Holographic equation of state in fluid/gravity duality

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Abstract

We establish a precise relation between mixed boundary conditions for scalar fields in asymptotically anti de Sitter spacetimes and the equation of state of the dual fluid. We provide a detailed derivation of the relation in the case of five bulk-dimensions for scalar fields saturating the Breitenlohner-Freedman bound. As a concrete example, we discuss the five dimensional scalar-tensor theories describing a constant speed of sound.

It is now widely accepted that there is a precise correspondence between observables in a \((D−1)\)-dimensional gauge field theory and a \(D\)-dimensional gravity theory. Indeed, since this duality was precisely formulated by Maldacena [1], there has been an increasing amount of conceptual understanding of its meaning. In particular, the long wavelength regime of the duality, known as the fluid/gravity correspondence, has attracted much attention, e.g. [2–5]. It follows from the fact that a relativistic form of the Navier-Stokes equations governing the hydrodynamic limit of a field theory in \((D−1)\) dimensions, on a fixed background \(\gamma_{ab}\), is equivalent to the dynamics of \(D\)-dimensional Einstein gravity with a negative cosmological constant with \(\gamma_{ab}\) as its conformal boundary.

This correspondence allows one to pick a fluid dynamical solution, with an equation of state dictated by the tracelessness of the boundary energy momentum tensor, and reconstruct a bulk solution of the full Einstein equations (for a review see [6]). This opened the possibility of modeling fluid dynamics using general relativity. For instance the elusive description of turbulence has been considered within the fluid/gravity duality, and it has been proposed that gravitational dynamics can become turbulent when its dual fluid is at large Reynolds number [7–9]. This has led to the definition of a “gravitational Reynolds number” constructed in terms of the black hole quasinormal modes [10].

The problem with this description is that, in asymptotically anti de Sitter (AdS) spacetimes with a flat boundary, it is intrinsically limited to traceless energy momentum tensors and so the actual fluid is a very particular one. For understanding realistic fluids by using the fluid/gravity duality, one must be able to describe their dynamics by a general equation of state that is experimentally determined. The holographic relation between the real world systems, which are not conformally invariant in the ultraviolet (UV), and gravity, requires that the conformal symmetry in the boundary should be broken.
The main goal of this letter is to propose a concrete relation between the equation of state of a (non-conformal) fluid and the asymptotic fall-off behaviour of a scalar field in the AdS bulk. We treat the case of a single scalar field with mass $m^2 = -4l^{-2}$, which is the mass of some of the scalars of type IIB supergravity on $AdS_5 \times S^5$. This case is also interesting because the mass saturates the Breitenlohner-Freedman (BF) bound in five dimensions [11,12] and so the logarithmic branches exist [13–15]. We find that, in general, the coefficients of the leading terms in the asymptotic expansion of a scalar field in AdS gravity determine the relationship between the pressure and density of a perfect fluid on the conformal boundary.

This can be traced back to the existence, in any dimension, of two normalizable modes for scalar fields with masses $m^2$ in the window [16]

$$-\frac{(D-1)^2}{4l^2} = m^2_{BF} \leq m^2 < m^2_{BF} + l^{-2}. \quad (1)$$

From the existence of two normalizable modes it follows that these theories admit mixed boundary conditions; most of them break the conformal invariance at the boundary. This implies that the dual energy momentum tensor is not traceless [17,18] and so the hydrodynamic limit of the field theory is described by a non-conformal fluid. We are going to obtain a general holographic equation of state for a time dependent scalar field using a counterterm method similar in spirit with the one in [18] (the work of [17] is based on the Hamilton-Jacobi equation). It follows that our results are easily generalizable to any dimension and theories with scalars satisfying (1).

Interest in the holographic descriptions of arbitrary equations of state emerged naturally some years after the AdS/CFT correspondence was proposed. However, the proposals are restricted to the idea that one scalar field potential allows the description of one equation of state [19–23]. The main result of this paper is that every scalar field potential describes an infinite number of equations of state. Namely, given a single scalar field potential, every boundary condition that provides a regular black hole corresponds to an equation of state. Indeed, when the scalar field mass is in the BF window (1) it provides two integration constants ($\alpha, \beta$) to the system. A static metric provides one extra integration constant, $\mu$. So, asymptotically, the solution space of the fully back-reacted metric plus the scalar field is characterized by three integration constants [24]. Requiring the existence of a regular horizon implies that the bulk configuration is completely determined by two parameters. Integrating the system from the horizon to infinity constrains the boundary data such that a codimension-1 surface contains black hole solutions. Schematically, the solution of the non-linear system of differential equations is the map

$$\alpha = \alpha(\phi_h, A), \quad \beta = \beta(\phi_h, A), \quad \mu = \mu(\phi_h, A), \quad (2)$$

where $(\phi_h, A)$ are the horizon data, namely the value of the scalar field at the horizon, $\phi_h$, and the normalized black hole area $A$, and $(\alpha, \beta, \mu)$ are the boundary data.

Indeed, a classical field theory is well defined when the field equations and boundary conditions are provided, which correspond to a relation of the form $\beta = \beta(\alpha)$. Therefore, if one is given a theory defined by a Lagrangian and some boundary condition, knowledge of (2) provides an immediate answer to the question on the existence of black holes for the given boundary condition: there will be a black hole whenever the boundary condition intersects the surface defined by (2). This intersection is a line. It can be characterized by a single parameter that corresponds to the unique integration constant associated with the static black hole. It turns out that when the boundary condition is fixed the map (2) can be written as

$$\mu = \mu(\alpha) \quad \beta = \beta(\alpha). \quad (3)$$

1The normalized area is the total area divided by the unit-radius area. This will save us some factors in the body of the paper.
We shall see how the regularity condition (3) is the central object that connects the gravitational description with the equation of state of the holographic fluid.

Our conventions are defined by the action principle

$$I [g, \phi] = \int_M d^5 x \sqrt{-g} \left( \frac{R}{2\kappa} - \frac{1}{2} (\partial \phi)^2 - V(\phi) \right) + \frac{1}{\kappa} \int_{\partial M} K \sqrt{-h} + I_{ct},$$

where $\kappa = 8\pi G$ is the reduced Newton constant in five dimensions. The potential $V(\phi)$ is required to have at least one local maximum, where it attains a negative value, so that asymptotically AdS solutions exist. The counterterms $I_{ct}$ are known from well-established results and render the action principle finite [26,27] and well defined for mixed boundary conditions on the scalar field [18]. We are interested in describing a timelike boundary and so the induced metric on $\partial M$ is $h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$. The extrinsic curvature of the surface with metric $h_{\mu\nu}$ is $2K_{\mu\nu} = \nabla_\alpha n_\mu \nabla_\beta n_\nu$ where $n_\mu$ is the outwards-pointing normal and $K = h^{\mu\nu} K_{\mu\nu}$. We use below the Einstein equations as defined by the relation $G_{\mu\nu} = \kappa T_{\mu\nu}$, where $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left[ \frac{1}{2} (\partial \phi)^2 + V(\phi) \right]$. Working in units where the speed of light and Planck’s constant are set to 1, we consider the class of metrics

$$ds^2 = e^A ( - f dt^2 + d\Sigma_k ) + e^B dr^2 ,$$

where $d\Sigma_k$ is a constant Ricci scalar space, $R(\Sigma_k) = 6k$, with $k = \pm 1$ or 0, and volume $\sigma_k$. The boundary metric is $h_{\mu\nu} dx^a dx^b = e^A ( - f dt^2 + d\Sigma_k )$, and the field theory dual metric, which is related by a conformal transformation to $h_{\mu\nu}$, is

$$\gamma_{ab} dx^a dx^b = -dt^2 + l^2 d\Sigma_k .$$

The counterterms $I_{ct}$ are constructed so that the action principle is well-posed and to obtain a finite action. The quasilocal formalism of Brown and York [25] provides a concrete way to compute the action and stress tensor, from which one can directly obtain the mass of the system.

Let us quickly see how one can gain knowledge of the map (2). For static metrics one can introduce the variables $Z = \frac{d\phi}{dA} \equiv \dot{A}^{-1}$ and $Y = \left( \dot{A} f \right)^{-1} f$. When $k = 0$ it is straightforward to see that the Einstein equations are

$$\dot{Z} = \frac{3\dot{V} + 4\kappa Z V}{12 \kappa V Z Y} \left( 2\kappa Z^2 Y - 6Y - 3Z^2 \right), \quad \dot{Y} = \frac{3\dot{V} + 2\kappa Z V}{6 \kappa V Z^2} \left( 2\kappa Z^2 Y - 6Y - 3Z^2 \right).$$

It follows that $\dot{Z}$ is finite at the horizon, located at $Y(\phi_h) = 0$, if and only if $Z(\phi_h) = -3\dot{Y}(\phi_h)$. Furthermore, we readily see that these equations decouple and reduce to the single master equation

$$-3 \left( 3\dot{V} + 4\kappa Z V \right) Z \dot{Z} + \left( -9\dot{V} + 12\kappa ZV \right) \dot{Z}^2 + \left[ 8\kappa^2 Z^2V + \kappa \left( 24ZV + 18Z^2\dot{V} \right) + 18\dot{V} + 9Z\dot{V} \right] \dot{Z} = 0 .$$

The regularity condition imply that $Z$ is completely determined by the value of $\phi$ at the horizon, and so

$$A(r) = \int_{\phi_h}^{\phi(r)} \frac{d\phi}{Z} + \frac{2}{3} \ln \left( A \right) ,$$

which follows from the definition of $Z$ and the value of $A(r_h)$. When the scalar field saturates the BF bound, $m^2 = -\frac{1}{r^2}$ its fall-off is

$$\phi = \alpha \frac{\ln(r)}{r^2} + \beta \frac{1}{r^2} + O\left( \frac{\ln(r)}{r^3} \right) .$$
It is a consequence of the fall-off of the scalar (10) that, asymptotically, \( e^A \phi \sim \alpha \ln(r) \). The derivative of this relation with respect to \( \ln r \sim \frac{1}{2} \) yields \( 2e^A \left( \phi + \frac{d\phi}{dr} \right) \sim \alpha \). Thus,

\[
\alpha (\phi_h, A) = \lim_{\phi \to 0} 2(Z + \phi) e^A = \alpha_h (\phi_h) \mathcal{A}^2.
\]

(11)

We see that \( \alpha \) is generically a function of \( \phi_h \) times a very precise function of the normalized black hole area. What is remarkable here is that, since the map (2) is coming from the complete integration of Einstein equations, one might have expected that the boundary data is a very complicated function of the horizon data, but that is not the case. It follows from the straightforward derivation of (11) that the same sort of relation exists for scalar fields with masses in the BF window (1).

To construct the dual energy momentum tensor we need to identify the total energy of the system. We shall use then the Regge-Teitiboim approach [28, 29]. We emphasize below that the coefficients \( \alpha, \beta \) and \( \mu \) can be generalized from integration constant to be time-dependent. The calculation can be done in full generality with all the boundary coordinate dependence however, for the sake of simplicity, we restrict it to time dependence. When the metric matches (locally) AdS at infinity the relevant fall-off is

\[
g_{tt} = \frac{r^2}{l^2} + k - \frac{\mu(t)}{r^2} + O(r^{-3}) \quad , \quad g_{ij} = r^2 \Sigma_{ij} + O(r^{-3}) ,
\]

(12)

\[
g_{rr} = \frac{l^2}{r^2} - \frac{t^4 k}{r^4} + \frac{1}{3} \frac{M_h(t)l^2 + 3k^2 r^4 + \kappa \alpha(t) (\alpha(t) - 4 \beta(t)) \ln(r) - 2 \kappa \alpha(t)^2 \ln(r)^2}{r^6} + O \left( \frac{\ln(r)^2}{r^7} \right) ,
\]

(13)

where \( \Sigma_{ij} \) is the metric associated with the “angular” part, \( d\Sigma_k \). Inserting these expansions in the Einstein-scalar field equations we find that the boundary conditions (12)-(13) are compatible with the field equations provided

\[
M_h(t) = \mu(t) - \frac{\kappa l^{-2}}{12} \left( \alpha(t)^2 - 4 \alpha(t) \beta(t) + 8 \beta(t)^2 \right) .
\]

(14)

Using the fall-off of the metric and scalar field we obtain

\[
\delta H = \left[ \frac{3 \delta M_h(t)}{2 \kappa} - \frac{1}{l^2} (\alpha(t) \delta \beta(t) - 2 \beta(t) \delta \beta(t)) \right] \sigma_k ,
\]

(15)

and so the Hamiltonian is finite\(^2\). To remove the variations from these equations we need to impose boundary conditions on the scalar field. If we write \( \beta = \frac{dW(\alpha)}{d\alpha} \) then the right-hand side of (15) is a total variation and using the field equations (14) it yields

\[
H = \left[ \frac{3 \mu(t)}{2 \kappa} + \frac{1}{l^2} \left( -\frac{1}{8} \alpha(t)^2 - \frac{1}{2} \alpha(t) \beta(t) + W(\alpha) \right) \right] \sigma_k + H_h ,
\]

(17)

where \( \mu(t) \) is the \( O(r^{-2}) \) coefficient of the \( g_{tt} \) and \( \delta H_h = 0 \).

It was originally pointed out in [16] that the evolution of scalar fields in AdS is well defined for Robin boundary conditions for scalar fields with masses that satisfy \( m_{BF}^2 \leq m^2 < m_{BF}^2 + l^{-2} \) where

\[^2\text{The gravitational and scalar contributions to the Hamiltonian are universal when given in terms of its variations}
\]

\[
\delta H = \delta Q_G + \delta Q_\phi ,
\]

(16)

and the concrete expressions can be found in [28] — exact solutions were studied in [30–32].
\( m_{BF}^2 \) is the Breitenlohner-Freedman bound, \( m_{BF}^2 = -\frac{4}{l^2} \) [12]. Indeed, it is possible to find this kind of formula in a number of places in the literature [33]. What is new here is that we have taken one order more in the fall off of \( g_{tt} \), namely the \( \mu/r^2 \) term, and shown how it connects with the standard definition of mass given in terms of the coefficient of \( O(r^{-2}) \) of \( g_{rr}^{-1} \) (see, also, [34]).

Due to integration over an infinite volume, the action suffers from infrared divergences that can be regulated by adding suitable boundary terms. With this in mind, the action can be naturally divided into the bulk part, the usual Gibbons-Hawking boundary term, the Balasubramanian-Kraus counterterm, an extrinsic scalar field counterterm, \( I^\phi_{ext} \), and an intrinsic scalar field counterterm, \( I^\phi_{ct} \):

\[
I = I_B + I_{GH} + I_{BK} + I^\phi_{ext} + I^\phi_{ct} .
\] (18)

The boundary conditions (10), (12), (13) and the field equations imply that it is possible to introduce the following counterterms that provide the correct result for the free energy

\[
I^\phi_{ext} = \frac{1}{2} \int_{\partial M} d^4x \sqrt{-h} n^\mu \delta \phi \partial_\mu \phi , \quad I^\phi_{ct} = \frac{1}{l^2} \int_{\partial M_r} d^4x \sqrt{-\gamma} \left[ \frac{\alpha \beta}{2} - W(\alpha) \right] ,
\]

where we have used the metric \( \gamma_{ab} \) of the dual field theory description. When the field equations hold, the variation of the total action (18) vanishes for Dirichlet boundary conditions for the metric and for scalar field boundary conditions of the form \( \beta(t) = \frac{dW}{d\alpha} \), namely

\[
\lim_{r \to \infty} \delta I = 0 .
\] (19)

Let us clarify this further for the scalar field. From (18) we obtain

\[
\delta I = \int_M -d^5x \partial_\mu \left( \sqrt{-g} g^{\mu \nu} \delta \phi \partial_\nu \phi \right) + \frac{1}{2} \int_{\partial M} d^4x \sqrt{-h} n^\mu \delta \phi \partial_\mu \phi + \frac{1}{2} \int_{\partial M} d^4x \sqrt{-h} n^\mu \phi \partial_\mu \delta \phi + \frac{1}{l^2} \int_{\partial M_r} d^4x \sqrt{-\gamma} \left( -\frac{\beta(t)}{2} + \frac{\alpha(t)}{2} \frac{d^2W}{d\alpha^2} \right) \delta \alpha(t) ,
\] (20)

when the field equations hold. Using

\[
\phi = \frac{\alpha(t) \ln(r)}{r^2} + \frac{\beta(t)}{r^2} + O\left( \frac{\ln(r)}{r^3} \right) \Rightarrow \delta \phi = \left( \ln(r) + \frac{d^2W}{d\alpha^2} \right) \frac{\delta \alpha(t)}{r^2} + O\left( \frac{\ln(r)}{r^3} \right) ,
\] (21)

and employing (10), (12), (13) and (14), it is straightforward to show that (19) indeed holds.

There is one remaining ambiguity in (18), namely that of adding finite counterterms quadratic in the Riemann tensor, Ricci tensor and Ricci scalar of the boundary metric. This is related to the regularization of the field theory dual as discussed in [26].

**Holographic Smarr formula and equation of state**

The expectation value of the dual energy-momentum tensor is related to the quasilocal stress tensor (including the counterterms):

\[
\langle T_{ab} \rangle = -\frac{2}{\sqrt{-\gamma}} \frac{\delta I}{\delta \gamma_{ab}} = \lim_{r \to \infty} \frac{r^2}{l^2} T_{BK}^{\mu \nu} + \lim_{r \to \infty} \frac{r^2}{l^2} T_{ext}^{\mu \nu} + T_{ct}^{\mu \nu} ,
\] (22)

where the first term is the Balasubramanian-Kraus part [26]

\[
T_{BK}^{\mu \nu} = -\frac{1}{\kappa} \left( K_{\mu \nu} - h_{\mu \nu} K + \frac{3}{l} h_{\mu \nu} - \frac{l}{2} G_{\mu \nu} \right) ,
\] (23)
with $\mathcal{G}_{\mu\nu}$ the Einstein tensor constructed with the metric $h$. The second term and the third term are the extrinsic scalar field term and the finite contribution contributions introduced in this paper:

\[
\mathcal{T}^\text{ext}_{\mu\nu} = \frac{1}{2} h_{\mu\nu} \phi \partial_\mu \phi , \quad \mathcal{T}^\text{ct}_{ab} = \frac{1}{l^3} \gamma_{ab} \left[ \frac{\alpha (t) \beta (t)}{2} - W (\alpha) \right]. \tag{24}
\]

The relevant divergence coming from the bulk and Gibbons-Hawking contributions is canceled out by the divergence from the counterterm and we obtain the following regularized stress tensor of the dual field theory:

\[
\langle \mathcal{T}_{ab} \rangle = \frac{\gamma_{ab}}{l^3} \left[ - \frac{3 M_h (t)}{2 \kappa} + \frac{k^2 l^2}{8 \kappa} + \frac{2 \mu (t)}{\kappa} + \frac{1}{l^2} \left( \alpha (t) \beta (t) - \beta (t)^2 - W (\alpha) \right) \right] \\
+ \frac{1}{l^3} \beta_0^a \beta_0^b \left[ \frac{k^2}{2} + \frac{2 \mu (t)}{l^2} \right]. \tag{25}
\]

Taking the trace and using the field equations (14), we obtain

\[
\rho \langle \mathcal{T}_{ab} \rangle = \frac{1}{l^3} \left[ \frac{3 M_h (t)}{2 \kappa} + \frac{3 k^2 l^2}{8 \kappa} - \frac{1}{l^2} \left( \alpha (t) \beta (t) - \beta (t)^2 - W (\alpha) \right) \right]. \tag{26}
\]

which vanishes for the AdS invariant boundary conditions. Using the normalized timelike vector $u^a = \partial_t$, the energy density of the fluid is

\[
\rho = u^a u^b \langle \mathcal{T}_{ab} \rangle = \frac{1}{l^3} \left[ \frac{3 M_h (t)}{2 \kappa} + \frac{3 k^2 l^2}{8 \kappa} - \frac{1}{l^2} \left( \alpha (t) \beta (t) - \beta (t)^2 - W (\alpha) \right) \right]. \tag{27}
\]

The total mass is the energy density integrated on a spacelike section

\[
M = \int_{\Sigma} \rho^a d^4 \Sigma = \left[ \frac{3 M_h (t)}{2 \kappa} + \frac{3 k^2 l^2}{8 \kappa} - \frac{1}{l^2} \left( \alpha (t) \beta (t) + W (\alpha) + \beta (t)^2 \right) \right] \sigma_k = \Sigma_k = H, \tag{27}
\]

where the last equality is to remark that this result is in agreement with the Hamiltonian with $H_h = \frac{3 k^2 l^2}{8 \kappa}$. The counterterm computation also provides the Casimir energy of the large N limit of $\mathcal{N} = 4$ Super Yang-Mills theory — a cross check of our computation is its exact agreement with the original paper of Balasubramanian-Kraus when the scalar field vanishes [26].

The introduction of the scalar field yields a dual perfect fluid with energy momentum tensor $\langle \mathcal{T}_{ab} \rangle = (\rho + p) u_a u_b + \rho g_{ab}$. Hence, we can identify

\[
\rho = \frac{1}{l^3} \left[ \frac{3 \mu (t)}{2 \kappa} + \frac{3 k^2 l^2}{8 \kappa} - \frac{1}{l^2} \left( \alpha (t)^2 + 4 \alpha (t) \beta (t) - 8 W (\alpha) \right) \right] \tag{28}
\]

\[
\rho = \frac{1}{l^3} \left[ \frac{3 \mu (t)}{2 \kappa} + \frac{3 k^2 l^2}{8 \kappa} - \frac{1}{l^2} \left( \alpha (t)^2 + 4 \alpha (t) \beta (t) - 8 W (\alpha) \right) \right] \tag{29}
\]

where we have used the relation (14). Note that when there is no scalar field we get a thermal gas of massless particles $\rho = 3 p$ [35]. When $k = 0$, an interesting implication of (28) and (29) is that the entropy density, $s = \frac{A}{4 \alpha}$, and temperature $T$ of a perfect fluid is defined by the relation

\[
T s = l^3 (\rho + p) = \frac{2 \mu (t)}{\kappa}, \tag{30}
\]

\[\text{footnote}{^3}\text{The } l^3 \text{ factor is due to our definition of the dual metric (6).}\]
which exactly coincides with the generalized Smarr formula of [24]. Our calculation shows that the same formula holds when the gravitational configuration is time dependent. When $k = 0$, the temperature of the configuration has the form $T = \frac{8G}{kT} \mu_h (\phi_h, A)^\frac{1}{2}$. Using (30) we find

$$\mu (\phi_h, A) = \frac{\mu_h (\phi_h) A^\frac{1}{2}}{l^2}. \quad (31)$$

Inserting (2) in the first law of black hole thermodynamics, $\delta H = T \delta S$, with the knowledge of (31) and (11) shows that the terms proportional $\delta A$ cancel, provided that

$$\beta (\phi_h, A) = -\frac{1}{2} \alpha \ln (\alpha) - \frac{1}{2} \alpha \beta_h (\phi_h) \quad . \quad (32)$$

Moreover, the terms proportional to the variation of $\delta \phi_h$ cancel if and only if

$$6d\mu_h + \kappa \alpha_h^2 d\beta_h = 0. \quad (33)$$

By the implicit function theorem we can take the independent variable to be $\alpha_h$ and hence reduce the problem to that of finding only one function, for instance $\beta_h$ obtaining then $\mu_h$ by a direct integration of (33). Finally, note that fixing the boundary condition implies a functional relation between $\phi_h$ and $A$. This is equivalent to say that the black hole is characterized by a single integration constant.

A useful consequence of the general considerations made so far, it that there are two $A$–independent functions

$$\frac{\beta}{\alpha} + \frac{1}{2} \ln (\alpha), \quad \frac{\mu}{\alpha^2}. \quad (34)$$

Whenever there is a hairy black hole with AdS invariant conditions at $\phi_h = \phi_s$ then the functions $(\beta_h, \alpha_h, \mu_h)$ admit a Taylor expansion around $\phi_h = \phi_s$. It follows then that one can obtain the generic form of the surface of existence of hairy black holes around a given regular point

$$\frac{\beta}{\alpha} + \frac{1}{2} \ln (\alpha) = -\frac{1}{2} \beta_s + \frac{C}{\kappa} \left( \frac{\mu_s^2}{\alpha^2} - \frac{\mu_s^2}{\alpha_s^2} \right) + O \left( \frac{\mu_s^2}{\alpha^4} \right), \quad (35)$$

where it was used that exists a hairy black hole with AdS invariant boundary conditions at $(\beta_h, \alpha_h, \mu_h) = (\beta_s, \alpha_s, \mu_s)$ and $C$ is a constant that depends on the theory. When higher order corrections are neglected, insertion of equation (35) in (33) yields

$$\mu_h = \mu_s \left( \frac{\alpha_h}{\alpha_s} \right)^\frac{2C}{\alpha_s^2 + C}, \quad \beta_h = \beta_s - \frac{2C}{\kappa} \mu_s^2 \left( \frac{\alpha_h}{\alpha_s} \right)^\frac{6}{\alpha_s^2 + C} - 1 \right) . \quad (36)$$

upon using (32), the integration constant being fixed by requiring $\alpha_h = \alpha_s \implies \mu_h = \mu_s$.

We have now the tools to further analyze the connection between infrared regularity and the equation of state. When the boundary conditions are fixed, $\beta = \frac{dW(\alpha)}{d\alpha}$, the density and pressure are specified by the choice of $W(\alpha)$. This is tantamount to defining an equation of state. Conversely, specification of an equation of state $p = p(\rho)$ necessarily determines $W(\alpha)$ from equations (23) and (29). Indeed, we need the function $\mu(\alpha)$ to determine the exact boundary condition associated to a given equation of state of the dual fluid. However, we will keep the discussion general and treat a simple case.

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The numerical integration of the field equations done in [33] shows that for a scalar field belonging to a consistent truncation of a type IIB supergravity Lagrangian it is possible to obtain black holes with AdS invariant boundary conditions for any finite constant value of $\beta_h$. 

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For instance, we find that the equation of state \( p = c_s^2 \rho \) where \( c_s^2 \) is the (constant) speed of sound squared, is equivalent to the following one-parameter family of boundary conditions

\[
W(\alpha) = \frac{l^2}{\kappa (c_s^2 + 1)} \left( \alpha^2 \omega(\alpha) - \frac{k^2 l^2}{8} \right) - \frac{\alpha^2}{4} \left( \ln(\alpha) + \alpha_1 - \frac{1}{2} \right),
\]

where we have parameterized \( \mu = \alpha^3 \frac{d\omega(\alpha)}{d\alpha} \) and \( \alpha_1 \) is an integration constant. We readily see that when \( c_s^2 = \frac{1}{3} \) we recover the description of the gas of massless particles and the AdS invariant boundary conditions. Now, let us assume that there is a black hole with AdS invariant boundary conditions at \( \alpha_1 = \beta_+ \). Then, we can use (35) to find \( \mu(\alpha) \) and finally to find \( \beta(\alpha) : \)

\[
\beta(\alpha) = \alpha \left[ \left( - \frac{1}{2} + \frac{3\mu_+ (3c_s^2 - 1)}{2\kappa \alpha_+^2} \right) \ln(\alpha) - \frac{\beta_+}{2} - \frac{\mu_+ (3c_s^2 - 1)(C - 3)}{4\kappa \alpha_+^2} \right] + O \left( (3c_s^2 - 1)^2 \right)
\]

where we linearize around \( c_s^2 = \frac{1}{3} \) to be consistent with the fact that we have neglected \( O \left( \frac{\mu^2}{\alpha^4} \right) \) in (35). The dependence of the boundary condition on \( C \) shows that for any theory defined by the surface one can find the corresponding boundary condition that yields the desired equation of state. This is valid in a neighborhood of the AdS invariant boundary condition. Hence, it is valid for equations of state that can be made arbitrarily close to the gas of massless particles.

Although we have worked in five dimensions for scalar fields saturating the Breitenlohner-Freedman bound, our results are easily generalized to any spacetime dimension for any scalar field with masses between this bound and the unitarity bound. This can be of particular use to the holographic description of metals, superconductors and different kind of materials [36]. The holographic description of condensed matter systems has recently been discussed in the hydrodynamic regime [37]. The results brought in here allow to actually introduce a detailed description of the condensed matter system through its equation of state, in the holographic picture.

The formalism introduced here has a direct application on the exact, time dependent hairy black hole solutions in Einstein-dilaton gravity with general moduli potential, recently constructed in [38–42]. Indeed, all these collapsing black holes are dual to some process in fluid/gravity with a very precise equation of state that can now be unveiled.

We have seen that contrary to the common belief, there are many dual fluid equations of state associated to one theory. This is particularly relevant for string theory. The construction of the map along the lines described in this letter for type IIB supergravity will provide an holographic description of the fluid dynamics associated to the deformations of \( \mathcal{N} = 4 \) super Yang-Mills well behaved in the infrared.

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