Hermitian structures on a class of quaternionic Kähler manifolds

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Abstract

Any quaternionic Kähler manifold $(\tilde{N},g_N,Q)$ equipped with a Killing vector field $X$ with nowhere vanishing quaternionic moment map carries an integrable almost complex structure $J_1$ that is a section of the quaternionic structure $Q$. Using the HK/QK correspondence, we study properties of the almost Hermitian structure $(g_N,\tilde{J}_1)$ obtained by changing the sign of $J_1$ on the distribution spanned by $X$ and $J_1X$. In particular, we derive necessary and sufficient conditions for its integrability and for it being conformally Kähler. We show that for a large class of quaternionic Kähler manifolds containing the one-loop deformed c-map spaces, the structure $\tilde{J}_1$ is integrable. We do also show that the integrability of $\tilde{J}_1$ implies that $(g_N,\tilde{J}_1)$ is conformally Kähler in dimension four, but not in higher dimensions. In the special case of the one-loop deformation of the quaternionic Kähler symmetric spaces dual to the complex Grassmannians of two-planes we construct a third canonical Hermitian structure $(g_N,\tilde{J}_1)$. Finally, we give a complete local classification of quaternionic Kähler four-folds for which $\tilde{J}_1$ is integrable and show that these are either locally symmetric or carry a cohomogeneity 1 isometric action generated by one of the Lie algebras $\mathfrak{o}(2) \ltimes \mathfrak{heis}(\mathbb{R})$, $\mathfrak{u}(2)$, or $\mathfrak{u}(1,1)$.

Keywords: quaternionic Kähler manifolds, HK/QK correspondence, c-map, complex structure, conformally Kähler, cohomogeneity one.

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Contents

1 Introduction 2

2 Quaternionic Kähler manifolds from the HK/QK correspondence 4
  2.1 Twists and the HK/QK correspondence ......................... 4
  2.2 The c-map ............................................ 6

3 A Hermitian structure inducing the opposite orientation 7

4 Examples carrying a third Hermitian structure 10

5 The conformal Kähler property 11
1 Introduction

Despite decades of effort, quaternionic Kähler manifolds remain arguably the most elusive manifolds of special holonomy. In particular, for a long time, only a small number of constructions of complete examples beyond locally symmetric spaces was known [Ale75, LeB91]. Though it is conjectured that no such examples are forthcoming in the case of positive scalar curvature [LS94, BWW20], much progress has been made in recent years in the case of negative scalar curvature. The main driving force behind these developments has been a construction known as the (supergravity) $c$-map, and its (one-loop) deformation (see, for instance, [ACDM15] and references therein), which gives rise to many explicit examples of complete quaternionic Kähler manifolds of negative scalar curvature.

The input for the $c$-map is what is known as a projective special Kähler (PSK) manifold, or equivalently a conical affine special (pseudo-)Kähler (CASK) manifold $M$. CASK manifolds have the interesting property that their cotangent bundle $N = T^*M$ naturally admits the structure of a pseudo-hyper-Kähler manifold and moreover comes equipped with a nowhere-vanishing rotating Killing field $Z$, which means that it fixes one of the complex structures defining the pseudo-hyper-Kähler structure but rotates the remaining two.

In order to obtain a quaternionic Kähler manifold, one applies what is called the HK/QK correspondence to $N$ [Hay08, ACM13, ACDM15]. This construction, which applies to any pseudo-hyper-Kähler manifold with rotating Killing field $Z$ along with a choice of Hamiltonian for $Z$ with respect to the invariant complex structure, produces a pseudo-quaternionic Kähler manifold endowed with a Killing field $X$. The conditions for the resulting metric to be positive-definite are known in general [ACM13] and are satisfied in the case of the cotangent bundles of CASK manifolds.

Since the Killing field $X$, present on any quaternionic Kähler manifold $(\tilde{N}, g_{\tilde{N}}, Q)$ in the image of the HK/QK correspondence, has nowhere-vanishing quaternionic moment map [Dyc15], $\tilde{N}$ always carries an integrable almost complex structure [Sal99, § 7]. This complex structure, which we call $\tilde{J}_1$, is compatible with the quaternionic structure $Q$. Changing the sign of $\tilde{J}_1$ on the two-dimensional distribution spanned by $X$ and $J_1X$ yields another natural almost complex structure, which we call $\tilde{J}_1$. Note that $\tilde{J}_1$ is never compatible with $Q$ since it induces the opposite orientation.

In this paper, we study the properties of the almost Hermitian structure $(\tilde{J}_1, g_{\tilde{N}})$. In particular, we are interested to know when $\tilde{J}_1$ is also integrable, in which case we refer to the quaternionic Kähler manifold as a doubly integrable HK/QK manifold. Accordingly, our first main result, Theorem 3.1, is a necessary and sufficient condition for the integrability of $\tilde{J}_1$:

**Theorem 1.** Let $(\tilde{N}, g_{\tilde{N}}, Q)$ be a (pseudo-)quaternionic Kähler manifold that arises from the HK/QK correspondence applied to a pseudo-hyper-Kähler manifold $(N, g_N, I_1, I_2, I_3)$ endowed with rotating Killing field $Z$. Then the almost complex structure $\tilde{J}_1$ on $\tilde{N}$ is
integrable if and only if there exists a smooth function $\psi$ on the dual pseudo-hyper-Kähler manifold $N$ such that the identity $\nabla_Z Z = \psi I_Z$ holds.

In the case where $\tilde{N}$ arises from the $c$-map, the equation $\nabla_Z Z = \psi I_Z$ is satisfied with $\psi \equiv -1$ (see [ACM13, Prop. 2]) and therefore the class of doubly integrable HK/QK manifolds includes the (one-loop deformed) $c$-map spaces. This extends earlier results from [CDMV15].

Given a doubly integrable HK/QK manifold, one may inquire as to further properties of the Hermitian structure $(\tilde{J}_1, g_{\tilde{N}})$. On the one hand, it is known that no quaternionic Kähler manifold, with exception of manifolds locally isometric to the Grassmannian of complex 2-planes or to its non-compact dual, is Kähler. On the other hand, based on the study of Einstein metrics conformal to Kähler metrics, see [DM03] and references therein, one may suspect that, at least in some cases, the quaternionic Kähler metric could be conformal to a Kähler metric. We prove that this is indeed the case in dimension four (cf. Theorem 5.5), but never so in higher dimensions (cf. Theorem 5.6):

**Theorem 2.** Let $(\tilde{N}, g_{\tilde{N}}, Q)$ be a doubly integrable HK/QK manifold. Then $g_{\tilde{N}}$ is conformal to a Kähler metric compatible with the complex structure $\tilde{J}_1$ and invariant under the canonical Killing field $X$ if and only if $\tilde{N}$ is a four-fold.

While (deformed) $c$-map spaces are doubly integrable HK/QK manifolds, with $(\tilde{J}_1, g_{\tilde{N}})$ moreover being conformally Kähler in dimension four by virtue of the above result, they do not exhaust the class of doubly integrable HK/QK manifolds. To prove this, we show in Theorem 6.1 that every four-fold in this class is locally isometric to a member of a three-dimensional family $g^{a,b,c}$ of metrics. Since there is only a single $c$-map space in dimension four (carrying a one-parameter family of metrics), this gives us non-trivial examples beyond the $c$-map, which we analyze in some detail. In particular, we prove that they are either locally symmetric or are of cohomogeneity 1. In the latter case, we give an explicit action generated by a Lie algebra of Killing fields isomorphic to either $\mathfrak{o}(2) \ltimes \mathfrak{heis}_3(\mathbb{R})$, $\mathfrak{u}(2)$, or $\mathfrak{u}(1,1)$ (cf. Proposition 6.4).

The paper is organized as follows. We recall some preliminaries regarding the HK/QK correspondence and $c$-map in Section 2, after which we prove Theorem 1 in Section 3. In Section 4, we study a distinguished family of examples $(\tilde{N}_n, g_n^c)$, $n \in \mathbb{N}$, $c \geq 0$, for which we have a third integrable almost complex structure. The metrics $g_n^c$ are a one-parameter deformation of the non-compact symmetric space $SU(n,2)/SU(n) \times U(2)$ (which corresponds to $c = 0$) through complete quaternionic Kähler metrics. They have been extensively studied, see e.g. [CRT21, CST22]. We take up the question of finding a Kähler metric in the same conformal class as the quaternionic Kähler metric in Section 5, proving Theorem 2. Finally, Section 6 is dedicated to an in-depth analysis of the four-dimensional case, for which we give a local classification. We conclude with a detailed examination of the local models.

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2 Quaternionic Kähler manifolds from the HK/QK correspondence

In this preparatory section we recall the HK/QK correspondence in some detail as well as the properties of the resulting quaternionic Kähler manifolds that we will need later on. For more details see [ACDM15, MS15, CST21] and references therein.

2.1 Twists and the HK/QK correspondence

The starting point for the HK/QK correspondence is a pseudo-hyper-Kähler manifold $(N, g^N, I_k)$, $k = 1, 2, 3$, endowed with a distinguished vector field.

**Definition 2.1.** A vector field $Z$ on a pseudo-hyper-Kähler manifold $(N, g_N, I_k)$ such that $g^N(Z, Z)$ is nowhere-vanishing is called a rotating Killing field if it satisfies $L_Z g^N = 0$, $L_Z I_1 = 0$ and $L_Z I_2 = I_3$.

In order to apply the HK/QK correspondence, we ask that $(N, g_N, I_k)$ comes equipped with a rotating Killing field $Z$ which is Hamiltonian with respect to the two-form $\omega_H = \omega_1 + d(g^N(Z, -))$, where $\omega_1$ is the Kähler form associated to $I_1$. Furthermore, we will require that the Hamiltonian function, which we denote by $f_H$, is nowhere-vanishing.

We remark that, under these assumptions, $Z$ is automatically Hamiltonian with respect to $\omega_1$, with Hamiltonian function $f_Z := f_H - g(Z, Z)$. We will also assume that $f_Z$ is nowhere-vanishing.

Under the assumption that $\omega_1$ is an integral form, the triple $(Z, \omega_H, f_H)$ forms twist data on $N$ in the sense of A. Swann [Swa10]. In particular, there exists a principal circle bundle $\pi_N : P \to N$ with connection $\eta$ whose curvature is $\omega_H$, and $Z$ lifts to a vector field $Z_P$ on $P$ that preserves $\eta$. Explicitly, we have

$$Z_P = \tilde{Z} + \pi_N^* f_H X_P$$

where $X_P$ generates the principal circle action on $P$ and $\tilde{Z}$ is the $\eta$-horizontal lift of $Z$. The quotient space $\bar{N} := P/\langle Z_P \rangle$ is then called the twist of $N$ with respect to the twist data $(Z, \omega_H, f_H)$. Under the assumption that $Z$ and $Z_P$ act freely and properly, it is a smooth manifold. Note that $\bar{N}$ comes equipped with a distinguished vector field $X$, obtained by pushing down $X_P$.

The power of the twist construction lies in the fact that $Z$-invariant structures on $N$ can be carried over to $\bar{N}$. For instance, if $\varphi$ is an invariant function on $N$, then $\pi_N^* \varphi \in C^\infty(P)$ is invariant under $Z_P$ and therefore induces a well-defined function $\psi$ on $\bar{N}$, which we will call the twist of $\varphi$. Similarly, we can push down the $\eta$-horizontal lift $\tilde{U}$ of an invariant vector field $U$ on $N$ to obtain its twist $V = d\pi_N(\tilde{U})$ on $\bar{N}$. Twists of arbitrary tensor fields are then defined by demanding compatibility of twisting with tensor products and contractions. By comparing lifts to $P$, one can work out explicit formulas for the twist of a tensor field without much effort.

The following Lemmata, whose proofs can be found in [Swa10], follow from such computations:
Lemma 2.2. If α is an invariant p-form on N and β denotes its twist with respect to twist data (Z,ω_H,f_H), then dβ is the twist of dα − f_H^{-1}ω_H ∧ ι_Zα. □

Lemma 2.3. The twist of an invariant complex structure I on N with respect to twist data (Z,ω_H,f_H) is integrable if and only if ω_H of type (1,1) with respect to I. □

Returning to the setting of the HK/QK correspondence, let (N,g,I_k,Z), k = 1,2,3, be a pseudo-hyper-Kähler manifold endowed with a rotating Killing field Z, and consider the twist data (Z,ω_H,f_H) as above.

Proposition 2.4 ([CST22, §2.3]). Let ∇ denote the Levi-Civita connection of (N,g). Then the endomorphism field I_H = I_1 + 2∇Z satisfies the relation g(I_H−,−) = ω_H, is skew-symmetric with respect to g, and commutes with each I_k. In particular, the two-form ω_H is of type (1,1) with respect to each I_k. □

The quaternionic structure bundle ⟨I_1,I_2,I_3⟩ ⊂ End(TN) is invariant under Z and therefore induces an almost quaternionic structure Q on the resulting twist manifold N (with respect to twist data (Z,ω_H,f_H) as above). In fact, it is known [ACDM15, MS15] that (N,Q) carries a compatible quaternionic Kähler metric g_N. This metric, however, is not the twist of the pseudo-hyper-Kähler metric g_N, but rather of its so-called elementary deformation g_H [MS15]. The elementary deformation g_H is related to g_N by the formula

$$g_H = \frac{1}{f_Z} g_N|_{(HZ)^\perp} + \frac{f_H}{f_Z} g_N|_{HZ} \tag{2}$$

where HZ = ⟨Z,I_1Z,I_2Z,I_3Z⟩ denotes the quaternionic span of Z. We say that the quaternionic Kähler manifold (N,g_N,Q) arises from the HK/QK correspondence applied to N. Since the signature of a metric is preserved by twisting, the quaternionic Kähler metric g_N is positive definite if and only if g_H is. From (2), we see that this may happen even if g_N has signature (4k,4), as long as f_Z > 0 and f_H < 0. This observation is crucial in applying the HK/QK correspondence to the c-map [ACM13]. Since we are interested primarily in positive-definite quaternionic Kähler metrics, we will from now on assume that g_H is Riemannian (though our results remain valid, mutatis mutandis, in the indefinite case).

Note that the above construction depends on our choice of Hamiltonian function f_H, which appears explicitly in the expression for the elementary deformation g_H, whose twist is the quaternionic Kähler metric g_N. Since there is always a one-parameter freedom in choosing a Hamiltonian function, the HK/QK correspondence in fact yields a one-parameter family of quaternionic Kähler metrics on N.

We note the following property of quaternionic Kähler manifolds resulting from the HK/QK correspondence:

Proposition 2.5. Every quaternionic Kähler manifold that arises from the HK/QK correspondence admits an integrable complex structure compatible with the metric and quaternionic structure.

Proof. This follows immediately from applying Lemma 2.3 to the invariant complex structure I_1 on the hyper-Kähler manifold, using Proposition 2.4, and noting that I_1 is compatible with the elementary deformation and quaternionic structure on the hyper-Kähler side of the correspondence. □
2 Quaternionic Kähler manifolds from the HK/QK correspondence

2.2 The c-map

In order to put the HK/QK correspondence to use we need a way to construct pseudo-hyper-Kähler manifolds endowed with a rotating Killing field. The theory of special Kähler manifolds provides a plentiful source of examples.

Definition 2.6. An affine special Kähler manifold is a pseudo-Kähler manifold \((M, g_M, J_M)\) endowed with a flat, torsion-free and symplectic connection \(\nabla^\text{SK}\) such that \(\nabla^\text{SK} J_M\) is symmetric. An affine special Kähler manifold is called conical (or a CASK manifold) if it admits a vector field \(\xi\) such that \(\nabla^\text{SK} \xi = \nabla^g M \xi = \text{id}_{TM}\), where \(\nabla^g M\) denotes the Levi-Civita connection of \(g_M\). Moreover, we require that \(\{\xi, J_M \xi\}\) generate a principal \(\mathbb{C}^*\)-action, and that \(g_M(\xi, \xi) < 0\) while \(g_M\) is positive-definite on \(\langle \xi, J_M \xi \rangle^1\).

We recall how to construct a pseudo-hyper-Kähler manifold \(N\) with rotating Killing field from a CASK manifold \((M, g_M, J_M, \nabla^\text{SK}, \xi)\) of real dimension \(2n + 2\). The smooth manifold underlying \(N\) is \(T^* M\). Now note that the special Kähler connection \(\nabla^\text{SK}\) induces a splitting \(T(T^* M) \cong \pi^* T^* M \oplus \pi^* TM\). With respect to this splitting, we define the tensor fields

\[
g_N = \begin{pmatrix} g_M & 0 \\ 0 & g_M^{-1} \end{pmatrix}, \quad I_1 = \begin{pmatrix} J_M & 0 \\ 0 & J_M^* \end{pmatrix}, \quad I_2 = \begin{pmatrix} 0 & -\omega_M^{-1} \\ \omega_M & 0 \end{pmatrix}, \quad I_3 = I_1 I_2 \tag{3}\]

where we have omitted pullbacks throughout to simplify notation. It is well-known (see e.g. [ACD02] and references therein) that these tensor fields define a pseudo-hyper-Kähler structure on \(N = T^* M\) of quaternionic signature \((n, 1)\). This construction is known as the rigid \(c\)-map. Moreover, the \(\nabla^\text{SK}\) horizontal lift of \(-J_M \xi\) to \(N\) endows it with a canonical rotating Killing field \(Z\) [ACM13].

This construction gives rise to many examples, since there is an abundance of CASK manifolds. Indeed, if \(U \subset \mathbb{C}^n\) is any \(\mathbb{C}^*\)-invariant domain, there is a CASK manifold associated to any generic holomorphic function on \(U\) which is homogeneous of degree 2 under the \(\mathbb{C}^*\)-action, see [ACD02] for more detailed statements.

It follows easily from the definitions that the vector field \(-J_M \xi\) on a CASK manifold \(M\) is Killing and Hamiltonian with negative Hamiltonian function \(\frac{1}{2} g_M(\xi, \xi)\), so we may consider the Kähler quotient \(M//S^1 = \mu^{-1}(\{ -\frac{1}{2}\})/S^1\) with respect to the circle action it generates.

Definition 2.7. Let \((M, g_M, J_M, \nabla^\text{SK}, \xi)\) be a CASK manifold. Then the Kähler quotient \(\bar{M} = M//S^1\) with respect to the Hamiltonian circle action generated by \(-J_M \xi\) is called a projective special Kähler (PSK) manifold.

Note that the induced Kähler metric on a PSK manifold is always positive-definite.

By this (extrinsic) definition, giving a PSK structure on a manifold is equivalent to giving the corresponding CASK manifold, and the two manifolds can be used interchangeably for our purposes. Thus, we may say that, starting from a PSK manifold, we may pass to the associated CASK manifold and apply the rigid \(c\)-map construction and the HK/QK correspondence to obtain a quaternionic Kähler manifold. Thus, we have obtained a method to associate quaternionic Kähler manifolds to PSK manifolds (cf. [ACDM15]). This construction recovers what is known as the (supergravity) \(c\)-map [FS90].

Recall that we obtain not just one, but an entire one-parameter family of quaternionic Kähler metrics from a single hyper-Kähler manifold with rotating Killing vector field.
by varying the corresponding Hamiltonian function. In the case of the c-map, there is a distinguished choice of Hamiltonian. We call the corresponding quaternionic Kähler metric the undeformed c-map metric and denote it by $g_0^N$. The quaternionic Kähler metrics that arise from different shifting the Hamiltonian function by a constant $c \in \mathbb{R}$ are known as (one-loop) deformed c-map metrics [RSV06] and denoted by $g_c^N$. It turns out that these metrics have good completeness properties only if we restrict to $c \geq 0$ [CDS17], which we will do from now on.

3 A Hermitian structure inducing the opposite orientation

Let $(\bar{N}, g_N, Q)$ be a quaternionic Kähler manifold obtained by the HK/QK correspondence from a pseudo-hyper-Kähler manifold $(N, g_N, I_1, I_2, I_3)$ endowed with rotating Killing field $Z$, as described in Section 2. Recall that it inherits a distinguished vector field $X$ from the generator of the principal circle action on $\pi_N : P \to N$. This vector field, which we can also view as the twist of the vector field $-f_{\bar{H}}^{-1}Z$ on $N$ (cf. Equation (1)), is Killing with respect to $g_N$. Its quaternionic moment map $\mu_X$ is nowhere-vanishing and therefore defines an integrable complex structure $J_1 := -\frac{\mu_X}{||\mu_X||} \in \Gamma(Q)$, see [Dyc15, Rem. 4.1.6]. This is, in fact, just another way of describing the complex structure whose existence is guaranteed by Proposition 2.5. We shall refer to the vector field $X$ as the canonical Killing field on $\bar{N}$.

Now let $D$ be the rank-two distribution spanned by $X$ and $J_1X$. We define a new skew-symmetric almost complex structure $\tilde{J}_1$ by

$$\tilde{J}_1|_D := -J_1|_D, \quad \tilde{J}_1|_{D^\perp} := J_1|_{D^\perp}.$$  

The orientation defined by $\tilde{J}_1$ is opposite to the one defined by $Q$.

**Theorem 3.1.** Let $\bar{N}$ be obtained from $N$ by the HK/QK correspondence as summarized above. Then the almost complex structure $\tilde{J}_1$ is integrable if and only if the vector field $\nabla_Z Z$ on $N$ is a section of the vector bundle $(Z, I_1 Z)$. In this case, it defines a Hermitian structure $(\bar{g}_N, \tilde{J}_1)$ compatible with the opposite orientation.

Since $Z$ is Killing and $g_N(Z, Z)$ is nowhere-vanishing, the above condition is equivalent to requiring that $\nabla_Z Z = \psi I_1 Z$ for some smooth function $\psi$ on $N$. Contracting with $g_N$ and again using that $Z$ is Killing, this becomes $\frac{1}{2}d(g_N(Z, Z)) = -\psi d f_Z$.

It is known that this holds when the pseudo-hyper-Kähler manifold $N$ with rotating Killing field $Z$ arises by applying the rigid c-map to a CASK manifold $M$. Indeed, in [ACM13, Prop. 2] it is shown that, in this case, the identity $\nabla_Z Z = -I_1 Z$ holds (with our conventions). Let $\bar{M}$ denote the PSK manifold associated to $M$. Then the corresponding one-parameter family of quaternionic Kähler manifolds $(\bar{N}, g_N^\bar{N})$ arise from the (deformed) c-map applied to $\bar{M}$. Therefore, the following result is a corollary of the previous theorem.

**Theorem 3.2.** Any quaternionic Kähler manifold $(\bar{N}, g_N)$ obtained from the (deformed) c-map construction admits a Hermitian structure $(\bar{g}_N, \tilde{J}_1)$ compatible with the opposite orientation.
Proof of Theorem 3.1. We denote by $D^{1,0}$ and $(D^\perp)^{1,0}$ the $i$-eigenbundles of $\tilde{J}_1$ on $D_C$ and $D^{\perp}_C$, respectively. $D^{0,1}$ and $(D^\perp)^{0,1}$ signify the corresponding $-i$-eigenbundles. The distribution $D^{1,0} = \langle X + iJ_1X \rangle$ is involutive for dimensional reasons. We first prove that $(D^\perp)^{1,0}$ is also involutive.

Let $Y,Z \in \Gamma(D^\perp)$. Then $[Y - iJ_1 Y, Z - iJ_1 Z]$ is of type $(1,0)$ with respect to $J_1$ by integrability of $J_1$. To prove that it is also of type $(1,0)$ with respect to $\tilde{J}_1$ it suffices to check that its projection onto $D^{0,1}$ vanishes, which in turn follows from

$$g_N(X + iJ_1X, [Y - iJ_1 Y, Z - iJ_1 Z]) = 0$$

or, equivalently,

$$[Y, Z] - [J_1Y, J_1 Z] - J_1([J_1Y, Z] + [Y, J_1 Z]) \perp D.$$

Defining the one-forms $\alpha = g_N(X, -)$ and $\beta = g_N(J_1X, -)$, this amounts to showing that $\partial \alpha|_{A^2D}$ and $d\beta|_{A^2D}$ are of type $(1,1)$ with respect to $J_1$. For $\alpha$, this follows by combining the fact that $\mu^X$ is proportional to $(\nabla X)^{sp}$ and the definition of $J_1$ (see (4)) to deduce that $\nabla X$ commutes with $J_1$. To show it for $\beta$, we remark that the vanishing of the Nijenhuis tensor of $J_1$ implies

$$J_1([X, Y] - [J_1X, J_1Y]) = [X, J_1Y] + [J_1X, Y].$$

Therefore

$$\beta([Y, Z] - [J_1Y, J_1 Z]) = -\alpha([Y, J_1 Z] + [J_1Y, Z]).$$

So the claimed property for $\beta$ follows from that for $\alpha$.

To establish the integrability of $\tilde{J}_1$ it now suffices to prove that $X + iJ_1X$ preserves $(D^\perp)^{1,0}$. Note that $X$ and $J_1X$ are real holomorphic vector fields with respect to both $J_1$ and $\tilde{J}_1$. In addition, $X$ is Killing and therefore preserves not only $D$ but also $D^\perp$. As a consequence $X$ preserves the complex distribution $(D^\perp)^{1,0}$. To show that this distribution is also $J_1X$-invariant it suffices to show that $J_1X$ preserves $D^\perp$. This means that we have to verify that $\alpha([J_1X, Y]) = \beta([J_1X, Y]) = 0$ for all $Y \perp D$. Using that $J_1X$ is $J_1$-holomorphic, one can easily see that the second condition actually follows from the first.

Decomposing $\nabla X$ into its part $(\nabla X)^{sp} = \mu^X$ in $Q$, which is proportional to $J_1$, and the part $(\nabla X)^{sp}$ ($n$ being the quaternionic dimension of $\tilde{N}$) in the centralizer of $Q$ we see that (for all $D \perp D$)

$$\alpha([J_1X, Y]) = -d\alpha(J_1X, Y) = -2g_N((\nabla X)^{sp} J_1X, Y) = 2g_N((\nabla X)^{sp} X, J_1Y).$$

Thus, $\tilde{J}_1$ is integrable if and only if $(\nabla X)^{sp} X$ is a section of $D$ or, equivalently, if $\nabla X X$ is. The latter is equivalent to $d(g_N(X, X))|_{(J_1X)^\perp} = 0$, since $Xg_N(X, X) = 0$.

Lemma 3.3. The condition $\nabla Z = \Gamma((Z, I_1Z))$ is equivalent to $d(g_N(X, X))|_{(J_1X)^\perp} = 0$.

Proof. Since $g(\nabla Z, Z) = 0$ by the Killing equation, the condition $\nabla Z \in \Gamma((Z, I_1Z))$ is equivalent to $d(g_N(Z, Z))|_{(J_1Z)^\perp} = 0$. To see that the latter is in turn equivalent to $d(g_N(X, X))|_{(J_1X)^\perp} = 0$ it suffices to go through the HK/QK correspondence.

It follows from (1) that $X$ is the twist of the vector field $-f^\perp_{\mathbb{H}} Z$ on $N$. Since $g_N$ is the twist of $g_H$, it follows that the function $g_N(X, X)$ is the twist of $\frac{1}{\mathbb{H}} g_H(Z, Z)$. Recall that $g_H$ differs from $g_N$ by a rescaling on $\mathbb{H}Z$ and on $\langle \mathbb{H}Z \rangle^\perp$, with factors involving only the
$Z$-invariant functions $f_Z$ and $f_H$ (cf. Equation (2)). Since $d f_Z = -\iota_Z \omega_1$ and $d(g_N(Z, Z))$ vanish on $(I_1 Z)^\perp$, the same holds for $d(\frac{1}{f_H} g_H(Z, Z))$. Now, the fact that the complex structure $J_1$ is the twist of $I_1$ implies that the twist $g_N(X, X)$ of the function $\frac{1}{f_H} g_H(Z, Z)$ satisfies $d(g_N(X, X))|_{(J_1 X)^\perp} = 0$ as well.

This establishes the equivalence between the integrability of $\tilde{J}_1$ and the condition given in the statement of Theorem 3.1.

It was already known that $\tilde{J}_1$ is integrable if the quaternionic Kähler manifold $(\tilde{N}, g_{\tilde{N}})$ arises from the undeformed $c$-map [CDMV15, Thm. 1 (a)], so we may think of Theorem 3.2 as an extension of this result to the one-loop deformed $c$-map. Theorem 3.1 and its proof put it in the natural, broader context of the HK/QK correspondence.

There are a number of reformulations of the integrability condition, some of which will be of use later on:

**Lemma 3.4.** The following conditions are equivalent:

a) $\nabla_Z Z$ is a section of the rank-two distribution $\langle Z, I_1 Z \rangle$.

b) $d(g_N(Z, Z)) \wedge d f_Z = 0$.

c) $d f_H \wedge d f_Z = 0$.

d) There exists a smooth function $\psi$ such that one of the following equivalent identities holds:

   (i) $\nabla_Z Z = \psi I_1 Z$.

   (ii) $I_H Z = (1 + 2\psi) I_1 Z$.

   (iii) $d(g_N(Z, Z)) = 2\psi d f_Z$.

   (iv) $d f_H = (1 + 2\psi) d f_Z$.

**Proof.** First, let us check the equivalence of the conditions listed in d). That (i) and (ii) are equivalent follows from $I_H = I_1 + 2\nabla Z$. For an arbitrary vector field $Y$, the following identity holds:

$$g_N(\nabla_Z Z, Y) = -g_N(\nabla_Y Z, Z) = -\frac{1}{2} d(g_N(Z, Z))(Y).$$

Now (i) holds if and only if $g_N(\nabla_Z Z, Y) = -\psi d f_Z(Y)$, which is to say $d(g_N(Z, Z)) = 2\psi d f_Z$, i.e. condition (iii) holds. Moreover, (iii) $\iff$ (iv) since $f_H = f_Z + g_N(Z, Z)$.

It is obvious that b) and c) are equivalent to d), (iii) and d), (iv), respectively. Finally, the equivalence of a) and d), (i) follows directly from the Killing equation.
4 Examples carrying a third Hermitian structure

Among the simplest examples of PSK manifolds are the complex hyperbolic spaces $\mathbb{C}H^n$. The corresponding CASK manifolds are

$$M_{n+1} := \left\{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \left| |z_0|^2 > \sum_{j=1}^{n} |z_j|^2 \right. \right\}$$

equipped with the flat Kähler structure defined by the Hermitian form

$$h = -dz_0 \otimes d\bar{z}_0 + \sum_{j=1}^{n} dz_j \otimes d\bar{z}_j.$$

The conical structure is induced by the standard $\mathbb{C}^*$-action by complex multiplication. In particular, we have

$$-J\xi = -i \sum_{j=0}^{n} \left( z_j \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \right).$$

Applying the c-map to $\mathbb{C}H^n$, we obtain a one-parameter family of complete quaternionic Kähler manifolds $(\bar{N}_{n+1}, g_{c_{n+1}})$, $c \geq 0$, of dimension $4n + 4$. The undeformed c-map metric yields a a quaternionic Kähler symmetric space, while the deformed c-map metrics are of cohomogeneity one \cite{CST21, CST22}. We denote the corresponding hyper-Kähler manifold by $N_{n+1}$.

**Proposition 4.1.** The endomorphism field $I_H$ (defined in Proposition 2.4) on $N_n$ is an integrable almost complex structure inducing the same orientation as $Q$ if $n$ is even and the opposite orientation if $n$ is odd.

**Proof.** Observe that $N_n \cong M_n \times \mathbb{C}^n$ is nothing but an open subset of a quaternionic vector space, endowed with the standard hyper-Kähler structure of quaternionic signature $(n-1,1)$. In particular, $N_n$ is a product of complex manifolds and $I_1$ is the induced complex structure corresponding to this product structure. Using coordinates $(z_0, \ldots, z_{n-1}, w_0, \ldots, w_{n-1})$ adapted to the product structure, we see that the rotating Killing field on $N_n$ is given by

$$Z = -i \sum_{j=0}^{n-1} \left( z_j \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \right).$$

Decomposing $TN_n \cong TM_n \oplus T\mathbb{C}^n$, one now easily sees that

$$\nabla Z = \begin{pmatrix} -I_1|_{TM_n} & 0 \\ 0 & 0 \end{pmatrix}$$

and correspondingly

$$I_H = I_1 + 2\nabla Z = \begin{pmatrix} -I_1|_{TM_n} & 0 \\ 0 & I_1|_{T\mathbb{C}^n} \end{pmatrix}.$$
Theorem 4.2. For any $n \geq 2$, the quaternionic Kähler manifolds $(\mathcal{N}_n, g^c_n)$, $c \geq 0$, that arise from the $c$-map applied to $\mathbb{C}H^{n-1}$ admit a third complex structure $\tilde{J}$. The associated Hermitian structure $(g^c_n, \tilde{J})$ induces the same orientation as $\mathcal{Q}$ if and only if $n$ is even.

Remark 4.3. In the case $n = 1$, where one applies the $c$-map to a single point, the complex structures $\tilde{J}$ and $\tilde{J}_1$ coincide. In higher dimensions they do not, since $\tilde{J}$ disagrees with $J_1$ by a sign on a half-dimensional distribution while $\tilde{J}_1$ disagrees with $J_1$ on a two-dimensional distribution.

Proof of Theorem 4.2. It should not be surprising that we define $\tilde{J}$ as the twist of the integrable complex structure $I_H$. Since $I_H$ is skew with respect to $g_N$ by Proposition 2.4, the twisting two-form $\omega_H$ is of type $(1, 1)$ with respect to $I_H$. By Lemma 2.3, $\tilde{J}$ is then integrable. Furthermore, $I_H$ is compatible with $g_H$, so $\tilde{J}$ is compatible with its twist $g^c_N$. The claim regarding the induced orientation follows immediately from the corresponding statement for $I_H$, since twisting preserves algebraic properties.

Again, in the case of the undeformed $c$-map, the existence of the Hermitian structure defined by $\tilde{J}$ had already been established in [CDMV15, Thm. 2 (a)]. Theorem 4.2 generalizes it to the deformed $c$-map and provides an interpretation in terms of the HK/QK correspondence.

5 The conformal Kähler property

Proposition 5.1. The members of the one-parameter family $g^c_1$, $c \geq 0$, of quaternionic Kähler metrics on the self-dual Einstein four-manifold $\mathcal{N}_1$ are all conformal to a Kähler metric compatible with the complex structure $\tilde{J}_1$. In other words, the Hermitian structures $(g^c_1, J_1)$, $c \geq 0$, are conformally Kähler.

Proof. We start by considering the hyper-Kähler manifold $\mathcal{N}_1$. Because we are in the four-dimensional case, we have $TN_1 = \mathbb{H}Z$, so that in particular the elementary deformation consists only of a conformal rescaling: $g_H = f_H^Z g_N$. Correspondingly, the quaternionic Kähler metric $g^c_1$, with $c \geq 0$ arbitrary, is a conformal rescaling of the twist of $g_{N_1}$, which we will denote by $h^c$.

Recall that a Hermitian structure is conformally Kähler if and only if the associated fundamental two-form $\sigma$ satisfies $d\sigma = \theta \wedge \sigma$ for an exact one-form $\theta$. We will now show that this is the case for the structure $(h^c, \tilde{J}_1)$. It follows from Remark 4.3 that $\tilde{J}_1$ is the twist of $I_H$, which satisfies $g_{N_1}(I_H - , -) = \omega_H$. This shows that the fundamental two-form $\sigma$ associated with the Hermitian structure $(h^c, \tilde{J}_1)$ is nothing but the twist of $\omega_H$.

We now use Lemma 2.2 and conclude that the exterior derivative of $\sigma$ is the twist of the form

$$d\omega_H - \frac{1}{f_H^Z} i_Z \omega_H \wedge \omega_H = d(\log f_H) \wedge \omega_H.$$ 

Now write $\psi$ for the twist of the $Z$-invariant function $\log f_H$. Then, again by Lemma 2.2, $d\psi$ is the twist of $d \log f_H$ and we conclude that

$$d\sigma = d\psi \wedge \sigma.$$
Since $h^c$ and $g^c_1$ are conformally equivalent, this proves that the Hermitian structure $(g^c_1, \tilde{J}_1)$ is conformally Kähler.

**Remark 5.2.** In dimensions $4n$, $n \geq 2$, the hyper-Kähler metric $g_{N_n}$ is indefinite of signature $(4n - 4, 4)$ and therefore so is its twist $h^c$. The above proof still shows that $h^c$ is conformal to a pseudo-Kähler metric, but we can no longer use this to draw the same conclusion about $g^c_n$ since the relationship between $g_{N_n}$ and $g_H$ is more complicated for $n > 1$ (cf. Equation (2)).

The quaternionic Kähler four-manifolds $(\tilde{N}_1, g^c_1)$, $c \geq 0$, are examples of half conformally-flat Einstein four-manifolds that admit a principal isometric action of a three-dimensional Heisenberg group. These have recently been classified in arbitrary signature [CM21]. It turns out that such manifolds are always conformally Kähler or conformally para-Kähler, in accordance with Proposition 5.1.

In light of the above, it is natural to ask whether every doubly integrable HK/QK metric is conformal to a Kähler metric compatible with the integrable almost complex structure $\tilde{J}_1$. In the rest of this section, we shall show that this is indeed true in dimension four but false in higher dimensions.

As earlier, our main tool is the twist interpretation of the HK/QK correspondence and our first task is to interpret the conformal Kähler condition on the doubly integrable HK/QK manifold in terms of data on the dual pseudo-hyper-Kähler manifold $(N, g_N, I_1, I_2, I_3)$ endowed with a rotating Killing field $Z$, which we know satisfies $\nabla_Z Z = \psi I_1 Z$ for some $\psi \in C^\infty(N)$, thanks to Theorem 3.1.

Denoting the fundamental two-form associated to the Hermitian structure $(\tilde{g}_N, \tilde{J}_1)$ by $\tilde{\tau}$, the condition that $(\tilde{g}_N, \tilde{J}_1)$ is conformally Kähler means that there exists a smooth function $\varphi \in C^\infty(\tilde{N})$ such that $d\tilde{\tau} = -d\varphi \wedge \tilde{\tau}$. Since $\tilde{\tau}$ is non-degenerate, the fact that $\tilde{\tau}$ is invariant under the canonical Killing field $X$ implies that the same holds for $d\varphi$.

This does not, in full generality, imply that $\varphi$ itself is invariant, but it is true under the weak assumption that the action generated by $X$ has at least one closed orbit. Indeed, since $L_X d\varphi = d(L_X \varphi) = 0$, we know that $L_X \varphi$ is a constant (assuming that $M$ is connected) and therefore the restriction of $\varphi$ to any orbit is an affine function of the parameter along this orbit. In case of a closed orbit, this forces $\varphi$ to be constant, so $L_X \varphi = 0$ on this orbit and thus everywhere. Since in our setting the canonical vector field $X$ on $\tilde{N}$ is induced by the fundamental vector field $X_P$ of the principal $S^1$-bundle $P \to N$, all of its orbits are closed. Therefore $\varphi$ (and not only $d\varphi$) is invariant under $X$.

Using the invariant function $\varphi$, we can pass to its twist $\phi$ on the dual pseudo-hyper-Kähler manifold $N$. It follows from Lemma 2.2 that $d\phi$ is the twist of $d\varphi$; denoting the twist of $\tilde{\tau}$ by $\tilde{\sigma}$, Lemma 2.2 moreover shows that the conformal Kähler condition on $\tilde{N}$ is equivalent to the equation

$$d\phi \wedge \tilde{\sigma} + d\tilde{\sigma} = \frac{1}{\tilde{f}_H} \omega_H \wedge \iota_Z \tilde{\sigma}$$

(5)

on $N$. To make further progress, it will be helpful to have explicit expressions at hand for $\tilde{\sigma}$ and its exterior derivative. We introduce some convenient notation for this purpose.
Notation. Set $I_0 = \text{id}_{TN}$ and $\omega_0 = g_N$, and define $\alpha_\mu = \omega_\mu(Z, -)$, where $\mu = 0, 1, 2, 3$. Introducing the orthogonal projection operator

$$\mathcal{P}_{HZ} = \frac{1}{g_N(Z, Z)} \sum_{\mu=0}^3 \alpha_\mu \otimes I_\mu Z : \mathfrak{X}(N) \to \Gamma(\mathbb{H}Z)$$

and $\mathcal{P}_{(HZ)^\perp} = 1 - \mathcal{P}_{HZ}$, we define

$$(\omega_1)_{HZ} = \omega_1(\mathcal{P}_{HZ}, \mathcal{P}_{HZ}),$$

$$(\omega_1)_{(HZ)^\perp} = \omega_1(\mathcal{P}_{(HZ)^\perp}, \mathcal{P}_{(HZ)^\perp}).$$

Note that, since $I_1$ preserves $\mathbb{H}Z$, $\omega_1 = (\omega_1)_{HZ} + (\omega_1)_{(HZ)^\perp}$.

Lemma 5.3. The two-form $\bar{\sigma}$ and its exterior derivative are given by the following expressions:

$$\bar{\sigma} = \frac{f_H}{f_Z^2 g_N(Z, Z)} (- \alpha_0 \wedge \alpha_1 + \alpha_2 \wedge \alpha_3) + \frac{1}{f_Z} (\omega_1)_{(HZ)^\perp}$$

where $\omega_0 = g$ and $\alpha_\mu = \omega_\mu(Z, -)$ for $j = 0, 1, 2, 3$, and

$$d\bar{\sigma} = \frac{f_H}{f_Z^2 g_N(Z, Z)} \left( \left( - \frac{1 + 2\psi}{f_H} + \frac{2\psi}{f_H - f_Z} \right) \alpha_2 \wedge \alpha_3 - d\alpha_0 \right) \wedge \alpha_1$$

$$+ \frac{f_H}{f_Z^2 g_N(Z, Z)} \left( \alpha_2 \wedge \omega_2 + \alpha_3 \wedge \omega_3 \right) + \frac{1}{f_Z} \alpha_1 \wedge (\omega_1)_{(HZ)^\perp} + \frac{1}{f_Z} d(\omega_1)_{(HZ)^\perp}$$

where $\nabla_Z \omega = \psi I_1 Z$.

Proof. We start from

$$\omega_1 = (\omega_1)_{HZ} + (\omega_1)_{(HZ)^\perp} = \frac{1}{g_N(Z, Z)} (\alpha_0 \wedge \alpha_1 + \alpha_2 \wedge \alpha_3) + (\omega_1)_{(HZ)^\perp}$$

From here, we can obtain $\bar{\sigma} = g_H(\bar{I}_1, -)$ by elementary deformation and subsequently flipping the sign on the term that evaluates non-trivially on the distribution $\langle Z, I_1 Z \rangle$. This immediately yields the claimed expression for $\bar{\sigma}$.

Now we compute its exterior derivative.

$$d\bar{\sigma} = \frac{f_H}{f_Z^2 g_N(Z, Z)} \left( \frac{df_H}{f_H} - \frac{2}{f_Z} df_Z - \frac{1}{g_N(Z, Z)} d(g_N(Z, Z)) \right) \wedge (- \alpha_0 \wedge \alpha_1 + \alpha_2 \wedge \alpha_3)$$

$$+ \frac{f_H}{f_Z^2 g_N(Z, Z)} \left( - d(\alpha_0 \wedge \alpha_1) + d(\alpha_2 \wedge \alpha_3) \right) - \frac{1}{f_Z} df_Z \wedge (\omega_1)_{(HZ)^\perp} + \frac{1}{f_Z} d(\omega_1)_{(HZ)^\perp}.$$

From $df_Z = -\alpha_1$ and Lemma 3.4, we know that $df_Z \wedge \alpha_1$ and $d\alpha_1$ vanish while $df_H = (1 + 2\psi) df_Z$ and $d(g_N(Z, Z)) = 2\psi df_Z$. This yields

$$d\bar{\sigma} = \frac{f_H}{f_Z^2 g_N(Z, Z)} \left( \left( - \frac{1 + 2\psi}{f_H} + \frac{2\psi}{f_H - f_Z} \right) \alpha_2 \wedge \alpha_3 - d\alpha_0 \right) \wedge \alpha_1$$

$$+ \frac{f_H}{f_Z^2 g_N(Z, Z)} d(\alpha_2 \wedge \alpha_3) + \frac{1}{f_Z} \alpha_1 \wedge (\omega_1)_{(HZ)^\perp} + \frac{1}{f_Z} d(\omega_1)_{(HZ)^\perp}.$$

The fact that $Z$ is a rotating Killing field furthermore implies $d\alpha_2 = \omega_3$ and $d\alpha_3 = -\omega_2$. The claimed expression for $d\bar{\sigma}$ is directly obtained from these substitutions. □
Lemma 5.4. Assume there exists a smooth function \( \varphi \in C^\infty(\mathbb{N}) \), invariant under the canonical Killing field on \( \mathbb{N} \), such that \( (e^\varphi g_N, \tilde{J}_1) \) is a Kähler structure. Then, if \( \phi \) denotes the twist of \( \varphi \), there exists a smooth function \( \xi \) on \( \mathbb{N} \) such that \( d\phi = \xi df_Z \).

Proof. Recall (5), which shows that the conformal Kähler condition is equivalent to the equation

\[
d\phi \wedge \tilde{\sigma} + d\tilde{\sigma} = f_Z^{-2} \omega_H \wedge df_Z
\]

where we used that \( \iota_Z \tilde{\sigma} = \frac{f_H}{f_Z} df_Z \) by Lemma 5.3.

Contracting with \( Z \), making use of the \( Z \)-invariance of \( \phi \) and recalling that \( \iota_Z \omega_H = -dH \), we obtain

\[
d\iota_Z \tilde{\sigma} = d\left( \frac{f_H}{f_Z} \right) \wedge df_Z = \frac{1}{f_Z} (df_H - f_H \phi) \wedge df_Z.
\]

Since \( df_H \wedge df_Z = 0 \) (by the integrability of \( \tilde{J}_1 \), cf. Lemma 3.4) and \( f_H \) is nowhere-vanishing, this is equivalent to \( d\phi \wedge df_Z = 0 \) or in other words \( d\phi = \xi df_Z \) for some function \( \xi \).

\[\square\]

Theorem 5.5. Let \( (\mathbb{N}, g_N, Q) \) be a doubly integrable HK/QK four-fold. Then \( g_N \) is conformal to a Kähler metric which is compatible with \( \tilde{J}_1 \) and invariant under the canonical Killing field on \( \mathbb{N} \).

Proof. In the four-dimensional case, \( TN = \mathbb{H}Z \). The assumption \( \nabla_Z Z = \psi I_1 Z \), together with the fact that \( I_H \) commutes with the complex structures \( I_k, k = 1, 2, 3 \), then implies that \( I_H = -(1 + 2\psi)I_1 \) (this generalizes Remark 4.3, which corresponds to the case \( \psi = -1 \)). Contracting with \( g_N \), which in the four-dimensional case equals \( \frac{f_Z^2}{f_H} g_N \), we see that this is equivalent to

\[
g_N(I_H -, -) = \omega_H = -\frac{f_Z^2}{f_H} (1 + 2\psi) \tilde{\sigma}.
\]

Now substituting (7) into (6), writing \( d\phi = \xi df_Z \) (cf. Lemma 5.4) and using the above observation, we get

\[
d\tilde{\sigma} = -(\xi + \frac{f_H}{f_Z} (1 + 2\psi)) df_Z \wedge \tilde{\sigma}.
\]

We can obtain another expression for \( d\tilde{\sigma} \) by either restricting the result of Lemma 5.3 to dimension four or by realizing that, in dimension four, we have

\[
\tilde{\sigma} = \frac{f_H}{f_Z} \omega_1 - \frac{2 f_H}{f_Z} \frac{df_Z \wedge \alpha_0}{f_H - f_Z}.
\]

Taking the exterior derivative and once again applying \( df_H \wedge df_Z = 0 \), as well as the identity \( d\alpha_0 = \omega_H - \omega_1 \), this yields:

\[
d\tilde{\sigma} = \frac{f_H}{f_Z} \left( \left( \frac{df_H}{f_H} - \frac{2 df_Z}{f_Z} \right) \wedge \omega_1 + 2 \frac{df_Z \wedge (\omega_H - \omega_1)}{f_H - f_Z} \right)
\]

\[
= \frac{f_H}{f_Z} \left( \left( \frac{df_H}{f_H} - \frac{2 df_Z}{f_Z} - 2 \frac{df_Z}{f_H - f_Z} \right) \wedge \omega_1 + 2 \frac{df_Z \wedge \omega_H}{f_H - f_Z} \right)
\]

\[
= \left( \frac{1 + 2\psi}{f_H} - \frac{2}{f_Z} - \frac{4 + 4\psi}{f_H - f_Z} \right) df_Z \wedge \tilde{\sigma}
\]

where, in the last step, we used \( df_Z \wedge \omega_1 = \frac{f_Z^2}{f_H} df_Z \wedge \tilde{\sigma} \) and (7), as well as the fact that \( df_H = (1 + 2\psi) df_Z \) by Lemma 3.4.
Comparing the two expressions for $d\tilde{\sigma}$ yields an equation which we can solve for $\xi$:

$$
\xi = \frac{2}{f_Z} - \frac{2 + 4\psi}{f_H} + \frac{4 + 4\psi}{f_H - f_Z}.
$$

(8)

Integrating $d\phi = \xi df_Z$ now yields $\phi$ and, after twisting, the conformal factor required to obtain a Kähler metric.

The obvious next question is whether this phenomenon persists in higher dimensions. The following result shows that this is never the case:

**Theorem 5.6.** Let $(\bar{N}, g_{\bar{N}}, Q)$ be a doubly integrable HK/QK manifold of dimension $\dim \bar{N} > 4$. Then there is no function $\varphi$ invariant under the canonical Killing field on $\bar{N}$ such that $(e^\varphi g_{\bar{N}}, \bar{J}_1)$ is Kähler.

**Proof.** Assume that $(e^\varphi g_{\bar{N}}, \bar{J}_1)$ is Kähler for some function $\varphi$ invariant under the canonical Killing field on $\bar{N}$. Then by Lemma 5.4 we know that $d\phi = \xi df_Z$ for a smooth function $\xi$. By contracting equation (6) with a triple of vector fields, we may derive an expression for $\xi$. Let us consider any section $V \in \Gamma((\mathbb{H}Z)^\perp)$ with $g_N(V, V) \neq 0$ and evaluate on the triple $(I_1Z, V, f_ZI_1V)$. Since $(df_Z \wedge \beta)(I_1Z, A, B) = -g_N(Z, Z)\beta(A, B)$ for any two-form $\beta$ and any two vector fields $A, B$ perpendicular to $I_1Z$, we have to compute $\tilde{\sigma}(V, f_ZI_1V)$ and $\varpi_H(V, f_ZI_1V)$. They are given by

$$
\tilde{\sigma}(V, f_ZI_1V) = g_N(V, V),
$$

$$
\varpi_H(V, f_ZI_1V) = f_Z(\omega_1(V, I_1V) + d\alpha_0(V, I_1V)) = f_Z(g_N(V, V) + d\alpha_0(V, I_1V)).
$$

The final term that needs to be computed is

$$
d\tilde{\sigma}(I_1Z, V, f_ZI_1V) = -\frac{f_H}{f_Z} d\alpha_0(V, I_1V) + \frac{1}{f_Z} g_N(Z, Z) g_N(V, V) + d(\omega_1)((\mathbb{H}Z)^\perp)(I_1Z, V, I_1V).
$$

Applying these identities, (6) yields

$$
\xi = \frac{1}{f_Z} \left( 1 + \frac{d\alpha_0(V, I_1V)}{g_N(V, V)} \right) - \frac{f_H}{f_Z g_N(Z, Z)} \frac{d\alpha_0(V, I_1V)}{g_N(V, V)}
$$

$$
+ \frac{1}{f_Z} + \frac{1}{g_N(Z, Z)} \frac{d(\omega_1)((\mathbb{H}Z)^\perp)(I_1Z, V, I_1V)}{g_N(V, V)}
$$

$$
= \frac{2}{f_Z} - \frac{2}{g_N(Z, Z)} \frac{g_N(\nabla_{VZ}, I_1V)}{g_N(V, V)} + \frac{1}{g_N(Z, Z)} \frac{d(\omega_1)((\mathbb{H}Z)^\perp)(I_1Z, V, I_1V)}{g_N(V, V)},
$$

where we have used $d\alpha_0 = 2g_N(\nabla Z, -)$ and $f_H = g_N(Z, Z) + f_Z$.

We can further simplify this expression by rewriting the final term. Note that, since $\nu_{I_1Z}(\omega_1)((\mathbb{H}Z)^\perp) = 0$, we have

$$
d(\omega_1)((\mathbb{H}Z)^\perp)(I_1Z, V, I_1V) = (I_1Z)(g_N(V, V)) - g_N([I_1Z, V], V) - g_N([I_1Z, I_1V], I_1V)
$$

$$
= 2g_N(\nabla_{I_1Z}V, V) - g_N([I_1Z, V], V) - g_N([I_1Z, I_1V], I_1V).
$$

Now, we use torsion-freeness and the fact that $\nabla I_1 = 0$ to find

$$
d(\omega_1)((\mathbb{H}Z)^\perp)(I_1Z, V, I_1V)
$$

$$
= 2g_N(\nabla_{I_1Z}V, V) - g_N(\nabla_{I_1Z}V - I_1\nabla Z, V) - g_N(I_1\nabla_{I_1Z}V - I_1\nabla_{I_1V}Z, I_1V)
$$

$$
= g_N(I_1\nabla Z, V) + g_N(\nabla_{I_1V}Z, V)
$$

$$
= 2g_N(I_1\nabla Z, V),
$$
where the final step follows from the Killing equation for \( Z \).

In conclusion, we have shown that
\[
\xi = \frac{2}{f_Z} + \frac{4}{g_N(Z, Z)} \frac{g_N(I_1 \nabla_V Z, V)}{g_N(V, V)}.
\]

The fact that \( \xi \) does not depend on our choice of \( V \) now imposes the existence of a smooth function \( \lambda \) such that
\[
g_N(I_1 \nabla_V Z, V) = -\lambda \cdot g_N(V, V).
\]

Next, observe that the endomorphism field \( I_1 \circ \nabla Z \) is symmetric with respect to \( g_N \). Indeed, both \( I_1 \) and \( \nabla Z \) are skew-symmetric with respect to \( g_N \) – for the latter, this is the Killing equation and furthermore commute since \([I_1, \nabla Z] = \frac{1}{2}[I_1, I_H - I] = 0\) by Proposition 2.4. It now follows from the polarisation identity that
\[
g_N(I_1 \nabla_V Z, W) = -\lambda \cdot g_N(V, W)
\]
for any other \( W \in \Gamma((\mathbb{H}Z)^\perp) \). This shows that \( I_1 \circ \nabla Z |_{(\mathbb{H}Z)^\perp} = -\lambda \, \mathrm{id}_{(\mathbb{H}Z)^\perp} \) or, equivalently, \( \nabla Z |_{(\mathbb{H}Z)^\perp} = \lambda I_1 |_{(\mathbb{H}Z)^\perp} \). Since \( I_H = I_1 + 2\nabla Z \), we deduce that
\[
I_H |_{(\mathbb{H}Z)^\perp} = (1 + 2\lambda) I_1 |_{(\mathbb{H}Z)^\perp}.
\]

To avoid running afoul of Proposition 2.4, which guarantees that \( I_H \) commutes with \( I_2 \) and \( I_3 \), we must have \( \lambda \equiv -\frac{1}{2} \), so in particular
\[
\xi = \frac{2}{f_Z} + \frac{2}{g_N(Z, Z)} = \frac{2}{f_Z} + \frac{2}{f_H - f_Z}
\]
and \( I_H |_{(\mathbb{H}Z)^\perp} = 0 \). Meanwhile since \( I_H \) commutes with all the \( I_i \) and we know its action on \( Z \), the restriction \( I_H |_{\mathbb{H}Z} \) must be \( -(1 + 2\psi) I_1 |_{\mathbb{H}Z} \). This gives us
\[
\omega_H = -\frac{1 + 2\psi}{g_N(Z, Z)} (-\alpha_0 \wedge \alpha_1 + \alpha_2 \wedge \alpha_3).
\]

Now, evaluating the exterior derivative of \( \omega_H \) on \( I_2 Z, V, I_2 V \), where \( V \) is, as earlier, a section \( V \in \Gamma((\mathbb{H}Z)^\perp) \) with \( g_N(V, V) \neq 0 \), we get
\[
d\omega_H(I_2 Z, V, I_2 V) = \frac{1 + 2\psi}{g_N(Z, Z)} \alpha_2(I_2 Z) d\alpha_3(V, I_2 V)
= -(1 + 2\psi) \omega_2(V, I_2 V) = -(1 + 2\psi) g_N(V, V).
\]

Here we have used that \( d\alpha_3 = -\omega_2 \), as in the proof of Lemma 5.3. Since \( \omega_H \) is closed, the above must vanish, giving us \( 1 + 2\psi = 0 \). So, \( \omega_H \) and \( I_H \) are zero, implying \( \nabla Z = -\frac{1}{2} I_1 \) and that \( f_H \) is constant (since \( df_H = -i \omega_H = 0 \)). Thus, we may explicitly solve \( d\phi = \xi \, df_Z \) to obtain
\[
e^\phi = \frac{C f_Z^2}{(f_H - f_Z)^2},
\]
for some positive constant \( C \). Meanwhile (6) just becomes the statement that \( e^\phi \tilde{\sigma} \) is closed. We shall now verify using our explicit expressions for \( e^\phi \) and \( \tilde{\sigma} \) that this is not the case.

First of all, we write \( \tilde{\sigma} \) as
\[
\tilde{\sigma} = \frac{1}{f_Z} \left( \frac{f_H}{f_Z (f_H - f_Z)} (-\alpha_0 \wedge \alpha_1 + \alpha_2 \wedge \alpha_3) + (\omega_1)_{(\mathbb{H}Z)^\perp} \right)
= \frac{1}{f_Z} \left( -\frac{f_H + f_Z}{f_H - f_Z} \alpha_0 \wedge \alpha_1 + \alpha_2 \wedge \alpha_3 \right) + \frac{1}{f_Z} \omega_1.
\]
We see that evaluating the exterior derivative of \( e^\phi \tilde{\sigma} \) on \( I_2 Z, V, I_2 V \) gives

\[
d(e^\phi \tilde{\sigma})(I_2 Z, V, I_2 V) = -\frac{C}{(f_H - f_Z)^2} \alpha_2(I_2 Z) d\alpha_3(V, I_2 V)
\]

\[
= \frac{C \omega_2(V, I_2 V)}{f_H - f_Z} = \frac{C g_N(V, V)}{f_H - f_Z} \neq 0.
\]

Thus \( d(e^\phi \tilde{\sigma}) \neq 0 \) and our result follows. \( \square \)

6 Four-dimensional examples beyond the \( c \)-map

6.1 Local classification from the Przanowski–Tod ansatz

The one-parameter family \((\tilde{N}_1, g^1)\) is known as the (one-loop) deformed universal hypermultiplet in the physics literature and its members are the only four-dimensional quaternionic Kähler manifolds that arise from the \( c \)-map (namely, by applying it to a point). Mathematically, \((\tilde{N}_1, g^1)\) is a one-parameter deformation of the Hermitian symmetric space \( \mathbb{C}H^2 \cong (\tilde{N}_1, g^0) \). In this section, we construct further examples belonging to the class described in Theorem 3.1. In order to do so, we widen our scope to arbitrary quaternionic Kähler four-manifolds endowed with a Killing field.

Via HK/QK correspondence, any quaternionic Kähler manifold of dimension four equipped with a Killing field (locally) arises from a hyper-Kähler four-manifold \((N, g_N)\) with rotating Killing field. Such a hyper-Kähler metric is locally homothetic to a metric satisfying the Boyer–Finley ansatz \([BF82]\). With respect to local coordinates \((\rho, x, y, t)\), this means that

\[
g_N = K(\partial_\rho u)(d\rho^2 + 2e^u(dx^2 + dy^2)) + \frac{4K}{\partial_\rho u}
\]

\[
\left(d\rho - \frac{1}{2}(\partial_y u dx - \partial_x u dy)\right)^2
\]

with \( u \) a smooth function of \( \rho, x, y \) (but not \( t \)) satisfying the continuous Toda equation

\[
(\partial_x^2 + \partial_y^2)u = -2\partial_\rho^2(e^u).
\]

The rotating Killing vector field is \( \partial_t \) and the invariant Kähler form is

\[
\omega_1 = 2K d\rho \wedge \left(d\rho - \frac{1}{2}(\partial_y u dx - \partial_x u dy)\right) 2Ke^{\Theta} \partial_\rho dx \wedge dy.
\]

The Killing vector field \( \partial_t \) is Hamiltonian. Without loss of generality, the Hamiltonian function can be taken to be \( f_Z = 2K \rho \) (shifts in \( \rho \) can be adsorbed into a redefinition of the function \( u \)).

The result of applying the HK/QK correspondence to the Boyer–Finley ansatz is the Przanowski–Tod ansatz \([Prz91]\). Up to constant scaling, any four-dimensional quaternionic Kähler metric admitting a Killing field is locally given by

\[
g_N = \frac{1}{4\rho^2}
\]

\[
\left(P d\rho^2 + 2Pe^u(dx^2 + dy^2) + \frac{1}{P}(dt + \Theta)^2\right)
\]

where \( P \) is a \( \partial_\rho \)-invariant smooth function and \( \Theta \) is a \( \partial_t \)-invariant 1-form satisfying

\[
P = K(\rho \partial_\rho u - 2) > 0, \quad d\Theta = (\partial_y P dx - \partial_x P dy) \wedge d\rho - 2\partial_\rho(Pe^u) dx \wedge dy.
\]
We have normalized \( g_\bar{N} \) so that the reduced scalar curvature \( \nu \) equals \( \frac{2}{K} \). In particular, since \( P \) is required to be positive, the function \( \rho \partial_\rho u - 2 \) must have a definite sign over the domain of definition. This sign is furthermore equal to the sign of \( K \) and hence the sign of the scalar curvature.

Note that if we let \( \zeta = x + iy \) and \( f \) be a holomorphic function of \( \zeta \) whose derivative is non-vanishing in the domain of definition, then the replacement

\[
 u(\rho, \zeta, \bar{\zeta}) \mapsto u(\rho, f(\zeta), \bar{f}(\bar{\zeta})) - \ln \left| \frac{df}{d\zeta} \right|^2
\]

doesn’t change the isometry class of the metrics \( g_N \) and \( g_\bar{N} \). Such a replacement can therefore be regarded as a gauge transformation.

Now we can explicitly which of such quaternionic Kähler metrics are conformal to a Kähler metric compatible with \( \tilde{J}_1 \) in terms of the function \( u \) up to gauge transformations.

**Theorem 6.1.** A quaternionic Kähler four-manifold which arises from the HK/QK correspondence has the structure \( \tilde{J}_1 \) integrable if and only if it is locally isometric to a metric of the form

\[
 g^{a,b,c} = \frac{K}{4\rho^2} \left( \frac{bp + 2c}{a\rho^2 + bp + c} \, d\rho^2 + \frac{2(bp + 2c)|d\zeta|^2}{(1 + \frac{a}{2}|\zeta|^2)^2} \right) + \frac{\rho^2 + bp + c}{bp + 2c} \left( \frac{dt}{K} + \frac{b \text{Im}(\zeta \, d\bar{\zeta})}{1 + \frac{a}{2}|\zeta|^2} \right)^2
\]

where \( a, b, c \) are real constants, and \( \rho \) and \( \zeta \) satisfy

\[
 \rho > 0, \quad a\rho^2 + bp + c > 0, \quad bp + 2c \neq 0, \quad 1 + \frac{a}{2}|\zeta|^2 > 0
\]

and \( K \) is a non-zero constant for which the metric is positive-definite. These metrics arise by applying the Przanowski–Tod ansatz to

\[
 e^u = \frac{a\rho^2 + bp + c}{(1 + \frac{a}{2}|\zeta|^2)^2}
\]

up to gauge transformation.

**Proof.** Any quaternionic Kähler four-manifold which arises from the HK/QK correspondence is locally homothetic to the Przanowski–Tod ansatz, so we restrict to this case. By Theorem 5.5 and Lemma 3.4, it suffices to show that \( d(g_N(Z, Z)) \wedge df_Z = 0 \) if and only if (13) holds. The Hamiltonian \( f_Z \) is \( 4K\rho \) in our case while \( g_N(Z, Z) \) can be seen to be \( 4K/\partial_\rho u \). So, the condition above amounts to saying that the differential of \( \partial_\rho u \) is proportional to \( d\rho \), which is to say \( u \) is a separable function

\[
 u = F(\rho) + G(\zeta, \bar{\zeta}).
\]

Substituting this into the continuous Toda equation, we obtain

\[
 \partial_\zeta \partial_{\bar{\zeta}} G = -ae^G, \quad \partial_\rho^2 e^F = 2a.
\]

The equation for \( F \) implies that \( e^F \) is a quadratic polynomial in \( \rho \) with leading coefficient \( a \). That is, its general solutions are given by

\[
 e^F = a\rho^2 + bp + c.
\]
Note in particular that the function \( \rho \) is nowhere vanishing and so has a definite sign if we assume connectedness. Since the sign can always be absorbed into the constant \( b \), we can assume without loss of generality \( \rho > 0 \).

Meanwhile, the equation for \( G \) is just the 2d Liouville equation whose general solutions are known to be of the form

\[
e^G = \frac{4}{(1 + 2a|f(\zeta)|^2)^2} \left| \frac{df}{d\zeta} \right|^2
\]

where \( f \) is some holomorphic function which is nonvanishing in the domain of definition. The freedom to choose \( f \) may be absorbed into the gauge transformation \((10)\). In particular, we may set \( f(\zeta) = \frac{1}{2} \zeta \). This gives us the required metric. Moreover, we may additionally assume without any loss of generality that \( 1 + \frac{a}{2} |\zeta|^2 > 0 \). To see this, note that when \( a \geq 0 \), this is automatic, and when \( a < 0 \), the replacement \( \zeta \mapsto \frac{2}{a} \zeta \) leaves \( e^G \) (and hence the metric) unchanged but swaps the regions \( \pm (1 + \frac{a}{2} |\zeta|^2) > 0 \).

**Remark 6.2.** As explained earlier, the sign of the scalar curvature of a metric satisfying the Przanowski–Tod ansatz is equal to the sign of the (definite) function \( \rho \partial \rho u - \frac{2}{\rho^2 + b \rho + c} \), which in the case at hand satisfies \( \rho \partial \rho u - \frac{2}{\rho^2 + b \rho + c} = -\frac{b \rho + 2c}{\rho^2 + b \rho + c} \). The denominator, being equal to \( e^F \) in \((15)\), is positive, so the sign of the scalar curvature of \( g^{a,b,c} \) equals the sign of the function \(- (b \rho + 2c)\).

### 6.2 Case-by-case analysis

The metrics \( g^{a,b,c} \) have previously been studied by Ketov [Ket01]. There are many ranges for the parameters in which we recover well-known examples of quaternionic Kähler four-manifolds, as can be seen from the following case-by-case analysis of the metrics.

(i) When \( a = b = 0 \) we get the real hyperbolic 4-space \( \text{SO}(4,1)/\text{O}(4) \) presented as solvable Lie group \( \mathbb{R}_{>0} \times \mathbb{R}^3 \) (the Iwasawa subgroup of \( \text{SO}_0(4,1) \)) with a left-invariant metric.

(ii) For \( a = 0, b \neq 0 \), the metrics \( g^{a,b,c} \) are isometric to the (one-loop) deformed universal hypermultiplet metrics, i.e. yield a one-parameter deformation of the complex hyperbolic plane \( \text{SU}(2,1)/\text{U}(2) \) \((c = 0)\) through quaternionic Kähler metrics. For \( c \neq 0 \), the full isometry group of the deformed universal hypermultiplet is known to be \( \text{O}(2) \times \text{Heis}_3 \), acting with cohomogeneity one [CST21].

(iii) For \( a, b > 0 \), we may perform the following change of coordinates:

\[
\rho = \frac{b}{2a} \left( \frac{1}{\xi^2} - 1 \right), \quad \zeta = \sqrt{\frac{2}{a}} \xi, \quad t = \frac{bK}{a} \theta
\]

where \( 0 < \rho < 1 \). In these coordinates, after making the identification \( \theta \sim \theta + 2\pi \) and smoothly extending to \( \zeta = \infty \) and \( \rho = 0 \), the metric \( g^{a,b,c} \) becomes the Pedersen metric [Ped86] on the open unit ball in \( \mathbb{R}^4 \):

\[
g^{a,b,c} = \frac{-1}{\nu(1 - \rho^2)^2} \left( \frac{1 + k \rho^2}{1 + k \rho^4} d\rho^2 + \rho^2(1 + k \rho^2)(s_1^2 + s_2^2) + \frac{\rho^2 (1 + k \rho^4)}{1 + k \rho^2} s_3^2 \right).
\]
Here, \( k \) is given by \( k = \frac{4ac}{d^2} - 1 \). The Pedersen metric is known to be invariant under the standard action of \( U(2) \) on \( S^3 \), as confirmed by the appearance of the \( SU(2) \)-invariant 1-forms \( \xi_1, \xi_2, \xi_3 \), which are given by

\[
\begin{align*}
\xi_1 &= \frac{\text{Re}(e^{i\theta}d\xi)}{1 + |\xi|^2}, & \xi_2 &= \frac{\text{Im}(e^{i\theta}d\xi)}{1 + |\xi|^2}, & \xi_3 &= \frac{1}{4} d\theta + \frac{\text{Im}(\xi d\bar{\xi})}{1 + |\xi|^2}
\end{align*}
\]

with respect to the Hopf parametrisation. The combination in which they appear in the metric ensures that it is even \( U(2) \)-invariant.

(iv) As a limiting case of the above, we have \( a > 0 \) and \( b = 0 \). If \( c > 0 \), we carry out a change of coordinate \( \rho = \frac{1}{\rho'} \sqrt{\frac{a}{b}} \) to obtain the following metric on \( \mathbb{R} \times \mathbb{R}_{>0} \times S^2 \cong \mathbb{R} \times (\mathbb{R}^3 \setminus \{0\}) \subset \mathbb{R}^4 \):

\[
g^{a,b,c} = -\frac{1}{\nu} \left( \frac{d\rho^2}{1 + \rho^2} + (1 + \rho'^2) \left( \frac{dt}{2K} \right)^2 + \rho'^2 g_{S^2} \right)
\]

where \( g_{S^2} \) is the standard metric on a unit \( S^2 \). By including a line parametrised by \( t \) at \( \rho' = 0 \), this can be extended to the real hyperbolic 4-space metric on \( \mathbb{R}^4 \).

On the other hand, if \( c > 0 \), we carry out a change of coordinate \( \rho = \frac{1}{\rho'} \sqrt{\frac{a}{b}} \) to obtain the following metric on \( \mathbb{R} \times (0,1) \times S^2 \subset \mathbb{R}^4 \):

\[
g^{a,b,c} = -\frac{1}{\nu} \left( \frac{d\rho^2}{1 - \rho^2} + (1 - \rho'^2) \left( \frac{dt}{2K} \right)^2 + \rho'^2 g_{S^2} \right).
\]

Upon making the identification \( t \sim t + 4\pi K \), the first two terms give the standard \( S^2 \) metric restricted to a hemisphere minus the pole, with \( \sin^{-1}(\rho') \) and \( \frac{1}{\rho'} \) being the latitude and longitude respectively. With this identification, the full metric \( g^{a,b,c} \) can be extended to \( S^4 \) to give the standard \( O(5) \)-invariant metric on it.

(v) When \( b < 0 < a \) and \( 2\alpha \rho + b < 0 \), the same change of coordinates as in case (iii) can be carried out, but now we have \( \varrho > 1 \), as can be seen from

\[
\varrho^2 = \frac{b}{2a\rho + b} = \frac{|b|}{|b| - 2a\rho} > 1.
\]

We are therefore dealing with a metric on the complement of the closed unit ball in \( \mathbb{R}^4 \). The scalar curvature remains negative since

\[
-(b\rho + 2c) = (2a\rho + b)\rho - 2(a\rho^2 + b\rho + c) < 0.
\]

This does not change the fact that the metric is \( U(2) \)-invariant, since it is given by the same formal expression as the Pedersen metric.

(vi) When \( b < 0 < a \) and \( 2\alpha \rho + b > 0 \), we instead set

\[
\rho = -\frac{b}{2a} \left( \frac{1}{\varrho^2} + 1 \right), \quad \zeta = \sqrt{\frac{2}{a}} \xi, \quad t = \frac{bK}{a} \theta.
\]

The metric now becomes

\[
g^{a,b,c} = \frac{1}{\nu(1 + \varrho^2)^2} \left( \frac{1 - k\varrho^2}{1 + k\varrho^4} d\varrho^2 + \varrho^2 (1 - k\varrho^2)(s_1^2 + s_2^2) + \varrho^2 (1 + k\varrho^4)(s_3^2 + s_3^2) \right).
\]

For \( k \leq 0 \), the scalar curvature is positive since

\[
-(b\rho + 2c) = -\frac{b}{2a}(2a\rho + b + \varrho) > 0.
\]
These metrics are U(2)-invariant in general (for the same reason as the Pedersen metric), but for particular values of $k$ we can say more. Setting $k = 0$ gives the standard O(5)-invariant metric on the complement of a point in $S^4$. If we set $k = -1$ instead (which corresponds to $c = 0$) we obtain

$$g^{a,b,c} = \frac{1}{\nu(1 + \vartheta^2)^2} \left( \frac{d\vartheta^2}{1 - \vartheta^2} + \vartheta^2 (1 + \vartheta^2) (\varsigma_1^2 + \varsigma_2^2) + \vartheta^2 (1 - \vartheta^2) \varsigma_3^2 \right).$$

This is the U(3)-invariant Fubini–Study metric on the complement of the point $[z_0 : z_1 : z_2] = [1 : 0 : 0]$ in the complex projective plane with

$$\frac{1}{\vartheta^2} = \frac{2}{z_1^2 + z_2^2} + 1.$$

(vii) For $a, b < 0$, we may perform the following change of coordinates:

$$\rho = \frac{b}{2a} \left( \frac{1}{\vartheta^2} - 1 \right), \quad \zeta = \sqrt{-\frac{2}{a}} \xi, \quad t = \frac{bK}{a} \theta$$

where $0 < \vartheta < 1$. In these coordinates, after making the identification $\theta \sim \theta + 2\pi$, the metric $g^{a,b,c}$ becomes

$$g^{a,b,c} = \frac{-1}{\nu(1 - \vartheta^2)^2} \left( \frac{1 + k\vartheta^2}{1 + k\vartheta^4} \frac{d\vartheta^2}{1 - \vartheta^2} - \vartheta^2 (1 + k\vartheta^2) (\varsigma_1^2 + \varsigma_2^2) - \frac{\vartheta^2 (1 + k\vartheta^4)}{1 + k\vartheta^4} \varsigma_3^2 \right).$$

This is formally very similar to the Pedersen metric, except that we now have SU(1,1)-invariant 1-forms $\varsigma_1, \varsigma_2, \varsigma_3$ on $B^2 \times S^1$ given by

$$\varsigma_1 = \frac{\text{Re}(e^{i\theta} d\xi)}{1 - |\xi|^2}, \quad \varsigma_2 = \frac{\text{Im}(e^{i\theta} d\xi)}{1 - |\xi|^2}, \quad \varsigma_3 = \frac{1}{4} d\theta - \frac{\text{Im}(\xi d\overline{\xi})}{1 - |\xi|^2}.$$

Correspondingly, the metric is invariant under the transitive action of U(1,1) on $B^2 \times S^1$. Note that $k$ is necessarily negative in order for the metric to be definite. This can also be seen from the fact that, were $k$ non-negative, the discriminant of $a\varrho^2 + b\varrho + c$ would be positive. Since $a < 0$, this would then imply that $a\varrho^2 + b\varrho + c$ is nowhere positive.

(viii) As a limiting case of the above, we have $a < 0$ and $b = 0$. Noting that $c > 0$ necessarily in order for $a\varrho^2 + c$ to be positive, and carrying out the change of variable $\rho = \frac{1}{\varrho} \sqrt{-\frac{2}{\pi}}$, we get

$$g^{a,b,c} = \frac{1}{\nu} \left( (1 - \rho^2) \left( \frac{dt^2}{2K} \right) + \frac{d\rho^2}{1 - \rho^2} + \frac{4\rho^2 |d\xi|^2}{(1 - |\xi|^2)^2} \right).$$

Note that as in case (iv) with $c < 0$, upon making the identification $t \sim t + 4\pi K$, the first two terms give the standard $S^2$ metric restricted to a hemisphere minus the pole, with $\sin^{-1}(\rho)$ and $\frac{t}{2K}$ being the latitude and longitude respectively, while the last term is the disc form of the real hyperbolic plane metric. The full metric $g^{a,b,c}$ is in fact the real hyperbolic 4-space, presented in terms of the fibration over $S^2$ given by sending a point $(x_0, x_1, x_2, x_3, x_4)$ on the hyperboloid $x_0^2 - \sum_{i=1}^{4} x_i^2 = (-\nu)^{-1/2}$ sitting inside $\mathbb{R}^{1,4}$ to $(\sqrt{x_0^2 - x_1^2 - x_2^2}, x_3, x_4) \in S^2 \subset \mathbb{R}^3$. 

6 Four-dimensional examples beyond the c-map
(ix) When \(a < 0 < b\) and \(2a\rho + b > 0\), the same change of coordinates as in case (vii) can be done, but now we have \(\varrho > 1\), as can be seen from

\[
\varrho^2 = \frac{b}{2a\rho + b} = \frac{b}{b - 2|a|\rho} > 1.
\]

Because the metric is formally identical to the metric from case (vii), it is also \(U(1,1)\)-invariant.

(x) In the final case, \(a < 0 < b\) and \(2a\rho + b < 0\), we instead set

\[
\rho = -\frac{b}{2a}\left(\frac{1}{\varrho^2} + 1\right), \quad \zeta = \sqrt{-\frac{2}{a}}\xi, \quad t = \frac{bK}{a}\theta.
\]

The metric now becomes

\[
g^{a,b,c} = \frac{1}{\nu(1 + \varrho^2)^2} \left(1 - \frac{k\varrho^2}{1 + k\varrho^4} d\varrho^2 - \varrho^2(1 - k\varrho^2)(\xi_1^2 + \xi_2^2) + \frac{\varrho^2(1 + k\varrho^4)}{1 - k\varrho^2} \xi_3^2\right).
\]

which is once again \(U(1,1)\)-invariant because of its general form.

A careful inspection of our list shows that, in every case, we have either identified the metric \(g^{a,b,c}\) as locally symmetric or given an explicit action of a group of isometries whose orbits are three-dimensional. This means, in the latter case, that the metrics are of cohomogeneity at most one.

It is known that the (one-loop) deformed universal hypermultiplet is of cohomogeneity (exactly) 1 for non-zero values of the deformation parameter. It is therefore natural to ask whether this holds for the metrics \(g^{a,b,c}\) which are not locally symmetric. We answer this question affirmatively:

**Proposition 6.3.** The metric \(g^{a,b,c}\) of (11) is locally symmetric when \(bc(b^2 - 4ac) = 0\) and of cohomogeneity exactly 1 otherwise. In the cohomogeneity 1 case, there is a transitive locally isometric action of \(\mathfrak{o}(2) \ltimes \mathfrak{heis}_3(\mathbb{R})\) if \(a = 0\), \(u(2)\) if \(a > 0\), and \(u(1,1)\) if \(a < 0\) on the constant \(\rho\) hypersurfaces.

**Proof.** The condition \(bc(b^2 - 4ac) = 0\) implies that either we have \(b = 0\) or \(c = 0\) or \(b^2 - 4ac = 0\). In the last case, if we assume that \(b \neq 0\), this is the same as saying \(k := \frac{4ac}{b^2} - 1 = 0\). From the case-by-case analysis in the previous subsection, we see that these yield locally symmetric spaces.

To see that the cohomogeneity of the metric is at least 1 when \(bc(b^2 - 4ac) \neq 0\), we note that its curvature norm

\[
\text{tr}(\mathfrak{R}^2) = 6\nu^2 \left(1 + b^2(b^2 - 4ac)^2 \left(\frac{\rho}{b\rho + 2c}\right)^6\right)^2
\]

is an injective function of \(\rho > 0\) whenever \(b(b^2 - 4ac) \neq 0\) and \(c \neq 0\).

Though we have already given explicit actions of groups of isometries whose orbits are of codimension 1 in every case, we can provide a direct proof that such a group must exist. When \(b(b^2 - 4ac) \neq 0\), in the notation of Theorem 6.1, the level sets of \(\rho\) admit
four Killing fields which span a three-dimensional distribution (at every point). Their explicit expressions are the following:

\[
\text{Re}\left(\partial_\zeta + \frac{a}{2} \zeta^2 \partial_\xi - \frac{i}{2} Kb \partial_t\right), \quad -a \text{ Im}(\zeta \partial_\zeta) - \frac{1}{2} Kb \partial_t,
\]

\[
\text{Im}\left(\partial_\zeta + \frac{a}{2} \zeta^2 \partial_\xi - \frac{i}{2} Kb \partial_t\right), \quad \text{Im}(\zeta \partial_\zeta).
\]

These are generators of \(\mathfrak{o}(2) \ltimes \mathfrak{heis}_3(\mathbb{R})\) if \(a = 0\), \(u(2)\) if \(a > 0\), and \(u(1,1)\) if \(a < 0\), in complete agreement with our conclusions from the case-by-case analysis.

The attentive reader may also have noticed that the examples with positive scalar curvature are incomplete, and well-known spaces like \(S^4\) or \(\mathbb{CP}^2\) are recovered only upon adding points. This is no accident: A quaternionic Kähler manifold of positive scalar curvature that arises from the HK/QK correspondence is necessarily incomplete. Indeed, it is known that no complete quaternionic Kähler manifold of positive scalar curvature carries an almost complex structure compatible with the quaternionic structure \([\text{AMP}98]\) or in fact, with the exception of the Grassmannians of complex 2-planes, any almost complex structure at all \([\text{GMS}11]\). Proposition 2.5, on the other hand, shows that any example that arises from the HK/QK correspondence admits an (integrable) complex structure compatible with the quaternionic structure.

Using the expression (17) for the curvature norm, we can rule out the possibility of finding a smooth completion in many cases:

**Proposition 6.4.** The metric \(g^{a,b,c}\) of (11) cannot be extended to a complete metric whenever \(bc(b^2 - 4ac) \neq 0\) and \(bρ + 2c \leq 0\) anywhere in the closure of the allowed range of the coordinate \(ρ\).

**Proof.** First of all, by the previous proposition, the condition \(bc(b^2 - 4ac) \neq 0\) implies that the metric \(g^{a,b,c}\) is not locally symmetric.

If there is a solution to \(bρ + 2c < 0\) in the closure of the allowed range of the coordinate \(ρ\), then, because the condition is open, there is a solution \(bρ + 2c < 0\) in the allowed range of the coordinate \(ρ\). As the sign of the scalar curvature is the opposite of the sign of \(bρ + 2c\), the scalar curvature is positive. It is well-known that there are no complete quaternionic Kähler metrics of dimension 4 of positive scalar curvature which are not locally symmetric \([\text{FK}82, \text{Hit}81]\). Thus, \(g^{a,b,c}\) cannot be extended to a complete metric.

On the other hand, if there is a solution to \(bρ + 2c = 0\) in the closure of the allowed range of the coordinate \(ρ\), we can find a \(ρ_0\) in the neighborhood of \(-\frac{2c}{b}\) such that the half-open interval between \(ρ_0\) and \(-\frac{2c}{b}\) (with \(ρ_0\) included) is contained in the allowed range of the coordinate \(ρ\). At \(ρ = -\frac{2c}{b}\), the polynomial \(aρ^2 + bρ + c\) takes the value \(\frac{c^2}{b}(4ac - b^2) \neq 0\). As a result, we have

\[
\int_{ρ_0}^{-\frac{2c}{b}} \frac{bρ + 2c}{\sqrt{aρ^2 + bρ + c}} \frac{dρ}{ρ} < \infty.
\]

By (17), \(ρ = -\frac{2c}{b}\) is a curvature singularity. It follows from the finiteness of the above integral that the curvature singularity is a finite distance away from \(ρ = ρ_0\). So, \(g^{a,b,c}\) cannot be extended to a complete metric in this case either.
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