Dynamic winding number for exploring band topology

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Topological invariants and their detections play a key role in the characterization of topological state of matter. Recently, non-Hermitian topological states and their topological invariants have attracted great attentions and interests. Due to the intrinsic non-Hermiticity, non-Hermitian topological invariants and their relations to Hermitian counterparts are less clear, leaving alone their detections. The call for general understanding of Hermitian and non-Hermitian topological invariants is summoned. Here, we put forward a dynamic winding number, the winding of observables in long-time average, for exploring band topology in both Hermitian and non-Hermitian systems. We build a concrete relation between dynamic winding number and equilibrium topological invariants. The dynamical winding number directly gives the conventional winding number, while the Chern number relates to the winding number of phase singularity points. We also reveal the relation between Hermitian and non-Hermitian Chern number. This work not only gives a clear physics meaning of topological invariants, but also opens a new direction to detect topological phase by general dynamic evolution.

I. INTRODUCTION

Topological states of matter, robust against local defects/perturbations, go beyond the phase of matter characterized by local order parameter and spontaneous symmetry breaking. Due to their potential applications in quantum materials and quantum computing, etc, topological states have ignited extensive researches in the overall physics community [1, 2]. Many novel topological states have been discovered in condensed matter systems, cold atoms systems, photonic crystals and mechanical metamaterials, etc. [3–6]. Topological band theory provides a unified framework for a wide range of topological states of quantum matter [7–11]. In this theory, topological phases are classified by topological invariants associated with eigenstates of bulk band in periodic structure. Topological phase transitions happen when the topological invariant changes. Topological edge states are protected by the nontrivial bulk topological invariants. Thus, topological invariants and theirs detections play key roles in exploring the band topology.

There are various kind of topological invariants in Hermitian systems, depending on the symmetry and dimension. Winding number and Chern number are two represented topological invariants in the Z topological classification [10]. In odd dimensional systems with chiral symmetry, the winding number can be used to characterize topology, which is closely related to the quantized Zak phase [12, 13]. Recently, theoretical works show that winding number can be also defined in the absence of chiral symmetry [14]. In even dimensional systems breaking time-reversal symmetry, Chern number or TKNN number successfully explains the integer quantum hall effect and topological transport [7]. The difference of spin Chern number also gives Z_2 invariant in the spin conserved systems [15]. Detecting Z topological invariants have been widely explored in photonic systems and cold atom systems. For example, the winding number (or Zak phase) has been directly measured by mean chiral displacement in the photonic quantum walk [16, 17], and Ramsey interferometry of cold atoms in superlattices [18]; The Chern number has been detected via hall response [19, 20], Thouless pumping [21, 22], reconstruction of Berry curvature [23], spin polarization [24, 25], linking number [26] and emerging ring structure in quenched dynamics [27]. In particular, the quenched-dynamics approach require that initial state is prepared as the ground state of a trivial Hamiltonian and then quenched under nontrivial Hamiltonian, enabling a high-precision determination of Chern number.

Topological states not only widely exist in Hermitian systems, but also may arise in non-Hermitian systems [28–40]. One of the most outstanding discovery is the topological protected lasing in both one and dimensional active photonic devices [41–44]. Different from Hermitian systems, non-Hermitian systems exhibit exceptional points (EPS) where spectral singularities coalesce and the Hamiltonian is not diagonalizable [40, 45, 46]. Recently, topological nature of exceptional points have been revealed in a one-dimensional (1D) non-Hermitian model [32]. Winding numbers for
exceptional points can be defined with and without chiral symmetry, and may take fractional values of 1/2 [14, 47]. The existence of zero-mode edge states is closely related to the winding number. In two dimension, Chern number have also been generalized to ‘gapped’ non-Hermitian bands [48]. Different from its Hermitian counterpart, equivalent non-Hermitian Chern numbers can be defined with either left or right Bloch eigenstates, and expected to predict the edge states [49]. Besides, there also exists non-Bloch definition of Chern numbers predicting the numbers of chiral edge modes [50]. Theory framework has been established to systematically classify the non-Hermitian topological phases [51, 52].

Although intensive studies are devoted to characterize the non-Hermitian topological phases [32, 33, 37, 38], how to detect topological invariants in non-Hermitian systems is seldom addressed. Non-Hermitian winding number has been measured in a special waveguide arrays with bending losses [53]. The dynamic topological phenomena has been observed in parity-time-symmetric non-unitary quantum walks [54]. Due to intrinsic difference between Hermitian and non-Hermitian systems, some measurement approaches succeed in Hermitian systems but fail in non-Hermitian systems. For example, Hall conductivity is no longer quantized despite being classified as a Chern insulator based on non-Hermitian topological band theory [55]. Up to now, there is no systematical scheme to explore topological invariants in both Hermitian and non-Hermitian systems. It will give us a clear geometric meaning if we understand the relations between Hermitian and non-Hermitian topological invariants in the same framework. One may ask: can we use a common observable to extract winding number and Chern number in both Hermitian and non-Hermitian systems?

In this paper, we study a generic two-band (or spin-1/2) model which supports both Hermitian and non-Hermitian topological invariants in different parameter settings. In generic time evolution, the long time average of spin polarization will tend to a stable value associated with the properties of the systems. We construct a dynamic winding number in the space forming by the long time average of spin polarization. Then, we prove that the dynamic winding number directly gives the conventional winding number in both Hermitian and non-Hermitian systems. We reveal that Chern number relates to the winding numbers around the phase singularity points in the Hermitian case. In the non-Hermitian case, each phase singularity point is split into two exceptional points. The non-Hermitian Chern number relates to the winding number around the exceptional points. Because the phase singularity points double while the winding number of each singularity point becomes half, thus the non-Hermitian Chern number is the same as its Hermitian counterpart. We numerically examine the dynamical winding number for extracting the topological invariants, and discuss the possibility of experimental setups for the measurement. Dynamic winding number provides a general approach for exploring topology in both Hermitian and non-Hermitian systems.

II. DYNAMIC WINDING NUMBER

We consider a general two-band model for d-dimensional topological systems. The Hamiltonian in momentum space is composed of three Pauli matrices, that is,

$$H(k) = h_x(k)\sigma_x + h_y(k)\sigma_y + h_z(k)\sigma_z.$$  (1)

Here, $k$ is the quasi-momentum, $h_x$, $h_y$ and $h_z$ are periodic functions of $k$. We study both the Hermitian Hamiltonian $H^\dagger = H$ and non-Hermitian Hamiltonian $H^\dagger \neq H$ via biorthogonal quantum mechanics [56, 57]. The right and left eigenvector are respectively defined as $H(k)|\varphi_{k\mu}\rangle = \varepsilon_{k\mu}|\varphi_{k\mu}\rangle$ and $H^\dagger(k)|\chi_{k\mu}\rangle = \varepsilon^*_{k\mu}|\chi_{k\mu}\rangle$, where $\varepsilon_{k\mu} = \mu/\sqrt{h_x^2 + h_y^2 + h_z^2}$ is the eigenenergy and $\mu = \pm$. For the Hermitian system, the eigenstates $|\varphi_{k\mu}\rangle = |\chi_{k\mu}\rangle$ and $\varepsilon_{k\mu} = \varepsilon^*_{k\mu}$. For the non-Hermitian system, neither the eigenstates $|\varphi_{k\mu}\rangle$ nor $|\chi_{k\mu}\rangle$ are orthogonal. We adopt biorthogonal vectors which fulfill $\langle \chi_{k\nu}|\varphi_{k\mu}\rangle = \delta_{\nu\mu}$ and $\sum_{\mu} |\varphi_{k\mu}(\chi_{k\mu}) = 1$ by normalizing $|\varphi_{k\mu}/\sqrt{N_{k\mu}}$ and $|\chi_{k\mu}/\sqrt{N^*_{k\mu}}$ with $N_{k\mu} = \sqrt{\langle \chi_{k\mu}|\varphi_{k\mu}\rangle}$, this is,

$$|\varphi_{k\mu}\rangle = \frac{1}{\sqrt{2\varepsilon_{k\mu}(\varepsilon_{k\mu} - h_z)}}(h_x - ih_y, \varepsilon_{k\mu} - h_z)^T,$n

$$|\chi_{k\mu}\rangle = \frac{1}{\sqrt{2\varepsilon_{k\mu}(\varepsilon_{k\mu} - h_z)}}(h_x + ih_y, \varepsilon_{k\mu} - h_z),$$  (2)

where the superscript $T$ is the transpose operation.

For an arbitrary initial state $|\psi_{k}(0)\rangle = \sum_{\mu} c_{k\mu}|\varphi_{k\mu}\rangle$, the associated state is defined as $|\tilde{\psi}_{k}(0)\rangle = \sum_{\mu} e^{i\varepsilon_{k\mu}t} c_{k\mu}|\chi_{k\mu}\rangle$. The dynamics of $|\psi_{k}(t)\rangle$ and $|\tilde{\psi}_{k}(t)\rangle$ satisfy

$$|\psi_{k}(t)\rangle = \sum_{\mu} c_{k\mu}e^{-i\varepsilon_{k\mu}t}|\varphi_{k\mu}\rangle,$n

$$|\tilde{\psi}_{k}(t)\rangle = \sum_{\mu} c^*_{k\mu}e^{-i\varepsilon_{k\mu}t}|\chi_{k\mu}\rangle.$$  (3)

According to the biorthogonal quantum mechanics, the expectation values of Pauli matrices are given as

$$\langle \sigma_j(k, t) \rangle = \frac{\langle \tilde{\psi}_{k}(t)|\sigma_j|\psi_{k}(t)\rangle}{\langle \tilde{\psi}_{k}(t)|\psi_{k}(t)\rangle},$$  (4)

where $j = x, y, z$. The long time average of $\langle \sigma_j(k, t) \rangle$ is given as

$$\sigma_j(k) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \langle \sigma_j(k, t) \rangle dt.$$  (5)

With the help of $\sigma_j(k)$, we can make a mapping $k \to (\sigma_j, \sigma_j')$ from the quasimomentum space to the $\sigma_j - \sigma_j'$ plane. As the quasimomentum continuously varies,
(\sigma_i, \sigma_j) will form a trajectory in the plane. We can define the dynamic winding number of the vector \( (\sigma_i, \sigma_j) \) around the original point as
\[
w_d = \frac{1}{2\pi} \oint k \partial_k \eta_{ij}(k) dk,
\]
with
\[
\eta_{ij}(k) = \arctan[|\sigma_j(k)|/|\sigma(k)|].
\]
It seems that the dynamic winding number depends on the initial state, and it is unclear whether such number is convergent in the long time. Here, we prove that the initial state can be rather general and the dynamic winding number is convergent. The time-average of \( \langle \sigma_j \rangle \) is given as
\[
\bar{\sigma}_j(k) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{\sum_{\mu, \mu'} c_{k \mu} \psi_{k \mu} e^{-i(\epsilon_{k \mu} - \epsilon_{k \mu'}) t} \langle \chi_{k \mu} | \sigma_j | \varphi_{k \mu'} \rangle}{\sum_{\mu} |c_{k \mu}|^2 e^{-i(\epsilon_{k \mu} - \epsilon_{k \mu'}) t}} dt.
\]
For Hermitian systems, \( \varepsilon_{k \mu} = \varepsilon_{k \mu}^* \), and the periodic terms vanish in the long time average and only the diagonal terms preserve. The above equation can be simplified as
\[
\bar{\sigma}_j(k) = \sum_{\mu} |c_{k \mu}|^2 \langle \chi_{k \mu} | \sigma_j | \varphi_{k \mu} \rangle = \left( |c_{k+}|^2 - |c_{k-}|^2 \right) \frac{h_j(k)}{\varepsilon_{k+}}.
\]
For the non-Hermitian systems, we assume that the eigenenergy \( \varepsilon_{k \mu} = \mu(A + iB) \), where \( A \) and \( B \) is a real number. When \( B > 0 \), Eq. (8) is approximately given as
\[
\bar{\sigma}_j(k) = \langle \chi_{k+} | \sigma_j | \varphi_{k+} \rangle = \frac{h_j(k)}{\varepsilon_{k+}}.
\]
Similarly, when \( B < 0 \), Eq. (8) is approximately given as
\[
\bar{\sigma}_j(k) = \langle \chi_{k-} | \sigma_j | \varphi_{k-} \rangle = -\frac{h_j(k)}{\varepsilon_{k+}}.
\]
Combining with Eqs. (8), (9) and (10), we can also obtain
\[
\bar{\sigma}_j(k) = \frac{h_j(k)}{\varepsilon_{k+}}.
\]
in the conditions \( |c_{k+}|^2 \neq |c_{k-}|^2 \) for Hermitian systems and \( c_{k+} \neq 0 \) for \( B > 0 \) or \( c_{k-} \neq 0 \) for \( B < 0 \) in non-Hermitian systems. It means that the dynamic winding number also converges in the long time limits when the initial state satisfies a few constraint. According to the Eq. (11), one can also obtain
\[
\eta_{ij} = \phi_{ij} = \arctan \left( \frac{\langle \chi_{k \mu} | \sigma_j | \varphi_{k \mu} \rangle}{\langle \chi_{k \mu} | \sigma_i | \varphi_{k \mu} \rangle} \right),
\]
where the azimuthal angle \( \phi_{ij} = \arctan |h_j(k)|/h_i(k) \).
Generally, for the non-Hermitian case the azimuthal angle \( \phi_{ij} \) is a complex angle, so that they do not represent physical observables in the biorthogonal system. This problem can be fixed by decomposing the azimuthal angle into two parts, \( \phi_{ij} = \phi_{ij}^R + i\phi_{ij}^I \), where \( \phi_{ij}^R = \text{Re}(\phi_{ij}) \) and \( \phi_{ij}^I = \text{Im}(\phi_{ij}) \). The azimuthal angle satisfies
\[
e^{i2\phi_{ij}^R} e^{-2\phi_{ij}^I} = \frac{1 + i \tan(\phi_{ij})}{1 - i \tan(\phi_{ij})} = \frac{h_i + ih_j}{h_i - ih_j},
\]
\[
e^{-2\phi_{ij}^I} = \frac{h_i + ih_j}{h_i - ih_j},
\]
\( \phi_{ij}^R \) and \( \phi_{ij}^I \) contribute to the argument and amplitude, respectively. \( \phi_{ij}^R \) is a real continuous periodic function of \( k \), so that \( \oint k \partial_k \phi_{ij}^R(k) dk = 0 \). It means that only the real part of azimuthal angle contribute to the dynamic winding number,
\[
w_d = \frac{1}{2\pi} \oint k \partial_k \psi_{ij}^R dk = \frac{1}{2\pi} \oint k \partial_k \psi_{ij}^R dk,
\]
where \( \psi_{ij}^R = \text{Re}(\eta_{ij}) \). Next, we will show that the real part of azimuthal angle is a physical observable.
According to the Eq. (13), the real part of azimuthal angle satisfies
\[
\tan(2\phi_{ij}^R) = \frac{\text{Im}(\psi_{ij}^R)}{\text{Re}(\psi_{ij}^R)} = \frac{\phi_{ij}^R + \phi_{ij}^I}{1 - \tan(\phi_{ij}^R) \tan(\phi_{ij}^I)} = \tan(\phi_{ij}^R + \phi_{ij}^I),
\]
where
\[
\tan(\phi_{ij}^R) = \frac{\text{Re}(\psi_{ij}^R) + \text{Im}(\psi_{ij}^R)}{\text{Re}(\psi_{ij}^R) - \text{Im}(\psi_{ij}^R)},
\]
\[
\tan(\phi_{ij}^I) = \frac{\text{Re}(\psi_{ij}^I) - \text{Im}(\psi_{ij}^I)}{\text{Re}(\psi_{ij}^I) + \text{Im}(\psi_{ij}^I)},
\]
which define two real angles \( \phi_{ij}^R \) and \( \phi_{ij}^I \), respectively. It is worth noting that the two real angles \( \phi_{ij}^R \) and \( \phi_{ij}^I \) will be changed by different parameters \( \mathcal{L} \), but maintain the values of \( \phi_{ij}^I \). The relation between \( \phi_{ij}^R \) and \( \phi_{ij}^a, \phi_{ij}^b \) satisfies
\[
\phi_{ij}^R = \eta_{ij}^R = \frac{1}{2}(\phi_{ij}^a + \phi_{ij}^b) + n \frac{\pi}{2},
\]
where \( n \) is an integer, means the dynamic winding number
\[
w_d = \frac{1}{2}(w_d^a + w_d^b). \]
Here, \( w_d^f = \frac{1}{2\pi} \oint k \partial_k \phi_{ij}^f dk \) and \( f \in (a, b) \). Interestingly, the two real angles \( \phi_{ij}^a \) and \( \phi_{ij}^b \) can be respectively replaced by spin polarization corresponding to \( |\psi_k(t)\rangle \) and \( |\phi_k(t)\rangle \),
\[
\phi_{ij}^a = \arctan \left( \frac{\langle \psi_k(t) | \sigma_j | \psi_k(t) \rangle}{\langle \psi_k(t) | \sigma_i | \psi_k(t) \rangle} \right),
\]
\[
\phi_{ij}^b = \arctan \left( \frac{\langle \psi_k(t) | \sigma_j | \psi_k(t) \rangle}{\langle \psi_k(t) | \sigma_i | \psi_k(t) \rangle} \right),
\]
where \( \langle \bullet \rangle = \lim_{T \to \infty} \frac{1}{T} T_0^T \langle \bullet \rangle dt \). The Eq. (18) means that the two real angles \( \phi^\mu_y \) and \( \phi^\nu_y \) are physical observables. For simplicity, we prove the relation by two real \( |x| \) and the left-left spin polarization corresponding to \( |\psi_k(t)\rangle \) satisfies
\[
\frac{\langle \psi_k(t) | \sigma_y | \psi_k(t) \rangle}{\langle \psi_k(t) | \sigma_x | \psi_k(t) \rangle} = \frac{\langle \varphi_{k^+} | \sigma_y | \varphi_{k^+} \rangle}{\langle \varphi_{k^+} | \sigma_x | \varphi_{k^+} \rangle},
\]
and the left-left spin polarization corresponding to \( |\psi_k(t)\rangle \) satisfies
\[
\frac{\langle \psi_k(t) | \sigma_y | \psi_k(t) \rangle}{\langle \psi_k(t) | \sigma_x | \psi_k(t) \rangle} = \frac{\langle \chi_{k^+} | \sigma_y | \chi_{k^+} \rangle}{\langle \chi_{k^+} | \sigma_x | \chi_{k^+} \rangle}.
\]
According to Eq. (2), (19) and (20), we can immediately obtain
\[
\frac{\langle \varphi_{k^+} | \sigma_y | \varphi_{k^+} \rangle}{\langle \varphi_{k^+} | \sigma_x | \varphi_{k^+} \rangle} = \frac{\text{Re}(h_y \mathcal{E} \mu_1) + \text{Im}(h_z \mathcal{E} \mu_1)}{\text{Re}(h_y \mathcal{E} \mu_1) - \text{Im}(h_z \mathcal{E} \mu_1)},
\]
\[
\frac{\langle \chi_{k^+} | \sigma_y | \chi_{k^+} \rangle}{\langle \chi_{k^+} | \sigma_x | \chi_{k^+} \rangle} = \frac{\text{Re}(h_y \mathcal{E} \mu_1) - \text{Im}(h_z \mathcal{E} \mu_1)}{\text{Re}(h_y \mathcal{E} \mu_1) + \text{Im}(h_z \mathcal{E} \mu_1)},
\]
where \( \mathcal{E} \mu_1 = h^*_x + \varepsilon^*_k \). Similarly, one can obtain \( \mathcal{E} \mu_1 = h^*_x - \varepsilon^*_k \) for the case of \( B < 0 \). Combining with Eq. (16) and (21), one can easily obtain the relations of Eq. (18).

In summary, we put forward the dynamic winding number as the integral of azimuthal angle in both Hermitian and non-Hermitian systems. In Hermitian systems, the dynamic winding number can be directly probed by long time average of spin polarization; In non-Hermitian systems, the dynamic winding number is only determined by the real part of azimuthal angle, which can also be physically observed by time evolution of left-left and right-right spin polarizations respectively governed by \( \hat{H} \) and \( \hat{H}^\dagger \). The dynamic winding number provides a powerful tool to uncover the topology of static Hamiltonian, namely, the equilibrium winding number in one dimension and the Chern number in two dimension.

### III. CONNECTION BETWEEN DYNAMIC WINDING NUMBER AND CONVENTIONAL WINDING NUMBER

Winding number has been widely used for characterize the topology of Hermitian systems with chiral symmetry. In one dimension, winding number can be applied to both Hermitian and non-Hermitian systems, with or without chiral symmetry. In this section, we consider a 1D two-band topological system governed by the Hamiltonian,
\[
H(k) = h_x(k) \sigma_x + h_y(k) \sigma_y + h_z(k) \sigma_z.
\]
The winding number for each band is defined as [14, 47],
\[
w_\mu = \frac{1}{\pi} \oint \limits_c dk \langle \chi_\mu | i \partial_k | \varphi_\mu \rangle = \frac{1}{2\pi} \oint \limits_c dk \frac{h_\mu \partial_k h_y - h_y \partial_k h_\mu}{\varepsilon_\mu(\varepsilon_\mu - h_z)}.
\]
where \( c \) is a closed loop with \( k \) varying from 0 to \( 2\pi \). Next, we will build relations between equilibrium winding number to the dynamic winding number in different situations.

#### A. Chiral symmetric systems

When \( h_z = 0 \), the Hamiltonian (22) has chiral symmetry \( \Gamma H(k) \Gamma = -H(k) \) with \( \Gamma = i \sigma_z \sigma_y \). The winding number for each band are the same, and we denote as
\[
w_\pm = \frac{1}{2\pi} \int \limits_c dk \frac{h_x \partial_k h_y - h_y \partial_k h_x}{h_x^2 + h_y^2}.
\]
The expression reduces to the Hermitian cases when \( \langle \chi_{\mu k} | = \langle \varphi_{\mu k} \rangle \). If we define an azimuthal angle as \( \phi_{yx} = \arctan(h_y/h_x) \), except as EPS, the above equation equals to
\[
w_\pm = \frac{1}{2\pi} \int \limits_c \partial_k \phi_{yx} dk,
\]
According to Eq. (14), we can immediately conclude that the equilibrium winding number equals to the dynamic winding number,
\[
w_e = w_d,
\]
under a few constraints of initial state: \( |c_{k^+}|^2 \neq |c_{k^-}|^2 \) for Hermitian systems and \( |c_{k^+}|^2 \neq 0 \cap |c_{k^-}|^2 \neq 0 \) for non-Hermitian systems.

![Figure 1](image)

**Figure 1.** Winding number \( w = 1 \). (a) and (b) indicate the dynamics evolution of the spin polarizations \( \langle \sigma^x \rangle \) and \( \langle \sigma^y \rangle \), respectively. (c) Time-averaged spin polarizations \( \overline{\sigma^x} \) (red line) and \( \overline{\sigma^y} \) (black line). (d) Angle \( \eta_{yx} \) with a function of \( k \), \( k_a \) and \( k_b \) is a discontinuity point.

To numerically verify our theory, we first consider the systems: \( h_x = t_0 + t_1 \cos(k) + t_2 \cos(2k) \) and \( h_y = t_1 \sin(k) + t_2 \sin(2k) - i \delta \). In the Hermitian case, the parameters are chosen as \( t_1 = 1, t_2 = 0 \) and \( \delta = 0 \).
As parameter $t_0$ varies from from $-\infty$ to $+\infty$, the system can appear topological phase transitions at $t_0 = t_1$ and $t_0 = -t_1$. The winding number is $w_c = 1$ for $|t_0| < t_1$, corresponding to equilibrium topological states with zero-energy edge states; while $w_c = 0$ for $|t_0| > t_1$, corresponding the equilibrium trivial states. According to the dynamic approach, we calculate the time evolutions of $(\sigma_x(k))$ and their long-time averages, see Fig. 1 (a)-(d) with $t_0 = 0.5t_1$. The expectation values of Pauli matrices $(\sigma_x)$ and $(\sigma_y)$ oscillate with time, with a momentum-dependent period $\tilde{t} = \pi/|\varepsilon_\mu|$, see Fig. 1 (a) and (b), respectively. After a long time evolution ($T = 100$), the time-average of $\overline{\sigma_x}$ and $\overline{\sigma_y}$ depend on quasi-momentum, see the red line and black line in Fig. 1 (c), respectively. According to Fig. 1(c), we calculate the angle $\eta_{yx}$ with a function of $k$ in Fig. 1 (d), where two discontinuity points $k_a$ and $k_b$ exist. Thus, the dynamic winding number can be obtained via section integration, 

$$w_d = \frac{1}{2\pi} (\int_{-\pi}^{\pi} \partial_k \eta_{yx} dk + \int_{k_a}^{k_b} \partial_k \eta_{yx} dk + \int_{k_b}^{\pi} \partial_k \eta_{yx} dk) \approx 1. \quad (27)$$

Obviously, the dynamic winding number agree well with the equilibrium winding number $w_c = 1$.

Figure 2. (a) The phase diagram of the chiral symmetry 1D topological systems with $t_2 = 1$. The colors from shallow to deep represent $v = 0, 1/2, 1, 3/2$, and 2 marked. (b)-(f) Angle $\eta_{yx}^R$ diagrams with different parameters $(t_0, \delta)$, here (b)(1.5, 0.2), (c)(1.5, 1), (d)(0.5, 1), (e)(0.2, 0.5) and (f)(0.5, 0.2), which corresponding to points b, c, d, e and f in (a), respectively.

In the non-Hermitian case with $\delta \neq 0$, the equilibrium winding numbers $w_c$ can appear half integer values in some parameter ranges, different from the integer values in the Hermitian systems. For simplicity, we take $t_1 = 1$, and $t_0, t_2, \delta$ is real. The dispersion of this Hamiltonian is

$$\varepsilon_\mu = \mu \sqrt{(t_0 - \delta - e^{-ik} + t_2e^{-2ik})(t_0 + \delta + e^{ik} + t_2e^{2ik})}. \quad (28)$$

The energy is symmetric about zero energy, which is ensured by the chiral symmetry. Since the energy gap must close at phase transition points, we can determine the phase boundaries by the band-crossing condition $\varepsilon_\mu(k) = 0$, which yields $t_0 = \pm \delta + 1 - t_2$ and $t_0 = \pm \delta - 1 - t_2$ for arbitrary $t_2$. Particularly, $t_0 = t_2 \pm \delta$ if $|t_2| > 0.5$. Fixing $t_2 = 1$ and changing both $\delta$ and $t_0$, we calculate topological phase diagram distinguished by their winding numbers, see Fig. 2 (a). Here, the white, blue, green, deep yellow and light yellow regions possess winding number $w_c = 0, 1/2, 1, 3/2$ and 2, respectively. In Figs. 2 (b)-(f), we also give the angle $\eta_{yx}^R$ versus the quasi-momentum $k$ with different parameters $(t_0, \delta)$ marked as b, c, d, e, f in the Figs. 2 (a). The dynamic winding number are 0, 1, 1/2, 3/2 and 2, respectively. The numerical results are in well agreement with the theoretical prediction as well, which prove the validity for our dynamic approach once again.

B. Non-chiral symmetric systems

When $h_z \neq 0$, the Hamiltonian (22) breaks the chiral symmetry. Unlike the systems with chiral symmetry, the winding number for each band is not a quantized number, suggesting that $\omega_\pm$ is no longer a topological invariant. However, the summation of the winding number of different bands, $w^\ell_c = \omega_+ + \omega_-$, has been demonstrated to be a topological invariant[14]. So that one can obtain the new topological invariant

$$w^\ell_c = \omega_+ + \omega_- = \frac{1}{\pi} \int_{-\pi}^{\pi} \partial_k \phi_{yx} dk, \quad (29)$$

The topological invariant $w_c$ is independent of $h_z$, although its definition is related to the eigenvector of $H(k)$. The parameters $h_x$ and $h_y$ become very important for the definition of topological invariant. Except for the exceptional point $h_x^2 + h_y^2 = 0$, we introduce a complex angle $\phi_{yx}$ satisfying $\tan(\phi_{yx}) = h_y/h_x$. In terms of $\phi_{yx}$, $w^\ell_c$ can be represented as

$$w^\ell_c = \frac{1}{\pi} \int_{-\pi}^{\pi} \partial_k \phi_{yx} dk, \quad (30)$$

where the integral is also taken along a loop with $k$ from $0$ to $2\pi$. According to Eq. (14), we can relate the equilibrium winding number to the dynamic winding number

$$w^\ell_c = 2w_d, \quad (31)$$

under a few constraints of initial state: $|e_{k+}|^2 \neq |e_{k-}|^2$ for Hermitian systems and $|e_{k+}|^2 \neq 0 \cap |e_{k-}|^2 \neq 0$ for
non-Hermitian systems. Fixing $h_z = 0.5$ and changing both $\delta$ and $t_0$, we calculate topological phase diagram distinguished by their winding numbers, see Fig. 3 (a). Here, the white, green, and light yellow regions possess winding number $w_c = 0, 1$ and 2, respectively. In Figs. 3 (b)-(f), we also give the angle $\eta_{ji}^R$ versus the quasi-momentum $k$ with different parameters $(t_0, \delta)$ marked as b, c, d, e, f in the Fig. 3 (a). The dynamic winding number are 0, 1/2, 1, 1/2 and 0, respectively. The numerical results are in well agreement with the theoretical prediction as well, which demonstrate the validity for our dynamic approach.

Figure 3. (a) The phase diagram of the without chiral symmetry 1D topological systems with $t_2 = 0$. The colors from shallow to deep represent $w_c^b = 0, 1$ and 2 marked. (b)-(f) Angle $\eta_{ji}^R$ diagrams with different parameters $(t_0, \delta)$, here (b) $(1.7, 0.3)$, (c) $(1, 0.3)$, (d) $(0.3, 0.3)$, (e) $(0.3, 1)$ and (f) $(0.3, 1.7)$, which corresponding to points b, c, d, e and f in (a), respectively.

C. Physical observables by spin polarization

For both chiral and non-chiral symmetric systems, the topological invariant $w_c$ and $w_z^c$ can be obtain by dynamic winding number $w_d$. In the non-Hermitian system, the dynamical azimuthal angles $\eta_{ji} = \phi_{ji}$ generally do not represent physical observables for non-Hermitian case. However, only the real part of azimuthal angle is important and can be directly measured via $\phi_{ji}^a$ and $\phi_{ji}^b$. It requires to evolve the initial state under two different Hamiltonians $H$ and $H^\dagger$ and measure the long time average of spin polarization corresponding to $|\psi_k(t)\rangle$ and $|\psi_k(t)\rangle$. According to Eqs. (17) and (18), one can finally obtain the real angles $\phi_{ji}^a$ and $\phi_{ji}^b$. Here, we find that the two real angles satisfy $\phi_{ji}^a = \phi_{ji}^b$ for the 1D chiral symmetric systems, which can be easily proved by Eqs. (21). It means that we only need to measure the real angles $\phi_{ji}^a$ in experimentally. However, the two real angles are no longer equal for the 1D non-chiral symmetric systems. As an example, we calculate the real angles $\phi_{yx}^a$ and $\phi_{yx}^b$, as a function of $k$ with $h_z = 0$ for chiral symmetric systems and $h_z = 0.5$ for non-chiral symmetric systems, see Fig. 4. The other parameters are the same as Fig. 3 (c). The dynamic winding number $w_d = 1/2$ for both the chiral or non-chiral symmetric systems. For the chiral symmetric systems, one can clearly see the real angles $\phi_{yx}^a = \phi_{yx}^b$, and $w_d^c = w_d^b = 1/2$. However, for the non-chiral symmetric systems, $\phi_{yx}^a(k)$ and $\phi_{yx}^b(k)$ do not cover with each other, and $w_d^c = 1$ and $w_d^b = 0$.

Figure 4. The real angles $\phi_{yx}^a$ and $\phi_{yx}^b$ as a function of $k$. (a) chiral symmetric systems with $h_z = 0$. (b) Non-chiral symmetric systems with $h_z = 0.5$. The other parameters same as the Fig. 3 (c).

IV. CONNECTION BETWEEN DYNAMIC WINDING NUMBER AND CHERN NUMBER

In two dimensional parameter space, Chern number is an important topological invariant to explain many important topological phenomena, such as integer quantum hall effect, Thouless pumping, chiral edge states, etc. Chern number can also be utilized to character topology in non-Hermitian systems. In this section, we build a concrete relation between Chern number and the dynamic winding number for both the Hermitian and non-Hermitian two-band models. The concrete relation provide a powerful method to detect Chern number via pure quantum dynamics.
A. Hermitian systems

In the 2D Hermitian two-band models, the quasi-momentum $\mathbf{k}$ is lying in $(k_x, k_y)$ Brillouin zone (BZ). The Hamiltonian $H(\mathbf{k})$ defines a mapping from the BZ to the unconstrained Bloch vectors $\hat{\mathbf{h}}(\mathbf{k}) = (h_x, h_y, h_z)$. The Bloch functions only depend on the normalized vector field $\vec{n}(\mathbf{k}) = |\hat{\mathbf{h}}(\mathbf{k})|$, a point in the surface of Bloch sphere. Thus, the topology of Bloch functions in the quasi-momentum space can be mapped to the topology of the vector in Bloch sphere. Further, the normalized vector $\vec{n}(\mathbf{k})$ can be reshaped as

$$\vec{n}(\mathbf{k}) = (\sin(\theta) \cos(\phi), \sin(\theta) \sin(\phi), \cos(\theta)).$$  (32)

Here, $\theta_i = \arccos(h_i/|\hat{h}(\mathbf{k})|)$ is the polar angle between the Bloch vector and reference $(\mathbf{z})$ axis. The azimuthal angle denotes as $\phi = \phi_{jl} = \arctan(h_j/h_i)$, $i,j,l \in (x,y,z)$ and $i \neq j \neq l$. The choice of reference axis does not affect the calculation of Chern number, since different references can be conversed to each other by a unitary transformation. Thus, the eigenstate with the lowest energy is given as

$$|\varphi_-(\theta_i, \phi_{jl})\rangle = \left( -e^{-i\phi_{jl}/2} \cos\left(\frac{\theta_i}{2}\right), e^{i\phi_{jl}/2} \sin\left(\frac{\theta_i}{2}\right) \right).$$  (33)

$\vec{n}(\mathbf{k}) = (0, 0, 1)$ and $(0, 0, -1)$ respectively correspond to the "north" and "south" pole of the Bloch spherical surface. These points are singularity points where the azimuthal angle is not well defined. When the topological invariant is non-zero, the vector $\vec{n}(\mathbf{k})$ necessarily maps at least one point in the BZ to the "south" or "north" poles [10]. From the Bloch wavefunctions, the Chern number assigned to the lowest energy band is defined as

$$C = \frac{1}{2\pi} \int dk F(\mathbf{k}),$$  (34)

where the associated Berry curvature $F(\mathbf{k})$ and the Berry connection $A_{k_{x(y)}}$ are given as [7, 58]

$$F(\mathbf{k}) = \partial_{k_x} A_{k_y} - \partial_{k_y} A_{k_x},$$

$$A_{k_{x(y)}} = i \langle \varphi_- | \partial_{k_{x(y)}} | \varphi_- \rangle = \frac{\cos(\theta_i)}{2} \partial \phi_{jl}.$$  (35)

We discrete the parameter space $(k_x, k_y)$ by $N \times M$ mesh grids in the first Brillouin zone [59], thus that the Chern numbers can be expressed as

$$C = \frac{1}{2\pi} \sum_{l_x=1}^{N} \sum_{l_y=1}^{M} \int_{k_{x,l_x}}^{k_{x,l_x+1}} \int_{k_{y,l_y}}^{k_{y,l_y+1}} dk F(\mathbf{k}).$$  (36)

For each grid, a direct application of the two-dimensional Stokes theorem implies [60]

$$C = \frac{1}{2\pi} \sum_{l_x=1}^{N} \sum_{l_y=1}^{M} \oint_{s}^{+} (A_{k_x} dk_x + A_{k_y} dk_y),$$  (37)

where $s_{l_x,l_y}^{+}$ represents the counterclockwise path integration of the $(l_x, l_y)$ grid. The major contributions to the Chern number come from the winding number near the poles. Around the "north" poles, we have $\theta_i \approx 0$ and Berry connection $A_{k_{x(y)}} \approx \frac{1}{2} \partial \phi_{jl}$. Around the "south" poles, we have $\theta_i \approx \pi$ and Berry connection $A_{k_{x(y)}} \approx -\frac{1}{2} \partial \phi_{jl}$. The Chern number can be reshaped as

$$C = \frac{1}{4\pi} \sum_{k_0 \in \text{poles}} \langle \text{sgn}(h_i(k_0)) \oint_{k_0} \partial_k \phi_{jl}(\mathbf{k}) dk \rangle$$

$$= \frac{1}{2} \sum_{k_0 \in \text{poles}} \langle \text{sgn}(h_i(k_0)) w_d(k_0) \rangle,$$  (39)

where $w_d(k_0)$ is the dynamic winding number around the pole at $k_0$.

B. Non-Hermitian systems

In the presence of dissipation, the systems described by non-Hermitian Hamiltonians, whose energy spectra are generally complex. By generalizing the notion of gapped band structures to the non-Hermitian case, the Chern number for an energy separable band can be constructed from these eigenstates in a similar way as in Hermitian systems [48]. In contrast to Hermitian systems, there are left-right, right-right, left-left, right-left Chern numbers in non-Hermitian systems, depending on the Berry connection $A^L_R = i \langle \chi_\mu | \partial_k | \varphi_\mu \rangle$, $A^R_L = i \langle \varphi_\mu | \partial_k | \chi_\mu \rangle$, $A^L_L = i \langle \chi_\mu | \partial_k | \chi_\mu \rangle$ or $A^R_R = i \langle \varphi_\mu | \partial_k | \varphi_\mu \rangle$. Although the corresponding Berry curvatures are indeed locally different quantities, the four kinds of Chern numbers are the same. Here, we only focus on analyzing the Chern number defined with left-right Berry connection $A^L_R = i \langle \chi_\mu | \partial_k | \varphi_\mu \rangle$.

We map the Hamiltonian to normalized Bloch vector $\vec{n}$ as Eq. (32). Due to $\hat{H} \neq \hat{H}^\dagger$, the normalized Bloch vector satisfies $\vec{n} \neq \vec{n}^\dagger$. Then, the left and right eigenstates with the lowest energy are respectively given as

$$\langle \chi_- | \theta_i, \phi_{jl} \rangle = \left( -e^{i\phi_{jl}/2} \cos\left(\frac{\theta_i}{2}\right), e^{-i\phi_{jl}/2} \sin\left(\frac{\theta_i}{2}\right) \right),$$

$$|\varphi_- | \theta_i, \phi_{jl} \rangle = \left( e^{i\phi_{jl}/2} \cos\left(\frac{\theta_i}{2}\right), e^{-i\phi_{jl}/2} \sin\left(\frac{\theta_i}{2}\right) \right).$$  (40)

The right and left eigenstates have a phase singularity at $\vec{n}(\mathbf{k}) = (0, 0, \pm 1)$, corresponding to the north and south exceptional points (EPS) of the virtual Bloch spherical...
vector $\hat{h}(k) = (h_x, h_y, h_z)$, where

$$h_x = t_x \sin(k_x); h_y = t_y \sin(k_y),$$

$$h_z = m_z - t_z \cos(k_x) - t_z \cos(k_y) - i \delta.$$  

Here, $t_x(y)$ and $t_z$ denote spin-orbit coupling parameters. $m_z$ is the effective magnetization. $\delta$ is gain or loss strength. For $\delta = 0$, the system reduces to a quantum anomalous Hall (QAH) model [61], which has been realized in recent experiments [27, 62].

According to the topological band theory for Non-Hermitian system [48], Chern number can be defined for ‘gap’ bands, where the real part of bands are separated. Otherwise, the Chern number is not well defined. We first show the topological phase diagram in the parameter plane $(m_z, \delta)$, see Fig. 5 (a). The other parameters are chosen as $t_x = t_y = t_z = 1$. The Chern numbers of the first band are $C = 0$ in the green region, $C = 1$ in the grey region, and not well defined in the white region. The boundaries satisfy $(m_z - 1)^2 + \delta^2 < 1$ for the grey region and $m_z > 2$ for the green region. Varying $\delta$ along the dashed red arrow in Fig. 5 (a), we explore the corresponding relation between the Chern number and energy modes under open boundary condition, see Fig. 5 (b)–(e). The parameters of Fig. 5 (b)–(e) correspond to the point $a$–$d$ in Fig. 5(a), where $m_z = 1$. In the Hermitian case ($m_z = 1, \delta = 0$), one can clearly see the two bulk bands (red and blue regions) lying in the real energy axis, and the edge-state modes (black line) connecting the two bulk bands. In the Non-Hermitian case ($m_z = 1, \delta = 0.5$), one can see that the complex bulk bands are still gapped and the edge-state modes still preserved in the real energy axis. In the phase boundary and white region, both the real gap of the bulk bands and edge-state modes disappear, see Fig. 5 (d) and (e), respectively. Similar to the Hermitian cases, the edge state is still a clear signature of the topological phase in non-Hermitian systems. Adding small dissipation term $\delta$ do not change the topology of the original Hermitian systems. However, large dissipation will destroy the topological phases. In the following, we mainly illustrate how to relate the dynamical winding number to the nontrivial
Chern number in Hermitian case \((m_z = 1, \delta = 0)\) and non-Hermitian case \((m_z = 1, \delta = 0.5)\).

In the Hermitian case \((m_z = 1, \delta = 0)\), we first determine the north and south poles in the parameter space \((k_x, k_y)\). Since the poles are related to the oriental axis, we select \(\theta = \theta_y = \arccos(h_y/|\tilde{h}(k)|)\) and \(\phi = \phi_{xz} = \arctan(h_x/h_z)\). In the parameter space \((k_x, k_y)\), by solving \(h_x^2 + h_y^2 = 0\), we find the location of north and south poles at \(k_0 = (k_x, k_y) = (0, \pm \pi/2)\), see the blue and red dot in Fig. 6 (a), respectively. Here, \(sgn[h_y(\pm \pi/2)] = \pm 1\) respectively correspond to the ‘north’ pole and ‘south’ pole. Once the north and south poles are known, we need to extract the dynamical winding number around the two poles. We randomly choose an initial states \(|\psi_k(0)\rangle = \sum_{\mu} c_{k\mu} |\varphi_{k\mu}\rangle\) with \(|c_{k+}|^2 > |c_{k-}|^2\) and calculate the instantaneous expectation values \(\langle \sigma_z(k,t) \rangle\) and \(\langle \sigma_x(k,t) \rangle\) in the long time evolution, \(i\partial_t |\psi(t)\rangle = H(k)|\psi(t)\rangle\). The dynamical azimuthal angle \(\eta_{xz}(k) = \arctan(\langle \sigma_z(k)/\sigma_x(k) \rangle)\) can be extracted via long time average values \(\langle \sigma_{xz}(k) \rangle = 1/T \int_0^T \langle \sigma_{xz}(k,z) \rangle \, dt\) \((T = 100\) in this case), see Fig. 6 (b). We also calculate the static azimuthal angle \(\phi_{xz}(k)\) via the eigenstates, see Fig. 6 (c). The dynamical azimuthal angle \(\eta_{xz}(k)\) and equilibrium azimuthal angle \(\phi_{xz}(k)\) are almost the same, in other words, \(\eta_{xz}(k)\) converges to \(\phi_{xz}(k)\) for long time. Then, we can respectively obtain the dynamical winding numbers for the north and south poles via integral the azimuthal angle \(\eta_{xz}(k)\) over the blue and red trajectories in Fig. 6 (a). The dynamical winding numbers for the north and south poles are respectively given as \(w_a = \pm 1\). Applying Eq. (39), we can obtain the Chern number as 1. Similarly, based on azimuthal angle \(\phi_{xz}\) of in Fig. 6 (c), we also give the integral around the north and south poles, obtaining the equilibrium winding number \(w_c = \pm 1\), respectively. With the equation (39), sum of the equilibrium winding number adhered by a sign factor gives the exactly same Chern number \(C = 1\).

In the non-Hermitian case, we can obtain the left-right Chern number by the similar procedure. We also select \(\theta = \theta_y = \arccos(h_y/|\tilde{h}(k)|)\) and \(\phi = \phi_{xz} = \arctan(h_x/h_z)\). Here, \(\theta_y\) is real and \(\phi_{xz}\) is generally complex. In the parameter space \((k_x, k_y)\), by solving \(h_x^2 + h_z^2 = 0\), the original two poles in Hermitian case are split into four EPS as \(\delta\) increases, see Fig. 6 (d). The blue and red points represent ‘north’ and ‘south’ EPS. Similarly, the “north” EPSs have \(sgn[h_y(\pm \pi/2)] = 1\) and the “south” EPSs have \(sgn[h_y(\pm \pi/2)] = -1\). In Figs. 6 (e) and 6 (f), we respectively give the azimuthal angle \(\eta_{xz}^R\) and the equilibrium azimuthal angle \(\phi_{xz}^R\) in parameter space \((k_x, k_y)\), which are almost the same. One can also easily obtain the dynamic winding number by integral the azimuthal angle \(\eta_{xz}^R\) over the paths encircling the north and south EPS. In Fig. 6 (d), the blue and red circles denote the four integral paths around the north and south EPS. The dynamical winding numbers for the north and south EPS are \(1/2\) and \(-1/2\), respectively. Finally we obtain the left-right Chern number \(C^{LR} = 1\) via equation (41), consistent with the one obtained via eigenstates.

For completeness, we also show the dynamic azimuthal angles defined with the right-right and left-left spin polarizations, see Fig. 7. The dynamic azimuthal angles are quite different from each other, corresponding to \(\phi_{xz}^L\) and \(\phi_{xz}^R\), respectively. Nevertheless, the north and south EPSs are the same as those in the Fig. 6 (d). Around an EP, we can extract the right-right and left-left dynamic winding number \(w_d^R\) and \(w_d^L\), which satisfy \(w_d = \frac{1}{2}(w_d^R + w_d^L)\). One
important thing is that the dynamic azimuthal angles \( \phi^x_x \) and \( \phi^y_x \), defined with the real left-left and right-right spin polarizations are accessible in experimental measures.

V. CONCLUSION AND DISCUSSION

In summary, we put forward a new concept ‘dynamic winding number’ and connect it to topological invariants in both Hermitian and non-Hermitian two-band models. In 1D systems with chiral symmetry, the conventional winding number equals to the dynamic winding number. In 1D systems without chiral symmetry, the conventional winding number is twice as the dynamic winding number. In 2D Hermitian systems, the Chern number is related to the sum of dynamic winding number of the ‘north’ and ‘south’ poles with opposite sign. In 2D non-Hermitian systems, the Chern number is related to the sum of dynamic winding number of the ‘north’ and ‘south’ poles with opposite sign. Dynamic winding number provides a general dynamical approach in exploring static band topology, paving the way to detect topological invariants in both Hermitian and non-Hermitian systems.

There are several advantages of the dynamical winding number approach to extract topological invariants. First, the initial states can be easily prepared since there is less constraints on the initial state. There is no need to prepare perfect ground state uniformly filling topological band, or quenching from ground states of a trivial band. Second, our methods are capable in both Hermitian and non-Hermitian systems. Most traditional methods are workable in the Hermitian systems but fails to extract topological invariants in non-Hermitian systems [55, 63]. Third, the dynamical approach can be easily applied to the systems with larger Chern number. The larger Chern number is always associated with small energy gap, which makes it hard to detect by adiabatically sweeping the whole Brillouin zone. However, our method does not need to satisfy the adiabatic condition. Although there will be more than one pair of ‘north’ and ‘south’ poles, it will not increase much more effort to extract Chern number. The generalizations to multi-band/higher dimensional/interacting systems are deserved further study.

One can immediately apply the dynamical approach for topological Hermitian systems. Cold atom systems is an excellent platform to realize topological band models and detect topological invariants. One and two dimensional spin-orbit couplings have been realized in a highly controllable Raman lattice [27, 64–66]. Initial states are quite easily prepared by loading the atoms into the lattices. Here, the initial constraint \( |c_{k+}|^2 \neq |c_{k-}|^2 \) may be not satisfied for some specific momentum \( k \), but the occurred probability is so small that the global dynamical azimuthal angle is not affected due to the topological nature. The spin population \( N_{\uparrow}(k) \) with different momentum can be measured by spin-resolved time-of-flight (TOF) absorption imaging [27]. Thus, one can obtain the spin population difference \( \langle \psi_k(t)\sigma_z|\psi_k(t) \rangle = (N_{\uparrow}(k) - N_{\downarrow}(k))/(N_{\uparrow}(k) + N_{\downarrow}(k)) \). The spin polarization \( \langle \psi_k(t)\sigma_x|\psi_k(t) \rangle \) can be transferred to the spin population difference by applying \( \pi/2 \) pulse, that is, \( \langle \psi_k(t)\sigma_x|\psi_k(t) \rangle = \langle \psi_k(t)e^{-i\frac{\pi}{2}\sigma_z}|\psi_k(t) \rangle \). Because the cold atom systems have long coherent time, there is no obstacle to extract the dynamic winding number via long time average of the spin polarization.

To apply the dynamical approach in topological non-Hermitian systems, we should first consider how to realize the topological non-Hermitian models in experiments. Since two-level non-Hermitian models have been widely realized in optical systems, such as two coupled optical cavities [67, 68], optical waveguides [53, 69, 70], optomechanical cavity [71–73] etc. We mainly discuss how to extract dynamic winding number with two optical waveguides with tunable parameters. A two-level non-Hermitian system can be realized by introducing gain and loss in the two waveguides. The coupling strength can be tuned by the waveguide separation. We regard the two different waveguides as two spin components. The initial states can be prepared by randomly split the light injecting into the two waveguides. One can obtain \( \langle \psi(l)\sigma_z|\psi(l) \rangle \) by measuring the intensity difference between two waveguides at propagating distance \( l \). Here, the distance \( l \) plays the role of time. Actually, the final states will collapse into one of the eigenstate in the long distance. Thus, the output intensity difference of the waveguides is sufficient and long distance average of the intensity difference is not necessary. One can also obtain \( \langle \psi(l)\sigma_y|\psi(l) \rangle \) by inserting a beam splitter before intensity measurement. By designing the waveguide separation and gain and loss rates, one can simulate the two-band model. Repeating the above operations, one can finally construct the dynamics winding number.

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