On rich and poor directions determined by a subset of a finite plane

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Abstract

We generalize to sets with cardinality more than \( p \) a theorem of Rédei and Szönyi on the number of directions determined by a subset \( U \) of the finite plane \( \mathbb{F}_p^2 \). A \( U \)-rich line is a line that meets \( U \) in at least \( \#U/p + 1 \) points, while a \( U \)-poor line is one that meets \( U \) in at most \( \#U/p - 1 \) points. The slopes of the \( U \)-rich and \( U \)-poor lines are called \( U \)-special directions. We show that either \( U \) is contained in the union of \( n = \lceil \#U/p \rceil \) lines, or it determines “many” \( U \)-special directions. The core of our proof is a version of the polynomial method in which we study iterated partial derivatives of the Rédei polynomial to take into account the “multiplicity” of the directions determined by \( U \).

Keywords: finite incidence geometry, polynomial method, affine Galois plane; special directions, lacunary polynomials, multiplicities

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1. Introduction

In this paper we fix a prime number \( p \) and we study the (affine, finite) Galois plane \( \mathbb{F}_p^2 \) from the point of view of Incidence Geometry. Our aim is to provide an
extension to a theorem of Rédei and Szőnyi and to stimulate some research on
the “polynomial method” in situations where “multiplicities” are allowed. Our
main result and the Rédei-Szőnyi theorem pertain to the intersections between
a fixed set \( U \subseteq F_p^2 \) and all the lines \( \ell \subseteq F_p^2 \) considered “up to parallelism”,
as follows. For every straight line in the finite plane \( F_p^2 \) we may consider its
direction (or “slope”) so that parallel lines have the same direction. A set
\( U \subseteq F_p^2 \) determines a direction \( m \) if there exists a line \( \ell \subseteq F_p^2 \) with slope \( m \)
passing through two distinct points of \( U \). Notice that there are \( p+1 \) possible
directions, naturally identified with the elements of \( F_p \cup \{ \infty \} \). The following
theorem is well-known:

**Theorem 1.1** (Rédei-Szőnyi). Let \( U \subseteq F_p^2 \) with \( \#U \leq p \). If \( U \) is not contained
in the support a line, then it determines at least \( \lceil \frac{\#U+3}{2} \rceil \) directions.

Theorem 1.1 was proved in a seminal monograph [1, Par. 36] by Rédei, with a
contribution of Megyesi, in the case \( \#U = p \). It was later extended to \( \#U \leq p \)
by Szőnyi, see [2, Remark 5] and [3, Sec. 3 and 5]. Our objective is to obtain
an analogous statement for \( \#U > p \) as well. Since for every given direction
\( m \in F_p \cup \{ \infty \} \) there are only \( p \) parallel lines with slope \( m \), it is clear by a
Pigeonhole argument that any set \( U \subseteq F_p^2 \) with \( \#U > p \) determines all directions
of the plane. However, we propose the following natural definitions:

**Definition 1.2.** Let \( U \subseteq F_p^2 \) and let \( \theta := \#U/p \). We say that a line \( \ell \) is \( U \)-rich if
it meets \( U \) in at least \( \theta+1 \) points, and that \( \ell \) is \( U \)-poor if instead \( \#(\ell \cap U) \leq \theta - 1 \).
We say that a direction \( m \in F_p \cup \{ \infty \} \) is \( U \)-special if there is a line \( \ell \) with slope \( m \)
that is either \( U \)-rich or \( U \)-poor.

Notice that, when \( \theta \leq 1 \), a \( U \)-special direction is nothing but a direction deter-
mined by \( U \). Our main result reads as follows

**Theorem 1.3.** Let \( U \subseteq F_p^2 \) with \( \#U = np - r \), for some \( 1 \leq n \leq p \) and
\( 0 \leq r < p \). Then there are at least \( \lceil \frac{p+n+2-r}{n+1} \rceil \) \( U \)-special directions, if \( U \) is not
contained in the union of \( n \) lines.

Notice that for \( n = 1 \) we recover the theorem of Rédei and Szőnyi. The case
\( n = 2 \) and \( r = 0 \) was examined by the author in [4]. The directions determined
by a set \( U \subseteq F_q^2 \) with \( \#U \leq q \), where \( q \) is a power of a prime, have been studied
in [5, 6, 7]. We expect that a generalization of Theorem 1.3 for \( U \)-generic
directions in the Galois plane \( F_q^2 \) might be obtained with similar ideas. See also
[8, 9] for results on the directions determined by a set \( U \subseteq F_p^d \) in dimension
\( d \geq 3 \).

A widely applied technique in Incidence Galois Geometry is the so-called poly-
nomial method, introduced by Rédei in his pioneering work [1]. In order to keep
the article self-contained, we will briefly outline in sections 3 and 4 the main
definitions and results that will be needed in our application of the method.
Our Proposition 4.3 on lacunary polynomials of high degree is apparently new,
although its proof is completely elementary. For the interested reader we refer
to [10, 11, 12, 13, 14] for other expositions of the polynomial method of Rédei and to [15, 16, 17, 18] for other uses of polynomials for incidence problems in Combinatorics and Number Theory.

In this work we are concerned with lines that meet \( U \subseteq \mathbb{F}_p^2 \) in multiple points, possibly more than two. Thus, we were naturally led to look for a way to exploit these “multiplicities” in the polynomial method. Our crucial idea is to consider the iterated derivatives of the Rédei-Szőnyi polynomial with respect to its second variable (see sections 3.2 and 4.2). Multiplicities have been considered also by the works on multiple blocking sets, such as [19, 20, 21, 22, 23]. However, in these papers the set \( U \) is supposed to intersect, with a given multiplicity, lines coming from all directions (as opposed to some directions), and so the use of \( y \)-derivatives is not required.

The conclusion of Theorem 1.3 is especially strong when \( n \) is small compared to \( p \), but in general it is unclear whether this result is optimal. If \( n \ll 1 \) is fixed, \( p - r \gg p \) and \( U \) is a set with cardinality \( \#U = np - r \) not contained in \( n \) lines, then our theorem predicts that a positive proportion of all directions is \( U \)-special. In this range the conclusion is best-possible up to multiplication by a constant. On the other hand, the extreme opposite case \( n = p \) is trivial because every set \( U \subseteq \mathbb{F}_p^2 \) is contained in the union of \( p \) lines.

As a final note, we remark that when \( n \leq \frac{p + 1}{2} \) and \( p \neq 2 \) it is possible to find a set \( U_p^{(n)} \subseteq \mathbb{F}_p^2 \) with cardinality \( \#U_p^{(n)} = np \), and not contained in \( n \) lines, that determines exactly \( \frac{p + 3}{2} \) \( U \)-special directions, as follows. We know by [24] that the set \( U_p^{(1)} := \{ (k, k^{(p-1)/2}) : k \in \mathbb{F}_p \} \) determines exactly \( \frac{p + 3}{2} \) directions of \( \mathbb{F}_p^2 \). This set is not contained in one line, but we have \( U_p^{(1)} \subseteq \ell_1 \cup \ell_{-1} \), where \( \ell_{\pm 1} := \{ (k, \pm k) : k \in \mathbb{F}_p \} \). Then, if \( n \leq \frac{p + 1}{2} \), we can form the disjoint union \( U_p^{(n)} = U_p^{(1)} \sqcup_{i=1}^{n-1} \ell^{(i)} \) with \( n - 1 \) lines parallel to \( \ell_1 \). The resulting set \( U_p^{(n)} \) has the properties mentioned above. It would be interesting to know if better constructions are possible, so we propose the following:

**Problem 1.4.** Let \( U \subseteq \mathbb{F}_p^2 \) with \( \#U = np - r \), for some \( 1 \leq n \leq p \) and \( 0 \leq r < p \), as in Theorem 1.3. Suppose that \( U \) is not contained in the union of \( n \) lines. Then, are there necessarily at least \( \lceil \frac{p + 3 - r}{2} \rceil \) \( U \)-special directions?

### 2. Number of special directions in a general case

In this section we are going to prove Theorem 1.3 under the assumption \( p - r \geq n + 1 \). The remaining cases will be examined in section 5. Here we also assume as a blackbox the results coming from the polynomial method. These are recorded in Lemma 2.4 below and will be proved in section 3 and section 4.

#### 2.1. Setup

First we prepare the stage with some notation: we fix a prime number \( p \) and a set \( U \subseteq \mathbb{F}_p^2 \) with cardinality \( N := \#U = np - r \) for some \( 1 \leq n < p \) and \( 0 \leq r < p - n \). Moreover, we assume that \( U \) is not contained in the union of
n lines. To make the terminology less cumbersome, we will simply call special a direction that is \( U \)-special. Similarly, we say that a line \( \ell \subseteq \mathbb{F}_p^2 \) is rich (resp. poor, special) if it is \( U \)-rich (resp. \( U \)-poor, \( U \)-special). Moreover, we complete Definition 1.2 with the following one.

**Definition 2.1.** We say that a direction \( m \in \mathbb{F}_p \cup \{\infty\} \) is rich (or \( U \)-rich) if there is some rich line with slope \( m \). A direction will be called generic (or \( U \)-generic) if it is not special, and we call it poor if it is special but not rich.

We let \( D \) be the number of special directions, let \( E \) be the number of rich directions and let \( W \) the number of rich lines. In particular, notice that \( E \leq W \) and that there are \( D - E \) poor directions. Finally, we assume that there are strictly less than \( \left\lceil \frac{p+n+2-r}{n+1} \right\rceil \) special directions or, equivalently, that

\[
D \leq 1 + \frac{p-r}{n+1}.
\]  

(2.1)

Given the above assumptions, we want to find a contradiction. To do so, we introduce the following quantities.

**Definition 2.2.** For every direction \( m \in \mathbb{F}_p \cup \{\infty\} \) we let \( c_m \) be the number of unordered pairs of points \( u, v \in U \) such that the line joining them has slope \( m \).

The idea will be to estimate \( c_m \) separately for generic, poor and rich directions and then to compare these inequalities with the following identity:

\[
\sum_{m \in \mathbb{F}_p \cup \{\infty\}} c_m = \binom{N}{2} = \frac{1}{2} \left( n^2p^2 - np - np + r^2 + r \right).
\]  

(2.2)

### 2.2. Preliminary considerations

Our first observation is that \( U \) is contained in the union of the rich lines.

**Lemma 2.3.** If \( U \subseteq \mathbb{F}_p^2 \) has cardinality \( N = np - r \) with \( p - r \geq n + 1 \), then every point of \( U \) is contained in a rich line.

**Proof.** Pick \( v \in U \) arbitrarily and notice that \( U \setminus \{v\} \) is the disjoint union of the sets \((\ell \cap U) \setminus \{v\}\) as \( \ell \) ranges through the \( p + 1 \) lines passing through \( v \). If \( v \) is not contained in a rich line, then \( \#(\ell \cap U) \leq n \) for each such line, and so

\[
N - 1 \leq (p + 1)(n - 1) = np - p + n - 1,
\]

which contradicts the assumption \( p - r \geq n + 1 \). \( \square \)

For every rich direction \( m \) we denote by \( w_m \) the number of rich lines with slope \( m \), so that \( W = \sum_{\text{rich}} w_m \). The polynomial method furnishes the following result.
Lemma 2.4. Let $U \subseteq \mathbb{F}_p^2$ be as before, such that eq. (2.1) holds and $U$ is not contained in the union of $n$ lines, and let $m$ be a rich direction. Then every non-rich line with slope $m$ meets $U$ in at most $n - w_m$ points and each rich line meets $U$ in at least $w_m(n - w_m) \geq n - 1$.

Since $U$ is contained in the union of the rich lines by Lemma 2.3 and $U$ is not contained in the union of $n$ lines by assumption, we have that $W \geq n + 1$.

Moreover, by Lemma 2.4 we have $w_m \leq n - 1$ for every rich direction $m$, and so $W \leq (n - 1)E$. Since $E \leq D$ we get

$$n + 1 \leq W \leq (n - 1)D$$

(2.3)

For every rich direction $m$ we let $\ell_1^{(m)}, \ldots, \ell_{w_m}^{(m)}$ be all the rich lines with slope $m$ and we let $L_i^{(m)} := \#(\ell_i^{(m)} \cap U)$ for $i = 1, \ldots, w_m$. Then, we define

$$z_m := \sum_{i=1}^{w_m} L_i \quad \text{and} \quad Z := \sum_{\text{rich}} z_m.$$ 

Since $U$ is contained in the union of the rich lines, we can estimate the sum of the lengths of the rich lines as follows.

Lemma 2.5. Let $Q := \sum_{\text{rich}} w_m^2$. Then

$$Z \leq N + \frac{1}{2} W^2 - \frac{1}{2} Q.$$  

(2.4)

Proof. Let $\ell_1, \ldots, \ell_W$ be all the rich lines. By the Principle of Inclusion-Exclusion we have

$$\#U \geq \sum_{i=1}^{W} \#(\ell_i \cap U) - \sum_{1 \leq i < j \leq W} \#(\ell_i \cap \ell_j \cap U).$$

Since two lines meet in a point only when they have different slopes, we deduce that

$$N \geq Z - \frac{1}{2} \sum_{\text{rich}} (W - w_m)w_m,$$

which is equivalent to eq. (2.4).

2.3. The key computations

We are now ready to study the quantities $c_m$ introduced in Definition 2.2 and to work them out to reach a contradiction. The following lemma is useful to estimate the sums that implicitly appear in the definition of the $c_m$’s.

Lemma 2.6. Let $L_1, \ldots, L_W$ be integers satisfying $A \leq L_i \leq B$ for all $i = 1, \ldots, W$ for some $A, B \geq 0$, and let $\sum_{i=1}^{W} L_i = Z$. Then

$$\sum_{i=1}^{W} \binom{L_i}{2} \leq \frac{1}{2} [ (A + B - 1)Z - ABW ],$$

with equality if and only if $L_i \in \{ A, B \}$ for all $i = 1, \ldots, W$.
Proof. For every $i = 1, \ldots, W$ we have $L_i(L_i - B) \leq A(L_i - B)$, with equality if and only if $L_i \in \{A, B\}$. Therefore we have

$$L_i(L_i - 1) \leq (A + B - 1)L_i - AB$$

by adding $L_i(B - 1)$ on both sides. The lemma now follows summing over $i = 1, \ldots, W$ and dividing by 2. □

We first consider the generic and poor directions.

**Proposition 2.7.** We have

$$\sum_{m \text{ rich}} c_m \geq \frac{p - n}{2}N - \left(1 - \frac{1}{n + 1}\right)r(p - r) + \frac{n - 1}{2}NE. \quad (2.5)$$

**Proof.** The lines with a generic slope $m$ meet $U$ in either $n$ or $n - 1$ points, so

$$c_m = (p - r)\left(\frac{n}{2}\right) + r\left(\frac{n - 1}{2}\right) = \frac{n - 1}{2}(np - 2r). \quad (2.6)$$

The lines with a poor slope $m$ meet $U$ in at most $n$ points, so we see that

$$c_m \leq \frac{N}{n}\left(\frac{n}{2}\right) = \frac{n - 1}{2}(np - r). \quad (2.7)$$

If we compare these estimates with eq. (2.2) we obtain

$$\sum_{m \text{ rich}} c_m \geq \frac{N}{2} - \frac{n - 1}{2}[(p + 1 - D)(np - 2r) + (D - E)(np - r)] - \frac{n - 1}{2}(p + 1)(np - 2r) - \frac{n - 1}{2}Dr + \frac{n - 1}{2}NE.$$ 

Using the fact that $D - 1 \leq (p - r)/(n + 1)$ by eq. (2.1) and

$$\left(\frac{N}{2}\right) - \frac{n - 1}{2}(p + 1)(np - 2r) - \frac{n - 1}{2}r = \frac{p - n}{2}N + \frac{r(p - r)}{2},$$

we get the inequality 2.5. □

Next we consider the estimate, in the other direction, coming from the rich lines.

**Proposition 2.8.** We have

$$\sum_{m \text{ rich}} c_m \leq \frac{1}{2}\left((n - 1)NE + (2p - r - 1)N + f(W)\right), \quad (2.8)$$

where

$$f(T) := \left(p - \frac{r}{2} - \frac{1}{2}\right)T^2 - (n - 1)\left(p - \frac{r}{2} - \frac{1}{2}\right)T - p(p - r)T - NT. \quad (2.9)$$
Proof. For every rich direction $m$ let $\ell_1^{(m)}, \ldots, \ell_{w_m}^{(m)}$ be all the rich lines with slope $m$ and let $\ell_{w_m+1}^{(m)}, \ldots, \ell_p^{(m)}$ be the non-rich lines with slope $m$. Then for every $i = 1, \ldots, p$ we let $L_i^{(m)} = \#(\ell_i^{(m)} \cap U)$. For every $i = 1, \ldots, w_m$ we have $p - r + n - 1 \leq L_i^{(m)} \leq p$ by Lemma 2.4, so by Lemma 2.6 we have

$$2 \sum_{i=1}^{w_m} \left( \frac{L_i^{(m)}}{2} \right) \leq (2p - r + n - 2)z_m - p(p - r + n - 1)w_m.$$  

Again by Lemma 2.4 we have $L_i^{(m)} \leq n - w_m$ for every $i = w_m + 1, \ldots, p$. Therefore

$$2 \sum_{i=w_m+1}^{p} \left( \frac{L_i^{(m)}}{2} \right) \leq (n - w_m - 1)(N - z_m) \leq (n - 1)N - w_mN - (n - 1)z_m + w_m^2p,$$

where we also noticed that $w_mz_m \leq w_m^2p$. By the above two estimates and summing over $m$ we obtain

$$\sum_m c_m \leq \frac{1}{2} \left( (n - 1)NE - NW + (2p - r - 1)Z - p(p - r + n - 1)W + pQ \right). \quad (2.10)$$

By Lemma 2.4 we have $w_m \leq n - 1$ for all rich direction $m$, and so $Q \leq (n - 1)W$. Using Lemma 2.5 to simplify the $Z$ in eq. (2.10) and then using this inequality $Q \leq (n - 1)W$, we finally get eq. (2.8). \qed

Next we observe that the worst-case scenario (with respect to the goal of getting a contradiction from the two above estimates) is represented by the case $W = n + 1$.

Lemma 2.9. Let $f(T)$ be as in eq. (2.9). Then $f(W) \leq f(n + 1)$.

Proof. If $W = n + 1$ there is nothing to prove. Otherwise, we have

$$n + 1 < W \leq (n - 1)D \leq n - 1 + \frac{n - 1}{n + 1}(p - r) \leq n - 3 + p - r$$

by eqs. (2.1) and (2.3) and the assumption $p - r \geq n + 1$. Moreover the leading coefficient $p - (r + 1)/2 = p/2 + (p - r - 1)/2$ of $f(T)$ is positive, hence

$$\frac{f(W) - f(n + 1)}{W - n - 1} = \left( p - \frac{r}{2} - \frac{1}{2} \right) \left( W + n + 1 - (n - 1) \right) - p(p - r) - N$$

$$\leq \left( p - \frac{r}{2} - \frac{1}{2} \right) \left( p - r + n - 1 \right) - p(p - r) - N$$

$$= -\frac{3}{2}(p - r) - \frac{n - 1}{2}(r + 1) - \frac{r(p - r)}{2}.$$  

This last expression is manifestly nonnegative, so $f(W) \leq f(n + 1)$. \qed
By Propositions 2.7 and 2.8 and Lemma 2.9 we get
\[(p - n)N - 2r(p - r) + \frac{2r(p - r)}{n + 1} \leq (2p - r - 1)N + f(n + 1),\]  
where, after some simplification:
\[f(n + 1) = (n + 1)[-p(p - r) - np + 2p - 1].\]
We notice that
\[p(p - r)(n + 1) - (2p - r)N + pN - 2r(p - r) = (p - r)^2\]
so eq. (2.11) implies
\[(p - r)^2 + \frac{2r(p - r)}{n + 1} \leq - (n + 1)(n - 2) - (n + 1).\]  
If \(n \geq 2\) the right-hand side of eq. (2.12) is negative, so this inequality is impossible. Also for \(n = 1\) it is, because in this case \(p - r \geq n + 1 = 2\) and so eq. (2.12) implies
\[2p \leq (p - r)^2 + 2r(p - r) \leq 2p - 2.\]

3. The polynomial method

In this section introduce important tools in the polynomial method applied to “direction problems” of finite affine geometry.

3.1. Rédei polynomial and Szőnyi complement

The starting point of the method is the following polynomial that was introduced by Rédei [1]. This polynomial is used to encode algebraically the multiplicity of intersection between the set \(U\) and all the lines \(\ell \subseteq \mathbb{F}_p^2\) of the plane.

**Definition 3.1.** Given \(U \subseteq \mathbb{F}_p^2\) nonempty, we define the (inhomogeneous affine) Rédei polynomial \(R_U(x, y) \in \mathbb{F}_p[x, y]\) by
\[R_U(x, y) := \prod_{(a, b) \in U} (x - ay + b).\]

It has the following remarkable property: if \((a, b) \neq (a', b')\), we have
\[(x - ay + b) = (x - a'y + b') \iff \frac{b - b'}{a - a'} = y.\]  
In other words, two linear factors of \(R_U\) are equal when \(y\) is the slope of the line connecting \((a, b)\) and \((a', b')\). In particular:

**Remark 3.2.** For all \(k, m \in \mathbb{F}_p\), the multiplicity of the linear polynomial \(x - k\) within the factorization of \(R_U(x, m) \in \mathbb{F}_p[x]\) equals the number of points of \(U\) on the line \(\ell_{m, k} = \{(u, v) : v = mu - k\}\).
We observe that $R_U$ is a (non homogeneous) completely reducible (i.e. it factors completely as a product of linear polynomials) polynomial in two variables of total degree equal to $\#U$. When $\#U < p$, Szönyi found an ingenious and meaningful way to complete $R_U$ to a polynomial of degree $p$. Our objective now is to define an analogous natural “complement to degree $np$” when $(n-1)p < \#U < np$, for some $n \in \mathbb{N}$.

**Definition 3.3.** Let $A$ denote the ring $A := \mathbb{F}_p[y]$. Given $U \subseteq \mathbb{F}_p^2$ nonempty and $n \in \mathbb{N}$ with $\#U \leq np$ we define $S_{U,n}(x,y), T_{U,n}(x,y) \in \mathbb{F}_p[x,y]$ be respectively the quotient and the remainder of the univariate polynomial long division in $A[x]$ of $(x^p-x)^n$ by $R_U$: 

$$(x^p-x)^n = R_U(x,y)S_{U,n}(x,y) + T_{U,n}(x,y).$$

We call $S_{U,n}(x,y)$ the (nth generalized) Szőnyi complement of $U$. Notice that, as a polynomial in $x$, $R_U$ is a monic polynomial of degree $\text{deg}_x R_U = \#U$, so the long division is well-defined and $S_{U,n}$ is again a monic polynomial in $x$, with degree $\text{deg}_x S_{U,n} = np - \#U$.

### 3.2. The RS-polynomial and its y-derivatives

We are now able to introduce our main object of study.

**Definition 3.4.** Let $U \subseteq \mathbb{F}_p^2$ be nonempty and let $n \in \mathbb{N}$ with $np \geq \#U$. We define the nth Rédei-Szőnyi polynomial $H_{U,n}(x,y) = x^{np} + h_1(y)x^{np-1} + \cdots + h_{np}(y)$ (in short RS-polynomial) of $U$ by

$$H_{U,n}(x,y) := R_U(x,y)S_{U,n}(x,y).$$

By inspection of the long division algorithm in Definition 3.3, it is not difficult to check that the Szőnyi complement $S_{U,n}(x,y)$ is a polynomial in two variables with total degree equal to $np - \#U$. Therefore we have

**Remark 3.5.** For all $U \subseteq \mathbb{F}_p^2$ and $n \in \mathbb{N}$ with $np \geq \#U$ the RS-polynomial $H_{U,n}$ is a polynomial in two variables with total degree equal to $np$. In particular its $x$-coefficients $h_j(y)$, for $1 \leq j \leq np$, are polynomials in $y$ of degree at most $j$.

The following is the most fundamental property of the RS-polynomial, namely its interaction with the non-vertical non-$U$-rich directions.

**Proposition 3.6.** Let $U \subseteq \mathbb{F}_p^2$ be nonempty, let $n \in \mathbb{N}$ with $np \geq \#U$ and let $m \neq \infty$. If $m$ is not a $U$-rich direction, then $H_{U,n}(x,m) = (x^p-x)^n$.

**Proof.** By Definition 3.3 we have

$$(x^p-x)^n = R_U(x,m)S_{U,n}(x,m) + T_{U,n}(x,m)$$

and from Remark 3.2 we have $R_U(x,m) \mid (x^p-x)^n$. Hence $R_U(x,m) \mid T_{U,n}(x,m)$. However, $R_U(x,m)$ is monic with $\text{deg}_x R_U(x,m) = \#U$, while $\text{deg}_x T_{U,n} < \#U$ by definition. The only way this can happen is with $T_{U,n}(x,m) = 0$, so $H_{U,n}(x,m) = (x^p-x)^n$ by Definition 3.4. \qed
The following is another important observation that we will exploit for the proof of our main theorem: when \( m \) is a \( U \)-generic direction, the \( y \)-derivatives of the RS-polynomial are divisible by suitable powers of \( x^p - x \).

**Proposition 3.7.** Let \( U \subseteq \mathbb{F}_p^2 \) with \( (n-1)p < \#U < np \) for some \( n \in \mathbb{N} \). Suppose that \( m \neq \infty \) is not a \( U \)-special direction. Then for every \( \alpha \leq n \) we have

\[
(x^p - x)^{n-\alpha} | (\partial_y^\alpha H_{U,n})(x, m).
\]

**Proof.** Recalling Definition 1.2, we notice that \( H_{U,n}(x, m) = (x^p - x)^n \) by Proposition 3.6. Moreover, by Remark 3.2, we have that \( R_U(x, m) \) is divisible by \( (x^p - x)^{n-1} \) (or even \( (x^p - x)^n \) if \( \#U = np \)). Therefore it is possible to find some \( V \subset U \) such that, for

\[
P(x, y) := \prod \limits_{(a, b) \in V} (x - ay + b),
\]

we have \( P(x, m) = (x^p - x)^{n-1} \). We write \( H_{U,n}(x, y) = P(x, y)Q(x, y)S_{U,n}(x, y) \) for \( Q = R_U/P \) and we notice that

\[
Q(x, m)S_{U,n}(x, m) = x^p - x. \tag{3.2}
\]

By Leibniz' formula, any iterated \( y \)-derivative \( \partial_y^\alpha P \) is a linear combination of polynomials of the form

\[
P_W := \prod \limits_{(a, b) \in W} (x - ay + b)
\]

where \( W \subset V \) satisfies \( \#W = (n-1)p - j \). For every such \( W \) we clearly have \( (x^p - x)^{n-1-j} | P_W(x, m) \) and so \( (x^p - x)^{n-1-j} | \partial_y^\alpha P \) for every \( \leq n - 1 \). Again by Leibniz’ formula we have

\[
\partial_y^\alpha H_{U,n} = (\partial_y^\alpha P) \cdot Q \cdot S_{U,n} + \sum \limits_{j=0}^{\alpha-1} \binom{\alpha}{j} (\partial_y^j P) \cdot \partial_y^{\alpha-j} (Q \cdot S_{U,n}).
\]

If we evaluate this identity at \( y = m \) and we remember eq. (3.2), the previous discussion about \( \partial_y^\alpha P \) implies that \( (x^p - x)^{n-\alpha} | (\partial_y^\alpha H_{U,n})(x, m) \) as required. \( \square \)

### 4. Lacunary polynomials

The work of Rédei includes beautiful and remarkable, albeit elementary, results on “lacunary” polynomials over finite fields. Here we state and generalize his main observation. Then we will give sufficient conditions for the RS-polynomial to be lacunary, and we will deduce combinatorial consequences from this result.
4.1. Lacunary polynomials and reducibility

It is well-known that the polynomials $x^p - x$ and $x^p - \alpha$ (for any $\alpha \in \mathbb{F}_p$) are completely reducible: in fact, we have

$$x^p - x = \prod_{k \in \mathbb{F}_p} (x - k) \quad \text{and} \quad x^p - \alpha = (x - \alpha)^p.$$ 

These polynomials are “lacunary” in the sense that most of their coefficients are zero. In other words, the products above give rise to massive cancellation coefficient-wise. Rédei noticed that these are the only cases in which this happens:

**Proposition 4.1.** (Rédei). If $x^p + g(x) \in \mathbb{F}_p[x]$ with $\deg g < \frac{p}{2}$ is completely reducible, then either $g(x)$ is constant or $g(x) = -x$.

This result is best-possible as the example $x^p + g(x) = x(x^{(p-1)/2} - 1)^2$ shows. The proof (see Proposition 4.3 below) is simple and it makes use of the following lemma (the case $S(x) = 1$ suffices).

**Lemma 4.2.** Let $P(x) \in \mathbb{F}_p[x]$ be a polynomial that factors over $\mathbb{F}_p[x]$ as $P(x) = R(x)S(x)$ with $R(x)$ completely reducible. Then

$$P(x) \text{ divides } P'(x) \cdot \overline{P}(x) \cdot S(x),$$

where $P'$ is the derivative of $P$ and $\overline{P}$ is the polynomial of degree $< p$ such that $P(x) \equiv \overline{P}(x) \mod x^p - x$.

**Proof.** Let $Q(x) = \gcd(R, x^p - x)$ be the “square-free part” of $R(x)$. It is clear from the Leibniz expansion that $\frac{R(x)}{Q(x)}$ divides $P'(x)$. On the other hand $Q(x)$ divides both $P(x)$ and $X^p - x$, so $Q(x)$ divides $\overline{P}(x)$. Therefore $P = Q \overline{P} S [P'(x) S]$. $\square$

In Lemma 4.2 we included the “possibly non reducible” factor $S(x)$ because, as Szönyi discovered [2], this allows for more general applications to combinatorics. In the following proposition we recover the Rédei-Szőnyi proposition [1, 2] for degree=$p$ almost-reducible lacunary polynomials, and we extend it naturally to degree=$np$ via an iterative procedure.

**Proposition 4.3.** Let $H(x) = x^{np} + g_1(x)x^{(n-1)p} + \cdots + g_{n-1}xp + g_n(x) \in \mathbb{F}_p[x]$ such that $\deg g_1 \leq A + 1$ and $\deg g_k \leq B + k$ for all $1 \leq k \leq n$ and some $A, B, n \in \mathbb{N}$. Suppose also that $H(x) = R(x)S(x)$ with $R(x)$ completely reducible and that $A + B + n + \deg S < p$. Then $H(x)$ is a product of $n$ factors of the form $x^p - x$ or $x^p - \alpha$, $\alpha \in \mathbb{F}_p$.

**Proof.** We proceed by induction on $n$. We have that $\deg H' \leq (n-1)p + A$ and $\deg \overline{P} \leq n + B$, hence

$$\deg H = np > (n - 1)p + A + B + n + \deg S \geq \deg (H' \cdot \overline{P} \cdot S).$$
However we also have \( H \mid H' \cdot \overline{H} \cdot S \) by Lemma 4.2, so we must either have \( H' = 0 \) or \( \overline{H} = 0 \). If \( H' = 0 \) then \( H = G^p \) for some \( G \in \mathbb{F}_p[x] \) of degree \( n \). This polynomial \( G \) must be completely reducible; otherwise \( S \) would be divisible by the \( p \)-th power of some non-linear polynomial, but this is impossible because \( \deg S < p \). We conclude, in case \( H' = 0 \), that \( H = \prod_{i=1}^n (x - \alpha_i)^p \) for some \( \alpha_1, \ldots, \alpha_n \in \mathbb{F}_p \). If \( \overline{H} = 0 \) instead, then \( H(x) = (x^p - x) \cdot \tilde{H}(x) \) for some \( \tilde{H}(x) \in \mathbb{F}_p[x] \). If \( n = 1 \) then \( \tilde{H}(x) = 1 \) and we are done. Otherwise, we use an induction on \( n \) as follows. By polynomial long division we have

\[
\tilde{H}(x) = x^{(n-1)p} + \tilde{g}_1(x)x^{(n-2)p} + \cdots + \tilde{g}_{n-2}(x)x^p + \tilde{g}_{n-1}(x)
\]

with

\[
\tilde{g}_k(x) = x^k + \sum_{j=1}^{k} g_j(x)x^{k-j}
\]

for all \( 1 \leq k \leq n - 1 \). We notice that \( \deg \tilde{g}_1 \leq A + 1 \) and \( \deg \tilde{g}_k \leq B + k \) for all \( 1 \leq k \leq n - 1 \). Moreover \( \tilde{H}(x) = \tilde{R}(x) \cdot S(x) \) for some \( \tilde{R} \mid R \) and \( S \mid S \). In particular \( A + B + (n - 1) + \deg S < p \) and so by induction we have that \( \tilde{H} \) is a product of \( n - 1 \) factors of the form \( x^p - x \) or \( x^p - \alpha \), \( \alpha \in \mathbb{F}_p \). Since \( H(x) = (x^p - x) \cdot \tilde{H}(x) \), the proposition follows.

4.2. The RS polynomial is lacunary

We now prove that the RS-polynomial is “lacunary” (i.e. many of its \( x \)-coefficients vanish) if \( U \) has few \( U \)-special directions. The induction in the following proof is a little technical, but it is executed according to the following principles

- The first proposition of section 3.2 implies that the specialization of the RS-polynomial at \( y = m \) is “lacunary” when \( m \) is not a \( U \)-rich direction. In other words, \( h_j(m) = 0 \) for most \( j \leq np \).
- We use the second proposition of that section to show that \( \partial^\ell_y h_j(m) = 0 \) for suitable \( \ell \) and \( j \), when \( m \) is a \( U \)-generic direction. In other words, that these \( h_j \) vanish with a certain multiplicity.
- By Remark 3.5 we have an upper bound for the degree: \( \deg_y h_j(y) \leq j \). As a consequence, if the \( U \)-generic directions are numerous enough, then several \( x \)-coefficients \( h_j \) of \( H_{U,n} \) vanish identically.

We recall that a direction \( m \) is non-vertical if \( m \neq \infty \).

**Proposition 4.4.** Let \( U \subseteq \mathbb{F}_p^2 \) with \( (n - 1)p < \#U \leq np \) for some \( n \in \mathbb{N} \), write

\[
H_{U,n}(x, y) = x^{np} + G_1(x, y)x^{(n-1)p} + \cdots + G_n(x, y)
\]

for some \( G_j(x, y) \) with \( \deg_x G_j < p \), and let \( D^* \) (resp. \( E^* \)) be the number of the non-vertical \( U \)-special (resp. \( U \)-rich) directions. Then for all \( 1 \leq j \leq n \) we have

\[
\deg_x G_j \leq E^* + (j - 1)D^*, \tag{4.2}
\]

where \( D^* := \max\{D^*, 1\} \) and \( E^* := \max\{E^*, 1\} \).

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Proof. If \( m \) is not a \( U \)-rich direction, then by Proposition 3.6 we have \( H_{U,n}(x, m) = (x^p - x)^n \), and so \( \deg G_j(x, m) \leq j \) for all \( j \leq n \). In particular for every \( 1 \leq \beta \leq p - 2 \) and every non-\( U \)-rich direction \( m \) we have that \( h_\beta(m) = 0 \), because \( \deg G_j(x, m) \leq 1 \). Therefore these polynomials \( h_\beta \) vanish at \( p - E^\dagger \) distinct elements of \( \mathbb{F}_p \). However, we have \( \deg h_\beta(y) \leq \beta \), and so \( h_\beta(y) \) is the zero polynomial for \( \beta = 1, \ldots, p - E^\dagger - 1 \). In other words

\[
\deg_x G_1(x, y) \leq E^\dagger. \tag{4.3}
\]

We are now going to prove eq. \((4.2)\) by induction on \( j \), but for clarity first we show it for \( j = 2 \). We consider the derivative \( \partial_y H_{U,n} \) of the RS-polynomial:

\[
(\partial_y H_{U,n})(x, y) = (\partial_y G_1)(x, y) \cdot x^{(n-1)p} + \cdots + (\partial_y G_n)(x, y) \cdot x^0.
\]

We know by Proposition 3.7 that \((x^p - x)^{n-1} \mid (\partial_y H_{U,n})(x, m)\) for all non-\( U \)-special direction \( m \neq \infty \), so that we can write \((\partial_y H_{U,n})(x, m) = K_1(x) \cdot (x^p - x)^{n-1} \) or

\[
(\partial_y H_{U,n})(x, m) = K_1(x) \cdot x^{(n-1)p} + \cdots + (-1)^n K_1(x) x^n,
\]

for some \( K_1(x) \) that depends on \( m \). If we compare the first term of the two displayed equations above, we get \( K_1(x) = \partial_y G_1(x, m) \) and so \( \deg K_1(x) \leq E^\dagger \) because of eq. \((4.3)\). By comparing the other terms of the expansions above we deduce that \( \deg \partial_y G_j(x, m) \leq E^\dagger + j - 1 \) for all \( 1 \leq j \leq n \). In particular, this estimate for \( j = 2 \) implies that \( \partial_y h_{p+\beta}(m) = 0 \) for all \( 1 \leq \beta \leq p - E^\dagger - 2 \) and all non-vertical non-\( U \)-special direction \( m \). Summing up, for these values of \( \beta \), the polynomial \( h_{p+\beta} \) vanishes at \( p - E^\dagger \) elements \( m \in \mathbb{F}_p \) (i.e. the non-\( U \)-rich directions, by the initial discussion, because \( \deg G_2(x, m) \leq 2 \leq E^\dagger + 1 \)) and it vanishes with double multiplicity at \( p - D^\dagger \) elements \( m \in \mathbb{F}_p \) (i.e. the non-\( U \)-special directions, by the last discussion, because \( \partial_y G_2(x, m) \leq E^\dagger + 1 \)). Since \( \deg h_{p+\beta} \leq p + \beta \) by Remark 3.5, this forces \( h_{p+\beta} \equiv 0 \) for all \( 1 \leq \beta \leq p - E^\dagger - D^\dagger - 1 \). In other words,

\[
\deg_x G_2(x, y) \leq E^\dagger + D^\dagger.
\]

We now prove by induction on \( \alpha = 2, \ldots, n \) that

(i) \( \deg_x G_\alpha(x, y) \leq E^\dagger + (\alpha - 1)D^\dagger \);

(ii) \( \deg(\partial_y^{\alpha - 1} G_j)(x, m) \leq E^\dagger + (\alpha - 2)D^\dagger + j - \alpha + 1 \) for all \( \alpha \leq j \leq n \) and all non-\( U \)-special direction \( m \neq \infty \).

We already proved these statements for \( \alpha = 2 \), so now we suppose they are true up to a certain \( \alpha = \alpha_0 \) with \( 2 \leq \alpha_0 \leq n - 1 \) and we aim to prove them for \( \alpha = \alpha_0 + 1 \), with the same strategy as before. For all non-\( U \)-special direction we know by Proposition 3.7 that \((x^p - x)^{n - \alpha_0} \mid (\partial_y^{\alpha_0} H_{U,n})(x, m)\), so

\[
(\partial_y^{\alpha_0} H_{U,n})(x, m) = (x^p - x)^{n - \alpha_0} \cdot (K_1(x)x^{(\alpha_0 - 1)p} + \cdots + K_{\alpha_0}(x)) \tag{4.4}
\]

\[
= \sum_{j=1}^{n} \min_{\{j, \alpha_0\}} \left( \sum_{i=0}^{\min\{j, \alpha_0\}} (-1)^{j-i} \binom{n - \alpha_0}{j - i} K_1(x)x^{j-i} \right) \cdot x^{(n-j)p} \tag{4.5}
\]

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for some $K_j \in \mathbb{F}_p[x]$ with $\deg K_j \leq p - 1$. On the other hand we have

$$(\partial_y^{\alpha} H_{U,n})(x, m) = (\partial_y^{\alpha} G_1)(x, m) \cdot x^{(n-1)p} + \cdots + (\partial_y^{\alpha} G_n)(x, m) \cdot x^0$$

(4.6)

and since $\deg_x G_j(x, y) \leq E^\dagger + (j - 1)D^\dagger$ for $j \leq \alpha_0$ by statement (i), we also have

$$\deg(\partial_y^{\alpha} G_j)(x, m) \leq E^\dagger + (j - 1)D^\dagger$$

for $j \leq \alpha_0$. If we compare the first $\alpha_0$ terms of eqs. (4.5) and (4.6) we discover that $\deg K_j \leq E^\dagger + (j - 1)D^\dagger$ as well. By comparing the remaining terms we obtain $\deg(\partial_y^{\alpha} G_j)(x, m) \leq E^\dagger + (\alpha_0 - 1)D^\dagger + (j - \alpha_0)$ for all $j \geq \alpha_0 + 1$, that is statement (ii) for $\alpha = \alpha_0 + 1$. Now we consider the polynomial $h_{\alpha_0 \beta}(y) \in \mathbb{F}_p[y]$ for

$$1 \leq \beta \leq p - E^\dagger - \alpha_0 D^\dagger - 1.$$ 

We have that $\deg h_{\alpha_0 \beta} \leq \alpha_0 p + \beta$, but also that $h_{\alpha_0 \beta}$ has at least $p - E^\dagger$ zeros in $\mathbb{F}_p$, of which at least $p - D^\dagger$ have multiplicity at least $\alpha_0 + 1$. We have

$$\alpha_0 p + \beta < p - E^\dagger + \alpha_0(p - D^\dagger),$$

so these $h_{\alpha_0 \beta}$ are identically zero as polynomials. This is equivalent to statement (i) for $\alpha = \alpha_0 + 1$. □

### 4.3. Outcome of the polynomial method

We now combine Proposition 4.3 and Proposition 4.4 to get some nontrivial information on the number of intersections between the set $U$ and the lines along a U-rich direction.

As in section 2.1 let $U \subseteq \mathbb{F}_p^2$ be a set with cardinality $\#U = np - r$ for some $1 \leq n < p$ and $0 \leq r < p - n$. We let $D$ be the number of $U$-special directions, let $E$ be the number of $U$-rich directions and assume that

$$D \leq 1 + \frac{p - r}{n + 1}$$

(4.7)

as in eq. (2.1). We now further assume that $E \geq 2$ and that the vertical direction $\infty$ is a $U$-rich direction. Notice that $D \geq E \geq 2$ and that we have $E^\dagger = E* = E - 1$ and $D^\dagger = D* = D - 1$ in Proposition 4.4. Let now $m \neq \infty$ be another $U$-rich direction and seek to apply Proposition 4.3 with $H = H_{U,n}(x, m)$. We have $H(x) = R(x)S(x)$ where $R(x) = R_U(x, m)$ is the Rédei polynomial and $S_{U,n}(x, m)$ is the Szőnyi complement. In particular, $R(x)$ is completely reducible by its definition. We write $H_{U,n}$ as in eq. (4.1) and we notice, using Proposition 4.4 and the inequality $E \leq D$, that

$$\deg G_1(x, m) \leq A + 1 \quad \text{and} \quad \deg G_j(x, m) \leq B + j$$

(4.8)

for $A = D - 2$, $B = n(D - 2)$ and all $j = 1, \ldots, n$. Moreover we have that $\deg S = r$ and so

$$A + B + n + \deg S = (n + 1)(D - 2) + r < p$$

(4.9)
by eq. (4.7). To sum up, we have that the hypotheses of Proposition 4.3 are fulfilled. Therefore the polynomial \( H(x) \) can be factored as

\[
H(x) = (x^p - x)^{n - w_m} \prod_{i=1}^{w_m} (x - \alpha_i)^p
\]  

(4.10)

for some \( 0 \leq w_m \leq n \) and some \( \alpha_i \in \mathbb{F}_p \). It is easy to see that

**Lemma 4.5.** The elements \( \alpha_i \) are pairwise distinct and the lines \( \ell_i = \{(u, v) : v = mu - \alpha_i\} \) for \( i = 1, \ldots, w_m \) are the \( U \)-rich lines with slope \( m \).

**Proof.** By Remark 3.2 we have that for all \( t \in \mathbb{F}_p \), the multiplicity of \( x - t \) in the factorization of \( R(x) \) is the number of points of \( U \) on the line \( \ell_{m,t} = \{(u, v) : v = mu - t\} \). Since moreover \( R(x) | H(x) \), we have that every line \( \ell_{m,t} \) other than the \( \ell_i \)’s meets \( U \) in at most \( n \) points, so it is not \( U \)-rich. Conversely, we notice that \( S(x) \) has degree \( r \), therefore the multiplicity of \( x - \alpha_i \) in the factorization of \( R(x) \) is at least

\[ p - r \geq n + 1 \]

because by hypothesis \( r < p - n \). This means that each line \( \ell_i \) for \( i = 1, \ldots, w_m \) is \( U \)-rich. In addition to this, every line has no more than \( p \) points, so we have that the multiplicity of every linear factor of \( R(x) \) is at most \( p \). Since \( \deg S(x) = r < p \), we deduce that all linear factor of \( H(x) \) appear with multiplicity at most \( p + r < 2p \). This implies that we cannot have \( \alpha_i = \alpha_j \) for \( i \neq j \), otherwise by eq. (4.10) the multiplicity of \( x - \alpha_i \) in the factorization of \( H(x) \) would be at least \( 2p \).

In particular, \( w_m \) is the number of \( U \)-rich lines with slope \( m \), as in section 2.2. Since \( m \) is a \( U \)-rich direction, we have that \( w_m \geq 1 \). We already remarked that \( w_m \leq n \), moreover we can have \( w_m = n \) only if the whole of the set \( U \) is contained in the union of the \( w_m = n \) parallel lines \( \ell_{1, \ldots, w_m} \) described in Lemma 4.5. Finally, we observe that

\[
r \geq w_m(n - w_m).
\]  

(4.11)

Indeed, every factor \( x - \alpha_i \) appears with multiplicity \( p + n - w_m \) in the factorization of \( H(x) \). Since every linear factor of \( R(x) \) has multiplicity at most \( p \), we have that \( (x - \alpha_i)^{n - w_m} | S(x) \) for all \( i = 1, \ldots, w_m \). Since \( \deg S = r \), we get eq. (4.11). We are now able to prove the result we used as a blackbox in section 2.

**Proof of Lemma 2.4.** Let \( U \subseteq \mathbb{F}_p^2 \) and let \( m \) be any \( U \)-rich direction. If \( E \geq 2 \) then there is some other \( U \)-rich direction \( m' \) and by a linear change of coordinates we may assume that \( m' = \infty \), and so also \( m \neq \infty \). Then the discussion above implies the lemma. If \( E = 1 \) but \( D \geq 2 \), we choose some \( U \)-special direction \( m' \neq m \) and by a linear change of coordinates we assume that \( m' = \infty \). Then we repeat the above discussion, the main difference being that now \( E^\dagger = E^* = E \) in Proposition 4.4. Moreover we have \( E \leq D - 1 \), hence
we still have eq. (4.8) with $A = D - 2$ and $B = n(D - 2)$ and so also the inequality eq. (4.9). The rest of the discussion is the same as in the case before. Finally, if $E = D = 1$ we simply assume by a linear change of coordinates that $m \neq \infty$. Again, we repeat the reasoning of this subsection. Now we have, by Proposition 4.4, that eq. (4.8) holds with $A = 0$ and so eq. (4.9) is replaced by the following computation

$$A + B + n \deg S = n + r < p,$$

which holds by the assumption $r < p - n$. Therefore, the hypotheses of Proposition 4.3 are still fulfilled and the lemma follows also in this case. \qed

**Remark 4.6.** The trick of reducing to the case where $\infty$ is $U$-special is the contribution by Megyesi mentioned in the introduction.

### 5. Completing the proof of the main theorem

In this section we will prove Theorem 1.3 when $p - r \leq n$. Since in this case

$$\left\lfloor \frac{p - r + n + 2}{n + 1} \right\rfloor = 2,$$

we need to show that $U$ is either contained in the union of $n$ lines or that there are at least two $U$-special directions. We can assume that $n < p$ because otherwise $U$ is trivially contained in the union of $n = p$ lines. In particular the assumptions $n < p$ and $p - r \leq n$ imply that $r \neq 0$. Now, we suppose that there is at most one $U$-special direction $m_0 \in \mathbb{F}_p \cup \{\infty\}$. If there is no $U$-special direction, we choose $m_0$ arbitrarily. We make the following observation.

**Lemma 5.1.** Every line with slope $m_0$ is either contained in $U$ or it meets $U$ in at most $p - r$ points.

**Proof.** Let $\ell_{m_0} \subseteq \mathbb{P}_p^2$ be a line with slope $m_0$, not completely contained in $U$, and let $v \in \ell \setminus U$. By the choice of $m_0$, we notice that every line passing through $v$, other than $\ell_{m_0}$, is not $U$-special. Therefore, if $\ell_m$ denotes the line through $v$ with slope $m$, we have

$$N = \sum_{m \in \mathbb{F}_p \cup \{\infty\}} \#(\ell_m \cap U) \geq \#(\ell_{m_0} \cap U) + p(n - 1).$$

This shows that $\#(\ell_{m_0} \cap U) \leq p - r$. \qed

We recall the definition of $c_m$ from Definition 2.2 and the fact that $2c_m = (n - 1)(np - 2r)$ for all $m \in \mathbb{F}_p \cup \{\infty\} \setminus \{m_0\}$ by eq. (2.6). Therefore we have

$$c_{m_0} = \left( \frac{N}{2} \right) - p \cdot \frac{n - 1}{2} \cdot (np - 2r) \quad (5.1)$$

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by eq. (2.2). On the other hand by Lemma 5.1 we have

\[ c_{m_0} \leq W \left( \frac{p}{2} \right) + \frac{p - r - 1}{2} (N - W p), \tag{5.2} \]

where \( W \) is the number of the lines (necessarily with slope \( m_0 \)) that are contained in \( U \). Moreover, according to Lemma 2.6, we have the equality in eq. (5.2) if and only if we have \( \#(\ell \cap U) \in \{0, p - r\} \) for each line \( \ell \) with slope \( m_0 \) not contained in \( U \). If we compare eq. (5.1) and eq. (5.2), after due simplification, we get

\[ (n - 1)pr \leq W pr. \tag{5.3} \]

This forces \( W = n - 1 \), because \( Wp \leq \#U < np \) and \( r \neq 0 \). Consequently, we have equality in eq. (5.3), and so also in eq. (5.2). This is only possible if \( U \) consists of the union of \( n - 1 \) parallel lines along \( m_0 \), plus \( p - r \) additional points on another line with same slope. In particular we have that \( U \) is contained in the union of \( n \) lines, as we wanted to prove.

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