Some Unusual Properties of Turbulent
Convection and Dynamos in Rotating Spherical
Shells

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1 Introduction

Convection of an electrically conducting fluid in a rotating system represents a basic
dynamical process in planetary interiors and in stars. Astrophysicists and geophysicists have long been interested in the mechanisms that govern the convective heat transport and the generation of magnetic fields by convection in those systems. The availability in recent years of large scale computer capacities has permitted numerical simulations of detailed models for those processes. Only the larger length scales can be taken into account in those computations, of course, and eddy diffusivities are usually introduced to model the influence of the smaller unresolved scales of the turbulent flows.

A difficulty arises from the fact that it cannot generally be assumed that all eddy diffusivities are equal. First, they apply to scalar as well as to vector quantities, such as temperature and magnetic fields. Secondly, the diffusivities in the absence of turbulence differ enormously such that the turbulence may not be sufficiently strong to equalize them. In the Earth’s liquid core, for instance, the magnetic diffusivity is large enough to be taken into account without the consideration of an eddy contribution while a comparable eddy viscosity would have to exceed the probable molecular viscosity value by a factor of at least $10^6$. As has been demonstrated in the past [1] the dynamics of convection in rotating spherical shells and its dynamo action are very sensitive to ratios of diffusivities, especially to the Prandtl number around its usually assumed value of unity.

More complex methods for treating effects of turbulence could eventually be used, such as $k-\varepsilon$-models, and the undoubtedly important anisotropy of turbulent
diffusivities in rotating systems could also be considered. Since these influences are
difficult to evaluate, however, we shall restrict the attention in this paper to a min-
imum of physical parameters. Inspite of this restriction a number of characteristic
features of convection and its dynamos in rotating spherical shells can be demon-
strated that are likely to exist as coherent spatio-temporal structures in natural sys-
tems.

2 Mathematical formulation of the problem and methods of
solution

We consider a rotating spherical fluid shell of thickness \(d\) and assume that a static
state exists with the temperature distribution \(T_S = T_0 - \frac{\beta d^2}{r^2}\). Here \(rd\) is the
length of the position vector with respect to the center of the sphere. The grav-
ity field is given by \(g = -\frac{d \gamma}{r}\). In addition to \(d\), the time \(\frac{d^2}{\nu}\), the temperature
\(\frac{\nu^2}{\gamma \alpha d^4}\) and the magnetic flux density \(\frac{\nu (\mu \rho)^{1/2}}{d}\) are used as scales for the di-
imensionless description of the problem where \(\nu\) denotes the kinematic viscosity of
the fluid, \(\kappa\) its thermal diffusivity, \(\rho\) its density and \(\mu\) is its magnetic permeability.

Since we shall assume the Boussinesq approximation material properties are re-
garded as constants except for the temperature dependence of the density described
by \(\alpha \equiv -\frac{(d \rho / dT)}{\rho}\) which is taken into account only in the gravity term. Both,
the velocity field \(v\) and the magnetic flux density \(B\), are solenoidal vector fields for
which the general representation

\[
v = \nabla \times (\nabla u \times r) + \nabla w \times r, \quad B = \nabla \times (\nabla h \times r) + \nabla g \times r, \tag{1}
\]

can be employed. By multiplying the (curl)\(^2\) and the curl of the Navier-Stokes equa-
tions of motion by \(r\) we obtain two equations for \(u\) and \(w\),

\[
[(\nabla^2 - \partial_t) L_2 + \tau \partial_\phi] \nabla^2 u + \tau Q w - L_2 \Theta = -r \cdot \nabla \times [\nabla \times (v \cdot \nabla v - B \cdot \nabla B)], \tag{2}
\]

\[
[(\nabla^2 - \partial_t) L_2 + \tau \partial_\phi] w - \tau Q u = r \cdot \nabla \times (v \cdot \nabla v - B \cdot \nabla B), \tag{3}
\]

where \(\partial_t\) denotes the partial derivative with respect to time \(t\) and where \(\partial_\phi\) is the
partial derivative with respect to the angle \(\phi\) of a spherical system of coordinates
\(r, \theta, \phi\). For further details we refer to [11]. The operators \(L_2\) and \(Q\) are defined by

\[
L_2 \equiv -\nu^2 \nabla^2 + \partial_r (r^2 \partial_r),
\]

\[
Q \equiv r \cos \theta \nabla^2 - (L_2 + r \partial_r) (\cos \theta \partial_r - r^{-1} \sin \theta \partial_\theta).
\]

The heat equation for the dimensionless deviation \(\Theta\) from the static temperature
distribution can be written in the form

\[
\nabla^2 \Theta + R L_2 u = P (\partial_t + v \cdot \nabla) \Theta, \tag{4}
\]
and the equations for $h$ and $g$ are obtained through the multiplication of the equation of induction and of its curl by $r$,

\begin{align}
\nabla^2 \mathcal{L}_2 h &= P_m [\partial_t \mathcal{L}_2 h - r \cdot \nabla \times (v \times B)], \\
\nabla^2 \mathcal{L}_2 g &= P_m [\partial_t \mathcal{L}_2 g - r \cdot \nabla \times (\nabla \times (v \times B))].
\end{align}

The Rayleigh number $R$, the Coriolis number $\tau$, the Prandtl number $P$ and the magnetic Prandtl number $P_m$ are defined by

\begin{align}
R &= \frac{\alpha \gamma \beta d^6}{\nu \kappa}, \quad \tau = \frac{2 \Omega d^2}{v}, \quad P = \frac{v}{\kappa}, \quad P_m = \frac{v}{\lambda},
\end{align}

where $\lambda$ is the magnetic diffusivity. For the static temperature distribution we have chosen the case of a homogeneously heated sphere. This state is traditionally used for the analysis of convection in self-gravitating spheres and offers the numerical advantage that for Rayleigh numbers close to the critical value $R_c$ the strength of convection does not differ much near the inner and outer boundaries. As the Rayleigh number increases beyond $R_c$, heat enters increasingly at the inner boundary and is delivered by convection to the outer boundary. When $R$ reaches a high multiple of $R_c$ the heat generated internally in the fluid becomes negligible in comparison to the heat transported by convection from the inner to the outer boundary.

Fixed temperatures and stress-free boundaries,

$$u = \partial^2_{rr} u = \partial_r (w/r) = \Theta = 0 \quad \text{at } r = r_i \equiv \eta/(1 - \eta) \text{ and } r = r_o \equiv 1/(1 - \eta),$$

will be assumed where $\eta$ denotes radius ratio, $\eta = r_i/r_o$. In the following only $\eta = 0.4$ will be used. For the magnetic field electrically insulating boundaries are assumed such that the poloidal function $h$ must be matched to the function $\hat{h}$ which describes the potential fields outside the fluid shell.
Fig. 2 Time series of energy densities of convection for \( P = 1, \tau = 10^{4} \) and \( R = 2.8 \times 10^{5}, 3.0 \times 10^{5}, 3.5 \times 10^{5}, 7 \times 10^{5}, 12 \times 10^{5} \), (from top to bottom). Solid and dashed lines indicate \( E_t \) and \( \tilde{E}_t \), respectively. The Nusselt number \( Nu_i \) is indicated by dotted lines and measured at the right ordinate.

\[
g = h - \hat{h} = \partial_r (h - \hat{h}) = 0 \quad \text{at} \quad r = r_i = \eta/(1 - \eta) \quad \text{and} \quad r = r_o = 1/(1 - \eta). \quad (9)
\]

But computations for the case of an inner boundary with no-slip conditions and an electrical conductivity equal to that of the fluid have also been done [11]. The numerical integration of the equations together with the boundary conditions proceeds with the pseudo-spectral method as described in [13] and [14] which is based on an expansion of all dependent variables in spherical harmonics for the \( \theta, \phi \)-dependences, i.e.

\[
u = \sum_{l,m} U_l^m (r,t) P_l^m (\cos \theta) \exp \{im\phi\} \quad (10)
\]

and analogous expressions for the other variables, \( w, \Theta, h \) and \( g \). \( P_l^m \) denotes the associated Legendre functions. For the \( r \)-dependence expansions in Chebychev polynomials are used. For further details see also [11]. For the non-magnetic convection calculations to be reported in the following a minimum of 33 collocation points in the radial direction and spherical harmonics up to the order 64 have been used. The resolution has been increased to 41 or 55 collocation points and spherical harmonics up to the order 96 or 128 in the case of dynamo simulations.
Fig. 3 Localized convection for $R = 7 \times 10^5$, $\tau = 1.5 \times 10^4$, $P = 0.5$. The streamlines, $r \partial u / \partial \phi = \text{const.}$ (upper row) and the isotherms, $\Theta = \text{const.}$ (lower row), are shown in the equatorial plane for equidistant times (from left to right) with $\Delta t = 0.05$.

3 Convection in rotating spherical shells

For an introduction to the problem of convection in spherical shells we refer to the review [2] and to the respective chapter in the book [5]. Additional information can be found in the papers by Grote and Busse [6], Jones et al. [7], Christensen [4] and Smitnev and Busse [10]. Typically the onset of convection occurs in the form of progradely propagating thermal Rossby waves as illustrated in figure 1. Only for low Prandtl numbers $P$, i.e. $P < 10 / \sqrt{\tau}$ according to [1], the onset occurs in the form of inertial waves attached to the outer equatorial boundary of the fluid shell as is discussed in [15], [16], [11].

Because of the symmetry of the velocity field with respect to the equatorial plane it is sufficient to plot streamlines in this plane, given by $r \partial u / \partial r = \text{const.}$, to characterize the convection flow. This has been done in figures 3 and 5. Even in the case of turbulent convection the part of the velocity field that is antisymmetric with respect to the equatorial plane is rather small as long as the parameter $\tau$ is sufficiently large.

As the Rayleigh number $R$ grows beyond its critical value $R_c$, the thermal Rossby waves become modified by a sequence of bifurcations similar to those found in other problems of convection. First, oscillations of the amplitude are observed, then another bifurcation causes a low wavenumber modulation as function of the azimuth [10]. Finally, a chaotic state of convection is obtained.
4 Chaotic Convection

The sequence of transitions can also be visualized through the time dependence of average quantities such as the contributions to the kinetic energy density. These are defined by

\[
\bar{E}_p = \langle |\nabla \times (\nabla \bar{u} \times \mathbf{r})|^2 \rangle / 2, \quad \bar{E}_i = \langle |\nabla \bar{w} \times \mathbf{r}|^2 \rangle / 2 \tag{11a}
\]

\[
\tilde{E}_p = \langle |\nabla \times (\nabla \tilde{u} \times \mathbf{r})|^2 \rangle / 2, \quad \tilde{E}_i = \langle |\nabla \tilde{w} \times \mathbf{r}|^2 \rangle / 2 \tag{11b}
\]

where the angular brackets indicate the average over the fluid shell and where \( \bar{u} \) refers to the azimuthally averaged component of \( u \) and \( \tilde{u} \) is given by \( \tilde{u} = u - \bar{u} \). At low supercritical Rayleigh numbers the energy densities corresponding to steadily drifting thermal Rossby waves are constant in time and are not included in figure 2. The onset of vacillations manifests itself in the sinusoidal oscillations of the kinetic energies as shown in the top plot of figure 2. Also plotted in figure 2 is the Nusselt number \( Nu_i \) measuring the efficiency of the convective heat transport at the inner boundary,

\[
Nu_i = 1 - \frac{P_r}{\frac{d\Theta}{dr}} \bigg|_{r=r_i} \tag{12}
\]

where the double bar indicates the average over the spherical surface. \( \bar{E}_i \) describes the energy density of the differential rotation which increases strongly with increasing \( R \) as can be noticed in the lower plots of figure 2. This increase is caused by the strong azimuthal Reynolds stress exerted by the convection eddies resulting from their inclination with respect to radial direction as is apparent in figure 1. The increasing shear of the differential rotation tends to inhibit convection, however, in that it shears off the convection eddies. This is a consequence of the nearly two-dimensional nature of the dynamics in a rotating system: Because of the requirement that the structure of convection approaches closely the Taylor-Proudman condition, there is no possibility for a reorientation of the convection rolls as happens in non-rotating systems. In the rotating sphere convection thus generates the agent that tends to destroy it. A precarious balance in the form localized convection is the result. As shown in figure 3 convection occurs only in a restricted azimuthal section of the spherical shell where its amplitude is strong enough to overcome the inhibiting influence of the shear. For the geostrophic zonal flow it does not matter whether it is driven locally or more uniformly around the azimuth. Since in the non-convecting region thermal buoyancy accumulates, the advection of this buoyancy by the differential rotation strengthens and stabilizes the localized convection.

At even higher Rayleigh numbers this balance no longer works and instead of a localization in space the localization of convection in time is initiated as is shown in figures 4 and 5. Here convection exist only for a short period while the differential rotation is sufficiently weak. As the amplitude of convection grows, the differential rotation grows even more strongly since the Reynolds stress increases with the square of the amplitude. Soon the shearing action becomes strong enough to cut off
The energy densities $E_t$ (solid line), $E_r$ (dashed line), $E_p$ (dotted line) and the Nusselt number (dash-dotted line, right ordinate) are shown as function of time.

Convection. Now a viscous diffusion time must pass before the differential rotation has decayed sufficiently such that convection may start growing again. It is remarkable to see how the chaotic system exhibits its nearly periodic relaxation oscillations as shown in figure 4.

In figure 5 a sequence of plots is shown at four instances around the time of a convection peak. At first there is hardly any convection, - the dotted lines just indicate zero. At the next instance the differential rotation as shown by the upper row has decayed sufficiently such that convection columns can grow reaching nearly their maximum amplitude in the third plot. At the same time the differential rotation has grown as well and begins to exert its inhibiting effect such that convection decays at the fourth instance of the sequence, while the differential rotation reaches its maximum. It should be mentioned that localized convection and relaxation oscillations occur at moderate Prandtl numbers of the order unity or less. At higher values of $P$ Reynolds stresses are no longer sufficiently powerful to generate a strong differential rotation. Instead variations of the temperature field caused by the dependence of the convective heat transport on latitude induce a differential rotation in the form of a thermal wind.

The convective heat transport in the case of localized convection as well as in the case of relaxation oscillations is much reduced, of course, relative to a case without strong differential rotation. This causes the magnetic field to enter the problem in a crucial way provided the electrical conductivity is sufficiently high. By putting brakes on the differential rotation through its Lorentz force the magnetic field permits a much higher heat transport than would be possible in an electrically-insulating fluid. This is the basic reason that the Earth’s core as well as other planets with convecting cores and rotating stars exhibit magnetic fields. A demonstration of this effect is seen in figure 6 where by chance the convection driven dynamo was just marginal such that it could not recover after a downward fluctuation of the magnetic field. Hence the relaxation oscillations with their much reduced average heat flux take over from the dynamo state.
Fig. 5 Sequence of plots starting at $t = 2.31143$ and equidistant in time ($\Delta t = 0.01$) for the same case as in Fig. 8. Lines of constant $\bar{u}_\phi$ and streamlines $r \sin \theta \partial_\theta \bar{h} = \text{const.}$ in the meridional plane, are shown in the left and right halves, respectively, of the upper row. The lower row shows corresponding streamlines, $r \partial u/\partial \phi = \text{const.}$, in the equatorial plane.

Fig. 6 Transition from a dynamo state to a state of chaotic relaxation oscillations for $\tau = 1.5 \times 10^4, R = 1.2 \times 10^6, P = P_m = 0.5$. The energy density $E_t$ (dotted line), the total magnetic energy density (dashed line) and the Nusselt number $Nu_i$ (solid line, right ordinate) are shown as function of time.

5 Distinct turbulent dynamos at identical parameter values

Convection driven dynamos in rotating spherical fluid shells are often subcritical as is apparent in figure 6, for instance. At somewhat higher Rayleigh numbers convection with a strong magnetic field will persist. On the other hand the dynamo will decay when the magnetic field is artificially reduced to, say, a quarter of its averaged energy. There thus exists the possibility of a convection driven dynamo state and of a non-magnetic convection state at identical values of the external parameters $R, \tau, P$. 
Fig. 7 Two distinct dynamos at identical parameter values, $\tau = 5 \times 10^3, R = 5 \times 10^5, P = P_m = 1$. Upper (lower) plots show time series of quadrupolar (dipolar) magnetic energy densities. Thick lines indicate mean toroidal (solid lines) and poloidal (dashed lines) energy components. Thin lines indicate the same for the fluctuating components.

The bistable coexistence of a non-magnetic convection state and a dynamo state is typical for subcritical bifurcations as in the analogous coexistence of laminar and turbulent states in shear flows.

More surprising is the fact that two different turbulent dynamo states can exist at identical values of the external parameters which now should include the magnetic Prandtl number $P_m$. An example is shown in figure 7 where a spherical dynamo of mixed parity, i.e. with dipolar and quadrupolar components, and a purely quadrupolar dynamo evolve in time at identical external parameters. In the latter case the exponential decay of dipolar disturbances is clearly demonstrated. In figure 7 magnetic energy densities have been plotted that are defined in analogy to expressions (12),

$$\overline{M}_q = \langle | \nabla \times (\nabla h \times \mathbf{r}) |^2 \rangle / 2, \quad \overline{M}_t = \langle | \nabla g_t \times \mathbf{r} |^2 \rangle / 2 \quad (13a)$$

$$\overline{\tilde{M}}_q = \langle | \nabla \times (\nabla \tilde{h}_q \times \mathbf{r}) |^2 \rangle / 2, \quad \overline{\tilde{M}}_t = \langle | \nabla \tilde{g}_t \times \mathbf{r} |^2 \rangle / 2. \quad (13b)$$

An even more surprising case is that of two convection driven dynamos without any distinction in symmetry, just with differences in the magnitude of various energy densities as is apparent from figure 8. While a strong mean poloidal magnetic field as shown in the left half of figure 8 acts as an efficient brake on the differential rotation
Fig. 8 Time series of two different chaotic attractors are shown - a MD (left column (a,b)) and a FD dynamo (right column (c,d)) both in the case $R = 3.5 \times 10^6$, $\tau = 4 \times 10^4$, $P = 0.5$ and $P_m = 1$. The top two panels (a,c) show magnetic energy densities, and the bottom two panels (b,d) show kinetic energy densities in the presence of the magnetic field. The components $M_p$ are shown by thick solid black lines, while $X_t$, $\tilde{X}_p$, and $\tilde{X}_t$ are shown in red green and blue respectively. $X$ stands for either $M$ or $E$.

Fig. 9 The upper row shows the hysteresis effect in the ratio of magnetic to kinetic energy, $E/M$, at $\tau = 3 \times 10^4$ (a) as a function of the Prandtl number in the case of $R = 3.5 \times 10^6$, $P/P_m = 0.5$; (b) as a function of the ratio $P/P_m$ in the case of $R = 3.5 \times 10^6$, $P = 0.75$ and (c) as a function of the Rayleigh number in the case $P = 0.75$, $P_m = 1.5$. Full and empty symbols indicate FD and MD dynamos, respectively and circles and squares indicate the two hysteresis branches. The critical value of $R$ for the onset of thermal convection for the cases shown in (c) is $R_c = 659145$. A transition from FD to MD dynamos as $P/P_m$ decreases in (b) is expected, but is not indicated owing to lack of data. The lower row shows the value $Nu_t$ of the Nusselt number at $r = r_t$ for the same dynamo cases. Values for non-magnetic convection are indicated by triangles for comparison.
as measured by $\mathcal{E}_t$, it also inhibits convection. The alternative dynamo on the right side of the figure is characterized by a relatively weak mean magnetic field and dominant fluctuating components. Here the kinetic energy densities of convection are larger, but the differential rotation is still much weaker than it would be in the non-magnetic case.

The bistability in the form of two different types of dynamos is not a singular phenomenon, but exists over an extended region of the parameter space. Since the region includes values of the Prandtl number of the order unity which has been the preferred value of $P$ in many simulations of convection driven dynamos the phenomenon of bistable dynamos is of considerable importance. The first results of our extensive numerical simulations can be found in [12]. A typical diagram is shown in figure 9. The extended regime of coexistence of the two types of dynamo is bounded by transitions where one of the two dynamos ceases to be stable and evolves into the other one within a magnetic diffusion time. A basic reason for the competitiveness of both dynamos is that they exhibit essentially the same convective heat transport as measured by the Nusselt number $N_u$. The lower half of figure 9 not only demonstrates the surprising coincidence of the heat transports of the two dynamo types, but also indicates that these heat transports by far exceed those found in the absence of a magnetic field.

Recently the computations have been extended to higher rotation rates as shown in figure 10. At $\tau = 4 \times 10^4$ the same phenomenon of bistability has been found as at $\tau = 3 \times 10^4$. Again, the two kinds of dynamo exhibit the same heat transport as indicated in the lower part of figure 10.
6 Concluding Remarks

The existence of two distinct turbulent states is a rare phenomenon, although examples exist in non-magnetic hydrodynamics, see, for instance, [9] [8]. In magnetohydrodynamics an electrically conducting fluid in the presence of a magnetic field offers new degrees of freedom which allow more than a single balance between the various forces operating in turbulent states. Initial conditions thus determine which of the competing states is actually realized.

The possibility of bistability could be of interest for the interpretation of planetary and stellar magnetism. Magnetic hysteresis effects associated with stellar oscillations may eventually be explained in this way. Anyone involved with numerical simulations of convection driven dynamos should be aware that his solutions could change drastically after different initial conditions have been introduced.

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