Quasilinear evolution of non-thermal distributions in ion cyclotron resonance heating of tokamak plasmas

E F Jaeger\(^1\), R W Harvey\(^2\), V E Lynch\(^1\), N Ershov\(^2\), L A Berry\(^1\), P T Bonoli\(^3\), V Tang\(^3\), R D Moore\(^1\) and the members of the SciDAC Center for Simulation of Wave-Plasma Interactions (CSWPI)

\(^1\)Oak Ridge National Laboratory, P.O. Box 2008, Oak Ridge, TN 37831-6169, USA
\(^2\)CompX, P.O. Box 2672, Del Mar, CA 92014-5672, USA
\(^3\)Plasma Fusion Center, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

E-mail: jaegere@ornl.gov

Abstract. The AORSA global-wave solver is combined with the CQL3D bounce-averaged Fokker-Planck code to simulate the quasilinear evolution of non-thermal distributions in ion cyclotron resonance heating of tokamak plasmas. A novel re-formulation of the quasilinear operator enables calculation of the velocity space diffusion coefficients directly from the global wave fields. To obtain self-consistency between the wave fields and particle distribution function, AORSA and CQL3D have been iteratively coupled using Python. The combined self-consistent model is applied to minority ion heating in the Alcator C-Mod tokamak. Results show the formation of a 70 keV ion tail near the minority ion cyclotron resonance layer in approximate agreement with measurements from charge exchange neutral particle analyzers.

1. Introduction

A self-consistent simulation of plasma heating in the ion cyclotron range of frequencies (ICRF) requires a description of two different aspects of the wave-plasma interaction: (1) wave propagation and absorption in the plasma, and (2) the quasilinear response of the plasma to the wave heating. This is a nonlinear problem in which the energetic ions generated by the waves can significantly alter the wave propagation and absorption in the plasma. In addition, the energetic ions can absorb power at high harmonics of the ion cyclotron frequency where finite Larmor radius expansions are not valid. Thus, we require an electromagnetic field solver that is valid to all orders in the ion Larmor radius, as well as for arbitrary non-Maxwellian distribution functions. In this work, the all-orders global-wave solver AORSA [1] is combined with the CQL3D bounce-averaged Fokker-Planck code [2] to simulate the quasilinear evolution of non-thermal ion distributions in ICRF heating. A novel re-formulation of the quasilinear operator [3] enables calculation of the velocity space diffusion coefficients directly from the global wave fields. To obtain self-consistency between the wave fields and particle distribution function, AORSA and CQL3D are iteratively coupled (using Python) in a stand-alone system in which both codes communicate and interact automatically on the same computing platform (the Cray XT-3 “Jaguar” at ORNL). The combined self-consistent model has been applied to minority ion heating in the Alcator C-Mod tokamak where numerical results show the formation of a 70 keV ion tail near the resonance layer in approximate agreement with measurements from charge exchange neutral particle analyzers [4]. These calculations would not be possible without the massively parallel architectures required for the global-wave solver.

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Time-dependent processes in fusion plasmas are governed by the Maxwell-Boltzmann system of equations. The plasma state is described by a distribution function \( f_s(r,v,t) \) representing the density of species \( s \) in a six dimensional phase space of position and velocity. This function evolves in time according to the Boltzmann equation by convection in the 6-D phase space while under forces exerted by the electric and magnetic fields, \( E \) and \( B \), respectively. For ICRF applications, the wave time scale is by far the fastest time scale in the system. Thus, the fields and distribution function can be separated into a time-average, or slowly varying part, \( \langle E_0, B_0, f_0^0 \rangle \), and a rapidly oscillating, or rapidly varying part, \( \{E(r)\exp(-\omega_0 t), B(r)\exp(-\omega_0 t), f_{s}^{1}(r,v)\exp(-\omega_0 t)\} \) where \( \omega_0 \) is the frequency of the wave. The time-harmonic wave fields are small compared to the equilibrium fields, and Boltzmann’s equation can be linearized with respect to these amplitudes. Solving the linearized Boltzmann equation gives the rapidly varying part of the distribution function \( f_{s}^{1} \) in terms of the equilibrium part \( f_{s}^{0} \).

For the rapidly oscillating, time harmonic wave fields, Maxwell’s equations reduce to a generalization of the Helmholtz equation,

\[
-\nabla \times \nabla \times E + \frac{\omega^2}{c^2} \left( E + \frac{i}{\omega e_0} J_p \right) = -i \epsilon \omega \mu_0 J_{\text{ant}},
\]

where \( J_{\text{ant}} \) is an externally driven antenna current that acts as a source for the waves, and the fluctuating plasma current \( J_p \) can be derived directly from the rapidly varying part of the distribution function \( f_{s}^{1}(r,v) \). In general, \( J_p \) is a non-local, integral operator on the wave electric field,

\[
J_p(r,v,t) = \sum_{i,j} \int dr' \int dt' \sigma(f_{s}^{0}(E),r',t',t) \cdot E(r',t').
\]

where \( \sigma(f_{s}^{0}, r', t', t') \) is the “plasma conductivity kernel.”

The numerical solution of equations (1-2) is a very intensive task because of the non-local nature of the plasma current and the enormous range of spatial scales that must be treated. New wave solvers called “all-orders spectral algorithms” (AORSA) [1] have been developed that take advantage of computational techniques for today’s parallel computers. These solvers include the general integral form of equations (1-2), with no restriction on wavelength relative to orbit size and no limit on the number of cyclotron harmonics.

The long time response of the plasma distribution function \( f_{s}^{0}(r,v,t) \) is obtained from a time-averaged form of the Boltzmann equation, or bounce-averaged Fokker-Planck equation,

\[
\frac{\partial}{\partial t} \langle f_{s}^{0} \rangle = \nabla_u \cdot \Gamma_u + \langle \langle R \rangle \rangle + \langle \langle S \rangle \rangle
\]

where \( f_{s}^{0} \) is the equilibrium part of the bounce-averaged distribution function evaluated at the outer equatorial plane, and expressed as a function of one spatial variable that labels a flux surface (\( p \)) and two velocity space variables that are constants of the motion: particle speed (\( u \)) and midplane pitch angle (\( \theta \)). In equation (3), \( \langle \langle R \rangle \rangle \) is a bounce-averaged radial diffusion operator that is set to zero for the calculations in this paper, and \( \langle \langle S \rangle \rangle \) is a bounce-averaged particle source/sink operator, for example, from neutral deposition or magnetic ripple loss [2]. The coefficient \( \lambda \) is defined by \( \lambda = |u| v_0 \), where \( v_0 \) is the bounce time. The divergence term in equation (3) includes two parts: \( \nabla_u \cdot \Gamma_u = C(f_{s}^{0}) + Q(f_{s}^{0},E) \) where \( C(f_{s}^{0}) \) is the collision operator, and \( Q(f_{s}^{0},E) \) is the quasi-linear operator [5] describing diffusion of \( f_{s}^{0} \) in velocity space,

\[
Q(f_{s}^{0},E) = C_0 \frac{\partial}{\partial u_0} \left( B_0 \frac{\partial f_{s}^{0}}{\partial u_0} + \frac{1}{u_0^2} \sin \vartheta \frac{\partial}{\partial \vartheta_0} \left( E_0 \frac{\partial f_{s}^{0}}{\partial u_0} + F_0 \frac{\partial f_{s}^{0}}{\partial \vartheta_0} \right) \right).
\]

In (4), \( B_0, C_0, E_0, \) and \( F_0 \) are the bounce-averaged quasi-linear diffusion coefficients. The subscript “0” is used for bounce-averaged quantities and indicates a reference location that is typically at the outer midplane. Equations (3-4) are solved by CQL3D [2], a 3-D, bounce-averaged Fokker-Planck solver in which particle orbits are tied to a flux surface (zero orbit width approximation).
2. Power absorption and the quasilinear diffusion coefficients

For an arbitrary non-Maxwellian plasma component, the normalized plasma conductivity (susceptibility) is derived by Stix [6], and the local power absorption can be expressed as a double sum over Fourier wave numbers \( \mathbf{k}_1 \) and \( \mathbf{k}_2 \) [7]

\[
P_{RF} = \frac{1}{2} \text{Re} \left\{ \frac{\varepsilon_0 \omega^2}{i} \sum_{k_1, k_2} e^{i(k_1 - k_2) \cdot r} \mathbf{E}_{k_2}^* \cdot \mathbf{W}_r \cdot \mathbf{E}_{k_1} \right\} 
\]

where \( \mathbf{W}_r \) is the energy absorption kernel,

\[
\mathbf{W}_r = 2\pi \frac{\omega_p^2}{\omega} \sum_{l=\infty}^{\infty} e^{i(\beta_2 - \beta_1)} \mathbf{C}^{-1}(\beta_2) \left[ \int_{-\infty}^{\infty} \frac{du_u}{1 - \frac{\eta\mu}{\omega}} \int_{0}^{\infty} d\nu u U \mathbf{S}' \right] \cdot \mathbf{C}(\beta_1) .
\]

The factor \( U \) contains velocity space derivatives of the distribution function,

\[
U = \frac{\partial f}{\partial u_\perp} - \frac{n||}{\sqrt{\mu}} \left( u_\parallel \frac{\partial f}{\partial u_\parallel} - u_\perp \frac{\partial f}{\partial u_\perp} \right).
\]

In equations (5-6), \( l \) is the harmonic number, \( \Omega \) is the cyclotron frequency, \( \omega_p \) is the plasma frequency, and \( n|| = k|/c/\omega \), where \( k| \) is the wave number parallel to the equilibrium magnetic field. The tensor \( \mathbf{S}' \) is defined as,

\[
\mathbf{S}'_l = \begin{pmatrix}
\frac{1}{2} u_\perp^2 J_{l+1}(\xi_2) J_{l+1}(\xi_1) & \frac{1}{2} u_\perp^2 J_{l+1}(\xi_2) J_{l-1}(\xi_1) & \frac{1}{\sqrt{2}} u_\perp u_\parallel J_{l+1}(\xi_2) J_{l}(\xi_1) \\
\frac{1}{2} u_\perp^2 J_{l+1}(\xi_2) J_{l-1}(\xi_1) & \frac{1}{2} u_\perp^2 J_{l-1}(\xi_2) J_{l-1}(\xi_1) & \frac{1}{\sqrt{2}} u_\perp u_\parallel J_{l-1}(\xi_2) J_{l}(\xi_1) \\
\frac{1}{\sqrt{2}} u_\perp u_\parallel J_{l}(\xi_2) J_{l}(\xi_1) & \frac{1}{\sqrt{2}} u_\perp u_\parallel J_{l}(\xi_2) J_{l}(\xi_1) & u_\parallel^2 J_{l}(\xi_2) J_{l}(\xi_1)
\end{pmatrix},
\]

where the argument of the Bessel functions is \( \xi = k|v_\perp/\Omega = (k|u_\perp/\Omega) c/\sqrt{\mu} \). The velocity \( u \) is normalized to \( v_e = c/\sqrt{\mu} \), \( c \) is the speed of light, and \( \mu = mc^2/2eE_{\text{norm}} \) where \( E_{\text{norm}} \) is the maximum energy in eV at which the numerical distribution function is evaluated. The velocity components perpendicular and parallel to the magnetic field are \( u_\perp \) and \( u_\parallel \), respectively, and the distribution function \( f \) is normalized to \( n/v_e^3 \), where \( n \) is the density. The rotational matrix \( \mathbf{C}(\beta) \) transforms the electric field from local magnetic coordinates to the \( (E_\perp, E_\parallel, E_z) \) frame [6].

The summation over Fourier wave numbers \( \mathbf{k}_1 \) and \( \mathbf{k}_2 \) in equation (5) can be extremely time consuming to evaluate. In 2-D, four nested do loops are required, and in 3-D, six loops are required. Even for Maxwellians, these sums can take an order of magnitude more time than the wave solution itself, and for non-Maxwellians, the time is totally prohibitive. A more efficient way to calculate the power absorption is to bring the velocity space integrals in equation (6) outside of the sum over \( \mathbf{k}_1 \) and \( \mathbf{k}_2 \) in which case \( P_{RF} \) can be expressed as a product of sums rather than as nested sums,

\[
P_{RF} = -\frac{\pi \varepsilon_0 \omega_p^2}{2 \omega} \text{Re} \left\{ \int_{0}^{\infty} \left[ \sum_{l=\infty}^{\infty} \frac{\sqrt{\mu}}{n||} \sum_{k_1} \sum_{k_2} \varepsilon_{k_2} \cdot \mathbf{a}_l \cdot \mathbf{a}_l^{(2l)} \right] \varepsilon_{k_1}^T \right\}.
\]

In equation (9), \( \varepsilon \) is the rotated electric field, and \( \mathbf{a}_l = \left( u_\perp J_{l+1}(\xi), u_\parallel J_{l+1}(\xi), \sqrt{2} u_\parallel J_{l}(\xi) \right) \).

Because the sums over \( \mathbf{k}_1 \) and \( \mathbf{k}_2 \) are separate, there is an enormous savings in computation time, and equation (9) can be evaluated for non-Maxwellians in approximately the same time as required to calculate the plasma current. The quasi-linear diffusion coefficients required for the solution of the Fokker Planck equations can be deduced from the power absorption. For example, the local value for \( B \) can be expressed as,
with similar expressions for $C$, $E$, and $F$ [3]. The bounce-averaged form of the diffusion coefficients $(B_0, C_0, E_0, \text{and } F_0)$ can be expressed as a flux surface average of the local values.

3. Quasilinear evolution of minority ion distributions in Alcator C-Mod

Using Python, AORSA and CQL3D have been iteratively coupled in a stand-alone system in which both codes communicate and interact automatically on the same computing platform (the Cray XT-3 at ORNL). The combined, self-consistent model has been applied to simulate the quasilinear evolution of the minority hydrogen (H) distribution function during ICRF heating experiments in the Alcator C-Mod tokamak [4]. Figure 1 shows five iterative steps in the self-consistent solution for the wave fields and minority ion distribution function with 600 kW of fast wave power absorbed in the plasma. For a spatial mesh $(R, Z)$ of 256×256 and a velocity space mesh $(u_{\perp,0}, u_{\parallel,0})$ of 65×129, the solution in figure 1 requires about 8 hours of clock time using 1024 processors on the Cray XT3. Plots of the distribution function $f$ are shown at the radial location where the power deposition is maximum $(r/a = 0.46)$. In the $0^{\text{th}}$ iteration, CQL3D calculates the distribution function for the hydrogen ions with no ICRF power absorbed. Using this distribution function, AORSA calculates the wave electric field, power absorption, and quasi-linear diffusion coefficients for the minority H, assuming that the electrons and majority deuterium ions are Maxwellian. The quasi-linear diffusion coefficients are then passed to CQL3D and used in the next iteration to recalculate the distribution function. The iteration procedure is continued until the results stop changing. CQL3D independently calculates the power absorption profile for the non Maxwellian minority ions, and the resulting profiles agree closely with those calculated by AORSA.

![Figure 1](image-url)
For the converged solution in figure 1, approximately 0.51 MW or 85% of the total ICRF power is absorbed by the minority hydrogen, with the remaining fraction being absorbed by the electrons and the majority deuterium ions. This power induces a high energy tail on the hydrogen distribution function that is clearly evident in figure 1(a). The tail is most extended in velocity space near the trapped-passing boundary where some of the ions have their turning points very close to the resonant surface and can therefore gain large amounts of energy. To estimate the approximate temperature of this tail, an equivalent Maxwellian has been fit to the distribution function in this region of velocity space. For the converged solution, the equivalent temperature is about 72 keV in approximate agreement with charge exchange neutral measurements in Alcator C-Mod [4].

Figure 2 shows a visualization of the bounce-averaged hydrogen distribution function for the 0th and 4th iterations of figure 1. Contours in $u_{\perp,0}$ and $u_{\parallel,0}$ are shown along the minor radius of the tokamak with the centre of the plasma at the lower left side of the image and the edge of the plasma at the upper right. The white colour represents regions of high density in velocity space, while the yellow colour corresponds to regions of low density. Notice that the formation of the energetic tail is localized in minor radius at about $r/a = 0.4$ where the power deposition is maximum.

**Figure 2.** The bounce-averaged hydrogen distribution function in Alcator C-Mod shot 1051206002 at $f = 80$ MHz: (a) 0th iteration; (b) 4th iteration.

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