MARTINGALE-COBOUNDARY DECOMPOSITION FOR STATIONARY RANDOM FIELDS
Dalibor Volny

To cite this version:
Dalibor Volny. MARTINGALE-COBOUNDARY DECOMPOSITION FOR STATIONARY RANDOM FIELDS. Stochastics and Dynamics, World Scientific Publishing, 2018. hal-01878176

HAL Id: hal-01878176
https://hal.archives-ouvertes.fr/hal-01878176
Submitted on 20 Sep 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers. L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
MARTINGALE-COBOUNDARY DECOMPOSITION FOR STATIONARY RANDOM FIELDS

DALIBOR VOLNÝ

ABSTRACT. We prove a martingale-coboundary representation for random fields with a completely commuting filtration. For random variables in $L^2$ we present a necessary and sufficient condition which is a generalization of Heyde’s condition for one dimensional processes from 1975. For $L^p$ spaces with $2 \leq p < \infty$ we give a necessary and sufficient condition which extends Volný’s result from 1993 to random fields and improves condition of El Machkouri and Giraudo from 2016. A new sufficient condition is presented which for dimension one improves Gordin’s condition from 1969. In application, new weak invariance principle and estimates of large deviations are found.

1. Introduction.

Let $(\Omega, \mathcal{A}, \mu)$ be a probability space and $(T_i)_{i \in \mathbb{Z}^d}$ a $\mathbb{Z}^d$ action on $(\Omega, \mathcal{A}, \mu)$ generated by commuting invertible and measure-preserving transformations $T_{\epsilon_q}$, $1 \leq q \leq d$. By $\epsilon_q$, we denote the vector from $\mathbb{Z}^d$ which has 1 at $q$-th place and 0 elsewhere. By $U_{\epsilon_q}$ we denote the operator in $L^p$ ($1 \leq p < \infty$) defined by $U_{\epsilon_q} f = f \circ T_{\epsilon_q}, \epsilon_q \in \mathbb{Z}^d$.

By $0 \leq j \leq d$, we understand $0 \leq j \leq d$. The vectors $(0, \ldots, 0)$ and $(1, \ldots, 1)$ will be denoted 0 and 1 respectively.

By $\mathbb{Z}$, we denote the vector from $\mathbb{Z}^d$ with $1$ at $q$-th place and 0 elsewhere.

We suggest that there is a completely commuting filtration $(\mathcal{F}_i)_{i \in \mathbb{Z}^d}$, i.e. there is a $\sigma$-algebra $\mathcal{F}$ such that $\mathcal{F}_i = T_{\epsilon_i} \mathcal{F}$, for $0 \leq j \leq d$, we have $\mathcal{F}_{\epsilon_q} \subset \mathcal{F}_1$, and for an integrable $f$ it is

\[
E \left( E(f | \mathcal{F}_{i_1,i_2,\ldots,i_d}) | \mathcal{F}_{j_1,j_2,\ldots,j_d} \right) = E(f | \mathcal{F}_{i_1 \wedge j_1,i_2 \wedge j_2,\ldots,i_d \wedge j_d})
\]

where $i \wedge j = \min\{i, j\}$ (cf. [VWa14]). As a frequent (cf. [VWa14], [WaWo13] and references therein) but not exclusive example, let us introduce a Bernoulli $\mathbb{Z}^d$ action: the $\sigma$-algebra $\mathcal{A}$ is generated by iid random variables $e_i = U_{\epsilon_i}$. The filtration $\mathcal{F}_{\epsilon_i} = \sigma\{e_i : \epsilon_i \leq \epsilon_j\}$ is completely commuting.

By $\mathcal{F}_{\epsilon_i}^{(q)}$ we denote the $\sigma$-algebra generated by all $\mathcal{F}_{\epsilon_i}$ with $i_q \leq l$ ($i_q \in \mathbb{Z}$ for $1 \leq q \leq d$, $l \neq q$), $1 \leq q \leq d$.

For $\sigma$-algebras $\mathcal{G} \subset \mathcal{F} \subset \mathcal{A}$ and $1 \leq p < \infty$, we by $L^p(\mathcal{F}) \subset L^p(\mathcal{G})$ denote the space of $f \in L^p(\mathcal{G})$ for which $E(f | \mathcal{G}) = 0$. Similarly as in the one dimensional case we can define projection operators $P_{\epsilon_i}^{(q)}$ onto $L^p(\mathcal{F}_{\epsilon_i}^{(q)}) \subset L^p(\mathcal{F}_{\epsilon_i}^{(q)})$ by $P_{\epsilon_i}^{(q)} f = E(f | \mathcal{F}_{\epsilon_i}^{(q)}) - E(f | \mathcal{F}_{\epsilon_i}^{(q)})$. These operators commute and for $l \neq k$, $P_{\epsilon_i}^{(q)} P_{\epsilon_i}^{(q)} = 0$. We define projections $P_{j_1,j_2,\ldots,j_d}^{(1)} = P_{j_1}^{(1)} \ldots P_{j_d}^{(1)}$ onto $\bigcap_{1 \leq q \leq d} L^p(\mathcal{F}_{\epsilon_i}^{(q)}) \subset L^p(\mathcal{F}_{\epsilon_i}^{(q)})$ (cf. [VWa14]). Let us notice that $T_{\epsilon_i}^{-1} \mathcal{F}_{j_1,j_2,\ldots,j_d} = \mathcal{F}_{j_1,j_2,\ldots,j_d}$, where $j_q = j_q + 1$ and...
\(j'_i = j_i\) for \(i \neq q\). We thus have \(T_{\epsilon_q}^{-1}F_i^{(q)} = F_i^{(q)}\) and \(T_{\epsilon_q}^{-1}F_l^{(q')} = F_l^{(q')}\) for \(q' \neq q\); 
\(U_{\epsilon_q}P_i^{(q)} = P_i^{(q)}U_{\epsilon_q}\) and \(U_{\epsilon_q}P_l^{(q')} = P_l^{(q')}U_{\epsilon_q}\).

An integrable function \(f\) is called regular if it is \(F_{\infty}\)-measurable and for every 
\(1 \leq q \leq d, E(f \mid F_{\infty}^{(q)}) = 0\), where \(F_{\infty}^{(q)} = \cap_{k \in \mathbb{Z}} F_{-k}^{(q)}\). A function \(f\) is adapted if it is 
\(F_0\)-measurable. \(f \in L_p, 1 \leq p < \infty\), is thus regular if and only if 
\(f = \sum_{i \in \mathbb{Z}^d} P_i f\).

In this paper all functions will be supposed to be regular. A random field \(U_i f\) 
generated by a regular function \(f\) will be called regular.

A useful tool in proving limit theorems for one dimensional (strictly) stationary 
processes \((f \circ T^i)\), (i.e. for \(d = 1\)) has been the martingale-coboundary 
decomposition

\[(1) \quad f = m + g - g \circ T\]

where \((m \circ T^i)\) is a martingale difference sequence. The decomposition \((1)\) was one 
of the first conditions giving a CLT for stationary sequences of random variables 
by martingale approximation; for \(f, m, g \in L^2\) it was introduced already in Gordin’s 
1969 paper [Go69]. Even if the cobounding function \(g\) is just measurable, \((1)\) still 
implies a central limit theorem. A paper proving a CLT for \(f, g \in L^1\) was proved 
in [Go73] (cf. also [EJ85]; see that square integrability of the martingale differences 
\(m\) needed to be proved). Square integrability of \(m, g\) guarantees a weak invariance 
principle (WIP) and a functional law of iterated logarithm ([He75]). Notice that 
in general, a central limit theorem does not imply WIP. For strictly stationary and 
ergodic processes this has been shown e.g. in [VSa00]. In [GV14] a beta mixing 
process satisfying the CLT but not WIP is found.

The condition \((1)\) provides a very close martingale approximation and for central 
limit theorems it is sometimes suboptimal, e.g. the conditions of Dedecker-Rio and 
of Maxwell-Woodroofe imply the weak invariance principle (cf. [DR00], [MW00]).
The conditions mentioned above follow from \((1)\) (with \(m, g \in L^2\)) but not vice versa, cf. e.g. [DuV08]. \((1)\) is independent of the Hannam’s criterium (cf. [V93]) and 
remains useful in the study of central limit theorems for Markov chains (one of the 
first papers on the subject is [GoLi]). The martingale-coboundary decomposition 
\((1)\) can be used in proving other limit theorems like estimates of large deviations 
(cf. [LV01]) where other conditions do not apply. This motivates study of \((1)\) in \(L^p\) 
spaces with \(p > 2\). Probably the most exhaustive study of \((1)\) in various spaces is 
in [V06].

In this paper we will extend the martingale-coboundary decomposition to ran-
dom fields. In dimension \(d \geq 2\) the decomposition appears more complicated: we 
are interested in the existence of the (martingale-coboundary) representation

\[(2) \quad f = \sum_{S \subseteq \{1, \ldots, d\}} \prod_{q \in S^c} (I - U_{\epsilon_q})g_S \quad (f \in L^p, 1 \leq p < \infty)\]

where for \(S \subseteq \{1, \ldots, d\}, g_S \in \bigcap_{q \in S} L^p(F_0^{(q)}) \oplus L^p(F_{-1}^{(q)}); \prod_{q \in \emptyset} (I - U_{\epsilon_q})\) is defined 
as \(I\), the identity operator. For \(q' \in S, U_{\epsilon_q' q}, \prod_{q \in S^c} (I - U_{\epsilon_q})g_S, i \in \mathbb{Z}\), are thus 
martingale differences while for \(q' \in S^c, \prod_{q \in S^c} (I - U_{\epsilon_q})g_S\) are coboundaries for the 
transformation \(T_{\epsilon_q'}\).
As an illustration, consider $d = 2$. (2) then becomes
\[
f = m + [g_1 - U_{1,0}g] + [g_2 - U_{0,1}g] + [g - U_{1,0}g - U_{0,1}(g - U_{1,0}g)]
\]
where $m, g \in L^p$, $P_{0,0}m = m$ (i.e. $U_{i,j}m$ are martingale differences), $g_1 \in L^p(\mathcal{F}_0^{(2)} \ominus L^p(\mathcal{F}_0^{(2)}))$, $g_2 \in L^p(\mathcal{F}_0^{(1)} \ominus L^p(\mathcal{F}_0^{(1)}))$. The term $g_1 - U_{1,0}g_1 \in L^p(\mathcal{F}_0^{(2)} \ominus L^p(\mathcal{F}_0^{(2)}))$ is thus a martingale difference sequence for the transformation $T_{0,1}$ and a coboundary for $T_{1,0}$; the last term is a coboundary.

The aim of this paper is to study both sufficient, and necessary and sufficient conditions for the decomposition (2) for regular functions. In the same setting as here, a sufficient condition was recently found by El Machkouri and Giraudo in [ElG16]. Their main result (cf. Theorem 5 here) is a multiparameter version of Gordin’s one-dimensional sufficient condition from 1969 (cf. [Go69]). In 2009, the problem was studied by Gordin in [Go09] where, instead of $\mathbb{Z}^d$, he used semigroup $\mathbb{Z}^d_+$ and instead of martingale differences he got reversed martingale differences.

For $d = 1$ his condition becomes the Poisson equation. Our results can be easily converted to the setting applied in Gordin’s paper.

We prove a multiparameter version of necessary and sufficient conditions from [He75] (Theorem 2) and [V93] (Theorem 4). In Theorem 6 we present a sufficient condition which seems to be easier to verify than assumptions of Theorem 2 and Theorem 4.

The martingale-coboundary representation will be used in proving limit theorems, in particular a weak invariance principle (WIP) and estimates of probabilities of large deviations. We will extend similar results from [ElG16].

2. Main results.

Let us recall that in the paper we suppose regularity of the function $f$. We will need

**Proposition 1.** For $1 \leq p < \infty$, the decomposition (1) with $f, m, g \in L^p$ is equivalent to the convergence of

\[
\sum_{j=0}^{\infty} E(U_j f | \mathcal{F}_{-j}), \quad \sum_{j=1}^{\infty} [(U^{-j} f - E(U^{-j} f) | \mathcal{F}_{-1})]
\]

in $L^p$. The transfer function $g$ can be regular and we can fix $m = P_{0}m$. In such a case

\[
m = \sum_{i \in \mathbb{Z}} P_{0}U_i f, \quad g = \sum_{j=0}^{\infty} E(U_j f | \mathcal{F}_{-1}) - \sum_{j=1}^{\infty} [(U^{-j} f - E(U^{-j} f) | \mathcal{F}_{-1})].
\]

A proof can be found in [V93] (for $p = 1, 2$) and in [V06].

In the case of $d \geq 2$ we will prove a necessary and sufficient condition for (2) in $L^2$.

**Theorem 2.** The martingale-coboundary decomposition (2) holds in $L^2$ if and only if for every $S \subset \{1, \ldots, d\}$ and $S' \subset S^c$

\[
\sum_{j_u \geq 1, u \in S'} \sum_{i_v \geq 1, v \in S^c \setminus S'} \left\| \sum_{i_r \in \mathbb{Z}, r \in S} \sum_{i_u \geq j_u, u \in S'} \sum_{i_v \geq j_v, v \in S^c \setminus S'} P_{0}U_{i_r, i_u, -i_v} f \right\|_2^2 < \infty;
\]
this is equivalent to

\[(4b) \sum_{S \subset \{1, \ldots, d\}} \sum_{j_k \geq 0, k \in S} \sum_{j_l \leq 0, l \in S^c} \left\| \sum_{i_k \geq j_k, k \in S} \sum_{i_l \leq j_l, l \in S^c} P_2 U_{i_1, \ldots, i_d} f \right\|_2^2 < \infty.\]

If (2), (4) are valid then the functions \(g_S\) can be regular and we then get

\[(5) g_S = \sum_{S' \subset S^c} (-1)^{|S^c\setminus S'|} \sum_{i_r \in \mathbb{Z}, r \in S^c} \sum_{j_v \geq 1, j_u \geq 0, u \in S^c} \sum_{j_v \geq 1, j_v \geq 0, v \in S^c} P_{0, -j_u, j_v} U_{i_r, i_u, -i_v} f.\]

In the formula (5), \(P_{0, -j_u, j_v}\) is the projection operator \(P_j\) where \(j_i = 0\) for \(i \in S\), \(j_i \leq -1\) for \(i \in S'\), and \(j_i \geq 0\) for \(i \in S^c \setminus S'\); the expression \((\text{operator})\) \(U_{i_r, i_u, -i_v}\) is to be understood similarly.

For \(d = 1\) the condition (4b) is equivalent to

\[(6) \sum_{j=1}^\infty \| \sum_{i=j}^\infty P_0 U^i f \|_2^2 + \sum_{j=1}^\infty \| \sum_{i=j}^\infty P_0 U^{-i} f \|_2^2 < \infty.\]

The condition (6) was found as sufficient for (1) by C.C. Heyde ([He75], cf. also [HaHe, Theorem 5.5]) while in [V93] it was proved necessary and sufficient.

From (5) and orthogonality it follows

\[\|g_S\|^2_2 = \sum_{S' \subset S^c} \sum_{j_u \geq 1, u \in S^c} \sum_{j_v \geq \mathbb{Z}, v \in S' \setminus S'} \left\| \sum_{i_r \in \mathbb{Z}, i_u \geq 0, u \in S^c} \sum_{i_v \geq 1, v \in S^c} P_{0, -j_u, j_v} U_{i_r, i_u, -i_v} f \right\|_2^2,\]

\(S \subset \{1, \ldots, d\}\).

Let us recall that we use regularity assumption; if we take all functions \(g_S\) regular then they are unique.

To prove Theorem 2 we can use a superlinear random field representation (cf. [VWoZ11, Theorem 1] for \(d = 1\), [CCo13] in the general case): there exist \(e_k = P_0 e_k\) and real numbers \(a_{k, l}\) such that

\[(7) \|e_k\|^2_2 = 1, \quad k \geq 0, \quad \sum_{k=0}^\infty \sum_{l \in \mathbb{Z}^d} a_{k, l}^2 < \infty, \quad \text{and} \quad f = \sum_{k=0}^\infty \sum_{l \in \mathbb{Z}^d} a_{k, l} U_{-l} e_k.\]

To see this, let’s notice that without loss of generality we can suppose that the \(\sigma\)-algebra \(\mathcal{A}\) is countably generated. Then there exists a countable orthonormal basis \(\{e_k : k = 0, 1, \ldots\}\) of \(\bigcap_{1 \leq q \leq d} L^2(F_0^{(q)}) \oplus L^2(F_1^{(q)})\), and \(U_{-l} e_k, k \geq 0, l \in \mathbb{Z}^d\), is the orthonormal basis of the Hilbert space of regular functions from \(L^2\).

If \(f = \sum_{l \in \mathbb{Z}^d} a_{l} U_{-l} e, e \in \bigcap_{1 \leq q \leq d} L^2(F_0^{(q)}) \oplus L^2(F_1^{(q)})\), we will speak of a stationary linear field.

Let \(L_k, k = 0, 1, \ldots, d\), denote the Hilbert space generated by \(U_{l} e_k, l \in \mathbb{Z}^d\), and \(\Pi_k\) the orthogonal projection operator on \(L_k\). The space of regular elements from
$L^2$ is the direct sum $\bigoplus_{k \geq 0} L_k$. Each of the spaces $L_k$ is invariant w.r.t. $U_{i, \ell} \in \mathbb{Z}^d$, and $\Pi_k$ commutes with the operators $U_{i, \ell}$. In (2) we thus get

$$f = \sum_{k=0}^{\infty} f_k = \sum_{k \geq 0} \sum_{S \subseteq \{1, \ldots, d\}} \prod_{q \in S^c} (I - U_{x_q}) g_{k, S}$$

where $f_k, g_{k, S} \in L_k$ and $g_S = \sum_{k=0}^{\infty} g_{k, S}, \|g_S\|_2^2 = \sum_{k=0}^{\infty} \|g_{k, S}\|_2^2$.

Heyde’s condition (5) can be deduced from Proposition 1. We prove it in a form useful for proving Theorem 2.

**Proposition 3 (C.C. Heyde).** For $f \in L^2$ regular, (6) is equivalent to the martingale-coboundary decomposition (1) where for $f$ represented by (7)

$$m = \sum_{k=0}^{\infty} \sum_{i \in \mathbb{Z}} a_{k,i} e_k, \quad g = \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \sum_{i=j} a_{k,j} U^{-j} e_k - \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \sum_{i=j+1} a_{k,-i} U^j e_k.$$

**Proof.**

For $f$ represented by (7), (6) is equivalent to

$$\sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \left( \sum_{i=j}^{} a_{k,i} \right)^2 + \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \sum_{i=j} \left( \sum_{i=j} a_{k,-i} \right)^2 < \infty.$$

For simplicity sake let’s suppose that $f$ is adapted, i.e. $f = \sum_{k=0}^{\infty} \sum_{i \geq 0} a_{k,i} U^{-i} e_i$. By Proposition 1, (1) is then equivalent to the convergence (in $L^2$) of

$$\sum_{j=0}^{\infty} E(U^j f | \mathcal{F}_{-1}) = \sum_{i<0} P_i \sum_{j=0}^{\infty} E(U^j f | \mathcal{F}_{-1}) = \sum_{k=0}^{\infty} \sum_{i=1}^{\infty} \left( \sum_{j=0}^{\infty} a_{k,i+j} \right) U^{-i} e_k,$$

i.e. to the convergence of $\sum_{k=0}^{\infty} \sum_{i=1}^{\infty} \left( \sum_{j=0}^{\infty} a_{k,i+j} \right)^2$, which, for $f$ adapted, is equivalent to the condition (6b).

The proof of the general non adapted case is similar.

square

**Proof of Theorem 2.** We suppose that $f$ is represented by (7).

The condition (4a) is then equivalent to

$$\sum_{k=0}^{\infty} \sum_{j_u > 0, u \in S'} \sum_{i_v < 0, v \in S \setminus S'} \left( \sum_{i_r \in \mathbb{Z}} \sum_{u \in S'} \sum_{v \in S' \setminus \{u \}} a_{k,i_1,...,i_d} \right)^2 < \infty$$

and (4b) is equivalent to

$$\sum_{k=0}^{\infty} \sum_{j_u \geq 0, u \in S} \sum_{i_v \leq 0, v \in S'} \left( \sum_{i_r \in \mathbb{Z}} \sum_{u \in S} \sum_{v \in S' \setminus \{u \}} a_{k,i_1,...,i_d} \right)^2 < \infty.$$

By using elementary equality $\sum_{j=0}^{\infty} \left( \sum_{i=j}^{} a_i \right)^2 = \left( \sum_{i=0}^{\infty} a_i \right)^2 + \sum_{j=1}^{\infty} \left( \sum_{i=j}^{} a_i \right)^2$ and induction we can prove that the sum of all (4c) over $S \subset \{1, \ldots, d\}$ and $S' \subset S^c$...
equals the sum of all \((4d)\) over \(S \subset \{1, \ldots, d\}\). The conditions \((4a)\) and \((4b)\) are thus equivalent.

(5) becomes

(5a) \[ g_S = \sum_{k=0}^{\infty} \sum_{i_r \in \mathbb{Z}} \sum_{S' \subset S} (-1)^{|S'\setminus S|} \sum_{j_u \geq 1, i_u \geq j_u, u \in S'} \sum_{i_v \geq j_u + 1, v \in |S'\setminus S|} a_{i_r, i_u, -i_v} U_{0, -j_u, j_v} e_k. \]

Let us prove equivalence of (2) and (4).

For \(d = 1\), Theorem 2 becomes the Heyde’s theorem mentioned above (cf. Proposition 3).

Let’s suppose that \(d \geq 2\) and that for \(d = 1\) the theorem is true.

Because the operators \(U_i\) commute with the projections \(\Pi_k\) and \(g_S = \sum_{k=0}^{\infty} g_k, S (= \sum_{k=0}^{\infty} \Pi_k g_S), \|g_S\|^2 = \sum_{k=0}^{\infty} \|g_k, S\|^2\), (2) holds in \(L^2\) if and only if it holds in all spaces \(L_k\). It is thus sufficient to prove the theorem for a stationary linear process.

For simplicity of notation we suppose that \(f\) is adapted. The expression (5a) then becomes

(5b) \[ g_S = \sum_{i_r \geq 0, i_u \geq 1, u \in S'} \sum_{i_v \geq j_u + 1, v \in S'} a_{i_r, i_u, -i_v} U_{0, -j_u, j_v} e, \]

\(S \subset \{1, \ldots, d\}\) and \((0, -j_u)\) is a vector from \(\mathbb{Z}^d\); (4d) becomes

(4e) \[ \sum_{j_1=0}^{\infty} \cdots \sum_{j_d=0}^{\infty} \left( \sum_{i_1=j_1}^{\infty} \cdots \sum_{i_d=j_d}^{\infty} a_{i_1, \ldots, i_d} \right)^2 < \infty. \]

Let’s suppose (4e) (we have a stationary adapted random field). We will prove (2).

If we apply Proposition 3 to \(U_{\varepsilon_d}\) and the filtration \((\mathcal{F}_i^{(d)})_i\), we by using (4e) deduce

(8) \[ f = m_d + g_d - U_{\varepsilon_d} g_d \]

where

\[ m_d = \sum_{i_1=0}^{\infty} \cdots \sum_{i_{d-1}=0}^{\infty} \sum_{i_d=0}^{\infty} a_{i_1, \ldots, i_{d-1}, i_d} U_{-i_1, \ldots, -i_{d-1}, 0} e, \]

\[ g_d = \sum_{i_1=0}^{\infty} \cdots \sum_{i_{d-1}=0}^{\infty} \sum_{j_d=1}^{\infty} \sum_{i_d=j_d}^{\infty} a_{i_1, \ldots, i_{d-1}, i_d} U_{-i_1, \ldots, -i_{d-1}, -j_d} e. \]

For \(i_d \geq 0\) let us denote

\[ f_{i_d} = \sum_{i_1=0}^{\infty} \cdots \sum_{i_{d-1}=0}^{\infty} a_{i_1, \ldots, i_{d-1}, i_d} U_{-i_1, \ldots, -i_{d-1}, 0} e. \]
We thus have
\[ m_d = \sum_{i_d=0}^{\infty} f_{i_d}, \quad g_d = \sum_{j_d=1}^{\infty} \sum_{i_d=j_d}^{\infty} U_{e_d}^{-j_d} f_{i_d}. \]

When applying Theorem 2 to an action of \( T_{e_1}, \ldots, T_{e_{d-1}} \) (notice that (4e) remains satisfied) we get
\[ f_{i_d} = \sum_{S \subset \{1, \ldots, d-1\}} \prod_{q \in S^e \setminus \{d\}} (I - U_{e_q}) g_{S,i_d} \]
where \( g_{S,i_d} \) is defined by (5b) applied to an action of \( T_{e_1}, \ldots, T_{e_{d-1}} \). By (8) we thus have
\[ f = m_d + (I - U_{e_d}) g_d = \sum_{S \subset \{1, \ldots, d-1\}} \prod_{q \in S^e \setminus \{d\}} (I - U_{e_q}) \sum_{i_d=0}^{\infty} g_{S,i_d} + \]
\[ \sum_{S \subset \{1, \ldots, d-1\}} (I - U_{e_d}) \prod_{q \in S^e \setminus \{d\}} (I - U_{e_q}) \sum_{j_d=1}^{\infty} \sum_{i_d=j_d}^{\infty} U_{e_d}^{-j_d} g_{S,i_d} = \]
\[ \sum_{S \subset \{1, \ldots, d-1\}, d \in S} \sum_{q \in S^c} (I - U_{e_q}) g_S + \sum_{S \subset \{1, \ldots, d-1\}} \prod_{q \in S^c} (I - U_{e_q}) U_{e_d}^{-i_d} g_S \]
where for \( S \subset \{1, \ldots, d-1\}, \)
\[ g_S = \sum_{j_d=1}^{\infty} \sum_{i_d=j_d}^{\infty} U_{e_d}^{-j_d} g_{S,i_d} \text{ and } g_{S \cup \{d\}} = \sum_{i_d=0}^{\infty} g_{S,i_d}. \]
This proves (2) with \( g_S \) defined by (5b), for \( d \) parameters; (4e) guarantees convergence.

Now, let’s suppose (2). For \( d = 1 \), (4) follows by Proposition 3. Let us assume that the implication \( (2) \implies (4) \) is true for \( d - 1, d \geq 2 \). We consider the random field defined by \( U_{e_1}, \ldots, U_{e_{d-1}} \) and
\[ f = \sum_{i_d=0}^{\infty} \left( \sum_{i_1=0}^{\infty} \cdots \sum_{i_{d-1}=0}^{\infty} a_{i_1, \ldots, i_{d-1}, i_d} U_{e_{i_1}, \ldots, e_{i_{d-1}}, e_{i_d}} (U_{e_d}^{-i_d} e) \right); \]
\( i_d \) is a parameter. We thus have
\[ f = \sum_{S \subset \{1, \ldots, d\}} \prod_{q \in S^c} (I - U_{e_q}) g_S = \sum_{S \subset \{1, \ldots, d-1\}} \prod_{q \in S^c \setminus \{d\}} (I - U_{e_q}) \bar{g}_S \]
where
\[ \bar{g}_S = g_{S \cup \{d\}} + (I - U_{e_d}) g_S, \quad S \subset \{1, \ldots, d-1\}; \]
by (5b) we have
\[ \bar{g}_S = \sum_{i_d=0}^{\infty} \sum_{i_r \geq 0} \sum_{j_u \geq 1} \sum_{u \in S^c \setminus \{d\}} a_{i_r, i_u, i_d} U_{e_{j_u}, e_{i_d}} \]
By using Heyde’s condition (6) we get, for $S \subset \{1, \ldots, d-1\}$,

$$\infty > \sum_{j_d \geq 1} \| \sum_{i_d \geq j_d} P_{0}^{(d)} U_{i_d}^{j_d} \hat{g}_S \|_{2}^{2} =$$

$$\sum_{j_d \geq 1} \| \sum_{i_r, r \in S} \sum_{j_u \geq 1, u \in S \setminus \{d\}} \sum_{i_u \geq j_u, u \in S \setminus \{d\}} \sum_{i_d \geq j_d} a_{i_r, i_u, i_d} U_{0, -j_u, 0} \|_{2}^{2} =$$

$$\sum_{j_d \geq 1} \sum_{j_u \geq 1, u \in S \setminus \{d\}} \left( \sum_{i_r, r \in S} \sum_{i_u \geq j_u, u \in S \setminus \{d\}} \sum_{i_d \geq j_d} a_{i_r, i_u, i_d} \right)^{2}.$$  

In (6) we can sum the $j_d$ from 0, hence we also have

$$\sum_{j_u \geq 1, u \in S \setminus \{d\}} \left( \sum_{i_r, r \in S} \sum_{i_u \geq j_u, u \in S \setminus \{d\}} a_{i_r, i_u} \right)^{2} < \infty;$$

we thus have proved (4e). By the preceding implication we also get (5b).

In the same way we can prove the theorem for non adapted random fields. The proof is thus accomplished. □

In [V06] it was proved that for $1 \leq p < \infty$, convergence of (3) in $L^p$ is a necessary and sufficient condition for the decomposition (1) in $L^p$. The result can be extended to $\mathbb{Z}^d$ actions. Let us denote

$$Q_{j}^{(q)} f = E(f | F_{j}^{(q)}) = \sum_{i \leq j} P_{i}^{(q)} f.$$

**Theorem 4.** Let $1 \leq p < \infty$, $f \in L^p$. (2) is equivalent to the convergence in $L^p$ of

\[ (3a) \]

$$\sum_{i_r, r \in S} \sum_{i_u, u \in S} \sum_{i_v, v \in S^c} \prod_{r \in S} \prod_{u \in S^c} \prod_{v \in S^c} P_{0}^{(r)} Q_{-1}^{(u)} (I - Q_{-1}^{(v)}) U_{i_r}^{i_u} U_{i_u}^{i_v} f =$$

$$\sum_{i_r, r \in S} \sum_{i_u, u \in S} \sum_{i_v, v \in S^c} P_{i_r, -j_u, j_v} U_{i_u, -i_v} f$$

for all $S \subset \{1, \ldots, d\}$.

If $f$ is adapted then a necessary and sufficient condition for (2) is the convergence in $L^p$ of

\[ (3b) \]

$$\sum_{i_1 = 0}^{\infty} \cdots \sum_{i_d = 0}^{\infty} E(U_{i_1, \ldots, i_d} f | F_{0}).$$

Like in Theorem 2, (5) gives us the transfer functions $g_S$.

**Proof.** We prove the theorem for adapted functions only. For $f$ adapted, (3a) means the convergence of

$$\sum_{i_r, r \in S} \sum_{i_u, u \in S^c} \sum_{i_v, v \in S^c} \prod_{r \in S} \prod_{u \in S^c} \prod_{v \in S^c} P_{0}^{(r)} Q_{-1}^{(u)} U_{i_r}^{i_u} U_{i_u}^{i_v} f.$$
S \subset \{1, \ldots, d\}. In the same way as in the proof of Theorem 2 we can prove equivalence with (3b). A generalization to nonadapted (regular) random fields is straightforward.

Let us suppose the convergence in (3b). As shown in the Introduction, the functions $U_{i_1}^i f$ are $\mathcal{F}_0^{(q)}$-measurable for $2 \leq q \leq d$, hence $E(U_{i_1}^i f | \mathcal{F}_0^{(1)}) = E(U_{i_1}^i f | \mathcal{F}_0)$, $i \in \mathbb{Z}$. By (3b) and Proposition 1 we thus get

$$f = m_1 + g_1 - U_{i_1} g_1, \quad g_1 = \sum_{i=0}^{\infty} E(U_{i_1}^i f | \mathcal{F}_0), \quad m_1 = \sum_{i=0}^{\infty} P_0^{(i)} U_{i_1}^i f;$$

(3b) applies to $g_1, m_1$. By iterating the procedure we get (2).

Now let us suppose that (2) is true. In the same way as in the proof of Theorem 2 we can see that $f = m_1 + g_1 - U_{i_1} g_1$ where $g_1, m_1$ can be decomposed by (2), and by Proposition 1 we have $g_1 = \sum_{i=0}^{\infty} E(U_{i_1}^i f | \mathcal{F}_0), \quad m_1 = \sum_{i=0}^{\infty} P_0^{(i)} U_{i_1}^i f$. After having repeated the procedure for $g_1, m_1$ we will get the convergence of

$$\sum_{i_1=0}^{\infty} \cdots \sum_{i_d=0}^{\infty} E(U_{i_1}^{i_1}, \ldots, i_d f | \mathcal{F}_0)$$

is, however, not sufficient for (2). As an example we can consider a two dimensional adapted stationary random field (7) where $a_{i,j} = 0$ for $i \geq 1$, $a_{0,0} = 0$, and $a_{0,j} = 1/j$ for $j \geq 1$.

In what follows we will present two conditions which are sufficient (but not necessary) for (2). The first was proved by M. El Machkouri and D. Giraudo in [ElM16] and is formulated for adapted functions. The result can, nevertheless, be extended to the general (regular) case.

**Theorem 5 (El Machkouri, Giraudo).** If $f$ is adapted and

$$\sum_{i_1=0}^{\infty} \cdots \sum_{i_d=0}^{\infty} \|E(U_{i_1}^{i_1}, \ldots, i_d f | \mathcal{F}_0)\|_p < \infty$$

then the martingale-coboundary representation (2) holds.

Theorem 5 extends Theorem 2 from [Go69] to random fields. Its relation to Theorem 4 is the same as the relation of Theorem 2 from [Go69] to Theorem 2 from [V93].

Let us denote

$$\tilde{i} = \begin{cases} i & \text{if } i \leq -1, \\ i + 1 & \text{if } i \geq 0. \end{cases}$$
Theorem 6. Let $p \geq 2$. If
\begin{equation}
\sum_{i_1 \in \mathbb{Z}} \cdots \sum_{i_d \in \mathbb{Z}} i_1^2 i_2^2 \cdots i_d^2 \| P_{i_1, i_2, \ldots, i_d} f \|_p^2 < \infty
\end{equation}
then the martingale-coboundary decomposition (2) in $L^p$ holds.

Before proving Theorem 6, let us prove

Lemma 7. Let $d \in \mathbb{N}$ and $a_{i_1}, \ldots, a_{i_d} \in \mathbb{R}$, be real numbers for which
\[ \sum_{i_1=1}^{\infty} \cdots \sum_{i_d=1}^{\infty} i_{1,j}^2 i_{2,j}^2 \cdots i_{d,j}^2 < \infty. \]
Then for every $j \in \mathbb{N}^d$ the sum
\[ \sum_{i_1=j_1}^{\infty} \cdots \sum_{i_d=j_d}^{\infty} a_{i_1} a_{i_2} \cdots a_{i_d} \]
converges and
\begin{equation}
\sum_{j=1}^{\infty} \sum_{i=j}^{\infty} (\sum_{i=1}^{\infty} a_i )^2 \leq C \sum_{i=1}^{\infty} i_{1,j}^2 i_{2,j}^2 \cdots i_{d,j}^2
\end{equation}
where $C < \infty$ is a universal constant.

Remark 1. From Lemma 7 we deduce that (10) implies $\sum_{j \in \mathbb{Z}} \| P_{f,j} f \|_p < \infty$, i.e. Hannan’s condition in $L^p$ (cf. [WV14]). For $p = 2$ and $d = 1$ Hannan’s condition is independent of decomposition (2) (cf. [V93]).

Proof of Lemma 7. First, let us suppose $d = 1$, $j \geq 1$, and $1/2 < \alpha < 1$. We then have
\[ (\sum_{i=1}^{\infty} a_i )^2 = \left( \sum_{i=j}^{\infty} \frac{1}{i^{2\alpha}} a_i \right)^2 \leq \left( \sum_{i=j}^{\infty} \frac{1}{j^{2\alpha}} \right) \left( \sum_{i=j}^{\infty} i^{2\alpha} a_i^2 \right) \leq \frac{c}{j^{2\alpha-1}} \sum_{i=j}^{\infty} i^{2\alpha} a_i^2 \]
hence
\[ \sum_{j=1}^{\infty} (\sum_{i=j}^{\infty} a_i )^2 \leq c \sum_{j=1}^{\infty} \frac{1}{j^{2\alpha-1}} \sum_{i=j}^{\infty} i^{2\alpha} a_i^2 = c \sum_{i=1}^{\infty} \left( \sum_{j=1}^{\infty} \frac{1}{j^{2\alpha-1}} \right) i^{2\alpha} a_i^2 \leq C \sum_{i=1}^{\infty} i^{2\alpha} a_i^2 \]
for some constants $c, C$.

For $d \geq 2$ we can prove (11) by induction. We present this just for $d = 2$:
\[ \sum_{i,j} (\sum_{i=j}^{\infty} a_{i,j} )^2 \leq \frac{c}{j^{2\alpha-1}} \sum_{i,j} i_1^{2\alpha} (\sum_{i,j} a_{i,j} )^2 = \]
\[ c \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} i_1^{2\alpha} \left( \sum_{i=j}^{\infty} a_{i,j} \right)^2 \leq C \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} i_1^{2\alpha} a_{i,j}^2 \]
\[ \sum_{i,j} (\sum_{i=j}^{\infty} a_{i,j} )^2 \leq C' \sum_{i,j} i_1^{2\alpha} a_{i,j}^2. \]

Proof of Theorem 6. Let $d = 1$, suppose that $f$ is adapted. Let $J \subset \mathbb{Z}_+ = \{0, 1, \ldots\}$. Recall
\[ \sum_{j \in J} E(U^j f | F_0) = \sum_{i \leq 0} P_i \sum_{j \in J} E(U^j f | F_0) = \sum_{i \leq 0} (\sum_{j \in J} P_i E(U^j f | F_0)) = \sum_{i \leq 0} \sum_{j \in J} P_i U^j f. \]
By Burkholder’s inequality there is a $C_1$ such that

$$
\|\sum_{i \leq 0} \left( \sum_{j \in J} P_i U^j f \right) \|_p \leq C_1 \|\sum_{i \leq 0} \left( \sum_{j \in J} P_i U^j f \right)^2 \|_p^{1/2} \leq C_1 \left( \sum_{i \leq 0} \| \sum_{j \in J} P_i U^j f \|_p \right)^{1/2} \leq C_1 \left( \sum_{i \leq 0} \left( \sum_{j \in J} \| P_i U^j f \|_p^2 \right)^{1/2} \right)
$$

and by Lemma 7 there is a $C_2$ such that

$$
\sum_{i=0}^{\infty} \left( \sum_{j=1}^{\infty} \| P_0 U^j f \|_p \right)^2 \leq C_2 \sum_{i=0}^{\infty} i^2 \| P_0 U^j f \|_p^2.
$$

From this we deduce the convergence of $\sum_{j=0}^{\infty} E(U^j f | \mathcal{F}_0)$ (in $L^p$). In the same way we can prove the theorem for a (regular) non adapted $f$.

By induction the result can be extended to all $d \geq 1$. To show this, let us consider $d = 2$.

Denote $f_j = P^{(2)}_{-j} f$, $j \geq 0$. Thus $f = \sum_{j=0}^{\infty} f_j$ and assumption (10) implies

$$
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} i^2 j^2 \| P^{(1)}_{-i+1} f_{j-1} \|_p^2 < \infty.
$$

In particular, for $j \geq 0$ we have $\sum_{i=1}^{\infty} i^2 \| P^{(1)}_{-i+1} f_j \|_p^2 < \infty$. From the proof of Theorem 6 for $d = 1$ it follows that there exist $m_j, g_j \in L^p$, $j \geq 0$, such that

$$
f_j = m_j + g_j - U_{e_1} g_j, \quad P^{(2)}_{-j} m_j = m_j, \quad P^{(2)}_{-j} g_j = g_j, \quad P^{(1)}_{0} m_j = m_j;
$$

$$
\|g_j\|_p^2 \leq C \sum_{i=1}^{\infty} i^2 \| P^{(1)}_{-i} f_j \|_p^2.
$$

Because $\| P^{(1)}_{-1} f_j \|_p = || P^{(1)}_{0} U_{e_1} f_j \|_p = || P^{(1)}_{0} U_{-1} f \|_p$, and because by (10) we have $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} i^2 j^2 \| P^{(1)}_{-i} U_{-1} f_j \|_p^2$, we by Lemma 7 get $\sum_{j=0}^{\infty} \| g_j \|_p < \infty$. Therefore the series $g = \sum_{j=0}^{\infty} g_j$ converges in $L^p$.

Because $g_j = P^{(2)}_{-j} g$ and $\sum_{j=1}^{\infty} j^2 \| g_{j-1} \|_p^2 < \infty$ we get a martingale-coboundary decomposition $g = G - U_{e_2} G$, $G \in L^p$.

In the same way we get $m = \sum_{j=0}^{\infty} m_j = M - U_{e_2} M$, $M \in L^p$. This results in a martingale-coboundary representation (2) for $f$.

\[ \square \]

**Remark 2.** For $p = 2$, the assumption of Theorem 6 follows from Gordin - El Machkouri - Giraudo’s condition (assumption of Theorem 5).

**Proof.** It is sufficient to study the adapted case only because the proof of the general case is similar. Let $f = \sum_{i=0}^{\infty} \sum_{i=0}^{\infty} a_{t,i} U^{-i} e_t$, $\sum_{i=0}^{\infty} \sum_{i=0}^{\infty} a_{t,i}^2 < \infty$, $P_0 e_t = e_t$, $\| e_t \|_2 = 1$, and $e_t$ are mutually orthogonal. Let us suppose

$$
\sum_{j=0}^{\infty} \| E(U^j f | \mathcal{F}_0) \|_2 = \sum_{j=0}^{\infty} \left( \sum_{i=0}^{\infty} \sum_{i=j}^{\infty} a_{t,i}^2 \right)^{1/2} < \infty.
$$
we have
\[ \sum_{i=0}^{\infty} \bar{i}^2 \| P_{-i} f \|_2^2 = \sum_{l=0}^{\infty} \sum_{i=0}^{\infty} \bar{i}^2 a_{i,i}^2 \]
where \( \bar{i} = i + 1 \). For \( k \geq 0 \) let us denote \( b_k = (\sum_{i=0}^{\infty} \sum_{i=k}^{\infty} a_{i,i}^2)^{1/2} \); we thus have \( \| E(U^1 f | F_0) \|_2 = b_j \). Then

(12) \[
\sum_{k=0}^{\infty} \bar{k}b_k^2 = \sum_{l=0}^{\infty} \sum_{i=0}^{\infty} (\sum_{k=1}^{\bar{i}} k) a_{i,i}^2 \geq C \sum_{l=0}^{\infty} \sum_{i=0}^{\infty} \bar{i}^2 a_{i,i}^2
\]
and

(13) \[
\sum_{k=0}^{\infty} \bar{k}b_k^2 = \sum_{l=0}^{\infty} \sum_{i=0}^{\infty} \bar{k}b_k^{3/2} b_k^{1/2} \leq \sqrt{\sum_{k=0}^{\infty} \bar{k}b_k^3} \sqrt{\sum_{k=0}^{\infty} b_k}.
\]
Let us suppose \( \sum_{k=0}^{\infty} b_k < \infty \). By Abel’s summation

(14) \[
\sum_{k=1}^{n} ((k + 1) - k) b_k = (n + 1) b_{n+1} - b_1 - \sum_{k=1}^{n} (k + 1)(b_{k+1} - b_k).
\]
The sequence of \( b_n \) is decreasing and the sums on the left converge. If \( \sup_n nb_n = \infty \) then the series \( \sum_{k=1}^{n} (k + 1)(b_{k+1} - b_k) \) diverges and so \( nb_n \to \infty \). This contradicts \( \sum_{k=1}^{\infty} b_k < \infty \). Therefore, \( \sup_n nb_n < \infty \). From this and (13) it follows \( \sum_{k=1}^{\infty} k b_k^2 < \infty \).

Before passing to higher dimension let us notice that \( \sum_{k=1}^{\infty} b_k < \infty \) implies \( \lim \inf_{k \to \infty} k b_k = 0 \). In (14) we thus get \( \sum_{k=1}^{\infty} (k + 1)(b_{k+1} - b_k) = b_1 + \sum_{k=1}^{\infty} b_k \) hence

\[ nb_n \leq 2b_1 + 2 \sum_{k=1}^{\infty} b_k \leq 4 \sum_{k=1}^{\infty} b_k, \quad n \geq 1. \]

To present a proof for \( d > 1 \), let us consider \( d = 2 \) (for \( d > 2 \) the proof is analogical). We suppose that \( f = \sum_{l=0}^{\infty} \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} a_{l_1,i_1,i_2} U_{-i_1,i_2} e_l \) where \( e_l \) are mutually orthogonal, \( F_0 e_l = e_l, \| e_l \|_2 = 1 \), and \( \sum_{l=0}^{\infty} \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} a_{l_1,i_1,i_2}^2 < \infty \). Like in the one dimensional case we denote

\[ b_{j_1,j_2}^2 = \sum_{l=0}^{\infty} \sum_{i_1=j_1}^{\infty} \sum_{i_2=j_2}^{\infty} a_{l_1,i_1,i_2}^2 = \| E(U_{j_1,j_2} f | F_0,0) \|_2^2. \]

We have

\[ \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \bar{i_1}_1^2 \bar{i_2}_2^2 \| P_{-i_1,i_2} f \|_2^2 = \sum_{l=0}^{\infty} \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \bar{i_1}_1^2 \bar{i_2}_2^2 a_{l_1,i_1,i_2}^2 \]
and using the same idea as in (12) and (13)

\[ \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \bar{j}_1 \bar{j}_2 b_{j_1,j_2}^2 = \sum_{l=0}^{\infty} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \bar{j}_1 \bar{j}_2 \sum_{i_1=j_1}^{\infty} \sum_{i_2=j_2}^{\infty} a_{l_1,i_1,i_2}^2 = \]

\[ \sum_{l=0}^{\infty} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \bar{j}_1 \sum_{j_2=0}^{\infty} \sum_{i_2=j_2}^{\infty} a_{l_1,i_1,i_2}^2 \geq \sum_{l=0}^{\infty} \sum_{j_1=0}^{\infty} \sum_{i_1=j_1}^{\infty} \bar{j}_1 (C \sum_{i_2=0}^{\infty} \sum_{j_2=0}^{\infty} a_{l_1,i_1,i_2}^2 \geq \]

\[ C^2 \sum_{l=0}^{\infty} \sum_{j_1=0}^{\infty} \sum_{i_2=0}^{\infty} \bar{j}_1 \bar{j}_2 a_{l_1,i_1,i_2}, \]

12
\[
\sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} \tilde{e}_{j_1} \tilde{e}_{j_2} b_{j_1,j_2}^2 = \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} \tilde{e}_{j_1} \tilde{e}_{j_2} b_{j_1,j_2}^{3/2} b_{j_1,j_2}^{1/2} \leq \left( \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} \tilde{e}_{j_1} \tilde{e}_{j_2} b_{j_1,j_2}^3 \right)^{1/2} \left( \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} b_{j_1,j_2} \right)^{1/2}.
\]

Let us suppose that \( \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} b_{j_1,j_2} < \infty \). In the same way as before we prove that for every \( j_1 \geq 1 \), \( j_1 \sup_{j_2 \geq 1} j_2 b_{j_1,j_2} \leq 4 j_1 \sum_{j_2=1}^{\infty} b_{j_1,j_2} = j_1 B_{j_1} \). Because \( \sum_{j_1=1}^{\infty} B_{j_1} < \infty \) we in an analogous way as in the one dimensional case deduce \( \sup_{j_1,j_2 \geq 1} \tilde{e}_{j_1} \tilde{e}_{j_2} b_{j_1,j_2} < \infty \). Therefore, \( \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \tilde{e}_{j_1} \tilde{e}_{j_2} b_{j_1,j_2}^2 < \infty \).

\[ \square \]

**Remark 3.** There exists a dynamical system \((\Omega, \mathcal{A}, T, \mu)\) with an increasing filtration \( F_i = T^{-i} \mathcal{F} \) and an \( f \in L^2 \) such that \( f = \sum_{i=0}^{\infty} a_i U^{-i} e, \sum_{i=0}^{\infty} a_i^2 < \infty, e \in L^2 \), and \((U^i e)\) is a martingale difference sequence,

\[
\sum_{k=0}^{\infty} \|E(f|\mathcal{F}_{-k})\|_2 = \infty, \text{ and } \sum_{k=0}^{\infty} k^2 \|P_{-k} f\|_2^2 = \sum_{k=0}^{\infty} k^2 a_k^2 < \infty.
\]

**Proof.** Let \((U^i e)\) be a martingale difference sequence. We define \( n_k = 2^k, k \geq 0 \), \( \epsilon_0 = 1 \), and \( \epsilon_k = 2^{-k}/k, k \geq 1 \). Let \( a_i = \epsilon_k > 0 \) for \( i = n_k \) and \( a_i = 0 \) for all other \( i \). For \( f = \sum_{i \geq 0} a_i U^{-i} e \) it then holds \( \sum_{i \leq 0} i^2 \|P_i f\|_2^2 = \sum_{i = 0}^{\infty} i^2 a_i^2 = \sum_{k=0}^{\infty} n_k^2 \epsilon_k^2 < \infty \) and \( \sum_{i = 0}^{\infty} \|E(f|\mathcal{F}_{-i})\|_2 \geq \sum_{k=0}^{\infty} n_k \epsilon_k = \infty \).

\[ \square \]

3. Applications.

Martingale-coboundary decomposition has played an important role in the study of limit theorems for stationary sequences of random variables. For random fields, the martingale-coboundary representation has proved useful as well.

1. Invariance principle. We are interested in the weak convergence of normalized sums \( (1/n^d) \sum_{1 \leq i \leq n^d} U_i f, n^d = (n_1 t, \ldots, n_d t) \), to a Brownian sheet in \( D[0, 1]^d \). We will call this case the weak invariance principle (WIP).

For proving a WIP we first need a central limit theorem for fields of martingale differences. If \( d \geq 2 \) the CLT, however, does not need to hold even in the case of an ergodic field of orthomartingale differences (cf. [WaWo13]). The CLT is true if one of the generating transformations \( T_{e_j} \) is ergodic, cf. [V15]. In particular, this assumption is valid if the \( \mathbb{Z}^d \) action is Bernoulli, i.e. the \( \sigma \)-algebra \( \mathcal{A} \) is generated by iid random variables \( e \circ T_{i}^j, j \in \mathbb{Z}^d \), and the filtration is defined by \( \mathcal{F}_{i} = \sigma\{ e \circ T_{j} : j \leq i \} \). (The filtration is completely commuting then.)

A WIP under Hannan’s condition for random fields was proved in [VWa14]. In [EIG16] El Machkouri and Giraudo proved that assumptions of Theorem 5 imply a WIP. The next theorem extends their result.
Theorem 8. If the representation (2) applies and all components are square integrable then the WIP holds.

The proof is, in fact, included in [VWa14], the proof of Theorem 5.1. In that paper a martingale-coboundary representation (2) with all terms in $L^2$ is proved for $f = \sum P_i f$ where in the sum, only finitely many terms are nonzero (this follows from each of the Theorems 2, 4, 5, 6 here). The WIP for such functions is proved and the proof uses square integrability and (2) only.

In [V93] it is exposed that martingale-coboundary representation and Hannan’s condition are independent. Therefore, Theorem 7 does not follow from the WIP proved in [VWa14].

For $d = 1$ the martingale-coboundary representation is “suboptimal” because WIP follows from Dedecker-Rio’s and Maxwell-Woodroofe’s conditions which are weaker. For $d \geq 2$ we, nevertheless, do not know an analogue to the conditions of Dedecker-Rio or Maxwell-Woodroofe. The version of Maxwell-Woodroofe’s condition used in [WaWo13] implies Hannan’s condition.

2. Probabilities of large deviations. Let $(X_i)$ be a strictly stationary sequence of martingale differences, $X_i \in L^p$, $p \geq 2$. By [D01, Proposition 1(a)]

$$E|\sum_{i=1}^{n} X_i|^p \leq (2p)^{p/2} n^{p/2} \|X_1\|_p^p.$$  

This improves Burkholder inequality as presented in [HaHe] and also the inequality $E|\sum_{i=1}^{n} X_i|^p \leq (18p^{1/2})^{p/2} n^{p/2} E|X_1|^p$ in [LV01, p.150].

Let $X_i$, $1 \in \mathbb{Z}^d$, be strictly stationary orthomartingale differences. For $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$, $\sum_{i_1=1}^{n_1} \cdots \sum_{i_{d-1}=1}^{n_{d-1}} X_{i_1,\ldots,i_{d-1}}$, $i_d = 1, \ldots, n_d$, are stricly stationary martingale differences. Consequently, by (15)

$$E|\sum_{i_d=1}^{n_d} (\sum_{i_1=1}^{n_1} \cdots \sum_{i_{d-1}=1}^{n_{d-1}} X_{i_1,\ldots,i_{d-1}})|^p \leq (2p)^{p/2} n^{p/2} \|X_1\|_p^p \cdots \|X_{i_{d-1},\ldots,i_{d-1}}\|_p^p.$$  

By iterating this approach we deduce

$$E|\sum_{i_d=1}^{n_d} (\sum_{i_1=1}^{n_1} \cdots \sum_{i_{d-1}=1}^{n_{d-1}} X_{i_1,\ldots,i_{d-1}})|^p \leq (2p)^{dp/2} n^{dp/2} \|X_1\|_p^p \cdots \|X_{i_{d-1},\ldots,i_{d-1}}\|_p^p.$$  

Let us suppose (2) with $g_S \in L^p$, $S \subset \{1, \ldots, d\}$. Let $S = \{q_1, \ldots, q_r\} \subset \{1, \ldots, d\}$. Because for each $q \in S$, $U_{\epsilon_q} g_S$ are martingale differences, for $1 \leq n_1, \ldots, n_r < \infty$ it holds

$$E|\sum_{i_1=1}^{n_1} \cdots \sum_{i_r=1}^{n_r} U_{\epsilon_{q_1}} \cdots U_{\epsilon_{q_r}} g_S|^p \leq (2p)^{rp/2} (n_1 \cdots n_r)^{p/2} \|g_S\|_p^p$$  

so that

$$\|\prod_{q \in S^c} (I - U_{\epsilon_q}) g_S\|_p \leq (2p)^{rp/2} 2^{d-r} n^{1/2} \|g_S\|_p.$$  

We thus get the following theorem (the second statement is deduced with the help of Markov Inequality).
Theorem 9.

\begin{align}
&\| \sum_{1 \leq i \leq n} U_i f \|_p \leq 2^{2d} d^{d/2} |\mathbb{N}|^{1/2} \sum_{S \subset \{1, \ldots, d\}} \| g_S \|_p, \tag{17} \\
&\mu\left( \sum_{1 \leq i \leq n} U_i f > x |\mathbb{N}| \right) \leq 2^{2dp} d^{dp/2} \left( \sum_{S \subset \{1, \ldots, d\}} \| g_S \|_p \right)^p x^{-p} |\mathbb{N}|^{-p/2}. \tag{18}
\end{align}

For the dimension \( d = 1 \) an estimate similar to (18) was found in [LV01]; stationarity is not needed there (only uniformly bounded \( L^p \) norms). Under the assumptions of Theorem 5, estimates similar to (17), (18) were proved in [ElG16].

Using the same ideas as in [ElG16] the results can be extended to Orlicz spaces.

Let us recall (cf. [KR]) that for a Young function \( \psi \) (a real convex nondecreasing function on \( \mathbb{R}_+ \), \( \psi(0) = 0 \), \( \lim_{x \to \infty} \psi(x) = \infty \)) the Orlicz space \( L_\psi \) associated to \( \psi \) is the space of all random variables \( Z \) such that for some \( c > 0 \), \( E(\psi(Z/c)) < \infty \). The Luxemburg norm \( \| Z \|_\psi \) of \( Z \) is then defined by \( \| Z \|_\psi = \inf\{ c > 0 : E(\psi(Z/c)) \leq 1 \} \).

\( (L_\psi, \| \cdot \|_\psi) \) is a Banach space.

Let us define \( h_\alpha = ((1 - \alpha)/\alpha)^{1/\alpha}1_{0 < \alpha < 1} \) and a Young function \( \psi_\alpha(x) = \exp((x + h_\alpha)^\alpha) - \exp(h_\alpha^\alpha) \).

In the same way as in [ElVWu13, Lemma 4] we can deduce, for \( \psi_\alpha \) defined above, \( 0 < q < 2/d \), \( \beta(q) = 2q/(2 - dq) \), and for a positive constant \( C \) depending only on \( d \) and \( q \), the following estimates.

Theorem 10.

\begin{align}
&\| \sum_{1 \leq i \leq n} U_i f \|_q \leq C |\mathbb{N}|^{1/2} \sum_{S \subset \{1, \ldots, d\}} \| g_S \|_{\psi_\beta q}, \\
&\mu\left( \sum_{0 \leq i \leq n} U_i f > x \right) \leq (1 + e^{h_q}) \exp\left( -\left( \frac{x}{C |\mathbb{N}|^{1/2} \sum_{S \subset \{1, \ldots, d\}} \| g_S \|_{\psi_\beta q} + h_q} \right)^q \right).
\end{align}

If \( x = |\mathbb{N}| \) and \( q = 2/3 \) we get an estimate of ordre \( C \exp(-c|\mathbb{N}|^{1/3}) \). For \( d = 1 \) it is \( \mu(S_n \geq n) \leq C \exp(-cn^{1/3}) \) which was found, like (18), in [LV01] where only uniform boundedness of moments is needed. In [LV01] it is proved that for strictly stationary sequences of martingale differences both estimations are (up to a constant and for \( d = 1 \)) optimal.

More applications can be found in the field of reverse martingale approximation for noninvertible commuting transformations, cf. [DeGo14] and [CDV15].

REFERENCES

[CCo13] Cohen, G. and Conze, J.-P., The CLT for rotated ergodic sums and related processes, Discrete and Continuous Dynamical Systems - Series A, American Institute of Mathematical Sciences 33(9) (2013), 3981-4002.

[CuDV15] Cuny, Ch, Dedecker, J., and Volný, D., A functional CLT for fields of commuting transformations via martingale approximation, Zap. Nauchn. Sem. POMI 441 (2015), 239-262.
Dedecker, J., *Exponential inequalities and functional central limit theorem for random fields*, ESAIM Probability and Statistics 5 (2001), 77-104.

Dedecker, J. and Rio, E., *On the functional limit theorem for stationary processes*, Ann. IHP 36 (2000), 1-34.

Denker, M. and Gordin, M.I., *Limit theorems for von Mises statistics of a measure preserving transformation*, Probability Theory and Related Fields 160(1) (2014), 1-45.

Durieu, O. and Volný, D., *Comparison between criteria leading to the weak invariance principle*, Ann. IHP 44 (2008), 324-340.

El Machkouri, M. and Giraudo, D., *Orthomartingale-coboundary decomposition for stationary random fields*, Stoch. Dyn. 16 (2016).

El Machkouri, M., Volný, D., and Wu, W.B., *A central limit theorem for stationary random fields*, Stochastic Process. Appl. 123 (2013), 1-14.

Esseen, C.G. and Janson, S., *On moment conditions for sums of independent variables and martingale differences*, Stoch. Proc. Appl. 19 (1985), 173-182.

Giraudo, D. and Volný, D., *A strictly stationary $\beta$-mixing process satisfying the central limit theorem but not the weak invariance principle*, Stochastic Process. Appl. 124 (2014), 3769-3781.

Gordin, M.I., *The central limit theorem for stationary processes*, Sov. Math. Dokl. 10 (1969), 174-176.

Gordin, M.I., *Martingale-coboundary representation for a class of random fields*, Zap. Nauchn. Sem. POMI 364 (2009), 88-108.

Gordin, M.I., *The Central Limit Theorem for stationary processes without the finiteness of variance assumption*, The First Vilnius International Conference on Probab. Theory and Math. Stat., Vilnius 1 (1973), 173-174.

Gordin, M.I. and Lifshits, *A remark on Markov processes with normal transition operators (in Russian)*, The Third Vilnius International Conference on Probab. Theory and Math. Stat., Vilnius 1 (1981), 147-148.

Hall, P and Heyde, C.C., *Martingale Limit Theory and its Application*, Academic Press, New York, 1980.

Heyde, C.C., *On the central limit theorem and iterated logarithm law for stationary processes*, Bull. Austral. Math. Soc. 12 (1975), 1-8.

Lesigne, E. and Volný, D., *Large deviations for martingales*, Stoch. Proc. Appl. 96 (2001), 143-159.

Krasnosel’ski, M.A. and Rutickii, Y.B., *Convex functions and Orlicz Spaces*, P. Noordhoff LTD-Groningen-The Netherlands, 1961.

Maxwell, M. and Woodroofe, M., *Central limit theorems for additive functionals of Markov chains*, Ann. Probab. 28 (2000), 713-724.

Volný, D., *Approximating martingales and the CLT for strictly stationary processes*, Stochastic Process. Appl. 44 (1993), 41-74.

Volný, D. and Samek, P., *On the invariance principle and the law of iterated logarithm for stationary processes*, Mathematical Physics and Stochastic Analysis. (Essays in Honour of Ludwig Streit) Eds. S. Albeverio, Ph. Blanchard, L. Ferreira, T. Hida, Y. Kokndratiev, R. Vilela Mendes (2000), World Scientific Publ. Co., Singapore, Hong Kong, 424-438.

Volný, D., *Martingale approximation of non-stationary stochastic processes*, Stoch. Dyn. 6(2) (2006), 173-183.

Volný, D., *A central limit theorem for fields of martingale differences C. R. Math. Acad. Sci. Paris 353* (2015), 1159-1163.

Volný, D., Woodroofe, M. and Zhao, O., *Central limit theorems for superlinear processes*, Stoch. Dyn. 11 (2011), 71-80.

Volný, D. and Wang, Y., *An invariance principle for stationary random fields under Hannan’s condition*, Stoch. Proc. Appl. 124(12) (2014), 4012-4029.

Wang, Y. and Woodroofe, M., *A new condition for the invariance principle for stationary random fields*, Statistica Sinica 23 (2013), 1673-1696.