\]

\[
\tilde{F}_{5mn11} = - \left[ \frac{\sin \theta}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}} \right] F_{5mn11}, \quad (m, n) = (6, 7), (6, 8), (6, 9), (7, 8), (7, 9),
\]

\[
\tilde{F}_{57mn} = - \left[ \frac{\cos \theta_6 \sin \theta}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}} \right] F_{57mn} - \left[ \frac{\cos \theta_6}{1 - \sin^2 \theta \sin^2 \theta_6} \right] F_{7mn11},
\]

\[
(m, n) = (8, 9), (8, 10), (9, 10),
\]

\[
\tilde{F}_{56np11} = - \left[ \frac{\sin \theta \cos \theta_6}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}} \right] F_{56np11} - \left[ \frac{\sin \theta \cos \theta_6 \sin \theta}{1 - \sin^2 \theta \sin^2 \theta_6} \right] F_{56np11},
\]

\[
(m, n, p) = 6, \ldots, 9,
\]

\[
\tilde{F}_{5mn1011} = - \left[ \frac{\sin \theta \cos \theta_6}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}} \right] F_{5mn1011} - \left[ \frac{\sin \theta \cos \theta_6 \sin \theta}{1 - \sin^2 \theta \sin^2 \theta_6} \right] F_{5mn1011},
\]

\[
(m, n) = (6, 7), (6, 8), (6, 9), (7, 8), (7, 9).
\]

The $\tilde{F}_{12311}$ is new, compared to the $SU(2) \times U(1) \times U(1)_R$-invariant flow. At the IR fixed point in four dimensions, the components $\tilde{F}_{1235}, \tilde{F}_{12311}, \tilde{F}_{45mn}$ and $\tilde{F}_{46mn}$ vanish.

The 4-form field strengths with upper indices can be obtained from those with lower indices (D.2) by multiplying the 11-dimensional inverse metric (3.1) and they are given as
follows:
\[
\tilde{F}^{1234} = F^{1234}, \quad \tilde{F}^{1235} = - \left[ \frac{\cos \theta \sin \theta}{\sqrt{1 - \sin^2 \theta \sin^2 \theta}} \right] F^{1235},
\]
\[
\tilde{F}^{12311} = - \left[ \frac{\csc \theta \cos \theta}{\sqrt{1 - \sin^2 \theta \sin^2 \theta}} \right] F^{1235},
\]
\[
\tilde{F}^{45mn} = [\cos \theta_0] F^{4mn11} - \left[ \frac{\cos \theta \sin \theta}{\sqrt{1 - \sin^2 \theta \sin^2 \theta}} \right] F^{45mn},
\]
\[
(m, n) = (6, 8), (6, 9), (6, 10), (7, 8), (7, 9), (7, 10), (8, 9), (8, 10), (9, 10),
\]
\[
\tilde{F}^{4mn10} = F^{4mn10}, \quad (m, n) = (6, 8), (6, 9), (7, 8), (7, 9), (8, 9),
\]
\[
\tilde{F}^{4mn11} = - [\sin \theta_0 \cot \theta] F^{4mn11} - \left[ \frac{\csc \theta \cos \theta}{\sqrt{1 - \sin^2 \theta \sin^2 \theta}} \right] F^{45mn},
\]
\[
(m, n) = (6, 8), (6, 9), (7, 8), (7, 9), (7, 10), (8, 9), (8, 10), (9, 10),
\]
\[
\tilde{F}^{5mn10} = - \cos \theta_0 F^{5mn11} - \left[ \frac{\cos \theta \sin \theta}{\sqrt{1 - \sin^2 \theta \sin^2 \theta}} \right] F^{5mn10},
\]
\[
(m, n) = (6, 8), (6, 9), (7, 8), (7, 9), (8, 9),
\]
\[
\tilde{F}^{5mn11} = \left[ \csc \theta \sqrt{1 - \sin^2 \theta \sin^2 \theta} \right] F^{5mn11},
\]
\[
(m, n) = (6, 8), (6, 9), (7, 8), (7, 9), (7, 10), (8, 10), (9, 10),
\]
\[
\tilde{F}^{mn10} = - [\cot \theta_0 \sin \theta] F^{mn10} + \left[ \frac{\csc \theta \cos \theta}{\sqrt{1 - \sin^2 \theta \sin^2 \theta}} \right] F^{5mn10},
\]
\[
(m, n) = (6, 8), (6, 9), (7, 8), (7, 9), (8, 9).
\]
\[(D.3)\]

There is a new 4-form \(\tilde{F}^{12311}\), compared to the 4-form field strengths \((C.3)\) for the SU(2) x U(1) \(\mu\)-invariant flow. At the IR fixed point in four dimensions, the components \(\tilde{F}^{1235}, \tilde{F}^{12311}, \tilde{F}^{44mn}\) and \(\tilde{F}^{45mn}\) vanish. By substituting the Ricci tensor \((D.1)\) and 4-form field strengths \((D.2)\) and \((D.3)\) into the Einstein equation in \((1.3)\), one can check that the LHS coincides with the RHS exactly. In doing this, one uses the fact that the solution characterized by \((C.1)\), \((C.2)\) and \((C.3)\) for the SU(2) x U(1) \(\mu\)-invariant flow satisfies the Einstein equation. Therefore, one concludes that the Einstein equation for the SU(2) x U(1) \(\mu\)-invariant flow is satisfied.

\[D.3. \text{The left-hand side of Maxwell equation}\]

The nonzero components of the left-hand side of Maxwell equations in terms of \((C.2)\) are given as follows:
\[
(123) = F_{47911} F_{56810} - F_{47910} F_{56811} - F_{47811} F_{56910} + F_{47810} F_{56911} - F_{46811} F_{57810} + F_{46910} F_{57811} + F_{46811} F_{57910} - F_{46810} F_{57911} + F_{4579} F_{58910} - F_{4578} F_{69101} - F_{4569} F_{78101} + F_{4568} F_{79101},
\]
\[
(468) = - F_{1235} F_{791011}, \quad (469) = F_{1235} F_{781011}, \quad (4610) = - F_{1235} F_{78911},
\]
\[
(478) = F_{1235} F_{691011}, \quad (479) = - F_{1235} F_{681011}, \quad (4710) = F_{1235} F_{68911},
\]
\[
(489) = F_{1235} F_{671011}, \quad (4810) = - F_{1235} F_{67911}, \quad (4910) = F_{1235} F_{67811},
\]
\[
(568) = - [\cos \theta_0] (F_{1235} F_{47910} - F_{1234} F_{57910}) - \left[ \frac{\cos \theta \sin \theta}{\sqrt{1 - \sin^2 \theta \sin^2 \theta}} \right] F_{1234} F_{791011},
\]

31
(569) = [\cos \theta_6] (F_{1235} F_{47810} - F_{1234} F_{57810}) + \left[ \frac{\cos \theta \sin \theta_6}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}} \right] F_{1234} F_{781011}.

(5610) = - [\cos \theta_6] (F_{1235} F_{4789} - F_{1234} F_{5789}) - \left[ \frac{\cos \theta \sin \theta_6}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}} \right] F_{1234} F_{7811}.

(578) = [\cos \theta_6] (F_{1235} F_{46910} - F_{1234} F_{56910}) + \left[ \frac{\cos \theta \sin \theta_6}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}} \right] F_{1234} F_{691011}.

(579) = - [\cos \theta_6] (F_{1235} F_{46810} - F_{1234} F_{56810}) - \left[ \frac{\cos \theta \sin \theta_6}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}} \right] F_{1234} F_{681011}.

(5710) = [\cos \theta_6] (F_{1235} F_{4689} - F_{1234} F_{5689}) + \left[ \frac{\cos \theta \sin \theta_6}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}} \right] F_{1234} F_{68911}.

(589) = [\cos \theta_6] (F_{1235} F_{46710} - F_{1234} F_{56710}) + \left[ \frac{\cos \theta \sin \theta_6}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}} \right] F_{1234} F_{671011}.

(5810) = - [\cos \theta_6] (F_{1235} F_{4679} - F_{1234} F_{5679}) - \left[ \frac{\cos \theta \sin \theta_6}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}} \right] F_{1234} F_{67911}.

(5910) = [\cos \theta_6] (F_{1235} F_{4678} - F_{1234} F_{5678}) + \left[ \frac{\cos \theta \sin \theta_6}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}} \right] F_{1234} F_{67811}.

(6810) = F_{1325} F_{47911} - F_{1234} F_{57911},

(6811) = [\cot \theta \sin \theta_6] (F_{1235} F_{47910} - F_{1234} F_{57910}) - \left[ \frac{\csc \theta \cos \theta_6}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}} \right] F_{1234} F_{791011}.

(6910) = - F_{1235} F_{47811} + F_{1234} F_{57811},

(6911) = [\cot \theta \sin \theta_6] (-F_{1235} F_{47810} + F_{1234} F_{57810}) + \left[ \frac{\csc \theta \cos \theta_6}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}} \right] F_{1234} F_{781011}.

(61011) = [\cot \theta \sin \theta_6] (F_{1235} F_{4789} - F_{1234} F_{5789}) - \left[ \frac{\csc \theta \cos \theta_6}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}} \right] F_{1234} F_{78911}.

(7810) = - F_{1235} F_{46911} + F_{1234} F_{56911},

(7811) = [\cot \theta \sin \theta_6] (-F_{1235} F_{46910} + F_{1234} F_{56910}) + \left[ \frac{\csc \theta \cos \theta_6}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}} \right] F_{1234} F_{691011},

(7910) = F_{1235} F_{46811} - F_{1234} F_{56811},

(7911) = [\cot \theta \sin \theta_6] (F_{1235} F_{46810} - F_{1234} F_{56810}) - \left[ \frac{\csc \theta \cos \theta_6}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}} \right] F_{1234} F_{681011}.

(71011) = [\cot \theta \sin \theta_6] (-F_{1235} F_{4689} + F_{1234} F_{5689}) + \left[ \frac{\csc \theta \cos \theta_6}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}} \right] F_{1234} F_{68911},

(8910) = - F_{1235} F_{46711} + F_{1234} F_{56711},

(8911) = [\cot \theta \sin \theta_6] (-F_{1235} F_{46710} + F_{1234} F_{56710}) + \left[ \frac{\csc \theta \cos \theta_6}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_6}} \right] F_{1234} F_{671011}.
(8 10 11) = \left[ \cot \theta \sin \theta_0 \right] (F_{1235} F_{4679} - F_{1234} F_{5678}) - \left[ \frac{\csc \theta \cos \theta_0}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_0}} \right] F_{1234} F_{5678} 11 - \left[ \frac{\csc \theta \cos \theta_0}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_0}} \right] F_{1234} F_{5678} 11.

(9 10 11) = \left[ \cot \theta \sin \theta_0 \right] (-F_{1235} F_{4678} + F_{1234} F_{5678}) + \left[ \frac{\csc \theta \cos \theta_0}{\sqrt{1 - \sin^2 \theta \sin^2 \theta_0}} \right] F_{1234} F_{5678} 11.

(D.4)

One can easily check that the RHS of the Maxwell equation (1.3) with (D.2) and (3.1) is exactly coincident with the above LHS of the Maxwell equation (D.4).

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