REVERSE SUPERPOSITION ESTIMATES IN SOBOLEV SPACES

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Abstract. We study when and how the norm of a function \( u \) in the homogeneous Sobolev spaces \( \dot{W}^{s,p}(\mathbb{R}^n, \mathbb{R}^m) \), with \( p \geq 1 \) and either \( s = 1 \) or \( s > 1/p \), is controlled by the norm of composite function \( f \circ u \) in the same space.

1. Introduction

The absolute value preserves weak differentiability despite its non-differentiability at \( 0 \) [5] (see also [3, lemma 7.6; 8, corollary 6.1.4; 9, corollary 2.1.8]). More precisely, if \( u \) belongs to the homogeneous first-order Sobolev space \( \dot{W}^{1,p}(\Omega, \mathbb{R}) \) for some \( p \in [1, \infty) \), that is, if the function \( u : \Omega \to \mathbb{R} \) is weakly differentiable on the open set \( \Omega \subseteq \mathbb{R}^n \) and its weak derivative \( Du \) satisfies the integrability condition \( \int_{\Omega} |Du|^p < +\infty \), then \( |u| \in \dot{W}^{1,p}(\Omega, \mathbb{R}) \); moreover, one has then
\[
|Du| = \text{sgn}(u)Du \quad \text{almost everywhere in } \Omega, (1.1)
\]
where the signum function \( \text{sgn} \) is defined by \( \text{sgn}(t) = -1 \) when \( t < 0 \), \( \text{sgn}(0) = 0 \) and \( \text{sgn}(t) = 1 \) when \( t > 0 \). A consequence of the identity (1.1) and of the fact that \( Du = 0 \) almost everywhere on \( u^{-1}(\{0\}) \) is the integral identity
\[
\int_{\Omega} |D|u|^p = \int_{\Omega} |Du|^p, (1.2)
\]
which can be interpreted either as an estimate in the homogeneous Sobolev space \( \dot{W}^{1,p}(\Omega) \) for \( |u| \) in terms of \( u \), or conversely as an a priori estimate for \( u \) in terms of \( |u| \), provided it is known a priori that \( u \in \dot{W}^{1,p}(\Omega, \mathbb{R}) \). We will adopt the latter point of view.

This result about the absolute value is a particular case of reverse estimates for superposition operators \( u \mapsto f \circ u \), for \( u \in \dot{W}^{1,p}(\mathbb{R}^n, \mathbb{R}^m) \) and \( f : \mathbb{R}^m \to \mathbb{R}^\ell \). We state in theorem 2.1 below a wide condition on the function \( f \) which ensures that \( u \in \dot{W}^{1,p}(\mathbb{R}^n, \mathbb{R}^m) \) is controlled by \( f \circ u \in \dot{W}^{1,p}(\mathbb{R}^n, \mathbb{R}^\ell) \); this condition does not require that \( f \circ u \in \dot{W}^{1,p}(\mathbb{R}^n, \mathbb{R}^\ell) \) when \( u \in \dot{W}^{1,p}(\mathbb{R}^n, \mathbb{R}^m) \).

We next consider the question whether such reverse superposition estimate extend to the homogeneous fractional Sobolev space
\[
\dot{W}^{s,p}(\Omega, \mathbb{R}^m) := \left\{ u : \Omega \to \mathbb{R}^m \right\} \int_{\Omega \times \Omega} \frac{|u(y) - u(x)|^p}{|y - x|^{n+sp}} dy dx < +\infty, (1.3)
\]
with \( 0 < s < 1 \) and \( 1 \leq p < +\infty \). Although there is no identity such as (1.1) for fractional Sobolev spaces, we prove that when \( sp > 1 \) there exists a constant such that for every
and measurable, if

Remark 2.2

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Theorem 2.1.

(2.2)

\[
\lim sup_{y \to x} \frac{|y - u(x)|}{|f(y) - f(u(x))|} = 1
\]

\[
\sup \left\{|Df(u(x))[k]| / |k| : k \in \mathbb{R}^m \setminus \{0\} \right\}.
\]

Remark 2.3. If for each \( y \in \mathbb{R} \) the function \( f \) is defined as \( f(y) := |y| \), then we have for every \( z \in \mathbb{R} \),

\[
\lim_{y \to z} \frac{|y - z|}{||y| - |z||} = 1,
\]

and (2.1) is then in this particular case a consequence of (1.1).

The proof of theorem 2.1 follows the strategy of the general chain rule for weakly differentiable functions [1].

Proof of theorem 2.1 when \( n = 1 \). By the characterisation of weakly differentiable functions on an interval (see for example [4, theorem 7.13]), for almost every \( x \in \Omega \) there exists a sequence \( (h_j)_{j \in \mathbb{N}} \) in \( \mathbb{R} \setminus \{0\} \) converging to 0 such that both

\[
\lim_{j \to \infty} \frac{u(x + h_j) - u(x)}{h_j} = u'(x)
\]

and

\[
\lim_{j \to \infty} \frac{f(u(x + h_j)) - f(u(x))}{h_j} = (f \circ u)'(x).
\]
Assuming without loss of generality that
\[ \limsup_{y \to u(x)} \frac{|y - u(x)|}{|f(y) - f(u(x))|} < +\infty \]
we have for \( j \in \mathbb{N} \) large enough \( f(u(x) + h_j) \neq f(u(x)) \); it then follows from the limits (2.4) and (2.5) that
\[
|u'(x)| = |(f \circ u)'(x)| \lim_{j \to \infty} \frac{|u(x + h_j) - u(x)|}{|f(u(x + h_j)) - f(u(x))|} \leq |(f \circ u)'(x)| \limsup_{y \to u(x)} \frac{|y - u(x)|}{|f(y) - f(u(x))|} \tag{3.1} \]

\[ \square \]

**Proof of theorem 2.1 when \( n \geq 2 \).** The proof goes by noting that the restrictions of \( u \) and \( f \circ u \) to almost every one-dimensional line \( L \) are weakly differentiable (see for example [4, theorem 10.35]), applying the one-dimensional case and concluding by Fubini’s theorem. \[ \square \]

### 3. Fractional Sobolev spaces

In the fractional case, we have the following counterpart of theorem 2.1.

**Theorem 3.1.** For every \( s \in (0, 1) \) and \( p \in [1, +\infty) \) satisfying \( sp > 1 \), there exists a constant \( C \) such that for every convex set \( \Omega \subseteq \mathbb{R}^n \) and every \( f : \mathbb{R}^m \to \mathbb{R}^\ell \), if \( u \in \dot{W}^{s,p}(\Omega, \mathbb{R}^m) \) and if \( f \circ u \in \dot{W}^{s,p}(\Omega, \mathbb{R}^\ell) \), then
\[
\left( \iint_{\Omega \times \Omega} \frac{|u(y) - u(x)|^p}{|y - x|^{n+sp}} \, dy \, dx \right)^{1/p} \leq C \left( \iint_{\Omega \times \Omega} \frac{|f(u(y)) - f(u(x))|^p}{|y - x|^{n+sp}} \, dy \, dx \right)^{1/p}.
\]

Here \( \text{ess rg } u \) denotes the essential range of the function \( u : \Omega \to \mathbb{R}^m \) with respect to Lebesgue’s \( n \)-dimensional measure \( \mathcal{L}^n \), defined as
\[
\text{ess rg } u := \left\{ y \in \mathbb{R}^m \mid \text{for each } \varepsilon > 0, \mathcal{L}^n(u^{-1}(B_\varepsilon(y))) > 0 \right\}.
\]

Our main tool to prove theorem 3.1 is the following reverse oscillation inequality [6].

**Proposition 3.2.** If the set \( \Omega \subseteq \mathbb{R}^n \) is convex and if \( sp > 1 \), then there exists a constant \( C \) such that for every \( u \in \dot{W}^{s,p}(\Omega, \mathbb{R}^m) \) one has
\[
\left( \iint_{\Omega \times \Omega} \frac{(\text{ess osc}_{x,y} u)^p}{|y - x|^{n+sp}} \, dy \, dx \right)^{1/p} \leq C \left( \iint_{\Omega \times \Omega} \frac{|u(y) - u(x)|^p}{|y - x|^{n+sp}} \, dy \, dx \right)^{1/p}.
\]
Here we have defined the segment \([x, y] = \{(1 - t)x + ty \mid 0 \leq t \leq 1\}\) and the **essential oscillation**

\[
\text{ess osc } u := \text{ess sup}_{[x, y]} \sup_{t, r \in [0, 1]} |u((1 - r)x + ry) - u((1 - t)x + ty)|
\]

Proposition 3.2 is proved for \(n = 1\) and extended by Fubini-type arguments to higher dimension \([6]\); we give here a direct proof in all dimensions.

**Proof of proposition 3.2.** Since \(sp > 1\), we can fix \(\sigma \in \mathbb{R}\) so that \(\frac{1}{p} < \sigma < s\). There exists a constant \(C_1\) such that for every \(x, y \in \mathbb{R}^n\), we have

\[
(3.5) \quad (\text{ess osc } u)^p \leq C_1 \int_{[0, 1] \times [0, 1]} \frac{|u((1 - t)x + ty) - u((1 - r)x + ry)|^p}{|t - r|^{1 + \sigma p}} \, dt \, dr.
\]

Indeed, since \(\sigma p > 1\), by the fractional Morrey–Sobolev embedding there exists a constant \(C_1\) such that (see \([2, \S 8]\)) for almost every \(\rho, \tau \in [0, 1]\),

\[
(3.6) \quad |u((1 - \rho)x + \rho y) - u((1 - \tau)x + \tau y)|^p
\]

\[
\leq C_1 \int_{[0, 1] \times [0, 1]} \frac{|u((1 - t)x + ty) - u((1 - r)x + ry)|^p}{|t - r|^{1 + \sigma p}} \, dt \, dr,
\]

and (3.5) follows from the definition of essential oscillation (3.4) and from the estimate (3.6).

Integrating (3.5) with respect to \(x, y \in \Omega\) we get

\[
(3.7) \quad \iint_{\Omega \times \Omega} \frac{(\text{ess osc } u)^p}{|y - x|^{n + sp}} \, dy \, dx
\]

\[
\leq C_1 \iint_{\Omega \times [0, 1] \times [0, 1]} \frac{|u((1 - t)x + ty) - u((1 - r)x + ry)|^p}{|t - r|^{1 + \sigma p}|y - x|^{n + sp}} \, dt \, dr \, dy \, dx.
\]

Applying in the right-hand side of (3.5) the change of variable \((x, y) \mapsto (w, z) = ((1 - t)x + ty, (1 - r)x + ry)\), we get

\[
(3.8) \quad \iint_{\Omega \times \Omega} \frac{(\text{ess osc } u)^p}{|y - x|^{n + sp}} \, dy \, dx \leq C_1 \iint_{\Omega \times [0, 1] \times [0, 1]} \frac{|u(z) - u(w)|^p}{|t - r|^{1 - (s - \sigma)p} |z - w|^{n + sp}} \, dt \, dr \, dz \, dw.
\]

where for each \(z, w \in \Omega\) we have defined the set

\[
(3.9) \quad \Sigma_{z, w} := \{(t, r) \in [0, 1]^2 \mid \frac{z - tw}{r - t} \in \Omega \text{ and } \frac{(1 - r)z - (1 - t)w}{r - t} \in \Omega\}.
\]

We conclude by estimating the innermost integral in the right-hand side of (3.9) by monotonicity of the integral as

\[
(3.10) \quad \iint_{\Sigma_{z, w}} \frac{1}{|t - r|^{1 - (s - \sigma)p}} \, dt \, dr \leq \iint_{[0, 1] \times [0, 1]} \frac{1}{|t - r|^{1 - (s - \sigma)p}} \, dt \, dr
\]

\[
= \frac{1}{(s - \sigma)p} \int_0^1 |1 - r|^{(s - \sigma)p} - |r|^{(s - \sigma)p} \, dr < +\infty,
\]
since $\sigma > s$. The conclusion follows from (3.8) and (3.10).

\[ \square \]

**Proof of theorem 3.1.** Since $sp > 1$, for almost every $[x, y]$, by the fractional Morrey embedding, the closed set $\text{ess rg}_{[x, y]} u \subset \text{ess rg}_\Omega u$ is compact and connected, and $\text{ess rg}_{[x, y]} f \circ u = f(\text{ess rg}_{[x, y]} u)$, we have thus for almost every $x, y \in \Omega$,

\begin{align*}
|u(y) - u(x)| &\leq \text{ess osc}_{[x, y]} u = \text{diam}(\text{ess rg}_{[x, y]} u) \\
&\leq \lambda \text{diam}(\text{ess rg}_{[x, y]} f \circ u) = \lambda \text{ess osc}_{[x, y]} f \circ u.
\end{align*}

(3.11)

where

\[ \lambda := \sup \left\{ \frac{\text{diam}(K)}{\text{diam}(f(K))} \mid K \subset \text{ess rg} u \text{ compact, connected and } \text{diam}(K) > 0 \right\}. \]

(3.12)

By (3.11) and the reverse oscillation inequality proposition 3.2, we conclude that there exists a constant $C$ such that

\[ \int_{\Omega \times \Omega} \frac{|u(y) - u(x)|^p}{|y - x|^{n+sp}} \, dy \, dx \leq C \lambda^p \int_{\Omega \times \Omega} \frac{|f(u(y)) - f(u(x))|^p}{|y - x|^{n+sp}} \, dy \, dx. \]

(3.13)

\[ \square \]

4. **Counterexamples**

The following example shows that in the rough case $sp < 1$, the fractional reverse estimate theorem 3.1 fails as soon as the function $f$ is not injective.

**Proposition 4.1.** Let $\Omega \subseteq \mathbb{R}^n$, $s \in (0, 1)$ and $p \in [1, +\infty)$. If $sp < 1$ and if the function $f : \mathbb{R}^m \to \mathbb{R}^\ell$ is not injective, then there exists a sequence $(u_j)_{j \in \mathbb{N}}$ in $W^{s,p}(\Omega, \mathbb{R}^m)$ such that for every $j \in \mathbb{N}$, the function $f \circ u_j$ is constant $\Omega$ and such that

\[ \lim_{j \to \infty} \int_{\Omega \times \Omega} \frac{|u_j(y) - u_j(x)|^p}{|y - x|^{n+sp}} \, dy \, dx = +\infty. \]

(4.1)

**Proof.** We consider the case $\Omega = (0, 1) \subset \mathbb{R}$; the other cases are similar. By assumption, there exist two points $b_0, b_1 \in \mathbb{R}^m$ such that $f(b_0) = f(b_1)$. For each $j \in \mathbb{N}$ we define the function $u_j : (0, 1) \to \mathbb{R}^m$ for every $x \in (0, 1)$ by

\[ u_j(x) := \begin{cases} 
 b_0 & \text{if } jx \in [2k, 2k + 1) \text{ for some } k \in \mathbb{Z}, \\
 b_1 & \text{if } jx \in [2k + 1, 2(k + 1)) \text{ for some } k \in \mathbb{Z}.
\end{cases} \]

(4.2)
By construction, for each \( j \in \mathbb{N} \), we have \( f(u_j) = f(b_0) = f(b_1) \) everywhere in the interval \((0, 1)\). Estimating

\[
\iint_{(0,1) \times (0,1)} \left| \frac{u_j(y) - u_j(x)}{|y-x|^{1+sp}} \right|^p \, dy \, dx = \sum_{\ell=0}^{j-1} \iint_{(\frac{(\ell+1)}{j}) \times (0,1)} \left| \frac{u_j(y) - u_j(x)}{|y-x|^{1+sp}} \right|^p \, dy \, dx
\]

we infer that for each \( j \in \mathbb{N} \), we have \( u_j \in W^{s,p}((0, 1), \mathbb{R}^m) \). Finally, we have if \( j \in \mathbb{N}_* \),

\[
\iint_{(0,1) \times (0,1)} \left| \frac{u_j(y) - u_j(x)}{|y-x|^{1+sp}} \right|^p \, dy \, dx \geq \sum_{\ell=1}^{j-1} \iint_{(\frac{(\ell+1)}{j}) \times (\frac{\ell}{j})} \left| \frac{b_0 - b_1}{|y-x|^{1+sp}} \right|^p \, dy \, dx
\]

\[
= (1 - \frac{1}{j}) j^{sp} \int_{(0,1) \times (0,1)} \left| \frac{b_0 - b_1}{|y-x|^{1+sp}} \right|^p \, dy \, dx
\]

\[
= \frac{2(1 - \frac{1}{j}) j^{sp}}{sp(1-sp)} |b_1 - b_0|^p,
\]

which goes to \(+\infty\) as \( j \to \infty \). \( \square \)

Finally, in the critical case \( sp = 1 \), the fractional reverse estimate of theorem 3.1 fails when the function \( f \) is Lipschitz continuous and not injective.

**Proposition 4.2.** Let \( \Omega \subseteq \mathbb{R}^n \), \( s \in (0, 1) \) and \( p \in [1, +\infty) \). If \( sp = 1 \), if the function \( f : \mathbb{R}^m \to \mathbb{R}^l \) is Lipschitz-continuous and is not injective, then there exists a sequence \((u_j)_{j \in \mathbb{N}} \) in \( W^{s,p}(\Omega, \mathbb{R}^m) \) such that

\[
\lim_{j \to \infty} \iint_{\Omega \times \Omega} \left| \frac{u_j(y) - u_j(x)}{|y-x|^{n+sp}} \right|^p \, dy \, dx = +\infty
\]

and

\[
\sup_{j \in \mathbb{N}} \iint_{\Omega \times \Omega} \left| \frac{f \circ u_j(y) - f \circ u_j(x)}{|y-x|^{n+sp}} \right|^p \, dy \, dx < +\infty.
\]

**Proof.** We concentrate on the case \( \Omega = (-1, 1) \subseteq \mathbb{R} \), the other cases being similar. By our assumption, there are two points \( b_0, b_1 \in \mathbb{R}^m \) such that \( f(b_0) = f(b_1) \). We define the function \( u_\ast : \mathbb{R} \to \mathbb{R}^n \) for each \( t \in \mathbb{R} \) by

\[
u_\ast(t) := \begin{cases} b_0 & \text{if } t \leq -1, \\ \frac{1}{2}b_0 + \frac{1}{2}t b_1 & \text{if } -1 < t < 1, \\ b_1 & \text{if } t \geq 1. \end{cases}
\]
and we define for every \( j \in \mathbb{N} \) the function \( u_j : (-1,1) \to \mathbb{R}^m \) by setting for each \( x \in (-1,1), \ u_j(x) := u(jx) \). Since the function \( u_* \) is Lipschitz-continuous, we have \( u_j \in W^{s,p}((-1,1),\mathbb{R}^m) \). Since \( sp = 1 \), we have for every \( j \in \mathbb{N} \),

\[
\int_{\mathbb{R}\times\mathbb{R}} \frac{|f(u_j(y)) - f(u_* y)|^p}{|y-x|^{1+sp}} \, dy \, dx \leq \int_{\mathbb{R}\times\mathbb{R}} \frac{|f(u_* y) - f(u_* x)|^p}{|y-x|^{2}} \, dy \, dx,
\]

which blows up as \( j \to \infty \).

On the other hand, we have for every \( j \in \mathbb{N} \),

\[
\int_{(-1,1)\times(-1,1)} \frac{|u_j(y) - u_j(x)|^p}{|y-x|^{1+sp}} \, dy \, dx \geq 2 \int_{-1}^{1} \frac{1}{j} \int_{-1}^{1} \frac{|b_1 - b_0|^p}{|y-x|^{2}} \, dy \, dx = 2|b_1 - b_0|^p \ln \frac{(j+1)^2}{4j},
\]

which blows up as \( j \to \infty \).

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