On \( m \)-th root metrics with special curvature properties

\section{Introduction}

The \( m \)-th root Finsler metrics originated from Riemann’s celebrated address “On the hypothesis, which lie the foundation of geometry”, made in 1854. This class of metrics is regarded as a direct generalization of the class of Riemannian metrics in the sense that the second root metric is a Riemannian metric. The third and fourth root metrics are called the cubic metric and quartic metric, respectively [2–4].

Recent research has shown that such metrics have important applications in Biology, Ecology, Physics and information theory. In two papers [5,6], V. Balan and N. Brinzei study the Einstein equations for some relativistic models relying on such metrics. Y. Yu and Y. You show that an \( m \)-th root Einstein Finsler metric is Ricci-flat [9]. The authors characterize locally flat \( m \)-th root Finsler metrics as well as \( m \)-th root \( \gamma \)-Berwald metrics in [8].

In this Note, we prove that every \( m \)-th root Finsler metric with isotropic Landsberg curvature reduces to a Landsberg metric. Then, we show that every \( m \)-th root metric with almost vanishing \( H \)-curvature has vanishing \( H \)-curvature.

Let \( M \) be an \( n \)-dimensional \( C^\infty \) manifold. Denote by \( TM = \bigcup_{x \in M} T_x M \) the tangent space of \( M \). Let \( TM_0 = TM \setminus \{0\} \). Let \( F = \sqrt[m]{A} \) be a Finsler metric on \( M \), where \( A \) is given by

\[
A := a_{i_1 \ldots i_m}(x) y^{i_1} y^{i_2} \ldots y^{i_m}
\]  

(1)

with \( a_{i_1 \ldots i_m} \) symmetric in all its indices [8]. Then \( F \) is called an \( m \)-th root Finsler metric. Let \( F \) be an \( m \)-th root Finsler metric on an open subset \( U \subset \mathbb{R}^n \). Put

\[
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Suppose that \((A_{ij})\) is a positive definite tensor and \((A^i_j)\) denotes its inverse. Then the following hold:

\[
g_{ij} = \frac{A^{-\frac{1}{2}}}{m^2} \left[ m A A_{ij} + (2 - m) A_{ij} \right], \quad g^{ij} = A^{\frac{1}{2}} \left[ m A A^{ij} + \frac{m}{m - 1} y^i y^j \right].
\]

(2)

\[
y^i A_i = m A, \quad y^i A_{ij} = (m - 1) A_j, \quad y_i = \frac{1}{m} A^{\frac{1}{2} - 1} A_{ij}, \quad A^{1/2} A^{ij} = \frac{m}{m - 1} A.
\]

(3)

Let \((M, F)\) be a Finsler manifold. The second derivatives of \(\frac{1}{2} F^2\) at \(y \in T_x M_0\) are the components of an inner product \(g_y\) on \(T_x M\). The third order derivatives of \(\frac{1}{2} F^2\) at \(y \in T_x M_0\) are a symmetric trilinear form \(C_y\) on \(T_x M\). We call \(g_y\) and \(C_y\) the fundamental form and the Cartan torsion, respectively. The rate of change of the Cartan torsion along geodesics is the mean Berwald curvature, which gives rise to the mean Berwald curvature \(G_i\) and of almost vanishing Finsler metric. Then \(F\) is said to be Landsbergian if \(L = 0\). The quotient \(L/C\) is regarded as the relative rate of change of Cartan torsion \(C\) along Finslerian geodesics. Then \(F\) is said to be isotropic Landsberg metric if \(L = c F C\), where \(c = c(x)\) is a scalar function on \(M\). In this paper, we prove the following:

**Theorem 1.** Let \((M, F)\) be an \(n\)-dimensional \(m\)-th root Finsler manifold. Suppose that \(F\) is a non-Riemannian isotropic Landsberg metric. Then \(F\) reduces to a Landsberg metric.

Let \(F\) be a Finsler metric on a manifold \(M\). The geodesics of \(F\) are characterized locally by the equations \(\frac{d^2 x^i}{dt^2} + 2G^i(x, \frac{dx}{dt}) = 0\), where \(G^i = \frac{1}{2} g^{ik} |\frac{d}{dt} x^k - \frac{\partial g_{ij}}{\partial x^k} y^k|^2 y^k\) are coefficients of the spray associated with \(F\). A Finsler metric \(F\) is called a Berwald metric if \(G^i = \frac{1}{2} \Gamma^i_j(x) y^j y^k\) are quadratic in \(y \in T_x M\) for any \(y \in M\). Taking the trace of Berwald curvature gives rise to the mean Berwald curvature \(E\). In [1], Akbar-Zadeh introduces the non-Riemannian quantity \(H\) which is obtained from the mean Berwald curvature by the covariant horizontal differentiation along geodesics. More precisely, the non-Riemannian quantity \(H = H_{ij} dx^i \otimes dx^j\) is defined by \(H_{ij} := E_{ijk} y^k\). He proves that for a Finsler manifold of scalar flag curvature \(K\) with dimension \(n \geq 3\), \(K = \text{constant}\) if and only if \(H = 0\). It is remarkable that the Riemann curvature \(R_y = R^k dx^k \otimes \frac{d}{dt} x^i : T_x M \to T_x M\) is a family of linear maps on tangent spaces, defined by

\[
R^i_k = 2 \frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2 G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}.
\]

A Finsler metric \(F\) is said to be of scalar curvature if there is a scalar function \(K = K(x, y)\) such that \(R^i_k = K(x, y) F^2 h^i_k\). If \(K = \text{constant}\), then \(F\) is called of constant flag curvature.

A Finsler metric is called of almost vanishing \(H\)-curvature if \(H_{ij} = \frac{n + 1}{2} \theta h_{ij}\), for some 1-form \(\theta \) on \(M\), where \(h_{ij}\) is the angular metric. It is remarkable that in [7], Z. Shen with the authors prove that every Finsler metric of scalar flag curvature \(K\) and of almost vanishing \(H\)-curvature has almost isotropic flag curvature, i.e., the flag curvature is in the form \(K = \frac{3}{r} + \sigma\), for some scalar function \(\sigma\) on \(M\).

**Theorem 2.** Let \((M, F)\) be an \(n\)-dimensional \(m\)-th root manifold with \(n \geq 2\). Suppose that \(F\) has almost vanishing \(H\)-curvature. Then \(H = 0\).

2. Proof of the Main Theorems

**Lemma 3.** (See [9].) Let \(F\) be an \(m\)-th root Finsler metric on an open subset \(U \subset \mathbb{R}^n\). Then the spray coefficients of \(F\) are given by

\[
G^i = \frac{1}{2} (A_{ij} - A_{ij}) A^{ij}.
\]

(4)

**Proof of Theorem 1.** Let \(F = \sqrt{A}\) be an \(m\)-th root isotropic Landsberg metric, i.e., \(L_{ijk} = c F C_{ijk}\), where \(c = c(x)\) is a scalar function on \(M\). The Cartan tensor of \(F\) is given by the following:

\[
C_{ijk} = \frac{1}{m} A^{\frac{2}{m} - 3} \left[ A^2 A_{ijk} + \left( \frac{2}{m} - 1 \right) \left( \frac{2}{m} - 2 \right) A_{ij} A_{kj} + \left( \frac{2}{m} - 1 \right) A_{ij} A_{kj} + A_{ij} A_{kl} + A_{jk} A_{ki} + A_{ik} A_{ji} \right].
\]

(5)

Since \(L_{ijk} = -\frac{1}{2} y^k G^i_{y^i y^j y^k}\), then we have \(L_{ijk} = -\frac{1}{2m} A^{\frac{2}{m} - 1} A^2 G^i_{y^i y^j y^k}\). Therefore, we get

\[
A^2 G^i_{y^i y^j y^k} = -2c A^{\frac{2}{m} - 2} \left[ A^2 A_{ijk} + \left( \frac{2}{m} - 1 \right) \left( \frac{2}{m} - 2 \right) A_{ij} A_{kj} + (2 A_{ij} A_{kj} + A_{ij} A_{kl} + A_{jk} A_{ki} + A_{ik} A_{ji}) \right].
\]

(6)
By Lemma 3, the left-hand side of (6) is a rational function in $y$, while its right-hand side is an irrational function in $y$. Thus, either $c = 0$ or $A$ satisfies the following PDEs

$$A^2 A_{ijk} + \left(\frac{2}{m} - 1\right) \left(\frac{2}{m} - 2\right) A_i A_j A_k + \left(\frac{2}{m} - 1\right) A_i A_j \kappa_k A_i A_k + A_i A_{jk} = 0. \tag{7}$$

Plugging (7) into (5) implies that $C_{ijk} = 0$. Hence, by Deicke’s theorem, $F$ is Riemannian metric, which contradicts our assumption. Therefore, $c = 0$. This completes the proof.

**Proof of Theorem 2.** Let $F = \sqrt[2m]{A}$ be of almost vanishing $H$-curvature, i.e.,

$$H_{ij} = \frac{n + 1}{2F} \theta h_{ij}, \tag{8}$$

where $\theta$ is a 1-form on $M$. The angular metric $h_{ij} = g_{ij} - F^2 y_i y_j$ is given by the following

$$h_{ij} = \frac{A^{\frac{1}{2m}}}{m^2} \left[ m A A_{ij} + (1 - m) A_i A_j \right]. \tag{9}$$

Plugging (9) into (8), we get

$$H_{ij} = \frac{(n + 1)}{2m^2} A^{\frac{1}{2m}} \theta \left[ m A A_{ij} + (1 - m) A_i A_j \right]. \tag{10}$$

By (4), one can see that $H_{ij}$ is rational with respect to $y$. Thus, (10) implies that $\theta = 0$ or $m A A_{ij} + (1 - m) A_i A_j = 0$. \tag{11}

By (9) and (11), we conclude that $h_{ij} = 0$, which is impossible. Hence $\theta = 0$ and $H_{ij} = 0$. \qed

By the Schur Lemma, Theorem 2 and Theorem 1.1 of [7], we have the following:

**Corollary 4.** Let $(M, F^n)$ be an $n$-dimensional $m$-th root Finsler manifold of scalar flag curvature $K$ with $n \geq 3$. Suppose that the flag curvature is given by $K = 3\theta F + \sigma$, where $\theta$ is a 1-form and $\sigma = \sigma(x)$ is a scalar function on $M$. Then $K = 0$.

**References**

[1] H. Akbar-Zadeh, Sur les espaces de Finsler à courbures sectionnelles constantes, Bull. Acad. Roy. Belg. Cl. Sci. (5) LXXXIV (1988) 281–322.
[2] V. Balan, Numerical multilinear algebra of symmetric $m$-root structures. Spectral properties and applications, in: Geometric Approaches to Symmetry, Symmetry in the History of Science, Art and Technology, Part 2, Symmetry Festival 2009, Budapest, Hungary, Symmetry: Culture and Science 21 (1-3) (2010) 119–131.
[3] V. Balan, CMC and minimal surfaces in Berwald–Moor spaces, Hypercomplex Numbers in Geometry and Physics 3 (2(6)) (2006) 113–122.
[4] V. Balan, N. Perminov, Applications of resultants in the spectral $m$-root framework, Applied Sciences 12 (2010) 20–29.
[5] V. Balan, N. Brinzei, Einstein equations for $(h, v)$–Berwald–Moor relativistic models, Balkan. J. Geom. Appl. 11 (2) (2006) 20–26.
[6] V. Balan, N. Brinzei, Berwald–Moor-type $(h, v)$-metric physical models, Hypercomplex Numbers in Geometry and Physics 2 (2(4)) (2005) 114–122.
[7] B. Najafi, Z. Shen, A. Tayebi, Finsler metrics of scalar flag curvature with special non-Riemannian curvature properties, Geom. Dedicata 131 (2008) 87–97.
[8] A. Tayebi, B. Najafi, On $m$-th root Finsler metrics, J. Geom. Phys. 61 (8) (2011) 1479–1484.
[9] Y. Yu, Y. You, On Einstein $m$-th root metrics, Differential Geometry and its Applications 28 (2010) 290–294.