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Asymptotic form of level density distributions for a class of inhomogeneous 1D vertex models

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Abstract. Level density distributions for some Haldane-Shastry like spin chains associated with \(A_{N-1}\) root system have been computed recently by Enciso \textit{et al.}, using the connection of these spin chains with inhomogeneous one-dimensional vertex models. The energy functions of such vertex models are given by specific polynomials of first or second degree. Here we consider a much broader class of one-dimensional vertex models whose energy functions are given by arbitrary polynomials of any possible degree and show that the level densities for this class of vertex models asymptotically follow the Gaussian distribution for large number of vertices.

One-dimensional (1D) quantum integrable spin chains are usually classified into following two types depending on their range of interaction. One such class consists of spin chains having only local interactions, like nearest or next-to-nearest neighbour interaction. Isotropic and anisotropic versions of Heisenberg spin-\(\frac{1}{2}\) chain, Hubbard model etc. are prominent examples of this type of spin models with short range interaction \([1–3]\). However, it is also possible to construct 1D quantum integrable spin systems like Haldane-Shastry (HS) spin chain \([4,5]\), Polychronakos spin chain (also known as Polychronakos-Frahm (PF) spin chain) \([6–8]\) and Frahm-Inozemstsev (FI) spin chain chain \([9,10]\), for which all constituent spins mutually interact with each other through long range forces. This class of spin chains with long range interactions have attracted a lot of attention in recent years due to their exact solvability and applicability in a wide range of subjects like generalized exclusion statistics, fractional quantum Hall effect, SUSY Yang-Mills theory and Yangian quantum groups \([11–24]\). For the simplest case of \(A_{N-1}\) root system, the Hamiltonians of su\((m)\) invariant, ferromagnetic type HS, PF and FI spin chains can be written in a general form like \([4–10]\)

\[
H = \sum_{1 \leq j < k \leq N} \frac{1 - P_{jk}}{f(\xi_j - \xi_k)}. \tag{1}
\]

Here \(\xi_j\) denotes the position of the \(j\)-th lattice site, spins can take \(m\) possible values on each lattice site, \(P_{jk}\) represents the exchange operator which interchanges the spins on the \(j\)-th and \(k\)-th lattice sites and the strength of this exchange interaction is determined by the function \(f(\xi_j - \xi_k)\). Supersymmetric generalizations of the spin chains (1) have also been studied in the literature \([12, 25–27]\).

As is well known, the Hamiltonians of 1D quantum integrable spin chains with short range interaction are related to the transfer matrices of some exactly solvable 2D vertex models in statistical mechanics \([28, 29]\). However, there exists no simple relation between the energy levels.
of 1D spin chains of short range interaction and the energy functions of corresponding 2D vertex models. On the other hand, a direct connection between the energy levels of 1D quantum integrable spin chains with long range interaction and energy functions of 1D classical vertex models can be inferred from some works in the literature [20,22,30]. In this context it should be noted that, being motivated by the Yangian quantum group symmetry of HS like spin chains with long range interactions, a large class of 1D inhomogeneous vertex models have been proposed recently [31]. Moreover, as an application of a rather general type of equivalence relation, it has been shown that the energy levels of HS, PF and FI spin chains associated with the $A_{N-1}$ root system exactly coincide with those of three vertex models belonging to the above mentioned class, whose energy functions depend on the vertex index through specific polynomials of first or second degree. Such a relation between spin chains with long range interactions and 1D vertex models might be applied to explore various physical properties of these spin chains. For example, by using the transfer matrix approach, it has been proved analytically that the level densities of the three vertex models related to HS, PF and FI spin chains asymptotically tend to the Gaussian distribution as the number of vertices become very large [30]. Hence, it automatically follows that the level densities of the HS, PF and FI spin chains also tend to the Gaussian distribution with the increase in number of lattice sites. However, it is worth noting that, the energy functions for the class of 1D inhomogeneous vertex models proposed in Ref. [31] could be taken as arbitrary functions of the vertex index. Therefore, it is interesting to ask whether the Gaussian nature of level density distribution is a generic feature for all of these vertex models. In a recent paper the present authors have considered a large subclass of these vertex models, where the energy functions are taken as arbitrary polynomials of any possible degree, and shown that the corresponding level densities always follow the Gaussian pattern [32]. The aim of this short review article is to recapitulate some main results of Refs. [30] and [31], and subsequently discuss how these results have been applied in Ref. [32] to find out the asymptotic form of level densities for 1D vertex models with arbitrary polynomial type energy functions.

At first let us discuss how, being motivated by the Yangian quantum group symmetry of HS like spin chains with long range interactions, a class of 1D inhomogeneous vertex models have been proposed in Ref. [31]. For the sake of simplicity, we shall restrict ourselves to the special case of nonsupersymmetric $su(m)$ ferromagnetic spin chains. A class of irreducible representations of the $Y(\mathfrak{sl}_m)$ Yangian algebra, known as ‘border strips’ or ‘motifs’, completely span the Hilbert space of HS like spin systems with long range interaction [12, 17–24, 33]. For the case of a spin system with $N$ number of lattice sites, border strips are denoted by $\langle k_1, k_2, \ldots, k_r \rangle$, where each $k_i$ is a positive integer satisfying the relation $\sum_{i=1}^{r} k_i = N$. Thus the Hilbert space of this spin chain may be decomposed as

$$V = \bigoplus_{k_1+k_2+\ldots+k_r=N} V_{\langle k_1, k_2, \ldots, k_r \rangle},$$

where $V_{\langle k_1, k_2, \ldots, k_r \rangle}$ denotes the irreducible vector space associated with $\langle k_1, k_2, \ldots, k_r \rangle$. Let us assume that there exists a class of Yangian invariant spin systems, for which the complete set of energy levels can be written in the following way:

$$E_{\langle k_1, k_2, \ldots, k_r \rangle} = \sum_{l=1}^{r-1} E_N(j)|j=K_l,$$

where $K_l = \sum_{i=1}^{l} k_i$ and $E_N(j)$ is an arbitrary function of the discrete variables $j$ and $N$. Due to Yangian symmetry of this class spin systems, the multiplicity of the eigenvalue $E_{\langle k_1, k_2, \ldots, k_r \rangle}$ in Eq. (3) coincides with the dimensionality of the vector space $V_{\langle k_1, k_2, \ldots, k_r \rangle}$. It should be noted that, all energy levels in the spectra of $su(m)$ ferromagnetic HS, PF and FI spin chains can be
generated through Eq. (3) as special cases, where \( E_N(j) \)’s are given by ‘dispersion relations’ of the form \([17–24, 30]\)

\[
E_N(j) = \begin{cases} 
  j(N - j), & \text{for the HS chain} \\
  j, & \text{for the PF chain} \\
  j(c + j - 1), & \text{with } c > 0, \text{ for the FI chain.} 
\end{cases}
\] (4)

Next, we consider some 1D classical vertex models with \((N + 1)\) number of vertices, which are connected through \(N\) number of intermediate bonds. The ‘colour’ of each of these bonds can be chosen in \(m\) number of different ways. Therefore, any state for such a vertex model can be represented by a path configuration of finite length:

\[ \bar{s} \equiv \{s_1, s_2, \ldots, s_N\}, \] (5)

where \(s_i \in \{1, 2, \cdots, m\}\) denotes the colour of the \(i\)-th bond. We define an ‘energy function’ corresponding to the state \(\bar{s}\) as

\[
E(\bar{s}) = \sum_{j=1}^{N-1} E_N(j) \delta(s_j, s_{j+1}),
\] (6)

where \(\delta(x, y)\) is taken as

\[
\delta(x, y) = \begin{cases} 
  0, & \text{if } x \geq y \\
  1, & \text{if } x < y.
\end{cases}
\] (7)

A crucial point to note here is that, the same function \(E_N(j)\) has appeared in Eqs. (6) and (3). It can be shown that, for any possible functional form of \(E_N(j)\), the spectrum of the \(su(m)\) ferromagnetic spin chain generated by eigenvalues (3) (along with the degeneracy factors of these energy levels) completely matches with that of the corresponding vertex model generated by energy functions (6) \([31]\). In particular, the full spectra of \(su(m)\) ferromagnetic HS, PF and FI spin chains can be reproduced from the energy function (6), by substituting to it the corresponding forms of dispersion relations (4).

Applying the transfer matrix approach it can be shown analytically that the level densities of the three vertex models, associated with the dispersion relations (4) of HS, PF and FI spin chains, follow the Gaussian distribution for large number of lattice sites \([30]\). However, one can easily construct a much larger class of vertex models, by taking \(E_N(j)\) in Eq. (6) as an arbitrary polynomial function of the discrete variables \(j\) and \(N\):

\[
E_N(j) = \sum_{\alpha=0}^{r} \sum_{\beta=0}^{s} f_{\alpha,\beta} N^\alpha j^\beta,
\] (8)

where \(r\) and \(s\) are some non-negative integers and \(f_{\alpha,\beta}\)’s are arbitrary real numbers \([32]\). For the sake of uniquely expressing a polynomial type dispersion relation in the form (8) with minimum possible values of \(r\) and \(s\), we assume that (without any loss of generality) there exist at least one value of \(\beta \in \{0, \ldots, s\}\) such that \(f_{\beta,\beta} \neq 0\) and at least one value of \(\alpha \in \{0, \ldots, r\}\) such that \(f_{\alpha,\alpha} \neq 0\). The degree of the two-variable polynomial \(E_N(j)\) in Eq. (8) may be denoted by \(\chi\), which is an integer taking value within the range \(0 \leq \chi \leq r + s\). Evidently, \(f_{\alpha,\beta}\)’s appearing in Eq. (8) satisfy the conditions

\[
f_{\alpha,\beta} = 0, \text{ for } \alpha + \beta > \chi, \\
\neq 0, \text{ for at least one pair of } \alpha, \beta \text{ with } \alpha + \beta = \chi.
\] (9)
By using these conditions, it is easy to show that
\[ \chi \geq \max\{r, s\}. \]  
(10)

It may be noted that the general form of dispersion relation (8) yields all dispersion relations in (4) as special cases. Therefore, it is interesting to ask whether the Gaussian nature of level density distribution is a generic feature for all 1D vertex models defined through the energy function (6) and the polynomial type dispersion relation (8).

For the purpose of finding out an answer to the above mentioned question, in the following we shall briefly discuss the transfer matrix approach used in Ref. [30] for calculating the level density distributions of the three vertex models related to ferromagnetic $su(m)$ HS, PF and FI spin chains. A key role in this approach is played by the normalised characteristic function, which may be defined as
\[ \hat{\phi}_N(t) = \langle e^{\frac{itE(\tilde{s}) - \mu}{\sigma}} \rangle, \]  
(11)

where $E(\tilde{s})$ is the energy function (6) of the vertex model, $\mu$ and $\sigma$ denote the mean and the variance of this energy function, respectively, and the notation $\langle O \rangle$ represents the average of $O$ over all possible states of the vertex model. If this characteristic function satisfies the asymptotic relation
\[ \lim_{N \to \infty} \hat{\phi}_N(t) = e^{-\frac{t^2}{2}}, \]  
(12)

then it can be shown that [34] at $N \to \infty$ limit the normalised level density approaches Gaussian distribution of the form
\[ G(E) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(E-\mu)^2}{2\sigma^2}}. \]  
(13)

Therefore, instead of directly working with the level density of a vertex model, one can study the $N \to \infty$ limit of the related characteristic function. Inserting the form of $E(\tilde{s})$ given in Eq. (6) to Eq. (11), it is easy to express $\hat{\phi}_N(t)$ as
\[ \hat{\phi}_N(t) = m^{-N} e^{-\frac{it\mu}{\sigma}} Z_N(e^{\frac{it}{\sigma}}), \]  
(14)

where the ‘partition function’ $Z_N(e^{\frac{it}{\sigma}})$ is defined through a product of transfer matrices as
\[ Z_N(e^{\frac{it}{\sigma}}) = m^{N-1} \sum_{n,n'=1}^{m} [T(\omega_1)T(\omega_2)\ldots T(\omega_{N-1})]_{nn'}. \]  
(15)

Here the transfer matrix $T(\omega_j)$ is given by $T(\omega_j) = T(\omega)|_{\omega = \omega_j}$, with
\[ T(\omega) = \frac{1}{m} \begin{pmatrix} 1 & \omega & \ldots & \omega^m \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \ldots & \ldots & 1 \end{pmatrix}, \]  
(16)
\[ \omega_j = e^{i\gamma_j t} \]  
and
\[ \gamma_j = \frac{\xi_N(j)}{m\sigma}. \]  
(17)

From the above discussions it is clear that, the $N \to \infty$ limit of $\gamma_j$ would play a significant role in determining the asymptotic form of $\hat{\phi}_N(t)$. In this context it should be noted that, the
asymptotic form of a function can be estimated in several possible ways. For example, one may use the ‘big O notation’ like \( f(N) = O(N^a) \), which implies that
\[
\lim_{N \to \infty} |N^{-a} f(N)| = \tau ,
\] (18)
where \( \tau \) is a non-negative real number. More generally, the notation \( f(N) = O(N^a) \) is used when \( f(N) \) is asymptotically bounded above by \( N^a \) (up to a constant factor). On the other hand, in ‘big Θ notation’, \( f(N) = \Theta(N^a) \) implies that
\[
\lim_{N \to \infty} |N^{-a} f(N)| = \tau ,
\] (19)
where \( \tau \) is a positive real number. More generally, the notation \( f(N) = \Theta(N^a) \) is used when \( f(N) \) is asymptotically bounded above and below by \( N^a \) (up to constant factors) \([35–37]\). Hence, \( f(N) = \Theta(N^a) \) automatically implies \( f(N) = O(N^a) \), but the reverse is not true. A key result of Ref. [30] is to show that, if the asymptotic forms of \( \gamma_j \) and \( \gamma_j - \gamma_{j+1} \) are given by
\[
\gamma_j = O(N^{-1/2})
\] (20)
and
\[
\gamma_j - \gamma_{j+1} = O(N^{-3/2})
\] (21)
for \( j \in \{1, 2, \ldots, N - 1\} \), then the characteristic function \( \hat{\phi}_N(t) \) in Eq. (14) satisfies the relation (12) and, consequently, the normalized level density approaches the Gaussian distribution (13) at \( N \to \infty \) limit. The mean and the variance of the energy function \( E(\vec{s}) \) in Eq. (6) can be expressed as
\[
\mu \equiv \langle E(\vec{s}) \rangle = \frac{1}{2} \left( 1 - \frac{1}{m} \right) \sum_{j=1}^{N-1} \mathcal{E}_N(j) ,
\] (22)
and
\[
\sigma^2 \equiv \langle (E(\vec{s}) - \mu)^2 \rangle = \left( 1 - \frac{1}{m^2} \right) \left[ \frac{1}{4} \sum_{j=1}^{N-1} \mathcal{E}_N(j)^2 - \frac{1}{6} \sum_{j=1}^{N-2} \mathcal{E}_N(j) \mathcal{E}_N(j+1) \right] .
\] (23)
Computing \( \sigma \) exactly by using (23) for the three vertex models related to dispersion relations (4), and substituting those expressions of \( \sigma \) in Eq. (17), it is found that the asymptotic forms (20) and (21) are satisfied for all of these cases. Thus it is proved that, the level densities of the three vertex models related to ferromagnetic \( su(m) \) HS, PF and FI spin chains asymptotically follow the Gaussian distribution.

Next, our goal is to show that the asymptotic forms of \( \gamma_j \) and \( \gamma_j - \gamma_{j+1} \) are in fact universal, i.e., Eqs. (20) and (21) are satisfied even if \( \mathcal{E}_N(j) \) is chosen as an arbitrary polynomial function of the form (8). Instead of finding out the full expression of \( \sigma^2 \), at present our strategy is to directly compute the order of \( \sigma^2 \) by using Eq. (23). To this end, let us examine the order of \( \mathcal{E}_N(j) \) in two possible cases: i) \( j = \Theta(1) \), where \( N \to \infty \) limit is taken by keeping \( j \) fixed, and ii) \( j = \Theta(N) \), where \( N \to \infty \) limit is taken in such a way that \( j/N \) tends to a fixed value. Using Eq. (8), we easily obtain the order of \( \mathcal{E}_N(j) \) for case i) as
\[
\mathcal{E}_N(j) = \Theta(N^{r_j}).
\] (24)
Next we consider the case ii), which is defined through the limiting procedure
\[
\lim_{N \to \infty} \frac{j}{N} = \rho ,
\] (25)
where $\rho$ is a real parameter taking value within the range: $0 < \rho \leq 1$. Due to Eq. (25), one may write $\mathcal{E}_N(j) \approx \mathcal{E}_N(\rho N)$ for large values of $N$. Substituting $\rho N$ in the place of $j$ in Eq. (8), we obtain

$$\mathcal{E}_N(\rho N) = \sum_{\alpha=0}^{r} \sum_{\beta=0}^{s} f_{\alpha, \beta} \rho^\beta N^{\alpha+\beta}. \quad (26)$$

Let us define a new variable as $\xi \equiv \alpha + \beta$. Making a change of the summation variables in Eq. (26), it can be expressed as

$$\mathcal{E}_N(\rho N) = \sum_{\xi=0}^{r+s} g_\xi(\rho) N^\xi, \quad (27)$$

where $g_\xi(\rho)$ is a polynomial of $\rho$ given by

$$g_\xi(\rho) = \sum_{\beta=\max\{0, \xi-r\}}^{\min\{\xi, s\}} f_{\xi-\beta, \beta} \rho^\beta. \quad (28)$$

Since from Eq. (9) it follows that $g_\xi(\rho) = 0$ for $\xi > \chi$, one can rewrite Eq. (27) as

$$\mathcal{E}_N(\rho N) = \sum_{\xi=0}^{\chi} g_\xi(\rho) N^\xi. \quad (29)$$

With the help of Eqs. (25) and (29), we obtain

$$\lim_{N \to \infty} |N^{-\chi} \mathcal{E}_N(j)| = \lim_{N \to \infty} |N^{-\chi} \mathcal{E}_N(\rho N)| = |g_\chi(\rho)|. \quad (30)$$

Since $g_\chi(\rho)$ is a polynomial of $\rho$, it is bounded within the range $0 < \rho \leq 1$. If $g_\chi(\rho) \neq 0$ for some value of $\rho$, then by using Eqs. (19) and (30) we obtain the order of $\mathcal{E}_N(j)$ as

$$\mathcal{E}_N(j) = \Theta(N^\chi). \quad (31)$$

On the other hand, if $g_\chi(\rho) = 0$ for some value of $\rho$, then Eq. (29) reduces to $\mathcal{E}_N(\rho N) = \sum_{\xi=0}^{\tilde{\chi}} g_\xi(\rho) N^\xi$, where $\tilde{\chi} < \chi$ and $g_{\tilde{\chi}}(\rho) \neq 0$. Hence, for the case $g_\chi(\rho) = 0$, we obtain the order of $\mathcal{E}_N(j)$ as

$$\mathcal{E}_N(j) = \Theta(N^{\tilde{\chi}}), \quad (32)$$

where $\tilde{\chi} < \chi$. Replacing $\xi$ by $\chi$ in Eq. (28) and using Eq. (10), one can get an explicit expression for $g_\chi(\rho)$ as

$$g_\chi(\rho) = \sum_{\beta=\chi-r}^{s} f_{\chi-\beta, \beta} \rho^\beta. \quad (33)$$

Since $g_\chi(\rho)$ is a polynomial in $\rho$ of degree less than or equal to $s$, it can have at most $s$ number of distinct zeros within the interval $0 < \rho \leq 1$. For such zero points of $g_\chi(\rho)$ the order of $\mathcal{E}_N(j)$ is evidently given by Eq. (32). However, the zero points of $g_\chi(\rho)$ clearly form a set of zero measure within the range $0 < \rho \leq 1$, and, therefore, the order of $\mathcal{E}_N(j)$ is given by Eq. (31) for generic values of $\rho$. Let us now compare the orders of $\mathcal{E}_N(j)$ given in Eqs. (24), (31) and (32) for various possible cases. Since from Eq. (10) it follows that $\chi \geq r$, the order of $\mathcal{E}_N(j)$ given by (31) is clearly the highest one among all these cases. Furthermore, it can be shown that, there exists at least one value of $\rho$ within the range $0 < \rho \leq 1$ for which Eq. (31) is satisfied and this highest order of $\mathcal{E}_N(j)$ is preserved even if the index $j$ is varied within a range of order $\Theta(N)$ [32].
By following the above mentioned procedure, one can also find out the asymptotic form for the difference of energy functions like \( E'_{N}(j) \equiv E_{N}(j + 1) - E_{N}(j) \). With the help of Eq. (8), we express \( E'_{N}(j) \) as

\[
E'_{N}(j) = \sum_{\alpha=0}^{r'} \sum_{\beta=0}^{s'} f'_{\alpha,\beta} N^{\alpha} j^{\beta},
\]

where \( r' \leq r \), \( s' = s - 1 \), and \( f'_{\alpha,\beta} \) is given by

\[
f'_{\alpha,\beta} = \sum_{\nu=\beta+1}^{s} ^\nu C_{\beta} f_{\alpha,\nu},
\]

with \( ^\nu C_{\beta} \equiv \frac{\nu!}{\beta!(\nu-\beta)!} \). Since \( E'_{N}(j) \) in Eq. (34) is expressed in the same form as \( E_{N}(j) \) in Eq. (8), our analysis for finding out the order of \( E_{N}(j) \) would also be applicable to the case of \( E'_{N}(j) \). Let us denote the degree of \( E'_{N}(j) \) in Eq. (34) by \( \chi' \). With the help of Eq. (35) and the analog of Eq. (9) for the present case, it is easy to find that

\[
\chi' \leq \chi - 1.
\]

In analogy with the polynomial \( g_{\chi}(\rho) \) in Eq. (33), one can define the polynomial \( \tilde{g}_{\chi'}(\rho) \) for the present case as

\[
\tilde{g}_{\chi'}(\rho) = \sum_{\beta=\chi'-r'}^{s'} f'_{\chi'-\beta,\beta} \rho^{\beta}.
\]

Proceeding in the same way as we have done to find out the maximum possible order of \( E_{N}(j) \) in Eq. (31), it is easy to show that the order of \( E'_{N}(j) \) becomes maximum when \( j \) satisfies the relations \( j = \Theta(N) \) and \( \tilde{g}_{\chi'}(\rho) \neq 0 \), and this order of \( E'_{N}(j) \) is given by

\[
E'_{N}(j) = \Theta(N^{\chi'}).\]

Finally, for the purpose of computing the orders of \( \sigma^{2} \), \( \gamma_{j} \) and \( \gamma_{j} - \gamma_{j+1} \) for any polynomial type dispersion relation of the form (8), we rewrite Eq. (23) as

\[
\sigma^{2} = \frac{1}{12} \left( 1 - \frac{1}{m^{2}} \right) \left[ \sum_{j=1}^{N-1} E_{N}^{2}(j) - 2 \sum_{j=1}^{N-2} E_{N}(j) E'_{N}(j) + 2 E_{N}(N - 1)^{2} \right].
\]

Applying Eqs. (31), (38) and (36), and also noting that Eq. (31) continues to be satisfied if the index \( j \) is varied in some range of order \( \Theta(N) \) [32], it is easy to see that the dominant contribution in the r.h.s. of Eq. (39) comes from the positive definite terms within the first summation. Thus, we obtain the order of \( \sigma^{2} \) as

\[
\sigma^{2} = \Theta(N^{2\chi+1}).
\]

Since the parameter \( \tau \) in Eq. (19) takes strictly nonzero value, it is possible to ‘invert’ the relation (40) and calculate the order of \( 1/\sigma \) as

\[
\frac{1}{\sigma} = \Theta(N^{-\chi-\frac{1}{2}}).
\]
Next, for the case \( j = \Theta(N) \), \( g_N(\rho) \neq 0 \), we use Eqs. (17), (31) and (41) to find out the order of \( \gamma_j \) as
\[
\gamma_j = \Theta(N^{-\frac{1}{2}}) \tag{42}
\]
Since the order of \( E_N(j) \) becomes maximum for the above mentioned case, it is evident that the order of \( \gamma_j \) can not exceed the value given in Eq. (42) for any possible choice of \( j \). Translating this result in terms of the ‘big O’ notation defined in (18), we find that the asymptotic form (20) of \( \gamma_j \) is satisfied for all values of \( j \). Subsequently, for calculating the order of \( \gamma_j - \gamma_{j+1} \), we proceed in exactly the same way and consider the case \( j = \Theta(N) \), \( g_N(\rho) \neq 0 \). Using Eqs. (17), (38) and (41), we obtain the order of \( \gamma_j - \gamma_{j+1} \) for this case as
\[
\gamma_j - \gamma_{j+1} = \Theta(N^{\chi-\chi^{-\frac{1}{2}}}) \tag{43}
\]
As the order of \( \gamma_j - \gamma_{j+1} \) can not exceed the value given in the above equation for any possible choice of \( j \), and from Eq. (36) it follows that \( \chi' - \chi - \frac{1}{2} \leq -\frac{3}{2} \), we find that the asymptotic form (21) of \( \gamma_j - \gamma_{j+1} \) is satisfied for all values of \( j \). Since the asymptotic forms (20) and (21) do not depend at all on the parameters or coefficients present in the dispersion relation (8), they are indeed universal in nature. Consequently, we are able to establish that the level density for all 1D vertex models associated with the polynomial type dispersion relation (8) follow the Gaussian distribution (13) at \( N \to \infty \) limit.

Next, we wish to compare the above mentioned result with numerical calculations, by choosing some specific form of the dispersion relation (8). The numerical data for the level density corresponding to any given dispersion relation can be obtained by starting from the energy function \( E(\hat{s}) \) defined in Eq. (6). In general, one can choose the state \( \hat{s} \) of a vertex model in many different ways leading to the same value of the energy function, say, \( E_i \). Let us denote the number of such different states for a vertex model (where the colour of each bond can be chosen in \( m \) possible ways) by \( D^{(m)}(E_i) \). Choosing values of \( N \) and \( m \) within some range, and using a symbolic software package like Mathematica, it is possible to obtain values of all \( E_i \) and \( D^{(m)}(E_i) \) for a given dispersion relation. Consequently one can draw a histogram for the corresponding level density and compare it with the Gaussian distribution. Furthermore, for the purpose of eliminating the effect of local fluctuations, one may also use these data to calculate the cumulative level density given by
\[
F(E) = \frac{1}{m^N} \sum_{E_i \leq E} D^{(m)}(E_i) \tag{44}
\]
and check whether this \( F(E) \) agrees with the error function:
\[
C(E) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{E} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{1}{2} \left[ 1 + \text{erf} \left( \frac{E - \mu}{\sqrt{2\sigma}} \right) \right] \tag{45}
\]
To get some specific form of dispersion relation, let us choose \( r = 1 \) and \( s = 2 \) in Eq. (8), and set the values of three coefficients viz. \( f_{0,0}, f_{1,0} \) and \( f_{1,2} \) equal to zero. Thus, we obtain
\[
E_N(j) = f_{0,1}j + f_{0,2}j^2 + f_{1,1}Nj \tag{46}
\]
where it is assumed that \( f_{0,1}, f_{0,2} \) and \( f_{1,1} \) are nonzero real parameters. Furthermore, we take a particular case of the dispersion relation (46) by setting \( f_{0,1} = 1, f_{0,2} = -1, f_{1,1} = 1 \), and also choose the values of \( m \) and \( N \) for the corresponding vertex model as \( m = 2, N = 40 \). Then we plot a histogram for the level density of the above mentioned vertex model in Fig. 1. It is evident that this histogram matches very well with the Gaussian distribution \( G(E) \) in Eq. (13),
Figure 1. Continuous curve represents the Gaussian distribution $G(E)$ and the histogram represents the normalized level density for the dispersion relation (46) with coefficients $f_{0,1} = 1$, $f_{0,2} = -1$, $f_{1,1} = 1$, drawn for the case $N = 40$ and $m = 2$.

where the values of $\mu$ and $\sigma^2$ are obtained by using Eqs. (22) and (23) respectively. Inserting the numerical values of $D^{(m)}(E_i)$ in Eq. (44), subsequently we calculate the cumulative level density $F(E)$ for the above mentioned case. Again, it turns out that this $F(E)$ is in excellent agreement with the error function $C(E)$ given in Eq. (45). We also compute the mean square error (MSE) for the deviation of $F(E)$ with respect to $C(E)$ by choosing some equidistant points of $E$, and find this MSE to be as low as $4.10 \times 10^{-7}$. In Table 1 we present the values of such MSE, which are calculated by taking different values of $N$ (while keeping $m$ unchanged) and also by taking different sets of values of the three parameters in the dispersion relation (46). From this table we find that the MSE decreases steadily with the increase of $N$, for any particular choice of these three parameters. This numerical result clearly indicates that the level densities of the corresponding vertex models asymptotically follow the Gaussian distribution.

| Sets of Parameters | $N = 20$ | $N = 40$ | $N = 50$ | $N = 60$ | $N = 70$ |
|-------------------|----------|----------|----------|----------|----------|
| $f_{0,1}$ | $f_{0,2}$ | $f_{1,1}$ | $f_{0,1}$ | $f_{0,2}$ | $f_{1,1}$ | $f_{0,1}$ | $f_{0,2}$ | $f_{1,1}$ | $f_{0,1}$ | $f_{0,2}$ | $f_{1,1}$ |
| 1 | 1 | 1 | 81.76 | 11.82 | 6.56 | 4.08 | 2.74 |
| 1 | -1 | 1 | 37.04 | 4.10 | 2.29 | 1.44 | 0.98 |
| 1 | 2 | 1 | 101.52 | 14.38 | 7.95 | 4.93 | 4.93 |
| 1 | -1 | 2 | 24.63 | 4.09 | 2.33 | 1.48 | 1.00 |
| 1 | -1 | 3 | 30.75 | 5.10 | 2.90 | 1.83 | 1.24 |
symmetric quantum spin chain is discovered in future, whose spectrum can be written in the form (3) with $E_N(j)$ being a polynomial function of $j$ and $N$. In future, we also hope to explore the density of spacings between consecutive energy levels of unfolded spectra for the above mentioned class of vertex models. It should be noted that, for calculating the density of spacings, the energy levels are usually taken from the unfolded spectrum which is generated from the raw spectrum through a transformation involving its level density distribution [38]. Since the level densities of all 1D vertex models associated with polynomial type dispersion relations satisfy the Gaussian distribution, one can easily construct the corresponding unfolded spectra and analyze their density of spacings. Finally, it might be interesting to study various thermodynamical properties for this type of vertex models by using the transfer matrix approach.

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