We generalize recent study of the stability of isotropic (spherical) rotating membranes to the anisotropic ellipsoidal membrane. We find that while the stability persists for deformations of spin \( l = 1 \), the quadrupole and higher spin deformations \((l \geq 2)\) lead to instabilities. We find the relevant instability modes and the corresponding eigenvalues. These indicate that the ellipsoidal rotating membranes generically decay into finger-like configurations.

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I. INTRODUCTION

The eleven dimensional classical and quantum super-membrane is one of the poorly understood elements of M-Theory \( \square \). The study of this sector is important both for the understanding of the strongly coupled string theories as well as for the quantization of the supermembrane. Recent interest for the classical solutions of the Matrix theory representing \( D_0 \)-branes attached to spherical membranes is explained as a first step to a formulation of Matrix theory in weak external gravitational and gauge backgrounds \( \square \). Particular solutions of the classical matrix equations representing rotating ellipsoidal configurations of \( N D_0 \) branes attached to a membrane which exhibit stability properties have been proposed and their semiclassical spectrum has been studied \( \square \).

In a recent paper \( \square \) we found that the isotropic (spherical) rotating membrane is stable under all modes of small multipole deformations and we found explicitly the corresponding spectrum and the eigenmodes. Here we generalize this study by finding in detail the stability properties of the rotating ellipsoidal membrane which is an inhomogeneous solution of the bosonic part of the supermembrane equations restricted to six spatial dimensions. We find that while the stability persists for deformations of spin \( l = 1 \), the higher spin deformations \((l \geq 2)\) lead to instabilities. We find the relevant modes and the corresponding eigenvalues.

It is well known that spherical membrane solutions are isomorphic \( \square \). Moreover the linearized problems for the fluctuations preserve the same isomorphism due to the specific spin 1 form of the solution. The matrix solution is a bound state of \( N D_0 \) branes attached on a \( D_2 \) brane whose stability properties is obtained using our continuous membrane investigation. Indeed one only has to replace the perturbations \( \delta X_i = \sum_m \epsilon_i^m Y_{lm} \) by the matrix fluctuations \( \delta \hat{X}_i = \sum_m \epsilon_i^m \hat{Y}_{lm} \) where \( \hat{Y}_{lm} \) are the \( SU(2) \) tensor spherical harmonics \( \square \).

The isomorphism with the stability analysis of matrix solutions persists also in the anisotropic case. The interpretation of the instability modes implies that the ellipsoidal membranes generically decay into finger-like configurations.

We now make a quick review of the formalism for the bosonic sector of the theory relevant to the present work. After fixing the gauge and using reparametrization invariance of the Nambu-Gotto Lagrangian we find that the eqs of motion for the 9 bosonic coordinates \( X_i(t, \sigma_1, \sigma_2) \), \( i = 1, 2, \ldots, 9 \) in the light cone frame are:

\[
\dot{X}_i = \{X_k, \{X_k, X_i\}\} \quad (1.1)
\]

where the Poisson bracket of two functions, \( f \) and \( g \) on \( S^2 \) is defined as

\[
\{f, g\} = \frac{\partial f}{\partial \cos \theta} \frac{\partial g}{\partial \phi} - \frac{\partial f}{\partial \phi} \frac{\partial g}{\partial \cos \theta} \quad (1.2)
\]

and the remaining area preserving symmetry generated by the constraint

\[
\{X_i, \dot{X}_i\} = 0 \quad (1.3)
\]

In the matrix model the above coordinates are replaced by \( N \times N \) Hermitian and traceless matrices and the corresponding equations of motion and constraint are found by exchanging Poisson brackets with commutators.

The first connection between the \( SU(N) \) Suss Yang-Mills truncation of the supermembrane with the recent nonperturbative studies of string theories was discovered by Witten \( \square \) representing the Yang-Mills mechanics as a low energy effective theory of bound states of \( N D_0 \) branes. The \( D_0 \) branes carry RR charge. Now it is understood how to couple the \( SU(N) \) matrix model with weak background fields either directly using supergravity arguments or truncating supermembrane Lagrangians in weak background fields \( \square \). There is an expectation that taking appropriate limits of \( N \to \infty \) for special bound states of \( N D_0 \) branes one could recover the supermembrane or its magnetic dual, the super-five brane \( \square \).
In the next section we turn our attention to the analysis of the stability properties of specific classical solutions which are spherical rotating membranes. Recent work in the matrix model presented such a time dependent solution representing a bound system of $N D_0/D_2$ branes.

II. STABILITY

The equation of motion for the supermembrane in six dimensions may be written as

$$\ddot{X}_i = \{X_j, \{X_j, X_i\}\}$$

(2.1)

where summation is implied in the $j$ indices and $\{\}$ stands for the Poisson bracket with respect to the angular coordinates $\theta, \phi$. The Gauss constraint that also needs to be satisfied is

$$\left\{ \ddot{X}_i, X_i \right\} = 0$$

(2.2)

where $i, j = 1, 2, 6$. We now define $Y_i \equiv X_{i+3}$ with $i = 1, 2, 3$. This constraint is preserved by the equations of motion and therefore if it is initially obeyed (as is the case in what follows) it will be obeyed at all times. The equations of motion are

$$\ddot{X}_i = \{X_j, \{X_j, X_i\}\} + \{Y_j, \{Y_j, X_i\}\}
$$

$$\ddot{Y}_i = \{X_j, \{X_j, Y_i\}\} + \{Y_j, \{Y_j, Y_i\}\}$$

(2.3)

We now use the ansatz of a rotating spherical membrane in analogy with the matrix membrane ansatz given in [6]:

$$X_i = r_i(t) e_i(\theta, \phi)$$

$$Y_i = s_i(t) e_i(\theta, \phi)$$

(2.4)

where the generators $e_i(\theta, \phi)$ are defined as

$$e_1 = \sin \theta \sin \phi$$

$$e_2 = \sin \theta \sin \phi$$

$$e_3 = \cos \theta$$

(2.5)

satisfy the relations

$$\left\{ e_i, e_j \right\} = -\epsilon_{ijk} e_k$$

(2.6)

Using now the ansatz (2.4) in the equations of motion (2.3) we obtain the differential equations obeyed by the functions $r(t)$, $s(t)$

$$\ddot{r}_i = -(r^2 + s^2 - r_i^2 - s_i^2) r_i$$

(2.7)

$$\ddot{s}_i = -(r^2 + s^2 - r_i^2 - s_i^2) s_i$$

(2.8)

where $r^2 = r_1^2 + r_2^2 + r_3^2$, $s^2 = s_1^2 + s_2^2 + s_3^2$. A particular class of solutions of (2.7) is of the form

$$r_i = R_i \cos(\omega_i t + \phi_i)$$

(2.9)

$$s_i = R_i \sin(\omega_i t + \phi_i)$$

(2.10)

with

$$\omega_i^2 = R^2 - R_i^2$$

(2.11)

where $R^2 = R_1^2 + R_2^2 + R_3^2$.

We observe that all the relations we obtained for the ansatz (2.4) are identical with those of ref. [6] for the matrix model solution of a bound state of $N D_0/D_2$-branes where the three functions $e_i(\theta, \phi)$ are replaced by $N$-dimensional representational matrices $J_i (i = 1, 2, 3)$ of $SU(2)$. This unique isomorphism is due to the existence of an $SU(2)$ subgroup of the infinite dimensional area preserving group of the sphere ($Dif f(S^2)$). It is known that there is no other finite dimensional subalgebra of ($Dif f(S^2)$). As we shall see the stability analysis of the anisotropic ($R_i \neq R_j$) membrane solution follows an isomorphic pattern with the matrix model solution. We point out that in ref. [6] the anisotropic ($R_i \neq R_j$) matrix solution was found to be stable under a restricted set of the $l = 1$ perturbations. In the following we extend their analysis for every value of $l$ and we complete also the case $l = 1$. The variational equations that correspond to the splitting in eq. (4.3) between $X_i$ and $Y_i$ are:

$$\delta X_i = \{\delta X_j, \{X_j, X_i\}\} + \{X_j, \{\delta X_j, X_i\}\}
$$

$$+ \{X_j, \{X_j, \delta X_i\}\} + \{\delta X_j, \{Y_j, X_i\}\}
$$

$$+ \{Y_j, \{\delta X_j, X_i\}\} + \{Y_j, \{Y_j, \delta X_i\}\}$$

(2.12)

The corresponding perturbation for $\delta Y_i$s and $Y_i$s satisfy equations that are obtained by exchanging $\delta X_i \leftrightarrow \delta Y_i$, $X_i \leftrightarrow Y_i$ in eq (2.12). The equations of motion imply the validity of the constraint at all times

$$\{\ddot{X}_i, X_i\} + \{\ddot{Y}_i, Y_i\} = 0$$

(2.13)

This is obtained by taking the time derivative of eq. (2.13) and by applying the equations of motion and the Jacobi identity. By expanding a configuration which at $t = 0$ is consistent with the constraint (2.13) around any classical solution we see (by using only the linearized eqs. (2.12)) that the variation $\delta X_i$ and $\delta Y_i$ satisfy the constraint

$$\{\delta \ddot{X}_i, X_i\} + \{\ddot{X}_i, \delta X_i\} + \{\delta Y_i, Y_i\} + \{\ddot{Y}_i, \delta Y_i\} = 0$$

(2.14)

for all times.

In order to study the stability of this solution we consider the following general form of perturbations

$$\delta X_i(t) = \sum_{l,m} \zeta_{lm}^i(t) Y_{lm}(\theta, \phi)$$

(2.15)

$$\delta Y_i(t) = \sum_{l,m} \zeta_{lm}^i(t) Y_{lm}(\theta, \phi)$$
We now use the fact that
\[ \{e_i, Y_{lm}(\theta, \phi)\} = i\hat{L}_i Y_{lm}(\theta, \phi) \quad (2.16) \]
where \( \hat{L}_i \) is the angular momentum differential operator in spherical coordinates. This implies that
\[ \{e_i, Y_{lm}(\theta, \phi)\} = \sum_{m'} a_{lm'} Y_{lm'}(\theta, \phi) = i \sum_{m'} (L_i)_{mm'} Y_{lm'} \quad (2.17) \]
where \( L_i \) are the angular momenta in the representation \( l = (N - 1)/2 \). A crucial observation is that the sum involves spherical harmonics of the same \( l \) as the spherical harmonic in the Poisson bracket. This decouples the various \( l \) fluctuation modes and simplifies the differential equations obeyed by the modes \( \epsilon_i \) as well as the gauge constraint. This feature is specific to the particular background solution of the spherical membrane.

The equations obeyed by the fluctuation modes \( \epsilon \) and \( \zeta \) may be written as
\[ \dot{\epsilon}_i + L_R^2 \epsilon_i = \cos \omega_i t T_{ij}[\epsilon_j \cos \omega_j t + \zeta_j \sin \omega_j t] \]
\[ \dot{\zeta}_i + L_R^2 \zeta_i = \sin \omega_i t T_{ij}[\epsilon_j \cos \omega_j t + \zeta_j \sin \omega_j t] \quad (2.18) \]
where repeated indices are summed and the \( 3(2l + 1) \times 3(2l + 1) \) matrix \( T_{ij} \) is defined as
\[ T_{ij} = R_i R_j (L_i L_j - 2i \epsilon_{ijk} L_k) \quad (2.19) \]
\( L_i \) is the angular momentum operator and \( L_R^2 \equiv R_1^2 L_1^2 + R_2^2 L_2^2 + R_3^2 L_3^2 \).

We now perform a rotation and define the new variables \( \theta_i \) and \( \eta_i \)
\[ \theta_i \equiv \epsilon_i \cos \omega_i t + \zeta_i \sin \omega_i t \quad (2.20) \]
\[ \eta_i \equiv -\epsilon_i \sin \omega_i t + \zeta_i \cos \omega_i t \quad (2.21) \]
The equations obeyed by the \( 3(2l + 1) \) vectors \( \Theta = (\theta_i) \), \( H = (\eta_i) \) may now be shown to be
\[ \ddot{\Theta} - 2\Omega \dot{H} + [L_R^2 - \Omega^2] \Theta - T\Theta = 0 \quad (2.22) \]
\[ \dot{H} + 2\Omega \dot{\Theta} + [L_R^2 - \Omega^2] H = 0 \]
where \( \Omega = (\omega_i \delta_{ij}) \), \( T = (T_{ij}) \) and from the equation of motion of the background solution we have \( \omega_i^2 = R_1^2 + R_2^2 + R_3^2 \) (with all cyclic rotations of indices).

Perturbations of the classical solutions along the \( 7, 8, 9 \) dimensions can be parametrized as
\[ \delta Z_i = \delta X_{i+6} \quad (2.23) \]
with \( i = 1, 2, 3 \) and
\[ \delta Z_i = \{X_j, \{X_j, \delta Z_i\}\} + \{Y_j, \{Y_j, \delta Z_i\}\} \quad (2.24) \]
With the definition
\[ \delta Z_i(t) = \sum_{l,m} q_{lm}^i(t) Y_{lm}(\theta, \phi) \quad (2.25) \]
we obtain
\[ \ddot{q}_i = -L_R^2 q_i \quad (2.26) \]
which implies stability along the \( 7, 8, 9 \) dimensions since \( L_R^2 \) is positive definite.

In order to study the stability of the system \( (2.22) \) we must convert it to an eigenvalue problem. Assuming \( \Theta = e^{i\lambda t} a \) and \( H = e^{i\lambda t} b \) we obtain
\[ \begin{pmatrix} M a \\ (L_R^2 - \Omega^2) b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix} \quad (2.27) \]
This is a complicated \( 6(2l + 1) \) dimensional quadratic eigenvalue problem for which there is no standard mathematical theory. It can be formulated as a noncommutative quadratic equation (with matrix coefficients) \([14]\). In what follows we shall bypass this problem applying a method introduced by N. Papanicolaou \([14]\) and which transforms the problem to a linear eigenvalue one.

We define the new variables \( \Theta_1, \Theta_2, H_1 \) and \( H_2 \) as follows:
\[ \dot{\Theta}_1 = \Theta_2 \quad (2.29) \]
\[ H_1 = H_2 \quad (2.30) \]
Thus the system \( (2.22) \) may be written as
\[ \dot{\Theta}_2 - 2\Omega H_2 + [L_R^2 - \Omega^2] \Theta_1 - T \Theta_1 = 0 \quad (2.31) \]
\[ \dot{H}_2 + 2\Omega \Theta_2 + [L_R^2 - \Omega^2] H_1 = 0 \quad (2.32) \]

or
\[ \dot{X} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\Omega^2 - T & 0 & 0 & 2\Omega \\ 0 & -(L_R^2 - \Omega^2) & -2\Omega & 0 \end{pmatrix} X \quad (2.33) \]
where
\[ X = \begin{pmatrix} \Theta_1 \\ H_1 \\ \Theta_2 \\ H_2 \end{pmatrix} = e^{i\lambda t} \begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{pmatrix} = e^{i\lambda t} A \quad (2.34) \]
Thus we are led to the eigenvalue problem
\[ i \lambda A = M A \]  
(2.35)

where \( M \) is defined by eq. (2.33). The membrane solution is stable if and only if all the eigenvalues \( \lambda \) are real. The matrix \( M \) is a well defined matrix of dimension \( 12(2l + 1) \times 12(2l + 1) \) and therefore it is straightforward to solve the eigenvalue problem (2.35) numerically for any \( l \).

The stability analysis for the isotropic case was performed in ref \[7,12\] and the eigenvalues along with their degeneracies are shown in Table I

In this case it is obvious that all \( \lambda^2 \) are non-negative and therefore the eigenfrequencies \( \lambda \) are all real. This implies that the isotropic membrane solution studied is stable to first order in perturbation theory.

The 2\( l + 1 \) zero modes are due to gauge degrees of freedom and survive in the anisotropic case. For the gauge zero modes the corresponding constant vectors \( \theta_i, \eta_i \) are \( \theta_i = R_i L_i v, \eta_i = 0 \) for any 2\( l + 1 \) dimensional vector \( v \). These modes satisfy the equation

\[ (L_R^2 - \Omega^2 - T)_{ij} \theta_j = 0 \]  
(2.36)

The physical modes must satisfy the constraint equation (2.14) which may be written as

\[ R_i L_i (2 \omega_i \eta_i - \dot{\theta}_i) = 0 \]  
(2.37)

where summation is implied in \( i \).

We have checked numerically the expected orthogonality between the physical modes and the gauge zero modes. The modes orthogonal to the gauge zero modes satisfy the constraint equations i.e are physical.

| Degeneracy | \( \lambda^2 \) | \( \lambda^2 \) |
|------------|----------------|----------------|
| 2\( l + 1 \) | 0 | \( l^2 + l + 6 \) |
| 2\( l + 3 \) | \( l^2 - 3l + 2 \) | \( l^2 + 3l + 2 \) |
| 2\( l - 1 \) | \( l^2 - l \) | \( l^2 - 5l + 6 \) |

TABLE I. The eigenvalues along with their degeneracy for the isotropic membrane stability problem.

| TABLE II. The maximum imaginary eigenvalue for various degrees of anisotropy (we have set \( R_1 = R_2 = 1 \) and \( R_3 = 1 - \epsilon \)). |
|-----------------|----------------|----------------|----------------|
| \( \epsilon = 1 \) | \( \epsilon = 0.9 \) | \( \epsilon = 0.4 \) | \( \epsilon = 0.1 \) |
| \( l = 1 \) | 0 | 0 | 0 |
| \( l = 2 \) | 0 | 0.44 | 0.92 | 0.99 |
| \( l = 3 \) | 0 | 0 | 1.16 | 1.4 |

This can be proven by taking the inner product of the left hand side of the constraint equation (2.37) with an arbitrary 2\( l + 1 \) dimensional vector \( v \) and using the properties of the gauge zero modes discussed above.

In the general complete analysis of the anisotropic membrane, for \( l = 1 \), we find that all modes are stable in agreement with ref \[12\] where only some of the \( l = 1 \) modes were studied. However we have found unstable modes for \( l \geq 2 \) and therefore we conclude that the membrane configuration is unstable in this case. In particular, we have found that the instability for \( l = 2 \) persists for all values of non-zero anisotropy. However, for \( l > 2 \) there is a minimum anisotropy below which we have stability of the corresponding modes. In Table II we show the imaginary part of the most unstable eigenvalue for various \( l \) values as a function of the anisotropy while in Fig. 1 we show the dependence of the most unstable zero modes on the anisotropy.

The derived instability of the anisotropic membrane is reminiscent of the Bohr atom where the classical electron spirals towards the nucleus emitting electromagnetic radiation. Thus the stability analysis we have performed has implications for the semiclassical quantization of the supermembrane indicating that there is an additional principle required in order to stabilize quantum mechanically the rotating anisotropic membranes against emission of spikes and D0 branes. In a later publication we will present a detailed description of the findings presented here.
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