Multiple choice minority game with different publicly known histories

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Abstract. In the standard minority game (MG), players use historical minority choices as the sole public information to pick one out of the two alternatives. However, publishing historical minority choices is not the only way to present global system information to players when more than two alternatives are available. Thus, it is instructive to study the dynamics and co-operative behaviours of this extended game as a function of the global information provided. We numerically find that, although the system dynamics depend on the kind of public information given to the players, the degree of co-operation follows the same trend as that of the standard MG. We also explain most of our findings by the crowd–anticrowd theory.

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1. Introduction

Many phenomena in a variety of fields including biology and economics can be modelled by agent-based complex adaptive systems (CAS) [1]–[3]. In fact, CAS can be used to gain empirical understanding, normative understanding, qualitative insight and theory generation together with methodological advancement in economic systems. This agent-based approach focuses on the dynamics and the effects of the initial or boundary conditions on an economic system, as opposed to the conventional economic methodology which concentrates mainly on the equilibrium state of the system [3]. In this respect, techniques in statistical physics and nonlinear dynamics can be applied in the study of economic systems. This is the goal of the rapidly growing field of econophysics.

The minority game (MG) [4] is perhaps the simplest agent-based econophysical model that captures the minority seeking behaviour of independent selfish players. In the original version of the MG, each player picks one out of two alternatives in each time step based on the publicly posted minority choices of the previous $M$ turns. Those correctly picking the minority choice are awarded one dollar while the others are deducted one dollar. Although players in the MG are selfish and only aim at maximizing their own wealth, they do so in a co-operative manner. In addition, the MG exhibits a second-order phase-transition point dividing the parameter space into the so-called symmetric and asymmetric phases [4, 5]. Besides, the co-operation phenomenon and the phase-transition point appear to be very common as they are also observed in several variants of the MG that use more than two alternatives [6, 7], evolving strategies [8, 9], different pay-off functions [10, 11], different network topology [12] and a mixed population of players [13, 14].

In the MG, some public information is given to players for reference in making their decisions. In both the original MG and many of its variants, the public information is the minority choices of the previous $M$ turns. In other words, this public information gives a complete description of historical winning choices of the previous $M$ turns. Of course, one may give other public information to players. Perhaps the most well-known case comes from a series of studies concerning the relevance of history in the MG that was initiated by a paper by Cavagna [15]. This series of studies investigated the effect of replacing the actual historical winning choices by some
fake ones on the dynamics of the MG. For instance, Challet and Marsili extensively studied the effect of substituting a randomly and independently chosen bit string for each historical minority choice on the dynamics of the MG. They found that, although the modified game still shows phase transition right at the same point in the parameter space, the dynamics are markedly different from the original MG in the asymmetric phase [16]. Their findings were echoed by a recent work of Ho et al [17], who discovered that the dynamics of the original and the modified game also differ in the symmetric phase. Other examples of using different public information came indirectly from the investigations of players with different memory sizes [4, 18] and players acting only on local information [12, 19]. These studies showed that the dynamics in many variants of the MG depend on the historical outcomes of the game.

Nonetheless, publishing the historical winning choices of the previous $M$ turns is not the only way to present certain real global information of the system to players. For example, it is instructive to investigate what will happen if the publicly known historical minority choice is replaced by the publicly known historical majority choice. In the case of the original MG [4, 5], the statistical properties of this majority history model are identical to that of the original MG, as the knowledge of the historical majority choice is equivalent to that of the historical minority choice. In contrast, the situation is radically different when the number of alternatives in the model is greater than two. An extension of the MG that allows more than two alternatives was proposed by Ein-Dor et al [20]. They regarded the alternatives as states of Potts spins and replaced player’s strategies by feedforward neural networks with Hebbian learning rules. Their model is not suitable to study the effect of global information replacement as the method used by players to decide their choices is also changed. Thus, our study is based on a simpler extension of the standard MG proposed by Chau and Chow [6], which allows players to choose from $N_c > 2$ equally capable alternatives with different types of publicly posted real histories without changing the algorithm of player’s decision. Our numerical simulations show that the general trend of the co-operative behaviour does not depend on the kind of common real historical data used. In addition, we find that the location of the second-order phase-transition point separating the symmetric and asymmetric phases is independent of the publicly posted histories. Most of our findings can be understood by the crowd–anticrowd theory proposed by Hart et al [21, 22].

2. The game MG ($N_c$) and its extension

Our study focuses on an extension of the MG known as MG($N_c$) proposed by Chau and Chow in [6]. In MG($N_c$), each of the $N$ players picks one out of $N_c$ alternatives independently in each turn where $N_c$ is a prime power. The choice picked by the least nonzero number of players is said to be the (1st) minority choice in that turn. Those who have picked the minority choice will be awarded one dollar while the others will be deducted one dollar. (In this respect, the $N_c$ alternatives are treated on equal footing in this game.) The minority choices of the previous $M$ turns are publicly announced. To aid each player making their decisions, each of them is randomly

\[2 \text{ There are two reasons why we restrict } N_c \text{ to be a prime power. (Definitions of various terms in this footnote can be found later in the text.) Firstly, finding the maximal reduced strategy space size is a very difficult combinatorial problem whose solution is not known for a general } N_c \text{ to date. Secondly, even if the maximal reduced strategy space is found, it is possible that such space is not uniformly distributed in the full strategy space. Hence, the expression of the control parameter } \alpha \text{ is not known when } N_c \text{ is not a prime power.} \]
and independently assigned once and for all \( S \)-deterministic strategies before the game begins. A strategy is a table that assigns every possible history (in this case, the 1st minority choice of the previous \( M \) turns) to a choice. In other words, it is a map from the set of all possible histories \( H \) to the finite field of \( N_c \) elements \( GF(N_c) \). Clearly, there are totally \( N_c^{N_M} \) different possible strategies and this collection of strategies is called the full strategy space. To evaluate the performance of a strategy, a player looks at the virtual score which is the current hypothetical wealth if that strategy were used throughout the game. Every player follows the choice of his/her current best-working strategy, namely, the one with the largest virtual score, to pick an alternative [6]. (In case of a tie, the player randomly picks one from his/her pool of best-working strategies.)

Now, we consider a generalization of the MG(\( N_c \)) model known as MG\(_{sub}(N_c)\), where the subscript ‘\( sub \)’ describes the kind of historical choices used. When \( sub = \text{min}(q) \), we publicly announce the historical \( q \)th minority choices of the past \( M \) turns instead of the historical 1st minority choices. (More precisely, we arrange those alternatives chosen by nonzero number of players in ascending order of the number of players chosen. Those alternatives with equal number of players chosen are arranged randomly in this sequence. The \( q \)th minority choice is the \( q \)th alternative in this sequence. In the event that the number of alternatives chosen by nonzero number of players is less than \( q \), we define the \( q \)th minority choice as the last entry in this sequence.) Similarly, when \( sub = \text{maj}(q) \), we publish the historical \( q \)th majority choices of the past \( M \) turns. We call the publicly announced alternatives the history string irrespective of the state ‘\( sub \)’. Moreover, we stress that apart from the global information released, all the rules in MG\(_{sub}(N_c)\) are the same as those of MG(\( N_c \)). Thus, MG(\( N_c \)) = MG\(_{\text{min}(1)}(N_c)\).

3. Numerical results

Following [4, 6], we measure the degree of player co-operation by considering the mean variance of attendance over all alternatives (or simply the mean variance)

\[
\Sigma^2 = \frac{1}{N_c} \sum_{\Omega \in GF(N_c)} \langle \{(A_i(t)^2)_{t} - \langle A_i(t)^2 \rangle_t\}_\Omega^2 \rangle = \langle \{(A_0(t)^2)_{t} - \langle A_0(t)^2 \rangle_t\}_\Omega^2 \rangle, \tag{1}
\]

where the attendance of an alternative \( A_i(t) \) is the number of players picking the alternative \( i \) at turn \( t \). Note that \( \langle \cdots \rangle_t \) and \( \langle \cdots \rangle_\Omega \) are the expectation values averaged over time and over strategies initially assigned to the players respectively. Since all the \( N_c \) alternatives are treated on equal footing in MG\(_{sub}(N_c)\), we may arrive at the second line in equation (1). The smaller the \( \Sigma^2 \), the better the player co-operates. More importantly, for a fixed \( S \) and up to first-order approximation, \( \Sigma^2 \) depends only on the control parameter \( \alpha = N_c^{M+1}/NS \) which measures the relative diversity of the strategies used in the system [23].

Furthermore, to investigate the phase diagram of MG\(_{sub}(N_c)\), we follow [7, 24] to study the order parameter

\[
\theta = \frac{1}{N_c} \sum_{\mu} \left\{ \sum_{\Omega} \left[ \langle p(\Omega|\mu) \rangle_t - \frac{1}{N_c} \right]^2 \right\}, \tag{2}
\]

where \( \langle p(\Omega|\mu) \rangle_t \) denotes the time average of the probability such that the current minority choice is \( \Omega \) conditioned on a global history string \( \mu \).

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Figure 1. The variance of attendance per player $\frac{\Sigma^2}{N}$ and the order parameter $\theta$ versus the control parameter $\alpha$ for (a) $\text{MG}_{\text{sub}}(5)$ and (b) $\text{MG}_{\text{sub}}(7)$. The values of $M$ and $S$ used in all the figures in this paper are 3 and 2, respectively.

Figure 1 plots $\frac{\Sigma^2}{N}$ and $\theta$ as a function of $\alpha$ for different ‘sub’ and $N_c$. Each data point presents the average value over 1000 different runs and the value for each run is averaged over 25 000 iterations after discarding the first 20 000 iterations to allow equilibration. We find that within the range of parameters we have simulated, the system indeed equilibrates well before the first 20 000 iterations. All $\frac{\Sigma^2}{N}$ curves in figure 1 show a similar trend and have cusps around $\alpha \approx 1$, irrespective of the values of $N_c$ and ‘sub’ used; and the $\theta$ curves indicate second-order phase transitions around the cusps separating the symmetric and the asymmetric phases. (The small drop in $\frac{\Sigma^2}{N}$ for large value of $\alpha$ in figure 1(a) is due to finite size effect as the number of players $N$ is about 30.) Although, we only show those curves for $N_c = 5$ and 7, similar behaviours are observed for other values of $N_c > 2$. In fact, our numerical simulations show that the critical points of all curves with the same $N_c$ and $S$ coincide. That is to say, the critical value $\alpha_c$ is found to be a function of $N_c$ and $S$ only and is independent of the kind of history string used. Besides, $\frac{\Sigma^2}{N}|_{\alpha_c}$ is a function of $N_c$ and $S$ only. The behaviour of $\frac{\Sigma^2}{N}$ away from the phase-transition point $\alpha_c$ is also worth mentioning. For $\alpha \ll \alpha_c$, the variance per player for sub $= \text{min}(1)$ is consistently greater than those obtained in MGs using other public information. In contrast, for $\alpha \gtrsim \alpha_c$, $\frac{\Sigma^2}{N}$ for sub $= \text{maj}(1)$ is consistently smaller than those obtained in MGs using other globally announced information.
4. The possibility of an analytical or semi-analytical solution

Three major approaches to study the MG analytically or semi-analytically are known to date. We briefly discuss their potentials in solving the MG$_{\text{sub}}(N_c)$ one by one below.

4.1. Replica trick

The replica trick was used by Challet et al to solve the MG analytically [24]–[26]. By using the known fact that the majority choice automatically implies the knowledge of the minority choice when $N_c = 2$, they wrote down a simple Hamiltonian quadratic in the random spin variable. The standard replica trick can then be used to compute the probability distribution of the average action of a player. Subsequently, the quantities such as variance per player, $\Sigma^2/N$, can be computed analytically. Although the value of $\Sigma^2/N$ obtained by the replica trick agrees reasonably well with the numerical findings in the asymmetric phase of the standard MG, the averaging procedure in the replica trick makes this method impossible to recover the dynamics and the value of $\Sigma^2/N$ in the symmetric phase.

The replica trick faces a further problem when $N_c > 2$. The Hamiltonian involves a term in the form $\delta_{i,i_0}$ for some $i, i_0 \in GF(N_c)$. This term is equal to the degree $(N_c - 1)$ polynomial $\prod_{j \neq i_0} (i - j)/(i - i_0)$ in random spin variable and hence the associated saddle point equation is much harder to solve.

4.2. Generating functional method

Coolen pioneered the use of the generating functional method to solve the standard MG [27, 28]. This method is mathematically rigorous and exact. In fact, it can be used to compute the variance per player $\Sigma^2/N$ in the asymmetric phase of the MG efficiently. Nonetheless, the presence of long-time scale periodic dynamics in the symmetric phase makes the computation of $\Sigma^2/N$ in this phase using the generation functional method not very successfully to date [29].

Applying the generation functional method to solve MG$_{\text{sub}}(N_c)$ could face two more problems. Firstly, the equation governing the evolution from one turn to the next involves a term in the form $\delta_{i,i_0}$ for some spin variables $i, i_0 \in GF(N_c)$ and hence is equal to a polynomial of degree $(N_c - 1)$. Secondly, computing the relative popularity of the $N_c$ alternatives are required when sub $\neq \min(1)$ or maj(1). These two requirements add further complexity to the generating functional when $N_c > 2$.

4.3. Crowd–anticrowd theory

Developed by Hart et al [21, 22], crowd–anticrowd theory is a semi-analytical method to explain the dynamics and variance observed in various variants of MG. This method provides an intuitive understanding of the origin of player co-operation. Although it predicts the existence of phase transition around $\alpha \approx 1$, it does not give us a simple way to calculate the value of $\alpha_c$.

Chau and Chow [6, 7] have successfully extended the crowd–anticrowd theory to study the case when the number of alternatives $N_c$ is a prime power. This makes the theory a good choice to investigate the dynamics of MG$_{\text{sub}}(N_c)$.
5. The crowd–anticrowd explanation

Most of our numerical simulation results can be explained by the crowd–anticrowd theory. Recall that two strategies are said to be uncorrelated if the probability for them to make the same choice equals \( \frac{1}{N_c} \) when averaged over the set of all possible history string. And two strategies are called anti-correlated if they make different choices for every input history string. Besides, two strategies are said to be significantly different if they are either anti-correlated or uncorrelated. In fact, one can form a subset of \( N_c^{M+1} \) strategies from the full strategy space in such a way that any two distinct strategies in this subset are significantly different. Besides, the size of this subset is maximal in the sense that no such subset with more than \( N_c^{M+1} \) strategies exists. This subset is called the maximal reduced strategy space \([5, 6, 21, 22]\). Most importantly, numerical simulations show that the dynamics of \( \text{MG}_{\text{sub}}(N_c) \) for strategies taken from the full or from the maximal reduced strategy spaces are similar.

From the discussions in \([6]\), one may label a strategy in the maximal reduced strategy space by \((\lambda, \beta) \in GF(N_c^M) \times GF(N_c)\), in such a way that two strategies \((\lambda, \beta)\) and \((\lambda', \beta')\) are uncorrelated if and only if \( \lambda \neq \lambda' \). They are anti-correlated if and only if \( \lambda = \lambda' \) and \( \beta \neq \beta' \). They are the same if and only if \( \lambda = \lambda' \) and \( \beta = \beta' \). According to the crowd–anticrowd theory, the mean variance of attendance \( \Sigma^2 \) in \( \text{MG}(N_c) \) is governed by an ensemble of mutually uncorrelated sets of anti-correlated strategies \([6, 21, 22]\). That is to say,

\[
\Sigma^2 \approx \left( \frac{1}{N_c^{M+2}} \sum_{\lambda, \beta} \left\{ \sum_{\beta' \neq \beta} \left[ N_{\lambda, \beta}(t) - N_{\lambda, \beta'}(t) \right] \right\} \right)_{t, \Sigma}^2 , \tag{3}
\]

where \( N_{\lambda, \beta}(t) \) denotes the number of players making decision according to the strategy \((\lambda, \beta)\) in the anti-correlated strategy set \( \lambda \) in turn \( t \).

When \( \alpha \ll \alpha_c \), known as the symmetric phase, there is a periodic dynamics in the time series of the minority choice. According to the crowd–anticrowd theory, these dynamics lead to a large mean variance of attendance per player \( \Sigma^2 / N \) in \( \text{MG}_{\text{min}}(N_c) \) \([21]–[23], [30, 31]\). Let us briefly review the origin of this periodic dynamics in \( \text{MG}_{\text{min}}(N_c) \). When the number of strategies at play is much larger than the maximal reduced strategy space size \( N_c^{M+1} \), it is very likely for players to employ similar strategies. Initially, for a given history string \( \mu \), every alternative has equal probability of being the minority. Moreover, the virtual score of a strategy \((\lambda, \beta)\) that gives the correct prediction of the minority is increased while that of its anti-correlated strategies are decreased. As there are more players than the maximal reduced strategy space size, in the next occurrence of the same history string \( \mu \), more players may use \((\lambda, \beta)\) to pick their alternatives. Thus, \((\lambda, \beta)\) is less likely to correctly predict the minority choice due to overcrowding of strategies. Inductively, overcrowding of strategies leads to the existence of a period-\( N_c \) dynamics in the minority choice as well as the attendance time series conditioned on an arbitrary but fixed history in \( \text{MG}_{\text{min}}(N_c) \) \([16, 17, 21, 22, 30, 31]\). Another periodic dynamics coming from a slightly different origin is also present in \( \text{MG}_{\text{min}}(N_c) \). Recall that the history string gives complete information of the winning choices in the past \( M \) turns in \( \text{MG}_{\text{min}}(N_c) \), making its minority choice time series highly correlated. By extending the analysis of Challet and Marsili in \([16]\) from \( N_c = 2 \) to a general prime power \( N_c \), we conclude that the minority choice time series from the \((N_c^{M+1} k + 1)\)th to the \([N_c^{M+1} (k + 1)]\)th turn is likely to form a
Figure 2. The power spectral density of the auto-correlation function of the attendance $A(t)$ against frequency for a typical run in (a) $\mu_{\text{min}}^{(1)}(5)$, (b) $\mu_{\text{min}}^{(2)}(5)$, (c) $\mu_{\text{maj}}^{(1)}(5)$ and (d) $\mu_{\text{maj}}^{(2)}(5)$ averaged by 50 runs for $\alpha = 0.05$. The period-$N_{c}^{M+1}$ dynamics is pronounced only in (a).

de Bruijn sequence$^3$ of length $N_{c}^{M+1}$ for all $k \in \mathbb{N}$, resulting in a period $N_{c}^{M+1}$ peak in the Fourier transform of both the minority choice and the attendance time series [31]. Besides, it is likely that between the above $N_{c}^{M+1}$ turns, each strategy wins exactly $N_{c}^{M}$ times. We follow the convention in [17] by calling this correlation in the minority choice and attendance time series the period-$N_{c}^{M+1}$ dynamics. Note that because of the period-$N_{c}^{M+1}$ dynamics, the virtual score difference between any two strategies is likely to be zero in the $(N_{c}^{M+1}k + 1)$th turn for all $k \in \mathbb{N}$. We call this phenomenon virtual score reset [17].

However, for $\mu_{\text{sub}}(N_{c})$ other than $\mu_{\text{min}}^{(1)}(N_{c})$ or $\mu_{\text{maj}}^{(1)}(2)$, the knowledge of the history string does not give a player complete information on the minority choice. Suppose again that initially the strategy $(\lambda, \beta)$ correctly predicts the minority choice for a given history string $\mu$. Since the virtual score calculation is still based only on the historical minority choices, in the next occurrence of $\mu$, strategy $(\lambda, \beta)$ may be well able to correctly predict the minority choice as it is chosen by only a few players. Therefore, the publicly announced histories from the $(N_{c}^{M+1}k + 1)$th to the $(N_{c}^{M+1}(k + 1))$th turn no longer tend to form a de Bruijn sequence and the virtual score difference between the two distinct strategies is unlikely to reset [17, 31]. That is why the period-$N_{c}^{M+1}$ dynamics almost completely disappears as shown in figure 2. (Although we only present

$^3$ A de Bruijn sequence of length $q^n$ over an alphabet of size $q$ is defined as a sequence that contains all the $q^n$ possible $n$-tuples as its subsequence. (We allow wraparound when defining a subsequence. For example, 0110 is a de Bruijn sequence of length 4 over the set of alphabets $\{0, 1\}$.)
Figure 3. The power-spectral density of the auto-correlation function of the attendance $A(t)$ conditioned on an arbitrary but fixed history against frequency shows the slight strengthening of period-$N_c$ dynamics when $N_c > 2$. Parameters used in this plot is the same as that in figure 2.

the periodic dynamics and distribution of history strings for the case of $N_c = 5$ in figures 2–4, our simulations for other values of $N_c$ are consistent with our crowd–anticrowd explanation in this section.) Nonetheless, from figure 3, we observe that the period-$N_c$ dynamics is slightly strengthened. To understand why, let us recall that in $\text{MG}_{\text{min}}(N_c)$ and $\text{MG}_{\text{maj}}(2)$, the virtual score reset mechanism implies that the $(N_c k + 1)$th terms for all $k \in \mathbb{N}$ in the attendance time series conditioned on an individual history is positively correlated. This is the major contributor to the period-$N_c$ dynamics. In contrast, for other $\text{MG}_{\text{sub}}(N_c)$ and for a fixed $\ell = 1, 2, \ldots, N_c$, the absence of a virtual score reset mechanism implies that correlations among the $(N_c k + \ell)$th terms for all $k \in \mathbb{N}$ in the time series of attendance conditioned on an individual history all pay about the same contribution to the period-$N_c$ dynamics, resulting in a stronger correlation. However, the strength of this auto-correlation conditioned on a particular history does not give complete information on the degree of overcrowding. It is the disappearance of period-$N_c^{M+1}$ dynamics and the absence of virtual score reset mechanism that make a player more likely to stick to a strategy. Hence, players co-operate slightly better leading to a slightly smaller mean variance of attendance in other $\text{MG}_{\text{sub}}(N_c)$ [17]. In this way, crowd–anticrowd theory explains not only why all the $\Sigma^2 / N$ versus $\alpha$ curves in figure 1 follow the same trend in the symmetric phase, but also attributes their slight differences to the strength of period-$N_c$ dynamics.

We move on to discuss the situation of $\alpha \gtrsim \alpha_c$, namely, the asymmetric phase. In this phase, the number of strategies at play $N_S$ is less than the maximal reduced strategy space size $N_c^{M+1}$. 


Figure 4. The frequency of occurrence of history strings sorted in descending order and then averaged over 50 runs for (a) $\mathrm{MG}_{\text{maj}}(1)(5)$, (b) $\mathrm{MG}_{\text{maj}}(2)(5)$, (c) $\mathrm{MG}_{\text{min}}(1)(5)$ and (d) $\mathrm{MG}_{\text{min}}(2)(5)$ at $\alpha = 1.5$.

Thus, the probability that a particular alternative can never be picked by at least $N/N_c$ players increases as the number of players $N$ decreases. By the pigeonhole principle, for an alternative to be the majority choice, the number of players choosing that alternative must be at least $N/N_c$. Consequently, whenever $\alpha \gtrsim \alpha_c$, some alternatives may never have a chance to be the majority choice. In contrast, there is no such type of constraint preventing an alternative from being a non-maj(1) choice. Therefore, among all the history string generation methods we have studied, the sub = maj(1) one will generate the most non-uniformly distributed history strings. This assertion is confirmed in figure 4, which plots the frequency count of the history occurrence arranged in descending order and then averaged over 50 runs. As the non-uniformity of history string leads to a reduction of the effective strategy space size, the crowd–anticrowd cancellation is strengthened in the asymmetric phase [16]. Thus, the mean variance of attendance obtained by using a majority history string is consistently lower than those obtained by using other history string generation methods. Again, crowd–anticrowd theory is able to explain the slight difference in $\Sigma^2/N$ between maj(1) and non-maj(1) history posting methods in the asymmetric phase.

Finally, we study the phase-transition point $\alpha = \alpha_c$. The crowd–anticrowd analysis reported earlier attributes the increase in variance per player by decreasing $\alpha$ in the symmetric phase (increasing $\alpha$ in the asymmetric phase) to overcrowding of strategies (insufficient sampling). In this respect, crowd–anticrowd theory predicts the existence of a single-phase-transition point separating the regions $\theta = 0$ and $\theta > 0$ around $\alpha \approx 1$ although it does not provide an efficient way to compute its value $\alpha_c$. Near the point of maximal co-operation, the number of strategies at play is approximately equal to $N_c^{M+1}$. In this regime, the number of players using strategies $(\lambda, \beta)$
and $(\lambda, \beta')$ are always about the same for all $\beta \neq \beta'$. Thus, from equation (3), the small variance at the point of maximal co-operation is the result of an optimal crowd–anticrowd cancellation [6, 21, 22]. Recall that two anti-correlated strategies always give different suggestions irrespective of the history string. So, it seems reasonable that near the point of maximal co-operation, the degree of co-operation and hence the values of $\alpha_c$ and $\Sigma^2 / N |_{\alpha_c}$ are independent of the history string generation method ‘sub’. However, one should not regard this argument as a firm proof for the precise values of $\alpha_c$ and $\Sigma^2 / N |_{\alpha_c}$ may depend in general on the complex adaptive dynamics of the system.

6. Discussions

We have introduced a modification of the MG called $MG_{\text{sub}}(N_c)$ and studied its properties numerically. We argued under the framework of crowd–anticrowd theory that the general trend of the $\Sigma^2 / N$ versus $\alpha$ curve is independent of the real history string generation method although the dynamics of the system depend on the kind of common history string used. We also numerically found that $\alpha_c$ and $\Sigma^2 / N |_{\alpha_c}$ are functions of $N_c$ and $S$ only and are independent of the type of global history ‘sub’ used. This finding is consistent with crowd–anticrowd theory. We remark that although all numerical simulations reported in the paper are performed by picking the strategies from the full strategy space, the same conclusions are reached if the strategies are taken from the maximal reduced strategy space. To summarize, these findings together with that in [17] show that the level of co-operation among players does not change significantly if the minority history string is replaced by a variety of global data such as the historical majority and a fake history string.

It is instructive to further extend our analysis to the case that the rankings of all the $N_c$ alternatives in each of the previous $M$ turns are used as public information. We believe that both kinds of periodic dynamics should be present in the symmetric phase. Verification of these hypotheses by numerical simulation is, however, a very computational intensive task as the maximal reduced strategy space size is prohibitively large except for $N_c \lesssim 4$.

Lastly, our findings imply that one must alter the original MG in some other ways in order to significantly change the trend of the $\Sigma^2 / N$ versus $\alpha$ curve. Possibilities include the use of thermal updating rules [32] and the introduction of initial bias in a virtual score [33, 34].

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