Relative mechanical quantities in spaces with a transport along paths

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Abstract
The concepts of relative velocity and acceleration, deviation velocity and acceleration and relative momentum of point particles in spaces (manifolds), the tangent bundle of which is equipped with a transport along paths, are introduced. If the tangent bundle is endowed also with a metric, it gives rise also to the notion of a relative energy. Certain ties between these quantities are considered. The cases of massless particles and of special relativity are presented in this context.

1. INTRODUCTION
The introduced in [1] transports along paths, which in particular can be linear [2], are applied in the present paper to defining certain mechanical quantities

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in spaces (manifolds), the tangent bundle of which is endowed with such a transport. Analogous problem has been considered in [3] but, in fact, in this work only linear transports along paths without self-intersections are used which is not generally necessary everywhere. We closely follow [3] without presupposing such restrictions.

All considerations in the present work are made in a (real) differentiable manifold $M[4, 5]$ whose tangent bundle $(T(M), \pi, M)$ is endowed with a transport along paths [1]. Here $T(M) := \bigcup_{x \in M} T_x(M),$ $T_x(M)$ being the tangent to the $M$ space at $x \in M$ and $\pi : T(M) \to M$ is such that if $V \in T_x(M),$ then $\pi(V) := x.$

By $J$ and $\gamma : J \to M$ are denoted, respectively, an arbitrary real interval and a path in $M.$ If $\gamma$ is of class $C^1,$ its tangent vector is written as $\dot{\gamma}.$

The transport along paths in $(T(M), \pi, M)$(cf.[1]) is a map $I : \gamma \to I^\gamma, I^\gamma : (s, t) \to \gamma^\gamma_{s \to t}, s, t \in J$ being the transport along $\gamma$, where $\gamma^\gamma_{s \to t} : T_\gamma(s)(M) \to T_\gamma(t)(M),$ satisfy the equalities

$$I^\gamma_{t \to r} \circ I^\gamma_{s \to t} = I^\gamma_{s \to r}, \quad r, s, t \in J.$$ (1.1)

$$I^\gamma_{s \to s} = id T_\gamma(s), \quad s \in J.$$ (1.2)

Here $id_X$ is the identity map of the set $X.$

A linear transport ($L$-transport) along paths $L$ in $(T(M), \pi, M)$ satisfies besides (1.1) and (1.2) also the equality (cf.[2])

$$L^\gamma_{s \to t}(\lambda U + \mu V) = \lambda L^\gamma_{s \to t}U + \mu L^\gamma_{s \to t}V, s, t \in J, U, V \in T_\gamma(s)(M).$$ (1.3)

In Sect. 2, the concepts of relative velocity, deviation velocity and the corresponding to them accelerations between two point particles are introduced. Sect. 3 is devoted to the relative momentum of these particles. The central role in this investigation belongs by Sect. 4. In spaces, the tangent bundle of which is endowed with a metric and a transport along paths, the relative energy of two point particles is introduced and investigated. Certain connections between the mentioned concepts are studied and the notion of a proper (rest) energy is naturally obtained. A note on the zero-mass particles case is made. Sect. 5 illustrates the considered general concepts in the case of special theory of relativity. The paper ends with some concluding remarks in Sect. 6.

2. RELATIVE VELOCITY AND RELATIVE ACCELERATION

Let there be given paths $x_a : J_a \to M,$ $a = 1, 2$ and $x : J \to M.$ Let there be fixed one-to-one maps $\tau_a : J \to J_a,$ $a = 1, 2.$ (The maps $\tau_1$ and $\tau_2$ always exist because all real intervals are equipollent.) Let also be given the one parameter families of paths $\{\gamma_s : \gamma_s : J \to M, s \in J\}$ and $\{\eta_s : \eta_s : J \to M, s \in J\}$ having the properties $\gamma_s(r) := x_1(\tau_1(s)) =: \eta_s(t),$ $\gamma_s(r) := x_2(\tau_2(s))$ and $\eta_s(t) := x(s)$ for some $r, x \in J$ and $t, t \in J,$ $s \in J.$

Physically the paths $x_1, x_2$ and $x$ are interpreted as trajectories (world lines) of, respectively, observed point particles 1 and 2 and a point observer observing them. The parameters $s \in J, s_1 = \tau_1(s)$ and $s_2 = \tau_2(s)$ are interpreted as proper times of the corresponding particles (cf.[8], sect. 2).
If the particles 1 and 2 are moving along the paths \( x_1 \) and \( x_2 \) respectively, then their velocities are \([6,7]\)

\[
V_a := \dot{x}_a, \quad a = 1, 2. \tag{2.1}
\]

The vectors \( V_1 \) and \( V_2 \) can not be compared as they are defined at different points. To compare them, we put

\[
(V_2)_1 := I_{t_1}^{s_{1_{r_2}}} V_2 \in T_{x_1}(M). \tag{2.2}
\]

As \((V_2)_1\) and \( V_1 \) are defined at one and the same point, the vector

\[
\Delta V_{21} := \Delta V_{21}(s; x) := I_{t_1}^{s_{1_{r_2}}}((V_2)_1 - V_1) = I_{t_1}^{s_{1_{r_2}}}(I_{r_2}^{s_{1_{r_2}}} V_2 - V_1) \in T_{x(s)}(M) \tag{2.3}
\]

is uniquely defined and represents their difference defined at \( x(s) \) with the help of I. This vector is called a relative velocity of the second observed particle with respect to the first one (as it is “seen” from the observer) at the point \( x(s) \).

This definition of a relative velocity is a natural generalization of the Newtonian concept for a relative velocity which can be simply defined as a difference of the 3-vectors representing the velocities of the corresponding particles.

Let the paths \( \gamma_s, s \in J \) be of class \( C^1 \) and such that the maps

\[
d_s : J \to T_{\gamma(s)}(M) = \pi^{-1}(\gamma(s)), \quad s \in J, \tag{2.4a}
\]

defined by

\[
d_s(t) := \int_a^t (I_{u \to s}^{r_2} \gamma(u))du, \quad s, t \in J, \tag{2.4b}
\]

be homeomorphisms from \( J \) into \( d_s(J) \) for every \( s \in J \). (cf.[8], sect. 2).

According to [8], definition 2.3 the deviation vector of \( x_2 \) with respect to \( x_1 \) relatively to \( x \) at the point \( x(s), s \in J \), i.e. between the investigated particles, is

\[
h_{21} := h_{21}(s; x) := \left( I_{t_1}^{s_{1_{r_2}}} \circ d_s^{r_2} \right)(r)
\]

\[
= I_{t_1}^{s_{1_{r_2}}} \int_{r_2}^{r_2} (I_{u \to r_2}^{r_2} \gamma_s(u))du \in T_{x(s)}(M). \tag{2.5}
\]

Let in the manifold \( M \) be given also a covariant differentiation \( \nabla \) and the deviation vector \( h_{21} \) of \( x_2 \) with respect to \( x_1 \) have a\( C^1 \) dependence on \( s \). Then there arises the concept for a deviation velocity \( V_{21} \) between the observed particles:

\[
V_{21} := \left. \frac{D}{ds} \right|_x h_{21}, \tag{2.6}
\]

where \( D/\partial_s \mid x \) is the covariant differentiation along \( x \) and the deviation vector is given by (2.5). This velocity has a direct physical meaning because
it can be measured. For example, if the observer defines somehow (e.g. by radiolocation) the relative position \( h_{21} \) of the observed particles, then he can find the deviation velocity from (2.6) in which \( s \) is now interpreted as observer’s "proper time".

Generally speaking, the vectors \( \Delta V_{21} \) and \( V_{21} \) do not coincide even in the Euclidean case (see [8], sect.4) in which we evidently have

\[
\Delta V_{21}|_{E^n} = V_2 - V_1, \quad (2.7)
\]

\[
V_{21}|_{E^n} = \frac{d}{ds}(x_2(\tau_2(s)) - x_1(\tau_1(s))) = \frac{dr_2(s)}{ds} \cdot V_2 - \frac{dr_1(s)}{ds} \cdot V_1. \quad (2.8)
\]

Nevertheless, in the Newtonian mechanics, where we have an Euclidean world with an absolute simultaneity \((\tau_1 = \tau_2 = \text{id}_J)\), these velocities coincide.

Let the manifold \( M \) be endowed with a transport of vectors along paths and a covariant differentiation. If \( x_1 \) and \( x_2 \) are \( C^2 \) paths, then the accelerations of the observed particles are

\[
A_a := \frac{D}{ds}|_{x_a} V_a, \quad a = 1, 2 \quad (2.9)
\]

and we can define in an analogous way the relative acceleration and the deviation acceleration between them and the observer, respectively, by the equalities

\[
\Delta A_{21} := I^{\mu}_{\tau_2 \rightarrow \tau_1}(x_2 - x_1), (A_2)_1 := I^{\mu}_{\tau_2 \rightarrow \tau_1} A_2, \quad (2.10)
\]

\[
A_{21} := \frac{D}{ds}|_x V_{21} = \frac{D^2}{ds^2}|_x h_{21}. \quad (2.11)
\]

The treatment of \( \Delta A_{21} \) and \( A_{21} \) is similar to the one of \( \Delta V_{21} \) and \( V_{21} \).

3. RELATIVE MOMENTUM

Let a point particle with a (rest) mass \( m \) be moving along the path \( \gamma : J \rightarrow M \). Then by definition (see [6], ch. III, §3) its momentum at the point \( \gamma(s) \) is

\[
p := p(s) := \mu(s) \dot{\gamma}(s), \quad s \in J
\]

where \( \mu : J \rightarrow \mathbb{R} \setminus \{0\} \) is a scalar function with a dimension of mass. If \( m \neq 0 \), then \( \mu(s) := m \). If \( m = 0 \), which is the case, e.g., with the photons, then the momentum \( p \) is considered as a primary defined quantity and \( \mu \) is obtained from the above equation. It is important to be noted that in both the cases \( \mu(s) \neq 0, s \in J \). (The case \( m = \mu(s) = 0 \) describes the vacuum but not a particle.)

So, the momenta of the observed particles are

\[
p_a := p_a(s_a) := \mu_a(s_a) V_a, \quad s_a = \tau_a(s), \quad a = 1, 2, \quad s \in J, \quad (3.1)
\]

where \( \mu_a : J_a \rightarrow \mathbb{R} \setminus \{0\}, a = 1, 2 \) are scalar functions.
As the vector
\[ (p_2)_1 := T_{\gamma^0_{\gamma}} \rightarrow r'_p p_2 \in T_{x_1}(M) \] (3.2)
is in \( T_{x_1}(M) \), it can be compared with \( p_1 \). In accordance with this, (the vector of) the relative momentum of the second particle with respect to the first one as it is "seen" from the observer at \( x(s) \) is defined by
\[
\Delta p_{21} := \Delta p_{21}(s; x) := I_{\gamma^0_{\gamma}} \rightarrow r'_p ((p_2)_1 - p_1) = I_{\gamma^0_{\gamma}} \rightarrow r'_p p_2 - p_1 \in T_{x(s)}(M). \tag{3.3}
\]
It is clear that in the Euclidean case (see [8], sect. 4) the relative momentum takes its well known Newtonian form
\[
\Delta p_{21}\big|_{E^n} = p_2 - p_1. \tag{3.4}
\]
If the used above transport in \( (T(M), \pi, M) \) is linear (see Sect. 1, eq.(1.3) or [2]), then due to (3.1) – (3.3) and (2.2) – (2.3) the following equalities are valid
\[
(p_2)_1 = \mu_2(s_2)(V_2)_1, \tag{3.5}
\]
\[
\Delta p_{21} = \mu_2(s_2)\Delta V_{21} + [\mu_2(s_2)/(\mu_1(s_1) - 1)] I_{\gamma^0_{\gamma}} \rightarrow r'_p p_1. \tag{3.6}
\]

4. RELATIVE ENERGY

Let in the tangent bundle \( (T(M), \pi, M) \) be given a transport along paths \( I \) and a real bundle metric \( g \), i.e., [5] a map \( g : x \rightarrow g_x, x \in M \), where the maps \( g_x : T_x(M) \otimes T_x(M) \rightarrow \mathbb{R} \) are bilinear, nondegenerate and symmetric. For brevity, the defined by \( g \) scalar products of \( X, Y \in T_y(M), y \in M \) will be denoted by a dot (\( \cdot \)), i.e. \( X \cdot Y := g_y(X, Y) \). The scalar square of \( X \) will be written as \( (X)^2 \) for it has to be distinguished from the second component \( X^2 \) of \( X \) in some local basis (in the case when \( \dim(M) > 1 \)). As \( g \) is not supposed to be positively defined, \( (X)^2 \) can take any real values.

By definition the relative energy of the second particle with respect to the first one is called the (scalar) quantity
\[
E_{21} := E_{21}(s) := \epsilon((V_1(s_1))^2)p_{21} \cdot V_1(s_1) = \epsilon((V_1(s_1))^2)(I_{\gamma^0_{\gamma}} \rightarrow r'_p p_2(s_2)) \cdot V_1(s_1), \tag{4.1}
\]
where \( \epsilon(\lambda) := -1 \) for \( \lambda < 0 \) and \( \epsilon(\lambda) := +1 \) for \( \lambda \geq 0 \). The introduction of the multiplier \( \epsilon \) is due to the fact that if the particles coincide, i.e., if we apply (4.1) to one and the same particle, then the so obtained quantity has a meaning of a proper energy of that particle (see below) and according to the accepted opinion [6,7] it must be positive.

If there exists \( s_0 \in J \) such that \( x_1(\gamma(s_0)) = x_2(\tau(s_0)) \), i.e. if at the "moment" \( s = s_0 \) the trajectories of the observed particles intersect each other, then from (4.1) and (1.2) we get
\[
E_{21}(s_0) = \epsilon((V_1(\gamma(s_0)))^2)p_2(\tau(s_0)) \cdot V_1(\gamma(s_0)). \tag{4.2}
\]
In the case of the space-time of general relativity, this expression coincides with the given in [6], ch. III, §6, eq.(23) definition for a relative energy which has the "bad" property that it is valid only for the "moment" $s = s_0$. So it does not allow the evolution of the relative energy in time to be studied. Evidently, our definition (4.1) is free from this deficiency.

Analogously to (4.1), the relative energy of the first particle with respect to the second one is

$$E_{12} := E_{12}(s) := \epsilon((V_2(s_2))^2)p_{12} \cdot V_2(s_2)$$

$$= \epsilon((V_2(s_2))^2)(I_{\gamma_{s} \rightarrow t} \tau_{s} p_1(s_1)) \cdot V_2(s_2). \quad (4.3)$$

If we use arbitrary transports along paths, then, generally, the quantities $E_{12}, E_{21}, \Delta p_{21}$ and $\Delta p_{12}$ are not connected somehow with each other. From the viewpoint of the existence of a certain connection between them an essential role is played by the transports along paths which are consistent (at least along the paths $\gamma_s$ and $\eta_s$, $s \in J$) with the fibred metric $g$, i.e. for which (c.f. [9, 10])

$$I_{\gamma_{s} \rightarrow t} (U \cdot V) = (I_{\eta_{s} \rightarrow t} U) \cdot (I_{\gamma_{s} \rightarrow t} V) \quad (4.4)$$

for arbitrary $\gamma : J \rightarrow M, s, t \in J$ and $U, V \in T_{\gamma(s)}(M)$ (For further considerations it is enough that this equality be valid only for $\gamma \in \{ \gamma_s, \eta_s : s \in J \}$.)

If $I$ and $g$ are consistent, then the relative momentum (3.3) and the relative energy (4.1), as one can easily prove, are connected by the relation

$$E_{21}(s) = \epsilon((V_1(s_1))^2)[\Delta p_{21} \cdot I_{\gamma_{s} \rightarrow t} p_1(s_1) + p_1 \cdot V_1(s_1)]. \quad (4.5)$$

If the transport along paths $I$ is consistent with the operation multiplication with real numbers (see [10], example 3.2), i.e.

$$I_{\gamma_{s} \rightarrow t}(\lambda U) = \lambda I_{\gamma_{s} \rightarrow t} U, \quad \lambda \in \mathbb{R}, \quad U \in T_{\gamma(s)}(M), \quad (4.6)$$

then (3.5) holds and after its substitution into (4.1), one gets

$$E_{21} = \epsilon((V_1(s_1))^2)\mu_2(s_2)/(V_2)_{2} \cdot V_1(s_1). \quad (4.7)$$

If the equalities (4.4) and (4.6) are simultaneously valid, then with a direct verification we confine ourselves to that the relative energies $E_{21}$ and $E_{12}$ are connected by

$$\epsilon((V_2(\tau_2(s))^2))\mu_1(\tau_1(s))E_{21}(s) = \epsilon((V_1(\tau_1(s))^2))\mu_2(\tau_2(s))E_{12}(s). \quad (4.8)$$

In particular, this equality is true for every $L$-transport consistent with the metric.

Let us apply definition (4.1) only to the first observed particle, for which it is enough to put in it $r = r, \ x_2 = x_1$ and $\tau_2 = \tau_1$, or equivalently to replace the subscript 2 with 1. Using (1.2), we see that the energy of this particle (with respect to itself) is

$$E_{11}(s) = \epsilon((V_1(s_1))^2)p_1(s_1) \cdot V_1(s_1) = \mu_1(s_1) | (V_1(s_1))^2 |
\[ \lambda := \epsilon(\lambda) \lambda \] is the absolute value of \( \lambda \in \mathbb{R} \).

The quantity \( E_{11} \) may be called a proper (or rest) energy of the considered particle. If \( m_1 > 0 \), then \( \mu_1(s_1) = m_1 \) and consequently \( E_{11} \geq 0 \). If \( m_1(V_1(s_1))^2 \neq 0 \), then \( E_{11} > 0 \) which corresponds to the most popular case of massive material particle.

If \( m_1 \neq 0 \), then \( \mu_1(s_1) := m_1 \) and due to (4.9) the proper energy \( E_{11} \) is proportional to \( m_1 \), so \( E_{11} \) is a \( C^\infty \) function of \( m_1 \) for \( m_1 \in \mathbb{R} \setminus \{0\} \). From here comes the mind on \( E_{11} \) to be imposed the additional restriction for continuous dependence of \( m_1 \) at the point \( m_1 = 0 \), i.e. one may want

\[ E_{11} = 0 \quad \text{for } m_1 = 0, \quad (4.10) \]

or, equivalently,

\[ \lim_{m_1 \to 0} E_{11} = 0, \quad (4.11) \]

which has far going physical corollaries. In fact, (4.9) shows the equivalence of (4.11) with

\[ (V_1(s_1))^2 = 0 \quad \text{for } m_1 = 0, \quad (4.12) \]

or, which is all the same, with

\[ (p_1(s_1))^2 = 0 \quad \text{for } m_1 = 0. \quad (4.13) \]

These relations are a direct generalization of the well known fact from the special and general relativity that the massless particles are moving with the velocity of light, i.e. that their world lines lie on the light cone described by (4.12).

We want to note that without further assumptions \((V_1(s_1))^2 = 0\) does not imply \( m_1 = 0 \).

The energies \( E_{21} \) (or \( E_{12} \)) and \( E_{11} \) may be connected with the components of \( \Delta p_{21} \) (or \( \Delta p_{12} \)) and \( p_{12} \) in some local bases in the following way.

Let \((V_1)^2 \neq 0 \). Along \( x \) we define a field of basis \( \{\lambda_i\} \), i.e. the vectors \( \lambda_i \mid_{\gamma(s) \in T_{\gamma(s)}(M)} \) form a basis in \( T_{\gamma(s)}(M) \), such that \( \lambda_1 := V_1 \cdot (V_1)^2 \) \( |_{-1/2} \) and \( \lambda_1 \cdot \lambda_i = 0 \) for \( i \neq 1 \) (if \( \dim(M) > 1 \)). Here and henceforth the Latin indices run from 1 to \( \dim(M) \). (In this case the concrete choice of \( \lambda_i \) for \( i \neq 1 \) is insignificant.) So \( (\lambda_1)^2 = \epsilon((V_1)^2) \), due to which the component \( A^1 \) of any vector field \( A = A^i \lambda_i \) along \( x \) in \( \{\lambda_i\} \) is

\[ A^1 = A \cdot \lambda_1/(\lambda_1)^2 = \epsilon((V_1)^2)(A \cdot V_1) \mid (V_1)^2 \mid^{-1/2}. \quad (4.14) \]

Applying this equality to \( p_1, p_{21} \) and \( \Delta p_{21} := \Delta p_{21, x=x_1} = (p_2)_1 - p_1 \), the last vector being the relative momentum of the second particle with respect to the first one as it is "seen" from the latter, and using (4.9) and (4.1), we find:

\[ p_1^i = E_{11} \mid (V_1)^2 \mid^{-1/2}, p_i^i = 0 \quad \text{for } i \neq 1, \quad (4.15) \]

\[ (p_{21})_1^1 = E_{21} \mid (V_1)^2 \mid^{-1/2}, \quad (4.16) \]
\[
\Delta \pi_{21}^i = (p_{21}^i)^2 - p_{1}^i = (E_{21} - E_{11}) | (V_{1})^2 |^{-1/2}. \tag{4.17}
\]

Let \((V_{1})^2 \neq 0\) and \(g\) and \(I\) be consistent, i.e. (4.4) be valid. Defining along \(x_{a}\) basis \(\{l_{i}\}\) such that \(l_{i} := l_{i}^{\mu}_{\nu} \rightarrow l_{i}^{\nu}_{\mu} \lambda_{1}\) and \(l_{i} \cdot l_{i} = 0\) for \(i \neq 1\), we see that the first component of \(\Delta p_{21}\) in \(\{l_{i}\}\) is
\[
\Delta p_{21}^1 = \Delta p_{21} \cdot l_{i}/(l_{i})^2 = \Delta \pi_{21} \cdot \lambda_{1}/(\lambda_{1})^2 = \Delta \pi_{21}^1
\]

If \((V_{1})^2 = 0\), then (see (4.9)) \(E_{11} = 0\) and the invariant \((p_{2})_{1} \cdot V_{1} = \Delta \pi_{21} \cdot V_{1} = E_{21}\) cannot be connected with some component of \((p_{2})_{1}\) or \((\Delta \pi_{21})\) in a given local basis. In this case, we can say that the relative energy \(E_{21}\) is spread over all the components of \((p_{2})_{1}\) and \((\Delta \pi_{21})\) and in basis transformations it cannot be connected with a single component of that vector.

If \((V_{2})^2 \neq 0\) and \(I\) and \(g\) are consistent, i.e. (4.4) holds, then defining along \(x_{1}a\) basis \(\{\lambda_{i}'\}\) such that \(\lambda_{i}' := \lambda_{i}' \rightarrow \lambda_{i}' (V_{2})_{2} | (V_{2})^2 |^{-1/2}\) and \(\lambda_{1}' \cdot \lambda_{i}' = 0\) for \(i \neq 1\), we find the first component of \(p_{1}\) in \(\{\lambda_{i}\}\) as
\[
p_{i}' = p_{1} \cdot \lambda_{1}/(\lambda_{1})^2 = p_{1} \cdot (\lambda_{i}' \rightarrow \lambda_{i}')(V_{2} | (V_{2})^2 |^{-1/2}) \cdot \epsilon((V_{2})^2)
\]

At the end of this section we want to stress the fact that all relative energies \(E_{11}, E_{21}, E_{12}\) and \(E_{22}\), connected with the considered observed particles, are not arbitrary, as they are connected with the invariant \((\Delta \pi_{21})^2\) by
\[
(\Delta \pi_{21})^2 = \epsilon((V_{1})^2) \mu_{1} E_{11} + \epsilon((V_{2})^2) \mu_{2} E_{22} - 2 \epsilon((V_{1})^2) \mu_{1} E_{21}. \tag{4.20}
\]

This follows from \((\Delta \pi_{21})^2 = ((p_{2})_{1})^2 - 2(p_{2})_{1} \cdot p_{1} + (p_{1})^2\) and the definitions of the corresponding energies. If the transport and the metric are consistent, then this equality can be written in a more symmetric form as
\[
(\Delta p_{21})^2 = (\Delta \pi_{21})^2 = \epsilon((V_{1})^2) \mu_{1} E_{11} + \epsilon((V_{2})^2) \mu_{2} E_{22}
\]
\[
- \epsilon((V_{1})^2) \mu_{1} E_{21} - \epsilon((V_{2})^2) \mu_{2} E_{12}, \tag{4.21}
\]
where we have used (4.8).

5. EXAMPLE: SPECIAL RELATIVITY

The purpose of this section is to find explicit forms of the introduced relative quantities in the concrete case of special relativity. (As a standard reference to the problems of this theory see, e.g., [6, 7].)

Let there be given a standard (4-dimensional, flat, with signature (+---)) Minkowski’s space-time \(M^{4}\), in which as a concrete realization of the general transport along paths the parallel transport along them will be used. Let two point particles 1 and 2 with masses \(m_{1} \neq 0\) and \(m_{2} \neq 0\) be moving in \(M^{4}\) with constant 3-velocities \(v_{1}\) and \(v_{2}\), respectively, with respect to a given frame of reference. Then, their world lines are \(x_{a}(s_{a}) = (ct, lv_{a}) + y_{a}, a = 1, 2\), where \(c\)
is the velocity of light in vacuum, \( t \) is the time in the used frame, \( s_\alpha := \tau_\alpha(t) := t(1 - v_\alpha^2/c^2)^{1/2}, \ a = 1, 2 \) are the corresponding proper times and \( y_1, y_2 \in M^4 \) are fixed.

According to (4.1), the 4-velocities [6] of the particles are

\[
V_\alpha = (c, v_\alpha)(1 - v_\alpha^2/c^2)^{-1/2}, \ a = 1, 2 \quad (5.1)
\]

and hence

\[
(V_\alpha)^2 = (c^2 - v_\alpha^2)((1 - v_\alpha^2/c^2)^{-1/2})^2 = c^2, \ \epsilon((V_\alpha)^2) = +1, \ a = 1, 2. \quad (5.2)
\]

Due to this by using (3.1) – (3.3), (4.1) and (4.9), we get:

\[
p_\alpha = m_\alpha(c, v_\alpha)(1 - v_\alpha^2/c^2)^{-1/2}, \ a = 1, 2 (\mu_1 = m_1, \mu_2 = m_2), \quad (5.3a)
\]

\[
(p_1)_2 = p_1, (p_2)_1 = p_2, \Delta p_{21} = \Delta \pi_{21} = p_2 - p_1, \quad (5.3b)
\]

\[
E_{21} = m_2 c^2[(1 - v_1 \cdot v_2/c^2)(1 - v_1^2/c^2)]^{-1/2}, \quad (5.4a)
\]

\[
E_{12} = m_1 c^2[(1 - v_1 \cdot v_2/c^2)(1 - v_1^2/c^2)]^{-1/2}, \quad (5.4b)
\]

\[
E_{11} = m_1 c^2, \quad E_{22} = m_2 c^2. \quad (5.4c)
\]

Evidently, \( E_{11} \) and \( E_{22} \) are the proper (rest) energies of the particles. If, e.g., \( v_1 = 0 \), then \( E_{21} = m_2 c^2(1 - v_2^2/c^2)^{-1/2} = E_2 \) is the energy of the second particle with respect to the used frame [6].

If \( m_1 \neq 0 \) and \( m_2 = 0 \), then in the above considerations one has to replace \( x_2 \) and \( s_2 \), respectively, by \( x_2(s_2) = (ct, ct \mathbf{n}_2) + y_2 \) and \( s_2 = t \), where \( \mathbf{n}_2 \) is a unit 3-vector (\( \mathbf{n}_2^2 = 1 \)) showing the direction of movement of the second particle, i.e. \( \mathbf{v}_2 = c \mathbf{n}_2 \), and, consequently

\[
V_2 = c(1, \mathbf{n}_2), \ (V_2)^2 = 0, \ \epsilon((V_2)^2) = +1. \quad (5.5)
\]

If \( E_2 \) is the energy of the second particle with respect to the given frame, then its 4-momentum is [6]

\[
p_2 = (E_2/c, \mathbf{p}_2) = (E_2/c, (E_2/c) \mathbf{n}_2) = (E_2/c)(1, \mathbf{n}_2) = (E_2/c^2)V_2 \quad (5.6)
\]

and due to (3.1), we have

\[
\mu_2 = \mu_2(s_2) = E_2/c^2. \quad (5.7)
\]

In this case, (5.3b) is also true and (5.4) take the form:

\[
E_{21} = E_2(1 - v_1 \cdot \mathbf{n}_2/c)(1 - v_1^2/c^2)^{-1/2}, \quad (5.8a)
\]

\[
E_{12} = m_1 c^2(1 - v_1 \cdot \mathbf{n}_2/c)(1 - v_1^2/c^2)^{-1/2}, \quad (5.8b)
\]

\[
E_{11} = m_1 c^2, \quad E_{22} = 0. \quad (5.8c)
\]

the last of which is in accordance with (4.10).
Evidently, \( E_{21} = E_2 \) for \( v_1 = 0 \), due to which (5.8a) expresses the usual Doppler effect in terms of energies of the corresponding particles [6]. In fact, if we have a moving with \( v_1 = v \) source of massless particles (e.g. photons) with 3-velocities \( v_2 = c n \) and energy (with respect to the source) \( E_{21} = E_0 \), which are registered by immovable in this frame observer, we will find that the particles are with energy \( E = E_2 \), which due to (5.8a) is

\[
E = E_0 (1 - v \cdot n/c)^{-1} (1 - v^2/c^2)^{1/2}.
\]  

(5.9)

The corresponding formulae for \( m_1 = 0 \) and \( m_2 \neq 0 \) are obtained from the above ones by means of the change \( 1 \rightarrow 2 \rightarrow 1 \) in the subscripts in them.

In the case when \( m_1 = m_2 = 0 \), we have \( x_a(s_a) = (ct, ct, a) + y_a, s_a = t, a = 1, 2 \), so

\[
v_a = cn_a, n_a^2 = 1, V_a = c(1, n_a), (V_a)^2 = 0, \epsilon((V_a)^2) = +1,
\]  

(5.10)

\[
p_a = (E/c)(1, n_a), \mu_a = \mu_a(t) = E_a/c^2, \quad a = 1, 2,
\]  

(5.11)

and the equations (5.3b) remain the same. Hence:

\[
E_{21} = E_2 (1 - n_1 \cdot n_2), E_{12} = E_1 (1 - n_1 \cdot n_2),
\]  

(5.12a)

\[
E_{11} = E_{22} = 0.
\]  

(5.12b)

So, if \( n_1 = n_2 \), then \( E_{21} = E_{12} = 0 \) and vice versa.

At the end, we shall consider the concepts of relative velocity and deviation velocity in special relativity.

Let \( K \) be a fixed inertial frame of reference in which an arbitrary moving particle 2 has a 4-radius-vector \( x_2(s_2) |_K = (ct, x_2(t)) \), \( s_2 = t(1 - v_2^2/c^2)^{1/2}, v_2 = 2 \), where \( t \) is the time in \( K \). Let the inertial frame \( K' \) be attached to the particle 1 having in \( Ka \) world line \( x_1(s_1) |_K = (ct, t, v_1) = \text{const} \), \( s_1 = t(1 - v_1^2/c^2)^{1/2} \). The world line of the observer is completely arbitrary.

In the frame \( K \), we have

\[
V_a|_K = \frac{dx_a}{ds_a} = (c, v_a)(1 - v_a^2/c^2)^{-1/2}, \quad a = 1, 2,
\]  

(5.13a)

and in \( K' \), we get

\[
V_1|_{K'} = (c, 0), \quad V_2|_{K'} = (c, v')(1 - v'^2/c^2)^{-1/2},
\]  

(5.13b)

where \( v \) is the 3-velocity of the particle 2 in \( K' \) (i.e. with respect to the particle 1) in a sense of special relativity (see [6]).

Consequently, as we are working in a pseudo-Euclidean case, due to (2.6) the relative velocity is \( \Delta V_{21} = V_2 - V_1 \). So, we get:

\[
\Delta V_{21}|_K = (1 - v_2^2/c^2)^{-1/2}(c, v_2) - (1 - v_1^2/c^2)^{-1/2}(c, v_1),
\]  

(5.14a)

\[
\Delta V_{21}|_{K'} = (1 - v'^2/c^2)^{-1/2}(c, v) - (c, 0).
\]  

(5.14b)
Besides, in the pseudo-Euclidean case $h_{21} = x_2(s_2) - x_1(s_1)$, so that:

$$h_{21}|_K = (0, x_2(t) - tv_1), \quad (5.15a)$$

$$h_{21}|_{K'} = (0, x), \quad (5.15b)$$

where $x$ is obtained from $x_2(t)$ by a Lorentz transformation describing the transition from $K$ to $K'[6]$.

Due to (2.6) the deviation velocity is

$$V_{21} = dh_{21}/ds = (ds_1/ds)(dh_{21}/ds_1) = (dt/ds)(dh_{21}/dt),$$

where $s_1 = t'$ is the time in $K'[6]$, from where, we get:

$$V_{21}|_K = \left. \frac{dt}{ds} \right|_s (0, v_2 - v_1), \quad (5.16a)$$

$$V_{21}|_{K'} = \left. \frac{ds_1}{dt} \right|_v (0, v). \quad (5.16b)$$

So, if the observer coincides with the first particle, then $s_1 = s$ and $V_{21}|_{K'} = (0, v)$. This shows that in fact the deviation velocity is a direct generalization of the relative velocity in a sense of special relativity.

6. COMMENTS

In this work many times was met the problem for comparing (defining the difference of) two defined at different points vectors. Below is presented a general scheme for the used in the present work method in a manifold $M$ endowed with a transport along paths $I$ in its tangent bundle.

Let $A_a \in T_{z_a}(M), a = 1, 2, \gamma : J \to M$ and $\gamma(s_a) = z_a, a = 1, 2$ for some $s_1, s_2 \in J$ and $z_1, z_2 \in M$. Let

$$(A_2)_1 := I^\gamma_{s_2} A_2, (A_1)_2 := I^\gamma_{s_1} A_1. \quad (6.1)$$

Now instead of $A_1$ and $A_2$ one can compare the vectors $(A_2)_1$ and $(A_2)_1$, or equivalently the vectors $A_2$ and $(A_1)_2$. The corresponding differences (defined by 1), by definition, are

$$\Delta A_{21} := (A_2)_1 - A_1, \Delta A_{12} := (A_1)_2 - A_2. \quad (6.2)$$

Evidently, for a linear transport along paths $L$ these two quantities are connected by

$$\Delta A_{12} = L^\gamma_{s_1 \to s_2} \Delta A_{21}, \quad \Delta A_{21} = L^\gamma_{s_2 \to s_1} \Delta A_{12}. \quad (6.3)$$

In manifolds with a transport of vectors along paths and a covariant differentiation there arises a "mixed" acceleration $D_{\Delta x}/ds_x \Delta V_{21}$, but there are not physical reasons that it plays some significant role.

Let $X_a$ denote one of the vector fields of velocity, acceleration or momentum of the $a$-th, $a = 1, 2$ particles. In sections 2 and 3 we introduce the quantities

$$(X_2)_1 := I^\gamma_{s_2 \to s_1} X_2 \in T_{z_1}(M), \quad (6.4)$$
\[ \Delta X_{21} := \Delta X_{21}(s; x) := I_{t_{1}' \to t_{2}'}^\rho ((X_2)_1 - X_1) \]
\[ = I_{t_{1}' \to t_{2}'}^\rho (I_{r_{2}' \to r_{1}'}^\gamma X_2 - X_1) \in T_{x(s)}(M). \quad (6.5) \]

Analogously, if \( \eta_\ast^s : J \ast \to M, s \in J, \) \( \eta_\ast^s(t_\ast^s) \) for some \( t_\ast^s, t_{\ast*}^s \in J_\ast^s \) and using the same paths \( \gamma_s, s \in J, \) we can define the quantities
\[ (X_{12})_2 := I_{r_{2}' \to r_{1}'}^\rho X_1 \in T_{x_2}(M), \quad (6.6) \]
\[ \Delta X_{12} := \Delta X_{12}(s; x) := I_{r_{2}' \to r_{1}'}^\rho ((X_1)_2 - X_2) \]
\[ = I_{r_{2}' \to r_{1}'}^\rho (I_{r_{2}' \to r_{1}'}^\gamma X_1 - X_2) \in T_{x(s)}(M), \quad (6.7) \]
the latter of which, in a case of linear transport along paths \( L, \) due to (1.1) and (1.3) is connected with \( \Delta X_{21} \) by
\[ \Delta X_{12} = -L_\gamma \circ L_{r_{2}' \to r_{1}'}^\gamma \circ I_{r_{2}' \to r_{1}'}^\rho (\Delta X_{21}). \quad (6.8) \]

If \( L_\gamma \) does not depend on \( \gamma \) or if \( \eta_\ast^s \) is a product of \( \gamma_s \) and \( \eta_s \) and the equalities (2.6) and (2.7) of [1] are true, then according to [1] proposition 3.4 the last equality reduces to
\[ \Delta X_{12} = -\Delta X_{21}. \quad (6.9) \]

For some purposes, in (4.1) one can put \( \epsilon(0) = -1 \) instead of \( \epsilon(0) = +1. \) Our general results do not depend on that choice.

Opposite to (4.11), if we admit that \( \lim_{m_2 \to 0} E_{21} = 0, \) i.e. a continuous dependence of \( E_{21} \) on \( m_2, \) then we arrive at an explicit contradiction with the physical reality. Namely, if this is so, then due to \( \mu_2(s_2) \neq 0 \) from (4.1), we get \( V_2(s_2) = 0 \) for \( m_2 = 0, \) which contradicts the fact that we are dealing with a material particle but not with the vacuum. Besides, the equality \( E_{21} = 0 \) for \( m_2 = 0 \) means that any massless particle, e.g. a photon, has zero (relative) energy with respect to any other particle, something which, evidently, is not true.

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Relative Mechanical Quantities
in Spaces with a Transport along Paths

The concepts of relative velocity and acceleration, deviation velocity and acceleration and relative momentum of point particles in spaces (manifolds), the tangent bundle of which is equipped with a transport along paths, are introduced. If the tangent bundle is endowed also with a metric, it gives rise also to the notion of a relative energy. Certain ties between these quantities are considered. The cases of massless particles and of special relativity are presented in this context.

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