Abstract. Let $\tilde{X}$ be a locally finite complete Gromov hyperbolic metric graph with the geometric boundary consisting of infinite points. Suppose that there is a discrete subgroup of the isometry group $Iso(\tilde{X})$ acting geometrically on $\tilde{X}$. The $\lambda$-Martin boundary is the boundary of the image of an embedding from $\tilde{X}$ to the space of $\lambda$-superharmonic functions.

We show that the $\lambda$-Martin boundary coincides with the geometric boundary for any $\lambda \in [0, \lambda_0]$, in particular at the bottom of the spectrum $\lambda_0$.

1. Introduction

Let $\tilde{X}$ be a locally finite Gromov hyperbolic metric graph with the geometric boundary consisting of infinite points. Suppose that there exists a group $\Gamma$ acting isometrically and geometrically on $\tilde{X}$.

In this article, we study Brownian motion on a hyperbolic metric graph $\tilde{X}$. Brownian motion is a random process of which the probability density $p(t, x, y)$ of going from $x$ to $y$ at time $t$ is given by the heat equation (see Definition 2.13). Therefore, to define a Brownian motion, we first need to define a Laplacian operator. For Riemannian manifolds, the canonical choice is the Laplace-Beltrami operator.

We start with a more general metric spaces, namely strip complexes. A strip complex $\tilde{X}\tilde{M}$ is the product space of a graph $\tilde{X}$ and a manifold $\tilde{M}$. On strip complexes, we choose the Laplace operator defined by Bendikov, Saloff-Coste, Salvatori and Woess, which is related to Dirichlet forms [BSSW]. It is a generalization of Laplace-Beltrami operator in the sense that the restriction of the operator on the product of an open edge of the graph and the manifold is the Laplacian on the product space seen as a manifold.

We define the Sobolev space $W^{1,p}(O)$ on a relatively compact open set $O$ of the strip complex $\tilde{X}\tilde{M}$ and a Dirichlet form $E$ whose domain is $W^{1}(O)$. There exists a one-to-one correspondence between the family of closed forms and the non-negative definite self-adjoint operators. Using the one-to-one correspondence, we define the graph version of the Laplacian $\Delta, Dom_{O}(\Delta)$ on any open set $O$.

Let a discrete group $\Gamma$ acts geometrically on $\tilde{X}\tilde{M}$. The bottom of the spectrum of Laplacian depends on the group $\Gamma$. The group $\Gamma$ is non-amenable if and only if the bottom of the spectrum of the Laplacian is positive ([SW] Theorem 8.5).

We construct the heat kernel $p(t, x, y)$ on the strip complex as the limit of the heat kernel $p_{O_i}(t, x, y)$ for an increasing sequence of relatively compact connected open sets $O_i$ covering $\tilde{X}\tilde{M}$. This construction enables us to obtain the convergence of the Green function and the existence of a harmonic function. For that purpose, we observe two properties of the eigenfunctions of the Laplacian $\Delta, Dom_{O}(\Delta)$. The first property is that eigenvalues are discrete and the dimension of each eigenspace is finite. The second property is the existence of the orthonormal basis $L^{2}(O)$ consisting of the eigenfunctions of the Laplacian $\Delta, Dom_{O}(\Delta)$.
After constructing the heat kernel, we consider the \( \lambda \)-Green function \( \int_0^x e^{\lambda t} p(t, x, y) dt \) for any two distinct points \( x, y \in \tilde{X} \tilde{M} \). The \( \lambda \)-Green function converges for all \( \lambda \in [0, \lambda_0) \). Using Sullivan’s idea [Su], we show the existence of the positive \( \lambda \)-harmonic functions on the graph for any \( \lambda \in [0, \lambda_0] \). Following the proof in [LS], we prove the convergence of the \( \lambda_0 \)-Green function.

In the second part, we restrict our attention to hyperbolic graphs and their Martin boundary. Our main result is the uniform Ancona inequality, also called Ancona-Gouëzel inequality on hyperbolic graphs (see [GL] and [G] for random walks).

**Theorem 1.1.** Let \( \tilde{X} \) be a locally finite complete Gromov hyperbolic metric graph with the geometric boundary consisting of infinite points. Suppose that a graph \( \Gamma \) acts isometrically and geometrically on \( \tilde{X} \). For all \( \lambda \in [0, \lambda_0] \) and for three points \( x, y, z \) on the same geodesic \( [x, z] \) with \( d(x, y) \geq 1 \) and \( d(y, z) \geq 1 \),

\[
C^{-1} G_\lambda(x, y) G_\lambda(y, z) \leq G_\lambda(x, z) \leq CG_\lambda(x, y) G_\lambda(y, z).
\]

(1.1)

The most non-trivial part is the uniformity of the constant \( C \) in Equation (1.1) on \( \lambda \), which implies equation (1.1) for \( \lambda = \lambda_0 \). To show (1.1), we first prove a graph version of Harnack inequality: there exists a constant \( C_n \) such that for all \( \lambda \in [0, \lambda_0] \) and for two distinct points \( x, y \) with \( 1 \leq d(x, y) \leq n + 1 \),

\[
\frac{G_\lambda(x, z)}{G_\lambda(y, z)} \leq C_n.
\]

(1.2)

The Brownian motion associated with a Dirichlet form on a graph is a Hunt process, in particular a Markov process. Using strong Markov properties of our Brownian motion, we show that the relative \( \lambda \)-Green function \( G_\lambda(x, z : B(x, r)^c) \) decays super-exponentially fast, from which the uniform Ancona-Gouëzel inequality follows, an idea due to S. Gouëzel [G].

We remark that there exists a constant \( C_m \) such that for all \( \lambda \in [0, \lambda_0] \) and for two distinct points \( y, z \) with \( 1 \leq d(y, z) \leq m \),

\[
C_m^{-1} \leq G_\lambda(y, z) \leq C_m.
\]

(1.3)

However, (1.2) and (1.3) together implies only an inequality similar to (1.1) with constant depending on the distance between points \( x, y \) and \( z \).

Using Ancona-Gouëzel inequality for \( \lambda \in [0, \lambda_0] \), we show the next main theorem

**Theorem 1.2.** Let \( \tilde{X} \) be a locally finite complete Gromov hyperbolic metric graph with the geometric boundary consisting of infinite points. Suppose that a graph \( \Gamma \) acts isometrically and geometrically on \( \tilde{X} \). The \( \lambda \)-Martin boundary of \( \tilde{X} \) coincides with the geometric boundary for all \( \lambda \in [0, \lambda_0] \).

An important question in the study of Brownian motions is whether the local limit theorem on a hyperbolic graph holds: does there exist a function \( c : \tilde{X} \times \tilde{X} \to \mathbb{R} \) such that for two distinct points \( x, y \in \tilde{X} \),

\[
\lim_{t \to \infty} t^{3/2} e^{\lambda t} p(t, x, y) = c(x, y)?
\]

The proof of the local limit theorem for random walks or Brownian motion on Riemannian manifolds uses various strategies (see [Bo], [GL], [G] and [LL]). In particular, in [LL], one uses Gibbs measure associated to a pressure which is defined using \( \lambda \)-Green function (see [LL]). We hope that Theorem 1.1 will enable us to apply thermodynamics formalism of the geodesic flow on a hyperbolic graph and Gibbs measures on the \( \lambda_0 \)-Martin boundary.

The article is organized as follows. In Section 2, we define the Laplacian and construct the heat kernel on strip complexes.
In Section 3, we observe properties of positive $\lambda$-harmonic functions on strip complexes. Using these properties, we show that if the group $\Gamma$ is non-amenable, then the $\lambda_0$-Green function converges.

In Section 4, we first prove Ancona-Gouëzel inequality (1.1) (Theorem 4.9 and Theorem 4.15). Using the inequality (1.1), we show Theorem 1.2.

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2. LAPLACIAN AND HEAT KERNEL

Let $\hat{X} = (V, E)$ be a locally finite connected complete graph. Let $d$ be a metric on $\hat{X}$ and $l$ be the length function of $d$. Let $l_e$ be the length of the edge $e$. Let $\hat{M}$ be a Riemannian manifold. The strip complex $\hat{X}\hat{M}$ is defined as the product metric space $\hat{X} \times \hat{M}$. Let $e$ be an edge of the graph $\hat{X}$. Denote the open edge of $e$ by $e^o$. Denote by $S_e = e \times \hat{M}$ and $S_e^c = e^o \times \hat{M}$ the closed strip and the open strip of edge $e$, respectively. If two edges $e$ and $e'$ intersect a vertex $v$, then the strips $S_e$ and $S_{e'}$ meet along $\hat{M}_v = \{v\} \times \hat{M}$. The space $\hat{M}_v$ is called the bifurcation manifold of the vertex $v$. Choose an orientation of the edge $e$ once for all. Denote by $i(e)$ and $t(e)$ the initial vertex and the terminal vertex of the edge $e$, respectively.

Let $\Gamma$ be a non-amenable discrete group. Suppose that $\Gamma$ acts isometrically and geometrically (i.e. properly and cocompactly) on $\hat{X}\hat{M}$. Using the barycenter subdivision if necessary, we may assume that $\Gamma$ acts without inversions. Denote $X\hat{M} = \hat{X}\hat{M}/\Gamma$ and fix a fundamental domain $F$ of $\hat{X}\hat{M}$. Choose $S = \{g \in \Gamma | F \cap gF \neq \phi\}$ as a set of generators of $\Gamma$, which is finite by Proposition I.8.19 in [BH]. We denote the word distance of $\Gamma$ with respect to $S$ by $d_{\Gamma}$.

To define the transition probability of the Brownian motion on a strip complex, we take a strip complex version of the Laplacian defined in [BSSW].

2.1. Dirichlet forms. This section is devoted to the theory of Dirichlet forms needed to define the Laplacian on strip complexes in Section 2.2 (see [FOT] for general Hilbert space).

Let $\mu$ be the length Lebesgue measure on $\hat{X}\hat{M}$ defined as follows: for any measurable function $f$,

$$\int_{\hat{X}\hat{M}} f d\mu := \sum_{e \in E} \int_{S_e} f|_e(s) ds dvol_{\hat{M}}.$$  

Definition 2.1. Let $H = L^2(\hat{X}\hat{M}, d\mu)$ and let $Dom(\mathcal{E})$ be a dense subspace of $H$. Denote the standard $L^2$-inner product by $\langle \cdot , \cdot \rangle$. The map $\mathcal{E} : Dom(\mathcal{E}) \times Dom(\mathcal{E}) \rightarrow \mathbb{R}$ is a symmetric form and $Dom(\mathcal{E})$ is the domain of $\mathcal{E}$ if the following properties hold: for all $u, v, w \in Dom(\mathcal{E})$ and $\alpha \in \mathbb{R}$,

$$\mathcal{E}(\alpha u + v, w) = \alpha \mathcal{E}(u, w) + \mathcal{E}(v, w)$$

$$\mathcal{E}(u, u) \geq 0$$

$$\mathcal{E}(u, v) = \mathcal{E}(v, u).$$

Let $(\mathcal{E}, Dom(\mathcal{E}))$ be a symmetric form. For any $\alpha > 0$, we define another symmetric form $\mathcal{E}_{\alpha}$ for all $\alpha > 0$ as follows:

$$\mathcal{E}_{\alpha}(u, v) := \mathcal{E}(u, v) + \alpha \langle u, v \rangle$$

for all $u, v \in Dom(\mathcal{E})$ and $Dom(\mathcal{E}_{\alpha}) = Dom(\mathcal{E})$.

In particular, when $\alpha = 1$, we call $\sqrt{\mathcal{E}_1(u, u)}$ the $\mathcal{E}_1$-norm of a function $u$ in $Dom(\mathcal{E})$. 
For all $u, v \in \mathcal{H}$, denote by $\wedge$ and $\lor$ the minimum and the maximum functions:

$$u \wedge v(x) = \min \{u(x), v(x)\} \quad \text{and} \quad u \lor v(x) = \max \{u(x), v(x)\}.$$

Then, we define the Dirichlet form as follows.

**Definition 2.2** (Dirichlet form). Let $\mathcal{E}$ be a symmetric form with domain $\text{Dom}(\mathcal{E}) \subset \mathcal{H}$.

1. Let $C_0(\tilde{X} \tilde{M})$ be the space of continuous functions on $\tilde{X} \tilde{M}$ that vanish at infinity. A subspace $\mathcal{C}$ of $\text{Dom}(\mathcal{E}) \cap C_0(\tilde{X} \tilde{M})$ is a core if $\mathcal{C}$ is dense in $\text{Dom}(\mathcal{E})$ with $\mathcal{E}_1$-norm and $\mathcal{C}$ is dense in $C_0(\tilde{X} \tilde{M})$ with uniform norm $\| \cdot \|_X$. The symmetric form $\mathcal{E}$ is regular if $\mathcal{E}$ has a core.

2. A symmetric form $\mathcal{E}$ is strongly local if for all compactly supported functions $u, v \in \text{Dom}(\mathcal{E})$, $\mathcal{E}(u, v) = 0$, when $v$ is constant on a neighborhood of $\text{supp}(u)$.

3. A symmetric form $\mathcal{E}$ is closed if for any sequence of functions $u_n$ in $\text{Dom}(\mathcal{E})$ satisfying $\lim_{n \to \infty} \mathcal{E}_1(u_n - u, u_n - u) = 0$, there exists a function $u$ in $\text{Dom}(\mathcal{E})$ such that $\lim_{n \to \infty} \mathcal{E}_1(u_n - u, u_n - u) = 0$.

4. A closed symmetric form $\mathcal{E}$ is Markovian if the following hold: for all $u \in \text{Dom}(\mathcal{E})$, if $v = (0 \wedge u) \lor 1$, then $\mathcal{E}(v, v) \leq \mathcal{E}(u, u)$.

5. A symmetric form $\mathcal{E}$ is a Dirichlet form if $\mathcal{E}$ is closed and Markovian.

An operator $(A, \text{Dom}(A))$ is non-negative definite if for any $u \in \text{Dom}(A)$, $(Au, u) \geq 0$ and it is self-adjoint if $A$ has a transpose operator $A^t$ such that $\text{Dom}(A) = \text{Dom}(A^t)$.

**Lemma 2.3.** ([FOT], Corollary 1.3.1) There is a one-to-one correspondence between the family of closed symmetric forms $\mathcal{E}$ on $\mathcal{H}$ and the family of non-negative definite self-adjoint operators $-A$ on $\mathcal{H}$. The correspondence is characterized by the following properties:

$$\begin{cases} 
\text{Dom}(A) \subset \text{Dom}(\mathcal{E}) \\
\mathcal{E}(u, v) = -(Au, v) \quad \text{for all} \quad u \in \text{Dom}(A) \quad \text{and} \quad \text{for all} \quad v \in \text{Dom}(\mathcal{E}).
\end{cases}$$

**Lemma 2.3** is used to define the Laplacian on strip complexes in Definition 2.6.

### 2.2. Laplacian on the strip complex

Using Lemma 2.3, we define the Laplacian operator for strip complexes. Using the orientation of the edge $e$, the edge $e$ can be parametrized by the interval $[0, e]$. Then one obtains a Riemannian metric $g = ds^2 + g_\tilde{M}$, where $s$ is the parametrization of $e$ and $g_\tilde{M}$ is the Riemannian metric on $\tilde{M}$. Let $f|_e$ be the restriction of $f$ on $S^o_e$.

**Definition 2.4.** Let $(U, \varphi)$ be a relatively compact chart of $\tilde{M}$. Let $\kappa = (\kappa_0, \ldots, \kappa_n)$ be an $(n + 1)$-non-negative tuple of integers. Let $\tilde{X} \tilde{M}^o$ be the union of all open strips of $\tilde{X} \tilde{M}$.

1. The partial derivatives $\partial_\kappa f|_e$ over $e^o \times U$ is defined as follows: for any $x = (x, x_\tilde{M}) \in e^o \times U$,

$$\partial_\kappa f|_e(x, x_\tilde{M}) = \partial_\kappa^{\kappa_0} \partial_\kappa^{(\kappa_1, \ldots, \kappa_n)} f|_e(s, \varphi(x_\tilde{M})), $$

where $s = d(i(e), x)$ and $\partial_\kappa^{(\kappa_1, \ldots, \kappa_n)}$ is the partial derivative for $(\kappa_1, \ldots, \kappa_n)$ on $\tilde{M}$.

2. A vector space $S^k(\tilde{X} \tilde{M}^o)$ is the space of $k$-th differentiable functions $f$ on $\tilde{X} \tilde{M}^o$ such that for any edge $e$ of $\tilde{X}$, for any relatively compact coordinate chart $U$ in $\tilde{M}$, and for any $\kappa$ with $\kappa_0 + \cdots + \kappa_n \leq k$, the partial derivative $\partial_\kappa f|_e$ is continuous and

$$(2.1) \quad \sup \{ \| \partial_\kappa f|_e(x) \| : x \in e^o \times U \} < \infty.$$

Set $S^\kappa(\tilde{X} \tilde{M}^o) = \bigcap_{k \geq 1} S^k(\tilde{X} \tilde{M}^o)$.
(3) Let $C^\infty(\tilde{X}\tilde{M})$ be the space of continuous functions whose restriction on $\tilde{X}\tilde{M}$ is contained in $S^\infty(\tilde{X}\tilde{M})$. Let $C_c^\infty(\tilde{X}\tilde{M})$ be the space of compactly supported functions in $C^\infty(\tilde{X}\tilde{M})$.

(4) The gradient of a function $f$ in $C^\infty(\tilde{X}\tilde{M})$ at a point $x \in S_e^\circ$ is defined as follows:

$$\nabla f|_e(x) = (\partial_x f|_e(x), \nabla_{\tilde{M}} f|_e(x)),\$$

where $\nabla_{\tilde{M}}$ is the gradient on the manifold $\tilde{M}$.

Given $f \in C^\infty(\tilde{X}\tilde{M})$, every partial derivative of $f|_e$ on $S_e^\circ$ can be extended to $S_e$. Let $e$ and $e'$ be edges intersecting at vertex $v$. If $\kappa_0 = 0$, $x \in \tilde{M}_v$, then $\partial^e f|_e = \partial^e f|_{e'}$ on $\tilde{M}_e$. Otherwise, $\partial^e f|_e$ may be different from $\partial^e f|_{e'}$ on $\tilde{M}_e$.

Given an edge $e$, $W^1(S_e^\circ)$ is the subspace of $L^2(S_e^\circ)$-functions whose first weak derivative is also in $L^2(S_e^\circ)$. Let $W^1_{\text{loc}}(S_e^\circ)$ be the space of functions $f$ such that for any compact set $K$ in $S_e^\circ$, there is a function $g \in W^1(S_e^\circ)$ such that $f|_K = g|_K$. By the extension of the restriction operator from $C^\infty(S_e^\circ)$ to $C^\infty(\tilde{M}_e)$, one obtains a continuous linear operator where $v$ is a vertex of $e$:

$$\text{Tr}_{\tilde{M}_v} : W^1_{\text{loc}}(S_e^\circ) \to L^2_{\text{loc}}(\tilde{M}_v, \text{Vol}_{\tilde{M}}).$$

**Definition 2.5.** Let $O$ be an open set in $\tilde{X}\tilde{M}$.

1. The Sobolev space $W^1(O)$ is the set of functions such that
   (i) for every $f \in W^1(O)$, $f|_{e^\circ \cap O} \in W^1_{\text{loc}}(S_e^\circ \cap O)$,
   (ii) $||\nabla f||^2_{L^2(O)} = \int_O ||\nabla f||^2 d\mu < \infty$,
   (iii) $\text{Tr}_{\tilde{M}_v} f|_e = \text{Tr}_{\tilde{M}_v} f|_{e'}$ for $e$ and $e'$ intersecting at vertex $v$.

Denote $||f||_{W^1(O)} := (||f||_{L^2(O)} + ||\nabla f||_{L^2(O)})^{\frac{1}{2}}$. Let $W^1_0(O)$ be the closure of $C^\infty_c(O)$ in $W^1(O)$.

2. Using the Sobolev space $W^1(O)$ in $\tilde{X}\tilde{M}$ and the integration on $\tilde{X}\tilde{M}$, let us define a symmetric form $\mathcal{E}$ on $W^1(O)$ as follows: for all $f,g \in W^1(O)$,

$$\mathcal{E}(f,g) := \sum_{e \in E} \int_{S_e \cap O} g(\nabla f, \nabla g)_{\tilde{X}} dsd\text{Vol}_{\tilde{M}}.$$

The symmetric form $\mathcal{E}$ on $W^1_0(O)$ is a strongly local regular Dirichlet form and $(\mathcal{E}, W^1_0(\tilde{X}\tilde{M}))$ coincides with $(\mathcal{E}, W^1(\tilde{X}\tilde{M}))$ ([BSSW] Theorem 3.29 and 3.30). The Laplacian operator of the strip complex is defined as follows.

**Definition 2.6** (Laplacian). Let $O$ be a relatively compact connected open set.

1. The domain $\text{Dom}_O(\Delta)$ of Laplacian $\Delta$ on an open set $O$ is the space of functions $f$ in $W^1_0(O)$ such that there exists a constant $C_f$ such that $\mathcal{E}(f,g) \leq C_f ||g||_{L^2(O)}$ for all $g \in W^1_0(O)$. Denote $\text{Dom}(\Delta) = \text{Dom}_\tilde{X}(\Delta)$.

2. Let $f$ be an element of $\text{Dom}_O(\Delta)$. By Lemma 2.3, there exists a function $h$ such that

$$\mathcal{E}(f,g) = -\int_O gh d\mu.$$

We define $\Delta f$ to be the function $h$. The Laplacian $\Delta$ is defined on $\text{Dom}_O(\Delta)$. 
2.3. Properties of eigenfunctions of $(\Delta, Dom_0(\Delta))$. Let $O$ be a relatively compact open set in $\tilde{X}M$. In this section, we observe properties of the eigenfunctions of $(\Delta, Dom_0(\Delta))$ which will be used to construct the heat kernel on $\tilde{X}M$ in Section 2.4.

Let $\{u_n\}_{n \geq 1}$ be a Cauchy sequence with respect to $E_1$-norm. By the definition of Dirichlet forms, there exists an $E_1$-limit $u \in W_0^1(\tilde{X}M)$ of the sequence $\{u_n\}_{n \geq 1}$. If $E(u, u) > 0$ for any nonzero function $u \in W_0^1(\tilde{X}M)$, then $\{u_n\}_{n \geq 1}$ is a Cauchy sequence with respect to $E$ and $\lim_{n \to \infty} E(u_n - u, u_n - u) = 0$. This means that $(W_0^1(\tilde{X}M), E)$ is a Hilbert space. It follows that $W_0^1(\tilde{X}M)$ decomposes into $W_0^1(O) \oplus H(O)$, where $f \in W_0^1(O)$ is considered as a function in $W_0^1(\tilde{X}M)$ by setting $f(x) = 0$ for all $x \notin O$. By Theorem 5.2 in [BSSW], if $g = g_1 + g_2 \in W_0^1(\tilde{X}M) \oplus H(O)$ and $g \geq 0$, the function $g_2$ is also non-negative $\mu$-almost everywhere for any non-negative function $g \in W_0^1(\tilde{X}M)$ and $g \geq g_1 \mu$-almost everywhere.

For any $f \in Dom_0(\Delta)$ and $g \geq 0$ in $W_0^1(\tilde{X}M)$,

$$|E(f, g)| = |E(f, g_1)| \leq C_f\|g_1\|_{L^2(\tilde{X}M)} \leq C_f\|g\|_{L^2(\tilde{X}M)}.$$  

(2.2)

Denote $g^+(x) = \max\{g(x), 0\}$ and $g^- = \max\{-g(x), 0\}$. By definition, for any $g \in W_0^1(\tilde{X}M)$, $g^+$ and $g^-$ are also contained in $W_0^1(\tilde{X}M)$. Since $g^+$ and $g^-$ satisfy the inequality (2.2), for any $f \in Dom_0(\Delta)$ and $g \in W_0^1(\tilde{X}M)$,

$$|E(f, g)| \leq C_f(\|g^+\|_{L^2(\tilde{X}M)} + \|g^-\|_{L^2(\tilde{X}M)}) \leq 2C_f\|g\|_{L^2(\tilde{X})}.$$  

Thus, $Dom_0(\Delta)$ is a subspace of $Dom(\Delta)$.

**Definition 2.7.** $[(D, D^\infty)]$

1. The operator $D : C^\infty(\tilde{X}M) \to S^\infty(\tilde{X}M^\circ)$ is defined by,

$$Df|_e(x) = (e^2 + \Delta_M)(f|_e)(x)$$

for all $e \in E$ and $x \in e^\circ$, where $\Delta_M$ is Laplace-Beltrami operator on $\tilde{M}$.

2. The space $D^\infty$ is the subspace of $C^\infty(\tilde{X}M)$ consisting of functions $f$ such that
   (i) $T^e_{\tilde{M}}D^k|_e = T^e_{\tilde{M}}D^k|_e$ for any edges $e$ and $e'$ intersecting a vertex $v$ and
   (ii) [Kirchhoff’s law] for any positive integer $k$ and for any vertex $v$,

$$\sum_{e \in E_v} g(n_{v,e}, \nabla D^k(f|_e)) = 0,$$

where $E_v$ is the set of edges intersecting $v$ and $n_{v,e}$ is the outwards unit vector normal to $S_e$ along $\tilde{M}_v$.

The restriction of Laplacian $\Delta$ on $D^\infty$ coincides with $D$ ([BSSW] Remark 7.3).

**Lemma 2.8.** Let $O$ be a relatively compact open subset of $\tilde{X}M$. The eigenfunctions of $(\Delta, Dom_0(\Delta))$ is contained in $D^\infty$.

**Proof.** Let $p$ be an eigenfunction in $Dom_0(\Delta)$. The restriction of $p$ on an open strip $S_e^\circ$ satisfies $\Delta p|_e(x) = Dp|_e(x)$. Let $U$ be an open set in $\tilde{M}$ with $\{v\} \times U \subset O \cap \tilde{M}_v$, where $v$ is a vertex of $e$. Choose $r > 0$ with $B(v, r) \times U \subset O$. Let $f$ be a compactly supported smooth function on $U$. A smooth function $F$ on $B(v, r) \times U$ is defined by $F(x, x\tilde{M}) = f(x\tilde{M})$. Let $e_v$ be a point in $e$ satisfying $d_\tilde{X}(v, e_v) = \epsilon$ and $(e_{\epsilon_1}, e_{\epsilon_2})$ be the set of points $x$ in $e$ with $\epsilon_1 < d_\tilde{X}(v, x) < \epsilon_2$. Denote $R^e_v = \{e_v\} \times U$ and $R^e_{\epsilon_1, \epsilon_2} = (e_{\epsilon_1}, e_{\epsilon_2}) \times U$. Using Equation 5.6
in [BSSW], one obtains

\[
\left| \int_{\mathcal{R}_{c_{1},c_{2}}} p(x)(\lambda F(x) + \Delta_{\tilde{M}} F(x))d\mu(x) \right|
\]

(2.3) \quad \frac{1}{2} = \left| \int_{U} \partial_{s}p(c_{1}, x_{\tilde{M}})f(x_{\tilde{M}})d\nu(x_{\tilde{M}}) - \int_{U} \partial_{s}p(c_{2}, x_{\tilde{M}})f(x_{\tilde{M}})d\nu(x_{\tilde{M}}) \right|

Since \( p \) and \( \Delta_{\tilde{M}} f \) is uniformly bounded, by (2.3), for all \( f \in C_{c}^{\infty}(U) \),

\[
\lim_{\epsilon \to 0} \int_{U} \partial_{s}p(\epsilon_{t}, x_{\tilde{M}})f(x_{\tilde{M}})d\nu(x_{\tilde{M}}) = \int_{U} \partial_{s}p(\epsilon, x_{\tilde{M}})f(x_{\tilde{M}})d\nu(x_{\tilde{M}})
\]

exists. As in the Section 5.B in [BSSW], using the Green formula and the definition of Laplacian, for all \( f \in C_{c}^{\infty}(U) \), one obtains

\[
\sum_{v \in \mathcal{V}} \int_{U} \partial_{s}p(v, x_{\tilde{M}})f(x_{\tilde{M}})d\nu(x_{\tilde{M}}) = 0.
\]

Theorem 5.12 in [BSSW] means that \( p \) is smooth and satisfies the Kirchoff’s law. \( \square \)

The first property of eigenfunctions of the Laplacian is about the spectrum.

**Theorem 2.9.** ([Sc] Theorem A.3) The spectrum \( \sigma(A) \) of a compact operator \( A \) on a Hilbert space \( \mathcal{H} \) is at most countable and has no nonzero accumulation point. If the dimension of \( \mathcal{H} \) is infinite, \( 0 \in \sigma(A) \). The eigenspace of any eigenvalue \( \lambda \neq 0 \) of \( A \) is finite dimensional.

By the definition of the Laplacian and Cauchy inequality, for any \( f \in \text{Dom}(\Delta) \),

\[
\|(-\Delta + I)f\|_{L^{2}(O)} \geq \|f\|_{L^{2}(O)}.
\]

Using Proposition 2.1 in [Sc], \((-\Delta + I)^{-1}\) is a bounded operator. Since the embedding \( \iota: W_{0}^{1}(O) \to L^{2}(O) \) is a compact operator, \( \iota \circ (-\Delta + I)^{-1}: L^{2}(O) \to L^{2}(O) \) is compact.

**Corollary 2.10.** Let \( O \) be a relatively compact open subset of \( \tilde{M} \). The spectrum of the Laplacian \( \Delta, \text{Dom}_{O}(\Delta) \) is discrete and every eigenspace is finite dimensional.

Let \( Y \) and \( Z \) be normed vector spaces. A function \( A \) from an open set \( U \) of \( Y \) to \( Z \) is Fréchet differentiable if for all \( y \in U \), there exists a bounded linear operator \( DA(y) \) from \( Y \) to \( Z \) such that

\[
\lim_{t \to 0} \frac{\|A(y + t) - A(y) - DA(y)t\|_{Z}}{\|t\|_{Y}} = 0.
\]

The operator \( DA(y) \) is Fréchet derivative at \( y \). The function \( A \) is \( C^{1} \)-function if the Fréchet derivative \( DA \) on \( U \) is continuous.

To observe the second property of eigenfunctions, we need the following theorem.

**Theorem 2.11.** ([Mc] 7.2.a Theorem 1, Lagrange multiplier for Banach space) Let \( Y \) be a Banach spaces and let \( A, B : Y \to \mathbb{R} \) be \( C^{1} \)-functions. Let \( DA \) and \( DB \) be the Fréchet derivatives. If \( f \in B^{-1}(0) \) is an extreme point of \( A \) and \( DB(f) \) is a nontrivial linear functional, then there exists a Lagrange multiplier \( \lambda \in \mathbb{R} \) such that

\[
DA(f) = \lambda DB(f).
\]

Let \( O \) be a relatively compact connected open set in \( \tilde{M} \). Define functions

\[
A(f) = ||f'||_{L^{2}(\tilde{M})}^{2}, \quad B(f) = ||f||_{L^{2}(\tilde{M})}^{2} - 1
\]

on \( W_{0}^{1}(O) \). Then the Fréchet derivatives of \( A \) and \( B \) are \( DA(f)g = 2\mathcal{E}(f, g) \) and \( DB(f)g = 2(f, g) \), respectively.
Suppose that $A(f)$ is an extreme value of $A$ on $B^{-1}(0)$. Since the Fréchet derivative $DB(f)$ is nontrivial, there exists a constant $\lambda$ such that for all $g \in W^1_0(O)$,

$$E(f, g) = \lambda(f, g).$$

By Cauchy inequality, $f$ is contained in $Dom\sigma(\Delta)$ and $f$ is an eigenfunction of $-\Delta$ with eigenvalue $\lambda$. Using this fact, we find an orthonormal basis of $L^2(O)$ which consists of the eigenfunctions of $\Delta$:

**Theorem 2.12.** ([Mc] 7.2.b) Let $O$ be a relatively compact connected open subset of $\tilde{X}\tilde{M}$. There is a maximal orthonormal set of eigenfunctions $\{p_i^O\}$ such that for all $f \in L^2(O)$,

$$f = \sum_{i=0}^{\infty} \langle f, p_i^O \rangle p_i^O. \quad (2.4)$$

Let $O$ be a relatively compact open subset of $\tilde{X}\tilde{M}$. Using the eigenfunctions of $(\Delta, Dom\sigma(\Delta))$, let us define a function $p_O : (0, \infty) \times O \to O$ as follows:

$$p_O(t, x, y) = \sum_i e^{-\lambda_i^O t} p_i^O(x)p_i^O(y),$$

where $\{p_i^O\}$ is the orthonormal basis of $L^2(O)$ from the equation (2.4) and each constant $-\lambda_i^O$ is the eigenvalue of the eigenfunction $P_i^O$.

2.4. Constructing the heat kernel.

**Definition 2.13.** The heat kernel of a strip complex $\tilde{X}\tilde{M}$ is the fundamental solution of the heat equation i.e. $\Delta_x p(t, x, y) = \partial p(t, x, y)/\partial t$ and $p(t, x, y) \to \delta_{x-y}$ as $t \to 0$.

Using the eigenfunctions of Laplace-Beltrami operator of relatively compact open subsets of a manifold $M$, Dodziuk built the heat kernel on $M$ ([D]). Let us construct the heat kernel $p(t, x, y)$ on $\tilde{X}\tilde{M}$ analogously, and show that $p(t, x, y)$ is stochastically complete, i.e. $\int_{\tilde{X}\tilde{M}} p(t, x, y)d\mu(y) = 1$ for all $t \in (0, \infty)$.

**Lemma 2.14.** Let $I = [0, T] \subset \mathbb{R}$ and let $O$ be a relatively compact connected open subset of $\tilde{X}\tilde{M}$. Let $u : I \times O \to \mathbb{R}$ be a function such that $u(\cdot, x) : I^0 \to \mathbb{R}$ is differentiable for any $x \in O$ and $u(t, \cdot)$ is in $D^2_c(O)$ for any $t \in I^0$. If

$$\Delta u(t, x) - \frac{d}{dt} u(t, x) \geq 0$$

for all $(t, x) \in I^0 \times O$, and if there is a point $(t_0, x_0) \in (0, T) \times O$ such that $M = u(t_0, x_0)$ is the maximum of $u$ in $I^0 \times O$, then $u$ is a constant function on $I^0 \times O$.

**Proof.** Let $S_e$ be a strip of an edge intersecting the open set $O$. Suppose there exists a point $(t_0, x_0) \in (0, T) \times S_e$ such that $u(t_0, x_0)$ is the maximum of $u$. By the maximum principle (for example Theorem 3.3.5 in [PW]), the function $u|_{(t_0, T) \times S_e \cap O}$ is constant. Then, it is enough to show that $u$ is a constant function when the maximum of $u$ appears at a point $x_0 = (v_0, x_{\tilde{M}, 0})$ in $\tilde{M}_{v_0} \cap O$.

Suppose that there is a point $(t_0, x_0)$ in $(0, T) \times (\tilde{M}_{v_0} \cap O)$ such that $u(t_0, x_0)$ is the maximum of $u$ on $(0, T) \times O$. Choose a positive number $\varepsilon$ satisfying $B(v_0, \varepsilon) \times B(x_{\tilde{M}, 0}, \varepsilon) \subset O$. To use maximum principle for manifolds, we define a function on $I \times (-\varepsilon, \varepsilon) \times B(x_{\tilde{M}, 0}, \varepsilon)$ as follows:

$$u_{v_0}(t, a, x_{\tilde{M}}) = \begin{cases} \frac{1}{\|e_{-i(e)=v_0}\|} \sum_{t(e)=v_0} u(t, e_{l+a}, x_{\tilde{M}}) & \text{if } a < 0 \\ u(t, v_0, x) & \text{if } a = 0 \\ \frac{1}{\|e_{i(e)=v_0}\|} \sum_{\iota(e)=v_0} u(t, e_{a}, x_{\tilde{M}}) & \text{if } a > 0, \end{cases}$$
where \( e_s \) is a point in \( e \) satisfying \( d(e_s, i(e)) = s \). Since \( u(t, \cdot) \) satisfies Kirchhoff’s rule for all \( t \), \( u_{v_0} \) is smooth. By the maximum principle on manifolds, \( u_{v_0} \) is constant. Hence, \( u \) is constant on \( I \times (\widehat{M}_{v_0} \cap O) \). If \( v_0 = i(e) \) (\( v_0 = t(e) \), resp.) for some edge \( e \), then \( \partial_s u_e \leq 0 \) on \( I \times (\widehat{M}_{v_0} \cap O) \) (\( \partial_s u_e \geq 0 \), resp.).

Since \( u(t, x) \in D^C_0(\overline{O}) \) for all \( t \in [0, T] \),
\[
\sum_{i(e)=v} \partial_s u_e = \sum_{t(e)=v} \partial_s u_e = 0
\]
on \( I \times (\widehat{M}_{v_0} \cap O) \). Suppose that there are an edge \( e \) and a point \( (t, x) \in I \times (\widehat{M}_{v_0} \cap O) \) such that \( \partial_s u_e(t, x) \neq 0 \) and \( i(e) = v \ (t(e) = v) \), resp.). Since \( \sum_{i(e)=v} \partial_s u_e = 0 \) on \( I \times (\widehat{M}_{v_0} \cap O) \), there is an edge \( e \) such that \( \partial u_e(t, x) > 0 \) and \( i(e) = v_0 \) (\( \partial_s u_e(t, x) < 0 \) and \( t(e) = v_0 \), resp.). This is a contradiction. Therefore, each edge \( e \) intersecting \( v_0 \) satisfies that \( \partial_s u_e = 0 \). Let \( e' \) be an edge with initial vertex \( v_0 \). The function \( u_{v_0, e'} \) on \( I \times (-\varepsilon, \varepsilon) \times B(x_{M, 0}, \varepsilon) \) is defined as follows:
\[
u_{v_0, e'}(t, a, x_M) = \begin{cases} u_{v_0}(t, a, x_M) & \text{if } a \leq 0 \\ u(t, e'a', x_M) & \text{if } a > 0. \end{cases}
\]
By the maximum principle, the restriction \( u|_{I \times (S_{e'} \cap B(v, \varepsilon) \times B(x_{M, 0}, \varepsilon))} \) is a constant function. Similarly, the restriction \( u|_{I \times (S_e \cap B(v, \varepsilon) \times B(x_{M, 0}, \varepsilon))} \) is a constant function when \( v_0 \) is the terminal vertex of an edge \( e'' \). Thus, the function \( u \) is constant on the star of \( v_0 \). Since \( u \) has the maximum at the boundary points of the star of \( v_0 \), we conclude that \( u \) is constant.

Since \( \Gamma \) is non-amenable and the bottom of the spectrum is non-zero by Theorem 8.5 in [SW], \( \lambda_{0,i} > 0 \). The function \( p_O(t, x, y) \) satisfies the assumption of Lemma 2.14.

**Proposition 2.15.** For any relatively compact connected open set \( O \), the function \( p_O(t, x, y) \) satisfies the following:

(i) \( p_O(t, x, y) = p_O(t, y, x) \) and \( \frac{d}{dt} p_O(t, x, y) = \Delta_y p_O(t, x, y) \),

(ii) \( \int_O p_O(t, x, z) \omega(t, s, z, y) d\mu(z) = p_O(t + s, x, y) \) for all \( x, y \in O \),

(iii) \( p_O(t, x, y) > 0 \) for all \( x, y \in O \) and for all \( t > 0 \),

(iv) \( \int_O p_O(t, x, y) d\mu(y) \leq 1 \) for all \( t \geq 0 \).

**Proof.** As the proof of Lemma 3.2 in [D], the parts (1), (2) and (3) are proved by the construction of \( p_O(t, x, y) \) and Lemma 2.14.

Since \( \lim_{t \to 0} \int_O p_O(t, x, y)d\mu(y) = 1 \), it is enough to show that \( \frac{d}{dt} \int_O p_O(t, x, y)d\mu(y) \leq 0 \).
Since \( p_0(t, x, \cdot) \) satisfies Kirchhoff’s law, by Stokes’ theorem, we obtain the following:

\[
\int_O \frac{d}{dt} p_0(t, x, y) d\mu(y) = \int_O \Delta_g p_0(t, x, y) d\mu(y) = \sum_{S^O_e \cap t \neq 0} \int_{S^O_e \cap \partial O} g(n_y, \nabla_g p_0(t, x, y)) d\text{Vol}_{S^O_e \cap \partial O}(y),
\]

where \( \text{Vol}_{S^O_e \cap \partial O} \) is the volume form of \( S^O_e \cap \partial O \) and \( n_y \) is the outward normal vector at \( y \) relative to \( S^O_e \cap \partial O \). Since for all \((t, x, y) \in (0, \infty) \times O \times O\), \( p_0(t, x, y) > 0 \), and \( p_0(t, x, y) \) vanishes at the boundary of \( O \), \( g(n_y, \nabla p_0(t, x, y)) \) \( \leq 0 \). Hence, (4) holds.

Given two relatively compact connected open sets \( O_1 \) and \( O_2 \) with \( \overline{O}_1 \subset O_2 \), for all \( x \in O_1 \) and \( y \in \partial O_1 \), \( p_{O_1}(t, x, y) - p_{O_2}(t, x, y) \leq 0 \). Therefore, we conclude that for all \( x, y \in O_1 \),

\[
(2.6) \quad p_{O_1}(t, x, y) \leq p_{O_2}(t, x, y).
\]

**Definition 2.16.** Let \( \{O_i\}_{i \geq 1} \) be an increasing sequence of relatively compact connected open subsets such that \( \overline{O}_i \subset O_{i+1} \) for all \( i \) and \( \bigcup_{i=1}^{\infty} O_i = \tilde{X}M \). Since \( p_{O_i}(t, x, y) \) is bounded for all \((t, x, y)\) and for all \( i \), we define the function \( p(t, x, y) \) as follows:

\[
p(t, x, y) = \lim_{i \to \infty} p_{O_i}(t, x, y).
\]

Note that \( p(t, x, y) \) does not depend on the choice of \( \{O_i\} \), since given a relatively compact connected open set \( O \), there exists a relatively compact connected open set \( O_i \) such that \( O \subset O_i \). Therefore, we conclude that

\[
p(t, x, y) = \sup_{O \in \Omega} p_O(t, x, y),
\]

where the supremum is taken over the set of all relatively compact connected open subsets of \( \tilde{X}M \).

**Lemma 2.17.** Let \( \{u_i(t, x)\} \) be an increasing sequence of functions satisfying the assumption in Lemma 2.14. Suppose that \( u_4(t, x) \) satisfies the heat equation for all \( i \), i.e., \( \frac{du_i}{dt} = \Delta u_i \) and the sequence \( \{u_i(t, x)\} \) converges pointwise to a function \( u(t, x) \). Suppose that for all compact subset \( K \), there exists a constant \( C_K \) such that

\[
\int_K |u_i(t, x)| d\mu(x) \leq C_K.
\]

Then \( \{u_i(t, x)\} \) converges uniformly to a function \( u(t, x) \) on compact sets and \( u(t, x) \) satisfies the heat equation. Furthermore, \( u(\cdot, x) \) is in \( C^\infty((0, \infty)) \) for all \( x \in \tilde{X}M \) and \( u(t, \cdot) \) is in \( D^\infty \) for all \( t \in (0, \infty) \).

**Proof.** We follow the proof of Lemma 3.7 in [D]. For any \((t, x)\), there exists a connected compact set \( K \) in \( \tilde{X}M \) such that \((t, x)\) is an interior point of \([t_1, t_2] \times K\). Let \( O_1 \) and \( O_2 \) be open sets such that \( x \in O_1 \subset O_2 \subset K \) and let \( h \) be a positive function of \( D^\infty_{\tilde{X}M} \) such that \( h|_{O_1} \equiv 1 \) and \( h|_{K \setminus O_2} \equiv 0 \). Then we obtain the same integration as Lemma 3.7 in [D]. By Kirchhoff’s law and Stokes’ theorem, all the derivatives of \( u_i(t, x) \) are described by the integration of products of \( u_i \), the derivatives of \( h \) and the derivatives of \( p_{K^e}(t, x, y) \). Since \( \{u_i\} \) is an increasing sequence of functions and the sequence \( \{u_i\} \) converges pointwise to a function \( u \), \( u_i \) converges uniformly to \( u \) on compact sets. Thus, given a compact set \( K \), the derivative of \( u \) is obtained by the limit of integration.

By Lemma 2.17, for all \( t \in (0, \infty) \) and for all edge \( e \), the restriction \( p(t, x, \cdot)|_{e^e} \) has all the derivatives. The derivatives can be extended to the bifurcation manifolds \( M_{i(e)} \) and \( M_{l(e)} \).
of the strip $S_e$. As a limit, $p(t, x, \cdot)$ inherits properties of $p_O(t, x, y)$ described in Proposition 2.15.

Proposition 2.18. The function $p(t, x, y)$ satisfies the following:

(i) $p(t, x, y) = p(t, y, x)$ and $\frac{\partial}{\partial t} p(t, x, t) = \Delta p(t, x, y)$.

(ii) $\int_{\tilde{\mathcal{M}}_{\lambda}} p(t, x, y) p(s, y, z) d\mu(y) = p(t + s, x, z)$.

(iii) $p(t, x, y) > 0$ for all $(t, x, y) \in (0, \infty) \times \tilde{\mathcal{M}} \times \tilde{\mathcal{M}}$.

We now show the stochastically completeness

$$\int_{\tilde{\mathcal{M}}} p(t, x, y) d\mu(y) = 1.$$ 

By Proposition 2.15, for all $t \in (0, \infty)$, $\int_{\tilde{\mathcal{M}}} p(t, x, y) d\mu(y) \leq 1$. For any bounded function $u(x)$ on $\tilde{\mathcal{M}}$, and any relatively compact connected open set $O$, the restriction $u|_O$ is contained in $L^2(O)$. For all $x \in O$, the following holds:

$$\lim_{t \to 0} \int_{\tilde{\mathcal{M}}} \rho_O(t, x, y) u(y) d\mu(y) = u(x).$$

We conclude that

$$\lim_{t \to 0} \int_{\tilde{\mathcal{M}}} p(t, x, y) u(y) d\mu(y) = u(x).$$

This implies that $p(t, x, y)$ is a heat kernel.

Theorem 2.19. ([BSSW] Theorem 4.3) If $\tilde{\mathcal{M}}$ is complete and for all $x \in \tilde{\mathcal{M}}$,

$$\int_1^\infty \frac{r}{\ln(\mu(B(x, r)))} dr = \infty,$$

then the solution of the bounded Cauchy problem on $(0, T) \times \tilde{\mathcal{M}}$ is unique. In particular, $e^\Delta 1 = \int_{\tilde{\mathcal{M}}} p(t, x, y) d\mu(y) = 1$.

Since $\Gamma$ acts cocompactly on $\tilde{\mathcal{M}}$, the sectional curvature of $\tilde{\mathcal{M}}$ is bounded. Hence, there is a constant $C$ such that the volume of $B_{\tilde{\mathcal{M}}}(x, r)$ and the sum of edge length in $B_{\tilde{x}}(x, r)$ is also bounded above by $e^{Cr}$. Since $B(x, r)$ is in $B_{\tilde{x}}(x, r) \times B_{\tilde{\mathcal{M}}}(x, r)$, the equation (2.7) holds and we obtain the following proposition.

Proposition 2.20. The heat kernel $p(t, x, y)$ satisfies the following:

$$e^{t\Delta} f(x) = \int_{\tilde{\mathcal{M}}} p(t, x, y) f(y) d\mu(y) \quad \text{and} \quad \int_{\tilde{\mathcal{M}}} p(t, x, y) d\mu(y) = 1.$$

3. Green functions of strip complexes

3.1. Convergence of Green function. In the section, we study the $\lambda$-Green function on the strip complex $\tilde{\mathcal{M}}$.

Definition 3.1. The bottom of the spectrum $\lambda_0$ of $-\Delta$ is defined as follows:

$$\lambda_0 := \inf \left\{ \frac{||f||^2_{L^2(\tilde{\mathcal{M}})}}{||f||^2_{L^2(\tilde{\mathcal{M}})}} : f \in W^1_0(\tilde{\mathcal{M}}) \setminus \{0\} \right\}.$$

The $\lambda$-Green function is defined as follows:

$$G_\lambda(x, y) := \int_0^\infty e^{\lambda t} p(t, x, y) dt.$$

The Green region is the set of $\lambda \in \mathbb{R}$ such that the $\lambda$-Green function $G_\lambda$ is finite.
The generator $A$ of a semigroup $\{T_t\}_{t \geq 0}$ of operators is an operator defined by

$$ (3.1) \quad Au := \lim_{t \to 0} \frac{T_t u - u}{t}. $$

The domain of $A$ is $\text{Dom}(A) := \{ u \in \mathcal{H} : Au \text{ exists as a strong limit} \}$.

**Definition 3.2.** The semigroup $P_t^\lambda : L^2(\tilde{X}\tilde{M}) \to L^2(\tilde{X}\tilde{M})$ generated by $(\Delta + \lambda I)$ is defined by the following: for all $f \in L^2(\tilde{X}\tilde{M}),$

$$ P_t^\lambda f(x) := \int_{\tilde{X}\tilde{M}} e^{\lambda t} p(t, x, y) f(y) d\mu(y) = e^{(\Delta + \lambda)t} f(x). $$

Recall that $\lambda_0 > 0$. Consider

$$ \mathcal{E}_\lambda(u, v) = \mathcal{E}(u, v) - \lambda \langle u, v \rangle \text{ for all } \lambda \in (-\infty, \lambda_0]. $$

Using $0 \leq \mathcal{E}_{\lambda}(u, u) < \mathcal{E}(u, u)$ for all $u \in W^1_0(O)$, one can show that $\mathcal{E}_{-\lambda}$ is a closed form for any $\lambda \in [0, \lambda_0]$. By Lemma 1.3.3 in [FOT], the semigroup $P_t^\lambda$ associated with $\mathcal{E}_\lambda$ is contracting for all $\lambda \in [0, \lambda_0], \text{i.e. } \|P_t^\lambda u\|_{L^2(\tilde{X}\tilde{M})} \leq \|u\|_{L^2(\tilde{X}\tilde{M})}$ for all $u \in L^2(\tilde{X}\tilde{M})$.

Suppose that there is a function $\phi \in W^1_0(\tilde{X})$ such that $-\Delta \phi = \lambda \phi \text{ almost everywhere } x$. Since $|\phi(x)| = |P_t^\lambda \phi(x)| \leq |P_t^\lambda_0 \phi(x)|$, it follows that for all $t \in (0, \infty), \|P_t^\lambda \phi(x)\|_{L^2(\tilde{X}\tilde{M})} \geq \|\phi\|_{L^2(\tilde{X}\tilde{M})}$. As in Section 7 of [Su], we conclude that $\phi$ is positive $\mu$-almost everywhere, in particular, not compactly supported. Hence, every eigenfunction of $-\Delta$ with compact support has an eigenvalue greater than $\lambda_0$.

**Remark 3.3.** The dimension of the eigenspace of $\lambda_0$ in $L^2(\tilde{X}\tilde{M})$ is at most 1. Let $O$ be a relatively compact connected open subset of $\tilde{X}\tilde{M}$. Similar to the simplicity of $(-\lambda_0)$, the largest eigenvalue $-\lambda_0^O$ of $(\Delta, \text{Dom}_O(\Delta))$ is simple.

**Definition 3.4.** The resolvent set of $-\Delta$ on $\text{Dom}(\Delta)$ is the set of $\lambda \in \mathbb{C}$ such that $(-\Delta - \lambda I)$ has a bounded inverse operator on $\mathcal{H}$. The spectrum of $-\Delta$ is the complement of the resolvent set.

**Proposition 3.5.** The closure of the spectrum of the operator $-\Delta$ is contained in $[\lambda_0, \infty)$. 

**Proof.** By definition, for any $\lambda \in (\lambda_0, \infty)$, there exists a function $f$ in $C_c^\infty(\tilde{X}\tilde{M})$ such that

$$ \lambda_0 < \frac{\|f\|^2_{L^2(\tilde{X}\tilde{M})}}{\|f\|^2_{L^2(\tilde{X}\tilde{M})}} < \lambda. $$

Let $K$ be the support of $f$. As in Section 2.3, any eigenvalue of the eigenfunction $p_0^K$ with smallest eigenvalue in the equation (2.4) is smaller than $\lambda$. Hence, it is enough to show that the open set $(-\infty, \lambda_0)$ is contained in the resolvent set of $-\Delta$. Thus, for all $f \in \text{Dom}(\Delta)$ and for all $\lambda \in (-\infty, \lambda_0)$,

$$ \langle -\Delta f, f \rangle - \lambda \langle f, f \rangle = \mathcal{E}(f, f) - \lambda \langle f, f \rangle > (\lambda_0 - \lambda) \langle f, f \rangle. $$

The dimension of the kernel of $(-\Delta - \lambda I)$ is zero. By Proposition 1.6 in [Sc], $(-\Delta - \lambda I)$ is surjective. By Proposition 2.1 in [Sc], $(-\Delta - \lambda)$ has an inverse operator. Hence, the open set $(-\infty, \lambda_0)$ is in the resolvent set of $\Delta$. \hfill \Box

As in [Su], if $\lambda$ is in Green region, for any compact set $K$ and $x \in \tilde{X}\tilde{M}$, $\lim_{t \to \infty} P_t^\lambda 1_K(x) = 0$. For any $\lambda > \lambda_0$, there exists a connected relatively compact open set $O$ in $\tilde{X}\tilde{M}$ such that $\lambda > \lambda_0^O$. Since $\lim_{t \to \infty} e^{\lambda t} p_0(t, x, y) = p_0^O(x)p_0^O(y)$, $\lim_{t \to \infty} P_t^\lambda 1_K(x) = \infty$, thus one obtain the following corollary.
Corollary 3.6. If $\lambda$ is an element of the resolvent set of $-\Delta$, then the inverse operator of $(-\Delta - \lambda I)$ is described by the following integral: for all $f \in L^2(\tilde{X}\tilde{M})$,

$$(-\Delta - \lambda I)^{-1}(f)(x) = \int_{\tilde{X}\tilde{M}} G_\lambda(x,y)f(y)d\mu.$$ 

Furthermore, the Green region is $(-\infty, \lambda_0)$ or $(-\infty, \lambda_0]$.

Definition 3.7. Let $O$ be an open subset of $\tilde{X}\tilde{M}$. Let $\Omega_x$ be the set of continuous paths in $\tilde{X}\tilde{M}$ starting at $x$ and $Y_t$ be a Brownian path in $\Omega_x$.

(1) A function $f$ in $W^1_{loc}(O)$ is $\lambda$-harmonic on $O$ if $f$ is a weak solution of the following equation: for all $g \in W^1_{c}(O)$,

$$\mathcal{E}(f, g) = \lambda(f, g).$$

(2) The probability measure $\mathbb{P}_x$ on $\Omega_x$ is defined as follows: for any $0 < t_1 < \cdots < t_n$ and any Borel sets $B_n$ in $\tilde{X}\tilde{M}$,

$$\mathbb{P}_x[Y_{t_1} \in B_1, \ldots, Y_{t_n} \in B_n] = \int_{B_1 \times \cdots \times B_n} p(t_1, x, y_1)p(s_1, y_1, y_2)\cdots p(s_n, y_{n-1}, y_n)d\mu^n(y_1, \ldots, y_n),$$

where $\mu^n = \mu \times \cdots \times \mu$ and $s_i = t_i - t_{i-1}$.

The probability measure $\mathbb{P}_x$ describes the Brownian motion related to our Laplacian on the strip complex ([BSSW], [FOT]). Following Sullivan’s argument ([Su]), we obtain Lemma 3.8 and Theorem 3.11.

Lemma 3.8. Let $O$ be a connected relatively compact open subset of $\tilde{X}\tilde{M}$. Then for all $\lambda \in (-\infty, \lambda_0]$, there exists a positive $\lambda$-harmonic function on $O$.

Proof. Let $\sigma_O$ be the exit time, i.e. $\sigma_O(Y) = \inf\{t \geq 0 : Y_t \in O^c\}$, where $Y_t$ is a continuous path in $\tilde{X}\tilde{M}$.

The expectation of a function $f$ on $\Omega_x$ is defined by $\mathbb{E}_x(f) = \int_{\Omega_x} f\mathbb{P}_x$. The expectation of $e^{\lambda t}$ over the set of the random paths in $\Omega_x$ staying in $O$ reaching $B$ at time $t$ satisfies

$$\mathbb{E}_x[e^{\lambda t}1_{\{t < \sigma_O, Y_t \in B\}}] = \int_B e^{\lambda t} p_O(t,x,y)d\mu(y).$$

Since $\lambda < \lambda_{O,i}$ for all $i$ and $p_O(t,x,y) = \sum_{i=1}^{\infty} e^{-\lambda_{O,i}t}p^O_i(x)p^O_i(y)$, we define a function $f_\lambda$ associated with a function on $f$ on $\partial O$ as in the proof of Theorem 2.1 in [Su]. Remark 3.3 implies that the hitting distribution on $\partial O$ depends on $p^O_i$, thus the proof in [Su] varied. Given a positive function $f$ on $\partial O$ and $\lambda \in (-\infty, \lambda_0]$, we extend the function $f_\lambda$ as follows: for all $x \in O$,

$$f_\lambda(x) := \mathbb{E}_x[e^{\lambda \sigma_o} f(Y_{\sigma_o})].$$

By Theorem 4.3.1 in [FOT] and the proof of Theorem 5.2 in [BSSW], $f_\lambda \in W^1_{loc}(O)$ is a weak solution of the following equation

$$\mathcal{E}_\lambda(f_\lambda, u) = 0 \quad \text{for all } u \in W^1_0(O),$$

and $f(x) = f_\lambda(x)$ for all $x \in 4\partial O$. By Proposition 5.12 in [BSSW], $f_\lambda \in D^{\infty}(\bar{O})$. □

Proposition 3.9. The set of $\lambda \in \mathbb{R}$ such that there exists a positive $\lambda$-harmonic function coincides with $(-\infty, \lambda_0]$. 

\[ \square \]
Proof. Let \( \{K_{\tilde{X},n}\} \) be an increasing sequence of connected compact sets of \( \tilde{X} \) whose boundary consists of vertices of \( \tilde{X} \) and let \( \{K_{\tilde{M},n}\} \) be an increasing sequence of connected compact sets of \( \tilde{M} \). Denote by \( K_n = K_{\tilde{X},n} \times K_{\tilde{M},n} \). Suppose that \( \bigcup_{n=1}^{\infty} K_n = \tilde{X} \tilde{M} \). By Lemma 3.8, for any \( \lambda \in (-\infty, \lambda_0) \), there exists a sequence of positive functions \( f_n \) such that \( f_n \in \mathcal{D}^{\infty}(K_n) \) and \( -\Delta f_n = \lambda f_n \). Let \( v_0 \) be a vertex in \( \bigcap_{n=1}^{\infty} K_{\tilde{X},n} \) and let \((v_0, x_0)\) be a point in \( M_{v_0} \cap K_1 \).

We may assume that \( f_n(v_0, x_0) = 1 \) for all \( n \). Let us consider an edge \( e \) with \( v \) as a vertex of \( e \). By Harnack’s principle, there exists a subsequence \( f_{e,i} \) of \( f_n \) that converges uniformly to a \( \lambda \)-harmonic function on \( K_n \cap S_n^e \). Using Harnack’s principle to strips connected to the strips \( S_e \), we find a subsequence \( f_{1,k} \) of \( f_n \) that converges uniformly to a \( \lambda \)-harmonic function on \( K_1 \). Inductively, we obtain a subsequence \( f_{n,k} \) of \( f_{n-1,j} \) which converges to a \( \lambda \)-harmonic function on \( K_n \). As \( n \) goes to infinity, we find a subsequence of \( f_n \) which converges to a \( \lambda \)-harmonic function \( f \) uniformly on compact sets.

Let \( f \) be a positive \( \lambda \)-harmonic function. See the proof of Theorem 2.1 in [Su]. Then \( \lambda \) must be contained in \( (-\infty, \lambda_0] \). \( \square \)

As in [Su], if the compact convex set of positive \((-\lambda_0)\)-harmonic functions \( f \) satisfying \( f(x_0) = 1 \) for some \( x_0 \) has more than one element, the \( \lambda_0 \)-Green function is finite.

Proposition 3.10. ([Su]) If the Green region is \((-\infty, \lambda_0)\), then there exists a unique positive \((-\lambda_0)\)-harmonic function up to a constant multiple.

Following the proof of Theorem 3 in [LS], we show the following theorem.

Theorem 3.11. If \( \Gamma \) is non-amenable, then the \( \lambda_0 \)-Green function is finite.

Proof. By Lemma 3.8, there exists a positive \((-\lambda_0)\)-harmonic function \( f \). Denote

\[
q(t, x, y) := e^{\lambda_0 t} p(t, x, y) f(y)/f(x).
\]

Let us define a semigroup \( \{Q_t\} \) as follows: for any \( g \in L^\infty(\tilde{X} \tilde{M}) \),

\[
Q_t g(x) := \int_B q(t, x, y) g(y) d\mu(y).
\]

Denote \( u_{t_0}(t) = u(t + t_0) \). Since \( \mathbb{R}_+ \) is amenable, there exists a linear functional \( \varphi \) on \( C_b(\mathbb{R}_+) \) such that for all \( t_0 \in \mathbb{R}_+ \) and for all \( u \in L^\infty(\mathbb{R}_+) \), \( \varphi(u_{t_0}(t)) = \varphi(u) \) and \( \|\varphi\|_{op} \leq 1 \).

Denote \( \mathcal{H}_{\lambda_0} = \{g \in C_b(\tilde{X} \tilde{M}) : Q_t g = g\} \). The map \( \tilde{\varphi} \) from \( L^\infty(\tilde{X} \tilde{M}) \) to the subspace \( \mathcal{H}_{\lambda_0} \) is defined as follows: for any \( g \in L^\infty(\tilde{X} \tilde{M}) \),

\[
\tilde{\varphi}(f)(x) := \varphi(Q_{t+t_0} f(x)).
\]

Suppose that the \( \lambda_0 \)-Green function diverges. By Proposition 3.10, the space \( \mathcal{H}_{\lambda_0} \) consists of constant functions. Any function \( g \) in \( L^\infty(\Gamma) \) is regarded as a function in \( L^\infty(\tilde{X} \tilde{M}) \), by defining as follows \( g(x) = g(\gamma) \), where \( x \in \gamma F \).

As in [LS], the functional \( \tilde{\varphi} \) on \( L^\infty(\Gamma) \) is \( \Gamma \)-invariant. Hence, the group \( \Gamma \) is amenable. \( \square \)

4. MARTIN BOUNDARY OF HYPERBOLIC GRAPHS

Let \( \tilde{X} \) be a locally finite connected complete Gromov hyperbolic graph whose Gromov boundary \( \partial \tilde{X} \) is an infinite set. The graph \( \tilde{X} \) itself, as the product of \( \tilde{X} \) with one point is a strip complex. Let \( l_m \) and \( l_M \) be the smallest and the largest edge length of \( \tilde{X} \). Let \( B(x, r) \) and \( S(x, r) \) be the ball and the sphere of radius \( r \) centered at \( x \). Suppose that a discrete group \( \Gamma \) acts isometrically and geometrically on \( \tilde{X} \). Since group \( \Gamma \) is quasi-isometric to \( \tilde{X} \), \( \Gamma \) is hyperbolic and non-amenable.
Ancona showed that for any Riemannian manifold with negatively pinched curvature, there exists a constant $C_c$ satisfying the inequality (1.1) when $\lambda \in [0, \lambda_0 - \varepsilon)$ ([A]). Using the inequality (1.1), Ancona proved that the geometric boundary of a hyperbolic group coincides with the $\lambda$-Martin boundary for all $\lambda \in [0, \lambda_0)$. Using the ideas of [GL] and [G], Ledrappier and Lim showed that the geometric boundary of a universal cover of a negatively curved closed manifold coincides with its $\lambda_0$-boundary ([LL]). We extend their results to hyperbolic graphs.

4.1. Harnack inequality for graphs. Since the function $G_\lambda(x, y)$ is $\lambda$-harmonic on $\tilde{X}\setminus\{x\}$, Harnack inequality is used to show Ancona-Gouézel inequality. In this section, we show the graph version of Harnack inequality, which is the analog of the result of Cheng and Yao ([CY]). Denote the counting measure of a discrete subset $Y \subset \tilde{X}(A \subset \Gamma, \text{resp.})$ by $|Y|$ ([A], resp.).

**Proposition 4.1.** Let $f$ be a positive $\lambda$-harmonic function on $B(x, r + 1)$. Then there exists a constant $r_s \in [r, r + 1]$ such that

\[(4.1) \quad \int_{B(x,r)} |(\log f)'|^2 \, d\mu \leq 2|\partial B(x, r_s)|.\]

**Proof.** Let $r = r_0 < r_1 < \cdots < r_n = r + 1$ be the maximal sequence of radii in which branching appears between $r$ and $r + 1$.

**Step 1.** We observe each interval $(r_i, r_{i+1}]$. Since $f$ is a $\lambda$-harmonic function, $f$ is contained in the space $D^\infty$. Hence, $(\log f)'$ is well-defined on each closed edges and $\Delta \log f$ coincides with $(\log f)''$. Thus

\[(4.2) \quad (\log f)'' = f''/f - |(\log f)'|^2.\]

For any $s$ and $\delta$ such that $(s - \delta, s] \subset (r_i, r_{i+1}]$, choose $\varphi \in C^\infty(B(x, r + 1))$ satisfying

\[\varphi|_{B(x,s-\delta)} = 1 \quad \text{and} \quad \varphi|_{B(x,s)} = 0 \quad \text{and} \quad |\varphi'| \leq \frac{2}{\delta}.\]

By the equation (4.2) and Kirchhoff’s law, we obtain

\[(4.3) \quad -\int_{C_{s,\delta}} \varphi'(\log f)' \, d\mu = \int_{B(x,s)} -\lambda \varphi \, d\mu - \int_{B(x,s)} |(\log f)'|^2 \varphi \, d\mu,\]

where $C_{s,\delta} = B(x, s) \setminus B(x, s - \delta)$. Let $r_{i,s}$ be the middle point of the interval $[r_i, r_{i+1}]$. By Hölder’s inequality and (4.3),

\[\int_{B(x,s)} |(\log f)'|^2 \varphi \, d\mu \leq \left(\frac{2}{\delta}\right) \int_{C_{s,\delta}} |(\log f)'|^2 \, d\mu \left(\frac{2}{\delta}\right) \frac{1}{2} \left|\partial B(x, r_{i,s})\right|^2.\]

For any measurable function $f$ on $B(x, r_{i+1})$ and for any $s \in (r_i, r_{i+1})$, define a differentiable function $\tilde{f}$ by

\[\tilde{f}(s) = \int_{B(x,s)} f \, d\mu.\]

Since $B(x, r_{i+1}) \setminus B(x, r_i)$ is a disjoint union of $|\partial B(x, r_{i,s})|$ distinct intervals of length $r_{i+1} - r_i$, we have:

\[\frac{1}{|\partial B(x, r_{i,s})|} \leq \frac{2F'(s)}{F(s)^2},\]

where $F(r') = \int_{B(x,r')} |(\log f)'|^2 \, d\mu$. Integrating over the interval $(r_i, r_{i+1})$,

\[\frac{r_{i+1} - r_i}{|\partial B(x, r_{i,s})|} = \int_{r_i}^{r_{i+1}} \frac{1}{|\partial B(x, r_{i,s})|} \leq \int_{r_i}^{r_{i+1}} \frac{2F'(s)}{F(s)^2} \, ds = \frac{2}{F(r_i)} - \frac{2}{F(r_{i+1})}.\]
Step 2. Using the inequality \((4.4)\), we have:
\[
\sum_{i=0}^{n-1} \frac{r_{i+1} - r_i}{|\partial B(x, r_i)|} \leq \sum_{i=0}^{n-1} \left( \frac{2}{F(r_i)} - \frac{2}{F(r_{i+1})} \right) \leq \frac{2}{F(r)}.
\]
Let \(r_\ast\) be a number such that \(|\partial B(x, r_\ast)| = \max_{0 \leq i \leq n-1} |\partial B(x, r_i)|\). Then
\[
\frac{1}{|\partial B(x, r_\ast)|} \leq \frac{\sum_{i=0}^{n-1} r_{i+1} - r_i}{|\partial B(x, r_\ast)|} \leq \frac{2}{F(r)},
\]
thus the inequality \((4.1)\) holds. \(\square\)

Using Proposition \(4.1\), we obtain the graph version of Harnack inequality.

**Corollary 4.2.** Let \(u\) be a positive \(\lambda\)-harmonic function in \(C(B(x, r + 1))\). Then there exists an explicit constant \(D_r = (2|\partial B(x, r_\ast)|\mu(B(x, r)))^\frac{1}{2}\) such that for all \(y, z \in B(x, r)\),
\[
\frac{u(y)}{u(z)} \leq D_r.
\]

### 4.2. Proof of Ancona-Gouëzel inequality

Using the properties of Markov process as in [LL], we prove the Ancona-Gouëzel inequality. In this section, the constant \(C\) may vary from line to line. In this section, we mainly use the strong Markov property.

**Definition 4.3.** Let \((\Omega, \mathcal{M}, \{Y_t\}_{t \geq 0}, \mathbb{P})\) be a Markov process with state space \((S, \mathcal{B})\). Let \(\mathcal{M}_t\) be a \(\sigma\)-algebra generated by \(\{Y_t\}\).

1. A function \(\sigma\) on \(\Omega\) is a stopping time if \(\{x \in \Omega : \sigma(x) \leq t\} \in \mathcal{M}_t\) for all \(t\).
2. Let \(\sigma\) be a stopping time. Denote \(\mathcal{M}_\sigma = \{A \in \mathcal{M} : A \cap \{x \in \Omega : \sigma(x) \leq t\} \in \mathcal{M}_t, \forall t \geq 0\}\). The Markov process has strong Markov property if \(\mathcal{M}_{t+} = \bigcap_{t' > t} \mathcal{M}_{t'}\) and for any \(x \in S, t \geq 0, B \in \mathcal{B}\) and stopping time \(\sigma, \mathbb{P}_x(Y_{\sigma+s} \in E|\mathcal{M}_\sigma) = \mathbb{P}_{Y_s}(Y_s \in E)\).

The exit time \(\sigma_O\) of a relatively compact connected open set \(O\) in Lemma \(3.8\) is the example of stopping time. The process derived from a strongly local regular Dirichlet form is a strong Markov process on continuous paths with probability 1 ([FOT] Theorem 4.5.3 and Theorem 7.3.1).

Using the exit time \(\sigma_O\) of an open set \(O\), the relative Green function \(G_\lambda(x, y : O)\) is defined as follows: for any Borel measurable function \(f\) and for any distinct points \(x, y \in O\),
\[
\int_O G_\lambda(x, y : O)f(y)d\mu = \mathbb{E}_x \left[ e^{\int_0^{\tau} f(Y_t)dt} \right].
\]

By the strong Markov property, we have the following proposition as in [LL].

**Proposition 4.4.** Let \(O_1\) and \(O_2\) be connected open sets in \(\tilde{X}\) which are not disjoint. Then we have the following: for all \(x \in O_1 \setminus \tilde{O}_2\) and \(y \in O_1\) and for all \(\lambda \in [0, \lambda_0]\),
\[
(4.5) \quad G_\lambda(x, y : O_1) = \mathbb{E}_x[1_{\tau < \sigma} e^{\lambda \tau} G_\lambda(Y_\tau, y : O_1)] + G_\lambda(x, y : O_1 \setminus \tilde{O}_2),
\]
where \(\sigma = \sigma_{O_1}\) and \(\tau = \sigma_{O_1 \setminus \tilde{O}_2}\) are the exit time of \(O_1\) and \(O_1 \setminus \tilde{O}_2\), respectively.

**Remark 4.5.** To show Proposition 4.4, we need Lebesgue’s theorem, which is clear if a point \(x\) is in an open edge \(e^o\). Let us verify Lebesgue theorem for vertices of \(\tilde{X}\). Let \(\delta\) be smaller than the smallest length of edges in the graph \(\tilde{X}\). For a vertex \(x\) in \(\tilde{X}\),
\[
\frac{1}{B(x, \delta)} \int_{B(x, \delta)} f d\mu = \frac{1}{\deg(x)} \left( \sum_{i(e)=x} \frac{1}{\delta} \int_0^{\delta} f dt + \sum_{i(e)=x} \frac{1}{\delta} \int_{t_\delta}^{t_e} f dt \right).
\]
Using Lebesgue theorem on each edge, we obtain the graph version of Lebesgue’s theorem.
Lemma 4.6. There exists a constant $A_r$ such that for all $\lambda \in [0, \lambda_0]$ and for all $(y, z)$ in $B(x, r)^2$ with $d(y, z) \geq r/2$, 

$$G_\lambda(y, z : B(x, 2r)) \geq A_r.$$ 

Proof. Suppose on the contrary that there exists a sequence of $(\lambda_n, y_n, z_n) \in [0, \lambda] \times B(x, r)^2$ such that 

$$G_{\lambda_n}(y_n, z_n : B(x, 2r)) \leq \frac{1}{n}. $$

Since $[0, \lambda] \times B(x, r)^2$ is compact, the sequence $\{(\lambda_n, y_n, z_n)\}_{n=1}^\infty$ has a convergent subsequence $\{(\lambda_{n_k}, y_{n_k}, z_{n_k})\}_{k=1}^\infty$ with the limit $(\lambda, y, z)$. The point $(\lambda, y, z)$ satisfies $G_{\lambda}(y, z : B(x, 2r)) = 0$. By Theorem 4.4.1 in [FOT], for all $t > 0$, 

$$G_{\lambda}(y, z : B(x, 2r)) = \int_0^\infty e^{\lambda t} p_{B(x, 2r)}(t, x, y) dt = 0,$$

which contradicts Proposition 2.15 (3). \qed

Let us consider the equation (4.5). The measure $\eta^{\lambda, O_1 \cap \partial O_2}_x$ on $O_1 \cap \partial O_2$ is defined as follows: for all $y \in O_1$,

$$\int_{O_1 \cap \partial O_2} G_{\lambda}(x, z) f(z)dz = C \int_{O_1 \cap \partial O_2} f(z) d\eta^{\lambda, O_1 \cap \partial O_2}_x(z) = \mathbb{E}_x[1_{\tau < \sigma} e^{\lambda \tau} G_{\lambda}(Y_\tau, y : O_1)].$$

Using the strong Markov property, we obtain the following lemma.

Lemma 4.7. Let $O_1$ and $O_2$ be connected open sets in $\tilde{X}$. Suppose that for any $z \in O_1 \cap \partial O_2$, $\mu((B(y, 1) \cap O_1 \cap O_2) \setminus B(y, 1/2)) \geq \frac{1}{2}$. Let $f$ be a bounded positive function on $\partial O_2$. There exists a constant $C$ such that for all $x \in O_1 \setminus O_2$

$$\sum_{z \in O_1 \cap \partial O_2} G_{\lambda}(x, z) f(z) \geq C \int_{O_1 \cap \partial O_2} f(z) d\eta^{\lambda, O_1 \cap \partial O_2}_x.$$

Proof. The measure of $(B(y, 1) \cap O_1 \cap O_2) \setminus B(y, 1/2)$ is contained in the interval $[1/2, d_M/2]$, where $d_M$ is the maximal degree of $\tilde{X}$. By Harnack inequality, there exists a constant $C$ such that for all $y \in O_1 \cap \partial O_2$,

$$\int_{(B(y, 1) \cap O_1 \cap O_2) \setminus B(y, 1/2)} G_{\lambda}(x, y') d\mu(y') \leq C G_{\lambda}(x, y).$$

By Lemma 4.6, we have the following inequality:

$$\frac{1}{2} A_2 < \int_{(B(y, 1) \cap O_1 \cap O_2) \setminus B(y, 1/2)} G_{\lambda}(y, y' : O_1) d\mu(y').$$

Denote $\sigma = \sigma_{O_1}$ and $\tau = \sigma_{O_1 \setminus O_2}$. By Proposition 4.4, we have the following:

$$\sum_{z \in O_1 \cap \partial O_2} f(z) \int_{(B(z, 1) \cap O_1 \cap O_2) \setminus B(z, 1/2)} G_{\lambda}(x, y') d\mu(y') \geq \mathbb{E}_x \left[1_{\tau < \sigma} \int_0^{\sigma} e^{\lambda t} 1_{(B(Y_{\tau}, 1) \cap O_1 \cap O_2) \setminus B(Y_{\tau}, 1/2)} (Y_t) f(Y_t) dt \right]$$

$$= \mathbb{E}_x \left[1_{\tau < \sigma} e^{\lambda \tau} f(Y_\tau) \mathbb{E}_{Y_\tau} \left[ \int_0^{\tau} 1_{(B(Y_\tau, 1) \cap O_1 \cap O_2) \setminus B(Y_\tau, 1/2)} (Y_t) dt \right] \right]$$

$$= \mathbb{E}_x \left[1_{\tau < \sigma} e^{\lambda \tau} f(Y_\tau) \int_{(B(Y_\tau, 1) \cap O_1 \cap O_2) \setminus B(Y_\tau, 1/2)} G_{\lambda}(Y_\tau, y' : O_1) d\mu(y') \right].$$
By the inequalities (4.8) and (4.9), and (4.10), we have the inequality (4.7).

Using the proof of Theorem 3.5 in [LL], we have the following theorem.

**Lemma 4.8.** Let $O_1$ be an open set in $\tilde{X}$ and let $O_2$ be a relatively compact open subset of $O_1$ with $d(\partial O_1, O_2) \geq 1$. There exists a constant $C$ such that for any $x \in O_1 \cap O_2$ and for any $y \in O_2$,

\[
C \int_{O_1 \cap \partial O_2} f(z) d\eta^{\lambda x} \geq \sum_{z \in \partial O_2} G_\lambda(x, z) f(z).
\]

**Theorem 4.9.** Let $y$ be a point on a geodesic $[x, z]$ between $x$ and $z$. Suppose that $d(x, y) \geq 1$ and $d(y, z) \geq 1$. There exists a constant $C$ such that for all $\lambda \in (0, \lambda_0]$,

\[
C^{-1} G_\lambda(x, y) G_\lambda(y, z) \leq G_\lambda(x, z).
\]

**Proof.** By equation (4.5), we obtain the following inequality:

\[
G_\lambda(x, z) \geq \int_{S(y, 1/2)} G_\lambda(x, y') d\eta^{\lambda S(y, 1/2)} = E_{\lambda} [e^{\lambda \sigma} G_\lambda(x, Y_\sigma)],
\]

where $\sigma = \sigma_{B(y, 1/2)}$. By the Lemma 4.8, there exists a constant $C$ such that

\[
\int_{S(y, 1/2)} G_\lambda(x, y') d\eta^{\lambda S(y, 1/2)} \geq C^{-1} \sum_{y' \in S(y, 1/2)} G_\lambda(x, y') G_\lambda(y', z).
\]

Since $\Gamma$ acts cocompactly, we can choose $C$ independent of $y \in X_0$. By Harnack inequality, we obtain the inequality (4.12).

By Proposition 2.18 and the integration by substitution, the following proposition holds as in [LL].

**Proposition 4.10.** For $\lambda \in [0, \lambda_0)$, for any two distinct points $x, y$ in $\tilde{X}$,

\[
\frac{\partial}{\partial \lambda} G_\lambda(x, y) = \int_{\tilde{X}} G_\lambda(x, z) G_\lambda(z, y) dz.
\]

The inverse operator of $(-\Delta - \lambda I)$ is described by the Green function $G_\lambda(x, y)$ (see Corollary 3.6). Hence, for two distinct points $x, y$ in $\tilde{X}$, the derivative $\frac{\partial}{\partial \lambda} G_\lambda(x, y)$ converges at $\lambda \in (0, \lambda_0)$.

Denote by $\gamma_x$ the element satisfying $x \in \gamma_x X_0$. Then we obtain the following lemma.

**Lemma 4.11.** ([G], Lemma 2.4) There exists a positive constant $D$ such that for all $x, y$, and $z$ in $\tilde{X}$, there exists an element $\gamma(y, z) \in \Gamma$ such that $d(e, \gamma(y, z)) < D$ and

\[
d(x, y) + d(x, z) - 3 \text{diam}(X_0) \leq d_T(x, \gamma(y, z) z).
\]

**Proof.** By Theorem 2.12 in [GH], there exists a constant $C$ such that for all points $x_1, x_2, x_3, x_4 \in \tilde{X}$, there exists a map $\Phi$ from $\{x_1, x_2, x_3, x_4\}$ to some metric tree $T$ satisfying

\[
d(x_i, x_j) - C \leq d_T(\Phi(x_i), \Phi(x_j)) \leq d(x_i, x_j)
\]

for all $x_i, x_j \in \{x_1, x_2, x_3, x_4\}$.

As in the proof of Lemma 2.4 in [G], since $\Gamma$ is non-elementary, there exist two elements $\alpha$ and $\beta$ in $\Gamma$ such that the sequences $\left\{\alpha^N x\right\}$ and $\left\{\beta^N x\right\}$ converge to 4 distinct points $\alpha_+ \pm \epsilon \in \tilde{X}$, respectively. Put $D = \max\{d(e, a^N) : a \in \{\alpha^\pm, \beta^\pm\}\}$. Denote $V(\alpha^\pm)$ and $V(\beta^\pm)$ be the disjoint neighborhood of $\alpha_+ \pm \epsilon$ and $\beta_+ \pm \epsilon$ in $\tilde{X} \cup \partial X$, respectively. For any $a \in \{\alpha^\pm, \beta^\pm\}$, there exists a constant $K$ such that for any $\gamma x \in V(a^\pm)^c$, the distance of $\Phi(x)$ and the branching point in $T$ is bounded above by $K$. Choose $\alpha \in \{\alpha^\pm, \beta^\pm\}$ with $\gamma^{-1} x \in V(a^\pm)^c$ and $\gamma x \in V(a^-)^c$. Choose $N$ such that $d(a^N x, x) \geq 4K + 3C$ for all $a \in \{\alpha^\pm, \beta^\pm\}$.
As in [G], \( d(x, \gamma y x) + d(x, \gamma z x) \leq d(x, \gamma y a^N \gamma z x) \). By the triangle inequality, we complete the proof of the lemma.

Let \( l_m \) be the minimal edge length of \( \tilde{X} \). Denote by \( A_{x,n} = \{ e \in E : e \cap S(x, nl_m) \neq \emptyset \} \) and \( B_{x,n} = \bigcup_{e \in A_{x,n}} e \). Using Lemma 4.11, we prove the following which is the analog of Lemma 2.5 in [G].

**Proposition 4.12.** There is a constant \( C > 0 \) such that for all integer \( n \geq 3 \text{diam}(X_0) + l_M + 1 \) and for all \( \lambda \in [0, \lambda_0] \),

\[
\int_{B_{x,n}} G^2_\lambda(x, y) d \mu(y) \leq C.
\]

**Proof.** By Harnack inequality and (4.13), for all \( \lambda \in [0, \lambda_0] \) and for all \( y \in B(x,1) \),

\[
\sum_{n > \text{diam}(X_0)} \int_{B_{x,n}} G^2_\lambda(x, y) d \mu(y) \leq l_M \int_{X - B(x,1)} G^2_\lambda(x, z) d \mu(z) \leq C l_M \int_{X - B(x,1)} G_\lambda(x, z) G_\lambda(z, y) d \mu(z) < \infty,
\]

By the inequality (4.15), the integration \( \int_{B_{x,n}} G^2_\lambda(x, z) d \mu(z) \) is bounded for all \( \lambda \in [0, \lambda_0] \).

By Lemma 4.11, there exists a constant \( C \) such that for any \( z \in B_{x,m} \) and \( w \in B_{x,n} \),

\[
G_\lambda(x, \gamma(z, w)x) \geq C.
\]

Using Proposition 4.1, (4.16) and Theorem 4.9 in order, we obtain

\[
G^2_\lambda(x, z) G^2_\lambda(x, w) = G^2_\lambda(x, z) G^2_\lambda(\gamma z \gamma(z, w)x, \gamma z \gamma(z, w)w) \leq C G^2_\lambda(x, \gamma z x) G^2_\lambda(\gamma z x, \gamma z \gamma(z, w)x) G^2_\lambda(\gamma z \gamma(z, w)x, \gamma z \gamma(z, w)w) \leq C G^2_\lambda(x, \gamma z \gamma(z, w)x, \gamma z \gamma(z, w)w).
\]

Since \( d_F(e, \gamma(z, w)) < D \) for all \( z, w \in \tilde{X} \),

\[
(m + n) l_m - 3 \text{diam}(X_0) - l_M - 2 \leq d(x, \gamma z \gamma(z, w)w) \leq (m + n) l_m + 2 + D \text{diam}(X_0) + 2 l_M.
\]

The point \( z \) is contained in the neighborhood of the geodesic from \( x \) to \( \gamma z \gamma(z, w)w \). By the inequality (4.17),

\[
\int_{B_{x,m}} \int_{B_{x,n}} G^2_\lambda(x, z) G^2_\lambda(x, w) d \mu(z) d \mu(w) \leq C \sum_{i=1}^{T_2} \int_{B_{x,i+T_1}} G^2_\lambda(x, z) d \mu(z),
\]

where \( T_1 = \max\{ l \in \mathbb{N} : l \leq (m + n) l_m - 2 \text{diam}(X_0) - l_M \} \) and \( T_2 = \min\{ n \in \mathbb{N} : n \geq (D + 1) \text{diam}(X_0) + 3 l_M + 2 \} \). Let \( M_\lambda \) be the supremum of \( \int_{B_{x,n}} G^2_\lambda(x, z) d \mu(z) \). By the
inequality (4.18), the following holds:

$$M_\lambda^2 \leq C \sum_{i=1}^{T_2} M_\lambda = CT_2 M_\lambda.$$  

Hence, $M_\lambda \leq CT_2$ for any $\lambda \in [0, \lambda_0]$. Suppose that $M_{\lambda_0} > CT_2$. There exists a constant $n$ satisfying

$$\int_{B_{x,n}} G_{\lambda_0}^2 (x,y) d\mu(y) > CT_2.$$  

For any $x, y \in \tilde{X}$, $G_{\lambda}(x, y)$ is continuous with respect to $\lambda$. There exists a constant $\lambda$ in $[0, \lambda_0)$ such that $\int_{B_{x,n}} G_{\lambda}^2 (x,y) d\mu(y) > CT_2$. This is a contradiction. □

Using the strong Markov property as in [LL], we obtain the following proposition.

**Proposition 4.13.** ([LL], Proposition 8.6) Let $O_1$, $O_2$, and $O_3$ be open sets in $\tilde{X}$ satisfying $O_3 \subset O_2 \subset O_1$. Then the following equation holds: for any $x \in O_1 \setminus \bar{O}_2$,

$$\int_{O_2 \cap \partial O_3} f(z) \eta_{O_1 \cap \partial O_3} (z) = \int_{O_1 \cap \partial O_2} \left( \int_{O_2 \cap \partial O_3} f(z) \eta_{O_1 \cap \partial O_3} (z) \right) \eta_{O_1 \cap \partial O_2} (y).$$

The following lemma is analogous to Lemma 2.6 in [G], which is the main technical part of the proof of Theorem 4.15.

**Lemma 4.14.** Let $y$ be a point on a geodesic $[x, z]$ between $x$ and $z$. There exist constants $\varepsilon > 0$ and $R_0 > 0$ such that for all $d(x, y) \geq r$, $d(y, z) \geq r$ and $r \geq R_0$,

$$G_{\lambda_0}(x, z : B(y, r)) \leq 2e^{-r}.$$  

**Proof.** Step 1. Geometric argument using quasi isometry: Since $G_{\lambda_0}(x, z : B(y, r)) = 0$ if $B(y, r)$ is disconnected, we may assume that $B(x, r)$ is connected for all $r$. Since $\Gamma$ is cocompact, by Theorem 10.2 in [BS], there exist a map $\Psi$ from $\tilde{X}$ to a convex subset $Y$ of a hyperbolic space $\mathbb{H}^n$ and positive constants $L$ and $k$ such that for all $x, y \in \tilde{X}$,

$$|Ld(x, y) - d_{\mathbb{H}^n}(\Psi(x), \Psi(y))| \leq k$$

and $Y$ is contained in the $k$-neighborhood of $\Psi(\tilde{X})$. Since the image of the geodesic $[x, z]$ is a quasi-geodesic in $\mathbb{H}^n$, it is contained in the $K$-neighborhood of the geodesic $g$ from $\Psi(x)$ to $\Psi(z)$ in $\mathbb{H}^n$ (see Figure 2). Choose a point $o$ on the geodesic $g$ in $\mathbb{H}^n$ such that $d_{\mathbb{H}^n}(\Psi(y), o) < K$.

Let $a, b$ be points in $\tilde{X}$ and let $i_{a,b}$ be the radius of the inscribed circle of the geodesic triangle $\Delta_o \Psi(a) \Psi(b)$ in $\mathbb{H}^n$, which is bounded by a universal constant, say $R_1$ (Proposition II.1.17 in [BH]). Choose a positive constant $\kappa < L$. Denote

$$R_2 = \max \left\{ R_1, \frac{3k + 2K}{2L} + 3M, \frac{3l M + 3k + K + \log(8 \tanh R_1)}{2(L - \kappa)} \right\}.$$  

By Theorem 7.11.2.(i) in [Be],

$$\tanh i_{a,b} = \sinh((\Psi(a) | \Psi(b))_o) \tan \frac{1}{2} \angle_o \Psi(a) \Psi(b).$$

By (4.20), for all $a, b$ in $\tilde{X}$ with $(\Psi(a) | \Psi(b))_o > \log \sqrt{16 + 8 \tanh^2 R_2}$,

$$\tan \angle O \Psi(a) \Psi(b) = \frac{2 \sinh((\Psi(a) | \Psi(b))_o) \tanh i_{a,b}}{\sinh^2((\Psi(a) | \Psi(b))_o) - \tanh^2 i_{a,b}} \leq 8(\tanh i_{a,b}) e^{-(\Psi(a) | \Psi(b))_o}.$$
By the triangle inequality, for all edge \( e \) with \( d(i(e), y) > R_2 \) and points \( a, b \) in \( e \),

\[
2(\Psi(a)\Psi(b))_o = (d_{\mathbb{H}^n}(\Psi(a), o) + d_{\mathbb{H}^n}(\Psi(b), o) - d_{\mathbb{H}^n}(\Psi(a), \Psi(b))) \\
\geq L(d(a, y) + d(b, y) - d(a, b)) - 3k - 2K \\
(4.22) \\
\geq L(2d(i(e), y) - 3l_M) - 3k - 2K > 0.
\]

By (4.21) and (4.22), for all edge \( e \) with \( d(i(e), y) > R_2 \),

\[
(4.23) \quad \theta_c := \max\{\angle_o \Psi(a)\Psi(b) : a, b \in e\} \leq \tan \theta_c \leq e^{-\kappa d(i(e), y)}.
\]

\[
\hat{X} \\
\partial A_i(\theta) \\
\psi \\
\Psi(x) \\
\Psi(z) \\
\Psi(y) \\
\partial \mathbb{H}^n \\
\mathbb{H}^n \\
0 \\
K \\
x \\
y \\
z
\]

**Figure 3. Ancona-Gouëzel inequality**

Fix \( \epsilon < \kappa \) and \( r > R_2 \) satisfying \( e^{-\kappa r} < e^{-\epsilon r}/4 \). Denote \( N = e^{\epsilon r} \). Denote \( A_0 = \{x\} \), \( A_{N+1} = \{z\} \) and let \( A_i(\theta) \) be the connected component containing \( z \) of the set

\[
\{u \in \overline{B(y, r)}^c : \angle_o \Psi(x)\Psi(u) > (2i - 1)/N + \theta\}.
\]

Fix an \( N \)-tuple \((\theta_1, \cdots, \theta_N) \in [0, 1/N]^N \). Denote \( A_i = A_i(\theta_i) \). Let \( u_i \) be a point in \( \partial A_i \). Then there exists a point \( w \) such that \( \angle_o \Psi(x)\Psi(w) < \theta_i + (2i - 1)/N \) and \( w \) and \( u_i \) are in the same edge \( e \). By (4.23),

\[
(4.24) \quad \angle_o \Psi(x)\Psi(u_i) \leq \angle_o \Psi(x)\Psi(w) + \angle_o \Psi(w)\Psi(u_i) \leq 2i/N + \theta_c \leq (8i + 1)/4N.
\]

Hence, \( u_i \) is not in the interior point of \( A_{i+1} \).

**Step 2. Decomposition of \( G_{\lambda_0}(x, z : \overline{B(x, r)}^c) \)** Using Proposition 4.4 and (4.6), we have:

\[
(4.25) \quad G_{\lambda_0}(u_i, z : \overline{B(x, r)}^c) = \int_{\partial A_{i+1}} G_{\lambda_0}(u_{i+1}, z : \overline{B(x, r)}^c) d\eta_{\lambda_0, \partial A_{i+1} \cap \overline{B(x, r)}}(u_{i+1}).
\]
By Proposition 4.13 and (4.25),

\[
G_{\lambda_0}(x, z : \overline{B(x, r)}^c) = \int_{\partial A_1} G_{\lambda_0}(u_1, z : \overline{B(x, r)}^c) d\eta_{u_1}^{\lambda_0, \partial A_1 \cap \overline{B(x, r)}}(u_1)
\]

\[
= \int_{\partial A_1} \int_{\partial A_2} G_{\lambda_0}(u_2, z : \overline{B(x, r)}^c) d\eta_{u_1}^{\lambda_0, \partial A_2 \cap \overline{B(x, r)}}(u_2) d\eta_{u_1}^{\lambda_0, \partial A_1 \cap \overline{B(x, r)}}(u_1)
\]

\[
= \int_{\partial A_1} \cdots \int_{\partial A_N} G_{\lambda_0}(u_N, z : \overline{B(x, r)}^c) d\eta_{u_{N-1}}^{\lambda_0, \partial A_N \cap \overline{B(x, r)}}(u_N) \cdots d\eta_{u_1}^{\lambda_0, \partial A_1 \cap \overline{B(x, r)}}(u_1)
\]

(4.26) \leq \int_{\partial A_1} \cdots \int_{\partial A_N} G_{\lambda_0}(u_N, z) d\eta_{u_{N-1}}^{\lambda_0, \partial A_N \cap \overline{B(x, r)}}(u_N) \cdots d\eta_{u_1}^{\lambda_0, \partial A_1 \cap \overline{B(x, r)}}(u_1).

Since \( A_1 \) is connected, \( \mu((B(u_i, 1) \cap \partial A_1))(B(u_i, 1/2)) \geq \frac{1}{2} \). By Proposition 4.7, for any \( u_i \in \partial A_1 \),

\[
(4.27) \int_{\partial A_{i+1}} G_{\lambda_0}(u_{i+1}, z) d\eta_{u_i}^{\lambda_0, \partial A_{i+1} \cap \overline{B(x, r)}}(u_{i+1}) \leq C \sum_{u_{i+1} \in \partial A_{i+1}} G_{\lambda_0}(u_i, u_{i+1}) G_{\lambda_0}(u_{i+1}, z).
\]

As in [G] and [LL], the operator \( L_i : l^2(\partial A_{i+1}) \to l^2(\partial A_{i}) \) is defined by

\[
L_i f(u_i) = \sum_{u_{i+1} \in \partial A_{i+1}} G_{\lambda_0}(u_i, u_{i+1}) f(u_{i+1}).
\]

Let \( ||L_i||_{op} \) be the operator norm of \( L_i \). Applying (4.26) and (4.27),

\[
G_{\lambda_0}(x, z : \overline{B(x, r)}^c) \leq C^N \sum_{u_1 \in \partial A_1} \cdots \sum_{u_N \in \partial A_N} G_{\lambda_0}(x, u_1) G_{\lambda_0}(u_1, u_2) \cdots G_{\lambda_0}(u_N, z)
\]

\[
= C^N \left[ G_{\lambda_0}(x, \cdot) ||L_1 \cdots L_N G_{\lambda_0}(\cdot, z)||_{l^2(\partial A_1)} \right] \leq C^N \left[ ||G_{\lambda_0}(x, \cdot)||_{l^2(\partial A_1)} ||L_1 \cdots L_N G_{\lambda_0}(\cdot, z)||_{l^2(\partial A_1)} \right]
\]

\[
\leq C^N \left[ ||G_{\lambda_0}(x, \cdot)||_{l^2(\partial A_1)} ||L_1||_{op} \cdots ||L_N||_{op} ||G_{\lambda_0}(\cdot, z)||_{l^2(\partial A_N)} \right].
\]

By Cauchy inequality,

\[
||L_i f||^2_{l^2(\partial A_i)} \leq \sum_{u_i \in \partial A_i} \sum_{u_{i+1} \in \partial A_{i+1}} G_{\lambda_0}^2(u_i, u_{i+1}) ||f||^2_{l^2(\partial A_{i+1})}.
\]

Denote

\[
f_i(\theta_1, \cdots, \theta_N) = \left( \sum_{u_i \in \partial A_i(\theta_i)} G_{\lambda_0}^2(u_i, u_{i+1}) \right)^{1/2}.
\]

To complete the proof, it remains to find an \( N \)-tuple \((\theta_1, \cdots, \theta_N) \in [0, 1/N]^N\) such that for all \( i \),

\[
||L_i||_{op} \leq f_i(\theta_1, \cdots, \theta_N) < \frac{1}{4C^2}.
\]

Step 3. Contribution of edges in \( f_i(\theta_1, \cdots, \theta_N) \): Given edges \( e \) and \( e' \), the function \( G|_{e,e'}(\theta_i, \theta_{i+1}) \) is defined by

\[
G|_{e,e'}(\theta_i, \theta_{i+1}) = \begin{cases} G_{\lambda_0}(x, y) & \text{if } x \in e \cap \partial A_i(\theta_i) \text{ and } y \in e' \cap \partial A_{i+1}(\theta_{i+1}) \\ 0 & \text{otherwise.} \end{cases}
\]

By applying Corollary 4.2 twice to edges \( e \) and \( e' \),

\[
G|_{e,e'}(\theta_i, \theta_{i+1}) \leq D_{i,m}^2 G_{\lambda_0}(i(e), i(e')).
\]
Let $db_\theta$ be the Lebesgue measure on the interval $[0, 1/2N]$. By (4.23) and (4.28),

$$
\int_0^{1/2N} \int_0^{1/2N} G_{\theta, e', e}(\phi, \theta, \phi+1) d\phi d\theta \leq D_4 G_\lambda(\phi, \phi+1)
$$

(4.29)

**Step 4. Counting $\gamma \in \Gamma_i(v)$:** Let $E_i$ be the set of edges in $\tilde{X}$ that intersect $\partial A_i(\phi)$ for some $\phi \in [0, 1/2N]$. By (4.29),

$$
\int_0^{1/2N} \int_0^{1/2N} f_\phi d\phi d\theta \leq \sum_{e_i \in E_i} \int_0^{1/2N} \int_0^{1/2N} G_{\theta, e_i, e}(\phi, \phi+1) d\phi d\theta
$$

(4.30)

Let $e_i$ be an edge in $E_i$. By (4.23), for any $w \in e_i$,

$$
\angle_0 \Psi(x) \Psi(w) \geq (2i - 1)/N - \theta_{e_i} \geq (8i - 5)/4N.
$$

Denote $C_i = \{u \in \overline{B(x, r)} : \angle_0 \Psi(x) \Psi(u) \in [(8i - 5)/4N, (8i + 1)/4N]\}$. If $e_i \in E_i$, then $e_i \subset C_i$. Let $[y, v]$ be a geodesic segment from $y$ to a vertex $v$ in $\tilde{X}$. Denote

$$
\Gamma_i(v) = \{\gamma \in \Gamma : \gamma y \in C_i \text{ and } \gamma v \in C_{i+1}\}.
$$

By Corollary 4.2 and (4.30), we have

$$
\int_0^{1/2N} \int_0^{1/2N} f_\phi d\phi d\theta \leq C \sum_{e_i \in E_i} \sum_{e_i \in E_i} |\Gamma_i(v)| e^{-\kappa(d(y, v) - \text{diam}(X_0))} G_\lambda(\phi, \phi+1) d\phi d\theta.
$$

(4.31)

Similar to (4.23), for any $\gamma \in \Gamma_i(v)$,

$$
e^{-\kappa(g(y | \gamma y)p)} \geq \angle_0 \Psi(\gamma y) \Psi(\gamma v) \geq 1/4N.
$$

This means $\kappa(g | \gamma y)p \leq cr + \log 4$. Since $d(y, \gamma y) \geq r$ and $d(y, \gamma v) \geq r$, for sufficiently small $\epsilon$, $(\gamma y | \gamma y)p \leq 2r$ and $(\gamma y, \gamma v) \geq r$. Hence, the length of $[y, v]$ is at least $r$ and there exists a point $w$ in $[y, v]$ such that $\gamma w \in B(y, 2r)$. Denote $h(r) = (r + \text{diam}(X_0))/l_m$. Let $d^f$ be a word metric defined by $S = \{\gamma \in \Gamma : \gamma X_0 \cap \overline{X_0}\}$. Let $id_\Gamma$ be the identity of $\Gamma$. Since $d(w, \gamma w) \geq 2r$ for all $\gamma \in \Gamma$ with $d^f(id_\Gamma, \gamma) \geq h(2r)$, $B(id_\Gamma, h(2r)) \overline{X_0}$ contains $B(y, 2r)$. If $\gamma w$ is in $B(id_\Gamma, h(2r)) \overline{X_0}$ for some $w \in [y, v]$, then $\gamma \in B(id_\Gamma, h(2r)) \gamma w$. The geodesic $[y, v]$ intersects at most $h(d(y, v))$ orbits of $\overline{X_0}$. Hence, there exists a constant $C'$ such that for all $v$ with $d(y, v) \geq r$, $|\Gamma_i(v)| \leq (1 + h(d(y, v))) e^{2C' r d(y, v)}$.

**Step 5. Finding $\epsilon$ and $m$ using $\kappa$ in (4.23):** Denote $V_n = \{v \in V : n l_m < d(y, v) \leq (n + 1) l_m\}$.
For any \( v \in V_n \), if \( v \) is a vertex of an edge \( e \), then \( e \cap S(y, nl_m) \neq \emptyset \) or \( e \cap S(y, (n + 1)l_m) \neq \emptyset \). Choose \( R_3 > 0 \) so that \( e^{\kappa r/2} \geq (1 + h(r + 1))^2 \) for all \( r > R_3 \). By (4.31),

\[
\int_0^\infty \int_0^\infty f_i d\theta_i d\theta_{i+1} \leq C \sum_{n=r/l_m}^\infty \sum_{v \in V_m} \sum_{i \in E \mid i(e) = v} \{1 + h(n + 1)^2\}^2 e^{-\kappa n + 2C'\epsilon(n+1)} \int_0^\infty C^2_M(\lambda \epsilon) d\iota d\theta
\leq 2C \sum_{n=r/l_m}^\infty e^{-\kappa n/2 + 2C'\epsilon(n+1)} M_{\lambda_0},
\]

where \( e_i \) is a point in the edge \( e_i \) such that \( d(\iota(e), e_i) = t \). For sufficiently small \( \epsilon \), \( \max\{4\epsilon(C + 1), 4\epsilon\} \leq \kappa/4 \). Using Proposition 4.12, we have:

\[
N^2 \int_0^\infty \int_0^\infty f_i d\theta_i d\theta_{i+1} \leq C \sum_{n=r/l_m}^\infty e^{-\kappa n/2 + 2(C + 1)(n+1)} M_{\lambda_0}
\leq C \sum_{n=r/l_m}^\infty e^{-\kappa n/4} \leq Ce^{-\kappa r/2}.
\]

Choose \( R_4 > 0 \) satisfying \( e^{-\kappa r}/(4C^3) > e^{-\kappa r/2} \) for all \( r > R_4 \).

\[
N^N \int \sum_{i} f_i d\theta_i \cdots d\theta_N \leq 1/(4C^2).
\]

Put \( R_0 = \max\{R_2, R_3, R_4\} \). Then there exists an \( N \)-tuple \( (\theta_1, \cdots, \theta_N) \) such that for any \( i \in N \), \( f_i(\theta_1, \cdots, \theta_N) \leq 1/(4C^2) \). By Harnack inequality (4.2), the inequality (4.19) holds. \( \Box \)

Using Harnack inequality, Lemma 4.8, and Lemma 4.14, we obtain Ancona inequality as in [LL].

**Theorem 4.15.** Let \( y \) be a point on a geodesic \([x, z]\) between \( x \) and \( z \). Suppose that \( d(x, y) \geq 1 \) and \( d(y, z) \geq 1 \). There exists a constant \( C \) such that for all \( \lambda \in (0, \lambda_0) \),

\[
G_\lambda(x, z) \leq CG_\lambda(x, y)G_\lambda(y, z).
\]

**4.3. Martin boundary.** In this section, we show that the geometric boundary coincides with its \( \lambda \)-Martin boundary.

**Definition 4.16.** The \( \lambda \)-Martin kernel \( K \) of \( \tilde{X} \) is defined as follows:

\[
K_\lambda(x_0, x, y) = \frac{G_\lambda(x, y)}{G_\lambda(x_0, y)}.
\]

The \( \lambda \)-Martin boundary is the boundary of the image of the embedding \( y \mapsto K_\lambda(x_0, \cdot, y) \) on \( \tilde{X} \).

Let \( f \) and \( g \) be functions on \( \tilde{X} \). Denote by \( f \equiv_c g \) if there exists a constant \( c \) such that \( f \leq cg \) and \( g \leq cf \). The following theorem is the analog of Theorem 4.6 in [GL]. Unlike the proof in [GL], we prove the theorem without harmonic functions.

**Theorem 4.17.** Let \([x, y]\) be a geodesic segment of length \( n \geq 3 \). Suppose that \([x, y]\) is contained in the \( r \)-neighborhood of a geodesic segment \([x', y']\) and \( d(x, y) \leq d(x', y') \). Then there exist positive constant \( C(x, x') \) and \( \rho < 1 \) such that for all \( \lambda \in (0, \lambda_0) \),

\[
\left| \frac{G_\lambda(x, y)}{G_\lambda(x', y')} - \frac{G_\lambda(x, y')}{G_\lambda(x', y)} \right| \leq C \rho^\lambda.
\]
Proof. Let \( x_1 \) be the point on the geodesic segment \([x, y]\) such that
\[
\min\{d(x, x_1), d(x', x_1)\} = 2.
\]
Let \( x_k \) be the point on the geodesic \([x, y]\) with \( d(x_1, x_k) = k - 1 \) for all integer \( k < d(x_1, y) \) (see Figure 4).

Using (4.12), (4.32), and Harnack inequality, there exists a constant \( C \) such that
\[
\frac{G_\lambda(x, y)}{G_\lambda(x', y')} \leq C \frac{G_\lambda(x, x_k)}{G_\lambda(x', x_k)} \quad \text{and} \quad \frac{G_\lambda(x', y')}{G_\lambda(x', x_k)} \leq C \frac{G_\lambda(x, x_k)}{G_\lambda(x', x_k)}.
\]
We first claim that for all integer \( k < d(x_1, y) \),
\[
A(x, x', k) := \frac{G_\lambda(x, y)}{G_\lambda(x', y)} \left( 1 - \frac{1}{C} \right) \frac{G_\lambda(x, x_k)}{G_\lambda(x', x_k)} \quad \text{and} \quad A(x, x', y, k) \geq - (1 - \frac{1}{C})^k \frac{G_\lambda(x, y)}{G_\lambda(x', y_k)}.
\]
By (4.34), (4.35) and (4.36), it follows that
\[
\frac{G_\lambda(x, y)}{G_\lambda(x', y') \left( 1 - \frac{1}{C} \right)} \leq |A(x, x', y, k) - A(x, x', y', k)| \leq |A(x, x', y, k)| + |A(x, x', y', k)| \leq 2C(1 - \frac{1}{C})^k \frac{G_\lambda(x, x_k)}{G_\lambda(x', x_k)}.
\]
It remains to prove the claim. It is clear when \( n = 1 \). Suppose that the inequality (4.35) holds for all \( k \leq n \). By induction, the following inequality holds:
\[
A(x, x', y, k + 1) = A(x, x', y, k) \left( 1 - \frac{1}{C} \right) \frac{G_\lambda(x, x_{k+1})}{G_\lambda(x', x_{k+1})} \leq (1 - \frac{1}{C})^k \frac{G_\lambda(x, y)}{G_\lambda(x', y')} \left( 1 - \frac{1}{C} \right) \frac{G_\lambda(x, x_k)}{G_\lambda(x', x_k)} = (1 - \frac{1}{C})^{k+1} \frac{G_\lambda(x, y)}{G_\lambda(x', y')}.
\]
Similarly, (4.36) holds for all \( k \).

\[\square\]

**Lemma 4.18.** For any \( \lambda \in [0, \lambda_0] \), \( G_\lambda(x, y) \) goes to zero as \( y \) goes to infinity.

**Proof.** Suppose that there exist a constant \( c > 0 \) and a sequence \( \{y_n\} \) such that
\[
G_\lambda(x, y_n) \geq c \quad \text{and} \quad \lim_{n \to \infty} d(x, y_n) = \infty.
\]
By the choosing a subsequence, we may assume that for all distinct two points \( y_n \) and \( y_m \),
\[
d(y_n, y_m) > 2r \quad \text{for some} \quad r.
\]
Let us define the stopping time as follows:
\[
T_1 = \inf \{ t : X_t \in B(y_n, r) \text{ for some} \ n \}.
\]
Inductively, we define the stopping times \( T_j \) as follows:
\[
T_{j+1} = \inf \{ t > T_j : X_t \in B(y_n, r) \text{ for some} \ n \}.
\]
Denote $B = \bigcup_{n=1}^{\infty} B(y_n, r)$. By Proposition 4.4, for any $x, y \in \tilde{X}$ with $d(x, y) = r$,

$$G_\lambda(x, y) = \mathbb{E}_x\left[1_{T_1 \leq \infty} e^{\lambda T_1} G_{\lambda_0}(X_{T_1}, y)\right] + G_\lambda(x, y : \overline{B})$$

$$= \mathbb{E}_x\left[1_{T_1 \leq \infty} e^{\lambda T_1} \mathbb{E}_{X_{T_1}}\left[e^{\lambda(T_2-T_1)} G_{\lambda_0}(X_{T_2}, y)\right]\right] + \mathbb{E}_x\left[1_{T_1 \leq \infty} e^{\lambda T_1} G_{\lambda_0}(X_{T_1}, y : \overline{B})\right] + G_\lambda(x, y : \overline{B})$$

$$= \sum_{j=1}^{\infty} \mathbb{E}_x\left[1_{T_j \leq \infty} e^{\lambda T_1} \mathbb{E}_{X_{T_1}}\left[e^{\lambda(T_2-T_1)} \cdots e^{\lambda(T_j-T_{j-1})} G_{\lambda_0}(X_{T_1}, y : \overline{B})\right]\right] \cdots$$

(4.37) \quad + \quad \mathbb{E}_x\left[1_{T_1 \leq \infty} e^{\lambda T_1} G_{\lambda_0}(X_{T_1}, y : \overline{B})\right] + G_\lambda(x, y : \overline{B})$$

For any $j > 0$, the following holds:

$$\mathbb{E}_x\left[1_{T_j \leq \infty} e^{\lambda T_1} \mathbb{E}_{X_{T_1}}\left[e^{\lambda(T_2-T_1)} \cdots e^{\lambda(T_j-T_{j-1})} G_{\lambda_0}(X_{T_1}, y : \overline{B})\right]\right] \cdots$$

(4.38) = \sum_{n=1}^{\infty} \mathbb{E}_x\left[1_{\tau_n = T_j \leq \infty} e^{\lambda T_1} \mathbb{E}_{X_{T_1}}\left[e^{\lambda(T_2-T_1)} \cdots e^{\lambda(T_j-T_{j-1})} G_{\lambda_0}(X_{T_1}, y : \overline{B})\right]\right] \cdots$$

where $\tau_n = \inf\{t : X_t \in B(y_n, r)\}$.

$$\mathbb{E}_x\left[1_{\tau_n \leq \infty} e^{\lambda \tau_n} G_{\lambda}(X_{\tau_n}, y)\right] = \mathbb{E}_x\left[1_{\tau_n = T_j \leq \infty} e^{\lambda T_1} G_{\lambda}(X_{T_1}, y : \overline{B})\right]$$

$$= \mathbb{E}_x\left[1_{\tau_n = T_j \leq \infty} e^{\lambda T_1} \mathbb{E}_{X_{T_1}}\left[e^{\lambda(T_2-T_1)} G_{\lambda}(X_{T_1}, y)\right]\right]$$

$$= \mathbb{E}_x\left[1_{\tau_n = T_j \leq \infty} e^{\lambda T_1} \mathbb{E}_{X_{T_1}}\left[e^{\lambda(T_2-T_1)} G_{\lambda}(X_{T_1}, y : \overline{B})\right]\right]$$

(4.39) = \sum_{j=1}^{\infty} \mathbb{E}_x\left[1_{\tau_n = T_j \leq \infty} e^{\lambda T_1} \mathbb{E}_{X_{T_1}}\left[e^{\lambda(T_2-T_1)} \cdots e^{\lambda(T_j-T_{j-1})} G_{\lambda_0}(X_{T_1}, y : \overline{B})\right]\right] \cdots$$

By the equations (4.37), (4.38), and (4.39), Lemma 4.8 and Harnack inequality, the following inequality holds:

$$G_\lambda(x, y) \geq \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \mathbb{E}_x\left[1_{\tau_n = T_j \leq \infty} e^{\lambda T_1} \mathbb{E}_{X_{T_1}}\left[e^{\lambda(T_2-T_1)} \cdots e^{\lambda(T_j-T_{j-1})} G_{\lambda_0}(X_{T_1}, y : \overline{B})\right]\right] \cdots$$

$$= \sum_{n=1}^{\infty} \mathbb{E}_x\left[1_{\tau_n \leq \infty} e^{\lambda \tau_n} G_{\lambda}(X_{\tau_n}, y)\right] \geq C \sum_{n=1}^{\infty} \sum_{z_n \in \partial B(y_n, r)} G_\lambda(x, z_n) G_\lambda(z_n, y)$$

(4.40) \quad \geq \quad C \sum_{n=1}^{\infty} G_\lambda(x, y_n)^2 = \infty,$n

which is a contradiction. Hence, the $\lambda$-Green function $G_\lambda(x, y)$ converges to zeros as $y$ goes to infinity. \hfill \square

Let $x_0$ be a point of $\tilde{X}$ and let $\{y_n\}$ and $\{y_n'\}$ be sequences converging to a point $\xi$ of the geometric boundary of $\tilde{X}$. By Theorem 4.17, for all $x \in \tilde{X}$, the functions $K_\lambda(x_0, x, y_n)$ and $K_\lambda(x_0, x, y_n')$ converge pointwise to the same function $K_\lambda(x_0, x, \xi)$. The map from the geometric boundary to the $\lambda$-Martin boundary is defined by

$$\xi \mapsto K_\lambda,\xi(x) = K_\lambda(x_0, x, \xi).$$

For two different points $\xi_1, \xi_2$ in the geometric boundary, let $\gamma_1$ and $\gamma_2$ be the geodesic rays that converge to $\xi_1$ and $\xi_2$, respectively. Let $\gamma_3$ be the geodesic from $\xi_1$ to $\xi_2$. Since $\tilde{X}$ is hyperbolic, there exist a point $p$ and a constant $C$ such that for all $i \in \{1, 2, 3\}$, $p$ is in the
C-neighborhood of $\gamma_i$. By Harnack inequality and Ancona inequality, for sufficiently large $n$,

$$K_\lambda(x_0, y_n, \xi_2) = O(G_\lambda(p, y_n)),$$

where $\{y_n\}$ is a sequence converging to $\xi_1$. By Lemma 4.18, \(\lim_{n\to \infty} K_\lambda(x_0, y_n, \xi_2) = 0\). Hence, two distinct points in the geometric boundary converge to the distinct points in $\lambda$-Martin boundary.

**Theorem 4.19.** For any $\lambda \in [0, \lambda_0]$, the geometric boundary coincides with the $\lambda$-Martin boundary.

**Proof.** Suppose that a sequence $\{y_n\}$ in $\hat{X}$ converges to a function $K_\lambda(x_0, x, \zeta)$ of $\lambda$-Martin boundary. Let us consider the geodesic $g_n$ from $x_0$ to $y_n$. By Arzelà-Ascoli’s theorem ([(BH] Theorem I.3.10), for any integer $m$, the sequence of geodesics $g_{n_k}[0,m]$ has a subsequence that converges to a geodesic of length $m$. By the induction on the length of geodesics, we have a subsequence of $g_{n_k}$ that converges to a geodesic ray $g$. Let $\xi$ be a point satisfying $\xi = \lim_{t\to \infty} g(t)$. Then the subsequence $\{g_{n_k}\}$ converges to $\xi$. Since the subsequence $\{K\lambda(x_0, x, y_{n_k})\}$ converges pointwise to $K\lambda(x_0, x, \xi)$, $K\lambda(x_0, x, \xi) = K\lambda(x_0, x, \xi)$ for all $x \in \hat{X}$. Hence, the map from the geometric boundary to $\lambda$-Martin boundary is surjective. \(\square\)

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