REPRESENTATIONS OF $GL_N$ OVER FINITE LOCAL PRINCIPAL IDEAL RINGS - AN OVERVIEW

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Abstract. We give a survey of the representation theory of $GL_N$ over finite local principal ideal rings via Clifford theory, with an emphasis on the construction of regular representations. We review results of Shintani and Hill, and the generalisation of Takase. We then summarise the main features, with some details but without proofs, of the recent constructions of regular representations due to Krakovski–Onn–Singla and Stasinski–Stevens, respectively.

1. Introduction

This paper is a survey of the (complex) representation theory of the group $GL_N(o)$, where $o$ is a compact discrete valuation ring, or equivalently, the ring of integers in a non-Archimedean local field with finite residue field $\mathbb{F}_q$ of characteristic $p$. Since $GL_N(o)$ is a profinite group, we consider its continuous representations, and a representation is continuous if and only if it is smooth if and only if it factors through a finite quotient $GL_N(o_r)$, where $o_r := o/p^r$, $p$ is the maximal ideal in $o$, and $r \geq 1$. We therefore focus on the representations of the finite groups $GL_N(o_r)$.

The representation theory of $GL_N(o_r)$ has a relatively long history (see the historical notes in Section 2), and has very recently seen intensified activity from several directions. We will focus mostly on the recent developments regarding so-called regular representations, studied via Clifford theory. Regular representations roughly correspond to regular conjugacy classes of matrices in the Lie algebra $g_r = M_N(o_r)$, that is, matrices whose centralisers mod $p$ have dimension $N$. The first construction of this kind goes back to Shintani [28], who constructed all the regular representations when $r$ is even. This was followed by work of Hill [12], who rediscovered Shintani’s construction and also provided a partial construction of so-called split regular representations for $r$ odd. As we will see in subsequent sections, the representation theory of $GL_N(o_r)$ is much harder when $r$ is odd compared to when $r$ is even. Very recently it was realised by Takase [33] that Hill’s construction does not actually produce all the split regular representations. Furthermore, Takase gave a construction of all regular representations which correspond to conjugacy classes with separable characteristic polynomial mod $p$, assuming the residue characteristic $p$ of $o$ is not 2. At the same time, and independently, two general constructions of regular representations have been found. One is by Krakovski, Onn and Singla [18], which works whenever $p$ is not 2, and the other is by Stasinski and Stevens [32]. The latter works for any $o$, and we therefore now have a complete construction of all the regular representations of $GL_N(o_r)$. This is currently the most general uniform construction of irreducible representations of $GL_N(o_r)$ available.
In Section 3 we give an introduction to the Clifford theory approach to the representations of $GL_N(o_r)$. In Section 4 we define regular representations and give the construction when $r$ is even. In Sections 5-7 we then focus on the various constructions of regular representations for $r$ odd. Section 5 contains a summary of Hill’s and Takase’s constructions of regular semisimple representations. In Section 6 we give an outline of the construction of Krakovski, Onn and Singla. Finally, in Section 7 we elaborate on the main steps in the construction of Stasinski and Stevens. In the final Section 8 we mention some open problems.

Throughout the paper we have omitted most proofs, apart from the proofs of some occasional lemmas. On the other hand, we have tried to provide detailed explanations of many of the arguments.

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2. Historical overview

The characters of $GL_N(o_1) = GL_N(F_q)$ were determined in a classical paper of Green [9] and the representations can be constructed via Deligne-Lusztig theory. The representations of the finite groups $GL_2(o_r)$, for all $r \geq 1$, have in one form or another been known to some mathematicians since the late 70s. There have been at least two different approaches to this problem. On the one hand, there is the Weil representation approach of Nobs and Wolfart (applied to the case $o = Z_p$ in [23]). On the other hand, there is the approach via orbits and Clifford theory due to Kutzko (unpublished), and independently to Nagornyj [20]; see also [30].

The related case of $SL_2(Z_p)$, $p \neq 2$, was studied by Kloosterman [16], Tanaka [34], Kutzko (thesis; unpublished), and Shalika (for general $o$ and $p \neq 2$) [27]. The Clifford theoretic approach of Kutzko and Shalika was rediscovered by Jakin-Zapirain in [15] Section 7. Another description of the representations of $SL_2(Z_p)$ (including the much more difficult case $p = 2$) was obtained by Nobs and Wolfart [22], using Weil representations. The case $PGL_2(o)$, again with $p \neq 2$, was treated by Silberger [29].

The representations of $GL_3(o)$ were studied by Nagornyj in [21], but the construction of representations was left incomplete. However, it was shown in [21] that the classification of representations of $GL_N(o)$ is a so-called wild problem, and in general one can therefore not expect an explicit and surveyable parametrisation of all the representations.

Recently, thorough and in-depth work on the representations of $GL_3(o)$ and $SL_3(o)$ (and related groups) has appeared in a series of papers by Avni, Kloppersch, Onn and Voll; see, for example, [1] [2]. The results in [1] are based, among other things, on the Kirillov orbit method, which works for principal congruence subgroups of $SL_3(o)$ of index large enough compared to $p$, and only when $o$ has characteristic 0. In [2] the authors employ Clifford theoretic methods to count the representations of $GL_3(o_r)$ and $SL_3(o_r)$ (when char $o = 0$ or char $o = p$ and $p$ is large enough relative to $r$) and of $SL_3(o)$ (when char $o = 0$ and $p$ is large enough relative to the absolute ramification index of $o$). Analogous results are obtained for unitary groups corresponding to an unramified extension of $o$. 
For the groups $\text{GL}_N(\mathfrak{o}_r)$ with $N \geq 2, r \geq 2$, the first general results seem to be due to Shintani in 1968 [28], who constructed the so-called regular representations when $r$ is even. This construction was rediscovered in independent work of Hill around 1995 [12]. The series of papers by Hill [10, 11, 12, 13] use the method of orbits and Clifford theory to study and construct some of the representations of $\text{GL}_N(\mathfrak{o}_r)$. In particular, in addition to Shintani’s construction of regular representations for $r$ even, Hill went on to construct certain regular representations when $r$ is odd. Over twenty years after Hill’s work, it was realised by Takase [33] that Hill’s construction of split regular representations is not exhaustive when the orbit is not semisimple. As mentioned in the introduction, recent constructions of regular representations have led to successively more general results, so that we now have a complete construction of all the regular representations of $\text{GL}_N(\mathfrak{o}_r)$. We will give a more detailed description of this work in subsequent sections.

Another approach to the representation theory of $\text{GL}_N(\mathfrak{o}_r)$ is based on viewing this group as the automorphism group of a rank $N$ $\mathfrak{o}$-module. This was initiated by Onn [25], who defined a new type of induction functor, called infinitesimal induction, for general automorphism groups of $\mathfrak{o}$-modules of residual rank $N$. Infinitesimal induction complements the classical induction from parabolic subgroups, which in [24] is referred to as geometric induction. Decomposing these induced representations and using the known construction of (strongly) cuspidal representations for $\text{GL}_2(\mathfrak{o})$, leads to another classification of the representations of this group.

Finally, we mention a different approach to the representations of $\text{GL}_N(\mathfrak{o}_r)$, or more generally, for reductive groups over $\mathfrak{o}_r$. This approach is a cohomological construction of certain irreducible representations attached to characters of finite maximal tori. It was given by Lusztig in [19] in the case where $\mathfrak{o}$ has positive characteristic, and for arbitrary $\mathfrak{o}$ in [31]. This is a higher level generalisation of the classical construction of Deligne and Lusztig [6], which corresponds to the case $r = 1$. Another, “purely algebraic” (non-cohomological) construction of representations of certain split reductive groups, also attached to characters of finite maximal tori, was given by Gérardin [8]. In [19, Section 1] Lusztig suggested the problem of whether these representations are in fact the same as those given by the higher Deligne-Lusztig construction. This was recently answered in the affirmative for $r$ even by Chen and Stasinski [5].

### 3. Clifford Theory for $\text{GL}_N(\mathfrak{o}_r)$

If $G$ is a finite group, we will write $\text{Irr}(G)$ for the set of isomorphism classes of complex irreducible representations of $G$. For convenience, we will always consider an element $\rho \in \text{Irr}(G)$ as a representation, rather than an equivalence class of representations, that is, we identify $\rho \in \text{Irr}(G)$ with any representative of the isomorphism class $\rho$. One can view $\text{Irr}(G)$ as the set of irreducible characters of $G$, but we prefer to work with representations when possible. If $G$ is abelian, we will often refer to a one-dimensional representation of $G$ as a character. If $H \subseteq G$ is a subgroup and $\rho$ is any representation of $G$ we write $\rho|_H$ for the restriction of $\rho$ to $H$.

Let $\mathfrak{o}$ be a compact discrete valuation ring, that is, the ring of integers in a non-Archimedean local field with finite residue field, say $\mathbb{F}_q$, of characteristic $p$. Denote by $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$, and by $\varpi$ a fixed generator of $\mathfrak{p}$. For any integer $r \geq 1$ we write $\mathfrak{o}_r$ for the finite ring $\mathfrak{o}/\mathfrak{p}^r$. We will also use $\mathfrak{p}$ and $\varpi$ to denote the
corresponding images of \( p \) and \( \varpi \) in \( \mathfrak{o}_r \). Fix an integer \( N \geq 2 \) and, for any \( r \geq 1 \), put

\[
G_r = \text{GL}_N(\mathfrak{o}_r),
\]
\[
\mathfrak{g}_r = M_N(\mathfrak{o}_r),
\]
where, for a commutative ring \( R \), we use \( M_N(R) \) to denote the algebra of \( N \times N \) matrices over \( R \). From now on, we consider a fixed \( r \geq 2 \). For any integer \( i \) such that \( r \geq i \geq 1 \), let \( \rho_i = \rho_{r,i} : G_r \to G_i \) be the surjective homomorphism induced by the canonical map \( \mathfrak{o}_r \to \mathfrak{o}_i \), and write \( K^i = K^i_r = \text{Ker} \rho_i \). We also write \( \rho_i \) for the corresponding homomorphism \( \mathfrak{g}_r \to \mathfrak{g}_i \). We thus have a descending chain of subgroups

\[
G_r \supset K^1 \supset \cdots \supset K^r = \{1\},
\]
where

\[
K^i = 1 + p^i \mathfrak{g}_r.
\]
With this description of the kernels, it is easy to show the commutator relation \([K^i, K^j] \subseteq K^{\min(i+j,r)}\), for \( r \geq i, j \geq 1 \). In particular, if \( i \geq r/2 \), then \( K^i \) is abelian, and if we let \( \lambda = \left[ \frac{r}{2} \right] \), then \( K^\lambda \) is the maximal abelian group among the kernels \( K^i \). From now on, let \( r \geq i, r \geq 2 \), that is, \( i \geq 1 \). Then the map \( x \mapsto 1 + \varpi^r x \) induces an isomorphism

\[
\mathfrak{g}_{r-i} \xrightarrow{\sim} K^i.
\]
The group \( G_r \) acts on \( \mathfrak{g}_{r-i} \) by conjugation, via its quotient \( G_{r-i} \). This action is transformed by the above isomorphism into the action of \( G_r \) on its normal subgroup \( K^i \). Let \( F \) be the fraction field of \( \mathfrak{o} \). Fix an additive character \( \psi : F \to \mathbb{C}^\times \) which is trivial on \( \mathfrak{o} \) but not on \( p^{-1} \). For each \( r \geq 1 \) we can view \( \psi \) as a character of the group \( F/p^r \) whose kernel contains \( \mathfrak{o}_r \). We will use \( \psi \) and the trace form \( (x,y) \mapsto \text{tr}(xy) \) on \( \mathfrak{g}_r \) to set up a duality between the groups \( \text{Irr}(K^i) \) and \( \mathfrak{g}_{r-i} \). For \( \beta \in M_N(\mathfrak{o}_r) \), define a homomorphism \( \psi_\beta : K^i \to \mathbb{C}^\times \) by

\[
\psi_\beta(1 + x) = \psi(\varpi^{-r} \text{tr}(\beta x)),
\]
for \( x \in p^i \mathfrak{g}_r \). Note that \( \varpi^{-r} \text{tr}(\beta x) \) is a well defined element of \( F/p^r \). Since \( \psi \) is trivial on \( \mathfrak{o}_r \), \( \psi_\beta \) only depends on \( x \) mod \( p^{-i} \) (as it must in order to be well defined). Moreover, the map \( \beta \mapsto \psi_\beta \) is a homomorphism whose kernel is \( p^i \mathfrak{g}_r \), thanks to the non-degeneracy of the trace form. Hence it induces an isomorphism

\[
\mathfrak{g}_r/p^{r-i} \mathfrak{g}_r \xrightarrow{\sim} \text{Irr}(K^i),
\]
where we will usually identify \( \mathfrak{g}_r/p^{r-i} \mathfrak{g}_r \) with \( \mathfrak{g}_{r-i} \). For \( g \in G_r \) we have

\[
\psi_{\beta \gamma^{-1}}(x) = \psi(\varpi^{-r} \text{tr}(g \beta \gamma^{-1} x)) = \psi(\varpi^{-r} \text{tr}(\beta \gamma^{-1} x g)) = \psi_\beta(g^{-1} x g).
\]
Thus the isomorphism (3.1) transforms the action of \( G_r \) on \( \mathfrak{g}_r \) into (the inverse) conjugation of characters.

Remark 3.1. In the above we have used adjoint orbits (i.e., conjugacy classes) in \( \mathfrak{g}_r/p^{r-i} \mathfrak{g}_r \) to parametrise orbits of characters of \( K^i \). From some points of view it is more natural to use co-adjoint orbits in the dual

\[
\mathfrak{g}_r^* := \text{Hom}_{\mathfrak{o}_r}(\mathfrak{g}_r, \mathfrak{o}_r).
\]
Indeed the pairing \( \langle \cdot, \cdot \rangle : \mathfrak{g}_r^* \times \mathfrak{g}_r \to \mathfrak{o}_r \) given by \( \langle f, \beta \rangle = f(\beta) \) is non-degenerate and one can define

\[
\psi_f(1 + x) = \psi(\varpi^{-r}(f, x)),
\]
where \( f \in \mathfrak{g}^* \). This induces an isomorphism \( \mathfrak{g}^*/p^r \cong \mathfrak{g}^*_r \cong \mathfrak{g}^*_r / \mathfrak{g}^*_r \) to \( \text{Irr}(K^i) \), and has the advantage of generalising to Chevalley groups other than \( \text{GL}_N \) (where the trace form may be degenerate); see [8]. However, for \( \text{GL}_N \) we prefer to work with elements in \( \mathfrak{g}_r \) rather than elements in its dual, and we can translate between the two by means of the \( G_r \)-equivariant bijection induced by the trace form.

If \( G \) is a finite group, \( H \subseteq G \) is a subgroup and \( \rho \in \text{Irr}(H) \), we will write \( \text{Irr}(G \mid \rho) \) for the set of \( \pi \in \text{Irr}(G) \) such that \( \pi \) contains \( \rho \) on restriction to \( H \), that is,

\[
\text{Irr}(G \mid \rho) = \{ \pi \in \text{Irr}(G) \mid \langle \pi|_H, \rho \rangle \neq 0 \}.
\]

Moreover, if \( N \) is a normal subgroup of \( G \), then \( G \) acts on \( \text{Irr}(N) \) by \( \rho \mapsto g \rho \), where \( g \rho(n) := \rho(gng^{-1}) \), for \( g \in G \), \( n \in N \). In this case, we define the stabiliser of \( \rho \in \text{Irr}(N) \) to be \( G(\rho) = \{ g \in G \mid g\rho \cong \rho \} \). We will subsequently make use of the following well known results from Clifford theory of finite groups:

**Theorem 3.2.** Let \( G \) be a finite group, and \( N \) a normal subgroup. Then the following hold:

(i) (Clifford’s theorem) If \( \pi \in \text{Irr}(G) \), then \( \pi|_N = \epsilon \bigoplus \rho \in \Omega \rho \), where \( \Omega \subseteq \text{Irr}(N) \) is an orbit under the action of \( G \) on \( \text{Irr}(N) \) by conjugation, and \( \epsilon \) is a positive integer.

(ii) Suppose that \( \rho \in \text{Irr}(N) \). Then \( \theta \mapsto \text{Ind}_{G(\rho)}^G \theta \) is a bijection from \( \text{Irr}(G \mid \rho) \) to \( \text{Irr}(G \mid \rho) \).

(iii) Let \( H \) be a subgroup of \( G \) containing \( N \), and suppose that \( \rho \in \text{Irr}(N) \) has an extension \( \tilde{\rho} \) to \( H \) (i.e., \( \tilde{\rho}|_N = \rho \)). Then

\[
\text{Ind}_N^H \rho = \bigoplus_{\chi \in \text{Irr}(H/N)} \tilde{\rho}\chi,
\]

where each \( \tilde{\rho}\chi \) is irreducible, and where we have identified \( \text{Irr}(H/N) \) with \( \{ \chi \in \text{Irr}(H) \mid \chi(N) = 1 \} \).

For proofs of the above, see for example [14], 6.2, 6.11 and 6.17, respectively. The above results [i] and [ii] show that in order to obtain a classification of the representations of \( G_r \), it is enough to classify the orbits of characters \( \psi_\beta \) of a normal subgroup \( K^i \), and to construct all the elements in \( \text{Irr}(G_r(\psi_\beta) \mid \psi_\beta) \), that is, to decompose \( \text{Ind}_{K^i(\psi_\beta)}^{G_r(\psi_\beta)} \psi_\beta \) into irreducible representations. This is what we shall do in the following, taking \( K^i = K^i \).

**Remark 3.3.** By an (algebraic) construction of some irreducible representations (or characters) of \( G_r \) via Clifford theory, we will always mean a general (i.e., valid for all \( G_r \)) finite sequence of extensions and inductions of characters, starting from the one-dimensional characters of \( K^i \). Note that the existence of an extension of a representations is allowed to be a non-constructive fact.

In order to have a complete understanding of representations constructed via Clifford theory, it is necessary to have an understanding of the \( G_r \) conjugacy classes (or orbits) in \( \mathfrak{g}_r \), because \( \text{Irr}(G_r(\psi_\beta) \mid \psi_\beta) = \text{Irr}(G_r(\psi_\beta') \mid \psi_\beta') \) if \( \beta \) and \( \beta' \) are conjugate. One cannot expect to have an explicit understanding of all the orbits, but we do have an explicit normal form for regular orbits, as we will see next.
4. Regular representations, \( r \) even

An irreducible representation \( \pi \) of \( G_r \) is called regular if \( \pi|_{K^r} \) contains \( \psi_\beta \) with \( \beta \in \mathfrak{g}_r \) regular. By a result of Hill [12, Theorem 3.6] \( \beta \in \mathfrak{g}_r \) is regular if and only if its image \( \bar{\beta} \in \mathfrak{g}_1 = M_N(F) \) is regular, that is, if \( \dim C_{\mathfrak{g}_1}(\bar{\beta}) = N \). There are several equivalent characterisations of regular elements in \( \mathfrak{g}_1 \); in particular, \( \beta \in \mathfrak{g}_1 \) is regular iff \( C_{\mathfrak{g}_1}(\bar{\beta}) \) is abelian iff the characteristic polynomial of \( \bar{\beta} \) equals the minimal polynomial iff \( \bar{\beta} \) is conjugate to a companion matrix. Note that \( \beta \) depends on the choice of \( \psi \), but for any other choice \( \psi' \) we have \( \psi_\beta = \psi'_a \beta \), for some \( a \in \sigma_r^x \), and since \( \beta \) is regular if and only if \( a\beta \) is regular, regularity is an intrinsic property of a representation \( \pi \in \text{Irr}(G_r) \).

There are three special properties of regular elements which will allow us to construct and completely classify all the regular representations:

(i) We can tell explicitly when two regular elements are \( G_r \)-conjugate, namely, if and only if their companion matrices coincide.

(ii) The centraliser \( C_{G_r}(\beta) \) of a regular element \( \beta \in \mathfrak{g}_r \) is abelian.

(iii) For any \( 1 \leq s \leq r \), the map \( \rho_s : C_{G_r}(\beta) \to C_{G_s}(\beta_s) \) is surjective, where \( \beta_s \) is the image of \( \beta \) under \( \rho_s : \mathfrak{g} \to \mathfrak{g}_s \).

We will illustrate this in the construction of all regular representations of \( G_r \) when \( r \) is even, given below.

Remark 4.1. If \( \pi \in \text{Irr}(G_r \mid \psi_\beta) \), then [12] implies that \( \pi \) has \( K^{r-1} \) in its kernel if and only if \( \bar{\beta} = 0 \). Thus \( \pi \) factors through \( G_{r-1} \) if and only if \( \bar{\beta} = 0 \). If this is the case, \( \pi \) is called imprimitive. If \( \pi \) does not factor through \( G_{r-1} \) it is called primitive. Note that a regular representation is necessarily primitive. On the other hand, there exist irreducible representations of \( G_r \) which are not regular, because they factor through \( G_{r-1} \), but are regular when viewed as characters of \( G_{r-1} \). For example, take the representations of \( \text{GL}_2(\mathfrak{o}_r) \) with \( \beta = (\begin{smallmatrix} 0 & a \\ \bar{a} & 0 \end{smallmatrix}) \).

From now on, let \( \psi_\beta \in \text{Irr}(K^r) \) with \( \beta \in \mathfrak{g}_r \) regular. Let \( l' = r - l \), so that \( l = l' \) when \( r \) is even and \( l' = l-1 \) when \( r \) is odd. As indicated in the previous section, the stabiliser \( G_r(\psi_\beta) \) plays an important role in the construction of representations of \( G_r \). The formula \( \psi_\beta(g^{-1}xg) = \psi_{g\beta^{-1}g^{-1}}(x) \), together with the fact that \( \psi_\beta = \psi_{\beta'} \Leftrightarrow \bar{\beta} \equiv \beta' \mod p') \), implies that

\[
G_r(\psi_\beta) = C_{G_r}(\beta + p' \mathfrak{g}_r).
\]

An important corollary of [12] Theorem 3.6 is that for regular \( \beta \), and any \( s \) such that \( r \geq s \geq 1 \), the natural reduction map

\[
C_{G_r}(\beta) \to C_{G_s}(\beta_s)
\]

is surjective. Another corollary of [12] Theorem 3.6 is that for regular \( \beta \) we have \( C_{G_r}(\beta) = \sigma_r[\beta]^* \), so that in particular, the centraliser is abelian. Together with [11] these two results imply that

\[
G_r(\psi_\beta) = C_{G_r}(\beta)K^{l'} = \sigma_r[\beta]^*K^{l'}.
\]

We now give the construction of regular representations of \( G_r \) in the case when \( r \) is even. Suppose that \( r \) is even, so that \( l = l' \). Let \( \theta \in \text{Irr}(C_{G_r}(\beta)) \) be any irreducible component of \( \text{Ind}_{C_{G_r}(\beta) \cap K^r}(\psi_\beta|_{C_{G_r}(\beta) \cap K^r}) \). Since \( C_{G_r}(\beta) \) is abelian \( \theta \) is
one-dimensional, and hence it agrees with $\psi_\beta$ on $C_{G_r}(\beta) \cap K^l$. It is then easy to check that

$$\tilde{\psi}_\beta(ck) := \theta(c)\psi_\beta(k)$$

is a well defined one-dimensional representation of $G_r(\psi_\beta)$, and by construction it is an extension of $\psi_\beta$. By Theorem 3.2 (iii) we obtain

$$\text{Irr}(G_r(\psi_\beta) \mid \psi_\beta) = \{\tilde{\psi}_\beta \chi \mid \chi \in \text{Irr}(C_{G_l}(\beta_l))\},$$

where $\beta_l \in g_l$ is the image of $\beta$. Hence Theorem 3.2 (ii) implies that there is a bijection

$$\text{Irr}(C_{G_l}(\beta_l)) \rightarrow \text{Irr}(G_r \mid \psi_\beta)$$

$$\chi \mapsto \text{Ind}_{G_r(\psi_\beta)}^{G_r} \tilde{\psi}_\beta \chi.$$  

Note that this is not canonical, but depends on the choice of $\tilde{\psi}_\beta$. We have thus constructed the irreducible representations of $G_r$ containing $\psi_\beta$, in terms of the irreducible representations of the abelian group $C_{G_l}(\beta_l)$ (which we consider known; cf. Remark 3.3). Note that if we start with another element in the conjugacy class of $\beta$, we obtain the same set of irreducible representations of $G_r$. Thus, when $r$ is even, running through a set of representatives for the regular conjugacy classes in $g_l$, yields all the regular representations of $G_r$ exactly once.

As far as the author is aware, the above construction is due to Shintani [28, §2, Theorem 2], although Shintani does not prove that every regular element in $g_l$ is regular mod $p$. The construction was rediscovered by Hill [12, Theorem 4.1].

It remains to construct the regular representations of $G_r$ when $r$ is odd. This requires additional methods, due to the fact that $G_r(\psi_\beta) = C_{G_r}(\beta)K^l'$, and it is not possible to extend $\psi_\beta$ from $K^l$ to $G_r(\psi_\beta)$. Instead, one has to take several intermediate steps consisting of extensions and inductions. In the following we will give an exposition of the currently known constructions (sometimes partial) of regular representations of $G_r$, for $r$ odd.

5. The constructions of Hill and Takase

From now on and until the end of Section 7 we will assume that $r$ is odd, so that $l' := r - l = l - 1$. In this case, Hill [12] claimed to give a construction of so-called split regular representations, that is, those for which the characteristic polynomial of $\bar{\beta} \in g_1$ splits into linear factors over $\mathbb{F}_q$. Takase [33] recently pointed out a gap in the proof of Hill’s result [12, Theorem 4.6] and proved that the construction exhausts at most the split regular semisimple representations, but does not exhaust all split regular representations. We give a summary of Hill’s construction following [12], point out two problems in the proof, and state the correction/generalisation due to Takase.

We have an isomorphism $K^l/K^l \cong g_1$, and we can identify any subgroup of $K^l'$ which contains $K^l$ with a sub-vectorspace of $g_1$. Define the alternating bilinear form

$$B_\beta : K^l' / K^l \times K^l' / K^l \rightarrow \mathbb{F}_q$$

$$B_\beta((1 + \pi^{l'} x)K^l, (1 + \pi^{l'} y)K^l) = \text{tr}(\bar{\beta}(\bar{x}y - \bar{y}x)).$$
where the bars denote reductions mod $p$. The following is [12] Lemma 4.5, rewritten in our notation.

**Lemma 5.1.** Suppose that $\beta \in g_r$ is split regular. Then there exists a subgroup $H_\beta$ of $K^\ell$ such that $H_\beta$ contains $K^l$ and such that $H_\beta/K^l$ is a maximal isotropic subspace of $K^\ell/K^l$ with respect to the form $B_\beta$. Moreover, $H_\beta$ is a normal subgroup of $G_r(\psi_\beta)$.

We recall that a subspace $U$ of a vector space $V$ with a bilinear form $B(\cdot, \cdot)$ is called isotropic (or sometimes totally isotropic) if $U \subseteq U^\perp$, that is, if $B(U, U) = 0$. Furthermore, $U$ is called maximal isotropic (or sometimes Lagrangian) if it is not properly contained in any isotropic subspace, or equivalently, if $U = U^\perp$.

The proof of the above lemma consists of taking $H_\beta = (B \cap K^\ell)K^l$, where $B$ is the upper-triangular subgroup of $G_r$, and showing that it has the required properties, using the assumption that $\beta$ is upper-triangular. Thus in Hill’s construction, $H_\beta$ is in fact independent of $\beta$.

Hill’s main theorem [12] Theorem 4.6] regarding the construction of split regular representations for $r$ odd claims that if $\beta \in g_r$ is split regular, then for every $\pi \in \text{Irr}(G_r \mid \psi_\beta)$, there exists a subgroup $H_\beta$ as in Lemma 5.1 and an extension $\psi_\beta$ of $\psi_\beta$ to $C_{G_r}(\beta)H_\beta$ such that

$$\pi = \text{Ind}_{C_{G_r}(\beta)H_\beta}^{G_r} \psi_\beta.$$ 

Unfortunately, Hill’s proof of [12] Theorem 4.6] suffers from two problems. One is that a certain counting argument only goes through when $\beta$ is assumed to be semisimple (see [33] Proposition 2.1.1), so that Hill’s construction does not exhaust the split regular representations. The other problem is that, in the second paragraph of the proof, it is asserted that a result of Brauer implies that the number of $C_{G_r}(\beta)H_\beta/N$-stable characters of $H_\beta/N$ is equal to the number of $C_{G_r}(\beta)H_\beta/N$-stable conjugacy classes of $H_\beta/N$, where $N = \text{Ker} \psi_\beta$. However, the quoted result of Brauer holds only for characters/conjugacy classes fixed by a single element in a group, and does not necessarily apply to the whole group $C_{G_r}(\beta)H_\beta/N$. We remark that by results of Glauberman and Isaacs (see [14] (13.24)) the appropriate generalisation of Brauer’s result holds for coprime group actions, but may fail otherwise. Since $p$ divides the orders of both $C_{G_r}(\beta)H_\beta/N$ and $H_\beta/N$, the crucial step in Hill’s proof which asserts the existence of an extension of $\psi_\beta$ to $C_{G_r}(\beta)H_\beta$ remains unclear.

In addition to the split regular representations, there are many regular representations which are not split, in particular the cuspidal representations, that is, those where $\beta$ has irreducible characteristic polynomial. In [13] Hill gave a construction of so-called strongly semisimple representations, that is, those for which $\bar{\beta}$ is semisimple and $\bar{\beta} \tau \in g_r$ has additive Jordan decomposition $\bar{\beta} \tau = s + n$, with $n$ in the centre of the algebra $C_{g_r}(s)$.

**Example 5.2.** Consider the function $\iota : \mathbb{F}_q \rightarrow \mathfrak{o}_r$, which is the multiplicative section extended by setting $\iota(0) = 0$. This induces an injective function $\mathfrak{g}_1 \rightarrow \mathfrak{g}_r$. An element in $\mathfrak{g}_r$ is called semisimple if it is the image of a semisimple element in $\mathfrak{g}_1$ under the map $\mathfrak{g}_1 \rightarrow \mathfrak{g}_r$. Then any $\beta \in \mathfrak{g}_r$ has a unique Jordan decomposition $\beta = s + n$, where $s$ is semisimple, $n$ is nilpotent and $sn = ns$ (see [13] Proposition 2.3]). If $n = 0$, then $\beta$ is strongly semisimple, so in particular, there are strongly semisimple
representations which are not regular. The strongly semisimple representations include the cuspidal ones (see [13, Proposition 4.4]).

Hill’s construction of strongly semisimple representations for \( r \) odd is summarised in the following (cf. [13, Proposition 3.6]) result:

**Theorem 5.3.** Let \( \pi \in \text{Irr}(G_r | \psi_\beta) \) be strongly semisimple. Then there exists a \( \rho \in \text{Irr}(K'' | \psi_\beta) \) and an extension \( \tilde{\rho} \) of \( \rho \) to \( G_r(\psi_\beta) \) such that

\[
\pi = \text{Ind}_{G_r(\psi_\beta)}^{G_r} \tilde{\rho}.
\]

Note that the only non-trivial part of this theorem is that \( \rho \) has an extension. In fact, it follows from the proof in [13] that every \( \rho \in \text{Irr}(K'' | \psi_\beta) \) extends to \( G_r(\psi_\beta) \). Moreover, by Theorem 3.2 (ii), distinct extensions of \( \rho \) give rise to distinct representations \( \pi \).

The elements of \( \text{Irr}(K'' | \psi_\beta) \) are constructed in [12, Proposition 4.2 (3)], so that together with the above theorem, this gives a complete construction of strongly semisimple representations, up to a knowledge of the elements in \( \text{Irr}(G_r(\psi_\beta)/K'') \cong \text{Irr}(C_{G_r(\beta)}) \). A version of Theorem 5.3 holds also when \( r \) is even; see [13, Proposition 3.3].

We see that out of the regular representations, Hill’s constructions cover at most those which are semisimple (i.e., where \( \bar{\beta} \) is semisimple). The next step was taken recently by Takase, who proved the following (see [33, Theorem 3.2.2, 5.2.1 and 5.3.1]):

**Theorem 5.4.** Let \( \pi \in \text{Irr}(G_r | \psi_\beta) \) be a regular character and suppose that \( \bar{\beta} \) satisfies at least one of the following properties:

(i) \( \bar{\beta} \) has separable characteristic polynomial and \( p > 2 \),

(ii) \( \bar{\beta} \) has Jordan blocks of size at most 4 and \( p > 7 \).

Then there exists a \( \rho \in \text{Irr}(K'' | \psi_\beta) \) and an extension \( \tilde{\rho} \) of \( \rho \) to \( G_r(\psi_\beta) \) such that

\[
\pi = \text{Ind}_{G_r(\psi_\beta)}^{G_r} \tilde{\rho}.
\]

Just as for Hill’s theorem on strongly semisimple representations above, the difficulty in Takase’s proof lies in showing that every \( \rho \in \text{Irr}(K'' | \psi_\beta) \) extends to \( G_r(\psi_\beta) \). The existence of an extension follows from the vanishing of the cohomology group \( H^2(F_q[\bar{\beta}], C^\times) \), the so-called Schur multiplier. When \( \bar{\beta} \) has irreducible characteristic polynomial, \( F_q[\bar{\beta}] \) is a finite field, so \( F_q[\bar{\beta}]^\times \) is cyclic. In this case it is well known that \( H^2(F_q[\bar{\beta}], C^\times) \) is trivial. For \( p > 2 \) Takase reduces the separable case to the irreducible, and thus proves Theorem 5.4 when \( \bar{\beta} \) satisfies the first condition; cf. [33, Theorem 4.3.2]. The existence of an extension when \( \bar{\beta} \) satisfies the second condition is proved in [33] by explicit computation of the relevant cocycles.

These results led Takase to conjecture that a certain element in the Schur multiplier is always trivial for \( p \) large enough; see [33, Conjecture 4.6.5].

6. THE CONSTRUCTION OF KRakovski, ONn AND SINGLA

We will now describe the construction of regular representations of \( G_r, r \) odd, due to Krakovski, Onn and Singla [18]. This gives a construction of all the regular representations, provided the residue characteristic \( p \) of \( \psi \) is odd. Furthermore, [18] also contains constructions and enumeration of all the regular representations
of $\text{SL}_N(\mathfrak{o}_r)$ when $p > N$, and of the unitary groups $\text{SU}_N(\mathfrak{o}_r)$ and $\text{GU}_N(\mathfrak{o}_r)$ with respect to a quadratic unramified extension of $\mathfrak{o}$ (with some restrictions on $p$).

The construction in [18] was inspired by a construction of Jaikin-Zapirain for $\text{SL}_2(\mathfrak{o}_r)$, $p > 2$; see [15] Section 7. We continue to assume that $r = l + l'$ is odd. The following result is [18, Theorem 3.1], which is a more detailed statement of [18, Theorem A]. We state this only for $\text{GL}_N$, in a form slightly adapted to our present notation.

**Theorem 6.1.** Assume that $\mathfrak{o}$ has residue characteristic $p > 2$. Let $\sigma \in \text{Irr}(K^l \mid \psi_{\beta})$ with $\beta$ regular. Then $\sigma$ has an extension $\tilde{\sigma}$ to $G_r(\psi_{\beta})$, and thus any $\pi \in \text{Irr}(G_r \mid \psi_{\beta})$ is of the form $\pi = \text{Ind}_{G_r(\psi_{\beta})}^{G_r} \tilde{\sigma}$, for some extension $\tilde{\sigma}$.

In particular, this proves a strengthened form of Takase’s conjecture mentioned above, namely for all $p > 2$ (another proof of this, for all $p$, follows from the construction of Stasinski and Stevens). We elaborate on the proof of [18, Theorem 3.1] in order to provide some of the details of the construction. As we have already remarked in previous sections, the main difficulty is to show that every $\sigma \in \text{Irr}(K^l \mid \psi_{\beta})$ extends to $G_r(\psi_{\beta})$. We will mainly formulate things in our present notation, but use the notation of [18] where possible.

### 6.1 Characters

Assume that $p > 2$. For $i$ such that $r/2 \leq i < r$, the exponential map $\exp : x \mapsto 1 + x$ gives an isomorphism $\mathfrak{p}^i\mathfrak{g}_r \to K^i$ (we already saw this in Section 3 and it works for any $p$). Moreover, when $p > 2$ and $r/3 \leq i < r/2$, the exponential map $\exp : x \mapsto 1 + x + \frac{1}{2}x^2$ gives a bijection $\mathfrak{p}^i\mathfrak{g}_r \to K^i$, which is however not an isomorphism in general. As usual, the inverse of this exponential map is given by the logarithm $\log : 1 + x \mapsto x - \frac{1}{2}x^2$. Every $\beta \in \mathfrak{g}_r$ defines a character

$$\varphi_{\beta} : \mathfrak{g}_r \to \mathbb{C}^\times, \quad \varphi_{\beta}(x) = \psi(\omega^{-r} \text{tr}(\beta x)).$$

The corresponding map $\beta \mapsto \varphi_{\beta}$ is an isomorphism. Any $\theta \in \text{Irr}(\mathfrak{p}^i\mathfrak{g}_r)$ can be pre-composed with the logarithm map $\log : K^i \to \mathfrak{p}^i\mathfrak{g}_r$, $1 + x \mapsto x$, to give a character $\log^* \theta := \theta \circ \log \in \text{Irr}(K^i)$, such that, for $1 + x \in K^i$,

$$(\log^* \theta)(1 + x) = \varphi_{\beta}(x),$$

where $\beta$ is determined by $\theta$. Note that $\log^* \theta = \psi_{\beta}$, where $\psi_{\beta}$ is as in (3.2). In particular, $\varphi_{\beta}$ restricts to $\theta$ on $\mathfrak{p}^i\mathfrak{g}_r$, but for a given $\theta$, there is more than one $\beta$ such that $\varphi_{\beta}$ restricts to $\theta$, since the restriction only depends on $\beta$ mod $\mathfrak{p}^i$.

A crucial step in [18] (due to Jaikin-Zapirain for $\text{SL}_2$), is to extend the above definition of $\log^* \theta$, in order to give a useful description of certain characters on any subgroup $K^l \subseteq J_{\beta} \subseteq K^l$ such that $J_{\beta}/K^l$ is a maximal isotropic subspace for the form $B_\beta$ defined in Section 5. This is the motivation behind [18, Lemma 3.2], and the essential reason for the assumption $p > 2$. The following result gives a summary of the key facts involved (see [18, Lemma 3.2 and Section 3.2]).

**Lemma 6.2.** Let $J_{\beta}$ be such that $J_{\beta}/K^l$ is a maximal isotropic subspace. Let $\theta''$ be the restriction of a character $\varphi_{\beta} \in \text{Irr}(\mathfrak{g}_r)$ to $\mathfrak{p}^i\mathfrak{g}_r$. Then the function $\log^* \theta'' : K^l \to \mathbb{C}^\times$ defines a multiplicative character when restricted to $J_{\beta}$.

**Proof.** Let $1 + \omega^r x$ and $1 + \omega^r y$ be elements in $J_{\beta}$. Direct computation yields the commutator

$$g := [(1 + \omega^r x), (1 + \omega^r y)] = 1 + \omega^{2r}(xy - yx).$$
Since we are assuming that \( p > 2 \), we have a unique square root \( g^{1/2} = 1 + \frac{1}{2} \omega^r(xy - yx) \). In particular, since \( 2l' = r - 1 \), \( g^{1/2} \) is in the centre of \( K' \). Thus,

\[
\log((1 + \omega^r x)(1 + \omega^r y)) = \log((1 + \omega^r x)(1 + \omega^r y)g^{-1/2}g^{1/2})
\]

\[
= \log((1 + \omega^r x)(1 + \omega^r y)g^{-1/2}) + \log(g^{1/2})
\]

\[
= \log(1 + \omega^r (x + y) + \frac{1}{2} \omega^{2l'} (xy + yx)) + \log(g^{1/2})
\]

\[
= \log(1 + \omega^r x) + \log(1 + \omega^r y) + \log(g^{1/2}),
\]

where the second equality follows from the fact that \( g^{1/2} \) is central. Applying \( \theta'' \), we get

\[
\theta''(\log((1 + \omega^r x)(1 + \omega^r y))) = \theta''(\log(1 + \omega^r x)) + \theta''(\log(1 + \omega^r y)) + \theta''(\log(g^{1/2}))
\]

\[
= \theta''(\log(1 + \omega^r x)) + \theta''(\log(1 + \omega^r y)) + \psi(\frac{1}{2} \omega^{-1} \text{tr}(\beta(xy - yx)))
\]

\[
= \theta''(\log(1 + \omega^r x)) + \theta''(\log(1 + \omega^r y)),
\]

where the last equality follows from the fact that \( g^{1/2} \) is central. Applying \( \theta'' \), we get

\[
\theta''(\log((1 + \omega^r x)(1 + \omega^r y))) = \theta''(\log(1 + \omega^r x)) + \theta''(\log(1 + \omega^r y)) + \theta''(\log(g^{1/2}))
\]

\[
= \theta''(\log(1 + \omega^r x)) + \theta''(\log(1 + \omega^r y)) + \psi(\frac{1}{2} \omega^{-1} \text{tr}(\beta(xy - yx)))
\]

\[
= \theta''(\log(1 + \omega^r x)) + \theta''(\log(1 + \omega^r y)),
\]

where the last equality follows from the fact that \( \text{tr}(\beta(x\tilde{y} - \tilde{y}x)) = B_\beta(1 + \omega^r x, 1 + \omega^r y) = 0 \), since \( J_\beta/K' \) is isotropic.

The crucial corollary of this lemma is that any \( \log^* \theta \in \text{Irr}(K') \) extends to \( J_\beta \) by the same formula, that is, \( \log^* \theta'' = \log^* \varphi_\beta \). We emphasise that the key is not just that \( \log^* \theta \) has an extension to \( J_\beta \) (this is true for any \( p \), by [12 Proposition 4.2]), but that there is an extension given by an explicit formula which makes it evident that the extension is stabilised by any \( g \in C_G(\beta) \) which normalises \( J_\beta \). As we will explain below, the \( p \)-Sylow subgroup \( P_\beta \) of \( C_G(\beta) \) normalises \( J_\beta \), so the extension \( \log^* \theta'' \) of \( \log^* \theta \) to \( J_\beta \) is stabilised by \( P_\beta \). Note that it is not known whether all of \( C_G(\beta) \) normalises \( J_\beta \), in general.

We now describe the representations of the non-abelian group \( K' \), following Hill [12 Proposition 4.2]. It is easy to check that the radical of the bilinear form \( B_\beta \) introduced in Section 4, is \( (C_G(\beta) \cap K')K'/K' \). There is then a subgroup \( K' \subseteq J_\beta \subseteq K' \) such that \( J_\beta/K' \) is a maximal isotropic subspace. The radical and maximal isotropic subspace correspond to two subspaces of \( M_N(F_q) \cong K'/K' \), and we let

\[
\nu_\beta \quad \text{and} \quad i_\beta
\]

denote the inverse images in \( p^l \mathfrak{g}_r \) of these two subspaces, respectively, under the map \( p\mathfrak{g}_r \to \text{M}_N(F_q) \), \( \omega^r x \to \tilde{x} \). Clearly \( \nu_\beta \) and \( i_\beta \) only depend on \( \beta \). Let \( \theta \in \text{Irr}(p^l \mathfrak{g}_r) \), and let \( \theta' \) be an extension of \( \theta \) to \( \nu_\beta \) (here we are just talking about characters of abelian groups). Then \( \theta' \) determines a unique irreducible representation of \( K' \), which arises as follows. Let \( \theta'' \) be an extension of \( \theta' \) to \( i_\beta \). Then \( \log^* \theta'' \) is a character of the group \( J_\beta \) thanks to Lemma 6.2 and

\[
\text{Ind}_{J_\beta}^{K'} (\log^* \theta'')
\]

can be shown to be irreducible. In fact, it is the unique element in \( \text{Irr}(K'| \log^* \theta') \).
6.2. Construction of representations. From now on, let \( \theta \in \text{Irr}(p'g_r) \) be a character that corresponds to a regular element, that is \( \log^* \theta = \psi_\beta \), where \( \beta \in g_r \) is regular (recall that \( \psi_\beta \) only depends on the coset \( \beta + p'g_r \)).

**Lemma 6.3.** Let \( \sigma \in \text{Irr}(K^l \mid \log^* \theta) \). Then \( G_r(\sigma) = G_r(\psi_\beta) \).

**Proof.** Let \( \theta' \in \text{Irr}(r_\beta) \) be the unique extension of \( \theta \) that corresponds to \( \sigma \). Choose \( \beta' \in g_r \) such that \( \varphi_\beta' \in \text{Irr}(g_r) \) is an extension of \( \theta' \). Then \( \varphi_\beta' \) is also an extension of \( \theta \), so \( \beta' \equiv \beta \mod p'g_r \), and by (1.2) we have

\[
G_r(\sigma) \subseteq G_r(\log^* \theta) = C_{G_r}(\beta)K^l = C_{G_r}(\beta')K^l,
\]

where the first inclusion follows from the fact that \( \log^* \theta \) is the unique irreducible character of \( K^l \) contained in \( \sigma \) (the orbit of the restriction of \( \sigma \) to \( K^l \) consists of copies of \( \psi_\beta \) since \( K^l \) stabilises \( \psi_\beta \)).

For the reverse inclusion, note that \( C_{G_r}(\beta') \) stabilises \( \varphi_\beta' \), hence its restriction \( \theta' \), and hence the character \( \log^* \theta' \). Since \( \sigma \) is the unique representation in \( \text{Irr}(K^l \mid \log^* \theta') \), \( \sigma \) is stabilised by \( C_{G_r}(\beta') \), and so \( C_{G_r}(\beta')K^l \subseteq G_r(\sigma) \).

We now explain how to show that \( \sigma \) extends to the stabiliser \( G_r(\psi_\beta) \). For this, it will be enough (by Lemma 6.3 and [14, Corollary 11.31]) to show that \( \sigma \) extends to the \( p \)-Sylow subgroup of \( G_r(\psi_\beta) \) (which is unique since \( G_r(\psi_\beta) \) is abelian modulo the \( p \)-group \( K^l \)). Let \( P_\beta \) denote the \( p \)-Sylow subgroup of \( C_{G_r}(\beta) \). The following crucial lemma, see [15, Lemma 3.4], goes back to Howe:

**Lemma 6.4.** Let \( V \) be a finite dimensional \( \mathbb{F}_p \)-vector space and \( \alpha \) an antisymmetric bilinear form on \( V \). Suppose that \( P \) is a \( p \)-group which acts on \( V \) and preserves \( \alpha \). Then there exists a maximal isotropic subspace \( U \) of \( V \) which is \( P \)-invariant.

The group \( P_\beta \) acts on \( K^l \) and \( K^l \) by conjugation, and hence induces an action on the vector space \( K^l / K^l \). By the above lemma, there exists a maximal isotropic subspace of \( K^l / K^l \) which is stable under this action of \( P_\beta \), that is, there is a subgroup \( K^l \subseteq J_\beta \subseteq K^l \), such that the image of \( J_\beta \) in \( K^l / K^l \) is a maximal isotropic subspace and such that \( J_\beta \) is normalised by \( P_\beta \). As in the proof of Lemma 6.3, let \( \theta' \in \text{Irr}(r_\beta) \) be the unique extension of \( \theta \) that corresponds to \( \sigma \) and \( \varphi_\beta' \in \text{Irr}(g_r) \) an extension of \( \theta' \). Then the restriction \( \varphi_\beta'|_{J_\beta} \) is stabilised by \( P_\beta \) (because it is stabilised by all of \( C_{G_r}(\beta) \)), and thus

\[
\log^* (\varphi_\beta'|_{J_\beta})
\]

is a character of \( J_\beta \) (by Lemma 6.2), which is stabilised by \( P_\beta \). Here we again see the crucial role played by Lemma 6.2 as well as the order in which choices are made: For any \( \sigma \in \text{Irr}(K^l \mid \psi_\beta) \), there is a unique \( \theta' \in \text{Irr}(r_\beta) \), and this extends to a \( \theta'' \in \text{Irr}(r_\beta) \) such that \( \log^* \theta'' \in \text{Irr}(J_\beta) \) is stabilised by \( C_{G_r}(\beta) \).

Since \( \log^* (\varphi_\beta'|_{J_\beta}) \) is one-dimensional and \( P_\beta \) is abelian, this character extends to a character \( \omega \in \text{Irr}(P_\beta J_\beta) \). The induced representation

\[
\sigma' := \text{Ind}_{P_\beta J_\beta}^{P_\beta K^l} \omega
\]

has dimension

\[
[P_\beta K^l : P_\beta J_\beta] = |P_\beta| / |J_\beta| = |P_\beta K^l| / |P_\beta J_\beta|
\]

and

\[
[P_\beta J_\beta] = [P_\beta K^l, J_\beta] / [P_\beta, J_\beta]
\]

where

\[
[P_\beta, J_\beta] = [P_\beta \cap J_\beta] / [P_\beta \cap K^l] K^l / J_\beta]
\]
Since \( J_β \) contains the group \((C_{G_r}(β) \cap K^{l'})K^l \) (since every maximal isotropic subspace contains the radical of the form), we have \( P_β \cap J_β \supseteq P_β \cap K^{l'} \). The reverse inclusion is trivial, so we have \( \dim \sigma' = [K^{l'} : J_β] = \dim \sigma \). Since \( \sigma' \) must contain \( \log \theta' \) on restriction to \( K^{l'} \) (because \( \sigma' \) contains \( \log \theta' \)), \( \sigma' \) must be an extension of \( \sigma \) (so in particular, \( \sigma' \) must be irreducible). Thus \( \sigma \) extends to the \( p \)-Sylow in \( G_r(ψ_β) \) and hence to all of \( G_r(ψ_β) \), by the above remarks. This concludes the proof of Theorem 6.1.

7. The construction of Stasinski and Stevens

In this section we summarise forthcoming work of Stasinski and Stevens \[32\] which gives a construction of all the regular representations of \( G_r = \text{GL}_N(\mathbb{O}_r) \), without any restriction on the residue characteristic. As in the previous two sections, we assume that \( r = l + l' \) is odd.

One of the key distinguishing features of the present approach is the systematic use of the subgroup structure of \( G_r \) provided by lattice chains. In particular, for a given regular orbit, two specific associated parahoric subgroups and their filtrations will play a crucial role. The construction is somewhat analogous to the construction of supercuspidal representations of Bushnell and Kutzko \[4\], but with the difference that for us everything takes place inside \( G_r \) and all relevant centralisers are abelian (because we consider only regular representations).

7.1. Subgroup structure. Let \( \mathfrak{A} \subseteq \mathfrak{g}_r = M_N(\mathfrak{a}_r) \) be a parahoric subalgebra, that is, the preimage under the reduction mod \( p \) map of a parabolic subalgebra of \( \mathfrak{g}_1 = M_N(\mathbb{F}_q) \). Let \( \mathfrak{P} \) denote the preimage of the corresponding nilpotent radical of the parabolic subalgebra. A parabolic subalgebra of \( \mathfrak{g}_1 \) is the stabiliser of a flag, and as such is \( G_1 \)-conjugate to a block upper triangular subalgebra of \( \mathfrak{g}_1 \). The nilpotent radical of a parabolic subalgebra in block form is the subalgebra obtained by replacing each diagonal block by a \( \mathfrak{0} \)-block of the same size. Define the following subgroups of \( G_r \):

\[
U = U^0 = \mathfrak{A}^\times, \quad U^m = 1 + \mathfrak{P}^m, \quad \text{for } m \geq 1.
\]

Let \( e = e(\mathfrak{A}) \) be the length of the flag in \( \mathfrak{g}_1 \) defining \( \mathfrak{A} \). Then it can be shown that

\[
\mathfrak{p}\mathfrak{A} = \mathfrak{A}\mathfrak{p} = \mathfrak{P}^e
\]

and one can think of \( e \) as a ramification index. We have a filtration

\[
U \supset U^1 \supset \cdots \supset U^{cr-1} \supset U^{cr} = \{1\},
\]

where the inclusions can be shown to be strict. It is also convenient to define \( U^i = \{1\} \) for all \( i > cr \). Since \( \mathfrak{P} \) is a (two-sided) ideal in \( \mathfrak{A} \), each group \( U^i \) is normal in \( U \). Moreover, we have the commutator relation

\[
[U^i, U^j] \subseteq U^{i+j}.
\]

Thus in particular, the group \( U^i \) is abelian whenever \( i \geq cr/2 \).

From now on, let \( β \in \mathfrak{g}_r \) be a regular element and write \( \bar{β} \) for its image in \( \mathfrak{g}_1 \). We will associate a certain parahoric subalgebra to \( β \) (or rather, to the orbit of \( β \)), which will be denoted by \( \mathfrak{A}_m \). Let

\[
\prod_{i=1}^{h} f_i(x)^{m_i} \in \mathbb{F}_q[x]
\]
be the characteristic polynomial of $\tilde{\beta}$, where the $f_i(x)$ are distinct and irreducible of degree $d_i$, for $i = 1, \ldots, h$. This determines a partition of $n$:

$$
\lambda = (d_1^{m_1}, \ldots, d_h^{m_h}) = (d_1, d_1, \ldots, d_h, d_h, \ldots, d_h).
$$

We define $\mathfrak{A}_m \subseteq \mathfrak{g}_r$ to be the preimage of the standard parabolic subalgebra of $\mathfrak{g}_1$ corresponding to $\lambda$ (i.e., the block upper-triangular subalgebra whose block sizes are given by $\lambda$, in the order given above). Moreover, we let $\mathfrak{A}_M = \mathfrak{g}_r = M_N(\mathfrak{o}_r)$ be the full matrix algebra. Let $\mathfrak{P}_m$ and $\mathfrak{P}_M$ be the corresponding ideals in $\mathfrak{A}_m$ and $\mathfrak{A}_M$, respectively. For $* \in \{m, M\}$ we have the corresponding groups

$$
U_* = U_*^0 = \mathfrak{A}_*^N, \quad U_*^i = 1 + \mathfrak{P}_*^i, \quad \text{for } i \geq 1,
$$

and the filtration

$$
U_* \supset U_*^1 \supset \cdots \supset U_*^{e_*} = \{1\},
$$

where $e_* = e(\mathfrak{A}_*)$. Note that $U_M^1 = K^1$. and $e_M = 1$. The label $m$ here stands for “minimal”, while $M$ stands for “maximal”. From the definitions, we have

$$
U_m/U_m^1 \cong \prod_{i=1}^{h} \text{GL}_{d_i}(\mathbb{F}_q)^{m_i},
$$

$$
U_M/U_M^1 \cong \text{GL}_N(\mathbb{F}_q).
$$

Note that if $\tilde{\beta}$ has irreducible characteristic polynomial, then $\mathfrak{A}_m = \mathfrak{g}_r$, and $U_m^i = K^i$ are the normal subgroups defined earlier.

By definition, we have $\mathfrak{A}_M \supset \mathfrak{A}_m$, and therefore $\mathfrak{P}_m \supset \mathfrak{P}_M$. The relations $\mathfrak{A}_M \supset \mathfrak{A}_m \supset \mathfrak{P}_m \supset \mathfrak{P}_M$ imply that for every $i \geq 1$, $\mathfrak{P}_i^M$ is a two-sided ideal in $\mathfrak{A}_m$, so $U_m$ normalises $U_M^i$. For $* \in \{m, M\}$, we can therefore define the following groups

$$
C = C_{G_*}(\beta),
$$

$$
J_* = (C \cap U_*)U_*^{e_*-1},
$$

$$
J_1^* = (C \cap U_1^*)U_*^{e_*-1},
$$

$$
H_1^* = (C \cap U_1^*)U_*^{e_*-1+1}.
$$

Recall that since $\beta$ is regular, $C$ is abelian. Since $[U_*^1, U_*^{e_*-1}] \subseteq U_*^{e_*-1+1}$ and $\mathfrak{A}_*^N$ normalises $U_*^{e_*-1}$, the group $J_*$ normalises both $J_1^*$ and $H_1^*$. Moreover, we define the group

$$
J_{m,M} = (C \cap U_m^1)K^{e_*}.
$$

We have the following diagram of subgroups, where the vertical and slanted lines denote inclusions (we have only indicated the inclusions which are relevant to us.
and repeat the definitions of the groups, for the reader’s convenience).

\[
\begin{array}{c}
CK^{l'} \\
| J_{m,M} \\
| J_m \\
| H_m \\
\vdots \\
K^l
\end{array}
\]

\[
J_{m,M} = (C \cap U_m)K^{l'},
\]

\[
J_m = (C \cap U_m)U_m^{e_m l'},
\]

\[
H_m = (C \cap U_m)U_m^{e_m l' + 1},
\]

\[
J_M = (C \cap K^1)K^{l'},
\]

\[
H_M = (C \cap K^1)K^l.
\]

We explain the non-trivial inclusions in the above diagram. Since \(P_m \supseteq P_M\), we have \(U_m^1 \supseteq K_1^1\) and \(U_m^{e_m} \supseteq 1 + \beta_m^l\P_m \supseteq 1 + \beta_m^l\P_M = K^l\); thus \(H_m^1 \supseteq H_M^1\). Moreover, \(U_m^{e_m l' + 1} = 1 + \beta_m^{l'}A_m \subseteq 1 + \beta_m^{l'}A_M = K^{l'}\), so \(J_{m,M}\) contains both \(J_m^1\) and \(J_M^1\) as subgroups. We remark that \(J_m^1\) is normal in \(CK^{l'}\) since \(C\) normalises both \(K_1^1\) and \(K^{l'}\), and \([K^{l'}, K^1] \subseteq K^l \subseteq K^{l'}\).

The following lemma will be used in Step ?? of the construction we will outline below, and is the main reason why we work with the algebra \(A_m\) and its associated subgroups.

**Lemma 7.1.** There exists a \(G_r\)-conjugate of \(\beta\) such that the group \(J_{m,M}\) is a normal \(p\)-Sylow subgroup of \(CK^{l'}\).

We sketch the proof of this lemma. We first show that \(J_{m,M}\) is normal in \(J_M\). Since \(C \cap A_m^\times\) normalises \(J_{m,M}\) (\(C\) being abelian), it is enough to observe that \(U_m^{e_m l'}\) normalises \(J_{m,M}\) (in any finite group \(G\) with a normal subgroup \(N\) and a subgroup \(H\), the group \(HN\) is normalised by \(N\); here \(G\) would be \(U_M\)). Write \(\beta_m\) for the image of \(\beta\) in \(U_m/U_m^1\). Then, up to conjugating \(\beta\), we have

\[
\beta_m = \underbrace{\beta_1 + \cdots + \beta_1}_{m_1 \text{ times}} \oplus \underbrace{\beta_h + \cdots + \beta_h}_{m_h \text{ times}},
\]

where \(\beta_i \in M_{d_i}(\mathbb{F}_q)\), and \(d_i\) and \(m_i\) are as in the partition \(\lambda\) above. With \(\beta_m\) of the above form, one can show that \(\beta\) being regular implies that \(C \subseteq U_m\), so we have an isomorphism

\[
CK^{l'}/J_{m,M} \cong \frac{C}{(C \cap U_m^1)(C \cap K^{l'})} = \frac{C \cap U_m}{(C \cap U_m^1)}.
\]
Then the isomorphism $U_m/U_m^1 \cong \prod_{i=1}^h \GL_d(F_q)^{m_i}$ induces an isomorphism

$$\frac{C \cap U_m}{C \cap U_m^1} \cong \prod_{i=1}^h \GL_d(F_q)(\beta_i)^{m_i}.$$  

Each $\beta_i$ has irreducible characteristic polynomial over $F_q$, so $F_q[\beta_i]/F_q$ is an extension of degree $d_i$. Since $\GL_d(F_q)(\beta_i) = F_q[\beta_i]^{\times}$, we conclude that $p$ does not divide the order of $\GL_d(F_q)(\beta_i)$. Therefore, $p$ does not divide the order of $\frac{C \cap U_m}{C \cap U_m^1}$, so $J_{m,M}$ is a $p$-Sylow subgroup of $CK^r$ (in fact the unique $p$-Sylow subgroup, since it is normal).

### 7.2. Characters

Let $\psi : F \to C^\times$ be as in Section 3. Let $\mathfrak{N}, \mathfrak{g}$, and $U_m$, $m \geq 0$ be the objects associated to an arbitrary flag of length $e$, as in Section 7.1. Let $n$ and $m$ be two integers such that $e(r-1)+1 \geq n > m \geq n/2 > 0$. Then $U_m/U^n$ is abelian, and we have an isomorphism

$$\mathfrak{g}^m/\mathfrak{g}^n \cong U^m/U^n, \quad x + \mathfrak{g}^n \mapsto (1+x)U^n.$$  

Each $g \in \mathfrak{g}_r$ defines a character $\psi_g : C^\times \to C^\times$ via $x \mapsto \psi(\text{tr}(ax))$, and this defines an isomorphism $\psi_g : \Irr(\mathfrak{g}_r)$. For any subgroup $S$ of $\mathfrak{g}_r$, define

$$S^\perp = \{x \in \mathfrak{g}_r \mid \psi(\text{tr}(xS)) = 1\}.$$  

Using the isomorphism $\mathfrak{g}_r \to \Irr(\mathfrak{g}_r)$, we can identify $S^\perp$ with the group of characters of $\mathfrak{g}_r$ which are trivial on $S$.

For any $\beta \in \mathfrak{g}^{e(r-1)+1-n}$ define a character $\psi_\beta : U^m \to C^\times$ by

$$\psi_\beta(1+x) = \psi(\text{tr}(\beta x)).$$

#### Lemma 7.2

Let $e(r-1)+1 \geq n > m \geq n/2 > 0$. Then

(i) For any integer $i$ such that $0 \leq i \leq e(r-1)+1$, we have

$$\mathfrak{g}^i = \mathfrak{g}^{e(r-1)+1-i}.$$  

(ii) The map $\beta \mapsto \psi_\beta$ induces an isomorphism

$$\mathfrak{g}^{e(r-1)+1-n}/\mathfrak{g}^{e(r-1)+1-m} \cong \Irr(U^m/U^n).$$  

We omit the proof of this lemma, and only remark that the first part essentially follows from the observation that $j = e(r-1)+1$ is the smallest integer such that $\mathfrak{g}^j$ is strictly block-upper triangular mod $p$. Indeed, $\mathfrak{g}^{e(r-1)+1} = p^{-1}\mathfrak{g}$, and $\mathfrak{g}^0$ is strictly block-upper mod $p$. This implies that $\mathfrak{g}^j = \mathfrak{g}^{e(r-1)+1}$, and the general case follows similarly.

As a special case of the above, suppose that $e = 1$, so that $\mathfrak{N} = \mathfrak{g}_r$ and $U^r = K^r = 1 + \mathfrak{g}_r$. For any $r = n > m \geq r/2$ and $\beta \in \mathfrak{g}_r$, we have a character $\psi_\beta : K^r \to C^\times$ defined as above, and the isomorphism of Lemma 7.2(ii) becomes

$$\mathfrak{g}_r/\mathfrak{p}^{-m}\mathfrak{g}_r \cong \Irr(K^m),$$

which agrees with the considerations in Section 3.
7.3. Construction of representations. For our fixed arbitrary regular element \( \beta \in g_r \), we start with the character \( \psi_\beta \) of \( K^l \), and construct all the irreducible representations of \( CK^l \) which contain \( \psi_\beta \). Theorem 7.2(ii) then yields all the irreducible representations of \( G \) with \( \beta \) in their orbits. The construction consists of a number of steps. For each step we indicate some of the details involved.

Some of the steps can be carried out for the groups arising from the algebras \( \mathfrak{A}_m \) and \( \mathfrak{A}_M \) simultaneously. For this purpose, we will let \( \mathfrak{A} \) denote either \( \mathfrak{A}_m \) or \( \mathfrak{A}_M \), and let \( \mathfrak{P} \) be the radical in \( \mathfrak{A} \), with “ramification index” \( e \). The associated subgroups will be denoted by \( U^\ell, H^1, J^1 \).

**Step 1:** Show that \( \psi_\beta \) has an extension \( \theta_m \) to \( H^1_M \). Show that \( \theta_M \) has an extension \( \theta_m \) to \( H^1_M \).

By Lemma 7.2(ii) if we take

\[
m = e\ell + 1, \quad n = 2m - 1 = e(r - 1) + 1,
\]
then \( \beta \), or rather the coset \( \beta + \mathfrak{P}^{e\ell} \), defines a character on \( U^m \), trivial on \( U^n \) by the same formula as the one defining \( \psi_\beta \). Since \( \mathfrak{P}^{e\ell} = p^{e} \mathfrak{A} \), we have a map

\[\mathfrak{A}/\mathfrak{P}^{e\ell} \to g_r/p^{e} g_r,\]
which sends the coset \( \beta + \mathfrak{P}^{e\ell} \) to \( \beta + p^{e} g_r \). Thus the different choices of lift of the latter coset give the different choices of extension of \( \psi_\beta \) to \( U^{e\ell + 1} \). Our element \( \beta \in \mathfrak{A} \) therefore gives rise to an extension (which we still denote by \( \psi_\beta \)) of \( \psi_\beta \) to \( U^{e\ell + 1} \), defined by

\[\psi_\beta(1 + x) = \psi(\varpi^{-r} \mathrm{tr}(\beta x)), \quad \text{for } x \in \mathfrak{P}^{e\ell + 1}.
\]

We now show the existence of the extensions \( \theta_M \) and \( \theta_m \). If \( c \in C \cap U^1 \) and \( x \in \mathfrak{P}^{e\ell + 1} \), then

\[\{c, 1 + x\} \in c(1 + x)c^{-1}(1 - x + \mathfrak{P}^{e(r - 1)+2}) = 1 + cc^{-1} - x + \mathfrak{P}^{e(r - 1)+2}.
\]

By Lemma 7.2(i) since \( \beta \in \mathfrak{A} \), we have

\[U^{e(r-1)+1} \subseteq \ker \psi_\beta,
\]
so

\[\psi_\beta([c, 1 + x]) = \psi(\varpi^{-r} \mathrm{tr}(\beta(cxc^{-1} - x))) = \psi(\varpi^{-r} \mathrm{tr}(c\beta xc^{-1} - \beta x)) = 1,
\]
where we have used that \( c \) commutes with \( \beta \).

Thus \( C \cap U^1 \) stabilises the character \( \psi_\beta \) on \( U^{e\ell + 1} \), and since \( C \cap U^1 \) is abelian, this implies that \( \psi_\beta \) extends to \( H^1 = (C \cap U^1)U^{e\ell + 1} \). We fix an extension \( \theta_M \) to \( H^1_M \) and an extension of \( \theta_m \) to \( H^1_M \), denoted \( \theta_m \).

**Step 2:** For \( * \in \{m, M\} \), construct the irreducible representations \( \eta_\ast \) of \( J^1_M \) containing \( \theta_\ast \). In particular, show that there exists a unique representation \( \eta_M \) of \( J^1_M \) containing \( \theta_M \).

As in the previous step, we will treat both cases simultaneously, denoting either \( \theta_m \) or \( \theta_M \) by \( \theta \). We outline the ingredients needed for this. First note that \( \theta \) is
This step can be seen as the reason for involving the “auxiliary” path through representation

Step 3: Show that there exists an extension 

The order of \( R_{\beta,m} \), can be used to calculate the dimension of \( \eta_m \), indeed \( \dim \eta_m = \left[ J_m^1 : R_{\beta,m}\right]^{1/2} \), so 

\[
\dim \eta = \left[ J_m^1 : R_{\beta,m}\right]^{1/2} [J_m^1 : J_m^1].
\]
Comparing this with the dimension of \( \eta_{M} \), which is \( [J_{M}^{1} : H_{M}^{1}]^{1/2} = q^{N(N-1)/2} \), it turns out that \( \dim \eta_{M} = \dim \eta \). Then, since the restricted representation \( \eta|_{J_{M}^{1}} \) contains \( \theta_{M} \) on further restriction to \( H_{M}^{1} \), and \( \eta_{M} \) is the unique representation of \( J_{M}^{1} \) with this property, it follows that \( \eta \) contains \( \eta_{M} \) on restriction to \( J_{M}^{1} \). The equality of the dimensions then forces \( \eta|_{J_{M}^{1}} = \eta_{M} \) (and in particular, \( \eta \) is irreducible).

Furthermore, one can show that all of \( CK' \) stabilises the character \( \theta_{M} \). Since \( J_{m,M} \) is a \( p \)-Sylow subgroup in \( CK' \) by Lemma 7.1 and \( \eta_{M} \) extends to \( J_{m,M} \), it then follows from [14, Corollary 11.31] and a theorem of Gallagher [7, Theorem 6] that \( \eta_{M} \) has an extension \( \hat{\eta}_{M} \) to \( CK' \) (the same extension result was used in the end of Section 6 for the extension from \( J_{\beta} \) to \( C_{G_{\beta}} \)).

Note that \( \eta_{m} \) is not the only representation containing \( \theta_{m} \), and therefore \( \eta \) is not unique. This does not matter for us, since we are only interested in proving that \( \eta_{M} \) has an extension, so we only need one representation \( \eta \).

We also remark that even though both \( \eta \) and \( \hat{\eta}_{M} \) are extensions of \( \eta_{M} \), we do not know (and do not need to know) whether \( \hat{\eta}_{M} \) is an extension of \( \eta \).

**Step 4:** The final step in the construction is to note that every irreducible representation of \( CK' \) which contains \( \psi_{\beta} \) is of the form \( \hat{\eta}_{M} \) for some choice of extension \( \theta_{M} \) of \( \psi_{\beta} \) and some choice of extension \( \hat{\eta}_{M} \) of \( \eta_{M} \), and that distinct choices of \( \theta_{M} \), as well as distinct choices of extensions \( \hat{\eta}_{M} \) of \( \eta_{M} \), give rise to distinct representations of \( CK' \).

By a standard result in Clifford theory (Lemma 3.2) we have a one to one correspondence between \( \text{Irr}(CK' \mid \psi_{\beta}) \) and \( \text{Irr}(G_{\beta} \mid \psi_{\beta}) \) given by induction. Thus, we have constructed all the irreducible representations of \( G_{\beta} \) with \( \beta \) in their orbits.

Schematically, the construction is illustrated by the following diagrams (dotted lines are extensions, dashed are Heisenberg lifts, and solid one between \( \eta_{m} \) and \( \eta \) is an induction):

```
\begin{figure}
\centering
\includegraphics[width=\textwidth]{diagram.png}
\caption{Diagram illustrating the construction process.}
\end{figure}
```

8. **Open problems**

We close with a non-exhaustive list of open problems in the representation theory of \( G_{r} = \text{GL}_{N}(\mathfrak{o}_{r}) \). Several other problems are suggested in [2, Section 1.6].
8.1. Beyond GL\(_N\). It is natural to ask whether it is possible to construct regular representations of reductive groups over \(\mathfrak{o}\), other than GL\(_N\). As we have already mentioned, [18] constructs regular representations for SL\(_N(\mathfrak{o})\), \(p \nmid N\), as well as for unitary groups. These cases are relatively close to GL\(_N\), but one may expect that it is possible to construct the regular representations of \(G(\mathfrak{o})\) whenever \(G\) is a sufficiently nice reductive group scheme over \(\mathfrak{o}\), for example when the derived group of \(G\) is simply connected and \(p\) is a very good prime. The first step is to show that under some hypotheses on \(G\), any \(\beta \in \text{Lie}(\mathfrak{o})\) such that \(\tilde{\beta} \in \text{Lie}(\mathfrak{F}_q)\) is regular, will have abelian centraliser in \(G(\mathfrak{o})\) and the surjective mapping property of centralisers under reduction maps.

8.2. Beyond regular representations. Hill’s construction of strongly semisimple representations (see Section 5) shows that Clifford theoretic methods can be used to construct some non-regular representations of \(GL_N(\mathfrak{o})\), up to knowledge of all the irreducible representations of \(GL_N(\mathfrak{o})\) for \(N' < N\), \(r' < r\). Is there a uniform construction which includes the regular representations and the strongly semisimple representations (and perhaps others)?

8.3. Relation with supercuspidal types. Henniart [3] and Paskunas [26] have shown that every supercuspidal representation of \(GL_N(F)\) has a unique type on \(GL_N(\mathfrak{o})\). It would be interesting to identify the regular representations which are supercuspidal types and determine what they map to under the inertial Langlands correspondence.

8.4. Onn’s conjectures. For each integer \(n \geq 1\), let

\[ r_n = r_n(G_r) = \# \{ \pi \in \text{Irr}(G_r) \mid \dim \pi = n \}. \]

The experience with the known cases of \(GL_2(\mathfrak{o})\) [31, 25], \(GL_3(\mathfrak{o})\) [2] and the regular representations of \(GL_N(\mathfrak{o})\), suggests that \(r_n\), as a function of \(\mathfrak{o}\), is rather well behaved. More precisely, in all known cases, it is a polynomial over \(\mathbb{Q}\) in the size \(q\) of the residue field, independent of the compact DVR \(\mathfrak{o}\), as long as the residue field is \(F_q\). Moreover, the dimensions of the known representations of \(GL_N(\mathfrak{o})\) are given by polynomials in \(q\), and one may ask whether this is true in general. In [25] Onn made the following conjectures, which we paraphrase slightly and state only for \(GL_N(\mathfrak{o})\):

**Conjecture (Onn).**

(i) Suppose \(\mathfrak{o}\) and \(\mathfrak{o}'\) are two compact DVRs with maximal ideals \(p\) and \(p'\), respectively, such that \(|\mathfrak{o}/p| = |\mathfrak{o'}/p'|\). Then there is an isomorphism of group algebras

\[ \mathbb{C}[GL_N(\mathfrak{o})] \cong \mathbb{C}[GL_N(\mathfrak{o}')]. \]

(ii) For any \(n \geq 1\) there exists a polynomial \(p_n(x) \in \mathbb{Q}[x]\) such that for any compact DVR \(\mathfrak{o}\) we have

\[ r_n(GL_N(\mathfrak{o})) = p_n(q), \]

where \(q = |\mathfrak{o}/p|\).

(iii) There exist finitely many polynomials \(d_1(x), \ldots, d_h(x) \in \mathbb{Z}[x]\) with \(\deg d_i \leq \binom{N}{2} r\), such that for any compact DVR \(\mathfrak{o}\) we have

\[ \{ \dim \pi \mid \pi \in \text{Irr}(GL_N(\mathfrak{o})), \pi \text{ primitive} \} = \{d_1(q), \ldots, d_h(q)\}, \]

where \(q = |\mathfrak{o}/p|\).
Note that part (ii) of this conjecture implies part (i).

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