Dynamical typicality of isolated many-body quantum systems

Peter Reimann
Fakultät für Physik, Universität Bielefeld, 33615 Bielefeld, Germany

Dynamical typicality refers to the property that two pure states, which initially exhibit (almost) the same expectation value for some given observable $A$, are very likely to exhibit also very similar expectation values when evolving in time according to the pertinent Schrödinger equation. We unify and generalize a variety of previous findings of this type for sufficiently high dimensional quantum mechanical model systems. Particular emphasis is put on the necessary and sufficient conditions, which the initial expectation value and the spectrum of $A$ have to fulfill.

I. INTRODUCTION

The following quite remarkable feature of isolated, high dimensional quantum systems has been discovered and named “dynamical typicality” in a hallmark paper by Bartsch and Gemmer [1]. The vast majority of all pure states with very similar expectation values of some observable at a given initial time, will yield very similar expectation values of the same observable also at any later time. Most notably, this prediction does not depend on any details of the Hamiltonian which governs the time evolution. However, the restrictions in Ref. [1] regarding the initial expectation value and also the considered observable itself were still quite significant. In an entirely unrelated work, Müller, Gross, and Eisert explored by means of concentration of measure concepts the typical properties of random pure states with a fixed expectation value of some observable [2]. Somewhat similar investigations were independently carried out even earlier by Fine in Ref. [3]. Yet another related finding is contained in the last paragraph of Ref. [4]: If pure initial states are randomly sampled according to any statistical ensemble that corresponds to a density operator of low purity, then most of them exhibit very similar expectation values at any later time. A less general version of the same result was also established and numerically exemplified in Ref. [5].

The main objective of the present work is to unify and extend all those previously independent approaches and results. In particular, Ref. [2] closes with “the hope that methods similar to the ones established here help answering questions of typicality in the context of quantum dynamics and addressing key open problems in the theory of relaxation of non-equilibrium complex quantum systems.” Progress along these lines is exactly at the focus of our present explorations.

II. GENERAL FRAMEWORK

We consider an isolated quantum system, which is described by means of some Hilbert space $\mathcal{H}$ of large but finite dimension $N$, and whose dynamics is governed by an arbitrary Hamiltonian $H : \mathcal{H} \to \mathcal{H}$. Choosing any normalized $|\phi\rangle \in \mathcal{H}$ as initial state (at $t = 0$), its time evolution can be written as $|\phi(t)\rangle = U_t|\phi\rangle$, where $U_t := e^{-iHt/\hbar}$. Going over from the Schrödinger to the Heisenberg picture of quantum mechanics, the resulting expectation value for an arbitrary observable $O$ at time $t$ thus follows as

$$\langle \phi(t)|O|\phi(t)\rangle = \langle \phi|O_t|\phi\rangle,$$  

$$O_t := U_t^* O U_t.$$  

The main objective of our paper is to establish dynamical typicality in the following sense: For the vast majority of all initial states $|\phi\rangle \in \mathcal{H}$, which exhibit (almost) identical expectation values $\langle \phi|A|\phi\rangle$ for some given observable $A$, also the expectation values for any arbitrary but fixed observable $O$ and time point $t$ in (1) will be very similar to each other. Unlike in Ref. [1], the two observables $A$ and $O$ are thus not any more required to be identical. A more precise meaning of the term “vast majority” will be provided later. We also note that for any given $t$ and $O$, there may still be a small set of “untypical” initial states $|\phi\rangle$, entailing significantly different expectation value in (1). Moreover, this set will usually differ for different time points $t$ and/or different observables $O$.

Quantitatively, the temporal evolution of the expectation values in (1) as well as the above mentioned $t$- and $O$-dependent sets of “untypical” $|\phi\rangle$’s are determined in a very complicated way by many details of the Hamiltonian $H$. Given that $H$ can still be chosen arbitrarily (see above), it is obvious that apart from establishing dynamical typicality per se, no further conclusion regarding the actual time evolution and the untypical sets will be possible.

Note that a priori, the pertinent Hilbert space $\mathcal{H}_0$ of a given model system is often of infinite dimension, and the actual Hamiltonian and observables are Hermitian operators on that Hilbert space $\mathcal{H}_0$. Our above introduced, $N$ dimensional Hilbert space $\mathcal{H}$ then represents, for instance, an energy shell, i.e., it is spanned by $N$ eigenvectors of the original Hamiltonian, whose eigenvalues are contained in an energy interval, which is microscopically large (thus $N$ is large) but macroscopically small (well defined system energy). Since we are only interested in vectors $|\phi\rangle$ with support in $\mathcal{H}$, also the support of the Hamiltonian and the observables can be readily projected (restricted) from $\mathcal{H}_0$ to $\mathcal{H}$ without any change of the dynamics and expectation values. More generally, $\mathcal{H}$ may be any high but finite dimensional subspace of $\mathcal{H}$.
of such a transition from $\mathcal{H}_0$ to $\mathcal{H}$ is invariant under the original system Hamiltonian. The straightforward and well known formal details of such a transition from $\mathcal{H}_0$ to $\mathcal{H}$ are provided, e.g., in Ref. [6].

III. MAIN IDEA AND RESULT

A. Basic definitions and properties

The observable $A$ introduced below (2) can be rewritten in terms of its eigenvalues $a_n$ and eigenvectors $|n\rangle$ as

$$A = \sum_{n=1}^{N} a_n |n\rangle\langle n| .$$

(3)

Furthermore, the largest and smallest eigenvalues of $A$ will be denoted as $a_{\text{max}}$ and $a_{\text{min}}$, respectively. Their difference

$$\Delta A := a_{\text{max}} - a_{\text{min}}$$

(4)

thus represents the measurement range of the observable $A$. Finally, we define the microcanonical ensemble as

$$\rho_{\text{mc}} := \frac{1}{N} \sum_{n=1}^{N} |n\rangle\langle n|$$

(5)

and the concomitant microcanonical expectation value of $A$ from (3) as

$$a_{\text{mc}} := \text{Tr} \{ \rho_{\text{mc}} A \} = \frac{1}{N} \sum_{n=1}^{N} a_n .$$

(6)

From now on, we tacitly exclude the trivial case $a_{\text{min}} = a_{\text{max}}$. It thus follows that $a_{\text{min}} < a_{\text{mc}} < a_{\text{max}}$.

The main objective of the present Sec. III will be to construct an ensemble of normalized vectors $|\phi\rangle \in \mathcal{H}$, most of which exhibit expectation values of $A$ very close to some preset value $a$, and which furthermore entail dynamical typicality properties. Our starting point in constructing this ensemble consists in the following key observation: Given an arbitrary but fixed real number

$$a \in (a_{\text{min}}, a_{\text{max}}) ,$$

(7)

there exists a unique $y \in \mathbb{R}$ so that the quantities

$$p_n := \frac{1}{N} \frac{1}{1 + y (a - a_n)}$$

(8)

have the properties that

$$p_n > 0 \text{ for all } n = 1, \ldots, N ,$$

(9)

$$\sum_{n=1}^{N} p_n = 1 ,$$

(10)

$$\sum_{n=1}^{N} a_n p_n = a .$$

(11)

The derivation of these properties is provided in full detail in Appendix A, and can be summarized as follows: Considering the function

$$g(x) := \frac{1}{N} \sum_{n=1}^{N} \frac{1}{1 + x (a - a_n)} ,$$

(12)

one readily verifies that $g(x) \to \infty$ as $x$ approaches $x_{\text{max}} := 1/(a_{\text{max}} - a) > 0$ from below, and likewise as $x$ approaches $x_{\text{min}} := 1/(a_{\text{min}} - a) < 0$ from above. Moreover, one can show that $g''(x) > 0$ for all $x \in (x_{\text{min}}, x_{\text{max}})$, i.e., $g(x)$ is a convex function. Upon observing that $g(0) = 1$, $g'(0) = a_{\text{mc}} - a$ it thus follows that the equation $g(x) = 1$ is solved by $x = 0$ for any given $a$ in (7). In the case $a = a_{\text{mc}}$, there can be no further solution of $g(x) = 1$, hence we set $y := 0$, and (9)-(11) readily follow. In the case $a \neq a_{\text{mc}}$, there must exist exactly one further $x \in (x_{\text{min}}, x_{\text{max}}) \setminus \{0\}$ which solves $g(x) = 1$, and this $x$-value is now identified with $y$. Again, (9) and (10) are easily verified, while (11) is recovered upon concluding from (8) that $\sum_{n=1}^{N} [1 + y (a - a_n)] p_n = 1$. With (11) this yields $1 + y [a - \sum_{n=1}^{N} a_n p_n] = 1$ and due to $y \neq 0$ we arrive at (10).

Note that the value of $y$ depends on the choice of $a$ in (7). It is thus sometimes more appropriate to view $y$ as a function $y(a)$ of $a$. On the other hand, $a$ is often considered as arbitrary but fixed, hence it is justified to omit the $a$ dependence of $y$.

From (10) and (11) one can infer that

$$\rho := \sum_{n=1}^{N} p_n |n\rangle\langle n|$$

(13)

is a well defined density operator and that

$$\rho^{1/2} := \sum_{n=1}^{N} \sqrt{p_n} |n\rangle\langle n|$$

(14)

is Hermitian and satisfies $(\rho^{1/2})^2 = \rho$. Adopting the usual definition

$$f(A) := \sum_{n=1}^{N} f(a_n) |n\rangle\langle n|$$

(15)

for an arbitrary function $f(x) : \mathbb{R} \to \mathbb{R}$, Eq. (13) can be rewritten by means of (8) and (15) as

$$\rho = \frac{1}{N} \frac{1}{1 + y (a - A)} .$$

(16)

Next we consider an ensemble of uniformly distributed and normalized random vectors $|\psi\rangle \in \mathcal{H}$. In other words, the probability distribution of those vectors $|\psi\rangle$ is invariant under arbitrary unitary transformations and thus all vectors $|\psi\rangle$ are equally likely. For any given Hermitian operator $B : \mathcal{H} \to \mathcal{H}$ it follows that the “Hilbert space
average” (HA), i.e., the average of \( \langle \psi | B | \psi \rangle \) over the above ensemble of vectors \(|\psi\rangle\), amounts to

\[
\text{HA}\left[ \langle \psi | B | \psi \rangle \right] = \frac{\text{Tr}\{B\}}{N}.
\]  

(17)

Likewise, the corresponding “Hilbert space variance” (HV) is given by

\[
\text{HV}\left[ \langle \psi | B | \psi \rangle \right] := \text{HA}\left[ \left( \langle \psi | B | \psi \rangle - \text{HA}\left[ \langle \psi | B | \psi \rangle \right] \right)^2 \right] = \frac{1}{N+1}\left( \frac{\text{Tr}\{B^2\}}{N} - \left( \frac{\text{Tr}\{B\}}{N} \right)^2 \right).
\]  

(18)

The appearance of the denominator \(N+1\) on the right hand side of (18) will play a key role in what follows. A simple intuitive explanation of its origin is not known to this author, while the formal mathematical derivation can be found, e.g., in Appendix C.1 of Ref. \[8\] or in the Appendix of Ref. \[5\].

Next we consider (not necessarily normalized) vectors \(|\varphi\rangle \in \mathcal{H}\) of the form

\[
|\varphi\rangle := \sqrt{N} \rho^{1/2} |\psi\rangle
\]  

(19)

with \(\rho^{1/2}\) from (12). For the ensemble of random vectors \(|\psi\rangle\) as defined above, we thus obtain a modified ensemble of vectors \(|\varphi\rangle\) via (12). Denoting their ensemble average by an overbar, we thus can conclude that

\[
\langle \varphi | B | \varphi \rangle = \text{HA}\left[ \langle \psi | N \rho^{1/2} B \rho^{1/2} | \psi \rangle \right]
\]  

(20)

and with (17) that

\[
\langle \varphi | B | \varphi \rangle = \text{Tr}\{\rho^{1/2} B \rho^{1/2}\} = \text{Tr}\{\rho B\}.
\]  

(21)

Likewise, the corresponding variance

\[
\sigma_B^2 := \left( \langle \varphi | B | \varphi \rangle - \langle \varphi | B | \varphi \rangle \right)^2
\]  

(22)

can be rewritten with the help of (18) as

\[
\sigma_B^2 = \text{HV}\left[ \langle \psi | N \rho^{1/2} B \rho^{1/2} | \psi \rangle \right] = N \text{Tr}\{\rho B\} - \left( \text{Tr}\{\rho B\} \right)^2
\]  

(23)

It readily follows that

\[
\sigma_B^2 \leq \text{Tr}\{\rho^2 B^2\}.
\]  

(24)

Considering \(\text{Tr}\{C_1^\dagger C_2\}\) as a scalar product between two arbitrary (not necessarily Hermitian) linear operators \(C_{1,2} : \mathcal{H} \rightarrow \mathcal{H}\), the Cauchy-Schwarz inequality takes the form \(|\text{Tr}\{C_1^\dagger C_2\}|^2 \leq \text{Tr}\{C_1^\dagger C_1\} \text{Tr}\{C_2^\dagger C_2\}\). Choosing \(C_2 = C_1^\dagger = \rho B\) it follows that

\[
\text{Tr}\{(\rho B)^2\} \leq \text{Tr}\{(\rho B)(B\rho)\} = \text{Tr}\{\rho^2 B^2\}.
\]  

(25)

Evaluating the trace by means of the eigenbasis of \(B\), one finds that

\[
\text{Tr}\{\rho^2 B^2\} \leq \|B\|^2 \text{Tr}\{\rho^2\},
\]  

(26)

where \(\|B\|\) indicates the operator norm of \(B\) (largest eigenvalue in modulus). Combining (24), (26) thus yields

\[
\sigma_B^2 \leq \|B\|^2 P,
\]  

(27)

\[
P := \text{Tr}\{\rho^2\} = \sum_{n=1}^N p_n^2
\]  

(28)

\[
= \frac{1}{N^2} \sum_{n=1}^N \left[ 1 + y(a - a_n) \right]^2
\]  

(29)

\[
= \frac{1}{N^2} \text{Tr}\left\{ \left[ 1 + y(a - A) \right]^{-2} \right\}
\]  

(30)

where we have exploited (3), (13), and (19).

B. Establishing dynamical typicality

Throughout this section we take for granted that \(P\) from (28) satisfies

\[
P \ll 1.
\]  

(31)

A more detailed discussion of this assumption is postponed to the following sections. Choosing for \(B\) the identity operator, it follows from (21)-(23) and (28) that

\[
\langle \varphi | \varphi \rangle = 1,
\]  

(32)

\[
\frac{\langle \varphi | \varphi \rangle - 1}{\langle \varphi | \varphi \rangle - 1} = \frac{NP - 1}{N + 1} \leq P.
\]  

(33)

Invoking the Chebyshev inequality from probability theory, one thus can infer that

\[
\text{Prob}\left( \left| \langle \varphi | \varphi \rangle - 1 \right| \leq P^{1/3} \right) \geq 1 - P^{1/3},
\]  

(34)

where the left hand side denotes the probability that \(|\langle \varphi | \varphi \rangle - 1| \leq P^{1/3}\) when randomly sampling vectors \(|\varphi\rangle\) according to (19). Due to (31), the overwhelming majority of all vectors \(|\varphi\rangle\) in (19) thus have norms very close to unity.

Choosing for \(B\) the operator \(A\) from (3), it follows with (11), (13), and (21) that

\[
\langle \varphi | A | \varphi \rangle = a
\]  

(35)

and with (22) and (27) that

\[
\frac{\langle \varphi | A | \varphi \rangle - a}{\langle \varphi | A | \varphi \rangle - a} \leq \|A\|^2 P.
\]  

(36)

Clearly, adding an arbitrary constant \(c \in \mathbb{R}\) to \(A\) and simultaneously to \(a\) does not change the variance on the left hand side of (36), while \(A\) on the right hand side is replaced by \(A + c\mathbb{1}\). The best possible upper bound is
thus obtained by minimizing $\|A + cI\|$ over all $c \in \mathbb{R}$. A straightforward calculation shows that this minimal value will be given by $\Delta_A/2$, where $\Delta_A$ is the measurement range of $A$ from (3). Invoking Chebyshev's inequality, it follows that

$$\text{Prob} \left( |\langle \phi | A | \phi \rangle - a | \leq P^{1/3} \Delta_A/2 \right) \geq 1 - P^{1/3}.$$  \hfill (37)

In view of (31), the overwhelming majority of all vectors $|\phi\rangle$ in (19) thus exhibit expectation values $\langle \phi | A | \phi \rangle$, whose deviations from the preset value $a$ in (7) are very small compared to full range $\Delta_A$ over which those expectation values in principle could vary.

Finally, by choosing for $B$ the operator $O_t$ in (2) and observing that $\|O_t\| = \|O\|$, one can infer from (22), (27), and (31) along the very same line of reasoning as before that most vectors $|\phi\rangle$ in (19) exhibit very similar expectation values $\langle \phi | O_t | \phi \rangle$.

So far, the initial states $|\phi\rangle$ in (19) are in general not normalized. But, as mentioned below (34), the vast majority among them is almost of unit length. Hence, if we replace for every given $|\psi\rangle$ the concomitant $|\phi\rangle$ in (19) by its strictly normalized counterpart

$$|\phi\rangle := \frac{|\phi\rangle}{\sqrt{\langle \phi| \phi \rangle}} = \frac{\rho^{1/2}|\psi\rangle}{\sqrt{\langle \psi| \rho |\psi\rangle}},$$  \hfill (38)

then the “new” expectation values $\langle \phi | A | \phi \rangle$ will mostly remain very close to the “old” ones, i.e., to $\langle \phi | A | \phi \rangle$. Essentially, this is a consequence of the relation

$$\langle \phi | A | \phi \rangle = \frac{\langle \phi | A | \phi \rangle}{\langle \phi | \phi \rangle} \quad \hbox{(39)}$$

following from (35) and of the fact that $\langle \phi | \phi \rangle$ is very close to unity for most $|\phi\rangle$’s according to (34). More precisely, with the help of (39) and

$$q(\phi) := |1 - \langle \phi | \phi \rangle|,$$  \hfill (40)

we can rewrite $|\langle \phi | A | \phi \rangle - \langle \phi | A | \phi \rangle| = q(\phi)|\langle \phi | A | \phi \rangle|$. Exploiting the triangle inequality, it follows that

$$|\langle \phi | A | \phi \rangle - a| \leq q(\phi)|\langle \phi | A | \phi \rangle| + |\langle \phi | A | \phi \rangle - a|.$$  \hfill (41)

Since $|\langle \phi | A | \phi \rangle| \leq \|A\|$, an argument analogous to the one below (30) then yields

$$|\langle \phi | A | \phi \rangle - a| \leq q(\phi)\Delta_A/2 + |\langle \phi | A | \phi \rangle - a|.$$  \hfill (42)

According to (37), the probability that $|\langle \phi | A | \phi \rangle - a| \leq P^{1/3} \Delta_A/2$ is at least $1 - P^{1/3}$, and due to (34), (40) the probability that $q(\phi) \leq P^{1/3}$ is at least $1 - P^{1/3}$. Therefore, the probability that both $|\langle \phi | A | \phi \rangle - a| \leq P^{1/3} \Delta_A/2$ and $q(\phi) \leq P^{1/3}$ are simultaneously fulfilled must be at least $1 - 2P^{1/3}$. Together with (42) we thus can conclude that

$$\text{Prob} \left( |\langle \phi | A | \phi \rangle - a| \leq P^{1/3} \Delta_A \right) \geq 1 - 2P^{1/3}.$$  \hfill (43)

In conclusion, by sampling random initial conditions according to (35), the vast majority of them exhibits expectation values $\langle \phi | A | \phi \rangle$ very close the preset value $a$ from (7). Along the same line of reasoning one sees that for most of them also the expectation values $\langle \phi | O_t | \phi \rangle$ in (1) will be very similar. Altogether, we thus recover dynamical typicality as announced below (2).

As an aside, we note that the possible values of $\langle \phi | A | \phi \rangle$ are bounded by the smallest and largest eigenvalues of $A$, hence the restriction of admissible $a$-values in (4) still covers all physically meaningful cases.

### C. Necessary and sufficient conditions

According to (28), $P$ is the purity of $\rho$ and thus satisfies $0 \leq P \leq 1$. The demonstration of dynamical typicality in the preceding subsection is based on the additional assumption (31). In the opposite case that $P$ is not small (but still taking for granted $N \gg 1$, as required above Eq. (11)), also the left hand side of (33) is not any more a small quantity. As a consequence, one expects that the norm of most $|\phi\rangle$’s is not any more close to unity and the arguments below Eq. (38) are no longer valid. In particular, the denominator on the right hand side of (39) now exhibits non-negligible random fluctuations, hence the same property will be inherited by the left hand side, since there is no reason why the numerator should (almost) exactly compensate the fluctuations of the denominator. It is therefore reasonable to expect that (31) is in fact not only a sufficient but also a necessary prerequisite for dynamical typicality, provided the quite trivial cases with $P$-values very close to unity are tacitly excluded. Without providing the details we mention that this conjecture can also be confirmed more rigorously.

One readily concludes from (39), (11), and (28) that

$$p_{\text{max}} \leq P \leq p_{\text{max}},$$ \hfill (44)

$$p_{\text{max}} := \max_N p_{\text{n}}.$$ \hfill (45)

The two relations in (41) imply that $P$ is small if and only if $p_{\text{max}}$ is small. As a consequence, (31) is equivalent to the following necessary and sufficient condition for dynamical typicality:

$$p_{\text{max}} \ll 1.$$ \hfill (46)

In passing we note that (11) implies $p_{\text{max}} \geq 1/N$, hence the condition $N \gg 1$ (see above (11)) is automatically guaranteed if (40) is fulfilled.

### IV. COMPARISON WITH REF. 2

We recall that the ensemble of random vectors $|\phi\rangle$ on the left hand side of (35) derives from a uniformly distributed ensemble of normalized $|\psi\rangle$’s on the right hand side. This specific ensemble of initial states $|\phi\rangle$ was
demonstrated in Sec. [11] to exhibit dynamical typicality, provided condition (10) is fulfilled. In particular, the expectation value $\langle \phi | A | \phi \rangle$ turned out to be almost equal to the preset value $a$ from (7) for most of those $| \phi \rangle$’s. Apart from that, the actual properties of the ensemble at hand are not very clear. Specifically, it is not at all obvious how these findings for the ensemble $S_a$ can be translated into statements about the more natural but quite different ensemble, where all normalized vectors, whose expectation value is strictly equal to $a$, are realized with equal probability (and all other vectors are excluded). It is exactly the latter ensemble which is at the focus of the explorations by Müller, Gross, and Eisert in Ref. [2].

Accordingly, our subsequent resolution of the above issue will be largely based on their results.

To begin with we note that the set of all possible vectors $| \psi \rangle$ on the right hand side of (19) amounts to a unit sphere in $\mathbb{C}^N$, or equivalently, to a $2N - 1$ sphere in $\mathbb{R}^{2N}$. Accordingly, the set of vectors $| \varphi \rangle$ on the left hand side of (19) may be viewed as a $2N - 1$ dimensional ellipsoid in $\mathbb{R}^{2N}$. After an additional normalization of each vector, see (25), the latter ellipsoid is deformed back into a $2N - 1$ sphere, henceforth denoted as $S_a$, where the index $a$ refers to the preset $a$-value from (7).

As mentioned above, the main focus in Ref. [2] is put on a different set of vectors $| \varphi \rangle$, namely (in our present notation)

$$M_a := \{ | \varphi \rangle \mid \langle \varphi | A | \varphi \rangle = a \text{ and } \langle \varphi | \varphi \rangle = 1 \}.$$  \hspace{1cm} (47)

Due to the presence of two constraints on the right hand side, the set $M_a$ generically amounts to a $2N - 2$ dimensional submanifold of $\mathbb{R}^{2N}$, namely the intersection of an ellipsoid and a sphere. In other words, the two sets $S_a$ and $M_a$ are quite different.

To go over from those two sets to the corresponding statistical ensembles, we also need to specify the probability measures on each set.

The probability measure on $M_a$ considered in Ref. [2] is the natural “Hausdorff measure” inherited from $\mathbb{R}^{2N}$, i.e., the probability measure of any $2N - 2$ dimensional “surface element” on $M_a$ is proportional to its “size”. In other words, the distribution may be viewed as “uniform” on $M_a$: Every vector $| \varphi \rangle \in M_a$ is “equally likely” in a very natural sense.

In contrast, the uniformly distributed vectors $| \psi \rangle$ in $S_a$ generally induce a quite non-trivial (far from uniform) probability measure on the above specified $2N - 1$ dimensional submanifold $S_a$.

From now on, the symbols $S_a$ and $M_a$ refer not only to the corresponding submanifolds (sets) but also to the above specified probability measures for each of them, i.e., they amount to full fledged statistical ensembles of normalized random vectors $| \varphi \rangle$.

Given dynamical typicality as detailed in the previous section is fulfilled for the ensemble $S_a$, does a similar property also hold for the ensemble $M_a$, and vice versa? The answer to this highly non-trivial question is “yes”, as can be inferred from Section 6 of Ref. [2]. However, the detailed arguments and calculations are quite tedious and are therefore postponed to Appendix B. In particular, it can be shown that the main prerequisite in the previous section, namely the condition (10), is at the same time the crucial prerequisite in the pertinent Theorem 1 of Ref. [2]. Moreover, the average of $\langle \varphi | B | \varphi \rangle$ over the ensemble $S_a$ agrees very well with the corresponding average over the ensemble $M_a$ for arbitrary $B$ and any given $a$-value, for which $\rho$ in (19) satisfies condition (10).

In conclusion, the two ensembles $S_a$ and $M_a$ are largely equivalent as far as their dynamical typicality properties are concerned. Since the ensemble $M_a$ is in some sense considerably more “natural” than $S_a$, this equivalence notably enhances the physical significance of the findings in the remainder of our paper.

The intuitive picture is that – in spite of the fact that the two sets $S_a$ and $M_a$ are very different – the extremely non-uniform probability measure on $S_a$ essentially only leaves over vectors $| \varphi \rangle$ “close to $M_a$”.

It should be emphasized that the physical motivation and the interpretation of the results in Ref. [2] are very different from our present dynamical typicality viewpoint. Accordingly, our above conclusions are not at all immediate consequences of the actual findings in Ref. [2] itself, but rather should be considered as significant new insights of the present paper (see also Appendix B).

Technically speaking, the above mentioned typicality result for the ensemble $M_a$ were established in Ref. [2] by means of concentration of measure concepts, and may be viewed as an extension of Levy’s Lemma [9, 10]. In particular, since $M_a$ in (47) does not amount to a hypersphere, one cannot directly apply Levy’s Lemma itself to this case.

V. EXPLORING THE CONDITIONS FOR DYNAMICAL TYPICALITY

A. General considerations

In principle, once the observable $A$ in (5) and the value of $a$ in (7) have been fixed, $y$ is uniquely determined according to the discussion below (12). Hence, all $p_n$ in (3) and their maximum in (15) follow, and one can decide whether or not the necessary and sufficient condition for dynamical typicality in (10) is fulfilled. In practice, it is usually impossible to “directly” solve all the necessary equations, hence one has to figure out “indirect criteria” whether or not the given $A$ and $a$ satisfy (10). This is the main goal of the present section.

A first important special case arises by considering the microcanonical ensemble from (5) and the concomitant microcanonical expectation value from (6). Namely, for the particular choice $a = a_{mc}$ we have $y = 0$ (see below (13)), and hence $p$ from (10) must coincide with $\rho_{mc}$. In other words, we find that

$$a = a_{mc} \iff y = 0 \iff \rho = \rho_{mc} \iff p_n = 1/N \forall n.$$  \hspace{1cm} (48)
Hence, \(a_0\) is satisfied if an only if \(N\) is large. Moreover, \(19\) and \(38\) simplify to \(|\phi\rangle = |\varphi\rangle = |\psi\rangle\), i.e., we recover the original “non-dynamical” or “microcanonical” typicity results from \(3\), \(14\) (see also \(4\), \(2\)).

From now on, we mainly focus on \(a\)-values with

\[
 a_{\text{nc}} < a < a_{\text{max}} .
\]

The corresponding results for \(a_{\text{min}} < a < a_{\text{nc}}\) are quite obvious (consider \(-A\) instead of \(A\)) and will only be briefly mentioned.

Denoting by \(D_{\text{max}}\) the multiplicity of the largest eigenvalue of \(A\) (i.e., there are \(D_{\text{max}}\) indices \(n\) with \(a_n = a_{\text{max}}\), we can infer from \(19\) and Eq. \((A15)\) in Appendix A that

\[
 0 < y < \frac{1 - D_{\text{max}}/N}{a_{\text{max}} - a} .
\]

With the help of the first relation, \(y > 0\), in \(60\), one can conclude from \(8\) and \(13\) that

\[
 p_{\text{max}} = \frac{1}{N} \frac{1}{1 - y(a_{\text{max}} - a)} .
\]

So far, \(a\) in \(7\) was considered as arbitrary but fixed. Next, we are interested in how things change upon variation of \(a\), hence \(y\) and thus \(p_{\text{max}}\) in \(61\) become functions of \(a\) (see also the discussion above Eq. \((13)\)). A first general picture of how \(p_{\text{max}}\) depends on \(a\) within the range \(19\) is provided by the following three observations: First, \(p_{\text{max}}\) is an increasing function of \(a\), as shown in Appendix C. Second, \(p_{\text{max}}\) approaches \(1/N\) for \(a \to a_{\text{max}}\), see \(18\). Third, one readily confirms that when \(a\) approaches \(a_{\text{max}}\), the unique \(y\)-value in \(8\) compatible with \(9\) - \(11\) approaches \((1 - D_{\text{max}}/N)/(a_{\text{max}} - a)\). With \(51\) it follows that

\[
 p_{\text{max}} \to 1/D_{\text{max}} \quad \text{for} \quad a \to a_{\text{max}} .
\]

If \(D_{\text{max}} \gg 1\) we can conclude \(15\) that the condition \(40\) is satisfied for any \(a\) within the range \(19\). (If \(a_{\text{min}} < a < a_{\text{nc}}\) the same conclusion holds on condition that the smallest eigenvalue of \(A\) is highly degenerate.)

From now on, we restrict ourselves to cases, for which \(D_{\text{max}} \gg 1\) is not satisfied, including the most common case \(D_{\text{max}} = 1\). In view of \(52\), \(a\)-values sufficiently close to \(a_{\text{max}}\) thus inevitably must lead to a violation of \(40\). “How closely” \(a\) may approach \(a_{\text{max}}\) without violating \(40\) depends on many details of the spectrum of \(A\), as we will see in the following.

### B. Eigenvalue density approximation

To begin with, we assume that the spectrum of \(A\) gives rise to an approximately constant density of eigenvalues close to its upper limit \(a_{\text{max}}\). More precisely, when rewriting \(3\), \(10\) as

\[
 \int_a^b dx \, w(x) \frac{1}{1 + y(a - x)} = 1 ,
\]

\[
 w(x) := \frac{1}{N} \sum_{n=1}^N \delta(a_n - x)
\]

we assume that \(w(x)\) can be approximated by some constant, positive value \(w_0\) for \(x\) close to \(a_{\text{max}}\). Moreover, we assume that \(a\) is sufficiently close to \(a_{\text{max}}\) so that the latter approximation applies to all \(x \in [a, a_{\text{max}}]\). Note that the function \(w(x)\) is normalized to unity and thus may be viewed as an eigenvalue probability distribution. A more rigorous approach, which does not exploit the above assumptions, is worked out in Appendix D, see also Sec. \(\text{V.D}\) below.

As a first consequence of the above assumptions, the number of eigenvalues of \(A\) which are larger than (or equal to) \(a\) is given by

\[
 N_a := N \int_a^{a_{\text{max}}} dx \, w(x)
\]

and can be approximated as

\[
 N_a \approx N (a_{\text{max}} - a) w_0 .
\]

Second, upon restricting the integration domain in \(53\) to \(x \geq a\) and observing that \(w(x) = 0\) for \(x > a_{\text{max}}\) we can conclude that

\[
 \int_a^{a_{\text{max}}} dx \, w(x) \frac{1}{1 + y(a - x)} \leq 1 .
\]

Approximating \(w(x)\) by \(w_0\) and performing the integration yields

\[
 \frac{w_0}{y} \ln \left( \frac{1}{1 - y(a_{\text{max}} - a)} \right) \leq 1 .
\]

With \(51\) it follows that

\[
 \frac{w_0}{y} \ln(p_{\text{max}} N) \leq 1
\]

and with \(50\) that

\[
 \frac{N_a}{N} \frac{\ln(p_{\text{max}} N)}{y(a_{\text{max}} - a)} \leq 1 .
\]

By taking into account \(52\), the inequality \(60\) implies

\[
 \ln(p_{\text{max}} N) \leq \frac{N}{N_a} .
\]

Observing that \(\alpha, \beta \in \mathbb{R}\) satisfy \(\alpha \leq \beta\) if and only if \(e^\alpha \leq e^\beta\) it follows that

\[
 p_{\text{max}} \leq \frac{1}{N} \exp \left\{ \frac{N}{N_a} \right\} .
\]
we see that very large $a$ values in these examples. For an analogous property near $a = 0$ (see Appendix C), it follows that if the above conditions are met by some $a$ close to $a_{\text{mc}}$, then (46) remains valid also for all smaller $a$-values down to $a_{\text{mc}}$. A similar conclusion applies for $a < a_{\text{mc}}$ provided the eigenvalues $a_n$ of $A$ are sufficiently numerous and of approximately constant density near $a_{\text{min}}$.

Referring to Appendix D for the detailed calculations, we remark that when $a$ approaches $a_{\text{max}}$ so closely that (63) is no longer satisfied then one can show that $p_{\text{max}}$ is no longer small. In other words, if the eigenvalue probability distribution $w(x)$ assumes a constant, positive value near $a_{\text{max}}$, then (63) (together with an analogous property near $a_{\text{min}}$) is not only a sufficient but also a necessary condition for (46).

A numerical illustration of these analytical predictions is provided in Fig. 1. It exemplifies the case when $w(x)$ can be well approximated by $w_0 = 1/2$ for all $x \in (a_{\text{min}}, a_{\text{max}}) = (-1,1)$ in the sense explained below (64). Note that $a_{\text{mc}}$ in (62) is always very close to zero in these examples. For $a$-values close to $a_{\text{max}} = 1$, we see that $p_{\text{max}}$ in Fig. 1 becomes small only for very large $N$ (and likewise for $a$ close to $a_{\text{min}} = -1$).

To better understand this numerical finding, we temporarily consider the inequalities in (57)-(62) as approximate equalities. With (50) we thus can rewrite (62) as

$$p_{\text{max}} \approx N^{-1} \exp\{2(1-a)^{-1}\}.$$  

For a given (fixed) value of $p_{\text{max}}$ (e.g., $p_{\text{max}} = 0.01$) the corresponding value of $1-a$ thus scales as $1/\ln N$, i.e., $a$ approaches $a_{\text{mc}} = 1$ only very slowly upon increasing $N$. Closer inspection shows that (57)-(62) indeed become approximate equalities in the above considered case that both $p_{\text{max}}$ and $1-a$ are small. We may also recall that it is the purity $P$ which actually counts in the dynamical typicality considerations in Sec. III B and that $P$ is bounded by $p_{\text{max}}$ according to (44). Quantitatively, $P$ may thus be considerably smaller than $p_{\text{max}}$. However, also $P$ will still decrease very slowly with $N$ (at most as $(1/\ln N)^2$).

In passing we note that $p_{\text{max}}$ is given by (51) for $a \geq a_{\text{mc}}$, while for $a \leq a_{\text{mc}}$ the corresponding formula reads

$$p_{\text{max}} = \frac{1}{N} \frac{1}{1 - y(a - a_{\text{min}})}.$$  

As a consequence, $p_{\text{max}}$ turns out to be a continuous but not differentiable function of $a$ at $a = a_{\text{mc}}$, see Fig. 1.

C. Large $N$ limit

So far we considered the Hilbert space dimension $N$ as large but fixed. In order to draw conclusions about how things (in particular $p_{\text{max}}$) change upon variations of $N$, it is necessary (and sufficient) to specify how the spectrum of the observable $A$ in (5) changes with $N$, which a priori appears to be a quite subtle problem in itself. A physically natural way would be to model how the “same” measurement device acts on different systems. Here, we adopt an alternative approach which is physically somewhat abstract but mathematically quite common: Namely, we are mainly interested in the large $N$ limit and we assume that the eigenvalue probability distribution from (51) approaches a well-defined limit for asymptotically large $N$ (with slightly washed-out delta functions on the right hand side of (51)). In particular, $a_{\text{max}}$ and $a_{\text{min}}$ are assumed to become (asymptotically) $N$-independent. Moreover, the symbol $w(x)$ is henceforth used for this limiting function, and likewise for $w_0$ in (46) and (50). For an example, see also Fig. 1.

For any given $a$ close to $a_{\text{max}}$, the solution $y$ of (63) is thus (asymptotically) $N$-independent. From (59) it follows that $p_{\text{max}}$ approaches zero as $N$ tends to infinity. Since $p_{\text{max}}$ is monotonically increasing with $a$ (see above or Appendix C), we thus find that (46) will be satisfied for any given $a$-value from (49) when $N \to \infty$.

Yet another conclusion is mentioned here without detailed proof since it is quite plausible in view of the so far results: If the eigenvalue probability distribution $w(x)$ tends to zero for $x \to a_{\text{max}}$ then (46) is violated for all $a$ beyond some threshold $a_{\text{th}} < a_{\text{max}}$. Analogous conclusions apply to the lower end of the spectrum of $A$.  

FIG. 1: Dependence of $p_{\text{max}}$ from (51), (64) on the parameter $a$ in (67)-(68) by numerically solving (10) for $y$ (see also Sec. VI) with $N = 10^2$, $N = 10^3$, and $N = 10^6$ (top down). In each case, the $N$ eigenvalues $a_n$ in (8) were randomly generated so that the difference between neighboring eigenvalues are Wigner distributed (10) and $a_{\text{min}} = -1$, $a_{\text{max}} = 1$. Under the assumption that

$$N_a \gg \frac{N}{\ln N}$$  

one readily concludes that $(1/N) \exp\{N/N_a\} \ll 1$ and with (72) that $p_{\text{max}} \ll 1$. Altogether we find that (46) is guaranteed under the following sufficient conditions: The number $N_a$ of eigenvalues $a_n$ larger than $a$ must satisfy (63) and the density of levels between $a$ and $a_{\text{max}}$ must be (approximately) constant. For large $N$ one expects that in many cases there will be $a$-values close to $a_{\text{max}}$ which satisfy both conditions. Recalling that $p_{\text{max}}$ is an increasing function of $a$ (see Appendix C), it follows that if the above conditions are met by some $a$ close to $a_{\text{max}}$, then (46) remains valid also for all smaller $a$-values down to $a_{\text{mc}}$. A similar conclusion applies for $a < a_{\text{mc}}$ provided the eigenvalues $a_n$ of $A$ are sufficiently numerous and of approximately constant density near $a_{\text{min}}$. 

For any given $a$ close to $a_{\text{max}}$, the solution $y$ of (63) is thus (asymptotically) $N$-independent. From (59) it follows that $p_{\text{max}}$ approaches zero as $N$ tends to infinity. Since $p_{\text{max}}$ is monotonically increasing with $a$ (see above or Appendix C), we thus find that (46) will be satisfied for any given $a$-value from (49) when $N \to \infty$. Yet another conclusion is mentioned here without detailed proof since it is quite plausible in view of the so far results: If the eigenvalue probability distribution $w(x)$ tends to zero for $x \to a_{\text{max}}$ then (46) is violated for all $a$ beyond some threshold $a_{\text{th}} < a_{\text{max}}$. Analogous conclusions apply to the lower end of the spectrum of $A$.  

FIG. 1: Dependence of $p_{\text{max}}$ from (51), (64) on the parameter $a$ in (67)-(68) by numerically solving (10) for $y$ (see also Sec. VI) with $N = 10^2$, $N = 10^3$, and $N = 10^6$ (top down). In each case, the $N$ eigenvalues $a_n$ in (8) were randomly generated so that the difference between neighboring eigenvalues are Wigner distributed (10) and $a_{\text{min}} = -1$, $a_{\text{max}} = 1$. Under the assumption that

$$N_a \gg \frac{N}{\ln N}$$  

one readily concludes that $(1/N) \exp\{N/N_a\} \ll 1$ and with (72) that $p_{\text{max}} \ll 1$. Altogether we find that (46) is guaranteed under the following sufficient conditions: The number $N_a$ of eigenvalues $a_n$ larger than $a$ must satisfy (63) and the density of levels between $a$ and $a_{\text{max}}$ must be (approximately) constant. For large $N$ one expects that in many cases there will be $a$-values close to $a_{\text{max}}$ which satisfy both conditions. Recalling that $p_{\text{max}}$ is an increasing function of $a$ (see Appendix C), it follows that if the above conditions are met by some $a$ close to $a_{\text{max}}$, then (46) remains valid also for all smaller $a$-values down to $a_{\text{mc}}$. A similar conclusion applies for $a < a_{\text{mc}}$ provided the eigenvalues $a_n$ of $A$ are sufficiently numerous and of approximately constant density near $a_{\text{min}}$. 

Referring to Appendix D for the detailed calculations, we remark that when $a$ approaches $a_{\text{max}}$ so closely that (63) is no longer satisfied then one can show that $p_{\text{max}}$ is no longer small. In other words, if the eigenvalue probability distribution $w(x)$ assumes a constant, positive value near $a_{\text{max}}$, then (63) (together with an analogous property near $a_{\text{min}}$) is not only a sufficient but also a necessary condition for (46).

A numerical illustration of these analytical predictions is provided in Fig. 1. It exemplifies the case when $w(x)$ can be well approximated by $w_0 = 1/2$ for all $x \in (a_{\text{min}}, a_{\text{max}}) = (-1,1)$ in the sense explained below (64). Note that $a_{\text{mc}}$ in (62) is always very close to zero in these examples. For $a$-values close to $a_{\text{max}} = 1$, we see that $p_{\text{max}}$ in Fig. 1 becomes small only for very large $N$ (and likewise for $a$ close to $a_{\text{min}} = -1$).
Closely related rigorous results are derived in Appendix D.

A particularly prominent example arises when the spectrum satisfies a so-called semicircle law, e.g., due to a corresponding random matrix character of $A$. It seems reasonable to expect that after the projection (restriction) of the original observable to an energy shell as described at the end of Sec. II the resulting reduced observable $A$ may possibly exhibit such features reminiscent of a random matrix.

Further instructive examples are obtained by considering any model of $M$ identical subsystems with negligible interactions and finite dimensional Hilbert spaces. Namely, when choosing any observable of the form $A = (1/M) \sum_{m=1}^{M} A_m$, where the $A_m$ act on the single subsystems but are otherwise identical, one can infer that $w(x)$ approaches $\delta(a_{mc} - x)$ for $M \to \infty$ by exploiting the central limit theorem. It follows that (10) is violated for any $a \neq a_{mc}$. A particular example of this type is originally due to Ref. [2] (see Sec. 3 therein). However, it should be emphasized that the energy of the system is only recently due to Ref. [2] (see Sec. 3 therein). However, it should be emphasized that the energy of the system is not confined to some non-trivial energy window in these examples, in contrast to the physical setup we mainly have in mind in the present paper (see end of Sec. II).

D. Outlook

Generally speaking, the idea to replace the exact distribution in (43) by a “washed out” approximation appears quite reasonable and also the so obtained results seem decent. However, in cases where the levels $a_n$ are very far from being approximately equally spaced (at least locally), such a smooth local eigenvalue probability distribution may still be well-defined, but the conclusions may become questionable. An extreme example arises when the levels $a_n$ can be partitioned into groups, so that each group contains the same number $D$ of elements and the eigenvalues $a_n$ are equal within each group ($D$-fold degeneracies). If the levels belonging to different groups are still approximately equally spaced, then a smoothed distribution $w(x)$ still appears to be reasonably well-defined ($D$ possibly large but fixed, $N$ sufficiently large so that the gaps are still small). However, if $D \gg 1$ we know from the discussion below (52) that (46) will be satisfied without any further restriction regarding $a$ and $a_n$, in contradiction to the above conclusions under the assumption of a smooth $w(x)$.

In view of such concerns, some further results are derived in Appendix D by means of more rigorous, but technically also more involved calculations.

In summary, the range of $a$-values, for which dynamical typicality applies, crucially depends on the detailed spectral properties of $A$, in particular near the upper and lower ends of the spectrum.

VI. FURTHER REMARKS AND COMPARISON WITH REF. [1]

Conceptually, a dynamical typicality result is particularly satisfying when referring to the set of normalized vectors from (47), i.e., when it amounts to a statement about all initial states $|\phi\rangle$ whose expectation values $\langle \phi|A|\phi\rangle$ are exactly equal to $a$.

As detailed in Ref. [2] and in Sec. II the set of vectors $|\phi\rangle$ arising via (38) is very different from the set (47), and also the probability measures on the two sets are very different. Nevertheless, the typicality statements for the two ensembles of random vectors are largely equivalent.

In turn, the main virtue of (38) is its constructive characteristic: It is an explicit recipe to generate “typical initial states” $|\phi\rangle$. The only remaining disadvantage is that all eigenvalues and eigenvectors of $A$ are explicitly needed, see (3) and (14). Moreover, in order to determine $y$, the transcendental equation $g(x) = 1$ has to be solved as discussed below (12). Since $g(x)$ in (43) consists of $N$ summands, the numerical effort to solve $g(x) = 1$ scales linearly with $N$. Generically, the diagonalization of $A$ is thus numerically much more expensive than the determination of $y$. At least for $a$-values sufficiently close to $a_{mc}$, this diagonalization of $A$ as well as the numerical determination of $y$ can be circumvented as follows.

Considering $y = y(a)$ as a function of $a$, our first goal is to expand $y(a)$ about $a_{mc}$. From (48), we can infer that $y(a_{mc}) = 0$. Focusing on small $\delta a := a - a_{mc}$, it follows that $y(a)$ is small and (5) can be rewritten as

$$p_n = \frac{1}{N} \sum_{k=0}^{\infty} (-y(a) (a - a_n))^k.$$ (65)

Replacing $a$ by $a_{mc} + \delta a$ and introducing the expansion of $y(a)$ about $a_{mc}$ on the right hand side of (65), plugging in the resulting expressions for the $p_n$ into (10), and finally comparing terms with equal powers of $\delta a$, one obtains equations for the derivatives of $y(a)$ at $a = a_{mc}$ which can be iteratively solved. The resulting series expansion of $y(a)$ reads

$$y(a) = \frac{1}{m_2} (a - a_{mc}) - \frac{m_3}{m_2^2} (a - a_{mc})^2 + \ldots ,$$ (66)

$$m_k := \frac{1}{N} \sum_{n=1}^{N} (a_n - a_{mc})^k = \frac{1}{N} \text{Tr} \{ A - a_{mc} \}^k ,$$ (67)

where we exploited (49) and (150) in the last identity. In the same vein, one can deduce from (5) for small values of $y = y(a)$ the expansion

$$\sqrt{p_n} = 1 - \frac{1}{2} y(a) (a - a_n) + \frac{3}{N} [y(a) (a - a_n)]^2 + \ldots .$$ (68)

Accordingly, (14) takes the form

$$\rho^{1/2} = 1 + \frac{1}{2} y(a) (A - a) + \frac{3}{N} [y(a) (A - a)]^2 + \ldots .$$ (69)
For $a$-values sufficiently close to $a_{\text{mc}}$, one can introduce (66) into (69) and truncate the series after a few terms, yielding very good approximations for the exact ensemble in (67) without the need to diagonalize the operator $A$.

The latter expansions amounts to a systematic generalization of the original dynamical typicality explorations by Bartsch and Gemmer in Ref. [1]. Indeed, as they pointed out below their formula (3), their approach is meant to be restricted to small values of their so-called disequilibrium parameter $d$, which in turn is essentially equivalent to our present expansion parameter $\delta d$ (17). Accordingly, beyond the linear approximation of (66) and (67) adopted by Bartsch and Gemmer [1], notable differences may generically arise between their ensemble of initial states $|\phi\rangle$ and either of the two ensembles from (68) and (71), as considered in our present work and in Ref. [2], respectively.

VII. SUMMARY

By unifying and extending previous works (mainly Refs. [1] [2], but also Refs. [3] [4]), dynamical typicality has been established in the sense that the vast majority of all pure states, which initially exhibit the same expectation value from (6), by $\delta d$ and $\delta d_{\text{max}}$, respectively (i.e., there are $D_{\text{max}}$ indices $n$ with $a_n = a_{\text{max}}$, and likewise for $D_{\text{min}}$). Introducing the open interval

$$I_A := (a_{\text{min}}, a_{\text{max}}) ,$$

we choose an arbitrary but fixed $a \in I_A$ and define the interval

$$I_a := \left( \frac{-1}{a - a_{\text{min}}}, \frac{1}{a_{\text{max}} - a} \right) .$$

From now on, the symbol $x$ indicates a real variable and we always silently take for granted that its value is restricted according to

$$x \in I_a .$$

It follows that the

$$q_n(x) := \frac{1}{1 + x(a - a_n)}$$

are well defined, positive functions for all $x \in I_A$, and likewise for the function $g(x)$ defined in (12). From these definitions, one readily can infer that

$$g(0) = 1 ,$$

$$g'(0) = a_{\text{mc}} - a ,$$

$$g(x) \to \infty \text{ for } x \uparrow \frac{1}{a_{\text{max}} - a} ,$$

$$g(x) > 1 \text{ for } x > \frac{1 - D_{\text{max}}/N}{a_{\text{max}} - a} ,$$

$$g(x) \to \infty \text{ for } x \downarrow -\frac{1}{a - a_{\text{min}}} ,$$

$$g(x) > 1 \text{ for } x < -\frac{1 - D_{\text{min}}/N}{a - a_{\text{min}}} ,$$

$$g''(x) = \frac{2}{N} \sum_{n=1}^{N} (a - a_n)^2 q_n^2(x) > 0 \text{ for all } x \in I_a ,$$

where $x \in I_a$ is tacitly assumed in (A8) and (A10).

Acknowledgments

Numerous extremely enlightening discussions with Jochen Gemmer and Ben N. Balz are gratefully acknowledged. Special thanks is due to Lennart Dabelow for carefully reading the manuscript. This work was supported by the Deutsche Forschungsgemeinschaft (DFG) under Grant No. RE 1344/10-1 and within the Research Unit FOR 2692 under Grant No. RE 1344/12-1.

Appendix A

As in the main text, we consider an observable $A$ of the form (3), and we denote by $a_{\text{mc}}$ its microcanonical expectation value from (6), by $a_{\text{max}}$ and $a_{\text{min}}$ its largest and smallest eigenvalues, and by $D_{\text{max}}$ and $D_{\text{min}}$ the degeneracy of $a_{\text{max}}$ and $a_{\text{min}}$, respectively (i.e., there are $D_{\text{max}}$ indices $n$ with $a_n = a_{\text{max}}$, and likewise for $D_{\text{min}}$).
Next we consider the equation
\[ g(x) = 1 \] (A12)
for an arbitrary but fixed \( a \in I_A \). According to (A12), one solution is \( x = 0 \). If \( a \neq a_{mc} \), then (A5)–(A11) imply that there exists exactly one further solution of (A12) within the interval \( I_A \), henceforth denoted as \( y(a) \). Moreover, this solution \( y(a) \) satisfies
\[ -\frac{1 - D_{\min}/N}{a - a_{\min}} \leq y(a) \leq \frac{1 - D_{\max}/N}{a_{\max} - a} \] (A13)
and the sign of \( y(a) \) must agree with the sign of \( a - a_{mc} \). In particular, if \( a = a_{mc} \) then \( y(a) := 0 \) is the only solution of (A12).

In summary, for any given \( a \in I_A \) there exists a unique real number \( y(a) \in I_A \) with the properties
\[ a = a_{mc} \Rightarrow y(a) = 0 \] (A14)
\[ a \in (a_{mc}, a_{\max}) \Rightarrow 0 < y(a) \leq \frac{1 - D_{\max}/N}{a_{\max} - a} \] (A15)
\[ a \in (a_{\min}, a_{mc}) \Rightarrow 0 > y(a) > -\frac{1 - D_{\min}/N}{a - a_{\min}} \] (A16)
\[ p_n(a) := \frac{1}{N} \left(1 + y(a)(a - a_n)\right) > 0 \text{ for all } n \] (A17)
\[ \sum_{n=1}^{N} p_n(a) = 1 \] (A18)
It follows that
\[ 1 = \sum_{n=1}^{N} \left(1 + y(a)(a - a_n)\right) p_n(a) \]
\[ = [1 + y(a)a] \sum_{n=1}^{N} p_n(a) - y(a) \sum_{n=1}^{N} a_n p_n(a) \]
\[ = 1 + y(a) \left[a - \sum_{n=1}^{N} a_n p_n(a)\right] \] (A19)
If \( y(a) \neq 0 \), we can conclude that
\[ \sum_{n=1}^{N} a_n p_n(a) = a \] (A20)
If \( y(a) = 0 \), the same result follows from (16) and (A14).

In turn, one readily sees that there exists no further real number \( y(a) \) so that all three properties (A17), (A18), and (A20) are still satisfied. Omitting the argument \( a \) in \( y(a) \) and in \( p_n(a) \), one thus recovers the statement from Eqs. (4)–(11) in the main text.

**Appendix B**

This appendix provides the omitted details in the comparison at the end of Sec. [IV] between our present approach and the one from Ref. [2].

We start out from the submanifolds \( S_a \) and \( M_a \) and the probability measure on each of them, see Sec. [IV]. In our present work, we demonstrate that dynamical typicality, as specified in Sec. [III], holds for \( S_a \) if condition (46) is satisfied. In Ref. [2] it is shown that the same kind of typicality holds for \( M_a \) on conditions which are specified in Theorem 1 therein. Moreover, one can infer from Section 6 of Ref. [2] that dynamical typicality for \( M_a \) implies the same for \( S_a \). In particular, the ensemble averaged expectation values of any given observable \( B \) are very close to each other for the two ensembles \( M_a \) and \( S_a \). Strictly speaking, three additional steps are required to arrive at this conclusion: (i) The notation in Ref. [2] has to be translated into our present notation. This will be done in the second succeeding paragraph. (ii) Rather than the ensemble \( S_a \), which is generated via (38), the ensemble generated via (19) is actually considered in Ref. [2]. Their equivalence has been pointed out below Eq. (48). (iii) One has to show that the mapping of a vector \(|\varphi\rangle \in \mathcal{H} \) to its expectation value \( \langle \varphi | B | \varphi \rangle \) amounts to a Lipschitz continuous function (see p. 820 in [2]). This has been demonstrated, e.g., in Ref. [10] (see Lemma 5 therein).

In the remainder of this appendix we show that the prerequisites and/or the applicability of Theorem 1 in Ref. [2] are essentially tantamount to the condition (46) required by our present approach. The details of the pertinent Theorem 1 in Ref. [2] are quite involved and therefore not reproduced here. Most of the subsequent discussion can be followed without knowledge of those details.

To begin with, we point out the main notational differences: Our present \( A = \sum_{n=1}^{N} a_n |n\rangle\langle n| \) corresponds to \( H = \sum_{k=1}^{n} E_k |k\rangle\langle k| \) in Ref. [2]. Likewise, our \( a_{\max}, a_{\min}, \) and \( a \) correspond to \( E_{\max}, E_{\min}, \) and \( E, \) respectively. Furthermore, \( M_E \) and \( E_A \) in Ref. [2] correspond to \( M_a \) from [17] and \( a_{mc} \) from [18], respectively. Similarly to Theorem 1 in [2], only \( a < a_{mc} \) are considered from now on (the case \( a \geq a_{mc} \) then readily follows, see also below Eq. (19)). Finally, the definitions \( E' := E + s \) and \( E_k' := E_k + s \) from [2] are translated into \( a + s \) and \( a_n + s \), respectively.

According to Equation (2) in Ref. [2], the parameter \( s \) is defined as the solution of the equation
\[ E' = (1 + \delta_E) \left[ \frac{1}{n} \sum_{k=1}^{n} (E_k')^{-1} \right]^{-1} \] , (B1)
\[ \delta_E := \frac{1}{n} + \frac{\epsilon}{\sqrt{n}} \left[ 1 + \frac{1}{n} \right] \] , (B2)
where \( \epsilon \) will be specified later. Furthermore, \( s \) must satisfy \( s > -E_{\min} \). Introducing
\[ y(a, \delta) := -\frac{1}{a + s} \] , (B3)
these conditions translate into
\[ g(y(a, \delta)) = (1 + \delta) , \]  
\[ \delta := \frac{1}{N} + \frac{\epsilon}{\sqrt{N}} \left( 1 + \frac{1}{N} \right) , \]  
\[ 0 > y(a, \delta) > -\frac{1}{a_{\min}} , \]  
where \( g(x) \) is defined in (12). For \( \delta \to 0 \) we thus recover \( A12 \) and hence \( y(a, \delta) \) goes over into \( y(a) \) from \( A13 \).

The most important condition in Theorem 1 of Ref. [2] reads \( \epsilon > E'/E'_Q \) with \( E'_Q := (n \sum_{k=1}^{n}[E'_k]^{-2})^{-1/2} \) and translates along the same lines as above into
\[ \frac{1}{N} \sum_{n=1}^{N} \left| q_n(y(a, \delta)) \right|^2 < \epsilon^2 \]  
with \( q_n(x) \) from \( A4 \). This condition suggests that the so far unspecified parameter \( \epsilon \) should be chosen as large as possible. Since \( \epsilon \) also enters via \( B4 \), \( B5 \), a somewhat more elaborate consideration is needed to show that the latter statement is indeed true.

Finally, the definition \( c := 3E'_{\min} / (32E') \) in \( \epsilon \) translates into \( c = 3/[32N p_{\max}(a, \delta)] \), where \( p_{\max}(a, \delta) := \max_n q_n(y(a, \delta))/N \). Since \( y(a, \delta) < 0 \), see \( B6 \), it follows that \( p_{\max}(a, \delta) = N^{-1}[1 + y(a, \delta)(a - a_{\min})]^{-1} \). The largest possible choice of \( \epsilon \) so that the exponential term in Equation (1) of Ref. [2] is – as requested – a large positive number even for quite small \( t \)-values turns out to be \( \epsilon \approx \sqrt{N p_{\max}(a, \delta)} \), where \( \epsilon \) is \( N \) independent and satisfies \( \epsilon \approx 1 \). For large \( N \) it follows that \( \delta \) in \( B5 \) can be approximated as zero, hence \( y(a, \delta) \approx y(a) \) (see above) and \( p_{\max}(a, \delta) \approx p_{\max}(a) \) (see \( A4 \) and \( A6 \)). Accordingly, by exploiting \( A6 \), \( A13 \), \( A4 \) and \( A17 \) the condition \( B7 \) can be rewritten as
\[ N^2 p_{\max}^2 P \ll 1 \]  
With \( A4 \) it follows that
\[ p_{\max} \ll 1/N^{2/3} \]  
will be a somewhat stronger condition (i.e., a sufficient but possibly not necessary condition for \( B8 \) to hold true). For large \( N \), this is a slightly (but not dramatically) stronger condition than the requirement \( H0 \) of our present approach. In turn, when \( p_{\max} \) is not small then it immediately follows with \( A4 \) that \( B8 \) is not satisfied.

Comparing \( A6 \) and \( B9 \) shows that the present approach is actually valid under slightly more general conditions than the corresponding Theorem 1 in Ref. [2], but for practical purposes this generalization seems only of minor relevance.

Appendix C

In this appendix, we show that \( p_{\max} \) is a monotonically increasing function of \( a \) for \( a > a_{\text{mc}} \), and monotonically decreasing for \( a < a_{\text{mc}} \).

By differentiating \( A13 \) with respect to \( a \) and introducing \( A17 \) one can infer that
\[ \sum_{n=1}^{N} \frac{1}{N} \frac{y'(a)(a - a_n) + y(a)}{[1 + y(a)(a - a_n)]^2} = 0 . \]  

Multiplying this equation by \( y(a) \) and rewriting the resulting numerator on the left hand side as \( y'(a)[1 + y(a)(a - a_n) - 1] + y^2(a) \) yields
\[ \sum_{n=1}^{N} \frac{1}{N} \frac{y'(a)}{1 + y(a)(a - a_n)} = [y'(a) - y^2(a)]S(a) , \]  
\[ S(a) := \sum_{n=1}^{N} \frac{1}{1 + y(a)(a - a_n)}^2 . \]  

With \( A14 \) and \( A18 \) we can rewrite \( C2 \) as
\[ y'(a)[S(a) - 1] = y^2(a)S(a) . \]  

From \( A17 \) and \( C3 \) we can conclude that
\[ S(a) = N \sum_{n=1}^{N} p_n^2(a) = NP , \]  
where \( P \) is the purity of \( \rho \) from Eqs. \( 28 \) of the main text. Observing \( A17 \) and \( A18 \) one can infer that \( S(a) = 1 \) if \( y(a) = 0 \) and \( S(a) > 1 \) in any other case. (Equivalently, the purity \( P \) takes its minimal possible value \( 1/N \) if an only if \( \rho \) is the microcanonical density operator.) With \( A14 \) we thus can conclude that
\[ y'(a) = y^2(a) \frac{S(a)}{S(a) - 1} > 0 \text{ for all } a \neq a_{\text{mc}} . \]  

For continuity reason it follows that \( y'(a_{\text{mc}}) \geq 0 \) (see also \( 63 \)).

Next we address the dependence of the quantity
\[ p_{\max}(a) := \max_n p_n(a) \]  
on the argument \( a \in I_A \). We first focus on \( a \)-values so that \( A15 \) is fulfilled. Exploiting \( A17 \) it follows that
\[ p_{\max}(a) = \frac{1}{N} \frac{1}{1 - y(a)(a_{\max} - a)} . \]  

Differentiating and introducing \( C6 \) yields
\[ p'_{\max}(a) = \frac{y'(a)(a_{\max} - a) - y(a)}{N[1 - y(a)(a_{\max} - a)]^2} = \frac{N p_{\max}^2(a) y(a) h(a)}{S(a) - 1} , \]  
\[ h(a) := y(a)S(a)(a_{\max} - a) - (S(a) - 1) = 1 - S(a)[1 - y(a)(a_{\max} - a)] . \]  

(C10)
With Eq. (C8) we can conclude that

\[ S(a)[1 - y(a)(a_{\text{max}} - a)] = \sum_{n=1}^{N} p_n(a) b_n(a), \quad (C11) \]

\[ b_n(a) := \frac{1 - y(a)(a_{\text{max}} - a)}{1 - y(a)(a_n - a)} \]

\[ = \frac{p_{\text{max}}(a)}{p_{\text{max}}(a) + y(a)(a_{\text{max}} - a_n)/N}, \quad (C12) \]

where we exploited (C8) in the last step. Observing that \( y(a) > 0, p_{\text{max}}(a) > 0, \text{and } a_{\text{max}} - a_n \geq 0 \) for all \( n \) it follows that \( 0 \leq b_n(a) \leq 1 \) and with (A18) that \( \sum_{n=1}^{N} p_n(a) b_n(a) \in [0, 1] \). With (C10) and (C11) we can conclude that \( h(a) \geq 0 \) and with (C9) that

\[ p'_{\text{max}}(a) \geq 0 \text{ for all } a \in (a_{mc}, a_{\text{max}}), \quad (C13) \]

i.e., \( p_{\text{max}}(a) \) is a monotonically increasing function of \( a \). For \( a \)-values so that (A16) is fulfilled, one finds in the same way that

\[ p'_{\text{max}}(a) \leq 0 \text{ for all } a \in (a_{\text{min}}, a_{mc}), \quad (C14) \]

i.e., \( p_{\text{max}}(a) \) is a monotonically decreasing function of \( a \).

### Appendix D

In this Appendix, the necessary and sufficient condition (10) for dynamical typicality is explored by means of more rigorous calculations than those presented in Sec. V A.

Without loss of generality we assume that the eigenvalues \( a_n \) of \( A \) from (5) are ordered by magnitude:

\[ a_1 \leq a_2 \leq \ldots \leq a_N =: a_{\text{max}}. \quad (D1) \]

As in Sec. V A, we can and will focus on \( a \)-values within the range from (49), implying (50) and (51).

Similarly as below Eq. (52), we furthermore disregard the trivial case that \( D_{\text{max}} \gg 1 \), i.e., we take for granted that the largest eigenvalue of \( A \) is not highly degenerate. As a consequence, the case \( a \rightarrow a_{\text{max}} \) is then also trivial: (40) is always violated according to (52). We thus can and will restrict ourselves to the case

\[ \Delta a := a_{\text{max}} - a > 0. \quad (D2) \]

From (50) and (52) it follows that

\[ \delta := y \Delta a > 0. \quad (D3) \]

Moreover, (51) can be rewritten as

\[ p_{\text{max}} = \frac{1}{N} \frac{1}{1 - \delta}. \quad (D4) \]

Finally, we restrict ourselves to cases where \( a \) coincides with one of the eigenvalues \( a_n \), say

\[ a = a_\nu. \quad (D5) \]

Since \( p_{\text{max}} \) is an increasing function of \( a \) (see Appendix C), this is a quite minor restriction: If we can show that (40) is fulfilled for \( a = a_\nu \), then the same conclusion remains true for all \( a \leq a_\nu \) (as long as (49) is fulfilled). Likewise, if (40) is violated for \( a = a_\nu \), then the same applies for all \( a \geq a_\nu \).

### D1. Derivation of a sufficient condition

Our first goal will be to derive a sufficient criterion under which (40) is fulfilled. To begin with, we define the linear function

\[ f(x) := a_{\text{max}} - b(N - x), \quad (D6) \]

where the slope \( b \) is chosen as follows

\[ b := \max_{\nu \leq n < N} b_n, \quad (D7) \]

\[ b_n := \frac{a_N - a_n}{N - n}. \quad (D8) \]

One thus can conclude that

\[ f(n) \leq a_n \text{ for all } n \in \{\nu, \ldots, N\} \quad (D9) \]

and that (D7) is the smallest possible slope in (D6) which exhibits this property. In other words, \( f(n) \) represents a tight linear lower bound for all eigenvalues \( a_n \geq a \) with the extra constraint that \( f(N) = a_{\text{max}} \).

Note that the slope of a linear function through the two points \( a = a_\nu \) and \( a_{\text{max}} = a_N \) would be given by \( b_\nu \). The ratio between the two slopes,

\[ \beta := b/b_\nu \quad (D10) \]

quantifies how well the true \( a_\nu \)'s (with \( \nu \leq n \leq N \)) can be approximated by a linear function in the above sense of a rigorous lower bound. In particular, one readily sees that \( \beta \geq 1 \). On the other hand, not too large \( \beta \)'s may be expected in many cases.

In the following, we will show that

\[ N_\alpha \gg \beta \frac{N}{\ln N} \quad (D11) \]

is a sufficient condition for (40), where \( \beta \) is defined in (D10) and where, analogously as in (55),

\[ N_\alpha := N - \nu \quad (D12) \]

is the number of eigenvalues \( a_n \) which are larger than \( a \). This is the rigorous counterpart of the heuristic condition (63) in the main text.

In order to prove this claim, we exploit (S4)-(10) to conclude

\[ 1 \geq S := \sum_{n=\nu}^{N} p_n = \frac{1}{N} \sum_{n=\nu}^{N} \frac{1}{1 + y(a - a_n)} \quad (D13) \]
Defining an auxiliary function \( a(x) \) via

\[
a(x) := a_n \text{ for } x \in (n-1, n], n = 1, \ldots, N .
\]

it follows that

\[
S = \frac{1}{N} \int_{\nu-1}^{\nu} dx \frac{1}{1 + y(a - a(x))} .
\]

(D15)

Since all summands on the right hand side of (D13) are positive (see (D9)), the same applies to the integrand on the right hand side of (D15), implying that

\[
S \geq \frac{1}{N} \int_{x_0}^{N} dx \frac{1}{1 + y(a - a(x))} .
\]

(D16)

for any

\[
x_0 \in [\nu - 1, N] .
\]

(D17)

From (D9) and (D14) one can deduce that \( f(x) \leq a(x) \) for all \( x \in (\nu - 1, N] \). Since \( y > 0 \) (see (D10)) it follows that \( 1 + y(a - f(x)) \geq 1 + y(a - a(x)) \). Hence we can conclude from (D16) that

\[
S \geq \frac{1}{N} \int_{x_0}^{N} dx \frac{1}{1 + y(a - f(x))} .
\]

(D18)

Recalling that \( f(x) \) in (D6) is a linear function, the integral on the right hand side can be readily evaluated. Choosing \( x_0 \) so that \( f(x_0) = a \) one can infer from (D6) that \( x_0 \) is uniquely fixed in this way and from (D5), (D9) that (D17) is satisfied. By means of a straightforward but somewhat lengthy calculation and exploiting the relations (D2), (D3), (D6), (D8), (D10), (D12) one finally obtains

\[
S \geq \frac{N_a}{N^3} \ln \left( \frac{1}{1 - \delta} \right) .
\]

(D19)

It remains to be shown that if (D11) is taken for granted, then (D16) follows: Given (D11) is valid, and observing that \( N_\delta \leq N \) and \( \beta \geq 1 \), it follows that \( N \gg 1 \). If \( \delta \) is not close to unity, say \( \delta \leq 1/\sqrt{N} \), then (D1) implies that \( p_{\max} \leq 1/\sqrt{N} \), i.e., (D16) is fulfilled. Hence we are left with the case \( \delta > 1/\sqrt{N} \). Exploiting (D4) in the argument of the logarithm in (D19) and approximating the remaining factor \( \delta \) by unity, we can conclude with (D13) and (D19) that

\[
1 \geq \frac{N_a}{N^3} \ln \left( \frac{p_{\max} N}{\kappa \Delta} \right) =: Q .
\]

(D20)

Solving for \( p_{\max} \) yields

\[
p_{\max} = \frac{1}{N} \exp \{ Q N \beta / N_a \} .
\]

(D21)

It follows from (D21) that \( p_{\max} \ll 1 \), i.e., (D16) is fulfilled if and only if \( \exp \{ Q N \beta / N_a \} \ll N \). Taking logarithms on both sides, and exploiting that \( N \gg 1 \), this is tantamount to \( Q N \beta / N_a \ll \ln N \) and hence to \( Q N \beta / \ln N \ll N_a \). Since \( Q \leq 1 \) according to (D20), the latter relation is guaranteed by our premise (D11).

**D2. Lower bound for \( p_{\max} \)**

We consider the following subsets of \( G := \{1, \ldots, N\} \):

\[
G_1 := \{ n \in G \mid a_n \leq a - \Delta a \} ,
\]

(D22)

\[
G_2 := \{ n \in G \mid a_n > a - \Delta a \text{ and } a_n \leq a \} ,
\]

(D23)

\[
G_3 := \{ n \in G \mid a_n > a \} .
\]

(D24)

Hence, \( G \) is the disjoint union of the subsets \( G_k \), \( k = 1, \ldots, 3 \). Denoting the number of elements contained in the subset \( G_k \) as

\[
N_k := |G_k| ,
\]

(D25)

it follows that

\[
N_1 + N_2 + N_3 = N .
\]

(D26)

In particular, \( N_3 \) is identical to \( N_a \) from (D3) and (D12),

\[
N_3 = N - \nu = N_a .
\]

(D27)

Moreover, defining

\[
S_k := \sum_{n \in G_k} p_n
\]

(D28)

we can conclude with (D10) that

\[
S_1 + S_2 + S_3 = 1 .
\]

(D29)

Finally, (D3), (D2), (D6), and (D22)-(D24) can be exploited to conclude

\[
p_n \leq \frac{1}{N} \frac{1}{1 + \delta} \text{ for all } n \in G_1 ,
\]

(D30)

\[
p_n \leq \frac{1}{N} \text{ for all } n \in G_2 ,
\]

(D31)

\[
p_n \leq \frac{1}{N} \frac{1}{1 - \delta} \text{ for all } n \in G_3 .
\]

(D32)

With (D25) and (D28) it follows that

\[
S_1 \leq \frac{N_1}{N} \frac{1}{1 + \delta} ,
\]

(D33)

\[
S_2 \leq \frac{N_2}{N} ,
\]

(D34)

\[
S_3 \leq \frac{N_3}{N} \frac{1}{1 + \delta} .
\]

(D35)

In combination with (D26) and (D29) this yields after a short calculation the result

\[
S_1 + S_2 + S_3 = 1 \leq 1 + \frac{\delta}{N(1 - \delta^2)} R .
\]

(D36)

\[
R := \delta(N_1 + N_3) - N_1 + N_3 .
\]

(D37)

With (D3) and (D4) it follows that \( R \) in (D36) must be non-negative and with (D37) that

\[
\delta \geq \frac{N_1 - N_3}{N_1 + N_3} .
\]

(D38)
For instance, if \( a \) is sufficiently close to \( a_{\text{max}} \) so that \( N_3 \leq N_3/3 \) then (D38) implies \( \delta \geq 1/2 \) and with (D4) we obtain the lower bound \( p_{\text{max}} \geq 2/N \). Upon further increasing \( a \) towards \( a_{\text{max}} \), the ratio \( N_3/N_1 \) decreases, and the lower bounds for \( \delta \) and \( p_{\text{max}} \) increase towards limits not too far from their largest possible values compatible with (50) and (52).

**D3. Derivation of a necessary condition**

The lower bound for \( p_{\text{max}} \) implied by (D38) implicitly amounts to a necessary criterion for (46), which, however does not substantially go beyond the findings around (52) in the main text. The basic idea in the following considerations is to proceed like in the previous Sec. D2, except that a better upper bound for \( S_3 \) than in (D35) will be derived. The latter in turn is achieved by similar arguments as in Sec. D1.

To begin with, we define the function

\[
 f(x) := a_{\text{max}} - b(N - 1 - x)^\gamma \tag{D39}
\]

with an arbitrary but fixed exponent

\[
 \gamma > 0 , \tag{D40}
\]

and where the prefactor \( b \) is chosen as follows

\[
 b := \min_{\nu \leq x < N} b_n , \tag{D41}
\]

\[
 b_n := \frac{a_N - a_n}{(N - n)^\gamma} . \tag{D42}
\]

It follows that

\[
 f(n) \geq a_{n+1} \quad \text{for all} \quad n \in \{ \nu - 1, \ldots, N - 1 \} \tag{D43}
\]

and hence

\[
 f(x) \geq a(x) \quad \text{for all} \quad x \in [\nu - 1, N - 1] , \tag{D44}
\]

where \( a(x) \) is defined in (D14). Similarly as in (D10) we define

\[
 \sigma := b/b_\nu \tag{D45}
\]

but now it follows from (D41) and (D12) that \( \sigma \leq 1 \). For the rest, the intuitive meaning of \( f(x) \) and \( \sigma \) is analogous to the discussion below (D9).

With (8) and (49) it follows that \( p_N = p_{\text{max}} \) and with (D14), (D5), (D24), and (D28) that

\[
 S_3 = p_{\text{max}} + S_3' , \tag{D46}
\]

\[
 S_3' := \frac{1}{N} \sum_{n=\nu+1}^{N-1} \frac{1}{1 + y(a - a_n)} . \tag{D47}
\]

Similarly as in (D15)–(D18) one can exploit (D44) to conclude

\[
 S_3' \leq \frac{1}{N} \int_{x_0}^{N-1} \frac{1}{1 + y(a - f(x))} \tag{D48}
\]

for any \( x_0 \leq \nu \). In particular, we can choose \( x_0 = 0 \). Replacing the integration variable \( x \) via substitution by \( u := (N - 1 - x)/\eta \), where

\[
 \eta := N_3 \left( \frac{1 - \delta}{\sigma \delta} \right)^{1/\gamma} , \tag{D49}
\]

it follows with (D2), (D3), (D27), (D39), (D42), (D45) that

\[
 S_3' \leq \frac{\eta}{N(1 - \delta)} I \left( \frac{\delta}{1 - \delta} \right)^{1/\gamma} , \tag{D50}
\]

\[
 I(v) := \int_0^v \frac{1}{1 + u^\gamma} . \tag{D51}
\]

For \( \gamma = 1 \), i.e., the function \( f(x) \) from (D39) amounts to a linear upper bound in (D43), one finds with (D19)–(D51) that

\[
 S_3' \leq \frac{N_3}{N \sigma \delta} \ln \left( \frac{1}{1 - \delta} \right) . \tag{D52}
\]

Introducing (D26), (D33), (D34), (D46), (D49) into (D29) implies

\[
 \frac{N_1 + N_2 + N_3}{N} = S_1 + S_2 + S_3 \leq \frac{N_1}{N} \frac{1}{1 + \delta} + \frac{N_2}{N} + \frac{p_{\text{max}}}{N \sigma \delta} \ln \left( \frac{1}{1 - \delta} \right) . \tag{D53}
\]

By means of (D4) we thus obtain

\[
 p_{\text{max}} \geq \frac{N_1}{N} \delta + \frac{N_3}{N} \left( 1 - \ln(p_{\text{max}} N) / \sigma \delta \right) . \tag{D54}
\]

In the following, we restrict ourselves to the case

\[
 N_2, N_3 \leq N_1/3 . \tag{D55}
\]

With (D26) and (D38) it follows that

\[
 N_1 \geq 3 N/5 , \tag{D56}
\]

\[
 \delta \geq 1/2 . \tag{D57}
\]

One readily verifies that this implies \( \delta/(1 - \delta) \geq 1/3 \) and with (D54) and \( p_{\text{max}} \leq 1 \) (see (10)) that

\[
 p_{\text{max}} \geq \frac{1}{5} - \frac{N_3}{N} \frac{2 \ln N}{\sigma} = (1 - R)/5 , \tag{D58}
\]

\[
 R := \frac{N_3}{N} \frac{10}{\sigma} \ln N . \tag{D59}
\]

It follows that a necessary prerequisite for (46) is \( R \geq 1/2 \) and thus

\[
 N_3 \geq \frac{\sigma}{2} \frac{N}{\ln N} . \tag{D60}
\]

Upon comparison of the sufficient condition (D11) for (46) with the necessary condition (D60) (and recalling
bounded by unity for

Observing that in (D51) the integrand is upper

not much larger than unity. In other words, the break-

down of (46) is expected to happen when

words, our above derivation of (D60) is self consistent in

In other words, the above derivation of (D60) is self consistent in

the non-trivial regime, i.e., whenever the stronger condi-

tion (11) is violated.

Finally, we turn to exponents \( \gamma \) in (D39) with \( 1 > \gamma > 0 \) (see also (D40); exponents \( \gamma > 1 \) are of minor interest). As before, we focus on the case (D56). Hence, \( \delta \) satisfies (D57) and the argument of \( I \) in (D50) is lower bounded by unity. Observing that in (D51) the integrand is upper bounded by unity for \( u \leq 1 \) and by \( u^{-\gamma} \) for \( u \geq 1 \) then yields

\[
I(u) \leq 1 + \frac{v^{1-\gamma} - 1}{1-\gamma} = \frac{v^{1-\gamma} - \gamma}{1-\gamma} \leq \frac{v^{1-\gamma}}{1-\gamma}.
\]  

(D61)

Introducing (D49) and (D61) into (D50) results in

\[
S_3' \leq \frac{N_3}{N(1-\gamma)\sigma^{1/\gamma}\delta} \leq \frac{N_3}{N(1-\gamma)\sigma^{1/\gamma}}.
\]  

(D62)

where we exploited (D57) in the last step. Similarly as in (D55) one can conclude that

\[
p_{\text{max}} \geq \frac{(1-R')/5}{R'} := \frac{N_3}{N(1-\gamma)\sigma^{1/\gamma}}.
\]  

(D63)

It follows that a necessary prerequisite for (46) is \( R' \geq 1/2 \) and thus

\[
N_3 \geq \frac{(1-\gamma)\sigma^{1/\gamma}}{2} N.
\]  

(D65)

In contrast to the result (D60) for \( \gamma = 1 \), the result (D65) for \( \gamma < 1 \) predicts a violation of (16) at a certain fraction \( N_3/N \) of eigenvalues \( a_n \) near \( a_{\text{max}} \), which does not approach zero even in the limit \( N \to \infty \).

[1] C. Bartsch and J. Gemmer, Phys. Rev. Lett. 102, 110403 (2009)
[2] M.P. Müller, D. Gross, and J. Eisert, Commun. Math. Phys. 303, 785 (2011)
[3] B. V. Fine, Phys. Rev. E 80, 051130 (2009)
[4] P. Reimann, Phys. Rev. Lett. 99, 160404 (2007)
[5] G. A. Alvarez, E. P. Danieli, P. R. Levstein, and H. M. Pastawski, Phys. Rev. Lett. 101, 120503 (2008)
[6] P. Reimann, Phys. Rev. Lett. 115, 010403 (2015)
[7] J. Gemmer, M. Michel, and G. Mahler, Quantum Thermodynamics (1st edition, Springer, Berlin, Heidelberg, 2004)
[8] S. Lloyd, Ph.D. Thesis, The Rockefeller University (1988), Chapter 3, arXiv:1307.0378
[9] S. Popescu, A. J. Short, and A. Winter, Nature Phys. 2, 754 (2006)
[10] S. Popescu, A. J. Short, and A. Winter, The foundations of statistical mechanics from entanglement: Individual states vs. averages, arXiv:quant-ph/0511225 (2016).
[11] S. Goldstein, J. L., Lebowitz, R. Trumkula, and N. Zanghi, Phys. Rev. Lett. 96, 050403 (2006)
[12] A. Sugita, Nonlinear Phenom. Complex Syst. 10, 192 (2007)
[13] S. Sugiuira and A. Shimizu, Phys. Rev. Lett. 108, 240401 (2012)
[14] H. Tasaki, J. Stat. Phys. 163, 937 (2016)
[15] A simpler way to arrive at the same conclusion is as follows: Summing on the left hand side of (10) only over the \( D_{\text{max}} \) indices \( n \) with \( a_n = a_{\text{max}} \) yields \( D_{\text{max}}p_{\text{max}} \leq 1 \). Thus \( D_{\text{max}} \geq 1 \) implies \( p_{\text{max}} \ll 1 \).
[16] T. A. Brody, J. Flores, J. B. French, P. A. Mello, A. Pandey, and S. S. M. Wong, Rev. Mod. Phys. 53, 385 (1981)
[17] With respect to the \( m_k \) from (37), it is assumed in Ref. [1] that \( a_{\text{mc}} = 0 \) and \( m_2 = 1 \), which can always be achieved by means of a trivial “renormalization” of \( A \). In addition, it is required in [1] that the \( m_k \) are at most of the order of unity (compared to \( N \)) for all \( k = 3, \ldots, 8 \). This is a non-trivial restriction with respect to \( A \) and notably stronger than our present conditions in Sec. [V] and Appendix D.