All Symmetries
of Non-Einsteinian Gravity in $d = 2$

Thomas Strobl\textsuperscript{*}
Institut für Theoretische Physik
Technische Universität Wien
Wiedner Hauptstr. 8-10, A-1040 Wien
Austria

Abstract

The covariant form of the field equations for two-dimensional $R^2$--gravity with torsion as well as its Hamiltonian formulation are shown to suggest the choice of the light-cone gauge. Further a one-to-one correspondence between the Hamiltonian gauge symmetries and the diffeomorphisms and local Lorentz transformations is established, thus proving that there are no hidden local symmetries responsible for the complete integrability of the model. Finally the constraint algebra is shown to have no quantum anomalies so that Dirac’s quantization should be applicable.

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\textsuperscript{*}e-mail: tstrobl@email.tuwien.ac.at
1 Introduction

One of the unsolved problems of quantum field theory is to find a quantized version of general relativity. A promising step in this direction is the investigation of some simpler models in lower dimensions, which was one of the reasons\(^1,2\) for considering the Lagrangian

\[
\mathcal{L} = -e \left( \frac{\gamma}{4} R^{ab\mu\nu} R_{ab\mu\nu} + \frac{\beta}{4} T^{a\mu\nu} T_{a\mu\nu} + \lambda \right)
\]

in \(d = 2\). \(\Box\) is an example of a theory with quadratic terms in curvature and torsion. In two dimensions it is the unique diffeomorphism invariant action, up to surface terms, yielding second order differential equations for the zweibein \(e_\mu^a = g_\mu^a\) and the spin connection \(\omega^a_{ba}\). Supplementing it e.g. by the Einstein–Hilbert term \(eR\) does not change the classical field equations since it is a trivial total divergence here. In Katanaev et. al.\(^1\) the integrability of \(\Box\) was proven using the conformal gauge, thereafter Kummer et. al.\(^2\) showed that the light–cone (LC) gauge enables one to find the explicit form of the general solution. Nevertheless, although the LC gauge is ideally apt to solve the field eqs. (cf. also sec. 2 below), the study of properties concerning geodesics, such as global completeness\(^3\), is preferably done within the conformal gauge.

It is one of the purposes of this paper to show that the choice of the LC gauge, as intuitively introduced in Kummer et. al.\(^2\), is almost compelling when starting from the covariant form of the field equations (sec. 2) and even more so when using the Dirac–Hamiltonian formulation (sec. 3). As a by–product of this we will find that there is no direct analogue of decoupling field eqs. for the Euclidian version of the theory, except possibly when using complex coordinates on an intermediary level. In contrast to previous work\(^2,4\) we shall try to stay as covariant as possible in all of our calculations, keeping e.g. the covariance in the Lorentz indices within the Hamiltonian formulation. In sec. 3 we also propose a solution to the problem of surface terms on the Hamiltonian level\(^11\), at least for the case of this model.

The integrability of the system inevitably raises the question of symmetries. In a recent publication\(^4\) it was related to the fact that the Poisson bracket relations between the first class (FC) constraints and the momenta may be written as a deformed \(iso(2,1)\)–algebra, which was said to correspond to some 'novel symmetry' visible only on the Hamiltonian level.

It is, however, the point of view of the present paper that the Poisson bracket relations between the constraints (and momenta) alone are of little relevance for the integrability and the symmetries of the system. Due to a general theorem\(^5\) it is always possible to reformulate the first class constraints such that their Poisson brackets vanish (strongly)\(^6\). Then the information about the symmetries

\(^6\)Such an abelianization of the constraints of the present model has been actually accomplished in Ref. 7.
is completely hidden in the specific form of the constraints as functions on phase space. To our mind the real content of the symmetries corresponding to FC constraints becomes visible only when being translated to the Lagrangian level.

In sec. 4 we will show that there is a one-to-one correspondence between the symmetry transformations generated by the FC constraints on the Hamiltonian level and the diffeomorphisms and local Lorentz transformations leaving the action invariant. With the results of Henneaux et. al. it follows, moreover, that there are no further local symmetry transformations hidden in the model.

The Hamiltonian formulation of sec. 3 also serves as a preparation for a non-perturbative quantization of the model. In sec. 5 we will show that the constraint algebra has no quantum anomalies. Therefore Dirac’s procedure of imposing the constraints on a Hilbert space should be applicable. Some of the related problems will be addressed in sec. 6; they will be tackled in a further publication.

2 Covariant Field Equations and Light–Cone Gauge

Let us first fix some of the notation: An index from the middle of the Greek alphabet (µ, ν,...) shall correspond to a holonomic frame ∂µ, one of the beginning (α, β,...) to an arbitrary frame ea, and a Latin index (a, b,...) to an 'orthonormal' frame ea, i.e. a frame fulfilling g(ea,eb) = gab in which we will restrict gab only to be constant and 'normalized' to det gab = (−1)M (M = 1 in the Minkowskian case). The values of the world–indices µ are taken from \{0, 1\}, whereas those of Lorentz–indices a from \{^\hat{0}, ^\hat{1}\}; for the special case that gab is of the form (11) below we write also \{±\} instead of \{^\hat{0}, ^\hat{1}\}. The ε–tensor (not pseudo–tensor) is defined by εαβ = (−1)Oα √ det g(αβ) ε(αβ), ε(αβ) being the ε–symbol (ε(01) = 1 etc.) and (−1)Oα the orientation of the frame ea. For convenience restricting ourselves to a positively oriented orthonormal frame ea in the following, we have εab = ε(ab), εab = (−1)M ε(ab), εμν = e ε(μν), and εμν = (−1)M (1/e) ε(μν), the sign of e giving the orientation of ∂µ. Furthermore, we will set ωabµ = εab εabµ, which now is invariant only against global Lorentz–transformations, but the full covariance can be regained immediately by the substitution ωµν = (−)M (1/2) εab εabµ.

The first step in order to find the covariant form of the field equations is to express (1) in terms of the curvature scalar $R = R^{\alpha\beta}_{\alpha\beta}$ and the Hodge dual of the torsion two–form $T^a = \frac{1}{2} \varepsilon^{a\mu\nu} T_{\mu\nu}$. This is possible in d = 2 since one has e.g. $R^{\alpha\beta\gamma\delta} = (−1)^M (\hat{R}/2) \varepsilon^{\alpha\beta\gamma\delta}$ following from the symmetries of the curvature tensor and the frequently used contractions of $\varepsilon^{\alpha\beta} \varepsilon_{\gamma\delta} = (−)M (\delta^\alpha_{\gamma} \delta^\beta_{\delta} − \delta^\alpha_{\delta} \delta^\beta_{\gamma})$. Thus one obtains

$$\mathcal{L} = − e \left[ \frac{\gamma}{4} R^2 + (−1)^M \frac{\beta}{2} T^2 + \lambda \right]$$

with $T^2 \equiv T^c T_c$. Using Cartan’s structural equations, the second one of which is
$R = -2 \varepsilon^{\mu \nu} \omega_{\mu \nu}$ in $d = 2$, and relations such as $\delta e = e e_a^{\mu} \delta e_{\mu}^{a}$, $e_a^{\mu} = g_a^{\mu}$ being the inverse of $e_{\mu}^{a}$, the variational principle yields for (3) (covariant derivatives are denoted by indices after a semicolon):

$$\beta T_{a;b} = \varepsilon_{ab} E(R, T^2)$$

(3)

$$\gamma R_{ia} = (-)^M \beta \varepsilon_{ab} T^b$$

(4)

with

$$E \equiv \frac{\gamma}{4} R^2 + \frac{(-)^M \beta}{2} T^2 - \lambda.$$  

(5)

As a natural extension of the usual variational principle in classical point mechanics we have required the variation of the fields $e_{\mu}^{a}$ and $\omega_{\mu}$ to vanish at the boundary $\partial B$ of the parameter area $B$ so that we were allowed to drop the surface term

$$\int_{\partial B} e \varepsilon^{\mu \nu} [\gamma R \delta_{\nu} + (-)^{M+1} \beta T_a \delta e^a_{\nu}] \, dx^\mu.$$  

(6)

Since (3) and (4) are tensor equations all Latin indices can be replaced by Greek ones. The field equations have to be supplemented by the additional geometric requirement $e \equiv \det e_{\mu}^{a} \neq 0$.

The equations (3), (4) immediately show that for $\operatorname{sgn}(\lambda) = \operatorname{sgn}(\gamma)$ there exists a solution with $T^a \equiv 0$ and $R \equiv \pm \sqrt{4\lambda/\gamma}$, which has been called the deSitter solution.

In order to find the complete solution, we write the equations of motion (e.o.m.) explicitly in the $e_a$-basis, still for an arbitrary (constant and normalized) reference metric $g_{ab}$:

$$T_{k;1} = T_{\dot{0};\dot{0}} = 0$$

(7)

$$-T_{0;1} = T_{1;\dot{0}} = -\frac{1}{\beta} E(R, T^2)$$

(8)

$$R_{\dot{0}0} = \frac{\beta}{\gamma} (-g_{01} T_0 + g_{00} T_1)$$

(9)

$$R_{1\dot{0}} = \frac{\beta}{\gamma} (-g_{1\dot{0}} T_0 + g_{\dot{0}0} T_1)$$

(10)

To integrate at least part of the above equations directly, it is suggestive to replace a covariant derivative, say $\frac{\partial}{\partial 0}$, by a normal coordinate derivative ($\partial_0$). That such a gauge is attainable at all — in $d = 2$ and the covariant derivative $\frac{\partial}{\partial 0}$ following Lorentz indices — shall be shown in the Appendix A. There we will see also that we cannot choose this gauge in a topology of a torus (staying within one chart by the requirement of periodic continuation). Furthermore, for the topology of a cylinder the above coordinate $x^0$ has to be the one $\in R$. Now, of the eqs. (7) to (10) there are three which are to be integrated for $x^0$. The first one gives $T_{\dot{0}} = -\frac{1}{\beta} A(x^1)$ with an arbitrary function $A$. To integrate (7) next (and thereafter the second equation of (3)), one obviously has to choose
\( g_{00} = 0 \). Staying with a real tangent space, this can only be achieved for the case \( M = 1 \). Thus, at least without introducing complex coordinates, the field eqs. corresponding to the Euclidean version of (1) cannot be decoupled as easily as in the Minkowskian case. Therefore in the following we will consider only \( M = 1 \).

What we have demanded so far is already equivalent to the LC gauge as introduced in Kummer et. al.\(^2\). To show this, we first note that the coordinate change\(^2\,^4\) \( x^\pm = \left( 1 / \sqrt{2} \right) (x^0 \pm x^1) \), performed before having introduced any gauge, comes down to a mere relabelling (since it is a diffeomorphism). Further one notices that the above requirement \( e_0^\mu = \delta(0\mu) \) implies for the inverse matrix, i.e. the zweibein, \( e_0^a = \delta(0a) \) and vice versa. This last equation still corresponds to a reference metric with an arbitrary \( g_{11} \), but there exists a (global) frame transformation from this \( g_{ab} \) to a light–cone metric

\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]  

so that \( e_0^a = \delta(0a) \) remains unchanged. This would be e.g. not the case, if it originally corresponded to \( g_{ab} = \text{diag}(1, -1) \). Finally, \( \omega_{ab0} = 0 \) is invariant under any global frame transformation (cf. (22) below). So, the natural requirement of replacing a covariant derivative by a normal one and the requirement of an obvious decoupling of the covariant field eqs. (3), (4) or (7) to (9) automatically leads to the intuitively introduced light–cone gauge of Kummer et. al.\(^2\).

At this point let us observe that the above gauge already fixes the space–time character of the coordinate \( x^0 \) to be a light–cone coordinate: \( g_{00} = e_0^a e_0^b g_{ab} = g_{00} = 0 \). So when referring the zweibein to (11), it is not so severe that the LC gauge is not attainable for \( x^0 \in S^1 \), since this corresponds to an ‘unphysical world’ anyway. \( g_{11} = 2e_1^0 e_1^1 \left( e_1^a \right)^2 g_{11} \), on the other hand, is to be determined by the field eqs. and in general will differ from zero.

For reasons of completeness and because our formulation differs from previous work, we will briefly sketch how to solve the remaining field equations. Since there is no sense in keeping \( g_{11} \) arbitrary for this purpose, the zweibein will refer to the LC metric (11) for the rest of this section. Knowing the \( x^0 \)–dependence of \( T^- = T_+ \) as well as of \( R \) and \( T^+ \), one first determines the dependence of \( e_-^\mu = (1/e_1^-)(-e_1^+, 1) \) and \( \omega_1 \) on this coordinate, in order to reformulate the remaining eqs. of (7) to (10) as ordinary differential eqs. in \( x^1 \). To this end one uses the defining eqs. for torsion and curvature which in this gauge are \( T^a = -\left( 1/e_1^- \right) \left[ \partial_0 e_1^a + \delta(\alpha+) \omega_1 \right] \) and \( R = -2(\partial_0 \omega_1) / e_1^- \). Thus one obtains for \( A(x^1) \neq 0 \):

\[
\begin{align*}
R &= Ax^0 + B + \frac{\beta}{\gamma} \\
T^+ &= C \exp\left( -\frac{2}{\beta} A x^0 \right) - \frac{1}{4A}(Ax^0 + B)^2 + \frac{\lambda - (\beta^2/4\gamma)}{\gamma A}
\end{align*}
\]
\( T^- = -\frac{\gamma}{\beta} A \)

\( \omega_1 = F - \frac{\beta}{2\gamma} D \exp\left(\frac{\gamma}{\beta} A x^0\right) (A x^0 + B) \)

\( e_1^+ = G - (F + CD) x^0 + \frac{\beta D}{4\gamma A^2} [(A x^0 + B)^2 + \frac{\beta^2}{\gamma^2} - \frac{4\lambda}{\gamma}] \exp\left(\frac{\gamma}{\beta} A x^0\right) \)

\( e_1^- = D \exp\left(\frac{\gamma}{\beta} A x^0\right) \) \hspace{1cm} (12)

with \( A, B, C, D, F, G \) being functions of \( x^1 \) (\( D(x^1) \neq 0 \) because of \( e_1^- = e \)). Likewise for \( A(x^1) = 0 \) one gets:

\( R = B + \frac{\beta}{\gamma} \)

\( T^+ = \tilde{C} + I x^0 \)

\( T^- = 0 \)

\( \omega_1 = \tilde{F} - \frac{\beta}{2\gamma} D (1 + \frac{\gamma}{\beta} B) x^0 \)

\( e_1^+ = \tilde{G} - (\tilde{F} + \tilde{C} D) x^0 + \frac{D}{4}(B + \frac{\beta}{\gamma} - I) (x^0)^2 \)

\( e_1^- = D \) \hspace{1cm} (13)

with

\( I \equiv \frac{\lambda}{\beta} - \frac{\gamma}{4\beta} (B + \frac{\beta}{\gamma})^2 \)

and with the arbitrary functions \( B, \tilde{C}, D \neq 0, \tilde{F}, \tilde{G} \). For the case \( A = 0 \) everywhere the remaining eqs. of \( (\tilde{F}) \) to \( (14) \) can be shown to lead to \( B = \pm \sqrt{(4\lambda/\gamma) - (\beta/\gamma)} \) (\( \Rightarrow I = 0 \)) and \( \tilde{C} = 0 \) everywhere, which is the deSitter part of the solution. For \( T^- \neq 0 \) everywhere it suffices to regard only the first eq. of \( (\tilde{F}) \), the other two eqs. being redundant. It yields in this case (the prime denoting a derivative):

\( A' + A(CD + F) = 0 \) \hspace{1cm} (14)

\( B' - \frac{\beta}{\gamma} CD - AG = 0 \) \hspace{1cm} (15)

\( C' - C(F - \frac{\gamma}{\beta} AG) = 0 \) \hspace{1cm} (16)

Plugging \( (14, 15) \) into \( (16) \) one can easily integrate the latter equation to give

\[ C = \frac{C_1}{A} \exp\left[-\frac{\gamma}{\beta} B - 1\right] \] \hspace{1cm} (17)

for an arbitrary constant \( C_1 \). Eqs. \( (14) \) and \( (15) \) finally can be solved by expressing \( F \) and \( G \) in terms of the remaining three functions.
Calculating the Lorentz invariant \( T^2 = 2 T^+ T^- \) and making the result explicit in \( C_1 \), one regains the quantity \( Q(R, T^2) \propto C_1 \)

\[
Q = \exp\left(\frac{2\gamma}{\beta} R\right) \left[\frac{2\gamma}{\beta} T^2 - (\frac{\gamma}{\beta} R - 1)^2 - 1 + \frac{4\gamma \lambda}{\beta^2}\right] \equiv 2 \exp\left(\frac{\gamma}{\beta} R\right) \left[-\frac{2\gamma \lambda}{\beta^2} E + \frac{\gamma}{\beta} R - 1\right]
\]

(18)
of Kummer et. al.\(^2\) (up to a proportionality constant). Being a covariant function, we have gauge–independently

\[ Q; a = 0, \]

(19)

which also follows immediately from taking the covariant derivative of the first equation of (18) and the subsequent usage of (3), (4).

There exist also solutions with isolated zeros of \( A(x^1) \) as outlined in Ref. 7.

### 3 Hamiltonian Formulation

Again \( g_{ab} \) will be restricted only by \( g_{ab} = const \) and \( \det g_{ab} = -1 \) in the following. Further we will use the notation \( \partial_0 f =: \dot{f}, \ x^1 =: x, \ \partial_1 f =: \partial f \) and choose, without loss of generality, \( x^0 =: t \) as evolution parameter; clearly the classical system is reproduced by the Hamiltonian formalism irrespective of the space–time character of \( t \equiv x^0 \), which is not fixed at this stage. We observe first that (1) or (2) is of the simple form

\[
\mathcal{L} = \frac{1}{2} (\dot{\varphi}^i - f^i) W_{ij} (\dot{\varphi}^j - f^j) - V
\]

(20)

with

\[
\varphi^A \equiv (\varphi^i, \bar{\varphi}^i) \equiv (e^a_1, \omega_1, e^a_0, \omega_0) \quad i = (0, 1, \omega),
\]

\[
f^i \equiv (\dot{e}^a_1 - T^a_{01}, \dot{\omega}_0) = f^i(\varphi^A, \partial \bar{\varphi}^j)
\]

\[
V \equiv \lambda e, \quad W_{ij} \equiv \frac{1}{\beta} \begin{pmatrix} \beta g_{ab} & 0 \\ 0 & -2\gamma \end{pmatrix}.
\]

(21)

Therefore \( \mathcal{L} \) is completely regular in the three fields \( \varphi^i \) — i.e. the Legendre transformation is bijective in these fields — and completely singular in \( \bar{\varphi}^i \). So the Legendre transformation in the regular sector \( \pi_i \equiv (\pi_a, \pi_\omega) = W_{ij}(\dot{\varphi}^j - f^j) = (-\beta T_a, \gamma R) \) gives just the inverse to the Hamiltonian equations for \( \dot{\varphi}^i \) (cf. eq. 33 below) and the primary constraints are simply \( \bar{\pi}_i \approx 0 \). Since we have \( (W^{-1})^{ij} = e^{\text{'diag'}(g^{ab}/\beta, -1/2\gamma)} \), the canonical Hamiltonian

\[
\mathcal{H}_C = \frac{1}{2} \pi_i (W^{-1})^{ij} \pi_j + f^i \pi_i + V
\]

(22)
is polynomial in the basic fields, in contrast to the Lagrangian and also in contrast to the Ashtekar formulation of the latter — cf. Isham\textsuperscript{12}. This seems to indicate an advantage of the Hamiltonian path integral formulation as opposed to the Lagrangian one. However, when formally integrating out the momenta\textsuperscript{8} of a BRST–version of the canonical Hamiltonian (with $\bar{\phi}^i$ interpreted as Lagrange multipliers – cf. comments following (31) below), one just regains\textsuperscript{9} the Lagrangian as used in Kummer et. al.;\textsuperscript{9} the four–vertex ghost couplings\textsuperscript{10}, typical for theories with a non–closing constraint algebra, disappear in this model.

To get e.o.m. and constraints which are equivalent to the Lagrangian field eqs., one has to use the primary Hamiltonian $H_P$, which is the sum of $H_C$ and the product of all primary constraints with Lagrange multiplier (LM) functions. Usually this is proven for discrete systems (cf. Sundermeyer\textsuperscript{11} and references therein), and a formal translation to the continuous case would imply $H_P = H_C + \lambda_i \bar{\pi}_i$. But in our opinion this is not enough to reproduce exactly what we have done on the Lagrangian sector (sec. 2). There we have required the variation of the basic fields to vanish at the boundary of the parameter space $B$ in order to get the field equations (3), (4). This can be viewed as a minimalization of the corresponding action with ‘temporarily fixed’ boundary conditions on $\partial B$, finally asking for the set of all solutions compatible with some boundary condition. Now, any of the usual proofs working in a discrete version of a continuous Lagrangian system correspond to a rectangular $B \subset \mathbb{R}^2$. But for such a $B$ the ‘temporary’ fixation of the fields at $t_{min}$ and $t_{max}$ have to be supplemented by ‘temporary’ boundary conditions for the field variables at $x_{min}$ and $x_{max}$. According to (6) it suffices to prescribe such boundary conditions only for the $\phi^i$ (and not necessarily for the $\bar{\phi}^i$) in our case; they are explicitly $t$–dependent and have to be added as additional primary constraints to the Hamiltonian formulation of the system.

As a next step one has to calculate the secondary constraints. Considering $\phi^i \big|_{x–{boundary}} \approx g^i(t)$ first, the requirement that its Poisson brackets with $H_P = \int dx \mathcal{H}_P$ shall vanish reproduces just $\pi_i \big|_{x–{boundary}} \approx (\beta T_\alpha, \gamma R) \big|_{x–{boundary}}$ (cf. (22)) when identifying $\phi^i \big|_{x–{boundary}}$ with $\dot{g}^i(t)$. All further tertiary, etc. constraints can be reformulated to be only restrictions to the possible choice for $\dot{g}^i(t)$. So we are left with a set of $t$–dependent second class constraints at the boundary. The corresponding degrees of freedom can be eliminated by means of Dirac brackets so that we will be allowed to set all surface terms strongly equal to zero below.

A simple calculation, using $e = \varepsilon(ab)e_0^a e_1^b$, shows that the conservation of $\bar{\pi}_i \approx 0$ gives ($G_i := \{\bar{\pi}_i, H_P\}$):

\begin{align*}
G_a &= \varepsilon_{ab} e_1^b E + \partial \pi_a - \varepsilon_b^a \omega_1 \pi_b \approx 0 \quad (23)
G_\omega &= \partial \pi_\omega + \varepsilon_a^b e_1^b \pi_a \approx 0 \quad (24)
\end{align*}

\textsuperscript{c}Up to a factor in the measure of the Lagrangian path integral which basically originates from the fact that the Hessian (3.2) is not constant.
with
\[ E \equiv \frac{1}{4\gamma} (\pi_\omega)^2 - \frac{1}{2\beta} \pi^2 - \lambda, \quad \pi^2 \equiv \pi^a \pi_a. \] (25)

On shell (using the e.o.m. for \( \dot{\varphi}^i \)) (25) becomes just (8). It should be mentioned, furthermore, that on deriving (23, 24) we have replaced \( \int dy \partial_y [\delta(y - x)] \pi_i(y) \) by \( -\partial \pi_i(x) \), which is possible only since \( \pi_i \) vanishes strongly at \( x_{\min}, x_{\max} \).

At this point it is convenient to calculate the brackets of the basic fields with the \( G_i \); the only nonvanishing ones are (we suppress the arguments \( x \) and \( y \) and write \( \delta' \) for \( \frac{\partial}{\partial y} [\delta(x - y)] \) etc.):
\[
\{ e^a_1, G_b \} = \left( -\frac{1}{\beta} \varepsilon_{bc} e^c e^a - \varepsilon^a \omega_1 \right) \delta - \varepsilon^{a\beta} e^b \delta', \\
\{ \omega_1, G_a \} = \frac{1}{2\gamma} \varepsilon_{ab} e^b \pi_\omega \delta, \\
\{ \pi_a, G_b \} = \varepsilon_{ab} E \delta. 
\] (26)

By means of this one derives
\[
\{ G_a, G_b \} = -\varepsilon_{ba} G_b \delta, \\
\{ G_a, G_b \} = \varepsilon_{ab} \left( -\frac{1}{\beta} \pi^c G_c + \frac{1}{2\gamma} \pi_\omega G_\omega \right) \delta. 
\] (29, 30)

(28) to (30) is the Lorentz-covariant version of the algebra analyzed in Grosse et. al.\(^4\).

Dropping \( \partial(\bar{\varphi}^i \pi_i) \) since it does not contribute to \( H_P \) by the above argument, we can express our Hamiltonian by means of (22) to (25) as
\[
H_P = -\bar{\varphi}^i G_i(\varphi, \pi, \partial \pi) + \bar{\lambda}^i \bar{\pi}_i. 
\] (31)

If we had not employed the argument with the Dirac brackets, (31) would be still valid since, as alluded to above, one would have to include the (not completely well-defined) term \( -\delta(x - x_{\text{boundary}}) \pi_i(x) \) in the formulas (28, 29) for the secondary constraints \( G_i \).

That the Hamiltonian vanishes weakly is a feature in common to all parameterization invariant theories\(^11\). In this specific case, though, the structure of (31) reveals that from the Hamiltonian point of view one could also consider \( \mathcal{H} = \mu_i G_i \) instead of \( H_P \), which is obtained from (31) in the gauge \( \varphi^i \approx -\mu^i \), dropping the barred fields by the introduction of Dirac brackets. This is also completely analogous to the usual Hamiltonian formulation of general relativity in four dimensions (cf. e.g. Isham\(^12\)), where one regards the lapse function and the shift vector as mere Lagrange multipliers. Because of (31) the relations (29, 30) show that there are no further secondary constraints. Moreover, all of the 6 \( \infty \) constraints \( (G_i, \bar{\pi}_i) =: G_A \) are first class (FC).

\(^d\) For a detailed comparison one has to set \( \gamma = 1/2 \), to replace \( \beta \) by \( 2\beta \), and to replace \( i = (0, 1, \omega) \) or \((+, -, \omega)\) by \( i = (3, 2, -1) \) [i.e. \( \varphi^i \) by \( (q_3, q_2, -q_1) \), etc.].
Before turning to the symmetries generated by the FC constraints, let us comment on the flow generated by (31), i.e. the ‘t–evolution’ of the coordinates. Supplementing them by the constraints, we have (cf. (23,24)):

\[
\dot{\bar{\phi}}_i \approx \bar{\lambda}_i, \quad \bar{\pi}_i \approx 0 \quad (32)
\]

\[
\dot{\phi}^i \approx \{\phi^i, H_P\} \iff \pi_i \approx (-\beta T_a, \gamma R) \quad (33)
\]

\[
\partial_\mu \pi_\omega \approx \varepsilon_{ab} e^a_\mu \pi^b \quad (34)
\]

\[
\partial_\mu \pi_a \approx -\varepsilon_{ab} (e^b_\mu E + \omega_\mu \pi^b) \quad (35)
\]

Multiplying the latter two eqs. by \(e^b_\mu\), one obviously regains just the covariant e.o.m. (3), (4).

Beside this reformulation of the covariant field equations on the Lagrangian level the Hamiltonian formulation (using \(H_P\)) provides some additional information and insight. On the one hand, the equations (32) practically force one to use a LC–like gauge. Furthermore, the constraint eqs. (23,24) serve as constants of the motion, thus saving one the integration of three of the e.o.m. (cf. Ref. 4 or Ref. 7 for more details). On the other hand, the knowledge that (34), (35) with \(\mu = 1\) are first class constraints reveals\(^{11}\) that on shell the choice of the LC gauge does not fix the gauge freedom completely. From the Hamiltonian point of view the LC gauge turns only \(\bar{\pi}_i \approx 0\) into second class constraints. This shows that it should be possible to gauge away still \(3 \infty\) phase space coordinates, i.e the three arbitrary functions of \(x \equiv x^1\) in the general solution obtained at the end of sec. 2 - except possibly for a finite number of constants. Since we will show in the following section that the gauge symmetries generated by the FC constraints are just diffeomorphisms and Lorentz transformations, this elimination could already have been carried through in sec. 2. The corresponding steps on the Lagrangian level can be found in Grosse et. al.\(^4\). According to their result

\[-4(\gamma/\beta)^2 \bar{C}_1 = Q(R, T^2)\]

is the only gauge independent (physical) parameter left in the model. For the topology of a cylinder, however, the reduced phase space is two dimensional (cf. Ref. 7).

One could also try to build up a Hamiltonian formulation with the extended Hamiltonian \(H_E\) as proposed by Dirac\(^{13}\), which emerges from \(H_c\) by adding all FC constraints via Lagrange multipliers (LM). When absorbing \(-\bar{\phi}^i\) into the definition of new LMs \(\lambda^i\), which will prove to be a mathematical shortcut especially in the following section since it formally corresponds to a strongly vanishing canonical Hamiltonian, one obtains (\(\lambda^A := (\lambda^i, \bar{\lambda}^i)\)):

\[
H_E = \lambda^A G_A. \quad (36)
\]

In this way the e.o.m. for the unbarred fields become proportional to LMs, too. Setting all the \(\lambda^A\) zero as the simplest choice, one is left with \(\varphi^A \approx \varphi^A(x), \pi_A \approx (\pi_i(x), 0)\), in which the nine functions of \(x\) are restricted by the three contraints \(G_i\). The FC character of the six constraints \(G_A\) will lead to a gauge identification of the six functions needed as ‘initial data’ on a hyperline.
\[ t = \text{const} \] so that the (superficial) degrees–of–freedom count gives the same as in the case of \( \mathcal{H}_P \). That the fixation of all LMs in \( \mathcal{H} \) does not fix the gauge completely can be understood by considering a hyperline for some smaller \( t \). Starting from such a line with different choices for the LMs leads to different but physically equivalent points on the hyperline used above. Nevertheless, without having gone into further calculational detail, we believe that it would be difficult to extract geometrically interpretable results comparable to the results of sec. 2 when using \( \mathcal{H}_E = 0 \) as above. (Note also that the gauge choice \( \lambda^A = 0 \) in (3.17) corresponds to a gauge with \( e = 0 \) in the original formulation). However, for the gauge independent quantity \( Q(R, T^2) \) the e.o.m. are unchanged: \( \dot{Q} \approx 0 \) is trivial and \( \partial Q \approx 0 \) follows from \( G_i \approx 0 \).

4 All local symmetries of the model

Thanks to the work of Henneaux et. al. the relationship between the gauge symmetries on the Hamiltonian level and the ones on the Lagrangian level has become much more precise than it had been before (cf. e.g. Sundermeyer). The assumptions are quite general, but excluding e.g. ineffective constraints so as to evade counterexamples of ’Dirac’s conjecture’ as the one given in Gotay. The main idea of Henneaux et. al. is to restrict the symmetries of the action \( S_E = \int d^2x (\dot{\phi}^A \pi_A - \mathcal{H}_E) \) successively to the ones of \( S_P = \int d^2x (\dot{\phi}^A \pi_A - \mathcal{H}_P) \) and \( S_L = \int d^2x L \). The present section is a nice illustration for the general considerations made in that work as well as for the usefulness of \( \mathcal{H}_E \), i.e. of ’Dirac’s conjecture’, when analyzing the symmetries of a model.

With

\[ \{ G_B(x), G_C(y) \} = \delta(x - y) C_{BC}^A G_A \]  

(37)

it is easy to verify that the gauge transformations

\[ \delta f(x,t) = \int dy \{ f(x,t), G_A(y,t) \} \epsilon^A(y,t), \]  

(38)

in which \( f = f(\phi^A(x,t), \pi_A(x,t)) \) is an arbitrary function on the phase space and \( \epsilon^A(x,t) \) an arbitrary infinitesimal parameter, are a symmetry transformation of \( S_E \) (up to surface terms), iff

\[ \delta \lambda^A = \epsilon^A + C_{BC}^A \lambda^B \epsilon^B. \]  

(39)

If we had not absorbed \( \mathcal{H}_c \) by the definition of the LMs, we had to add \( -V_B^A \epsilon^B \) to (39) \( (\{ \mathcal{H}_c(t), G_A(x,t) \} =: V_B^A G_B). \) Now, according to Henneaux et. al. these are all local transformations leaving \( S_E \) invariant.

In our case the above equations become (cf. (26) to (28)):

\[ \delta \bar{\phi}^i = \bar{\epsilon}^i, \quad \delta \bar{\pi}_i = 0, \quad \delta \bar{\lambda}^i = \bar{\epsilon}^i \]  

(40)
\[ \delta e^a_1 = -\frac{1}{\beta} \varepsilon_{bc} e^b_1 \pi^a \varepsilon^b - \varepsilon^a_1 b_\omega \varepsilon^b - \partial \varepsilon^a + \varepsilon^a_1 c_\omega \varepsilon^c \]  

(41)

\[ \delta \omega_1 = \frac{1}{2\gamma} \varepsilon_{bc} e^b_1 \pi_\omega \varepsilon^b - \partial \varepsilon^b \]  

(42)

\[ \delta \pi_a = \varepsilon_{ab} E \varepsilon^b + \varepsilon_{ac} \pi^c \varepsilon^\omega \]  

(43)

\[ \delta \pi_\omega = -\varepsilon_{bc} \pi^c \varepsilon^b \]  

(44)

\[ \delta \lambda^a = \varepsilon^{a} + \frac{1}{\beta} \varepsilon_{bc} e^0_1 \pi^a \varepsilon^b + \varepsilon^a c_\omega \varepsilon^c \varepsilon^\omega \]  

(45)

\[ \delta \lambda^\omega = \varepsilon^\omega + \frac{1}{2\gamma} \varepsilon_{bc} e^0_1 \pi_\omega \varepsilon^b \]  

(46)

Setting \( \lambda^i = -\bar{\phi}^i \), the action \( S_E \) becomes equal to \( S_P \). Thus, all local symmetries leaving \( S_P \) invariant are obtained when restricting (40) to (46) by \( \delta \lambda^i = -\delta \bar{\phi}^i \equiv -\bar{\epsilon}^i \). This gives among others:

\[ \delta e^a_\mu = -\partial^\mu \varepsilon^a - \frac{1}{\beta} \varepsilon_{bc} e^\mu_1 \pi^a \varepsilon^b - \varepsilon^a b_\omega \varepsilon^b + \varepsilon^a c_\omega \varepsilon^c \varepsilon^\omega \]  

(47)

\[ \delta \omega^\mu = \varepsilon^\mu + \frac{1}{2\gamma} \varepsilon_{bc} e^0_1 \pi_\omega \varepsilon^b - \partial^\mu \varepsilon^\omega, \]  

(48)

whereas the transformations for the momenta remain unchanged. The transition from \( S_P \) to \( S_L \) is accomplished by the second equations of (32) and (33). Using the latter to eliminate the momenta from (47), (48) as well as the identities (T\(^a\), R/2) \( \varepsilon_{\nu\mu} = (T^a_{\mu\nu}, 2\omega_{[\nu\mu]} \) and the abbreviation \( e^\mu_\nu \varepsilon^\nu = \varepsilon^\nu \), one verifies easily:

\[ \delta e^a_\mu = -\partial^\mu \varepsilon^a + 2e^b_{[\nu\mu]} \varepsilon^\nu + \varepsilon^a c_\omega \varepsilon^c (\epsilon^\omega - \omega^\nu \varepsilon^\nu) \]  

(49)

\[ \delta \omega^\mu = \omega^\mu + \varepsilon^\nu \varepsilon^\nu - \omega^\nu \varepsilon^\nu_{,\mu} - \partial^\mu (\epsilon^\omega - \omega^\nu \varepsilon^\nu) \]  

(50)

These are the transformations fulfilling \( \delta S_L = 0 \).

Before turning to some consistency checks regarding the momenta, let us verify that (47, 48) are nothing but the infinitesimal form of diffeomorphisms and Lorentz transformations. The well-known parametrization of the Lorentz group in 1 + 1 dimensions with \( \cosh \alpha, \sinh \alpha = \alpha(x^\mu) \in R \) gives in its infinitesimal form

\[ \delta L^\alpha e^a_\mu = \varepsilon^a b_\alpha \varepsilon^b \]  

(51)

The inhomogeneity in \( \omega^a_{\bar{b}\mu} = e^a c_{e^d \omega_{\bar{a}b\mu}} + e_{ac} e^b c_\mu \), in which \( e^a \) denotes the Lorentz boost inverse to the one used in (51), leads to the transformation property

\[ \bar{\omega}^\mu = \omega^\mu - \alpha_{,\mu}, \]  

(52)

\[ \text{e} \]  

11
or $\delta^L_\alpha \omega_\mu = -\alpha_\mu$. The usual transformation under diffeomorphisms $\tilde{x}^\mu := x^\mu + \xi^\mu(x^\nu)$ for a field $\psi_\nu$ reads infinitesimally: $\delta^D_\xi \psi_\mu = -\psi_\nu \xi^{\nu \mu} - \psi_{\mu \nu} \xi^\nu$. Since obviously $\delta^{\nu \mu}_D e_\mu^a = -(e_\nu^a \xi^\nu)_\mu - 2\epsilon_{\mu \nu} \omega_a \xi^\nu$ the comparison of (49, 50) to the above equations yields:

$$\delta \varphi^A = (\delta^D_\epsilon + \delta^{L - \omega} e) \varphi^A. \quad (53)$$

Thus it is found that the flow generated by $G_a$ on the regular sector is a specific combination of a diffeomorphism and a Lorentz transformation, whereas $G_\omega$ corresponds directly to the generator of the Lorentz group in this sector.

There are no momenta living on the purely Lagrangian sector. Nevertheless, we do not see a general reason within the above procedure$^7$ why the transformation properties of the momenta on the $S_P$–level could not lead to a restriction of the parameters in (53) (similar to the transition from $S_E$ to $S_P$, in which the number of free parameters has been halved). To show explicitly that this is not the case, we note that $\delta \bar{\pi}_a = 0$ is consistent with the second equation of (32). Furthermore, by means of (33) the eqs. (43) and (44) can be written as:

$$\delta(-\beta T_a) = [\varepsilon^{ab} E(R, T^2) - \beta T_{a;b}] e^b - \beta [\varepsilon_{ac} T^c (\epsilon^{\nu} - \omega_\nu \epsilon^\nu) - T_{a;\nu} e^\nu]$$

$$\equiv [\delta^\epsilon + \delta^D_\nu + \delta^{L - \omega)}_\epsilon e^\nu] (-\beta T_a) \quad (54)$$

$$\delta(\gamma R) = (\beta \varepsilon_{bc} T^c + \gamma R_{b;}) e^b - \gamma R_{\nu} e^\nu$$

$$\equiv [\delta^\epsilon + \delta^D_\nu + \delta^{L - \omega)}_\epsilon e^\nu] (\gamma R) \quad (55)$$

with $\delta = 0$ on shell (cf. (3)!) This completes the proof for the one–to–one correspondence of the gauge symmetries on the Lagrangian and the Hamiltonian level, and, as consequence, the absence of any further local symmetry.

5 Constraint Algebra without Anomalies

The first step towards a quantized version of the model is to turn the $\varphi^A$ and $\pi_A$ into operators on some — at this stage unspecified — Hilbert space and to replace the fundamental Poisson brackets $\{\varphi^A(x), \pi_B(y)\} = \delta^A_B \delta(x-y)$ by the commutator relations $[\varphi^A(x), \pi_B(y)] = \delta^A_B \delta(x-y)$ by the commutator relations $[\varphi^A(x), \pi_B(y)] = \delta^A_B \delta(x-y)$. Next, although not mandatory (cf. e.g. Isham$^{12}$), it is ‘natural’ to require the constraints $G_A$ to become hermitean operators. This is accomplished replacing $e_1^B E$ in (23) by the anticommutator $\{1/2 \vert e_1^B, E \vert = \delta^A_B \delta(x-y)$. Next, although not mandatory (cf. e.g. Isham$^{12}$), it is ‘natural’ to require the constraints $G_A$ to become hermitean operators. This is accomplished replacing $e_1^B E$ in (23) by the anticommutator $\{1/2 \vert e_1^B, E \vert = \delta^A_B \delta(x-y)$. G\_ is already hermitean since we have

$$\varepsilon^{ab} [e_1^b(x), \pi_a(x)] = \varepsilon^{ab} i \delta(0) = 0 \quad (56)$$

in any regularization for $\delta(0)$. To employ Dirac’s approach

$$G_A \psi_{\text{phys}} = 0, \quad (57)$$

$^7$ Certainly we know already that $S_L$ is invariant under all of (33).
it is common to demand that one cannot produce further constraints by applying 
$G_B$ to the lefthand side of (55). This is guaranteed, if the constraint algebra 
(29,30) has no quantum anomalies, i.e. if the quantum version of (29,30) is 
obtained from these equations by the mere replacement 
\[
\{ , \} \to -i/\hbar [ , ]
\] 
without reordering of the (hermitean) operators $G_i$ on the r.h.s. so that some 
$G_i$ would be placed on the left of $\pi_i$. To show that there are indeed no such 
anomalies for the case of our model, one first observes that the constraints are 
only linear in the fields $\varphi^i$ (cf. (23,24)). It is easy to verify, by dropping a 
similar term as the one in (56), that the commutator between two operator 
valued functions 
\[
f = \frac{1}{2} [\varphi^i(x), f_i(\pi_j(x))]_+ \quad \text{and} \quad g = \frac{1}{2} [\varphi^i(x), g_i(\pi_j(x))]_+
\] 
gives 
\[
[f(x), g(y)] = i\hbar \frac{1}{2} [\varphi^k, (\frac{\partial g_k}{\partial \pi_i} f_i - \frac{\partial f_k}{\partial \pi_i} g_i)]_+ \delta(x - y), \tag{59}
\]
which, except for the factor $i\hbar$, is just the antisymmetrization of the 
classical result. Applying this to (29,30), we are left to show that 
\[(1/2) [e^b_1(x), \pi^a(x) \varepsilon_{ab} E(x)]_+ = (1/2) \pi^a(x) \varepsilon_{ab} [e^b_1(x), E(x)]_+ \]
but this is true because of (56).

6 Outlook

To carry through the Dirac quantization there are still two main problems to be 
solved at this stage. The first one is the appearance of a term proportional to 
$\delta(0)$ when one solves (57) with hermitean $G_A$ by representing $e^b_1$ as a functional 
derivative operator. The other one is the well-known problem of ‘frozen time’ 
(cf. Isham\textsuperscript{12}): Due to (57) $H_P$ vanishes on all physical states so that a normal 
Schroedinger equation does not make sense. These problems, among others, 
are treated in Ref. 7. — What makes the quantization of the present model 
interesting is that there appear similar conceptual problems as in $d = 4$, but 
that the mathematical difficulties are much simpler to be overcome. Therefore it 
is possible to explicitly check some of the basic approaches to quantum gravity.

Another promising step seems to be the coupling of matter fields to the 
action (1), which, according to our results, is not restricted by any ‘hidden’ 
symmetry.

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A Attainability of the LC–Gauge

In this appendix we will prove the local attainability of the gauge

\[ e_0^a = \delta(0a) \]  
\[ \omega_0 = 0 \]  

which is the LC gauge as introduced in Kummer et. al.\textsuperscript{2} when (60) is referred to the LC-metric (11). We will show further that (60), (61) cannot be obtained for the topology of a torus.

As can be shown easily, the attainability of (60), (61) is independent of the choice of the reference metric \( g_{ab} \). For convenience we will restrict ourselves to (11) since the corresponding local 'Lorentz' transformations can be parametrized in the simple form \( e_{\tilde{a}}^b = \text{diag}(\exp(\alpha), \exp(-\alpha)) \) with \( \alpha = \alpha(x) \in \mathbb{R} \), where we have written simply \( x \) for \( x^\mu \) (in contrast to sec. 3). According to (52), which is certainly also valid when using (11), the complete gauge freedom can be expressed as

\[ \tilde{\omega}_{\tilde{\mu}}(\tilde{x}) = \frac{\partial x^\nu}{\partial x^{\tilde{\mu}}}(\tilde{x}) (\omega_\nu - \alpha_\nu)(x(\tilde{x})) \]  
\[ e_{\tilde{\mu}}^a(\tilde{x}) = \frac{\partial x^a}{\partial x^{\tilde{\mu}}}(\tilde{x}) e_{\tilde{b}}^a e_\nu^b(x(\tilde{x})), \]  

with \( e_{\tilde{a}}^b \) being the inverse to \( e_{\tilde{a}}^b \).

Now, (60) is equivalent to \( e_0^a = \delta(0a) \). This in turn implies that one has chosen the flow of the vector field \( e_{\tilde{0}} \) as a coordinate \( x^0 \), and this is always possible. Plugging (63) into both sides of (62), one gets the residual gauge freedom available to obtain \( \tilde{\omega}_{\tilde{0}} = 0 \) in a second step. In this way it is straightforward to show (using \( e \neq 0 \)) that (61) is attainable under the assumption that (60) is already fulfilled, iff

\[ x^1 = x^1(\tilde{x}^1) \]  
\[ \frac{\partial x^0}{\partial x^{\tilde{0}}}(\tilde{x}(x)) = \exp(\alpha(x)) \]  
\[ \alpha_{00}(x) = \omega_0(x). \]

Because of (64) the lefthand side of (65) is just \( 1/(\partial x^0/\partial x^0)(x) \) so that (65, 66) are solved by

\[ \alpha(x) = \int_{f(x^1)}^{x^0} dy \omega_0(y, x^1) \]  
\[ \tilde{x}^0 = \int_{g(x^1)}^{x^0} dy \exp(-\alpha(y, x^1)) \]
with some arbitrary functions $f$ and $g$.

In a manifold with topology $S^1 \times S^1$ it is possible to use just one chart when requiring that all fields are periodic in $x^0$ and $x^1$ (e.g. with the period normalized to $2\pi$). One concludes from (66) by integrating this equation over a period in $x^0$ that even if (60) is obtainable globally (61) is not.

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