Some remarks on bielliptic and trigonal curves

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Abstract

We prove some results on algebraic curves $X$ of genus $g \geq 2$ in characteristic 0. For example: Assume that $X$ has an automorphism $\sigma$ of prime order $p \geq 5$. If $\sigma$ has no fixed points, then $X$ cannot be trigonal. On the other hand, if $\sigma$ has fixed points, then $X$ is bielliptic only if it belongs to one of three extremal types of curves of small genus.

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1. Introduction and basic facts

Unless said otherwise, in this paper curve means a connected, smooth, projective, algebraic curve over the complex numbers. Equivalently, one can think of a compact Riemann surface.

In [M], using the uniformization of compact Riemann surfaces of genus $g \geq 2$ by Fuchsian groups, Maclachlan proved, among other results, the following theorem.

**Theorem 1.1. (Maclachlan) [M, Theorem 2]** Let $X$ be a hyperelliptic curve and $G$ a subgroup of $\text{Aut}(X)$ such that the covering $X \to X/G$ is totally unramified. Then $|G|$ divides 4 and $G$ has exponent 2.

We first give a short and elementary proof of this theorem. Then we work out similar and related results for other classes of curves, notably for bielliptic and trigonal curves.

**Proof.** Denote the genus of $X$ by $g$. The hyperelliptic involution $\tau$ lies in the center of $\text{Aut}(X)$. So $G$ acts on the $2g + 2$ fixed points of $\tau$, and as the action of $G$ is without fixed points, $|G|$ divides $2g + 2$. On the other hand, since the covering is

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unramified, the Hurwitz formula implies that $|G|$ divides $2g - 2$. Subtraction already shows that $|G|$ divides 4.

With an eye towards generalization, we prove the second claim independently of the first by counting the ramified points on $X$ of the covering $X \to X/\langle \tau, G \rangle$.

The subcover $X/\langle G \rangle \to X/\langle \tau, G \rangle$ is the hyperelliptic one and has $2h + 2$ ramified points, where $h$ is the genus of $X/\langle G \rangle$. They all split completely in $X$, which gives $|G|(2h + 2) = 2g + 2 + 4|G| - 4$ ramified points, each with ramification index 2.

Taking the $2g - 2$ ramified points of $X \to X/\langle \tau \rangle$ into account, there must still be $2|G| - 2$ ramified points in $X/\langle \tau \rangle \to X/\langle \tau, G \rangle$, each with ramification index 2 and splitting into 2 points in $X$. Note that the covering $X/\langle \tau \rangle \to X/\langle \tau, G \rangle$ is Galois with group $\tilde{G}$ induced by (and isomorphic to) $G$, but not necessarily fixed-point-free. The stabilizer in $\tilde{G}$ of each of these $2|G| - 2$ points on $X/\langle \tau \rangle$ is an involution. Since $X/\langle \tau \rangle$ has genus 0, each such involution has exactly 2 fixed points. So $\tilde{G}$ (and hence $G$) must have at least $|G| - 1$ involutions, i.e. every non-trivial element is an involution.

Maclachlan’s theorem says that hyperelliptic curves cannot have automorphisms of certain types, for example fixed-point-free of prime order $p \geq 3$. In the sequel we want to generalize this to other types of curves.

On the other hand, it is clear that the curve $Y = X/G$ in Theorem 1.1 must also be hyperelliptic. So another interpretation is that certain coverings $X$ of a hyperelliptic curve $Y$ cannot be hyperelliptic. In this form we also want to generalize it, keeping the curve $Y$ hyperelliptic, but taking other types of curves $X$.

One generalization of a hyperelliptic involution is the notion of a $b$-hyperelliptic involution, that is, an involution such that the genus of the quotient curve is $b$. We concentrate on the case $b = 1$; then the involution and the curve are called bielliptic. Some of the literature also uses the expression elliptic-hyperelliptic for this.

Another generalization of $X$ being hyperelliptic is the existence of an $n$-gonal map. This is a surjective morphism of degree $n$ from $X$ to a projective line. Note that for $n > 2$ such a morphism is not necessarily Galois, i.e. does not necessarily come from an automorphism of $X$. This makes it much more difficult to establish or exclude the existence of such a map. We are mainly interested in the case $n = 3$; then the map and the curve are called trigonal.

Throughout the paper we will freely use the following easy consequences of the Castelnuovo inequality:

- The hyperelliptic involution is unique.
- On a bielliptic curve of genus $g \geq 6$ the bielliptic involution is unique.
- A curve of genus $g \geq 5$ has at most one trigonal map.
- A hyperelliptic curve of genus $g \geq 3$ cannot be trigonal.
A hyperelliptic curve of genus \( g \geq 4 \) cannot be bielliptic.

Finally, we’ll need the following results.

**Theorem 1.2.** Let \( Y \) be a hyperelliptic curve of genus \( h \), and let \( X \rightarrow Y \) be a smooth, cyclic covering of degree \( n \).

(a) *(Bujalance) [B, Theorem]* If \( n = 2 \), then \( X \) is \( b \)-hyperelliptic for some \( b \) with \( b \leq \lfloor \frac{h-1}{2} \rfloor \).

In particular, a smooth degree 2 cover of a genus 2 curve is a genus 3 curve that is hyperelliptic (and bielliptic).

(b) *(Accola) [A, Lemma 2]* If \( n \) is odd, then the hyperelliptic involution of \( Y \) lifts to \( n \) different involutions on \( X \) that are all \( b \)-hyperelliptic where \( b = \frac{(n-1)(h-1)}{2} \).

Together with the automorphism of order \( n \) from \( \text{Gal}(X/Y) \) they generate a dihedral group \( D_n \) of order 2\( n \).

In particular, a smooth degree 3 Galois cover of a genus 2 curve is a genus 4 curve that has at least 3 bielliptic involutions.

2. Bielliptic curves

We start with a quick corollary to the known results from the previous section.

**Corollary 2.1.** Let \( X \) be a curve of genus 5. If \( X \) has an automorphism \( \sigma \) of order 4 such that \( \sigma \) has no fixed points, then \( X \) is bielliptic.

**Proof.** As \( X/\langle \sigma^2 \rangle \) is an unramified degree 2 cover of \( X/\langle \sigma \rangle \), the genus of \( X/\langle \sigma^2 \rangle \) can only be 1 (if \( \sigma^2 \) has fixed points) or 3. In the first case we are done.

In the second case we get a chain \( X \rightarrow X/\langle \sigma^2 \rangle \rightarrow X/\langle \sigma \rangle \) of smooth coverings of degree 2 with genera 5, 3, and 2. By Theorem 1.2 (a) the genus 3 curve is hyperelliptic, and the genus 5 curve is hyperelliptic or bielliptic. But it cannot be hyperelliptic by Theorem 1.1. \( \square \)

Ultimately Theorem 1.2 and Corollary 2.1 are based on lifting the hyperelliptic involution to an unramified cover. Our next result requires lifting the hyperelliptic involution to a ramified cover, which is a much more tricky problem. The paper [CT] discusses conditions under which this is possible. Compared to our previous statements we change the names of the curves and their automorphisms so that they fit precisely with those in the somewhat technical conditions in [CT].

**Proposition 2.2.** Let \( Y \) be a non-hyperelliptic curve of genus 4. If \( Y \) has an automorphism \( \tau \) of order 4, then \( Y \) is bielliptic.
Proof. Since \( Y \) is non-hyperelliptic, the curve \( X = Y/\langle \tau^2 \rangle \) can only have genus 1 or 2. In the first case we are done.

In the second case \( \tau^2 \) has exactly 2 fixed points \( P \) and \( Q \) on \( Y \). From the Hurwitz formula we see that \( Y/\langle \tau \rangle \) has genus 1 and \( \tau \) also fixes \( P \) and \( Q \). So \( \tau \), the involution induced by \( \tau \) on \( X \) is different from \( \sigma \), the hyperelliptic involution of \( X \).

As \( \sigma \) commutes with \( \tau \), it acts on its fixed points \( \tilde{P} \) and \( \tilde{Q} \), the images of \( P \) and \( Q \) on \( X \). Moreover, different involutions must have disjoint fixed points. Hence \( \sigma(\tilde{P}) = \tilde{Q} \). Since the Weierstrass points of \( X \) are exactly the fixed points of \( \sigma \), we see that \( \tilde{P} \) and \( \tilde{Q} = \sigma(\tilde{P}) \), the only two points on \( X \) that ramify in \( Y \), are not Weierstrass points of \( X \). So our constellation satisfies the conditions of [CT, Theorem 2.1], which allows us to conclude that \( \sigma \) lifts to an automorphism of \( Y \).

Actually, \( \sigma \) cannot lift to an automorphism \( \psi \) of order 4, because then we would have \( \psi^2 = \tau^2 \) and as before \( X/\langle \sigma \rangle = Y/\langle \psi \rangle \) would have genus 1. So \( \sigma \) lifts to an involution \( \psi \) on \( Y \) that commutes with \( \tau^2 \). Then the other two intermediate curves between \( Y \) and the genus zero curve \( Y/\langle \tau^2, \psi \rangle \) have genera 2 and 0 (which would contradict non-hyperellipticity of \( Y \)) or 1 and 1, which implies that in this case \( Y \) actually has at least 2 bielliptic involutions. \( \square \)

Here is an analogue of Maclachlan’s result for bielliptic curves.

**Proposition 2.3.** Let \( Y \) be a hyperelliptic curve of genus at least 4 and let \( X \to Y \) be a smooth Galois cover with Galois group \( G \) such that \( X \) is bielliptic. Then

(a) At least half of the elements of \( G \) are involutions.

(b) If \( G \) is abelian, then \( G \) has exponent 2 and \( |G| \) divides 8.

**Proof.**

(a) Since \( g(X) \geq 6 \), the bielliptic involution \( \tau \) is unique and hence commutes with \( G \). As \( Y \) cannot be bielliptic, \( \tau \) induces the hyperelliptic involution on \( Y \). Counting ramified points in the covering \( X \to X/\langle \tau, G \rangle \) as in the proof of the second claim of Theorem 1.1, one sees that \( G \) must have at least \( |G|/2 \) involutions.

(b) If \( G \) is abelian, part (a) implies that \( G \) must be 2-elementary abelian. In general, \( G \) is isomorphic to a subgroup of \( Aut(X/\langle \tau \rangle) \). But the biggest 2-elementary abelian group contained in the automorphism group of a genus one curve is \( \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \). \( \square \)

If in Proposition 2.3 we put the condition that the curve \( Y \) is bielliptic instead of hyperelliptic, then the number of fixed points of the bielliptic involution fits with the Hurwitz formula for smooth covers, and the smoothness of the cover fits with the fact that a surjective map between two curves of genus 1 is unramified. In short, the method of proof does not seem to imply any restrictions.

On the other hand, for hyperelliptic \( Y \) we can even say something about the non-smooth case. But we need a little preparation.
Lemma 2.4. Let $X$ be a bielliptic curve of genus $g \geq 6$ with an automorphism $\sigma$ of prime order $p \geq 3$.

(a) If $\sigma$ has fixed points, then necessarily $p = 3$.

(b) If $\sigma$ has no fixed points, then the curve $X/\langle \sigma \rangle$ is also bielliptic.

Proof. As $g \geq 6$, the bielliptic involution $\tau$ is unique. So $\tau$ and $\sigma$ commute.

For (a) we use that $\sigma$ induces an automorphism $\tilde{\sigma}$ of order $p$ with fixed points on the genus one curve $X/\langle \tau \rangle$, which is only possible for $p = 3$.

For (b) note that any fixed point of $\tilde{\sigma}$ on $X/\langle \tau \rangle$ would lift to one or two fixed points of $\sigma$ on $X$. So $\tilde{\sigma}$ is fixed-point-free and hence $X/\langle \tau, \sigma \rangle$ has genus 1. This means that $\tau$ induces a bielliptic involution on $X/\langle \sigma \rangle$.

Proposition 2.5. Let $Y$ be a hyperelliptic curve of genus at least 4. If $X \to Y$ is a Galois cover of degree relatively prime to 6, then $X$ cannot be bielliptic.

Proof. Assume that $X$ is bielliptic. As $g(X) \geq 6$, the bielliptic involution $\tau$ is unique and lies in the center of $\text{Aut}(X)$. Moreover, because of $g(Y) \geq 4$ the Galois group $G$ of the covering $X \to Y$ cannot contain $\tau$. So $G$ induces an isomorphic group of automorphisms on $X/\langle \tau \rangle$. As finite subgroups of automorphisms on a genus 1 curve are solvable, $G$ is solvable. (Alternatively, we could argue here with the highly non-trivial fact that groups of odd order are solvable.) So there exists a chain of coverings

$$X = X_1 \to X_2 \to \cdots \to X_s = Y$$

such that each covering $X_i \to X_{i+1}$ is Galois of prime degree. By Lemma 2.4 (a) the biellipticity of $X_i$ implies that the covering $X_i \to X_{i+1}$ is unramified, and by Lemma 2.4 (b) this implies that $X_{i+1}$ is bielliptic, too. By iteration we get that $Y$ is bielliptic, which cannot be for a hyperelliptic curve of genus bigger than 3.

Remark 2.6. With practically the same proof Proposition 2.5 holds under slightly more general conditions, for example if the degree of the Galois cover is odd and none of the ramification indices is divisible by 3.

A bielliptic curve $X$ of genus $g(X) \leq 5$ can have more than one bielliptic involution. We recall some curves that reach the maximally possible number of bielliptic involutions for the given genus.

The maximal possible number of automorphisms on a curve of genus 3 is 168. The unique genus 3 curve that realizes this bound is the modular curve $X(7)$. Its automorphism group is the simple group $PSL_2(\mathbb{F}_7)$. This group is also isomorphic to $GL_3(\mathbb{F}_2)$. The projective model of this curve is

$$x^3 y + y^3 z + z^3 x = 0,$$
which is why it is also called the **Klein quartic**. Since its automorphism group is simple, this curve cannot be hyperelliptic. So by Theorem 1.2 (b) every involution must be bielliptic. Thus it has 21 bielliptic involutions, the maximum possible for a curve of genus 3.

By [CD, Corollary 6.9] a curve of genus 4 can have at most 10 bielliptic involutions, and there is exactly one curve of genus 4 with 10 bielliptic involutions. It is called **Bring’s curve**, and it is isomorphic to the modular curve $X_1(5,10)$. Its automorphism group is isomorphic to the symmetric group $S_5$. Actually, 120 is also the maximum number of automorphisms for a curve of genus 4.

By [KMV] a curve of genus 5 can have 0, 1, 2, 3 or 5 bielliptic involutions, and the genus 5 curves with 5 bielliptic involutions form a 2-dimensional family. They are called **Humbert curves**. Their automorphism groups have order 160. This does not quite reach the maximally possible number of automorphisms on a curve of genus 5, which is 192.

For the proof of the next result we have to introduce yet another type of curve with an obvious automorphism of order $p$, namely for each prime $p \geq 5$ the **Lefschetz curve**

$$y^p = x(x - 1)$$

of genus $\frac{p-1}{2}$. Its full automorphism group is isomorphic to $\mathbb{Z}/2p\mathbb{Z}$. The unique involution $x \mapsto 1-x$ is hyperelliptic, and hence the curve is not bielliptic.

In [JKS2] we showed that there is only one bielliptic curve of genus 4 with an automorphism of order 5, namely Bring’s curve. Now we want to generalize this.

**Theorem 2.7.** Let $X$ be a bielliptic curve. Assume that $X$ has an automorphism $\sigma$ of prime order $p \geq 5$ with fixed points. Then one of the following holds:

(a) $g = 5$, $p = 5$ and $X$ is a Humbert curve;

(b) $g = 4$, $p = 5$ and $X$ is Bring’s curve;

(c) $g = 3$, $p = 7$ and $X$ is the Klein quartic.

**Proof.** By Lemma 2.4 (a) we must have $g \leq 5$. Moreover, $\sigma$ acts by conjugation on the bielliptic involutions. If it commutes with a bielliptic involution, it induces an automorphism of order $p$ with fixed points on the elliptic quotient curve, which is impossible. Hence the number of bielliptic involutions must be divisible by $p$.

If $g = 5$, the maximal number of bielliptic involutions is 5 [KMV], so necessarily $p = 5$ and $X$ has 5 bielliptic involutions, i.e. $X$ is a Humbert curve.

If $g = 4$, the possible numbers of bielliptic involutions are 0, 1, 2, 3, 4, 6, 10 [CD, p.600]. So $p = 5$ and 10 bielliptic involutions, i.e. $X$ is Bring’s curve.
If \( g = 3 \), the possible prime divisors of \( |Aut(X)| \) are 2, 3 and 7. By [RR, Theorem 1] there are exactly two curves of genus 3 with an automorphism of order 7. One is the Klein quartic, the other one is the Lefschetz curve (which is not bielliptic).

If \( g = 2 \), the biggest possible prime divisor of \( |Aut(X)| \) is 5. By [RR, p.199] the only genus 2 curve with an automorphism of order 5 is the Lefschetz curve, which is not bielliptic. □

3. Trigonal curves

We give another generalization of Maclachlan’s theorem.

**Theorem 3.1.** Let \( p \) be an odd prime. Let \( X \) be a \( p \)-gonal curve of genus \( g > (p-1)^2 \) and \( G \leq Aut(X) \) a subgroup of fixed-point-free automorphisms. Then \( |G| \) divides \( p \).

**Proof.** Since \( g > (p-1)^2 \), by the Castelnuovo inequality the \( p \)-gonal map \( \pi : X \to Y \) is unique. So \( G \) induces an isomorphic group of automorphisms on the genus zero curve \( Y \). Then \( G \) acts without fixed points on the ramification points \( P \) of \( \pi \) on \( X \). Consequently, they come in batches of size \( |G| \) with the same ramification index \( e_P \). Hence \( |G| \) divides \( \sum (e_P - 1) \). Thus by the Hurwitz formula \( |G| \) divides \( 2g - 2 + 2p \).

On the other hand, as the covering \( X \to X/G \) is smooth, \( |G| \) divides \( 2g - 2 \). Subtracting this, we see that \( |G| \) divides \( 2p \).

So what is left is showing that every involution \( \sigma \) on \( X \) must have fixed points. Let \( P \) be one of the two fixed points of the induced involution \( \tilde{\sigma} \) on \( Y \). Let \( P_1, \ldots, P_r \) be the points on \( X \) lying above \( P \) and let \( e_1, \ldots, e_r \) be their ramification indices. Then \( \sigma \) acts on these points. If \( \sigma \) fixes no \( P_i \), then they come in pairs with the same ramification index. So \( p = \sum_{i=1}^{r} e_i \) would be even, a contradiction. □

Theorem 3.1 applies in particular to trigonal curves of genus \( g \geq 5 \).

**Proposition 3.2.** Let \( Y \) be a hyperelliptic curve of genus \( h \geq 3 \), and let \( X \to Y \) be a Galois cover. Then \( X \) cannot be trigonal.

**Proof.** If \( X \) is trigonal, then, because of \( g(X) \geq 5 \), the trigonal map \( X \to Z \) is unique. So \( G \), the Galois group of \( X \to Y \), induces an isomorphic group of automorphisms on \( Z \). This implies the existence of a trigonal map \( Y \to Z/G \). But by Castelnuovo \( Y \) cannot be trigonal. □

**Corollary 3.3.** Let \( X \to Y \) be a smooth Galois cover of degree \( n \) where \( Y \) is a hyperelliptic curve of genus \( h \). Then \( X \) is trigonal if and only if \( n = 3 \), \( h = 2 \), and \( X \) has genus 4.

**Proof.** Assume that \( X \) is trigonal. If \( g(X) \geq 5 \), then \( n = 3 \) by Theorem 3.1...
and \( h = 2 \) by Proposition 3.2; but this contradicts \( g(X) \geq 5 \). So for trigonal \( X \) we must have \( g(X) \leq 4 \). Now the Hurwitz formula leaves only the possibilities \( g = 4, n = 3, h = 2 \) or \( g = 3, n = 2, h = 2 \). But by Theorem 1.2 (a) the second possibility would imply the contradiction that \( X \) is hyperelliptic.

Conversely, a curve of genus 4 is either hyperelliptic or trigonal. But by Theorem 1.1 a smooth Galois cover of degree 3 cannot be hyperelliptic.

Concerning involutions on trigonal curves we have the following result.

**Lemma 3.4.** Let \( X \) be a trigonal curve of genus \( g \) and \( \sigma \) an involution on \( X \).

(a) If \( g \) is odd, then \( \sigma \) has exactly 4 fixed points.

(b) If \( g \) is even, then \( \sigma \) has 2 or 6 fixed points.

**Proof.** If \( g \geq 5 \), the trigonal map \( X \to Y \) is unique, and hence \( \sigma \) induces an involution \( \tilde{\sigma} \) on \( Y \). Since \( \tilde{\sigma} \) has exactly two fixed points on \( Y \), \( \sigma \) can have at most 6 fixed points on \( X \). By the Hurwitz formula this means 2 or 6 fixed points if \( g \) is even, and 0 or 4 if \( g \) is odd. But 0 is excluded by Theorem 2.1.

If \( g = 4 \), then a priori there could be 2, 6 or 10 fixed points. But 10 fixed points would mean that the involution is hyperelliptic, which is not possible for a trigonal curve of genus bigger than 2.

For the same reason 8 fixed points are not possible if \( g = 3 \). But 0 fixed points are not possible either for \( g = 3 \), as by Theorem 1.2 this would also imply that \( X \) is hyperelliptic.

**Corollary 3.5.** Let \( X \) be a curve of genus \( g \equiv 1 \mod 4 \). If \( \text{Aut}(X) \) has a subgroup \( H \) isomorphic to \( \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \), then \( X \) cannot be trigonal.

**Proof.** Assume that \( X \) is trigonal. Then by Lemma 3.4 each of the 3 involutions in \( H \) has exactly 4 fixed points. But then applying the Hurwitz formula to the covering \( X \to X/H \) would give \( g \equiv 3 \mod 4 \).

The famous Theorem of Faltings says that a curve \( X \) of genus \( g \geq 2 \) over a number field \( K \) has only finitely many \( K \)-rational points. Likewise, it has only finitely many \( L \)-rational points for any fixed cubic extension \( L \) of \( K \). But since \( K \) has infinitely many cubic extensions, \( X \) could have infinitely many cubic points in total. Here a cubic point over \( K \) means a point \( P \) that is \( L \)-rational for a cubic extension \( L \) of \( K \) where \( L \) might depend on \( P \).

For example, if \( X \) is trigonal over \( K \), i.e. if there is a degree 3 cover \( X \to Y \), defined over \( K \), such that \( Y \) is a projective line over \( K \), then \( X \) has infinitely many cubic points over \( K \), namely at least one over each of the infinitely many \( K \)-rational points of \( Y \).

This is the reason why trigonality is one of the key points to examine, if for a given ensemble of curves one wants to decide which of them have infinitely many
cubic points. See for example [JKS1], where we investigated certain modular curves
under this aspect to determine which finite groups occur infinitely often as torsion
groups of elliptic curves over cubic number fields.

Now a curve of genus 2 is always trigonal. If \( P \) is any non-Weierstrass point,
there is a rational function with a triple pole at \( P \). But the question is whether
there is a \( K \)-rational trigonal map. By what was just said, a \( K \)-rational point that
is not a Weierstrass point would suffice for this. But not every genus 2 curve has
such a point. In [JKS1, Lemma 2.1] we also showed that if a genus 2 curve has 3 or
more \( K \)-rational points (Weierstrass or not), then it is trigonal over \( K \), and hence
has infinitely many cubic points over \( K \).

Considering points that are genuinely cubic over \( K \) (i.e. cubic over \( K \) but not
\( K \)-rational), we now prove that a curve of genus 2 either has infinitely many such
points or none.

**Theorem 3.6.** For a curve \( X \) of genus 2 over a number field \( K \) the following
are equivalent:

(a) \( X \) has a cubic point \( P \) over \( K \) that is not \( K \)-rational.

(b) \( X \) has a trigonal map that is defined over \( K \).

(c) \( X \) has infinitely many cubic points over \( K \).

**Proof.** Let \( L \) be the cubic extension of \( K \) generated by \( P \). Let \( \iota_1, \iota_2, \iota_3 \) be the
three different \( K \)-embeddings of \( L \) into \( \mathbb{C} \).

If there is a \( \overline{K} \)-rational function on \( X \) with pole divisor \( \iota_i(P) + \iota_j(P) \) \((i \neq j)\), this
divisor must lie in the canonical class. We can apply a suitable automorphism from
\( \text{Gal}(\overline{K}/K) \) that permutes the three embeddings cyclically, and get that \( \iota_j(P) + \iota_k(P) \)
also lies in the canonical class. But then \( \iota_i(P) - \iota_k(P) \) would be a principal divisor,
which is impossible on a curve of positive genus.

Thus the Riemann-Roch space of the \( K \)-rational divisor \( D = \iota_1(P) + \iota_2(P) + \iota_3(P) \)
contains a \( K \)-rational function whose pole divisor actually is \( D \). This function is a
\( K \)-rational trigonal map.

The conclusion \( (b) \rightarrow (c) \) is clear, and \( (c) \rightarrow (a) \) is trivial. \( \square \)

Finally we show that among the genus 2 curves that have no \( \mathbb{Q} \)-rational points
some are trigonal over \( \mathbb{Q} \) and some are not.

**Example 3.7.** The genus 2 curve

\[ y^2 = -x^6 - 1 \]

has no \( \mathbb{Q} \)-rational points and no cubic points over \( \mathbb{Q} \). In fact, it has no real points
at all. The points lying over the point at infinity of the \( x \)-line are not real, as \(-1\) is
a square in their residue field.
Admittedly, the degree 3 map from the curve to $C : y^2 = -x^2 - 1$ is defined over $\mathbb{Q}$. But $C$ is a genus 0 curve without rational points. So over $\mathbb{Q}$ this map is not a trigonal map. Over $\mathbb{Q}(i)$, where $C$ is a projective line, the map will of course be trigonal.

**Example 3.8.** The genus 2 curve

$$y^2 = -x^6 + 13$$

also has no $\mathbb{Q}$-rational points. Here we are using that the elliptic curve $y^2 = x^3 + 13$ has rank 0 and trivial torsion, and that the points at infinity are not $\mathbb{Q}$-rational for exactly the same reasons as in the previous example.

Again, there is an obvious map of degree 3, defined over $\mathbb{Q}$, to the genus 0 curve $y^2 = -x^2 + 13$. But this curve has $\mathbb{Q}$-rational points, for example $(\pm2, \pm3)$ or $(\pm3, \pm2)$, so it is a projective line over $\mathbb{Q}$, and the degree 3 map to it is a trigonal map over $\mathbb{Q}$. Indeed, $(\pm\sqrt{2}, \pm3)$ and $(\pm\sqrt{3}, \pm2)$ are genuine cubic points on our genus 2 curve.

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