**K-theory of weight varieties**

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**Abstract.** Let $T$ be a compact torus and $(M, \omega)$ a Hamiltonian $T$-space. We give a new proof of the $K$-theoretic analogue of the Kirwan surjectivity theorem in symplectic geometry (see [HL1]) by using the equivariant version of the Kirwan map introduced in [G1]. We compute the kernel of this equivariant Kirwan map, and hence give a computation of the kernel of the Kirwan map. As an application, we find the presentation of the kernel of the Kirwan map for the $T$-equivariant $K$-theory of flag varieties $G/T$ where $G$ is a compact, connected and simply-connected Lie group. In the last section, we find explicit formulae for the $K$-theory of weight varieties.

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1. Introduction

For $M$ a compact Hamiltonian $T$-space, where $T$ is a compact torus, we have a moment map $\phi: M \to t^*$. For any regular value $\mu$ of $\phi$, $\phi^{-1}(\mu)$ is a submanifold of $M$ and has a locally free $T$-action by the invariance of $\phi$. The symplectic reduction of $M$ at $\mu$ is defined as $M//T(\mu) := \phi^{-1}(\mu)/T$. The parameter $\mu$ is suppressed when $\mu = 0$. Kirwan [K] proved that the natural map, which is now called the Kirwan map,

$$\kappa: H^*_T(M; \mathbb{Q}) \to H^*_T(\phi^{-1}(0); \mathbb{Q}) \cong H^*(M//T; \mathbb{Q})$$

induced from the inclusion $\phi^{-1}(0) \subset M$ is a surjection when $0 \in t^*$ is a regular value of $\phi$. This result was done in the context of rational Borel equivariant cohomology. In the context of complex $\mathcal{K}$-theory, a theorem of Harada and Landweber [HL1] showed that

$$\kappa: K^*_T(M) \to K^*_T(\phi^{-1}(0))$$

is a surjection. This result was done over $\mathbb{Z}$.

In Section 2 we give another proof of this theorem by using equivariant Kirwan map, which was first introduced by Goldin [G1] in the context of rational cohomology. It can also be seen as an equivariant version of the Kirwan map.

Theorem 1.1. Let $T$ be a compact torus and $M$ be a compact Hamiltonian $T$-space with moment map $\phi: M \to t^*$. Let $S$ be a circle in $T$, and $\phi|_S := M \to \mathbb{R}$ be the corresponding component of the moment map. For a regular value $0 \in t^*$ of $\phi|_S$, the equivariant Kirwan map

$$\kappa_S: K^*_T(M) \to K^*_T(\phi|_S^{-1}(0))$$

is a surjection.

As an immediate corollary of a result in [HL1], we also find the kernel of this equivariant Kirwan map.

In Section 3 for the special case $G = SU(n)$, we find an explicit formula for the $K$-theory of weight varieties, the symplectic reduction of flag varieties $SU(n)/T$. The main result is Theorem 3.3. The results in this section are the $K$-theoretic analogues of [G2].

2. Equivariant Kirwan map in $K$-theory

First we recall the basic settings of the subject. Let $G$ be a compact connected Lie group. A compact Hamiltonian $G$-space is a compact symplectic manifold $(M, \omega)$ on which $G$ acts by symplectomorphisms, together with a $G$-equivariant moment map $\phi: M \to \mathfrak{g}^*$ satisfying Hamilton’s equation:

$$(d\phi, X) = \iota_X \omega, \forall X \in \mathfrak{g}$$

where $G$ acts on $\mathfrak{g}^*$ by the coadjoint action and $X'$ denotes the vector field on $M$ generated by $X \in \mathfrak{g}$. In this paper, we only deal with a compact torus
action, so we will use the $T$-action on $M$ as our notation instead. Let $T'$ be a subtorus in $T$, $\phi|_{T'}: M \to t'^*$ is the restriction of the $T$-action to the $T'$-action. We call $\phi|_{T'}$ the component of the moment map corresponding to $T'$ in $T$.

We fix the notations about Morse theory. Let $f: M \to \mathbb{R}$ be a Morse function on a compact Riemannian manifold $M$. Consider its negative gradient flow on $M$, let $\{C_i\}$ be the connected components of the critical sets of $f$. Define the stratum $S_i$ to be the set of points of $M$ which flow down to $C_i$ by their paths of steepest descent. There is an ordering on $I$: $i \leq j$ if $f(C_i) \leq f(C_j)$. Hence we obtain a smooth stratification of $M = \cup S_i$. For all $i, j \in I$, denote

$$M_i^+ = \bigcup_{j \leq i} S_j, \quad M_i^- = \bigcup_{j < i} S_j$$

As we are working in the equivariant category, we require that the Morse function and the Riemannian metric to be $T$-invariant.

In the following, we will consider the norm square of the moment map. In general, it is not a Morse function due to the possible presence of singularities of the critical sets. But the norm square of the moment map still yields a smooth stratifications and the results of Morse-Bott theory still holds in this general setting (Such functions are now called the Morse-Kirwan functions). For the descriptions and properties of these functions, see [K]. Kirwan proved that the Morse-Kirwan functions are equivariantly perfect in the context of rational cohomology. For more results in this direction, see [K] and [L]. In the context of equivariant $K$-theory, the following result is shown in [HL1]:

**Lemma 2.1** (Harada and Landweber). Let $T$ be a compact torus and $(M, \omega)$ be a compact Hamiltonian $T$-space with moment map $\phi: M \to t^*$. Let $f = ||\phi||^2$ be the norm square of the moment map. Let $\{C_i\}$ be the connected components of the critical sets of $f$ and the stratum $S_i$ be the set of points of $M$ which flow down to $C_i$ by their paths of steepest descent. The inclusion $C_i \to S_i$ of a critical set into its stratum induces an isomorphism $K_i^+(S_i) \cong K_i^+(C_i)$.

For a smooth stratification $M = \cup S_i$ defined by a Morse-Kirwan function $f$, i.e. the strata $S_i$ are locally closed submanifolds of $M$ and each of them satisfies the closure property $\overline{S_i} \subseteq M_i^+$. We have a $T$-normal bundle $N_i$ to $S_i$ in $M$. By excision, we have

$$K_i^+(M_i^+, M_i^-) \cong K_i^+(N_i, N_i \setminus S_i)$$

If $N_i$ is complex, by Thom Isomorphism we have

$$K_i^+(N_i, N_i \setminus S_i) \cong K_i^* \otimes d(i)(S_i)$$

where the degree $d(i)$ of the stratum is the rank of its normal bundle $N_i$. Since the collection of the sets $M_i^+$ gives a filtration of $M$, we obtain a
filtration of $K_T^*(M)$ and a spectral sequence

$$E_1 = \bigoplus_{i \in I} K_T^*(M^+_i, M^-_i) = \bigoplus_{i \in I} K_T^{-d(i)}(S_i), \quad E_\infty = \text{Gr}K_T^*(M)$$

which converges to the associated graded algebra of the equivariant $K$-theory of $M$. By Lemma 2.1, the spectral sequence becomes

$$E_1 = \bigoplus_{i \in I} K^*_{T}(C_i), \quad E_\infty = \text{Gr}K_T^*(M)$$

**Definition 2.2.** The function $f$ is called **equivariantly perfect** for equivariant $K$-theory if the above spectral sequence for equivariant $K$-theory collapses at the $E_1$ page, or equivalently speaking, we have the following short exact sequences:

$$0 \rightarrow K^*-d(i)(C_i) \rightarrow K_T^*(M^+_i) \rightarrow K_T^*(M^-_i) \rightarrow 0$$

for each $i \in I$.

In [HL1], Harada and Landweber showed the following theorem. (Indeed, they showed it for a compact Lie group $G$. But in our paper, we only need to consider the abelian case.)

**Theorem 2.3 (Harada and Landweber).** Let $T$ be a compact torus and $(M, \omega)$ be a compact Hamiltonian $T$-space with moment map $\phi: M \rightarrow \mathfrak{t}^*$. The norm square of the moment map $f = ||\phi||^2$ is an equivariantly perfect Morse-Kirwan function for equivariant $K$-theory. By Bott periodicity in complex equivariant $K$-theory, we can rewrite the short exact sequences as:

$$0 \rightarrow K^*_T(C_i) \rightarrow K_T^*(M^+_i) \rightarrow K_T^*(M^-_i) \rightarrow 0$$

Let $\phi|_S: M \rightarrow \mathbb{R}$ be the component of the moment map $\phi$ corresponding to a circle $S$ in $T$. Equivalently we are considering a compact Hamiltonian $S$-manifold with the moment map $\phi|_S$. By Theorem 2.3 above, the norm square of the moment map $||\phi|_S||^2$ is an equivariantly perfect Morse-Kirwan function for equivariant $K$-theory. We can now give our proof of Theorem 1.1.

**Proof of Theorem 1.1.** Our proof is essentially the $K$-theoretic analogue of Theorem 1.2 in [G1]. For the Morse-Kirwan function $f = ||\phi|_S||^2$, denote $C_0 = f^{-1}(0) = \phi|_S^{-1}(0)$.

First, we need to show that $K_T^*(M^+_1) \rightarrow K_T^*(\phi|_S^{-1}(0))$ is surjective for all $i \in I$. We will show it by induction.

Notice that $K_T^*(M^+_0) \cong K_T^*(C_0) = K_T^*(\phi|_S^{-1}(0))$ by Theorem 2.3. Assume the inductive hypothesis that $K_T^*(M^+_i) \rightarrow K_T^*(C_0)$ is surjective for $0 \leq i \leq k - 1$. By the equivariant homotopy equivalence, we have

$$K_T^*(M^-_k) \cong K_T^*(M^+_k)$$
By Theorem 2.3, we know that
\[ (1) \]
Hence, we now have the surjection of each induction works.

Proof. Choose a splitting of \( T = S_1 \times S_2 \times \ldots \times S_{\text{dim}T} \) where each \( S_i \) is quotiented out one at a time. Since \( T \) acts freely on the zero level set of the moment map. Then we have the surjection result for \( \kappa \): \( K_T(M) \to K_T(C_0) = K_T(\phi|_{S_i}^{-1}(0)) \), as desired.

Corollary 2.4. Let \( T \) be a compact torus and \( M \) be a compact Hamiltonian \( T \)-space with moment map \( \phi: M \to \mathfrak{t^*} \). Suppose that \( T \) acts freely on the zero level set of the moment map. Then
\[ \kappa: K_T^*(M) \to K^*(M//T) \]
is a surjection.

Proof. Choose a splitting of \( T = S_1 \times S_2 \times \ldots \times S_{\text{dim}T} \) where each \( S_i \) is quotiented out one at a time. Since \( T \) acts freely on the zero level set of the moment map, by Theorem 1.1, we have
\[ \kappa_{S_i}: K_T^*(M) \to K_T^*(\phi|_{S_i}^{-1}(0)) \equiv K_{T/S_i}^*(M//S_1) \]
is a surjection. By reduction in stages, we have
\[ K_T^*(M) \to K_{T//S_1}^*(M//S_1) \to K_{T/(S_1 \times S_2)}^*(M//(S_1 \times S_2)) \to \ldots \to K_{T/T}^*(M//T) = K^*(M//T) \]
as desired.

We compute the kernel of our equivariant Kirwan map, which can be seen as a \( K \)-theoretic analogue of [G1].

Theorem 2.5. Let \( T \) be a compact torus and \( M \) be a compact Hamiltonian \( T \)-space with moment map \( \phi: M \to \mathfrak{t^*} \). Let \( T' \) be a subtorus in \( T \). Let \( \phi|_{T'} \) be the corresponding moment map for the Hamiltonian \( T' \)-action on \( M \). For \( 0 \) a regular value of \( \phi|_{T'} \), the kernel of the equivariant Kirwan map
\[ \kappa_{T'}: K_T^*(M) \to K_T^*(\phi|_{T'}^{-1}(0)) \]
is the ideal \( \langle K_T^{T'} \rangle \) generated by \( K_T^{T'} = \cup_{\xi \in \mathfrak{t'}} K_T^\xi \), where
\[ K_T^\xi = \{ \alpha \in K_T^*(M) \mid \alpha|_C = 0 \text{ for all connected components } C \text{ of } M^T \text{ with } \langle \phi(C), \xi \rangle \leq 0 \} \]

Proof. Choose a splitting of \( T' = S_1 \times S_2 \times \ldots \times S_{\text{dim}T'} \) where each \( S_i \) is quotiented out one at a time. By Theorem 3.1 in [HL2], the kernel of the equivariant Kirwan map \( \kappa_{S_i} \) is generated by \( K_T^\xi \) and \( K_T^{-\xi} \) for a choice of generator \( \xi \in s_i \). By successive application of this result to each \( S_i \) where \( i = 1, 2, 3, \ldots \cdot \text{dim}T' \), we get our desired result.
3. \(K\)-theory of weight variety

3.1. Weight varieties. If \(G = SU(n)\), we can naturally identify the set of Hermitian matrices \(H\) with \(g^*\) by the trace map, i.e. \(tr: (H) \to g^*\) defined by \(A \mapsto i.tr(A)\). So \(\lambda \in t^*\) is a real diagonal matrix with entries \(\lambda_1, \lambda_2, ..., \lambda_n\) in the diagonal. Through this identification, \(M = \mathcal{O}_{\lambda}\) is an adjoint orbit of \(G\) through \(\lambda\). The moment map corresponding to the \(T\)-action on \(\mathcal{O}_{\lambda}\) takes a matrix to its diagonal entries, call it \(\mu \in t^*\). Hence, \(\mathcal{O}_{\lambda}/T(\mu), \mu \in t^*\) is the symplectic quotient by the action of diagonal matrices at \(\mu \in t^*\). The symplectic quotient consists of all Hermitian matrices with spectrum \(\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)\) and diagonal entries \(\mu = (\mu_1, \mu_2, ..., \mu_n)\). We call this symplectic quotient \(\mathcal{O}_{\lambda}/T(\mu)\) a weight variety.

If \(\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)\) has the property that all entries have distinct values, then \(\mathcal{O}_{\lambda}\) is a generic coadjoint orbit of \(SU(n)\). It is symplectomorphic to a complete flag variety in \(\mathbb{C}^n\). In this section, we mainly deal with the generic case unless otherwise stated. For more about the properties of weight varieties, see [Kn]. For the Weyl element action of any \(\gamma \in W\) on \(\lambda \in t^*\), we are going to use the notation \(\lambda_{\gamma} = (\lambda_{\gamma} - 1(1), ..., \lambda_{\gamma} - 1(n))\) in our proofs for our notational convenience.

3.2. Divided difference operators and double Grothendieck polynomials. Let \(f\) be a polynomial in \(n\) variables, call them \(x_1, x_2, ..., x_n\) (and possibly some other variables), the divided difference operator \(\partial_i\) is defined as

\[
\partial_i f(\ldots, x_i, x_{i+1}, \ldots) = \frac{f(\ldots, x_i, x_{i+1}, \ldots) - f(\ldots x_{i+1}, x_i, \ldots)}{x_i - x_{i+1}}
\]

The isobaric divided difference operator is

\[
\pi_i(f) = \partial_i(x_i f) = \frac{x_i f(\ldots, x_i, x_{i+1}, \ldots) - x_{i+1} f(\ldots, x_{i+1}, x_i, \ldots)}{x_i - x_{i+1}}
\]

The top Grothendieck polynomial is

\[
G_{id}(x, y) = \prod_{i<j} (1 - \frac{y_j}{x_i})
\]

Note that the isobaric divided difference operator acts on \(G_{id}\) naturally by \(\pi_i(G_{id})\). And \(\pi_i(P, Q) = \pi_i(P)Q\) provided that \(Q\) is a symmetric polynomial in \(x_1, x_2, ..., x_n\). So this operator preserves the ideal generated by all differences of elementary symmetric polynomials \(e_i(x_1, ..., x_n) - e_i(y_1, ..., y_n)\) for all \(i = 1, ..., n\), denote this ideal by \(I\). That is, the operator \(\pi_i\) acts on the ring \(R\) defined by

\[
R = \mathbb{Z}[x_1^{\pm 1}, ..., x_n^{\pm 1}, y_1^{\pm 1}, ..., y_n^{\pm 1}] / I
\]
For any element $\omega \in S_n$, $\omega$ has reduced word expression $\omega = s_{i_1}s_{i_2}...s_{i_l}$ (where each $s_{i_j}$ is a transposition between $i_j, i_{j+1}$). We can define the corresponding operator:

$$\pi_{s_{i_1}s_{i_2}...s_{i_l}} = \pi_{s_{i_1}}...\pi_{s_{i_l}}$$

which is independent of the choice of the reduced word expression.

For any $\mu \in S_n$, the double Grothendieck polynomial $G_{\mu}$ is:

$$\pi_{\mu^{-1}}G_{id} = G_{\mu}$$

Define the permuted double Grothendieck polynomials $G_{\omega}^{(\gamma)}$ by

$$G_{\omega}^{(\gamma)}(x, y) = G_{\gamma^{-1}\omega}(x, y_\gamma) = \pi_{\omega^{-1}\gamma}G_{id}(x, y_\gamma)$$

where $y_\gamma$ means the permutation of the $y_1, ..., y_n$ variables by $\gamma$.

**Example.** For $G = SU(3), W = S_3$, we have

$$G_{id} = (1 - \frac{y_2}{x_1})(1 - \frac{y_3}{x_1})(1 - \frac{y_3}{x_2})$$

$$G_{(12)}^{(23)} = \pi_{(23)(12)}G_{id}(x, y_{(12)})$$

$$= \pi_{(23)(12)} \left( 1 - \frac{y_3}{x_1} \right) \left( 1 - \frac{y_1}{x_1} \right) \left( 1 - \frac{y_3}{x_2} \right)$$

$$= \pi_{(23)} \left( x_1 \left( 1 - \frac{y_3}{x_1} \right) \left( 1 - \frac{y_1}{x_1} \right) \left( 1 - \frac{y_3}{x_2} \right) - x_2 \left( 1 - \frac{y_3}{x_2} \right) \left( 1 - \frac{y_3}{x_2} \right) \right)$$

$$= \pi_{(23)} \left( 1 - \frac{y_3}{x_1} \right) \left( 1 - \frac{y_3}{x_2} \right)$$

$$= \left( 1 - \frac{y_3}{x_1} \right)$$

### 3.3. $T$-equivariant $K$-theory of flag varieties.

We have the following formula for $K^*_T(SU(n)/T)$ (see [F]):

$$K^*_T(SU(n)/T) \cong R(T) \otimes_{R(G)} R(T) \cong R(T) \otimes_{\mathbb{Z}} R(T)/J$$

where $R(G) \cong R(T)^W$ and $R(T)$ are the character rings of $G, T$, where $G = SU(n)$, respectively. $J \subset R(T) \otimes_{\mathbb{Z}} R(T)$ is the ideal generated by $a \otimes 1 - 1 \otimes a$ for all elements $a \in R(T)^W$. $R(T)^W$ is the Weyl group invariant of $R(T)$.

$R(T)$ can be written as a polynomial ring:

$$R(T) = K^*_T(pt) \cong \mathbb{Z}[a_1^{\pm 1}, ..., a_{n-1}^{\pm 1}]$$

In the equation $K^*_T(X) = R(T) \otimes_{\mathbb{Z}} R(T)/J$, denotes the first copy of $R(T)$ by $\mathbb{Z}[y_1^{\pm 1}, ..., y_{n-1}^{\pm 1}]$ and the second copy of $R(T)$ by $\mathbb{Z}[x_1^{\pm 1}, ..., x_{n-1}^{\pm 1}]$. Then the ideal $J$ is generated by $e_i(y_1, ..., y_{n-1}) - e_i(x_1, ..., x_{n-1}), i = 1, ..., n - 1,$
where \( e_i \) is the \( i \)-th symmetric polynomial in the corresponding variables. Equivalently,

\[
K^*_T(FL(\mathbb{C}^n)) \cong \frac{\mathbb{Z}[y_1^{\pm 1}, ..., y_n^{\pm 1}, x_1, ..., x_n]}{(J, \prod_{i=1}^n y_i) - 1)}
\]

Notice that \( x_i^{-1}, i = 1, ..., n \) can be generated by some elements in the ideal \( J \), where \( J \) is the ideal generated by \( e_i(y_1, ..., y_n) - e_i(x_1, ..., x_n) \), for all \( i = 1, ..., n \).

Let \( G^C \) be the complexification of a compact Lie group \( G \) and \( B \subset G^C \) be a Borel subgroup. In our case, \( G = SU(n), G^C = SL(n, \mathbb{C}) \). Then \( G/T \approx G^C/B \). \( G^C/B \) consists of even-real-dimensional Schubert cells, \( C_\omega \) indexed by the elements in the Weyl Group \( W \). That is,

\[
C_\omega = B\omega B/B, \omega \in W
\]

The closures of these cells are called Schubert varieties:

\[
X_\omega = \overline{B\omega B/B}, \omega \in W
\]

For each Schubert variety \( X_\omega, \omega \in W \), denotes the \( T \)-equivariant structure sheaf on \( X_\omega \subset G^C/B \) by \( [\mathcal{O}_{X_\omega}] \). It extends to the whole of \( G^C/B \) by defining it to be zero in the complement of \( X_\omega \). Since \( [\mathcal{O}_{X_\omega}] \) is a \( T \)-equivariant coherent sheaf on \( G^C/B \), it determines a class in \( K_0(T, G^C/B) \), the Grothendieck group constructed from the semigroup whose elements are the isomorphism classes of \( T \)-equivariant locally free sheaves. The elements \( [\mathcal{O}_{X_\omega}]_{\omega \in W} \) form a \( R(T) \)-basis for the \( R(T) \)-module \( K_0(T, G^C/B) \). Since there is a canonical isomorphism between \( K_0(T, G^C/B) \) and \( K_T(G^C/B) = K_T(G/T) \) (see [KK]), by abuse of notation we also denote \( [\mathcal{O}_{X_\omega}]_{\omega \in W} \) as a linear basis in \( K^*_T(G/T) \) over \( R(T) \).

On the other hand, the double Grothendieck polynomials \( G_\omega, \omega \in W \), as Laurent polynomials in variables \( x_i, y_i, i = 1, 2, ..., n \) form a basis of \( K_{T \times B}(pt) \cong R(T) \otimes_\mathbb{Z} R(T) \) over \( K_T(pt) \cong R(T) \). By the equivariant homotopy principle,

\[
K_{T \times B}(pt) = K_{T \times B}(M_{n \times n})
\]

where \( M_{n \times n} \) denote the set of all \( n \times n \) matrices over \( \mathbb{C} \). By a theorem of [KM], we are able to identify the classes generated by matrix Schubert varieties in \( K_{T \times B}(M_{n \times n}) \) (matrix Schubert varieties form a cell decomposition of \( M_{n \times n}/B \)) with the double Grothendieck polynomials in \( K_{T \times B}(pt) \). The open embedding \( \iota: GL(n, \mathbb{C}) \to M_{n \times n} \) induces a map in equivariant \( K \)-theory:

\[
\iota^*: K_{T \times B}(M_{n \times n}) \to K_{T \times B}(GL(n, \mathbb{C})) = K_T(GL(n, \mathbb{C})/B) = K_T(SU(n)/T)
\]

Under this map, the classes generated by the matrix Schubert varieties in \( K_{T \times B}(M_{n \times n}) \) are mapped to the classes, \( [\mathcal{O}_{X_\omega}] \in K_T(SU(n)/T) \), of the corresponding Schubert varieties in \( SU(n)/T \). By identifications of the double Grothendieck polynomials in \( K_{T \times B}(pt) \) and the classes generated by the matrix Schubert varieties in \( K_{T \times B}(M_{n \times n}) \), the map \( \iota^* \) sends the double
3.4. Restriction of \( T \)-equivariant structure sheaves \([O_{X_w}]\), as a \( R(T) \)-basis in \( K_T(G/T) \equiv R(T) \otimes_{R(G)} R(T) \). For more results about the geometry and combinatorics of double Grothendieck polynomials and matrix Schubert varieties, see [KM].

By abuse of notation, from now on, we will take the double Grothendieck polynomials \( G_\omega(x, y) \), \( \omega \in W \) as a basis in \( K_T^*(SU(n)/T) \) over \( R(T) \). Under our notations, notice that the top double Grothendieck polynomial \( G_{id}(x, y) \) corresponds to the \( T \)-equivariant structure sheaf \([O_{X_{\omega_0}}] \), where \( \omega_0 \in W \) is the permutation with the longest length, i.e. \( \omega_0 = s_n s_{n-1} \ldots s_3 s_2 s_1 \).

For more about \( K \)-theory and \( T \)-equivariant \( K \)-theory of flag varieties, for example, see [F] and [KK].

Let \( \text{Theorem 3.1} \).

Proof. Let \( \pi_{\omega} : p_{\omega} \rightarrow Fl(\mathbb{C}^n) \) be a fixed point in \( Fl(\mathbb{C}^n)^T \) as above. The inclusion \( \iota_\omega : p_{\omega} \rightarrow Fl(\mathbb{C}^n) \) induces a restriction

\[
\iota_\omega^* : K_T^*(Fl(\mathbb{C}^n)) \rightarrow K_T^*(p_{\omega}) = R(T) = \mathbb{Z}[y_1^{\pm 1}, \ldots, y_n^{\pm 1}]
\]

such that \( \iota_\omega^* : y_i^{\pm 1} \rightarrow y_i^{\pm 1}, \iota_\omega^* : x_i \rightarrow y_{\omega(i)}, i = 1, \ldots, n \). Also, the inclusion map \( \iota : Fl(\mathbb{C}^n)^T \rightarrow Fl(\mathbb{C}^n) \) induces a map

\[
\iota^* : K_T^*(Fl(\mathbb{C}^n)) \rightarrow K_T^*(Fl(\mathbb{C}^n)^T) = \oplus_{\omega \in W} \mathbb{Z}[y_1^{\pm 1}, \ldots, y_n^{\pm 1}]
\]

whose further restriction to each component in the direct sum is just the map \( \iota_\omega^* \).

**Proof.** Consider \( K_T^*(Fl(\mathbb{C}^n)) \) as a module over \( K_T^*(pt) = \mathbb{Z}[y_1^{\pm 1}, \ldots, y_n^{\pm 1}] \), the map

\[
K_T^*(Fl(\mathbb{C}^n)) \rightarrow K_T^*(p)
\]

induced by mapping any point \( p \) into \( Fl(\mathbb{C}^n) \) is a surjective \( R(T) \)-module homomorphism and \( K_T^*(Fl(\mathbb{C}^n)) \) has a linear basis over \( K_T^*(p) = R(T) = \mathbb{Z}[y_1^{\pm 1}, \ldots, y_n^{\pm 1}] \). Hence we must have \( \iota_\omega^* : y_i^{\pm 1} \rightarrow y_i^{\pm 1}, i = 1, \ldots, n, \) for all
\( \omega \in W \). To find the image of \( x_i \) under \( \iota^*_\omega \), first, notice that in \( K_T^*(pt) \), \( y_i = [pt \times C_i] \). \( C_i \) corresponds to the action of \( T = S^1 \times \ldots \times S^1 \) on the \( i \)-th copy of \( C^n = C \times \ldots \times C \) with weight 1 and acting trivially on all the other copies of \( C \). More generally, \( y_{\omega(i)} = [pt \times C_{\omega(i)}] \). In \( K_T^*(pt) \), \( y_{\omega(i)} = [p_\omega \times C_{\omega(i)}] \), where \( p_\omega \times C_{\omega(i)} \) is the \( T \)-line bundle over the point \( p_\omega \). By the Hodgkin’s result (see \( \text{(H\text{O\text{}})} \)), \( K_T^*(G/T) = R(T) \otimes_{R(G)} K^*_G(G/T)(\cong R(T) \otimes_{R(G)} R(T)) \). Following our use of notations in \text{3.3} \( x_i \) comes from the second copy of \( R(T) \) (which is isomorphic to \( K^*_G(G/T) \) under our identification). Hence, each \( x_i \) is the class represented by the \( G \)-line bundle \( G \times_T C_i \) over \( G/T \). \( T \) acts on \( G \times C_i \) diagonally and \( G \times_T C_i \) is the orbit space of the \( T \)-action. In particular, \( x_i \) is a \( T \)-line bundle over \( G/T \) by restriction of \( G \)-action to \( T \)-action. So, \( \iota^*_\omega(x_i) \) is simply the pullback \( T \)-line bundle of the map \( \iota_\omega : p_\omega \to Fl(C^n) \). For \( i = 1 \), we have \( \iota^*_\omega(x_1) = [p_\omega \times C_{\omega(1)}] = y_{\omega(1)} \). Similarly, \( \iota^*_\omega(x_i) = y_{\omega(i)} \) for \( i = 2, \ldots, n \). And hence the result. \( \square \)

### 3.5. Relations between double Grothendieck polynomials and the Bruhat Ordering

Recall our definition of the permuted double Grothendieck polynomials \( G^\omega_\gamma \) in Section \text{3.2}:

\[
G^\omega_\gamma(x, y) = G_{\gamma^{-1}\omega}(x, y_\gamma) = \pi_{\omega^{-1}\gamma}G_{id}(x, y_\gamma)
\]

where \( y_\gamma \) indicates the permutation of \( y_1, \ldots, y_n \) by \( \gamma \). For \( \gamma \in W \), define the permuted Bruhat ordering by

\[
v \leq_\gamma \omega \iff \gamma^{-1}v \leq \gamma^{-1}\omega
\]

Notice that the permuted Bruhat ordering is related to the Schubert varieties in the following way: Each of the \( T \)-fixed points of a Schubert variety \( X_\omega \) sits in one Schubert cell \( C_v \) (the interior of a Schubert variety) for \( v \leq \omega \). So the \( T \)-fixed point set can be identified as:

\[
(X_\omega)_T = \{v \mid v \leq \omega\}
\]

For a fixed \( \gamma \in W \), we can define the permuted Schubert varieties by

\[
X^\omega_\gamma = \gamma B \gamma^{-1}B / B
\]

for any \( \omega \in W \). Then the \( T \)-fixed point set of \( X^\omega_\gamma \) is

\[
(X^\omega_\gamma)_T = \{v \mid v \leq_\gamma \omega\}
\]

Notice that \( \{X^\omega_\gamma\}_{\omega \in W} \) also forms a cell decomposition of \( G^C / B \approx G / T \).

We define the support of the permuted double Grothendieck polynomials by

\[
\text{Supp}(G^\omega_\gamma) = \{z \in W \mid G^\omega_\gamma|_z \neq 0\}
\]

Here we consider \( G^\omega_\gamma \) as an element in \( K_T^*(Fl(C^n)) \) (see Section \text{3.3}). So \( G^\omega_\gamma|_z \) is the image of \( G^\omega_\gamma \) under the restriction of the Kirwan injective map at the point \( z \in W \). That is,

\[
\iota^*_z : K_T^*(Fl(C^n)) \to K_T^*(pt)
\]
Notice that the restriction rule follows Theorem 3.1. That is,
\[ G_\omega^T(x, y)|_z = G_\omega^T(x_1, x_2, ..., x_n, y_1, ..., y_n)|_z = G_\omega(y_{z(1)}, y_{z(2)}, ..., y_{z(n)}; y_1, ..., y_n) \]

**Example.** Using the same notations as in the example in 3.2, \( G_{(23)}^{(12)} = (1 - \frac{y_4}{y_2}) \in K^*_T(\text{Fl}(\mathbb{C}^3)) \). There are six fixed points for each element in \( S_3 \),
\[ G_{(23)}^{(12)}|_{(23)} \neq 0, G_{(23)}^{(12)}|_{(123)} \neq 0, G_{(23)}^{(12)}|_{(13)} = 0 \]
\[ G_{(23)}^{(12)}|_{(132)} = 0, G_{(23)}^{(12)}|_{(12)} \neq 0, G_{(23)}^{(12)}|_{id} \neq 0 \]

So the support of a permuted double Grothendieck polynomial contains \( id, (12), (23), (123) \). On the other hand,
\[ (X_{(23)}^{(12)})^T = \{ v \in S_3 | (12)v \leq (12)(23) = (123) \} \]
\[ = \{ v \in S_3 | (12)v \leq id, (12), (23) \text{ or } (123) \} \]
\[ = \{ v \in S_3 | v \leq (12), id, (123) \text{ or } (23) \} \]

which is the same as \( \text{Supp}(G_{(23)}^{(12)}) \).

Now we will show a fundamental relation between the permuted double Grothendieck polynomials and the permuted Bruhat Orderings:

**Theorem 3.2.** The support of a permuted double Grothendieck polynomial \( G_\omega^T \) is \( \{ v | v \leq_\gamma \omega \} \)

**Proof.** We need to show \( \text{Supp}(G_\omega) = (X_\omega)^T \) first. We do it by induction on the length of \( v \in W, l(v) \), which stands for the minimum number of transpositions in all the possible choices of word expressions of \( v \).

For \( \omega = id \), \( G_{id} \) is just the top Grothendieck polynomial. It is non-zero only at the identity and zero at all the other elements. Assume the inductive hypothesis that \( \text{Supp}(G_\omega) = (X_\omega)^T \) for all \( l(\omega) \leq l - 1 \). Consider \( v \in W, l(v) = l \), write \( v = s_{i_1}s_{i_2}...s_{i_l} \) where each \( s_{i_j} \) is a transposition of elements \( i_j, i_j + 1 \), let \( \omega = vs_{i_l} = s_{i_1}...s_{i_{l-1}} \), so \( l(\omega) = l - 1 \) and
\[ G_v|_z = \pi_{v^{-1}}G|_z = \pi_i \pi_{i-1} ... \pi_{i_l}G|_z = \pi_iG|_z \]
\[ = \frac{x_{i_l}G_\omega(x, y) - x_{i_l+1}G_\omega(x_{s_{i_l}}, y)}{x_{i_l} - x_{i_l+1}}|_z \]
\[ \frac{y_{z(i_l)}G_\omega(y_z, y) - y_{z(i_l+1)}G_\omega(y_{z(s_{i_l})}, y)}{y_{z(i_l)} - y_{z(i_l+1)}} \]

(3)

First, to prove that \( \text{Supp}(G_v) \subset (X_v)^T \), suppose that \( z \not\in (X_v)^T \), then \( z \not\in (X_\omega)^T \) since \( \omega \leq v \). Since \( l(\omega) = l - 1 \), we have \( z \not\in \text{Supp}(G_\omega) \). That is \( G_\omega(y_z, y) = 0 \). Hence,
\[ G_v|_z = \frac{-y_{z(i_l+1)}G_\omega(y_{z(s_{i_l})}, y)}{y_{z(i_l)} - y_{z(i_l+1)}} \]
We claim that it is zero. If it were not zero, then $G_\omega(yzs_{i_1}, y) = G_\omega(x, y)|zs_{i_1} \neq 0$. Equivalently, $zs_{i_1} \in \text{Supp}(G_\omega) = (X_\omega)^T$. If $z < zs_{i_1}$, then $z \in (X_\omega)^T$ which contradicts $z \notin \text{Supp}(G_\omega)$ shown before. If $z > zs_{i_1}$, then $s_{i_1}$ increases the length of $zs_{i_1}$. Then $zs_{i_1} \in (X_\omega)^T$ implies that $z \in (X_\omega)^T$ which contradicts $z \notin (X_\omega)^T$. So the claim is proved. i.e. $z \notin (X_\omega)^T \Rightarrow G_v|_z = 0 \Rightarrow z \notin \text{Supp}(G_v)$.

Second, we need to prove that $(X_v)^T \subset \text{Supp}(G_v)$. Suppose that $z \notin \text{Supp}(G_v)$, i.e. $G_v|_z = 0$. Assume that $z \in (X_\omega)^T$. From (3),

\begin{equation}
y_{z(i_j)}G_\omega(yz, y) = y_{z(i_j+1)}G_\omega(yzs_{i_j}, y)
\end{equation}

Now there are two cases, $z = v$ and $z \neq v$. We consider these two cases separately.

If $z = v$, then $z \leq w$ (since $l(\omega) = l - 1$ and $l(z) = l(v) = l) \Leftrightarrow z \notin (X_\omega)^T = \text{Supp}(G_\omega) \Leftrightarrow G_\omega|_z = 0 \Leftrightarrow G_\omega(yz, y) = 0 \Leftrightarrow G_\omega(yzs_{i_j}, y) = 0$. The last equality is by (4). So we now have $G_\omega(x, y)|zs_{i_j} = 0 \Leftrightarrow zs_{i_j} \notin \text{Supp}(G_\omega) = (X_\omega)^T$. Since $zs_{i_j} = vs_{i_j} = \omega \in (X_\omega)^T$, it’s a contradiction.

If $z \neq v$, then $l(z) < l(v)$, then $l(z) \leq l - 1$. Let $t \in W$ with $l(t) = l - 1$ such that $z \leq t$. Although $t$ may not be the same as $\omega$ but $t = v's_{i_j}$ for some $j \in 1, ..., l$ ($v'$ is another word expression for $v$) By our inductive hypothesis, $\text{Supp}(G_t) = (X_t)^T$, so

\begin{equation}
z \in \text{Supp}(G_t) \Leftrightarrow G_t(yz, y) = G_t(x, y)|_z \neq 0
\end{equation}

But $zs_{i_j} \leq t$ implies that $zs_{i_j} \notin (X_t)^T = \text{Supp}(G_t)$. By (4), (but now we have $\omega$ replaced by $t$), $G_t(yzs_{i_j}, y) = 0$. By (3) and (5), we have $G_v|_z \neq 0$ contradicting our initial assumption that $z \notin \text{Supp}(G_v)$.

Hence, we have $z \notin \text{Supp}(G_v) \Rightarrow z \notin (X_v)^T$. The induction step is done.

Then we need to show that the statement holds for the permuted double Grothendieck polynomials, i.e. $\text{Supp}(G_\omega) = (X_\omega)^T$. By definition, $G_\omega(x, y) = G_{\gamma^{-1}\omega}(x, y_\gamma)$, so

\begin{equation}
\text{Supp}G_{\gamma^{-1}\omega}(x, y) = (X_{\gamma^{-1}\omega})^T = \{v \in W \mid v \leq \gamma^{-1}\omega\}
\end{equation}

By permuting the $y$’s variables by $\gamma$, we obtain

\begin{align*}
\text{Supp}(G_\omega) &= \text{Supp}G_{\gamma^{-1}\omega}(x, y_\gamma) \\
&= \{\gamma v \in W \mid v \leq \gamma^{-1}\omega\} \\
&= \{v \in W \mid \gamma^{-1}v \leq \gamma^{-1}\omega\} \\
&= \{(X_\omega)^T\}
\end{align*}

\[\Box\]

3.6. Main theorem. In this subsection, we prove the following result:

Theorem 3.3. Let $O_\lambda$ be a generic coadjoint orbit of $SU(n)$. Then

\[K^*(O_\lambda/\mathcal{T}(\mu)) \cong \frac{\mathbb{Z}[x_1, ..., x_n, y_1^{\pm 1}]}{(I, (\prod_{i=1}^n y_i) - 1), \pi_0 G(x, y_\gamma))}\]
for all \(v, r \in S_n\) such that \(\sum_{i=k+1}^{n} \lambda_v(i) < \sum_{i=k+1}^{n} \mu_r(i)\) for some \(k = 1, \ldots, n - 1\). \(I\) is the difference between \(e_i(x_1, \ldots, x_n) - e_i(y_1, \ldots, y_n)\) for all \(i = 1, \ldots, n\), where \(e_i\) is the \(i\)-th elementary symmetric polynomial.

It is a \(K\)-theoretic analogue of the main result in [G2].

To make the symplectic picture more explicit, we denote \(M = \mathcal{O}_\lambda \approx SU(n)/T\) to be the generic coadjoint orbit. So we have \(K_\mathcal{O}(M) = K_\mathcal{T}(\mathcal{O}_\lambda) = K_\mathcal{T}(Fl(\mathbb{C}^n))\). For \(\lambda \in \mathfrak{t}^*\), \(\lambda = (\lambda_1, \ldots, \lambda_n)\), assume that \(\lambda_1 > \lambda_2 > \ldots > \lambda_n\), and \(\lambda_1 + \ldots + \lambda_n = 0\). Since \(M = \mathcal{O}_\lambda\) is compact, \(\mathcal{M}\) has only a finite number of points. The kernel of the Kirwan map \(\kappa\) is generated by a finite number of components, see Theorem 2.5 and [H12]. More specifically, let \(M^\mu_\xi\subset M, \xi \in \mathfrak{t}\) be the set of points where the image under the moment map \(\phi\) lies to one side of the hyperplane \(\xi^\perp\) through \(\mu = (\mu_1, \ldots, \mu_n)\in \mathfrak{t}^*\), i.e.

\[
M^\mu_\xi = \{m \in M \mid \langle \xi, \phi(m) \rangle \leq \langle \xi, \mu \rangle\}
\]

Then the kernel of \(\kappa\) is generated by

\[
K_\xi = \{\alpha \in K_\mathcal{T}(M) \mid \text{Supp}(\alpha) \subset M^\mu_\xi\}
\]

That is,

\[
\ker(\kappa) = \sum_{\xi \in \mathfrak{t}} K_\xi
\]

Now, we are going to compute the kernel explicitly. Our proof is similar to the results in [G2]. In [G2], Goldin proved a very similar result in rational cohomology by using the permuted double Schubert polynomials as a linear basis of \(H_\mathcal{T}(M)\) over \(H_\mathcal{T}(pt)\). In \(K\)-theory, the permuted double Grothendieck polynomials are used as a linear basis of \(K_\mathcal{T}(M)\) over \(K_\mathcal{T}(pt) \cong R(T)\). The following lemma will be used in our proof of Theorem 3.3.

**Lemma 3.4.** Let \(\mathcal{O}_\lambda\) be a generic coadjoint orbit of \(SU(n)\) through \(\lambda \in \mathfrak{t}^*\). Let \(\alpha \in K_\mathcal{T}(\mathcal{O}_\lambda)\) be a class with \(\text{Supp}(\alpha) \subset (\mathcal{O}_\lambda)^\mu_\xi\). Then there exists some \(\gamma \in W\) such that if \(\alpha\) is decomposed in the \(R(T)\)-basis \(\{G^\gamma_\omega\}_{\omega \in W}\) as

\[
\alpha = \sum_{\omega \in W} a^\gamma_\omega G^\gamma_\omega
\]

where \(a^\gamma_\omega \in R(T)\), then \(a^\gamma_\omega \neq 0\) implies \(\text{Supp}(G^\gamma_\omega) \subset (\mathcal{O}_\lambda)^\mu_\xi\). Indeed, \(\gamma\) can be chosen such that \(\xi\) attains its minimum at \(\phi(\lambda_\gamma)\), where \(\lambda_\gamma = (\lambda_\gamma^{-1}(1), \ldots, \lambda_\gamma^{-1}(n)) \in \mathfrak{t}^*\).

**Proof.** The proof is essentially the same as Theorem 3.1 in [G2]. □

**Proof of Theorem 3.3**: Let \(e_i\) be the coordinate functions on \(\mathfrak{t}^*\). That is, for \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathfrak{t}^*\), \(e_i(\lambda) = \lambda_i\). For \(\gamma \in S_n\), define \(\eta^\gamma_k\) by

\[
\eta^\gamma_k = \sum_{i=k+1}^{n} e_\gamma(i)
\]
We compute \( M_{\eta_k}^\mu \) explicitly:

\[
M_{\eta_k}^\mu = \{ m \in M \mid \langle \eta_k^\gamma, \phi(m) \rangle \leq \langle \eta_k^\gamma, \mu \rangle \} \\
= \{ m \in M \mid \eta_k^\gamma(\phi(m)) \leq \eta_k^\gamma(\mu) \} \\
= \{ m \in M \mid \eta_k^\gamma(\phi(m)) \leq \sum_{i=k+1}^n \mu_{\gamma(i)} \}
\]

For any \( \omega \in W \),

\[
\eta_k^\gamma(\lambda_\omega) = \sum_{i=k+1}^n e_{\gamma(i)}(\lambda_\omega) = \sum_{i=k+1}^n e_{\gamma(i)}(\lambda_{\omega^{-1}(1)}, \ldots, \lambda_{\omega^{-1}(n)}) \\
= \sum_{i=k+1}^n \lambda_{\omega^{-1}\gamma(i)}
\]

Notice that \( \eta_k^\gamma \) attains minimum at \( \lambda_\gamma \) (due to our assumption that \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \)) and respects the permuted Bruhat ordering, i.e.

\[
\eta_k^\gamma(\lambda_v) \leq \eta_k^\gamma(\lambda_\omega)
\]

if \( v \leq \gamma \). By restriction to the domain \( \text{Supp}(G_\omega^\gamma) = (X_\omega^\gamma)^T = \{ v \in W \mid v \leq \gamma \} \), \( \eta_k^\gamma \) attains its maximum at \( \lambda_\omega \) and minimum at \( \lambda_\gamma \). If \( \eta_k^\gamma(\lambda_\omega) = \sum_{i=k+1}^n \lambda_{\omega^{-1}\gamma(i)} \leq \sum_{i=k+1}^n \mu_{\gamma(i)} \), then for \( v \in (X_\omega^\gamma)^T \),

\[
\eta_k^\gamma(\lambda_v) = \sum_{i=k+1}^n \lambda_{v^{-1}\gamma(i)} \leq \sum_{i=k+1}^n \lambda_{\omega^{-1}\gamma(i)} < \sum_{i=k+1}^n \mu_{\gamma(i)}
\]

and hence

\[
\text{Supp}(G_\omega^\gamma) = (X_\omega^\gamma)^T = \{ v \in W \mid \gamma^{-1}v \leq \gamma^{-1}\omega \} \subset M_{\eta_k}^\mu
\]

Since \( G_\omega(x, y) = \pi_{\omega^{-1}\gamma}G(x, y_\gamma) \), we have \( \pi_\varphi G(x, y_\gamma) \in \ker(\kappa) \) if \( \sum_{i=k+1}^n \lambda_{\varphi(i)} < \sum_{i=k+1}^n \lambda_{\omega(i)} \).

For the other direction, we need to show that the classes \( \pi_{\varphi} G(x, y_\gamma) \) with \( \varphi, \gamma \in W \) having the property that \( \sum_{i=k+1}^n \lambda_{\varphi(i)} < \sum_{i=k+1}^n \lambda_{\omega(i)} \) for some \( k \in \{1, \ldots, n-1\} \) actually generate the whole kernel. Let \( \alpha \in K_\gamma^T(M) \) be a class in \( \ker(\kappa) \), so \( \text{Supp}(\alpha) \subset M_k^\mu \) for some \( \xi \in \mathfrak{t} \). We take \( \gamma \in W \) such that \( \xi(\lambda_\gamma) \) attains its minimum. Decompose \( \alpha \) over the \( R(T) \)-basis \( \{ G_\omega^\gamma \}_{\omega \in W} \),

\[
\alpha = \sum_{\omega \in W} a_\omega^\gamma G_\omega^\gamma
\]

where \( a_\omega^\gamma \in R(T) \). By Lemma 3.3 we need to show that \( \text{Supp}(G_\omega^\gamma) \subset M_{\eta_k}^\mu \) for some \( k \). Since \( \eta_k^\gamma \) is preserved by the permuted Bruhat ordering and attains its maximum at \( \lambda_\omega \) in the domain \( \text{Supp}(G_\omega^\gamma) \), we just need to show that

\[
\eta_k^\gamma(\lambda_\omega) < \eta_k^\gamma(\mu)
\]
for some $k$. It is actually purely computational: Suppose (6) does not hold for all $k$. We have

$$\lambda_{\omega^{-1}\gamma(n)} \geq \mu_{\gamma(n)}$$

$$\vdots$$

$$\lambda_{\omega^{-1}\gamma(2)} + \ldots + \lambda_{\omega^{-1}\gamma(n)} \geq \mu_{\gamma(2)} + \ldots + \mu_{\gamma(n)}$$

For $\xi = \sum_{i=1}^{n} b_{i}e_{i}$, $b_{1}, \ldots, b_{n} \in \mathbb{R}$ (recall that $\xi$ attains its minmum at $\lambda_{\gamma}$ by our choice of $\gamma \in W$), we have $\xi(\lambda_{\gamma}) \leq \xi(\lambda_{s_{i}\gamma})$ where $s_{i}$ is a transposition of $i$ and $i + 1$. And hence

$$b_{i}\lambda_{\gamma^{-1}(i)} + b_{i+1}\lambda_{\gamma^{-1}(i+1)} \leq b_{i}\lambda_{\gamma^{-1}(i+1)} + b_{i+1}\lambda_{\gamma^{-1}(i)}$$

By our assumption that $\lambda_{i} > \lambda_{i+1}$, we get $b_{\gamma(i)} \leq b_{\gamma(i+1)}$. And hence $b_{\gamma(1)} \leq b_{\gamma(2)} \leq \ldots \leq b_{\gamma(n)}$. Then,

$$(b_{\gamma(n)} - b_{\gamma(n-1)})\lambda_{\omega^{-1}\gamma(n)} \geq (b_{\gamma(n)} - b_{\gamma(n-1)})\mu_{\gamma(n)}$$

$$(b_{\gamma(n-1)} - b_{\gamma(n-2)})\lambda_{\omega^{-1}\gamma(n-1)} + \lambda_{\omega^{-1}\gamma(n)} \geq (b_{\gamma(n-1)} - b_{\gamma(n-2)})\mu_{\gamma(n-1)} + \mu_{\gamma(n)}$$

$$\vdots$$

$$(b_{\gamma(2)} - b_{\gamma(1)})\lambda_{\omega^{-1}\gamma(2)} + \ldots + \lambda_{\omega^{-1}\gamma(n)} \geq (b_{\gamma(2)} - b_{\gamma(1)})\mu_{\gamma(2)} + \ldots + \mu_{\gamma(n)}$$

Using $\sum_{i=1}^{n} \lambda_{i} = 0 = \sum_{i=1}^{n} \mu_{i}$ and summing up all the above inequalities to get

$$\sum_{i=1}^{n} b_{\gamma(i)}\lambda_{\omega^{-1}\gamma(i)} \geq \sum_{i=1}^{n} b_{i}\mu_{i}$$

$$\Leftrightarrow \sum_{i=1}^{n} b_{i}\lambda_{\omega^{-1}(i)} \geq \sum_{i=1}^{n} b_{i}\mu_{i}$$

$$\Leftrightarrow \xi(\lambda_{\omega}) \geq \xi(\mu)$$

the last inequality contradicts $\text{Supp}(\alpha) \subset M_{\xi}^\mu$ since $\omega$ has the property that $\omega \in \text{Supp}(\alpha)$. So (6) is true.

So the kernel $\ker(\kappa)$ is generated by the set $\pi_{v}G(x, y_{\gamma})$ for $v, \gamma \in W$ satisfying $\sum_{i=k+1}^{n} \lambda_{v(i)} < \sum_{i=k+1}^{n} \mu_{\gamma(i)}$ for some $k = 1, \ldots, n - 1$. By (2) and the surjectivity of the Kirwan map $\kappa$,

$$\kappa: K_{T}^{*}(SU(n)/T) = K_{T}^{*}(\mathcal{O}_{\lambda}) \to K_{T}^{*}(\phi^{-1}(\mu)) \cong K^{*}(\mathcal{O}_{\lambda//T}(\mu))$$

It implies that

$$K^{*}(\mathcal{O}_{\lambda//T}(\mu)) = K_{T}^{*}(\mathcal{O}_{\lambda})/\ker(\kappa)$$

With $\ker(\kappa)$ explicitly computed and by (2), Theorem 3.3 is proved. \qed
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