Properties of polynomial bases used in a line-surface intersection algorithm

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Abstract

In [5], Srijuntongsiri and Vavasis propose the Kantorovich-Test Subdivision algorithm, or KTS, which is an algorithm for finding all zeros of a polynomial system in a bounded region of the plane. This algorithm can be used to find the intersections between a line and a surface. The main features of KTS are that it can operate on polynomials represented in any basis that satisfies certain conditions and that its efficiency has an upper bound that depends only on the conditioning of the problem and the choice of the basis representing the polynomial system.

This article explores in detail the dependence of the efficiency of the KTS algorithm on the choice of basis. Three bases are considered: the power, the Bernstein, and the Chebyshev bases. These three bases satisfy the basis properties required by KTS. Theoretically, Chebyshev case has the smallest upper bound on its running time. The computational results, however, do not show that Chebyshev case performs better than the other two.

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1 The line-surface intersection problem and the required basis properties

Let $\phi_0, \ldots, \phi_n$ denote a basis for the set of univariate polynomials of degree at most $n$. For example, the power basis is defined by $\phi_i(t) = t^i$. The line-surface intersection problem can be reduced to the problem of finding all zeros of

$$f(u,v) \equiv \sum_{i=0}^{m} \sum_{j=0}^{n} c_{ij} \phi_i(u)\phi_j(v), \quad 0 \leq u, v \leq 1,$$

where $c_{ij} \in \mathbb{R}^2$ ($i = 0, 1, \ldots, m; j = 0, 1, \ldots, n$) denote the coefficients.

For this article, let the notation $\|\cdot\|$ refer specifically to infinity norm. Other norms are explicitly notated so. The Kantorovich-Test Subdivision algorithm (KTS in short), proposed by Srijuntongsiri and Vavasis, can be used to solve (1). KTS works with any polynomial basis $\phi_i(u)\phi_j(v)$ provided that the following properties hold:

1. There is a natural interval $[l, h]$ that is the domain for the polynomial. In the case of Bernstein polynomials, this is $[0, 1]$, and in the case of power and Chebyshev polynomials, this is $[-1, 1]$.

2. It is possible to compute a bounding polytope $P$ of $S = \{f(u,v) : l \leq u, v \leq h\}$, where $f(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} c_{ij} \phi_i(u)\phi_j(v)$ and $c_{ij} \in \mathbb{R}^d$ for any $d \geq 1$, that satisfies the following properties:

   (a) Determining whether $0 \in P$ can be done efficiently (ideally in $O(mn)$ operations).

   (b) The polytope $P$ is affinely invariant. In other words, the bounding polytope of $\{A f(u,v) + b : l \leq u, v \leq h\}$ is $\{A x + b : x \in P\}$ for any nonsingular matrix $A \in \mathbb{R}^{d \times d}$ and any vector $b \in \mathbb{R}^d$.

   (c) For any $y \in P$,

   $$\|y\| \leq \theta \max_{l \leq u, v \leq h} \|f(u,v)\|,$$

   where $\theta$ is a function of $m$ and $n$.

   (d) If $d = 1$, then the endpoints of $P$ can be computed efficiently (ideally in $O(mn)$ time).
3. It is possible to reparametrize with \([l, h]^2\) the surface \(S_1 = \{ f(x) : x \in \overline{B}(x_0, r) \}\), where \(x_0 \in \mathbb{R}^2\) and \(r \in \mathbb{R} > 0\). In other words, it is possible (and efficient) to compute the polynomial \(\hat{f}\) represented in the same basis such that \(S_1 = \{ \hat{f}(\hat{x}) : \hat{x} \in [l, h]^2 \}\).

4. Constant polynomials are easy to represent.

5. Derivatives of polynomials are easy to determine in the same basis. (preferably in \(O(mn)\) operations).

We are generally interested in the case where \(d = 2\). In this case, we call \(P\) a bounding polygon. Recall that \(P\) is a bounding polygon of \(S\) if and only if \(x \in S\) implies \(x \in P\).

2 The Kantorovich-Test Subdivision algorithm

The description of KTS, as well as the definitions of the quantities mentioned in the description, are given below. More details can be found in [5].

For a given zero \(x^*\) of polynomial \(f\), let \(\omega_*(x^*)\) and \(\rho_*(x^*)\) be quantities satisfying the conditions that, first, \(\omega_*(x^*)\) is the smallest Lipschitz constant for \(f'(x^*)^{-1} f'\), i.e.,

\[
\| f'(x^*)^{-1} (f'(x) - f'(y)) \| \leq \omega_*(x^*) \cdot \| x - y \| \quad \text{for all } x, y \in \overline{B}(x^*, \rho_*(x^*))
\]

and, second,

\[
\rho_*(x^*) = \frac{2}{\omega_*(x^*)}. \tag{4}
\]

Define

\[
\gamma(\theta) = 1 / \left( 4 \sqrt{\theta(4\theta + 1) - 8\theta} \right),
\]

where \(\theta\) is as in (2). Define \(\omega_{D'}\) to be the smallest nonnegative constant \(\omega\) satisfying

\[
\| f'(x^*)^{-1} (f'(y) - f'(z)) \| \leq \omega \cdot \| y - z \|, \quad y, z \in D', x^* \in [0, 1]^2 \quad \text{satisfying } f(x^*) = 0,
\]

where

\[
D' = [-\gamma(\theta), 1 + \gamma(\theta)]^2. \tag{6}
\]
Denote $\omega_f$ as the maximum of $\omega_{D'}$ and all $\omega_*(x^*)$

$$\omega_f = \max\{\omega_{D'}, \max_{x^* \in \mathbb{C}^2: f(x^*) = 0} \omega_*(x^*)\}.$$ 

Finally, define the condition number of $f$ to be

$$\text{cond}(f) = \max\{\omega_f, \max_{x^* \in \mathbb{C}^2: f(x^*) = 0, y \in [0, 1]^2} \| f'(x^*)^{-1} f'(y) \| \}.$$  

We define the Kantorovich test on a region $X = B(x^0, r)$ as the application of Kantorovich’s Theorem on the point $x^0$ using $\bar{B}(x^0, 2\gamma(\theta)r)$ as the domain (refer to [1, 4] for the statement of Kantorovich’s Theorem). The region $X$ passes the Kantorovich test if $\eta \omega \leq 1/4$ and $B(x^0, \rho \theta) \subseteq D'$.

The other test KTS uses is the exclusion test. For a given region $X$, let $\hat{f}_X$ be the polynomial in the basis $\phi_i(u) \phi_j(v)$ that reparametrizes with $[l, h]^2$ the surface defined by $f$ over $X$. The region $X$ passes the exclusion test if the bounding polygon of $\{ \hat{f}_X(u, v) : l \leq u, v \leq h \}$ excludes the origin.

Having defined the above prerequisites, the description of KTS can now be given.

**Algorithm KTS:**

- Let $Q$ be a queue with $[0, 1]^2$ as its only entry. Set $S = \emptyset$.
- Repeat until $Q = \emptyset$
  
  1. Let $X$ be the patch at the front of $Q$. Remove $X$ from $Q$.
  2. If $X \not\subseteq X_S$ for all $X_S \in S$,
     - Perform the exclusion test on $X = \bar{B}(x^0, r)$
     - If $X$ fails the exclusion test,
       - (a) Perform the Kantorovich test on $X$
       - (b) If $X$ passes the Kantorovich test,
         - i. Perform Newton’s method starting from $x^0$ to find a zero $x^*$.
         - ii. If $x^* \not\subseteq X_S$ for any $X_S \in S$ (i.e., $x^*$ has not been found previously),
           * Compute $\rho_*(x^*)$ and its associated $\omega_*(x^*)$ by binary search.
Set $S = S \cup \{ \overline{B}(x^*, \rho_*(x^*)) \}$.

(c) Subdivide $X$ along both $u$ and $v$-axes into four equal subregions. Add these subregions to the end of $Q$.

The following theorem shows that the efficiency of KTS has an upper bound that depends only on the conditioning of the problem and the choice of the basis.

**Theorem 2.1.** Let $f(x) = f(u, v)$ be a polynomial system in basis $\phi_i(u)\phi_j(v)$ in two dimensions with generic coefficients whose zeros are sought. Let $X = \overline{B}(x^0, r)$ be a patch under consideration during the course of the KTS algorithm. The algorithm does not need to subdivide $X$ if

$$r \leq \frac{1}{2} \cdot \min \left\{ \frac{1 - 1/\gamma(\theta)}{\omega_{D'}} \cdot \frac{1}{2\theta \text{cond}(f)^2} \right\}.$$  

(8)

**Proof.** See [5].

**Remark:** Both terms in the bound on the right-hand side of (8) are increasing as a function of $1/\theta$. Therefore, our theorem predicts that the KTS algorithm will be more efficient for $\theta$ as small as possible (close to 1).

3 Properties of the power, Bernstein, and Chebyshev bases

As mentioned above, the basis used to represent the polynomial system must satisfy the properties listed in Section 1 for KTS to work efficiently. Three bases, the power, Bernstein, and Chebyshev bases are examined in detail. The power basis for polynomials of degree $n$ is $\phi_k(t) = t^k$ ($0 \leq k \leq n$). The Bernstein basis is $\phi_k(t) = Z_{k,n}(t) = \binom{n}{k} (1-t)^{n-k} t^k$ ($0 \leq k \leq n$). The Chebyshev basis is $\phi_k(t) = T_k(t)$ ($0 \leq k \leq n$), where $T_k(t)$ is the Chebyshev polynomial of the first kind generated by the recurrence relation

$$T_0(t) = 1,$$
$$T_1(t) = t,$$
$$T_{k+1}(t) = 2tT_k(t) - T_{k-1}(t) \text{ for } k \geq 1.$$  

(9)
Another way to define the Chebyshev polynomials of the first kind is through the identity
\[ T_k(\cos \alpha) = \cos k\alpha. \]  
(10)
This second definition shows, in particular, that all zeros of \( T_k(t) \) lies in \([-1, 1]\). It also shows that \(-1 \leq T_k(t) \leq 1\) for any \(-1 \leq t \leq 1\).

The rest of this article shows that the power, Bernstein, and Chebyshev bases all satisfy these basis properties. The values \( \theta \)'s of the three bases and their corresponding bounding polygons are also derived as these values dictate the efficiency of KTS operating on such bases. The upper bound of the efficiency of KTS is lowest when it operates on the basis with the smallest \( \theta \).

3.1 Bounding polygons

The choices of \( l \) and \( h \) and the definitions of bounding polygons of the surface \( S = \{ f(u, v) : l \leq u, v \leq h \} \), where \( f(u, v) \) is represented by one of the three bases, that satisfy the required properties are as follows: For Bernstein basis, the convex hull of the coefficients (control points), call it \( P_1 \), satisfies the requirements for \( l = 0 \) and \( h = 1 \). The convex hull \( P_1 \) can be described as
\[
P_1 = \left\{ \sum_{i,j} c_{ij} s_{ij} : \sum_{i,j} s_{ij} = 1, 0 \leq s_{ij} \leq 1 \right\}.
\]  
(11)
For power and Chebyshev bases, the bounding polygon
\[
P_2 = \left\{ c_{00} + \sum_{i+j>0} c_{ij} s_{ij} : -1 \leq s_{ij} \leq 1 \right\}
\]  
(12)
satisfies the requirements for \( l = -1 \) and \( h = 1 \). Note that \( P_2 \) is a bounding polygon of \( S \) in the Chebyshev case since \( |T_k(t)| \leq 1 \) for any \( k \geq 0 \) and any \( t \in [-1, 1] \). Determining whether \( 0 \in P_2 \) is done by solving a small linear programming problem. To determine if \( 0 \in P_1 \), the convex hull is constructed by conventional method and is tested if it contains the origin.

The affine and translational invariance of \( P_1 \) and \( P_2 \) for their respective bases can be verified as follows: Let
\[
g(u, v) = Af(u, v) + b = \sum_{i=0}^{m} \sum_{j=0}^{n} c_{ij} \phi_i(u) \phi_j(v).
\]
For the Bernstein basis, by using the property that \( \sum_{k=0}^{n} Z_{k,n}(t) = 1 \), it is seen that \( c'_{ij} = Ac_{ij} + b \) for all \( c_{ij} \)'s. Therefore, the bounding polygon of \( \{g(u,v) : 0 \leq u, v \leq 1\} \) is

\[
P_1' = \left\{ \sum_{i,j} c'_{ij}s_{ij} : \sum_{i,j} s_{ij} = 1, 0 \leq s_{ij} \leq 1 \right\}
\]

\[
= \left\{ \sum_{i,j} (Ac_{ij} + b)s_{ij} : \sum_{i,j} s_{ij} = 1, 0 \leq s_{ij} \leq 1 \right\}
\]

\[
= \left\{ A\sum_{i,j} c_{ij}s_{ij} + b\sum_{i,j} s_{ij} : \sum_{i,j} s_{ij} = 1, 0 \leq s_{ij} \leq 1 \right\}
\]

\[
= \left\{ A\sum_{i,j} c_{ij}s_{ij} + b : \sum_{i,j} s_{ij} = 1, 0 \leq s_{ij} \leq 1 \right\}
\]

\[
= \{ Ax + b : x \in P_1 \}.
\]

For the power and the Chebyshev bases, note that \( \phi_0(u)\phi_0(v) = 1 \) for both bases. Hence, \( c'_{00} = Ac_{00} + b \) and \( c'_{ij} = Ac_{ij} \) for \( i + j > 0 \). The bounding polygon of \( \{g(u,v) : 0 \leq u, v \leq 1\} \) for this case is

\[
P_2' = \left\{ c'_{00} + \sum_{i+j>0} c'_{ij}s_{ij} : -1 \leq s_{ij} \leq 1 \right\}
\]

\[
= \left\{ Ac_{00} + b + \sum_{i+j>0} Ac_{ij}s_{ij} : -1 \leq s_{ij} \leq 1 \right\}
\]

\[
= \left\{ A\left( c_{00} + \sum_{i+j>0} c_{ij}s_{ij} \right) + b : -1 \leq s_{ij} \leq 1 \right\}
\]

\[
= \{ Ax + b : x \in P_2 \}.
\]

3.2 The size of the bounding polygons compared to the size of the bounded surface

Item 2c of the basis properties in effect ensures that the bounding polygons are not unboundedly larger than the actual surface itself lest the bounding polygons lose their usefulness. The value \( \theta \) also can be used as a measure
of the tightness of the bounding polygon. Recall from Theorem 2.1 that the efficiency of KTS depends on θ.

Since the bounding polygons $P_1$ and $P_2$ are defined by the coefficients of $f$, our approach to derive $\theta$ is to first derive $\xi$, a function of $m$ and $n$, satisfying

$$
\|c_{ij}\| \leq \xi \max_{t \leq u, v \leq h} \|f(u, v)\|
$$

for any coefficient $c_{ij}$ of $f$. But the following lemma shows that one needs only derive the equivalent of $\xi$ for univariate polynomial to derive $\xi$ itself.

**Lemma 3.1.** Assume there exists a function $h(n)$ such that

$$
\|b_i\| \leq h(n) \max_{t \leq u \leq h} \|g(t)\| 
$$

for any $b_i$ ($i = 0, 1, \ldots, n$), and any univariate polynomial $g(t) = \sum_{i=0}^{n} b_i \phi_i(t)$. Then

$$
\|c_{ij}\| \leq h(m)h(n) \max_{t \leq u, v \leq h} \|f(u, v)\|
$$

for any $c_{ij}$ ($i = 0, 1, \ldots, m; j = 0, 1, \ldots, n$), and any bivariate polynomial $f(u, v) = \sum_{i=0}^{m} \sum_{j=0}^{n} c_{ij} \phi_i(u) \phi_j(v)$.

**Proof.** Let $f(u, v) = \sum_{i=0}^{m} \sum_{j=0}^{n} c_{ij} \phi_i(u) \phi_j(v)$ be an arbitrary bivariate polynomial. For any $i_0 = 0, 1, \ldots, m$, define $g_{i_0}(v) = \sum_{j=0}^{n} c_{i_0,j} \phi_j(v)$. Applying (13) to $g_{i_0}(v)$ yields

$$
\|c_{i_0,j}\| \leq h(n) \max_{t \leq v \leq h} \|g_{i_0}(v)\|
$$

for any $j = 0, 1, \ldots, n$. Let $v_{i_0}^* = \arg \max_{t \leq v \leq h} \|g_{i_0}(v)\|$. Define $l_{i_0}(u) = \sum_{i=0}^{m} \sum_{j=0}^{n} c_{ij} \phi_i(u) \phi_j(v_{i_0}^*)$. Applying (13) to $l_{i_0}(v)$ yields

$$
\left\|\sum_{j=0}^{n} c_{ij} \phi_j(v_{i_0}^*)\right\| \leq h(m) \max_{t \leq u \leq h} \|l_{i_0}(u)\|
$$

for any $i = 0, 1, \ldots, m$. Consequently, by combining (15) and (16),

$$
\|c_{i_0,j}\| \leq h(n) \max_{t \leq v \leq h} \|g_{i_0}(v)\| = h(n) \|g_{i_0}(v_{i_0}^*)\| = h(n) \left\|\sum_{j=0}^{n} c_{i_0,j} \phi_j(v_{i_0}^*)\right\| \leq h(m)h(n) \max_{t \leq u, v \leq h} \|f(u, v)\|.
$$
We proceed to derive $\xi$ of the three bases, starting with the Bernstein basis. The following lemma regarding the product of two polynomials in Bernstein basis is needed to find $\xi$ for Bernstein case.

**Lemma 3.2.** Let

$$f(t) = \sum_{i=0}^{n} c_i Z_{i,n}(t), \quad 0 \leq t \leq 1$$

and

$$g(t) = \sum_{i=0}^{n'} c_{i}' Z_{i,n'}(t), \quad 0 \leq t \leq 1.$$

Then

$$f(t)g(t) = \sum_{i=0}^{n+n'} b_i Z_{i,n+n'}(t),$$

where

$$|b_i| \leq \max_i |c_i| \cdot \max_i |c_{i}'|.$$

**Proof.** Straightforward arithmetic shows that

$$b_i = \sum_{k=\max(0,i-n')}^{\min(n,i)} \binom{n}{k} \binom{n'}{i-k} \frac{c_k c_{i-k}'}{n+n'}.$$ 

Taking absolute value on both sides and bounding $|c_k|$ (resp. $|c_{i-k}'|$) with $\max_i |c_i|$ (resp. $\max_i |c_{i}'|$) gives

$$|b_i| \leq \max_i |c_i| \cdot \max_i |c_{i}'| \sum_{k=\max(0,i-n')}^{\min(n,i)} \frac{\binom{n}{k} \binom{n'}{i-k}}{\binom{n+n'}{i}}.$$

Recall the combinatorial identity

$$\binom{n+n'}{i} = \sum_{k=\max(0,i-n')}^{\min(n,i)} \binom{n}{k} \binom{n'}{i-k}.$$ 

Hence, the lemma follows. \qed
With the above lemma, we are ready to derive $\xi$ of the Bernstein basis.

**Theorem 3.3.** Let $f(t)$ be a polynomial system

$$f(t) = \sum_{i=0}^{n} c_i Z_{i,n}(t), \ 0 \leq t \leq 1,$$

where $c_i \in \mathbb{R}^d$. The norm of the coefficients can be bounded by

$$\|c_i\| \leq \xi_B(n) \max_{0 \leq t \leq 1} \|f(t)\|,$$

(17)

where

$$\xi_B(n) = \sum_{i=0}^{n} \prod_{j=0,\ldots,i-1,i+1,\ldots,n} \frac{\max\{|n-j|,|j|\}}{|i-j|} = O(n^{n+1}).$$

**Remark.** An inequality in the other direction, namely, that

$$\max_{0 \leq t \leq 1} \|f(t)\| \leq \max \|c_i\|,$$

is a well-known consequence of the convex hull property of Bernstein polynomials [2].

**Proof.** By definition of infinity norm, it suffices to prove the lemma for the case $c_i \in \mathbb{R}$. Therefore, it is assumed that $d = 1$ for the rest of this proof.

Let $t_j = j/n \ (j = 0, 1, \ldots, n)$. Define a matrix $A \in \mathbb{R}^{(n+1) \times (n+1)}$ having element

$$A_{j+1,i+1} = Z_{i,n}(t_j).$$

Define the vectors $c = (c_0, c_1, \ldots, c_n)^T$ and $f = (f(t_0), f(t_1), \ldots, f(t_n))^T$. Observe that

$$Ac = f. \quad \text{(18)}$$

We claim that $A$ is invertible. In particular, we show that the linear system $Ax = b$ has solution for any arbitrary $b \in \mathbb{R}^{n+1}$. Due to the definition of $A$, solving the system $Ax = b$ is equivalent to finding the coefficients of the polynomial

$$g(t) = \sum_{i=0}^{n} x_{i+1} Z_{i,n}(t) \quad \text{(19)}$$
with the property that \( g(t_0) = b_1, g(t_1) = b_2, \ldots, g(t_n) = b_{n+1} \). The polynomial \( g \) satisfying such property is the Lagrange interpolant

\[
g(t) = \sum_{j=0}^{n} \left( b_{j+1} \prod_{j'=0,\ldots,j-1,j+1,\ldots,n} \frac{t-t_{j'}}{t_{j}-t_{j'}} \right). \tag{20}
\]

Transforming (20) to the Bernstein basis yields the solution \( x \).

Knowing that \( A \) is invertible, we multiply both sides of (18) by \( A^{-1} \),

\[
c = A^{-1} f, \tag{21}
\]

and hence, for any \( i = 0, 1, \ldots, n \),

\[
|c_i| \leq \|c\| \leq \|A^{-1}\| \cdot \|f\| \leq \|A^{-1}\| \cdot \max_{0 \leq t \leq 1} |f(t)|. \tag{22}
\]

Comparing (17) to (22), it is seen that the final step is to show that \( \|A^{-1}\| \leq \xi_B(n) \).

Observe that the \( i \)th column of \( A^{-1} \) is \( A^{-1} e_i \), where \( e_i \) denotes the \( i \)th column of the identity matrix. Let \( g_i(t) \) be a polynomial in the Bernstein basis and let \( \{c'_{i'}\} \) be its coefficients. With similar reasoning as the above,

\[
\begin{pmatrix}
  c'_0 \\
  \vdots \\
  c'_n
\end{pmatrix}
= A^{-1}
\begin{pmatrix}
  g_i(t_0) \\
  \vdots \\
  g_i(t_n)
\end{pmatrix}.
\tag{23}
\]

But (23) implies that the \( i \)th column of \( A^{-1} \), \( A^{-1} e_i \), are the coefficients of \( g_i \) such that, for \( j = 0, 1, \ldots, n \),

\[
g_i(t_j) = \begin{cases}
  1, & j = i, \\
  0, & j \neq i.
\end{cases}
\tag{24}
\]

The following Lagrange interpolant \( g_i \) satisfies (24):

\[
g_i(t) = \prod_{j=0,\ldots,i-1,i+1,\ldots,n} \frac{t-t_j}{t_i-t_j} = \prod_{j=0,\ldots,i-1,i+1,\ldots,n} \left( \frac{n-j}{i-j} t - \frac{j}{i-j} (1-t) \right). \tag{25}
\]
Note that each term of the product in (25) is a polynomial in Bernstein basis with coefficients \((n - j)/(i - j)\) and \(j/(i - j)\). Applying Lemma 3.2 to (25) shows that

\[
\|A^{-1}c_i\| \leq \prod_{j=0,1,...,i-1,i+1,...,n} \max\{|n-j|,|j|\}/|i-j|.
\] (26)

Since (26) holds for any column \(i\) of \(A^{-1}\), the lemma follows. \(\square\)

Next is the derivation of \(\xi\) of the Chebyshev basis. The following identity is useful for this derivation:

\[
\sum_{k=1}^{n} T_i(t_k)T_j(t_k) = \begin{cases} 0 & i \neq j \\ n & i = j = 0 \\ n/2 & i = j \neq 0, \end{cases}
\] (27)

for \(i, j = 0, \ldots, n - 1\), where \(t_k (k = 1, 2, \ldots, n)\) are the \(n\) zeros of \(T_n(t)\).

**Theorem 3.4.** Let \(f(t)\) be a polynomial system

\[
f(t) = \sum_{i=0}^{n} c_i T_i(t),
\]

where \(c_i \in \mathbb{R}^d\). The norm of the coefficients can be bounded by

\[
\|c_i\| \leq \sqrt{2} \max_{t: -1 \leq t \leq 1} \|f(t)\|.
\] (28)

**Proof.** By definition of infinity norm, it suffices to prove the lemma for the case \(c_i \in \mathbb{R}\). Therefore, it is assumed that \(d = 1\) for the rest of this proof.

Let \(t_j (j = 1, 2, \ldots, n + 1)\) be the \(n + 1\) zeros of \(T_{n+1}(t)\), which lie in \([-1, 1]\). Define a matrix \(A \in \mathbb{R}^{(n+1)\times(n+1)}\) having element

\[
A_{j+1,i+1} = T_i(t_j).
\]

Define the vectors \(c = (c_0, c_1, \ldots, c_n)^T\) and \(f = (f(t_0), f(t_1), \ldots, f(t_n))^T\). Observe that

\[
Ac = f.
\] (29)

By (27),

\[
A^T A = \text{diag} (n + 1, (n + 1)/2, (n + 1)/2, \ldots, (n + 1)/2),
\]
which implies that $A$ is invertible and

$$A^{-1}A^{-T} = \text{diag}(1/(n + 1), 2/(n + 1), 2/(n + 1), \ldots, 2/(n + 1)).$$  \hspace{1cm} (30)

The equation (30) implies

$$\|A^{-1}\|_2 = \sqrt{2/(n + 1)}.$$  \hspace{1cm} (31)

Finally, from (29) and (31),

$$|c_i| \leq \|c\|_2 \leq \|A^{-1}\|_2 \|f\|_2 \leq \sqrt{n + 1} \|A^{-1}\|_2 \|f\| \leq \sqrt{n + 1} \|A^{-1}\|_2 \max_{-1 \leq t \leq 1} |f(t)| = \sqrt{2} \max_{-1 \leq t \leq 1} |f(t)|. \hspace{1cm} \square$$

Last is the power basis. Our approach to derive $\xi$ of power basis is to derive the relationship between the coefficients of a polynomial in power basis and the coefficients of the same polynomial but written in Chebyshev basis. By using this relationship and Theorem 3.4, $\xi$ of the power basis can be computed.

**Lemma 3.5.** Let $f$ be a univariate polynomial such that

$$f(t) = \sum_{i=0}^{n} a_i t^i = \sum_{i=0}^{n} c_i T_i(t).$$

In other words, $\{a_i\}$ are the coefficients of $f$ when written in the power basis and $\{c_i\}$ are the coefficients of $f$ when written in the Chebyshev basis. Then

$$|a_i| \leq \frac{3^{n+1} - 1}{2} \max_{j=0,\ldots,n} |c_j|,$$

for any $i = 0, \ldots, n$.

**Proof.** Let $D = [d_{i,j}]$ be the $n + 1$-by-$n + 1$ matrix such that

$$a = Dc,$$
where \( a = (a_0, a_1, \ldots, a_n)^T \) and \( c = (c_0, c_1, \ldots, c_n)^T \). Note that

\[
T_j(t) = \sum_{i=0}^{j} d_{i+1,j+1} t^i.
\]

Recall the recurrence relation \( T_j(t) = 2tT_{j-1}(t) - T_{j-2}(t) \). It follows from this recurrence that

\[
|d_{i+1,j+1}| \leq 3^j.
\]  \hspace{1cm} (32)

That is, when \( T_j(t) \) is written in the power basis, the resulting coefficients (of power basis) is less than or equal to \( 3^j \). The inequality (32) can be verified by induction on the recurrence relation. Since the entries in the \((j + 1)\)th column of \( D \) is bounded by \( 3^j \), we have, from geometric sum,

\[
\|D\| \leq (3^{n+1} - 1)/2.
\]

The lemma follows from \( \|a\| \leq \|D\| \|c\| \). \hfill \Box

**Theorem 3.6.** Let \( f(t) \) be a polynomial system

\[
f(t) = \sum_{i=0}^{n} c_i t^i,
\]

where \( c_i \in \mathbb{R}^d \). The norm of the coefficients can be bounded by

\[
\|c_i\| \leq \frac{3^{n+1} - 1}{\sqrt{2}} \max_{-1 \leq t \leq 1} \|f(t)\|.
\]  \hspace{1cm} (33)

**Proof.** Follow directly from Theorem 3.4 and Lemma 3.5. \hfill \Box

Having \( \xi \) for each of the three bases, the values of \( \theta \) for the three bases can now be derived.

**Corollary 3.7.** Let

\[
f(u, v) = \sum_{i=0}^{m} \sum_{j=0}^{n} c_{ij} Z_{i,m}(u) Z_{j,n}(v),
\]

where \( c_{ij} \in \mathbb{R}^2 \) (\( i = 0, 1, \ldots, m; j = 0, 1, \ldots, n \)). Let \( P_1 \) be the convex hull of \( \{c_{ij}\} \). Then, for any \( y \in P_1 \),

\[
\|y\| \leq \xi_{B}(m) \xi_{B}(n) \max_{0 \leq u,v \leq 1} \|f(u, v)\|.
\]
Proof. By the convex hull property of Bernstein polynomials, \( \|y\| \leq \max_{i,j} \|c_{ij}\| \) for any \( y \in P_1 \). The corollary then follows from Theorem 3.3 and Lemma 3.1.

Corollary 3.8. Let

\[
f(u, v) = \sum_{i=0}^{m} \sum_{j=0}^{n} c_{ij} u^i v^j,
\]

where \( c_{ij} \in \mathbb{R}^2 \) \((i = 0, 1, \ldots, m; j = 0, 1, \ldots, n)\). Let

\[
P_2 = \left\{ c_{00} + \sum_{i+j>0} c_{ij} s_{ij} : -1 \leq s_{ij} \leq 1 \right\}.
\]

Then, for any \( y \in P_2 \),

\[
\|y\| \leq \frac{(m+1)(n+1)(3^{m+1}-1)(3^{n+1}-1)}{2} \max_{-1 \leq u, v \leq 1} \|f(u, v)\|.
\]

Proof. For any \( y \in P_2 \),

\[
\|y\| \leq \sum_{i=0}^{m} \sum_{j=0}^{n} \|c_{ij}\|.
\]

The corollary then follows from Theorem 3.6 and Lemma 3.1.

Corollary 3.9. Let

\[
f(u, v) = \sum_{i=0}^{m} \sum_{j=0}^{n} c_{ij} T_i(u) T_j(v),
\]

where \( c_{ij} \in \mathbb{R}^2 \) \((i = 0, 1, \ldots, m; j = 0, 1, \ldots, n)\). Let

\[
P_2 = \left\{ c_{00} + \sum_{i+j>0} c_{ij} s_{ij} : -1 \leq s_{ij} \leq 1 \right\}.
\]

Then, for any \( y \in P_2 \),

\[
\|y\| \leq 2(m+1)(n+1) \max_{-1 \leq u, v \leq 1} \|f(u, v)\|.
\]

Proof. For any \( y \in P_2 \),

\[
\|y\| \leq \sum_{i=0}^{m} \sum_{j=0}^{n} \|c_{ij}\|.
\]

The corollary then follows from Theorem 3.4 and Lemma 3.1.
Let $P_p^2$ denote the bounding polygon $P_2$ computed from the power basis representation of a polynomial and $P_c^2$ denote $P_2$ computed from the Chebyshev basis representation of it. The results from previous section show that the value $\theta$ of $P_c^2$ is smaller than $\theta$ of $P_p^2$. This only implies that the worst case of $P_c^2$ is better than the worst case of $P_p^2$. Comparing the values of $\theta$’s of the two does not indicate that $P_c^2$ is always a better choice than $P_p^2$ for every polynomial. The following results show, however, that for any given polynomial, its bounding polygon $P_c^2$ is a subset of its bounding polygon $P_p^2$. Specifically, this section shows that for any given polynomial, its bounding polygon $P_c^2$ is a subset of its bounding polygon $P_p^2$.

The following two lemmas show that when representing monomials $t^k$ in Chebyshev basis, each coefficient is nonnegative, and the sum of all coefficients are exactly 1. These results are useful in relating $P_p^2$ to $P_c^2$.

Lemma 4.1. Let $d_{ki}$’s ($k = 0, 1, \ldots; i = 0, 1, \ldots, k$) be the numbers satisfying $t^k = \sum_{i=0}^{k} d_{ki} T_i(t)$. Then

$$d_{ki} \geq 0,$$

for any $k = 0, 1, \ldots$ and any $i = 0, 1, \ldots, k$.

Proof. We prove the lemma by induction on $k$. The base cases $k = 0$ and $k = 1$ are trivial. For the inductive step, for any $k \geq 1$,

$$t^{k+1} = t \cdot t^k = t \sum_{i=0}^{k} d_{ki} T_i(t) = \sum_{i=0}^{k} \frac{d_{ki}}{2} (2tT_i(t) - T_{i-1}(t)) + \sum_{i=1}^{k} \frac{d_{ki}}{2} T_{i-1}(t) + d_{k0} t T_0(t).$$

By (9) and noting that $t T_0(t) = t = T_1(t)$,

$$t^{k+1} = \sum_{i=1}^{k} \frac{d_{ki}}{2} T_{i+1}(t) + \sum_{i=1}^{k} \frac{d_{ki}}{2} T_{i-1}(t) + d_{k0} T_1(t) = \sum_{i=k}^{k+1} \frac{d_{k,i-1}}{2} T_i(t) + \sum_{i=2}^{k-1} \frac{d_{k,i-1} + d_{k,i+1}}{2} T_i(t) + \left( \frac{d_{k2}}{2} + d_{k0} \right) T_1(t) + \frac{d_{k1}}{2} T_0(t).$$
Hence,
\[
d_{k+1,i} = \begin{cases} 
    d_{k,i-1}/2, & i = k, k + 1, \\
    (d_{k,i-1} + d_{k,i+1})/2, & i = 2, \ldots, k - 1, \\
    d_{k2}/2 + d_{k0}, & i = 1, \\
    d_{k1}/2, & i = 0.
\end{cases}
\] (34)

But since \(d_{ki} \geq 0\) for any \(i = 0, \ldots, k\) by the induction hypothesis, (34) shows that \(d_{k+1,i} \geq 0\) for any \(i = 0, \ldots, k + 1\).

**Lemma 4.2.** Let \(d_{ki}\)'s \((k = 0, 1, \ldots; i = 0, 1, \ldots, k)\) be the numbers satisfying \(t^k = \sum_{i=0}^{k} d_{ki}T_i(t)\). Then
\[
\sum_{i=0}^{k} d_{ki} = 1,
\]
for any \(k = 0, 1, \ldots\) and any \(i = 0, 1, \ldots, k\).

**Proof.** We prove the lemma by induction on \(k\). The base cases \(k = 0\) and \(k = 1\) are trivial. For the inductive step, the same reasoning as in the proof of Lemma 4.1 shows that \(d_{k+1,i}\) is as in (34) for any \(k \geq 1\). Therefore,
\[
\sum_{i=0}^{k+1} d_{k+1,i} = \sum_{i=k}^{k+1} \frac{d_{k,i-1}}{2} + \sum_{i=2}^{k-1} \frac{d_{k,i-1} + d_{k,i+1}}{2} + \left(\frac{d_{k2}}{2} + d_{k0}\right) + \frac{d_{k1}}{2}
\]
\[
= \sum_{i=0}^{k} d_{ki} = 1,
\]
by the induction hypothesis. \(\square\)

**Theorem 4.3.** Let \(f : \mathbb{R}^2 \rightarrow \mathbb{R}^2\) be a bivariate polynomial. Its bounding polygon \(P^c_2\) is a subset of its bounding polygon \(P^p_2\).

**Proof.** Let \(f = f(u, v) = \sum_{k=0}^{m} \sum_{l=0}^{n} a_{kl} u^k v^l = \sum_{i=0}^{m} \sum_{j=0}^{n} c_{ij} T_i(u) T_j(v)\), where \(a_{kl} \in \mathbb{R}^n\) \((k = 0, 1, \ldots, m; l = 0, 1, \ldots, n)\) are the coefficients of \(f\) when written in the power basis and \(c_{ij} \in \mathbb{R}^n\) \((i = 0, 1, \ldots, m; j = 0, 1, \ldots, n)\) are the coefficients of \(f\) when written in the Chebyshev basis. Let \(d_{ki}\)'s \((k = 0, 1, \ldots; i = 0, 1, \ldots, k)\) be the numbers satisfying \(t^k = \sum_{i=0}^{k} d_{ki}T_i(t)\).
Hence,
\[
f(u, v) = \sum_{k=0}^{m} \sum_{l=0}^{n} a_{kl} \left( \sum_{i=0}^{k} d_{ki} T_i(u) \right) \left( \sum_{j=0}^{l} d_{lj} T_j(v) \right)
\]
\[
= \sum_{i=0}^{m} \sum_{j=0}^{n} \sum_{k=0}^{m} \sum_{l=0}^{n} a_{kl} d_{ki} d_{lj} T_i(u) T_j(v).
\]

Therefore, \( c_{ij} = \sum_{k=i}^{m} \sum_{l=j}^{n} a_{kl} d_{ki} d_{lj} \). This means that \( P^c_2 \) can be written as
\[
P^c_2 = \left\{ a_{00} d_{00} + \sum_{k=1}^{m} \sum_{l=1}^{n} a_{kl} d_{k0} d_{l0} + \sum_{i=0}^{m} \sum_{j=0}^{n} a_{kl} d_{kj} d_{lj} s_{ij} : -1 \leq s_{ij} \leq 1 \right\},
\]

But since \( d_{00} = 1 \),
\[
P^c_2 = \left\{ a_{00} + \sum_{k=1}^{m} \sum_{l=1}^{n} a_{kl} d_{k0} d_{l0} + \sum_{i=0}^{m} \sum_{j=0}^{n} a_{kl} d_{kj} d_{lj} s_{ij} : -1 \leq s_{ij} \leq 1 \right\}
\]
\[
= \left\{ a_{00} + \sum_{k=1}^{m} \sum_{l=1}^{n} a_{kl} d_{k0} d_{l0} + \sum_{i=0}^{m} \sum_{j=0}^{n} a_{kl} d_{kj} d_{lj} s_{ij} : -1 \leq s_{ij} \leq 1 \right\}
\]
\[
= \left\{ a_{00} + \sum_{l=1}^{n} a_{0l} \sum_{j=1}^{l} d_{lj} s_{0j} + \sum_{k=0}^{m} a_{k0} \sum_{i=1}^{k} d_{ki} s_{i0} + \sum_{k=1}^{m} \sum_{l=1}^{n} a_{kl} \left( d_{k0} d_{l0} + \sum_{j=1}^{l} d_{ij} s_{0j} + \sum_{i=1}^{k} d_{ki} s_{i0} + \sum_{i=1}^{k} \sum_{j=1}^{l} d_{ij} s_{ij} \right) : -1 \leq s_{ij} \leq 1 \right\}.
\]

By Lemma 4.1 and Lemma 4.2 it is seen that \(-1 \leq \sum_{j=1}^{l} d_{lj} s_{0j} ; \sum_{i=1}^{k} d_{ki} s_{i0} \leq 1 \). In addition, using the fact that \( |s_{ij}| \leq 1 \), for any \( i = 0, \ldots, m \) and any
\[ j = 0, \ldots, n, \text{ together with Lemma 4.1 and Lemma 4.2, it is seen that} \]
\[
\left| d_{k0}d_{l0} + d_{k0} \sum_{j=1}^{l} d_{ij}s_{0j} + d_{l0} \sum_{i=1}^{k} d_{ki}s_{0i} + \sum_{i=1}^{k} d_{ki} \sum_{j=1}^{l} d_{ij}s_{ij} \right| \leq |d_{k0}d_{l0}| + |d_{k0}| \sum_{j=1}^{l} |d_{ij}| + |d_{l0}| \sum_{i=1}^{k} |d_{ki}| + \sum_{i=1}^{k} |d_{ki}| \sum_{j=1}^{l} |d_{ij}| = d_{k0}d_{l0} + d_{k0} \sum_{j=1}^{l} d_{ij} + d_{l0} \sum_{i=1}^{k} d_{ki} + \sum_{i=1}^{k} d_{ki} \sum_{j=1}^{l} d_{ij} = \left( \sum_{i=0}^{k} d_{ki} \right) \left( \sum_{j=0}^{l} d_{ij} \right) = 1. \text{ Therefore,} \]
\[ P^c_{2} \subseteq P^p_{2}. \]

4.1 Reparametrization

The last nontrivial basis property that warrants detailed discussion is the issue of efficient reparametrization. Reparametrizing polynomials in power basis is straightforward from the binomial theorem. Polynomials in other bases, on the other hand, may not be as simple to reparametrize. The details of the process for polynomials in Bernstein and Chebyshev bases are covered in this section.

4.1.1 Reparametrization of polynomials in Bernstein basis

There is more than one algorithm to compute the reparametrization with \([0, 1]^2\) of a bivariate polynomial in Bernstein basis. We describe one method here. Our method makes use of a program that, given \(\alpha_{ij}\)'s, \(c, d, e, f, g, h, k, \) and \(l\), computes \(\beta_{ij}\)'s satisfying

\[
\sum_{i=0}^{m} \sum_{j=0}^{n} \alpha_{ij}(cy + d)^i(ey + f)^{m-i}(gz + h)^j(kz + l)^{n-j} = \sum_{i=0}^{m} \sum_{j=0}^{n} \beta_{ij}y^i z^j.
\]

Such conversion can be done in \(O((mn)^2)\) by generalizing Horner’s rule. We leave the details of the conversion to the reader. Let \(X\) denote \{(\(u, v\) : \(u^0 - r \leq u \leq u^0 + r, v^0 - r \leq v \leq v^0 + r\)}. To compute the coefficients \(\{\hat{c}_{ij}\}\) of \(\{\hat{f}_X(\hat{u}, \hat{v}) : 0 \leq \hat{u}, \hat{v} \leq 1\}\), the \([0, 1]^2\)-reparametrized surface of \(\{f(\(u, v\) : \(u^0 - r \leq u \leq u^0 + r, v^0 - r \leq v \leq v^0 + r\)), first substitute \(u = 2r\hat{u} + u^0 - r\)

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and \( v = 2r\hat{v} + v^0 - r \) into \( f \), yielding

\[
f(u, v) = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} c_{ij} u^i (1 - u)^{m-i} v^j (1 - v)^{n-j}.
\]

\[
= \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} c_{ij} (2r\hat{u} + u^0 - r)^i (1 - (2r\hat{u} + u^0 - r))^{m-i}.
\]

\[
= \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} c_{ij} (2r\hat{v} + v^0 - r)^j (1 - (2r\hat{v} + v^0 - r))^{n-j}.
\]

(35)

Substituting \( \hat{u} = \tilde{u}/(\tilde{u} + 1) \) and \( \hat{v} = \tilde{v}/(\tilde{v} + 1) \) into (35) yields

\[
f(u, v) = \frac{1}{(\tilde{u} + 1)^m(\tilde{v} + 1)^n} \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} c_{ij} ((u^0 + r)\tilde{u} + u^0 - r)^i.
\]

\[
= \frac{1}{(\tilde{u} + 1)^m(\tilde{v} + 1)^n} \sum_{i=0}^{m} \sum_{j=0}^{n} \gamma_{ij} \hat{u}^i \hat{v}^j.
\]

(37)

where (37) is obtained from (35) by the conversion program mentioned above.

Substituting \( \hat{u} \) and \( \hat{v} \) back into (37) to see that

\[
f(u, v) = \sum_{i=0}^{m} \sum_{j=0}^{n} \gamma_{ij} \hat{u}^i \hat{v}^j(1 - \hat{u})^{m-i}\hat{v}^j(1 - \hat{v})^{n-j}.
\]

(38)

Therefore, \( \hat{c}_{ij} = \gamma_{ij}/(C(m, i)C(n, j)) \) are the control points of \( \hat{f}_X \) where

\[
C(m, i) = \binom{m}{i}.
\]
4.1.2 Reparametrization of polynomials in Chebyshev basis

Let $a, b, d$ and $e$ be scalar constants. The reparametrization with $[-1, 1]^2$ of a bivariate polynomial in Chebyshev basis can be computed if the values of $\lambda_{ik}$’s ($i = 0, 1, \ldots, m$) satisfying

$$T_i(at + b) = \sum_{k=0}^{i} \lambda_{ik} T_k(t)$$

and the values of $\mu_{jk}$’s ($j = 0, 1, \ldots, n$) satisfying

$$T_j(dt + e) = \sum_{k=0}^{j} \mu_{jk} T_k(t)$$

are known. Note that

$$T_i(au + b)T_j(dv + e) = \sum_{k=0}^{i} \sum_{k'=0}^{j} \lambda_{ik}\mu_{jk'}T_k(u)T_{k'}(v),$$

which is adequate to find the reparametrization. The values of $a$, $b$, $d$, and $e$ are determined by the $uv$-domain of the surface to be reparametrized.

To compute $\lambda_{ik}$’s, observe that for $i \geq 1$, by (9),

$$T_{i+1}(at + b) = 2(at + b)T_i(at + b) - T_{i-1}(at + b)$$

$$= 2(at + b) \sum_{k=0}^{i} \lambda_{ik} T_k(t) - \sum_{k=0}^{i-1} \lambda_{i-1,k} T_k(t)$$

$$= \sum_{k=0}^{i} 2a \lambda_{ik} T_k(t) + \sum_{k=0}^{i} 2b \lambda_{ik} T_k(t) - \sum_{k=0}^{i-1} \lambda_{i-1,k} T_k(t)$$

$$= 2a \lambda_{i0} T_0(t) + \sum_{k=1}^{i} a \lambda_{ik} (2tT_k(t) - T_{k-1}(t)) +$$

$$\sum_{k=0}^{i-1} a \lambda_{i,k+1} T_k(t) + \sum_{k=0}^{i} 2b \lambda_{ik} T_k(t) - \sum_{k=0}^{i-1} \lambda_{i-1,k} T_k(t)$$

$$= 2a \lambda_{i0} T_1(t) + \sum_{k=1}^{i} a \lambda_{ik} T_{k+1}(t) +$$

$$\sum_{k=0}^{i-1} a \lambda_{i,k+1} T_k(t) + \sum_{k=0}^{i} 2b \lambda_{ik} T_k(t) - \sum_{k=0}^{i-1} \lambda_{i-1,k} T_k(t), (39)$$

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and

\[ T_0(at + b) = T_0(t), \quad (40) \]
\[ T_1(at + b) = aT_1(t) + bT_0(t). \quad (41) \]

The equalities (39), (40), and (41) yield a recurrence relation of \( \lambda_{jk} \)'s that can be used to compute their values. The values of \( \mu_{jk} \)'s can be computed similarly.

## 5 Computational results

Three versions of KTS algorithms are implemented in Matlab; one operating on the polynomials in power basis, one on Bernstein basis, and one on Chebyshev basis. They are tested against a number of problem instances with varying condition numbers. Most of the test problems are created by using the normally distributed random numbers as the coefficients \( c_{ij} \)'s of \( f \) in Chebyshev basis. For some of the test problems especially those with high condition number, some coefficients are manually entered. The resulting Chebyshev polynomial system is then transformed to the equivalent system in the power and the Bernstein bases. Hence the three versions of KTS solve the same polynomial system and the efficiency of the three are compared.

The degrees of the test polynomials are between biquadratic and biquartic.

For the experiment, we use the algorithm by Jónsson and Vavasis [3] to compute the complex zeros required to estimate the condition number. Table 1 compares the efficiency of the three versions of KTS for the test problems with differing condition numbers. The total number of subpatches examined by KTS during the entire computation and the width of the smallest patch among those examined are reported. The results do not show any one version to be particularly more efficient than the others although the Chebyshev basis has better theoretical bound than the other two.

Since the types of test polynomials may affect the relative efficiency of the three versions of KTS, another experiment is performed on degree 6 univariate polynomials generated by different methods. Since Section 4 shows that the Chebyshev basis always gives tighter bounding polygons than the power basis, this experiment only compares between the Chebyshev and Bernstein bases. Table 2 and Table 3 show the results of this experiment. The polynomials are generated as follows. The “rand” polynomials are generated by interpolating points whose x-coordinates are evenly spaced between \(-1\) and
Table 1: Comparison of the efficiency of KTS algorithm operating on the power, the Bernstein, and the Chebyshev bases. The number of patches examined during the course of the algorithm and the width of the smallest patch examined are shown for each version of KTS.

| $\text{cond}(f)$ | Power basis | Bernstein basis | Chebyshev basis |
|------------------|-------------|-----------------|-----------------|
|                  | Num. of patches | Smallest width | Num. of patches | Smallest width | Num. of patches | Smallest width |
| $3.8 \times 10^3$ | 29 | .125 | 17 | .0625 | 21 | .125 |
| $1.3 \times 10^4$ | 13 | .125 | 17 | .0625 | 13 | .125 |
| $2.5 \times 10^5$ | 49 | .0625 | 21 | .0625 | 45 | .0625 |
| $1.1 \times 10^6$ | 97 | .0313 | 65 | .0313 | 85 | .0313 |
| $3.9 \times 10^7$ | 89 | .0313 | 81 | .0313 | 89 | .0313 |

1, inclusive, and whose y-coordinates are normally distributed random numbers. The “sin” ones are interpolations of $\sin(ax + b)$ at evenly spaced points between $-1$ and 1, inclusive, where $a$ and $b$ are normally distributed random numbers. The “sin-L” ones are the same as the “sin” ones except that least-squares interpolation is used instead. The “sinw” (resp. “sinw-L”) ones are generated in the same way as the “sin” (resp. “sin-L”) ones but with the function $\sin(6ax + b)$. Table 2 compares the number of test polynomials of each type where one basis yields tighter bounding intervals than the other. Table 3 shows the number of test polynomials of each type that bounding intervals of each basis have at least one endpoint exactly at the boundary of the ranges of the polynomials. The results show that the Chebyshev basis is decidedly better for “rand” polynomials, is about the same for “sin” ones, but is worse for the rests of the polynomials than the Bernstein basis.

6 Conclusion

Three common bases, the power, the Bernstein, and the Chebyshev bases, are shown to satisfy the required properties for KTS to perform efficiently. In particular, the values of $\theta$ for the three bases are derived. These values are used to calculate the time complexity of KTS when that basis is used to represent the polynomial system. The Chebyshev basis has the smallest $\theta$.
Table 2: The numbers of test polynomials out of 1000 that bounding intervals associated with the Bernstein basis is tighter than the those associated with the Chebyshev basis, and *vice versa*.

| Poly. type | Num. that Bernstein is tighter | Num. that Chebyshev is tighter |
|------------|-------------------------------|-------------------------------|
| rand       | 1                             | 999                           |
| sin        | 963                           | 37                            |
| sin-L      | 960                           | 40                            |
| sinw       | 436                           | 564                           |
| sinw-L     | 998                           | 2                             |

Table 3: The numbers of test polynomials out of 1000 that bounding intervals associated with the Bernstein basis and those associated with the Chebyshev basis having at least one endpoint exactly at the boundary of the ranges of the polynomials.

| Poly. type | Num. Bernstein with exact endpoint | Num. Chebyshev with exact endpoint |
|------------|------------------------------------|------------------------------------|
| rand       | 2                                  | 13                                 |
| sin        | 965                                | 0                                  |
| sin-L      | 972                                | 0                                  |
| sinw       | 330                                | 0                                  |
| sinw-L     | 999                                | 658                                |
among the three, which shows that using KTS with the Chebyshev basis has the smallest worst-case time complexity. The computational results, however, show no significant differences between the performances of the three versions of KTS operating on the three bases. It appears that, in average case, choosing any of the three bases do not greatly affect the efficiency of KTS. The experiment on univariate polynomials show that the Bernstein basis is more suitable for certain types of polynomials while the Chebyshev basis is better suited for other types.

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