The $k$-anonymity Problem is Hard

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Abstract

The problem of publishing personal data without giving up privacy is becoming increasingly important. An interesting formalization recently proposed is the $k$-anonymity. This approach requires that the rows in a table are clustered in sets of size at least $k$ and that all the rows in a cluster become the same tuple, after the suppression of some records. The natural optimization problem, where the goal is to minimize the number of suppressed entries, is known to be NP-hard when the values are over a ternary alphabet, $k=3$ and the rows length is unbounded. In this paper we give a lower bound on the approximation factor that any polynomial-time algorithm can achieve on two restrictions of the problem, namely (i) when the records values are over a binary alphabet and $k=3$, and (ii) when the records have length at most 8 and $k=4$, showing that these restrictions of the problem are APX-hard.

1 Introduction

In many research fields, for example in epidemic analysis, the analysis of large amounts of personal data is essential. However, a relevant issue in the management of such data is the protection of individual privacy. One approach to deal with such problem is the $k$-anonymity model [9, 10, 8], where a single table is given. The rows of the table represent records belonging to different individuals. Then some of the entries in the table are suppressed so that, for each record $r$ in the resulting table, there exist at least $k−1$ other records identical to $r$. At the end of this process, identical rows can be clustered together; clearly the resulting data is not sufficient to identify each individual. Different versions of the problem have also been introduced [1], for example allowing the generalization of entry values (an entry value can be replaced with a less specific value) [3]. However, in this paper we will focus only on the suppression model.

A simple parsimonious principle leads to the optimization problem where the number of entries in the table to be suppressed (or generalized) has to be minimized. The $k$-anonymity problem is known to be NP-hard for rows of unbounded length with values over ternary alphabet and $k=3$ [2]. Moreover, a polynomial-time $O(k)$-approximation algorithm on arbitrary input alphabet, as well as some other approximation algorithms for some restricted cases, are known

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Recently, approximation algorithms with factor $O(\log k)$ have been proposed \[7\], even for generalized versions of the problem \[6\].

In this paper, we further investigate the approximation and computational complexity of the $k$-anonymity problem, settling the APX-hardness for two interesting restrictions of the problem: (i) when the matrix entries are over a binary alphabet and $k = 3$, or (ii) when the matrix has 8 columns and $k = 4$. We notice that these are the first inapproximability results for the $k$-anonymity problem. More precisely, we prove the two inapproximability results by designing two $L$-reductions \[5\] from the Minimum Vertex Cover problem to 3-anonymity problem over binary alphabet and 4-anonymity problem when the rows are of length 8 respectively. Those two restrictions are of particular interests as some data can be inherently binary (e.g. gender) and publicly disclosed usually have only a few columns, therefore solving such restrictions could help for most practical cases.

The rest of the paper is organized as follows. In Section 2 we introduce some preliminary definition and we give the formal definition of the $k$-anonymity problem. In Section 3 we show that the 3-anonymity is APX-hard, even when the matrix is restricted to binary data, while in Section 4 we show that the 4-anonymity problem is APX-hard, even when the rows have length bounded by 8.

## 2 Preliminary Definitions

In this section we introduce some preliminary definitions that will be used in the rest of the paper. A graph $G = (V, E)$ is cubic when each vertex in $V$ has degree three.

Given an alphabet $\Sigma$, a row $r$ is a vector of elements taken from the set $\Sigma$, and the $j$-th element of $r$ is denoted by $r[j]$. Let $r_1, r_2$ be two equal-length rows. Then $H(r_1, r_2)$ is the Hamming distance of $r_1$ and $r_2$, i.e. $|\{i : r_1[i] \neq r_2[i]\}|$. Let $R$ be a set of $l$ rows, then a clustering of $R$ is a partition $P = (P_1, \ldots, P_t)$ of $R$. Since all rows in a table have the same number of elements and the order of the elements of a rows is important, we may think of a row over the set $\Sigma$ as a string over alphabet $\Sigma$.

Given a clustering $P = (P_1, \ldots, P_t)$ of $R$, we define the cost of the row $r$ belonging to a set $P_i$, as $|\{j : \exists r_1, r_2 \in P_i, r_1[j] \neq r_2[j]\}|$, that is the number of entries of $r$ that have to be supressed so that all rows in $P_i$ are identical. Similarly we define the cost of a set $P_i$, denoted by $c(P_i)$, as $|P_i||\{j : \exists r_1, r_2 \in P_i, r_1[j] \neq r_2[j]\}|$. The cost of $P$, denoted by $c(P)$, is defined as $\sum_{P_i \in P} c(P_i)$. Notice that, given a clustering $P = (P_1, \ldots, P_t)$ of $R$, the quantity $|P_i| \max_{r_1, r_2 \in P_i} |H(r_1, r_2)|$ is a lower bound for $c(P_i)$, since all the positions for which $r_1$ and $r_2$ differ will be deleted in each row of $P_i$. We are now able to formally define the $k$-Anonymity Problem ($k$-AP) as follows:

**Problem 1.** $k$-AP.

**Input:** a set $R$ of rows over an alphabet $\Sigma$.

**Output:** a clustering $P = (P_1, \ldots, P_t)$ of $R$ such that for each set $P_i$, $|P_i| \geq k$

**Goal:** to minimize $c(P)$.

The following Property will be used in several proofs.

**Proposition 1.** \[2\] Let $R$ be an instance of $k$-AP, and let $P$ be a solution of $k$-AP over instance $R$. Then we can compute in polynomial time a solution $P'$, with $c(P') \leq c(P)$, such that for each cluster $P'_i$ of $P'$, $k \leq |P'_i| \leq 2k - 1$. 


We will study two restrictions of the \( k \)-anonymity problem. In the first restriction, denoted by \( 3\text{-ABP} \), the rows are over a binary alphabet \( \Sigma = \{ 0_b, 1_b \} \) and \( k = 3 \). In the second restriction, denoted by \( 4\text{-AP}(8) \), \( k = 4 \) and the rows are over an arbitrary alphabet and have length 8.

In the remaining of the paper we will prove the APX-hardness of both restrictions, presenting two different reductions from the Minimum Vertex Cover on Cubic Graphs (MVCC) problem, which is known to be APX-hard \([4]\). Consider a cubic graph \( G = (V, E) \), where \( |V| = n \) and \( |E| = m \), the MVCC problem asks for a subset \( C \subseteq V \) of minimum cardinality, such that for each edge \((v_i, v_j) \in E\), at least one of \( v_i \) or \( v_j \) belongs to \( C \).

### 3 APX-hardness of 3-ABP

In this section we will show that 3-ABP is APX-hard via an L-reduction from Minimum Vertex Cover on Cubic Graphs (MVCC), which is known to be APX-hard \([4]\). From Proposition 1 it follows Remark 2, that shows that we can restrict ourselves to solutions of 3-ABP where each cluster contains at most 5 rows.

**Remark 2.** Let \( R \) be an instance of 3-ABP, and let \( P \) be a solution of 3-ABP over instance \( R \). Then we can compute in polynomial time a solution \( P' \), with \( c(P') \leq c(P) \), such that for each cluster \( P'_i \) of \( P' \), \( 3 \leq |P'_i| \leq 5 \).

Let \( \mathcal{G} = (V, E) \) be an instance of MVCC, the reduction builds an instance \( R \) of 3-ABP associating with each vertex \( v_i \in V \) a set of rows \( R_i \), and with each \( e = (v_i, v_j) \in E \) a row \( r_{i,j} \). Actually, starting from the cubic graph \( \mathcal{G} \), the reduction builds an intermediate multigraph,

![Figure 1: Gadgets for \( v_i, v_j, (v_i, v_j) \)](image)

denoted as *gadget graph* \( \mathcal{VG} \) – a snippet of a gadget graph obtainable through our reduction is represented in Fig. 1. The reduction associates with each vertex \( v_i \) of \( \mathcal{G} \) a vertex gadget \( \mathcal{VG}_i \) and some other vertices and edges called respectively jolly vertices and jolly edges. More precisely, the vertex-set of a core vertex gadget \( \mathcal{CVG}_i \) consists of the seven vertices \( c_{i,1}, c_{i,2}, c_{i,3}, c_{i,4}, c_{i,5}, c_{i,6}, c_{i,7} \). The vertices \( c_{i,1}, c_{i,2} \) and \( c_{i,3} \) of \( \mathcal{CVG}_i \) are called *docking vertices*. The edge-set of \( \mathcal{CVG}_i \) consists of nine edges between vertices of \( \mathcal{CVG}_i \).
(see Fig. 1). Such a set of edges is defined as the set of core edges of $VG_i$. The vertex-set of a vertex gadget consists of the seven vertices of $CVG_i$ and of three more vertices $J_{i,1}, J_{i,2}, J_{i,3}$, called jolly vertices of $VG_i$. The edge-set of $VG_i$ consists of the edge-set of $CVG_i$ and of three sets of four parallel edges (see Fig. 1). More precisely, for each docking vertex $c_{i,z}$ adjacent to a jolly vertex $J_{i,z}$, we define a set $E'_{i,z}$ of four parallel edges between $c_{i,z}$ and $J_{i,z}$. The set of edges $E'_i = \bigcup_{z \in \{1,2,3\}} E'_{i,z}$ is called the set of jolly edges of $VG_i$.

Each edge $(v_i, v_j)$ of $G$ is encoded by an edge gadget $EG_{ij}$ consisting of a single edge that connects a docking vertex of $VG_i$ with one of $VG_j$, so that in the resulting graph each docking vertex is an endpoint of exactly one edge gadget (this can be achieved trivially as the original graph is cubic.) The resulting graph, denoted by $V\mathcal{G}$, is called gadget graph. An edge gadget is said to be incident on a vertex gadget $VG_i$ if it is incident on a docking vertex of $VG_i$. In our reduction we will associate a row with each edge of the graph gadget. Therefore 3-ABP is equivalent to partitioning the edge set of the gadget graph into sets of at least three edges. Hence in what follows we may use edges of $V\mathcal{G}$ to denote the corresponding rows. Before giving some details, we present an overview of the reduction.

First, the input set $R$ of rows is defined, so that each row corresponds to an edge of the gadget graph. Then, it is shown that, starting from a general solution, we can restrict ourselves to a canonical solution, where there exist only two possible partitions of the rows of a vertex gadget (and possibly some edge gadgets). Such partitions are denoted as type $a$ and type $b$ solution. Finally, the rows of a vertex gadget that belongs to a type $b$ (type $a$ resp.) solution are related to vertices in the cover (not in the cover, respectively) of the graph $G$.

We are now able to introduce our reduction. All the rows in $R$ are the juxtaposition of $n+2$ blocks, where the $i$-th block, for $1 \leq i \leq n$, is associated with vertex $v_i \in V$, the $(n+1)$-th block is called jolly block, and the $(n+2)$-th block is called edge block. The first $n$ blocks are called vertex blocks, and each vertex block has size 21. The jolly block has size 6n, and the edge block has size 3n.

The rows associated with edges of the gadget graph $V\mathcal{G}$ are obtained by introducing the following operations on rows (also called encoding operations). For simplicity’s sake we will use a string-based notation.

**Definition 1** (Encoding operations). Let $VG_i$ be a vertex gadget, $CVG_i$ be a core vertex gadget, $c_{i,j}$ be a vertex of $CVG_i$, $1 \leq i \leq n$, $1 \leq j \leq 7$, and let $r$ be a row. Then the vertex encoding of $c_{i,j}$ applied to $r$, denoted by $v\text{-}enc_{i,j}(r)$, is obtained by assigning 1b to the positions $3j-2$, $3j-1$, $3j$ of the $i$-th block of $r$ (and leaving all other entries as in $r$). The gadget encoding of $VG_i$ applied to $r$, denoted by $g\text{-}enc_i(r)$, is obtained by assigning $1_b$ to the positions $3i-2$, $3i-1$, $3i$ of the edge block of $r$ (and leaving all other entries as in $r$). Finally, let $J_{i,x}$ be a jolly vertex of $VG_i$, $1 \leq i \leq n$, $1 \leq x \leq 3$, and let $r$ be a row, then the jolly encoding $j\text{-}enc_{i,x}$ of $J_{i,x}$ applied to $r$, denoted by $j\text{-}enc_{i,x}(r)$, is obtained by assigning $1_b$ to the to the positions $6(i-1)+x$, $6(i-1)+x+1$ of the jolly block of row $r$.

Notice that the vertex encoding and the gadget encoding operations set to $1_b$ at most 3 entries of any row, while the jolly encoding operation sets to $1_b$ at most 2 entries of any row. Let $c_{i,x}$ be a docking vertex of $VG_i$, and let $c_{i,y}, c_{i,z}$ be the two core vertices of $VG_i$ adjacent to $c_{i,x}$, let $(c_{i,x}, c_{i,y})$ be a core edge, and let $J_{i,x}$ be a jolly vertex adjacent to $c_{i,x}$. Then the row $r_{i,x,y}$ associated with $(c_{i,x}, c_{i,y})$ is $g\text{-}enc_i(v\text{-}enc_{i,y}(v\text{-}enc_{i,x}(0_{30n})))$, each row associated with a jolly edge $(c_{i,x}, J_{i,x})$, denoted by $r_{i,x,y,z}$ (for $1 \leq z \leq 4$) is
Table 1: Summary of the encoding operations

| Operation | Positions of the i-th vertex blocks set to 1_b | Positions of the edge blocks set to 1_b | Positions of the jolly block set to 1_b |
|-----------|-----------------------------------------------|----------------------------------------|----------------------------------------|
| v-enc_{i,j}(r) | 3j − 2, 3j − 1, 3j | 3i − 2, 3i − 1, 3i | 6(i−1)+x, 6(i−1)+x+1 |
| g-enc_i(r) | | | |
| j-enc_{i,x}(r) | | | |

Table 2: Encodings of the edges

For example consider the row r_{i,1,4} associated with the core edge (c_{i,1},c_{i,4}). Observe that v-enc_{i,1} sets to 1_b the first three positions of the i-th block of r_{i,1,4}, while v-enc_{i,4} sets to 1_b the positions 10, 11, 12 of the i-th block of r_{i,1,4}. Finally, g-enc_i sets to 1_b the positions 3i − 2, 3i − 1 and 3i of the edge block of r_{i,1,4}. Edge (c_{i,1},c_{i,4}) is associated with the following row r_{i,1,4}:

\[
\begin{align*}
0_b0_b\ldots0_b & \quad 1_b1_b1_b & 0_b0_b0_b & 0_b0_b0_b & 1_b1_b1_b & 0_b0_b0_b & \ldots & 0_b0_b\ldots0_b0_b0_b & \ldots & 0_b0_b\ldots0_b0_b0_b & \ldots & 0_b0_b0_b\ldots0_b1_b1_b & \ldots & 0_b0_b0_b
\end{align*}
\]

Observe that by construction only jolly rows may have a 1_b in a position of the jolly block. It is immediate to notice that clustering together three or more jolly rows associated with parallel edges has cost 0. We recall that we may use edges of V G to denote the corresponding rows.

**Proposition 3.** Let e_1, e_2 be two edges of CV G, let e_3 be an edge of VG_j (with i ≠ j), let e_j be a jolly edge of VG_i, let e_5 be a jolly edge of VG_z, and let EG_{ix}, EG_{jl} be two edge gadgets. Then:

1. \( H(e_1, e_3) \geq 18; \)
2. \( H(EG_{ix}, e_j), H(EG_{jl}, e_j) \geq 14; \)
3. If \( e_1 \) and \( e_2 \) are incident on the same vertex, then \( H(e_1, e_2) = 6; \)
4. If \( e_1, e_2 \) are not incident on the same vertex, then \( H(e_1, e_2) = 12; \)
5. If \( e_1 \) and \( e_j \) are incident on the same vertex, then \( H(e_1, e_j) = 5; \)
6. If $e_1$ and $e_j$ are not incident on the same vertex, then $H(e_1, e_j) \geq 11$;

7. If $e_1$ and $EG_{ix}$ are incident on the same vertex, then $H(EG_{ix}, e_1) = 9$;

8. If $e_1$ and $EG_{ix}$ are not incident on the same vertex, then $H(EG_{ix}, e_1) \geq 15$;

9. $H(EG_{ix}, EG_{jl}) \geq 18$;

10. If $i, x, j, l$ are all distinct (i.e. the two edge gadgets are not incident on the same vertex gadget), then $H(EG_{ix}, EG_{jl}) = 24$.

11. If $e_j, e_5$ are not in the same jolly set, then $H(e_j, e_5) \geq 12$.

12. Let $r$ be a row not incident on a common vertex with $e_j$. Then $H(e_j, r) \geq 11$.

Proof. Observe that all the cases can be easily proved by observing that each row is obtained applying 3 kinds of encoding operations, where the vertex encoding and gadget encodings assign values $1_b$ in three positions, while the jolly encoding assigns $1_b$ into two positions. We now prove the various cases, following the order of the statement.

1. Since $e_1$ and $e_3$ are incident on different vertices, the associated rows are the result of applying once the encoding operations with different values of $i$. Therefore the positions where a $1_b$ is set are disjoint. Since there are at least 12 such positions of the vertex blocks and 6 positions of the edge block, we obtain $H(e_1, e_3) \geq 18$.

2. Notice that there are 2 positions of the jolly block that are set to $1_b$ as a result of applying the jolly encoding to $e_j$, while the whole block is set to $0_b$ in $EG_{ix}$. Since $EG_{ix}$ is subject to two gadget encodings, while $e_j$ is subject only to one gadget encoding, $EG_{ix}$ and $e_j$ have 3 different entries also in an edge block. Moreover at most two of the overall five vertex encoding operations have the same arguments (those corresponding to a shared docking vertex), resulting in an additional 9 different entries.

3. Since $e_1$ and $e_2$ are incident on a common vertex, they share a gadget encoding operation and a vertex encoding operation, therefore there are two different vertex encoding operations that result in 6 different entries.

4. Since $e_1$ and $e_2$ are not incident on a common vertex, but are in the same vertex gadget, they share only a gadget encoding operation, therefore there are four different vertex encoding operations that result in 12 different entries.

5. Since $e_1$ and $e_j$ are incident on a common vertex, they share a gadget encoding operation and two vertex encoding operations, while they differ for a jolly encoding operation and a vertex encoding operation, resulting in 5 different entries.

6. Since $e_1$ and $e_j$ are not incident on a common vertex, they share a gadget encoding operation (since by hypothesis $e_1$ and $e_j$ are in the same vertex gadget), and at most one vertex encoding operations (if $e_1$ is incident on a vertex adjacent to the docking vertex on which $e_j$ is also incident), while they differ for a jolly encoding operation and three vertex encoding operations, resulting in at least 11 different entries.
7. Since $e_1$ and $EG_{i,x}$ are incident on a common (docking) vertex, they share a gadget encoding operation and a vertex encoding operation, while they differ for a gadget encoding operation and two vertex encoding operations, resulting in 9 different entries.

8. Since $e_1$ and $EG_{i,x}$ are not incident on a common vertex, they share a gadget encoding operation (since $EG_{i,x}$ is incident on a docking vertex of $VG_i$), while they differ for a gadget encoding operation and four vertex encoding operations, resulting in 15 different entries.

9. Since $EG_{i,x}$ and $EG_{j,l}$ are not incident on a common vertex, they might share a gadget encoding operation (if the two edge gadget are incident on the same vertex gadget), while they differ for four vertex encoding operations and two gadget encoding operations, resulting in at least 18 different entries.

10. Since $EG_{i,x}$ and $EG_{j,l}$ are not incident on a common vertex gadget, they share no encoding operations, resulting in at 24 different entries.

11. Since $e_j$ and $e_5$ are not in the same jolly set, they might share a gadget encoding operation and a vertex encoding operations (if the two jolly edges are in the same vertex gadget), while the differ for four vertex encoding operations, resulting in at least 12 different entries.

12. The results follows from the previous cases (case 2, 6 and 11), as row $r$ is either a row of a $CVG_z$, for some $z$, a jolly row in a jolly row set of a different vertex of $VG$, or an edge gadget.

The cost of a solution $S$ is specified by introducing the notion of virtual cost of a single row $r$ of $R$. Let $S$ be a solution of 3-ABP, and let $C$ be the cluster of $S$ to which $r$ belongs. Let $r$ be a non-jolly row, we define the virtual cost of $r$ in the solution $S$, denoted as $\text{virt}_S(r)$, as the cost of $C$ divided by the number of non-jolly rows in $C$. Otherwise, if $r$ is a jolly row, then $\text{virt}_S(r) = 0$. Given the above notion, observe that the cost $c(C)$ of set $C$ is equal to $\sum_{r \in C} \text{virt}_S(r)$ and that for a solution $S$, the cost $c(S)$ of set $S$ is equal to $\sum_{r \in R} \text{virt}_S(r)$.

In the following we will consider only canonical solutions of 3-ABP, that is solutions where the rows for each vertex gadget $VG_i$ and edge gadgets eventually incident on $VG_i$ are clustered into type $a$ and type $b$ solutions constructed as follows.

The type $a$ solution defines the partition of the rows for vertex gadget $VG_i$ and consists of six clusters: three clusters of rows of $CVG_i$, each one is made of the three edges incident on vertex $v$, where $v$ is one of the three vertices $c_{i,4}$, $c_{i,5}$ and $c_{i,7}$, and three more clusters, each one consisting of the jolly rows associated with one of the three docking vertices of $VG_i$.

The type $b$ solution defines the partition of the rows for a vertex gadget $VG_i$ and some edge gadgets incident on $VG_i$. It consists of four clusters containing rows of $CVG_i$. One of them consists of the three edges incident on $c_{i,6}$. The remaining three clusters are associated with the three docking vertices of $VG_i$. For each docking vertex $c_{i,x}$, the cluster associated with $c_{i,x}$ consists of the two core edges of $CVG_i$ that are incident on $c_{i,x}$, together with either the edge gadget incident on $c_{i,x}$ or one jolly edge incident in $c_{i,x}$. Finally, there are three more clusters, each one consisting of all remaining jolly edges associated with parallel edges incident on one
of the three docking vertices of $V G_i$. Notice that in a *type b* solution each cluster associated with a docking vertex may contain an edge gadget or not, the only requirement is that at least one of the clusters contains an edge gadget. Notice that type *a* and type *b* solutions cluster together edges incident on a common vertex (by an abuse of language, we will call canonical such a cluster): the common vertex of a canonical cluster is called the *center* of the cluster.

**Proposition 4.** Let $S$ be a canonical solution of an instance of 3-ABT associated with an instance of MVCC, and let $V G_i, V G_j$ be two vertex gadgets such that the rows of $V G_i$ are clustered in a *type a* solution in $S$ and rows of $V G_j$ are clustered in a *type b* solution in $S$. Then each edge gadget has a virtual cost of 12 in $S$, the rows of $V G_i$ have a total cost of 81, while the rows of $V G_j$ have a total cost of 99.

**Proof.** Let $E G_{ij}$ be an edge gadget of $V G$. Observe that in a canonical solution each edge gadget belongs to a *type b* solution. Consider a *type b* solution for $V G_i$ containing $E G_{ij}$. By definition of *type b* solution, $E G_{ij}$ is co-clustered with two rows of $C V G_i$, so that those rows are incident on a common docking vertex with $E G_{ij}$. It follows that 12 entries (9 of the vertex blocks, 3 of the edge block) are deleted in each of these rows. Now, consider a cluster of a *type b* solution of $V G_i$ consisting of two rows $r_1, r_2$ of $C V G_i$ and a jolly row incident on a common docking vertex. Then 8 entries (6 of the vertex blocks, 2 of the jolly blocks) are deleted in each of these rows, hence this cluster has a total cost of 24. Since the virtual cost of the jolly row is 0, each of $r_1, r_2$ has virtual cost 12. A *type b* solution of $V G_i$ contains four clusters, three clusters containing row incident on the docking vertices (as described above) and a cluster of three rows incident on $c_{i,6}$, that has a virtual cost equal to 27 (9 for each row incident on $c_{i,6}$).

Each row of $C V G_i$ incident on a docking vertex has a virtual cost of 12 in a *type b* solution, hence the total virtual cost of the rows in $C V G_i$ of a *type b* solution is $27 + 12 \cdot 6 = 99$.

Consider a $C V G_i$ associated with a *type a* solution. Observe that a *type a* solution consists
Figure 3: A type b solution for the rows associated with $V_{G_i}$ and $E_{G_{ij}}$, where the dashed lines represent borders among clusters. Recall that each edge $e$ corresponds to a row.
of three cluster, each of cost 27. Indeed, each cluster of type a solution consists of three rows incident on a common vertex.

In the following we state two basic results that will be used to show the L-reduction from MVCC to 3-ABP: (i) each solution $S$ of 3-ABP can be modified in polynomial time into a canonical solution $S'$ whose cost is at most that of $S$ (Lemma 16); (ii) the graph $G$ has a vertex cover of size $p$ iff the 3-ABP problem has a canonical solution of cost $99 \cdot p + 81 \cdot (n - p) + 12m$, (we recall that 81 is the total virtual cost of the rows of a type a solution, and 99 is the total virtual cost of the rows of a vertex gadget in a type b solution – see Theorem 17). We will first introduce some basic Lemmas that will help in excluding some possible solutions.

**Lemma 5.** Let $S$ be a solution of an instance of 3-ABT associated with an instance of MVCC and let $C$ be a cluster of $S$ consisting of rows of $CVG_i$. Then $\text{virt}_S(r) \geq 9$ for each row $r$ of $C$, and $\text{virt}_S(r) \geq 12$ if $C$ is not a canonical cluster.

*Proof.* First notice that by construction $VG$ does not contain any cycle of length 4. It follows that if $C$ is not a canonical cluster, then $C$ contains two rows $e_1$ and $e_2$ not incident on a common vertex. By case 4 of Prop. 3, $e_1$ and $e_2$ have Hamming distance 12.

Assume now that $C$ is a canonical cluster, and let $W$ be the set of vertices incident on the edges of $C$, except for the center of $C$. By definition of canonical cluster, $C$ contains no cycles, moreover $|C|, |W| \geq 3$, therefore for each vertex $v_{i,x} \in W$, there exists one edge in $C$ not incident in $v_{i,x}$. Since the vertex encoding $\text{enc}_{i,x}$ is applied only to edge incident in $v_{i,x}$, the three entries set to 1 by $\text{enc}_{i,x}$ are deleted in each row of $C$, for each $v_{i,x} \in V'$, and the lemma follows.

**Lemma 6.** Let $S$ be a solution of an instance of 3-ABT associated with an instance of MVCC and let $C$ be a cluster of $S$ consisting of rows of $VG_i$, such that $C$ contains at least one jolly row of $VG_i$ and at least one row of $CVG_i$, then the virtual cost of each non-jolly row in $C$ is at least 12.

*Proof.* Assume first that $C$ contains exactly one non-jolly row $r_1$. Then by case 5 of Prop. 3, $H(r_1, r_j) \geq 5$ for each jolly row $r_j$ and, since $|C| \geq 3$, the cost of $C$ is at least 15. Since $r_1$ is the only non-jolly row of $C$, then the virtual cost of $r_1$ is at least 15. Assume now that $C$ contains at least two non-jolly rows, $r_1$, $r_2$ and let $r_j$ be a jolly row in $C$. If $C$ contains exactly two rows of $VG_i$ by cases 3,4 of Prop. 3 $H(r_1, r_2) \geq 6$ and by construction there are two positions $h_1, h_2$ in the jolly block where $r_1[h_z] = r_2[h_z] = 0_b$, while $r_j[h_z] = 1_b$, with $z \in \{1, 2\}$. Hence the total cost of $C$ is at least $8|C|$. Since $C$ contains exactly two non-jolly rows $r_1$, $r_2$, the virtual cost of $r_1$ and $r_2$ is at least $8|C|/2$. Since $|C| \geq 3$, the virtual cost of $r_1$ and $r_2$ will be at least 12.

Assume that $C$ contains more than two non-jolly rows. Then $C$ contains a set $C'$, where $C'$ consists of at least 3 rows of $CVG_i$. Notice that $C'$ can be a cluster of a feasible solution of 3-ABT, therefore Lemma 5 applies also to $C'$ and an immediate consequence is that the same 9 entries in the vertex blocks must be suppressed also in the same position in $C$. Furthermore, by construction, there exist two positions $h_1, h_2$ in the jolly block where the non-jolly rows have $0_b$, while some of the jolly rows has value $1_b$. Hence the total cost of $C$ is at least $11|C|$ and the virtual cost of each non-jolly row in $C$ is at least $11|C|/(|C| - 1)$. But Remark 2 implies $|C| \leq 5$, therefore $11|C|/(|C| - 1) \geq 12$ and the virtual cost of each non-jolly row in $C$ is at least 12.
Lemma 7. Let $S$ be a solution of an instance of 3-ABT associated with an instance of MVCC and let $C$ be a cluster of $S$ containing a row of $V G_i$. Then the virtual cost of each non-jolly row of $C$ is at least 9.

Proof. Notice that if $C$ contains a jolly row, then the lemma is a consequence of Prop. 3 (Cases 2 and 12), Lemmas 5 and 6. Hence assume that $C$ contains no jolly row. If $C$ contains at least two edge gadgets then, by case 9 of Prop. 3, the virtual cost of each non-jolly row of $C$ is at least 18, therefore we can assume that there is exactly one edge gadget in $C$.

If $C$ is not a canonical cluster, there are two rows that not incident on a common vertex, therefore by cases 1, 4, 8 of Prop. 3 and by construction of $V G_i$, each non-jolly row of $C$ has a virtual cost of at least 12. The final case that we have to consider for $C$ is when $C$ contains an edge gadget and two edges of a core vertex gadget and all edges are incident on a common vertex: in this case we can apply case 7 of Prop. 3 to obtain the lemma. 

An immediate consequence of Lemma 7 and of the construction of $V G_i$, is that a type a solution is the optimal solution for the rows associated with edges of $V G_i$.

Lemma 8. Let $S$ be a solution of an instance of 3-ABT associated with an instance of MVCC and let $C$ be a cluster of $S$ containing exactly two edge gadgets $E G_1$ and $E G_2$. Then each of the virtual costs $\text{virt}_S(E G_1)$, $\text{virt}_S(E G_2)$ is at least 21. If the edge gadgets are not incident on a common vertex gadget, then $\text{virt}_S(E G_1), \text{virt}_S(E G_2) \geq 27$.

Proof. Notice that all 1's in a vertex block of $E G_1$ correspond to 0's of $E G_2$. The same fact holds for 3 1's in the edge block of $E G_1$, if $E G_1$ and $E G_2$ are incident on a common vertex gadget, otherwise 6 1's in the edge block of $E G_1$ are deleted. By symmetry of $E G_1$, $E G_2$ the number of deleted columns is at least 18, when $E G_1$ and $E G_2$ are incident on a common vertex gadget, otherwise the number of deleted columns is at least 24.

Let $r_3 \in C$ different from $E G_1$, $E G_2$. Since $r_3$ is not an edge gadget, there is a vertex block of $r_3$ containing 6 1's, while both $E G_1$ and $E G_2$ have at most 3 1's in that block. It is immediate to notice from the construction of $E G_1$ and $E G_2$ that this fact leads to at least 3 additional columns that must be deleted.

Lemma 9. Let $S$ be a solution of an instance of 3-ABT associated with an instance of MVCC and let $C$ be a cluster of $S$ containing two edge gadgets $E G_1$, $E G_2$ that are not incident on a common vertex gadget and a row $r$ belonging to a vertex gadget, such that $r$ is not adjacent to $E G_1$ nor to $E G_2$. Then $\text{virt}_S(E G_1), \text{virt}_S(E G_2), \text{virt}_S(r) \geq 30$.

Proof. Since $E G_1$ and $E G_2$ are not incident on a common vertex gadget, by case 10 of Prop. 3 we know that $H(E G_1, E G_2) \geq 24$. Now consider the row $r$ and assume w.l.o.g. that $r$ belongs to vertex gadget $V G_i$. Since $r$ is not adjacent to $E G_1$ nor to $E G_2$ there are at least 6 positions in the $i$-th block, where $E G_1$ and $E G_2$ have both value 0, while $r$ has value 1.

Lemma 10. Let $S$ be a solution of an instance of 3-ABT associated with an instance of MVCC and let $C$ be a cluster of $S$ containing three edge gadgets $E G_1$, $E G_2$, $E G_3$. Then each of the virtual costs $\text{virt}_S(E G_1), \text{virt}_S(E G_2), \text{virt}_S(E G_3)$ is at least 27. If there is no pair of edge gadgets in $\{E G_1, E G_2, E G_3\}$ incident on a common vertex gadget, then $\text{virt}_S(E G_1), \text{virt}_S(E G_2), \text{virt}_S(E G_3) \geq 36$. 

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Proof. Observe that $EG_1, EG_2, EG_3$ have minimum virtual cost either when they are all incident on the same vertex gadget or the set of vertex gadgets to which $EG_1, EG_2, EG_3$ are incident consists of three vertex gadgets (that is $EG_1, EG_2, EG_3$ encode a cycle of length 3). Therefore 9 entries for each of $EG_1, EG_2, EG_3$ are deleted in the edge block, while 18 entries of the vertex blocks are deleted for each of $EG_1, EG_2, EG_3$, since $EG_1, EG_2, EG_3$ represent edges incident on a set of 6 different docking vertices. Hence the virtual cost of each $EG_i, i = \{1, 2, 3\}$, is at least 27.

Observe that when there is no pair of edge gadgets in $\{EG_1, EG_2, EG_3\}$ incident on the same vertex gadget, the positions of the edge block with value 1 in $EG_1, EG_2, EG_3$ are all different, hence at least 18 entries are deleted for each of $EG_1, EG_2, EG_3$. Hence the virtual cost of each $EG_i, i = \{1, 2, 3\}$, is at least 36.

Lemma 11. Let $S$ be a solution of an instance of 3-ABT associated with an instance of MVCC and let $C$ be a cluster of $S$ containing more than three edge gadgets. Then the virtual cost of each edge gadget in $S$ is at least 36.

Proof. Consider 4 edge gadgets in $C$: $EG_1, EG_2, EG_3, EG_4$. First observe that 24 entries of the vertex blocks are deleted for each of $EG_1, EG_2, EG_3, EG_4$, since $EG_1, EG_2, EG_3, EG_4$ represent edges incident on a set of 8 different docking vertices.

A simple argument shows that at least two of such edge gadgets are not incident on a common vertex gadget. Indeed, the set of vertex gadgets on which $EG_1, EG_2, EG_3, EG_4$ are incident contains at least four vertex gadgets, for otherwise two edge gadgets must be incident on the same two vertex gadgets. Hence 12 entries of the edge block will be deleted from each row in $C$. □

Lemma 12. Let $S$ be a solution of an instance of 3-ABT associated with an instance of MVCC and let $C$ be a cluster of $S$ containing an edge gadget $EG_{ij}$ incident on vertex gadgets $VG_i$ and $VG_j$, two rows $r_x, r_y$ adjacent to $EG_{ij}$, where $r_x$ belongs to $VG_i$ and $r_y$ belongs to $VG_j$. Then the cost of $C$ in $S$ is at least $18|C|$.

Proof. It is an immediate consequence of case 1 of Prop. □

Lemma 13. Let $S$ be a solution of an instance of 3-ABT associated with an instance of MVCC and let $C$ be a cluster of $S$ containing an edge gadget $EG_{ij}$ and a jolly row $j_i$. Then $\nu_r S(EG_{ij}) \geq 18$.

Proof. Observe that by case 2 of Prop. $H(EG_{ij}, j_i) \geq 14$. Let $\vec{J}$ be the subset of $C$ consisting of all rows of $C$ that are not jolly rows. Moreover, we can assume that $|\vec{J}| \leq 4$, as $|C| \leq 5$. If $|s(j)| = 4$, then there is at least one row $r$ in $\vec{J}$ such that there exist at least 3 positions where $EG_{ij}$ and $j_i$ have the same value, while $r$ has a different value. Hence the virtual cost of $EG_{ij}$ is at least $\frac{17|C|}{|J|} > 18$. If $|s(j)| \leq 3$, then the virtual cost of $EG_{ij}$ is at least $\frac{14|C|}{|\vec{J}|} \geq 18$ and the lemma holds. □

The following Lemma 14 is a consequence of cases 1, 2, 7, 8 of Prop. and the construction of the gadget graph.

Lemma 14. Let $S$ be a solution of an instance of 3-ABT associated with an instance of MVCC and let $C$ be a cluster of $S$ with at least an edge gadget $EG_{ij}$. Then $\nu_r S(EG_{ij}) \geq 12$.  

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Now, we will show our key transformation of a generic solution into a canonical solution without increasing its cost. The proof is based on the fact that, whenever a solution $S$ is not a canonical one, it can be transformed into a canonical one by applying Alg. 1.

Let us denote by $S_1$ and $S_2$ respectively, the solution before and after applying Alg. 1. Observe that, by construction, in the solution $S_2$ computed by Alg. 1 each edge gadget belongs to a type $b$ solution.

**Algorithm 1**: ComputeCanonical($S_1$)

**Data**: a solution $S_1$ consisting of the set \{ $C_1, \ldots, C_k$ \} of clusters

1. Unmark all edge gadgets and all vertex gadgets;
2. $S_2 \leftarrow \emptyset$;
3. **while** there is a cluster $C$ in \{ $C_1, \ldots, C_k$ \} with an unmarked edge gadget **do**
4. \hspace{1em} $U(C) \leftarrow$ the set of unmarked edge gadgets in $C$;
5. \hspace{1em} $V(C) \leftarrow$ a smallest possible set of vertex gadgets such that each edge gadget in $U(C)$ has at least one endpoint in $V(C)$; /* $|C| \leq 5$ by Remark 2, hence we can compute $V(C)$ in polynomial time. We assume that if a cluster $C$ contains only an edge gadget $EG_{i,j}$ and rows of vertex gadget $VG_i$, then $V(C) = \{ VG_i \}$.
6. \hspace{1em} $E' \leftarrow$ all unmarked edge gadgets incident on some vertices of $V(C)$;
7. \hspace{1em} Add to $S_2$ a type $b$ solution for all vertex gadgets in $V(C)$ and all edge gadgets in $E'$; /* Notice that $E' \supseteq U(C)$ */
8. \hspace{1em} Mark the edge gadgets in $E'$ and the vertex gadgets in $V(C)$;
9. **end**
10. Add to $S_2$ a type $a$ solution for each unmarked vertex gadget;
11. **return** $S_2$

Notice that at the end of the execution of the algorithm, each vertex gadget is assigned either a type $a$ or a type $b$ solution, and that each row is assigned to one of those solutions. As type $a$ solutions are optimal, we can concentrate on type $b$ solutions.

Clusters corresponding to type $b$ solutions are built iteratively at line 3-9 of Alg. 1. More precisely at each iteration the algorithm examines a set of clusters $C_1, \ldots, C_l$ of $S_1$ and it extracts a cluster $C$ containing at least one unmarked edge gadget. Then the algorithm imposes a type $b$ solution on a set $V(C)$ of vertex gadgets and on a set $E'$ of edge gadgets so that $U(C) \subseteq E'$. Such step is the only one where the virtual cost of some rows can be modified. More precisely only edges of the core vertex gadgets in $V(C)$ and edge gadgets in $U(C)$ may have in $S_2$ a virtual cost different from that in $S_1$.

Notice that by Lemma 14, each edge gadget in $E' - U(C)$, has virtual cost of at least 12 in solution $S_1$, and virtual cost 12 in solution $S_2$. Hence the virtual cost of such edge gadget is not increased by Alg. 1.

Now, we consider the rows associated with core vertex gadgets in $V(C)$ and edge gadgets in $U(C)$. For simplicity’s sake, let us denote by $\text{virt}_S(V(C))$ the sum of the virtual cost of the set of rows associated with core vertex gadgets in $V(C)$ in a solution $S$ and similarly, let us denote by $\text{virt}_S(U(C))$ the sum of the virtual cost of the set of rows associated with unmarked edge gadgets in $U(C)$.

Observe that, by construction, the sets $U(C)$ considered in different iterations of Alg. 1 are
pairwise disjoint, therefore it makes sense to analyze each iteration separately. Consequently, it is immediate to conclude that the correctness of the Alg. 1 can be proved by showing that, in a generic iteration of Alg. 1, the following lemma holds:

**Lemma 15.** Let $C$ be the cluster containing an unmarked edge gadget found at line 3 of Alg. 1. Then $\text{virt}_{S_2}(V(C)) - \text{virt}_{S_1}(V(C)) \leq \text{virt}_{S_1}(U(C)) - \text{virt}_{S_2}(U(C))$.

**Proof.** The proof consists of several cases. For each case it will suffice to determine that one of the following conditions hold:

(i) $\text{virt}_{S_1}(U(C)) - \text{virt}_{S_2}(U(C)) \geq 18|V(C)|$;

(ii) $\text{virt}_{S_1}(V(C)) + \text{virt}_{S_1}(U(C)) \geq 99|V(C)| + 12|U(C)|$;

(iii) $\text{virt}_{S_2}(V(C)) - \text{virt}_{S_1}(V(C)) \leq \text{virt}_{S_1}(U(C)) - \text{virt}_{S_2}(U(C))$.

Notice that conditions (i) and (ii) imply condition (iii). First we show that condition (i) implies condition (iii). Assume that condition (i) holds, that is $\text{virt}_{S_1}(U(C)) - \text{virt}_{S_2}(U(C)) \geq 18|V(C)|$. Solution $S_2$ builds a type $a$ or a type $b$ solution for $V(C)$ (whose cost is at most 99 for each vertex gadget), while we know that the optimal solution for $V(C)$ has cost at least 81 (Lemma 7), which implies that $\text{virt}_{S_2}(V(C)) - \text{virt}_{S_1}(V(C)) \leq 18|V(C)|$, and hence $\text{virt}_{S_2}(V(C)) - \text{virt}_{S_1}(V(C)) \leq \text{virt}_{S_1}(U(C)) - \text{virt}_{S_2}(U(C))$.

Now we show that condition (i) implies condition (iii). Assume that conditions (ii) holds, that is $\text{virt}_{S_1}(V(C)) + \text{virt}_{S_1}(U(C)) \geq 99|V(C)| + 12|U(C)|$. As by construction of solution $S_2$ $\text{virt}_{S_2}(V(C)) + \text{virt}_{S_2}(U(C)) = 99|V(C)| + 12|U(C)|$, it follows that $\text{virt}_{S_2}(V(C)) - \text{virt}_{S_1}(V(C)) \leq \text{virt}_{S_1}(U(C)) - \text{virt}_{S_2}(U(C))$.

We will distinguish several cases, depending on the structure of $U(C)$. Recall that Remark 2 implies $|C| \leq 5$, hence $|U(C)| \leq 5$. Notice also that, by construction, $|V(C)| \leq |U(C)|$, and that $\text{virt}_{S_1}(V(C)) - \text{virt}_{S_2}(V(C)) \leq 18|V(C)|$ as the set of rows of each vertex gadget in $V(C)$ has a total cost of at least 81 in solution $S_1$ and at most 99 in solution $S_2$.

- Assume that $|U(C)| > 3$. By Lemma 11 the virtual cost (in $S_1$) of each edge gadget in $U(C)$ is at least 36. Therefore $\text{virt}_{S_1}(V(C)) + \text{virt}_{S_1}(U(C)) \geq 81|V(C)| + 36|U(C)| = 81|V(C)| + 12|U(C)| + 24|U(C)| \geq 81|V(C)| + 12|U(C)| + 24|V(C)| = 12|U(C)| + 105|V(C)| \geq 12|U(C)| + 99|V(C)|$, as required by condition (ii).

- Assume that $|U(C)| = 3$ and no two gadgets in $U(C)$ are incident on a common vertex gadget. By Lemma 10 the virtual cost (in $S_1$) of each edge gadget in $U(C)$ is at least of 36. We can apply the same analysis of case $|U(C)| > 3$ to show that $\text{virt}_{S_1}(V(C)) + \text{virt}_{S_1}(U(C)) \geq 12|U(C)| + 105|V(C)| \geq 12|U(C)| + 99|V(C)|$, as required by condition (ii).

- Assume that $|U(C)| = 3$ and two gadgets in $U(C)$ are incident on a common vertex gadget. By Lemma 10 the virtual cost (in $S_1$) of each edge gadget in $U(C)$ is at least of 27, but notice that $|V(C)| \leq 2$. Therefore $\text{virt}_{S_1}(V(C)) + \text{virt}_{S_1}(U(C)) \geq 27 \cdot 3 + 81|V(C)|$. It is immediate to notice that $27 \cdot 3 + 81|V(C)| \geq 12 \cdot 3 + 99|V(C)|$ when $|V(C)| \leq 2$, as required by condition (ii).
• Assume that $|U(C)| = 2$ and such two gadgets $EG_1$, $EG_2$ in $U(C)$ are not incident on a common vertex gadget. By Lemma 8 the virtual cost in $S_1$ of each edge gadget in $U(C)$ is at least 27, while by Prop. 4 the virtual cost in $S_2$ of each edge gadget in $U(C)$ is exactly 12. Hence $\text{virt}_{S_1}(U(C)) - \text{virt}_{S_2}(U(C)) \geq 27 - 12 = 15$. Therefore we are only interested in the case $|V(C)| \geq 2$. Since $|V(C)| \leq |U(C)| = 2$, we can assume that $|V(C)| = 2$ and, consequently, $\text{virt}_{S_2}(V(C)) - \text{virt}_{S_1}(V(C)) \leq 15$.

Let $r$ be a row in $C$ which is not an edge gadget (one must exist because $|C| \geq 3$ and there are exactly two edge gadgets in $C$). We have to distinguish two cases, according to the fact that $r$ is a jolly row or a row of a vertex gadget.

First consider the case when $r$ is a jolly row. By Lemma 8 at least 27 entries of the vertex blocks in the rows of $C$ are deleted. Since $r$ is a jolly row $\text{virt}_{S_1}(EG_1), \text{virt}_{S_1}(EG_2) \geq 27 |C|/(|C|-1) \geq 33$, as $|C| \leq 5$. Therefore $\text{virt}_{S_1}(U(C)) - \text{virt}_{S_2}(U(C)) = \text{virt}_{S_1}(EG_1) + \text{virt}_{S_1}(EG_2) - \text{virt}_{S_2}(EG_1) - \text{virt}_{S_2}(EG_2) \geq 33 - 2 \cdot 12 \cdot 2 = 42$, which implies that $\text{virt}_{S_2}(V(C)) + \text{virt}_{S_2}(U(C)) \leq \text{virt}_{S_1}(V(C)) + \text{virt}_{S_1}(U(C))$, as required by condition (iii).

Assume now that $r$ is a row of a vertex gadget $X$, we have to consider two subcases depending on the fact that $r$ is adjacent or not to $EG_1$ or $EG_2$. Assume that $r$ is not adjacent to $EG_1$ nor to $EG_2$. By Lemma 8 $\text{virt}_{S_1}(U(C)) - \text{virt}_{S_2}(U(C)) \geq 27$. As we have assumed that $|V(C)| = 2$ and $\text{virt}_{S_2}(V(C)) - \text{virt}_{S_1}(V(C)) \leq 12$, we can immediately prove that $\text{virt}_{S_2}(V(C)) + \text{virt}_{S_2}(U(C)) \leq \text{virt}_{S_1}(V(C)) + \text{virt}_{S_1}(U(C))$, as required by condition (iii). The last subcases that we have to consider is when $r$ is adjacent to $EG_1$ or $EG_2$. Assume w.l.o.g. that $r$ is adjacent to $EG_1$ and that $X \subseteq V(C)$. Observe that $r$ is co-clustered in $S_2$ in a type a or in a type b solution, hence by Prop. 4 $\text{virt}_{S_2}(r) \leq 12$, while $\text{virt}_{S_1}(r) \geq 27$, as $\text{virt}_{S_1}(EG_1) \geq 27$ by Lemma 8. Taking into account the fact that $\text{virt}_{S_1}(r) - \text{virt}_{S_2}(r) \geq 15$, as $r$ is a row of $X \subseteq V(C)$, we immediately obtain that $\text{virt}_{S_2}(V(C)) - \text{virt}_{S_1}(V(C)) \leq 18 \cdot 2 - 15 = 21$ which can be coupled with $\text{virt}_{S_1}(U(C)) - \text{virt}_{S_2}(U(C)) \geq 30$ proved before to obtain $\text{virt}_{S_2}(V(C)) + \text{virt}_{S_2}(U(C)) \leq \text{virt}_{S_1}(V(C)) + \text{virt}_{S_1}(U(C))$, as required by condition (iii).

• Assume that $|U(C)| = 2$ and that the edge gadgets $EG_1$, $EG_2$ in $U(C)$ are incident on a common vertex gadget. Hence, by construction of the algorithm, $|V(C)| \leq 1$, therefore $\text{virt}_{S_2}(V(C)) - \text{virt}_{S_1}(V(C)) \leq 18$. By Lemma 8 the virtual cost (in $S_1$) of each edge gadget in $U(C)$ is 21, therefore $\text{virt}_{S_1}(U(C)) - \text{virt}_{S_2}(U(C)) \geq (21 - 12) \cdot 2 = 18$, as required by condition (i).

• Assume that $U(C) = \{EG_{l,h}\}$ and $EG_{l,h}$ is clustered with two rows $r_1$, $r_2$ from different vertex gadgets $VG_i$, $VG_j$. Since $|U(C)| = 1$, $|V(C)| \leq 1$, hence $\text{virt}_{S_2}(V(C)) - \text{virt}_{S_1}(V(C)) \leq 18$. By Lemma 12 $\text{virt}_{S_2}(EG_{l,h}), \text{virt}_{S_2}(r_1), \text{virt}_{S_2}(r_2) \geq 18$, while by Prop. 4 $\text{virt}_{S_2}(EG_{l,h}), \text{virt}_{S_2}(r_1), \text{virt}_{S_2}(r_2) \leq 15$, hence $\text{virt}_{S_1}(EG_{l,h}) - \text{virt}_{S_2}(EG_{l,h}) \geq 6$ while $\text{virt}_{S_2}(V(C)) - \text{virt}_{S_1}(V(C)) \leq 6$ which immediately implies $\text{virt}_{S_2}(V(C)) + \text{virt}_{S_2}(U(C)) \leq \text{virt}_{S_1}(V(C)) + \text{virt}_{S_1}(U(C))$, as required by condition (iii).

• Assume that $U(C) = \{EG_{l,h}\}$ and that $C$ contains the edge gadget $EG_{l,h}$ and (at least)
two rows \(r_1, r_2\) of \(VG_j\), with \(j \neq i, l\). Since \(|U(C)| = 1\), \(|V(C)| \leq 1\), hence \(\text{virt}_{S_2}(V(C)) = 21\).

Initially we will prove that for each row \(r\) of \(C\) the virtual cost \(\text{virt}_{S_1}(r) \geq 21\). In fact \(r_1\) and \(r_2\) may share a gadget encoding operation and a vertex encoding operation (if they are incident on a common vertex), therefore there are three distinct gadget encoding operations and four distinct gadget encoding operations overall, and none of such operation is shared by all edges in \(C\). An immediate consequence is that at least 21 entries of each row of \(C\) must be suppressed, therefore \(\text{virt}_{S_1}(\{r_1, r_2, EG_{l,h}\}) = 21\) and \(\text{virt}_{S_2}(\{r_1, r_2, EG_{l,h}\}) = 3\), as by Prop. 4 each row of \(C\) in \(S_2\) has virtual cost at most 12. For bookkeeping purposes we attribute the entire value of \(\text{virt}_{S_1}(\{r_1, r_2, EG_{l,h}\})\) to \(EG_{l,h}\) (this bookkeeping trick is possible as a row \(r\) is allowed to give its “credit” only to an edge gadget with which it is co-clustered in \(S_1\)), therefore we obtain \(\text{virt}_{S_1}(U(C)) = 21\) and \(\text{virt}_{S_2}(U(C)) = 3\), as required by condition (i).

- Assume that \(U(C) = \{EG_{i,l}\}\) and that \(C\) contains the edge gadget \(EG_{i,l}\) and (at least) two rows \(r_1, r_2\) of \(VG_i\), with \(r_2\) not adjacent to \(EG_{i,l}\). Notice that, by case 8 of Property 3 \(H(r_2, EG_{i,l}) \geq 15\). Therefore the \(\text{virt}_{S_1}(\{EG_{i,l}, r_1, r_2\}) \geq 45\), while \(\text{virt}_{S_2}(\{EG_{i,l}, r_1, r_2\}) \leq 36\) by Prop. 4.

In what follows we will consider the virtual costs of the rows in \(VG_i\). More precisely, let \(T\) be the set \(VG_i - \{r_1, r_2\}\); we will show that there exists a row \(r \in T\) such that \(\text{virt}_{S_1}(r) \geq 12\).

Assume initially that there exists a row \(r \in T\) that is clustered with an edge gadget or with a row belonging to a different vertex gadget \(VG_j\). Then \(\text{virt}_{S_1}(r) \geq 12\) by Lemma 4 and case 1 of Prop. 3. Hence, we can assume that the rows in \(T\) are clustered only with rows of \(VG_i\), which implies that \(r_1\) and \(r_2\) are not clustered together with any row in \(T\), as \(r_1\) and \(r_2\) are clustered with \(EG_{i,l}\).

Since \(T\) contains 7 rows, a trivial counting argument shows that 4 rows of \(T\) are clustered together, or there is a row \(r \in T\) that is clustered with a jolly row of \(VG_i\). Indeed, if four rows of \(T\) are clustered together, by construction there are two of those four rows that are not incident on a common vertex, therefore an immediate application of case 4 of Prop. 3 gives the desired result. If \(r\) is clustered with a jolly row then, by Lemma 4 \(\text{virt}_{S_1}(r_w) \geq 12\).

Now we know that there is a row \(r \in VG_i - \{r_1, r_2\}\) such that \(\text{virt}_{S_1}(r) \geq 12\). Moreover \(\text{virt}_{S_1}(\{EG_{i,l}, r_1, r_2\}) \geq 45\) and \(\text{virt}_{S_2}(\{EG_{i,l}, r_1, r_2\}) \leq 36\).

By Lemma 4 the virtual cost of any row of \(VG_i\) different from \(r, r_1, r_2\) is at least 9. Since \(\text{virt}_{S_1}(r_1) = \text{virt}_{S_1}(r_2) \geq 15\), \(\text{virt}_{S_1}(VG_i) = 6 \cdot 9 + 12 + 15 \cdot 2 = 96\), while \(\text{virt}_{S_2}(VG_i) = 99\) (since \(S_2\) has a type b solution for the rows in \(VG_i\)).

Since \(\text{virt}_{S_1}(EG_{i,j}) = 15\) and \(\text{virt}_{S_2}(EG_{i,j}) = 12\) by Prop. 3, it is immediate to obtain that \(\text{virt}_{S_1}(V(C)) + \text{virt}_{S_1}(U(C)) \geq \text{virt}_{S_2}(V(C)) + \text{virt}_{S_2}(U(C))\) as required by condition (iii).

- Assume that \(U(C) = \{EG_{i,j}\}\) and that \(C\) contains at least a jolly row \(e_j\) of a vertex gadget \(VG_i\) or \(VG_h\) (w.l.o.g. \(VG_i\)). We assume that \(VG_i \subseteq V(C)\). Furthermore,
we can assume that no other row of a different vertex gadget belongs to $C$ otherwise the previous cases hold. By case 2 of Prop. 3 the Hamming distance of $EG_{l,h}$ and $e_j$ is at least 14, therefore $virt_{S_1}(C) \geq 14|C|$, while $virt_{S_2}(C) \leq 12(|C| - 1)$, hence $virt_{S_1}(C) - virt_{S_2}(C) \geq 2(|C| - 1) + 14$ and, since $|C| \geq 3$, $virt_{S_1}(C) - virt_{S_2}(C) \geq 18$. Once again, we attribute the entire value of $virt_{S_1}(C) - virt_{S_2}(C) \geq 18$ to $E_{G_{l,h}}$ (this bookkeeping trick is possible as a row $r$ is allowed to give its “credit” only to an edge gadget with which it is co-clustered in $S_1$), therefore $virt_{S_1}(U(C)) - virt_{S_2}(U(C)) \geq 18$, as required by condition (i).

The proof is completed by the observation that the only possible case that is not explicitly considered in the above cases is when an unmarked edge gadget is clustered in $S_1$ only with rows of the core vertex gadget $VG_i$ and all rows share a common vertex. In such case, Alg. 1 does not modify the clustering, as $V(C)$ is made only by the vertex gadget $VG_i$ and $V(C)$ has a type b solution in $S_2$.

Lemma 16. Let $S$ be a solution of an instance of 3-ABT associated with an instance of MVCC, then we can compute in polynomial time a canonical solution $S_c$ such that $c(S_c) \leq c(S)$.

Theorem 17. Let $G = (V, E)$ be an instance of MVCC. Then $G$ has a cover of size $p$ if and only if the corresponding instance $R$ of 3-ABT has a (canonical) solution $S$ of cost $99p + 81(n - p) + 12m$.

Proof. Let us show that if $G$ has a vertex cover $V_c$ of size $p$, then $R$ has a solution $S$ of cost $99p + 81(n - p) + 12m$. Since $V_c$ is a vertex cover then it is possible to construct a canonical solution $S$ for $R$ consisting of a type b solution for all vertex gadgets associated with vertices in $V_c$ and a type a solution for all other vertex gadgets. Indeed each edge gadget can be clustered in a type b solution of a vertex gadget to which the edge is incident, choosing arbitrarily whenever there is more than one possibility. Finally, for each docking vertex, its jolly rows that are not used in some type b solution are clustered together. The cost derives immediately by previous observations.

Let us consider now a solution $S$ of 3-ABP over instance $R$ with cost $99p + 81(n - p) + 12m$. By Lemma 16 we can assume that $S$ is canonical solution, therefore $R$ has a set $V_c$ of $p$ vertex gadgets that are associated with a type b solution. By construction, each edge gadget must be in a type b solution, for otherwise $S$ is a not canonical solution. Hence the set of vertices of $G$ associated with vertex gadgets in $V_c$ is a vertex cover of $G$ of size $p$.

Since the cost of a canonical solution of 3-ABP and the size of a vertex cover of the graph $G$ are linearly related, the reduction is an L-reduction, thus completing the proof of APX-hardness.

4 APX-hardness of 4-AP(8)

In this section we prove that the 4-anonymity problem is APX-hard even if all rows of the input table have 8 entries (this restriction is denoted by 4-AP(8)). More precisely, we give an L-reduction from Minimum Vertex Cover on Cubic Graphs (MVCC) to 4-AP(8).

Given a cubic graph $G = (V, E)$, with $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$, we will construct an instance $R$ of 4-AP(8) consisting of a set $R_i$ of 5 rows for each vertex $v_i \in V$, an
edge row \( r(i, j) \) for each edge \( e = (v_i, v_j) \in E \) and a set \( F \) of 4 rows. The 8 columns are divided in 4 blocks of two columns each. For each vertex \( v_i \), all the rows in \( R_i \) have associated a block called edge block, denoted as \( b(R_i) \), so that \( b(R_i) \neq b(R_j) \) for each \( v_j \) adjacent to \( v_i \) in \( G \). The latter property can be easily enforced in polynomial time as the graph is cubic.

The entries of the rows in \( R_i = \{r_{i,1}, \ldots, r_{i,5}\} \), are over the alphabet \( \Sigma(R_i) = \{a_{i,1}, \ldots, a_{i,5}, a_i\} \). The entries of the columns corresponding to the edge block \( b(R_i) \), as well as to the odd columns are set to \( a_i \) for all the rows in \( R_i \). The entries of the even columns not in \( b(R_i) \) of each row \( r_{i,h} \) are set to \( a_{i,h} \).

For each edge \( e = (v_i, v_j) \), we define a row \( r(i, j) \) (called edge row) of \( R \). Row \( r(i, j) \) has value \( a_j \) (equal to the values of the rows in \( R_i \)) in the two columns corresponding to the edge block \( b(R_i) \), value \( a_j \) (equal to the values of the rows in \( R_j \)) in the two columns corresponding to the edge block \( b(R_j) \), and value \( t_{i,j} \) in all other columns. Given a set of rows \( R_i \), we denote by \( E(R_i) \) the set of rows \( r(i, j), r(i, l), r(i, h) \), associated with edges of \( G \) incident in \( v_i \). Finally, we introduce in the instance \( R \) of 4-AP(8) a set of 4 rows \( F = \{f_1, f_2, f_3, f_4\} \), over alphabet \( \Sigma(F) = \{u_1, \ldots, u_4\} \). Each row \( f_j \) is called a free row and all its entries have value \( u_i \).

Since all tables have 8 entries, w.l.o.g. we can assume that there exists only one cluster \( F_c \), called the filler cluster, whose cost is equal to \( 8|F_c| \). In fact, if there exists two clusters \( F_c, F_c' \) exist, whose cost is equal to \( 8|F_c| \) and \( 8|F_c'| \) respectively, then we can merge them without increasing the cost of the solution. The free rows must belong to \( F_c \), as each free row has Hamming distance 8 with all other rows of \( R \). Notice that, by construction, \( \Sigma(R_i) \cap \Sigma(R_j) = \emptyset \), hence two rows have Hamming distance smaller than 8 only if they both belong to \( R_i \cup E(R_i) \) for some \( i \). This observation immediately implies the following proposition.

**Proposition 18.** Let \( S \) be a solution of an instance of 4-AP(8) associated with an instance of MVC and let \( C \) be a cluster of \( S \) where each row in \( C \) has cost strictly less than 8. Then \( C \subseteq R_i \cup E(R_i) \).

Since in \( R_i \cup E(R_i) \) there are 8 rows, there can be at most two sets having rows in \( R_i \cup E(R_i) \) and satisfying the statement of Proposition 18. Consider a solution \( S \) and a set of rows \( R_i \). We will say that \( S \) is a black solution for \( R_i \) if in \( S \) there is a cluster containing 4 rows of \( R_i \) and a cluster containing one row of \( R_i \) and the three rows of \( E(R_i) \). We will say that \( S \) is a red solution for \( R_i \) if in \( S \) there is a cluster consisting of all 5 elements of \( R_i \). By an abuse of language we will say respectively that \( R_i \) is black (resp. red) in \( S \). Given an instance \( R \) of 4-AP(8), a solution where each set \( R_i \) is either black or red is called a canonical solution. Notice that a canonical solution consists of a filler cluster and a red or black solution for each \( R_i \). The main technical step in our reduction consists of proving Lemma 22 which states that, starting from a solution \( S \), it is possible to compute in polynomial time a canonical solution \( S' \) with cost not larger than that of \( S \). To achieve such goal we need some technical results.

Next we show that moving the rows of \( R_i \) that are in the filler cluster to another existing cluster that contains some rows of \( R_i \) (if possible) or to a new cluster, does not increase the cost of the solution.

**Lemma 19.** Let \( S \) be a solution of an instance of 4-AP(8) associated with an instance of MVCC, and let \( r \) be a row of \( R_i \). Then at least three even entries of \( r \) that are not in the edge block are deleted.

**Proof.** The lemma follows from the property that a row \( r \in R_i \) must be co-clustered with at
least three other rows, and that \( r \) disagrees with any other row of \( R \) in the even entries not in the edge block of \( R_i \).

**Lemma 20.** Let \( S \) be a solution of an instance of 4-\( AP(8) \) associated with an instance of MVCC. Then we can compute in polynomial time a solution \( S' \) with cost not larger than that of \( S \) and such that in \( S' \) there exist at most two sets containing some rows of \( R_i \).

**Proof.** Consider a generic set of rows \( R_i \); clearly if at most two sets of \( S \) contain some rows of \( R_i \), then \( S \) satisfies the lemma, therefore assume that in \( S \) there are at least three sets containing some rows of \( R_i \). Let \( C_i^1, \ldots, C_i^3 \) be the clusters of \( S \) containing rows of \( R_i \). By a simple counting argument, at most one of the clusters \( C_i^j \) (w.l.o.g. let \( C_i^1 \) be such cluster), can be a subset of \( R_i \cup E(R_i) \), all other clusters \( C_i^j \) contain some rows not in \( R_i \cup E(R_i) \) (as well as some rows in \( R_i \cup E(R_i) \) by construction), therefore the cost of each row of \( C_i^2, \ldots, C_i^3 \) is \( 8 \). Move all the rows of \( C_i^2, \ldots, C_i^3 \) to the filler cluster to obtain a solution whose cost is not larger than that of the original solution. At the same time the resulting solution has exactly two clusters containing some rows of \( R_i \). Repeating the process for all sets \( R_i \) completes the proof.

Hence, in what follows we assume that in any solution there are at most two sets containing rows of each set \( R_i \).

**Lemma 21.** Let \( S \) be a solution of an instance of 4-\( AP(8) \) associated with an instance of MVCC. Then it is possible to compute in polynomial time a solution \( S' \), whose cost is not larger than that of \( S \), such that the filler cluster \( F_c \) of \( S' \) consists of all free rows and some (possibly zero) edge rows. Moreover in \( S' \) there are at most two clusters containing rows of \( R_i \).

**Proof.** Consider a generic set \( R_i \). By Lemma 20 we already know that there are at most two clusters of \( S \) containing some rows of \( R_i \), assume initially that there exists only one cluster of \( S \) containing some rows of \( R_i \). If such cluster is the filler cluster, then remove all rows of \( R_i \) from the filler cluster and make \( R_i \) a cluster of \( S' \). In the resulting solution none of the rows of \( R_i \) are in the filler cluster.

Consider now the case that there are exactly two clusters \( C_1, C_2 \) of \( S \) containing some rows of \( R_i \). If one of those clusters (say \( C_2 \)) is the filler cluster \( F_c \), then move all rows of \( R_i \) that are in \( F_c \) to \( C_1 \), obtaining two clusters \( C_1 \cup R_i \) and \( F_c - R_i \). Notice that before moving the rows of \( C_2 \) to \( C_1 \), at least one even position not in \( b(R_i) \) is deleted for each row in \( C_2 \). As each row moved from \( C_2 \) to \( C_1 \) differs from any other rows of \( C_1 \) in at most two not yet deleted entries, and all the entries of block \( b(R_i) \) are equal for the rows in \( R_i \cup E(R_i) \), it follows that this change does not increase the cost of the solution.

Now we are ready to prove Lemma 22.

**Lemma 22.** Let \( S \) be a solution of an instance of 4-\( AP(8) \) associated with an instance of MVCC. Then it is possible to compute in polynomial time a canonical solution \( S' \) with cost not larger than that of \( S \).

**Proof.** Consider a generic set of rows \( R_i \). By Lemma 21 no row of \( R_i \) belongs to the filler cluster. Therefore, if \( R_i \) is neither red nor black in \( S \), then the rows of \( R_i \) can be partitioned in \( S \) in one of the following two ways: (i) a cluster \( C_1 \) contains three rows of \( R_i \) and a row
of $E(R_i)$, while $C_2$ contains two rows of $R_i$ and two rows of $E(R_i)$, or (ii) a cluster $C$ of $S$ contains all rows of $R_i$ and some rows of $E(R_i)$.

In the first case, replace $C_1$ and $C_2$ with two clusters $C'_1$, $C'_2$, where $C'_1$ consists of 4 rows of $R_i$ and $C'_2$ consists of a row of $R_i$ and all rows of $E(R_i)$ (it is immediate to notice that $C'_1$, $C'_2$ have cost 12 and 24 respectively, while $C_1$ and $C_2$ have both cost 24).

In the second case let $X$ be the set $C \cap E(R_i)$, replace $C$ with cluster $C' = R_i$ and move all rows in $X$ to the filler cluster. Let $x = |X|$, then the cost of $C$ in $S$ is $6(5 + x)$, while the cost of $C'$ and $X$ in the new solution is equal to $3 \cdot 5 + 8 \cdot x$. Since $x \leq 3$, the cost of the new solution is strictly smaller than that of $S$. \hfill \Box

Notice that, given a canonical solution $S$, each red set $R_i$ in $S$ has a cost of 15, each black set $R_i$ in $S$ has a cost of 36 (that is a cost of 12 associated with the rows of $R_i$ and a cost of 24 associated with 3 edge rows in the black solution of $R_i$), and the filler cluster $F_c$ has cost $8|F_c|$. Now, it is easy to see that Lemma 23 holds.

**Lemma 23.** Let $S$ be a canonical solution with $k$ red sets $R_i$ of an instance of 4-AP(8) associated with an instance of MVCC. Then $S$ has cost $12(|V| - k) + 15k + 8|E| + 32$.

Now, we can show that the sets of rows $R_i$ that are red in a canonical solution $S$ corresponds to a cover of the graph $G$.

**Lemma 24.** Let $S$ be a canonical solution of cost $12(|V| - k) + 15k + 8|E| + 32$ of an instance of 4-AP(8) associated with an instance of MVCC. Then it can be computed in polynomial time a vertex cover of $G$ of size $k$.

**Proof.** Since $S$ is a canonical solution of 4-AP(8) of cost $12(|V| - k) + 15k + 8|E|$, then all the sets $R_i$ must be associated with either a red or a black solution. Furthermore, since all the edge rows have a cost of 8 in $S$, then there must exist $k$ sets $R_i$ associated with a red solution, and $|V| - k$ sets associated with a black solution.

Notice that, given two black sets $R_i$ and $R_j$, there cannot be an edge between two vertices $v_i$ and $v_j$ of $G$ associated with $R_i$ and $R_j$, by definition of black solution. Hence, the set of vertices associated with black sets of $S$ is an independent set of $G$, which in turn implies that the vertices associated with red sets are a vertex cover of $G$. \hfill \Box

**Theorem 25.** The 4-AP(8) problem is APX-hard.

**Proof.** Let $C$ be a vertex cover of graph $G$. Then, it is easy to see that a canonical solution $S$ of the instance of 4-AP(8) associated with $G$ such that $S$ has cost at most $12|V| + 3|C| + 8|E| + 32$ can be computed in polynomial time by defining a black solution for each set $R_i$ associated with a vertex $v_i \in V \setminus C$, a red solution for each set $R_i$ associated with a vertex $v_i \in C$, and assigning all the remaining rows to the filler cluster $F_c$.

On the other side, by Lemma 24, starting from a canonical solution of 4-AP(8) with size $12(|V| - k) + 15k + 8|E| + 32$, we can compute in polynomial time a cover of size $k$ for $G$. Since the cost of a canonical solution of 4-AP(8) and the size of a vertex cover of the graph $G$ are linearly related, the reduction is an L-reduction, thus completing the proof. \hfill \Box
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