EQUIVARIANT A-THEORY

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Abstract. We give a new construction of the equivariant K-theory of group actions of Barwick et al., by producing an infinite loop G-space for each Waldhausen category with G-action, for a finite group G. On the category R(X) of retractive spaces over a G-space X, this produces an equivariant lift of Waldhausen’s functor A(X), and we show that the H-fixed points are the bivariant A-theory of the fibration X_{hH} → BH. We then use the framework of spectral Mackey functors to produce a second equivariant refinement A_G(X) whose fixed points have tom Dieck type splittings. We expect this second definition to be suitable for an equivariant generalization of the parametrized h-cobordism theorem.

Contents

1. Introduction 2
Acknowledgements 6
2. Equivariant K-theory of Waldhausen G-categories 6
2.1. The category Cat(EG, C) 6
2.2. Rectification of symmetric monoidal G-categories 7
2.3. Rectification of Waldhausen G-categories 8
2.4. Delooping symmetric monoidal G-categories 10
2.5. Delooping Waldhausen G-categories 12
3. The Waldhausen G-category of retractive spaces R(X) 14
3.1. Action of G on R(X) 15
3.2. Homotopy fixed points of R(X) 15
3.3. Definition of A_G^{coarse}(X) 16
3.4. Relation to bivariant A-theory 17
4. Transfers on Waldhausen G-categories 23
4.1. Review of spectral Mackey functors 23
4.2. Categorical transfer maps 26
4.3. Construction of A_G(X) 31
References 35
1. Introduction

Waldhausen’s celebrated \(A(X)\) construction, and the “parametrized \(h\)-cobordism” theorem relating it to the space of \(h\)-cobordisms \(\mathcal{H}^\infty(X)\) on \(X\), provides a critical link in the chain of homotopy-theoretic constructions relating the behavior of compact manifolds to that of their underlying homotopy types [Wal78] [WJR13]. While the \(L\)-theory assembly map provides the primary invariant that distinguishes the closed manifolds in a given homotopy type, \(A(X)\) provides the secondary information that accesses the diffeomorphism and homeomorphism groups in a stable range [WW88]. And in the case of compact manifolds up to stabilization, \(A(X)\) accounts for the entire difference between the manifold and its underlying homotopy type with tangent information [DWW03]. As a consequence, calculations of \(A(X)\) have immediate consequences for the automorphism groups of high-dimensional closed manifolds, and of compact manifolds up to stabilization.

When the manifolds in question have an action by a group \(G\), there is a similar line of attack for understanding the equivariant homeomorphisms and diffeomorphisms. One expects to replace \(\mathcal{H}^\infty(X)\) with an appropriate space \(\mathcal{H}^\infty(X)^G\) of \(G\)-isovariant \(h\)-cobordisms on \(X\), stabilized with respect to representations of \(G\). The connected components of such a space would be expected to coincide with the equivariant Whitehead group of [Lüc89], which splits as

\[
\text{Wh}^G(X) \cong \bigoplus_{(H) \leq G} \text{Wh}(X^H_{hWH})
\]

where \((H) \leq G\) denotes conjugacy classes of subgroups. This splitting is reminiscent of the tom Dieck splitting for genuine \(G\)-suspension spectra

\[
(\Sigma_\infty^G X^+_+)^G \cong \bigvee_{(H) \leq G} \Sigma_\infty^H X^H_{hWH}
\]

and suggests that the variant of \(A\)-theory most directly applicable to manifolds will in fact be a genuine \(G\)-spectrum, whose fixed points have a similar splitting.

In this paper we begin to realize this conjectural framework. We define an equivariant generalization \(A_G(X)\) of Waldhausen’s \(A\)-theory functor, when \(X\) is a space with an action by a finite group \(G\), whose fixed points have the desired tom Dieck style splitting.

**Theorem 1.1** (Theorem 4.1). For \(G\) a finite group, there exists a functor \(A_G\) from \(G\)-spaces to genuine \(G\)-spectra with fixed points

\[
A_G(X)^G \cong \prod_{(H) \leq G} A(X^H_{hWH}),
\]
and a similar formula for the fixed points of each subgroup $H$.

To be more specific, the fixed points are the $K$-theory of the category $\mathcal{R}_{hf}^G(X)$ of finite retractive $G$-cell complexes over $X$, with equivariant weak homotopy equivalences between them. The splitting of this $K$-theory is a known consequence of the additivity theorem, and an explicit proof appears in [BD16].

In a subsequent paper, we plan to explain how $A_G(X)$ fits into a genuinely $G$-equivariant generalization of Waldhausen’s parametrized $h$-cobordism theorem. The argument we have in mind draws significantly from an analysis of the fixed points of our $A_G(X)$ carried out by Badzioch and Dora-biala [BD16], and a forthcoming result of Goodwillie and Igusa that defines $H^\infty(X)^G$ and gives a splitting that recovers (1). We emphasize that lifting these theorems to genuine $G$-spectra permits the tools of equivariant stable homotopy theory to be applied to the calculation of $H^\infty(X)^G$, in addition to the linearization and trace techniques that have been used so heavily in the nonequivariant case.

Most of the work in this paper is concerned with constructing equivariant spectra out of category-theoretic data. One approach is to generalize classical delooping constructions such as the operadic machine of May [May72] or the $\Gamma$-space machine of Segal [Seg74] to allow for deloopings by representations of $G$. Using the equivariant generalization of the operadic infinite loop space machine from [GMa], we show how this approach generalizes to deloop Waldhausen $G$-categories.

The theory of Waldhausen categories with $G$-action is subtle. Even when the $G$-action is through exact functors, the fixed points of such a category do not necessarily have Waldhausen structure (Observation 2.4). Define $\mathcal{E}G$ be the category with objects the elements of $G$ and precisely one morphism between any two objects, whose classifying space is $EG$. Let $\text{Cat}(\mathcal{E}G, \mathcal{C})$ be the category of all functors and all natural transformations with $G$ acting by conjugation; we define the homotopy fixed points $\mathcal{C}^{hG}$ of a $G$-category $\mathcal{C}$ as the fixed point category $\text{Cat}(\mathcal{E}G, \mathcal{C})^G$, and we explain in §2.3 how this category does have a Waldhausen structure.

The “equivariant $K$-theory of group actions” of Barwick, Glasman, and Shah produces a genuine $G$-spectrum (using the framework of [Bar]) whose $H$-fixed points are $K(\mathcal{C}^{hH})$ [BGS, §8]. We complement this with a result that shows the $G$-space $|\text{Cat}(\mathcal{E}G, \mathcal{C})|$ may be directly, equivariantly delooped.

**Theorem 1.2** (Theorem 2.13 and Proposition 2.15). If $\mathcal{C}$ is a Waldhausen $G$-category then the $K$-theory space defined as $K_G(\mathcal{C}) := \Omega|wS, \text{Cat}(\mathcal{E}G, \mathcal{C})|$,
where $S$, is Waldhausen’s construction from [Wal85], is an equivariant infinite loop space. The $H$-fixed points of the resulting $\Omega-G$-spectrum are equivalent to the $K$-theory of the Waldhausen category $\mathcal{C}^{hH}$ for every subgroup $H$.

The downside of this approach is that one does not have much freedom to modify the weak equivalences in the fixed point categories. Note that if $X$ is a $G$-space, then the category $R_{hf}(X)$ of homotopy finite retractive spaces over $X$ has a $G$-action. For a retractive space $Y$, $gY$ is defined by precomposing the inclusion map by $g^{-1}$ and postcomposing the retraction map by $g$. We can apply Theorem 1.2 to this category, and the resulting theory $A^\text{coarse}_G(X)$ has as its $H$-fixed points the $K$-theory of $H$-equivariant spaces over $X$, as we expect, but the weak equivalences are the $H$-maps which are nonequivariant homotopy equivalences. Thus, Theorem 1.2 does not suffice to prove Theorem 1.1.

Although $A^\text{coarse}_G(X)$ does not match our expected input for the $h$-cobordism theorem, it does have a surprising connection to the bivariant $A$-theory of Williams [Wil00]:

**Theorem 1.3** (Proposition 3.7 and Proposition 3.8). There is a natural equivalence of spectra

$$A^\text{coarse}_G(X)^H \simeq A(EG \times_H X \to BH)$$

Under this equivalence, the coassembly map for bivariant $A$-theory agrees up to homotopy with the map from fixed points to homotopy fixed points:

$$A^\text{coarse}_G(X)^H \xrightarrow{\sim} A^\text{coarse}_G(X)^{hH}$$

In order prove Theorem 1.1 it is necessary to modify the weak equivalences in the fixed point categories giving $A^\text{coarse}_G(X)^H$, and to do this we use the framework of spectral Mackey functors. These are diagrams over a certain spectral variant of the Burnside category, denoted $\mathcal{GB}$. By celebrated work of Guillou and May, the homotopy theory of $\mathcal{GB}$-diagrams is equivalent to that of genuine $G$-spectra [GMb]. Moreover, there are by now a few different ways to pass from combinatorial, category-theoretic data to diagrams of spectra over $\mathcal{GB}$ [Bar, BGS, BO15, BO]. In essence, one is allowed to give separately
for each \( H \leq G \) some permutative category, Waldhausen category, or \( \infty \)-category \( R^H \) whose algebraic \( K \)-theory will become the \( H \)-fixed points. The rest of the glue that creates the \( G \)-spectrum is generated by a large collection of exact functors giving the restrictions, transfers, and sums thereof, between the categories \( \{ R^H : H \leq G \} \).

Barwick’s approach to managing this large collection of data is to define certain adjoint pairs of functors between the categories \( R^H \), satisfying Beck-Chevalley isomorphisms [Bar, §10]. These may then be “unfurled” to create suitably coherent actions of spans on the categories \( R^H \), giving a spectral Mackey functor on the \( K \)-theory spectra \( K(R^H) \). In §4.2, we describe concretely how spans act on the categories \( \{ C^{hH} : H \leq G \} \) – this is essentially the application of Barwick’s “unfurling” construction found in [BGS, §8], but formulated for ordinary Waldhausen categories with a \( G \)-action. Our variant of this construction is then a “Mackey functor of Waldhausen categories” in the sense of Bohmann and Osorno [BO], which combined with the theorem of Guillou and May [GMb] gives a genuine \( G \)-spectrum. This in particular allows an alternative “spectral Mackey functor” definition of \( A_G^{\text{coarse}}(X) \) when one plugs in the category \( R(X) \) with the \( G \)-action described above.

However, as we pointed out, the categories \( R^{h}_hf(X) \) are not of the form \( C^{hH} \) – they have the same objects and maps as \( R(X)^{hH} \) but more restricted weak equivalences. In order to get the desired tom Dieck style splittings of the fixed points, in §4.3, we descend the action of spans on the categories \( R(X)^{hH} \) to get a “Mackey functor of Waldhausen categories” with values \( G/H \mapsto R^{h}_hf(X) \), thereby proving Theorem 1.1. Though we work in the framework of [GMb] and [BO] to build \( A_G(X) \), the same constructions appear to also make \( R^{h}_hf(X) \) into a Mackey functor of Waldhausen categories within Barwick’s framework.

**Remark 1.5.** There is a “Cartan” map

\[
A_G(X) \longrightarrow A_G^{\text{coarse}}(X).
\]

This becomes a map of genuine \( G \)-spectra if we define \( A_G^{\text{coarse}}(X) \) using the Mackey structure on \( K(C^{hH}) \). We believe that this Mackey structure gives the same \( G \)-spectrum as the one produced by delooping the space \( K_G(C) \) using Theorem 1.2, and that more generally the \( K \)-theory of group actions from [BGS, §8] gives the same \( G \)-spectrum as Theorem 1.2. The argument we have in mind for the former claim depends on multifunctorial properties of equivariant \( K \)-theory that have not yet been carefully established.

Our constructions are inspired by, but distinct from, the construction of Real algebraic \( K \)-theory by Hesselholt and Madsen [HM13]. We consider
Waldhausen categories with (covariant) actions by $G$ through exact functors, whereas the basic input for Real $K$-theory is categories with a contravariant involution. We do not formulate an equivariant version of $S$, here, but we consider this to be a problem of significant importance for future work.

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2. Equivariant $K$-theory of Waldhausen $G$-categories

Let $G$ be a finite group. In this first section we recall from [Mer15] the construction $\text{Cat}(\mathcal{E}G, -)$, and how it rectifies pseudo-equivariant functors into equivariant ones. We give several applications, and end by proving Theorem 1.2.

2.1. The category $\text{Cat}(\mathcal{E}G, \mathcal{C})$. For $G$-categories $\mathcal{A}$ and $\mathcal{B}$, we define $\text{Cat}(\mathcal{A}, \mathcal{B})$ to be the category of all functors and natural transformations, with $G$ acting by conjugation. Therefore the fixed point category $\text{Cat}(\mathcal{A}, \mathcal{B})^G$ is the category of equivariant functors and equivariant natural transformations.

Definition 2.1. Define $\mathcal{E}G$ to be the $G$-groupoid with objects the elements of $G$ and a unique morphism between any two objects. Up to $G$-isomorphism, this is the translation category of $G$, and its classifying space is the space $EG$. Define the homotopy $G$-fixed points $\mathcal{C}^{hG}$ of a $G$-category $\mathcal{C}$ as $\text{Cat}(\mathcal{E}G, \mathcal{C})^G$.

Remark 2.2. For each $H \leq G$, there is an equivalence of $H$-categories $\mathcal{E}G \simeq \mathcal{E}H$. So, we can unambiguously define the homotopy $H$-fixed points $\mathcal{C}^{hH}$ as $\text{Cat}(\mathcal{E}G, \mathcal{C})^H \simeq \text{Cat}(\mathcal{E}H, \mathcal{C})^H$.

Recall from [Mer15] the following explicit description of the homotopy fixed point category $\text{Cat}(\mathcal{E}G, \mathcal{C})$. Its objects of $\mathcal{C}$ together with isomorphisms $\psi_g : C \xrightarrow{\sim} gC$ for all $g \in G$, such that $\psi_e = \text{id}_C$ and and the following cocycle condition is satisfied:

\[(2) \quad \psi_{gh} = (g\psi_h)\psi_g.\]
A morphism is given by a morphism \( \alpha : C \to C' \) in \( \mathcal{C} \) such that the following diagram commutes for any \( g \in G \):

\[
\begin{array}{ccc}
C & \xrightarrow{\psi_g} & gC \\
\downarrow{\alpha} & & \downarrow{g\alpha} \\
C' & \xrightarrow{\psi'_g} & gC'
\end{array}
\]

We may think of a \( G \)-category as a functor \( BG \to \text{Cat} \), where \( BG \) is the groupoid with one object and morphism group \( G \). Then an equivariant functor is just a natural transformation between the corresponding functors \( BG \to \text{Cat} \), and a pseudo equivariant functor is a pseudo natural transformation. So, a functor between \( G \)-categories \( \Theta : \mathcal{C} \to \mathcal{D} \) is pseudo equivariant if we have coherent isomorphisms \( \theta_g : \Theta(gC) \xrightarrow{\cong} g\Theta(C) \). (The needed coherence is spelled out explicitly in [Mer15].)

**Proposition 2.3** ([Mer15]). A pseudo equivariant functor \( \Theta : \mathcal{C} \to \mathcal{D} \) naturally induces an on the nose equivariant functor 

\[
\tilde{\Theta} : \text{Cat}(\mathcal{E}G, \mathcal{C}) \to \text{Cat}(\mathcal{E}G, \mathcal{D}),
\]

where for a functor \( F : \mathcal{E}G \to \mathcal{C} \), the functor \( \tilde{\Theta}(F) : \mathcal{E}G \to \mathcal{D} \) is defined on objects by 

\[
\tilde{\Theta}(F)(g) = g\Theta((g^{-1}F)(e)) = g\Theta(g^{-1}F(g))
\]

and on morphisms \( g \to g' \) it is defined as the composite

\[
g\Theta(g^{-1}F(g)) \xrightarrow{\Theta(g^{-1}F(g))} \Theta(\mathcal{E}G \to \mathcal{D})
\]

The induced map on homotopy fixed points \( \tilde{\Theta}^H : \mathcal{C}^{hH} \to \mathcal{D}^{hH} \) takes an object \( C \) with choices of isomorphisms \( \psi_g : C \xrightarrow{\cong} gC \) to \( \Theta(C) \) with isomorphisms \( \Theta(C) \xrightarrow{\cong} g\Theta(C) \) defined as the composites

\[
\Theta(C) \xrightarrow{\Theta(\psi_g)} \Theta(gC) \xrightarrow{\theta_g} g\Theta(C).
\]

It was checked explicitly in [Mer15] that these composites satisfy the required cocycle condition.

**2.2. Rectification of symmetric monoidal \( G \)-categories.** One application of Proposition 2.3 is to strictify \( G \)-actions on symmetric monoidal categories. Suppose \( \mathcal{C} \) is a symmetric monoidal category, with a \( G \)-action that preserves the symmetric monoidal structure \( \oplus \) up to coherent isomorphism. In other words, \( \mathcal{C} \) is a functor \( BG \to \text{Sym Cat}^{\text{strong}} \) from \( BG \) to the category of strict symmetric monoidal categories and strong monoidal functors. Then the symmetric monoidal structure map

\[
\mathcal{C} \times \mathcal{C} \xrightarrow{\oplus} \mathcal{C}
\]
is pseudoequivariant, where the $G$-action on $\mathcal{C} \times \mathcal{C}$ is diagonal. In addition, we get coherent isomorphisms $gI \cong I$ for every $g \in G$, where $I$ is the unit object of $\mathcal{C}$.

If $\mathcal{C}$ is such a symmetric monoidal category, then $\text{Cat}(\mathcal{E}G, \mathcal{C})$ is a symmetric monoidal category whose sum and unit are strictly $G$-equivariant. This is because Proposition 2.3 gives an on the nose equivariant functor

$$\oplus: \text{Cat}(\mathcal{E}G, \mathcal{C} \times \mathcal{C}) \cong \text{Cat}(\mathcal{E}G, \mathcal{C}) \times \text{Cat}(\mathcal{E}G, \mathcal{C}) \longrightarrow \text{Cat}(\mathcal{E}G, \mathcal{C})$$

which we take as the sum in $\text{Cat}(\mathcal{E}G, \mathcal{C})$. The unit is the functor $F_I: \mathcal{E}G \longrightarrow \mathcal{C}$ defined by $F_I(g) = gI$, where $I$ is the unit of $\mathcal{C}$. Explicitly, $F_1 \oplus F_2$ in $\text{Cat}(\mathcal{E}G, \mathcal{C})$ is defined on objects as

$$(F_1 \oplus F_2)(g) = g(g^{-1}F_1(g) \oplus g^{-1}F_2(g)).$$

which, of course, is the same as $F_1(g) \oplus F_2(g)$ when the $G$-action on $\mathcal{C}$ preserves $\oplus$ strictly, and a morphism $g \rightarrow g'$, it is defined as

$$(F_1 \oplus F_2)(g) \overset{\cong}{\longrightarrow} F_1(g) \oplus F_2(g) \xrightarrow{F_1(g \rightarrow g') \oplus F_2(g \rightarrow g')} F_1(g') \oplus F_2(g') \overset{\cong}{\longrightarrow} (F_1 \oplus F_2)(g').$$

These sum formulas motivate our definition of transfers on $\mathcal{C}^H$ in §4.2 below.

When we take the $K$-theory of $\mathcal{C}$ below, we will actually want to strictify $\mathcal{C}$ in two ways: we will want to make the $G$-action commute with the sum strictly, but we will also want to strictify the symmetric monoidal category $\mathcal{C}$ to a monoidally $G$-equivalent permutative category with $G$-action. We give the details in the discussion before Proposition 2.10.

2.3. Rectification of Waldhausen $G$-categories. Now suppose that $\mathcal{C}$ is a Waldhausen category with $G$-action through exact functors. In other words, for each $g \in G$ the functor $g: \mathcal{C} \rightarrow \mathcal{C}$ preserves cofibrations, weak equivalences, the zero object, and pushouts along cofibrations. (In fact, the last two are automatic since $g$ is an equivalence of categories.) However, we emphasize that $g$ preserves the zero object and pushouts only up to unique isomorphism, and not on the nose.

Observation 2.4. In general, the fixed point category $\mathcal{C}^H$ is not a Waldhausen category, because it is not closed under pushouts. A pushout diagram in $\mathcal{C}^H$ has a pushout in $\mathcal{C}$, but it is only preserved by the $H$-action up to isomorphism, and so in general it does not lie in $\mathcal{C}^H$.

We will get around this by showing that the homotopy fixed points $\mathcal{C}^{hH}$ form a Waldhausen category (Theorem 2.7). First we check that $\text{Cat}(\mathcal{E}G, \mathcal{C})$ is a Waldhausen $G$-category, by defining the cofibrations and weak equivalences pointwise. More precisely, for $F_1, F_2 \in \text{Cat}(\mathcal{E}G, \mathcal{C})$,

$$F_1 \overset{\eta}{\longrightarrow} F_2$$
is a cofibration or a weak equivalence if for every \( g \in \mathcal{E}G \), the map \( F_1(g) \to F_2(g) \) is a cofibration or a weak equivalence, respectively, in \( \mathcal{C} \). If we define the zero object and pushouts in a pointwise manner, they will not be fixed, so we show a little more care:

**Lemma 2.5.** There is a zero object in \( \text{Cat}(\mathcal{E}G, \mathcal{C}) \), which is \( G \)-fixed.

**Proof.** Consider the functor \( Z : \ast \to \mathcal{C} \) from the one object category \( \ast \) to \( \mathcal{C} \), which picks out the zero object 0 of \( \mathcal{C} \). Note that this functor is not equivariant since \( 0 \neq g \cdot 0 \), but for every \( g \) we have a unique isomorphism \( \theta_g : 0 \cong g \cdot 0 \). Since these isomorphisms are unique, it must be that the isomorphisms \( 0 \xrightarrow{\theta_g} g \cdot 0 \xrightarrow{g \theta_h} (gh) \cdot 0 \) and \( 0 \xrightarrow{\theta_{gh}} (gh) \cdot 0 \) coincide, and therefore \( Z \) is pseudo equivariant.

By Proposition 2.3, since \( Z \) is pseudo equivariant, there is an induced on the nose equivariant functor \( \ast \cong (\mathcal{E}G, \ast) \to (\mathcal{E}G, \mathcal{C}) \), which sends the one object of \( \ast \) to the functor \( F_0 \in \text{Cat}(\mathcal{E}G, \mathcal{C}) \) defined on objects by \( F_0(g) = g \cdot 0 \), and defined on the unique morphism from \( g \) to \( h \) by composing the unique isomorphisms \( 0 \cong g \cdot 0 \) and \( 0 \cong h \cdot 0 \) to get an isomorphism \( g \cdot 0 \cong h \cdot 0 \) in \( \mathcal{C} \). Since the functor \( * \to (\mathcal{E}G, \mathcal{C}) \) with value \( F_0 \) is equivariant by Proposition 2.3, the object \( F_0 \) of \( (\mathcal{E}G, \mathcal{C}) \) lies in the \( G \)-fixed point subcategory. It is easy to check that this is a zero object in \( \text{Cat}(\mathcal{E}G, \mathcal{C}) \).

**Lemma 2.6.** There exist pushouts along cofibrations in \( \text{Cat}(\mathcal{E}G, \mathcal{C}) \), so that pushouts of \( H \)-fixed diagrams are \( H \)-fixed.

**Proof.** The same argument as in the previous proof applies: if one considers the category of \( \mathcal{P}(\mathcal{C}) \) pushout diagrams along cofibrations, and a functor \( \mathcal{P}(\mathcal{C}) \to \mathcal{C} \) which assigns each diagram to a choice \( P \) of pushout, this functor is not equivariant. However, the unique isomorphisms \( P \cong g \cdot P \) that exist for any pushout \( P \) in \( \mathcal{C} \) and any \( g \in G \), ensure that the functor is pseudo equivariant. Therefore by Proposition 2.3, we get the nose equivariant functor

\[
\mathcal{P}(\text{Cat}(\mathcal{E}G, \mathcal{C})) \simeq \text{Cat}(\mathcal{E}G, \mathcal{P}(\mathcal{C})) \to \text{Cat}(\mathcal{E}G, \mathcal{C})
\]

Since pushouts of functors are defined objectwise, this assigns to each diagram in \( \text{Cat}(\mathcal{E}G, \mathcal{C}) \) a pushout, and if the diagram is \( H \)-fixed then the pushout is \( H \)-fixed as well. \( \square \)

From the construction of the corresponding equivariant functor from a pseudo equivariant functor in the proof of Proposition 2.3, we get an explicit
description for the pushouts in $\text{Cat}(\mathcal{E}G, \mathcal{C})$. For a diagram

$$
\begin{array}{ccc}
F_1 & \xrightarrow{f} & F_2 \\
\downarrow & & \downarrow \\
F_3 & \xrightarrow{g} & F_4
\end{array}
$$

in $\text{Cat}(\mathcal{E}G, \mathcal{C})$, the pushout $P : \mathcal{E}G \to \mathcal{C}$ is defined on objects by

$$
P(g) = g \cdot (g^{-1}F_3(g) \coprod g^{-1}F_1(g)) g^{-1}F_2(g)).
$$

If the pushout diagram is $G$-fixed, then the pushout $P$ is defined by $P(e) = P$, where $P$ is a pushout of the above diagram evaluated at $e$, and $P(g) = g \cdot P$. On morphisms, $P(g, g')$ is the composite of the unique isomorphisms $P \sim g \cdot P$ and $P \sim g' \cdot P$.

**Theorem 2.7.** Let $\mathcal{C}$ be a $G$-equivariant Waldhausen category, and let $H$ be a subgroup of $G$. Then $\mathcal{C}^hH$ is a Waldhausen category with cofibrations and weak equivalences the $H$-fixed cofibrations and weak equivalences in $\text{Cat}(\mathcal{E}G, \mathcal{C})$.

**Proof.** Note that composition of $H$-fixed maps is $H$-fixed, thus the classes of cofibrations and weak equivalences in $\text{Cat}(\mathcal{E}G, \mathcal{C})^H$ are closed under composition, and an $H$-fixed isomorphism is in particular a $H$-fixed cofibration and weak equivalence.

By Lemma 2.5, there is a zero object $F_0$ in $\text{Cat}(\mathcal{E}G, \mathcal{C})^H$. Moreover, for any functor $F$ in $\text{Cat}(\mathcal{E}G, \mathcal{C})$, each map $F_0(g) \to F(g)$ is a cofibration since it is the composite of $g \cdot 0 \cong 0$ and the unique map $0 \to F(g)$, which are both cofibrations. Thus the map $F_0 \to F$ is by definition a cofibration.

By Lemma 2.6, for a pushout diagram along a cofibration in $\text{Cat}(\mathcal{E}G, \mathcal{C})^H$, there exists a pushout in this fixed point subcategory. The gluing axiom for weak equivalences is inherited from $\mathcal{C}$. 

Note that the equivalence from Remark 2.2 $\text{Cat}(\mathcal{E}G, \mathcal{C})^H \simeq \text{Cat}(\mathcal{E}H, \mathcal{C})^H$ is an equivalence of Waldhausen categories.

**2.4. Delooping symmetric monoidal $G$-categories.** Classical operadic infinite loop space theory [May72] gives a machine for constructing, from a space $X$ with an action by an $E_\infty$ operad, an $\Omega$-spectrum whose zeroth space is the group completion of $X$. If in addition $X$ has an action of a finite group $G$ through $E_\infty$ maps, then the resulting spectrum has a $G$-action, namely it is a naïve $\Omega$-$G$-spectrum. By definition, “naïve” means that the deloopings are only for spheres with trivial $G$-action.

In order to get deloopings by representation spheres $S^V$ for all finite-dimensional representations $V$ of $G$, the $G$-space $X$ needs to be an algebra
over a genuine $E_\infty$-$G$-operad. The difference between a naïve and a genuine $E_\infty$ operad $\mathcal{O}$ lies in the fixed points of the $G \times \Sigma_n$-space $\mathcal{O}(n)$ for each $n$. For each subgroup $\Lambda \leq G \times \Sigma_n$ we have:

|                     | $\Lambda \cap \Sigma_n \neq \{1\}$ | $\Lambda \cap G = \{1\}$ | $\Lambda \cap G \neq \{1\}$ |
|---------------------|--------------------------------------|---------------------------|---------------------------|
| naïve $E_\infty$ operad | $\mathcal{O}(n)^\Lambda = \emptyset$ | $\mathcal{O}(n)^\Lambda = \emptyset$ | $\mathcal{O}(n)^\Lambda \simeq \ast$ |
| genuine $E_\infty$ operad | $\mathcal{O}(n)^\Lambda = \emptyset$ | $\mathcal{O}(n)^\Lambda \simeq \ast$ | $\mathcal{O}(n)^\Lambda \simeq \ast$ |

**Remark 2.8.** In a naïve $E_\infty$ operad $\mathcal{O}$, the spaces $\mathcal{O}(n)$ are the total spaces of universal principal $\Sigma_n$-bundles with $G$-action, whereas in a genuine $E_\infty$ operad $\mathcal{O}_G$, the spaces $\mathcal{O}_G(n)$ are the total spaces of equivariant universal principal $G\Sigma_n$-bundles. For a thorough discussion of equivariant bundle theory, see [May96, Ch.VII].

There are a few different machines that produce these equivariant deloopings, though they are all equivalent [MMO]. We will focus on the machine of Guillou and May [GMa]. Consider the categorical Barratt-Eccles operad $\mathcal{O}(j) = \mathcal{E}\Sigma_j$, and apply $\text{Cat}(\mathcal{E}G, -)$ levelwise. Since $\text{Cat}(\mathcal{E}G, -)$ preserves products, this gives an operad

$$\mathcal{O}_G(j) = \text{Cat}(\mathcal{E}G, \mathcal{E}\Sigma_j)$$

in $G$-categories. Guillou and May show that the levelwise realizations $|\mathcal{O}_G(j)|$ then form a genuine $E_\infty$-operad in unbased $G$-spaces.

**Theorem 2.9** ([GMa]). There is a functor $K_G(-)$ from $|\mathcal{O}_G|$-algebras $X$ to orthogonal $G$-spectra, whose output is an $\Omega$-$G$-spectrum in the sense that the maps

$$K_G(X)(V) \to \Omega^{W-V} K_G(X)(W)$$

are equivariant equivalences. There is a natural equivariant group completion map

$$X \to K_G(X)(0)$$

and a natural weak equivalence of nonequivariant orthogonal spectra

$$K(X^H) \to (K_G X)^H$$

for all subgroups $H$ of $G$.

Recall that an equivariant group completion is a map that is a group completion on the $H$-fixed points for all subgroups $H$ of $G$. In particular, if the fixed points $X^H$ are connected for all subgroups $H$, then the map $X \to K_G(X)(0)$ is an equivalence.

Since realization is a monoidal functor, if $\mathcal{C}$ is a $G$-category with an action of $\mathcal{O}_G$, its classifying space $|\mathcal{C}|$ is an algebra over $|\mathcal{O}_G|$ in $G$-spaces.
We are therefore interested in constructing examples of $O_G$-algebras $\mathcal{C}$. We first recall that a category $\mathcal{C}$ with an action of the Barratt-Eccles operad $O$ in $\text{Cat}$ is a permutative category, i.e., it is symmetric monoidal with strict unit and strict associativity [May78]. Any symmetric monoidal category $\mathcal{C}$ can be rectified to an equivalent permutative category by a well known trick of MacLane [ML98]. The MacLane strictification functor $(-)_{\text{str}} : \text{Sym Cat}^{\text{strong}} \to \text{Sym Cat}^{\text{strict}}$ is the left adjoint of the forgetful map $U$. The category $\mathcal{C}_{\text{str}}$ has as objects lists $(c_1, \ldots, c_n)$ of objects in $\mathcal{C}$ with sum given by concatenation, and morphisms between $(c_1, \ldots, c_n)$ and $(d_1, \ldots, d_m)$ are given by morphisms $c_1 \oplus \ldots \oplus c_n \to d_1 \oplus \ldots \oplus d_m$ in $\mathcal{C}$.

If $\mathcal{C}$ has a coherent $G$-action as in §2.2, then the composition $BG \to \text{Sym Cat}^{\text{strong}} \to \text{Sym Cat}^{\text{strict}}$ describes $\mathcal{C}_{\text{str}}$ as a category with a $G$-action that commutes with the symmetric monoidal product strictly. This action is defined on objects by $g(c_1, \ldots, c_n) = (gc_1, \ldots, gc_n)$, and on morphisms by $gc_1 \oplus \ldots \oplus gc_n \cong g(c_1 \oplus \ldots \oplus c_n) \to g(d_1 \oplus \ldots \oplus d_m) \cong (gd_1 \oplus \ldots \oplus gd_m)$.

The components of the unit of the adjunction $\eta : \mathcal{C} \to \mathcal{U}\mathcal{C}_{\text{str}}$ are monoidal equivalences of symmetric monoidal categories with inverses $\eta^{-1}$ sending the list $(c_1, \ldots, c_n)$ to $c_1 \oplus \ldots \oplus c_n$. We have observed in [Mer15] that the equivalence of $\mathcal{C}$ and $\mathcal{C}_{\text{str}}$ is through $G$-equivariant functors, when the action on $\mathcal{C}$ commutes with $\oplus$ strictly. However, now we are assuming that $g$ commutes with $\oplus$ only up to coherent isomorphism. In this case, $\eta$ is still equivariant, but the inverse equivalence $\eta^{-1}$ is only pseudo-equivariant. After applying $\text{Cat}(\mathcal{E}G, -)$, we conclude by Proposition 2.3 that $\eta$ and $\eta^{-1}$ give a $G$-equivariant monoidal equivalence of categories

$$\text{Cat}(\mathcal{E}G, \mathcal{C}) \simeq \text{Cat}(\mathcal{E}G, \mathcal{C}_{\text{str}}).$$

We summarize this discussion in the next proposition.

**Proposition 2.10.** Let $\mathcal{C}$ be a symmetric monoidal category with $G$-action given through strong monoidal endofunctors. Then the symmetric monoidal $G$-category $\text{Cat}(\mathcal{E}G, \mathcal{C})$ is $G$-equivalent to the $O_G$-algebra $\text{Cat}(\mathcal{E}G, \mathcal{C}_{\text{str}})$.

We may therefore deloop the classifying space $|\text{Cat}(\mathcal{E}G, \mathcal{C})|$ by representations, simply by applying Theorem 2.9 to the equivalent classifying space $|\text{Cat}(\mathcal{E}G, \mathcal{C}_{\text{str}})|$.

2.5. **Delooping Waldhausen $G$-categories.** Recall that the algebraic $K$-theory of the Waldhausen category $\mathcal{C}$ is defined as $\Omega|w\mathcal{S}\mathcal{C}|$, where $\mathcal{S}\mathcal{C}$ is the simplicial Waldhausen category constructed in [Wal85]. The $w$ means that we restrict to the subcategory of weak equivalences when we take the nerves of the categories $w\mathcal{S}_n\mathcal{C}$ for varying $n$, before taking the realization of the resulting bisimplicial set $w\mathcal{N}\mathcal{S}\mathcal{C}$. 
This is an infinite loop space whose deloopings are given by iterations of the $S^\ast$-construction. However Waldhausen remarks that it is enough to apply $S^\ast$ once, which has the effect of splitting the exact sequences, and then to use an alternate infinite loop space machine with the group completion property on the space $|wS\mathcal{C}|$. Waldhausen notes that the comparison can be achieved by fitting the two resulting spectra into a bispectrum, and a detailed proof of this result is written down in [Mal15a]. We will use this idea to produce equivariant deloopings of Waldhausen $G$-categories.

Suppose that $\mathcal{C}$ is a Waldhausen category, with an action of $G$ through exact functors. We give $\text{Cat}(\mathcal{E}G, \mathcal{C})$ the Waldhausen category structure defined in §2.3. The $G$-action on $\text{Cat}(\mathcal{E}G, \mathcal{C})$ induces a $G$-action on the simplicial Waldhausen category $S\mathcal{C}, \text{Cat}(\mathcal{E}G, \mathcal{C})$, which commutes with fixed points:

$$(S, \text{Cat}(\mathcal{E}G, \mathcal{C}))^H \cong S((\text{Cat}(\mathcal{E}G, \mathcal{C}))^H).$$

**Remark 2.11.** It does not make sense to ask whether $S_\ast$ commutes with fixed points in general, because the fixed point categories $\mathcal{C}^H$ do not in general have Waldhausen structure.

**Definition 2.12.** We define the algebraic $K$-theory $G$-space of a Waldhausen $G$-category $\mathcal{C}$ as

$$K_G(\mathcal{C}) := \Omega|wS\mathcal{C}, \text{Cat}(\mathcal{E}G, \mathcal{C})|$$

From the above discussion, the $H$-fixed points of this space coincide with the algebraic $K$-theory space of the Waldhausen category $\mathcal{C}^\ast_H$.

**Theorem 2.13.** The space $K_G(\mathcal{C})$ is an infinite loop $G$-space.

**Proof.** By forgetting structure, each Waldhausen $G$-category $\mathcal{C}$ is a symmetric monoidal $G$-category under the coproduct $\vee$. The $G$-coherence is automatic because each $g$ acts by exact endomorphisms of the category, and therefore preserves coproducts up to canonical isomorphism.

By Proposition 2.10 we obtain an $O_G$-algebra $\text{Cat}(\mathcal{E}G, \mathcal{C}^{\text{str}})$ that is monoidally $G$-equivalent to $\mathcal{C}$. Since we have an actual $G$-equivalence of categories between

$$\text{Cat}(\mathcal{E}G, \mathcal{C}) \simeq \text{Cat}(\mathcal{E}G, \mathcal{C}^{\text{str}}),$$

$\text{Cat}(\mathcal{E}G, \mathcal{C}^{\text{str}})$ has Waldhausen structure obtained by transporting the Waldhausen structure of $\text{Cat}(\mathcal{E}G, \mathcal{C})$ along the equivalence, so that the functors in the equivalence are exact. By applying $S_\ast$, we obtain a simplicial $O_G$-algebra $S_\ast \text{Cat}(\mathcal{E}G, \mathcal{C}^{\text{str}})$. By the gluing lemma, a coproduct of weak equivalences is also a weak equivalence, so the subcategories of weak equivalences $wS_\ast \text{Cat}(\mathcal{E}G, \mathcal{C}^{\text{str}})$ also form a simplicial $O_G$-algebra.
Since the nerve and geometric realization functors are monoidal, the space \(|wS\text{Cat}(EG,C)\) is an \(|O_G|-algebra. Furthermore, since geometric realization and \(S\) commute with taking fixed points of \(\text{Cat}(EG,C)\), we get a homeomorphism
\[
|wS\text{Cat}(EG,C)|^H \simeq |wS\text{Cat}(EG,C)|^H.
\]
These spaces are all connected, so the \(G\)-space \(|wS\text{Cat}(EG,C)|\) is already group complete in the equivariant sense. By Theorem 2.9 it is therefore an infinite loop \(G\)-space. □

Definition 2.14. For a Waldhausen \(G\)-category \(\mathcal{C}\), define \(K_G(\mathcal{C})\) as the orthogonal \(\Omega\)-\(G\)-spectrum with zeroth space \(K_G(\mathcal{C})\) obtained by looping once the spectrum given by applying Theorem 2.9.

Proposition 2.15. For every subgroup \(H\) of \(G\), the orthogonal fixed point spectrum \(K_G(\mathcal{C})^H\) is equivalent to the prolongation to orthogonal spectra of the Waldhausen \(K\)-theory symmetric spectrum of \(C^hH\) defined by iterating the \(S\)-construction.

Proof. By Theorem 2.9, we get that
\[
K_G(\mathcal{C})^H \simeq \Omega K(|wS(n)\text{Cat}(EG,C)|^H),
\]
where \(K\) is the nonequivariant operadic infinite loop space machine landing in orthogonal spectra. By [Mal15a, Thm 3.11.], the orthogonal spectrum above is equivalent to the prolongation of the symmetric spectrum of \(C^hH\) defined by \(\Omega|wS(n)\text{Cat}(EG,C)|\), which is Waldhausen’s \(K\)-theory spectrum of the Waldhausen category \(C^hH\).

Remark 2.16. The argument [Mal15a, Thm 3.11.] applies verbatim for a Waldhausen category with \(G\)-action to give an equivalence of naïve \(G\)-spectra. In particular, by applying the argument to the category with \(G\)-action \(\text{Cat}(EG,C)\), we can conclude that the underlying naïve orthogonal \(G\)-spectrum of \(K_G(\mathcal{C})\) is \(G\)-equivalent to the prolongation of the symmetric spectrum with \(\text{G-action} \Omega|wS(n)\text{Cat}(EG,C)|\). On fixed points \(C^hH\), the equivalences are obtained by repeating the nonequivariant argument for each \(H\), since \(S\) commutes with taking fixed points of \(\text{Cat}(EG,C)\).

3. The Waldhausen \(G\)-category of retractive spaces \(R(X)\)

Let \(G\) be a finite group and let \(X\) be an unbased space with a continuous left \(G\)-action. Let \(R(X)\) be the category of non-equivariant retractive spaces over \(X\). That is, an object of \(R(X)\) is an unbased space \(Y\) and two maps
\[
X \xrightarrow{\gamma_Y} Y \xrightarrow{\nu_Y} X
\]
which compose to the identity on $X$. A morphism $f$ in $R(X)$ is given by a commutative diagram

```
\begin{array}{c}
Y \\
\downarrow f \\
X
\end{array}
```

3.1. **Action of $G$ on $R(X)$.** The category $R(X)$ inherits a left action by $G$, which we describe explicitly. For any $g \in G$, the functor $g: R(X) \to R(X)$ sends an object

```
X \xrightarrow{i_Y} Y \xrightarrow{p_Y} X
```

to the object

```
X \xrightarrow{g^{-1}} X \xrightarrow{i_Y} Y \xrightarrow{p_Y} X \xrightarrow{g} X,
```

and for a map $f: (Y, i_Y, p_Y) \to (Y', i_{Y'}, p_{Y'})$, the map $gf$ is defined as the diagram

```
\begin{array}{c}
Y' \\
\downarrow gf \\
X
\end{array}
```

which clearly also commutes.

We take the weak equivalences in $R(X)$ to be the weak homotopy equivalences, and the cofibrations to be the maps that have the fiberwise homotopy extension property (FHEP). Then the subcategory of cofibrant objects is a Waldhausen category. By abuse of notation, we will also call this subcategory $R(X)$. It is easy to check that the $G$-action we defined above is through exact functors.

3.2. **Homotopy fixed points of $R(X)$.** Recall from Observation 2.4 that the fixed point categories $R(X)^H$ may not be Waldhausen. In fact, if $X$ has a nontrivial $G$-action, the category $R(X)^G$ is not Waldhausen because it is empty and hence fails to contain a zero object. However by Theorem 2.7 the homotopy fixed point categories $R(X)^{hH}$ have a Waldhausen category structure. In this case they admit a more explicit description.

**Proposition 3.1.** The homotopy fixed point category $R(X)^{hH}$ is equivalent to the Waldhausen category of retractive objects over $X$ in the category of $H$-equivariant spaces. The cofibrations and weak equivalences are the $H$-equivariant maps which are nonequivariantly cofibrations, resp. weak equivalences.
Proof. By Remark 2.2, it is enough to prove the result for $H = G$. The objects of the homotopy fixed point category $R(X)^hG = \text{Cat}(\mathcal{E}G, R(X))^G$ are retractive spaces $(Y, i_Y, p_Y)$ together with isomorphisms $\psi_g: Y \xrightarrow{\cong} Y$ for all $g$ making the following diagram commute:

\[
\begin{array}{ccc}
X & \xrightarrow{i_Y} & Y & \xrightarrow{p_Y} & X \\
\downarrow{g^{-1}} & & \downarrow{\psi_g} & & \downarrow{g} \\
X & \xrightarrow{i_Y} & Y & \xrightarrow{p_Y} & X
\end{array}
\]

The maps $\psi_g: Y \rightarrow Y$ define the $G$-action on $Y$ and from the commutativity of the above diagram it follows that $i_Y$ and $r_Y$ are equivariant. It is then clear that the maps in $R(X)^hG$ are the $G$-equivariant maps. \hfill \Box

Before taking $K$-theory, we will restrict to a subcategory of finite objects. Let $R_{hf}(X)$ denote the subcategory of retractive spaces that are homotopy finite. Recall that a retractive space $Y$ is homotopy finite if it is a retract in the homotopy category of an actual finite relative cell complex over $X$.

Clearly the action of $G$ on $R(X)$ respects this condition, and so restricts to a $G$-action on $R_{hf}(X)$. The proof of Proposition 3.1 applies verbatim to give us that $R_{hf}(X)^hH = \text{Cat}(\mathcal{E}G, R_{hf}(X))^H$ is the Waldhausen category of retractive $H$-equivariant spaces over $X$ whose underlying space is homotopy finite.

Remark 3.2. By an easy application of Waldhausen’s approximation theorem, if we restrict to the subcategory of spaces that are homotopy equivalent to cell complexes, with the homotopy equivalences on the total space and the HEP cofibrations, we get equivalent $K$-theory.

3.3. Definition of $A_G^{\text{coarse}}(X)$. Applying Definition 2.12 and Theorem 2.13 to the category of retractive spaces $R_{hf}(X)$ provides our first equivariant generalization of Waldhausen’s functor.

Definition 3.3. We define the $G$-space $A_G^{\text{coarse}}(X) := \Omega|wS, \text{Cat}(\mathcal{E}G, R_{hf}(X))|.$

Corollary 3.4. The $G$-space $A_G^{\text{coarse}}(X)$ is the zeroth space of a $\Omega$-$G$-spectrum $A_G^{\text{coarse}}(X)$.

The upper script “coarse” indicates that the $H$-fixed point spectrum is the nonequivariant $K$-theory of the category of $H$-equivariant retractive spaces over $X$ with the coarse equivalences. We will proceed to explain how this fixed point spectrum is related to Williams’s bivariant $A$-theory functor $A(E \rightarrow B)$. 
3.4. **Relation to bivariant $A$-theory.** For each fibration $p : E \to B$ into a cell complex $B$, form a Waldhausen category $R_{hf}(E \xrightarrow{p} B)$ whose objects are retractive spaces

$$E \xrightarrow{i_Y} Y \xrightarrow{p_Y} E \xrightarrow{p} B$$

for which $p \circ p_Y$ is a fibration, and over each point $b \in B$ the retractive space $Y_b$ over the fiber $E_b$ is homotopy finite. The weak equivalences are the maps giving weak homotopy equivalences on $Y$. The cofibrations are the maps with the fiberwise homotopy extension property (FHEP) over $E$.

**Definition 3.5.** The bivariant $A$-theory of a fibration is defined as

$$A(E \xrightarrow{p} B) := K(R_{hf}(E \xrightarrow{p} B)).$$

This contains as a special case both Waldhausen’s $A(X) = A(X \to *)$ and the contravariant analog $V(X) = A(X \xrightarrow{id} X)$.

**Remark 3.6.** This definition of bivariant $A$-theory is equivalent to the one given in [RS14] by an easy application of the approximation property. Their cofibrations are the maps having the homotopy extension property (HEP) on each fiber separately.

We may now begin proving Theorem 1.3. We regard $A^\text{coarse}_G(X)$ as a symmetric spectrum with a $G$-action by iteration of the $S$-construction. We are therefore only considering its underlying a naive $G$-spectrum.

**Proposition 3.7.** There is a natural equivalence of symmetric spectra

$$A^\text{coarse}_G(X)^H \simeq A(EG \times_H X \to BH)$$

In particular,

$$A^\text{coarse}_G(X)^{\{e\}} \simeq A(X), \quad A^\text{coarse}_G(\ast)^H \simeq V(BH)$$

**Proof.** From Proposition 2.15, the fixed points $A^\text{coarse}_G(X)^H$ are given by the Waldhausen $K$-theory of the category $R_{hf}(X)^{hH}$, which we identify with the category of retractive $H$-equivariant spaces over $X$ with underlying homotopy finite space, as in Proposition 3.1. As explained in Remark 3.2, we may restrict $R_{hf}(X)$ to the spaces with the homotopy type of relative cell complexes, with strong homotopy equivalences and HEP cofibrations. We do so in this proof.

We adopt the shorthand

$$E = EG \times_H X = B(\ast, G, G \times_H X), \quad B = BH = B(\ast, G/H)$$

We freely use the result from [May90], Cor 8.3 that for a well-based topological group $H$ the map $B(\ast, H, H) \to B(\ast, H, \ast)$ is a principal $H$-bundle. This implies that $B(\ast, H, X) \to B(\ast, H, \ast)$ is a fiber bundle with fiber $X$. 

Since realization of simplicial spaces commutes with strict pullbacks, our desired map \( p : E \to B \) is a pullback of this fiber bundle, hence also a fiber bundle.

The equivalence of \( K \)-theory spectra will be induced by the functor

\[
\Phi : R_{hf}(X)^{hH} \to R_{hf}(EG \times_H X \xrightarrow{p} BH)
\]

that applies \( EG \times_H - \) to the retractive space \((Y, i_Y, p_Y)\) over \( X \), obtaining a retractive space over \( EG \times_H X \):

\[
EG \times_H X \xrightarrow{EG \times_H i_Y} EG \times_H Y \xrightarrow{EG \times_H p_Y} EG \times_H X
\]

The composite map \( EG \times_H Y \to BH \) is a fiber bundle with fiber \( Y \), which is assumed to be a homotopy finite retractive space over \( X \). Therefore \( \Phi \) indeed lands in the Waldhausen category \( R_{hf}(EG \times_H X \xrightarrow{p} BH) \). It is elementary to check that weak equivalences and cofibrations are preserved, and therefore \( \Phi \) induces a map on \( K \)-theory.

To prove that this map is an equivalence we verify the approximation property from [Wal85]. We observe that the category \( \text{Cat}(\mathcal{E}H, R_{hf}(X)) \) has a tensoring with unbased simplicial sets sending the \( H \)-space \( Y \) over \( X \) and a simplicial set \( K \) to the external smash product \( Y \wedge |K| \). This has the pushout-product property, by the usual formula for an NDR-pair structure on a product of NDR-pairs. Therefore \( \text{Cat}(\mathcal{E}H, R_{hf}(X)) \) has a cylinder functor.

For the approximation property, we use the right adjoint \( F(EG, -) \) of the functor \( \Phi \) when regarded as a functor from \( H \)-equivariant spaces under \( X \) to spaces under \( \Phi(X) = EG \times_H X \). Given a cofibrant retractive \( H \)-space \( Y \) and a map of retractive \( \Phi(X) \)-spaces \( \Phi(Y) \to Z \), we factor the adjoint into a mapping cylinder

\[
Y \longrightarrow Y' = Y \wedge I_+ \cup_{Y \times 1} F_{BH}(EG, Z) \sim F_{BH}(EG, Z) \longrightarrow F(EG, Z)
\]

We observe that this map is under \( X \) and over \( F(EG, EG \times_H X) \). The map \( Y \to Y' \) is a cofibration of spaces under \( X \) and over \( F(EG, EG \times_H X) \) by the pushout-product property. Pushing \( Y' \) back through the adjunction, we get a factorization of retractive spaces over \( \Phi(X) \)

\[
\Phi(Y) \longrightarrow \Phi(Y') \sim Z
\]

The map \( \Phi(Y') \to Z \) is an equivalence because it is a map of fibrations whose induced map of fibers is measured by the equivalence \( Y' \to F_{BH}(EG, Z) \) from above. This finishes the proof.

The following proposition implies the rest of Theorem 1.3, and its proof occupies the remainder of the section.
Proposition 3.8. Under the equivalence of Proposition 3.7, the coassembly map for bivariant $A$-theory agrees up to homotopy with the map from fixed points to homotopy fixed points:

$$A^\text{coarse}_G(X)^H \xrightarrow{\sim} A^\text{coarse}_G(X)^{hH}$$

$$A(EG \times_H X \to BH) \xrightarrow{\sim} \Gamma_{BH}(A_{BH}(EG \times_H X))$$

In fact, we prove a much more general fact about coassembly maps agreeing with homotopy fixed point maps. Our approach takes three steps. We first observe:

Proposition 3.9. The coassembly map of a functor on spaces over $BG$, and the homotopy fixed point map for a naive $G$-spectrum, are each given by the unit of an adjunction of homotopy categories.

To be more specific, for each subgroup $H$, the map $(-)^H \to (-)^{hH}$ is the unit of the following adjunction evaluated on the $H$-fixed points.

$$\text{naive } G\text{-spectra} \xrightarrow{\text{localize}} \perp \xrightarrow{F(EG, -)} \text{coarse } G\text{-spectra}$$

For coassembly, the relevant adjunction is

$$\text{Homotopy functors } \xrightarrow{\text{restrict}} \perp \xrightarrow{\text{holim } F(\Delta^p)}$$

where $U^\text{op}_{BG}$ is the opposite of the category of unbased spaces over $BG$, and the subcategory $\Delta_{BG}$ consists of only the simplices $\Delta^p \to BG$ for varying $p$ and the compositions of face maps. A homotopy functor is one which sends weak equivalences of spaces to equivalences of spectra.

Proof. In the category of symmetric or orthogonal $G$-spectra, the functor $EG_+ \wedge -$ is equivalent to the localization to coarse $G$-spectra, and preserves cofibrant objects. Its right adjoint is the function spectrum $F(EG, -)$, which preserves fibrant objects. The adjunction on the homotopy category immediately follows. We may then modify the unit by collapsing away the extra $EG$ inside, and we are left with the map from fixed points to homotopy fixed points.

The claim about the coassembly map follows from the standard fact that the homotopy right Kan extension is the right adjoint of restriction, on the homotopy category of all functors. Furthermore, the canonical map of $F$ into the extension of the restriction of $F$ is the unit of this adjunction.
By [Mal15b], this particular model for the homotopy right Kan extension preserves homotopy functors, so the adjunction on all functors descends to the subcategories of homotopy functors, with the same unit.

Next we observe that every homotopy functor over $BG$ gives a naïve $G$-spectrum, because it may be restricted to the spaces of the form $BH = EG/H$, and the mapping spaces between these spaces over $BG$ are homeomorphic to the mapping spaces of the orbit category of $G$:

$$\text{Map}_{BG}(EG/H, EG/K) \cong \text{Map}_G(G/H, G/K)$$

This gives a contravariant functor on the orbit category $O(G)$. Since Elmendorf’s theorem applies to symmetric and orthogonal spectra (cf. [Ste13]), the homotopy category of such diagrams is equivalent to the homotopy category of naïve $G$-spectra. Therefore we compare our two constructions as $O(G)^{op}$-diagrams.

**Proposition 3.10.** The equivalence of Proposition 3.7 gives a map of diagrams of symmetric spectra over $O(G)^{op}$.

**Proof.** We give an explicit model for the pullback of $EG \times_G X \to BG$ to $EG/H$ for all $H$ that illuminates the $O(G)^{op}$-action. Each map of $G$-sets $f : G/H \to G/K$ and $K$-space $X$ gives a pullback square

$$
\begin{array}{ccc}
B(\ast, G \times_H X) & \longrightarrow & B(\ast, G \times_K X) \\
\downarrow & & \downarrow \\
B(\ast, G/H) & \longrightarrow & B(\ast, G/K)
\end{array}
$$

Recall that a map $f : G/H \to G/K$ of $G$-sets is determined by the image $f(eH) = g^{-1}K$ where $g \in G$ has the property that $H \leq g^{-1}Kg$. We define the action of each $h \in H$ on $X$ to be the given action of $ghg^{-1} \in K$. In the pullback above, vertical maps collapse $X$ to a point, and the map $G \times_H X \to G \times_K X$ sends $(\gamma, x)$ to $(\gamma g^{-1}, x)$. The functoriality of bivariant $A$-theory with respect to pullbacks gives the dashed map

$$A_G^{\text{coarse}}(X)^K \overset{\Phi}{\longrightarrow} A(EG \times_K X \to EG/K) \overset{f^*}{\longrightarrow} A(EG \times_H Y \to EG/H)$$

This defines the action of $O(G)^{op}$, which is strict by functoriality of bivariant $A$-theory (see [RS14], Remark 3.2.1).

We first prove that the above square commutes up to canonical homotopy for each $f : G/H \to G/K$. The top leg takes each retractive $K$-space $Y$ to $EG \times_H Y$, where the $H$-action on $Y$ is defined as above. For the other leg, we
need to describe the action of the orbit category on the symmetric-spectrum variant of $A^\text{coarse}_G(X)^H$. If we let $G/H$ denote the category on the object set $G/H$ with only identity morphisms, we get isomorphisms of symmetric spectra

$$|w_*S^{(n)} \text{Cat}(EG, R_{hf}(X))|^H \cong |w_*S^{(n)} \text{Cat}(G/H, \text{Cat}(EG, R_{hf}(X)))|^G$$

under which the $O(G)^{\text{op}}$-action becomes composition of functors out of $G/K$ with $f : G/H \to G/K$. By the proof of Proposition 4.14 below, the resulting exact functor takes retractive $K$-spaces $Y$ to retractive $H$-spaces given by $Y$, with $h \in H$ acting by $ghg^{-1} \in K$, precisely as above. Therefore the two legs give canonically isomorphic exact functors, which give homotopic maps on $K$-theory.

To eliminate the homotopies we strictify $A^\text{coarse}_G(X)^H$ and $A(EG \times_H X \to EG/H)$ in the following way. We define a new category $\tilde{R}_{hf}(X)^{hH}$ whose objects are triples $(f, K, Y)$ where $K \leq G$, $f : G/H \to G/K$, and $Y$ is a retractive $K$-space over $X$. The morphisms are defined so that the map which applies $f$ to $Y$ to obtain a retractive $H$-space, gives an equivalence of categories $\tilde{R}_{hf}(X)^{hH} \to R_{hf}(X)^{hH}$. This is an equivalence of strict $O(G)^{\text{op}}$-diagrams, if we have $O(G)^{\text{op}}$ act on triples simply by composition with the map $f$.

There is a strictification $\tilde{R}_{hf}(EG \times_H X \to EG/H)$ defined in a similar way. We lift $\Phi$ to the unique functor $\tilde{\Phi}$ between strictifications which on the objects sends the triple $(f, K, Y)$ to $(f, K, \Phi(Y))$, and which agrees with $\Phi$ along our equivalences of categories. Then $\tilde{\Phi}$ clearly commutes with the action of $O(G)^{\text{op}}$ on the objects. That $\Phi$ commutes on morphisms is a straightforward diagram-chase boiling down to the fact that our canonical isomorphisms above respect compositions of maps of $G$-sets.

The final step is to formally identify the two adjunctions we gave above. Once this is done, the units must agree, and this proves that coassembly agrees with the homotopy fixed point map up to homotopy.

We observe that our passage from homotopy functors to $O(G)^{\text{op}}$ diagrams becomes an equivalence of homotopy categories if we localize our homotopy functors by measuring the weak equivalences on the spaces $EG/H$. We have already remarked that naïve $G$-spectra are equivalent to diagrams over $O(G)^{\text{op}}$. These equivalences of homotopy categories are collected in the figure below.

**Proposition 3.11.** This figure commutes, as a diagram of adjunctions of homotopy categories.

*Proof.* For this proof, $Sp$ will denote orthogonal spectra, so we use the Quillen equivalence from [MMSS01] to push our earlier constructions from
symmetric spectra forward to orthogonal spectra. The adjunction at the top comes from factoring the adjunction of Proposition 3.9 through localization. The middle adjunction will fall out from the rest of the proof.

First we show that the left adjoints in the topmost region commute. As before, the category $\Delta_{BG}$ consists of simplices over $BG$; its classifying space $|\Delta_{BG}|$ admits a weak equivalence to $BG$. The construction $|\Delta_X| = |N \Delta_X|$ preserves covering spaces, so that $|\Delta_{EG}| \to |\Delta_{BG}|$ is a covering map, equivalent to $EG \to BG$. The homotopy colimit over $\Delta_{BG}$ is taken in the unbased sense on each spectrum level, giving a parametrized spectrum over $|\Delta_{BG}|$. By Quillen’s Theorem B the levels of this spectrum are quasifibrations. It is well-known that this gives an equivalence of homotopy categories; an explicit proof for the spectrum case can be found in [LM].

In addition, when $F$ is a functor on spaces over $BG$, $F(EG)$ has a left $G$-action coming from the right $G$-action on $EG$. We use the functoriality of $F$ to form a map on each spectrum level of $F(EG)$

$$|\Delta_{EG}^{op}| \times F(EG) = \underset{\Delta_{EG}^{op}}{\text{holim}} F(EG) \longrightarrow \underset{\Delta^p \in \Delta_{EG}^{op}}{\text{holim}} F(\Delta^p)$$

and observe that it descends to the quotient $|\Delta_{EG}^{op}| \times_G F(EG)$, where it becomes an equivalence of quasifibrations. This gives a natural levelwise equivalence of parametrized spectra, and proves that the left adjoints in the topmost region commute.
In the bottom half of the diagram, it is elementary to check that both the claimed left adjoints and the claimed right adjoints give commuting squares. Therefore the claimed adjunction in the middle is actually an adjunction, which agrees with the one at the bottom. Since the left adjoints in the top region all commute, the right adjoints must commute as well by the universal property.

This finishes the proof of Proposition 3.8 and concludes our analysis of the functor $A^\text{coarse}_G(X)$.

4. Transfers on Waldhausen $G$-categories

In this section, we give the construction of $A_G(X)$ and prove the following main theorem (Theorem 1.1 from the Introduction.)

**Theorem 4.1.** For $G$ a finite group, there exists a functor $A_G$ from $G$-spaces to genuine $G$-spectra with the property that on $G$-fixed points,

$$A_G(X)^H \simeq \prod_{(H) \leq G} A(X^H_{hWH}),$$

and a similar formula for the fixed points of each subgroup $H$.

We construct $A_G(X)$ as a spectral Mackey functor because we need the flexibility to refine the weak equivalences in each of the homotopy fixed point categories $R_{hf}(X)^{hH}$. We describe the framework of spectral Mackey functors as models of $G$-spectra, developed by Guillou and May in [GMb], followed by the work of Bohmann and Osorno [BO], which constructs categorical input that directly feeds into their theorem. We then construct this categorical input by a 1-categorical variant of a general construction due to Barwick, Glasman and Shah [BGS, §8]. In particular, Proposition 4.12 should be viewed as a reinterpretation of [BGS, 8.1]. Finally, we construct $A_G(X)$ by descending the structure to the Waldhausen categories with refined weak equivalences for each $H \subseteq G$.

4.1. Review of spectral Mackey functors. We start with a description of the framework in broad strokes. By a general result of Schwede and Shipley [SS03b], if $C$ is a stable model category with a finite set of generators $\{X_1, \ldots, X_n\}$, then the derived mapping spectra $C(X_i, X_j)$ form a spectrally enriched category $B(C)$ on the objects $\{X_1, \ldots, X_n\}$. Thinking of such a spectral category as the many-objects version of a ring spectrum, and spectrally-enriched functors into spectra as modules over that ring, there is a model category $\text{Mod}_B(C)$ of modules over $B(C)$ and a Quillen equivalence $\text{Mod}_B(C) \simeq C$ given by coend with $\{X_i\}$ and its right adjoint:

$$L(\{M_i\}) = \{X_i\} \wedge_{B(C)} \{M_i\}, \quad R(Y)_i = C(X_i, Y)$$
This is the spectral analog of classical Morita theory; when $R$ is a ring and $M$ a perfect $R$-module generator, this construction gives a Quillen equivalence between $R$-modules and $\text{End}_R(M)$-modules.

Taking $\mathcal{C}$ to be the category of orthogonal $G$-spectra for a finite group $G$, $\mathcal{C}$ is generated by the suspension spectra $\Sigma_\infty^+ G/H$ for conjugacy classes of subgroups $(H) \leq G$. By the self-duality of the orbits $\Sigma_\infty^+ G/H$, the mapping spectrum from $G/H$ to $G/K$ may be written as the genuine fixed points of a suspension spectrum

$$(\Sigma_\infty^+ G/H \times G/K)^G$$

and the compositions are given by stable $G$-maps

$$G/H \times G/L \times G/L \times G/K \longrightarrow G/H \times G/L \times G/K \longrightarrow G/H \times G/K$$

which collapse away the complement of the diagonal of $G/L$ and then fold that diagonal to a single point.

Guillou and May prove that this spectral category is equivalent to a spectral version of the Burnside category, namely a spectral category $\mathcal{GB}$ with objects $G/H$, and morphisms spectra $\mathcal{GB}(G/H, G/K)$ given by the $K$-theory of the permutative category of finite equivariant spans from $G/H$ to $G/K$. The composition is by pullback of spans, which can be made strictly associative by using a skeleton of the category of finite $G$-sets and by picking explicit models for pullbacks of spans (cf. [GMb]).

**Theorem 4.2** (Guillou-May). *There is a string of Quillen equivalences between $\mathcal{GB}$-modules $\{M_H\}$ and genuine $G$-spectra $X$, and the fixed points $X^H$ are equivalent to the spectrum $M_H$ for every subgroup $H$.***

Therefore, by Theorem 4.2, to create a $G$-spectrum whose $H$-fixed points are $K(R^H_{hf}(X))$, it is enough to show that the spectra $K(R^H_{hf}(X))$ form a module over the “ring on many objects” $\mathcal{GB}$. The spectral category $\mathcal{GB}$ from [GMb] is built using permutative categories; following [BO], we give an alternate version $\mathcal{GB}_{Wald}$ using Waldhausen categories.

**Definition 4.3.** For each pair of subgroups $H, K \leq G$ let $S_{H,K}$ denote the category of finite $G$-sets containing $G/H \times G/K$ as a retract. Such sets are of the form $\text{SI}(G/H \times G/K)$, which we abbreviate to $S_+$ when $H$ and $K$ are understood. This is a Waldhausen category in which the weak equivalences are isomorphisms and the cofibrations are injective maps. Of course, the coproduct is disjoint union along $G/H \times G/K$. We adopt the conventions of [GMb, Def. 1.2, Def. 1.5], assuming that each of the $G$-sets $S$ is one of the standard sets $\{1, \ldots, n\}$ with a $G$-action, so that the coproduct, product, and pullback are given by specific formulas that make them associative and unital on the nose.
Define a pairing
\[ S_{H,L} \times S_{L,K} \rightarrow S_{H,K} \]
by sending each pair of composable spans \( S_+ = S \amalg (G/H \times G/L) \) and \( T_+ = T \amalg (G/L \times G/K) \) to the span \( (S * T)_+ = (S * T) \amalg (G/H \times G/K) \), where \( (S * T) \) is the pullback span

This pairing is biexact and strictly associative by our adopted conventions. The units are given by the spans \( (G/H)_+ = G/H \amalg (G/H \times G/H) \) with identity projections

**Definition 4.4.** Let \( \mathcal{GB}_{Wald} \) be the spectrally-enriched category on the objects \( G/H, (H) \leq G \) whose mapping spectra are the Waldhausen \( K \)-theory spectra \( K(S_{H,K}) \).

We use the following formulation of a hard and technical result of Bohmann-Osorno, which will appear in [BO], in order to translate Theorem 4.2 into something that interacts more readily with Waldhausen categories.

**Theorem 4.5 (Bohmann-Osorno).** There is an equivalence of spectrally enriched categories \( \mathcal{GB} \) and \( \mathcal{GB}_{Wald} \).

Since equivalences of spectral categories induce Quillen equivalences on their module categories ([SS03a], Thm 6.1), by Theorem 4.5, it is now enough to show that the spectra \( K(R^H_{Hf}(X)) \) form a module over \( \mathcal{GB}_{Wald} \). We will spell out explicit categorical conditions that will imply this and then verify these conditions in examples.

**Proposition 4.6.** Suppose we are given

1. a Waldhausen category \( R^H \) for each \( H \leq G \),
2. an exact functor \( (\ast S): R^H \rightarrow R^K \) for each retractive span \( S_+ \) in the category \( S_{H,K} \),
3. a natural transformation of functors \( \bar{f}: (\ast S) \Rightarrow (\ast T) \) for each map of retractive spans \( f: S_+ \rightarrow T_+ \),

subject to the conditions
4. for fixed $A \in R^H$, the assignment $S_+ \mapsto A \ast S$ defines a functor $S_{H,K} \to R^K$.

5. we have $A \ast \emptyset \cong \ast$ and $(A \ast S) \vee (A \ast T) \to A \ast (S \amalg T)$ is an isomorphism in $R^K$ for all spans $S_+, T_+$.

6. the unit span action $(- \ast G/H) : R^H \to R^H$ is the identity.

7. if $(S \ast T)$ is the horizontal composition of $S$ and $T$ as above then $(- \ast (S \ast T)) = ((- \ast S) \ast T)$ as functors $R^H \to R^K$.

Then the spectra $K(R^H)$ form a module over $GB_{Wald}$, and therefore, also over $GB$.

Basically, we are asking for a “right action” map of spans on the categories $R^H$, such that the action map is a bi-exact functor, associative and unital.

**Proof.** By (5) the functor $S_+ \mapsto A \ast S$ preserves all sums. Observe that every cofibration $S_+ \to T_+$ is a coproduct of the identity of $S_+$ and the map $\emptyset_+ \to (T - S)_+$. Therefore $A \ast S \to A \ast T$ is isomorphic to a sum of the identity of $A \ast S$ and the inclusion $\ast \to A \ast (T - S)$ which is a cofibration. Of course, equivalences of spans are isomorphisms, which go to isomorphisms in $R^K$. Therefore the pairing $R^H \times S_{H,K} \to R^K$ is exact in the span variable, and it is exact in the $R^H$ variable by condition 2.

To complete the verification of biexactness, note given an inclusion $S_+ \to T_+$ and a cofibration $A \to B$ in $R^H$, the map $A \ast T \cup_{A \ast S} B \ast S \to B \ast T$ is a pushout of the map $A \ast (T - S) \to B \ast (T - S)$, which is a cofibration because $(T - S)$ acts by an exact functor.

Therefore we have biexact pairings $R^H \times S_{H,K} \to R^K$ with a strict associativity property. By the multifunctoriality of Waldhausen $K$-theory, this gives us a module over $GB_{Wald}$. □

In the next section we show how to give such data for $C^{hH}$ when $C$ is any Waldhausen $G$-category.

**4.2. Categorical transfer maps.** Suppose that $C$ is a $G$-category with a chosen sum bifunctor $\oplus$ isomorphic to the categorical coproduct $\amalg$. Since $G$ acts through equivalences of categories, it preserves $\oplus$ up to coherent isomorphism. Let $f : S \to T$ be a map of finite $G$-sets. Note that a finite $G$-set $S$ can be regarded as a category with objects the elements of $S$ and only identity morphisms, so the functor category $\text{Cat}(S, C)$ is isomorphic to the $S$-indexed product $\prod_S C$. We can define a map $f_! : \text{Cat}(S, C) \to \text{Cat}(T, C)$, by $$(f_! F)(t) := \bigoplus_{i \in f^{-1}(t)} F(i),$$
or equivalently,

\[ f_1 : \prod_S \mathcal{C} \to \prod_T \mathcal{C} \]

\[ (c_1, \ldots, c_j) \mapsto \left( \bigoplus_{i \in f^{-1}(1)} c_i, \ldots, \bigoplus_{i \in f^{-1}(k)} c_i \right), \]

where \( j = |S| \) and \( k = |T| \). Note that this map is not on the nose equivariant even if the sum \( \oplus \) in \( \mathcal{C} \) commutes with the \( G \)-action strictly, but it is only pseudo-equivariant. When we apply the \( \text{Cat}(\mathcal{E}G, -) \), by Proposition 2.3, we get an on the nose equivariant functor

\[ f_1 : \text{Cat}(S \times \mathcal{E}G, \mathcal{C}) \cong \text{Cat}(\mathcal{E}G, \text{Cat}(S, \mathcal{C})) \to \text{Cat}(\mathcal{E}G, \text{Cat}(T, \mathcal{C})) \cong \text{Cat}(T \times \mathcal{E}G, \mathcal{C}), \]

which upon taking \( G \)-fixed points gives a transfer (or pushforward map) along the map of \( G \)-sets \( f : S \to T \). We make this more explicit in the following definition.

**Definition 4.7.** Let \( \mathcal{C} \) be a \( G \)-category with coproduct \( \oplus \), and let \( f : S \to T \) be a map of unbased finite \( G \)-sets. Define a **pullback (restriction) functor**

\[ f^* : \text{Cat}(T \times \mathcal{E}G, \mathcal{C})^G \to \text{Cat}(S \times \mathcal{E}G, \mathcal{C})^G \]

for objects and for maps by the formulas

\[ (f^*F)(s, g) = F(f(s), g) \]

\[ (f^*F)(s, g \to h) = F(f(s), g \to h) \]

Define a **pushforward (transfer) functor**

\[ f_! : \text{Cat}(S \times \mathcal{E}G, \mathcal{C})^G \to \text{Cat}(T \times \mathcal{E}G, \mathcal{C})^G \]

on objects by

\[ (f_! F)(t, g) := g \left( \bigoplus_{i \in f^{-1}(g^{-1}t)} F(i, 1) \right) \]

To get the morphisms we use the canonical isomorphism

\[ g \left( \bigoplus_{i \in f^{-1}(g^{-1}t)} F(i, 1) \right) \cong \bigoplus_{i \in f^{-1}(g^{-1}t)} F(g i, g) = \bigoplus_{j \in f^{-1}(t)} F(j, g) \]

Under this isomorphism, the morphism \( (f_! F)(t, g \to h) \) is chosen to be the coproduct

\[ \bigoplus_{j \in f^{-1}(t)} F(j, g \to h) \]
Remark 4.8. In the special case where $H$ is a subgroup of $K$ and $f : G/H \rightarrow G/K$ is the quotient map, $f_!$ defines a transfer map $\mathcal{C}^hH \rightarrow \mathcal{C}^hK$.

More generally, for a span

$$
\begin{array}{c}
p \downarrow \\
S & \xrightarrow{q} & G/K \\
p \uparrow & & \uparrow \\
G/H & \xrightarrow{f} & G/K
\end{array}
$$

one can define a functor $\mathcal{C}^hH \rightarrow \mathcal{C}^hK$ by $qp^*$. To prove that this defines a bifunctor that respects compositions of spans, one needs the following formal properties of $f_!$ and $f^*$ (cf. [Bar, §10]).

Proposition 4.9. For each equivariant map $f : S \rightarrow T$ of finite $G$-sets, the functors $(f_!, f^*)$ form an adjoint pair.

Proof. We have a natural bijection between natural transformations $f_!F \Rightarrow F'$ of functors $T \times E^G \rightarrow \mathcal{C}$ which send $(t', g)$ to the zero object for $t' \in T'$ and natural transformations $F \Rightarrow f^*F'$ of functors $S \times E^G \rightarrow \mathcal{C}$ which send $(s', g)$ to the zero object if $s' \in S'$.

The transformation $f_!F \Rightarrow F'$, at $(t, g)$ is just $\oplus \rightarrow *$ if $t \in T'$, and by applying the canonical isomorphism from the above definition, for $t \notin T'$, it is given by

$$
\bigoplus_{s \in f^{-1}(t)} F(s, g) \rightarrow F'(t, g)
$$

Since $\oplus$ is a coproduct, these are in canonical bijection with maps

$$
F(s, g) \rightarrow F'(f(s), g)
$$

for all $(s, g)$ for which $f(s) \notin T'$, which specify the natural transformation $F \Rightarrow f^*F'$ at these values. This natural transformation, at values $(s, g)$ for which $f(s) \in T'$ is specified by the unique maps $F(s, g) \rightarrow *$. \qed

Proposition 4.10. Given a pullback square of finite $G$-sets

$$
\begin{array}{c}
A \xrightarrow{k} B \\
\downarrow k \downarrow f \\
C \xrightarrow{j} D
\end{array}
$$

there is a “Beck-Chevalley” isomorphism

$$
\begin{array}{c}
\text{Cat}(B \times E^G, C)^G \xrightarrow{h_!k^*} \text{Cat}(C \times E^G, C)^G \\
\downarrow \text{Cat}(h_! \downarrow BC) \\
\downarrow j^*f_!
\end{array}
$$
defined as the composite
\[
h_k^* \xrightarrow{\eta} j^* j_k^* \xrightarrow{\cong} j^* f_k^* \xrightarrow{\epsilon} j^* f_1
\]

**Proof.** Unwinding the definitions gives a natural transformation between the two functors on \( C \times E \) defined by
\[
h_k^* F(c, g) = g \left( \bigoplus_{a \in h^{-1}(g^{-1}c)} F(k(a), 1) \right), \quad j^* f_1 F(c, g) = g \left( \bigoplus_{b \in f^{-1}(g^{-1}j(c))} F(b, 1) \right)
\]
that sends each \( F(k(a), 1) \) to the \( F(b, 1) \) where \( b = k(a) \), by an identity map. Since the square is a pullback, \( k \) defines a bijection \( h^{-1}(c) \to f^{-1}(j(c)) \) for all \( c \in C \), so this is a natural isomorphism. □

**Proposition 4.11.** Each diagram of \( G \)-sets
\[
\begin{array}{ccc}
S & \xrightarrow{f} & V \\
\downarrow{p} & & \downarrow{q} \\
U & \xrightarrow{r} & T
\end{array}
\]
induces a natural transformation \( f_\# : qp^* \to sr^* \). These natural transformations depend in a functorial way on the maps \( f \).

**Proof.**
\[
\begin{array}{ccc}
\text{Cat}(U \times E \mathcal{G}, \mathcal{C})^\mathcal{G} & \xrightarrow{p^*} & \text{Cat}(S \times E \mathcal{G}, \mathcal{C})^\mathcal{G} \\
\downarrow{r^*} & & \downarrow{f^*} \\
\text{Cat}(T \times E \mathcal{G}, \mathcal{C})^\mathcal{G} & \xrightarrow{s_1} & \text{Cat}(V \times E \mathcal{G}, \mathcal{C})^\mathcal{G}
\end{array}
\]
The identities \( p = rf, q = sf \) and the counit of \( (f_1, f^*) \) gives a natural transformation
\[
qp^* = s_1 f_1 f^* r^* \xrightarrow{\epsilon} sr^*
\]
which we take as the definition of \( f_\# \). Functoriality follows from the fact that the counit of a composite adjunction is equal to the composite of the counits. □

We conclude this section by checking that the action of spans on \( \mathcal{C}^{hH} \) gives a spectral Mackey functor, in the sense of the previous section, for any Waldhausen \( G \)-category \( \mathcal{C} \). As mentioned earlier, our argument appears to be a concrete special case of the general unfurling construction of [Bar, §11].

**Proposition 4.12.** (cf. [BGS, 8.1]) Let \( \mathcal{C} \) be a Waldhausen \( G \)-category. The spectra \( K(\mathcal{C}^{hH}) \) form a module over \( \mathcal{G} \).
Proof. By Proposition 4.6, it suffices to check the following seven points.

1. We set \( R^H = \text{Cat}(G/H \times \mathcal{E}G, C)^G \cong C^hH \).

2. Given \( S_+ \in S_{H,K} \) with maps \( G/H \xrightarrow{p} S \xrightarrow{q} G/K \), we define \((-) \ast S : R^H \to R^K \) by \( q \circ p \).

3. Given a map of retractive spans

\[
\begin{array}{ccc}
S \times (G/H \times G/K) & \xrightarrow{p \Pi \pi_1} & G/H \\
& f & \xrightarrow{q \Pi \pi_2} & G/K \\
& \xrightarrow{r \Pi \pi_1} & T \times (G/H \times G/K)
\end{array}
\]

we recognize canonical isomorphisms

\[
(q \Pi \pi_2) \ast (p \Pi \pi_1) \ast \cong q \circ p \ast \ast (\pi_2) \ast \pi_1
\]

and take \( \overline{f} \) to be the summand of \( f_2 \) taking \( q \circ p \ast \) to \( s_1 r \ast \). Note that because \( f \) restricts to the identity of \( G/H \times G/K \), we have the commuting diagram

\[
\begin{array}{ccc}
(\pi_2) \ast \pi_1 & \xrightarrow{q \circ p \ast \ast (\pi_2) \ast \pi_1} & q \circ p \ast \\
& \xrightarrow{f_2} & \overline{f} \\
(\pi_2) \ast \pi_1 & \xrightarrow{s_1 r \ast \ast (\pi_2) \ast \pi_1} & s_1 r \ast
\end{array}
\]

4. Given two maps of spans

\[
\begin{array}{ccc}
S \times (G/H \times G/K) & \xrightarrow{p \Pi \pi_1} & G/H \\
& f & \xrightarrow{q \Pi \pi_2} & G/K \\
& \xrightarrow{r \Pi \pi_1} & T \times (G/H \times G/K) \\
& \xrightarrow{m \Pi \pi_1} & U \times (G/H \times G/K) \\
& & \xrightarrow{n \Pi \pi_2} & G/K
\end{array}
\]

Proposition 4.11 tells us that \((h_2 f_2) = h_2 f_2\). A simple chase of the diagram below confirms that \( \overline{h_2 f} = \overline{h} \circ \overline{f} \).
5. If \( i \) is the inclusion of the empty set then \( i! \) always gives a zero object. The isomorphism \( A \ast S \vee A \ast T \to A \ast (S \amalg T) \) is immediate from the definition of the transfer \( q_i \).

6. and 7. For these last two we change our choice of \( RH \) up to equivalence; by formal arguments this does not destroy any of the previous properties. We let \( RH \) have as objects the triples \((Y, K, (S_+, p, q))\) where \( K \leq G \), \( S_+ \) is a retractive span from \( G/K \) to \( G/H \), and \( Y \) is an object of \( RK = \text{Cat}(G/K \times EG, \mathcal{C})^G \). The morphisms are defined so that the map to \( RH = \text{Cat}(G/H \times EG, \mathcal{C})^G \) is an equivalence of categories; more explicitly,

\[
RH((K, Y, S_+), (K', Y', S'_+)) := RH(Y \ast S, Y' \ast S')
\]

The action of \( SH,L \) on \( RH(X) \) is defined on objects by

\[
T(K, Y, S_+) = (K, Y, (S \ast T)_+)
\]

This is clearly associative and unital by our conventions on compositions of retractive spans.

The action of \( SH,L \) on morphisms in \( RH \) is subtle and uses the Beck-Chevalley isomorphisms from Proposition 4.10. We define the action of the span \( T \) on morphisms by

\[
RH((K, Y, S_+), (K', Y', S'_+)) = RH(Y \ast S, Y' \ast S') \xrightarrow{T} RH(Y \ast S \ast T, (Y' \ast S') \ast T) \xrightarrow{BC} RH((K, Y, (S \ast T)_+), (K', Y', (S' \ast T)_+))
\]

By a straightforward diagram chase, these actions strictly respect composition and identity maps. The only needed ingredient is the standard fact that the Beck-Chevalley natural transformations agree with composition along pasting of pullback squares.

\[ \square \]

4.3. **Construction of** \( A_G(X) \). By the previous section, the spectra \( K(R_{hf}(X)^{hH}) \) form a spectral Mackey functor. To construct \( A_G(X) \) we simply need to check that the structure thus defined on \( R_{hf}(X)^{hH} \) respects the subcategory of retractive \( H \)-cell complexes and the equivariant weak equivalences and cofibrations between them.

**Definition 4.13.** Let \( RH(X) \) be the category of relative \( H \)-cell complexes \( X \to Y \). The weak equivalences are those inducing weak equivalences rel \( X \) on the fixed points for all subgroups of \( H \). The cofibrations are the maps \( Y \to Z \) with the \( H \)-equivariant FHEP: there is an \( H \)-equivariant, fiberwise retract

\[
Z \times I \longrightarrow Y \times I \cup_{Y \times 1} Z \times 1.
\]
In particular, when \( L \leq H \), the \( L \)-fixed points of a cofibration are a cofibration in \( R(X^L) \). Finally, let \( R^H_{hf}(X) \) be the subcategory of objects which are retracts in the homotopy category of \( R^H(X) \) of finite relative complexes.

We want to define an action of spans on \( R^H_{hf}(X) \). From the previous section, each span \( S \) over \( G/H \) and \( G/K \) acts on the larger category \( R^H(X) \simeq R(X)^{hH} \). It will suffice to describe this action explicitly, and check that it respects the subcategory \( R^H_{hf}(X) \) and its Waldhausen structure.

**Proposition 4.14.** The functor \( (\ - \ast S) : R(X)^{hH} \to R(X)^{hK} \) restricts to an exact functor \( R^H_{hf}(X) \to R^K_{hf}(X) \).

**Proof.** From the definition of \( q \circ p^* \) it is clear that up to isomorphism the resulting retractive space over \( X \) is a coproduct of the spaces one would get from considering each orbit of \( S \) separately. Therefore, without loss of generality, we assume \( S \cong G/L \). Recall that there is a \( G \)-map \( G/L \to G/H \) if and only if \( L \) is subconjugate to \( H \). So it is enough to show:

1. if \( L \leq H \) is a subgroup, the pullback of \( G/L \to G/K \) gives an exact functor \( R^H_{hf}(X) \to R^K_{hf}(X) \),
2. if \( L \) and \( L' \) are conjugate by \( L' = gLg^{-1} \) the pullback and pushforward of the isomorphism \( f : G/L \xrightarrow{\cong} G/L' \) gives and exact functor \( f^*: R^L_{hf}(X) \to R^L_{hf}(X) \).
3. if \( L \leq K \) is a subgroup, the pushforward of \( G/L \to G/K \) gives an exact functor \( f_!: R^H_{hf}(X) \to R^K_{hf}(X) \).

To show (1), suppose \( L \leq H \), and let \( p : G/L \to G/H \) be the map \( gL \mapsto gH \).

Through the equivalence of categories

\[
R^H(X) \simeq \text{Cat}(\mathcal{E}G, R(X))^H \cong \text{Cat}(G/H \times \mathcal{E}G, R(X))^G
\]

a retractive space \( Y \) in \( R^H(X) \) is naturally isomorphic to \( F(eH, e) \) for some \( G \)-equivariant functor \( F : G/H \times \mathcal{E}G \to R(X) \), and through the reverse chain of equivalences of categories

\[
\text{Cat}(G/L \times \mathcal{E}G, R(X))^G \cong \text{Cat}(\mathcal{E}G, R(X))^L \simeq R^L(X)
\]

the object \((p^*F)(eL,e)\) gets sent to the retractive space \( Y \) with \( L \)-action restricted from \( H \). So the induced map

\[
R^H(X) \to R^L(X)
\]

is the forgetful functor which restricts from the \( H \)-action to the \( L \)-action, which preserves our cofibrations and weak equivalences. Since all groups are finite, it also preserves finite complexes, so it restricts to an exact functor \( R^H_{hf}(X) \to R^L_{hf}(X) \).
For (2) consider $L' = gLg^{-1}$, and let $f : G/L \overset{\cong}{\rightarrow} G/L'$ be the isomorphism of $G$-sets given by $hL \mapsto hg^{-1}L'$. Given a retractive space $X \overset{i_Y}{\rightarrow} Y \overset{p_Y}{\rightarrow} X$ that is $L'$-equivariant, it is naturally isomorphic to $F(eL',e)$ for some $F \in \text{Cat}(G/L' \times \mathcal{E}G, R(X))^G$, and under the the functor $\Phi : \text{Cat}(G/L \times \mathcal{E}G, R(X))^G \rightarrow \text{Cat}(G/L \times \mathcal{E}G, R(X))^G$

it maps to

$$(f^*F)(eL, e) = F(g^{-1}L', e) = g^{-1}F(eL', g).$$

with action of $\ell \in L$ given by the usual recipe applied to the morphism

$$(f^*F)(eL, e \rightarrow \ell) = F(g^{-1}L', e \rightarrow \ell) = g^{-1}F(eL', g \rightarrow g\ell).$$

We observe that the map $-g^{-1} : \mathcal{E}G \rightarrow \mathcal{E}G$ that multiplies on the right by $g^{-1}$ is a $G$-equivariant isomorphism of categories, and that any $G$-equivariant functor $\Phi : \mathcal{E}G \rightarrow \mathcal{C}$ is $G$-equivariantly isomorphic to $\Phi = \Phi \circ -g^{-1}$. The components of the natural transformation that give the isomorphism $\Phi \Rightarrow \tilde{\Phi}$ are just $\Phi$ applied to the unique isomorphisms $g_0 \overset{\cong}{\rightarrow} g_0g^{-1}$ in $\mathcal{E}G$. We precompose $f^*$ with the automorphism of the category $\text{Cat}(G/L' \times \mathcal{E}G, R(X))^G$ induced by $-g^{-1} : \mathcal{E}G \rightarrow \mathcal{E}G$, thus $(f^*\tilde{F})(eL, e)$ is naturally isomorphic to

$$(f^*\tilde{F})(eL, e) = (f^*F)(eL, g^{-1}) = F(g^{-1}L', g^{-1}) = g^{-1}F(eL', e).$$

with action of $\ell \in L$ given by the usual recipe applied to the morphism

$$(f^*\tilde{F})(eL, e \rightarrow \ell) = (f^*F)(eL, g^{-1} \rightarrow \ell g^{-1}) = F(g^{-1}L', g^{-1} \rightarrow \ell g^{-1})$$

$$= g^{-1}F(eL', e \rightarrow g\ell g^{-1}) = g^{-1}F(eL', e) \rightarrow \ell g^{-1}F(eL', e).$$

Since $F(eL', e) = X \overset{i_Y}{\rightarrow} Y \overset{p_Y}{\rightarrow} X$, diagrammatically, this morphism is

$$
\begin{array}{cccccc}
X & \overset{g}{\rightarrow} & X & \overset{i_Y}{\rightarrow} & Y & \overset{p_Y}{\rightarrow} & X & \overset{g^{-1}}{\rightarrow} & X \\
\downarrow \ell^{-1} & & \downarrow \ell^{-1} & & \downarrow \ell & & \downarrow \ell & & \downarrow \ell \\
X & \overset{g}{\rightarrow} & X & \overset{i_Y}{\rightarrow} & Y & \overset{p_Y}{\rightarrow} & X & \overset{g^{-1}}{\rightarrow} & X 
\end{array}
$$

which is equivalent to the diagram

$$
\begin{array}{cccccc}
X & \overset{g^{-1}}{\rightarrow} & X & \overset{i_Y}{\rightarrow} & Y & \overset{p_Y}{\rightarrow} & X & \overset{g}{\leftarrow} & X \\
\downarrow \ell^{-1} & & \downarrow \ell^{-1} & & \downarrow \ell & & \downarrow \ell & & \downarrow \ell \\
X & \overset{g}{\rightarrow} & X & \overset{i_Y}{\rightarrow} & Y & \overset{p_Y}{\rightarrow} & X & \overset{g^{-1}}{\rightarrow} & X 
\end{array}
$$

where the middle vertical map defines the $\ell$-action on $Y$. 

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**EQUIVARIANT A-THEORY**

33
Now, for \( \ell' = g\ell g^{-1} \), this diagram is equal to the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{iv} & Y \\
\downarrow \ell' & & \downarrow \ell' \\
X & \xrightarrow{iv} & Y
\end{array}
\]

which is the diagram that defines the \( \ell' \)-action on \( Y \). Therefore pullback functor \( f^* \) is naturally isomorphic to the functor that sends a retractive \( L' \)-space \( Y \) to the retractive \( L \)-space \( Y \) with action defined by letting \( \ell \in L \) act as \( \ell' = g\ell g^{-1} \in L' \). This functor clearly preserves cofibrations, weak equivalences and finite cell complexes, and therefore, so does \( f^* \), so we get an exact functor \( f^* : R_{L'}(X) \rightarrow R_L(X) \).

Now consider \( L \leq K \). Since the pushforward along \( f : G/L \rightarrow G/K \) is the left adjoint to the pullback, and left adjoints are unique up to natural isomorphism, it must induce on \( R^L(X) \rightarrow R^K(X) \) the left adjoint to the forgetful functor which restricts the group action from \( K \) to \( L \). On each retractive \( L \)-equivariant space \( Y \), this left adjoint \( f! \) is given by the pushout

\[
\begin{array}{ccc}
K \times_L Y & \xrightarrow{f!} & f!Y \\
\downarrow & & \downarrow \\
K \times_L X & \xrightarrow{f!} & X
\end{array}
\]

We recall that if \( H \leq K \) then the \( H \)-fixed points of \( K \times_L Y \) can be computed as

\[
(K \times_L Y)^H \cong \prod_{\{kL \in K/L \mid k^{-1}Hk \leq L\}} Y^{k^{-1}Hk}
\]

Since fixed points commute with pushouts along a closed inclusion, we get the pushout square

\[
\begin{array}{ccc}
\prod_{\{kL \in K/L \mid k^{-1}Hk \leq L\}} Y^{k^{-1}Hk} & \xrightarrow{(f!Y)^H} & (f!Y)^H \\
\downarrow & & \downarrow \\
\prod_{\{kL \in K/L \mid k^{-1}Hk \leq L\}} X^{k^{-1}Hk} & \xrightarrow{X^H} & X^H
\end{array}
\]

From this it is clear that if \( Y \rightarrow Z \) is a map of \( L \)-spaces giving an equivalence on all fixed points, it induces an equivalence of pushouts. Similarly, these constructions all commute up to isomorphism with mapping cylinder, so this construction preserves cofibrations. Finally we check that it preserves finite complexes by an induction on the number of cells. For the base case, we observe that if \( N \leq L \) is any subgroup, \( X \amalg (L/N \times D^n) \) is sent to \( X \amalg (K/N \times D^n) \), and similarly with \( S^{n-1} \) in the place of \( D^n \). Therefore cells are sent to cells. For the inductive step, we observe that each cell...
attaching diagram is sent to a cell attaching diagram, because by exactness the pushouts along cofibrations are preserved. Thus the pushforward of \( G/L \to G/K \) gives an exact functor \( f_! : R^L_{hf}(X) \to R^K_{hf}(X) \).

\[ \square \]

This establishes the first two conditions from Proposition 4.6. The remaining five conditions automatically descend from \( R(X)^{hH} \) to any a full subcategory with the same coproducts. Therefore the spectra \( K(R^H_{hf}(X)) \) form a spectral Mackey functor, so there exists a \( G \)-spectrum \( A_G(X) \) whose fixed points are \( A_G(X)^H \simeq K(R^H_{hf}(X)) \). It has been long known that the \( K \)-theory of the Waldhausen category \( R^H_{hf}(X) \) has a splitting

\[ K(R^H_{hf}(X)) \simeq \prod_{(H) \leq G} A(X^H_{hWH}). \]

A proof of this can be found in [BD16]. Therefore, we can conclude that the fixed points of the genuine \( G \)-spectrum \( A_G(X) \) have a tom Dieck type splitting:

\[ A_G(X)^H \simeq \prod_{(H) \leq G} A(X^H_{hWH}), \]

This finishes the proof of Theorem 4.1.

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