Second-order perturbations of cosmological fluids: 
Relativistic effects of pressure, multi-component, curvature, and rotation

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We present general relativistic correction terms appearing in Newton’s gravity to the second-order perturbations of cosmological fluids. In our previous work we have shown that to the second-order perturbations, the density and velocity perturbation equations of general relativistic zero-pressure, irrotational, single-component fluid in a flat background coincide exactly with the ones known in Newton’s theory. We also have shown the effect of gravitational waves to the second-order, and pure general relativistic correction terms appearing in the third-order perturbations. Here, we present results of second-order perturbations relaxing all the assumptions made in our previous works. We derive the general relativistic correction terms arising due to (i) pressure, (ii) multi-component, (iii) background curvature, and (iv) rotation. In case of multi-component zero-pressure, irrotational fluids under the flat background, we effectively do not have relativistic correction terms, thus the relativistic result again coincides with the Newtonian ones. In the other three cases we generally have pure general relativistic correction terms. In case of pressure, the relativistic corrections appear even in the level of background and linear perturbation equations. In the presence of background curvature, or rotation, pure relativistic correction terms directly appear in the Newtonian equations of motion of density and velocity perturbations to the second order. In the small-scale limit (far inside the horizon), relativistic equations including the rotation coincide with the ones in Newton’s gravity. All equations in this work include the cosmological constant in the background world model. We also present the case of multiple minimally coupled scalar fields, and properly derive the large-scale conservation properties of curvature perturbation variable in various temporal gauge conditions to the second order.

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I. INTRODUCTION

Large amount of cosmological data on the large-scale structures and motions of galaxies \[1,2\], and the temperature and polarization anisotropies of cosmic microwave background radiation \[3,4\] have been accumulating recently. In current standard cosmological scenario such structures are explained as small (linear) or large (nonlinear) deviations from spatially homogeneous and isotropic Friedmann background world model. In order to explain these data theoretically, researchers rely on the linear perturbation theory based on relativistic gravity, and quasi-linear perturbation theories and nonlinear simulations based on Newton's gravity. To the linear order in perturbation the general relativistic result was first derived by Lifshitz in 1946 \[5\], and later shown to coincide with the Newtonian result in a zero-pressure medium \[6\]. The same is also known to be true for the background world model. That is, the general relativistic result was first derived by Friedmann in 1922 \[7\], and later shown to coincide with Newtonian result in a zero-pressure medium \[8\].

The observed large-scale distribution of galaxies shows that in the largest observed scale (say, larger than several hundred mega-parsec scale) the distribution may not be inconsistent with the linear assumption around the Friedmann background. However, as the scale becomes smaller the distribution apparently shows quasi-linear to fully nonlinear structures. The fully nonlinear processes occur in small scale where the relativistic effects characterized by \(GM/rc^2 \sim v^2/c^2\) are quite small. If we could ignore such relativistic effects, Newton's gravity would be sufficient to handle the relevant nonlinear processes. If we need to consider the weakly relativistic correction terms in fully nonlinear stage, instead of the relativistic perturbation approach which can handle the fully relativistic processes under weakly nonlinear assumption, we can use the post-Newtonian approximation developed in the context of cosmology in \[9\].

For structures in the quasi-linear evolution phase, previous researches were based on Newton's gravity especially assuming the single component zero-pressure fluid without rotational perturbation \[10\]. In our previous works in...
we have shown that, in the single component zero-pressure fluid without rotational perturbation, cosmological scalar-type perturbation equations in a spatially flat background coincide exactly with the Newtonian ones up to the second order in perturbation. In \[11, 12\] we also have shown the contribution of gravitational wave perturbations to the hydrodynamic parts in the second-order perturbations. In Newton’s gravity the hydrodynamic equations of zero-pressure fluid contain only the quadratic order nonlinearity. In \[13\] we presented pure general relativistic correction terms appearing in the third-order perturbation, and showed that all third-order correction terms are \(10^{-5}\) times smaller than the second-order relativistic/Newtonian terms, and independent of the horizon scale.

In this work we will take into account of the pure general relativistic effects appearing in the second-order perturbations which were ignored in our previous work in \[12\]. We will consider general relativistic effects of (i) pressure, (ii) multi-component, (iii) background curvature, (iv) rotation in cosmological fluids to the second-order perturbations. As results we will show that, although in \[12\] we have shown the exact relativistic/Newtonian correspondence to the second-order perturbations by ignoring the above four conditions and the gravitational waves, as we take these four effects into account we often encounter pure general relativistic effects appearing in the corresponding Newtonian equations even to the second order in perturbations. Our results will show that the relativistic/Newtonian correspondence continues even in the multi-component situation assuming zero-pressure irrotational fluid in a flat background, but in the presence of the cosmological constant. This is a practically useful result because the matter content of present universe is dominated by collisionless dark matter and baryon both of which practically have zero-pressure. In the other three cases, relaxing any of the assumptions about pressure, rotation, and background curvature generally leads to pure general relativistic correction terms to the second order. We will present such correction terms in the context of Newtonian hydrodynamics. One additional relativistic/Newtonian correspondence occurs in the case of rotation in small-scale (sub-horizon-scale) limit which is another practically important result. This correspondence allows us to use the Newtonian equations safely in such a small-scale limit even in the presence of rotational perturbation to the second order.

In Sec. \(\text{II}\) we summarize Newtonian hydrodynamic perturbation equations valid to fully nonlinear order. In Sec. \(\text{III}\) we briefly summarize our previous result of relativistic/Newtonian correspondence to the second order, and pure general relativistic correction terms appearing in the third order. In Sec. \(\text{IV}\) we present parts of the covariant and the ADM (Arnowitt-Deser-Misner) equations which are valid in multi-component situation. In Sec. \(\text{V}\) we present the basic perturbation equations valid to second order. In \(\text{VI}\) the basic set of equations was presented using fluid quantities based on the normal-frame four-vector. The fluid quantities in the present work are based on the energy-frame four-vector, and in this section we present the basic equations using such fluid quantities. In Secs. \(\text{VII} \sim \text{XI}\) we analyse the effects of the pressure, the multi-components, the background curvature, and the rotational perturbation, respectively. In Sec. \(\text{XII}\) we present equations in the scalar fields and generalized gravity theories using the energy-frame fluid quantities. In Sec. \(\text{XIII}\) we properly derive conservation properties of curvature perturbation in various temporal gauge (hypersurface) conditions to the second order in perturbations. Section \(\text{XIV}\) is a discussion. In this work we follow notations used in \(\text{[11, 12]}\). We set \(c \equiv 1\), but when we compare with Newtonian case we often recover the speed of light \(c\).

\section{II. Newtonian Nonlinear Perturbations}

In order to compare properly the relativistic results with the Newtonian ones, in this section we summarize the Newtonian cosmological perturbation theory in fully nonlinear context. We consider multi-component fluids in the presence of isotropic pressure. In case of \(n\)-fluids with the mass densities \(\varrho_i\), the pressures \(p_i\), the velocities \(v_i\) \((i = 1, 2, \ldots n)\), and the gravitational potential \(\Phi\), we have

\begin{align}
\dot{\varrho}_i + \nabla \cdot (\varrho_i v_i) &= 0, \\
\dot{v}_i + v_i \cdot \nabla v_i &= -\frac{1}{\varrho_i} \nabla p_i - \nabla \Phi, \\
\nabla^2 \Phi &= 4\pi G \sum_{j=1}^{n} \varrho_j.
\end{align}

Assuming the presence of spatially homogeneous and isotropic but temporally dynamic background, we introduce fully nonlinear perturbations as

\begin{align}
\varrho_i &= \bar{\varrho}_i + \delta \varrho_i, \quad p_i = \bar{p}_i + \delta p_i, \quad v_i = H r + u_i, \quad \Phi = \bar{\Phi} + \delta \Phi,
\end{align}

where \(H \equiv \dot{a}/a\), and \(a(t)\) is a cosmic scale factor. We move to the comoving coordinate \(x\) where

\begin{align}
r = a(t) x.
\end{align}
thus
\[ \nabla = \nabla_r = \frac{1}{a} \nabla_x, \]
\[ \frac{\partial}{\partial t} \nabla_r = \frac{\partial}{\partial t} \nabla_x + \left( \frac{\partial}{\partial t} \nabla_x \right) \cdot \nabla_x - H \nabla_x \cdot \nabla_x. \]  
\[ (6) \]

In the following we neglect the subindex x. To the background order we have
\[ \dot{\theta}_i + 3H \theta_i = 0, \quad \frac{\dot{a}}{a} = -\frac{4\pi G}{3} \sum_j \theta_j, \quad H^2 = \frac{8\pi G}{3} \sum_j \theta_j + \frac{2E}{a^2}, \]  
\[ (7) \]

where E is an integration constant which can be interpreted as the specific total energy in Newton’s gravity; in Einstein’s gravity we have \( 2E = -Kc^2 \) where \( K \) can be normalized to be the sign of spatial curvature. To the perturbed order we have \[ (13) \]
\[ \delta_i + \frac{1}{a} \nabla \cdot u_i = -\frac{1}{a} \nabla \cdot (\delta_i u_i), \]  
\[ (8) \]
\[ \dot{u}_i + Hu_i + \frac{1}{a} u_i \cdot \nabla u_i = -\frac{1}{a} \nabla \delta p_i - \frac{1}{a} \nabla \delta \Phi, \]  
\[ (9) \]
\[ \frac{1}{a^2} \nabla^2 \delta \Phi = 4\pi G \sum_j \tilde{\theta}_j \delta_j. \]  
\[ (10) \]

By introducing the expansion \( \theta_i \) and the rotation \( \bar{\omega}_i \) of each component as
\[ \theta_i = \frac{1}{a} \nabla \cdot u_i, \quad \bar{\omega}_i = \frac{1}{a} \nabla \times u_i, \]  
\[ (11) \]
Eq. \[ (12) \] gives
\[ \dot{\theta}_i + 2H \theta_i + 4\pi G \sum_j \tilde{\theta}_j \delta_j = -\frac{1}{a^2} \nabla \cdot (u_i \cdot \nabla u_i) - \frac{1}{a^2 \tilde{\theta}_i} \nabla \left( \frac{\nabla \delta p_i}{1 + \delta_i} \right), \]  
\[ (12) \]
\[ \tilde{\omega}_i + 2H \bar{\omega}_i = -\frac{1}{a^2} \nabla \times (u_i \cdot \nabla u_i) + \frac{1}{a^2 \tilde{\theta}_i} \left( \nabla \delta_i \times \nabla \delta p_i \right). \]  
\[ (13) \]

By introducing decomposition of perturbed velocity into the potential- and transverse parts as
\[ u_i \equiv \nabla u_i + u_i^{(v)}, \quad \nabla \cdot u_i^{(v)} \equiv 0; \quad \theta_i = \Delta \frac{\delta_i}{a} u_i, \quad \bar{\omega}_i = \frac{1}{a} \nabla \times u_i^{(v)}, \]  
\[ (14) \]

instead of Eq. \[ (13) \] we have
\[ \dot{u}_i^{(v)} + Hu_i^{(v)} = -\frac{1}{a} \left[ u_i \cdot \nabla u_i + \frac{1}{\tilde{\theta}_i} \nabla \delta p_i - \Delta^{-1} \nabla \cdot \left( u_i \cdot \nabla u_i + \frac{1}{\tilde{\theta}_i} \nabla \delta p_i \right) \right]. \]  
\[ (15) \]

We note that the pure \( u_i \) contributions in the right-hand-side of Eq. \[ (13) \] or Eq. \[ (14) \] vanish. Thus, under vanishing pressure, pure irrotational perturbation cannot generate the rotational perturbation. Equation \[ (13) \] shows that presence of pressure perturbation oblique (i.e., non-parallel) to the density perturbation can generate rotational perturbation. Combining Eqs. \[ (8) \] \[ \text{and} \] \[ (12) \] we can derive
\[ \ddot{\delta}_i + 2H \dot{\delta}_i - 4\pi G \sum_j \tilde{\theta}_j \delta_j = -\frac{1}{a^2} \left[ a \nabla \cdot (\delta_i u_i) \right] + \frac{1}{a^2 \tilde{\theta}_i} \nabla \cdot (u_i \cdot \nabla u_i) + \frac{1}{a^2 \tilde{\theta}_i} \nabla \cdot \left( \nabla \delta p_i \right). \]  
\[ (16) \]

Equations \[ (8) \]-\[ (10) \] are valid to fully nonlinear order. Notice that for vanishing pressure these equations have only quadratic order nonlinearity in perturbations.
III. SUMMARY OF PREVIOUS WORK

In [11, 12] we have derived second-order perturbation equations valid for the single component, irrotational, and zero-pressure medium in zero-curvature background. These are

\[
\dot{\delta} + \frac{1}{a} \nabla \cdot \mathbf{u} = -\frac{1}{a} \nabla \cdot (\delta \mathbf{u}),
\]

(17)

\[
\frac{1}{a} \nabla \cdot (\mathbf{u} + H \mathbf{u}) + 4\pi G \rho \delta = -\frac{1}{a^2} \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) - \dot{\zeta}^{(i)}_{\alpha\beta} \left( \frac{2}{a} u_{\alpha|\beta} + \dot{C}^{(i)}_{\alpha\beta} \right),
\]

(18)

\[
\ddot{\delta} + 2H \dot{\delta} - 4\pi G \rho \delta = \frac{1}{a^2} \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) - \frac{1}{a^2} [a \nabla \cdot (\delta \mathbf{u})] + \dot{\zeta}^{(i)}_{\alpha\beta} \left( \frac{2}{a} u_{\alpha|\beta} + \dot{C}^{(i)}_{\alpha\beta} \right).
\]

(19)

Except for the presence of tensor-type perturbation, Eqs. (17)-(19) are exactly the same as the ones known in the Newtonian theory. We note that these equations are valid in the presence of \(\Lambda\). To the linear order, these are valid in the presence of general \(K\), see Sec. VIII. We have correctly identified the relativistic density and velocity perturbation variables which correspond to the Newtonian counterparts to the second order. In the relativistic context, our \(\delta\) and \(\mathbf{u}\) are the density perturbation and (related to) the perturbed expansion scalar, respectively, in the comoving gauge; the variables are equivalently gauge-invariant. However, we were not able to identify relativistic variable which corresponds to the Newtonian gravitational potential to the second order; this is understandable if we consider the factor of two difference between Einstein’s (post-Newtonian) and Newton’s gravity theories in predicting the light bending under the gravitational field. To the linear order the spatial curvature perturbation in the zero-shear gauge can be identified as the perturbed Newtonian potential [13, 16]. Equations (17)-(19) include effects of gravitational waves to the density and velocity perturbations. Equations of the gravitational waves can be found in [12].

To the third order, we have [13]

\[
\ddot{\delta} + \frac{1}{a} \nabla \cdot \mathbf{u} = -\frac{1}{a} \nabla \cdot (\delta \mathbf{u}) + \frac{1}{a} [2\varphi \mathbf{u} - \nabla (\Delta^{-1} X)] \cdot \nabla \delta,
\]

(20)

\[
\frac{1}{a} \nabla \cdot \left( \mathbf{u} + \frac{\dot{\alpha}}{a} \mathbf{u} \right) + 4\pi G \mu \delta = -\frac{1}{a^2} \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u})
\]

\[
-\frac{2}{3a^2} \varphi \mathbf{u} \cdot \nabla (\nabla \cdot \mathbf{u}) + \frac{4}{a^2} \nabla \cdot \left[ \varphi \left( \mathbf{u} \cdot \nabla \mathbf{u} - \frac{1}{3} \mathbf{u} \nabla \cdot \mathbf{u} \right) \right] - \frac{\Delta}{a^2} \left[ \mathbf{u} \cdot \nabla (\Delta^{-1} X) \right] + \frac{1}{a^2} \mathbf{u} \cdot \nabla X + \frac{2}{3a^2} X \nabla \cdot \mathbf{u},
\]

(21)

\[
\ddot{\delta} + 2 \frac{\dot{\alpha}}{a} \dot{\delta} - 4\pi G \mu \delta = -\frac{1}{a^2 \dot{\delta}} \left[ \varphi \cdot (\delta \mathbf{u}) \right] + \frac{1}{a^2} \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) + \frac{1}{a^2} \dot{\partial} \left\{ a [2\varphi \mathbf{u} - \nabla (\Delta^{-1} X)] \cdot \nabla \delta \right\}
\]

\[
+ \frac{2}{3a^2} \varphi (\mathbf{u} \cdot \nabla (\nabla \cdot \mathbf{u})) - \frac{4}{a^2} \nabla \cdot \left[ \varphi \left( \mathbf{u} \cdot \nabla \mathbf{u} - \frac{1}{3} \mathbf{u} \nabla \cdot \mathbf{u} \right) \right] + \frac{\Delta}{a^2} \left[ \mathbf{u} \cdot \nabla (\Delta^{-1} X) \right] - \frac{1}{a^2} \mathbf{u} \cdot \nabla X + \frac{2}{3a^2} X \nabla \cdot \mathbf{u},
\]

(22)

where

\[
X \equiv 2\varphi \nabla \cdot \mathbf{u} - \mathbf{u} \cdot \nabla \varphi + \frac{3}{2} \Delta^{-1} \nabla \cdot [\mathbf{u} \cdot \nabla (\nabla \varphi) + \mathbf{u} \Delta \varphi].
\]

(23)

In these equations we ignored the role of tensor-type perturbation; for a complete set of equations, see [13]. The variable \(\varphi\) is a perturbed-order metric (spatial curvature) variable in the comoving gauge condition, see later.

All the third-order correction terms in Eqs. (20)-(22) are simply of \(\varphi\)-order higher than the second-order relativistic/Newtonian terms. Thus, the pure general relativistic effects are at least \(\varphi\)-order higher than the relativistic/Newtonian ones in the second order equations. Thus, we only need the behavior of \(\varphi\) to the linear order which is related to the other hydrodynamic variables as

\[
\varphi = -\delta \Phi + \dot{\Phi} \Delta^{-1} \nabla \cdot \mathbf{u}.
\]

(24)

It also satisfies [17]

\[
\dot{\varphi} = 0,
\]

(25)

thus \(\varphi = C(\mathbf{x})\) with no decaying mode; this is true considering the presence of the cosmological constant, see [17].
IV. RELATIVISTIC FULLY NONLINEAR EQUATIONS

In this section, for convenience, we present some additional covariant or ADM equations not available in [11]. In the multi-component situation, we have

\[ \bar{T}_{ab} = \sum_j \bar{T}_{(j)ab}. \]  

(26)

The energy-momentum conservation gives

\[ \bar{T}_{(i)a}^b \equiv \bar{I}_{(i)a}^b, \quad \sum_j \bar{I}_{(j)a}^b = 0. \]  

(27)

Tildes indicate the covariant quantities.

A. Covariant equations

We introduce the fluid quantities as

\[ \bar{T}_{(i)ab} \equiv \tilde{\mu}_{(i)} \tilde{u}_{(i)a} \tilde{u}_{(i)b} + \tilde{p}_{(i)} (\tilde{g}_{ab} + \tilde{u}_{(i)a} \tilde{u}_{(i)b}) + \tilde{q}_{(i)a} \tilde{u}_{(i)b} + \tilde{q}_{(i)b} \tilde{u}_{(i)a} + \tilde{\pi}_{(i)ab}, \]  

(28)

where

\[ \tilde{u}_{(i)a}^0 \equiv -1, \quad \tilde{u}_{(i)a}^a \tilde{q}_{(i)a}^a \equiv 0 \equiv \tilde{u}_{(i)a}^b \tilde{\pi}_{(i)ab}, \quad \tilde{\pi}_{(i)a}^a = 0. \]  

(29)

The fluid quantities of each component are based on the fluid four-vector \( \tilde{u}_{(i)a} \) as

\[ \tilde{\mu}_{(i)} \equiv \tilde{T}_{(i)ab} \tilde{u}_{(i)a} \tilde{u}_{(i)b}, \quad \tilde{p}_{(i)} \equiv \frac{1}{3} \tilde{T}_{(i)ab} \tilde{h}_{(i)a}^b, \quad \tilde{q}_{(i)a} \equiv - \tilde{T}_{(i)c} \tilde{u}_{(i)b} \tilde{h}_{(i)c}^a, \quad \tilde{\pi}_{(i)ab} \equiv \tilde{T}_{(i)c} \tilde{h}_{(i)a}^c \tilde{h}_{(i)b}^d - \tilde{p}_{(i)} \tilde{h}_{(i)ab}. \]  

(30)

where \( \tilde{h}_{(i)ab} \equiv \tilde{g}_{ab} + \tilde{u}_{(i)a} \tilde{u}_{(i)b} \). Equation (27) gives

\[ \tilde{\mu}_{(i)} + (\tilde{\mu}_{(i)} + \tilde{p}_{(i)}) \tilde{\theta}_{(i)}^b + \tilde{q}_{(i)a} \tilde{\theta}_{(i)}^a + \tilde{q}_{(i)b} \tilde{\theta}_{(i)}^b + \tilde{\pi}_{(i)ab} = - \tilde{u}_{(i)a} \tilde{h}_{(i)ab}, \]  

(31)

\[ (\tilde{\mu}_{(i)} + \tilde{p}_{(i)}) \tilde{\theta}_{(i)} + \tilde{h}_{(i)ab} (\tilde{\theta}_{(i)ab} + \tilde{\pi}_{(i)ab} + \frac{4}{3} \tilde{\theta}_{(i)} \tilde{h}_{(i)ab}) = \tilde{h}_{(i)ab} \tilde{\theta}_{(i)ab}, \]  

(32)

where the kinematical quantities are also based on the fluid four-vector \( \tilde{u}_{(i)a} \) as

\[ \tilde{h}_{(i)ab} \tilde{u}_{(i)c} = \tilde{h}_{(i)ab} \tilde{u}_{(i)c} + \tilde{h}_{(i)bc} \tilde{u}_{(i)c} = \tilde{\omega}_{(i)ab} + \tilde{\theta}_{(i)ab} = \tilde{u}_{(i)a} \tilde{u}_{(i)b} + \tilde{\pi}_{(i)ab}, \]  

(33)

In the multi-component situation, we can derive the corresponding equation of Raychaudhury equation for the individual component. From \( \tilde{u}_{(i)ab} - \tilde{u}_{(i)a} \tilde{u}_{(i)b} = \tilde{u}_{(i)} dR_{abc}^d \), we can derive

\[ \tilde{\theta}_{(i)} + \frac{1}{3} \tilde{\theta}_{(i)}^a \tilde{\theta}_{(i)}^a + \tilde{\pi}_{(i)ab} = 4\pi G \left( \tilde{\mu} - 3 \tilde{p} - 2 \tilde{T}_{ab} \tilde{u}_{(i)a} \tilde{u}_{(i)b} \right) + \Lambda. \]  

(34)

In the energy-frame we take \( \tilde{q}_{(i)a} = 0 \) for each component of the fluids without losing any physical degree of freedom.

In a single component situation, taking the energy-frame, the energy conservation equation, the momentum conservation equation, and the Raychaudhury equation are [18]

\[ \tilde{\mu} + (\tilde{\mu} + \tilde{p}) \tilde{\theta} + \tilde{\pi}_{ab} \tilde{\sigma}_{ab} = 0, \]  

(35)

\[ (\tilde{\mu} + \tilde{p}) \tilde{a}_a + \tilde{h}_b^b (\tilde{\theta}_b + \tilde{\pi}_{bc}) = 0, \]  

(36)

\[ \tilde{\theta} + \frac{1}{3} \tilde{\theta}_{(i)} = \tilde{\pi}_{ab} \tilde{\sigma}_{ab} - \tilde{\omega}_{ab} \tilde{\omega}_{ab} + 4 \pi G (\tilde{\mu} + 3 \tilde{p}) - \Lambda = 0. \]  

(37)
By combining Eqs. (35)–(37) we can derive
\[
\left( \frac{\tilde{\mu} + \tilde{\pi}^{ab}\tilde{\sigma}_{ab}}{\mu + \tilde{\rho}} \right) \tilde{\mu} + \frac{1}{3} \left( \frac{\tilde{\mu} + \tilde{\pi}^{ab}\tilde{\sigma}_{ab}}{\mu + \tilde{\rho}} \right)^2 = 4\pi G \left( \tilde{\mu} + 3\tilde{\rho} \right) - \Lambda + \tilde{\sigma}^{ab}\tilde{\sigma}_{ab} - \tilde{\omega}^{ab}\tilde{\omega}_{ab} + \left[ \frac{\tilde{h}^{ab}(\tilde{p}_{ab} + \tilde{\pi}^{ce})}{\mu + \tilde{\rho}} \right]_{;a}. \tag{38}
\]

This equation was derived in Eq. (88) of [19], see also [20]. In the multi-component case, in the energy-frame, combining Eqs. (31)–(34) we can derive
\[
\left( \frac{\tilde{\mu}(i) + \tilde{\pi}^{ab}(i)\tilde{\sigma}_{ab}(i) + \tilde{u}^a_{(i)}\tilde{\pi}^{b}_{(i)}(i)}{\mu(i) + \tilde{p}(i)} \right) \tilde{\mu}(i) + \frac{1}{3} \left( \frac{\tilde{\mu}(i) + \tilde{\pi}^{ab}(i)\tilde{\sigma}_{ab}(i) + \tilde{u}^a_{(i)}\tilde{\pi}^{b}_{(i)}(i)}{\mu(i) + \tilde{p}(i)} \right)^2 = -4\pi G \left( \tilde{\mu} - 3\tilde{\rho} - 2\tilde{T}_{ab}\tilde{u}^a_{(i)}\tilde{u}^b_{(i)} \right) - \Lambda + \tilde{\sigma}^{ab}(i)\tilde{\sigma}_{ab}(i) - \tilde{\omega}^{ab}(i)\tilde{\omega}_{ab}(i) + \left[ \frac{\tilde{h}^{ab}(i)(\tilde{p}_{(i),ab} + \tilde{\pi}_{(i),bc} - \tilde{I}_{(i),b})}{\mu(i) + \tilde{p}(i)} \right]_{;a}. \tag{39}
\]

In [21] Langlois and Vernizzi derived a simple covariant relation which leads to one of the conserved variable in the large-scale limit. These authors introduced
\[
\tilde{\zeta}_a \equiv \tilde{h}_a^b \begin{pmatrix} \tilde{\alpha}_{(i),b} + \frac{\tilde{\mu}_{(i),ab}}{3(\mu + \tilde{\rho})} \end{pmatrix}, \quad \tilde{\alpha} \equiv \tilde{\alpha}_{(i),a}\tilde{u}^a_{(i)} \equiv \frac{1}{3} \tilde{\theta}. \tag{40}
\]

Using only the energy conservation in Eq. (39) we can derive
\[
\mathcal{L}_u \tilde{\zeta}_a = -\frac{\tilde{\theta}}{3(\mu + \tilde{\rho})} \tilde{\mu}_a + \frac{\tilde{\theta}}{3(\mu + \tilde{\rho})} \tilde{\mu}_a - \left[ \tilde{u}_a \tilde{\pi}^{bc}\tilde{\sigma}_{bc} \right]_{\tilde{\mu}_a} - \left[ \tilde{u}_a \tilde{\pi}^{bc}\tilde{\sigma}_{bc} \right]_{\tilde{\mu}_a} - \left[ \tilde{\mu}_a \tilde{\pi}^{bc}\tilde{\sigma}_{bc} \right]_{\tilde{\mu}_a} + \frac{\tilde{\mu}_a \tilde{\pi}^{bc}\tilde{\sigma}_{bc}}{3(\mu + \tilde{\rho})^2}. \tag{41}
\]

where \( \mathcal{L}_u \) is a Lie derivative along \( \tilde{u}_a \) with \( \mathcal{L}_u \tilde{\zeta}_a \equiv \tilde{\zeta}_a, b \tilde{u}^b + \tilde{\zeta}_a, b \tilde{u}^b \). Thus, for vanishing anisotropic pressure, we have the Langlois-Vernizzi relation [21]
\[
\mathcal{L}_u \tilde{\zeta}_a = -\frac{\tilde{\theta}}{3(\mu + \tilde{\rho})} \tilde{h}_a^b \left( \tilde{\mu}_b - \frac{\tilde{\theta}}{\mu} \tilde{\mu}_b \right). \tag{42}
\]

These equations are valid in a single component fluid, or in multiple component fluids for the collective fluid variables. We can easily extend the relation to the individual fluid component as follows. We introduce
\[
\tilde{\zeta}_{(i),a} \equiv \tilde{h}_{(i),a}^b \left( \tilde{\alpha}_{(i),b} + \frac{\tilde{\mu}_{(i),ab}}{3(\mu + \tilde{\rho})} \right), \quad \tilde{\alpha}_{(i),a} \equiv \tilde{\alpha}_{(i),a}\tilde{u}^a_{(i)} \equiv \frac{1}{3} \tilde{\theta}_{(i)}. \tag{43}
\]

Using only Eq. (31) we can derive
\[
\mathcal{L}_{\tilde{u}_{(i)}} \tilde{\zeta}_{(i),a} = -\frac{\tilde{\theta}_{(i)}}{3(\mu + \tilde{\rho})} \tilde{\mu}_{(i),a} + \frac{\tilde{\theta}_{(i)}}{3(\mu + \tilde{\rho})} \tilde{\mu}_{(i),a} - \left[ \tilde{u}_{(i),a} \tilde{\pi}_{(i)}^{bc}\tilde{\sigma}_{bc} \right]_{\tilde{\mu}_{(i),a}} - \left[ \tilde{u}_{(i),a} \tilde{\pi}_{(i)}^{bc}\tilde{\sigma}_{bc} \right]_{\tilde{\mu}_{(i),a}} + \tilde{\mu}_{(i),a} \tilde{\pi}_{(i)}^{bc}\tilde{\sigma}_{bc} - \tilde{\mu}_{(i),a} \tilde{\pi}_{(i)}^{bc}\tilde{\sigma}_{bc} \tag{44}
\]
\[
\mathcal{L}_{\tilde{u}_{(i)}} \tilde{\zeta}_{(i),a} = -\frac{\tilde{\theta}_{(i)}}{3(\mu + \tilde{\rho})} \tilde{h}_{(i),a}^b \left( \tilde{\mu}_{(i),b} - \frac{\tilde{\theta}_{(i)}}{\mu} \tilde{\mu}_{(i),b} \right). \tag{45}
\]

Application of these compact relations to large-scale conservation properties to the second order will be studied in Sec. X1.
B. ADM equations

The ADM formulation \[22\] is presented in Eqs. (2)-(13),(47),(48) of \[11\]. Interpretation of the ADM fluid quantities in Eqs. (45),(46) of \[11\] was based on the normal-frame fluid quantities; for relations to the energy-frame fluid quantities, see Eq. (57) below. The ADM fluid quantities of individual component are introduced as

\[ E_{(i)} = \frac{\tilde{\eta}_a \tilde{\eta}_b \tilde{T}^{ab}_{(i)}}{N^2}, \quad J_{(i)\alpha} = -\tilde{\eta}_b \tilde{T}^{b}_{(i)\alpha}, \quad S_{(i)\alpha\beta} = \tilde{T}_{(i)\alpha\beta}, \quad S_{(i)} = \tilde{h}^\alpha_{\alpha} S_{(i)\alpha\beta} = S_{(i)\alpha\beta} - \frac{1}{3} \tilde{h}_{\alpha\beta} S_{(i)}. \] (46)

Equation \[27\] gives Eqs. (12),(13),(47),(48) in \[11\]. Equations (10),(12),(47) in \[11\] can be arranged as

\[ \frac{1}{N} \left( \partial_\theta - N^\alpha \partial_\alpha \right) K_{\alpha} - \frac{1}{3} \left( K_{\alpha} \right)^2 = 4\pi G \left( E + S \right) - \Lambda + K^{\alpha\beta} \tilde{K}_{\alpha\beta} - \frac{1}{N} \tilde{\eta}^\alpha_{\alpha}, \] (47)

\[ K_{\alpha} = \left( E + \frac{1}{3} S \right)^{-1} \left[ \frac{1}{N} \left( \partial_\theta - N^\alpha \partial_\alpha \right) E + \frac{1}{N^2} \left( N^2 J^\alpha \right)_{\alpha} - \tilde{S}^{\alpha\beta} \tilde{K}_{\alpha\beta} \right], \] (48)

\[ K_{\alpha} = \left( E_{(i)} + \frac{1}{3} S_{(i)} \right)^{-1} \left[ \frac{1}{N} \left( \partial_\theta - N^\alpha \partial_\alpha \right) E_{(i)} + \frac{1}{N^2} \left( N^2 J^\alpha_{(i)} \right)_{\alpha} - \tilde{S}^{\alpha\beta}_{(i)} \tilde{K}_{\alpha\beta} + \frac{1}{N} \left( \tilde{I}_{(i)\alpha} - \tilde{I}_{(i)} N^\alpha \right) \right]. \] (49)

Momentum conservation equations for the collective and individual components can be found in Eqs. (13),(48) of \[11\]. By combining Eqs. \[47-49\] we can derive the ADM counterpart of the density perturbation equations in Eqs. \[35,39\].

V. SECOND-ORDER PERTURBATIONS

We use a metric convention in Eq. (49) of \[11\]

\[ \tilde{g}_{00} = -a^2 (1 + 2A), \quad \tilde{g}_{0\alpha} = -a^2 B_{\alpha}, \quad \tilde{g}_{\alpha\beta} = a^2 \left( g^{(3)}_{\alpha\beta} + 2C_{\alpha\beta} \right). \] (50)

The subindex 0 indicates the conformal time \( \eta \) with \( ad\eta \equiv cdt \). To the second-order in perturbation we introduce the fluid four-vector of individual component as

\[ \tilde{u}^\alpha_{(i)} = \frac{1}{a} V^\alpha_{(i)}, \quad \tilde{u}^0_{(i)} = \frac{1}{a} \left( 1 - A + \frac{3}{2} A^2 + \frac{1}{2} V^\alpha_{(i)} V_{(i)\alpha} - B^\alpha V_{(i)\alpha} \right); \]

\[ \tilde{u}_{(i)\alpha} = a \left( V_{(i)\alpha} - B_{\alpha} + AB_{\alpha} + 2V^\beta_{(i)} C_{\alpha\beta} \right), \quad \tilde{u}_{(i)0} = -a \left( 1 + A - \frac{1}{2} A^2 + \frac{1}{2} V^\alpha_{(i)} V_{(i)\alpha} \right). \] (51)

In this definition of fluid four-vector we follow the notation in Eq. (53) of \[11\]. If we introduce \( \tilde{u}^\alpha_{(i)} = \tilde{V}^\alpha_{(i)} \tilde{u}^0_{(i)} \), we have \( V^\alpha_{(i)} = (1 - A) \tilde{V}^\alpha_{(i)} \). The fluid quantities of individual component are introduced as

\[ \tilde{\mu}_{(i)} \equiv \mu_{(i)} + \delta \mu_{(i)}, \quad \tilde{p}_{(i)} \equiv p_{(i)} + \delta p_{(i)}, \quad \tilde{\pi}_{(i)\alpha\beta} \equiv a^2 \Pi_{(i)\alpha\beta}, \quad \tilde{\pi}_{(i)0\alpha} = -a^2 \Pi_{(i)\alpha\beta} V^\beta_{(i)}, \quad \tilde{\pi}_{(i)00} = 0, \] (52)

where from \( \tilde{\pi}^{a}_{(i)\alpha} = 0 \) we have

\[ \Pi_{(i)\alpha} - 2C^{\alpha\beta} \Pi_{(i)\alpha\beta} = 0. \] (53)

The energy-momentum tensor of individual component in the energy-frame follows from Eq. \[28\] as

\[ \tilde{T}^0_{(i)\alpha} = -\mu_{(i)} + \delta \mu_{(i)} - (\mu_{(i)} + p_{(i)}) \left( V_{(i)\alpha} - B_{\alpha} \right), \]

\[ \tilde{T}_{(i)\alpha} = (\mu_{(i)} + p_{(i)}) \left( V_{(i)\alpha} - B_{\alpha} + AV_{(i)\alpha} + 2AB_{\alpha} + 2V^\beta_{(i)} C_{\alpha\beta} \right) + (\delta \mu_{(i)} + \delta p_{(i)}) \left( V_{(i)\alpha} - B_{\alpha} \right) + \Pi_{(i)\alpha\beta} \left( V^\beta_{(i)} - B^\beta \right), \]

\[ \tilde{T}_{(i)\beta} = (p_{(i)} + \delta p_{(i)}) \delta^\beta_{\beta} + (\mu_{(i)} + p_{(i)}) V^\alpha_{(i)} \left( V_{(i)\beta} - B_{\beta} \right) + \Pi_{(i)\beta\gamma} - 2C^{\alpha\beta} \Pi_{(i)\beta\gamma}. \] (54)
Using $\vec{T}_b^a = \sum_j \vec{T}_{(j)k}^a$, and the total fluid quantities in Eq. (82) of [1] we have

$$\mu = \sum_j \mu_j, \quad p = \sum_j p_j,$$

(55)

for the background order fluid quantities, and

$$\delta \mu = \sum_j \left[ \delta \mu_j + (\mu_j + p_j) \left( V_{(j)}^\alpha - B^\alpha \right) (V_{(j)\alpha} - V_\alpha) \right],$$

$$\delta p = \sum_j \left[ \delta p_j + \frac{1}{3} (\mu_j + p_j) \left( V_{(j)}^\alpha - B^\alpha \right) (V_{(j)\alpha} - V_\alpha) \right],$$

$$(\mu + p) V_\alpha = \sum_j \left[ (\mu_j + p_j) V_{(j)\alpha} + (\delta \mu_j + \delta p_j) (V_{(j)\alpha} - V_\alpha) + \Pi_{(j)\beta} (V_{(j)\beta} - V_\beta) \right],$$

$$\Pi_{\beta} = \sum_j \left\{ \Pi_{(j)\beta} + (\mu_j + p_j) \left[ (V_{(j)\alpha} - B_\alpha) (V_{(j)\beta} - B_\beta) - \frac{1}{3} \delta_{\beta} (V_{(j)}^{\gamma} - B^{\gamma}) (V_{(j)\gamma} - V_\gamma) \right] \right\},$$

(56)

for perturbed order fluid quantities to the second-order. From Eq. (40) we have the ADM fluid quantities based on the energy-frame fluid quantities

$$E_{(i)} = \mu_{(i)} + \delta \mu_{(i)} + (\mu_{(i)} + p_{(i)}) \left( V_{(i)}^\alpha - B^\alpha \right) (V_{(i)\alpha} - B_\alpha),$$

$$J_{(i)\alpha} = a \left[ (\mu_{(i)} + p_{(i)}) \left( V_{(i)\alpha} - B_\alpha + A B_\alpha + 2V_{(i)}^\beta C_{\alpha\beta} \right) + (\delta \mu_{(i)} + \delta p_{(i)}) (V_{(i)\alpha} - B_\alpha) + \Pi_{(i)\alpha\beta} (V_{(i)\beta} - B_\beta) \right],$$

$$S_{(i)} = 3 (p_{(i)} + \delta p_{(i)}) + (\mu_{(i)} + p_{(i)}) \left( V_{(i)}^\alpha - B^\alpha \right) (V_{(i)\alpha} - B_\alpha),$$

$$\dot{S}_{(i)\alpha\beta} = a^2 \left\{ \Pi_{(i)\alpha\beta} + (\mu_{(i)} + p_{(i)}) \left[ (V_{(i)\alpha} - B_\alpha) (V_{(i)\beta} - B_\beta) - \frac{1}{3} \delta_{\beta} (V_{(i)}^{\gamma} - B^{\gamma}) (V_{(i)\gamma} - B_\gamma) \right] \right\}. $$

(57)

We can compare Eq. (57) with the ADM fluid quantities based on the normal-frame fluid quantities in Eq. (76) of [1]; in a single component case we simply delete (i) subindices.

In [1] the fluid quantities are based on the normal-frame vector. By taking $\tilde{u}_{(i)\alpha} \equiv 0, \tilde{u}_{(i)\alpha}$ becomes the normal-frame vector $\tilde{n}_\alpha$, see Eq. (54) in [1]. Based on the normal-frame vector, to the second order, the fluid quantities have contributions due to the frame choice: for example, even in the zero-pressure fluid, the perturbed pressure based on the normal-frame does not necessarily vanish to the second-order, see Eq. (58) below. By comparing Eq. (57) with Eq. (76) of [1] we have

$$\delta \mu_{(i)} = \delta \mu_{(i)} + (\mu_{(i)} + p_{(i)}) \left( V_{(i)}^\alpha - B^\alpha \right) (V_{(i)\alpha} - B_\alpha),$$

$$\delta p_{(i)} = \delta p_{(i)} + \frac{1}{3} (\mu_{(i)} + p_{(i)}) \left( V_{(i)}^\alpha - B^\alpha \right) (V_{(i)\alpha} - B_\alpha),$$

$$Q_{(i)\alpha} = (\mu_{(i)} + p_{(i)}) \left( V_{(i)\alpha} - B_\alpha + A B_\alpha + 2V_{(i)}^\beta C_{\alpha\beta} \right) + (\delta \mu_{(i)} + \delta p_{(i)}) (V_{(i)\alpha} - B_\alpha) + \Pi_{(i)\alpha\beta} (V_{(i)\beta} - B_\beta),$$

$$\Pi_{(i)\alpha\beta} = \Pi_{(i)\alpha\beta} + (\mu_{(i)} + p_{(i)}) \left[ (V_{(i)\alpha} - B_\alpha) (V_{(i)\beta} - B_\beta) - \frac{1}{3} \delta_{\beta} (V_{(i)}^{\gamma} - B^{\gamma}) (V_{(i)\gamma} - B_\gamma) \right].$$

(58)

For the total fluid quantities the relations between the two frames are presented in Eq. (87) of [1]. Thus, by replacing all fluid quantities in Eqs. (99)-(107) of [1] using Eq. (58) and Eq. (87) of [1] we have the equations in the energy frame. Using $Q_{(i)\alpha}$ in Eq. (58), Eq. (51) gives

$$\vec{T}_{(i)\alpha} = (1 - A) Q_{(i)\alpha}.$$

(59)

As the fluid four-velocity of i-th component we can use either $Q_{(i)\alpha}$ or $V_{(i)\alpha} - B_\alpha$ related by Eq. (58).

The kinematic quantities in the energy-frame are presented in Eqs. (63)-(66) of [1]. In Eq. (33) we introduced
kinematic quantities for the individual component. To the second order we can show

\[ \hat{\theta}_i = \frac{3a'}{a^2} - \frac{3a'}{a^2} A + \frac{1}{a} C_{\alpha'} + \frac{1}{a} V_i(a) + \frac{9a'}{2a^2} A^2 - 3\frac{a'}{a^2} B^\alpha V_i(a) - \frac{1}{a} A C_{\alpha'} + \frac{1}{a} V_i(\alpha) \left( A_{\alpha} + C_{\beta|\alpha}^\beta \right) \]

\[ + \frac{3a'}{2a^2} V_i(a) \left( V_i(\alpha) - B^\alpha \right) (V_i(\alpha) - B_\alpha)^{'} - \frac{2}{a} C_{\alpha\beta} C_{\alpha\beta}, \]

\[ \hat{\sigma}_{(i)\alpha} = a \left[ (V_i(a) - \beta - A^\alpha C_{\alpha}) + (V_i(a) - B^\alpha) (V_i(\beta) - B_\beta)^{'} + V_i(a) A_{\alpha} + V_i(a) C_{\alpha\gamma} + 2V_i(a) C_{\gamma \beta} \right] \]

\[ - \frac{2}{3} C_{\alpha\beta} \left( C_{\gamma}^{\gamma} + V_i(\alpha) \right) \]

\[ - \frac{1}{3} C_{\alpha\beta} \left( V_i(\alpha) + C_{\gamma}^{\gamma} + (V_i(\gamma) - B_\gamma)^{'} + V_i(a) A_{\alpha} + V_i(a) C_{\alpha\gamma} - 2C_{\alpha\beta} C_{\gamma}^{\gamma} \right), \]

\[ \hat{\omega}_{(i)\beta} = a \left( V_i(a) - B^\alpha + A B^\alpha + 2V_i(a) C_{\alpha \beta} \right) - a \left( V_i(a) - B^\alpha \right) \left[ (V_i(\beta) - B_\beta)^{'} + A_{\beta} \right], \]

\[ \hat{a}_{(i)\alpha} = A_{\alpha} + \frac{1}{a} \left[ a \left( V_i(\alpha) - B_\alpha + A B^\alpha + 2V_i(a) C_{\alpha \beta} \right) + 2A A_{\alpha} - A_{\alpha} \right] \left( V_i(a) - B_\alpha \right) \]

\[ + V_i(a) \left( V_i(a) - B_\alpha \right) \left[ A_{\alpha} + B_\beta \right], \]

\[ \hat{\sigma}_{(i)\alpha} = -V_i(a) \hat{a}_{(i)\alpha}, \quad \hat{\sigma}_{(i)0} = 0; \quad \hat{\omega}_{(i)0} = -V_i(a) \hat{a}_{(i)\alpha}, \quad \hat{\omega}_{(i)0} = 0; \quad \hat{a}_{(i)0} = -V_i(a) \hat{a}_{(i)\alpha}, \quad (60) \]

where a prime indicates the time derivative based on \( \eta \).

The gauge transformation properties of the fluid quantities are presented in Eqs. (232)-(235) of \[11\] for the normal-frame, and Eqs. (238) of \[11\] for the energy-frame. A prescription to get the gauge transformation properties for individual fluid quantities is also presented below Eq. (235) of \[11\]. Under the gauge transformation we have

\[ \delta \mu_i = \delta \mu_i \left( \rho_i(a), \xi^0, \xi \right), \]

\[ \delta \rho_i = \delta \rho_i \left( \rho_i(a), \xi^0, \xi \right), \]

\[ \delta V_i(a) - B_\alpha + \delta A B_\alpha + 2V_i(a) \delta A B_\alpha = V_i(a) - B_\alpha + A B_\alpha + 2V_i(a) C_{\alpha \beta} + \xi_0(a) - (V_i(a) - B_\alpha) \xi_0 - \frac{d'}{a} (V_i(a) - B_\alpha) \xi_0 \]

\[ + (V_i(\beta) - B_\beta) \xi_0(a) \xi_{\beta} - (V_i(a) - B_\alpha) \xi_0 - A \xi_0(a) \xi_0 - \xi_0(a) \xi_0 - \xi_0(a) \xi_0 - \xi_0(a) \xi_0, \]

\[ \hat{\Pi}_{(i)\alpha} = \hat{\Pi}_{(i)\alpha}, \quad (61) \]

A. Basic equations in the energy-frame

The basic set of equations with fluid quantities based on the normal-frame is presented in Eqs. (99)-(107) of \[11\]. By using Eq. (638) we can recover the equations with fluid quantities based on the energy frame. For convenience, in the following we present the complete set of equations with fluid quantities in the energy frame. These equations are written without taking any gauge conditions yet, thus in a sort of gauge-ready form. To the linear order this method was suggested by Bardeen \[23, 24\].

Definition of \( \delta K \):

\[ \tilde{K} + 3H + \delta K - 3HA + \hat{C}_\alpha^\alpha + \frac{1}{a} B^\alpha |_\alpha \]

\[ = -A \left( \frac{9}{2} H A - C_\alpha^\alpha - \frac{1}{a} B^\alpha |_\alpha \right) + \frac{3}{2} H B^\alpha B_\alpha + \frac{1}{a} B^\alpha \left( 2C_\alpha|\beta - C_\beta|\alpha \right) + 2C_{\alpha\beta} \left( \hat{C}_{\alpha\beta} + \frac{1}{a} B_\alpha|\beta \right) \equiv n_0. \]
Energy constraint equation:

\[
16\pi G\mu + 2\Lambda - 6H^2 - \frac{1}{a^2}R^{(3)} + 16\pi G\delta\mu + 4H\delta K - \frac{1}{a^2} \left( 2C^\beta_\alpha|\beta - 2C^\alpha_{\alpha|\beta} - \frac{2}{3}R^{(3)}C_\alpha \right) \\
= \frac{2}{3} \delta K^2 - 16\pi G(\mu + p)(V^\alpha - B^\alpha)(V_\alpha - B_\alpha) - \left( C_\alpha\beta + \frac{1}{a}B_{\alpha|\beta}\right) \left( C^\alpha_{\beta} + \frac{1}{a}B^\alpha_{\beta}\right) \left( C^\alpha_{\beta} + \frac{1}{a}B^\alpha_{\beta}\right) + \left( \frac{1}{a}C^\alpha_{\alpha|\beta} \right)^2 \\
+ \frac{1}{a^2} \left[ 4C^{\alpha\beta} \left( -C^\gamma_{\alpha|\beta\gamma} - C^\gamma_{\alpha|\beta\gamma} + C^\beta_{\alpha|\gamma\beta} + C^\gamma_{\alpha|\gamma\beta} \right) + \left( \frac{4}{3}R^{(3)}C^\alpha_{\alpha} \right)^2 \\
- \left( 2C^\beta_{\beta|\gamma} - C^\gamma_{\gamma|\beta} \right) \left( 2C^\alpha_{\beta|\alpha} - C^\alpha_{\alpha|\beta} \right) + C^\alpha_{\beta|\gamma} \left( 3C^\alpha_{\alpha|\gamma} - 2C^\alpha_{\alpha|\beta} \right) \right] \equiv n_1. \tag{63}
\]

Momentum constraint equation:

\[
\left[ \dot{C}^\beta_\alpha + \frac{1}{2a} \left( B^\beta_{\alpha|\alpha} + B^\alpha_{\beta|\alpha}\right) \right]_{\beta} - \frac{1}{3} \left( \dot{C}^\gamma_{\gamma} + \frac{1}{a}B^\gamma_{\gamma} \right)_{\alpha} + \frac{2}{3} \delta K_{\alpha} + 8\pi Ga (\mu + p)(V_\alpha - B_\alpha + AB_\alpha + 2V^\beta C_{\alpha\beta}) \\
= -\frac{2}{3} A\delta K_{\alpha} - 8\pi Ga \left[ (\delta\mu + \delta p)(V_\alpha - B_\alpha) + (\mu + p) A(V_\alpha - B_\alpha) + \Pi_{\alpha\beta} (V^\beta - B^\beta) \right] \\
+ A_{\alpha\beta} \left[ \dot{C}^\beta_\alpha + \frac{1}{2a} \left( B^\beta_{\alpha|\alpha} + B^\alpha_{\beta|\alpha}\right) \right] + \left( 2C^\beta_\gamma - C^\beta_{\beta|\gamma} \right) \left[ \dot{C}^\gamma_{\gamma} + \frac{1}{2a} \left( B^\gamma_{\gamma} + B^\gamma_{\alpha|\alpha}\right) \right] + 2C^\beta_{\gamma} \left( \dot{C}^\alpha_{\gamma\alpha} + \frac{1}{a}B^\beta_{(\alpha|\gamma)} \right) \right]_{\beta} \\
+ \frac{1}{a} \left[ B^\gamma \left( C^\beta_{\beta|\alpha} + C^\beta_{\alpha|\beta} - C^\beta_{\alpha|\gamma} \right) \right]_{\beta} + \frac{1}{3} C^\beta_{\alpha\beta} \left( \dot{C}^\gamma_{\gamma} + \frac{1}{a}B^\beta_{\gamma|\gamma} \right) \\
- \frac{1}{3} \left\{ A_{\alpha\beta} \left( \dot{C}^\gamma_{\gamma} + \frac{1}{a}B^\gamma_{\gamma} \right) + 2C^\gamma_{\beta|\gamma} \left( C^\alpha_{\gamma\alpha} + \frac{1}{a}B^\beta_{\gamma|\alpha} \right) + \frac{1}{a} \left[ B^\beta \left( 2C^\gamma_{\beta|\gamma} - C^\gamma_{\gamma|\beta} \right) \right]_{\alpha} \right\} \equiv n_{2\alpha}. \tag{64}
\]

Trace of the ADM propagation equation:

\[
- \left[ 3\dot{H} + 3H^2 + 4\pi G (\mu + 3p) - \Lambda \right] + \delta K + 2H\delta K - 4\pi G(\delta\mu + 3\delta p) + \left( 3\dot{H} + \frac{\Delta}{a^2} \right) A \\
= A\delta K - \frac{1}{a} \delta K_{\alpha} B^\alpha + \frac{1}{3} \delta K^2 + 8\pi G(\mu + p)(V^\alpha - B^\alpha)(V_\alpha - B_\alpha) \\
+ \frac{3}{2} \dot{H} (3A^2 - B^\alpha B_\alpha) + \frac{1}{a^2} \left[ 2A\Delta A + A^\alpha A_\alpha - \frac{1}{2} \Delta (B^\alpha B_\alpha) + A^\alpha \left( 2C^\beta_{\alpha|\beta} - C^\beta_{\beta|\alpha} \right) + 2C^\alpha_{\alpha|\beta} A_{\alpha|\beta} \right] \\
+ \left( \dot{C}^\alpha_{\alpha} + \frac{1}{a}B^\alpha_{(\alpha|\beta)} \right) \left( \dot{C}^\beta_{\alpha} + \frac{1}{a}B^\beta_{\alpha|\beta} \right) - \frac{1}{3} \left( \dot{C}^\alpha_{\alpha} + \frac{1}{a}B^\alpha_{(\alpha|\beta)} \right)^2 \equiv n_3. \tag{65}
\]
Tracefree ADM propagation equation:

\[
\left[ C^\alpha_\beta + \frac{1}{2a} \left( B^\alpha_{\beta|\gamma} + B^\beta_{\alpha|\gamma} \right) \right] + 3H \left[ \dot{C}^\alpha_\beta + \frac{1}{2a} \left( B^\alpha_{\beta|\gamma} + B^\beta_{\alpha|\gamma} \right) \right] - \frac{1}{a^2} A^\alpha_{\beta|\gamma}
\]

\[
- \frac{1}{3} \delta^\alpha_\beta \left[ \left( C^\gamma_\beta + \frac{1}{a} B^\gamma_{|\gamma} \right) \right] + 3H \left( C^\gamma_\beta + \frac{1}{a} B^\gamma_{|\gamma} \right) - \frac{1}{a^2} A^\gamma_\gamma
\]

\[
+ \frac{1}{a^2} \left[ C^\alpha_{\beta|\gamma} + C^\beta_{\alpha|\gamma} - C^\alpha_{|\gamma} - 2 \frac{B^\gamma_{|\gamma}}{3} R(3) C^\delta_\gamma - \frac{1}{3} \delta^\alpha_\beta \left( 2C^\gamma_{\delta|\gamma} - 2C^\gamma_{\gamma|\delta} - \frac{2}{3} R(3) C^\gamma_\gamma \right) \right] - 8\pi G \Pi^\alpha_\beta
\]

\[
= \left\{ \left[ \dot{C}^\alpha_\beta + \frac{1}{2a} \left( B^\alpha_{\beta|\gamma} + B^\beta_{\gamma|\alpha} \right) \right] A + 2C^\alpha_\gamma \left( \dot{C}^\gamma_\beta + \frac{1}{a} B_{(\gamma|\beta)} \right) + \frac{1}{a} B_\gamma \left( C^\alpha_{\gamma|\beta} + C^\beta_{\gamma|\alpha} - C^\gamma_{\alpha|\beta} \right) \right\}
\]

\[
+ 3H \left\{ \left[ C^\beta_\gamma + \frac{1}{2a} \left( B^\alpha_{\beta|\gamma} + B^\gamma_{\alpha|\beta} \right) \right] A + 2C^\beta_\gamma \left( \dot{C}^\gamma_\beta + \frac{1}{a} B_{(\gamma|\beta)} \right) + \frac{1}{a} B_\gamma \left( C^\beta_{\gamma|\beta} + C^\gamma_{\gamma|\alpha} - C^\gamma_{\alpha|\beta} \right) \right\}
\]

\[
+ \left( C^\gamma_\beta + \frac{1}{a} B^\gamma_{|\gamma} \right) A - \frac{1}{a} \left[ C^\alpha_\beta + \frac{1}{2a} \left( B^\alpha_{\beta|\gamma} + B^\gamma_{\alpha|\beta} \right) \right] B^\beta + \delta K \left[ \dot{C}^\gamma_\beta + \frac{1}{2a} \left( B^\alpha_{\beta|\gamma} + B^\gamma_{\alpha|\beta} \right) \right]
\]

\[
- \frac{1}{a^2} \left[ -AA^\alpha_{\beta|\gamma} - \frac{1}{2} \left( 1 - 2A^\alpha + 2B^\beta B_\gamma \right) - 2C^\alpha_{\beta|\gamma} A_{\gamma|\beta} - \left( C^\alpha_{\beta|\gamma} + C^\beta_{\gamma|\alpha} - C^\gamma_{\alpha|\beta} \right) A_{\gamma|\beta} \right]
\]

\[
+ \frac{1}{a^2} \left\{ \left[ \frac{1}{a} \left( C^\alpha_\beta + \frac{1}{a} B^\gamma_{|\gamma} \right) A + 2C^\gamma_\beta \left( \dot{C}^\alpha_\beta + \frac{1}{a} B_{(\beta|\alpha)} \right) + \frac{1}{a} B^\beta \left( 2C^\gamma_{|\gamma} - C^\gamma_{\gamma|\beta} \right) \right] \right\}
\]

\[
+ \left( C^\gamma_\beta + \frac{1}{a} B^\gamma_{|\gamma} \right) A - \frac{1}{a} \left( C^\beta_\gamma + \frac{1}{a} B^\beta_{|\gamma} \right) B^\gamma + \delta K \left( C^\gamma_\beta + \frac{1}{a} B^\gamma_{|\gamma} \right)
\]

\[
+ \frac{1}{a^2} \left[ -AA^\gamma_{|\gamma} + \frac{1}{2} \left( 1 - 2A^\beta + 2B^\beta B_\gamma \right) - 2C^\gamma_{|\gamma} A_{\gamma|\delta} - \left( 2C^\gamma_{|\gamma} - C^\gamma_{\gamma|\delta} \right) A_{\delta|\gamma} \right]
\]

\[
\frac{1}{a} B^\alpha_{\gamma|\gamma} \left[ \dot{C}^\alpha_\beta + \frac{1}{2a} \left( B^\gamma_{|\beta} + B^\beta_{|\gamma} \right) \right] - \frac{1}{a} B^\beta_{|\gamma} \left[ \dot{C}^\gamma_\beta + \frac{1}{2a} \left( B^\gamma_{|\beta} + B^\beta_{|\gamma} \right) \right]
\]

\[
+ \frac{1}{a^2} \left\{ 2C^\alpha_{\delta|\gamma} \left( C^\delta_{\gamma|\beta} + C^\gamma_{\delta|\beta} - C^\gamma_{\delta|\gamma} - C^\gamma_{\alpha|\beta} \right) + 2C^\alpha_{\gamma|\delta} \left( C^\delta_{\gamma|\beta} + C^\beta_{|\delta} - C^\beta_{\gamma|\delta} - C^\delta_{\gamma|\beta} \right) - \frac{4}{3} R(3) C^\delta_\gamma \gamma \beta
\]

\[
+ 2C^\gamma_{\delta|\gamma} \left( C^\delta_{\gamma|\beta} + C^\gamma_{\delta|\beta} - C^\gamma_{\delta|\gamma} - C^\gamma_{\alpha|\beta} \right) - C^\delta_{\gamma|\beta} C^\gamma_{\gamma|\delta} - 2C^\gamma_{\delta|\gamma} \left( C^\delta_{\gamma|\beta} - C^\gamma_{\gamma|\beta} \right)
\]

\[
- \frac{1}{3} \delta^\beta_\delta \left[ 4C^\gamma_{\delta|\epsilon} \left( C^\epsilon_{|\delta|\epsilon} + C^\epsilon_{|\delta|\epsilon} - C^\epsilon_{|\delta|\epsilon} - C^\epsilon_{|\delta|\epsilon} \right) - \frac{1}{4} R(3) C^\gamma_\gamma \gamma \beta
\]

\[
+ 2C^\gamma_{\delta|\epsilon} \left( C^\gamma_{|\gamma} - C^\gamma_{\gamma|\delta} \right) \right\} + C^\gamma_{\delta|\epsilon} \left( 2C^\gamma_{|\gamma} - 3C^\gamma_{\gamma|\epsilon} \right)
\]

\[
- 16\pi G C^\alpha_{\beta|\gamma} + 8\pi G (\mu + p) \left[ (V^\alpha - B^\alpha) (V_\beta - B_\beta) - \frac{1}{2} \right]
\]

\[
\textrm{Energy conservation equation:}
\]

\[
[\dot{\mu} + 3H (\mu + p)] + \delta \dot{\mu} + 3H (\delta \mu + \delta p) - (\mu + p) (\delta K - 3HA) + \frac{1}{a} (\mu + p) \left[ V^\alpha - B^\alpha + AB^\alpha + 2V^\beta C^\alpha_\beta \right]_{|\alpha}
\]

\[
= - \frac{1}{a} \delta\mu_{\mu} B^\alpha + (\delta \mu + \delta p) (\delta K - 3HA) + (\mu + p) A\delta K + \frac{3}{2} H (\mu + p) (A^2 - B^\alpha B_\alpha)
\]

\[
- \frac{1}{a^2} \left[ \left( (\mu + p) (V^\alpha - B^\alpha) \right) - \frac{1}{a} \left[ (\delta \mu + \delta p) (V^\alpha - B^\alpha) + \Pi^\alpha_{\beta} (V^\beta - B^\beta) \right]_{|\alpha}
\]

\[
+ \frac{1}{a} (\mu + p) \left[ -A (V^\alpha - B^\alpha)_{|\alpha} - 2A_{\alpha} (V^\alpha - B^\alpha) + 2 \left[ C^\alpha_{\beta} (V^\beta - B^\beta) \right]_{|\alpha} - C^\alpha_{\alpha|\beta} (V^\beta - B^\beta) \right]
\]

\[
- \Pi^\alpha_{\beta} \left( C^\alpha_{\beta} + \frac{1}{a} B_{a|\beta} \right) \equiv n_5.
\]
Momentum conservation equation:

\[
\frac{1}{a^4} \left[ a^4 (\mu + p) \left( V_i - B_\alpha + A_B \alpha + 2V^{\gamma}_{\alpha\beta} C_{\alpha\beta} \right) \right] + \frac{1}{a} (\mu + p) A_{\alpha\alpha} + \frac{1}{a} \left( \delta p_{\alpha\alpha} + \Pi_{\alpha\alpha}^\beta \right) \\
= (\mu + p) \left( \delta K - 3HA \right) (V_i - B_\alpha - B_{\alpha\alpha}) - \frac{1}{a^4} \left[ a^4 \left[ (\delta \mu + \delta p) (V_i - B_\alpha) + \Pi_{\alpha\alpha}^\beta (V_\beta - B_\beta) \right] \right] \\
+ \frac{1}{a} \left\{ - \left( \delta p_{\alpha\alpha} + \Pi_{\alpha\alpha}^\beta \right) A - (\delta \mu + \delta p) A_{\alpha\alpha} \\
- (\mu + p) \left[ -AA_{\alpha\alpha} + B_{\beta\alpha} V^\beta_{\alpha\beta} + (V_i - B_\alpha)_{|\beta} V^\beta_{\alpha\beta} + (V_i - B_\alpha) (V_\beta - B_\beta)_{|\beta} \right] \\
+ 2 \left( C_{\beta\gamma}^\alpha \Pi_{\alpha\alpha}^\beta \right)_{|\beta} = n_{6\alpha} \right\}.
\]

In the multi-component situation we additionally have the energy and the momentum conservation of individual component. Using the energy-frame fluid quantities, Eqs. (106), (107) in [11] become

\[
\left[ \dot{\mu}_{(i)} + 3H \left( \mu_{(i)} + p_{(i)} \right) + \frac{1}{a} I_{(i)0} \right] + \frac{1}{a} \left( \mu_{(i)} + p_{(i)} \right) (\delta K - 3HA) \\
+ \frac{1}{a^4} \left[ a^4 (\mu_{(i)} + p_{(i)} \left( V_{i\alpha} - B^\alpha + A_B^\alpha + 2V^{\gamma}_{\alpha\beta} C_{\alpha\beta} \right) \right]_{|\alpha} + \frac{1}{a} \delta I_{(i)0} \\
= -\frac{1}{a} \delta \mu_{(i)\alpha} B^\alpha + (\delta \mu_{(i)} + \delta p_{(i)}) (\delta K - 3HA) + (\mu_{(i)} + p_{(i)}) A \delta K + \frac{3}{2} H (\mu_{(i)} + p_{(i)}) (A^2 - B^\alpha B_\alpha) \\
- \frac{1}{a^4} \left[ a^4 (\mu_{(i)} + p_{(i)} \left( V_{i\alpha}^\alpha - B^\alpha \right) (V_{i\alpha} - B_\alpha) \right]_{|\alpha} - \frac{1}{a} \left[ (\delta \mu_{(i)} + \delta p_{(i)}) \left( V_{i\alpha}^\alpha - B^\alpha \right) + \Pi_{(i)\alpha}^\beta (V_{i\beta} - B_{\beta}) \right]_{|\alpha} \\
+ \frac{1}{a} \left( \mu_{(i)} + p_{(i)} \right) \left\{ -A \left( V_{i\alpha}^\alpha - B^\alpha \right) - 2A_{\alpha\alpha} \left( V_{i\alpha} - B^\alpha \right) + 2 \left[ C_{\alpha\beta} (V_i - B_\beta) \right]_{|\alpha} - C_{\alpha\beta} (V_i - B_\beta) \right\} \\
- \Pi_{(i)\alpha}^\beta \left( \delta K_{\alpha\beta}^\gamma + \frac{1}{a} B_{\alpha\beta} \right) - \frac{1}{a} \delta I_{(i)\alpha} B^\alpha = n_{(i)5},
\]

\[
\frac{1}{a^4} \left[ a^4 (\mu_{(i)} + p_{(i)} \left( V_{i\alpha} - B_\alpha + A_B^\alpha + 2V^{\gamma}_{\alpha\beta} C_{\alpha\beta} \right) \right]_{|\alpha} + \frac{1}{a} (\mu_{(i)} + p_{(i)}) A_{\alpha\alpha} + \frac{1}{a} \left( \delta p_{(i)\alpha} + \Pi_{(i)\alpha}^\beta - \delta I_{(i)\alpha} \right) \\
= (\mu_{(i)} + p_{(i)}) \left( \delta K - 3HA \right) (V_{i\alpha} - B_\alpha - B_{\alpha\alpha}) - \frac{1}{a^4} \left[ a^4 \left[ (\delta \mu_{(i)} + \delta p_{(i)}) \left( V_{i\alpha} - B_\alpha \right) + \Pi_{(i)\alpha}^\beta (V_{i\beta} - B_{\beta}) \right] \right]_{|\alpha} \\
+ \frac{1}{a} \left\{ - \left( \delta p_{(i)\alpha} + \Pi_{(i)\alpha}^\beta - \delta I_{(i)\alpha} \right) A - (\delta \mu_{(i)} + \delta p_{(i)}) A_{\alpha\alpha} \\
- (\mu_{(i)} + p_{(i)}) \left[ -AA_{\alpha\alpha} + B_{\beta\alpha} V^\beta_{\alpha\beta} + (V_{i\alpha} - B_\alpha)_{|\beta} V^\beta_{\alpha\beta} + (V_{i\alpha} - B_\alpha) (V_{i\beta} - B_{\beta})_{|\beta} \right] \\
+ 2 \left( C_{\beta\gamma}^\alpha \Pi_{(i)\alpha\gamma}^\beta - C_{\beta\gamma}^\alpha \Pi_{(i)\alpha}^\beta + C_{\beta\gamma}^\alpha \Pi_{(i)\alpha\gamma}^\beta - A_{\beta\gamma} \Pi_{(i)\alpha}^\beta \right) \right\} = n_{(i)6\alpha}.
\]

By removing indices indicating the components in Eqs. (69), (70) we recover equations for the collective component which coincide with the equations in a single component situation in Eqs. (67), (68).

To the background order, from Eqs. (69), (65) we have

\[
\mu_{(i)} + 3H \left( \mu_{(i)} + p_{(i)} \right) = -\frac{c}{a} I_{(i)0},
\]

\[
\mu + 3H (\mu + p) = 0,
\]

\[
3H + 3H^2 = -\frac{4\pi G}{c^2} \sum_j (\mu_{(j)} + 3p_{(j)}) + \Lambda c^2,
\]

\[
H^2 = \frac{8\pi G}{3c^2} \sum_j \mu_{(j)} - \frac{K c^2}{a^2} + \frac{\Lambda c^2}{3},
\]

where we recovered the speed of light c. Dimensions are

\[
[G_\beta] = T^{-2}, \quad [c] = LT^{-1}, \quad [\eta] = 1, \quad [p] = [\mu] = [gc^2], \quad [a] = L, \quad [K] = 1, \quad [\Lambda] = L^{-2}, \quad [I_{(i)0}] = [\mu].
\]
Equation (72) follows from the sum of Eq. (71) over components. Equation (74) follows from integrating Eq. (73) where \( K \)-term can be regarded as an integration constant; in Einstein’s gravity \( K \)-term can be normalized as the sign of spatial curvature. Compared with the Newtonian background equations in Eq. (4), ignoring the direct interaction terms in Eq. (71), the presence of pressure terms in Eqs. (71)-(74) is the pure general relativistic effect. The cosmological constant \( \Lambda \) can be introduced by hand even in the Newtonian case.

B. Decomposition

We decompose the metric to three perturbation types

\[
A \equiv \alpha, \quad B_\alpha \equiv \beta_\alpha + B^{(v)}_\alpha, \quad C_{\alpha\beta} \equiv \varphi g^{(3)}_{\alpha\beta} + \gamma_{,\alpha|\beta} + C^{(v)}_{(\alpha|\beta)} + C^{(t)}_{\alpha\beta},
\]

where superscripts \((v)\) and \((t)\) indicate the transverse vector-type, and transverse-tracefree tensor-type perturbations, respectively. We introduce

\[
\chi \equiv a \left( \beta + c^{-1} a \dot{\gamma} \right), \quad \Psi^{(v)}_\alpha \equiv B^{(v)}_\alpha + c^{-1} a \dot{C}^{(v)}_\alpha,
\]

which are spatially gauge-invariant to the linear order. We set

\[
K^\alpha_\alpha \equiv -3H + \kappa.
\]

We will identify \( \kappa \) with Newtonian velocity variable which will be an important step in our analysis, see Eqs. (120), (194), (215).

For the fluid quantities we decompose

\[
\dot{u}_{(i)\alpha} = a \left( V_{(i)\alpha} - B_\alpha + AB_\alpha + 2V^{\beta}_{(i)} C_{\alpha\beta} \right) \equiv av_{(i)\alpha} \equiv a \left( -v_{(i),\alpha} + v^{(v)}_{(i)\alpha} \right),
\]

\[
\Pi_{(i)\alpha\beta} \equiv \frac{1}{a^2} \left( \Pi_{(i),\alpha|\beta} - \frac{1}{3} g^{(3)}_{\alpha\beta} \Delta \Pi_{(i)} \right) + \frac{1}{a} \Pi^{(v)}_{(i)|\alpha|\beta} + \Pi^{(t)}_{(i)\alpha\beta}, \quad \delta I_{(i)\alpha} \equiv \delta I_{(i),\alpha} + \delta I^{(v)}_{(i)\alpha}.
\]

The perturbed fluid velocity variables \( v_{(i)} \) and \( v^{(v)}_{(i)\alpha} \) subtly differ from the ones introduced in [11]; see Sec. V C. For the collective fluid component or for a single component case, we simply delete \((i)\) subindices. For isotropic pressure we introduce

\[
\delta p \equiv c_s^2 \delta \mu + e, \quad c_s^2 \equiv \frac{\dot{\mu}}{\mu}.
\]

The perturbation variable \( e \) is called an entropic perturbation. Defined in this way \( e \) is gauge-invariant only to the linear order. To the second order, from Eq. (61) we can derive the following gauge-invariant combination

\[
\delta p_{\delta \mu} \equiv e - \frac{\delta \mu}{\mu} \left[ \dot{e} + \frac{1}{2} \left( c_s^2 \right) \delta \mu \right].
\]

In our notation, \( \delta p_{\delta \mu} \) is a gauge-invariant combination which is the same as \( \delta \mu \) in the \( \delta \mu = 0 \) slicing (temporal gauge) condition to the second order; as the spatial gauge we take \( \gamma \equiv 0 \) to the second order; for the derivation, see the prescription below Eq. (266) of [11]. In the multi-component case, we similarly have

\[
\delta p_{(i)\delta \mu_{(i)}} \equiv e_{(i)} - \frac{\delta \mu_{(i)}}{\mu_{(i)}} \left[ \dot{e}_{(i)} + \frac{1}{2} \left( c_s^2_{(i)} \right) \delta \mu_{(i)} \right],
\]

where

\[
\delta p_{(i)} \equiv c_{s;i}^2 \delta \mu_{(i)} + e_{(i)}, \quad c_{s;i}^2 \equiv \frac{\dot{\mu}_{(i)}}{\mu_{(i)}}.
\]

We have [25]

\[
e = e_{\text{rel}} + e_{\text{int}}, \quad e_{\text{rel}} \equiv \sum_j \left( c_{s;j}^2 - c_s^2 \right) \delta \mu_{(j)}, \quad e_{\text{int}} \equiv \sum_j e_{(j)}.
\]
Equation (50) gives
\[ \delta \mu = \sum_j \left[ \delta \mu(j) + (\mu(j) + p(j)) v_i^\alpha (v(j)_{\alpha} - v_{\alpha}) \right], \]
\[ \delta p = \sum_j \left[ \delta p(j) + \frac{1}{3} (\mu(j) + p(j)) v_i^\alpha (v(j)_{\alpha} - v_{\alpha}) \right], \]
\[ (\mu + p) v_a = \sum_j \left[ (\mu(j) + p(j)) v_a(j) + (\delta \mu(j) + \delta p(j)) (v(j)_{\alpha} - v_{\alpha}) + \Pi_{j,a} (v(j)_{\beta} - v_{\beta}) \right], \]
\[ \Pi^\alpha_{\beta} = \sum_j \left\{ \Pi_{j,a}^\alpha + (\mu(j) + p(j)) \left[ v_i^\alpha (v(j)_{\beta} - v_{\beta}) - \frac{1}{3} \delta p_{\gamma} (v(j)_{\gamma} - v_{\gamma}) \right] \right\}, \] (86)

By recovering \( c \), dimensions of the variables are
\[ [g_{ab}] = [g^{ab}] = [\tilde{u}_a] = 1, \quad [\tilde{T}_{ab}] = [\mu], \quad [g^{(3)}_{a\beta}] = 1, \quad [\nabla] = [\Delta] = 1, \quad [w] = [v^2] = 1, \]
\[ [A] = [B_a] = [C_{a\beta}] = 1, \quad [\alpha] = [\varphi] = [\beta] = [\gamma] = [B_\alpha^{(v)}] = [C_\alpha^{(v)}] = [C_{a\alpha \beta}^{(v)}] = 1, \quad [\chi] = L, \quad [\kappa] = T^{-1}, \]
\[ [\delta \mu] = [\delta p] = [e] = [\Pi_{a\beta}] = [\Pi^{(v)}_{a\beta}] = [\mu], \quad [\delta] = [V_a] = 1, \quad [v] = [v^{(v)}] = 1, \quad [\Pi] = L^2 [\mu], \quad [\Pi^{(v)}_a] = L [\mu]. \] (87)

Scalar-type perturbation equations can be derived from Eqs. (62)-(70)
\[ \kappa - 3H \alpha + 3 \dot{\varphi} + \frac{\Delta}{a^2} \chi = n_0, \] (88)
\[ 4\pi G \delta \mu + H \kappa + \frac{\Delta + 3K}{a^2} \varphi = \frac{1}{4} n_1, \] (89)
\[ \kappa + \frac{\Delta + 3K}{a^2} \chi - 12\pi G (\mu + p) v_{\alpha} = n_2 \equiv \frac{3}{2} \Delta^{-1} \nabla^\alpha n_{2\alpha}, \] (90)
\[ \dot{\kappa} + 2H \kappa - 4\pi G (\delta \mu + 3 \delta p) + \left( 3 \dot{H} + \frac{\Delta}{a^2} \right) \alpha = n_3, \] (91)
\[ \dot{\chi} + H \chi - \dot{\varphi} - \alpha - 8\pi G \Pi = n_4 \equiv \frac{3}{2} a^2 (\Delta + 3K)^{-1} \Delta^{-1} \nabla^\alpha \nabla_\beta n_{4\beta}, \] (92)
\[ \delta \dot{\mu} + 3H (\delta \mu + \delta p) - (\mu + p) \left( \kappa - 3H \alpha + \frac{3}{2} \alpha v \right) = n_5, \] (93)
\[ \frac{\left[ a^4 (\mu + p) v \right]}{a^4 (\mu + p) v} - \frac{1}{4} a - \frac{1}{4} \left( \frac{2 \Delta + 3K}{a^2} \right) = n_6 \equiv - \frac{1}{\mu + p} \Delta^{-1} \nabla^\alpha n_{6\alpha}, \] (94)
\[ \delta \dot{\mu}(i) + 3H (\delta \mu(i) + \delta p(i)) - (\mu(i) + p(i)) \left( \kappa - 3H \alpha + \frac{3}{2} \alpha v \right) + \frac{1}{a} I = n_{5(i)}, \] (95)
\[ \frac{\left[ a^4 (\mu(i) + p(i)) v_i^{(v)} \right]}{a^4 (\mu(i) + p(i)) v_i^{(v)}} - \frac{1}{4} a - \frac{1}{4} \left( \frac{2 \Delta + 3K}{a^2} \right) \Pi(i) \delta I = n_{6(i)}, \] (96)

Equations for the vector-type perturbation follow from Eqs. (61), (66), (68), (70)
\[ \frac{\Delta + 2K}{2a^2} \Psi^{(v)}_\alpha + 8\pi G (\mu + p) v_i^{(v)} = \frac{1}{a} (n_{2\alpha} - \nabla \alpha \Delta^{-1} \nabla^\beta n_{2\beta}) \equiv v_i^{(v)}, \] (97)
\[ \dot{\Psi}^{(v)}_\alpha + 2H \Psi^{(v)}_\alpha - 8\pi G \Pi^{(v)}_a = 2a (\Delta + 2K)^{-1} \left( \nabla_\beta n_{4\beta} - \nabla \alpha \Delta^{-1} \nabla^\gamma \nabla_\beta n_{4\gamma} \right) \equiv n_{4\alpha}, \] (98)
\[ \frac{\left[ a^4 (\mu + p) v_i^{(v)} \right]}{a^4 (\mu + p) v_i^{(v)}} + \frac{\Delta + 2K}{2a^2} \Psi^{(v)}_\alpha = \frac{1}{\mu + p} \left( n_{6\alpha} - \nabla \alpha \Delta^{-1} \nabla^\beta n_{6\beta} \right) \equiv n^{(v)}_{6\alpha}, \] (99)
\[ \frac{\left[ a^4 (\mu(i) + p(i)) v_i^{(v)} \right]}{a^4 (\mu(i) + p(i)) v_i^{(v)}} + \frac{\Delta + 2K}{2a^2} \Pi_{i,a}^{(v)} = \frac{1}{\mu(i) + p(i)} \left( n_{6(i)\alpha} - \nabla \alpha \Delta^{-1} \nabla^\beta n_{6(i)\beta} \right) \equiv n^{(v)}_{6(i)\alpha}. \] (100)
Equations for the tensor-type perturbation follow from Eq. (60)
\[ C_{\alpha\beta}^{(t)} + 3H C_{\alpha\beta}^{(t)} - \frac{\Delta - 2K}{a^2} C_{\alpha\beta}^{(t)} - 8\pi G \Pi_{\alpha\beta}^{(t)} = n_{4\alpha\beta}, \]
\[ = \frac{3}{2} \left( \nabla_\alpha \nabla_\beta - \frac{1}{3} g_{\alpha\beta} \Delta \right) (\Delta + 3K)^{-1} \Delta^{-1} \nabla^\gamma \nabla_\gamma n_{4\gamma}^\beta - 2\nabla_\alpha (\Delta + 2K)^{-1} (\nabla^\gamma n_{4\beta}\gamma - \nabla_\beta \Delta^{-1} \nabla^\gamma \nabla_\gamma n_{4\gamma}) \equiv n_{4\alpha\beta}^{(t)}. \] (101)

In order to derive eqs. (92,98,101) it is convenient to show
\[ \frac{1}{a^2} \left( \nabla_\alpha \nabla_\beta - \frac{1}{3} g_{\alpha\beta} \Delta \right) (\chi + H \chi - \varphi - \alpha - 8\pi G \Pi) + \frac{1}{a^2} \left( a^2 \Psi_{(v)}^{(v)} \right) - 8\pi G \frac{1}{a} \Pi_{(v)}^{(v)} \]
\[ + \frac{\Delta - 2K}{a^2} C_{\alpha\beta}^{(t)} - 8\pi G \Pi_{\alpha\beta}^{(t)} = n_{4\alpha\beta}, \] (102)
which follows from Eq. (60). Quadratic combinations of linear-order perturbation variables of all three types of perturbations contribute to all three types of perturbation to the second order.

C. Comoving gauge and irrotational condition

In Eq. (180) of [11] we introduced
\[ Q_{(i)\alpha} = (\mu_{(i)} + p_{(i)}) \left( -\tilde{v}_{(i)\alpha} + \tilde{v}_{(i)\alpha}^{(v)} \right), \] (103)
where we put overbars to \( \tilde{v}_{(i)\alpha} \) and \( \tilde{v}_{(i)\alpha}^{(v)} \) in order to distinguish these from our new notations to be used in this paper; in [11] we didn’t have overbars. From Eq. (58) we have
\[ -\tilde{v}_{(i)\alpha} + \tilde{v}_{(i)\alpha}^{(v)} = V_{(i)\alpha} - B_{\alpha} + AB_{\alpha} + 2V_{(i)\alpha}^{\beta} C_{\alpha\beta}^{(t)} + \frac{\delta \mu_{(i)} + \delta p_{(i)}}{\mu_{(i)} + p_{(i)}} (V_{(i)\alpha} - B_{\alpha}) + \frac{\Pi_{(i)\alpha\beta}}{\mu_{(i)} + p_{(i)}} (V_{(i)}^{\beta} - B^{\beta}). \] (104)
It is more convenient to introduce the decomposition in Eq. (70). Thus, we have
\[ -\tilde{v}_{(i)\alpha} + \tilde{v}_{(i)\alpha}^{(v)} = v_{(i)\alpha} + v_{(i)\alpha}^{(v)} + \frac{\delta \mu_{(i)} + \delta p_{(i)}}{\mu_{(i)} + p_{(i)}} (v_{(i)\alpha} + v_{(i)\alpha}^{(v)}) + \frac{\Pi_{(i)\alpha\beta}}{\mu_{(i)} + p_{(i)}} (-v_{(i)}^{\beta} + v_{(i)}^{(v)\beta}). \] (105)

The variable \( \tilde{v}_{(i)\alpha}^{(v)} \) introduced in [11] cannot be regarded as a proper vector-type perturbation. However, if we also take the temporal comoving gauge in [11] which sets \( \tilde{v} \equiv 0 \) together with \( \tilde{v}_{(i)\alpha}^{(v)} = 0 \), we have \( v = 0 = v_{(i)\alpha}^{(v)} \); these are the same as taking the proper irrotational condition \( (v_{(i)\alpha}^{(v)} = 0) \) and the temporal comoving gauge \( (v = 0) \). Our analyses in [12,13] are, in fact, based on taking these two conditions together. In the irrotational fluids, the temporal comoving gauge \( v \equiv 0 \) leads to \( \dot{u}_{\alpha} = 0 \), thus \( u_{\alpha} \) coincides with the normal frame four-vector \( n_{\alpha} \).

From Eq. (60) we have
\[ \tilde{\omega}_{(i)\alpha\beta} = a v_{(i)\alpha\beta}^{(v)} + \frac{a}{\mu_{(i)} + p_{(i)}} \left( -v_{(i),\alpha}^{(v)} + v_{(i)\alpha}^{(v)} \right) \left[ (\delta p_{(i)} + \frac{2 \Delta + 3K}{a^2} \Pi_{(i)} - \delta I_{(i)}) \right]_{\alpha\beta} + \frac{\Delta + 3K}{2a} \Pi_{(i)\alpha\beta} - \delta I_{(i)\beta}, \] (106)
where we used Eq. (70) to the linear order. For vanishing vector-type perturbation we set \( v_{(i)\alpha}^{(v)} \equiv 0 \), etc. In this case, we have
\[ \tilde{\omega}_{(i)\alpha\beta} = -\frac{a}{\mu_{(i)} + p_{(i)}} v_{(i),\alpha}^{(v)} \left( \delta p_{(i)} + \frac{2 \Delta + 3K}{a^2} \Pi_{(i)} - \delta I_{(i)} \right)_{\alpha\beta}, \] (107)
which vanishes for \( \delta p_{(i)} = 0 = \Pi_{(i)} \) and \( \delta I_{(i)} = 0 \).
VI. EFFECTS OF PRESSURE

We consider a single component situation without rotational perturbations. Equations (65), (67), (68), and Eq. (64) to the linear order provide a complete set of equations we need in the following.

A. Irrotational case

As an irrotational fluid we ignore all vector-type perturbations, thus \( u^{(v)}_a = B^{(v)}_a = C^{(v)}_a = \Pi^{(v)}_a = 0 \). Quadratic combinations of linear-order perturbations of all three types of perturbations contribute to each type of second-order perturbations. Thus, concerning the second-order scalar-type and vector-type perturbations, ignoring the pure vector-type perturbation only has a decaying solution in the expanding phase. Assuming the background equations combinations of linear-order perturbations of all three types of perturbations contribute to each type of second-order to the linear order provide a complete set of equations we need in the following.

As the temporal and spatial gauge conditions we set

\[
\dot{\alpha} + 2H (1 + w) \left( \alpha + \frac{\Delta}{a} - \frac{3H}{a} \right) = 1 - \frac{\Delta}{a},
\]

\[
\frac{\dot{\delta}}{\mu} - \frac{3H}{\mu} (1 + w) \left( \alpha - \beta \right) = \frac{1}{\mu} \left( \frac{\Delta}{a} - \frac{\Delta}{a} \right) \left( \delta \alpha - \beta \right),
\]

\[
\dot{\varphi} = \alpha \left( H + \frac{\delta p}{\mu} + \frac{\delta p}{\mu} \right),
\]

\[
\dot{\chi} + 2H (1 + w) \left( \alpha - \beta \right) + \frac{1}{a} \left( \delta \mu + \delta p \right) v + \Pi \delta v = 0.
\]

Equation (111) is valid to the linear order.

B. Comoving gauge

As the temporal and spatial gauge conditions we set

\[
v \equiv 0 \equiv \gamma,
\]

thus, \( \beta = \chi/a \). Under these gauge conditions Eq. (109) gives

\[
\alpha = -\frac{1}{2} \frac{\Delta}{a^2} \chi \alpha - \frac{c_s^2}{1 + w} \delta \left[ 1 - \frac{1}{2} \frac{3c_s^2}{1 + w} \right] + \alpha \Pi,
\]
where the imperfect fluid contributions (stresses) are denoted by $\alpha_{II}$ with

$$
\alpha_{II} \equiv \frac{1}{\mu + p} \left( e + \frac{2}{3} \frac{\Delta + 3K}{a^2} \Pi \right) \left[ -1 + \frac{2}{1 + w} \delta + \frac{1}{\mu + p} \left( e + \frac{2}{3} \frac{\Delta + 3K}{a^2} \Pi \right) \right] \\
+ \frac{1}{\mu + p} \Delta^{-1} \nabla^\alpha \left[ \frac{1 + c_s^2}{1 + w} \delta \left( e_{,\alpha} + \Pi_{\alpha|\beta}^\beta \right) - \left( \frac{c_s^2}{1 + w} \delta_{,\beta} + \frac{e_{,\beta} + \Pi_{\beta|\gamma}^\gamma}{\mu + p} \right) \Pi_{\alpha} + 2 \left( C_{\beta\gamma} \Pi_{\alpha\gamma} \right)_{|\beta} - \Pi_{\alpha} C_{\beta|\gamma} + \Pi_{\gamma} C_{\beta|\alpha} \right].
$$

(114)

Using this, Eqs. (110) - (111) give

$$
\dot{\delta} = -3H \delta - (1 + w) \kappa = -\frac{1}{a^2} \delta_{,\alpha} \chi^{\alpha} + \kappa \delta + \frac{3}{2} \frac{H c_s^2}{1 + w} \delta^2 + \delta_{\Pi},
$$

(115)

$$
\dot{\kappa} + 2H \kappa - 4\pi G \mu \delta - \frac{c_s^2}{a^2} \left( \Delta + 3K \right) \delta = -\frac{1}{a^2} \delta_{,\alpha} \chi^{\alpha} + \kappa \delta + \frac{c_s^2}{2(1 + w)} \left( 4\pi G \mu - \frac{1 + 2c_s^2}{a^2} \frac{\Delta + 3K}{1 + w} \right) \delta^2 \\
+ \frac{c_s^2}{1 + w} \left[ 2H \delta \kappa + \frac{1}{a^2} \left( \varphi_{,\alpha} \delta_{,\alpha} - 2\varphi \Delta \delta - 2\delta_{,\alpha|\beta} C_{\alpha|\beta} \right) \right] + \left( \frac{1}{a^2} \chi_{,\alpha|\beta} + \dot{C}_{\alpha|\beta} \right) \left( \frac{1}{a^2} \chi_{,\alpha|\beta} + \dot{C}_{,\alpha|\beta} \right) - \frac{1}{3} \left( \frac{\Delta}{a^2} \chi \right)^2,
$$

(116)

$$
\kappa = -\frac{\Delta + 3K}{a^2} \chi,
$$

(117)

where Eq. (117) is valid only to the linear order, and

$$
\delta_{II} = 2H \frac{\Delta + 3K}{a^2} \Pi \frac{\mu}{\mu + p} \left( \kappa + 3H \frac{c_s^2}{1 + w} \delta \right) - \frac{\Pi_{\alpha\beta}}{\mu} \left( \frac{1}{a^2} \chi_{,\alpha|\beta} + \dot{C}_{\alpha|\beta} \right) \\
+ \frac{1}{\mu + p} \left( e + \frac{2}{3} \frac{\Delta + 3K}{a^2} \Pi \right) \left[ 3H \delta - (1 + w) \kappa + \frac{3}{2} \frac{H}{\mu} \left( e - \frac{2}{3} \frac{\Delta + 3K}{a^2} \Pi \right) \right] \\
- 3H \frac{1}{\mu} \Delta^{-1} \nabla^\alpha \left[ \frac{1 + c_s^2}{1 + w} \delta \left( e_{,\alpha} + \Pi_{\alpha|\beta}^\beta \right) - \left( \frac{c_s^2}{1 + w} \delta_{,\beta} + \frac{e_{,\beta} + \Pi_{\beta|\gamma}^\gamma}{\mu + p} \right) \Pi_{\alpha} + 2 \left( C_{\beta\gamma} \Pi_{\alpha\gamma} \right)_{|\beta} - \Pi_{\alpha} C_{\beta|\gamma} + \Pi_{\gamma} C_{\beta|\alpha} \right].
$$

(118)

$$
\kappa_{II} \equiv 12\pi G e \left( 1 - \frac{c_s^2}{1 + w} \delta \right) \\
- \left( 3H + \frac{\Delta}{a^2} \right) \left\{ \frac{1}{\mu + p} \left( e + \frac{2}{3} \frac{\Delta + 3K}{a^2} \Pi \right) \left[ -1 + \frac{2}{1 + w} \delta + \frac{1}{\mu + p} \left( e + \frac{2}{3} \frac{\Delta + 3K}{a^2} \Pi \right) \right] \\
+ \frac{1}{\mu + p} \Delta^{-1} \nabla^\alpha \left[ \frac{1 + c_s^2}{1 + w} \delta \left( e_{,\alpha} + \Pi_{\alpha|\beta}^\beta \right) - \left( \frac{c_s^2}{1 + w} \delta_{,\beta} + \frac{e_{,\beta} + \Pi_{\beta|\gamma}^\gamma}{\mu + p} \right) \Pi_{\alpha} + 2 \left( C_{\beta\gamma} \Pi_{\alpha\gamma} \right)_{|\beta} - \Pi_{\alpha} C_{\beta|\gamma} + \Pi_{\gamma} C_{\beta|\alpha} \right] \right\} \\
- \left[ -2H \kappa + 4\pi G \left( 1 + 6c_s^2 \right) \mu \delta + 12\pi G e \right. \\
\times \frac{1}{\mu + p} \left( e + \frac{2}{3} \frac{\Delta + 3K}{a^2} \Pi \right) + \frac{\Delta}{\mu + p} \left\{ \left( e + \frac{2}{3} \frac{\Delta + 3K}{a^2} \Pi \right) \left[ \frac{1}{2} \frac{1}{\mu + p} \left( e + \frac{2}{3} \frac{\Delta + 3K}{a^2} \Pi \right) + \frac{c_s^2}{1 + w} \delta \right] \right\} \\
\left. + \frac{1}{\mu + p} \frac{1}{a^2} \left( e + \frac{2}{3} \frac{\Delta + 3K}{a^2} \Pi \right)^{,\alpha} \varphi_{,\alpha} - 2 \left( e + \frac{2}{3} \frac{\Delta + 3K}{a^2} \Pi \right)^{,\alpha|\beta} C_{\alpha|\beta} \right].
$$
Combining Eqs. (115), (116) we can derive

\[
\frac{1 + w}{a^2 H} \left[ \frac{H^2}{(\mu + p)a} (a^3 \mu H) \right] - c^2_{s} \frac{\Delta}{a^2} = (1 + w) \left\{ \frac{1}{a} \chi^{a|\beta}_{,\alpha|\beta} - \frac{1}{a^2} \kappa_{,\alpha} \chi^{\alpha} + \frac{1}{3} \left[ \kappa^2 - \left( \frac{\Delta}{a} \right)^2 \right] \right\} \\
+ \frac{1 + w}{a^2} \left[ \frac{2H^2}{1 + w} \left( a^2 \kappa_\delta - \delta_\alpha \chi^{\alpha} \right) + \frac{1}{2} c^2_\delta \left( 4\pi G \mu - \frac{1 + 2c^2_\delta}{a^2} \Delta + 3K \right) \delta^2 \right] \\
+ c^2_\delta \left( 2a^2 H \delta \kappa + \Phi^{a_\delta} \delta_\alpha - 2\Phi^{a_\delta} \Delta_\delta - 2\delta^{a|\beta} C^{(t)}_{a\alpha} \right) (1 + w) \dot{C}^{(t)}_{a\alpha} \left( \frac{2}{a^2} \chi_{a|\alpha} + \dot{C}^{(t)}_{a\alpha} \right) \\
+ (1 + w) \kappa_\Pi + \frac{1 + w}{a^2} \left( \frac{a^2}{1 + w} \delta_\Pi \right) - \frac{1}{a^2} \delta_{\Pi,a} \chi^{\alpha}.
\]

(119)

Notice that the equations above are valid in the presence of general \( K \) and \( \Lambda \).

C. Newtonian correspondence

Guided by our success in the zero-pressure case, we continue to identify

\[
\kappa \equiv -\frac{1}{a} \nabla \cdot u,
\]

(120)
to the second order. Using Eq. (117), assuming \( K = 0 \), we can identify

\[
\kappa = -c \frac{\Delta}{a^2} \chi = -\frac{1}{a} \nabla \cdot u, \quad u \equiv \frac{c}{a} \nabla \chi = -c \nabla v_\chi,
\]

(121)
to the linear order. We have recovered the speed of light \( c \). We have \( |u| = LT^{-1} \). Equation (113) becomes

\[
\alpha = -\frac{1}{2c^2} u^2 - \frac{c^2_\delta}{1 + w} \delta \left[ 1 - \frac{1}{2(1 + w)} \right] + \alpha_\Pi.
\]

(122)

Equations (115), (116) give

\[
\frac{\dot{\delta}}{3wH\delta + (1 + w) \frac{1}{a} \nabla \cdot u} - \frac{1}{a} \nabla \cdot (\delta u) + \frac{3}{2} \frac{c^2_\delta}{1 + w} H \delta^2 + \delta_\Pi,
\]

(123)

\[
\frac{1}{a} \nabla \cdot (\dot{u} + H u) + \frac{4\pi G \mu}{c^2} \delta + \frac{c^2_\delta}{1 + w} \frac{\Delta}{a^2} \delta = -\frac{1}{a^2} \nabla \cdot (u \cdot \nabla u) - \dot{C}^{(t)}_{a\alpha} \left( \frac{2}{a} u_{a|\beta} + \dot{C}^{(t)}_{a\alpha} \right) \\
+ \frac{1}{2} c^2_\delta \left( \frac{4\pi G \mu}{c^2} - \frac{2}{1 + w} \frac{c^2_\delta}{a^2} \Delta \right) \delta^2 + 2H \delta \frac{1}{a} \nabla \cdot u + c^2_\delta \left[ 2\phi \Delta \delta - (\nabla \phi) \cdot \nabla \delta + 2\delta^{a|\beta} C^{(t)}_{a\alpha} \right] - \kappa_\Pi.
\]

(124)

We have \( |\alpha_{\Pi}| = 1 \), \( |\delta_{\Pi}| = T^{-1} \), and \( |\kappa_{\Pi}| = T^{-2} \). Combining these equations or Eq. (119) gives

\[
\frac{1 + w}{a^2 H} \left[ \frac{H^2}{(\mu + p)a} (a^3 \mu H) \right] - c^2_{s} \frac{\Delta}{a^2} \delta \\
= \frac{1 + w}{a^2} \nabla \cdot (u \cdot \nabla u) - \frac{1 + w}{a^2} \left\{ \frac{a}{1 + w} \left[ \nabla \cdot (\delta u) - \frac{3}{2} aH \frac{c^2_\delta}{1 + w} \delta^2 \right] \right\} + (1 + w) \dot{C}^{(t)}_{a\alpha} \left( \frac{2}{a} u_{a|\beta} + \dot{C}^{(t)}_{a\alpha} \right) \\
+ \frac{1}{2} c^2_\delta \left( \frac{4\pi G \mu}{c^2} - \frac{2}{1 + w} \frac{c^2_\delta}{a^2} \Delta \right) \delta^2 - c^2_{s} \frac{1}{a^2} \left[ 2aH \delta \nabla \cdot u + 2\phi c^2 \Delta \delta - (\nabla \phi) \cdot \nabla \delta + 2c^2 \delta^{a|\beta} C^{(t)}_{a\alpha} \right] \\
+ (1 + w) \kappa_{\Pi} + \frac{1 + w}{a^2} \left( \frac{a^2}{1 + w} \delta_{\Pi} \right) - \frac{1}{a} u \cdot \nabla \delta_{\Pi}.
\]

(125)

We note that, to the linear order, Eq. (125) is valid in the presence of general \( K \) and \( \Lambda \), see Eq. (119).
D. Linear-order relativistic pressure corrections

To the linear order, ignoring the entropic perturbation $e$ and the anisotropic stress $\Pi$, Eqs. (123)-(125) become

\[ \dot{\delta} - 3wH\delta + (1 + w) \frac{1}{a} \nabla \cdot {\bf u} = 0, \]
\[ \frac{1}{a} \nabla \cdot \left( \dot{{\bf u}} + H{\bf u} \right) + 4\pi G\rho\delta = -\frac{1}{1 + w} \Delta \frac{\delta p}{\rho}, \]
\[ 1 + w \frac{a^2 H^2}{a(1 + w)\rho} \left( \frac{a^2 \rho}{H^2} \delta \right) = \Delta \frac{\delta p}{\rho}. \]

where ignoring the specific internal energy density $\epsilon$ we used $\mu = \rho c^2$, thus $\delta = \delta \rho / \rho$; in general we have $\mu = \rho (c^2 + \epsilon)$ [18]. Equation (128) can be expanded to give

\[ \frac{1 + w}{a^2 H} \left[ \frac{H^2}{a(1 + w)\rho} \left( \frac{a^2 \rho}{H^2} \delta \right) \right] = \frac{\Delta \delta p}{\rho} = \ddot{\delta} + (2 - 6w + 3c_s^2) H\delta - \left[ (1 + 8w - 6c_s^2 - 3w^2) 4\pi G\rho - 12 (w - c_s^2) Kc^2 a^2 + (5w - 3c_s^2) \Lambda c^2 \right] \delta, \]

which is valid in the presence of $K$ and $\Lambda$. Equation of density perturbation in the comoving gauge was first derived by Nariai in [26].

In a single component case the Newtonian equations in Eqs. (8),(12),(16) give

\[ \dot{\delta} + \frac{1}{a} \nabla \cdot {\bf u} = 0, \]
\[ \frac{1}{a} \nabla \cdot \left( \dot{{\bf u}} + H{\bf u} \right) + 4\pi G\rho\delta = -\frac{1}{1 + w} \Delta \frac{\delta p}{\rho}, \]
\[ \ddot{\delta} + 2H\dot{\delta} - 4\pi G\rho\delta = \frac{1}{a^2 H^2} \left[ a^2 H^2 \left( \frac{1}{H^2} \delta \right) \right] = \frac{\Delta \delta p}{\rho}. \]

Comparing Eqs. (126), (127) with Eqs. (130), (131), we notice the presence of $w = p / (\rho c^2)$ term in three places in the relativistic equations. Even to the linear order the presence of these terms should be regarded as pure general relativistic effect of the isotropic pressure. The effects of pressures to the second order compared with the Newtonian equations can be found in Eqs. (129)-(125) which should be compared with Newtonian equations in Eqs. (8), (12), (16). Pressure has the genuine relativistic role in cosmology even in the background level.

VII. EFFECTS OF MULTI-COMPONENT

We assume zero-pressure medium, thus set

\[ p_{(i)} = 0, \quad \delta p_{(i)} = 0 \equiv \Pi_{(i)\alpha\beta}. \]

From Eq. (80) we notice that to the second order the collective fluid quantities differ from simple sum of individual fluid quantities. Even in the zero-pressure mediums, we have

\[ \delta \mu = \sum_j \left[ \delta \mu_{(j)} + \mu_{(j)} v^\alpha_{(j)} (v_{(j)\alpha} - v_\alpha) \right], \]
\[ \delta p = \frac{1}{3} \sum_j \mu_{(j)} v^\alpha_{(j)} (v_{(j)\alpha} - v_\alpha), \]
\[ \mu v_\alpha = \sum_j \left[ \mu_{(j)} v_{(j)\alpha} + \delta \mu_{(j)} (v_{(j)\alpha} - v_\alpha) \right], \]
\[ \Pi^\alpha_{\beta} = \sum_j \mu_{(j)} \left[ v^\alpha_{(j)} (v_{(j)\beta} - v_\beta) - \frac{1}{3} \delta^\alpha_\beta v^\gamma_{(j)} (v_{(j)\gamma} - v_\gamma) \right], \]

thus $\delta p \neq 0 \neq \Pi^\alpha_{\beta}$ to the second order.
where Eq. (140) is valid to the linear order. Assuming an irrotational condition we ignore all vector-type perturbations. As the spatial gauge condition we take

\begin{equation}
\dot{\delta} = -\frac{1}{a} \left[ (1 + \delta(i)) v_\alpha \right]_\alpha = -\frac{1}{a} \delta(i,\alpha) B^\alpha + \delta(i) (\delta K - 3HA) + A\delta K + \frac{3}{2} H (A^2 - B^\alpha B_\alpha)
\end{equation}

\begin{equation}
\begin{aligned}
&\dot{v}_\alpha - \frac{1}{a} \left[ (1 + \delta(i)) v_\alpha \right]_\alpha = \frac{1}{a} A_{\alpha} - \delta(i,\alpha) B^\beta B_\beta |_\alpha - \left( v_\alpha v_\beta \right)_\beta - \frac{1}{a} C^\alpha_\beta v_\beta - \frac{1}{a} C^\beta_\alpha v_\beta,
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
&\frac{1}{a} \left[ (1 + \delta(i)) v_\alpha \right]_\alpha + \frac{1}{a} A_{\alpha} = (\delta K - 3HA) v_\alpha
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
&\dot{\delta} = -\frac{1}{a} \left[ (1 + \delta(i)) v_\alpha \right]_\alpha = -\frac{1}{a} \delta(i) B^\alpha + \delta(i)(\delta K - 3HA) + A\delta K + \frac{3}{2} H (A^2 - B^\alpha B_\alpha)
\end{aligned}
\end{equation}

where Eq. (140) is valid to the linear order.

\section{Irrotational case}

Assuming an irrotational condition we ignore all vector-type perturbations. As the spatial gauge condition we take

\begin{equation}
\gamma \equiv 0,
\end{equation}
thus, \( \beta \equiv \chi/a \). Equations (135)-(140) become

\[
\dot{\chi}_i - \kappa + 3H \alpha - \frac{1}{a^2} \left[ \left( 1 + \delta_i \right) v^\alpha_{(i)} \right]_\alpha = \frac{1}{a^2} \delta_{(i),\alpha} \chi^\alpha + \delta_i \left( \kappa - 3H \alpha \right) + \kappa \alpha + \frac{3}{2} H \left( \alpha^2 - \frac{1}{a^2} \chi^\alpha \chi_\alpha \right).
\]

\[
\dot{v}_{(i),\alpha} v_{(i),\alpha} - \frac{1}{a} \dot{\alpha} \Delta v_{(i),\alpha} - \frac{1}{a} \varphi \Delta v_{(i),\alpha} - \frac{2}{a} \varphi \Delta v_{(i),\alpha} - \frac{2}{a} C^{(t)\alpha\beta} v_{(i),\alpha\beta},
\]

\[
\frac{1}{a} \left[ a^2 \left( 1 + \delta_i \right) v^\alpha_{(i)} \right]_\alpha = -\frac{1}{a^2} \alpha \chi^\alpha + \delta_i \left( \kappa - 3H \alpha \right) + \kappa \chi^\alpha + \frac{3}{2} H \left( \alpha^2 - \frac{1}{a^2} \chi^\alpha \chi_\alpha \right).
\]

Equations (143) become

\[
\dot{\chi}_i - \kappa + 3H \alpha - \frac{1}{a} \left[ \left( 1 + \delta_i \right) v^\alpha_{(i)} \right]_\alpha = \frac{1}{a^2} \alpha \chi^\alpha + \delta_i \left( \kappa - 3H \alpha \right) + \kappa \alpha + \frac{3}{2} H \left( \alpha^2 - \frac{1}{a^2} \chi^\alpha \chi_\alpha \right).
\]

\[
\dot{v}_{(i),\alpha} v_{(i),\alpha} - \frac{1}{a} \dot{\alpha} \Delta v_{(i),\alpha} - \frac{1}{a} \varphi \Delta v_{(i),\alpha} - \frac{2}{a} \varphi \Delta v_{(i),\alpha} - \frac{2}{a} C^{(t)\alpha\beta} v_{(i),\alpha\beta} = 0.
\]

Equation (147) is valid to the linear order.

**B. Linear perturbations**

To the linear order Eqs. (142)-(146) give

\[
\dot{\chi}_i - \kappa + 3H \alpha - \frac{\Delta}{a} v_{(i)} = 0,
\]

\[
\dot{v}_{(i)} + H v_{(i)} - \frac{c}{a} \alpha = 0,
\]

\[
\dot{\kappa} + 2H \kappa - 4\pi G \sum_j \theta_{(j)} \delta_{(j)} + \left( 3H + c^2 \frac{\Delta}{a^2} \right) \alpha = 0,
\]

\[
\frac{\Delta + 3K}{a^2} \chi + \kappa = \frac{12\pi G \alpha}{c} \sum_j \theta_{(j)} v_{(j)},
\]

\[
\dot{\delta} - \kappa + 3H \alpha - \frac{\Delta}{a} v = 0,
\]

\[
\dot{v} + H v - \frac{c}{a} \alpha = 0.
\]
where we have recovered the speed of light $c$. The following additional equations can be found in Eqs. (195),(196),(199) of [11]

$$\kappa - 3H\alpha + 3\dot{\varphi} + c^2 \frac{\Delta}{a^2} \chi = 0,$$

(154)

$$4\pi G\rho \delta + H\kappa + c^2 \frac{\Delta + 3K}{a^2} \varphi = 0,$$

(155)

$$\frac{1}{c}(\dot{\chi} + H\chi) - \varphi - \alpha = 0.$$

(156)

Equation (156) gives

$$\alpha \chi = -\varphi \chi.$$

(157)

Equation (158) gives

$$\alpha_v = 0.$$

(158)

Equations (151), (154) give

$$\dot{\varphi}_v = \frac{Kc}{a^2} \chi_v.$$

(159)

Thus, for $K = 0$ we have

$$\dot{\varphi}_v = 0,$$

(160)

which is valid even in the presence of multi-components and the cosmological constant. Equations (149), (153) give

$$\dot{\chi}_v + 2H\dot{\chi}_v - 4\pi G \sum_j \varrho(j) \delta_v = 0.$$  

(161)

$$\dot{v}(i)_v + H\dot{v}(i)_v + \frac{c}{a} \Phi(v)(i)_v = 0.$$  

(162)

Equations (151), (155) give

$$c^2 \frac{\Delta + 3K}{a^2} \varphi = -4\pi G \rho \delta_v - 4\pi G \sum_j \varrho(j) \delta_v = -4\pi G \sum_j \varrho(j) \delta_v.$$  

(163)

We can derive density perturbation equation in many different temporal gauge (hypersurface) conditions all of which naturally correspond to gauge-invariant variables. In a single component case, density perturbation in the comoving gauge ($v = 0$) is known to give Newtonian result. In the multi-component situation we have many different comoving gauge conditions. Here, we consider two such gauges for $\delta_v$ variable: one based on $v = 0$ gauge, and the other based on $v(\ell) = 0$ gauge for a specific $\ell$.

(I) Equation (148) evaluated in the $v = 0$ gauge, and using Eq. (151) we can derive

$$\delta_{(i)v} = \frac{c^2}{a^2} \varphi_v,$$

where we used $\chi_v \equiv \chi - av \equiv -av$, $v_v(i) \equiv v - v(i)$, and $\alpha_v \equiv \alpha - c^{-1} (av) = 0$. Using Eqs. (161), (162), (163), we have

$$\delta_{(i)v} + 2H\delta_{(i)v} - 4\pi G \sum_j \varrho(j) \delta_{(i)v} = 0.$$  

(164)

This coincides exactly with the Newtonian result in Eq. (16) to the linear order, even in the presence of $K$. Thus, we may identify $\delta_{(i)v}$ as the Newtonian density perturbation $\delta_i$ to the linear order even in the presence of $K$. For $K = 0$ we may also identify $-c\nabla v(i)_\chi$ as the Newtonian velocity perturbation $u_i$. However, in the presence of $K$, we cannot identify the relativistic variables which correspond to the Newtonian velocity perturbation of individual component. Therefore, to the linear order we have the following Newtonian correspondences

$$\delta_i = \delta_{(i)v}, \quad u_i = -c\nabla v(i)_\chi.$$  

(166)
where the latter one is valid only for $K = 0$.

(II) Evaluating Eq. (148) in the $v(\ell) = 0$ gauge for a specific $\ell$, and using Eq. (151) we can derive

$$\delta_{(i)v(\ell)} - c \frac{\Delta + 3K}{a} v(\ell) \chi = \frac{12\pi G}{c} a \sum_j \theta(j) \left( v(j) - v(\ell) \right) + \frac{c}{a} \Delta \left( v(i) - v(\ell) \right),$$  \hspace{1cm} (167)$$

where we used $\chi v(\ell) \equiv \chi - av(\ell)$, $v(i)v(\ell) \equiv v(i) - v(\ell)$, and $\alpha v(\ell) \equiv \alpha - c^{-1} (av(\ell))' = 0$. Using Eqs. (161), (162), (163), we have

$$\delta_{(i)v(\ell)} + 2H \delta_{(i)v(\ell)} - 4\pi G \sum_j \theta(j) \delta_{(j)v(\ell)} = \frac{12\pi G}{c} aH \sum_j \theta(j) \left( v(\ell) - v(j) \right).$$  \hspace{1cm} (168)$$

The terms in right-hand-sides of Eqs. (167), (168) look like relativistic correction terms present even to the linear order based on the variable $\delta_{(i)v(\ell)}$. Since no such correction terms appear in Eq. (165) based on the variable $\delta_{(i)v}$, the relativistic correction terms in Eq. (168) can be regarded as being caused by a complicated hypersurface (gauge) choice.

1. Exact solutions

Assuming $K = 0$, we can identify Newtonian perturbation variables as

$$\delta \equiv \delta_v, \quad \kappa_v \equiv -\frac{1}{a} \nabla \cdot u, \quad u \equiv -c \nabla v, \quad \delta \Phi \equiv -c^2 \varphi, \quad \delta_i \equiv \delta_{(i)v}, \quad u_i \equiv -c \nabla v_{(i)v}. \hspace{1cm} (169)$$

Equations (152), (150), (148), (149), (163) become

$$\dot{\delta} = -\frac{1}{a} \nabla \cdot u, \quad \hspace{1cm} (170)$$

$$\dot{u} + Hu = -\frac{1}{a} \nabla \delta \Phi, \quad \hspace{1cm} (171)$$

$$\dot{\delta}_i = -\frac{1}{a} \nabla \cdot u_i, \quad \hspace{1cm} (172)$$

$$\dot{u}_i + Hu_i = -\frac{1}{a} \nabla \delta \Phi, \quad \hspace{1cm} (173)$$

$$\frac{\Delta}{a^2} \delta \Phi = 4\pi G \rho \delta = 4\pi G \sum_j \theta_j \delta_j. \quad \hspace{1cm} (174)$$

Under the identification in Eq. (169) these equations are valid in both Newton’s and Einstein’s gravity theories. Equations (170), (171), (174), and Eqs. (172), (173), (174), respectively, give

$$\ddot{\delta} + 2H \dot{\delta} - 4\pi G \rho \delta = \frac{1}{a^2 H} \left[ a^2 H^2 \left( \frac{\delta}{H} \right) \right]' = 0, \quad \hspace{1cm} (175)$$

$$\ddot{\delta}_i + 2H \dot{\delta}_i - 4\pi G \sum_j \theta_j \delta_j = 0. \hspace{1cm} (176)$$

Equation (175) has an exact solution

$$\delta(x, t) = H \left[ c_g(x) \int_0^t \frac{dt}{a^2 H^2} + c_d(x) \right], \quad \hspace{1cm} (177)$$

where $c_g$ and $c_d$ are integration constants which indicate the relatively growing and decaying solutions in expanding phase; we do not consider the lower bound of integration which is absorbed to the $c_d$ mode. Equations (173), (176), and the solution in Eq. (177) are valid considering general $K$ and $\Lambda$ in the background world model. Equation (174) can be solved to give

$$\delta \Phi = -G \rho a^2 \int \frac{\delta(x', t)}{|x' - x|} d^3 x'. \hspace{1cm} (178)$$
From Eqs. (170), (171), (174) we can show [14]

\[ u = -a \left( \frac{\nabla \delta \Phi}{4\pi G \rho a^2} \right) + \frac{1}{a} D(x), \quad \nabla \cdot D = 0, \]  

(179)

where the \( D \) term is the solution of the homogeneous part of Eq. (171); it decoupled from the density inhomogeneity and corresponds to the peculiar velocity in the background world model. Since the \( D \) term is not connected to the density inhomogeneity and simply decays, we may ignore it to the linear order.

Now, for the individual component, from Eqs. (172), (173) we have

\[ u_i = -a \left( \frac{\nabla \delta \Phi}{4\pi G \rho a^2} \right) + \frac{1}{a} \nabla d_i(x) + \frac{1}{a} D_i(x), \quad \nabla \cdot D_i \equiv 0, \]

(180)

\[ \delta_i = \delta + c_i(x) - \Delta d_i(x) \int^t dt' \frac{d}{a^2(t')} , \]

(181)

with

\[ \sum_j \delta_j d_j \equiv 0 \equiv \sum_j \delta_j c_j . \]

(182)

The \( c_i \) and \( d_i \) are the two isocurvature-type (\( \delta = 0 \), thus \( \delta \Phi = 0 \)) solutions. It happens that the relatively decaying isocurvature-type solution, i.e., \( d_i \)-mode, temporally behaves the same as the peculiar velocity in the background, i.e., \( D_i \)-mode. The relatively growing isocurvature-type solution \( c_i \) does not contribute to the \( u_i \), see Eq. (172). From Eqs. (170), (180) we have

\[ u_i - u = \frac{1}{a} \left[ \nabla d_i(x) + D_i(x) - D(x) \right] , \]

(183)

which simply decays; the \( D_i \) and \( D \) solutions are divergence-free and decoupled from the density perturbation, and are the peculiar velocity perturbation present in the background world model.

C. Comoving gauge

To the linear order, only the hypersurface condition \( v = 0 \) (the comoving temporal gauge) allows the density perturbation equation presented in the Newtonian form. Thus, we take

\[ v \equiv 0, \]

(184)
even to the second order. We take \( \gamma = 0 \) in Eq. (141) as the spatial gauge condition. Equation (146) gives

\[ \alpha = -\frac{1}{2a^2} \chi^\alpha \chi_\alpha + \frac{\kappa}{2a^2} \sum_j \frac{\mu_{(j)}}{\mu} \left[ \frac{1}{2} \psi_{(j)\beta}^\alpha \psi_{(j)\beta}^\alpha + \Delta^{-1} \nabla_\alpha \left( \psi_{(j)\beta}^\alpha \psi_{(j)\beta}^\beta \right) \right] . \]

(185)

Using this, Eqs. (142) - (145) give

\[ \dot{\delta} - \kappa = -\frac{c}{a^2} \delta_\alpha \chi^\alpha + \delta \kappa + \frac{1}{2} H \sum_j \mu_{(j)} \psi_{(j)\alpha}^{\alpha} \psi_{(j)\alpha}^\alpha + 3H \sum_j \mu_{(j)} \Delta^{-1} \nabla_\alpha \left( \psi_{(j)\beta}^\alpha \psi_{(j)\beta}^\beta \right), \]

(186)

\[ \dot{\kappa} - 2HK - 4\pi G \rho \delta = -\frac{c}{a^2} \kappa_\alpha \chi^\alpha + \frac{\kappa^2}{2} + \left( \dot{\psi}_{(t)\alpha}^\beta + \frac{c}{a^2} \chi_{(t)\alpha \beta} \right) \left( \dot{\psi}_{(t)\alpha}^\beta + \frac{c}{a^2} \chi_{(t)\alpha \beta} \right) - \frac{1}{3} \left( \frac{\Delta}{a^2} \right)^2 , \]

(187)

\[ \dot{\delta}_{(i)} - \kappa + \frac{c}{a} \left[ \left( 1 + \delta_{(i)} \right) \psi_{(i)\alpha}^{\alpha} \right]_\alpha = -\frac{c}{a^2} \delta_{(i),\alpha} \chi^{\alpha} + \delta_{(i)} \kappa + H \psi_{(i)\alpha} \psi_{(i)\alpha}^\alpha \]

\[ - \frac{c}{a} \left( \dot{\psi}_{(i)\alpha}^{\alpha} \psi_{(i)\alpha}^\alpha - 2 \dot{\varphi}_{(i)\alpha}^\alpha - 2 \psi_{(i)\beta}^\alpha \psi_{(i)\beta}^\beta \right) \right] \frac{3}{2} H \sum_j \mu_{(j)} \psi_{(j)\beta}^\alpha \psi_{(j)\beta}^\alpha + 3H \sum_j \mu_{(j)} \Delta^{-1} \nabla_\alpha \left( \psi_{(j)\beta}^\alpha \psi_{(j)\beta}^\beta \right) , \]

(188)

\[ \dot{v} \equiv 0 , \]

\[ \dot{\psi}_{(i)\alpha}^{\alpha} - \psi_{(i)\beta}^\alpha \psi_{(i)\beta}^\beta = \frac{1}{a} \left[ \psi_{(i)\beta}^\alpha \psi_{(i)\beta}^\beta \right] \]

\[ + \frac{c}{a^2} \sum_j \mu_{(j)} \nabla_\alpha \left( \psi_{(j)\beta}^\alpha \psi_{(j)\beta}^\beta \right) + \frac{c}{a} \sum_j \frac{\mu_{(j)}}{\mu} \nabla_\alpha \Delta^{-1} \nabla_\gamma \left( \psi_{(j)\beta}^\alpha \psi_{(j)\beta}^\beta \right) . \]

(189)
From Eqs. (186)-(187), and Eqs. (187)-(189) we can derive, respectively,
\[
\frac{1}{a^2} \left[ a^2 \left( \delta + \frac{c}{a^2} \delta \alpha \chi^\alpha \right) \right] - 4 \pi G \varrho \delta (1 + \delta) = -\frac{c}{a^2} \kappa \alpha \chi^\alpha + \frac{4}{3} \kappa^2 \\
+ \left( C^{(t)\alpha\beta} + \frac{c}{a^2} \chi^\alpha \chi^\beta \right) \left( C^{(t)} + \frac{c}{a^2} \chi^\alpha \chi^\beta \right) - \frac{1}{3} \left( \frac{c}{a^2} \chi \right)^2 \\
+ \left( 2 \dot{H} + 4 \pi G \varrho + \frac{1}{2} c^2 \frac{\Delta}{a^2} \right) \sum \mu_{(j)} v_{(j)\alpha}^\alpha v_{(j)\beta}^\beta + \left( 6 \dot{H} + c^2 \frac{\Delta}{a^2} \right) \sum \mu_{(j)} \Delta^{-1} \nabla_\alpha \left( v_{(j)\alpha}^\alpha v_{(j)\beta}^\beta \right)
\]
(190)
\[
\frac{1}{a^2} \left[ a^2 \left( \delta_{(i)} + \frac{c}{a^2} \delta_{(i),\alpha} \chi^\alpha \right) \right] - 4 \pi G \varrho \delta (1 + \delta_{(i)}) = -\frac{c}{a^2} \kappa_{(i)} \chi^\alpha + \frac{4}{3} \kappa^2 \\
+ \left( C^{(t)\alpha\beta} + \frac{c}{a^2} \chi^\alpha \chi^\beta \right) \left( C^{(t)} + \frac{c}{a^2} \chi^\alpha \chi^\beta \right) - \frac{1}{3} \left( \frac{c}{a^2} \chi \right)^2 - \frac{c}{a} \left[ (\kappa + \dot{\varphi})^\alpha v_{(i)\alpha} + 2 (\kappa - \dot{\varphi}) v_{(i)\alpha}^\alpha \right] \\
+ c^2 \frac{\Delta}{a^2} \left( v_{(i)\alpha} \chi^\alpha \right) + \frac{c^2}{a^2} \left( v_{(i)}^\alpha v_{(i)}^\beta \right) \dot{C}^{(t)}_{\alpha\beta} + \dot{H} v_{(i)\alpha} v_{(i)}^\alpha + \frac{2c}{a} v_{(i)}^\alpha \dot{C}^{(t)}_{\alpha\beta} \\
+ \left( 3 \dot{H} + 4 \pi G \varrho \right) \sum \mu_{(j)} v_{(j)\alpha} v_{(j)\beta} + 6 \dot{H} \sum \frac{\mu_{(j)}}{\mu} \Delta^{-1} \nabla_\alpha \left( v_{(j)\alpha}^\alpha v_{(j)\beta}^\beta \right)
\]
(191)
Notice the \( O(v_{(i)\alpha} v_{(i)\beta}) \) correction terms are present in Eqs. (186), (187), (190) even in the single component situation. Except for these \( O(v_{(i)\alpha} v_{(i)\beta}) \) terms, the remaining parts of these equations coincide with the ones in the single component situation.

D. Newtonian correspondence

For \( K = 0 \), to the linear order, we have
\[
\chi_v = \chi - a v \equiv -a v_x, \quad \kappa_v = -\frac{c}{a^2} \chi_v = \frac{c}{a} v_x, \quad v_{(i)v} \equiv v_{(i)} - v = v_{(i)x} - v_x, \quad \dot{\varphi}_v = 0.
\]
(192)
To the linear order we identify
\[
u \equiv -c v_x, \quad u_i \equiv -c v_{(i)x}.
\]
(193)
Now, to the second order, we attempt identifying the Newtonian perturbation variables \( \delta, \delta_i, u, \) and \( u_i \) as
\[
\kappa_v \equiv -\frac{1}{a} \nabla \cdot u, \quad \chi_v \equiv \frac{a}{c} u, \quad u \equiv \nabla u, \quad v_{(i)v} \equiv \frac{1}{c} (u_i - u), \quad \delta \equiv \delta_v, \quad \delta_i \equiv \delta_{(i)v}.
\]
(194)
Using these identifications Eqs. (186) - (191) can be written as

\[
\dot{\mathbf{u}} + \frac{1}{a} \nabla \cdot \mathbf{u} = -\frac{1}{a} \nabla \cdot (\mathbf{u} \mathbf{u}) + H \sum_j \frac{\theta_j}{c^2} \left\{ \frac{1}{2} \mathbf{u}_j - \mathbf{u}_j^2 + 3 \Delta^{-1} \nabla \cdot [(\mathbf{u}_j - \mathbf{u}) \nabla \cdot (\mathbf{u}_j - \mathbf{u})] \right\},
\]

\[
\frac{1}{a} \nabla \cdot (\dot{\mathbf{u}} + H \mathbf{u}) + 4\pi G \rho \delta = -\frac{1}{a^2} \nabla \cdot (\mathbf{u} \nabla \mathbf{u}) - \hat{C}^{(t)\alpha\beta} \left( \frac{2}{a} u_{\alpha|\beta} + \hat{C}^{(t)}_{\alpha\beta} \right)
+ \sum_j \frac{\theta_j}{c^2} \left\{ \frac{1}{2} \left( 4\pi G \rho - c^2 \Delta \right) |\mathbf{u}_j - \mathbf{u}|^2 + \left( 12\pi G \rho - c^2 \Delta \right) \Delta^{-1} \nabla \cdot [(\mathbf{u}_j - \mathbf{u}) \nabla \cdot (\mathbf{u}_j - \mathbf{u})] \right\},
\]

\[
\frac{1}{a} \nabla \cdot (\dot{\mathbf{u}} + H \mathbf{u}) + 4\pi G \rho \delta = -\frac{1}{a^2} \nabla \cdot (\mathbf{u} \nabla \mathbf{u}) - \hat{C}^{(t)\alpha\beta} \left( \frac{2}{a} u_{\alpha|\beta} + \hat{C}^{(t)}_{\alpha\beta} \right)
+ \sum_j \frac{\theta_j}{c^2} \left\{ \frac{1}{2} \left( 4\pi G \rho - c^2 \Delta \right) |\mathbf{u}_j - \mathbf{u}|^2 + \left( 12\pi G \rho - c^2 \Delta \right) \Delta^{-1} \nabla \cdot [(\mathbf{u}_j - \mathbf{u}) \nabla \cdot (\mathbf{u}_j - \mathbf{u})] \right\},
\]

\[
\frac{1}{a^2} (a^2 \delta) - 4\pi G \rho \delta = -\frac{1}{a^2} [a \nabla \cdot (\mathbf{u} \mathbf{u})] + \frac{1}{a^2} \nabla \cdot (\mathbf{u} \nabla \mathbf{u}) + \hat{C}^{(t)\alpha\beta} \left( \frac{2}{a} u_{\alpha|\beta} + \hat{C}^{(t)}_{\alpha\beta} \right)
- \sum_j \frac{\theta_j}{c^2} \left\{ \left( 4\pi G \rho - c^2 \Delta \right) |\mathbf{u}_j - \mathbf{u}|^2 + \left( 12\pi G \rho - c^2 \Delta \right) \Delta^{-1} \nabla \cdot [(\mathbf{u}_j - \mathbf{u}) \nabla \cdot (\mathbf{u}_j - \mathbf{u})] \right\},
\]

\[
\frac{1}{a^2} (a^2 \delta) - 4\pi G \rho \delta = -\frac{1}{a^2} [a \nabla \cdot (\mathbf{u} \mathbf{u})] + \frac{1}{a^2} \nabla \cdot (\mathbf{u} \nabla \mathbf{u}) + \hat{C}^{(t)\alpha\beta} \left( \frac{2}{a} u_{\alpha|\beta} + \hat{C}^{(t)}_{\alpha\beta} \right)
+ \frac{1}{a^2} \left( \Delta |\mathbf{u}_j - \mathbf{u}_j - \mathbf{u}_j| - \nabla \cdot [(\mathbf{u}_j - \mathbf{u}_j) \nabla \mathbf{u} + \mathbf{u}_j \nabla \mathbf{u}_j] \right)
- \frac{4\pi G \rho}{c^2} |\mathbf{u}_j - \mathbf{u}_j|^2 - 8\pi G \sum_j \frac{\theta_j}{c^2} \left\{ |\mathbf{u}_j - \mathbf{u}_j|^2 + 3 \Delta^{-1} \nabla \cdot [(\mathbf{u}_j - \mathbf{u}_j) \nabla \cdot (\mathbf{u}_j - \mathbf{u}_j)] \right\}.
\]

In Sec. VII.B.1, we have shown that, to the linear order, \((\mathbf{u}_i - \mathbf{u})\) simply decays \((\propto a^{-1})\) in an expanding phase. From Eq. (183) we have

\[
\mathbf{u}_i - \mathbf{u} = \frac{1}{a} [\nabla d_i(x) + \mathbf{D}_i(x) - \mathbf{D}(x)],
\]

\[
\sum_j \theta_j d_j \equiv 0, \quad \nabla \cdot \mathbf{D} \equiv 0 \equiv \nabla \cdot \mathbf{D}_i.
\]

Thus, \((\mathbf{u}_i - \mathbf{u})\) simply decays in an expanding background. If we ignore these contributions from the velocity differences, except for the presence of the tensor-type perturbation, Eq. (200) coincides exactly with the zero-pressure limit of Newtonian result in Eq. (110). Terms in the last two lines of Eq. (200) are relativistic correction terms which vanish for a single component case leading to Eq. (119); the second line in Eq. (119) also vanishes in the single component case.

Ignoring quadratic combination of \((\mathbf{u}_i - \mathbf{u})\) terms, we have

\[
\dot{\mathbf{u}} + \frac{1}{a} \nabla \cdot \mathbf{u} = -\frac{1}{a} \nabla \cdot (\mathbf{u} \mathbf{u}) ,
\]

\[
\frac{1}{a} \nabla \cdot (\dot{\mathbf{u}} + H \mathbf{u}) + 4\pi G \rho \delta = -\frac{1}{a^2} \nabla \cdot (\mathbf{u} \nabla \mathbf{u}) - \hat{C}^{(t)\alpha\beta} \left( \frac{2}{a} u_{\alpha|\beta} + \hat{C}^{(t)}_{\alpha\beta} \right),
\]

\[
\frac{1}{a^2} (a^2 \delta) - 4\pi G \rho \delta = -\frac{1}{a^2} [a \nabla \cdot (\mathbf{u} \mathbf{u})] + \frac{1}{a^2} \nabla \cdot (\mathbf{u} \nabla \mathbf{u}) + \hat{C}^{(t)\alpha\beta} \left( \frac{2}{a} u_{\alpha|\beta} + \hat{C}^{(t)}_{\alpha\beta} \right),
\]
and
\[\dot{\delta}_i + \frac{1}{a} \nabla \cdot u_i = -\frac{1}{a} \nabla \cdot (\dot{\delta}_i u_i) + \frac{1}{a} \left[ 2\varphi \nabla \cdot (u_i - u) - (u_i - u) \cdot \nabla \varphi + 2 (u_i^0 - u^0) |^\beta C_{\alpha \beta}^{(t)} \right], \tag{205}\]
\[\frac{1}{a} \nabla \cdot (\dot{u}_i + H u_i) + 4\pi G \rho \delta = -\frac{1}{a^2} \nabla \cdot (u_i \cdot \nabla u_i) - \dot{C}_{\alpha \beta}^{(t)} \left( \frac{2}{a} u_{\alpha |\beta} + \dot{C}_{\alpha \beta}^{(t)} \right), \tag{206}\]
\[\frac{1}{a^2} \left( a^2 \dot{\delta}_i \right) - 4\pi G \rho \delta = -\frac{1}{a^2} \left[ (u_i \cdot \nabla u_i) \right] + \frac{1}{a^2} \nabla \cdot (\dot{u}_i \cdot \nabla u_i) + \dot{C}_{\alpha \beta}^{(t)} \left( \frac{2}{a} u_{\alpha |\beta} + \dot{C}_{\alpha \beta}^{(t)} \right), \tag{207}\]

Equations (202)-(204) coincide with the density and velocity perturbation equations of a single component medium [12]; thus, except for the contribution from gravitational waves, these equations coincide with ones in the Newtonian context.

If we further ignore \((u_i - u)\) terms appearing in the pure second-order combinations, Eqs. (205)-(207) become
\[\dot{\delta}_i + \frac{1}{a} \nabla \cdot u_i = -\frac{1}{a} \nabla \cdot (\dot{\delta}_i u_i), \tag{208}\]
\[\frac{1}{a} \nabla \cdot (\dot{u}_i + H u_i) + 4\pi G \rho \delta = -\frac{1}{a^2} \nabla \cdot (u_i \cdot \nabla u_i) - \dot{C}_{\alpha \beta}^{(t)} \left( \frac{2}{a} u_{\alpha |\beta} + \dot{C}_{\alpha \beta}^{(t)} \right), \tag{209}\]
\[\frac{1}{a^2} \left( a^2 \dot{\delta}_i \right) - 4\pi G \rho \delta = -\frac{1}{a^2} \left[ (u_i \cdot \nabla u_i) \right] + \frac{1}{a^2} \nabla \cdot (\dot{u}_i \cdot \nabla u_i) + \dot{C}_{\alpha \beta}^{(t)} \left( \frac{2}{a} u_{\alpha |\beta} + \dot{C}_{\alpha \beta}^{(t)} \right), \tag{210}\]

Notice that, by ignoring \(i\)-indices, Eqs. (208)-(210) coincide with Eqs. (202)-(204). In this context, except for the contribution from gravitational waves, the above equations coincide exactly with ones in the Newtonian context even in the multi-component case; compare with Eqs. (8), (9), (16) without pressure. In the single component situation such a relativistic/Newtonian correspondence to the second order was shown in [11, 12]. In the present case, the same equation valid in the single component is now valid in the multi-component case for the collective fluid variables. This justifies our identifications of Newtonian perturbation variables in Eq. (194).

**VIII. EFFECTS OF CURVATURE**

We consider a single zero-pressure, irrotational fluid. We take the temporal comoving gauge \((v = 0)\) and the spatial \(\gamma = 0\) gauge. In the presence of background curvature the basic equations are presented in Eqs. (115)-(117) for nonvanishing pressure, or Eqs. (185)–(191) for zero-pressure multiple component fluids. By setting pressures equal to zero in Eqs. (115)-(117), or from Eqs. (187)-(190), we have
\[\dot{\delta} - \kappa = -\frac{c}{a^2} \delta_{\alpha} \chi^{\alpha} + \delta \kappa, \tag{211}\]
\[\kappa + 2H \kappa - 4\pi G \rho \delta = -\frac{c}{a^2} \kappa_{\alpha} \chi^{\alpha} + \frac{1}{3} \kappa^2 - \frac{1}{3} \left( \frac{\Delta}{a^2} \chi \right)^2 - \left( \dot{C}_{\alpha \beta}^{(t)} + \frac{c}{a^2} \chi_{\alpha |\beta} \right), \tag{212}\]
\[\ddot{\delta} + 2H \dot{\delta} - 4\pi G \rho \delta = \frac{c}{a^2} \left( \delta_{\alpha} \chi^{\alpha} \right) - \frac{c}{a^2} \kappa_{\alpha} \chi^{\alpha} + \frac{4}{3} \kappa^2 - \frac{1}{3} \left( \frac{\Delta}{a^2} \chi \right)^2 \]
\[+ \left( \dot{C}_{\alpha \beta}^{(t)} + \frac{c}{a^2} \chi_{\alpha |\beta} \right), \tag{213}\]
\[\frac{\Delta + 3K}{a^2} \chi + \kappa = 0, \tag{214}\]

where Eq. (214) is valid to the linear order. Compared with the situation with vanishing curvature, the effects of curvature in the above perturbed set of equations appear only in the linear-order relation between \(\kappa\) and \(\chi\) in Eq. (214).

**A. Newtonian correspondence**

Considering the successful Newtonian correspondence to the linear order even in the presence of the background curvature, we assume the identification in Eq. (120) is valid to the second order. Then, to the linear order, from Eq.
we have
\[ \kappa \equiv -\frac{1}{a} \nabla \cdot \mathbf{u} \equiv -\frac{\Delta}{a} u = -c \frac{\Delta + 3K}{a^2} \chi, \] (215)
where \( \mathbf{u} \equiv \nabla u \) and \( \chi = \chi_v = \frac{a v}{c} \chi \). Thus, to the linear order, formally we have
\[ \frac{c}{a} \chi = \left( 1 - \frac{3K}{\Delta + 3K} \right) u. \] (216)

In the presence of curvature, the scalar-type perturbation can be handled by solving Eqs. (211), (212), (214) together with the identifications made above. We can formally separate the effects of pure curvature contribution. Using Eqs. (215), (216), Eqs. (211), (212), (214) become
\[ \dot{\delta} + \frac{1}{a} \nabla \cdot \mathbf{u} = -\frac{1}{a} \nabla \cdot (\delta \mathbf{u}) + \frac{1}{a} (\nabla \delta) \cdot \nabla \left( \frac{3K}{\Delta + 3K} u \right), \] (217)
\[ \frac{1}{a} \nabla \cdot (\dot{\mathbf{u}} + H \mathbf{u}) + 4\pi G \rho \delta = -\frac{1}{a^2} \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) - \dot{C}^{(t)\alpha\beta} \left( C_{\alpha\beta} + \frac{2}{a} u_{\alpha|\beta} \right) + \frac{1}{a^2} \nabla \cdot \left[ \left( \frac{3K}{\Delta + 3K} \mathbf{u} \right) \cdot \nabla \left( \frac{3K}{\Delta + 3K} u \right) \right] - \frac{1}{3} \frac{1}{a^2} \left( \frac{3K}{\Delta + 3K} \nabla \cdot \mathbf{u} \right) \left( \frac{2\Delta + 3K}{\Delta + 3K} \nabla \cdot \mathbf{u} \right) + \frac{1}{a} \left( \frac{3K}{\Delta + 3K} u \right)^{\alpha|\beta} \left[ 2 \dot{C}^{(t)\alpha\beta} + \frac{1}{a} \left( \frac{\Delta}{\Delta + 3K} \mathbf{u} \right)_{\alpha|\beta} \right]. \] (218)

From Eq. (74), the \( K \) term can be written as
\[ K = \left( \frac{aH}{c} \right)^2 (\Omega_t - 1), \quad \Omega_t \equiv \Omega + \Omega_\Lambda, \quad \Omega \equiv \frac{8\pi G \rho}{3H^2}, \quad \Omega_\Lambda \equiv \frac{\Lambda c^2}{3H^2}. \] (219)

**IX. EFFECTS OF VECTOR-TYPE PERTURBATION**

The spatial \( C \)-gauge sets
\[ \gamma \equiv 0 \equiv C^{(v)}_\alpha. \] (220)
The remaining variables under this gauge condition are completely free of the spatial gauge modes and have unique spatially gauge-invariant counterparts. If we simultaneously take any temporal gauge which also removes the temporal gauge mode completely, all the remaining variables have corresponding unique gauge-invariant counterparts. The above statements are true to all orders in perturbations, see Sec. VI of [11]. From Eq. (77) we have
\[ \beta \equiv \frac{1}{a} \chi, \quad B^{(v)}_\alpha \equiv \Psi^{(v)}_\alpha. \] (221)
Thus, we have
\[ B_\alpha = \frac{1}{a} \chi, \quad C_{\alpha\beta} \equiv \varphi_{\alpha\beta}^{(v)} + C^{(t)}_{\alpha\beta}. \] (222)
As the temporal comoving gauge we set
\[ v = 0. \] (223)
As we mentioned, the remaining variables under these gauge conditions are completely free of the gauge modes and have unique gauge-invariant counterparts to all orders in perturbation, see [11].

**A. Linear perturbations**

To the linear order, the three types of perturbations decouple, and evolve independently. The rotational perturbation is described by Eqs. (206)-(209) in [11]:
\[ \frac{\Delta + 2K}{2a^2} \Psi^{(v)}_\alpha + \frac{8\pi G}{c^4} (\mu + p) v^{(v)}_\alpha = 0, \] (224)
\[ \frac{a^4 (\mu + p) v^{(v)}_\alpha}{a^4 (\mu + p)} \frac{a^4 (\mu + p)}{a^4 (\mu + p)} = -\frac{\Delta + 2K}{2a^2} \frac{\Pi^{(v)}_\alpha}{\mu + p}, \] (225)
\[ \dot{\Psi}^{(v)}_\alpha + 2H \Psi^{(v)}_\alpha = \frac{8\pi G}{c^3} \Pi^{(v)}_\alpha. \] (226)
where the last equation follows from the first two. Compared with Bardeen’s notation in [16], we have

\[
B^{(1)}_\alpha = B^{(1)} Q^{(1)}_\alpha, \quad C^{(v)}_\alpha = - \frac{1}{k} H_T^{(1)} Q^{(1)}_\alpha, \quad \Psi^{(v)}_\alpha = \Psi Q^{(1)}_\alpha \equiv \left( B^{(1)} - \frac{a}{k} \dot{H}_T^{(1)} \right) Q^{(1)}_\alpha,
\]

\[
v^{(v)}_\alpha = v_s Q^{(1)}_\alpha \equiv \left( v^{(1)} - B^{(1)} \right) Q^{(1)}_\alpha, \quad v^{(v)}_\alpha = v_s Q^{(1)}_\alpha \equiv \left( v^{(1)} - \frac{a}{k} \dot{H}_T^{(1)} \right) Q^{(1)}_\alpha,
\]

where Bardeen’s \(v^{(1)}_\alpha\) and \(v_s^{(1)}\), thus our \(v^{(v)}_\alpha\) and \(v^{(v)}_\alpha + \Psi^{(v)}_\alpha\), are related to the vorticity and the shear, respectively. From Eq. (60), to the linear order, we have

\[
\tilde{\omega}_{\alpha\beta} = a v^{(v)}_{[\alpha]}/c, \quad \tilde{\sigma}_{\alpha\beta} = a \left[ -\left( \nabla_\alpha \nabla_\beta - \frac{1}{3} g_{\alpha\beta} \Delta \right) \left( v^{(1)} - \frac{1}{a} \chi \right) + \left( v^{(v)}_\alpha + \Psi^{(v)}_\alpha \right) \right] + \left[ C^{(v)}_\alpha + \left( \nabla_\alpha v^{(1)} \right) \right].
\]

(227)

Bardeen called \(\Psi^{(v)}_\alpha\) a ‘frame-dragging potential’. The difference between \(v^{(1)}_\alpha\) and \(v_s^{(1)}\) is crucially important to show Mach’s principle including the linear order rotational perturbation in [27].

In the absence of anisotropic stress which can act as a sink or source of the angular momentum, we have

Angular momentum \(\propto a^4 (\mu + p) v^{(v)}_\alpha \propto a^2 \Psi^{(v)}_\alpha \propto \text{constant in time}.\)

Thus, for vanishing anisotropic stress, we have

\[
v^{(v)}_\alpha \propto \frac{1}{a^4 (\mu + p)}, \quad \Psi^{(v)}_\alpha \propto \frac{1}{a^2}. \tag{230}\]

In the zero-pressure limit we have

\[
v^{(v)}_\alpha \propto \frac{1}{a}, \quad \Psi^{(v)}_\alpha \propto \frac{1}{a^2}. \tag{231}\]

B. Second-order perturbations

We consider a single component situation with general pressure. We set \(K \equiv 0\). To the linear-order we use

\[
\chi \equiv \chi_v \equiv -a v^{(1)}_\alpha, \quad \kappa \equiv \kappa_v = \frac{\Delta}{a} v^{(1)}_\alpha. \tag{232}\]

The scalar-type perturbation is described by Eqs. (67), (68) which give

\[
\dot{\chi} + 3 H (c_s^2 - w) \dot{\chi} + 3 H \frac{c}{\mu} \left( -1 + w \right) \kappa
\]

\[
= -3 H (1 + w) \left( \alpha - \frac{1}{2} \alpha^2 + \frac{1}{2} \left( v^{\alpha}_{,\alpha} - \Psi^{(v)}_\alpha \right) \left( v^{,\alpha}_{\chi,\alpha} - \Psi^{(v)}_\alpha \right) \right) + \frac{c}{\mu} \delta_{,\alpha} \left( v^{\alpha}_{,\alpha} - \Psi^{(v)}_\alpha \right) + \frac{c}{\mu} \left( \Delta a \nabla_\alpha - 3 H \alpha \right) + \frac{(1 + w) \alpha \Delta}{a} v^{,\alpha}_{\chi} + (1 + w) v^{(v)}_\alpha \left( 1 - 3c^2 a \right) H v^{(v)}_\alpha + \frac{c}{\mu + p} \frac{\Delta}{a^2} \left( \chi^{(1)}_\alpha - \frac{c}{a} v^{(1)}_{\chi,\alpha} - \frac{c}{a} \Psi^{(v)}_\alpha \right) \right]
\]

\[
+ \frac{c}{\mu} \left( 1 + w \right) \left[ -2 \alpha v^{(v)}_\alpha + 2 C^{(v)}_\alpha \right] - \frac{2}{\mu} \frac{1}{a} v^{(v)}_\alpha \left( \Pi^{(v)}_{\alpha \beta} v^{(v)}_\beta \right) - \frac{2}{\mu} \frac{1}{a} v^{(v)}_\alpha \left( \Pi^{(v)}_{\alpha \beta} v^{(v)}_\beta \right) - \frac{2}{\mu} \frac{1}{a} v^{(v)}_\alpha \left( \Pi^{(v)}_{\alpha \beta} v^{(v)}_\beta \right) \right]
\]

\[
- \frac{1}{a} \mu \left[ \left( \delta \mu + 3 \delta p \right) v^{(v)}_\alpha + \left( \Pi^{(v)}_{\alpha \beta} v^{(v)}_\beta \right) \right], \tag{233}\]

\[
\dot{\kappa} + 2 H \kappa - \frac{4 \pi G}{c^2} (\delta \mu + 3 \delta p) \kappa = - \left( 3 H + c^2 \Delta a \right) \left[ \alpha + \frac{1}{2} \left( v^{\alpha}_{,\alpha} - \Psi^{(v)}_\alpha \right) \left( v^{,\alpha}_{\chi,\alpha} - \Psi^{(v)}_\alpha \right) \right] - 2 H \alpha \frac{\Delta}{a} v^{(1)}_\chi
\]

\[
+ \frac{4 \pi G}{c^2} (\delta \mu + 3 \delta p) \alpha + \frac{3}{2} H \alpha^2 + (\alpha + 2 \varphi) c^2 \Delta a^2 + \frac{c}{a^2} \alpha - \varphi \right) v^{(v)}_\alpha \right] + \frac{2}{a} \frac{c}{a^2} \left( C^{(v)}_{\alpha \beta} \alpha_{\alpha \beta} + \frac{8 \pi G}{c^2} (\mu + p) v^{(v)}_\alpha v^{(v)}_\alpha \right)
\]

\[
+ \frac{1}{a} \left( v^{\alpha}_{,\alpha} - \Psi^{(v)}_\alpha \right) \left( c^2 \frac{\Delta}{a^2} v^{(1)}_\chi \right) + \left( \dot{\Psi}^{(v)}_{\alpha \beta} - \frac{c}{a} v^{(1)}_{\chi,\alpha} - \frac{c}{a} \Psi^{(v)}_{\alpha \beta} \right) \right] \right) \right] \right]. \tag{234}\]
The vector-type perturbation is described by Eq. (68) which gives

\[
\frac{1}{a^4} \left[ a^4 (\mu + p) v_\alpha^{(v)} \right] + \frac{c}{a} (\mu + p) \alpha_\alpha + \frac{c}{a} (\delta p_\alpha + \Pi_{\alpha\beta}^\beta) \\
= \frac{c}{a} \left\{ -\alpha \delta \rho + \frac{1}{2} (\mu + p) \left[ \alpha^2 - \left( v_{\chi,\beta} - \Psi^{(v)\beta} \right) \right] v_{\chi,\beta} - \Psi^{(v)\beta} \right\}_{\alpha,\alpha} - \frac{c}{a} \alpha_\alpha \delta \mu \\
+ (\mu + p) \left( \frac{c \Delta}{a} v_{\chi} - 3H \alpha \right) v_{\alpha}^{(v)} - \frac{c}{a} (\mu + p) \left( \frac{v_{\beta}^{(v)}}{v_{\alpha}^{(v)}} \beta = \Psi^{(v)\beta} \right)_{\alpha} + \frac{c}{a} (\mu + p) \left( v_{\alpha}^{(v)} \beta - v_{\chi,\beta} + \Psi^{(v)\beta} \right) \\
+ \frac{c}{a} \left( 2 \varphi \Pi_{\alpha\beta}^\beta - \varphi_{\alpha\beta} \Pi^\beta + \varphi_{\alpha} \Pi_{\alpha\beta}^\beta - (\alpha \Pi_{\alpha})_{\beta} + 2 C^{(t)\gamma\beta} \Pi_{\alpha\beta\gamma} + C^{(t)\gamma\beta} \Pi_{\gamma}^\beta \right) \\
- \frac{1}{a^4} \left( a^4 f \left( (\delta \mu + \delta p) v_{\alpha}^{(v)} + \Pi_{\alpha\beta}^\beta v_{\beta}^{(v)} \right) \right) \\
= \mu c A_\alpha. 
\]

(235)

We have \([A_\alpha] = L^{-1}\). From this we have

\[
\frac{1}{a^4} \left[ a^4 (\mu + p) v_\alpha^{(v)} \right] + 4 \frac{\Delta}{3 a^2} \Pi = a \mu \Delta^{-1} \nabla \cdot A, 
\]

(236)

\[
\frac{1}{a^4} \left[ a^4 (\mu + p) v_\alpha^{(v)} \right] + \frac{\Delta}{2 a^2} \Pi = \mu c \left[ A_\alpha - (\Delta^{-1} \nabla \cdot A)_{\alpha} \right]. 
\]

(237)

Equation for the tensor-type perturbation (gravitational waves) follows from Eqs. (66), (102).

C. Zero-pressure case

In the zero-pressure limit, Eqs. (236), (237) give

\[
\alpha = a \Delta^{-1} \nabla \cdot A, \quad a A_\alpha = - \frac{1}{2} \left( v_{\chi,\beta} - \Psi^{(v)\beta} \right) \left( v_{\chi,\beta} - \Psi^{(v)\beta} \right)_{\alpha} + \frac{c}{a} v_{\alpha}^{(v)} \left( v_{\chi,\beta} - \Psi^{(v)\beta} \right) + v_{\beta}^{(v)} \left( v_{\chi,\beta} - \Psi^{(v)\beta} \right)_{\alpha} - v_{\alpha}^{(v)} v_{\beta}^{(v)\beta}, \quad (238)
\]

thus, \(\alpha\) is purely second-order, and

\[
\alpha + \frac{1}{2} \left( v_{\chi,\beta} - \Psi^{(v)\beta} \right) \left( v_{\chi,\beta} - \Psi^{(v)\beta} \right) = \Delta^{-1} \nabla_{\alpha} \left[ v_{\alpha}^{(v)} \left( v_{\chi,\beta} - \Psi^{(v)\beta} \right) + v_{\beta}^{(v)} \left( v_{\chi,\beta} - \Psi^{(v)\beta} \right)_{\alpha} - v_{\alpha}^{(v)} v_{\beta}^{(v)\beta} \right]. \quad (240)
\]

Equations (233), (234), (237) give

\[
\dot{\delta} - \kappa = \frac{c}{a} \left( v_{\chi} - \Psi^{(v)} \right) \left( v_{\chi,\beta} - \Psi^{(v)\beta} \right)_{\alpha} + \frac{c}{a} v_{\alpha}^{(v)} \left( v_{\chi} - \Psi^{(v)} \right) + \frac{c}{a} \left( A_{\alpha}^{(v)} v_{\beta}^{(v)\beta} + 2 C^{(t)\alpha\beta} v_{\beta}^{(v)\beta} \right) \\
- 3 H \Delta^{-1} \nabla_{\alpha} \left[ v_{\alpha}^{(v)} \left( v_{\chi,\beta} - \Psi^{(v)\beta} \right) + v_{\beta}^{(v)} \left( v_{\chi,\beta} - \Psi^{(v)\beta} \right)_{\alpha} - v_{\alpha}^{(v)} v_{\beta}^{(v)\beta} \right], \quad (241)
\]

\[
\dot{\kappa} + 2 H \kappa - 4 \pi G \dot{\delta} = \frac{c^2}{a^2} \left[ v_{\alpha}^{(v)} \left( v_{\chi} - \Psi^{(v)} \right) \left( v_{\chi,\beta} - \Psi^{(v)\beta} \right)_{\alpha} \right] + \frac{c}{a} \left( v_{\alpha}^{(v)} \left( v_{\chi} - \Psi^{(v)} \right) \right) + \frac{c}{a} \left( 2 C^{(t)\alpha\beta} v_{\beta}^{(v)\beta} \right) \\
+ 8 \pi G \dot{v}_{\alpha}^{(v)} - \left( 3 \dot{H} + \frac{2 \Delta}{a^2} \right) \Delta^{-1} \nabla_{\alpha} \left[ v_{\alpha}^{(v)} \left( v_{\chi} - \Psi^{(v)} \right) + v_{\beta}^{(v)} \left( v_{\chi,\beta} - \Psi^{(v)\beta} \right)_{\alpha} - v_{\alpha}^{(v)} v_{\beta}^{(v)\beta} \right]. \quad (242)
\]

\[
\dot{v}_{\alpha}^{(v)} + H v_{\alpha}^{(v)} = \frac{c}{a} \left[ v_{\alpha}^{(v)} \left( v_{\chi,\beta} - \Psi^{(v)\beta} \right) + v_{\beta}^{(v)} \left( v_{\chi,\beta} - \Psi^{(v)\beta} \right)_{\alpha} - v_{\alpha}^{(v)} v_{\beta}^{(v)\beta} \right] \\
- \frac{c}{a} \nabla_{\alpha} \Delta^{-1} \nabla_{\gamma} \left[ v_{\alpha}^{(v)} \left( v_{\chi} - \Psi^{(v)} \right) + v_{\beta}^{(v)} \left( v_{\chi,\beta} - \Psi^{(v)\beta} \right)_{\alpha} - v_{\alpha}^{(v)} v_{\beta}^{(v)\beta} \right]. \quad (243)
\]

D. Newtonian correspondence

In order to compare with Newtonian equations we continue identifying \(\kappa\) as in Eq. (120) to the second order. To the linear order we identify

\[
u \equiv - c \left( \nabla v_{\chi} - v^{(v)} \right) \equiv \nabla u + u^{(v)}, \quad (244)
\]
thus
\[ u \equiv -c\varphi, \quad u^{(v)} \equiv c\Psi^{(v)}. \tag{245} \]

We introduce the following notations
\[ U_\alpha \equiv u_\alpha + c\Psi^{(v)}_\alpha, \quad \tilde{U}_\alpha \equiv u_{\alpha\beta} + c\Psi^{(v)}_\alpha. \tag{246} \]

As mentioned below Eq. (224), to the linear order, \( u_\alpha \) and \( U_\alpha \) are related to the vorticity and the shear, respectively. Equations (241)-(243) become
\[ \dot{\delta} + 2 \frac{c}{a} \nabla \cdot u = - \frac{1}{a} \nabla \cdot (\delta U) + \frac{1}{a} C^{(t)}_{\alpha\beta} u^{(v)}_{\alpha\beta} - \frac{1}{a} u^{(v)} \cdot \nabla \varphi + \frac{1}{c^2} H \left[ u^{(v)} \cdot u^{(v)} + 3 \Delta^{-1} \nabla \cdot \left( u^{(v)} U^{\beta} + u^{(v)} \tilde{U}^{\beta}_{\alpha\beta} \right) \right], \tag{247} \]
\[ \frac{1}{a} \nabla \cdot (\dot{u} + H u) + 4\pi G \rho \delta = - \frac{1}{a^2} \nabla \cdot (U \cdot \nabla U) - \dot{C}^{(t)}_{\alpha\beta} \left( \dot{\delta} + \frac{2}{a} \tilde{U}_{\alpha\beta} \right) + \frac{8\pi G \rho}{c^2} \left[ -u^{(v)} \cdot u^{(v)} + \frac{3}{2} \Delta^{-1} \nabla \cdot \left( u^{(v)} U^{\beta} + u^{(v)} \tilde{U}^{\beta}_{\alpha\beta} \right) \right], \tag{248} \]
\[ u^{(v)} + H u^{(v)} = - \frac{1}{a} \left[ U \cdot \nabla u - \nabla \Delta^{-1} \nabla \cdot (U \cdot \nabla u) \right] - \frac{1}{a} \left[ U^{\beta} e\Psi^{(v)}_{\beta\alpha} - \nabla \Delta^{-1} \nabla \cdot \left( U^{\beta} e\Psi^{(v)}_{\beta\gamma} \right) \right]. \tag{249} \]

From Eq. (249) we notice that the tensor-type perturbation does not affect the vector-type perturbation to the second order. The pure scalar-type perturbation also cannot generate the vector-type perturbation to the second-order; the same is true in the Newtonian case, see below Eq. (15).

From Eq. (241), to the linear order, we have
\[ \varphi^{(v)} = \frac{1}{6} \left( \frac{ck}{aH} \right)^2 \Psi^{(v)}_\alpha, \tag{250} \]
where \( k \) is a comoving wavenumber with \( \Delta \equiv -k^2 \), thus \( |k| = 1 \). Since \((ck)/(aH) \sim \text{(visual-horizon)/(scale)}\), we have
\[ \begin{align*}
\text{far inside horizon:} & \quad v^{(v)} \gg \Psi^{(v)}_\alpha, \quad U \simeq u = \nabla u + u^{(v)}, \quad \tilde{U} \simeq \nabla u, \\
\text{far outside horizon:} & \quad v^{(v)} \ll \Psi^{(v)}_\alpha, \quad U \simeq u_{\alpha\beta} + c\Psi^{(v)}_\alpha. \tag{251} \end{align*} \]

Apparently, contributions of vector-type perturbation to the second order depend on the visual-horizon scale.

Far inside the horizon, we can ignore \( c\Psi^{(v)}_\alpha \) compared with \( u^{(v)} \). In the matter dominated era we have
\[ \varphi_v = \frac{5}{3} \varphi_x = \frac{5}{3} \frac{a^2 4\pi G \rho}{k^2} \Delta_v = \frac{5}{2} \left( \frac{aH}{ck} \right)^2 \delta_v, \tag{252} \]
where we used Eqs. (293), (329), (330) of [11]. Thus, the third term in the right-hand-side of Eq. (247) is \((aH/ck)^2\)-order smaller than the first term. The fourth (and last) term in the right-hand-side of Eq. (247) is \((aH/ck)[u^{(v)}/(c\delta)]\)-order smaller than the first term. The third (and last) term in the right-hand-side of Eq. (248) is also \((aH/ck)^2\)-order smaller than the first term. Thus, Eqs. (247), (249) give
\[ \begin{align*}
\dot{\delta} + \frac{1}{a} \nabla \cdot u & = - \frac{1}{a} \nabla \cdot (\delta U) + \frac{1}{a} C^{(t)}_{\alpha\beta} u^{(v)}_{\alpha\beta}, \tag{253} \\
\frac{1}{a} \nabla \cdot (\dot{u} + H u) + 4\pi G \rho \delta & = - \frac{1}{a^2} \nabla \cdot (U \cdot \nabla U) - \dot{C}^{(t)}_{\alpha\beta} \left[ \dot{\delta} + \frac{2}{a} \tilde{U}_{\alpha\beta} \right], \tag{254} \\
\dot{u}^{(v)} + H u^{(v)} & = - \frac{1}{a} \left[ u \cdot \nabla u - \nabla \Delta^{-1} \nabla \cdot (u \cdot \nabla u) \right]. \tag{255} \end{align*} \]

Thus, if we could ignore the tensor-type combination in Eqs. (253), (241), Eqs. (253)-(255) coincide exactly with the Newtonian equations: see Eqs. [3], [12], [15] ignoring the pressure terms and the subindices \( i \). The vector-tensor combinations in Eqs. (253)-(241) are new relativistic contributions of the vector-type perturbations; compare these two equations with Eqs. [17], [18] which are valid in the absence of the vector-type perturbations. Notice the form of last term \( u_{\alpha\beta} + c\Psi^{(v)}_{\alpha\beta} \) in Eq. (253) which subtly differs from the expression \( u_{\alpha\beta} \) in Eq. (18).

Contributions from the vector-type perturbation become more complicated near and outside the horizon scale. The presence of vector-type metric perturbation \( \Psi^{(v)}_\alpha \), the scalar-type curvature perturbation \( \varphi \), and the tensor-type perturbation \( C^{(t)}_{\alpha\beta} \) coupled with the vector-type perturbation give additional effects.
E. Pure vector-type perturbations

In Sec. VII-E of [11] we have considered a situation with pure vector-type perturbation. As the analysis was made based on the fluid quantities in the normal frame, in the following we present the case based on the fluid quantities in the energy frame. Considering only the vector-type perturbation of a fluid, Eq. (68) gives

$$\frac{1}{a^4(\mu + p)} \left\{ a^4 \left( \mu + p \right) u^{(v)}_\alpha + \frac{1}{a} \hat{\Pi}^{(v)}_{\alpha\beta} v^{(v)}_\beta \right\}.$$  \hspace{1cm} (256)

Thus, for $\Pi^{(v)}_\alpha = 0$ we have

$$\frac{1}{a^4(\mu + p)} a^4 \left( \mu + p \right) u^{(v)}_\alpha = - \frac{c}{a} \left( \left( v^{(v)}_\alpha + \Psi^{(v)}_\alpha \right) \left( v^{(v)}_\alpha + \Psi^{(v)}_\alpha \right) - \frac{2}{3} v^{(v)}_\alpha v^{(v)}_\alpha \right).$$  \hspace{1cm} (257)

This differs from Eq. (365) of [11] which is due to the difference in the frame choice. The momentum constraint equation in Eq. (101) of [11] becomes

$$\frac{\Delta + 2K}{2a^2} \Psi^{(v)}_\alpha + \frac{8\pi G}{c^4} (\mu + p) v^{(v)}_\alpha = - \frac{8\pi G}{c^4} \frac{1}{a} \hat{\Pi}^{(v)}_{\alpha\beta} v^{(v)}_\beta.$$  \hspace{1cm} (258)

Under the gauge transformation, from Eq. (61) we have

$$\hat{v}^{(v)}_\alpha = v^{(v)}_\alpha - v^{(v)}_\beta \xi^{(v)}_{\beta\epsilon} - v^{(v)}_\epsilon \xi^{(v)}_{\alpha\beta}.$$  \hspace{1cm} (259)

To the linear order, from Eq. (230) of [11] we have $B^{(v)}_{\alpha} = B^{(v)}_{\alpha} + \xi^{(v)}_{\alpha}$. We consider a gauge transformation from the $C$-gauge ($\xi^{(v)}_{\alpha} \equiv 0$, without hat) to the $B$-gauge ($B^{(v)}_{\alpha} \equiv 0$, with hat). We have $B^{(v)}_{\alpha} \equiv 0$, and $B^{(v)}_{\alpha}|_{C\text{-gauge}} = -\xi^{(v)}_{\alpha}$. Thus,

$$\xi^{(v)}_{\alpha} = - \int^\eta B^{(v)}_{\alpha}|_{C\text{-gauge}} d\eta = -a^2 \Psi^{(v)}_\alpha \int^\eta \frac{d\eta}{a^2},$$  \hspace{1cm} (260)

where we used $B^{(v)}_{\alpha}|_{C\text{-gauge}} = \Psi^{(v)}_\alpha \propto a^{-2}$. Thus, Eq. (259) gives

$$v^{(v)}_{\alpha}|_{B\text{-gauge}} = \left[ v^{(v)}_\alpha - \left( v^{(v)}_\beta \Psi^{(v)}_\beta \right) - v^{(v)}_\alpha \Psi^{(v)}_\beta \right] a^2 \int^\eta \frac{d\eta}{a^2} |_{C\text{-gauge}}.$$  \hspace{1cm} (261)

X. EQUATIONS WITH FIELDS

A. A minimally coupled scalar field

Equations in the case of a minimally coupled scalar field are presented in Eqs. (112)-(114) of [11]. The equation of motion in Eq. (112) and the full Einstein’s equations in Eqs. (99)-(105) expressed using the normal-frame fluid quantities together with the normal-frame fluid quantities for the scalar field in Eqs. (114) all in [11] provide a complete set of equations we need to the second-order. The fluid quantities in Eq. (114) of [11] are presented in the normal-frame four-vector and it is convenient to know the conventionally used fluid quantities which are based on the energy-frame four-vector. These latter quantities can be read from Eqs. (88), (114) of [11] and Eq. (79) as

$$Q^{N,\phi}_\alpha = - \frac{1}{a} \left[ \dot{\phi} \delta \phi, \phi + \delta \phi, \phi \left( \delta \phi - \phi A \right) \right]$$

$$\hat{\Pi}^{N,\phi}_\alpha = - \frac{1}{a^2} \left( \delta \phi, \phi \delta \phi, \phi + \frac{1}{3} \delta \phi, \phi \delta \phi, \phi \right).$$  \hspace{1cm} (262)
The uniform-field gauge gives \( v \equiv 0 \) which is the comoving gauge, and vice versa. Thus, \( \delta \phi = 0 \equiv v = 0 \). (264)

We also have

\[
\delta \mu^{(\phi)} - \delta p^{(\phi)} = 2 V_{,\phi} \delta \phi + V_{,\phi\phi} \delta \phi^2,
\]

and under the uniform-field gauge we have

\[
\dot{\delta \mu}^{(\phi)} = \dot{\delta p}^{(\phi)} = -\dot{\phi}^2 \left[ A (1 - 2A) + \frac{1}{2} B^\alpha B_\alpha \right]_{\phi}.
\]

Equation (263) apparently shows that the vector-type perturbation \( v^{(\phi,v)} \) does not vanish to the second-order. However, the second-order quantities in right-hand-side depend on the temporal gauge condition for the scalar-type perturbations, and trivially vanish for the uniform-field gauge. We can also show that it vanishes for the uniform-density gauge or the uniform-pressure gauge where we have \( \delta \phi \equiv v = 0 \). Thus, using Eq. (114) of [11], we have

\[
v^{(\phi)} = \frac{1}{a \dot{\phi}} \delta \phi - \frac{1}{a \dot{\phi}^2} \Delta^{-1} \nabla^\alpha \left( (\delta \phi - \dot{\phi} A) \delta \phi_{,\alpha} \right),
\]

\[
v^{(\phi,v)} = \frac{1}{a \dot{\phi}^2} \left\{ (\delta \phi - \dot{\phi} A) \delta \phi_{,\alpha} - \nabla_\alpha \Delta^{-1} \nabla^\beta \left[ (\delta \phi - \dot{\phi} A) \delta \phi_{,\beta} \right] \right\},
\]

\[
\delta \mu^{(\phi)} = \dot{\delta \phi} - \dot{\phi}^2 A + V_{,\phi} \delta \phi + \frac{1}{2} \dot{\phi}^2 - \frac{1}{2 a^2} \delta \phi^2 \delta \phi_{,\alpha} + \frac{1}{2} V_{,\phi\phi} \delta \phi^2 - 2 \dot{\phi} \delta \phi A + \frac{1}{a} \delta \phi \delta \phi_{,\alpha} B^\alpha + 2 \dot{\phi}^2 A^2 - \frac{1}{2} \dot{\phi}^2 B^\alpha B_\alpha,
\]

\[
\delta p^{(\phi)} = \dot{\delta \phi} - \dot{\phi}^2 A - V_{,\phi} \delta \phi + \frac{1}{2} \dot{\phi}^2 - \frac{1}{2 a^2} \delta \phi^2 \delta \phi_{,\alpha} - \frac{1}{2} V_{,\phi\phi} \delta \phi^2 - 2 \dot{\phi} \delta \phi A + \frac{1}{a} \delta \phi \delta \phi_{,\alpha} B^\alpha + 2 \dot{\phi}^2 A^2 - \frac{1}{2} \dot{\phi}^2 B^\alpha B_\alpha,
\]

\[
\Pi_{\alpha\beta}^{(\phi)} = 0.
\]

Notice that no anisotropic stress is caused by a minimally coupled scalar field even to the second order in perturbations. The uniform-field gauge takes \( \delta \phi \equiv 0 \) as a temporal gauge (slicing) condition to the second-order in perturbation. Thus, we conclude that a minimally coupled scalar field does not contribute to the rotational perturbation to the second order in perturbations. In fact, we can show that a single scalar field does not support vector-type perturbations to all orders in perturbations. As we take the energy frame, thus \( q_a \equiv 0 \), from Eq. (23) of [11], we can show

\[
\tilde{\omega}^{(\phi)}_{\alpha\beta} = 0,
\]

to the second order. Thus, we conclude that a minimally coupled scalar field does not contribute to the rotational perturbation to the second order in perturbations. In fact, we can show that a single scalar field does not support vector-type perturbations to all orders in perturbations. As we take the energy frame, thus \( q_a \equiv 0 \), from Eq. (23) of [11], we can show

\[
\tilde{u}_a = \frac{\tilde{\phi}_{,\alpha}}{\sqrt{-\tilde{\phi}^a \tilde{\phi}_{,c} \tilde{\phi}^{bc}}}
\]

Using the definition of the vorticity tensor in Eq. (33) we can show that \( \tilde{\omega}_{ab} = 0 \).

Using the fluid quantities in Eq. (263) we can handle the scalar field using our non-ideal fluid formulation. The fluid equations in the energy frame, like Eqs. (62)-(68), remain valid in the case of scalar field with the fluid quantities expressed as in Eq. (263). The anisotropic stress vanishes and the entropic perturbation \( e \) is given as

\[
e^{(\phi)} \equiv \delta p^{(\phi)} - c_s^2 \delta \mu^{(\phi)}, \quad c_s^2 \equiv \frac{\dot{\phi}^{(\phi)}}{\mu^{(\phi)}} = \frac{\dot{\phi} - V_{,\phi}}{\phi + V_{,\phi}}.
\]

Under the comoving gauge \( v \equiv 0 \), using Eq. (266) we have

\[
\delta \mu^{(\phi)} = \delta \mu^{(\phi)}_{\phi}, \quad \delta p^{(\phi)} = \delta p^{(\phi)}_{\phi}, \quad \delta p^{(\phi)} = \left( 1 - c_s^2 \right) \delta \mu^{(\phi)}_{\phi},
\]

which is a well known relation, now valid to the second order in perturbations. A fluid formulation of the scalar field to the linear order is presented in [24, 29]. Using Eq. (270) together with vanishing anisotropic stress, Eqs. (62)-(68) or (88)-(94) provide the fluid formulation for a minimally coupled scalar field to the second order in perturbations. The perturbed equation of motion of the scalar field is presented in Eq. (112) of [11].
B. Minimally coupled scalar fields

In the case of multiple minimally coupled scalar fields, the equation of motions in Eq. (119) and the full Einstein’s equations in Eqs. (99)-(105) expressed using the normal-frame fluid quantities together with the normal-frame fluid quantities for the scalar field in Eqs. (121) all in [11] provide a complete set of equations we need to the second-order. The fluid quantities in Eq. (121) of [11] are presented in the normal-frame four-vector and it is convenient to know the conventionally used fluid quantities which are based on the energy-frame four-vector. These latter quantities can be read from Eqs. (121) of [11] and Eq. (58) as

\[
v^{(\phi)} = \frac{1}{a} \sum_{n} \phi^2(n) \sum_{k} \left\{ \phi(k) \delta \phi(k) + \Delta^{-1} \nabla \alpha \left[ \left( \delta \phi(k) - \hat{\phi}(k) A \right) \left( \delta \phi(k) - 2 \hat{\phi}(k) \sum_{l} \frac{\hat{\phi}(l) \delta \phi(l)}{\sum_{m} \delta^{2}(m)} \right) \right] \right\},
\]

\[
v^{(\phi,\nu)} = -\frac{1}{a} \sum_{n} \phi^2(n) \sum_{k} \left\{ \left( \delta \phi(k) - \hat{\phi}(k) A \right) \left( \delta \phi(k) - 2 \hat{\phi}(k) \sum_{l} \frac{\hat{\phi}(l) \delta \phi(l)}{\sum_{m} \delta^{2}(m)} \right) \right\},
\]

\[
\nabla \alpha \Delta^{-1} \nabla \beta \left[ \left( \delta \phi(k) - \hat{\phi}(k) A \right) \left( \delta \phi(k) - 2 \hat{\phi}(k) \sum_{l} \frac{\hat{\phi}(l) \delta \phi(l)}{\sum_{m} \delta^{2}(m)} \right) \right],
\]

\[
\delta \mu^{(\phi)} = \mu^{N(\phi)} - \frac{1}{a^2} \sum_{m} \phi^2(m) \left( \sum_{k} \phi(k) \delta \phi(k) \right) \left( \sum_{l} \hat{\phi}(l) \delta \phi(l) \right),
\]

\[
\delta p^{(\phi)} = p^{N(\phi)} - \frac{1}{3a^2} \sum_{m} \phi^2(m) \left( \sum_{k} \phi(k) \delta \phi(k) \right) \left( \sum_{l} \hat{\phi}(l) \delta \phi(l) \right),
\]

\[
\Pi^{(\phi)} = \Pi^{N(\phi)} - \frac{1}{a^2} \sum_{m} \phi^2(m) \left[ \left( \sum_{k} \phi(k) \delta \phi(k) \right) \left( \sum_{l} \hat{\phi}(l) \delta \phi(l) \right) - \frac{1}{3} g^{(3)}_{\alpha \beta} \left( \sum_{k} \phi(k) \delta \phi(k) \right) \left( \sum_{l} \hat{\phi}(l) \delta \phi(l) \right) \right],
\]

(271)

where the normal-frame fluid quantities are presented in Eq. (121) of [11]. Thus, by moving into the energy-frame fluid quantities from the normal-frame ones we have rather complicated terms which do not cancel out nicely any term in the normal-frame quantities. In single field case we had such cancelations, for example we have \( \Pi^{(\phi)} = 0 \) in Eq. (263). But, we do not have such a luxury in the multi-component situation. Apparently \( \Pi^{(\phi)} \) in the energy-frame does not vanish and looks more complicated. Thus, in the multi-field situation we had better use both the fluid quantities and Einstein’s equations all expressed in the normal frame: these are Eqs. (99)-(105),(121) in [11].

As the temporal gauge condition we can set any one field perturbation, say the specific \( \ell \)-th one \( \delta \phi_{\ell}(k) \), equal to zero which might be called the uniform-\( \phi_{\ell}(k) \) gauge to the second order. This apparently differs from the comoving gauge which sets \( v^{(\phi)} \equiv 0 \). In the multi-component situation we cannot take a gauge condition which makes \( v^{(\phi,\nu)} = 0 \). Thus, the multiple scalar fields source the vector-type perturbation to the second order in perturbations. The scalar-, vector-, and tensor-type decomposition of the anisotropic stress \( \Pi^{(\phi)} \) can be read by using decomposition formulae in Eq. (177) of [11].

In multiple-field situation, it is ad hoc and cumbersome (if not impossible) to introduce individual fluid quantity for each field variable even to the background and linear order perturbations, see [30].

C. Generalized gravity case

In Sec. IV.D of [11] we presented the equation of motion and effective fluid quantities in a class of generalized gravity theories together with additional presence of fluids and fields to the second order. The equation of motion is in Eq. (128) of [11], and the full Einstein’s equations in Eqs. (99)-(105) expressed using the normal-frame fluid quantities together with the normal-frame effective fluid quantities in Eq. (130) of [11] provide a complete set of equations we need to the second-order. The effective fluid quantities in Eq. (130) of [11] are presented in the normal-frame four-vector and using Eq. (88) in [11] we can easily derive the effective fluid quantities based on the energy-frame four-vector. As in the multiple field case in a previous subsection, by moving into the energy-frame, the effective fluid quantities become more complicated compared with the ones in the normal-frame. Thus, in these class of generalized gravity theories we had better use both the fluid quantities and Einstein’s equations all expressed in the normal frame: these are Eqs. (99)-(105),(130) in [11].
XI. CURVATURE PERTURBATIONS AND LARGE-SCALE CONSERVATIONS

In the large-scale limit the spatial curvature perturbation $\varphi$ in several different gauge conditions is known to remain constant in expanding phase. Often the conservation properties are shown based on the first time derivative of the curvature perturbation. In order to show the conservation properties properly we have to construct the closed form second-order differential equations for the curvature perturbation. In the following we will derive such first-order and second-order differential equations for $\varphi$, $\dot{\varphi}$, $\varphi_\kappa$, and $\dot{\varphi}_\kappa$. First we will derive equations for the linear perturbation including the background curvature and non-ideal fluid properties. Then we will derive equations for the second order perturbation assuming a flat background; including the background curvature is trivial, though. We consider a single-component fluid.

A. Linear-order equations

We introduce a combination

$$\Phi \equiv \varphi_v - \frac{K/a^2}{4\pi G (\mu + p)} \varphi_\kappa.$$  \hspace{1cm} (272)

This combination was first introduced by Field and Shepley in [28]. From Eqs. [88, 90, 92], Eqs. [88-90, 92, 94], Eqs. [88, 93], and Eqs. [88, 89, 91], respectively, we can derive

$$\frac{H}{a} \left( \frac{a}{H} \dot{\varphi}_\kappa \right) = 4\pi G (\mu + p) \Phi - 8\pi G H \Pi,$$  \hspace{1cm} (273)

$$\dot{\Phi} = \frac{H c_s^2}{4\pi G (\mu + p)} \frac{\Delta}{a} \dot{\varphi}_\kappa - \frac{H}{\mu + p} \left( e + \frac{2}{3} \frac{\Delta}{a^2} \Pi \right),$$ \hspace{1cm} (274)

$$\dot{\varphi}_\delta = \frac{\Delta}{3a} \dot{v}_\chi - \frac{He}{\mu + p},$$ \hspace{1cm} (275)

$$\varphi_\kappa = -\frac{H}{3H + \Delta/a^2} \left[ (1 + 3c_s^2) \frac{\Delta + 3K}{a^2} \varphi_\kappa - 12\pi Ge \right] - \frac{\Delta}{3a^2} \Pi.$$ \hspace{1cm} (276)

These equations were presented in [24, 31].

We can derive closed form second-order differential equations for $\Phi$, $\varphi_\kappa$, $\varphi_\delta$, and $\varphi_\kappa$

$$\frac{\mu + p}{H} \left[ \frac{H^2 c_s^2}{(\mu + p) a^3} \left( \frac{\mu + p}{H} \dot{\varphi}_\kappa \right) + \frac{8\pi G H^2}{(\mu + p) \Pi} \right] = -c_s^2 \frac{\Delta}{a^2} \dot{\varphi}_\kappa - 4\pi G \left( e + \frac{2}{3} \frac{\Delta}{a^2} \Pi \right),$$ \hspace{1cm} (277)

$$\frac{H \varphi_\delta}{\mu + p} \left[ \frac{H}{a} \frac{\dot{\varphi}_\delta}{\varphi_\kappa} + \frac{a^3}{H c_s^2} \left( e + \frac{2}{3} \frac{\Delta}{a^2} \Pi \right) \right] = c_s^2 \frac{\Delta}{a^2} \left( \Phi - \frac{2H^2}{\mu + p} \right),$$ \hspace{1cm} (278)

$$\frac{3H + \Delta/a^2}{(\mu + p) a^3} \left[ \frac{\mu + p}{a^3} \frac{\varphi_\kappa}{H c_s^2} \right] = \frac{1}{a^3} \left[ a^{\frac{3}{2}} \left( \frac{\varphi_\kappa}{3H + \Delta/a^2} \left( \frac{1 + 3c_s^2}{a^2} \frac{\Delta + 3K}{a^2} \varphi_\kappa - 12\pi Ge \right) \right) \right] = -\frac{\Delta}{3a^2} \left[ \varphi_\kappa - \frac{1}{3H + \Delta/a^2} \left( (1 + 3c_s^2) \frac{\Delta + 3K}{a^2} \varphi_\kappa - 12\pi Ge + 8\pi G \Pi \right) \right].$$ \hspace{1cm} (279)

Equations (277), (278) follow by combining Eqs. (273), (274). Equations (279) and (280) follow from Eqs. [88, 93, 91], (282). Equations (279) and (280) follow from Eqs. [88, 93, 91], (282). Equations (279) and (280) follow from Eqs. [88, 93, 91], (282).

From Eqs. [89, 91] we can derive

$$\varphi_\delta = \left[ 1 - \frac{\Delta + 3K}{12\pi G (\mu + p) a^2} \right] \varphi_\kappa = \varphi_v - \frac{\Delta + 3K}{12\pi G (\mu + p) a^2} \varphi_\chi = \Phi - \frac{\Delta}{12\pi G (\mu + p) a^2} \varphi_\lambda.$$ \hspace{1cm} (281)

These relations were presented in [17]. In the large-scale limit and in near flat background, thus ignoring $\Delta$ and $K$ terms, we have

$$\Phi \simeq \varphi_v \simeq \varphi_\delta \simeq \varphi_\kappa,$$ \hspace{1cm} (282)
to the leading order in the large-scale expansion.

In the large-scale limit, ignoring $\Delta$ terms, in near flat background and for an ideal fluid case, thus setting $K = 0$ and $e = 0 = \Pi$, Eqs. (277)-(280) give

$$\left(\frac{a}{H} \chi\right)^\prime \propto \frac{(\mu + p) a}{H^2} \dot{\varphi}, \quad \varphi_v \propto \frac{H^2 \dot{\varphi}}{(\mu + p) a}, \quad \varphi_\delta \propto \varphi_\kappa \propto \frac{1}{a^3},$$

(283)

(284)

(285)

Notice that if we simply ignore the $\Delta$ terms in Eqs. (274)-(276) we simply have $\dot{\varphi}_v = \dot{\varphi}_\delta = \dot{\varphi}_\kappa = 0$. In such a way we cannot recover the terms in the right-hand-side of Eqs. (283), (285); these terms lead to decaying solutions (in an expanding era) in the large-scale limit and are higher order in the large-scale expansion compared with the decaying solution of $\varphi_\chi$, see below. From Eqs. (283), (285) we have general large-scale asymptotic solutions

$$\varphi_\chi = 4\pi GC(x) \frac{H}{a} \int t \left(\frac{a (\mu + p)}{H^2}\right) dt + d(x) \frac{H}{a},$$

(286)

$$\varphi_v = C(x) + \frac{\Delta}{4\pi G} d(x) \int t \frac{c_3^2 H^2}{(\mu + p) a^3} dt,$$

(287)

$$\varphi_\delta = \varphi_\kappa = C(x) + \frac{\Delta}{3} d(x) \int t \frac{dt}{a^3},$$

(288)

where $C(x)$ and $d(x)$ are integration constants which correspond to the relatively growing and decaying modes, respectively, in an expanding phase; in a collapsing phase the roles are reversed. The coefficients are fixed using the relations in Eqs. (273), (274), (271). Notice that for the $C$-mode the relation in Eq. (282) is satisfied, and simply remain constant. For the $d$-mode, $\varphi_v$, $\varphi_\delta$, and $\varphi_\kappa$ are $\Delta/(aH)^2$-order higher compared with the $d$-mode of $\varphi_\chi$. The $\varphi_\chi$ is one of the well known conserved quantity in the large-scale even in the context of generalized gravity theories [24, 32].

In order to evaluate the solutions to the second order in the next section, we need complete sets of linear order solutions for different gauge conditions. For an ideal fluid, and for a minimally coupled scalar field such complete sets of solutions are presented in tabular forms in [29, 31]. In the following we summarize such sets of solutions in an ideal fluid case for four different gauge conditions. From Table 8 of [31] we have

$$\varphi_\chi = -\alpha_\chi = C \left(1 - \frac{H}{a} \int t dt\right), \quad \delta_\chi = -\frac{2 \kappa_\chi}{3 H} = -2C \frac{H}{aH} \int t dt, \quad v_\chi = -\frac{C}{a^2} \int t dt;$$

$$\varphi_v = C, \quad H\chi_v = C \frac{H}{a} \int t dt, \quad \delta_v = \frac{1 + w}{c_3^2} \alpha_v = -\frac{2}{3} \frac{\Delta}{a^2 H^2} C \left(1 - \frac{H}{a} \int t dt\right), \quad \kappa_v = -\frac{\Delta}{3 a^2 H^2} \frac{C}{a} \int t dt;$$

$$\varphi_\delta = C, \quad H\chi_\delta = C \frac{H}{a} \int t dt, \quad \kappa_\beta = 3 \alpha_\delta = -\frac{\Delta}{3 a^2 H^2} C, \quad v_\delta = \frac{2}{3} \frac{\Delta}{a^2 H^2} C \frac{H}{aH} \int t dt;$$

$$\varphi_\kappa = C, \quad H\chi_\kappa = C \frac{H}{a} \int t dt, \quad \delta_\kappa = -\frac{1 + w}{1 + 3 c_3^2} \alpha_\kappa = -2 \frac{\Delta}{3 a^2 H^2} C, \quad v_\kappa = -\frac{\Delta}{3 a^2 H^2} \frac{C}{a} \int t dt.$$  

(289)

For corresponding sets of solutions for a minimally coupled scalar field, see Table 1 of [29]. Compared with the notation used in [31] we have $\gamma = -8\pi G (\mu + p) av$. The lower bounds of integration of solutions in Eq. (289) give behaviors of $d$-modes. For solutions without integration, the $d$-modes are $\Delta/(aH)^2$-order higher than the non-vanishing $d$-mode, for example, see the solutions in Table 2 of [31]. Thus, the $d$-modes are

$$\varphi_\chi = -\alpha_\chi = \frac{d_H}{a}, \quad \delta_\chi = -\frac{2 \kappa_\chi}{3 H} = 2d_H \frac{H}{aH}, \quad v_\chi = \frac{d}{a^2};$$

$$\varphi_v = \frac{\Delta}{4\pi G} d(x) \int t \frac{c_3^2 H^2}{(\mu + p) a^3} dt, \quad H\chi_v = -\frac{H}{a} d, \quad \delta_v = -\frac{1 + w}{c_3^2} \alpha_v = -\frac{2}{3} \frac{\Delta}{a^2 H^2} \frac{H}{aH}, \quad \kappa_v = \frac{\Delta}{a^3 H};$$

$$\varphi_\delta = \frac{\Delta}{3} d(x) \int t \frac{dt}{a^3}, \quad H\chi_\delta = -\frac{H}{a} d, \quad \delta_\delta = -\frac{1 + w}{1 + 3 c_3^2} \alpha_\delta = -\frac{2}{9} \frac{\Delta}{a^2 H^2} d \int t \frac{dt}{a^3}, \quad v_\delta = \frac{1}{3 a^4 H};$$

(290)

$$\varphi_\kappa = \frac{\Delta}{3} d(x) \int t \frac{dt}{a^3}, \quad H\chi_\kappa = -\frac{H}{a} d, \quad \delta_\kappa = -\frac{1 + w}{1 + 3 c_3^2} \alpha_\kappa = -\frac{2}{9} \frac{\Delta}{a^2 H^2} d \int t \frac{dt}{a^3}, \quad v_\kappa = \frac{1}{3 a^4 H}.$$
B. Second-order equations

We assume $K = 0$. From Eqs. (88), (90), (92), Eqs. (88), (91), (94), Eqs. (88), (93), and Eqs. (88), (89), (91), respectively, we can derive

$$
\frac{H}{a} \left[ \frac{a}{H} (\varphi - H \chi) \right] = \frac{4\pi G (\mu + p)}{H} (\varphi - aHv) - 8\pi G\Pi + \frac{1}{3} (n_0 - n_2) - Hn_4, \tag{291}
$$

$$
(\varphi - aHv) = \frac{Hc_2}{4\pi G (\mu + p)} \left[ \frac{\Delta}{a^2} (\varphi - H\chi) - \frac{1}{4} n_1 + Hn_2 \right] - \frac{H}{\mu + p} \left( e + \frac{2}{3} a^2 \Pi \right) + \frac{1}{3} (n_0 - n_2) - aHn_6, \tag{292}
$$

$$
\dot{\varphi}_\delta = \frac{\Delta}{3a} \left( v - \frac{1}{\alpha} \right) - \frac{He}{\mu + p} + \frac{1}{3} \left( n_0 + \frac{n_5}{\mu + p} \right), \tag{293}
$$

$$
\dot{\varphi}_\kappa = -\frac{H}{3H + \Delta/a^2} \left[ (1 + 3c_2^2) \frac{\Delta}{a^2} \varphi_\kappa - 12\pi Ge - \frac{1 + 3c_2^2}{4} n_1 - n_3 \right] - \frac{\Delta}{3a^2} \kappa + \frac{1}{3} n_0. \tag{294}
$$

The perturbed order variables in Eqs. (293) (294) are evaluated in the uniform-density gauge ($\delta \equiv 0$), and the uniform-expansion gauge ($\kappa \equiv 0$), respectively. Equation (293) also follows from Eq. (11) evaluated to the second order.

We can derive closed form second-order differential equations for $\varphi_v$, $\varphi_\chi$, $\varphi_\delta$, and $\varphi_\kappa$:

$$
\frac{\mu + p}{H} \left\{ \frac{H}{\mu + p} \left[ \frac{a}{H} (\varphi - H \chi) \right] + 8\pi G\Pi - \frac{1}{3} (n_0 - n_2) + Hn_4 \right\}.
$$

$$
= c_2^2 \frac{\Delta}{a^2} (\varphi - H\chi) - 4\pi G \left( e + \frac{2}{3} a^2 \Pi \right) + c_2^2 \left( - \frac{1}{4} n_1 + Hn_2 \right) + \frac{4\pi G (\mu + p)}{H} \left[ \frac{1}{3} (n_0 - n_2) - aHn_6 \right], \tag{295}
$$

$$
\frac{H^2c_2^2}{4\pi G (\mu + p) a^3} \left\{ \frac{H}{3H + \Delta/a^2} \left[ (\varphi - aHv) - \frac{H}{\mu + p} \left( e + \frac{2}{3} a^2 \Pi \right) - \frac{1}{3} (n_0 - n_2) + aHn_6 \right]
$$

$$
+ \frac{a^2}{H} \left( n_1 - Hn_2 \right) \right\} = c_2^2 \frac{\Delta}{a^2} \left( \varphi - aHv - \frac{2H^2}{\mu + p} \right) + \frac{Hc_2^2 \Delta}{4\pi G (\mu + p) a^2} \left[ \frac{1}{3} (n_0 - n_2) - Hn_4 \right], \tag{296}
$$

$$
1 + \frac{\Delta}{(3a^2 H)} \left\{ \frac{a^3}{1 + \Delta/(3a^2 H)} \left( \dot{\varphi}_\delta + \frac{H}{\mu + p} e \right) + \frac{a^3}{3H + \Delta/a^2} \left[ \frac{1}{4} n_1 + \frac{1}{4} \left( 2H + \frac{\Delta}{3Ha^2} \right) n_1 - Hn_3 \right] - \frac{1}{3} a^3 n_0 \right\}.
$$

$$
= - \frac{1 - c_2^2 \Delta/(a^2 H)}{3 + \Delta/(a^2 H)} \frac{\Delta}{a^2} \varphi_\delta + \frac{1}{3H + \Delta/a^2} \left[ e + \frac{2}{3} \left( 3H + \frac{\Delta}{a^2} \right) \Pi \right]
$$

$$
+ \frac{\Delta}{3Ha^2} \left[ \frac{1}{4a^2} \frac{a^2}{H} n_1 \right] - n_3 - \left( 3\dot{H} + \frac{\Delta}{a^2} \right) n_4, \tag{297}
$$

$$
\frac{1}{a^3} \left\{ a^3 \left[ \varphi_\kappa + \frac{H}{3H + \Delta/a^2} \left( (1 + 3c_2^2) \left( \frac{\Delta}{a^2} \varphi_\kappa - \frac{1}{4} n_1 \right) - 12\pi Ge - n_3 \right) - \frac{1}{3} n_0 \right]\right\}.
$$

$$
= \frac{\Delta}{3a^2} \left[ -\varphi_\kappa + \frac{1}{3H + \Delta/a^2} \left( (1 + 3c_2^2) \left( \frac{\Delta}{a^2} \varphi_\kappa - \frac{1}{4} n_1 \right) - 12\pi Ge - n_3 \right) - 8\pi G\Pi - n_4 \right]. \tag{298}
$$

Equations (295), (296) follow by combining Eqs. (291), (292). Equation (297) follows from Eqs. (88), (89), (91), (92). Equation (298) follows from Eqs. (88), (89), (91), (92). The perturbed order variables in Eqs. (297), (298) are evaluated in the uniform-density gauge ($\delta \equiv 0$), and the uniform-expansion gauge ($\kappa \equiv 0$), respectively.
C. Large-scale solutions

Now, we assume an ideal fluid, thus set \( e = 0 = \Pi \). In the large-scale limit, thus ignoring the \( \Delta / (aH)^2 \)-order higher terms, Eqs. (296)-(298) give

\[
\dot{\varphi}_v = -\frac{1}{3} (n_0 - n_2) + aH n_6 + \frac{H c_s^2}{4\pi G (\mu + p)} \left( -\frac{1}{4} n_1 + H n_2 \right) \propto \frac{H^2 c_s^2}{4\pi G (\mu + p) a^3}, \tag{299}
\]

\[
\dot{\varphi}_\delta + \frac{1}{3H} \left( \frac{1}{4} n_1 + \frac{1}{2} H n_1 - H n_3 \right) - \frac{1}{3} n_0 \propto \frac{1}{a^3}, \tag{300}
\]

\[
\dot{\varphi}_\kappa - \frac{H}{3H} \left( \frac{1 + 3 c_s^2}{4} n_1 + n_3 \right) - \frac{1}{3} n_0 \propto \frac{1}{a^3}, \tag{301}
\]

where the perturbed order variables in Eq. (299) are evaluated in the comoving gauge \((v = 0)\). We already used the behavior of linear order solutions in Eqs. (289)-(290) in order to show that the right-hand-side of Eqs. (296)-(298) vanish. Using the solutions in Eqs. (289),(290) we can show that

\[
(\varphi_v - \varphi_v^2) + O(\Delta C^2, \Delta^2 d^2) \propto \frac{H^2 c_s^2}{(\mu + p) a^3}, \tag{302}
\]

\[
(\varphi_\delta - \varphi_\delta^2) + O(\Delta C^2, \Delta^2 d^2) \propto (\varphi_\kappa - \varphi_\kappa^2) + O(\Delta C^2, \Delta^2 d^2) \propto \frac{1}{a^3}. \tag{303}
\]

Thus, we have general large-scale asymptotic solutions

\[
\varphi_v - \varphi_v^2 = C(x) + \frac{\Delta}{4\pi G} d(x) \int^t \frac{c_s^2 H^2}{(\mu + p) a^3} dt, \tag{304}
\]

\[
\varphi_\delta - \varphi_\delta^2 = \varphi_\kappa - \varphi_\kappa^2 = C(x) + \frac{\Delta}{3} d(x) \int^t \frac{dt}{a^3}, \tag{305}
\]

where \( C(x) \) and \( d(x) \) are integration constants now including the second-order contributions, i.e., \( C = C^{(1)} + C^{(2)} \), etc. Ignoring the transient solutions in an expanding phase we have

\[
\varphi_v = \varphi_\delta = \varphi_\kappa = C(x), \tag{306}
\]

even to the second order in perturbations in the large-scale limit.

XII. DISCUSSION

In this work we presented pure general relativistic effects of second-order perturbations in Friedmann cosmological world model. In our previous work we have shown that to the second-order perturbations, the density and velocity perturbation equations of general relativistic zero-pressure, irrotational, single-component fluid in a flat background coincide exactly with the ones known in Newton’s theory. \[12\]. We also have shown the effect of gravitational waves to the second-order, and pure general relativistic correction terms appearing in the third-order perturbations, \[12\] \[13\]. Here, we presented results of second-order perturbations relaxing all the assumptions made in our previous work in \[12\]. We derived the general relativistic correction terms arising due to (i) pressure, (ii) multi-component, (iii) background curvature, and (iv) rotation. We also presented a general proof of large-scale conserved behaviors of curvature perturbation variable in several gauge conditions, now to the second order.

Effects of pressure can be found in Eqs. (123)-\(125\). As we emphasized, the effect of pressure is generically relativistic even in the background world model and the linear order perturbations. Still, our equations show the pure general relativistic effects of pressure (including stresses) appearing in the second-order perturbations. Effects of multi-component fluids can be found in Eqs. (195)-(200). Although these equations apparently show deviations from Newtonian situation, in Sec. \[\text{VII D}\] we showed that if we ignore purely decaying terms in an expanding phase the equations are effectively the same as in the Newtonian situation. Effects of background spatial curvature \( K \) can be read from Eqs. (217), (218) or Eqs. (211)-\(214\). Effects of vector-type perturbation can be read from Eqs. (247)-\(249\). In the small-scale limit we showed that, if we ignore the tensor-type perturbation, the equations coincide with the Newtonian ones.

Our results may have important practical implications in cosmology and the large-scale structure formation. Our new result showing relativistic/Newtonian correspondence in the zero-pressure irrotational multi-component fluids.
is practically relevant in currently favored cosmology where baryon and dark matter are two important ingredients of the current matter content in addition to the cosmological constant. All equations in our work are valid in the presence of the cosmological constant. A related important result is the relativistic/Newtonian correspondence valid in the presence of rotational perturbation far inside horizon. Thus, inside the horizon scale, even in the presence of rotational perturbations we can still rely on the Newtonian equations to handle quasi-linear evolution of large-scale structures. As the spatial curvature in the present cosmological era is known to be small, the possible presence of small spatial curvature may not be important in the second-order perturbations. Still, while the Newtonian equation is exactly valid to the linear order even in the presence of the spatial curvature, we have nontrivial general relativistic correction terms present to the second order in perturbations. Our second-order perturbation equations in the presence of pressure may have an interesting role as we approach early stage of universe where the effect of radiation becomes important. The importance of pressure to the second-order perturbations, of course, depends on whether nonlinear effects are significant in the early evolution stage of the large-scale structure during the radiation era and in the early matter dominated era. Realistic estimations of the diverse pure general relativistic contributions using the complete set of equations presented in this work are left for future investigations.

In an accompanying paper we will investigate the effects of third-order perturbations of zero-pressure irrotational multi-component fluids in a flat background. This is one obvious remaining issue in our series of investigation of nonlinear cosmological perturbations where nontrivial general relativistic effects are expected. In the case of a single fluid we presented the pure general relativistic effects appearing in the third order in [13]. Corresponding results in the case of multi-component will be presented in [34].

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