Shifted Poisson Structures on Differentiable Stacks

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Abstract

The purpose of this paper is to investigate shifted (+1) Poisson structures in context of differential geometry. The relevant notion is shifted (+1) Poisson structures on differentiable stacks. More precisely, we develop the notion of Morita equivalence of quasi-Poisson groupoids. Thus isomorphism classes of (+1) Poisson stack correspond to Morita equivalence classes of quasi-Poisson groupoids. In the process, we carry out the following programs of independent interests:

(1) We introduce a \(\mathbb{Z}\)-graded Lie 2-algebra of polyvector fields on a given Lie groupoid and prove that its homotopy equivalence class is invariant under Morita equivalence of Lie groupoids, thus can be considered as polyvector fields on the corresponding differentiable stack \(\mathcal{X}\). It turns out that shifted (+1) Poisson structures on \(\mathcal{X}\) correspond exactly to elements of the Maurer-Cartan moduli set of the corresponding dgla.

(2) We introduce the notion of tangent complex \(T_X\) and cotangent complex \(L_X\) of a differentiable stack \(\mathcal{X}\) in terms of any Lie groupoid \(\Gamma \rightarrow M\) representing \(\mathcal{X}\). They correspond to homotopy class of 2-term homotopy \(\Gamma\)-modules \(A[1] \rightarrow TM\) and \(T^\vee M \rightarrow A^\vee[-1]\), respectively. We prove that a (+1)-shifted Poisson structure on a differentiable stack \(\mathcal{X}\), defines a morphism \(L_X[1] \rightarrow T_X\). We rely on the tools of theory of VB-groupoids including homotopy and Morita equivalence of VB-groupoids.

1 Introduction

Recently, there has been increasing interest in derived algebraic geometry for shifted symplectic and Poisson structures on moduli spaces as they have proved to be extremely important in understanding fascinating theories such as Donaldson–Thomas invariants [32] and quantum field theory [11]. The symplectic case was addressed first in [32] and later shifted Poisson structures were developed (see [10, 27, 28, 33, 34, 36]). Although very powerful post-Grothendieck machinery has been developed to deal with bad singularities — both derived and stacky — in the context of algebraic geometry, we believe it is valuable to develop a purely differential geometric approach to issues pertaining to...
symplectic and Poisson geometry that are specific to the smooth context. Both derived and stacky types of singularities occur spontaneously in problems from classical symplectic and Poisson geometry [37]. There, many existing tools from differential geometry can be used and results can be sharpened. In this paper, we will thus focus on shifted (+1) Poisson structures on differentiable Artin 1-stacks (or differentiable stacks in short).

Classical Poisson manifolds and shifted (+1) Poisson stacks are fundamentally different in nature. While classical Poisson manifolds arise as phase spaces of Hamiltonian systems in classical mechanics, shifted (+1) Poisson stacks are abstract mathematical constructions capturing the symmetry of Hamiltonian systems featuring momentum maps. The word 'momentum' denotes quantities whose conservation under the time evolution of a physical system is related to some symmetry of the system. The shifted (+1) symplectic stack \([g^\ast/G]\) was perhaps the first instance (in a hidden form) of a shifted (+1) Poisson stack encountered in the study of Hamiltonian systems. It can be credited to Mikami–Weinstein [29] who showed that the usual Hamiltonian momentum map theory can in fact be reformulated as a symplectic action of the symplectic groupoid \(g^\ast \ltimes G \rightarrow g^\ast\). The latter is indeed a presentation of the shifted (+1) symplectic stack \([g^\ast/G]\).

In the late 1980's, Weinstein introduced the notion of Poisson groupoids [38] in order to unify Drinfeld’s theory of Poisson groups [12] with the theory of symplectic groupoids [39]. The introduction of Poisson groupoids has led to many new developments in Poisson geometry in the last three decades, one of them being the theory of quasi-Poisson groupoids developed by two of the authors together with Iglesias-Ponte [18]. Roughly speaking, a quasi-Poisson groupoid is a Lie groupoid endowed with a multiplicative bivector field whose Schouten bracket with itself is 'homotopic to zero.' In the present paper, we adopt the viewpoint that quasi-Poisson groupoids ought to be understood as shifted (+1) differentiable Poisson stacks, a notion yet to be developed.

It is well known that differentiable stacks — more precisely isomorphism classes of differentiable stacks — can be construed as Morita equivalence classes of Lie groupoids [7]. Hence one is naturally led to define shifted (+1) differentiable Poisson stacks as Morita equivalence classes of quasi-Poisson groupoids. This immediately raises the following double problem:

**Problem 1.**

- What is Morita equivalence for quasi-Poisson groupoids?
- Given a quasi-Poisson structure on a Lie groupoid, is it possible to transfer it to any other Morita equivalent Lie groupoid?

While the notion of Morita equivalence of Lie groupoids was easily extended to quasi-symplectic groupoids [40], it does not admit a straightforward extension to quasi-Poisson groupoids. Indeed, unlike differential forms, which are contravariant tensors, polyvector fields are covariant tensors and cannot be pulled back. To overcome this difficulty, we develop an alternative description of quasi-Poisson structures on a given groupoid \(\Gamma \rightarrow M\) as Maurer–Cartan elements of the dgla determined by a \(\mathbb{Z}\)-graded Lie 2-algebra \(\Sigma^\ast(A) \rightarrow T^\ast_{\text{mult}}\Gamma\) constructed in a canonical way from the groupoid \(\Gamma \rightarrow M\) — the construction is explained in Section 2 and the Appendix.

This re-characterization of quasi-Poisson structures is closely related to the following important question, which is of independent interest:

**Problem 2.** What are polyvector fields on a differentiable stack and how can we describe them efficiently?
Berwick–Evans and Lerman [5] proved that, given a presentation of a differentiable stack \( X \) by a Lie groupoid \( \Gamma \Rightarrow M \), the vector fields on \( X \) can be understood in terms of a Lie 2-algebra consisting of multiplicatives vector fields [24] on \( \Gamma \) and sections of the Lie algebroid \( A \) associated with the Lie groupoid \( \Gamma \Rightarrow M \).

Inspired by [5], we associate a \( \mathbb{Z} \)-graded Lie 2-algebra \( \Sigma^\bullet(A) \xrightarrow{d} T^\bullet \text{mult}_\Gamma \) of ‘polyvector fields’ with every Lie groupoid \( \Gamma \Rightarrow M \). Here \( T^\bullet \text{mult}_\Gamma \) denotes the space of multiplicative polyvector fields on \( \Gamma \) and \( \Sigma^\bullet(A) \) denotes the space of sections of the exterior powers of the Lie algebroid \( A \). We prove that the \( \mathbb{Z} \)-graded Lie 2-algebra associated in this way to Morita equivalent Lie groupoids are necessarily homotopy equivalent. Consequently, we define the space of polyvector fields on a differentiable stack \( X \) to be the homotopy equivalence class of the \( \mathbb{Z} \)-graded Lie 2-algebras associated with the Lie groupoids representing the differentiable stack \( X \). For vector fields, a very similar idea is used in [31].

A shifted (+1) Poisson structure on a differentiable stack \( X \) is then simply an element of the Maurer–Cartan moduli set of the dgla determined by the homotopy equivalence class of \( \mathbb{Z} \)-graded Lie 2-algebras corresponding to \( X \). The choice of a presentation of the stack \( X \) by a Lie groupoid \( \Gamma \Rightarrow M \) identifies the shifted (+1) Poisson structures on \( X \) with gauge equivalence classes of quasi-Poisson structures on \( \Gamma \Rightarrow M \). Modulo the choice of an Ehresmann connection, such gauge equivalence classes of quasi-Poisson structures can be passed along uniquely from one Lie groupoid to any other Morita equivalent Lie groupoid.

Thus, we obtain a satisfying definition of Morita equivalence of quasi-Poisson groupoids.

The second goal of the paper is aimed to explore where the degree shifting comes from for a quasi-Poisson groupoid, which is crucial for introducing the rank and non-degeneracy of shifted (+1)-Poisson stacks. Our construction is certainly inspired by derived algebraic geometry, but is of a different nature. Recall that, in classical Poisson geometry, a Poisson structure \( \pi \) on a smooth manifold \( X \) determines a morphism \( \pi^! : T_X^\vee \to T_X \) from the cotangent bundle \( T_X^\vee \) to the tangent bundle \( T_X \). One expects an analogue statement holds for shifted (+1) Poisson stacks. Before one can attempt to address this issue, one must first investigate the following

**Problem 3.** What are the analogues of the tangent and cotangent bundles for differentiable stacks?

In (derived) algebraic geometry, talking about the cotangent complex [19] requires enormous preparation work. This seems neither practical nor necessary when dealing with differentiable stacks. Here we propose a different approach introducing these notions in terms of presentations of the differentiable stack by Lie groupoids. The following short answer was suggested to us by Behrend (private communication): the **tangent complex** \( T_X \) of a stack \( X \) admitting a presentation by a Lie groupoid \( \Gamma \Rightarrow M \) ought to be the **homotopy equivalence class of the homotopy \( \Gamma \)-module** \( \rho : A[1] \to T_M \) [13, 15, 4], where \( A \) designates once again the Lie algebroid of \( \Gamma \Rightarrow M \) and \( \rho \) denotes its anchor map. Its dual, the **cotangent complex** \( L_X \) of \( X \), would be the **homotopy equivalence class of the homotopy \( \Gamma \)-module** \( \rho^! : T^\vee_M \to A^\vee[-1] \). Homotopy \( \Gamma \)-modules were independently introduced by Gracia-Saz-Mehta, called “flat superconnection” [15], and by Abad-Crainic [4], called “representations up to homotopy”, both of which were inspired by the work of Evens-Lu-Weinstein [14].

Of course, one must fully justify this short answer and verify that tangent and cotangent complexes are well defined by investigating the precise relation between the homotopy \( \Gamma \)-modules arising from different presentations of the stack \( X \), i.e. different Morita equiv-
alent groupoids $\Gamma \equiv M$.

Homotopy $\Gamma$-modules have been studied extensively in the literature. In their pioneering work [16], Gracia-Saz and Mehta established a dictionary between VB groupoids over a fixed Lie groupoid $\Gamma \equiv M$, and 2-term homotopy $\Gamma$-modules. Here we enrich the dictionary by investigating Morita equivalence. Morita equivalence of VB groupoids is closely related to homotopy equivalence of VB groupoids. Two VB groupoids $V_1 \equiv E_1$ and $V_2 \equiv E_2$ over $\Gamma_1 \equiv M_1$ and $\Gamma_2 \equiv M_2$, respectively, are Morita equivalent if and only if there exists a $\Gamma_1$-$\Gamma_2$-bitorsor $M_1 \xrightarrow{\phi_1} X \xrightarrow{\phi_2} M_2$ such that the pullback VB groupoids $V_1[\phi_1^*E_1]$ and $V_2[\phi_2^*E_2]$ are homotopy equivalent. Making use of the dictionary of Gracia-Saz and Mehta [16], this definition can be transposed to homotopy $\Gamma$-modules: a homotopy $\Gamma$-module $E_\Gamma$ is Morita equivalent to a homotopy $\Gamma_1$-$\Gamma_2$-module $E_2$ if and only if there exists a $\Gamma_1$-$\Gamma_2$-bitorsor $M_1 \xrightarrow{\phi_1} X \xrightarrow{\phi_2} M_2$ and a homotopy equivalence of homotopy $\Gamma_1[X](\equiv \Gamma_2[X])$-modules from $E_\Gamma[X]$ to $E_2[X]$.

It is easily established that, if $\Gamma_1 \equiv M_1$ and $\Gamma_2 \equiv M_2$ are Morita equivalent Lie groupoids, then $TT_1 \equiv TM_1$ and $TT_2 \equiv TM_2$ are Morita equivalent VB groupoids and, similarly, $T^\vee \Gamma_1 \equiv A^\vee_1$ and $T^\vee \Gamma_2 \equiv A^\vee_2$ are Morita equivalent VB groupoids. It immediately follows that the homotopy $\Gamma_1$-module $A_1[1] \rightarrow TM_1$ is Morita equivalent to the homotopy $\Gamma_2$-module $A_2[1] \rightarrow TM_2$, while the homotopy $\Gamma_1$-module $T^\vee M_1 \rightarrow A^\vee_1[-1]$ is Morita equivalent to the homotopy $\Gamma_2$-module $T^\vee M_2 \rightarrow A^\vee_2[-1]$. This concludes the justification of our definition of the tangent and cotangent complexes $T_X$ and $L_X$ of a differentiable stack $X$.

Given a quasi-Poisson groupoid $(\Gamma, \Pi, \Lambda)$, the associated map $\Pi^\Gamma: T^\vee \Gamma \rightarrow TT$ is VB groupoid morphism. Moreover, if $(\Gamma_1, \Pi_1, \Lambda_1)$ and $(\Gamma_2, \Pi_2, \Lambda_2)$ are Morita equivalent quasi-Poisson groupoids, then the associated VB groupoid morphisms $\Pi_1^\Gamma: T^\vee \Gamma_1 \rightarrow TT_1$ and $\Pi_2^\Gamma: T^\vee \Gamma_2 \rightarrow TT_2$ are homotopic. As an immediate consequence, we prove that a $(+1)$-shifted Poisson structure on a differentiable stack $X$ indeed determines a morphism $\Pi^\Gamma: L_X[1] \rightarrow T_X$ of 2-term complexes from the shifted $(+1)$-cotangent complex to the tangent complex. This, in turn, allows us to introduce the rank of a shifted $(+1)$ Poisson stack $X$ as an integer-valued map defined on its coarse moduli space $|X|$. We are thus led to a natural definition of non-degenerate shifted $(+1)$ Poisson stacks.

We conclude with a few remarks. It is natural to expect that non-degenerate shifted $(+1)$ Poisson stacks are shifted $(+1)$ symplectic stacks, which can be thought of as Morita equivalence classes of quasi-symplectic groupoids [30]. This is indeed true. In a forthcoming paper [8], we establish an explicit one-one correspondence between non-degenerate shifted $(+1)$ Poisson stacks and shifted $(+1)$ symplectic stacks and apply it to momentum map theory. In particular, we prove that the momentum map theory of quasi-Poisson groupoids in [13] is stacky in nature and that Hamiltonian reductions can be carried out, which agrees with the derived symplectic geometry principle that the derived intersection of coisotropics of a shifted $(+1)$ Poisson stack gives rise to a Poisson structure. It also enables us to merge the quasi-Hamiltonian momentum map theory of Alekseev-Malkin-Meinrenken [1] with the quasi-Poisson theory of Alekseev, Kosmann-Schwarzbach and Meinrenken [2, 3]. One of the authors announced some of the results set forth in the present paper at the conference Derived algebraic geometry with a focus on derived symplectic techniques held at the University of Warwick in April 2015. He wishes to thank the organizers for providing him the opportunity to disseminate the results of our work.
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2 Polyvector fields on a differentiable stack

The purpose of this section is to introduce the notion of polyvector fields on a differentiable stack. They can be represented by a dgla when the stack is represented by a Lie groupoid. Different Lie groupoids representing the same stack shall give rise to homotopy equivalent dglas. In fact, more precisely, they are represented by homotopy equivalence classes of \( \mathbb{Z} \)-graded Lie 2-algebras.

2.1 The \( \mathbb{Z} \)-graded Lie 2-algebra of polyvector fields on a Lie groupoid

We first recall a few basic facts about multiplicative polyvector fields on a Lie groupoid [18].

Let \( \Gamma \Rightarrow M \) be a Lie groupoid with, respectively, source and target map \( s \) and \( t \). Let \( A \) be its Lie algebroid, with anchor map \( \rho : A \to TM \). Let us denote the graph of the multiplication by \( \text{graph}(\Gamma) = \{(g_1, g_2, g_1 g_2) | g_1, g_2 \in \Gamma, s(g_1) = t(g_2)\} \subset \Gamma \times \Gamma \times \Gamma \).

We say that a \( k \)-vector field \( P \in \Gamma(\bigwedge^k T\Gamma) \) on \( \Gamma \) is multiplicative if \( \text{graph}(\Gamma) \) is coisotropic with respect to \( P \oplus P \oplus (-1)^{k+1} P \). For any \( k \geq 0 \), we denote by \( T^k_{\text{mult}} \Gamma \) the space of multiplicative \((k+1)\)-vector fields on \( \Gamma \). Let \( T^*_{\text{mult}} \Gamma := \bigoplus_{k \geq -1} T^k_{\text{mult}} \Gamma \) be the graded vector space of multiplicative polyvector fields on a Lie groupoid \( \Gamma \Rightarrow M \).

For any \( k \geq -1 \), denote by \( \Sigma^k(A) \) the space \( \Gamma(\bigwedge^{k+1} A) \) of sections of the exterior vector bundle \( \bigwedge^{k+1} A \to M \). When equipped with the Schouten-Nijenhuis bracket, \( \Sigma^*(A) := \bigoplus_{k \geq -1} \Sigma^k(A) \) is a \( \mathbb{Z} \)-graded Lie algebra.

For every \( a \in \Sigma^k(A) \), we denote the corresponding right and left invariant \((k+1)\)-vector fields by \( \overrightarrow{a} \) and \( \overleftarrow{a} \), respectively. It is easy to check that \( \overrightarrow{a} - \overleftarrow{a} \) is a multiplicative \((k+1)\)-vector field [15], called an exact multiplicative \((k+1)\)-vector field. We recall from [15] some well-known facts about multiplicative polyvector fields.

Lemma 2.1. i) The space of multiplicative polyvector fields \( T^*_{\text{mult}} \Gamma \) on \( \Gamma \Rightarrow M \) is closed under the Schouten-Nijenhuis bracket, and is therefore a \( \mathbb{Z} \)-graded Lie algebra.

ii) The map \( d : \Sigma^*(A) \to T^*_{\text{mult}} \Gamma \),

\[
d(a) = \overrightarrow{a} - \overleftarrow{a}
\]  

(1)
is an homomorphism of graded Lie algebras.

iii) For each \( P \in \mathcal{T}_{mult}^k \Gamma \) and \( a \in \Sigma^1(A) \), there exists an unique section \( \delta_P(a) \in \Sigma^{k+1}(A) \) such that

\[
\delta_P(a) = [P, a].
\]

Moreover the correspondence \( P \mapsto \delta_P \) establishes a graded Lie algebra morphism between \( \mathcal{T}_{mult}^* \Gamma \) and \( \text{Der}^*(\Sigma(A)) \). The action satisfies the following properties:

1) \( \delta_{[P,a]}(b) = [a, b] \),

for any \( P \in \mathcal{T}_{mult}^* \Gamma \) and \( a, b \in \Sigma^* (A) \).

The following proposition follows immediately.

**Proposition 2.2.** Let \( \Gamma \rightrightarrows M \) be a Lie groupoid. Then \( \Sigma^*(A) \overset{\delta}{\longrightarrow} \mathcal{T}_{mult}^* \Gamma \) together with the Lie brackets and actions described in Lemma 2.1 is a \( \mathbb{Z} \)-graded Lie 2-algebra.

In other words, \( \Sigma^*(A) \overset{\delta}{\longrightarrow} \mathcal{T}_{mult}^* \Gamma \) is a crossed module of \( \mathbb{Z} \)-graded Lie algebras. To such a \( \mathbb{Z} \)-graded Lie 2-algebra or \( \mathbb{Z} \)-graded Lie algebra, one can associate in a standard way a dgla \( \mathcal{V}^*(\Gamma):= \oplus_{k=2} \mathcal{V}^k(\Gamma) \), where

\[
\mathcal{V}^k(\Gamma) = \left( \Sigma^{k+1}(A) \oplus T^k_{mult} \Gamma \right).
\]

See Appendix A.1 for details. This associated dgla \( \mathcal{V}^*(\Gamma) \) is called the dgla of polyvector fields on the Lie groupoid \( \Gamma \rightrightarrows M \).

**Remark 2.3.** Recall that for any dgla \( \mathcal{V}^* \) its cohomology \( H^*(\mathcal{V}) \) is a \( \mathbb{Z} \)-graded Lie algebra. The \( k \)-th cohomology \( H^k(\mathcal{V}(\Gamma)) \) of the dgla \( \mathcal{V}^*(\Gamma) \) of polyvector fields on the Lie groupoid \( \Gamma \rightrightarrows M \) is easily seen to be

\[
H^k(\mathcal{V}(\Gamma)) \cong \Sigma^{k+1}(A)^\Gamma \oplus \frac{T^k_{mult} \Gamma}{\{a - a[a \in \Sigma^k(A)]\}},
\]

where \( \Sigma^{k+1}(A)^\Gamma \) denotes the space of \( \Gamma \)-invariant sections of \( \Lambda^{k+1} \Gamma \).

### 2.2 Morita equivalence

In this section we discuss how the \( \mathbb{Z} \)-graded 2-term complex \( \Sigma^*(A) \overset{\delta}{\longrightarrow} \mathcal{T}_{mult}^* \Gamma \) changes under Morita equivalence of Lie groupoids. Note that if \( \Gamma_i \rightrightarrows M_i \), \( i = 1, 2 \), are two Lie groupoids with respective Lie algebroids \( A_i \), and \( \phi: \Gamma_1 \to \Gamma_2 \) is a Lie groupoid morphism over \( \varphi : M_1 \to M_2 \), in general there is no map at the level of chain complexes, from \( \Sigma^*(A_1) \overset{\delta}{\longrightarrow} \mathcal{T}_{mult}^* \Gamma_1 \) to \( \Sigma^*(A_2) \overset{\delta}{\longrightarrow} \mathcal{T}_{mult}^* \Gamma_2 \). However we will prove that when \( \phi \) is a Morita morphism of Lie groupoids, these \( \mathbb{Z} \)-graded 2-term complexes are homotopy equivalent.

Assume, indeed, that \( \Gamma[X] \rightrightarrows X \) is the pull-back groupoid of the Lie groupoid \( \Gamma \rightrightarrows M \) under a surjective submersion \( \varphi : X \to M \), where \( \Gamma[X] = X \times_{M,s} \Gamma \times_{M,s} X \). Let \( \phi : \Gamma[X] \to \Gamma \) be the natural projection which is a Morita morphism and let \( \phi_A : A[X] \to A \) be the corresponding Lie algebroid map. As \( \Sigma_3 \) does for vector fields we will consider the following spaces:

---

1 A \( \mathbb{Z} \)-graded 2-term complex is a 2-term complex \( A \overset{d}{\longrightarrow} B \), where both \( A \) and \( B \) are \( \mathbb{Z} \)-graded vector spaces and \( d \) is a graded map of degree zero. A morphism from \( A \overset{d}{\longrightarrow} B \) to \( A' \overset{d'}{\longrightarrow} B' \) is a chain map given by a pair of degree 0 maps from \( A \) to \( A' \) and from \( B \) to \( B' \). Homotopies between morphisms are just usual homotopies given by degree 0 maps from \( B \) to \( B' \).
1. By $\mathcal{T}^\bullet_{\text{mult}}\Gamma[X]_{\text{proj}}$, we denote the subspace of $\mathcal{T}^\bullet_{\text{mult}}\Gamma[X]$ consisting of projectable multiplicative polyvector fields on $\Gamma[X]$, namely those elements $P \in \mathcal{T}^\bullet_{\text{mult}}\Gamma[X]$ such that there exists $\bar{P} \in \mathcal{T}^\bullet_{\text{mult}}\Gamma$ satisfying the condition $\Phi_\times(P) = \bar{P}$.

2. By $\Sigma^\bullet(A[X])_{\text{proj}}$, we denote the subspace of $\Sigma^\bullet(A[X])$ consisting of projectable sections in $\Gamma(X; \Lambda^{\bullet+1}A[X])$, namely those sections $a \in \Sigma^\bullet(A[X])$ such that there exists $\bar{a} \in \Sigma^\bullet(A)$ satisfying the condition $\phi_A(a) = \bar{a}$.

We then have, clearly, two projection maps:

$$\text{pr} : \mathcal{T}^\bullet_{\text{mult}}\Gamma[X]_{\text{proj}} \to \mathcal{T}^\bullet_{\text{mult}}\Gamma, \quad P \mapsto \bar{P},$$

$$\text{pr} : \Sigma^\bullet(A[X])_{\text{proj}} \to \Sigma^\bullet(A), \quad a \mapsto \bar{a}.$$

**Proposition 2.4.** Assume that $\Gamma \equiv M$ is a Lie groupoid, $\varphi : X \to M$ a surjective submersion. Let $\phi : \Gamma[X] \to \Gamma$ be the corresponding Morita morphism. Then

1. $\Sigma^\bullet(A[X])_{\text{proj}} \to \mathcal{T}^\bullet_{\text{mult}}\Gamma[X]_{\text{proj}}$ is a $\mathbb{Z}$-graded Lie 2-subalgebra of $\Sigma^\bullet(A[X]) \to \mathcal{T}^\bullet_{\text{mult}}\Gamma[X]$;

2. the projection map $\text{pr}$ is a morphism of $\mathbb{Z}$-graded Lie 2-algebras from $\Sigma^\bullet(A[X])_{\text{proj}} \to \mathcal{T}^\bullet_{\text{mult}}\Gamma[X]_{\text{proj}}$ to $\Sigma^\bullet(A) \to \mathcal{T}^\bullet_{\text{mult}}\Gamma$.

Proposition 2.4 means that both horizontal maps in the diagram below are morphisms of $\mathbb{Z}$-graded Lie 2-algebras, where $i$ stands for the inclusion maps:

$$\begin{array}{ccc}
\mathcal{T}^\bullet_{\text{mult}}\Gamma & \to & \mathcal{T}^\bullet_{\text{mult}}\Gamma[X]_{\text{proj}} \\
\downarrow & & \downarrow \\
\Sigma^\bullet(A) & \to & \Sigma^\bullet(A[X])_{\text{proj}}
\end{array}$$

We now define horizontal lifts. By an Ehresmann connection $\nabla$ for a surjective submersion $\varphi : X \to M$, we mean an injective bundle map $\nabla : \varphi^*TM \to TX$ such that for any $x \in M$ the horizontal subspace $H^\nabla_x = \text{Im}\nabla_x \subseteq T_x X$ is isomorphic via $\varphi_{\times x}$ to $T_{\varphi(x)}M$. The subbundle $H^\nabla \subseteq TX$ is also called an horizontal lift.

Let now $x, y \in X$ and $\gamma \in \Gamma$ with $\varphi(x) = t(\gamma)$ and $\varphi(y) = s(\gamma)$: the connection $\nabla$ induces a pair of natural injections:

$$T_{\gamma}\Gamma \hookrightarrow T_{(x,\gamma,y)}(\Gamma[X]) \quad \text{and} \quad A_{\varphi(x)} \hookrightarrow A[X]_x$$

defined, respectively by,

$$T_{\gamma}\Gamma \hookrightarrow T_x X \oplus_{T_{\varphi(x)}M} T_{\gamma}\Gamma \oplus_{T_{\varphi(y)}M} T_y X \quad \text{and} \quad T_{(x,\gamma,y)}(\Gamma[X])$$

$$u \quad \text{and} \quad ((\nabla \circ s_{TT}) (u)) \oplus u \oplus ((\nabla \circ s_{TT}) (u))$$

where $s_{TT} (u) \in T_{s(\gamma)}M \hookrightarrow \varphi^*(TM)_y$ and $t_{TT} (u) \in T_{t(\gamma)}M \hookrightarrow \varphi^*(TM)_x$ are the source
and target map of the tangent groupoid $TT$ over $TM$, and

$$A_{\varphi(x) ↗ \varphi^* A \oplus T_{\varphi(x)} M \xrightarrow{\sim} As[X]}$$

(4)

$$a ↣ a \oplus (\nabla_{\rho(a)})$$

By dualizing the maps (3-4), we obtain a pair of vector bundle morphisms:

$$T^{\vee} \Gamma[X] \xrightarrow{\phi_{\varphi}} T^{\vee} \Gamma$$

and

$$A[X]^{\vee} \xrightarrow{\phi_{\varphi}} A^{\vee}$$

(5)

This maps can be further extended to exterior product bundles of any order; they give rise to an induced pair of maps on the sections of their dual bundles,

$$\lambda_{\varphi} : \Gamma(\wedge TT) \to \Gamma(\wedge TT[X]) \text{ and } \Gamma(\wedge A) \to \Gamma(\wedge A[X]).$$

(6)

Note that $T_{\text{mult}}^{\bullet} \Gamma \to \Sigma^\bullet(A), T_{\text{mult}}^{\bullet} \Gamma[X]_{\text{proj}} \to \Sigma^\bullet(A[X])_{\text{proj}},$ and $T_{\text{mult}}^{\bullet} \Gamma[X] \to \Sigma^\bullet(A[X])$ are all $\mathbb{Z}$-graded 2-term complexes. By forgetting, for the moment, their $\mathbb{Z}$-graded Lie brackets, we have the following proposition, whose proof is postponed to Section B.2.

**Proposition 2.5.** Let $\Gamma \rightrightarrows M$ be a Lie groupoid and $\varphi : X \to M$ a surjective submersion. Choose an Ehresmann connection $\nabla$ for $\varphi$. Then:

1. the map $pr$ is a left inverse of $\lambda_{\varphi}$, and there exists a chain homotopy $h_{\lambda_{\varphi}} : T_{\text{mult}}^{\bullet} \Gamma[X]_{\text{proj}} \to \Sigma^\bullet(A[X])_{\text{proj}}$ between $\lambda_{\varphi} \circ pr$ and the identity map:

$$\begin{array}{ccc}
T_{\text{mult}}^{\bullet} \Gamma & \xrightarrow{\lambda_{\varphi}} & T_{\text{mult}}^{\bullet} \Gamma[X]_{\text{proj}} \\
\downarrow d & & \downarrow d \\
\Sigma^\bullet(A) & \xrightarrow{pr} & \Sigma^\bullet(A[X])_{\text{proj}} \\
\end{array}$$

2. there exists a chain map $\psi$

$$\begin{array}{ccc}
T_{\text{mult}}^{\bullet} \Gamma[X]_{\text{proj}} & \xrightarrow{i} & T_{\text{mult}}^{\bullet} \Gamma[X] \\
\downarrow h_{\lambda_{\varphi}} & & \downarrow h_X \\
\Sigma^\bullet(A[X])_{\text{proj}} & \xrightarrow{\psi} & \Sigma^\bullet(A[X]),
\end{array}$$

such that both $\psi \circ i$ and $i \circ \psi$ are homotopy to the identities as chain maps.

**Remark 2.6.** In Proposition 2.5 the maps $\psi$, $h_X$ and $h_{\lambda_{\varphi}}$ are not simply algebraic maps. They can be described explicitly in terms of geometric data such as the connection $\nabla$ on $\varphi : X \to M$, a partition of unity with respect to an open cover $(U_i)_{i \in I}$ of $M$, and local sections $\sigma_i : U_i \to X$ of $\varphi$. Explicit formulas can be derived from Equations (65–67).
2.3 Polyvector fields on a differentiable stack

Let $\Gamma \rightrightarrows M$ be a Lie groupoid and $\varphi : X \to M$ a surjective submersion. Let $\text{pr}$ and $i$ be the morphisms of $\mathbb{Z}$-graded 2-term complexes as in Equation (2).

Choose an Ehresmann connection $\nabla$ for $\varphi : X \to M$. According to Proposition 2.5 (1), the horizontal lift $\lambda \nabla$ is an homotopy inverse of $\text{pr}$. According to Proposition 2.5 (2), there also exists a retraction $\psi$ which is a homotopy inverse of $i$. In the diagram below, therefore, all morphisms of graded 2-term complexes pointing on the left are homotopy inverses of those pointing on the right.

\[
\array{\text{T}^{\bullet}_{\text{mult}} \Gamma & \text{T}^{\bullet}_{\text{mult}} \Gamma[X]_{\text{proj}} & \text{T}^{\bullet}_{\text{mult}} \Gamma[X] \\ \Sigma^\bullet(A) & \Sigma^\bullet(A[X])_{\text{proj}} & \Sigma^\bullet(A[X])}
\]

In addition to being morphisms of $\mathbb{Z}$-graded 2-term complexes, both $\text{pr}$ and $i$ are strict morphisms of Lie 2-algebras. However, neither $\lambda \nabla$ nor $\psi$ is a strict morphism of $\mathbb{Z}$-graded Lie 2-algebras in general. They are, however, the linear parts of morphisms of $\mathbb{Z}$-graded Lie 2-algebras. Moreover, the latter are unique up to homotopy.

**Proposition 2.7.** Let $\Gamma \rightrightarrows M$ be a Lie groupoid, and $\varphi : X \to M$ a surjective submersion. Let us choose an Ehresmann connection $\nabla$ for $\varphi$. Then,

1. the map $\text{pr}$ admits an homotopy inverse as morphisms of $\mathbb{Z}$-graded Lie 2-algebras, whose linear part is the horizontal lift and the quadratic term depends only on $\nabla$ and $h \lambda \nabla$;

2. the map $i$ admits an homotopy inverse, as morphisms of $\mathbb{Z}$-graded Lie 2-algebras, whose linear part is the retraction $\psi$ and the quadratic term depends only on $\psi, h_X$ and $h \lambda \nabla$.

**Proof.** To prove (1), we shall apply Theorem A.9 to the morphisms as in Proposition 2.5 (1). Recall the notations as in Theorem A.9

\[
\array{A & \Phi_1 & B \\ A' & \Phi_1' & B'}
\]

Here we need to take the following data: (1) $A' \longrightarrow B'$ is $\Sigma^\bullet(A[X])_{\text{proj}}$; (2) $A \rightarrow B$ is $\Sigma^\bullet(A)$; (3) $\Phi_1$ is the projection $\text{pr}$; (4) $\Phi_1$ is the horizontal lift $\lambda \nabla$; (5) $h = 0$; and (6) $h'$ is the homotopy $h_X : \text{T}^{\bullet}_{\text{mult}} \Gamma[X]_{\text{proj}} \to \Sigma^\bullet(A[X])_{\text{proj}}$ that appears in Proposition 2.5 (1). It is easy to see that conditions in Theorem A.9 are all satisfied, and therefore Claim 1 is proved.

Claim 2 can be proved similarly by applying Theorem A.9 to the maps appearing in Proposition 2.5 (2).
It follows from the previous proposition that the \( \mathbb{Z} \)-graded Lie 2-algebras \( \Sigma^\bullet(A) \xrightarrow{d} T^\bullet_{\text{mult}} \Gamma \) and \( \Sigma^\bullet(A[X]) \xrightarrow{d} T^\bullet_{\text{mult}} \Gamma[X] \) are homotopy equivalent in a unique, up to homotopy, way; an explicit homotopy can be obtained by inverting \( pr \), then composing with the inclusion \( i \). The following result extends Theorem 7.4 in [31].

**Theorem 2.8.** Let \( \Gamma_1 \rightrightarrows M_1 \) and \( \Gamma_2 \rightrightarrows M_2 \) be Morita equivalent Lie groupoids. Then any \( \Gamma_1 \rightrightarrows \Gamma_2 \)-bitorsor \( M_1 \leftarrow X \rightarrow M_2 \) induces a homotopy equivalence between the \( \mathbb{Z} \)-graded Lie 2-algebra \( \Sigma^\bullet(A_2) \xrightarrow{d} T^\bullet_{\text{mult}} \Gamma_2 \) of polyvector fields on \( \Gamma_2 \rightrightarrows M_2 \) and the \( \mathbb{Z} \)-graded Lie 2-algebra \( \Sigma^\bullet(A_1) \xrightarrow{d} T^\bullet_{\text{mult}} \Gamma_1 \) of polyvector fields on \( \Gamma_1 \rightrightarrows M_1 \).

By construction, the assignment in Theorem 2.8 is functorial. More precisely, let \( \text{Gr} \) be the category whose objects are Lie groupoids, and arrows are Morita bitorsors up to isomorphisms, and \( \text{Lie}_2 \) be the category whose objects are \( \mathbb{Z} \)-graded Lie 2-algebras, and arrows are homotopy equivalences of morphisms of \( \mathbb{Z} \)-graded Lie 2-algebras.

**Corollary 2.9.** The assignment in Theorem 2.8 yields a functor from the category \( \text{Gr} \) to the category \( \text{Lie}_2 \).

Such a functor is called the **polyvector field functor**.

**Remark 2.10.** Note that, to any Morita morphism, there is associated a canonical bitorsor. In the sequel, we will use either of them interchangeably. Assume that \( \phi \) is a Morita morphism of Lie groupoids from \( \Gamma_1 \rightrightarrows M_1 \) to \( \Gamma_2 \rightrightarrows M_2 \). It is easy to check that

\[
\begin{array}{ccc}
\Gamma_1 & \xleftarrow{\sigma_1} & M_1 \times_{M_2 \times \Gamma_2} \Gamma_2 \\
\downarrow & & \downarrow \\
M_1 & & M_2 \\
\end{array}
\]

is a \( \Gamma_1 \rightrightarrows \Gamma_2 \)-bitorsor. Here \( \sigma_1(m, \gamma) = m \), \( \sigma_2(m, \gamma) = s_2(\gamma) \), \( \forall (m, \gamma) \in M_1 \times_{M_2 \times \Gamma_2} \Gamma_2 \). The left action of \( \Gamma_1 \rightrightarrows M_1 \) on \( M_1 \times_{M_2 \times \Gamma_2} \Gamma_2 \) is given by

\[
\gamma_1 \cdot (m_1, \gamma_2) = (t_1(\gamma_1), \phi(\gamma_1)\gamma_2),
\]

while the right action of \( \Gamma_2 \rightrightarrows M_2 \) on \( M_1 \times_{M_2 \times \Gamma_2} \Gamma_2 \) is given by

\[
(m_1, \gamma_2) \cdot \gamma' = (m_1, \gamma_2 \gamma'),
\]

whenever composable.

It follows from Theorem 2.8 that, for Morita equivalent Lie groupoids \( \Gamma_1 \rightrightarrows M_1 \) and \( \Gamma_2 \rightrightarrows M_2 \), there is an \( L_\infty \)-isomorphism, canonical up to homotopy, between their corresponding dgla \( \mathcal{V}^\bullet(\Gamma_1) \) and \( \mathcal{V}^\bullet(\Gamma_2) \). At the level of cohomology this induces a canonical isomorphism of \( \mathbb{Z} \)-graded Lie algebras. The following corollary then extends to polyvector fields Corollary 7.2 in [31] for vector fields:

**Corollary 2.11.** Under the same hypothesis as in Theorem 2.8 there is a canonical isomorphism \( H^\bullet(\mathcal{V}(\Gamma_1)) \approx H^\bullet(\mathcal{V}(\Gamma_2)) \) as graded Lie algebras.

We are now ready to introduce the following

**Definition 2.12.** Let \( \mathcal{X} \) be a differentiable stack. The space of polyvector fields on \( \mathcal{X} \) is defined to be the homotopy equivalence class of \( \mathbb{Z} \)-graded Lie 2-algebras \( \Sigma^\bullet(A) \xrightarrow{d} T^\bullet_{\text{mult}} \Gamma \), where \( \Gamma \rightrightarrows M \) is any Lie groupoid representing \( \mathcal{X} \).
3 (+1)-shifted Poisson structures on differentiable stacks

3.1 Quasi-Poisson groupoids

**Definition 3.1** ([18]). Let $\Gamma \Rightarrow M$ be a Lie groupoid.

1. A quasi-Poisson structure on $\Gamma \Rightarrow M$ is a pair $(\Pi, \Lambda)$, with $\Pi \in T^1_{\text{mult}} \Gamma$ a multiplicative bivector field on $\Gamma$ and $\Lambda \in \Sigma^2(A)$ a section of $\Lambda^3 A$, satisfying
   \[ \frac{1}{2}[\Pi, \Pi] = \Lambda - \delta \Pi(\Lambda) = 0. \]  
   (9)

2. Two quasi-Poisson structures $(\Pi_1, \Lambda_1)$ and $(\Pi_2, \Lambda_2)$ on $\Gamma \Rightarrow M$ are said to be twist equivalent if there exists a section $T \in \Sigma^1(A)$, called the twist, such that
   \[ \Pi_2 = \Pi_1 +dT, \quad \Lambda_2 = \Lambda_1 - \delta \Pi_1(T) - \frac{1}{2}[T, T]. \]
   (10)

In the sequel, we will denote the twisted quasi-Poisson structure $(\Pi+dT, \Lambda - \delta \Pi_1(T) - \frac{1}{2}[T, T])$ by $(\Pi_T, \Lambda_T)$. Quasi-Poisson structures and twist equivalences may be spelled completely in the language of $\mathbb{Z}$-graded Lie 2-algebra.

**Proposition 3.2.** Let $\Gamma \Rightarrow M$ be a Lie groupoid.

1. The set of quasi-Poisson structures on $\Gamma \Rightarrow M$ coincides with the set of Maurer-Cartan elements of the $\mathbb{Z}$-graded Lie 2-algebra $\Sigma^\bullet(A) \mapsto T_{\text{mult}} \Gamma$.

2. Twist equivalence of quasi-Poisson structures coincides with gauge equivalence of Maurer-Cartan elements of the $\mathbb{Z}$-graded Lie 2-algebra $\Sigma^\bullet(A) \mapsto T_{\text{mult}} \Gamma$.

The second statement follows immediately once we note that Equation (10) can be rewritten as:

\[ \Lambda_2 \oplus \Pi_2 = e^T \cdot (\Lambda_1 \oplus \Pi_1). \]

As a consequence, for a given Lie groupoid $\Gamma \Rightarrow M$, the Maurer-Cartan moduli set $\text{MC}(\Sigma^\bullet(A) \mapsto T_{\text{mult}} \Gamma)$ of the $\mathbb{Z}$-graded Lie 2-algebra $\Sigma^\bullet(A) \mapsto T_{\text{mult}} \Gamma$ coincides with the set of twist equivalence classes of quasi-Poisson structures on $\Gamma \Rightarrow M$. The composition of the polyvector field functor $\underline{\text{Gr}} \rightarrow \text{Lie}_2$ (Corollary 2.9) with the Maurer-Cartan functor (Definition A.17) is a new functor, denoted $\text{Pois}$ from $\underline{\text{Gr}}$ to the category $\text{Sets}$. We will call such functor the Poisson functor.

According to Proposition 3.2, the Poisson functor associates to a Lie groupoid its moduli set of quasi-Poisson structures up to twists $\text{Pois}(\Gamma) := \text{MC}(\Sigma^\bullet(A) \mapsto T_{\text{mult}} \Gamma)$, and to a Morita equivalence of Lie groupoids the induced bijection between the corresponding moduli sets. We denote by $\Lambda \oplus \Pi$ the class in $\text{Pois}(\Gamma)$ of a quasi-Poisson groupoid $(\Gamma \Rightarrow M, \Pi, \Lambda)$.

**Lemma 3.3.** Let $(\Gamma_1 \Rightarrow M_1, \Pi_1, \Lambda_1)$ and $(\Gamma_2 \Rightarrow M_2, \Pi_2, \Lambda_2)$ be quasi-Poisson groupoids. Let $\phi$ be a Morita morphism from $\Gamma_2 \Rightarrow M_2$ to $\Gamma_1 \Rightarrow M_1$. The following statements are equivalent:

1. Under the Poisson functor $\text{Pois}(\phi) : \text{Pois}(\Gamma_2) \simeq \text{Pois}(\Gamma_1)$, the class $(\Lambda_2 \oplus \Pi_2) \in \text{Pois}(\Gamma_2)$ corresponds to $(\Lambda_1 \oplus \Pi_1) \in \text{Pois}(\Gamma_1)$;
2. The following relation holds

\[ MC(pr)^{-1} \left( (\Lambda_1 \oplus \Pi_1) \right) = MC(i)^{-1} \left( (\Lambda_2 \oplus \Pi_2) \right) \cdot \]

**Proof.** The polyvector field functor assigns to the Morita morphism \( \phi : \Gamma_2 \to \Gamma_1 \) an homotopy equivalent class of \( Z \)-graded Lie 2-algebra morphisms from polyvector fields on \( \Gamma_2 \cong M_2 \) to polyvector fields on \( \Gamma_1 \cong M_1 \). The latter can be represented by the composition \( pr \circ i^{-1} \), where \( pr \) and \( i \) are as in Equation (2). Then our claim follows immediately by properties of the Maurer-Cartan functor as in Definition A.17.

It is a standard result [21] that, for a given dgla \((g,d,[\cdot,\cdot])\) and a Maurer-Cartan element \( \lambda \in g_1 \), the triple \((g,d+\lambda,\cdot],[\cdot,\cdot])\) is again a dgla, called the tangent dgla [21]. In our case, for a given quasi-Poisson structure \((\Pi,\Lambda)\) on \( \Gamma \cong M \), since \((\Pi,\Lambda)\) is a Maurer-Cartan element in \( V^\cdot(\Gamma) \), the resulting twisted differential is given as follows:

\[
\begin{align*}
\delta_{\Pi,A} : \mathcal{V}^k(\Gamma) &\to \mathcal{V}^{k+1}(\Gamma) \\
 a \oplus P &\mapsto (-\delta_1(a) - \delta_P(\Lambda)) \oplus ([\Pi, P] + da),
\end{align*}
\]

where \( P \in \mathcal{T}_{mult}^k \Gamma \) and \( a \in \Sigma^{k+1}(A) \). As in classical Poisson geometry, we introduce the following

**Definition 3.4.** Let \( (\Gamma \cong M, \Pi, \Lambda) \) be a quasi-Poisson groupoid. The complex \( (\mathcal{V}^\cdot(\Gamma), \delta_{\Pi,A}) \) is called the Lichnerowicz-Poisson (LP) cochain complex of the quasi-Poisson structure \((\Pi,\Lambda)\), and its cohomology is called the Lichnerowicz-Poisson cohomology of \((\Pi,\Lambda)\), denoted by \( H^\cdot_{LP}(\Gamma \cong M, (\Pi, \Lambda)) \).

Since twist equivalent quasi-Poisson structures are gauge equivalent according to Proposition 3.2, the following proposition is immediate.

**Proposition 3.5.** If two quasi-Poisson structures on a Lie groupoid \( \Gamma \cong M \) are twist equivalent their corresponding Lichnerowicz-Poisson cohomologies are canonically isomorphic.

### 3.2 Morita equivalence and \((+1)\)-shifted Poisson differentiable stacks

**Definition 3.6.** Let \( (\Gamma_1 \cong M_1, \Pi_1, \Lambda_1) \) and \( (\Gamma_2 \cong M_2, \Pi_2, \Lambda_2) \) be quasi-Poisson groupoids. By a Morita morphism of quasi-Poisson groupoids from \((\Gamma_1 \cong M_1, \Pi_1, \Lambda_1)\) to \((\Gamma_2 \cong M_2, \Pi_2, \Lambda_2)\) we mean a Morita morphism of Lie groupoids

\[
\begin{array}{ccc}
\Gamma_1 & \xrightarrow{\phi} & \Gamma_2 \\
\downarrow & & \downarrow \\
M_1 & \xrightarrow{\phi} & M_2
\end{array}
\]

such that

1. there exists a twist \( T \in \Sigma^1(A_1) \) such that \( e^T \cdot (\Lambda_1 \oplus \Pi_1) \) is a projectable quasi-Poisson structure on \( \Gamma_1 \cong M_1 \);

2. \( \phi_* (e^T \cdot (\Lambda_1 \oplus \Pi_1)) = \Lambda_2 \oplus \Pi_2 \), i.e., \( (\phi)_* \Pi_{1,T} = \Pi_2 \), and \( (\phi)_* \Lambda_{1,T} = \Lambda_2 \).
Lemma 3.7. Let \( (\Gamma_1 \rightrightarrows M_1, \Pi_1, \Lambda_1) \) and \( (\Gamma_2 \rightrightarrows M_2, \Pi_2, \Lambda_2) \) be quasi-Poisson groupoids, and

\[
\begin{array}{ccc}
\Gamma_1 & \xrightarrow{\phi} & \Gamma_2 \\
\downarrow & & \downarrow \\
M_1 & \xrightarrow{\varphi} & M_2
\end{array}
\]

a Morita morphism of Lie groupoids. Then the following statements are equivalent.

(i) \( \phi \) is a Morita morphism of quasi-Poisson groupoids;

(ii) There exists a twist \( T_1 \in \Sigma^1(A_1) \) such that \( e^{T_1} \cdot (\Lambda_1 \oplus \Pi_1) \) is projectable on \( \Gamma_1 \rightrightarrows M_1 \), and \( \phi_*(e^{T_1} \cdot (\Lambda_1 \oplus \Pi_1)) = e^{T_2} \cdot (\Lambda_2 \oplus \Pi_2) \) for some \( T_2 \in \Sigma^1(A_2) \).

(iii) The relation \( \text{Pois}(\phi)(\Lambda_1 \oplus \Pi_1) = \Lambda_2 \oplus \Pi_2 \) holds;

(iv) The relation \( \text{MC}(\text{pr}) \circ \text{MC}(\text{pr}^{-1})(\Lambda_1 \oplus \Pi_1) = (\Lambda_2 \oplus \Pi_2) \) holds.

Proof. First, we prove the equivalence of (i) and (ii).

To be consistent with notations introduced earlier, let us assume that \( M_1 = X \), and \( \Gamma_1 = \Gamma[X] \). Then \( \phi : \Gamma_1 \rightarrow \Gamma_2 \) is simply the projection map \( \text{pr} : \Gamma[X] \rightarrow \Gamma \). It is obvious that (ii) holds if \( \phi \) is a Morita morphism of quasi-Poisson groupoids as defined in Definition 3.6.

Conversely, assume that (ii) is valid. Let \( T' \in \Sigma^1(A[X])_{\text{proj}} \) be any projectable section such that \( \phi_*(T') = \text{pr}(T') = T_2 \). Such a section \( T' \) always exists since \( \phi : \Sigma_1 A[X] \rightarrow \Sigma_1 A \) is a surjective submersion. For instance, the image \( \lambda_\phi(T_2) \) of \( T_2 \) through the horizontal lift as defined in Equation (6) satisfies such a property. It is simple to check that

\[
\phi_*\left(e^{T_1 - T'}(\Lambda_1 \oplus \Pi_1)\right) = \Lambda_2 \oplus \Pi_2.
\]

Therefore, \( \phi \) is indeed a Morita morphism of quasi-Poisson groupoids.

Next we prove the equivalence between (ii) and (iv). Let \( \hat{\Lambda} \oplus \hat{\Pi} \) be any representative of \( \text{MC}(i^{-1})(\Lambda_1 \oplus \Pi_1) \). By definition, \( \hat{\Lambda} \oplus \hat{\Pi} \) is projectable and twist equivalent to \( \Lambda_1 \oplus \Pi_1 \). Moreover \( \Gamma[X] \rightrightarrows \hat{X}, \hat{\Pi}, \hat{\Lambda} \) is a quasi-Poisson groupoid. The condition \( \text{MC}(\text{pr})(\Lambda_1 \oplus \Pi_1) = (\Lambda_2 \oplus \Pi_2) \) is then equivalent to \( \text{pr}_*(\hat{\Lambda} \oplus \hat{\Pi}) = \phi_*(\Lambda_1 \oplus \Pi_1) = e^{T_2} \cdot (\Lambda_2 \oplus \Pi_2) \) for some \( T_2 \in \Sigma^1(A_2) \).

Therefore (ii) and (iv) are indeed equivalent.

Finally, Lemma 3.3 implies that (iii) and (iv) are equivalent. This concludes the proof of the lemma.

We are now ready to introduce the Morita equivalence of quasi-Poisson groupoids.

Definition 3.8. Two quasi-Poisson groupoids \( (\Gamma_1 \rightrightarrows M_1, \Pi_1, \Lambda_1) \) and \( (\Gamma_2 \rightrightarrows M_2, \Pi_2, \Lambda_2) \) are Morita equivalent if there exist a third quasi-Poisson groupoid \( (\Sigma \rightrightarrows X, \Pi_X, \Lambda_X) \) and two Morita morphisms of quasi-Poisson groupoids \( (\Sigma \rightrightarrows X, \Pi_X, \Lambda_X) \rightarrow (\Gamma_1 \rightrightarrows M_1, \Pi_1, \Lambda_1) \) and \( (\Sigma \rightrightarrows X, \Pi_X, \Lambda_X) \rightarrow (\Gamma_2 \rightrightarrows M_2, \Pi_2, \Lambda_2) \).

In order to prove that this is indeed an equivalence relation, we need to describe Definition 3.8 in terms of the Poisson functor \( \text{Pois} \). Recall that \( \Lambda \oplus \Pi \) stands for the class of \( \Lambda \oplus \Pi \) in the moduli set \( \text{Pois}(\Gamma) := \text{MC}(\Sigma^*(A) \rightrightarrows \mathcal{T}_{\text{mult}}^\bullet). \)

Proposition 3.9. Two quasi-Poisson groupoids \( (\Gamma_1 \rightrightarrows M_1, \Pi_1, \Lambda_1) \) and \( (\Gamma_2 \rightrightarrows M_2, \Pi_2, \Lambda_2) \) are Morita equivalent if and only if there exists a Morita bitorsor \( M_1 \rightrightarrows X \rightarrow M_2 \) between \( \Gamma_1 \rightrightarrows M_1 \) and \( \Gamma_2 \rightrightarrows M_2 \) such that

\[
\text{Pois}(M_1 \rightrightarrows X \rightarrow M_2)(\Pi_1 \oplus \Lambda_1) = \Pi_2 \oplus \Lambda_2.
\]
Proof. Assume that we are given two Morita equivalent quasi-Poisson groupoids \((\Gamma_1 \equiv M_1, \Pi_1, \Lambda_1)\) and \((\Gamma_2 \equiv M_2, \Pi_2, \Lambda_2)\). By definition there exists a third quasi-Poisson groupoid \((\Xi \equiv X, \Pi_X, \Lambda_X)\) and two Morita morphisms of quasi-Poisson groupoids \(\phi_1 : (\Xi \equiv X, \Pi_X, \Lambda_X) \to (\Gamma_1 \equiv M_1, \Pi_1, \Lambda_1)\) and \(\phi_2 : (\Xi \equiv X, \Pi_X, \Lambda_X) \to (\Gamma_2 \equiv M_2, \Pi_2, \Lambda_2)\). According to Lemma 3.7, we have
\[
\text{Pois}(\phi_1)(\Lambda_X \oplus \Pi_X) = \Lambda_1 \oplus \Pi_1 \quad \text{and} \quad \text{Pois}(\phi_2)(\Lambda_X \oplus \Pi_X) = \Lambda_2 \oplus \Pi_2.
\]
This implies that:
\[
\text{Pois}(\phi_2) \circ \text{Pois}(\phi_1)^{-1} (\Lambda_1 \oplus \Pi_1) = \Lambda_2 \oplus \Pi_2.
\]
But the composition \(\text{Pois}(\phi_2) \circ \text{Pois}(\phi_1)^{-1}\) is exactly \(\text{Pois}(M_1 \leftarrow X \to M_2)\) since \(\text{Pois}\) is a functor.

Conversely, assume that we are given a bitorsor \(M_1 \leftarrow X \to M_2\) between the Lie groupoids \(\Gamma_1 \equiv M_1\) and \(\Gamma_2 \equiv M_2\) and quasi-Poisson structures \((\Pi_i, \Lambda_i)\) on \(\Gamma_i \equiv M_i, i = 1, 2\) such that:
\[
\text{Pois}(M_1 \leftarrow X \to M_2) (\Lambda_1 \oplus \Pi_1) = \Lambda_2 \oplus \Pi_2. \tag{13}
\]
Let \(\Gamma_i[X] \equiv X\) be the pull-back Lie groupoids of \(\Gamma_i \equiv M_i\). It is well known, then, that \(\Gamma_1[X] \equiv X\) is canonically isomorphic to \(\Gamma_2[X] \equiv X\) and let us denote such groupoid by \(\Xi \equiv X\). Both projections from \(\Xi \equiv X\) onto \(\Gamma_1 \equiv M_1\), and onto \(\Gamma_2 \equiv M_2\) are Morita morphisms that we denote by \(\phi_1\) and \(\phi_2\), respectively. By functoriality, we have
\[
\text{Pois}(M_1 \leftarrow X \to M_2) = \text{Pois}(\phi_2) \circ \text{Pois}(\phi_1)^{-1}.
\]
Then Equation (20) implies that
\[
\text{Pois}(\phi_1)^{-1} (\Lambda_1 \oplus \Pi_1) = \text{Pois}(\phi_2)^{-1} (\Lambda_2 \oplus \Pi_2).
\]
Let \((\Pi_X, \Lambda_X)\) be any quasi-Poisson structure on \(\Xi \equiv X\) whose class in \(\text{Pois}(\Xi)\) is
\[
\text{Pois}(\phi_1)^{-1} (\Lambda_1 \oplus \Pi_1) = \text{Pois}(\phi_2)^{-1} (\Lambda_2 \oplus \Pi_2).
\]
By construction, we have:
\[
\text{Pois}(\phi_1)(\Lambda_X \oplus \Pi_X) = \Lambda_1 \oplus \Pi_1 \quad \text{and} \quad \text{Pois}(\phi_2)(\Lambda_X \oplus \Pi_X) = \Lambda_2 \oplus \Pi_2,
\]
and therefore \(\phi_1\) and \(\phi_2\) are Morita morphisms of quasi-Poisson groupoids. Hence the quasi-Poisson groupoids \((\Gamma_1 \equiv M_1, \Pi_1, \Lambda_1)\) and \((\Gamma_2 \equiv M_2, \Pi_2, \Lambda_2)\) are Morita equivalent.

\[\square\]

**Corollary 3.10.** Morita equivalence in Definition 3.8 is indeed an equivalence relation among quasi-Poisson Lie groupoids.

**Proof.** It is an obvious consequence of Proposition 3.9 together with the fact that \(\text{Pois}\) is a functor. \[\square\]

**Theorem 3.11.** Let \((\Gamma_1 \equiv M_1, \Pi_1, \Lambda_1)\) be a quasi-Poisson groupoid. Assume that \(\Gamma_2 \equiv M_2\) is any Lie groupoid Morita equivalent to \(\Gamma \equiv M\) as Lie groupoids. Then there exist a quasi-Poisson structure \((\Pi_2, \Lambda_2)\), unique up to twists, on \(\Gamma_2 \equiv M_2\) such that \((\Gamma_2 \equiv M_2, \Pi_2, \Lambda_2)\) and \((\Gamma_1 \equiv M_1, \Pi_1, \Lambda_1)\) are Morita equivalent quasi-Poisson groupoids.
Proof. This is an immediate consequence of Proposition 3.9. Indeed, it suffices to choose \((\Pi_2, \Lambda_2)\) as any representative of the element in \(\text{Pois}(\Gamma_2)\), which is the image of \(\Pi_1 \oplus \Lambda_1 \in \text{Pois}(\Gamma_1)\) under a bitorsor \(\text{Pois}(M_1 \leftarrow X \rightarrow M_2)\).

We are now ready to introduce

**Definition 3.12.** A (+1) shifted Poisson differentiable stack, up to isomorphisms, is a Morita equivalence class of quasi-Poisson groupoids.

We will use the notation \((X, P)\) to denote a (+1)-shifted Poisson differentiable stack.

**Lemma 3.13.** Assume that \(\phi\) is a Morita morphism from the quasi-Poisson groupoid \((\Gamma_1 \rightrightarrows M_1, \Pi_1, \Lambda_1)\) to the quasi-Poisson groupoid \((\Gamma_2 \rightrightarrows M_2, \Pi_2, \Lambda_2)\). Then \(\phi\) induces a canonical isomorphism of Lichnerowicz-Poisson cohomology

\[ \phi_* : H^*_{LP}(\Gamma_1 \rightrightarrows M_1, (\Pi_1, \Lambda_1)) \xrightarrow{\sim} H^*_{LP}(\Gamma_2 \rightrightarrows M_2, (\Pi_2, \Lambda_2)) \]

**Proof.** There are two general facts regarding dgla’s and \(L_\infty\)-morphisms: (1) gauge equivalent MC elements have isomorphic cohomologies; (2) given two dglas \(V_1\) and \(V_2\) and an \(L_\infty\)-morphism \(\Psi\) from \(V_1\) to \(V_2\), for any MC element \(\alpha\) in \(V_1\), \(\Psi\) induces a morphism from the tangent cohomology associated to \(\alpha\) to the one associated to \(\Psi(\epsilon^\alpha)\). This morphism only depends on the homotopy class of \(\Psi\). Our result essentially follows from the above general facts combining with Corollary 2.8.

Since Lichnerowicz-Poisson cohomology of quasi-Poisson groupoids is invariant under Morita equivalence, the following definition is well-posed.

**Definition 3.14.** Let \((X, P)\) be a +1-shifted Poisson differentiable stack. Its Lichnerowicz-Poisson cohomology is

\[ H^*_{LP}(X, P) := H^*_{LP}(\Gamma \rightrightarrows M, (\Pi, \Lambda)) \]

where \((\Gamma \rightrightarrows M, \Pi, \Lambda)\) is any quasi-Poisson groupoid representing \((X, P)\).

## 4 Homotopy and Morita equivalence of VB groupoids

### 4.1 Homotopy equivalence of VB groupoids

We recall first basic facts of VB groupoids, following [16, 22, 23].

**Definition 4.1.** A VB groupoid is a groupoid object in the category of vector bundles.

In more concrete terms, a VB groupoid is the collection of Lie groupoids \(V \rightrightarrows E\) and \(\Gamma \rightrightarrows M\), where \(V \to \Gamma\) and \(E \to M\) are vector bundles, such that in the following commutative diagram

\[ \begin{array}{ccc}
V & \longrightarrow & E \\
\downarrow & & \downarrow \\
\Gamma & \longrightarrow & M
\end{array} \]  \hspace{1cm} (14)

1) source and target maps \(s_V, t_V\) of \(V\) are vector bundle morphisms over source and targets \(s, t\) of \(\Gamma\).
ii) \( k \)-composable arrows define a vector bundle \( V^{(k)} \to \Gamma^{(k)} \) for any \( k \in \mathbb{N} \);

iii) the multiplication map \( m_V : V^{(2)} \to V \) is a vector bundle morphism over the multiplication map \( m : \Gamma^{(2)} \to \Gamma \);

iv) the inverse map \( \text{inv}_V : V \to V \) is a vector bundle morphism over the inverse \( \text{inv}_\Gamma \).

For each \( \gamma \in \Gamma, m \in M \) and \( c_m \in E_m \), we denote with \( 0^V_\gamma \in V_\gamma \) and \( 1_{c_m} \in V_{1m} \) the zero of the vector bundle and the identity of the groupoid structure, respectively. We have that \( 0^V_m = 1_{0E_m} \) for each \( m \in M \). The core \( C \to M \) of \( V \) is the vector bundle over \( M \) defined as \( \text{Ker} \ s_V|_M \subset V|_M \). The core can be embedded by \( R_V : t^*C \to V \) and \( L_V : s^*C \to V \) defined as

\[
L_V(c_{s(\gamma)}) = -0^V_\gamma \cdot c_{s(\gamma)}^{-1}, \quad R_V(c_{t(\gamma)}) = c_{t(\gamma)} \cdot 0^V_\gamma, \tag{15}
\]

where \( \gamma \in \Gamma \). The core embedding by \( R_V \) fits into the following exact sequence of vector bundles over \( \Gamma \):

\[
0 \to t^*C \to V \to s^*E \to 0; \tag{16}
\]

we omit the analogous one for \( L_V \). The restriction of this sequence to \( M \) is canonically split by the embedding of unities: \( V|_M = C \oplus E \). Every section \( \pi : s^*E \to V \) of \( s_V : V \to s^*E \) defines a splitting of \( (16) \); a splitting that coincides with the canonical one when restricted to \( M \) is called a \textit{right decomposition}. Every right decomposition induces a vector bundle isomorphism \( \pi : V \to t^*C \times s^*E \) over the identity, which in turn defines a VB groupoid structure on \( t^*C \times s^*E \) over \( \Gamma \). The latter is referred to as a \textit{split VB groupoid}. See [16] for explicit structure maps.

**Definition 4.2.** Let \( V_1 \to \Gamma_1 \) and \( V_2 \to \Gamma_2 \) be VB groupoids over \( \Gamma_1 \equiv M_1 \) and \( \Gamma_2 \equiv M_2 \) respectively. A VB groupoid morphism \( \Phi : V_1 \to V_2 \) over \( \phi : \Gamma_1 \to \Gamma_2 \) is

- a vector bundle morphism over \( \phi : \Gamma_1 \to \Gamma_2 \);
- a groupoid morphism.

The induced map \( \phi : \Gamma_1 \to \Gamma_2 \) is a groupoid morphism. Moreover, \( \Phi \) induces vector bundle morphisms between units and cores that we denote with the same letter \( \Phi \).

Let \( V^\vee \to \Gamma \) be the dual bundle of a VB groupoid \( V \to \Gamma \) with units \( E \) and core \( C \). It inherits from \( V \) the structure of VB groupoid

\[
\begin{array}{c}
\xrightarrow{V^\vee} \\
\xleftarrow{\Gamma} \\
\xrightarrow{M}
\end{array}
\]

where the source and target maps \( s_{V^\vee}, t_{V^\vee} : V^\vee \to C^\vee \) are defined as

\[
(s_{V^\vee}(\eta), c) = -\langle \eta, 0^V_\gamma \cdot c^{-1} \rangle, \quad (t_{V^\vee}(\eta), c') = \langle \eta, c' \cdot 0^V_\gamma \rangle \tag{18}
\]

with \( c \in C_{s(\gamma)} \), \( c' \in C_{t(\gamma)} \) and \( \eta \in V^\vee_\gamma \). In particular, one sees that

\[
R_V = t_{V^\vee}^V : t^*C \to V, \quad L_V = s_{V^\vee}^V : s^*C \to V. \tag{19}
\]

Finally, the core of the dual VB-groupoid \( V^\vee \) is \( E^\vee \).
**Example 4.3.** For any Lie groupoid \( \Gamma \rightrightarrows M \), the tangent groupoid \( TT \rightrightarrows TM \) is a VB groupoid with core the Lie algebroid \( A \) of \( \Gamma \) and \( E = TM \):

\[
\begin{array}{c}
TT \\
\downarrow \\
\Gamma
\end{array} \longrightarrow \begin{array}{c}
TM \\
\downarrow \\
M
\end{array} \tag{20}
\]

We recall that all the structure maps of \( TT \) are the differentials of the structure maps of \( \Gamma \), e.g. \( s_{TT} = s_* \), \( t_{TT} = t_* \), and so on.

**Example 4.4.** The dual VB groupoid of the tangent groupoid defined in Example 4.3 is the cotangent groupoid \( T^\vee \Gamma \) with \( E = A^\vee \) and core \( C = T^\vee M \):

\[
\begin{array}{c}
T^\vee \Gamma \\
\downarrow \\
\Gamma
\end{array} \longrightarrow \begin{array}{c}
A^\vee \\
\downarrow \\
M
\end{array} \tag{21}
\]


It is a standard result [25] that any multiplicative bivector field \( \Pi \) induces a morphism of VB-groupoids from (21) to (20). Thus we have the following

**Proposition 4.5.** Let \( (\Gamma \rightrightarrows M, \Pi, A) \) be a quasi-Poisson groupoid. Then \( \Pi \) induces a morphism of VB-groupoids \( \Pi^\# : T^\vee \Gamma \rightarrow TT \) from the tangent VB groupoid to the cotangent VB groupoid.

The following result follows from a direct verification.

**Lemma 4.6.** Let \( \hat{\varphi} : E \rightarrow E \) be a bundle map that is a surjective submersion over \( \varphi : X \rightarrow M \). The pull-back Lie groupoid \( V[\mathcal{E}] = \mathcal{E} \times_E V \times_E \mathcal{E} \rightrightarrows \mathcal{E} \) is a VB groupoid over \( \Gamma[\mathcal{X}] = X \times_M \Gamma \times_M X \rightrightarrows \mathcal{X} \) and the middle term projection \( \Phi_{\hat{\varphi}} : V[\mathcal{E}] \rightarrow V \) is a VB groupoid morphism over the middle term projection \( \Phi_{\varphi} : \Gamma[\mathcal{X}] \rightarrow \Gamma \).

It is straightforward to check that the core \( C(V[\mathcal{E}]) \) of \( V[\mathcal{E}] \) is \( E \times_{\varphi^*E} E \text{ for } C \). In particular, for any surjective submersion \( \varphi : X \rightarrow M \), we can consider \( V[\varphi^*E] = X \times_M V \times_M X \rightrightarrows \varphi^*E \) and so on.

From now on, let \( \Gamma \rightrightarrows M \) be a Lie groupoid. Let \( V_1 \rightrightarrows E_1 \) and \( V_2 \rightrightarrows E_2 \) be VB groupoids over \( \Gamma \) with cores \( C_1 \) and \( C_2 \) respectively and let \( \Phi \) and \( \Psi \) be two VB groupoid morphisms from \( V_1 \) to \( V_2 \) over the identity on \( \Gamma \).

**Definition 4.7.** We say that \( \Phi \) is homotopic to \( \Psi \) if there exists a bundle map \( h : E_1 \rightarrow C_2 \) over the identity on \( M \) such that the following relation holds

\[
\Phi - \Psi = L_{V_2} h \circ s_{V_1} + R_{V_2} h \circ t_{V_1} \equiv J_h, \tag{22}
\]

where \( L_{V_2} : s^*(C_2) \rightarrow V_2, R_{V_2} : t^*(C_2) \rightarrow V_2 \) are the left and right inclusions of the core \( C_2 : M \) into \( V_2 \rightrightarrows E_2 \) [22] [23] [16].

**Remark 4.8.** A useful expression for \( J_h \) follows from (18,19). Indeed, if \( v_\gamma \in (V_1)_\gamma \), then \( J_h \) reads

\[
J_h(v_\gamma) = 0_\gamma \cdot h(s_{V_1}(v_\gamma))^{-1} + h(t_{V_1}(v_\gamma)) \cdot 0_\gamma. \tag{23}
\]

We call \( J_h \) the VB homotopy defined by \( h \). It is clear that \( J_h \) is a VB groupoid morphism over the identity.
Proposition 4.9. If Φ and Ψ are homotopic VB groupoid morphisms from V₁⇒E₁ to V₂⇒E₂ over the identity on Γ with VB homotopy Jₜ defined by h : E₁ → C₂, then also Φ₁ and Φ₂ are homotopic VB groupoid morphisms from V₁⇒C₂ to V₂⇒C₁ with VB homotopy Jₜ defined by h : C₂ → E₁.

Proof. Since Lᵥ = sᵥᵥ C → V and Rᵥ = tᵥᵥ C → V, then Equation (22) can be written as

\[ Jₜ = sᵥᵥ h + tᵥᵥ h, \]

so that the result follows.

Definition 4.10. An homotopy equivalence between VB groupoids V₁ over Γ₁ and V₂ over Γ₂, is a pair of VB groupoid morphisms Φ : V₁ → V₂ over φ : Γ₁ → Γ₂ and Ψ : V₂ → V₁ over φ⁻¹ : Γ₂ → Γ₁, such that Φ ∘ Ψ and Ψ ∘ Φ are homotopic to the identity map.

Let us extend Definition 4.7 of homotopy of VB morphisms to the case in which the sources (and the targets) of the two VB morphisms are homotopy equivalent VB groupoids.

Definition 4.11. Let V₁⇒E₁ and W₁⇒F₁ be VB groupoids, which are homotopically equivalent to V₂⇒E₂ and W₂⇒F₂, respectively, as VB groupoids over Γ. Let Φ : V₁ → W₁ and Ψ : V₂ → W₂ be VB groupoid morphisms over the identity of Γ. We say that Φ and Ψ are homotopic if the diagram

\[
\begin{array}{ccc}
V₁ & \xrightarrow{\Phi} & W₁ \\
\downarrow{Φ} & & \downarrow{Ψ} \\
V₂ & \xrightarrow{Ψ} & W₂
\end{array}
\]

is commutative up to VB-homotopy.

Remark 4.12. In Section 6 of [17], a VB homotopy is called an isomorphism and an homotopy equivalence is just an equivalence. It is then introduced the VB-groupoid derived category \( VB[Γ] \), whose objects are the VB groupoids over Γ and morphisms the classes of VB morphisms over idΓ up to homotopy equivalence. It was proven that a Morita morphism \( φ : Z → Γ \) defines an equivalence of categories \( \hat{φ} : VB[Γ] → VB[Z] \) (Theorem 6.7 [17]). Homotopic morphisms in Definition 4.11 should correspond to invertible morphisms in \( VB[Γ] \) in terms of terminologies in [17].

4.2 Morita equivalence of VB groupoids

We define Morita equivalence of VB groupoids in a similar fashion as Morita equivalence of Lie groupoids. First of all, we define Morita morphisms of VB groupoids.

Definition 4.13. Let W⇒E and V⇒E be VB groupoids with Z⇒X and Γ⇒M be the underlying base Lie groupoids. A VB morphism \( Φ : W → V \) is a VB Morita morphism if

i) the induced vector bundle morphism \( Φ₀ : E → E \) is a surjective submersion;

ii) the associated diagram

\[
\begin{array}{ccc}
W & \longrightarrow & E \times E \\
\downarrow & & \downarrow \\
V & \longrightarrow & E \times E
\end{array}
\]

is cartesian in the category of vector bundles.
Let $V[E] = \mathcal{E} \times_{E} V \times_{E} \mathcal{E}$ be the pull back VB groupoid defined in Lemma 4.6. Condition (ii) says that the VB groupoid morphism $W \to V[E]$, defined as $w \in W \to (tw(w), \Phi(w), sw(w)) \in V[E]$, is an isomorphism of vector bundles. In particular, the Lie groupoid morphism $Z \to \Gamma$ is a Morita morphism.

**Remark 4.14.** In [17] a Morita morphism is equivalently defined as a VB groupoid morphism that is also a Morita morphism between Lie groupoids.

The following Lemma indicates that a VB Morita morphism is encoded in the base Morita morphism and a VB homotopy.

**Lemma 4.15.** Let $V \subset E$ be a VB groupoid over $\Gamma \rightrightarrows M$ as in [14] and let $\varphi : X \to M$ be a surjective submersion.

i) The VB morphism $\Phi_{\varphi} : V[\varphi^{*}E] \to V$ is a VB Morita morphism.

ii) Any VB Morita morphism $\Phi : W \to V$, where $W \subset E$ is over $\Gamma[X] \rightrightarrows X$, inducing the same map $\varphi : X \to M$, factors through $\Phi_{\varphi}$, i.e. $\Phi = \Phi_{\varphi} \circ \Phi'$.

![Diagram](https://via.placeholder.com/150)

where $\Phi' : W \to V[\varphi^{*}E]$ is defined as $\Phi'(w) = (\Phi_{0}(tw(w)), \Phi_{1}(w), \Phi_{0}(sw(w)))$, $w \in W$.

iii) the VB groupoids $W$ and $V[\varphi^{*}E]$ in ii) are homotopically equivalent.

**Proof.** i) is obvious. Analogously ii) is proved by observing that $W$ is by hypothesis isomorphic to $V[E] = \mathcal{E} \times_{E} V \times_{E} \mathcal{E}$ that can be equally seen as $\mathcal{E} \times_{\varphi^{*}E} V[\varphi^{*}E] \times_{\varphi^{*}E} \mathcal{E}$.

Let us prove iii). Since $\Phi_{0} : \mathcal{E} \to E$ is a surjective vector bundle morphism over $\varphi$, we can choose an Ehresmann connection on it, i.e. a bundle map $\nabla : \varphi^{*}E \to \mathcal{E}$ satisfying $\Phi_{0} \circ \nabla = \text{id}$. Let $\Phi'_{E} : V[\mathcal{E}] \to V[\varphi^{*}E]$ be the VB morphism defined as $\Phi'_{E}(\epsilon_{x}, v_{\gamma}, \epsilon_{y}) = (x, v_{\gamma}, y)$. We denote with the same symbol $\nabla$ the horizontal lift $\nabla : V[\varphi^{*}E] \to V[\mathcal{E}]$ defined as $\nabla(x, v_{\gamma}, y) = (\nabla(tv(v_{\gamma})), v_{\gamma}, \nabla(sv(v_{\gamma})))$, satisfying $\Phi'_{E} \circ \nabla = \text{id}$. We compute

$$(\text{id} - \nabla \circ \Phi'_{E})(\epsilon_{x}, v_{\gamma}, \epsilon_{y}) = (\epsilon_{x}^{\text{vert}}, \epsilon_{y}^{\text{vert}}),$$

where $\epsilon_{x}^{\text{vert}} = \epsilon_{x} - \nabla(\Phi_{0}(\epsilon_{x}))$. Since vertical vectors $\epsilon_{x}^{\text{vert}} \in \text{Ker} \Phi_{0} \subset \mathcal{E}_{\Phi_{0} \times tv} \varphi^{*}C = C(V[\mathcal{E}])$ and $L_{V[E]}(w_{y}) = (0, 0, w_{y})$, $R_{V[E]}(v_{x}) = (v_{x}, 0, 0)$, the above formula reproduces (22) for $h : \mathcal{E} \to C(V[\mathcal{E}])$ defined as $h(\epsilon_{x}) = \epsilon_{x}^{\text{vert}}$. We conclude that $V[\varphi^{*}E]$ and $V[\mathcal{E}]$ are homotopically equivalent; the result then follows by composing it with the VB isomorphism between $V[E]$ and $W$.

**Definition 4.16.** Two VB groupoids $V_1$ and $V_2$ are VB-Morita equivalent if there exists a third VB groupoid $W$ and two VB Morita morphisms

$$V_1 \leftarrow W \rightarrow V_2.$$
Exactly as for Lie groupoids, the following result holds:

**Proposition 4.17.** Two VB groupoids $V_1$ and $V_2$ are VB-Morita equivalent if and only if there exists a $V_1$-$V_2$ bitorsor $T \rightarrow X$

\[
V_1 \leftarrow T \rightarrow V_2
\]

such that all structural maps defining the bitorsor $T$ are vector bundle morphisms.

**Proof.** Let $V_1$ and $V_2$ be two VB groupoids.

If a $V_1 - V_2$ bitorsor $T$ fulfilling the hypothesis of the proposition exists then the pull-back VB groupoids $V_1[T]$ and $V_2[T]$ are isomorphic as Lie-groupoids. Since the structural maps defining the bitorsor structures are vector bundle morphisms, the Lie groupoid isomorphism $V_1[T] \simeq V_2[T]$ is a VB-groupoid isomorphism and the conditions of Definition 4.16 are satisfied.

Conversely, given a Morita equivalence as in Definition 4.16, then $W \cong V_1[E]$, with $E$ the unit vector bundle of the VB-groupoid $W \Rightarrow E$. As a consequence, $T_1 := V_1 \times_{M_1} E$ is a $V_1$-$W$ bitorsor. For the same reason, $T_2 := E \times_{M_2} V_2$ is a $W$-$V_2$ bitorsor. Composing these bitorsors (i.e. considering $T_1 \times T_2$, with $W$ acting diagonally), we obtain a $V_1 - V_2$ bitorsor as in Proposition 4.17. This proves the claim.

**Remark 4.18.** The Lie groupoid morphism $\phi : Z \rightarrow \Gamma$ underlying $\Phi : W \rightarrow V$ in Definition 4.13 is a Morita morphism. As a consequence, if $\Gamma_1$ and $\Gamma_2$ are the underlying base groupoids of $V_1$ and $V_2$ in Definition 4.16 then $\Gamma_1$ and $\Gamma_2$ are Morita equivalent as Lie groupoids.

The following facts are direct consequences of the definition.

**Proposition 4.19.** VB-Morita equivalent VB groupoids are Morita equivalent as Lie groupoids.

**Proof.** It is clear from Definition 4.13 that a VB Morita morphism is a Morita morphism of Lie groupoids.

At this point, just repeating the steps of the proof for Lie groupoids (e.g. [7]), we can show the following:

**Proposition 4.20.** VB-Morita equivalence between VB groupoids defines an equivalence relation.

We now need a characterization of VB-Morita equivalence in terms of VB homotopies.

We are going first to prove the following result.

**Lemma 4.21.** If $V_1 \equiv E_1$ and $V_2 \equiv E_2$ are homotopy equivalent VB groupoids over $\Gamma \equiv M$ then they are VB-Morita equivalent.

**Proof.** Let $V_1 \xleftarrow{\psi} \xrightarrow{\phi} V_2$ be an homotopy equivalence with homotopies $h_1 : E_1 \rightarrow C_1$ and $h_2 : E_2 \rightarrow C_2$. Let $E_1 \times_M E_2 \rightarrow E_1$ be the natural projection and let $V_1[E_1 \times_M E_2] \equiv E_1 \times_M E_2$ be the pull back VB groupoid over $\Gamma$; analogously for $V_2[E_1 \times_M E_2] \equiv E_1 \times_M E_2$. Let us consider the
vector bundle morphisms $A : V_1[E_1 \times_M E_2] \to V_2[E_1 \times_M E_1]$ and $B : V_2[E_1 \times_M E_1] \to V_1[E_2 \times_M E_2]$ defined as

$$A(\tilde{e}_{t(\gamma)}, v_\gamma, \tilde{e}_{s(\gamma)}) = (\Psi_0(\tilde{e}_{t(\gamma)}) + t v_1(v_\gamma), \Phi_1(v_\gamma) + h_2(\tilde{e}_{t(\gamma)}) \cdot 0^1_\gamma + 0^2_\gamma \cdot h_2(\tilde{e}_{s(\gamma)})^{-1}, \Psi_0(\tilde{e}_{s(\gamma)}) + s v_1(v_\gamma))$$

and

$$B(\tilde{e}_{t(\gamma)}, \tilde{e}_\gamma, e_{s(\gamma)}) = (\Phi_0(\tilde{e}_{t(\gamma)}) - t v_2(\tilde{e}_\gamma), \Psi_1(\tilde{e}_\gamma) - h_1(e_{s(\gamma)}) \cdot 0^1_\gamma - 0^1_\gamma \cdot h_1(e_{s(\gamma)})^{-1}, \Phi_0(e_{s(\gamma)}) - s v_2(\tilde{e}_\gamma)).$$

It is easy to check that $A$ and $B$ commute with source and target maps over $A_0, B_0 : E_1 \times_M E_2 \to E_1 \times_M E_2$, where

$$A_0 = \begin{pmatrix} \text{id}_{E_1} & \Psi_0 \\ 0 & t v_2 \circ h_2 \end{pmatrix}, \quad B_0 = \begin{pmatrix} -t v_1 \circ h_1 & \Psi_1 \\ 0 & -\text{id}_{E_2} \end{pmatrix},$$

and that they preserve the multiplication. Let us check that $B \circ A = \text{id}$. Indeed, we compute

$$B \circ A(\tilde{e}_{t(\gamma)}, v_\gamma, \tilde{e}_{s(\gamma)}) = (\Phi_0(\Psi_0(\tilde{e}_{t(\gamma)}) + t v_1(v_\gamma)) - t v_2(\Phi_1(v_\gamma) - h_2(\tilde{e}_{t(\gamma)})), \Psi_1(\Phi_1(v_\gamma) + \Psi_1(h_2(\tilde{e}_{t(\gamma)})) \cdot 0^1_\gamma + 0^2_\gamma \cdot \Psi_1(h_2(\tilde{e}_{s(\gamma)})^{-1})$$

$$-h_1(\Psi_0(\tilde{e}_{t(\gamma)} + t v_1(v_\gamma))) \cdot 0^1_\gamma - 0^1_\gamma \cdot h_1(\Psi_0(\tilde{e}_{s(\gamma)}) + s v_1(v_\gamma))^{-1}, \quad (\Phi_0(\Psi_0(\tilde{e}_{s(\gamma)})) + s v_1(v_\gamma)) - s v_2(\Phi_1(v_\gamma) + 0^2_\gamma \cdot h_2(\tilde{e}_{s(\gamma)})^{-1})$$

$$= (\tilde{e}_{t(\gamma)}, v_\gamma + (\Psi_1 \circ h_2 - h_1 \circ \Psi_0)(\tilde{e}_{t(\gamma)}) \cdot 0^1_\gamma$$

$$+ 0^1_\gamma \cdot (\Psi_1 \circ h_2 - h_1 \circ \Psi_0)(\tilde{e}_{s(\gamma)})^{-1}, \tilde{e}_{s(\gamma)})$$

where we used the definition (23) of VB homotopy in order to get the second equality. We actually proved that

$$B \circ A = \text{id} + J_h$$

where $J_h$ is the VB homotopy with $\tilde{h} : E_1 \times E_2 \to C(V_1[E_1 \times_M E_2]) = C_1 \times_M E_2$ defined by

$$\tilde{h}(e_m, \tilde{e}_m) = ((\Psi_1 h_2 - h_1 \Psi_0)(\tilde{e}_m), 0_m).$$

The result then follows from the fact that $\text{id} + J_h$ is invertible, that is easily checked by verifying from the above computation that it is injective and surjective.

Proposition 4.22. Two VB groupoids $V_1 \equiv E_1$ and $V_2 \equiv E_2$ over $\Gamma_1 \equiv M_1$ and $\Gamma_2 \equiv M_2$ are VB-Morita equivalent if and only if there exist

a) a $\Gamma_1 \to \Gamma_2$ bitorsor $M_1 \xrightarrow{\varphi_1} X \xrightarrow{\varphi_2} M_2$;

b) an homotopy equivalence between pullback VB groupoids $V_1[\varphi_1^* E_1]$ and $V_2[\varphi_2^* E_2]$ along $\varphi_1$ and $\varphi_2$ respectively.
Proof. Let $V_1 \rightleftarrows E_1$ and $V_2 \rightleftarrows E_2$ be VB-Morita equivalent VB-groupoids and let $E_1 \leftarrow T \rightarrow E_2$ be a $V_1 - V_2$ bitorsor as in [4.17]. Recall from the proof of that proposition that $T$ induces a VB-groupoid isomorphism $\Phi_T : V_1[T] \cong V_2[T]$. Let $\Gamma_1 \rightleftarrows M_1, \Gamma_2 \rightleftarrows M_2$ and $\Gamma_W \rightleftarrows M_W$ be the base groupoids of $V_1, V_2$ and $W$ respectively. By definition, $T$ is a vector bundle and its base manifold is a $\Gamma_1 - \Gamma_2$ bitorsor $M_1 \xrightarrow{\phi_1} X \xrightarrow{\phi_2} M_2$. Now, by the third item in Lemma 4.15 the tangent VB-groupoids $V_1[\phi_1^* E_1]$ and $V_1[T]$ are homotopy equivalent and so are $V_2[T]$ and $V_2[\phi_1^* E_2]$. Composing these homotopy equivalences with the VB-groupoid isomorphism $\Phi_T : V_1[T] \cong V_2[T]$, one obtains a homotopy equivalence between $V_1[\phi_1^* E_1]$ and $V_2[\phi_1^* E_2]$. This proves one direction.

Assume now Diagram (22) is satisfied. Lemma [4.21] applied to $V_1[\phi_1^* E_1]$ and $V_2[\phi_2^* E_2]$ implies that these VB groupoids are VB-Morita equivalent. Since the Morita morphisms $V_1[\phi_1^* E_2] \rightarrow V_1$ and $V_2[\phi_2^* E_2] \rightarrow V_2$ are Morita equivalences of VB-groupoids, the result follows from Proposition 4.19.

Corollary 4.23. Two VB groupoids $V_1 \rightleftarrows E_1$ and $V_2 \rightleftarrows E_2$ over $\Gamma_1 \rightleftarrows M_1$ and $\Gamma_2 \rightleftarrows M_2$ are VB-Morita equivalent if and only if their duals $V_1^\vee \rightleftarrows C_1^\vee$ and $V_2^\vee \rightleftarrows C_2^\vee$ are VB-Morita equivalent.

Proof. Given a surjective submersion $\phi : X \rightarrow M$, then $V[\phi^* E^\vee] = V^\vee [\phi^* C^\vee]$. We can then apply Proposition 4.9 and conclude that $V_1^\vee [\phi_1^* C_1^\vee]$ and $V_2^\vee [\phi_2^* C_2^\vee]$ are homotopy equivalent. Therefore, $V_1^\vee$ and $V_2^\vee$ are VB-Morita equivalent according to Proposition 4.22.

Proposition 4.24. If $\Gamma_1$ and $\Gamma_2$ are Morita equivalent Lie groupoids, both the tangent VB groupoids $TT_1$ and $TT_2$ and the cotangent VB groupoids $T^\vee \Gamma_1$ and $T^\vee \Gamma_2$ are VB Morita equivalent.

Proof. Let $X$ be a $\Gamma_1 - \Gamma_2$ bitorsor; then $TX$ is a $TT_1 - TT_2$ bitorsor. Its structural maps are vector bundle morphisms by construction. This proves the claim by Proposition 4.17. The result for the cotangent groupoids follows from Corollary 4.23.
Remark 4.25. By applying Proposition 4.22 we can conclude that if $\varphi : X \to M$ is a surjective submersion then there exist homotopy equivalences

$$\xymatrix{T^\vee(\Gamma[X]) \ar[r]^{\nabla_\varphi} & (T^\vee\Gamma)[\varphi^\ast A^\vee] \ar[r]^{\nabla_{\Phi\varphi}} & (T^\vee\Gamma)[\varphi^\ast TM]}$$

where $\Phi_\varphi : \Gamma[X] \to \Gamma$ denotes the natural Morita morphism and $\nabla : \varphi^\ast TM \to TX$ is an Ehresmann connection for $\varphi : X \to M$; we denote with the same symbol $\nabla$ also the induced horizontal lifting $\nabla : (T\Gamma)[\varphi^\ast TM] \to T(\Gamma[X])$ for $\Phi_\varphi$. In particular the morphism $T^\vee\Gamma[X] \to T^\vee\Gamma$ obtained by composing $\nabla_\varphi$ with the canonical vector bundle map $T^\vee[\varphi^\ast A^\vee] \to T^\vee\Gamma$ is the Morita map.

Definition 4.26. Let $V_1 \equiv E_1$ and $W_1 \equiv F_1$ be VB groupoids over $\Gamma_1$, and $V_2 \equiv E_2$ and $W_2 \equiv F_2$ VB groupoids over $\Gamma_2$. Assume that $V_1$ and $W_1$ are Morita equivalent quasi-Poisson groupoids, which are Morita equivalent to $V_2$ and $W_2$, respectively. Let $\Phi : V_1 \to W_1$ and $\Psi : V_2 \to W_2$ be VB groupoid morphisms over the identity of $\Gamma_1$ and $\Gamma_2$ respectively. We say that $\Phi$ and $\Psi$ are homotopic if the pull back maps

$$\Phi : V_1[\varphi_1^\ast E_1] \to W_1[\varphi_1^\ast F_1], \quad \Psi : V_2[\varphi_2^\ast E_2] \to W_2[\varphi_2^\ast F_2]$$

are homotopic as VB groupoid maps over $\Gamma_1[X] \cong \Gamma_2[X]$, where $M_1 \xleftarrow{\varphi_1} X \xrightarrow{\varphi_2} M_2$ is a $\Gamma_1 - \Gamma_2$ bitorsor and the homotopies are as in Proposition 4.22.

The above definition means that in the following diagram

$$V_1[\varphi_1^* E_1] \cong \cong V_2[\varphi_2^* E_2] \quad (26)$$

the sub-diagram made of the dashed lines is commutative up to homotopy.

We are now ready to state the main result of this section.

Theorem 4.27. If $(\Gamma_1, \Pi_1, A_1)$ and $(\Gamma_2, \Pi_2, A_2)$ are Morita equivalent quasi-Poisson groupoids, then $\Pi_1^\# : T^\vee \Gamma_1 \to TT_1$ and $\Pi_2^\# : T^\vee \Gamma_2 \to TT_2$ are homotopic VB-groupoid morphisms in the sense of Definition 4.29.

Proof. Let $(\Gamma_1 \equiv M_1, \Pi_1, A_1)$ and $(\Gamma_2 \equiv M_2, \Pi_2, A_2)$ be two quasi-Poisson groupoids, which are Morita equivalent. We first prove the statement for two quasi-Poisson structures related by a twist. Let $\Gamma \equiv M$ be a Lie groupoid with algebroid $A$ and $(\Pi, A)$ be a quasi Poisson structure. Let $\Pi_T = \Pi + dT$ be the twist by $T \in A^2(A)$. We easily check that $\Pi_T - \Pi = J_T$, where $J_T : T^\vee \Gamma \to TT$ is the VB-homotopy defined by $T : A^\vee = E(T^\vee \Gamma) \to A = C(TT)$.

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Let now \( \varphi : X \to M \) be a surjective submersion and let us choose an Ehresmann connection \( \nabla \). Let us consider the homotopy equivalences as in Remark 1.29:

\[
T^\vee \Gamma[X] \xrightarrow{\nabla^\vee} (T^\vee \Gamma)[\varphi^* A^\vee] \quad \text{and} \quad T\Gamma[X] \xrightarrow{T\Phi_{\varphi}} (T\Gamma)[\varphi^* TM].
\]

We now have to prove that if \( \Pi \) is a multiplicative bivector field on \( \Gamma[X] \) that projects to \( \Pi' \) on \( \Gamma \) then

\[
T^\vee \Gamma[X] \xrightarrow{(T^\vee \Gamma)[\varphi^* A^\vee]} T\Gamma[X] \xrightarrow{T\Phi_{\varphi}} (T\Gamma)[\varphi^* TM]
\]

is commutative up to homotopy. Let \( h : \varphi^* A^\vee \to \text{Ker} T\varphi \subset A[X] = TX \times_{TM} \varphi^* A \) be defined as

\[
h = (1 - \nabla \circ T\varphi) \circ \Pi \circ (T\Phi_{\varphi})^\vee.
\]

One checks that:

\[
\Pi \circ (T\Phi_{\varphi})^\vee - \nabla \circ \Pi' = J_h.
\]

This completes the proof of the claim.

\( \square \)

5 2-term complexes over a differentiable stack

The aim of this section is to introduce 2-term complexes over a differentiable stack in terms of Morita equivalent classes of homotopy \( \Gamma \)-modules, and to build up the connection with Morita equivalent classes of VB-groupoids. First, we establish here the basic dictionary between VB-groupoids and 2-term homotopy \( \Gamma \)-modules, following [16]. We then translate the results established in the previous section in the language of homotopy \( \Gamma \)-modules. In this way, we are led naturally to the category of 2-term complexes over a given differentiable stack \( X \), and obtain an efficient way of studying this category in terms of VB-groupoids.

5.1 Homotopy \( \Gamma \)-modules

For a given Lie groupoid \( \Gamma \Rightarrow M \), let \((C^\bullet(\Gamma), \delta)\) denote the groupoid cohomology cochain complex \( \delta : C^\infty(\Gamma^{(p+1)}) \to C^\infty(\Gamma^{(p+1)}), p \geq 0 \), where \( \Gamma^{(p)} \) denotes the manifold consisting of \( p \)-composable arrows in \( \Gamma \Rightarrow M \). We recall that for \( f \in C^\infty(M) \) we have

\[
\delta f = s^* (f) - t^* (f)
\]

and for \( f \in C^\infty(\Gamma^{(p)}) \) and \((\gamma_0, \ldots, \gamma_p) \in \Gamma^{(p+1)}\)

\[
(\delta f)(\gamma_0, \ldots, \gamma_p) = f(\gamma_1, \ldots, \gamma_p) + \sum_{j=1}^{p} (-)^j f(\gamma_0, \ldots, \gamma_{j-1}\gamma_j \ldots, \gamma_p) + (-)^{p+1} f(\gamma_0, \ldots, \gamma_{p-1}).
\]

Let \( \mathcal{E} := \bigoplus E_r \) be a graded vector bundle over \( M \), and let \( C^p(\Gamma, E_r) = \Gamma((\pi^p_{\delta})^* E_r) \), where \( \pi^p_{\delta} : \Gamma^{(p)} \to \Gamma \) is defined as \( \pi^p_{\delta}(\gamma_1, \ldots, \gamma_q) = s(\gamma_1) \) for \( q > 0 \), and \( \pi^0_{\delta} = \text{id}_M \). Let \( C^p(\Gamma, \mathcal{E}) = \bigoplus_{s+r=p} C^p(\Gamma, E_r) \) and \( C(\Gamma, \mathcal{E}) = \bigoplus C^p(\Gamma, \mathcal{E}). \) There is a right \( C^\bullet(\Gamma) \)-module structure defined, for \( \omega \in C^p(\Gamma, E_r) \) and \( f \in C^\infty(\Gamma^{(0)}) \), as

\[
(\omega \cdot f)(\gamma_1, \ldots, \gamma_p) = \omega(\gamma_1, \ldots, \gamma_p) f(\gamma_{p+1}, \ldots, \gamma_{p+q}).
\]
Definition 5.1 ([4][16]). A representation up to homotopy is a degree +1 smooth operator $D : C^*(\Gamma, E) \to C^{*+1}(\Gamma, E)$ satisfying the equation $D^2 = 0$ and the Leibniz rule:

$$D(\omega \cdot f) = (D\omega) \cdot f + (-1)^{|\omega|} \omega \cdot (Df)$$

$\forall \omega \in C^*(\Gamma, E)$ and $f \in C^*(\Gamma)$.

We say that $E$ is an homotopy $\Gamma$-module. Here we are interested in 2-term homotopy modules when the graded vector bundle is concentrated only in two degrees, in particular when $E = C[1] \oplus E$, i.e. with $C$ and $E$ concentrated in degree $-1$ and $0$ respectively. In this case we say that $E$ is a 2-term homotopy $\Gamma$-module.

Remark 5.2. A 2-term homotopy $\Gamma$-module $E$ is given by the quadruple $(\rho, R^E, R^C, \Omega)$ where $\rho$ is a bundle map $\rho : C \to E$ over the identity of $M$, $R^E \in \Gamma(s^* E^* \otimes t^* E)$, $R^C \in \Gamma(s^* C^* \otimes t^* C)$, respectively, and $\Omega \in C^2(\Gamma, \text{Hom}(E, C))$ such that

1. $R^E_\gamma \circ \rho = \rho \circ R^C_\gamma$, for each $\gamma \in \Gamma$.
2. $R^E_{\gamma_1} \circ R^E_{\gamma_2} - R^E_{\gamma_1 \gamma_2} + \rho \circ \Omega(\gamma_1, \gamma_2) = 0$, for each $(\gamma_1, \gamma_2) \in \Gamma^{(2)}$.
3. $R^C_{\gamma_1} \circ R^C_{\gamma_2} - R^C_{\gamma_1 \gamma_2} + \Omega(\gamma_1, \gamma_2) \circ \rho = 0$, for each $(\gamma_1, \gamma_2) \in \Gamma^{(2)}$.
4. $\Omega(\gamma_1, \gamma_2, \gamma_3) - \Omega(\gamma_1, \gamma_2 \gamma_3) + R^E_{\gamma_1} \circ \Omega(\gamma_2, \gamma_3) - \Omega(\gamma_1, \gamma_2) \circ R^E_{\gamma_3} = 0$ for each $(\gamma_1, \gamma_2, \gamma_3) \in \Gamma^{(3)}$.

We will also use the notation $\rho : C[1] \to E$ to indicate the 2-term homotopy $\Gamma$-module $E = C[1] \oplus E$.

Definition 5.3. Let $E = C[1] \oplus E$ and $E' = C'[1] \oplus E'$ be 2-term homotopy $\Gamma$-modules. A morphism of homotopy $\Gamma$-modules from $E$ to $E'$ is a $C^*(\Gamma)$-linear chain map

$$\Phi : C^*(\Gamma, E) \to C^*(\Gamma, E').$$

(27)

It is simple to see that $\Phi$ is indeed generated by a pair of maps $(\phi_0, \mu)$, where $\phi_0 : E \to E'$ is a degree 0 bundle map, called the linear term of $\Phi$, and $\mu \in C^2(\Gamma, \text{Hom}(E, C'))$.

Recall that an invertible morphism $\Phi : (C(\Gamma, E), D) \to (C(\Gamma, E), D')$ is a gauge transformation [16] if $\mu \circ \Phi = \mu$ where $\mu : C(\Gamma, E) \to C(\Gamma, E)$ is defined as $\mu = \text{id}$ on $\Gamma(E_r) \subset C^{-1}(\Gamma, E)$ and zero elsewhere.

For every surjective submersion $\varphi : X \to M$ and every 2-term homotopy $\Gamma$-module $E = C[1] \oplus E$, we define the pull back homotopy $\Gamma[X]$-module $E[X] = (\varphi^* C)[1] \oplus (\varphi^* E)$ by simply pulling back all the data in Remark 5.2 by the Morita morphism $\phi_\varphi : \Gamma[X] \to \Gamma$.

Let $\Phi$ and $\Psi$ be morphisms of 2-term homotopy $\Gamma$-modules from $E_1$ to $E_2$. We say that $\Phi$ is homotopic to $\Psi$ if they are homotopic as chain maps $C^*(\Gamma, E_1) \to C^*(\Gamma, E_2)$, i.e. there exists a $C^*(\Gamma)$-module morphism $H : C^*(\Gamma, E_2) \to C^{*-1}(\Gamma, E_1)$ such that

$$\Phi - \Psi = D_2 \circ H + H \circ D_1.$$  

Definition 5.4. We say that two homotopy $\Gamma$-modules are homotopy equivalent if there exist morphisms $\Phi : C(\Gamma, E_1) \to C(\Gamma, E_2)$, $\Psi : C(\Gamma, E_2) \to C(\Gamma, E_1)$ such that $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are homotopic to the identity.
5.2 From VB groupoids to 2-term homotopy \(\Gamma\)-modules

Let us introduce the VB cohomology, following [10]. Let \(V \subseteq E\) be a VB groupoid over \(\Gamma \subseteq M\). For \(k > 0\), let \(C^k_{VB}(V)\) be the space of those sections \(\sigma \in \Gamma((\pi^k)^*V)\), where \(\pi^k : \Gamma^{(k)} \to \Gamma\) is defined as \(\pi^k(\gamma_1, \ldots, \gamma_k) = \gamma_1\), such that

\[
s_v(\sigma(\gamma_1, \ldots, \gamma_k)) = s_v(\sigma(1_{\pi(\gamma_1)}, \gamma_2, \ldots, \gamma_k))
\]

for all \((\gamma_1, \ldots, \gamma_k) \in \Gamma^{(k)}\). Moreover, let \(C^0_{VB}(V) = \Gamma(C)\). The graded vector space \(C^*_B(V)\) has the structure of a right \(C^*(\Gamma)\)-module defined as

\[
(\sigma \ast f)(\gamma_1, \ldots, \gamma_{k+n}) = \sigma(\gamma_1, \ldots, \gamma_k)f(\gamma_{k+1}, \ldots, \gamma_{k+n}),
\]

where \(\sigma \in C^k_{VB}(V), f \in C^\infty(\Gamma^n)\) and \((\gamma_1, \ldots, \gamma_{k+n}) \in \Gamma^{(k+n)}\). We denote by \((C^*(\Gamma^v), \delta)\) the Lie groupoid cohomology cochain complex of the dual VB-groupoid \(V^v \subseteq C^\infty\) as in [17].

**Proposition 5.5.** Let \(i : C^k_{VB}(V) \to C^k(V^v)\) be the map defined, for \(k > 0\), as

\[
i(\sigma)(\eta_1, \ldots, \eta_k) = \langle \eta_1, \sigma(\gamma_1, \ldots, \gamma_k) \rangle
\]

\(\forall (\eta_1, \ldots, \eta_k) \in (V^v)^{(k)}\) and \(\eta_i \in V_i^v\), and for \(k = 0\) as \(i(\sigma)(\alpha) = \langle \alpha, \sigma(m) \rangle \forall \alpha \in C^0_{V^v}\). Then \((C^*(\Gamma^v), \delta)\) is a subcomplex of \((C^*(\Gamma^v), \delta)\), which is also a \(C^*(\Gamma)\)-submodule.

Moreover, the coboundary differential applied to \(\sigma \in C^0_{VB}(V)\) reads

\[
\delta(\gamma_0, \ldots, \gamma_k) = -\sigma(0, \ldots, \gamma_k) \cdot \sigma(\gamma_1, \ldots, \gamma_k)^{-1} + \sum_{i=2}^{k} (-1)^{i}\sigma(0, \ldots, \gamma_{i-1}, \gamma_i, \ldots, \gamma_k) + (-1)^{k+1} \sigma(0, \ldots, \gamma_{k-1}).
\]

When \(k = 0\) for each \(\sigma \in C^0_{VB}(V) = \Gamma(C)\) and \(\gamma \in \Gamma\),

\[
\delta(\sigma)(\gamma) = -0_{\gamma} \cdot \sigma(s(\gamma))^{-1} - \sigma(t(\gamma)) \cdot 0_{\gamma}.
\]

**Proof.** The first statement is the content of Proposition 5.5 of [10]. Formula (29) follows from a direct computation. \(\square\)

**Lemma 5.6.**

i) Let \(\Phi : V_1 \to V_2\) be a VB groupoid morphism over \(id : \Gamma \to \Gamma\). Then \(\hat{\Phi} : C^*_B(V_1) \to C^*_B(V_2)\) defined as \(\hat{\Phi}(\sigma) = \Phi \circ \sigma\) for \(\sigma \in C^*_B(V_1)\), is a chain map, which is also a right \(C^*(\Gamma)\)-module map.

ii) If \(\Phi\) and \(\Psi : V_1 \to V_2\) are two homotopic VB groupoid morphisms with homotopy \(h : E_1 \to C_2\), then the chain maps \(\hat{\Phi}\) and \(\hat{\Psi}\) are homotopic with chain homotopy being the \(C^*(\Gamma)\)-linear morphism \(\hat{h} : C^{k+1}_{VB}(V_1) \to C^k_{VB}(V_2)\) defined as

\[
\hat{h}(\sigma)(\gamma_1, \ldots, \gamma_k) = -h(s_{V_1}(\sigma(1_{\pi(\gamma_1)}, \gamma_1, \ldots, \gamma_k))) \cdot 0_{\gamma_1}.
\]

**Proof.** The statement i) is trivial. In order to prove ii), it is sufficient to check it on \(C^0_{VB}(V_1)\) and \(C^0_{VB}(V_2)\). For \(\sigma \in C^0_{VB}(V_1)\), we compute that

\[
(\delta \hat{h} + \hat{h}\delta)(\sigma)(m) = \hat{h}(\delta\sigma)(m) = -h(s_{V_1}(\delta\sigma(1_m))) = h(s_{V_1}(\sigma(m)^{-1}))
\]

\[
= h(t_{V_1}(\sigma(m))) = J_h(\sigma)(m),
\]
where in the third equality of the first line we used (30) and in the second line we used (23) and the fact that $0_{1m} = 1_{0m}$.

Let $\sigma \in C^p_B(V)$; we compute

$$(\delta \hat{h} + \hat{h}\delta)(\sigma)(\gamma) = \delta(\hat{h}(\sigma))(\gamma) + \hat{h}(\delta\sigma)(\gamma) = -0_\gamma \cdot \hat{h}(\sigma)(s(\gamma))^{-1} - \hat{h}(\sigma)(t(\gamma)) \cdot 0_\gamma - \hat{h}(s_{V_1}(\delta\sigma(1_{t(\gamma)}),\gamma)) \cdot 0_\gamma = 0_\gamma \cdot h(s_{V_1}(\sigma(1_{s(\gamma)})))^{-1} + h(s_{V_1}(\sigma(1_{t(\gamma)}))) \cdot 0_\gamma - h(s_{V_1}(\sigma(1_{t(\gamma)})) - \sigma(\gamma)) \cdot 0_\gamma = 0_\gamma \cdot h(s_{V_1}(\sigma(\gamma)))^{-1} + h(t_{V_1}(\sigma(\gamma))) \cdot 0_\gamma = J_h(\sigma)(\gamma),$$

where in the last line we used the defining property (28) of VB cochains.

Let $V = t^*C \times s^*E$ be a split VB-groupoid, and let $\sigma = \sigma_C \oplus \sigma_E \in C^p_B(t^*C \times s^*E)$ be a VB cochain; condition 28 implies that $\sigma_E(\gamma_1,\ldots,\gamma_p)$ does not depend on $\gamma_1$, i.e. $\sigma_E \in \Gamma((\pi_0^{p-1})^*E)$, so that the map $\sigma \rightarrow (\sigma_C, \sigma_E)$ identifies $C^p_B(t^*C \times s^*E)$ with $C^{p-1}_E(\Gamma, C[1] \oplus E)$. This shows that if $V = t^*C \times s^*E$ is a split VB groupoid, then $t_V : C[1] \rightarrow E$ is a two term homotopy $\Gamma$-module.

The following result is proven in [16].

**Proposition 5.7.**

1. There is a one-to-one correspondence between VB-groupoids over $\Gamma$ equipped with right-decompositions and 2-term homotopy $\Gamma$-modules.

2. Two different choices of right-decompositions of a VB groupoid correspond to gauge equivalent homotopy $\Gamma$-modules.

**Lemma 5.8.** Let $V_1$ and $V_2$ be homotopy equivalent VB groupoids over $\Gamma$. For any choice of right decompositions of $V_1$ and $V_2$, the induced homotopy $\Gamma$-modules as in Proposition 5.7 are homotopy equivalent.

**Proof.** Let $J : V_1 \rightarrow V_2$ be a VB groupoid morphism, and let $\pi_i : V_i \rightarrow t^*C_i \times s^*C_i$ be two right decompositions. It is easy to observe that if $J$ is a VB homotopy over $h : E_2 \rightarrow C_1$, then the induced VB morphism $J_{\pi_1 \pi_2} = \pi_2 \circ J \circ \pi_1^{-1}$ is a VB homotopy between the split VB groupoids over the same $h$. The result then follows from Lemma 5.6.

Let us consider now the case of the tangent and cotangent groupoid.

**Lemma 5.9.** The following are equivalent:

1. right-decompositions for the tangent VB-groupoid $TT\Gamma$;
2. right-decompositions for the cotangent VB-groupoid $T^*\Gamma$;
3. Ehresmann connections on $s : \Gamma \rightarrow M$ compatible with the unit map $\epsilon : M \rightarrow \Gamma$ in the sense that the horizontal lifting at any point $\epsilon(m)$ coincides with $\epsilon_*T_mM$.

Any Ehresmann connection on $s : \Gamma \rightarrow M$ as in Lemma 5.9 is called compatible Ehresmann connection. As an immediate consequence, we have the following

**Corollary 5.10** ([16]). Let $\Gamma \cong M$ be a Lie groupoid, and let us choose a compatible Ehresmann connection on $s : \Gamma \rightarrow M$.

1. The tangent VB-groupoid (20) induces a homotopy $\Gamma$-module $\rho : A[1] \rightarrow TM$;
2. The cotangent VB-groupoid \[\mathbb{1}\] induces a homotopy \(\Gamma\)-module \(\rho^\vee : (T^\vee M)[1] \to A^\vee\), where \(\rho\) is the anchor map of the Lie algebroid \(A\).

5.3 2-term complexes over a differentiable stack

We interpretate here the results obtained in the previous subsections on Morita equivalence of VB groupoids in terms of homotopy \(\Gamma\)-modules.

**Definition 5.11.** A homotopy \(\Gamma_1\)-module \(E_1\) and a homotopy \(\Gamma_2\)-module \(E_2\) are said to be Morita equivalent if there exist

a) a \(\Gamma_1\) − \(\Gamma_2\) bitorsor \(M_1 \xleftarrow{\varphi_1} X \xrightarrow{\varphi_2} M_2\);

b) an homotopy equivalence between the pull-back representations up to homotopy \(\varphi_1^*E_1\) and \(\varphi_2^*E_2\) along \(\varphi_1\) and \(\varphi_2\) respectively.

In a diagram:

\[
\begin{array}{ccc}
\varphi_1^*E_1 & \xleftarrow{\sim} & \varphi_2^*E_2 \\
\downarrow & & \downarrow \\
E_1 & \xrightarrow{\sim} & E_2 \\
\Gamma_1[X] & \xrightarrow{\sim} & \Gamma_2[X] \\
\downarrow & & \downarrow \\
\Gamma_1 & \xrightarrow{\sim} & \Gamma_2 \\
\downarrow & & \downarrow \\
M_1 & \xrightarrow{\sim} & M_2
\end{array}
\]

(32)

**Proposition 5.12.** Let \(V_i, i = 1, 2\), be VB-Morita equivalent VB groupoids over \(\Gamma_i\) with \(C_i\) and \(E_i\) being the cores and the units, respectively. For any fixed choice of right decompositions the induced homotopy \(\Gamma_1\)-module \(C_1[1] \to E_1\) and homotopy \(\Gamma_2\)-module \(C_2[1] \to E_2\) are Morita equivalent.

*Proof.* This is a straightforward consequence of Lemma 5.6 and Theorem 4.22.

**Corollary 5.13.** Let \(\Gamma_1 \Rightarrow M_1\) and \(\Gamma_2 \Rightarrow M_2\) be Morita equivalent Lie groupoids. Choose right decompositions on \(T\Gamma_1\) and \(T\Gamma_2\); Then

i) the induced homotopy \(\Gamma_1\)-module \(A_1[1] \to TM_1\) and homotopy \(\Gamma_2\)-module \(A_2[1] \to TM_2\) are Morita equivalent;

ii) the induced homotopy \(\Gamma_1\)-module \((T^\vee M_1)[1] \to A_1^\vee\) and homotopy \(\Gamma_2\)-module \((T^\vee M_2)[1] \to A_2^\vee\) are Morita equivalent.

**Definition 5.14.** Let \(X\) be a differentiable stack. A 2-term complex over \(X\) is a class of Morita equivalent 2-term homotopy \(\Gamma\)-modules \(E\), where \(\Gamma \Rightarrow M\) is any representative of \(X\).

We denote the Morita equivalent class of the homotopy \(\Gamma\)-module \(E\) as \([E]\), and say that \(E\) represents \([E]\) on \(\Gamma \Rightarrow M\).
**Definition 5.15.** Let $E_1$ and $E'_1$ be homotopy $\Gamma_1$-modules. Let $E_2$ and $E'_2$ be homotopy $\Gamma_2$-modules, which are Morita equivalent to $E_1$ and $E'_1$, respectively. Let $\Phi : E_1 \to E'_1$ and $\Psi : E_2 \to E'_2$ be morphisms of homotopy $\Gamma_1$-modules and $\Gamma_2$-modules over $\Gamma_1$ and $\Gamma_2$, respectively. We say that $\Phi$ and $\Psi$ are homotopic if the pull back maps

$$
\Phi : \varphi^*_1 E_1 \to \varphi^*_1 E'_1, \quad \Psi : \varphi^*_2 E_2 \to \varphi^*_2 E'_2
$$

are homotopic morphisms of homotopy $\Gamma$-modules, where $M_1 \xleftarrow{\varphi_1} X \xrightarrow{\varphi_2} M_2$ is a $\Gamma_1 - \Gamma_2$ bitorsor and the homotopies are as in Definition 5.11.

The above definition means that in the following diagram

the sub-diagram made of the dashed lines is commutative up to homotopy. Together with Definition 5.14, this definition allows to make sense of morphisms of 2-term complexes over differentiable stacks.

**Definition 5.16.** Let $X$ be a differentiable stack. A morphism between 2-term complexes over $X$ is a homotopy equivalent class, in the sense of Definition 5.15, of morphisms between Morita equivalent 2-term homotopy groupoid modules.

It is clear that the 2-term complexes over a given differentiable stack $X$ is a category. Taking into account the Gracia-Saz-Mehta dictionary, as detailed in Section 5.2, the following is immediate.

**Proposition 5.17.** Homotopic VB-groupoid morphisms in the sense of Definition 4.26 give rise to homotopic morphisms between their Morita equivalent 2-term homotopy groupoid modules in the sense of Definition 5.15.

Therefore, we have the following

**Corollary 5.18.** An homotopy equivalent class of VB-groupoid morphisms in the sense of Definition 5.15 induces a morphism of the corresponding 2-term complexes over the stack.

### 6 The rank of a $(+1)$-shifted Poisson stack

#### 6.1 The tangent complex and cotangent complex

Now we are ready to introduce the tangent complex and the cotangent complex of a differentiable stack $X$. 
Definition 6.1. Let $\mathcal{X}$ be a differentiable stack.

1. By the tangent complex of $\mathcal{X}$, denoted by $T\mathcal{X}$, we mean the 2-term complex over $\mathcal{X}$ defined by the Morita equivalent class of the homotopy $\Gamma$-module $\rho : A[1] \to TM$.

2. By the cotangent complex of $\mathcal{X}$, denoted by $L\mathcal{X}$, we mean the 2-term complex over $\mathcal{X}$ defined by the Morita equivalent class of the homotopy $\Gamma$-module $\rho^\vee : T^\vee M \to A^\vee[-1]$, which is the dual to $A[1] \to TM$.

Here $\Gamma \Rightarrow M$ is any Lie groupoid representing $\mathcal{X}$, and $A$ is the Lie algebroid of $\Gamma$ with the anchor map $\rho$.

Note that $(A[1])^\vee \cong A^\vee[-1]$. Corollary 5.13 implies that the definition above is indeed justified. The representative $\rho : A[1] \to TM$ of $T\mathcal{X}$ is denoted $T\mathcal{X}|_M$, and the representative $\rho^\vee : T^\vee M \to A^\vee[-1]$ of $L\mathcal{X}$ by $L\mathcal{X}|_M$.

The following result is a consequence of Theorem 4.27.

Theorem 6.2. A $(+1)$-shifted Poisson structure on a differentiable stack $\mathcal{X}$ defines a morphism of 2-term complexes over $\mathcal{X}$ from the cotangent complex shifted by $(+1)$ to the tangent complex:

$$\Pi^\#: L\mathcal{X}[1] \to T\mathcal{X}$$

Choosing a compatible Ehresmann connection on $s : \Gamma \to M$ as in Lemma 5.9, one can compute explicitly the morphism of homotopy $\Gamma$-modules $\Pi^\#: L\mathcal{X}[1]_M \to T\mathcal{X}|_M$. In particular, we have the following

Proposition 6.3. Under the same hypothesis as in Theorem 6.2, for any presentation $\Gamma \Rightarrow M$ of $\mathcal{X}$, the linear term of the morphism of homotopy $\Gamma$-modules $\Pi^\#: L\mathcal{X}[1]_M \to T\mathcal{X}|_M$ is given as follows:

$$\begin{array}{c}
(T^\vee M)[1] \xrightarrow{-\rho^\vee} A[1] \\
\downarrow \rho^\vee \\
A^\vee \xrightarrow{\rho^*} TM
\end{array}$$

where $\rho^* : A^\vee \to TM$ is the map induced from $\Pi^\#: T^\vee \Gamma \to T\Gamma$ by restricting to $A^\vee \subset T^\vee \Gamma|_M$. [22]

6.2 Rank of a $(+1)$-shifted Poisson stack

The main purpose of this section is to introduce the rank of a $+1$-shifted Poisson structure on a differentiable stack.

Recall that the rank of an ordinary Poisson structure is defined at a point of the base manifold. For a $(+1)$-shifted Poisson structure on a differentiable stack, we shall define its rank at each element in the coarse moduli space of the differentiable stack. The latter can be defined as the quotient space $M/\Gamma$, where $\Gamma \Rightarrow M$ is any Lie groupoid representing the differentiable stack $\mathcal{X}$, which is known to be invariant under the Morita equivalence of $\Gamma$.

Our strategy is, first of all, to define the rank of a quasi-Poisson groupoid $(\Gamma \Rightarrow M, \Pi, \Lambda)$ at any given point $m \in M$. Then, we show that this rank is constant along Lie groupoid orbits. Furthermore, it is also invariant under twists of the quasi-Poisson structures and
indeed is invariant under Morita equivalence. In this way, we are led to a well defined map \( |X| \to \mathbb{Z} \), the rank of the \((+1)\)-shifted Poisson stack.

**Definition 6.4.** The rank of a quasi-Poisson groupoid \((\Gamma \rightrightarrows M, \Pi, \Lambda)\) at \( m \in M \) is defined to be

\[
dim(\rho|_m(A_m) + \rho^*_|_m(A^\vee_m)) - \text{rk}(A).
\]

**Remark 6.5.** Recall that the dimension of a differentiable stack \( X \) is \( \dim X = \dim(M) - \text{rk}(A) \). Hence the rank of the quasi-Poisson groupoid \((\Gamma \rightrightarrows M, \Pi, \Lambda)\) at \( m \in M \) can also be written as

\[
dim X - \dim(\ker(\rho^\vee|_m \cap \ker(\rho^*_|_m))
\]

For a Poisson groupoid \([25]\), the rank is full, i.e., equal to \( \dim X \) at a point \( m \in M \) if and only if the orbits of the Lie algebroid \( A \) and the dual Lie algebroid \( A^\vee \) intersect transversally at \( m \in M \).

**Proposition 6.6.** Let \((\Gamma \rightrightarrows M, \Pi, \Lambda)\) be a quasi-Poisson groupoid. The rank of the quasi-Poisson structure \((\Pi, \Lambda)\) is constant on any orbit of the groupoid.

**Proof.** Let us use Remark 6.5 and show that \( \dim(\ker(\rho^\vee|_m \cap \ker(\rho^*_|_m)) \) is constant along the Lie groupoid orbits.

We start with a few linear algebra facts. Let us call **butterfly** a commutative diagram \( C \) of the form below, where \( NW, NE, SW, SE, C \) are vector spaces and where both diagonal lines are short exact sequences.

\[
\begin{array}{ccc}
NW & \xrightarrow{i_N} & NE \\
\rho_W & & \rho_E \\
SW & \xleftarrow{i_S} & SE \\
\end{array}
\]

By diagram chasing, each butterfly induces vector spaces isomorphisms:

\[ C_{Ker} : \ker(\rho_W) \xrightarrow{\sim} \ker(\rho_E) \]

Let us be more explicit: \( a_N \in \ker(\rho_W) \) and \( a_S \in \ker(\rho_E) \) correspond one to the other through the isomorphism \( C_{Ker} \) if and only if \( i_N(a_N) = i_S(a_S) \).

By a **morphism** \( \pi^\otimes \) from a butterfly \( C \) to a butterfly \( C' \), we mean a family of five linear maps, as represented by dotted lines in diagram (35) below, making it commutative:
By diagram chasing, it is routine to check that $\pi_{SW}$ (resp. $\pi_{SE}$) maps $\text{Ker}(\rho_W)$ to $\text{Ker}(\rho'_W)$ (resp. $\text{Ker}(\rho_E)$ to $\text{Ker}(\rho'_E)$). Also, the commutativity of the diagrams above implies the commutativity of the following diagrams (where vertical lines are vector spaces isomorphisms):

\[
\begin{array}{ccc}
\text{Ker}(\rho_W) & \xrightarrow{\pi_{SW}} & \text{Ker}(\rho_W) \\
\downarrow & & \downarrow \\
\text{Ker}(\rho_E) & \xrightarrow{\pi_{SE}} & \text{Ker}(\rho'_E)
\end{array}
\]

In particular, the butterfly morphism $\pi^{\omega}$ induces a vector space isomorphism:

\[
\text{Ker}(\rho_W) \cap \text{Ker}(\pi_{SW}) \cong \text{Ker}(\rho_E) \cap \text{Ker}(\pi_{SE}).
\] (36)

To verify our claim we are going to apply this general fact to the following situation. For any $\gamma \in \Gamma$ with source $m$ and target $n$, the following commutative diagrams are butterflies:

\[
\begin{array}{ccc}
A^\vee_m & \xrightarrow{L^\vee} & A^\vee_n \\
\downarrow & & \downarrow \\
T^\vee_m M & \xrightarrow{T^\vee_{\gamma}} & T^\vee_n M
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
T^\vee_m M & \xrightarrow{\rho^{\vee}|_m} & T^\vee_{\gamma} M \\
\downarrow & & \downarrow \\
A^\vee_m & \xrightarrow{L} & A^\vee_n
\end{array}
\]

Now, for any multiplicative bivector field $\Pi$ on a Lie groupoid $\Gamma$, there is a natural butterfly morphism from the first butterfly above to the second one given by $\Pi^{\#}: T^\vee_{\gamma} \Gamma \to T^\vee_{\gamma} \Gamma$, while the four remaining arrows are (using notations of Equation (35)):

\[
\pi_{NW} = \rho^{\vee}|_m, \pi_{SW} = \rho^{\vee}|_m, \pi_{NE} = \rho^{\vee}|_n, \pi_{SE} = \rho^{\vee}|_n.
\]

Constance of rank along orbits then follows from Equation (36). \hfill \Box

For any $T \in \Gamma(\wedge^2 A)$, denote by $\rho^T_\gamma : A^\vee \to TM$ the map corresponding to the quasi-Poisson structure $(\Pi_T, \Lambda_T)$. More explicitly, we have:

\[
\rho^T_\gamma = \rho + \rho : T^{\#}.
\] (37)

**Lemma 6.7.** Let $(\Gamma \rightrightarrows M, \Pi, \Lambda)$ be a quasi-Poisson groupoid. Then for any $T \in \Gamma(\wedge^2 A)$,

\[
\dim(\ker \rho^{\vee}|_m \cap \ker(\rho^T_\gamma)^{\vee}|_m) = \dim(\ker \rho^{\vee}|_m \cap \ker \rho^{\vee}|_n)
\]

*Proof.* From Equation (37), it follows that $T^{\#} : A^\vee \to A$ is a homotopy between the chain maps $(-\rho^T_\gamma, \rho_\gamma)$ and $(-\rho^T_\gamma^{\vee}, \rho^{\vee}_T)$:

\[
\begin{array}{ccc}
T^\vee M[1] & \xrightarrow{-(\rho^T_\gamma)^{\vee}} & A[1] \\
\downarrow & & \downarrow \\
A^\vee & \xrightarrow{T^\#} & TM.
\end{array}
\]

The result then follows by elementary linear algebra. \hfill \Box
As an immediate consequence of Definition 6.4, Remark 6.5 and Lemma 6.7, we have

**Corollary 6.8.** The ranks at a given orbit of any two quasi-Poisson structures on a Lie groupoid $\Gamma \rightrightarrows M$, which are equivalent up to a twist, are equal.

Finally, we have the following

**Lemma 6.9.** Let $(\Gamma_1 \rightrightarrows M_1, \Pi_1, \Lambda_1)$ and $(\Gamma_2 \rightrightarrows M_2, \Pi_2, \Lambda_2)$ be quasi-Poisson groupoids. Assume that $\Gamma_1 \rightrightarrows M_1 \xrightarrow{\phi} \Gamma_2 \rightrightarrows M_2$ is a Morita morphism of quasi-Poisson groupoids from $(\Gamma_1 \rightrightarrows M_1, \Pi_1, \Lambda_1)$ to $(\Gamma_2 \rightrightarrows M_2, \Pi_2, \Lambda_2)$. Then, for any $m_1 \in M_1$, the rank of the quasi-Poisson structure $(\Gamma_1 \rightrightarrows M_1, \Pi_1, \Lambda_1)$ at $m_1$ is equal to the rank of the quasi-Poisson structure $(\Gamma_2 \rightrightarrows M_2, \Pi_2, \Lambda_2)$ at $\phi(m_1) \in M_2$.

Now, we are ready to introduce the rank of a $(+1)$-shifted Poisson structure on a differentiable stack $X$.

**Definition 6.10.** For a $(+1)$-shifted Poisson structure $P$ on a differentiable stack $X$, let $(\Gamma \rightrightarrows M, \Pi, \Lambda)$ be any quasi-Poisson groupoid representing it. Define the rank of $P$ as a map $|X| \to \mathbb{Z}$:

$$\text{rank} P = \dim X - \dim (\ker \rho^\vee |_m \cap \ker \rho^*_\vee |_m),$$

where $m$ is any point in the groupoid orbit representing the element in the coarse moduli space $|X|$ of the stack $X$.

According to Lemma 6.9, $\text{rank} P$ is indeed well defined. Let us now describe non-degenerate Poisson structures on a differentiable stack.

**Definition 6.11.** A $(+1)$-shifted Poisson structure $P$ on a differentiable stack $X$ is non-degenerate if and only if the linear term (34) of the morphism of homotopy $\Gamma$-modules $\Pi^\# : L_X[1] |_M \to T_X |_M$ is a quasi-isomorphism of 2-term complexes of vector bundles. That is, for any $m \in M$, the morphism defined by the horizontal arrows below

$$\begin{array}{ccc}
T^\vee_m M[1] & \xrightarrow{-\rho^\vee} & A_m[1] \\
\downarrow{\phi^\vee} & & \downarrow{\rho} \\
A^\vee_m & \xrightarrow{-\rho^*_\vee} & T_m M
\end{array}$$

is a quasi-isomorphism of the 2-term complexes.

Not all differentiable stacks admit $(+1)$-shifted non-degenerate Poisson structures.

**Lemma 6.12.** If $X$ is a $(+1)$-shifted non-degenerate Poisson stack, then $\dim X = 0$.

**Proof.** Since the 2-term complexes of vector bundles associated to $L_X[1] |_M$ and $T_X |_M$ are $T^\vee M[1] \xrightarrow{-\rho^\vee} A^\vee$ and $A[1] \xrightarrow{\rho} T M$, respectively, their Euler characteristics are $-\dim X$ and $\dim X$, respectively. Since quasi-isomorphic 2-term complexes have the same Euler characteristic, we have $\dim X = -\dim X$. Therefore, it follows that $\dim X = 0$.

The following proposition gives an alternative description of non-degenerate Poisson stacks.

---

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Proposition 6.13. A (+1)-shifted Poisson structure $\mathcal{P}$ on a differentiable stack $\mathcal{X}$ is non-degenerate if and only if \(\text{rank} \mathcal{P} = \text{dim} \mathcal{X} = 0\) uniformly on the coarse moduli space of the stack.

Proof. Assume that $\mathcal{P}$ is non-degenerate. By Lemma 6.12 we know that \(\text{dim} \mathcal{X} = 0\). From assumption, it follows that \(\text{dim}(\ker \rho^\vee_m \cap \ker \rho^\vee_m) = 0\). Therefore, \(\text{rank} \mathcal{P} = 0\) according to Remark 6.5.

Conversely, assume that \(\text{rank} \mathcal{P} = \text{dim} \mathcal{X} = 0\). It thus follows that all vector spaces in Diagram (39) have the same dimension. A simple linear algebra argument implies that the morphism defined by the horizontal arrows in (39) must be a quasi-isomorphism of the 2-term complexes. \(\blacksquare\)

6.3 Examples

Example 6.14. Let \((G, \Pi, \Lambda)\) be a quasi-Poisson group of dimension $n$ in the sense of Kosmann-Schwarzbach [20]. Then it defines a (+1)-shifted Poisson structure on \(\mathcal{X}/G\) of rank $-n$. Indeed (+1)-shifted Poisson structures on \(\mathcal{X}/G\) correspond exactly to equivalent classes of quasi-Poisson group structures on $G$, where the equivalence relation is given by “Drinfeld twists” [20]. In particular, they cannot be non-degenerate.

When $G$ is a Lie group whose Lie algebra $\mathfrak{g}$ is equipped with a symmetric $\mathfrak{g}$-invariant element $t \in S^2(\mathfrak{g})^G$, then \((G, \Pi, \Lambda)\), where $\Pi = 0$ and $\Lambda = -\frac{1}{4}[t_{12}, t_{23}] \in (\wedge^3 \mathfrak{g})^G$, defines a quasi-Poisson group. And therefore it induces a (+1)-shifted Poisson structure on \(\mathcal{X}/G\). It is well known that any quasi-triangular Poisson Lie group is twist-equivalent to a quasi-Poisson group on $G$ of the above form, and therefore its corresponding (+1)-shifted Poisson stack on \(\mathcal{X}/G\) is isomorphic to the (+1)-shifted Poisson stack on \(\mathcal{X}/G\) as described above. This viewpoint can certainly be traced back to Drinfeld [13].

If $G$ is a connected and simply connected semi-simple Lie group, it is known [13, 20] that any quasi-Poisson group structure on $G$ is twist-equivalent to one as above, where the twist $t \in S^2(\mathfrak{g})^G \cong S^2(\mathfrak{g}^{\vee})^G$ is a multiple of the Killing form. Therefore there is one to one correspondence between (+1)-shifted Poisson structures on \(\mathcal{X}/G\) and elements in \((\wedge^3 \mathfrak{g})^G\).

Another type of quasi-Poisson groupoid arise as integration of the so called Manin pairs. Let \((\mathfrak{d}, \mathfrak{g})\) be a Manin pair [12], that is, \(\mathfrak{d}\) is an even dimensional Lie algebra of signature \((n, n)\) with an invariant, non-degenerate symmetric bilinear form, and \(\mathfrak{g}\) is a maximal isotropic Lie subalgebra of \(\mathfrak{d}\). Let $D$ be the connected and simply connected Lie group with Lie algebra \(\mathfrak{d}\), and $G \subset D$ a closed Lie subgroup with Lie algebra \(\mathfrak{g}\). We call \((D, G)\) the group pair corresponding to the Manin pair. Denote by $S$, the homogeneous space $S = D/G$. Then the action of the Lie group $D$ on itself by left multiplication induces an action of $D$ on $S = D/G$, and in particular a $G$-action on $S$, which is called the dressing action.

Theorem 6.15. Let \((\mathfrak{d}, \mathfrak{g})\) be a Manin pair, and \((D, G)\) its corresponding group pair. Then the quotient stack \([S/G]\), where $S = D/G$ and $G$ acts on $S$ by the dressing action, is naturally a non-degenerate (+1)-shifted Poisson stack.

Theorem 6.15 follows from Proposition 6.16 and Proposition 6.17 below.
Let Proposition 6.16. Λ quasi-Poisson groupoid. Here to the 3-vector φ therefore, it induces a quasi-Lie bialgebra (g, h) complement G group with multiplicative bivector field denoted by Π in terms of the transformation groupoid G. Proposition 6.17. Π and g the dimension of d| is the bivector field on G × S given by ΠS = −∑n i=1 (ei)s ⊗ (ei)s, i.e.,

\[ \Pi_S(df, dg) = -\sum_{i=1}^{n} (e^i)_s(f(e^i)_s(g), \forall f, g \in C^\infty(S). \] (40)

Here \( \{e_i\} \) is a basis of g and \( \{e^i\} \) the dual basis of \( g^\vee \cong h \).

We have the following:

**Proposition 6.16.** Let \((\mathfrak{d}, g)\) be a Manin pair, and \( h \) an isotropic complement of \( g \) in \( \mathfrak{d} \). Let \((D, G)\) be its corresponding Lie group pair. Then \((G \ltimes S \rightrightarrows S, \Pi, \Lambda)\) is a non-degenerate quasi-Poisson groupoid. Here \( \Lambda \in \Gamma(S; \wedge^3(g) \times S) \) is the constant section corresponding to the 3-vector \( \phi \in \wedge^3 g \) defined by

\[ \phi(\xi, \eta, \zeta) = \langle [\xi, \eta], \zeta \rangle \quad \forall \xi, \eta, \zeta \in g^\vee, \]

and \( \Pi \) is the bivector field on \( G \times S \) defined by

\[ \Pi\big((\theta_\eta, \theta_\zeta), (\theta'_\eta, \theta'_\zeta)\big) = \Pi_C(\theta_\eta, \theta'_\eta) - \Pi_S(\theta_\zeta, \theta'_\zeta) + \langle \theta'_\zeta, (L^*_\eta \theta_\eta)_s \rangle - \langle \theta_\zeta, (L^*_\eta \theta'_\eta)_s \rangle, \] (41)

for any point \((g, s) \in G \times S\), and \((\theta_\eta, \theta_\zeta), (\theta'_\eta, \theta'_\zeta) \in T^*_g G \times T^*_s S \cong T^*_{(g, s)} (G \times S)\), where \((L^*_\eta \theta_\eta)_s\) denotes the vector field on S corresponding to the infinitesimal dressing action of \( L^*_\eta \theta_\eta \in g^\ast \cong h \subset \mathfrak{d} \), similarly for \((L^*_\eta \theta'_\eta)_s\).

**Proof.** It was proved in Theorem 4.22 [18] that this is a quasi-Poisson groupoid. By a direct verification, one can show that this quasi-Poisson groupoid is indeed non-degenerate in the sense of Definition 6.11.

According to Proposition 4.15 [18], a direct verification leads to the following

**Proposition 6.17.** Under the same hypothesis as in Proposition 6.16, the quasi-Poisson structures on the transformation groupoid \( G \ltimes S \rightrightarrows S \) corresponding to different isotropic complements \( h \) of \( g \) in \( \mathfrak{d} \) are equivalent up to a twist.

Let \( g \) be a Lie algebra endowed with a non-degenerate symmetric bilinear form \( K \). On the direct sum \( \mathfrak{d} = g \oplus \mathfrak{d} \), one constructs a scalar product of signature \((n, n)\) (with \( n \) being the dimension of \( g \)) \( \langle \cdot | \cdot \rangle \) by

\[ \langle (u_1, u_2), (v_1, v_2) \rangle = K(u_1, v_1) - K(u_2, v_2), \]

for \((u_1, u_2), (v_1, v_2) \in \mathfrak{d} \). Then, \((\mathfrak{d}, \Delta(g), \Delta_-(g))\) is a Manin quasi-triple, where \( \Delta(v) = (v, v) \) and \( \Delta_-(v) = (v, -v), \forall v \in g \) [2]. If \( G \) is the connected and simply connected Lie group with Lie algebra \( g \) then \( D = G \times G \) is the connected and simply connected Lie group with Lie algebra \( \mathfrak{d} \) and \( G \) is identified with the diagonal of \( D = G \times G \). It is simple to see that the map \([g', g]) \mapsto g' \cdot g^{-1} \) allows us to identify the homogeneous space \( D/G \) with \( G \), under which the dressing action of \( G \) becomes the conjugation. Hence the transformation groupoid \( G \ltimes S \rightrightarrows S \) in Proposition 6.16 becomes the transformation groupoid \( G \ltimes G \rightrightarrows G \).
Proposition 6.18. Assume that $\mathfrak{g}$ is a Lie algebra endowed with a non-degenerate symmetric bilinear form $K$ and $G$ is its corresponding connected and simply connected Lie group. Then $(G \ltimes G; \Pi, \Lambda)$, where $G$ acts on $G$ by conjugation, is a non-degenerate quasi-Poisson groupoid, where the multiplicative bivector field $\Pi$ on $G \times G$ is:

$$\Pi|_{(g,s)} = \frac{1}{2} \sum_{i=1}^{n} e_1^2 \wedge e_i^2 - e_i^2 \wedge e_1^2 - (Ad_{g^{-1}} e_i^2)^2 \wedge e_1^1,$$

(42)

and $\Lambda \in \Gamma(G; \wedge^3 \mathfrak{g} \times G)$ is the constant section corresponding to the 3-vector in $(\wedge^3 \mathfrak{g})^G$ identified by the Cartan 3-form $\frac{1}{3} K(\cdot, [\cdot, \cdot]) \in \wedge^3 \mathfrak{g}^*$. Here $\{e_i\}$ is an orthonormal basis of $\mathfrak{g}$ and the superscript refers to the respective $G$-component.

Proof. It was proved in Corollary 4.24 [18] that this is a quasi-Poisson groupoid. By a direct verification, one can show that $\Pi$ is indeed non-degenerate in the sense of Definition 6.11. \qed

In summary, we have the following

Theorem 6.19. Let $G$ be a connected and simply connected Lie group whose Lie algebra $\mathfrak{g}$ is endowed with a non-degenerate symmetric bilinear form. Then the quotient stack $[G/G]$, where $G$ acts on $G$ by conjugation, is naturally a non-degenerate $(+1)$-shifted Poisson stack.

In [8], we will explore the question how to invert a non-degenerate $(+1)$-shifted Poisson stack to a $(+1)$-shifted symplectic stack by “homotopy inverting” non-degenerate quasi-Poisson groupoids. In particular, we prove that by homotopy inverting the non-degenerate quasi-Poisson groupoid in Proposition 6.18, we obtain the AMM quasi-symplectic groupoid $G \ltimes G; \Pi, \Lambda$. Therefore one obtains AMM $(+1)$-shifted symplectic stack $[G/G]$ by inverting the non-degenerate $(+1)$-shifted Poisson stack $[G/G]$ in Theorem 6.19.

A Z-graded Lie 2-algebras

We discuss here the extension to the Z-graded case of the standard notions of Lie 2-algebras and their morphisms. This extension is straightforward, nevertheless since we could not find it in the literature we will give a self contained presentation.

A.1 Definitions

Definition A.1. A Z-graded Lie 2-algebra (or graded Lie algebra crossed-module) $\mathfrak{A} \rightarrow \mathfrak{G}$ is a pair $(\mathfrak{A}, \mathfrak{G})$ of graded Lie algebras, equipped with

1. a degree 0 graded Lie algebra morphism $d : \mathfrak{A} \rightarrow \mathfrak{G},$
2. a graded Lie algebra action of $\mathfrak{G}$ on the graded vector space $\mathfrak{A}$

$$\mathfrak{G} \times \mathfrak{A} \rightarrow \mathfrak{A}$$

$$(\pi, a) \mapsto \pi \cdot a,$$

such that:

a) for all $a \in \mathfrak{A}, \pi \in \mathfrak{G}_1$, the relation $d(\pi \cdot a) = [\pi, da]$ holds and
b) for all $a_1, a_2 \in \mathfrak{A}$, the relation $[a_1, a_2] = (da_1) \cdot a_2$ holds.

In the non-graded case, when $\mathfrak{A}$ and $\mathfrak{G}$ are ordinary Lie algebras, i.e. graded Lie algebras concentrated in degree 0, we recover the usual notion of a crossed module, which is also called a **strict Lie 2-algebra** ($\Pi$). Since the strict case is the only case we are interested in, we omitted the term strict in Definition A.1. For $\pi \in \mathfrak{G}_k, a \in \mathfrak{A}_l$, it will be practical to denote by $a \cdot \pi$ the element $(-1)^{k(l+1)} \pi \cdot a$.

**Remark A.2.** Items a) and b) in Definition A.1 imply that $\mathfrak{G}$ acts on $\mathfrak{A}$ by derivations of graded Lie algebras. Moreover, the fact that $d$ respects the Lie algebra bracket is a consequence of items a) and b) so that it could be omitted from Definition A.1.

To any $\mathbb{Z}$-graded Lie 2-algebra $\mathfrak{A} \xrightarrow{\phi} \mathfrak{G}$, there is an associated differential graded Lie algebra, denoted $\mathcal{V}(\mathfrak{A} \xrightarrow{\phi} \mathfrak{G})$, which is defined as follows:

1. as a graded vector space, $\mathcal{V} = \mathfrak{A}[1] \oplus \mathfrak{G}$, i.e. the component $V_k$ of degree $k \in \mathbb{Z}$ is the direct sum $\mathfrak{A}_{k+1} \oplus \mathfrak{G}_k$;
2. the differential is $d(a \oplus \pi) = 0 \oplus da$ for all $a \in \mathfrak{A}_{k+1} \subset V_k$ and $\pi \in \mathfrak{G}_k \subset V_k$;
3. the graded Lie bracket is given for all $a_1 \oplus \pi_1, a_2 \oplus \pi_2 \in V_l$ by:

\[
[a_1 \oplus \pi_1, a_2 \oplus \pi_2] := (-1)^{k} \pi_1 \cdot a_2 - (-1)^{l} a_1 \cdot \pi_2 \oplus [\pi_1, \pi_2].
\]

For usual crossed modules $\mathfrak{A} \xrightarrow{d} \mathfrak{G}$ (i.e. the non-graded case), the only non vanishing terms are $V_0 = \mathfrak{G}$ and $V_{-1} = \mathfrak{A}$. In this case, a $L_\infty$-morphism from the dgla $\mathcal{V}(\mathfrak{A} \xrightarrow{d} \mathfrak{G})$ to the dgla $\mathcal{V}'(\mathfrak{A}' \xrightarrow{d'} \mathfrak{G'})$ is determined by a pair of linear maps $\mathfrak{A} \rightarrow \mathfrak{A}'$ and $\mathfrak{G} \rightarrow \mathfrak{G}'$, respectively together with a bilinear skew-symmetric map $\wedge \mathfrak{G} \rightarrow \mathfrak{A}'$. For degree reasons, no other Taylor coefficients may exist. We extend this notion to the graded case by requiring that the only non-vanishing Taylor coefficients are of this particular type.

**Definition A.3.** A morphism of $\mathbb{Z}$-graded Lie 2-algebras from $\mathfrak{A} \xrightarrow{d} \mathfrak{G}$ to $\mathfrak{A}' \xrightarrow{d'} \mathfrak{G}'$ is a $L_\infty$-morphism $\Phi$ between their associated dglas $\mathcal{V}(\mathfrak{A} \xrightarrow{d} \mathfrak{G})$ and $\mathcal{V}(\mathfrak{A}' \xrightarrow{d'} \mathfrak{G}')$ whose Taylor coefficients $(\Phi_n)_{n \geq 1}$ satisfy the following properties:

1. the **linear Taylor coefficient** $\Phi_1$ maps $\mathfrak{A}$ to $\mathfrak{A}'$ and $\mathfrak{G}$ to $\mathfrak{G}'$;
2. the **quadratic Taylor coefficient** $\Phi_2$ is zero, except on $\mathfrak{G} \wedge \mathfrak{G}$ where it takes values in $\mathfrak{A}'$;
3. all higher Taylor coefficients $(\Phi_n)_{n \geq 3}$ are equal to zero.

When the quadratic Taylor coefficient $\Phi_2$ is zero, we call it a **strict** morphism of $\mathbb{Z}$-graded Lie 2-algebras. Strict morphisms are then simply pairs of maps from $\mathfrak{A}$ to $\mathfrak{A}'$ and from $\mathfrak{G}$ to $\mathfrak{G}'$ that preserve all the structures defining $\mathbb{Z}$-graded Lie 2-algebras.

**Remark A.4.** Let us spell out Definition A.3. A morphism of $\mathbb{Z}$-graded Lie 2-algebras is given by a pair of linear maps of degree 0, $\Phi_1 : \mathfrak{A} \rightarrow \mathfrak{A}'$ and $\Phi_2 : \mathfrak{G} \rightarrow \mathfrak{G}'$, called the linear terms, together with a graded skew-symmetric bilinear map $\Phi_2 : \mathfrak{G} \wedge \mathfrak{G} \rightarrow \mathfrak{A}'$ of degree +1, called the quadratic term, such that:

(a) $\Phi_1$ is a chain map:

\[
\mathfrak{A} \xrightarrow{d} \mathfrak{G} \xrightarrow{\Phi_1} \mathfrak{A}' \xrightarrow{d'} \mathfrak{G}'.
\]
(b) for all $\pi_1, \pi_2 \in \mathfrak{g}$, the relation $d' \circ \Phi_2(\pi_1, \pi_2) = \Phi_1([\pi_1, \pi_2]) - [\Phi_1(\pi_1), \Phi_1(\pi_2)]$ holds,
(c) for all $\pi \in \mathfrak{g}, a \in \mathfrak{a}$, the relation $\Phi_2(\pi, da) = \Phi_1(\pi \cdot a) - \Phi_1(\pi) \cdot \Phi_1(a)$ holds,
(d) the relation $(-1)^{||\pi_1||\pi_2} (\Phi_2(\pi_1, [\pi_2, \pi_3]) - \Phi_1(\pi_1) \cdot \Phi_2(\pi_2, \pi_3)) + \circ \pi_1 \pi_2 \pi_3 = 0$ holds for all $\pi_1, \pi_2, \pi_3 \in \mathfrak{g}$.

In the non-graded case, these are exactly the relations satisfied by the Taylor coefficients of an $L_\infty$-morphism between dglas concentrated in degrees 0 and $-1$, as can be seen in [30].

**Remark A.5.** Recall that morphisms of $L_\infty$-algebras can be composed; it is routine to check that morphisms of graded Lie 2-algebras are stable under composition. Indeed if $\Phi$ and $\Psi$ are two morphisms of $\mathbb{Z}$-graded Lie 2-algebras then $\Phi \circ \Psi$ is a morphism whose only non-vanishing terms are the linear and quadratic ones that read

$$
(\Phi \circ \Psi)_1 = \Phi_1 \circ \Psi_1 \quad \text{and} \quad (\Phi \circ \Psi)_2 = \Phi_1 \circ \Psi_2 + \Phi_2 \circ (\wedge^2 \Psi_1).
$$

\[43\]

The following definition generalizes to the graded case the notion of homotopy between morphisms of Lie 2-algebras (see for instance Definition 2.9 of [30]). Again, for the non-graded case, such homotopies (called natural transformations) are the only possible ones. In the graded case, we simply impose their form to mimic the non-graded case.

**Definition A.6.** Let $\Phi$ and $\Psi$ be morphisms of $\mathbb{Z}$-graded Lie 2-algebras from $\mathfrak{A} \to \mathfrak{G}$ to $\mathfrak{A'} \to \mathfrak{G'}$. An *homotopy* between $\Phi$ and $\Psi$ is a linear map $h : \mathfrak{G} \to \mathfrak{A'}$ of degree $0$ such that:

- (α) $h$ is a homotopy between the chain maps $\Phi_1$ and $\Psi_1$, i.e.,
  $$
  \Psi_1|_{\mathfrak{G}} = \Phi_1|_{\mathfrak{G}} + d' \circ h,
  \quad \Psi_1|_{\mathfrak{A}} = \Phi_1|_{\mathfrak{A}} + h \circ d
  $$

- (β) $$
  \Psi_2 = \Phi_2 + \Theta_h^\mathfrak{g},
  \quad (44)
  $$

where $\Theta_h^\mathfrak{g} : \mathfrak{A} \times \mathfrak{A'} \to \mathfrak{A'}$ is the map defined, $\forall \pi_1 \in \mathfrak{G}, \pi_2 \in \mathfrak{G'}$ by

$$
\Theta_h^\mathfrak{g}(\pi_1, \pi_2) := h([\pi_1, \pi_2]) - [h(\pi_1), h(\pi_2)] - \Phi_1(\pi_1) \cdot h(\pi_2) + (-1)^{i} h(\pi_1) \cdot \Phi_1(\pi_2).
$$

\[45\]

It is straightforward to check that this definition is compatible with the usual notion of homotopy of $L_\infty$-morphisms.

**Proposition A.7.**

- i) Homotopy is an equivalence relation on morphisms of $\mathbb{Z}$-graded Lie 2-algebras;
- ii) Composition of morphisms of $\mathbb{Z}$-graded Lie 2-algebras is compatible with homotopies.

**Proof.** In order to prove i), one easily sees that symmetry follows from $\Theta_h^\mathfrak{g} = \Theta_{-h}^\mathfrak{g}$, the non-trivial additional point is to check transitivity on quadratic Taylor coefficients. Indeed spelling out Definition A.6, in particular [44], we see that this is equivalent to

$$
\Theta_h^{\mathfrak{g} + \mathfrak{g}} = \Theta_h^{\mathfrak{g}} + \Theta_h^\mathfrak{g},
$$

\[46\]

\[\text{2}\] Notice that $h$ becomes of degree $-1$ if $\mathfrak{G}$ and $\mathfrak{A'}$ are seen as subspaces of their associated dglas.
with \( h, g : \mathfrak{G} \to \mathfrak{A}' \) and \( \Theta \) as in (45). Indeed let us compute for all \( \pi_1 \in \mathfrak{G}_k \) and \( \pi_2 \in \mathfrak{G}_l \)

\[
\Theta \Psi g (\pi_1, \pi_2) = g([\pi_1, \pi_2]) - [g(\pi_1), g(\pi_2)] + (-1)^l g(\pi_1) \cdot \Phi_1(\pi_2) - \Phi_1(\pi_1) \cdot g(\pi_2)
\]

\[
+ (-1)^l g(\pi_1) \cdot d' \circ h(\pi_2) - d' \circ h(\pi_1) \cdot g(\pi_2)
\]

\[
= g([\pi_1, \pi_2]) - [g(\pi_1), g(\pi_2)] + (-1)^l g(\pi_1) \cdot \Phi_1(\pi_2) - \Phi_1(\pi_1) \cdot g(\pi_2)
\]

\[
- [g(\pi_1), h(\pi_2)] - [h(\pi_1), g(\pi_2)]
\]

where we used \( \Psi_1(\pi) = \Phi_1(\pi) + d \circ h(\pi) \) for all \( \pi \in \mathfrak{G} \) to go from the first to the second equality and Item b) in Definition A.1 to go from the second to the last one. Equation (46) then follows from adding \( \Theta \Phi h \) as defined in (45) to the latter expression.

In order to prove ii) let now \( \Phi \) and \( \Phi' \) be two morphisms of \( \mathbb{Z} \)-graded Lie 2-algebras. If \( h \) is a homotopy between them, then for all morphisms \( \Psi, \Xi \), the degree 0 linear map \( \Xi_1 \circ h \circ \Psi_1 \) is a homotopy between \( \Xi \circ \Phi \circ \Psi \) and \( \Xi \circ \Phi' \circ \Psi \).

Proposition A.7 allows us to make sense of the following definition:

**Definition A.8.** A *homotopy equivalence* between the \( \mathbb{Z} \)-graded Lie 2-algebras \( \mathfrak{A}_d \to \mathfrak{G} \) and \( \mathfrak{A}_d' \to \mathfrak{G}' \) is a pair of morphisms of \( \mathbb{Z} \)-graded Lie 2-algebras:

\[
\begin{array}{ccc}
\mathfrak{G} & \xrightarrow{\Phi} & \mathfrak{G}' \\
\downarrow d & & \downarrow d' \\
\mathfrak{A} & \xleftarrow{\Psi} & \mathfrak{A}'
\end{array}
\]

such that \( \Phi \circ \Psi \) and \( \Psi \circ \Phi \) are homotopic to the identity map. The morphism \( \Phi \) (resp. \( \Psi \)) is said to be an *homotopy inverse* of \( \Psi \) (resp. \( \Phi \)).

**A.2 Homotopy inverses of morphisms of \( \mathbb{Z} \)-graded Lie 2-algebras**

Recall that if \( \Phi \) is a morphism of \( \mathbb{Z} \)-graded Lie 2-algebras from \( \mathfrak{A}_d \to \mathfrak{G} \) to \( \mathfrak{A}_d' \to \mathfrak{G}' \), its linear part \( \Phi_1 \) is a chain map. A homotopy inverse of \( \Phi_1 \) is a graded chain map \( \Psi_1 \) from \( \mathfrak{A}_d \to \mathfrak{G}' \) to \( \mathfrak{A}_d \to \mathfrak{G} \) together with an homotopy \( h \) between \( \Psi_1 \circ \Phi_1 \) and the identity, and an homotopy \( h' \) between \( \Phi_1 \circ \Psi_1 \) and the identity:

\[
\begin{array}{ccc}
\mathfrak{G} & \xrightarrow{\Phi_1} & \mathfrak{G}' \\
\downarrow d & & \downarrow d' \\
\mathfrak{A} & \xleftarrow{\Psi_1} & \mathfrak{A}'
\end{array}
\]

\[
\begin{array}{ccc}
\mathfrak{A} & \xrightarrow{h} & \mathfrak{A}'
\end{array}
\]

The following theorem states that a morphism of \( \mathbb{Z} \)-graded Lie 2-algebras is homotopy invertible as long as its linear part is homotopy invertible as a chain map. Moreover, the homotopy inverse is uniquely determined by the homotopy inverse of the linear part.
**Theorem A.9.** Let $\Phi$ be a morphism of $\mathbb{Z}$-graded Lie 2-algebras from $\mathfrak{A}^d \to \mathfrak{G}$ to $\mathfrak{A}^{d'} \to \mathfrak{G}'$ and let $\Phi_1$ be its linear Taylor coefficient. Assume that an homotopy inverse $\Psi_1$ of $\Phi_1$ is given with homotopies $h, h'$ as in \[48\]. Then, there exists a unique morphism $\Psi$ of $\mathbb{Z}$-graded Lie 2-algebras from $\mathfrak{A}^{d'} \to \mathfrak{G}'$ to $\mathfrak{A}^d \to \mathfrak{G}$ such that:

1. $\Psi_1$ is the linear Taylor coefficient of $\Psi$;
2. $h$ (resp. $h'$) is a homotopy of morphisms of graded 2-Lie algebras between the composition $\Phi \circ \Psi$ (resp. $\Psi \circ \Phi$) and the identity.

**Proof.** One has to prove existence and uniqueness of the $\mathbb{Z}$-graded Lie 2-algebra morphism $\Psi$. Since its linear Taylor coefficient is already imposed, one has to find and prove uniqueness of its quadratic term $\Psi_2 : \wedge^2 \mathfrak{G}' \to \mathfrak{A}$.

For all $\pi_1', \pi_2', \pi_1, \pi_2 \in \mathfrak{G}'$, $\pi_1, \pi_2 \in \mathfrak{G}$, $\Psi_2$ has to satisfy:

$$
\begin{align*}
\text{d} \Psi_2(\pi_1', \pi_2') &= \Psi_1([\pi_1', \pi_2']) - [\Psi_1(\pi_1'), \Psi_1(\pi_2')], \\
\Psi_2(\pi', \text{d}'a') &= \Psi_1(\pi' \cdot a') - \Psi_1(\pi') \cdot \Psi_1(a'), \\
\Phi_1 \circ \Psi_2(\pi_1', \pi_2') &= \Theta_h^d(\pi_1', \pi_2') - \Phi_2(\Psi_1(\pi_1'), \Psi_1(\pi_2')), \\
\Psi_1 \circ \Phi_2(\pi_1, \pi_2) &= \Theta_h^d(\pi_1, \pi_2) - \Psi_2(\Phi_1(\pi_1), \Phi_1(\pi_2))
\end{align*}
$$

(49)

where $\Theta_h^d$ and $\Theta_h^{d'}$ are defined as in \[45\]. The first two relations above say that $\Psi_2$ is the quadratic Taylor coefficient of a $\mathbb{Z}$-graded Lie 2-algebra morphism whose linear Taylor coefficient is $\Psi_1$. The third (and fourth) relations say that $h'$ and $h$ are the homotopies between $\Phi \circ \Psi$ (and $\Psi \circ \Phi$) and the identity.

In order to prove uniqueness it is enough to observe that, if $\Psi_2$ and $\Psi_3$ are two solutions of \[49\] then $\text{Im}(\Psi_2 - \Psi_3) \subset \text{Ker \, \text{d} \cap \text{Ker } \Phi_1 = 0}$, since $\Psi_1$ and $\Phi_1$ are inverse up to homotopy.

Existence is proved by an explicit bilinear assignment $\Psi_2$ that satisfies \[49\]. Let us consider:

$$
\Psi_2 := \Psi_1 \circ \Theta_h^d + h \circ \kappa_{\Psi_1} - \Psi_1 \circ \Phi_2 \circ \wedge^2 \Psi_1,
$$

(50)

where:

$$
\kappa_{\Psi_1}(\pi_1', \pi_2') = -\Psi_1([\pi_1', \pi_2']) + [\Psi_1(\pi_1'), \Psi_1(\pi_2')].
$$

The check that $\Psi_2$ defined in \[50\] solves \[49\] is a long and explicit computation. \(\square\)

### A.3 Maurer-Cartan moduli set of a $\mathbb{Z}$-graded Lie 2-algebra

Let $\mathfrak{A}^d \to \mathfrak{G}$ be a $\mathbb{Z}$-graded Lie 2-algebra and $\mathcal{V} := \mathcal{V}(\mathfrak{A}^d \to \mathfrak{G})$ its associated dgla. The **Maurer-Cartan elements** of $\mathfrak{A}^d \to \mathfrak{G}$ are the Maurer-Cartan elements of its associated dgla. The set of Maurer-Cartan elements is denoted by $MC(\mathfrak{A}^d \to \mathfrak{G})$.

**Lemma A.10.** Maurer-Cartan elements of a $\mathbb{Z}$-graded Lie 2-algebra $\mathfrak{A}^d \to \mathfrak{G}$ are elements $\Lambda \oplus \Pi \in \mathcal{V}_1 = \mathfrak{A}_2 \oplus \mathfrak{G}_1$ satisfying

$$
d\Lambda + \frac{1}{2}[\Pi, \Pi] = 0 \text{ and } \Pi \cdot \Lambda = 0.
$$

For any Maurer-Cartan element $\Lambda \oplus \Pi \in \mathfrak{A}_2 \oplus \mathfrak{G}_1$ and any $T \in \mathfrak{A}_1$, define $\Lambda_T \oplus \Pi_T \in \mathfrak{A}_2 \oplus \mathfrak{G}_1$ by:

$$
\Pi_T := \Pi + dT \text{ and } \Lambda_T := \Lambda - \Pi \cdot T - \frac{1}{2}[T, T].
$$

(51)
We say that \((\Lambda \oplus \Pi)_T := \Lambda_T \oplus \Pi_T\) defined in \([51]\) is the **twist** of \(\Lambda \oplus \Pi\) by \(T\). Twist transformations are related to gauge transformations of dglas. Recall that two Maurer-Cartan elements \(m\) and \(m'\) in a dglas are said to be **gauge-equivalent**, if there exists an element \(b\) of degree 1 (called a **gauge**), such that \(m' = \exp(b) \cdot m\) where

\[
\exp(b) := m - \sum_{i \geq 0} \frac{\text{ad}^i_T(b)}{i!}(db + [m, b]).
\]  

(52)

See, for instance, Equation (3.7) in \([9]\). In general, there is a convergence issue. However when the gauge element \(b\) is a nilpotent element for the graded Lie algebra, the RHS of Equation (52) is well defined. In our situation, it is clear that \(A_1 \subset V_0 = A_1 \oplus G_0\) is an abelian Lie subalgebra; in particular the above gauge transformations by \(T \in A_1\) makes sense.

**Proposition A.11.** Let \(A \xrightarrow{d} \mathcal{G}\) be a \(\mathbb{Z}\)-graded Lie 2-algebra and \(V := \mathcal{V}(\mathcal{A}[1] \xrightarrow{d} \mathcal{G})\) its associated dgl. For every Maurer-Cartan element \(\Lambda \oplus \Pi\) and \(T \in A_1\)

\[
\exp(-T) \cdot (\Lambda \oplus \Pi) = (\Lambda \oplus \Pi)_T.
\]

**Proof.** A direct computation gives:

\[
d\left(T \oplus 0\right) + [T \oplus 0, \Lambda \oplus \Pi] = -\Pi \cdot T \oplus dT
\]

\[
ad^0_T\left(d\left(T \oplus 0\right) + [T \oplus 0, \Lambda \oplus \Pi]\right) = 0 \oplus [T, T]
\]

\[
ad^i_T\left(d\left(0 \oplus T\right) + [T \oplus 0, \Lambda \oplus \Pi]\right) = 0 \text{ for } i \geq 2
\]

The result then follows by using these relations to compute the right hand side of (52) and comparing it with (51).

**Corollary A.12.** For any Maurer-Cartan element \(\Lambda \oplus \Pi\) of a \(\mathbb{Z}\)-graded Lie 2-algebra \(A \xrightarrow{d} \mathcal{G}\) and any \(T \in A\), the element \((\Lambda \oplus \Pi)_T\) is still a Maurer-Cartan element. Moreover, twist transformations define an equivalence relation on \(MC(A \xrightarrow{d} \mathcal{G})\).

**Definition A.13.** The **Maurer-Cartan moduli set** \(MC(A \xrightarrow{d} \mathcal{G})\) is the quotient of \(MC(A \xrightarrow{d} \mathcal{G})\) by twist equivalence.

It is a general fact that if \(F : g_1 \rightarrow g_2\) is a morphism of \(L_\infty\) algebras and \(m \in g_1\) is a Maurer-Cartan element then

\[
\sum_{n=0}^{\infty} \frac{1}{n!} F_n(m, \ldots, m)
\]

is a Maurer-Cartan element of \(g_2\). By applying this formula to a morphism \(\Phi\) of \(Z\)-graded Lie 2-algebras from \(\mathcal{A} \xrightarrow{d} \mathcal{G}\) to \(\mathcal{A'} \xrightarrow{d'} \mathcal{G'}\), we obtain a map \(MC(\Phi) : MC(\mathcal{A} \xrightarrow{d} \mathcal{G}) \rightarrow MC(\mathcal{A'} \xrightarrow{d'} \mathcal{G'})\) that reads

\[
MC(\Phi)(\Lambda \oplus \Pi) = \left(\Phi_1(\Lambda) + \frac{1}{2} \Phi_2(\Pi, \Pi)\right) \oplus \Phi_1(\Pi).
\]

(53)

The following result is also a straightforward consequence of a general result valid for any \(L_\infty\)-morphisms. For completeness, we outline a proof below.

**Lemma A.14.** Let \(\Phi\) be a morphism of graded Lie 2-algebras from \(\mathcal{A} \xrightarrow{d} \mathcal{G}\) to \(\mathcal{A'} \xrightarrow{d'} \mathcal{G'}\). The images through \(MC(\Phi)\) of twist equivalent Maurer-Cartan elements are twist equivalent.
Proof. Let us prove that if $T \in \mathfrak{A}_1$ and $\Lambda \oplus \Pi \in MC(\mathfrak{A} \xrightarrow{d} \mathfrak{G})$ then
\[ MC(\Phi) \left( (\Lambda \oplus \Pi)_T \right) = (MC(\Phi) (\Lambda \oplus \Pi))_{\Phi_1(T)}. \] (54)

In view of the definition of twist equivalence [51], this last relation decomposes as the two following relations:
\[
\begin{align*}
\Phi_1(\Pi + dT) & = \Phi_1(\Pi) + d' \circ \Phi_1(T) \\
\Phi_1 \left( (\Lambda - \Pi \cdot T - \frac{1}{2}[T, T]) \right) & = \Phi_1(\Lambda) + \frac{1}{2} \Phi_2(\Pi, \Pi)
\end{align*}
\] (55)

The first of these relations follows from the fact that $\Phi_1$ is a chain map. The second relation can be checked by a direct computation. First, by definition of 2-graded Lie 2-algebra morphism, for all $P \in \mathfrak{G}$:
\[ \Phi_2(P, dT) = \Phi_1(P \cdot T) - \Phi_1(P) \cdot \Phi_1(T). \] (56)

In particular for $P = dT$, Equation (56) gives:
\[ \Phi_2(dT, dT) = \Phi_1(dT \cdot T) - \Phi_1(dT) \cdot \Phi_1(T) = \Phi_1((dT, T)) - [\Phi_1(T), \Phi_1(T)]. \]

The second relation in the system (55) follows from the previous relation and Equation (56), applying for $P = \Pi$. \qed

As a consequence, $MC(\Phi) : MC(\mathfrak{A} \xrightarrow{d} \mathfrak{G}) \to MC(\mathfrak{A}' \xrightarrow{d'} \mathfrak{G}')$ defines a map between Maurer-Cartan moduli sets:
\[ MC(\Phi) : MC(\mathfrak{A} \xrightarrow{d} \mathfrak{G}) \to MC(\mathfrak{A}' \xrightarrow{d'} \mathfrak{G}'). \]

We prove in the following Lemma that such a map depends only on the homotopy type of $\Phi$.

**Lemma A.15.** Let $\Phi$ and $\Psi$ be morphisms of 2-graded Lie 2-algebras from $\mathfrak{A} \xrightarrow{d} \mathfrak{G}$ to $\mathfrak{A}' \xrightarrow{d'} \mathfrak{G}'$ and let $h : \mathfrak{G} \to \mathfrak{A}'$ be an homotopy between $\Phi$ and $\Psi$; then
\[ MC(\Psi)(\Lambda \oplus \Pi) = MC(\Phi)(\Lambda \oplus \Pi)_{h(\Pi)} \]
for all $\Lambda \oplus \Pi \in MC(\mathfrak{A} \xrightarrow{d} \mathfrak{G})$. As a consequence $MC(\Phi) = MC(\Psi)$.

Proof. According to the definition $\Theta^h_{\Phi}$ given in (45) we get
\[ \Theta^h_{\Phi}(\Pi, \Pi) = h([\Pi, \Pi]) - [h(\Pi), h(\Pi)] - 2\Phi_1(\Pi) \cdot h(\Pi) = -2h(d\Lambda) - [h(\Pi), h(\Pi)] - 2\Phi_1(\Pi) \cdot h(\Pi), \] (57)

where we used the Maurer-Cartan condition $[\Pi, \Pi] = -2d\Lambda$.

Applying (53), and using the fact that $h$ is a homotopy between the chain maps $\Phi_1$ and $\Psi_1$ together with the relation between $\Phi_2$ and $\Psi_2$ given in (44), we obtain:
\[ MC(\Psi)(\Lambda \oplus \Pi) = \left( \Phi_1(\Lambda) + \frac{1}{2} \Phi_2(\Pi, \Pi) \right) \oplus \Psi_1(\Pi) \]
\[ = \left( \Phi_1(\Lambda) + h \circ d(\Lambda) + \frac{1}{2} \Phi_2(\Pi, \Pi) + \frac{1}{2} \Theta^h_{\Phi}(\Pi, \Pi) \right) \]
\[ \oplus \left( \Phi_1(\Pi) + d' \circ h(\Pi) \right). \]
By Equation (57), this gives:

\[
MC(Ψ)(Λ ⊕ Π) = \left( Φ_1(Λ) + \frac{1}{2} Φ_2(Π, Π) - \frac{1}{2} \left[ h(Π), h(Π) \right] - Φ_1(Π) \cdot h(Π) \right) \\
\oplus \left( Φ_1(Π) + d' \circ h(Π) \right)
\]

\[
= MC(Φ)(Λ ⊕ Π) h(Π).
\]

The above Lemmas imply immediately the following Corollary.

**Corollary A.16.** An homotopy equivalence between two strict Lie 2-algebras induces a one-to-one correspondence between their Maurer-Cartan moduli sets.

The previous result justify the following construction:

1. to any \( Z \)-graded Lie 2-algebra \( A \), we can associate the set \( MC(A \to \mathcal{G}) \) of Maurer-Cartan elements up to twists,

2. to any morphism \( Φ \) from \( A \to \mathcal{G} \) to \( A' \to \mathcal{G}' \), we have associated a map \( MC(Φ) : MC(A \to \mathcal{G}) \to MC(A' \to \mathcal{G}') \) (by Lemma A.14).

The relations \( MC(Φ \circ Ψ) = MC(Φ) \circ MC(Ψ) \) and \( MC(id) = id \) are of course satisfied. Moreover, if two morphisms \( Φ \) and \( Ψ \) are homotopic, then \( MC(Φ) = MC(Ψ) \) by Lemma A.15. This shows that the assignment \( MC \) induces a functor from the category Lie\(_{2}\) (where objects are \( Z \)-graded Lie 2-algebras and arrows are homotopy classes of \( Z \)-graded Lie 2-algebra morphisms) to the category of sets. It is in fact valued in the subcategory of sets where objects are sets and all arrows are bijections by Corollary A.16.

**Definition A.17.** We call Maurer-Cartan functor the above functor from Lie\(_{2}\) to the category of sets and denote it by \( MC \).

**B. \( Z \)-graded Lie groupoids and cohomology**

This section is devoted to establishing results which are necessary to prove Proposition 2.5.

**B.1 \( Z \)-graded Lie groupoids and truncated 2-term groupoid cohomology complexes**

\( Z \)-graded Lie groupoids are Lie groupoids in the category of \( Z \)-graded manifolds. For details, see [26]. Many standard notations and constructions of Lie groupoids have straightforward extensions to the context of \( Z \)-graded Lie groupoids, including groupoid cohomology, morphisms, Morita morphisms and so on. In particular, for a \( Z \)-graded Lie groupoid \( G \Rightarrow M \), the \( Z \)-graded groupoid cohomology complex is the \( Z \)-graded complex:

\[
C^\infty(M) \overset{δ}{\longrightarrow} C^\infty(G) \overset{δ}{\longrightarrow} C^\infty(G^{(2)}) \overset{δ}{\longrightarrow} \ldots
\]

(58)

where \( (G^{(k)})_{k \geq 0} \) denotes the space of composable \( k \)-arrows. Functions on \( G \) that are \( δ \)-closed are called multiplicative. The space of multiplicative functions on \( G \) is denoted by \( Z(G) \). The \( Z \)-graded 2-term complex we are interested in is the (shifted) truncation of (58) at degree 1.
Definition B.1. Let \( \mathcal{G} \rightrightarrows \mathcal{M} \) be a \( \mathbb{Z} \)-graded Lie groupoid. By its 2-term truncated (groupoid cohomology) complex, we mean the graded 2-term complex:

\[
C^\infty(\mathcal{M})[1] \xrightarrow{\delta} \mathcal{Z}(\mathcal{G})[1]
\]

A graded groupoid morphism \( \Phi : \mathcal{G}_1 \to \mathcal{G}_2 \) induces a cochain map \( \Phi^* \) between their \( \mathbb{Z} \)-graded groupoid cohomology complexes:

\[
C^\infty(\mathcal{M}_1) \xrightarrow{\delta} C^\infty(\mathcal{G}_1) \xrightarrow{\delta} C^\infty(\mathcal{G}_1^{(2)}) \xrightarrow{\delta} \ldots
\]

\[
\phi^* \quad \Phi^* \quad \phi^* \quad \Phi^* \quad \phi^*
\]

\[
C^\infty(\mathcal{M}_2) \xrightarrow{\delta} C^\infty(\mathcal{G}_2) \xrightarrow{\delta} C^\infty(\mathcal{G}_2^{(2)}) \xrightarrow{\delta} \ldots
\]

Therefore, \( \Phi^* \) induces a morphism of their corresponding truncated 2-term \( \mathbb{Z} \)-graded complexes:

\[
\mathcal{Z}(\mathcal{G})[1] \quad \mathcal{Z}(\mathcal{G'})[1]
\]

\[
\delta \quad \Phi^* \quad \delta
\]

\[
C^\infty(\mathcal{M})[1] \quad C^\infty(\mathcal{M'})[1]
\]

The main purpose of this section is to construct an explicit homotopy inverse of \( \Phi^* \) in (61) when \( \Phi \) is a Morita morphism. From now on, assume that \( \mathcal{G}_1 \rightrightarrows \mathcal{M}_1 \) is the pullback groupoid \( \mathcal{G}[\mathcal{X}] \rightrightarrows \mathcal{X} \), and \( \Phi : \mathcal{G}[\mathcal{X}] \to \mathcal{G} \) is the natural projection, where \( \phi : \mathcal{X} \to \mathcal{M} \) is a surjective submersion of \( \mathbb{Z} \)-graded manifolds.

Assume that \( \phi : \mathcal{X} \to \mathcal{M} \) admits a section \( \sigma : \mathcal{M} \to \mathcal{X} \). Introduce maps \( \tilde{\sigma} : \mathcal{G} \to \mathcal{G}[\mathcal{X}] \) and \( \tau : \mathcal{X} \to \mathcal{G}[\mathcal{X}] \), respectively by

\[
\tilde{\sigma} = (\sigma \circ t, \text{id}, \sigma \circ s)
\]

and

\[
\tau = (\text{id}, \epsilon \circ \phi, \sigma \circ \phi)
\]

where we identify \( \mathcal{G}[\mathcal{X}] \) with \( \mathcal{X} \times_{\mathcal{M}, \phi} \mathcal{G} \times_{\mathcal{M}, s} \mathcal{X} \) and \( \epsilon : \mathcal{M} \to \mathcal{G} \) is the embedding of units of \( \mathcal{G} \). It is simple to check that the pair of maps \( (\tilde{\sigma}, \sigma) \) defines a \( \mathbb{Z} \)-graded groupoid morphism from \( \mathcal{G} \rightrightarrows \mathcal{M} \) to \( \mathcal{G}[\mathcal{X}] \rightrightarrows \mathcal{X} \). Therefore, it induces a morphism \( \tilde{\sigma}^* \) of the truncated 2-term complexes from \( C^\infty(\mathcal{M})[1] \to C^\infty(\mathcal{G}[\mathcal{X}])[1] \) to \( C^\infty(\mathcal{M})[1] \to \mathcal{Z}(\mathcal{G})[1] \). Since \( \Phi^* \circ \tilde{\sigma}^* = \text{id} \), it follows that \( \tilde{\sigma}^* \circ \Phi^* = \text{id} \). Moreover, it is straightforward to check that \( \Phi^* \circ \tilde{\sigma}^* \) is homotopic to the identity, with \( \tau^* : \mathcal{Z}(\mathcal{G}(\mathcal{X}))[1] \to C^\infty(\mathcal{X})[1] \) being a homotopy map.

In general, global sections \( \sigma : \mathcal{M} \to \mathcal{X} \) may not exist. However, since \( \phi : \mathcal{X} \to \mathcal{M} \) is a surjective submersion, local sections always exist. The standard argument of partition of unity will enable us to construct a homotopy inverse of \( \Phi^* \). More precisely, denote by \( \mathcal{X} \) and \( \mathcal{M} \) the body of \( \mathcal{X} \) and \( \mathcal{M} \), respectively, and by the same \( \phi : \mathcal{X} \to \mathcal{M} \) the surjective submersions at the level of bodies. Choose a nice open cover \( (U_i)_{i \in S} \) of the body \( \mathcal{M} \) of \( \mathcal{M} \). Let \( \chi_i \) be a partition of unity subject to the cover \( (U_i)_{i \in S} \). Denote by \( \mathcal{U}_i \) the restriction of the graded manifold \( \mathcal{M} \) to \( U_i \), and \( \phi^{-1}(U_i) \) the restriction of \( \mathcal{X} \) to the open subset \( \phi^{-1}(U_i) \) of the body \( \mathcal{X} \). Therefore, for each \( i \in S \), there exists a local section.
Let \( \sigma_i : U_i \leftrightarrow \phi^{-1}(U_i) \) of \( \phi : X \to M \). Let \( \tau_i : \phi^{-1}(U_i) \to G[\chi_{\phi^{-1}(U_i)}] \) be the map defined as in Equation (63) with respect to the section \( \sigma_i : U_i \leftrightarrow \phi^{-1}(U_i) \). Similar to Equation (62), for any \( i, i_2 \in S \), denote by \( \hat{\sigma}_{i, i_2} : G[\|i_{i_2}|_{\phi^{-1}(U_{i_2})}] \to G[\chi_{\phi^{-1}(U_{i_2})}] \), the map

\[
\hat{\sigma}_{i, i_2} = \langle \sigma_{i, i_2}, t, id, \sigma_{i_2}, s \rangle,
\]

where \( G[\|i_{i_2}|_{\phi^{-1}(U_{i_2})}] = s^{-1}(U_{i_2}) \cap t^{-1}(U_{i_2}) \) with \( s \) and \( t \) being the source and target maps of \( G \sqsupset M \); similarly for \( G[\chi_{\phi^{-1}(U_{i_2})}] \).

Consider the maps

\[
I_1 : C^\infty(\hat{G}[\chi])[1] \to C^\infty(G)[1], \quad I_1 = \sum_{i_1, i_2 \in S} (s_i^* \chi_{i_1}) (t_i^* \chi_{i_2}) \hat{\sigma}_{i_1, i_2}^*
\]

(65)

\[
I_0 : C^\infty(\chi)[1] \to C^\infty(M)[1], \quad I_0 = \sum_{i \in S} \chi_i \sigma_i^*
\]

(66)

and

\[
H : Z(\hat{G}(\chi))[1] \to C^\infty(\chi)[1], \quad H = \sum_{i \in S} (\phi_i^* \chi_i) \tau_i^*
\]

(67)

The following proposition can be verified directly.

**Proposition B.2.** Let \( G \sqsupset M \) be a \( \mathbb{Z} \)-graded groupoid, \( \phi : X \to M \) a surjective submersion, and \( \Phi : G[\chi] \to G \) the natural projection. Then,

1. the pair \( I := (I_0, I_1) \) defines a morphism of the truncated 2-term complexes from \( C^\infty(G)[1] \to Z(\hat{G}(\chi))[1] \) to \( C^\infty(M)[1] \to Z(G)[1] \);
2. \( I \) is a left inverse of \( \Phi^* \);
3. the composition \( \Phi^* \circ I \) is homotopic to the identity with \( H \) being a homotopy map.

In summary, we have the following diagram:

\[
\begin{array}{ccc}
Z(G)[1] & \xrightarrow{\Phi^*} & Z(G[\chi])[1] \\
\downarrow{\delta} & & \downarrow{H} \\
C^\infty(M)[1] & \xrightarrow{I} & C^\infty(\chi)[1].
\end{array}
\]

(68)

**B.2 Proof of Proposition 2.5**

Every VB groupoid \( V \sqsupset E \) defines a \( \mathbb{Z} \)-graded Lie groupoid \( V[1] : \sqsupset E[1] \). The space of multiplicative functions \( Z(V[1]) \subset \Gamma(\Lambda^* V^*) \) inherits the \( \mathbb{N} \)-grading, i.e. \( Z(V[1]) = \oplus_k Z^k(V[1]) \). The following straightforward Lemma gives an useful characterization.

**Lemma B.3.** Let \( V \sqsupset E \) be a VB groupoid over \( \Gamma \sqsupset E ; \ P \in \Gamma(\Lambda^k V^*) \) is a multiplicative function, i.e. \( P \in Z^k(V[1]) \), if and only if the function

\[
F_P(\mu_1, \ldots, \mu_k) = \langle P, \mu_1 \wedge \ldots \wedge \mu_k \rangle, \quad (\mu_1, \ldots, \mu_k) \in V \times_\Gamma \ldots \times_\Gamma V
\]

is a one cocycle of \( V \times_\Gamma \ldots \times_\Gamma V \sqsupset E \times_M \ldots \times_M E \) considered as a subgroupoid of the direct product groupoid \( V \times \ldots \times V \sqsupset E \times \ldots \times E \).
For any Lie groupoid $\Gamma \Rightarrow M$ with Lie algebroid $A$, the cotangent groupoid $T^\vee \Gamma \Rightarrow A^\vee$ is a VB-groupoid as seen in Example 4.4. Therefore, it gives rise to a $\mathbb{Z}$-graded Lie groupoid $T^\vee \Gamma \Rightarrow A^\vee$ in Lemma $B.3$ and the characterization of multiplicative polyvector fields given in Proposition 2.7 of [15].

**Lemma B.4.** Let $\Gamma \Rightarrow M$ be a Lie groupoid. The truncated 2-term complex of the $\mathbb{Z}$-graded groupoid $T^\vee \Gamma \Rightarrow A^\vee$ coincides with the $\mathbb{Z}$-graded 2-term complex $\Sigma^\bullet(A) \longrightarrow T^\bullet_{\text{mult}} \Gamma$ in Lemma $2.1$.

Now assume that $\varphi : X \to M$ is a surjective submersion and let $\nabla$ be an Ehresmann connection for $\varphi$. It is simple to see that $(\Phi_\nabla, \phi_\nabla)$ in Equation (5) indeed defines a VB-groupoid morphism:

\[
\begin{array}{ccc}
T^\vee \Gamma & \longrightarrow & A^\vee \\
\downarrow & \Phi_\nabla & \downarrow \phi_\nabla \\
\Gamma[X] & \longrightarrow & X \\
\downarrow & & \downarrow \\
M & & M
\end{array}
\]

and this in turn induces a $\mathbb{Z}$-graded groupoid morphism from $T^\vee \Gamma(X) \Rightarrow (A[X])^\vee$ to $T^\vee \Gamma \Rightarrow A^\vee$.

The following result is just a rephrasing of the discussion of Remark 4.25 into the language of graded groupoids.

**Proposition B.5.** Let $\Gamma \Rightarrow M$ be a Lie groupoid, $\varphi : X \to M$ a surjective submersion and let $\nabla$ be an Ehresmann connection for $\varphi$. The pair $(\Phi_\nabla, \phi_\nabla)$ defined in Equation (5) is a Morita morphism of $\mathbb{Z}$-graded groupoid from $T^\vee \Gamma(X) \Rightarrow (A[X])^\vee$ to $T^\vee \Gamma \Rightarrow A^\vee$.

The following lemma can be verified in a straightforward manner.

**Lemma B.6.** Let $\Gamma \Rightarrow M$ be a Lie groupoid, $\varphi : X \to M$ a surjective submersion and let $\nabla$ be an Ehresmann connection for $\varphi$. Under the identifications of Lemma B.4, the morphism of 2-term truncated complexes associated to the $\mathbb{Z}$-graded groupoid morphism $\Phi_\nabla$ as in Proposition B.5 coincides with the horizontal lift:

\[
\begin{array}{ccc}
\Sigma^\bullet(A) & \longrightarrow & \Sigma^\bullet(A[X]) \\
\downarrow & \Phi_\nabla & \downarrow \phi_\nabla \\
T^\bullet_{\text{mult}} \Gamma & \longrightarrow & T^\bullet_{\text{mult}} \Gamma[X] \\
\downarrow & & \downarrow \\
\Sigma^\bullet(A) & \longrightarrow & \Sigma^\bullet(A[X]).
\end{array}
\]

**Proof.** It is straightforward to see that the dual of the maps $\Phi_\nabla$ and $\phi_\nabla$ are the horizontal lifts $\lambda_\nabla$ defined in Equation (6). Since $(\Phi_\nabla, \phi_\nabla)$ is a $\mathbb{Z}$-graded groupoid morphism, its dual $\lambda_\nabla$ is a morphism of $\mathbb{Z}$-graded 2-term complexes. This completes the proof. 

We are now ready to prove Proposition 2.5.
Proof of Proposition 2.5. Applying Proposition 2.2 to the Morita morphism described in Lemma B.5 we obtain a morphism $I = (I_0, I_1)$ of 2-term complexes from $\Sigma^*(A[X]) \to \Gamma: \mathcal{T}_{A^{\text{mult}}} \to \Sigma^*(A[X])$ to $\mathcal{T}_{A^{\text{mult}}} \to \Sigma^*(A[X])$. These maps depend on the choice of local sections of $\phi: (A[X])^{\text{pr}} \to A^{\text{pr}}$. Indeed, choose a nice open cover $(U_i)_{i \in S}$ of $M$ so that the image of $\lambda \varphi$ lies in $\Sigma^*(A[X])^{\text{proj}} \to \mathcal{T}_{A^{\text{mult}}} \to \Sigma^*(A[X])^{\text{proj}}$. In order to prove the first statement, we have to show that it is possible to choose local sections so that: (i) the restriction of $I$ to projectable elements is given by the natural projection $\text{pr}$; (ii) the restriction of the homotopy map $h_X$ to $\mathcal{T}_{A^{\text{mult}}} \to \Sigma^*(A[X])^{\text{proj}}$ yields a homotopy map $h_{\lambda \varphi}: \mathcal{T}_{A^{\text{mult}}} \to \Sigma^*(A[X])^{\text{proj}}$. Also, choose a nice open cover $(U_i)_{i \in S}$ of $M$ so that $\varphi: X \to M$ admits a family of local sections $\sigma_i: U_i \to \varphi^{-1}(U_i)$. Denote by $U_i$ the restriction of the graded manifold $A[i]^{\text{pr}}$ to $U_i$, and $\varphi^{-1}(U_i)$ the restriction of $A[X]^{\text{pr}}$ to the open subset $\varphi^{-1}(U_i) \subset X$. Therefore, for each $i \in S$, there is an induced local section $\sigma_i: U_i \to \varphi^{-1}(U_i)$ of the submersion $\varphi: (A[X])^{\text{pr}} \to A^{\text{pr}}$. It is now straightforward to check that the maps $I$ and $h_X$ defined in the proof of Proposition B.2 with these local sections satisfy (i) and (ii).

For the second part of the proposition, let $\psi := \lambda \varphi \circ I$. By construction, $\psi$ is a chain map from $\Sigma^*(A[X]) \to \mathcal{T}_{A^{\text{mult}}} \to \Sigma^*(A[X])$ to $\Sigma^*(A[X])^{\text{proj}} \to \mathcal{T}_{A^{\text{mult}}} \to \Sigma^*(A[X])^{\text{proj}}$. According to Proposition 2.2, $\psi$ is homotopic to the identity (when seen as a chain map from $\Sigma^*(A[X]) \to \mathcal{T}_{A^{\text{mult}}} \to \Sigma^*(A[X])$ to itself) with respect to the homotopy map $h_X$, i.e.,

$$i \circ \psi = \text{id} + d \circ h_X + h_X \circ d.$$ 

Also,

$$\psi \circ i = \lambda \varphi \circ \text{pr} = \text{id} + d \circ h_{\lambda \varphi} + h_{\lambda \varphi} \circ d.$$ 

This concludes the proof. 

□

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