A Short Note On The Wilson-Loop Average And The AdS/CFT Correspondence.

Somdatta Bhattacharya

Saha Institute of Nuclear Physics, 1/AF Bidhannagar, Calcutta-700 064, India
E-Mails: som@theory.saha.ernet.in

Abstract

In hep-th/9803002, Maldacena argued that in the light of the AdS/CFT correspondence as formulated by Witten and Gubser, Klebanov and Polyakov as a relation between partition functions, the expectation value of the Wilson Loop in $\mathcal{N}=4$ SU(N) SYM is given by the world sheet partition function with the action formulated on an $AdS_5 \times S^5$ background and the world sheet ending on the loop on the boundary of $AdS_5$. What we propose to do in this paper is to give some alternative arguments as to why it should be so.
1 Introduction

The AdS/CFT correspondence [1, 2, 3] has come a long way in elucidating the long conjectured duality between gauge theories and string theories. The case most studied and with respect to which the most evidence has accumulated is the case of $\mathcal{N}=4$ SU($N$) Super Yang Mills being dual to type IIB string theory on $\text{AdS}_5 \times S^5$ [[4] and references therein]. However, there is a caveat on most of these evidences, because the duality relates the weakly coupled regime of one theory to the strongly coupled one of the other and vice versa. For instance, matching of the correlation functions on both sides makes use of non-renormalization theorems, which keeps one wondering whether these are really evidences or are dictated by symmetry alone.

Gauge theory, however, has a formal non-perturbative formulation, in terms of the loop equation [5], which allows one to define if not compute quantities at arbitrarily large couplings. The loop equation is an equation for the expectation value of the Wilson loop and it is argued that the following ansatz is the solution [6, 7]

$$\langle W \rangle = \int \frac{D XDqD\theta}{Vol(\text{diff} \times \text{Weyl})} e^{-S}$$

(1.1)

where $S$ is the Green-Schwarz superstring action on an $\text{AdS}_5 \times S^5$ background as has been formulated in [8] and the path integral is over all embeddings of the string into $\text{AdS}_5 \times S^5$ with proper boundary conditions (the string world surface should end along the loop at the boundary of $\text{AdS}_5$).

Here $W$ is given by

$$W = \frac{1}{N} Tr P e x p i \int (A_{\mu} \dot{X}^\mu - i |\dot{X} | \Phi \theta^i) ds$$

(1.2)

where $A_{\mu}$ are the gauge fields and $\Phi_i$ are the six scalars in the adjoint representation. The $i$ in front of the second term in the integral is due to the metric being Euclidean. The $\theta$ here are not the same as the $\theta$ in the Green Schwarz action. They are angular coordinates of magnitude 1 and can be regarded as coordinates on $S_5$. The former denote the Green Schwarz fermions.

Polyakov has argued for such a relationship to hold for the case of the non-supersymmetric Yang Mills with the corresponding string path-integral being that for just $\text{AdS}_5$, from a totally different viewpoint [9]. Moreover, Maldacena [6] has given some arguments based on the AdS/CFT correspondence in the form as given in [2, 3]. See [10] for related developments.

What we propose to do is to forward some alternative arguments as to why such a
relationship should follow from the AdS/CFT correspondence. Our arguments will be based upon certain reasonable assumptions which are hard to prove and we will not try to prove them here but simply assume them to be true. We will point out where we will make these assumptions.

2 The Arguments

On one side of the AdS/CFT correspondence, one has the string theory partition function, and on the other side one has the Yang-Mills partition function. We shall be interested in the Yang-Mills partition function without any sources. On the string theory side, this means that no fields are kept fixed on the boundary of $AdS_5$ which is $R^4$.

Thus the string theory partition function is

$$Z = \int D\phi_i e^{-S[\phi_i]} \quad (2.1)$$

where $\phi_i$ are the various fields in the entire spectrum of type IIB superstring theory and the action is formulated about the $AdS_5 \times S^5$ background.

Now it is well known that the one-loop free energy for conformal backgrounds (in the sense of world-sheet conformal invariance) is just the string world sheet partition function on the torus, formulated with the background acting as the sigma model couplings, i.e., for bosonic string theories,

$$F = \ln Z = \int_{\text{torus}} \frac{DXDg}{Vol(diff \times Weyl)} e^{-[G_{MN}g^{ab}x^M \partial_a x^N + \ldots]} \quad (2.2)$$

where $G_{MN}$ is part of the conformal background.

Thus, the free energy for type IIB superstring theory on the $AdS_5 \times S^5$ background (which is a conformal background as the $AdS_5 \times S^5$ background is a solution of the supergravity equations of motion) can be written as

$$F = \int_{\text{torus}} \frac{DXDgD\theta}{Vol(diff \times Weyl)} e^{-S_{G.S}[X^M, g^{ab}, \theta]} \quad (2.3)$$

where $S_{G.S}[X^M, g^{ab}, \theta]$ is the Green-Schwarz action for type IIB superstrings on an $AdS_5 \times S^5$ background as has been formulated by Metsaev and Tseytlin \[8\]. The RR fiveform is incorporated into one of the fermionic terms in the action. This is assumption no.1 as there is no well-defined quantum theory as yet of the Green-Schwarz action formulated over the $AdS_5 \times S^5$ background. However if the Metsaev-Tseytlin action is the correct world sheet action for the Type IIB theory on an $AdS_5 \times S^5$ background then
such a relationship between the one-loop free energy and the torus partition function has to hold.

Now the Yang-Mills partition function can be reformulated in terms of the master variables which can be taken as the Wilson loops. What the exact path integral formulation will be is not known but we will not need the exact form here. We will assume that the YM partition function is expressible in the form

\[ Z = \int D\Psi e^{-S[\Psi]} \]  

(2.4)

where \( \Psi = Tr P e^{i \oint (A_\mu X^\mu - i|X|\theta^i)ds} \) and \( S[\Psi] \) is some unknown functional in terms of the \( \Psi \) variables, which includes the Jacobian factor in transforming from the \( A_\mu \) to the \( \Psi \) variable.

The point to be noted is that since the \( \Psi \) variable is a function of the curve \((X^\mu(s), \theta^i(s))\), it is like a string field in 10-d. That a Wilson loop can be interpreted as a string field had been conjectured in the early eighties [8] and [11] and references therein. In the more modern treatment of Polyakov [9], such a relationship is implicit. Also, such a relationship has been used in [12]. If we denote \(|\dot{X}|\theta_i\) by \( Y_i \), the resultant 10-d coordinates obey \( \dot{X}^2 = \dot{Y}^2 \). Thus the metric in this space has signature \((4,6)\) and is hence not the signature of \( AdS_5 \times S^5 \), but of the space where the loops are defined. As has been shown in [13], this is related to the fact that the six loop variables \( Y_i(s) \) correspond to T-dual coordinates on the string worldsheet. Loops obeying this condition are BPS loops as they are invariant under half the supersymmetries in super-loop space. We refer the reader to [13] for further details. The resulting interpretation is thus that if the AdS/CFT correspondence is correct, the string field theory action corresponding to type IIB string theory on an \( AdS_5 \times S^5 \) background is given by the unknown action functional \( S[\Psi] \) in (2.4). For all practical purposes, it is an interacting string field theory incorporating splitting-joining interactions and zig-zag symmetry. And according to this interpretation, these interactions are precisely taken care of by the non-trivial background on the string side. This is assumption no.2 and it is a very strong one but it would hold if indeed the \( \Psi \) can be assumed to be a string field and if the AdS/CFT correspondence is valid.

Thus if this interpretation is correct, the free energy corresponding to this string field theory partition function as obtained through the AdS/CFT correspondence is given by (2.3).

Now it is a well known fact in string field theory that given a torus partition function for the free energy, the correlator for two string fields/Wilson Loops is given by the partition function over a cylinder with the same action, with the strings forming the boundary of the cylinder acting as the coordinates for the string fields, i.e.
\[ \langle \Psi(C_f) \Psi(C_i) \rangle = \int_{C_f}^{C_i} \frac{DX Dg D\theta}{Vol(\text{Diff} \times \text{Weyl})} e^{-S[X^M(\sigma, \tau), g^{ab}(\sigma, \tau), \theta^\alpha(\sigma, \tau)]} \]

In the above \( C_f \) and \( C_i \) are both parametrised by some \( X_\mu(s) \) and \( \theta_i(s) \).

Once the propagator is known, the vev for a single field follows in a straightforward fashion. It is just the “one-point” function and is hence given by the same partition function over a disc, with the boundary being formed by the coordinates of the string field/Wilson loop whose vev is being taken, i.e.

\[ \langle \Psi(C) \rangle = \int_C \frac{DX Dg D\theta}{Vol(\text{Diff} \times \text{Weyl})} e^{-S[X^M, g^{ab}, \theta^\alpha]} \]

This is the expression for the Wilson Loop average that appears in \([7]\). When evaluated it yields the area of the minimal surface bounded by the loop in the classical limit, calculated with the \( AdS_5 \times S^5 \) metric.

## 3 Discussion

Thus in this note we have shown how under reasonable assumptions based on the AdS/CFT correspondence the Wilson loop average can be shown to be the world sheet partition function on an \( AdS_5 \times S^5 \) background bounded by the loop which in the classical limit yields the exponential of the minimal surface area bounded by the loop. In the process we have conjectured an equivalence between the Yang-Mills partition function and the type IIB string field theory partition function on an \( AdS_5 \times S^5 \) background. Though such a relationship is hard to prove, given the non-availability of a path-integral formulation of Yang-Mills theory in terms of Wilson loops, it appears that once available it would have to conform to such a relationship in the face of the AdS/CFT correspondence. Polyakov has conjectured that the relationship between the Wilson loop average and the world sheet partition function bounded by the loop is true for the non-supersymmetric case, from a totally different perspective. He has argued that the solution of the loop equation which is an equation of the loop average should yield the world sheet partition function on an \( AdS_5 \) background bounded by the loop and hence the exponential of the area of the minimal surface bounded by the loop \([3]\). He and Rychkov have shown that the above is true in the “WKB approximation” \([14]\). Drukker, Gross and Ooguri have worked with the \( N = 4 \) case and have shown that for the above mentioned BPS loops, the above holds good to a certain extent, in the sense that the loop average is given by the minimal surface for smooth non-intersecting loops \([13]\). If indeed the Wilson loop average can be
shown to be given by the minimal area surface for all kinds of loops, we can turn the
previous arguments around to reason that if the expectation value of the Wilson loop is
indeed given by the minimal area surface in $AdS_5 \times S^5$, then the relationship between
the partition functions also holds good via the conjectured relationship between string
theory partition function on $AdS_5 \times S^5$ and the string field theory interpretation coming
from the Yang-Mills side. This would then constitute some evidence for the AdS/CFT
correspondence beyond the supergravity approximation.

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