VARIATIONS OF HODGE–DE RHAM STRUCTURE
AND ELLIPTIC MODULAR UNITS

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Introduction

According to the general motivic folklore, one expects the group of one-extensions of $\mathbb{Q}(\kappa)$ by $\mathbb{Q}(\kappa)$ in the category $\mathcal{MM}_\mathbb{Q}(B)$ of mixed motivic sheaves on a scheme $B$ to be given by

$$\text{Ext}^1_{\mathcal{MM}_\mathbb{Q}(B)}(\mathbb{Q}(\kappa), \mathbb{Q}(\kappa)) = \Gamma(B, \mathcal{O}_B^\ast) \otimes \mathbb{Q}.$$  

If $B$ is a smooth and separated scheme over a field embeddable into $\mathbb{C}$, then it is possible to define, as a first approximation to $\mathcal{MM}_\mathbb{Q}(B)$, a smooth sheafified variant $\mathcal{MS}^\circ_B(B)$ of mixed realizations à la Deligne–Jannsen, and a functorial monomorphism

$$\Gamma(B, \mathcal{O}_B^\ast) \otimes \mathbb{Q} \longrightarrow \text{Ext}^1_{\mathcal{MS}^\circ_B(B)}(\mathbb{Q}(\kappa), \mathbb{Q}(\kappa)).$$

Its cokernel is enormous. Its image is expected to consist of the geometrically motivated one-extensions.

As far as $\text{Ext}^1(\mathbb{Q}(\kappa), \mathbb{Q}(\kappa))$ is concerned, it turns out that a much more precise approximation to $\mathcal{MM}_\mathbb{Q}(B)$ is obtained by actually forgetting part of the data of objects of $\mathcal{MM}_\mathbb{Q}(B)$: the category $\mathcal{HDR}_\mathbb{Q}^\circ(B)$ of variations of $\mathbb{Q}$-Hodge–de Rham structure comes about by leaving away the $l$-adic components of $\mathcal{MS}^\circ_B(B)$.

A variant of the definition used here, in the case when $B$ is a point, appeared already in [H], section 1. There, the interested reader also finds a detailed account of Hodge–de Rham structures in the context of motives, and their $L$-functions.

The first aim of the present article is to popularize the variational point of view, and to illustrate the flexibility of the resulting formalism by a concrete example.

For an elliptic curve $\mathcal{E}$ over $B$ with zero section $i$, we denote, letting $\tilde{\mathcal{E}} := \mathcal{E} - i(B)$, by $\mathcal{L}(\mathcal{E})$ the $\mathbb{Q}$-vector space with basis ($\{s\} \mid s \in \tilde{\mathcal{E}}(B)$). Furthermore, define

$$d : \mathcal{L}(\mathcal{E}) \longrightarrow \mathcal{E}(B) \otimes_\mathbb{Z} \mathcal{E}(B) \otimes_\mathbb{Z} \mathbb{Q}, \quad \{\sim\} \longmapsto \sim \otimes \sim.$$  

In [W2], we constructed, using the so-called polylogarithmic extension on $\tilde{\mathcal{E}}$, a homomorphism

$$\varphi : \ker(d) \longrightarrow \text{Ext}^1_{\mathcal{HDR}_\mathbb{Q}}(\mathbb{Q}(\kappa), \mathbb{Q}(\kappa)).$$
whose image consists of extensions one will consider as geometrically motivated.

The main result of this work shows that \( \varphi \) factors through the natural monomorphism

\[
\kappa_B : \Gamma(B, \mathcal{O}_B^*) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \text{Ext}^1_{\text{HDR}}(B)(\mathbb{Q}(\kappa), \mathbb{Q}(\kappa)),
\]

thus obtaining a proof of the elliptic Zagier conjecture on the lowest step \( k = 2 \). The proof is the logical continuation of the sheaf-theoretical approach developed in \([W2]\), sections 2 and 3. It is thus independent of the one sketched in \([W2]\), 1.9 (which makes use of the Poincaré line bundle, but not of the material contained in later sections of loc. cit.). It is also independent of the proof given in \([W3]\), 3.2, 3.9 (which gives a geometrical rather than sheaf-theoretical construction). For the precise statement of our main result, we refer to Theorem 3.2.

The proof of 3.2 is in three steps.

1. Our main technical tool will be Theorem 2.4. It states that the cokernel of \( \kappa_B \) remains unchanged under pullback morphisms of a quite general type.

2. Consider the following special elements of \( \ker(d) \):

   o) \( \{s\} \) for a torsion section \( s \in \tilde{E}(B) \).

   i) \( \{s\} - \{s - t\} \) for \( s, s - t \in \tilde{E}(B) \) and \( t \in \mathcal{E}(B)_{\text{tors}} \).

   ii) \( \{s + t\} + \{s - t\} - 2\{s\} - 2\{t\} \) for \( s, t, s + t, s - t \in \tilde{E}(B) \).

   We prove that given \( S \in \ker(d) \), then the restriction of \( S \) to some open dense subscheme \( B' \) of \( B \) will lie in the subspace of \( \ker(d_{B'}) \) generated by such elements.

   Because of Theorem 2.4, we need to show \( \varphi(S) \in \text{im } (\kappa_B) \) only for the special expressions o), i), ii).

3. In order to do so, we identify explicitly the one-extensions of variations on \( B \otimes_k \mathbb{C} \) in terms of holomorphic functions. For this we use the formulae of \([W1V]\), chapter 3. Depending on which kind of relation they come from, we call the resulting functions on \( B \) elliptic modular units of the zeroeth, first and second kind respectively.

   The explicit description over \( \mathbb{C} \) of these functions is in fact the second aim of this work. The functions occurring in the image of \( \varphi \) can safely be expected to be of arithmetic interest: for the units of the zeroeth
kind, this is well-known: they are specialization of the Siegel units studied in [KL] and elsewhere. In particular, the classical elliptic units occur in this framework.

More recently, elliptic modular units of the first kind appeared in Kato’s construction of Euler systems for modular curves.

The plan of the paper is as follows:

In section 1, we define the Siegel function, which will turn out to generate all the functions in the image of $\varphi$.

Section 2 can be read independently of the rest of this article. It contains a self-contained introduction of the category $HDR_{\mathbb{Q}}$ of variations of Hodge–de Rham structure, which will hopefully turn out to be of interest in its own right.

In section 3, polylogarithms enter. We review the construction of one-extensions of $[W2]$ in the case of interest to us (3.1–3.3). In sections 3.4–3.7, we describe, following the treatment of [W1V], chapter 3, the Hodge–de Rham incarnation of the polylog in the case when the elliptic curve $E$ is the universal object over some modular curve. We need to slightly modify the explicit description given in loc. cit. in order to be able to transfer easily the methods developed in section 4 of [BD] to the elliptic case. The main result of this section is Theorem 3.11, where we identify the extension of variations underlying $\varphi(S)$. In 3.12, we compare the formula to the “naïve” one obtained by averaging the Siegel function over the divisor $S$.

Sections 4–6 are concerned with elliptic modular units of the zeroeth, first and second kind respectively. We need to show that for $S$ of the special type $o$, $i$, or $ii$) above, the holomorphic function $\varphi(S)^{MHS}$ of 3.11 descends to the field of definition $k$ of $B$. For $o$, we are able, thanks to the explicit description of our functions, to connect to the classical theory of Siegel units. For $i$ and $ii$, we use 2.4 to restrict to the case of torsion sections, which then follows from case $o$.

In section 7, we conclude the proof of Theorem 3.2.

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Notation: We denote by $\mathcal{H}^+$ the complex upper half plane.

1 The Siegel function

1.1 We start by defining the following elementary functions on $\mathbb{C} \times \mathcal{H}^+$:

$r_1 : \mathbb{C} \times \mathcal{H}^+ \rightarrow \mathbb{R}$, \( (z, \tau) \mapsto \Re(z) - \frac{\Re(\tau) \cdot \Im(z)}{\Im(\tau)} \),

$r_2 : \mathbb{C} \times \mathcal{H}^+ \rightarrow \mathbb{R}$, \( (z, \tau) \mapsto -\frac{\Im(z)}{\Im(\tau)} \),

$c_\mathcal{H} : \mathbb{C} \times \mathcal{H}^+ \rightarrow \mathcal{H}^+$, \( (z, \tau) \mapsto \tau \),

$c_\mathbb{C} : \mathbb{C} \times \mathcal{H}^+ \rightarrow \mathbb{C}$, \( (z, \tau) \mapsto z \).

So we have the equality

$c_\mathbb{C} = -r_2c_\mathcal{H} + r_1$.

Furthermore, we let

$q_\mathcal{H} := \exp(2\pi ic_\mathcal{H})$, \( q_\mathbb{C} := \exp(2\pi ic_\mathbb{C}) \).

1.2 Definition: (cmp. [Ku], (2.14).) The Siegel function

$Si : \mathbb{C} \times \mathcal{H}^+ - \{(f, \tau) \mid f \in \mathbb{Z} \oplus \mathbb{Z}\tau\} \rightarrow \mathbb{C}$

is given by

$Si := -\exp(\pi i \cdot B_2(-r_2)c_\mathcal{H}) \exp(-\pi i \cdot r_1(2r_1 + 1))(1-q_\mathbb{C})\prod_{n=1}^{\infty}(1-q_\mathcal{H}q_\mathbb{C})(1-q_\mathcal{H}^n/q_\mathbb{C})$.

Recall the shape of the second Bernoulli polynomial:

$B_2(X) = X^2 - X + \frac{1}{6}$.

1.3 The proof of the following is left to the reader:

Lemma: For $\tau \in \mathcal{H}^+$ and $z = -r_2\tau + r_1 \in \mathbb{C}$, we have

$Si(z + 1, \tau) = \exp(-\pi i \cdot (r_2 + 1))Si(z, \tau)$,

$Si(z + \tau, \tau) = \exp(-\pi i \cdot (r_1 + 1))Si(z, \tau)$. 

2 Variations of Hodge–de Rham structure

2.1 As far as this article is concerned, the natural Tannakian category in which the relevant one-extensions live is that of variations of Hodge–de Rham structure.

Definition: Let \( k \) be a field which is embeddable into \( \mathbb{C} \), \( X/k \) smooth, separated and of finite type, \( F \subset \mathbb{R} \) a field. \( HDR^*_F(X) \), the category of variations of mixed \( F \)-Hodge–de Rham structure on \( X \) consists of families

\[
(V_{DR}, V_\sigma, I_{DR,\sigma}, F_\sigma \mid \sigma : k \hookrightarrow \mathbb{C}),
\]

where

a) \( V_{DR} \) is a vector bundle on \( X \), equipped with a flat connection \( \nabla \) which is regular at infinity in the sense of [D], II, remark following Définition 4.5. Further parts of the data are an ascending filtration \( W \) by flat subbundles, called the weight filtration, and a descending filtration \( F \) by subbundles, the so-called Hodge filtration.

b) \( V_\sigma \) is a variation of mixed \( F \)-Hodge structure (\( F\text{-MHS} \)) on \( X_\sigma(\mathbb{C}) \) which is admissible in the sense of [Ka].

c) Denote by \( \text{For}_O \) the forgetful functor assigning to a variation of \( F\text{-MHS} \) the underlying flat bifiltered vector bundle. \( I_{DR,\sigma} \) is an isomorphism

\[
\text{For}_O(V_\sigma) \sim \to V_{DR} \otimes_{k,\sigma} \mathbb{C}
\]

of flat bifiltered vector bundles.

d) For any \( \sigma : k \hookrightarrow \mathbb{C} \), complex conjugation defines a diffeomorphism

\[
c_\sigma : X_\sigma(\mathbb{C}) \sim \to X_{\overline{\sigma}}(\mathbb{C}).
\]

For a variation of \( F\text{-MHS} \) \( W \) on \( X_{\overline{\sigma}}(\mathbb{C}) \), we define a variation \( c^*_\sigma(W) \) on \( X_\sigma(\mathbb{C}) \) as follows: the local system and the weight filtration are the pullbacks via \( c_\sigma \) of the local system and the weight filtration on \( W \), and the Hodge filtration is the pullback of the conjugate of the Hodge filtration on \( W \). The functor \( c^*_\sigma \) preserves admissibility.

\( F_\sigma \) is an isomorphism of variations

\[
V_\sigma \sim \to c^*_\sigma(V_{\overline{\sigma}})
\]

such that \( c^*_\sigma(F_\sigma) = F_{\overline{\sigma}}^{-1} \).
Furthermore, we require the following: for each $\sigma$, let $\iota_\sigma$ be the antilinear involution of $\text{For}_{\text{diff}}(V_\sigma)$, the $C^\infty$-bundle underlying $V_\sigma$, given by complex conjugation of coefficients. Likewise, let $\iota_{\text{DR},\sigma}$ be the antilinear isomorphism $\text{For}_{\text{diff}}(V_\sigma) \sim \rightarrow c_{-1}(\sigma)^{-1} \text{For}_{\text{diff}}(V_\sigma)$ given by complex conjugation of coefficients on the right hand side of the isomorphism in $c$). Our requirement is the formula

$$\text{For}_{\text{diff}}(F_\sigma) = \iota_{\text{DR},\sigma} \circ \iota_\sigma = c_{-1}(\sigma)^{-1} \circ \iota_{\text{DR},\sigma}.$$ 

In the category of these data, it is straightforward to define Tate twists $F(n)$ for $n \in \mathbb{Z}$: on $F(n)$, the involution $F_\sigma$ acts by multiplication by $(-1)^n$. The last condition we impose is the existence of a system of polarizations: there are compatible morphisms

$$\text{Gr}_n^W V_{\text{DR}} \otimes_{\mathcal{O}_X} \text{Gr}_n^W V_{\text{DR}} \rightarrow F_{\text{DR}}(-n), \quad n \in \mathbb{Z}$$

of flat vector bundles on $X$, and polarizations

$$\text{Gr}_n^W V_\sigma \otimes F \text{Gr}_n^W V_\sigma \rightarrow F(-n), \quad \sigma : k \hookrightarrow \mathbb{C}, \quad \kappa \in \mathbb{Z}$$

of variations such that the $I_{\text{DR},\sigma}$ and $F_\sigma$ and the corresponding morphisms for $F(-n)$ form commutative diagrams.

2.2 For $k$ and $X$ as in 2.1, we define a map

$$\kappa_X : \Gamma(X, \mathcal{O}_X^*) \otimes_{\mathbb{Z}} F \rightarrow \text{Ext}^1_{\text{HDR}_k}(X)(F(0), F(1))$$

as follows:

The underlying bifiltered vector bundle is the trivial bundle with basis $(e_0, \frac{1}{2\pi i} \cdot e_1)$, and

$$\mathcal{F}^0 := \langle e_0 \rangle_{\mathcal{O}_X},$$

$$W_{-1} := \left\langle \frac{1}{2\pi i} \cdot e_1 \right\rangle_{\mathcal{O}_X}.$$ 

For $g \in \Gamma(X, \mathcal{O}_X^*)$, the flat regular connection is trivial on $\frac{1}{2\pi i} \cdot e_1$, and maps $e_0$ to

$$\frac{dg}{g} \cdot \frac{1}{2\pi i} \cdot e_1.$$ 

For any embedding $\sigma$ of $k$ into $\mathbb{C}$, the rational structure is given by

$$\left( e_0 - \log g_\sigma \cdot \frac{1}{2\pi i} \cdot e_1, e_1 \right).$$
2.3 Recall the situation in the setting of variations of Hodge structure: Let $Z/\mathbb{C}$ be smooth, and denote by $\text{Var}_F(Z)$ the category of admissible variations of $F$–\text{MHS} on $Z(\mathbb{C})$. By the same construction as in 2.2, we get a map

$$\kappa^\text{MHS}_Z = \Gamma(Z, \mathcal{O}_Z^*) \otimes_{\mathbb{Z}} F \to \text{Ext}^1_{\text{Var}_F(Z)}(F(0), F(1)).$$

**Theorem:** $\kappa^\text{MHS}_Z$ is an isomorphism.

**Proof:** e.g. [WITV], Theorem 3.7. q.e.d.

From the theorem, we already conclude that the map $\kappa_X$ of 2.2 is injective.

Let us describe the inverse of $\kappa^\text{MHS}_Z$: assume given an extension

$$(*) \quad 0 \to F(1) \to E \to \mathcal{F}(\mathcal{F}) \to \mathcal{F}$$

of variations on $Z$. We get an isomorphism of vector bundles

$$\mathcal{F}^0(E) \sim \mathcal{F}(\mathcal{F}) \otimes_F \mathcal{O}_X,$$

hence a splitting of $(*)$ on the level of bifiltered vector bundles. Denote by $e_1$ the base vector “$2\pi i$” of the constant variation $F(1)$, by $e_0$ the global section of $\mathcal{F}^0(E)$ mapping to $1 \in F(0)$, and by $\tilde{e}_0$ some multivalued rational flat section of $E$ mapping to $1$. We have

$$e_0 - \tilde{e}_0 \in F(1) \otimes_F \mathcal{O}_X.$$

Then the theorem tells us that $e_0 - \tilde{e}_0$ is necessarily of the form

$$\left(\frac{1}{2\pi i} \cdot f \log g + f'\right) \cdot e_1$$

for some $g \in \Gamma(Z, \mathcal{O}_Z^*) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $f, f' \in F$. We have

$$g \otimes f = (\kappa^\text{MHS}_Z)^{-1}(E).$$

**Proposition:** Let $k$ be a field which is embeddable into $\mathbb{C}$, $X/k$ smooth, separated and of finite type. For

$$E \in \text{Ext}_\mathbb{HDR}_X^\kappa(\mathcal{F}(\mathcal{F}), \mathcal{F}(\mathcal{F})), $$

the following are equivalent:
i) $E$ lies in the image of $\kappa_X$.

ii) The collection

$$\text{For}(E) \in \prod_{\sigma : \kappa \to \mathbb{C}} \Gamma(X_\sigma, \mathcal{O}^*_X) \otimes_{\mathbb{Z}} F = \prod_{\sigma : \kappa \to \mathbb{C}} \text{Ext}^{\kappa}_{\text{Var}_{\mathbb{Z}}(X_\sigma)}(F(\kappa), F(\kappa'))$$

of extensions of variations underlying $E$ lies in the image of

$$\Delta : \Gamma(X, \mathcal{O}^*_X) \otimes_{\mathbb{Z}} F \longrightarrow \prod_{\sigma : \kappa \to \mathbb{C}} \Gamma(X_\sigma, \mathcal{O}^*_X) \otimes_{\mathbb{Z}} F.$$ 

If i) and ii) are fulfilled, then $\kappa_X^{-1}(E) = \Delta^{-1}(\text{For}(E)).$

2.4 As for $\text{coker} \ k_X$, we have the following

**Theorem:** Let $f : X \to Y$ be a morphism of smooth, separated $k$-schemes of finite type inducing an isomorphism of the schemes of geometrically connected components:

$$f_{\text{conn}} : X_{\text{conn}} \sim \to Y_{\text{conn}}.$$ 

Then the diagram

$$
\begin{array}{ccc}
\Gamma(Y, \mathcal{O}^*_Y) \otimes_{\mathbb{Z}} F & \xrightarrow{f^*} & \Gamma(X, \mathcal{O}^*_X) \otimes_{\mathbb{Z}} F \\
\kappa_Y \downarrow & & \kappa_X \downarrow \\
\text{Ext}^1_{\text{HDR}_{\mathbb{Z}}(Y)}(F(0), F(1)) & \xrightarrow{f^*} & \text{Ext}^1_{\text{HDR}_{\mathbb{Z}}(X)}(F(0), F(1))
\end{array}
$$

is cartesian. In other words, an extension $E \in \text{Ext}^{\kappa}_{\text{HDR}_{\mathbb{Z}}(Y)}(F(\kappa), F(\kappa'))$ lies in the image of $\kappa_Y$ if and only if $f^*E$ lies in the image of $\kappa_X$.

**Proof:** By [WTV], Proposition 3.35, we have

$$\text{coker} \ k_Y = \text{coker} \ k_{Y_{\text{conn}}}, \quad \text{coker} \ k_X = \text{coker} \ k_{X_{\text{conn}}}.$$ 

Hence our claim follows from the assumption $f_{\text{conn}} : X_{\text{conn}} \sim \to Y_{\text{conn}}$ and the snake lemma.

q.e.d.

2.5 We conclude with another interpretation of the isomorphism $\kappa^M_{\text{MHS}}$ of 2.3.
By [CKS], Theorem 2.13, for any variation of Hodge structure \( V \), there is a unique decomposition of the bifiltered \( C^\infty \)-bundle \( V^\infty \) underlying \( V \),

\[ V^\infty = \bigoplus_{p,q} H^{p,q}, \]

such that

i) \( W_k V^\infty = \bigoplus_{p+q \leq k} H^{p,q} \),

ii) \( F^p V^\infty = \bigoplus_{p' \geq p} H^{p',q} \),

iii) \( \overline{H}^{p,q} = H^{p,q} \mod \bigoplus_{p'<q, q'<q} H^{p',q'} \).

It is easily checked that this decomposition is compatible with the tensor structure of the category of variations.

We thus get a functorial isomorphism

\[ \Theta_v : V^\infty = \bigoplus_{p,q} H^{p,q} \stackrel{\sim}{\longrightarrow} (\text{Gr}^W V)^\infty, \]

which is compatible with tensor products and formation of duals, and which satisfies

\[ \text{Gr}^W \Theta_v = \text{id}_{(\text{Gr}^W V)^\infty}. \]

**2.6** Now assume given a smooth scheme \( Z \) over \( \mathbb{C} \), and an exact sequence

\[ 0 \longrightarrow F(1) \longrightarrow V \longrightarrow F(0) \longrightarrow 1 \]

of admissible variations of \( F-MHS \) on \( Z \). We claim that there is a close connection between \( \Theta_v \) and the class of \( V \) in

\[ \Gamma(Z, \mathcal{O}_Z^\ast) \otimes \mathbb{F}_{\text{Var}} \rightarrow \text{Ext}^1_{\text{Var}(Z)}(F(0), F(1)). \]

In this case, the decomposition is already uniquely characterized by axioms i) and ii), and exists on the level of holomorphic bundles. We have

\[ H^{0,0} = F^0 V, \]

\[ H^{-1,-1} = W^{-2} V, \]
and the isomorphism $\Theta_V$ can be described by expressing the images of a basis of rational flat multivalued sections of $V$ in the basis of $\mathbb{Q}(\kappa) \oplus \mathbb{Q}(\kappa)$ given by $e_0$ and $e_1$. The result is a matrix of the shape
\[
\begin{pmatrix}
1 & 0 \\
\ast & 1
\end{pmatrix}.
\]

By Theorem 2.3, the $\ast$ is of the shape
\[
-\frac{1}{2\pi i} \cdot f \log g,
\]
and we have $g \otimes f = (\kappa_Z^{MHS})^{-1}(V)$.

3 Elliptic polylogarithms

3.1 In [W2], 3.1, we axiomatized some formal properties of stacks $\mathcal{T}$ on certain schemes, which allowed to construct one-extensions in $\mathcal{T}(B)$ from specific linear combinations of symbols on the Mordell–Weil group $\mathcal{E}(B)$ of an elliptic curve $E$ over $B$.

Let $C$ denote the category of schemes $B$ which are smooth, separated and of finite type over some field, which is embeddable into $\mathbb{C}$. Then by loc. cit., 3.2 c),
\[
B \mapsto HDR^s_Q(B)
\]
is such a stack on $C$. In particular, we have, using the notation of loc. cit., 3.1:

(C) For any elliptic curve $\pi : E \to B$, there is given an object of rank two, $R^1\pi_*\mathbb{Q}$ in $HDR^s_Q(B)$. The formation of $R^1\pi_*\mathbb{Q}$ is compatible with base change. Write
\[
V_2 := R^1\pi_*\mathbb{Q}(\kappa),
\]
and use the same symbol for the pullback to $E$, or to the complement $\overline{E}$ of the zero section.

(E) For any elliptic curve $E/B$, there is given an Abel–Jacobi map
\[
[\cdot] : \mathcal{E}(B) \otimes_{\mathbb{Z}} \mathbb{Q} \to \text{Ext}^{1}_{\text{HDR}^s_Q(B)}(\mathbb{Q}(\kappa), \mathbb{V}_\kappa)
\]
which is compatible with base change.

(F) ($N = 2$ in loc. cit.) Consider
\[
[\Delta] \in \text{Ext}^{1}_{\text{HDR}^s_E}(F(0), V_2)
\]
as a variation of Hodge–de Rham structure on $\mathcal{E}$. There is given an extension $\text{pol}^2$ in $HDR_{\mathbb{Q}}^s(\tilde{\mathcal{E}})$ of $V_2$ by $[\Delta](1)|_{\tilde{\mathcal{E}}}$, the (small) polylogarithmic extension, such that

$$
\text{pol}^1 := \text{pol}^2/V_2(1) \in \text{Ext}^1_{\text{HDR}_{\mathbb{Q}}^s(\tilde{\mathcal{E}})}(V_2, \mathbb{Q}(k'))
= \text{Ext}^1_{\text{HDR}_{\mathbb{Q}}^s(\tilde{\mathcal{E}})}(\mathbb{Q}(k'), V_2')(k')
= \text{Ext}^1_{\text{HDR}_{\mathbb{Q}}^s(\tilde{\mathcal{E}})}(\mathbb{Q}(k'), V_{k'})
$$

equals the restriction of $[\Delta]$ to $\tilde{\mathcal{E}}$. Here, the last equality is induced by the isomorphism

$$
V_2 \sim \rightarrow V_2'(1)
$$

coming from Poincaré duality.

Furthermore, a certain norm compatibility (loc. cit., 3.1 (G)) is satisfied. Let us remark that the extension $\text{pol}^2$ is unique if one requires as in loc. cit. that it be part of a whole projective system $(\text{pol}^N)_N$ of extensions.

Note that in loc. cit., 3.2 c), we restricted our attention to the smaller category $\mathcal{C}'$ of schemes which are smooth and quasi-projective over some number field. This assumption was not used in the proof of (E). As for (F) and norm compatibility, we note that since everything is supposed to be compatible with change of the base $B$, the construction in (F), as well as norm compatibility carry over to the general case because the relevant moduli spaces of elliptic curves together with finitely many sections are smooth and quasi-projective over $\mathbb{Q}$.

3.2 For a scheme $B$ which is smooth, separated and of finite type over some field of characteristic 0, and an elliptic curve $\mathcal{E}$ over $B$, we denote, slightly modifying the notation of [W2], by $\mathcal{L}(\mathcal{E})$ the $\mathbb{Q}$-vector space with basis $\{s\} | s \in \tilde{\mathcal{E}}(B)$). Furthermore, define

$$
d = d(\mathcal{E}) : \mathcal{L}(\mathcal{E}) \longrightarrow \mathcal{E}(B) \otimes_{\mathbb{Z}} \mathcal{E}(B) \otimes_{\mathbb{Z}} \mathbb{Q}, \quad \{\sim\} \mapsto \sim \otimes \sim .
$$

The rest of this article will be concerned with the proof of the following

**Theorem:** There is a homomorphism

$$
\varphi = \varphi(\mathcal{E}) : \ker(d) \longrightarrow \Gamma(B, \mathcal{O}_B^*) \otimes_{\mathbb{Z}} \mathbb{Q}
$$

with the following properties:
a) $\varphi$ is functorial with respect to change of the base $B$.

b) $\varphi$ satisfies norm compatibility: for any isogeny $\psi : \mathcal{E}_1 \to \mathcal{E}_2$, whose kernel consists of sections of $\mathcal{E}_1$, and any $s_{1,\alpha} \in (\mathcal{E}_1 - \ker(\psi))(B)$, $q_\alpha \in \mathbb{Q}$:

$$d \left( \sum_\alpha q_\alpha \{\psi(s_{1,\alpha})\} \right) = 0 \iff d \left( \sum_\alpha q_\alpha \sum_{t \in \ker(\psi)(B)} \{s_{1,\alpha} + t\} \right) = 0.$$ 

If this is the case, then the equality

$$\varphi \left( \sum_\alpha q_\alpha \{\psi(s_{1,\alpha})\} \right) = \varphi \left( \sum_\alpha q_\alpha \sum_{t \in \ker(\psi)(B)} \{s_{1,\alpha} + t\} \right)$$

holds.

c) If $\mathcal{E} = E$ is an elliptic curve over the spectrum $B$ of a finite field extension $K$ of either $\mathbb{Q}$, or $\mathbb{R}$, then for any $S = \sum_\alpha q_\alpha \{s_\alpha\}$ in the kernel of $d$, the absolute value of $\varphi(S) \in K^* \otimes \mathbb{Q}$ satisfies

$$\log \left\| \varphi \left( \sum_\alpha q_\alpha \{s_\alpha\} \right) \right\| = \sum_\alpha q_\alpha \lambda_K(s_\alpha),$$

where $\lambda_K$ equals the local Néron height function (see e.g. [5], VI).

**Remark:** It suffices to prove 3.2 for schemes $B$ which are smooth, separated and of finite type over some field, which is embeddable into $\mathbb{C}$.

3.3 As a first approximation to Theorem 3.2, we recall that due to the axioms of 3.1, one can construct from $\text{pol}^2$ certain one-extensions of $\mathbb{Q}(\mathcal{K})$ by $\mathbb{Q}(\mathcal{K})$ in $HDR^n_{\mathbb{Q}}(B)$. We have:

**Theorem:** There is a homomorphism

$$\varphi = \varphi(\mathcal{E}) : \ker(d) \to \text{Ext}^1_{HDR^n_{\mathbb{Q}}(B)}(\mathbb{Q}(\mathcal{K}), \mathbb{Q}(\mathcal{K}))$$

with the following properties:

a) as in Theorem 3.2 a).

b) as in Theorem 3.2 b).
c) If $\mathcal{E} = E$ is an elliptic curve over $\mathbb{C}$, then for any $S = \sum_{\alpha} q_{\alpha} \{ s_{\alpha} \}$ in \(\ker(d)\), the extension of Hodge structures

$$\varphi(S)^{\text{MHS}} := \varphi(S)_{\sigma = \text{id} : \mathbb{C} \to \mathbb{C}} \in \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(\kappa), \mathbb{Q}(\kappa)) \cong \mathbb{C}^* \otimes_{\mathbb{Z}} \mathbb{Q}$$

satisfies

$$\log \left\| \varphi \left( \sum_{\alpha} q_{\alpha} \{ s_{\alpha} \} \right) \right\| = \sum_{\alpha} q_{\alpha} \lambda_{\mathbb{C}}(s_{\alpha}) .$$

**Proof:** a) and b) is [W2], Corollary 3.5. c) is [W2], Theorem 4.2, together with [Si], VI, Theorem 3.4, and [L], chapter 20, §5. q.e.d.

### 3.4

Let us describe in explicit terms the variation of Hodge structure underlying $\text{pol}^2$. We follow the treatment of chapter 3 of [WTV], where the case of the universal elliptic curve over some modular curve was considered.

Fix $n \geq 3$, and let

$$\pi_n : \mathcal{E}_n \to Y(n)$$

denote the universal elliptic curve over the modular curve $Y(n)$ “of full level $n$”. $Y(n)$ is a smooth affine scheme over $\mathbb{Q} \left( \frac{\mathbb{Z}}{\mathbb{Z}} \right)$. We shall also consider it as a scheme over $\mathbb{Q}$. The scheme $Y(n)_{\text{conn}}$ of geometrically connected components equals $\text{Spec} \left( \mathbb{Q} \left( \frac{\mathbb{Z}}{\mathbb{Z}} \right) \right)$.

Let us describe $\pi_n(\mathbb{C}) : \mathcal{E}_\kappa(\mathbb{C}) \to \mathcal{Y}(\kappa)(\mathbb{C})$, and simultaneously connect to the notation of [WTV]. We let

$$L := L_n := \ker(\text{GL}_2(\hat{\mathbb{Z}}) \to \text{GL}_2(\mathbb{Z}/\kappa \mathbb{Z})) ,$$

$$N := 1;$$

$$V_2(\hat{\mathbb{Z}}) := \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \mid a, b \in \hat{\mathbb{Z}} \right\} ,$$

and $K := K_{a,1} := V_2(\hat{\mathbb{Z}}) \rtimes L$, the semidirect product with respect to the natural action of $L$ on $V_2(\hat{\mathbb{Z}})$. We let

$$P_2 := P_{2,a} := \begin{pmatrix} 1 & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix} \leq \text{GL}_3 ,$$
and consider $K$ as a subgroup of $P_2(A_{13})$. It is open and compact. We have a natural action of $P_2(\mathbb{R})$ on $\mathbb{C} \times \mathcal{H}^+$:

$$\begin{pmatrix} 1 & 0 & 0 \\ a & \alpha & \beta \\ b & \gamma & \delta \end{pmatrix} \in P_2(\mathbb{R})$$

acts by sending $(z, \tau)$ to

$$\left( (\alpha\delta - \beta\gamma) \cdot \frac{z}{\gamma\tau + \delta} + \left( -\frac{b\alpha\tau + \beta\gamma}{\gamma\tau + \delta} + a \right), \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right).$$

Then writing $P'_2 := P_2 \cap SL_3$, we have

$$Y(n)(\mathbb{C}) = SL_2(\mathbb{Z}) \setminus (\mathcal{H}^+ \times (GL_\mathbb{R}(\hat{\mathbb{Z}})/L)),$$

$$E_n(\mathbb{C}) = P'_2(\mathbb{Z}) \setminus (\mathbb{C} \times \mathcal{H}^+ \times (P_\mathbb{R}(\hat{\mathbb{Z}})/K)),$$

and $\pi_n(\mathbb{C})$ is induced by the natural projections

$$c_{\mathcal{H}} : \mathbb{C} \times \mathcal{H}^+ \longrightarrow \mathcal{H}^+ \quad \text{and} \quad P_2(\hat{\mathbb{Z}}) \longrightarrow GL_2(\hat{\mathbb{Z}})$$

respectively. As for the connected components of $Y(n)(\mathbb{C})$ and $E_n(\mathbb{C})$, one defines

$$\Gamma := SL_2(\mathbb{Z}) \cap L \leq GL_\mathbb{R}(\mathbb{Q}) \quad \text{and} \quad \Lambda := \begin{pmatrix} 1 & 0 & 0 \\ \frac{z}{z} & 0 & 0 \\ \frac{z}{z} & 0 & \Gamma \end{pmatrix} \leq P_2(\mathbb{Q}).$$

Note that the determinant induces isomorphisms

$$P'_2(\mathbb{Z}) \setminus P_\mathbb{R}(\hat{\mathbb{Z}})/K \xrightarrow{\sim} SL_\mathbb{R}(\mathbb{Z}) \setminus GL_\mathbb{R}(\hat{\mathbb{Z}})/L \xrightarrow{\sim} (\mathbb{Z}/\mathbb{Z})^*.$$

Choose a set of representatives $R \subset GL_2(\hat{\mathbb{Z}})$ for $SL_2(\mathbb{Z}) \setminus GL_\mathbb{R}(\hat{\mathbb{Z}})/L$, and write

$$p_f := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & g_f \end{pmatrix} \in P_2(\hat{\mathbb{Z}})$$

for any $g_f \in R$. Then we have:

$$E_n(\mathbb{C}) = \coprod_{g_f \in R} \Lambda \setminus (\mathbb{C} \times \mathcal{H}^+),$$

$$Y(n)(\mathbb{C}) = \coprod_{g_f \in R} \Gamma \setminus \mathcal{H}^+,$$
and the inclusion of the connected component indexed by \( g_f \in R \) into

\[
\mathcal{E}_n(\mathbb{C}) = P'_n(\mathbb{Z}) \setminus (\mathbb{C} \times H^+ \times (\mathbb{P}_\mathbb{K}(\hat{\mathbb{Z}})/\mathbb{K})) \quad \text{and} \\
Y(n)(\mathbb{C}) = SL_2(\mathbb{Z}) \setminus (H^+ \times (\mathbb{G}_\mathbb{L}(\hat{\mathbb{Z}})/\mathbb{L}))
\]

respectively is given by assigning to the classes of \((z, \tau) \in \mathbb{C} \times H^+\) and \(\tau \in H^+\) the classes of \((z, \tau, p_f)\) and \((\tau, g_f)\), respectively.

3.5 We have \( Gr^W_{pol^2} = V_2 \oplus \mathbb{Q}(\mathcal{K}) \oplus \mathbb{V}_\mathbb{K}(\mathcal{K}) \). One way to describe the canonical isomorphism \( \Theta_{pol^2} \) of 2.5 is to express a basis of rational flat multivalued sections of \( pol^2 \) in the corresponding basis of \( Gr^W_{pol^2} \).

As in [WTV], chapter 3, we use the parameterization of any of the connected components \( \mathcal{E}_n(\mathbb{C})^{\mathcal{K}} \) of \( \mathcal{E}_n(\mathbb{C}) \) given by the universal covering map

\[
pr = pr_1 : \mathbb{C} \times H^+ \longrightarrow \mathcal{E}_x(\mathbb{C})^{\mathcal{K}}.
\]

We write \((e_1, e_2)\) for the basis of the homology sheaf \( V_2 \), whose value at \(\tau \in H^+\) is given by the lines connecting 0 and 1, resp. 0 and \(-\tau\). In order to distinguish the basis \((e_1, e_2)\) of \( V_2 \subset Gr^W_{pol^2} \) from the basis \((2\pi i \cdot e_1, 2\pi i \cdot e_2)\) of \( V_2(1) \subset Gr^W_{pol^2} \), we follow the notation of loc. cit. and write \((\varepsilon'_1, \varepsilon'_2)\) for the basis of \( V_2 \). We end up with a basis

\[
\mathcal{B} := (\varepsilon'_1, \varepsilon'_2, 2\pi i \cdot 1, 2\pi i \cdot e_1, 2\pi i \cdot e_2)
\]

of flat rational multivalued sections of \( Gr^W_{pol^2} \).

3.6 A basis of flat rational multivalued section of \( pol^2 \subset (pol^2)^\infty \) is given by the columns of the matrix \( P_1^W \) of [WTV], Lemma 3.13. We first define its entries:

Definition: The \((0,1)\)-th elliptic higher logarithm is defined as

\[
\text{Li}_{0,1} := \frac{1}{2\pi i} \left( \sum_{j=0}^{\infty} \log(1 - q^j_H q_C) + \sum_{j=1}^{\infty} \log(1 - q^j_H / q_C) \right) + \frac{1}{2} B_2(-r_2)c_H.
\]

Theorem: Let \( \mathcal{B} = (\varepsilon'_1, \varepsilon'_2, 2\pi i \cdot 1, 2\pi i \cdot e_1, 2\pi i \cdot e_2) \) be the basis of 3.5 of flat rational multivalued sections of \( Gr^W_{pol^2} \). There is a basis of
flat rational multivalued sections of $pol^2$ whose image under $\Theta_{pol^2}$ is described by the columns of the matrix

$$P := \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \varepsilon'_1 \\
0 & 1 & 0 & 0 & 0 & \varepsilon'_2 \\
-r_2 & r_1 & 1 & 0 & 0 & 2\pi i \cdot 1 \\
\text{Li}_{0,1} - \frac{1}{2}r_1 & -\frac{1}{2}r_1^2 - \frac{1}{12} & -r_1 & 1 & 0 & 2\pi i \cdot e_1 \\
\frac{1}{2}r_2^2 + \frac{1}{12} & \text{Li}_{0,1} - r_2r_1 - \frac{1}{2}r_1 + \frac{1}{4} & -r_2 & 0 & 1 & 2\pi i \cdot e_2
\end{pmatrix}$$

**Proof:** This is [W1V], Lemma 3.13. There, we used a matrix called $P^W_1$, which is best suited as far as norm compatibility is concerned (see loc. cit., Corollary 3.16). The matrix $P$ is obtained from $P^W_1$ by adding $\frac{1}{2}$ times the third column to the first and second columns. We thus get a basis of flat rational sections which makes the equality of extensions $pol^1 = [s]$ of $\mathcal{F}$ more transparent. q.e.d.

### 3.7

The matrix $P$ plays a role analogous to the one of the matrix $L(z)$ used in [BD].

Following [W2], we set

$$c_1 := \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix},$$

$$c_2 := \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix},$$

$$d_1 := \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},$$

$$d_2 := \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}.$$
Define $v$ as the Lie algebra generated by $c_1, c_2, d_1, d_2$. So

$$v = \begin{pmatrix}
0 & 0 \\
0 & 0 \\
* & * & 0 \\
* & * & 0 & 0 \\
* & * & 0 & 0 \\
* & * & 0 & 0
\end{pmatrix} \subset \mathfrak{sl}_5,$$

a basis being given by

$$(c_1, c_2, (\text{ad } c_2)^l (\text{ad } c_1)^m (d_i) | 0 \leq l, m \leq 1, m + l \leq 1).$$

Inside $v$, consider the Lie algebra

$$w := \langle c_1 - d_2, c_2 + d_1, (\text{ad } c_2)^l (\text{ad } c_1)^m (d_i) | m + l = 1 \rangle.$$

The Lie algebras $w \subset v$ correspond to unipotent subgroups $W \leq V$ of $SL_5$, and $P$ is a multivalued function with values in $W(\mathbb{C})$. Writing $P = (p_{ij})_{1 \leq i, j \leq 5}$, we have in particular

$$p_{41} = \text{Li}_{0,1} - \frac{1}{2} r_1,$$

$$p_{52} = \text{Li}_{0,1} - r_1 r_2 - \frac{1}{2} r_1 + \frac{1}{4}.$$

3.8 Let $B$ be smooth over $\mathbb{C}$, $\pi : E \longrightarrow B$ an elliptic curve such that for some $n \geq 3$, the whole $n$-torsion of $E$ consists of sections of $\pi$. Then we get a cartesian diagram

$$\begin{array}{ccc}
E & \xrightarrow{f} & E_{n,\mathbb{C}} \\
\pi \downarrow & & \downarrow \pi_{n,\mathbb{C}} \\
B & \xrightarrow{f} & Y(n)_{\mathbb{C}}
\end{array}$$

and in particular, a lift of $f$, also denoted $f : \tilde{X} \longrightarrow \coprod_{[\theta] \in \mathbb{R}} (\mathbb{C} \times \mathcal{H}^+)$, from the universal cover $\tilde{X}$ of $E(\mathbb{C})$.

Since the formation of $\text{pol}^2$ is compatible with base change, the matrix $f^* P$ describes the isomorphism $\Theta_{\text{pol}^2}$. By abuse of notation, we again write $P$ and $p_{ij}, 1 \leq i, j \leq 5$ for $f^* P$ and its entries.

For any locally closed submanifold $C$ of $\tilde{E}(\mathbb{C})$, any basis of flat rational sections of $\text{pol}^2|_C$ respecting the weight filtration and inducing the basis
\( \mathfrak{B} \) of \( \text{Gr}^W \text{pol}^2 |_C \) gives a multivalued function \( P' \) satisfying \( P' = P \cdot U \) for a \( V(\mathbb{Q}) \)-valued function \( U \) on \( C \), which is therefore constant on connected components. Writing \( P' = (p'_{ij})_{1 \leq i,j \leq 5} \), we have the relations

\[
\begin{align*}
    p'_{31} &= p'_{53}, \\
    p'_{32} &= -p'_{43},
\end{align*}
\]

if and only if \( U \) is actually \( W(\mathbb{Q}) \)-valued.

**Definition:** (cmp. [BD], 4.1.) A multivalued function \( P' \) with values in \( \mathbb{W}(C) \) is called a *generalized determination* of \( P \) if \( P \) and \( P' \) induce the same function with values in \( \mathbb{W}(C)/\mathbb{W}(Q) \).

### 3.9
For the entries of the generalized determination \( P' = P \cdot U \), we have:

\[
\begin{align*}
    a_1) & \quad p'_{32} = -p'_{43} = r_1 + u_1, \\
    a_2) & \quad -p'_{31} = -p'_{53} = r_2 + u_2, \\
    b_1) & \quad p'_{41} = p_{41} + u_2 r_1 + x, \\
    b_2) & \quad p'_{52} = p_{52} - u_1 r_2 + y.
\end{align*}
\]

Call a 4-tuple \( (R_1, R_2, P_{41}, P_{52}) \) of functions a generalized determination of \( (r_1, r_2, p_{41}, p_{52}) \) if its entries occur as \( p'_{32} = -p'_{43}, -p'_{31} = -p'_{53}, p'_{41}, \) and \( p'_{52} \) of a generalized determination of \( P \).

### 3.10
We now imitate the construction of [BD], 4.2. With the notation of 3.9, assume given a finite subset \( \{s_\alpha | \alpha \in I\} \) of \( \tilde{\mathcal{E}}(B) \), and consider the group

\[
\langle s_\alpha \rangle_{\alpha \in I} \leq \mathcal{E}(B)
\]

generated by the \( s_\alpha \). Let \( \Delta \subset B(\mathbb{C}) \) be a simply connected locally closed submanifold, e.g., an open ball. For each \( s \in \langle s_\alpha \rangle_{\alpha \in I} \), choose (one-valued!) generalized determinations \( R_1 \) and \( R_2 \) of \( r_1 \) and \( r_2 \) on \( s(\Delta) \subset \mathcal{E}(\mathbb{C}) \),

i.e., functions \( R_i : s(\Delta) \to \mathbb{R} \) inducing the same functions modulo \( \mathbb{Q} \) as \( r_1 \) and \( r_2 \) respectively. Furthermore, ensure that these choices are made in a way compatible with the group structure, i.e.,

\[
R_i(s(z)) + R_i(t(z)) = R_i((s + t)(z))
\]

for all \( s, t \in \langle s_\alpha \rangle_{\alpha \in I} \), and \( z \in \Delta \).
Next, choose for any of the $s_\alpha, \alpha \in I$ a generalized determination $(R_1, R_2, P_{41}, P_{52})$ of $(r_1, r_2, p_{41}, p_{52})$ on 

$$s_\alpha(\Delta) \subset \tilde{E}(C),$$

which is compatible with the choices already made.

Observe that any other choice of $R_i$ is of the form $R_i + u_i$ for a $\mathbb{Q}$-valued function $u_i$ satisfying 

$$u_i(s(z)) + u_i(t(z)) = u_i((s + t)(z)).$$

Different choices therefore lead to the replacements 

$$P_{41} \mapsto P_{41} + u_2 R_1 + x,$$
$$P_{52} \mapsto P_{52} - u_1 R_2 + y,$$

and one concludes:

**Lemma:** If a linear combination $\sum_\alpha q_\alpha \{s_\alpha\}$ satisfies 

$$d \left( \sum_\alpha q_\alpha \{s_\alpha\} \right) = \sum_\alpha q_\alpha \cdot s_\alpha \otimes s_\alpha = 0 \in \mathcal{E}(B) \otimes_\mathbb{Z} \mathcal{E}(B) \otimes_\mathbb{Z} \mathbb{Q},$$

then the sums 

$$\sum_\alpha q_\alpha P_{41} \circ s_\alpha \quad \text{and} \quad \sum_\alpha q_\alpha P_{52} \circ s_\alpha,$$

considered as functions 

$$\Delta \mapsto \mathbb{C}/\mathbb{Q},$$

are independent of the choice of 

$$(R_1, R_2, P_{41}, P_{52}).$$

**3.11** With the notation of [3.10], assume given 

$$S = \sum_\alpha q_\alpha \{s_\alpha\} \in \ker(d).$$

By the lemma just proved, the functions 

$$\frac{1}{2} \sum_\alpha q_\alpha (P_{41} + P_{52}) \circ s_\alpha$$
on open balls in $B(\mathbb{C})$ glue together to a multivalued function on the whole of $B(\mathbb{C})$, which is well-defined modulo $\mathbb{Q}$. So if we define

$$g_S := \exp \left( -2\pi i \cdot \frac{1}{2} \sum_\alpha q_\alpha (P_{41} + P_{52})^* s_\alpha \right),$$

then $g_S$ is a multivalued function

$$B(\mathbb{C}) \rightarrow \mathbb{C}^*,$$

which is well-defined as a function to $\mathbb{C}^* \otimes_{\mathbb{Z}} \mathbb{Q}$. On simply connected open subsets of $B(\mathbb{C})$, it is representable by functions which are differentiable.

From Theorem 3.3, we recall that $S$ defines an element

$$\varphi(S) \in \text{Ext}^1_{\text{HDR}_0^q(B)}(\mathbb{Q}(\kappa), \mathbb{Q}(\kappa)),$$

and in particular, an element, denoted by $\varphi(S)^{\text{MHS}}$, in

$$\text{Ext}^1_{\text{Var}_q(B)}(\mathbb{Q}(\kappa), \mathbb{Q}(\kappa)) \rightarrow \Gamma(\mathcal{B}, \mathcal{O}^*(\mathcal{B})) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

**Theorem:** $\varphi(S)^{\text{MHS}}$ and $g_S$ agree as functions from $B(\mathbb{C})$ to $\mathbb{C}^* \otimes_{\mathbb{Z}} \mathbb{Q}$.

**Remark:** In particular, our local construction of $g_S$ gives a holomorphic function, which can be continued to the whole universal cover of $B(\mathbb{C})$, and which represents $\varphi(S)^{\text{MHS}}$.

**Proof of Theorem 3.11:** We imitate the proof of [BD], Proposition 4.6. Fix an arbitrary point $b \in B(\mathbb{C})$, and denote by $\omega$ the fibre functor on $\text{Var}_q(B)$ associating to a variation the vector space underlying its fibre at $b$. There is a second fibre functor $\omega_0$ on $\text{Var}_q(B)$ given by

$$\omega_0 : \mathcal{V} \mapsto \omega(\text{Gr}^W \mathcal{V}).$$

$\omega_0$ and $\omega$ coincide on the subcategory $\text{Var}^\text{pure}_q(B)$ of variations with split weight filtration. If $G$ denotes the Tannakian dual $G$ of $\text{Var}^\text{pure}_q(B)$, then the dual of $\text{Var}_q(B)$ with respect to $\omega_0$ is a semidirect product

$$W \rtimes G,$$

with a pro-unipotent group $W$ (compare [W2], 2.5).

By [DM], Theorem 2.13, $\omega$ defines an element in $H^1(\mathbb{Q}, W \rtimes G)$ mapping to zero in $H^1(\mathbb{Q}, G)$. Since $H^1(\mathbb{Q}, W) = \mathbb{K}$, there is an isomorphism of fibre functors

$$\omega \xrightarrow{\sim} \omega_0.$$
(see loc. cit.). On the other hand, the construction of 2.3 gives an isomorphism
\[ \omega_C \sim \omega_{0,C}, \]
which is the identity on Var^\text{pure}_\mathbb{Q}(B).

Comparing these isomorphisms, we get an automorphism of \( \omega_{0,C} \), i.e., an element
\[ w \cdot g \in (W \rtimes G)(\mathbb{C}) \]
such that \( g \in G(\mathbb{Q}) \). We may assume \( g = 1 \). Then \( w \in W(\mathbb{C}) \) determines generalized determinants \( R_1 \) and \( R_2 \) of \( r_1 \) and \( r_2 \) on
\[ s(b) \subset \mathcal{E}(\mathbb{C}) \]
for any \( s \in \mathcal{E}(B) \), which behave additively: for an extension \( V \) of \( \mathbb{Q}(\mathcal{H}) \) by \( V_2 \), the image of \( w \) in \( \text{GL}(\omega_0(V)) \) is of the shape
\[
\begin{pmatrix}
1 & 0 & 0 \\
x & 1 & 0 \\
y & 0 & 1
\end{pmatrix}
\]
where \( x \) and \( y \) are additive in \( \mathbb{E} \), and we set
\[ R_1(s(b)) := -x, \quad R_2(s(b)) := -y \]
for \( V = [s] \).

Similarly, if \( s \in \tilde{\mathcal{E}}(B) \), then the image \( w_s = (w_{ij,s})_{1 \leq i,j \leq 5} \) of \( w \) in \( \text{GL}(\omega_0(s^*pol^2)) \) is a generalized determination of the matrix \( P(s(b)) \).

Now by definition of \( \varphi(S)^{MHS} \), the element \( \log(w) \) of \( \text{Lie}(W) \) acts on the corresponding extension by the matrix
\[
\begin{pmatrix}
1 & 0 \\
* & 1
\end{pmatrix},
\]
where the * equals
\[
\frac{1}{2} \sum_{\alpha} q_\alpha (w_{41,s} + w_{52,s}).
\]
By the recipe given in 2.4, we have
\[ \varphi(S)^{MHS}(b) = \exp(-2\pi i \cdot *) . \]
q.e.d.
3.12 We remark that we have the equality
\[ \exp \left( -2\pi i \cdot \left( \frac{1}{2}(p_{41} + p_{52}) \right) \right) = Si^{-1}, \]
at least up to an eight root of unity.

So if \( S = \sum_{\alpha} q_{\alpha} \{ s_{\alpha} \} \in \ker(d) \), then the function \( \varphi(S)^{MHS} \) differs from the multivalued function
\[ \prod_{\alpha} (Si^{-1} \circ s_{\alpha})^{q_{\alpha}} \]
by a multivalued function of the shape
\[ \prod_{\alpha} (\exp(2\pi i \cdot F_{\alpha}) \circ s_{\alpha})^{q_{\alpha}} \]
where the \( F_{\alpha} \) are polynomials in \( \mathbb{Q}[\mathbb{Q}, \mathbb{Q}] \) of total degree smaller or equal to one.

4 Elliptic modular units of the zeroeth kind

4.1 The most visible elements of the kernel of
\[ d : \mathcal{L}(E) \longrightarrow \mathcal{E}(B) \otimes_{\mathbb{Z}} \mathcal{E}(B) \otimes_{\mathbb{Z}} \mathbb{Q}, \quad \{ \sim \} \longmapsto \sim \otimes \sim \]
are certainly those of the shape \( \{ s \} \), for a torsion section \( s \) of \( \mathcal{E} \) disjoint from the zero section.

As in 3.8, let \( B \) be smooth over \( \mathbb{C} \), and assume that for some \( n \geq 3 \),
the whole \( n \)-torsion of \( \mathcal{E} \) consists of sections of \( \pi \). So we get a cartesian diagram
\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{f} & \mathcal{E}_{n,\mathbb{C}} \\
\pi \downarrow & & \downarrow \pi_{n,\mathbb{C}} \\
B & \xrightarrow{f} & Y(n)_{\mathbb{C}}
\end{array}
\]

**Theorem:** Let \( s \in \mathcal{E}(B) \) be a torsion section. Then
\[ \varphi(\{ s \})^{MHS} = 1/Si \circ f \circ s \in \Gamma(B, \mathcal{O}^*(B)) \otimes_{\mathbb{Z}} \mathbb{Q}. \]

**Proof:** By 3.12, both sides differ multiplicatively by a function of the shape \( \exp(2\pi i \cdot F) \circ s \), where \( F \) is a polynomial in \( \mathbb{Q}[\mathbb{Q}, \mathbb{Q}] \). But since \( s \) is a torsion section, \( r_1 \) and \( r_2 \) are rational constants. \( \text{q.e.d.} \)
4.2 The functions classically known as *Siegel units* come about as specializations of $\text{Si}$ to “torsion sections” of
\[ c_H : \mathbb{C} \times \mathcal{H}^+ \longrightarrow \mathcal{H}^+ . \]

**Definition:** Let $v \in \mathbb{Q}^\neq - \mathbb{Z}^\neq$. The function $S_i^v : \mathcal{H}^+ \longrightarrow \mathbb{C}$ is given by
\[ S_i^v(\tau) := S_i(-v_2 \tau + v_1, \tau) . \]
So $S_i^v$ coincides with the classical Siegel function $g_{(-v_2,v_1)}$, as defined on page 29 of [KL]. It is holomorphic, and we have

**Theorem:** If $n \in \mathbb{Z}_{>\nu}$ is such that
\[ v \in \left( \frac{1}{n} \mathbb{Z} \right)^2 - \mathbb{Z}^\neq , \]
then the $(12n)$-th power of $S_i^v$ is a non-vanishing algebraic function on $Y(n)_\mathbb{C}$. It descends to $Y(n)$, viewed as a geometrically connected scheme over the subfield $\mathbb{Q} \left( \frac{\nu \pi \mathbb{Z}}{\mathbb{Q}} \right)$ of $\mathbb{C}$:
\[ S_i^v \in \Gamma(Y(n), \mathcal{O}_{Y(n)}^\ast) \otimes_{\mathbb{Z}} \mathbb{Z} \left[ \frac{\nu}{\nu \pi \mathbb{Z}} \right] . \]
As an element of $\Gamma(Y(n), \mathcal{O}_{Y(n)}^\ast) \otimes_{\mathbb{Z}} \mathbb{Q}$, the function $S_i^v$ only depends on $v \mod \mathbb{Z}^\neq$.

**Proof:** The first statement is [KL], II, Theorem 1.2 – but note from Theorem [4.1], we know that *some* power of $S_i^v$ is an algebraic function on $Y(n)_\mathbb{C}$. For the descent to $Y(n)$, one uses the $q$-expansion principle. The last statement follows from Lemma [1.3] – again, the independence of $S_i^v$ in
\[ \Gamma(Y(n), \mathcal{O}_{Y(n)}^\ast) \otimes_{\mathbb{Z}} \mathbb{Q} \]
is also predicted by Theorem [4.1]. q.e.d.

4.3 Let $v \in \left( \frac{1}{n} \mathbb{Z} \right)^2 - \mathbb{Z}^\neq$, and consider the section
\[ \mathcal{H}^+ \longrightarrow \mathbb{C} \times \mathcal{H}^+ , \]
\[ \tau \longmapsto (-v_2 \tau + v_1, \tau) . \]
It descends to the level of $Y(n)_{\mathbb{C}}$ and $E_{n,\mathbb{C}}$, defining a non-zero $n$-torsion section $i_{v,\mathbb{C}}$ of $\tau_{n,\mathbb{C}}$. Via the canonical embedding of $Q\left(\frac{\mathbb{A}}{\mathbb{F}}\right)$ into $\mathbb{C}$, we get

$$i_v : Y(n) \rightarrow E_n.$$ 

By 4.1 and [KL], II, Proposition 1.3, the element

$$\varphi(\{i_v\}) \in \text{Ext}^1_{HDR_{Q}(Y(n)_{\mathbb{C}})}(Q(\mathbb{K}), Q(\mathbb{K}))$$

satisfies: the underlying collection

$$\text{For}(\varphi(\{i_v\})) \in \prod_{\sigma : Q(\mathbb{K}) \rightarrow \mathbb{C}} \text{Ext}^1_{\text{Var}_{Q}(Y(n)_{\mathbb{C}})}(Q(\mathbb{K}), Q(\mathbb{K}))$$

of extensions of variations underlying $\varphi(\{i_v\})$ lies in the image of

$$\Delta : \Gamma(Y(n), \mathcal{O}_{Y(n)}^{*}) \otimes_{\mathbb{Z}} Q \rightarrow \prod_{\sigma} \Gamma(Y(\mathbb{K}), \mathcal{O}_{Y(\mathbb{K})}^{*}) \otimes_{\mathbb{Z}} Q.$$ 

From Proposition 2.3, we conclude:

**Proposition:** For any $v \in (\mathbb{Z}/n\mathbb{Z})^2 - \mathbb{Z}^2$, we have

$$\varphi(\{i_v\}) \in \Gamma(Y(n), \mathcal{O}_{Y(n)}^{*}) \otimes_{\mathbb{Z}} Q.$$ 

Via the canonical embedding of $Q\left(\frac{\mathbb{A}}{\mathbb{F}}\right)$ into $\mathbb{C}$, we have the equality

$$\varphi(\{i_v\}) = (Si_v)^{-1}.$$ 

4.4 We now work over an arbitrary base field $k$ which is embeddable into $\mathbb{C}$, but still assume that for some $n \geq 3$, the whole $n$-torsion of

$$\pi : E \rightarrow B$$

consists of sections. Let $s \in \tilde{E}(B)$ be an $n$-torsion section. It comes about as the base change by some $f : B \rightarrow Y(n)$ of a section

$$i_v : Y(n) \rightarrow E_n$$

of $\pi_n$. Because of the functoriality statement in Theorem 3.3, we know that

$$\varphi(\{s\}) = f^* \varphi(\{i_v\}).$$

$\varphi(\{s\})$ is therefore an algebraic function on $B$. 

24
4.5 We now remove the hypothesis on the $n$-torsion in 4.4. So let $\mathcal{E}$ be an arbitrary elliptic curve over $B$, and $s \in \tilde{\mathcal{E}}(B)$ a torsion section. Choose a multiple $n \geq 3$ of the order of $s$, and a finite étale Galois covering $C$ of $B$ such that the whole $n$-torsion of $\mathcal{E} \times_B C$ consists of sections. For the base change $s_C$ of $s$ to $C$, we have, by 4.4

$$\varphi(\{s_C\}) \in \Gamma(C, \mathcal{O}_C^*) \otimes \mathbb{Z} \cdot \mathbb{Q}.$$ 

Using the functoriality of $\varphi$ with respect to base change under the automorphisms of $C$ over $B$, one concludes purely formally:

**Theorem:** Let $s \in \tilde{\mathcal{E}}(B)$ a torsion section. Then

$$\varphi(\{s\}) \in \text{Ext}^1_{HDR}(\mathbb{Q}(\mathcal{K}), \mathbb{Q}(\mathcal{K}))$$ 

lies in the image of $\kappa_B$:

$$\varphi(\{s\}) \in \Gamma(B, \mathcal{O}_B^*) \otimes \mathbb{Z} \cdot \mathbb{Q}.$$

4.6 For a scheme $B$, which is smooth, separated, connected and of finite type over some field of characteristic 0, fix a geometric point $\bar{b}$, and consider the projective system $\{B_\alpha \mid \alpha \in I\}$ of pointed finite étale coverings of $B$. We have

$$\Gamma(B, \mathcal{O}_B^*) \otimes \mathbb{Z} \cdot \mathbb{Q} = \left( \lim_{\longrightarrow} \Gamma(B_\alpha, \mathcal{O}_{B_\alpha}^*) \otimes \mathbb{Z} \cdot \mathbb{Q} \right)^{\pi_1(B, \bar{b})}.$$ 

Let $\mathcal{E}$ be an elliptic curve over $B$. We denote by $\mathcal{L}_0(\mathcal{E})$ the subspace of $\ker(d)$ of divisors supported on torsion sections.

**Definition:** The subspace of elliptic modular units of the zeroth kind on $B$ is defined as

$$\left( \lim_{\longrightarrow} \varphi(\mathcal{L}_0(\mathcal{E} \times_B B_\alpha)) \right)^{\pi_1(B, \bar{b})} \subset \Gamma(B, \mathcal{O}_B^*) \otimes \mathbb{Z} \cdot \mathbb{Q}.$$ 

It is denoted by $EM_0(\mathcal{E})$.

4.7 It is natural to ask for the size of $EM_0(\mathcal{E})$ inside $\Gamma(B, \mathcal{O}_B^*) \otimes \mathbb{Z} \cdot \mathbb{Q}$. 

25
Examples:  a) For $E_n, n \geq 3$, the group

$$EM_0(E_n) \subset \Gamma(Y(n), \mathcal{O}_{Y(N)}) \otimes \mathbb{Z} \mathbb{Q}$$

contains the Siegel units $Si^\nu$. By [KL], IV, Theorem 1.1, we thus have

$$\mathbb{Q} \left( \frac{\mathbb{Q}}{\mathbb{Q}} \right) \cdot EM_\mathcal{F}(E_\infty) = \Gamma(\mathbb{Y}(\infty), \mathcal{O}_{\mathbb{Y}(\infty)}^*) \otimes \mathbb{Z} \mathbb{Q}.$$ 

The norm compatibility statement 3.2 b) translates into a distribution relation modulo roots of unity of the Siegel units. For the precise distribution relation, see [Kul], Theorem 2.2.

b) Let $E = \mathcal{E}$ be an elliptic curve over a number field $F$ with complex multiplication by $K \subset F$, such that every torsion point of $E$ is defined over the maximal abelian extension $K^{ab}$ of $K$. Then $EM_0(E \otimes_F F^{ab})$ contains the classical elliptic units modulo torsion (see [dSh], II, § 2).
5 Elliptic modular units of the first kind

5.1 Suppose given a section \( s \in \tilde{E}(B) \), and a divisor

\[
D = \sum_t q_t \cdot (t) \in \text{Div}^0(\mathcal{E}) \otimes_{\mathbb{Z}} \mathbb{Q},
\]

where \( t \) runs through the torsion sections of \( \pi : \mathcal{E} \rightarrow B \). Assume that \( q_t = 0 \) if \( s - t \) is not disjoint from the zero section. Then

\[
S_{D,s} := \sum_t q_t \{s - t\} \in \mathcal{L}(\mathcal{E})
\]

is an element of the kernel of \( d \).

**Theorem:** \( \varphi(S_{D,s}) \in \text{Ext}^1_{\text{HDR}_B}((\mathcal{O}(\mathcal{E})), \mathbb{Q}(l)) \) lies in the image of \( \kappa_B \):

\[
\varphi(S_{D,s}) \in \Gamma(B, \mathcal{O}_B^* \otimes_{\mathbb{Z}} \mathbb{Q}).
\]

**Proof:** Our situation arises via the base change \( s : B \rightarrow \tilde{E} \) from the projection

\[
pr_1 : \tilde{E} \times_B \mathcal{E} \rightarrow \tilde{E},
\]

with the torsion sections \( t \) replaced by

\[
t : \tilde{E} \rightarrow \tilde{E} \times_B \mathcal{E},
\]

\[
x \mapsto (x, t),
\]

and \( s \) replaced by \( \Delta \). So we need to show the statement for \( pr_1 \) and \( S_{pr_1^*D,\Delta} \).

As in the proof of 4.3, we may assume that there is a torsion section \( t' \) not contained in the support of \( D \). Then base change of

\[
pr_1 : \tilde{E} \times_B \mathcal{E} \rightarrow \tilde{E}
\]

via \( t' \) gives back the original situation, with \( s \) replaced by \( t' \).

Our claim then follows from 4.3 and 2.4. q.e.d.

5.2 We want to write down explicit formulae for the \( \varphi(S_{D,s}) \). In section 4, our geometrical object of study was the moduli space \( Y'(n) \) for
$n$-torsion sections. The moduli space for $n$-torsion sections plus an additional section is the universal elliptic curve $\mathcal{E}_n$ itself.

So let $n \geq 3$, and

$$pr_1 : \mathcal{E}_n \times_{Y(n)} \mathcal{E}_n \longrightarrow \mathcal{E}_n.$$ 

Recall from [4.3] that we parameterized the $n$-torsion sections of $\pi_n$ by $v \in \left(\frac{1}{n}\mathbb{Z}\right)^2 / \mathbb{Z}^\times$.

$$i_v : Y(n) \longrightarrow \mathcal{E}_n.$$ 

Via base change, we get $n$-torsion sections $i_v$ of $pr_1$.

Assume given a divisor

$$D = \sum_v q_v \cdot (i_v)$$

of degree 0. Then

$$S_D := S_{D, \Delta} = \sum_v q_v \cdot (\Delta - i_v) \in \mathcal{L}(U_D \times_{Y(n)} \mathcal{E}_n),$$

where $U_D \subset \mathcal{E}_n$ is the open subscheme of $\mathcal{E}_n$ complementary to the support of $D$. We have $S_D \in \ker(d)$.

We apply base change to $\mathbb{C}$ and determine

$$\varphi(S_D)^{MHS} \in \Gamma(U_{D, \mathbb{C}}, \mathcal{O}_{D, \mathbb{C}}^* \otimes_{\mathbb{Z}} \mathbb{Q}).$$

By [3.12], it differs from the inverse of the multivalued function

$$S_{i_D} : (z, \tau) \longmapsto \prod_v S_i(z + v_2\tau - v_1, \tau)^{q_v}$$

by a multivalued function of the shape $\exp(2\pi i \cdot F)$, where $F$ is a polynomial in $\mathbb{Q}[\leq \emptyset, \in \emptyset]$ of total degree smaller or equal to one.

**Definition:** A multivalued function is called a holomorphic modification of $S_{i_D}$, if it is of the shape

$$S_{i_D} \cdot \exp(2\pi i \cdot F)$$

for a polynomial $F$ in $\mathbb{Q}[\leq \emptyset, \in \emptyset]$ of total degree smaller or equal to one.

Since expressions of the form $\exp(2\pi i \cdot F)$ are holomorphic on $\mathbb{C} \times \mathcal{H}^+$ if and only if $F = 0$, we have:

**Proposition:** There exists a unique holomorphic modification of $S_{i_D}$. It is equal to the inverse of $\varphi(S_D)^{MHS}$. 

28
In practical terms, this means: in order to find \( (\varphi(S_D)^{\text{MHS}})^{-1} \), write down the formula for \( S_D \) and cancel the factors

\[
\exp(2\pi i \cdot (u_1 r_1 + u_2 r_2))
\]

of non-holomorphicity.

5.3 Now let again \( \mathcal{E} \) be an elliptic curve over a \( k \)-scheme \( B \), \( k \) being of characteristic 0. We define \( \mathcal{L}_1(\mathcal{E}) \) as the subspace of \( \ker(d) \) generated by divisors of the shape \( S_{D,s} \) as in 5.1.

**Definition:** The subspace of *elliptic modular units of the first kind on \( B \) is defined as

\[
\left( \lim_{\alpha \to} \varphi(\mathcal{L}_1(\mathcal{E} \times_B B_{\alpha})) \right)^{\pi_1(B \mathcal{B})} \subset \Gamma(B, \mathcal{O}_B^*) \otimes_{\mathbb{Z}} \mathbb{Q}.
\]

It is denoted by \( EM_1(\mathcal{E}) \).

**Example:** Let \( s \in \tilde{\mathcal{E}}(B) \) and \( N \geq 1 \) such that \( [N]s \) is still in \( \tilde{\mathcal{E}}(B) \), i.e., disjoint from the zero section. Then

\[
S := \{[N]s\} - N^2\{s\}
\]

is in \( \ker(d) \). We claim that \( \varphi(S) \) is actually an elliptic modular unit of the first kind.

In order to see this, we may assume that the whole \( N \)-torsion of \( \mathcal{E} \) consists of sections of \( \pi \). Then \( T := \{[N]s\} - \sum_{t \in \mathcal{E}[N](B)} \{s - t\} \) is in \( \ker(d) \). So our claim follows if we show *strong norm compatibility:*

\[
\varphi(T) = 1 \in \Gamma(B, \mathcal{O}_B^*) \otimes_{\mathbb{Z}} \mathbb{Q}.
\]

This is achieved, as in the proof of Theorem 5.1, by reducing to the case when \( s \) is torsion.
6  Elliptic modular units of the second kind

6.1 The last special elements of \( \ker(d) \) we want to consider are the parallelograms

\[
\{s + t\} + \{s - t\} - 2\{s\} - 2\{t\},
\]

for \( s, t, s + t, s - t \in \tilde{\mathcal{E}}(B) \).

**Theorem:** Let \( S_P := \{s + t\} + \{s - t\} - 2\{s\} - 2\{t\} \), for \( s \) and \( t \) as above. Then

\[
\varphi(S_P) \in \Ext^1_{H^0(B)}(\mathcal{K}, \mathcal{K}^*)
\]

lies in the image of \( \kappa_B \):

\[
\varphi(S_P) \in \Gamma(B, \mathcal{O}_B^*) \otimes \mathbb{Z} \mathcal{Q}.
\]

**Proof:** If \( s \) and \( t \) are torsion sections, then the claim holds by Theorem 4.5. The general case follows as in Theorem 5.1. We leave the details to the reader. q.e.d.

6.2 Again, we write down an explicit formula for \( \varphi(S) \). Following the procedure of 5.2, we work on the moduli space for \( n \)-torsion sections, plus two additional sections.

Let \( n \geq 3 \), and

\[
pr_{12} : \mathcal{E}_n \times_{Y(n)} \mathcal{E}_n \times_{Y(n)} \mathcal{E}_n \longrightarrow \mathcal{E}_n \times_{Y(n)} \mathcal{E}_n.
\]

There are two sections \( s \) and \( t \) of \( pr_{12} \):

\[
s : (x, y) \mapsto (x, y, x),
\]

\[
t : (x, y) \mapsto (x, y, y).
\]

Write

\[
S_P := \{s + t\} + \{s - t\} - 2\{s\} - 2\{t\}.
\]

It is an element of \( \mathcal{L}(V \times_{Y(n)} \mathcal{E}_n) \), where \( V \subset \mathcal{E}_n \times_{Y(n)} \mathcal{E}_n \) is the complement of the union of the two zero sections, of the diagonal \( \Delta \), and of the anti-diagonal

\[
\Delta^a := \{(x, -x) \in \mathcal{E}_n \times_{Y(n)} \mathcal{E}_n \mid x \in \mathcal{E}_n\}.
\]
We apply base change to \( C \) and determine 
\[
\varphi(S_P)^{MHS} \in \Gamma(V, \mathcal{O}_V^*) \otimes_{\mathbb{Z}} \mathbb{Q}.
\]

In the following, write 
\[
f : X \rightarrow Y
\]
for a morphism of real manifolds if it is clear from the context which closed subset of \( X \) one would have to remove in order to get the domain of definition for \( f \).

**Definition:** Define 
\[
S_{\varphi} : C \times C \times \mathcal{H}^+ \rightarrow C,
\]
\[
(z_1, z_2, \tau) \mapsto \frac{Si(z_1 + z_2, \tau)(Si(z_1 - z_2, \tau)}{Si(z_1, \tau)^2 Si(z_2, \tau)^2}.
\]

**Proposition:** \( S_{\varphi} \) is holomorphic on its domain of definition. It is equal to the inverse of \( \varphi(S_P)^{MHS} \).

**Proof:** The proof of the first claim uses Lemma [13]. The second claim follows from \[3.12\] q.e.d.

6.3 Let \( B \) be connected, smooth and separated of finite type over some field of characteristic 0, 
\[
\pi : \mathcal{E} \rightarrow B
\]
an elliptic curve. We define \( \mathcal{L}_2(\mathcal{E}) \) as the subspace of \( \ker(d) \) generated by divisors of the shape 
\[
\{s + t\} + \{s - t\} - 2\{s\} - 2\{t\},
\]
for \( s, t, s + t, s - t \in \tilde{\mathcal{E}}(B) \).

Again, write \( \{B_\alpha \mid \alpha \in I\} \) for the projective system of pointed finite étale coverings of \( B \).

**Definition:** The subspace of *elliptic modular units of the second kind* on \( B \) is defined as 
\[
\left( \lim_{\alpha} \varphi(\mathcal{L}_2(\mathcal{E} \times_B B_\alpha)) \right)^{\pi_1(B)} \subset \Gamma(B, \mathcal{O}_B^*) \otimes_{\mathbb{Z}} \mathbb{Q}.
\]
It is denoted by \( EM_2(\mathcal{E}) \).
7 Proof of Theorem 3.2

7.1 Let \( A \) be an abelian group

\[
\delta_A : \mathbb{Q}[A] \longrightarrow A \otimes_{\mathbb{Z}} A \otimes_{\mathbb{Z}} \mathbb{Q} , \quad \{ \mathcal{O} \} \longmapsto \mathcal{O} \otimes \mathcal{O} ,
\]

and denote by \( d_A \) the restriction of \( \delta_A \) to \( \mathcal{L}(A) := \mathbb{Q}[A - \{ \mathcal{K} \}] \).

**Proposition:** The kernel of \( d_A \) is generated by expressions of the following shape:

- **o)** \( \{ a \} \) for \( a \in A_{\text{tors}}, a \neq 0 \).
- **i)** \( \{ a \} - \{ a - b \} \) for \( a, b \in A_{\text{tors}}, a - b \neq 0 \);
  \( \{ Na \} - N^2\{ a \} \) for \( a \in A, N \geq 1, Na \neq 0 \).
- **ii)** \( \{ a + b \} + \{ a - b \} - 2\{ a \} - 2\{ b \} \) for \( a, b \in A, a, b, a + b, a - b \neq 0 \).

**Proof:** First show that the set of expressions as in o)–ii), with the condition “\( \neq 0 \)” removed, generates \( \ker(\delta_A) \). Then observe that \( \delta_A \) factors through the projection

\[
\mathbb{Q}[A] \longrightarrow \mathbb{Q}[A - \{ \mathcal{K} \}]
\]

given by sending \( \{ 0 \} \) to 0. \( \text{q.e.d.} \)

7.2 We are finally able to show:

**Theorem:** The morphism

\[
\varphi = \varphi(\mathcal{E}) : \ker(\delta) \longrightarrow \text{Ext}^1_{HDR^*_{Q}}(\mathbb{Q}(\mathcal{K}), \mathbb{Q}(\mathcal{K}))
\]

factors through \( \Gamma(B, \mathcal{O}_B^*) \otimes_{\mathbb{Z}} \mathbb{Q} \).

**Proof:** We may assume that \( B \) is connected. This implies that if \( S \) is a non-zero section of

\[
\pi : \mathcal{E} \longrightarrow B ,
\]

then it is actually disjoint from the zero section \( i \) on an open dense subscheme of \( B \).

So if \( S \in \ker(d) \), Proposition 4.3, 5.3, 5.1, and 6.1 tell us that the restriction of \( \varphi(S) \) to some open dense subscheme \( B' \) of \( B \) lies in

\[
\Gamma(B', \mathcal{O}_{B'}^*) \otimes_{\mathbb{Z}} \mathbb{Q} .
\]

Our claim follows from Theorem 2.4. \( \text{q.e.d.} \)
We thus get the proof of parts a) and b) of Theorem 3.2.

7.3 Using the explicit formulae of 4.2, 5.2, and 5.2, and the classical relations between the Siegel function and the Weierstraß function and its derivatives, one identifies $\varphi$ on the special elements $\{s, t\} \in \ker(d)$ of [GL], 4.5 (cf. e.g. [R], V.4, proof of Proposition 4.4). One gets a relative version of [W2], Proposition 1.9.1, which actually shows that the present construction and that of loc. cit. produce the same functions.

In order to show the relation to the non-archimedian local heights in 3.2.c), one then proceeds as in loc. cit.

7.4 Let $B$ be a connected scheme, which is smooth and separated of finite type over some field of characteristic 0,

$$\pi : \mathcal{E} \longrightarrow B$$

an elliptic curve. Write $\{B_\alpha | \alpha \in I\}$ for the projective system of pointed finite étale coverings of $B$.

**Definition:** The subspace of *elliptic modular units on $B$* is defined as

$$\left( \lim_{a} \varphi(\ker(d(\mathcal{E} \times_B B_\alpha))) \right)^{\pi_1(B, \tilde{\mathcal{E}})} \subset \Gamma(B, \mathcal{O}_B^* ) \otimes_{\mathbb{Z}} \mathbb{Q} .$$

It is denoted by $EM(\mathcal{E})$.

7.5 In the case when $B$ is a point, the following observation is due to Goncharov and Levin ([GL], Corollary 4.5):

**Theorem:** Zariski-locally on $B$, every non-vanishing function is an elliptic modular unit.

**Proof:** Assume given $s, t \in \tilde{\mathcal{E}}$ such that $s$ is disjoint from both $t$ and $-t$. Then we have

$$\{s, t\} = \{s + t\} + \{s - t\} - 2\{s\} - 2\{t\} \in \ker(d) .$$

According to the relative version of [W2], Proposition 1.9.1,

$$\varphi(\{s, t\}) = ((x(t) - x(s))\Delta^{-1/6})^{-1}$$

for any local Weierstraß equation $y^2 = x^3 + ax + b$ of $\mathcal{E}$. q.e.d.
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