On the Jacobian conjecture in characteristic zero

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Abstract

We study the Jacobian conjecture for Keller maps $f : X_0 := \mathbb{A}^n \to Y_0 := \mathbb{A}^n$ in characteristic 0 and attempt to prove it. We are quite aware of the fact that many people have tried to prove the Jacobian conjecture before us and hence we stress that this manuscripts is only an attempt. Our approach is to study the finiteness variety $V_f \subset Y_0$ of $f$, the set of points of $Y_0$ over which $f$ fails to be proper. We study a general component $V \subset V_f$ of this set by introducing a suitable representation of $X_0$ which we call the $u - \gamma$ representation. This view of $X_0$ has the advantage that it allows us to explicitly write down the Jacobian matrix of $f$. We then turn our attention to the condition that $|J(f)| = 1$, which we interpret as a partial differential equation in one unknown function. We study the characteristics of this equation and prove that the dynamics of this is strongly related to the ramification $K$ above $V$. We then study the action of a certain cyclic Galois group induced by the $u - \gamma$ action on these differential equations and prove that the growth of the functions are bounded. Alternatively we prove the same result via a vector field argument where we deduce that if $K > 0$ then the function $u^{-K}$ is in fact an analytic function around $v \in V \subset Y_0$. This leads to $K = 1$ and as $\pi_1(Y_0 - S) \cong \pi_1(Y_0)$ if $S$ is of codimension at least two, the Jacobian conjecture follows immediately.

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1 Introduction

The Jacobian conjecture in characteristic 0 and dimension \( n \geq 1 \) states the following.

**Conjecture 1 (Jacobian conjecture in dimension \( n \))** Let \( f : \mathbb{C}^n \to \mathbb{C}^n \) be a polynomial map such that \( |J(f)| = 1 \) where \( J(f) \) is the Jacobian matrix of \( f \) and \( |J(f)| \) its determinant. Then \( f \) is an isomorphism.

For a rich account of the history of this problem and what is known about it see the book of Arno van den Essen ([21]). In the current manuscript we shall attempt to prove this conjecture. We stress once again that this here is only an attempt, even if there should be mistakes then perhaps they can be corrected.

Let us give some flavour of our approach. The main insight is the Jacobian condition itself. It is most well illustrated in the case of dimension two. Let \( f = (f_1, f_2) \) be a Keller map \( f : X_0 := \mathbb{A}^2 \to Y_0 := \mathbb{A}^2 \). Consider now the set of ordinary differential equations given by

\[
\frac{dx_1}{dr} = \frac{\delta f_2}{\delta x_2}
\]

and

\[
\frac{dx_2}{dr} = -\frac{\delta f_2}{\delta x_1}
\]

Notice that this has the solution \( \frac{dx_2}{dr} = 0 \) and \( \frac{dy_1}{dy_2} = 1 \). In other words the trajectory followed by \( (x_1, x_2) \) leads to a straight line in \( Y_0 \). If we now chose the starting values of \( x_1 \) and \( x_2 \) such that the \( (y_1 = f_1, y_2 = f_2) \) trajectory approach a point \( v \in V_f \), where \( V_f \) is the finiteness variety of \( f : X_0 \to Y_0 \), i.e. the set of points over which \( f \) is not proper, then we see that the trajectory breaks...
through the finiteness variety $V_f$ at the point $v$. In particular the tangent of
the $(y_1, y_2)$-trajectory does not vanish at $v$ and the trajectory goes through it
at constant speed.

Yet the initial values of $(x_1, x_2)$ can be chosen so that the trajectory $(x_1, x_2) \to e \in \Gamma$ where $\Gamma$ is a component of $\hat{X}$ at infinity which maps to a component $V \subset V_f$. Here $\hat{X} \to X = \mathbb{P}^2$ is a suitable blowup induced by a representation which we shall introduce.

We have done some numerical experiments for maps which are not Keller and
we have always noted that in the case where $f$ is not Keller, the $(y_1, y_2)$ trajec-
tory tends to a point $v \in V_f$ but never quite reaches it, i.e. the speed of the trajectory approaches 0 as we approach the point $v$.

The point is however, in the case of $|J(f)| = 1$ one approaches $v$ with constant speed. One can now ask the opposite question: starting at $v$, what does the dynamics of $x_1$ and $x_2$ look like? In general one expects a sort of reversability of the trajectory, as its image $(y_1, y_2)$ is reversible.

The problem is however that the map $f : \hat{X} \to Y_0$ could have ramification along $\Gamma$. Denote this ramification by $K$. Considering the reversibility of the dynamics of $(x_1, x_2)$ one sees that there will in general by $K$-different branches that the $(x_1, x_2)$ can follow.

The reason this happens is because of the explicit dynamics of $x_1, x_2$ which we shall explain later. To give an idea we shall introduce a new representation of $X_0$ which we call the $u - \gamma$ representation. Here $u$ is related to $x_1$ by $u^m = x_1$ where $m > 0$ is an integer.

We shall see that the induced dynamics on $u$ is such that

$$\frac{du}{dr} = u^{K+1} \sum q_i(\gamma) u^{-i}.$$ 

Furthermore as $(x_1, x_2) \to e \in \Gamma$, where $e \to v \in V_f$, we shall see that $u^{-1} \to 0$.

The problem is now that for the reversibility, the function $u^{-K}$ has the dynamics

$$\frac{du^{-K}}{dr} = -K \sum q_i(\gamma) u^{-i}.$$ 

Apart from the fact that the $\gamma$ might have branches, we see that the dynamics of $u^{-K}$ is dependend on the dynamics of $u^{-1}$.

However, if we can prove that the $q_i = 0$ if $K$ does not divide $i$, then we see
that the dynamics of $u^{-K}$ can be reversed uniquely (of course, if it is the case for $\gamma$). This would imply that $u^{-K}$ and hence $x_1^K$ is uniquely determined by the
trajectory in $Y_0$.

This single idea, that the dynamics of $x_1$ and $x_2$ given by the Jacobian of $f$, leads to a $(y_1, y_2)$-trajectory that breaks through the finiteness variety, is crucial throughout our entire work. In particular it allows us to relate the ramification above $V_f$ with the growth of the dynamics of $u^{-K}$ and $\gamma$.

Our manuscript is divided into two parts, the first part focuses on the two-dimensional case. In some sense things are easier in this case (although the word easy must be taken with some salt). Throughout we shall illustrate some ideas with examples of maps which are not Keller. In the second part of the paper we shall study the higher dimensional case. The arguments are mostly the same.

Let us now give an outline of our manuscript. After fixing some notation in Section 2 we give an overview of what is known about the two-dimensional Jacobian conjecture. In this section we focus in particular on the results of the work of N. Van Chau (20), the reader can find much more complete overviews in the literature, see for instance the book of A. Van den Essen (21). We also mention an interesting idea of van den Dries and McKenna (15) which relates the distance of images of rational integral points to the finiteness variety.

In Section 3 we shall start our approach for the two-dimensional case. In this section we shall assume that $f$ admits a finiteness variety $V_f$ which is not empty. Here we understand with finiteness variety $V_f \subset Y_0$ the set of points of the image over which $f$ is not proper. Note that $V_f$ is of codimension one, as $Y_0$ is four dimensional as a real manifold and hence simply connected if we omit a finite number of points.

Notice, that as we are dealing with surfaces, the theory of regular blowups at closed points are well understood. Furthermore, any rational map between smooth surfaces can be blown up so that the map is defined everywhere. These blowups are blowups at closed points, the so-called monoidal transformations.

We shall isolate a codimension-one component $V$ of $V_f$ and construct a blowup $\pi : \hat{X} \to X$ of the projective closure $X = \mathbb{P}^2$ of the domain $X_0 = \mathbb{A}^2$ of $f$ until we have a line $\Gamma$ in $\hat{X}$ which maps onto the studied component $V$ of $V_f$. In general $\Gamma$ will be an exceptional curve mapping to a point $x_1 \in X - X_0$, or the line at infinity of $X_0$.

Thereafter we shall approach the points of $\Gamma$ via affine curves in $X_0$. To do so, we shall write down a local chart of a point $e \in \Gamma$ using the exact structure theory of monoidal transformations in surfaces. In this chart we shall approach $e$ by a line which is parametrized by the parameter $t$. 

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Next we study the image of this line in \( X \), i.e. under the map \( \pi : \hat{X} \to X \). By introducing then a suitable local parater change \( t \to s \), we shall obtain our parametrization in \( X \) of the form \( [X_1 = 1 : X_2 = P(e, s) : T = s^m] \) where \( P \) is in general a power series in \( s \) and \( e \). In general the group \( \mathbb{Z}/m\mathbb{Z} \) will act on this representation, a fact which we shall exploit later on.

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By introducing the parameter \( u^{-1} = s \) we thus obtain a curve \( \hat{C}(u, e) \), depending on \( u \) and \( e \), which tends to \( e \in \Gamma \) as \( u \to \infty \).

The next problem is however, that \( P(e, s) \) could be quite complicated. Our main theorem for this section is that we may truncate \( P = \sum \beta_i(e)s^i \) directly after the first index \( N \) where \( \beta_N \) depends on \( e \). In some sense, this suggests that all the information is contained in the first nontrivial term of \( P \). Thus we obtain a representation \( x_1 = u^m, x_2 = h_0u^m + .. + \gamma u^{m-N} \) of \( X_0 \) in terms of \( u \) and \( \gamma = \beta_N(e) \) which has the property that as \( u \to \infty \),

\[
(x_1, x_2) \in X_0 \to e \in \Gamma.
\]

Note, this section is quite general and does not require the assumption that \( |J(f)| = 1 \).

In Section 4 we shall then give an explicit form for the Jacobian \( J(f) \) of \( f \) in terms of the \( u, \gamma \) representation. Although the hypothesis \( |J(f)| = 1 \) will simplify our calculations, it will strictly speaking not be needed in this section. Of particular interest of this section is the relation of the entries of \( J(f) \) in terms of the \( \gamma \)-derivatives of \( a_0 \) and \( b_0 \), where \( \gamma \to (a_0(\gamma), b_0(\gamma)) \) is a representation of the component \( V \) of \( V_f \subset Y_0 \) that we are studying.

In Section 5 we shall then use our explicit form of \( J(f) \) to derive some differential equation like conditions on the component \( V \) of \( V_f \). We consider the image \( D(u, \gamma) := f(x_1(u, \gamma), x_2(u, \gamma)) \). We can write \( D(u, \gamma) = (a(u, \gamma), b(u, \gamma)) \) where \( a(u, \gamma) = \sum a_i(\gamma)u^{-i} \) and similarly for \( b \). Here we shall start using the assumption that \( |J(f)| = 1 \), both as a formula condition but also its interpretation on volumes. The main result of Section 5 is a differential equation which relates the terms \( a_k \) and \( b_k \) with \( a_0 \) and \( b_0 \), where \( k \) is the first positive index where one of \( a_k \) or \( b_k \) is not identically zero.

In particular we shall prove that \( k \leq N - 2m \) and if \( k = N - 2m \) then we have the identity

\[
b_k(\gamma) \frac{da_0(\gamma)}{d\gamma} - a_k(\gamma) \frac{db_0(\gamma)}{d\gamma} = \frac{m}{k}.
\]

Notice thus, that if \( k = N - 2m \) then we will have the surprising fact that \( \frac{da_0}{d\gamma} \) and \( \frac{db_0}{d\gamma} \) can never share a root, in particular this would imply that the map \( \Gamma \to V \) when considered as a map of manifolds in their own manner is etale. Notice that this would imply that \( V \) can have only ordinary singularities (i.e.
self intersections), but in a general sense is smooth.

The results in Section 5 will become useful in Section 6, where we shall study the condition $|J(f)| = 1$ and derive a system of ordinary coupled differential equations. Our representation of $J(f)$ in terms of $u$ and $\gamma$ will allow us to gain insight into this system of coupled differential equations.

To give an idea of what we shall do, let $x_1$ and $x_2$ be coordinates on the domain $X_0 = \mathbb{A}^2$ of $f = (f_1, f_2)$. Notice that the Jacobian condition implies that

$$\frac{\delta f_1 \delta f_2}{\delta x_1 \delta x_2} = 1.$$

Our idea will be to interpret this as a partial differential equation. As it is linear and of first order, one is encouraged to study this system via its characteristics. As we have an exact description of $J(f)$ from Section 4, we can write down equations for the characteristics quite explicitly.

More specifically, starting from an initial value $u = u(0)$ and $\gamma = \gamma(0)$ we shall write down the evolution equations of $u$ and $\gamma$ in terms of the time parameter $r$. In particular we shall set

$$\frac{dx_1}{dr} = \frac{\delta f_2}{\delta x_2}$$

and

$$\frac{dx_2}{dr} = -\frac{\delta f_2}{\delta x_1}.$$

Notice that these equations imply that $f_2$ is constant on the trajectory $(x_1, x_2)$. Furthermore, we can solve for $f_1$ on this trajectory and one sees that $f_1$ is of the form $r + C$.

This implies in particular, that if $(z_1, z_2)$ is a point on the finiteness variety component $(a_0, b_0)$, then starting from $(z_1 - \epsilon, z_2)$ the trajectory $(x_1, x_2)$ tends to a point on $\Gamma$, if the initial conditions are chosen correctly.

We shall see in Section 7 that the growth of $u$ is strongly related to the ramification of $f$ above $V$. Denote this ramification by $K$. First of all we shall see that $K = N - 2m$. In what follows we shall follow two ways to study $K$, the one is a Galois theoretic approach and the other is a differential-geometric approach which uses vector fields and the fact that $\Gamma \to V$ is etale almost everywhere when considered as manifolds on their own. Our main result will be to prove that if $K > 0$ then $K = 1$. With the Galois theoretic approach we shall actually prove directly that $K = 0$.

To prove that $K = 0$ we shall study the action of $\mathbb{Z}/m\mathbb{Z} \to \zeta_m$ on the $u$ and $\gamma$ in Section 8. Using this action we shall prove that $K \equiv 0 \pmod{m}$. However, we can change $m$ to a constant $cm$ by composing $f : X_0 \to Y_0$ with maps of the
form \([x_1, x_2 \to x_1 + x_2^c, x_2]\) which are clearly Keller maps and which leaves the ramification above \(V\) fixed. We shall prove that for such maps the \(m\) is transformed to \(cm\) by applying L'Hospital's rule. This implies that \(K \equiv 0 \pmod{cm}\) for all \(c \in \mathbb{N}\) implying that \(K = 0\).

In Section \(9\) we shall follow a different approach by proving directly that if \(K > 1\) then \(K = 1\). In order to do so we shall study the pullbacks of differential \(n\)-forms on open balls under the map \(f\). Indeed, let \(v \in V\) and let \(e \in \Gamma\) map to \(v\). We can regard \(f\) as a map from an open two-dimensional ball \(B\) around \(e\) to an open two-dimensional ball \(C\) around \(v\). The dynamics of \(u\) and \(\gamma\) allows us to explicitly write down the pullback \(f^*(\omega_0)\) of the differential two-form \(\omega_0 := dy_1 \wedge dy_2\).

We shall then study the action of the group \(\mathbb{Z}/K\mathbb{Z}\) given by \(r \to [u \to \zeta_K^ru]\). We shall prove that the pullback \(f^*(\omega_0)\) of \(\omega_0\) descends to a differential 2-form \(\omega_1\) on the quotient space \(B \xrightarrow{\pi} A\) which is nonsingular.

Using the local etaleness of the map \(\Gamma \to V\) (considered as submanifolds on there own, and not as subschemes of \(X\) where \(f\) is ramified) we see that we can find differential \(n - 1\) forms on \(\Gamma\) and these can be pulled back uniquely to differential \(n - 1\) forms on \(V\) and vice-versa.

Using this and some local calibration we define a map

\[ f_* : H^0(B, \pi^*T_A) \to H^0(B, f^*T_C) \]

on vector bundles over \(B\). We shall prove that the existence of this map, which is infact an isomorphism, implies immediately that \(k = K\) on the one hand, and on the other hand that for the dynamics of \(u\) the \(q_i = 0\) if \(K\) does not divide \(i\).

This implies in particular, that if \(K > 0\), then the function \(u^{-K}\) is completely determined by a trajectory in \(Y_0\). As such \(x_1^K\) is predetermined by trajectories in \(Y_0\). Hence monodromy changes \(x_1\) to \(\zeta_K x_1\). However, all our arguments would have applied to \(f(x_1, \ldots, x_n) = f(x_1 + c_1, \ldots, x_n + c_n)\) where the \(c_i\) are arbitrary constants. This implies that \(K = 1\) and that the ramification above \(V\) is trivial.

As \(\pi_1(Y_0 - S) \cong \pi_1(Y_0) \cong \{0\}\) if \(S\) is of codimension two, the conjecture in dimension two follows immediately. Hence \(K \leq 0\).

But this immediately leads to a contradiction, as this implies that the growth of the differential equations are severely bounded. In particular it would imply that zero-tangent vectors map to non-zero tangent vectors, a contradiction. This shows that the finiteness variety cannot be of codimension one and the
two-dimensional Jacobian conjecture follows at once.

Our main idea in this manuscript is the fact that the trajectories constructed earlier break through the finiteness variety. In Section 10 we shall show some numerical experiments for why this is not the case in non-Keller maps.

Finally in Section 12 we move on to the higher dimensional case. In this section we shall study the finiteness variety of higher dimensional Keller maps in exactly the same fashion as in the two-dimensional case.

After fixing some notation in Section 11 we shall construct the higher dimensional $u - \gamma$ representation starting in Section 12 by studying blowups of $P^n$. In Section 13 we shall then introduce a system of coupled differential equations which follows the line of thought that we used in the two-dimensional case. In particular we shall explicitly write down the dynamics of $u$ and $\gamma$.

Similarly as in the two-dimensional case we shall relate the growth of $u$ with the ramification of $f$ above $V$, where $V$ is a component of the finiteness variety $V_f \subset Y_0 = A^n$.

Finally, analogous to the two-dimensional case, we shall prove that $K = 0$ in two ways in Sections 15 and 16. As a result the higher dimensional Jacobian conjecture will follow immediately.

After scanning the Arxiv (arxiv.org), we could not find work that was similar to our own. If however, there is work in the literature that follows a similar approach as ours, we would be most grateful to know about this.

2 Notation for the two-dimensional case

$K$ will denote a field of characteristic 0 which will be allowed to extend as we continue our studies. For all purposes one may take $K = \mathbb{C}$.

$X_0$ will always denote the affine plane $A^2_K$ and so will $Y_0$. $X = P^2_K$ and $Y = P^2_K$ will denote the projective spaces which are the projective closures of $X_0$ respectively $Y_0$. We shall denote by $H_X$ the line at infinity in $X$, i.e. $X - X_0$ and similarly by $H_Y$ the line at infinity of $Y$. $X_1, X_2$ and $T$ will denote projective coordinates of $X$, where $x_1$ and $x_2$ will denote the standard coordinates of $X_0$ and $T = 0$ will denote the line of infinity $H_X \subset X$. Similarly for $Y_1, Y_2$ and $Z$. When choosing $X_1 = 1$, we shall write $X_2$ respectively $T$ for the coordinates $\frac{X_2}{X_1}$ respectively $\frac{T}{X_1}$. Similarly for $Y$.

$f : X_0 \to Y_0$ will always denote a polynomial map. When the determinant of the Jacobian matrix of $f$ is identically one, i.e. the map is globally etale,
then we shall call $f$ a Keller map. $V_f \subset Y_0$ will denote the finiteness variety of $f$, i.e. the set of points of $Y_0$ over which $f$ fails to be proper.

**Remark 1** Because a Keller map is etale, emptiness of the finiteness variety implies that $f$ is in fact an isomorphism, as $\mathbb{A}^2_\mathbb{C}$ is simply connected.

We shall often use the homogenous decomposition of $f$, i.e. writing $f = (f_1, f_2)$ we shall consider

$$f_i = \sum_{j \leq d} F_j^i(x_1, x_2)$$

where the $F_j$ are homogenous.

Let $W \subset Y_0$ be a closed subset of $Y_0$. For a point $y \in Y_0$, we shall write $d(y, W)$ for the distance between $y$ and $W$, i.e. the infimum of $||y - w||$ where $w$ ranges over $W$. Note the infimum is in fact a minimum, as $W$ is closed inside $Y_0$.

The following is known about the irreducible components $V \subset Y_0$ of the finiteness variety.

**Theorem 1 (Van Chau ([20], [19]))** The components of $V_f$ are pure of codimension one. Furthermore, an irreducible component $V$ is the image of polynomial trajectory $\gamma \to (p_1(\gamma), p_2(\gamma))$.

In his paper Van Chau ([20], see also [19] in particular Theorems 3 and 4) the author provides an exact description of the degrees of $p_1$ and $p_2$. Indeed, let $d_1$ and $d_2$ be the degrees of $f_1$ and $f_2$, where $f = (f_1, f_2)$. Then the author proves that

$$\frac{d_1}{d_2} = \frac{\text{deg}(p_1)}{\text{deg}(p_2)}$$

Following Van Chau, in particular this implies that $V_f$ cannot contain a smooth complex projective line as per the Abhyankar-Moh Theorem (Abhyankar-Moh [1]) this would imply that one can transform this component to $y_1 = 0$, contradicting Van Chau’s theorem.

Bass, Connell and Wright ([3]) proved that it suffices to study Keller maps of degree maximally three. Of special interest to us will be the following theorem of Druzkowski.

**Theorem 2 (Druzkowski ([5], [6], [7]))** It suffices to consider maps of the form

$$x \to x + (Ax)^3,$$

where $A$ is a square nilpotent matrix

**Remark 2** See also the work Gorni and Zampieri ([8]).

Lastly we state an interesting observation of Van den Dries and McKenna ([15]) regarding the images of integral points.
Theorem 3 (Van den Dries and McKenna ([15])) Assume that \( f \) is Keller and defined over \( \mathbb{Z} \). Then we have the following inequality for all points \( x \in \mathbb{Z}^n \):
\[
d(f(x), V_f) < 1.
\]

3 Blowups of \( \mathbb{P}^2 \)

In this section we shall start our studies of the finiteness variety of a Keller map
\[
f : X_0 = \mathbb{A}^2 \to Y_0 = \mathbb{A}^2.
\]
Our main theorem will be the following.

**Theorem 4** Let \( V \) be a component of the finiteness variety. Then \( V \) can be parametrized by
\[
V = (a_0(\gamma), b_0(\gamma))
\]
where \( a_0 \) and \( b_0 \) are polynomials. Furthermore, we can find representation parameters \( u, \gamma \) and constants \( m, N \) and \( h_0, \ldots, h_{N-1} \in \mathbb{C} \) such that the curve
\[
\hat{C}(\gamma, u) = (x_1 = u^m, x_2 = h_0 u^m + \ldots + h_{N-1} u^{1+m-N} + \gamma u^{m-N})
\]
is such that
\[
f(\hat{C}(\gamma, u)) \to (a_0(\gamma), b_0(\gamma))
\]
as \( u \to \infty \).

Consider the induced rational map
\[
f : X = \mathbb{P}^2 \to \mathbb{P}^2 = Y.
\]
Let \( V \subset Y_0 = \mathbb{A}^2 \) be an irreducible component of the finiteness variety \( V_f \subset Y_0 \) of \( f \). As it is of codimension one, we can blow up \( \hat{X} \to X \) at regular closed points such that there exists an irreducible curve \( \Gamma \to \hat{X} \) and a rational morphism
\[
f : \hat{X} \to Y\n\]
such that \( f \) is defined almost everywhere on \( \Gamma \) and such that \( f \) maps \( \Gamma \) densely onto \( V \). We summarize this in the following theorem.

**Theorem 5** There exists an extension of number fields \( K_1/K \) and a proper birational morphism \( \pi : \hat{X} \to X \) and a divisor \( \Gamma \to \hat{X} \) such that the map \( f \) extends to a rational morphism \( \hat{X} \to Y \) which is defined almost everywhere on \( \Gamma \). The morphism \( f \) maps \( \Gamma \) densely onto \( V \) and induces a finite cover of curves
\[
f|_\Gamma : \Gamma \to V.
\]
Furthermore, the morphism \( \hat{X} \to X \) is a sequence of blowups
\[
\hat{X} = X_n \to X_{n-1} \to \ldots \to X_2 \to X_1 = X
\]
where each
\[
X_{i+1} \to X_i
\]
is a regular blowup at the closed point \( x_i \in X_i(K_1) \). The image of \( \Gamma \) under the map \( \pi : \hat{X} \to X \) is a point \( x \in H_X \) where \( F_d(x) = 0 \).
In terms of diagrams we can summarize the situation with the following.

\[ \Gamma \xrightarrow{i_{|\Gamma}} \hat{X} \xrightarrow{\pi} X \]
\[ V \xrightarrow{i_{|V}} Y \xrightarrow{f} Y \]

**Remark 3** See for instance Liu [14] Theorem 9.2.7.

We shall now study the construction of \( \Gamma \) in some more detail. Let us start by studying the blowup \( X_2 \to X_1 = X \). This a regular closed blowup at the point \( x_1 \in H_X \), in particular at a point \( x_1 \) where \( F_d(x_1) = 0 \).

**Assumption 1** We shall assume without loss of generality that

\[ x_1 = [X_1 = 1 : X_2 = \alpha_1 : T = 0] \in H_X. \]

Notice that

\[ X_2 \to X_1 \]

can be covered by affine open subvarieties of the form

\[ U_2 = \text{spec}(K_1[X_2,T][t_1]) = \text{spec}(K_1[g_1,t_1]) \]

where \( f_1 \) is either \( X_2 - \alpha_1 \) or \( T \), and \( g_1 \) is the other of \( X_2 - \alpha_1 \) and \( T \).

The next blowup \( X_3 \to X_2 \) is the regular closed blowup at the point

\[ x_2 = [g_1 = 0; t_1 = \alpha_2]. \]

Again we can cover \( X_3 \) by affine opens of the form

\[ U_3 = \text{spec}(K_1[g_1,t_1][t_2]) \]

where \( f_2 \) is either \( g_1 \) or \( t_1 - \alpha_2 \) and \( g_2 \) is the other of these two.

We can continue in this fashion until we eventually arrive at

\[ U_n = \text{spec}(K_1[g_{n-1},t_{n-1}]). \]

In this case the divisor \( \Gamma \) corresponds to the divisor \( V(g_{n-1} = 0) \). We arrive immediately at the following.

**Lemma 1** We have that \( \Gamma \) is isomorphic to \( \mathbb{P}^1 \) and that \( V \) is an affine line (which might intersect itself and may have singularities).

**Remark 4** Compare to Hartshorne [2] Proposition V.3.1.
Let \( e \in \hat{Q} \) be a free parameter. Consider now the curve

\[ C_e(t) = [g_{n-1} = t, t_{n-1} = e]. \]

Notice that \( C_e(t) \) tends to a point \( e \in \Gamma \) as \( t \to 0 \). As we vary \( e \), we vary the points on \( \Gamma \) and hence its image varies on \( V \).

We see that

\[ \cup_{e,t} C_e(t) \cap \pi^{-1}(X_0) \neq \emptyset. \]

Hence the image \( C_e(t) \) inside \( X_0 \) induces a curve \( X(e,t) : U \to X_0 \) which is defined on some open subset \( U \subset C^2 \). Furthermore, for almost all \( e \in \mathbb{C} \) there exists a punctured interval \( I_e = (-a_e, 0) \cup (0, a_e) \) such that \( (e, I_e) \subset U \).

Notice that the image \( X_e(t) \) in \( X_0 \) is such that its image under \( f \) tends to \( f(e) \in V \) as \( t \to 0 \).

We shall now try to explicitly write down \( X_e \). We consider the image of \( C_e(t) \) under \( \pi : \hat{X} \to X \).

This is a curve, where we shall abuse notation and also write \( C_e(t) \), given by

\[ C_e(t) : t \to [X_1 = 1 : X_2 = Q_e(t) : T = Z_e(t)]. \]

**Lemma 2** For almost all \( e \), we have that in a neighborhood \( W \subset \mathbb{C} \) of \( t = 0 \) that \( T \neq 0 \) for \( t \neq 0 \).

**Proof:** Notice that \( Z_e(t) \) is a polynomial in \( e \) and \( t \). Furthermore for \( t = 0 \) we have that \( Z_e(t) = 0 \). Hence \( Z_e(t) = t^r W_e(t) \) where \( r \) is some integer and \( W_e(t) \) is some polynomial in \( e \) and \( t \) with \( W_e(0) \neq 0 \). [End-Of-Proof]

Hence we see that the curve maps partly into \( \mathbb{A}^2 = X_0 \). Now we have a fundamental observation: notice that as \( \hat{X} \to X \) is a regular sequence of blowups above \( H_X \), we have that \( X(e,t) \) is the same as

\[ [x_1 = \frac{1}{Z_e(t)}, x_2 = \frac{Q_e}{Z_e(t)}] \]

inside \( X_0 \). Hence we arrive at the following.

**Theorem 6** Consider the curve

\[ \hat{C}_e(t) = [x_1 = \frac{1}{Z_e(t)}, x_2 = \frac{Q_e(t)}{Z_e(t)}] \subset X_0 \]

for \( t \in W \) and \( t \neq 0 \). Then under the morphism

\[ \pi : \hat{X} \to X \]
we have that
\[ \hat{C}_e(t) \to e \in \Gamma_i \]
and
\[ f(\hat{C}_e(t)) \to f(e) \in V \]
as \( t \to 0 \).

**Proof:** If we can prove that
\[ \hat{C}_e(t) \to e \in \Gamma_i \]
then
\[ f(\hat{C}_e(t)) \to f(e) \in V \]
would follow immediately. Notice that for \( t \neq 0 \) we have that \( \lim_{t \to 0} \hat{C}_e(t) \to e \) in \( \hat{X} \) as \( \hat{X} \to X \) is a blowup above a point in \( H_X \) and for \( t \neq 0 \) we have that \( \hat{C}_e(t) \in X_0 \). Hence the result follows. [End-Of-Proof]

Now we consider the curve \( C_e(t) \) inside \( X \) again. The key observation now is that we can change the parameter \( t \) to bring \( C_e(t) \) in a suitable form. However, this will depend on \( e \). Fix an \( e \in C \) and consider the change of parameter \( t \to s_e \) such that \( Z_e(t) = s_e^m \). Notice that this change \( Q_e(t) \) into a power series
\[ \hat{Q}_e(s_e) \in \mathbb{C}[[s_e]] \]
which we would now like to explain.

Notice that \( Q_e(t) \) and \( Z_e(t) \) induces two functions
\[ Q_e : \mathbb{P}^1 \to \mathbb{P}^1 \]
and
\[ Z_e : \mathbb{P}^1 \to \mathbb{P}^1. \]
Here the domains of the two morphisms are parametrized by the parameter \( t \). Notice that for \( m > 1 \), \( t = 0 \) is a branch point of \( Z_e \). Hence locally at least, as we are in characteristic 0, the morphism is of the form \( s_e \to s_e^m \). In such a neighbourhood, the function \( Q_e \) is a local power series in \( s_e \) which has a positive radius of convergence. Furthermore, the values of \( Q_e \) can be computed for a specific \( \zeta_m s_e \). For \( m = 1 \), the morphism \( Z_e \) is etale at \( t = 0 \) and a similar argument holds (in which case \( Q_e \) has only one branch in a neighbourhood of \( t = 0 \)).

Hence we can write
\[ C_e(t) = [X_1 = 1 : X_2 = \hat{Q}_e(s_e) : T = s_e^m] \]
which we summarize in the following theorem.
Theorem 7 For a small neighborhood $W_e \subset C$ of $s_e = 0$ we have that
\[ f(C_e(s_e)) \to f(e) \in V \]
as $s_e \to 0$.

Now consider the $s_e^i$ terms of $\hat{Q}_e(s_e)$. Let $i = N$ be the first index of $\hat{Q}_e(s_e)$ where the coefficient changes as $e$ changes, i.e. writing
\[ \hat{Q}_e(s_e) = \sum \beta_i s_e^i \]
we choose $i = N$ such that $\beta_N$ changes as $e$ changes, but that $\beta_i$ are independent of $e$ for $i < N$. We now come to our main theorem for this section. Let
\[ \hat{C}_e(s_e) : s_e \to [X_1 = 1 : X_2 = P_e(s_e) : T = s_e^m] \]
where
\[ P_e(s_e) = \sum_{i \leq N} \beta_i s_e^i, \]
i.e. $P_e$ is the power series $\hat{Q}$ truncated at index $N$.

**Theorem 8** We have that
\[ f(\hat{C}_e(s_e)) \to f(e) \in V \]
as $s_e \to 0$.

**Proof:** We consider again the notation of the $t_i$, $f_i$ and $g_i$ of when we regarded the sequence of blowups $X_n \to ... \to X_1$. Notice that the value of $t_i$ is given by
\[ \lim_{s_e \to 0} \frac{\hat{f}_i(\hat{Q}_e(s_e), s_e^m)}{\hat{g}_i(\hat{Q}_e(s_e), s_e^m)} = \lim_{s_e \to 0} \frac{\hat{f}_i(\hat{Q}_e(s_e), s_e^m)}{\hat{g}_i(\hat{Q}_e(s_e), s_e^m)} \]
where $\hat{f}_i$ and $\hat{g}_i$ are polynomials in $X_2$ and $T$. Let us study this limit explicitly.

Consider $\hat{f}_i(\hat{Q}_e(s_e), s_e^m)$. We can write this as a sum
\[ \hat{f}_i(\hat{Q}_e(s_e), s_e^m) = \sum_{i,j,k} \delta_{i,j,k} s_e^{i+mj+Nk}(\gamma + s_e G)^k \]
where $\delta_{i,j,k} \in \mathbb{C}$ are constants and $G(s_e) \in \mathbb{C}[[s_e]]$ is a power series which depends on the terms $\beta_{N+i}s_e^{N+i}$ of $\hat{Q}_e$, i.e. the terms after index $N$. Here we wrote $\gamma := \beta_N(e)$.

For indices $i, j, k$ define $D(i,j,k) := i + mj + Nk$. Notice we can write
\[ \hat{f}_i(\hat{Q}_e(s_e), s_e^m) = \sum_r \sum_k \sum_{D(i,j,k)=r} \delta_{i,j,k} s_e^{r}(\gamma + s_e G)^k. \]
Let $r = D_0$ be the smallest index such that the expression
\[
\sum_k \sum_{D(i,j,k) = r = D_0} \delta_{i,j,k} s_e^r (\gamma + s_e G)^k
\]
is nonzero.

Assume that for all $k > 0$, $\delta_{i,j,k} = 0$ when $D(i,j,k) = D_0$. Then we notice that
\[
\hat{f}_i(\tilde{Q}_e(s_e), s_e^m) = \Delta s_e^{D_0} + O(s_e^{D_0+1})
\]
where $\Delta \neq 0$ is a constant and independent of $e$ or $\gamma$ and hence the limit
\[
\lim_{s_e \to 0} \frac{\hat{f}_i}{s_e^{D_0}}
\]
is independent from $\gamma$ and $G$.

Assume now that for some $D(i,j,k) = D_0$ with $k > 0$ we have that $\delta_{i,j,k} \neq 0$. We consider the sum
\[
\sum_k \sum_{i,j:D(i,j,k) = D_0} \delta_{i,j,k} (\gamma + s_e G)^k.
\]
Fix a $k$ and consider
\[
\sum_{i,j:D(i,j,k) = D_0} \delta_{i,j,k} (\gamma + s_e G)^k.
\]
If
\[
\sum_{i,j:D(i,j,k) = D_0} \delta_{i,j,k} = 0
\]
then we notice that neither $\gamma$ nor $G$ will play a role in it, as it is 0.

Assume now, that for some $k$, the expression
\[
\sum_{i,j:D(i,j,k) = D_0} \delta_{i,j,k}
\]
is not zero. Then the expression
\[
\sum_k \sum_{i,j:D(i,j,k) = D_0} \delta_{i,j,k} (\gamma + s_e G)^k
\]
will be of the form $p(\gamma) + s_e H$, where $p(\gamma)$ will be a nontrivial polynomial in $\gamma$ and $H$ will be some power series in $s_e$. Notice in this case the factor $s_e$ in front of the $H$. Furthermore, the information of $G$ will only go into the construction of $H$, $p(\gamma)$ is not affected by $G$. 

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In this case we note that
\[
\lim_{s_e \to 0} \frac{\hat{f}_i}{s_e^{0}}
\]
will depend and change as \( \gamma \) changes, but that it will be independent of \( G \), because \( G \) always occurs with the \( s_e \) in front of it.

Lastly, assume that for all \( k > 0 \), we have that
\[
\sum_{i,j:D(i,j,k)=D_0} \delta_{i,j,k} = 0.
\]
Then neither \( \gamma \) nor \( G \) will play a role in the limit, only the constants \( \sum_{i,j} \delta_{i,j} \), which is assumed nonzero as we have chosen \( D_0 \) to be the smallest nonzero such expression.

A similar analysis applied to \( \hat{g}_i \). Hence we see that the limit \( \lim_{s_e} \) is either not affected by the term \( s_e^{N} \gamma \) and in this case also not by the higher terms, or if it is affected by \( \gamma \), then only by \( \gamma \) and not by the higher terms. We are done.

[End-Of-Proof]

The importance of this theorem is that we may study the finiteness variety component \( V \) as an approximation by an affine line (which might intersect itself). Indeed, setting \( u = s_e^{-1} \), we note that we can write
\[
\hat{C}_e(s_e) = \hat{C}(e, u)
\]
where
\[
\hat{C}(e, u) : u \to [x_1 = u^m; x_2 = p_e(u)]
\]
with \( p_e(u) \) a polynomial in \( u \) and \( u^{-1} \). Here \( p_e(u) \) is essentially a constant polynomial, except for the term \( E(e)u^{m-N} \) which changes as \( e \) changes. Here \( E(e) = \beta_X(e) \) is some algebraic function of \( e \) (which might have branches).

Set \( \gamma = E(e) \). Note that for a specific \( \gamma \), there may be several \( e \) such that \( E(e) = \gamma \). However, changing \( \gamma \) also changes the branches of \( E : e \to \gamma \). Notice however, that we may view \( p_e(u) \) solely as a polynomial in \( u, u^{-1} \) and \( \gamma \) and hence also \( \hat{C} = \hat{C}(\gamma, u) \).

Consider now the image of \( \hat{C}(\gamma, u) \) under \( f \). It maps to a curve \( D(\gamma, u) \) which has the property that \( D(\gamma, u) \to V \) as \( u \to \infty \) and this limit point changes as \( \gamma \) changes.

Hence we see that \( D(\gamma, u) \) involves only terms of \( \gamma \) and \( u^{-1} \). Furthermore, we see that \( V \) is parametrized by two polynomials in \( \gamma \), i.e.
\[
V = (a_0(\gamma), b_0(\gamma)).
\]

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In particular we can write

\[ f(\hat{C}(\gamma, u)) = \left[ \sum a_i(\gamma)u^{-i}, \sum b_i(\gamma)u^{-i} \right] \in Y_0. \]

Hence we see that we have a map

\[ f: \text{spec}(C[u^{-1}, \gamma]) \to Y_0 \]

and for \( u^{-1} \) this maps to \( V \). In particular, for generic \( \gamma \), where \( \frac{d a_0(\gamma)}{d \gamma} \) and \( \frac{d b_0(\gamma)}{d \gamma} \) do not both vanish simultaneously, we see that \( \gamma \) is a local parameter for \( \Gamma \) around \( e \in \Gamma \), here local parameter means in the analytic sense (note \( \gamma \) need not be in \( K_X/C(X_0) = C(\hat{X}) \) but is in the analytic completed local ring of \( e \)).

Let us now consider again the relation between \( e \) and \( \gamma \). From the representation \( x_1 = u^m \) and \( x_2 = \sum \beta_i u^{m-i} + \gamma u^m - N \) we see that \( C(X_0) \subset C(u, \gamma) \) and the latter is an extension of degree \( m \). In particular we see that \( e = \Omega(\gamma, u^{-1}) \) for some rational function \( \Omega \in C(\gamma, u^{-1}) \). Now let \( e \in \Gamma \). Notice that this corresponds to \( u^{-1} = 0 \) and hence as \( e \) varies on \( \Gamma \) we see that \( e = \Omega(\gamma, 0) \). Let \( e \in \Gamma \cong \mathbb{P}^1 \) be a point such that the map \( \Omega(-, 0): \mathbb{P}^1 \to \mathbb{P}^1 \) is not ramified above \( e \). Let \( \gamma_1, \ldots, \gamma_r \) be the values of \( \gamma \) above \( e \) and assume that \( e \) has been chosen such that all of the \( \gamma_i \) are finite.

**Lemma 3** For generic \( e \) we have that \( u^{-1} \) and \( \gamma - \gamma_i \) are local parameters for the completed local ring of \( \hat{X} \) at \( e \in \Gamma \subset \hat{X} \).

**Proof:** Consider \( Z_{e}(t) \). For our curve we chose the representation \( g_n - 1 = t \) and \( t_{n-1} = e \). Hence the function \( t \) is a local parameter together with \( e \) at \( \gamma_0 \) (not just in the completed local ring but also in \( \hat{X} \) itself).

Write \( Z_e(t) = t^m Z_0(e, t) \). Notice that for generic \( e_0 \) with \( Z_0(e_0, 0) \neq 0 \) we have that

\[ \frac{\delta Z}{\delta t} = Z_0(e, t) \frac{d}{d t} \in C[[e - e_0, t]]. \]

Furthermore, for almost all \( e \) we have that

\[ \gamma - \gamma_i \in C[[e - e_0, u^{-1}]] \]

We have that both \( \frac{\delta Z}{\delta t} \) and \( \frac{\delta Z}{\delta e} \) exist. We see thus that locally in around \( \gamma_0 \)

\[ \frac{d Z}{d u^{-1}} = \left[ \frac{\delta Z}{\delta t} \right] dt + i \frac{\delta Z}{\delta e} de. \]

The result follows. [End-Of-Proof]
Remark 5 Let $\Psi(r) : \mathbb{R} \to \hat{X}$ be a curve in $\hat{X}$. If $\Psi(c) = e \in \Gamma \subset \hat{X}$ and is smooth there in the sense that its derivatives exist in the tangent space around $\gamma_0$, then after choosing a branch of $u, \gamma$ we can write $u^{-1} = u^{-1}(r)$ and $\gamma = \gamma(r)$. It needs to be stressed that this can only be done once AFTER choosing a branch of $u$ and $\gamma$. In this case we see that both $\frac{du}{dr}|_c$ and $\frac{dv}{dr}|_c$ exist as $u^{-1}$ and $\gamma$ are analytic functions around $e \in \Gamma$ and hence analytic functions of $r$ if $\Psi(r) \in C^1$.

We would like to remark on the above and the blowup $U_n \to U_{n-1}$. We had that $t_{n-1}$ was defined by $t_{n-1}g_{n-1} = f_{n-1}$. However, we could also have studied a different chart, namely the chart defined by $v_{n-1}f_{n-1} = g_{n-1}$ where $v_{n-1} = \frac{1}{t_{n-1}}$. In this case we see that the curve $C_e(t)$ induces $v_{n-1} = \frac{1}{r}$ and $f_{n-1} = et$. We thus see that the tangent of $C_e(t)$ at $t = 0$ is in $\mathcal{C}\overline{f_{n-1}}$ and its projection onto $\mathcal{C}\overline{v_{n-1}}$ is 0.

Lastly, before we end this section we would like to comment on trajectories in $X_0$ which tend to a point $e \in \Gamma \subset \hat{X}$.

Lemma 4 Assume that

$$\Psi(r) : \mathbb{R} \to X_0 \subset \hat{X}, r \to (x_1(r), x_2(r))$$

is a trajectory in $X_0$ which tends to a point

$$e \in \Gamma \subset \hat{X}$$

as $r \to c$ and such that $x_1(r)$ does not tend to 0. Let $\gamma_0$ be a value of $\gamma$ which maps to $e$. Then there exists a branch $u, \gamma$ such that $\Psi(r)$ lifts to a curve in the $u - \gamma$ plane which tends to $\gamma_0$ as $r \to c$.

We would now like to comment on the condition that $x_1(r)$ does not tend to 0. Notice that we can adjust our map $f$ to arrange that this is always the case. Indeed, $\Gamma$ is constructed from blowups in the plane at infinity $T = 0$ and the first blowup takes place at $[X_1 = 1 : X_2 = h_0 : T = 0]$. Hence if $(x_1, x_2) \to e \in \Gamma$, then $x_1$ cannot tend to 0 (see Assumption \[4\]).

We would also like to illustrate by example how to obtain the $u - \gamma$ representation. For this we shall calculate three examples which we shall always refer to throughout this paper.

Let $X = \mathbb{P}^2$ and denote by $X_1, X_2, T$ homogenous coordinates for $X$. Consider the blowup $\hat{X}_1$ of $X$ at $[X_1 = 1 : X_2 = 0 : T = 0]$. This induces a line parametrized by $u$ where $x_2 = tu$, and $x_2 = \frac{t}{X_1}$ and $t = \frac{X_1}{X_1}$. Now blowup $\hat{X}_1$ at $u = 2$ and $t = 0$ to obtain $\hat{X}_2$. The exceptional curve here is parametrized by $w$ where $t = w(u - 2)$. Lastly blowup $\hat{X}_2$ at $u = 1$ and $u = 2$ to obtain $\hat{X}_3$ with exceptional curve $\Gamma$ parametrized by $v$ where $w = 1 = v(u - 2)$.
Now consider the $s$-curve $(u = 2 + s, v = e + s)$ approaching a point $(u = 2, v = e)$ on $\Gamma$. We can backtrack to find a $s$-curve in terms of $X_2$ and $T$. Indeed, $w$ is given by

$$w(s) = 1 + es + s^2.$$

Furthermore, backtracking further, we have

$$t(s) = w(u - 2) = s + es^2 + s^3$$

and lastly we obtain for

$$x_2(s) = ut = 2s + s^2 + 2es^2 + es^3 + 2s^3 + s^4.$$  

The first of order of business is to write $t(s) = z$ where $z$ is some parameter. Notice that we have

$$s = z - ez^2 - z^3 + 2ez^3 + O(z^4).$$

Substituting this into $x_2(s)$ we obtain $x_2 = 2z + z^2 - ez^3 + O(z^4)$. We would now like to illustrate we can truncate $x_2$ at degree 3, i.e. only consider the trajectory $[X_1 = 1 : X_2 = 2z + z^2 - ez^3 : T = z]$.

Indeed, one sees immediately that $u(z)$ is given by $2 + z - ez^2$ and notice $u \to 2$ as $z \to 0$. Furthermore,

$$w = \frac{t}{u - 2} = \frac{z}{z - ez^2} \to 1$$

as $z \to 0$. Lastly

$$v = \frac{w - 1}{u - 2} \to e$$

as $z \to 0$.

The example above was for the projective space $\mathbb{P}^2$ on its own. Let us now consider an example where we have a morphism.

Let $f_0 : X_0 \to Y_0$ be given by $[x_1, x_2] \to [x_1 x_2 + x_2^3, x_1 x_2]$. Notice that this map admits the finiteness variety $V_{f_0} = V = (\epsilon, \epsilon)$ where $\epsilon \in \mathbb{C}$.

We blowup $X = \mathbb{P}^2$ once at $X_1 = 1; X_2 = 0; T = 0$ to obtain the component $\Gamma$ which maps to $V$. We obtain the representation $x_1 = u$ and $x_2 = \epsilon u^{-1}$.

Consider now the map $f_1 : Z_0 = \mathbb{A}^2 \xrightarrow{g_0} X_0 \xrightarrow{f_0} Y_0$ where

$$g_0(z_1, z_2) = [z_1, z_2 + z_1^2].$$

Notice that $g_0$ is Keller. Blowing up we obtain the representation

$$x_1 = v + \gamma v^{-2}, x_2 = -v^2$$
where $\gamma = \frac{1}{2}\epsilon$. One can also deduce this representation in another way, without blowing up.

Indeed notice that $z_2 = x_2 - x_1^2 = \epsilon u^{-1} - u^2$ and $z_1 = x_1 = u$. We shall see later on that $z_2$ must have the representation $z_2 = -v^2$ for some parameter $v$. Hence we see

$$-v^2 = -u^2[1 - \epsilon u^{-3}]$$

and hence

$$v = \pm u\sqrt{1 - \epsilon u^{-3}} = \pm u[1 - \frac{\epsilon}{2}u^{-3} + ..].$$

Hence $u = \pm v[1 + \frac{\epsilon}{2}v^{-3} + ..]$. Substituting this into $z_1$ and applying the truncation theorem we arrive at $z_1 = v + \frac{\epsilon}{2}v^{-2}$.

4 The Keller condition and the Jacobian matrix

In this section we shall continue our study of the components of the finiteness variety. Before we state our main results, we recall our main result of the previous section:

**Theorem 9** There exists an affine curve, parametrized by $\gamma$, given by

$$\hat{C}(\gamma, u) : u \to [x_1 = u^m : x_2 = h_0.u^m + h_1.u^{m-1} + .. + \gamma.u^{m-N}]$$

which is such that

$$f(\hat{C}(\gamma, u)) \to v_\gamma \in V$$

as $u \to \infty$. Furthermore, $v_\gamma \in V$ changes as $\gamma$ changes.

We can write

$$D(\gamma, u) := f(\hat{C}(\gamma, u)) = [a(\gamma, u), b(\gamma, u)]$$

where $a$ respectively $b$ are polynomials in $\gamma$ and $u^{-1}$.

Let us fix some notation before stating the main results. We may write

$$a(\gamma, u) = a_0(\gamma) + a_1(\gamma)u^{-1} + a_2(\gamma)u^{-2} + ...$$

and similarly for

$$b(\gamma, u) = b_0(\gamma) + b_1(\gamma)u^{-1} + b_2(\gamma)u^{-2} + ....$$

The points of $V$ are given by

$$\gamma \to (a_0(\gamma), b_0(\gamma)).$$

**Definition 1** We shall say that a choice $\gamma = \gamma_0$ is admissible if $\frac{da_0}{d\gamma}$ and $\frac{db_0}{d\gamma}$ are both nonzero.
For an admissible \( \gamma = \gamma_0 \), we can write
\[
a(\gamma_0 + \delta, u) = a(\gamma_0, u) + \delta \chi_1(\gamma_0, u) + \delta^2 \chi_2(\gamma_0, u) + ...
\]
and similarly for
\[
b(\gamma_0 + \delta, u) = b(\gamma_0, u) + \delta \psi_1(\gamma_0, u) + \delta^2 \psi_2(\gamma_0, u) + ....
\]
where \( \chi_i \) and \( \psi_i \) are some polynomials in \( \gamma, u \) and \( u^{-1} \). Notice that the \( \chi_i \) and \( \psi_i \) only depend on the \( \gamma_0 \) and \( u \) and are all polynomials in \( \gamma_0 \) and \( u^{-1} \).

Furthermore, \( \chi_1 = \frac{\delta a}{\delta \gamma} \) and \( \psi_1 = \frac{\delta b}{\delta \gamma} \).

Let us state our main result.

**Theorem 10** We have the following representation of \( J(f) \) in terms of \( u \) and \( \gamma \):

\[
J(f) = \begin{bmatrix} u^{m-N} + \frac{\delta h_1}{\delta \gamma} r_3 & \chi_1 u^{N-m} \\ \psi_1 r_3 & \psi_1 u^{N-m} \end{bmatrix}
\]

where \( r_3 \) is given by the following expression:

\[
r_3 = \frac{\delta h - \psi_1 u^{N-m} [mh_0 u^{m-1} + (m-1)h_1 u^{n-2} + ...]}{mu^{m-1}}.
\]

We devote the rest of this section to proving this.

Fix a \( u \). Let \( g := g_{\gamma_0, u}(Y_1, Y_2) \) be the local inverse of \( f \) above \( \hat{C}(\gamma_0, u) \), i.e. a local inverse of \( f \) such that
\[
g(D(\gamma_0, u)) = \hat{C}(\gamma_0, u).
\]

Let \( p \) be a prime such that all coefficients of \( \hat{C} \) are \( p \)-adically integral.

**Lemma 5** There exists an \( \epsilon > 0 \) such that for \( v(\delta) > \epsilon \) we have that
\[
g(D(\gamma_0 + \delta, u)) = \hat{C}(\gamma_0 + \delta, u).
\]

**Proof:** \( f \) is etale and hence for some small \( p \)-adic neighborhood \( U \) around \( \hat{C}(\gamma_0, u) \) we have that \( g(f(U)) = U \). \( \text{[End-Of-Proof]} \)

Now we consider the Jacobian \( J(f) \) of \( f \) at \( \hat{C}(\gamma_0, u) \). Write
\[
J(f) = \begin{bmatrix} r_1 & r_2 \\ r_3 & r_4 \end{bmatrix}.
\]

**Remark 6** As \( |J(f)| = 1 \) we have explicitly that
\[
J^{-1} = \begin{bmatrix} r_4 & -r_2 \\ -r_3 & r_1 \end{bmatrix}.
\]
Furthermore, write
\[ g(a_0(\gamma_0) + Y_1, b_0(\gamma_0) + Y_2) = C_0 + C_1(Y_1, Y_2) + C_2(Y_1, Y_2) + ... \]
where the \( C_i \) is the degree \( i \) homogenous terms of \( g \).

**Lemma 6** We have that \( C_0 = \hat{C}(\gamma_0, u) \).

**Lemma 7** We have that
\[
C_1 = J^{-1} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} r_4 & -r_2 \\ -r_3 & r_1 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}
\]
where \( J \) is the Jacobian \( J(f) \) of \( f \) at \( \hat{C}(\gamma_0, u) \).

By Kramer’s theorem we have that
\[
J^{-1}(Y_1, Y_2) = (Y_1 r_4 - Y_2 r_2, Y_2 r_1 - Y_1 r_3) .
\]

Setting
\[
Y_1 = \delta \chi_1(\gamma_0, u) + \delta^2 \chi_2(\gamma_0, u) + ...
\]
and
\[
Y_2 = \delta \psi_1(\gamma_0, u) + \delta^2 \psi_2(\gamma_0, u) + ...
\]
we now study what \( g(Y_1, Y_2) \) could be. We keep \( \delta \) a free parameter.

**Lemma 8** We have that
\[
g(a_0(\gamma_0) + Y_1, b_0(\gamma_0) + Y_2) = (u^m, \sum_{m \geq i > m-N} h_i u^i + (\gamma_0 + \delta)u^{m-N})
\]
\[
= \hat{C}(\gamma_0, u) + (0, u^{m-N}\delta).
\]
Notice however, that we can choose \( \delta \) freely, as long as \( v(\delta) > \epsilon \). This places a large restriction on \( J^{-1} \) which we now explain. Indeed, notice that \( C_i(Y_1, Y_2) \) are all expressions which involve \( \delta^2 \) for \( i \geq 2 \). Hence we can write
\[
g(Y_1, Y_2) = \hat{C}(\gamma_0, u) + J^{-1}(Y_1, Y_2) + O(\delta^2).
\]
Here we note that
\[
J^{-1} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} \delta(\chi_1 r_4 - \psi_1 r_2) \\ \delta(\psi_1 r_1 - \chi_1 r_3) \end{bmatrix} + O(\delta^2).
\]

**Lemma 9** We have that \( \chi_1 r_4 - \psi_1 r_2 = 0 \).

**Proof:** This follows from Lemma 5. Indeed, if the expression above was nonzero then this would imply a change in \( x_1 = u^m \) which is a contradiction.

[End-Of-Proof]
**Lemma 10** We have that $\psi_1 r_1 - \chi_1 r_3 = u^{m-N}$.

**Proof:** This is similar as above, where we note that the terms $C_i$ for $i \geq 2$ only involves $\delta^2$ terms. [End-Of-Proof]

Furthermore, we know that $|J(f)| = 1$, hence $r_1 r_4 - r_2 r_3 = 1$. Hence we can solve for $r_1, r_2, r_3$ and $r_4$. We obtain

**Theorem 11** We have that $r_4 = \psi_1 u^{N-m}$ and $r_2 = \chi_1 u^{N-m}$. Furthermore, $r_3$ is free and $r_1$ is then given by

$$r_1 = \frac{\chi_1 r_3 + u^{m-N}}{\psi_1}.$$ 

**Remark 7** In general we have

$$r_1 = \frac{\chi_1 r_3 + |J(f)|u^{m-N}}{\psi_1}.$$ 

Now we shall study the free parameter $r_3$. Indeed, we can write

$$J(f) = \begin{bmatrix} \frac{u^{m-N+\chi_1 r_3}}{\psi_1} & \chi_1 u^{N-m} \\ \chi_1 u^{N-m} & \psi_1 u^{N-m} \end{bmatrix}.$$ 

Define

$$\frac{\delta D(\gamma, u)}{\delta u} = [a_u(\gamma, u), b_u(\gamma, u)]$$

to be the partial derivative of $D$ in $u$ (notice that it is a vector). In this case we have that

$$a_u(\gamma, u) = -a_1(\gamma)u^{-2} - 2a_2(\gamma)u^{-3} - ...$$

and similarly for

$$b_u(\gamma, u) = -b_1(\gamma)u^{-2} - 2b_2(\gamma)u^{-3} - ....$$

For a fixed $\gamma$ we thus see that

$$\begin{bmatrix} a_u \\ b_u \end{bmatrix} = \frac{dD(\gamma, u)}{du} = J(f). \frac{d\vec{C}(\gamma, u)}{du} = J(f) \begin{bmatrix} mu^{m-1} \\ mh_0 u^{m-1} + (m-1)h_1 u^{m-2} + ... \end{bmatrix}$$

$$= \begin{bmatrix} \frac{u^{m-N+\chi_1 r_3}}{\psi_1} & \chi_1 u^{N-m} \\ \chi_1 u^{N-m} & \psi_1 u^{N-m} \end{bmatrix} \begin{bmatrix} mu^{m-1} \\ mh_0 u^{m-1} + (m-1)h_1 u^{m-2} + ... \end{bmatrix}.$$ 

We see thus that

$$mu^{m-1}r_3 + \psi_1 u^{N-m}[mh_0 u^{m-1} + (m-1)h_1 u^{m-2} + ...] = b_u$$

hence we can solve for $r_3$ as

$$r_3 = \frac{b_u - \psi_1 u^{N-m}[mh_0 u^{m-1} + (m-1)h_1 u^{m-2} + ...]}{mu^{m-1}}.$$
Lastly, we would like to deal with some corner case, i.e. for instance when one of \( \frac{da}{d\gamma} \) or \( \frac{db}{d\gamma} \) is identically zero, or if \( m = 0 \).

If one of \( \frac{da}{d\gamma} \) or \( \frac{db}{d\gamma} \) is identically zero, then this would imply that the component \( V \) is a straight line. However by Van Chau (20) this cannot be the case. Lastly, consider what happens if \( m = 0 \). Then \( (a_0(\gamma), b_0(\gamma)) \) would be a single point, which can also not happen.

Let us illustrate our equations with an example. Recall the map \( f_0 \) constructed in the examples of Section 3. Consider \( f_0(x_1, x_2) = (x_1 x_2, x_1 x_2 + x_2^2) \). Notice that in this case there is a finite variety in \( Y_0 \), namely the line \( C := (\epsilon, \epsilon) \) as \( \epsilon \in \mathbb{C}^2 \).

Notice that we can approach \( C \) with the curve \( x_1 = t, x_2 = \epsilon t^{-1} \). In our notation, this would imply that \( m = 1 \) and \( L = 1 \). The Jacobian of \( f \) in this representation is given by

\[
J(f) = \begin{bmatrix}
x_2 & x_1 \\
x_2 & x_1 + 2x_2
\end{bmatrix}
= \begin{bmatrix}
et^{-1} & t \\
et^{-1} & t + 2\epsilon t^{-1}
\end{bmatrix}.
\]

Furthermore, the determinant of the Jacobian is given by

\[
|J(f)| = 2x_2^2 = 2\epsilon^2 t^{-2}.
\]

Let us bring this into relation with our standard \( u, \gamma \) representation of \( J(f) \), that we deduced earlier on. We have that \( a(\gamma, u) = \epsilon \) and \( b(\gamma, u) = \epsilon + \epsilon^2 t^{-2} \). Hence \( a_u = 0 \) and \( \chi_1 = 1 \). Furthermore we have that \( b_u = -2\epsilon^2 t^{-3} \) and \( \psi_1 = 1 + 2\epsilon t^{-2} \).

Hence we see that if we write

\[
J(f) = \begin{bmatrix}
r_1 & r_2 \\
r_3 & r_4
\end{bmatrix}
\]

then \( r_2 = t^L \chi_1 \) and \( r_4 = t^L \psi_1 \), as expected. Let us study \( r_3 \). Notice that

\[
r_3 := \epsilon t^{-1} = -2\epsilon^2 t^{-3} - t(1 + 2\epsilon t^{-2})[-\epsilon t^{-2}] = b_u - t^L \psi_1 [-L \epsilon t^{-L-1}]
\]

which fits our expectation. Lastly, note that we have

\[
r_1 := \epsilon t^{-1} = \frac{2\epsilon^2 t^{-2} t^{-1} + 1 \epsilon t^{-1}}{1 + 2\epsilon t^{-2}}
\]

which thus shows that \( r_1 = \frac{|J(f)|t^{-L} + r_3 \chi_1}{\psi_1} \), which is as expected.
5 Differential equations describing the finiteness variety

In the previous sections we constructed the curves $\hat{C}(\gamma, u)$, $D(\gamma, u)$ and also found explicit form for the Jacobian matrix at $\hat{C}(\gamma, u)$ in terms of $\gamma$ and $u$. In this section we carry our analysis further. Our main theorems are the following.

Theorem 12 We have the following relation on $a(\gamma, u)$ and $b(\gamma, u)$:
\[
\frac{\delta b}{\delta \gamma} \frac{\delta a}{\delta u} - \frac{\delta a}{\delta \gamma} \frac{\delta b}{\delta u} = mu^{2m-N-1}.
\]

Theorem 13 Let $k$ be smallest index larger than 0 such that one of $a_k$ or $b_k$ is nonzero. Then either $k < N - 2m$ and
\[
\frac{da_0}{d\gamma} b_k - \frac{db_0}{d\gamma} a_k = 0
\]
or $k = N - 2m$ and
\[
\frac{da_0}{d\gamma} b_k - \frac{db_0}{d\gamma} a_k = m \cdot k.
\]

Fix a $\gamma$ and consider the curve
\[
u \to f(\hat{C}(\gamma, u)) = D(\gamma, u) = \begin{bmatrix} a(\gamma, u) \\ b(\gamma, u) \end{bmatrix}
\]
We may ask what $\frac{dD}{du}$ is, i.e. the derivative vector of $u \to D(\gamma, u)$. Note that this is given by
\[
\begin{bmatrix} \frac{da}{du} \\ \frac{db}{du} \end{bmatrix} = \frac{dD}{du} = J(f) \begin{bmatrix} \frac{dC_1}{du} \\ \frac{dC_2}{du} \end{bmatrix}
\]
where we wrote $\hat{C} = [C_1, C_2]$.

Hence we have the relations
\[
\frac{\delta a}{\delta u} = r_1 \frac{dC_1}{du} + r_2 \frac{dC_2}{du}
\]
and
\[
\frac{\delta b}{\delta u} = r_3 \frac{dC_1}{du} + r_4 \frac{dC_2}{du}.
\]
However we can now use our knowledge about the $r_i$ that we gained in the previous section. Indeed, we have
\[
\frac{\delta a}{\delta u} = r_1 \frac{dC_1}{du} + \chi_1 u^{N-m} \frac{dC_2}{du}
\]
and
\[
\frac{\delta b}{\delta u} = (\frac{\psi_1 r_1 - u^{m-N}}{\chi_1}) \frac{dC_1}{du} + \psi_1 u^{N-m} \frac{dC_2}{du}.
\]
We can now multiply the top equation by \( \psi_1 \) and the bottom equation by \( \chi_1 \) to obtain
\[
\psi_1 \frac{\delta a}{\delta u} = \psi_1 r_1 \frac{dC_1}{du} + \psi_1 \chi_1 u^{N-m} \frac{dC_2}{du}
\]
and
\[
\chi_1 \frac{\delta b}{\delta u} = (\psi_1 r_1 - u^{m-N}) \frac{dC_1}{du} + \chi_1 \psi_1 u^{N-m} \frac{dC_2}{du}.
\]
Thus we finally arrive at the relation
\[
\psi_1 \frac{\delta a}{\delta u} - \chi_1 \frac{\delta b}{\delta u} = u^{m-N} \frac{dC_1}{du}.
\]
Now we notice that \( \frac{dC_1}{du} = mu^{m-1} \). Hence we arrive at the following

**Lemma 11** We have that
\[
\psi_1 \frac{\delta a}{\delta u} - \chi_1 \frac{\delta b}{\delta u} = mu^{2m-N-1}.
\]
Recall that we expanded \( a \) and \( b \) as follows:
\[
a(\gamma, u) = a_0(\gamma) + a_1(\gamma)u^{-1} + a_2(\gamma)u^{-2} + \ldots + a_{d_1}(\gamma)u^{-d_1}
\]
and similarly for
\[
b = b_0(\gamma) + b_1(\gamma)u^{-1} + \ldots + b_{d_2}(\gamma)u^{-d_2}.
\]
Similarly we defined \( \chi_1 \) and \( \psi_1 \) via the derivatives of \( a \) and \( b \) under \( \gamma \), indeed recall that :
\[
a(\gamma_0 + \delta, u) = a(\gamma_0, u) + \delta a_1(\gamma_0, u) + \delta^2 a_2(\gamma_0, u) + \ldots
\]
and similarly for
\[
b(\gamma_0 + \delta, u) = b(\gamma_0, u) + \delta b_1(\gamma_0, u) + \delta^2 b_2(\gamma_0, u) + \ldots
\]
Notice that we can write
\[
\chi_i = \frac{d a_0}{d \gamma} + \frac{d a_1}{d \gamma} u^{-1} + \ldots + \frac{d a_{D_1}}{d \gamma} u^{-D_1}
\]
and similarly
\[
\psi_i = \frac{d b_0}{d \gamma} + \frac{d b_1}{d \gamma} u^{-1} + \ldots + \frac{d b_{D_2}}{d \gamma} u^{-D_2}
\]
where \( D_1, D_2 \in \mathbb{N} \). Grouping the coefficients of \( u \) and the differential equation relation we obtain some partial differential like relations between the \( a_i \) and \( b_i \).

Interesting is the case of the first \( a_k, b_k \) such that one of \( a_k \) or \( b_k \) is nonzero for \( k > 0 \). In this case we either have that
\[
\frac{d a_0}{d \gamma} b_k - \frac{d b_0}{d \gamma} a_k = 0
\]
and $-k > 2m - N$ or
\[ \frac{da_0}{d\gamma} b_k - \frac{db_0}{d\gamma} a_k = \frac{m}{k} \]
and $k = N - 2m$.

We can summarize in the following theorem.

**Theorem 14** We have that $k \leq N - 2m$. Furthermore, in case $k < N - 2m$ then
\[ \frac{da_0}{d\gamma} b_k - \frac{db_0}{d\gamma} a_k = 0. \]
If $k = N - 2m$ then
\[ \frac{da_0}{d\gamma} b_k - \frac{db_0}{d\gamma} a_k = \frac{m}{k}. \]

**Remark 8** In this section we subtly used the assumption $|J(f)| = 1$ in another way. The culprit is the term $r_1$. In general one has
\[ r_1 = \frac{|J(f)| u^{m-N} + r_3 \chi_1}{\psi_1}. \]

### 6 A coupled system of ordinary differential equations

Now consider again the fundamental relation
\[ \frac{\delta f_1 \delta f_2}{\delta x_1 \delta x_2} - \frac{\delta f_1 \delta f_2}{\delta x_2 \delta x_1} = 1. \]

Consider a system of coupled ordinary differential equations given by
\[ \frac{dx_1}{dr} = \frac{\delta f_2}{\delta x_2} = \psi_1 u^{N-m} \]
and
\[ \frac{dx_2}{dr} = -\frac{\delta f_2}{\delta x_1} = -r_3. \]

**Definition 2** We shall refer to the above system as the inverse dynamics coupled system.

Notice that we can solve for $u$ and $\gamma$ from $x_1$ and $x_2$.

**Lemma 12** $f_2$ is constant along such a trajectory.
PROOF: Indeed we have that
\[
\frac{df_2}{dr} = \frac{\delta f_2}{\delta x_1} \frac{dx_1}{dr} + \frac{\delta f_2}{\delta x_2} \frac{dx_2}{dr} = 0.
\]
We see thus that \( f_2 \) must be constant along the trajectory of \( x_1 \) and \( x_2 \). [End-Of-Proof]

**Lemma 13** We have that \( f_1(r) = r + C \).

**Remark 9** Here the assumption that \( |J(f)| = 1 \) was used. In general
\[
\frac{df_1}{dr} \to |J(f)|.
\]

**Lemma 14** There are no cycles in the trajectory \( (x_1(r), x_2(r)) \).

Let us translate these equations into the unknowns \( u \) and \( \gamma \). For the following set
\[ L := N - m. \]
Then we have \( x_1 = u^m \) and
\[ x_2 = h_0 u^m + h_1 u^{m-1} + \ldots + \gamma u^{-L}. \]
Hence we obtain
\[
\frac{du}{dr} = \frac{\zeta_m}{m} u^{1-\frac{m}{m}} \frac{dx_1}{dr} = \frac{\zeta_m}{m^2} u^{1-m \frac{1}{2}} \frac{dx_2}{dr} = \frac{\zeta_m}{m^2} u^{1-m \frac{L}{2}} \psi_1
\]
as \( \frac{dx_2}{dr} = \psi_1 u^{N-m} \). Furthermore for \( \gamma \) we obtain that
\[ \gamma = x_2 u^L - h_0 u^{m+L} - h_1 u^{m+L-1} - \ldots \]
and hence
\[
\frac{d\gamma}{dr} = Lu^{L-1} \frac{x_2}{dr} \frac{du}{dr} + u^L \frac{dx_2}{dr}
\]
\[
-[(m + L) h_0 u^{m+L-1} + (m + L - 1) h_1 u^{m+L-2} + \ldots] \frac{du}{dr}
\]
\[
= Lu^{L-1} \frac{x_2}{dr} \frac{du}{dr} + u^L \frac{dx_2}{dr}
\]
\[
- \frac{\zeta_m}{m} [(m + L) h_0 u^{m+L-1} + (m + L - 1) h_1 u^{m+L-2} + \ldots] u^{1-L} \psi_1.
\]
Hence we see that
\[
\frac{d\gamma}{dr} = \frac{L \zeta_m}{m} u^{2L-m} \psi_1 [h_0 u^m + h_1 u^{m-1} + \ldots]
\]
\begin{align*}
&+ u^L u^L \psi_1 (m \eta_0 u^{m-1} + (m - 1) h_1 u^{m-2} + \ldots) - b_u \\
&- \frac{\zeta_m}{m} [(m + L) h_0 u^{m+L-1} + (m + L - 1) h_1 u^{m+L-2} + \ldots] \psi_1 u^{1+L-m}
\end{align*}

as \( \frac{dx}{dr} = -r_3 \).

Let us consider the general term of each of the expressions that go into the above. Notice that we have

\[ u^L (m \eta_0 u^{m-1} + h_1 u^{m-1} + \ldots) = \psi_1 \sum_{0 \leq i \leq m+L-1} \frac{L}{m} h_i u^{2L-i} + \frac{L}{m} \psi_1 h_m + L \gamma u^{L-m}. \]

Furthermore, if we ignore the terms involving \( b_u \) we obtain that

\[ u^L \frac{L}{m} u^{2L-m} \psi_1 [h_0 u^m + h_1 u^{m-1} + \ldots] = \psi_1 \sum_{0 \leq i \leq m+L-1} \frac{L}{m} h_i u^{2L-i} - L \gamma u^{L-m}. \]

Lastly we obtain

\[ - \frac{1}{m} [(m + L) h_0 u^{m+L-1} + (m + L - 1) h_1 u^{m+L-2} + \ldots] \psi_1 u^{1+L-m} = \frac{1}{m} \psi_1 \sum_{0 \leq i \leq m+L-1} (m + L - i) u^{2L-m}. \]

**Theorem 15** We have that \( \zeta_m = 1 \).

**Proof:** Indeed we have the relation \( u^m = x_1 \). Consider now a small perturbation \( \epsilon \) in \( x_1 \). Choose a branch \( \zeta u \) of \( u \) and consider the perturbation in \( \zeta u \) needed to result in \( x_1 \). Let this be \( \eta \). We see that \( (\zeta u + \eta)^m = x_1 + \epsilon \)

and hence \( \eta = \frac{\epsilon (\zeta u)^{1-m}}{m} + \mathcal{O}(\epsilon^2) \). Notice that the dominating term we can write as \( \frac{\epsilon (\zeta u)^{1-m}}{m} \). The result follows. \( \text{[End-Of-Proof]} \)

**Corollary 1** We have the following dynamics for \( u \) and \( \gamma \): we have that

\[ \frac{du}{dr} = \frac{1}{m} u^{L-m+1} \psi_1 \]

and

\[ \frac{d\gamma}{dr} = \frac{1}{m} u^{L-m+1} b_u. \]
Remark 10 The dominating term of $b_u = \frac{db_u}{dr}$ is the term $-kb_k(\gamma)u^{-k-1}$.

Theorem 16 Consider the inverse dynamics coupled system again. Let $(u_i, \gamma_i) = \{(u_j', \gamma_j')\}$ be all possible pairs of $u, \gamma$ values such that $(u_i, \gamma_i)$ corresponds to at least one $w_j$. Then for almost all $e \in \Gamma$ we have the property that for at least one $i$, the inverse dynamics coupled system with initial values $u(0) = u_i$ and $\gamma(0) = \gamma_i$ is such we have that $\gamma \to \gamma_0$ and $u^{-1} \to 0$ as $r \to 0$ increases from the left.

**Proof:** Indeed, let $R$ be the ramification index of $f$ along $\Gamma$. Then we see that for almost all $e \in \Gamma$ we can lift the trajectory $f_1 = r$ to $R$ distinct trajectories $\Psi_1, ..., \Psi_R$ in $\hat{X}$ such $\Psi_i \to e = \beta_N(\gamma_0) \in \Gamma$. Let $\Psi_1$ be any one of these trajectories.

For almost all $e$ we can lift $\Psi_1$ uniquely to a trajectory $\dot{\Psi}_1$ such that $\gamma \to \gamma_0$ as we approach $e \in \Gamma$. In this branch $\gamma - \gamma_0$ together with $u^{-1}$ are local parameters for $\hat{X}$ around $e \in \Gamma$. As the dynamics of $\gamma - \gamma_0$ is governed by the inverse dynamics coupled system the result follows. \[ \text{End-Of-Proof} \]

Remark 11 It is important to note that the argument above only applies AFTER a choice of the branch $u, \gamma$ has been made.

Remark 12 We shall prove later that $k \geq K$.

7 Analytic functions

Write $K = L - m$. Notice that at least formally

$$\frac{du^{-K}}{dr} = \frac{-K}{m} \psi_1 = \frac{db_0}{d\gamma} + \frac{db_k}{d\gamma} u^{-k} + ....$$

Furthermore,

$$\frac{d\gamma}{dr} = \frac{-k}{m} u^{-K-k} [b_k(\gamma) + u^{-1}b_{k+1} + ...]$$

It is not a priori true that the derivative of $u^{-K}$ exists at all.

Theorem 17 We have that the derivative $\frac{du^{-K}}{dr}$ exists at $r = 0$.

**Proof:** Consider the unit circle $S^1 \subset \mathbb{C}$. For $\zeta \in S^1$ define the linear map $\mathcal{L}_\zeta : Y_0 \to Y_0$ which maps $y_1 \to \zeta^{-1}y_1$ and $y_2 \to \zeta y_2$. Notice that $\mathcal{L} \circ f$ is still Keller.

Denote now by $u_1(s)$ the function $u^{-K}(r = s)$ for $s \in [-\epsilon, 0]$.
Consider now the smooth continuous trajectory \( l : [-1, 1] \to S^1 \) which starts at \( l(-1) = 1 \) and ends at \( l(1) = -1 \). Furthermore, let \((u^0, \gamma^0)\) be starting conditions which tends to \( u^{-1} = 0 \) and \( \gamma_0 \in \Gamma \). The trajectory \( l \) corresponds to a trajectory \( \tilde{l} : [-1, 1] \to Y_0 \) which starts at \( \tilde{l}(-1) = (-\epsilon, 0) \) and ends at \( \tilde{l}(1) = (\epsilon, 0) \). In general \( \tilde{l}(\zeta) = (\epsilon l(\zeta), 0) \).

As \( \tilde{l} \) does not wind around \( V_f \) (we can make \( \epsilon \) small enough such that this does not happen) we see that we can lift the trajectory \( \tilde{l} \) to a trajectory \( \hat{l} \) in \( u - \gamma \) space such that for a pair \( u, \gamma \) at \( \hat{l}(1) \) the trajectory of \( u^{-K} \) and \( \gamma \) still ends at \( \gamma_0 \) and \( u^{-K} = 0 \) for the map \( L \circ f \). Denote the value of \( u^{-K}(r) \) along this branch (where \( \zeta = -1 \)) for \( r \in [0, \epsilon] \) by \( u_2(s) \) where \( s = -r \).

Notice that the right limit \( \lim_{s \to 0^+} \frac{du}{ds} \) as \( s \to 0 \) from the right exists and is equal to left limit \( \lim_{s \to 0^-} \frac{du}{ds} \).

Let now \( v \) denote the function \( u_1(s) \) if \( s < 0 \) and \( u_2(s) \) if \( s > 0 \). Let \( v(0) = 0 \). Notice that \( v \) is continuous on the interval \([-\epsilon, \epsilon] \). Furthermore, as \( s \to 0 \), the limit \( \lim_{s \to 0^-} \frac{du}{ds} \) exists and is finite. Hence by Apostol Exercise 5.16 we see that \( \frac{dv}{ds} \) exists and is in fact equal to \( \lim_{s \to 0^-} \frac{du}{ds} \). Hence the derivative of \( u^{-K} \) exists.

[End-Of-Proof]

Next we consider the structure of \( u^{-K} \) and in particuar the structure of the extension of discrete valuation rings \( O_V \subset O_\Gamma \). Notice that \( u^{-1} \) is a local parameter for the completed discrete valuation ring \( \hat{O}_\Gamma \). As such there is a unit \( \mu \in \hat{O}_\Gamma \) such that \( \mu u^{-R} = \chi \) where \( \chi \in O_V \subset C(Y_0) \) is a local parameter for \( O_V \). Here \( R \) is the ramification index of \( \Gamma/V \).

For generic \( v \in V \), \( \chi \) is regular holomorphic function at \( v \in V \subset Y_0 \) and in fact we can find a local parameter \( \hat{\gamma} \) such that \( \chi \) together with \( \hat{\gamma} \) are local parameters for \( Y_0 \) at \( v \). Furthermore, we can choose \( \hat{\gamma} \) such that \( u^{-1}, \hat{\gamma} \) are local analytic parameters for \( X \) at \( e \in \Gamma \). In other words we have that

\[
\hat{O}_{Y_0,v} \simeq C[[\chi, \hat{\gamma}]]
\]

and

\[
\hat{O}_{X,e} \simeq C[[u^{-1}, \hat{\gamma}]].
\]

There exists a local parameter \( g \in O_\Gamma \subset C(X_0) \) together with a unit \( h \in O_\Gamma \) such that \( g^Rh = \chi \). But for generic \( e \in \Gamma \) we have that \( g \) and \( h \) are regular functions in the local ring of \( X \) at \( e \), hence they both admit power series expansions \( g = G(u^{-1}, \hat{\gamma}) \) and \( h = H(u^{-1}, \hat{\gamma}) \).

In general we can write

\[
g = u^{-1}[g_0(\hat{\gamma}) + g_1(\hat{\gamma})u^{-1} + .].
\]

If we choose \( e \) such that \( g_0(\hat{\gamma}) \neq 0 \) then we see that

\[
[g_0(\hat{\gamma}) + g_1(\hat{\gamma})u^{-1} + .]^{-1} \in C[[u^{-1}, \hat{\gamma}]].
\]
and hence
\[ \mu = \frac{h}{[g_0(\hat{\gamma}) + g_1(\hat{\gamma})u^{-1} + \ldots] R} \in \mathbb{C}[[u^{-1}, \hat{\gamma}]]. \]

Thus we may write \( \mu = \sum p_i(\hat{\gamma})u^{-i} \) where \( p_i \in \mathbb{C}[[\hat{\gamma}]]. \)

Consider now the relation \( u^{-R} \mu = \chi \in \mathbb{C}^\infty \). As \( \chi \) together with the \( \hat{\gamma} \) form local parameters at \( v \in V \), we can choose a combination of a translation (so that \( v = 0 \) in \( Y_0 \)) and a rotation \( L \) of \( Y_0 \) such that \( \frac{\delta \chi}{\delta \hat{\gamma}} \neq 0 \) along the real line \( f_1 = r \) and \( f_2 = 0 \).

In particular this implies that \( \frac{du^{-R} \mu}{dr} \) exists and is different from 0. Originally we used the parameter \( \gamma \) but it is not hard to see that \( \frac{du}{dr} \) can also be expressed in terms of the parameter \( \hat{\gamma} \). In general we obtain
\[ \frac{du}{dr} = u^{\hat{K}+1} \sum q_i(\hat{\gamma})u^{-i} \]
for some constant \( \hat{K} \).

The point is however that \( \hat{K} = K \). One sees this by noting that we can write
\[ \gamma = \sum \omega_i(\hat{\gamma})u^{-i} \]
and that the determinant of the map \( [u^{-1}, \hat{\gamma}] \rightarrow [u^{-1}, \gamma] \) is simply \( \frac{\delta \gamma}{\delta \hat{\gamma}} \) at points \( v \) where \( \Gamma \rightarrow V \) is etale.

**Remark 13** Note \( \hat{\gamma} \) can also be considered as a tangent vector in \( Y_0 \). In the above we regard the map \( [u^{-1}, \hat{\gamma}] \rightarrow [u^{-1}, \gamma] \) and as \( \gamma = \sum \omega_i(\hat{\gamma})u^{-i} \) we see that \( \frac{\delta \gamma}{\delta \gamma} \) exists. However consider a map \( [u^{-1}, \gamma] \rightarrow [\chi, \hat{\gamma}] \) does not imply that \( \frac{\delta \gamma}{\delta \gamma} \) exists. Indeed, the notion of a derivative always depends on the choice of coordinate system.

Notice that \( q_0 \neq 0 \) for generic \( v \in V \).

**Lemma 15** We have that \( R \leq K \).

**Proof:** Assume that \( R > K \). Then
\[ \frac{d\chi}{dr} = \frac{du^{-R} \mu}{dr} = u^{-R} \left[ \sum \frac{dp_i}{dr} u^{-i} + \sum \sum -ip_i q_j u^{K-i-j} \right] \]
\[ -Ru^{K-R} \sum q_j p_i u^{-i-j}. \]
As \( R > K \) one sees that this tends to 0 as \( r \rightarrow 0 \), a contradiction as \( \frac{d\gamma}{dr} \neq 0 \).

[End-Of-Proof]

**Lemma 16** We have that \( R \geq K \).
Proof: Assume that $R < K$. Consider again the expression

$$\frac{d\chi}{dr} = \frac{du^{-R}\mu}{dr} = u^{-R}[\sum \frac{dp_i}{dr}u^{-i} + \sum \sum -i p_i q_j u^{K-i-j}]$$

$$-Ru^{K-R} \sum \sum q_j p_i u^{-i-j}.$$

Note that $q_0(\hat{\gamma}) \neq 0$ and $p_0(\hat{\gamma}) \neq 0$ at $e \in \Gamma$. Hence the expression above which tends to $\infty$ as $r \to 0$ unless $R \geq K$. [End-Of-Proof]

Hence we arrive at the following:

**Theorem 18** For generic rotations $\mathcal{L}$ the $K$ is the ramification index of $\Gamma/V$.

As by-product from our work earlier on we obtain the following.

**Lemma 17** There exists a function $\mu : [-\epsilon, 0] \to \mathbb{C}$ such that $\mu = \sum p_i(\hat{\gamma})u^{-i}$ and such that $u^{-K}\mu \in \mathbb{C}^\infty$. Furthermore $\mu(0) \neq 0$.

Next we consider the unit $\mu$.

The following theorem is useful.

**Theorem 19** Let $\psi : [-\epsilon, 0] \to \mathbb{C}$ be in $\mathbb{C}^\infty$ and let $\epsilon \in \mathbb{Q}_+ - \mathbb{N}$, i.e. $\epsilon$ is a proper positive fraction. Then for all $\phi : [-\epsilon, 0] \to \mathbb{C}$ with $\phi \in \mathbb{C}^\infty$ we have that

$$\lim_{r \to 0} \frac{\psi - \phi}{r^\epsilon} = \begin{cases} 0 & \text{if } \epsilon = 0 \\ \infty & \text{if } \epsilon = 0 \end{cases}.$$.

In other words, the limit is either $0$ or diverges.

Proof: We can write $\psi - \phi = \sum c_i r^i$ where the $c_i \in \mathbb{C}$. The theorem follows at once from the fact that $\epsilon$ is not integral by inspecting the first $c_i \neq 0$. [End-Of-Proof]

Set $\chi = u^{-K}\mu$. As

$$\frac{du}{dr} = u^{K+1} \sum q_j u^{-j}$$

we see that

$$\frac{du^{-K}}{dr} = -K \sum q_j u^{-j}.$$

Notice that we have

$$u^{-K} = \int \left[ \frac{du^{-K}}{dr} \right] dr = -K \int \left[ \sum_{j \geq 0} q_j(\gamma)u^{-j}\right] dr$$

and hence we see that

$$\chi = -K \sum_{i} \sum_{j} p_i(\gamma)u^{-i} \int \left[ q_j(\gamma)u^{-j}\right] dr.$$
8 Galois action of $Gal(K_X(u)/K_X) \simeq \mathbb{Z}/m\mathbb{Z}$ on $u$ and the $\gamma$

In this section we shall study the action of $Gal(K_X(u)/K_X) \simeq \mathbb{Z}/m\mathbb{Z}$ on $u$ and the $\gamma$. Let $\sigma \in Gal(K_X(u)/K_X) \simeq \mathbb{Z}/m\mathbb{Z}$ denote the action $u \rightarrow \zeta_m u$. Here we regard $\mathbb{Z}/m\mathbb{Z}$ as the group action on the branches of $u, \gamma$. Notice that the $x_1, x_2$ are fixed by this.

Consider now the $\sigma u = \zeta_m u$-branch of $u, \gamma$. We consider the dynamics of $\sigma u = \zeta_m u$ along this branch. We consider the system where $f_1 = r$ and $f_2 = 0$.

Let $P(y_1, y_2, u) = 0$ be the minimal polynomial of $u$ in $K_Y(u)$. Notice that

$$\frac{\delta P}{\delta y_1} = 0.$$ 

An $\frac{\delta y_2}{\delta y_1} = 1$ if $i = 1$ respectively $0$ if $i > 1$ we see that the function $\frac{\delta u}{\delta y_1} \in K_Y(u)$. In fact we have $\frac{\delta y_1}{\delta y_1} = Q(y_1, y_2, u)$ where $Q$ is some rational function in the $y_i$ and $u$.

Furthermore $P$ is also a minimal polynomial for $\sigma u$. It follows at once that

$$\frac{\delta \sigma u}{\delta y_1} = Q(y_1, y_2, \sigma u) = \sigma Q(y_1, y_2, u) = \sigma \frac{\delta u}{\delta y_1}.$$ 

Hence we arrive at the following lemma:

**Lemma 18** We have that the Galois conjugate

$$\sigma \frac{\delta u}{\delta y_1} = \frac{\delta \sigma u}{\delta y_1}.$$ 

Notice however that

$$\frac{d\sigma u}{dr} = \zeta_m \frac{du}{dr}.$$ 

We have proven earlier that

$$\frac{\delta u}{\delta y_1} = \frac{1}{m} u^{K+1} \frac{\delta y_2}{\delta \gamma}.$$ 

Hence we have

$$\zeta_m u^{K+1} \frac{\delta y_2}{\delta \gamma} = \zeta_m \frac{du}{dr} = \frac{d\zeta u}{dr}$$ 

$$= \frac{d\sigma u}{dr} = \sigma \frac{du}{dr} = \sigma [u^{K+1} \frac{\delta y_2}{\delta \gamma}] .$$

It follows that

$$\frac{\delta y_2}{\delta \gamma} = \zeta_m^{K} \frac{\delta y_2}{\delta \gamma}.$$ 


Notice also that the reduction of $\frac{\delta y_2}{\delta \gamma}$ along $u^{-1}$ is simply $\frac{\delta y_2}{\delta \gamma}$ where $\bar{\gamma}$ respectively $\bar{\gamma}$ denotes the classes of the reductions of $y_2$ respectively $\gamma$. To see this we note that $y_2 \in C[[u^{-1}, \bar{\gamma}]]$.

Hence we see in the reduction the relation

$$\sigma \frac{\delta y_2}{\delta \gamma} = \zeta - K \frac{\delta y_2}{\delta \gamma}.$$ 

We shall now prove that $K = 0$.

**Theorem 20** We have that $K = 0$.

We shall devote the rest of this section to proving this.

Let $c \in \mathbb{N}$. Consider the map $X_0 \rightarrow X_0 =: Z_0$ where

$$[x_1,..,x_n] \rightarrow [z_1 = x_1 + x_2, z_2 = x_2].$$

Notice that this induces a map $Z_0 \rightarrow X_0 \rightarrow Y_0$. We can apply exactly the same analysis to $Z_0 \rightarrow Y_0$ to obtain a $v - \hat{\gamma}$ representation. In general $z_1 = v^k$ etc.

Consider the representation $x_i = x_i(u, \gamma)$. By introducing a rotation on $X_0$ we may assume that all $x_2 = c_2 u^m + O(u^{m-1})$ where $c_2 \neq 0$.

Consider now the map $\omega : Z_0 := \mathbb{A}^n \rightarrow X_0 := \mathbb{A}^n$ given by

$$[z_1, z_2] \rightarrow [x_1 = z_1 + z_2, x_2 = z_2]$$

where $c \in \mathbb{N}$.

Notice that $\omega$ is a Keller map and indeed an isomorphism. We may apply all of the above to the map $f_\omega := f \circ \omega : Z_0 \rightarrow Y_0$. Let $\Omega$ denote the component above $V$ in $\hat{Z}$.

**Lemma 19** The ramification index of $f_\omega$ along $\Omega$ is $K$.

**Proof:** As $\omega$ is an isomorphism we see that $C(X_0) = C(Z_0)$. Hence the ramification of $\omega$ along $\Omega$ is trivial. [End-Of-Proof]

We may introduce exactly as above a parameter $v$, a constant $m_1$ and parameter $\omega$ such that we have a representation $z_1 = v^{m_1}$ and in general $z_2 = \sum d_i v^{m_{1-i}} + \omega v^{L_i}$. Consider now the inverse dynamical system along $f_1 = r$, $f_2 = 0$ for both $Z_0$ and $X_0$.

**Theorem 21** We have that $m_1 = cm$.

Let us prove the theorem above. Indeed as $K$ is the ramification index of both $f_\omega$ along $\Omega$ and $f$ along $\Gamma$ we see that we have that both $\frac{dm_1}{m}^K$ and $\frac{dm}{m}$ exist.
Furthermore we can arrange by a rotation on $Y_0$ that both are nonzero. Hence we see that

$$\lim_{r \to 0} \frac{v-K}{u-K} = \lim_{r \to 0} \frac{du-K}{dv-K} \neq 0$$

by L'Hospital's rule.

Hence we see that $\lim \frac{u}{v}$ exists and is nonzero. This implies immediately that $m_1 = cm$ which proves the theorem.

Our next order of business is study the relation between the $\gamma$ and the $\omega$.

**Theorem 22** On $\Gamma(v)$ we have that the reductions are $\gamma \equiv \alpha \omega$ for some nonzero $\alpha \in \mathbb{C}$.

**Proof:** Indeed we have that

$$v^{cm} = u^m + (c_2 u^m + .. + \gamma u^{-J_1})^c.$$  

We see that

$$(c_2 u^m + .. + \gamma u^{-J_1-m})^c = [c_2 u^m]^c[1 + .. + \gamma u^{-J_1-m}]^c.$$  

Hence

$$v^{cm} = u^{cm}[1 + .. + k_0 \gamma u^{-J_1-m} + ..].$$  

Thus

$$v = c_{cm}^{r} u[1 + .. + k_1 \gamma u^{-J_1-m} + ..]$$

and it follows that

$$u = c^{-r}_{cm}[1 + .. + k_2 \gamma^{r} u^{-J_1-m} + ..].$$

Hence we see that $\omega = p_1(\gamma)$ in the reduction on $\Gamma(v)$ for some polynomial $p_1$.

But we could have done exactly the same analysis to obtain $\gamma = q_1(\omega)$ on $\Gamma(v)$. Hence we see that at least locally at $\hat{e}$ we have $\omega = p_1(q_1(\omega))$ where $p_1$ and $q_1$ are polynomials. It follows that $\omega = k \gamma$ on $\Gamma(v)$ and the expressions for other parameters follow.  

[End-Of-Proof]

But this implies immediately that $\zeta^{-K} = \zeta^{-cK}$ must be $\zeta^{-cK}$ which implies that $(c - 1)K \equiv 0 \pmod{cm}$. To see this notice that in the reduction we have

$$\frac{\delta \tau}{\delta \zeta} = \frac{\delta \tau}{\delta \gamma} \frac{\delta \gamma}{\delta \zeta}.$$  

As we could have chosen $c$ freely this implies that $K = 0$.

**Theorem 23** We have that $K = 0$. 

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Corollary 2 We have that
\[ \frac{\delta u^{-1}}{\delta y_i} = 0 \]
and
\[ \frac{\delta \gamma}{\delta y_i} = 0 \]
at \( e \to v = f(e) \). It follows that \( u^{-1}(r) \) and \( \gamma(r) \) are \( \mathcal{C}^\infty \) functions.

In the following section we shall follow a slightly different route to prove that \( K \leq 0 \).

Before we continue to the following section, we would like to illustrate the Galois action of \( \mathbb{Z}/m\mathbb{Z} \) on a concrete example. Consider again the map \( f_1 \) constructed in the examples in Section 3. Recall that the map \( f_1 \) was given by \( [z_1, z_2] \to [x_1 = z_1 : x_2 = z_1^2 + z_2] \to [y_1 = x_1 x_2 + x_2^2 : y_2 = x_1 x_2] \). We also obtained the representation \( z_2 = -v^2 \) and \( z_1 = v + \gamma v^{-2} \).

Notice in this case \( m = L = 2 \) and \( K = 0 \). Furthermore the Galois action \( \mathbb{Z}/m\mathbb{Z} \) sends \( v \to -v \) and \( \gamma \to 2v^3 + \gamma \). For \( y_2 = z_1 z_2 + z_1^3 \) we have the following expression in terms of \( v \) and \( \gamma \):
\[ y_2 = 2\gamma + 3\gamma^2 v^{-3} + \gamma^3 v^{-6}. \]
Hence we see that
\[ \frac{\delta y_2}{\delta \gamma} = 2 + 6\gamma v^{-3} + 3\gamma^2 v^{-6}. \]
We expect this expression to remain the same if we apply \( \sigma \in \mathbb{Z}/2\mathbb{Z} \). Let us calculate. We have that \( v \to -v \) and \( \gamma \to 2v^3 + \gamma \). We obtain
\[ \sigma[2 + 6\gamma v^{-3} + 3\gamma^2 v^{-6}] = 2 - 6\sigma\gamma v^{-3} + 3[\sigma\gamma]^2 v^{-6} \]
\[ = 2 - 12 - 6\gamma v^{-3} + 3[2v^3 + \gamma]^2 v^{-6} \]
\[ = 2 - 12 - 6\gamma v^{-3} + 12 + 12\gamma v^{-3} + 3\gamma v^{-6} \]
\[ = 2 + 6\gamma v^{-3} + 3\gamma^2 v^{-6} \]
confirming our expectations.

9 Differential 2-forms and vector fields

In this section we shall follow a different route to prove that \( K \leq 0 \). We shall use the language of vector fields and differential forms in order to do this. We follow Lang [12] especially his account on Moser’s Theorem (Lang [12] XVIII §2 and in particular Proposition XVIII.2.1). In particular we shall use the following theorem (see loc. cit.) which we shall prove in Section 16 Theorem 35 for full generality in all dimensions.
Theorem 24 Let $B$ be the complex open ball parametrized by $z_1,..,z_n$. Let $\omega \in H^0(B, \wedge^n \Omega^1)$ be the differential $n$-form $\omega = dz_1 \wedge .. \wedge dz_n$ and let $\chi \in H^0(B, \wedge^{n-1} \Omega^1)$ be a differential $(n-1)$-form. Then there exists a holomorphic vector field $\eta \in H^0(B, T_B)$ such that $\omega \circ \eta = \chi$.

Consider the morphisms

$$spec(C[[y_1, y_2]]) \xrightarrow{f} spec(C[[u^{-1}, \gamma]]) \xrightarrow{\pi} spec(C[[u^{-K}, \gamma]]).$$

Here $\pi$ denotes the morphism $u^{-1} \rightarrow u^{-K}$:

$$spec(C[[u^{-1}, \gamma]]) \xrightarrow{\pi} spec(C[[u^{-K}, \gamma]]) \xrightarrow{f} spec(C[[y_1, y_2]])$$

Notice that at least locally we can regard these as morphisms

$$\mathcal{C} \xleftarrow{f} B \xrightarrow{\pi} A$$

where $A, B$ and $C$ are complex analytic $n$-dimensional open balls, centered infinitesimally around $v$ and $e$.

Furthermore the map $\pi : B \rightarrow A$ is a Galois extension with Galois group $G \simeq \mathbb{Z}/K\mathbb{Z}$ acting on $u^{-1} \rightarrow \zeta_K u^{-1}$.

Lemma 20 The pullback $f^*(dy_1 \wedge dy_2)$ of the $n$-form $\omega = dy_1 \wedge dy_2$ is fixed by $G$.

Proof: Indeed $f^*(dy_1 \wedge dy_2) = m[u^{-1}]^{K-1}d[u^{-1}] \wedge d\gamma$ which is fixed by $G$. [End-Of-Proof]

Hence we see that on $A$ we have a differential $2$-form $\omega_1 = mdv \wedge d\gamma$ which is such that $f^*(\omega) = \pi^*(\omega_1)$. Here $v = u^{-K}$.

Let $T_A$ respectively $T_B$ respectively $T_C$ denote the tangent bundles of $A$ respectively $B$ respectively $C$. We shall now construct a map $\pi^*(T_A) \rightarrow f^*(T_C)$ as a bundle map on $B$.

Notice that

$$H^0(B, \pi^* T_A) \simeq H^0(A, T_A) \otimes_{\mathcal{O}_A} \mathcal{O}_B.$$ 

As such we see that $H^0(B, \pi^* T_A)$ is generated by the pullbacks of the vector fields $\zeta_v$ and $\zeta_0$ where $\zeta_v$ is the constant vector field along $v$ and $\zeta_0$ that along $\gamma$.

For each $\zeta_v$ and $\zeta_0$ we shall define a section $\eta_v$ and $\eta_0$ in $H^0(B, f^* T_C)$ as follows.

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Consider $\zeta_v$. Let $\psi_v = \omega_1 \circ \zeta_v$. Notice that $\psi_v$ is a differential $n-1$-form on $\mathcal{A}$. Denote by $\overline{\psi}_v$ the reduction of $\psi_v$ to $\pi(\Gamma(u))$. As $\Gamma(u) \to V$ is etale, there is a unique differential 1-form on $V$ which maps to $\overline{\psi}_v$. Denote this form by $\overline{\psi}_v = \overline{\psi}_v$.

Let $\alpha_v$ be a differential 1-form on $\mathcal{C}$ which reduces to $\overline{\psi}_v$. As such we can find a unique holomorphic vector field $\beta_v \in H^0(\mathcal{C}, T\mathcal{C})$ such that $\omega_1 \circ \beta_v = \alpha_v$.

Notice now that $\pi^*(\psi_v) - f^*(\alpha_v) = \pi^*(\omega_1 \circ \zeta_v) - f^*(\alpha_v)$
is a differential $n-1$-form on $\mathcal{B}$ with a zero on $\Gamma(u)$.

**Lemma 21** There exists a vector field $\chi_v$ on $\mathcal{B} - \Gamma(u)$ such that $f^*(\omega_0) \circ \chi = \pi^*(\omega_1) \circ \chi_v = \pi^*(\omega_1 \circ \zeta_v) - f^*(\alpha_v)$ where $\chi_v$ has a pole of order $K' < K - 1$ on $\mathcal{B}$.

**Proof:** Indeed, we can find a vector field $\chi_v'$ such that $du^{-1} \wedge d\gamma \circ \chi_v' = \pi^*(\omega_1 \circ \zeta_v) - f^*(\alpha_v)$.

We now simply set $\chi = [u^{-1}]^{1-K} \chi_v'$. As $\pi^*(\omega_1 \circ \zeta_v) - f^*(\alpha_v)$ has a zero on $\Gamma(u)$ then so does $\chi'$ and the result follows. [End-Of-Proof]

Similarly we can find a vector field $\chi_0$ such that $f^*(\omega_0) \circ \chi_0 = \pi^*(\omega_1) \circ \chi_0 = \pi^*(\omega_1 \circ \zeta_0) - f^*(\alpha_0)$ where the $\alpha_0$ are defined similarly as the $\alpha_v$. Similarly we can define the vector field $\beta_0$ on $\mathcal{C}$.

We now define a map $f_\pi : H^0(\mathcal{B} - \Gamma(u), \pi^* T\mathcal{A}) \to H^0(\mathcal{B} - \Gamma(u), f^* T\mathcal{C})$ by setting $f_\pi(\pi^* \zeta_v) = f^*(\beta_v) + f^*(\chi_v)$.

Similarly we define $f_\pi(\pi^* \zeta_0) = f^*(\beta_0) + f^*(\chi_0)$.

We need to show that the map $f_\pi$ is well defined, i.e. independent of the choice of $\alpha_v$ and $\alpha_0$.

To do so we shall work locally. Indeed, let $U \subset \mathcal{A}$ be an open subset away from $\pi(\Gamma(u))$ and let $V \subset \pi^{-1}(U)$ be such that $\pi|_V : V \to U$ is a bi-holomorphic
map. I.e. \( V \to U \) represents a local branch of \( \pi \). Let \( W \) be the image of \( V \) in \( C \) and assume that \( U \to V \to W \) is an biholomorphic isomorphism (note by making \( U \) small we can assume this).

We denote by \( \pi_U : U \to V \) the local isomorphism and by \( g = f \circ \pi_U \).

Notice that we have a map

\[
H^0(U, \mathcal{T}_V) \xrightarrow{g^\ast} H^0(U, g^\ast \mathcal{T}_W) \xrightarrow{t_w} H^0(W, \mathcal{T}_W) \xrightarrow{t_f} H^0(V, f^\ast \mathcal{T}_W) \xrightarrow{i} H^0(V, \mathcal{T}_V).
\]

Now

\[
f^\ast(\omega_0 \circ t^{-1}_f[f^\ast \beta_v + f_\ast(\chi_v)]) = f^\ast(\omega_0) \circ [i \circ t_f(\beta_v)] + f^\ast(\omega_0) \circ \chi_v
\]

\[
= \pi^\ast(\omega_1 \circ \zeta_v) - f^\ast(\alpha_v) + f^\ast(\omega_0 \circ \beta_v)
\]

\[
= \pi^\ast(\omega_1 \circ \zeta_v).
\]

As the differential \( n \)-forms \( \omega_0 \) and \( \omega_1 \) are nonsingular on \( W \) and \( U \) we see that at least locally on the branch the map

\[
f_\pi|_V : \pi^\ast(\zeta_v)|_V \to [f^\ast(\beta_v) + f_\ast(\chi_v)]|_V
\]

is well defined.

However the map \( f_\pi \) was defined independent from the choice of branch, and it follows that it is well-defined globally on \( B - \Gamma(u) \).

Consider now again the expressions \( y_i = \sum Y_i^j(\gamma)u^{-j} \) where the \( Y_i \) are power series in the \( \gamma \). For \( i = 1, 2 \) define \( k_i \) to be the smallest index \( j > 0 \) such that \( Y_i^j \neq 0 \).

**Lemma 22** We can arrange that \( k_1 = k_2 \).

**Proof:** Indeed it suffices to consider Keller maps of the form

\[
[y_1, y_2] \to [y_1 + a_1 y_2, b_1 y_1 + c_2 y_2]
\]

where

\[
\left| \begin{array}{cc}
1 & a_1 \\
b_1 & c_1
\end{array} \right| = 1.
\]

[End-Of-Proof]

Let \( k = k_1 = k_2 \). Notice that this corresponds to the \( k \) that we introduced in Section 5. Notice that \( \frac{\partial y_i}{\partial u^{-1}} = Y_i^k[u^{-1}]^{k-1} + \mathcal{O}([u^{-1}]^k) \).

Furthermore, the section \( f_\ast(\chi_v) \in H^0(B - \Gamma(u), f^\ast(\mathcal{T}_C)) \) is given by

\[
f_\ast(\chi_v) = \left[ \begin{array}{c}
\frac{\partial y_1}{\partial u} \\
\frac{\partial y_2}{\partial u}
\end{array} \right] \chi_v.
\]

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It follows that the section \( f_*(\chi_v) \) can have a pole of at most order \( K' - (k-1) < K - k \) on \( \Gamma(u) \) and as such the section \( f^*(\beta_v) + f_*(\chi_v) \) can have a pole of at most order \( K - k - 1 \) on \( \Gamma(u) \).

Now consider the map \( \pi^{-1} : [v, \gamma] \to [u^{-1}, \gamma] \). Notice that we can use this to express \( y_i = y_i(u^{-1}, \gamma) \) locally away from \( \Gamma(u) \) as a function of \( y_i = y_i(v, \gamma) \). In particular we have that

\[
\begin{bmatrix}
\frac{\delta y_1}{\delta u} & \frac{\delta y_2}{\delta u} & \frac{\delta y_1}{\delta \gamma} & \frac{\delta y_2}{\delta \gamma}
\end{bmatrix}
\begin{bmatrix}
[u^{-1}]^{1-K} & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\frac{\delta y_1}{\delta u} & \frac{\delta y_2}{\delta u} & \frac{\delta y_1}{\delta \gamma} & \frac{\delta y_2}{\delta \gamma}
\end{bmatrix}^{-1}.
\]

It follows that \( f_*(\pi^*\zeta_v) \) has a pole of order \( K - k \) for generic \( \gamma \). But \( f_*(\pi^*\zeta_v) = f^*(\beta_v) + f_*(\chi_v) \) which has a pole of order at most \( K - k - 1 \).

It follows that \( k \geq K \) and that \( f_*(\pi^*\zeta_v) = f^*(\beta_v) + f_*(\chi_v) \) is in fact a holomorphic section of \( H^0(B, f^*(\mathcal{T}_C)) \).

**Lemma 23** We have that \( k = K \).

**Proof:** See Section 5 Theorem 13.

The map

\[ f_\pi : H^0(B, \pi^*\mathcal{T}_A) \to H^0(C, f^*\mathcal{T}_C) \]

constructed above is independent from the local branches of \( \pi \), infact the map is well defined globally on \( B \) as we saw.

However writing

\[ y_i = \sum Y_j^i(\gamma) u^{-j} \]

we see that locally the values of \( f_\pi(\pi^*\zeta_v) \) and \( f_\pi(\pi^*\zeta_i) \) will depend on the branch unless all \( Y_j^i = 0 \) if \( j \) does not divide \( K \). Indeed locally \( f_\pi(\pi^*\zeta_v) \) is given by

\[
\begin{bmatrix}
\frac{\delta y_1}{\delta u} & \frac{\delta y_2}{\delta u} & \frac{\delta y_1}{\delta \gamma} & \frac{\delta y_2}{\delta \gamma}
\end{bmatrix}
\begin{bmatrix}
[u^{-1}]^{1-K} & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\frac{\delta y_1}{\delta u} & \frac{\delta y_2}{\delta u} & \frac{\delta y_1}{\delta \gamma} & \frac{\delta y_2}{\delta \gamma}
\end{bmatrix}^{-1}.
\]

We arrive at the following.
Theorem 25 Consider the dynamics of \( u^{-K} \) and \( \gamma \) along \( f_1 = r \) and \( f_2 = 0 \). Then the functions \( u^{-K}(r) \) and \( \gamma(r) \) are independent of the branch of \( u \) and are in fact smooth \( C^\infty \) functions.

Proof: The fact that \( k = K \) implies that all the \( \frac{d}{dr} \) exist. The fact that only powers of \( u^{-iK} \) can occur in the expansions of the \( y_i \) implies that

\[
\frac{du^{-K}}{dr} = -K \sum q_i K(\gamma) u^{-iK}
\]

and

\[
\frac{d\gamma}{dr} = \sum p_j K(\gamma) u^{-jK}
\]

where the \( q_i K \) and \( p_j K \) are power series in \( \gamma \). It follows from induction that the \( u^{-K} \) and \( \gamma \) are infinitely differentiable and are in fact in \( C^\infty \).

To prove that they are independent from the branch of \( u \) one notes that the coupled differential equations

\[
\frac{du^{-K}}{dr} = -K \sum q_i K(\gamma) u^{-iK}
\]

and

\[
\frac{d\gamma}{dr} = \sum p_j K(\gamma) u^{-jK}
\]

with boundary values \( u^{-K}(r = 0) = 0 \) and \( \gamma(r = 0) = \gamma^0 \) evolve uniquely (they depend only on the initial values of \( \gamma^0 \) which are the values of \( \gamma \) on \( \Gamma(u) \) and depend solely on the trajectory of \( u^{-K}(r) \) and \( \gamma(r) \) and not \( u^{-1}(r) \)). [End-Of-Proof]

Remark 14 Alternatively note that the \( y_1 \) and \( y_2 \) are functions of \( v = u^{-K} \) and \( \gamma \). As such we have a map \( A \to C \) which is volume preserving, holomorphic and compact. It follows that \( u^{-K} \) and \( \gamma \) are holomorphic functions defined on some neighbourhood of \( v \in Y_0 \) where \( v \in V \).

Notice that this implies for a route \( f_i = r \) and \( f_j = 0 \) for \( i \neq j \) we have that the values of the \( \gamma \) and \( u^{-K} \) is predetermined. It follows that the values of \( x_i^K = u^{mK} \) is predetermined. Hence the monodromy of winding around \( V \) in \( Y_0 \) changes \( x_1 \) to \( \zeta_K x_1 \).

But we could have chosen the representation \( x_i = x_i + c \) where \( c \) is some complex constant. It follows that monodromy around \( V \) changes \( x_1 \) to \( \zeta_K x_1 \) but at the same time changes \( x_1 + c \) to \( \zeta^*_K (x_1 + c) \). As \( c \) was arbitrary it follows that \( K = 1 \).

Hence the map \( f: \hat{X} \to Y \) is generically unramified along \( \Gamma \). As

\[
\pi_1(Y_0 - S) \simeq \pi_1(Y_0)
\]

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if $S \subset Y_0$ is of codimension two, the two-dimensional Jacobian conjecture follows. Hence $K \leq 0$.

The two-dimensional Jacobian conjecture now follows almost immediately. Indeed, notice that the growth of $\frac{du}{dr} - 1$ and $\frac{d\gamma}{dr}$ are bounded. In particular it implies that the functions $u^{-1}$ and $\gamma$ are smooth, i.e. they are in $C^\infty$. Furthermore we have $\frac{du}{dr} \to 0$ and $\frac{d\gamma}{dr} \to 0$. But this is a contradiction, as this would imply that a zero-tangent vector in $\hat{X}$ maps to non-zero tangent vector in $Y_0$ (recall that the trajectory in $Y_0$ has non-zero tangents), a contradiction. Hence it follows that $V_f$ cannot be of codimension one and the two-dimensional Jacobian conjecture follows at once.

One may ask if we really need the $|J(f)| = 1$ hypothesis for the argument above. The only important condition is that $f^*(\omega)$ remains fixed under $\mathbb{Z}/K\mathbb{Z}$. Let us illustrate this with the map $f_0$ constructed in Section 3. Recall that $f_0(x_1, x_2) = [x_1 x_2 + x_2^2, x_1 x_2]$. Notice that $f_0$ is not Keller.

Recall from Section 4 that $J(f_0) = 2x^2t^{-2}$. Let $K_1$ denote the ramification of $f_0$ which is two. One may ask if we can apply the argument above to $K_1 = 2$ instead of $K = 0$. Indeed, we can and we see thus that $t^{-2}$ must be an analytic function in $C[[y_1, y_2]]$. But this is the case for generic points, indeed notice that the map $f_1$ is invariant under the action $\sigma : [x_1, x_2] \to [-x_1, -x_2]$.

However, care needs to be taken. Given a polynomial map

$$ f : X^n_0 = \mathbb{A}^n \to Y^n_0 = \mathbb{A}^n $$

with

$$ f^*(dy_1 \wedge .. \wedge dy_n) = x_1^{K-1}dx_1 \wedge .. \wedge dx_n $$

is not enough to infer that $f_0$ induces a map from

$$ \text{spec}(C[x_1^K, x_2, .., x_n]) \to \text{spec}(C[y_1, .., y_n]). $$

To illustrate this we use the very same example as above, namely the map $f_0$. Note the argument we just gave applies to the finiteness variety.

However one may ask if we can do the same for $f$ as a map from $X_0 \to Y_0$ and not looking at what happens at infinity. Indeed, $|J(f_0)| = 2x^2_2$ and hence

$$ f^*(dy_1 \wedge dy_2) = 2x^2_2dx_1 \wedge dx_2. $$

But $f_0$ clearly does not descent to a map

$$ \text{spec}(C[x_1, x^2_2]) \to \text{spec}(C[y_1, y_2]). $$

The problem is the usage of differential 1-forms on the finiteness variety. Indeed, above we used the fact that $\Gamma(u) \to \Gamma \to V$ was locally etale, and hence locally
1-forms on $\Gamma(u)$ are in bijection with 1-forms on $V$.

However the critical locus of $f_0$ is where $x_2 = 0$. Notice however that the line $x_2 = 0$ maps to a single point $(0,0) \in Y_0$ and hence the argument in this section cannot be applied.

As a by-product we would like to come back to the constant $k$ that we introduced in Section 5. Recall there we proved that $k \leq K$ and if $k = K$ then we have the relation

$$b_K \frac{da_0}{d\gamma} - a_K \frac{db_0}{d\gamma} = \frac{m}{k}.$$  

As $k = K$ this implies that the map $\gamma \to (a_0(\gamma), b_0(\gamma))$ can never be ramified and that the finiteness variety component $V$ can only have ordinary singularities (i.e. coming from self intersections).

It seems that it cannot happen that $V = (a_0(\gamma), b_0(\gamma))_{\gamma \in \mathbb{C}}$ intersects itself and as $\frac{da_0}{d\gamma}$ and $\frac{db_0}{d\gamma}$ cannot vanish simultaneously, this would imply that the component $V$ is actually smooth as a line in $Y_0$ (which would also imply the plane Jacobian conjecture). We would like to explain this shortly.

For a point $y \in Y_0$ denote by $n(y) = |f^{-1}(y)|$, i.e. the number of points of $X_0$ which map to $y$. Consider the map $\hat{X} \to Y$ which is a projective morphism. As $X_0 \to Y_0$ is flat and flatness is an open property, there exists an open subset $U$ of $Y$ such that $\hat{U} := \hat{X} \otimes_Y U \to U$ is flat.

**Lemma 24** We have that $u \to \dim H^0(\hat{U} \otimes_U u)$ is constant.

This implies that on $U$ the function $u \to n(u)$ is constant and infact equals the degree of the field extension $K(X_0)/K(Y_0)$. Denote this by $n$.

**Lemma 25** For $y \in Y_0 - U$ we have that $n(y) \leq n$.

**Proof:** One notes simply that around every inverse image of $y$ we can find a neigbourhood which maps isomorphically onto a neigbourhood of $y$. The statement follows from the fact that $u \to n(u)$ is constant and equal to $n$ on $U$ which is dense in $Y_0$. [End-Of-Proof]

Consider now the curve $\Delta := V$ which as usual is a component of the finiteness variety. Let $\hat{\Delta}_1, ..., \hat{\Delta}_r \subset X_0$ be the divisors of $X_0$ which map onto $\Delta$ and let $\Gamma_1, ..., \Gamma_s$ be the exceptional curves of $\hat{X}$, i.e. the divisors at infinity, which map onto $\Delta$.

As $X_0 \to Y_0$ is etale we know that the ramification indices of the covers $\hat{\Delta}_i \to \Delta$ are all one. However, they might be proper covers, and denote the degrees of these covers by $f_i$.  

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Furthermore, in general the covers $\Gamma_i \to \Delta$ may be a composition of a ramified cover together with a proper of field extensions $\mathbb{C}(\Gamma_i)/\mathbb{C}(\Delta)$. Denote the ramification index of each $\Gamma_i \to \Delta$ as divisors of $X \to Y$ by $e_i$ and its corresponding inertial extension degree by $\tau_i$.

As we may regard the divisors of $\hat{X}$ and $\hat{Y}$ as discrete valuation rings, and as the degree of field extension $K(X_0)/K(Y_0)$ is $n$, we see that we have

$$\sum_i \hat{f}_i + \sum_i e_i\tau_i = n.$$ 

Lemma 26 We have that $\tau_i = 1$ for all $i$.

Proof: Indeed $\Gamma \to V$ is etale and a cover of lines. [End-Of-Proof]

Let us sketch how we think one might be able to prove the Jacobian conjecture using this insight. Assume that for some component $\Gamma_i$, we have that for two different $\gamma_1$ and $\gamma_2$ we have

$$(a_0(\gamma_1), b_0(\gamma_1)) = (a_0(\gamma_2), b_0(\gamma_2))$$

inside $Y_0$. Denote this point of $Y_0$ by $\bar{y}$. Our idea will be to study lines $L : \mathbb{R} \to Y_0$ where $L(0) = \bar{y}$ and consider the pullbacks in $\hat{X}$.

In general we expect that there are at least $e_i$ different pullbacks $\hat{L} : \mathbb{R} \to \hat{X}$ which pass through $\gamma_1$ and the same for $\gamma_2$. Furthermore, if $\overline{\Delta_i}$, the projective closure of $\Delta_i$ in $\hat{X}$ has the points $x_1,...,x_m$ with ramification indices $\overline{\tau}_1,...,\overline{\tau}_m$ mapping to $\bar{y}$, then we expect that one can construct $\overline{\tau}_1,...,\overline{\tau}_m$ pullbacks of $L$ which pass through $x_1,...,x_m$. In total it seems there will thus be more than $n$ pullbacks of $L$ in $\hat{X}$, which is a contradiction as the generic point on $L$ will have only $n$ inverse images.

Hence the components $V$ of $V_f$ would all be smooth curves and in fact rational curves. Hence we can apply an automorphism to $Y_0$ and we see that we can always arrange that one of the components of $V_f$ is a straight line, a contradiction.

10 Numerical simulations

From our argument above it was not always clear where we used the fact that $f$ was Keller. The most striking example is the fundamental coupled relation

$$\frac{\delta f_1}{\delta x_1} \frac{\delta f_2}{\delta x_2} - \frac{\delta f_1}{\delta x_2} \frac{\delta f_2}{\delta x_1} = |J(f)| = 1.$$ 

In general $|J(f)| \neq 1$ and is a polynomial in $x_1$ and $x_2$, and as such a function in $u$ and $\gamma$. When it is a constant, we can deduce that along the trajectory
\((x_1(r), x_2(r))\) the tangent of \((f_1, f_2)(r)\) does not vanish as \(r \to z_1 - C\), i.e. as we approach \((z_1, z_2)\), which lead to a fundamental contradiction in our studies. Let us illustrate this with a concrete example for the map \(f_0\) constructed in Section. Recall that \(f_0\) we given by

\[
f_0(x_1, x_2) = [x_1x_2 + x_2^2, x_1x_2].
\]

Now let us study the inverse dynamics system in this case. We shall do so numerically. Note that the system is described by

\[
\frac{dx_1}{dr} = t + 2\epsilon t^{-1}
\]

and

\[
\frac{dx_2}{dr} = -\epsilon t^{-1}
\]

where \(x_1 = t\) and \(x_2 = \epsilon t^{-1}\). Simulating this numerically we obtain the following results:

| step | \(x_1\) | \(x_2\) | \(t\) | \(\epsilon\) | \(f_1\) | \(f_2\) |
|------|--------|--------|------|---------|-------|-------|
| 1    | -7.000 | 3.000  | -7.000 | -21.000 | -21.000 | -12.000 |
| 10   | -7.000 | 3.000  | -7.000 | -21.000 | -21.000 | -12.000 |
| 100  | -7.001 | 2.999  | -7.001 | -20.982 | -20.999 | -12.000 |
| 1000 | -7.010 | 2.970  | -7.010 | -20.821 | -20.822 | -12.000 |
| 10000| -7.135 | 2.714  | -7.135 | -19.368 | -19.368 | -12.000 |
| 100000| -11.976| 1.103  | -11.976| -13.217 | -13.217 | -12.000 |
| 1000000| -88102.339| 0.000 | -88101.458| -11.998 | -11.998 | -11.998 |
| 10000000| -1.07e+44| 0.000 | -1.07e+44| -11.988 | -11.988 | -11.988 |

Notice that \(f_2\) stays constant and that \(\epsilon \to f_2\) as well as \(f_1 \to f_2\). Furthermore, notice that \(\frac{d\epsilon}{dr} \to 0\).

Let us now comment on the magical disappearance of \(\zeta_m\) in Section. There we deduced that for \(\mu > 0\) we must have \(\zeta_m = 1\), else \(\frac{d\gamma}{dr}\) would diverge. Let us study this for our map above.

We introduce the variable \(v\) given by \(v^2 = t\). In this case we would have \(x_1 = v^2\) and \(x_2 = \epsilon v^{-2}\). Our governing equations would be

\[
\frac{dx_1}{dr} = v^2 + 2\epsilon v^{-2}
\]

and

\[
\frac{dx_2}{dr} = -\epsilon v^{-2}.
\]

Furthermore \(\psi_1 = 1 + 2\epsilon v^{-4}\). Let us start with the initial conditions \(x_1(0) = 9\) and \(x_2(0) = 3\). In this case there are two possible starting values for \((v, \epsilon)\),
namely \((\pm 3, 9)\).

Let us now simulate this numerically. We shall compare the actual, numerically calculated, \(\frac{dv}{dt}\) with our analytic expression which is \(v\psi_1\).

We have simulated the system above with a step size of \(\delta = 0.000001\). The simulations were done in Microsoft Visual Studio Community 2015 for C\# .net (see \([10]\)).

We obtain the following for \(v(0) = 3\):

| step | \(v\)       | \(\epsilon\)     | \(\frac{dv}{dt}\) | \(v\psi_1\) | \(\frac{dv}{dt} - v\psi_1\) |
|------|-------------|------------------|------------------|--------------|-----------------------------|
| 1    | 3.00\(E + 000\) | 2.70\(E + 001\) | 2.50\(E + 000\) | 2.50\(E + 000\) | -4.586\(E - 007\)          |
| 2    | 3.00\(E + 000\) | 2.70\(E + 001\) | 2.50\(E + 000\) | 2.50\(E + 000\) | -4.579\(E - 007\)          |
| 3    | 3.00\(E + 000\) | 2.70\(E + 001\) | 2.50\(E + 000\) | 2.50\(E + 000\) | -4.586\(E - 007\)          |
| 4    | 3.00\(E + 000\) | 2.70\(E + 001\) | 2.50\(E + 000\) | 2.50\(E + 000\) | -4.583\(E - 007\)          |
| 10   | 3.00\(E + 000\) | 2.70\(E + 001\) | 2.50\(E + 000\) | 2.50\(E + 000\) | -4.579\(E - 007\)          |
| 100  | 3.00\(E + 000\) | 2.70\(E + 001\) | 2.50\(E + 000\) | 2.50\(E + 000\) | -4.581\(E - 007\)          |
| 1000 | 3.00\(E + 000\) | 2.70\(E + 001\) | 2.50\(E + 000\) | 2.50\(E + 000\) | -4.582\(E - 007\)          |
| 10000| 3.00\(E + 000\) | 2.70\(E + 001\) | 2.50\(E + 000\) | 2.50\(E + 000\) | -4.583\(E - 007\)          |
| 100000| 3.00\(E + 000\) | 2.70\(E + 001\) | 2.50\(E + 000\) | 2.50\(E + 000\) | -4.584\(E - 007\)          |
| 1000000| 3.00\(E + 000\) | 2.70\(E + 001\) | 2.50\(E + 000\) | 2.50\(E + 000\) | -4.585\(E - 007\)          |

For \(v(0) = -3\) we obtain:

| step | \(v\)       | \(\epsilon\)     | \(\frac{dv}{dt}\) | \(v\psi_1\) | \(\frac{dv}{dt} - v\psi_1\) |
|------|-------------|------------------|------------------|--------------|-----------------------------|
| 1    | -3.00\(E + 000\) | 2.70\(E + 001\) | -2.50\(E + 000\) | -2.50\(E + 000\) | 4.586\(E - 007\)          |
| 2    | -3.00\(E + 000\) | 2.70\(E + 001\) | -2.50\(E + 000\) | -2.50\(E + 000\) | 4.579\(E - 007\)          |
| 3    | -3.00\(E + 000\) | 2.70\(E + 001\) | -2.50\(E + 000\) | -2.50\(E + 000\) | 4.586\(E - 007\)          |
| 4    | -3.00\(E + 000\) | 2.70\(E + 001\) | -2.50\(E + 000\) | -2.50\(E + 000\) | 4.583\(E - 007\)          |
| 10   | -3.00\(E + 000\) | 2.70\(E + 001\) | -2.50\(E + 000\) | -2.50\(E + 000\) | 4.579\(E - 007\)          |
| 100  | -3.00\(E + 000\) | 2.70\(E + 001\) | -2.50\(E + 000\) | -2.50\(E + 000\) | 4.581\(E - 007\)          |
| 1000 | -3.00\(E + 000\) | 2.70\(E + 001\) | -2.50\(E + 000\) | -2.50\(E + 000\) | 4.610\(E - 007\)          |
| 10000| -3.00\(E + 000\) | 2.70\(E + 001\) | -2.49\(E + 000\) | -2.49\(E + 000\) | 4.842\(E - 007\)          |
| 100000| -3.00\(E + 000\) | 2.70\(E + 001\) | -2.46\(E + 000\) | -2.46\(E + 000\) | 6.926\(E - 007\)          |
| 1000000| -3.00\(E + 000\) | 2.70\(E + 001\) | -3.00\(E + 000\) | -3.00\(E + 000\) | 2.003\(E - 006\)          |
| 10000000| -3.00\(E + 000\) | 2.70\(E + 001\) | -2.57\(E + 002\) | -2.57\(E + 002\) | 1.928\(E - 004\)          |
| 100000000| -3.00\(E + 000\) | 2.70\(E + 001\) | -8.98\(E + 021\) | -8.98\(E + 021\) | 6.735\(E + 015\)          |

We think we can discard the last rows due to numerical instabilities. By making the step size smaller by a factor of ten, we obtain the following for \(v(0) = -3\) (and similar for \(v(0) = 3\)):
We would also like to cross-check Corollary 1 of Section 2. We have that $b_u = -4\epsilon^2v^{-5}$. Hence, according to our prediction, we should have

$$\frac{dc}{dr} = -\frac{1}{m=2} v^{1+L-m} b_u = 2\epsilon^2 v^{-4}.$$ 

Let us check this numerically for the starting value $v(0) = -3$ and $\epsilon = 9$ (with a step size of $10^{-5}$).

| step | $v$          | $\epsilon$          | $\frac{dv}{dr}$ | $2\epsilon^2 v^{-4}$ | $\frac{dv}{dr} - 2\epsilon^2 v^{-4}$ |
|------|--------------|----------------------|------------------|----------------------|---------------------------------------|
| 1    | $-3,00E+000$ | $2,70E+001$          | $-2,50E+000$     | $-2,50E+000$         | $4,56E-008$                           |
| 2    | $-3,00E+000$ | $2,70E+001$          | $-2,50E+000$     | $-2,50E+000$         | $4,50E-008$                           |
| 3    | $-3,00E+000$ | $2,70E+001$          | $-2,50E+000$     | $-2,50E+000$         | $4,89E-008$                           |
| 4    | $-3,00E+000$ | $2,70E+001$          | $-2,50E+000$     | $-2,50E+000$         | $4,38E-008$                           |
| 10   | $-3,00E+000$ | $2,70E+001$          | $-2,50E+000$     | $-2,50E+000$         | $4,47E-008$                           |
| 100  | $-3,00E+000$ | $2,70E+001$          | $-2,50E+000$     | $-2,50E+000$         | $4,84E-008$                           |
| 1000 | $-3,00E+000$ | $2,70E+001$          | $-2,50E+000$     | $-2,50E+000$         | $4,60E-008$                           |
| 10000| $-3,00E+000$ | $2,70E+001$          | $-2,50E+000$     | $-2,50E+000$         | $4,27E-008$                           |
| 100000| $-3,02E+000$| $2,72E+001$          | $-2,49E+000$     | $-2,49E+000$         | $4,96E-008$                           |

We would also like to cross-check Corollary 1 of Section 2. We have that $b_u = -4\epsilon^2v^{-5}$. Hence, according to our prediction, we should have

$$\frac{dc}{dr} = -\frac{1}{m=2} v^{1+L-m} b_u = 2\epsilon^2 v^{-4}.$$ 

Let us check this numerically for the starting value $v(0) = -3$ and $\epsilon = 9$ (with a step size of $10^{-5}$).

| step | $v$          | $\epsilon$          | $\frac{dv}{dr}$ | $2\epsilon^2 v^{-4}$ | $\frac{dv}{dr} - 2\epsilon^2 v^{-4}$ |
|------|--------------|----------------------|------------------|----------------------|---------------------------------------|
| 1    | $-3,00E+000$ | $2,70E+001$          | $1,80E+001$      | $1,80E+001$          | $-9,00E-005$                          |
| 2    | $-3,00E+000$ | $2,70E+001$          | $1,80E+001$      | $1,80E+001$          | $-9,00E-005$                          |
| 3    | $-3,00E+000$ | $2,70E+001$          | $1,80E+001$      | $1,80E+001$          | $-9,00E-005$                          |
| 4    | $-3,00E+000$ | $2,70E+001$          | $1,80E+001$      | $1,80E+001$          | $-9,00E-005$                          |
| 10   | $-3,00E+000$ | $2,70E+001$          | $1,80E+001$      | $1,80E+001$          | $-9,05E-005$                          |
| 100  | $-3,00E+000$ | $2,70E+001$          | $1,80E+001$      | $1,80E+001$          | $-9,05E-005$                          |
| 1000 | $-3,02E+000$ | $2,72E+001$          | $1,76E+001$      | $1,76E+001$          | $-9,54E-005$                          |
| 10000| $-3,25E+000$ | $2,86E+001$          | $1,47E+001$      | $1,47E+001$          | $-1,39E-004$                          |
| 100000| $-5,61E+000$| $3,48E+001$          | $2,44E+000$      | $2,44E+000$          | $-3,235E-004$                         |

With a step size of $10^{-8}$ we obtain the following.

| step | $v$          | $\epsilon$          | $\frac{dv}{dr}$ | $2\epsilon^2 v^{-4}$ | $\frac{dv}{dr} - 2\epsilon^2 v^{-4}$ |
|------|--------------|----------------------|------------------|----------------------|---------------------------------------|
| 1    | $-3,00E+000$ | $2,70E+001$          | $1,80E+001$      | $1,80E+001$          | $7,297E-008$                          |
| 2    | $-3,00E+000$ | $2,70E+001$          | $1,80E+001$      | $1,80E+001$          | $7,770E-008$                          |
| 3    | $-3,00E+000$ | $2,70E+001$          | $1,80E+001$      | $1,80E+001$          | $4,242E-007$                          |
| 4    | $-3,00E+000$ | $2,70E+001$          | $1,80E+001$      | $1,80E+001$          | $4,708E-007$                          |
| 10   | $-3,00E+000$ | $2,70E+001$          | $1,80E+001$      | $1,80E+001$          | $-1,695E-007$                         |
| 100  | $-3,00E+000$ | $2,70E+001$          | $1,80E+001$      | $1,80E+001$          | $-1,805E-007$                         |
| 1000 | $-3,00E+000$ | $2,70E+001$          | $1,80E+001$      | $1,80E+001$          | $-2,565E-007$                         |
| 10000| $-3,00E+000$ | $2,70E+001$          | $1,80E+001$      | $1,80E+001$          | $-3,834E-007$                         |
| 100000| $-3,02E+000$| $2,72E+001$          | $1,76E+001$      | $1,76E+001$          | $-2,730E-007$                         |
| 1000000| $-3,25E+000$| $2,86E+001$          | $1,47E+001$      | $1,47E+001$          | $1,789E-007$                          |
Let us now study the dynamics of the map \( f_1 \) introduced in Section 3. Recall that the map \( f_1 : Z_0 \to Y_0 \) is given by

\[
f_1 : Z_0 = \mathbf{A}^2 \xrightarrow{g_0} X_0 \xrightarrow{f_0} Y_0
\]

where

\[
g_0(z_1, z_2) = [z_1, z_2 + z_1^2]
\]

where \( f_0 : X_0 \to Y_0 \) is given by

\[
[x_1, x_2] \to [x_1 x_2 + x_2^2, x_1 x_2].
\]

Notice that \( g_0 \) is Keller. Blowing up we obtain the representation

\[
z_1 = v + \gamma v^{-2}, z_2 = -v^2.
\]

Let us now simulate this numerically. Notice the Jacobian of \( f_1 \) is given by

\[
J(f_1) = \begin{bmatrix}
x_2 + 2z_1(x_1 + 2x_2) & x_1 + 2x_2 \\
x_2 + 2z_1 x_1 & x_1
\end{bmatrix}.
\]

Now consider the dynamics

\[
\frac{dz_1}{dr} = x_1
\]

and

\[
\frac{dz_2}{dr} = -x_2 - 2z_1 x_1.
\]

We shall solve this numerically starting at \( z_1 = 5 \) and \( z_2 = -11 \) (simply random values) and study the evolution of \( z_1 \) and \( z_2 \) over time. Note we choose the branch

\[
v = \sqrt{-z_2} = +\sqrt{11} \approx 3.317
\]

and

\[
\gamma \approx 18.517
\]

corresponding to these values. In each step we shall solve for \( v \) and \( \gamma \). We obtain the following:
We used a step size of $\Delta = 10^{-9}$. At step $i = 2100000000$ we started running into numerical instabilities. However we can see that $y_1 \to 70$ and $y_2$ is fixed at 70. Also notice that $\gamma \to 35 = \frac{1}{2}y_2$.

### 11 The higher dimensional case: notation

Through the rest of this paper we shall use the following notation. $X_0$ will denote the affine space $\mathbb{A}^n$ and so will $Y_0$. $f : X_0 \to Y_0$ will always denote a Keller map, i.e. a map defined over $\mathbb{C}$ which is globally etale. $X$ will denote the projective closure $\mathbb{P}^n$ of $X_0$ and similarly $Y = \mathbb{P}^n$ for $Y_0$.

$x_1, \ldots, x_n$ will denote coordinates for $X_0$ and similarly $y_1, \ldots, y_n$ for $Y_0$. $X_1, \ldots, X_n, T$ will denote projective coordinates for $X$ corresponding $x_1, \ldots, x_n$ where $T = 0$ is the projective divisor at infinity, i.e. $X = X_0$. Similarly for $Y_1, \ldots, Y_n, T$.

We shall denote by $K_X$ respectively $K_Y$ the function fields $K_X := \mathbb{C}(X_0) = \mathbb{C}(X)$ and $K_Y := \mathbb{C}(Y_0) = \mathbb{C}(Y)$.

By the finiteness variety of $f$ in $Y_0$ we mean the subvariety $V_f \subset Y_0$ over which $f$ fails to be proper. Denote by $\text{deg}(X/Y)$ and similarly $\text{deg}(X_0/Y_0)$ to be the
degree of the field extension $K_X/K_Y$. For a generic point $y \in Y_0$ we have
$\text{deg}(X/Y) = \text{deg}(X_0/Y_0)$ inverse points in $X_0$ (with exceptions over $V_f$).

12 Higher dimensional Keller maps and blowups of $P^n$

Let $f : X_0 := A^n \to Y_0 := A^n$ be a Keller map which is not an isomorphism. Let $X_1$ be the normalization of $Y_0$ inside $K_X := C(X_0)$. Let $V_f \subset Y_0$ denote the finiteness variety of $f$.

**Lemma 27** We have that $V_f$ is of codimension one.

**Proof:** Assume that $V$ is of codimension at least two. Then we have the isomorphism of fundamental groups

$$\pi_1(Y_0) \simeq \pi_1(Y_0 - V_f)$$

and hence $Y_0 - V_f$ is simply connected. The result follows. \[End-Of-Proof\]

Notice that Lemma 27 does not imply that $V_f$ is pure of codimension one, it simply states that there is at least one irreducible component $V$ of $V_f$ which is of codimension one. From now on fix such a component $V$ and let $v$ respectively $O \subset K_Y$ denote the discrete valuation respectively discrete valuation ring of $K_Y$ associated to $V$.

Notice as $X_1 \to Y_0$ is finite, there are discrete valuations $w_1,..,w_r$ of $K_X$ with discrete valuation rings $A_i \subset K_X$ such that each induced rational map $\text{spec}(A_i) \to X_1$ is actually a morphism and such that the center of $w_i$ is of codimension one inside $X_1$. Denote the closure of $\text{spec}(A_i)$ inside $X_1$ by $\Gamma_i$.

**Lemma 28** The subvariety $\Gamma_i$ is of codimension one and the map $f : X_1 \to Y_0$ induces a finite cover of varieties $\Gamma_i \to V$. Furthermore, the map $f$ could be ramified along $\Gamma_i$.

Denote by $e_i$ the ramification index of $f$ along $\Gamma_i$. Notice this corresponds to the ramification index of the extension of discrete valuation rings $O \subset A_i$. Furthermore, the map $f|_{\Gamma_i} : \Gamma_i \to V$ induces an extension of function fields $C(V) \subset C(\Gamma_i)$ the degree of which we denote by $f_i$.

**Lemma 29** We have that $\sum e_i f_i = m$ where $m = \text{deg}(K_X/K_Y)$.

Let $X$ denote the projective closure of $X_0$ and we choose this to be $X := P^n$. We also denote by $Y$ the space $P^n$ understood as the projective closure of $Y_0$.

Notice that $K_X \simeq C(X) \simeq C(X_1)$. Hence we have a rational map (which need not be defined everywhere) $j : X_1 \to X$. In particular this induces rational maps $j_i : \text{spec}(A_i) \to X$ which by the valuative property of $X$ are actually morphisms.
Theorem 26 By blowing up $X$ and taking the normalization we obtain a proper morphism $\pi : \hat{\hat{X}} \to X$ where $\hat{\hat{X}}$ is normal and such that each $w_i$ corresponds to codimension one divisor which we also denote by $\Gamma_i \subset \hat{\hat{X}}$. Furthermore, the map $f$ extends to a rational map $f : \hat{\hat{X}} \to Y$ (which need not be defined everywhere). Lastly, $f$ is defined on the complement of a codimension two subvariety of $\hat{\hat{X}}$ and induces generically finite and etale rational maps $f|_{\Gamma_i} : \Gamma_i \to V$.

Proof: See Liu ([14]) Theorem 8.3.26 and Exercise 8.3.14. [End-Of-Proof]

By Hironaka’s theorem ([10]) we can even arrange that $\hat{\hat{X}}$ is smooth.

Our order of business now is to restrict $\hat{\hat{X}}$ to make things easier later on. In particular we shall construct an open subset of $\hat{\hat{X}}$ with some desired properties.

Let us first consider the domain of definition of $f : \hat{\hat{X}} \to X$.

Lemma 30 There exists an open subset $U_0 \subset \hat{\hat{X}}$ such that $f|_{U_0}$ is defined everywhere on $U_0$ and such that $U_0$ contains the generic points of the $\Gamma_i$.

Proof: The $\Gamma_i$ are of codimension one and hence the lemma follows. [End-Of-Proof]

Consider now the maps $f|_{\Gamma_i} : \Gamma_i \to V$.

Lemma 31 There exists an open subset $U_1 \subset U_0$ such that each $f|_{\Gamma_i}$ considered as a map of reduced irreducible schemes $\Gamma_i \to V$ is etale on $U_1 \cap \Gamma_i$.

Remark 15 Note that $f$ could be ramified along $\Gamma_i$. We do not consider $f|_{\Gamma_i}$ as the restriction of $f$ to $\Gamma_i$ but rather the induced map of reduced subvarieties as such as a map of manifolds.

Proof: This follows as $f|_{\Gamma_i}$ is generically etale (we are in characteristic 0) and generically finite and each $\Gamma_i$ is of codimension one inside $U_0$. [End-Of-Proof]

Lemma 32 There exists an open subset $U_2 \subset U_1$ such that each $\Gamma_i \cap U_2$ is smooth as considered a variety over $\mathbb{C}$.

Proof: Note the $\Gamma_i$ need not be normal. They are however normal at their generic points and the lemma follows. [End-Of-Proof]

Lemma 33 There exists an open subset $U_3$ of $U_2$ such that there are nontrivial sections $s_i$ of $H^0(\Gamma_i \cap U_3, \mathcal{N}_{\Gamma_i/U_3})$ where $\mathcal{N}_{\Gamma_i/U_3}$ denotes the normal sheaf of $\Gamma_i \cap U_3$ inside $U_3$.

Lemma 34 There exists an analytic open subset $U_4$ of $U_3$ such that each $U_i^4 := \Gamma_i \cap U_4$ admits an etale morphism $\phi_i : U_i^4 \to \mathbb{A}^{n-1}$. 

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Proof: Each $\Gamma_i \cap U_3$ is smooth. \hspace{1cm} [End-Of-Proof]

Consider again the sections $s_i \in H^0(U_4 \cap \Gamma_i, \mathcal{N}_{U_4 \cap \Gamma_i/U_4})$. Fix a component $\Gamma := \Gamma_i$ and fix a point $\gamma \in \Gamma \cap U_4$.

Let $W \subset U_4$ be an open neighbourhood of $\gamma$ together with an analytic chart $\chi: W \simeq Z \subset \mathbb{C}^n$ such that $\gamma$ maps to 0. Notice that $\Gamma$ corresponds to a closed smooth subset $\Gamma^Z \subset Z$ and that the section $s := s_i$ transforms to a section $s^Z$ of the pushforward of the normal sheaf of $\Gamma$ inside $W$ to $Z$. Now define a map $\psi: T \otimes W \cap \Gamma \to W$ as

$$\psi: (t, e) \to \chi^{-1}(\chi(e) + t[s^Z|_e]).$$

Lemma 35 We have that $\psi$ is holomorphic.

Proof: Indeed, the section $s$ is holomorphic. \hspace{1cm} [End-Of-Proof]

Lemma 36 For $t = 0$ we have that $\psi(0, e) = e \in W \cap \Gamma$.

Proof: This follows from the definition of $\psi$. \hspace{1cm} [End-Of-Proof]

Lemma 37 By restricting $W$ we can arrange that for $t \neq 0$ we have $\psi(t, e) \in X_0$.

Proof: Indeed it suffices to consider a single point such that $\chi(e) + t[s^Z]|_e$ does not intersect $\Gamma^Z$ for $t \neq 0$. This can be arranged by noting that $\chi$ maps the tangent vectors of $\Gamma$ to the tangent vectors of $\Gamma^Z$ and hence does not map $s^Z$ to a tangent vector of $\Gamma^Z$. Hence for some punctured disc $D_0$ around 0 in $\mathbb{C}$ we see that $\chi(e) + t[s^Z]|_e$ does not intersect $\Gamma^Z$ for $t \in D_0$. This stays true in a small neighbourhood of $e$. \hspace{1cm} [End-Of-Proof]

Consider now the map $\pi: \tilde{X} \to \mathbb{P}^n$. In general it would map $e \in \Gamma$ to a point $[X_1 : X_2 : \ldots : X_n : T = 0]$. Without loss of generality we may assume that $X_1 = 1$. Consider now the map $\psi(t, e): T \otimes W \cap \Gamma \to W \hookrightarrow \tilde{X}$ and its projection onto $\mathbb{P}^n$. Fixing a set of local parameters $\gamma_1, \ldots, \gamma_{n-1}$ of $\Gamma$ around $e$ we see that we have map $\pi \circ \psi(t, \gamma_1, \ldots, \gamma_{n-1}): T \otimes D \to \mathbb{P}^n$ such that $\hat{\psi} := \pi \circ \psi(t, \gamma_1, \ldots, \gamma_{n-1}) \in X_0$ for $t \neq 0$. Here $D$ is an open analytic subset of $\mathbb{C}^{n-1}$ around 0 where the parameters of $\mathbb{C}^{n-1}$ are the $\gamma_i$.

In general the image of $\hat{\psi}$ will be given by power series of the form $X_1 = 1$, $X_2 = Q_2(t, \gamma_1, \ldots, \gamma_{n-1})$, ..., $X_n = Q_n(t, \gamma_1, \ldots, \gamma_{n-1})$ and $T = T(t, \gamma_1, \ldots, \gamma_{n-1})$ where the $Q_i$ and $T$ are elements of $\mathbb{C}[[t, \gamma_1, \ldots, \gamma_{n-1}]]$.
Notice that by restricting $\mathcal{D}$ we can arrange that the $t$-order of $T(t, \gamma_1, ..., \gamma_{n-1})$ is constant on $\mathcal{D}$, i.e.

$$T(t, \gamma_1, ..., \gamma_{n-1}) = t^m \tilde{T}(t, \gamma_1, ..., \gamma_{n-1})$$

where

$$\tilde{T}(0, \gamma_1, ..., \gamma_{n-1}) \neq 0.$$  

Hence, similar as in the two-dimensional case, we can find an element $s$ such that $T(t, \gamma_1, ..., \gamma_{n-1}) = s^m$. Note that this might branch above $\gamma_1, ..., \gamma_{n-1}$ however by restricting $\mathcal{D}$ we can arrange that $C[[t, \gamma_1, ..., \gamma_{n-1}]] \simeq C[[s, \gamma_1, ..., \gamma_{n-1}]]$.

Similar as in the two-dimensional case we thus obtain a representation, which we call also the $u - \gamma$ representation, such that

$$x_1 = u^m, \quad x_2 = \sum X_i^j u^{m-i}$$

and in general $x_j = \sum X_i^j u^{m-i}$ where the $X_i^j \in C[[\gamma_1, ..., \gamma_{n-1}]]$ with the property that for fixed $\gamma^0 := [\gamma_1^0, ..., \gamma_{n-1}^0]$ we have that $f(x) \to f(\gamma^0) \in V$ as $u \to \infty$.

Furthermore, we see that $f$ and $u - \gamma$ representation leads to a map

$$f : B = Spf(C[[u-1, \gamma_1, ..., \gamma_{n-1}]]) \to Y_0$$

where $B$ is an open complex $n$-dimensional ball parametrized by $u^{-1}$ and the $\gamma_i$ such that

$$f(u^{-1}, \gamma_1, ..., \gamma_{n-1}) = [y_1, ..., y_n]$$

where

$$y_i = \sum Y_i^j(\gamma_1, ..., \gamma_{n-1})u^{-j}.$$  

We see that the $\gamma_i$ are thus analytic local parameters around $e$.

In particular, if $C(r) : \mathbb{R} \to Y_0$ is a trajectory which tends to $v \in V \subset Y_0$ as $r \to 0$, then we can lift this trajectory locally around $e \in \Gamma$ such that the $\gamma_i \to \gamma_i^0$.

**Remark 16** Note that this representation is dependent on the branch of $s$ and hence $u$ chosen above.

Note that we have some freedom on the $\gamma_1, ..., \gamma_{n-1}$, indeed the $u - \gamma$ representation is not unique, as we could have chosen other parameters.

### 13 Higher dimensional inverse dynamical system

Choose a branch of the $u^{-1}$ above $e \in \Gamma$. Let $J$ be the Jacobian matrix of $f$ at a point $x$. Let $J_i$ be the resulting matrix where the first row and the $i$th column has been deleted and let $j_i = (-1)^{1+s(i)}|J_i|$ where $s(i)$ is the sign function i.e. $s(i) = 0$ if $i$ is even and 1 if odd. Notice that $j_i$ is a function of $x$. 

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Consider the system of coupled differential equations given by
\[ \frac{dx_i}{dr} = j_i = (-1)^{1+s(i)} |J_i|. \]

**Theorem 27** We have that \( f_1(r) = r + C_1 \) and \( f_i(r) = C_i, \ i \geq 2 \), along the trajectory of \( x_i(r) \) where \( C_1 \) and the \( C_i \) are constants.

**Proof:** Note that \( \frac{df_1}{dr} = \sum \frac{\delta f_1}{\delta x_i} \frac{dx_i}{dr} = 1. \)

Let \( L \) be an upper right \( n \) by \( n \) triangular matrix with only ones on the diagonal. Notice that \( \hat{f} := L \circ f \) is still Keller. Denote by \( \hat{J}_i \) respectively \( \hat{j}_i \) the corresponding \( J_i \)-matrices respectively their determinants for the map \( \hat{f} \). Denote by \( \hat{L} \) the matrix \( L \) with the first row and first column deleted. We have that \( \hat{J}_i = \hat{L}J_i \) and hence the \( j_i \) remain unchanged.

Consider now a point \( y = (-\epsilon, 0, \ldots, 0) \). Notice that \( Ly = y \) and hence \( L^{-1}y = y \). The fact that the \( j_i \) remain unchanged and \( y \) is fixed by \( L \) imply that the dynamics of the \( x_i \) do not change for \( f \) or \( \hat{f} \), they are identical. However, the \( Y_0 \)-trajectory is still given by \( \frac{df_1}{dr} = 1. \) As we could have chosen \( L \) freely this implies that \( \frac{df_i}{dr} = 0 \) where \( i \geq 2 \). [End-Of-Proof]

Notice that we have map \( \hat{f} : [u, \gamma_1, \ldots, \gamma_{n-1}] \rightarrow Y_0 \). We can write
\[ f_i(u, \gamma_1, \ldots, \gamma_{n-1}) = \sum a_j^i(\gamma_1, \ldots, \gamma_{n-1})u^{-i}. \]

Furthermore \( \hat{f} \) factors through the map
\[ [u, \gamma_1, \ldots, \gamma_{n-1}] \xrightarrow{\Delta} [x_1, \ldots, x_n] \xrightarrow{L} [f_1, \ldots, f_n] \]
and hence \( J(\hat{f}) = J(f)J(x) \) where \( \Delta \) denotes the map \((u, \gamma_1, \ldots, \gamma_{n-1}) \rightarrow (x_1, \ldots, x_n)\).

In particular this implies that \( |J(\hat{f})| = L(u, \gamma_1, \ldots, \gamma_{n-1}) \) which is a power series in \( u \), \( u^{-1} \) and the \( \gamma_i \). Hence we have the relation
\[ \sum \frac{\delta f_i}{\delta \gamma_i} \hat{T}_i + \frac{\delta f_1}{\delta u} \hat{T}_n = L \]
where the \( \hat{T}_i \) are the signed determinants of the corresponding submatrices of \( J(\hat{f}) \). Let us clarify this.

We may write
\[
J(\hat{f}) = \begin{bmatrix}
\frac{\delta f_1}{\delta \gamma_1} & \cdots & \frac{\delta f_1}{\delta \gamma_{n-1}} & \frac{\delta f_1}{\delta u} \\
\vdots & \ddots & \vdots & \vdots \\
\frac{\delta f_i}{\delta \gamma_1} & \cdots & \frac{\delta f_i}{\delta \gamma_{n-1}} & \frac{\delta f_i}{\delta u} \\
\end{bmatrix}.
\]
We now denote by $\overline{J}_i$ the determinant of the matrix formed from $J(\hat{f})$ with the first row and column $i$ deleted, multiplied by $(-1)^{s_1(i+1)}$.

Let $\gamma_j^i$ and $u^i$ denote the trajectories of the $\gamma_j$ and $u$ for the system given by $f_i = r$ and $f_l = 0$ for $l \neq i$. Note we assume here that $v \in V$ is $0 \in Y_0$.

**Theorem 28** We have the relation

$$\sum \frac{\delta f_i}{\delta \gamma_j} \frac{d \gamma_j^i}{dr} + \frac{\delta f_i}{\delta u} \frac{du^i}{dr} = 1.$$

In particular this implies that

$$\frac{d \gamma_j^i}{dr} = L^{-1} J_j^i$$

and

$$\frac{du^i}{dr} = L^{-1} f_n^i$$

where $J_j^i$ is the signed determinant of $J(\hat{f})$ with the $i$-th row and $l$-th column deleted.

In particular

$$J_1^1 = \begin{bmatrix} \frac{\delta f_1}{\delta \gamma_1} & \cdots & \frac{\delta f_2}{\delta \gamma_{n-1}} & \frac{\delta f_2}{\delta u} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\delta f_1}{\delta \gamma_1} & \cdots & \frac{\delta f_2}{\delta \gamma_{n-1}} & \frac{\delta f_2}{\delta u} \end{bmatrix}$$

and for instance

$$J_n^1 = \begin{bmatrix} \frac{\delta f_1}{\delta \gamma_1} & \cdots & \frac{\delta f_2}{\delta \gamma_{n-1}} \\ \vdots & \ddots & \vdots \\ \frac{\delta f_1}{\delta \gamma_1} & \cdots & \frac{\delta f_2}{\delta \gamma_{n-1}} \end{bmatrix}.$$

**Remark 17** Notice that the analysis of the dynamics above for $u$ and the $\gamma_i$ can be carried through for any choice of parameter $\gamma_i$. The only hypothesis that we need is that the dynamics in the image is given by $f_i = r$ and $f_l = 0$ for $l \neq i$. In the following we shall change our choice of parameters $\gamma_i$ and we shall use the same analysis.

### 14 Transformation of the $\gamma_i$

We shall now introduce new parameters $\overline{\gamma}_1, \ldots, \overline{\gamma}_{n-1}$ which will facilitate our computations.

Write

$$x_i = \sum_{j \geq 0} X_j^i(\gamma) u^{m-j}$$

where the $X_j^i$ are power series in $\gamma_1, \ldots, \gamma_{n-1}$. Notice that $x_1 = u^m$. The following trivial remark is crucial.
Lemma 38 The $x_2, \ldots, x_n$ are algebraically independent.

Consider $x_2$. Let $j = J_1$ be the first index of $j$ where $X^2_{J_1} (\gamma)$ is not a constant in $\mathbb{C}$. Define $\tau_1 = \sum_{j \geq J_1} X^2_{J_1} (\gamma) u^{m-J_1-j}$. 

Now consider $x_3$. As we have defined we have that $x_3 = \sum_{j \geq 0} X^3_j (\gamma) u^{m-j}$. Now let $\eta_1, \ldots, \eta_{n-1}$ be generic constants in $\mathbb{C}$. Let $\tau_1$ denote the value of $X^3_{J_1} (\eta_1, \ldots, \eta_{n-1})$.

Lemma 39 For generic $\eta_i$ we have that

$$\mathbb{C}[[u^{-1}, \gamma_1 - \eta_1, \ldots, \gamma_{n-1} - \eta_{n-1}]] = \mathbb{C}[[u^{-1}, \tau_1, \gamma_2 - \eta_2, \ldots, \gamma_{n-1} - \eta_{n-1}]].$$

Hence by introducing a translation we see that we can write

$$x_2 = \sum_{j \geq 0} X^3_j (\tau_1, \gamma_2, \ldots, \gamma_{n-1}) u^{m-j}.$$ 

Now let $j = J_2$ be the first index of $j$ such that

$$\tau_j (\gamma, \gamma_2, \ldots, \gamma_{n-1}) \neq \tau_j (\tau_1),$$

i.e. that $\tau_j$ is only a function of $\tau_1$ for $j < J_1$.

Lemma 40 We have that $J_2 \leq \infty$.

Proof: Assume that $J_2 = \infty$. Then we see that $x_1, x_2, x_3$ are only functions of $u, u^{-1}$ and $\tau_1$, a contradiction as we can choose the values of $x_1, x_2$ and $x_3$ arbitrarily and independent of each other. [End-Of-Proof]

Set

$$\tau_2 = \sum_{j \geq J_2} X^3_j (\tau_1, \gamma_2, \ldots, \gamma_{n-1}) u^{m-J_2-j}.$$ 

Hence we see that we can write

$$x_2 = \sum_{j < J_2} X^3_j (\tau_1) u^{m-j} + \tau_2 u^{m-J_2}.$$ 

Continuing in this way we see that we can find parameters $\tau_1, \ldots, \tau_{n-1}$ which are such that $x_1 = u^m$, $x_2 = x_2 (u, \tau_1)$ and in general $x_i = x_i (u, \tau_1, \ldots, \tau_{i-1})$. Furthermore we can write

$$x_i = \sum_{j < J_i} X^3_j (\tau_1, \ldots, \tau_{i-1}) u^{m-j} + \tau_i u^{m-J_i}.$$ 

Theorem 29 For generic $e \in \Gamma$ we have that $\tau_1, \ldots, \tau_{n-1}$ are local parameters of $\Gamma$ at $e$. Furthermore, for fixed $\tau_1, \ldots, \tau_{n-1}$ we have that $x \to e$ as $u \to \infty$ where $e$ corresponds to $\tau_1, \ldots, \tau_{n-1}$.
In particular one sees that $f$ induces a map

$$f : \mathcal{B} = \text{Spf}(\mathcal{C}[[u^{-1}, \tau_1, \ldots, \tau_{n-1}]]) \rightarrow Y_0$$

where $\mathcal{B}$ is an open complex $n$-dimensional ball parametrized by $u^{-1}$ and the $\tau_i$ such that

$$f(u^{-1}, \tau_1, \ldots, \tau_{n-1}) = [y_1, \ldots, y_n]$$

where

$$y_i = \sum Y_i^j(\tau_1, \ldots, \tau_{n-1})u^{-j}.$$ 

Notice that $L^{-1} = u^{\sum J_i+1-mn} = u^{K+1}$. Furthermore for generic $e$ we have that $\Gamma \rightarrow V$ is etale at $e$.

We would now like to carry out a similar analysis for another set of parameters namely for the parameter $y_1, \ldots, y_n$ and in general any set of parameters that are contained in $\mathcal{C}[[y_1, \ldots, y_n]]$. Notice that locally around $v = f(e) \in V \subset Y_0$ we have the parameters $y_1, \ldots, y_n$. Let $\chi \in \mathcal{C}[[y_1, \ldots, y_n]]$ be a local parameter such that $V$ is locally defined as $\chi = 0$.

**Lemma 41** For almost all $v$ we may assume that $\mathcal{P}_i := \{\chi, y_1, \ldots, \hat{y}_i, \ldots, y_n\}$ are sets of local parameters at $v$ where $\hat{y}_i$ denotes the set where $y_i$ has been omitted.

Notice that $\Gamma \rightarrow V$ is etale above $v$ for generic $v$ and $\Gamma(u) \rightarrow \Gamma$ is etale for generic $e \rightarrow v$ we see that $\mathcal{P}_i(u) = \{u^{-1}, y_1, \ldots, \hat{y}_i, \ldots, y_n\}$ for a local set of parameters at $e_u \rightarrow e \rightarrow v$ where $e_i \in \Gamma(u)$ maps to $e$.

Hence we can carry out exactly the same analysis as in Section 13 to calculate the partial derivatives $\frac{\delta u}{\delta y_i}$.

We obtain

$$\frac{\delta u}{\delta y_1} = u^{K+1} \left[ \begin{array}{ccc} \frac{\delta f_1}{\delta y_2} & \cdots & \frac{\delta f_1}{\delta y_n} \\ \frac{\delta f_2}{\delta y_2} & \cdots & \frac{\delta f_2}{\delta y_n} \\ \vdots & \ddots & \vdots \\ \frac{\delta f_n}{\delta y_2} & \cdots & \frac{\delta f_n}{\delta y_n} \end{array} \right]^{-1} \left[ \begin{array}{c} \frac{\delta f_1}{\delta y_1} \\ \frac{\delta f_2}{\delta y_1} \\ \vdots \\ \frac{\delta f_n}{\delta y_1} \end{array} \right].$$

Hence by a suitable coordinate change we can arrange that

$$\frac{du}{dr} = u^{K+1}[\sum q_i(y_1, \ldots, \hat{y}_i, \ldots, y_n)u^{-i}]$$

where $q_0 \neq 0$ above $e$.

**Theorem 30** We have that $\hat{K}$ is the ramification index of $f$ along $\Gamma$.

**PROOF:** Our proof is similar to the two-dimensional case. Indeed, let $\hat{\gamma}_1, \ldots, \hat{\gamma}_{n-1}$ be parameters in $\mathcal{C}[[y_1, \ldots, y_n]]$ of $V \subset Y_0$ at $v = f(e) = 0$ and assume that $\Gamma \rightarrow V$
is étale above $v$. We also assume that the branch of $u$ above $e$ is unramified.

Notice that the inverse dynamical coupled system implies that the dynamics of $\hat{\gamma}_i$ are smooth functions in $r$, i.e. $\hat{\gamma}_i \in C^\infty$ and that $\frac{d\hat{\gamma}_i}{dr}$ all exist at $r = 0$.

Let $R$ be the ramification index of $f$ along $\Gamma$. Exactly as in the two-dimensional case we can find a unit $\mu = \sum p_i(\hat{\gamma}_1, ..., \hat{\gamma}_{n-1})$ and a holomorphic function $\chi$ at $v \in Y_0$ such that locally $u^{-R}\mu = \chi$ where $p_0(0) \neq 0$.

By a rotation we can ensure that $\frac{d\chi}{dr} \neq 0$ along $f_1 = r$, $f_i = 0$ for $i \geq 2$. We see that

$$0 \neq \frac{d\chi}{dr} = \frac{du^{-R}\mu}{dr} = u^{-R}\frac{d\mu}{dr} - Ru^{-R}(\hat{\gamma}_1, ..., \hat{\gamma}_{n-1})u^{-i}$$

where the $\hat{q}_i \in C[[\hat{\gamma}_1, ..., \hat{\gamma}_{n-1}]]$. Now if $R > \hat{K}$ we see that the expression above would be $0$, a contradiction. If $R < \hat{K}$ then a calculation on the expression shows that $\frac{d\chi}{dr} \to \infty$ which is also a contradiction. Hence $\hat{K} = R$. [End-Of-Proof]

It remains to relate the $\hat{K}$ with the $K$ obtained for the parameters $\gamma_1, ..., \gamma_{n-1}$.

**Lemma 42** We have that $\hat{K} = K$.

**Proof:** One notes that we can write

$$\gamma_i = \sum \alpha_j^i(\hat{\gamma}_1, ..., \hat{\gamma}_{n-1})u^{-j}$$

and hence we can write

$$y_i = y_i(u^{-1}, \gamma_1, ..., \gamma_{n-1}) = y_i(u^{-1}, \hat{\gamma}_1, ..., \hat{\gamma}_{n-1}).$$

By rotating $Y_0$ we can assume that the direction $f_1 = r$, $f_i = 0$ for $i \neq 1$ is not a tangent direction to $V$ at $v$. As such we see that

$$\begin{bmatrix}
\frac{\delta f_1}{\delta \gamma_1} & \cdots & \frac{\delta f_1}{\delta \gamma_{n-1}} \\
\frac{\delta f_2}{\delta \gamma_1} & \cdots & \frac{\delta f_2}{\delta \gamma_{n-1}} \\
\vdots & \ddots & \vdots \\
\frac{\delta f_n}{\delta \gamma_1} & \cdots & \frac{\delta f_n}{\delta \gamma_{n-1}}
\end{bmatrix}$$

and

$$\begin{bmatrix}
\frac{\delta f_1}{\delta \gamma_1} & \cdots & \frac{\delta f_1}{\delta \gamma_{n-1}} \\
\frac{\delta f_2}{\delta \gamma_1} & \cdots & \frac{\delta f_2}{\delta \gamma_{n-1}} \\
\vdots & \ddots & \vdots \\
\frac{\delta f_n}{\delta \gamma_1} & \cdots & \frac{\delta f_n}{\delta \gamma_{n-1}}
\end{bmatrix}$$

are both nonzero. The result follows (recall $\Gamma \to V$ is étale above $v$). [End-Of-Proof]

One can prove directly that $K$ is the ramification index of $f$ along $\Gamma \to V$. 

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Indeed, let \( y_i = \sum Y_j (\overline{\tau}_1, ..., \overline{\tau}_{n-1}) u^{-j} \). Let \( k \) be the smallest integer larger than 0 such that \( Y_k \neq 0 \). We see from Theorem 28 that the growth of the \( \frac{d\gamma}{dr} \) are bounded by \( K - k \), i.e. we can write

\[
\frac{d\gamma_i}{dr} = u^{K-k} \sum \beta_j^i (\overline{\tau}_1, ..., \overline{\tau}_{n-1}) u^{-j}.
\]

Hence considering

\[
\frac{dx}{dr} = \frac{d\mu}{dr}
\]

we see that

\[
\frac{dx}{dr} = u^{-R} \frac{du}{dr} - Ru^{K-R} \sum q_i (\overline{\tau}_1, ..., \overline{\tau}_{n-1}) u^{-i}.
\]

Here we abused notation and wrote

\[
\mu = \sum p_i (\overline{\tau}_1, ..., \overline{\tau}_{n-1}) u^{-i}
\]

and

\[
\frac{du}{dr} = u^{K+1} \sum q_i (\overline{\tau}_1, ..., \overline{\tau}_{n-1}) u^{-i}.
\]

We see that

\[
\frac{d\mu}{dr} = \sum \frac{dp_i}{\delta \overline{\tau}_j} u^{-i} + \sum p_i \frac{du^{-i}}{dr}
\]

\[
= \sum \sum \frac{\delta p_i}{\delta \overline{\tau}_j} u^{-i} + \sum p_i q_j u^{K-j}.
\]

One sees thus that \( u^{-R} \frac{du}{dr} \) can maximally have a dominating term \( u^{K-R-1} \) and the result follows.

15 Galois action of \( Gal(K_X(u)/K_X) \simeq \mathbb{Z}/m\mathbb{Z} \) on \( u \) and the \( \overline{\tau}_1, ..., \overline{\tau}_{n-1} \)

In this section we shall study the action of \( Gal(K_X(u)/K_X) \simeq \mathbb{Z}/m\mathbb{Z} \) on \( u \) and the \( \overline{\tau}_i \). Let \( \sigma \in Gal(K_X(u)/K_X) \simeq \mathbb{Z}/m\mathbb{Z} \) denote the action \( u \to \zeta_m u \). Here we regard \( \mathbb{Z}/m\mathbb{Z} \) as the group action on the branches of \( u, \overline{\tau}_i \). Notice that the \( x_1, ..., x_n \) are fixed by this.

**Lemma 43** We have that

\[
\sigma \overline{\tau}_1 = \zeta_m \overline{\tau}_1 + Q_1(u)
\]

where \( Q_1(u) \in \mathbb{C}[[u]] \). In general we have that

\[
\sigma \overline{\tau}_i = \zeta_m \overline{\tau}_i + Q_i(u, \overline{\tau}_1, ..., \overline{\tau}_{i-1})
\]

where

\[
Q_i \in \mathbb{C}[[u, \overline{\tau}_1, ..., \overline{\tau}_{i-1}]].
\]
Indeed the lemma above follows from the fact that the $x_1, \ldots, x_n$ are fixed by $\sigma$.

Consider now the $\sigma u = \zeta_m u$-branch of $u, \overline{\gamma}_1, \ldots, \overline{\gamma}_{n-1}$. We consider the dynamics of $\sigma u = \zeta_m u$ along this branch. We consider the system where $f_1 = r$ and $f_2 = \ldots = f_n = 0$.

Let $P(y_1, \ldots, y_n, u) = 0$ be the minimal polynomial of $u$ in $K[y][u]$. Notice that $\frac{\delta P}{\delta y_1} = 0$. As $\frac{\delta u}{\delta y_1} = 1$ if $i = 1$ respectively $0$ if $i > 1$ we see that the function $\frac{\delta u}{\delta y_1} \in K_X(u)$. In fact we have $\frac{\delta u}{\delta y_1} = Q(y_1, \ldots, y_n, u)$ where $Q$ is some rational function in the $y_i$ and $u$.

Furthermore $P$ is also a minimal polynomial for $\sigma u$. It follows at once that

$$\frac{\delta \sigma u}{\delta y_1} = Q(y_1, \ldots, y_n, \sigma u) = \sigma Q(y_1, \ldots, y_n, u) = \sigma \frac{\delta u}{\delta y_1}.$$ 

Hence we arrive at the following lemma:

**Lemma 44** We have that the Galois conjugate

$$\sigma \frac{\delta u}{\delta y_1} = \frac{\delta \sigma u}{\delta y_1}.$$ 

Notice however that

$$\frac{d\sigma u}{dr} = \zeta_m \frac{du}{dr}.$$ 

It follows at once that

$$\sigma \frac{\delta u}{\delta y_1} = \zeta_m \frac{\delta u}{\delta y_1}.$$ 

Hence we see that

$$\zeta_m u^{K+1} \begin{bmatrix} \frac{\delta y_2}{\delta y_1} & \ldots & \frac{\delta y_2}{\delta y_{n-1}} \\ \frac{\delta y_3}{\delta y_1} & \ldots & \frac{\delta y_3}{\delta y_{n-1}} \\ \vdots & \ddots & \vdots \\ \frac{\delta y_n}{\delta y_1} & \ldots & \frac{\delta y_n}{\delta y_{n-1}} \end{bmatrix} = \zeta \frac{\delta u}{\delta y_1} = \frac{\delta \zeta_m u}{\delta y_1}$$

$$= \frac{\delta \sigma u}{\delta y_1} = \sigma \frac{\delta u}{\delta y_1} = \sigma \begin{bmatrix} \frac{\delta y_2}{\delta y_1} & \ldots & \frac{\delta y_2}{\delta y_{n-1}} \\ \frac{\delta y_3}{\delta y_1} & \ldots & \frac{\delta y_3}{\delta y_{n-1}} \\ \vdots & \ddots & \vdots \\ \frac{\delta y_n}{\delta y_1} & \ldots & \frac{\delta y_n}{\delta y_{n-1}} \end{bmatrix}.$$ 

It follows that

$$\sigma \begin{bmatrix} \frac{\delta y_2}{\delta y_1} & \ldots & \frac{\delta y_2}{\delta y_{n-1}} \\ \frac{\delta y_3}{\delta y_1} & \ldots & \frac{\delta y_3}{\delta y_{n-1}} \\ \vdots & \ddots & \vdots \\ \frac{\delta y_n}{\delta y_1} & \ldots & \frac{\delta y_n}{\delta y_{n-1}} \end{bmatrix} = \zeta_m u^{K+1} \begin{bmatrix} \frac{\delta y_2}{\delta y_1} & \ldots & \frac{\delta y_2}{\delta y_{n-1}} \\ \frac{\delta y_3}{\delta y_1} & \ldots & \frac{\delta y_3}{\delta y_{n-1}} \\ \vdots & \ddots & \vdots \\ \frac{\delta y_n}{\delta y_1} & \ldots & \frac{\delta y_n}{\delta y_{n-1}} \end{bmatrix}.$$ 

Before we continue we would like to get rid of a technicality. Earlier one we wrote

$$x_i = \sum_j \overline{\gamma}_j (\overline{\gamma}_1, \ldots, \overline{\gamma}_{i-1}) u^{m-j} + \overline{\gamma}_i u^{m-j_i}.$$
However it can happen that the $X_i^0$ are not constants in $C$. However notice that we can arrange for this by adding the power $x_i^c$ to the $x_i$ and then considering maps of the form $x_j \to x_j + cx_i$. From now one we shall assume that all the $X_i^0$ are constants in $C$.

We shall now prove that $K = 0$.

**Theorem 31** We have that $K = 0$.

We shall devote the rest of this section to proving this.

Consider the representation $x_i = x_i(u, \gamma_1, ..., \gamma_{n-1})$. By introducing a rotation on $X_0$ we may assume that all $x_i = c_i u + O(u^{m-1})$ where $c_i \neq 0$ for all $i$.

Consider now the map $\omega : Z_0 := A^n \to X_0 := A^n$ given by

$$[z_1, ..., z_n] \to [x_1 = z_1 + z_2^c, x_2 = z_2, ..., x_n = z_n]$$

where $c \in \mathbb{N}$.

Notice that $\omega$ is a Keller map and indeed an isomorphism. We may apply all of the above to the map $f_\omega := f \circ \omega : y_0 \to Y_0$. Let $\Omega$ denote the component above $V$ in $\hat{Z}$.

**Lemma 45** The ramification index of $f_\omega$ along $\Omega$ is $K$.

**Proof:** As $\omega$ is an isomorphism we see that $C(X_0) = C(y_0)$. Hence the ramification of $\omega$ along $\Omega$ is trivial. [End-Of-Proof]

We may introduce exactly as above a parameter $v$, a constant $m_1$ and parameter $\omega_1, ..., \omega_{n-1}$ such that we have a representation $z_1 = v^{m_1}$ and in general $z_i = \sum Z_j^i(\omega_1, ..., \omega_{i-1})v^{-j} + \omega_i v^{-L_i}$. Consider now the inverse dynamical system along $f_1 = r$, $f_2 = 0$, .., $f_n = 0$ for both $y_0$ and $X_0$.

**Theorem 32** We have that $m_1 = cm$.

Let us prove the theorem above. Indeed as $K$ is the ramification index of both $f_\omega$ along $\Omega$ and $f$ along $\Gamma$ we see that we have that both $\frac{du-K}{dr}$ and $\frac{dw-K}{dr}$ exist.

Furthermore we can arrange by a rotation on $Y_0$ that both are nonzero. Hence we see that

$$\lim_{r \to 0} \frac{v^{-K}}{u^{-K}} = \lim_{r \to 0} \frac{du-K}{dr} \neq 0$$

by L’Hospital’s rule.

Hence we see that $\lim \frac{v}{u}$ exists and is nonzero. This implies immediately that $m_1 = cm$ which proves the theorem.

Our next order of business is study the relation between the $\gamma_i$ and the $\omega_i$. 62
Theorem 33 On \( \Gamma(v) \) we have that the reductions are \( \gamma_1 \equiv k\omega \) and \( \gamma_i \equiv \omega_i + P_i(\omega_1, \ldots, \omega_{i-1}) \) in the reductions.

**Proof:** Indeed we have that

\[
v^{\text{cm}} = u^m + (c_2u^m + \ldots + \gamma_1u^{-J_1})^c.
\]

We see that

\[
(c_2u^m + \ldots + \gamma_1u^{-J_1-m})^c = [c_2u^m]^c[1 + \ldots + \gamma_1u^{-J_1-m}]^c.
\]

Hence

\[
v^{\text{cm}} = u^{\text{cm}}[1 + \ldots + k_0\gamma_1u^{-J_1-m} + \ldots].
\]

Thus

\[
v = \zeta^{\text{cm}}u[1 + \ldots + k_1\gamma_1u^{-J_1-m} + \ldots]
\]

and it follows that

\[
u = \zeta^{-\text{cm}}u[1 + \ldots + k_2\gamma_1v^{-J_1-m} + \ldots].
\]

Hence we see that \( \omega_1 = p_1(\gamma_1) \) in the reduction on \( \Gamma(v) \) for some polynomial \( p_1 \).

But we could have done exactly the same analysis to obtain \( \gamma_1 = q_1(\omega_1) \) on \( \Gamma(v) \). Hence we see that at least locally at \( \hat{e} \) we have \( \omega_1 = p_1(q_1(\omega_1)) \) where \( p_1 \) and \( q_1 \) are polynomials. It follows that \( \omega_1 = k\gamma_1 \) on \( \Gamma(v) \) and the expressions for other parameters follow. \[\text{[End-Of-Proof]}\]

But this implies immediately that \( \zeta^{-\epsilon K} = \zeta^{-K} \) must be \( \zeta^{-K} \) which implies that \( (c-1)K \equiv 0 \pmod{cm} \). As we could have chosen \( c \) freely this implies that \( K = 0 \).

Theorem 34 We have that \( K = 0 \).

16 Vector fields on \( n \)-dimensional complex open balls

Before we start we need the following theorem.

Theorem 35 Let \( B \) be the complex open ball parametrized by \( z_1, \ldots, z_n \). Let \( \omega \in H^0(B, \bigwedge^n \Omega^1) \) be the differential \( n \)-form \( \omega = dz_1 \wedge \ldots \wedge dz_n \) and let \( \chi \in H^0(B, \bigwedge^{n-1} \Omega^1) \) be a differential \((n-1)\)-form. Then there exists a vector field \( \eta \in H^0(B, T_B) \) such that \( \omega \circ \eta = \chi \).

**Proof:** Notation that the contraction induces a morphism

\[
\phi : H^0(B, \bigwedge^n \Omega^1) \otimes H^0(B, T_B) \to H^0(B, \bigwedge^{n-1} \Omega^1).
\]
Let $w = [w_1, ..., w_n] \in \mathcal{B}$ be a point and consider the local power series ring $A := \mathbb{C}[[z_1 - w_1, ..., z_n - w_n]]$. Notice that we have a commutative diagram

$$
\begin{array}{ccc}
H^0(\mathcal{B}, \wedge^n \Omega^1) \otimes H^0(\mathcal{B}, \mathcal{T}_B) & \xrightarrow{\phi} & H^0(\mathcal{B}, \wedge^{n-1} \Omega^1) \\
\downarrow i & & \downarrow i \\
H^0(\text{Spec } f, \wedge^n \Omega^1_A) \otimes H^0(\text{Spec } f, \mathcal{T}_A) & \xrightarrow{\phi \otimes_B A} & H^0(\text{Spec } f, \wedge^{n-1} \Omega^1_A)
\end{array}
$$

Notice that $H^0(\text{Spec } f, \wedge^n \Omega^1_A) \simeq A$ as it is free of rank one and furthermore both $H^0(\text{Spec } f, \mathcal{T}_A)$ and $H^0(\text{Spec } f, \wedge^{n-1} \Omega^1_A)$ are free of rank $n$ as $A$-modules. In particular $H^0(\text{Spec } f, \wedge^{n-1} \Omega^1_A)$ is generated by $\{dz_1, ..., dz_i, ..., dz_n\}$ and $H^0(\text{Spec } f, \mathcal{T}_A)$ is generated by $\delta z_1, ..., \delta z_n$. However notice that by considering the constant vector fields $\zeta_i = [0, 0, ..., 1, 0]$ we see that $\phi \otimes_B A$ induces an isomorphism of $A$-modules

$$H^0(\text{Spec } f, \wedge^n \Omega^1_A) \otimes H^0(\text{Spec } f, \mathcal{T}_A) \to H^0(\text{Spec } f, \wedge^{n-1} \Omega^1_A).$$

In particular this implies that given $\chi \in H^0(\mathcal{B}, \wedge^{n-1} \Omega^1)$ we can find a local vector field $\eta_w$ locally around $w$ such that $\omega \circ \eta_w = \chi$, at least locally around $w$.

But we can do this for any $w \in \mathcal{B}$. The result follows by patching the $\eta_w$ together.

[End-Of-Proof]

In the previous section we proved that $K = 0$. Starting from now we shall actually prove again that $K = 0$. Our method will be to assume that $K > 0$. We shall shows that this implies that $K = 1$.

Consider the morphisms

$$\text{spec } \mathbb{C}[[z_1, ..., z_n]] \xleftarrow{f} \text{spec } \mathbb{C}[[u^{-1}, \overline{T}_1, ..., \overline{T}_{n-1}]] \xrightarrow{\pi} \text{spec } \mathbb{C}[[u^{-K}, \overline{T}_1, ..., \overline{T}_{n-1}]].$$

Here $\pi$ denotes the morphism $u^{-1} \to u^{-K}$:

$$\text{spec } \mathbb{C}[[u^{-1}, \overline{T}_1, ..., \overline{T}_{n-1}]] \xrightarrow{\pi} \text{spec } \mathbb{C}[[u^{-K}, \overline{T}_1, ..., \overline{T}_{n-1}]].$$

Notice that at least locally we can regard these as morphisms

$$\mathcal{C} \xleftarrow{f} \mathcal{B} \xrightarrow{\pi} A$$

where $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are complex analytic $n$-dimensional open balls, centered infinitesimally around $v$ and $e$.

Furthermore the map $\pi : \mathcal{B} \to \mathcal{A}$ is a Galois extension with Galois group $G \simeq \mathbb{Z}/K\mathbb{Z}$ acting on $u^{-1} \to \zeta_K u^{-1}$.
Lemma 46  The pullback \( f^*(dy_1 \wedge dy_2 \wedge \ldots \wedge dy_n) \) of the \( n \)-form \( \omega = dy_1 \wedge dy_2 \wedge \ldots \wedge dy_n \) is fixed by \( G \).

**Proof:**  Indeed \( f^*(dy_1 \wedge dy_2 \wedge \ldots \wedge dy_n) = m[u^{-1}]K^{-1}d[u^{-1}] \wedge d\gamma_1 \wedge \ldots \wedge d\gamma_{n-1} \) which is fixed by \( G \).

Hence we see that on \( \mathcal{A} \) we have a differential \( n \)-form \( \omega_1 = dv \wedge d\gamma_1 \wedge \ldots \wedge d\gamma_{n-1} \) which is such that \( f^*(\omega_0) = \pi^*(\omega_1) \). Here \( v = u^{-K} \).

Let \( \mathcal{T}_A \) respectively \( \mathcal{T}_B \) respectively \( \mathcal{T}_C \) denote the tangent bundles of \( \mathcal{A} \) respectively \( \mathcal{B} \) respectively \( \mathcal{C} \). We shall now construct a map \( \pi^*(\mathcal{T}_A) \rightarrow f^*(\mathcal{T}_C) \) as a bundle map on \( \mathcal{B} \).

Notice that

\[
H^0(\mathcal{B}, \pi^*\mathcal{T}_A) \simeq H^0(\mathcal{A}, \mathcal{T}_A) \otimes_{\mathcal{O}_A} \mathcal{O}_B.
\]

As such we see that \( H^0(\mathcal{B}, \pi^*\mathcal{T}_A) \) is generated by the pullbacks of the vector fields \( \zeta_i \) and \( \zeta_i \) where \( \zeta_i \) is the constant vector field along \( v \) and \( \zeta_i \) that along \( \gamma_i \).

For each \( \zeta_i \) and \( \zeta_i \) we shall define a section \( \eta_i \) and \( \eta_i \) in \( H^0(\mathcal{B}, f^*\mathcal{T}_C) \) as follows.

Consider \( \zeta_i \). Let \( \psi_i = \omega_1 \circ \zeta_i \). Notice that \( \psi_i \) is a differential \( n-1 \)-form on \( \mathcal{A} \). Denote by \( \overline{\psi}_v \) the reduction of \( \psi_i \) to \( \pi(\Gamma(u)) \). As \( \Gamma(u) \rightarrow V \) is etale, there is a unique differential \( n-1 \)-form on \( V \) which maps to \( \overline{\psi}_v \). Denote this form by \( \overline{\psi}_{\alpha} = \overline{\psi}_v \).

Let \( \alpha \) be a differential \( n-1 \)-form on \( \mathcal{C} \) which reduces to \( \overline{\psi}_v \). As such we can find a unique holomorphic vector field \( \beta_i \in H^0(\mathcal{C}, \mathcal{T}_C) \) such that \( \omega_0 \circ \beta_i = \alpha_i \).

Notice now that

\[
\pi^*(\psi_i) - f^*(\alpha_i) = \pi^*(\omega_1 \circ \zeta_i) - f^*(\alpha_i)
\]

is a differential \( n-1 \)-form on \( \mathcal{B} \) with a zero on \( \Gamma(u) \).

**Lemma 47**  There exists a vector field \( \chi_i \) on \( \mathcal{B} - \Gamma(u) \) such that

\[
f^*(\omega_0) \circ \chi = \pi^*(\omega_1) \circ \chi = \pi^*(\omega_1 \circ \zeta_i) - f^*(\alpha_i)
\]

where \( \chi_i \) has a pole of order \( K' < K - 1 \) on \( \mathcal{B} \).

**Proof:**  Indeed, we can find a vector field \( \chi_i' \) such that

\[
du^{-1} \wedge d\gamma_1 \wedge \ldots \wedge d\gamma_{n-1} \circ \chi_i' = \pi^*(\omega_1 \circ \zeta_i) - f^*(\alpha_i).
\]

We now simply set \( \chi = [u^{-1}]^{1-K} \chi_i' \). As \( \pi^*(\omega_1 \circ \zeta_i) - f^*(\alpha_i) \) has a zero on \( \Gamma(u) \) then so does \( \chi \) and the result follows.  \([\text{End-Of-Proof}]\)
Similarly we can find vector fields \(\chi_i, i = 1, \ldots, n - 1\), such that
\[
f^*(\omega_0) \circ \chi_i = \pi^*(\omega_1) \circ \chi_i = \pi^*(\omega_1 \circ \zeta_i) - f^*(\alpha_i)
\]
where the \(\alpha_i\) are defined similarly as the \(\alpha_v\). Similarly we can define the vector fields \(\beta_i\) on \(C\).

We now define a map
\[
f_c : H^0(\mathcal{B} - \Gamma(u), \pi^*\mathcal{T}_A) \to H^0(\mathcal{B} - \Gamma(u), f^*\mathcal{T}_C)
\]
by setting
\[
f_c(\pi^*\zeta_v) = f^*(\beta_v) + f_*(\chi_v).
\]
Similarly we define
\[
f_c(\pi^*\zeta_i) = f^*(\beta_i) + f_*(\chi_i).
\]

We need to show that the map \(f_c\) is well defined, i.e. independent of the choice of \(\alpha_v\) and \(\alpha_i\).

To do so we shall work locally. Indeed, let \(U \subset \mathcal{A}\) be an open subset away from \(\pi(\Gamma(u))\) and let \(V \subset \pi^{-1}(U)\) be such that \(\pi|_V : V \to U\) is a bi-holomorphic map. I.e. \(V \to U\) represents a local branch of \(\pi\). Let \(W\) be the image of \(V\) in \(\mathcal{C}\) and assume that \(U \to V \to W\) is an biholomorphic isomorphism (note by making \(U\) small we can assume this).

We denote by \(\pi_U : U \to V\) the local isomorphism and by \(g = f \circ \pi_U\).

Notice that we have a map
\[
H^0(U, \mathcal{T}_U) \xrightarrow{g_*} H^0(U, g^*\mathcal{T}_W) \xrightarrow{f_*} H^0(W, \mathcal{T}_W) \xrightarrow{f_*} H^0(V, f^*\mathcal{T}_W)) \xrightarrow{\pi_*} H^0(V, \mathcal{T}_V).
\]
Now
\[
f^*(\omega_0 \circ t_f^{-1}[f^*\beta_v + f_*(\chi_v)]) = f^*(\omega_0) \circ [i \circ t_f(\beta_v)] + f^*(\omega_0) \circ \chi_v
\]
\[
= \pi^*(\omega_1 \circ \zeta_v) - f^*(\alpha_v) + f^*(\omega_0 \circ \beta_v)
\]
\[
= \pi^*(\omega_1 \circ \zeta_v).
\]
As the differential n-forms \(\omega_0\) and \(\omega_1\) are nonsingular on \(W\) and \(U\) we see that at least locally on the branch the map
\[
f_c|_V : \pi^*(\zeta_v)|_V \to [f^*(\beta_v) + f_*(\chi_v)]|_V
\]
is well defined.
However the map $f_\pi$ was defined independent from the choice of branch, and it follows that it is well-defined globally on $B - \Gamma(u)$.

Consider now again the expressions $y_i = \sum_j Y_j^i (\gamma_1, \ldots, \gamma_{n-1}) u^{-j}$ where the $Y_j$ are power series in the $\gamma_1, \ldots, \gamma_{n-1}$. For $i = 1, \ldots, n$ define $k_i$ to be the smallest index $j > 0$ such that $Y_j^i \neq 0$.

**Lemma 48** We can arrange that $k_1 = \ldots = k_n$.

**Proof:** Indeed it suffices to consider maps of the form $[y_1, \ldots, y_n] \to [y_1 + c_1 y_i, y_2 + c_2 y_i, \ldots, y_n + c_n y_i]$ where $i$ is such that $k_i$ is the smallest of the $k_j$.

[End-Of-Proof]

Let $k = k_1 = \ldots = k_n$. Notice that $\frac{\delta y_i}{\delta u} = Y_i^k [u^{-1}]^{k-1} + \mathcal{O}([u^{-1}]^k)$.

Furthermore, the section $f_\pi(\chi_v) \in H^0(B - \Gamma(u), f^*(\mathcal{T}_C))$ is given by

$$f_\pi(\chi_v) = \begin{bmatrix} \frac{\delta y_1}{\delta u} & \frac{\delta y_2}{\delta v} & \ldots & \frac{\delta y_n}{\delta v_n} \\ \delta y_1 & \delta y_2 & \ldots & \delta y_n \\ \ldots & \ldots & \ldots & \ldots \\ \delta y_1 & \delta y_2 & \ldots & \delta y_n \\ \delta y_1 & \delta y_2 & \ldots & \delta y_n \end{bmatrix} \chi_v.$$

It follows that the section $f_\pi(\chi_v)$ can have a pole of at most order $K' - (k-1) < K - k$ on $\Gamma(u)$ and as such the section $f^*(\beta_v) + f_\pi(\chi_v)$ can have a pole of at most order $K - k - 1$ on $\Gamma(u)$.

Now consider the map $\pi^{-1} : [v, \gamma_1, \ldots, \gamma_{n-1}] \to [u^{-1} = v^{\gamma_1}, \gamma_1, \ldots, \gamma_{n-1}]$. Notice that we can use this to express $y_i = y_i(u^{-1}, \gamma_1, \ldots, \gamma_{n-1})$ locally away from $\Gamma(u)$ as a function of $y_i = y_i(v, \gamma_1, \ldots, \gamma_{n-1})$. In particular we have that

$$\begin{bmatrix} \frac{\delta y_1}{\delta u} & \frac{\delta y_2}{\delta v} & \ldots & \frac{\delta y_n}{\delta v_n} \\ \delta y_1 & \delta y_2 & \ldots & \delta y_n \\ \ldots & \ldots & \ldots & \ldots \\ \delta y_1 & \delta y_2 & \ldots & \delta y_n \\ \delta y_1 & \delta y_2 & \ldots & \delta y_n \end{bmatrix} = \begin{bmatrix} [u^{-1}]^{-1-K} & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 \end{bmatrix} \begin{bmatrix} \delta y_1 \\ \delta y_2 \\ \ldots \\ \delta y_n \end{bmatrix} = [C|B]$$

where $C$ is a column matrix and $B$ is $n$ by $n-1$ matrix.

It follows that $f_\pi(\pi^*\zeta_v)$ has a pole of order $K - k$ for generic $\gamma_i$. But $f_\pi(\pi^*\zeta_v) = f^*(\beta_v) + f_\pi(\chi_v)$ which has a pole of order at most $K - k - 1$.

It follows that $k \geq K$ and that $f_\pi(\pi^*\zeta_v) = f^*(\beta_v) + f_\pi(\chi_v)$ is in fact a holomorphic section of $H^0(B, f^*(\mathcal{T}_C))$. With an argument exactly similar as in the two-dimensional case of Section 5 one proves an analog of Theorem 5.3 for the
higher dimensional case, namely that \( k \leq K \). It follows that \( k = K \).

The map
\[
f_\pi : H^0(\mathcal{B}, \pi^*\mathcal{T}_A) \to H^0(\mathcal{C}, f^*\mathcal{T}_C)
\]
constructed above is independent from the local branches of \( \pi \), in fact the map is well defined globally on \( \mathcal{B} \) as we saw.

However writing
\[
y_i = \sum Y_i (\gamma_1, \ldots, \gamma_{n-1}) u^{-j}
\]
we see that locally the values of \( f_\pi(\pi^*\zeta_v) \) and \( f_\pi(\pi^*\zeta_i) \) will depend on the branch unless all \( Y_i \neq 0 \) if \( j \) does not divide \( K \). Indeed locally \( f_\pi(\pi^*\zeta_v) \) is given by
\[
\begin{bmatrix}
\frac{\delta u_1}{\delta u} & \frac{\delta u_2}{\delta u} & \cdots & \frac{\delta u_r}{\delta u} \\
\frac{\delta u_1}{\delta \gamma_1} & \frac{\delta u_1}{\delta \gamma_2} & \cdots & \frac{\delta u_r}{\delta \gamma_r} \\
\frac{\delta u_1}{\delta \gamma_{n-1}} & \frac{\delta u_1}{\delta \gamma_{n-2}} & \cdots & \frac{\delta u_r}{\delta \gamma_{n-1}} \\
\end{bmatrix}
\cdot
\begin{bmatrix}
[u^{-1}]^{-K} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\cdot
\zeta_v
\]

We arrive at the following.

**Theorem 36** Consider the dynamics of \( u^{-K} \) and \( \gamma_i \) along \( f_i \) for \( i \neq j \). Then the functions \( u^{-K}(r) \) and \( \gamma_i(r) \) are independent of the branch of \( u \) and are in fact smooth \( C^\infty \) functions.

**Proof:** The fact that \( k = K \) implies that all the \( \frac{\partial \gamma}{\partial r} \) exist. The fact that only powers of \( u^{-iK} \) can occur in the expansions of the \( y_i \) implies that
\[
\frac{du^{-K}}{dr} = -K \sum q_i K(\gamma_1, \ldots, \gamma_{n-1}) u^{-iK}
\]
and
\[
\frac{d\gamma_i}{dr} = \sum p^j_{iK}(\gamma_1, \ldots, \gamma_{n-1}) u^{-jK}
\]
where the \( q_i K \) and \( p^j_{iK} \) are power series. It follows from induction that the \( u^{-K} \) and \( \gamma_i \) are infinitely differentiable and are in fact in \( C^\infty \).

To prove that they are independent from the branch of \( u \) one notes that the coupled differential equations
\[
\frac{du^{-K}}{dr} = -K \sum q_i K(\gamma_1, \ldots, \gamma_{n-1}) u^{-iK}
\]
\[ \frac{\partial \gamma_i}{dr} = \sum p_j^i K(\gamma_1, \ldots, \gamma_{n-1}) u^{-j} K \]

with boundary values \( u^{-K}(r) = 0 \) and \( \gamma_i = \gamma_i^0 \) evolve uniquely (they depend only on the initial values of \( \gamma_i^0 \) which are the values of \( \gamma_i \) on \( \Gamma(u) \)). [End-Of-Proof]

Indeed, notice that this implies for a route \( f_i = r \) and \( f_j = 0 \) for \( i \neq j \) we have that the values of the \( \gamma_i \) and \( u^{-K} \) are predetermined. It follows that the values of \( x^K_i = u^{nK} \) are predetermined. Hence the monodromy of winding around \( V \) in \( Y_0 \) changes \( x_1 \) to \( \zeta_K x_1 \).

But we could have chosen the representation \( x_i = x_i + c \) where \( c \) is some complex constant. It follows that monodromy around \( V \) changes \( x_1 \) to \( \zeta_K x_1 \) but at the same time changes \( x_1 + c \) to \( \zeta'_K (x_1 + c) \). As \( c \) was arbitrary it follows that \( K = 1 \).

Hence the map \( f : X \to Y \) is generically unramified along \( \Gamma \). As
\[ \pi_1(Y_0 - S) \cong \pi_1(Y_0) \]

if \( S \subset Y_0 \) is of codimension two, it follows that \( f \) must be an isomorphism. Hence \( K \leq 0 \). The Jacobian conjecture follows at once.

We are done.

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