Strong Hopf modules for weak Hopf quasigroups

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Abstract In this paper we introduce the category of strong Hopf modules for a weak Hopf quasigroup $H$ in a braided monoidal category. We also prove that this category is equivalent to the category of right modules over the image of the target morphism of $H$.

Keywords. Weak Hopf algebra, Hopf quasigroup, bigroupoid, Strong Hopf module, Fundamental theorem of Hopf modules.

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1. Introduction

Let $\mathbb{F}$ be a field and $\mathcal{C} = \mathbb{F} - \text{Vect}$. Let $M$ be a right $H$-module and a right $H$-comodule. If, for all $m \in M$ and $h \in H$, we write $m.h$ for the action and we use the Sweedler notation $\rho_M(m) = m_{[0]} \otimes m_{[1]}$ for the coaction, we will say that $M$ is a Hopf module if the equality

$$\rho_M(m.h) = m_{[0]}h(1) \otimes m_{[1]}h(2)$$

holds, where $\delta_H(h) = h(1) \otimes h(2)$ is the coproduct of $H$ and $m_{[1]}h(2)$ the product in $H$ of $m_{[1]}$ and $h(2)$. A morphism between two Hopf modules is $\mathbb{F}$-linear and $H$-colinear. Hopf modules and morphisms of Hopf modules constitute the category of Hopf modules denoted by $\mathcal{M}_H$. In 1969 Larson and Sweedler proved a result, called Fundamental Theorem of Hopf Modules, that asserts the following: If $M \in \mathcal{M}_H$ and $M^{coH} = \{m \in M \mid \rho_M(m) = m \otimes 1_H\}$ are the coinvariants of $H$ in $M$, $M$ is isomorphic to $M^{coH} \otimes H$ as Hopf modules (see [11] and [16]). On the other hand, if $N$ is a $\mathbb{F}$-vector space, the tensor product $N \otimes H$, with the action and coaction induced by the product and the coproduct of $H$, is a Hopf module. This construction is functorial, so we have a functor $F = - \otimes H : \mathcal{C} \to \mathcal{M}_H$. Also, for all $M \in \mathcal{M}_H$, the construction of $M^{coH}$ is functorial and we have a functor $G = (\cdot)^{coH} : \mathcal{M}_H \to \mathcal{C}$ and $F \dashv G$. Moreover, $F$ and $G$ is a pair of inverse equivalences, and therefore, $\mathcal{M}_H$ is equivalent to the category of $\mathbb{F}$-vector spaces.

The Fundamental Theorem of Hopf Modules also holds for weak Hopf algebras as was proved by Böhm, Nill and Szlachányi in [5]. In this case, if $H$ is a weak Hopf algebra, the category of Hopf modules is defined in the same way as in the Hopf algebra setting. For $M \in \mathcal{M}_H$, the coinvariants of $H$ in $M$ are defined by $M^{coH} = \{m \in M \mid \rho_M(m) = m_{[0]} \otimes \Pi_H(m_{[1]})\}$, where $\Pi_H$ is the target morphism associated to $H$. Then, Böhm, Nill and Szlachányi proved that $M$ is isomorphic to $M^{coH} \otimes H_L H$ as Hopf modules, where $H_L$ is the image of $\Pi_H$. Moreover, if $\mathcal{C}_{H_L}$ is the category of right $H_L$-modules, there exist two functors $F = - \otimes_{H_L} H : \mathcal{C}_{H_L} \to \mathcal{M}_H$ and $G = (\cdot)^{coH} : \mathcal{M}_H \to \mathcal{C}_{H_L}$ such that $F$ is left adjoint of $G$ and they induce a pair of inverse equivalences (see [7]). Therefore, in the weak setting, $\mathcal{M}_H$ is equivalent to $\mathcal{C}_{H_L}$. In this case, is a relevant fact the following property: the tensor product $M^{coH} \otimes H$ is isomorphic as Hopf modules to $M^{coH} \times H$ where $M^{coH} \times H$ is the image of a suitable idempotent $\nabla_M = M^{coH} \otimes H \to M^{coH} \otimes H$. Note that, as a consequence, in the weak framework the Fundamental Theorem of Hopf Modules can be written using $M^{coH} \times H$ instead of $M^{coH} \otimes H$. 
In the two previous paragraphs we spoke about associative algebraic structures like Hopf algebras and weak Hopf algebras. Recently Klim and Majid introduced in [10] the notion of Hopf quasigroup as a generalization of Hopf algebras in the context of non-associative algebra, in order to understand the structure and relevant properties of the algebraic 7-sphere. A Hopf quasigroup is a particular instance of unital coassociative $H$-bialgebra in the sense of Pérez Izquierdo [14], and it includes as example the enveloping algebra of a Malcev algebra, when the base ring has characteristic not equal to 2 nor 3. In this sense Hopf quasigroups extend the notion of Hopf algebra in a parallel way that Malcev algebras extend the one of Lie algebra. On the other hand, it also contains as an example the notion of quasigroup algebra of an I.P. loop. Therefore, Hopf quasigroups unify I.P. loops and Malcev algebras in the same way that Hopf algebras unify groups and Lie algebras. For these non-associative algebraic structures, Brzeziński introduced in [6] the notion of Hopf module and he proved a version of the Fundamental Theorem of Hopf Modules. In this case, the main difference appears in the definition of the category of Hopf modules $\mathcal{M}_H$, because the notion of Hopf module reflects the non-associativity of the product defined on $H$, and the morphisms are $H$-quasilinear and $H$-colinear (see Definition 3.4 of [6]). In Lemma 3.5 of [6], we can find that, if $M \in \mathcal{M}_H^H$ and $M^{coH}$ is defined like in the Hopf algebra setting, $M$ is isomorphic to $M^{coH} \otimes H$ as Hopf modules. Moreover, there exist two functors $F = - \otimes H : C \to \mathcal{M}_H^H$ and $G = (\ )^{coH} : \mathcal{M}_H^H \to C$ such that $F \dashv G$, and they induce a pair of inverse equivalences. Therefore, in this non-associative context $\mathcal{M}_H^H$ is equivalent to the category of $\mathbb{F}$-vector spaces as in the Hopf algebra ambit.

Working in a monoidal setting, in [2] we introduce the notion of weak Hopf quasigroup as a new Hopf algebra generalization that encompasses weak Hopf algebras and Hopf quasigroups. A family of non-trivial examples of these algebraic structures can be obtained working with bigroupoids, i.e., bicategories where every 1-cell is an equivalence and every 2-cell is an isomorphism (see Example 2.3 of [2]). For a weak Hopf quasigroup $H$ in a braided monoidal category $C$ with tensor product $\otimes$, using the ideas proposed by Brzeziński for Hopf quasigroups, in [2] we introduce the notion of Hopf module and the category of Hopf modules $\mathcal{M}_H^H$. In this case, if we define $M^{coH}$ in the same way as in the weak Hopf algebra setting, we obtain a version of the Fundamental Theorem of Hopf Modules in the following way: all Hopf module $M$ is isomorphic to $M^{coH} \times H$ as Hopf modules, where $M^{coH} \times H$ is the image of the same idempotent $\nabla_M$ used for Hopf modules associated to a weak Hopf algebra. Moreover, in [3] we proved that $H_L$, the image of the target morphism, is a monoid and then it is possible to take into consideration the category $C_{H_L}$, to construct the tensor product $M^{coH} \otimes_{H_L} H$, and, if the functor $- \otimes H$ preserves coequalizers, to endow this object with a Hopf module structure. Unfortunately, it is not possible to assure that $M^{coH} \otimes_{H_L} H$ is isomorphic to $M^{coH} \times H$ as in the weak Hopf algebra case. In this paper we find the conditions under which these objects are isomorphic in $\mathcal{M}_H^H$. Then, as a consequence, we introduce the category of strong Hopf modules, denoted by $SM_{H_L}^H$ and we obtain that there exist two functors $F = - \otimes_{H_L} H : C_{H_L} \to SM_{H_L}^H$ and $G = (\ )^{coH} : SM_{H_L}^H \to C_{H_L}$ such that $F$ is left adjoint of $G$ and they induce a pair of inverse equivalences. In the Hopf quasigroup setting all Hopf module is strong, and then our results are the ones proved by Brzeziński in [6]. Also, in the weak Hopf case, all Hopf module is strong and then we generalize the theorem proved by Böhm, Nill and Szlachányi in [5].

2. Weak Hopf Quasigroups

Throughout this paper $C$ denotes a strict braided monoidal category with tensor product $\otimes$, unit object $K$ and braiding $c$. For each object $M$ in $C$, we denote the identity morphism by $id_M : M \to M$ and, for simplicity of notation, given objects $M$, $N$ and $P$ in $C$ and a morphism $f : M \to N$, we write $P \otimes f$ for $id_P \otimes f$ and $f \otimes P$ for $f \otimes id_P$. We want to point out that there is no loss of generality in assuming that $C$ is strict because by Theorem 3.5 of [9] (which implies the Mac Lane’s coherence theorem) every monoidal category is monoidally equivalent to a strict one. This lets us to treat monoidal categories as if they were strict and, as a consequence, the results proved in this paper hold for every non-strict symmetric monoidal category.

From now on we also assume in $C$ that every idempotent morphism splits, i.e., if $\nabla : Y \to Y$ is such that $\nabla = \nabla \circ \nabla$, there exist an object $Z$ and morphisms $i : Z \to Y$ and $p : Y \to Z$ such that $\nabla = i \circ p$
and \( p \circ i = id_Z \). Note that, in these conditions, \( Z, p \) and \( i \) are unique up to isomorphism. There is no loss of generality in assuming that \( C \) admits split idempotents, taking into account that, for a given category \( C \), there exists a universal embedding \( C \rightarrow \tilde{C} \) such that \( C \) admits split idempotents, as was proved in \cite{8}. The categories satisfying this property constitute a broad class that includes, among others, the categories with epimonic decomposition for morphisms and categories with equalizers or coequalizers.

**Definition 2.1.** By a unital magma in \( C \) we understand a triple \( A = (A, \eta_A, \mu_A) \) where \( A \) is an object in \( C \) and \( \eta_A : K \rightarrow A \) (unit), \( \mu_A : A \otimes A \rightarrow A \) (product) are morphisms in \( C \) such that \( \mu_A \circ (A \otimes \eta_A) = id_A = \mu_A \circ (\eta_A \otimes A) \). If \( \mu_A \) is associative, that is, \( \mu_A \circ (A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes A) \), the unital magma will be called a monoid in \( C \). Given two unital magmas (monoids) \( A = (A, \eta_A, \mu_A) \) and \( B = (B, \eta_B, \mu_B) \), \( f : A \rightarrow B \) is a morphism of unital magmas (monoids) if \( \mu_B \circ (f \otimes f) = f \circ \mu_A \) and \( f \circ \eta_A = \eta_B \).

By duality, a counital comagma in \( C \) is a triple \( D = (D, \varepsilon_D, \delta_D) \) where \( D \) is an object in \( C \) and \( \varepsilon_D : D \rightarrow K \) (counit), \( \delta_D : D \rightarrow D \otimes D \) (coproduct) are morphisms in \( C \) such that \( (\varepsilon_D \otimes D) \circ \delta_D = id_D = (D \otimes \varepsilon_D) \circ \delta_D \). If \( \delta_D \) is coassociative, that is, \( (\delta_D \otimes D) \circ \delta_D = (D \otimes \delta_D) \circ \delta_D \), the counital comagma will be called a comonoid. If \( D = (D, \varepsilon_D, \delta_D) \) and \( E = (E, \varepsilon_E, \delta_E) \) are counital comagmas (comonoids), \( f : D \rightarrow E \) is a morphism of counital comagmas (comonoids) if \( (f \otimes f) \circ \delta_D = \delta_E \circ f \) and \( \varepsilon_E \circ f = \varepsilon_D \).

If \( A, B \) are unital magmas (monoids) in \( C \), the object \( A \otimes B \) is a unital magma (monoid) in \( C \) where \( \eta_{A \otimes B} = \eta_A \otimes \eta_B \) and \( \mu_{A \otimes B} = (\mu_A \otimes \mu_B) \circ (A \otimes B, A \otimes B) \). In a dual way, if \( D, E \) are counital comagmas (comonoids) in \( C \), \( D \otimes E \) is a counital comagma (comonoid) in \( C \) where \( \varepsilon_{D \otimes E} = \varepsilon_D \otimes \varepsilon_E \) and \( \delta_{D \otimes E} = (D \otimes \delta_D, E) \circ (\delta_D \otimes \delta_E) \).

Finally, if \( D \) is a comagma and \( A \) a magma, given two morphisms \( f, g : D \rightarrow A \) we will denote by \( f * g \) its convolution product in \( C \), that is

\[ f * g = \mu_A \circ (f \otimes g) \circ \delta_D. \]

Now we recall the notion of weak Hopf quasigroup in a braided monoidal category that we introduced in \cite{2}.

**Definition 2.2.** A weak Hopf quasigroup \( H \) in \( C \) is a unital magma \( (H, \eta_H, \mu_H) \) and a comonoid \( (H, \varepsilon_H, \delta_H) \) such that the following axioms hold:

\[
\begin{align*}
(a1) \quad & \delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ \delta_H. \\
(a2) \quad & \varepsilon_H \circ \mu_H \circ (\mu_H \otimes H) = \varepsilon_H \circ \mu_H \circ (H \otimes \mu_H) \\
& = ((\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)) \circ (H \otimes \delta_H \otimes H) \\
& = ((\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)) \circ (H \otimes (\varepsilon_H \circ \delta_H) \otimes H). \\
(a3) \quad & (\delta_H \otimes H) \circ \delta_H \circ \eta_H = (H \otimes \delta_H \otimes H) \circ ((\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H)) \\
& = (H \otimes (\mu_H \circ c_{H,H}^H) \otimes H) \circ ((\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H)). \\
(a4) \quad & \text{There exists } \lambda_H : H \rightarrow H \text{ in } C \text{ (called the antipode of } H) \text{ such that, if we denote the morphisms} \\
& \text{id}_H \ast \lambda_H \text{ by } \Pi^L_H \text{ (target morphism) and } \lambda_H \ast \text{id}_H \text{ by } \Pi^R_H \text{ (source morphism),} \\
& \text{we have the monoidal version of the original definition of weak Hopf algebra introduced by Böhm,} \\
& \text{Nill and Szlachányi in } \cite{5}. \text{ On the other hand, under these conditions, if } \varepsilon_H \text{ and } \delta_H \text{ are morphisms of} \\
& \text{unital magmas (equivalently, } \eta_H, \mu_H \text{ are morphisms of counital comagmas), } \Pi^L_H = \Pi^R_H = \eta_H \otimes \varepsilon_H. \text{ As} \\
& \text{a consequence, conditions } (a2), (a3), (a4)-(a4-3) \text{ trivialize, and we get the notion of Hopf quasigroup} \\
& \text{defined by Klim and Majid in } \cite{10} \text{ in the category of vector spaces over a field } \mathbb{F}. 
\end{align*}
\]
Below we will summarize the main properties of weak Hopf quasigroups. There are more, and the interested reader can see a complete list with the proofs in [2].

First note that, by Propositions 3.1 and 3.2 of [2], the following equalities
\[ \Pi_H^L \ast id_H = id_H \ast \Pi_H^R = id_H, \]
hold. Moreover, the antipode is unique, \( \lambda_H \circ \eta_H = \eta_H \circ \lambda_H = \varepsilon_H \), and, by Theorem 3.19 of [2], we have that it is anticommutative and anticomultiplicative. Also, if we define the morphisms \( \Pi_H^L \) and \( \Pi_H^R \) by \( \Pi_H^L = (H \otimes (\varepsilon_H \circ \mu_H)) \circ ((\delta_H \circ \eta_H) \otimes H) \), \( \Pi_H^R = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)) \), we proved in Proposition 3.4 of [2], that \( \Pi_H^L, \Pi_H^R, \Pi_H^L, \) and \( \Pi_H^R \) are idempotent. On the other hand, Propositions 3.5, 3.7 and 3.9 of [2] assert that
\[ \mu_H \circ (H \otimes \Pi_H^L) = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H), \]
\[ \mu_H \circ (\Pi_H^R \otimes H) = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes \delta_H), \]
\[ \mu_H \circ (H \otimes \Pi_H^L) = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (\delta_H \otimes H), \]
\[ \mu_H \circ (\Pi_H^R \otimes H) = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes \delta_H), \]
\[ (H \otimes \Pi_H^L) \circ \delta_H = (\mu_H \circ H) \circ ((\delta_H \circ \eta_H) \otimes H), \]
\[ (\Pi_H^R \otimes H) \circ \delta_H = (\lambda_H \circ \mu_H) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)), \]
\[ (H \otimes \Pi_H^R) \circ \delta_H = (\mu_H \circ H) \circ (H \otimes (\delta_H \circ \eta_H)), \]
hold. Also, it is possible to prove the following identities involving the idempotent morphisms \( \Pi_H^L, \Pi_H^R, \Pi_H^L, \) and \( \Pi_H^R \) and the antipode \( \lambda_H \) (see Propositions 3.11 and 3.12 of [2]):
\[ \Pi_H^L \circ \Pi_H^L = \Pi_H^L, \]
\[ \Pi_H^L \circ \Pi_H^R = \Pi_H^R, \]
\[ \Pi_H^L \circ \Pi_H^L = \Pi_H^L, \]
\[ \Pi_H^R \circ \Pi_H^L = \Pi_H^L, \]
\[ \Pi_H^R \circ \Pi_H^R = \Pi_H^R, \]
\[ \lambda_H \circ \Pi_H^L = \Pi_H^L, \]
\[ \lambda_H \circ \Pi_H^R = \Pi_H^R, \]
\[ \lambda^R = \lambda^L. \]

Moreover, by Proposition 3.16 of [2], we have
\[ \mu_H \circ (\mu_H \otimes H) \circ (H \otimes (\Pi_H^L \otimes H) \circ \delta_H) = \mu_H = \mu_H \circ (\mu_H \otimes \Pi_H^L) \circ (H \otimes \delta_H), \]
\[ \mu_H \circ (\Pi_H^L \circ \lambda_H) \circ (\delta_H \circ H) = \mu_H = \mu_H \circ (\mu_H \circ (\Pi_H^L \otimes H)) \circ (\delta_H \otimes H). \]

On the other hand, if \( H_L = \text{Im}(\Pi_L^L), p_L : H \to H_L, \) and \( i_L : H_L \to H \) are the morphisms such that \( \Pi_L^L = i_L \circ p_L \) and \( p_L \circ i_L = id_{H_L} \),
\[ H_L \xrightarrow{i_L} H \xrightarrow{\delta_H} (H \otimes \Pi_H^L) \circ \delta_H \]
is an equalizer diagram and
\[ H \otimes H \xrightarrow{\mu_H} H \xrightarrow{p_L} H_L \]
is a coequalizer diagram. As a consequence, \( (H_L, \eta_H \circ i_L, \mu_H \circ (\varepsilon_H \circ i_L), \delta_H = (p_L \otimes p_L) \circ \delta_H \circ i_L) \) is a unital magma in \( C \) and \( (H_L, \varepsilon_H \circ i_L, \delta_H = (p_L \otimes p_L) \circ \delta_H \circ i_L) \) is a comonoid in \( C \) (see Proposition 3.13 of [2]). Surprisingly, the product \( \mu_H \) is associative because, by Proposition 2.4 of [3], we have that
\[ \delta_H \circ \mu_H \circ (i_L \otimes H) = (\mu_H \otimes H) \circ (i_L \otimes \delta_H), \]
\[ \delta_H \circ \mu_H \circ (H \otimes i_L) = (\mu_H \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes i_L). \]
and, as a consequence, the following identities hold

$$\mu_H \circ ((\mu_H \circ (i_L \otimes H)) \otimes H) = \mu_H \circ (i_L \otimes \mu_H),$$

(18)

$$\mu_H \circ (H \otimes (\mu_H \circ (i_L \otimes H))) = \mu_H \circ ((\mu_H \circ (H \otimes i_L)) \otimes H),$$

(19)

$$\mu_H \circ (H \otimes (\mu_H \circ (H \otimes i_L))) = \mu_H \circ (\mu_H \otimes i_L).$$

(20)

By Proposition 3.9 of [2], [19] and the equality $\Pi^L_H \circ \mu_H \circ (\Pi^L_H \otimes \Pi^L_R) = \mu_H \circ (\Pi^L_H \otimes \Pi^L_R)$, it is easy to show that $\mu_{HL} \circ (H_L \otimes \mu_{HL}) = \mu_{HL} \circ (\mu_{HL} \otimes H_L)$ and therefore the unital magma $H_L$ is a monoid in the category $\mathcal{C}$.

**Definition 2.3.** If $B$ is a monoid in $\mathcal{C}$, we will say that $B$ is separable if there exists a morphism $q_B : K \rightarrow B \otimes B$ satisfying $(\mu_B \otimes B) \circ (B \otimes q_B) = (B \otimes \mu_B) \circ (q_B \otimes B)$ and $\mu_B \circ q_B = \eta_B$. The morphism $q_B$ is called the Casimir morphism of $B$. If the first equality of the previous line holds and there exists a morphism $\varepsilon_B : B \rightarrow K$ such that $(B \otimes \varepsilon_B) \circ q_B = \eta_B = (\varepsilon_B \otimes B) \circ q_B$, we will say that $B$ is Frobenius.

**Proposition 2.4.** Let $H$ be a weak Hopf quasigroup. The monoid $H_L$ is Frobenius separable. Therefore, if $\mathcal{C}$ is the category of vector spaces over a field $\mathbb{F}$, $H_L$ is semisimple.

**Proof.** Let $q_{HL} : K \rightarrow H_L \otimes H_L$ and be the morphism defined by $q_{HL} = (\rho_L \circ \lambda_H \otimes p_L) \circ \delta_H \circ \eta_H$. Then, using the same proof of the similar result proved for weak braided Hopf algebras in [1] (see Proposition 2.19.), we obtain that $q_{HL}$ is the Casimir morphism of $H_L$ because $(\mu_L \otimes H_L) \circ (H_L \otimes q_{HL}) = \delta_{HL} = (H_L \otimes \mu_H) \circ (q_{HL} \otimes H_L)$ and $\mu_{HL} \circ q_{HL} = \eta_{HL}$. Also, $(H_L \otimes \varepsilon_{HL}) \circ q_{HL} = \eta_{HL} = (\varepsilon_{HL} \otimes H_L) \circ q_{HL}$, and $H_L$ is Frobenius. Finally, if $\mathcal{C}$ is the category of vector spaces over a field $\mathbb{F}$, the semisimple character for $H_L$ follows from [13].

Finally, if $H_R = \text{Im}(\Pi^R_H)$, $p_R : H \rightarrow H_R$, and $\varepsilon_R : H_R \rightarrow H$ are the morphisms such that $\Pi^R_H = \varepsilon_R \circ p_R$ and $p_R \circ i_R = \text{id}_{H_R}$, the pair $(H_R, i_R)$ is the equalizer of $\delta_H$ and $(\Pi^R_H \otimes H) \circ \delta_H$. Also the pair $(H_R, p_R)$ is the coequalizer of $\mu_H$ and $\mu_H \circ (\Pi^R_H \otimes H)$. As a consequence, $(\varepsilon_R, \pi_{HR}) = (p_R \circ \varepsilon_H, \mu_{HR} = p_R \circ \mu_H \circ (i_R \otimes i_R))$ is a unital magma in $\mathcal{C}$ and $(H_R, \varepsilon_{HR} = \varepsilon_H \circ i_R, \delta_H = (p_R \otimes p_R) \circ \delta_H \circ i_R)$ is a comonoid in $\mathcal{C}$. In a similar way to [18]–[20], we can obtain

$$\mu_H \circ ((\mu_H \circ (i_R \otimes H)) \otimes H) = \mu_H \circ (i_R \otimes \mu_H),$$

(21)

$$\mu_H \circ (H \otimes (\mu_H \circ (i_R \otimes H))) = \mu_H \circ ((\mu_H \circ (H \otimes i_R)) \otimes H),$$

(22)

$$\mu_H \circ (H \otimes (\mu_H \circ (H \otimes i_R))) = \mu_H \circ (\mu_H \otimes i_R).$$

(23)

Then, it is easy to show that $\mu_{HR} \circ (H_R \otimes \mu_{HR}) = \mu_{HR} \circ (\mu_{HR} \otimes H_R)$ and therefore the unital magma $H_R$ is a monoid in $\mathcal{C}$. As a consequence, $H_R$ is Frobenius separable with Casimir morphism $q_{HR} = (p_R \otimes (p_R \circ \lambda_H)) \circ \delta_H \circ \eta_H$. Therefore, if $\mathcal{C}$ is the category of vector spaces over a field $\mathbb{F}$, $H_R$ is semisimple.

3. **Hopf modules, strong Hopf modules and categorical equivalences**

The definition of right-right $H$-Hopf module for a weak Hopf quasigroup $H$ was introduced in [24]. If $H$ is a Hopf quasigroup and $\mathcal{C}$ is the symmetric monoidal category $\mathbb{F} \otimes Vect$, we get the notion defined by Brzeziński in [6] for Hopf quasigroups.

**Definition 3.1.** Let $H$ be a weak Hopf quasigroup and $M$ an object in $\mathcal{C}$. We say that $(M, \phi_M, \rho_M)$ is a right-right $H$-Hopf module if the following axioms hold:

\begin{enumerate}
\item[(b1)] The pair $(M, \rho_M)$ is a right $H$-comodule, i.e. $\rho_M : M \rightarrow M \otimes H$ is a morphism such that $(M \otimes \varepsilon_H) \circ \rho_M = \text{id}_M$ and $(\rho_M \otimes H) \circ \rho_M = (M \otimes \delta_H) \circ \rho_M$.
\item[(b2)] The morphism $\phi_M : M \otimes H \rightarrow M$ satisfies:
\item[(b2-1)] $\phi_M \circ (M \otimes \delta_H) = \text{id}_M$.
\item[(b2-2)] $\rho_M \circ \phi_M = (\phi_M \circ \rho_M) \circ (M \otimes c_{H,H} \otimes H) \circ (\rho_M \otimes \delta_H)$, i.e. $\phi_M$ is a morphism of right $H$-comodules with the codiagonal coaction on $M \otimes H$.
\item[(b3)] $\phi_M \circ (\phi_M \otimes \lambda_H) \circ (M \otimes \delta_H) = \phi_M \circ (M \otimes \Pi^R_H)$.
\item[(b4)] $\phi_M \circ (\phi_M \otimes \lambda_H) \circ (M \otimes \delta_H) = \phi_M \circ (M \otimes \Pi^R_H)$.
\item[(b5)] $\phi_M \circ (\phi_M \otimes \lambda_H) \circ (M \otimes \Pi^R_H) \circ (M \otimes \delta_H) = \phi_M$.
\end{enumerate}
Obviously, if $H$ is a weak Hopf quasigroup, the triple $(H, \phi_H = \mu_H, \rho_H = \delta_H)$ is a right-right $H$-Hopf module. Moreover, if $(M, \phi_M, \rho_M)$ is a right-right $H$-Hopf module, the axiom (b5) is equivalent to $\phi_M \circ (\phi_M \otimes \Pi^H) \circ (M \otimes \delta_H) = \phi_M$. Also, composing in (b2-2) with $M \otimes \eta_H$ and $M \otimes \varepsilon_H$ we have that $\phi_M \circ (M \otimes \Pi^H) \circ \rho_M = id_M$, and if $(M, \phi_M, \rho_M)$, $(N, \phi_N, \rho_N)$ are right-right $H$-Hopf modules, and there exists a right $H$-comodule isomorphism $\alpha : M \to N$, the triple $(M, \phi_M^\alpha = \alpha^{-1} \circ \phi_N \circ (\alpha \otimes H), \rho_M)$ is a right-right $H$-Hopf module (see Proposition 4.7 of [2]).

By Proposition 4.3 of [2], we have that for all $(M, \phi_M, \rho_M)$ right-right $H$-Hopf module, the morphism
$q_M := \phi_M \circ (M \otimes \lambda_H) \circ \rho_M : M \to M$ satisfies $\rho_M \circ q_M = (M \otimes \Pi^H) \circ \rho_M \circ q_M$ and, as a consequence, $q_M$ is idempotent. Moreover, if $M^{coH}$ (object of coinvariants) is the image of $q_M$ and $p_M : M \to M^{coH}$, $i_M : M^{coH} \to M$ are the morphisms such that $q_M = i_M \circ p_M$ and $id_{M^{coH}} = p_M \circ i_M$,

\[ M^{coH} \xrightarrow{i_M} M \xrightarrow{\rho_M} M \otimes H \]

is an equalizer diagram. Also,

\[ M^{coH} \xrightarrow{i_M} M \xrightarrow{\rho_M} (M \otimes \Pi^H) \circ \rho_M \]

is an equalizer diagram. Moreover, the following identities hold (see Remark 4.4 of [2]):

\[ \phi_M \circ (q_M \otimes H) \circ \rho_M = id_M, \]

\[ \rho_M \circ \phi_M \circ (i_M \otimes H) = (\phi_M \otimes H) \circ (i_M \otimes \delta_H), \]

\[ p_M \circ \phi_M \circ (i_M \otimes H) = p_M \circ \phi_M \circ (i_M \otimes \Pi^H). \]

On the other hand, the morphism

\[ \nabla_M := (p_M \otimes H) \circ \rho_M \circ \phi_M \circ (i_M \otimes H) : M^{coH} \otimes H \to M^{coH} \otimes H \]

is idempotent and the equalities

\[ \nabla_M = ((p_M \circ \phi_M) \otimes H) \circ (i_M \otimes \delta_H), \]

\[ (M^{coH} \otimes \delta_H) \circ \nabla_M = (\nabla_M \otimes H) \circ (M^{coH} \otimes \delta_H). \]

hold (see Proposition 4.5 of [2]). If we define the morphisms

\[ \omega_M : M^{coH} \otimes H \to M, \quad \omega'_M : M \to M^{coH} \otimes H, \]

by $\omega_M = \phi_M \circ (i_M \otimes H)$ and $\omega'_M = (p_M \otimes H) \circ \rho_M$. Then, $\omega_M \circ \omega'_M = id_M$ and $\nabla_M = \omega'_M \circ \omega_M$. Also, we have a commutative diagram

\[ \begin{array}{c}
M \\
M^{coH} \otimes H \\
M^{coH} \times H
\end{array} \quad \begin{array}{c}
\omega_M \\
\omega'_M \\
\nabla_M
\end{array} \quad \begin{array}{c}
p_{M^{coH} \otimes H} \\
i_{M^{coH} \otimes H} \\
p_{M^{coH} \times H}
\end{array} \]

where $M^{coH} \times H$ denotes the image of $\nabla_M$ and $p_{M^{coH} \otimes H}$, $i_{M^{coH} \otimes H}$ are the morphisms such that $p_{M^{coH} \otimes H} \circ i_{M^{coH} \otimes H} = id_{M^{coH} \times H}$ and $i_{M^{coH} \otimes H} \circ p_{M^{coH} \otimes H} = \nabla_M$. Therefore, the morphism

\[ \alpha_M = p_{M^{coH} \otimes H} \circ \omega'_M : M \to M^{coH} \times H \]
is an isomorphism of right $H$-modules (i.e., $\rho_{M \cdot M} \circ \alpha_M = (\alpha_M \otimes H) \circ \rho_M$) with inverse $\alpha_M^{-1} = \omega_M \circ i_{M \cdot M} \otimes H$. The comodule structure of $M \otimes H \times H$ is the one induced by the isomorphism $\alpha_M$ and it is equal to

$$\rho_{M \cdot M} \times H = (\rho_{M \cdot M} \otimes H) \circ (M \otimes \delta_H) \circ i_{M \cdot M} \otimes H.$$ 

As a consequence, the triple $(M \otimes H \times H, \phi_{M \cdot M} \otimes H, \rho_{M \cdot M} \otimes H)$

where

$$\phi_{M \cdot M} \otimes H = p_{M \cdot M} \otimes H \circ (M \otimes \mu_R) \circ (i_{M \cdot M} \otimes H \otimes H),$$

is a right-right $H$-Hopf module (see Proposition 4.8 of [2]).

Finally, following Proposition 4.9 of [2], for the isomorphism of right $H$-comodules $\alpha_M$, the triple $(M, \phi_M^\alpha, \rho_M)$ is a right-right $H$-Hopf module with the same object of coinvariants of $(M, \phi_M, \rho_M)$. Moreover, the identity $\phi_M^\alpha = \phi_M \circ (q_M \otimes \mu_R) \circ (\rho_M \otimes H)$ holds and

$$q_M^\alpha = q_M,$$ (29)

where $q_M^\alpha = \phi_M^\alpha \circ (M \otimes \lambda_H) \circ \rho_M$ is the idempotent morphism associated to the Hopf module $(M, \phi_M^\alpha, \rho_M)$. Finally, if $\nabla_M^\alpha$, denotes the idempotent morphism associated to $(M, \phi_M^\alpha, \rho_M)$, we have that $\nabla_M^\alpha = \nabla_M$ and then, for $(M, \phi_M^\alpha, \rho_M)$, the associated isomorphism between $M$ and $M \otimes H$ is $\alpha_M$. Finally, $(\phi_M^\alpha)_{\otimes M} = \phi_M^\alpha_{\otimes M}$ holds. Note that the triple $(H, \phi_H = \mu_H, \rho_H = \delta_H)$ is a right-right $H$-Hopf module and $\phi_H^\alpha = \phi_H$.

**Proposition 3.2.** Let $H$ be a weak Hopf quasigroup and let $(M, \phi_M, \rho_M)$ be a right-right $H$-Hopf module. The following equality holds:

$$\phi_M \circ (i_M \otimes \mu_R) = \phi_M \circ (i_M \otimes \mu_H) \circ (\nabla_M \otimes H).$$ (30)

**Proof.** The equality holds because

1. $\phi_M \circ (i_M \otimes \mu_R) \circ (\nabla_M \otimes H) = \phi_M \circ (q_M \otimes \mu_H) \circ ((\rho_M \circ \phi_M \circ (i_M \otimes H)) \otimes H)$
2. $\phi_M \circ (q_M \otimes \mu_H) \circ (((\rho_M \circ \phi_M \circ (i_M \otimes H)) \otimes H) \otimes H)$
3. $\phi_M \circ ((\rho_M \circ \phi_M \circ (i_M \otimes H)) \otimes (M \otimes \mu_R)) \circ (i_M \otimes \delta_R)$
4. $\phi_M \circ ((\rho_M \circ \phi_M \circ (i_M \otimes H)) \otimes (M \otimes \delta_R)) \otimes (i_M \otimes \mu_R)$
5. $\phi_M \circ ((\rho_M \circ \phi_M \circ (i_M \otimes H)) \otimes (M \otimes \delta_R)) \otimes (i_M \otimes \mu_H)$
6. $\phi_M \circ ((\rho_M \circ \phi_M \circ (i_M \otimes H)) \otimes (M \otimes \delta_R)) \otimes (i_M \otimes \mu_R)$
7. $\phi_M \circ (i_M \otimes \mu_R)$

where the first equality follows by the definition of $\nabla_M$, the second and the fourth ones follow by (27), the third one relies on the definition of $q_M$, and the fifth one is a consequence of (b3) of Definition 3.1. In the sixth one we used the definition of $\Pi_H^2$ and the seventh and the eleventh ones follow by the naturality of $\alpha$. The equalities eighth and tenth are consequence of (5), and the ninth one follows by $\mu_M \circ ((\mu_M \circ (\Pi_H^2 \otimes H)) \otimes H) = \mu_H \circ (\Pi_H^2 \otimes \mu_H)$. The equality holds by (18) and by (19), more concretely, by $\Pi_H^2 \circ \Pi_H^2 = \Pi_H^2$. Finally, the last one relies on (b5) of Definition 3.1. 

**Definition 3.3.** Let $H$ be a weak Hopf quasigroup and let $(M, \phi_M, \rho_M)$ and $(N, \phi_N, \rho_N)$ be right-right $H$-Hopf modules. A morphism $f : M \rightarrow N$ in $C$ is said to be $H$-quasilineal if the following identity holds:

$$\phi_N^{\otimes M} \circ (f \otimes H) = f \circ \phi_M^\alpha.$$ (31)

A morphism of right-right $H$-Hopf modules between $M$ and $N$ is a morphism $f : M \rightarrow N$ in $C$ such that is both a morphism of right $H$-comodules and $H$-quasilineal. The collection of all right $H$-Hopf modules with their morphisms forms a category which will be denoted by $\mathcal{M}_H^R$. 

If \((M, \phi_M, \rho_M)\) is an object in \(\mathcal{M}_M^H\), for \((M^{co}H \times H, \phi_{M^{co}H \times H}, \rho_{M^{co}H \times H})\) the identity
\[
\phi_{M^{co}H \times H}^{\alpha_{M^{co}H \times H}} = \phi_{M^{co}H \times H}
\] (32)
holds (see Proposition 4.12 of [2]). Then, as a consequence, we can prove (see Theorem 4.13 of [2])

**Theorem 3.4. (Fundamental Theorem of Hopf modules)** Let \(H\) be a weak Hopf quasigroup and assume that \((M, \phi_M, \rho_M)\) is an object in the category \(\mathcal{M}_M^H\). Then, the right-right \(H\)-Hopf modules \((M, \phi_M, \rho_M)\) and \((M^{co}H \times H, \phi_{M^{co}H \times H}, \rho_{M^{co}H \times H})\) are isomorphic in \(\mathcal{M}_M^H\).

Since now on we assume that \(C\) admits coequalizers. With \(C_{HL}\) we will denote the category of right \(H_L\)-modules, i.e., the category whose objects are pairs \((N, \psi_N)\) with \(N\) an object in \(C\) and \(\psi_N : N \otimes H_L \to N\) a morphism such that \(\psi_N \circ (N \otimes \mu_{HL}) = \psi_N \circ (\psi_N \otimes H_L)\), \(\psi_N \circ (N \otimes \eta_{HL}) = \text{id}_N\). A morphism \(f : (N, \psi_N) \to (P, \psi_P)\) in \(C_{HL}\) is a morphism \(f : N \to P\) in \(C\) such that \(\psi_P \circ (f \otimes H) = f \circ \psi_N\). Note that the pair \((H, \psi_H = \mu_H \circ (H \otimes i_L))\) is a right \(H_L\)-module.

Let \((N, \psi_N)\) be an object in \(C_{HL}\) and consider the coequalizer diagram
\[
\begin{array}{ccc}
N \otimes H_L \otimes H & \xrightarrow{\psi_N \otimes H} & N \otimes H & \xrightarrow{n_N} & N \otimes H_L H
\end{array}
\] (33)
where \(\varphi_H = \mu_H \circ (i_L \otimes H)\). By (16) we have
\[
(n_N \otimes H) \circ (\psi_N \otimes \delta_H) = ((n_N \otimes (N \otimes \varphi_H)) \otimes H) \circ (N \otimes H_L \otimes \delta_H) = (n_N \otimes H) \circ (N \otimes (\delta_H \otimes \varphi_H))
\]
and, as a consequence, there exists a unique morphism \(\rho_{N \otimes H_L H} : N \otimes H_L H \to (N \otimes H_L H) \otimes H\) such that
\[
\rho_{N \otimes H_L H} \circ n_N = (n_N \otimes H) \circ (N \otimes \delta_H).
\] (34)

The pair \((N \otimes H_L H, \rho_{N \otimes H_L H})\) is a right \(H\)-comodule. Indeed: Trivially, \(((N \otimes H_L H) \otimes \varepsilon_H) \circ \rho_{N \otimes H_L H} = \text{id}_{N \otimes H_L H}\) because composing with \(n_N\) we have
\[
((N \otimes H_L H) \otimes \varepsilon_H) \circ \rho_{N \otimes H_L H} \circ n_N = (n_N \otimes \varepsilon_H) \circ (N \otimes \delta_H) = n_N.
\]
Moreover, \(((\rho_{N \otimes H_L H} \otimes H) \circ \rho_{N \otimes H_L H} \circ (\rho_{N \otimes H_L H} \otimes H) \circ (\rho_{N \otimes H_L H} \circ n_N \otimes H_L H) \otimes H) \circ (\rho_{N \otimes H_L H} \circ n_N \otimes H_L H) \circ n_N\times H_L H\)

On the other hand, by (18) we have
\[
n_N \circ (\psi_N \otimes \mu_H) = n_N \circ (N \otimes (\mu_H \circ (i_L \otimes \mu_H))) = n_N \circ (N \otimes (\mu_H \circ (\varphi_H \otimes H)))
\]
and then, if the function \(- \otimes H\) preserves coequalizers, there exists a unique morphism
\[
\phi_{N \otimes H_L H} : (N \otimes H_L H) \otimes H \to N \otimes H_L H
\]
such that
\[
\phi_{N \otimes H_L H} \circ (n_N \otimes H) = n_N \circ (N \otimes \mu_H).
\] (35)

Trivially, \(\phi_{N \otimes H_L H} \circ ((N \otimes H_L H) \otimes \eta_H) = \text{id}_{N \otimes H_L H}\) because
\[
\phi_{N \otimes H_L H} \circ (n_N \otimes \eta_H) = n_N \circ (N \otimes (\mu_H \circ (H \otimes \eta_H))) = n_N.
\]

Then, if the function \(- \otimes H\) preserves coequalizers, the triple \((N \otimes H_L H, \phi_{N \otimes H_L H}, \rho_{N \otimes H_L H})\) is a right-right \(H\)-Hopf module. Indeed: By the previous reasoning conditions (b1) and (b2-1) of Definition \(\mathcal{M}_M^H\) hold. Composing with \(n_N \otimes H\) and using (31), (35) and (a1) of Definition \(\mathcal{M}_M^H\) we have
\[
\rho_{N \otimes H_L H} \circ (\rho_{N \otimes H_L H} \circ (n_N \otimes H) = (n_N \otimes H) \circ (N \otimes (\delta_H \otimes \mu_H))) = (n_N \otimes H) \circ (N \otimes ((\mu_H \otimes \mu_H) \circ \delta_H))
\]
\[
= (\phi_{N \otimes H_L H} \circ \mu_H) \circ ((N \otimes H_L H) \otimes c_{HL} H) \circ (\rho_{N \otimes H_L H} \circ n_N) \otimes \delta_H,
\]
and then (b2-1) of Definition \(\mathcal{M}_M^H\) holds. Also, by (a4-6) of Definition \(\mathcal{M}_M^H\) we obtain
\[
\phi_{N \otimes H_L H} \circ (\phi_{N \otimes H_L H} \circ \lambda_H) \circ (n_N \otimes \delta_H) = n_N \circ (N \otimes (\mu_H \circ (H \otimes \lambda_H)) \circ (H \otimes \delta_H))) = n_N \circ (N \otimes (\mu_H \circ (H \otimes \Pi_H^L)))
\]
\[
= \phi_{N \otimes H_L H} \circ (n_N \otimes \Pi_H^L).
\]
and then (b3) of Definition 3.1 holds. Similarly, by (35) and (a4-7) of Definition 2.2 we get (b4) of Definition 3.1. The equality (b5) of this definition is a consequence of (35) and (14).

Note that, by (34), (35), we obtain that

\[ q_{N \otimes H_L H} \circ n_N = n_N \circ (N \otimes \Pi^L_H). \]  

(36)

Also,

\[ \phi_{N \otimes H_L H}^{\otimes \otimes H} = \phi_{N \otimes H_L H} \]  

(37)

because by (34), (35) and (15),

\[ \phi_{N \otimes H_L H}^{\otimes \otimes H} \circ (n_N \otimes H) = n_N \circ (N \otimes (\mu_H \circ (\Pi^L_H \otimes \mu_H) \circ (\delta_H \otimes H))) = n_N \circ (N \otimes \mu_H) = \phi_{N \otimes H_L H} \circ (n_N \otimes H). \]

On the other hand, if \( f : N \rightarrow P \) is a morphism in \( C_{H_L} \), we have that

\[ n_P \circ (f \otimes H) \circ (\psi_N \otimes H) = n_P \circ (f \otimes H) \circ (N \otimes \varphi_H) \]

and, as a consequence, there exists an unique morphism \( f \otimes H_L H : N \otimes H_L H \rightarrow P \otimes H_L H \) such that

\[ n_P \circ (f \otimes H) = (f \otimes H_L H) \circ n_N. \]  

(38)

The morphism \( f \otimes H_L H \) is a morphism in \( \mathcal{M}^H_H \) because by (34), (35), (37) and (38)

\[ \rho_{P \otimes H_L H} \circ (f \otimes H_L H) \circ n_N = (n_P \otimes H) \circ (f \otimes \delta_H) = ((f \otimes H_L H) \otimes H) \circ \rho_{N \otimes H_L H} \circ n_N \]

and

\[ \phi_{P \otimes H_L H}^{\otimes \otimes H} \circ ((f \otimes H_L H) \otimes H) \circ (n_N \otimes H) = \phi_{P \otimes H_L H} \circ ((f \otimes H_L H) \otimes H) \circ (n_N \otimes H) \]

\[ = (f \otimes H_L H) \circ \phi_{N \otimes H_L H} \circ (n_N \otimes H) = (f \otimes H_L H) \circ \phi_{N \otimes H_L H}^{\otimes \otimes H} \circ (n_N \otimes H). \]

Summarizing, we have the following proposition:

**Proposition 3.5.** Let \( H \) be a weak Hopf quasigroup such that the functor \( - \otimes H \) preserves coequalizers. There exists a functor

\[ F : C_{H_L} \rightarrow \mathcal{M}^H_H, \]

called the induction functor, defined on objects by \( F((N, \psi_N)) = (N \otimes H_L H, \phi_{N \otimes H_L H}, \rho_{N \otimes H_L H}) \) and for morphisms by \( F(f) = f \otimes H_L H \).

**Definition 3.6.** Let \( H \) be a weak Hopf quasigroup. With \( SM^H_H \) we will denote the full subcategory of \( \mathcal{M}^H_H \) whose objects are the right-right \( H \)-Hopf modules \( (M, \phi_M, \rho_M) \) such that the following equality hold:

\[ \phi_M \circ ((\phi_M \circ (M \otimes i_L)) \otimes H) = \phi_M \circ (M \otimes (\mu_H \circ (i_L \otimes H))). \]  

(39)

The objects of \( SM^H_H \) will be called right-right strong \( H \)-Hopf modules.

By (19) we obtain that \( (H, \phi_H = \mu_H, \rho_H = \delta_H) \) is a right-right strong \( H \)-Hopf module. Note that if \( H \) is a Hopf quasigroup, (39) holds because \( i_L = \eta_H \) (see Theorem 1 of [12]). Then in this particular setting \( SM^H_H = \mathcal{M}^H_H \). Also the previous equality holds trivially for any Hopf module associated to a weak (braided) Hopf algebra (see Section 3 of [1]).

**Proposition 3.7.** Let \( H \) be a weak Hopf quasigroup such that the functor \( - \otimes H \) preserves coequalizers. The induction functor \( F : C_{H_L} \rightarrow \mathcal{M}^H_H \) factorizes through the category \( SM^H_H \).

**Proof.** We must show that for any \( (N, \psi_N) \in C_{H_L} \), the triple \( (N \otimes H_L H, \phi_{N \otimes H_L H}, \rho_{N \otimes H_L H}) \) is an object in \( SM^H_H \). First note that if \( - \otimes H \) preserves coequalizers then \( - \otimes H_L H \) preserves coequalizers, and (39) holds because by (35) and (19)

\[ \phi_{N \otimes H_L H} \circ ((\phi_{N \otimes H_L H} \circ (n_N \otimes i_L)) \otimes H) = n_N \circ (N \otimes (\mu_H \circ ((\mu_H \circ (H \otimes i_L)) \otimes H))) \]

\[ = n_N \circ (N \otimes (\mu_H \circ (H \otimes (\mu_H \circ (i_L \otimes H))))) = \phi_{N \otimes H_L H} \circ (n_N \otimes (\mu_H \circ (i_L \otimes H))). \]

□
Let \((M, \phi_M, \rho_M)\) be a right-right \(H\)-Hopf module. If \(M\) is strong, the pair
\[
(M^{\text{co}H}, \psi_M^{\text{co}H} = p_M \circ \phi_M \circ (i_M \otimes i_L))
\]
is a right \(H_L\)-module. Indeed: Trivially \(\psi_M^{\text{co}H} \circ (M^{\text{co}H} \otimes \eta_{H_L}) = id_{M^{\text{co}H}}\). Moreover,
\[
\psi_M^{\text{co}H} \circ (\psi_M^{\text{co}H} \otimes H_L) \\
= p_M \circ \phi_M \circ ((\phi_M \circ (\phi_M \otimes \lambda_H) \circ (i_M \otimes (\delta_H \circ i_L))) \otimes i_L) \\
= p_M \circ \phi_M \circ ((\phi_M \circ (i_M \otimes i_L)) \otimes i_L) \\
= p_M \circ \phi_M \circ (i_M \otimes (\mu_H \circ (i_L \otimes i_L))) \\
= \psi_M^{\text{co}H} \circ (M^{\text{co}H} \otimes \mu_{H_L}).
\]

The first equality follows by (24), the second one by (b3) of Definition 3.1, the third one by (39), and the last one by the properties of \(\mu_{H_L}\).

Let \(g : M \to T\) be a morphism in \(SM_H^H\). Using the comodule morphism condition we obtain that \(\rho_T \circ g \circ i_M = (T \otimes \eta_{H_L}) \circ \rho_T \circ g \circ i_M\) and this implies that there exists a unique morphism \(\eta^H : M^{\text{co}H} \to T^{\text{co}H}\) such that
\[
i_T \circ \eta^H = g \circ i_M.
\]
Then, by (10) and (29),
\[
i_T \circ \eta^H \circ p_M = g \circ q_M = g \circ q_M^{\alpha_M} = q_T^{\alpha_T} \circ g = q_T \circ g
\]
and, as a consequence,
\[
\eta^H \circ p_M = p_T \circ g.
\]

On the other hand, for any right-right \(H\)-Hopf module \(M\), by (29), we know that \(\nabla_M = \nabla_M^{\alpha_M}\). Then composing with \(\phi_M \circ (i_M \otimes H)\) in this equality and using (24) we get the equality
\[
\phi_M \circ (i_M \otimes H) = \phi_M^{\alpha_M} \circ (i_M \otimes H).
\]

Therefore, by (11), (12) and (10) we obtain that \(\eta^H\) is a morphism of right \(H_L\)-modules because:
\[
\eta^H \circ \psi_M^{\text{co}H} = p_T \circ g \circ \phi_M \circ (i_M \otimes i_L) = p_T \circ g \circ \phi_M^{\alpha_M} \circ (i_M \otimes i_L) = p_T \circ \phi_T^{\alpha_T} \circ ((g \circ i_M) \otimes i_L) \\
= p_T \circ \phi_T^{\alpha_T} \circ ((i_T \circ \eta^H) \otimes i_L) = p_T \circ \phi_T \circ ((i_T \circ \eta^H) \otimes i_L) = \psi_T^{\text{co}H} \circ (\eta^H \otimes H_L).
\]

Thus, in this setting we have the following result.

**Proposition 3.8.** Let \(H\) be a weak Hopf quasigroup. There exists a functor
\[
G : SM_H^H \to C_H,
\]
called the functor of coinvariants, defined on objects by \(G((M, \phi_M, \rho_M)) = (M^{\text{co}H}, \psi_M^{\text{co}H})\) and for morphisms by \(G(g) = g^{\text{co}H}\).

**Proposition 3.9.** Let \(H\) be a weak Hopf quasigroup such that the functor \(- \otimes H\) preserves coequalizers. For any \((M, \phi_M, \rho_M) \in SM_H^H\), the objects \(M^{\text{co}H} \otimes_{H_L} H_\) and \(M^{\text{co}H} \times H\) are isomorphic right-right \(H\)-Hopf modules.

**Proof.** First note that \(p_{M^{\text{co}H} \otimes H} \circ (\psi_{M^{\text{co}H}} \otimes H) = p_{M^{\text{co}H} \otimes H} \circ (M^{\text{co}H} \otimes \varphi_H)\) because
\[
\nabla_M \circ (\psi_{M^{\text{co}H}} \otimes H) \\
= (p_M \otimes H) \circ \rho_M \circ \phi_M \circ ((q_M \circ \phi_M \circ (i_M \otimes i_L)) \otimes H) \\
= (p_M \otimes H) \circ \rho_M \circ \phi_M \circ ((\phi_M \circ (\phi_M \otimes \lambda_H) \circ (i_M \otimes (\delta_H \circ i_L))) \otimes H) \\
= (p_M \otimes H) \circ \rho_M \circ \phi_M \circ ((\phi_M \circ (i_M \otimes i_L)) \otimes H) \\
= (p_M \otimes H) \circ \rho_M \circ \phi_M \circ (i_M \otimes (\mu_H \circ (i_L \otimes H))) \\
= \nabla_M \circ (M^{\text{co}H} \otimes \varphi_H).
\]

The first and the last equalities are consequence of the definition of \(\nabla_M\). The second one follows by (25), the third one by (b3) of Definition 3.1 and the properties of \(\Pi^H_L\). Finally, the fourth one relies on
Let $t : M^{coH} \otimes H \to P$ be a morphism such that $t \circ (\psi_{M^{coH}} \otimes H) = t \circ (M^{coH} \otimes \varphi_H)$. Put $t' : M^{coH} \times H \to P$ defined by $t' = t \circ i_{M^{coH} \otimes H}$. Then

$$t' \circ p_{M^{coH} \otimes H} = t \circ \nabla_M = t$$

because

$$
t \circ \nabla_M = t \circ ((p_M \circ \phi_M) \otimes H) \circ (i_M \otimes \delta_H)
= t \circ ((p_M \circ \phi_M) \otimes H) \circ (i_M \otimes (\Pi_H \otimes H) \circ \delta_H))
= t \circ (M^{coH} \otimes ((p_L \otimes H) \circ \delta_H))
= t \circ (M^{coH} \otimes (\Pi_H \ast id_H))
= t.
$$

Applying (27) we obtain the first equality. The second one relies on (26). The third one follows by the definition of $\psi_{M^{coH}}$ and the fourth one by the properties of $t$. Finally the last one is a consequence of (1).

The morphism $t'$ is the unique such that $t' \circ p_{M^{coH} \otimes H} = t$ because if $r : M^{coH} \times H \to P$ satisfies $r \circ p_{M^{coH} \otimes H} = t$, composing with $i_{M^{coH} \otimes H}$, we obtain $r = t \circ i_{M^{coH} \otimes H} = t'$. Therefore,

$$M^{coH} \otimes H_L \otimes H \xrightarrow{\psi_{M^{coH}} \otimes H} M^{coH} \otimes H \xrightarrow{p_{M^{coH} \otimes H}} M^{coH} \times H$$

is a coequalizer diagram and as a consequence there exists an isomorphism

$$s_M : M^{coH} \otimes H_L H \to M^{coH} \times H$$

such that

$$s_M \circ n_{M^{coH}} = p_{M^{coH} \otimes H}.$$  \hspace{1cm} (43)

The morphism $s_M$ belongs to the category of right-right $H$-Hopf modules. Indeed: It is a morphism of right $H$-comodules because composing with $n_{M^{coH}}$ and using the equalities (33), (28) we have

$$
\rho^{M^{coH} \times H} \circ s_M \circ n_{M^{coH}} = \rho^{M^{coH} \times H} \circ p_{M^{coH} \otimes H} = (p_{M^{coH} \otimes H} \circ (M^{coH} \otimes \delta_H) \circ \nabla_M
= (p_{M^{coH} \otimes H} \circ H) \circ (M^{coH} \otimes \delta_H) = ((s_M \circ n_{M^{coH}}) \circ H) \circ (M^{coH} \otimes \delta_H) = (s_M \circ H) \circ \rho_{M^{coH} \otimes H_L H} \circ n_{M^{coH}}.
$$

Moreover, by (32) and (27) we know that $\phi^{M^{coH} \times H} = \phi^{M^{coH} \times H}_{M^{coH} \otimes H_L H}$ and $\phi^{M^{coH} \otimes H_L H} = \phi^{M^{coH} \otimes H_L H}$. As a consequence, $s_M$ is $H$-quasilineal because composing with the coequalizer $n_{M^{coH}} \otimes H$ and the equalizer $i_{M^{coH} \otimes H}$ we obtain

$$i_{M^{coH} \otimes H} \circ s_M \circ \phi^{M^{coH} \otimes H_L H} \circ (n_{M^{coH}} \otimes H)
= \omega_M^{ \ast} \circ \phi_M \circ (i_M \otimes \mu_H)
= \omega_M \circ \phi_M \circ (i_M \otimes \mu_H) \circ (\nabla_M \otimes H)
= i_{M^{coH} \otimes H} \circ \phi^{M^{coH} \times H_L H} \circ ((s_M \circ n_{M^{coH}}) \otimes H),
$$

where the first equality follows by (34), the second one by (30), and the last one by (13).

\[ \square \]

**Theorem 3.10.** For any Hopf quasigroup $H$ such the the functor $- \otimes H$ preserves coequalizers, the category $SM^H$ is equivalent to the category $C_{H_L}$.

**Proof.** To prove the theorem we will obtain that the induction functor $F$ is left adjoint to the coinvariants functor $G$ and that the unit and counit associated to this adjunction are natural isomorphisms. Then we divide the proof in three steps.

**Step 1:** In this step we will define the unit of the adjunction. For any right $H_L$-module $(N, \psi_N)$ define $u_N : N \to GF(N) = (N \otimes_{H_L} H)^{coH}$ as the unique morphism such that

$$i_{N \otimes_{H_L} H} \circ u_N = n_N \circ (N \otimes \eta_H).$$  \hspace{1cm} (44)

This morphism exists and is unique because by (34) and (9) we have

$$((N \otimes_{H_L} H) \otimes \Pi^L_H) \circ \rho_{N \otimes_{H_L} H} \circ n_N \circ (N \otimes \eta_H) = (n_N \otimes \Pi^L_H) \circ (N \otimes (\delta_H \otimes \eta_H)) = (n_N \otimes H) \circ (N \otimes (\delta_H \otimes \eta_H)).$$
= \rho_{N \otimes H_L} \circ n_N \circ (N \otimes \eta_H).

Also, it is a morphism in \(C_{H_L}\). Indeed: Composing with the equalizer \(i_{N \otimes H_L} H\) we have
\[
\begin{align*}
i_{N \otimes H_L} H \circ \psi_{N \otimes H_L} H \circ (u_N \otimes H_L)
= q_{N \otimes H_L} H \circ \phi_{N \otimes H_L} H \circ ((n_N \circ (N \otimes \eta_H)) \otimes i_L)
= q_{N \otimes H_L} H \circ n_N \circ (N \otimes (\mu_H \circ (\eta_H \otimes i_L)))
= q_{N \otimes H_L} H \circ n_N \circ (N \otimes i_L)
= n_N \circ (N \otimes (\mu_H \circ i_L))
= n_N \circ (N \otimes i_L)
= n_N \circ (N \otimes (\eta_H))
= \psi_{N \otimes H_L} H \circ u_N \circ \psi_N,
\end{align*}
\]

where the first and last equalities follow by (44), the second one by (35), the third and the sixth one by the unit properties, the fourth one by (39), the fifth one by the properties of \(\Pi^L_H\), and the seventh one is a consequence of the definition of \(N \otimes H_L\).

The morphism \(u_N\) is natural in \(N\) because if \(f : N \to P\) is a morphism in \(C_{H_L}\) by (44), (35), (39) we have
\[
\begin{align*}
i_{P \otimes H_L} H \circ (f \otimes H_L)^{coH} \circ u_N
= (f \otimes H_L) \circ i_{N \otimes H_L} H \circ u_N
= (f \otimes H_L H) \circ n_N \circ (N \otimes \eta_H)
= n_P \circ (f \otimes \eta_H) = i_{P \otimes H_L} H \circ u_P \circ f,
\end{align*}
\]
and then \((f \otimes H_L H)^{coH} \circ u_N = u_P \circ f\).

Finally, we will prove that \(u_N\) is an isomorphism for all right \(H_L\)-module \(N\). First note that \(\psi_N \circ (\psi_N \otimes p_L) = \psi_N \circ (N \otimes (p_L \otimes \varphi_H))\) and then there exists a unique morphism \(m_N : N \otimes H_L \to N\) such that
\[
m_N \circ n_N = \psi_N \circ (N \otimes p_L). \tag{45}
\]
Define \(x_N = m_N \circ i_{N \otimes H_L} H : (N \otimes H_L H)^{coH} \to N\). Then, composing with \(i_{N \otimes H_L} H\) and \(p_{N \otimes H_L} H \circ n_N\) and using (39), (45), (44) and the properties of \(\Pi^L_H\) we have
\[
\begin{align*}
i_{N \otimes H_L} H \circ u_N \circ x_N \circ p_{N \otimes H_L} H \circ n_N
= i_{N \otimes H_L} H \circ u_N \circ m_N \circ q_{N \otimes H_L} H \circ n_N
= i_{N \otimes H_L} H \circ u_N \circ \psi_N \circ (N \otimes (p_L \otimes \Pi^L_H))
= n_N \circ ((\psi_N \circ (N \otimes p_L)) \otimes \eta_H)
= n_N \circ (N \otimes (\mu_H \circ (\Pi^L_H \otimes \eta_H)))
= q_{N \otimes H_L} H \circ n_N.
\end{align*}
\]
Therefore, \(u_N \circ x_N = id_{(N \otimes H_L H)^{coH}}\). Moreover, by (44) and (45),
\[
x_N \circ u_N = \psi_N \circ (N \otimes (p_L \otimes \eta_H)) = id_N.
\]

\textbf{Step 2}: For any \((M, \phi_M, \rho_M) \in SM^H_{\Pi^L_H}\) the counit is defined by
\[
v_M = \alpha^{-1}_M \circ \sigma_M : M^{coH} \otimes H_L H \to M,
\]
where \(\alpha^{-1}_M = \omega_M \circ i_{M^{coH} \otimes H_L}^H\) is the inverse of the isomorphism \(\alpha_M\) defined in Theorem 3.4 and \(\sigma_M\) the isomorphism defined in Proposition 3.9. Note that \(\alpha^{-1}_M\) and \(\sigma_M\) are isomorphisms in \(SM^H_{\Pi^L_H}\) and then \(v_M\) is an isomorphism in \(SM^H_{\Pi^L_H}\). Also, \(v_M\) is the unique morphism such that
\[
v_M \circ n_{M^{coH}} = \phi_M \circ (i_M \otimes \varphi_H), \tag{46}
\]
because by (29), (b3) of Definition 3.1, the properties of \(\Pi^L_H\) and (39),
\[
\begin{align*}
\phi_M \circ ((i_M \circ \psi_{M^{coH}}) \otimes H)
= \phi_M \circ ((\phi_M \circ (\lambda_H)) \circ (i_M \otimes (\delta_H \circ i_L))) \otimes H)
= \phi_M \circ ((\phi_M \circ (i_M \otimes (\Pi^L_H \circ i_L))) \otimes H)
= \phi_M \circ ((\phi_M \circ (i_M \otimes i_L)) \otimes H)
= \phi_M \circ (i_M \otimes \varphi_H),
\end{align*}
\]
and, on the other hand, by (13) \[
v_M \circ n_{M^{coH}} = n_M^{-1} \circ s_M \circ n_{M^{coH}} = \omega_M \circ \omega_M' \circ \omega_M = \phi_M \circ (i_M \otimes H).
\]

**Step 2:** Now we prove the triangular identities for the unit and the counit that we defined previously. Indeed: The first triangular identity holds because composing with \(n_N\) we have
\[
v_N \otimes_{HL} H \circ (u_N \otimes_{HL} H) \circ u_N = v_N \otimes_{HL} H \circ (u_N \otimes H) = \phi_N \otimes_{HL} H \circ ((u_N \otimes_{HL} H) \circ u_N) \otimes H = n_N \circ (N \otimes (\mu_H \circ (\eta_H \otimes H))) = n_N,
\]
where the first equality follows by (38), the second one by (46), the third one by (44) and the fourth one by (35). The last one follows by the properties of the unit \(\eta_H\). Finally, we compose with \(i_M\), applying (40), (41) and (46) we obtain:
\[
i_M \circ v_M^\iota_{coH} \circ u_M^{coH} = v_M \circ i_M \otimes_{HL} H \circ u_M^{coH} = v_M \circ n_{M^{coH}} \circ (M^{coH} \otimes \eta_H) = \phi_M \circ (i_M \otimes \eta_H) = i_M,
\]
and then \(v_M^\iota_{coH} \circ u_M^{coH} = id_{M^{coH}}\).

\[\Box\]

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