Viscosity Solutions to Second Order Elliptic Hamilton-Jacobi-Bellman Equation with infinite delay

Jianjun Zhou *

Abstract
This paper introduces a notion of viscosity solutions for second order elliptic Hamilton-Jacobi-Bellman (HJB) equations with infinite delay associated with infinite-horizon optimal control problems for stochastic differential equations with infinite delay. We identify the value functional of optimal control problems as unique viscosity solution to associated second order elliptic HJB equation with infinite delay. We also show that our notion of viscosity solutions is consistent with the corresponding notion of classical solutions, and satisfies a stability property.

Keywords: Second order elliptic Hamilton-Jacobi-Bellman equations; Infinite delay; Viscosity solutions; Optimal control; Stochastic differential equations

AMS subject classifications. 93C23; 93E20; 60H30; 49L20; 49L25.

1 Introduction
Let \( \{W(t), t \geq 0\} \) be an \( n \)-dimensional standard Wiener process on a complete probability space \((\Omega, \mathcal{F}, P)\), and \( \{\mathcal{F}_s\}_{s \geq 0} \) its natural filtration, augmented with the family \( N \) of \( P \)-null of \( \mathcal{F} \). The process \( u(\cdot) = (u(s))_{s \in [0, \infty)} \) is \( \mathcal{F}_s \)-progressively measurable and take values in some Polish space \((U, d)\) (subsequently called \( u(\cdot) \in U_0 \)). \( C_0 \) is the totality of all continuous \( \mathbb{R}^d \)-valued functions defined on \((-\infty, 0]\) with \( \lim_{\theta \to -\infty} y(\theta) = 0 \). Define a norm on \( C_0 \) as follows:

\[
|x|_C := \sup_{-\infty < \theta \leq 0} |x(\theta)|, \quad x \in C_0.
\]

We assume the coefficients \( b : C_0 \times U \to \mathbb{R}^d \) and \( \sigma : C_0 \times U \to \mathbb{R}^{d \times n} \) satisfy Lipschitz condition under \( |\cdot|_C \) with respect to the continuous function.

Let us consider a controlled stochastic differential equation with infinite delay:

\[
\begin{aligned}
    &dX^{x,u}(s) = b(X_s^{x,u}, u(s))ds + \sigma(X_s^{x,u}, u(s))dW(s), \quad s > 0, \\
    &X_0^{x,u} = x \in C_0,
\end{aligned}
\]

where

\[
X_s^{x,u}(\theta) = X_s^{x,u}(s + \theta), \quad \theta \in (-\infty, 0].
\]

In the above equation, the unknown \( X^{x,u}(s) \), representing the state of the system, is an \( \mathbb{R}^d \)-valued process.

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*College of Science, Northwest A&F University, Yangling 712100, Shaanxi, P. R. China. Partially supported by the National Natural Science Foundation of China (Grant No. 11401474), the Natural Science Foundation of Shaanxi Province (Grant No. 2021JM-083) and the Fundamental Research Funds for the Central Universities (Grant No. 2452019075, 2452021063). Email: zhoujianjun@nwsuaf.edu.cn
One tries to minimize a utility functional of the form:

\[ J(x, u(\cdot)) = \mathbb{E} \int_0^\infty e^{-\lambda s} q(X_{s-, u(s)}) ds, \quad x \in C_0, \]  

(1.2)

over \( U_0 \). Here, the term \( q \) is a given real function on \( C_0 \times U \), and the constant \( \lambda \) is sufficiently large.

We introduce value functional of the optimal control problem as follows:

\[ V(x) := \inf_{u(\cdot) \in U_0} J(x, u(\cdot)), \quad x \in C_0. \]  

(1.3)

We aim at characterizing this value functional \( V \). We consider the following second order elliptic Hamilton-Jacobi-Bellman (HJB) equation with infinite delay:

\[ -\lambda V(x) + \partial_t V(x) + H(x, \partial_x V(x), \partial_{xx} V(x)) = 0, \quad x \in C_0, \]  

(1.4)

where

\[ H(x, p, l) = \inf_{u \in U} [(p, b(x, u))]_{\mathbb{R}^d} + \frac{1}{2} \text{tr}[\sigma(x, u)\sigma^T(x, u)] + q(x, u)], \quad (x, p, l) \in C_0 \times \mathbb{R}^d \times \mathcal{S}(\mathbb{R}^d). \]

Here, \( \sigma^T \) is the transpose of the matrix \( \sigma \), \( \mathcal{S}(\mathbb{R}^d) \) the set of all \((d \times d)\) symmetric matrices, \((\cdot, \cdot)_{\mathbb{R}^d}\) the scalar product of \( \mathbb{R}^d \), and \( \partial_t \), \( \partial_x \) and \( \partial_{xx} \) the so-called pathwise (or functional or Dupire; see [7, 3, 4]) derivatives, where \( \partial_t \) is known as horizontal derivative, while \( \partial_x \) and \( \partial_{xx} \) are first and second order vertical derivatives, respectively.

The type of problem described above arises in many fields (for an overview on their applications see Kolmanovskii and Shaikhet [21]). We refer to Elsanousi, Øksendal and Sulem [12], Øksendal and Sulem [28] and Øksendal, Sulem and Zhang [29] for applications to mathematical finance, to Gozzi and Marinelli [18] and Gozzi, Marinelli and Savin [19] for advertising models with delayed effects and to Frederico [15] and Gabay and Grasselli [17] for Pension funds.

Crandall and Lions [5] first introduced the notion of viscosity solutions for first order HJB equations in finite dimension, and later Lions [22], [23], [24] extended the notion to second order case. Reader can also refer to the survey paper of Crandall, Ishii and Lions [6] and the monographs of Fleming and Soner [16] and Yong and Zhou [36] for a detailed account for the theory of viscosity solutions. For viscosity solutions of second order HJB equations in infinite dimensional space, we refer to Fabbri, Gozzi and Świȩch [13], Gozzi, Rouy and Świȩch [20], Lions [25], [26], [27] and Świȩch [33], [34]. One of the structural assumption is that the state space has to be a Hilbert space or certain Banach space with smooth norm, not including the continuous function space. For the delay case, the theory of viscosity solutions is more difficult. Our paper [37] studied a class of infinite-horizon optimal control problems for stochastic differential equations with finite delay, in which the term \( X(\cdot - \tau) \) only occurs in the drift term in the state equation.

In this paper, we want to develop a concept of viscosity solutions to second order elliptic HJB equations with infinite delay. We adopt the natural generalization of the well-known Crandall-Lions definition in terms of test functions and then show that the value functional \( V \) defined in (1.3) is the unique viscosity solution to HJB equation given in (1.4) when coefficients \( b, \sigma \) and \( q \) only satisfy Lipschitz conditions under \( | \cdot |_C \).

The standard approach to the treatment of delay optimal control problems is to reformulate them as optimal control problems of the evolution equation in a Hilbert space by inserting the delay term in the linear unbounded operator of the evolution equation (see, e.g., Carlier and Tahraoui [2] and Federico, Goldys and Gozzi [14] for deterministic cases, and Gozzi and Marinelli [18] for stochastic cases). Since coefficients \( b \) and \( \sigma \) are genuinely nonlinear functions about \( X_{x-, u} \), the
standard techniques are not applicable in our case. By reformulating the original control problem as infinite dimensional stochastic control problem in a suitable Banach space, Federico [18] studied the optimal control of a stochastic differential equation with delay arising in the management of a pension fund with surplus, in which the delay in the state equation concentrates on a point of the past and appears in a nonlinear way. However, as the author said that the uniqueness of viscosity solutions is a difficult topic and it is not clear whether their definition of solution is strong enough to guarantee such a result or not.

Our paper [32] introduced a Crandall-Lions definition of viscosity solutions for second order parabolic path-dependent HJB equations associated with optimal control problems for path-dependent stochastic differential equations (PSDEs), and identified the value functional of optimal control as unique viscosity solution to associated path-dependent HJB equations when coefficients only satisfy $d_{\infty}$-Lipschitz conditions with respect to path function. However, this path-dependent case does not include our delay case since the coefficients of PSDEs depend essentially on time $t$.

Our main results are as follows. For every $m \in \mathbb{N}^+$ and $M \in \mathbb{R}$, we define functional $\Upsilon^{m,M} : [0, \infty) \times C_0 \times [0, \infty) \times C_0 \to \mathbb{R}$ by

$$\Upsilon^{m,M}(t,x,s,y) = S_m(t,x,s,y) + M|x(0) - y(0)|^{2m}, \quad (t,x),(s,y) \in [0, \infty) \times C_0,$$

and

$$S_m(t,x,s,y) = \begin{cases} \frac{|(x(s-t)v_0 - y(t-s)v_0)|^{2m} - |x(0) - y(0)|^{2m}}{|x(s-t)v_0 - y(t-s)v_0|^{2m}}, & |x(s-t)v_0 - y(t-s)v_0|_C \neq 0; \\ 0, & |x(s-t)v_0 - y(t-s)v_0|_C = 0, \end{cases}$$

where $x_h \in C_0$ and $x_h(\theta) := x(0)1_{[-h,0]}(\theta) + x(\theta + h)1_{(-\infty,-h)}(\theta), \quad \theta \in (-\infty,0]$ for every $(h,x) \in [0, \infty) \times C_0$.

This key functional is the starting point for the proof of uniqueness result. First, we define

$$\Upsilon^{3,3}(t,x,s,y) := \Upsilon^{3,3}(t,x,s,y) + |s-t|^2, \quad (t,x),(s,y) \in [0, \infty) \times C_0.$$  

Unfortunately, $\Upsilon^{3,3}$ is not a gauge-type function in space $[0, \infty) \times C_0$ with usual norm $\cdot_1$, where $|(t,x)| = |t| + |x|_C$ for every $(t,x) \in [0, \infty) \times C_0$ (see Remark 3.2 (iv)). To overcome this difficulty, we define a metric on $[0, \infty) \times C_0$ as follows: for any $0 \leq t \leq s < \infty$ and $(t,x),(s,y) \in [0, \infty) \times C_0$,

$$d_\infty((t,x),(s,y)) = d_\infty((s,y),(t,x)) := |s-t| + |x_{s-t} - y|_C,$$  

and show that $\Upsilon^{3,3}$ is a smooth gauge-type function in space $([0, \infty) \times C_0, d_\infty)$. Then a modification of Borwein-Preiss variational principle (see Theorem 2.5.2 in Borwein & Zhu [1]) can be used to get a maximum of a perturbation of the auxiliary function $\Psi$.

Second, for every fixed $(t,x) \in [0, \infty) \times C_0$, define $f : [t, \infty) \times C_0 \to \mathbb{R}$ by

$$f(s,y) := \Upsilon^{3,3}(t,x,s,y), \quad (s,y) \in [t, \infty) \times C_0.$$  

We study its regularity in the horizontal/vertical sense and show that it satisfies a functional Itô formula. Then the test function in our definition of viscosity solutions and the auxiliary function $\Psi$ in the proof of uniqueness can include $f$. More importantly, $f(s,y)$ is equivalent to $|y - x_{s-t}|^6$. By this, the uniqueness result is established when the coefficients only satisfy Lipschitz assumption under $\cdot_1$.  

Finally, notice that the key functional includes time term $t$, different from the definition of viscosity solutions to classic elliptic HJB equations (see Definition 2.2 in Crandall, Ishii and Lions...
our definition of viscosity solutions includes time term $t$ and horizontal derivative $\partial_t$. However, we show that our definition of viscosity solutions is a natural extension of viscosity solutions to classic elliptic HJB equations.

Regarding existence, we show that the value functional $V$ defined in (1.3) is a viscosity solution to HJB equation with infinite delay given in (1.4) by functional Itô formula and dynamic programming principle (DPP). Such a formula was firstly provided in Dupire [7] (see also Cont and Fournié [3], [4]). We point out that the functional $\Upsilon$ is also the key in the proof of existence. In infinite-horizon optimal control problem, it is an important problem to find a suitable $\Upsilon$ such that (1.2) is well-defined. Applying functional Itô formula to $\Upsilon$, there exists a constant $\Theta > 0$, such that (1.2) is well-defined when $\lambda > \Theta$ (see Theorem 4.5). This constant is smaller than that obtained by Itô formula.

We also mention that a new concept of viscosity solutions for semi-linear path-dependent partial differential equations was introduced by Ekren, Keller, Touzi and Zhang [9] in terms of a nonlinear expectation, and further extended to fully nonlinear parabolic equations by Ekren, Touzi, and Zhang [10, 11], elliptic equations by Ren [30], obstacle problems by Ekren [8] when the Hamilton function $H$ is uniformly nondegenerate, and degenerate second-order equations by Ren, Touzi, and Zhang [31] and Ren and Rosestolato [32] when the nonlinearity $H$ is $d_p$-uniformly continuous in the path function. However, none of the results we know are directly applicable to our situation as in our case the Hamilton function $H$ may be degenerate and is only required to have continuity properties under supremum norm $| \cdot |_C$.

The outline of this article is as follows. In the following section, we introduce the framework of [4] and [7] and a modification of Borwein-Preiss variational principle. We present the smooth functionals $S_n$, which are the key to proving the stability and uniqueness results of viscosity solutions in Section 3. In Section 4, we introduce preliminary results on stochastic delay optimal control problems. In Section 5, we define classical and viscosity solutions to our HJB equations and prove that the value functional $V$ defined by (1.3) is a viscosity solution to equation (1.4). We also show the consistency with the notion of classical solutions and the stability result. A maximum principle for delay case is given and the uniqueness of viscosity solutions for (1.4) is proved in Section 6.

2 Preliminaries

2.1. Notations and Spaces. We list some notations that are used in this paper. For the vectors $x, y \in \mathbb{R}^d$, the scalar product is denoted by $(x, y)_{\mathbb{R}^d}$ and the Euclidean norm $(x, x)_{\mathbb{R}^d}^\frac{1}{2}$ is denoted by $|x|$ (we use the same symbol $\cdot |$ to denote the Euclidean norm on $\mathbb{R}^k$, for any $k \in \mathbb{N}$). If $A$ is a vector or matrix, its transpose is denoted by $A^\top$; For a matrix $A$, denote its operator norm and Hilbert-Schmidt norm by $|A|$ and $|A|_2$, respectively. Let $D := D((-\infty, 0], \mathbb{R}^d)$, the set of càdlàg $\mathbb{R}^d$-functions on $(-\infty, 0]$, and $C := C((-\infty, 0], \mathbb{R}^d)$ the family of continuous functions from $(-\infty, 0]$ to $\mathbb{R}^d$. Let $D_0 = \{f \in D : \lim_{\theta \to -\infty} f(\theta) = 0\}$ and $C_0 = \{f \in C : \lim_{\theta \to -\infty} f(\theta) = 0\}$. We define a norm on $D_0$ as follows:

$$|x|_C = \sup_{\theta \in (-\infty, 0]} |x(\theta)|, \quad x \in D_0.$$  

Then, $(D_0, | \cdot |_C)$ and $(C_0, | \cdot |_C)$ are Banach spaces. Following Dupire [7], for $x \in D_0$, $h \geq 0$, we define $x_h \in D_0$ as

$$x_h(\theta) := x(0)1_{[-h, 0]}(\theta) + x(\theta + h)1_{(-\infty, -h)}(\theta), \quad \theta \in (-\infty, 0].$$

For each $t \in [0, \infty)$, define $\hat{\Lambda}^t := [t, \infty) \times D_0$ and $\Lambda^t := [t, \infty) \times C_0$. Let $\hat{\Lambda}$ and $\Lambda$ denote $\hat{\Lambda}^0$ and $\Lambda^0$ respectively.
\( \hat{\Lambda}^0 \), respectively. We define a metric on \( \hat{\Lambda}^t \) as follows: for any \( t \leq s \leq l < \infty \) and \( (s,x), (l,y) \in \hat{\Lambda}^t \),

\[
d_\infty((s,x), (l,y)) = d_\infty((l,y), (s,x)) := |l-s| + |x_{l-s} - y|_C.
\] (2.1)

Then \((\hat{\Lambda}^t, d_\infty)\) and \((\Lambda^t, d_\infty)\) are complete metric spaces.

Now we define the pathwise derivatives of Dupire \cite{Dupire.2001}.

**Definition 2.1.** (Pathwise derivatives) Let \( t \in [0, \infty) \) and \( f : \hat{\Lambda}^t \to \mathbb{R} \).

(i) Given \( (s,x) \in \hat{\Lambda}^t \), the horizontal derivative of \( f \) at \( (s,x) \) (if the corresponding limit exists and is finite) is defined as

\[
\partial_t f(s,x) := \lim_{h \to 0, h > 0} \frac{1}{h} [f(s + h, x) - f(s,x)].
\] (2.2)

If the above limit exists and is finite for every \( (s,x) \in \hat{\Lambda}^t \), the map \( \partial_t f : \hat{\Lambda}^t \to \mathbb{R} \) is called the horizontal derivative of \( f \) with domain \( \hat{\Lambda}^t \).

(ii) Given \( (s,x) \in \hat{\Lambda}^t \), the vertical derivatives of first and second order of \( f \) at \( (s,x) \) (if the corresponding limit exists and is finite) are defined as

\[
\partial_x f(s,x) := (\partial_{x_1} f(s,x), \partial_{x_2} f(s,x), \ldots, \partial_{x_d} f(s,x)),
\] (2.3)

and

\[
\partial_{xx} f(s,x) := (\partial_{x_1 x_1} f(s,x))_{i=1,2,\ldots,d},
\] (2.4)

where

\[
\partial_{x_i} f(s,x) := \lim_{h \to 0} \frac{1}{h} \left[ f(s,x + he_i 1_{\{0\}}) - f(s,x) \right], \quad i = 1, 2, \ldots, d,
\] (2.5)

and

\[
\partial_{x_i x_j} f(s,x) := \partial_{x_i} (\partial_{x_j} f)(s,x), \quad i, j = 1, 2, \ldots, d,
\] (2.6)

with \( e_1, e_2, \ldots, e_d \) is the standard orthonormal basis of \( \mathbb{R}^d \). If the above limits in (2.5) exist and are finite for every \( (s,x) \in \hat{\Lambda}^t \), the map \( \partial_x f := (\partial_{x_1} f, \partial_{x_2} f, \ldots, \partial_{x_d} f)^T : \hat{\Lambda}^t \to \mathbb{R}^d \) is called the first order vertical derivative of \( f \) with domain \( \hat{\Lambda}^t \), and if the above limits in (2.6) exist and are finite for every \( (s,x) \in \hat{\Lambda}^t \), the map \( \partial_{xx} f := (\partial_{x_i x_j} f)_{i,j=1,2,\ldots,d} : \hat{\Lambda}^t \to \mathcal{S}^{d \times d} \) is called the second order vertical derivative of \( f \) with domain \( \hat{\Lambda}^t \), where \( \mathcal{S}^{d \times d} \) is the space of all \( d \times d \) symmetric matrices.

**Definition 2.2.** Let \( t \in [0, \infty) \) and \( f : \hat{\Lambda}^t \to \mathbb{R} \) be given.

(i) We say \( f \in C(\hat{\Lambda}^t) \) if \( f \) is continuous in \( (s,x) \) on \((\hat{\Lambda}^t, d_\infty)\);

(ii) We say \( f \in C^{1,2}(\hat{\Lambda}^t) \subset C(\hat{\Lambda}^t) \) if \( \partial_t f, \partial_x f \) and \( \partial_{xx} f \) exist and are continuous in \( (s,x) \) on \((\hat{\Lambda}^t, d_\infty)\);

(iii) We say \( f \in C^{1,2}_p(\hat{\Lambda}^t) \subset C^{1,2}(\hat{\Lambda}^t) \) if \( f, \partial_t f, \partial_x f \) and \( \partial_{xx} f \) grow in a polynomial way.

For every \( t \in [0, \infty) \), \( f : \Lambda^t \to \mathbb{R} \) and \( \hat{f} : \hat{\Lambda}^t \to \mathbb{R} \) are called consistent on \( \Lambda^t \) if \( f \) is the restriction of \( \hat{f} \) on \( \Lambda^t \).

**Definition 2.3.** Let \( t \in [0, \infty) \) and \( f : \Lambda^t \to \mathbb{R} \) be given.
Lemma 2.5. \( f \in C(\Lambda^t) \) if \( f \) is continuous in \((s,x)\) on \((\Lambda^t,d_\infty)\).

(ii) We say \( f \in C_p^{1,2}(\Lambda^t) \subset C(\Lambda^t) \) if there exists \( \hat{f} \in C_p^{1,2}(\hat{\Lambda}^t) \) which is consistent with \( f \) on \( \Lambda^t \).

**Definition 2.4.** Let \( f : C_0 \to \mathbb{R} \) be given. Define \( \hat{f} : \Lambda \to \mathbb{R} \) by \( \hat{f}(t,x) := f(x) \), \((t,x) \in \Lambda \).

(i) We say \( f \in C(C_0) \) if \( f \) is continuous in \( x \) on \((C_0,|\cdot|_C)\).

(ii) We say \( f \in C(\Lambda) \) if \( \hat{f} \in C(\Lambda) \).

(iii) We say \( f \in C_p^{1,2}(C_0) \) if \( \hat{f} \in C_p^{1,2}(\Lambda) \); and we define \( \partial_t f \), \( \partial_x f \) and \( \partial_{xx} f \) by

\[
\partial_t f(x) := \partial_t \hat{f}(0,x), \quad \partial_x f(x) := \partial_x \hat{f}(0,x), \quad \partial_{xx} f(x) := \partial_{xx} \hat{f}(0,x), \quad x \in C_0.
\]

We also have

**Lemma 2.5.** \( f \in C(C_0) \) if and only if \( f \in C(\Lambda) \).

**Proof.** For every \((t,x),(s,y) \in \Lambda\), if \( t \leq s \),

\[
|x - y|_C \leq |x_{s-t} - y|_C + |x - x_{s-t}|_C \leq d_\infty((t,x),(s,y)) + |x - x_{s-t}|_C;
\]

if \( t > s \),

\[
|x - y|_C \leq |z - y|_C + |x - z|_C \leq d_\infty((t,x),(s,y)) + \sup_{\theta \in (-\infty,0]}|x(\theta) - x(\theta + s - t)|,
\]

where \( z(\theta) = x(\theta + s - t) \) for all \( \theta \in (-\infty,0] \). Letting \((s,y) \to (t,x)\) in \((\Lambda,d_\infty)\), we have \( y \to x \) in \((C_0,|\cdot|_C)\) and \( s \to t \). Then if \( f \in C(C_0) \), we get \( \hat{f}(s,y) = f(y) \to f(x) = \hat{f}(t,x) \) as \((s,y) \to (t,x)\) in \((\Lambda,d_\infty)\). Thus, \( f \in C(\Lambda) \). It is clear that \( f \in C(C_0) \) if \( f \in C(\Lambda) \). The proof is now complete. \( \Box \)

Let \( \{W(t), t \geq 0\} \) be a \( n \)-dimensional standard Wiener process defined on a complete probability space \((\Omega,\mathcal{F},\mathbb{P})\). Let \( \{\mathcal{F}_t\}_{t \geq 0} \) be the natural filtration of \( W(t) \), augmented with the family \( \mathcal{N} \) of \( \mathbb{P} \)-null of \( \mathcal{F} \). The filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) satisfies the usual condition. For every \([t,s] \subset [0,\infty)\), we also use the notations:

\[
\mathcal{F}^{s,0}_t = \sigma(W(l) - W(t) : l \in [t,s]), \quad \mathcal{F}^s_t = \mathcal{F}^{s,0}_t \lor \mathcal{N}.
\]

We also write \( \mathcal{F}^{t,0}_s \) for \( \{\mathcal{F}^{s,0}_t\}_{t \geq 0} \) and \( \mathcal{F}^t_s \) for \( \{\mathcal{F}^{s,0}_t\}_{s \geq t} \).

Next we define several classes of random variables or stochastic processes with values in a Banach space \((K,|\cdot|_K)\).

- \( L^p(\Omega,\mathcal{F}_t;K) \) defined for all \( t \geq 0 \) and \( p \geq 1 \), denotes the space of all \( \mathcal{F}_t \)-measurable maps \( \xi : \Omega \to K \) satisfying \( \mathbb{E}|\xi|^p_K < \infty \).

- \( L^p_p(\Omega \times [t,T];K) \) defined for all \( 0 \leq t < T < \infty \), denotes the space of equivalence classes of processes \( y \in L^2(\Omega \times [t,T];K) \), admitting a predictable version. \( L^2_p(\Omega \times [t,T];K) \) is endowed with the norm

\[
|y|^2 = \mathbb{E} \int_t^T |y(s)|^2_K ds.
\]

- \( L^2_p(\Omega \times [t,\infty);K) \) defined for all \( t \geq 0 \), denotes the space of equivalence classes of processes \( \{y(s), s \geq t\} \), with values in \( K \) such that \( y|_{[t,T]} \in L^2_p(\Omega \times [t,T];K) \) for all \( T > t \), where \( y|_{[t,T]} \) denotes the restriction of \( y \) to the interval \([t,T]\).

2.2. **Functional Itô formula.** Assume that \( \vartheta \in L^2_p(\Omega \times [0,\infty);\mathbb{R}^d), \varpi \in L^2_p(\Omega \times [0,\infty);\mathbb{R}^{d \times n}) \) and \((\tau,\sigma) \in \Lambda\), then the following process

\[
X(s) = x(0) + \int_\tau^s \vartheta(\sigma) d\sigma + \int_\tau^s \varpi(\sigma) dW(\sigma), \quad s \geq \tau \geq 0,
\]

(2.7)
Lemma 2.6. Suppose \( f \in C^{1,2}_p(\Lambda t) \) for some \( t \in [0, \infty) \). Then, under the above conditions, \( \mathbb{P} \)-a.s., for all \( \tau \leq t \leq s < \infty \):

\[
f(s, X_s) = f(i, X_i) + \int_t^s \left[ \partial_t f(\sigma, X_\sigma) + (\partial_x f(\sigma, X_\sigma), 0) \right] d\sigma + \frac{1}{2} \int_t^s \text{tr}(\partial_{xx} f(\sigma, X_\sigma) \omega(\sigma) \omega^\top(\sigma)) d\sigma + \int_t^s \partial_x f(\sigma, X_\sigma) \omega(\sigma) dW(\sigma).
\]

(2.8)

Here and in the following, for every \( s \in \mathbb{R} \), \( X(s) \) denotes the value of \( X \) at time \( s \), and \( X_s \) the function from \((-\infty, 0)\) to \( \mathbb{R}^d \) by \( X_s(\theta) = X(s + \theta) \), \( \theta \in (-\infty, 0) \).

The proof is similar to Theorem 4.1 in Cont & Fournie [1] (see also Dupire [7]). Here we omit it.

By the above Lemma, we have the following important results.

Lemma 2.7. Let \( f \in C^{1,2}_p(\Lambda t) \) and \( \hat{f} \in C^{1,2}_p(\hat{\Lambda} t) \) such that \( \hat{f} \) is consistent with \( f \) on \( \Lambda t \), then the following definition

\[
\partial_t \hat{f} := \partial_t \hat{f}, \quad \partial_x \hat{f} := \partial_x \hat{f}, \quad \partial_{xx} \hat{f} := \partial_{xx} \hat{f} \quad \text{on} \quad \Lambda^t
\]

is independent of the choice of \( \hat{f} \). Namely, if there is another \( \hat{f}' \in C^{1,2}_p(\hat{\Lambda} t) \) such that \( \hat{f}' \) is consistent with \( f \) on \( \Lambda^t \), then the derivatives of \( \hat{f}' \) coincide with those of \( \hat{f} \) on \( \Lambda^t \).

\[
\text{Proof.} \quad \text{By the definition of the horizontal derivative, it is clear that} \quad \partial_t \hat{f}(l, x) = \partial_t \hat{f}'(l, x) \quad \text{for every} \quad (l, x) \in \Lambda^t. \quad \text{Next, for every} \quad (l, x) \in \Lambda^t, \quad \text{let} \quad \varpi = 0, \quad \tau = l \quad \text{and} \quad \vartheta \equiv h \in \mathbb{R}^d \quad \text{in (2.7)}, \quad \text{by Lemma 2.6}
\]

\[
\int_l^s (\partial_x \hat{f}(\sigma, X_\sigma), h) d\sigma = \int_l^s (\partial_\vartheta \hat{f}'(\sigma, X_\sigma), h) d\sigma, \quad s \in [l, \infty),
\]

where \( X_\sigma(r) = x((\sigma + r - l) \land 0) + (\sigma + r - l) h \mathbb{1}_{[l, \sigma]}(r), r \in (-\infty, 0] \). Here and in the sequel, for notational simplicity, we use \( 0 \) to denote elements, or functions which are identically equal to zero.

By the continuity of \( \partial_x \hat{f}, \partial_\vartheta \hat{f}' \) and the arbitrariness of \( h \in \mathbb{R}^d \), we have \( \partial_x \hat{f}(l, x) = \partial_\vartheta \hat{f}'(l, x) \) for every \( (l, x) \in \Lambda^t \). Finally, let \( \vartheta = 0, \quad \tau = l \) and \( \varpi \equiv a \in \mathcal{S}(\mathbb{R}^d) \) in (2.7), by Lemma 2.6,

\[
\int_l^s \text{tr}(\partial_{xx} \hat{f}(\sigma, X_\sigma) aa^*) d\sigma = \int_l^s \text{tr}(\partial_{xx} \hat{f}'(\sigma, X_\sigma) aa^*) d\sigma, \quad s \in [l, \infty).
\]

By the continuity of \( \partial_{xx} \hat{f}, \partial_{xx} \hat{f}' \) and the arbitrariness of \( a \in \Gamma(\mathbb{R}^d) \), we also have \( \partial_{xx} \hat{f}(l, x) = \partial_{xx} \hat{f}'(l, x) \) for every \( (l, x) \in \Lambda^t \). \( \square \)

2.3. Borwein-Preiss variational principle. In this subsection we introduce a modification of Borwein-Preiss variational principle, which plays a crucial role in the proof of the comparison Theorem. We firstly recall the definition of gauge-type function for the space \( (\Lambda^t, d_\infty) \).

Definition 2.8. Let \( t \in [0, \infty) \) be fixed. We say that a continuous functional \( \rho: \Lambda^t \times \Lambda^t \to [0, \infty) \) is a gauge-type function under \( d_\infty \) provided that:

(i) \( \rho((s, x), (s, x)) = 0 \) for all \( (s, x) \in \Lambda^t \),

(ii) for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that, for all \( (s, x), (l, y) \in \Lambda^t \), we have \( \rho((s, x), (l, y)) \leq \delta \) implies that \( d_\infty((s, x), (l, y)) < \varepsilon \).
Lemma 2.9. Let $t \in [0, \infty)$ be fixed and $f : \Lambda^t \to \mathbb{R}$ be an upper semicontinuous functional and bounded from above. Suppose that $\rho$ is a gauge-type function under $d_\infty$ and $\{\delta_i\}_{i \geq 0}$ is a sequence of positive number, and suppose that $\varepsilon > 0$ and $(t_0, x^0) \in \Lambda^t$ satisfy
\[ f(t_0, x^0) \geq \sup_{(s, x) \in \Lambda^t} f(s, x) - \varepsilon. \]
Then there exist $(\hat{t}, \hat{x}) \in \Lambda^t$ and a sequence $\{(t_i, x^i)\}_{i \geq 1} \subset \Lambda^t$ such that
(i) $f((\hat{t}, \hat{x}), (t_0, x^0)) \leq \frac{\varepsilon}{2^i}, \rho((\hat{t}, \hat{x}), (t_i, x^i)) \leq \frac{\varepsilon}{2^{i+1}}$ and $t_i \uparrow \hat{t}$ as $i \to \infty$,
(ii) $f(\hat{t}, \hat{x}) - \sum_{i=0}^{\infty} \delta_i \rho((\hat{t}, \hat{x}), (t_i, x^i)) \geq f(t_0, x^0)$, and
(iii) $f(s, x) - \sum_{i=0}^{\infty} \delta_i \rho((s, x), (t_i, x^i)) < f(\hat{t}, \hat{x}) - \sum_{i=0}^{\infty} \delta_i \rho((\hat{t}, \hat{x}), (t_i, x^i))$ for all $(s, x) \in \Lambda^t \setminus \{(\hat{t}, \hat{x})\}$.

The proof is completely similar to Lemma 2.13 in our paper [38] (see also Theorem 2.5.2 in Borwein & Zhu [1]). Here we omit it.

3 Smooth gauge-type functions.

In this section we introduce the functionals $S_m$, which are the key to proving the uniqueness and stability of viscosity solutions.

For every $m \in \mathbb{N}^+$, we define $S_m : \hat{\Lambda} \times \hat{\Lambda} \to \mathbb{R}$ as follows: for every $(t, x), (s, y) \in \hat{\Lambda}$,
\[ S_m(t, x, s, y) = \begin{cases} \frac{((x(s-t) \cup_0 y(t-s)) \circ x)^{2m} - |x(0) - y(0)|^{2m}}{|x(s-t) \cup_0 y(t-s) \circ x|^{4m}}, & |x(s-t) \cup_0 y(t-s) \circ x| \neq 0; \\ 0, & |x(s-t) \cup_0 y(t-s) \circ x| = 0. \end{cases} \]

For every $M \in \mathbb{R}$, define $\Upsilon^{m,M}$ and $\overline{\Upsilon}^{m,M}$ by
\[ \Upsilon^{m,M}(t, x, s, y) = S_m(t, x, s, y) + M|x(0) - y(0)|^{2m}, \quad (t, x), (s, y) \in \hat{\Lambda}, \]
and
\[ \overline{\Upsilon}^{m,M}(t, x, s, y) = \Upsilon^{m,M}(t, x, s, y) + |s - \hat{t}|^2 \quad (t, x), (s, y) \in \hat{\Lambda}. \]

For notational simplicity, we let $S_m(x, y)$, $\Upsilon^{m,M}(x, y)$ and $\overline{\Upsilon}^{m,M}(x, y)$ denote $S_m(t, x, t, y)$, $\Upsilon^{m,M}(t, x, t, y)$ and $\overline{\Upsilon}^{m,M}(t, x, t, y)$ for all $(t, x, y) \in [0, \infty) \times \mathcal{D}_0 \times \mathcal{D}_0$, respectively. Moreover, we let $S_m(x)$ and $\Upsilon^{m,M}(x)$ denote $S_m(x, x)$ and $\Upsilon^{m,M}(x, x)$ respectively when $y(\theta) \equiv 0$ for all $\theta \in (-\infty, 0]$. we also let $S$, $\Upsilon$ and $\overline{\Upsilon}$ denote $S_3$, $\Upsilon^{3,3}$ and $\overline{\Upsilon}^{3,3}$, respectively.

Now we study the regularity of $S_m$.

Lemma 3.1. For every fixed $(\hat{t}, a) \in \hat{\Lambda}$, define $S_m^{\hat{t},a} : \hat{\Lambda} \to \mathbb{R}$ by
\[ S_m^{\hat{t},a}(t, x) := S_m(t, x, \hat{t}, a), \quad (t, x) \in \hat{\Lambda}. \]
Then $S_m^{\hat{t},a}(\cdot, \cdot) \in C_p^{1,2}(\hat{\Lambda}^i)$. Moreover, for every $M \geq 3$,
\[ |x|^{2m} \leq \Upsilon^{m,M}(x) \leq M|x|^{2m}, \quad x \in \mathcal{D}_0. \] (3.1)
Proof. First, by the definition of $S^t, a_m$, it is clear that $S^t, a_m(\cdot, \cdot) \in C(\hat{\Lambda}^t)$ and $\partial_t S^t, a_m(t, x) = 0$
for $(t, x) \in \hat{\Lambda}^t$. Second, we consider $\partial_x S^t, a_m(\cdot, \cdot)$. For every $x \in D_0$, let $|x|_{C^\infty} := \sup_{-\infty < s < 0} |x(s)|$
and $x_i(0) := (x(0), e_i)_{R^2}$, $i = 1, 2, \ldots, d$. If $|x(0) - a(0)| < |x - a_{t-i}|_{C^\infty}$,

$$
\partial_x S^t, a_m(t, x) = \lim_{h \to 0} \frac{S_m(t, x + he_i1_{\{0\}}, \hat{t}, a) - S_m(t, x, \hat{t}, a)}{h}
= \lim_{h \to 0} \frac{|x - a_{t-i}|_{C^\infty}^2 - |x(0) + he_i - a(0)|^2m^3 - |x(0) - a(0)|^2m^3}{h |x - a_{t-i}|_{C^\infty}^4 |x - a(0)|^2m^2 (x_i(0) - a_i(0))};
$$

(3.2)

if $|x(0) - a(0)| > |x - a_{t-i}|_{C^\infty}$,

$$
\partial_x S^t, a_m(t, x) = 0;
$$

(3.3)

if $|x(0) - a(0)| = |x - a_{t-i}|_{C^\infty} > 0$, since

$$
|x(0) - a(0)|^2m - |x(0) + he_i - a(0)|^2m
= \begin{cases} 0, & |x(0) + he_i - a(0)| \geq |x(0) - a(0)|, \\ |x(0) - a(0)|^2m - |x(0) + he_i - a(0)|^2m, & |x(0) + he_i - a(0)| < |x(0) - a(0)|, \end{cases}
$$

(3.4)

we have

$$
0 \leq \lim_{h \to 0} \frac{|x(0) + he_i - a(0)|^2m}{h |x - a_{t-i}|_{C^\infty}^4 |x - a(0)|^2m^2 (x_i(0) - a_i(0))} = 0;
$$

(3.5)

if $|x(0) - a(0)| = |x - a_{t-i}|_{C^\infty} = 0$,

$$
\partial_x S^t, a_m(t, x) = 0.
$$

(3.6)

From (3.2), (3.3), (3.5) and (3.6) we obtain that

$$
\partial_x S^t, a_m(t, x) = \begin{cases} -\frac{6m(|x - a_{t-i}|_{C^\infty}^2 - |x(0) - a(0)|^2m^2 |x(0) + he_j - a(0)|^2m - 2(x_i(0) - a_i(0))}{|x - a_{t-i}|_{C^\infty}^4}, & |x - a_{t-i}|_{C^\infty} \neq 0, \\ 0, & |x - a_{t-i}|_{C^\infty} = 0. \end{cases}
$$

(3.7)

It is clear that $\partial_x S^t, a_m(\cdot, \cdot) \in C(\hat{\Lambda}^t)$.

We now consider $\partial_{x_i} S^t, a_m(\cdot, \cdot)$. If $|x(0) - a(0)| < |x - a_{t-i}|_{C^\infty}$,

$$
\partial_{x_i} S^t, a_m(t, x) = \lim_{h \to 0} \frac{-6m(|x - a_{t-i}|_{C^\infty}^2 - |x(0) + he_j - a(0)|^2m^2 |x(0) + he_j - a(0)|^2m^2 - 2(x_i(0) - a_i(0))}{h |x - a_{t-i}|_{C^\infty}^4}
\times (x_i(0) - a_i(0) + h1_{\{i=j\}})
+ \frac{6m(|x - a_{t-i}|_{C^\infty}^2 - |x(0) - a(0)|^2m^2 |x(0) - a(0)|^2m^2 (x_i(0) - a_i(0))}{h |x - a_{t-i}|_{C^\infty}^4} \bigg]\bigg]
$$

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Thus, we have (3.1) holds true. The proof is now complete.

if $|x(0) - a(0)| > |x - a_{t-\xi}|_{C-}$,

$$\partial_{x\cdot x\cdot t\cdot a\cdot m\cdot}(t, x) = 0; \quad (3.9)$$

if $|x(0) - a(0)| = |x - a_{t-\xi}|_{C-} \geq 0$, by (3.4), we have

$$0 \leq \lim_{h \to 0} \left| \frac{\partial_{x\cdot x\cdot t\cdot a\cdot m\cdot}(t, x + he_j 1_{\{0\}}) - \partial_{x\cdot x\cdot t\cdot a\cdot m\cdot}(t, x)}{h} \right| \leq \lim_{h \to 0} \frac{6m(|x(0) - a(0)|^{2m} - |x(0) + he_j - a(0)|^{2m})^2 |x(0) + he_j - a(0)|^{2m-2}}{h|x - a_{t-\xi} + he_j 1_{\{0\}}|^{4m}} \times \left(x_i(0) - a_i(0) + h 1_{\{i=j\}}\right) = 0; \quad (3.10)$$

if $|x(0) - a(0)| = |x - a_{t-\xi}|_{C-} = 0,$

$$\partial_{x\cdot x\cdot t\cdot a\cdot m\cdot}(t, x) = 0. \quad (3.11)$$

Combining (3.8), (3.9), (3.10) and (3.11) we obtain

$$\partial_{x\cdot x\cdot t\cdot a\cdot m\cdot}(t, x) = \begin{cases} 
\frac{24m^2(|x(a_{t-\xi}|_{C-}^2m - |x(0) - a(0)|^{2m})|x(0) - a(0)|^{4m-4} (x_i(0) - a_i(0))(x_j(0) - a_j(0))}{|x - a_{t-\xi}|^{4m}_{C-}} \\
- \frac{12m(m-1)(|x(a_{t-\xi}|_{C-}^2m - |x(0) - a(0)|^{2m})^2 |x(0) - a(0)|^{2m-4} (x_i(0) - a_i(0))(x_j(0) - a_j(0))}{|x - a_{t-\xi}|^{4m}_{C-}} \\
- \frac{6m(|x(a_{t-\xi}|_{C-}^2m - |x(0) - a(0)|^{2m})^2 |x(0) - a(0)|^{2m-2} 1_{\{i=j\}}}{|x - a_{t-\xi}|^{4m}_{C-}}, & |x - a_{t-\xi}|_{C-} \neq 0, \\
0, & |x - a_{t-\xi}|_{C-} = 0. \end{cases} \tag{3.12}$$

It is clear that $\partial_{x\cdot x\cdot t\cdot a\cdot m\cdot}(\cdot, \cdot) \in C(\hat{\Lambda}^\dagger)$. By simple calculation, we can see that $S_{m\cdot}(\cdot, \cdot)$ and all of its derivatives grow in a polynomial way. Thus, we have show that $S_{m\cdot}(\cdot, \cdot) \in C_{p^{1/2}}(\hat{\Lambda}^\dagger)$.

Now we prove (3.1). If $|x|_{C} = 0$, it is clear that (3.1) holds. Then we may assume that $|x|_{C} \not= 0$. Letting $\alpha := |x(0)|^{2m}$, we have

$$\Upsilon^{M\cdot}(x) = \frac{|x_{C}^{2m} - |x(0)|^{2m}|^3}{|x_{C}^{4m}|} + M|x(0)|^{2m} := f(\alpha) = \frac{|x_{C}^{2m} - \alpha|^3}{|x_{C}^{4m}|} + M\alpha.$$ By

$$f'(\alpha) = -3\frac{|x_{C}^{2m} - \alpha|^2}{|x_{C}^{4m}|} + M \geq 0, \quad M \geq 3, \quad 0 \leq \alpha \leq |x_{C}^{2m}|,$$

we get that

$$|x_{C}^{2m}| = f(0) \leq \Upsilon^{M\cdot}(x) = f(\alpha) \leq f(|x_{C}^{2m}|) = M|x_{C}^{2m}|, \quad M \geq 3, \quad x \in D_{0}.$$ Thus, we have (3.1) holds true. The proof is now complete. \quad \Box
Remark 3.2. (i) For every fixed \(m \in \mathbb{N}^+\) and \(a_0 \in \mathbb{R}^d\), define \(f : \hat{\Lambda} \to \mathbb{R}\) by
\[
f(t, x) := |x(0) - a_0|^{2m}, \quad (t, x) \in \hat{\Lambda}.
\]
Notice that
\[
\partial_t f(t, x) = 0; \tag{3.13}
\]
\[
\partial_x f(t, x) = 2m|x(0) - a_0|^{2m-2}(x(0) - a_0); \tag{3.14}
\]
\[
\partial_{xx} f(t, x) = 2m|x(0) - a_0|^{2m-2}I + 4m(m-1)|x(0) - a_0|^{2m-4}(x(0) - a_0)(x(0) - a_0)^\top. \tag{3.15}
\]
Then \(f \in C^{1,2}_p(\hat{\Lambda})\), and by the above lemma, \(\Upsilon^{m,M}(:, :, t, a) \in C^{1,2}_p(\hat{\Lambda}^i)\) for all \(m \in \mathbb{N}^+, M \in \mathbb{R}\) and \((\hat{t}, a) \in \hat{\Lambda}_i^i\).

(ii) Since \(| \cdot |_C^6\) does not belong to \(C^{1,2}_p(\hat{\Lambda})\), then, for every \((\hat{t}, a) \in \Lambda, \ |x - a_{-\hat{t}}|_C^6\), cannot appear as an auxiliary functional in the proof of the uniqueness and stability of viscosity solutions. However, by the above lemma and (i) of this remark, we can replace \(|x - a_{-\hat{t}}|_C^6\) with its equivalent functional \(\Upsilon(t, x, t, a)\).

(iii) It follows from (3.1) that, for all \((l, x), (s, y) \in \hat{\Lambda}\),
\[
\Upsilon(\gamma_t, \eta_s) = \Upsilon(\gamma_{t \lor s} - \eta_{s \lor t} - \gamma_{s \lor t} + \eta_{t \lor s}) + |s - t|^2 \geq |x(s - t)\lor 0 - y(t - s)\lor 0|_C^6 + |s - t|^2. \tag{3.16}
\]
Thus \(\Upsilon\) is a gauge-type function. We can apply it to Lemma 2.9 to get a maximum of a perturbation of the auxiliary function in the proof of uniqueness.

(iv) \(\Upsilon\) is not a gauge-type function under usual norm \(| \cdot |_1\), where \(|(t, x)|_1 = |t| + |x|_C\) for every \((t, x) \in \Lambda\). As a matter of fact, consider the following example. Take \((t_n, x^n), (s_n, y^n) \in \Lambda\) as follows:
\[
t_n = 0, \quad s_n = \frac{1}{n}, \quad x^n(\theta) = (1 + n\theta)1_{[\frac{-1}{n}, 0]}(\theta), \quad y^n(\theta) = [(2 + n\theta) \land 1]1_{[\frac{2}{n}, 0]}(\theta), \quad \theta \in (-\infty, 0].
\]
It is clear that \(\Upsilon(t_n, x^n, s_n, y^n) = \frac{1}{n} \to 0\) as \(n \to \infty\). However,
\[
|(t_n, x^n) - (s_n, y^n)|_1 = |t_n - s_n| + |x^n - y^n|_C = \frac{1}{n} + 1 \geq 1.
\]
This shows that \(\Upsilon\) does not satisfy condition (ii) of Definition 2.3 with norm \(| \cdot |_1\).

In the proof of uniqueness of viscosity solutions, in order to apply Theorem 8.3 in [6], we also need the following lemma. Its proof is completely similar to the path-dependent case (see Lemma 3.3 in [8]). Here we omit it.

Lemma 3.3. For \(m \in \mathbb{N}^+\) and \(M \geq 3\), we have
\[
\Upsilon^{m,M}(x + y) \leq 2^{2m-1}(\Upsilon^{m,M}(x) + \Upsilon^{m,M}(y)) \quad x, y \in \mathcal{D}_0. \tag{3.17}
\]
4 A DPP for optimal control problems.

In this section, we consider the controlled state equation \(4.1\) and cost functional \(4.2\). We introduce the admissible control. Let \(t\) be a deterministic time, \(0 \leq t < \infty\).

**Definition 4.1.** An admissible control process \(u(\cdot) = \{u(r), r \in [t, \infty)\}\) on \([t, \infty)\) is an \(\mathcal{F}_t^\mathbb{F}\)-progressing measurable process taking values in some Polish space \((U,d)\). The set of all admissible controls on \([t, \infty)\) is denoted by \(\mathcal{U}_t\). We identify two processes \(u(\cdot)\) and \(\tilde{u}(\cdot)\) in \(\mathcal{U}_t\) and write \(u(\cdot) \equiv \tilde{u}(\cdot)\) on \([t, \infty)\), if \(\mathbb{P}(u(\cdot) = \tilde{u}(\cdot)\ a.e.\ in\ [t, \infty)) = 1\).

We make the following assumption.

**Hypothesis 4.2.** \(b : C_0 \times U \to \mathbb{R}^d,\ \sigma : C_0 \times U \to \mathbb{R}^{d \times n}\) and \(q : C_0 \times U \to \mathbb{R}\) are continuous, and there exists a constant \(L > 0\) such that, for all \((x, u), (y, u) \in C_0 \times U,\)
\[
|b(x, u)|^2 \vee |\sigma(x, u)|^2 \vee |q(x, u)|^2 \leq L^2(1 + |x|_C^2),
\]
\[
|b(x, u) - b(y, u)| \vee |\sigma(x, u) - \sigma(y, u)|_2 \vee |q(x, u) - q(y, u)| \leq L|x - y|_C.
\]

For given \(t \in [0, \infty), \mathcal{F}_t\)-measurable map \(\xi : \Omega \to C_0\) and admissible control \(u(\cdot) \in \mathcal{U}_0\), consider the following stochastic differential equation (SDE) with infinite delay:
\[
\begin{cases}
\text{d}X^{t,\xi,u}(s) = b(X^{t,\xi,u}_s, u(s))ds + \sigma(X^{t,\xi,u}_s, u(s))dW(s), & s \in [t, \infty),
\end{cases}
\]
\[
X^{t,\xi,u}_t = \xi.
\]

We first recall a result on the solvability of \(4.2\) on a bounded interval.

**Lemma 4.3.** Take \(p \geq 2\) and assume that Hypothesis \(4.2\) holds. Then for every \(T > 0\) and \((t, \xi, u(\cdot)) \in [0, T) \times L^p(\Omega, \mathcal{F}_t; C_0) \times \mathcal{U}_0\), equation \(4.2\) admits a unique strong solution \(X^{t,\xi,u}(\cdot)\) on \([t, T]\). Furthermore, let \(X^{t,\xi',u}(\cdot)\) be the solution of equation \(4.2\) corresponding \((t, \xi', u(\cdot)) \in [0, T) \times L^p(\Omega, \mathcal{F}_t; C_0) \times \mathcal{U}_0\). Then the following estimates hold:
\[
\mathbb{E}\left[\sup_{t \leq s \leq T} |X^{t,\xi,u}_s(s) - X^{t,\xi',u}_s(s)|^p\right] \leq C_p \mathbb{E}[|\xi - \xi'|_C^p];
\]
\[
\mathbb{E}\left[\sup_{t \leq s \leq T} |X^{t,\xi,u}_s(s)|^p\right] \leq C_p (1 + \mathbb{E}[|\xi|_C^p]).
\]

The constant \(C_p\) depending only on \(p, T\) and \(L\).

**Proof.** In the case of \(p = 2\), we refer to Theorem 3.1 in \(35\) for existence and uniqueness of the solution to equation \(4.2\) on \([t, T]\) and to Lemma 3.2 in \(35\) for the Linear growth of this solution on the initial datum. By the similar proving process of Lemma 3.2 in \(35\), we obtain \(4.3\). The proof in the case of \(p > 2\) can be performed in a similar way. \(\square\)

Now we study equation \(4.2\) and consider certain continuities for the solution \(X^{t,\xi,u}(\cdot)\). These properties will be used in the proof of Theorem 4.5.

**Theorem 4.4.** Take \(p \geq 2\) and suppose that Hypothesis \(4.2\) is satisfied, then for every \((t, \xi, u(\cdot)) \in [0, \infty) \times L^p(\Omega, \mathcal{F}_t; C_0) \times \mathcal{U}_0\) and \(\beta < -(\frac{2}{L^2} + L)\), equation \(4.2\) has a unique solution \(X^{t,\eta,v}(\cdot)\). Moreover, if we let \(X^{t,\eta,v}(\cdot)\) be the solutions of \(4.2\) corresponding \((t, \eta, v(\cdot)) \in [0, \infty) \times L^p(\Omega, \mathcal{F}_t; C_0) \times \mathcal{U}_0\). Then the following estimates hold:
\[
\sup_{s \geq t} e^{2\beta s} \mathbb{E}\left[|X^{t,\xi,u}_s|^2|_{C_0}^2| \mathcal{F}_t\right] + \int_t^\infty e^{2\beta l} \mathbb{E}[|X^{t,\xi,u}_l|^2|_{C_0}^2| \mathcal{F}_l]dl \leq C(1 + |\xi|_C^2);
\]
and

\[
\sup_{s \geq t} e^{2\beta s} \mathbb{E} \left[ |X_s^{t, \xi, u} - X_s^{t, \eta, v}|^2 \right] + \int_t^\infty e^{2\beta l} \mathbb{E} \left[ |X_l^{t, \xi, u} - X_l^{t, \eta, v}|^2 \right] dl \\
\leq C |\xi - \eta|_C^2 + C \int_t^\infty e^{2\beta l} \mathbb{E} \left[ |b(X_l^{t, \eta, v}, u(l)) - b(X_l^{t, \eta, v}, v(l))|^2 \right] dl \\
+ C \int_t^\infty e^{2\beta l} \mathbb{E} \left[ |\sigma(X_l^{t, \eta, v}, u(l)) - \sigma(X_l^{t, \eta, v}, v(l))|^2 \right] dl. \\
\tag{4.6}
\]

The constant C depends only on \(\beta\) and \(L\). Moreover, for all \(s \geq t\), there exists some constant \(C_0 > 0\) depending only on \(\beta\) and \(L\) such that

\[
\mathbb{E} \left[ |X_s^{t, \xi, u} - \xi_{s-t}|_C^2 \right] \leq C_0 (1 + \mathbb{E} |\xi|_C^2) e^{-2\beta s} ((s-t) + 1)(s-t). \\
\tag{4.7}
\]

**Proof.** Existence and uniqueness are satisfied by Lemma 4.3. We only need to prove that (4.5) and (4.6) hold true. Applying Itô formula to \(3e^{2\beta l}|X_l^{t, \xi, u}|^2\) and functional Itô formula (2.8) to \(e^{2\beta l}S_1(X_l^{t, \xi, u})\) on \([t, s]\), we obtain

\[
e^{2\beta s} \mathbb{E} \left[ |X_s^{t, \xi, u}|^2 \right] \leq e^{2\beta s} \mathbb{E} \left[ |X_t^{t, \xi, u}|^2 \right] \\
\leq 3e^{2\beta s} |\xi|_C^2 + 6\beta \int_t^s e^{2\beta l} \mathbb{E} \left[ |X_l^{t, \xi, u}|^2 \right] dl \\
+ 6L \int_t^s e^{2\beta l} \mathbb{E} \left[ \left( |X_l^{t, \xi, u}|^2 \right) |F_l \right] dl + 15L^2 \int_t^s e^{2\beta l} (1 + \mathbb{E} |X_l^{t, \xi, u}|^2) dl \\
\leq 3|\xi|_C^2 + (15L^2 + 6L + 6\beta + \varepsilon) \int_t^s e^{2\beta l} \mathbb{E} \left[ |X_l^{t, \xi, u}|^2 \right] dl - \frac{9L^2}{2\beta\varepsilon} - \frac{15L^2}{2\beta}.
\]

For every \(\beta < -\left(\frac{5}{2}L^2 + L\right)\), we can let \(\varepsilon\) be small enough such that \(15L^2 + 6L + 6\beta + \varepsilon < 0\), then there exists \(C > 0\) depending only on \(\beta, L\) such that

\[
e^{2\beta s} \mathbb{E} \left[ |X_s^{t, \xi, u}|^2 \right] + \int_t^s e^{2\beta l} \mathbb{E} \left[ |X_l^{t, \xi, u}|^2 \right] dl \leq C(1 + |\xi|_C^2).
\]

Taking the supremum over \(s \in [t, \infty)\), we obtain (4.5). By the similar process, we can show (4.6) holds true. Now let us prove (4.7). By (4.1) and (4.5), we obtain the following result:

\[
\mathbb{E} \left[ |X_s^{t, \xi, u} - \xi_{s-t}|_C^2 \right] \leq 2L^2 \mathbb{E} \left[ \int_t^s (1 + |X_l^{t, \xi, u}|) dl \right]^2 + 8L^2 \mathbb{E} \int_t^s (1 + |X_l^{t, \xi, u}|^2) dl \\
\leq 4L^2 ((s-t) + 2)(s-t)(1 + \mathbb{E} |X_s^{t, \xi, u}|^2)_C
\]
\[
L^2((s-t) + 2)(s-t)(1 + C e^{-\beta s}(1 + \mathbb{E}|\xi|^2_c)).
\]

Then there exists a suitable constant \(C_0 > 0\) depending only on \(\beta\) and \(L\) such that (4.7) holds true. The proof is now complete.

For the particular case of a deterministic \(\xi\), i.e. \(\xi = x \in C_0\), we let \(X^{t,x,u}(\cdot)\) denote the solution of equation (4.2) corresponding \((t, x, u(\cdot)) \in [0, \infty) \times C_0 \times U_0\). For simplicity, if \(t = 0\) we denote \(X^{0,x,u}(\cdot)\) by \(X^{x,u}(\cdot)\). Our first result for the value functional defined in (1.3) includes the local boundedness and the continuity.

**Theorem 4.5.** Suppose that Hypothesis 4.2 holds. Then, for every \(\lambda > \Theta := \frac{5}{2}L^2 + L\), there exists a constant \(C_1 > 0\) such that

\[
|V(x)| \leq C_1(1 + |x|_C), \quad x, y \in C_0,
\]

and

\[
|V(x) - V(y)| \leq C_1|x - y|_C, \quad x, y \in C_0.
\]

**Proof.** From the definition of \(J\) and from Hypothesis 4.2, we know that, for all \(u(\cdot) \in U_0\),

\[
|J(x, u(\cdot)) - J(y, u(\cdot))| \leq \int_0^\infty e^{-\lambda t}\mathbb{E}|q(X^{x,u}_t, u(l)) - q(X^{y,u}_t, u(l))|dl \leq \int_0^\infty e^{-\lambda t}\mathbb{E}|X^{x,u}_t - X^{y,u}_t|_C dl.
\]

According to Theorem 4.4, we obtain that for a constant \(C_1 > 0\),

\[
|V(x) - V(y)| \leq \sup_{u(\cdot) \in U_0} |J(x, u(\cdot)) - J(y, u(\cdot))| \leq C_1|x - y|_C.
\]

By a similar procedure, we can show that (4.8) holds true, which completes the proof.

Now we present the following result, which is called the dynamic programming principle (DPP) for optimal control problems (1.1) and (1.2).

**Theorem 4.6.** Assume that Hypothesis 4.2 holds true. Then, for every \(\lambda > \Theta\), we know that

\[
V(x) = \inf_{u(\cdot) \in U_0} \left\{ \int_0^t e^{-\lambda t}\mathbb{E}q(X^{x,u}_t, u(l))dl + e^{-\lambda t}\mathbb{E}V(X^{x,u}_t) \right\}, \quad (t, x) \in \Lambda.
\]

The proof is completely similar to Theorem 2.31 in Fabbri, Gozzi and Świżch [13]. The only difference being that here the value functional \(V\) is defined on a separable metric space \((C_0, \cdot |_C)\), while the latter is defined on a separable Hilbert space. Here we omit it.

In Theorem 4.5 we have already seen that the value functional \(V(x)\) is Lipschitz continuous in \(x\). With the help of Theorem 4.6 now show that another continuity property of \(V(x)\).

**Theorem 4.7.** Under Hypotheses 4.2 for every \(\delta \in [0, \infty), x \in C_0\) and \(\lambda > \Theta\), there is a constant \(C' > 0\) such that

\[
|V(x) - V(x_\delta)| \leq C'(1 + |x|_C)(\delta + \delta^{\frac{1}{2}} + 1 - e^{-\lambda \delta}).
\]
Proof. Let \( x \in C_0 \) and \( \delta \geq 0 \) be arbitrarily given. From Theorem 4.6 it follows that for any \( \varepsilon > 0 \) there exists an admissible control \( u^\varepsilon(\cdot) \in U_0 \) such that

\[
V(x) \geq \int_0^\delta e^{-\lambda l} E q(X^x_0, u^\varepsilon(l)) dl + e^{-\lambda \delta} EV(X^x_0, u^\varepsilon) - \varepsilon.
\]

Therefore,

\[
|V(x) - V(x_\delta)| \leq |I^1_\delta| + |I^2_\delta| + \varepsilon, \tag{4.12}
\]

where

\[
I^1_\delta = \int_0^\delta e^{-\lambda l} E q(X^x_0, u^\varepsilon(l)) dl,
\]

\[
I^2_\delta = e^{-\lambda \delta} EV(X^x_0, u^\varepsilon) - V(x_\delta).
\]

By (4.5), we have that

\[
|I^1_\delta| \leq L \delta \left( 1 + \sup_{0 \leq l \leq \delta} e^{-\lambda l} |X^x_0, u^\varepsilon|_C \right) \leq L \delta \left( 1 + C^2 \frac{1}{2} (1 + |x|_C) \right).
\]

Now we consider second term \( I^2_\delta \). By (4.7) and Theorem 4.5,

\[
|I^2_\delta| \leq e^{-\lambda \delta} \left| EV(X^x_0, u^\varepsilon) - V(x_\delta) \right| + \left| (e^{-\lambda \delta} - 1) V(x_\delta) \right|
\]

\[
\leq C_1 e^{-\lambda \delta} |X^x_0, u^\varepsilon|_C + C_1 (1 - e^{-\lambda \delta}) (1 + |x|_C)
\]

\[
\leq C_1 C^2 \frac{1}{2} (1 + |x|_C) (1 + \delta) \frac{1}{2} \delta^\frac{1}{2} + C_1 (1 - e^{-\lambda \delta}) (1 + |x|_C).
\]

Hence, from (4.12), we then have, for some suitable constant \( C' \) independent of the control \( u^\varepsilon \),

\[
|V(x) - V(x_\delta)| \leq C' (1 + |x|_C) (\delta + \frac{1}{2} \delta^\frac{1}{2} + 1 - e^{-\lambda \delta}) + \varepsilon,
\]

and letting \( \varepsilon \downarrow 0 \) we get (4.11) holds true. The proof is complete. \( \square \)

5 Viscosity solutions to HJB equations: Existence theorem.

In this section, we consider second order elliptic HJB equation with infinite delay (1.4). As usual, we start with classical solutions.

Definition 5.1. (Classical solution) A functional \( v \in C^{1,2}_p (C_0) \) is called a classical solution to equation (1.4) if it satisfies equation (1.4) pointwise.

We will prove that the value functional \( V \) defined by (1.3) is a viscosity solution of equation (1.4). We give the following definition for the viscosity solutions. For every \((t,x) \in \Lambda \) and \( w \in C(C_0)\), define

\[
\mathcal{A}^+(t,x,w) := \left\{ \varphi \in C^{1,2}_p (\Lambda^t) : 0 = w(x) - \varphi(t,x) = \sup_{(s,y) \in \Lambda^t} (w(y) - \varphi(s,y)) \right\},
\]

and

\[
\mathcal{A}^-(t,x,w) := \left\{ \varphi \in C^{1,2}_p (\Lambda^t) : 0 = w(x) + \varphi(t,x) = \inf_{(s,y) \in \Lambda^t} (w(y) + \varphi(s,y)) \right\}.
\]
Definition 5.2. \( w \in C(C_0) \) is called a viscosity subsolution (resp., supersolution) to equation (1.4) if whenever \( \varphi \in A^+(s,x,w) \) (resp., \( \varphi \in A^-(s,x,w) \)) with \( (s,x) \in \Lambda \), we have

\[
-\lambda w(x) + \partial_t \varphi(s,x) + H(x,\partial_x \varphi(s,x),\partial_{xx} \varphi(s,x)) \geq 0,
\]

(resp., \( -\lambda w(x) - \partial_t \varphi(s,x) + H(x,-\partial_x \varphi(s,x),-\partial_{xx} \varphi(s,x)) \leq 0 \)).

\( w \in C(C_0) \) is said to be a viscosity solution to equation (1.4) if it is both a viscosity subsolution and a viscosity supersolution.

Remark 5.3. Assume that coefficients \( b(x,u) = \overline{b}(x(0),u), \sigma(x,u) = \overline{\sigma}(x(0),u), q(x,u) = \overline{q}(x(0),u) \) for all \( (x,u) \in C_0 \times U \). Then there exists a function \( \overline{V} : \mathbb{R}^d \to \mathbb{R} \) such that \( V(x) = \overline{V}(x(0)) \) for all \( x \in C_0 \), and equation (1.4) reduces to the following HJB equation:

\[
-\lambda \overline{V}(x) + \overline{H}(x,\nabla_x \overline{V}(x),\nabla_{xx} \overline{V}(x)) = 0, \quad x \in \mathbb{R}^d,
\]

where

\[
\overline{H}(x,p,l) = \sup_{u \in U} [(p,\overline{b}(x,u))_{\mathbb{R}^d} + \frac{1}{2} \text{tr} [\sigma(x,u)\sigma^+(x,u)] + \overline{q}(x,u)], \quad (x,p,l) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{S}(\mathbb{R}^d).
\]

Here and in the sequel, \( \nabla_x \) and \( \nabla_{xx} \) denote the standard first and second order derivatives with respect to \( x \).

The following theorem show that our definition of viscosity solutions to equation (1.4) is a natural extension of classical viscosity solution to equation (5.1).

Theorem 5.4. Consider the setting in Remark 5.3. Assume that \( V \) is a viscosity solution of equation (1.4) in the sense of Definition 5.2. Then \( \overline{V} \) is a viscosity solution of equation (5.1) in the standard sense (see Definition 2.2 in [D]).

Proof. Without loss of generality, we shall only prove the viscosity subsolution property. Let \( \overline{\varphi} \in C^2(\mathbb{R}^d) \) and \( x \in \mathbb{R}^d \) such that

\[
0 = (\overline{V} - \overline{\varphi})(x) = \sup_{y \in \mathbb{R}^d} (\overline{V} - \overline{\varphi})(y).
\]

We can modify \( \overline{\varphi} \) such that \( \overline{\varphi}, \nabla_x \overline{\varphi} \) and \( \nabla_{xx} \overline{\varphi} \) grow in a polynomial way. Define \( \varphi : \hat{\Lambda} \to \mathbb{R} \) by

\[
\varphi(s,\gamma) = \overline{\varphi}(\gamma(0)), \quad (s,\gamma) \in \hat{\Lambda},
\]

and define \( \hat{\gamma} \in C_0 \) by

\[
\hat{\gamma}(\theta) = e^\theta x, \quad \theta \in (-\infty, 0].
\]

It is clear that,

\[
\partial_t \varphi(s,\gamma) = 0, \quad \partial_x \varphi(s,\gamma) = \nabla_x \overline{\varphi}(s,\gamma(0)), \quad \partial_{xx} \varphi(s,\gamma) = \nabla_{xx} \overline{\varphi}(s,\gamma(0)), \quad (s,\gamma) \in \hat{\Lambda}.
\]

Thus we have \( \varphi \in C^{1,2}_p(\Lambda) \). Moreover, by the definitions of \( V \) and \( \varphi \), for every fixed \( t \geq 0 \),

\[
0 = V(\hat{\gamma}) - \varphi(t,\hat{\gamma}) = (\overline{V} - \overline{\varphi})(x) = \sup_{y \in \mathbb{R}^d} (\overline{V} - \overline{\varphi})(y) = \sup_{(s,\gamma) \in \Lambda^t} (V(\gamma) - \varphi(s,\gamma)).
\]
Therefore, \( \varphi \in \mathcal{A}^+(t, \gamma, V) \) with \((t, \gamma) \in \Lambda \). Since \( V \) is a viscosity subsolution of equation (1.4), we have
\[
-\lambda V(\gamma) + \partial_t \varphi(t, \gamma) + H(\gamma, \partial_x \varphi(t, \gamma), \partial_{xx} \varphi(t, \gamma)) \geq 0.
\]
Thus,
\[
-\lambda \nabla(x) + \Pi(x, \nabla x \varphi(x), \nabla xx \varphi(x)) \geq 0.
\]
By the arbitrariness of \( \varphi \in C^2(\mathbb{R}^d) \), we see that \( \nabla \) is a viscosity subsolution of equation (5.1), and thus completes the proof. \( \square \)

We are now in a position to give the existence and consistency results for the viscosity solutions.

**Theorem 5.5.** Suppose that Hypothesis 4.2 holds. Then, for every \( \lambda > \Theta \), the value functional \( V \) defined by (1.3) is a viscosity solution to equation (1.4).

**Theorem 5.6.** Suppose Hypothesis 4.2 holds, \( \lambda > \Theta \) and \( v \in C^{1,2}(\mathcal{C}_0) \). Then \( v \) is a classical solution of equation (1.4) if and only if it is a viscosity solution.

The proof of Theorems 5.5 and 5.6 is rather standard. For the sake of the completeness of the article and the convenience of readers, we give their proof in the appendix A.

We conclude this section with the stability of viscosity solutions.

**Theorem 5.7.** Let \( b, \sigma, q \) satisfy Hypothesis 4.2 and \( v \in C(\mathcal{C}_0) \). Assume

(i) for any \( \varepsilon > 0 \), there exist \( b^\varepsilon, \sigma^\varepsilon, q^\varepsilon \) and \( v^\varepsilon \in C(\mathcal{C}_0) \) such that \( b^\varepsilon, \sigma^\varepsilon, q^\varepsilon \) satisfy Hypothesis 4.2 and \( v^\varepsilon \) is a viscosity subsolution (resp., supersolution) of equation (1.4) with generators \( b^\varepsilon, \sigma^\varepsilon, q^\varepsilon \);

(ii) as \( \varepsilon \to 0 \), \( (b^\varepsilon, \sigma^\varepsilon, q^\varepsilon, v^\varepsilon) \) converge to \( (b, \sigma, q, v) \) uniformly in the following sense:
\[
\lim_{\varepsilon \to 0} \sup_{(x,u) \in \mathcal{C}_0 \times U} [(|b^\varepsilon - b| + |\sigma^\varepsilon - \sigma| + |q^\varepsilon - q|)(x,u) + |v^\varepsilon - v|(x)] = 0. \tag{5.2}
\]

Then \( v \) is a viscosity subsolution (resp., supersolution) of equation (1.4) with generators \( b, \sigma, q \).

**Proof.** Without loss of generality, we shall only prove the viscosity subsolution property. Let \( \varphi \in \mathcal{A}^+(\hat{t}, \hat{x}, v) \) with \((\hat{t}, \hat{x}) \in \Lambda \). By (5.2), there exists a constant \( \delta > 0 \) such that for all \( \varepsilon \in (0, \delta) \),
\[
\sup_{(t,x) \in \Lambda^t} (v^\varepsilon(x) - \varphi(t,x)) \leq 1.
\]
Denote \( \varphi_1(t,x) := \varphi(t,x) + \Upsilon(t,x,\hat{t},\hat{x}) \) for all \((t,x) \in \Lambda \). For every \( \varepsilon \in (0, \delta) \), by Lemma 2.5, it is clear that \( v^\varepsilon - \varphi_1 \) is an upper semicontinuous functional and bounded from above on \( \Lambda^t \). Define a sequence of positive numbers \( \{\delta_i\}_{i \geq 0} \) by \( \delta_i = 1 \) for all \( i \geq 0 \). Since \( \Upsilon \) is a gauge-type function, from Lemma 2.9, it follows that, for every \( \varepsilon \in (0, \delta) \) and \((t_0, x^0) \in \Lambda^t \) satisfy
\[
v^\varepsilon(x^0) - \varphi_1(t_0, x^0) \geq \sup_{(s,y) \in \Lambda^t} (v^\varepsilon(y) - \varphi_1(s,y)) - \varepsilon, \quad \text{and} \quad v^\varepsilon(x^0) - \varphi_1(t_0, x^0) \geq v^\varepsilon(\hat{x}) - \varphi_1(\hat{t}, \hat{x}),
\]
there exist \((t_\varepsilon, x^\varepsilon) \in \Lambda^t \) and a sequence \( \{(t_i, x^i)\}_{i \geq 1} \subset \Lambda^t \) such that

(i) \( \Upsilon(t_0, x^0, t_\varepsilon, x^\varepsilon) \leq \varepsilon, \Upsilon(t_i, x_i, t_\varepsilon, x^\varepsilon) \leq \frac{\varepsilon}{2^i} \) and \( t_i \uparrow t_\varepsilon \) as \( i \to \infty \),

(ii) \( v^\varepsilon(x^\varepsilon) - \varphi_1(t_\varepsilon, x^\varepsilon) - \sum_{i=0}^{\infty} \Upsilon(t_i, x_i, t_\varepsilon, x^\varepsilon) \geq v^\varepsilon(x^0) - \varphi_1(t_0, x^0) \), and

(iii) \( v^\varepsilon(y) - \varphi_1(s,y) - \sum_{i=0}^{\infty} \Upsilon(t_i, x_i, s, y) < v^\varepsilon(x^\varepsilon) - \varphi_1(t_\varepsilon, x^\varepsilon) - \sum_{i=0}^{\infty} \Upsilon(t_i, x_i, t_\varepsilon, x^\varepsilon) \) for all \((s,y) \in \Lambda^t \setminus \{(t_\varepsilon, x^\varepsilon)\} \).
Indeed, if not, by (3.1), we can assume that there exists an \(\nu_0 > 0\) such that

\[
\overline{Y}(t_\varepsilon, x_\varepsilon, \hat{t}, \hat{x}) \geq \nu_0.
\]

Thus, we obtain that

\[
0 = v(\hat{x}) - \varphi(\hat{t}, \hat{x}) = \lim_{\varepsilon \to 0} v^\varepsilon(\hat{x}) - \varphi_1(\hat{t}, \hat{x})
\]

\[
\leq \limsup_{\varepsilon \to 0} \left[ v^\varepsilon(x_\varepsilon) - \varphi_1(t_\varepsilon, x_\varepsilon) - \sum_{i=0}^{\infty} \overline{Y}(t_i, x_i, t_\varepsilon, x_\varepsilon) \right]
\]

\[
\leq \limsup_{\varepsilon \to 0} \left[ v(x_\varepsilon) - \varphi(t_\varepsilon, x_\varepsilon) + (v^\varepsilon - v)(x_\varepsilon) - \sum_{i=0}^{\infty} \overline{Y}(t_i, x_i, t_\varepsilon, x_\varepsilon) \right] - \nu_0
\]

\[
\leq v(\hat{x}) - \varphi(\hat{t}, \hat{x}) - \nu_0 = -\nu_0,
\]

contradicting \(\nu_0 > 0\). We notice that, by (3.7), (3.12), (3.14), (3.15), the definition of \(Y\) and the property (i) of \(((t_\varepsilon, x_\varepsilon))\), there exists a generic constant \(C > 0\) such that

\[
2 \sum_{i=0}^{\infty} (t_\varepsilon - t_i) \leq 2 \sum_{i=0}^{\infty} \left( \frac{\varepsilon}{2^i} \right)^{\frac{1}{2}} \leq C\varepsilon^{\frac{1}{4}};
\]

\[
|\partial_x Y(x_\varepsilon - \hat{x}_{t_\varepsilon-i})| \leq C|\hat{x}(0) - x_\varepsilon(0)|^5,
\]

\[
|\partial_{xx} Y(x_\varepsilon - \hat{x}_{t_\varepsilon-i})| \leq C|\hat{x}(0) - x_\varepsilon(0)|^4;
\]

\[
\left| \partial_x \sum_{i=0}^{\infty} Y(x_\varepsilon - x_i - t_{t_\varepsilon-i}) \right| \leq 18 \sum_{i=0}^{\infty} |x_i(0) - x_\varepsilon(0)|^5 \leq 18 \sum_{i=0}^{\infty} \left( \frac{\varepsilon}{2^i} \right)^{\frac{5}{4}} \leq C\varepsilon^{\frac{5}{4}};
\]

and

\[
\left| \partial_{xx} \sum_{i=0}^{\infty} Y(x_\varepsilon - x_i - t_{t_\varepsilon-i}) \right| \leq 306 \sum_{i=0}^{\infty} |x_i(0) - x_\varepsilon(0)|^4 \leq 306 \sum_{i=0}^{\infty} \left( \frac{\varepsilon}{2^i} \right)^{\frac{4}{3}} \leq C\varepsilon^{\frac{4}{3}}.
\]

Then for any \(\rho > 0\), by (5.2) and (5.3), there exists \(\varepsilon > 0\) small enough such that

\[
2|t_\varepsilon - \hat{t}| + 2 \sum_{i=0}^{\infty} (t_\varepsilon - t_i) \leq \frac{\rho}{4},
\]

and

\[
\lambda |v^\varepsilon(x_\varepsilon) - v(\hat{x})| + |\partial_t \varphi(t_\varepsilon, x_\varepsilon) - \partial_t \varphi(\hat{t}, \hat{x})| \leq \frac{\rho}{4},
\]

where

\[
I = H(x_\varepsilon, \partial_x \varphi_2(t_\varepsilon, x_\varepsilon), \partial_{xx} \varphi_2(t_\varepsilon, x_\varepsilon)) - H(x_\varepsilon, \partial_x \varphi_2(t_\varepsilon, x_\varepsilon), \partial_{xx} \varphi_2(t_\varepsilon, x_\varepsilon)),
\]

\[
II = H(x_\varepsilon, \partial_x \varphi_2(t_\varepsilon, x_\varepsilon), \partial_{xx} \varphi_2(t_\varepsilon, x_\varepsilon)) - H(\hat{x}, \partial_x \varphi(\hat{t}, \hat{x}), \partial_{xx} \varphi(\hat{t}, \hat{x})),
\]

\[
\varphi_2(t_\varepsilon, x_\varepsilon) = \varphi_1(t_\varepsilon, x_\varepsilon) + \sum_{i=0}^{\infty} \overline{Y}(t_i, x_i, t_\varepsilon, x_\varepsilon),
\]

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and
\[ H^\epsilon(x,p,l) = \sup_{u \in U} \{ (p, b^\epsilon(x,u))_{\mathbb{R}^d} + \frac{1}{2} \text{tr} \sigma^\epsilon(x,u) \sigma^\epsilon \top(x,u) + q^\epsilon(x,u) \}, \quad (x, p, l) \in C_0 \times \mathbb{R}^d \times \mathcal{S}(\mathbb{R}^d). \]

Since \( v^\epsilon \) is a viscosity subsolution of equation (1.4) with generators \( b^\epsilon, \sigma^\epsilon, q^\epsilon \), we have
\[
- \lambda v^\epsilon(x^\epsilon) + \partial_t \varphi(t^\epsilon, x^\epsilon) + 2(t^\epsilon - \hat{t}) + 2 \sum_{i=0}^{\infty} (t^\epsilon - t_i) + H^\epsilon(x^\epsilon, \partial_x \varphi_2(t^\epsilon, x^\epsilon), \partial_{xx} \varphi_2(t^\epsilon, x^\epsilon)) \geq 0.
\]

Thus
\[
0 \leq - \lambda v^\epsilon(x^\epsilon) + \partial_t \varphi(t^\epsilon, x^\epsilon) + 2(t^\epsilon - \hat{t}) + 2 \sum_{i=0}^{\infty} (t^\epsilon - t_i) + H(\hat{x}, \partial_x \varphi(\hat{t}, \hat{x}), \partial_{xx} \varphi(\hat{t}, \hat{x})) + I + II
\]
\[
\leq - \lambda v(\hat{x}) + \partial_t \varphi(\hat{t}, \hat{x}) + H(\hat{x}, \partial_x \varphi(\hat{t}, \hat{x}), \partial_{xx} \varphi(\hat{t}, \hat{x})) + q.
\]

Letting \( \varepsilon \downarrow 0 \), we show that
\[
- \lambda v(\hat{x}) + \partial_t \varphi(\hat{t}, \hat{x}) + H(\hat{x}, \partial_x \varphi(\hat{t}, \hat{x}), \partial_{xx} \varphi(\hat{t}, \hat{x})) \geq 0.
\]

Since \( \varphi \in A^+(\hat{t}, \hat{x}, v) \) is arbitrary, we see that \( v \) is a viscosity subsolution of equation (1.4) with generators \( b, \sigma, q, \) and thus completes the proof. \( \square \)

6 Viscosity solution to HJB equation: Uniqueness theorem.

6.1. Maximum principle. In this subsection we extend Crandall-Ishii maximum principle to infinite delay case. It is the cornerstone of the theory of viscosity solutions, and will be used to prove comparison theorem in next subsection.

**Definition 6.1.** Let \((\hat{t}, \hat{x}_0, T) \in (0, \infty) \times \mathbb{R}^d \times (\hat{t}, \infty)\) and \(f : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}\) be an upper semicontinuous function bounded from above. We say \(f \in \Phi(\hat{t}, \hat{x}_0, T)\) if there is a constant \(r > 0\) such that for every \(L > 0\) and \(\varphi \in C^{1,2}(0, T) \times \mathbb{R}^d\) be a function such that \(f(s, y_0) - \varphi(s, y_0)\) has a maximum over \([0, T] \times \mathbb{R}^d\) at a point \((t, x_0) \in (0, T) \times \mathbb{R}^d\), there is a constant \(C > 0\) such that
\[
\varphi_t(t, x_0) \geq C \text{ whenever } \|t - \hat{t}\| + \|x_0 - \hat{x}_0\| < r, \quad |f(t, x_0)| + |\nabla_x \varphi(t, x_0)| + |\nabla^2_x \varphi(t, x_0)| \leq L.
\]

**Definition 6.2.** Let \(\hat{t} \in [0, \infty)\) be fixed and \(w : \Lambda \to \mathbb{R}\) be an upper semicontinuous function bounded from above. Define, for \((t, x_0) \in [0, \infty) \times \mathbb{R}^d\),
\[
\tilde{w}^\hat{t}(t, x_0) := \sup_{\xi \in C_0, \xi(0) = x_0} [w(t, \xi)], \quad t \in [\hat{t}, \infty); \quad \bar{w}^\hat{t}(t, x_0) := \tilde{w}^\hat{t}(t, x_0) - (\hat{t} - t)^{\frac{3}{2}}, \quad t \in [0, \hat{t}).
\]

Let \(\tilde{w}^{\hat{t}}\) be the upper semicontinuous envelope of \(\tilde{w}^\hat{t}\) (see Definition D.10 in [12]), i.e.,
\[
\tilde{w}^{\hat{t}}_{\hat{s}}(t, x_0) = \limsup_{(s, y_0) \in [0, \infty) \times \mathbb{R}^d, (s, y_0) \to (t, x_0)} \tilde{w}^\hat{t}(s, y_0).
\]

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In what follows, by a modulus of continuity, we mean a continuous function \( \rho_1 : [0, \infty) \to [0, \infty) \), with \( \rho_1(0) = 0 \) and subadditive: \( \rho_1(t + s) \leq \rho_1(t) + \rho_1(s) \), for all \( t, s > 0 \); by a local modulus of continuity, we mean a continuous function \( \rho_1 : [0, \infty) \times [0, \infty) \to [0, \infty) \), with the properties that for each \( r > 0 \), \( t \to \rho_1(t, r) \) is a modulus of continuity and \( \rho_1 \) is increasing in second variable.

**Theorem 6.3.** *(Crandall-Ishii maximum principle)* Let \( \kappa > 0 \). Let \( w_1, w_2 : \Lambda \to \mathbb{R} \) be upper semicontinuous functions bounded from above and such that

\[
\limsup_{t + |x| C \to \infty} \frac{w_1(t, x)}{t + |x| C} < 0; \quad \limsup_{t + |x| C \to \infty} \frac{w_2(t, x)}{t + |x| C} < 0. \tag{6.2}
\]

Let \( \varphi \in C^2(\mathbb{R}^d \times \mathbb{R}^d) \) be such that

\[
w_1(t, x) + w_2(t, y) - \varphi(x(0), y(0))
\]

has a maximum over \([\hat{t}, \infty) \times C_0 \times C_0\) at a point \((\hat{t}, \hat{x}, \hat{y})\). Assume, moreover, \( \hat{w}_1^{i*} \in \Phi(\hat{t}, \hat{x}(0), T) \) and \( \hat{w}_2^{i*} \in \Phi(\hat{t}, \hat{y}(0), T) \) for some \( T \in (\hat{t}, \infty) \), and there exists a local modulus of continuity \( \rho_1 \) such that, for all \( \hat{t} \leq t \leq \hat{s} \leq T \), \( x \in C_0 \),

\[
w_1(t, x) - w_1(s, x_{s-t}) \leq \rho_1(|s - t|, |x| C), \quad w_2(t, x) - w_2(s, x_{s-t}) \leq \rho_1(|s - t|, |x| C). \tag{6.3}
\]

Then there exist the sequences \((t_k, x_k), (s_k, y_k) \in \Lambda \) and the sequences of functionals \( \varphi_k \in C_p^{1,2}(\Lambda^{t_k}), \psi_k \in C_p^{1,2}(\Lambda^{s_k}) \) such that \( \varphi_k, \partial_t \varphi_k, \partial_x \varphi_k, \partial_{xx} \varphi_k, \psi_k, \partial_t \psi_k, \partial_x \psi_k, \partial_{xx} \psi_k \) are bounded and uniformly continuous, and such that

\[
w_1(t, x) - \varphi_k(t, x)
\]

has a strict global maximum \( 0 \) at \((t_k, x_k)\) over \( \Lambda^{t_k} \),

\[
w_2(t, y) - \psi_k(t, y)
\]

has a strict global maximum \( 0 \) at \((s_k, y_k)\) over \( \Lambda^{s_k} \), and

\[
\begin{align*}
\left( t_k, x_k(0), w_1(t_k, x_k), \partial_t \varphi_k(t_k, x_k), \partial_x \varphi_k(t_k, x_k), \partial_{xx} \varphi_k(t_k, x_k) \right) \\
k \to \infty \left( \hat{t}, \hat{x}(0), w_1(\hat{t}, \hat{x}), b_1, \nabla_{x_1} \varphi(\hat{x}(0), \hat{y}(0)), X \right), \tag{6.4}
\end{align*}
\]

\[
\begin{align*}
\left( s_k, y_k(0), w_2(s_k, y_k), \partial_t \psi_k(s_k, y_k), \partial_x \psi_k(s_k, y_k), \partial_{xx} \psi_k(s_k, y_k) \right) \\
k \to \infty \left( \hat{t}, \hat{y}(0), w_2(\hat{t}, \hat{y}), b_2, \nabla_{x_2} \varphi(\hat{x}(0), \hat{y}(0)), Y \right), \tag{6.5}
\end{align*}
\]

where \( b_1 + b_2 = 0 \) and \( X, Y \in \mathcal{S}(\mathbb{R}^d) \) satisfy the following inequality:

\[
- \left( \frac{1}{\kappa} + |A| \right) \leq \left( \begin{array}{cc} X & 0 \\ 0 & Y \end{array} \right) \leq A + \kappa A^2, \tag{6.6}
\]

and \( A = \nabla_{x_2}^2 \varphi(\hat{x}(0), \hat{y}(0)) \). Here \( \nabla_{x_1} \varphi \) and \( \nabla_{x_2} \varphi \) denote the standard first order derivative of \( \varphi \) with respect to the first variable and the second variable, respectively.

**Proof.** By the following Lemma 6.5 we have that

\[
\hat{w}_1^{i*}(t, x_0) + \hat{w}_2^{i*}(t, y_0) - \varphi(x_0, y_0) \text{ has a maximum over } [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \text{ at } (\hat{t}, \hat{x}(0), \hat{y}(0)). \tag{6.7}
\]
Moreover, we have $\hat{w}_{1}^{t,s}(\hat{t},\hat{x}(0)) = w_{1}(\hat{t},\hat{x})$, $\hat{w}_{2}^{t,s}(\hat{t},\hat{y}(0)) = w_{2}(\hat{t},\hat{y})$. Then, by $\hat{w}_{i}^{t,s} \in \Phi(\hat{t},\hat{x}(0),T)$, $\hat{w}_{2}^{t,s} \in \Phi(\hat{t},\hat{y}(0),T)$ for some $T \in (\hat{t},\infty)$ and Remark 6.4 the Theorem 8.3 in [6] can be used to obtain sequences of bounded functions $\hat{\varphi}_{k}, \hat{\psi}_{k} \in C^{1,2}(0,T) \times \mathbb{R}^{d}$ with bounded and uniformly continuous derivatives such that $\hat{w}_{1}^{t,s}(t,x_{0}) - \hat{\varphi}_{k}(t,x_{0})$ has a strict global maximum 0 at some point $(t_{k},x_{0}^{k}) \in (0,T) \times \mathbb{R}^{d}$ over $[0,T] \times \mathbb{R}^{d}$, $\hat{w}_{2}^{t,s}(s,y_{0}) - \hat{\psi}_{k}(s,y_{0})$ has a strict global maximum 0 at some point $(s_{k},y_{0}^{k}) \in (0,T) \times \mathbb{R}^{d}$ over $[0,T] \times \mathbb{R}^{d}$, and that such that

$$
\begin{aligned}
(t_{k},x_{0}^{k},\hat{w}_{1}^{t,s}(t_{k},x_{0}^{k}), (\hat{\varphi}_{k})_{t}(t_{k},x_{0}^{k}), \nabla_{x}\hat{\varphi}_{k}(t_{k},x_{0}^{k}), \nabla_{x}^{2}\hat{\varphi}_{k}(t_{k},x_{0}^{k})) \\
\kappa \rightarrow \infty \ (t,\hat{t}(0), w_{1}(t,\hat{t}), b_{1}, \nabla_{x}\varphi(\hat{t}(0), \hat{y}(0)), \mathcal{X}),
\end{aligned}
$$

(6.8)

$$
\begin{aligned}
(s_{k},y_{0}^{k},\hat{w}_{2}^{t,s}(s_{k},y_{0}^{k}), (\hat{\psi}_{k})_{t}(s_{k},y_{0}^{k}), \nabla_{x}\hat{\psi}_{k}(s_{k},y_{0}^{k}), \nabla_{x}^{2}\hat{\psi}_{k}(s_{k},y_{0}^{k})) \\
\kappa \rightarrow \infty \ (\hat{t},\hat{y}(0), w_{2}(\hat{t},\hat{y}), b_{2}, \nabla_{x}\varphi(\hat{x}(0), \hat{y}(0)), \mathcal{Y}),
\end{aligned}
$$

(6.9)

where $b_{1} + b_{2} = 0$ and (6.6) is satisfied.

We claim that we can assume the sequences $\{t_{k}\}_{k \geq 1} \in [\hat{t},T]$ and $\{s_{k}\}_{k \geq 1} \in [\hat{t},T]$. Indeed, if not, for example, there exists a subsequence of $\{t_{k}\}_{k \geq 1}$ still denoted by itself such that $t_{k} < \hat{t}$ for all $k \geq 0$. Since $\hat{w}_{1}^{t,s}(t,x_{0}) - \hat{\varphi}_{k}(t,x_{0})$ has a maximum at $(t_{k},x_{0}^{k})$ on $[0,T] \times \mathbb{R}^{d}$, we obtain that

$$
(\hat{\varphi}_{k})_{t}(t_{k},x_{0}^{k}) = \frac{1}{2}(\hat{t} - t_{k})^{-\frac{3}{2}} \rightarrow \infty, \text{ as } k \rightarrow \infty,
$$

which contradicts that $(\hat{\varphi}_{k})_{t}(t_{k},x_{0}^{k}) \rightarrow b_{1}$, $(\hat{\psi}_{k})_{t}(s_{k},y_{0}^{k}) \rightarrow b_{2}$ and $b_{1} + b_{2} = 0$.

We can modify $\hat{\varphi}_{k}, \hat{\psi}_{k}$ such that $\check{\varphi}_{k}, \check{\psi}_{k} \in C^{1,2}(0,\infty) \times \mathbb{R}^{d}$ with bounded and uniformly continuous derivatives, $\check{w}_{1}^{t,s}(t,x_{0}) + \check{w}_{2}^{t,s}(s,y_{0}) - \check{\varphi}_{k}(t,x_{0}) - \check{\psi}_{k}(s,y_{0})$ has a strict global maximum 0 at $(t_{k},x_{0}^{k},s_{k},y_{0}^{k})$ on $[0,\infty) \times \mathbb{R}^{d} \times [0,\infty) \times \mathbb{R}^{d}$, and (6.8) and (6.9) hold true. Now we consider the functional, for $(t,x),(s,y) \in \Lambda^{\hat{t}}$,

$$
\Gamma_{k}(t,x,s,y) = w_{1}(t,x) + w_{2}(s,y) - \hat{\varphi}_{k}(t,x(0)) - \hat{\psi}_{k}(s,y(0)).
$$

(6.10)

It is clear that $\Gamma_{k}$ is an upper semicontinuous functional bounded from above on $\Lambda^{\hat{t}} \times \Lambda^{\hat{t}}$. Define a sequence of positive numbers $\{\delta_{i}\}_{i \geq 0}$ by $\delta_{i} = \frac{1}{2^{i}}$ for all $i \geq 0$. For every $k$ and $j > 0$, from Lemma 2.9 it follows that, for every $(\hat{t}_{0},\hat{t}^{0}), (\hat{s}_{0},\hat{y}^{0}) \in \Lambda^{\hat{t}}$ satisfying

$$
\Gamma_{k}(\hat{t}_{0},\hat{t}^{0},\hat{s}_{0},\hat{y}^{0}) \geq \sup_{(t,s),(y,s) \in \Lambda^{\hat{t}}} \Gamma_{k}(t,s,y) - \frac{1}{j},
$$

(6.11)

there exist $(t_{k,j},x_{k,j}^{s}), (s_{k,j},y_{k,j}^{s}) \in [\hat{t},\infty) \times \Lambda^{\hat{t}}$ and two sequences $\{(\hat{t}_{i},\hat{t}^{i})\}_{i \geq 1}, \{(\hat{s}_{i},\hat{y}^{i})\}_{i \geq 1} \subset [\hat{t},\infty) \times \Lambda^{\hat{t}}$ such that

(i) $\mathbb{T}(\hat{t}_{i},\hat{t}^{i},t_{k,j},x_{k,j}^{s}) + \mathbb{T}(\hat{s}_{i},\hat{y}_{i},s_{k,j},y_{k,j}^{s}) \leq \frac{1}{j}, \mathbb{T}(\hat{t}_{i},\hat{t}^{i},t_{k,j},x_{k,j}^{s}) + \mathbb{T}(\hat{s}_{i},\hat{y}_{i},s_{k,j},y_{k,j}^{s}) \leq \frac{1}{2^{j}}$ and

$$
\hat{t}_{i} \uparrow t_{k,j}, \hat{s}_{i} \uparrow s_{k,j} \text{ as } i \rightarrow \infty,
$$

(ii) $\Gamma_{k}(t_{k,j},x_{k,j}^{s},s_{k,j},y_{k,j}^{s}) - \sum_{i=0}^{\infty} \frac{1}{2^{i}} \mathbb{T}(\hat{t}_{i},\hat{t}^{i},t_{k,j},x_{k,j}^{s}) + \mathbb{T}(\hat{s}_{i},\hat{y}_{i},s_{k,j},y_{k,j}^{s}) \geq \Gamma_{k}(\hat{t}_{0},\hat{t}^{0},\hat{s}_{0},\hat{y}^{0}),$ and

(iii) for all $(t,x,s,y) \in \Lambda^{t_{k,j}} \times \Lambda^{s_{k,j}} \setminus \{(t_{k,j},x_{k,j}^{s},s_{k,j},y_{k,j}^{s})\},$

$$
\Gamma_{k}(t,x,s,y) \geq \sum_{i=0}^{\infty} \frac{1}{2^{i}} \mathbb{T}(\hat{t}_{i},\hat{t}^{i},t,x) + \mathbb{T}(\hat{s}_{i},\hat{y}_{i},s,y)
$$

$$
< \Gamma_{k}(t_{k,j},x_{k,j}^{s},s_{k,j},y_{k,j}^{s}) - \sum_{i=0}^{\infty} \frac{1}{2^{i}} \mathbb{T}(\hat{t}_{i},\hat{t}^{i},t_{k,j},x_{k,j}^{s}) + \mathbb{T}(\hat{s}_{i},\hat{y}_{i},s_{k,j},y_{k,j}^{s})],
$$

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By the following Lemma 6.6 we have
\[
(t_{k,j}, x^{k,j}(0)) \to (t_k, x^k_0), \quad (s_{k,j}, y^{k,j}(0)) \to (s_k, y_0^k) \text{ as } j \to \infty,
\] (6.12)
\[
\tilde{w}_1^{i,*}(t_{k,j}, x^{k,j}(0)) \to \tilde{w}_1^{i,*}(t_k, x_k^0), \quad \tilde{w}_2^{j,*}(s_{k,j}, y^{k,j}(0)) \to \tilde{w}_2^{j,*}(s_k, y_0^k) \text{ as } j \to \infty,
\] (6.13)
and
\[
w_1(t_{k,j}, x^{k,j}) \to \tilde{w}_1^{i,*}(t_k, x_k^0), \quad w_2(s_{k,j}, y^{k,j}) \to \tilde{w}_2^{j,*}(s_k, y_0^k) \text{ as } j \to \infty.
\] (6.14)

Using these and (6.8) and (6.9) we can therefore select a subsequence \(j_k\) such that
\[
\begin{aligned}
(t_{k,j_k}, x^{k,j_k}(0), w_1(t_{k,j_k}, x^{k,j_k}), ((\tilde{\varphi}_k) t, \nabla_x \tilde{\varphi}_k, \nabla_x^2 \tilde{\varphi}_k)(t_{k,j_k}, x^{k,j_k}(0))) \\
k \to \infty \left( \tilde{t}, \tilde{x}(0), w_1(\tilde{t}, \tilde{x}), (b_1, \nabla_x \varphi(\tilde{x}(0), \tilde{y}(0)), X) \right),
\end{aligned}
\]
and
\[
\begin{aligned}
(s_{k,j_k}, y^{k,j_k}(0), w_2(s_{k,j_k}, y^{k,j_k}), ((\tilde{\psi}_k) t, \nabla_x \tilde{\psi}_k, \nabla_x^2 \tilde{\psi}_k)(s_{k,j_k}, y^{k,j_k}(0))) \\
k \to \infty \left( \tilde{t}, \tilde{y}(0), w_2(\tilde{t}, \tilde{y}), (b_2, \nabla_x \varphi(\tilde{x}(0), \tilde{y}(0)), Y) \right).
\end{aligned}
\]

We notice that, by (3.7), (3.12), (3.14), (3.15), the definition of \(Y\) and the property (i) of \((t_{k,j}, x^{k,j}, s_{k,j}, y^{k,j})\), there exists a generic constant \(C > 0\) such that
\[
2 \sum_{i=0}^{\infty} \frac{1}{2^i} \left( |\varphi(t_{k,j} - \tilde{t}_i) + (t_{k,j} - \tilde{t}_i)| \right) \leq C j_k^{-\frac{1}{2}};
\]
\[
\left| \partial_x \left[ \sum_{i=0}^{\infty} \frac{1}{2^i} Y(x^{k,j_k} - \tilde{x}_{t_{k,j_k}-\tilde{t}_i}) \right] \right| + \left| \partial_x \left[ \sum_{i=0}^{\infty} \frac{1}{2^i} Y(y^{k,j_k} - \tilde{y}_{s_{k,j_k}-\tilde{s}_i}) \right] \right| \leq C j_k^{-\frac{1}{8}};
\]
and
\[
\left| \partial_{xx} \left[ \sum_{i=0}^{\infty} \frac{1}{2^i} Y(x^{k,j_k} - \tilde{x}_{t_{k,j_k}-\tilde{t}_i}) \right] \right| + \left| \partial_{xx} \left[ \sum_{i=0}^{\infty} \frac{1}{2^i} Y(y^{k,j_k} - \tilde{y}_{s_{k,j_k}-\tilde{s}_i}) \right] \right| \leq C j_k^{-\frac{1}{8}}.
\]

Therefore the lemma holds with \(\varphi_k(t, x) := \varphi_k(t, x(0)) + \sum_{i=0}^{\infty} \frac{1}{2^i} \bar{Y}(\tilde{t}_i, \tilde{x}_i, t, x), \psi_k(s, y) := \tilde{\psi}_k(s, y(0)) + \sum_{i=0}^{\infty} \frac{1}{2^i} \bar{Y}(\tilde{s}_i, \tilde{y}_i, s, y)\) and \(t_k := t_{k,j_k}, x^k := x^{k,j_k}, s_k := s_{k,j_k}, y^k := y^{k,j_k}\). \(\Box\)

**Remark 6.4.** As mentioned in Remark 6.1 in Chapter V of [6], Condition (6.1) is stated with reverse inequality in Theorem 8.3 of [6]. However, we immediately obtain results (6.8)-(6.9) from Theorem 8.3 of [6] by considering the functions \(u_1(t, x) := \tilde{w}_1^{i,*}(T-t, x)\) and \(u_2(t, x) := \tilde{w}_2^{j,*}(T-t, x)\).

To complete the proof of Theorem 5.3 it remains to state and prove the following two lemmas.

**Lemma 6.5.** \(\tilde{w}_1^{i,*}(t, x_0) + \tilde{w}_2^{j,*}(t, y_0) - \varphi(x_0, y_0)\) has a maximum over \([0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d\) at \((\tilde{t}, \tilde{x}(0), \tilde{y}(0))\). Moreover, we have
\[
\tilde{w}_1^{i,*}(\tilde{t}, \tilde{x}(0)) = w_1(\tilde{t}, \tilde{x}), \quad \tilde{w}_2^{j,*}(\tilde{t}, \tilde{y}(0)) = w_2(\tilde{t}, \tilde{y}).
\] (6.15)
We claim that we can assume that there exists a constant $C_{x_0} > 0$ depending only on $x_0$ such that

$$\bar{w}_1^*(t,x_0) - \bar{w}_1^*(s,x_0) = \sup_{\xi \in C_0, |\xi| \leq C_{x_0}, \xi(0) = x_0} [w_1(t,\xi)] - \sup_{\eta \in C_0, \eta(0) = x_0} [w_1(s,\eta)].$$

Indeed, if not, for every $l$, without loss of generality, we may assume

$$\bar{w}_1^*(t,x_0) - \bar{w}_1^*(s,x_0) \leq \sup_{\xi \in C_0, |\xi| \leq C_{x_0}, \xi(0) = x_0} [w_1(t,\xi)] - \sup_{\eta \in C_0, \eta(0) = x_0} [w_1(s,\eta)] \leq \sup_{\xi \in C_0, |\xi| \leq C_{x_0}, \xi(0) = x_0} [w_1(t,\xi) - w_1(s,\xi_{s-t})].$$

By (6.3) we have that

$$\bar{w}_1^*(t,x_0) - \bar{w}_1^*(s,x_0) \leq \sup_{\xi \in C_0, |\xi| \leq C_{x_0}, \xi(0) = x_0} \rho_1(|s-t|, |\xi|, C_{x_0}).$$

(6.16)

Clearly, if $0 \leq t \leq s \leq \hat{t}$, we have

$$\bar{w}_1^*(t,x_0) - \bar{w}_1^*(s,x_0) = -(\hat{t} - t)^\frac{1}{2} + (\hat{t} - s)^\frac{1}{2} \leq 0,$$

(6.17)

and, if $0 \leq t \leq \hat{t} \leq s < \infty$, we have

$$\bar{w}_1^*(t,x_0) - \bar{w}_1^*(s,x_0) \leq \bar{w}_1^*(\hat{t},x_0) - \bar{w}_1^*(s,x_0) \leq \rho_1(|s-\hat{t}|, C_{x_0}).$$

(6.18)

On the other hand, for every $(t,x_0,y_0) \in [0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$, by the definitions of $\bar{w}_1^*(t,x_0)$ and $\bar{w}_2^*(t,y_0)$, there exist sequences $(l_i, x_i, (\tau_i, y_i)) \in [0,\infty) \times \mathbb{R}^d$ such that $(l_i, x_i) \to (t,x_0)$, $(\tau_i, y_i) \to (t,y_0)$ as $i \to \infty$ and

$$\bar{w}_1^*(t,x_0) = \lim_{i \to \infty} \bar{w}_1^*(l_i, x_i), \quad \bar{w}_2^*(t,y_0) = \lim_{i \to \infty} \bar{w}_2^*(\tau_i, y_i).$$

(6.19)

Without loss of generality, we may assume $l_i \leq \tau_i$ for all $i > 0$. By (6.16)-(6.18), we have

$$\bar{w}_1^*(t,x_0) = \lim_{i \to \infty} \bar{w}_1^*(l_i, x_i) \leq \lim_{i \to \infty} [\bar{w}_1^*(\tau_i, x_i) + \rho_1(|\tau_i - l_i|, C_{x_i})].$$

(6.20)

We claim that we can assume that there exists a constant $M_1 > 0$ such that $C_{x_i} \leq M_1$ for all $i \geq 1$. Indeed, if not, for every $n$, there exists $i_n$ such that

$$\bar{w}_1^*(l_{i_n}, x_{i_n}) = \begin{cases} \sup_{\xi \in C_0, |\xi| > n, \xi(0) = x_{i_n}} [w_1(l_{i_n}, \xi)], & i_n \geq \hat{t}; \\ \sup_{\xi \in C_0, |\xi| > n, \xi(0) = x_{i_n}} [w_1(\hat{t}, \xi)] - (\hat{t} - l_{i_n})^\frac{1}{2}, & i_n < \hat{t}. \end{cases}$$

(6.21)

Letting $n \to \infty$, by (6.2), we get that

$$\bar{w}_1^*(l_{i_n}, x_{i_n}) \to -\infty \text{ as } n \to \infty,$$

which contradicts the convergence that $\bar{w}_1^*(t,x) = \lim_{i \to \infty} \bar{w}_1^*(l_i, x_i)$. Then, by (6.20),

$$\bar{w}_1^*(t,x) \leq \lim_{i \to \infty} [\bar{w}_1^*(\tau_i, x_i) + \rho_1(|\tau_i - l_i|, M_1)] = \lim_{i \to \infty} \bar{w}_1^*(\tau_i, x_i).$$

(6.22)

Therefore, by (6.19), (6.20) and the definitions of $\bar{w}_1^*$ and $\bar{w}_2^*$,

$$\bar{w}_1^*(t,x_0) + \bar{w}_2^*(t,y_0) - \varphi(x_0, y_0)$$

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\[
\begin{align*}
&\leq \liminf_{i \to \infty}[\hat{w}_1^i(t, x_i) + \hat{w}_2^i(t, y_i) - \varphi(x_i, y_i)] \\
&\leq \sup_{(l, x, y) \in [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d} [\hat{w}_1^i(l, x) + \hat{w}_2^i(l, y) - \varphi(x, y)] \\
&= \sup_{(l, x, y) \in [l, \infty) \times \mathbb{R}^d \times \mathbb{R}^d} [\hat{w}_1^i(l, x) + \hat{w}_2^i(l, y) - \varphi(x, y)]. \quad (6.23)
\end{align*}
\]

We also have, for \((l, x, y) \in [l, \infty) \times \mathbb{R}^d \times \mathbb{R}^d\),
\[
\hat{w}_1^i(l, x) + \hat{w}_2^i(l, y) - \varphi(x, y)
= \sup_{\gamma, \eta \in C_0, \gamma(0) = x, \eta(0) = y} [w_1(l, \gamma) + w_2(l, \eta) - \varphi(\gamma(0), \eta(0))]
\leq w_1(l, \hat{x}) + w_2(l, \hat{y}) - \varphi(\hat{x}(0), \hat{y}(0)), \quad (6.24)
\]
where the inequality becomes equality if \(l = \hat{l}\) and \(x = \hat{x}(0), y = \hat{y}(0)\). Combining (6.23) and (6.24), we obtain that
\[
\limsup_{i \to \infty}[\hat{w}_1^i(t, x_i) + \hat{w}_2^i(t, y_i) - \varphi(x_i, y_i)] \leq w_1(\hat{l}, \hat{x}) + w_2(\hat{l}, \hat{y}) - \varphi(\hat{x}(0), \hat{y}(0)). \quad (6.25)
\]

By the definitions of \(\hat{w}_1^i\) and \(\hat{w}_2^i\), we have \(\hat{w}_1^i(t, x_i) \geq \hat{w}_1^i(t, x_0), \hat{w}_2^i(t, y_i) \geq \hat{w}_2(t, y_0)\). Then by also (6.24) and (6.25), for every \((l, x, y) \in [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d\),
\[
\hat{w}_1^i(t, x_0) + \hat{w}_2^i(t, y_0) - \varphi(x_0, y_0) \leq \hat{w}_1^i(t, \hat{x}) + \hat{w}_2(t, \hat{y}) - \varphi(\hat{x}(0), \hat{y}(0))
= \hat{w}_1^i(t, \hat{x}(0)) + \hat{w}_2(t, \hat{y}(0)) - \varphi(\hat{x}(0), \hat{y}(0))
\leq \hat{w}_1^i(t, \hat{x}(0)) + \hat{w}_2^i(t, \hat{y}(0)) - \varphi(\hat{x}(0), \hat{y}(0)). \quad (6.26)
\]
Thus we obtain that (6.15) holds true, and \(\hat{w}_1^i(t, x_0) + \hat{w}_2^i(t, y_0) - \varphi(x_0, y_0)\) has a maximum at \((\hat{t}, \hat{x}(0), \hat{y}(0))\) on \([0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d\). The proof is now complete. \(\square\)

**Lemma 6.6.** The maximum points \((t_{k,j}, x^{k,j}, s_{k,j}, y^{k,j})\) of \(\Gamma_k(t, x, s, y) - \sum_{i=0}^{\infty} \frac{1}{2^i} \bigl(\mu(i, i, t, x) + \bigl(\hat{s}_i, \hat{y}, s, y\bigr)\bigr)\) satisfy conditions (6.12), (6.13) and (6.14).

**Proof.** Recall that \(\hat{w}_1^i(t, x_j^0) \geq \hat{w}_1^i, \hat{w}_2^i \geq \hat{w}_2, \) by the definitions of \(\hat{w}_1^i\) and \(\hat{w}_2^i\), we get that
\[
\hat{w}_1^i(t_{k,j}, x^{k,j}(0)) + \hat{w}_2^i(s_{k,j}, y^{k,j}(0)) - \check{\varphi}_k(t_{k,j}, x^{k,j}(0)) \geq \psi_k(s_{k,j}, y^{k,j}(0))
\geq w_1(t_{k,j}, x^{k,j}) + w_2(s_{k,j}, y^{k,j}) - \check{\varphi}_k(t_{k,j}, x^{k,j}(0)) - \psi_k(s_{k,j}, y^{k,j}(0)) = \Gamma_k(t_{k,j}, x^{k,j}, s_{k,j}, y^{k,j}).
\]
We notice that, from (6.11) and the property (ii) of \((t_{k,j}, x^{k,j}, s_{k,j}, y^{k,j})\),
\[
\Gamma_k(t_{k,j}, x^{k,j}, s_{k,j}, y^{k,j}) \geq \Gamma_k(t_0, x_0, s_0, y_0) \geq \sup_{(t, x, s, y) \in \Lambda^i} \Gamma_k(t, x, s, y) - \frac{1}{j},
\]
and by the definitions of \(\hat{w}_1^i\) and \(\hat{w}_2^i\),
\[
\sup_{(t, x, s, y) \in \Lambda^i} \Gamma_k(t, x, s, y) \geq \hat{w}_1^i(t_{k,j}, x^{k,j}(0)) + \hat{w}_2^i(s_{k,j}, y^{k,j}(0)) - \check{\varphi}_k(t_{k,j}, x^{k,j}(0)) - \psi_k(s_{k,j}, y^{k,j}(0)).
\]
Therefore,
\[
\hat{w}_1^i(t_{k,j}, x^{k,j}(0)) + \hat{w}_2^i(s_{k,j}, y^{k,j}(0)) - \check{\varphi}_k(t_{k,j}, x^{k,j}(0)) - \psi_k(s_{k,j}, y^{k,j}(0))
\]
\[ \geq \Gamma_k(t_{k,j}, x_{k,j}^{k,j}, s_{k,j}, y_{k,j}^{k,j}) \geq \tilde{w}_1^{k,j} - \tilde{\psi}_k(s_{k,j}, y_{k,j}^{k,j}) - \tilde{\psi}_k(s_{k,j}, y_{k,j}^{k,j}) - \frac{1}{k}. \] (6.27)

By (6.2) and \( \tilde{\psi}_k, \tilde{\psi}_k \) are bounded, there exists a constant \( M_2 > 0 \) that is sufficiently large that

\[ \Gamma_k(t, x, s, y) < \sup_{(l, z^1), (r, z^2) \in \Lambda} \Gamma_k(l, z^1, r, z^2) - 1 \]

for all \( (t + |x|, s + |y|) \geq M_2 \). Thus, we have

\[ (t_{k,j} + |x_{k,j}|) \vee (s_{k,j} + |y_{k,j}|) < M_2. \]

In particular, \( t_{k,j} \vee s_{k,j} \geq |x_{k,j}|(0) \vee |y_{k,j}(0)| < M_2. \)

We note that \( M_2 \) is independent of \( j \). Then letting \( j \to \infty \) in (6.27), we obtain (6.12). Indeed, if not, we may assume that there exist \( (t, \hat{x}, \hat{y}) \in [0, \infty) \times \mathbb{R}^d \times [0, \infty) \times \mathbb{R}^d \) and a subsequence of \( (t_{k,j}, x_{k,j}(0), s_{k,j}, y_{k,j}(0)) \) still denoted by itself such that

\[ (t_{k,j}, x_{k,j}, s_{k,j}, y_{k,j}(0)) \to (t, \hat{x}, \hat{y}) \neq (t_k, x_0^k, s_0^k, y_0^k). \]

Letting \( j \to \infty \) in (6.27), by the upper semicontinuity of \( \tilde{w}_1^{k,\ast} + \tilde{w}_2^{\ast} - \tilde{\psi}_k - \tilde{\psi}_k \), we have

\[ \tilde{w}_1^{k,\ast}(t, \hat{x}) + \tilde{w}_2^{\ast}(\hat{y}) - \tilde{\psi}_k(\hat{x}, \hat{y}) \geq \tilde{w}_1^{k,\ast}(t, x_0^k) + \tilde{w}_2^{\ast}(s_0^k, y_0^k) - \tilde{\psi}_k(t, x_0^k) - \tilde{\psi}_k(s_0^k, y_0^k), \]

which contradicts that \( (t_k^k, x_0^k, s_0^k, y_0^k) \) is the strict maximum point of \( \tilde{w}_1^{k,\ast}(t, x_0^k) + \tilde{w}_2^{\ast}(s_0^k, y_0^k) - \tilde{\psi}_k(t, x_0^k) - \tilde{\psi}_k(s_0^k, y_0^k) \).

By (6.12), the upper semicontinuity of \( \tilde{w}_1^{k,\ast} \) and \( \tilde{w}_2^{\ast} \) and the continuity of \( \tilde{\psi}_k \) and \( \tilde{\psi}_k \), letting \( j \to \infty \) in (6.27), we obtain (6.13), and then also (6.14). The proof is now complete. \( \square \)

6.2. Uniqueness. This subsection is devoted to a proof of uniqueness of viscosity solutions to equation (1.4). This result, together with the results from the previous section, will be used to characterize the value functional defined by (1.3).

We now state the main result of this subsection.

Theorem 6.7. Suppose Hypothesis 4.2 holds and \( \lambda \geq (12 + 15L) \). Let \( W_1 \in C(C_0) \) (resp., \( W_2 \in C(C_0) \)) be a viscosity subsolution (resp., supersolution) to equation (1.4) and let there exist a constant \( \tilde{C} > 0 \) such that for \( h \in [0, \infty) \), \( x, y \in C_0 \),

\[ |W_1(x)| \vee |W_2(x)| \leq \tilde{C}(1 + |x|); \quad |W_1(x) - W_1(y)| \vee |W_2(x) - W_2(y)| \leq \tilde{C}|x - y|; \]

and

\[ |W_1(x) - W_1(x_h)| \vee |W_2(x) - W_2(x_h)| \leq \tilde{C}(1 + |x|)(h + h^{\frac{1}{2}} + 1 - e^{-\lambda h}). \] (6.29)

Then \( W_1 \leq W_2 \).

Theorems 5.5 and 6.7 lead to the result (given below) that the viscosity solution to the HJB equation with infinite delay given in (1.3) corresponds to the value functional \( V \) of our optimal control problem given in (1.3) and (1.2).

Theorem 6.8. Let Hypothesis 4.2 hold and \( \lambda \geq (12 + 15L) \). Then the value functional \( V \) defined by (1.3) is the unique viscosity solution to equation (1.4) in the class of functionals satisfying (4.3) and (4.11).

Proof. Theorem 5.5 shows that \( V \) is a viscosity solution to (1.4). Thus, our conclusion follows from Theorems 4.5, 4.7, and 6.7. \( \square \)

We are now in a position to prove Theorem 6.7.

Proof of Theorem 6.7 The proof of this theorem is rather long. Thus, we split it into several steps.
Step 1. Definitions of auxiliary functionals.

To prove the theorem, we assume the converse result that $\tilde{x} \in C_0$ exists such that $\tilde{m} := W_1(\tilde{x}) - W_2(\tilde{x}) > 0$.

Consider that $\varepsilon > 0$ is a small number such that

$$W_1(\tilde{x}) - W_2(\tilde{x}) - 2\varepsilon \Upsilon^{1,3}(\tilde{x}) > \frac{\tilde{m}}{2},$$

and

$$\varepsilon \leq \frac{\tilde{m}}{16}. \quad (6.30)$$

Next, we define for any $(t, x, y) \in [0, \infty) \times C_0 \times C_0$,

$$\Psi(t, x, y) = W_1(x) - W_2(y) - \beta \Upsilon(x, y) - \beta^4|x(0) - y(0)|^2 - \varepsilon(\Upsilon^{1,3}(x) + \Upsilon^{1,3}(y)).$$

By (3.1) and (6.28), it is clear that $\Psi$ is bounded from above on $[0, \infty) \times C_0 \times C_0$. Moreover, by Lemma 2.9, $\Psi$ is an upper semicontinuous functional on $[0, \infty) \times C_0 \times C_0$ where $d_1, \infty((t, x^1, x^2), (s, y^1, y^2)) = |s - t| + |x^1_{(s-t)\wedge 0} - y^1_{(s-t)\wedge 0}|+|x^2_{(s-t)\wedge 0} - y^2_{(s-t)\wedge 0}|$ for all $(t, x^1, x^2), (s, y^1, y^2) \in [0, \infty) \times C_0 \times C_0$. Define a sequence of positive numbers $\{\delta_i\}_{i \geq 0}$ by $\delta_i = \frac{1}{2^i}$ for all $i \geq 0$. Since $\Upsilon$ is a gauge-type function, from Lemma 2.9 it follows that, for every $(1, x^0, y^0) \in [0, \infty) \times C_0 \times C_0$ satisfy

$$\Psi(1, x^0, y^0) \geq \sup_{(t, x, y) \in [1, \infty) \times C_0 \times C_0} \Psi(t, x, y) - \frac{1}{\beta},$$

and

$$\Psi(1, x^0, y^0) \geq \Psi(1, \tilde{x}, \tilde{x}) > \frac{\tilde{m}}{2},$$

there exist $(\tilde{t}, \tilde{x}, \tilde{y}) \in [1, \infty) \times C_0 \times C_0$ and a sequence $\{(t_i, x^i, y^i)\}_{i \geq 1} \subset [1, \infty) \times C_0 \times C_0$ such that

(i) $\Upsilon(1, x^0, \tilde{t}, \tilde{x}) + \Upsilon(1, y^0, \tilde{t}, \tilde{y}) + |\tilde{t} - 1|^2 \leq \frac{1}{\beta^2}, \ Upsilon(t_i, x^i, \tilde{t}, \tilde{x}) + \Upsilon(t_i, y^i, \tilde{t}, \tilde{y}) + |\tilde{t}_i - t_i|^2 \leq \frac{1}{\beta^2}$ and $t_i \uparrow \tilde{t}$ as $i \to \infty$,

(ii) $\Psi_1(\tilde{t}, \tilde{x}, \tilde{y}) \geq \Psi_1(1, x^0, y^0)$, and

(iii) for all $(s, x, y) \in [\tilde{t}, \infty) \times C_0 \times C_0 \setminus \{(\tilde{t}, \tilde{x}, \tilde{y})\},$

$$\Psi_1(s, x, y) < \Psi_1(\tilde{t}, \tilde{x}, \tilde{y}), \quad (6.31)$$

where we define

$$\Psi_1(s, x, y) = \Psi(s, x, y) - \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon(t_i, x^i, s, x) + \Upsilon(t_i, y^i, s, y) + |s - t_i|^2, \quad (s, x, y) \in [0, \infty) \times C_0 \times C_0.$$

We should note that the point $(\tilde{t}, \tilde{x}, \tilde{y})$ depends on $\beta$ and $\varepsilon$.

Step 2. There exists $M_0 > 0$ independent of $\beta$ such that

$$|\tilde{x}|_C \vee |\tilde{y}|_C < M_0, \quad (6.32)$$

and the following result holds true:

$$\beta^{\frac{1}{3}}|\tilde{x} - \tilde{y}|_C^2 + \beta^{\frac{1}{2}}|\tilde{x}(0) - \tilde{y}(0)|^2 \to 0 \text{ as } \beta \to \infty. \quad (6.33)$$

Let us show the above. First, noting $\varepsilon$ is independent of $\beta$, by the definition of $\Psi$, there exists an $M_0 > 0$ that is sufficiently large that $\Psi(t, x, y) < 0$ for all $t \geq 0$ and $|x|_C \vee |y|_C \geq M_0$. Thus, we
have $|\dot{x}|_C \lor |\dot{y}|_C \lor |x^0|_C \lor |y^0|_C < M_0$.

Second, by (6.31), we have

$$2\Psi_1(\hat{t}, \hat{x}, \hat{y}) \geq \Psi_1(\hat{t}, \hat{x}, \hat{x}) + \Psi_1(\hat{t}, \hat{y}, \hat{y}).$$

(6.34)

This implies that

$$2\beta \Upsilon(\hat{x}, \hat{y}) + 2\beta^\frac{1}{2}|\dot{x}(0) - \dot{y}(0)|^2$$

$$\leq |W_1(\dot{x}) - W_1(\dot{y})| + |W_2(\dot{x}) - W_2(\dot{y})| + \sum_{i=0}^{\infty} 2^{\frac{i}{2}}[\Upsilon(t_i, y^i, \hat{t}, \hat{x}) + \Upsilon(t_i, x^i, \hat{t}, \hat{y})].$$

(6.35)

On the other hand, by Lemma 3.3 and the property (i) of $(\hat{t}, \hat{x}, \hat{y})$,

$$\sum_{i=0}^{\infty} 2^{\frac{i}{2}}[\Upsilon(t_i, y^i, \hat{t}, \hat{x}) + \Upsilon(t_i, x^i, \hat{t}, \hat{y})] \leq 2^5 \sum_{i=0}^{\infty} 2^{\frac{i}{2}}[\Upsilon(t_i, y^i, \hat{t}, \hat{y}) + \Upsilon(t_i, x^i, \hat{t}, \hat{y})] + 2\Upsilon(\hat{x}, \hat{y})$$

$$\leq \frac{2^6}{\beta} + 2^7 \Upsilon(\hat{x}, \hat{y}).$$

(6.36)

Combining (6.35) and (6.36), from (6.28) and (6.32) we have

$$(2\beta - 2^7) \Upsilon(\hat{x}, \hat{y}) + 2\beta^\frac{1}{2}|\dot{x}(0) - \dot{y}(0)|^2 \leq |W_1(\dot{x}) - W_1(\dot{y})| + |W_2(\dot{x}) - W_2(\dot{y})| + \frac{2^6}{\beta}$$

$$\leq 2L(2 + |\dot{x}|_C + |\dot{y}|_C) + \frac{2^6}{\beta} \leq 4L(1 + M_0) + \frac{2^6}{\beta}.$$  

(6.37)

Letting $\beta \to \infty$, we get

$$\Upsilon(\hat{x}, \hat{y}) \leq \frac{1}{2\beta - 2^7} \left[ 4L(1 + M_0) + \frac{2^6}{\beta} \right] \to 0 \text{ as } \beta \to \infty.$$  

From (3.1) it follows that

$$|\dot{x} - \dot{y}|_C \to 0 \text{ as } \beta \to \infty.$$  

(6.38)

Combining (3.1), (6.28), (6.35), (6.36) and (6.38), we see that

$$\beta|\dot{x} - \dot{y}|_C^6 + \beta^\frac{1}{2}|\dot{x}(0) - \dot{y}(0)|^2 \leq \beta \Upsilon(\hat{x}, \hat{y}) + \beta^\frac{1}{2}|\dot{x}(0) - \dot{y}(0)|^2$$

$$\leq \frac{1}{2}|W_1(\dot{x}) - W_1(\dot{y})| + |W_2(\dot{x}) - W_2(\dot{y})| + \frac{2^5}{\beta} + 2^6 \Upsilon(\hat{x}, \hat{y})$$

$$\leq C|\dot{x} - \dot{y}|_C + \frac{2^5}{\beta} + 2^8|\dot{x} - \dot{y}|_C^6 \to 0 \text{ as } \beta \to \infty.$$  

(6.39)

Multiply the leftmost and rightmost sides of inequality (6.39) by $\beta^\frac{1}{2}$, we obtain that

$$\beta^\frac{1}{2}|\dot{x}(0) - \dot{y}(0)|^2 \leq C^\frac{1}{2}|\dot{x} - \dot{y}|_C + \frac{2^5}{\beta^\frac{1}{2}} + 2^8 \beta^\frac{1}{2}|\dot{x} - \dot{y}|_C^6.$$  

(6.40)

By also (6.39), the right side of above inequality converges to 0 as $\beta \to \infty$. Then we have that

$$\beta^\frac{1}{2}|\dot{\gamma}(\hat{t}) - \dot{\gamma}(\hat{t})|^2 \to 0 \text{ as } \beta \to \infty.$$  

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Combining with (6.39), we get (6.33) holds true.

**Step 3. Maximum principle.**

We put, for \((t, x), (t, y) \in \Lambda^1\),

\[
w_1(t, x) = W_1(x) - 2^5 \beta \Upsilon(t, x, \hat{t}, \hat{x}) - \varepsilon \Upsilon^{1,3}(x) - \varepsilon \Upsilon(\hat{t}, \hat{x}, t, x) - \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon(t_i, x^i, t, x),
\]

\[
w_2(t, y) = -W_2(y) - 2^5 \beta \Upsilon(t, y, \hat{t}, \hat{\xi}) - \varepsilon \Upsilon^{1,3}(y) - \varepsilon \Upsilon(\hat{t}, \hat{\xi}, t, y) - \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon^3(t_i, y^i, t, y),
\]

where \(\hat{\xi} = \frac{\hat{x} + \hat{y}}{2}\). We note that \(w_1, w_2\) depend on \(\hat{\xi}_i\), and thus on \(\beta\) and \(\varepsilon\). By the following Lemma 6.9, \(w_1\) and \(w_2\) satisfy the conditions of Theorem 6.3. Then by Theorem 6.3, there exist the sequences \((l_k, x^k)\), \((s_k, y^k)\) \(\in \Lambda^i\) and the sequences of functionals \(\varphi_k \in C_p^{1,2}(\Lambda^i), \psi_k \in C_p^{1,2}(\Lambda^i)\) such that \(\varphi_k, \partial_t \varphi_k, \partial_x \varphi_k, \partial_{xx} \varphi_k, \psi_k, \partial_t \psi_k, \partial_x \psi_k, \partial_{xx} \psi_k\) are bounded and uniformly continuous, and such that

\[
w_1(t, x) - \varphi_k(t, x)
\]

has a strict global maximum 0 at \((l_k, \bar{x}^k)\) over \(\Lambda^i\),

\[
w_2(t, y) - \psi_k(t, y)
\]

has a strict global maximum 0 at \((s_k, \bar{y}^k)\) over \(\Lambda^s\), and

\[
\begin{align*}
(l_k, \bar{x}^k(0), w_1(l_k, \bar{x}^k), \partial_{t} \varphi_k(l_k, \bar{x}^k), \partial_x \varphi_k(l_k, \bar{x}^k), \partial_{xx} \varphi_k(l_k, \bar{x}^k)) \rightarrow \infty \quad & (\hat{t}, \hat{x}(0), w_1(\hat{t}, \hat{x}), b_1, \nabla x_1 \varphi(\hat{x}(0), \hat{y}(0)), X), \\
(s_k, \bar{y}^k(0), w_2(s_k, \bar{y}^k), \partial_{t} \psi_k(s_k, \bar{y}^k), \partial_x \psi_k(s_k, \bar{y}^k), \partial_{xx} \psi_k(s_k, \bar{y}^k)) \rightarrow \infty \quad & (\hat{t}, \hat{y}(0), w_2(\hat{t}, \hat{y}), b_2, \nabla x_2 \varphi(\hat{x}(0), \hat{y}(0)), Y),
\end{align*}
\]

where \(b_1 + b_2 = 0\) and \(X, Y \in S(\mathbb{R}^d)\) satisfy the following inequality:

\[
-6\beta \left( \begin{array}{cc} I & 0 \\ 0 & I \end{array} \right) \leq \left( \begin{array}{cc} X & 0 \\ 0 & Y \end{array} \right) \leq 6\beta \left( \begin{array}{cc} I & -I \\ -I & I \end{array} \right).
\]

(6.47)

We note that (6.47) follows from (6.6) choosing \(\kappa = \beta^{-\frac{1}{2}}\), and sequence \((\bar{x}^k, \bar{y}^k, l_k, s_k, \varphi_k, \psi_k)\) and \(b_1, b_2, X, Y\) depend on \(\beta\) and \(\varepsilon\). By the following Lemma 6.12, we have

\[
\lim_{k \to \infty} [d_\infty(l_k, \bar{x}^k, \hat{t}, \hat{x}) + d_\infty(s_k, \bar{y}^k, \hat{t}, \hat{y})] = 0.
\]

(6.48)

For every \((t, x), (s, y) \in \Lambda^T-\hat{a}\), let

\[
\chi^k(t, x) := \varepsilon \Upsilon^{1,3}(x) + \varepsilon \Upsilon(t, x, \hat{t}, \hat{x}) + \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon(t_i, x^i, t, x) + 2^5 \beta \Upsilon(t, x, \hat{t}, \hat{\xi}) + \varphi_k(t, x),
\]

\[
h^k(s, y) := \varepsilon \Upsilon^{1,3}(y) + \varepsilon \Upsilon(s, y, \hat{t}, \hat{y}) + \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon(t_i, y^i, s, y) + 2^5 \beta \Upsilon(s, y, \hat{t}, \hat{\xi}) + \psi_k(s, y).
\]
It is clear that \( \chi^k(\cdot) \in C_p^{1,2}(\Lambda^{l_k}), h^k(\cdot) \in C_p^{1,2}(\Lambda^{s_k}) \). Moreover, by \((6.43), (6.44)\) and definitions of \( w_1 \) and \( w_2 \),

\[
(W_1 - \chi^k)(l_k, \bar{x}^k) = \sup_{(t,x) \in \Lambda^{l_k}} (W_1 - \chi^k)(t,x),
\]

\[
(W_2 + h^k)(s_k, \bar{y}^k) = \inf_{(s,y) \in \Lambda^{s_k}} (W_2 + h^k)(s,y).
\]

Now, for every \( \beta > 0 \) and \( k > 0 \), from the definition of viscosity solutions it follows that

\[
-\lambda W_1(\bar{x}^k) + \partial_t \chi^k(l_k, \bar{x}^k) + H(l_k, \bar{x}^k, \partial_x \chi^k(l_k, \bar{x}^k), \partial_{xx} \chi^k(l_k, \bar{x}^k)) \geq 0,
\]

and

\[
-\lambda W_2(\bar{y}^k) - \partial_t h^k(s_k, \bar{y}^k) + H(s_k, \bar{y}^k, -\partial_x h^k(s_k, \bar{y}^k), -\partial_{xx} h^k(s_k, \bar{y}^k)) \leq 0,
\]

where, for every \( (t,x) \in \Lambda^{l_k} \) and \( (s,y) \in \Lambda^{s_k} \), from Remark \( \text{3.2 (i)} \),

\[
\partial_t \chi^k(t,x) = \partial_t \varphi_k(t,x) + 2\varepsilon(t - \hat{t}) + 2 \sum_{i=0}^{\infty} \frac{1}{2^i}(t - t_i),
\]

\[
\partial_x \chi^k(t,x) = \varepsilon \partial_x \Upsilon^{1,3}(x) + \partial_x \varphi_k(t,x) + \varepsilon \partial_x \Upsilon(x - \hat{x}_{t-l}) + 2^5 \beta \partial_x \Upsilon(x - \hat{t}_{l-t}) + \partial_x \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon(x - \gamma_{l-t}),
\]

\[
\partial_{xx} \chi^k(t,x) = \varepsilon \partial_{xx} \Upsilon^{1,3}(x) + \partial_x \varphi_k(t,x) + \varepsilon \partial_{xx} \Upsilon(x - \hat{x}_{t-l}) + 2^5 \beta \partial_{xx} \Upsilon(x - \hat{t}_{l-t}) + \partial_{xx} \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon(x - \gamma_{l-t}),
\]

\[
\partial_t h^k(s,y) = \partial_t \psi_k(s,y) + 2\varepsilon(s - \hat{t}),
\]

\[
\partial_x h^k(s,y) = \varepsilon \partial_x \Upsilon^{1,3}(y) + \partial_x \psi_k(s,y) + \varepsilon \partial_x \Upsilon(y - \hat{y}_{s-l}) + 2^5 \beta \partial_x \Upsilon(y - \hat{t}_{s-l}) + \partial_x \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon(y - \gamma_{s-l}),
\]

\[
\partial_{xx} h^k(s,y) = \varepsilon \partial_{xx} \Upsilon^{1,3}(y) + \partial_x \psi_k(s,y) + \varepsilon \partial_{xx} \Upsilon(y - \hat{y}_{s-l}) + 2^5 \beta \partial_{xx} \Upsilon(y - \hat{t}_{s-l}) + \partial_{xx} \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon(y - \gamma_{s-l}).
\]

**Step 4. Calculation and completion of the proof.**

We notice that, by \((5.7), (5.12), (5.14), (3.15)\) and the definition of \( \Upsilon \), there exists a generic constant \( C > 0 \) such that

\[
|\partial_x \Upsilon(\bar{x}^k - \bar{x}_{l-t})| + |\partial_x \Upsilon(\bar{y}^k - \bar{y}_{s-l})| \leq C|\bar{x}(0) - \bar{x}^k(0)|^5 + C|\bar{y}(0) - \bar{y}^k(0)|^5;
\]
\[ |\partial_{xx} \Upsilon(\hat{x}^k - \hat{x}_{t_k - \ell})| + |\partial_{xx} \Upsilon(\hat{y}^k - \hat{y}_{s_k - \ell})| \leq C|\hat{x}(0) - \hat{x}^k(0)|^4 + C|\hat{y}(0) - \hat{y}^k(0)|^4. \]

Letting \( k \to \infty \) in (6.49) and (6.50), and using (6.45), (6.46) and (6.48), we obtain

\[ -\lambda W_1(\hat{x}) + b_1 + 2 \sum_{i=0}^{\infty} \frac{1}{2i}(\hat{t} - t_i) + \mathbf{H}(\hat{x}, \partial_x \chi(\hat{t}, \hat{x}), \partial_{xx} \chi(\hat{t}, \hat{x})) \geq 0; \quad (6.51) \]

and

\[ -\lambda W_2(\hat{y}) - b_2 + \mathbf{H}(\hat{y}, -\partial_x h(\hat{t}, \hat{y}), -\partial_{xx} h(\hat{t}, \hat{y})) \leq 0, \quad (6.52) \]

where

\[ \partial_x \chi(\hat{t}, \hat{x}) := 2\beta^\frac{1}{2}(\hat{x}(0) - \hat{y}(0)) + 2^5 \beta_0 \Upsilon(\hat{x} - \hat{\xi}) + \varepsilon \partial_x \Upsilon^{1.3}(\hat{x}) + \partial_x \sum_{i=0}^{\infty} \frac{1}{2} \Upsilon(\hat{x} - x_{i-t_i}^i), \]

\[ \partial_{xx} \chi(\hat{t}, \hat{x}) := X + 2^5 \beta_0 \partial_{xx} \Upsilon(\hat{x} - \hat{\xi}) + \varepsilon \partial_{xx} \Upsilon^{1.3}(\hat{x}) + \partial_{xx} \sum_{i=0}^{\infty} \frac{1}{2} \Upsilon(\hat{x} - x_{i-t_i}^i), \]

\[ \partial_x h(\hat{t}, \hat{y}) := -2\beta^\frac{1}{2}(\hat{x}(0) - \hat{y}(0)) + 2^5 \beta_0 \partial_x \Upsilon(\hat{y} - \hat{\xi}) + \varepsilon \partial_x \Upsilon^{1.3}(\hat{y}) + \partial_x \sum_{i=0}^{\infty} \frac{1}{2} \Upsilon(\hat{y} - y_{i-t_i}^i) \]

and

\[ \partial_{xx} h(\hat{t}, \hat{y}) := Y + 2^5 \beta_0 \partial_{xx} \Upsilon(\hat{y} - \hat{\xi}) + \varepsilon \partial_{xx} \Upsilon^{1.3}(\hat{y}) + \partial_{xx} \sum_{i=0}^{\infty} \frac{1}{2} \Upsilon(\hat{y} - y_{i-t_i}^i). \]

Notice that \( b_1 + b_2 = 0 \) and \( \hat{\xi} = \frac{\hat{x} + \hat{y}}{2} \), combining (6.51) and (6.52), we have

\[ \lambda(W_1(\hat{x}) - W_2(\hat{y})) - 2 \sum_{i=0}^{\infty} \frac{1}{2i}(\hat{t} - t_i) \leq \mathbf{H}(\hat{x}, \partial_x \chi(\hat{t}, \hat{x}), \partial_{xx} \chi(\hat{t}, \hat{x})) - \mathbf{H}(\hat{y}, -\partial_x h(\hat{t}, \hat{y}), -\partial_{xx} h(\hat{t}, \hat{y})). \quad (6.53) \]

On the other hand, via a simple calculation we obtain

\[ \mathbf{H}(\hat{x}, \partial_x \chi(\hat{t}, \hat{x}), \partial_{xx} \chi(\hat{t}, \hat{x})) - \mathbf{H}(\hat{y}, -\partial_x h(\hat{t}, \hat{y}), -\partial_{xx} h(\hat{t}, \hat{y})) \leq \operatorname{sup}_{u \in U}(J_1 + J_2 + J_3), \quad (6.54) \]

where

\[ J_1 = \left( b(\hat{x}, u), 2\beta^\frac{1}{2}(\hat{x}(0) - \hat{y}(0)) + 2^5 \beta_0 \partial_x \Upsilon(\hat{x} - \hat{\xi}) + \varepsilon \partial_x \Upsilon^{1.3}(\hat{x}) \right) \]  

\[ + \partial_x \sum_{i=0}^{\infty} \frac{1}{2} \Upsilon(\hat{x} - x_{i-t_i}^i) \right)_{\mathbb{R}^d} - \left( b(\hat{y}, u), 2\beta^\frac{1}{2}(\hat{x}(0) - \hat{y}(0)) - 2^5 \beta_0 \partial_x \Upsilon(\hat{y} - \hat{\xi}) \right) \]

\[ - \varepsilon \partial_x \Upsilon^{1.3}(\hat{y}) - \partial_x \sum_{i=0}^{\infty} \frac{1}{2} \Upsilon(\hat{y} - y_{i-t_i}^i) \right)_{\mathbb{R}^d} \]
\[
\begin{align*}
&\leq 2\beta^\frac{1}{2} L |\dot{x}(0) - \dot{y}(0)| |\dot{x} - \dot{y}|_C + 18\beta L |\dot{x}(0) - \dot{y}(0)|^5 L |\dot{x} - \dot{y}|_C \\
&+ 18L \sum_{i=0}^{\infty} \frac{1}{2^i} ||x^i(0) - \dot{x}(0)||^5 + |y^i(0) - \dot{y}(0)||^5 ||y^i(0) + |\dot{x}||C + |\dot{y}||C \\
&+ 12\varepsilon L (1 + |\dot{x}|^2_w + |\dot{y}|^2_w);
\end{align*}
\] (6.55)

\[
J_2 = \frac{1}{2} \text{tr} \left[ \left( X + 2^5 \beta \partial_{xx} \Upsilon(\dot{x} - \dot{\xi}) + \varepsilon \partial_{xx} \Upsilon^{1,3}(\dot{x}) + \frac{1}{2} \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon(\dot{x} - x^i_{t-i}) \right) \sigma(\dot{x}, u) \sigma^T(\dot{x}, u) \right] \\
- \frac{1}{2} \text{tr} \left[ \left( -Y - 2^5 \beta \partial_{xx} \Upsilon(\dot{y} - \dot{\xi}) - \varepsilon \partial_{xx} \Upsilon^{1,3}(\dot{y}) - \frac{1}{2} \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon(\dot{y} - y^i_{t-i}) \right) \sigma(\dot{y}, u) \sigma^T(\dot{y}, u) \right]
\]
\[
\leq 3\beta^\frac{1}{2} ||\sigma(\dot{x}, u) - \sigma(\dot{y}, u)||^2 + 306\beta |\dot{x}(0) - \dot{y}(0)|^4 (|\sigma(\dot{x}, u)||^2_w + |\sigma(\dot{y}, u)||^2_w) \\
+ 15\varepsilon (|\sigma(\dot{x}, u)||^2_w + |\sigma(\dot{y}, u)||^2_w) + 153\sum_{i=0}^{\infty} \frac{1}{2^i} |x^i(0) - \dot{x}(0)|^4 |\sigma(\dot{x}, u)||^2_w \\
+ 153\sum_{i=0}^{\infty} \frac{1}{2^i} |y^i(0) - \dot{y}(0)|^4 |\sigma(\dot{y}, u)||^2_w
\]
\leq 3\beta^\frac{1}{2} L^2 |\dot{x} - \dot{y}|^2_w + 306\beta |\dot{x}(0) - \dot{y}(0)|^4 L^2 (2 + |\dot{x}|^2_w + |\dot{y}|^2_w) + 15\varepsilon L^2 (2 + |\dot{x}|^2_w + |\dot{y}|^2_w) \\
+ 153\sum_{i=0}^{\infty} \frac{1}{2^i} |x^i(0) - \dot{x}(0)|^4 + |y^i(0) - \dot{y}(0)||^4 \right] L^2 (1 + |\dot{x}|^2_w + |\dot{y}|^2_w);
\] (6.56)

\[
J_3 = q(\dot{x}, u) - q(\dot{y}, u) \leq L |\dot{x} - \dot{y}|_C;
\] (6.57)

We notice that, by the property (i) of \((\hat{t}, \hat{x}, \hat{y})\),
\[
2 \sum_{i=0}^{\infty} \frac{1}{2^i} (\hat{t} - t_i) \leq 2 \sum_{i=0}^{\infty} \frac{1}{2^i} \left( \frac{1}{2^i \beta} \right) \leq 4 \left( \frac{1}{\beta} \right)^{\frac{1}{2}},
\]
\[
\sum_{i=0}^{\infty} \frac{1}{2^i} |x^i(0) - \dot{x}(0)|^5 + |y^i(0) - \dot{y}(0)||^5 \leq 2 \sum_{i=0}^{\infty} \frac{1}{2^i} \left( \frac{1}{2^i \beta} \right)^{\frac{5}{2}} \leq 4 \left( \frac{1}{\beta} \right)^{\frac{5}{2}},
\]
and
\[
\sum_{i=0}^{\infty} \frac{1}{2^i} |x^i(0) - \dot{x}(0)|^4 + |y^i(0) - \dot{y}(0)||^4 \leq 2 \sum_{i=0}^{\infty} \frac{1}{2^i} \left( \frac{1}{2^i \beta} \right)^{\frac{4}{2}} \leq 4 \left( \frac{1}{\beta} \right)^{\frac{4}{2}}.
\]

Combining (6.53)-(6.57), and by (6.52) and (6.33) we can let \(\beta > 0\) be large enough such that,
\[
\lambda(W_1(\hat{x}) - W_2(\hat{y})) \leq (12 + 15L)\varepsilon L (2 + |\dot{x}|^2_w + |\dot{y}|^2_w) + (12 + 15L)\frac{\hat{m}}{8}.
\] (6.58)

Recalling \(\frac{\hat{m}}{2} \leq \Psi(\hat{t}, \hat{x}, \hat{y}) \leq W_1(\hat{x}) - W_2(\hat{y}) - \varepsilon (\Upsilon^{1,3}(\hat{x}) + \Upsilon^{1,3}(\hat{y}))\) and \(\lambda \geq (12 + 15L)L\), by (3.1) and (6.30), the following contradiction is induced:
\[
\frac{\hat{m}}{2} \leq \varepsilon (2 + |\dot{x}|^2_w + |\dot{y}|^2_w) + \frac{\hat{m}}{8} - \varepsilon (\Upsilon^{1,3}(\hat{x}) + \Upsilon^{1,3}(\hat{y})) \leq 2\varepsilon + \frac{\hat{m}}{8} \leq \frac{\hat{m}}{4}.
\]

The proof is now complete. \(\square\)

To complete the previous proof, it remains to state and prove the following lemmas. In the following Lemmas of this subsection, let \(\tilde{w}_1, \tilde{w}_1^{L, *}, \tilde{w}_2, \tilde{w}_2^{L, *}\) be the definitions in Definition 6.7 with respect to \(w_1\) defined by (6.41) and \(w_2\) defined by (6.42), respectively.
Lemma 6.9. The functionals $w_1$ and $w_2$ defined by (6.41) and (6.42) satisfy the conditions of Theorem 6.3.

Proof. From (3.31) and (6.28), $w_1$ and $w_2$ are upper semicontinuous functions bounded from above and satisfy (6.2). By the following Lemmas 6.10 and 6.11, $w_1$ and $w_2$ satisfy condition (6.3), and $\tilde{w}_{1,*}^i \in \Phi(\tilde{t}, \tilde{x}(0), T)$ and $\tilde{w}_{2,*}^i \in \Phi(\tilde{t}, \tilde{y}(0), T)$ for some $T \in (\tilde{t}, \infty)$. Moreover, by Lemma 6.9 and (6.31) we obtain that, for all $(t, x, y) \in [\tilde{t}, \infty) \times C_0 \times C_0$,

$$w_1(t, x) + w_2(t, y) - \beta_1^i |x(0) - y(0)|^2 \leq \Psi_1(t, x, y) = w_1(\tilde{t}, \tilde{x}) + w_2(\tilde{t}, \tilde{y}) - \beta_1^i |\tilde{x}(0) - \tilde{y}(0)|^2,$$

where the last inequality becomes equality if and only if $t = \tilde{t}$, $x = \tilde{x}$, $y = \tilde{y}$. Then we obtain that $w_1(t, x) + w_2(t, y) - \beta_1^i |x(0) - y(0)|^2$ has a maximum over $[\tilde{t}, \infty) \times C_0 \times C_0$ at a point $(\hat{t}, \hat{x}, \hat{y})$ with $\hat{t} > 0$. Thus $w_1$ and $w_2$ satisfy the conditions of Theorem 6.3.

Lemma 6.10. For every fixed $T > \tilde{t}$, there exists a local modulus of continuity $\rho_1$ such that the functionals $w_1$ and $w_2$ defined by (6.41) and (6.42) satisfy condition (6.3).

Proof. From (6.20) and the definition of $w_1$, we have that, for every $\hat{t} \leq t \leq s \leq T$ and $x \in C_0$,

$$w_1(t, x) - w_1(s, x_{s-t}) = W_1(x) - 2^5 \beta \Upsilon(t, x, \tilde{t}, \hat{x}) - \varepsilon \Upsilon^1,3(x) - \varepsilon \Upsilon(t, \tilde{x}, t, x) - \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon(t_i, x_i, t, x)$$

$$- W_1(x_{s-t}) + 2^5 \beta \Upsilon(s, x_{s-t}, \tilde{t}, \hat{x}) + \varepsilon \Upsilon^1,3(x_{s-t}) + \varepsilon \Upsilon(\tilde{t}, \hat{x}, s, x_{s-t}) + \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon(t_i, x_i, s, x_{s-t})$$

$$= W_1(x) - W_1(x_{s-t}) + \varepsilon((s - \tilde{t})^2 - (t - \tilde{t})^2) + \sum_{i=0}^{\infty} \frac{1}{2^i}((s - t_i)^2 - (t - t_i)^2)$$

$$\leq \bar{C}(1 + |x|C)((s - t) + (s - t)^{\frac{1}{2}} + 1 - e^{-\lambda(s-t)} + (2\varepsilon + 4)T(s-t)).$$

Taking $\rho_1(h, z) = \bar{C}(1 + z)(h + h^{\frac{1}{2}} + 1 - e^{-\lambda h}) + (2\varepsilon + 4)Th$, $(h, z) \in [0, \infty) \times [0, \infty)$, it is clear that $\rho_1$ is a local modulus of continuity and $w_1$ satisfies condition (6.3) with it. In a similar way, we show that $w_2$ satisfies condition (6.3) with this $\rho_1$. The proof is now complete.

Lemma 6.11. $\tilde{w}_{1,*}^i \in \Phi(\tilde{t}, \tilde{x}(0), T)$ and $\tilde{w}_{2,*}^i \in \Phi(\tilde{t}, \tilde{y}(0), T)$ for every $T \in (\tilde{t}, \infty)$.

Proof. We only prove $\tilde{w}_{1,*}^i \in \Phi(\tilde{t}, \tilde{x}(0), T)$ for every $T \in (\tilde{t}, \infty)$. $\tilde{w}_{2,*}^i \in \Phi(\tilde{t}, \tilde{y}(0), T)$ can be obtained by a symmetric way. Set $r = 1$, for a given $L_1 > 0$, let $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^d)$ be a function such that $\tilde{w}_{1,*}^i(t, x_0) - \varphi(t, x_0)$ has a maximum at $(\tilde{t}, \tilde{x}_0) \in (0, \infty) \times \mathbb{R}^d$, moreover, the following inequalities hold true:

$$|\tilde{t} - \hat{t}| + |\tilde{x}_0 - \hat{x}(0)| < r,$$

$$|\tilde{w}^i_{1,*}(\tilde{t}, \tilde{x}_0)| + |\nabla_x \varphi(\tilde{t}, \tilde{x}_0)| + |\nabla_{xx} \varphi(\tilde{t}, \tilde{x}_0)| \leq L_1.$$
We may assume that $\varphi$ grow quadratically at $\infty$. By (6.11) and (6.28), it is clear that $\Gamma$ is bounded from above on $\Lambda^I$. Moreover, by Lemma 3.1, $\Gamma$ is an upper semicontinuous functional. Define a sequence of positive numbers $\{\delta_i\}_{i \geq 0}$ by $\delta_i = \frac{1}{2^i}$ for all $i \geq 0$. For every $0 < \delta < 1$, by Lemma 2.9 we have that, for every $(\bar{t}_0, \bar{x}^0) \in \Lambda^I$ satisfying
\begin{align}
\Gamma(\bar{t}_0, \bar{x}^0) \geq \sup_{(s,x) \in \Lambda^I} \Gamma(s, x) - \delta,
\end{align}
there exist $(\bar{t}, \bar{x}) \in \Lambda^I$ and a sequence $\{((\bar{t}_i, \bar{x}_i))\}_{i \geq 1} \subset \Lambda^I$ such that
(i) $\Upsilon(\bar{t}_0, \bar{x}^0, \bar{t}, \bar{x}) \leq \delta$, $\Upsilon(\bar{t}_i, \bar{x}_i, \bar{t}, \bar{x}) \leq \frac{\delta}{2^i}$ and $t_i \uparrow \bar{t}$ as $i \to \infty$,
(ii) $\Gamma(\bar{t}, \bar{x}) - \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon(\bar{t}_i, \bar{x}_i, \bar{t}, \bar{x}) \geq \Gamma(\bar{t}_0, \bar{x}^0)$, and
(iii) for all $(s, x) \in \Lambda^I \setminus \{(\bar{t}, \bar{x})\},$
\begin{align}
\Gamma(s, x) - \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon(\bar{t}_i, \bar{x}_i, s, x) \leq \Gamma(\bar{t}, \bar{x}) - \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon(\bar{t}_i, \bar{x}_i, \bar{t}, \bar{x}).
\end{align}
We should note that the point $(\bar{t}, \bar{x})$ depends on $\delta$. By Lemma 6.10, $w_1$ satisfies condition (6.3).

Then, by the definitions of $\tilde{w}_1^I$ and $\tilde{w}_1^{I,*}$, we have
\begin{align}
\tilde{w}_1^{I,*}(\bar{t}, \bar{x}) - \varphi(\bar{t}, \bar{x}) &= \limsup_{s \geq \bar{t}, (s,y) \to (\bar{t},\bar{x})} (\tilde{w}_1^I(s, y) - \varphi(s, y)) = \limsup_{s \geq \bar{t}, (s,y) \to (\bar{t},\bar{x})} \left( \sup_{\gamma \in C_0 \gamma(0) = y} [w_1(s, \gamma)] - \varphi(s, y) \right) \\
&= \limsup_{s \geq \bar{t}, (s,y) \to (\bar{t},\bar{x})} \sup_{\gamma \in C_0 \gamma(0) = y} [w_1(s, \gamma) - \varphi(s, \gamma(0))] \leq \sup_{(s,x) \in \Lambda^I} \Gamma(s, x).
\end{align}
Combining with (6.60),
\begin{align}
\Gamma(\bar{t}_0, \bar{x}^0) \geq \sup_{(s,x) \in \Lambda^I} \Gamma(s, x) - \delta \geq \tilde{w}_1^{I,*}(\bar{t}, \bar{x}_0) - \varphi(\bar{t}, \bar{x}_0) - \delta.
\end{align}
Recall that $\tilde{w}_1^{I,*} \geq \tilde{w}_1^I$. Then, by the definition of $\tilde{w}_1^I$ and the property (ii) of $(\bar{t}, \bar{x})$,
\begin{align}
\tilde{w}_1^{I,*}(\bar{t}, \bar{x}(0)) - \varphi(\bar{t}, \bar{x}(0)) &\geq \tilde{w}_1^I(\bar{t}, \bar{x}(0)) - \varphi(\bar{t}, \bar{x}(0)) \geq w_1(\bar{t}, \bar{x}) - \varphi(\bar{t}, \bar{x}(0)) \\
&\geq \Gamma(\bar{t}_0, \bar{x}^0) \geq \tilde{w}_1^{I,*}(\bar{t}, \bar{x}_0) - \varphi(\bar{t}, \bar{x}_0) - \delta.
\end{align}
Noting $\varepsilon$ is independent of $\delta$ and $\varphi$ grows quadratically at $\infty$, by the definition of $\Gamma$, there exists a constant $M_3 > 0$ independent of $\delta$ that is sufficiently large that $\Gamma(t, x) < \sup_{(s,x) \in \Lambda^I} \Gamma(s, x) - 1$ for all $t + |x|_C \geq M_3$. Thus, we have $\bar{t} \lor |\bar{x}|_C \lor \bar{t}_0 \lor |\bar{x}^0|_C < M_3$. In particular, $|\bar{x}(0)| < M_3$. Letting $\delta \to 0$, by the similar proof procedure of (6.12), we obtain
\begin{align}
\bar{t} \to \bar{t}, \; \bar{x}(0) \to \bar{x}_0 \text{ as } \delta \to 0.
\end{align}
Thus, the definition of the viscosity subsolution can be used to obtain the following result:
\begin{align}
-\lambda W_1(\bar{x}) + \partial_t \mathcal{S}(\bar{t}, \bar{x}) + H(\bar{x}, \partial_x \mathcal{S}(\bar{t}, \bar{x}), \partial_{xx} \mathcal{S}(\bar{t}, \bar{x})) \geq 0.
\end{align}
where, for every \((t, x) \in \Lambda^i\),
\[
\exists(t, x) := \varepsilon \Upsilon^{1,3}(x) + \varepsilon \Upsilon(t, x, \hat{t}, \hat{x}) + \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon(t_i, x^i, t, x) + 2^5 \beta \Upsilon(t, x, \hat{t}, \hat{x})
\]
\[
+ \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon(t_i, x^i, t, x) + \varphi(t, x(0)),
\]
\[
\partial_t \exists(t, x) := 2\varepsilon(t - \hat{t}) + 2 \sum_{i=0}^{\infty} \frac{1}{2^i} [(t - t_i) + (t - \bar{t}_i)] + \varphi_t(t, x(0)),
\]
\[
\partial_x \exists(t, x) := \varepsilon \partial_x \Upsilon^{1,3}(x) + \varepsilon \partial_x \Upsilon(x - \hat{x}_{t-i}) + 2^5 \beta \partial_x \Upsilon(x - \hat{t}_{t-i})
\]
\[
+ \partial_x \left[ \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon(x - x^i_{t-i}) + \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon(x - \check{x}^i_{t-i}) \right] + \nabla_x \varphi(t, x(0)),
\]
\[
\partial_{xx} \exists(t, x) := \varepsilon \partial_{xx} \Upsilon^{1,3}(x) + \varepsilon \partial_{xx} \Upsilon(x - \hat{x}_{t-i}) + 2^5 \beta \partial_{xx} \Upsilon(x - \hat{t}_{t-i})
\]
\[
+ \partial_{xx} \left[ \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon(x - x^i_{t-i}) + \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon(x - \check{x}^i_{t-i}) \right] + \nabla_x \varphi(t, x(0)).
\]
We notice that \(\bar{t} \cup |\bar{x}| \leq M_3\). Then letting \(\delta \to 0\) in (6.63), by the definition of \(H\), it follows that there exists a constant \(C\) such that \(b = \varphi(t, \bar{x}_0) \geq C\). The proof is now complete. \(\square\)

**Lemma 6.12.** The maximum points \((l_k, \check{x}_k, s_k, \check{y}_k)\) satisfy condition (6.48).

**Proof.** Without loss of generality, we may assume \(s_k \leq l_k\), by (3.17), (6.29), (6.31) and the definitions of \(w_1\) and \(w_2\), we have that
\[
\begin{align*}
    &w_1(l_k, \check{x}_k) + w_2(s_k, \check{y}_k) - \beta^\frac{1}{2} |\check{x}(0) - \check{y}(0)|^2 \\
    &\leq \Psi_1(l_k, \check{x}_k, \check{y}_k) - W_2(\check{y}(k)) + W_2(\check{y}(0) - l_k) - \varepsilon [\Upsilon(l_k, \check{x}_k, \hat{t}, \check{x}) + \Upsilon(s_k, \check{y}_k, \hat{t}, \check{y})] \\
    &\leq \Psi_1(\hat{t}, \check{x}, \check{y}) + \hat{C}(1 + |\check{y}|) ((l_k - s_k) + \frac{1}{2} - 1 - e^{-\lambda(l_k - s_k)}) \\
    &- \varepsilon [\Upsilon(l_k, \check{x}_k, \hat{t}, \check{x}) + \Upsilon(s_k, \check{y}_k, \hat{t}, \check{y})].
\end{align*}
\]
(6.64)

By (3.49), \(w_2(s_k, \check{y}_k) \to w_2(\hat{t}, \check{y})\) as \(k \to \infty\). Then by that \(w_2\) satisfies condition (6.2), there exists a constant \(M_4 > 0\) that is sufficiently large that
\[
|\check{y}_k| \leq M_4, \text{ for all } k > 0.
\]

Letting \(k \to \infty\) in (6.64), by (6.45) and (6.46) we have that
\[
\begin{align*}
    \Psi_1(\hat{t}, \check{x}, \check{y}) &= w_1(\hat{t}, \check{x}) + w_2(\hat{t}, \check{y}) - \beta^\frac{1}{2} |\check{x}(0) - \check{y}(0)|^2 \\
    &\leq \Psi_1(\hat{t}, \check{x}, \check{y}) - \varepsilon \limsup_{k \to \infty} [\Upsilon(l_k, \check{x}_k, \hat{t}, \check{x}) + \Upsilon(s_k, \check{y}_k, \hat{t}, \check{y})].
\end{align*}
\]

Thus,
\[
\lim_{k \to \infty} [\Upsilon(l_k, \check{x}_k, \hat{t}, \check{x}) + \Upsilon(s_k, \check{y}_k, \hat{t}, \check{y})] = 0.
\]

Then by (3.1) we get (6.48) holds true. The proof is now complete. \(\square\)
Appendix A  Existence and consistency for viscosity solutions.

Proof (of Lemma 5.5). First, let φ ∈ \( \mathcal{A}^+(t, x, V) \) with \((t, x) \in \Lambda\), then for fixed \( u \in U \), by the DPP (Theorem 4.6), we obtain the following result:

\[
\varphi(t, x) = V(x) \leq \int_0^s e^{-\lambda t} \mathbb{E}q(X^{x, u}_t, u) dl + e^{-\lambda s} \mathbb{E}V(X^{x, u}_s) 
\]

\[
\leq \int_0^s e^{-\lambda t} \mathbb{E}q(X^{x, u}_t, u) dl + e^{-\lambda s} \mathbb{E}\varphi(t + s, X^{x, u}_s), \quad s \geq 0.
\]

Thus,

\[
0 \leq \frac{1}{s} \int_0^s e^{-\lambda t} \mathbb{E}q(X^{x, u}_t, u) dl + \frac{1}{s} e^{-\lambda s} \mathbb{E}\varphi(t + s, X^{x, u}_s) - \varphi(t, x).\]

Now, applying functional Itô formula (2.8) to \( e^{-\lambda s} \varphi(t + s, X^{x, u}_s) \), we have that

\[
0 \leq \frac{1}{s} \int_0^s e^{-\lambda t} [\mathbb{E}q(X^{x, u}_t, u) - \lambda \mathbb{E}\varphi(t + l, X^{x, u}_l)] dl,
\]

where, for every \((s, x, u) \in \Lambda^t \times U \) and \( \varphi \in \mathcal{C}^{1,2}_p(\Lambda^t) \),

\[
(\mathcal{L}\varphi)(s, x, u) = \partial_t \varphi(s, x) + (\partial_x \varphi(s, x), b(x, u))_{\mathbb{R}^d} + \frac{1}{2} \text{tr}[\partial_{xx} \varphi(s, x) \sigma(x, u) \sigma^T(x, u)].
\]

Letting \( s \to 0 \),

\[
0 \leq q(x, u) - \lambda V(x) + (\mathcal{L}\varphi)(t, x, u).
\]

Taking the infimum over \( u \in U \), we see that \( V \) is a viscosity subsolution of (1.4).

Let \( \varphi \in \mathcal{A}^-(t, x, V) \) with \((t, x) \in \Lambda\), then any \( \varepsilon > 0 \) and \( s > 0 \), by the DPP (Theorem 4.6), one can find a control \( u^\varepsilon(\cdot) \equiv u^\varepsilon(\cdot) \in \mathcal{U}_0 \) such that the following result holds:

\[
\varepsilon s \geq \int_0^s e^{-\lambda l} \mathbb{E}q(X^{x, u^\varepsilon}_l, u^\varepsilon(l)) dl + e^{-\lambda s} \mathbb{E}V(X^{x, u^\varepsilon}_s) - V(x) 
\]

\[
\geq \int_0^s e^{-\lambda l} \mathbb{E}q(X^{x, u^\varepsilon}_l, u^\varepsilon(l)) dl - e^{-\lambda s} \mathbb{E}\varphi(t + s, X^{x, u^\varepsilon}_s) + \varphi(t, x).
\]

Applying also (2.8) to \( e^{-\lambda s} \varphi(t + s, X^{x, u^\varepsilon}_s) \), we have that

\[
\varepsilon \geq \frac{1}{s} \int_0^s e^{-\lambda l} [\mathbb{E}q(X^{x, u^\varepsilon}_l, u^\varepsilon(l)) + \lambda \mathbb{E}\varphi(t + l, X^{x, u^\varepsilon}_l) - \mathbb{E}(\mathcal{L}\varphi)(t + l, X^{x, u^\varepsilon}_l, u^\varepsilon(l))] dl 
\]

\[
= -\lambda V(x) + \frac{1}{s} \int_0^s e^{-\lambda l} [\mathbb{E}q(x, u^\varepsilon(l)) - \mathbb{E}(\mathcal{L}\varphi)(t + l, x, u^\varepsilon(l))] dl + o(1) 
\]

\[
\geq -\lambda V(x) + [\partial_t \varphi(t, x) + \mathbf{H}(x, -\partial_x \varphi(t, x), -\partial_{xx} \varphi(t, x))] \frac{1}{s} \int_0^s e^{-\lambda l} dl + o(1).
\]

Letting \( s \to 0^+ \), we obtain the following inequality:

\[
\varepsilon \geq -\lambda V(x) - \partial_t \varphi(x) + \mathbf{H}(x, -\partial_x \varphi(t, x), -\partial_{xx} \varphi(t, x)).
\]

By the arbitrariness of \( \varepsilon \), we show \( V \) is a viscosity subsolution to (1.4). This step completes the proof. \( \square \)
Proof (of Lemma 5.6). Assume \( v \) is a viscosity solution. For any \( x \in C_0 \), since \( v \in C^{1,2}_p(C_0) \), by definition of viscosity solutions we see that
\[
-\lambda v(x) + \partial_t v(x) + \mathbf{H}(x, \partial_x v(x), \partial_{xx} v(x)) = 0.
\]
On the other hand, assume \( v \) is a classical solution. For every \((t, x) \in \Lambda\), let \( \varphi \in \mathcal{A}^+(t, x, v) \). For every \( \alpha \in \mathbb{R}^d \) and \( \gamma \in \mathbb{R}^{d \times n} \), let \( \tau = t, \theta(\cdot) \equiv \alpha \) and \( \varpi(\cdot) \equiv \gamma \) in (2.4), applying functional formula (2.8) and noticing that \( v(x) - \varphi(t, x) = 0 \), we have, for every \( \delta > 0 \),
\[
0 \leq \mathbb{E}(\varphi - v)(X_{t+\delta}) = \mathbb{E} \int_t^{t+\delta} \left[ \partial_t \varphi(l, X_l) - \partial_t v(X_l) + (\partial_x \varphi(l, X_l) - \partial_x v(X_l), \alpha)_{\mathbb{R}^d} \right] dl
\]
\[
+ \mathbb{E} \int_t^{t+\delta} \frac{1}{2} \text{tr}((\partial_{xx} \varphi(l, X_l) - \partial_{xx} v(X_l))\gamma\gamma^\top) dl
\]
\[
= \mathbb{E} \int_t^{t+\delta} \tilde{\mathcal{H}}(l, X_l) dl,
\]
(A.1)
where
\[
\tilde{\mathcal{H}}(s, y) = \partial_t \varphi(s, y) - \partial_t v(y) + (\partial_x \varphi(s, y) - \partial_x v(y), \alpha)_{\mathbb{R}^d} + \frac{1}{2} \text{tr}((\partial_{xx} \varphi(s, y) - \partial_{xx} v(y))\gamma\gamma^\top), \ (s, y) \in \Lambda.
\]
Letting \( \delta \to 0 \),
\[
\tilde{\mathcal{H}}(t, x) \geq 0.
\]
(A.2)
Let \( \gamma = 0 \), by the arbitrariness of \( \alpha \),
\[
\partial_t \varphi(t, x) \geq \partial_t v(x), \quad \partial_x \varphi(t, x) = \partial_x v(x).
\]
For every \( u \in U \), let \( \gamma = \sigma(x, u) \) in (A.2),
\[
\partial_t \varphi(t, x) + \frac{1}{2} \text{tr}(\partial_{xx} \varphi(t, x)\sigma(x, u)\sigma(x, u)^\top) \geq \partial_t v(x) + \frac{1}{2} \text{tr}(\partial_{xx} v(x)\sigma(x, u)\sigma^\top(x, u)).
\]
Noting that \( \varphi(t, x) = v(x) \) and \( \partial_x \varphi(t, x) = \partial_x v(x) \), we show that
\[
-\lambda \varphi(t, x) + \partial_t \varphi(t, x) + (\partial_x \varphi(t, x), b(x, u))_{\mathbb{R}^d} + \frac{1}{2} \text{tr}(\partial_{xx} \varphi(t, x)\sigma(x, u)\sigma^\top(x, u)) + q(x, u)
\]
\[
\geq -\lambda v(x) + \partial_t v(x) + (\partial_x v(x), b(x, u))_{\mathbb{R}^d} + \frac{1}{2} \text{tr}(\partial_{xx} v(x)\sigma(x, u)\sigma^\top(x, u)) + q(x, u),
\]
Taking the infimum over \( u \in U \), we see that
\[
-\lambda \varphi(t, x) + \partial_t \varphi(t, x) + \mathbf{H}(x, \partial_x \varphi(t, x), \partial_{xx} \varphi(t, x)) \geq -\lambda v(x) + \partial_t v(x) + \mathbf{H}(x, \partial_x v(x), \partial_{xx} v(x)).
\]
Note that \( -\lambda v(x) + \partial_t v(x) + \mathbf{H}(x, \partial_x v(x), \partial_{xx} v(x)) = 0 \). Thus,
\[
-\lambda \varphi(t, x) + \partial_t \varphi(t, x) + \mathbf{H}(x, \partial_x \varphi(t, x), \partial_{xx} \varphi(t, x)) \geq 0.
\]
We have that \( v \) is a viscosity subsolution of (1.4). In a symmetric way, we show that \( v \) is also a viscosity supersolution to equation (1.3). \( \square \)
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