A TORSION-FREE GROUP WITH UNIQUE C*-NORM

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Abstract. Let \( p \) and \( q \) be multiplicatively independent integers. We show that the complex group ring of \( \mathbb{Z}[\frac{1}{pq}] \rtimes \mathbb{Z}^2 \) admits a unique C*-norm. The proof uses a characterization, due to Furstenberg, of closed \( \times p \)- and \( \times q \)-invariant subsets of \( \mathbb{T} \).

1. Introduction

Given a group \( G \), there are in general many C*-norms on its complex group ring \( C[G] \). For example, if \( G = \mathbb{Z} \), identifying \( C[\mathbb{Z}] \) with complex polynomials \( p(z) = \sum \alpha_n z^n \) defined on \( \mathbb{T} \), we have that any infinite closed subset \( F \subset \mathbb{T} \) gives rise to a distinct C*-norm \( \| \cdot \| \) on \( C[\mathbb{Z}] \), defined by \( \| p \| := \sup_{z \in F} |p(z)| \), for \( p \in C[\mathbb{Z}] \) (this was noted in [6, Chapter 19] and [8]).

Following [4], we say that a group \( G \) is C*-unique if \( C[G] \) admits a unique C*-norm (note, however, that this term has been used in the literature with different meanings; see [2] for historical remarks).

Clearly, any C*-unique group must be amenable. In [8], Grigorchuk, Musat and Rørdam proved that any locally finite group is C*-unique and asked whether the converse holds. Furthermore, Alekseev and Kyed obtained in [2] large classes of amenable groups which are not C*-unique.

In [4], Caspers and Skalski studied the question of C*-uniqueness in the context of discrete quantum groups, and showed that there exists a C*-unique discrete quantum group which is not locally finite.

As told by Alekseev in [1], Ozawa, during the 2019 workshop C*-algebras at Oberwolfach, pointed out that the lamplighter group \((\bigoplus_{\mathbb{Z}} \mathbb{Z}_2) \rtimes \mathbb{Z}\) is C*-unique and not locally finite, thus answering in the negative the question in [8]. Alekseev asked then in [1] whether there exists a torsion-free group which is C*-unique.

In this paper, by adapting Ozawa’s argument, we provide an example of a torsion-free C*-unique group \( G \). The proof of C*-uniqueness of \( G \) uses a certain result by Furstenberg ([7]) on diophantine approximation.

2. A torsion-free C*-unique group

Fix \( p \) and \( q \) integer such that \( p, q \geq 2 \) and such that there are no \( r, s \in \mathbb{N} \) such that \( p^r = q^s \) (this means that \( p \) and \( q \) are multiplicatively independent).

Let \( \mathbb{Z}[\frac{1}{pq}] \) be the additive group \( \{ \frac{a}{(pq)^n} : a \in \mathbb{Z}, n \in \mathbb{N} \} \), and \( \alpha \) the action of \( \mathbb{Z}^2 \) on \( \mathbb{Z}[\frac{1}{pq}] \) given by \( \alpha((n,m)) (x) := p^n q^m x \), for \( n, m \in \mathbb{Z} \) and \( x \in \mathbb{Z}[\frac{1}{pq}] \). We will show that the torsion-free group \( \mathbb{Z}[\frac{1}{pq}] \rtimes \mathbb{Z}^2 \) is C*-unique. By [2] Lemma 2.2,

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this is equivalent to showing that, given an ideal \( I \trianglelefteq C^*(\mathbb{Z}[\frac{1}{pq}] \rtimes \mathbb{Z}^2) \) such that \( I \cap \mathbb{C}[\mathbb{Z}[\frac{1}{pq}] \rtimes \mathbb{Z}^2] = \{0\} \), we have that \( I = \{0\} \).

**Remark 2.1.** In [9], Huang and Wu studied unitary representations of \( \mathbb{Z}[\frac{1}{pq}] \rtimes \mathbb{Z}^2 \) in connection with Furstenberg’s conjecture on \( \times p - \) and \( \times q - \) invariant probability measures on \( \mathbb{T} \).

Notice that, because \( p \) and \( q \) are multiplicatively independent, we have that, if \( n, m \in \mathbb{Z} \) are not both zero, then \( p^n q^m \neq 1 \). This implies that the action of \( \mathbb{Z}^2 \) on \( \mathbb{Z}[\frac{1}{pq}] \) is faithful.

Recall that an action of a group \( G \) on a locally compact Hausdorff space \( X \) is **topologically free** if, for each \( g \in G \setminus \{e\} \), the set of points of \( X \) fixed by \( g \) has empty interior.

The following lemma follows easily from [5, Lemma 2.1]. For the sake of completeness, we include a proof.

**Lemma 2.2.** Let \( A \) be a torsion-free discrete abelian group and \( \beta : G \curvearrowright A \) a faithful action. Then \( \hat{\beta} : G \curvearrowright \hat{A} \) is topologically free.

**Proof.** Take \( g \in G \) such that the set \( F_g \) of points of \( \hat{A} \) fixed by \( \hat{\beta}_g \) has non-empty interior, and we will show that \( g = e \).

Since \( F_g \) is a subgroup of \( \hat{A} \), the fact that \( F_g \) has non-empty interior implies that \( F_g \) is open. Moreover, since \( \hat{\beta}_g \) is continuous, we also have that \( F_g \) is closed.

From the fact that \( A \) is torsion-free, we obtain that \( \hat{A} \) is connected, hence \( F_g = \hat{A} \). Since \( \beta \) is a faithful action, we conclude that \( g = e \).

□

From the lemma above, we obtain that the action \( \hat{\alpha} : \mathbb{Z}^2 \rtimes \mathbb{Z}[\frac{1}{pq}] \) is topologically free. Hence, given a non-zero ideal \( I \trianglelefteq C^*(\mathbb{Z}[\frac{1}{pq}] \rtimes \mathbb{Z}^2) \simeq C(\mathbb{Z}[\frac{1}{pq}] \rtimes \mathbb{Z}^2) \), we have that \( I \cap C^*(\mathbb{Z}[\frac{1}{pq}]) \neq \{0\} \) (for a proof of this general fact about topologically free actions, see, for instance, [6, Theorem 29.5]).

Let

\[
\varphi : \mathbb{T} \to \mathbb{T} \\
z \mapsto z^{pq}
\]

and \( X := \varprojlim (\mathbb{T}, \varphi) = \{(x_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} \mathbb{T} : \forall n \in \mathbb{N}, x_n = \varphi(x_{n+1})\} \).

**Lemma 2.3.** There is an isomorphism \( \tilde{\psi} : C^*(\mathbb{Z}[\frac{1}{pq}]) \to C(X) \) such that

\[
\tilde{\psi}(\delta_{x \in \mathbb{Z}[\frac{1}{pq}]})(x) = (x_m)^a,
\]

for \( a \in \mathbb{Z}, m \in \mathbb{N} \) and \( x = (x_n)_{n \in \mathbb{N}} \in X \).

**Proof.** Let

\[
\text{Ev} : C^* \left( \mathbb{Z} \left[ \frac{1}{pq} \right] \right) \to C \left( \mathbb{Z} \left[ \frac{1}{pq} \right] \right)
\]

be the isomorphism given by point-evaluation, i.e., given \( u \in \mathbb{Z}[\frac{1}{pq}] \) and \( \tau \in \mathbb{Z}[\frac{1}{pq}] \), we have that \( \text{Ev}(\delta_u)(\tau) = \tau(u) \).
Let $H : \mathbb{Z}[\frac{1}{pq}] \to X$ be the continuous map given by $H(\tau) := (\tau(\frac{1}{pq}))_{n \in \mathbb{N}}$, for $\tau \in \mathbb{Z}[\frac{1}{pq}]$. Also let $\tilde{H} : C(X) \to C(\mathbb{Z}[\frac{1}{pq}])$ be the homomorphism induced by $H$.

Let $\psi : \mathbb{Z}[\frac{1}{pq}] \to C(X)$ be given by $\psi(\frac{a}{pq})(x) = (x_m)^a$, for $a \in \mathbb{Z}$, $m \in \mathbb{N}$ and $x = (x_n)_{n \in \mathbb{N}} \in X$. It is straightforward to check that $\psi$ is a well-defined unitary representation of $\mathbb{Z}[\frac{1}{pq}]$. Let $\hat{\psi} : C^*([\frac{1}{pq}]) \to C(X)$ be the canonical extension of $\psi$. An application of the Stone-Weierstrass theorem shows that $\hat{\psi}$ is surjective.

Furthermore, it can be readily checked that $\hat{H} \circ \hat{\psi} = \text{Ev}$. Since we know that $\text{Ev}$ is an isomorphism, we conclude that $\hat{\psi}$ is also an isomorphism.

We denote by $\tilde{\alpha}$ the action of $\mathbb{Z}$ on $C^*([\frac{1}{pq}])$ induced by $\alpha$. There is an action $\beta : \mathbb{Z} \curvearrowright X$ such that, for $f \in C(X)$ and $(r, s) \in \mathbb{Z}^2$, we have that $\hat{\psi} \circ \tilde{\alpha}(r,s) \circ \hat{\psi}^{-1}(f) = f \circ \beta^{-1}_{r,s}$. One can readily check that, for $x \in X$ and $r, s \in \mathbb{Z}$ non-negative integers, it holds that $\beta^{-1}_{r,s}(x) = x^{p^r q^s}$. Furthermore, $\beta_{(1,1)}$ is the left-sided shift on $X$.

We will need the following result of Furstenberg ([7]), whose precise formulation we take from [3, Theorem 1.2]:

**Theorem 2.4.** If $B \subset \mathbb{T}$ is an infinite closed set which is $x p$- and $x q$-invariant, then $B = \mathbb{T}$.

**Theorem 2.5.** The group $\mathbb{Z}[\frac{1}{pq}] \rtimes \mathbb{Z}^2$ is $C^*$-unique.

**Proof.** Let $I \lhd C(X) \rtimes \mathbb{Z}^2$ be under the identification given in Lemma 2.3, we have that $I \cap C(\mathbb{Z}[\frac{1}{pq}]) = \{0\}$. We will show that $I \cap C(X) = \{0\}$, and therefore $I = \{0\}$.

Notice that $C(X) \cap I$ is a $\mathbb{Z}^2$-invariant ideal of $C(X)$, hence there is $F \subset X$ a $\mathbb{Z}^2$-invariant closed set such that $C(X) \cap I = C_0(F^c)$.

For $n \in \mathbb{N}$, let $\pi_n : X \to \mathbb{T}$ be the canonical projection. Let $B := \pi_1(F)$. The fact that $F$ is $\mathbb{Z}^2$-invariant implies that $B = \pi_n(F)$ for $n \in \mathbb{N}$ and that

$$ B = \{ z^p : z \in B \} = \{ z^q : z \in B \}. $$

Since $I \cap C(\mathbb{Z}[\frac{1}{pq}]) = \{0\}$, we have that $B$ contains infinitely many points, for otherwise there would be a non-zero polynomial vanishing on $B$.

Using Theorem 2.5, we conclude that $B = \mathbb{T}$, hence $F = X$ and $I = \{0\}$.

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