Nonaxisymmetric patterns in the linear theory of MHD Taylor-Couette instability

D.A. Shalybkov\textsuperscript{1,2}, G. Rüdiger\textsuperscript{1}, and M. Schultz\textsuperscript{1}

\textsuperscript{1} Astrophysikalisches Institut Potsdam, An der Sternwarte 16, D-14482 Potsdam, Germany
\textsuperscript{2} A.F. Ioffe Institute of Physics and Technology, 194021 St. Petersburg, Russia

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Abstract. The linear stability of MHD Taylor-Couette flow of infinite vertical extension is considered for various magnetic Prandtl numbers Pm. The calculations are performed for a wide gap container with $\hat{\eta} = 0.5$ with an axial uniform magnetic field excluding counterrotating cylinders. For both hydrodynamically stable and unstable flows the magnetorotational instability produces characteristic minima of the Reynolds number for certain (low) magnetic field amplitudes and Pm > 0.01. For Pm $\lesssim 1$ there is a characteristic magnetic field amplitude beyond which the instability sets in in form of nonaxisymmetric spirals with the azimuthal number $m = 1$. Obviously, the magnetic field is able to excite nonaxisymmetric configurations despite of the tendency of differential rotation to favor axisymmetric magnetic fields which is known from the dynamo theory. If Pm is too big or too small, however, the axisymmetric mode with $m=0$ appears to be the most unstable one possessing the lowest Reynolds numbers – as it is also true for hydrodynamic Taylor-Couette flow or for very weak fields. That the most unstable mode for modest Pm proves to be nonaxisymmetric must be considered as a strong indication for the possibility of dynamo processes in connection with the magnetorotational instability.

Key words. magnetohydrodynamics – accretion disks – turbulence

1. Introduction

In order to discuss possible experimental realizations of the magnetorotational instability as the main transporter of angular momentum in all kinds of accretion disks there are several recent studies of Taylor-Couette flow for electro-conducting fluids between rotating cylinders under the influence of an uniform axial magnetic field (Ji et al. 2001; Rüdiger & Zhang 2001; Willis & Barenghi 2002). The numbers describing the geometry of the container and the magnetic Prandtl number of the fluid have been considered as the free parameters. For a given magnetic field amplitude (the Hartmann number) the critical angular velocity of the inner cylinder (the critical Reynolds number) is computed for the onset of an instability of the rotation law between the cylinders.

In Rüdiger & Shalybkov (2002) the instability pattern is considered as axisymmetric. The main result for resting outer cylinder is that for high magnetic Prandtl number for weak magnetic field the excitation of the instability is easier than without magnetic field but for strong magnetic field the excitation of the instability is more complicated. The effect, however, disappears for small magnetic Prandtl number, i.e. for lower electric conductivity of the fluid as it may be realized in protoplanetary disks.

On the other hand for rotating outer cylinder, when no instability without magnetic field exists, the magnetic field always produces critical Reynolds numbers which, however, are running with $1/Pm$. For Pm of order $10^{-5}$ the critical Reynolds number is of order $10^{6}$ which is just the experimental limit.

In the present paper the nonaxisymmetric perturbations are included into the consideration. This is of particular relevance for the question whether the Cowling theorem for dynamo action can be fulfilled, after which a dynamo can only work with nonaxisymmetric fields. We shall find that indeed for certain parameters – despite the smoothing action of the differential rotation – nonaxisymmetric modes can be excited easier than axisymmetric modes. This is in great contrast to earlier results of Taylor-Couette flow without magnetic fields where always the axisymmetric modes possess the lowest Reynolds numbers (Roberts 1965; DiPrima 1961\textsuperscript{1}).

Here, the dependence of a real Taylor-Couette flow on the magnetic Prandtl number and on the azimuthal ‘quantum number $m$’ is investigated. The simple model of uniform density fluid contained between two vertically-infinite rotating cylinders is used with a constant magnetic field parallel to the ro-

\textsuperscript{1} For counterrotating cylinders, however, the preference of nonaxisymmetric modes is already known, see Krüger et al. 1966; Chen & Chang 1998.
2. Basic equations

The MHD equations which have to be solved are

$$\frac{\partial u}{\partial t} + (u \nabla) u = -\frac{1}{\rho} \nabla p + \nu \Delta u + J \times B$$  \hspace{1cm} (4)

and

$$\frac{\partial B}{\partial t} = \text{rot}(u \times B) + \eta \Delta B$$  \hspace{1cm} (5)

with $\text{div} u = \text{div} B = 0$. They are considered in cylindrical geometry with $R$, $\phi$, and $z$ as the cylindrical coordinates. A viscous electrically-conducting incompressible fluid between two rotating infinite cylinders in the presence of a uniform magnetic field parallel to the rotation axis leads to the basic solution $U_R = U_z = B_R = B_\phi = 0, B_z = B_0 = \text{const.}$, and $\Omega = a + b/R^2$, with $U$ as the flow and $B$ as the magnetic field. We are interested in the stability of this solution. The perturbed state of the flow may be described by $u'_R, u'_\phi, u'_z, B'_R, B'_\phi, B'_z, p'$, with $p'$ as the pressure perturbation.

Here only the linear stability problem is considered. By analyzing the disturbances into normal modes the solutions of the linearized magnetohydrodynamical equations are of the form

$$u'_R = u_R(R)e^{i(m\phi + kz - \omega t)}, \quad B'_R = B_R(R)e^{i(m\phi + kz - \omega t)}$$
$$u'_\phi = u_\phi(R)e^{i(m\phi + kz - \omega t)}, \quad B'_\phi = B_\phi(R)e^{i(m\phi + kz - \omega t)}$$
$$u'_z = u_z(R)e^{i(m\phi + kz - \omega t)}, \quad B'_z = B_z(R)e^{i(m\phi + kz - \omega t)}.$$  \hspace{1cm} (6)

Only marginal stability will be considered where the imaginary part of $\omega$ vanishes. Let $d = R_{\text{out}} - R_{\text{in}}$ be the gap between the cylinders. We use

$$H = (R_0d)^{1/2}$$  \hspace{1cm} (7)

as unit of length, the $\eta/H$ as unit of perturbed velocity and $B_0$ as unit of perturbed magnetic field with the magnetic Prandtl number

$$P_m = \frac{\nu}{\eta},$$  \hspace{1cm} (8)

$\nu$ is the kinematic viscosity, $\eta$ is the magnetic diffusivity. Note $H^{-1}$ as the unit of wave numbers and $\nu/H^2$ as the unit of frequencies. After elimination of both pressure fluctuations and the fluctuations of the vertical magnetic field, $B'_z$, the equations are

$$\frac{\partial u_R}{\partial R} + \frac{u_R}{R} + \frac{i m}{R} u_\phi + i k u_z = 0,$$  \hspace{1cm} (9)

$$\frac{\partial^2 u_\phi}{\partial R^2} + \frac{1}{R} \frac{\partial u_\phi}{\partial R} - u_\phi - \frac{m^2}{R^2} + k^2 u_\phi -$$
$$-i \left( m \text{Re} \frac{\Omega}{\Omega_{m}} - \omega \right) u_\phi + \frac{2 m}{R^2} u_R - \text{Re} \frac{1}{R} \frac{\partial}{\partial R} \left( R^2 \frac{\Omega}{\Omega_{m}} \right) u_R$$
$$- \frac{m}{k} \left( \frac{1}{R} \frac{\partial u_z}{\partial R^2} + \frac{1}{R^2} \frac{\partial u_z}{\partial R} - \frac{m^2}{R^2} + k^2 u_z \right) R$$
$$- i \left( m \text{Re} \frac{\Omega}{\Omega_{m}} - \omega \right) u_z + \frac{m}{k} \text{Ha}^2 \left( \frac{1}{R} \frac{\partial B_R}{\partial R} + \frac{B_R}{R} \right)$$
$$+ \frac{i}{k} \text{Ha}^2 \left( \frac{m^2}{R^2} + k^2 \right) B_\phi = 0,$$  \hspace{1cm} (10)
\[ \frac{\partial^3 u_z}{\partial R^3} + \frac{1}{R} \frac{\partial^2 u_z}{\partial R^2} - \frac{1}{R^2} \frac{\partial u_z}{\partial R} - \left( \frac{m^2}{R^2} + k^2 \right) \frac{\partial u_z}{\partial R} + \]
\[ + \frac{2m^2}{R^3} u_z - \frac{i}{m} \left( \frac{\partial u}{\partial \Omega} - \omega \right) \frac{\partial u}{\partial R} - i \frac{\partial \Omega}{\partial R} \left( \frac{\partial \Omega}{\partial m} \right) u_z \]
\[ - \text{Ha}^2 \left[ \frac{\partial^2 B_R}{\partial R^2} + \frac{1}{R} \frac{\partial B_R}{\partial R} - \frac{B_R}{R^2} - k^2 B_R^+ + \]
\[ + \left( \frac{i}{m} \frac{\partial B_\phi}{\partial R} - \frac{i}{m} B_\phi \right) \right] - \frac{1}{2} k^2 \frac{\partial u_R}{\partial R} + \frac{1}{R} \frac{\partial u_R}{\partial R} u_R - \]
\[ - \left( \frac{k^2 + m^2}{R^2} \right) u_R \right] \frac{m}{m} \left( \frac{\partial \Omega}{\partial \Omega} - \omega \right) u_R - \]
\[ - 2 \frac{ikm}{R^2} u_\phi - 2ik Re \frac{\Omega}{\Omega_m} u_\phi = 0, \tag{11} \]
\[ \frac{\partial^2 B_R}{\partial R^2} + \frac{1}{R} \frac{\partial B_R}{\partial R} - \frac{B_R}{R^2} - \left( \frac{m^2}{R^2} + k^2 \right) B_R^- \]
\[ - \frac{2im}{R^2} B_\phi - i \text{Pm} \left( \frac{\partial \Omega}{\partial \Omega} - \omega \right) B_R + i ku_R = 0, \tag{12} \]
\[ \frac{\partial^2 B_\phi}{\partial R^2} + \frac{1}{R} \frac{\partial B_\phi}{\partial R} - \frac{B_\phi}{R^2} - \left( \frac{m^2}{R^2} + k^2 \right) B_\phi^+ + \]
\[ + \frac{2im}{R^2} B_R - i \text{Pm} \left( \frac{\partial \Omega}{\partial \Omega} - \omega \right) B_\phi + i ku_\phi + \]
\[ + \text{Pm} Re \frac{\partial \Omega}{\partial R} \left( \frac{\partial \Omega}{\partial m} \right) B_R = 0. \tag{13} \]

The Reynolds number Re and the Hartmann number Ha are defined as
\[ \text{Re} = \frac{\Omega_m R^2}{\nu}, \quad \text{Ha} = \frac{B_0 H}{\sqrt{\mu_0 \mu_r \eta}}. \tag{14} \]

For given Hartmann number and Prandtl number in the present paper we shall derive in a linear theory the critical Reynolds number of the rotation of the inner cylinder for various mode numbers \( m \).

### 3. Boundary conditions, numerics

An appropriate set of ten boundary conditions is needed to solve the system \( \hat{\mathcal{L}} \). Always no-slip conditions for the velocity on the walls are used, i.e., \( u_R = u_\phi = du_R/\partial R = 0 \). The boundary conditions depend on the electrical properties of the walls. The transverse currents and the perpendicular component of the magnetic field vanish on conducting walls hence \( dB = du_R = 0 \). These boundary conditions hold both for \( R = R_{in} \) and for \( R = R_{out} \).

The homogeneous set of equations \( \hat{\mathcal{L}} \) with the boundary conditions for conducting walls determine the eigenvalue problem of the form \( \mathcal{L}(k, m, \text{Re}, \text{Ha}, \mathcal{R}(\omega)) = 0 \) for given \( \text{Pm} \). The real part of \( \omega \), \( \mathcal{R}(\omega) \), describes a drift of the pattern along the azimuth which only exists for nonaxisymmetric flows. \( \mathcal{L} \) is a complex quantity, both its real part and its imaginary part must vanish for the critical Reynolds number (Fig. 3). The latter is minimized by choice of the wave number \( k \). \( \mathcal{R}(\omega) \) is the second quantity which is fixed by the eigeneqation.

The system is approximated by finite differences with typically 81 gridpoints. The resulting determinant, \( \mathcal{L} \), takes the

### 4. Results for conducting walls

Only a container is considered in the present paper with one and the same gap geometry, i.e., \( \hat{\eta} = 0.5 \). Then the flow between the cylinders is hydrodynamically unstable between \( \hat{\mu} = 0 \) and \( \hat{\mu} = 0.25 \). We shall work with both the hydrodynamically unstable container with \( \hat{\mu} = 0 \) and with the hydrodynamically stable container with \( \hat{\mu} = 0.33 \).

#### 4.1. Resting outer cylinder (steep rotation law)

We start with the results for containers with resting outer cylinders (Fig. 3). Provided a critical rotation rate of the inner cylinder is exceeded they are hydrodynamically unstable. Of course, for \( \text{Ha} = m = 0 \) the known critical Reynolds number \( \text{Re} = 68 \) is reproduced. For \( m > 0 \) the critical Reynolds numbers exceed the value for \( m = 0 \). Without magnetic field the instability yields rolls. The critical Reynolds number for \( m = 1 \) is 73 and for \( m = 2 \) it is 101 (Roberts 1965).

With magnetic fields (\( \text{Ha} > 0 \)) the magnetic Prandtl number comes into the game. Results for \( \text{Pm} = 10, 1, 0.1 \) and 0.01 are presented in Fig. 3. For \( \text{Pm} \geq 1 \) the electrical conductivity is so high that the magnetorotational instability (Balbus & Hawley 1991; Brandenburg et al. 1995; Ziegler & Rüdiger 2000) for \( \text{Ha} \approx 5 \) produces a characteristic minimum of the critical Reynolds numbers but for stronger magnetic fields the suppressing action of the magnetic field starts to dominate. In contrast to the expectations, however, for the magnetic Prandtl numbers which are not too high and not too low the mode with \( m = 1 \) be-
comes more and more dominant. This is a new and interesting result: The linear instability of the Taylor-Couette flow without magnetic field is formed by axisymmetric rolls but the magnetic field favors the excitation of bisymmetric spirals. For $Ha > 10 \ldots 20$ the instability sets in in form of a drifting pattern with maximum and minimum separated by $180^\circ$. However, as can be seen in Fig. 3 (last plot) for small magnetic Prandtl number (here $Pm=0.01$) again the axisymmetric pattern with $m = 0$ again starts to dominate with the lowest critical Reynolds number.

The modes with $m = 2$, which we have also computed, do never possess the lowest Reynolds numbers, they are not important for the discussion of the pattern of the instability. What we have found is that in contrast to the hydrodynamic case ($Pm=0$) there are experimental combinations where the nonaxisymmetric mode with $m = 1$ has a lower Reynolds number than the axisymmetric mode with $m = 0$. This is one of the most surprising structure-forming consequences of the inclusion of magnetic fields to the Taylor-Couette flow experiment found first in astrophysical simulations.

4.2. Rotating outer cylinder (flat rotation law)

If the outer cylinder rotates with an angular velocity $\hat{\mu} \geq \hat{\eta}^2$ than the linear instability without magnetic field disappears and the critical Reynolds number for $Ha = 0$ moves to infinity. However, for finite Hartmann number (again of order 10) the instability survives practically for the same Reynolds numbers. The consequence is the occurrence of typical minima in the stability diagram (Fig. 4 for $\hat{\mu} = 0.33$).

The minima also occur for the nonaxisymmetric solutions with $m = 1$. For very high electrical conductivity ($Pm=10$) there seems to be no intersection between both the bifurcation profiles. The ring-like structure with $m = 0$ always possesses the lowest critical Reynolds number.

This is not true, however, for smaller magnetic Prandtl numbers, i.e. for lower electrical conductivity. For $Pm \lesssim 1$ we always find intersections between the lines for $m = 0$ and $m = 1$. Again there is a critical Hartmann number at which the ring geometry ($m = 0$) of the excited flow and field pattern changes to a nonaxisymmetric geometry with $m = 1$.

Hence, also in experiments with rotating outer cylinder the magnetic field is able to produce nonaxisymmetric structures. After the Cowling theorem which requires the existence of nonaxisymmetric magnetic modes for the existence of a dynamo, a selfexcited dynamo might thus exist, but only for certain magnetic Prandtl numbers, i.e. for $Pm \lesssim 1$. The magnetic Prandtl number for experiments with liquid metals like sodium or gallium with $Pm$ of order $10^{-(5\ldots6)}$ are still smaller than the considered values.

5. Wave number and drift frequencies

The wave numbers have been discussed in detail for the axisymmetric modes in a foregoing paper (Rüdiger & Shalybkov 2002). Generally, the cells for the nonaxisymmetric modes become more and more elongated in the vertical direction. Here we only add remarks about the drift velocity $\dot{\phi} = \frac{R(\omega)\Omega_{\alpha}}{mRe}$, so that for $m = 1$ the drift period in units of the rotation period runs as $Re/R(\omega)$. A typical value for this ratio is 2. In Fig. 5

\[ \dot{\phi} = \frac{G(\omega)}{mRe}, \] (15)

Fig. 3. Resting outer cylinder: Stability lines for axisymmetric ($m = 0$, solid lines) and nonaxisymmetric instability modes ($m = 1$ (dashed lines), $m = 2$ (dashed-dotted lines). Results are given for $Pm = 10$, 1, 0.1 and 0.01. Note that for $Pm = 1$ and for $Pm = 0.1$ for certain magnetic fields the nonaxisymmetric modes with $m = 1$ possess the lowest Reynolds numbers.
the expression (15) is given normalized with the rotation rate $\Omega_{\text{in}}$ for the solutions with $m = 1$ and for the two low magnetic Prandtl numbers that we have considered.

6. Conclusions

We have shown that a Taylor-Couette flow which is stable in the hydrodynamic regime ($\hat{\mu} \geq \hat{\eta}^2$) is destabilized by a weak axial magnetic field. Below a critical Hartmann number of order $10...100$ the instability sets in in form of axisymmetric rolls while above this value the instability forms nonaxisymmetric field and flow modes. This phenomenon exists despite of the observation (e.g. in dynamo theory) that differential rotation is known as suppressing the formation of nonaxisymmetric magnetic fields.

On the other hand, after the Cowling theorem of dynamo theory a magnetic field can only be maintained if it is nonaxisymmetric. Considering a number of typical magnetic Prandtl numbers we find that for our container with conducting cylinders the dominance of the nonaxisymmetric modes only occurs for not too high and not too low magnetic Prandtl number. Obviously, the dissipation processes are more important for nonaxisymmetric modes rather than axisymmetric modes. Hence the dissipation allows nonaxisymmetric modes only to be preferred if both the dissipation values have nearly the same order of magnitude.

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