TWISTED N=2 SUPERGRAVITY

AS

TOPOLOGICAL GRAVITY IN FOUR DIMENSIONS

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We show that the BRST quantum version of pure D=4 N=2 supergravity can be topologically twisted, to yield a formulation of topological gravity in four dimensions. The topological BRST complex is just a rearrangement of the old BRST complex, that partly modifies the role of physical and ghost fields: indeed, the new ghost number turns out to be the sum of the old ghost number plus the internal U(1) charge. Furthermore, the action of N=2 supergravity is retrieved from topological gravity by choosing a gauge fixing that reduces the space of physical states to the space of gravitational instanton configurations, namely to self-dual spin connections. The descent equations relating the topological observables are explicitly exhibited and discussed. Ours is a first step in a programme that aims at finding the topological sector of matter coupled N=2 supergravity, viewed as the effective Lagrangian of type II superstrings and, as such, already related to 2D topological field-theories. As it stands the theory we discuss may prove useful in describing gravitational instantons moduli-spaces.
I. INTRODUCTION

Recently Topological Field Theories \[\text{[1]}\] have attracted a lot of interest, both for their own sake and in connection with string theory. Particularly interesting, because of their relation with \(N=2\) superconformal theories \[\text{[2]}\] and with Calabi-Yau moduli spaces \[\text{[3]}\] are topological theories in \(D=2\) \[\text{[4]}\]. In two dimensions the relation between \(N=2\) supersymmetry and topological field-theory is established via a topological twist that redefines a new Lorentz group \(SO(2)'\) as the diagonal of the old Lorentz group with the \(U(1)\) automorphism group of the supersymmetry algebra \[\text{[5]}\]. In particular it implies that a whole class of \(N=2\) correlation functions is topological in nature and, as such, both independent of the space-time points where the operators are localized and exactly calculable with geometrical techniques \[\text{[6]}\].

Notwithstanding the interest of the \(D=2\) case, topological theories are worth considering also in four-dimensions. Actually they were originally introduced in \(D=4\) with the discovery by Witten of topological Yang-Mills theory \[\text{[7]}\] and of its relation with the mathematical theory of Donaldson invariants \[\text{[8]}\] and with \(N=2\) super Yang-Mills theory. Indeed Witten’s original form of Topological Yang-Mills theory, which is already gauged fixed, was obtained via a suitable twist from \(D=4\), \(N=2\) Yang-Mills theory. The twist consists of the redefinition of a new Lorentz group \(SO(4)' = SU(2)_L \otimes SU(2)'_R\) where the factor \(SU(2)'_R\) is the diagonal of the old \(SU(2)_R\) with the \(SU(2)_I\) automorphism group of the supersymmetry algebra. The general BRST-approach to this theory was developped only later by Beaulieu and Singer \[\text{[9]}\], uncovering some of the subtleties hidden in Witten’s twist approach.

From the experience of this example a general lesson is anyhow learnt: just as in \(D=2\), also in \(D=4\) any \(N=2\) theory is liable to a topological twist and, as such, it should contain a topological sector where the correlation functions are independent of the space-time points and exactly calculable. In particular this should apply to \(N=2\) supergravity, whose topological twist must yield a gauge-fixed version of \(D=4\) topological gravity. It should also apply to the hypermultiplets, that are the \(D=4\) counterparts of the \(N=2\) Wess-Zumino multiplets. Actually, to state the conjecture in its most general form, the entire matter coupled \(N=2\)
supergravity, whose general form has been obtained in [10], further generalizing the results of conformal tensor calculus [11], should be liable to a topological twist and have a topological sector.

Although, the systematic programme of topologically twisting D=4, N=2 theories has not yet been carried through. In this paper we try to fill the gap beginning with pure N=2 supergravity.

Before addressing some of the technical and conceptual details of our derivation, let us spend few words on motivations. They are essentially three:

i) The construction and the analysis of a well founded four-dimensional topologically gravity may furnish a gravitational analogue of Donaldson theory. In other words, it may provide a new tool to study intersection theory on the moduli space of gravitational instantons.

ii) The topological interpretation should provide new calculational tools in N=2 supergravity.

Finally, to our taste the most exciting, although still vague motivation is the third

iii) The special Kaehler geometry [12] of Calabi-Yau moduli-space is related, as we already recalled, to D=2 topological field-theories. On the other hand, it also follows from the requirement of N=2 supersymmetry in D=4. From the superstring point of view, this is understood in terms of the h-map [13], stating that on the same Calabi-Yau manifold we can compactify both the heterotic and the type II string. The latter has N=2 matter coupled supergravity as an effective lagrangian. Hence the topological interpretation of this theory should shed new light on the relation between topological field-theories in two and in four dimensions.

Let us now outline the conceptual set up and the contents of our paper.

Our purpose is to show that the topological twist of N=2 pure supergravity defines a gauge-fixed version of pure topological gravity where the gauge-fixing condition is \( \omega^{-ab} = 0 \), \( \omega^{-ab} \) denoting the antiselfdual part of the spin connection. To this effect we utilize the BRST-approach, having, as final goal, the comparison of the abstract gauge-theory a la
Beaulieu-Singer [14] with the gauge-fixed approach a la Witten [7].

Our viewpoint on the construction of a BRST-theory is the following. First one singles out the classical symmetries and constructs an abstract BRST-algebra involving only the classical fields and the ghost, with the exclusion of the antighosts. We name this algebra the gauge-free BRST algebra, since, at this level no commitment is made on the gauge fixing terms and on the lagrangian. Next, in the BRST-algebra, one introduces the antighosts and the auxiliary fields. The choice of these latter is motivated by the gauge fixings one wants to consider. Finally one constructs the BRST quantum action with the given gauge-fixings.

In the case of topological gravity the gauge-free BRST algebra is the specialization to the Poincaré group of the gauge-free algebra for a topological Yang-Mills theory. The general form of this algebra is [9]:

\[
\begin{align*}
    sA &= -(Dc + J \psi), \\
    sc &= \phi - \frac{1}{2} [c, c], \\
    sF &= D\psi - [c, F], \\
    s\psi &= D\phi - [c, \psi], \\
    s\phi &= -[c, \phi],
\end{align*}
\]

(1.1)

where \( A = A_\mu dx^\mu \) is the classical 1-form gauge-field, \( c \) are the 0-form ghosts (corresponding to ordinary gauge transformations \( \delta A_\mu = D_\mu \varepsilon \)), \( \psi = \psi_\mu dx^\mu \) is the 1-form ghost associated with the topological symmetry \( \delta A_\mu = u_\mu \) and \( \phi \) is the 0-form ghost for ghosts that has ghost number \( g = 2 \), while the previous ghosts have \( g = 1 \). All fields are Lie algebra-valued. The BRST operation \( s \) in (1.1) is manifestly nilpotent \( (s^2 = 0) \) and anticommutes with the exterior derivative \( (sd + ds = 0) \).

One important ingredient of our discussion will be the relation between the gauge-free BRST algebra for the ordinary theory and for the topological theory. It can be understood in general terms as it follows. As it is more explicitly discussed in section II, one can extend the concept of differential forms to that of ghost-forms, by setting
\[ \hat{A} = A + c, \quad (1.2) \]

where \( A \) and \( c \) are the (1,0) and (0,1) parts of \( \hat{A} \) (a generic object of form degree \( f \) and ghost number \( g \) will be described by \((f,g)\)). One can also extend the concept of exterior differentiation defining

\[ \hat{d} = d + s, \quad (1.3) \]

where \( d \) and \( s \) are the (1,0) and (0,1) parts of \( \hat{d} \). With these notations one finds that, expanding the extended field-strength

\[ \hat{F} = \hat{d}\hat{A} + \frac{1}{2} [\hat{A}, \hat{A}] \quad (1.4) \]

in its \((f,g)\) sectors, the following identifications are possible: \( \hat{F}_{(2,0)} = F, \hat{F}_{(1,1)} = \psi \) and \( \hat{F}_{(0,2)} = \phi \). Indeed the first two equations in (1.1) amount precisely to these identifications, while the last three are sectors of the extended Bianchi identity

\[ \hat{d}\hat{F} + [\hat{A}, \hat{F}] = 0. \quad (1.5) \]

Hence the gauge-free topological BRST algebra corresponds to a parametrization of the extended curvature \( \hat{F} \) where no constraints are imposed on the extra components \( \hat{F}_{(1,1)} \) and \( \hat{F}_{(0,2)} \).

On the other hand the ordinary gauge-free BRST algebra

\[
\begin{align*}
    sA &= -\mathcal{D}c, \\
    sc &= -\frac{1}{2} [c, c], \\
    sF &= -[c, F],
\end{align*}
\quad (1.6)
\]
correspond to imposing the horizontality conditions \( \hat{F}_{(1,1)} = 0 \) and \( \hat{F}_{(0,2)} = 0 \).

This is interpreted in the framework of rheonomy [13] as follows: the topological BRST algebra is the off-shell solution of the extended Bianchi identity (1.3) where all the outer components are kept on equal footing with the inner ones. The ordinary gauge-free BRST
algebra is instead provided by the *quantum rheonomic* solution of the extended Bianchi identity \((1.5)\). By definition the *quantum rheonomic parametrization* is obtained from the classical rheonomic parametrization by replacing the classical cotangent basis of differential forms with the corresponding extended one. In this way the components of the extended curvatures in the extended basis are the same as the components of the classical curvatures in the classical basis. For instance in the case of Yang-Mills theory the classical basis is given by \(A\) and \(V^a\), the last being the vierbein; the classical rheonomic parametrization is:

\[
F = F_{ab} V^a \wedge V^b, \tag{1.7}
\]

so that the quantum rheonomic parametrization is

\[
\hat{F} = F_{ab} \hat{V}^a \wedge \hat{V}^b = F_{ab} V^a \wedge V^b. \tag{1.8}
\]

Indeed, in this case \(\hat{V}^a = V^a\), the ghost part being attached only to the gauge field \(A\), according to \((1.2)\), since only the gauge transformations are symmetries, not the diffeomorphisms.

In the case of pure gravity the classical curvatures are \([13]\)

\[
R^a = D V^a = d V^a - \omega^a_{\ b} \wedge V^b, \]
\[
R^{ab} = d \omega^{ab} - \omega^c_{\ a} \wedge \omega^{cb}. \tag{1.9}
\]

Their classical rheonomic parametrization is

\[
R^a = 0, \]
\[
R^{ab} = R^{ab}_{\ cd} V^c \wedge V^d, \tag{1.10}
\]

so that the corresponding quantum rheonomic parametrization is

\[
\hat{R}^a = 0, \]
\[
\hat{R}^{ab} = R^{ab}_{\ cd} \hat{V}^c \wedge \hat{V}^d. \tag{1.11}
\]

This time the vielbein being quantum extended
\[ \hat{V}^a = V^a + \varepsilon^a. \]  

Eq.s \((1.11)\) lead to the BRST algebra associated with diffeomorphisms and Lorentz rotations. On the other hand, if we relax \((1.11)\) and we keep all the outer components of \(\hat{R}^{ab}\) as independent fields, we obtain a gauge-free BRST algebra that includes also the ghosts for the topological symmetry \(\delta V^a_\mu = \xi^a_\mu, \xi^a_\mu\) being an arbitrary infinitesimal vierbein. This is our definition of gravitational topological BRST algebra. We want to compare it with the BRST algebra associated with twisted N=2 supergravity. Indeed in order to make a successful twist we must already start at the quantum BRST-level.

Our logical development is the following.

In section II we consider N=2 supergravity in the rheonomy framework and we construct its BRST quantization. In particular we discuss its gauge-free BRST algebra prior to the introduction of antighosts.

In section III we define D=4 topological gravity along the lines discussed above and we introduce the gauge-free topological BRST algebra. We also discuss the descent equations arising from the topological observables associated with the Pontriagin and Euler characteristic classes.

In sections IV and V we discuss the topological twist of N=2 supergravity, introducing also the concept of topological shift that is instrumental for a correct interpretation of the resulting theory. We identify the ghosts and antighosts and from the latter identification we conclude that the gauge-fixing implicit in the theory is \(\omega^{-ab} = 0\).

In section VI we show that the action of N=2 supergravity can be obtained as a topological term plus the BRST-variation of a gauge fermion \(\Psi\) that implements the gauge-fixing \(\omega^{-ab} = 0\). Some subtleties related with the redundancy of this gauge-fixing and with the appearance of extraghosts are also discussed.

Finally section VII contains our conclusions.
II. BRST-QUANTUM VERSION OF D=4 N=2 SUPERGRAVITY

In this section we construct the BRST quantization of D=4 N=2 supergravity. As anticipated in section I, we think it convenient to employ the formalism of differential forms and rheonomic parametrizations [15]. The concepts of rheonomy is applied to the construction of the BRST-quantum version of the theory in the way explained in Ref. [14].

D=4 N=2 simple supergravity is described by the following curvatures

\begin{align*}
R^a &= \mathcal{D}V^a - \frac{i}{2} \bar{\psi}_A \wedge \gamma^a \psi_A = dV^a - \omega^a{}_b \wedge V^b - \frac{i}{2} \bar{\psi}_A \wedge \gamma^a \psi_A, \\
R^{ab} &= d\omega^{ab} - \omega^a{}_c \wedge \omega^{cb}, \\
\rho_A &= \mathcal{D}\psi_A = d\psi_A - \frac{1}{2} \omega^{ab} \wedge \sigma_{ab} \psi_A, \\
R^\otimes &= F + \epsilon_{AB} \bar{\psi}_A \wedge \psi_B, \quad (2.1)
\end{align*}

where Lorentz indices are denoted by latin letters, $V^a$ is the one form representing the vierbein, $\omega^{ab}$ the one form representing the spin connection, $\psi_A$ ($A=1,2$) is the couple of gravitinos (one forms as well), while $F \equiv dA$, $A$ being the one form representing the graviphoton. $d$ denotes the operation of exterior derivative, while $\mathcal{D}$ represents the covariant exterior derivative. Finally, $\sigma^{ab} \equiv \frac{1}{4} [\gamma^a, \gamma^b]$ and $\epsilon_{AB}$ is the completely antisymmetric tensor with two indices.

The above curvatures satisfy the following Bianchi identities

\begin{align*}
\mathcal{D}R^a + R^a{}_b \wedge V^b - i \bar{\psi}_A \wedge \gamma^a \rho_A &= 0, \\
\mathcal{D}R^{ab} &= 0, \\
\mathcal{D}\rho_A + \frac{1}{2} R^{ab} \wedge \sigma_{ab} \psi_A &= 0, \\
\mathcal{D}R^\otimes + 2\epsilon_{AB} \bar{\psi}_A \wedge \rho_B &= 0. \quad (2.2)
\end{align*}

The rheonomic parametrizations of the four curvatures (2.1) that are compatible with the Bianchi identities (2.2), at least on shell, since we do not introduce auxiliary fields, are [15]
\[ R^a = 0, \]
\[ R^{ab} = R^{ab}_{\text{cd}} V^c \wedge V^d + \tilde{\theta}_{A[c}^{ab} \psi_{A} \wedge V^c - \frac{1}{2} \tilde{\psi}_A \wedge F^{ab} \psi_B \epsilon_{AB}, \]
\[ \rho_A = \tilde{\rho}_A^{ab} V_a \wedge V_b + \frac{1}{2} i \gamma^a F^{ab} \psi_B \wedge V^b \epsilon_{AB}, \]
\[ R^\otimes = F_{ab} V^a \wedge V^b, \]  \hspace{2cm} (2.3)

where \( F^{ab} \equiv F^{ab} + \frac{i}{2} \gamma_5 F^{cd} \varepsilon^{abcd} \) and \( \tilde{\theta}_A^{ab} = 2i \tilde{\rho}_A^{[a} \gamma^{b]} - i \tilde{\rho}_A^{ab} \gamma^c \) \cite{13}, where the square brackets denote antisymmetrization. These parametrizations are found by expanding the curvatures (2.1) in a basis of differential forms in superspace (which can be written as exterior products of \( V^a \) and \( \psi_A \)) and then imposing the Bianchi identities (2.2) on shell \cite{15}.

For completeness, we write here the lagrangian of N=2 supergravity, because it will be useful later on.

\[ \mathcal{L} = R^{ab} \wedge V^c \wedge V^d \varepsilon_{abcd} + 4 \tilde{\rho}_A \wedge \gamma_5 \gamma_a \psi_A \wedge V^a + 2i R^\otimes \wedge \tilde{\psi}_A \wedge \gamma_5 \psi_B \epsilon_{AB} + \\
- 2i \tilde{\psi}_A \wedge \psi_B \wedge \tilde{\psi}_A \wedge \gamma_5 \psi_B - F^{ab} V^c \wedge V^d \wedge R^\otimes \epsilon_{abcd} + \\
+ \frac{1}{12} F_{ab} F^{ab} V^i \wedge V^j \wedge V^k \wedge V^l \varepsilon_{ijkl}. \]  \hspace{2cm} (2.4)

The operation of BRST transformation is denoted by \( s \). We introduce ghost-number and \( s \) has ghost-number one. In such a way we have two natural gradations: form-number \( f \) and ghost-number \( g \). As anticipated in the introduction, a generic object is described by the couple \( (f, g) \). When permuting two objects it is the sum \( f + g \) that determines the correct sign (but note that some fields, like the gravitinos and their ghosts also have a fermionic number and when permuting two of them, the preceding rule must by suitably amended). So, \( f + g \) is a gradation of primary importance. We shall call it the ghost-form-number. Any object must have a well defined ghost-form-number and so the first part of BRST quantization consists in extending any differential form (of form-number \( f \), say) to a ghost-form of ghost-form-number \( f \). Let

\[ \hat{V}^a = V^a + \varepsilon^a, \]
\[ \hat{\omega}^{ab} = \omega^{ab} + \varepsilon^{ab}, \]
\[ \hat{\psi}_A = \psi_A + c_A, \]
\[ \hat{A} = A + c, \] (2.5)

where \( \varepsilon^a, \varepsilon^{ab}, c_A \) and \( c \) (form-number zero, ghost-number one) are the ghosts of diffeomorphisms, Lorentz rotations, supersymmetries and Maxwell transformations, respectively. For the time being, the spin connection is treated as an independent variable: later on we shall go over to second order formalism. It is useful to similarly extend the operation of exterior differentiation, as already mentioned in (1.3). The curvatures are extended to ghost-forms of ghost-form-number two, that are the sum of a (2,0)-piece (the original curvature) plus a (1,1)-term and a (0,2)-term. These extra-terms will be fixed by the rheonomic parametrizations. Let it be

\[ \hat{R}^a = R^a + \psi^a + \phi^a, \]
\[ \hat{R}^{ab} = R^{ab} + \chi^{ab} + \eta^{ab}, \]
\[ \hat{\rho}_A = \rho_A + \xi_A + \zeta_A, \]
\[ \hat{R}^\otimes = R^\otimes + \psi + \phi. \] (2.6)

The same curvatures can be written by suitably extending the definitions (2.1) (that is to say by replacing nonhatted quantities with the corresponding hatted version). For example,

\[ \hat{R}^a = R^a + \psi^a + \phi^a = \hat{\nabla}^a - \hat{\omega}^{ab} \wedge \hat{V}_b - \frac{i}{2} \bar{\psi}_A \wedge \gamma^a \hat{\psi}_A. \] (2.7)

After explicit substitution and separation of the various \((f,g)\)-parts, one can read, besides the definition of \( R^a \) itself,

\[ sV^a = \psi^a - \mathcal{D} \varepsilon^a + \varepsilon^{ab} \wedge V_b + \frac{i}{2} (\bar{\psi}_A \wedge \gamma^a \rho_A + \bar{c}_A \wedge \gamma^a \psi_A), \]
\[ s\varepsilon^a = \phi^a + \varepsilon^{ab} \wedge \varepsilon_b + \frac{i}{2} \bar{c}_A \wedge \gamma^a \rho_A. \] (2.8)

These are the BRST variations of \( V^a \) and \( \varepsilon^a \) (at least upon fixing \( \psi^a \) and \( \phi^a \)). As \( d^2 = 0 \) and the extension to hatted quantities preserves all the algebraic manipulations (as one can easily convince oneself), we are guaranteed that this property is extended to \( \hat{d}^2 = 0 \), that is to say
\[d^2 = 0,\]
\[ds + sd = 0,\]
\[s^2 = 0.\]  \hspace{1cm} (2.9)

In particular we are guaranteed to find a well defined BRST algebra \((s^2 = 0)\). By analysing the remaining curvatures in a similar way, one gets the rest of the BRST algebra. We give only the results. The complete BRST algebra is

\[s V^a = \psi^a - D\epsilon^a + \epsilon^{ab} \wedge V_b + i\bar{c}_A \wedge \gamma^a \psi_A,\]
\[s\omega^{ab} = \chi^{ab} - D\epsilon^{ab},\]
\[s\epsilon^a = \phi^a + \epsilon^{ab} \wedge \epsilon_b + \frac{i}{2} \bar{c}_A \wedge \gamma^a c_A,\]
\[s\epsilon^{ab} = \eta^{ab} + \epsilon^a_c \wedge \epsilon^{cb},\]
\[s\psi_A = \xi_A - Dc_A + \frac{1}{2} \epsilon^{ab} \sigma_{ab} \psi_A,\]
\[s\epsilon^{cb} = \eta^{ab} + \epsilon^a_c \wedge \epsilon^{cb},\]
\[sA = \psi - dc - 2\epsilon_{AB}\bar{c}_A \wedge \psi_B,\]
\[sc = \phi - \epsilon_{AB}\bar{c}_A \wedge c_B.\]  \hspace{1cm} (2.10)

In a similar way one can also analyse the content of the hatted extensions of the Bianchi identities (2.2). One then finds two sets of variations: i) the variations of the curvatures themselves, i.e.

\[s R^a = -D\psi^a + \epsilon^{ab} \wedge R_b - R^{ab} \wedge \epsilon_b - \chi^{ab} \wedge V_b + i\bar{\psi}_A \wedge \gamma^a \xi_A + i\bar{c}_A \wedge \gamma^a \rho_A,\]
\[s R^{ab} = -D\chi^{ab} + \epsilon^a_c \wedge R^{cb} - R^{ab} \wedge \epsilon^c_b,\]
\[s\rho_A = -D\xi_A + \frac{1}{2} \epsilon^{ab} \sigma_{ab} \rho_A - \frac{1}{2} R^{ab} \sigma_{ab} c_A - \frac{1}{2} \chi^{ab} \sigma_{ab} \psi_A,\]
\[s R^{\otimes} = -d\psi - 2\epsilon_{AB}(\bar{\psi}_A \wedge \xi_B + \bar{c}_A \wedge \rho_B),\]  \hspace{1cm} (2.11)

that are consistent with their definitions (2.1) and with (2.10); ii) the variations of the free parameters \(\psi^a, \phi^a, \chi^{ab}, \eta^{ab}, \xi_A, \rho_A, \psi \) and \(\phi,\)

\[s\psi^a = -D\phi^a + \epsilon^{ab} \wedge \psi_b - \chi^{ab} \wedge \epsilon_b - \eta^{ab} \wedge V_b + i\bar{\psi}_A \wedge \gamma^a \xi_A + i\bar{c}_A \wedge \gamma^a \rho_A,\]
\[
\begin{align*}
\delta \phi^a &= \varepsilon^{ab} \wedge \phi_b - \eta^{ab} \wedge \varepsilon_b + i\bar{c}_A \land \gamma^a \zeta_A,
\delta \chi^{ab} &= -\mathcal{D} \eta^{ab} + \varepsilon^{ac} \wedge \chi_{bc} - \chi^{ac} \wedge \varepsilon_c^b,
\delta \eta^{ab} &= \varepsilon^{ac} \land \eta_c^b - \eta^{ac} \land \varepsilon^b_c,
\delta \xi_A &= -\mathcal{D} \zeta_A + \frac{1}{2} \xi^{ab} \sigma_{ab} \xi_A - \frac{1}{2} \chi^{ab} \sigma_{ab} \xi_A - \frac{1}{2} \eta^{ab} \sigma_{ab} \psi_A,
\delta \zeta_A &= \frac{1}{2} \xi^{ab} \sigma_{ab} \zeta_A - \frac{1}{2} \eta^{ab} \sigma_{ab} \zeta_A,
\delta \psi &= -d\phi - 2\epsilon_{AB}(\bar{\psi}_A \land \chi_B + \bar{\psi}_A \land \zeta_B),
\delta \phi &= -2\epsilon_{AB}\bar{c}_A \land \zeta_B.
\end{align*}
\] (2.12)

Eqs. (2.12) and (2.11) are the specialization to the case of the BRST quantum algebra of N=2 supergravity of the last three equations in (1.1).

As a next step, we fix the free parameters \( \psi^a, \phi^a, \chi^{ab}, \eta^{ab}, \xi_A, \zeta_A, \psi \) and \( \phi \) by means of the rheonomic conditions \([14,15]\). These conditions state that the parametrizations of the hatted curvatures are obtained by the old ones (see (2.3)) upon substitution of the forms \( V^a \) and \( \psi_A \) (the basis of forms in superspace) by their hatted quantities. For example, according to this prescription, \( \hat{\rho}_A \) is equal to

\[
\hat{\rho}_A = \rho_{ab}^A \hat{V}^a \land \hat{V}^b + \frac{1}{2} i\gamma^a \mathcal{F}_{ab} \hat{\psi}_B \land \hat{V}^b \epsilon_{AB}.
\] (2.13)

After use of (2.3) and (2.6) and separation of the various \( (f,g) \)-parts, one can read the definitions of \( \xi_A \) and \( \zeta_A \). In a similar way one proceeds for the other curvatures and free parameters. We report here only the final result, that is

\[
\begin{align*}
\psi^a &= 0,
\phi^a &= 0,
\chi^{ab} &= 2\rho^{ab}_{cd} V^c \land \varepsilon^d + \bar{g}^{ab}_{A|c}(c_A \land V^c + \psi_A \land \varepsilon^c) - \bar{c}_A \mathcal{F}^{ab} \psi_B \epsilon_{AB},
\eta^{ab} &= R^{ab}_{cd} \varepsilon^c \land \varepsilon^d + \bar{g}^{ab}_{A|c} c_A \land \varepsilon^c - \frac{1}{2} \bar{c}_A \mathcal{F}^{ab} \bar{c}_B \epsilon_{AB},
\xi_A &= 2\rho_{ab}^A \varepsilon^a \land V^b + \frac{i}{2} \gamma^a \mathcal{F}_{ab} (c_B \land V^b + \psi_B \land \varepsilon^b) \epsilon_{AB},
\zeta_A &= \rho_{ab}^A \varepsilon^a \land \varepsilon^b + \frac{i}{2} \gamma^a \mathcal{F}_{ab} c_B \land \varepsilon^b \epsilon_{AB},
\end{align*}
\]
\[ \psi = 2F_{ab}V^a \wedge \varepsilon^b, \]
\[ \phi = F_{ab}\varepsilon^a \wedge \varepsilon^b. \]  

(2.14)

One can verify that (2.12) are consistent with (2.14) on shell. This requires no further computational work than the one which is required to prove that the rheonomic parametrizations (2.3) are consistent with the Bianchi identities (2.2) [15] (the formal manipulations are the same).

By means of suitable redefinitions one can put formulas (2.10) in a more familiar form, i.e. to write diffeomorphisms in terms of Lie derivatives [14]. To this purpose, let \( \varepsilon' = \varepsilon^aV_a^\mu \), \( V_a^\mu \) being the inverse vierbein, so that \( \varepsilon^a = i_\varepsilon V^a \), where \( i \) denotes contraction. The Lie derivative \( \mathcal{L}_\varepsilon \) is equal to \( i_\varepsilon d - di_\varepsilon \), where the minus sign is due to the fact that \( \varepsilon \) is a ghost [14]. If we define \( \varepsilon'^{ab} = \varepsilon^{ab} - i_\varepsilon \omega^{ab} \), \( c'_A = c_A - i_\varepsilon \psi_A \) and \( c' = c - i_\varepsilon A \), then we get

\[ \begin{align*}
sv^a &= \mathcal{L}_\varepsilon v^a + \varepsilon'^{ab} \wedge v^b + i c'_A \wedge \gamma^a \psi_A, \\
s\psi_A &= \mathcal{L}_\varepsilon \psi_A + \frac{1}{2} \varepsilon'^{ab} \sigma_{ab} \psi_A - d c'_A + \frac{i}{2} \gamma^a F_{ab}c'_B \wedge v^b \epsilon_{AB}, \\
sA &= \mathcal{L}_\varepsilon A - d c' - 2e_{AB}c'_{A} \wedge v_{B}. 
\end{align*} \]

(2.15)

We see that the variations of the main fields \( v^a \), \( \psi_A \) and \( A \) are the sum of diffeomorphisms, Lorentz rotations and supersymmetries, as it must be.

The last point regards the possibility of employing the second order formalism, that is to say of expressing \( \omega^{ab} \) in terms of the vierbein \( (R^a = 0) \). For consistency, we must also have \( sR^a = 0 \), and this gives a condition on \( \chi^{ab} \). Consequently, we should expect to have a condition on \( s\chi^{ab} \), however \( s\chi^{ab} \) turns out to be automatically consistent with (2.10) and so it imposes no further constraint.

N=2 supergravity has an internal \( SU(2)_I \) symmetry holding off-shell and an internal \( U(1) \) symmetry, which, however, holds only on shell [16] (that is to say it is a symmetry of the equations of motion, but not of the lagrangian). This \( U(1) \) internal symmetry combines chirality of the gravitinos with duality of the graviphoton in the following way [16]

\[ \delta\psi_A = i\alpha \gamma_5 \psi_A, \]
\[
\delta F_{ab} = -2i\alpha \tilde{F}_{ab} = \alpha \epsilon_{abcd} F^{cd}.
\] (2.16)

One easily verifies that the equations of motion derived from (2.4) are all invariant under the chiral-dual transformations (2.16). The less trivial case is the one of the field equation coming from the variation of the graviphoton $A$, that is

\[
4i\epsilon_{AB} \bar{\rho}_A \wedge \gamma_5 \bar{\psi}_B - \mathcal{D}(F^{ab} V^c \wedge V^d) \epsilon_{abcd} = 0.
\] (2.17)

Its variation under (2.16) is the last of the Bianchi identities (2.2) and viceversa, thus proving $U(1)$ on shell invariance (the remaining Bianchi identities are trivially invariant).

### III. TOPOLOGICAL GRAVITY

In this section we discuss the gauge-free BRST algebra of topological gravity. As already pointed out this BRST algebra involves only ghosts (and not antighosts). In the following sections we show that this algebra stands to the algebra of twisted topological gravity determined in Section II as the Beaulieu-Singer approach \[9\] stands to the Witten approach \[7\]. The procedure resembles the construction of the BRST quantum version of supergravity, but the difference is that, according to the discussion of Section I, we impose no rheonomic parametrization. We show that this prescription gives automatically a topological theory. Similarly, (2.10) and (2.12), without imposition of (2.14), are the gauge-free BRST algebra of topological $N=2$ supergravity.

Let us start from the curvatures of the theory, that are given by (1.9). Similarly with respect to before, we define hatted quantities

\[
\hat{d} = d + s,
\]
\[
\hat{V}^a = V^a + \epsilon^a,
\]
\[
\hat{\omega}^{ab} = \omega^{ab} + \epsilon^{ab},
\]
\[
\hat{R}^a = R^a + \psi^a + \phi^a,
\]
\[
\hat{R}^{ab} = R^{ab} + \chi^{ab} + \eta^{ab},
\] (3.1)
but now $\psi^a$, $\phi^a$, $\chi^{ab}$ and $\eta^{ab}$ remain independent fields. From the definitions of the curvatures (1.9), extended to hatted expressions as before, and the Bianchi identities

$$\mathcal{D}R^a + R^a_b \wedge V^b = 0,$$
$$\mathcal{D}R^{ab} = 0,$$

also extended to hatted quantities, one obtains the BRST algebra

$$sV^a = \psi^a - \mathcal{D}\varepsilon^a + \varepsilon^{ab} \wedge V_b,$$
$$s\omega^{ab} = \chi^{ab} - \mathcal{D}\varepsilon^{ab},$$
$$s\varepsilon^a = \phi^a + \varepsilon^{ab} \wedge \varepsilon_b,$$
$$s\varepsilon^{ab} = \eta^{ab} + \varepsilon^a_c \wedge \varepsilon^{cb},$$
$$s\psi^a = -\mathcal{D}\phi^a + \varepsilon^{ab} \wedge \psi_b - \chi^{ab} \wedge \varepsilon_b - \eta^{ab} \wedge V_b,$$
$$s\phi^a = \varepsilon^{ab} \wedge \phi_b - \eta^{ab} \wedge \varepsilon_b,$$
$$s\chi^{ab} = -\mathcal{D}\eta^{ab} + \varepsilon^{ac} \wedge \chi^{b}_c - \chi^{ac} \wedge \varepsilon^{b}_c,$$
$$s\eta^{ab} = \varepsilon^{ac} \wedge \eta^b_c - \eta^{ac} \wedge \varepsilon^{b}_c.$$

(3.3)

Once more this is the specialization to the case of the Poincaré algebra of Eq.s (1.1). Of course, this algebra is also obtainable by reduction to N=0 of the N=2 algebra of Eq.s (2.10) and (2.12) (with no imposition of (2.14)).

We have used the same symbols as before, for similar, but different, quantities. Whenever necessary, we shall distinguish objects belonging to the BRST algebra of N=2 supergravity (2.10) from those of the BRST algebra of topological gravity (3.3) by an index, which will be 2 in the former case, 0 in the latter. For example, $\omega^{ab}_2$ will be the superconnection (coming from $R^a_2 = 0$), while $\omega^{ab}_0$ will be the usual connection (coming from $R^a_0 = 0$). The transformations (2.10) will be denoted by $s_2$, the transformations (3.3) by $s_0$. Similarly, we shall write $\psi^a_2$ and $\psi^a_0$, $\phi^a_2$ and $\phi^a_0$, et cetera.

As before, we are guaranteed that $s^2 = 0$, but now $s_0^2 = 0$ holds off-shell (it is the imposition of a rheonomic parametrization holding only on shell that forces $s^2_2 = 0$ to hold

15
only on shell). Let us analyse (3.3) in more detail. As we see, \( \psi^a_0 \) represents the topological ghost and the variation of \( V^a \) is equal to the topological variation \( \psi^a_0 \) plus diffeomorphisms plus Lorentz rotations. \( \phi^a_0 \) and \( \eta^{ab}_0 \) are ghosts for ghost, the former corresponding to diffeomorphisms, the latter corresponding to Lorentz rotations. As for \( \chi^{ab}_0 \), in the second order formalism (\( R^a = 0 \)) the condition \( s_0 R^a_0 = 0 \) (which can be read from the first formula of (2.11) upon reduction to \( N=0 \)) implies \( \chi^{ab}_0 \wedge V_b = -D_0 \psi^a_0 - R^{ab}_0 \wedge \varepsilon_b \), which can be solved in the same manner as the condition defining \( \omega^{ab}_0 \) (i.e. \( \omega^{ab}_0 \wedge V_b = d V^a \)). As noted in Section II, the fact that \( \chi^{ab}_0 \) depends on the other fields does not impose further constraints and the BRST algebra is well defined. From now on we shall employ the second order formalism.

The procedure here followed to determine a BRST algebra for topological gravity does not introduce any antighost. This is because we are not choosing any particular gauge-fixing. The topological twist, on the other hand, will give automatically a preferred gauge-fixing for the topological symmetry, as we shall see in the following section.

Now we describe the observables of the theory, which are related to the Pontriajn \( \mathcal{P} = R^{ab}_0 \wedge R^{ab}_0 \) and Euler characteristic classes \( \mathcal{E} = R^{ab}_0 \wedge R^{cd}_0 \varepsilon_{abcd} \). \( s \phi^a_0 \) and \( s \eta^{ab}_0 \) should be compared with the variation of the ghost for ghost \( \phi \) that appears in (1.1), \( s \phi = -[c, \phi] \). As we see, the transformation of \( \phi \) is nothing but a gauge transformation and so all gauge invariants constructed from \( \phi \) are BRST invariants and can lead to the descent equations that give the observables of the theory [7,9]. In our case it is \( \eta^{ab}_0 \) that has a BRST variation which is only a gauge transformation (Lorentz rotation). \( \eta^{ab}_0 \) is a \( 4 \times 4 \) antisymmetric matrix. Any \( 4 \times 4 \) matrix has the four invariants \( \text{tr}[\eta_0], \text{tr}[\eta_0^2], \text{tr}[\eta_0^3] \) and \( \text{tr}[\eta_0^4] \). In our case only \( \text{tr}[\eta_0^2] \) and \( \text{tr}[\eta_0^4] \) are nonvanishing. What are the corresponding descent equations and to what topological invariants do they correspond? It will be soon proved that they correspond to the Pontriajn number and to the Euler number.

We start by noticing that the proof that the form \( R^{ab}_0 \wedge R^{ab}_0 \) is closed works with hatted quantities, exactly as with nonhatted ones:

\[
\hat{d}(\hat{R}_0^{ab} \wedge \hat{R}_0^{ab}) = -2 \hat{d}\hat{R}_0^{ab} \wedge \hat{R}_0^{ab} = -2(\hat{\omega}_0^{ab} \wedge \hat{R}_0^{bc} \wedge \hat{R}_0^{cd} - \hat{R}_0^{ab} \wedge \hat{\omega}_0^{bc} \wedge \hat{R}_0^{cd}) = 0. \quad (3.4)
\]
We have used the hatted Bianchi identity \(\hat{D}\hat{R}^{ab}_{0} = 0\). After explicit substitution and separation of the various \((f,g)\)-parts, one can read the descent equations

\[
\begin{align*}
    s_0 \text{tr}[\eta_0 \wedge \eta_0] &= 0, \\
    s_0 \text{tr}[\eta_0 \wedge \chi_0 + \chi_0 \wedge \eta_0] &= -d \text{tr}[\eta_0 \wedge \eta_0], \\
    s_0 \text{tr}[\eta_0 \wedge R_0 + \chi_0 \wedge \chi_0 + R_0 \wedge \eta_0] &= -d \text{tr}[\eta_0 \wedge \chi_0 + \chi_0 \wedge \eta_0], \\
    s_0 \text{tr}[R_0 \wedge \chi_0 + \chi_0 \wedge R_0] &= -d \text{tr}[\eta_0 \wedge R_0 + \chi_0 \wedge \chi_0 + R_0 \wedge \eta_0], \\
    s_0 \text{tr}[R_0 \wedge R_0] &= -d \text{tr}[R_0 \wedge \chi_0 + \chi_0 \wedge R_0], \\
    0 &= -d \text{tr}[R_0 \wedge R_0],
\end{align*}
\]

where the trace refers to the Lorentz indices. So, we have the following observables

\[
\begin{align*}
    \mathcal{O}^{(0)} &= \text{tr}[\eta_0 \wedge \eta_0], \\
    \mathcal{O}^{(1)}_{\gamma} &= \int_{\gamma} \text{tr}[\eta_0 \wedge \chi_0 + \chi_0 \wedge \eta_0], \\
    \mathcal{O}^{(2)}_{S} &= \int_{S} \text{tr}[\eta_0 \wedge R_0 + \chi_0 \wedge \chi_0 + R_0 \wedge \eta_0], \\
    \mathcal{O}^{(3)}_{V} &= \int_{V} \text{tr}[R_0 \wedge \chi_0 + \chi_0 \wedge R_0], \\
    \mathcal{O}^{(4)}_{\mathcal{M}} &= \int_{\mathcal{M}} \text{tr}[R_0 \wedge R_0],
\end{align*}
\]

where \(\mathcal{M}\) is the four dimensional manifold where the theory is defined, and \(\gamma, S,\) and \(V\) are generic one-, two- and three-dimensional cycles on \(\mathcal{M}\). So we have proved that \(\text{tr}[\eta_0^2]\) corresponds to the Pontriagin number. In precisely the same way, one can deduce descent equations and construct observables associated to the Euler form \(\mathcal{E} = R_0^{ab} \wedge R_0^{cd} \epsilon_{abcd}\). These observables will be denoted by \(\hat{\mathcal{O}}^{(n)}\) and correspond to \(\text{tr}[\eta_0 \wedge \tilde{\eta}_0]\). As \(\text{tr}[\eta_0^4] = \frac{1}{16} (\text{tr}[\eta_0 \wedge \tilde{\eta}_0])^2 + \frac{1}{2} (\text{tr}[\eta_0^2])^2\), we see that we have exhausted the two invariants discussed before.

**IV. TOPOLOGICAL TWIST OF N=2 SUPERGRAVITY**

The topological twist of N=2 supergravity is performed in a similar way as the topological twist of Yang Mills theories \([7]\). Nevertheless, some generalizations and specifications are
needed. We identify the internal symmetry group $SU(2)_I$ with $SU(2)_R$, the right handed part of the Lorentz group, that is to say we define a twisted $SU(2)'_R$ as the diagonal subgroup of $SU(2)_R \otimes SU(2)_I$. Let us fix a bit of notation. Every field will be classified, before the twist, by an expression like $c(L,R,I)^g_f$, where $L$, $R$ and $I$ are the representation labels for $SU(2)_L$, $SU(2)_R$ and $SU(2)_I$ respectively, $c$ is the $U(1)$ charge, $g$ is the ghost number and $f$ is the form degree. Some fields (the graviphoton and the corresponding ghosts) have not a well defined $U(1)$ charge and so $c$ will be replaced by a dot in these cases. After the twist, each field will be denoted by $(L,R')^{g+c}_f$, where $R' = R \otimes I$. The new ghost number is the sum of the old ghost number and the old $U(1)$ charge. So, for some fields the new ghost number is not defined off-shell, but only on shell. However, we do not think this is a problem, rather one of the new features of ghost number conservation in topological theories.

We note that ghost number conservation has particular features even in twisted Yang Mills theories\cite{7}, because the chiral anomaly of the untwisted theory appears as a ghost number anomaly in the twisted version of the theory. In two dimensional topological theories, the same phenomenon is represented by the appearance of a charge at infinity after the twist \cite{5}. We think that the new features of ghost number conservation that appear in twisted N=2 supergravity deserve further investigation.

The fields are also characterized by a fermionic number, however it will not play an important role in the twisted theory. We shall explain this fact in the following section.

In Table I we list the fields of N=2 supergravity and their twisted counterparts. We see that the twisted version of $\psi_A$ has a $(\frac{1}{2}, \frac{1}{2})^1_1$ component. This is substantially the ghost of topological variations of the vierbein (the exact identification will be given in the following section). The components $(0,1)^{-1}_1$ and $(0,0)^{-1}_1$ become the corresponding antighosts. The variation of the $(0,1)^{-1}_1$ component, in particular, gives the gauge-fixing of the topological symmetry, precisely as in Yang-Mills theories. The twisted version of $A$ represents ghosts for ghosts and antighosts for ghosts. This is because the tensor $F^{ab}$ has two components of $U(1)$ charge $\pm 2$, and so the twisted version of $F^{ab}$ has two components of ghost number $\pm 2$.

In Table II we list the antighosts and Lagrange multipliers of N=2 supergravity and their
twisted counterparts. $\varepsilon^a$ and $\varepsilon^{ab}$ are the antighosts of diffeomorphisms and Lorentz rotations, respectively; $\pi^a$ and $\pi^{ab}$ are the corresponding Lagrange multipliers; $\bar{c}^a_A$ are the antighosts of supersymmetries and $P_A$ are their Lagrange multipliers; $\bar{c}$ and $P$ are the antighost and Lagrange multiplier of the Maxwell gauge-symmetry. In Table III we give a summary of all the fields involved in the BRST quantum algebra of N=2 supergravity, their twisted version and their meaning.

The explicit twist can be realized by interpreting the internal indices $A, B$ as dotted indices $\dot{\alpha}, \dot{\beta}$. Refer to Appendix A for the notation. The left handed and right handed components of $\psi_A$ are twisted as follows

$$
\begin{align*}
\psi_{\alpha A} &\rightarrow \psi_{\dot{\alpha} A}, \\
\psi_{\dot{\alpha} A} &\rightarrow \psi_{\dot{\alpha} \dot{A}},
\end{align*}
$$

while $\epsilon_{AB} \rightarrow \epsilon_{\dot{A}\dot{B}} = -\epsilon_{\dot{A}\dot{B}}$. Let us now consider the supersymmetry transformations (which can be read from (2.15) when $\varepsilon^a = 0$ and $\varepsilon^{ab} = 0$)

$$
\begin{align*}
\delta V^a &= i\bar{c}^a_A \wedge \gamma^a \psi_A, \\
\delta \psi_A &= -\mathcal{D}c_A + \frac{i}{2} \gamma^a \mathcal{F}_{ab}c_B \wedge V^b \epsilon_{AB}, \\
\delta A &= -2\epsilon_{AB}\bar{c}^a_A \wedge \psi_B. 
\end{align*}
$$

We now twist these transformations and specialize the twisted version of the supersymmetry ghost $c_A$ to its $(0, 0)^0_0$-component. This component is $C \equiv c_{\dot{\alpha}} \dot{A} \dot{\delta}_A$ (see Appendix A). We set it equal to a constant and precisely $+i$, for convenience. The twisted version of $\psi_A$ consists of a $(\frac{1}{2}, \frac{1}{2})^1_1$-component, which will be denoted by $\tilde{\psi}^m = -\frac{1}{2} \psi_{\dot{\alpha} \dot{A}} (\bar{\sigma}^m)^{\dot{A}\dot{\alpha}}$ (the ghost of topological symmetry), a $(0, 1)^1_1$-component, which will be denoted by $\tilde{\psi}^{ab} = (\bar{\sigma}^{ab})^{\dot{A}\dot{a}} \psi_{\dot{\alpha} \dot{A}}$ (the antighost corresponding to the gauge breaking of the topological symmetry) and a $(0, 0)^1_1$-component, denoted by $\tilde{\psi} = \psi_{\dot{\alpha} \dot{A}} \delta_{\dot{A}}$ (see Table III and Appendix A). The transformations (4.2) become

$$
\begin{align*}
\delta V^a &= \tilde{\psi}^a, \\
\delta \tilde{\psi}^a &= \frac{1}{4} F^{-ab} \wedge V_b. 
\end{align*}
$$
\[
\delta \tilde{\psi}^{ab} = \frac{i}{4} \omega^{-ab}, \quad \delta \tilde{\psi} = 0, \quad \delta A = i \tilde{\psi}.
\] (4.3)

These transformations should be compared with those of topological Yang-Mills theories, as found in Ref. [7]. As we see, the topological gauge-fixing is the antiselfdual part of the spin connection. Its vanishing describes the gravitational instantons of the theory of topological gravity that we are studying.

The square of the transformation (4.3) is not zero, but it is a Lorentz rotation with field-dependent parameters \(\frac{1}{4} F^{-ab}\). This can be immediately deduced from the fact that \(\delta^2 V^a = \delta \tilde{\psi}^a = \frac{1}{4} F^{-ab} \wedge V_b\). A phenomenon like the present one also happens in topological Yang-Mills theories [7]. It is only when dealing with the complete BRST algebra [14] that the square of the transformations is zero (at least on shell).

In [14] we see that the complete BRST symmetry of topological Yang-Mills theories derives from a composition of the BRST symmetry of the untwisted \(N=2\) super Yang-Mills theory and the \((0,0)_0\)-component of the supersymmetric transformations. We want to consider the analogue of this mechanism in twisted topological gravity. Here we have to deal with the fact that now supersymmetry is a local symmetry and nevertheless we expect that the twisted BRST symmetry is in some sense a composition of the twisted version of the transformations (2.10) and the transformations (4.3). At the same time we need to be sure that the new BRST symmetry closes on shell. We cannot simply specialize the twisted version of supersymmetry transformations to their \((0,0)_0\)-component, because this would require to set some ghosts equal to zero, thus not guaranteeing \(s^2 = 0\). A simple way to overcome all this is to shift the \((0,0)_0\)-component \(C\) of the twisted version of the ghosts \(c_A\) by a constant, namely \(C \to C + i\). In such a way the new BRST transformations of the main fields are the old ones plus the transformations (4.3), as we would like, and closure on shell is automatically assured. The procedure of shifting \(C\) will be called topological shift. The topological shift should be considered as a mere trick to reach our purpose to define a
suitable new BRST symmetry and should not be regarded as substantial.

The twisted-shifted BRST symmetry will be denoted by $s'$. It will not be explicitly written down here; we only make some observations. The topological twist and the topological shift make the new BRST transformations appear as follows

$$
\begin{align*}
 s'V^a &= \tilde{\psi}^a - d\xi^a + \varepsilon^{ab} \wedge V_b + \cdots, \\
 s'\omega^{ab} &= \chi^{ab} - d\varepsilon^{ab} + \cdots, \\
 s'\varepsilon^a &= C^a + \cdots, \\
 s'\varepsilon^{ab} &= \frac{1}{4} F^{-ab} + \cdots, \\
 s'\eta^{ab} &= 0 + \cdots, \\
 s'\tilde{\psi}^a &= -dC^a + \frac{1}{4} F^{-ab} \wedge V_b + \cdots, \\
 s'\tilde{\psi}^{ab} &= -dC^{ab} + \frac{i}{4} \omega^{-ab} + \cdots, \\
 s'\tilde{\psi} &= -dC + \cdots, \\
 s'C^a &= 0 + \cdots, \\
 s'C^{ab} &= \frac{i}{4} \varepsilon^{-ab} + \cdots, \\
 s'C &= 0 + \cdots, \\
 s'A &= i\tilde{\psi} - dc + \cdots, \\
 s'c &= -\frac{1}{2} + iC + \cdots.
\end{align*}
$$

(4.4)

where $\chi^{ab} \wedge V_b = -d\tilde{\psi}^a + \cdots$ and the dots refer to interactions terms, i.e. terms involving products of two or more fields. $C^a = -\frac{1}{2}c_{\alpha A}(\sigma^a)^{\dot{A} \alpha}$, $C^{ab} = (\sigma^{ab})^{\dot{A} \dot{\alpha}} c_{\dot{\alpha} \dot{\alpha}}$ and $C = c_{\alpha} \hat{A}_{\dot{\alpha}}$ (see Table 1 and Appendix A). The BRST transformation of the Maxwell ghost $c$ contains a constant $-\frac{1}{2}$. This constant is inessential and can be suppressed. In fact $c$ appears only in $s' A$ as $dc$ (not even the dots contain $c$). Consequently, $s'^2 = 0$ is assured even if we write $s'c = iC + \cdots$. The transformations (4.4) will be useful for the computations of the next section; in particular, note that, according to (2.14), $\eta^{ab} = -\frac{1}{4} F^{-ab} + \cdots$, so (4.4) shows that $s'\varepsilon^{ab} = \eta^{ab} + \cdots$, in agreement with (3.3).
V. MATCHING BETWEEN TWISTED N=2 SUPERGRAVITY AND TOPOLOGICAL GRAVITY

In this section we give the correspondence between the transformations \( s' \) (i.e. the topologically twisted and topologically shifted version of \( s_2 \)) and the transformations \( s_0 \), at least for what regards the sector that not includes antighosts. First of all, let us compare

\[
s_2 V^a = -\mathcal{D}_2 \varepsilon^a + \varepsilon^{ab} \wedge V_b + i \bar{c}_A \wedge \gamma^a \psi_A \tag{5.1}
\]

with

\[
s_0 V^a = \psi_0^a - \mathcal{D}_0 \varepsilon^a + \varepsilon^{ab} \wedge V_b. \tag{5.2}
\]

Let us put \( \omega_{2}^{ab} = \omega_0^{ab} - A^{ab} \), where \( A^{ab} \) is determined by the condition \( A^{ab} \wedge V_b = \frac{i}{2} \bar{\psi}_A \wedge \gamma^a \psi_A \).

We can identify the two variations of \( V^a \) (i.e. impose \( s_2 V^a = s_0 V^a \)) if we put

\[
\psi_0^a = i \bar{c}_A \wedge \gamma^a \psi_A - A^{ab} \wedge \varepsilon^b = \bar{\psi}_a + \cdots. \tag{5.3}
\]

As we see, there is no need to make the topological twist and the topological shift explicit. By comparing

\[
s_2 \varepsilon^a = \varepsilon^{ab} \wedge \varepsilon_b + \frac{i}{2} \bar{c}_A \wedge \gamma^a c_A \tag{5.4}
\]

with

\[
s_0 \varepsilon^a = \phi_0^a + \varepsilon^{ab} \wedge \varepsilon_b, \tag{5.5}
\]

we deduce

\[
\phi_0^a = \frac{i}{2} \bar{c}_A \wedge \gamma^a c_A = C^a + \cdots. \tag{5.6}
\]

By comparing \( s_2 \varepsilon^{ab} \) and \( s_0 \varepsilon^{ab} \), one deduces \( \eta_{2}^{ab} = \eta_0^{ab} \). After making these identifications, one can check by a direct but tedious computation that \( s_2 \psi_0^a \), \( s_2 \phi_0^a \), \( s_2 \chi_0^{ab} \), \( s_2 \eta_0^{ab} \) and \( s_2 \omega_0^{ab} \) automatically match with the corresponding \( s_0 \)-transformations.
Let us make a comment to the fact that the above identifications involve bilinear terms in the fields. This problem is promptly solved by the topological shift that reduces the above products of fields to a term linear in the fields plus interactions. The presence of these interactions is the consequence of the already noted fact that, since supersymmetry is local, we cannot specialize it to its \((0,0)^{0}_{0}\) component. Moreover, a little insight shows that the appearance of the bilinear terms is all but a problem. In fact, we must remember that in the topological twist chirality adds to ghost number; since the commutation properties between \(s\) and the fields is regulated by ghost-form number, it could happen, in general, that a field changes its properties of commutation with \(s\) during the twist. This would be surely dangerous, because it is important to preserve all formal manipulations to guarantee \(s^2 = 0\) on shell. Having to deal with bilinear terms, we are sure that the commutation properties do not change, since chirality is always even. An analogous observation can be done about fermion number; furthermore, since in the twisted theory fermion number has no importance, we can simply forget about it. By means of bilinear terms it is also possible to define the antighosts \(\psi_{0}^{ab}\) and \(\psi_{0}\), at least up to interaction pieces. The bilinear terms corresponding to them are \(\bar{c}_{A} \wedge c_{B}^{ab}\) and \(\bar{c}_{A} \wedge \psi_{B}\).

The above identifications permit to get explicitly the observables of the twisted theory, by simply taking the definitions (3.6), rewriting them in the twisted notation and shifting the gost \(C\). All that is needed is \(\chi_{0}^{ab}\) and \(\eta_{0}^{ab}\), which are given by

\[
\eta_{0}^{ab} = R_{2}^{cd} c^{c} \wedge \varepsilon^{d} + \bar{\theta}_{A ; c}^{ab} c_{A} \wedge \varepsilon^{c} - \frac{1}{2} \bar{c}_{A} \mathcal{F}^{ab} c_{B} \varepsilon_{AB},
\]

\[
\chi_{0}^{ab} \wedge V_{b} = -D_{0} \psi_{0}^{a} - R_{0}^{ab} \wedge \varepsilon_{b}. \tag{5.7}
\]

VI. THE LAGRANGIAN OF TOPOLOGICAL GRAVITY

In this section we discuss the twisted-shifted version of the lagrangian (2.4) of N=2 supergravity. In particular, we want to show that it can be written as the BRST variation of a gauge fermion \(\Psi\). We shall be satisfied of a gauge fermion \(\Psi\) that reproduces the kinetic
terms of the twisted-shifted N=2 supergravity lagrangian. In fact any gauge fermion can be corrected by adding to it interaction terms and indeed, even if they are not explicitly written, they are in general required in perturbation theory (a similar remark is made in [14]). In any case, the main requirement for a good gauge fermion is that it must remove all degeneracies of the kinetic terms and permit the definition of propagators, so we are justified in concentrating our attention on the kinetic terms, a restriction that simplifies considerably the computational effort.

First of all, we re-write the gravitational action in the second order formalism in a convenient way. As a matter of fact, one easily verifies that

\[ L = R^{ab} \wedge V^c \wedge V^d \epsilon_{abcd} = \]
\[ = 2\omega^{ab} \wedge \omega^c_e \wedge V^e \wedge V^d \epsilon_{abcd} + \]
\[ - \omega^{ae} \wedge \omega^b_e \wedge V^c \wedge V^d \epsilon_{abcd} + d(\omega^{ab} \wedge V^c \wedge V^d \epsilon_{abcd}), \]

(6.1)

and that

\[ A \equiv \omega^{-ab} \wedge M_{ab,cd} \wedge \omega^{-cd} \equiv \]
\[ \equiv \omega^{-ab} \wedge \omega^{-c}_e \wedge V^e \wedge V^d \epsilon_{abcd} - \frac{1}{2} \omega^{-ae} \wedge \omega^{-b}_e \wedge V^c \wedge V^d \epsilon_{abcd} = \]
\[ = L - d(\omega^{ab} \wedge V^c \wedge V^d \epsilon_{abcd} - 2iV^a \wedge dV_a). \]

(6.2)

In other words, we have written the gravitational lagrangian as quadratic in the antiselfdual part of the spin connection, which is our gauge-fixing, plus a total derivative. \( M_{ab,cd} \) is a two form and is independent from derivatives of the vierbein. This way of expressing the gravitational lagrangian (up to a topological term) should be compared with the expression \(-\frac{1}{4}\text{tr}[F_{\mu\nu}F^{-\mu\nu}]\) for the lagrangian of Yang-Mills theories, which is the square of the gauge-fixing of topological Yang-Mills theories [14]. Making space-time components explicit, we can write

\[ A \equiv \omega^{-ab}_\mu \wedge M_{\nu,\rho ab,cd} \wedge \omega^{-cd}_\sigma \epsilon^{\mu\nu\rho\sigma}. \]

(6.3)

\( M_{\nu,\rho ab,cd} \) is a matrix which is antysimmetric and antiselfdual in \( ab \) and \( cd \). One can easily verify that there exist only two such matrices in flat space, namely the identity
\[ I_{\nu,\rho,ab,cd} = \frac{1}{4} \eta_{\nu \rho} (\eta_{ac} \eta_{bd} - \eta_{ad} \eta_{bc} - i \epsilon_{abcd}), \]
where \( \eta_{ab} = \text{diag}(1, -1, -1, -1) \), and \( M_{\nu,\rho,ab,cd} \) itself.
Furthermore, \( M \) is invertible (one proves that \( M^2 \) is not proportional to \( M \), so it must be a nontrivial linear combination of \( M \) and \( I \)).

First of all, we introduce a Lagrange multiplier \( B^{ab} \) for the topological symmetry (a one form, antisymmetric and antiselfdual in \( ab \)), such that \( s' \psi^{ab} = B^{ab} \) and \( s'B^{ab} = 0 \). In this section we omit the subscript 0 in \( \psi_0^a, \psi_0^{ab} \) and \( \psi_0 \).

By comparing with the old expression for \( s' \psi^{ab} \), (4.4), we see that the linear part of the gauge-fixing term is \( \frac{i}{4} \omega^{-ab} - dC^{ab} \), that is to say there is a ghost term besides the expected term \( \frac{i}{4} \omega^{-ab} \). We shall explain in a short time the reason for this presence. In any case, we expect that the gauge-fermion \( \Psi \) contains a term

\[ \Psi_1 = 8i(2iB^{ab} + \omega^{-ab} + 4idC^{ab}) \wedge M_{ab,cd} \wedge \psi^{cd}. \]  

(6.4)

Indeed, the BRST variation of \( \Psi_1 \), i.e. \( s'\Psi_1 \), contains a term

\[ -8i(2iB^{ab} + \omega^{-ab} + 4idC^{ab}) \wedge M_{ab,cd} \wedge B^{ab}, \]  

(6.5)

which, upon integration over \( B^{ab} \) gives

\[ (\omega^{-ab} + 4idC^{ab}) \wedge M_{ab,cd} \wedge (\omega^{-cd} + 4idC^{cd}). \]  

(6.6)

So, the gravitational lagrangian (6.1) is correctly reproduced (at least up to a topological term), but there are two more terms, namely \( 8idC^{ab} \wedge M_{ab,cd} \wedge \omega^{-cd} \) and \( -16dC^{ab} \wedge M_{ab,cd} \wedge dC^{cd} \).

The first one is zero (or better, it is a total derivative), because

\[ d(M_{ab,cd} \wedge \omega^{-cd}) \equiv 0, \]  

(6.7)

as can be promptly checked. The second term looks like, at first sight, the kinetic lagrangian for the ghosts \( C^{ab} \), however this is not true, because the kinetic part of \( -16dC^{ab} \wedge M_{ab,cd} \wedge dC^{cd} \) turns out to be zero. From these remarks we can deduce two considerations: i) the gauge-fixing \( \omega^{-ab} = 0 \) is redundant, because the one forms \( \omega^{-ab} \) are not independent, but are related by the condition (6.7), which holds identically, without imposition of \( \omega^{-ab} = 0 \); ii) the
ghost \( C^{ab} \) is an extraghost, i.e., a ghost the presence of which is due to a redundancy of the gauge-fixing; of course, it is associated to the redundancy \((6.7)\) of the topological gauge-fixing conditions \( \omega^{-ab} = 0 \). A good treatment of such nontrivial ghosts of vanishing ghost number can be found in Ref. \[17\], where the case of the antisymmetric tensor, call it \( B_{\mu\nu} \), is explicitly exhibited. In that case the BRST variation of the antighost \( \bar{C}_\mu \) (\( \delta \bar{C}_\mu = \partial^\nu B_{\mu\nu} - \partial_\mu c_1 \)) contains, as well as the expected gauge-fixing term, \( \partial^\nu B_{\mu\nu} \), a term involving the extraghost \( c_1 \) and giving information about the redundancy (which is \( \partial_\mu \partial_\nu B^{\mu\nu} \equiv 0 \)), precisely as it happens in our case. However, in the simple example of the antisymmetric tensor \( B_{\mu\nu} \) the analogous term of \( -16 dC^{ab} \wedge \mathcal{M}_{ab,cd} \wedge dC^{cd} \) does give the kinetic lagrangian of the extraghost \( c_1 \) and so there is no further problem. In our case, instead, this does not happen (the reason is the richness of symmetries of our theory, in particular local supersymmetry). Since, as previously noted, only one matrix with the properties of \( \mathcal{M} \) exists besides \( \mathcal{M} \) itself, that is the identity \( \mathcal{I} \), there is little to do: to give a kinetic term to \( C^{ab} \), it is necessary to have a further extraghost, say \( C^{*ab} \) (of ghost number zero, antiselfdual in \( ab \)) and a kinetic term \( dC^{*ab} \wedge \mathcal{I}_{ab,cd} \wedge dC^{cd} \), that is to say \( C^{*ab} \Box C_{ab} \) plus interactions. However, since we are only reinterpreting a theory and we cannot construct it by hand, such a field and such a kinetic term must already be present. In particular \( C^{*ab} \) can only come from the twist of the antighost \( \bar{c}_A^* \) of \( N=2 \) local supersymmetry and as a matter of fact, Table II shows that such a field is indeed present and it is precisely the \((0,1)^0_0\)-component of the twisted version of \( \bar{c}_A^* \). So, all we have to do is to check that the gauge-fermion that breaks supersymmetry, say \( \Psi_S \), gives the correct kinetic lagrangian for \( C^{ab} \) and \( C^{*ab} \). Let us choose the most common expression for \( \Psi_S \), i.e.,

\[
\Psi_S \equiv \bar{c}_A^* \mathcal{D}(\gamma^\alpha V^\mu_{\alpha} \psi_{\mu A} + \alpha P_A),
\]

where \( P_A \) is the lagrange multiplier of supersymmetries (\( s'c_A^* = P_A, s'P_A = 0 \)) and \( \alpha \) is a constant that is usually determined in order to conveniently simplify the kinetic term of gravitinos (\( \alpha \) will be of no importance for our purposes). In any case, the BRST variation of \( \Psi_S \) contains a term of the kind
\( \partial^* \gamma^a V^\mu_a \partial_\mu c_A. \) \hspace{1cm} (6.9)

As we expect, after the twist, the quadratic term in \( C^{*ab} - C^{cd} \) is \( C^{*ab} \Box C_{ab}. \)

Let us now come back to the analysis of the BRST variation of the gauge-fermion \( \Psi_1 \), (5.3). The terms \(-8i(2iB^{ab} + \omega^{-ab} + 4idC^{ab}) \wedge s'(\mathcal{M}_{ab,cd}) \wedge \psi^{cd} \) are only interaction terms and so we discard them. Then there are the terms \( 8is'(\omega^{-ab} + 4idC^{ab}) \wedge \mathcal{M}_{ab,cd} \wedge \psi^{cd} \). By looking at (4.4), one sees that the term with \( dC^{ab} \) in \( \Psi_1 \) is required in order to restore invariance under Lorentz rotations (i.e. in order to avoid kinetic terms like \( d\varepsilon^{-ab} \wedge \mathcal{M}_{ab,cd} \wedge \psi^{cd} \)). So, the only kinetic term coming from \( \Psi_1 \) that remains to be discussed is \( 8i\chi^{-ab} \wedge \mathcal{M}_{ab,cd} \wedge \psi^{cd} \). This term reproduces the twisted version of the Rarita-Schwinger action, precisely the part that contains \( \psi^a \) and \( \psi^{ab} \), which turns out to be

\[ -16d\psi^a \wedge \psi_{ab} \wedge V^b. \] \hspace{1cm} (6.10)

The remaining piece of the Rarita-Schwinger action, namely

\[ -8d\psi^a \wedge \psi \wedge V_a, \] \hspace{1cm} (6.11)

can be retrieved by means of a further piece \( \Psi_2 \) to be added to the gauge fermion \( \Psi_1 \). \( \Psi_2 \) must also give account of the kinetic term of the graviphoton and turns out to be (remember \( \eta^{ab} = -\frac{1}{4}F^{-ab} + \cdots \) and \( R^\otimes = dA + \cdots \))

\[ \Psi_2 = 8iR^\otimes \wedge \psi^a \wedge V_a + \frac{2}{3}\eta^{ab}\varepsilon_{ab}V^i \wedge V^j \wedge V^k \wedge V^l \varepsilon_{ijkl}. \] \hspace{1cm} (6.12)

Summarizing, the total gauge-fermion is

\[ \Psi = 8i(2iB^{ab} + \omega^{-ab} + 4idC^{ab}) \wedge \mathcal{M}_{ab,cd} \wedge \psi^{cd} + \]
\[ + 8iR^\otimes \wedge \psi^a \wedge V_a + \frac{2}{3}\eta^{ab}\varepsilon_{ab}V^i \wedge V^j \wedge V^k \wedge V^l \varepsilon_{ijkl}, \] \hspace{1cm} (6.13)

plus the usual terms that break diffeomorphisms, Lorentz rotations, supersymmetries (this one being in part already discussed) and Maxwell gauge-symmetry.

27
VII. CONCLUSIONS AND OUTLOOK

Having shown that twisted N=2 supergravity is a formulation of D=4 topological gravity it remains to be seen which of the N=2 supergravity correlators are topological and how they are accordingly calculated using some version of intersection theory on instanton moduli-space. We postpone this investigation to a future publication. The same we do with the other obvious problem, namely the topological twisting of the hypermultiplets and their coupling to topological gravity. It is by now clear that, notwithstanding several quite important subtleties, there is a completely parallel development of the topological twisting programme in two and four dimensions and we think it worth to explore all its consequences.

APPENDIX A: NOTATION AND CONVENTIONS

In this appendix we give the notation for spinor algebra. The algebra of $\gamma$-matrices is represented by

$$\gamma^m = \begin{pmatrix} 0 & \sigma^m \\ \tilde{\sigma}^m & 0 \end{pmatrix}, \quad (A1)$$

where

$$\sigma^0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (A2)$$

and

$$\epsilon^{12} = \epsilon_{21} = 1, \quad \epsilon_{12} = \epsilon^{21} = -1. \quad (A4)$$

A Lorentz vector $v^m$ is represented by

$$v_{\dot{a}a} = (\sigma^m)_{\dot{a}a}v_m, \quad (A5)$$
and the inverse formula is

\[ v^m = -\frac{1}{2} v_{\alpha\dot{\alpha}} (\bar{\sigma}^m)^{\dot{\alpha}}. \]  

(A6)

An antisymmetric tensor \( F^{ab} \) is represented by

\[ F^{ab} = \frac{1}{2} (F^{+ab} + F^{-ab}) = \frac{1}{2} (f^{+\alpha} (\sigma^{ab})_\alpha^\beta + f^{-\beta} (\sigma^{ab})_\alpha^\dot{\beta}), \]  

(A7)

where

\[ F^{\pm ab} = F^{ab} \pm \frac{i}{2} \varepsilon^{abcd} F_{cd}, \]

\[ (\sigma^{ab})_\alpha^\beta = \frac{1}{4} (\sigma^a_{\alpha\dot{\alpha}} \bar{\sigma}^{b\dot{\alpha}\beta} - \sigma^b_{\alpha\dot{\alpha}} \bar{\sigma}^{a\dot{\alpha}\beta}), \]

\[ (\bar{\sigma}^{ab})_\dot{\beta}^{\dot{\alpha}} = \frac{1}{4} (\sigma^a_{\alpha\dot{\alpha}} \sigma^b_{\dot{\beta} \dot{\alpha}} - \sigma^b_{\alpha\dot{\alpha}} \sigma^a_{\dot{\beta} \dot{\alpha}}) \]  

(A8)

A generic spinor \( \psi_A \) is written as

\[ \psi_A = \begin{pmatrix} \psi_{\alpha A} \\ \psi_{\dot{\alpha} A} \end{pmatrix}, \]  

(A9)

while

\[ \bar{\psi}_A = (\psi^\alpha_A, \psi^{\dot{\alpha}}_A), \]  

(A10)

and indices are raised and lowered by means of \( e^{\alpha\beta} \) and \( e^{\dot{\alpha}\dot{\beta}} \). For other details, see [18].
TABLES

TABLE I. Topological twist

| Field | Before the twist | After the twist |
|-------|------------------|-----------------|
| $V^a$ | $0(\frac{1}{2}, \frac{1}{2}, 0)^0_1$ | $(\frac{1}{2}, \frac{1}{2})^0_1$ |
| $\varepsilon^a$ | $0(\frac{1}{2}, \frac{1}{2}, 0)^0_0$ | $(\frac{1}{2}, \frac{1}{2})^0_0$ |
| $\varepsilon^{ab}$ | $0(1, 0, 0)^0_0 \oplus 0(0, 1, 0)^0_0$ | $(1, 0)^0_0 \oplus (0, 1)^0_0$ |
| $\psi_A$ | $1(\frac{1}{2}, 0, \frac{1}{2})^0_1 \oplus -1(0, \frac{1}{2}, \frac{1}{2})^0_1$ | $(\frac{1}{2}, \frac{1}{2})^0_0 \oplus (0, 1)^0_1 \oplus (0, 0)^0_0$ |
| $c_A$ | $1(\frac{1}{2}, 0, \frac{1}{2})^0_0 \oplus -1(0, \frac{1}{2}, \frac{1}{2})^0_0$ | $(\frac{1}{2}, \frac{1}{2})^2_0 \oplus (0, 1)^0_0 \oplus (0, 0)^0_0$ |
| $A$ | $(0, 0, 0)^0_1$ | $(0, 0)^0_1$ |
| $c$ | $(0, 0, 0)^0_0$ | $(0, 0)^0_0$ |

TABLE II. Twist of antighosts and Lagrange multipliers

| Field | Before the twist | After the twist |
|-------|------------------|-----------------|
| $\bar{\varepsilon}^a$ | $0(\frac{1}{2}, \frac{1}{2}, 0)^0_0^{-1}$ | $(\frac{1}{2}, \frac{1}{2})^0_0^{-1}$ |
| $\bar{\varepsilon}^{ab}$ | $0(1, 0, 0)^0_0^{-1} \oplus 0(0, 1, 0)^0_0^{-1}$ | $(1, 0)^0_0^{-1} \oplus (0, 1)^0_0^{-1}$ |
| $\bar{c}_A$ | $-1(\frac{1}{2}, 0, \frac{1}{2})^0_0^{-1} \oplus 1(0, \frac{1}{2}, \frac{1}{2})^0_0^{-1}$ | $(\frac{1}{2}, \frac{1}{2})^2_0^{-1} \oplus (0, 1)^0_0^{-1} \oplus (0, 0)^0_0^{-1}$ |
| $\bar{c}$ | $(0, 0, 0)^0_0^{-1}$ | $(0, 0)^0_0$ |
| $\pi^a$ | $0(\frac{1}{2}, \frac{1}{2}, 0)^0_0$ | $(\frac{1}{2}, \frac{1}{2})^0_0$ |
| $\pi^{ab}$ | $0(1, 0, 0)^0_0^{-1} \oplus 0(0, 1, 0)^0_0^{-1}$ | $(1, 0)^0_0^{-1} \oplus (0, 1)^0_0^{-1}$ |
| $P_A$ | $-1(\frac{1}{2}, 0, \frac{1}{2})^0_0 \oplus 1(0, \frac{1}{2}, \frac{1}{2})^0_0$ | $(\frac{1}{2}, \frac{1}{2})^0_0^{-1} \oplus (0, 1)^0_0^{-1} \oplus (0, 0)^0_0$ |
| $P$ | $(0, 0, 0)^0_0$ | $(0, 0)^0_0$ |
| Field | Meaning | Classification | Tw.-field | Twisted meaning | Tw.-classification |
|-------|---------|----------------|-----------|-----------------|-------------------|
| $V^a$ | vierbein | $^0(\frac{1}{2}, \frac{1}{2}, 0)_1$ | $V^a$ | vierbein | $^0(\frac{1}{2}, \frac{1}{2})_1$ |
| $\psi^a$ | topological ghost | | | | $(\frac{3}{2}, \frac{1}{2})_1$ |
| $\psi^a_A$ | gravitinos | $^1(\frac{1}{2}, 0, \frac{1}{2})_0 \oplus -^1(0, \frac{1}{2}, \frac{1}{2})_1$ | $\psi^{ab}$ | topological antighost | $(0, 1)^{-1}_1$ |
| $\bar{\psi}$ | antighost | | | | $(0, 0)^{-1}_1$ |
| $A$ | graviphoton | $(0, 0, 0)_0^0$ | $A$ | ghosts for ghosts | $(0, 0)_1^1$ |
| $\varepsilon^a$ | ghost | $^0(\frac{1}{2}, \frac{1}{2}, 0)_0^1$ | $\varepsilon^a$ | ghost | $(\frac{3}{2}, \frac{1}{2})_0^1$ |
| $\pi^a$ | L. multiplier | $^0(\frac{1}{2}, \frac{1}{2}, 0)_0^0$ | $\pi^a$ | L. multiplier | $(\frac{3}{2}, \frac{1}{2})_0^0$ |
| $\varepsilon^{ab}$ | antighost | $^0(\frac{1}{2}, \frac{1}{2}, 0)_0^{-1}$ | $\varepsilon^{ab}$ | antighost | $(\frac{3}{2}, \frac{1}{2})_0^{-1}$ |
| $\varepsilon^{ab}$ | ghost | $^0(1, 0, 0)_0^1 \oplus ^0(0, 1, 0)_0^1$ | $\varepsilon^{ab}$ | ghost | $(1, 0)_0^1 \oplus (0, 1)_0^1$ |
| $\pi^{ab}$ | L. multiplier | $^0(1, 0, 0)_0^{-1} \oplus ^0(0, 1, 0)_0^{-1}$ | $\pi^{ab}$ | L. multiplier | $(1, 0)_0^{-1} \oplus (0, 1)_0^{-1}$ |
| $\varepsilon^{ab}$ | antighost | $^0(1, 0, 0)_0^{-1} \oplus ^0(0, 1, 0)_0^{-1}$ | $\varepsilon^{ab}$ | antighost | $(1, 0)_0^{-1} \oplus (0, 1)_0^{-1}$ |

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| $C^a$ | ghost for ghost | $(\frac{1}{2}, \frac{1}{2})_0^1$ |
| $c_A$ | ghost | $\frac{1}{2}(\frac{1}{2}, 0, \frac{1}{2})_0^1 \oplus -1(0, \frac{1}{2}, \frac{1}{2})_0^1$ |
| $C^{ab}$ | extraghost | $(0, 1)_0^0$ |
| $C$ | extraghost | $(0, 0)_0^0$ |
| $P^a$ | antighost | $(\frac{1}{2}, \frac{1}{2})_0^{-1}$ |
| $P_A$ | L. multiplier | $-1(\frac{1}{2}, 0, \frac{1}{2})_0^0 \oplus 1(0, \frac{1}{2}, \frac{1}{2})_0^0$ |
| $P^{ab}$ | ghost | $(0, 1)_0^1$ |
| $P'$ | ghost | $(0, 0)_0^1$ |
| $C^{*a}$ | antighost for ghost | $(\frac{1}{2}, \frac{1}{2})_0^{-2}$ |
| $c^*_A$ | antighost | $-1(\frac{1}{2}, 0, \frac{1}{2})_0^{-1} \oplus 1(0, \frac{1}{2}, \frac{1}{2})_0^{-1}$ |
| $C^{*ab}$ | extraghost | $(0, 1)_0^0$ |
| $C^*$ | extraghost | $(0, 0)_0^0$ |
| $c$ | ghost | $(0, 0, 0)_0^1$ |
| $P$ | L. multiplier | $(0, 0, 0)_0^0$ |
| $\bar{c}$ | antighost | $(0, 0, 0)_0^{-1}$ |
| $\bar{c}$ | ghost | $(0, 0)_0^0$ |
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