WEIGHTED INEQUALITIES FOR COMMUTATORS OF
SCHRÖDINGER-RIESZ TRANSFORMS

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ABSTRACT. In this work we obtain weighted $L^p$, $1 < p < \infty$, and weak $L \log L$ estimates for the commutator of the Riesz transforms associated to a Schrödinger operator $-\Delta + V$, where $V$ satisfies some reverse Hölder inequality. The classes of weights as well as the classes of symbols are larger than $A_p$ and $BMO$ corresponding to the classical Riesz transforms.

1. Introduction

Let $V : \mathbb{R}^d \to \mathbb{R}$, $d \geq 3$, be a non-negative locally integrable function that belongs to a reverse-Hölder class $RH_q$ for some exponent $q > d/2$, i.e. there exists a constant $C$ such that

$$(1) \quad \left( \frac{1}{|B|} \int_B V(y)^q \, dy \right)^{1/q} \leq \frac{C}{|B|} \int_B V(y) \, dy,$$

for every ball $B \subset \mathbb{R}^d$.

For such a potential $V$ we consider the Schrödinger operator

$$\mathcal{L} = -\Delta + V,$$

and the associated Riesz Transform vector

$$\mathcal{R} = \nabla \mathcal{L}^{-1/2}. $$

Boundedness results of $\mathcal{R}$ have been obtained in [10] by Shen, where he shows that they are bounded on $L^p(\mathbb{R}^d)$ for $1 < p < p_0$, with $p_0$ depending on $q$. When $V \in RH_q$ with $q \geq d$, $\mathcal{R}$ and its adjoint $\mathcal{R}^*$ are in fact Calderón-Zygmund operators (see [10]).

We denote by $T$ either $\mathcal{R}$ or $\mathcal{R}^*$. For some function $b$ we will consider the commutator operator

$$T_b f(x) = T(bf)(x) - b(x)Tf(x), \quad x \in \mathbb{R}^d.$$

It is well known (see [3]) that for the classical case (that is $V \equiv 0$) the corresponding commutators $T_b$ are of strong type $(p, p)$ for $1 < p < \infty$ whenever $b$ belongs to $BMO$. However, for the case we deal with in this article, the operators $\mathcal{R}$ have better properties related to their decay. This behavior was the key point to get a significant improvement about the commutators $T_b$. In fact, in [2], it was obtained strong type $(p, p)$, $1 < p < \infty$, for $b$ in a wider space than $BMO$, that is

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the space $BMO_{\infty}(\rho) = \cup_{\theta > 0} BMO_{\theta}(\rho)$, where for $\theta > 0$ the space $BMO_{\theta}(\rho)$ is the set of locally integrable functions $f$ satisfying
\[
(3) \quad \frac{1}{|B(x, r)|} \int_{B(x, r)} |b(y) - b_B| \, dy \leq C \left(1 + \frac{r}{\rho(x)}\right)^{\theta},
\]
for all $x \in \mathbb{R}^d$ and $r > 0$, with $b_B = \frac{1}{|B|} \int_B b$. A norm for $b \in BMO_{\theta}(\rho)$, denoted by $\|b\|_{BMO_{\theta}(\rho)}$, is given by the infimum of the constants in (3).

The present article is devoted to obtain weighted boundedness for $T_b$. Once again, the special behavior of $\mathcal{R}$ allows us to get better results than in the classical case.

Particularly, we get strong $(p, p)$ inequalities for $b \in BMO_{\infty}(\rho)$ and weights in a class larger than Muckenhoupt’s. Such classes already appeared in connection with the $L^p$-boundedness of $\mathcal{R}$ (see [1]).

Moreover, we obtain weighted weak type inequalities for $T_b$. Related to this, it is important to remember that weak type $(1, 1)$ is not true in the case of classical singular integrals (see [3]). Nevertheless in that situation we are able to prove an $L \log L$ weak estimate but for $b$ in $BMO_{\infty}(\rho)$ and weights in a class larger than $A_1$. These results are completely new even in the unweighted case.

In order to get the results for $1 < p < \infty$ we use basically the same comparison techniques developed in [1]. However, this method fails for the extreme case $p = 1$, so we adapt the techniques in [9], based on some appropriate Calderón-Zygmund decomposition. Also, since the kernels of $\mathcal{R}$ may not have point-wise smoothness, we have to work with a Hörmander type condition instead.

The article is organized as follows. In sections 2 and 3 we review some properties concerning the critical radius function and the space $BMO_{\infty}(\rho)$. Section 4 is devoted to the class of weights where, in particular, we give a method to construct $A_1^{\infty, \rho}$ weights using a maximal function. In Section 5 we collect some estimates of the kernels of the Schrödinger-Riesz transforms, including a Hörmander type inequality, which slightly improves Lemma 4 in [6]. The main results concerning the boundedness of the commutators are presented in sections 6 and 7.

In the sequel, when $B = B(x, r)$ and $C > 0$, we shall use the notation $CB$, to denote the ball with the same center $x$ and radius $Cr$.

2. The critical radius function

The notion of locality is given by the critical radius function
\[
(4) \quad \rho(x) = \sup \left\{ r > 0 : \frac{1}{r^{d+2}} \int_{B(x, r)} V \leq 1 \right\}, \quad x \in \mathbb{R}^d,
\]
which, under our assumptions, satisfies $0 < \rho(x) < \infty$ (see [10]).

**Proposition 1** ([10]). If $V \in RH_{d/2}$, there exist $c_0$ and $N_0 \geq 1$ such that
\[
(5) \quad c_0^{-1} \rho(x) \left(1 + \frac{|x - y|}{\rho(x)}\right)^{-N_0} \leq \rho(y) \leq c_0 \rho(x) \left(1 + \frac{|x - y|}{\rho(x)}\right)^{\frac{N_0}{N_0 + 1}},
\]
for all $x, y \in \mathbb{R}^d$.

**Corollary 1.** Let $x, y \in B(x_0, R_0)$. Then,
(i) There exists $C > 0$ such that
\[
1 + \frac{R_0}{\rho(y)} \leq C \left( 1 + \frac{R_0}{\rho(x_0)} \right)^{N_0}.
\]

(ii) There exists $C > 0$ such that
\[
1 + \frac{r}{\rho(y)} \leq C \left( 1 + \frac{R_0}{\rho(x_0)} \right) \gamma \left( 1 + \frac{r}{\rho(x)} \right),
\]
for all $r > R_0$, where $\gamma = N_0(1 + \frac{N_0}{N_0+1})$.

Proof. Inequality (6) is a straightforward consequence of the left hand side of (5). Inequality (7) follows from the right hand side of (5) and then (6).

\[ \square \]

Proposition 2 (See [5]). There exists a sequence of points $x_j$, $j \geq 1$, in $\mathbb{R}^d$, so that the family $Q_j = B(x_j, \rho(x_j))$, $j \geq 1$, satisfies
\[
(i) \bigcup_j Q_j = \mathbb{R}^d.
\]
\[
(ii) \text{ For every } \sigma \geq 1 \text{ there exist constants } C \text{ and } N_1 \text{ such that, } \sum_j \chi_{\sigma Q_j} \leq C \sigma N_1.
\]

Lemma 1. Let $V \in RH_q$ with $q > d/2$ and $\epsilon > \frac{d}{q}$. Then for any constant $C_1$ there exists a constant $C_2$ such that
\[
\int_{B(x,C_1r)} \frac{V(u)}{|u-x|^{d-\epsilon}} du \leq C_2 r^{2-d/q},
\]
if $0 < r \leq \rho(x)$.

3. The space $BMO_{\infty}(\rho)$

From the definition given in the introduction, it is clear that $BMO \subset BMO_\theta(\rho) \subset BMO_{\theta'}(\rho)$ for $0 < \theta < \theta'$, and hence $BMO \subset BMO_{\infty}(\rho)$. Moreover, it is in general a larger class. For instance, when $\rho$ is constant (which corresponds to $V$ a positive constant) the functions $b_j(x) = |x_j|$, $1 \leq j \leq d$, belong to $BMO_{\infty}(\rho)$ but not to $BMO$. Also, when $V(x) = |x|^2$ and $\mathcal{L}$ becomes the Hermite operator, we obtain $\rho(x) \approx \frac{1}{1+|x|}$ and we may take $b(x) = |x_j|^2$.

Given a Young function $\varphi$ and a locally integrable $f$ we consider the $\varphi$-average over a ball or a cube (denoted by $Q$) defined as
\[
\|f\|_{\varphi,Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \varphi\left( \frac{|f|}{\lambda} \right) \leq 1 \right\}.
\]

If we denote by $\tilde{\varphi}$ the conjugate Young function of $\varphi$, it is well known that the following version of Hölder inequality holds
\[
\frac{1}{|Q|} \int_Q |fg| \leq 2 \|f\|_{\varphi,Q} \|g\|_{\tilde{\varphi},Q}.
\]

Let us remind that for a function $b \in BMO(Q)$, as a consequence of the John-Nirenberg inequality (see for example [4] p.151), we have
\[
\|b\|_{BMO(Q)} \simeq \sup_{B \subset Q} \|b - b_B\|_{\varphi,B},
\]
for certain Young functions $\varphi$. For instance $\varphi(t) = t^s$, $1 < s < \infty$, or $\varphi(t) = e^t - 1$.

For the spaces $BMO_{\infty}(\rho)$, we have a weaker version of this fact that will be enough to our purposes.
Lemma 2. Let $b \in BMO_\theta(\rho)$ and $\varphi$ such that (10) holds. Then there exist constants $C$ and $\theta'$ such that for every ball $B = B(x, r)$ we have
\[ \|b - b_{2k}B\|_{\varphi, B} \leq C k |b|_{\theta} \left( 1 + \frac{2^k r}{\rho(x)} \right)^{\theta'}. \]

Proof. For $k = 1$ the proof follows the same lines than that of Proposition 3 in [2]. The case $k > 1$ is a consequence of the case $k = 1$ and the inequality
\[ \|b - b_{2}B\|_{\varphi, B} \leq \|b - b_{1}B\|_{\varphi, B} + \frac{1}{\varphi^{-1}(1)} \sum_{i=1}^{k} |b_{2^i}B - b_{2^{i-1}}B|. \]
\[ \square \]

4. Weights

As in [1], we need classes of weights that are given in terms of the critical radius function (4). Given $p > 1$, we define $A_{\rho, \infty}^p = \cup_{\theta \geq 0} A_{\rho, \theta}^p$, where $A_{\rho, \theta}^p$ is the set of weights $w$ such that
\[ \left( \frac{1}{|B|} \int_B w \right)^{1/p} \left( \frac{1}{|B|} \int_B w^{-\frac{1}{p'-1}} \right)^{1/p'} \leq C \left( 1 + \frac{r}{\rho(x)} \right)^{\theta}, \]
for every ball $B = B(x, r)$.

For $p = 1$ we define $A_{1, \infty}^\infty = \cup_{\theta \geq 0} A_{1, \theta}^\infty$, where $A_{1, \theta}^\infty$ is the set of weights $w$ such that
\[ \frac{1}{|B|} \int_B w \leq C \left( 1 + \frac{r}{\rho(x)} \right)^{\theta} \inf_B w, \]
for every ball $B = B(x, r)$.

Remark 1. It is not difficult to see that in (11) it is equivalent to consider cubes instead of balls, due to Proposition [1].

These classes of weights, that contain Muckenhoupt weights, were introduced in [1], where the next property is proven.

Proposition 3. If $w \in A_{p, \infty}^p$, $1 < p < \infty$, then there exists $\epsilon > 0$ such that $w \in A_{p-\epsilon, \infty}^p$.

The following results are extensions of very well known properties of $A_1$ weights.

Lemma 3. If $u \in A_{1, \infty}^\infty$, then there exists $\nu > 1$ such that $u^\nu \in A_{1, \infty}^\infty$.

Proof. This result follows immediately from the reverse Hölder type inequality valid for $A_{p, \infty}^\infty$ weights (see Lemma 5 in [1]).

For $\theta > 0$ let us introduce the maximal function $M^\theta$ by
\[ M^\theta f(x) = \sup_{r > 0} \frac{1}{\left( 1 + \frac{r}{\rho(x)} \right)^{\theta}} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f|. \]

Remark 2. Observe that a weight $u$ belongs to $A_{1, \infty}^\infty$ if and only if there exists $\theta > 0$ such that $M^\theta u \lesssim u$.

Lemma 4. Let $g \in L^{1}_{\text{loc}}$, $\theta \geq 0$ and $0 < \delta < 1$, then $(M^\theta g)^\delta \in A_{1, \infty}^\infty$. 

Proof. It is enough to prove that there exists $\beta \geq 0$ such that for every ball $B_0 = B(x_0, R_0)$,

\begin{equation}
\label{eq:12}
\frac{1}{|B_0|} \int_{B_0} (M^\theta g)^\delta \lesssim (1 + \frac{R_0}{\rho(x_0)})^\beta \inf_{B_0} (M^\theta g)^\delta.
\end{equation}

We split $g = g_1 + g_2$, with $g_1 = g \chi_{2B_0}$.

For $g_1$ we use the weak type $(1, 1)$ of $M^\theta$ and Kolmogorov inequality to get for any $x \in B_0$,

\[ \frac{1}{|B_0|} \int_{B_0} (M^\theta g_1)^\delta \lesssim \left( \frac{1}{|B_0|} \int_{2B_0} |g| \right)^\delta \lesssim \left( 1 + \frac{R_0}{\rho(x)} \right)^{\delta \delta} (M^\theta g(x))^\delta. \]

Using (10) we arrive to the right hand side of (12).

For the term with $g_2$ we have that for any $x$ and $y$ in $B(x_0, R_0)$

\begin{equation}
\label{eq:13}
M^\theta g_2(x) \lesssim \left( 1 + \frac{R_0}{\rho(x_0)} \right)^{\gamma \theta} M^\theta g_2(y),
\end{equation}

where $\gamma$ is the constant appearing in (7).

In fact, considering a ball $B(x, r)$ with $r \geq R_0$ (otherwise the average of $g_2$ is zero), and using (7) it follows

\[ \frac{1}{|B_0|} \int_{B(x, r)} |g_2| \lesssim \left( \frac{1}{|B_0|} \int_{2B_0} |g| \right)^\delta \left( 1 + \frac{R_0}{\rho(x)} \right)^{\delta \delta} \left( 1 + \frac{r}{\rho(y)} \right)^{\gamma \theta} \frac{1}{|B(y, C\rho)|} \int_{B(y, C\rho)} |g_2| \]

for any $y \in B_0$, leading to (13).

Raising (13) to the $\delta$ power and taking averages over $B_0$ respect to $x$ we arrive to the right hand side of (12) with $\beta = \gamma \theta \delta$.

Finally, collecting the estimates for $g_1$ and $g_2$ the proof of the Lemma is finished. \qed

5. Estimates of the Kernels

The operators $R$ and $R^*$ have singular kernels with values in $\mathbb{R}^d$ that will be denoted by $K$ and $K^*$ respectively. For such kernels, we have the following estimates that are basically proved in [10] and [6] (see also Lemma 3 in [2]).

**Lemma 5.** Let $V \in RH_q$ with $q > d/2$.

(i) For every $N$ there exists a constant $C_N$ such that

\begin{equation}
\label{eq:14}
|K^*(x, y)| \leq C_N \left( 1 + \frac{|x-y|}{\rho(x)} \right)^{-N} \frac{V(u)}{|u-y|^{d-1}} du + \frac{1}{|x-y|}. \end{equation}

Moreover, the last inequality also holds with $\rho(x)$ replaced by $\rho(y)$.

(ii) For every $N$ and $0 < \delta < \min\{1, 2 - d/q\}$ there exists a constant $C$ such that

\[ |K^*(x, z) - K^*(y, z)| \leq \]

\begin{equation}
\label{eq:15}
C \frac{|x-y|^\delta}{|x-z|^{d-1+\delta}} \left( \frac{1}{|B(z, |x-z|/4)} \int_{B(z, |x-z|/4)} \frac{V(u)}{|u-z|^{d-1}} du + \frac{1}{|x-z|} \right). \end{equation}
whenever $|x - y| < \frac{2}{3}|x - z|$. Moreover, the last inequality also holds with $\rho(x)$ replaced by $\rho(z)$.

(iii) If $K^*$ denotes the $\mathbb{R}^d$ vector valued kernel of the adjoint of the classical Riesz operator, then

$$|K^*(x, z) - K^*(x, z)| \leq \frac{C}{|x - z|^{d-1}} \left( \int_{B(|z|/4)} |u - z|^{d-1} |u|^{1/2} \left( \frac{|x - z|}{\rho(x)} \right)^{2 - \frac{d}{q}} \right), \tag{16}$$

whenever $|x - z| \leq \rho(x)$.

(iv) When $q > d$, the term involving $V$ can be dropped from inequalities (14) and (16).

(v) If $q > d$, the term involving $V$ can be dropped from inequalities (14), (15) and (16).

The following lemma improves a result appearing in [6].

**Lemma 6.** Let $V \in RH_q$ with $d/2 < q < d$ and $s$ such that $\frac{1}{q} = \frac{1}{s} - \frac{1}{d}$. Then the kernel $K$ satisfies the following Hörmander type inequality

$$\sum_k k(2^k r)^{d/s'} \left( 1 + \frac{2^k r}{\rho(x_0)} \right) \theta \left( \int_{|x - x_0| \sim 2^k r} |K(x, y) - K(x, x_0)|^s dx \right)^{1/s} \leq C_0, \tag{17}$$

whenever $|y - x_0| < r$, and $r \geq 0$.

**Proof.** We follow the lines of the proof of Lemma 4 in [6] but performing a more careful estimate.

Using (15) we get

$$\left( \int_{|x - x_0| \sim 2^k r} |K(x, y) - K(x, x_0)|^s dx \right)^{1/s} \leq (2^k r)^{(1-d)2^{-k\delta}} \left( 1 + \frac{2^k r}{\rho(x_0)} \right)^{-N} \left( \|I_1(V\chi_{B(x_0, 2^k r)})\|_s + (2^k r)^{\frac{d}{q} - 1} \right),$$

where $I_1$ stands for the fractional integral operator of order one.

The estimate of (17) involving the second term above follows easily. Now, from the boundedness of $I_1$ and the fact that $V \in RH_q$, we get

$$\|I_1(V\chi_{B(x_0, 2^k r)})\|_s \leq (2^k r)^{-\frac{d}{q}} \int_{B(x_0, 2^k r)} V,$$

where the last integral can be estimated as

$$\int_{B(x_0, 2^k r)} V \leq (2^k r)^{d-2} \left( \frac{2^k r}{\rho(x_0)} \right)^\beta$$

with $\beta = 2 - \frac{d}{q}$ when $2^k r \leq \rho(x_0)$ and $\beta = \mu d$, $\mu \geq 1$ in other case (see [1]).

Therefore we can bound the left hand side of (15) by either $\rho(x_0)^{\frac{d}{q} - 2}$ or $(2^k r)^{\frac{d}{q} - 2} \left( \frac{2^k r}{\rho(x_0)} \right)^\mu$ with $\mu \geq 1$. 

Now, to finish the estimate of the sum on the left hand side of (17) we first sum over \( k \in J_1 = \{ k \in \mathbb{N} : 2^k r \leq \rho(x_0) \} \). For such sum, using the above estimates and that \( 2 - \frac{d}{q} > 0 \) we get the bound
\[
\sum_k k 2^{-k\delta} (2^k r)^{1-d+\frac{d}{q}+\frac{s}{d}} \rho(x_0)^{\frac{s}{d}} \lesssim \sum_k k 2^{-k\delta} \lesssim 1.
\]
Similarly, the other sum can be bounded by
\[
\sum_k k 2^{-k\delta} (2^k r)^{1-d+\frac{d}{q}+\frac{s}{d}} \left( \frac{2^k r}{\rho(x_0)} \right)^{N+\theta+\mu d} \lesssim \sum_k k 2^{-k\delta} \lesssim 1,
\]
choosing \( N \) large enough.

\[\square\]

**Lemma 7.** Let \( V \in RH_{q, \frac{d}{q}} \), \( \frac{d}{q} < q < d \), and \( \frac{d}{s} = \frac{1}{q} - \frac{1}{d} \). Then, for all \( N \) there exists \( C_N \) such that for any ball \( B = B(z, r) \) with \( r \geq \rho(z) \), \( y \in B \) and \( B^k = 2^k B \), the inequality
\[
(20) \quad \left( \int_{B^k \setminus B^{k-1}} |K(x, y)|^s \, dx \right)^{1/s} \leq C_N (2^k r)^{1-d+\frac{d}{q}+\frac{s}{d}} \left( \frac{\rho(z)}{2^k r} \right)^{N-\mu d}
\]
holds for some \( \mu \geq 1 \), which depends only on the constants appearing in the doubling condition that \( V \) satisfies.

**Proof.** From Lemma 5 we know
\[
(21) \quad |K(x, y)| \leq C_N \left( 1 + \frac{|x-y|}{\rho(z)} \right)^{-N} \left( \int_{B(y, 2|x-y|)} \frac{V(u)}{|u-y|^{d-1}} \, du + \frac{1}{|x-y|} \right).
\]
Now, for \( B \) and \( y \) as in the statement and \( x \in B^k \setminus B^{k-1} \) we have \( B(y, 2|x-y|) \subseteq B^{k+1} \). Also, since \( x \in B^{k+1} \) we may use Corollary 1 to reduce (21), with perhaps a different \( N \), to
\[
|K(x, y)| \leq C_N (2^k r)^{1-d} \left( \frac{\rho(z)}{2^k r} \right)^N \left( \frac{1}{2^k r} + I_1(\chi_{B^{k+1}} V)(y) \right).
\]
Therefore,
\[
\left( \int_{B^k \setminus B^{k-1}} |K(x, y)|^s \, dx \right)^{1/s} \lesssim (2^k r)^{1-d} \left( \frac{\rho(z)}{2^k r} \right)^N \left( (2^k r)^{\frac{s}{d}} + \|I_1(\chi_{B^{k+1}} V)\|_s \right).
\]
According to inequalities (18) and (19) we have
\[
\|I_1(\chi_{B^{k+1}} V)\|_s \lesssim (2^k r)^{-\frac{d}{s}+d-2} \left( \frac{2^k r}{\rho(z)} \right)^{\mu d}
\]
for some \( \mu \geq 1 \). Thus, plugging this estimate and using that \( r \geq \rho(z) \) and that \( \frac{d}{s} - 1 = \frac{d}{q} - 2 = -\frac{d}{q} + d - 2 \), we arrive to (20).

\[\square\]
6. \(L^p\) INEQUALITIES

For an operator \(T\) we associate the local and global operators of \(T\) as
\[
T_{\text{loc}}f(x) = T(f \chi_{B(x, \rho(x))})(x)
\]
and
\[
T_{\text{glob}}f(x) = T(f \chi_{B(x, \rho(x))})(x)
\]
respectively, where the first integral should be understood, if necessary, in the sense of principal value.

In the following Theorem, we use a larger classes of weights \(A^p_{\text{loc}}\) already defined in [1] as those weights that satisfy the classical \(A_p\) condition for balls \(B(x, r)\) with \(r \leq \rho(x)\). From the well known proof for \(A_p\) classes it is easy to derive the following result.

**Proposition 4.** If \(w \in A^p_{\text{loc}}, 1 < p < \infty\), then there exists \(\epsilon > 0\) such that \(w \in A^{p-\epsilon}_{\text{loc}}\).

Let us observe that a function \(b \in BMO_{\text{loc}}(\rho)\) has bounded mean oscillations over sub-critical balls, that is balls \(B(x, r)\) with \(r \leq \rho(x)\). For the next result we shall denote \(BMO_{\text{loc}}(\rho)\) the set of functions with the latter property.

**Remark 3.** Notice that using Proposition 2 it is possible to prove that for each constant \(C\) we have \(BMO_{\text{loc}}(\rho) = BMO_{\text{loc}}(C\rho)\) and the norms are equivalent with a constant that depends on \(C\).

**Theorem 1.** Let \(\rho\) a function satisfying \(\beta\) and \(b \in BMO_{\text{loc}}(\rho)\), then \((R_b)_{\text{loc}}\) are bounded on \(L^p(w)\), \(1 < p < \infty\), for \(w \in A^p_{\text{loc}}\).

**Proof.** Let \(\{Q_j\}_j\) be a covering by critical balls as in Proposition 2. It is possible to find a constant \(\beta\) such that if \(\tilde{Q}_j = \beta Q_j\) then \(\cup_{x \in Q_j} B(x, \rho(x)) \subset \tilde{Q}_j\).

From Lemma 1 in [1] a weight in \(A^p_{\text{loc}}\) when restricted to some \(\tilde{Q}_j\) can be extended to \(\mathbb{R}^d\) as an \(A_p\) weight preserving the \(A_{p,\text{loc}}\) constant. This kind of result can be extended also to \(BMO\) functions because of their well known relationship with \(A_p\) weights [1]. Therefore, given \(b \in BMO_{\text{loc}}(\rho)\) and any \(\tilde{Q}_j\), there is an extension of \(b_{\chi_{\tilde{Q}_j}}\) to the whole \(\mathbb{R}^d\) that we call \(b_j\) belonging to \(BMO\) with \(\|b_j\|_{BMO_{\text{loc}}}\) bounded by \(\|b\|_{\text{loc}}\), the natural norm in \(BMO_{\text{loc}}(\rho)\).

For \(x \in Q_j\), since \(b = b_j\) on \(\tilde{Q}_j\), we have
\[
|(R_b)_{\text{loc}}f(x)| \leq |(R_b)_{\text{loc}}f(x) - (R_b)(\chi_{\tilde{Q}_j}f)(x)| + |(R_b)(\chi_{\tilde{Q}_j}f)(x)|.
\]

The first term can be bounded as
\[
|(R_b)_{\text{loc}}f(x) - (R_b)(\chi_{\tilde{Q}_j}f)(x)| \lesssim \int_{Q_j \setminus B(x, \rho(x))} \frac{|f(y)| |b_j(x) - b_j(y)|}{|x - y|^d} dy \lesssim \|b\|_{\text{loc}} M_{s,\text{loc}}(f)(x),
\]
for each \(s > 1\), with \(M_{s,\text{loc}}f(x) = \sup_B \left(\frac{1}{|B|} \int_B |f|^s\right)^{1/s}\) where the sup is taken over sub-critical balls respect to the function \(\beta\rho\). Notice that to get the last inequality we made use of Remark 3. Then, since \(w = w_j\) on \(\tilde{Q}_j\),
\[
\int_{\mathbb{R}^d} |(R_b)_{\text{loc}}f|^p w \leq \sum_j \left(\|b\|_{\text{loc}} \int_{\tilde{Q}_j} |M_{s,\text{loc}}(f)|^p w + \int_{\tilde{Q}_j} |(R_b)(\chi_{\tilde{Q}_j}f)|^p w_j\right).
\]
By Proposition 1 and the boundedness of \( M_{1,\text{loc}} \) with \( A_{p,\text{loc}}^\rho \) weights (see Theorem 1 in [1]), we obtain the desired estimate for the first term.

For the second term we use that, since \( b_j \) belongs to \( BMO \), the commutator \( R_{b_j} \) is bounded on \( L^p(w_j) \) where \( w_j \) is an \( A_p \) extension of \( w \chi_{Q_j} \) to all \( \mathbb{R}^d \) with \( A_p \) constant depending only on the \( A_{p,\text{loc}}^\rho \) constant of the original weight \( w \). We also notice that the operator norm of \( R_{b_j} \) is independent of \( j \).

□

Now, give a technical Lemma that we will need in the proof of Theorem 3.

**Lemma 8.** Let \( \rho \) a function satisfying (1) and \( b \in BMO_\rho(\rho) \). Let \( w \) verifying the reverse Hölder’s inequality (1) for \( q = \delta \) and \( B \) any sub-critical ball. Then, given any \( p, \nu > 0 \), there exists a constant \( M > 0 \) such that

\[
\int_B w(x) \left( \int_{\lambda B} |b(x) - b(y)|^\nu dy \right)^{p/\nu} dx \lesssim \lambda^M \|b\|_p^p |B|^{\nu/p} w(B)
\]

for any sub-critical ball \( B \) and all \( \lambda \geq 1 \).

**Proof.** Let \( B = B(x_0, r) \) with \( r \leq \rho(x_0) \). The left side of (23) can be bounded by

\[
\lambda^{dp/\nu} |B|^{p/\nu} \int_B w(x) |b(x) - b_{\lambda B}|^p dx + w(B) \left( \int_{\lambda B} |b(y) - b_{\lambda B}|^\nu dy \right)^{p/\nu}
\]

For the first term of the last expression we use Hölder’s inequality with exponent \( \delta \) and the assumption on \( w \) and Lemma 2 to bound it by

\[
\lambda^{dp/\nu} |B|^{p/\nu - 1/\delta'} w(B) \left( \int_{\lambda B} |b(x) - b_{\lambda B}|^{p+\delta'} dx \right)^{1/\delta'}
\]

and the proof is finished proceeding as before.

□

**Theorem 2.** Let \( V \in RH_\delta \) and \( b \in BMO_\infty(\rho) \).

(i) If \( q \geq d \), the operators \( R_b \) and \( R_b^\star \) are bounded on \( L^p(w) \), \( 1 < p < \infty \), for \( w \in A_p^{\rho,\infty} \).

(ii) If \( d/2 < q < d \), and \( s \) is such that \( \frac{1}{s} = \frac{1}{q} - \frac{1}{d} \), the operator \( R_b^\star \) is bounded on \( L^p(w) \), \( 1 < p < \infty \) and \( w \in A_{p/s}^{\rho,\infty} \) and hence by duality \( R_b \) is bounded on \( L^p(w) \), \( 1 < p < s \), with \( w \) satisfying \( w^{-\frac{1}{p'-1}} \in A_{p'/s'}^{\rho,\infty} \).

**Proof.** First of all, notice that there is no need to consider \( q = d \) since in that case there exists an \( \epsilon > 0 \) such that \( V \in RH_{d+\epsilon} \). We begin giving estimates for \( R_b^\star \).

Now we write

\[
(R_b^\star)f = (R_b^\star)_{\text{loc}}f + (R_b^\star)_{\text{glob}}f + [(R_b^\star)_{\text{loc}} - (R_b)_{\text{loc}}]f.
\]
As a consequence of Theorem 1 the first term is bounded on $L^p(w)$ for $w \in A_{p,0}^{0,\text{loc}}$, $1 < p < \infty$. Since $w \in A_{p,\infty}^{0,\text{loc}} \subset A_{p}^{0,\text{loc}}$, $1 \leq p < \infty$, and $w \in A_{p/\nu}^{0,\text{loc}} \subset A_{p}^{0,\text{loc}}$, $s' < p < \infty$, all the conclusions for $(R^b_k)_{\text{loc}}$ hold.

For the second term of (25) we use (14) to obtain

$$\left| (\mathcal{R}^b_k)_{\text{glob}} f(x) \right| \leq \int_{B(x,\rho(x))} |b(y) - b(x)| |\mathcal{K}^*(x,y)||f(y)| \, dy \lesssim g_1(x) + g_2(x),$$

with

$$g_1(x) = \sum_{k=0}^{\infty} 2^{-kN} g_{1,k}(x),$$

where $g_{1,k}(x) = \frac{1}{(2^k \rho(x))^{d-1}} \int_{B(x,2^k \rho(x))} |b(y) - b(x)| |f(y)| \, dy$, and

$$g_2(x) = \sum_{k=0}^{\infty} 2^{-kN} g_{2,k}(x).$$

$$g_{2,k}(x) = \frac{1}{(2^k \rho(x))^{d-1}} \int_{B(2^k \rho(x))} \left( \int_{B(x,2^k \rho(x))} \frac{V(u)}{|u-y|^{d-1}} \, du \right) |b(y) - b(x)| |f(y)| \, dy$$

To deal with $g_1$, let $\sigma = c_0 2^{-N_0}$, with $N_0$ and $c_0$ as in Proposition 4. Let $\{Q_j\}$ be the family given by Proposition 2 and set $\hat{Q}_j = \sigma Q_j$. Clearly, we have

$$\bigcup_{x \in Q_j} B(x, \rho(x)) \subset \hat{Q}_j.$$

Denoting $\hat{Q}_j^k = 2^k \hat{Q}_j$, then $2^k B_x \subset \hat{Q}_j^k$ and $\rho(x) \simeq \rho(x_j)$, whenever $x \in Q_j$. Therefore, by Hölder’s inequality with $\gamma$ and $\nu$ such that $\frac{1}{p} + \frac{1}{\gamma} + \frac{1}{\nu} = 1$,

$$\int_{Q_j} (g_{1,k})^p w \lesssim \int_{Q_j} \left( \frac{1}{|Q_j^k|} \int_{\hat{Q}_j^k} |b(x) - b(y)| |f(y)| \, dy \right)^p w(x) \, dx$$

$$\lesssim \frac{1}{|Q_j^k|^p} \left( \int_{Q_j^k} w^{-\gamma/p} \right)^{p/\gamma} \left( \int_{Q_j^k} |f|^p w \right) \times \int_{Q_j} w(x) \left( \int_{Q_j^k} |b(x) - b(y)|^\nu \, dy \right)^{p/\nu} \, dx,$$

$$\lesssim 2^{kM} [b]_\theta^p w(\hat{Q}_j^k) \left( \int_{Q_j^k} w^{-\gamma/p} \right)^{p/\gamma} \int_{Q_j^k} |f|^p w$$

for some $M > 0$, where in the last inequality we have used Lemma 8 for $\theta$ such that $b \in BMO^\theta(\rho)$.

From Proposition 3 we can choose $\gamma$ close enough to $p'$ in such a way that $w \in A_{1+p'/\gamma}^{\rho,\eta}$ for some $\eta > 0$. Therefore, for some $M_1 > 0$, we get

$$\int_{Q_j} (g_{1,k})^p w \lesssim 2^{kM_1} [b]_\theta^p \int_{\hat{Q}_j^k} |f|^p w$$
and hence for $M'_1 > 0$,

$$\|g_1\|_{L^p(w)} \lesssim \sum_k 2^{-kN} \|g_{1,k}\|_{L^p(w)}$$

$$\lesssim \sum_k 2^{-kN} \left( \sum_j \int_{Q_j^k} g_{1,k}^p w \right)^{1/p}$$

$$\lesssim |b| \sum_k 2^{k(-N + M'_1)} \left( \sum_j \int_{Q_j^k} |f|^p w \right)^{1/p}$$

$$\lesssim |b| \|f\|_{L^p(w)} \sum_k 2^{k(-N + M'_1 + N_1)},$$

where in the last inequality is due to Proposition 2. Choosing $N$ large enough the last series is convergent.

Regarding $g_2$, according to Lemma 5, we only have to consider $\frac{d}{2} < q < d$. Observe that for $x \in Q_j^k$ we have

$$\int_{B(x, 2^k \rho(x))} \frac{V(u)}{|u - y|^{d-1}} du \lesssim I_1(\chi_{\tilde{Q}_j^k} V)(y),$$

where $I_1$ is the classical Fractional Integral of order 1.

Therefore, by Hölder’s inequality with $\gamma$ and $\nu$ such that $\frac{1}{p} + \frac{1}{s} + \frac{1}{\gamma} + \frac{1}{\nu} = 1$,

$$\int_{Q_j^k} (g_{2,k})^p w \lesssim \int_{Q_j^k} \left( \frac{1}{|Q_j^k|^{1-1/d}} \int_{\tilde{Q}_j^k} |b(x) - b(y)| |f|^p \left( I_1(\chi_{\tilde{Q}_j^k} V)(y) \right) dy \right)^p w(x) dx$$

$$\lesssim \frac{1}{|Q_j^k|^{p(1-1/d)}} \left( \int_{Q_j^k} w^{-\gamma/p} \right)^{p/\gamma} \|\chi_{\tilde{Q}_j^k} f\|_{L^p(w)} \|I_1(\chi_{\tilde{Q}_j^k} V)\|_{L^s}^p \times$$

$$\int_{Q_j^k} w(x) \left( \int_{\tilde{Q}_j^k} |b(x) - b(y)|^\nu dy \right)^{p/\nu} dx.$$

Recall that $V \in RH_q$ for some $q > 1$ implies that $V$ satisfies the doubling condition, i.e., there exist constants $\mu \geq 1$ and $C$ such that

$$\int_{tB} V \leq C t^{d\mu} \int_B V,$$

holds for every ball $B$ and $t > 1$. Therefore, due to the boundedness of $I_1$ from $L^q$ into $L^s$, and the assumptions on $V$,

$$\|I_1(\chi_{\tilde{Q}_j^k} V)\|_{L^s} \lesssim \|\chi_{\tilde{Q}_j^k} V\|_{L^q} \lesssim |\tilde{Q}_j^k|^{-1/q'} \int_{\tilde{Q}_j^k} V$$

$$\lesssim 2^{kd\mu} |\tilde{Q}_j^k|^{-1/q'} \int_{\tilde{Q}_j^k} V \lesssim 2^{kd(\mu - 1 + \frac{d}{q'})} |\tilde{Q}_j^k|^{\frac{1}{s} - \frac{1}{q'}}$$

where the last inequality follows from the definition of $\rho$ (see (4)). With this estimate and using the claim, we proceed as in the case of $g_1$, choosing this time $\gamma$
such that $1 + \frac{p}{q}$ is close enough to $\frac{p}{s}$, to obtain

$$
\int_{Q_j} (g_{2,k})^p w \lesssim 2^{kM_2} \int_{Q_j^*} |f|^p w
$$

for some $M_2$, leading to the desired estimate.

Now we have to deal with the term \((R_b^*)_{loc} - (R_b^*)_{loc}\)\( f \) of (25). By using estimate (16), we have

$$
\|((R_b^*)_{loc} - (R_b^*)_{loc}) f(x)\| \lesssim h_1(x) + h_2(x)
$$

where

$$
h_1(x) = \sum_k 2^{-k(2-d/q)} h_{1,k}(x),
$$

with

$$
h_{1,k}(x) = 2^{kd} \rho(x)^{-d} \int_{B(x,2^{-k}\rho(x))} |f(y)| |b(x) - b(y)| dy.
$$

and

$$
h_2(x) \lesssim \sum_{k=0}^{\infty} 2^{(d-1)k} h_{2,k}(x),
$$

where

$$
h_{2,k}(x) = \rho(x)^{-d+1} \int_{B(x,2^{-k}\rho(x))} |f(y)| |b(x) - b(y)| \left( \int_{B(y,|x-y|/4)} \frac{V(u)}{|u-y|^{d-1}} du \right) dy.
$$

Let us take a covering \(\{Q_j\}\) as before. For each \(j\) and \(k\) there exist $2^{dk}$ balls of radio $2^{-k}\rho(x_j)$, $B_{j,k}^{i,j,k} = B(x_{j,k}^{i,j,k},2^{-k}\rho(x_j))$ such that $Q_j \subset \bigcup_{l=1}^{2^{dk}} B_{j,k}^{i,j,k} \subset 2Q_j$ and $\sum_{l=1}^{2^{dk}} \chi_{B_{j,k}^{i,j,k}} \leq 2^d$. Moreover, this construction can be done in a way that for each \(k\) the family of a fixed dilation \(\{\tilde{B}_{j,k}^{i,j,k}\}_{j,l}\) is a covering of $\mathbb{R}^d$ such that

$$
\sum_j \sum_{l=1}^{2^{dk}} \chi_{\tilde{B}_{j,k}^{i,j,k}} \leq C,
$$

with the constant $C$ independent of \(k\). To our purpose we take the dilation $\tilde{B}_{j,k}^{i,j,k} = 5c_0 B_{j,k}^{i,j,k}$ (where $c_0$ appears in (14)).

Observe that if $x \in B_{j,k}^{i,j,k}$, $B(x,2^{-k}\rho(x)) \subset \tilde{B}_{j,k}^{i,j,k}$ and $\rho(x) \simeq \rho(x_j)$. Then

$$
h_{1,k}(x) \lesssim 2^{kd} \rho(x_j)^{-d} \int_{\tilde{B}_{j,k}^{i,j,k}} |f(y)| |b(x) - b(y)| dy,
$$
By Hölder’s inequality with $\gamma$ and $\nu$ as for $g_1$, and using Lemma 8 and Proposition 3, we have

\[
\int_{B_{l,k}^1} (h_{1,k})^p w \lesssim \int_{B_{l,k}^1} \left( 2^{kd\rho(x_j)} \int_{B_{l,k}^1} |b(x) - b(y)||f(y)| dy \right)^p w(x) dx \\
\lesssim 2^{kd\rho(x_j)} \int_{B_{l,k}^1} w^{-\gamma/p} \left( \int_{B_{l,k}^1} |f|^p w \right)^{p/\nu} dx \\
\lesssim 2^{kd\rho(x_j)} \int_{B_{l,k}^1} w(x) \left( \int_{B_{l,k}^1} |b(x) - b(y)|^\nu dy \right)^{p/\nu} dx \\
\lesssim \|b\|_\theta^p \int_{B_{l,k}^1} |f|^p w.
\]

Adding over $j$ and $l$, and using the bounded overlapping property (27),

\[
\|h_{1,k}\|_{L^p(w)} \lesssim \|b\|_\theta \|f\|_{L^p(w)},
\]

and thus we obtain the desired estimate for $h_1$.

To deal with $h_2$, we use that $I_1$ is bounded from $L^q$ into $L^s$, together with Lemma 4 to get

\[
\|I_1(\chi_{B_{l,k}^1}V)\|_s \lesssim \|\chi_{B_{l,k}^1}V\|_q \\
\lesssim |B_{l,k}^1|^{-1+1/q} \int_{B_{l,k}^1} V \\
\lesssim \rho(x_j)^{-2+d/q}.
\]

Now, we proceed as for $h_1$ but this time we apply Hölder’s inequality with $\gamma$ and $\nu$ such that $\frac{1}{p} + \frac{1}{s} + \frac{1}{1} + \frac{1}{\gamma} = 1$,

\[
\int_{B_{l,k}^1} (h_{2,k})^p w \lesssim \frac{1}{\rho(x_j)^{d-1}} \left( \int_{B_{l,k}^1} w^{-\gamma/p} \right)^{p/\gamma} \|\chi_{B_{l,k}^1} f\|_{L^p(w)} \|I_1(\chi_{B_{l,k}^1}V)\|_s^p \times \\
\int_{B_{l,k}^1} w(x) \left( \int_{B_{l,k}^1} |b(x) - b(y)|^\nu dy \right)^{p/\nu} dx \\
\lesssim |b\|_\theta^p 2^{-kd(1-\frac{1}{q}+\frac{1}{\gamma})} \|\chi_{B_{l,k}^1} f\|_{L^p(w)}^p
\]

Therefore, with the same argument as for $h_1$, and adding over $k$,

\[
\|h_{2,k}\|_{L^p(w)} \lesssim \|b\|_\theta \|f\|_{L^p(w)},
\]

and we finish the proof of the theorem.

7. AN ORLICZ WEAK ESTIMATE FOR THE CASE $p = 1$

In the next lemma we will use the notation $P(x,r)$ to denote the cube of center $x$ and side $2r$.

**Lemma 9.** Let $\rho$ be a function satisfying (3) and $\theta \geq 0$ fixed. Then for any $\lambda > 0$ there exists an at most countable family of cubes $\{P_j\}$, $P_j = P(x_j, r_j)$ such that

\[
\left( 1 + \frac{r_j}{\rho(x_j)} \right)^{\theta} \lambda \leq \frac{1}{|P_j|} \int_{P_j} |f| \leq C \lambda \left( 1 + \frac{r_j}{\rho(x_j)} \right)^{\sigma},
\]

\[
\lambda \leq \frac{1}{|P_j|} \int_{P_j} |f| \leq C \lambda \left( 1 + \frac{r_j}{\rho(x_j)} \right)^{\gamma}
\]

\[
\lambda \leq \frac{1}{|P_j|} \int_{P_j} |f| \leq C \lambda \left( 1 + \frac{r_j}{\rho(x_j)} \right)^{\delta}
\]
for some $\sigma \geq \theta$, depending only on the constants appearing in (5), and

\begin{equation}
(30) \quad |f(x)| \leq \lambda, \quad \text{a.e. } x \notin \bigcup P_j,
\end{equation}

Proof. First, let us observe that for any cube $P = P(x, r)$,

\begin{equation*}
\left(1 + \frac{r}{\rho(x)}\right)^{-\theta} \frac{1}{|Q|} \int_Q |f| \leq \frac{1}{|Q|} \int_{\mathbb{R}^d} |f|,
\end{equation*}

and the right hand side tends to zero when $r$ goes to infinity. Therefore we may start the Calderon-Zygmund decomposition process with some $r_0$-grid such that

\begin{equation}
(31) \quad \left(1 + \frac{r_0}{\rho(z)}\right)^{-\theta} \frac{1}{|P(z, r_0)|} \int_{P(z, r_0)} |f| \leq \lambda,
\end{equation}

for any cube in the grid. We divide dyadically the cubes selecting those for which the average on the left turns greater than $\lambda$.

Continuing dividing those cubes that have not been selected we obtain a sequence of $P_j$ satisfying the left inequality of (30).

To check the other inequality, observe that if $P_j = P(x_j, r_j)$ was selected, then $P_j$ is contained in a cube $P(y, 2r_j)$ satisfying (31) for some $y$. Hence

\begin{equation*}
\frac{1}{|P_j|} \int_{P_j} |f| \lesssim \left(1 + \frac{2r_j}{\rho(y)}\right)^{\theta} \lesssim \left(1 + \frac{r_j}{\rho(x_j)}\right)^{N_0\theta},
\end{equation*}

where in the last inequality we used (5).

Next, if $x \notin \bigcup P_j$ there exists a sequence of cubes containing $x$ and with radius tending to zero satisfying (31). Since $\rho$ is continuous and positive (30) follows from the Lebesgue’s differentiation theorem.

\[ \square \]

**Theorem 3.** Let $V \in RH_q$ and $b \in BMO_\infty(\rho)$.

(i) If $q \geq d$ and $w \in A_1^{\infty, \infty}$, then there exists a constant $C$ such that for every $f \in L_{1, \text{loc}}$ and $\lambda > 0$,

\begin{equation}
(32) \quad w(|\mathcal{R}_b f| > \lambda) \leq C \int_{\mathbb{R}^d} \frac{|f|}{\lambda} \left(1 + \log^+ \frac{|f|}{\lambda}\right) w.
\end{equation}

(ii) If $d/2 < q < d$ and $w^s \in A_1^{\infty, \infty}$, with $\frac{1}{q} = \frac{1}{b} - \frac{1}{d}$, inequality (32) holds.

Proof. First, let us observe that (31) can be deduced from (30). In fact, if $w \in A_1^{\infty, \infty}$ there exist $\gamma_0 > 1$ such that $w^\gamma \in A_1^{\infty, \infty}$, for $1 \leq \gamma \leq \gamma_0$, according to Lemma 5. Oh the other hand if $V \in RH_q$ for $q \geq d$ it certainly belongs to $RH_s$ for any $s < d$. In particular we may choose $\frac{d}{2} < s < d$ and $\gamma > \gamma_0$ such that $1 - \frac{1}{\gamma} = \frac{1}{s} - \frac{1}{d}$, to get the desired estimate.

Assume then $V \in RH_q$, $\frac{d}{2} < q < d$. Let $w$ be such that $w^s \in A_1^{\infty, \infty}$ and therefore $w^s \in A_1^{\infty, 0}$, for some $\beta > 0$. In this case it is also true that $w \in A_1^{\infty, \theta}$ with $\theta = \beta/s$.

Given $f \in L^1$, let us consider $P_j = P(x_j, r_j)$ the Calderón-Zygmund decomposition given in Lemma 9 associated to $\theta$. We define the set of indexes

$J_1 = \{j : r_j \leq \rho(x_j)\}, \quad J_2 = \{j : r_j > \rho(x_j)\},$

and

$\Omega_1 = \bigcup_{j \in J_1} P_j, \quad \Omega_2 = \bigcup_{j \in J_2} P_j.$
Now we split \( f = g + h + h' \), as
\[
g(x) = \begin{cases} \frac{1}{|P_j|} \int_{P_j} f, & \text{if } x \in P_j, \ j \in J_1, \\ 0, & \text{if } x \in P_j, \ j \in J_2, \\ f(x), & \text{if } x \notin \Omega, \end{cases}
\]
with \( \Omega = \Omega_1 \cup \Omega_2 \).
\[
h(x) = \begin{cases} f(x) - \frac{1}{|P_j|} \int_{P_j} f, & \text{if } x \in P_j, \ j \in J_1, \\ 0, & \text{otherwise,} \end{cases}
\]
and therefore \( h'(x) = \chi_{\Omega_2} f \).

Let \( P_j = P_j(x_j, 2r_j) \) and \( \tilde{\Omega} = \cup_j \tilde{P}_j \). Now,
\[
w(\{ x : |\mathcal{R}_b f|(x) > \lambda \}) \leq w(\tilde{\Omega}) + w(\{ x \notin \tilde{\Omega} : |\mathcal{R}_b f|(x) > \lambda \}).
\]
The first term of the last expression, can be controlled using (24) and that \( w \in A_1^{\rho,0} \) (see Remark 1), as
\[
w(\tilde{\Omega}) \lesssim \frac{1}{\lambda} \sum_j w(\tilde{P}_j) \lesssim \frac{1}{\lambda} \left( \sum_j \frac{w(\tilde{P}_j)}{|\tilde{P}_j|} \right) \left( 1 + \frac{r_j}{\rho(x_j)} \right)^{-\theta} \int_{\tilde{P}_j} |f| \leq \frac{1}{\lambda} \int_{\mathbb{R}^d} |f| w.
\]

For the second term of (33), it is enough to arrive to the right hand side of inequality (22) estimating \( II_1 = w(\{ x : |\mathcal{R}_b g(x)| > \lambda \}) \), \( II_2 = w(\{ x \notin \tilde{\Omega} : |\mathcal{R}_b h(x)| > \lambda \}) \) and \( II_3 = w(\{ x \notin \tilde{\Omega} : |\mathcal{R}_b h'(x)| > \lambda \}) \).

To deal with \( II_1 \) notice that, from Lemma 4 it follows that \( |g| \leq \lambda \). On the other hand, from Theorem 2 it follows that \( \mathcal{R}_b \) is bounded on \( L^p(w) \) for some \( p \) close enough to one.

In fact, from \( w^{s'} \in A_1^{\rho,\infty} \) we get \( w^{s'\nu} \in A_1^{\rho,\infty} \) for some \( \nu > 1 \) (see Lemma 3) and taking \( p \) such that \( p(1 - s') + s' = \frac{1}{r} \) it is easy to check that \( \frac{w}{s^{s'\nu}} \in A_1^{\rho,\infty} \). Therefore, since strong type implies weak type \((p, p)\), we get
\[
w(\{ x : |\mathcal{R}_b g(x)| > \lambda \}) \lesssim \frac{1}{\lambda^p} \int_{\mathbb{R}^d} |g|^p w \lesssim \frac{1}{\lambda} \left( \sum_{j \in J_1} \frac{w(\tilde{P}_j)}{|\tilde{P}_j|} \int_{\tilde{P}_j} |f| + \int_{\mathbb{R}^d} |f| w \right).
\]
Since \( w \in A_1^{\rho,\infty} \) and for \( j \in J_1, r_j \leq \rho(x_j) \) we have \( \frac{w(\tilde{P}_j)}{|\tilde{P}_j|} \lesssim \inf_{\tilde{P}_j} w \), and hence the last expression in (35) can be easily bounded by \( \frac{1}{\lambda} \int_{\mathbb{R}^d} |f| w \).

To take care of \( II_2 \) we observe that
\[
\mathcal{R}_b h(x) = \int_{\mathbb{R}^d} \mathcal{K}(x, y)[b(x) - b(y)] h(y) \, dy = \sum_{j \in J_1} \int_{\tilde{P}_j} \mathcal{K}(x, y)[b(x) - b(y)] h(y) \, dy.
\]
Adding and subtracting \( b_{P_j} \) inside the integral we write
\[
\mathcal{R}_b h(x) = -A(x) + B(x),
\]
where
\[
A(x) = \sum_{j \in J_1} \int_{\tilde{P}_j} \mathcal{K}(x, y)[b(y) - b_{P_j}] h(y) \, dy,
\]
and
and
\[ B(x) = \sum_{j \in J_1} \int_{P_j} K(x, y)[b(x) - b_{P_j}]h(y) \, dy. \]

So we need to estimate
\[ II_{2,1} = w(\{ x \notin \tilde{\Omega} : |A(x)| > \lambda \} \text{ and } II_{2,2} = w(\{ x \notin \tilde{\Omega} : |B(x)| > \lambda \}) \]

To deal with the first expression let \( \nu > 1 \) be such that \( w^{s,\nu} \in A^{p,\infty}_1 \). Hence, according to Remark 2 there exists \( \sigma \geq 0 \) such that
\[ M^\sigma(w^{s,\nu}) \lesssim w^{s,\nu}. \]

We set \( w_* = w_{\chi_{\tilde{\Omega}}} \) and since \( w_*^{s,\nu} \in L^1_{\text{loc}} \) we may apply Lemma 4. If \( g = w_*^{s,\nu} \), \( \theta = \sigma \) and \( \delta = 1/\nu \) to get that the weight \( u = (M^\sigma w_*^{s,\nu})\frac{1}{\nu} \) is such that \( u^{s,\nu} \) belongs to \( A^{p,\infty}_1 \). Also, from differentiation, \( w_* \leq u \) and from (36) we have \( u \lesssim w \). Moreover, notice that for \( y, z \in P_j, j \in J_1 \) we have \( u(x) \simeq u(y) \). This is due to the facts that \( w_* = 0 \) in \( P_j \) and that \( \rho(x) \simeq \rho(y) \).

Then since \( A(x) = -\mathcal{R}(\sum_{j \in J_1} (b - b_{P_j})\chi_{P_j} h)(x) \) and \( u^{s,\nu} \in A^{p,\infty}_1 \) (see Theorem 3 in [1]),
\[ II_{2,1} = w_*(\{ x : |A(x)| > \lambda \}) \]
\[ \lesssim u(\{ x : |A(x)| > \lambda \}) \]
\[ \lesssim \frac{1}{\lambda} \sum_{j \in J_1} \int_{P_j} [b(y) - b_{P_j}]|h(y)|u(y) \, dy \]
\[ \lesssim \frac{1}{\lambda} \sum_{j \in J_1} \inf_{P_j} u \int_{P_j} [b(y) - b_{P_j}]|f(y)| \, dy \]
\[ + \frac{1}{\lambda} \sum_{j \in J_1} \inf_{P_j} u \frac{1}{|P_j|} \int_{P_j} [b(y) - b_{P_j}] \int_{P_j} |f(y)| \, dy. \]

Clearly, the last sum is controlled by \( |b|_{\theta} \| f \|_1 \) since \( u \leq w \). For the first term we apply Hölder’s inequality [9] and Lemma 22
\[ II_{2,1} \lesssim \frac{1}{\lambda} |b|_{\theta} \sum_{j \in J_1} \inf_{P_j} u |P_j| \| f \|_{\varphi, P_j}. \]

We remind that from [7], p. 92, we get that for any cube \( Q \)
\[ \| f \|_{\varphi, Q} \simeq \inf_{t > 0} \left\{ t + \frac{t}{|Q|} \int_{Q} \varphi \left( \frac{|f|}{t} \right) \right\}. \]

Now, taking \( t = \lambda \),
\[ \frac{|P_j|}{\lambda} \| f \|_{\varphi, P_j} \lesssim |P_j| + \int_{P_j} \varphi \left( \frac{|f|}{\lambda} \right). \]

But since \( P_j \) satisfies (20), we have
\[ |P_j| \lesssim \int_{P_j} |f|. \]

Inserting these estimates and using again that \( u \simeq w \) it follows
\[ II_{2,1} \lesssim |b|_{\theta} \left( \int_{\mathbb{R}^d} |f| \, w + \int_{\mathbb{R}^d} \varphi \left( \frac{|f|}{\lambda} \right) \, w \right). \]
For $II_{2,2}$ we apply Tchebycheff inequality to get

$$II_{2,2} \lesssim \frac{1}{\lambda} \int_{\mathcal{W}} |B|w$$

$$\lesssim \frac{1}{\lambda} \sum_{j \in J_1} \int_{P_j^c} |b(x) - b_{P_j}| \left( \int_{P_j} |\mathcal{K}(x, y) - \mathcal{K}(x, x_j)| |h(y)| dy \right) w(x)\,dx$$

$$\lesssim \frac{1}{\lambda} \sum_{j \in J_1} \int_{P_j^c} |h(y)| \left( \int_{P_j} |b(x) - b_{P_j}| |\mathcal{K}(x, y) - \mathcal{K}(x, x_j)| w(x)\,dx \right) dy.$$

The inner integrals may be estimate splitting into annuli and applying Hölder’s inequality with $s, s', \gamma$ with $\nu > 1$ such that $w^{s'} \in A^{\gamma}_{\infty} \cap A_{1}^{\gamma}$ and $\frac{1}{s'} + \frac{1}{\gamma} + \frac{1}{\lambda} = 1$. In this way, setting $P_j^k = 2^k P_j$ we have

$$\int_{P_j} |b(x) - b_{P_j}| |\mathcal{K}(x, y) - \mathcal{K}(x, x_j)| w(x)\,dx$$

$$\lesssim \sum_{k=2}^{\infty} \left( \int_{P_j^k} |b(x) - b_{P_j}|^\gamma dx \right)^{1/\gamma}$$

$$\times \left( \int_{P_j^{k-1}} |\mathcal{K}(x, y) - \mathcal{K}(x, x_j)|^s dx \right)^{1/s} \left( \int_{P_j^k} w^{s'}\,dx \right)^{s'/s}. \tag{39}$$

Next, observe that if $b \in BMO^\theta_\infty(\rho)$, using Lemma [2] for some $\eta \geq \theta$ we have

$$\left( \int_{P_j^k} |b(x) - b_{P_j}|^\gamma dx \right)^{1/\gamma}$$

$$\lesssim \left( \int_{P_j^k} |b(x) - b_{P_j}|^\gamma dx \right)^{1/\gamma} + |P_j^k|^{1/\gamma} \sum_{i=0}^{k-1} \frac{1}{|P_j^i|} \int_{P_j^i} |b(x) - b_{P_j}|$$

$$\lesssim [b]_\rho|P_j^k|^{1/\gamma} \left[ \left( 1 + \frac{2^kr_j}{\rho(x_j)} \right)^\eta + \sum_{i=0}^{k-1} \left( 1 + \frac{2^ir_j}{\rho(x_j)} \right) \theta^\gamma \right]$$

$$\lesssim k[b]_\rho|P_j^k|^{1/\gamma} \left( 1 + \frac{2^kr_j}{\rho(x_j)} \right)^\eta.$$

Also, since $w^{s'} \in A^{\gamma}_{\infty}$, for some $\sigma > 0$ we have

$$\left( \int_{P_j^k} w^{s'} dx \right)^{1/s} \lesssim \inf_{P_j} w |P_j^{k}|^{1/s'} \left( 1 + \frac{2^r_j}{\rho(x_j)} \right)^\sigma. \tag{40}$$

Therefore, since $\frac{1}{s'} + \frac{1}{\gamma} = \frac{1}{\lambda}$ and $P_j \subset P_j^k$, the right hand side of (39) can be bounded by a constant times

$$[b]_\rho \inf_{P_j} \sum_{k=2}^{\infty} k(2^kr_j)^{d/s'} \left( 1 + \frac{2^r_j}{\rho(x_j)} \right)^{\eta + \sigma} \left( \int_{P_j^{k-1} \setminus P_j^{k-1}} |\mathcal{K}(x, y) - \mathcal{K}(x, x_j)|^s dx \right)^{1/s}$$
but for $P_j$, we have $|y - x_j| < r_j$ so we may apply Hörmander’s type condition of Lemma \[6\]. Therefore,

$$II_{2,2} \lesssim \frac{b_d}{\lambda} \inf_{j \in J_1} \frac{\int_{P_j} |h|}{P_j} \lesssim \frac{1}{\lambda} \sum_{j \in J_1} \int_{P_j} |f|w \lesssim \frac{1}{\lambda} \int_{\mathbb{R}^d} |f|w.$$ 

Finally, we take care of $III$ which involves $h' = f\chi_{\Omega_2}$. By Tchebycheff inequality and proceeding as for $II_{2,2}$,

$$III \lesssim \frac{1}{\lambda} \sum_{j \in J_2} \int_{P_j} |f(y)| \left( \int_{P_j} |b(x) - b_{P_j}| |K(x,y)|w(x) \right) dy.$$

Now, for each $j \in J_2$ we bound the inner integral splitting into annuli and applying Hölder’s inequality as in (39). With the same notation there we get using Lemma \[7\].

$$\int_{P_j} |b(x) - b(y)||K(x,y)|w(x)\,dx$$

$$\lesssim \sum_{k=2}^{\infty} (2^k r_j)^{-1-q} \left( \frac{\rho(x_j)}{2^k r_j} \right)^{N-\mu d} \left( \int_{P_j} |b(x) - b(y)|^\gamma \,dx \right)^{1/\gamma} \left( \int_{P_j} \eta \,dx \right)^{1-1/\gamma}.$$

For the factor with $w$ we use estimate (40), and the one concerning $b$ we can add and subtract $b_{P_j}$ to obtain

$$\left( \int_{P_j} |b(x) - b_{P_j}| \,dx \right)^{1/s} \lesssim (2^k r_j)^{-1-\eta} \left[ b_0 \left( \frac{2^k r_j}{\rho(x_j)} \right)^\eta + |b_{P_j} - b(y)| \right].$$

Collecting estimates, setting $\alpha = N - \mu d - \eta - \sigma$ and using that $r_j \geq \rho(x_j)$ for $j \in J_2$, we get

$$III \lesssim \frac{1}{\lambda} \sum_{k} 2^{-k\alpha} \sum_{j \in J_2} \left( \frac{r_j}{\rho(x_j)} \right)^{\alpha} \inf_{P_j} \int_{P_j} |f(y)| \left[ b_0 \left( \frac{2^k r_j}{\rho(x_j)} \right)^\eta + |b_{P_j} - b(y)| \right] dy.$$

For the term with $[b]_\theta$ choosing $N$ such that $N - \mu d - \eta - \sigma > 0$ and using that $r_j \geq \rho(x_j)$ for $j \in J_2$, to obtain that it is bounded by a constant times $\frac{1}{\lambda} \int \eta \,dx$.

For the other term we apply as before Hölder with $\phi$ and $\tilde{\phi}$ to get

$$\int_{P_j} |f(y)||b_{P_j} - b(y)| \,dy \lesssim |P_j| \left\| f(y) \right\|_{\phi, P_j} \left\| b_{P_j} - b \right\|_{\tilde{\phi}, P_j}.$$

Then, we apply Lemma \[2\] to bound the last factor. For the first factor we use (37) and (38). Therefore, choosing $N$ large enough in (42) such that $N - \mu d - \eta - \sigma - M > 0$ we obtain that expression bounded by

$$\int_{P_j} \phi \left( \frac{f}{\lambda} \right) w.$$
Remark 4. We want to point out that inequality (32) is also true for $\mathcal{R}_b^*$ with weights in $A^{\infty}_{\infty}$ provided the potential $V$ belongs to $RH_d$. On the other hand, when $V \in RH_q$ for some $q > d/2$ but $V \notin RH_d$, we can not expect this kind of result for $\mathcal{R}_b^*$ since $\mathcal{R}_b^*$ is not of weak type $(1,1)$ for $w = 1$ (see [10]). Therefore, in order to get (32) for $\mathcal{R}_b^*$ when $V \in RH_d$ we can not argue as we did for $\mathcal{R}_b$ in that case. Nevertheless, a close look at the proof of the case $q < d$ reveals that the same pattern could be followed in this case.

In fact, notice that the only instances in the argument where we use properties of $\mathcal{R}_b$, $\mathcal{R}$ or of the kernel $K$ are the following:

(i) Strong type $(p,p)$ of $\mathcal{R}$ with the weight $w$ for some $p > 1$ (see (35)).
(ii) Weak type $(1,1)$ of $\mathcal{R}$ with the weight $u = (M^* w_{\nu^s})^{1/2}$, when estimating $II_{2,1}$.
(iii) Hörmander’s like property of $K$ (see (17)) to bound $II_{2,2}$.
(iv) Estimates of the size of $\mathcal{K}$ given by Lemma 7 to obtain inequality (41).

When $V \in RH_d$ all these properties are true for $\mathcal{R}_b^*$, $\mathcal{R}^*$ and $K^*$ for the corresponding value $s = \infty$. In fact, that (i) and (ii) are true is a consequence of Theorem 2 of Section 6 and Theorem 3 in [1] together with Lemma 2 above.

Regarding (iii) and (iv), it is known that $K^*$ is a Calderón-Zygmund kernel when $V \in RH_d$ and moreover it satisfies the stronger inequalities

$$|K^*(x,y)| \leq C_N \left(1 + \frac{|x-y|}{\rho(x)}\right)^{-N} \frac{1}{|x-y|^{d+1}},$$

and

$$|K^*(x,y) - K^*(x,z)| \leq C_N \left(1 + \frac{|x-y|}{\rho(x)}\right)^{-N} \frac{|y-z|^\delta}{|x-y|^{d+\delta}},$$

whenever $2|y-z| \leq |x-y|$, for some $\delta > 0$ and any $N \geq 0$. Also, $\rho(x)$ can be substituted by $\rho(y)$ in all instances (see Lemma 4 in [2]).

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