A CRITICAL REGULARITY CONDITION ON THE ANGULAR VELOCITY OF AXIALLY SYMMETRIC NAVIER-STOKES EQUATIONS

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Abstract. Let $v$ be the velocity of Leray-Hopf solutions to the axially symmetric three-dimensional Navier-Stokes equations. It is shown that $v$ is regular if the angular velocity $v_\theta$ satisfies an integral condition which is critical under the standard scaling. This condition allows functions satisfying

$$|v_\theta(x,t)| \leq \frac{C}{r \ln r^2}, \quad r < 1/2,$$

where $r$ is the distance from $x$ to the axis, $C$ and $\epsilon$ are any positive constants.

Comparing with the critical a priori bound

$$|v_\theta(x,t)| \leq \frac{C}{r}, \quad 0 < r \leq 1/2,$$

our condition is off by the log factor $\ln r^{2+\epsilon}$ at worst. This is inspired by the recent interesting paper [2] where H. Chen, D. Y. Fang and T. Zhang establish, among other things, an almost critical regularity condition on the angular velocity. Previous regularity conditions are off by a factor $r^{-1}$.

The proof is based on the new observation that, when viewed differently, all the vortex stretching terms in the 3 dimensional axially symmetric Navier-Stokes equations are critical instead of supercritical as commonly believed.

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1. INTRODUCTION

In rectangular coordinates, the incompressible Navier-Stokes equations are

$$\Delta v - (v \cdot \nabla)v - \nabla p - \partial_t v = 0, \quad \text{div} \ v = 0,$$

where $v = (v_1(x,t), v_2(x,t), v_3(x,t)) : \mathbb{R}^3 \times [0, T] \to \mathbb{R}^3$ is the velocity field and $p = p(x, t) : \mathbb{R}^3 \times [0, T] \to \mathbb{R}$ is the pressure. In cylindrical coordinates $r, \theta, x_3$ with $(x_1, x_2, x_3) = \ldots$
\((r \cos \theta, r \sin \theta, x_3)\), axially symmetric solutions are of the form
\[
v(x, t) = v_r(r, x_3, t) \hat{e}_r + v_\theta(r, x_3, t) \hat{e}_\theta + v_3(r, x_3, t) \hat{e}_3.
\]
The components \(v_r, v_\theta, v_3\) are all independent of the angle of rotation \(\theta\). Here \(\hat{e}_r, \hat{e}_\theta, \hat{e}_3\) are the basis vectors for \(\mathbb{R}^3\) given by
\[
\hat{e}_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right), \quad \hat{e}_\theta = \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0\right), \quad \hat{e}_3 = (0, 0, 1).
\]
It is known (see [5] for example) that \(v_r, v_3\) and \(v_\theta\) satisfy the equations
\[
\begin{aligned}
(\Delta - \frac{1}{r^2}) v_r - (b \cdot \nabla) v_r + \frac{v_3^2}{r^2} - \partial_r p - \partial_t v_r &= 0, \\
(\Delta - \frac{1}{r^2}) v_\theta - (b \cdot \nabla) v_\theta - \frac{v_3 v_\theta}{r} - \partial_t v_\theta &= 0, \\
\Delta v_3 - (b \cdot \nabla) v_3 - \partial_3 p - \partial_t v_3 &= 0, \\
\frac{1}{r} \partial_r (rv_r) + \partial_3 v_3 &= 0,
\end{aligned}
\] (1.2)
where \(b(x, t) = (v_r, 0, v_3)\) and the last equation is the divergence-free condition. Here, \(\Delta\) is the cylindrical, scalar Laplacian and \(\nabla\) is the cylindrical gradient field:
\[
\Delta = \partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2 + \partial_3^2, \quad \nabla = \left(\partial_r, \frac{1}{r} \partial_\theta, \partial_3\right).
\]
Observe that the equation for \(v_\theta\) does not depend on the pressure. Let \(\Gamma = rv_\theta\), then
\[
\Delta \Gamma - (b \cdot \nabla) \Gamma - \frac{2}{r} \partial_r \Gamma - \partial_t \Gamma = 0, \quad \text{div } b = 0.
\] (1.3)
The vorticity \(\omega = \text{curl } v\) for axially symmetric solutions
\[
\omega(x, t) = \omega_r \hat{e}_r + \omega_\theta \hat{e}_\theta + \omega_3 \hat{e}_3
\]
is given by
\[
\omega_r = -\partial_3 v_\theta, \quad \omega_\theta = \partial_3 v_r - \partial_r v_3, \quad \omega_3 = \partial_r v_\theta + \frac{v_3}{r}.
\] (1.4)
The equations of vorticity \(\omega = \text{curl } v\) in cylindrical form are (again, see [5] for example):
\[
\begin{aligned}
(\Delta - \frac{1}{r^2}) \omega_r - (b \cdot \nabla) \omega_r + \omega_r \partial_r v_r + \omega_3 \partial_3 v_r - \partial_t \omega_r &= 0, \\
(\Delta - \frac{1}{r^2}) \omega_\theta - (b \cdot \nabla) \omega_\theta + \frac{2}{r} \partial_3 \omega_\theta + \omega_\theta \frac{v_3}{r} - \partial_t \omega_\theta &= 0, \\
\Delta \omega_3 - (b \cdot \nabla) \omega_3 + \omega_3 \partial_3 v_3 + \omega_\theta \partial_r v_3 - \partial_t \omega_3 &= 0.
\end{aligned}
\] (1.5)
Although the axially symmetric Navier-Stokes equations is a special case of the full 3 dimensional one, our level of understanding had been roughly the same, with essential difficulty unresolved. One quick explanation of the difficulty goes as follows. Viewing (1.1) as a reaction diffusion equation. The standard theory for regularity requires the velocity to be bounded in suitable function space whose norm is invariant under standard scaling, such as \(L^{p,q}\) with \(\frac{2}{p} + \frac{3}{q} = 1\). However the only general a priori bound available is the energy estimate, which scales as \(-1/2\). So there is a positive gap between the two which makes the equations supercritical.

Equation (1.2) has been studied by many authors in recent years. The following is a list which is far from complete. If the swirl \(v_\theta = 0\), then long time ago, O. A. Ladyzhenskaya [11], M. R. Uchoviskii and B. I. Yudovich [20], proved that finite energy solutions to (1.2)
are smooth for all time. See also the paper by S. Leonardi, J. Malek, J. Necas, and M. Pokorný [14].

In the presence of swirl, it is not known in general if finite energy solutions blow up in finite time. However a lower bound for the possible blow up rate is known by the recent results of C.-C. Chen, R. M. Strain, T.-P. Tsai, and H.-T. Yau in [5], [6], G. Koch, N. Nadirashvili, G. Seregin, and V. Sverak in [10]. See also the work by G. Seregin and V. Sverak [18] for a localized version. These authors prove that if \( |v(x,t)| \leq C_r \), then solutions are smooth for all time. Here \( C \) is any positive constant. Their result can be rephrased as: type I solutions are regular. See also the papers [12], [13] on further results in this direction. J. Neustupa and M. Pokorny [16] proved that the regularity of one component (either \( v_r \) or \( v_\theta \)) implies regularity of the other components of the solution. See more refined results in [17] and the work of Ping Zhang and Ting Zhang [22]. Also proving regularity is the work of Q. Jiu and Z. Xin [9] under an assumption of sufficiently small zero-dimension scaled norms. D. Chae and J. Lee [4] also proved regularity results assuming finiteness of another certain zero-dimensional integral. G. Tian and Z. Xin [19] constructed a family of singular axially symmetric solutions with singular initial data. T. Hou and C. Li [7] found a special class of global smooth solutions. See also a recent extension: T. Hou, Z. Lei and C. Li [8].

Define

\[
J = \frac{\omega_r}{r}, \quad \Omega = \frac{\omega_\theta}{r}.
\]

Then the triple \( J, \Omega, \omega_3 \) satisfy the system

(1.6)

\[
\begin{align*}
\Delta J - (b \cdot \nabla) J + \frac{2}{r} \partial_r J + (\omega_r \partial_r + \omega_3 \partial_3) \frac{2}{r} - \partial_t J &= 0, \\
\Delta \Omega - (b \cdot \nabla) \Omega + \frac{2}{r} \partial_r \Omega - \frac{2}{r} \partial_\theta J - \partial_t \Omega &= 0, \\
\Delta \omega_3 - (b \cdot \nabla) \omega_3 + w_r \partial_r v_3 + w_3 \partial_3 v_3 - \partial_t \omega_3 &= 0.
\end{align*}
\]

Here, in the second equation, we used the identity \( rJ = w_r = -\partial_3 v_\theta \).

A great observation by Hui Chen, Daoyuan Fang and Ting Zhang in [2] is that the first two equations in (1.6) form a critical system under the standard scaling. Using this and a "magic formula" relating \( \nabla (v_r/r) \) with \( w_\theta/r \) by Changxing Miao and Xiaoxin Zheng [15], they obtained, among other things, an almost critical regularity condition on \( v_\theta \). For example it is proven that if \( |v_\theta(x,t)| \leq C/r^{2-\epsilon} \) with \( \epsilon > 0 \), then solutions are regular.

In this paper we observe further that, all three equations are critical when viewed in a suitable way. Therefore the vorticity equation of 3 dimensional axially symmetric Navier-Stokes equations are critical instead of supercritical as commonly believed. This, together with a localization method in [21], allow us to prove Theorem 1.1 below, which provides a localized critical regularity condition on \( v_\theta \). It is tantalizing that our condition differs with the critical a priori bound ([4] or [16])

\[
|v_\theta(x,t)| \leq \frac{C}{r}, \quad 0 < r \leq 1/2,
\]

by the log factor \( |\ln r|^{2+\epsilon} \) at worst. See the remarks below.
Now we introduce the function class where \( v_0 \) lives. It is defined in an integral way which is usually called the form boundedness condition, which is more general than the corresponding \( L^{p,q} \) condition.

**Definition 1.1.** We say the angular velocity \( v_\theta \) is in the \( \lambda_1 \) critical class if there is a positive number \( a < 1 \) and another positive number \( \lambda_2 \) such that the inequality

\[
\int_0^t \int \left( \frac{|v_\theta|}{r} + v_\theta^2 \right) \psi^2 dyds \leq \lambda_1 \int_0^t \int |\nabla \psi|^2 dyds + \frac{\lambda_2}{a^2} \int_0^t \int \psi^2 dyds
\]

holds for all \( t \geq 0 \) and for all smooth \( \psi = \psi(y,s) \), \( s \in [0,t] \), satisfying the conditions (1) \( \psi \) is axially symmetric in \( y \); (2) \( \psi(\cdot, s) \) is supported in the cylinder \( D_{a,l} = \{(r,\theta,x_3) | 0 \leq r < a, -l < x_3 < l, 0 \leq \theta < 2\pi \} \) for some \( l \geq a \).

**Remark 1.1.** Clearly the class is scaling invariant. A function \( v_\theta \) is the \( \lambda_1 \) critical class for all \( \lambda_1 > 0 \) if it satisfies \( |v_\theta(x,t)| \leq \frac{C}{r} \ln\frac{1}{r} \), \( r < 1/2 \). Here \( C > 0, \epsilon > 0 \) are arbitrary positive constant. This claim will be proven at the end of the paper. One may also take \( \epsilon = 0 \) but replace \( r \) by \( r/a \) and \( C \) by a small constant in the bound, by virtue of the 2 dimensional Hardy’s inequality.

Here is the main result of the paper.

**Theorem 1.1.** Let \( v \) be a Leray-Hopf axially symmetric solution of the three-dimensional Navier-Stokes equations in \( \mathbb{R}^3 \times (0,\infty) \) with initial data \( v_0 = v(\cdot,0) \in L^2(\mathbb{R}^3) \). Assume further \( rv_0,\theta \in L^\infty(\mathbb{R}^3) \).

There exists a positive number \( \lambda_1 \). Suppose \( v_\theta \) is in the \( \lambda_1 \) critical class. Then \( v \) is smooth for all time.

**Remark 1.2.** The size of \( \lambda_1 \) is estimated in (2.36). It is an absolute constant depending on the \( L^2 \) norm of the Riesz operators. There is no size restriction on \( \lambda_2 \). Also the \( a^2 \) in the definition can be replaced by any positive continuous function of \( a \). But this may break the scaling invariance.

The theorem will be proven in the next section. The following are some notations to be frequently used. We use \( x = (x_1,x_2,x_3) \) to denote a point in \( \mathbb{R}^3 \) for rectangular coordinates, and in the cylindrical system we use \( r = \sqrt{x_1^2 + x_2^2} \), \( \theta = \tan^{-1} \frac{x_2}{x_1} \). We will use \( S(v_0,...),C(v_0,...) \) to denote positive constants which depend on the initial velocity \( v_0 \) etc. Also \( C \) denotes absolute constant which may change value.

Let us explain why the vortex stretching terms in (1.6) are critical. For example the term \( \partial_3 w_3 \) where \( \partial_3 v_3 \) being viewed as a potential of the unknown function \( w_3 \) is certainly supercritical. However, we view \( w_3 = \partial_r v_\theta + \frac{\nu}{r} \) as the potential and \( \partial_3 v_3 \) as the unknown. Since it is known that \( |v_\theta| \leq C/r \), we see that \( w_3 \) now scales as \(-2\) power of the distance. This scaling shows \( w_3 \) is a critical potential function. The unknown function \( \partial_3 v_3 \) scales the same way as the vorticity \( w \). By exploiting the integral relations between \( v \) and \( w \), we can convert \( \partial_3 v_3 \) into \( w_r, w_3, w_\theta \). This, combined with the observation [2] about the first two equations in (1.6), imply that all the vortex stretching terms are critical. Next we carry a local energy estimate for \( (J, \Omega, w_\phi) \) via equations (1.6). Once we know the potential terms are critical, the drift terms can be treated by an old small trick in [21], the proof thus goes through.
2. Proof of the theorem

The proof is divided into several steps. We may assume that \( v \) is smooth up to a given time \( t \).

**Step 1. Choose suitable test functions for equations (1.6).**

It is well known that singularity can possibly appear only on a finite segment of the \( x_3 \) axis (\( \mathbb{R}^3 \) for suitable solutions and \( \mathbb{R} \) for general ones). So by picking any positive number \( a \leq 1 \) and another positive number \( l > a \), which may depend on the initial velocity \( v_0 \), we can ensure that \( v \) is regular outside of the domain \( D_1 = \{(r, \theta, x_3) | 0 \leq r < a/2, -l/2 < x_3 < l/2, 0 \leq \theta < 2\pi\} \) for all time. Let \( \phi = \phi(r, x_3) \) be a axially symmetric cut off function in \( D_2 = \{(r, \theta, x_3) | 0 \leq r < a, -l < x_3 < l, 0 \leq \theta < 2\pi\} \) such that \( \phi = 1 \) on \( D_3 = \{(r, \theta, x_3) | 0 \leq r < 2a/3, -2l/3 < x_3 < 2l/3, 0 \leq \theta < 2\pi\} \) and \( \phi = 0 \) on \( D_2^c \) and also \( \frac{\nabla \phi}{\phi} \leq C/a, |\nabla^2 \phi| \leq C/a^2 \).

Use \( J\phi^2, \Omega \phi^2 \) and \( w_3 \phi^2 \) as test functions in equations 1, 2 and 3 in (1.6) respectively.

After integration on the region \( D_2 \times [0, t] \) for \( t > 0 \) we find that

\[
L_1 \equiv -\int_0^t \int \Delta J J\phi^2 dyds - \int_0^t \int \frac{2}{r} \partial_r J J\phi^2 dyds + \int_0^t \int \partial_t J J\phi^2 dyds
\]

(2.1)

\[
= -\int_0^t \int b \nabla J J\phi^2 dyds + \int_0^t \int (w_r \partial_r \frac{v_r}{r} + w_3 \partial_3 \frac{v_r}{r}) J\phi^2 dyds
\]

\[
\equiv R_1 + T_1.
\]

\[
L_2 \equiv -\int_0^t \int \Delta \Omega \Omega \phi^2 dyds - \int_0^t \int \frac{2}{r} \partial_r \Omega \Omega \phi^2 dyds + \int_0^t \int \partial_t \Omega \Omega \phi^2 dyds
\]

(2.2)

\[
= -\int_0^t \int b \nabla \Omega \Omega \phi^2 dyds - \int_0^t \int \frac{2v_\theta}{r} \Omega \phi^2 dyds
\]

\[
\equiv R_2 + T_2
\]

\[
L_3 \equiv -\int_0^t \int \Delta w_3 w_3 \phi^2 dyds + \int_0^t \int \partial_t w_3 w_3 \phi^2 dyds
\]

(2.3)

\[
= -\int_0^t \int b \nabla w_3 w_3 \phi^2 dyds + \int_0^t \int (w_3 \partial_3 v_3 + w_r \partial_r v_3) w_3 \phi^2 dyds
\]

\[
\equiv R_3 + T_3
\]

The left hand side of the three equalities \( L_1, L_2 \) and \( L_3 \) can be treated by routine integration by parts which shows:

\[
L_1 = \int_0^t \int |\nabla J|^2 \phi^2 dyds + \int_0^t \int J^2(0, y_3, t) \phi^2 dyds + \int_0^t \int J^2 \phi^2 dy dt + \frac{1}{2} \int \int J^2 \phi^2 dy dt
\]

\[
- \int_0^t \int \nabla J J \nabla \phi^2 dyds + \int_0^t \int J^2 \partial_r \phi^2 r dyds.
\]
Therefore
\[
L_1 \geq \frac{1}{2} \int_0^t \int |\nabla J|^2 \phi^2 dy ds + \frac{1}{2} \int J^2 \phi^2 dy \bigg|_0^t - 2 \int_0^t \int J^2 |\nabla \phi|^2 dy ds + \int_0^t \int \frac{J^2 \partial_r \phi^2}{r} dy ds.
\]

By our choice of the cut off function $\phi$, we know $v$ is regular in the supports of $\nabla \phi$ and $\partial_r \phi$, which is bounded away from the singular set by a distance $a/6$. So there is a positive constant $S = S(v_0, a, l)$ such that

\begin{align}
L_1 &\geq \frac{1}{2} \int_0^t \int |\nabla J|^2 \phi^2 dy ds + \frac{1}{2} \int J^2 \phi^2 dy \bigg|_0^t - CtS(v_0, a, l). \\
L_2 &\geq \frac{1}{2} \int_0^t \int |\Omega|^2 \phi^2 dy ds + \frac{1}{2} \int \Omega^2 \phi^2 dy \bigg|_0^t - CtS(v_0, a, l), \\
L_3 &\geq \frac{1}{2} \int_0^t \int |\nabla w_3|^2 \phi^2 dy ds + \frac{1}{2} \int w_3^2 \phi^2 dy \bigg|_0^t - CtS(v_0, a, l).
\end{align}

We remark that $S(v_0, a, l)$ may blow up when $a \to 0$. But we will make $a$ small and fixed.

Substituting (2.4), (2.5) and (2.6) into (2.1), (2.2) and (2.3) respectively, we deduce

\begin{align}
\int (J^2 + \Omega^2 + w_3^2) \phi^2 dy \bigg|_0^t + \int_0^t \int (|\nabla J|^2 + |\nabla \Omega|^2 + |\nabla w_3|^2) \phi^2 dy ds \\
\leq 2(R_1 + R_2 + R_3) + 2(T_1 + T_2 + T_3) + CS(v_0, a, l).
\end{align}

We are going to bound the right hand side in the next few steps.

**Step 2. bounds on $R_1 + R_2 + R_3$, the drift terms.**

These terms are generated by $b = v_r \vec{e}_r + v_3 \vec{e}_3$ which is supercritical. However since these are given by divergence free drift terms, they can be bounded as done in [21]. We present a proof for completeness.

Since $\text{div} \ b = 0$, we have
\[
R_1 = - \int_0^t \int b \cdot (\nabla J)(J\phi^2) dy ds
\]
\[
= \int_0^t \int b \cdot (\nabla \phi) \phi J^2 dy ds
\]
\[
\leq \left| \int \left( b \phi^{3/2} |J|^{3/2} \right) \left( \frac{\nabla \phi}{\phi^{1/2}} |J|^{1/2} \right) dy ds \right|.
\]

By Hölder’s inequality with exponents $\frac{4}{3}$ and 4,
\[
R_1 \leq \left( \int_0^t \int |b|^{\frac{4}{3}} \left( \phi^{3/2} |J|^{3/2} \right)^{\frac{4}{3}} dy ds \right)^{\frac{3}{4}} \left( \int_0^t \int \left( \frac{\nabla \phi}{\phi^{1/2}} |J|^{1/2} \right)^{\frac{4}{3}} dy ds \right)^{\frac{1}{4}}.
\]
Using properties of the cutoff function we find:

\[ R_1 \leq \left( \int_0^t \int |b|^{4/3}(J\phi)^2 \, dy \, ds \right)^{2/3} \frac{4}{3} C \left( \int_0^t \int_{\text{supp} |\nabla \phi|} J^2 \, dy \, ds \right)^{1/3} a. \]

Next we fix \( \epsilon_1 > 0 \) and we apply Young’s inequality, with exponents \( \frac{4}{3} \) and 4:

\[ R_1 \leq \frac{4}{3} \epsilon_1 \left( \int_0^t \int |b|^{4/3}(J\phi)^2 \, dy \, ds \right)^{2/3} \frac{4}{3} C \left( \int_0^t \int_{\text{supp} |\nabla \phi|} J^2 \, dy \, ds \right)^{1/3} \]

\[ \leq \epsilon_1 \int_0^t \int |b|^{4/3}(J\phi)^2 \, dy \, ds + \frac{C \epsilon_1^{-3}}{a^4} \int_0^t \int_{\text{supp} |\nabla \phi|} J^2 \, dy \, ds. \]

Thus,

(2.8) \[ |R_1| \leq \epsilon_1 c_0 \| b \|_{2,\infty}^{4/3} \int_0^t \int |\nabla (J\phi)|^2 \, dy \, ds + \frac{C \epsilon_1^{-3}}{a^4} \int_0^t \int_{\text{supp} |\nabla \phi|} J^2 \, dy \, ds. \]

This last inequality holds as a result of the standard energy estimate, Hölder’s inequality with exponents \( \frac{3}{2} \) and 3, and the 3 dimensional Sobolev Inequality,

\[ \int_0^t \int |b|^{4/3}(J\phi)^2 \, dy \, ds \leq \int_0^t \left( \int |b|^2 \, dy \right)^{2/3} \left( \int (J\phi)^6 \, dy \right)^{1/3} ds \]

\[ \leq c_0 \| b \|_{2,\infty}^{4/3} \int_0^t \int |\nabla (J\phi)|^2 \, dy \, ds. \]

By choosing \( \epsilon_1 \) suitably, we deduce

(2.9) \[ |R_1| \leq \frac{1}{8} \int_0^t \int |\nabla J|^2 \phi^2 \, dy \, ds + CS(v_0, a, l), \]

where we have used the fact that \( v \) is regular in the support of \( \nabla \phi \) for all time. In exactly the same manner, we find that

(2.10) \[ |R_1| + |R_2| + |R_3| \leq \frac{1}{8} \int_0^t \int \left( |\nabla J|^2 + |\nabla \Omega|^2 + |\nabla w_3|^2 \right) \phi^2 \, dy \, ds + CS(v_0, a, l), \]

Step 3. bounds on \( T_1 \) and \( T_2 \).

In this step we follow the idea in [CFZ] with one modification, namely a localized version of a formula of Miao and Zheng which relates \( \frac{w}{r} \) with \( \frac{w_3}{r} \). The rest of the step is divided into a few sub steps.

step 3.1

First we work on the easy one \( T_2 \) defined in (2.2).

\[ T_2 = - \int_0^t \int \frac{2v_0}{r} J\Omega \phi^2 \, dy \, ds \]

\[ \leq \int_0^t \int \frac{|v_0|}{r} (J\phi)^2 \, dy \, ds + \int_0^t \int \frac{|v_0|}{r} (\Omega_\theta)^2 \, dy \, ds. \]
By our assumption on $v_θ$, this implies
\[ T_2 ≤ \lambda_1 \int_0^t \int (|\nabla(Jφ)|^2 + |∇(Ωφ)|^2)dyds + \lambda_2 \int_0^t \int [(Jφ)^2 + (Ωφ)^2]dyds. \]

Let us write $∇(Jφ) = ∇Jφ + J∇φ$. As mentioned earlier, $J$ is regular in the support of $∇φ$. Hence

(2.11) \[ T_2 ≤ 2\lambda_1 \int_0^t \int (|∇J|^2 + |∇Ω|^2)φ^2dyds + \lambda_2 \int_0^t \int [(Jφ)^2 + (Ωφ)^2]dyds + CtS(v_0, a, l). \]

Here we also did the same argument for $∇(Ωφ)$.

**step 3.2**

Next we turn to $T_1$. From (2.1),
\[
\frac{dT_1}{dt} = \int (w_r \partial_r \frac{v_r}{r} + w_3 \partial_3 \frac{v_r}{r})Jφ^2dy
\]

Using the relation $w_r = -\partial_3 v_θ$, $w_3 = \frac{1}{r} \partial_r (rv_θ)$ and integration by parts, we see that
\[
\frac{dT_1}{dt} = -\int \partial_3 v_θ \partial_r \frac{v_r}{r} Jφ^2 dy + \int \frac{1}{r} \partial_r (rv_θ) \partial_3 \frac{v_r}{r} Jφ^2 dy
\]
\[
= \int v_θ \partial_3 \partial_r \frac{v_r}{r} Jφ^2 dy + \int v_θ \partial_r \frac{v_r}{r} \partial_3 (Jφ^2) dy
\]
\[
- \int v_θ \partial_3 \partial_r \frac{v_r}{r} Jφ^2 dy - \int v_θ \partial_3 \frac{v_r}{r} \partial_r (Jφ^2) dy.
\]

Notice that the first and third term on the right hand side of the last equality cancel. Therefore, we deduce
\[
\frac{dT_1}{dt} = \int v_θ \partial_r \frac{v_r}{r} (\partial_3 J)φ^2 dy - \int v_θ \partial_3 \frac{v_r}{r} (\partial_r J)φ^2 dy
\]
\[
+ \int v_θ \partial_r \frac{v_r}{r} J∂_r φ^2 dy - \int v_θ \partial_3 \frac{v_r}{r} J∂_r φ^2 dy.
\]

This implies, since the last two terms in the above identity are bounded, that
\[
T_1 ≤ \frac{1}{8} \int_0^t \int |∂_3 J|^2 φ^2 dy + 2 \int_0^t \int v_θ^2 |∂_r \frac{v_r}{r}|^2 φ^2 dy
\]
\[
+ \frac{1}{8} \int_0^t \int |∂_r J|^2 φ^2 dy + 2 \int_0^t \int v_θ^2 |∂_3 \frac{v_r}{r}|^2 φ^2 dy + CtS(v_0, a, l).
\]

By our condition on $v_θ$ again, we find that
\[
T_1 ≤ \frac{1}{8} \int_0^t \int |∇J|^2 φ^2 dy + CtS(v_0, a, l) + 2\lambda_1 \int_0^t \int |∇(φ∂_r \frac{v_r}{r})|^2 dy + 2\lambda_2 \int_0^t \int (φ∂_r \frac{v_r}{r})^2 dy
\]
\[
+ 2\lambda_1 \int_0^t \int |∇(φ∂_3 \frac{v_r}{r})|^2 dy + 2\lambda_2 \int_0^t \int (φ∂_3 \frac{v_r}{r})^2 dy.
\]
This implies, after using again the fact that $v$ is smooth in the support of $\nabla \phi$, that

\[
T_1 \leq \frac{1}{8} \int_0^t \int |\nabla J|^2 \phi^2 \, dy + CtS(v_0, a, l) + 4\lambda_1 \int_0^t \int |\nabla(\partial_r(v_r \frac{\phi}{r}))|^2 \, dy + 4\lambda_2 \int_0^t \int (\partial_r(v_r \frac{\phi}{r}))^2 \, dy
+ 4\lambda_1 \int_0^t \int |\nabla(\partial_3(v_r \frac{\phi}{r}))|^2 \, dy + 4\lambda_2 \int_0^t \int (\partial_3(v_r \frac{\phi}{r}))^2 \, dy.
\]

Here the constant $C$ may have changed. We need to bound the last 4 terms on the preceding inequality. For this purpose, we first need to prove the following localized version of a nice identity by Miao and Zheng. For any $q \in (1, \infty)$, there is a positive constant $c_q$ such that

\[
\|\nabla(\phi \partial_r(v_r \frac{\phi}{r}))\|_q \leq c_q \|\Omega \phi\|_q + S(v_0, a, l),
\]

\[
\|\nabla^2(\phi \partial_r(v_r \frac{\phi}{r}))\|_q \leq c_q \|\nabla(\Omega \phi)\|_q + S(v_0, a, l).
\]

Here, as always $\Omega = w_{\theta}/r$. The proof of these inequalities is given in step 3.3. From the identity

\[
\Delta b = -\nabla \times (w_{\theta} \nabla \phi) = \left( \partial_3(w_{\theta} \frac{x_1}{r}), \partial_3(w_{\theta} \frac{x_2}{r}), \partial_1(w_{\theta} \frac{x_1}{r}) - \partial_2(w_{\theta} \frac{x_2}{r}) \right),
\]

and $b = v_r(\frac{x_1}{r}, \frac{x_2}{r}, 0) + v_3(0,0,1)$, we see that

\[
\Delta(v_r \frac{x_1}{r}) = \partial_3(x_1 \Omega), \quad \Delta(v_r \frac{x_2}{r}) = \partial_3(x_2 \Omega).
\]

Therefore

\[
\Delta(v_r \frac{x_1}{r} \phi) = \partial_3(x_1 \Omega \phi) - x_1 \Omega \partial_3 \phi + 2\nabla(v_r \frac{x_1}{r}) \nabla \phi + v_r \frac{x_1}{r} \Delta \phi.
\]

Likewise

\[
\Delta(v_r \frac{x_2}{r} \phi) = \partial_3(x_2 \Omega \phi) - x_2 \Omega \partial_3 \phi + 2\nabla(v_r \frac{x_2}{r}) \nabla \phi + v_r \frac{x_2}{r} \Delta \phi.
\]

Inverting the Laplace operator, we infer

\[
v_r \frac{x_1}{r} \phi = \Delta^{-1} \partial_3(x_1 \Omega \phi) - \Delta^{-1} [x_1 \Omega \partial_3 \phi - 2\nabla(v_r \frac{x_1}{r}) \nabla \phi + v_r \frac{x_1}{r} \Delta \phi],
\]

\[
v_r \frac{x_2}{r} \phi = \Delta^{-1} \partial_3(x_2 \Omega \phi) - \Delta^{-1} [x_2 \Omega \partial_3 \phi - 2\nabla(v_r \frac{x_2}{r}) \nabla \phi + v_r \frac{x_2}{r} \Delta \phi].
\]

Multiplying \((2.17)\) by $x_1$, \((2.18)\) by $x_2$ and taking the sum, we arrive at

\[
v_r \phi = \sum_{i=1}^2 \frac{x_i}{r} \Delta^{-1} \partial_3(x_i \Omega \phi) - \sum_{i=1}^2 \Delta^{-1} [x_i \Omega \partial_3 \phi - 2\nabla(v_r \frac{x_i}{r}) \nabla \phi + v_r \frac{x_i}{r} \Delta \phi].
\]

Since $\phi$ is axially symmetric and $x_1/r = \cos \theta$, $x_2/r = \sin \theta$, we can write, for $i = 1, 2$, that

\[
\nabla(v_r \frac{x_i}{r}) \nabla \phi = \frac{x_i}{r} (\partial_r v_r \phi + \partial_3 v_r \partial_3 \phi).
\]
This turns (2.19) into

\[ v_r \phi = \sum_{i=1}^{2} \frac{x_i}{r} \Delta^{-1} \partial_3 (x_i \Omega \phi) - \sum_{i=1}^{2} \frac{x_i}{r} \Delta^{-1} (x_i f), \]

(2.20)

\[ f \equiv \Omega \partial_3 \phi - 2 \frac{\partial_r v_r}{r} \partial_r \phi - 2 \frac{\partial_3 v_r}{r} \partial_3 \phi - \frac{v_r}{r} \Delta \phi. \]

Note the function \( f \) is compactly supported, axially symmetric and point-wise bounded, due to the choice of the cut off function \( \phi \).

According to [15], the following operator identity holds, at least when acting on compactly supported functions,

\[ \sum_{i=1}^{2} \frac{x_i}{r} \Delta^{-1} x_i = r \Delta^{-1} - 2 \partial_r \Delta^{-2}. \]

(2.21)

Since their proof is very sharp and cute, we repeat it here for completeness. Notice that

\[ \sum_{i=1}^{2} x_i [x_i, \Delta^{-1}] = \sum_{i=1}^{2} x_i^2 \Delta^{-1} - \sum_{i=1}^{2} x_i \Delta^{-1} x_i = r^2 \Delta^{-1} - \sum_{i=1}^{2} x_i \Delta^{-1} x_i. \]

Hence

\[ \sum_{i=1}^{2} \frac{x_i}{r} \Delta^{-1} x_i = r \Delta^{-1} - \sum_{i=1}^{2} \frac{x_i}{r} [x_i, \Delta^{-1}]. \]

(2.22)

On the other hand

\[ \Delta [x_i, \Delta^{-1}] = \Delta (x_i \Delta^{-1}) - \Delta \Delta^{-1} x_i = 2 \partial_i \Delta^{-1}, \]

which implies

\[ [x_i, \Delta^{-1}] = 2 \partial_i \Delta^{-2}. \]

Substituting this to the last term in (2.22), one obtains (2.21). Plugging (2.21) into the first identity in (2.20), we find that

\[ \frac{v_r}{r} \phi = (\Delta^{-1} \partial_3 - 2 \frac{\partial_r}{r} \Delta^{-2} \partial_3) (\Omega \phi) - (\Delta^{-1} - 2 \frac{\partial_r}{r} \Delta^{-2}) f. \]

(2.23)

Recall that both \( \Omega \phi \) and \( f \) are axially symmetric. When the operator \( \frac{\partial_r}{r} \) acts on these functions, it can be written as

\[ \frac{\partial_r}{r} = \Delta - \partial_r^2 - \partial_3^2. \]

Plugging this into (2.23), we deduce

\[ \nabla (\frac{v_r}{r} \phi) = \Pi_1 (\Omega \phi) + \Pi_0 f, \]

(2.24)

where \( \Pi_1 \) and \( \nabla \Pi_0 \) are Riesz type singular integral operators that map \( L^q \) to \( L^q \), \( q \in (1, \infty) \) and \( \Pi_0 \) is a smoothing integral operator. Since \( f \) is bounded and compactly supported, this proves (2.13). We have used the fact that the gradient \( \nabla \) does not involve the derivative in \( \vec{e}_\theta \) direction, when acting on axially symmetric functions.

**step 3.4.**
Now we can take \( q = 2 \) in (2.13) and substitute it to (2.12) to obtain (2.25)

\[
T_1 \leq \frac{1}{8} \int_0^t \int \left| \nabla J \right|^2 \phi^2 dy + C t S(v_0, a, l) + 4 \lambda_1 c_2 \int_0^t \int |\nabla (\Omega \phi)|^2 dy + 4 \lambda_2 c_2 \int_0^t \int (\Omega \phi)^2 dy + 4 \lambda_1 c_2 \int_0^t \int |\nabla (\Omega \phi)|^2 dy + 4 \lambda_2 c_2 \int_0^t \int (\Omega \phi)^2 dy.
\]

This, together with (2.11), yield

\[
T_1 + T_2 \leq \left( \frac{1}{8} + 2 \lambda_1 + 9 \lambda_1 c_2 \right) \int_0^t \int \left( |\nabla J|^2 + |\nabla \Omega|^2 \right) \phi^2 dy ds + (\lambda_2 + 8 \lambda_2 c_2) \int_0^t \int (J \phi)^2 + (\Omega \phi)^2 \phi^2 dy ds + C t S(v_0, a, l).
\]

In the above we have used the product formula \((\nabla \Omega) \phi = \nabla (\Omega \phi) - \Omega \nabla \phi\). This completes Step 3.

**Step 4. bounds on** \( T_3 \).

Using \( w_3 = \frac{1}{4} \partial_r (rv \theta) \), we compute

\[
\int w_3 \partial_r v_3 \phi^2 dy = \int \int_0^\infty \partial_r (rv \theta) \partial_r v_3 \phi^2 dr dy_3
\]

\[
= - \int \int_0^\infty rv \partial_r \partial_r v_3 \phi^2 dr dy_3 - \int \int_0^\infty rv \partial_r v_3 \partial_r w_3 \phi^2 dr dy_3 - \int \int_0^\infty rv \partial_r v_3 \partial_r \phi^2 dr dy_3
\]

\[
= - \int v_\theta \partial_r \partial_r v_3 \phi^2 dy - \int v_\theta \partial_r v_3 \partial_r v_3 \phi^2 dy - \int v_\theta v_3 \partial_r v_3 \partial_r \phi^2 dy.
\]

Next, using \( w_r = - \partial_r v_\theta \), we have

\[
\int w_r \partial_r v_3 \phi^2 dy = \int \partial_r v_\theta \partial_r v_3 \phi^2 dy
\]

\[
= \int v_\theta \partial_r v_3 \phi^2 dy + \int v_\theta v_3 \partial_r v_3 \phi^2 dy + \int v_\theta v_3 \partial_r v_3 \phi^2 dy.
\]

Adding the previous two equalities and noting that the first terms on the right hand sides cancel, we obtain

\[
T_3 = - \int_0^t \int v_\theta \partial_r v_3 \partial_r v_3 \phi^2 dy ds - \int_0^t \int v_\theta v_3 \partial_r v_3 \partial_r \phi^2 dy ds
\]

\[
+ \int_0^t \int v_\theta v_3 \partial_r v_3 \phi^2 dy ds + \int_0^t \int v_\theta v_3 \partial_r v_3 \phi^2 dy ds.
\]

As before, all terms involving derivatives of \( \phi \) are bounded by \( C t S(v_0, a, l) \). Thus

\[
T_3 \leq - \int_0^t \int v_\theta \partial_r v_3 \partial_r v_3 \phi^2 dy ds + \int_0^t \int v_\theta v_3 \partial_r v_3 \phi^2 dy ds + C t S(v_0, a, l)
\]

\[
\equiv I_1 + I_2 + C t S(v_0, a, l).
\]
We will bound $I_1$ first. By our condition on $v_\theta$,

$$I_1 \leq \frac{1}{8} \int_0^t \int |\partial_r w_3|^2 \phi^2 \, dy \, ds + 2 \int_0^t \int v_\theta^2 |\partial_3 v_3|^2 \phi^2 \, dy \, ds$$

$$\leq \frac{1}{8} \int_0^t \int |\partial_r w_3|^2 \phi^2 \, dy \, ds + 2 \lambda_1 \int_0^t \int |\nabla (\phi \partial_3 v_3)|^2 \, dy \, ds + 2 \lambda_2 \int_0^t \int |\partial_3 v_3|^2 \phi^2 \, dy \, ds.$$

Consequently

$$I_1 \leq \frac{1}{8} \int_0^t \int |\partial_r w_3|^2 \phi^2 \, dy \, ds + \lambda_1 \int_0^t \int |\nabla v_3|^2 \phi^2 \, dy \, ds + \lambda_2 \int_0^t \int |\partial_3 v_3|^2 \phi^2 \, dy \, ds + C t S(v_0, a, l, \lambda_2).$$

We need to bound the second term on the right hand side. To this end we call the relation for the full three dimensional velocity and vorticity:

$$-\Delta \partial_i v = \nabla \times \partial_i w,$$

where $i = 1, 2, 3$. Using $\partial_i v \phi^2$ as a test function and integrate, we know that

$$\int |\nabla \partial_i v|^2 \phi^2 \, dy + \int \partial_j \partial_i v \partial_j v \phi^2 \, dy = \int (\nabla \times \partial_i w) \partial_i v \phi^2 \, dy$$

$$= - \int (\nabla \times w) \partial_i v \partial_i v \phi^2 \, dy - \int (\nabla \times w) \partial_i v \partial_i v \phi^2 \, dy$$

$$\leq \frac{1}{2} \int |\nabla \partial_i v|^2 \phi^2 \, dy + \frac{1}{2} \int |\nabla v|^2 \phi^2 \, dy - \int (\nabla \times w) \partial_i v \partial_i v \phi^2 \, dy.$$

Since the terms involving derivatives of $\phi$ are bounded, this shows

$$\int_0^t \int |\nabla \partial_3 v_3|^2 \phi^2 \, dy \, ds \leq \int_0^t \int |\nabla v|^2 \phi^2 \, dy \, ds + C t S(v_0, a, l),$$

(2.29)

and

$$\int_0^t \int |\nabla \partial_r v_3|^2 \phi^2 \, dy \, ds \leq \int_0^t \int |\nabla v|^2 \phi^2 \, dy \, ds + C t S(v_0, a, l),$$

(2.30)

Here the constant $C$ may have changed when we drop the cross product, which can be done through integration by parts that produces extra bounded terms involving $\nabla \phi$.

Substituting (2.29) into the second term on the right hand side of (2.28), we reach

$$I_1 \leq \frac{1}{8} \int_0^t \int |\partial_r w_3|^2 \phi^2 \, dy \, ds + 3 \lambda_1 \int_0^t \int |\nabla v|^2 \phi^2 \, dy \, ds + C t S(v_0, a, l, \lambda_1, \lambda_2).$$

(2.31)
Similarly, by our condition on \( v_\theta \),

\[
I_2 \leq \frac{1}{8} \int_0^t \int |\partial_3 w_3|^2 \phi^2 dyds + 2 \int_0^t \int v_\theta^2 |\partial_r v_3|^2 \phi^2 dyds
\]

\[
\leq \frac{1}{8} \int_0^t \int |\partial_3 w_3|^2 \phi^2 dyds + 2 \lambda_1 \int_0^t \int |\nabla \partial_r v_3|^2 \phi^2 dyds + 2 \lambda_2 \int_0^t \int |\partial_r v_3|^2 \phi^2 dyds.
\]

This with (2.30) imply that

\[
(2.32) \quad I_2 \leq \frac{1}{8} \int_0^t \int |\partial_3 w_3|^2 \phi^2 dyds + 3 \lambda_1 \int_0^t \int |\nabla w|^2 \phi^2 dyds + C(tS(v_0, a, l, \lambda_1, \lambda_2)).
\]

Substituting (2.31) and (2.32) into (2.27), we deduce the bound for \( T_3 \), i.e.

\[
T_3 \leq \frac{1}{8} \int_0^t \int |\nabla w_3|^2 \phi^2 dyds + 6 \lambda_1 \int_0^t \int |\nabla w|^2 \phi^2 dyds + C(tS(v_0, a, l, \lambda_1, \lambda_2)).
\]

**Step 5. Conclusion of the proof.**

Combining (2.26) with (2.33), we get

\[
(2.34) \quad T_1 + T_2 + T_3 \leq \left( \frac{1}{8} + 2 \lambda_1 + 9 \lambda_1 c_2 \right) \int_0^t \int (|\nabla J|^2 + |\nabla \Omega|^2) \phi^2 dyds
\]

\[
+ (\lambda_2 + 8 \lambda_2 c_2) \int_0^t \int [(J \phi)^2 + (\Omega \phi)^2] dyds + \frac{1}{8} \int_0^t \int |\nabla w_3|^2 \phi^2 dyds
\]

\[
+ 6 \lambda_1 \int_0^t \int |\nabla w|^2 \phi^2 dyds + C(tS(v_0, a, l, \lambda_1, \lambda_2)).
\]

This, (2.10) and (2.7) together give

\[
\int (J^2 + \Omega^2 + w_3^2) \phi^2 dy \bigg|_0^t + \int_0^t \int (|\nabla J|^2 + |\nabla \Omega|^2 + |\nabla w_3|^2) \phi^2 dyds
\]

\[
\leq \frac{1}{4} \int_0^t \int (|\nabla J|^2 + |\nabla \Omega|^2 + |\nabla w_3|^2) \phi^2 dyds
\]

\[
+ \left( \frac{1}{4} + 4 \lambda_1 + 18 \lambda_1 c_2 \right) \int_0^t \int (|\nabla J|^2 + |\nabla \Omega|^2) \phi^2 dyds
\]

\[
+ 2(\lambda_2 + 8 \lambda_2 c_2) \int_0^t \int [(J \phi)^2 + (\Omega \phi)^2] dyds + \frac{1}{4} \int_0^t \int |\nabla w_3|^2 \phi^2 dyds
\]

\[
+ 12 \lambda_1 \int_0^t \int |\nabla w|^2 \phi^2 dyds + C(tS(v_0, a, l, \lambda_1, \lambda_2)).
\]
Hence
\[
\int (J^2 + \Omega^2 + w_3^2) \phi^2 dy \bigg|_0^t + \frac{1}{4} \int_0^t \int (|\nabla J|^2 + |\nabla \Omega|^2 + |\nabla w_3|^2) \phi^2 dy ds
\]
(2.35)
\[
\leq (4 + 18c_2)\lambda_1 \int_0^t \int (|\nabla J|^2 + |\nabla \Omega|^2 + |\nabla w_3|^2) \phi^2 dy ds + 12\lambda_1 \int_0^t |\nabla w|^2 \phi^2 dy ds
+ 2\lambda_2(1 + 8c_2) \int_0^t \int [(J\phi)^2 + (\Omega\phi)^2] dy ds + Ct.S(v_0, a, l, \lambda_1, \lambda_2).
\]

There is still a little work to do, namely to bound the second term on the right hand side by the left hand side. Notice that \(w\) is axially symmetric. Hence
\[
|\nabla w|^2 = |\partial_r w_r|^2 + |\partial_r w_\theta|^2 + |\partial_3 w_r|^2 + |\partial_3 w_\theta|^2 + |\nabla w_3|^2
= |\partial_r (Jr)|^2 + |\partial_r (\Omega r)|^2 + r^2|\partial_3 J|^2 + r^2|\partial_3 \Omega|^2 + |\nabla w_3|^2
= |r \partial_r J + J| \phi^2 + |r \partial_r \Omega + \Omega| \phi^2 + r^2|\partial_3 J|^2 + r^2|\partial_3 \Omega|^2 + |\nabla w_3|^2
\leq 2r^2|\partial_r J|^2 + 2J^2 + 2r^2|\partial_r \Omega|^2 + 2\Omega^2 + r^2|\partial_3 J|^2 + r^2|\partial_3 \Omega|^2 + |\nabla w_3|^2.
\]

Hence
\[
|\nabla w|^2 \leq 2r^2(|\nabla J|^2 + |\nabla \Omega|^2) + 2J^2 + 2(\Omega^2).
\]

Plugging this to the second term on the right hand side of (2.35), we arrive at
\[
\int (J^2 + \Omega^2 + w_3^2) \phi^2 dy \bigg|_0^t + \frac{1}{4} \int_0^t \int (|\nabla J|^2 + |\nabla \Omega|^2 + |\nabla w_3|^2) \phi^2 dy ds
\]
(2.36)
\[
\leq (28 + 18c_2)\lambda_1 \int_0^t \int (|\nabla J|^2 + |\nabla \Omega|^2 + |\nabla w_3|^2) \phi^2 dy ds
+ 2[\lambda_2(1 + 8c_2) + 24\lambda_1] \int_0^t \int [(J\phi)^2 + (\Omega\phi)^2] dy ds + Ct.S(v_0, a, l, \lambda_1, \lambda_2).
\]

Here we have used the assumption that \(r \leq a \leq 1\). Choosing
\[
\lambda_1 = \frac{1}{4(28 + 18c_2)}.
\]

Here \(c_2\) is given in (2.13) with \(q = 2\). We reduce the last inequality to
\[
\int (J^2 + \Omega^2 + w_3^2) \phi^2 dy \bigg|_0^t
\]
\[
\leq 2[\lambda_2(1 + 8c_2) + 24\lambda_1] \int_0^t \int [(J\phi)^2 + (\Omega\phi)^2] dy ds + Ct.S(v_0, a, l, \lambda_1, \lambda_2).
\]

By Gronwall’s inequality
\[
\int_{0 \leq r \leq a/2, -l/2 < y < l/2} \left( \frac{w_r}{r} \right)^2 + \left( \frac{w_\theta}{r} \right)^2 + w_3^2 \phi^2(y, t) dy \leq C(t, v_0, a, l, \lambda_1, \lambda_2).
\]

By standard theory this is more than enough to imply the regularity of \(v\) for all time. The reason is that it implies \(w\) is locally \(L^{2, \infty}\) in any finite time. □
Finally we verify the claim that $v_\theta$ is in the $\lambda_1$ critical class for any fixed $\lambda_1 > 0$, if it satisfies $|v_\theta(x,t)| \leq \frac{C}{r|\ln r|^{2+\epsilon}}$, $r < 1/2$.

Let $\psi = \psi(y,s)$ be any test function in Definition 1.1 with $a > 0$ to be specified later. Fixing $s$, we compute

$$
\int \frac{\psi^2}{r^2|\ln r|^{2+\epsilon}} dy = 2\pi \int_0^\infty \frac{1}{r|\ln r|^{2+\epsilon}} \psi^2 dr dy
$$

$$
= \frac{2\pi}{1+\epsilon} \int_0^\infty (|\ln r|^{-1-\epsilon})' \psi^2 dr dy = -\frac{2\pi}{1+\epsilon} \int_0^\infty \frac{1}{|\ln r|^{1+(\epsilon/2)}} \frac{2\psi}{\sqrt{r}} \frac{1}{|\ln r|^{\epsilon/2}} \sqrt{r} dr dy
$$

$$
\leq \frac{2\pi}{1+\epsilon} \int_0^\infty \frac{\psi^2}{r|\ln r|^{2+\epsilon}} dr dy + \frac{2\pi}{1+\epsilon} \int_0^\infty \frac{1}{|\ln r|^{\epsilon}} \psi^2 dr dy
$$

$$
\leq \frac{1}{1+\epsilon} \int \frac{\psi^2}{r^2|\ln r|^{2+\epsilon}} dy + \frac{1}{1+\epsilon} \int \frac{1}{|\ln r|^{\epsilon}} \psi^2 dr dy.
$$

Therefore

$$
\int \frac{\psi^2}{r^2|\ln r|^{2+\epsilon}} dy \leq \frac{1}{\epsilon|\ln a|^{\epsilon}} \int |\partial_r \psi|^2 dy,
$$

which shows

$$
\int \left( \frac{|v_\theta|}{r} + v_\theta^2 \right) \psi^2 dy \leq \frac{C + C^2}{\epsilon|\ln a|^{\epsilon}} \int |\partial_r \psi|^2 dy.
$$

Since $C$, $\epsilon$ and $\lambda_1$ are fixed positive numbers, we can always choose $a > 0$ sufficiently small so that, for all $t \geq 0$,

$$
\int_0^t \int \left( \frac{|v_\theta|}{r} + v_\theta^2 \right) \psi^2 dy ds \leq \lambda_1 \int_0^t \int |\partial_r \psi|^2 dy ds.
$$

Therefore $v_\theta$ is in the $\lambda_1$ critical class.

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