Massive Majorana neutrinos in matter and a magnetic field

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Abstract

We summarize our recent results on the description of the evolution of Majorana neutrinos in external fields. First we discuss the purely quantum method which involves the secondly quantized Weyl spinors. In frames of this formalism we find exact solutions of the classical Hamilton equations for Weyl fields in background matter, which then are used to canonically quantize the system. The analog of the effective Hamiltonian for the description of spin-flavor oscillations of Majorana neutrinos in matter and a magnetic field is also obtained. Finally we discuss another approach for the treatment of Majorana neutrinos in external fields, which is based on the relativistic quantum mechanics. The latter method does not require the quantization of the neutrino wave function. Nevertheless we demonstrate that the relativistic quantum mechanics approach also appropriately describes spin-flavor oscillations of relativistic Majorana neutrinos.

1 Introduction

It is known that the neutrino interaction with external fields can significantly influence the evolution of astrophysical [1] and cosmological media [2]. For example, the interaction of supernova neutrinos with a strong magnetic field of a protoneutron star (PNS) may cause a macroscopic effect like the kick of a pulsar [3]. The neutrino scattering off the matter of a rotating PNS may result in a spin-down of a star [4]. However the recent estimates [5] showed that the role of neutrinos in PNS spin-down is significantly less important than that in the possible explanation of great linear velocities of neutron stars. The neutrino interaction with primordial plasma is essential for the observed spectrum of the cosmic microwave background [6] and for the generation of large scale magnetic fields [7].

It was experimentally confirmed that neutrinos are massive particles and there is a mixing between different neutrino generations [8]. These neutrino properties result in the appearance of neutrino flavor oscillations [9]. It is also known that the presence of background matter can cause the resonance enhancement of neutrino oscillations [10]. Moreover the combined action of a background matter and an external magnetic field leads to the resonant transition like, $\nu_\alpha^- \leftrightarrow \nu_\beta^+$ (see, e.g., Ref. [11]), where the indexes $\pm$ denote different helicity states. Hence active neutrinos of the flavor $\alpha$ can be converted into sterile neutrinos of another flavor $\beta$.

Neutrinos can interact with an external electromagnetic field due to the presence of anomalous magnetic moments [12]. Note that the structure of the magnetic moments is completely different for Dirac and Majorana neutrinos (see, e.g., Ref. [13]). Despite the fact that nowadays there is no universally recognized confirmation of the nature of neutrinos [14], in the present...
work we shall suppose that neutrinos are Majorana particles. Note that in various scenarios for the generation of elementary particles masses it is predicted that neutrinos should acquire Majorana masses [15]. It should be also mentioned that numerous experimental attempts are made to investigate whether neutrinos are Dirac or Majorana particles [16].

In the present work we summarize our recent results [17,18] on the theory of Majorana neutrinos oscillations in matter and electromagnetic fields. First, in Sec. 2 following the discussion of Ref. [20], we review the general properties of the neutrino mixing in vacuum. Then, in Sec. 3 we recall the basic equation for a massive Majorana neutrino propagating in background matter and electromagnetic field. In Sec. 4 using the results of our recent works [18,19] we propose the classical field theory treatment of massive Weyl fields propagating in a background matter and interacting with an external electromagnetic field. Exact solutions of the wave equation for Weyl fields in a background matter are found in Sec. 5. In Sec. 6 we use these solutions to canonically quantize the Weyl fields. In Sec. 7 in frames of our method we re-derive the effective Hamiltonian for the description of spin-flavor oscillations of Majorana neutrinos in matter and a magnetic field. Finally, in Sec. 8 we consider the alternative approach for the description of spin-flavor oscillations of Majorana neutrinos based on the relativistic quantum mechanics and compare it with the results of Sec. 7. In Sec. 9 we discuss our results.

2 Neutrino mixing in vacuum

Experimentally it was found [21] that the number of active neutrinos is equal to three with the high level of accuracy. Nevertheless in various theoretical and phenomenological scenarios (see, e.g., Ref. [22]) some additional sterile neutrinos are discussed. Thus the general dynamics of the system of flavor neutrinos can be formulated in terms of the $3 + N_s$ spinor fields, where $N_s \geq 0$ is the number of sterile neutrinos. It was experimentally confirmed [23] that active neutrinos correspond to the left-handed chiral projections, $\nu^L_\lambda$, $\lambda = e, \mu, \tau$, whereas sterile neutrinos should be expressed in terms of the right-handed chiral projections, $\nu^R_\lambda$, $\lambda = 1, \ldots, N_s$. Here $\nu^L,R_\lambda = (1 \mp \gamma^5) / 2 \times \nu$.

The system of wave equations for the spinors $\nu^{L,R}_\lambda$ will involve both Dirac and Majorana types of the mass terms. The simultaneous diagonalization of the mass terms can be performed with help of the matrix transformation [24],

$$N^L_\lambda = \sum_{a=1}^{3+N_s} U_{\lambda a} \chi^L_a,$$

where $N^T_\lambda = (\nu^L_\epsilon, \nu^L_\mu, \nu^L_\tau, [\nu^R_1]^c, \ldots, [\nu^R_{N_s}]^c)$ is the multiplet of the neutrino fields, $(U_{\lambda a})$ is the unitary vacuum mixing matrix of the size $(3 + N_s) \times (3 + N_s)$, and $\chi_a$ are the neutrino mass eigenstates. The charge conjugation is defined in the standard manner: $\nu^c = i \gamma^2 \nu^*$.

The fields $\chi_a$ are neither Dirac nor Majorana. However we may always construct Majorana fields of the the chiral projections of $\chi_a$ as

$$\psi_a = [\varkappa^* \chi^L_a + \tilde{\varkappa}^* (\chi^L_a)^c],$$

where $\varkappa$ and $\tilde{\varkappa}$ are the phase factors having unit absolute values. Now we can see that the fields $\psi_a$ satisfy the Majorana condition in an extended sense,

$$\psi^c_a = \varkappa_c \psi_a,$$

where $\varkappa_c = \varkappa \tilde{\varkappa}$. In the following we shall suppose that $\varkappa_c = 1$. 

2
3 Electrodynamics of Majorana neutrinos in a background matter

The evolution of Majorana fields $\psi_a$, defined in Eq. (4), in a background matter under the influence of an external electromagnetic field is governed by the following wave equation:

\[(i\gamma^\mu \partial_\mu - m_a)\psi_a - \frac{\mu_{ab}}{2} \sigma_{\mu\nu} F^{\mu\nu} \psi_b + g_{ab}^\mu \gamma^5 \psi_b = 0,\]

where $m_a$ are the masses of the particles and $\sigma_{\mu\nu} = (i/2)[\gamma_\mu, \gamma_\nu]$. Note that we will formulate the dynamics of the system (1) in the mass eigenstates basis rather than in the flavor basis, as it is usually done when neutrino oscillations are considered, since, as we demonstrated in Sec. 2 only in the mass eigenstates basis one can distinguish between Dirac and Majorana masses.

The matrix of the neutrino interaction with a background matter, $(g_{ab})$, introduced in Eq. (4) in the mass eigenstates basis, is related to the analogous matrix in the flavor basis as follows:

\[g_{ab}^\mu = \sum_{\lambda, \lambda' = e, \mu, \tau} U_{a\lambda}^\dagger f_{\lambda\lambda'}^{\mu} U_{\lambda' b},\]

where the mixing matrix $(U_{\lambda a})$ is defined in Eq. (1). In general case the matrix $(g_{ab}^\mu)$ is hermitian. However we shall discuss the situation when the CP invariance is conserved. In this case the matrix $(g_{ab}^\mu)$ is symmetric. Despite a current attempt to detect CP violating terms in the neutrino sector [25], no definite results have been obtained yet.

Note that in Eq. (4) we do not exclude the possibility of the existence of nonstandard neutrino interactions, which correspond to the nondiagonal elements of the matrix $(f_{\lambda\lambda}')$ (see, e.g., Ref. [17]). In the situation when only the standard model interactions with matter are present, the matrix $(f_{\lambda\lambda}')$ is diagonal: $f_{\lambda\lambda'}^\mu = \delta_{\lambda\lambda'} f_{\lambda}^\mu$, where $f_{\lambda}^\mu$ are the effective potentials of interactions with background fermions. The zero components of these potentials, $f_{\lambda}^0$, are proportional to the effective matter density of non-moving and unpolarized background fermions, whereas the vector components, $f_{\lambda}$, are the linear combinations of the averaged matter velocity and the polarization. The explicit form of the effective potentials and the details of the statistical averaging can be found in Ref. [26].

Note that the vector term in the neutrino matter interaction $\sim g_{ab}^\mu \gamma_\mu \psi_b$ is omitted in Eq. (4) since it is washed out for Majorana neutrinos. The contribution of the axial-vector interaction with matter $\sim g_{ab}^\mu \gamma_\mu \gamma^5 \psi_b$ to the wave equation (4) is twice the analogous contribution for Dirac particles since both neutrinos and antineutrinos equally interact with a background matter (see, e.g., Ref. [27]).

Neutrinos can interact with the external electromagnetic field $F_{\mu\nu} = (E, B)$ due to the presence of the anomalous magnetic moments ($\mu_{ab}$). It is known (see, e.g., Ref. [28]) that the matrix $(\mu_{ab})$ should be hermitian and pure imaginary, i.e. $\mu_{ab} = -\mu_{ba}$ and $\mu_{ab}^* = -\mu_{ab}$. Unlike the interaction with matter, which is generically defined for flavor neutrinos, the computation of the neutrino magnetic moments has to be done in the mass eigenstates basis. The detailed example of such a calculation is given in Ref. [29]. We shall discuss the situation when no admixture of sterile neutrinos is in the mass eigenstates $\psi_a$, i.e. when the eigenstates $\nu^R_{\lambda}$ are absent. In this case the neutrino electric dipole moments are equal to zero [28].

In Ref. [30] is was rigorously proved that four component Majorana spinors are equivalent to two component Weyl spinors. Thus we may re-express the wave function $\psi_a$ in Eq. (5) in two ways,

\[\psi^{(\eta)}_a = \begin{pmatrix} i\sigma_2 \eta^a \\ \eta_a \end{pmatrix}, \quad \text{or} \quad \psi^{(\xi)}_a = \begin{pmatrix} \xi_a \\ -i\sigma_2 \xi^*_a \end{pmatrix}.\]

Both representations in Eq. (6) satisfy the Majorana condition (3). Using Eq. (6) we can rewrite Eq. (4) as

\[\hat{\eta}_a - (\sigma \nabla) \eta_a + m_a \sigma_2 \eta^*_a - \mu_{ab} \sigma (B - iE) \sigma_2 \eta^*_b + i(g_{ab}^0 + \sigma g_{ab}) \eta_b = 0,\]

\[\hat{\xi}_a - (\sigma \nabla) \xi_a + m_a \sigma_2 \xi^*_a - \mu_{ab} \sigma (B - iE) \sigma_2 \xi^*_b + i(g_{ab}^0 + \sigma g_{ab}) \xi_b = 0,\]
or
\[
\dot{\xi}_a + (\sigma \nabla) \xi_a - m_a \sigma_2 \xi_a^* + \mu_{ab} \sigma(B + iE) \sigma_2 \xi_b^* - i(g_{ab}^0 - \sigma^* g_{ab}) \xi_b = 0. \tag{8}
\]

In the following we shall postulate these equations. Note that the analog of Eq. (7) was previously derived in Ref. [31].

## 4 Classical field theory

In Ref. [19] we demonstrated that the classical dynamics of a massive Weyl field in vacuum should be described only in frames of the Hamilton formalism. Generalizing the results of Ref. [19] to include the interactions with a background matter and an electromagnetic field we arrive to the following Hamiltonian (see also Ref. [18]):

\[
H = \int d^3r \left[ \sum_a \left\{ \pi_a^T (\sigma \nabla) \eta_a - (\eta_a^*)^T (\sigma \nabla) \pi_a^* + m_a \left[ (\eta_a^*)^T \sigma_2 \pi_a + (\pi_a^*)^T \sigma_2 \eta_a \right] \right\}
+ \sum_{ab} \left\{ \mu_{ab} \left[ \pi_a^T \sigma(B - iE) \sigma_2 \eta_b^* + \eta_a^T \sigma_2 \sigma(B + iE) \pi_b^* \right]
- i \left[ \eta_a^T (g_{ab}^0 + \sigma g_{ab}) \eta_b - (\eta_a^*)^T (g_{ab}^0 + \sigma g_{ab}) \pi_b^* \right] \right\}, \tag{9}
\]

where \(\pi_a\) are the canonical momenta conjugate to the “coordinates” \(\eta_a\). Using the aforementioned properties of the matrices \((\mu_{ab})\) and \((g_{ab}^\mu)\) we find that the functional \((9)\) is real as it should be for a classical Hamiltonian.

Applying the field theory version of the canonical equations to the Hamiltonian \(H\),
\[
\dot{\eta}_a = \frac{\delta H}{\delta \pi_a} = (\sigma \nabla) \eta_a - m_a \sigma_2 \eta_a^* + \mu_{ab} \sigma(B - iE) \sigma_2 \eta_b^* - i(g_{ab}^0 + \sigma g_{ab}) \eta_b, \tag{10}
\]

\[
\dot{\pi}_a = - \frac{\delta H}{\delta \eta_a} = (\sigma^* \nabla) \pi_a + m_a \sigma_2 \pi_a^* - \mu_{ab} \sigma_2 \sigma(B + iE) \pi_b + i(g_{ab}^0 + \sigma^* g_{ab}) \pi_b, \tag{11}
\]

one can see that in Eq. (10) we reproduce Eq. (7) for Weyl particles, which correspond to left-handed neutrinos, interacting with matter and an electromagnetic field. If we introduce the new variable \(\xi_a = i \sigma_2 \pi_a\), we can show that Eq. (11) is equivalent to Eq. (8) for right-handed neutrinos.

## 5 Exact solution of the wave equation

In the following we shall suppose that the background matter in average is at rest and unpolarized, i.e. \(g_{ab} = 0\). This approximation is valid in almost all realistic cases (see the detailed discussion in Ref. [18]). Let us decompose the Hamiltonian \((9)\) into two terms \(H = H_0 + H_{\text{int}}\). The former term, \(H_0\), contains the vacuum Hamiltonian as well as the matter term diagonal in neutrino types,

\[
H_0 = \int d^3r \sum_a \left\{ \pi_a^T [(\sigma \nabla) - i g_{aa}^0] \eta_a - (\eta_a^*)^T [(\sigma \nabla) - i g_{aa}^0] \pi_a^* \right\}
+ m_a \left[ (\eta_a^*)^T \sigma_2 \pi_a + (\pi_a^*)^T \sigma_2 \eta_a \right]. \tag{12}
\]

The latter term in this decomposition,

\[
H_{\text{int}} = \int d^3r \sum_{a \neq b} \left\{ \mu_{ab} \left[ \pi_a^T \sigma(B - iE) \sigma_2 \eta_b^* + \eta_a^T \sigma_2 \sigma(B + iE) \pi_b^* \right] - i g_{ab}^0 \left[ \pi_a^T \eta_b - (\eta_a^*)^T \pi_b^* \right] \right\}, \tag{13}
\]

has the nondiagonal matter interaction and the interaction with an electromagnetic field which is nondiagonal by definition.
Analogously to Eqs. (10) and (11) we define the reduced Hamilton equations which contain only the Hamiltonian $H_0$: $\dot{\eta}^{(0)}_a = \delta H_0 / \delta \pi^{(0)}_a$ and $\dot{\pi}^{(0)}_a = -\delta H_0 / \delta \eta^{(0)}_a$. Using the results of Refs. [18, 31] we can find the solutions of these equations in the form,

\[
\dot{\eta}^{(0)}_a(r,t) = \frac{1}{2} \int \frac{d^3p}{(2\pi)^{3/2}} \left\{ \left[ a_a^- w_- e^{-iE_a^- t} - \frac{m_a}{E_a^+ + |p| + g^0_{aa}} a_a^+ w_+ e^{iE_a^+ t} \right] e^{ipr} + \left[ (a_a^+)^* w_- e^{iE_a^+ t} + \frac{m_a}{E_a^- + |p| + g^0_{aa}} (a_a^-)^* w_+ e^{-iE_a^- t} \right] e^{-ipr} \right\},
\]

\[
\dot{\xi}^{(0)}_a(r,t) = \frac{i}{2} \int \frac{d^3p}{(2\pi)^{3/2}} \left\{ \left[ b_a^+ w_+ e^{-iE_a^+ t} + \frac{m_a}{E_a^- + |p| + g^0_{aa}} b_a^- w_- e^{iE_a^- t} \right] e^{ipr} + \left[ (b_a^-)^* w_+ e^{iE_a^- t} - \frac{m_a}{E_a^+ + |p| + g^0_{aa}} (b_a^+)^* w_- e^{-iE_a^+ t} \right] e^{-ipr} \right\},
\]

where we introduce the new variable $\xi^{(0)}_a = i\sigma_2 \pi^{(0)}_a$, $w_{\pm}$ are the helicity amplitudes defined in Ref. [32], and

\[
E_a^{(\xi)} = \sqrt{m_a^2 + (|p| - \zeta g^0_{aa})^2}.
\]

is the energy of a Weyl field [27], $\zeta = \pm 1$ is the particle helicity. To derive Eqs. (14) and (15) we suppose that the external field $g^0_{aa}$ is spatially constant.

6 Quantization

Now we can carry out the canonical quantization of the Weyl fields $\eta_a$ and $\pi_a$. For this purpose we suggest that the expansion coefficients $a_a^\pm(p)$ and $b_a^\pm(p)$ in Eq. (14) are operators. Note that the operators in the expansion of $\eta_a$ and $\xi_a$ are independent since in Sec. 4 we showed that these fields evolve independently.

Using the following relation:

\[
a_a^\pm(p)(E_a^\pm + |p| + g^0_{aa}) = b_a^\pm(p)(|p| + g^0_{aa}), \quad (16)
\]

and suggesting that the operators $a_a^\pm(p)$ obey the anti-commutation properties,

\[
\{a_a^\pm(k); (a_b^\pm(p))^*\}_+ = \delta_{ab}\delta^3(k-p),
\]

(17)

with all the rest of the anticommutators being equal to zero, we can cast Eq. (12) into the form,

\[
H_0 = \int d^3p \sum_a [E_a^-(a_a^-)^* a_a^- + E_a^+(a_a^+)^* a_a^+] + \text{divergent terms},
\]

(18)

which shows that the total energy of a massive Weyl field is a sum of energies corresponding to elementary oscillators of positive and negative helicities. Note that the details of the derivation of Eq. (18) can be found in Ref. [18].

Using the results Ref. [19] we can also quantize the total momentum of a Weyl field defined as

\[
P_0 = \int d^3r \sum_a \left( \eta_a^{(0)} \right)^T \nabla \pi_a^{(0)} - \left( \pi_a^{(0)} \right)^T \nabla \eta_a^{(0)}.
\]

(19)

With help of Eqs. (14), (16), and (17) we rewrite Eq. (19) in the following form:

\[
P_0 = \int d^3p \sum_a p [(a_a^-)^* a_a^- + (a_a^+)^* a_a^+] + \text{divergent terms},
\]

(20)

which has the similar structure as Eq. (18).
It is interesting to mention that a massive Weyl field in vacuum can be quantized in the two independent ways (see Ref. [19]) because of the degeneracy of the neutrino energy levels: 
\[ E_{a} = E_{a}' = \sqrt{m_a^2 + |p|^2}. \]
On the contrary, in matter only one of the possibilities for the quantization gives the correct result for the total energy [18] since the energy levels are no longer degenerate, cf. Eq. (15).

7 Nondiagonal interaction with matter and a magnetic field

To quantize the Hamiltonian \( H_{\text{int}} \) we shall use the forward scattering approximation. It means that one has to account for only the terms conserving the number of particles [33]. Using Eqs. (14), (16), and (17) we rewrite Eq. (13) in the form,

\[
H_{\text{int}} = \int d^3p \sum_{a \neq b} [M_{ab}^-(a^\ast_b e^{i\delta_{ab}t})^\ast + M_{ab}^+(a^+ b e^{i\delta_{ab}t})^\ast + F_{ab}^-(a^\ast_b e^{i\sigma_{ab}t})^\ast + F_{ab}^+(a^+ b e^{i\sigma_{ab}t})],
\]

where \( \delta_{ab}^\pm = E_a^\pm - E_b^\pm \) and \( \sigma_{ab}^\pm = E_a^\pm - E_b^\pm \).

The general form of the coefficients \( M_{ab}^\pm \) and \( F_{ab}^\pm \) is quite cumbersome. Therefore we describe the dynamics of the system in the ultrarelativistic approximation, \( |k| \gg \max(m_a, g_{a0}^0) \), where \( k \) is the initial momentum of neutrinos. Moreover we discuss the simplest case of the two neutrino eigenstates, \( a = 1, 2 \), and suppose that \( E = 0 \), since it is difficult to create a large scale electric field. In this limit we get for the coefficients \( M_{ab}^\pm \approx \mp g_{ab} \) and \( F_{ab}^\pm \approx -\mu_{ab} B \sin \vartheta_{kB} \), where \( \vartheta_{kB} \) is the angle between the vectors \( k \) and \( B \). The explicit form of these coefficients for the arbitrary neutrino momentum can be found in Ref. [18].

Now we define the neutrino density matrix as

\[
\delta^3(p - k)\rho_{AB}(k) = \langle a^\ast_B(p)a_A(k) \rangle,
\]

where \( A = (\zeta, a) \) is a composite index and \( \langle \ldots \rangle \) is the statistical averaging over the neutrino ensemble. Applying the quantum Liouville equation for the description of the density matrix evolution,

\[
i\dot{\rho} = [\rho, H_{\text{int}}],
\]

we can rewrite it as \( i\dot{\rho} = [H, \rho] \), where

\[
H = \begin{pmatrix}
M_{ab}^+ e^{i\delta_{ab}t} & F_{ab}^+ e^{i\sigma_{ab}t} \\
F_{ab}^- e^{i\sigma_{ab}t} & M_{ab}^- e^{i\delta_{ab}t}
\end{pmatrix},
\]

is the effective quantum mechanical Hamiltonian.

To study the evolution of our system we use Eq. (14), where the wave functions already contain time dependent exponential factors. The energies in Eq. (14) correspond to the total diagonal Hamiltonian (12), cf. Eqs. (15) and (18), which contains both the mass term and the diagonal interaction with a background matter rather than only a kinetic term as in Ref. [34].

Our treatment is similar to the Dirac picture of the quantum theory. That is why in Eq. (23) it is sufficient to commute the density matrix only with \( H_{\text{int}} \) rather than with the total Hamiltonian \( H = H_0 + H_{\text{int}} \).

To eliminate the time dependence in the effective Hamiltonian (24) we make the transformation of the density matrix [31],

\[
\rho_{\text{qm}} = \mathcal{U} \rho \mathcal{U}^\dagger,
\]

\[
\mathcal{U} = \text{diag} \{ e^{-i(\Phi + g_{11}^0) t}, e^{i(\Phi + g_{22}^0) t}, e^{-i(\Phi - g_{11}^0) t}, e^{i(\Phi - g_{22}^0) t} \},
\]

where \( \Phi = \delta m^2/4|k| \) is the phase of vacuum oscillations and \( \delta m^2 = m_1^2 - m_2^2 \) is the mass squared difference.
The evolution of the transformed density matrix can be represented as $i\dot{\rho}_{\text{qm}} = [H_{\text{qm}}, \rho_{\text{qm}}]$, where the new effective Hamiltonian has the form,

$$H_{\text{qm}} = \mathcal{U} H \mathcal{U}^\dagger + i \mathcal{U} \dot{\mathcal{U}}^\dagger$$

$$= \begin{pmatrix}
\Phi + g_{11}^0 & g_{12}^0 & 0 & -\mu_{12} |B| \sin \vartheta_{kB} \\
g_{21}^0 & -\Phi + g_{22}^0 & -\mu_{21} |B| \sin \vartheta_{kB} & 0 \\
-\mu_{21} |B| \sin \vartheta_{kB} & -\mu_{12} |B| \sin \vartheta_{kB} & \Phi - g_{11}^0 & -g_{12}^0 \\
0 & 0 & -g_{21}^0 & -\Phi - g_{22}^0
\end{pmatrix}.$$  \quad (26)

Recalling the properties of the magnetic moments matrix for Majorana neutrinos, $\mu_{12} = i \mu$ and $\mu_{21} = -i \mu$, with $\mu$ being a real number, we can see that Eq. (26) reproduces the well known quantum mechanical Hamiltonian for spin-flavor oscillations of Majorana neutrinos in matter and a magnetic field $[11]$.

8 Relativistic quantum mechanics

In Sec. 7 we showed that the dynamics of neutrino spin-flavor oscillations in matter and a magnetic field can be described in frames of the purely quantum approach based on the quantum Liouville equation (23) for the neutrino density matrix (22). There is, however, an alternative formalism for the treatment of neutrino oscillations which involves the relativistic quantum mechanics method $[17,31]$. This approach is based on the exact solutions of wave equations for massive neutrinos in an external field, which were obtained in Sec. 5.

The main idea of the relativistic quantum mechanics method is the following: for the given initial wave functions of flavor neutrinos one should find the wave functions at subsequent moments of time. Note that this approach is similar to the relativistic wave packets description of neutrinos oscillations $[35]$. However we study the propagation of neutrino wave packets which exactly account for external fields. If one defines some known initial distribution of active neutrinos, then we can calculate the transition and survival probabilities for other neutrino flavors. Note that using Eq. (11) we can always express the initial condition for flavor neutrinos via that for the neutrino mass eigenstates, i.e. we shall suppose that $\eta_\alpha(r, t = 0) = \eta_\alpha^{(0)}(r)$ are the given functions.

In our analysis we shall assume that the expansion coefficients in the decomposition of the wave functions $[11]$ are c-numbers rather than operators as it was supposed in Sec. 6. Thus, since we do not quantize the Weyl fields, we no longer need the additional degree of freedom represented by the spinor $\xi$. Moreover we may change the normalization of the spinors at our convenience.

We will look for a general solution of Eq. (7) in the following form (see Eq. (14) and Ref. [31]):

$$\eta_\alpha(r, t) = \int \frac{d^3p}{(2\pi)^{3/2}} \left\{ \left( a^-_\alpha(p, t) w_- e^{-iE^-_\alpha t} - \frac{m_\alpha}{E^-_\alpha + |p|} g_{aa}^- (p, t) w_+ e^{-iE^+_\alpha t} \right) e^{ipr} \\
+ \left( a^+_\alpha (p, t)^* w_- e^{iE^-_\alpha t} + \frac{m_\alpha}{E^-_\alpha + |p|} g_{aa}^- [a^-_\alpha (p, t)]^* w_+ e^{iE^+_\alpha t} \right) e^{-ipr} \right\}, \quad (27)$$

where the expansion coefficients $a^\pm_\alpha(p, t)$ are the functions of time to account for the nondiagonal terms in Eq. (7). The energy levels $E^\pm_\alpha$ and the helicity amplitudes $w_\pm$ are defined in Sec. 5.

We choose the plane wave initial wave functions of massive neutrinos, $\eta^{(0)}_\alpha(r) = A_\alpha e^{ikr}$, where $A_\alpha$ is the amplitude of the initial neutrino field distribution. If we are not interested in the next-to-leading effects in neutrino oscillations, we may discuss the ultrarelativistic approximation from the very beginning: $|k| \gg \max(m_\alpha, g_{aa}^-)$. As in Sec. 7 we suggest that $E = 0$ as well as adopt the following geometry: $B = Be_z$ and $k = |k| \times (\sin \vartheta_{kB}, 0, \cos \vartheta_{kB})$. 

7
Inserting the ansatz \((27)\) in Eq. \((7)\) and after the straightforward calculations we get the system of the ordinary differential equations for \(a^\pm_0(p,t)\). It is convenient to rewrite it in a conventional form as the Schrödinger-like equation, \(i\Psi = \mathcal{H}\Psi\), introducing the quantum mechanical “wave function” of neutrinos, \(\Psi_T = (a^-_1, a^-_2, \ldots, a^+_1, a^+_2, \ldots)\), and the effective Hamiltonian \(\mathcal{H}\), which has the same form as in Eq. \((24)\). As in Sec. \(7\) here we shall discuss the situation of two neutrino eigenstates, \(a = 1, 2\). Then we make the transformation, \(\Psi_{qm} = \mathcal{U}\Psi\), where the matrix \(\mathcal{U}\) is defined in Eq. \((25)\). The evolution of the transformed quantum mechanical “wave function” \(\Psi_{qm}\) is governed by the effective Hamiltonian \(\mathcal{H}_{qm}\) which coincides with that given in Eq. \((26)\). Therefore we obtain that the relativistic quantum mechanics method is consistent with the standard quantum mechanical description of neutrino spin-flavor oscillations in matter and a magnetic field.

9 Conclusion

In the present work we have summarized our recent results on the theory of spin-flavor oscillations of Majorana neutrinos in matter and a magnetic field. After the brief introduction to the theory of neutrino mixing in vacuum in Sec. \(2\) and the discussion of the Majorana neutrinos electrodynamics in Sec. \(3\) we have mainly considered two approaches: the purely quantum method, Secs. \(4\)-\(7\), and the relativistic quantum mechanics approach, Sec. \(8\).

The former method is based on the canonical quantization of Weyl fields in presence of a background matter. The procedure of the canonical quantization requires the construction of the classical field theory Hamiltonian \(\mathcal{H}\), in which all the dynamical observables should then be replaced with operators. Previously it was claimed \((37)\) that massive Weyl fields are essentially quantum objects to be expressed via anticommuting operators. Nevertheless, in Sec. \(4\) we have constructed the classical field theory of massive Weyl fields in a background matter and an electromagnetic field, which is based on the Hamilton formalism. Note that one does not doubt that generically our world is quantum. However there are numerous processes which may be also described within the classical physics (see Ref. \((38)\) for many interesting examples).

In Sec. \(5\) we have found the exact solution of the wave equation for a Weyl field in presence of a background matter. This solution has been used in Sec. \(6\) to canonically quantize the system. Then, in Sec. \(7\) we have completed the quantization of the system by taking into account the nondiagonal interaction with matter and a magnetic field using the density matrix formalism. For ultrarelativistic particles we have re-derived the effective Hamiltonian for the description of spin-flavor oscillations of Majorana neutrinos in matter and a magnetic field.

Finally, in Sec. \(8\) we have considered the alternative approach for the description of the evolution of Majorana neutrinos which is based on the relativistic quantum mechanics. In frames of this method we do not quantize the neutrino wave functions. Instead of operators in the expansion of the fields, we deal with \(c\)-number functions. These functions evolve in time in such a way to satisfy the given initial fields distribution. We have demonstrated that for ultrarelativistic neutrinos this method also gives the appropriate description of spin-flavor oscillations of Majorana neutrinos in external fields.

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