GEOMETRY OF THE CASSINIAN METRIC AND ITS INNER METRIC

Z. IBRAGIMOV, M. R. MOHAPATRA, AND S. K. SAHOO

Abstract. The Cassinian metric and its inner metric have been studied for subdomains of the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) \((n \geq 2)\) by the first named author. In this paper we obtain various inequalities between the Cassinian metric and other related metrics in some specific subdomains of \( \mathbb{R}^n \).

2010 Mathematics Subject Classification. 30C35, 30C20, 30F45, 51M10.

Key words and phrases. Möbius transformation, the hyperbolic metric, the Cassinian metric, the distance ratio metric, the visual angle metric, the triangular ratio metric, inner metric.

1. Introduction

One of the aspects of hyperbolic geometry deals with the comparison of the hyperbolic metric with the so-called hyperbolic-type metrics. Secondly, invariance and distortion properties of hyperbolic-type metrics under conformal maps (Möbius transformations in higher dimensions) also play significant roles in geometric function theory. In recent years, many authors have contributed to the study of hyperbolic-type metrics. Some of the familiar hyperbolic-type metrics are the quasihyperbolic metric [7, 8], the distance ratio metric [22], the Apollonian metric [3, 4, 10, 11, 12, 16, 17], the Seittenranta metric [21], the Ferrand metric [6, 13, 14], the K–P metric [14, 15, 20], the Cassinian metric [18], the visual angle metric [19], and the triangular ratio metric [5]. These metrics are also referred to as the relative metrics since they are defined in a proper subdomain of the Euclidean space \( \mathbb{R}^n, n \geq 2, \) relative to its boundary. A more general form of relative metrics has been considered by P. Hästö in [9, Lemma 6.1]. In this paper we study geometric properties of the Cassinian metric by comparing it with the hyperbolic, distance ratio, and visual angle metrics. For a quick overview on these metrics, the reader can refer to the next section. We also discuss the quasi-invariance (distortion) property of the Cassinian metric under Möbius transformations of the unit ball. Finally, we compute the inner metric of the Cassinian metric, the so-called inner Cassinian metric, in some specific subdomains of \( \mathbb{R}^n \) and study some of its basic properties.

Date: December 15, 2014.
2. Preliminaries

Throughout the paper $D$ denotes an arbitrary, proper subdomain of the Euclidean space $\mathbb{R}^n$, i.e., $D \subsetneq \mathbb{R}^n$. The Euclidean distance between $x, y \in \mathbb{R}^n$ is denoted by $|x - y|$. The standard Euclidean norm of a point $x \in \mathbb{R}^n$ is denoted by $|x|$. Given $x \in \mathbb{R}^n$ and $r > 0$, the open ball centered at $x$ and of radius $r$ is denoted by $B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$. The unit ball in $\mathbb{R}^n$ is denoted by $B^n$. Given $x \in D$, the distance $\delta_D(x)$ from $x$ to the boundary $\partial D$ of $D$ is given by

$$\delta_D(x) = \inf \{|x - \xi| : \xi \in \partial D\}.$$ 

For real numbers $r$ and $s$, we set $r \vee s = \max\{r, s\}$ and $r \wedge s = \min\{r, s\}$.

The Cassinian metric $c_D$ of the domain $D$ is defined as

$$c_D(x, y) = \sup_{p \in \partial D} \frac{|x - y|}{|x - p||p - y|}.$$ 

This metric was first introduced and studied in [18]. However, a more general form of this metric was considered by P. Hästö (see [9, Lemma 6.1]).

The distance ratio metric $j_D$ is defined by

$$j_D(x, y) = \log \left(1 + \frac{|x - y|}{\delta_D(x) \wedge \delta_D(y)}\right).$$ 

The above form of the metric $j_D$, which was first considered in [22], is a slight modification of the original distance ratio metric introduced in [7, 8]. This metric has been widely studied in the literature; see, for instance, [23].

The hyperbolic metric $\rho_{B^n}$ of the unit ball $B^n$ is given by

$$\rho_{B^n}(x, y) = \inf_{\gamma} \int_{\gamma} \frac{2|dz|}{1 - |z|^2},$$ 

where the infimum is taken over all rectifiable curves $\gamma \subset B^n$ joining $x$ and $y$.

The visual angle metric $v_D$, introduced in [19], is defined by

$$v_D(x, y) = \sup\{\angle(x, z, y) : z \in \partial D\}.$$ 

We also consider the quantity $p_D$,

$$p_D(x, y) = \frac{|x - y|}{\sqrt{|x - y|^2 + 4\delta_D(x)\delta_D(y)}}.$$ 

Note that the quantity $p_D$, which was first considered in [5], does not define a metric (see [5, Remark 3.1]). However, it has a nice connection with the
The hyperbolic metric, $p_{\mathbb{H}^2}$, of the upper half-plane $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$. Namely,

$$p_{\mathbb{H}^2}(z_1, z_2) = \tanh \frac{\rho_{\mathbb{H}^2}(z_1, z_2)}{2} = \frac{|z_1 - z_2|}{|z_1 - \bar{z}_2|}, \quad z_1, z_2 \in \mathbb{H}^2,$$

where $\bar{z}_2$ is the reflection of $z_2$ with respect to the real line $\mathbb{R}$ (see [5]). Hence it is natural to ask whether the quantity $p_D$ is comparable with hyperbolic-type metrics such as, the Cassinian metric $c_D$, in more general domains $D$.

### 3. Comparison of the Cassinian metric with other related quantities

This section is devoted to finding upper and lower bounds for the Cassinian metric in terms of the quantities, defined in Section 2, in some specific domains. We begin with the comparison of the Cassinian and hyperbolic metrics of the unit ball $\mathbb{B}^n$. Recall that

$$\tanh \left( \frac{\rho_{\mathbb{B}^n}(x, y)}{2} \right) = \frac{|x - y|}{\sqrt{|x - y|^2 + (1 - |x|^2)(1 - |y|^2)}}.$$ 

(See, for example, [2, p. 40].) One can also show that

$$\frac{|x - y|}{\sqrt{|x - y|^2 + (1 - |x|^2)(1 - |y|^2)}} = \frac{|x - y|}{|x||x^* - y|} = \frac{|x - y|}{|y||y^* - x|},$$

for all $x, y \in \mathbb{B}^n \setminus \{0\}$, where

$$x^* = \frac{x}{|x|^2} \quad \text{and} \quad y^* = \frac{y}{|y|^2}.$$

Consequently,

$$(3.1) \quad \tanh \left( \frac{\rho_{\mathbb{B}^n}(x, y)}{2} \right) = \frac{|x - y|}{|x||x^* - y|}$$

for all $x, y \in \mathbb{B}^n \setminus \{0\}$.

**Theorem 3.1.** For $x, y \in \mathbb{B}^n$, we have

$$\tanh \left( \frac{\rho_{\mathbb{B}^n}(x, y)}{2} \right) \leq 2c_{\mathbb{B}^n}(x, y).$$

**Proof.** Assume first that $x \neq 0$ and $y \neq 0$. Denote by $I := \inf_{z \in \partial \mathbb{B}^n} |x - z||z - y|$. Comparing the Cassinian metric with (3.1), we see that it is enough to show

$$I \leq 2|x||x^* - y|.$$
Without loss of generality we can assume that \(|y| \leq |x|\). Then \(|y| = s|x|\) for some \(s \in [0,1]\). Let \(\gamma \in [0,\pi)\) be the angle between the line segments joining \([0,x]\) and \([0,y]\). We consider two cases.

**Case 1:** \(\gamma \geq \pi/2\). In this case, we have \(2|x||x^* - y| \geq 2|x||x^*| = 2\). Set \(z_1 = x/|x|\). Then

\[
I \leq |x - z_1||z_1 - y| \leq (1 - |x|)\sqrt{|z_1|^2 + |y|^2} - (z_1\cdot y) = (1 - |x|)\sqrt{1 + s^2|x|^2 - |z_1||y|\cos(\gamma)} \leq (1 - |x|)\sqrt{1 + s^2|x|^2 + 2s|x|} = (1 - |x|)(1 + s|x|) \leq 2 \leq 2|x||x^* - y|.
\]

**Case 2:** \(\gamma < \pi/2\). Let \(z_1 = x/|x|\) and \(z_2 = y/|y|\). Clearly, \(2|x||x^* - y| = 2||y||x - z_2|| = 2||x||y - z_1||\). Let \(z\) be a point on the unit sphere with \(\angle(z_1,0,z) = \angle(z_2,0,z) = \gamma/2\). Then \(I \leq |x - z||z - y|\). We next show that

\[
\frac{p}{r} \geq 1, \quad p = |z_2 - |y||x||, \quad r = |z - x|.
\]

By applying the properties of the right triangle \(\triangle(0,z_2,(\cos \gamma)z_1)\) and the law of cosines, we obtain

\[
p \geq |z_2 - (\cos \gamma)z_1| = \sin \gamma \geq |z - (\cos \gamma)z_1|.
\]

If \(r < \sin(\gamma)\), then \(p/r \geq 1\). Otherwise, we have \(x \in [0,(\cos \gamma)z_1]\). Hence, the angle between \([x,z_2]\) and \([0,x]\) is greater than \(\pi/2\) so that \(p = |z_2 - |y||x|| > |z_2 - x| \geq |z - x|\). Similarly, we can show that \(|z - y| \leq |z_1 - |x||y||. Therefore,

\[
I \leq |z_2 - |y||x||z_1 - |x||y|| = (|x||x^* - y|) = (|x||x^* - y|)^2 \leq |x||x^* - y| \quad (\because |x||x^* - y| \leq 1)
\]

Finally, we assume that either \(x = 0\) or \(y = 0\), say \(x = 0\). Then

\[
\tanh\left(\frac{\rho_B^n(0,y)}{2}\right) = \tanh\left(\frac{1}{2}\log\left(\frac{1 + |y|}{1 - |y|}\right)\right) = |y|.
\]

On the other hand,

\[
c_B^n(0,y) = \sup_{\xi \in \partial B^n} \frac{|y|}{|\xi||\xi - y|} \geq \frac{|y|}{|\xi_y||\xi_y - y|} \geq \frac{|y|}{1 + |y|} \geq |y|/2,
\]

where \(\xi_y = y/|y|\). Consequently,

\[
\tanh\left(\frac{\rho_B^n(0,y)}{2}\right) \leq 2c_B^n(0,y)
\]

completing the proof. \(\square\)

Next, we compare the Cassinian and the distance ratio metrics of the unit ball.
Theorem 3.2. The following are true.

(i) For all \( x, y \in \mathbb{B}^n \), we have
\[
j_{\mathbb{B}^n}(x, y) \leq 2c_{\mathbb{B}^n}(x, y).
\]

(ii) For all \( x, y \in \mathbb{B}^n \) with \( |x| \vee |y| < \lambda < 1 \) we have
\[
c_{\mathbb{B}^n}(x, y) \leq \frac{1 + 2\lambda}{(1 - \lambda)^2} j_{\mathbb{B}^n}(x, y).
\]

Proof. To prove (i), let \( x, y \in \mathbb{B}^n \). Without loss of generality we can assume that \( \delta_{\mathbb{B}^n}(x) \leq \delta_{\mathbb{B}^n}(y) \). Let \( x_0 \) be the point on the boundary \( \partial \mathbb{B}^n \) with \( \delta_{\mathbb{B}^n}(x) = |x - x_0| \).

Then \( |y - x_0| \leq 2 \). Now,
\[
c_{\mathbb{B}^n}(x, y) = \sup_{p \in \partial \mathbb{B}^n} \frac{|x - y|}{|x - p||p - y|} \geq \frac{|x - y|}{|x - x_0||x_0 - y|} \geq \frac{|x - y|}{2\delta_{\mathbb{B}^n}(x)}.
\]

On the other hand,
\[
j_{\mathbb{B}^n}(x, y) = \log \left( 1 + \frac{|x - y|}{\delta_{\mathbb{B}^n}(x)} \right) \leq \frac{|x - y|}{\delta_{\mathbb{B}^n}(x)}.
\]

Hence \( j_{\mathbb{B}^n}(x, y) \leq 2c_{\mathbb{B}^n}(x, y) \), as required.

We next prove (ii). Since \( |x| \vee |y| < \lambda \), it is evident that
\[
|x - y| \leq 2\lambda.
\]

Moreover, for any \( w \in \partial \mathbb{B}^n \), we have
\[
|x - w||w - y| \geq (1 - \lambda)^2.
\]

Combining the above two inequalities, we get
\[(3.4) \quad (1 - \lambda)^2 c_{\mathbb{B}^n}(x, y) \leq |x - y| \leq 2\lambda.
\]

Now,
\[
j_{\mathbb{B}^n}(x, y) = \log \left( 1 + \frac{|x - y|}{\delta_{\mathbb{B}^n}(x) \wedge \delta_{\mathbb{B}^n}(y)} \right) \geq \log \left( 1 + \frac{|x - y|}{(1 + \lambda)} \right)
\]
\[
\geq \frac{2|x - y|}{(1 + \lambda)} \left( \log(1 + t) \geq \frac{2t}{2 + t} \text{ for } t > 0 \right)
\]
\[
= \frac{|x - y|}{(1 + \lambda) + \frac{|x - y|}{2}} \geq \frac{|x - y|}{(1 + 2\lambda)} \geq \frac{(1 - \lambda)^2}{(1 + 2\lambda)} c_{\mathbb{B}^n}(x, y),
\]

where the last two inequalities follow from (3.4). \( \Box \)
The next lemma describes the relations between the distance ratio metric and the visual metric of the unit ball.

**Lemma 3.3.** [5, Theorem 3.9]

1. For \( x, y \in \mathbb{B}^2 \) we have
   \[ v_{\mathbb{B}^2}(x, y) \leq 2 j_{\mathbb{B}^2}(x, y). \]
2. For \( x, y \in \mathbb{B}^2 \) with \( |x| \vee |y| < \lambda < 1 \) we have
   \[ j_{\mathbb{B}^2}(x, y) \leq \frac{2(3 + \lambda^2)}{3(1 - \lambda^2)} v_{\mathbb{B}^2}(x, y). \]

As an immediate corollary of Lemma 3.3 and Theorem 3.2 we obtain

**Corollary 3.4.** The following inequalities hold.

1. For \( x, y \in \mathbb{B}^2 \) we have
   \[ v_{\mathbb{B}^2}(x, y) \leq 4 c_{\mathbb{B}^2}(x, y). \]
2. For \( x, y \in \mathbb{B}^2 \) with \( |x| \vee |y| < \lambda < 1 \) we have
   \[ c_{\mathbb{B}^2}(x, y) \leq \frac{2(3 + \lambda^2)(1 + 2\lambda)}{3(1 - \lambda^2)(1 - \lambda)} v_{\mathbb{B}^2}(x, y). \]

Next, we compare the Cassinian metric \( c_D \) with the quantity \( p_D \).

**Theorem 3.5.** Let \( x, y \in D \subseteq \mathbb{R}^n \). Then
\[ p_D(x, y) \leq \sqrt{2} \left( \delta_D(x) \wedge \delta_D(y) \right) c_D(x, y). \]

**Proof.** Fix \( x, y \in D \) and let \( s = \delta_D(x) \wedge \delta_D(y) \). Then
\[
\begin{align*}
p_D(x, y) & = \frac{|x - y|}{\sqrt{|x - y|^2 + 4\delta_D(x)\delta_D(y)}} \leq \frac{|x - y|}{\sqrt{|x - y|^2 + (2s)^2}} \\
& \leq \frac{\sqrt{2}|x - y|}{|x - y| + 2s} \leq \frac{\sqrt{2}|x - y|}{|x - y| + s} \\
& = \sqrt{2} \left( \delta_D(x) \wedge \delta_D(y) \right) \frac{|x - y|}{\delta_D(x) \wedge \delta_D(y) \left( |x - y| + \delta_D(x) \wedge \delta_D(y) \right)} \\
& = \sqrt{2} \left( \delta_D(x) \wedge \delta_D(y) \right) \left[ \frac{|x - y|}{\delta_D(x) (|x - y| + \delta_D(x))} \vee \frac{|x - y|}{\delta_D(y) (|x - y| + \delta_D(y))} \right] \\
& \leq \sqrt{2} \left( \delta_D(x) \wedge \delta_D(y) \right) c_D(x, y),
\end{align*}
\]
where the second inequality follows from [11, 1.58 (13)] and the last inequality follows from [18, Lemma 3.4]. \( \square \)
Remark 3.6. Observe that if we take the domain $D$ in Theorem 3.2 to be the unit ball $\mathbb{B}^n$, then we can see that $p_D(x, y) \leq \sqrt{2}c_D(x, y)$. In fact, if $D$ is a bounded domain in $\mathbb{R}^n$, then $p_D(x, y) \leq (\text{diam}(D)/\sqrt{2})c_D(x, y)$.

4. Distortion of the Cassinian metric under Möbius transformations of the unit ball

In this section we study distortion properties of the Cassinian metric $c_{\mathbb{B}^n}$ of the unit ball $\mathbb{B}^n$ under Möbius transformations of $\mathbb{B}^n$. Note that the Möbius transformations of $\mathbb{B}^n$ preserve the hyperbolic metric $\rho_{\mathbb{B}^n}$.

Let $\phi$ be a Möbius transformation with $\phi(\mathbb{B}^n) = \mathbb{B}^n$ and put $a = \phi(0)$. If $a = 0$, then $\phi$ is an orthogonal matrix, i.e., $|\phi(x)| = |x|$ for each $x \in \mathbb{B}^n$. In particular, $\phi$ preserves the Cassinian metric. That is,

$$c_{\mathbb{B}^n}(\phi(x), \phi(y)) = c_{\mathbb{B}^n}(x, y) \quad \text{for all} \quad x, y \in \mathbb{B}^n.$$ (4.1)

Suppose now that $a \neq 0$. Let $\sigma$ be the inversion in the sphere $S^{n-1}(a^*, r)$, where

$$a^* = \frac{a}{|a|^2} \quad \text{and} \quad r = \sqrt{|a^*|^2 - 1} = \frac{\sqrt{1 - |a|^2}}{|a|}.$$ Note that the sphere $S^{n-1}(a^*, r)$ is orthogonal to $\partial \mathbb{B}^n$ and that $\sigma(a) = 0$. In particular, $\sigma$ is a Möbius transformation with $\sigma(\mathbb{B}^n) = \mathbb{B}^n$ and $\sigma(a) = 0$. Recall that

$$\sigma(x) = a^* + \left(\frac{r}{|x - a^*|}\right)^2 (x - a^*).$$

Then $\sigma \circ \phi$ is an orthogonal matrix (see, for example, [2, Theorem 3.5.1(i)]). In particular,

$$\left|\sigma(\phi(x)) - \sigma(\phi(y))\right| = |x - y|. \quad \text{(4.2)}$$

We will need the following property of $\sigma$ (see, for example, [2, p. 26]):

$$|\sigma(x) - \sigma(y)| = \frac{r^2|x - y|}{|x - a^*||y - a^*|}. \quad \text{(4.3)}$$

It follows from (4.2) and (4.3) that

$$|x - y| = \left|\sigma(\phi(x)) - \sigma(\phi(y))\right| = \frac{r^2|\phi(x) - \phi(y)|}{|\phi(x) - a^*||\phi(y) - a^*|} = \frac{|a^*|^2 - 1)|\phi(x) - \phi(y)||\phi(x) - a^*||\phi(y) - a^*|},$$

or equivalently,

$$|\phi(x) - \phi(y)| = \frac{|\phi(x) - a^*||\phi(y) - a^*|}{|a^*|^2 - 1}|x - y|. $$
In particular, for all \( x, y \in \mathbb{B}^n \) and \( \eta \in \partial \mathbb{B}^n \) we have
\[
\frac{|\phi(x) - \phi(y)|}{|\phi(x) - \phi(\eta)||\phi(y) - \phi(\eta)|} = \frac{|x - y|}{|x - \eta||y - \eta|} \cdot \frac{|a^*|^2 - 1}{|\phi(\eta) - a^*|^2}.
\]
Note that since \( \phi(\eta) \in \partial \mathbb{B}^n \) and \( |a^*| > 1 \), we have
\[
|a^*| - 1 \leq |\phi(\eta) - a^*| \leq |a^*| + 1
\]
and hence
\[
\frac{1 - |a|}{1 + |a|} = \frac{|a^*| - 1}{|a^*| + 1} \leq \frac{|a^*|^2 - 1}{|\phi(\eta) - a^*|^2} \leq \frac{|a^*| + 1}{|a^*| - 1} = \frac{1 + |a|}{1 - |a|}.
\]

Now given \( x, y \in \mathbb{B}^n \), there exist \( \eta_1 \in \partial \mathbb{B}^n \) and \( \eta_2 \in \partial \mathbb{B}^n \) such that
\[
c_{\mathbb{B}^n}(\phi(x), \phi(y)) = \frac{|\phi(x) - \phi(y)|}{|\phi(x) - \phi(\eta_1)||\phi(y) - \phi(\eta_1)|} \quad \text{and} \quad c_{\mathbb{B}^n}(x, y) = \frac{|x - y|}{|x - \eta_2||y - \eta_2|}.
\]
Using (4.4) and (4.5) we obtain
\[
c_{\mathbb{B}^n}(\phi(x), \phi(y)) = \frac{|x - y|}{|x - \eta_1||y - \eta_1|} \cdot \frac{|a^*|^2 - 1}{|\phi(\eta_1) - a^*|^2} \leq \frac{1 + |a|}{1 - |a|} c_{\mathbb{B}^n}(x, y)
\]
and
\[
c_{\mathbb{B}^n}(x, y) = \frac{|\phi(x) - \phi(y)|}{|\phi(x) - \phi(\eta_2)||\phi(y) - \phi(\eta_2)|} \cdot \frac{|\phi(\eta_2) - a^*|^2}{|a^*|^2 - 1} \leq \frac{1 + |a|}{1 - |a|} c_{\mathbb{B}^n}(\phi(x), \phi(y)).
\]

Thus, we have proved the following theorem.

**Theorem 4.1.** Let \( \phi \) be a Möbius transformation with \( \phi(\mathbb{B}^n) = \mathbb{B}^n \). Then
\[
\frac{1 - |\phi(0)|}{1 + |\phi(0)|} c_{\mathbb{B}^n}(x, y) \leq c_{\mathbb{B}^n}(\phi(x), \phi(y)) \leq \frac{1 + |\phi(0)|}{1 - |\phi(0)|} c_{\mathbb{B}^n}(x, y)
\]
for all \( x, y \in \mathbb{B}^n \).

5. **The inner Cassinian metric**

Let \( D \subseteq \mathbb{R}^n \) and \( \gamma \) be a rectifiable curve in \( D \). We define the Cassinian length of \( \gamma \) as
\[
c_D(\gamma) = \sup \left\{ \sum_{i=0}^{n-1} c_D(\gamma(t_i), \gamma(t_{i+1})) \right\}
\]
where the supremum is taken over all partitions \( (t_i)_{i=1}^n \) of \( I = [a, b] \) with \( t_1 = a \) and \( t_n = b \). Then the inner Cassinian metric is defined as
\[
\tilde{c}_D(x, y) = \inf_{\gamma} c_D(\gamma) = \inf_{\gamma} \int_{\gamma} \frac{|dz|}{(\delta_D(z))^2},
\]
where the infimum is taken over all rectifiable curves $\gamma \subset D$ connecting $x$ and $y$ (see [18]). First, we establish the monotonicity property of the inner Cassinian metric.

Lemma 5.1. The inner Cassinian metric is monotonic with respect to domains. That is, if $D \subset D'$, then $\tilde{c}_{D'}(x, y) \leq \tilde{c}_D(x, y)$ for all $x, y \in D$.

Proof. Given $x, y \in D$, we have
\[
\tilde{c}_D(x, y) = \inf_{\gamma} c_D(\gamma),
\]
where the infimum is taken over all rectifiable curves $\gamma \subset D$ connecting $x$ and $y$. Since the Cassinian metric is monotonic ([18, Corollary 3.2]), $c_D(\gamma) \geq c_{D'}(\gamma)$ for all such $\gamma$ and, consequently,
\[
\inf_{\gamma} c_D(\gamma) \geq \inf_{\gamma} c_{D'}(\gamma).
\]
Since each such $\gamma$ also connects $x$ and $y$ in $D'$, we have
\[
\tilde{c}_{D'}(x, y) = \inf_{\gamma} c_{D'}(\gamma) \leq \inf_{\gamma} c_D(\gamma) = \tilde{c}_D(x, y),
\]
completing the proof.

Next, we compute the inner Cassinian metrics in some special cases.

Example 5.2. For the punctured space $D = \mathbb{R}^n \setminus \{0\}$, the inner Cassinian metric $\tilde{c}_D$ is same as the Cassinian metric $c_D$ and is given by the formula
\[
\tilde{c}_D(x, y) = c_D(x, y) = \frac{|x - y|}{|x||y|}.
\]
To see this, let $f(\xi) = \xi/|\xi|^2$ be the inversion about the unit sphere $S^{n-1}(0, 1) = \{\xi \in \mathbb{R}^n : |\xi| = 1\}$. Then $f(D) = D$ and that $f$ is an isometry between $(D, c_D)$ and $(D, |\cdot|)$, where $|\cdot|$ is the Euclidean distance in $D$ (see, [18, Example 3.9(A)]). Since the inner metric of the Euclidean metric in $D$ is the same as the Euclidean metric itself and since $(D, c_D)$ is isometric to $(D, |\cdot|)$, we conclude that $(D, \tilde{c}_D)$ is isometric to $(D, |\cdot|)$. Hence $\tilde{c}_D$ is same as the Cassinian metric $c_D$. In particular, it follows from [2, (3.1.5)] that
\[
\tilde{c}_D(x, y) = c_D(x, y) = |f(x) - f(y)| = \frac{|x - y|}{|x||y|}
\]
for all $x, y \in D$, as required.

Example 5.3. For each $x \in \mathbb{B}^n$, we have
\[
\tilde{c}_{\mathbb{B}^n}(0, x) = c_{\mathbb{B}^n}(0, x) = \frac{|x|}{1 - |x|}.
\]
Indeed, it follows from [18, Example 3.9(B)] that
\[ c_{B^n}(0,x) = \frac{|0-x|}{|1-0||1-x|} = \frac{|x|}{1-|x|} \quad \text{for all } x \in B^n. \]
Also, it follows from [18, Theorem 3.8] that the line segment \([0,x]\) is a Cassinian geodesic so that its Cassinian length is equal to \(c_{B^n}(0,x)\). That is,
\[ c_{B^n}([0,x]) = c_{B^n}(0,x) = \frac{|x|}{1-|x|}. \]
Therefore,
\[ c_{B^n}(0,x) \leq \tilde{c}_{B^n}(0,x) = \inf_{\gamma} c_{B^n}(\gamma) \leq c_{B^n}([0,x]). \]
Hence \(\tilde{c}_{B^n}(0,x) = c_{B^n}(0,x)\), as required.

The following corollary is an easy consequence of Lemma 5.1 and Example 5.3.

**Corollary 5.4.** Given \(x \in D \setminus \{\infty\}\), we have
\[ \tilde{c}_D(x,y) \leq \frac{|x-y|}{\delta_D(x)(\delta_D(x) - |x-y|)} \]
for all \(y \in D\) with \(|x-y| < \delta_D(x)\).

**Proof.** Set \(B = B(x, \delta_D(x))\). Then as in Example 5.3 we obtain
\[ \tilde{c}_B(x,y) = \frac{|x-y|}{\delta_D(x)(\delta_D(x) - |x-y|)} \]
for any \(y \in B\). Now the conclusion follows from Lemma 5.1. \(\Box\)

**References**

[1] G. D. Anderson, M. K. Vamanamurthy, and M. K. Vuorinen, *Conformal Invariants, Inequalities, and Quasiconformal Maps*, John Wiley & Sons, Inc., 1997.

[2] A. F. Beardon, *Geometry of discrete groups*, Springer-Verlag New York, 1995.

[3] A. F. Beardon, *The Apollonian metric of a domain in \(\mathbb{R}^n\)*. In: P. Duren, J. Heinonen, B. Osgood, and B. Palka (Eds.) *Quasiconformal mappings and analysis* (Ann Arbor, MI, 1995), pp. 91–108. Springer-Verlag, New York, 1998.

[4] M. Borovikova and Z. Ibragimov, *Convex bodies of constant width and the Apollonian metric*, Bulletin, Malaysian Math. Sci. Soc. 31 (2) (2008), 1–12.

[5] J. Chen, P. Hariri, R. Klén and M. Vuorinen, *Lipschitz conditions, Triangular ratio metric and Quasiconformal maps*, arXiv:1403.6582v1.

[6] J. Ferrand, *A characterization of quasiconformal mappings by the behavior of a function of three points*. In: I. Laine, S. Rickman, and T. Sorvali (Eds.) *Proceedings of the 13th Rolf Nevanlinna Colloquium* (Joensuu, 1987), pp. 110–123. Lecture Notes in Mathematics Vol. 1351, Springer-Verlag, New York, 1988.

[7] F. W. Gehring and B. G. Osgood, *Uniform domains and quasihyperbolic metric*, J. Anal. Math., 36 (1979), 50–74.
 GEOMETRY OF THE CASSINIAN METRIC AND ITS INNER METRIC 11

[8] F. W. Gehring and B. P. Palka, *Quasiconformally homogeneous domains*, J. Anal. Math., 30 (1976), 172–199.

[9] P. Hästö, *A new weighted metric, the relative metric I*, J. Math. Anal. Appl., 274 (2002), 38–58.

[10] P. Hästö, *The Apollonian metric: uniformity and quasiconvexity*, Ann. Acad. Sci. Fenn. Math., 28 (2003), 385–414.

[11] P. Hästö and Z. Ibragimov, *Apollonian isometries of plane domains are Möbius mappings*, Journal of Geometric Analysis, 15 (2) (2005), 229–237.

[12] P. Hästö and Z. Ibragimov, *Apollonian isometries of regular domains are Möbius mappings*, Ann. Acad. Sci. Fenn., Ser. A I Math., 32 (1) (2007), 83–98.

[13] P. Hästö, Z. Ibragimov and H. Linden, *Isometries of relative metrics*, Computational Methods and Function Theory, 6 (1) (2006), 15–28.

[14] P. Hästö, Z. Ibragimov, D. Minda, S. Ponnusamy and S. K. Sahoo, *Isometries of some hyperbolic-type path metrics, and the hyperbolic medial axis*, In the tradition of Ahlfors-Bers, IV, Contemporary Math., 432 (2007), 63–74.

[15] D. Herron, Z. Ibragimov and D. Minda, *Geodesics and curvature of Möbius invariant metrics*, Rocky Mountain J. of Math., 38 (3) (2008), 891–921.

[16] Z. Ibragimov, *On the Apollonian metric of domains in \(\mathbb{R}^n\)*, Complex Var. Theory Appl., 48 (2003), 837–855.

[17] Z. Ibragimov, *Conformality of the Apollonian metric*, Computational Methods and Function Theory, 3 (2) (2003), 397–411.

[18] Z. Ibragimov, *The Cassinian metric of a domain in \(\mathbb{R}^n\)*, Uzbek Math. Journal, 1 (2009), 53–67.

[19] R. Klén, H. Linden, M. Vuorinen and G. Wang, *The visual angle metric and Möbius transformations*, Comput. Meth. Funct. Theory (to appear), 1208.2871math.MG

[20] R. Kulkarni and U. Pinkall, *A canonical metric for Möbius structures and its applications*, Math. Z., 216 (1994), 89–129.

[21] P. Seittenranta, *Möbius-invariant metrics*, Math. Proc. Cambridge Philos. Soc., 125 (1999), 511–533.

[22] M. Vuorinen, *Conformal invariants and quasiregular mappings*, J. Anal. Math., 45 (1985), 69–115.

[23] M. Vuorinen, *Conformal geometry and quasiregular mappings*, Lecture note in mathematics, Springer-Verlag, Berlin, 1988.

ZAIR IBRAГIMOV, DEPARTMENT OF MATHEMATICS, CALIFORNIA STATE UNIVERSITY, FULLERTON, CA, 92834

E-mail address: zibragimov@fullerton.edu

MANAS RANJAN MOHAPATRA, DISCIPLINE OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY INDORE, INDORE 452 017, INDIA

E-mail address: mrm.iiti@gmail.com

SWADESH KUMAR SAHHOO, DISCIPLINE OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY INDORE, INDORE 452 017, INDIA

E-mail address: swadesh@iiti.ac.in