THE $\mu$-PERMANENT REVISITED

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Abstract. Let $A = (a_{ij})$ be an $n$-by-$n$ matrix. For any real number $\mu$, we define the polynomial

$$P_\mu(A) = \sum_{\sigma \in S_n} a_{1\sigma(1)} \cdots a_{n\sigma(n)} \mu^{\ell(\sigma)},$$

as the $\mu$-permanent of $A$, where $\ell(\sigma)$ is the number of inversions of the permutation $\sigma$ in the symmetric group $S_n$. In this note, we review several less known results of the $\mu$-permanent, recalling some of its interesting properties. Some determinantal conjectures are considered and extended to that polynomial. A correction to a previous note is presented as well.

1. Introduction

Given an $n \times n$ matrix $A = (a_{ij})$ and a real number $\mu$, we define the $\mu$-permanent of $A$ as the polynomial

$$P_\mu(A) = \sum_{\sigma \in S_n} \left( \prod_{i=1}^n a_{i\sigma(i)} \right) \mu^{\ell(\sigma)},$$

where $\ell(\sigma)$ is the number of inversions of the permutation $\sigma$ in the symmetric group $S_n$ of degree $n$, i.e., the number of interchanges of consecutive elements necessary to arrange $\sigma$ in its natural order [16, p.1], i.e.,

$$\ell(\sigma) = \#\{(i, j) \in \{1, \ldots, n\}^2 \mid i < j \text{ and } \sigma(i) > \sigma(j)\}.$$

For example,

$$P_\mu \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} + a_{12}a_{21}\mu$$

and

$$P_\mu \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{11}a_{23}a_{32}\mu + a_{12}a_{21}a_{33}\mu + a_{13}a_{22}a_{31}\mu^3.$$

The $\mu$-permanent of a square matrix seems fairly trivial to manipulate, but it always leads into lengthy and tedious calculations, and it is often notoriously hard to compute [4, p.190].

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Among the linear algebra community, this notion was first introduced in 1991 by Ravindra Bapat [2] as a possible interpolation of the determinant ($\mu = -1$) and the permanent ($\mu = 1$). Bapat et al. also called it later as the $q$-permanent of $A$ \[3, 10\]. Actually, this has become the most common designation for it. However, as we will see, by the same time, this notion emerged independently in other fields of mathematics: Grassmann algebras and quantum groups. In any case, it should not be confused with the $\alpha$-determinant defined by D. Vere-Jones [24], where $\ell(\sigma)$ is $n$ minus the number of cycles in $\sigma \in S_n$.

In this note, we clarify this notion presenting some less-known basic results. We correct a typo and some consequences in our previous paper [8]. We also discuss some conjectures on determinants and permanents for the $\mu$-permanent and prove them for particular families of matrices. A more recent conjecture for the $\mu$-permanent of a tridiagonal matrix is also considered.

2. The $q$-determinant

In June, 1989, Yang submitted a note [25] where he defined the $q$-determinant of a matrix $A$, $\det_q(A)$, in the same way as in (1.1), where the entries are in a certain commutative ring. No particular previous motivation was given. The main aim was to extend the analysis of the determinant to $q$-Grassmann algebras. Briefly, a $q$-Grassmann algebra is the associative $K[q]$-algebra, where $K$ is a field of characteristic 0, generated by $x_1, \ldots, x_n$, satisfying the relations $x_i^2 = 0$ and $x_ix_j = qx_jx_i$, if $i < j$. A Grassmann algebra is also known as exterior algebra.

Coincidentally - or not - exactly at the same time, Noumi, Yamada, and Mimachi, led by representations of the quantum groups, announced in [18] the same definition where $q$ is replaced by $-q$. Here the quantum determinant is defined over a $C$-algebra where the canonical generators satisfied certain relations, containing the relations studied by Yang. Later on, the same authors found several useful properties in [17], namely the Laplace expansion. To the best of our knowledge, this is the first time the concept of $\mu$-permanent was defined. Its motivation is clearly independent from the one of Bapat and Lal (cf. [2–4, 10, 11]).

Probably the main result one can find in [25] is Theorem 1(4), where the Laplace expansion through the first/last row/column is stated. Namely,

$$\det_q(A) = \sum_{j=1}^{n} q^{j-1}a_{1j} \det_q(A_{1j}) = \sum_{j=n}^{1} q^{n-j}a_{nj} \det_q(A_{1j})$$

$$= \sum_{i=1}^{n} q^{i-1}a_{i1} \det_q(A_{i1}) = \sum_{i=n}^{1} q^{n-i}a_{in} \det_q(A_{in}),$$

where $A_{ij}$ is the $(n - 1) \times (n - 1)$ submatrix obtained from deleting the $i$th row and the $j$th column of $A$. This observation is very important to avoid confusion in the future. Moreover, the original definition of $A_{ij}$ as the "$(i, j)$-minor of $A$" in [25] Theorem 1(4) is not very accurate.

Shortly after, Tagawa [23] considered the same concept for commutative rings. Basically, the definition was the same as (1.1). To be more precise, we have $\det_{-\mu}(A) = P_{\mu}(A)$. This author [22] designated exactly $\det_{-\mu}(A)$ by $q$-permanent denoting it by $\text{per}_q(A)$. Furthermore, Tagawa extended even more the concept defining the multivariable quantum determinant. For a sequence $q = (q_1, q_2, \ldots, q_n)$, the multivariable quantum determinant,
or simply \( q \)-determinant, of \( A \) is defined by

\[
\det_q = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{\sigma(i)} (-q_i)^{\ell_i(\sigma)},
\]

where \( \ell_i(\sigma) \) is the number of inversions of \( \sigma \) at \( i \). For example, we have

\[
\det_q \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}q_1
\]

and

\[
\det_q \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}q_2 - a_{12}a_{21}a_{33}q_1 + a_{12}a_{23}a_{31}q_2 + a_{13}a_{21}a_{32}q_1^2 - a_{13}a_{22}a_{31}q_1^2 q_2.
\]

Clearly, from the definition, the determinant and the permanent as well as the \( q \)-determinant are particular cases of the \( q \)-determinant. For the \( q \)-determinant, the multilinearities with respect to the rows and the columns are valid as in the case of the ordinary determinant. This allows us to state the same for the \( q \)-determinant or the \( \mu \)-permanent. However, in general, \( \det_q(A^T) \neq \det_q(A) \), but \( \det_q(A^T) = \det_q(A) \) is always true. These observations can be found in \[22\,\[23\,\[25\].

We find in \[23\] the first properties for the \( q \)-determinant as particular cases of the corresponding relation involving the \( q \)-determinant and later extended in contemporary papers \[17\,\[22\]. We will translate next some of them to the \( \mu \)-permanent. If \( A = (a_{ij}) \) is an \( n \)-by-\( n \) matrix, we define the \((i, j)\)-\( q \)-complementary matrix \( A_{ij}(q) \) of \( A \) by

\[
A_{ij}(q) = \begin{pmatrix} a_{11} & \cdots & a_{1,j-1} & qa_{1,j+1} & \cdots & qa_{1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & qa_{i-1,j+1} & \cdots & qa_{i-1,n} \\ qa_{i+1,1} & \cdots & qa_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ qa_{n,1} & \cdots & qa_{n,j-1} & a_{n,j+1} & \cdots & a_{n,n} \end{pmatrix}.
\]

Then, from \[23\] Lemma 2.3, we have

\[
P_\mu(A_{ij}(\mu)) = P_\mu \begin{pmatrix} a_{1,1} & \cdots & a_{1,j-1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n,1} & \cdots & a_{n,j-1} & 0 & \cdots & a_{n,n} \end{pmatrix}.
\]

Therefore we have the Laplace expansion (cf. \[22\] Proposition 1.2.2 or \[23\] Proposition 2.2)

\[
P_\mu(A) = \sum_{j=1}^{n} a_{ij} P_\mu(A_{ij}(\mu)) = \sum_{i=1}^{n} a_{ij} P_\mu(A_{ij}(\mu)).
\]

Observe that \[2.2\] coincides with the corollary of Proposition 1.1 in \[17\], due to the fundamental relations (1.1) of the generators of the \( \mathbb{C} \)-algebra in that paper.
3. Corrections of previous results

It is clear that, under similarity, the \( \mu \)-permanent does not keep the same value, in general, i.e., the polynomial \( P_\mu(A) \) is not necessarily the same as \( P_\mu(BAB^{-1}) \), for \( B \) nonsingular. In particular, for permutation similarity, this means that interchanging rows and columns of the same indices leads to possible different \( \mu \)-permanents. Indeed, if \( P(\tau) \) is the permutation matrix \( (\delta_{i,\tau(i)}) \), then

\[
P_\mu(P(\tau)^{-1}AP(\tau)) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)} \mu^{\ell(\tau\sigma^{-1})}.
\]

In the note \[8\], we first analyzed the \( \mu \)-permanent of a matrix in terms of the underlying graph of the matrix, with special focus on the (symmetric) matrices whose graph is a tree. Interchanging rows and columns does not change the the underlying graph (or digraph) of the matrices involved. However the labeling of the vertices changes. Saying that, some corrections are needed.

First we need to clarify a notion in \[8\] which is misleading. When we have a digraph \( D \), in the sense of (2.1), \( D \setminus i \) is obtained by deleting all arcs including \( i \), but not exactly the vertex \( i \) as it was stated. This is implicit, for example, in the discussion of the example in Section 5. The existence of isolated vertices do not interfere in any result. So, for example, in the definition of a tree, we have a connected acyclic graph with eventually isolated vertices.

Next, we need an additional condition on the labeling of all graphs: given two disjoint edges \( ij \) and \( k\ell \) (say \( i < j, k < \ell, \) and \( i < k \)), then one of the conditions is satisfied:

(i) \( i < j < k < \ell \)
(ii) \( i < k < \ell < j \).

Clearly, for any tree, such labeling is always possible. For example, the results are valid for any matrices whose graph is

For paths, we have for example

For more general trees,
It is important now to clarify that Corollary 3.3, Lemma 4.1, and Theorem 4.3 in [8] are valid whenever the edges of the underlying graph of the matrix satisfy either (i) or (ii).

Finally, a large number of computational experiments for several families of graphs lead us to conjecture (cf. Conjecture [2]) that $P_\mu(A)$ is strictly increasing for $\mu \in [-1, \infty)$, assuming that $A$ is any positive definite, extending [8, Theorem 4.3].

4. A conjecture over the positive definiteness

We have seen that the $\mu$-permanent is a parametric generalization of both the determinant and the permanent, making $\mu = -1$ and $\mu = 1$, respectively. Note also that $P_0(A) = a_{11} \cdots a_{nn}$.

The Schur power matrix of $A$, denoted by $\Pi(A)$ [15, 20], is the $n! \times n!$ matrix whose rows and columns are indexed by $S_n$, where the $(\sigma, \tau)$-entry is

$$\prod_{i=1}^{n} a_{\sigma(i)\tau(i)};$$

$\Gamma_\mu$ is a matrix of same type with the $(\sigma, \tau)$-entry defined as

$$\mu^{(\tau\sigma^{-1})}.$$

For more results and several open problems on $\Gamma_\mu$ the reader is referred to [6, 21, 26]. If we set

$$\Pi_\mu(A) = \Pi(A) \circ \Gamma_\mu,$$

where $\circ$ denotes the Hadamard product, then we have

$$(4.1) \quad P_\mu(A) = \frac{1}{n!} \langle \Pi_\mu(A)1, 1 \rangle,$$

where $1$ denotes the column vector of all ones.

Bożejko and Speicher [5] proved that $\Gamma_\mu$ is positive semidefinite, for $\mu \in [-1, 1]$, and it is known that if $A$ is positive semidefinite matrix, then $\Pi(A)$ is also positive semidefinite [3]. Therefore, from (4.1), we may state:

**Lemma 4.1.** [2,3] For any Hermitian positive semidefinite matrix $A$,

$$P_\mu(A) \geq 0, \quad \text{if} \quad \mu \in [-1, 1].$$

Remark that, although the assumption of the complex (semi)definite matrix being Hermitian is unnecessary, we will include it, since we often consider real matrices, and in that case the symmetry assumption should be incorporated.

Bapat and Lal also established several conjectures in [2,3]. One of them is the following:

**Conjecture 1.** [2,3] Given an $n \times n$ Hermitian positive definite nondiagonal matrix $A$, $P_\mu(A)$ is a strictly increasing function of $\mu \in [-1, 1]$. 
This conjecture was motivated by the classical Hadamard inequality and the permanental analogue proved by Marcus [13]. It has been proved for \( n \leq 3 \) in [2], for tridiagonal positive definite matrices in [11], and in [8] for symmetric positive definite matrices under the graph labeling discussed in the previous section.

More recently, Conjecture 1 was extended [7].

**Conjecture 2.** [7] For a given matrix \( A > 0 \), there exists \( \epsilon < -1 \) such that \( P_\mu(A) \) is a strictly increasing function of \( \mu \in (\epsilon, +\infty) \).

Using orthogonal polynomials and chain sequences the author proved Conjecture 2 for tridiagonal matrices [7]. In this section we prove this conjecture for other classes of matrices. We start with the case of a 3 \( \times \) 3 positive definite matrix

\[
A = \begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{12} & a_{22} & a_{23} \\
a_{13} & a_{23} & a_{33}
\end{pmatrix}.
\]

From (1.1), we have

\[
P_\mu(A) = a_{11}a_{22}a_{33} + a_{11}a_{23}^2\mu + a_{13}^2a_{33}\mu + 2a_{12}a_{13}a_{23}\mu^2 + a_{13}^2a_{22}\mu^3.
\]

Therefore

\[
\frac{d}{d\mu} P_\mu(A) = a_{11}a_{23}^2 + a_{13}^2a_{33} + 4a_{12}a_{13}a_{23}\mu + 3a_{13}a_{22}\mu^2
\]

is always positive, which implies \( P_\mu(A) \) is a strictly increasing function. Since \( A \) is positive definite, \( P_{-1}(A) > 0 \) and, consequently, the only zero of \( P_\mu(A) \) is less than \(-1\). Moreover

\[
P_\mu(A) > 0, \quad \text{for any } \mu \in (-1, \infty).
\]

A similar conclusion can be reached, for example, for matrices whose graph is a star, with the central vertex labeled with 1, i.e.,

\[
A = \begin{pmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
a_{12} & a_{22} & 0 & \cdots & 0 \\
a_{13} & 0 & a_{33} & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 \\
a_{1n} & 0 & \cdots & 0 & a_{nn}
\end{pmatrix}.
\]

From [8, Corollary 3.3]

\[
P_\mu(A) = a_{11}a_{22} \cdots a_{nn} + \sum_{k=2}^{n} a_{1k}^2 \left( \prod_{1 \neq i \neq k}^{n} a_{ii} \right) \mu^{2k-3}.
\]

Hence

\[
\frac{d}{d\mu} P_\mu(A) = \sum_{k=2}^{n} (2k-3)a_{1k}^2 \left( \prod_{1 \neq i \neq k}^{n} a_{ii} \right) \mu^{2k-4}
\]

which is always positive, for any \( \mu \).
5. SOME INEQUALITIES

The next open problem is based on the analogue of Lieb’s inequality (cf. [12]) to $\mu$-permanental forms.

**Conjecture 3.** [3] Given an $n \times n$ Hermitian positive semidefinite matrix $A$, let $S$ be a nonempty subset of $\{1, \ldots, n\}$. Then for $\mu \in [0, 1]$,

$$P_\mu(A) \geq \sum_{\sigma \in S_n \atop \sigma(T) = S} \prod_{i=1}^{n} a_{i\sigma(i)} \mu^{\ell(\sigma)} .$$

Conjecture 3 can be easily verified for matrices whose graph is a tree under the labeling previously established. In fact, since $A$ is Hermitian positive semidefinite ($n \times n$) and $\mu \in [0, 1]$, the sum (1.1) is nonnegative and we get

$$\sum_{\sigma \in S_n \atop \sigma(T) = S} \prod_{i=1}^{n} a_{i\sigma(i)} \mu^{\ell(\sigma)} - \sum_{\sigma \in S_n \atop \sigma(T) \neq S} \prod_{i=1}^{n} a_{i\sigma(i)} \mu^{\ell(\sigma)} = \sum_{\sigma \in S_n \atop \sigma(T) = S} \prod_{i=1}^{n} a_{i\sigma(i)} \mu^{\ell(\sigma)} \geq 0 .$$

Lal [11] proved the simplest version of Conjecture 3 which is in fact a generalization of Fischer’s inequality for determinants:

**Theorem 5.1.** [11] Let $A$ be an $n \times n$ Hermitian positive semidefinite matrix which is partitioned as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix}$$

where $A_{11}, A_{22}$ are Hermitian. If $\mu \in [0, 1]$, then

$$P_\mu(A) \geq P_\mu(A_{11}) P_\mu(A_{22}) .$$

Conjecture 3 is also true for $n = 3$. We only have to check the case when $S = \{2\}$. The other cases are applications of Theorem 5.1. So, we have

$$a_{11}|a_{23}|^2 + |a_{12}|^2 a_{33} + \bar{a}_{12} a_{13} a_{23} \mu + a_{12} \bar{a}_{13} a_{23} \mu \geq 0 .$$

In fact,

$$\begin{pmatrix} \bar{a}_{23} & a_{12} \end{pmatrix} \begin{pmatrix} a_{11} & a_{13} \mu \\ \bar{a}_{13} & a_{33} \end{pmatrix} \begin{pmatrix} a_{23} \\ \bar{a}_{12} \end{pmatrix} \geq 0 .$$

Bapat and Lal also extended the longstanding Soules’ conjecture [15][19][20] to a $\mu$-permanent form.

**Conjecture 4.** [3] Given an $n \times n$ Hermitian positive semidefinite matrix $A$, let $\mu \in [0, 1]$. Then the largest eigenvalue of $\Pi_\mu(A)$ is $P_\mu(A)$.

Conjecture 4 has been proved for $n \leq 3$ in [11]. For higher orders no progress has been made so far. However, for acyclic matrices, Conjecture 4 can be easily shown. In fact, since $\Pi_\mu(A)$ is a Hermitian positive semidefinite $n \times n$ matrix, all eigenvalues are real and nonnegative. The eigenvectors associated to different eigenvalues are therefore orthogonal. Note also that, since the graph of $A$ is a tree, in each column (and in each row) of $\Pi_\mu(A)$, we have all terms of $P_\mu(A)$. Moreover, each term of the sum (1.1) appears in each column (and in each row) of $\Pi_\mu(A)$ only once.

From (1.1), $u$ is an eigenvector of $\Pi_\mu(A)$ associated to the eigenvalue $P_\mu(A)$. Supposing that $u$ is an eigenvector of $\Pi_\mu(A)$ associated to the eigenvalue $\lambda(\neq P_\mu(A))$, then

$$u_1 + \cdots + u_n = 0 .$$
Let us assume that \( u_k \) is largest (positive) coordinate of \( u \). Since
\[
\Pi_\mu(A) u = \lambda u,
\]
we have
\[
\sum_{j=1}^{n} \Pi_\mu(A)_{kj} u_j = \lambda u_k,
\]
and, finally,
\[
\lambda u_k = \sum_{j=1}^{n} \Pi_\mu(A)_{kj} u_j \leq \sum_{j=1}^{n} \Pi_\mu(A)_{kj} u_k = P_\mu(A) u_k,
\]
because all entries of \( \Pi_\mu(A) \) are nonnegative. Hence
\[
\lambda \leq P_\mu(A).
\]
Thus, we may state:

**Theorem 5.2.** Given an \( n \times n \) Hermitian positive semidefinite matrix \( A \) whose graph is a tree, let \( \mu \in [0, 1] \). Then the largest eigenvalue of \( \Pi_\mu(A) \) is \( P_\mu(A) \).

Again, the previous theorem is established for trees with the labeling satisfying the discussed conditions.

### 6. Final remarks

In [3], Bapat and Lal started the analysis of a \( \mu \)-permanental analogue of Gram’s equality. Later on, Lal [10] considered the Gram’s inequality for the Schur power matrix, inequalities using induced matrices, and inequalities of Schwarz type providing generalizations to the \( \mu \)-permanent of some results by Ando [1], and by Marcus and Minc [14]. Note that some inequalities of Minkowski type presented by Ando [1] can also be extended to the \( \mu \)-permanent.

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