Further results on skew monoid rings of a certain free monoid (I)

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Abstract: Let \( R \) be a ring with an endomorphism \( \sigma \), \( F \) be a free monoid and \( S \) be a factor of \( F \cup \{0\} \) such that \( (S \setminus \{1\})^n = 0 \) for some positive integer \( n \geq 2 \). The second author and Moussavi [Annihilator properties of skew monoid rings, Comm. Algebra, 42 (2) (2014), 842–852] started studying the skew monoid ring \( R[S;\sigma] \). In this paper, we continue the study of these rings.

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1. Introduction

Throughout this article, all rings are associative with identity, and 1 will always stand for the identity of the monoid \( S \) and the ring \( R \). For a ring \( R \), we denote by \( U(R) \) and \( J(R) \) the set of invertible elements and the Jacobson radical of \( R \), respectively. For a non-empty subset \( X \subset R \), \( rR(X) \) and \( /u1D4C1R(X) \) denote the right and left annihilators of \( X \) in \( R \), respectively. Also \( Z_r(R) \) and \( Z_r(R) \) denote the left and right singular ideals of \( R \), respectively.

Let \( n \geq 2 \) be a positive integer number and \( S \) be a free monoid generated by \( \{u_1, \ldots, u_t\} \) with 0 added, and the relations \( u_{i_1} \cdots u_{i_k} = 0 \) for all \( 1 \leq i_1, \ldots, i_k \leq t \). Note that maybe \( u_{i_1} \cdots u_{i_k} \) is equal to zero or not, for \( k < n \) but the product of \( n \) elements of \( \Omega \) is certainly zero. Let \( R \) be a ring with an endomorphism \( \sigma \) with \( \sigma(1) = 1 \). Then

\[
R[S;\sigma] = \{r_1s_1 + \cdots + r ks_k : r_1 \in R, s_i \in S, k \in \mathbb{N}\}
\]

is a ring with usual addition and multiplication subject to the relation

\[
u_ir = \sigma(r)u_i,
\]
for each \( u_i \in \mathbb{U} \) and \( r \in R \). The skew monoid ring \( R[S; \sigma] \) was introduced by Habibi and Moussavi (2014). They characterized various radicals of \( R[S; \sigma] \). These rings are perhaps the most interesting class of non-semiprime rings and may provide many surprising examples and counterexamples in ring theory. Motivated by results in Habibi and Moussavi (2014), Paykan (2017), and Paykan and Arjomandfar (2017), we will obtain criteria for \( R[S; \sigma] \) to satisfy various conditions on rings.

The main purpose of this paper is to continue the study of the skew monoid rings \( R[S; \sigma] \). In Section 2, we give some examples of these rings in order to familiarize the reader with the concept. This section, the reader shall realize the importance of this structure in matrix theory. In Section 3, we characterize when \( R[S; \sigma] \) is left quasi-duo, clean, exchange, Hermite, semiregular and \( J \)-ring, respectively. Also, we calculate the right singular ideal of \( R[S; \sigma] \) under appropriate conditions. As a corollary we show that, \( R \) is right weakly continuous if and only if \( R[S; \sigma] \) is right weakly continuous. In particular, prove that several properties, including the semiboolean, stable finite, 2-good, and stable range one property, transfer between \( R \) and the extension \( R[S; \sigma] \). Also, we show that \( R[S; \sigma] \) is a (strongly) nil-clean ring if and only if \( R \) is a (strongly) nil-clean ring. As an application, we provide (apparently) new examples of the aforementioned ring constructions.

2. Examples

For a positive integer \( n \), a ring \( R \) and an endomorphism \( \sigma \), in Chen, Yang and Zhou (2006) defined the skew triangular matrix ring \( T_n(R, \sigma) \) as the set of all upper triangular \( n \times n \) matrices over \( R \) with pointwise addition and a new multiplication subject to the condition

\[
E_{ij} r = a^{i-1}(r)E_{ij},
\]

where \( E_{ij} \) is the matrix unit for each \( i \) and \( j \). So for any \( (a_{ij}) \) and \( (b_{ij}) \) in \( T_n(R, \sigma) \) we have \( (a_{ij})(b_{ij}) = (c_{ij}) \), where

\[
c_{ij} = a_i b_j + a_{i+1} \sigma(b_{i+1}) + \cdots + a_j \sigma^{j-i}(b_j),
\]

for each \( i \leq j \). The subring of the ring \( T_n(R, \sigma) \) consisting of triangular matrices with constant main diagonal is denoted by \( S(R, n, \sigma) \), whereas the subring of \( T_n(R, \sigma) \) consisting of triangular matrices with constant diagonals is denoted by \( T(R, n, \sigma) \). We can denote \( A = (a_{ij}) \in T(R, n, \sigma) \) by \( (a_{11}, a_{12}, \ldots, a_{nn}) \). Then \( T(R, n, \sigma) \) is a ring with pointwise addition and with multiplication given by:

\[
(a_{0}, \ldots, a_{n-1})(b_{0}, \ldots, b_{n-1}) = (a_0 b_0, a_0 * b_1 + a_1 * b_0, \ldots, a_0 * b_{n-1} + \cdots + a_{n-1} * b_0),
\]

with \( a_i * b_j = a_i \sigma^j(b_j) \), for each \( i \) and \( j \).

We consider the following two subrings of \( S(R, n, \sigma) \), as follow (see Habibi, Moussavi, & Mokhtari, 2012):

\[
A(R, n, \sigma) = \left\{ \sum_{j=1}^{n-1} \sum_{i=1}^{n} a_{ij} E_{ij} \mid a_{ij} \in R \right\};
\]

\[
B(R, n, \sigma) = \{ A + rE_{1k} \mid A \in A(R, n, \sigma) \text{ and } r \in R \} \quad n = 2k \geq 4.
\]

For example:

\[
A(R, 3, \sigma) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\}.
\]
\[
A(R,4,\sigma) = \left\{ \begin{pmatrix}
 a_1 & a_2 & a & b \\
 0 & a_1 & a_2 & c \\
 0 & 0 & a_1 & a_2 \\
 0 & 0 & 0 & a_1 \\
\end{pmatrix} \mid a_1, a_2, a, b, c \in R \right\}.
\]

Below we show how the aforementioned classical ring constructions can be viewed as special cases of the skew monoid ring construction of a certain free monoid.

**Example 2.1** Let \( R \) be a ring with an endomorphism \( \sigma \) and \( u \) be a matrix in \( T_n(R,\sigma) \) as follows:

\[ u = \begin{pmatrix}
 0 & 1 & 0 & \cdots & 0 \\
 0 & 0 & 1 & \cdots & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & \ddots & \ddots & 1 \\
 0 & 0 & \cdots & \cdots & 0 \\
\end{pmatrix}. \]

So we have

\[ u^l = E_{1,1} + E_{2,2} + \cdots + E_{n,n} \]

for each \( 2 \leq l \leq n - 1 \) and \( u^0 = 0 \). Let \( S \) be a free monoid generated by \( \{u\} \) with 0 added and the relation \( u^0 = 0 \). Thus

\[ S = \{0, I_n, u, u^2, \ldots, u^{n-1}\}, \]

where \( I_n \) is the identity matrix of order \( n \). Therefore, we have

\[ R[S,\sigma] = \{ a_0 I_n + a_1 u + \cdots + a_{n-1} u^{n-1} \mid a_i \in R \ \forall i = 0, \ldots, n - 1 \}. \]

So

\[ R[S,\sigma] = \left\{ \begin{pmatrix}
 a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \\
 0 & a_0 & a_1 & \cdots & a_{n-2} \\
 0 & 0 & a_0 & \ddots & \vdots \\
 \vdots & \vdots & \ddots & \ddots & a_1 \\
 0 & 0 & \cdots & 0 & a_0 \\
\end{pmatrix} \mid a_i \in R \ \forall i = 0, \ldots, n - 1 \right\}. \]

and hence \( R[S,\sigma] = T(R, n, \sigma) \).

**Example 2.2** Let \( R \) be a ring with an endomorphism \( \sigma \). It is not hard to see that

\[ \varphi: R[x_\sigma]/(x^n) \to T(R, n, \sigma), \]

given by \( \varphi(\sum_{i=0}^{n-1} a_i x^i) = (a_0, a_1, \ldots, a_{n-1}) \), with \( a_i \in R \), \( 0 \leq i \leq n - 1 \) is a ring isomorphism, where \( R[x_\sigma] \) is the skew polynomial ring with multiplication subject to the condition

\[ x_\sigma = \sigma(x) \]

for each \( r \in R \), and \( (x^n) \) is the ideal generated by \( x^n \). So \( R[x_\sigma]/(x^n) \cong T(R, n, \sigma) \) and consequently \( R[x_\sigma]/(x^n) \) fit into the structure introduced in introduction.

**Example 2.3** Let \( R \) be a ring with an endomorphism \( \sigma \), and let \( S \) be a free monoid generated by \( \{u, v\} \) with 0 added and the relation

\[ u^2 = v^3 = vu = 0. \]

Thus \( R[S,\sigma] = \{ a + bv + cv^2 + du + euv + fuv^2 \mid a, b, c, d, e, f \in R \} \).
Example 2.4 Let \( R \) be a ring with an endomorphism \( \sigma \) and \( n \geq 2 \) be a positive integer number. Assume that \( S \) is a free monoid generated by the set of all elements \( t_{i,j} \) where \( i = 1, \ldots, n - 1 \) with 0 added and the relations \( E_{i,i+1} \cdots E_{k,k+1} = 0 \) for all \( 1 \leq i, k \leq n - 1 \). Then we have
\[
R[S;\sigma] = \begin{pmatrix}
\alpha & a_{1,2} & \cdots & a_{1,n-1} & a_{1,n} \\
0 & a & \cdots & a_{2,n-1} & a_{2,n} \\
0 & 0 & a & \cdots & a_{3,n-1} & a_{3,n} \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 0 & a
\end{pmatrix}
\]
and hence \( R[S;\sigma] = S(R, n, \sigma) \).

Example 2.5 Let \( R \) be a ring with an endomorphism \( \sigma \) and \( n \geq 2 \) be a positive integer number. Suppose \( u = E_{1,2} + \cdots + E_{n-1,n} \) is a matrix in \( T_n(R, \sigma) \) as mentioned as in Example 2.1. Let \( \mathcal{U} = \{u, E_{1,2} + E_{2,3} + \cdots + E_{n-1,n}\} \). Put \( k = \frac{n}{2} \). Now assume \( F \cup \{0\} \) is a free monoid generated by \( \mathcal{U} \) with 0 added, and that \( S \) is a factor of \( F \) setting certain monomials in \( \mathcal{U} \) to 0, enough so that, \( (S \setminus \{1\})^0 = 0 \). Then we have
\[
R[S;\sigma] = \begin{pmatrix}
a_1 & \cdots & a_k & a_{k+1} & \cdots & a_{1,n} \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & a_1
\end{pmatrix}
\]
and hence \( R[S;\sigma] = A(R, n, \sigma) \).

Example 2.6 Let \( R \) be a ring with an endomorphism \( \sigma \) and \( n = 2k \) be a positive integer number. Let \( \mathcal{U} = \{u, E_{1,2} + E_{2,3} + \cdots + E_{n-k,n}\} \). Now assume \( F \cup \{0\} \) is a free monoid generated by \( \mathcal{U} \) with 0 added, and that \( S \) is a factor of \( F \) setting certain monomials in \( \mathcal{U} \) to 0, enough so that, \( (S \setminus \{1\})^0 = 0 \). Then, we have
\[
R[S;\sigma] = \begin{pmatrix}
a_1 & \cdots & a_k & a_{k+1} & \cdots & a_{1,n} \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & a_1
\end{pmatrix}
\]
and hence \( R[S;\sigma] = B(R, n, \sigma) \).

The above examples mentioned only for familiarizing the reader with the structure \( R[S;\sigma] \). Although various examples of this structure exist in the Pure algebra, but as far as in the above example was observed, only applied examples are related to matrix theory. Therefore, the results obtained in this paper can be widely used in matrix theory.

3. Main Results
Recall that a ring \( R \) is local if \( RJ(R) \) is a division ring, and \( R \) is semilocal if \( RJ(R) \) is a semisimple ring. A ring \( R \) is said to be matrix local if \( RJ(R) \) is a simple Artinian ring. A ring \( R \) is said to be semiperfect if \( R \) is a semilocal ring and all idempotents of the Artinian ring \( RJ(R) \) can be lifted to idempotents of the ring \( R \). Due to Nicholson (1975), a ring \( R \) is said to be an \( I \)-ring if every right ideal of the ring \( R \) not contained in \( J(R) \) contains a nonzero idempotent and all idempotents of the ring \( RJ(R) \) can be lifted
to idempotents of the ring \( R \). Recall from Nicholson (1976) that a ring \( R \) is semiregular if \( R/J(R) \) is a (von Neumann) regular ring and idempotents can be lifted modulo \( J(R) \). According to Nicholson (2004), a ring \( R \) is called a clean (uniquely clean) ring if every element \( r \in R \) can be written (uniquely) in the form \( r = u + e \) where \( u \) is a unit in \( R \) and \( e^2 = e \in R \).

Throughout this section, we assume that all the monoids \( S \) is a free monoid generated by \( \{u_1, \ldots, u_l\} \) with 0 added, and the relations \( u_i \cdots u_i = 0 \) for all \( 1 \leq i_1, \ldots, i_n \leq t \), where \( n \geq 2 \) is a natural number. We start with the following lemma, which plays a key role in the sequel.

**Lemma 3.1** Let \( R \) be a ring and \( \sigma \) an endomorphism of \( R \). Then

\[
U(R[S;\sigma]) = \left\{ \sum_{s \in S} r_s s \in R[R;\sigma] \mid r_s \in U(R) \right\}.
\]

**Proof** Assume that \( I = \langle u_1, \ldots, u_l \rangle \) is the ideal of \( T = R[S;\sigma] \) generated by \( u_1, \ldots, u_l \). Then \( I \) is nilpotent and so \( I \subseteq J(T) \). Note also that \( R \cong T/I = T \). Now, let

\[
\alpha = a + \sum_{\sigma \in S} a_{\sigma} u_{\sigma} + \sum_{\sigma \in S} a_{\sigma} u_{\sigma} u_{\sigma} + \cdots + \sum_{\sigma \in S} a_{\sigma} u_{\sigma} u_{\sigma} \cdots u_{\sigma}
\]

be an element of \( T \) with \( a \in U(R) \). Thus, there exists \( b \in R \) such that \( ab = 1 \). Therefore, \( \bar{\alpha} \bar{b} = \bar{1} \). Then \( 1 - ab \in I \subseteq J(T) \) and so \( ab \in 1 + J(T) \subseteq U(T) \). Clearly, this implies that \( \alpha \) has a right inverse in \( T \). Analogously as above, one can show that \( \alpha \) has a left inverse in \( T \) and thus \( a \in U(T) \). Conversely, if

\[
\alpha = a + \sum_{\sigma \in S} a_{\sigma} u_{\sigma} + \sum_{\sigma \in S} a_{\sigma} u_{\sigma} u_{\sigma} + \cdots + \sum_{\sigma \in S} a_{\sigma} u_{\sigma} u_{\sigma} \cdots u_{\sigma}
\]

is an element of \( U(T) \), then one can easily verify that \( a \in U(R) \) and the result follows. \( \square \)

**Lemma 3.2** [Habibi & Moussavi, 2014 Theorem 2.9(i)] Let \( R \) be a ring and \( \sigma \) an endomorphism of \( R \). Then

\[
J(R[S;\sigma]) = \left\{ \sum_{s \in S} r_s s \in R[R;\sigma] \mid r_s \in J(R) \right\}.
\]

**Proof** Assume that \( \alpha = \sum_{s \in S} r_s s \) is an element of \( T = R[S;\sigma] \) with \( r_s \in J(R) \) and \( \beta = \sum_{s \in S} p_s s \) an arbitrary element of \( T \). Thus \( 1 - p_s r_s \in U(R) \) and by Lemma 3.1, \( 1 - \beta \in U(T) \) which means that \( \alpha \in J(T) \). Conversely, suppose that \( \alpha \in J(I) \). So \( 1 - \alpha a \in U(I) \) for each \( a \in R \) and hence \( 1 - \alpha r_s \in U(R) \), by Lemma 3.1. Therefore, \( r_s \in J(R) \) and we are done. \( \square \)

**Lemma 3.3** Let \( R \) be a ring and \( \sigma \) an endomorphism of \( R \), and \( T = R[S;\sigma] \). Then, we have the following:

1. The factor ring \( R/J(R) \) is naturally isomorphic to the factor ring \( T/J(T) \).
2. All idempotents of \( T/J(T) \) can be lifted to \( T \) if and only if all idempotents of the factor ring \( R/J(R) \) can be lifted to \( R \).

**Proof** It is easy to show that the map \( \psi : T \to R/J(R) \) via \( \psi \left( \sum_{s \in S} r_s s \right) = r_1 + J(R) \) is a ring epimorphism with \( \ker(\psi) = J(T) \), from Lemma 3.2.

2. The result follows from (1).

**Theorem 3.4** Let \( R \) be a ring and \( \sigma \) an endomorphism of \( R \). Then:

1. \( R[S;\sigma] \) is a local ring if and only if \( R \) is a local ring.
2. \( R[S;\sigma] \) is a semilocal ring if and only if \( R \) is a semilocal ring.
3. \( R[S;\sigma] \) is a matrix local ring if and only if \( R \) is a matrix local ring.
(4) \( R[S;\sigma] \) is a semiperfect ring if and only if \( R \) is a semiperfect ring.

(5) \( R[S;\sigma] \) is isomorphic to a full matrix ring over a local ring if and only if \( R \) is isomorphic to a full matrix ring over a local ring.

(6) \( R[S;\sigma] \) is an \( I \)-ring if and only if \( R \) is an \( I \)-ring.

(7) \( R[S;\sigma] \) is a semiregular ring if and only if \( R \) is a semiregular ring.

(8) \( R[S;\sigma] \) is a clean ring if and only if \( R \) is a clean ring.

**Proof** (1–3) follow from the fact that the ring \( R / J(R) \) is isomorphic to the ring \( R[S;\sigma] / J(R[S;\sigma]) \), by Lemma 3.3.

(4) This result is a direct consequence of (2) and Lemma 3.3.

(5) By [Lam, 1991, Theorem 23.10] the class of all matrix local semiperfect rings coincides with the class of all rings that are isomorphic to full matrix rings over local rings. Therefore, part (5) follows from (3) and (4).

(6) The result follows from [Nicholson, 1975, Proposition 1.4] and Lemma 3.3, since \( R/J(R) \) is an \( I \)-ring and all idempotents of the ring \( R/J(R) \) can be lifted to idempotents of the ring \( R \).

(7) This result is a consequence of Lemma 3.3.

(8) The result follows from [Camillo & Yu, 1994, Proposition 7] and Lemma 3.3, since \( R/J(R) \) is a clean ring if and only if \( R/J(R) \) is a clean ring and all idempotents of the ring \( R/J(R) \) can be lifted to idempotents of the ring \( R \).

A ring \( R \) is called right (left) quasi-duo if every maximal right (left) ideal of \( R \) is two-sided or, equivalently, every right (left) primitive homomorphic image of \( R \) is a division ring. Examples of right quasi-duo rings include, for instance, commutative rings, local rings, rings in which every non-unit has a (positive) power that is central, endomorphism rings of uniserial modules, power series rings and rings of upper triangular matrices over any of the above-mentioned rings (see Yu, 1995). But the \( n \)-by-\( n \) full matrix rings over right quasi-duo rings are not right quasi-duo (for more details see Lam & Alex, 2005; Leroy, Matczuk, & Puczylowski, 2008; Yu, 1995).

A ring \( R \) is said to be Dedekind finite if every element is the sum of two units. The ring of all \( n \)-by-\( n \) matrices over an elementary divisor ring is Dedekind-finite (for more details see Montgomery, 1983). Recall that a module \( \mu M \) has the (full) exchange property if for every module \( \mu A \) and any two decompositions \( A = M \bigoplus N = \bigoplus A_i \) with \( M \cong M \), there exist submodules \( A_i \subseteq A_i \) such that \( A = M \bigoplus (\bigoplus A_i) \). A module \( \mu M \) has the finite exchange property if the above condition is satisfied whenever the index set \( I \) is finite.

Recall that a ring \( R \) is semiboolean if and only if \( R/J(R) \) is Boolean and idempotents of \( R \) lift modulo \( J(R) \). According to [Nicholson & Zhou, 2004, Theorem 19], \( R \) is a Boolean ring if and only if \( R \) is uniquely clean and \( J(R) = 0 \). By Lemma 3.2, for an arbitrary (Boolean) ring \( R \), the ring \( R[S;\sigma] \) is not Boolean. But we will show that \( R[S;\sigma] \) is semiboolean if and only if \( R \) is semiboolean.

According to Vámos (2005), a ring \( R \) is said to be 2-good if every element is the sum of two units. The ring of all \( n \)-by-\( n \) matrices over an elementary divisor ring is 2-good (where the ring \( R \) is elementary divisor if, for every positive integer \( n \), every element of all \( n \)-by-\( n \) matrices with entries from \( R \), is equivalent to a diagonal matrix). A (right) self-injective von Neumann regular ring is 2-good provided it has no 2-torsion. In Wang and Ren (2013), showed that the 2-good property is preserved in extensions such as skew power series rings, full matrix rings, formal triangular matrix rings, upper triangular matrix rings, and trivial extension rings. (for more details see Vámos, 2005; Wang & Ren, 2013).
Theorem 3.5 Let \( R \) be a ring and \( \sigma \) an endomorphism of \( R \). Then:

1. \( R[S, \sigma] \) is left quasi-duo if and only if \( R \) is left quasi-duo.
2. \( R[S, \sigma] \) is stably finite if and only if \( R \) is stably finite.
3. \( R[S, \sigma] \) is exchange if and only if \( R \) is exchange.
4. \( R[S, \sigma] \) is 2-good if and only if \( R \) is 2-good.
5. \( R[S, \sigma] \) is semiboolean if and only if \( R \) is semiboolean.

Proof (1) The result follows from Lemma 3.3, since a ring \( R \) is left quasi-duo if and only if so is \( R/J(R) \).

(2) Let \( T = R[S, \sigma] \) be stably finite. Clearly, the subring \( R \) is also stably finite. Conversely, suppose that \( R \) is stably finite. Consider the ideal \( < u_1, \ldots, u_n > \) in \( T \). By Lemma 3.2, \( < u_1, \ldots, u_n > \subseteq J(T) \). We have \( T/ < u_1, \ldots, u_n > \cong R \), so by [Montgomery, 1983, Lemma 2], the fact that \( R \) is stably finite implies that \( T \) is stably finite.

(3) The result follows from [Nicholson, 1977, Corollary 2.4] and Lemma 3.3, since \( R \) is an exchange ring if and only if \( R/J(R) \) is an exchange ring and all idempotents of the ring \( R/J(R) \) can be lifted to idempotents of \( R \).

(4) Let \( R \) be 2-good and \( a = \sum r_i s_i \in R[S, \sigma] \). Write \( r_i = a + b \), where \( a, b \in U(R) \). Then, by Lemma 3.1, \( \beta, \beta \in U(R[S, \sigma]) \) and so \( R[S, \sigma] \) is 2-good. Conversely, if \( R[S, \sigma] \) is 2-good, then by analogy with the proof of part (2), we can show that the ring \( R \) is a homomorphic image of \( R[S, \sigma] \). By [Wang & Ren, 2013, Proposition 2.15], every homomorphic image of a 2-good ring is again 2-good, and therefore, the result follows.

(5) The result follows from Lemma 3.3.

Let \( S' \) be the free monoid generated by \( U = \{ u \} \) with 0 added, with the relation \( u^0 = 0 \). In the following, we calculate the right singular ideal of \( R[S', \sigma] \) under an appropriate condition.

Proposition 3.6 Let \( R \) be a ring and \( \sigma \) an epimorphism of \( R \). Then

\[
Z_r(R[S'; \sigma]) := \left\{ \sum_{i \in S} r_is_i \in R[S'; \sigma] \mid r_i \in Z_r(R) \right\}.
\]

Proof Suppose that \( a = r_1 + r_2 u + r_3 u^2 + \cdots + r_n u^{n-1} \) be an element in \( R[S, \sigma] \). First, let \( a \in Z_r(R[S'; \sigma]) \). Then \( aI = 0 \) for some essential right ideal \( I \) of \( R[S'; \sigma] \). Let \( I_0 \) be the set of all leading coefficients of elements of \( I \). Then \( I_0 \) is a right ideal of \( R \). On the other hand, we also have \( r_i I_0 = 0 \). We show that \( I_0 \) is an essential right ideal of \( R \). Let \( a \) be an arbitrary nonzero element of \( R \). There exists \( \beta \in R[S, \sigma] \) such that \( 0 \neq \alpha \beta \in I \). Hence, \( 0 \neq ab \in I_0 \) for some \( b \in R \). Thus \( I_0 \) is an essential right ideal of \( R \). Conversely, let \( r_i \in Z_r(R) \). There exists an essential right ideal \( J \) of \( R \) such that \( r_i J = 0 \). We show that \( I = Ju^{n-1} \) is an essential right ideal of \( R[S'; \sigma] \). Assume that \( \beta = b_1 + b_2 u + b_3 u^2 + \cdots + b_n u^{n-1} \) is an arbitrary nonzero element of \( R[S', \sigma] \). Also, let \( j \) be the minimum index such that \( b_j \neq 0 \). There exists \( y \in R \) such that \( 0 \neq b_j y \in J \). Since \( \sigma \) is an epimorphism of \( R \), there is \( x \) such that \( \sigma^{j-1}(x) = y \). Set \( r := xu^{n-j-1} \in R[S', \sigma] \). Then \( 0 \neq r \beta = b_j xu^{n-1} \in I \). So \( I \) is an essential right ideal of \( R[S, \sigma] \). On the other hand, we have \( aI = 0 \). Thus \( a \in Z_r(R[S'; \sigma]) \), and the proof is complete.

Recall the a ring \( R \) is said a right Rickart ring if every principal right ideal in \( R \) is projective (as a right \( R \)-module). A ring \( R \) is called right non-singular if \( Z_r(R) = 0 \). It is well known that the right Rickart ring is right non-singular.

Corollary 3.7 Let \( T = R[S'; \sigma] \) where \( R \) is a nonzero ring with an epimorphism \( \sigma \). Then \( T \) is not right non-singular. (In particular, the Rickart property of \( R \) is not preserved by \( R[S'; \sigma] \).)
COROLLARY 3.8 Let R be a ring and σ an epimorphism of R. If R is right non-singular, then \( Z_{tr}(R[S;\sigma]) \) is nilpotent.

A ring R is called right weakly continuous if R is semiregular and \( Z_{tr}(R) = \mathcal{J}(R) \). By [Nicholson & Yousif, 2003, Theorem 7.40] right weakly continuous is a Morita invariant property of rings. In the following Theorem 3.9, we prove that the right weakly continuous property of R preserves by \( R[S;\sigma] \).

Theorem 3.9 Let R be a ring and σ an epimorphism of R. Then \( R[S;\sigma] \) is right weakly continuous if and only if R is right weakly continuous.

Proof The result follows from Lemma 3.2, Theorem 3.4 (7) and Proposition 3.6.

COROLLARY 3.10 [Nasr-Isfahani, 2013, Corollary 3.8] Let R be a ring and σ an epimorphism of R. Then R is right weakly continuous if and only if \( R[x;\sigma]/(x^n) \) is right weakly continuous, for each positive integer n ≥ 2.

From Cohn (cohn), a ring R is said to be projective-free if every finitely generated projective left (equivalently right) R-module is free of unique rank. A ring homomorphism is called local if every non-unit is mapped to a non-unit.

THEOREM 3.11 Let R be a projective-free ring and σ an endomorphism of R. Then \( R[S;\sigma] \) is a projective-free ring.

Proof There is a ring epimorphism \( \psi: R[S;\sigma] \rightarrow R \) which sends \( \sum a_i s \) to \( a_i \). By Lemma 3.1, it follows that \( \psi \) is local. By [Cohn, 2003, Corollary 4], any ring with a surjective local homomorphism to a projective-free ring is projective-free, and so, the result follows.

As an immediate consequence of Theorem 3.11, we obtain the following.

COROLLARY 3.12 Let R be a projective-free ring and σ an endomorphism of R. Then the rings \( S(R, n, \sigma) \), \( T(R, n, \sigma) \), \( A(R, n, \sigma) \), and \( B(R, n, \sigma) \) are projective-free, for each positive integer n ≥ 2.

A ring R is called a Hermite ring provided for every \( (r_1, \ldots, r_m) \in R^m \), if there exists \( (p_1, \ldots, p_m) \in R^m \) such that \( r_1 p_1 + \cdots + r_m p_m = 1 \), then there exists a \( m \times m \) matrix \( M \) over R with first row \( (r_1, \ldots, r_m) \) and \( \det(M) \) is a unit in \( R \).

THEOREM 3.13 \( R[S;\sigma] \) is a Hermite ring if and only if R is a Hermite ring.

Proof Let \( R[S;\sigma] \) be a Hermite ring. Suppose that \( (r_1, \ldots, r_m) \) and \( (p_1, \ldots, p_m) \) are in \( R^m \) such that \( r_1 p_1 + \cdots + r_m p_m = 1 \). Since \( R[S;\sigma] \) is a Hermite ring, then exists \( m \times m \) matrix \( M \) over \( R[S;\sigma] \) with first row \( (r_1, \ldots, r_m) \) and \( \det(M) \) is a unit in \( R[S;\sigma] \). Suppose that

\[
M = \begin{pmatrix}
  r_1 & r_2 & \cdots & r_m \\
  a_{21} & a_{22} & \cdots & a_{2m} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mm}
\end{pmatrix}
\]

where

\[
a_{ui} = a_{i(1,0)}^{(1,0)} + \sum_{1 \leq i_1 < \cdots < i_k \leq n} a_{i_1i_2}^{(1,0)} u_{i_1} + \sum_{1 \leq i_1 < \cdots < i_k \leq n} a_{i_1i_2}^{(1,0)} u_{i_1} u_{i_2} + \cdots + \sum_{1 \leq i_1 < \cdots < i_k \leq n} a_{i_1i_2}^{(n-1)} u_{i_1} \cdots u_{i_{n-1}}
\]
is the element of $R[S;r]$. Denote

$$N = \begin{pmatrix}
    a^{(1,0)}_1 & r_2 & \cdots & r_m \\
    a^{(2,0)}_1 & \ddots & \cdots & \vdots \\
    \vdots & \ddots & \ddots & \vdots \\
    a^{(m,0)}_1 & \cdots & \cdots & a^{(m,m,0)}
\end{pmatrix}$$

Since

$$\det(N) = \sum_{i=1}^{m} \prod_{j=1}^{n} (-1)^{r_j^i} a^{(i,0)}_j \in U(R[S;r])$$

by Lemma 3.1, it follows that

$$\det(M) = \sum_{i=1}^{m} \prod_{j=1}^{n} (-1)^{r_j^i} a^{(j,0)}_i \in U(R)$$

Thus $R$ is a Hermite ring. Conversely, let $R$ be a Hermite ring. Assume that $(a_1, \ldots, a_m)$ and $(b_1, \ldots, b_m)$ are in $(R[S;r])^m$ such that $a_1 b_1 + \cdots + a_m b_m = 1$, then

$$a_k = a^{(k,0)} + \sum_{i=1}^{m} a^{(k,1)}_i u^i_i + \sum_{i=1}^{m} a^{(k,2)} u^i_i + \cdots + \sum_{i=1}^{m} a^{(k,m-1)} u^i_i \cdots u^i_{m-1}$$

and

$$b_k = b^{(k,0)} + \sum_{i=1}^{m} b^{(k,1)}_i u^i_i + \sum_{i=1}^{m} b^{(k,2)} u^i_i + \cdots + \sum_{i=1}^{m} b^{(k,m-1)} u^i_i \cdots u^i_{m-1}.$$ 

Then $a^{(1,0)} + a^{(2,0)} + \cdots + a^{(m,0)} = 1$. Since $R$ is a Hermite ring, there exists a $m \times m$ matrix

$$P = \begin{pmatrix}
    a^{(1,0)} & a^{(2,0)} & \cdots & a^{(m,0)} \\
    a_{11} & a_{22} & \cdots & a_{2m} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & \cdots & a_{mm}
\end{pmatrix}$$

over $R$ with first row $(a^{(1,0)}, a^{(2,0)}, \ldots, a^{(m,0)})$ and $\det(P) \in U(R)$. Let

$$Q = \begin{pmatrix}
    a_1 & a_2 & \cdots & a_m \\
    a_{11} & a_{22} & \cdots & a_{2m} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & \cdots & a_{mm}
\end{pmatrix}$$

Then, it is easy to see that the constant coefficient of $\det(Q)$ is equal to $\det(P)$ and thus in $U(R)$. Hence, by Lemma 3.1, it follows that $\det(Q) \in U(R[S;r]).$ This proves that $R[S;r]$ is a Hermite ring.

Recall that a $R$-module $P$ is a stably free module if there exist $m, n \in \mathbb{N}$ such that $P \oplus R^n = R^n$. Clearly free modules are stably free. It is well known that $R$ is a Hermite ring if and only if every stably free $R$-module is free (see, for example, Zhou, 1988, pp. 357–358). Thus, by Theorem 3.13, we have:

**Corollary 3.14** The following conditions are equivalent:

1. Every stably free $R$-module is free.
2. Every stably free $R[S;r]$-module is free.

Recall that a module is said to be uniserial if any two of its submodules are comparable with respect to inclusion, i.e. any two of its cyclic submodules are comparable by set inclusion. A ring $R$ is called a right (resp. left) uniserial ring, if $R_R$ (resp. $R_L$) is a uniserial module. Right uniserial rings are also called right chain rings, or right valuation rings, since they are obvious generalizations of
commutative valuation domains. Like commutative valuation domains, right uniserial rings have a rich theory and they offer remarkable examples (e.g. a right and left uniserial ring exists which is prime but not a domain).

For a right $R$-module $M$, let $M[S]$ denotes the monoid module formed by all finite formal combinations $\sum m_s S$ such that $m_s \in M$ for each $s \in S$. Then $M[S]$ is an abelian group under an obvious addition operation. Moreover, $M[S]$ becomes a right module over the skew monoid ring $R[S,\sigma]$ by allowing monomial from $R[S,\sigma]$ to act on monomial in $M[S]$ in the obvious way, and applying the above “twist” whenever necessary.

**Theorem 3.15** Let $R$ be a ring and $\sigma$ an endomorphism of $R$, and $M$ a right $R$-module. If $M[S]$ is a uniserial right $R[S,\sigma]$-module, then $M$ is a simple right $R$-module.

**Proof** Suppose that $0 \neq m \in M$. We will show that $M = mR$. Set $A = mR[S,\sigma]$. Let $B$ denotes the elements $\phi = \sum m_s S$ of $M[S]$ such that $m_1 = 0$. It is easy to show that $B$ is right $R[S,\sigma]$-submodule of $M[S]$ and $B \subseteq A$, since $M[S]$ is a uniserial right $R[S,\sigma]$-module and $0 \neq m \in M$. Now, let $n \in M$. Then for any $s \neq 1$, $\phi = ns \in B$, and so $\phi \in A$. Hence $n \in mR$. This implies that $M = mR$. Hence, $M$ is a simple right $R$-module.

In the following, we state that the converse of Theorem 3.15 is not true, in general.

**Example 3.16** Let $K$ be a field with an automorphism $\sigma$, and let $S$ be a free monoid generated by $\mathbb{U} = \{u, v\}$ with 0 added and the relation $u^2 = v^2 = vu = 0$.

Thus $K$ is simple, as a right $K$-module, but the submodules $Ku$ and $Kv$ are not comparable with respect to inclusion.

**Theorem 3.17** Let $R$ be a ring and $\sigma$ an epimorphism of $R$, and $M$ a right $R$-module. Then the following conditions are equivalent:

1. $M$ is a simple right $R$-module.
2. $M[S]$ is a uniserial right $R[S,\sigma]$-module.

**Proof** (1) $\Rightarrow$ (2) Suppose $\phi = p_1 + p_2 u + \cdots + p_n u^{n-1}$ is elements of $M[S']$. Let $j$ be an smallest index with property $p_j \neq 0$. Hence $p_j R = M$, since $M$ is a simple right $R$-module. Thus,

$$\phi R[S',\sigma] = \{m_{j_1} u^1 + m_{j_2} u^2 + \cdots + m_{j_k} u^{j_k} : m_{j_1}, m_{j_2}, \ldots, m_{j_k} \in M\}.$$

Now, let $\phi_1 = a_1 + a_2 u + \cdots + a_{n-1} u^{n-1}$ and $\phi_2 = b_1 + b_2 u + \cdots + b_{n-1} u^{n-1}$ be two elements of $M[S']$. Without loss of generality, one can assume that $j_1 \leq j_2$ where $j_1, j_2$ are smallest indices of $\phi_1, \phi_2$ with property $a_{j_1} \neq 0$ ($b_{j_2} \neq 0$). Thus $\phi_2 R[S',\sigma] \subseteq \phi_1 R[S',\sigma]$. It follows that $M[S]$ is a uniserial right $R[S,\sigma]$-module.

(2) $\Rightarrow$ (1) is clear, by Theorem 3.15.

Let $M$ be a right $R$-module. Recall that $M$ is a serial module if $M$ is a direct sum of uniserial modules. A ring $R$ is called a right (resp. left) serial ring, if $R$ (resp. $R$) is a serial module. A ring $R$ is called a serial ring, if $R$ is both a right and a left serial ring. It is well known that every serial Noetherian ring satisfies the restricted minimum condition. In particular, following Warfield (1975), such a ring is a direct sum of an Artinian ring and hereditary prime rings.
Proposition 3.18 Let $R$ be a ring and $\sigma$ an epimorphism of $R$. If $M$ is a semisimple Artinian right $R$-module, then $M[S']$ is a serial right $R[S';\sigma]$-module.

Proof Assume $M = M_1 \oplus \cdots \oplus M_n$, where $M_i$ is a simple right $R$-module, $i = 1, \ldots, n$ and $n \in \mathbb{N}$. Then $M[S] \cong M_1[S] \oplus \cdots \oplus M_n[S']$. By Theorem 3.17, $M_1[S]$ is a serial right $R[S';\sigma]$-module. Therefore, $M[S']$ is a serial right $R[S';\sigma]$-module, and the proof is complete.

A ring $R$ is said to have right stable range one if, whenever $aR + bR = R$, for $a, b \in R$, there exists $c \in R$ such that $a + bc \in U(R)$. The stable range one condition is especially interesting because of Evans’ Theorem (1973), which states that a module $M$ cancels from direct sums whenever $\text{End}_R(M)$ has stable range one.

Proposition 3.19 Let $R$ be a ring and $\sigma$ an endomorphism of $R$. Then $R[S;\sigma]$ has right stable range one if and only if $R$ has right stable range one.

Proof Let $R$ has right stable range one and $\alpha = \sum_{i \in S} a_i s_i$, $\beta = \sum_{i \in S} b_i s_i$ be elements of $T = R[S;\sigma]$ such that $\alpha T + \beta T = T$. Therefore $\alpha R + \beta R = R$, and hence there exists $c \in R$ such that $\alpha_i + bc_i \in U(R)$. Thus $\alpha + bc \in U(T)$, by Lemma 3.1. Conversely, suppose that $R[S;\sigma]$ has right stable range one and $a, b$ are elements of $R$ such that $aR + bR = R$. Thus $aT + bT = T$, and hence there exists $\alpha = \sum_{i \in S} r_i s_i \in T$ such that $\alpha + br_1 \in U(T)$. Thus $\alpha + br_1 \in U(R)$, and the result follows.

Following Diesl (2013), an element of a ring is called (strongly) nil-clean if it is a sum of an idempotent and a nilpotent (that commute with each other), and a ring is called (strongly) nil-clean if every element is (strongly) nil-clean. The following theorem can be used to produce more examples of nil-clean and strongly nil-clean rings.

Theorem 3.20 Let $R$ be a ring and $\sigma$ an endomorphism of $R$. Then:

1. $R[S;\sigma]$ is a nil-clean ring if and only if $R$ is a nil-clean ring.
2. $R[S;\sigma]$ is strongly nil-clean if and only if $R$ is strongly nil-clean.

Proof (1) This result is a direct consequence of [Diesl, 2013, Corollary 3.17], Lemma 3.2 and Lemma 3.3(1), since $R$ is a nil-clean ring if and only if $J(R)$ is nil and $R / J(R)$ is nil-clean.

(2) The result follows from [Koşan, Wang, & Zhou, 2016, Theorem 2.7], Lemma 3.3(1) and Lemma 3.2, since $R$ is strongly nil-clean if and only if $R / J(R)$ is boolean and $J(R)$ is nil.

Corollary 3.21 [Diesl, 2013, Corollary 3.23] Let $R$ be a ring and $\sigma$ an endomorphism of $R$. Then $R$ is Nil-clean (resp., strongly nil-clean) if and only if $R[x;\sigma]/(x^n)$ is nil-clean (resp., strongly nil clean), for each positive integer $n \geq 2$.

Due to Chen and Sheibani (2017), a ring $R$ is strongly 2-nil-clean if every element in $R$ is the sum of two idempotents and a nilpotent that commute. An element $e$ of the ring $R$ is a tripotent if $e^3 = e$.

According to [Chen & Sheibani, 2017, Theorem 3.3], the ring $R$ is strongly 2-nil-clean if and only if $J(R)$ is nil and $R / J(R)$ is tripotent. By an argument similar to the proof of Theorem 3.20 and using [Chen & Sheibani, 2017, Theorem 3.3], it follows that:

Theorem 3.22 Let $R$ be a ring and $\sigma$ an endomorphism of $R$. Then $R[S;\sigma]$ is a strongly 2-nil-clean ring if and only if $R$ is a strongly 2-nil-clean ring.
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