TWO-DIMENSIONAL ABELIAN $BF$ THEORY IN LORENZ GAUGE AS A TWISTED $\mathcal{N} = (2, 2)$ SUPERCONFORMAL FIELD THEORY

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Abstract. We study the two-dimensional topological abelian $BF$ theory in the Lorenz gauge and, surprisingly, we find that the gauged-fixed theory is a free type B twisted $\mathcal{N} = (2, 2)$ superconformal theory with odd linear target space, with the ghost field $c$ being the pullback of the linear holomorphic coordinate on the target. The $Q_{\text{BRST}}$ of the gauge-fixed theory equals the total $Q$ of type B twisted theory. This unexpected identification of two different theories opens a way for nontrivial deformations of both of these theories.

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Foreword

The note is designed as a self-contained exposition, with relevant background on (super)conformal field theory included in the text for the reader’s convenience. We refer to the sources [5, 7, 4] for details.

For the readers who are well acquainted with superconformal field theory, we suggest to look at the formula (7) and then read the subsections 1.2.1 and 2.7.1 for the quick gist of the story.

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1. Classical theory

We consider the abelian BF theory on a two-dimensional oriented surface Σ defined classically by the action functional

\[ S_0 = \int_\Sigma B dA \]

where the fields are a 1-form A and a 0-form B on Σ; d is the de Rham operator. The equations of motion read \( dA = 0, dB = 0 \) and the theory has gauge symmetry \( A \mapsto A + d\alpha, B \mapsto B \) with the 0-form \( \alpha \) being the generator of the gauge transformation.

1.1. Gauge-fixed model in BRST formalism. We impose the Lorenz gauge condition \( d^* A = 0 \) where \( d^* = - \ast d \ast \) is the Hodge dual of the de Rham operator associated to a choice of a Riemannian metric \( g \) on Σ, with \( \ast \) the Hodge star. The corresponding gauge-fixed action (the Faddeev-Popov action) is:

\[ S = \int_\Sigma B dA + \lambda d A + b d \ast dc \]

with scalar field \( \lambda \) the Lagrange multiplier imposing the Lorenz gauge condition and \( b, c \) the Faddeev-Popov ghosts – the odd scalar fields. Thus, the space of BRST fields of the model is:

\[ \mathcal{F} = \Omega^1_A \oplus \Omega^0_B \oplus \Omega^0_\lambda \oplus \Pi \Omega^0_b \oplus \Pi \Omega^0_c \]

with \( \Pi \) the parity reversal symbol. Equivalently, it is the space of sections \( \mathcal{F} = \Gamma(\Sigma, F) \) of the super vector bundle

\[ F = T^\ast \Sigma \oplus \mathbb{R}^2 \oplus \Pi \mathbb{R}^2 \]

over Σ, with last two terms the trivial even and odd rank 2 bundles\(^1\). The BRST operator acts as

\[ Q : \quad A \mapsto dc, \quad b \mapsto \lambda, \quad B, c, \lambda \mapsto 0 \]

One clearly has \( Q^2 = 0 \) and \( Q(S) = 0 \) – the gauge-invariance of the action. Also, the gauge-fixed action differs from \( S_0 \) by a \( Q \)-exact term:

\[ S = S_0 + Q(\Psi) \]

---

\(^1\) One can enhance the \( \mathbb{Z}_2 \)-grading on fields to a \( \mathbb{Z} \)-grading by the “ghost number”, where \( b, c \) are assigned degrees \(-1\) and \(+1\), respectively, and \( A, B, \lambda \) are assigned degree \(0\).
with

\[ \Psi = \int_{\Sigma} b \, d * A \]

the gauge-fixing fermion.

Equations of motion for the gauge-fixed action \( (2) \) read

\[ dA = 0, \quad d * A = 0, \quad dB - *d\lambda = 0, \quad \Delta b = 0, \quad \Delta c = 0 \]

with \( \Delta = *d *d \) the Laplacian acting on functions on \( \Sigma \).

1.2. Rewriting the action in terms of complex fields, conformal invariance. It is convenient to split the 1-form field \( A \) into its \((1, 0)\) and \((0, 1)\)-components

\[ A = a + \bar{a}, \]

where the splitting \( \Omega^1 = \Omega^{1,0} \oplus \Omega^{0,1} \) is inferred from the complex structure on \( \Sigma \) compatible with the chosen metric (in particular, \( \Omega^{1,0} \) and \( \Omega^{0,1} \) are the \(-i\) and \(i\)-eigenspaces for the Hodge star). We also combine the real scalar fields \( B, \lambda \) into a complex scalar field \( \gamma := \frac{1}{2}(\lambda + iB) \) with conjugate \( \bar{\gamma} = \frac{1}{2}(\lambda - iB) \).

Action \( (2) \) can then be written as

\[ S = 2i \int_{\Sigma} -\gamma \partial a + \bar{\gamma} \partial \bar{a} + b \partial \bar{\partial} c \]

with \( \partial, \bar{\partial} \) the holomorphic and anti-holomorphic Dolbeault operators (written as \( \partial = dz \partial_z, \bar{\partial} = d\bar{z} \partial_{\bar{z}} \) in local complex coordinates \( z, \bar{z} \); we reserve the non-boldface symbols \( \partial = \frac{\partial}{\partial z}, \bar{\partial} = \frac{\partial}{\partial \bar{z}} \) for the partial derivatives themselves). Written in this form, the action is manifestly dependent only on the complex structure induced by the metric \( g \), i.e. only on the conformal class of the metric \( g \) modulo Weyl transformations \( g \sim \Omega \cdot g \) with \( \Omega \) any positive function on \( \Sigma \). Thus the gauge-fixed abelian \( BF \) theory is conformal. The BRST operator \( Q \) written in terms of the new fields reads:

\[ Q : \quad a \mapsto \partial c, \quad \bar{a} \mapsto \bar{\partial} c, \quad b \mapsto \gamma + \bar{\gamma}, \quad \gamma, \bar{\gamma}, c \mapsto 0 \]

and the equations of motion are:

\[ \bar{\partial} a = 0, \quad \bar{\partial} \gamma = 0, \quad \partial a = 0, \quad \partial \bar{\gamma} = 0, \quad \partial \bar{\partial} b = 0, \quad \bar{\partial} \bar{\partial} c = 0 \]

Note that fields \( \gamma, \bar{\gamma} \) are more adapted to the action and the equations of motion, whereas fields \( B, \lambda \) are more adapted to the BRST operator.

Remark 1.1. Using local coordinates \( x^1, x^2 \) on \( \Sigma \) (such that \( g \) is in the conformal class of \((dx^1)^2 + (dx^2)^2 \)) and the corresponding complex coordinates \( z = x^1 + ix^2, \bar{z} = x^1 - ix^2 \), we can write \( a = dz a, \bar{a} = d\bar{z} \bar{a} \) with \( a, \bar{a} \) scalars. Then the action \( (7) \) reads

\[ S = 4 \int d^2x \, (\gamma \partial a + \bar{\gamma} \partial \bar{a} + b \partial \bar{\partial} c) \]

with \( d^2x = dx^1 dx^2 = \frac{1}{i} dz d\bar{z} \) the coordinate area form. In our conventions, for the fields \( a, \bar{a} \) one gets a sign in BRST transformations, \( Q : \quad a \mapsto -\partial c, \quad \bar{a} \mapsto -\bar{\partial} c \).

The BRST symmetry defines, via the Noether theorem, a current

\[ J_{\text{tot}} = 2i(\gamma \partial c - \bar{\gamma} \bar{\partial} c) = 2i \left( dz \gamma \partial c - d\bar{z} \bar{\gamma} \bar{\partial} c \right) \]
It is conserved modulo equations of motion: $d J^\text{tot}_{\text{e.o.m.}} \sim 0$. In fact, one has a stronger statement that both chiral parts of the current are conserved independently: $\bar{\partial} J \sim 0$ and $\partial J \sim 0$.

1.2.1. Abelian BF theory as a twisted superconformal field theory: an anticipation.

It is remarkable – see section [2.7] for details – that action (7) is a free type B twisted $\mathcal{N} = (2, 2)$ superconformal theory where the parity of the fields is changed (so scalars are fermions while the first order systems are constructed with bosons). The “holomorphic field” is just the Faddev-Popov ghost. As we will show later the $Q_{\text{BRST}}$ becomes a sum of two scalar charges, as usual in the twisted theory; their currents have changed the dimension from 3/2 to 1 in both holomorphic and antiholomorphic sectors. This unexpected property allows to pose questions in $BF$ theory that where prohibited by the naive understanding of allowed correlators – in particular, it is possible to study correlators of some gauge non-invariant observables, like the superpartner (BRST-primitive) of the energy-momentum tensor (see below). On the other hand non-abelian $BF$ theory in Lorenz gauge may serve as an example of a new conformal field theory (this question is clear classically and we will return to this important question on the quantum level in subsequent work).

1.3. Stress-energy tensor and its BRST-primitive. The stress-energy tensor is defined via the variational derivative of the action (2) with respect to metric. Explicitly, in a coordinate chart on $\Sigma$, one defines $T_{\mu \nu}$ via

$$
\delta g S = - \int_\Sigma \sqrt{\det g} \, d^2 x \, T_{\mu \nu} \delta g^{\mu \nu}
$$

where the left hand side is the variation w.r.t. the metric $g$; indices $\mu, \nu$ take values in $\{1, 2\}$ or $\{z, \bar{z}\}$. The total stress-energy tensor $T_{\text{tot}} = T_{\mu \nu} \, dx^\mu \cdot dx^\nu$ is a section of the symmetric square of the cotangent bundle of $\Sigma$; the dot stands for the symmetric tensor product in $\text{Sym}^* T^* \Sigma$. Note that $S$ depends on the metric only via the dependence of the gauge-fixing fermion $\Psi$ on the metric, entering via the Hodge star. Thus, from (13) we have that the components of the stress-energy tensor are exact w.r.t. the BRST operator,

$$
T_{\mu \nu} = Q G_{\mu \nu}
$$

where $G_{\mu \nu}$ is defined, similarly to (12), via

$$
\delta g \Psi = - \int_\Sigma \sqrt{\det g} \, d^2 x \, G_{\mu \nu} \delta g^{\mu \nu}
$$

Explicitly, in holomorphic coordinates $z, \bar{z}$, one obtains $\Psi$ (13)

$$
G_{\text{tot}} = (dz)^2 \left( \frac{\partial \bar{\partial} b}{G} + (d\bar{z})^2 \frac{\bar{\partial} \partial b}{G} \right) = a \cdot \partial b + \bar{a} \cdot \bar{\partial} b
$$

\begin{footnote}
\footnote{Here is the computation in local coordinates: Hodge star acts on 1-forms via $* dx^\mu = \sqrt{g} g^{\mu \nu} \epsilon_{\nu \rho} dx^\rho$ with $\epsilon_{\nu \rho}$ the Levi-Civita symbol and $\sqrt{g}$ the shorthand notation for $\sqrt{\det g}$. Variation w.r.t. the metric is thus $\delta g * dx^\mu = \sqrt{g} g^{\mu \nu} \epsilon_{\nu \rho} \delta g_{\rho \sigma} d^2 x^\rho$. Next, we use this to compute the variation of the gauge-fixing fermion: $\delta g \Psi = \int db \wedge \delta g * d A = \int \sqrt{g} \, db \wedge (\sqrt{g} g^{\mu \nu} \epsilon_{\nu \rho} \delta g_{\rho \sigma} + \delta g^{\rho \sigma} \epsilon_{\nu \rho}) dx^\rho \cdot A_{\rho} \delta g^{\rho \sigma}$. Note that the coefficient of $\sqrt{g} \delta g^{\rho \sigma}$ in the integrand is manifestly invariant under Weyl transformations $g \rightarrow \Omega \cdot g$. In particular, we can compute this coefficient in holomorphic coordinates $z, \bar{z}$, using the standard metric $g = dz \cdot d\bar{z}$. One obtains $\delta g \Psi = - \int d^2 x \wedge (db \cdot \partial g^{\rho \sigma} + \bar{d} b \cdot \bar{\partial} g^{\rho \sigma});$ reading off the coefficients of the variation of the metric, we obtain (13).}
\end{footnote}
and

\[ T_{\text{tot}} = QG_{\text{tot}} = (dz)^2 (-\partial_c \partial_b + a \partial \lambda) + (dz)^2 (-\partial_c \bar{\partial}b + \bar{a} \bar{\partial} \lambda) \]

\[ \frac{T_{z=\bar{z}}}{2} = (\partial_c \cdot \partial b + a \cdot \partial \lambda) + (\bar{\partial} c \cdot \bar{\partial} b + \bar{a} \cdot \bar{\partial} \lambda) \]

The components \( G_{z\bar{z}}, T_{z\bar{z}} \) vanish (equivalently, the traces \( G^α_μ, T^α_μ \) vanish), which is a manifestation of the conformal invariance of the theory. The stress-energy tensor and its primitive are conserved modulo equations of motion:

\[ \bar{\partial} G \sim 0, \quad \bar{\partial} G \sim 0; \quad \bar{\partial} T \sim 0, \quad \bar{\partial} T \sim 0 \]

2. Quantum Abelian BF Theory as a Conformal Field Theory

From now on we specialize to the case of the surface \( \Sigma \) being the plane \( \mathbb{R}^2 = \mathbb{C} \) with coordinates \( x^1, x^2 \) (or the complex coordinate \( z = x^1 + ix^2 \) and its conjugate \( \bar{z} = x^1 - ix^2 \)), endowed with the standard Euclidean metric \( g = (dx^1)^2 + (dx^2)^2 = dz \cdot d\bar{z} \).

2.1. Correlation functions.

We are interested in studying the normalized correlation functions

\[ \langle \Phi_1(z_1) \cdots \Phi_n(z_n) \rangle := \frac{1}{Z} \int_\mathcal{F} e^{-\frac{1}{4\pi} S} \Phi_1(z_1) \cdots \Phi_n(z_n) \]

Here:

- \( \Phi_1(z_1), \ldots, \Phi_n(z_n) \) are composite fields\(^3\): polynomials in the fields \( a, \bar{a}, B, \lambda, b, c \) and their derivatives of arbitrary order, evaluated at pairwise distinct points \( z_1, \ldots, z_n \in \mathbb{C} \).
- \( S \) is the action\(^1\).
- The normalization factor \( Z := \int_\mathcal{F} e^{-\frac{1}{4\pi} S} \) is the partition function. The r.h.s. of \((14)\) is a ratio of path integrals over \( \mathcal{F} \) which individually need a regularization (both ultraviolet and infrared) to be defined. However the ratio is independent of the regularization.
- The normalization factor \( \frac{1}{Z} \) in the exponential in the r.h.s. of \((14)\) is introduced to have a convenient normalization of propagators.

Since the action \( S \) is quadratic, the theory is free and the correlators are given by Wick’s lemma with the following basic propagators\(^4\).

\[ \langle c(w) b(z) \rangle = 2 \log |w - z| + C, \quad \langle a(w) \gamma(z) \rangle = \frac{1}{w - z}, \quad \langle \bar{a}(w) \bar{\gamma}(z) \rangle = \frac{1}{\bar{w} - \bar{z}} \]

\(^3\) We are using this terminology to emphasize the distinction between the “basic” BRST fields and the objects that can be used as decorations of punctures on \( \Sigma \) when calculating correlation functions. We call the latter composite fields. Another possible term is “observables”, or “0-observables” (though often one reserves the word “observable” only for the \( Q \)-closed expressions in fields).

\(^4\) One constructs the propagator \( \langle c(z) b(w) \rangle \) from the action\(^1\) as the Green’s function (the integral kernel of the inverse operator) for the operator \( \frac{1}{12\pi} \Delta \). Similarly, \( \langle a(w) \gamma(z) \rangle \) is the Green’s function for \( \frac{1}{2} \partial \). We implicitly fix the zero-mode for the Dolbeault operator by requiring that the fields \( a, \gamma, \bar{a}, \bar{\gamma} \) vanish at infinity; in other words we are considering the theory on the compactified plane \( \mathbb{C}P^1 \) relative to the point \( \{ \infty \} \). To fix the zero-mode of the Laplacian we do an infrared regularization by replacing the pair \( (\mathbb{C}P^1, \{ \infty \}) \) with a disk of large radius \( R \) relative to the boundary, i.e., we impose the Dirichlet boundary conditions on \( b, c \). The constant shift \( C \) in the propagator \( \langle c(w) b(z) \rangle \) depends on the infrared cut-off, \( C = -2 \log R \).
Propagators for all other pairs of fields from the set \( \{a, \bar{a}, \gamma, \bar{\gamma}, b, c\} \) vanish. In terms of fields \( A, B, \lambda \) this implies
\[
(A(w) B(z)) = 2 d_w \arg(w-z), \quad (A(w) \lambda(z)) = 2 d_w \log|w-z|
\]

When constructing the correlators (14) for composite fields by Wick’s lemma, we do not allow matchings of two basic fields in the same composite field \( \Phi_i \) (which would have led to an ill-defined expression) – this corresponds to the assumption that the composite fields \( \Phi_i \) are normally ordered.

Note also that if one of the fields \( \Phi_i \) vanishes modulo equations of motion (9), the correlator (14) vanishes identically.

2.2. The space of composite fields. One can formalize the notion of a composite field by considering the symmetric powers of the jet bundle of \( \text{Jet}_s^* = \mathbb{C} \otimes \text{Jet}_s^* \) – the complexified bundle dual to the bundle of BRST fields, and then taking a quotient by the ideal generated by the equations of motion (9) and their derivatives:
\[
\mathcal{F}_z := \text{Sym}^* \text{Jet}_s^*/\text{c.o.m.}
\]

It is a graded complex vector bundle over \( \Sigma \) with \( \mathbb{Z} \)-grading given by the ghost number (by assigning degree +1 to \( c \) and degree -1 to \( b \) and zero to all other basic fields). Thus, a composite field \( \Phi(z) \), regarded modulo equations of motion, is an element of the fiber \( \mathcal{F}_z \). Fibers of \( \mathcal{F}_z \) are differential graded commutative algebras, with the differential given by the BRST operator \( Q \). The n-point correlator (14) can then be regarded as bundle morphism from \( \iota^*(\mathcal{F}_z \otimes \cdots \otimes \mathcal{F}_z) \) to the trivial line bundle over \( \text{Conf}_n(\Sigma) \) – the (open) configuration space of \( n \) pairwise distinct ordered points on \( \Sigma \); here \( \iota: \text{Conf}_n(\Sigma) \to \Sigma^{\times n} \) is the tautological inclusion.

Explicitly, the space of composite fields \( \mathcal{F}_z \) is a free graded commutative algebra generated by the fields
\[
b, c; \{\partial^k b\}_{k \geq 1}, \{\partial^k c\}_{k \geq 1}, \{\partial^k a\}_{k \geq 0}, \{\partial^k \gamma\}_{k \geq 0};
\]

\[
\text{holomorphic sector}
\]

\[
\{\bar{\partial}^k b\}_{k \geq 1}, \{\bar{\partial}^k c\}_{k \geq 1}, \{\bar{\partial}^k a\}_{k \geq 0}, \{\bar{\partial}^k \gamma\}_{k \geq 0};
\]

\[
\text{anti-holomorphic sector}
\]

The BRST differential \( Q \) is a derivation defined on the generators by (8) together with the rule \( Q(D \phi) = DQ(\phi) \) for any differential operator \( D = \partial^k \bar{\partial}^l \) and \( \phi \) a basic field, and then extended to the whole \( \mathcal{F}_z \) by Leibniz identity.

The cohomology of \( Q \) acting on \( \mathcal{F}_z \) is calculated straightforwardly and yields the subalgebra of \( \mathcal{F}_z \) generated by fields \( B = \frac{\bar{\gamma}}{\gamma} \) and \( c \) (but not their derivatives)\(^5\)
\[
\mathcal{O}_z := H^\bullet_Q(\mathcal{F}_z) = \mathbb{C}[B, c]
\]

We denote the \( Q \)-cohomology by \( \mathcal{O}_z \) (for “observables”). Note that it is concentrated in degrees 0 and 1 only, since polynomials in \( c \) have degree at most 1. However, one can consider an \( N \)-component abelian \( BF \) theory, i.e., \( N \) non-interacting

\(^5\) Here is the calculation: denote by \( Y \) the linear span of the generators (17). Note that, by freeness of the theory, \( Q \) acts on \( Y \) as a differential and \( Q \) on \( \mathcal{F}_z \) is the extension of this action by Leibniz identity. Thus \( H^\bullet_Q(\mathcal{F}_z) = H^\bullet(\text{Sym} \ Y) = \text{Sym} \ H^\bullet(Y) \). To compute \( H^\bullet(Y) \), notice that \( Q \) maps \( \partial^{k-1} \bar{a} \to -\partial^k \bar{c}, \bar{\partial}^{k-1} \bar{a} \to -\bar{\partial}^k c, \partial^k \bar{b} \to \partial^k \bar{\gamma}, \bar{\partial}^k b \to \bar{\partial}^k \gamma \) for all \( k \geq 1 \). Thus we can remove the acyclic subcomplex spanned by these generators out of \( Y \) and we have \( H^\bullet(Y) = H^\bullet \text{Sym}(\bar{c}, c, \gamma, \bar{\gamma}) = H^\bullet \text{Sym}(\bar{b}, c, B, \lambda) \). Finally, since \( Q \) maps \( b \to \lambda \) and vanishes on \( B, c \), we have \( H^\bullet(Y) = \text{Sym}(\bar{b}, c) \) and hence \( H^\bullet_Q(\mathcal{F}_z) = \text{Sym} \text{Sym}(\bar{b}, c) \).
copies of the theory; in other words, one replaces the fiber of the bundle of BRST fields with
\[ F \mapsto F^{[N]} := F \otimes \mathbb{R}^N \]
(we will use the superscript \([N]\) when we want to emphasize that we work with \(N\)-component theory). In this case, there are \(N\) odd generators \(c^i\) and the polynomials in them can have degree up to \(N\) and thus the cohomology \(\mathcal{O}_{z}^{[N]} := H_{c}(\mathbb{F}^{[N]}) = \mathbb{C}[B_{1}, \ldots, B_{N}, c^{1}, \ldots, c^{N}]\) is spread across degrees \(0, 1, \ldots, N\).

One can geometrically interpret the space of observables as the space of polyvector fields
\[ \mathcal{O}^{[N]}_z = T_{\text{poly}}(\Pi V) \]
on the odd space \(\Pi V\) where \(V = \mathbb{R}^N\) is the space of coefficients in the \(N\)-component theory. Here the ghosts \(c^i\) are interpreted as coordinates on the base \(\Pi V\) and \(B_j = \frac{\partial}{\partial c^j} \in \Pi T_{c}(\Pi V)\) are interpreted as tangent vectors.

### 2.3. Operator product expansions

We are interested in analyzing the singular part of the asymptotics of the correlator
\[ \langle \Phi_1(w)\Phi_2(z)\phi_1(z_1)\cdots\phi_n(z_n) \rangle \]
as the point \(w\) approaches \(z\), with \(\phi_1(z_1), \ldots, \phi_n(z_n)\) being the “test fields”. This asymptotics is controlled by the operator product expansion (OPE) of the fields \(\Phi_1\) and \(\Phi_2\) which is an expression of the form
\[ \Phi_1(w)\Phi_2(z) \sim \sum_{j=1}^P f_j(w-z)\tilde{\Phi}_j(z) + \text{reg.} \]
with \(\tilde{\Phi}_j\) some fields and \(f_j\) some singular coefficient functions, typically a product of negative powers of \((w-z)\) and \((\bar{w}-\bar{z})\) and can also contain \(\log(w-z)\) and \(\log(\bar{w}-\bar{z})\); \text{reg.} stands for terms which are regular as \(w \to z\); the number \(P\) of singular terms depends on \(\Phi_1, \Phi_2\). Thus, \((22)\) means that in the correlator \((21)\) one can replace the first two fields with the expression on the r.h.s. of \((22)\), reducing the number of points by one.

**Example 2.1.** For instance, we have
\[ a(w)\gamma(z) \sim \frac{1}{w-z} + (a\gamma)_z + (w-z)\cdot(\partial a\gamma)_z + \frac{1}{2}(w-z)^2 \cdot (\partial^2 a\gamma)_z + \cdots \sim \frac{1}{w-z} + \text{reg.} \]
and
\[ a(w)\tilde{\gamma}(z) \sim (a\tilde{\gamma})_z + (w-z) \cdot (\partial a\tilde{\gamma})_z + \frac{1}{2}(w-z)^2 \cdot (\partial^2 a\tilde{\gamma})_z + \cdots \sim \text{reg.} \]
For brevity we put the point where the field is evaluated as a subscript (i.e. \(\Phi(z) = \Phi_z\)). Note that the OPE \(a(w)\tilde{\gamma}(z)\) is purely regular: correlators containing this pair of fields have a well-defined limit as \(w \to z\), whereas a correlator containing \(a(w)\gamma(z)\) will have a first order pole as \(w \to z\).

\[ \text{In terms of } \mathbb{Z}_2\text{-grading: } \mathcal{O}^{[N]}_z = T_{\text{poly}}(V[1]) \text{ – sections of the symmetric powers of the shifted tangent bundle } T[-1](V[1]) \text{ over } V[1], \text{ or, equivalently, functions on the graded space } T^{*}[1]V[1]. \]
Generally, since we are dealing with a free theory, the OPE \( \Phi_1(w) \Phi_2(z) \) for two generic composite fields is constructed by Wick’s lemma. In particular, for \( \Phi_1, \Phi_2 \) two monomials in (derivatives of) basic fields, the recipe for OPE is as follows. We consider all partial matchings of basic fields in \( \Phi_1 \) against basic fields in \( \Phi_2 \). For each matching, we replace matched pairs of basic fields by their propagators (acted on by respective derivatives in \( w, \bar{w}, z, \bar{z} \) that were acting on those basic fields). We multiply the result with the unmatched basic fields (acted on by respective derivatives), while also replacing basic fields evaluated at \( w \) with their Taylor expansion around \( z \). Finally, we sum these contributions over all partial matchings.

Using this recipe one obtains the following OPEs of distinguished fields – the stress-energy tensor \( T = \partial_b \partial_c + a \partial_a \lambda, \) its BRST primitive \( G = a \partial b \) and the BRST current \( J = \gamma \partial c \) (and their anti-holomorphic counterparts \( \bar{T}, \bar{G}, \bar{J} \)):

\[
T(w)T(z) \sim \frac{2T(z)}{(w-z)^2} + \frac{\partial T(z)}{w-z} + \text{reg.} \quad (23)
\]

\[
J(w)G(z) \sim -\frac{1}{(w-z)^3} + \frac{(\gamma a)z}{(w-z)^2} + \frac{T(z)}{w-z} + \text{reg.} \quad (24)
\]

\[
T(w)G(z) \sim \frac{2G(z)}{(w-z)^2} + \frac{\partial G(z)}{w-z} + \text{reg.}
\]

\[
T(w)J(z) \sim \frac{J(z)}{(w-z)^2} + \frac{\partial J(z)}{w-z} + \text{reg.}
\]

and their complex conjugates. Also, one has \( G(w)G(z) \sim \text{reg.} \) and \( J(w)J(z) \sim \text{reg.} \). All OPEs between holomorphic objects \( T, G, J \) and anti-holomorphic objects \( \bar{T}, \bar{G}, \bar{J} \) are regular. In particular, from the absence of 4-th order pole in (23), we see that the central charge of the theory vanishes \( c = 0 \). Also, we see that \( G \) and \( J \) are primary fields with conformal weights \( (2,0) \) and \( (1,0) \) respectively.

**Example 2.2.** For example, here is the computation of the OPE (24). By (15), we have the propagators \( \langle \gamma_w a_z \rangle = \frac{1}{w-z} \) and \( \langle (\partial c)_w (\partial b)_z \rangle = \partial_w \partial_z \langle c_w b_z \rangle = \partial_w \partial_z 2 \log |w-z| = \frac{1}{(w-z)^2} \). In the OPE \( \langle \gamma \partial c_w (a \partial b)_z \rangle \) one gets three singular terms from the Wick contractions of either \( \gamma_w \) with \( a_z \) or \( (\partial c)_w \) with \( (\partial b)_z \) or both. Thus,

\[
(\gamma \partial c)_w (a \partial b)_z \sim \frac{-1}{w-z} \cdot \frac{1}{(w-z)^2} \cdot \gamma_w a_z : + \frac{-1}{w-z} \cdot (\partial c)_w (\partial b)_z : + \text{reg.} \sim \frac{-1}{(w-z)^3} + \frac{(\gamma a)_z}{(w-z)^2} + \frac{1}{w-z} (\partial \gamma a + \partial b \partial c)_z + \text{reg.}
\]

Here in the last step we replaced fields at \( w \) with their Taylor expansions centered at \( z \). Note that the products of fields at \( w \) and at \( z \) occurring at the intermediate stage are normally ordered, i.e. Wick contractions inside them are prohibited. Finally, \( \gamma \partial c \) and \( \partial \gamma a \) vanish due to anomaly cancellation.

\[\text{Reg} = \text{regular terms} \]

\[\text{Ir} = \text{irregular terms} \]

\[\text{Reg} + \text{Reg} \sim \text{Reg} \quad \text{Reg} + \text{Ir} \sim \text{Ir} \]

\[\text{Reg} + \text{Reg} + \text{Ir} \sim \text{Reg} \quad \text{Reg} + \text{Reg} + \text{Reg} \sim \text{Reg} \]

\[\text{Reg} + \text{Reg} + \text{Ir} + \text{Reg} \sim \text{Reg} \quad \text{Reg} + \text{Reg} + \text{Reg} + \text{Ir} \sim \text{Reg} \]

\[\text{Reg} + \text{Reg} + \text{Reg} + \text{Ir} \sim \text{Reg} \quad \text{Reg} + \text{Reg} + \text{Reg} + \text{Reg} \sim \text{Reg} \]

\[\text{Reg} + \text{Reg} + \text{Reg} + \text{Reg} + \text{Ir} \sim \text{Reg} \quad \text{Reg} + \text{Reg} + \text{Reg} + \text{Reg} + \text{Reg} \sim \text{Reg} \]

[7] Recall that a conformal field theory with central charge \( c \) is characterized by the following OPE of the stress-energy tensor with itself: \( T(w)T(z) \sim \frac{\mathcal{C}}{(w-z)^4} + \frac{\mathcal{F}(z)}{(w-z)^3} + \frac{\mathcal{G}(z)}{(w-z)^2} + \text{reg.} \), plus the conjugate expression for \( \bar{T}T \), plus \( TT \sim \text{reg.} \). Also recall that a field \( \Phi \) is primary, of conformal weight \( (h, \bar{h}) \) iff its OPEs with the stress-energy tensor are: \( T(w)\Phi(z) \sim \frac{h \Phi(z)}{(w-z)^3} + \frac{\Phi(z)}{(w-z)^2} + \text{reg.} \), and the conjugate \( \bar{T}(w)\Phi(z) \sim \frac{\bar{h} \Phi(z)}{(w-z)^3} + \frac{\Phi(z)}{(w-z)^2} + \text{reg.} \).
notice that $\partial \gamma a + \partial b \partial c$ is equivalent to $T$ modulo equations of motion. Thus we obtain (21).

For $\Phi(z) = \Phi(B, c)z$ a $Q$-closed field \([13]\), we obtain

$$T(w)\Phi(z) \sim \frac{\partial \Phi(z)}{w-z} + \text{reg.}, \quad \bar{T}(w)\Phi(z) \sim \frac{\partial \Phi(z)}{\bar{w}-\bar{z}} + \text{reg.}$$

Thus all polynomials in $B, c$ are primary fields of weight $(0, 0)$.

### 2.4. Extended Virasoro algebra.

Every field $\alpha$ which is holomorphic, i.e. satisfies $\bar{\partial}\alpha \sim 0$, and has conformal weight $(h, 0)$, determines mode operators

$$\alpha_n^{(z)} : \Phi(z) \mapsto \oint_{C_z} \frac{dw}{2\pi i} (w-z)^{n+h-1} \alpha(w)\Phi(z)$$

on the space of fields $\mathbb{F}_z$ where on the r.h.s. one has the integral in variable $w$ over a contour $C_z$ going once counterclockwise around $z$. In other words, $\alpha_n^{(z)}$ acts on a field $\Phi(z)$ by taking the coefficient of $(w-z)^{-n-h}$ in the OPE $\alpha(w)\Phi(z)$. I.e. one has the mode expansion – the equality

$$\alpha(w) = \sum_{n \in \mathbb{Z}+h} (w-z)^{-n-h} \alpha_n^{(z)}$$

of $w$-dependent operators on $\mathbb{F}_z$. Here the left hand side acts on a field $\Phi(z)$ by sending it to the OPE $\alpha(w)\Phi(z)$. Similarly, an anti-holomorphic field $\bar{\alpha}$ (i.e. satisfying $\bar{\partial}\bar{\alpha} \sim 0$), of weight $(0, \bar{h})$, determines mode operators $\bar{\alpha}_n^{(z)} : \Phi(z) \mapsto \oint_{C_{\bar{z}}} \frac{d\bar{w}}{2\pi i} (\bar{w}-%z)^{n+\bar{h}-1} \bar{\alpha}(w)\Phi(z)$.

An important case of this construction is for $\alpha$ a conserved (holomorphic) Noether current – a primary field of conformal weight $(1, 0)$ satisfying $\bar{\partial}\alpha \sim 0$. Then

$$\hat{\alpha}^{(z)} : \Phi(z) \mapsto \oint_{C_z} \frac{dw}{2\pi i} \alpha(w)\Phi(z)$$

is the corresponding quantum Noether charge acting on $\mathbb{F}_z$.

In particular, it is a straightforward check that the operator $\hat{J}_{\text{tot}} := \hat{J} + \hat{J}$ associated to the total BRST current \( \hat{J}_{\text{tot}} = 2i(dz J - d\bar{z} \bar{J}) \) coincides with the classical BRST operator $Q$ acting on $\mathbb{F}_z$ \([13]\).

One defines the Virasoro generators $L_n^{(z)} := T_n^{(z)}$ with $n \in \mathbb{Z}$, as the mode operators for the stress-energy tensor $T$, defined by (26) with $h = 2$. Similarly, the anti-holomorphic Virasoro generators $\bar{L}_n^{(z)}$ are the mode operators for $\bar{T}$. We will

---

8The contour is supposed to be a boundary of a small neighborhood (e.g. a disk) of $z$, where “small” means that all the other field insertions in the correlators we are considering happen outside the neighborhood. Note that the holomorphic property $\bar{\partial}\alpha \sim 0$ implies that one can deform the contour as long as it does not intersect with field insertions.

9 Our normalization convention here is as follows: $J_{\text{tot}}\Phi(z) := -\frac{i}{4\pi} \oint_{C_z} (J_{\text{tot}}) w \Phi(z) = \oint_{C_z} \frac{dw}{2\pi i} J_w + \frac{dw}{2\pi i} J_{\bar{w}} \Phi(z)$. Here the factor $\frac{i}{4\pi}$ is the same as the factor accompanying the action in the path integral \([13]\).

10 For example: $(\gamma \partial c) w z \sim \frac{\partial c}{w-z} + \text{reg.}$ and $(\gamma \partial c) w z \sim \frac{\partial c}{w-z} + \text{reg.}$, hence $\delta b = \gamma, \delta c = \gamma$, and thus $(\delta + \bar{J})h = \gamma + \bar{\gamma} = \lambda = Q(b)$. Likewise, $(\gamma \partial c) w z \sim \frac{\partial c}{w-z} + \text{reg.}$ and $(\gamma \partial c) w z \sim \text{reg.}$, hence $J a = -\partial c, J a = 0$ and thus $(\delta + \bar{J})a = -\partial c = Q(a)$. Another example is $J_{\text{tot}} G$ which is given by the residue in the OPE \([24]\), thus $J_{\text{tot}} G = T$ which is a confirmation that in the quantum setting the classical relation $T = Q(G)$ still holds.
also need the mode operators $G_n^{(z)}$ of the BRST-primitive $G$ (which also has weight $h = 2$) and their conjugate counterparts $\bar{G}_n^{(z)}$ associated to $\bar{G}$.

**Example 2.3.** Operators $L_{-1}^{(z)}$ and $\bar{L}_{-1}^{(z)}$ act on $F_z$ as partial derivatives:

\begin{equation}
L_{-1}^{(z)} = \partial_z, \quad \bar{L}_{-1}^{(z)} = \partial_{\bar{z}}
\end{equation}

From the OPEs between $T, G, J$ and their conjugates, one obtains the following super commutation relations (Lie brackets) for the graded Lie algebra linearly generated by the operators $Q, \{L_n\}, \{G_n\}, \{\bar{L}_n\}, \{\bar{G}_n\}$:

\begin{equation}
\begin{aligned}
\{Q, Q\} &= 0, \quad [L_n, L_m] = (n - m)L_{n+m}, \\
\{Q, L_n\} &= 0, \quad \{Q, G_n\} = L_n, \quad [L_n, G_m] = (n - m)G_{n+m}, \quad [G_n, G_m] = 0
\end{aligned}
\end{equation}

plus the conjugate relations. Commutators involving a holomorphic generator $\in \{L_n, G_n\}$ and an anti-holomorphic generator $\in \{\bar{L}_n, \bar{G}_n\}$ vanish. The degrees (ghost numbers) of the generators are:

| Operator | Degree |
|----------|--------|
| $Q$      | +1     |
| $L_n, \bar{L}_n$ | 0     |
| $G_n, \bar{G}_n$ | -1     |

In particular, this is an extension of the direct sum of two copies (coming from holomorphic and anti-holomorphic sectors) of Virasoro algebra with central charge $c = 0$.

**Remark 2.4.** The theory contains a “logarithmic field” $(cb)_z$ whose OPE with $T$ is: $T(w)(cb)_z \sim \frac{1}{(w-z)^2} + \frac{\partial (cb)_z}{w-z} + \text{reg}$. Its presence implies that the Hamiltonian of the theory $H = L_0 + \bar{L}_0$ is not diagonalizable and has a Jordan block (with eigenvalue 0) consisting of the eigenvector 1 and a generalized eigenvector $\frac{1}{2}cb$.

**Remark 2.5** (Sugawara construction). Consider the holomorphic fields $a, \gamma, \partial b, \partial c$ and consider their Fourier modes around $z$ defined by (27). Note that the stress-energy tensor $T = a \partial \gamma + \partial b \partial c$, BRST current $J = \gamma \partial c$ and the primitive $G = a \partial b$ are explicitly written as quadratic expressions in the four fields $a, \gamma, \partial b, \partial c$. Thus, for the Fourier modes we have

\begin{equation}
\begin{aligned}
L_n &= \sum_{m} - (n - m) a_m \gamma_{n-m} + (\partial b)_m (\partial c)_{n-m}, \\
J_n &= \sum_{m} \gamma_m (\partial c)_{n-m}, \quad G_n = \sum_{m} a_m (\partial b)_{n-m}
\end{aligned}
\end{equation}

Commutation relations between the modes of $a, \gamma, \partial b, \partial c$ follow from OPEs $a_w \gamma_z \sim \frac{1}{w-z} + \text{reg}, (\partial b)_w (\partial c)_z \sim \frac{1}{(w-z)^2} + \text{reg}$ between these fields. Explicitly, one has the commutation relations

\begin{equation}
\begin{aligned}
[a_n, \gamma_m] &= \delta_{n,-m}, \quad [(\partial b)_n, (\partial c)_m] = m \delta_{n,-m}
\end{aligned}
\end{equation}

and all the other Lie brackets vanish. Formulæ (30) can be seen as an analog of the Sugawara construction in Wess-Zumino-Witten theory, expressing Virasoro generators as quadratic combinations of generators of the current algebra.

---

11We omit for brevity the superscript $(z)$, understanding that all operators here act on $F_z$ for a fixed point $z \in \Sigma$. 
2.5. Witten’s descent of observables. We are interested in constructing “p-observables” – composite fields $O^{(p)}$ with values in $p$-forms on $\Sigma$ with the property that

$$Q O^{(p)} = d O^{(p-1)}$$  \hspace{1cm} (32)

for some $O^{(p-1)}$ and $d$ the de Rham operator. This would imply that, for $\gamma \subset \Sigma$ any $p$-cycle, the integral $\int_{\gamma} O^{(p)}$ is $Q$-closed; in particular, a correlator of several such expressions is a gauge-independent quantity. Equation (32) is known as Witten’s descent equation for observables.

One can solve equation (32) using operators $G_{-1}, \tilde{G}_{-1}$. Namely, we introduce the operator

$$\Gamma = -dz G_{-1} - d\bar{z} \tilde{G}_{-1} : \mathbb{F}_{z} \otimes \wedge^{p} T_{z}^{*} \Sigma \to \mathbb{F}_{z} \otimes \wedge^{p+1} T_{z}^{*} \Sigma$$

It can be viewed as the contraction of the de Rham operator $d$ with Fourier mode $-1$ of $\Gamma_{\text{tot}}$. By virtue of (31) and (28), we have

$$[Q, \Gamma] = dz L_{-1} + d\bar{z} \tilde{L}_{-1} = d$$

We fix a $Q$-closed 0-observable $O^{(0)} \in \Omega_{z} - a$ polynomial in $B$ and $c$, cf. (18), and construct

$$O^{(1)} := \Gamma O^{(0)} = -(dz G_{-1} + d\bar{z} \tilde{G}_{-1}) O^{(0)},$$

$$O^{(2)} := \frac{1}{2} \Gamma^{2} O^{(0)} = -dz d\bar{z} G_{-1} \tilde{G}_{-1} O^{(0)}$$

Observables $O^{(0)}, O^{(1)}, O^{(2)}$ satisfy the descent equation (32) for $p = 0, 1, 2$.\footnote{Indeed, the descent equation for $p = 0$ reads $Q O^{(0)} = 0$ which is satisfied by assumption. Next, for $p = 1$, we have $Q O^{(1)} = [Q, \Gamma] O^{(0)} = d O^{(0)}$ by (18). Finally, for $p = 2$, we have $Q O^{(2)} = \frac{1}{2} [Q, \Gamma] O^{(1)} = \frac{1}{2} d (Q O^{(1)} + [Q, \Gamma] O^{(0)})$ which implies $Q O^{(2)} = \frac{1}{2} (d O^{(1)} + d O^{(0)}) = \frac{1}{2} (d O^{(1)} + d O^{(1)}) = d O^{(1)}$. Here we used that $\Gamma$ commutes with $d = dz L_{-1} + d\bar{z} \tilde{L}_{-1}$.}

Explicitly, $G_{-1}$ and $\tilde{G}_{-1}$ act on $\mathbb{F}_{z}$ as derivations defined on generators by

$$G_{-1} : \quad c \mapsto -a, \quad \gamma \mapsto \partial b, \quad \bar{\gamma}, b, a, \bar{a} \mapsto 0$$

$$\tilde{G}_{-1} : \quad c \mapsto -\bar{a}, \quad \tilde{\gamma} \mapsto \partial b, \quad \bar{\gamma}, b, a, \bar{a} \mapsto 0$$

Consider the case of an $N$-component theory \cite{19}. For $O^{(0)} \in \Omega_{z}^{[N]}$ a polynomial in $B_{1}, \ldots, B_{N}, c^{1}, \ldots, c^{N}$, the descended observables defined by (35,36) are:

$$O^{(1)} = (dz a^{j} + d\bar{z} \bar{a}^{j}) \frac{\partial}{\partial c^{j}} + (i dz \partial b - i d\bar{z} \partial \bar{b}) \frac{\partial}{\partial B_{j}} \right) O^{(0)}$$

and

$$O^{(2)} = -dz d\bar{z} \left( a^{j} \overline{a}^{k} \frac{\partial^{2}}{\partial c^{j} \partial c^{k}} + i a^{j} \partial \bar{b}_{k} + i \bar{a}^{j} \partial b_{k} \right) \frac{\partial}{\partial B_{j}} + \partial_{j} \partial \bar{b}_{k} \frac{\partial^{2}}{\partial B_{j} \partial B_{k}} \right) O^{(0)} \right.$$

$$= \left( -\frac{1}{2} A^{j} \wedge A^{k} \frac{\partial^{2}}{\partial c^{j} \partial c^{k}} - A^{j} \wedge *db_{k} \frac{\partial^{2}}{\partial c^{j} \partial B_{k}} + \frac{1}{2} *db_{j} \wedge db_{k} \frac{\partial^{2}}{\partial B_{j} \partial B_{k}} \right) O^{(0)}$$

Example 2.6. Taking $O^{(0)} = c$ (in 1-component theory), we get $O^{(1)} = dz a + d\bar{z} \bar{a} = A$ and $O^{(2)} = 0$. In particular, we can integrate this 1-observable along a closed oriented curve $\gamma \subset \Sigma$, obtaining a $Q$-closed expression $\oint_{\gamma} A$. Then one
can, e.g., consider a correlator $\langle B(z) \oint_\gamma A \rangle$. The expression in the correlator is $Q$-closed and thus the correlator is topological – invariant under isotopy. Using the propagator $\langle B(z) A(w) \rangle = 2d_w \arg(w - z)$, we can compute this correlator:

$$\langle B(z) \oint_\gamma A \rangle = 4\pi \text{lk}(\gamma, z)$$

where $\text{lk}(\gamma, z)$ is the “linking number” – the winding number of the curve $\gamma$ around the point $z$.

**Example 2.7.** In $N$-component theory, consider

$$\mathcal{O}^{(0)} = W(c)$$

a polynomial in variables $c^j$ containing only monomials of even degree. Then we have

$$\mathcal{O}^{(1)} = A^j \frac{\partial}{\partial c^j} W(c), \quad \mathcal{O}^{(2)} = -\frac{1}{2} A^j \wedge A^k \frac{\partial^2}{\partial c^j \partial c^k} W(c)$$

This 2-observable determines a deformation of the abelian theory analogous to the deformation of the Landau-Ginzburg model by a superpotential.

**Example 2.8.** In $N$-component theory, consider the cubic observable

$$\mathcal{O}^{(0)} = \frac{1}{2} f_{ijk} B_i c^j c^k$$

with $f_{ijk}$ arbitrary constant coefficients with $f_{ijk} = -f_{kji}$. Note that, from the viewpoint of interpretation of 0-observables as polyvectors, this is a quadratic vector field on $\Pi \mathbb{R}^N$. Then we have

$$\mathcal{O}^{(1)} = f_{ijk} B_i A^j c^k - \frac{1}{2} f_{ijk} \ast db_i c^j c^k, \quad \mathcal{O}^{(2)} = \frac{1}{2} f_{ijk} B_i A^j A^k - f_{ijk} \ast db_i A^j c^k$$

This 2-observable, in the case when $f_{ijk}$ are the structure constants of a Lie algebra, determines the deformation of the abelian $BF$ theory into the non-abelian $BF$ theory.

### 2.5.1. Descent vs. AKSZ construction.

Within Batalin-Vilkovisky formalism, the $N$-component abelian $BF$ theory is defined by the master action coming from the AKSZ construction.

$$S^{BV} = \int B_k dA^k + A^*_k dc^k + \lambda_k b^{*k} = \int_{S^{AKSZ}} B_k dA^k + \int_{S^{aux}} \lambda_k b^{*k}$$

– a function on the space of BV fields

$$\mathcal{F}^{BV} = \mathcal{F}^{AKSZ} \times \mathcal{F}^{aux} = \mathrm{Map}(T[1]\Sigma, T^* [1] V [1]) \times \left( \Omega^0_{\lambda^k} \oplus \Omega^0_{b^k} [-1] \oplus \Omega^2_{\lambda^*} \oplus \Omega^2_{b^*} \right)$$

Here $V = \mathbb{R}^N$ is the coefficient space of the theory; $\Omega^p$ is a shorthand for $\Omega^p(\Sigma) \otimes V$. The first factor above – the AKSZ mapping space is parameterized by the fields

---

13Here we are implicitly assuming that $z$ is not on $\gamma$. If $z \in \gamma$, the correlator also makes sense: the linking number in (41) then takes a half-integer value – the half-sum of the values obtained by displacing $z$ normally to $\gamma$ in two possible directions.
\(c^k, A^k, B_k\) and the respective anti-fields \(c^*_k, A^*_k, B^*_k\) which assemble into two AKSZ superfields – nonhomogeneous forms on \(\Sigma\),
\[A^k = c^k + A^k + B^*_k, \quad B_k = B_k + A^*_k + c^*_k\]
parameterizing the first and second term in \(\mathcal{F}_{\text{AKSZ}} = \Omega^\bullet(\Sigma, V[1]) \oplus \Omega^\bullet(\Sigma, V^*)\).

Thus the entire field content in BV setting is:

| field/antifield | \(c^k\) | \(A^k\) | \(B^*_k\) | \(B_k\) | \(A^*_k\) | \(c^*_k\) | \(\lambda_k\) | \(b_k\) | \(\lambda^*_k\) | \(b^*_k\) |
|----------------|--------|--------|--------|------|--------|--------|--------|------|--------|--------|
| form degree on \(\Sigma\) | 0 | 1 | 2 | 0 | 1 | 2 | 0 | 0 | 2 | 2 |
| ghost number | 1 | 0 | -1 | 0 | -1 | -2 | 0 | -1 | -1 | 0 |

Here objects without stars are the BRST fields and objects with stars are the corresponding anti-fields.

\(F_{\text{BV}}\) carries the symplectic form of ghost number \(-1\),
\[\omega_{\text{BV}} = \sum \phi \int \delta \phi \wedge \delta \phi^*\]
where the sum is over all species of BRST fields, \(\phi \in \{c^k, A^k, B_k, \lambda_k, b_k\}\). The action (42) satisfies the classical master equation
\[(S_{\text{BV}}, S_{\text{BV}}) = 0\]
with \((-,-)\) the Poisson bracket (the BV anti-bracket) on functions on \(F_{\text{BV}}\) associated to the symplectic structure \(\omega_{\text{BV}}\).

Imposing the Lorenz gauge-fixing corresponds in the BV language to restricting from the whole space of BV fields to a Lagrangian submanifold \(L \subset F_{\text{BV}} = T^*[-1]F\) defined as the graph of the exact 1-form \(\delta \Psi\) on the space of BRST fields, with \(\Psi\) the gauge-fixing fermion (6). Explicitly, \(L\) is given by
\[L : \begin{cases} c^k, A^k, B_k, \lambda_k, b_k & \text{are free} \\ A^*_k = -*db_k, \quad b^*_k = d*A^k, \quad c^*_k = B^*_k = \lambda^*_k = 0 \end{cases}\]

In particular the restriction \(S_{\text{BV}}|_L\) is exactly the gauge-fixed action (2).

Denote by \(X = T^*[1]V[1] = V[1] \oplus V^*\) the target of the AKSZ mapping space, appearing in (43). Let \(ev : F_{\text{AKSZ}} \times T[1]\Sigma \to X\) be the evaluation map for the AKSZ mapping space. Looking at (20) and our computation of the descent (39,40), we make the following observations:

(i) The space of 0-observables \(O_z(20)\) coincides with the space of functions on the AKSZ target \(X\).

(ii) For any 0-observable, \(O^{(0)} \in O_z\), adding to it its first and second descent, we obtain the pullback of \(O^{(0)}\), regarded as a function on the AKSZ target, by the evaluation map:
\[O^{(0)} + O^{(1)} + O^{(2)} = ev^*O^{(0)}|_L\]

For example, for \(O^{(0)} = c^k\), (15) yields \(c^k + A^k = A^k|_L\) – the restriction of the AKSZ superfield to the Lagrangian (14). Likewise, for \(O^{(0)} = B_k\), we get \(B_k - *db_k = B_k|_L\).

(iii) As immediately implied by the previous point, a deformation \(S \to S + g \int O^{(2)}\) of the abelian BF action by a 2-observable is the same as turning on the target Hamiltonian \(g O^{(0)}\) in AKSZ construction, i.e., adding to the BV action the term \(g \int ev^*O^{(0)}\) and imposing the Lorenz gauge by restricting to the Lagrangian (14).

Note that in (42,43) the BV system is presented as a sum of an AKSZ system and an auxiliary system which is not of AKSZ form. One can in fact cast the auxiliary
system in AKSZ form, too, by extending the four auxiliary fields \( b_k, \lambda_k, b^{*k}, \lambda^{*k} \) to a quadruple of AKSZ superfields:

\[
\begin{align*}
\hat{\lambda}_k &= \lambda_k + \mu_k + \nu_k \\
\hat{\lambda}^{*k} &= \nu^{*k} + \mu^{*k} + \lambda^{*k} \\
\hat{b}_k &= b_k + f_k^* + e_k^* \\
\hat{b}^{*k} &= e^k + f^k + b^{*k}
\end{align*}
\]

The form degrees and ghost numbers of the field components here are as follows.

| field/antifield | \( \lambda_k \) | \( \mu_k \) | \( \nu_k \) | \( \nu^{*k} \) | \( \mu^{*k} \) | \( \lambda^{*k} \) | \( b_k \) | \( f_k^* \) | \( e_k^* \) | \( e^k \) | \( f^k \) | \( b^{*k} \) |
|----------------|----------------|----------|----------|------------|------------|------------|--------|--------|--------|--------|--------|--------|
| form degree on \( \Sigma \) | 0 | 1 | 2 | 0 | 1 | 2 | 0 | 1 | 2 |
| ghost number | 0 | -1 | -2 | -1 | -2 | -3 | 2 | 1 | 0 |

The BV action \(^{14}\) in this setting is replaced with the full AKSZ action \(^{14}\)

\[
\tilde{S} = \int B_k dA^k + \hat{\lambda}^{*k} d\hat{\lambda}_k + \hat{b}^{*k} db_k + \hat{\lambda}_k \hat{b}^{*k}
\]

– a function on the full AKSZ mapping space

\[
\tilde{F} = \text{Map}(T[1]\Sigma, \Sigma^* \times T^*[1](V^*[1] \oplus V^*[1] \oplus V^*))
\]

The target \( X^{\text{full}} = \mathcal{X} \times \mathcal{X}^{\text{aux}} = T^*[1](V^*[1] \oplus V^*[1] \oplus V^*) \) is a shifted cotangent bundle with base coordinates \( e^k, b_k, \lambda_k \) (corresponding to the superfields \( A^k, \hat{b}_k, \hat{\lambda}_k \)) and fiber coordinates \( B_k, \nu^{*k}, \lambda^{*k} \) (corresponding to superfields \( B_k, \hat{b}^{*k}, \hat{\lambda}^{*k} \)). Kinetic term of (46) corresponds to the standard canonical 1-form on the target (as a cotangent bundle); term \( \hat{\lambda}_k \hat{b}^{*k} \) corresponds to the target Hamiltonian \( \Theta = \Delta_k A^k \)

The gauge-fixing Lagrangian is \( \tilde{L} = \text{graph}(\delta \Psi) \) in \( \tilde{F} \), regarded as the cotangent bundle to the space of non-starred fields, with \( \Psi \) as before \(^{15}\), (viewed as a constant function in fields \( \mu_k, \nu_k, e^k, f^k \)).

\[
\tilde{L} : \begin{cases} 
  c^k, A^k, B_k, \lambda_k, \mu_k, \nu_k, b_k, e^k, f^k \text{ are free,} \\
  A_k^* = - \ast dB_k, \quad b^{*k} = d \ast A^k, \\
  c_k^* = B^{*k} = \lambda^{*k} = \mu^{*k} = \nu^{*k} = f_k^* = e_k^* = 0
\end{cases}
\]

Restriction of the action \(^{16}\) to \( \tilde{L} \) yields

\[
\tilde{S}|_{\tilde{L}} = S + \int (\mu_k f^k + \nu_k e^k)
\]

with \( S \) the BRST action \(^{2}\). Integrating out the fields \( \mu_k, \nu_k, e^k, f^k \), we obtain the action \( S \).

\(^{14}\) In fact, one can write a simpler action \( \tilde{S} = \int B_k dA^k + \lambda_k \hat{b}^{*k} \), which is BV canonically equivalent to \( \tilde{S} \) by \( \tilde{S} = S + \left( S, \int \lambda^{*k} db_k \right) \).

\(^{15}\) Note that passing to the cohomology of the cohomological vector field \( Q_{\text{target}} = (\Theta, -) \) acting on functions on \( X^{\text{full}} \) contracts the auxiliary part of the target and yields functions on \( \mathcal{X} \), or the space of \( \theta \)-observables \( \mathcal{O}_\theta \).
2.5.2. Towards Gromov-Witten invariants: a toy example. Correlators of the form
\[ \langle G_{\text{tot}}(w_1) \cdots G_{\text{tot}}(w_p) \Phi_1(z_1) \cdots \Phi_n(z_n) \rangle \]
with \( Q \)-closed fields \( \Phi_1, \ldots, \Phi_n \) define closed \( p \)-forms on \( \mathcal{M}_{\Sigma,n} \) – the moduli space of conformal structures on \( \Sigma \) with \( n \) marked points \( z_1, \ldots, z_n \) (here \( \Sigma \) can be any surface). Integrating such a form over a \( p \)-cycle on \( \mathcal{M}_{\Sigma,n} \), one obtains interesting periods – a version of Gromov-Witten invariants. Example 2.6 above provides a simple example of such a period.

Example 2.9. For \( \Sigma = \mathbb{C} \), consider the correlator \( \langle G(w)c(z_0)B(z_1) \cdots B(z_n) \rangle \).
Integrating over \( w \) in a circle around \( z_0 \) and subsequently integrating over \( z_0 \) in a contour \( \gamma \), we obtain the “linking number” – the sum of the winding numbers of \( \gamma \) around \( z_1, \ldots, z_n \):

\[
\oint_{\gamma}dz_0 \oint_{C_{z_0}} \frac{dw}{2\pi i} \langle G_w c_{z_0} B_{z_1} \cdots B_{z_n} \rangle = -4\pi \text{lk}(\gamma, \{z_1, \ldots, z_n\})
\]

On the left hand side, under the integral over \( \gamma \), we have a 1-form \( \rho \) on \( \mathcal{M}_{\Sigma,n+1} \) or equivalently a \( PSL(2,\mathbb{C}) \)-invariant 1-form on the open configuration space \( \text{Conf}_{n+1}(\Sigma) \), defined by the property that, for \( (v_0, \ldots, v_n) \in T_{z_0} \Sigma \times \cdots \times T_{z_n} \Sigma \simeq T_{(z_0, \ldots, z_n)} \text{Conf}_{n+1}(\Sigma) \) a tangent vector to the configuration space, its contraction with \( \rho \) is

\[ \langle c_{v_0} \cdots c_{v_n} \rangle = -\frac{1}{2\pi i} \left( \int_{D_{z_0}} G_{w}^{\text{tot}} \bar{\partial} \nu \right) c_{z_0} B_{z_1} \cdots B_{z_n} \]

Here \( D_{z_0} \) is the disk around \( z_0 \) bounded by \( C_{z_0} \), \( \nu \) is the vector field on \( \Sigma \) constant and equal to \( v_0 \) inside the disk and extended by zero outside the disk; \( \bar{\partial} \nu \) is the corresponding Beltrami differential. Ultimately, the left hand side of \( (47) \) is the integral of a 1-form on \( \mathcal{M}_{\Sigma,n+1} \) over a 1-cycle \( \gamma \) in the fiber of the projection \( \mathcal{M}_{\Sigma,n+1} \to \mathcal{M}_{\Sigma,n} \) forgetting the point \( z_0 \) and yields the linking number as a simplest Gromov-Witten period.

2.6. BV algebra structure on the space of 0-observables. The space of 0-observables \( \mathcal{O}_z \) in addition to being a graded commutative algebra has a degree -1 Poisson bracket defined by

\[
\{ \mathcal{O}_1, \mathcal{O}_2 \} := \frac{(-1)^{|\mathcal{O}_1|}}{4\pi} \oint_{C_{\gamma}} (\Gamma \mathcal{O}_1)_{w}(\mathcal{O}_2)_{z}
\]

I.e. one descends the first 0-observable to a 1-observable and integrates over a contour encircling the second 0-observable.

Example 2.10. For \( \mathcal{O}_1 = c, \mathcal{O}_2 = B \), we have \( \Gamma(c) = A \) and we obtain

\[
\{ c, B \} = -\frac{1}{4\pi} \oint_{C_{z}} \frac{A_w B_z}{\sim 2d_w \arg(w-z)+\text{reg.}} = -1
\]

cf. example 2.6. In particular, \( c \) and \( B \) are conjugate variables for the Poisson bracket.
Explicitly, the Poisson bracket (15) is:

\[ \{ O_1, O_2 \} = \frac{\partial}{\partial B} O_1 \frac{\partial}{\partial c} O_2 + (-1)^{|O_1|} \frac{\partial}{\partial c} O_1 \frac{\partial}{\partial B} O_2 \]

Commutative multiplication together with this bracket comprise the structure of a "\( P_2 \) algebra" on \( O_z \) (the algebra over the homology of the operad \( E_2 \) of little 2-disks).

In addition to the bracket \( \{ -, - \} \), one has the operator

\[ G_0^- := \frac{1}{2i} (G_0 - \bar{G}_0) \]

the contraction of \( G_{\text{tot}} \) with the vector field corresponding to rotation about the point \( z \), acting on \( \mathbb{F}_z \) and in particular on \( O_z \).

**Example 2.11.** E.g. acting on \( O = Bc \), one has

\[
\frac{-i}{w - z} \frac{1}{w - z} \frac{\partial}{\partial B} (\partial B) c_1 + \frac{-i}{w - z} a_w B_z + \text{reg.} \sim \frac{i}{(w - z)^2} + \frac{(i \partial B - a B) z}{w - z} + \text{reg.}
\]

Thus, \( G_0(Bc) = i \) – the coefficient of the second order pole in the OPE above, and similarly one obtains \( \bar{G}_0(Bc) = -i \). Therefore, \( G_0^- (Bc) = 1 \).

Explicitly, the operator \( G_0^- \) acts on 0-observables by

\[
G_0^- : \ O \mapsto \frac{\partial^2}{\partial B \partial c} O
\]

Thus, one recognizes in \( G_0^- \) the Batalin-Vilkovisky Laplacian (of degree \(-1\)) and hence \( (O_z, \\{ -, - \}, \Delta) \) is a BV algebra with the bracket and the Laplacian of degree \(-1\). In other words, it is an algebra over the homology of the operad \( E_2^\text{fr} \) of *framed* little 2-disks.

Note that, from the standpoint of identification of 0-observables with polyvectors [20], this is the standard BV algebra structure on polyvectors.

**Remark 2.12.** In the example [23] we expect \( O^{(2)} \) to give a classically consistent deformation of the action \( S \mapsto S + g f_{ij}^{(2)} O^{(2)} \) (with \( g \) the deformation parameter) if and only if \( f_{ij} \) satisfy Jacobi identity, i.e., define a Lie algebra on the space of coefficients \( \mathbb{R}^N \) and we expect the deformation to be consistent on the quantum level if additionally the unimodularity property \( f_{ij}^{(2)} = 0 \) holds. Note that these two cases correspond to, respectively, classical and quantum BV master equation holding for \( O^{(0)} = \frac{1}{2} f_{jk} B_i c^j c^k \):

\[
\{ O^{(0)}, O^{(0)} \} = 0, \quad G_0^- O^{(0)} = 0
\]
Remark 2.13. One can consider $S^1$-equivariant version of BRST cohomology \[ \text{cohomology of the equivariant extension of the BRST operator} \]

\[ Q_{S^1} := Q + \epsilon G_0 \]

acting on the kernel of $L_0 \propto Q^{S_1}_{S^1}$ in $\mathbb{F}_z[\epsilon]$ (rotationally-invariant fields valued in polynomials in $\epsilon$), with $\epsilon$ the degree 2 equivariant parameter. This equivariant cohomology evaluates, in the context of $N$-component theory, to

\[ H_{S^1}(\mathbb{F}_z) \simeq H_{\epsilon G_0}(\mathbb{D}_z[\epsilon]) = T_{\text{poly-free}}(\mathbb{H}^N) \oplus \epsilon^1 \cdots \epsilon^N \cdot \epsilon \mathbb{C}[\epsilon] \]

- the space of divergence-free (or “unimodular”) polyvectors on $\mathbb{H}^N$, plus the $\mathbb{C}[\epsilon]$-linear span of the products of all ghosts times $\epsilon$.

2.6.1. Structure of an algebra over the framed $E_2$ operad on the space of composite fields. The space of composite fields itself $\mathbb{F}_z$ has the structure of an algebra over the operad $E_2^N$ of framed little 2-disks. Namely, given a configuration $\mathbf{c} \in E_2^N(n)$ of $n$ framed disks with centers at $z_1, \ldots, z_n$, radii $r_1, \ldots, r_n$ and rotation angles $\theta_1, \ldots, \theta_n$, one constructs a map

\[ \mathbf{o} : \mathbb{F}_{z_1} \otimes \cdots \otimes \mathbb{F}_{z_n} \to \mathbb{F}_0 \]

which sends an $n$-tuple of composite fields $\Phi_1(z_1), \ldots, \Phi_n(z_n)$ to a field $\Psi \in \mathbb{F}_0$ characterized by the property that

\[ \left\langle \prod_{j=1}^n r_j^{\hat{P}(z_j)} e^{i\theta_j \hat{P}(z_j)} \Phi_j(z_j) \cdot \phi_1(y_1) \cdots \phi_m(y_m) \right\rangle = \langle \Psi(0) \cdot \phi_1(y_1) \cdots \phi_m(y_m) \rangle \] (50)

for any test fields $\phi_1, \ldots, \phi_m$ inserted at points $y_1, \ldots, y_m$ outside the unit disk on $\mathbb{C}$. Here $\hat{H}(z) := L_0(\hat{z}) + L_0(\hat{\bar{z}})$ and $\hat{P}(z) := L_0(z) - L_0(\hat{z})$ are the energy and momentum operators acting on fields at $z$; in particular, for a field $\Phi(z)$ of conformal weights $(h, \bar{h})$, the rescaling factor in the l.h.s. of (51) is

\[ r^h \bar{r}^{\bar{h}} e^{i\theta(h - \bar{h})} \] (52)

Thus, operation (50) is an $n$-point version of an operator product expansion, rescaled appropriately to account for the size and orientation of the disks.

One calculates operations (50) explicitly using Wick’s lemma: one considers all partial contractions between basic fields in $\Phi_1, \ldots, \Phi_n$, replaces those with the appropriate propagators and replaces all the remaining fields with their Taylor expansion at zero. Finally, one rescales the result with the factors (52) (we are assuming for simplicity that fields $\Phi_j$, with $1 \leq j \leq n$, have well-defined conformal weights, i.e., are eigenvectors for the operators $L_0, \bar{L}_0$).

Example 2.14. Let $\mathbf{o}$ be a configuration of two disks centered at $z_1, z_2$ with radii $r_1, r_2$ and rotation angles $\theta_1, \theta_2$, and let $\Phi_1 = J = \gamma \partial c$ and $\Phi_2 = G = a \partial b$. Recall that the conformal weights are $(h, \bar{h}) = (1, 0)$ for $J$ and $(h, \bar{h}) = (2, 0)$ for $G$. We obtain from Wick’s lemma

\[ \mathbf{o}(J \otimes G) = \]

\[ = r_1 e^{i \theta_1} (r_2 e^{i \theta_2})^2 \left( -\frac{1}{z_1 - z_2} + \frac{\gamma_{1a} a_{22}}{(z_1 - z_2)^2} - \frac{(\partial c)_{z_1} (\partial b)_{z_2}}{z_1 - z_2} + (\gamma \partial c)_{z_1} (a \partial b)_{z_2} \right) \]
Here in the last expression all fields are evaluated at \( z = 0 \). Note that the Taylor series in \( k, l \) converges under the correlator with test fields inserted at points outside the unit disk, using that \( z_1, z_2 \) are inside the unit disk.

This way one constructs the \( E_2^{fr} \)-algebra structure on the space of composite fields \( \mathcal{F}_z \). Extending it by linearity, one gets the action of singular 0-chains of \( E_2^{fr} \) on \( \mathcal{F}_z \). In a similar way one constructs the action of all chains \( C_\bullet(E_2^{fr}) \) on \( \mathcal{F}_z \otimes \wedge T^*_z \Sigma \) – composite fields with values in differential forms: one constructs the following differential form on \( E_2^{fr}(n) \) with values in products of fields:

\[
\prod_{j=1}^{n} \frac{\zeta_j^{L_0}}{(1 - \frac{d\zeta_j}{\zeta_j} G_0)} \left( 1 - \frac{d\zeta_j}{\zeta_j} G_0 \right)^{\frac{1}{2}} \Phi_j(z_j)
= \exp \left[ Q - \sum \log \zeta_j \right]
\]

and integrates it over the chain in \( E_2^{fr} \). This construction is considered under the correlator with an arbitrary collection of test fields outside the unit disk, as in (51). Here \( \zeta_j = r_j e^{i\theta_j} \) and \( \Phi_j(z_j) \) are composite fields with values in differential forms on \( \Sigma \); we suppressed the superscripts \( (z_j) \) for the operators \( L_0, G_0 \) and their conjugates.

Further, one can restrict the construction above (53) to fields of form

\[
\Phi(z) = \Phi(z) + \Gamma \Phi(z) + \frac{1}{2} \Gamma^2 \Phi(z) = e^\Gamma \Phi(z)
\]

with \( \Gamma \) as in (53) – i.e. sums of an ordinary (not form-valued) composite field and its first and second descents. This way we obtain a representation of \( E_2^{fr} \) as a differential graded operad on the space of composite fields \( \mathcal{F}_z \) (not form-valued), viewed as a cochain complex with BRST differential \( Q^{fr} \).

Passing to (co)homology, we get the action of the homology \( H_\bullet(E_2^{fr}) \) on \( H_\bullet(\mathcal{F}_z) = \mathcal{O}_z \) – the BV algebra structure \((\mathcal{O}_z, \cdot, \{ - , - \}, G_0^-)\) described above.

**Remark 2.15.** In the discussion of the \( E_2^{fr} \)-action on composite fields and BV algebra structure on 0-observables, we used only a part of the extended Virasoro symmetry of the space of composite fields – only modes \( n = -1 \) and \( n = 0 \) (which displace and rotate/dilate the disks). Using the rest of the modes, one can infinitesimally reparameterize and deform the disks and thus, integrating the Virasoro action, one can extend the \( E_2^{fr} \)-action to the action of (the chains of) a larger operad of general genus zero conformal cobordisms \( S^1 \sqcup \cdots \sqcup S^1 \to S^1 \) with parameterized boundaries (more precisely, the operad of Riemannian spheres with \( n + 1 \) disjoint conformally embedded disks – Segal’s genus zero operad). In particular, in Segal’s picture of conformal field theory [6], the complex \((\mathcal{F}, Q)\) is the non-reduced space of states associated to a circle and \( \mathcal{O} \) is the reduced space of states.

\[
\begin{align*}
= r_1 e^{i\theta_1} (r_2 e^{i\theta_2})^2 \left( \frac{1}{(z_1 - z_2)^2} + \sum_{k,l \geq 0} \frac{z_1^k z_2^l}{k! l!} \left( \frac{\partial^k \gamma \partial^a c}{(z_1 - z_2)^2} - \frac{\partial^{k+1} c \partial^{l+1} b}{z_1 - z_2} + \partial^{k} (\gamma \partial c) \partial^{l} (a \partial b) \right) \right)
\end{align*}
\]
Remark 2.16. Note that the normally-ordered version of the expression (53) evaluated on fields (54) can be rewritten as follows:

\[ : \Xi(\Phi_1, \ldots, \Phi_n) := \prod_{j=1}^{n} \exp \left[ Q - d, \log \zeta_j G_0 + z_j G_{-1} + \log \bar{\zeta}_j \bar{G}_0 + \bar{z}_j \bar{G}_{-1} \right] \circ \Phi_j(0) \]

where \( d \) is the de Rham operator on \( E_2^{fr} \), i.e. the total de Rham operator in variables \( z_j, \zeta_j \) (and conjugates).

2.7. The \( U(1) \)-current and twisting back to a superconformal field theory.

Consider the field

\[ j = \gamma a \]

which we have encountered as a coefficient of the second order pole in the \( J(w)G(z) \) OPE (24). It is the Noether current for the \( U(1) \)-symmetry of the action, which rotates the phases of the fields \( a, \gamma \) in the opposite directions: \( a \mapsto e^{i\theta} a, \gamma \mapsto e^{-i\theta} \gamma \) and does not touch the fields \( b, c, \bar{a}, \bar{\gamma} \). This current is conserved modulo equations of motion, \( \partial j \sim 0 \), and satisfies the following OPEs:

\[ T_w j_z \sim \frac{1}{(w-z)^3} + \frac{j_z}{(w-z)^2} + \frac{\partial j_z}{w-z} + \text{reg.} \]

\[ j_w J_z \sim \frac{J_z}{w-z} + \text{reg.} \]

\[ j_w G_z \sim -\frac{G_z}{w-z} + \text{reg.} \]

\[ j_w j_z \sim -\frac{1}{(w-z)^2} + \text{reg.} \]

In particular, the fields \( J \) and \( G \) have charges +1 and −1 respectively w.r.t. the operator \( \hat{j} \). Similarly to \( j \), we have its anti-holomorphic counterpart \( \bar{j} = \bar{\gamma} \bar{a} \) which satisfies the same properties in the anti-holomorphic sector.

We can consider a new “untwisted” theory with same field content as before and the deformed stress-energy tensor

\[ \tilde{T} := T - \frac{1}{2} \partial j \]

With respect to the new stress-energy tensor, the fields change their (holomorphic) conformal weights as follows:

| field | weight w.r.t. \( T \) | weight w.r.t. \( \tilde{T} \) |
|-------|------------------|------------------|
| \( J \) | 1 | 3/2 |
| \( G \) | 2 | 3/2 |
| \( \bar{j} \) | 1 (not primary) | 1 |
| \( \gamma \) | 0 | 1/2 |
| \( a \) | 1 | 1/2 |
| \( \bar{\gamma}, \bar{a}, b, c \) | 0 | 0 |

Thus, fields \( \gamma, dz a \) become (even) Weyl spinors \((dz)^{\frac{1}{2}} \gamma, (dz)^{\frac{1}{2}} a\) in the untwisted theory. Similarly, \( \bar{\gamma}, d\bar{z} a \) become even spinors \((d\bar{z})^{\frac{1}{2}} \bar{\gamma}, (d\bar{z})^{\frac{1}{2}} \bar{a}\). The fields \( b, c \) are

\[ ^{19} \text{We call it “untwisted”, since the theory we started with is obtained from it by Witten’s topological twist of type B, cf. [7, 4, 2].} \]
unchanged. Thus, the bundle of basic fields, replacing (3) in the untwisted theory, is
\[ \mathcal{F} = K_{a\gamma}^{\frac{1}{2}} \oplus K_{\gamma}^{\frac{1}{2}} \oplus \bar{K}_{a\gamma}^{\frac{1}{2}} \oplus \bar{K}_{\gamma}^{\frac{1}{2}} \oplus \mathbb{R}^2 \]
with \( K = (T^{1,0})^* \Sigma \) and \( \bar{K} = (T^{0,1})^* \Sigma \) the canonical and anti-canonical line bundles on \( \Sigma \).

Note that \( j \) was not a primary field w.r.t. \( T \), due to the 3-rd order pole in (56). However, \( j \) is a primary field of weight \((1,0)\) in the untwisted theory. It is a conserved \( U(1) \)-current and its Fourier modes generate a Heisenberg Lie algebra due to (57).

One obtains the following OPE of the untwisted stress-energy tensor with itself:
\[ \bar{T}(w) \bar{T}(z) = \frac{-3/2}{(w-z)^4} + \frac{2 \bar{T}_z}{(w-z)^2} + \frac{(\partial \bar{T})_z}{w-z} + \text{reg}. \]

Thus, the untwisted theory has central charge \( c = -3 \) in \( N \)-component theory, the central charge becomes \( c = -3N \).

Consider the Fourier modes of the fields \( \bar{T}, J, G, j \), defined via
\[
\bar{T}_w = \sum_n (w-z)^{-n-2} \bar{L}_n, \quad J_w = \sum_r (w-z)^{-r-\frac{3}{2}} J_r, \quad G_w = \sum_s (w-z)^{-s-\frac{3}{2}} G_s, \quad j_p = \sum_{p} (w-z)^{-p-1} j_p
\]
with \( n, p \in \mathbb{Z} \) and \( r, s \in \mathbb{Z} + \frac{1}{2} \) for periodic (Neveu-Schwarz) boundary conditions on fermions and \( r, s \in \mathbb{Z} \) for anti-periodic (Ramond) boundary conditions. These Fourier modes satisfy the relations of \( N = 2 \) superconformal algebra with central charge \( c = -3 \):
\[
[\bar{L}_n, \bar{L}_m] = (n-m) \bar{L}_{n+m} - \frac{1}{4} (n^3 - 3n) \delta_{n,-m}, \quad [j_p, j_q] = -p \delta_{p,-q},
\]
\[
[\bar{L}_n, J_r] = \left( \frac{n}{2} - r \right) J_{n+r}, \quad \bar{L}_n, G_s = \left( \frac{n}{2} - s \right) G_{n+s}, \quad [\bar{L}_n, J_p] = -p J_{n+p},
\]
\[
[J_r, G_s] = L_{r+s} + \frac{r-s}{2} J_{r+s} - \frac{1}{2} \left( r^2 - \frac{1}{4} \right) \delta_{r,-s}, \quad [j_p, J_r] = J_{p+r}, \quad [j_p, G_s] = -G_{p+s},
\]
\[
[J_r, J_s] = 0, \quad [G_r, G_s] = 0
\]
plus the conjugate relation for the Fourier modes of \( \tilde{T}, \tilde{J}, \tilde{G}, \tilde{j} \). Lie brackets, involving one generator from holomorphic sector and one from anti-holomorphic sector, vanish. In the case of \( N \)-component theory, the three central extension terms in the commutation relations (58) – those proportional to \( \delta_{\cdot,-\cdot} \cdot 1 \) – get multiplied by \( N \).

\[ 20 \] Forgetting about the BRST structure, we can regard the action (10) as a superposition of three non-interacting theories: the second-order ghost system (the \( bc \) system) with holomorphic/anti-holomorphic central charges \( c_{bc} = c_{\bar{b}c} = -2 \) and two first order chiral systems with Lagrangians \( \gamma \delta a \) and \( \bar{\gamma} \delta \bar{a} \) with central charges \( c_{\gamma a} = -1, c_{\bar{\gamma} \bar{a}} = 0 \) and \( c_{\gamma \bar{a}} = 0, c_{\bar{\gamma} a} = -1 \) respectively (in the untwisted model, where \( a, \gamma, \bar{a}, \bar{\gamma} \) are even spinors). Thus the total central charge of the system is \( c = \varepsilon = (-2) + (-1) + (0) = -3 \). In the topological (twisted) model, central charges of the first-order systems change to \( c_{\gamma a} = 2, c_{\bar{\gamma} \bar{a}} = 0 \) and \( c_{\gamma \bar{a}} = 0, c_{\bar{\gamma} a} = 2 \), while the central charge of the ghost system remains \(-2\). Thus, \( c_{\text{top}} = c_{\text{top}} = (-2) + (2) + (0) = 0 \).
Thus, the fields $\tilde{T}, J, G, j$ together with their anti-holomorphic counterparts define on the untwisted abelian BF theory the structure of an $N = (2, 2)$ superconformal theory with supersymmetry currents $J, G$ and with $j$ the $R$-symmetry current (plus the conjugates).

2.7.1. Dictionary between abelian BF theory and the B model. Recall (see e.g. [7, 4]) that the free $N = (2, 2)$ supersymmetric sigma model (or Landau-Ginzburg model with zero superpotential) with target $\mathbb{C}^N$ is defined by the action

$$S = 2t \int_{\Sigma} d^2x \left( \bar{\phi}_k \partial \phi^k - i \bar{\psi}_+^k \partial \psi_+^k - i \bar{\psi}_-^k \partial \psi_-^k \right)$$

with scalar fields $\phi^k$, $\bar{\phi}_k$ corresponding to the holomorphic and anti-holomorphic coordinates on the target $\mathbb{C}^N$, respectively, and with $\psi_\pm^k, \bar{\psi}_\pm^k$ fermions of spin $1/2$; $t$ is a coupling constant (which is irrelevant in the free theory as it can be absorbed into the normalization of fields). Here, bar/no bar on fields corresponds to anti-holomorphic/holomorphic directions on the target and $\pm$ corresponds to holomorphic/anti-holomorphic directions on the source $\Sigma$. In the B-twisted sigma model the action is the same, however fields $\psi_\pm^k$ attain spin 1 and $\bar{\psi}_\pm^k$ attain spin 0.

Comparing (59) with (10), we see that the $N$-component abelian BF theory (i.e. with coefficient space $\mathbb{R}^N$) is the B-twisted supersymmetric sigma model with odd target $\mathbb{C}^N$: $\mathbb{R}^N \sslash \mathbb{Z}_2 = \mathbb{R}^N \cup \mathbb{R}^N$.

We have the following dictionary (we use the notations of [7, 4] for the B side).

| BF theory | B model |
|-----------|---------|
| coefficient space $V = \mathbb{R}^N$ | target $X = \mathbb{C}^N$ |
| $c^k$ | $\phi^k$ |
| $b_k$ | $\bar{\phi}_k$ |
| $a^k$ | $\psi_+^k$ |
| $i \gamma_k$ | $\psi_-^k$ |
| $\bar{a}^k$ | $\bar{\psi}_-^k$ |
| $i \gamma_k$ | $\bar{\psi}_+^k$ |
| $-\frac{1}{2} B_k = \frac{1}{2}(\gamma_k - \bar{\gamma}_k)$ | $\theta_k = \frac{i}{2}(\psi_+^k - \psi_-^k)$ |
| $i \lambda_k = i(\gamma_k + \bar{\gamma}_k)$ | $\bar{\eta}_k = \psi_+^k + \psi_-^k = "d \psi_{\bar{k}}"$ |
| $A^k = dz \bar{a}^k + d \bar{z} a^k$ | $\rho^k = dz \psi_+^k + d \bar{z} \psi_-^k$ |
| term $\lambda_k \partial \phi_k$ in $Q$ | Dolbeault differential on the target $\bar{\eta}_k \partial \phi_k$ |

0-observables

$O_z = T_{polym}(IV) = \mathbb{C}[c^k, B_k] \oplus \mathbb{R} p H^{0, p}(X, \wedge q T^{1,0} X) = \mathbb{C}[\phi^k, \theta_k]$

Supercurrents (in untwisted models)

| $J$ | $G_+$ |
| $\bar{J}$ | $G_-$ |
| $G$ | $G_+$ |
| $\bar{G}$ | $G_-$ |

total $U(1)$-current $J_{tot} = dz \bar{J} + d \bar{z} J$ axial R-symmetry current $J_A$
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