Stieltjes Functions and Hurwitz Stable Entire Functions

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Abstract The concept of stability, originally introduced for polynomials, will be extended to apply to the class of entire functions. This generalization will be called Hurwitz stability and the class of Hurwitz stable functions will serve as the main focus of this paper. A first theorem will show how, given a function of either of the Stieltjes classes, a Hurwitz stable function might be constructed. A second approach to constructing Hurwitz stable functions, based on using additional functions from the Laguerre-Pólya class, will be presented in a second theorem.

Keywords Stable polynomials · Stable entire functions · Routh–Hurwitz criterium · Stieltjes classes of holomorphic functions · Stieltjes moment problem · Polya-Schur composition Theorem · Laguerre-Pólya class of entire functions

Mathematics Subject Classification (2000) Primary 30D15 · 30C15 · 43A80; Secondary 70E50

1 Introduction

The main focus of this paper will be on a particular generalization of the idea of polynomial stability (here, we mean polynomials with complex coefficients).
Stable polynomials are polynomials which only have roots in the open left half-plane \( \{ z : \Re z < 0 \} \). These polynomials are important in automatic control theory. The well-known Routh–Hurwitz Criterion allows for a complete characterization of stable polynomials in terms of their coefficients.

There was already much interest in generalizing the Routh–Hurwitz Stability Criterion for suitable classes of entire functions in the early part of the last century. (See, for instance, the monographs [1,2].) We now introduce the stability concept for entire functions, which we will be using:

**Definition 1** An entire function \( \Phi(z) \) is called Hurwitz stable if \( \Phi(z) \) grows not more than exponentially, i.e.

\[
\lim_{z \to \infty} |z|^{-1} \ln |\Phi(z)| < \infty,
\]

and satisfies the conditions:

1. All roots of the function \( \Phi(z) \) lie in the open left half-plane \( \{ z : \Re z < 0 \} \);
2. \( h_\Phi(0) \geq h_\Phi(\pi) \), where \( h_\Phi(\theta) \) is the indicator function of \( \Phi \), i.e.

\[
h_\Phi(\theta) = \lim_{r \to \infty} r^{-1} \ln |\Phi(re^{i\theta})|.
\]

A variant of stability for entire functions, where the left half-plane is replaced by the upper half-plane, can be found in publications on Hilbert spaces of entire functions by L. de Branges (see, e.g., his monograph [3]).

The central goal of this paper is to present two methods for constructing particular classes of Hurwitz stable entire functions. At the outset, we begin with specific classes of functions holomorphic in \( \mathbb{C} \setminus (-\infty, 0] \), namely, the functions belonging to the Stieltjes classes \( S \) and \( S^{-1} \), introduced in Definition 2. The importance of these function classes first became apparent through Stieltjes’ classical work [4] on what is now known as the Stieltjes Moment Problem. Stieltjes’ method involved associating each non-negative finite measure \( d\sigma(\lambda) \) supported on \( [0, \infty) \) with a function \( \psi_\sigma(z) = \int_0^{+\infty} \frac{d\sigma(\lambda)}{\lambda + z} \), \( z \in \mathbb{C} \setminus (-\infty, 0] \), which is holomorphic in \( \mathbb{C} \setminus (-\infty, 0] \). This function \( \psi_\sigma(z) \) is called the Cauchy transform of the measure \( d\sigma \). The Cauchy transforms of finite non-negative measures supported on \( [0, \infty) \) are functions \( \psi \) holomorphic in the domain \( \mathbb{C} \setminus (-\infty, 0] \) and possessing certain positivity properties there. These positivity properties are:

\[
\psi(x) \geq 0 \quad \text{for } x > 0, \quad \text{Im} \psi(z) \leq 0 \quad \text{for } \text{Im} z > 0, \quad \text{Im} \psi(z) \geq 0 \quad \text{for } \text{Im} z < 0.
\]

It can be shown that the Stieltjes class coincides with the class of Cauchy transforms of finite non-negative measures supported on \( [0, +\infty) \).

The data for the Stieltjes Moment Problem is given as a sequence \( \{ s_k \}_{0 \leq k < \infty} \) of real numbers. A solution of the Stieltjes Moment Problem is any finite non-negative measure \( d\sigma(\lambda) \) supported on the positive half-axis \( [0, +\infty) \) such that the moments of this measure coincide with their respective terms of the data sequence, i.e.: \( \int_0^{+\infty} \lambda^k d\sigma(\lambda) = s_k, 0 \leq k < \infty \). The Stieltjes Moment Problem (which, in