Hadwiger’s conjecture for 3-arc graphs*

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Abstract

The 3-arc graph of a digraph $D$ is defined to have vertices the arcs of $D$ such that two arcs $uv, xy$ are adjacent if and only if $uv$ and $xy$ are distinct arcs of $D$ with $v \neq x, y \neq u$ and $u, x$ adjacent. We prove that Hadwiger’s conjecture holds for 3-arc graphs.

\textit{Keywords:} Hadwiger’s conjecture, graph colouring, graph minor, 3-arc graph

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1 Introduction

A graph $H$ is a \textit{minor} of a graph $G$ if a graph isomorphic to $H$ can be obtained from a subgraph of $G$ by contracting edges. An $H$-minor is a minor isomorphic to $H$. The \textit{Hadwiger number} $h(G)$ of $G$ is the maximum integer $k$ such that $G$ contains a $K_k$-minor, where $K_k$ is the complete graph with $k$ vertices.

In 1943, Hadwiger [10] posed the following conjecture, which is thought to be one of the most difficult and beautiful problems in graph theory:

\textbf{Hadwiger’s Conjecture.} For every graph $G$, $h(G) \geq \chi(G)$.

Hadwiger’s conjecture has been proved for graphs $G$ with $\chi(G) \leq 6$ [19], and is open for graphs with $\chi(G) \geq 7$. This conjecture also holds for particular classes of graphs, including powers of cycles [13], proper circular arc graphs [2], line graphs [18], and quasi-line graphs [6]. See [20] for a survey.

In this paper we prove Hadwiger’s conjecture for a large family of graphs. Such graphs are defined by means of a graph operator, called the 3-arc graph construction (see Definition 1), which bears some similarities with the line graph operator. This construction was first introduced by Li, Praeger and Zhou [15] in the study of a family of arc-transitive graphs whose automorphism group contains a subgroup acting imprimitively on the vertex set. (A graph is \textit{arc-transitive} if its automorphism group is transitive on the set of oriented edges.) It was used in classifying or characterizing certain families of arc-transitive graphs [1] [11] [13] [17] [23] [24] [25]. Recently, various graph-theoretic properties of 3-arc graphs have been investigated [1] [12] [13] [22].

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The original 3-arc graph construction \[15\] was defined for a finite, undirected and loopless graph \(G = (V(G), E(G))\). In \(G\), an arc is an ordered pair of adjacent vertices. Denote by \(A(G)\) the set of arcs of \(G\). For adjacent vertices \(u, v\) of \(G\), we use \(uv\) to denote the arc from \(u\) to \(v\), and \(\{u, v\}\) the edge between \(u\) and \(v\). We emphasise that each edge of \(G\) gives rise to two arcs in \(A(G)\). A 3-arc of \(G\) is a 4-tuple of vertices \((v, u, x, y)\), possibly with \(v = y\), such that both \((v, u, x)\) and \((u, x, y)\) are paths of \(G\). The 3-arc graph of \(G\) is defined as follows:

**Definition 1.** \([15, 24]\) Let \(G\) be an undirected graph. The 3-arc graph of \(G\), denoted by \(X(G)\), has vertex set \(A(G)\) such that two vertices corresponding to arcs \(uv\) and \(xy\) are adjacent if and only if \((v, u, x, y)\) is a 3-arc of \(G\).

The 3-arc graph construction can be generalised for a digraph \(D = (V(D), A(D))\) as follows \([12]\), where \(A(D)\) is a multiset of ordered pairs (namely, arcs) of distinct vertices of \(V(D)\). Here a digraph allows parallel arcs but not loops.

**Definition 2.** Let \(D = (V(D), A(D))\) be a digraph. The 3-arc graph of \(D\), denoted by \(X(D)\), has vertex set \(A(D)\) such that two vertices corresponding to arcs \(uv\) and \(xy\) are adjacent if and only if \(v \neq x, y \neq u\) and \(u, x\) are adjacent.

Let \(D\) be the digraph obtained from an undirected graph \(G\) by replacing each edge \(\{x, y\}\) by two opposite arcs \(xy\) and \(yx\). Then, \(X(D) = X(G)\).

Knor, Xu and Zhou \([12]\) introduced the notion of 3-arc colouring a digraph, which can be defined as a proper vertex-colouring of \(X(D)\). The minimum number of colours in a 3-arc colouring of \(D\) is called the 3-arc chromatic index of \(D\), and is denoted by \(\chi'_3(D)\). Then \(\chi(X(D)) = \chi'_3(D)\).

The main result of this paper is the following:

**Theorem 1.** Let \(D\) be a digraph without loops. Then \(h(X(D)) \geq \chi(X(D))\).

Note that in the case of the 3-arc graph of an undirected graph, we have obtained a much simpler proof of Theorem \([1]\).

## 2 Preliminaries

We need the following notation. Let \(D = (V(D), A(D))\) be a digraph. We denote by \(A_D\{x, y\}\) the set of arcs between vertices \(x\) and \(y\), and by \(A_D(x)\) the set of arcs outgoing from \(x\). Then vertices \(x\) and \(y\) are adjacent if and only if \(A_D\{x, y\} \neq \emptyset\). When \(|A_D\{x, y\}| = 1\), we misuse the notation \(A_D\{x, y\}\) to indicate the arc between \(x\) and \(y\). An in-neighbour (respectively, out-neighbour) of a vertex \(x\) of \(D\) is a vertex \(y\) such that \(yx \in A(D)\) (respectively, \(xy \in A(D)\)). The set of all in-neighbours (respectively, out-neighbours) of \(x\) is denoted by \(N_D^-(x)\) (respectively, \(N_D^+(x)\)). The in-degree \(d_D^-(x)\) (respectively, out-degree \(d_D^+(x)\)) is defined to be the number of in-neighbours (respectively, out-neighbours) of \(x\). A vertex \(x\) is called a sink if \(d_D^-(x) = 0\). A digraph is simple if \(|A_D\{x, y\}| \leq 1\) for all distinct vertices \(x\) and \(y\) of \(D\). A tournament is a simple digraph whose underlying undirected graph is complete.

For an undirected graph \(G\), the degree of a vertex \(v\) in \(G\) is denoted by \(d_G(v)\), and the minimum degree of \(G\) is denoted by \((G)\). We omit the subscript when there is no ambiguity.

A \(K_t\)-minor in \(G\) can be thought of as \(t\) connected subgraphs in \(G\) that are pairwise disjoint such that there is at least one edge of \(G\) between each pair of subgraphs. Each such subgraph is called a branch set.
Lemma 2. Let $D$ be a tournament on $n \geq 5$ vertices. Then $h(X(D)) \geq n$.

Proof. Since $D$ is a tournament, $A(x,y)$ is interpreted as a single arc. Denote $V(D) = \{x,v_0,v_1,\ldots,v_{n-2}\}$. We now construct a collection of $n$ branch sets. For $0 \leq i \leq n-2$, let $B_i := \{A(x,v_i), A\{v_{i+1},v_{i+2}\}\}$. Let $U := \{A\{v_i,v_{i+2}\} \mid 0 \leq i \leq n-2\}$, where all subscripts are taken modulo $n-1$. Clearly, these branch sets are pairwise disjoint.

Next we show that each branch set is connected. Note that each $B_i$ induces $K_2$ in $X(D)$. Since $A\{v_i,v_{i+2}\}$ is adjacent to $A\{v_{i+1},v_{i+3}\}$ in $X(D)$, $U$ induces a subgraph that contains an $(n-1)$-cycle passing through each element of $U$.

Let $v$ be a vertex of a digraph $D$. Let $A \subseteq A(v)$. An arc $xy$ is said to be $A$-feasible if $vx \in A$, $y \notin v$ and $(v,x,y)$ is a directed path. A set $A^f \subseteq A(D)$ is $A$-feasible if each arc in $A^f$ is $A$-feasible and no two arcs in $A^f$ share a tail. An arc $xy$ of $D$ is said to be $A$-compatible if $y \neq v$, $A(v,x) \neq 0$ and $vx \notin A$. A set $A^c \subseteq A(D)$ is $A$-compatible if each arc in $A^c$ is $A$-compatible. Note that each feasible arc $xy$ is adjacent in $X(D)$ to each arc in $A$ except $vx$, and each compatible arc $xy$ is adjacent to each arc in $A$.

Let $A^f$ be an $A$-feasible set, and $A^c$ be an $A$-compatible set. An $(A,A^f,A^c)$-net of size $p$ is a $K_p$-minor in $X(D)$ using only arcs in $A \cup A^f \cup A^c$ such that $p := |A|$ and each branch set has exactly one arc in $A$. An $(A,A^f,A^c)$-net is called a net at $v$ if $v$ is the common tail of all arcs in $A$. It may happen that one of $A^f$ and $A^c$ is empty. The following lemma provides some sufficient conditions for the existence of an $(A,A^f,A^c)$-net.

Lemma 3. Let $v$ be a vertex of a digraph $D$. Let $A \subseteq A(v)$ and $p := |A|$. Let $A^f$ be an $A$-feasible set. Let $A^c$ be an $A$-compatible set. Then, in the following cases, $D$ contains an $(A,A^f,A^c)$-net.

1. $p = 1$;
2. $|A^f| \geq 1$ and $p = 2$;
3. $|A^f| = 3$ and $p = 3$;
4. $|A^f| \geq 1$ and $|A^c| \geq 1$ and $p = 3$;
5. $|A^c| \geq 2$ and $p = 3$;
6. $|A^f| + |A^c| \geq p - 1$ and $p \geq 4$.

Proof. Denote $A = \{v_{v_0},v_{v_1},\ldots,v_{v_{p-1}}\}$, and without loss of generality, assume that $A(v_j) - \{v_jv\} \neq \emptyset$ for $0 \leq j \leq |A^f| - 1$. Denote the elements of $A^f$ by $v_0v'_0, v_1v'_1, \ldots, v_{|A^f| - 1}v'_{|A^f| - 1}$. Note that $(v,v_j,v'_j)$ is a directed path for $0 \leq j \leq |A^f| - 1$. Consider the following possibilities:

1. $p = 1$: Then $\{v_{v_0}\}$ is a trivial $(A,\emptyset,\emptyset)$-net of size 1.
2. $|A^c| \geq 1$ and $p = 2$: Let $ww'$ be an $A$-compatible arc and $A^c := \{ww'\}$. Since $ww'$ is adjacent to each arc of $A$, $\{v_{v_0}\}, \{vv_1,ww'\}$ is an $(A,\emptyset,\emptyset)$-net of size 2.
(3) \(|A^f| = 3| p = 3: \) Then \(\{vv_0, v_1v_1\}, \{vv_1, v_2v_2\}\) and \(\{vv_2, v_0v_0\}\) form an \((A, A^f, \emptyset)\)-net of size 3.

(4) \(|A^f| \geq 1| \) and \(|A^c| \geq 1|\) and \(p = 3: \) Let \(ww'\) be an \(A\)-compatible arc and \(A^c := \{ww'\}.\) Note that \(ww'\) is adjacent to each \(vv_i, \) and \(v_0v_0\) is adjacent to \(vv_2\) in \(X(D).\) So \(\{vv_0, ww'\}, \{vv_1, v_0v_0\}\) and \(\{vv_2\}\) form an \((A, A^f, A^c)\)-net of size 3.

(5) \(|A^c| \geq 2|\) and \(p = 3: \) Similar to case (4), \(\{vv_0, ww'\}, \{vv_1, yy'\}\) and \(\{vv_2\}\) form an \((A, A^f, A^c)\)-net of size 3, where \(A^c\) contains two \(A\)-compatible arcs \(yy'\) and \(ww'.\)

(6) \(|A^f| + |A^c| \geq p - 1|\) and \(p \geq 4: \) Let \(\beta_j := v_jv_j'\) for \(0 \leq j \leq |A^f| - 1.\) Since \(|A^c| \geq p - 1 - |A^f|,\) we can choose \(p - 1 - |A^f|\) arcs from \(A^c\) and name them as \(\beta_{i|A^f|}, \beta_{i|A^f|+1}, \ldots, \beta_{p-2}.\) Define \(B_j := \{vv_j, \beta_{j+1}\}\) for \(0 \leq j \leq p - 3, B_{p-2} := \{vv_{p-2}, \beta_0\},\) and \(B_{p-1} := \{vv_{p-1}\}.\) For \(0 \leq i < j \leq p - 2,\) observe that in \(X(D), vv_j \in B_j\) is adjacent to \(\alpha_i\) if \(i \neq j - 1;\) and \(vv_i \in B_i\) is adjacent to \(\alpha_j\) if \(i = j - 1,\) and \(\alpha_j \in B_j - \{vv_j\}\) and \(\alpha_i \in B_i - \{vv_i\}.\) Thus, \(B_j, B_i\) are adjacent. In addition, since \(vv_{p-1} \in B_{p-1}\) is adjacent in \(X(D)\) to every \(\beta_j, B_{p-1}\) is adjacent to \(B_j\) with \(j \leq p - 2.\) Thus, \(B_0, \ldots, B_{p-1}\) form an \((A, A^f, A^c)\)-net of size \(p.\)

Note that if \(D\) contains an \((A, A^f, A^c)\)-net, then \(X(D)\) contains a \(K_p\)-minor and \(h(X(D)) \geq p.\)

A graph \(G\) with chromatic number \(k\) is called \(k\)-critical if \(\chi(H) < \chi(G)\) for every proper subgraph \(H\) of \(G.\) The following result is well known:

**Lemma 4.** Let \(G\) be a \(k\)-critical graph. Then

(a) \(G\) has minimum degree at least \(k - 1,\) when \(k \geq 2\); [7];

(b) no vertex-cut of \(G\) induces a clique when \(k \geq 3\) and \(G\) is noncomplete [8].

Let \(D\) be a simple digraph. For each arc \(uv \in A(D),\) define \(S_D(uv) := d^+(u) + d^+(v) - 1.\)

**Lemma 5.** For a simple digraph \(D,

\[ \sum_{uv \in A(D)} S_D(uv) = \sum_{v \in V(D)} d^+(v)(d(v) - 1), \]

where \(d(v) = d^+(v) + d^-(v).\)

**Proof.**

\[ \sum_{uv \in A(D)} S_D(uv) = \sum_{uv \in A(D)} (d^+(u) + d^+(v) - 1) \]

\[ = \sum_{uv \in A(D)} d^+(u) + \sum_{uv \in A(D)} d^+(v) - \sum_{uv \in A(D)} 1 \]

\[ = \sum_{u \in V(D)} d^+(u)d^+(u) + \sum_{v \in V(D)} d^+(v)d^-(v) - \sum_{u \in V(D)} d^+(u) \]

\[ = \sum_{w \in V(D)} d^+(w)(d^+(w) + d^-(w) - 1) \]

\[ = \sum_{w \in V(D)} d^+(w)(d(w) - 1). \]
3 Proof of Theorem 1

In this proof, we assume that, for every pair of distinct vertices $u$ and $v$ of $D$, there is at most one arc from $u$ to $v$ and at most one arc from $v$ to $u$. That is, $A_D(u,v) \subseteq \{uv, vu\}$. That is because all the arcs from $u$ to $v$ can be assigned the same colour and deleting arcs does not increase $h(X(D))$.

Let $D$ be a digraph. An arc $uv$ of $D$ is called redundant if $A_D(u) \subseteq A_D(u,v)$ or $A_D(v) \subseteq A_D(u,v)$. Note that if $uv$ is redundant then so is $vu$ if it exists. Let $D'$ be the digraph obtained from $D$ by deleting all redundant arcs. Let $G$ be the (simple) underlying undirected graph of $D'$. We have the following claim:

**Claim 1.** $\chi(X(D)) \leq \chi(G)$.

*Proof.* Since $G$ is the underlying undirected graph of $D'$, $V(G) = V(D') = V(D)$. Let $c : V(G) \to \{1, 2, \ldots, \chi(G)\}$ be a $(G)$-colouring of $G$. For each arc $uv \in A(D)$, define $f(uv) := c(u)$. We now show that $f$ is a 3-arc colouring of $D$. For every pair of adjacent arcs $uv, xz \in A(D)$, we have that $A_D(u,x) \neq \emptyset$ (that is, $u, x$ are adjacent), and both $uv$ and $xz$ are not in $A_D(u,x)$. Thus, some arc between $u$ and $x$ is not redundant, and $u$ and $x$ are adjacent in $G$. So, $f(uv) = c(u) \neq c(x) = f(xz)$. It follows that $f$ is a 3-arc colouring of $D$ and $\chi(X(D)) \leq \chi(G)$. \hfill $\Box$

Hadwiger’s conjecture is true for $k$-chromatic graphs with $k \leq 6$. So assume that $\chi(X(D)) \geq 7$. Let $k := \chi(G)$ and let $H$ be a $k$-critical subgraph of $G$. By Lemma 3(a), $\delta(H) \geq k - 1$.

Let $F$ be an orientation of $H$ such that each arc $uv$ of $F$ inherits the orientation of an arc in $A_D(u,v)$ and the number of out-degree 1 vertices in $F$ is minimized. An arc $xy \in A(D)$ is called potential if $xy \notin A(F)$. In particular, every redundant arc is potential. $F$ has the following property:

**Property A.** If $\delta^+(F)(v) = 1$ and $A_F(v) = \{vw\}$, then there exists one potential arc $vz$ outgoing from $v$ in $D$ such that $vz \neq vw$, and $z \notin V(F)$ or $\delta^+(F)(z) \in \{0, 2\}$.

*Proof.* Since $vw$ is not redundant, $A_D(v) \not\subseteq A_D(v,w)$. Let $vz \in A_D(v,w) - A_D(v,w)$. Then $vz \neq vw$. Since $vw$ is the unique outgoing arc from $v$ in $F$, $vz$ is potential. Suppose that $z \in V(F)$. Suppose first that $v$ and $z$ are not adjacent in $F$. Then each arc between $v$ and $z$ in $D$ including $vz$ is redundant. Since $vw \in A(D)$, $A_D(z) \subseteq A_D(z,v) = \emptyset$. That is, no arc is outgoing at $z$ in $D$ except possibly $zv$. Thus, $\delta^+(z) = 0$ as desired. Suppose next that $v$ and $z$ are adjacent in $F$. By the assumption that $\delta^+(F)(v) = 1$, $A_F(z,v) = zv$. If $\delta^+(F)(z) \neq 2$, let $F'$ be obtained from $F$ by replacing $zv$ by $vz$. Then $\delta^+(F')(z) \neq 1$, $\delta^+(F')(v) = 2$ and the out-degree of every other vertex remains unchanged. Hence $F'$ is an orientation of $H$ with less out-degree 1 vertices than $F$, which is a contradiction. \hfill $\Box$

In addition, for each arc $xy$ of $F$, by the definition of $D'$, $A_D(x) \not\subseteq A_D(x,y)$. That is, there is an arc other than $yx$ outgoing from $y$ (hence, $\delta^+_D(y) \geq 1$) and there is a directed path in $D$ of length 2 starting from the arc $xy$, even if $\delta^+_D(y) = 0$. Note that $F$ is a simple digraph and $\delta^+_D(v) = \delta^+_F(v) + \delta^+_F(v) = \delta^+_H(v) \geq k - 1$ by Lemma 3(a).

By Claim 1, it suffices to prove that $h(X(D)) \geq k$.

Let $v \in V(F)$ be a vertex with maximum out-degree $\Delta^+_F(v)$. If $\Delta^+_F(v) \geq k$, let $A \subseteq A_F(v)$ with $|A| = k$, and let $A^f$ be a maximal $A$-feasible set. Then $|A^f| = k \geq 6$ since there exists a directed path of length 2 starting from every arc of $A$. By Lemma 3(6) with $p = k$ and $q = 0$, there exists an $(A, A^f, \emptyset)$-net of size $k$. Thus, $h(X(D)) \geq k$, and the result holds.
Now assume that $\Delta^+(F) \leq k - 1$. By Lemma 5 and since $F$ has minimum degree at least $k - 1$,

$$\sum_{uv \in A(F)} S_F(uv) = \sum_{v \in V(F)} d^+_F(v)(d_F(v) - 1) \geq (k - 2) \sum_{v \in V(F)} d^+_F(v) = (k - 2)e(F), \quad (1)$$

where $e(F)$ is the number of arcs of $F$.

If $\sum_{uv \in A(F)} S_F(uv) = (k - 2)e(F)$, then $d_H(x) = d_F(x) = k - 1$ for every $x \in V(F)$. Since $\chi(H) = k$, by Brooks’ Theorem [5], $H \cong K_k$ and $F$ is a tournament. By Lemma 2 $h(X(D)) \geq h(X(F)) \geq k$, the result follows.

Now assume that $\sum_{uv \in A(F)} S_F(uv) > (k - 2)e(F)$. We call a vertex $v$ of $F$ special if $d^+_F(v) = k - 2$ and $d^-_F(v) = 1$ and $d^-_F(v') = 0$ for each $vv' \in A_F(v)$. Let $W$ be the set of all special vertices of $F$, and let $W^+ := \{xy \in A(F) \mid x \in W\}$. Let $F'$ be the digraph obtained from $F$ by deleting the arcs in $W^+$. Then, for each vertex $v$ of $F'$ with $d^+_F(v) = d_F(v) - 1 = k - 2$, the head of (at least) one arc $vv' \in A(F')$ is not a sink in $F$; that is, $d^-_F(v') \geq 1$. Since this outgoing arc at $v'$ in $F$ is not redundant, $|d^-_F(v')| \geq 2$.

Denote by $Q$ the set of sinks of $F$. Then each arc of $W^+$ has its tail in $W$ and head in $Q$. Note that $W$ is independent in $F$, and $W \cap Q = \emptyset$. By Lemma 5

$$(k - 2)e(F) < \sum_{uv \in A(F)} S_F(uv)$$

$$= \sum_{v \in V(F)} d^+_F(v)(d_F(v) - 1)$$

$$= \sum_{v \in V(F) - (W \cup Q)} d^+_F(v)(d_F(v) - 1) + \sum_{v \in Q} d^+_F(v)(d_F(v) - 1) + \sum_{v \in W} d^+_F(v)(d_F(v) - 1)$$

$$= \left( \sum_{v \in V(F') - (W \cup Q)} d^+_F(v)(d_F(v) - 1) \right) + 0 + (k - 2)\left(|W^+| + \sum_{v \in W} d^-_F(v)\right).$$

Since vertices in $W \cup Q$ have outdegree 0 in $F'$,

$$(k - 2)e(F) < \left( \sum_{v \in V(F')} d^+_F(v)(d_F(v) - 1) \right) + |W^+|(k - 2)$$

$$= \left( \sum_{uv \in A(F')} S_F(uv) \right) + |W^+|(k - 2).$$

Thus $\sum_{uv \in A(F')} S_F(uv) > (k - 2)(e(F) - |W^+|) = (k - 2)e(F')$. Let $uv$ be an arc of $F'$ with maximum $S_F(uv)$. Thus, $S_F(uv) \geq S_F(uv) \geq k - 1$. If $v \in W$, then $d^+_F(v) = 0$ and $d^-_F(u) \geq k$, which contradicts the assumption that $\Delta^+(F) \leq k - 1$. Hence $v \notin W$.

Denote $A_F(u) = \{uv, uu_1, uu_2, \ldots, uu_i\}$ and $A_F(v) = \{vv_1, vv_2, \ldots, vv_j\}$, where $i + j = S_F(uv) \geq k - 1$. Set $T := \{u_1, u_2, \ldots, u_i\} \cap \{v_1, v_2, \ldots, v_j\}$. Denote $N_1 := N_F(u) - \{u\}$ and $N_2 := N_F(v) - \{v\}$. Say $N_1 = \{x_1, x_2, \ldots, x_r\}$, and $N_2 = \{y_1, y_2, \ldots, y_s\}$. Since $F$ has minimum degree at least $k - 1$, both $r$ and $s$ are at least $k - 2$.

Since the arc $A_F(u, x_1)$ is not redundant, $A_D(x_1) \not\subseteq A_D(u, x_1)$. Thus, for each $x_1 \in N_1$, to arc $A_F(u, x_1) \in A(F)$ we can associate an arc, denoted $\varphi(u, x_1)$, which is chosen from $A_D(x_1) - A_D(u, x_1)$. Similarly, for each $y_l \in N_2$, associate an arc, denoted $\varphi(v, y_l)$, in $A_D(y_l) - A_D(v, y_l)$ to arc $A_F(v, y_l) \in A(F)$. 

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Choose these arcs $\varphi(u, x_l)$ and $\varphi(v, y_i)$ such that if $\Sigma := \cup_{l=1}^{t} \varphi(u, x_l)$ and $\Pi := \cup_{i=1}^{t} \varphi(v, y_i)$ then $t := |\Sigma \cap \Pi|$ is minimized. We now prove that, for each $ww' \in \Sigma \cap \Pi$, $ww'$ is the unique arc outgoing from $w$ in $D$, $A_F\{u, w\} = uw$, $A_F\{v, w\} = vw$ and $w' \notin \{u, v\}$. Since $ww' = \varphi(u, w) = \varphi(v, w)$, we have $w' \notin \{u, v\}$. Suppose that $|A_D(w)| \geq 2$, and $ww''$ is an arc outgoing from $w$ other than $ww'$ in $D$. Then at least one of $u$ and $v$, say $u$, is not equal to $w''$. Now set $\varphi(u, w) := ww''$ and keep $\varphi(v, w) = ww'$. Then $|\Sigma \cap \Pi|$ is decreased. Thus, $ww'$ is the unique arc outgoing from $w$ in $D$. Since $A_D(w) = \{ww'\}$, we have that $A_F\{u, w\} = uw$ and $A_F\{v, w\} = vw$.

Denote $\Sigma \cap \Pi = \{w_1w_1', w_2w_2', \ldots, w_1w_1'\}$. Then $w_i \in T$ for each $i \in [1, t]$ and $t \leq |T| \leq \min\{i, j\}$. Consider the following cases:

**Case 1.** $S_F(ww) \geq k$.

In this case, we construct an $(A, A^f, A^c)$-net $A$ and a $(B, B^f, B^c)$-net $B$, for some $A \subseteq A_F(u) - \{uv\}$ and $B \subseteq A_F(v)$, such that $(A \cup A^f \cup A^c) \cap (B \cup B^f \cup B^c) = \emptyset$. Since each branch set in $A$ contains an outgoing arc at $u$ other than $uv$, and each branch set in $B$ contains an outgoing arc at $v$ other than $vu$, each branch set in $A$ is adjacent in $X(D)$ to each branch set in $B$. Since each branch set in $A$ is contained in $A \cup A^f \cup A^c$, and each branch set in $B$ is contained in $B \cup B^f \cup B^c$, no branch set in $\mathcal{A}$ intersects a branch set in $B$. Hence $A \cup B$ defines a complete minor in $X(D)$ on $|A| + |B|$ vertices. In most cases we construct $A$ and $B$ such that $|A| + |B| \geq k$, giving a $K_k$-minor in $X(D)$, as desired. Finally, we always choose $A^c \subseteq \Sigma$ and $B^c \subseteq \Pi$ in such a way that $A^c \cap B^c = \emptyset$.

Note that $i + j \geq k$. By the assumption that $\Delta^+(F) \leq k - 1$, we have $1 \leq i \leq k - 2$ and $2 \leq j \leq k - 1$.

**Case 1.1.** $j = k - 1$: Then $i \geq 1$. Let $B := A_F(v)$, and $B^f$ be a maximal $B$-feasible set in $D$. For $y_i \in N_2$, since $A_D\{y_i, v\}$ is not redundant, $A_D^+(y_i) - A_D\{y_i, v\} \neq \emptyset$. Thus, $|B^f| = |B| = k - 1 \geq 4$. By Lemma 3(6) with $p = |B^f| = k - 1$ and $|B^c| = 0$, there exists in $D$ a $(B, B^f, \emptyset)$-net $B$ of size $k - 1$. Then $B \cup \{\{uv_1\}\}$ forms the $k$ branch sets of a $K_k$-minor in $X(D)$, since each branch set of $B$ contains an outgoing arc at $v$ other than $vu$ and is thus adjacent to $uv_1$ in $X(D)$ (since $vu \notin B$).

**Case 1.2.** $j \leq k - 2$: Then $0 \leq t \leq k - 2$. Recall that $t = |\Sigma \cap \Pi| \leq |T|$.

**Case 1.2.1.** $t = k - 2 \geq 3$: Suppose first that $\Sigma - \Pi \neq \emptyset$. Let $x_lx_l' \in \Sigma - \Pi$. Since $|A_F(u) - \{uv\}| = i \geq t \geq 3$, there are distinct arcs $uu_a, uu_b$ in $A_F(u) - \{uv\}$ with $x_l \notin \{u_a, u_b\}$. Let $A := \{uu_a, uu_b\}$. Note that $x_lx_l'$ is $A$-compatible. Then $A := \{uu_a, uu_b\}$ is an $(A, \emptyset, \{x_lx_l'\})$-net of size 2. Let $B$ be a set of $k - 2$ arcs in $A_F(v)$. Then $B^f := \{\varphi(v, y) : vy \in B\}$ is a $B$-feasible set of $k - 2$ arcs in $\Pi$. By Lemma 3(6) with $p = |A^f| = k - 2$ and $|A^c| = 0$, there is a $(B, B^f, \emptyset)$-net $B$ of size $k - 2$. Each branch set in $A$ contains an outgoing arc at $u$ other than $vw$, and each branch set in $B$ contains an outgoing arc at $v$ other than $vu$. Thus each branch set in $A$ is adjacent in $X(D)$ to each branch set in $B$. Since $x_lx_l' \notin \Pi$ and $B^f \subseteq \Pi$, we have $(A \cup \{x_lx_l'\}) \cap (B \cup B^f) = \emptyset$. Thus, no branch set in $\mathcal{A}$ intersects a branch set in $B$. Hence $A \cup B$ is a $K_k$-minor in $X(D)$.

By symmetry and since $uv$ is not used in this case, if $\Pi - \Sigma \neq \emptyset$, then we obtain a $K_k$-minor in $X(D)$.

Now assume that $\Sigma = \Pi$. Then $|\Sigma| = |\Pi| = t = k - 2$. Set $w_0 := v$ and $w_0' := w_1$. For $0 \leq l \leq t$, let $B_l := \{uu_l, w_{l+1}w_{l+1}'\}$, where subscripts are taken modulo $t + 1$; and let
exists a \(\leq\) to \(\arcs\) in \(\Pi\). Since \(A\), \(\leq\) to \(\arcs\) in \(\Pi\) share a tail. Let \(\branch\) sets of \(A\) \(-\) compatible. If \(t\) \(-\) feasible \((\leq\) \(\) 1 \()\). Similarly, let \(B\) \(-\) compatible. Note that \(A\) is \(A\) - compatible and \(A\) is \(A\) compatible. By Lemma 3(6), there exists an \((A, A^f, A^c)\)-net \(A\) of size \(t\) in \(X(D)\).

Next, for \(1 \leq l \leq k - t - 1\), set \(\beta_l := w_l w'\). Choose \(k - 2 - t\) arcs \(\alpha_{l+1}, \alpha_{l+2}, \ldots, \alpha_{l-3}\) from \(\Sigma - \Pi\) (which exist since \(\Pi - \Sigma = s - t \geq k - 2 - t\)). Note that \(\Sigma = \Pi\) \(t\) \(-\) compatible and \(2k - 2t - 3 \geq k - t\). Let \(B := \{vw_1, vw_2, \ldots, vw_k\}\). Then \(\beta_l\) is \(B\) - compatible when \(1 \leq l \leq k - t + 1\), and \(\beta_l\) is \(B\) - compatible when \(k - t \leq l \leq 2k - 2t - 3\). Let \(B^f := \{\beta_1, \beta_2, \ldots, \beta_k\}\), and \(B^c := \{\beta_{k-1}, \beta_{k-2}, \ldots, \beta_k\}\). Note that \(B^f\) is \(B\) - compatible and \(B^c\) is \(B\) - compatible. If \(t \leq k - 3\), then by Lemma 3(4), there exists a \((B, B^f, B^c)\)-net \(B\) of size \(k - t\) in \(X(D)\). Otherwise \(t \leq k - 4\) and by Lemma 3(6) with \(p = k - t \geq 4\) and \(|A^f| = k - t - 1\) and \(|A^c| = k - t - 2\), there exists a \((B, B^f, B^c)\)-net \(B\) of size \(k - t\) in \(X(D)\).

**Case 1.2.3.** \([k/2]\] \(\leq t \leq k - 3\): For \(k - t \leq l \leq t\), set \(\alpha_l := w_l w'\). Choose \(k - 2 - t\) arcs \(\alpha_{l+1}, \alpha_{l+2}, \ldots, \alpha_{l-3}\) from \(\Pi - \Sigma\) (which exist since \(\Pi - \Sigma = s - t \geq k - 2 - t\)). Note that \(\Pi = \Sigma\) \(t\) \(-\) compatible and \(2k - 2t - 3 \geq k - t\). Let \(A := \{u_1, u_2, \ldots, u_{t}\}\). Note that each arc in \(\Sigma\) is either \(A\) - compatible or \(A\) - compatible, and no two arcs in \(\Sigma\) share a tail. Let \(A^f\) \((A^c, \) respectively) be the set of \(A\) - compatible \((A\) - compatible, respectively) arcs in \(\Sigma\). Then \(A^f\) is \(A\) - compatible and \(A^c\) is \(A\) compatible. Note that \(|A^f| + |A^c| = |\Sigma| \geq i\), and \(A^c \neq \emptyset\) if \(i \leq 2\) \((\Sigma\) contains an \(A\) - compatible arc since \(|\Sigma| = r \geq k - 2 \geq 3\)). If \(i \geq 3\), then by Lemma 3(3) or Lemma 3(6) with \(p = |A^f| = i\), there is an \((A, A^f, \emptyset)\)-net \(A\) of size \(i\). If \(i \leq 2\), then \(A^c \neq \emptyset\) \((\Pi = r \geq k - 2 \geq 3\)). By Lemma 3(1) or (2) with \(p = |A^f| = i\) and \(|A^c| \geq 1\), there is an \((A, \emptyset, A^c)\)-net \(A\) of size \(i\). Similarly, let \(B \subseteq A_F(v)\) with \(|B| = j\). Let \(B^f\) \((B^c, \) respectively) be the set of \(B\) - compatible \((B\) - compatible, respectively) arcs in \(\Pi\). Note that \(|B^f| + |B^c| = |\Pi| = s \geq k - 2 \geq j\). As in the construction of \(A\), by Lemma 3 there exists a \((B, B^f, B^c)\)-net \(B\) of size \(j\). \(A \cup B\) forms a \(k\) branch sets of a \(K_k\)-minor in \(X(D)\).

Suppose that \(t \geq 1\) and \(j = k - 2\). If \(t = 1\), then let \(A\) be a subset of \(A_F(u) - \{uv\}\) with \(uv \in A\) and \(|A| = 2\). Note that \(|\Sigma - \Pi| = r - t \geq k - 3 \geq 3\). Then at least one arc in \(\Sigma - \Pi\) is \(A\) - compatible. If \(t \geq 2\), then let \(A := \{uw_1, uw_2\}\). Then \(|\Sigma - \Pi| = r - t \geq k - 2 - \frac{2t}{2} + 1 = \frac{2t}{2} \geq 2\) because \(k \geq 6\). Again, at least one arc in \(\Sigma - \Pi\) is \(A\) - compatible. In both cases, by Lemma 3(2), there exists an \((A, \emptyset, A^c)\)-net \(A\) of size \(2\), where \(A^f\) is the set of \(A\) - compatible arcs in \(\Sigma - \Pi\). Let \(B := A_F(v)\). Note that each arc in \(A\) is either \(B\) - feasible or \(B\) - compatible, and no two arcs in \(A\) share a tail. Let \(B^f\) \((B^c, \) respectively) be the set of \(B\) - feasible \((B\) - compatible, respectively) arcs in \(\Pi\). Since \(|\Pi| = s \geq k - 2 = j \geq 4\), by Lemma 3(6), there is a \((B, B^f, B^c)\)-net \(B\) of size \(j\). Then \(A \cup B\) forms the \(k\) branch sets of a \(K_k\)-minor in \(X(D)\).

Suppose now that \(t \geq 1\) and \(j \leq k - 3\). Note that \(i \geq t\). Consider two possibilities: (i) \(i = t\), and (ii) \(i \geq t + 1\). If \(i = t\), then \(t = i \geq k - j \geq 6\). Let \(A := \{uw_1, uw_2, \ldots, uw_t\}\). Note that \(|\Sigma - \Pi| = r - t \geq k - 2 - t \geq (2t + 1) - 2 = t - 1 \geq 2\). Since \(\Sigma \neq \emptyset\), at least one arc in \(\Sigma - \Pi\) is \(A\) - compatible. Let \(A^f\) \((A^c, \) respectively) be the set of \(A\) - feasible \((A\) - compatible, respectively) arcs in \(\Sigma - \Pi\). By Lemma 3(2), (4) or (6), there exists an
\((A, A^f, A^c)\)-net \(A\) of size \(i\). Let \(B := \{vv_1, vv_2, \ldots, vv_j\}\). Let \(B^f\) (\(B^c\), respectively) be the set of \(B\)-feasible (\(B\)-compatible, respectively) arcs in \(\Pi\). Note that \(j' = k-t \geq k - \left[\frac{s}{2}\right] + 1 = \left[\frac{t}{2}\right] + 1 \geq 3\) and \(|\Pi| = s \geq k-2 \geq j \geq j'\). By Lemma 3 there is a \((B, B^f, B^c)\)-net \(B\) of size \(j'\).

If \(i \geq t+1\), then \(j' = k-i \leq k-t-1\). Let \(B := \{vv_1, vv_2, \ldots, vv_j\}\) be a subset of \(A_F(v)\) with \(vv_1 \in B\). By the assumption that \(j \leq k-3\), there is at least one incoming arc other than \(uv\) at \(v\). Thus, at least one arc in \(\Pi - \Sigma\) is \(B\)-compatible. Let \(B^f\) (\(B^c\), respectively) be the set of \(B\)-feasible (\(B\)-compatible, respectively) arcs in \(\Pi - \Sigma\). Note that \(|\Pi - \Sigma| = s-t \geq k-t-2 \geq j' - 1\). By Lemma 3(2), (4) or (6), there is a \((B, B^f, B^c)\)-net \(B\) of size \(j'\). Let \(A := \{uu_1, uu_2, \ldots, uu_i\}\). Let \(A^f\) (\(A^c\), respectively) be the set of \(A\)-feasible (\(A\)-compatible, respectively) arcs in \(\Sigma\). Since \(|\Sigma| = r \geq k-2 \geq i\), by Lemma 3(2), (3) or (6), there exists an \((A, A^f, A^c)\)-net \(A\) of size \(i\).

In each case above, \(A \cup B\) forms a \(K_k\)-minor in \(X(D)\).

**Case 2.** \(S(uv) = k-1\): Then \(i + j = k - 1\).

In this case, we construct an \((A, A^f, A^c)\)-net \(A\) and a \((B, B^f, B^c)\)-net \(B\) as in Case 1, except that \(|A| + |B| = k - 1\). We then define one further branch set \(B_0\) that, with \(A\) and \(B\), forms the desired \(K_k\)-minor in \(X(D)\).

**Case 2.1.** \(j = 1\): Then \(i = k - 2\). Let \(A := A_F(u) - \{uv\}\). Let \(A^f\) (\(A^c\), respectively) be the set of \(A\)-feasible (\(A\)-compatible) arcs in \(\Sigma - \Pi\). Since \(t \leq \min\{i, j\} = 1\) and \(r \geq k - 2\), we have \(|\Sigma - \Pi| \geq r - t \geq k - 3\). Since \(|A^f| + |A^c| = |\Sigma - \Pi| \geq k - 3\) and \(i = k - 2 \geq 5\), by Lemma 3(6), there exists an \((A, A^f, A^c)\)-net \(A\) of size \(i\). By Property A, there exists a potential arc \(vz \neq vv_1\) outgoing from \(v\) in \(D\), such that \(z \notin V(F)\) or \(d_F^k(z) \in \{0, 2\}\). Clearly, \(z \neq u\) since \(d_F^k(u) = i + 1 > 3\). Let \(B := \{vv_1, vz\}\), and \(\tau\) be an arc in \(\Pi - \Sigma\) such that \(\tau \neq \varphi(v, vz)\) and \(\tau \neq \varphi(v, z)\). \(\tau\) exists because \(|\Pi - \Sigma| = s-t \geq k-2-t \geq k-3 \geq 3\). Then \(B := \{vv_1, \{vz, \tau\}\}\) is a \((B, \emptyset, \{\tau\}\)-net of size 2. Thus, \(A \cup B\) forms a \(K_k\)-minor in \(X(D)\).

**Case 2.2.** \(2 \leq j \leq k-3\): Then \(2 \leq i \leq k - 3\). Let \(U := N_1 \cap N_2\) be the common neighbourhood of \(u\) and \(v\) in \(F\). Say \(U = \{a_1, a_2, \ldots, a_{|U|}\}\). Then \(T \subseteq U\) and \(t \leq |T| \leq |U|\). Recall that \(t = |\Sigma \cap \Pi|\).

**Case 2.2.1.** \(t \geq 2\): Let \(A := A_F(u) - \{uv\}\). Since \(2 \leq t \leq \min\{i, j\}\), we have \(i = k-1-j \leq k-1 - t\). Since there is at least one incoming arc at \(u\) (because \(i \leq k-3\)), at least one arc in \(\Sigma - \Pi\) is \(A\)-compatible. Let \(A^f\) (\(A^c\), respectively) be the set of \(A\)-feasible (\(A\)-compatible) arcs in \(\Sigma - \Pi\). Note that \(|A^f| + |A^c| = |\Sigma - \Pi| \geq k - 3\) and \(i = k - 2 \geq 5\). By Lemma 3(2), (4), (5) or (6), there exists an \((A, A^f, A^c)\)-net \(A\) of size \(i\). Let \(B := A_F(v)\). Let \(B^f\) (\(B^c\), respectively) be the set of \(B\)-feasible (\(B\)-compatible) arcs in \(\Pi - \Sigma\). Similarly, a \((B, B^f, B^c)\)-net \(B\) of size \(j\) exists (since \(2 \leq i, j \leq k-3\) and \(uv\) is not in \(A\)).

Let \(B_0 := \{w_1, w_2, w_3, uv\}\). Then \(B_0\) induces a connected subgraph in \(X(D)\) by noting that \(uv\) is adjacent to both \(w_1\) and \(w_2\). Each branch set of \(A\) and \(B\) contains an arc outgoing from \(u\) or \(v\), which is adjacent to \(w_1\) or \(w_2\). Thus \(B_0\) is adjacent to each branch set of \(A \cup B\). Hence \(A \cup B \cup \{B_0\}\) forms a \(K_k\)-minor in \(X(D)\).

**Case 2.2.2.** \(t \leq 1\) and \(U \cap N_2^c(v) \neq \emptyset\): That is, there is an arc \(av\) in \(F\) for some vertex \(a \in U\). If there exists an arc \(\bar{a}a\) in \(D\) with \(\bar{a} \notin \{u, v\}\), then let \(B_0 := \{uv, \bar{a}a\}\).

Suppose that there is no such arc \(\bar{a}a\). That is, \(A_D(a) \subseteq \{au, av\}\). Clearly, \(av \in A_D(a)\). Since \(A_F\{v, a\}\) is not redundant in \(F\), we have \(A_D(a) - A_F\{v, a\} \neq \emptyset\). Thus \(au \in A_D(a)\) and \(A_D(a) = \{au, av\}\). Let \(\bar{a}\) be an in-neighbour other than \(u, v\) of \(a\) in \(F\). Then \(A_F\{a, \bar{a}\} = \bar{a}a\).
Let $\bar{a} \neq a$ be an out-neighbour of $a$ in $F$. Note that $\bar{a}$ exists since $aa$ is not redundant. Then, by the minimality of $|\Sigma \cap \Pi|$, we have $\bar{a}a \notin \Sigma \cap \Pi$. Let $B_0 := \{uv, au, av, \bar{aa}\}$. Then
\[\max\{|B_0 \cap \Sigma|, |B_0 \cap \Pi|\} \leq 2 \text{ and } |B_0 \cap \Sigma| + |B_0 \cap \Pi| \leq 3.\]

Let $A := Af(u) - \{uv\}$ and $B := Af(v)$. We show that there is a net $A$ at $u$ of size $i$, and a net $B$ at $v$ of size $j$, such that $A \cup B \cup \{B_0\}$ forms a $K_k$-minor in $X(D)$.

First suppose that $3 \leq i, j \leq k - 4$. If $|B_0 \cap \Sigma| \leq 1$, let $A^f (A^c)$, respectively be the set of $A$-feasible ($A$-compatible) arcs in $\Sigma - \Pi - B_0$. If $|B_0 \cap \Sigma| = 2$, then $|B_0| = 4$ and $\bar{a}a \in \Sigma \cap B_0$. Thus, $\bar{a}$ is a neighbour of $u$ in $F$. Note that $\bar{a}a \notin \Sigma$ and $\bar{a}a$ is $A$-feasible or $A$-compatible. Let $A^f (A^c)$, respectively be the set of $A$-feasible ($A$-compatible) arcs in $(\Sigma - \Pi - B_0) \cup \{\bar{aa}\}$. In both cases, $|A^f| + |A^c| \geq r - t - 1 \geq k - 2 - 2 \geq i$. By Lemma 3(3), (4), (5) or (6), there exists an $\{A, A^f, A^c\}$-net $A$ of size $i$. Let $B^f$ ($B^c$, respectively) be the set of $B$-feasible ($B$-compatible) arcs in $\Pi - (B_0 \cup \{\bar{aa}\})$. Note that all arcs of $B_0 \cup \{\bar{aa}\}$ except $uv$ are outgoing from at most two vertices (that is, $a$ and $\bar{a}$). We have $|B^f| + |B^c| = |\Pi - (B_0 \cup \{\bar{aa}\})| \geq s - 2 \geq k - 4 \geq j$. Similarly, by Lemma 3 a $(B, B^f, B^c)$-net $B$ of size $j$ exists.

Next suppose that $i = k - 3$ and $j = 2$. If $|B_0 \cap \Sigma| \leq 1$, let $A^f (A^c)$, respectively be the set of $A$-feasible ($A$-compatible) arcs in $\Sigma - B_0$. If $|B_0 \cap \Sigma| = 2$, let $A^f (A^c)$, respectively be the set of $A$-feasible ($A$-compatible) arcs in $(\Sigma - B_0) \cup \{\bar{aa}\}$, where $a, a$ are as above. In both cases, we have $|A^f| + |A^c| \geq r - 1 \geq k - 3 = i$. By Lemma 3(6), there exists an $\{A, A^f, A^c\}$-net $A$ of size $i$. Let $B^f$ ($B^c$, respectively) be the set of $B$-feasible ($B$-compatible) arcs in $\Pi - \Sigma - (B_0 \cup \{\bar{aa}\})$. Since $v$ has in $F$ at least $k - 3 \geq 4$ in-neighbours, one of which is not in $\{u, a, \bar{a}\}$. Thus $B^c \neq \emptyset$. By Lemma 3(2), a $(B, B^f, B^c)$-net $B$ of size $2$ exists.

Suppose that $i = 2$ and $j = k - 3$. Let $B^f (B^c$, respectively) be the set of $B$-feasible ($B$-compatible) arcs in $\Pi - B_0$. Then $|B^f| + |B^c| = |\Pi - B_0| \geq s - 2 \geq k - 4 = j - 1$. By Lemma 3(6), there exists a $(B, B^f, B^c)$-net $B$ of size $j$. If $|B_0 \cap \Sigma| \leq 1$, let $A^f (A^c)$, respectively be the set of $A$-feasible ($A$-compatible) arcs in $\Sigma - \Pi - B_0$. If $|B_0 \cap \Sigma| = 2$, let $A^f (A^c)$, respectively be the set of $A$-feasible ($A$-compatible) arcs in $(\Sigma - \Pi - B_0) \cup \{\bar{aa}\}$, where $a, a$ are as above. In both cases, $|A^f| + |A^c| \geq r - t - 1 \geq k - 2 - 2 \geq 3$. Recall that $A = \{uu_1, uu_2\}$. Note that $|(A^f \cup A^c) - \{\varphi(u, u_1)\}| \geq 2$. Let $\tau_1, \tau_2$ be two arcs in $(A^f \cup A^c) - \{\varphi(u, u_1)\}$. Then, at least one arc, $\tau_2$ say, of $\tau_1, \tau_2$ is not equal to $\varphi(u, u_2)$. Note that $\tau_2$ is adjacent to both $uu_1$ and $uu_2$, and $\tau_1$ is adjacent to $uu_1$ in $X(D)$. Let $A := \{uu_1, \tau_1\}, \{uu_2, \tau_2\}$. Then, $A$ is a $(A, A^f, A^c)$-net of size 2.

In each case, $B_0$ induces a connected subgraph in $X(D)$. And $uv \in B_0$ is adjacent to each branch set of $A$, and an arc outgoing from $a$ other than $av$ is adjacent to each branch set of $B$. Hence $A \cup B \cup \{B_0\}$ forms a $K_k$-minor in $X(D)$.

**Case 2.2.3.** $t \leq 1$ and $U \cap N_F(v) = \emptyset$ and $|U| \geq 2$. That is, each arc in $F$ between a vertex of $U$ and $v$ is outgoing at $v$. Let $A := Af(u) - \{uv\}$ and $B := Af(v)$. We consider two situations.

First suppose that $U$ is not independent in $F$. That is, there is an arc $\tau$ in $F$ joining two vertices in $U$. Say, $\tau = a_1a_2$. Since $Af\{u, a_2\}$ is not redundant, in $D$ there is an arc $\gamma \neq a_2u$ outgoing from $a_2$. (It may happen that $\gamma \in \{a_2a_1, a_2v\}$.) Let $B_0 := \{uv, \tau, \gamma\}$. Since $uv$ is adjacent to both $\tau$ and $\gamma$, $B_0$ induces a connected subgraph in $X(D)$. Note that
\[\max\{|B_0 \cap \Sigma|, |B_0 \cap \Pi|\} \leq 2.\]

If $i \geq j$, then $j > \frac{k - i}{2} \leq k - 4$. Let $A^f (A^c)$, respectively be the set of $A$-feasible ($A$-compatible) arcs in $\Sigma - B_0$; and, let $B^f (B^c$, respectively be the set of $B$-feasible ($B$-compatible) arcs in $\Pi - \Sigma - B_0$. Then $|A^f| + |A^c| \geq r - 2 \geq k - 2 - 2 \geq i - 1$. By Lemma 3(6), there exists
an \((A, A^f, A^c)\)-net \(A\) of size \(i\). Also, \(|B^f| + |B^c| = |\Pi - \Sigma - B_0| \geq s - t - 2 \geq k - 5 \geq j - 1\). Note that there is at least one (in fact many) incoming arc \(v u\) at \(v\) with \(\varphi(v, v) \notin \Sigma \cup B_0\). Thus \(\varphi(v, v) \in B^c\) and \(|B^c| \geq 1\). By Lemma 3(2), (4) or (6), a \((B, B^f, B^c)\)-net \(B\) of size \(j\) exists. If \(i \leq j\), then \(i \leq \frac{1}{4} - 1 \leq k - 4\). Now let \(A^f\) (respectively) be the set of \(A\)-feasible \((A\)-compatible\) arcs in \(\Sigma - \Pi - B_0\); and let \(B^f\) \((B^c)\) be the set of \(B\)-feasible \((B\)-compatible\) arcs in \(\Pi - B_0\). Similarly, we obtain an \((A, A^f, A^c)\)-net \(A\) of size \(i\) and a \((B, B^f, B^c)\)-net \(B\) of size \(j\).

Since each arc outgoing from \(u\) or \(v\) is adjacent to \(\tau\) or \(\gamma\), each branch set of \(A \cup B\) is adjacent to \(B_0\). Thus, \(A \cup B \cup \{B_0\}\) forms a \(K_k\)-minor in \(X(D)\).

Next suppose that \(U\) is independent in \(F\). For each \(a_l \in U\), if \(D\) has an arc \(a_l a_l'\) other than \(a_l u\) or \(a_l v\), let \(Q_l := \{a_l a_l'\}\). Otherwise, we have \(A(D(a_l)) = \{a_l u, a_l v\}\). Let \(a_l\) be an in-neighbour other than \(u, v\) of \(a_l\) in \(F\). Then \(A(D(a_l)) = \{a_l u, a_l v\}\). Let \(a_l\) be an out-neighbour of \(a_l\) in \(F\). Let \(Q_l := \{a_l u, a_l v, a_l a_l\}\). Let \(a_l, a_m\) be distinct vertices in \(U\) such that \(w_l \in \{a_l, a_m\}\) when \(\pi = 1\) and \(Q_l \cup Q_m\) is minimised. Let \(B_0 := \{uv\} \cup Q_l \cup Q_m\). Note that in \(X(D)\) each of the subgraphs induced on \(Q_l\) and \(Q_m\) is connected and adjacent to \(uv\), \(B_0\) induces a connected subgraph.

Note that for each \(p \in \{l, m\}\), \(|Q_p \cap \Sigma| \leq 2\) and \(|Q_p \cap \Pi| \leq 2\). If \(|Q_p \cap \Sigma| = 2\), then \(Q_p := \{a_l u, a_l v, a_l a_l\}\) and \(a_l a_l' \in \Sigma\) and \(a_l'\) is adjacent to \(u\) (but not \(v\) because \(U\) is independent) in \(F\). Thus \(a_l a_l'\) is \(A\)-feasible \((A\)-compatible\) if \(a_l a_l'\) is \(A\)-feasible \((A\)-compatible\)). Let \(\Sigma'\) be obtained from \(\Sigma\) by replacing \(a_l a_l'\) with \(a_l a_l\). Then \(|Q_p \cap \Sigma'| \leq 1\) and \(|B_0 \cap \Sigma'| \leq 2\). In addition, each element in \(\Sigma'\) is \(A\)-feasible or \(A\)-compatible, and no two share a tail. Similarly, we can obtain \(\Pi'\) such that each of its elements is \(A\)-feasible or \(A\)-compatible, no two elements share a tail and \(|B_0 \cap \Pi'| \leq 2\).

Let \(A^f\) \((A^c)\) be the set of \(A\)-feasible \((A\)-compatible\) arcs in \(\Sigma' - B_0\); and let \(B^f\) \((B^c)\) be the set of \(B\)-feasible \((B\)-compatible\) arcs in \(\Pi' - B_0\). Then, \(|A^f| + |A^c| \geq r - 2 \geq k - 2 \geq i - 1\). Also, \(|B^f| + |B^c| = |\Pi' - B_0| \geq s - t - 2 \geq k - 4 \geq j - 1\). When \(i = 2\), since \(|A^f| + |A^c| \geq k - 4 \geq 3\), we have \(A^c \neq \emptyset\). Analogously, we have that \(B^c \neq \emptyset\) when \(j = 2\).

By Lemma 3(2)-(6), there exist an \((A, A^f, A^c)\)-net \(A\) of size \(i\) and a \((B, B^f, B^c)\)-net \(B\) of size \(j\).

Since each arc outgoing from \(u\) or \(v\) is adjacent to an arc in \(Q_l\) or \(Q_m\), each branch set of \(A \cup B\) is adjacent to \(B_0\). Thus, \(A \cup B \cup \{B_0\}\) forms a \(K_k\)-minor in \(X(D)\).

Case 2.2.4. \(U \cap N_F^c(v) = \emptyset\) and \(|U| \leq 1\) (hence \(t \leq 1\)): That is, \(u\) and \(v\) share at most one neighbour \(a_1\) in \(F\). If \(a_1\) exists, the arc between \(a_1\) and \(v\) in \(F\) is \(e_{a1}\). Let \(A := A_F(u) - \{uv\}\) and \(B := A_F(v)\).

Since \(\delta(F) \geq k - 1\) and \(j \leq k - 3\), \(v\) has at least \(k - 1 - j \geq 2\) in-neighbours in \(F\). Say, \(N_F^c(v) = \{u, y_j+1, y_j+2, \ldots, y_{k-2}\}\). Note that \(N_F^c(v) - \{u\} \neq \emptyset\). Recall that \(N_F^c(v) = \{v_1, v_2, \ldots, v_j\}\).

Let \(H\) be obtained from \(H\) by deleting vertices in \(U \cup \{u, v\}\). By Lemma 3(4b), \(H\) is connected. Let \(P_0 := (z_1, z_2, \ldots, z_m)\) be a shortest path in \(H\) between \(N_F(u) - \{v\} \cup U\) and \(N_F^c(v) - \{u\} \cup U\), where \(m \geq 2\) (because \(u\) and \(v\) share no common neighbour in \(H\), \(z_1 \in N_F(u) - \{v\} \cup U\) and \(z_m \in N_F^c(v) - \{u\} \cup U\). Then each internal vertex of \(P_0\) is not adjacent to \(u\) in \(F\).

If \(|V(P_0) \cap N_F(v)| = 1\), then \(z_m\) is the only neighbour of \(v\) in \(F\) which is on \(P_0\). Let \(P := P_0\) and set \(z := z_m\). If \(|V(P_0) \cap N_F(v)| \geq 2\), let \(P = (z_1, z_2, \ldots, z_i)\) be the subpath of \(P_0\) such that \(z_i \in N_F(v)\) and \(|V(P) \cap N_F(v)| = 2\).

We shall construct a branch set \(P'\) consisting of arcs alongside \(P\). Let \(z_0 = u\) and \(z_{i+1} = v\).

For \(1 \leq g \leq l\), we associate to \(z_g\) the set \(Q_g\) of arcs as follows. If \(A_D(z_g) - (A_D(z_{g-1}, z_g) \cup A_D(z_g, z_{g+1})) \neq \emptyset\), then let \(Q_g\) be a singleton set that contains exactly one arc, say, \(z_g, z_g' \in A_D(z_g) - (A_D(z_{g-1}, z_g) \cup A_D(z_g, z_{g+1}))\). Otherwise, \(A_D(z_g) - (A_D(z_{g-1}, z_g) \cup A_D(z_g, z_{g+1})) = \emptyset\).
Since the arc $A_F(\zeta_g, \zeta_{g+1}) \in A(F)$ is not redundant, $\zeta_g \zeta_{g+1} \in A_D(z_g)$. Similarly, $\zeta_g \zeta_{g+1} \in A_D(z_g)$ since $A_F(\zeta_{g+1}, \zeta_g) \in A(F)$ is not redundant. Let $z_g$ be an in-neighbour of $z_g$ in $F$. Then $z_g \zeta_g \in A(F)$. Let $\bar{z}_g \bar{z}_g$ with $\bar{z}_g \neq \zeta_g$ be an arc outgoing from $\bar{z}_g$ in $D$ (which exists because $\zeta_g \zeta_g$ is not redundant). Set $Q_g := \{z_g \bar{z}_g, z_g \zeta_{g+1}, z_g \bar{z}_g\}$. Note that $Q_g$ induces a connected subgraph in $X(D)$ since $\bar{z}_g \zeta_g$ is adjacent to both $\zeta_g \bar{z}_g$ and $\zeta_g \zeta_{g+1}$.

In the case where $V(P) \cap N_F(v) = \{z_p, z_q\}$ ($p < l$) and $Q_p = \{z_p v\}$, we slightly modify $Q_p$ as $\{z_p v, \gamma\}$, where $\gamma \in A_D(z_p) - \{z_p v\}$ (which exists because $A_F(z_p, v)$ is not redundant).

Let $P' := \bigcup_{g=1}^l Q_g$. Then, for $1 \leq g \leq l - 1$, since $Q_g$ contains an arc outgoing from $z_g$ other than $z_g \zeta_{g+1}$ and $Q_{g+1}$ contains an arc outgoing from $z_{g+1}$ other than $z_{g+1} \bar{z}_g$, each $Q_g$ is adjacent to $Q_{g+1}$ in $X(D)$. Thus, $P'$ induces a connected subgraph in $X(D)$. We call $P'$ a parallel set of $P$.

Let $\Sigma$ and $\Pi$ be as above. We have the following claim:

**Claim 2.** (a) There is a set $\Sigma'$ such that $|\Sigma'| \geq |\Sigma| - 1$ and $P' \cap \Sigma' = \emptyset$, and each element of which is $A$-feasible or $A$-compatible and no two elements share a tail;

(b) There is a set $\Pi'$ such that $|\Pi'| \geq |\Pi| - 2$ and $P' \cap \Pi' = \emptyset$, and each element of which is $B$-feasible or $B$-compatible and no two elements share a tail.

**Proof.** (a) Initially, set $\Sigma' := \Sigma - P'$. Clearly, all properties except $|\Sigma'| \geq |\Sigma| - 1$ in (a) are satisfied. If $|P' \cap \Sigma| \leq 1$, then we are done. Suppose that $|P' \cap \Sigma| \geq 2$. Since $P_0$ is a shortest path in $\bar{H}$ between $N_F(u) - \{v\} \cup U$ and $N_F(v) - \{u\} \cup U$, each vertex $z_g$ on $P$ with $g \geq 3$ is not adjacent to a vertex of $N_F(u) - \{v\} \cup U$. Thus, $Q_g \cap \Sigma = \emptyset$ for each $g \geq 3$. We now consider $g = 2$. Since $z_2$ is not adjacent to $u$ in $\bar{H}$, we have $|Q_2 \cap \Sigma| \leq 1$ and if $|Q_2 \cap \Sigma| = 1$ then $|Q_2| = 3$ and $Q_2 := \{z_2 z_1, z_2 z_3, \bar{z}_2 \bar{z}_2\}$, where $\bar{z}_2$ is an in-neighbour of $z_2$ in $F$. Since $z_2$ is not adjacent to $u$, $Q_2 \cap \Sigma = \{z_2 \bar{z}_2\}$, which means that $z_2$ is adjacent to $v$ in $F$ and $\varphi(u, z_2) = \bar{z}_2 \bar{z}_2$. In this case, update $\Sigma' := \Sigma' \cup \{\bar{z}_2 \bar{z}_2\}$. Note that $\bar{z}_2 \bar{z}_2$ is $A$-feasible or $A$-compatible.

If $|Q_1 \cap \Sigma| \leq 1$, then $\Sigma'$ is the desired set. Suppose that $|Q_1 \cap \Sigma| = 2$. Let $Q_1 := \{z_1 u, z_1 z_2, \bar{z}_1 \bar{z}_1\}$, where $\bar{z}_1$ is an in-neighbour of $z_1$ in $F$. Then, $Q_1 \cap \Sigma = \{z_1 z_2, \bar{z}_1 \bar{z}_1\}$, which means $\varphi(u, z_1) = z_1 z_2$ and $\varphi(u, \bar{z}_1) = \bar{z}_1 \bar{z}_1$. Note that $z_1 \bar{z}_1$ is $A$-feasible or $A$-compatible. By adding $z_1 \bar{z}_1$ into $\Sigma'$, we get that $|Q_1 \cap \Sigma'| \leq 1$. Then $|\Sigma'| \geq |\Sigma| - 1$, as desired.

(b) Initially, set $\Pi' := \Pi - P'$. Recall that $P$ contains at most two neighbours, $z_{g_1}$ and $z_{g_2}$, say, of $v$. Let $\gamma$ be an arc in $\Pi \cap P'$ such that there is a $Q_g$ containing $\gamma$ (there may be more than one $Q_g$ containing $\gamma$) and $g \notin \{g_1, g_2\}$. Since $z_g$ is not adjacent to $v$ in $\bar{H}$, we have $|Q_g| = 3$ and $Q_g = \{z_g \bar{z}_g, z_g \zeta_{g+1}, \bar{z}_g \bar{z}_g\}$, where $\bar{z}_g$ is an in-neighbour of $z_g$ in $F$ and $\bar{z}_g \bar{z}_g \neq \bar{z}_g \bar{z}_g$ is an arc outgoing from $\bar{z}_g$ in $D$. Further, $\bar{z}_g$ is a neighbour of $v$ in $F$ and $\varphi(v, \bar{z}_g) = \bar{z}_g \bar{z}_g$. Note that $\bar{z}_g \zeta_g \notin \Pi$ is $B$-feasible or $B$-compatible. Now update $\Pi'$ by adding $\bar{z}_g \zeta_g$. That is, $\Pi' := \Pi' \cup \{\bar{z}_g \zeta_g\}$. By repeating this procedure for all such $\gamma$, we obtain a $\Pi'$ with the same size as $\Pi - (Q_q \cup Q_g)$.

For each $g \in \{g_1, g_2\}$, if $|\Pi \cap Q_g| = 2$, we will add a $B$-feasible or $B$-compatible arc into $\Pi'$. Then $|\Pi'| \geq |\Pi| - 2$, as desired. Suppose that $|\Pi' \cap Q_g| = 2$ for some $g \in \{g_1, g_2\}$. Then $Q_g = \{z_g \bar{z}_g, z_g \zeta_{g+1}, \bar{z}_g \bar{z}_g\}$, where $\bar{z}_g$ is an in-neighbour of $z_g$ in $F$ and $\bar{z}_g \bar{z}_g \neq \bar{z}_g \bar{z}_g$. And, $\bar{z}_g$ is a neighbour of $v$ in $F$ with $\varphi(v, \bar{z}_g) = \bar{z}_g \bar{z}_g$. Note that $\bar{z}_g \zeta_g \notin \Pi$ is $B$-feasible or $B$-compatible. Set $\Pi' := \Pi' \cup \{\bar{z}_g \zeta_g\}$. Then $|\Pi'| \geq |\Pi| - 2$. Consequently, we get the desired $\Pi'$.

Let $B_0 := \{uv\} \cup P'$. Then $B_0$ induces a connected subgraph in $X(D)$ since $uv$ is adjacent to $Q_1$. 

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Next we show that there exists a net of size $i$ at $u$ and a net of size $j$ at $v$ such that none of their branch sets intersects $B_0$.

If $j = 2$ (hence $i = k - 3$), then at least one arc, say $\gamma$, in $\Pi' - \Sigma'$ is $B$-compatible (since there are more incoming arcs at $v$). Let $B'^c := \{ \gamma \}$. Since $|\Pi' - \Sigma'| \geq s - 2 + 1 \geq k - 5 \geq j = 2$, by Lemma 3(2), there exists a $(B, \emptyset, B'^c)$-net $B$ of size $j = 2$. Similarly, let $A^f$ (respectively) be the set of $A$-feasible ($A$-compatible, respectively) arcs in $\Sigma'$. Note that $|\Sigma'| \geq r - 1 \geq k - 3 = i \geq 4$. By Lemma 3(6), there exists an $(A, A^f, A^c)$-net $A$ of size $i$.

Suppose that $3 \leq j \leq k - 3$ (hence $2 \leq i \leq k - 4$). Let $B^f$ (respectively, $B'^c$) be the set of $B$-feasible ($B$-compatible) arcs in $\Pi'$. Since $|\Pi'| \geq s - 2 + 1 \geq k - 4 \geq j - 1$ and $B'^c \neq \emptyset$ when $j = 3$, by Lemma 3(4) or (6), there exists a $(B, B^f, B'^c)$-net $B$ of size $j$. Let $A^f$ (respectively) be the set of $A$-feasible ($A$-compatible, respectively) arcs in $\Sigma' - \Pi'$. We now show that there exists a net of size $i$ at $u$. If $i \geq 3$, then $|\Sigma' - \Pi'| \geq r - 1 - 1 \geq k - 4 \geq i \geq 3$. By Lemma 3(3) or (6), there exists an $(A, A^f, A^c)$-net $A$ of size $i$. Suppose that $i = 2$. Note that $|\Sigma' - \Pi'| \geq k - 4 \geq 3$ (because $k \geq 7$) and there are at least three incoming arcs at $u$ in $F$. $\Sigma' - \Pi'$ contains at least two $A$-compatible arcs, say, $\gamma_1$ and $\gamma_2$. Let $A := \{(uu_1, \gamma_1), (uu_2, \gamma_2)\}$. Then $A$ is a net of size $2$ at $u$.

Since each element of $A$ constructed above contains an arc $xx'$, which is outgoing from a neighbour $x \neq v$ of $u$ and $x' \neq u$, each element of $A$ is adjacent to $B_0$ because $uv \in B_0$ is adjacent to each $xx'$. Note that $|V(P) \cap N_F(v)| \in \{1, 2\}$. In the case when $|V(P) \cap N_F(v)| = 1$, $P'$ contains an arc $yy'$, which is outgoing from an in-neighbour $y \neq u$ of $v$ and $y' \neq v$. Since such a $yy'$ is adjacent to every arc of $A_F(v)$, it is adjacent to every element of $B$ constructed above.

In the case when $|V(P) \cap N_F(v)| = 2$, $P'$ contains two arcs $\alpha$ and $\beta$, each of them is outgoing from a neighbour of $v$ other than $u$ and heading to a vertex other than $v$. Then each arc of $A_F(v)$ is adjacent to either $\alpha$ or $\beta$. So every element of $B$ is adjacent to $P' \subseteq B_0$. Therefore, $\{B_0\} \cup A \cup B$ forms a $K_k$-minor in $X(D)$.

**Case 2.3.** $j = k - 2$: Then $i = 1$. Suppose first that $d^+_F(v) = 1$; that is, $uv$ is the only incoming arc at $v$ and $d^+_F(v) = k - 1$. Since $v$ is not special, one out-neighbour $v'$ of $v$ in $F$ is not a sink. Now consider the arc $vv'$. If $d^+_F(v') \geq 2$, then $S_F(vv') = d^+_F(v) + d^+_F(v') - 1 \geq k - 2 + 2 - 1 = k - 1$. This is a special case of Case 2.2 and thus can be treated similarly. If $d^+_F(v') = 1$, then by Property A, one potential arc $v'v''$ (not $v'v$) is outgoing from $v'$ in $D$ but not present in $F$ (since $d^+_F(v') = 1$). Let $F'$ be obtained from $F$ by adding $v'v''$. Again we have $S_{F'}(vv') = d^+_F(v) + d^+_F(v') - 1 \geq k - 2 + 2 - 1 = k - 1$, and this can also be treated similarly.

Suppose next that $d^+_F(v) \geq 2$. Then $t \leq 1$. This case can be dealt with by a similar way as in Cases 2.2.3 or 2.2.4.

**Case 2.4.** $j = k - 1$: Then $i = 0$, which implies $d^+_F(u) = 1$. By Property A, there exists a potential arc $uz \neq uv$ in $D$. Then $A := \{\{uz\}\}$ is a $(\{uz\}, \emptyset, \emptyset)$-net. Let $B := A_F(v)$. Let $B^f$ (respectively) be the set of $B$-feasible ($B$-compatible, respectively) arcs in $\Pi$. By Lemma 3(6), a $(B, B^f, B'^c)$-net $B$ of size $j$ exists. It is not hard to see that $A \cup B$ forms a $K_k$-minor in $X(D)$.

This completes the proof of Theorem 1.

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