On the phantom barrier crossing and the bounds on the speed of sound in non-minimal derivative coupling theories

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Abstract

In this paper we investigate the so-called ‘phantom barrier crossing’ issue in a cosmological model based on the scalar–tensor theory with non-minimal derivative coupling to the Einstein tensor. Special attention will be paid to the physical bounds on the squared sound speed. The numeric results are geometrically illustrated by means of a qualitative procedure of analysis that is based on the mapping of the orbits in the phase plane onto the surfaces that represent physical quantities in the extended phase space, that is: the phase plane complemented with an additional dimension relative to the given physical parameter. We find that the cosmological model based on the non-minimal derivative coupling theory—this includes both the quintessence and the pure derivative coupling cases—has serious causality problems related to superluminal propagation of the scalar and tensor perturbations. Even more disturbing is the finding that, despite the fact that the underlying theory is free of the Ostrogradsky instability, the corresponding cosmological model is plagued by the Laplacian (classical) instability related with negative squared sound speed. This instability leads to an uncontrollable growth of the energy density of the perturbations that is inversely proportional to their wavelength. We show that, independent of the self-interaction potential, for positive coupling the tensor perturbations propagate superluminally, while for negative
coupling a Laplacian instability arises. This latter instability invalidates the possibility for the model to describe the primordial inflation.

Keywords: scalar–tensor theory, phantom barrier crossing, gradient instability, causality, scalar perturbations, tensor perturbations, squared sound speed

(Some figures may appear in colour only in the online journal)

1. Introduction

Scalar–tensor theories [1, 2], among which the Brans–Dicke (BD) theory [3] is the prototype, have a long and hesitant history [4]. Despite that until very recently no fundamental scalar particle had been found in nature, these theories have found a variety of applications both in gravitational and in cosmological contexts. In the list of famous scalar fields (this includes the prototype BD scalar field) we encounter the Higgs particle of the standard model of particles [5], the dilaton—and other moduli fields—of the effective (low-energy) string theory [6], the inflaton, which accounts for the early inflationary stage of the cosmic evolution [7, 8], and the quintessence field that embodies the so-called dark energy that inflates the Universe at late times [9], among others. Starting in 2013 things changed and it seems that the first fundamental scalar particle had been finally discovered [10]. This means that scalars and, consequently, scalar–tensor theories, have to be taken seriously as feasible scenarios for physical phenomena.

The BD theory [3], as well as the more general scalar–tensor theories [1, 2], are classical theories of the gravitational field and as such these are not intended to describe quantum gravitational phenomena. However, there are indications that including higher order terms into the gravitational action makes the given theory of gravity more compatible with quantum (renormalizable) variants [11] whose predictions can be trusted back into the past. One example is the addition of four-order terms like $R_{\mu\nu\tau\rho}R^{\mu\nu\tau\rho}$, $R_{\mu\nu}R^{\mu\nu}$ and $R^2$ into the Einstein–Hilbert action that gives a class of multimass models of gravity [12] where, in addition to the usual massless excitations of the fields, there are massive scalar and spin-2 excitations with a total of eight degrees of freedom. In this vein it is interesting to complement the action of standard scalar–tensor theories with higher-order terms in order to have a theory more compatible with a would be quantum version. This modification would include not only terms quadratic in the curvature invariants but, also, higher-derivative terms like: $c_1 R_{\mu\nu\rho\sigma} \partial^\mu \phi \partial^\nu \phi$, $c_2 R_{\mu\nu} \partial^\mu \phi \partial^\nu \phi$, $c_3 R_{\mu\nu} \phi \nabla^\mu \nabla^\nu \phi$, $c_4 \nabla_{\mu} R \partial^\mu \phi$, $c_5 R \phi \nabla^2 \phi$, $c_6 \nabla^2 R \phi$, where $\nabla^2 \equiv \nabla_{\mu} \nabla^\mu$ and $c_1, \ldots, c_6$ are coupling constants with the dimensions of length squared.

The problem with the undiscriminated addition of higher-derivative terms is that the resulting equations of motion contain derivatives higher than second-order and this, in turn, leads to the appearance of awful and catastrophic Ostrogradsky ghosts in the theory, which makes it strongly unstable and untenable as an adequate model of gravitational phenomena. The most general possible scalar–tensor theories that contain higher order derivatives and derivative couplings in the Lagrangian and that, at the same time, lead to second-order motion equations—so that these are free of the Ostrogradsky instability—are called ‘Horndeski’ theories [14–17] (see [18] for a class of theories generalizing the Horndeski ones). These theories have been applied with success to describe the cosmological evolution of our Universe in different contexts [19–22]. An interesting subset of the Horndeski theories is composed of the

5 The unwanted (yet tractable) property of this theory is that the massive spin-2 mode is ghost-like [13].
so-called scalar–tensor theories with a non-minimal derivative (kinetic) coupling, in particular those where the kinetic coupling is to the Einstein tensor \[ G^{\mu \nu} \partial_\mu \phi \partial_\nu \phi \]. The latter theory is characterized by its relative mathematical simplicity when compared with other Horndeski theories and also by its ability to account for the early (transient) inflationary stage, since it is able to explain in a unique manner both a quasi-de Sitter phase and an exit from it without any fine-tuned potential \[23\].

The action for the typical theory with non-minimal derivative coupling of the scalar with the Einstein tensor:

\[
G_{\mu \nu} \equiv R_{\mu \nu} - g_{\mu \nu} R / 2,
\]

is given by:

\[
S = \int d^4 x \sqrt{|g|} \left[ R - (\epsilon g^{\mu \nu} - \alpha G^{\mu \nu}) \partial_\mu \phi \partial_\nu \phi - 2V(\phi) \right] + S_m, \tag{1}
\]

where we set \(8 \pi G_N = c = h = 1\), and the coupling constant \(\alpha\) is a real number. The parameter \(\epsilon\) can take the following values: \(\epsilon = +1\) (quintessence), \(\epsilon = -1\) (phantom cosmology), and \(\epsilon = 0\) (pure derivative coupling)\(^6\). In the above equation \(S_m\) is the action of the matter degrees of freedom rather than the scalar field.

Theories of the type (1) have been studied in different contexts. For instance, in \([30]\) static, spherically symmetric solutions to the gravitational field equations derived from (1) were explored and black hole solutions with a single regular horizon were found, and their thermodynamical properties were examined. Related work regarding asymptotically locally AdS and flat black holes can be found in \([31]\), while in \([32]\) the authors constructed the first neutron stars based on (1). The obtained construction may, in principle, constrain in a phenomenological way the free parameters of the model. Cosmological scenarios based on theories with kinetic coupling with the Einstein tensor have been studied in \([24]\) in order to examine quintessence (and phantom) models of dark energy with zero and constant self-interaction potentials. It has been shown that, in general, the universe transits from one de Sitter solution to another, depending on the coupling parameter. A variety of behaviors—including Big Bang and Big Crunch solutions, and also cosmological bounce—reveal the capabilities of the corresponding cosmological model. A dynamical systems analysis of the derivative coupling model using the Higgs-like potential can be found in \([25]\), while a similar study for the exponential potential was performed in \([33]\). It was found that, for the quintessence case, the stable fixed points are the same with and without the non-minimal derivative coupling, while for the pure derivative coupling (no standard canonical kinetic term) only the de Sitter attractor exists and the dark matter solution is unstable. Cosmology based on (1) was also investigated in \([34]\). The latter paper points out the existence of the Laplacian instability in the theory with kinetic coupling of the scalar field with the Einstein tensor in the context of reheating after inflation. Particle production after inflation in the model (1) tensor was also studied in \([35]\) by the same authors.

A very interesting—and in our opinion, central—aspect of the theory (1) was investigated in \([27]\). In that reference it was found that in the cosmological model based on (1) with pure derivative coupling to the Einstein tensor \(\epsilon = 0\) and with vanishing potential \(V = 0\)—in the absence of other matter sources \((S_m = 0)\)—the scalar behaves as pressureless matter with vanishing sound speed, so that it could be a candidate for cold dark matter. By also considering the scalar potential \((V \neq 0)\), it was found that the scalar field may play the role of both dark matter and dark energy. In this case, the effective equation of state (EOS) of the scalar field \(\omega_{\text{eff}}\) can cross the phantom divide \([36-40]\): \(\omega_{\Lambda} = -1\) (this is properly the EOS parameter of the cosmological constant), but this can lead to the sound speed becoming superluminal as

\(^6\)In this paper we refer to ‘pure derivative coupling’—independent of the presence or absence of the self-interacting potential—in the models based on the action principle (1) without the standard kinetic term \(\epsilon = 0\), i.e. there is only kinetic coupling to the Einstein tensor.
it crosses the divide, and so is physically forbidden\(^7\). The possibility of the phantom divide crossing in the model is in itself a very interesting finding; however, there are two results we find particularly interesting in this study: (i) that the crossing of the phantom divide may be linked with superluminal sound speed, and (ii) that the physical limits on the sound speed are used as basic criteria for rejection of a given cosmological model. The fact that the physical bounds on the speed of propagation of the perturbations of the field are to be taken carefully and seriously when Horndeski-type theories are under investigation was also understood by the authors of [41]. In that reference it was shown that, when the Dirac–Born–Infeld galileon is considered as a local modification to gravity, such as in the Solar system, the existing stable solutions always exhibit superluminality, casting doubt on the existence of a standard Lorentz invariant UV completion of that theory\(^8\).

In view of the importance of the above issue, and given that there does not exist in the bibliography a thorough discussion on the implications for cosmology of the physical bounds on the speed of sound in the theory with the kinetic coupling to the Einstein tensor\(^9\), in the present paper we shall be concentrating on the ‘\(\omega_\Lambda = -1\)’ barrier crossing issue in the model (1) by paying special attention to the physical bounds on the speed squared \(c^2_s\). These bounds are imposed by stability and causality, two fundamental principles of classical physical theories: the squared sound speed should be non-negative \(c^2_s \geq 0\) since otherwise, the cosmological model will be classically unstable against small perturbations of the background energy density, usually denoted as Laplacian—also gradient—instability. In addition, causality arguments impose that the mentioned small perturbations of the background should propagate at most at the local speed of light \(c^2_s \leq 1\).

In order to implement the numeric investigation we shall explore two specific potentials: that frequently encountered in cosmological applications, the exponential potential [33, 47, 48], \(V = V_0 \exp(\lambda \phi)\), and, also, the power-law potential \(V = V_0 \phi^{2n}\) [49]. The exponential potential

\[
V = V_0 e^{\lambda \phi} \Rightarrow V' = \lambda V,
\]

where \(V_0\) and \(\lambda\) are real constants \((V_0 \geq 0)\), can also be found in higher-order or higher-dimensional gravity theories [50], and in string or Kaluza–Klein type models, where the moduli fields may have effective exponential potentials [51]. Exponential potentials can also arise due to non-perturbative effects such as gaugino condensation [52]. In the present model the exponential potential has been investigated in [33], where a dynamical systems analysis was performed. The conclusion of the authors was that the derivative coupling to the Einstein tensor does not modify the phase space dynamics of the quintessence [48]. The power-law potential

\[
V = V_0 \phi^{2n} \Rightarrow V' = 2n V_0^{1/2n} \phi^{1-1/2n},
\]

\(^7\) It is well known that Horndeski theories all possess some configurations with a superluminal propagation.

\(^8\) There exist alternative points of view on this issue. For instance, in [42, 43] it is shown that \(k\)-essence and galileon theories, respectively, satisfy an analog of Hawking’s chronology protection conjecture, an argument that can be extended to include Horndeski theories in general. However, there are strong arguments that contradict such kinds of non-orthodox points of view on causality (for more on this issue see [44]). In this regard we recommend the clear and pedagogical discussion on this issue given in [45].

\(^9\) In [27] the subject was only partially investigated—only connection of the phantom barrier crossing with superluminality of the scalar perturbations was established—besides only the pure derivative coupling case \(\epsilon = 0\) was considered in that reference. The issue was also stated but not investigated in [46]. In this latter reference (see last paragraph of p 8) the authors state that the investigation of the instabilities and superluminality in the model with the kinetic coupling to the Einstein tensor lies beyond the scope of their paper. A similar statement can be found in [25] (see the top paragraph of p 3).
where \( V_0 \) is a non-negative constant and \( n \) is a real parameter, is also frequently found in the cosmological applications [49]. In the quintessence case the inverse-power law potential exhibits tracker behavior, a very desirable property for the quintessence if one wants to avoid the cosmic coincidence problem [53]. The origin of this potential might be associated with supersymmetry considerations [54].

As a qualitative support to the present discussion, a geometric procedure of analysis based on the properties of the dynamical system is developed. It provides a clear illustration of the failure of causality and/or of the development of Laplacian instability—as well as of the crossing of the phantom divide—along given phase space orbits. The mentioned procedure relies on the mapping of phase space orbits into the extended phase space, that is: the phase plane complemented with an additional dimension represented by the physical parameter of interest (the effective EOS or the squared sound speed, for instance). This is why we call the procedure \( P \)-embedding, where \( P \) refers to the given physical parameter. Although the numeric computations are performed for the exponential and for the power-law potentials only, the constant and vanishing potential cases are implicitly included as their particular cases. The embedding procedure is schematically represented in figure 1, where the \( c^2 \)-embedding is illustrated for the cosmological model of interest, for the positive coupling case (\( \alpha > 0 \)) and for the monotonically growing exponential potential (2) with \( \lambda = 5 \).

The numeric investigation is preceded by—and complimented by—a thorough analytic study. In this regard we shall go as far as we can before specifying the form of the self-interaction potential, so that our discussion is as independent as it can be of the specific choice of potential. Our results show that the cosmological models based on the scalar–tensor theory, with non-minimal derivative coupling to the Einstein tensor (1), develop severe causality problems related to superluminal propagation of the perturbations of the scalar field. These problems are critical whenever the crossing of the phantom divide happens; however, these may arise even in the absence of the crossing. More problematic than the violations of causality in the model is the finding that it is plagued by the classical (catastrophic) Laplacian instability, despite the fact that the theory (1) on which it is based is free of the Ostrogradsky instability. Our results just confirm the inappropriateness of the models based on the kinetic coupling theories of the type (1), as was discussed just recently in [55], in the light of the tight constraint on the difference in speed of photons and gravitons (\( c_T^2 - c_s^2 / c_s^2 \leq 6 \times 10^{-15} \) (\( c_T \) is the speed of the gravitational waves) implied by the announced detection of gravitational waves from the neutron star–neutron star merger GW170817 and the simultaneous measurement of the gamma-ray burst GRB170817A [56].

Before we go further, in order to unify the terminology and to be able to compare our results with other results in the bibliography, we want to comment on the sign of the coupling constant \( \alpha \). This constant was named \( \kappa \) in [23, 24], \( \alpha \) in [27], \( \zeta \) in [46] and \( \omega^2 \) in [33]. If compare the action in [23] (equation (8) of that reference)—the same action as in [24] but in this case the self-interacting potential for the scalar field is considered—we see that their \( \kappa \) corresponds to \( -\alpha \) of the present paper, so that when the authors of [23, 24] refer to negative coupling \( \kappa < 0 \) this means positive coupling in terms of our \( \alpha (\alpha > 0) \) and vice versa. We recall that in [23, 24] both cases, \( \kappa > 0 (\alpha < 0) \) and \( \kappa < 0 (\alpha > 0) \), were considered. In [27] it seems that there is a problem with the sign of the Lagrangian density in (1.5) of their work. While a straightforward comparison of the action (2.4) of [27]—with the substitution of the Lagrangian density (1.5)—with our equation (1) yields the correspondence \( \alpha \rightarrow -\alpha \) between the coupling constant in their work and in the present paper; a comparison of our cosmological field equation (5) with the corresponding equation (2.12) in [27] yields a direct correspondence \( \alpha \rightarrow \alpha \). Here we give preference to the cosmological field equations so that we shall assume that the sign of the coupling constant in [27] and in our paper coincides.
similar way the sign of the coupling constant \( \zeta \) in [46] and \( \omega^2 \) in [33] is the same as for our \( \alpha \).

The only difference is that in [33] the coupling constant \( \omega^2 \) is assumed to take positive values exclusively, while in other works (including ours) both signs are considered.

We have organized the paper in the following way. In section 2 we state the main assumptions on which the present work relies and we write down the basic expressions that will be useful in the subsequent study. Appropriate (dimensionless) variables of the phase space are introduced in order to study in a unified way both the positive and the negative coupling cases. A quite general discussion on the phantom barrier crossing in the model (1) is given in section 3. In section 4 we discuss the behavior of the sound speed squared \( c^2_s \)—the one that accounts for the speed of propagation of the perturbations of the energy density—in the present model. Particular emphasis is put on the possible violations of the physical bounds \( 0 \leq c^2_s \leq 1 \). Section 5 is dedicated to briefly exposing the main properties of the dynamical system corresponding to the present cosmological model in connection with the bounds on the squared sound speed. While in sections 3 and 4 we focus mainly in the quintessence case (\( \epsilon = 1 \)), in section 6 the pure derivative coupling case (\( \epsilon = 0 \)) is separately investigated. A thorough discussion of the results obtained in this paper is presented in section 7. In particular, the case with the constant potential that can be developed in a fully analytical way, is discussed as a simple illustration of the results analysis. Finally, brief conclusions are given in section 8. For completeness we have included an appendix. In the appendix an elementary discussion on the so-called Laplacian instability is included.

Throughout the paper we use the units system with \( 8\pi G_N = c^2 = 1 \), where \( G_N \) is the Newton constant and \( c \) is the speed of light in vacuum.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{The \( c^2_s \)-embedding schematically represented. The phase portrait of the dynamical system (39) and the plot of the surface \( c^2_s = c^2_s(x, u) \)—with contours—in the extended (three-dimensional) phase space that is spanned by the coordinates \( x, u \) and \( c^2_s \), are shown in the left-hand and in the middle figures, respectively. In the right-hand figure the \( c^2_s \)-embedding diagram is drawn: the orbits (red curves) appearing in the phase portrait (left) have been embedded into the surface \( c^2_s = c^2_s(x, u) \). The computations correspond to the cosmological model (1) with positive coupling (\( \alpha > 0 \)) and for the growing exponential potential (\( \lambda = 5 \)). The contours drawn in the right-hand figure mark the region where \( c^2_s < 0 \), i.e. where the Laplacian instability develops. The different embedded orbits correspond to whole cosmic evolutionary pathways that are associated with different sets of initial conditions. From the embedding diagram it is seen that, independent of the initial conditions chosen, the corresponding cosmological histories inevitably go through a stage where \( c^2_s < 0 \), so that the classical gradient instability destroys any chance for the Universe to evolve into its present state.}
\end{figure}
2. Basic equations and set up

The main hypothesis of this paper is that the physical bounds on the speed of sound (squared) are viable criteria to reject physical theories like the one being investigated here. Other assumptions considered in this paper are the following.

- For simplicity of the discussion we shall focus on the vacuum case, i.e. in (1) we set $S_m = 0$.
- For definiteness only expanding cosmologies ($H \geq 0$) will be considered and, in addition, the scalar field $\phi$ is assumed to be a monotone non-decreasing function of the cosmic time: $\dot{\phi} \geq 0$.
- We consider non-negative self-interacting potential $V \geq 0$ (non-negative energy density).
- Only the cases with $\epsilon = 1$ (quintessence) and $\epsilon = 0$ (pure derivative coupling) will be of interest.

As a model for the background spacetime here we assume the Friedmann–Robertson–Walker (FRW) metric with flat spatial sections, whose line element is given by:

$$d s^2 = -d t^2 + a^2(t) \delta_{ij} dx^i dx^j. \tag{4}$$

The cosmological field equations that can be derived from the action (1) read:

$$3H^2 = \rho_{eff} - 2\dot{H} = \rho_{eff} + p_{eff},$$

$$\ddot{\phi} + 3H\dot{\phi} = \frac{-6\alpha H\dot{H}\dot{\phi} - V_{\phi}}{\epsilon + 3\alpha H^2}, \tag{5}$$

where $V_{\phi} \equiv dV/d\phi$. The effective energy density and pressure of the scalar field are given by

$$\rho_{eff} = \frac{\epsilon + 9\alpha H^2}{2} \dot{\phi}^2 + V(\phi), \tag{6}$$

$$p_{eff} = \frac{\epsilon - 3\alpha H^2}{2} \dot{\phi}^2 - V(\phi) - \alpha \dot{\phi}^2 \dot{H} - 2\alpha H \phi \ddot{\phi}, \tag{7}$$

respectively. From the above equations one obtains that:

$$\rho_{eff} + p_{eff} = \left(\epsilon + 9\alpha H^2\right) \dot{\phi}^2 - \alpha \dot{\phi}^2 \dot{H} - 2\alpha H \phi \ddot{\phi} \left(\ddot{\phi} + 3\dot{\phi}\right). \tag{8}$$

An interesting property of the effective energy density $\rho_{eff}$ in (6), and of the effective pressure $p_{eff}$ in (7), is that these quantities depend not only on the scalar field matter degree of freedom $\phi$ and its derivatives $\dot{\phi}$ and $\ddot{\phi}$, but also on the curvature through $H^2$ and $\dot{H}$. In particular, the effective kinetic energy density of the scalar field in the right-hand side (RHS) of the Friedmann equation above, $(\epsilon + 9\alpha H^2) \dot{\phi}^2 / 2$, is contributed to not only by $\dot{\phi}$ but also by the curvature through the squared Hubble rate\(^\text{10}\).

With the help of the first equation in (5) and of (6) one can rewrite the Friedmann equation and, correspondingly, the effective energy density, in the following way:

$$3H^2 = \gamma^2 \rho_{\dot{\phi}} = \rho_{eff}, \quad \gamma = \frac{1}{\sqrt{1 - 3\alpha \dot{\phi}^2 / 2}}, \tag{9}$$

where $\gamma = \gamma(\dot{\phi})$ is the 'boost' function and

\(^{10}\text{Notice that, when in the above equations the non-minimal derivative coupling vanishes, } \alpha = 0, \text{ we recover the standard result of general relativity with minimally coupled scalar field matter.}\)
\[ \rho_\phi = \frac{\epsilon \dot{\phi}^2}{2} + V(\phi), \]

is the standard energy density of the scalar field. Written in the latter form \( \rho_{\text{eff}} \) is a function only of the scalar field degree of freedom \( \phi \), and of its derivative \( \dot{\phi} \) since the curvature effects are hidden in the non-canonical form of the effective energy density, i.e. in the boost function.

We point out that for negative coupling (\( \alpha < 0 \)), the boost function is bounded from below and also from above (\( 0 < \gamma \leq 1 \)), while for positive coupling (\( \alpha > 0 \)), \( 1 \leq \gamma < \infty \), i.e. it is bounded from below only.

2.1. Non-negative coupling and upper bound on \( |\dot{\phi}| \)

If we consider non-negative \( \alpha \geq 0 \), from (9)—given that we consider non-negative effective energy density exclusively—it follows that \( 1 - 3\alpha \dot{\phi}^2 / 2 \geq 0 \), i.e.

\[ 0 \leq \dot{\phi}^2 \leq \frac{2}{3\alpha} \Leftrightarrow -\frac{1}{3\alpha} \leq X \leq 0, \tag{10} \]

where \( X = \partial_\mu \phi \partial^\mu \phi / 2 = -\dot{\phi}^2 / 2 \).

We want to point out here the non-conventional nature of the ‘effective’ kinetic energy of the scalar field (6) under the derivative coupling when \( \alpha > 0 \). Actually, as just seen, the standard kinetic energy \( \propto \dot{\phi}^2 \) is bounded from above, a strange feature not arising in standard scalar tensor theories without self-coupling. Notwithstanding, the effective kinetic energy in (6), \( \propto (\epsilon + 9\alpha H^2)\dot{\phi}^2 \), is not bounded due to the curvature effects encoded in \( H^2 \).

In [23], since in that presentation the coupling \( \kappa \) is of opposite sign as compared with our \( \alpha \), \( \kappa = -\alpha \), the case where the standard kinetic term is bounded from above corresponds to the condition expressed by equation (27) in the mentioned reference (see also equations (19) and (21) of the same reference, recalling that in this paper we have chosen the units where \( 8\pi G_N = 1 \), while in [23] \( G_N = 1 \)).

2.2. New variables

In spite of the commonly used variable \( X \), in order to study both positive and negative coupling cases in a unified way in this paper we prefer to use the new variable:

\[ x := \alpha \dot{\phi}^2 / 2, \tag{11} \]

i.e. the new variable is properly the standard kinetic energy of the scalar field multiplied by the coupling constant. Hence, positive coupling entails that \( x \geq 0 \), while negative coupling means that \( x \leq 0 \). Vanishing \( x = 0 \) means that either the scalar field is a constant \( \phi = \phi_0 \), or there is no derivative coupling; \( \alpha = 0 \).

In the same way, in connection with the self-interaction potential term, it will be very useful to introduce the following variable:

\[ y := \alpha V, \tag{12} \]

where, for positive \( \alpha \), this variable takes non-negative values \( 0 \leq y < \infty \), while for negative coupling (\( \alpha < 0 \)) the variable takes non-positive values instead \( -\infty < y \leq 0 \).

We want to underline that, for positive coupling (\( \alpha > 0 \)), given that \( H^2 \),

\[ 3\alpha H^2 = \frac{\epsilon x + y}{1 - 3x}, \tag{13} \]
should be a non-negative quantity \((H^2 \geq 0)\), and the non-negative variable \(x\) should take values in the physically meaningful interval \(0 \leq x \leq 1/3\). Meanwhile, for negative coupling \((\alpha < 0)\) the variable \(x\) is non-positive: \(-\infty < x \leq 0\).

The above variables will allow us to write the equations in a more compact manner and to make our computations independent of the specific value of the coupling constant.

### 3. Phantom barrier crossing: general analysis

As mentioned in the introduction, one issue of interest when one explores cosmological models of dark energy is the possibility of crossing the so-called ‘phantom divide’ barrier \(\omega_\Lambda = -1\) [36–39]. Hence, it will be useful to look for the possibility of the crossing in the theory with non-minimal derivative (kinetic) coupling to the Einstein tensor [40].

If, under the assumptions exposed in section 2, we combine the second and third equations in (5), we obtain:

\[
-2\alpha \dot{H} = R_1 + R_2,
\]

with (recall that \(y_\phi = \alpha V_\phi = \alpha dV/d\phi\)):

\[
R_1 = \frac{2x(1-2x+y)(\epsilon + 3y)}{(1-3x)F_e}, \quad R_2 = \frac{2\sqrt{2x(1-3x)(\epsilon x + y)}y_\phi}{\sqrt{3}F_e},
\]

where, for compactness of writing, we have introduced the following definition:

\[
F_e \equiv F_e(x, y) : = \epsilon(1-3x+6x^2) + (1+3x)y.
\]

The effective EOS parameter of the scalar field is given by:

\[
\omega_{\text{eff}} = \frac{\rho_{\text{eff}}}{\rho_{\text{eff}}} = -1 - \frac{2\dot{H}}{H^2} = -1 + \frac{R_1 + R_2}{3\alpha H^2},
\]

where \(R_1\) and \(R_2\) are given by (15) and, in terms of the variables \(x, y\), the denominator \(3\alpha H^2\) is given by (13). Hence, for the effective EOS in the general case—unspecified \(\epsilon\)—we get:

\[
\omega_{\text{eff}} = -1 + \frac{2x(\epsilon + 3y)[\epsilon(1-2x+y)]}{(\epsilon x + y)F_e} + \frac{2\sqrt{2x(1-3x)^3}}{3(\epsilon x + y)}y_\phi.
\]

As can be seen from (17), the crossing of the phantom barrier is achieved only if \(-2\dot{H}\) can change sign during the cosmic evolution. In general \(-2\dot{H}\) is a non-negative quantity. This is specially true for the standard quintessence, where in equations (5)–(7) we set \(\alpha = 0\) and \(\epsilon = 1\). In this case \(-2\dot{H} = \dot{\phi}^2 \geq 0\), while the EOS parameter in (17) can be written as

\[
\omega_{\text{eff}} = -1 + \frac{\dot{\phi}^2}{3H^2},
\]

so that, given that \(\dot{\phi}^2/3H^2\) is always non-negative, then \(\omega_{\text{eff}} \geq -1\). In this case the phantom barrier crossing is not possible unless additional complications are considered, such as: (i) non-gravitational interaction of the dark energy and dark matter components [57], (ii) multiple dark energy fields like in quintom models [58, 59] or (iii) extra-dimensional effects [60]. Here we shall investigate the issue within the frame of the theory (1) where the derivatives of the scalar field are non-minimally coupled to the Einstein tensor.
3.1. Positive coupling ($\alpha > 0$)

For non-negative $x$, i.e. for positive coupling ($\alpha > 0$), the denominators of $R_1$ and of $R_2$ in (15) are always positive valued. So is the numerator of the term $R_1$, which means that this term is always non-negative. Meanwhile, the sign of the numerator of the term $R_2$ is determined by the slope of the self-interaction potential:

$$y \phi = \alpha V \phi = \alpha \frac{dV}{d\phi}.$$  

Consequently, for non-negative $0 \leq x \leq 1/3$, the term $R_2$ in (14) is the only one that may allow crossing of the phantom barrier.

In this case ($0 \leq x \leq 1/3$) two clear conclusions can be made: (i) the crossing is due to the derivative coupling with strength $\alpha$, and (ii) the crossing is allowed only if $\dot{\phi}V' = \dot{V} < 0$, i.e. if the self-interaction potential decays with the cosmic expansion. Assuming that this is indeed the case, the competition between the positive term $R_1$ and the negative one $R_2$ during the course of the cosmic evolution is what makes possible the flip of sign of $-2\dot{H} = R_1 + R_2$, and hence the crossing of the phantom barrier. Notice that for the constant potential $V_\phi = 0$, as well as for the monotonically growing potentials, the crossing is not possible. This is true, in particular, for the growing exponential potential: $V \propto \exp(\lambda \phi)$ with $\lambda > 0$ for $\dot{\phi} > 0$ or $\lambda < 0$ for $\dot{\phi} < 0$, and for the power-law $V \propto \phi^n$ with $n \geq 0$.

The above results are illustrated in figure 2 where a geometric representation of the quantity $\omega_{\text{eff}} + 1$ in the $xu$-plane is shown. Here we used the new (bounded) variable:

$$u = \frac{y}{y + 1}, \quad 0 \leq u \leq 1.$$  

This choice makes it possible to fit the whole (semi-infinite) phase plane $xy$ into a finite size box: $\{(x, u) : 0 \leq x \leq 1/3, 0 \leq u \leq 1\}$. The red-colored regions are the ones where $\omega_{\text{eff}} + 1 < 0$, i.e. where the scalar field behaves like phantom matter. It is appreciated that, for negative-slope potentials (the decaying exponential and the inverse power-law in the figure), both the phantom region with $\omega_{\text{eff}} + 1 < 0$ and the region where $\omega_{\text{eff}} + 1 > 0$ (gray color) coexist, so that the crossing of the phantom divide is possible.
3.2. Negative coupling ($\alpha < 0$)

For negative coupling, i.e. for $-\infty < x \leq 0$, $-\infty < y \leq 0$, the situation is a bit more complex. In this case it is more appropriate to go to a bounded set of variables:

$$v = \frac{x}{x-1}, \ w = \frac{y}{y-1},$$

where $0 \leq v \leq 1, 0 \leq w \leq 1$. In terms of the latter variables the whole plane $xy$ fits into the unit square $\{(v, w) : 0 \leq v \leq 1, 0 \leq w \leq 1\}$. We have that\(^{11}\)

$$\omega_{\text{eff}} = -1 + \frac{2v(4w - 1)(1 + v - 2w)}{(1-v)(1-w)(v + w - 2vw)F_1} + \frac{2}{F_1}\sqrt{\frac{2v(1 + 2v)^3(1 - w)}{3(1-v)^3(v + w - 2vw)\phi}},$$

where $y_\phi = -w_\phi/(1-w)^2$, and

$$F_1 \equiv F_{\epsilon=1} = \frac{1 + v + 4v^2 - 2(1 - 2v + 4v^2)w}{(1-v)^2(1-w)}.$$  \tag{23}

This latter quantity, as well as the second and the third terms on the RHS of (22), in general do not have a definite sign. As a consequence, there can be curves $w_c = w_c(v)$ where $F_1$ vanishes, meaning that the surface $\omega_{\text{eff}} = \omega_{\text{eff}}(v, w)$ tends to asymptotic large values, so that the surface literally ‘breaks off’ (slanted zigzagging curves in the top left hand corners in the figures in figure 3). The curves $w_c = w_c(v)$ that annihilate the function $F_1$, $F_1(w_c(v), v) = 0$, are in fact asymptotic separatrices in the $vw$-plane (the unit square). This means that any other curve in the unit square can only asymptotically approach to—or leave off—the curve $w_c$.

The competition between the second and the third terms on the RHS of (22) does not depend only on the slope of the potential $y_\phi$ (or $-w_\phi$), but also on whether the given $vw$-region is located in respect to the asymptote at $w_c = w_c(v)$ where $F_1 = 0$. From the plots in figure 3 it is seen that the only continuous regions in the $vw$-plane where the crossing of the phantom divide is possible are those located below and to the right of the separatrices (the slanted zigzagging curves in the top-left corner) for the monotonically decreasing potentials: the decaying exponential and the inverse power-law in the left-hand panels. In these regions there can be curves in the $vw$-plane that continuously join the domains where $\omega_{\text{eff}} > -1$ (gray color region at the bottom of the figures) with those where $\omega_{\text{eff}} < -1$ (red color). Hence, these curves can continuously cross the phantom divide: $\omega_{\text{eff}} + 1 = 0$.

One may conclude that, independent of the sign of the coupling constant $\alpha$, the crossing of the phantom divide can happen only for negative-slope potentials: $y_\phi < 0$ ($V_\phi < 0$).

4. Squared sound speed

In [61] the authors derived the evolution equations for the most general cosmological scalar, vector and tensor perturbations in a class of non-singular cosmologies derived from higher-order corrections to the low-energy bosonic string action:

$$\mathcal{L} = \frac{1}{2} f(\phi, R) - \frac{1}{2} \omega(\phi) \nabla^\mu \phi \nabla_\mu \phi - V(\phi) + \mathcal{L}_q.$$  \tag{24}

\(^{11}\) For definiteness here we set $\epsilon = 1$. The pure derivative coupling case $\epsilon = 0$ will be discussed separately in section 6.
where \( f(\phi, R) \) is an algebraic function of the scalar field \( \phi \) and of the curvature scalar \( R \), while \( \omega(\phi) \) and \( V(\phi) \) are functions of the scalar field. For our purposes it is enough to consider \( f(\phi, R) = R \) and \( \omega(\phi) = 1 \). Through \( L_q \) the inclusion of higher order derivative terms is allowed:

\[
L_q = -\frac{\lambda}{2} \left[ c_1 R_{\text{GB}}^2 + c_2 G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + c_3 \nabla^2 \phi \partial^\mu \phi \partial_\mu \phi + c_4 (\partial^\mu \phi \partial_\mu \phi)^2 \right],
\]

(25)

where \( \xi = \xi(\phi) \) is a function of the scalar field, \( R_{\text{GB}}^2 \equiv R_{\mu\nu\tau\lambda} R^{\mu\nu\tau\lambda} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \) is the Gauss–Bonnet combination, \( \lambda, c_1, \ldots, c_4 \) are constants, and we have chosen the units where \( \alpha' = 1 \). In this paper, without loss of generality we set \( \xi = 1 \).

Our action (1) is a particular case of (24), so that the results of [61] are easily applicable to the present model (see, for instance, [27]). The Einstein field equations that are derived from the Lagrangian (24) read:

\[
G_{\mu\nu} = T^{\text{eff}}_{\mu\nu} = T^{(\phi)}_{\mu\nu} + T^{(q)}_{\mu\nu}, \quad \nabla^2 \phi - T^{(q)} = V',
\]
where the comma stands for derivative with respect to $\phi$,

$$T_{\mu\nu}^{(\phi)} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} (\partial^\tau \phi \partial_\tau \phi) - g_{\mu\nu} V,$$

is the standard stress-energy tensor of a scalar field, while

$$T_{\mu\nu}^{(q)} = -2 \frac{\partial L_q}{\partial g^{\mu\nu}} - g_{\mu\nu} L_q,$$

and $T^{(\phi)}$ represent the contributions derived from the next to leading order corrections given by $L_q$ in equation (25) (equation (2) of [61]). These contribute towards the effective stresses and energy.

The perturbed line-element reads [61, 62]:

$$\mathrm{d}s^2 = -a^2(1 + 2\psi)\mathrm{d}t^2 - 2a^2(\beta_j + B_j)\mathrm{d}t\mathrm{d}x^j + a^2 \left[ g_{ij}(1 + 2\varphi) + 2\gamma_{ij} + 2C_{(ij)} + 2C_{ij} \right] \mathrm{d}x^i\mathrm{d}x^j,$$

where $d\eta = dt/a$. Latin letters denote space indices while $\psi = \psi(t, x)$, $\beta = \beta(t, x)$, $\varphi = \varphi(t, x)$ and $\gamma = \gamma(t, x)$ characterize the scalar-type perturbations. The traceless modes $B_i$ and $C_i$ $(B^i_i = C^i_i = 0)$ represent the vector-type perturbations, meanwhile, $C_{ij} = C_{ij}(t, x)$ are trace free and transverse; $C^i_{ij} = C^i_i = 0$, and correspond to the tensor-type perturbations. The vertical bar denotes covariant derivative defined in terms of the space metric $g_{ij}$.

Following [62] in [61] the uniform-field gauge ($\delta \phi = 0$) is chosen since this gauge admits the simplest analysis. In this case each variable is replaced by its corresponding gauge-invariant combination with $\delta \phi$, for instance, for the scalar perturbation the gauge-invariant combination

$$\varphi_{\delta \phi} \equiv \varphi - H \frac{\delta \phi}{\dot{\phi}},$$

is considered (in the uniform-field gauge $\varphi_{\delta \phi}$ is identified with $\varphi$ since $\delta \phi = 0$). The second-order differential (wave) equation for the scalar-metric perturbation $\varphi_{\delta \phi}$ in closed form reads [61]:

$$\frac{1}{a^2 Q_s} \frac{\partial}{\partial t} \left( a^2 Q_s \frac{\partial}{\partial t} \varphi_{\delta \phi} \right) - c_s^2 \nabla^2 \varphi_{\delta \phi} = 0,$$

where

$$Q_s = \frac{\dot{\phi}^2 + \frac{3\dot{\rho}^2}{2 + \dot{\rho}^2} + Q_e}{H + \frac{\dot{\rho}}{2 + \dot{\rho}^2}},$$

and the squared speed of propagation of the scalar perturbation is given by

$$c_s^2 = 1 + \frac{(2 + Q_b)Q_d + Q_c Q_e + \frac{\dot{\phi}^2}{2 + \dot{\rho}^2}}{(2 + Q_b)(\dot{\phi}^2 + Q_e) + 3Q_e^2},$$

with

$$Q_s = \lambda \dot{\phi}^2 \left( 2c_2H + c_3 \right), \quad Q_b = \lambda c_2 \dot{\phi}^2, \quad Q_c = -3\lambda \dot{\phi}^2 \left( c_2H^2 + 2c_3H + 2c_4 \dot{\phi}^2 \right), \quad Q_d = -2\lambda \dot{\phi}^2 \left[ c_2H + c_3 \left( \dot{\phi} - H \dot{\phi} \right) \right], \quad Q_e = 4\lambda \dot{\phi} \left[ c_2 \left( \ddot{\phi} - H \dot{\phi} \right) - c_3 \dot{\phi}^2 \right], \quad Q_f = 2\lambda \dot{c}_2 \dot{\phi}^2 = 2Q_b.$$
For the linearized tensor-type perturbations we obtain the following second order equation of motion [61]:

\[
\frac{1}{a^3 Q_T} \frac{\partial}{\partial t} \left( a^3 Q_T \frac{\partial}{\partial t} C^i j \right) - c_T^2 \frac{\nabla^2}{a^2} C^i j = \frac{1}{Q_T} \delta T^i j, \tag{30}
\]

where \( \delta T^i j \) includes contributions to the tensor-type energy-momentum tensor,

\[
Q_T = 1 + \frac{\lambda}{2} c_2 \dot{\phi}^2,
\]

and

\[
c_T^2 = \frac{2 - \lambda c_2 \dot{\phi}^2}{2 + \lambda c_2 \dot{\phi}^2}, \tag{31}
\]

is the squared speed of propagation of the gravitational waves perturbation. Notice that for \( c_T^2 > 0 \) and \( c_T^2 > 0 \) the wave equations (27) and (30), respectively, are hyperbolic differential equations—the Cauchy problem is well posed—while for negative \( c_T^2 < 0 \) and \( c_T^2 < 0 \), these equations are elliptic so there is no propagating mode (the Cauchy problem is not well posed). In this latter case a Laplacian instability develops (see the appendix).

In the present cosmological model based on (1) the Lagrangian (25) can be written in the following way:

\[
L_q = \frac{3}{2} \tilde{\phi}^2 H^2,
\]

where we have set \( \xi = 1 \), \( \lambda c_2 = -\alpha \) (the remaining constants in (25) vanish). Hence, we obtain that

\[
Q_a = -2\alpha H \dot{\phi}^2, \quad Q_b = -\alpha \dot{\phi}^2, \quad Q_c = 3\alpha H^2 \dot{\phi}^2, \quad Q_d = 2\alpha H \dot{\phi}^2, \quad Q_e = -4\alpha \dot{\phi} (\dot{\phi} - H \phi). \tag{32}
\]

For the squared speed of propagation of the gravitational waves perturbation (31) it is found that, for the present cosmological model:

\[
c_s^2 = \frac{1 + \alpha \dot{\phi}^2 / 2}{1 - \alpha \dot{\phi}^2 / 2}, \tag{33}
\]

where it is appreciated that, for the positive coupling \( \alpha > 0 \), the tensor perturbations propagate superluminally. A similar result was reported in [29] for the same model but under the slow-roll approximation, i.e. valid for primordial inflation. For negative coupling \( \alpha < 0 \), provided that \( \dot{\phi}^2 > 2/|\alpha| \), the squared sound speed of the tensor perturbations becomes negative, signaling the eventual occurrence of a Laplacian instability. For a detailed derivation of (28) and of (31) within the perturbative approach we recommend the [61].

Equation (28), with the substitution of the quantities (32), will be our master equation for determining the (squared) speed of propagation of the scalar perturbations of the energy density. In terms of the field variables \( x = \alpha \dot{\phi}^2 / 2 \) and \( y = \alpha V(\phi) \) we have that:

\[
c_s^2 = 1 + \frac{4x(e(3 - 11x + 6x^2) + (1 - 3x)y)}{3(1 - x)F_x} - \frac{3(1 - x)(ex + y)(\omega_{\text{eff}} + 1)}{F_x}, \tag{34}
\]

where \( \omega_{\text{eff}} \) is given by (18) and the function \( F_x \) was defined in (16).
4.1. Positive coupling

In this case we have that $0 \leq x \leq 1/3$ and $0 \leq y < \infty$. This means that $F_x$ is always a positive function. In addition, both the numerator and the denominator in the second term in the RHS of equation (34) are positive quantities. The same is true for the factor $(1 - x)(\epsilon x + y)/F_x$ in the third term in the RHS of the mentioned equation. Hence, while the second term always contributes towards superluminality of propagation of the scalar perturbations, the contribution of the third term depends on the sign of $\omega_{\text{eff}} + 1$. For $\omega_{\text{eff}} > -1$ the superluminal contribution of the second term in the RHS of (34) may be compensated by the third term. However, when $\omega_{\text{eff}} < -1$, both terms in the RHS of (34) contribute towards superluminality of the propagation of the scalar perturbations of the energy density. This means that, whenever the crossing of the phantom divide is allowed, then $\omega_{\text{eff}} + 1$ becomes necessarily negative during a given stage of the cosmic evolution and, consequently, causality violations are inevitable. This result is independent of the specific functional form of the self-interaction potential.

In general, from (34), it follows that whenever the condition

$$4x\epsilon(3 - 11x + 6x^2) + (1 - 3x)y_0 > \omega_{\text{eff}} + 1,$$

(35)

is fulfilled, the squared sound speed is superluminal ($c_s^2 > 1$). The latter condition may be satisfied only if $\omega_{\text{eff}} + 1 < 0$, i.e. if $\omega_{\text{eff}} < -1$. For positive $\omega_{\text{eff}} + 1 > 0$, the inequality (35) is never satisfied.

We want to point out that, although the condition $\omega_{\text{eff}} < -1$ boosts further superluminality of the propagation of the scalar perturbations, in general the $\omega = -1$ crossing is not required for the superluminality to arise in the present model. Actually, as seen from (34), given that the second term in the RHS of (34) is always a positive quantity, superluminality arises even if $\omega_{\text{eff}} + 1 = 0$.

The potential situation where $\omega_{\text{eff}} + 1 > 0$, i.e. where $\omega_{\text{eff}} > -1$, leads to another interesting and disturbing possibility, namely that

$$\omega_{\text{eff}} + 1 > \frac{(1 - x)F_x}{3(1 - x)^2(\epsilon x + y)} + \frac{4x\epsilon(3 - 11x + 6x^2) + (1 - 3x)y_0}{9(1 - x)^2(\epsilon x + y)},$$

(36)

that is, that $c_s^2 < 0$. Fulfilment of this latter bound leads to the development of the Laplacian/gradient instability. This is a classical instability associated with the uncontrolled growth of the amplitude of the scalar perturbations of the background density (see the appendix).  

In figure 4 we have geometrically represented the bound $c_s^2 \geq 0$ for the exponential (2) and for the power-law (3) potentials, for different values of the free parameters $\lambda$ and $\eta$, respectively$^{12}$. Meanwhile, in figure 1 we have drawn the surfaces $\omega_{\text{eff}} = \omega_{\text{eff}}(x, u)$ and $c_s^2 = c_s^2(x, u)$ for the growing exponential potential with $\lambda = 5$. In these figures we have used the bounded coordinate in (20):

$$u = \frac{y}{x + 1}, \quad 0 \leq u \leq 1,$$

instead of $y$ ($0 \leq y < \infty$), in order to fit the whole phase plane into a finite-size region. In the right-hand panel of figure 1 different orbits of the dynamical system corresponding to the present cosmological model have been mapped onto the surface $c_s^2 = c_s^2(x, u)$ in order to show, geometrically, that the choice of free parameters that is not compatible with the crossing of

$^{12}$ In this section we focus on the quintessence case $\epsilon = 1$ exclusively.
the phantom divide—in this case the growing exponential (positive slope)—leads eventually to the development of the Laplacian instability.

As already shown, the potentials that allow for the crossing of the phantom divide—potentials with negative slope—can also lead to causality problems. This finding is geometrically illustrated in the figure 5, where the EOS-embedding and $c^2_s$-embedding diagrams are shown for potentials with negative slope: (i) decaying exponential potential (2) with $\lambda = -5$ (top panels) and (ii) inverse power-law potential (3) with $n = -1$ (bottom panels), respectively.

4.2. Negative coupling

For $\alpha < 0$ we have that $-\infty < x \leq 0$, $-\infty < y \leq 0$, so it is recommended that the bounded variables $v$, $w$ in (21) are used. Under the latter choice the whole of the $xy$-plane is comprised within a unit square: $\{(v, w) : 0 \leq v \leq 1, 0 \leq w \leq 1\}$. In this case the analysis of the bounds on the squared sound speed: $0 \leq c^2_s \leq 1$, is a very complicated task and one has to heavily rely on the numeric investigation.

In the figures in figure 6 the red-colored regions in the unit square are the ones where $c^2_s < 0$, i.e. where the Laplacian instability develops. We concentrate on the first and second figures from left to the right—the plots corresponding to the decaying exponential (top) and to the inverse power-law (bottom) potentials, respectively—since only for these choices can the

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**Figure 4.** Geometric representation of the bound $c^2_s \geq 0$ in the $xu$-plane for positive coupling $\alpha > 0$. For illustrative purposes we consider the exponential potential $V = V_0 \exp(\lambda \phi)$—top panels—and the power-law potential $V = V_0 \phi^n$—bottom panels—for different values of the parameters $\lambda$ and $n$ respectively. As in figure 2, here we use the bounded variables $x = \alpha \dot{\phi}^2/2$ ($0 \leq x \leq 1/3$) and $u = y/y + 1$ ($0 \leq u \leq 1$) where $y = \alpha V$, so that the whole phase plane $xu$ fits into a finite size box. In the top panels, from left to the right: $\lambda = -5$, $\lambda = -2$, $\lambda = 2$ and $\lambda = 5$, while in the bottom panels: $n = -2$, $n = -1$, $n = 1$ and $n = 2$, respectively. The red-colored regions are the ones where the squared sound speed is negative ($c^2_s < 0$), i.e. where the Laplacian instability eventually develops. It is seen that, although the bound $c^2_s < 0$ is always met in some—even small—region in the $xu$-plane, for monotonically growing potentials ($\lambda > 0|n > 0$), i.e. for potentials that do not allow the crossing of the phantom divide, the region of the phase plane where the Laplacian instability arises is appreciably larger.
crossing of the phantom divide happen. It is obvious from the $c_s^2$-embedding diagrams in figure 7 that, independent of the choice of self-interaction potential (either the decaying exponential or the inverse power-law) and of the initial conditions, the development of a gradient instability is inevitable since, as the orbits in the unit (phase) square $\{(v, w) : 0 \leq v \leq 1, 0 \leq w \leq 1\}$ approach the global attractor, these necessarily enter the region where $c_s^2 < 0$. In addition, at the global attractor itself the squared sound speed is negative. We shall come back to this issue again in the next section where the basic properties of the corresponding dynamical system are discussed in connection with the bounds on the squared sound speed.

5. Squared sound speed and the dynamical system

Given that, in order to illustrate the main results of the present investigation we rely heavily on the properties of the dynamical system corresponding to the cosmological model of interest, here we give a compact exposition of the most elementary of these properties in connection with the bounds on the squared sound speed.
We want to underline that here we do not care about a detailed study of the critical points of the dynamical system and their stability. A detailed dynamical systems study of the present model can be found in [33]. Different orbits in the given phase space will correspond to possible patterns of cosmological evolution that are sustained by the dynamical system and, consequently, by the cosmological equations (5). Moreover, every possible orbit that can be generated by every possible choice of the initial conditions, represents a potential cosmic history for our universe. The critical points of the dynamical system correspond to ‘outstanding’ or generic cosmological solutions of (5).

5.1. Positive coupling $\alpha > 0$

Let us investigate the asymptotic properties of the dynamical system corresponding to the cosmological equation (5) in the phase plane

$$\psi = \{ (x, y) : 0 \leq x \leq 1/3, y \geq 0 \}.$$ 

It can be demonstrated that the second order cosmological field equations (5) can be traded by the following system of two ordinary differential equations on the variables $x, y$:

$$x' = \frac{x[\epsilon(1 - 2x) + y]}{1 - 3x} - \frac{(1 - x)(\epsilon x + y)(\omega_{\text{eff}} + 1)}{2(1 - 3x)},$$

$$y' = \gamma \sqrt{\frac{2x(\epsilon x + y)}{3(1 - 3x)}},$$

(37)
where the comma means derivative with respect to the time variable $d\tau = \alpha H dt$. The problem with (37) is that the phase plane is unbounded ($0 \leq y < \infty$) so that it may happen that one or several critical points of the dynamical system at infinity are unseen in a finite region of the phase plane. This is why in (20) we introduced the bounded variable $u = y/(y + 1)$ ($0 \leq u \leq 1$).

After this choice the whole phase plane is shrunk into the phase rectangle:

$$\psi_{\alpha > 0} = \{(x, u) : 0 \leq x \leq 1/3, 0 \leq u \leq 1\}, \quad (38)$$

and the ODE system (37) is rewritten as:

$$x' = \frac{x(1 - 2x)(1 - u) + u}{(1 - 3x)(1 - u)} - \frac{x(1 - x)[\epsilon(1 - u) + 3u][\epsilon(1 - 2x)(1 - u) + u]}{(1 - 3x)(1 - u)^2 F_{\epsilon}} - \sqrt{\frac{2x(1 - x)^2(1 - 3x)[\epsilon x(1 - u) + u]}{3(1 - u)}} \frac{u_{\phi}}{(1 - u)^2 F_{\epsilon}},$$

$$u' = u_{\phi} \sqrt{\frac{2x[\epsilon x(1 - u) + u]}{3(1 - 3x)(1 - u)}}, \quad (39)$$

Figure 7. Phase portraits (left) of the dynamical system (40), EOS-embedding diagrams (middle) and $c^2 s$-embedding diagrams (right) corresponding to the cosmological model (1) with the negative coupling ($\alpha < 0$). In the top panels the decaying exponential potential (2) ($\lambda = -5$) has been chosen, while in the bottom panels the inverse power-law potential (3) ($n = -1$) is considered. As in figure 5, in the EOS-embedding diagrams the thick contours are drawn for $\omega_{\text{eff}} = -1/3$ (upper contour) and for $\omega_{\text{eff}} = -1$ (lower contour), while in the $c^2 s$-embedding diagrams the drawn thick contours are for $c^2_s = 1$ (upper contour) and for $c^2_s = 0$ (lower contour). It is evident from the $c^2 s$-embedding diagrams that as the orbits approach the global attractor $P_A : (1, 0)$, these enter a domain on the surface $c^2_s = c^2_s(v, w)$ where the squared sound speed becomes negative, signaling the eventual development of a Laplacian instability.
where

\[ F_c = \epsilon \frac{(1 - 3x + 6x^2)(1 - u) + (1 + 3x)u}{1 - u}. \]

In the left-hand figures of figure 5 the phase portraits of the dynamical system (39) are drawn for the decaying exponential with \( \lambda = -5 \) (top) and for the inverse power-law with \( n = -1 \) (bottom), for a set of nine and eight different initial, conditions respectively.

A crude inspection of (39) reveals that, independent of the specific functional form of the self-interaction potential, among the equilibrium configurations of the dynamical system in the phase rectangle (38), there is a critical manifold:

\[ M_0 = \{ (0, u) : 0 \leq u \leq 1 \}. \]

Equilibrium points in this manifold have different stability properties. The origin \( P_0 : (0, 0) \) is a stable critical point. Moreover, it is the global future attractor. The remaining points \( P_1 \in M_0 \) represent unstable equilibrium configurations and can be only local sources. In the phase portraits (left-hand figures) in figure 5 the red-color orbits start at local sources in \( M_0 \) and end up at the global attractor \( P_0 \). For points \( P_\delta : (\delta, u) \in \) the neighborhood of \( M_0 \), where \( \delta \ll 1 \) is a small parameter, we have that:

\[ c_s^2 \approx 1 - 6\sqrt{2} \delta^{1/2} \sqrt{\frac{u}{1 - u} u_\delta}, \]

where the terms \( \propto \delta \) and of higher orders in the small parameter have been omitted. Hence, if we assume that \( u \neq 0 \), i.e. if exclude the global attractor at the origin, assuming potentials with the negative slope:

\[ V_\phi < 0 \Rightarrow y_\phi < 0 \Rightarrow u_\phi < 0, \]

for points in the neighborhood of the critical manifold \( M_0 \), the speed of sound becomes superluminal \( c_s^2 > 1 \). This behavior is illustrated in the \( c_s^2 \)-embedding diagrams in figure 5, where it is appreciated that as the red-colored orbits leave the source points the speed of sound becomes superluminal\(^{13} \). For orbits that start at points to the right of the phase rectangle \( (x = 1/3 - \delta) \), it is found that there are regions in the phase plane where the squared sound speed becomes negative, signaling the development of Laplacian instability. This is illustrated in the first and second figures (from left to right) in figure 4 where the small red-colored regions in the \( xu \)-plane represent the domains in the phase rectangle where \( c_s^2 < 0 \). In the \( c_s^2 \)-embedding diagrams in figure 5 it is appreciated that several of the mentioned orbits (continuous black curves) indeed meet the gradient instability regions.

5.2. Negative coupling \( \alpha < 0 \)

In terms of the variables \( v, w \) in (21) the autonomous system of ODE (37) can be written in the following way:

\[
\begin{align*}
    v' &= \left(1 - \frac{v}{1 - w}\right) \left(1 + v - 2w\right) - \frac{(1 - v)(v + w - 2vw)(\omega_{\text{eff}} + 1)}{2(1 + 2v)(1 - w)}, \\
    w' &= w_\phi \sqrt{\frac{2v(v + w - 2vw)}{3(1 - v)(1 - w)(1 + 2v)}},
\end{align*}
\]

(40)

where \( \omega_{\text{eff}} \) is given by (22). The phase portraits of the dynamical system (40) are shown in the left-hand figures of figure 7 for the decaying exponential with \( \lambda = -5 \) (top) and for the

\(^{13}\) At the source points, as well as at the global attractor at the origin, where \( \delta = 0 \), we have that \( c_s^2 = 1 \).
inverse power-law potential with \( n = -1 \) (bottom). The global (future) attractor at \( P_A : (1, 0) \) is sharply appreciated. If we make the replacement \( x \to v/v - 1 \) and \( y \to w/w - 1 \) in (34), and then we evaluate the squared sound speed at the attractor, we get:

\[
\lim_{(v,w) \to (1,0)} c_I^2(v, w) = -\frac{1}{3}.
\]

This means that, at least at the attractor, \( c_I^2 < 0 \), so that a Laplacian instability eventually develops. In the \( c_I^2 \)-embedding diagrams in figure 7 it is seen that, to a large extent, the embedded phase space orbits lie in domains on the surface \( c_I^2 = c_I^2(v, w) \) that are below the contour corresponding to \( c_I^2 = 0 \). Moreover, there are orbits that entirely lie in domains below the mentioned contour in the extended phase space, which means that the corresponding whole cosmic histories are classically unstable under scalar perturbations of the cosmic background.

6. Pure derivative coupling (\( \epsilon = 0 \))

The interest in the case where the kinetic coupling is exclusive to the Einstein tensor, i.e. where the term \( g_{\mu\nu} \partial^\mu \phi \partial^\nu \phi \) is removed from (1), is due to the significant simplification of the equations of the resulting cosmological model that allows one to discuss in a fully analytical way the phantom crossing and the bounds on the squared sound speed.

Actually, in this particular case where \( \epsilon = 0 \), the expressions for the effective EOS parameter and for the squared sound speed become

\[
\omega_{\text{eff}} + 1 = \frac{6x}{1 + 3x} + \frac{\sqrt{8x(1 - 3x)^3}}{3(1 + 3x)^{3/2}} y_\phi, \tag{41}
\]

and

\[
c_I^2 = 1 - \frac{2x(45x^2 - 54x + 29)}{3(1 - x)(1 + 3x)^2} - \frac{6(1 - x)}{(1 + 3x)^2} \sqrt{\frac{2x(1 - 3x)^3}{3y^3}} y_\phi, \tag{42}
\]

respectively. The analysis of the behavior of the above quantities is straightforward.

6.1. Positive coupling

From equation (41) it is seen that at the upper boundary, \( x = 1/3 \), the effective (background) fluid behaves like pressureless dust. It is seen also that, provided the slope of the potential is negative, \( y_\phi < 0 \), the second term on the RHS of (41) may compensate for the contribution of the first one. For the exponential potential \( y_\phi = \lambda y \), for instance, for

\[
y < \frac{2\lambda^2 (1 - 3x)^3}{27x},
\]

the crossing of the phantom divide may happen since \( \omega_{\text{eff}} + 1 \) becomes negative. For monotonically growing potentials the crossing is not possible.

Causality violations and the development of Laplacian instability in this case are apparent. Even for the constant potential \( y_\phi = 0 \) (this includes the vanishing potential case \( V = 0 \)) the instability issue is apparent. Actually, in this case (42) simplifies even more:

\[\text{Recall that we are interested in potentials that allow for the crossing of the phantom divide exclusively.}\]
\[ c_s^2 = 1 - \frac{2x(45x^2 - 54x + 29)}{3(1 - x)(1 + 3x)^2}. \] (43)

It is straightforward to show that the squared sound speed above is a monotone decreasing function of \( x \), and that it vanishes at \( x = 0.0897 \). In the interval \( 0.0897 < x \leq 1/3 \), \( c_s^2 \) is negative. In particular at \( x = 1/3 \) the squared sound speed \( c_s^2 = -1/3 \). The violation of causality in connection with superluminal propagation of the scalar perturbations may happen only for potentials with the negative slope \( y_\phi < 0 \). Only in this case the third term on the RHS of (42) may compensate for the contribution from the second one, and may contribute towards superluminality \( (c_s^2 > 1) \).

6.2. Negative coupling

In this case, \( -\infty < x \leq 0, -\infty < y \leq 0 \), so that both variables are unbounded. In terms of the bounded variables \( v, w \) in (21), for the simplest case when the potential is a constant \( (y_\phi = 0) \), the squared sound speed (43) can be written as:

\[ c_s^2 = 1 + \frac{2v(29 + 54v - 9v^2)}{3(1 - 4v)^2}, \] (44)

while the corresponding autonomous ODE is

\[ v' = -\frac{2y_0 v(1 - v)^2}{1 - 4v}, \] (45)

where \( y_0 = \alpha V_0 \) is a constant. The squared sound speed blows up at the asymptote \( v \to 1/4 \), so that a coarse violation of causality eventually occurs. In the phase line \( 0 \leq v \leq 1 \) the asymptote \( v = 1/4 \) represents a separatrix, since the orbits of (45) cannot cross from the left to the right of \( v = 1/4 \) and vice versa.

7. Discussion

Our results in the previous sections are clear and convincing. These show that, in general terms, without specifying the functional form of the self-interacting potential, the cosmological models based on the theory (1)—where the scalar field is kinetically coupled to the curvature—are unsatisfactory due to the occurrence of causality violations and, what is more problematic, of classical Laplacian instabilities, for a non-empty set of initial conditions. These results do not depend on the sign of the coupling constant \( \alpha \) in (1). We have shown this analytically and also numerically by specifying the form of the potential; we have done this for the exponential and for the power-law potentials. There is, however, a particular class of such models without the potential \( (V = 0) \) and with the constant potential \( (V = V_0) \) that deserves separate comment since these can be treated in a fully analytical way (see below).

In general terms theories with the kinetic coupling of the scalar field to the Einstein tensor—this is true also for more general Horndeski theories—all possess some configurations with superluminal propagation. In addition, these theories also have the speed of propagation of the gravity waves different from the speed of light. In particular, the speed of sound for the scalar perturbations can be subluminal while, simultaneously, the speed of propagation for the gravity waves can be superluminal [29]. In the later reference this has been shown for the theory (1) with positive coupling, for the quartic potential during inflation. In (31) the
squared speed of propagation of the gravity wave perturbations is given independent of the self-interaction potential:

\[ c_T^2 = \frac{1 + x}{1 - x}. \]  

(46)

This confirms that the speed of the gravitational waves is always superluminal if positive coupling \( \alpha > 0 \) is assumed. For negative coupling, in terms of the bounded variable \( v \ (0 \leq v \leq 1) \) we have that:

\[ c_T^2 = 1 - 2v. \]  

(47)

This means that for \( 0 \leq v \leq 1/2 \) the speed of propagation of the gravitational waves meets the bounds \( 0 \leq c_T^2 \leq 1 \); meanwhile, for \( v > 1/2 \), the squared sound speed of the tensor perturbations is a negative quantity that leads eventually to the development of a Laplacian instability.

In order to further illustrate our results, let us discuss in detail the constant potential case:

\[ V = V_0 \Rightarrow y = y_0 = \alpha V_0, \]

with the vanishing potential as the particular case when \( y_0 = 0 \), which can be studied analytically. We have that (for definiteness we consider \( \epsilon = 1 \)):

\[ 3\alpha H^2 = \frac{x + y_0}{1 - 3x}. \]  

(48)

Since for positive coupling \( 0 \leq x \leq 1/3 \), from (48) it follows that for \( \alpha > 0 \) the Hubble rate is unbounded from above and bounded from below: \( \sqrt{y_0/3\alpha} \leq H < \infty \).

For negative coupling \( \alpha < 0 \) \((-\infty < x < 0)\) the Hubble rate is bounded (in this case the constant \( y_0 \) should be a negative quantity as well):

\[ \frac{1}{3\sqrt{-\alpha}} \leq H \leq \sqrt{\frac{y_0}{3\alpha}} = \sqrt{\frac{V_0}{3}} \ (V_0 > 1/|3\alpha|), \]

\[ \sqrt{V_0/3} \leq H \leq 1/3\sqrt{-\alpha}, \ (V_0 < 1/|3\alpha|). \]  

(49)

For the constant potential the dynamical system (37) reduces to an ODE:

\[ x' = -\frac{2x(1 - 2x + y_0)}{1 - 3x} \left[ \frac{y_0 + (1 - 3y_0)x - 3x^2}{1 + y_0 + 3(y_0 - 1)x + 6x^2} \right]. \]  

(50)

For the positive coupling \( 0 \leq x < 1/3 \), one of the critical points of the ODE (50) is at the origin \( x = 0 \). This is a stable equilibrium point since linear perturbations \( \delta \) around it \( (x \to + \delta) \) exponentially decay with the time \( \tau = \alpha \ln a: \delta(\tau) \propto \exp(-2y_0\tau) \), or in terms of the scale factor of the Universe:

\[ \delta(a) \propto a^{-2y_0}, \]

the perturbations decay as an inverse power-law. The above means that the cosmic dynamics ends up at the de Sitter attractor \( x = 0 \), where \( H = H_0 = \sqrt{V_0/3} \). Consistent with the fact that, for the positive coupling, the late time dynamics is not modified by the kinetic coupling [23], the above is the standard late time behavior expected in any scalar field model with a constant potential. For the vanishing potential the asymptotic late time dynamics corresponds to the empty static universe \( H = 0 \), since for this particular case the origin \( (x = 0) \) is the attractor equilibrium configuration as well. The small linear perturbations around the origin decay like
This model is plagued by the Laplacian instability, as can be seen from the top figure in figure 8, where the squared sound speed is plotted against $x$. For negative coupling ($-\infty < x \leq 0$), it is better to use the bounded variable $v = x/(x - 1)$ ($0 \leq v < 1$). In this case the autonomous ODE (50) transforms into:

$$v' = -\frac{2\alpha(1-v)[1 + y_0 + (1-y_0)v]}{1 + 2v} \left[ \frac{y_0 - (1-y_0)v - 2(1+y_0)v^2}{1 + y_0 + (1-5y_0)v + 4(1+y_0)v^2} \right].$$

(51)

Two of the critical points of the ODE (51) are at the origin ($v = 0 \leftrightarrow x = 0$), and at $v = 1$ ($x \to \infty$). The dynamical equations for linear perturbations $\delta$ around these points read: $\delta' = -2y_0\delta$ and $\delta' = -\delta/3$, respectively. After integration, for perturbations around the origin $v = 0$, we get that $\delta(a) \propto a^{-2\alpha}$, i.e. given that both $\alpha$ and $y_0$ are negative for this case, then the perturbations decay with the cosmic expansion. Meanwhile, for perturbations around $v = 1$ we get that $\delta(a) \propto a^{-\alpha/3}$ and, since $\alpha$ is negative, then the corresponding perturbation grows with the expansion of the Universe. Hence the point $v = 1$ is unstable while the origin $v = 0$ is the attractor. Since in this case:

$$3\alpha H^2 = \frac{y_0 - (1+y_0)v}{1 + 2v},$$

Figure 8. Plot of $c_s^2$ versus $x$ (top) and of $c_s^2$ versus $v$ (bottom) for the model (1) with the constant potential ($y_0 = \alpha V_0$). The top figure is for the positive coupling case $\alpha > 0$ ($0 \leq x \leq 1/3$), while the bottom figure is for the negative coupling case $\alpha < 0$ ($0 \leq v < 1$). In the top we have arbitrarily set $y_0 = 10$, while in the bottom $y_0 = -0.01$. The dash-dot horizontal line marks the lower bound $c_s^2 = 0$ on the squared speed of sound. It is appreciated that, independent of the sign of the coupling, there always exist an interval in the $\psi$-$\nu$-coordinate where $c_s^2 < 0$, meaning that a Laplacian instability may eventually arise.

$$\delta(\tau) \propto \tau^{-1} \Rightarrow \delta(a) \propto \frac{1}{\alpha \ln a}.$$
in models with the constant potential (for the negative coupling) the Universe starts at the unstable de Sitter solution with $H = 1/3\sqrt{-\alpha}$ and ends up its history at the late-time de Sitter solution with

$$3\alpha H^2 = y_0 \Rightarrow H = H_0 = \sqrt{V_0/3}.$$ 

The asymptotic de Sitter state at $v = 1$, $H = 1/3\sqrt{-\alpha}$, is to be associated with primordial inflation [23] and the fact that it is an unstable equilibrium state warrants the natural (required) exit from the early times inflationary stage. Notice that for the above picture to make physical sense, in (49) we have to choose the bottom-line bound, i.e. $V_0 < 1/|3\alpha|$. Otherwise the attractor would be at higher curvature than the starting point of the cosmic expansion, which is a nonsense from the point of view of the inflationary history of our Universe.

In spite of the claims that this picture represents an appropriate description of primordial inflation, according to (47), in the neighborhood of the inflationary equilibrium point, $v = 1 \pm \delta$ ($\delta \ll 1$); for the squared speed of propagation of tensor perturbations we have that $c_T^2 \approx -1 \pm 2\delta$, so that the development of a Laplacian instability forbids the—otherwise unphysical—inflationary stage in the model.

The estimated value of the coupling constant in [63] is about:

$$|\alpha| \sim 10^{-74} s^2 \approx 10^{-24} \text{GeV}^{-2}, \quad (52)$$

where the authors chose the time at which inflation is assumed to start $t \approx 10^{-36}$ s. We may as well choose the time at which inflation is assumed to have ended: $t \approx 10^{-33}$ s. The estimated value for the coupling in this case is about four orders of magnitude larger:

$$|\alpha| \sim 10^{-70} s^2 \approx 10^{-20} \text{GeV}^{-2}. \quad (53)$$

If we combine the above estimates with the tight constraint on the difference in speed of photons and gravitons, $|c_s^2 - 1| \lesssim 10^{-15}$ (in this paper we have chosen the units where $c_s^2 = 1$), implied by the announced detection of gravitational waves from the neutron star–neutron star merger GW170817 and the simultaneous measurement of the gamma-ray burst GRB170817A [56], since according to (46):

$$c_T^2 - 1 = \frac{2x}{1-x} \Rightarrow 2x \lesssim 10^{-15},$$

we get that $\dot{\phi}^2 \lesssim 10^5 - 10^6 \text{GeV}^2$, i.e. $\dot{\phi}^2 \lesssim 10^{-33} - 10^{-29} M_{\text{Pl}}$, where $M_{\text{Pl}} \approx 10^{19} \text{GeV}$ is the Planck mass. These estimates leave not much freedom for the scalar field to behave different from an effective cosmological constant.

The above exposed—quite simple—picture is overshadowed by the stability problems associated with the scalar and tensor modes of the perturbations whose energy density grows without bound due to fact that, for these modes, it may happen that $c_s^2 < 0$ ($c_T^2 < 0$). In the bottom figure in figure 8 the plot of $c_T^2$ versus $v$ is drawn for $y_0 = -0.01$. The conditions for the development of the Laplacian instability ($c_T^2 < 0$) are evident in the figure, in particular for points in the neighborhood of (including) the source equilibrium configuration that can be associated with the primordial inflation. In addition, in the neighborhood of this point we have also that $c_T^2 < 0$, so that the tensor modes are classically unstable as well.

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15 Transient quasi-de Sitter phases of the cosmic evolution can be found also for other potentials than the constant one.
8. Conclusion

In this paper we have investigated several problems: (i) phantom barrier crossing, (ii) causality and (iii) classical Laplacian instability, and their potential interconnection in the model (1) where the scalar field has non-minimal derivative (kinetic) coupling to the Einstein tensor. As far as we know, this is the first time that the present model has been checked in all details against the physical bounds on the squared sound speed (see footnote 5 in the introductory section of this paper). We have also developed an illustrative procedure that allows us to show geometrically the evolution of given physical parameters (the effective EOS and the squared sound speed in the present work) along given phase space orbits. The resulting procedure—called here the ‘embedding diagram’—geometrically illustrates the way these parameters of physical interest evolve along potential cosmic histories. The power of the procedure relies, precisely, on the fact that each phase space orbit entails a potential cosmic history that is sustained by the dynamical system corresponding to the cosmological field equations of the model (5).

We have shown, both analytically and numerically, that violations of causality and—what is more disturbing—the occurrence of Laplacian instability during the propagation of the scalar and of the tensor perturbations, are distinctive features of the cosmological models based on the action (1), no matter what the sign of the coupling constant is. Moreover, even if the scalar perturbations can propagate subluminally, during inflation the gravitational waves travel with superluminal velocity (this is true for the positive coupling exclusively), as shown in [29] for the model (1) with the quartic potential $V \propto \phi^4$. In the general case—see (31), (46) or (47)—the situation cannot be more hopeless: independent of the self-interaction potential, for positive coupling the tensor perturbations propagate superluminally, while for negative coupling a Laplacian instability arises. This latter instability invalidates the possibility for the model to describe the primordial inflation.

It has been shown also that, in the positive coupling case ($0 \leq x \leq 1/3$), a sufficient (but not necessary) condition for superluminality to happen is that $\omega_{\text{eff}} + 1 < 0$, since in this case the third term in the RHS of (34) also adds to the unity. Since the crossing of the phantom barrier warrants that for some $x$-interval $\omega_{\text{eff}} + 1 < 0$, then it also warrants that superluminality will happen. However, as mentioned before, it is not necessary that the crossing occurs in order to have superluminal propagation of the perturbations of the background. One trivial example can be the situation when $\omega_{\text{eff}} + 1 < 0$ for all times. In this case violations of causality arise even when the crossing does not occur.

For the quintessence model with the kinetic coupling to the Einstein tensor, in the particular case when the potential is a constant $V = V_0$, eventual violations of the physical bounds of the squared sound speed are evident as well: no matter whether the coupling is positive or negative, the asymptotic dynamics at early times develops a (classical) Laplacian instability that makes impossible the formation of a cosmic structure. This makes it very improbable that the primordial inflationary stage can be described by this cosmological model as suggested, for instance, in [23, 25].

Although we lack a demonstration, we suspect that the violation of the bounds $0 \leq c_s^2 \leq 1$ on the squared sound speed are a feature of galileon models in general. In particular the cubic galileon model of [19, 64] seems to suffer from the same problems. A demonstration of the latter assumption will be the subject of forthcoming work.

16 For the tensor perturbations the violation of causality may happen only for the positive coupling case, while the Laplacian instability develops only for the negative coupling.
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Appendix. Classical instability due to imaginary sound speed

Even if the theory (1) is free of the Ostrogradsky instability (the equations of motion are second order in the derivatives), it may contain other kinds of instability since it is based on a non-standard Lagrangian. Here we shall discuss one such kind of instability that may arise in the theory with non-minimal derivative coupling with the Einstein tensor due to ‘imaginary’ sound speed.

Let $\rho_B$ and $p_B$ be the energy density and barotropic pressure of the FRW cosmological background. If we consider small perturbations of the background energy density, $\rho_B(t) + \delta \rho_B(x, t)$, the conservation of energy and stresses $\nabla^\mu T_{\mu\nu} = 0$ leads to the wave equation [65]:

$$\left(-\frac{\partial^2}{\partial t^2} + c_s^2 \nabla^2\right) \delta \rho_B = 0,$$

where $\nabla^2 = \partial^2/\partial x^2$ and $c_s^2 = dp_B/d\rho_B$ is the speed of sound squared. The solution of the wave equation (A.1) is given by $\delta \rho_B = \delta \rho_B^0 \exp(-i\omega t + ikx)$, so that the standard dispersion relation is found:

$$\omega^2 - c_s^2 k^2 = 0.$$  

(A.2)

For positive $c_s^2 > 0$ the solution is a free wave propagating with speed $c_s$, while for negative $c_s^2 < 0 \Rightarrow c_s = i|c_s|$, the frequency $\omega = \pm kc_s = \pm ik|c_s|$ is imaginary, so that the solution of (A.1) is not a propagating free wave but an exponentially growing spatial perturbation:

$$\delta \rho_B = \delta \rho_B^+ e^{2\pi|c_s|/\lambda} \exp(i kx) + \delta \rho_B^- e^{-2\pi|c_s|/\lambda} \exp(i kx),$$

(A.3)

where we have taken into account that $k = 2\pi/\lambda$ is the wave number of the perturbation ($a/k$ is the physical wavelength of the perturbation). Since the negative frequency part of the perturbation decreases with time, eventually the energy density of the perturbations uncontrollably grows, resulting in a classical instability of the cosmological model. As seen, the increment of instability is inversely proportional to the wavelength of the perturbations and the models where $c_s^2 < 0$ are violently unstable, so these should be rejected [66].

The situation is a bit more complex for a scalar field [45, 67] (see also [68]), which is the case considered in this paper. As an illustration, let us consider a general action of the form:

$$S = \frac{1}{2} \int d^4x \sqrt{|g|} R + \int d^4x \sqrt{|g|} p_\phi(X, \phi),$$

(A.4)

where $X \equiv (\partial \phi)^2/2$, $p_\phi = L_\phi$ is the parametric pressure of the scalar field and $p_\phi = 2X L_\phi X - L_\phi$ is its energy density, with $Z_X$ denoting the partial derivative with respect to $X$. Varying the scalar field Lagrangian $L_\phi$ with respect to the metric one gets the stress–energy tensor for the scalar field:

$$T^{(\phi)}_{\mu\nu} = (\rho_\phi + p_\phi) u_\mu u_\nu + p_\phi g_{\mu\nu},$$
where $u_{\mu} = \partial_{\mu} \phi / \sqrt{2X}$. As stated in [67], the Lagrangian $L_\phi$ can be used to draw a useful analogy with hydrodynamics. Indeed, if $p_\phi$ depends only on $X$, then $\rho_\phi = \rho_\phi(X)$. In many cases the equation $\rho_\phi = 2Xp_\phi X - p$ can be solved giving the equation of state $p_\phi = p_\phi(\rho_\phi)$ for an ‘isentropic’ fluid. In the general case, when $p_\phi = p_\phi(X, \phi)$, the pressure cannot be expressed only in terms of $\rho_\phi$. However, even in this case the hydrodynamical analogy is still useful.

If we consider small perturbations of the scalar field: $\phi(t, x) = \phi_0(t) + \delta \phi(t, x)$, and recalling that $\delta T^i_k \propto \delta_i^k$, one can write the perturbed FRW metric in the longitudinal gauge:

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)a^2(t)g_{ik}dx^i dx^k,$$

where $\Phi$ is the Newtonian gravitational potential. It is demonstrated in [67] that the wave equation for the fluctuations of the scalar field in a spatially flat FRW background can be written as:

$$v'' - c_s^2 \nabla^2 v - \frac{z''}{z} v = 0,$$

where

$$v = z \left( \Phi + H \frac{\delta \phi}{\phi} \right)$$

is the canonical quantization variable and

$$z \equiv \frac{a \sqrt{\rho_\phi + p_\phi}}{c_s H}.$$  \hfill (A.6)

In addition, in (A.5) the comma denotes a derivative with respect to the variable $\tau = \int dt / a$, while the quantity $c_s^2 = \frac{p_\phi X}{\rho_\phi X}$ plays the role of the effective speed of sound (squared) for the perturbations of the scalar field.

For negative $c_s^2 < 0$ the above equation (A.5) ceases to be a wave equation since it turns from hyperbolic ($c_s^2 > 0$) into elliptic. The imaginary effective sound speed ($c_s^2 < 0$) of the fluctuations of the scalar field is associated with the so-called gradient instability. Notice that if we set $v \propto v_k(\tau) \exp(i k x) (\nabla^2 v = -k^2 v)$, the wave equation (A.5) can be written as

$$v''_k + \left( c_s^2 k^2 - \frac{z''}{z} \right) v_k = 0.$$ \hfill (A.7)

During slow-roll inflation the Hubble rate $H$, $c_s$ and $\rho_\phi + p_\phi$ change much more slowly than the scale factor $a$, so that, under the reasonable assumption that $(\rho_\phi + p_\phi) / \rho_\phi \ll 1$, from (A.6) it follows that

$$\frac{z''}{z} \approx \frac{a''}{a} \approx 2(aH)^2.$$  

For a given wave number $k$ the term $z'' / z$ in (A.7) can be neglected at early times when the physical wavelength of the perturbations $a k$ is much smaller than the sound horizon $c_s / H$. Hence, $c_s k \gg aH$ and (A.7) can be written as: $v''_k + c_s^2 k^2 v_k = 0$, which is similar to (A.2) if set $v_k(\tau) \propto \exp(-i \omega \tau)$. 

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