Conformal bridge transformation, 
\( \mathcal{PT} \)- and super- symmetry

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Abstract

Supersymmetric extensions of the 1D and 2D Swanson models are investigated by applying the conformal bridge transformation (CBT) to the first order Berry-Keating Hamiltonian multiplied by \( i \) and its conformally neutral enlargements. The CBT plays the role of the Dyson map that transforms the models into supersymmetric generalizations of the 1D and 2D harmonic oscillator systems, allowing us to define pseudo-Hermitian conjugation and a suitable inner product. In the 1D case, we construct a \( \mathcal{PT} \)-invariant supersymmetric model with \( N \) subsystems by using the conformal generators of supersymmetric free particle, and identify its complete set of the true bosonic and fermionic integrals of motion. We also investigate an exotic \( N = 2 \) supersymmetric generalization, in which the higher order supercharges generate nonlinear superalgebras. We generalize the construction for the 2D case to obtain the \( \mathcal{PT} \)-invariant supersymmetric systems that transform into the spin-1/2 Landau problem with and without an additional Aharonov-Bohm flux, where in the latter case, the well-defined integrals of motion appear only when the flux is quantized. We also build a 2D supersymmetric Hamiltonian related to the “exotic rotational invariant harmonic oscillator” system governed by a dynamical parameter \( \gamma \). The bosonic and fermionic hidden symmetries for this model are shown to exist for rational values of \( \gamma \).

1 Introduction

The focus on non-Hermitian \( \mathcal{PT} \)-invariant systems has grown significantly following the pioneering studies of Bender and collaborators [1, 2, 3]. Such systems attract attention and find applications in various areas of physics, including optics [4, 5, 6, 7, 8], condensed matter physics [9, 10, 11], quantum field theory [12, 13, 14, 15, 16, 17, 18, 19], gravity and cosmology [20, 21], nonlinear waves [22], and theory of integrable models of finite [23, 24, 25, 26, 27] and infinite number of degrees of freedom [28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40].
The interest in them is based on the presence of non-Hermitian terms in their Hamiltonians, which, in particular, can be interpreted as the result of interaction of the respective systems with the environment [1, 41, 42, 43, 44, 45]. When such a Hamiltonian is pseudo-Hermitian (see Ref. [42] and Sec. 2 below), it can be related to a Hermitian system, that guarantees a unitarity of evolution.

A good starting point for studying \( \mathcal{PT} \)-invariant systems is the Swanson oscillator [46, 47, 48, 49, 50], a one-dimensional multi-parameter toy model. This system has all real eigenvalues for a certain choice of the parameters. Consequently, one can construct the Dyson map [51] that transforms the corresponding non-Hermitian Hamiltonian into a Hermitian one in such a case.

The present article aims to construct supersymmetric extensions of this model on the real line and of its generalization in the Euclidean plane. The resulting systems are interesting for their connection with the spin-1/2 Landau problem, the models in the presence of Aharonov-Bohm flux associated with anyons [52, 53, 54, 55], the physics of Bose-Einstein condensates in rotating harmonic traps [56, 57], and gravitoelectromagnetism [58].

The usual construction of one-dimensional supersymmetric quantum models is based on the Darboux [59, 60] and confluent Darboux transformations [8, 61], and in Refs. [62, 63] some interesting results related to super-extended generalized Swanson systems are presented. We will take a different approach here grounded on the peculiar property of the Swanson model admitting a representation in terms of the conformal \( \mathfrak{so}(2,1) \) generators of the free particle system, that allows us to use the recently developed conformal bridge transformation (CBT) technique [64]. This mapping relates the symmetry generators and eigenstates of asymptotically free and harmonically confined versions of the \( \mathfrak{so}(2,1) \cong \mathfrak{sl}(2,\mathbb{R}) \) conformal mechanics models (e.g., of the free particle and the harmonic oscillator in the simplest case). In this way, we identified hidden symmetries and spectra of different harmonically trapped systems in a monopole [65] and cosmic string backgrounds [66, 58], and constructed exotic rotational invariant harmonic oscillator (ERIHO) related to the physics of Bose-Einstein condensates [57] and gravitoelectromagnetism [58]. The CBT was also applied to the study of the black holes mechanics in ref. [67].

As a basis for the aim of the present research, we will go deep into the link between the CBT and the \( \mathcal{PT} \)-symmetric Swanson oscillator observed by us recently in [68].

The article is organized as follows. In Sec. 2 we briefly review the one-dimensional Swanson model and its connection with the CBT, which serves as the Dyson map. We also provide there some comments on the direct higher dimensional generalization of the model. The \( N \)-supersymmetric extension and a higher order \( N = 2 \) supersymmetric extension of the one-dimensional Swanson model are discussed in Sec. 3, where the first and higher order bosonic and fermionic integrals of motion for these systems are identified, and the corresponding superalgebraic structures are discussed. Sec. 4 is devoted to generalization of the construction to the two-dimensional case. First, we build a two-dimensional \( \mathcal{PT} \)-symmetric model using the generators of the planar supersymmetric free particle system obtained via a non-relativistic limit from the free Dirac and Klein-Gordon equations. The resulting model is related to the spin-1/2 Landau problem by applying the CBT. After that, we study two generalizations of this non-Hermitian \( \mathcal{PT} \)-invariant supersymmetric model. The first generalization is constructed by adding an Aharonov-Bohm flux, for
which integrals of motion are obtained from the free particle system only when the flux is quantized. Another generalized system is related by the CBT to the supersymmetric extension of the ERIHO system. In concluding Sec. 5 we discuss some related interesting open problems.

2 Swanson model and CBT

The non-Hermitian Swanson model is given by the $\mathcal{PT}$-symmetric Hamiltonian operator

$$\hat{H}_{\alpha,\beta,\omega} = \hbar \left[ \omega (\hat{a}^+_\omega \hat{a}_\omega + \frac{1}{2}) + \alpha (\hat{a}^-_\omega)^2 + \beta (\hat{a}^+_\omega)^2 \right],$$  

(2.1)

where $\omega > 0$, $\alpha$ and $\beta$ are real parameters with the dimension of frequency, subject to the relations $\omega^2 - 4\alpha\beta > 0$, $\alpha \neq \beta$, and $\hat{a}^\pm_\omega = \sqrt{\frac{\hbar \omega}{2m}} (\hat{x} \pm \frac{i\hbar}{m\omega} \frac{d}{dx})$ are the bosonic harmonic creation and annihilation operators, $[\hat{a}^-_\omega, \hat{a}^+_\omega] = 1$. By a generalized Bogolyubov transformation, operator (2.1) can be transformed into Hamiltonian of the harmonic oscillator with eigenvalues $\hbar \Omega (n + \frac{1}{2})$, $\Omega = \sqrt{\omega^2 - 4\alpha\beta}$, $n = 0, 1, \ldots, 1$. Some of such transformations are explicitly considered in Appendix A. This model gained popularity over the years as it allows to lay a good base for the study of $\mathcal{PT}$-symmetric systems in general, see for example Refs. [47, 48, 49, 50].

In terms of the Hamiltonian and dynamic generators of the $\mathfrak{so}(2,1)$ conformal symmetry $^2$ of the one-dimensional free particle system,

$$\hat{H} = \frac{1}{2m} \hat{p}^2, \quad \hat{D} = \frac{1}{2} \{ \hat{x}, \hat{p} \}, \quad \hat{K} = \frac{m}{\hbar} \hat{x}^2,$$

(2.2)

$$[\hat{D}, \hat{H}] = i\hbar \hat{K}, \quad [\hat{D}, \hat{K}] = -i\hbar \hat{K}, \quad [\hat{K}, \hat{H}] = 2i\hbar \hat{D},$$

(2.3)

operator $\hat{H}_{\alpha,\beta,\omega}$ reads as

$$\hat{H}_{\alpha,\beta,\omega} = (1 - \omega^{-1}(\alpha + \beta)) \hat{H} + (1 + \omega^{-1}(\alpha + \beta)) \omega^2 \hat{K} + 2i(\alpha - \beta) \hat{D}. $$

(2.4)

A system of this kind is related to the alternative non-Hermitian model

$$\hat{H}_\Omega = \hat{H} + 2i\Omega \hat{D} = -\hbar^2 \frac{d^2}{dx^2} + \frac{\Omega \hbar}{2} \left( x \frac{d}{dx} + \frac{1}{2} \right),$$

(2.5)

via a similarity transformation. This is easy to see since the eigenvalue equation corresponding to operator (2.5) is nothing else than the Hermite equation. Then, the similarity transformation $e^{-\frac{\Omega}{2}\hat{K}} \hat{H}_\Omega e^{\frac{\Omega}{2}\hat{K}}$ gives us the harmonic oscillator system with frequency $\Omega$, and $e^{-\frac{\Omega}{2}\hat{K}}$ to be the Gaussian factor appearing in the harmonic oscillator wave functions. In

$^1$In the original article [46] the parameters $\omega, \alpha$ and $\beta$ were restricted by the inequality $\omega^2 - 4\alpha\beta \geq 0$. We take a strict inequality in order to exclude the case $\Omega = 0$.

$^2$As dynamic symmetry generators we mean the integrals of motion that explicitly depend on time, $\frac{dA}{dt} = i\hbar [\hat{A}, \hat{H}] + \frac{\delta A}{\delta t} = 0$. They do not commute with the corresponding Hamiltonian operator, and throughout the article such generators are taken at $t = 0$. Their explicit dependence on time can be restored by applying a unitary transformation with the evolution operator. In contrast, the true integrals of motion are those operators that satisfy the equation $\frac{dA}{dt} = i\hbar [\hat{A}, \hat{B}] = 0$. In
the usual interpretation of $\mathcal{PT}$-symmetric systems, $e^{-\frac{i}{\hbar}K}$ generates the Dyson map and its square corresponds to the metric operator, which in this case is the weight function for Hermite polynomials inner product. Clearly, a composition of this last mapping and of the generalized Bogolyubov transformation mentioned above relates (2.4) and (2.5).

Now, the system (2.5) can be obtained from (2.1) in two alternative ways. First, this can be done by putting $\beta = -\frac{\alpha}{2}$, $\alpha = \frac{\omega}{2} + \epsilon^2$, $\omega = \epsilon$, and taking the limit $\epsilon \to 0$ in (2.4). This limit produces divergent operators in some constructions of Dyson maps for Swanson model considered in [48], and for that reason the system (2.5) is out of sight from the common point of view \footnote{In [48], the authors choose to work from the beginning with a Dyson operator of the form $S = \exp(i\tilde{a}^\dagger\tilde{a} + \eta(\tilde{a}^\dagger\tilde{a})^2 + \eta^*(\tilde{a}^\dagger\tilde{a})^2)$ with $\epsilon^2 - 4|\eta|^2 > 0$, that is incompatible with $e^{-\frac{i}{\hbar}K}$. The operator $S$ is rewritten then in terms of a new parameter $z = z(\alpha, \beta, \omega, \epsilon, \eta)$ that should take values in the interval $(-1, 1)$. For our limiting procedure, their parameter $z$ goes to infinity, see equations (9), (12) and (15) in [48] for the details.}.

Another way is via the similarity transform
\[ e^{\frac{i}{\hbar}\Omega}(\hat{H}_{\mathcal{PT}})e^{-\frac{i}{\hbar}\Omega} = \hat{H}_\Omega, \quad \hat{H}_{\mathcal{PT}} := 2i\Omega \hat{D} = \frac{\Omega}{2}(x\frac{d}{dx} + \frac{1}{2}) \],
(2.6)
applied to the $\mathcal{PT}$-invariant first order differential operator $\hat{H}_{\mathcal{PT}}$, that corresponds to the well known Weierstrass transformation, see [64] and references therein. However, the operator $\hat{H}_{\mathcal{PT}}$ cannot be obtained from (2.4) via the described limiting procedure applied to the parameters $\alpha$, $\beta$ and $\omega$. This can be done, nevertheless, if to generalize further the Swanson model by introducing a real parameter $\gamma$ in front of the first term in (2.1) and taking the limit $\gamma \to 0$, see [68]. The $\mathcal{PT}$-symmetric operator $\hat{H}_{\mathcal{PT}}$ can be transformed into the harmonic oscillator Hamiltonian via the CBT [64]
\[ \hat{S}_\Omega := e^{-\frac{i}{\hbar}K}e^{\frac{i}{\hbar}\Omega}\ln(2)\hat{D} = \exp\left(\frac{\Omega}{2\hbar}(\hat{H} - \Omega^2\hat{K})\right) = \exp\left(-\frac{\Omega}{\pi}(\hat{a}_0^2 + (\hat{a}_0^\dagger)^2)\right), \quad (2.7) \]
\[ \hat{S}_\Omega(\hat{H}_{\mathcal{PT}})\hat{S}_\Omega^{-1} = \hat{H} + \Omega^2\hat{K}. \quad (2.8) \]
Note that the two exponential operators in the first equality are just the composition of the explicit similarity transformations mentioned above, while the additional factor $e^{\frac{i}{\hbar}\ln(2)\hat{D}}$ does not change the result and helps to rewrite the CBT generator $\hat{S}_\Omega$ in the explicitly Hermitian form, which looks like the evolution operator of the inverted harmonic oscillator system for complex time $t = i\pi/4\Omega$. From this line of reasoning, it follows that models (2.1), (2.5), and $\hat{H}_{\mathcal{PT}}$ given by Eq. (2.6) are similarity equivalent, and which one of them we work with depends on the problem. This statement is rigorously re-enforced in Appendix A.

From now on, we will continue to work with (2.6) since this operator can easily be generalized to higher dimensions, as we will see below. For simplicity, we also change the notation
\[ \Omega \to \omega, \quad \hat{a}^\pm_\Omega \to \hat{a}^\pm, \quad \hat{S}_\Omega \to \hat{S}. \quad (2.9) \]

The advantage to take the operator (2.6) as a $\mathcal{PT}$-symmetric Hamiltonian is that its formal, single-valued but not square integrable on $\mathbb{R}$ eigenstates
\[ \phi_n(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{\omega}{2\pi}\right)^{\frac{1}{4}} \left(\sqrt{\frac{m\omega}{\hbar}} x\right)^n, \quad \hat{H}_{\mathcal{PT}}\phi_n(x) = \hbar\omega(n + \frac{1}{2})\phi_n(x), \quad n = 0, 1, \ldots, \quad (2.10) \]
have a simple monomial form. Furthermore, these functions are Jordan states of the free particle Hamiltonian as well \[69, 61\] \(^4\), that helps to compute their transformation under the action of \(\mathfrak{S}\),

\[
\hat{\mathfrak{S}} \phi_n(x) = \frac{1}{\sqrt{2^n n!}} \left( \frac{m\omega}{\hbar} \right)^{\frac{1}{2}} H_n \left( \sqrt{\frac{m\omega}{\hbar}} x \right) e^{-\frac{(m\omega x)^2}{2\hbar}} := \psi_n(x). \tag{2.11}
\]

Here, \(H_n(\eta)\) are the Hermite polynomials, and the resulting functions are the normalized eigenstates of the harmonic oscillator system.

On the other hand, it is easy to see that the operators \(\hat{H}\) and \(\hat{p}\) (\(\hat{K}\) and \(\hat{x}\)) are the second and first order lowering (raising) ladder operators for formal eigenstates (2.10) of the \(\mathcal{PT}\)-invariant Hamiltonian (2.6). Under the CBT generated by (2.7), they transform as

\[
\hat{\mathfrak{S}} \left( \frac{i}{\sqrt{m\omega}} \hat{p}, \sqrt{\frac{m\omega}{\hbar}} \hat{x}, -\frac{\hbar}{m\omega} \hat{H}, \frac{\omega}{\hbar} \hat{K}, \hbar^{-1} i \hat{D} \right) \hat{\mathfrak{S}}^{-1} = (\hat{a}^-, \hat{a}^+, \hat{J}_-, \hat{J}_+, \hat{J}_0), \tag{2.12}
\]

where

\[
\hat{J}_0 = \frac{i}{2}(\hat{a}^+ \hat{a}^- + 1) = \frac{1}{2\omega} \hat{H}_{\text{osc}}, \quad \hat{J}_\pm = \hat{J}_1 \pm i \hat{J}_2 = \frac{1}{2}(\hat{a}^\pm)^2 \tag{2.13}
\]

are the dimensionless generators of the \(\mathfrak{sl}(2, \mathbb{R})\) conformal algebra

\[
[\hat{J}_0, \hat{J}_\pm] = \pm \hat{J}_\pm, \quad [\hat{J}_-, \hat{J}_+] = 2 \hat{J}_0. \tag{2.14}
\]

In effect, the transformation (2.12) is nothing else than the Dyson map [51] that relates the non-Hermitian Hamiltonian (2.6) with the harmonic oscillator Hamiltonian \(\hat{H}_{\text{osc}}\). The CBT generator (2.7) allows to define the pseudo-Hermitian inner product

\[
(\phi_{n1}, \phi_{n2}) := \langle \phi_{n1} | \hat{\Theta} | \phi_{n2} \rangle = \langle \psi_{n1} | \psi_{n2} \rangle = \delta_{n1n2}, \quad \hat{\Theta} = \hat{\Theta}^\dagger = \hat{\mathfrak{S}}^2, \tag{2.15}
\]

which reduces to a usual scalar product for orthonormalized eigenfunctions of the harmonic oscillator.

Using definition (2.15), we introduce the pseudo-Hermitian conjugation \(\hat{\Theta}^\dagger\) of an operator \(\hat{O}\),

\[
(\phi_{n1}, \hat{O} \phi_{n2}) = \langle \hat{\Theta}^\dagger \phi_{n1}, \phi_{n2} \rangle. \tag{2.16}
\]

By developing the left hand side of this equation, one gets

\[
(\phi_{n1}, \hat{O} \phi_{n2}) = \langle \phi_{n1}, \hat{\Theta} \hat{O} \phi_{n2} \rangle = \langle \hat{\Theta}^\dagger \hat{\Theta} \phi_{n1}, \phi_{n2} \rangle = \langle \hat{\Theta}^{-1} \hat{\Theta}^\dagger \hat{\Theta} \phi_{n1}, \phi_{n2} \rangle = \langle \hat{\Theta}^{-1} \hat{\Theta} \phi_{n1}, \phi_{n2} \rangle,
\]

from where we identify

\[
\hat{\Theta}^\dagger = \hat{\mathfrak{S}}^{-2} \hat{\Theta}^\dagger \hat{\mathfrak{S}}^2. \tag{2.17}
\]

\(^4\)The Jordan states of \(\hat{H}\) are those functions that are annihilated by a certain polynomial of \(\hat{H}\). In this case \((\hat{H})^{k+1} \phi_{2k} = 0\) and \((\hat{H})^{k+1} \phi_{2k+1} = 0\), with \(k = 0, 1, \ldots\).
The application of this to the Hermitian operators ($\hat{x}$, $\hat{H}$, $\hat{K}$) and anti-Hermitian operators $i\hat{p}$ and $i\hat{D}$ gives us the (a priori) contra-intuitive relations with respect to the pseudo-Hermitian conjugation,

\[
\left(\frac{\hat{p}}{\sqrt{m_\omega}}\right)^\dagger = \sqrt{\frac{m_\omega}{\hbar}}\hat{x}, \quad \left(\frac{\hat{x}}{\sqrt{m_\omega}}\right)^\dagger = \frac{i}{\sqrt{m_\omega}}\hat{p}, \quad (2.19)
\]

\[
(-\frac{1}{\omega_n}\hat{H})^\dagger = \frac{\omega_n}{\hbar}\hat{K}, \quad \left(\frac{\omega_n}{\hbar}\hat{K}\right)^\dagger = -\frac{1}{\omega_n}\hat{H}, \quad (i\hat{D})^\dagger = i\hat{D}, \quad (2.20)
\]

which are just the analogues of the relations $\hat{a}_+^\dagger (\hat{a}^\dagger_\pm)$ and $\hat{J}_\pm = (\hat{J}_\pm)^\dagger$ and $\hat{J}_0 = (\hat{J}_0)^\dagger$ for the harmonic oscillator. Thus, the $\mathcal{PT}$-symmetric Hamiltonian (2.6) is pseudo-Hermitian with respect to the scalar product (2.15). Additionally, from (2.12) one can also see that our conformal bridge transformation generated by $\mathcal{S}$ effectively is the eighth-order root of the identity transformation, see Ref. [68] as well.

Having described the one-dimensional version of our system, let us now turn to its multidimensional generalizations. Before this, we note that, up to an imaginary factor $i$, the first-order operator $\hat{H}_{\mathcal{PT}}$ looks like the Berry-Keating Hamiltonian [70, 71], which has been considered in the literature in the context of the relation between the Riemann hypothesis on zeros of zeta-function and quantum mechanics [72, 73, 74, 75, 76]. We will return to this issue in the last section.

As it was shown in [57], it is simple to generalize this picture to $d$-dimensional systems. To do that, we just consider the changes (summation over a repeated index is assumed)

\[
\hat{x} \to \hat{x}_j, \quad \hat{p} \to \hat{p}_j = -i\hbar\frac{\partial}{\partial x_j}, \quad j = 1, \ldots, d, \quad (2.21)
\]

\[
\hat{H} = \frac{1}{2m}\rho_j\rho_j, \quad \hat{K} = \frac{\omega}{\hbar}\hat{x}_j\hat{x}_j, \quad \hat{D} = \frac{1}{4}\{\hat{x}_j, \hat{p}_j\}, \quad (2.22)
\]

\[
\hat{J}_0 = \frac{1}{2\hbar}\hat{H}_{\text{osc}}, \quad \hat{J}_\pm = \frac{1}{2}\hat{a}_j^\mp \hat{a}_j^\pm, \quad \hat{H}_{\text{osc}} = \hbar\omega(\hat{a}_j^\mp \hat{a}_j^\pm + \frac{d^2}{4}), \quad (2.23)
\]

and construct the CBT generator (2.7) using $\hat{H}$ and $\hat{K}$ defined in (2.22). The mapping is like (2.12), but with the inclusion of the index $j$ when it corresponds. Then for the $d$-dimensional $\mathcal{PT}$-symmetric Hamiltonian $\hat{H}_{\mathcal{PT}} = 2i\omega\hat{D}$, any manifestly scale invariant operator $(\hat{x}_j)^l (\hat{p}_k)^l$, where $l$ is any non-negative integer, is an integral of motion. At the same time, the angular momentum tensor

\[
\hat{M}_{jk} = \hat{x}_j\hat{p}_k - \hat{x}_k\hat{p}_j \quad (2.24)
\]

is invariant under the CBT since conformal generators (2.22) are rotationally invariant.

In the following sections, we generalize the described picture to generate the one- and two-dimensional supersymmetric systems.

3 CBT and one-dimensional supersymmetric systems

The aim of this section is to construct supersymmetric generalizations of the $\mathcal{PT}$-symmetric Hamiltonian (2.6) and its bosonic and fermionic integrals of motion. By means of an appropriate modification of the CBT we associate the obtained models with supersymmetric extensions of the one-dimensional harmonic oscillator. The developed construction will be used then to study the higher dimensional case.
3.1 Extension with $N$ subsystems

A generalized one-dimensional $N$-supersymmetric system can be given by the $N \times N$ matrix Hamiltonian operator and the set of intertwining operators,

$$\hat{H} = \text{diag}(\hat{H}_N, \hat{H}_{N-1}, \ldots, \hat{H}_1), \quad \hat{A}_q \hat{H}_q = \hat{H}_{q+1} \hat{A}_q, \quad q = 1, \ldots, N - 1. \quad (3.1)$$

In the conventional case with the second order matrix Schrödinger operator $\hat{H}$, the chain of subsystems’ Hamiltonians and the intertwining operators $\hat{A}_q$ can be obtained from a systematic procedure known as the Darboux transformation \[59, 60\]. The simplest example corresponds to the case

$$\hat{H}_1 = \ldots = \hat{H}_N = \hat{H}, \quad \hat{A}_q = \frac{1}{\sqrt{2m}} \hat{p}, \quad (3.2)$$

where $\hat{H}$ is the one-dimensional free particle Hamiltonian operator introduced in (2.2). The integrals of motion of this system are given by constant matrices $E_{jk}$ with elements

$$(E_{jk})_{lm} = \delta_{jl} \delta_{km}, \quad (3.3)$$

and the matrix first order differential operators $\hat{p} E_{ij}$. Identification of the integrals as even and odd generators of the corresponding superalgebra depends on the choice of the grading operator, and we will return to this point later. At the same time, the diagonal operators,

$$\hat{H} = \hat{H}_{I_{N \times N}}, \quad \hat{D} = \hat{D}_{I_{N \times N}}, \quad \hat{K} = \hat{K}_{I_{N \times N}}, \quad (3.4)$$

can be identified as generators of the $\mathfrak{so}(2,1)$ algebra, where $I_{N \times N}$ is the unit matrix of order $N$.

Let us take now the first order differential matrix operator

$$\hat{\mathcal{H}} = 2i\omega \hat{D} + \hbar \omega \vartheta, \quad \vartheta_{jk} = (N - j - \frac{1}{2}) \delta_{jk}, \quad j, k = 1, \ldots, N, \quad (3.5)$$

as the $\mathcal{PT}$-symmetric Hamiltonian. The constant diagonal matrix $\vartheta$ is included here to obtain the $N$-supersymmetric extension of the harmonic oscillator system after applying the CBT operators

$$\hat{\mathcal{J}} = \hat{\mathcal{S}}_{I_{N \times N}}, \quad \hat{\mathcal{J}}^{-1} = \hat{\mathcal{S}}^{-1}_{I_{N \times N}}, \quad \hat{\Theta} = \hat{\mathcal{J}}^2, \quad (3.6)$$

to the Hamiltonian (3.5) \[5\],

$$\hat{\mathcal{J}} (\hat{\mathcal{H}}), \hat{\mathcal{J}}^{-1} = \hat{H}_{\text{osc}} I_{N \times N} + \hbar \omega \vartheta. \quad (3.7)$$

It is convenient to represent $\vartheta$ in the equivalent form

$$\vartheta = (\frac{N}{2} - 1) I_{N \times N} + \tau, \quad \tau = \sum_{f=1}^{N-1} c_f h_f, \quad (3.8)$$

Such a system is obtained by choosing $H_1 = H_{\text{osc}} - \frac{1}{\hbar} \omega = \hbar \omega \hat{a}^+ \hat{a}$ and $\hat{A}_q = \hat{a}^-$ in (3.1) for all $q$. As a consequence, $H_{N-q+1} = H_{\text{osc}} + \hbar \omega (N - q - \frac{1}{2})$, and (3.1) coincides with (3.7).
where \( h_f \) are the \( N - 1 \) diagonal traceless matrices normalized by the inner product \( \text{Tr}(h_j \cdot h_k) = 2 \delta_{jk} \), and \( c_f \) are the real constants, \( c_f = \frac{1}{2} \text{Tr}(h_f \vartheta) \). The eigenstates of the operator (3.5), which diagonalize the \( h_f \) matrices as well, are given by

\[
\Phi_{n,k}(x) = \phi_n(x) e^{T}_{N-i}, \quad e_j = (0, \ldots, 0, 1, 0, \ldots, 0),
\]

\[
i = 0, \ldots, N - 1, \quad n = 0, 1, \ldots,
\]

where the functions \( \phi_n(x) \) correspond to (2.10), and \( e^{T}_{N-i} \) indicates the transposition of the unit vectors. These states satisfy the eigenvalue equation

\[
\mathcal{H} \Phi_{n,i}(x) = E_n \Phi_{n,k}(x), \quad E_n = \hbar \omega s, \quad s = n + i,
\]

from where we note that energy levels \( E_n \) with \( s = 0, 1, \ldots, N - 1 \) have degeneracy equal to \( s + 1 \). Degeneracy of all other states with \( s \geq N \) is equal to \( N \).

Having the \( PT \)-symmetric Hamiltonian (3.5) as well as its energy levels and eigenstates, we identify now the structure of its integrals of motion, and determine their corresponding bosonic or fermionic nature. For this we note that the traceless matrices \( h_f \) can be considered as generators of the Cartan sub-algebra \( \mathfrak{h} \) of \( \mathfrak{su}(N) \). The remaining \( N(N - 1) \) generators of this Lie algebra correspond to the constant matrices \( E_{jl} \) with \( j \neq l \) introduced in (3.3). To find their commutation relation with our Hamiltonian, we just need to compute

\[
[\tau, E_{jl}] = [\vartheta, E_{jl}] = (l - j) E_{jl},
\]

from where we revel the nature of these dynamic integrals as ladder operators in the second index of eigenfunctions, \( E_{jl} \Phi_{n,j} = \Phi_{n,i+j-l} \). Since \( \tau \in \mathfrak{h} \), relation (3.12) corresponds to the definition of a root vector of the corresponding simple Lie algebra of the Cartan series \( A_{N-1} \). Then, for \( j > k \) we have the \( N(N - 1) \) true integrals

\[
\hat{I}^{(k,j)}_+ = \left( \sqrt{\frac{1}{\hbar m}} \right)^{j-k} E_{kj}, \quad \hat{I}^{(k,j)}_- = \left( \sqrt{\frac{1}{\hbar m}} \right)^{j-k} E_{jk}, \quad j > k, \quad [\mathcal{H}, \hat{I}^{(k,j)}_\pm] = 0,
\]

\[
\hat{I}^{(k,j)}_+ \Phi_{n,i} \sim \Phi_{n-j+i+k-j}, \quad \hat{I}^{(k,j)}_- \Phi_{n,i} \sim \Phi_{n+j-i-k+j},
\]

which satisfy the pseudo-Hermitian conjugation relations \( (\hat{I}^{(k,j)}_\pm)^\dagger = \hat{I}^{(k,j)}_\mp \) in the sense of (2.18). Generally, these integrals generate a nonlinear polynomial superalgebra.

In order to determine the fermionic or bosonic nature of dynamic and true integrals \( E_{jl} \) and (3.13), we have to select the grading operator \( \Gamma \) such that \( \Gamma^2 = \mathbb{1}_{N \times N}, [\mathcal{H}, \Gamma] = 0 \). Here, we restrict ourselves to the case of even \( N \), and for the sake of definiteness choose \( \Gamma \) in the form of the diagonal traceless matrix

\[
\Gamma = \text{diag} \left( 1, -1, \ldots, 1, -1 \right),
\]

being a linear combination of generators of the \( \mathfrak{su}(N) \) Cartan sub-algebra \( \mathfrak{h} \). With this choice of the \( \mathbb{Z}_2 \)-grading operator, we find

\[
[\Gamma, E_{jk}] = 0, \quad j - k = 0 \text{ (mod 2)}, \quad \{\Gamma, E_{jk}\} = 0, \quad j - k = 1 \text{ (mod 2)}.
\]
So, the first subset of dynamic integrals $E_{jk}$ in (3.16) is identified as even (bosonic) generators of the superalgebra, while their second half is identified as odd (fermionic) operators of the system (3.5). Coherently with this, the even and odd nature of the true integrals (3.13) is identified.

We notice here that the permutation of the diagonal elements in (3.15) provides us with alternative choices for the grading operator $\Gamma$. This results in the change of identification of the even and odd nature of the generators $E_{jl}$ and nontrivial integrals (3.13). As a consequence, the concrete form of the corresponding (nonlinear) superalgebra of the system will be changed.

Finally, last but not least, from the free particle generators of translation and Galilean boost we identify the first order bosonic ladder operators of the system,

$$\hat{A}^- = \frac{i}{\sqrt{\text{mech}}} \hat{p} \mathbb{I}_{N \times N} \quad \hat{A}^+ = \sqrt{\frac{m\omega}{\hbar}} \hat{x} \mathbb{I}_{N \times N},$$

(3.17)

Their square yields us the second order ladder operators of the centrally extended $\mathfrak{s}(2, \mathbb{R})$ conformal symmetry of the model (3.5),

$$\hat{J}_0 = \frac{1}{2\omega} \hat{H}, \quad \hat{J}_- = -\frac{1}{\omega} \frac{\hat{J}_+}{\hbar}, \quad \hat{J}_+ = \frac{\hbar}{\omega} \hat{K},$$

(3.19)

Under the CBT, these generators map into the second and first order ladder operators of the superextended system (3.7) in accordance with (2.12).

Let us consider now two concrete examples.

- **The $N = 2$ case.**

In this simplest case, $\vartheta = \tau = \frac{1}{2} \sigma_3$, $\Gamma = \sigma_3$ and $E_{12} = \frac{1}{2} \sigma_+$, $E_{21} = \frac{1}{2} \sigma_-$, $\sigma_\pm = \sigma_1 \pm i \sigma_2$. The fermionic operators

$$\hat{Q}_- = \sqrt{\hbar \omega} \hat{I}_{12} = \frac{1}{2} \frac{\hbar \sigma_+}{\sqrt{m}}, \quad \hat{Q}_+ = \sqrt{\hbar \omega} \hat{I}_{21} = \frac{1}{2} \sqrt{m \omega^2} \hat{x} \hat{\sigma}_-$$

(3.21)

are the true integrals of motion, which satisfy the $\mathcal{N} = 2$ Poincaré superalgebra

$$[\hat{H}, \hat{Q}_\pm] = 0, \quad \{\hat{Q}_-, \hat{Q}_+\} = \hat{H}.$$  

(3.22)

The ground state of the system is non-degenerate, and all the excited states are doubly degenerate.

The inclusion of the integral $\sigma_3$ and second order ladder operators (3.19) extends the Poincaré supersymmetry to the $\mathfrak{osp}(2, 2)$ superalgebra, generating in the process the dynamical integrals $\hat{S}_- = \frac{1}{2} \sqrt{\text{mech}} \hat{p} \sigma_-$ and $\hat{S}_+ = \frac{1}{2} \sqrt{m \omega^2} \hat{x} \sigma_+$. By adding the first order ladder operators (3.17), one gets the super-Schrödinger symmetry, where the fermionic ladder operators $\sigma_\pm$ are also included. Accordingly, the application of the CBT to this $\mathcal{PT}$-symmetric model yields the usual harmonic super-oscillator system together with its entire symmetry superalgebra [77].
This model can be compared with supersymmetric generalizations of the Swanson model already treated in the literature by changing $\omega^2 \rightarrow \Omega_{\alpha,\beta}^2 = \omega^2 - 4\alpha\beta > 0$ and considering the system of units $m = \omega = \hbar = 1$ for a moment. By applying the similarity transformation $\hat{T}_a(\mathcal{H})\hat{T}_a^{-1}$ to (3.5) with $N = 2$, where $\hat{T}_a = \hat{T}_a^{-1}\hat{C}_{\Omega_{a,\beta}}\mathbb{I}_{2 \times 2}$, and $\hat{T}_a$ are defined in Appendix A in dependence on whether $\alpha + \beta < 1$ ($a = 1$) or $\alpha + \beta > -1$ ($a = 2$), one gets

$$
\hat{H}_{SS} = \text{diag}(\hat{H}_+, \hat{H}_-), \quad \hat{H}_\pm = \hat{a}_\mp \hat{a}^\pm + \alpha(\hat{a}_-^2) + \beta(\hat{a}_+^2). \tag{3.23}
$$

The supersymmetric partners $\hat{H}_\pm$ have the form of those in the generalized supersymmetric Swanson models discussed in Ref. [62, 63] with superpotential $\hat{W} = \hat{x}$. The intertwining operators are obtained from the action of the corresponding similarity transformation on $\hat{x}$ and $\hat{p}$.

- **The $N = 4$ case.**

For this case, the constant matrix $\vartheta$ takes the form

$$
\vartheta = \frac{1}{2}\text{diag}(5, 3, 1, -1) = \frac{1}{2}\mathbb{I}_{4 \times 4} + \frac{1}{2}h_1 + \sqrt{\frac{2}{3}}h_2 + \sqrt{\frac{2}{3}}h_3, \tag{3.24}
$$

where the three su(4) Cartan integrals are given by $h_1 = E_{11} - E_{22}$, $h_2 = \frac{1}{\sqrt{3}}(E_{11} + E_{22} - 2E_{33})$ and $h_3 = \frac{1}{\sqrt{3}}(E_{11} + E_{22} + E_{33} - 3E_{44})$. The dynamical integrals $E_{j,k}$ with $j, k = 1, 2, 3, 4$, $j \neq k$, are used to construct the true integrals $I_{\pm}^{(j,k)}$ and $I_{\pm}^{(j,k)}$. The $\mathbb{Z}_2$-grading operator

$$
\Gamma = \text{diag}(1, -1, 1, -1) = h_1 - \sqrt{\frac{2}{3}}h_2 + \sqrt{\frac{2}{3}}h_3 \tag{3.25}
$$

identifies the set of eight true integrals of motion $\mathcal{F} = (\hat{I}_{\pm}^{(1,2)}, \hat{I}_{\pm}^{(1,4)}, \hat{I}_{\pm}^{(2,3)}, \hat{I}_{\pm}^{(3,4)})$ as fermionic generators (supercharges), from which only $\hat{I}_{\pm}^{(1,4)}$ are of order three, and the rest are of order one operators in $\hat{p}$ and $\hat{x}$. Eight generators $\mathcal{B} = (\hat{\mathcal{H}}_{\mathcal{P}T}, h_1, h_2, h_3, \hat{I}_{\pm}^{(1,3)}, \hat{I}_{\pm}^{(2,4)})$ are the true bosonic integrals of motion. The entire superalgebra of this system is nonlinear due to the presence of the higher order operators. The ground state is singlet, while the first and the second excited energy levels have degeneracies two and three, respectively. Degeneracy of all other excited states is equal to four.

The inclusion of the first and second order ladder operators significantly increases the number of generators in this case, and essentially complicates the structure of the corresponding nonlinear superalgebra. Instead of the detailed analysis of its structure, we just note that the computation of the commutator between $\hat{I}_{\pm}^{(1,4)}$ and $\hat{\mathcal{J}}^\pm$ generates the four odd dynamical integrals $\hat{\mathcal{J}}^-E_{14}$, $\hat{\mathcal{J}}^+E_{14}$, $(\hat{\mathcal{J}}^-)E_{14}$ and $(\hat{\mathcal{J}}^+)E_{14}$. Then, the inclusion of the second order operators as well extends and complicates further the nonlinear superalgebraic structure.

### 3.2 Higher order $N = 2$ superextension

Let us consider again the case $N = 2$ and introduce the generalized model

$$
\hat{\mathcal{H}}_\gamma = 2i\omega \hat{D} + \frac{1}{2}\gamma \hbar \omega \sigma_3 + \frac{\omega \hbar}{2}(\gamma - 1) = \begin{pmatrix} \hat{H}_{\mathcal{P}T} + \hbar \omega (\gamma - \frac{1}{2}) & 0 \\ 0 & \hat{H}_{\mathcal{P}T} - \hbar \omega (\gamma - \frac{1}{2}) \end{pmatrix}, \tag{3.26}
$$

10
where $\gamma$ is, in principle, an arbitrary numerical parameter. Note that we have a copy of $\hat{H}_{PT}$ in each subsystem in (3.26), but they occur with a relative energy offset equal to $\hbar \omega \gamma$. Taking into account this shift and the nature of the spectrum of $\hat{H}_{PT}$, governed by a non-negative integer as it is shown in (2.10), the system $\hat{H}'_\gamma$ will be supersymmetric only when $\gamma = l \in \mathbb{Z}$, and without loss of generality we assume in the following $l \geq 0$.

The application of the CBT produces $\hat{\mathcal{S}}(\hat{H}_l)\hat{\mathcal{S}}^{-1} = \hat{H}_l$, where

$$\hat{H}_l = \hat{H}_\text{osc} \chi_{N \times N} + \frac{1}{2} \hbar \omega l \sigma_3 + \frac{\hbar \omega}{2} (l - 1) \left( \begin{array}{cc} \hat{H}_\text{osc} + \hbar \omega (l - \frac{1}{2}) & 0 \\ 0 & \hat{H}_\text{osc} - \hbar \omega \frac{l}{2} \end{array} \right).$$

(3.27)

When $l = 1$, operator $\hat{H}_1$ takes the form of the usual super-extended harmonic oscillator system, while for $l = 2, \ldots$, (3.27) corresponds to a higher-order supersymmetric extension of the model which is characterized by nonlinear supersymmetry [78]. Finally, in the case $l = 0$, one obtains the “order zero” supersymmetric extension [79, 77].

The eigenstates of (3.26) with $\gamma = l$ are provided by equations (3.9) with $N = 2$. However, it is convenient to present them in the basis

$$\Phi_{j}^{(0,l)}(x) = \left( \begin{array}{c} 0 \\ \phi_j(x) \end{array} \right), \quad j = 0, 1 \ldots, l - 1, \quad \Phi_{n+l}^{(\pm,l)} = \frac{1}{\sqrt{2}} \left( \phi_{n+l}(x) \pm \phi_n(x) \right),$$

(3.28)

which satisfy the eigenvalue equation

$$\hat{\mathcal{H}}_l \Phi_n^{(\lambda,j)}(x) = \hbar \omega n \Phi_n^{(\lambda,j)}(x), \quad \lambda = \pm, 0.$$  

(3.29)

From here it is clear that the value of $l$ is equal to the number of non-degenerate, singlet states in the system. Note that in the case $l = 0$, we just have two copies of the same operator $\hat{H}_{PT}$, and all energy levels $\hbar \omega n$ in its spectrum are doubly degenerate, with corresponding eigenstates given by $\Phi_n^{(\pm,0)}$.

Similar to the case $N = 2$ analyzed in the previous section, the only true fermionic integrals of motion of the system (3.26) with $l \geq 2$ are the two supercharges, but in this case their structure is given in terms of the higher order operators,

$$\hat{Q}_l^{-} = \frac{1}{2} (\sqrt{\hbar \omega n}) \hat{\sigma}_+, \quad \hat{Q}_l^{+} = \frac{1}{2} (\sqrt{\hbar \omega n}) \hat{\sigma}_-, \quad [\hat{\mathcal{H}}_l, \hat{Q}_l^{\pm}] = 0.$$  

(3.30)

These integrals are mapped by the CBT into the higher order supercharges of the $N = 2$ super-extended harmonic oscillator system,

$$\hat{\mathcal{S}}(\hat{Q}_l^{\pm})\hat{\mathcal{S}}^{-1} = \hat{Q}_l^{\pm}, \quad \hat{Q}_l^{\pm} = \frac{1}{2} (\hat{a}^{\pm}) \hat{\sigma}_\mp.$$  

(3.31)

It is convenient to present operators (3.30) in the pseudo-Hermitian basis (in the sense of (2.18))

$$\hat{Q}_l^{(1)} = \hat{Q}_l^{-} + \hat{Q}_l^{+} = \left( \begin{array}{c} 0 \\ (\sqrt{\hbar \omega n}) \hat{\sigma}_- \end{array} \right), \quad \hat{Q}_l^{(2)} = -i \sigma_3 \hat{Q}_l^{(1)},$$

(3.32)

since in this way we have their action on eigenstates (3.28) in a simple form

$$\hat{Q}_l^{(1)} \Phi_{n+l}^{(\pm,j)} = \pm \sqrt{(n+1) \ldots (n+l)} \Phi_{n+l}^{(\pm,j)}, \quad \hat{Q}_l^{(2)} \Phi_{n+l}^{(\pm,j)} = 0,$$

(3.33)

$$\hat{Q}_l^{(1)} \Phi_{n+l}^{(0,j)} = \pm i \sqrt{(n+1) \ldots (n+l)} \Phi_{n+l}^{(\mp,j)}.$$  

(3.34)
As the odd operators are the true integrals, and together with the corresponding Hamiltonian annihilate the singlet ground state $Φ^0_0$ (along with all other singlet states), we conclude that for a given $l \geq 2$, the system (3.26) is in the exact supersymmetric phase. The anti-commutator of supercharges produces a polynomial of order $l$ in the corresponding Hamiltonian operator,

$$\{Q_l^{(a)}, Q_l^{(b)}\} = 2δ_{ab} \Pi_{j=0}^{l-1} \left( \frac{1}{\hbar \omega} \mathcal{H} - j \right).$$  \hspace{1cm} (3.35)

Let us extend the picture by looking for the dynamical, with respect to the Hamiltonian $\mathcal{H}$, integrals of motion. These models also have the first and the second order ladder operators (3.17) and (3.19). The only difference appears here in the second commutation relation in (3.20), where $\vartheta$ has to be replaced by the matrix operator $\frac{1}{2}(l(σ_3 + 1) − 1)$. Together with them, we have the R-symmetry generator $\mathcal{R} = \frac{1}{2}σ_3$ and the higher order dynamical fermionic operators

$$\hat{S}_l^- = \frac{1}{2} (\sqrt{\frac{\hbar}{\omega}} \tilde{\vartheta})^l \sigma_-, \quad \hat{S}_l^+ = \frac{1}{2} (\sqrt{\frac{\hbar}{\omega}} \tilde{\vartheta})^l \sigma_+, \quad \hat{\Sigma}_± = \frac{1}{2} σ_±. \hspace{1cm} (3.36)$$

Some of the superalgebraic relations that these operators satisfy are

$$\{\hat{S}_l^+, \hat{S}_l^-\} = \Pi_{j=0}^{l-1} \left( \frac{1}{\hbar \omega} \mathcal{H} - \frac{\hbar}{\omega} \mathcal{R} - j \right), \quad \{\hat{S}_l^+, \hat{S}_l^\pm\} = 0 \hspace{1cm} (3.37)$$

$$[\mathcal{R}, \hat{Q}_l^\pm] = \pm \hbar \hat{Q}_l^\pm, \quad [\mathcal{H}, \hat{S}_l^\pm] = \mp \hbar \hat{S}_l^\pm, \hspace{1cm} (3.38)$$

$$[\mathcal{H}, \sigma_\pm] = \mp \hbar \omega σ_\pm, \hspace{1cm} (3.39)$$

$$\{\hat{\Sigma}_+, \hat{S}_l^\pm\} = \{\hat{\Sigma}_-, \hat{Q}_l^\pm\} = (\tilde{\vartheta}^+)^l, \quad \{\hat{\Sigma}_-, \hat{S}_l^+\} = \{\hat{\Sigma}_+, \hat{Q}_l^-\} = (\tilde{\vartheta}^-)^l, \hspace{1cm} (3.40)$$

$$\{\hat{S}_l^+, \hat{Q}_l^\pm\} = (2 \tilde{\vartheta}^+)^l, \quad \{\hat{S}_l^-, \hat{Q}_l^-\} = (2 \tilde{\vartheta}^-)^l. \hspace{1cm} (3.41)$$

The application of the CBT to the bosonic operators $\hat{\vartheta}_\pm$ and $\tilde{\vartheta}^\pm$ works as we explained in Sec. 3.1. On the other hand, the matrix operators $\mathcal{R}$ and $\Sigma_\pm$ are invariant under this mapping. The transformation of the supercharges $\hat{Q}_l^\pm$ is shown in Eq. (3.31), and for the dynamical fermionic integrals (3.36) we have

$$\tilde{\vartheta}(\tilde{\vartheta}^+) \tilde{\vartheta}^{-1} = \hat{S}_l^\pm, \quad \hat{S}_l^\pm = \frac{1}{2} (\tilde{\vartheta}^\pm)^l σ_\pm. \hspace{1cm} (3.42)$$

As noted in the previous subsection, the transformed generators produce the super-Schrödinger symmetry in the case $l = 1$, that completely describes the usual supersymmetric oscillator system, including the spectrum generating ladder operators [77]. In the case $l \geq 2$, the described set of generators produces a more complicated, nonlinearly deformed extension of the superconformal algebra, which involves some additional higher order generators, see Ref. [78].

The ideas applied in this subsection can be extended to the case of arbitrary number of subsystems by taking $\vartheta$ instead of $σ_3$ in (3.26). The appearance of even more complicated nonlinear superalgebraic structure is expected in this case.
4 CBT and two-dimensional supersymmetric systems

This section is devoted to the study of supersymmetric extensions of two-dimensional generalizations of the Swanson model (2.6). In subsection 4.1, an $\mathcal{N} = 4$ superconformal invariant 2D free particle model is obtained from the non-relativistic limit of the Dirac and Klein-Gordon equations in a $(2 + 1)$-dimensional Minkowski spacetime. In subsection 4.2, we use the dynamic symmetry generators of the resulting model to construct a two-dimensional non-relativistic $\mathcal{PT}$-invariant supersymmetric system. The latter model is related to the supersymmetric Landau problem via the rotationally invariant CBT. Two different extensions of the supersymmetric Landau problem are obtained then in subsections 4.3 and 4.4. The first of them corresponds to the inclusion of the Aharonov-Bohm flux. The second one is a modification that includes an angular momentum coupling term, and can be related with a supersymmetric extension of the exotic rotational invariant harmonic oscillator (ERIHO) studied in [57].

We employ here the notation (2.21)–(2.23) with $d = 2$ for the two-dimensional free particle and harmonic oscillator generators. However, instead of taking the Hermitian operators ($\hat{x}_i, \hat{p}_i$) and ($\hat{a}_+^\dagger, \hat{a}_-^\dagger$) as a basis, it is convenient to use

$$\hat{p}_\pm = \hat{p}_1 \pm i\hat{p}_2, \quad \hat{x}_\pm = \hat{x}_1 \pm i\hat{x}_2 \quad \text{and} \quad \hat{b}_j^\dagger = \frac{1}{\sqrt{2}}(\hat{a}_1^\dagger \mp i(-1)^j\hat{a}_2^\dagger).$$  \hspace{1cm} (4.1)

These operators are related via the CBT as follows,

$$\tilde{\mathbf{S}}(\sqrt{\frac{m}{2\hbar}}\hat{x}_+, \frac{i}{\sqrt{2\hbar m}}\hat{p}_-, \frac{i}{\sqrt{2\hbar m}}\hat{p}_+)\tilde{\mathbf{S}}^{-1} = (\hat{b}_1^\dagger, \hat{b}_1^\dagger, \hat{b}_2^\dagger, \hat{b}_2^\dagger),$$  \hspace{1cm} (4.2)

implying the pseudo-Hermitian conjugation relations ($\sqrt{\frac{m}{2\hbar}}\hat{x}_\pm)^\dagger = \frac{i}{\sqrt{2\hbar m}}\hat{p}_\pm$. The angular momentum operator $\hat{M}_{12} = \hat{x}_1\hat{p}_2 - \hat{x}_2\hat{p}_1 = \hbar(\hat{b}_1^\dagger \hat{b}_1 - \hat{b}_2^\dagger \hat{b}_2)$ is denoted as $\hat{\varphi}$.

4.1 Taking a non-relativistic limit

In a $(2 + 1)$-dimensional Minkowski spacetime, relativistic massive Dirac equation

$$i\hbar \frac{\partial}{\partial t} \Psi(t, x) = \hat{H}_{\text{Dirac}} \Psi(t, x), \quad \hat{H}_{\text{Dirac}} = c(\beta mc + \alpha_i\hat{p}_i),$$  \hspace{1cm} (4.3)

takes the form

$$i\hbar \frac{\partial}{\partial t} \Psi(t, x) = (\hat{\pi} + m\alpha_3)\Psi(t, x), \quad \hat{\pi} = \begin{pmatrix} 0 & \hat{p}_- \\ \hat{p}_+ & 0 \end{pmatrix},$$  \hspace{1cm} (4.4)

under the choice of representation $\beta = \sigma_3$, $\alpha_1 = \sigma_1$ and $\alpha_2 = \sigma_2$. Acting on equation (4.4) by $i\hbar \frac{\partial}{\partial t}$, one gets the Klein-Gordon equation

$$-\frac{\hbar^2}{c^2} \frac{\partial^2}{\partial t^2} \Psi(t, x) = (m^2c^2 + 2m\hat{\mathcal{H}})\Psi(t, x), \quad \hat{\mathcal{H}} = \hat{H}_{12} \times 2.$$  \hspace{1cm} (4.5)

The non-relativistic limit of this spin-1/2 system is obtained by taking the ansatz

$$\Psi(t, x) = e^{-\frac{i}{\hbar}(E + mc^2)t} \chi(x), \quad \chi(x) = \begin{pmatrix} \chi_1(x) \\ \chi_2(x) \end{pmatrix},$$  \hspace{1cm} (4.6)
and assuming \( E \ll mc^2 \). Then \((E + mc^2)^2 \approx m^2c^4 + 2mc^2E\), and equation (4.5) takes the form of the stationary Schrödinger equation of a two-dimensional supersymmetric free particle model

\[
\hat{H}\chi(x) = E\chi(x). \tag{4.7}
\]

Here, \(\hat{H}\) is a \(2 \times 2\) matrix Hamiltonian, whose diagonal elements are two copies of the usual two-dimensional non-relativistic free particle Hamiltonian. Taking \(\Gamma = \sigma_3\) as the \(\mathbb{Z}_2\)-grading operator, the following operators

\[
\begin{align*}
\hat{D} &= \hat{D}\mathbb{1}_{2\times 2}, & \hat{K} &= \hat{K}\mathbb{1}_{2\times 2}, & \hat{L}_0 &= \frac{1}{2}(\hat{p}_r\mathbb{1}_{2\times 2} + \frac{b}{2}\sigma_3), \\
\hat{T}_\pm &= \frac{1}{2m}(\hat{P}_\pm)^2, & \hat{X}_\pm^2 &= \frac{1}{2m}(\hat{X}_\pm)^2, & \hat{L}_\pm &= \frac{1}{2m}\hat{X}_\pm\hat{P}_\pm, \\
\hat{P}_\pm &= \hat{p}_\pm\mathbb{1}_{2\times 2}, & \hat{X}_\pm &= m\hat{x}_\pm\mathbb{1}_{2\times 2}, & \hat{M} &= m\mathbb{1}_{2\times 2},
\end{align*}
\tag{4.8}
\]

are identified as the even (bosonic) integrals of the matrix system (4.7). Among them, only \(\hat{L}_0\), \(\hat{P}_\pm\) and \(\hat{T}_\pm\) are the true integrals of motion that commute with the Hamiltonian operator, while \(\hat{M}\) is a central charge. All other generators are dynamical integrals of motion. Integrals (4.8), (4.9) generate the \(\mathfrak{sp}(4)\) symmetry with the nonzero commutators

\[
\begin{align*}
[\hat{D}, \hat{H}] &= i\hbar\hat{H}, & [\hat{D}, \hat{K}] &= -i\hbar\hat{K}, & [\hat{K}, \hat{H}] &= 2i\hbar\hat{D}, \\
[\hat{L}_0, \hat{L}_\pm] &= \pm\hbar\hat{L}_\pm, & [\hat{L}_-, \hat{L}_+] &= 2\hbar\hat{L}_0 - \hbar\hat{R}, \\
[\hat{L}_0, \hat{T}_\pm] &= \pm\hbar\hat{T}_\pm, & [\hat{L}_0, \hat{X}_\pm] &= \pm\hbar\hat{X}_\pm, \\
[\hat{L}_0, \hat{F}_\pm] &= -2i\hbar\hat{L}_\pm, & [\hat{L}_+, \hat{L}_\pm] &= -i\hbar\hat{T}_\pm, \\
[\hat{K}, \hat{T}_\pm] &= 2i\hbar\hat{L}_\pm, & [\hat{K}, \hat{X}_\pm] &= i\hbar\hat{F}_\pm, \\
[\hat{K}, \hat{F}_\pm] &= i\hbar\hat{T}_\pm, & [\hat{D}, \hat{F}_\pm] &= -i\hbar\hat{F}_\pm, \\
[\hat{F}_\pm, \hat{F}_\mp] &= \pm 4\hbar(\hat{L}_0 \pm i\hat{D}) + 2\hbar\hat{R}, \\
[\hat{L}_\pm, \hat{T}_\mp] &= 2i\hbar\hat{H}, & [\hat{L}_\pm, \hat{F}_\mp] &= 2i\hbar\hat{K}.
\end{align*}
\tag{4.11 - 4.18}
\]

The integral \(\hat{R} = \frac{b}{2}\sigma_3\) commutes with all generators (4.8)–(4.10), and as will be seen below, plays the role of the \(R\)-symmetry generator. Integrals (4.10) produce an ideal Heisenberg subalgebra,

\[
\begin{align*}
[\hat{X}_\pm, \hat{P}_\mp] &= 2i\hbar\hat{M}, \\
[\hat{H}, \hat{X}_\pm] &= -i\hbar\hat{P}_\pm, & [\hat{D}, \hat{X}_\pm] &= -\frac{i\hbar}{2}\hat{X}_\pm, & [\hat{L}_0, \hat{X}_\pm] &= \pm\frac{b}{2}\hat{X}_\pm, \\
[\hat{K}, \hat{P}_\pm] &= i\hbar\hat{X}_\pm, & [\hat{D}, \hat{P}_\pm] &= \frac{i\hbar}{2}\hat{P}_\pm, & [\hat{L}_0, \hat{P}_\pm] &= \pm\frac{b}{2}\hat{P}_\pm, \\
[\hat{T}_\pm, \hat{X}_\mp] &= -2i\hbar\hat{P}_\pm, & [\hat{F}_\pm, \hat{P}_\mp] &= 2i\hbar\hat{X}_\pm, \\
[\hat{L}_\pm, \hat{X}_\mp] &= -i\hbar\hat{X}_\pm, & [\hat{L}_\pm, \hat{P}_\mp] &= i\hbar\hat{P}_\pm.
\end{align*}
\tag{4.19 - 4.23}
\]

On the other hand, from the non-relativistic limit applied to the Dirac equation (4.4), one can read two superchargers

\[
\Pi_1 = \frac{1}{\sqrt{2m}}\hat{p}_-, \quad \Pi_2 = \frac{1}{\sqrt{2m}}i\sigma_3\hat{p}_+.
\tag{4.24}
\]

As the non-relativistic super-extended Hamiltonian \(\hat{H}\) is invariant under the transformation \(\sigma_1(\hat{H})\sigma_1\), one can identify two more superchargers,

\[
\Pi_3 = \sigma_1(\Pi_1)\sigma_1 = \frac{1}{\sqrt{2m}}\begin{pmatrix}
0 & \hat{p}_+ \\
\hat{p}_- & 0
\end{pmatrix}, \quad \Pi_4 = \sigma_1(\Pi_2)\sigma_1 = -i\sigma_3\Pi_3.
\tag{4.25}
\]
For a future application of the conformal bridge transformation, it is convenient to introduce the following non-Hermitian combinations of the operators (4.24) and (4.25),

\[
\hat{N}^{(1)}_\pm = \frac{1}{\sqrt{2m}}(\hat{N}_1 \pm i\hat{N}_2) = \frac{1}{\sqrt{2m}}\hat{p}_\pm \sigma_+ , \quad \hat{N}^{(2)}_\pm = \frac{1}{\sqrt{2m}}(\hat{N}_3 \pm i\hat{N}_4) = \frac{1}{\sqrt{2m}}\hat{p}_\pm \sigma_- .
\]

(4.26)

Then, the inclusion of these four odd true integrals together with the four dynamical generators

\[
\hat{\xi}^{(1)}_\pm = \sqrt{\frac{2}{s}}\hat{p}_\pm \sigma_+ , \quad \hat{\xi}^{(2)}_\pm = \sqrt{\frac{2}{s}}\hat{p}_\pm \sigma_- ,
\]

(4.27)

and the order zero in momenta true odd integrals \(\hat{\Sigma}_\pm = \sqrt{\frac{2}{s}}\sigma_\pm\) expands the bosonic algebra to its \(\cN = 4\) supersymmetric extension. The remaining non-vanishing superalgebraic relations are

\[
\begin{align*}
[\hat{H}, \hat{\xi}^{(j)}_\pm] &= -\hbar \hat{N}^{(j)}_\pm , \\
[\hat{\mathcal{K}}, \hat{N}^{(j)}_\pm] &= \hbar \hat{\xi}^{(j)}_\pm , \\
[\hat{L}_0, \hat{N}^{(j)}_\pm] &= \frac{\hbar}{2} \hat{N}^{(j)}_\pm , \\
[\hat{\mathcal{R}}, \hat{N}^{(j)}_\pm] &= \frac{\hbar}{2} \hat{\xi}^{(j)}_\pm , \\
[\hat{\mathcal{D}}, \hat{\xi}^{(j)}_\pm] &= -i\hbar \hat{\xi}^{(j)}_\pm , \\
[\hat{\mathcal{F}}, \hat{\xi}^{(j)}_\pm] &= i\hbar \hat{\xi}^{(j)}_\pm , \quad [\hat{\mathcal{F}}, \hat{\xi}^{(j)}_\pm] = i\hbar \hat{\xi}^{(j)}_\pm , \\
[\hat{\mathcal{F}}, \hat{\xi}^{(j)}_\pm] &= -i\hbar \hat{\xi}^{(j)}_\pm , \\
[\hat{\mathcal{F}}, \hat{\xi}^{(j)}_\pm] &= -i\hbar \hat{\xi}^{(j)}_\pm , \\
[\hat{\mathcal{F}}, \hat{\xi}^{(j)}_\pm] &= -i\hbar \hat{\xi}^{(j)}_\pm , \\
[\hat{\mathcal{F}}, \hat{\xi}^{(j)}_\pm] &= -i\hbar \hat{\xi}^{(j)}_\pm , \quad \{\hat{\xi}^{(1)}_\pm, \hat{\xi}^{(2)}_\pm\} = \{\hat{\xi}^{(2)}_\pm, \hat{\xi}^{(1)}_\pm\} = \hat{\mathcal{D}} ,
\end{align*}
\]

(4.28)

(4.29)

(4.30)

(4.31)

(4.32)

(4.33)

(4.34)

(4.35)

(4.36)

(4.37)

(4.38)

From here we note that the operators (\(\hat{H}, \hat{\mathcal{K}}, \hat{\mathcal{D}}, \hat{\mathcal{L}}_0, \hat{\mathcal{L}}_\pm, \hat{\mathcal{T}}_\pm, \hat{\mathcal{F}}_\pm, \hat{\mathcal{R}}, \hat{\mathcal{N}}^{(a)}_\pm, \hat{\xi}^{(a)}_\pm\)) with \(a = 1, 2\) satisfy the orthosymplectic \(D(1, 2) \cong \mathfrak{osp}(2, 4)\) superalgebra [80]. On the other hand, the generators (\(\hat{\xi}_\pm, \hat{\mathcal{P}}_\pm, \hat{\Sigma}_\pm, \hat{\mathcal{M}}\)) constitute an ideal sub-superalgebra, which in turn is a supersymmetric extension of the two-dimensional Heisenberg algebra with two additional fermionic ladder operators. In conclusion, the entire supersymmetry of the system is a semi-direct sum of these two superalgebras.

### 4.2 2D superextension of the Swanson model and the spin-1/2 Landau problem

To construct the Hamiltonian of a \(\mathcal{P}\mathcal{T}\)-invariant supersymmetric system, we just take a look at the anti-commutation relations (4.34) and consider the following four \(\mathcal{P}\mathcal{T}\)-invariant operators

\[
\begin{align*}
\hat{\mathcal{H}}^{(1, 1)}_{\mathcal{P}\mathcal{T}} &= 2\omega (i\hat{D} + (\hat{\mathcal{L}}_0 + \frac{1}{2}\hat{\mathcal{R}})) , \\
\hat{\mathcal{H}}^{(2, 2)}_{\mathcal{P}\mathcal{T}} &= 2\omega (i\hat{D} + (\hat{\mathcal{L}}_0 + \frac{1}{2}\hat{\mathcal{R}})) .
\end{align*}
\]

(4.39)
They satisfy relations
\[ \hat{H}_{PT}^{(\pm,1)} = - (\hat{H}_{PT}^{(\pm,2)})^\dagger, \quad \hat{H}_{PT}^{(\pm,2)} = - (\hat{H}_{PT}^{(\pm,2)})^\dagger, \]
\[ \sigma_1(\hat{H}_{PT}^{(\pm,1)})\sigma_1 = \hat{H}_{PT}^{(\pm,2)}, \]
and it is enough to identify one of them as the \( PT \)-invariant Hamiltonian. We choose
\[ \hat{H}_{PT}^{(\pm,1)} := \hat{H}_{PT}, \]
and relations (4.40), (4.41) can be used then to obtain the information for the alternative models.

As the CBT generator and the metric operator we take
\[ \hat{S} = \hat{S}_{I_{2 \times 2}}, \quad \hat{S}^{-1} = \hat{S}_{I_{2 \times 2}}, \quad \hat{\Theta} = (\hat{S})^2. \]
Operator (4.42) is mapped to the Hermitian supersymmetric Hamiltonian
\[ \hat{H}_L := \hat{S}(\hat{H}_{PT})\hat{S}^{-1} = \hat{H}_L I_{2 \times 2} - \hbar \omega \sigma_3, \]
with
\[ \hat{H}_L = \hat{H}_{osc} - \omega \hat{p}_\varphi = \hbar \omega (2\hat{b}_3^+ \hat{b}_2^- + 1), \]
see Eqs. (2.23) and (4.1). To interpret this system, we identify \( \omega \) with the Larmor frequency
\[ \omega = \frac{eB}{2mc}, \]
where \( q = -e \) is the electron charge and \( B \) is a homogeneous magnetic field. In this case, the operator \( \hat{H}_L \) takes the form of the Hamiltonian of a charged spin-1/2 non-relativistic particle with \( g = 2 \) Landé factor subjected to a homogeneous magnetic field, which is given by
\[ \hat{H}_L = \hat{H}_{osc} - \omega \hat{p}_\varphi = \hbar \omega (2\hat{b}_3^+ \hat{b}_2^- + 1), \]
where \( \mu_B = \frac{e\hbar}{2mc} \) is the Bohr magneton, and \( \hat{S}_j = \frac{\hbar}{2} \sigma_j \) is the vector spin operator. If we choose any other option in (4.39) as the Hamiltonian of the initial \( PT \)-invariant model, we get a similar system after the CBT but with a change in the sign of the electric charge or in the direction of the magnetic field, or with both changes simultaneously.

On the other hand, we note that the eigenstates of (4.42) are given by the spinors
\[ \phi_{n_1,n_2}^{(\pm)}(x_1, x_2) = \sqrt{\frac{m_\omega}{2n_1n_2}} \left( \begin{array}{c} \phi_{n_1,n_2}^0 \\ \phi_{n_1,n_2} \end{array} \right), \quad \phi_{n_1,n_2}^{(-)} = \left( \begin{array}{c} 0 \\ \phi_{n_1,n_2} \end{array} \right), \quad n_1, n_2 = 0, 1, \ldots, \]
where \( \phi_{n_1,n_2}(x_1, x_2) \) are the formal common eigenstates of the two-dimensional bosonic operators \( 2i\hat{D} \) and \( \hat{p}_\varphi \), that are mapped into the spin zero Landau problem eigenstates [57]. The basic properties that these functions satisfy and which are necessary to compute
the action of any operator on the spinor states (4.48) are summarized in Appendix B. In particular, from here we deduce the eigenvalue equations
\[
\mathcal{H}_{PT} \Phi_{n_1,n_2}^{(\pm)} = \hbar \omega (2n_2 + 1 \pm 1) \Phi_{n_1,n_2}^{(\pm)}, \quad \mathcal{L}_0 \Phi_{n_1,n_2}^{(\pm)} = \frac{\hbar}{2} (n_1 - n_2 \pm \frac{1}{2}) \Phi_{n_1,n_2}^{(\pm)}.
\] (4.50)

The spectrum of the system depends only on the quantum number \( n_2 \), and therefore is infinite degenerate in each of its energy levels, including the ground state with zero energy. Additionally, there is an obvious supersymmetry here since the states \( \Phi_{n_1,n_2+1}^{(+)} \) and \( \Phi_{n_1,n_2}^{(-)} \) have the same energy eigenvalue. Accordingly, the application of the CBT operator \( \hat{\mathcal{S}} \) to (4.48) yields the eigenstates of \( \hat{\mathcal{H}}_L \) with the same energy eigenvalues, see Appendix B.

All the true and dynamical symmetries of the system (4.42) can be identified by using the superalgebraic relations (4.11)–(4.18), (4.19)–(4.23) and (4.28)–(4.38). Therefore, the application of the CBT to them will give us the supersymmetry algebra of the spin-1/2 Landau problem (4.46). For the sake of simplicity, we restrict ourselves to look for the true integrals of motion related with the degeneracy of the spectrum. To do that, we take into account that the Hamiltonian is a complex linear combination of the generator of dilatations and the total angular momentum operator 2\( \hat{L}_0 \) + \( \mathcal{R} \) of the spin-1/2 particle. So, \( \hat{L}_0 \) and \( \mathcal{R} = \frac{\hbar}{2} \sigma_3 \) commute with \( \mathcal{H}_{PT} \). In addition to them, one finds that
\[
\mathcal{B}_+ = \sqrt{\frac{2m}{\hbar}} \hat{\mathcal{X}}_+ , \quad \mathcal{B}_- = \frac{1}{\sqrt{2m\hbar}} i \hat{\mathcal{P}}_-, \quad \mathcal{A}_+ = \frac{\hbar}{2\sigma} \hat{\mathcal{R}}_+, \quad \mathcal{A}_- = -\frac{\hbar}{2\sigma} \hat{\mathcal{R}}_-
\] (4.51)
are the true bosonic integrals of motion of \( \mathcal{H}_{PT} \). Under CBT, they are mapped into the bosonic operators
\[
\hat{\mathcal{S}}^{-1} (\mathcal{B}_+, \mathcal{B}_-), \hat{\mathcal{S}}^{-1} = (\hat{b}_1^+ \mathbb{I}_{N \times N}, \frac{1}{2} (\hat{b}_1^+)^2 \mathbb{I}_{N \times N} ),
\] (4.52)
where \( \hat{b}_1^\pm \) correspond to complex linear combinations of the integrals that represent the center of the circular orbit of the particle in the classical Landau problem. They are the first order ladder operators for the quantum number \( n_1 \) in the eigenfunctions, not appearing in the energy spectrum (4.50) \([57]\).

In order to identify the supercharges of the system, we just take a look at the first relations in (4.28) and (4.34), which with the identification
\[
\hat{\eta}_- = 2i \hat{\Pi}^{(1)} = \frac{\hbar}{\sqrt{2m}} \hat{\mathcal{P}}_+ \sigma_-, \quad \hat{\eta}_+ = 2\omega \hat{\xi}^{(1)} = \sqrt{\frac{m\omega^2}{2}} \hat{\mathcal{X}}_- \sigma_+
\] (4.53)
can be rewritten as the \( \mathcal{N} = 2 \) Poincaré superalgebra,
\[
[\mathcal{H}_{PT}, \hat{\eta}_\pm] = 0 , \quad \{ \hat{\eta}_-, \hat{\eta}_+ \} = 2 \mathcal{H}_{PT} , \quad \{ \hat{\eta}_\pm, \hat{\eta}_\pm \} = 0 .
\] (4.54)
The odd operators \( \hat{\eta}_\pm \) are recognised as the supercharges of the system, which are responsible for the supersymmetry mentioned above,
\[
\hat{\eta}_\pm \Phi_{n_1,n_2}^{(\pm)} = \sqrt{\hbar \omega (n_2 + \frac{1+\epsilon}{2})} \Phi_{n_1,n_2+\epsilon}^{(\mp)}, \quad \hat{\eta}_\pm \Phi_{n_1,n_2}^{(\pm)} = 0 .
\] (4.55)
Then under the CBT we have
\[
\hat{\mathcal{S}} (\hat{\eta}_\pm), \hat{\mathcal{S}}^{\pm} = \hat{\eta}_\pm, \quad \hat{\mathcal{S}}^{\pm} = \sqrt{\hbar \omega} \hat{b}_2^\pm \sigma_+ ,
\] (4.56)
where the operators $\sqrt{\hbar \omega} \hat{b}^\pm$ can be identified as the complex combinations of the components of the vector operator $\hat{p} - \frac{\mathbf{e}}{c} \mathbf{A}$ in the corresponding Landau problem [57].

Next, let us take a look at what happens with the other fermionic generators of the free particle system. First, due to the presence of $\hat{R}$ in the Hamiltonian $\hat{H}_{\mathcal{PT}}$, the operators $\hat{\Sigma}^\pm$ cannot be true integrals of the system. On the other hand, the application of the Hermitian conjugation and the unitary transformation generated by $\sigma_1$ to supercharges (4.53) produces

$$-(\hat{Q}_-)^\dagger = 2i \hat{\Pi}^{(1)}_+,$$

$$\sigma_1(\hat{Q}_-)\sigma_1 = 2i \hat{\Pi}^{(2)}_+,$$

$$-(\sigma_1(\hat{Q}_-)\sigma_1)^\dagger = 2i \hat{\Pi}^{(2)}_-, \\ -\sigma_1(\hat{Q}_+)\sigma_1)^\dagger = 2\omega \hat{\xi}_+(2).$$

In this way, in Eqs. (4.57), (4.58) and (4.59) there appear the remaining odd generators that come from the free particle system, and they are the supercharges of the Hamiltonians $\mathcal{H}_{\mathcal{PT}}^{(-1)}$, $\mathcal{H}_{\mathcal{PT}}^{(2)}$ and $\mathcal{H}_{\mathcal{PT}}^{(-2)}$, respectively, see (4.34). Using these arguments and the commutation relations introduced in the previous subsection, it is easy to see that none of these odd operators commutes with $\mathcal{H}_{\mathcal{PT}}$. So, all they are the dynamical integrals.

In conclusion, operators (4.51) and (4.53) are all the true symmetry generators that the system possesses, and they encode all the special properties of the spectrum. The remaining integrals are of the dynamical nature, and they work as the ladder operators. Unlike the free particle case, we have here the $N = 2$ instead of $N = 4$ supersymmetric extension as we explained above.

### 4.3 Inclusion of an Aharonov-Bohm flux

Let us consider now a spin zero electrically charged particle coupled to a curl-free in the exterior of a solenoid vector potential

$$A^\Delta \Phi = \frac{\Delta \Phi}{2\pi \rho^2}(-x_2, x_1, 0), \quad \nabla \times A^\Delta \Phi = 0, \quad \rho^2 = x_1^2 + x_2^2, \quad (4.60)$$

where $\Delta \Phi$ is the total magnetic flux inside the solenoid. The Hamiltonian operator of such a system has the form

$$\hat{H}^{(\alpha)} = \frac{1}{2m}(\hat{p}_i + \frac{e}{c} A_i^\Delta \Phi)^2 = \hat{H} + \frac{\hbar \alpha}{2m \rho^2}(\hbar \alpha + 2 \hat{p}_\varphi), \quad \alpha = \frac{\epsilon \Delta \Phi}{2\pi \hbar}. \quad (4.61)$$

As we will see below, the model is solvable for arbitrary values of the dimensionless parameter $\alpha$. However, it allows for different interpretations depending on the values of this number. The case $\alpha \in \mathbb{Z}$ appears in the study of superconductivity and quantum Hall effect, and corresponds to a quantized magnetic flux [54]. On the other hand, Hamiltonian (4.61) with arbitrary values of the parameter $\alpha$ describes the free dynamics in relative coordinate of the system of two identical anyons [52, 53, 55].

---

6This can be shown from the superalgebra presented in subsection 4.1, where one can see that each operator is an eigenstate of $\hat{D}$, $\hat{L}_0$ and $\hat{R}$ in the sense of the adjoint action $[O, \hat{A}_i] = \lambda_i \hat{A}_i$. 18
By solving the corresponding Schrödinger equation, we find the eigenstates of the system, as well as its energy eigenvalues
\[ \psi^{(\alpha)}_{\kappa,\ell} = \sqrt{\frac{\kappa}{2\pi}} J_{|\ell+\alpha|}(\kappa \rho)e^{i\ell \varphi}, \quad E_{\kappa} = \frac{\hbar^2 \kappa^2}{2m}, \quad \kappa > 0, \quad \ell = 0, \pm 1, \ldots \] (4.62)
where \( \varphi \) is the angular coordinate.

Such a model can be obtained from the usual two-dimensional free particle via the unitary singular at the origin transformation \[ \hat{U}_{\alpha} \] (4.42)

Unlike \( \hat{H} \) system \( \hat{H} \)

However, \( \hat{\alpha} \), which is different from \( \hat{\alpha} \) system \( \hat{\alpha} \),

where
\[ \hat{U}_{\alpha} = e^{-i\alpha \varphi}, \quad \hat{U}_{\alpha}^\dagger = e^{i\alpha \varphi}. \] (4.64)

Unlike \( \hat{H} \) and \( \hat{\varphi} \), operators \( \hat{K} \) and \( \hat{D} \) are invariant under this transformation, and the system \( \hat{H}_{\alpha} \) still maintains the \( \mathfrak{so}(2,1) \) conformal invariance.

Operator \( \hat{\alpha}^{(\alpha)} \) can be understood as the total angular momentum of the system, with eigenvalues \( \hbar (\ell + \alpha) \). In the same vein, the first order operators \( \hat{\varphi} \) and \( \hat{\varphi}^\dagger \) transform as
\[ \hat{U}_{\alpha}(\hat{\varphi})\hat{U}_{\alpha}^\dagger = \hat{\varphi}^{(\alpha)}, \quad \hat{U}_{\alpha}(\hat{\varphi}^\dagger)\hat{U}_{\alpha}^\dagger = \hat{\varphi}^{(\alpha)} \] (4.63)

However, \( \hat{\varphi}^{(\alpha)} \) are physical operators in the Hilbert space generated by the states (4.62) only when \( \alpha \in \mathbb{Z} \), see Appendix C. Despite these peculiarities associated with values of the parameter \( \alpha \) related to the mapping of symmetry generators, we will maintain it as an arbitrary real number for a while. Then, by considering the action of the matrix extension of this unitary transformation \( \hat{U}_{\alpha} \rightarrow \hat{U}_{\alpha} = \hat{U}_{\alpha} \otimes 2 \times 2 \) on \( \mathcal{PT} \)-symmetric Hamiltonian operator (4.42), we get
\[ \hat{H}_{\alpha}^{(\alpha)} = \hat{U}_{\alpha}(\hat{H}_{\alpha}^{(\alpha)})\hat{U}_{\alpha}^\dagger = 2\omega(i\hat{D} \mp (\hat{\varphi} + \frac{1}{2}\hat{\varphi}^\dagger)), \] (4.66)

which in effect, is just the \( \hat{H}_{\alpha}^{(\alpha)} \) shifted for the constant \( \hbar \omega \). To apply the conformal bridge transformation to the system (4.66), we use the modified operator
\[ \hat{\mathcal{J}}_{\alpha} = \hat{U}_{\alpha}(\hat{\mathcal{J}})\hat{U}_{\alpha}^\dagger = \hat{\mathcal{S}}_{\alpha} \otimes 2 \times 2, \quad \hat{\mathcal{S}}_{\alpha} = e^{i\alpha \varphi}(\hat{H}_{\alpha}^{(\alpha)} - \omega^2 \hat{R}), \] (4.67)

and its inverse. As a result we obtain the Hermitian Hamiltonian operator
\[ \hat{\mathcal{J}}_{\alpha}^{(\alpha)}(\hat{\mathcal{H}}_{\alpha}^{(\alpha)})\hat{\mathcal{J}}_{\alpha}^{-1} = \hat{H}_{\alpha}^{(\alpha)} = \hat{H}_{\alpha}^{(\alpha)} \otimes 2 \times 2 - \hbar \omega \sigma_3 := \hat{H}_{\alpha}^{(\alpha)}, \] (4.68)

where
\[ \hat{H}_{\alpha}^{(\alpha)} = \hat{U}_{\alpha}(\hat{H}_{\alpha}^{(\alpha)})\hat{U}_{\alpha} \] (4.69)

Here, (4.69) has the form of the Hamiltonian of a scalar charged particle in external magnetic field described by the total vector potential \( \mathbf{A}_T = \mathbf{A} + \mathbf{A}^{\alpha \phi} \). So, (4.68) is a

\(^7\)System (4.61) and the two-dimensional free particle are unitary equivalent iff \( \alpha \in \mathbb{Z} \). At \( \alpha \neq \mathbb{Z} \), operator \( \hat{U}_{\alpha} \) acting on the free particle eigenstate \( J_{|\ell|}(\kappa \rho)e^{i\ell \varphi} \) produces the multi-valued function \( J_{|\ell|}(\kappa \rho)e^{i(\ell + \alpha) \varphi} \) which is different from (4.62).
They satisfy the relations

\[ \hat{H}_L^{(\alpha)} = \frac{1}{2m} \left( \hat{p} + \frac{e}{c} A_T \right)^2 - \hat{\mu} \cdot \hat{B} . \]  

(4.70)

To construct the eigenstates, we look for the structure of the spinors in polar coordinates. For this, we introduce the functions

\[ \phi^{(\alpha)}_{n_{\rho}, \ell}(r, \varphi) = \mathcal{N}^{(\alpha)}_{n_{\rho}, \ell} 2^{n_{\rho} + |\ell + \alpha|} e^{i\ell \varphi}, \quad \mathcal{N}^{(\alpha)}_{n_{\rho}, \ell} = (-1)^{n_{\rho}} \frac{\left(\frac{m \omega}{2}\right)^{n_{\rho} + |\ell + \alpha|}}{\sqrt{\pi n_{\rho}! (n_{\rho} + |\ell + \alpha| + 1)}}, \]  

(4.71)

\[ n_{\rho} = 0, 1, \ldots, \ell = 0, \pm 1, \ldots . \]  

(4.72)

They satisfy the relations

\[ 2i \omega \hat{D} \phi^{(\alpha)}_{n_{\rho}, \ell}(r, \varphi) = \hbar \omega (2n_{\rho} + |\ell + \alpha| + 1) \phi^{(\alpha)}_{n_{\rho}, \ell}(r, \varphi), \]  

(4.73)

\[ \hat{p}_r \phi^{(\alpha)}_{n_{\rho}, \ell}(r, \varphi) = \hbar \ell \phi^{(\alpha)}_{n_{\rho}, \ell}(r, \varphi), \quad \frac{1}{\hbar \omega} \hat{H}^{(\alpha)} \phi^{(\alpha)}_{n_{\rho}, \ell} = \sqrt{n_{\rho}(n_{\rho} + |\ell + \alpha|)} \phi^{(\alpha)}_{n_{\rho} - 1, \ell}, \]  

(4.74)

and the last equation in (4.74) helps us to show that the corresponding CBT produces

\[ e^{\frac{\pi}{2} \hbar (\alpha - \omega^2 K)} \phi^{(\alpha)}_{n_{\rho}, \ell}(\rho, \varphi) = \psi^{(\alpha)}_{n_{\rho}, \ell}(\rho, \varphi), \]  

(4.75)

\[ \psi^{(\alpha)}_{n_{\rho}, \ell}(\rho, \varphi) = \left( \frac{m \omega}{\hbar^2} \right)^{\frac{1}{2}} \sqrt{\frac{n_{\rho}!}{\pi (n_{\rho} + |\ell + \alpha| + 1)!}} e^{i|\ell| L^{(\ell + \alpha)}_{n_{\rho}}(\rho \omega)} (\zeta^2) e^{-\frac{\rho^2}{\zeta^2}} e^{i\varphi}, \quad \zeta = \sqrt{\frac{m \omega}{\hbar}} \rho, \]  

(4.76)

where \( L^{(\beta)}_{\alpha}(\eta) \) are the generalized Laguerre polynomials. These functions are the eigenstates of the operator (4.69),

\[ \hat{H}^{(\alpha)}_{L} \psi^{(\alpha)}_{n_{\rho}, \ell}(\rho, \varphi) = \hbar \omega (2n_{\rho} + (1 - \text{sign}(\ell + \alpha))|\ell + \alpha| + 1) \psi^{(\alpha)}_{n_{\rho}, \ell}(\rho, \varphi). \]  

(4.77)

Functions (4.71) are useful to construct the spinor states of the \( \mathcal{PT} \)-invariant supersymmetric system (4.66), and to identify the real spectrum of the model,

\[ \Phi^{(+\alpha)}_{n_{\rho}, \ell}(\rho, \varphi) = \begin{pmatrix} \phi^{(\alpha)}_{n_{\rho}, \ell}(\rho, \varphi) \\ 0 \end{pmatrix}, \quad \Phi^{(-\alpha)}_{n_{\rho}, \ell}(\rho, \varphi) = \begin{pmatrix} 0 \\ \phi^{(\alpha)}_{n_{\rho}, \ell}(\rho, \varphi) \end{pmatrix}, \]  

(4.78)

\[ \mathcal{H}^{(\alpha)}_{\mathcal{PT}} \Phi^{(+\alpha)}_{n_{\rho}, \ell}(\rho, \varphi) = \hbar \omega (2n_{\rho} + (1 - \text{sign}(\ell + \alpha))|\ell + \alpha| + 1 \mp 1) \Phi^{(+\alpha)}_{n_{\rho}, \ell}(\rho, \varphi). \]  

(4.79)

All the true symmetry and supersymmetry generators of this \( \mathcal{PT} \)-invariant model with \( \alpha \in \mathbb{Z} \) can be obtained via the unitary transformation generated by \( \hat{U}_\alpha \) and applied to those of the system discussed in Sec. 4.2. For \( \alpha \notin \mathbb{Z} \), the action of the momenta operators (4.65) on states (4.71) is not well-defined as it is shown in Appendix C, and consequently linear symmetry generators are not allowed. If there exist supersymmetry generators for the non-trivial cases \( \alpha \notin \mathbb{Z} \), they have to be obtained using a different approach. We comment on this point in the last section.
4.4 Supersymmetric ERIHO system

We start this subsection by analyzing the $\mathcal{PT}$-invariant Hamiltonian operator

$$\hat{H}_\gamma = 2\omega \left( i\hat{D} + \gamma \left[ \hat{L}_0 + \frac{1}{2}\hat{R} \right] \right) - \hbar\omega(\gamma + 1), \quad (4.80)$$

where $\gamma$ is an arbitrary real parameter. In the case $\gamma = 0$ we obtain two copies of the dilatation generator, while the $\mathcal{PT}$-symmetric model (4.42) is recovered in the case $\gamma = -1$. The application of the CBT yields the model described by the Hamiltonian

$$\hat{H}_\gamma = \hat{S}(\hat{H}_\gamma)\hat{S}^{-1} = \hat{H}_{\text{osc}} + 2\gamma\omega \left[ \hat{L}_0 + \frac{1}{2}\hat{R} \right] - \hbar\omega(\gamma + 1), \quad (4.81)$$

which in matrix form is given by

$$\hat{H}_\gamma = \begin{pmatrix} \hat{H}_{\text{osc}} + \gamma\omega\hat{p}_\varphi - \hbar\omega & 0 \\ 0 & \hat{H}_{\text{osc}} + \gamma\omega\hat{p}_\varphi - \hbar\omega(2\gamma + 1) \end{pmatrix}. \quad (4.82)$$

This Hermitian Hamiltonian operator is a supersymmetric extension of the ERIHO system [57], whose spin zero version is described by the Hamiltonian operator

$$\hat{H}_\gamma = \hat{H}_{\text{osc}} + \gamma\omega\hat{p}_\varphi = \hbar\omega\left[ 1 + \gamma \right] \hat{b}_1^+ \hat{b}_1^- + (1 - \gamma) \hat{b}_2^+ \hat{b}_2^- + 1). \quad (4.83)$$

It can be related with the bosonic Landau problem subjected to an additional harmonic force, or with the harmonic oscillator system in a rigidly rotating reference frame, see [57]. As in the case $\gamma = -1$, the eigenstates of the system (4.80) are also given by Eq. (B.3), but now, they satisfy the eigenvalue equations

$$\hat{H}_\gamma \Phi^{(\gamma)}_{n_1,n_2} = E^{(\gamma)}_{n_1,n_2} \Phi^{(\gamma)}_{n_1,n_2}, \quad \hat{H}_\gamma \Phi^{(-)}_{n_1+1,n_2-1} = E^{(-)}_{n_1,n_2+1} \Phi^{(-)}_{n_1+1,n_2-1}, \quad \hat{H}_\gamma \Phi^{(-)}_{0,0} = E^{(-)}_{0,0} \Phi^{(-)}_{0,0}, \quad \hat{H}_\gamma \Phi^{(+)}_{0,0} = E^{(+)}_{0,0} \Phi^{(+)}_{0,0}, \quad E^{(+)}_{n_1,n_2} = \hbar\omega(n_1 + n_2 + \gamma(n_1 - n_2)). \quad (4.84)$$

Note that when $|\gamma| < 1$, the spectrum is bounded from below and in this case we can always redefine the Hamiltonian operator in order to have a ground energy level equal to zero when $0 < \gamma \leq 1$. Otherwise, the spectrum is not bounded from below, and even if there is a non-degenerated eigenstate with zero energy eigenvalue, this spinor cannot be identified as the ground state of the system (the model is ill-defined within the usual framework of quantum mechanics). On the other hand, from equation (4.84) we note the existence of supersymmetry for arbitrary values of $\gamma$. The fermionic (odd) integrals responsible for this phenomenon correspond to

$$\hat{I}_+ = \frac{1}{\sqrt{2}} \hat{x}_+ \hat{p}_+ \sigma_-, \quad \hat{I}_- = \frac{1}{\sqrt{2}} \hat{x}_- \hat{p}_- \sigma_+, \quad \hat{S}(\hat{I}_\pm)\hat{S}^{-1} = \hat{b}_1^+ \hat{b}_2^+ \sigma_+ := \hat{I}_\pm, \quad (4.86)$$

which simultaneously commute with $\hat{D}$ and $2\hat{L}_0 + \hat{R}$, and their action on states (B.3) yields

$$\hat{I}_+ \Phi^{(+)}_{n_1,n_2} = 2\sqrt{n_2(n_1 + 1)} \Phi^{(-)}_{n_1+1,n_2-1}, \quad \hat{I}_- \Phi^{(-)}_{n_1,n_2} = 2\sqrt{n_1(n_2 + 1)} \Phi^{(+)}_{n_1-1,n_2+1}, \quad \hat{I}_\pm \Phi^{(\pm)}_{n_1,n_2} = 0. \quad (4.87)$$

(4.88)
This supersymmetry does not enter in contradiction with the fact that we can have negative eigenvalues since the anti-commutator of supercharges $\hat{I}_\pm$ produces not the Hamiltonian (4.80) but the quadratic polynomial in it and in the even integral $2\hat{L}_0 + \hat{R}$,

$$ \{\hat{L}_-, \hat{I}_+\} = \frac{i}{\hbar} (2i\hat{D} - (2\hat{L}_0 + \hat{R}))(2i\hat{D} + (2\hat{L}_0 + \hat{R})),$$

$$ \{\hat{I}_+, \hat{L}_-\} \Phi_{n_1, n_2}^{(\pm)} = (2n_2 + 1 \mp 1)(2n_1 + 1 \mp 1)\Phi_{n_1, n_2}^{(\pm)}. $$

Note that in the case $\gamma = -1$ studied already in Sec. 4.2, the $\hat{I}_\pm$ are not independent integrals since

$$ \hat{I}_+ = \frac{1}{\sqrt{2n_0}} \hat{B}_+ \hat{Q}_- , \quad \hat{I}_- = \frac{1}{\sqrt{2n_0}} \hat{B}_- \hat{Q}_+ . $$

In Ref. [57], it was shown that the bosonic model (4.83) possesses hidden symmetry integrals of motion only when $\gamma$ is a rational number. So, the obvious question here is whether our extended model has additional fermionic integrals for particular values of this parameter. In the following, we will consider the case $0 < |\gamma| \leq 1$, and after that we study the case $|\gamma| > 1$. In both cases we obtain higher order fermionic operators whose action on the spinors (4.48) is reconstructed with the help of the basic formulas from Appendix B.

- i) The case \( \gamma = \frac{s_2 - s_1}{s_2 + s_1} \), with $s_1, s_2 = 1, 2, \ldots$, and $|\gamma| \leq 1$.

When we choose $\gamma$ in this way, one can show that the operators

$$ \hat{L}_{s_1, s_2}^+ = (\sqrt{\frac{\omega}{2m^2}} \hat{v}_+)^{s_1} (\frac{i}{\sqrt{2m^2} \hbar} \hat{p}_\sigma)^{s_2} \mathbb{I}_{2 \times 2} , \quad \hat{L}_{s_1, s_2}^- = (\frac{i}{\sqrt{2m^2} \hbar} \hat{p}_\sigma)^{s_2} (\sqrt{\frac{\omega}{2m^2}} \hat{v}_-)^{s_1} \mathbb{I}_{2 \times 2} , $$

(4.92) commute with our Hamiltonian (4.80). They together with $\hat{\mathcal{H}}_{\gamma}$ and $\hat{L}_0$ generate a nonlinear deformation of the $\mathfrak{su}(2)$ algebra in the general case [57], and by CBT generators (4.43) are transformed into

$$ \hat{\mathcal{H}} (\hat{L}_{s_1, s_2}^+, \hat{L}_{s_1, s_2}^-) = \hat{\mathcal{L}}_{s_1, s_2}^+ , \quad \hat{\mathcal{L}}_{s_1, s_2}^- = (\hat{b}_1^+)^{s_1} (\hat{b}_2^-)^{s_2} \mathbb{I}_{2 \times 2} . $$

(4.93)

The action of (4.92) on the eigenstates of the system is given by

$$ \hat{L}_{s_1, s_2}^+ \Phi_{n_1, n_2}^{(\pm)} = \sqrt{n_2!n_1!n_1(n_1 + s_1 - 1)} \Phi_{n_1 + s_2 - n_2 - s_2 + s_1 - s_2}^{(\pm)} , $$

(4.94)

$$ \hat{L}_{s_1, s_2}^- \Phi_{n_1, n_2}^{(\pm)} = \sqrt{n_2!n_1!n_2(n_2 - s_2 + 1)} \Phi_{n_1 - s_2 + n_2 + s_2 - s_1}^{(\pm)} . $$

(4.95)

To find additional supercharges for the choice of $\gamma$ under consideration, we compute the commutator of integrals (4.92) with the odd integrals $\hat{I}_{\pm}$. As a result one gets

$$ [\hat{L}_{s_1, s_2}^+, \hat{I}_+ \mp \frac{i(s_2 - s_1)}{\hbar} (2\hat{D} \mp i\frac{s_2 - s_1}{s_2 - s_1} \hat{p}_\sigma + i\hbar + 2\hbar s_1 s_2 \hat{v}_-)^{s_2 - s_1} \mathbb{I}_{2 \times 2} , \hat{I}_\pm = 0 , $$

(4.96)

where

$$ \hat{Q}_{s_1, s_2}^+ = \frac{1}{2} (\sqrt{\frac{\omega}{2m^2}} \hat{v}_+)^{s_1 - 1} (\frac{i}{\sqrt{2m^2} \hbar} \hat{p}_\sigma)^{s_2 - 1} \sigma_+ , $$

(4.97)

$$ \hat{Q}_{s_1, s_2}^- = \frac{1}{2} (\frac{i}{\sqrt{2m^2} \hbar} \hat{p}_\sigma)^{s_2 - 1} (\sqrt{\frac{\omega}{2m^2}} \hat{v}_-)^{s_1 - 1} \sigma_-, $$

(4.98)

(4.99)
which under CBT are mapped as \( \hat{\mathcal{N}}(\hat{Q}_{s_1,s_2}^\pm)\hat{\chi}^{-1} = \hat{Q}_{s_1,s_2}^+ \). \( \hat{Q}_{s_1,s_2}^\pm = \frac{1}{\sqrt{2}}(\hat{b}_1^\pm)^{s_1-1}(\hat{b}_2^\mp)^{s_2-1}\sigma^\pm \). Note that the existence of these operators assumes that \( s_1, s_2 \geq 1 \). The action of the resulting operators on the eigenstates produces

\[
\hat{Q}_{s_1,s_2}^- \Phi_{n_1,n_2}^{(+)} = \sqrt{\frac{n_1!\Gamma(n_2+s_2)}{n_2!\Gamma(n_1+s_1+1)}} \Phi_{n_1-1,n_2+s_2-1}^{(-)}; \quad \hat{Q}_{s_1,s_2}^+ \Phi_{n_1,n_2}^{(\pm)} = 0, \tag{4.100}
\]

\[
\hat{Q}_{s_1,s_2}^+ \Phi_{n_1,n_2}^{(-)} = \sqrt{\frac{n_2!\Gamma(n_1+s_1)}{n_1!\Gamma(n_2-s_2+2)}} \Phi_{n_1+s_1-1,n_2-s_2+2}^{(+)}. \tag{4.101}
\]

We do not explicitly calculate the anti-commutator between these two supercharges. Instead, let us comment on some properties of the resulting operator. First, due to the structure of (4.98), one sees that the anti-commutator has to be a polynomial operator in terms of \( \hat{x}_\pm, \hat{p}_\pm \) and \( \sigma_3 \) only. Besides, from relations (4.100) we can calculate the action of this anti-commutator on the eigenstates, which gives us

\[
\{\hat{Q}_{s_1,s_2}^+ , \hat{Q}_{s_1,s_2}^- \} \Phi_{n_1,n_2}^{(+)} = \frac{\Gamma(n_1+1)}{\Gamma(n_1-1)} \frac{\Gamma(n_2+s_2)}{\Gamma(n_2+s_2+2)} \Phi_{n_1,n_2}^{(+)}; \tag{4.102}
\]

\[
\{\hat{Q}_{s_1,s_2}^+ , \hat{Q}_{s_1,s_2}^- \} \Phi_{n_1,n_2}^{(-)} = \frac{\Gamma(n_2+1)}{\Gamma(n_2-1)} \frac{\Gamma(n_1+s_1)}{\Gamma(n_1+s_1+2)} \Phi_{n_1,n_2}^{(-)}. \tag{4.103}
\]

These equations show that the operator \( \{\hat{Q}_{s_1,s_2}^+ , \hat{Q}_{s_1,s_2}^- \} = \mathcal{O} \) is bounded from below and annihilates the eigenstates of the form \( \Phi_{n_1,n_2}^{(+)} \) with \( n_1 - s_1 + 2 \leq 0 \) and \( \Phi_{n_1,n_2}^{(-)} \) with \( n_2 - s_2 + 2 \leq 0 \). Finally, as a consequence of the relations

\[
[\mathcal{L}_0, \hat{Q}_{s_1,s_2}^\pm] = \pm \frac{1}{2} (s_1 + s_2 - 1) \hat{Q}_{s_1,s_2}^\pm, \quad [2i\hat{D}, \hat{Q}_{s_1,s_2}^\pm] = \pm \hbar (s_1 - s_2) \hat{Q}_{s_1,s_2}^\pm, \tag{4.104}
\]

\[
[\mathcal{R}, \hat{Q}_{s_1,s_2}^\pm] = \pm \hbar \hat{Q}_{s_1,s_2}^\pm, \tag{4.105}
\]

one can see that \([\mathcal{L}_0, \mathcal{O}] = [2i\mathcal{D}, \mathcal{O}] = [\mathcal{R}, \mathcal{O}] = 0 \). These relations imply that \( \mathcal{O} = \mathcal{O}(\mathcal{D}, \mathcal{R}, \mathcal{L}_0) \) since this structure is the only possible option that satisfies all the already mentioned properties by using the operators \( \hat{x}_\pm, \hat{p}_\pm \) and \( \sigma_3 \) as the building blocks.

On the other hand,

\[
[\hat{L}_{s_1,s_2}^\pm , \hat{Q}_{s_1,s_2}^\pm] = 0, \quad [\hat{L}_{s_1,s_2}^\pm , \hat{Q}_{s_1,s_2}^\mp] = P(\mathcal{H}_\gamma, \mathcal{L}_0, \mathcal{R}) \hat{I}_\pm, \quad \{\hat{Q}_{s_1,s_2}^\pm , \hat{I}_\pm\} = 2\hat{L}_{s_1,s_2}^\pm, \tag{4.106}
\]

where \( P(\mathcal{H}_\gamma, \mathcal{L}_0, \mathcal{R}) \) is another model dependent polynomial function. Therefore, we learn that the set of generators \( (\mathcal{H}_\gamma, \mathcal{L}_0, \mathcal{R}, \hat{I}_{s_1,s_2}^\pm, \hat{Q}_{s_1,s_2}^\pm) \) satisfy a nonlinear superalgebra that completely describes the degeneracy of the spectrum.

\textbf{ii) The case }\gamma = \frac{m_1 + m_2}{s_2 - s_1} \text{ with } s_1, s_2 = 2, \ldots, \text{ and } |\gamma| \geq 1 \text{ .}

Here one can show that the operators

\[
\hat{j}_{s_1,s_2}^+ = \left( \frac{\hbar}{2} \right)^{\frac{s_1+s_2}{2}} (\hat{x}_+)^s (\hat{p}_-)^{s_2}, \quad \hat{j}_{s_1,s_2}^- = \left( \frac{\hbar}{2} \right)^{\frac{s_1+s_2}{2}} (\hat{p}_+)^s (\hat{p}_-)^{s_2}, \tag{4.107}
\]

\[
\hat{\mathcal{N}}(\hat{j}_{s_1,s_2}^\pm)\hat{\mathcal{N}}^{-1} = \hat{j}_{s_1,s_2}^\pm, \quad \hat{j}_{s_1,s_2}^\pm = \left( \frac{\hbar}{2} \right)^{s_1}(\hat{b}_2^\mp)^{s_2} \hat{L}_{s_1,s_2}^\pm, \tag{4.108}
\]

commute with the Hamiltonian (4.80) for this choice of \( \gamma \). These integrals are responsible for the infinite degeneracy of each energy level since \( \hat{j}_{s_1,s_2}^\pm \) cannot annihilate any eigenstate,
Then we compute the commutators of these integrals with the odd integrals $\hat{I}_\pm$, which as a result give us four higher order fermionic integrals,

$$
[\hat{J}^\pm_{s_1,s_2}, \hat{I}_\pm] = \mp s_2 \hat{W}^\pm_{s_1,s_2}, \quad [\hat{J}^\pm_{s_1,s_2}, \hat{I}_\mp] = \mp s_1 \hat{T}^\pm_{s_1,s_2};
$$



$$
\hat{W}^+_{s_1,s_2} = (\frac{m_0}{2\hbar^2})^{s_1+s_2} (\hat{x}_+)^{s_1+1}(\hat{x}_-)^{s_2-1}\frac{\sigma_+}{2}, \quad \hat{T}^+_{s_1,s_2} = (\frac{m_0}{2\hbar^2})^{s_1+s_2} (\hat{x}_+)^{s_1-1}(\hat{x}_-)^{s_2+1}\frac{\sigma_+}{2},
$$



$$
\hat{W}^-_{s_1,s_2} = (\frac{1}{m_0\hbar})^{s_1+s_2} (\hat{p}_-)^{s_1+1}(\hat{p}_+)^{s_2-1}\frac{\sigma_-}{2}, \quad \hat{T}^-_{s_1,s_2} = (\frac{1}{m_0\hbar})^{s_1+s_2} (\hat{p}_-)^{s_1-1}(\hat{p}_+)^{s_2+1}\frac{\sigma_-}{2}.
$$

Acting on them, the CBT produces

$$
\mathcal{S}(\hat{W}^\pm_{s_1,s_2}, \hat{T}^\pm_{s_1,s_2})\mathcal{S}^{-1} = (\hat{W}^\pm_{s_1,s_2}, \hat{T}^\pm_{s_1,s_2}),
$$



$$
\hat{W}^\pm_{s_1,s_2} = \frac{1}{2}(\hat{b}^+_1)^{s_1+1}(\hat{b}^+_2)^{s_2-1}\sigma_\mp, \quad \hat{T}^\pm_{s_1,s_2} = \frac{1}{2}(\hat{b}^+_1)^{s_1-1}(\hat{b}^+_2)^{s_2+1}\sigma_\mp.
$$

Note that the existence of the operators $\hat{W}^\pm_{s_1,s_2}$ ($\hat{T}^\pm_{s_1,s_2}$) assumes that $s_2 > 0$ ($s_1 > 0$). We also note here that the case $s_2 = 0$ ($s_1 = 0$) corresponds to the case $\gamma = -1$ ($\gamma = 1$), which was analyzed in subsection 4.2.

The action of these operators produces

$$
\hat{W}^+_{s_1,s_2} \Phi^{(\pm)}_{n_1,n_2} = \sqrt{\frac{\Gamma(n_1+s_1+2)\Gamma(n_2+s_2)}{n_1!n_2!}} \Phi^{(-)}_{n_1+s_1+1,n_2+s_2-1};
$$



$$
\hat{W}^-_{s_1,s_2} \Phi^{(-)}_{n_1,n_2} = \sqrt{\frac{n_1!n_2!}{\Gamma(n_2+s_2+2)}} \Phi^{(\pm)}_{n_1+s_1-1,n_2+s_2+1};
$$



$$
\hat{T}^+_{s_1,s_2} \Phi^{(\pm)}_{n_1,n_2} = \sqrt{\frac{\Gamma(n_1+s_1+2)\Gamma(n_2+s_2+2)}{n_1!n_2!}} \Phi^{(-)}_{n_1+s_1+1,n_2+s_2+1};
$$



$$
\hat{T}^-_{s_1,s_2} \Phi^{(-)}_{n_1,n_2} = \sqrt{\frac{n_1!n_2!}{\Gamma(n_2+s_2+2)}} \Phi^{(\pm)}_{n_1+s_1-1,n_2+s_2+1}.
$$

We do not compute the anti-commutators, which are essentially nonlinear. Instead of that, we consider the action of the anti-commutators on the eigenstates,

$$
\{\hat{W}^+_{s_1,s_2}, \hat{W}^-_{s_1,s_2}\} \Phi^{(+)}_{n_1,n_2} = \frac{\Gamma(n_1+s_1+2)\Gamma(n_2+s_2)}{n_1!n_2!} \Phi^{(+)}_{n_1,n_2};
$$



$$
\{\hat{W}^+_{s_1,s_2}, \hat{W}^-_{s_1,s_2}\} \Phi^{(-)}_{n_1,n_2} = \frac{n_1!n_2!}{\Gamma(n_2+s_2+2)} \Phi^{(-)}_{n_1,n_2};
$$



$$
\{\hat{T}^+_{s_1,s_2}, \hat{T}^-_{s_1,s_2}\} \Phi^{(-)}_{n_1,n_2} = \frac{\Gamma(n_1+s_1+2)\Gamma(n_2+s_2+2)}{n_1!n_2!} \Phi^{(-)}_{n_1,n_2};
$$



$$
\{\hat{T}^+_{s_1,s_2}, \hat{T}^-_{s_1,s_2}\} \Phi^{(+)}_{n_1,n_2} = \frac{n_1!n_2!}{\Gamma(n_2+s_2+2)} \Phi^{(+)}_{n_1,n_2}.
$$

One can note that even if the eigenstate has a negative energy level, there is no negative eigenvalues in these four equations. This implies that the explicit form of the commutators have to be, similarly to the case $|\gamma| < 1$, a polynomial of the bosonic operators that does not have negative eigenvalues.
Some of the superalgebraic relations that describe the system are

\[
\begin{align*}
\{\hat{W}^{\pm}_{s_1 s_2}, \hat{T}^{\pm}_{s_1 s_2}\} &= (\hat{J}^{\pm}_{s_1 s_2})^2, & \{\hat{W}^{\pm}_{s_1 s_2}, \hat{T}^{\pm}_{s_1 s_2}\} &= 0, \\
\{\hat{W}^{\pm}_{s_1 s_2}, \hat{I}_{\pm}\} &= P_1(\hat{D}, \hat{L}_0, \hat{R})\hat{J}^{\pm}_{s_1 s_2}, & \{\hat{T}^{\pm}_{s_1 s_2}, \hat{I}_{\pm}\} &= P_2(\hat{D}, \hat{L}_0, \hat{R})\hat{J}^{\pm}_{s_1 s_2}, \\
[\hat{J}^{\pm}_{s_1 s_2}, \hat{W}^{\pm}_{s_1 s_2}] &= P_3(\hat{D}, \hat{L}_0, \hat{R})\hat{I}_{\pm}, & [\hat{J}^{\pm}_{s_1 s_2}, \hat{T}^{\pm}_{s_1 s_2}] &= P_4(\hat{D}, \hat{L}_0, \hat{R})\hat{I}_{\pm}, \\
[2i\hat{D}, \hat{W}^{\pm}_{s_1 s_2}] &= \pm \hbar(s_1 + s_2)\hat{W}^{\pm}_{s_1 s_2}, & [2i\hat{D}, \hat{T}^{\pm}_{s_1 s_2}] &= \pm \hbar(s_1 + s_2)\hat{T}^{\pm}_{s_1 s_2}, \\
[\hat{L}_0, \hat{W}^{\pm}_{s_1 s_2}] &= \pm \frac{1}{2}(s_1 - s_2 + 1)\hat{W}^{\pm}_{s_1 s_2}, & [\hat{L}_0, \hat{T}^{\pm}_{s_1 s_2}] &= \pm \frac{1}{2}(s_1 - s_2 - 1)\hat{T}^{\pm}_{s_1 s_2}, \\
[\hat{R}, \hat{W}^{\pm}_{s_1 s_2}] &= \mp \hbar\hat{W}^{\pm}_{s_1 s_2}, & [\hat{R}, \hat{T}^{\pm}_{s_1 s_2}] &= \pm \hbar\hat{T}^{\pm}_{s_1 s_2},
\end{align*}
\]

where \(P_k(\hat{D}, \hat{L}_0, \hat{R}), k = 1, 2, 3, 4\), are polynomial functions. From relations (4.128)–(4.130) one can verify that the fermionic operators \(\hat{W}^{\pm}_{s_1 s_2}\) and \(\hat{T}^{\pm}_{s_1 s_2}\) are integrals of motion of the system.

In conclusion of this section we note that all the 2D super-extended \(\mathcal{PT}\)-symmetric systems and their Hermitian analogs produced by applying the CBT that we have considered in this section are superintegrable. In particular case of the supersymmetrically extended Landau problem each of its two superpartners has four true integrals of motion. In the Hermitian version, these are the corresponding Hamiltonian, angular momentum, and the non-commuting operators being quantum analogs of the classical coordinates of the center of the circular orbit described by \(b^+_1\) and \(b^-_1\). These four integrals, however, are functionally dependent, while the Hamiltonian, angular momentum, and one of the two last integrals or their linear combination form a set of three independent integrals of motion. In the cases i) and ii) of the super-extended ERIHO systems we have a similar picture with components of the diagonal bosonic integrals (4.92) and (4.107), respectively, which play a role analogous to \(b^+_1\) and \(b^-_1\).

## 5 Discussion and Outlook

In the light of the presented results, we list here some problems related to them that may be interesting for a future research.

1. According to the Riemann hypothesis, nontrivial zeroes of the Riemann zeta function lie on the critical line and have the form \(\frac{1}{2} + iE_n\) with \(E_n \in \mathbb{R}\). The Hilbert–Pólya conjecture argues that the real values \(E_n\) correspond to eigenvalues of a self-adjoint operator. An important result that gave credibility to this conjecture was obtained by Selberg [81], who established a bridge (a kind of duality) between the quantum spectrum of the Laplacian on compact Riemann surfaces of constant negative curvature and the length spectrum of their prime geodesics. Selberg trace formula, which establishes that link, strongly resembles Riemann explicit formula. The next step was put forward by Berry who proposed the Quantum Chaos conjecture, according to which the Riemann zeros are the spectrum of a Hamiltonian obtained by quantization of a classical chaotic Hamiltonian, whose periodic orbits are labelled by the prime numbers [82]. In 1999 Berry and Keating [70, 71], and Connes [72] suggested that a spectral realization of the Riemann zeros could be achieved by quantizing the 1D classical Hamiltonian \(H = xp\), which generates dilatations in the phase
space \((x, p)\). Remarkably, generator of dilatations also plays a key role in constructing the compact hyperbolic Riemann surfaces of a constant curvature and geodesics on them from the hyperbolic Lobachevsky plane \([83, 84]\).

Different quantization schemes of the classical system \(H = xp\) were considered in the literature in the context of the Hilbert-Pólya conjecture, see, e.g. \([76]\) and further references therein. Probably, Bender and collaborators came closest to the solution of the problem in \([74, 75]\) by employing the theory and ideas of the \(\mathcal{PT}\)-symmetry. In their construction, the non-Hermitian operator (natural units are assumed here)

\[
\hat{\mathbb{H}} = \hat{\Delta}^{-1} \{\hat{x}, \hat{p}\} \hat{\Delta}, \quad \hat{\Delta} = 1 - e^{-i\hat{p}}, \quad x \in \mathbb{R}^+, \quad (5.1)
\]

is employed as an alternative quantization prescription of the Berry-Keating (B-K) Hamiltonian. Its eigenfunctions \(\psi_n(x), x \in (0, \infty),\) subject to the boundary condition \(\psi_n(x) = 0,\) are characterized by eigenvalues \(\{E_n\}\) such that \(\{\frac{1}{2}(1 - iE_n)\}\) are the nontrivial zeros of the Riemann zeta function in agreement with the B-K conjecture. On the other hand, the trivial zeros \(z = -2n, n = 1, 2, \ldots,\) of the Riemann zeta function are associated with real eigenvalues of the operator \(i\hat{\mathbb{H}}\) characterized by the unbroken \(\mathcal{PT}\)-symmetry. This can be seen from the explicit form of the eigenstates they use, which are of the form

\[
\psi_\zeta(x) = \hat{\Delta}^{-1} x^{-\zeta} = -\zeta(z, x + 1), \quad \hat{\mathbb{H}}\psi_\zeta(x) = i(2z - 1)\psi_\zeta(x), \quad (5.2)
\]

where \(\zeta(z, a)\) is the Hurwitz function that reduces to the Riemann zeta function when \(a = 1.\)

Note that the operator \((5.1)\) is (up to a numerical factor and a multiplication by \(-i\)) our \(\mathcal{PT}\)-symmetric Hamiltonian \((2.6)\) subjected to a similarity transformation generated by \(\hat{\Delta}\) and restricted to the half-axis. To relate these ideas with our constructions of Sec. 2, we realize the \(\mathfrak{so}(2, 1) \cong \mathfrak{sl}(2, \mathbb{R})\) symmetry generators as

\[
\hat{H}_\nu = \frac{1}{2} \hat{p}^2 + \frac{\nu(\nu + 1)}{x^2}, \quad \hat{D} = \frac{1}{2} \{\hat{x}, \hat{p}\}, \quad \hat{K} = \frac{1}{2} \hat{x}^2, \quad (5.3)
\]

\[
\nu \geq -\frac{1}{2}. \quad (5.4)
\]

In this way, the CBT produces the operator \([64, 68]\)

\[
\hat{\mathcal{S}}_\nu(2i\hat{D})\hat{\mathcal{S}}^{-1}_\nu = \frac{1}{2} \left( p^2 + \frac{\nu(\nu + 1)}{x^2} + x^2 \right) = \hat{H}_\nu^{\text{osc}}, \quad \hat{\mathcal{S}}_\nu = \exp \left( \frac{i}{2}(\hat{H}_\nu - \hat{K}) \right), \quad (5.5)
\]

corresponding to the harmonically confined version of the one-dimensional conformal mechanics model. As a consequence, one finds eigenvalues and relations between corresponding eigenstates,

\[
2i\hat{D}\phi_{n, \nu} = E_{n, \nu}\phi_{n, \nu} \Rightarrow \hat{H}_\nu^{\text{osc}}\psi_{n, \nu} = E_{n, \nu}\psi_{n, \nu}, \quad E_{n, \nu} = 2n + \nu + \frac{3}{2}, \quad (5.6)
\]

\[
\phi_{n, \nu} = x^{\nu+1+2n} \Rightarrow \psi_{n, \nu}(x) \propto \hat{\mathcal{S}}_\nu\phi_{n, \nu} \propto x^{\nu+1}L_n^{(\nu+\frac{3}{2})}(x^2)e^{-x^2/2}, \quad n = 0, 1, \ldots. \quad (5.7)
\]

On the other hand, since formally

\[
\hat{\mathcal{S}}_\nu \hat{\Delta}(i\hat{\mathbb{H}})\hat{\Delta}^{-1}\hat{\mathcal{S}}^{-1}_\nu = 2\hat{H}_\nu^{\text{osc}}, \quad (5.8)
\]

26
according to [74, 75] one gets that $\hat{\Delta}^{-1} \hat{\Delta}_{\nu}^{-1} \psi_{n,\nu} \propto \hat{\Delta}^{-1} x^{1+\nu+2n} = -\zeta(-2n - \nu - 1, x + 1)$ are the eigenstates of $i\hbar \hat{H}$ with eigenvalues $2E_{n,\nu}$. Then the trivial zeros of the Riemann zeta function are obtained at $x = 0$ when $\nu = -1$.

In this way, the equation (5.8) establishes a relationship between the study of the Riemann hypothesis and the model of the harmonically confined conformal mechanics, and we believe that this observation may contribute somehow to this area of research. One may also wonder if something interesting might happen for other values of the parameter $\nu$ distinguishing distinct $\mathfrak{su}(2, \mathbb{R})$ representations. This, however requires further investigation.

2. The geometric background on which the dynamics takes place can affect the physical properties of the system. In particular, the presence of explicit and hidden symmetries essentially depends on it [85, 86, 87, 88, 89]. In this context, one may wonder what forms the symmetries and supersymmetries of models like those discussed in the last section will take if we modify the space-time. For example, one can study the case of conical geometry, which formally can be obtained from our Hermitian systems (4.45) and (4.83) via the local canonical transformation [58, 66]

$$\rho \to \alpha \rho, \quad p_\rho \to \alpha^{-1} p_\rho, \quad \varphi \to \alpha^{-1} \varphi, \quad p_\varphi \to \alpha p_\varphi,$$

(5.9)

where $\alpha$ is a real parameter whose interpretation depends on its numerical value. Under this transformation, the Euclidean metric in polar coordinates $ds^2 = d\rho^2 + \rho^2 d\varphi^2$ takes the form

$$ds^2_\alpha = \alpha^2 d\rho^2 + \rho^2 d\varphi^2.$$

(5.10)

So, when $\alpha = \csc^2(\beta) > 1$, one interprets $\beta$ as the angular aperture of a cone. On the other hand, for $0 < \alpha < 1$, (5.10) can be understood as the metric of a background with a radial dislocation. Under further identification $\alpha = (1 - 4\mu c^{-2} G)^{-1}$, one can say that (5.10) represents the spatial part of a cosmic string background with a positive ($\alpha > 1$) or negative ($0 < \alpha < 1$) linear mass density $\mu$. For the bosonic cases of the planar harmonic oscillator and the ERIHO system in this space, we showed in [66] that the list of integrals of motion depends on the value of this parameter. Consequently, one can expect that these geometrical properties have to control the appearance of fermionic integrals in the supersymmetric case.

3. The following natural step is to explore generalizations of the two-dimensional picture discussed in Sec. 4 to higher dimensions. In this case one can start by looking the non-relativistic limit of the free Dirac and Klein-Gordon equations in $(d + 1)$ Minkowski space. An obvious generalization of the corresponding $\mathfrak{so}(2, 1)$ conformal symmetry operators, on which the CBT generators are based, will be

$$\hat{\mathcal{H}} = \hat{H}_{d' \times d'}, \quad \hat{\mathcal{D}} = \hat{D}_{d' \times d'}, \quad \hat{\mathcal{K}} = \hat{K}_{d' \times d'},$$

(5.11)

where $\hat{\mathcal{H}}$, $\hat{\mathcal{D}}$ and $\hat{\mathcal{K}}$ are the $d$-dimensional generators (2.22), and $d'$ is the dimension of representation of the corresponding Dirac matrices. Then the interesting point here is to find the fermionic operators by using the Clifford algebra generators as well as the momenta and position operators. We expect that the anti-commutators between the odd
generators will be similar to those in (4.34), that could help to identify possible candidates for a higher-dimensional $\mathcal{PT}$-invariant supersymmetric Hamiltonian operator. It would be interesting to identify the integrals of motion of the resulting model and their possible quantum mechanical interpretation in the light of the conformal bridge transformation.

4. Returning to the case of the Aharonov-Bohm flux addressed in Sec. 4.3, we believe that the non-existence of well-defined momenta operators for the non-trivial case $\alpha \notin \mathbb{Z}$ does not necessarily imply a lack of supersymmetry in our systems. The principal indication in this direction is the already known fact that the spinless case possesses hidden superconformal symmetry for arbitrary values of $\alpha$ [90]. That was shown by considering the reflection operator $\hat{R}$, corresponding to a rotation for angle $\pi$, as the grading operator. With respect to it, the well-defined in the integer case $\alpha = n$ operators $\hat{p}_\pm^{(\alpha)}$ are taken as the supercharges that anticommute for the Hamiltonian $\hat{H}^{(\alpha)}$. For $\alpha \neq n$, the von Neumann theory on self-adjoint extensions allowed there to identify the nonlocal operators $\hat{p}_1 + i\epsilon\hat{R}\hat{p}_2$ and $-\epsilon\hat{R}\hat{p}_1 + i\hat{R}\hat{p}_2$ as the well-defined supercharges of the system, where $\epsilon = \pm$ depends periodically on the value range of $\alpha \in \mathbb{R}$. Additionally, in the half-integer case the model has hidden supersymmetries generated by higher order nonlocal operators. On the other hand, it is also remarkable that if we restrict the particle to move on a circle, the system preserves the hidden Poincaré supersymmetry for $\alpha$ integer and half-integer as well [91]. Besides, it was shown then in [92, 77] that such hidden Poincaré supersymmetry and hidden superconformal symmetry in purely bosonic systems originate from the corresponding super-exended systems with fermion degrees of freedom by applying to them a special nonlocal unitary transformation followed by an appropriate reduction procedure. It was also observed that exotic supersymmetry appears in some self-isospectral [93, 94] and anyon [95] systems with fermion degrees of freedom under the choice of nonlocal grading operators. We speculate then that by taking into account the aspects of the theory of self-adjoint extensions used in ref. [90], and allowing nonlocality in grading operators, 1) our spin-1/2 $\mathcal{PT}$-invariant system (4.66) can be generalized to admit supersymmetry for arbitrary values of the parameter $\alpha$, and 2) that the hidden superconformal symmetry in some purely bosonic systems can be treated on the base of the $\mathcal{PT}$-symmetry and conformal bridge transformations.

5. Finally, it would be interesting to extend our construction to the relativistic case. In this context, let us return to the problem discussed in Sec. 4.2, and consider the fermionic operators (4.53) to construct the pseudo-Hermitian operator $\hat{Q} = ic\sqrt{m}\sigma_3(\hat{Q}_- + \hat{Q}_+)$. Under the conformal bridge transformation, this operator maps into

$$\hat{\mathcal{J}}(\hat{Q} + mc^2\sigma_3)\hat{\mathcal{J}}^{-1} = c\sigma_j(\hat{p}_j - \frac{\Phi}{c}\hat{A}_j) + mc^2\sigma_3, \quad \hat{A}_j = -\frac{mc}{c}c\epsilon_{jk}\hat{x}_k,$$

which corresponds to a Dirac Hamiltonian of the form (4.3), coupled to a homogeneous magnetic field with vector potential $A_j$. We could follow the logic of this article and postulate $\hat{Q} + mc^2\sigma_3$ as a first-order pseudo-Hermitian but not $\mathcal{PT}$-invariant Hamiltonian, even though this is not a usual form of a Dirac Hamiltonian since the CBT also works as a Dyson map for it. In order to have a more familiar structure, we can then consider another kind of similarity transformations to relate this operator with a Dirac Hamiltonian plus a non-Hermitian interaction term. An example of that could be

$$e^{\frac{2\Phi}{c}}(2^{-\frac{1}{2}}\hat{Q} + mc^2\sigma_3)e^{-\frac{2\Phi}{c}} = c\sigma_j\hat{p}_j + \frac{icmc}{c}\hat{x}_-\sigma_+ + mc^2\sigma_3,$$

28
which now looks like a pseudo-Hermitian Dirac Hamiltonian with a non-Hermitian interaction term. This new system can then be related to (5.12) via the composition of the transformation inverse to (5.13) and the CBT. Such an approach can be useful to generalize the ideas of Swanson and other authors on pseudo-Hermitian non-relativistic systems for the case of Dirac-like models.

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A  Equivalence through similarity transformations

Here we explicitly construct the similarity transformations that relate the $\mathcal{PT}$-symmetric operator $2i\Omega\hat{D}$ with the Swanson model (2.4) in dependence on the values of its parameters $\alpha$ and $\beta$. The construction only involves algebraic arguments, and so it is valid for any realization of the conformal generators employed in the article.

Based on algebra (2.3), we write down the relations

\[
e^{\alpha\hat{H}}\hat{D}e^{-\alpha\hat{H}} = \hat{D} - ith\hat{H}, \quad e^{b\hat{K}}\hat{D}e^{-b\hat{K}} = \hat{D} + ith\hat{K}, \quad (A.1)
\]

\[
e^{\alpha\hat{H}}\hat{K}e^{-\alpha\hat{H}} = \hat{K} - 2i\alpha\hat{D} - a^2h^2\hat{H}, \quad e^{b\hat{K}}\hat{D}e^{-b\hat{K}} = \hat{H} + 2ib\hat{H} - b^2h^2\hat{K}, \quad (A.2)
\]

\[
e^{c\hat{D}}\hat{H}e^{-c\hat{D}} = e^{ich\hat{H}}, \quad e^{c\hat{D}}\hat{K}e^{-c\hat{D}} = e^{-ich\hat{K}}, \quad (A.3)
\]

where the constant real parameters $a$, $b$, and $c$ have the dimensions $M^{-1}L^{-1}T^2$, $M^{-1}L^{-2}$ and $M^{-1}L^{-1}T$, respectively, in terms of units of mass, $M$, length, $L$, and time, $T$. Using them one finds

\[
\hat{S}_\Omega(2i\Omega\hat{D})\hat{S}_\Omega^{-1} = \hat{H} + \Omega^2\hat{K}, \quad \hat{S}_\Omega = e^{-\Omega\hat{K}}e^{\frac{\hbar}{2m}\Omega^2\ln(2)}e^{\hbar\Omega}, \quad \Omega \in \mathbb{R}^+, \quad (A.4)
\]

that is just an equivalent way to write the CBT.

Now, the generic Hamiltonian of the Swanson model (2.4) can be transformed into $\hat{H} + \Omega^2\alpha\beta\hat{K}$ with $\Omega^2_{\alpha,\beta} = (\omega^2 - 4\alpha\beta) > 0$ by employing the additional similarity transformation, whose form depends on the values of the parameters $\alpha$ and $\beta$. For this we first consider separately the cases $1) \alpha + \beta \in \mathcal{D}_1 = (-\infty, \omega)$ and $2) \alpha + \beta \in \mathcal{D}_2 = (-\omega, \infty)$, for which we introduce the operators

\[
\hat{T}_1 = e^{\frac{(\beta-\alpha)}{\omega}\hat{K}}e^{\frac{\hbar}{2m}\ln(1-\frac{(\alpha+\beta)}{\omega})}, \quad \hat{T}_2 = e^{\frac{\hbar}{2m}\ln(1-\frac{4\alpha\beta}{\omega})}e^{\frac{\hbar}{2m}\ln(1+\frac{(\alpha+\beta)}{\omega})}e^{\frac{\hbar}{2m}\ln(1)}.
\]

(A.5)

In both cases, the Swanson Hamiltonian is transformed into $\hat{H} + \Omega^2_{\alpha,\beta}\hat{K}$ with corresponding values of the parameters $\alpha$ and $\beta$, $\hat{T}_a\hat{H}_{\alpha,\beta}\hat{T}_a^{-1} = \hat{H} + \Omega^2_{\alpha,\beta}\hat{K}$, $a = 1, 2$. It is notable that both these transformations are well defined when $\alpha + \beta \in \mathcal{D}_1 \cap \mathcal{D}_2 = (-\omega, \omega)$, and they produce the same operator while acting on $2i\Omega_{a,\beta}\hat{D}$ for these values of the parameters. However, by taking the free particle realization of the conformal generators (2.2), and considering the corresponding mappings of operators $\hat{x}$ and $\hat{p}$,

\[
\hat{T}_1\hat{x}\hat{T}_1^{-1} = \sqrt{1 - \omega^{-1}(\alpha + \beta)}\hat{x}, \quad \hat{T}_1\hat{p}\hat{T}_1^{-1} = \frac{\hat{p} + im(\beta - \alpha)\hat{x}}{\sqrt{1 - \omega^{-1}(\alpha + \beta)}}, \quad (A.6)
\]

\[
\hat{T}_2\hat{x}\hat{T}_2^{-1} = \left(\frac{\omega^2 - 4\alpha\beta}{\omega^2 + \omega(\alpha + \beta)}\right)^{\frac{1}{2}}\left(\hat{x} - \frac{i}{m}\omega^{-1}(\alpha + \beta)\hat{p}\right), \quad \hat{T}_2\hat{p}\hat{T}_2^{-1} = \left(\frac{\omega^2 - 4\alpha\beta}{\omega^2 + \omega(\alpha + \beta)}\right)^{\frac{1}{2}}\hat{p}, \quad (A.7)
\]
one notes that these two transformations are not equivalent. Then, this is a particular example when the Dyson map, and so the metric operator, is not unique, compare with Ref. [48].

By combining these operators and the CBT one concludes that
\[
\hat{T}_a^{-1} \hat{\mathcal{S}}_{\Omega_{\alpha, \beta}} (2i \Omega_{\alpha, \beta} \hat{D}) \hat{\mathcal{S}}_{\Omega_{\alpha, \beta}}^{-1} \hat{T}_a = \hat{H}_{\alpha, \beta, \omega}
\]

(A.8)
is the similarity transformation which relates our \(\mathcal{P}\mathcal{T}\)-symmetric system with the Swanson model in the corresponding range of the parameters.

All the described similarity transformations can be extended to the supersymmetric cases we studied since the extra terms in the supersymmetric versions of \(\hat{H}_{\mathcal{P}\mathcal{T}}\) are conformal invariant.

### B CBT and \(\phi_{n_1, n_2}(x_1, x_2)\) functions

Functions \(\phi_{n_1, n_2}(x_1, x_2)\) given by Eq. (4.49) are mapped by the CBT into
\[
\psi_{n_1, n_2}(x_1, x_2) = \sqrt{\frac{m_\omega}{n_1 n_2}} H_{n_1, n_2} \left( \sqrt{\frac{m_\omega}{n_1}} x_1, \sqrt{\frac{m_\omega}{n_2}} x_2 \right) e^{-\frac{m_\omega}{2}} (x_1^2 + x_2^2), \tag{B.1}
\]

\[
H_{n_1, n_2}(\eta_1, \eta_2) = 2^{n_1 + n_2} \sum_{k, l=1}^{n_1, n_2} (i)^{n_1 + n_2 - l - k} H_{l+k}(\eta_1) H_{n_1 + n_2 - l - k}(\eta_2). \tag{B.2}
\]
The latter are the eigenstates of the operator \(\hat{H}_L\) satisfying the orthonormality relation
\[
\langle \psi_{n_1, n_2} | \psi_{n_1', n_2'} \rangle = \delta_{n_1 n_1'} \delta_{n_2 n_2'} \tag{57}.
\]
Application of the CBT operator \(\hat{\mathcal{S}}\) defined in (4.43) to spinors (4.48) yields
\[
\psi^{(+)}_{n_1, n_2} = \begin{pmatrix} \psi_{n_1, n_2} \\ 0 \end{pmatrix}, \quad \psi^{(-)}_{n_1, n_2} = \begin{pmatrix} 0 \\ \psi_{n_1, n_2} \end{pmatrix}, \tag{B.3}
\]
which are the eigenstates of \(\hat{H}_L\) with the eigenvalues \(\hbar \omega (2n_2 + 1 \mp 1)\).

The basic operators \(\hat{x}_\pm\) and \(\hat{p}_\pm\) act as ladder operators on functions \(\phi_{n_1, n_2}(x_1, x_2),\)
\[
\sqrt{\frac{m_\omega}{2\hbar}} \hat{x}_+ \phi_{n_1, n_2} = \sqrt{n_1 + 1} \phi_{n_1 + 1, n_2}, \quad \hat{p}_- \phi_{n_1, n_2} = \sqrt{\frac{m_\omega}{2\hbar} + n_1} \phi_{n_1, n_2}, \tag{B.4}
\]
\[
\sqrt{\frac{m_\omega}{2\hbar}} \hat{x}_- \phi_{n_1, n_2} = \sqrt{n_2 + 1} \phi_{n_1, n_2 + 1}, \quad \hat{p}_+ \phi_{n_1, n_2} = \sqrt{\frac{m_\omega}{2\hbar} + n_2} \phi_{n_1, n_2} \tag{B.5}
\]
Application of the CBT to these equalities produces the relations
\[
\hat{b}^+_1 \psi_{n_1, n_2} = \sqrt{n_1 + 1} \phi_{n_1 + 1, n_2}, \quad \hat{b}^-_2 \psi_{n_1, n_2} = \sqrt{n_1} \phi_{n_1 - 1, n_2}, \tag{B.6}
\]
\[
\hat{b}^+_2 \psi_{n_1, n_2} = \sqrt{n_2} \phi_{n_1, n_2 + 1}, \quad \hat{b}^-_1 \psi_{n_1, n_2} = \sqrt{n_2 + 1} \phi_{n_1, n_2 - 1}. \tag{B.7}
\]
From here one also notes that functions \(\psi_{n_1, n_2}\) are the eigenstates of the ERIHO Hamiltonian \(\hat{H}_L\) with eigenvalues \(\hbar \omega [\left(1 + \gamma\right)n_1 + \left(1 - \gamma\right)n_2 + 1]\).

Equations (B.4) and (B.5) are the basic formulas that allow to compute the action of any higher order operator appearing in Sec. 4.
C  Operators $\hat{p}_\pm^{(\alpha)}$

In this Appendix we explore the possibility of having well-defined momenta operators for the systems with the Aharonov-Bohm flux by looking for their action on physical eigenstates. First we compute

$$\hat{p}_\pm^{(\alpha)} \phi_{n,\ell}^{(\alpha)} = -i\hbar \kappa \sqrt{\frac{\ell+\alpha}{\ell+\alpha+1}} J_{\ell+\alpha}(\zeta) \pm \frac{\ell+\alpha}{\zeta} J_{\ell+\alpha}(\zeta) e^{i(\ell\pm 1)\varphi}$$

$$= -i\hbar \kappa \sqrt{\frac{\ell+\alpha}{\ell+\alpha+1}} \left[ (1 \pm \frac{\ell+\alpha}{\ell+\alpha+1}) J_{\ell+\alpha+1}(\zeta) \right] e^{i(\ell\pm 1)\varphi}, \quad (C.1)$$

where $\zeta = \kappa \rho$, and the second line is obtained by using the recurrence relations $\frac{2a}{\eta} J_{\beta}(\eta) = J_{\beta-1}(\eta) + J_{\beta+1}(\eta)$ and $2\frac{d}{d\eta} J_{\beta}(\eta) = J_{\beta-1}(\eta) - J_{\beta+1}(\eta)$. For any arbitrary value of $\alpha$, we choose two angular momentum quantum numbers $\ell_1$ and $\ell_2$ such that $\ell_1 + \alpha > 0$, $\ell_2 + \alpha < 0$ and $|\ell_1 + \alpha| < 1$. For these both cases one has

$$\hat{p}_-^{(\alpha)} \phi_{n,\ell_1}^{(\alpha)} = -i\hbar \kappa \sqrt{\frac{\ell_1+\alpha}{\ell_1+\alpha+1}} J_{\ell_1+\alpha-1}(\zeta) e^{i(\ell_1-1)\varphi}, \quad \hat{p}_+^{(\alpha)} \phi_{n,\ell_2}^{(\alpha)} = -i\hbar \kappa \sqrt{\frac{\ell_2+\alpha}{\ell_2+\alpha+1}} J_{\ell_2+\alpha-1}(\zeta) e^{i(\ell_2+1)\varphi}. \quad (C.2)$$

Since the index in the Bessel functions here are negative, one has that the obtained functions in (C.2) can be physical states if and only if $\alpha$ is an integer number. This follows from the relation $J_{-n}(\zeta) = (-1)^n J_n(\zeta)$, which is only available for $n \in \mathbb{Z}$. As a consequence, $\hat{p}_\pm^{(\alpha)}$ are physical operators for system $\hat{H}^{(\alpha)}$ only for integer $\alpha$.

Now we will test the possibility to implement these operators for the $\mathcal{PT}$-symmetric system (4.66). In the case $\alpha > 0$ one identifies two classes of states,

$$\phi_{n_{\rho},l}^{I(\alpha)} = N_{n_{\rho},l}^{(\alpha)} \rho^{2n_{\rho} + \alpha - l} e^{il\varphi}, \quad l = -[\alpha], -[\alpha] + 1, \ldots, \quad (C.3)$$

$$\phi_{n_{\rho},j}^{I(\alpha)} = N_{n_{\rho},j}^{(\alpha)} \rho^{2n_{\rho} + \alpha - j} e^{ij\varphi}, \quad j = -[\alpha] - 1, \ldots, \quad (C.4)$$

where $[\alpha]$ is the integer part of $\alpha$. The action of $\hat{p}_-^{(\alpha)}$ on the state $\phi_{n_{\rho},-[\alpha]}^{I(\alpha)}$ yields

$$\hat{p}_-^{(\alpha)} \phi_{n_{\rho},-[\alpha]}^{I(\alpha)} = -2i\hbar N_{n_{\rho},-[\alpha]}^{(\alpha)} (n_{\rho} + \alpha - [\alpha]) \rho^{2n_{\rho} + \alpha - [\alpha] - 1} e^{-(\alpha + 1)} e^{i(l\varphi + \alpha\varphi)}. \quad (C.5)$$

In the case in which $\alpha = [\alpha]$, the expression on the right hand side of this last equation becomes $\phi_{n_{\rho},-[\alpha]}^{I(\alpha)}(\alpha+1)$, and we have a transition between these two classes of eigenstates. For non-integer $\alpha$, however, the obtained functions do not correspond to what we understood as physical states: in the case $n_{\rho} = 0$ they are singular at zero. As a consequence, the action of conformal bridge transformation on such states will produce $\sim \rho^{-\alpha-1} e^{-\frac{4\alpha}{\hbar}} e^{-(\alpha + 1)} e^{i(l\varphi + \alpha\varphi)}$, that does not coincide with the form of physical eigenstates (4.76) with $n_{\rho} = 0$ and $l = -[\alpha] - 1$ of the Landau problem in the presence of the Aharonov-Bohm flux.

For $\alpha = -\beta$, $\beta > 0$ (that includes in this way the anyonic case) we also divide the eigenstates in two classes:

$$\phi_{n_{\rho},l}^{I(-\beta)} = N_{n_{\rho},l}^{(-\beta)} \rho^{2n_{\rho} + l - \beta} e^{il\varphi}, \quad l = [\beta], [\beta] + 1, \ldots, \quad (C.6)$$

$$\phi_{n_{\rho},j}^{I(-\beta)} = N_{n_{\rho},j}^{(-\beta)} \rho^{2n_{\rho} + j - \beta} e^{ij\varphi}, \quad j = [\beta] - 1, \ldots, \quad (C.7)$$

and the action of $\hat{p}_-^{(-\beta)}$ on $\phi_{n_{\rho},-[\beta]}^{I(-\beta)}$ leads to a conclusion similar to that in the case $\alpha > 0$.

So, one concludes that $\hat{p}_\pm^{(\alpha)}$ are physical operators only for $\alpha \in \mathbb{Z}$.
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