Research article

A hybrid collocation method for solving highly nonlinear boundary value problems

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A B S T R A C T

In this article, a hybrid collocation method for solving highly nonlinear boundary value problems is presented. This hybrid method combines Chebyshev collocation method with Laplace and differential transform methods to obtain approximate solutions of some highly nonlinear two-point boundary value problems of ordinary differential equations. The efficiency of the method is demonstrated by applying it to ordinary differential equations modelling Darcy-Brinkman-Forchheimer momentum problem, laminar viscous flow problem in a semi-porous channel subject to transverse magnetic field, fin problem with a temperature-dependent thermal conductivity, transformed equations modelling two-dimensional viscous flow problem in a rectangular domain bounded by two moving porous walls and two-dimensional constant speed squeezing flow of a viscous fluid between two approaching parallel plates. The results obtained are compared with the existing methods and the results show that the new method is quite reasonable, accurate and efficient.

1. Introduction

In this article, we consider the general nonlinear differential equation of order \( p \) given by

\[
\sum_{j=0}^{p} a_j(x)y^{(j)}(x) + N\left(y(x), y'(x), y''(x), \ldots, y^{(p)}(x)\right) = \Phi(x),
\]

with separated boundary conditions

\[
B_0(y(a)) = \alpha, \quad B_p(y(b)) = \beta,
\]

where \( B_j \) and \( B_p \) are linear operators, \( \rho \) is a positive integer, \( a_j(x) \), \( \Phi(x) \) are known functions of the independent variable \( x \), and \( y(x) \) is an unknown function (see [1]), and \( x \) is assumed to be solved in the domain \( x \in [a, b] \). The function \( N \) represents the nonlinear operator while the primes indicate differentiation with respect to \( x \).

In recent times, the studies on highly nonlinear boundary value problems have been the focus of considerable attention and this is as a result of the importance of these problems in modelling engineering, science, fluid mechanics and real life phenomena. Since most of these problems do not have exact analytic solutions, we resort to approximate and numerical techniques. Among the analytical and numerical methods that have been used to solve nonlinear boundary value problems are homotopy analysis method which was proposed by Liao [2] and [3], spectral homotopy analysis method, proposed by Motsa [4] and also improved spectral homotopy analysis method [5]. Homotopy analysis method is a non-perturbation method that is valid for linear and nonlinear problems with or without small/large parameters and the method is dependent upon four factors, namely, initial approximation, convergence-control parameter, auxiliary function, and auxiliary linear operator while spectral homotopy analysis method is a coupling technique of the traditional homotopy analysis method and Chebyshev spectral collocation method. Homotopy analysis method has been used by Ziabakhsh and Domairry [6] to give the solution of laminar viscous flow in a semi-porous channel in the presence of a uniform magnetic field. Other methods include optimal homotopy asymptotic method, proposed by Marinka and Herisanu [7], successive linearization method [8], tau homotopy analysis method [9], homotopy perturbation method [10], [11], spectral homotopy perturbation method [12], Adomian decomposition method [13], improved spectral homotopy analysis method [5], modified Adomian decomposition method [14], Parker-Sochacki method [15], variational iteration method [16], Laplace transform-homotopy perturbation method [17], finite difference method [18], etc.

Collocation approach was employed by Kantorovic in 1934 [19] and Frazer et al. [20] in 1937 for solving differential equations appears to have been first used by Slater. Other researchers who have also employed the approach include Oliveira, [21], Singh et al. [22], etc.

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The popularity of collocation method is due in part to their conceptual simplicity, wide applicability and ease of implementation [23]. In collocation method, a set of points are chosen which are called collocation points. The approach requires that at the chosen collocation points, the residual equation is satisfied, thus yielding a number of collocation equations. These equations in conjunction with the resulting equations from boundary conditions are solved to determine the unknown constant parameters which are thereafter substituted into the trial solution to yield the desired approximate solution.

Motivated by the robust and great advantage of simplicity of collocation methods in approximating solutions of linear and non-linear equations, we developed the hybrid method which combines Chebyshev Collocation method with Laplace and Differential Transform Methods. The method depends less on many restrictive factors as in Homotopy Analysis Method (HAM) and Spectral Homotopy Analysis Method. The method also has to its advantage better accuracy than the Chebyshev Collocation Method, Laplace Transform Collocation Method and the Differential Transform Collocation Method.

The remaining part of this paper is organized as follows. In Section 2, we discuss some basic ideas of Laplace and differential transforms and construction of Chebyshev polynomials. Also in Section 3, we give the description of the new method while Section 4 is devoted to the discussion of error and convergence analysis. Section 5 discusses the numerical stability of the proposed method while Section 6 gives details of implementation of the method on some problems and their results are presented. Finally, concluding remarks are given in Section 7.

2. Basic ideas of Laplace and differential transforms and properties Chebyshev polynomials

This section describes the basic ideas of Laplace and differential transforms method and employs the properties of first kind shifted Chebyshev polynomials.

2.1. Laplace Transform

The Laplace transform of a function \( y(x) \) for \( x \geq 0 \), is defined as

\[
Y(s) = \mathcal{L}[y(x)] = \int_0^\infty e^{-sx} y(x) dx,
\]

where \( s \) is real, and \( \mathcal{L} \) is called the Laplace transform operator. The conditions for the existence of a Laplace transform \( Y(s) \) may fail to exist. If \( y(x) \) has infinite discontinuities or if it grows up rapidly, then \( Y(s) \) does not exist. Thus the following are the Laplace transform of derivatives:

\[
\mathcal{L}[y'(x)] = sy(x) - y(0),
\]

\[
\mathcal{L}[y''(x)] = s^2y(x) - sy(0) - y'(0),
\]

\[
\mathcal{L}[y'''(x)] = s^3y(x) - s^2y(0) - sy'(0) - y''(0),
\]

\[
\vdots
\]

\[
\mathcal{L}[y^{(n)}(x)] = s^n y(x) - s^{n-1} y(0) - \ldots - s y^{(n-2)}(0) - y^{(n-1)}(0).
\]

Furthermore, if the Laplace transform of \( y(x) \) is \( Y(s) \), then the inverse Laplace transform of \( Y(s) \) is \( y(x) \). In other words, we write

\[
\mathcal{L}^{-1}[Y(s)] = y(x),
\]

where \( \mathcal{L}^{-1} \) is the operator of the inverse Laplace transform. For more details on other properties of Laplace transform (see [24]).

2.2. Differential transform

The differential transformation method of function \( y(x) \) is defined as

\[
Y(k) = \frac{1}{k!} \left. \frac{d^k y(x)}{dx^k} \right|_{x=x_0},
\]

where \( y(x) \) is the original function and \( Y(k) \) is the transformed function and the inverse differential transformation is defined as

\[
y(x) = \sum_{k=0}^{\infty} Y(k)x^k.
\]

In truncated series form, (2.4) can be written as [25]

\[
y(x) = \sum_{k=0}^{N} Y(k)x^k.
\]

The following properties hold for differential transformation:

1. If \( y(x) = g(x) \pm h(x) \), then \( Y(k) = G(k) \pm H(k) \).
2. If \( y(x) = cg(x) \), then \( Y(k) = cG(k) \), where \( c \) is a constant.
3. If \( y(x) = \frac{d^n y}{dx^n} \), then \( Y(k) = \frac{k+n}{k!}G(k+n) \).
4. If \( y(x) = g(x)h(x) \), then \( Y(k) = \sum_{k_1=0}^{k} G(k_1)H(k - k_1) \).
5. If \( y(x) = x^n \), then \( Y(k) = \delta(k-n) \), where \( \delta(k-n) = \begin{cases} 1, & k = n, \\ 0, & k \neq n. \end{cases} \)
6. If \( y(x) = x^m f(x) \) with \( m \in N \), then

\[
Y(k) = \begin{cases} 0, & k = n, \\ F(k-m), & k \geq m. \end{cases}
\]

2.3. Construction of Chebyshev polynomial

The Chebyshev polynomial of degree \( n \) over the interval \([-1,1]\] is defined by the relation

\[
T_n(x) = \cos(n \cos^{-1} x).
\]

Letting \( \theta = \cos^{-1} x \), so that \( x = \cos \theta \), then we have

\[
T_n(x) = \cos n\theta,
\]

which implies that

\[
T_0(x) = 1, \quad T_1(x) = x.
\]

But \( T_n(x) = \cos n\theta \), thus

\[
T_{n+1}(x) = \cos(n+1)\theta = \cos(n\theta) \cos \theta - \sin(n\theta) \sin \theta,
\]

and

\[
T_{n-1}(x) = \cos(n-1)\theta = \cos(n\theta) \cos \theta + \sin(n\theta) \sin \theta.
\]

From (2.9) and (2.10), we obtain the following recurrence relation

\[
T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n \geq 1,
\]

which yields

\[
T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x,
\]

etc. This could be converted into interval \([a,b]\) using the recursive relation

\[
T_n^*(x) = 2 \left( \frac{2x - (b+a)}{b-a} \right) T_n(x) - T_{n-1}(x), \quad n \geq 1,
\]

and this is called shifted Chebyshev polynomials (see [26]).
3. Description of Hybrid Collocation Method (HCM)

In this section, we give detailed description of the proposed method to solve numerically the \( p \)th-order nonlinear differential equation of the form (1.1) and the associated separated boundary conditions (1.2). The \( p \)th-order derivative is first sought in the truncated Chebyshev series form and then Laplace transform is carried out to obtain the expression for the function itself and thereafter differential transform is applied. Thus we have

\[
y^{(p)}(x) = \sum_{n=0}^{N} a_n T_n^p(x). \quad (3.1)
\]

Taking the Laplace transform of both sides of (3.1), we have

\[
s^p Y(x) - s^{p-1} y(0) - s^{p-2} y'(0) - \ldots - y^{p-1}(0) = \mathcal{L} \left( \sum_{n=0}^{N} a_n T_n^p(x) \right). \quad (3.2)
\]

Equation (3.2) is rearranged and simplified to give

\[
Y(x) = \frac{1}{s^p} \left( s^{p-1} y(0) + s^{p-2} y'(0) + \ldots + y^{p-1}(0) + \mathcal{L} \left( \sum_{n=0}^{N} a_n T_n^p(x) \right) \right). \quad (3.3)
\]

On taking the inverse Laplace transform of (3.3), we have

\[
y(x) = \mathcal{L}^{-1} \left[ \frac{1}{s^p} \left( s^{p-1} y(0) + s^{p-2} y'(0) + \ldots + y^{p-1}(0) + \mathcal{L} \left( \sum_{n=0}^{N} a_n T_n^p(x) \right) \right) \right]. \quad (3.4)
\]

Then taking the differential transform of (3.4), we obtain

\[
Y(k) = DT \left[ \mathcal{L}^{-1} \left[ \frac{1}{s^p} \left( s^{p-1} y(0) + s^{p-2} y'(0) + \ldots + y^{p-1}(0) + \mathcal{L} \left( \sum_{n=0}^{N} a_n T_n^p(x) \right) \right) \right] \right]. \quad (3.5)
\]

Thus the new approximate solution to \( y(x) \) is given as

\[
y(x) = \sum_{k=0}^{N+p} Y(k) x^k, \quad (3.6)
\]

where \( Y(k) \) is the value on the right-hand side of (3.5). Substituting (3.6) into (1.1) yields

\[
\sum_{j=0}^{p} \sum_{k=0}^{N+p} a_j(x_j) (Y(k) x^k)^{(j)}(x) + N \left( \sum_{k=0}^{N+p} Y(k) x^k \right)^{(p)} \left( \sum_{k=0}^{N+p} Y(k) x^k \right)^{(p-1)}, \quad (3.7)
\]

Collocating (3.7) at the points \( x = x_i \), we have

\[
\sum_{j=0}^{p} \sum_{k=0}^{N+p} a_j(x_i) (Y(k) x^k)^{(j)}(x_i) + N \left( \sum_{k=0}^{N+p} Y(k) x^k \right)^{(p)} \left( \sum_{k=0}^{N+p} Y(k) x^k \right)^{(p-1)}, \quad (3.8)
\]

where

\[
x_i = a + \frac{(b-a)}{N+2} i, \quad i = 1, 2, \ldots, N+1. \quad (3.9)
\]

Equation (3.8) together with \( p \) boundary conditions generates \( (N+p+1) \) system of nonlinear algebraic equations in \( (N+p+1) \) unknowns. The unknown constants are determined by using Newton’s method and these are thereafter substituted into (3.6) to give the required approximate solution.

4. Error and convergence analysis

This section is devoted to the study of error and convergence analysis of the method. Following [27], we state the following theorem:

**Theorem 4.1.** The error in approximating \( y(x) \) by the sum of its first \( m \) terms is bounded by the sum of the absolute values of all the neglected coefficients. In essence, if

\[
y_m(x) = \sum_{k=0}^{m} Y(k) x^k, \quad (4.1)
\]

then

\[
E_T(m) = |y(x) - y_m(x)| \leq \sum_{k=m+1}^{\infty} |Y(k)|, \quad (4.2)
\]

where \( m = N + p \), for all \( y(x) \), \( m \), and \( x \in [a, b] \).

Bataineh [28] and Ömür et al. [29] performed convergence studies on collocation methods by investigating the rates of convergence of Bernstein and Dickson polynomial solutions, respectively in Banach space. Following [28] and [29], we determine the behaviour of the solution by substituting \( y_m(x) \) into (1.1) since the residual function always shows distinctive reaction for different values of \( m \) and therefore a residual function \( R_m(x) \) is obtained. So, we have

\[
R_m(x) = \sum_{j=0}^{p} a_j(x_j) y_m^{(j)}(x) + N \left( \sum_{k=0}^{N+p} Y(k) x^k \right)^{p-1} \left( \sum_{k=0}^{N+p} Y(k) x^k \right)^{p-2} - \Phi(x). \quad (4.3)
\]

where \( R_m(x) \) is defined on the interval \([a, b] \), and it can be put in the form

\[
R_m(x) = Y(0) + Y(1)x + Y(2)x^2 \ldots Y(m)x^m = \sum_{k=0}^{m} Y(k) x^k. \quad (4.4)
\]

Thus we state and prove the following theorem.

**Theorem 4.2 ([28], [29]).** Let \( B \) be a Banach space. The residual function sequence \( \{R_m\}_{m=1}^{\infty} \) is convergent in \( B \) and the following inequality is satisfied so that \( 0 < \lambda_m < 1 \). Here \( \lambda_m \) is constant in \( B \):

\[
||R_m(x)|| \leq \lambda_m ||R_n(x)|| \quad (4.4)
\]

**Proof.** To show that \( \{R_m\}_{m=1}^{\infty} \) is a Cauchy sequence in \( B \), we consider ||\( R_{m+1}(x) || \) and write

\[
|| R_{m+1}(x) || = \sup \left\{ || \sum_{k=0}^{m+1} Y(k) x^k | : x \in [a, b] \right\} \quad (4.5)
\]

\[
\leq \sup \left\{ \sum_{k=0}^{m+1} |Y(k) x^k| : x \in [a, b] \right\} \quad (4.6)
\]

\[
= |R_{m+1}(x)| \quad (4.7)
\]

so the inequality in (4.4) can be written at the point \( b \) as

\[
|R_{m+1}(b)| \leq \lambda_m |R_n(b)|. \quad (4.8)
\]

From inequality (4.5), we have

\[
|R_{m+1}(b)| \leq |R_{m+1}(b) - R_m(b)| \leq (\lambda_m - 1)|R_m(b)|. \quad (4.9)
\]

The generalization of (4.6) gives
\[ |R_{m+1}(b) - R_m(b)| \leq |\lambda_m - 1| \| R_m(b) \|
\]
\[ \leq (\lambda_m - 1)^m |R_m(b)|
\]

For any \( m, q \in \mathbb{N}, m \geq q \), we have

\[ |R_{m}(b) - R_q(b)| \leq |R_{m}(b) - R_{m-1}(b)| + |R_{m-1}(b) - R_{m-2}(b)| + \cdots + |R_{q+1}(b) - R_q(b)|
\]
\[ \leq (\lambda_m - 1)^{m-q} |R_q(b)|
\]

If \( \lambda_m < 1 \) in the above inequality, then

\[ \lim_{m \to \infty} |R_{m}(b) - R_q(b)| = 0.
\]

Therefore, \( \{R_{m}(b)\}_{m=0}^{\infty} \) is a Cauchy sequence in \( B \) and so it is convergent.

**Remark 4.3.** In the proposed method, it should be noted that the theorem is valid for second-order problems and for higher order problems, \( m \) must take a value greater than twice the order of the problem, i.e., \( m \geq 2p + 1 \).

5. Numerical stability

Following [30], we demonstrate the stability of the proposed method by choosing and adding some random noises \( \delta_n, n = 1, 2, \ldots \) to the initial data \( y(0), y'(0), \ldots, y^{(p-1)}(0) \) such that

\[ y^{(n)}(0) = y(0) + \delta_n y^{(n)}(0) = y(0) + \delta_n, \ldots, y^{(p-1)}(0) = y^{(p-1)}(0) + \delta_n, \]

where \( \delta \) is a small quantity which can be taken as some certain percent of maximum absolute errors. Therefore the residual function together with associated separated boundary conditions with noises for (1.1) and (1.2) are given by

\[ R_n(y)(b) = \sum_{j=0}^{p} \sum_{k=0}^{N+n} a_j(x) Y(k)x^k \]

\[ + N \left( \sum_{k=0}^{N+n} Y(k)x^k \right) \]

\[ + \left( \sum_{k=0}^{N+n} Y(k)x^k \right)^{(m)} + \cdots + \left( \sum_{k=0}^{N+n} Y(k)x^k \right)^{(p-1)} \phi(x)
\]

and

\[ B_n(y)(a) = \delta + a, \quad B_n(y)(b) = \beta.
\]

Thus the approximate solution with noises \( \delta_n \) is given as

\[ y^n(x) = \sum_{k=0}^{N+n} \hat{Y}(k)x^k.
\]

where

\[ Y(k) = D [L^{-1} \left( \frac{1}{\nu} (s^{p-1}y^{(0)}(0) + \delta) + s^{p-2}y^{(1)}(0) + \delta) + \cdots + (s^{p-1}y^{(p-1)}(0) + \delta)
\]

\[ + L \left( \sum_{n=0}^{N} a_n T_n(x) \right) \right]

For the purpose of clarity, all the examples in the next section are solved with and without noises to establish and confirm the stability of the new method. Thus, we choose the noises \( \delta_1, \delta_2, \) and \( \delta_3 \) to be \( 10^{-2}, 10^{-3} \), and \( 10^{-4} \) respectively.

6. Numerical examples

In this section, we implement the new method described in Section 3 on some illustrative examples of second and fourth-order highly nonlinear boundary value problems of ordinary differential equations to test its efficiency and applicability.

**Example 6.1.** We first consider the Darcy-Brinkman-Forchheimer equation [31].

\[ y''(x) - s^2 y(x) - F s y'(x) + \frac{1}{M} = 0.
\]

subject to the symmetry boundary conditions

\[ y'(0) = 0, \quad y(1) = 0.
\]

where \( M \) is the viscosity ratio, \( F \) is the Forchheimer number and \( s \) is the porous media shape parameter.

This problem has been analytically and numerically solved by using different approaches, among them are homotopy perturbation method [10], spectral homotopy analysis method [1], tau homotopy analysis method [32], etc. We apply the proposed method that we described in Section 3 to solve this problem when \( N = 8 \) and \( N = 10 \). Thus for \( N = 8 \), we have

\[ y^n(x) = \sum_{n=0}^{8} a_n T_n(x).
\]

Taking the Laplace transform of (6.3), we obtain

\[ s^2 \mathcal{L}[y(x)] - sy(0) - y'(0) = \mathcal{L} \left( \sum_{n=0}^{8} a_n T_n(x) \right).
\]

By rearranging (6.4) and replacing \( y(0) \) and \( y'(0) \) with \( c_1 \) and \( c_2 \) respectively, we get

\[ \mathcal{L}[y(x)] = \frac{1}{s^2} \left( s c_1 + c_2 \right) \left( \sum_{n=0}^{8} a_n T_n(x) \right).
\]

where \( c_1 \) and \( c_2 \) are constants to be determined using the boundary conditions. Then taking the inverse Laplace transform of (6.5) and thereafter applying Property 5, we obtain

\[ Y(k) = \frac{16384}{45} a_0 \delta (k - 10) + \left( -\frac{16384}{9} a_9 + \frac{1024}{9} a_7 \right) \delta (k - 9)
\]

\[ + \left( \frac{26624}{7} a_8 + \frac{256}{7} a_6 - 512 a_7 \right) \delta (k - 8)
\]

\[ + \left( -\frac{90112}{21} a_9 + \frac{2816}{3} a_7 - \frac{1024}{7} a_6 + \frac{256}{7} a_5 \right) \delta (k - 7)
\]

\[ + \left( \frac{64}{15} a_8 + \frac{1152}{5} a_6 + 2816 a_7 - \frac{128}{3} a_5 - 896 a_6 \right) \delta (k - 6)
\]

\[ + \left( \frac{64}{5} a_8 + \frac{2352}{5} a_6 + 896 a_7 - \frac{5376}{5} a_5 + \frac{8}{3} a_4 + 56 a_5 \right) \delta (k - 5)
\]

\[ + \left( 70 a_6 + \frac{523}{4} a_4 + 40 a_3 - 4 a_2 + \frac{2}{3} a_2 - \frac{100}{3} a_3 + 224 a_4 \right) \delta (k - 4)
\]

\[ + \left( -\frac{16}{3} a_4 + \frac{25}{3} a_3 + \frac{4}{3} a_2 - 12 a_6 - \frac{64}{3} a_5 + \frac{49}{3} a_4 + \frac{1}{3} a_5 + 3 a_6 \right) \delta (k - 3)
\]

\[ \times \delta (k - 3)
\]

\[ + \left( \frac{1}{2} a_8 - \frac{1}{2} a_7 - \frac{1}{2} a_6 + \frac{1}{2} a_5 + \frac{2}{3} a_4 - \frac{1}{2} a_3 - \frac{1}{2} a_5 + \frac{1}{2} a_4 + \frac{1}{2} a_6 \right) \delta (k - 2)
\]

\[ \times \delta (k - 2)
\]

\[ + c_2 \delta (k - 1) + c_1 \delta (k - 0).
\]

From (6.6), we obtain the values of \( Y(k) \), for \( k = 0, 1, \ldots, 10 \) as follows:
Table 1
Numerical Results for Example 6.1 when $N = 8$, and $M = F = 1$.

| $s$ | $y(0)$ | $y'(1)$ |
|-----|--------|---------|
| 0.0 | 0.50000000000 | -1.00000000000 |
| 0.5 | 0.422685393494 | -0.880643124591 |
| 1.0 | 0.32384781999 | -0.721231155975 |
| 1.5 | 0.238385129133 | -0.579104135965 |
| 2.0 | 0.17443255240 | -0.469128653824 |
| 2.5 | 0.12917349995 | -0.38790626476 |

Table 2
Numerical Results for Example 6.1 when $N = 8$, and $s = M = 1$.

| $F$ | $y(0)$ | $y'(1)$ |
|-----|--------|---------|
| 0.0 | 0.35194572636 | -0.76159415595 |
| 1.0 | 0.32384781999 | -0.721231155975 |
| 2.0 | 0.302609200516 | -0.69043351776 |
| 3.0 | 0.285667347242 | -0.66560867848 |
| 4.0 | 0.271668792692 | -0.645032080484 |
| 5.0 | 0.259804135452 | -0.627419163998 |
| 6.0 | 0.249553269788 | -0.612097057509 |

Table 3
Numerical Results for Example 6.1 when $N = 8$, and $s = F = 1$.

| $M$ | $y(0)$ | $y'(1)$ |
|-----|--------|---------|
| 1.0 | 0.32384781999 | -0.721231155975 |
| 2.0 | 0.302609200516 | -0.69043351776 |
| 3.0 | 0.285667347242 | -0.66560867848 |
| 4.0 | 0.271668792692 | -0.645032080484 |
| 5.0 | 0.259804135452 | -0.627419163998 |
| 6.0 | 0.249553269788 | -0.612097057509 |

| $a_0 = 0$, $a_1 = 0$, $a_2 = 0$, $c_1 = 0.5$, $c_2 = 0$. |

Hence, the approximate solution is given as

$$y(x) = \frac{1}{2} - \frac{1}{2} x^2.$$ 

Tables 1, 2 and 3 show the comparison of the values obtained by the present method and the existing methods for different values of $s$, $M$ and $F$. Figs. 1a, 2a and 3a show the graphs of HCM solutions for different values of $s$, $F$, and $M$ when $N = 8$. Figs. 1b, 2b and 3b display the variations in the approximate solutions with and without noises. In each case, it is observed that the effect of the noise terms on the solutions is negligible. Hence, the proposed method is numerically stable.

Example 6.2. The transformed equation of the problem of laminar viscous flow in a semi- porous channel subject to a transverse magnetic field is given by [1], [5], [6]

$$y''(x) - Hu^2 y''(y(x)) + Re \left( y(x)y'''(y(x)) - y'(x)y''(y(x)) \right) = 0,$$

subject to the boundary conditions

$$y(0) = 0, \quad y'(0) = 0, \quad x = 0, \quad y(1) = 0, \quad y'(1) = 0, \quad x = 1,$$

where $Hu$ and $Re$ are the Hartmann and Reynolds numbers, respectively.

We apply the proposed method to solve this problem when $N = 8$ and $N = 10$, and we provide the numerical solutions at different values
of $x$. Also, an interesting comparison is made and it is observed that the results are in good agreement with the exact solution and also compare favourably with the existing results in the literature. In Fig. 4a the effect of Reynold’s number on the approximate solution is shown given $H a = 1$. The plot of the approximate solution with different noise terms as compared with the exact solution is shown in Fig. 4b. It is observed that the effect of noise terms on the approximate solution is negligible. Hence the proposed method is stable. Also, the numerical results of this problem are presented in Tables 4 and the results show that the method is efficient, reliable and accurate.

By applying Theorem 4.2, we give the convergence of the HCM solutions for Example (6.2). We consider cases $N = 5, 6, 7, 8$ which eventually give $m = 9, 10, 11, 12$ respectively since the problem is of order four. Thus we have

\[
\| R_{m+1}(x) \| = \sup \left\{ \sum_{k=0}^{n+1} Y(k)x^k : x \in [0, 1] \right\} \\
\leq \sup \left\{ \sum_{k=0}^{n+1} |Y(k)x^k| : x \in [0, 1] \right\} \\
= | R_{m+1}(x) |
\]
Therefore which implies

\[ \| R_{m+}(x) \| \leq \lambda_m \| R_m(x) \|. \]

Thus the residual functions sequence is as follows:

\[ \{ R_{m+}(1) \}_{m=0}^{\infty} = \left\{ |R_0(1)|, |R_2(1)|, |R_1(1)|, |R_2(1)|, \ldots \right\} = \{ 0.034455, 0.009894, 0.003948, 0.00165, \ldots \}. \]

Therefore

\[ \lambda_m = \left\{ \frac{|R_0(1)|}{|R_0(1)|}, \frac{|R_2(1)|}{|R_2(1)|}, \frac{|R_1(1)|}{|R_1(1)|}, \ldots \right\} = \{ 0.287157, 0.399030, 0.041793, \ldots \}. \]

which shows that

\[ \frac{|R_{m+}(1)|}{|R_m(1)|} < 1. \]

From the sequence obtained, it is obvious that \( \{ R_m(1) \}_{m=0}^{\infty} \) satisfies the inequality \( |R_{m+1}(1)| \leq \lambda_m |R_m(1)| \) with respect to \( \lambda_m < 1 \) at \( x = 1 \). Therefore it can be concluded that \( \{ R_m(1) \}_{m=0}^{\infty} \) is convergent in \( B \).

**Example 6.3.** We consider the following transformed equation of the problem of two-dimensional viscous flow in a rectangular domain bounded by two moving porous walls [1], [33].

\[ y''(x) + a \left( xy'''(x) + 3y''(x) \right) + Re \left( y(x)y'''(x) - y'(x)y'''(x) \right) = 0, \quad (6.9) \]

subject to the boundary conditions

\[ \begin{align*}
  y &= 0, \quad y'' = 0, \quad x = 0, \\
  y &= 1, \quad y' = 0, \quad x = 1.
\end{align*} \quad (6.10) \]

Here, \( a \) is the non-dimensional wall dilation rate defined to be positive for expansion and negative for contraction while \( Re \) is the permeation Reynolds number defined positive for injection and negative for suction through the walls (see [1]). Motsa [1], Dinarvand and Rashidi [33], etc., employed spectral homotopy analysis and homotopy analysis methods respectively to study Example 6.3. In this paper, we used our proposed method to solve the problem for different values of Reynolds number, \( Re \), and the non-dimensional wall dilation rate, \( a \). Table 5 presents the numerical results for cases \( Re = -2, 0, 2 \) when \( a = -1 \). Also, Fig. 5a shows the graph of \( y'(x) \) over a range of \( a \) at \( Re = 0 \). The plot of \( y'(x) \) with different noise terms as compared with the exact solution is shown in Fig. 5b. It is observed that the effect of noise terms on the approximate solution is negligible. Hence, the proposed method is stable.
Example 6.4. We also consider a transformed nonlinear equation governing two-dimensional constant speed squeezing flow of a viscous fluid between two approaching parallel plates [8, 17]

\[ y''(x) + M y(x)y'''(x) = 0, \]

subject to the boundary conditions

\[ y(0) = 0, \quad y''(0) = 0, \]

and

\[ y(1) = 1, \quad y'(1) = 0. \]

The proposed method is applied to solve this problem by using different values of \( N \) such as \( N = 5, 6, 7 \). The numerical results obtained are presented in Table 6 when \( M = 2 \). Fig. 6a shows the graph of the results of \( y'(x) \) for different values of \( M \). The plot for \( y'(x) \) with different noise terms as compared with the exact solution is shown in Fig. 6b. It is observed that the effect of noise terms on the approximate solution is negligible. Hence, the proposed method is stable.

Example 6.5. As the fifth example, we consider the following fin temperature distribution problem [15, 22]

\[ \theta''(\xi) + \beta \theta(\xi) \theta''(\xi) + \beta \big( \theta'(\xi) \big)^2 - \psi^2 \theta(\xi) = 0, \]

subject to the boundary conditions

\[ \theta' = 0 \quad \text{at} \quad \xi = 0, \quad \text{and} \quad \theta = 1 \quad \text{at} \quad \xi = 1. \]

Table 6

| Comparison of solutions | bvp4c | [8] | HCM solutions |
|-------------------------|-------|-----|---------------|
| \( x \)                |       |     | \( N = 5 \)   | \( N = 6 \)   | \( N = 7 \)   |
| 0.1                    | 0.15558330 | 0.15558330 | 0.15558441 | 0.15558378 | 0.15558330 |
| 0.2                    | 0.30735107 | 0.30735107 | 0.30735278 | 0.30735181 | 0.30735108 |
| 0.3                    | 0.45160336 | 0.45160336 | 0.45160524 | 0.45160417 | 0.45160337 |
| 0.4                    | 0.58485835 | 0.58485835 | 0.58486011 | 0.58485910 | 0.58485836 |
| 0.5                    | 0.70391178 | 0.70391178 | 0.70393232 | 0.70391238 | 0.70391379 |
| 0.6                    | 0.80598750 | 0.80598750 | 0.80598855 | 0.80598793 | 0.80598752 |
| 0.7                    | 0.88855853 | 0.88855853 | 0.88855917 | 0.88855877 | 0.88855854 |
| 0.8                    | 0.94954223 | 0.94954223 | 0.94954251 | 0.94954232 | 0.94954223 |
| 0.9                    | 0.98717592 | 0.98717592 | 0.98717597 | 0.98717592 | 0.98717592 |

where \( \psi \) is the thermo-geometric fin parameter and \( \beta \) is the thermal conductivity parameter.

This problem has been studied by several authors and different approaches have been used to obtain its approximate solutions. Among the methods are: differential transformation method [34], homotopy perturbation method [35], Taylor series method [36], numerical approach [37] and homotopy analysis method [38]. We apply our method to solve this problem and the results obtained are presented in Tables 7 and 8 when \( N = 8 \). The tables show the results of dimensionless temperature distribution when \( \beta = 0, \psi = 0.5 \) and \( \beta = 0, \psi = 1 \). Also, Fig. 7a shows the graph of dimensionless temperature when \( \psi = 0.5 \) with different thermal conductivity parameter \( \beta \). The plot of the approximate solution with different noise terms as compared with the exact solution
is shown in Fig. 7b. It is observed that the effect of noise terms on the approximate solution is negligible. Hence, the proposed method is stable.

Similarly, by using Theorem 4.2, the convergence of the proposed method for Example 6.5 is given as follows for cases $N = 2, 3, \ldots, 8$ which also mean $m = 4, 5, \ldots, 10$.

The residual functions sequence in this case is given as follows:

$$
\|R_{m+1}(x)\| = \left\{ \begin{array}{ll}
R_m(x) & |R_m(x)| \\
\frac{|R_m(x)|}{|R_{m+1}(x)|} & |R_{m+1}(x)| \\
\end{array} \right..
$$

Thus

$$
\lambda_m = \left\{ \begin{array}{ll}
|R_0(x)| & |R_0(x)| \\
\frac{|R_0(x)|}{|R_1(x)|} & |R_1(x)| \\
\end{array} \right..
$$

so

$$
\frac{|R_{m+1}(x)|}{|R_m(x)|} < 1.
$$

Since $|R_{m+1}(x)|$ satisfies the inequality $|R_{m+1}(x)| \leq \lambda_m |R_m(x)|$ with respect to $\lambda_m < 1$ at $x = 1$, it can therefore be concluded that the residual functions sequence is a Cauchy sequence in $B$ and thus convergent.

7. Conclusion

In this paper, we have proposed a new collocation method based on Laplace and differential transform methods for the solution of highly nonlinear boundary value problems of ordinary differential equations. Special attention is given to study the convergence and stability analysis of the proposed method. The method is employed to solve the Darcy-Brinkman-Forchheimer equation, laminar viscous flow problem in a semi-porous channel subject to a transverse magnetic field, two-dimensional viscous problem in a rectangular domain bounded by two moving porous walls and a fin temperature-dependent distribution problem. From Tables 1, 2, 3, 4, 5, 6 and 7, it is obvious that the new method is highly accurate, efficient and it is also observed that the new method is numerically stable. The significant advantage of the HCM is that it converges rapidly to the
References

[1] S.S. Motro, On the optimal auxiliary linear operator for the spectral homotopy analysis method solution of nonlinear ordinary differential equations, Math. Probl. Eng. (2014) 1–15.

[2] S.J. Liao, Beyond Perturbation: Introduction to the Homotopy Analysis Method, Chapman & Hall/CRC Press, Boca Raton, Fla, USA, 2003.

[3] S.J. Liao, Homotopy Analysis Method in Nonlinear Differential Equations, Springer, Berlin, Germany, 2012.

[4] S.S. Motro, P. Sibanda, S. Shateyi, A new spectral-homotopy analysis method for solving a nonlinear second order BVP, Commun. Nonlinear Sci. Numer. Simul. 15 (9) (2010) 2293–2302.

[5] S.S. Motro, S. Shateyi, G.T. Marewo, P. Sibanda, An improved spectral homotopy analysis method for MHD flow in a semi-porous channel, Numer. Algorithms 60 (2012) 463–481.

[6] Z. Ziabakhsh, G. Damoury, Solution of the laminar viscous flow in a semi-porous channel in the presence of a uniform magnetic field by using the homotopy analysis method, Commun. Nonlinear Sci. Numer. Simul. 14 (2009) 1284–1294.

[7] V. Marinca, N. Herisanu, The Optimal Homotopy Asymptotic Method, Engineering Applications, Springer International Publishing AG, Switzerland, 2015.

[8] Z. Makukula, S.S. Motro, P. Sibanda, On a new solution for the viscouselastic squeezing flow between two parallel plates, J. Adv. Res. Appl. Math. 2 (4) (2010) 31–38.

[9] S. Kazem, M. Shahab, Tau-homotopy analysis method for solving micropolar flow due to a linearly stretching of porous sheet, Commun. Numer. Anal. 2012 (2012) cn-00114.

[10] S.O. Akindeinde, Homotopy perturbation method for the strongly nonlinear Darcy-Furcheimer model, Math. Theory Model. 5 (2015) 78–84.

[11] J.H. He, A coupling method of a homotopy analysis method technique and a perturbation technique for nonlinear problems, Int. J. Non-Linear Mech. 35 (2000) 37–43.

[12] A.K. Ahmed, A new spectral homotopy perturbation method and its application to Jeffrey-Hamel nanofluid flow with high magnetic field, J. Comput. Methods Phys. (2013) 939143.

[13] G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer Academic Publishers, Boston, MA, 1994.

[14] S. Abbasbandy, Improving Newton-Raphson method for nonlinear equations by modified Adomian decomposition method, Appl. Math. Comput. 145 (2003) 887–899.

[15] S.O. Akindeinde, Parker-Sochacki method for the solution of convective straight fins problem with temperature-dependent thermal conductivity, Int. J. Nonlinear Sci. 25 (2) (2018) 119–125.

[16] J.H. He, Variational iteration method - a kind of non-linear analytical technique: some examples, Int. J. Non-Linear Mech. 34 (1999) 699–708.

[17] U. Filobello-Nino, H. Vazquez-Leal, J. Cervantes-Perez, B. Benhammouda, A. Prez-Sesma, L. Hernandez-Martinez, V.M. Jimenez-Fernandez, A.L. Herrera-May, D. Pereyra-Diaz, A. Marin-Hernandez, J.H. Chua, A handy approximate solution for a squeezing flow between two infinite plates by using of Laplace transform-homotopy perturbation method, SpringerPlus 3 (421) (2014) 1–10.

[18] M.G. Sobamowo, Analysis of convective longitudinal fin with temperature-dependent thermal conductivity and internal heat generation, Alex. Eng. J. 56 (2017) 1–11.

[19] L.V. Kantorovich, On a new method of approximate solution of partial differential equations, Dokl. Acad. Nauk SSSR 4 (1934) 522–526.

[20] R.A. Frazer, W.P. Jone, S.W. Skan, Approximations to functions and to the solutions of differential equations, Great Britain Air Ministry Aero. Res. Comm. Tech. Rep. 1, 1937, pp. 517–549.

[21] F.A. Oliveira, Collocation and residual correction, Numer. Math. 36 (1980) 27–31.

[22] S. Singh, D. Kumar, K.N. Rai, Wavelet collocation solution of non-linear fin problem with temperature dependent thermal conductivity and heat transfer coefficient, Int. J. Nonlinear Anal. Appl. 6 (1) (2015) 105–118.

[23] G. Fairweather, D. Meade, A survey of spline collocation methods for the numerical solution of differential equations, in: J.C. Diaz (Ed.), Mathematics for Large Scale Computation, in: Lectures in Pure and Applied Mathematics, vol. 120, Marcel Dekker, New York, 1989, pp. 297–341.

[24] A.M. Wazwaz, Linear and Nonlinear Integral Equations, Springer Higher Education Press, Beijing, 2011.

[25] J.K. Zhou, Differential Transformation Method and Its Applications for Electrical Circuits, Huazhong University Press, Wuhan, 1986.

[26] J.C. Mason, D.C. Handscomb, Chebyshev Polynomials, Chapman & Hall/CRC, Boca Raton, 2003.

[27] M.A. Snyder, Chebyshev Methods in Numerical Approximation, Prentice-Hall Inc., Englewood Cliffs, NJ, USA, 1966.

[28] A.S. Bataineh, Bernstein polynomials method and its error analysis for solving non-linear problems in the calculus of variations: convergence analysis via residual function, Filomat 32 (4) (2018) 1379–1393.

[29] K.K. Ömür, A. Erkin, S. Mehmet, A numerical method for solving model problems arising in science and convergence analysis based on residual function, Appl. Numer. Math. 121 (2017) 134–148.

[30] H. Singh, H.M. Srivastava, D. Kumar, A reliable algorithm for the approximate solution of the nonlinear Lane-Emden type equations arising in astrophysics, Numer. Methods Partial Differ. Equ. (2017) 1–33.

[31] S. Abbasbandy, E. Shivanian, I. Hashim, Exact analytical solution of forced convection in a porous-saturated duct, Commun. Nonlinear Sci. Numer. Simul. 16 (2011) 3981–3989.

[32] M. Shaban, S. Kazem, J.A. Rad, A modification of the homotopy analysis method based on Chebyshev operational matrices, Math. Comput. Model. 57 (2013) 1227–1239.

[33] S. Dinarvand, M.M. Rashidi, A reliable treatment of a homotopy analysis method for two-dimensional viscous flow in a rectangular domain bounded by two moving porous walls, Nonlinear Anal., Real World Appl. 11 (2010) 1502–1512.

[34] A.A. Joneidi, D.D. Ganji, M. Babaei, Differential transformation method to determine fin efficiency of convective straight fins with temperature dependent thermal conductivity, Int. Commun. Heat Mass Transf. 36 (7) (2009) 757–762.

[35] C. Erdem, M.C. Pinar, Homotopy perturbation method for temperature distribution, fin efficiency and fin effectiveness of convective straight fins, Proc. Inst. Mech. Eng., Part C, Mech. Eng. Sci. 227 (8) (2013) 1754–1760.

[36] K. Xu, H. Cheng-Hung, A series solution of the fin problem with a temperature-dependent thermal conductivity, J. Phys. D. Appl. Phys. 39 (2006) 4894–4901.

[37] C. Rafael, A numerical analysis to the nonlinear fin problem, J. Zhujiang Univ. Sci. A 9 (5) (2008) 648–653.

[38] M. Subba, L. Rajendran, K. Saravanakumar, V. Ananthaswamy, Analytical solution of nonlinear boundary value problem for fin efficiency of convective straight fins with temperature-dependent thermal conductivity, IJRNM Thermodyn. 2013 (3).