BOUNDING THE NUMBER OF MAXIMAL TORSION COSETS ON SUBVARIETIES OF ALGEBRAIC TORI

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Abstract. We obtain bounds on the number of maximal torsion cosets for algebraic subvarieties $V \subset \mathbb{G}_m^n$, defined over $\mathbb{Q}$, using model theoretic methods.

Let $V$ be an algebraic variety defined over the rationals $\mathbb{Q}$, with $V \subset \mathbb{G}_m^n(\mathbb{C})$, where $\mathbb{G}_m^n(\mathbb{C})$ is the multiplicative subgroup of $\mathbb{C}^n$. A cyclotomic point $\omega$ on $V$ is a point of the form $(\omega_1, \ldots, \omega_n)$, with $\omega_i$ a root of unity for $1 \leq i \leq n$. A torsion coset is a set of the form $\bar{\omega}T$ where $T$ is a connected algebraic subgroup of $\mathbb{G}_m^n(\mathbb{C})$. Any connected algebraic subgroup $T$ of $\mathbb{G}_m^n(\mathbb{C})$ is isomorphic to $\mathbb{G}_m^r(\mathbb{C})$ for some $0 \leq r \leq n$. We can write the isomorphism $\Phi$ in the following form;

$$\Phi : \mathbb{G}_m^r(\mathbb{C}) \to T$$

$$: (t_1, \ldots, t_r) \mapsto (t_1^{m_{11}} \cdots t_r^{m_{1r}}, \ldots, t_1^{m_{n1}} \cdots t_r^{m_{nr}})$$

We let $M = (m_{ij})_{1 \leq i \leq n, 1 \leq j \leq r}$ be the matrix defining the isomorphism $\Phi$, so we can write an element of a torsion coset in the form $\bar{\omega}t^M$, where $t = (t_1, \ldots, t_r) \in \mathbb{G}_m^r(\mathbb{C})$. We define a torsion coset $\bar{\omega}T \subset V$ to be maximal if for any torsion coset $\bar{\omega}'T' \subset V$, with $\bar{\omega}T \subset \bar{\omega}'T'$, we have that $\bar{\omega}T = \bar{\omega}'T'$.

We make the following straightforward observations about maximal torsion cosets. First,

Lemma 0.1. $\overline{V(K^\text{cycl})} = \bigcup_{i=1}^N \bar{\omega}_i T_i$

where $i$ runs over all the maximal torsion cosets on $V$.

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This is clear by noting that a cyclotomic point on $V$ is itself a torsion coset, hence we obtain the left hand inclusion. We obtain equality by the fact that cyclotomic points are Zariski dense in any torsion coset.

Secondly, if $\bar{a}S \subseteq V$ is a coset of a connected algebraic subgroup of $\mathbb{G}_m^n$, then;

$$\bar{a}S(K_{cyc}) = \bar{w}T$$

where $T$ is a connected algebraic subgroup of $\mathbb{G}_m^n(K)$ and $\bar{w}$ is a cyclotomic point.

In order to see this, observe that $\bar{a}S(K_{cyc}) = \bigcup_{i=1}^{N} \bar{w}_i T_i$, where $i$ runs over all the maximal torsion cosets on $\bar{a}S$. We claim that in fact $N = 1$. Let $\bar{w}$ be a cyclotomic point on $\bar{a}S$, then we have that $\bar{w}_i^{-1} \bar{w} \in S$ and $\bar{w} \in \bar{w}_1 S$. Hence $\bar{a}S(K_{cyc}) = (\bar{w}_1 S)(K_{cyc}) = \bar{w}_1 (S(K_{cyc}))$. We now claim that $S(K_{cyc})$ is connected, this follows easily from the fact that $S$ is a connected algebraic subgroup of $\mathbb{G}_m^n(K)$ and the classification of such groups. It follows that $\bar{a}S(K_{cyc}) = \bar{w}_1 T_1$, but, by maximality of $\bar{w}_1 T_1$, we have that $T_1 = T_1'$ as required.

An important consequence of the above result is the following;

**Lemma 0.2.** Let $\bar{w}_i T_i$, for $1 \leq i \leq n$, enumerate the maximal torsion cosets on $V$ and suppose that;

$$\bigcup_{i=1}^{N} \bar{w}_i T_i \subseteq \bigcup_{i=1}^{M} \bar{a}_i S_i \subseteq V$$

where the $S_i$ are connected algebraic subgroups of $\mathbb{G}_m^n(K)$. Then $N \leq M$.

**Proof.** We have that $\bigcup_{i=1}^{N} \bar{w}_i T_i = \bigcup_{i=1}^{M} \bar{a}_i S_i(K_{cyc}) = \bigcup_{i=1}^{M} \bar{a}_i S_i(K_{cyc})$. By the previous argument, this is a union of $M$ irreducible torsion cosets. As the irreducible decomposition of $\bigcup_{i=1}^{N} \bar{w}_i T_i$ is irredundant, by maximality of each torsion coset $\bar{w}_i T_i$, we have that $N \leq M$ as required.

$\square$

The following result is given in [4], (Theorem 8E*);

Suppose that $V$ is defined by equations of total degree $\leq d$, then there exist finitely many maximal torsion cosets on $V$ and, moreover,
the number is bounded by \( \exp(3N(d)^{2/3}\log(N(d))) \) where \( N(d) = C_d^{n+d} \).

The purpose of this paper is to find an alternative bound in the degree \( d \) and dimension \( n \), using methods from model theory.

We define the exponent of a torsion coset \( \bar{\omega}T \) to be any multiple of its order as an element of the group \( \mathbb{G}_m^n/T \). By the result given above, there must exist a single exponent for all maximal torsion cosets on \( V \).

In [3], (Theorem 4.2), the following result is proved:

Suppose that \( N(V) \), the Newton polygon associated to \( V \), has diameter \( D(V) \). Then every \((n-k)\)-dimensional maximal torsion coset on \( V \) has an exponent \( mP_N \) for \( m \leq D(V)^{2k/2} \), where \( N = \text{Card}(N(V)) \leq C_d^{n+d} \) and \( P_N \) is the product of all primes up to \( N \). (By results of Iskander Aliev, the bound on \( m \) can be improved to \( D(V)^{2k} \).)

Now let \( K \) be a uniform exponent for all the maximal torsion cosets on \( V \), for example we can take \( K = tP_N \), where \( t \leq D(V)^{n(n-1)} \), assuming the result of Aliev. Suppose that we are given a maximal torsion coset \( \bar{\omega}T \) of exponent \( K \), then we have that \( \bar{\omega}K \in T \). Suppose that \( \Phi(\bar{\omega}) = \bar{\omega}K \).

Clearly, \( \mathbb{G}_m^r(\mathbb{C}) \) is closed under taking \( K \)’th roots, so we can find \( \bar{t}_1 \in \mathbb{G}_m^r(\mathbb{C}) \) such that \( \bar{t}_1^K = \bar{t} \). Then \( \Phi(\bar{t}_1^K) = \Phi(\bar{t}_1)^K = \bar{\omega}K \).

Therefore, \((\Phi(\bar{t}_1)^{-1}\bar{\omega})^K = 1 \) and \( \Phi(\bar{t}_1)^{-1}\bar{\omega} \) represents the same torsion coset as \( \bar{\omega} \).

Enumerate elements \( \{\bar{w}_1, \ldots, \bar{w}_N\} \) representing the maximal torsion cosets on \( V \) such that \( \bar{w}_j^K = 1 \) for \( 1 \leq j \leq N \). We can write each \( \bar{w}_j \) as \( (\omega_{ij}, \ldots, \omega_{nj}) \) where \( \omega_{ij} \) is a primitive \( L_{ij} \)’th root of unity with \( L_{ij} | K \), for \( 1 \leq i \leq n \). Now choose a primitive \( K \)’th root of unity \( \xi \) with \( \xi^{K/L_{ij}} = \omega_{ij} \). We consider the following cyclotomic extensions:

(i). \( \mathbb{Q}(\xi)/\mathbb{Q} \):

In this case \( \text{Gal}(\mathbb{Q}(\xi)/\mathbb{Q}) = U_{p_1^{m_1}} \times \cdots \times U_{p_r^{m_r}} \), where \( K = p_1^{m_1} \cdots p_r^{m_r} \) is the prime factorisation of \( K \) and \( U_{p_j^{m_j}} \) is the cyclic group of units in the multiplicative group \( (\mathbb{Z}/p_j^{m_j}\mathbb{Z})^* \). This is an abelian group with generators \( \{\sigma_1, \ldots, \sigma_r\} \).

a. If \( K \) is odd, \( (2, K) = 1 \), hence \( 2 \in U_K \) and the map \( \sigma(\xi) = \xi^2 \) determines an element of \( \text{Gal}(\mathbb{Q}(\xi)/\mathbb{Q}) \). We then have that \( \sigma(\omega_{ij}) = \omega_{ij}^2 \) as well.
b. If $K = 2L$ with $L$ odd, then $(L + 2, 2L) = 1$ and $\sigma(\xi) = \xi^{L+2} = -\xi^2$ determines an element of $Gal(\mathbb{Q}(\xi)/\mathbb{Q})$, then $\sigma(\omega_{ij}) = (-1)^{K/L_{ij}}\omega_{ij}^2$.

c. If $K = 4L$, then $(2L + 1, 4L) = 1$ and $\sigma(\xi) = \xi^{2L+1} = -\xi$ determines an element of $Gal(\mathbb{Q}(\xi)/\mathbb{Q})$, then $\sigma(\omega_{ij}) = (-1)^{K/L_{ij}}\omega_{ij}$.

We extend $\sigma$ to a generic automorphism of $\mathbb{Q}(\xi) \subset K$, hence $(K, \sigma) \models ACFA$.

In the three cases, we construct a functional equation in $\sigma$. We denote the coordinates on $G_n^m(K)$ by $(x_1, \ldots, x_n)$

$$(a,b).((\sigma(x_1) - x_1^2)(\sigma(x_1) + x_1^2), \ldots, (\sigma(x_n) - x_n^2)(\sigma(x_n) + x_n^2)) = 0 \ (1)$$

$$(c). ((\sigma(x_1) - x_1)(\sigma(x_1) + x_1), \ldots, (\sigma(x_n) - x_n)(\sigma(x_n) + x_n)) = 0 \ (*)$$

If we denote the action by $\sigma$ on $G_n^m(K)$ as $\sigma(x_1, \ldots, x_n) = (\sigma(x_1), \ldots, \sigma(x_n))$, and for $m \in \mathbb{Z}$, define $m(x_1, \ldots, x_n) = (x_1^m, \ldots, x_n^m)$, then any polynomial $P(X)$ with integer coefficients defines an endomorphism of $G_n^m(K)$. We can then write the functional equation

$$(\sigma(x_1) - x_1^p, \ldots, \sigma(x_n) - x_n^p) = 0$$

as $Ker(P(\sigma))$ where $P(X)$ denotes the polynomial $X - p$.

We will denote the subgroups of $G_n^m(K)$ defined by the polynomial $X - p$ for $p \in \mathbb{Z}$ by $G_p$.

The functional equations from $(*)$ define subgroups of $G_n^m(K)$, which we denote by $G^1$ and $G^2$.

We need the following definitions;

Let $A$ be a $\sigma$-definable subgroup of $G_n^m(K)$. We say that $A$ is LMS (stable, stably embedded and 1-based) if every $\sigma$-definable subset $X$ of $A^r$ (possibly with parameters outside $A$) is a finite Boolean combination of cosets of definable subgroups of $A^r$. We say that $A$ is algebraically modular (ALM), if for every $\sigma$ definable $X \subset A^r$, the Zariski closure of $X$ is a finite union of cosets of algebraic subgroups of $G_n^m(K)$. 

The following result is proved in [2] (Corollary 4.1.13);

Let $p(T)$ be a polynomial with integer coefficients defining an endomorphism $p(\sigma)$ of $\mathbb{G}_m^n(K)$. Then $Ker(p(\sigma))$ is LMS iff $p(T)$ has no cyclotomic factors.

Applying this result to the polynomial $X - p$ for $p \in \mathbb{Z}$ gives that the groups $G_p$ are LMS iff $p \notin \{1, -1\}$.

If $A$ is LMS, then $A$ is ALM. This is almost immediate from the definitions. Let $X \subset A'$ be $\sigma$-definable. As $A$ is LMS, we can write;

$$X = \bigcup_{i=1}^{n}(C_i \setminus D_i)$$

where the $C_i$ are disjoint cosets of groups $H_i \subset A'$ and $D_i$ is contained in a finite union of cosets of subgroups of $H_i$ of infinite index. We then clearly have that;

$$X = \bigcup_{i=1}^{n}(C_i \setminus D_i) = \bigcup_{i=1}^{n} C_i$$

Each $C_i$ is clearly an algebraic subgroup of $\mathbb{G}_m^n(K)$, which gives the result.

In particular, the groups $G_p$ are ALM for $p \notin \{1, -1\}$.

We claim that the group $G^1$ is also ALM. We have that $G^1 = \bigcup_{\delta_1, \ldots, \delta_n} W_{\delta_1, \ldots, \delta_n}$ for $\delta_j \in \{0, 1\}$ where $W_{\delta_1, \ldots, \delta_n}$ is the $\sigma$-variety defined by the functional equation

$$(\sigma(x_1) + (-1)^{\delta_1}x_1^2, \ldots, \sigma(x_n) + (-1)^{\delta_n}x_n^2) = 0$$

Assuming $W_{\delta_1, \ldots, \delta_n} \neq \emptyset$, we can find $\bar{w} \in W_{\delta_1, \ldots, \delta_n} \neq \emptyset$, then the map $\theta : W_{\delta_1, \ldots, \delta_n} \to \mathbb{G}_2$ given by $\theta(x_1, \ldots, x_n) = (w_1x_1, \ldots, w_nx_n)$ is easily checked to be a definable bijection. It follows easily that the property of ALM is inherited by each $W_{\delta_1, \ldots, \delta_n}$, hence by $G^1$.

We now claim the following for the situations a. and b.;

$$\bigcup_{1 \leq j \leq N} \bar{w}_j T_j \subseteq V \cap G^1 = \bigcup_{1 \leq j \leq M} \bar{a}_j T_j;$$

where the left hand side consists of the union of all maximal torsion cosets on $V$ and the right hand side consists of a union of cosets of algebraic subgroups of $\mathbb{G}_m^n(K)$. 
By the property of ALM, we clearly have the right hand equality. For the left hand inclusion, suppose that \(\bar{w}_j T_j\) defines a maximal torsion coset on \(V\), where \(\bar{w}_j\) is chosen in the form given above. Then, by construction, \(\bar{w}_j \in \mathbb{G}^1\). Now consider the variety \(W \subset \mathbb{G}^m_r(K) \times \mathbb{G}^m_r(K)\) defined by the equations \(< y_1 - x_1^2, \ldots, y_r - x_r^2 >\). This has dimension \(r\) and projects dominantly onto the factors \(\mathbb{G}^m_r(K)\). We claim that there exists \((x_1, \ldots, x_r) \in \mathbb{G}^m_r(K)\), generic over \(acl(Q)\), with \((\bar{x}, \sigma(\bar{x})) \in W\). By compactness, it is sufficient to prove that for any proper closed subvariety \(Y\) of \(\mathbb{G}^m_r(K)\), defined over \(acl(Q)\), we can find \(\bar{x}\) with \((\bar{x}, \sigma(\bar{x})) \in W \setminus (Y \times \mathbb{G}^m_r(K)) \cap W\). This can be done exactly using the axiom scheme for \(ACFA\). Now let \(\Phi\) define the isomorphism of \(T_j\) with \(\mathbb{G}^m_r(K)\), then \(\bar{w}_j \Phi(\bar{x})\) belongs to \(\bar{w}_j T_j\) and is generic over the field defining the union of the maximal torsion cosets. By construction, the point \(\bar{w}_j \Phi(\bar{x})\) lies inside \(V \cap \mathbb{G}^1\). Hence, \(\bar{w}_j T_j \subseteq V \cap \mathbb{G}^1\) as required.

The functional equation \((c)\) does not define an ALM \(\sigma\)-variety, hence this approach fails (can it be defined using exponents \(\geq 1\)?)

Instead, we can use the method given in \([2]\) to handle this case;

Fix prime numbers \(p\) and \(q\) with \(p \neq q\). We claim there exists \(\sigma \in \text{Gal}(\overline{Q}/Q)\) with \(\sigma(\omega) = \omega^p\) for all primitive \(K\)'th roots of unity with \((K,p) = 1\) and \(\sigma(\omega) = \omega^q\) for all primitive \(K\)'th roots with \(K = p^s\). Let \(Q_p\) be the completion of \(Q\) at the prime \(p\) and \(Q_p^{unr} = \bigcup_{(K,p)=1}^\infty \overline{Q}(\xi_K)\) with \(\xi_K\) a primitive \(K\)'th root of unity. The residue field of \(Q_p(\xi_K)\) is \(F_p(\xi_K)\). The extension is unramified, so \(\text{Gal}(Q_p(\xi_K)/Q) = \text{Gal}(F_p(\xi_K)/F_p)\), with canonical generator \(\phi_p\) lifting Frobenius given by \(\phi_p(\xi_K) = \xi_K^p\). Clearly \(\phi_p\) lifts to an element of \(\text{Gal}(Q_p^{unr}/Q_p)\) such that \(\phi_p(\xi_K) = \xi_K^p\) for all primitive \(K\)'th roots with \((K,p) = 1\). Similarly, we can find \(\phi_q\) with \(\phi_q(\xi_K) = \xi_K^q\) for all primitive \(K\)'th roots with \((K,q) = 1\). As \(p \neq q\), we have \(\phi_q(\xi_K) = \xi_K^q\) for \(K = p^s\). By restriction, \(\phi_p\) and \(\phi_q\) define automorphisms of \(L\) and \(M\) where \(L\) is the extension of \(Q\) obtained by adding primitive roots of unity \(\xi_K\) with \((K,p) = 1\) and \(M\) is obtained by adding primitive roots of unity with \(K = p^s\).

These fields are linearly disjoint over \(Q\), hence we can find a single automorphism \(\sigma\) on \(\overline{Q}\) with \(\sigma|K = \phi_p\) and \(\sigma|L = \phi_q\). As usual, we can extend \(\sigma\) to a generic automorphism of \(K\). Now we consider the endomorphism on \(\mathbb{G}^m_r(K)\) defined by \(p(\sigma)\) where \(p(X)\) is the polynomial \((X - p)(X - q)\). We claim that \(\mathbb{G}^m_r(K^{cycl}) \subset Ker(p(\sigma))\). First note that if \(\bar{w}\) is a cyclotomic point of order \(K\) with \((K,p) = 1\), then we can write \(\bar{w} = (\omega_1, \ldots, \omega_n)\) with the \(\omega_i\) primitive roots of unity of
order prime to \( p \). By construction \( \sigma(\bar{w}) = p(\bar{w}) \), hence \((\sigma - p)(\bar{w}) = 1\). By the same argument, if \( \bar{w} \) is a cyclotomic point of order \( p^s \), then 
\((\sigma - q)(\bar{w}) = 1\). Now suppose that \( \bar{w} \) is an arbitrary cyclotomic point of order \( L \). Let \( L = p^sK \) where \( (K, p) = 1 \). We can find integers \( a \) and \( b \) with \( aK + bp^s = 1 \), hence \( \bar{w} = (\bar{w}_1)^a(\bar{w}_2)^b \) where \( \bar{w}_1 \) is cyclotomic of order \( p^s \) and \( \bar{w}_2 \) is cyclotomic of order \( K \). We then have that:

\[
p(\sigma)(\bar{w}) = (p(\sigma)(\bar{w}_1))^a(p(\sigma)(\bar{w}_2))^b = ((\sigma - p).1)^a.((\sigma - q).1)^b = 1
\]

as required. We now have the following explicit functional equation for \( \text{Ker}(p(\sigma)) \) using coordinates \((x_1, \ldots, x_n)\) on \( \mathbb{G}_m^n(K) \):

\[
p(\sigma).(x_1, \ldots, x_n) = 1 \quad (2)
\]

\[
\iff (\sigma - p).\frac{\sigma(x_1)\ldots\sigma(x_n)}{x_1\ldots x_n} = 1
\]

\[
\iff \frac{(\sigma^2(x_1)\ldots\sigma^2(x_n)).(x_1^{pq}\ldots x_n^{pq})}{(\sigma^2(x_1)\ldots\sigma^2(x_n)).(\sigma(x_1)^p\ldots\sigma(x_n)^p)} = 1
\]

\[
\iff ((\sigma^2x_1)(x_1^{pq}) - (\sigma^2x_1)^p(\sigma x_1)^q, \ldots, (\sigma^2x_n)(x_n^{pq}) - (\sigma x_n)^p(\sigma x_n)^q) = 0
\]

Let \( \mathbb{G}_{p,q} = \text{Ker}(p(\sigma)) \). Using the theorem above, we know that \( \mathbb{G}_{p,q} \) is ALM. Moreover, we again have that:

\[
\bigcup_{1 \leq j \leq N} \bar{w}_j T_j \subseteq \overline{V \cap \mathbb{G}_{p,q}} = \bigcup \bar{a}_j T_j \quad (**)
\]

where the left hand union is over all maximal torsion cosets and the right hand union consists of cosets of algebraic subgroups of \( \mathbb{G}_m^n(K) \). We again have the right hand equality by the ALM property. For the left hand inclusion, note that \( \bigcup_{1 \leq j \leq N} \bar{w}_j T_j = \overline{V(K_{cyc})} \) and \( V(K_{cyc}) \subset V \cap \mathbb{G}_{p,q} \) by construction of \( \mathbb{G}_{p,q} \).

We now use the functional equations to obtain explicit bounds on the number of maximal torsion cosets on \( V \). This is done by finding a bound \( N \) for the number of irreducible components of \( \overline{V \cap G} \) where \( G \) is one of the groups \( \mathbb{G}_{p,q}, \mathbb{G}_1 \). By (***) and Lemma 0.1, this will give a bound \( N \) for the number of maximal torsion cosets on \( V \). We require the following lemma (Proposition 2.2.1 in [2])
Lemma 0.3. Let \( P_n(K) \) be \( n \)-dimensional projective space, and \((K, \sigma)\) a difference closed difference field. Let \( S \) be a subvariety of \( P_n^l(K) \) defined over \( K \). Let

\[
Z = \text{Zariski closure of } \{ x \in P_n(K) : (x, \sigma(x), \ldots, \sigma^{l-1}(x)) \in S \}.
\]

Then \( \deg(Z) \leq \deg(S)^{2 \dim(S)} \). In particular \( Z \) has at most \( \deg(S)^{2 \dim(S)} \) irreducible components.

Here, \( \deg(S) \) is the sum of the multi-degrees of \( S \). It is a straightforward exercise to rephrase this result replacing \( P_n(K) \) by \( G_n^{n} \) and \( P_n^l(K) \) by \( G_m^{n \times l} \).

Now, in the case of the functional equation (1) for the cases \((a, b)\) and the functional equation (2) which covers cases \((a, b, c)\), we construct the following varieties \( W_1 \) and \( W_2 \).

For coordinates \((x_1, \ldots, x_n, y_1, \ldots, y_n)\) on \( G_m^{2n} \);
\[
W_1 = \langle (y_1 - x_1^2)(y_1 + x_1^2), \ldots, (y_n - x_n^2)(y_n + x_n^2) \rangle
\]

For coordinates \((x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n)\) on \( G_m^{3n} \);
\[
W_2 = \langle z_1x_1^{pq} - y_1^{pq+q}, \ldots, z_nx_n^{pq} - y_n^{pq+q} \rangle
\]

A straightforward calculation gives that \( \deg(W_1) = 3^n \) and \( \dim(W_1) = n \) whereas \( \deg(W_2) = (pq + 1)^n \) and \( \dim(W_2) = 2n \).

As \( \sigma \) fixes \( V \), we have in cases \((a, b)\) that;
\[
\bigcup_{1 \leq j \leq N} \tilde{w}_j T_j \subseteq \{ x \in G_m^n(K) : (x, \sigma x) \in V^2 \cap W_1 \}
\]

and in cases \((a, b, c)\) that;
\[
\bigcup_{1 \leq j \leq N} \tilde{w}_j T_j \subseteq \{ x \in G_m^n(K) : (x, \sigma x, \sigma^2 x) \in V^3 \cap W_2 \}
\]

We finally need the following version of Bezout’s theorem for counting the components of intersections in multi-projective space;

Lemma 0.4. \textit{Bezout’s Theorem}
Let $V, W$ be subvarieties of $\mathbb{P}_n^l(K)$. Let $Z_1, \ldots, Z_t$ be the irreducible components of $V \cap W$. Then $\sum_{i=1}^t \deg(Z_i) \leq \deg(V)\deg(W)$.

Applying this to the current situation, we have $\deg(V^2 \cap W_1) \leq \deg(V)^23^n$ and $\deg(V^3 \cap W_2) \leq \deg(V)^3(pq + 1)^n$. We also have that $\dim(V^2 \cap W_1) \leq 2\dim(V)$ and $\dim(V^3 \cap W_2) \leq 3\dim(V)$. Now, using Lemma 0.2, we have that:

$$N \leq (\deg(V)^23^n)^{2\dim(V)} = \exp[2^{2\dim(V)}\log(d^23^n)]$$

in cases $(a, b)$

and

$$N \leq (\deg(V)^3(pq + 1)^n)^{3\dim(V)}$$

in cases $(a, b, c)$.

where $d = \deg(V)$

Taking $p = 2$ and $q = 3$ gives

$$N \leq (\deg(V)^37^n)^{3\dim(V)} = \exp[2^{3\dim(V)}\log(d^37^n)]$$

in cases $(a, b, c)$.

Using the toric version of Bezout’s theorem in [1], we can replace $\deg(V)$ by $\text{vol}(N(V))$, where $N(V)$ is the Newton polytope associated to $V$. This gives a slightly more refined estimate for subvarieties $V$ of $\mathbb{G}_m^n(K)$. A comparison of the estimates from [4] shows that the estimate in this paper is better in certain situations and worse in others, depending on the dimension and degree of $V$.

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