Self-similar wave breaking in dispersive Korteweg-de Vries hydrodynamics

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We discuss the problem of breaking of a nonlinear wave in the process of its propagation into a medium at rest. It is supposed that the profile of the wave is described at the breaking moment by the function \((-x)^{1/n}\) \((x<0, \text{ positive pulse})\) or \(-x^{1/n}\) \((x>0, \text{ negative pulse})\) of the coordinate \(x\). Evolution of the wave is governed by the Korteweg-de Vries equation resulting in formation of a dispersive shock wave. In the positive pulse case, the dispersive shock wave forms at the leading edge of the wave structure, and in the negative pulse case at its rear edge. The dynamics of dispersive shock waves is described by the Whitham modulation equations. For power law initial profiles, this dynamics is self-similar and the solution of the Whitham equations is obtained in a closed form for arbitrary \(n > 1\).

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Wave breaking is a universal phenomenon which takes place in evolution of nonlinear waves in various media. In an idealized situation, when one neglects the dissipative and dispersive effects, it leads to formation of multi-valued solutions of the equations that describe evolution of the wave. Such a non-physical behavior is remedied by taking into account the viscosity or dispersion, so that the multi-valued region is replaced by a viscous or dispersive shock wave (DSW). If the nonlinear and dispersive effects are considered in the leading approximation, then the wave evolution is typically governed by the Korteweg-de Vries equation. In this paper, we consider DSWs which are generated after wave breaking of initial pulses with power-law profiles. The advantage of this particular class of initial data is that, on one hand, it approximates an arbitrary enough pulse near its edge and, on the other hand, the complete solution of the Whitham equations that govern evolution of the DSW can be obtained in a closed analytical form. This enables one to obtain elementary formulae for such important characteristics of DSWs as velocities of their edges accessible for experimental measurements.

I. INTRODUCTION

As is known, nonlinear waves can “break”, or suffer from “gradient catastrophe”, if one does not take into account such effects as viscosity or dispersion. This means that after some critical moment of time a formal solution of corresponding evolution equations becomes multi-valued. In compressible fluid dynamics, introduction into the theory of such irreversible processes as viscosity and heat conductivity permitted one to formulate a consistent theory of shock waves which has found many applications (see, e.g., [1, 2]). In framework of this theory, the multi-valued region is replaced by a narrow shock layer within which irreversible processes dominate. In typical situations, the width of this layer has the order of magnitude about a mean free path of molecules in the gas under consideration, and in the macroscopic description of continuous medium dynamics such a layer can be treated as a discontinuity in distributions of density, flow velocity and other physical parameters of the medium.

However, there is another possible way to overcome the difficulty of appearance of multi-valued solutions in wave dynamics. Namely, in many physical situations the dissipative effects are negligibly small compared with dispersion effects. In classical physics, such a situation was considered in the theory of “undular bores” in shallow water waves theory (see, e.g., [3]). The generality of this situation was underlined by Sagdeev [4] who indicated that in media with dispersion the macroscopic nonlinear wave structures are generated after the wave breaking moment, and this structure joins two flows with different values of physical parameters. At present, such wave structures are called dispersive shock waves and there exists considerable literature devoted to their theory (see, e.g., the review [5]). Typically, a dispersive shock wave (DSW) occupies some finite region which expands with time and consists of intensive nonlinear wave oscillations. At one its edge the DSW can be considered as a sequence of solitons and at another edge it degenerates into a linear harmonic wave propagating with a certain group velocity depending on the physical situation under consideration. The fundamental theoretical approach to the theoretical description of DSWs was suggested by Gurevich and Pitaevskii [6] and it was based on Whitham’s theory of modulations [7]. In this approach, the DSW is represented as a modulated periodic wave of the nonlinear wave equation and slow evolution of the modulation parameters is governed by the Whitham modulation equations obtained by averaging the conservation
laws over fast oscillations of the physical parameters in the wave. This idea was realized by Gurevich and Pitaevskii for waves whose evolution is described by the Korteweg-de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0,$$ (1)

which is derived in many typical situations in framework of the perturbation theory in the limit of long enough wavelength and small enough (but finite) wave amplitude.

Two typical problems about wave breaking were discussed by Gurevich and Pitaevskii in Ref. [6]. First, they gave complete analytical solution of the “dam-breaking” problem when the initial distribution of $u_0(x)$ has a step-like form,

$$u_0(x) = \begin{cases} 1, & x \leq 0, \\ 0, & x > 0. \end{cases}$$ (2)

Second, they found the main characteristics of the DSW near the generic wave breaking point when the initial distribution can be approximated at the wave breaking moment by a cubic parabola

$$u_0(x) = (-x)^{1/3}.$$ (3)

In this case, Gurevich and Pitaevskii found velocities of the edged of the DSW and wave amplitudes near the edges. Later, the full analytic solution of this problem was obtained in Ref. [8] with the use of the inverse scattering transform method applied to the KdV equation (see also [9, 10]).

However, as is known (see, e.g., [1]), if the wave propagates into medium at rest, then the wave profile at the wave breaking moment differs from (3) and, depending on the polarity of the pulse, can be approximated either by the function

$$u_0(x) = \begin{cases} (-x)^{1/n}, & x \leq 0, \\ 0, & x > 0. \end{cases}$$ (4)

or by the function

$$u_0(x) = \begin{cases} 0, & x \leq 0, \\ -x^{1/n}, & x > 0. \end{cases}$$ (5)

that is the wave amplitude vanishes either at the leading wave front, or at the rear edge, according to the power law with $n > 1$, and it equals to zero in the quiescent region of the space. This problem with a positive polarity was reduced in Ref. [11] in framework of the Gurevich-Pitaevskii method to solution of a certain ordinary differential equation for the dependence of the modulation parameters on the self-similar variable $z = x/t^{n/(n-1)}$, and in Ref. [12] the solution was obtained in a closed analytical form for the most typical case with $n = 2$ (see also [10]). A similar problem with arbitrary $n$ and positive polarity was considered in Ref. [13] in the context of supersonic flow past thin bodies in dispersive hydrodynamics. In this paper, we will present the detailed solution of the time-dependent evolution of the initial pulse for both cases (4) and (5) and for arbitrary value of $n > 1$.

II. WHITHAM MODULATION THEORY

At first, we shall present briefly the main necessary equations of the Whitham theory for the KdV equation. Periodic solution of the KdV equation can be written in the form (see, e.g., [10])

$$u(x,t) = r_3 + r_2 - r_1 - 2(r_2 - r_1)\text{sn}^2(\sqrt{r_3 - r_1}(x - Vt), m),$$ (6)

where

$$V = 2(r_1 + r_2 + r_3), \quad m = \frac{r_2 - r_1}{r_3 - r_1},$$ (7)

$\text{sn}$ is the elliptic Jacobi sine function. This solution depends on three parameters $r_1 \leq r_2 \leq r_3$, in terms of which we can represent the wave velocity $V$, amplitude of oscillations $a = r_2 - r_1$ in the wave and its wavelength

$$L = \frac{2K(m)}{\sqrt{r_3 - r_1}}.$$ (8)
where $K(m)$ is the complete elliptic integral of the first kind. In the limit $r_2 \rightarrow r_3$ we get the soliton solution

$$u(x,t) = r_1 + \frac{2(r_3 - r_1)}{\cosh^2[\sqrt{r_3 - r_1}(x - Vt)]}, \quad V = 2(r_1 + 2r_3), \quad (9)$$

and in the limit $r_2 \rightarrow r_1$ the solution \[\text{(6)}\] transforms into a harmonic linear wave

$$u(x,t) = r_3 + (r_2 - r_1) \cos[2\sqrt{r_3 - r_1}(x - Vt)], \quad V = 2(2r_1 + r_3), \quad (10)$$

with wavelength $L = \pi/\sqrt{r_3 - r_1}$.

In a modulated wave the parameters $r_1, r_2, r_3$ become slow functions of the space $x$ and time $t$ variables which change little in one wavelength $L$. Their evolution obeys the Whitham equations

$$\frac{\partial r_i}{\partial t} + v_i(r) \frac{\partial r_i}{\partial x} = 0, \quad i = 1, 2, 3, \quad (11)$$

where the Whitham velocities $v_i(r)$ can be expressed by the formula

$$v_i(r) = \left(1 - \frac{L}{\partial L/\partial r_i} \frac{\partial}{\partial x}\right) V = V - \frac{2L}{\partial L/\partial r_i}, \quad (12)$$

or, after substitution of \[\text{(8)}\], as

$$v_1 = 2(r_1 + r_2 + r_3) + \frac{4(r_2 - r_1)K(m)}{E(m) - K(m)},$$
$$v_2 = 2(r_1 + r_2 + r_3) - \frac{4(r_2 - r_1)(1 - m)K(m)}{E(m) - (1 - m)K(m)}, \quad (13)$$
$$v_3 = 2(r_1 + r_2 + r_3) + \frac{4(r_3 - r_1)(1 - m)K(m)}{E(m)},$$

where $E(m)$ is the complete elliptic integral of the second kind. In the soliton limit $r_2 \rightarrow r_3 \ (m \rightarrow 1)$ these velocities transform to

$$v_1|_{r_2=r_3} = 6r_1, \quad v_2|_{r_2=r_3} = v_3|_{r_2=r_3} = 2r_1 + 4r_3, \quad (14)$$

and in the harmonic linear limit $r_2 \rightarrow r_1 \ (m \rightarrow 0)$ to

$$v_1|_{r_2=r_1} = v_2|_{r_2=r_1} = 12r_1 - 6r_3, \quad v_3|_{r_2=r_1} = 6r_3. \quad (15)$$

Since the matrix of velocities in the system \[\text{(11)}\] has a diagonal form, the parameters $r_i$ are called Riemann invariants of the Whitham system \[\text{(7)}\].

### III. DISPERSIVE SHOCK WAVE FORMATION IN A POSITIVE PULSE

Now we can turn to our problem of evolution of the wave with the initial profile \[\text{(4)}\].

In dispersionless limit, when the pulse is supposed to be a smooth enough function of $x$, the dispersion term in Eq. \[\text{(1)}\] can be neglected and the pulse evolution is described by the Hopf equation

$$u_t + 6uu_x = 0 \quad (16)$$

with well-known general solution

$$x - 6ut = \varphi(u), \quad (17)$$

where $\varphi(u)$ is a function inverse to the initial distribution $u_0(x)$. For distributions with the form \[\text{(3)}\] the formal solution \[\text{(17)}\] with $u_0(x) = -u^n$ has a multi-valued region which is shown in Fig. \[\text{1(a)}\] by a thick dashed line. This nonphysical behavior is removed by means of a supposition that after wave breaking a DSW is generated which matches at its edges $x_\pm(t)$ with smooth single-valued solutions of the Hopf equation in agreement with the limiting expressions \[\text{(14)}\] and \[\text{(15)}\].

$$u = r_3 \quad \text{at} \quad x = x_- \quad (m = 0),$$
$$u = r_1 \quad \text{at} \quad x = x_+ \quad (m = 1), \quad (18)$$
which mean that the Whitham equations for the corresponding Riemann invariants \( r_3 \) and \( r_1 \) transform to the Hopf equation (16) at the edges of the DSW. Plots of the Riemann invariants as functions of the space coordinate \( x \) at fixed time \( t \) have the form shown in Fig. 1(a) by a solid line and they are similar qualitatively to the formal dispersionless solution. At the same time, finding the laws of motion of the DSW edges \( x^- (t) \) and \( x^+ (t) \) is part of the problem as well as determining of the functions \( r_i = r_i (x, t) \). As one can see in Fig. 1(a), we assume that in our case with the initial condition (4) the Riemann invariant \( r_1 = 0 \) is constant along the DSW in agreement with the condition \( u = 0 \) at the soliton edge. Consequently, the equation (11) for \( r_1 \) is satisfied by virtue of this assumption and our wave can be called “quasi-simple” in accordance with the Ref. [11], because only two Riemann invariants \( r_2 \) and \( r_3 \) change along it. They are governed by the equations (11) with

\[
V_2(m) = 2(1 + m) + \frac{4m(1 - m)K(m)}{E(m) - (1 - m)K(m)},
\]

\[
V_3(m) = 2(1 + m) + \frac{4(1 - m)K(m)}{E(m)},
\]

where \( m = r_2/r_3 \).

Now we take into account that a smooth part of the wave

\[
x - 6ut = -u^n,
\]

which is the solution of the Hopf equation, is a self-similar one: after introduction of the variables

\[
z = \frac{x}{t^{n/(n-1)}}, \quad u = \frac{1}{t^{1/(n-1)}} R(z)
\]

this solution can be rewritten as

\[
z - 6R = -R^n.
\]

The Whitham equations also admit the scaling transformation

\[
\gamma = t^\gamma R_i (xt^{-1-\gamma}),
\]

so that by taking \( \gamma = 1/(n-1) \) we arrive at the system of equations for \( R_2 \) and \( R_3 \) as functions of the self-similar variable \( z \):

\[
\frac{dR_2}{dz} = \frac{\gamma R_2}{(1 + \gamma)z - R_3 V_2}, \quad \frac{dR_3}{dz} = \frac{\gamma R_3}{(1 + \gamma)z - R_3 V_3}.
\]

After transformation to the variables

\[
m = \frac{R_2}{R_3}, \quad \zeta = \frac{z}{R_3},
\]

FIG. 1: (a) Formal solution (20) with \( n = 2 \) is shown by thick dashed line. It is replaced by a DSW with changing Riemann invariants \( r_2 \) and \( r_3 \) whose dependence on \( x \) and \( t \) is determined by the Whitham modulation equations. (b) The Riemann invariants for a DSW generated from a negative pulse for the case \( n = 2 \) (solid line) and a formal dispersionless solution (dashed thick line).
we obtain equation for the dependence $\zeta = \zeta(m)$:

$$
\frac{d\zeta}{dm} = \frac{[(1 + \gamma)\zeta - V_2(m)][\zeta - V_3(m)]}{\gamma m(V_2(m) - V_3(m))}.
$$

(26)

This equation has singular points $(0, -6/(1 + \gamma)), (0, 6), (1, 4/(1 + \gamma)), (1, 4)$ in the phase plane $(m, \zeta)$ which are shown in Fig. 2(a) with account of the fact that in our case $n > 1$ and $\gamma = 1/(n - 1) > 0$, what determines order of the points on the line $m = 1$. We are interested in the solution joining two edges of the DSW at $m = 0$ and $m = 1$, hence the solution must be a separatrix in the phase plane. At the small amplitude edge the variable $\zeta_+ = z_+/R_3$ satisfies the equation

$$
\zeta_+ - 6 = -R_3^{n-1},
$$

(27)

and since at this point the variable $R_3$ coincides with the value of $R$ in the smooth solution which is not equal to zero, we conclude that $\zeta(0) \neq 6$, and therefore the desired solution corresponds to the lower separatrix. Consequently, $\zeta_+ = -6/(1 + \gamma) = -6(1 - 1/n)$ and substitution of this expression into Eq. (22) gives $R_3^{n-1} = 6 - \zeta_+ = 6(2 - 1/n)$, that is at the matching point with the smooth solution at the small-amplitude edge we have

$$
R_- = \left[6\left(2 - \frac{1}{n}\right)\right]^{\frac{1}{n-1}}, \quad z_- = -6\left(1 - \frac{1}{n}\right)\left[6\left(2 - \frac{1}{n}\right)\right]^{\frac{1}{n-1}},
$$

(28)

and this edge propagated according to the law

$$
x_- = -6\left(1 - \frac{1}{n}\right)\left[6\left(2 - \frac{1}{n}\right)\right]^{\frac{1}{n-1}}t^{\frac{n}{n-1}}.
$$

(29)

For $n = 2$ this formula reproduces the known relationship $x_- = -27t^2$ (see [11] [12]).

For finding the general solution, we turn to the generalized hodograph method [14], according to which the solution of the Whitham equations is looked for in the form

$$
x - v_2(r)t = w_2(r), \quad x - v_3(r)t = w_3(r).
$$

(30)

By virtue of the complete integrability of the KdV equation by the Inverse Scattering Transform method, the functions $w_2(r), w_3(r)$ can be represented in the form

$$
w_i(r) = \left(1 - \frac{L}{\partial L/\partial r_i/\partial r_i}\right) W = W - \frac{1}{2}(V - v_i(r))\frac{\partial W}{\partial r_i}, \quad i = 2, 3,
$$

(31)

similar to Eq. [12]. Then, as was shown in Refs. [15] [17], the function $W$ must satisfy the Euler-Poisson equation

$$
\frac{\partial^2 W}{\partial r_2\partial r_3} - \frac{1}{2(r_2 - r_3)} \left(\frac{\partial W}{\partial r_2} - \frac{\partial W}{\partial r_3}\right) = 0.
$$

(32)

In our self-similar case it follows from Eq. [31] that $W$ is a uniform function of the order $n$, that is it can be represented as

$$
W(r_2, r_3) = r_3^n \tilde{W}(m), \quad m = r_2/r_3.
$$

(33)

Substitution of this expression into [32] yields the hypergeometric equation for $\tilde{W}$,

$$
m(1 - m)\frac{d^2 \tilde{W}}{dm^2} + \left[\frac{1}{2} - n - \left(\frac{3}{2} - n\right)m\right] \frac{d\tilde{W}}{dm} + \frac{n}{2}\tilde{W} = 0.
$$

(34)

As is known, (see, e.g., [18]), pairs of its basis solutions can be chosen by three different ways depending on behavior of the solution at the singular points, and we have to choose a linear combination of any pair such, that it corresponds to the above mentioned separatrix solution of Eq. (26). As we shall see, this condition is fulfilled for the basis solution

$$
\tilde{W}(m) = C_n F(-n, 1/2; 1; 1 - m),
$$

(35)
where $F$ is a standard hypergeometric function (see, e.g., [18]) and the constant $C_n$ is chosen according to the condition that for $m = 0$ the function $w_3$, which is determined by the formula (30), matches with the smooth solution (20) of the Hopf equation, that is $w_3(0, r_3) = -r_3^n$. As a result, we obtain

$$C_n = -\frac{4^n [\Gamma(n + 1)]^2}{\Gamma(2(n + 1))}. \quad (36)$$

Since the solution (35) is a regular function on the closed interval $0 \leq m \leq 1$, it is clear without any calculations that it corresponds to the separatrix solution, which joins the singular points at $m = 0$ and $m = 1$.

Representing $w_i$ in a self-similar form

$$w_i = r_3^n W_i(m), \quad i = 2, 3, \quad (37)$$

we find with the help of Eq. (31) the expressions

$$W_2 = C_n \left[ F(-n, 1/2; 1; 1 - m) - \frac{n}{2} \left( 1 + m - \frac{1}{2} V_2(m) \right) F(1 - n, 3/2; 2; 1 - m) \right],$$

$$W_3 = C_n \left[ F(-n, 1/2; 1; 1 - m) - \frac{n}{2} \left( 1 + m - \frac{1}{2} V_3(m) \right) \right] \times \left[ 2F(-n, 1/2; 1; 1 - m) - mF(1 - n, 3/2; 2; 1 - m) \right]. \quad (38)$$

As a result, we find the solution of our problem in a parametric form, where all variables are expressed as functions of $m$:

$$z(m) = \frac{(W_2 V_3 - W_3 V_2)(V_3 - V_2)^{1/(n-1)}}{(W_2 - W_3)^{n/(n-1)}}, \quad (39)$$

$$R_3(m) = \left( \frac{V_3 - V_2}{W_2 - W_3} \right)^{1/(n-1)}, \quad R_2(m) = mR_3(m), \quad (40)$$

$$\zeta(m) = \frac{W_2 V_3 - W_3 V_2}{W_2 - W_3}. \quad (41)$$

The last formula gives a closed analytic expression for the solution of Eq. (26). It is easy to see that for $m = 1$ it gives $\zeta(1) = 4(1 - 1/n)$, that is $\zeta(m)$ corresponds indeed to the lower separatrix in Fig. 2(a). Equations (39) and (40) allow us to obtain closed expressions for $z_+$ and $R_3^+$ at the leading soliton edge of the DSW propagating into a quiescent medium,

$$z_+ = \left( 1 - \frac{1}{n} \right) \left\{ \frac{\Gamma(2(n + 1))}{n[\Gamma(n + 1)]^2} \right\}^{1/(n-1)}.$$

As a result, we find the solution of our problem in a parametric form, where all variables are expressed as functions of $m$: 
\[
R_3^+ = \left\{ \frac{\Gamma(2(n+1))}{4^{2n-1}n(\Gamma(n+1))^2} \right\}^{1/(n-1)}.
\]

Thus, the leading edge propagates according to the law
\[
x_+(t) = \left(1 - \frac{1}{n}\right) \left\{ \frac{\Gamma(2(n+1))}{n(\Gamma(n+1))^2} \right\}^{1/(n-1)} t^{n/(n-1)}.
\]

For \( n = 2 \) this formula gives \( x_+ = (15/2)t^2 \) in agreement with the known result \[11, 12\].

The solution found here is correct for any value of \( n > 1 \). It simplifies for integer \( n \), when the hypergeometric function reduces to the Jacobi polynomials \( P_n^{(1/2-n,0)}(1-2m) \), that is
\[
W = (-1)^{n-1}r_3^n \cdot \frac{4^n[\Gamma(n+1)]^2}{\Gamma(2(n+1))} \cdot P_n^{(1/2-n,0)}(1-2m).
\]

In particular, for the case \( n = 2 \) we find
\[
W = -\frac{r_3^2}{5} \left(1 + \frac{2}{5}m + m^2\right) = -\frac{1}{5} \left( r_2^2 + \frac{2}{3}r_2r_3 + r_3^2 \right),
\]
\[
\zeta(m) = \frac{(1+m)(3+2m+3m^2)E(m) - (1-m)(3+14m-9m^2)K(m)}{(3+2m+3m^2)E(m) - (1-m)(3+m)K(m)},
\]
\[
R_3(m) = \frac{15[(1+m)E(m) - (1-m)K(m)]}{(3+2m+3m^2)E(m) - (1-m)(3+m)K(m)}.
\]

We illustrate the obtained solution by two more typical examples. For \( n = 3 \) we get
\[
x_- = -4\sqrt{10}t^{3/2}, \quad x_+ = \frac{4}{3} \sqrt{\frac{35}{3}} t^{3/2},
\]
and for \( n = 3/2 \) we obtain
\[
x_- = -128 t^{3}, \quad x_+ = \frac{1}{3} \left( \frac{256}{9\pi} \right)^{2/3} t^{3},
\]

The DSW arising as a result of wave breaking is described by the formulae obtained after substitution of the functions \( r_2(x,t) \), \( r_3(x,t) \), defined parametrically, into Eq. \[47\]. For example, the DSW for the case \( n = 3/2 \) is shown in Fig. 3(a). Thus, the solution obtained here provides a simple enough formulae for all most important parameters of the DSW evolved from the initial profile \[4\].

IV. DISPERSIVE SHOCK WAVE FORMATION IN A IN A NEGATIVE PULSE

The theory of DSW forming from a negative pulse is very similar to that for a positive pulse, but the details are different. Therefore we give here only a concise exposition of this theory.

In this case, the diagram of the Riemann invariants has the form depicted in Fig. 3(b), that is we have here a quasi-simple wave with \( r_3 = 0 \) and self-similar dependence of two other Riemann invariants
\[
r_1 = t^\gamma R_1(z), \quad r_2 = t^\gamma R_2(z), \quad z = x/t^{1+\gamma}, \quad \gamma = 1/(n-1),
\]
where \( R_1 \) and \( R_2 \) satisfy the equations
\[
\frac{dR_1}{dz} = \frac{\gamma R_1}{(1+\gamma)z - R_1V_1}, \quad \frac{dR_2}{dz} = \frac{\gamma R_2}{(1+\gamma)z - R_1V_2},
\]
and now
\[
V_1(m) = 2(2-m) - \frac{4mK(m)}{E(m) - K(m)},
\]
\[
V_2(m) = 2(2-m) + \frac{4m(1-m)K(m)}{E(m) - (1-m)K(m)}.
\]
FIG. 3: (a) Dispersive shock wave generated after wave breaking of a “positive” pulse with the profile (1) with $n = 3/2$. (b) Dispersive shock wave generated after wave breaking of a “negative” pulse with the profile (5) with $n = 3/2$.

where $m = 1 - R_2/R_1$. For the variable

$$\zeta = \frac{z}{R_1}$$

we obtain the differential equation

$$\frac{d\zeta}{dm} = \frac{[\zeta - V_1(m)][(1 + \gamma)\zeta - V_2(m)]}{\gamma(1 - m)(V_1(m) - V_2(m))},$$

and the self-similar solution, that we are looking for, corresponds to the separatrix solutions of Eq. (55), which joins two its singular points $(0, 12), (0, 12/(1 + \gamma)), (1, 6), (1, 2/(1 + \gamma))$. At the soliton edge of DSW we have now

$$6 - \zeta_+ = (-R_1)^{n-1},$$

where $R_1$ matches with the smooth solution with non equal to zero value or $R$. Hence we arrive at the conclusion that the self-similar solution corresponds to the lower separatrix in Fig. 2(b). This gives

$$\zeta_+ = \frac{2}{1 + \gamma} = 2 \left(1 - \frac{1}{n}\right), \quad R_+ = - \left(4 + \frac{2}{n}\right)^{\frac{1}{n-1}}, \quad z_+ = \zeta_+ R_+,$$

and, correspondingly, we find the low of motion of the leading edge of the DSW,

$$x_+ = -2 \left(1 - \frac{1}{n}\right) \left(4 + \frac{2}{n}\right)^{\frac{1}{n-1}} t^{\frac{n}{n+1}}.$$

To find the global solution of the Whitham equations including the law of motion of the small-amplitude edge, we assume that it has the form

$$x - v_i(r)t = w_i(r), \quad x - v_2(r)t = w_2(r)$$

with

$$w_i(r) = W - \frac{1}{2}(V - v_i(r)) \frac{\partial W}{\partial r_i}, \quad i = 1, 2,$$

and then the function $W$ must satisfy the Euler-Poisson equation

$$\frac{\partial^2 W}{\partial r_1 \partial r_2} - \frac{1}{2(r_1 - r_2)} \left(\frac{\partial W}{\partial r_1} - \frac{\partial W}{\partial r_2}\right) = 0.$$

In our self-similar case we look for its solution in the form

$$W(r_1, r_2) = (-r_1)^m \tilde{W}(m), \quad m = 1 - r_2/r_1.$$
Substitution of this expression into (61) yields the hypergeometric equation for $\tilde{W}$,

$$m(1 - m) \frac{d^2 \tilde{W}}{dm^2} + \left[ 1 - \left( \frac{3}{2} - n \right) m \right] \frac{d \tilde{W}}{dm} + \frac{n}{2} \tilde{W} = 0. \quad (63)$$

The separatrix solution corresponds now to

$$\tilde{W}(m) = C_n F(-n, 1/2; 1; m), \quad (64)$$

where the constant $C_n$ is determined by the matching condition $x - 6r_1 t = (-r_1)^n$ at the leading soliton edge what gives

$$C_n = \frac{4^n \Gamma(n + 1)^2}{\Gamma(2(n + 1))}. \quad (65)$$

Thus, we get the solution

$$w_i = (-r_1)^n W_i(m), \quad i = 1, 2, \quad (66)$$

where

$$W_1 = C_n \left[ F(-n, 1/2; 1; m) - \frac{n}{2} \left( 2 - m - \frac{1}{2} V_1(m) \right) \times \left( 2F(-n, 1/2; 1; m) - (1 - m)F(1 - n, 3/2; 2; m) \right) \right], \quad (67)$$

$$W_2 = C_n \left[ F(-n, 1/2; 1; m) - \frac{n}{2} \left( 2 - m - \frac{1}{2} V_2(m) \right) F(1 - n, 3/2; 2; m) \right].$$

As a result, the dependence of the Riemann invariants and other functions on the self-similar variable $z$ can be representer in a parametric form,

$$z(m) = \frac{(W_2 V_1 - W_1 V_2)(V_1 - V_2)^{1/(n-1)}}{(W_1 - W_2)^{n/(n-1)}}, \quad (68)$$

$$R_1(m) = -\left( \frac{V_1 - V_2}{W_1 - W_2} \right)^{1/(n-1)}, \quad R_2(m) = (1 - m)R_1(m), \quad (69)$$

$$\zeta(m) = \frac{V_2 W_1 - V_1 W_2}{W_1 - W_2}. \quad (70)$$

These expressions yield the values of the variable $R_1$ at the rear edge

$$R_1^- = -\frac{1}{4} \left\{ \frac{3\Gamma(2n + 1)}{n\Gamma(n + 1)^2} \right\}^{1/(n-1)} \quad (71)$$

and, hence, the law of motion of this edge is given by the formula

$$x_-(t) = -3 \left[ 1 - \frac{1}{n} \left\{ \frac{3\Gamma(2n + 1)}{n\Gamma(n + 1)^2} \right\}^{1/(n-1)} t^{n/(n-1)}. \quad (72)$$

For integer values of $n$ the function $W$ reduces to a polynomial form and these expressions can be simplified. In particular, for $n = 2$ we obtain

$$W = -\frac{1}{5} \left( r_1^2 + \frac{2}{3} r_1 r_2 + r_2^2 \right), \quad (73)$$

$$\zeta(m) = \frac{(2 - m)(3m^2 - 8m + 8)E(m) + 2(1 - m)(3m^2 + 8m - 8)K(m)}{(3m^2 - 8m + 8)E(m) - 4(1 - m)(2 - m)K(m)}, \quad (74)$$
\[ R_1(m) = -\frac{15[(2 - m)E(m) - 2(1 - m)K(m)]}{(3m^2 - 8m + 8)E(m) - 4(1 - m)(2 - m)K(m)}, \]  

(75)

We illustrate the solution found by several examples:

\[ x_- = -\left(\frac{8}{3}\right)^3 t^3, \quad x_+ = -\left(\frac{64}{3\pi}\right)^2 t^3 \quad \text{for} \quad n = 3/2; \]

\[ x_- = -\frac{27}{2} t^2, \quad x_+ = -5t^2 \quad \text{for} \quad n = 2; \]

\[ x_- = \frac{4}{3} \sqrt{\frac{14}{3}} t^{3/2}, \quad x_+ = -4\sqrt{5} t^{3/2}, \quad \text{for} \quad n = 3; \]

(76)

Again the DSW arising as a result of wave breaking of a negative pulse for the case \( n = 3/2 \) is shown in Fig. 3(b). As we see, it has a quite different form compared with the DSW generated by a positive pulse which is shown in Fig. 3(a).

V. CONCLUSION

We have obtained the complete solution of the wave breaking problem in the KdV equation theory for the case of power dependence of the initial positive or negative pulse profiles on the coordinate when this problem is self-similar. Since any initial monotonic profile can be represented as a sum of functions of the form (4), then the solution of the hodograph equations is a sum of the solutions obtained in this paper. Therefore they provide the method of description of DSW propagation in more general situations. Thus, the solution found can be used as a practical tool in comparison of experimental results with theoretical predictions of the KdV approximation. Besides that, the laws of motion of the DSW edges can serve as a basis for development of more general theory applicable to DSWs whose evolution is governed by non-integrable equations (see. Refs. [19, 20]).

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