ON THE MULTIPLICITY ONE CONJECTURE IN MIN-MAX THEORY

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ABSTRACT. We prove that in a closed manifold of dimension between 3 and 7 with a bumpy metric, the min-max minimal hypersurfaces associated with the volume spectrum introduced by Gromov, Guth, Marques-Neves, are two-sided and have multiplicity one. This confirms a conjecture by Marques-Neves.

We prove that in a bumpy metric each volume spectrum is realized by the min-max value of certain relative homotopy class of sweepouts of boundaries of Caccioppoli sets. The main result follows by approximating such min-max value using the min-max theory for hypersurfaces with prescribed mean curvature established by the author with Zhu.

0. INTRODUCTION

Let \((M^{n+1}, g)\) be a closed orientable Riemannian manifold of dimension \(3 \leq (n + 1) \leq 7\). In [2], Almgren proved that the space of mod-2 cycles \(Z_n(M, \mathbb{Z}_2)\) is weakly homotopic the Eilenberg-MacLane space \(K(\mathbb{Z}_2, 1) = \mathbb{RP}^{\infty}\); (see also [29] for a simpler proof). Later, Gromov [15, 16], Guth [18], Marques-Neves [28] introduced the notion of volume spectrum as a nonlinear version of spectrum for the area functional in \(Z_n(M, \mathbb{Z}_2)\). In particular, the volume spectrum is a non-decreasing sequence of positive numbers

\[0 < \omega_1(M, g) \leq \cdots \leq \omega_k(M, g) \leq \cdots \rightarrow +\infty,\]

which is uniquely determined by the metric \(g\) in a given closed manifold \(M\).

By adapting the celebrated min-max theory developed by Almgren [3], Pitts [31] (for \(3 \leq (n + 1) \leq 6\), and Schoen-Simon [33] (for \(n + 1 = 7\)), Marques-Neves [28, 27] proved that each \(\omega_k(M, g)\) is associated with an integral varifold \(V_k\) whose support is a disjoint collection of smooth, connected, closed, embedded, minimal hypersurfaces \(\{\Sigma^k_1, \cdots, \Sigma^k_{l_k}\}\), such that

\[\omega_k(M, g) = \sum_{i=1}^{l_k} m^k_i \cdot \text{Area}(\Sigma^k_i),\]

where \(\{m^k_1, \cdots, m^k_{l_k}\} \subset \mathbb{N}\) is a set of positive integers, usually called multiplicities. We refer to [39, 6, 10, 17, 9, 23, 7, 47, 50, 32] for other variants of this theory.

Our main theorem states that all these integer multiplicities are identically equal to one for a bumpy metric. A metric \(g\) is called bumpy if every closed immersed minimal hypersurface is non-degenerate. White proved that the set of bumpy metrics is generic in Baire sense [42, 44].

**Theorem A.** Given a closed manifold \(M^{n+1}\) of dimension \(3 \leq (n + 1) \leq 7\) with a bumpy metric \(g\), the min-max minimal hypersurfaces \(\{\Sigma^k_i : k \in \mathbb{N}, i = 1, \cdots, l_k\}\) associated with volume spectrum are all two-sided and have multiplicity one and index bounded by \(k\). That is \(m^k_i = 1\) for all \(k \in \mathbb{N}, 1 \leq i \leq l_k\),

\[\omega_k(M, g) = \sum_{i=1}^{l_k} \text{Area}(\Sigma^k_i), \quad \text{and} \quad \sum_{i=1}^{l_k} \text{index}(\Sigma^k_i) \leq k.\]
Remark 0.1. This solves the Multiplicity One Conjecture of Marques-Neves [29, 1.2]; (see also [27] for an earlier weaker version of this conjecture). We refer to Theorem 5.2 for a more detailed statement of this result. Note that by standard compactness analysis (see [35]), the same conclusion concerning two-sidedness and multiplicity one also holds true for a metric with positive Ricci curvature.

Remark 0.2. This conjecture was proved earlier for 1-parameter min-max constructions under positive Ricci curvature assumption by Marques-Neves [25], the author [48, 49], and Ketover-Marques-Neves [22]. Later it was fully proved for 1-parameter case by Marques-Neves [27]. Recently, Chodosh-Mantoulidis [5] proved this conjecture in dimension three \((n + 1) = 3\) for the Allen-Cahn setting; (see [12] for earlier works along this direction); they also proved that the total index is exactly \(k\) for their \(k\)-min-max solutions when \((n + 1) = 3\). After our results were poseted, Marques-Neves finished their program and also proved the same optimal index estimates for \(3 \leq (n + 1) \leq 7\) [29, Addendum].

One motivation of this conjecture is to prove the Yau’s conjecture [46] on existence of infinitely many closed minimal surfaces in three manifolds. Combining with the growth estimates of \(\{\omega_k(M, g)\}\) by Marques-Neves [28, Theorem 5.1 and 8.1] and the Frankel Theorem [11], we have

**Theorem B.** Let \(M^{n+1}\) be a closed manifold of dimension \(3 \leq (n + 1) \leq 7\).

(a) For each bumpy metric \(g\), there exists infinitely many smooth, connected, closed, embedded, minimal surfaces.

(b) If a metric \(g\) has positive Ricci curvature, then there exists a sequence of smooth, connected, closed, embedded, minimal hypersurfaces \(\{\Sigma_k\}_{k \in \mathbb{N}}\), such that

\[
\text{Area}(\Sigma_k) \sim k^{\frac{1}{n+1}}, \quad \text{as } k \to \infty.
\]

Remark 0.3. Result (a) was already known even without the bumpy assumption by combining Marques-Neves [28] and Song [40]. For a set of generic metrics, Irie-Marques-Neves [21] and Marques-Neves-Song [30] proved denseness and equi-distribution for the space of closed embedded minimal hypersurfaces, using the Weyl Law for volume spectrum by Liokumovich-Marques-Neves [24]. Their generic set in principle could be much smaller than the set of bumpy metrics.

Result (b) was also obtained by Chodosh-Mantoulidis [5] in dimension three \((n + 1) = 3\).

As a direct corollary of the compactness theory (see [35]), our multiplicity one result also gives a solution to the Weighted Morse Index Bound Conjecture by Marques-Neves.

**Theorem C.** Let \(M^{n+1}\) be a closed manifold of dimension \(3 \leq (n + 1) \leq 7\) with an arbitrary metric \(g\). In (0.1), we have

\[
\sum_{\Sigma^k_i: \text{orientable}} m^k_i \cdot \text{index}(\Sigma^k_i) + \sum_{\Sigma^k_i: \text{nonorientable}} \frac{m^k_i}{2} \cdot \text{index}(\Sigma^k_i) \leq k.
\]

0.1. **Sketch of the proof.** The key idea of our proof is to approximate the Area-functional by the weighted \(\mathcal{A}^h\)-functional used in the prescribing mean curvature (PMC) min-max theory developed by the author with Zhu [52]. Note that the \(\mathcal{A}^h\)-functional is only defined for boundaries of Caccioppoli sets; see (1.1). A smooth critical point of \(\mathcal{A}^h\) is a hypersurface whose mean curvature is prescribed by the restriction of \(h\) to itself. There are two crucial parts in the proof. In the first part, we consider min-max construction of minimal hypersurfaces using sweepouts of boundaries of Caccioppoli sets. We observe that in a bumpy metric if one approximates Area by a sequence \(\{\mathcal{A}^{\epsilon_k} h\}_{k \in \mathbb{N}}\) where \(\{\epsilon_k\}_{k \in \mathbb{N}} \to 0\), and if \(h : M \to \mathbb{R}\) is carefully chosen, then the limit min-max minimal hypersurfaces (of min-max PMC hypersurfaces associated with \(\mathcal{A}^{\epsilon_k} h\)) are all two-sided and have multiplicity one; see Theorem
In the second part, we show that in a bumpy metric the volume spectrum $\omega_k(M,g)$ can be realized by the area of some minimal hypersurfaces coming from min-max constructions using sweepouts of boundaries. We now elaborate the detailed ideas.

To implement the idea in the first part, we generalize the PMC min-max theory in [52] to multi-parameter families using continuous sweepouts. Since the space of Caccioppoli sets $C(M)$ is contractible, there is no nontrivial free homotopy class to do min-max, so we have to consider relative homotopy class. Heuristically, given a $k$-dimensional parameter space $X$, a subset $Z \subset X$, and a continuous map $\Phi_0 : X \to C(M)$, we can consider its relative $(X,Z)$-homotopy class $\Pi = \Pi(\Phi_0)$ consisting of all maps $\Phi : X \to C(M)$ that are homotopic to $\Phi_0$ and such that $\Phi|_Z \equiv \Phi_0|_Z$. If the min-max value $L^h = \inf\{\max_{x \in X} \mathcal{A}^h(\Phi(x)) : \Phi \in \Pi\}$ satisfies the nontriviality condition $L^h > \max_{x \in Z} \mathcal{A}^h(\Phi_0(x))$ with respect to the $\mathcal{A}^h$-functional, and if $h$ is chosen in a dense subset $S(g) \subset C^\infty(M)$ (depending on the metric $g$, see [52, Proposition 0.2]), we prove the existence of a smooth closed hypersurface $\Sigma^h$ of prescribed mean curvature $h$; moreover, it is represented as the boundary $\Sigma^h = \partial \Omega^h$ for some Caccioppoli set $\Omega^h$ and $\mathcal{A}^h(\Omega^h) = L^h$; hence $\Sigma^h$ is two-sided and have multiplicity one. $\Sigma^h$ is usually called a min-max PMC hypersurface. We also established Morse index upper bounds following Marques-Neves [27]. That is, we prove that the Morse index of $\Sigma^h$ is bounded from above by $k$ (the dimension of parameter space).

Given a relative homotopy class $\Pi$ as above, consider the min-max construction for the Area-functional and let $L = \inf\{\max_{x \in X} \text{Area}(\partial \Phi(x)) : \Phi \in \Pi\}$. If the nontriviality condition $L > \max_{x \in Z} \text{Area}(\partial \Phi_0(x))$ is satisfied, we can approximate $L$ by $L^{\epsilon h}$ for a fixed $h \in S(g)$ (to be chosen later) and small enough $\epsilon > 0$. We know that $\epsilon \cdot h$ also belongs to the dense subset $S(g)$. Denote $\Sigma_{\epsilon k}$ as the min-max PMC hypersurface associated with $L^{\epsilon h}$. As the family $\{\Sigma_{\epsilon k} : \epsilon > 0\}$ have uniformly bounded area and Morse index, we can pick a subsequence $\{\Sigma_{\epsilon k} = \Sigma_{k_0} \to 0\}$ which converges as varifolds and also locally smooth and graphically away from finitely many points to some limit minimal hypersurface $\Sigma_\infty$ with integer multiplicity such that $\text{Area}(\Sigma_\infty) = L$. The limit can be extended to a closed embedded minimal hypersurface $\Sigma_\infty$ across the bad points, and $\Sigma_\infty$ also has the same Morse index upper bound. Hence $\Sigma_\infty$ is a min-max minimal hypersurface associated with $L$. As a standard process, if the multiplicity is greater than one, or if a component is one-sided, one can obtain solutions of the Jacobi operator $L_{\Sigma_\infty}$ of $\Sigma_\infty$ by taking the limit of the renormalizations of the heights between the top and bottom sheets of $\Sigma_k$. In particular, there are two possibilities for the limit depending on the orientations of the top and bottom sheets. For simplicity, let us assume that $\Sigma_\infty$ is connected and two-sided. An easier case happens when the top and bottom sheets have the same orientation, and hence the limit is a nontrivial nonnegative solution $\varphi$ of the Jacobi equation $L_{\Sigma_\infty} \varphi = 0$ which cannot happen in a bumpy metric. When the top and bottom sheets have opposite orientations, the limit is either a nontrivial nonnegative solution to the Jacobi equation, or is a solution $\varphi$ of the following equation

$$L_{\Sigma_\infty} \varphi = 2h|_{\Sigma_\infty},$$

such that $\varphi$ does not change sign.

The key observation is that one can find a $h \in S(g)$ so that the unique solution (as $\Sigma_\infty$ is non-degenerate) of $L_{\Sigma_\infty} \varphi = 2h|_{\Sigma_\infty}$ must change sign, and hence $\Sigma_\infty$ must have multiplicity one; (see Lemma 4.2). Indeed, the set of minimal hypersurfaces with bounded area and Morse index in a bumpy metric is finite by the standard compactness results [35]. On each such $\Sigma$, we can construct a $h_{\Sigma} \in C^\infty(\Sigma)$ such that the unique solution $f_{\Sigma}$ of $L_{\Sigma} f_{\Sigma} = 2h_{\Sigma}$ must change sign, and we can further make the support of all such $h_{\Sigma}$ pairwise disjoint. Since $S(g)$ is open and dense, we can pick a $h \in S(g)$ that approximates all $h_{\Sigma}$ on $\Sigma$ as close as we want. Then the solution of $L_{\Sigma} \varphi = 2h|_{\Sigma}$ must also change sign. Up to here, we have elucidated how to construct two-sided min-max minimal hypersurfaces with multiplicity one for sweepouts of boundaries of Caccioppoli sets.
Lastly we apply the above multiplicity one result to the volume spectrum. Though the volume spectrum \( \omega_k(M,g) \) is defined using cohomological relations, Marques-Neves proved in [27], using their Morse index estimates, that in a bumpy metric \( \omega_k(M,g) \) is realized by the min-max value \( L(\Pi) \) for certain free homotopy class \( \Pi \) of maps \( \Phi: X \to \mathbb{Z}_n(M^{n+1},\mathbb{Z}_2) \), where \( X \) is some fixed \( k \)-dimensional parameter space and \( \mathbb{Z}_n(M^{n+1},\mathbb{Z}_2) \) is the space of mod-2 cycles. It was observed by Marques-Neves [29] that the space of Caccioppoli sets \( C(M) \) forms a double cover of \( \mathbb{Z}_n(M^{n+1},\mathbb{Z}_2) \) via the boundary map \( \partial: C(M) \to \mathbb{Z}_n(M^{n+1},\mathbb{Z}_2) \). Therefore, by lifting to the double cover, for each \( \Phi \in \Pi \), we can produce a map \( \tilde\Phi: \tilde{X} \to C(M) \), where \( \pi: \tilde{X} \to X \) is a double cover, such that \( \partial\tilde{\Phi}(x) = \Phi(\pi(x)) \). To produce a nontrivial relative homotopy class, we pick a map \( \Phi_0 \in \Pi \) such that \( \max_{x \in X} \text{Area}(\Phi_0(x)) \) is very close to \( L(\Pi) = \omega_k(M,g) \). Let \( Z \subset X \) to be the subset where each \( \Phi_0(x), x \in Z, \) is \( \epsilon \)-distance away from the set of smooth closed embedded minimal hypersurface \( \Sigma \) with \( \text{Area}(\Sigma) \leq L \) and \( \text{index}(\Sigma) \leq k \). Note that this set of minimal hypersurfaces is finite in a bumpy metric, hence for \( \epsilon \) small enough the complement \( Y = X \setminus Z \subset X \) is topologically trivial in the sense that \( Y \) does not detect the generator of the cohomological ring of \( \mathbb{Z}_n(M^{n+1},\mathbb{Z}_2) \). Therefore the pre-image \( \tilde{Y} = \pi^{-1}(Y) \subset \tilde{X} \) is homeomorphic to two disjoint identical copies of \( Y \), denoted as \( Y^+ \) and \( Y^- \). On the other hand, since no element in \( \Phi_0(Z) \) is regular, by Pitts’s combinatorial argument, one can homotopically deform \( \Phi_0 \) so that \( \max_{x \in Z} \text{Area}(\Phi_0(x)) < L \). Now consider the relative \((\tilde{X},\tilde{Z})\)-homotopy class \( \tilde{\Pi} \) generated by the map \( \tilde{\Phi}_0: \tilde{X} \to C(M) \). One key observation is that the min-max value \( L(\tilde{\Pi}) \geq L(\Pi) > \max_{x \in Z} \text{Area}(\Phi_0(x)) \). To see this, given any homotopic deformation \( \tilde{\Psi}: \tilde{X} \to C(M) \) of \( \tilde{\Phi}_0 \) relative to \( (\tilde{\Phi}_0)|_{\tilde{Z}} \), if \( \max_{x \in Y^+} \text{Area}(\partial\tilde{\Psi}(x)) < L(\Pi) \), then we can pass it to quotient and obtain a continuous map \( \Psi: X \to \mathbb{Z}(M,\mathbb{Z}_2) \) as \( Y^+ \) and \( Y^- \) are disjoint and \( \Psi|_{\tilde{Z}} \equiv (\tilde{\Phi}_0)|_{\tilde{Z}} \), so that \( \max_{x \in X} \text{Area}(\Psi(x)) < L(\Pi) \), but this is a contradiction as \( \Psi \) is homotopic to \( \Phi_0 \). Therefore, \( \tilde{\Pi} \) is a nontrivial relative homotopy class in \( C(M) \), and its associated min-max minimal hypersurfaces are two-sided and have multiplicity one. Finally, as the metric is bumpy, the min-max value \( L(\tilde{\Pi}) \) of \( \tilde{\Pi} \) is equal to \( L(\Pi) \) when \( \max_{x \in X} \text{Area}(\Phi_0(x)) \) is close enough to \( L(\Pi) = \omega_k(M,g) \). Hence we have explained how to construct two-sided min-max minimal hypersurfaces of multiplicity one whose areas realize the volume spectrum.

0.2. Outline of the paper. In Section [1] we establish the multi-parameter version of min-max theory for prescribing mean curvature hypersurfaces using continuous sweepouts. In Section [2] we prove several compactness results for prescribing mean curvature hypersurfaces with uniform area and Morse index upper bounds. In Section [3] we prove the Morse index upper bound for prescribing mean curvature hypersurfaces produced by our min-max theory. In Section [4] we prove that min-max minimal hypersurfaces associated with families of boundaries have multiplicity one in a bumpy metric. Finally, in Section [5] we prove the multiplicity one conjecture for volume spectrum.

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1. Multi-parameter Min-max Theory for Prescribing Mean Curvature Hypersurfaces

Here we present an adaption to multi-parameter families of the min-max theory for hypersurfaces with prescribed mean curvature (abbreviated as PMC) established by the author with Zhu [51, 52]. Let $S = S(g)$ (depending on the metric $g$) be the open and dense subset of $C^\infty(M)$ chosen as in Proposition 0.2. More precisely, $S(g)$ consists of all Morse functions $h$ such that the zero set $\Sigma_0 = \{h = 0\}$ is a smooth closed embedded hypersurface, and the mean curvature of $\Sigma_0$ vanishes to at most finite order. A hypersurface is almost embedded (sometimes also called strongly Alexandrov embedded) if it locally decomposes into smooth embedded sheets that touch but do not cross. By Theorem 3.11], any almost embedded hypersurface of prescribed mean curvature $h \in S$ has touching set $(n - 1)$-rectifiable, and no component is minimal.

Notations. We collect some notions. We refer to [36] and [31, §2.1] for further materials in geometric measure theory.

Let $(M^{n+1}, g)$ denote a closed, oriented, smooth Riemannian manifold of dimension $3 \leq (n + 1) \leq 7$. Assume that $(M, g)$ is embedded in some $\mathbb{R}^L$, $L \in \mathbb{N}$. $B_r(p)$ denotes the geodesic ball of $(M, g)$. We denote by $\mathcal{H}^k$ the $k$-dimensional Hausdorff measure; $\mathcal{I}_k(M)$ (or $\mathcal{I}_k(M, \mathbb{Z}_2)$) the space of $k$-dimensional integral (or mod 2) currents in $\mathbb{R}^L$ with support in $M$; $\mathcal{Z}_k(M)$ (or $\mathcal{Z}_k(M, \mathbb{Z}_2)$) the space of integral (or mod 2) currents $T \in \mathcal{I}_k(M)$ with $\partial T = 0$; $\mathcal{V}_k(M)$, the space, in the weak topology, of the space of $k$-dimensional rectifiable varifolds in $\mathbb{R}^L$ with support in $M$; $\mathcal{G}_k(M)$ the Grassmannian bundle of un-oriented $k$-planes over $M$; $\mathcal{F}$ and $\mathcal{M}$ respectively the flat norm [36, §31] and mass norm [36, 26.4] on $\mathcal{I}_k(M)$; $\textbf{F}$ the varifold $\mathcal{F}$-metric on $\mathcal{V}_k(M)$ and currents $\mathcal{F}$-metric on $\mathcal{I}_k(M)$ or $\mathcal{I}_k(M, \mathbb{Z}_2)$, [36, §2.1]; $\mathcal{C}(M)$ or $\mathcal{C}(U)$ the space of sets $\Omega \subset M$ or $\Omega \subset U \subset M$ with finite perimeter (Caccioppoli sets), [36, 14.2]; and $\mathfrak{X}(M)$ or $\mathfrak{X}(U)$ the space of smooth vector fields in $M$ or supported in $U$. $\partial \Omega$ denotes the (reduced)-boundary of $\Omega$ as an integral current, and $\nu_{\partial \Omega}$ denotes the outward pointing unit normal of $\partial \Omega$, [36, 14.2].

We also utilize the following definitions:

(a) Given $T \in \mathcal{I}_k(M)$, $|T|$ and $\|T\|$ denote respectively the integral varifold and Radon measure in $M$ associated with $T$;

(b) Given $c > 0$, a varifold $V \in \mathcal{V}_k(M)$ is said to have $c$-bounded first variation in an open subset $U \subset M$, if

$$|\delta V(X)| \leq c \int_M |X|d\mu_V, \text{ for any } X \in \mathfrak{X}(U);$$

here the first variation of $V$ along $X$ is $\delta V(X) = \int_{\mathcal{G}_k(M)} \text{div}_S X(x)d\nu(x, S)$,[36, §39];

(c) Given a smooth immersed, closed, orientable hypersurface $\Sigma$ in $M$, or a set $\Omega \subset \mathcal{C}(M)$ with finite perimeter, $[[\Sigma]], [[\Omega]]$ denote the corresponding integral currents with the natural orientation, and $[\Sigma]$ denotes the corresponding integer-multiplicity varifold.

As noted by Marques-Neves [29, Section 5], $\mathcal{C}(M)$ is identified with $\mathcal{I}_{n+1}(M, \mathbb{Z}_2)$. In particular, the flat $\mathcal{F}$-norm and the mass $\mathcal{M}$-norm are the same on $\mathcal{C}(M)$. Given $\Omega_1, \Omega_2 \in \mathcal{C}(M)$, the $\textbf{F}$-distance between them is:

$$\textbf{F}(\Omega_1, \Omega_2) = \mathcal{F}(\Omega_1 - \Omega_2) + \mathcal{F}(|\partial \Omega_1|, |\partial \Omega_2|).$$

Given $\Omega \in \mathcal{C}(M)$, we will denote $\mathcal{B}_\epsilon^\text{FC}(\Omega) = \{\Omega' \in \mathcal{C}(M) : \textbf{F}(\Omega', \Omega) \leq \epsilon\}$. 

We are interested in the following weighted area functional defined on \( C(M) \). Given \( h : M \to \mathbb{R} \), define the \( \mathcal{A}^h \)-functional on \( C(M) \) as

\[
\mathcal{A}^h(\Omega) = \mathcal{H}^n(\partial \Omega) - \int_{\Omega} h \, d\mathcal{H}^{n+1}.
\]

The first variation formula for \( \mathcal{A}^h \) along \( X \in \mathcal{X}(M) \) is (see [36, 16.2])

\[
\delta \mathcal{A}^h|_{\Omega}(X) = \int_{\partial \Omega} div_{\partial \Omega} X \, d\mu_{\partial \Omega} - \int_{\partial \Omega} h(X, \nu) \, d\mu_{\partial \Omega},
\]

where \( \nu = \nu_{\partial \Omega} \) is the outward unit normal on \( \partial \Omega \).

When the boundary \( \partial \Omega = \Sigma \) is a smooth immersed hypersurface, we have

\[
div_{\Sigma} X = H(X, \nu),
\]

where \( H \) is the mean curvature of \( \Sigma \) with respect to \( \nu \); if \( \Omega \) is a critical point of \( \mathcal{A}^h \), then (1.2) directly implies that \( \Sigma = \partial \Omega \) must have mean curvature \( H = h|_{\Sigma} \). In this case, we can calculate the second variation formula for \( \mathcal{A}^h \) along \( X \in \mathcal{X}(M) \) such that \( X = \varphi \nu \) along \( \partial \Omega = \Sigma \) where \( \varphi \in C^\infty(\Sigma) \), [4, Proposition 2.5],

\[
\delta^2 \mathcal{A}^h|_{\Omega}(X, X) = II_{\Sigma}(\varphi, \varphi) = \int_{\Sigma} (|\nabla \varphi|^2 - (Ric^M(\nu, \nu) + |A^\Sigma|^2 + \partial_v h) \varphi^2) \, d\mu_{\Sigma}.
\]

In the above formula, \( \nabla \varphi \) is the gradient of \( \varphi \) on \( \Sigma \); \( Ric^M \) is the Ricci curvature of \( M \); \( A^\Sigma \) is the second fundamental form of \( \Sigma \).

### 1.1. Min-max construction for \((X,Z)\)-homotopy class

In this part, we describe the setup for min-max theory for PMC hypersurfaces associated with multiple parameter families in \( C(M) \).

Let \( X^k \) be a cubical complex of dimension \( k \in \mathbb{N} \) in some \( I^m = [0, 1]^m \) and \( Z \subset X \) be a cubical subcomplex.

Let \( \Phi_0 : X \to (\mathcal{C}(M), F) \) be a continuous map (with respect to the \( F \)-topology on \( \mathcal{C}(M) \)). We let \( \Pi \) be the set of all sequences of continuous (in \( F \)-topology) maps \( \{ \Phi_i : X \to \mathcal{C}(M) \}_{i \in \mathbb{N}} \) such that:

1. each \( \Phi_i \) is homotopic to \( \Phi_0 \) in the flat topology on \( \mathcal{C}(M) \), and
2. there exist homotopy maps \( \{ \Psi_i : [0, 1] \times X \to \mathcal{C}(M) \}_{i \in \mathbb{N}} \) which are continuous in the flat topology, \( \Psi_i(0, \cdot) = \Phi_i \), \( \Psi_i(1, \cdot) = \Phi_0 \), and satisfy

\[
\lim_{i \to \infty} \sup_{t \in [0, 1]} \sup_{x \in X} \{ F(\Psi_i(t, x), \Phi_0(x)) : t \in [0, 1], x \in Z \} = 0.
\]

Note that a sequence \( \{ \Phi_i \}_{i \in \mathbb{N}} \) with \( \Phi_i = \Phi_0 \) for all \( i \in \mathbb{N} \) belongs to \( \Pi \).

**Definition 1.1.** Given a pair \((X,Z)\) and \( \Phi_0 \) as above, \( \{ \Phi_i \}_{i \in \mathbb{N}} \) is called a \((X,Z)\)-homotopy sequence of mappings into \( \mathcal{C}(M) \), and \( \Pi \) is called the \((X,Z)\)-homotopy class of \( \Phi_0 \).

**Remark 1.2.** \( \Pi \) can be viewed as the relative homotopy class for \( \Phi_0 \) in \((\mathcal{C}(M), \Phi_0|_Z)\). However, we cannot fix the values \( \Phi_i|_Z \) to be exactly \( \Phi_0|_Z \). In fact, in the later discretization/interpolation process, we will allow \( \Phi_i|_Z \) to deviate slightly from \( \Phi_0|_Z \); but the deviations will converge to zero as \( i \to \infty \).

**Definition 1.3.** The \( h \)-width of \( \Pi \) is defined by:

\[
L^h = L^h(\Pi) = \inf_{\{ \Phi_i \} \in \Pi} \lim_{i \to \infty} \sup_{x \in X} \{ \mathcal{A}^h(\Phi_i(x)) \}.
\]
**Theorem 1.7** (Min-max theorem). Let $(M^{n+1}, g)$ be a closed Riemannian manifold of dimension $3 \leq (n + 1) \leq 7$, and $h \in \mathcal{S}(g)$ which satisfies $\int_M h \geq 0$. Given a map $\Phi_0 : X \to (\mathcal{C}(M), \mathbf{F})$ continuous in the $\mathbf{F}$-topology and the associated $(X, Z)$-homotopy class $\Pi$, suppose

\begin{equation}
L^h(\Pi) > \max_{x \in Z} A^h(\Phi_0(x)).
\end{equation}

Let $\{\Phi_i\}_{i \in \mathbb{N}} \in \Pi$ be a min-max sequence for $\Pi$. Then there exists $V \in \mathcal{C}(\{\Phi_i\})$ induced by a nontrivial, smooth, closed, almost embedded hypersurface $\Sigma^n \subset M$ of prescribed mean curvature $h$ with multiplicity one.

Moreover, $V = \lim_{j \to \infty} |\partial \Phi_{i_j}(x_j)|$ for some $\{i_j\} \subset \{i\}, \{x_j\} \subset X \setminus Z$, with $\lim_{j \to \infty} A^h(\Phi_{i_j}(x_j)) = L^h(\Pi)$, and $\Phi_{i_j}(x_j)$ converges in the $\mathbf{F}$-topology to some $\Omega \in \mathcal{C}(M)$ such that $\Sigma = \partial \Omega$ where its mean curvature with respect to the unit outer normal is $h$, and

\[ A^h(\Omega) = L^h(\Pi). \]
1.2. Pull-tight. Now we describe the pull-tight process in [52] Section 5. Let \( c = \sup_M |h| \), and \( L^c = 2L^h + c \operatorname{Vol}(M) \). Denote
\[
A^c_\infty = \{ V \in \mathcal{V}_\infty(M) : \|V\|_c (M) \leq L^c, V \text{ has } c\text{-bounded first variation, or } V \in \partial \Phi_0|(Z) \}.
\]
We can follow [51] Section 4 or [52] Section 5 to construct a continuous map:
\[
H : [0, 1] \times (C(M), \mathcal{F}) \cap \{ M(\partial \Omega) \leq L^c \} \to (C(M), \mathcal{F}) \cap \{ M(\partial \Omega) \leq L^c \}
\]
such that:
(i) \( H(0, \Omega) = \Omega \) for all \( \Omega \);
(ii) \( H(t, \Omega) = \Omega \) if \( |\partial \Omega| \in A^c_\infty \);
(iii) if \( |\partial \Omega| \notin A^c_\infty \),
\[
A^h(H(1, \Omega)) - A^h(\Omega) \leq -L(F(|\partial \Omega|, A^c_\infty)) < 0;
\]
here \( L : [0, \infty) \to [0, \infty) \) is a continuous function with \( L(0) = 0 \), \( L(t) > 0 \) when \( t > 0 \);
(iv) for every \( \epsilon > 0 \), there exists \( \delta > 0 \) such that
\[
x \in Z, F(\Omega, \Phi_0(x)) < \delta \implies F(H(t, \Omega), \Phi_0(x)) < \epsilon, \text{ for all } t \in [0, 1];
\]
this is a direct consequence of (ii) since \( |\partial \Phi_0|(Z) \in A^c_\infty \).

Note that to construct \( H \), the only modification of [52] §5.1 is to add \( |\partial \Phi_0|(Z) \) into the definition of \( A^c_\infty \) as we want to fix the values assumed on \( Z \) in the tightening process; all other steps in [52] §5.1 carry out the same way. In particular, (using notions in [52] §5.1), \( H(t, \Omega) := (\Psi_{|\partial \Omega|(t)}(t))(\Omega) \).

**Lemma 1.8.** Given a min-max sequence \( \{ \Phi^*_i \}_{i \in \mathbb{N}} \in \Pi \), we define \( \Phi_i(x) = H(1, \Phi^*_i(x)) \) for every \( x \in X \). Then \( \{ \Phi_i \}_{i \in \mathbb{N}} \) is also a min-max sequence in \( \Pi \). Moreover, \( C(\{ \Phi_i \}) \subset C(\{ \Phi^*_i \}) \) and every element of \( C(\{ \Phi_i \}) \) either has \( c\)-bounded first variation, or belongs to \( |\partial \Phi_0|(Z) \).

**Proof.** By continuity of \( H \), we know that \( \Phi_i \) is homotopic to \( \Phi^*_i \) in the flat topology. By (iv), \( \{ \Psi_i(t, x) = H(t, \Phi^*_i(x)) \} \) satisfies (1.4), and hence \( \{ \Phi_i \} \in \Pi \). By (ii)(iii), \( A^h(\Phi_i(x)) \leq A^h(\Phi^*_i(x)) \) for every \( x \in X \), so \( \{ \Phi_i \} \) is also a min-max sequence. Finally, given any \( V \in C(\{ \Phi_i \}) \), then \( V = \lim_{j \to \infty} |\partial \Phi^*_i(x_j)| \) where \( \lim_{j \to \infty} A^h(\Phi^*_i(x_j)) = L^h \). Denote \( V^* = \lim_{j \to \infty} |\partial \Phi^*_i(x_j)| \). By (iii), \( \lim_{j \to \infty} F(|\partial \Phi^*_i(x_j)|, A^c_\infty) = 0 \) (as \( \lim_{j \to \infty} A^h(\Phi^*_i(x_j)) = \lim_{j \to \infty} A^h(\Phi^*_i(x_j)) = L^h \), so \( V^* \in A^c_\infty \). On the other hand,
\[
V = \lim_{j \to \infty} |\partial H(1, \Phi^*_i(x_j))| = H(1, \lim_{j \to \infty} |\partial \Phi^*_i(x_j)|) = H(1, V^*) = V^*.
\]
(Note that \( H \) is also well defined as a continuous map \( H : [0, 1] \times \{ V \in \mathcal{V}_n(M), \|V\|(M) \leq L^c \} \to \{ V \in \mathcal{V}_n(M), \|V\|(M) \leq L^c \} \). Hence \( C(\{ \Phi_i \}) \subset C(\{ \Phi^*_i \}) \) and the proof is finished. \( \square \)

**Definition 1.9.** Let \( c = \sup_M |h| \). Any min-max sequence \( \{ \Phi_i \}_{i \in \mathbb{N}} \in \Pi \) such that every element of \( C(\{ \Phi_i \}) \) has \( c\)-bounded first variation or belongs to \( |\partial \Phi_0|(Z) \) is called pulled-tight.

1.3. Discretization and interpolation results. We record several discretization and interpolation results developed by Marques-Neves [26] [28]. Though these results were proven for sweepouts in \( \mathcal{Z}_n(M, Z) \) or \( \mathcal{Z}_n(M, Z_2) \), they work well for sweepouts in \( C(M) \). We will point out necessary modifications.

We refer to Appendix A for the notion of cubic complex structure on \( X \). We refer to [52] Section 4 for the notion of discrete sweepouts. Though all definitions therein were made when \( X = [0, 1] \), there is no change for discrete sweepouts on \( X \).

Recall that given a map \( \phi : X(k)_0 \to C(M) \), the fineness of \( \phi \) is defined as
\[
f(\phi) = \sup \{ F(\phi(x) - \phi(y)) + M(\partial \phi(x) - \partial \phi(y)) : x, y \text{ are adjacent vertices in } X(k)_0 \}.
\]
Definition 1.10 (c.f. §3.7 in [28]). Given a continuous (in the flat topology) map \( \Phi : X \to C(M) \), we say that \( \Phi \) has no concentration of mass if

\[
\lim_{r \to 0} \sup \{ \| \partial \Phi(x) \| (B_r(p)), p \in M, x \in X \} = 0.
\]

The purpose of the next theorem is to construct discrete maps out of a continuous map in flat topology.

Theorem 1.11. Let \( \Phi : X \to C(M) \) be a continuous map in the flat topology that has no concentration of mass, and \( \sup_{x \in X} M(\partial \Phi(x)) < +\infty \). Assume that \( \Phi|_Z \) is continuous under the \( F \)-topology. Then there exist a sequence of maps

\[
\phi_i : X(k_i)_0 \to C(M),
\]

and a sequence of homotopy maps:

\[
\psi_i : I(k_i)_0 \times X(k_i)_0 \to C(M),
\]

with \( k_i < k_{i+1} \), \( \psi_i(0, \cdot) = \phi_{i-1} \circ n(k_i, k_{i-1}) \), \( \psi_i(1, \cdot) = \phi_i \), and a sequence of numbers \( \{ \delta_i \}_{i \in \mathbb{N}} \to 0 \) such that

(i) the fineness \( f(\psi_i) < \delta_i \);
(ii) \( \sup \{ F(\psi_i(t, x) - \Phi(x)) : t \in I(k_i)_0, x \in X(k_i)_0 \} \leq \delta_i \);
(iii) for some sequence \( l_i \to \infty \), with \( l_i < k_i \)

\[
M(\partial \psi_i(t, x)) \leq \sup \{ M(\partial \Phi(y)) : x, y \in \alpha, \text{ for some } \alpha \in X(l_i) \} + \delta_i;
\]

and this directly implies that

\[
\sup \{ M(\partial \psi_i(x)) : x \in X(k_i)_0 \} \leq \sup \{ M(\partial \Phi(x)) : x \in X \} + \delta_i.
\]

As \( \Phi|_Z \) is continuous in \( F \)-topology, we have from (iii) that for all \( t \in I(k_i)_0 \) and \( x \in Z(k_i)_0 \)

\[
M(\partial \psi_i(t, x)) \leq M(\partial \Phi(x)) + \eta_i
\]

with \( \eta_i \to 0 \) as \( i \to \infty \). Applying [26] Lemma 4.1 with \( S = \Phi(Z) \), we get by (ii) that

(iv)

\[
\sup \{ F(\psi_i(t, x), \Phi(x)) : t \in I(k_i)_0, x \in Z(k_i)_0 \} \to 0, \text{ as } i \to \infty.
\]

Now given \( h \in C^\infty(M) \), denoting \( c = \sup_M |h| \), then we have from (ii)(iii) that

(v)

\[
A^h(\phi_i(x)) \leq \sup \{ A^h(\Phi(y)) : \alpha \in X(l_i), x, y \in \alpha \} + (1 + c)\delta_i;
\]

and hence

\[
\sup \{ A^h(\phi_i(x)) : x \in X(k_i)_0 \} \leq \sup \{ A^h(\Phi(x)) : x \in X \} + (1 + c)\delta_i.
\]

Proof. [26] Theorem 13.1] and [28] Theorem 3.9] proved this result when \( C(M) \) is replaced by \( Z_n(M) \) and \( Z_n(M, Z_2) \) respectively. The adaption to \( C(M) \) was done in [49] Theorem 5.1] when \( X = [0, 1] \) and it is the same for general \( X \). □

The purpose of the next theorem is to construct a continuous map in the \( F \)-topology out of a discrete map with small fineness.
Theorem 1.12. There exist some positive constants $C_0 = C_0(M,m)$ and $\delta_0 = \delta_0(M,m)$ so that if $Y$ is a cubical subcomplex of $I(m,k)$ and

$$\phi : Y_0 \to \mathcal{C}(M)$$

has $f(\phi) < \delta_0$, then there exists a map

$$\Phi : Y \to \mathcal{C}(M)$$

continuous in the $F$-topology and satisfying

(i) $\Phi(x) = \phi(x)$ for all $x \in Y_0$;
(ii) if $\alpha$ is some $j$-cell in $Y$, then $\Phi$ restricted to $\alpha$ depends only on the values of $\phi$ restricted on the vertices of $\alpha$;
(iii)

$$\sup\{ M(\Phi(x), \Phi(y)) : x, y \text{ lie in a common cell of } Y \} \leq C_0 f(\phi).$$

Proof. [28 Theorem 3.10] proved this result when $\mathcal{C}(M)$ is replaced by $\mathcal{Z}_n(M,\mathbb{Z}_2)$. We can use the double cover $\partial : \mathcal{C}(M) \to \mathcal{Z}_n(M,\mathbb{Z}_2)$ (see [29 Section 5]) to lift the extension from $\mathcal{Z}_n(M,\mathbb{Z}_2)$ to $\mathcal{C}(M)$.

Let $C_0 = C_0(M,m)$ and $\delta_0 = \delta_0(M)$ be given in [28 Theorem 3.10]. Denote $\tilde{\phi} = \partial \circ \phi : Y_0 \to \mathcal{Z}_n(M,\mathbb{Z}_2)$ as the projection of $\phi$ into $\mathcal{Z}_n(M,\mathbb{Z}_2)$. Then $f(\tilde{\phi}) < \delta_0$, so by [28 Theorem 3.10], there exists a map:

$$\tilde{\Phi} : Y \to \mathcal{Z}_n(M,M,\mathbb{Z}_2)$$

continuous in the $\mathcal{M}$-topology and satisfying

(a) $\tilde{\Phi}(x) = \tilde{\phi}(x)$ for all $x \in Y_0$;
(b) if $\alpha$ is some $j$-cell in $Y$, then $\tilde{\Phi}$ restricted to $\alpha$ depends only on the values of $\tilde{\phi}$ restricted on the vertices of $\alpha$;
(c) 

$$\sup\{ M(\tilde{\Phi}(x), \tilde{\Phi}(y)) : x, y \text{ lie in a common cell of } Y \} \leq C_0 f(\phi).$$

By [29 Claim 5.2], $\tilde{\Phi}$ can be uniquely lifted to a continuous map $\Phi : Y \to \mathcal{C}(M)$ such that $\partial \circ \Phi = \tilde{\Phi}$ and $\Phi(x) = \phi(x)$ for all $x \in Y_0$. In fact, given a $j$-cell $\alpha$ and a fixed vertex $x_0 \in \alpha_0$, there is a unique lift $\Phi : \alpha \to \mathcal{C}(M)$ such that $\Phi(x_0) = \phi(x_0)$. By the construction in [29 Claim 5.2], $\mathcal{F}(\Phi(x), \Phi(x_0)) = \mathcal{F}(\tilde{\Phi}(x), \tilde{\Phi}(x_0)) \leq C_0 f(\phi)$ for every $x \in \alpha$, so we know by the Constancy Theorem that $\Phi(x) = \phi(x)$ for each vertex $x \in \alpha_0$ when $\delta_0$ is small enough. Thus $\Phi$ can be obtained by lifting $\tilde{\Phi}$ in each cell of $Y$.

Since $\partial \Phi(x)$ and $\tilde{\Phi}(x)$ represent the same varifold, $\Phi$ is continuous in the $F$-topology. So we have proved (i)(ii).

For (iii), we have

$$\mathcal{F}(\Phi(x), \Phi(y)) = \mathcal{F}(\Phi(x), \Phi(y)) + \mathcal{F}(|\partial \Phi(x)|, |\partial \Phi(y)|) \leq 2C_0 f(\phi).$$

Remark 1.13. Note that in general the mass of $\partial \Phi(x) - \partial \Phi(y)$ as element in $\mathcal{Z}_n(M)$ may not be equal to that of $\tilde{\Phi}(x) - \tilde{\Phi}(y)$, so we may not be able to prove the $M$-continuity for $\Phi$.

Following [28 3.10], we call the map $\Phi$ given in Theorem 1.12 the Almgren extension of $\phi$. We will record a few properties concerning the homotopy equivalence of Almgren’s extensions.
Before stating the next result, we first recall the notion of homotopic equivalence between discrete sweepouts. Let $Y$ be a cubical subcomplex of $I(m, k)$. Given two discrete maps $\phi_i : Y(l_i) \to C(M)$, we say $\phi_1$ is homotopic to $\phi_2$ with fineness less than $\eta$, if there exist $l \in \mathbb{N}$, $l > l_1, l_2$ and a map

$$\psi : I(1, k + l) \to Y(l_1) \to C(M)$$

with fineness $f(\psi) < \eta$ and such that

$$\psi([i - 1], y) = \phi_i(n(k + l, k + l_i)(y)), \ i = 1, 2, y \in Y(l_0).$$

The following result is analogous to [28, Proposition 3.11]. We provide a lightly different proof.

**Proposition 1.14.** With $\phi_1, \phi_2$ as above, if $\eta < \delta_0(M, m)$ in Theorem 1.12 then the Almgren extensions

$$\Phi_1, \Phi_2 : Y \to C(M)$$

of $\phi_1, \phi_2$, respectively, are homotopic to each other in the $F$-topology.

**Proof.** By Theorem 1.12 the Almgren extension $\Psi : I \times Y \to C(M)$ of $\psi$ is continuous in $F$-topology and is a homotopy between the Almgren extensions $\Phi'_1, \Phi'_2$ of $\phi'_1, \phi'_2 : Y(l_0) \to C(M)$ (given by $\phi'_i(y) = \psi([i - 1], y)$). Note that $\Phi'_i$ is just a reparametrization of the Almgren extension $\Phi_i$ of $\phi_i$ for $i = 1, 2$ respectively, so $\Phi_i$ is homotopic to $\Phi'_i$ in the $F$-topology. Now let us describe the reparametrization map. Given an arbitrary cell $\alpha$ and $k \in \mathbb{N}$, we take $\alpha_c$ to be the center cell of $\alpha(k)$.

We can define a map $n_{\alpha,k} : \alpha \to \alpha$ such that it maps $\alpha_c$ to $\alpha$ linearly, and for each $x \in \alpha \setminus \alpha_c$, if we denote by $x_c$ the nearest point projection of $x$ to $\partial \alpha_c$ then $n_{\alpha,k}$ maps $x$ to $n_{\alpha,k}(x_c)$. This map dilates $\alpha_c$ to $\alpha$ and compresses $\alpha \setminus \alpha_c$ to the boundary $\partial \alpha$, and is homotopic to the identity map. With this notion $\Phi'_i|_{\alpha} = \Phi_i|_{\alpha} \circ n_{\alpha,i-t}$ on each cell $\alpha \in Y(l_i)$. Hence we finish the proof. \hfill $\Box$

The following result is the counterpart of [28, Corollary 3.12].

**Proposition 1.15.** Let $\{\phi_i\}_{i \in \mathbb{N}}$ and $\{\psi_i\}_{i \in \mathbb{N}}$ be given by Theorem 1.11 applied to some $\Phi$ therein. Assume that $\Phi$ is continuous in the $F$-topology on $X$. Then the Almgren extension $\Phi_i$ is homotopic to $\Phi$ in the $F$-topology for sufficiently large $i$.

In particular, for $i$ large enough, there exist homotopy maps $\Psi_i : [0, 1] \times X \to C(M)$ continuous in the $F$-topology, $\Psi_i(0, \cdot) = \Phi_i$, $\Psi_i(1, \cdot) = \Phi$, and

$$\limsup_{i \to \infty} \sup_{t \in [0, 1]} \sup_{x \in X} F(\Psi_i(t, x), \Phi(x)) \to 0.$$  

Therefore for given $h \in C^\infty(M)$, we have

$$\limsup_{i \to \infty} \sup_{x \in X} A^h(\Phi_i(x)) \leq \sup_{x \in X} A^h(\Phi(x)).$$

**Proof.** For $i$ large enough such that $\delta_i < \delta_0$ in Theorem 1.12 we let $\tilde{\Psi}_i : I \times X \to C(M)$ be the Almgren extensions of $\psi_i$. By Theorem 1.11 iv (with $Z = X$) and Theorem 1.12 iii, we know that

$$\limsup_{i \to \infty} \sup_{t \in [0, 1]} \sup_{x \in X} F(\tilde{\Psi}_i(t, x), \Phi(x)) \to 0.$$  

As in the proof of the above Proposition, we can amend $\tilde{\Psi}_i$ with the reparametrization maps associated with the two pairs $(\Phi'_i, \Phi_i)$ and $(\Phi'_i, \Phi_i)$, and abuse the notation and still denote them by $\tilde{\Psi}_i$. Then $\tilde{\Psi}_i$ is a continuous (in the $F$-topology) homotopy between $\Phi_i$ and $\Phi_i$. Note that the reparametrizations are done is small cells with sizes converging to zero, so (1.6) still holds true for the amended maps by Theorem 1.12 iii again. For given $i$ large enough, to construct the homotopy from $\Phi_i$ to $\Phi$, we can just let $\Psi_i : [0, \infty] \times X \to C(M)$ be the gluing of all $\{\tilde{\Psi}_j\}_{j \geq i}$. Note that by (1.6), $\Psi_i(\infty, \cdot) = \Phi$
(we can identify $[0, \infty)$ with $[0, 1]$ in the definition of $\Psi_i$), and (1.6) holds true with $\bar{\Psi}_i$ replaced by $\Psi_i$. Hence we finish the proof. \hfill \Box

### 1.4. Proof of the min-max Theorem

One key ingredient in the Almgren-Pitts theory to prove regularity of min-max varifold is to introduce the “almost minimizing” concept. Given $h \in S(y)$, we refer to [52, Section 6] for the detailed notion of $h$-almost minimizing varifold and related properties. The existence of almost minimizing varifolds follows from a combinatorial argument of Pitts [31, page 165-page 174] inspired by early work of Almgren [3]. Pitts’s argument works well in the construction of min-max PMC hypersurfaces; see [52, Theorem 6.4]. Marques-Neves has generalized Pitts’s combinatorial argument to a more general form in [28, 2.12], and we can adapt their result to the PMC setting with no change. We now describe the adaption.

Consider a sequence of cubical subcomplexes $Y_i$ of $I(m, k_i)$ with $k_i \to \infty$, and a sequence $S = \{\varphi_i\}_{i \in \mathbb{N}}$ of maps

$$\varphi_i : (Y_i)_0 \to C(M)$$

with fineness $f(\varphi_i) = \delta_i$ converging to zero. Define

$$L_h^h(S) = \lim_{i \to \infty} \sup \{\mathcal{A}^h(\varphi_i(y)) : y \in (Y_i)_0\},$$

$$K(S) = \{V = \lim_{j \to \infty} |\partial \varphi_{ij}(y_j)| as varifolds : y_j \in (Y_j)_0\},$$

and

$$C(S) = \{V = \lim_{j \to \infty} |\partial \varphi_{ij}(y_j)| as varifolds : \text{with } \lim_{j \to \infty} \mathcal{A}^h(\varphi_{ij}(y_j)) = L_h^h(S)\}.$$

We say that an element $V \in C(S)$ is $h$-almost minimizing in small annuli with respect to $S$ (c.f. [52, Definition 6.3]), if for any $p \in M$ and any small enough annulus $A = A_{r_1, r_2}(p)$ centered at $p$ with radii $0 < r_1 < r_2$, there exist sequences $\{\epsilon_i\}_{i \in \mathbb{N}}$ and $\{\delta_i\}_{i \in \mathbb{N}}$ such that

$$V = \lim_{j \to \infty} |\partial \varphi_{ij}(y_j)|, \lim_{j \to \infty} \mathcal{A}^h(\varphi_{ij}(y_j)) = L_h^h(S), \text{ and } \varphi_{ij}(y_j) \in \mathcal{A}^h(A; \epsilon_i, \delta_i; M)$$

(see [52, Definition 6.1]) for some $\epsilon_i, \delta_i \to 0$. The last condition is usually called $(\epsilon, \delta, h)$-almost minimizing.

Note that by [52, Proposition 6.5], $V$ is also $h$-almost minimizing in small annuli in the sense of [52, Definition 6.3].

The following is a variant of [28, Theorem 2.13] and [31, Theorem 4.10].

**Theorem 1.16.** If no element $V \in C(S)$ is $h$-almost minimizing in small annuli with respect to $S$, then there exists a sequence $\tilde{S} = \{\tilde{\varphi}_i\}$ of maps

$$\tilde{\varphi}_i : Y_i(l_i)_0 \to C(M),$$

for some $l_i \in \mathbb{N}$, such that:

- $\tilde{\varphi}_i$ is homotopic to $\varphi_i$ with fineness converging to zero as $i \to \infty$;
- $L_h^h(\tilde{S}) < L_h^h(S)$.

**Proof.** By the assumption of the theorem, for each $V \in C(S)$, there exists a $p \in M$, such that for any $\tilde{r} > 0$, there exist $r, s > 0$, with $\tilde{r} > r + 2s > r - 2s > 0$ and $\epsilon > 0$, such that, if $\mathcal{A}^h(\varphi_i(y)) < L_h^h(S) - \epsilon$ and $F(|\partial \varphi_i(y)|, V) < \epsilon$, then $\varphi_i(y) \not\in \mathcal{A}^h(A_{r-2s, r+2s}(p); \epsilon, \delta; M)$ for any $\delta > 0$. As in the proof of [31, Theorem 4.10], we denote $c = (3^n)^{3^m}$. By the compactness of $C(S)$, we can find a uniform $\epsilon > 0$, and $I$, and finitely many points $p_1, \ldots, p_v \in M$, and for each $p_j$, we can find $c$ concentric annuli $A_{j,1} \supset \cdots \supset A_{j,c}$ (centered at $p_j$), such that, if $\mathcal{A}^h(\varphi_i(y)) > L_h^h(S) - \epsilon$ and $i > I$, then there exists some $j \in \{1, \ldots, v\}$, so that $\varphi_i(y) \not\in \mathcal{A}^h(A_{j,a}; \epsilon, \delta; M)$ for all $a \in \{1, \ldots, c\}$ and for any $\delta > 0$. From here the construction in [31, Page 165-174] can be applied to $S$ so as to produce the desired $\tilde{S}$. \hfill \Box
Now we are ready to prove Theorem 1.7 following closely that of [27, Theorem 3.8]. The only additional thing is to keep track of the volume term $\int_{\Omega} h dH^{n+1}$ in $A^h(\Omega)$ and the values of maps assumed on $\Omega$.

**Proof of Theorem 1.7.** Let $\{\Phi_i\}_{i \in \mathbb{N}}$ be a pulled-tight min-max sequence for $\Pi$. Given $\Phi_i : X \to C(M)$, it has no concentration of mass as it is continuous in the $F$-topology, so applying Theorem 1.11 gives a sequence of maps:

$$\phi_i^j : X(k_i^j) \to C(M),$$

with $k_i^j < k_i^{j+1}$ and a sequence of positive $\{\delta_i^j\}_{j \in \mathbb{N}} \to 0$, satisfying (i)–(v) in Theorem 1.11.

As $\Phi_i$ is continuous in the $F$-topology, by the same reasoning as Theorem 1.11(iii)(iv), we further have that for every $x \in X(k_i^j)_{0}$,

$$M(\partial \phi_i^j(x)) \leq M(\partial \Phi_i(x)) + \eta_i^j$$

with $\eta_i^j \to 0$ as $j \to \infty$, and

$$\sup \{F(\phi_i^j(x), \Phi_i(x)) : x \in X(k_i^j)_{0}\} \to 0, \text{ as } j \to \infty.$$

Now choose $j(i) \to \infty$ as $i \to \infty$, such that $\varphi_i = \phi_i^{j(i)} : X(k_i^{j(i)})_{0} \to C(M)$ satisfies:

- $\max \{F(\varphi_i(x), \Phi_i(x)) : x \in X(k_i^{j(i)})_{0}\} \leq a_i$ with $a_i \to 0$ as $i \to \infty$;
- $\max \{F(\Phi_i(x), \Phi_i(y)) : x, y \in \alpha, \alpha \in X(k_i^{j(i)})\} \leq a_i$;
- the fineness $f(\varphi_i) \to 0$ as $i \to \infty$;
- the Almgren extensions $\Phi_i^{j(i)} : X \to C(M)$ is homotopic to $\Phi_i$ in the $F$-topology with homotopy maps $\Psi_i^{j(i)}$, and

$$\limsup_{i \to \infty} \max \{F(\Psi_i^{j(i)}(t, x), \Phi_i(x)) : t \in [0, 1], x \in X\} = 0,$$

and

$$\limsup_{i \to \infty} A^h(\Phi_i^{j(i)}(x)) \leq \limsup_{i \to \infty} A^h(\Phi_i(x)) = L^h(\Pi),$$

by Proposition 1.15.

Therefore, if $S = \{\varphi_i\}$, then $L^h(S) = L^h(\{\Phi_i\})$ and $C(S) = C(\{\Phi_i\})$. By Theorem 1.16 if no element $V \in C(S)$ is $h$-almost minimizing in small annuli with respect to $S$, we can find a sequence $\tilde{S} = \{\tilde{\varphi}_i\}$ of maps:

$$\tilde{\varphi}_i : X(k_i^{j(i)} + l_i)_{0} \to C(M)$$

such that

- $\tilde{\varphi}_i$ is homotopic to $\varphi_i$ with fineness converging to zero as $i \to \infty$;
- $L^h(\tilde{S}) < L^h(S)$.

By Proposition 1.14 the Almgren extensions of $\varphi_i$, $\tilde{\varphi}_i$:

$$\Phi_i^{j(i)}, \tilde{\Phi}_i : X \to C(M),$$

respectively, are homotopic to each other in the $F$-topology for $i$ large enough, so $\tilde{\Phi}_i$ is homotopic to $\Phi_i$ in the $F$-topology.

By assumption 1.5 and 1.4, for $i$ large enough, $\tilde{\varphi}_i$ is the identical to $\varphi_i \circ n(k_i^{j(i)} + l_i, k_i^{j(i)})$ near $Z(k_i^{j(i)} + l_i)_{0}$; indeed, the deformation process in Theorem 1.16 was only made to those $\varphi_i(x)$
with $A^h(\varphi_i(x))$ close to $L^h(S)$. Therefore the homotopy maps $\tilde{\Phi}_i$ between $\Phi_i^{(i)}$ and $\Phi_i$ produced by Proposition 1.14 when restricted to $Z$ are just the reparametrization maps described therein. Hence
\[
\limsup_{i \to \infty} \sup \{ \Phi(\tilde{\Phi}_i(t, x), \Phi_i^{(i)}(x)) : t \in [0, 1], x \in Z \} = 0.
\]
Therefore $\{ \tilde{\Phi}_i \}_{i \in \mathbb{N}} \in \Pi$. However, by Theorem 1.12
\[
\limsup_{i \to \infty} \sup \{ A^h(\tilde{\Phi}_i(x)) : x \in X \} \leq L^h(\tilde{S}) < L^h(S) = L^h(\Pi).
\]
This is a contradiction. So some $V \in C(S) = C(\{ \Phi_i \})$ is $h$-almost minimizing in small annuli with respect to $S$, and hence is $h$-almost minimizing in small annuli in the sense of [52, Definition 6.3].

To finish the proof, we need to show that $V$ has $c$-bounded first variation, and then [52, Theorem 7.1 and Proposition 7.3] give the regularity of $V$ and the existence of $\Omega$. Indeed, by Definition 1.9 $V$ either has $c$-bounded first variation or belongs to $|\partial \Phi_0|(Z)$. Being $h$-almost minimizing in small annuli implies that $V$ has $c$-bounded first variation away from finitely many points by [52, Lemma 6.2]. If $V \in |\partial \Phi_0|(Z)$, then the proof of [19, Theorem 4.1] implies that $\|V\|$ has at most $r^{n-\frac{3}{2}}$-volume growth near these bad points, so the first variation extends across these points, and hence $V$ has $c$-bounded first variation in $M$. (Note that even if $V \in |\partial \Phi_0|(Z)$, the associated $\Omega \notin \Phi_0(Z)$, as $\Omega$ may be equal to $M \setminus \Phi_0(z)$ for some $z \in Z$.) So we finish the proof.

2. Compactness of PMC hypersurfaces with bounded Morse index

Now we present an adaption of Sharp’s compactness theorem [35] (for minimal hypersurfaces) to the PMC setting and necessary modifications of the proof. Given a closed Riemannian manifold $(M^{n+1}, g)$ and $h \in S(g)$, denote by $P^h$ the class of smooth, closed, almost embedded hypersurfaces $\Sigma \subset M$, such that $\Sigma$ is represented as the boundary of some open subset $\Omega \subset M$ (in the sense of current), and the mean curvature of $\Sigma$ with respect to the outer normal of $\Omega$ is prescribed by $h$, i.e.

$$H_\Sigma = h|_\Sigma.$$ 

In the following we will sometiem use the notation and identify $\Sigma$ with $\Omega$.

Note that when $h \in S(g)$, the min-max PMC hypersurfaces produced in Theorem 1.7 satisfy the above requirements. Indeed, such $\Sigma = \partial \Omega$ is a critical point of the weighted $A^h$ functional (1.1):

$$A^h(\Omega) = \text{Area}(\Sigma) - \int_\Omega h d\mathcal{H}^{n+1}.$$ 

The second variation formula for $A^h$ along normal vector field $X = \varphi \nu \in \mathcal{X}(M)$ is given by

$$\delta^2 A^h_{|\Omega}(X, X) = \int_\Sigma (|\nabla \varphi|^2 - (\text{Ric}^M(\nu, \nu) + |A^\Sigma|^2 + \partial_\nu h) \varphi^2) d\mu_\Sigma.$$

The classical Morse index for $\Sigma$ is defined as the number of negative eigenvalues of the above quadratic form. However, since we will deal with hypersurfaces with self-touching, a weaker version of index is needed. We adopt a concept used by Marques-Neves [27, Definition 4.1]. As we will see, this weaker index works well for proving both compactness theory and Morse index upper bound.

**Definition 2.1.** Given $\Sigma \in P^h$ with $\Sigma = \partial \Omega$, $k \in \mathbb{N}$ and $\epsilon \geq 0$, we say that $\Sigma$ is $k$-unstable in an $\epsilon$-neighborhood if there exists $0 < c_0 < 1$ an a smooth family $\{ F_v \}_{v \in B^k} \subset \text{Diff}(M)$ with $F_0 = \text{Id}$, $F_{-v} = F_v^{-1}$ for all $v \in B^k$ (the standard $k$-dimensional ball in $\mathbb{R}^k$) such that, for any $\Omega' \in \overline{B}^F_{2\epsilon}(\Omega)$, the smooth function:

$$A^h_{|\Omega'} : B^k \rightarrow [0, \infty), \quad A^h_{|\Omega'}(v) = A^h(F_v(\Omega'))$$
satisfies:

1. $A^h_{\Omega r}$ has a unique maximum at $m(\Omega') \in B_{c_0/\sqrt{40}}(0)$;
2. $-\frac{1}{c_0} \text{Id} \leq D^2 A^h_{\Omega r}(u) \leq -c_0 \text{Id}$ for all $u \in T^k$.

Since $\Sigma$ is a critical point of $A^h$, necessarily $m(\Omega) = 0$.

**Remark 2.2.** If a sequence $\Omega_i$ converges to $\Omega$ in the $F$-topology, then $A^h_{\Omega_i}$ tends to $A^h_{\Omega}$ in the smooth topology. Thus if a $\Sigma \in \mathcal{P}^h$ is $k$-unstable in a 0-neighborhood, then it is $k$-unstable in an $\epsilon$-neighborhood for some $\epsilon > 0$.

**Definition 2.3.** Given a $\Sigma \in \mathcal{P}^h$ and $k \in \mathbb{N}$, we say that its Morse index is bounded (from above) by $k$, denoted as

$$\text{index}(\Sigma) \leq k,$$

if it is not $j$-unstable in 0-neighborhood for any $j \geq k + 1$.

All the above concepts can be localized to an open subset $U \subset M$ by using $\text{Diff}(U)$ in place of $\text{Diff}(M)$. If $\Sigma$ has index equal to 0 in $U$, we say $\Sigma$ is weakly stable in $U$.

**Proposition 2.4.** If $\Sigma \in \mathcal{P}^h$ is smoothly embedded with no self-touching, then $\Sigma$ is $k$-unstable (in 0-neighborhood) if and only if its classical Morse index is $\geq k$.

**Proof.** The proof is the same as [27, Proposition 4.3].

We have the following curvature estimates as a variant of [52, Theorem 3.6] (with relatively weaker stability assumptions).

**Theorem 2.5** (Curvature estimates for weakly stable PMC). Let $3 \leq (n + 1) \leq 7$, and $U \subset M$ be an open subset. Let $\Sigma \in \mathcal{P}^h$ be weakly stable in $U$ with $\text{Area}(\Sigma) \leq C$, then there exists $C_1$ depending only on $n, M, \|h\|_{C^3}, C$, such that

$$|A^\Sigma|^2(x) \leq \frac{C_1}{\text{dist}_M^2(x, \partial U)}$$

for all $x \in \Sigma$.

**Proof.** The curvature estimates follow from standard blowup arguments together with the Bernstein Theorem [34, Theorem 2] and [33, Theorem 3]. In particular, being weakly stable in $U$ means that for any ambient vector field $X \in \mathfrak{X}(U)$ which generates the flow $\phi_t^X$, we have

$$\left.\frac{d^2}{dt^2}\right|_{t=0} A^h(\phi_t^X(\Omega)) \geq 0.$$  

Assume the conclusion were false, then there exists a sequence of weakly stable hypersurfaces $\{\Sigma_i\}_{i \in \mathbb{N}}$ with prescribing functions $\{h_i\}_{i \in \mathbb{N}}$ satisfying uniform bounds, but $\sup_U \text{dist}_M^2(\cdot, \partial U)|A_{\Sigma_i}^\Sigma|^2(\cdot) \to \infty$. By the standard blowup process (c.f. [41]), one can take a sequence of rescalings of $\Sigma_i$ which converges locally in $C^{3,\alpha}$ and graphically to a non-flat minimal hypersurface $\Sigma_\infty$ in $\mathbb{R}^{n+1}$. Note that the rescalings of $\{h_i\}$ converges to 0 locally uniformly in $C^3$. By the almost embedded assumption and the maximum principle for minimal hypersurfaces (8), $\Sigma_\infty$ is embedded and hence is 2-sided. By the classical monotonicity formula and area upper bound assumption on $\{\Sigma_i\}$, $\Sigma_\infty$ has polynomial volume growth. The key observation is that (2.1) is preserved under locally $C^{3,\alpha}$ convergence, and hence $\Sigma_\infty$ is a stable minimal hypersurface. Therefore it has to be flat by the Bernstein Theorem, but this is a contradiction.

Given $h \in S(g), 0 < \Lambda \in \mathbb{R}$ and $I \in \mathbb{N}$, let

$$\mathcal{P}^h(\Lambda, I) := \{\Sigma \in \mathcal{P}^h : \text{Area}(\Sigma) \leq \Lambda, \text{index}(\Sigma) \leq I\}.$$
Theorem 2.6 (Compactness for PMC’s with bounded index). Let \((M^{n+1}, g)\) be a closed Riemannian manifold of dimension \(3 \leq (n + 1) \leq 7\). Assume that \(\{h_k\}_{k \in \mathbb{N}}\) is a sequence of smooth functions in \(S(g)\) such that \(\lim_{k \to \infty} h_k = h_\infty\) in smooth topology. Let \(\{\Sigma_k\}_{k \in \mathbb{N}}\) be a sequence of hypersurfaces such that \(\Sigma_k \in \mathcal{P}^{h_k}(\Lambda, I)\) for some fixed \(\Lambda > 0\) and \(I \in \mathbb{N}\). Then,

(i) Up to a subsequence, there exists a smooth, closed, almost embedded hypersurface \(\Sigma_\infty\) with prescribed mean curvature \(h_\infty\), such that \(\Sigma_k \to \Sigma_\infty\) (possibly with integer multiplicity) in the varifold sense, and hence also in the Hausdorff distance by monotonicity formula.

(ii) There exists a finite set of points \(\mathcal{Y} \subset M\) with \(#\mathcal{Y} \leq I\), such that the convergence of \(\Sigma_k \to \Sigma_\infty\) is locally smooth and graphical on \(\Sigma_\infty \setminus \mathcal{Y}\).

(iii) If \(\Sigma_\infty \in S(g)\), then the multiplicity of \(\Sigma_\infty\) is 1, and \(\Sigma_\infty \in \mathcal{P}^{h_\infty}(\Lambda, I)\).

(iv) Assuming \(\Sigma_k \neq \Sigma_\infty\) eventually and \(h_k = h_\infty = h \in S(g)\) for all \(k\) such that every \(\Sigma \in \mathcal{P}^h\) is properly embedded with no self-touching, then \(\mathcal{Y} = \emptyset\), and the nullity of \(\Sigma_\infty\) with respect to \(\delta^2\mathcal{A}^h\) is \(\geq 1\).

(v) If \(h_\infty \equiv 0\), then the classical Morse index of \(\Sigma_\infty\) satisfies \(\text{index}(\Sigma_\infty) \leq I\) (without counting multiplicity).

Remark 2.7. One main goal of this result is to use PMC hypersurfaces with prescribing functions in \(S(g)\) to approximate PMC’s with prescribing functions lying in \(C^\infty(M) \setminus S(g)\). Therefore, it is natural to not assume \(h_\infty \in S(g)\). Indeed for some \(h \in C^\infty(M)\), a PMC \(\Sigma\) associated with \(h\) may have touching set containing a relative open subset \(W \subset \Sigma\), where \(h\) vanishes. For such hypersurfaces, \(\mathcal{A}^h\) is defined by viewing \(\Sigma\) as an Alexandrov immersed hypersurface, and so does the weak index.

Proof. The proof follows essentially the same way as [35] Theorem 2.3 once we use Theorem 2.5 to replace [35] Theorem 2.1; we will provide necessary modifications.

Part 1: We first have the following variant of [35] Lemma 3.1. Given any collection of \(I + 1\) pairwise disjoint open sets \(\{U_i\}_{i=1}^{I+1}\), we have that \(\Sigma_k\) (we drop the sub-index \(k\) in this paragraph) is weakly stable in \(U_i\) for some \(1 \leq i \leq I + 1\). Indeed, suppose this were false, then \(\Sigma = \partial \Omega\) is at least 1-unstable in each \(U_i\), hence there exist \(c_i \in (0, 1)\) and \(\{F_i^j\}_{i \in [-1, 1]} \subset \text{Diff}(U_i)\) with \(F_{i, t} = (F_i^j)_{i=-1}^1\), such that \(-\frac{1}{c_i} \leq \frac{d^2}{dt^2}\mathcal{A}^h(F_i^j(\Omega)) \leq -c_i\). Now for \(v = (v_0, \ldots, v_{I+1}) \in B_{I+1}\), let \(F_v(x) = F_{i,v_i+1} \circ \cdots \circ F_{i,1}(x)\). Since \(\{U_i\}\) are pairwise disjoint, it is easy to see that \(c_0 = \min\{c_i\}\) and \(\{F_i\}\) give an \((I + 1)\)-unstable pair for \(\Sigma\), and hence is a contradiction.

This fact together with Theorem 2.5 imply that (up to a subsequence) \(\Sigma_k\) converges locally smoothly and graphically to an almost embedded hypersurface \(\Sigma_\infty\) of prescribed mean curvature \(h_\infty\) (possibly with integer multiplicity) away from at most \(I\) points, which we denote by \(\mathcal{Y}\). Since as varifolds \(\Sigma_k\) have uniformly bounded first variation, by Allard’s compactness theorem [11], \(\Sigma_k\) also converges as varifolds to an integral varifold represented by \(\Sigma_\infty\).

Now we prove that \(\Sigma_\infty\) extends smoothly as an almost embedded hypersurface across the singular points \(\mathcal{Y}\), i.e. \(\mathcal{Y}\) are removable. By the argument in [35] Claim 2, page 326, for each \(y_i \in \mathcal{Y}\), there exists some \(r_i > 0\) such that \(\Sigma_\infty\) is weakly stable in \(B_{r_i}(y_i) \setminus \{y_i\}\) in the following sense. Denote \(\Omega_\infty\) as the weak limit of \(\Omega_k\) as Caccioppoli sets where \(\Sigma_k = \partial \Omega_k\). The associated functional for \(\Sigma_\infty\) is \(\mathcal{A}^{h_\infty}(\Sigma_\infty) = \text{Area}(\Sigma_\infty) - \int_{\Omega_\infty} h_\infty dH^{n+1}\). Note that the touching set of \(\Sigma_\infty\) may contain an open subset \(W \subset \Sigma_\infty\) and hence \(\partial \Omega_\infty = \Sigma_\infty \setminus \{\text{touching set of } \Sigma_\infty\}\) may only be a proper subset of \(\Sigma_\infty\). Nevertheless, we say \(\Sigma_\infty\) is weakly stable, if for any \(X \in \mathcal{X}(B_{r_i}(y_i) \setminus \{y_i\})\) with the associated flow \(\{\phi_t^X : t \in [-\epsilon, \epsilon]\}\), \(\frac{d}{dt}\big|_{t=0}\mathcal{A}^{h_\infty}(\phi_t^X(\Sigma_\infty)) \geq 0\). Note that if this were not true for some \(X \in \mathcal{X}(B_{r_i}(y_i) \setminus \{y_i\})\), as \(\mathcal{A}^{h_k}(\phi_t^X(\Sigma_k))\) converges to \(\mathcal{A}^{h_\infty}(\phi_t^X(\Sigma_\infty))\) smoothly as functions of \(t\), then \(\Sigma_k\) is not weakly stable in \(B_{r_i}(y_i) \setminus \{y_i\}\) for \(k\) sufficiently large. Following [35] Claim 2,
Part 2: If \( h_\infty \in S(g) \), [52] Theorem 3.20] implies that \( \Sigma_\infty \) has multiplicity 1, and is a boundary of some open set \( \Omega_\infty \); (note that when \( h_\infty \in S(g) \), only case (2) of [52] Theorem 3.20] will happen). In fact, fix a point \( p \in \Sigma_\infty \) where \( \Sigma_\infty \) is properly embedded. If the limit \( \Sigma_\infty \) has multiplicity \( \geq 2 \), then for \( i \) sufficiently large and inside a neighborhood of \( p \), \( \Sigma_i \) consists of several sheets with normal pointing to the same side of \( \Sigma_\infty \), but this can not happen when \( \Sigma_i \) bounds a region \( \Omega_i \). We refer to the proof of [52] Theorem 3.20] for more details.

If \( \text{index}(\Sigma_\infty) > I \), then there exist \( c_0 \in (0, 1) \) and \( \{F_v : v \in \overline{B}^{I+1}\} \subset \text{Diff}(M) \) such that \( -\frac{1}{c_0} \text{Id} \leq D^2 A^{h_\infty}(F_v(\Omega_\infty)) \leq -c_0 \text{Id} \) for all \( v \in \overline{B}^{I+1} \). Since \( \Sigma_k = \partial \Omega_k \) converges to \( \Sigma_\infty \) smoothly away from finitely many points, we know that \( \Omega_k \) converges to \( \Omega_\infty \) in the \( F \)-topology as Caccioppoli sets, then the sequence \( v \to A^{h_k}(F_v(\Omega_k)) \) converges to \( v \to A^{h_\infty}(F_v(\Omega_\infty)) \) smoothly as functions on \( \overline{B}^{I+1} \). Therefore, for \( k \) large enough, \( -\frac{1}{c_0} \text{Id} \leq D^2 A^{h_k}(F_v(\Omega_k)) \leq -c_0 \text{Id} \), so \( \Sigma_k \) is \((I+1)\)-unstable, which is a contradiction. This finishes the proof of (iii).

Part 3: Assuming \( \Sigma_k \neq \Sigma_\infty \) eventually and \( h_k = h_\infty = h \in S(g) \) such that every element in \( \mathcal{P}^h \) is properly embedded, we know \( \mathcal{Y} = \emptyset \) by multiplicity 1 convergence and the Allard regularity theorem [1]. Next we will produce a Jacobi field for the second variation \( \delta^2 A^h \) along \( \Sigma_\infty \); this implies the nullity is \( \geq 1 \).

By [1.3], the Jacobi operator associated with \( \delta^2 A^h \) along a PMC \( \Sigma \in \mathcal{P}^h \) is

\[
L^h_{\Sigma} \varphi = -\Delta_\Sigma \varphi - (\text{Ric}^M(\nu, \nu) + |A^\Sigma|^2 + \partial_u h) \varphi.
\]

The smooth graphical convergence of \( \Sigma_k \to \Sigma \) implies that for \( k \) sufficiently large, \( \Sigma_k \) can be written as a graph \( u_k \) in the normal bundle of \( \Sigma_\infty \), and \( u_k \to 0 \) uniformly in smooth topology. Subtracting the mean curvature operators between \( \Sigma_k \) and \( \Sigma_\infty \), we get:

\[
h(x, u_k) - h(x, 0) = H_{\Sigma_k} - H_{\Sigma_\infty} = L_{\Sigma_\infty} u_k + O(u_k),
\]

where \( L_{\Sigma_\infty} u = -\Delta u - (\text{Ric}^M(\nu, \nu) + |A^\Sigma|^2) u \) is the Jacobi operator for second variation of area, and the second equation follows from [37] and [55] page 331]; (note that though the calculation in [55] page 331] is done assuming \( h \equiv 0 \), it does not depend on \( h \). The left hand side equals to \( \partial_u h(x, t(x) u_k) \cdot u_k \) by the mean value theorem. Let \( \tilde{u}_k = u_k / \|u_k\|_{L^2(\Sigma_\infty)} \) be the renormalizations, then standard elliptic estimates imply that \( \tilde{u}_k \) converges smoothly to a nontrivial \( \varphi \in C^\infty(\Sigma_\infty) \) such that \( \partial_u h \cdot \varphi = L_{\Sigma_\infty} \varphi \). This is the same as \( L^h_{\Sigma_\infty} \varphi = 0 \), so we finish proving (iv).

Part 4: Assuming \( h_\infty \equiv 0 \), then \( \Sigma_\infty \) is an embedded minimal hypersurface. Assume without loss of generality that \( \Sigma_\infty \) is connected with multiplicity \( m \in \mathbb{N} \). Suppose the Morse index \( \text{index}(\Sigma_\infty) \geq I + 1 \), then by similar argument as in (iii), we can deduce a contradiction. In particular, by [27] Proposition 4.3], there exist \( c_0 \in (0, 1) \) and \( \{F_v : v \in \overline{B}^{I+1}\} \subset \text{Diff}(M) \) such that \( -\frac{1}{c_0} \text{Id} \leq D^2 \text{Area}(F_v(\Sigma_\infty)) \leq -c_0 \text{Id} \) for all \( v \in \overline{B}^{I+1} \). Since \( \Sigma_k \) converges to \( m \cdot \Sigma_\infty \) as varifolds, and since \( h_k \to 0 \) uniformly, we know that \( \mathcal{A}^{h_k}(F_v(\Omega_k)) \) converges to \( m \cdot \text{Area}(F_v(\Sigma_\infty)) \) smoothly as functions on \( \overline{B}^{I+1} \). Therefore, for \( k \) large enough, \( \Omega_k \) is \((I+1)\)-unstable, which is a contradiction. So we finish proving (v). \( \square \)
There is also a theorem analogous to the above one in the setting of changing ambient metrics on $M$; see \cite{35} Theorem A.6 for a similar result for minimal hypersurfaces. The proof proceeds the same way when one realizes that the constant $C_1$ in Theorem \ref{thm:2.3} depends only on the $\|g\|_{C^4}$ when $g$ is allowed to change.

**Theorem 2.8.** Let $M^{n+1}$ be a closed manifold of dimension $3 \leq (n + 1) \leq 7$, and $\{g_k\}_{k \in \mathbb{N}}$ be a sequence of metrics on $M$ that converges smoothly to some limit metric $g$. Let $\{h_k\}_{k \in \mathbb{N}}$ be a sequence of smooth functions with $h_k \in \mathcal{S}(g_k)$ that converges smoothly to some limit $h_\infty \in C^\infty(M)$. Let $\{\Sigma_k\}_{k \in \mathbb{N}}$ be a sequence of hypersurfaces with $\Sigma_k \in \mathcal{P}^{h_k}(\Lambda, I)$ for some fixed $\Lambda > 0$ and $I \in \mathbb{N}$. Then there exists a smooth, closed, almost embedded hypersurface $\Sigma_\infty$ with prescribing mean curvature $h_\infty$, such that all properties (i)(ii)(iii) in the above theorem are satisfied.

3. Morse index upper bound

In this part, we will establish Morse index upper bound for min-max PMC hypersurfaces obtained in Theorem \ref{thm:1.7}. We will follow closely the strategy of Marques-Neves \cite{27} Theorem 1.2, where they proved Morse index upper bound for min-max minimal hypersurfaces. Recall that the Morse index of an almost embedded PMC hypersurface $\Sigma$ is given in Definition \ref{def:2.3}.

**Theorem 3.1.** Let $(M^{n+1}, g)$ be a closed Riemannian manifold of dimension $3 \leq (n + 1) \leq 7$, and $h \in \mathcal{S}(g)$ which satisfies $\int_M h \geq 0$. Given a $k$-dimensional cubical complex $X$ and a subcomplex $Z \subset X$, let $\Phi_0 : X \to \mathcal{C}(M)$ be a map continuous in the $\mathcal{F}$-topology, and $\Pi$ be the associated $(X, Z)$-homotopy class of $\Phi_0$. Suppose

$$\text{(3.1)} \quad \mathbf{L}^h(\Pi) > \max_{x \in Z} \mathcal{A}^h(\Phi_0(x)).$$

Then there exists a nontrivial, smooth, closed, almost embedded hypersurface $\Sigma^N \subset M$, such that

- $\Sigma$ is the boundary of some $\Omega \in \mathcal{C}(M)$ where its mean curvature with respect to the unit outer normal of $\Omega$ is $h$, i.e.,
  $$H_{\Sigma} = h|_{\Sigma},$$
- $\mathcal{A}^h(\Omega) = \mathbf{L}^h(\Pi),$
- $\text{index}(\Sigma) \leq k.$

3.1. **Preliminary lemmas.** Let $h \in \mathcal{S}(g)$. Assume that $\Sigma_0 = \partial \Omega_0 \in \mathcal{P}^h$ is $k$-unstable in an $\epsilon$-neighborhood, $\epsilon > 0$. Let $\{F_v\}_{v \in \mathcal{B}^k}$ be the associated smooth family given in Definition \ref{def:2.1}.

The first lemma is a counterpart of \cite{27} Lemma 4.4].

**Lemma 3.2.** There exists $\bar{\eta} = \bar{\eta}(\epsilon, \Sigma_0, \{F_v\}) > 0$, such that if $\Omega \in \mathcal{C}(M)$ with $\mathcal{F}(\Omega, \Omega_0) \geq \epsilon$ satisfies

$$\mathcal{A}^h(F_v(\Omega)) \leq \mathcal{A}^h(\Omega) + \bar{\eta}$$

for some $v \in \mathcal{B}^k$, then $\mathcal{F}(F_v(\Omega), \Omega_0) \geq 2\bar{\eta}$.

**Proof.** Assume by contradiction that there exist $\Omega_i$, $\mathcal{F}(\Omega_i, \Omega_0) \geq \epsilon$ satisfying

$$\mathcal{A}^h(F_{v_i}(\Omega_i)) \leq \mathcal{A}^h(\Omega_i) + \frac{1}{i}$$

for some $v_i \in \mathcal{B}^k$, but $\mathcal{F}(F_{v_i}(\Omega_i), \Omega_0) \leq \frac{2}{i}$.

Denote $v = \lim_i v_i$, and pass to the limit as $i \to \infty$, then $\Omega_i \to F_v(\Omega_0)$ in $\mathcal{F}$-metric, and $\mathcal{A}^h(\Omega_0) \leq \mathcal{A}^h(F_{-v}(\Omega_0))$, which implies that $v = 0$; hence $\Omega_i \to \Omega_0$ in the $\mathcal{F}$-metric, which is a contradiction. \qed
For each \( \Omega \in \mathcal{B}_2^k(\Omega_0) \), consider the \( \phi \)-parameter flow \( \{ \phi^\Omega(\cdot, t) : t \geq 0 \} \subset \text{Diff}(\mathcal{B}^k) \) generated by the vector field

\[
u \rightarrow -(1 - |\nu|^2) \nabla A^h_\Omega(\nu), \quad \nu \in \mathcal{B}^k.
\]

When \( \nu \in \mathcal{B}^k \) is fixed, the function \( t \rightarrow A^h_\Omega(\phi^\Omega(\nu, t)) \) is non-increasing.

The following lemma is a variant of [27, Lemma 4.5], and the proof is recorded in Appendix C.

**Lemma 3.3.** For any \( \delta < 1/4 \) there exists \( T = T(\delta, \epsilon, \Omega_0, \{ F_v \}, c_0) \geq 0 \) such that for any \( \Omega \in \mathcal{B}_2^k(\Omega_0) \) and \( v \in \mathcal{B}^k \) with \( |v - m(\Omega)| \geq \delta \), we have

\[
A^h_\Omega(\phi^\Omega(v, T)) < A^h_\Omega(0) - \frac{c_0}{10} \quad \text{and} \quad |\phi^\Omega(v, T)| > \frac{c_0}{4}.
\]

### 3.2. Deformation theorem.

Taking a min-max sequence \( \{ \Phi_i \}_{i \in \mathbb{N}} \), we will prove a deformation theorem as an adaption of [27, Theorem 5.1] to our setting. Recall that \( \mathcal{P}^h \) denotes the class of smooth, closed, almost embedded hypersurface \( \Sigma \subset M \) represented as boundary \( \Sigma = \partial \Omega \), and of prescribed mean curvature \( h \).

Fix a \( \sigma > 0 \) such that \( \mathbf{L}^h - \sup_{x \in Z} \mathcal{A}^h(\Phi_0(x)) > 2\sigma \). Denote

\[
X_{i,\sigma} = \{ x \in X : \text{such that } \mathcal{A}^h(\Phi_i(x)) \geq \mathbf{L}^h - \sigma \}.
\]

Note that when \( i \) is sufficiently large, \( X_{i,\sigma} \subset X \setminus Z \).

Now we present the deformation theorem, and the proof follows closely that of [27, Theorem 5.1].

Given two subsets \( A, B \subset C(M) \), we denote

\[
F(A, B) := \inf \{ F(\Omega_A, \Omega_B) : \Omega_A \in A, \Omega_B \in B \}.
\]

**Theorem 3.4.** Suppose that

(a) \( \Sigma = \partial \Omega \in \mathcal{P}^h \) is \((k+1)\)-unstable;

(b) \( K \subset C(M) \) is a subset, so that \( F(\{ \Omega \}, K) > 0 \) and \( F(\Phi_i(X_{i,\sigma}), K) > 0 \) for all \( i \geq i_0 \);

(c) \( \mathcal{A}^h(\Omega) = \mathbf{L}^h \).

Then there exists \( \bar{\epsilon} > 0 \), \( j_0 \in \mathbb{N} \), and another sequence \( \{ \Psi_i \}_{i \in \mathbb{N}} \), \( \Psi_i : X \to (C(M), \mathbf{F}) \), so that

(i) \( \Psi_i \) is homotopic to \( \Phi_i \) in the \( \mathbf{F} \)-topology for all \( i \in \mathbb{N} \) and \( \Psi_i|_Z = \Phi_i|_Z \) for all \( i \geq j_0 \);

(ii) \( \mathbf{L}^h(\{ \Psi_i \}) \leq \mathbf{L}^h(\{ \Phi_i \}) \);

(iii) \( F(\Psi_i(X_{i,\sigma}), \mathcal{B}^F_{\bar{\epsilon}}(\Omega) \cup K) > 0 \) for all \( i \geq j_0 \).

**Proof.** Denote \( d = F(\{ \Omega \}, K) > 0 \).

By (a), \( \Sigma \) is \((k+1)\)-unstable in some \( \epsilon \)-neighborhood. Let \( \{ F_v \}_{v \in \mathcal{P}^{k+1}} \), \( c_0 \) be the associated family and constant as in Definition [27]. By possibly changing \( \epsilon \), \( \{ F_v \}, c_0 \), we can assume that

\[
(3.2) \quad \inf \{ F(F_v(\Omega'), K) : v \in \mathcal{B}^{k+1} \} > \frac{d}{2}, \quad \text{for all } \Omega' \in \mathcal{B}^F_{2\epsilon}(\Omega).
\]

Let \( X(k_i) \) be a sufficiently fine subdivision of \( X \) so that \( F(\Phi_i(x), \Phi_i(y)) < \delta_i \) for any \( x, y \) belonging to the same cell in \( X(k_i) \) with \( \delta_i = \min\{2^{-(i+k+2)}, \epsilon/4\} \). We can also assume that

\[
|m(\Phi_i(x)) - m(\Phi_i(y))| < \delta_i
\]

for any \( x, y \) with \( F(\Phi_i(x), \Omega) \leq 2\epsilon \), \( F(\Phi_i(y), \Omega) \leq 2\epsilon \), and belonging to the same cell in \( X(k_i) \).

For \( \eta > 0 \), let \( U_{i,\eta} \) be the union of all cells \( \sigma \in X(k_i) \) so that \( F(\Phi_i(x), \Omega) < \eta \) for all \( x \in \sigma \). Then \( U_{i,\eta} \) is a subcomplex of \( X(k_i) \). If a cell \( \beta \not\in U_{i,\eta} \), then \( F(\Phi_i(x'), \Omega) \geq \eta \) for some \( x' \in \beta \). Therefore, \( F(\Phi_i(x), \Omega) \geq \eta - \delta_i \) for all \( x \in \beta \). By (c) (after possibly shrinking \( \epsilon \)), we can assume

\[
U_{i,2\epsilon} \subset X_{i,\sigma}.
\]
For each $i \in \mathbb{N}$ and $x \in U_{i,2\varepsilon}$, we simply denote $A_{i,x}^h = A_{\phi_i(x)}^h$, $m_i(x) = m(\Phi_i(x))$ and $\phi_{i,x} = \phi^{\Phi_i(x)}$. The function $m_i : U_{i,2\varepsilon} \to \overline{B}^{k+1}$ is continuous, and the two families $\{A_{i,x}^h\}_{x \in U_{i,2\varepsilon}}$, $\{\phi_{i,x}\}_{x \in U_{i,2\varepsilon}}$ are continuous in $x$. Following [27, 5.1] we can define a continuous map

$$\hat{H}_i : U_{i,2\varepsilon} \times [0, 1] \to B_{\frac{k+1}{2}}^1(0),$$

so that $\hat{H}_i(x, 0) = 0$ for all $x \in U_{i,2\varepsilon}$ and

$$\inf_{x \in U_{i,2\varepsilon}} |\hat{H}_i(x, 1) - m_i(x)| \geq \eta_i > 0, \text{ for some } \eta_i > 0.$$  

The construction here is the same so we omit details. The crucial ingredient is the fact that $U_{i,2\varepsilon}$ has dimension less than or equal to $k$ while the image set $\overline{B}^{k+1}$ has dimension $k+1$.

Let $c : [0, \infty) \to [0, 1]$ be a cutoff function which is non-increasing, equals to 1 in a neighborhood of $[0, 3\varepsilon/2]$, and 0 in a neighborhood of $[7\varepsilon/4, +\infty)$. For $y \notin U_{i,2\varepsilon}$, $F(\Phi_i(y), \Omega) \geq 2\varepsilon - \delta_i \geq 7\varepsilon/4$. Hence

$$c(F(\Phi_i(y), \Omega)) = 0, \text{ for all } y \notin U_{i,2\varepsilon}.$$  

Consider the map $H_i : X \times [0, 1] \to B_{2^{-i}}^{k+1}(0)$ defined as

$$H_i(x, t) = \hat{H}_i(x, c(F(\Phi_i(x), \Omega))t), \text{ if } x \in U_{i,2\varepsilon}$$

and

$$H_i(x, t) = 0, \text{ if } x \in X \setminus U_{i,2\varepsilon}.$$  

Then $H_i$ is continuous.

With $\eta_i$ as given in (3.3), let $T_i = T(\eta_i, \varepsilon, \Omega, \{F_v\}, c_0) \geq 0$ be given by Lemma 3.3. Now we set $D_i : X \to \overline{B}^{k+1}$ such that

$$D_i(x) = \phi_{i,x}(H_i(x, 1), c(F(\Phi_i(x), \Omega))T_i), \text{ if } x \in U_{i,2\varepsilon}$$

and

$$D_i(x) = 0, \text{ if } x \in X \setminus U_{i,2\varepsilon}.$$  

Then $D_i$ is continuous.

Define

$$\Psi_i : X \to C(M), \quad \Psi_i(x) = F_{D_i(x)}(\Phi_i(x)).$$

In particular,

$$\Psi_i(x) = \Phi_i(x), \text{ if } x \in X \setminus U_{i,2\varepsilon}.$$  

Hence $\Psi_i|_Z = \Phi_i|_Z$ for $i$ sufficiently large.

Note that the map $D_i$ is homotopic to the zero map in $\overline{B}^{k+1}$, so $\Psi_i$ is homotopic to $\Phi_i$ in the $F$-topology for all $i \in \mathbb{N}$. Up to here, we proved (i).

**Claim 1:** $L^h(\{\Psi_i\}_{i \in \mathbb{N}}) \leq L^h$.

By the non-increasing property of $t \to A_{i,x}^h(\phi_{i,x}(u, t))$, we have that for all $x \in X$,

$$A^h(\Psi_i(x)) \leq A^h(F_{H_i(x, 1)}(\Phi_i(x))).$$

Using the fact that $H_i(x, 1) \in B_{\frac{k+1}{2}}^1(0)$ for all $x \in X$ and that $\|F_v - Id\|_{C^2} \to 0$ uniformly as $v \to 0$, we have that

$$\lim_{i \to \infty} \sup_{x \in X} |A^h(\Phi_i(x)) - A^h(F_{H_i(x, 1)}(\Phi_i(x)))| = 0,$$

and this finishes proving Claim 1.
Lemma 3.5. \( \text{the choice of a metric; (see [52, Proposition 3.8])}. \)

There are three cases. If \( x \in X \setminus U_{1,2\epsilon} \), then \( \Psi_i(x) = \Phi_i(x) \) and so \( F(\Psi_i(x), \Omega) \geq \frac{7\epsilon}{4} \).

If \( x \in U_{1,2\epsilon} \setminus U_{i,5\epsilon} \), then \( F(\Phi_i(x), \Omega) \geq \epsilon \). The non-increasing property of \( t \to A_v^h(\phi_i(x)(u,t)) \) implies

\[
A_v^h(\Psi_i(x)) = A_v^h(F_{D_i}(\Phi_i(x))) \leq A_v^h(F_{H_i(x,1)}(\Phi_i(x))).
\]

From (3.4), we have that for \( i \) large enough,

\[
A_v^h(F_{H_i(x,1)}(\Phi_i(x))) \leq A_v^h(\Phi_i(x)) + \tilde{\eta}, \quad \text{for all } x \in X,
\]

where \( \tilde{\eta} = \eta(\epsilon, \Omega, \{ F_v \}) > 0 \) is given by Lemma 3.2. Combining the two inequalities with Lemma 3.2 applied to \( \Phi_i(x), v = D_i(x), \) we get \( F(\Psi_i(x), \Omega) \geq 2\tilde{\eta} \).

Finally when \( x \in U_{i,5\epsilon} \), \( c(F(\Phi_i(x), \Omega)) = 1 \). Hence by Lemma 3.3 (with \( \delta = \eta_i, \Omega = \Phi_i(x), v = H_i(x,1) \)) we have

\[
A_v^h(\Psi_i(x)) = A_v^h(\phi_i(x)(H_i(x,1),T_i)) < A_v^h(0) - \frac{c_0}{10} = A_v^h(\Phi_i(x)) - \frac{c_0}{10}.
\]

Note that there exists \( \tilde{\gamma} = \tilde{\gamma}(\Omega, c_0) \) so that

\[
A_v^h(\Omega') \leq A_v^h(\Omega) - \frac{c_0}{20} \implies F(\Omega', \Omega) \geq 2\tilde{\gamma}.
\]

By assumption (c), we can choose \( i \) sufficiently large so that

\[
\sup_{x \in X} A_v^h(\Phi_i(x)) \leq A_v^h(\Omega) + \frac{c_0}{20}.
\]

So

\[
A_v^h(\Psi_i(x)) \leq A_v^h(\Omega) - \frac{c_0}{20}.
\]

This implies that \( F(\Psi_i(x), \Omega) \geq 2\tilde{\gamma} \), and hence ends the proof of Claim 2.

Claim 3: For all \( i, F(\Psi_i(X_{i,\sigma}), K) > 0 \).

If \( x \in X_{i,\sigma} \setminus U_{1,2\epsilon} \), then \( \Psi_i(x) = \Phi_i(x) \) and so \( F(\Psi_i(X_{i,\sigma} \setminus U_{i,2\epsilon}), K) > 0 \). If \( x \in U_{i,2\epsilon} \), then \( F(\Phi_i(x), \Omega) \leq 2\epsilon \), and by (3.2) we have \( F(\Psi_i(x), K) \geq \frac{4\epsilon}{2} \). So we finish proving Claim 3, and hence the theorem. \( \square \)

3.3. Proof of Morse index upper bound. Let \( M^{n+1} \) be a closed manifold of dimension \( 3 \leq (n + 1) \leq 7 \). A pair \((g, h)\) consisting of a Riemannian metric \( g \) and a smooth function \( h \in C^\infty(M) \) is called a good pair, if

- \( h \in S(g) \), i.e. \( h \) is Morse and the zero set \( \{ h = 0 \} \) is a smooth embedded hypersurface in \( M \) with mean curvature \( H \) vanishing to at most finite order, and
- \( g \) is bumpy for \( P^h \), i.e. every \( \Sigma \in P^h \) is properly embedded (no self-touching), and is nondegenerate (nullity equal to zero).

Denote \( S_0 \) as the class of smooth functions \( h \in C^\infty(M) \) such that \( h \) is Morse and the zero set \( \{ h = 0 \} \) is a smooth embedded hypersurface. \( S_0 \) is open and dense in \( C^\infty(M) \), and is independent of the choice of a metric; (see [52, Proposition 3.8]).

Lemma 3.5. Given \( h \in S_0 \), the set of Riemannian metrics \( g \) on \( M \) with \((g, h)\) as a good pair is generic in the Baire sense.
Proof. By the proof of [52] Proposition 3.8, we know that the set of metrics $g$ under which \{h = 0\} has mean curvature vanishing to at most finite order is an open and dense subset. In particular, openness follows as small smooth perturbations of $g$ will bound the order of vanishing of $H_{\{h=0\}}$. To show denseness, note that it is proved in [52] Proposition 3.8 for any $h \in S_0$ and any metric $g$, one can first perturb $g$ slightly so that \{h = 0\} is not a minimal hypersurface, and then there exists a flow $\{F_t : t \in (-\epsilon, \epsilon)\} \subset \text{Diff}(M)$ supported near \{h = 0\}, such that the zero set of $h \circ (F_t)^{-1}$ has mean curvature vanishing to at most finite order for $t > 0$. That is to say the zero set \{h = 0\} satisfies the requirement for the pull-back metrics $F_t^*g$.

In a series of celebrated papers [42][44][45], White proved that for a fixed $h \in S_0$, the set of metrics under which all closed, simple immersed CMC hypersurfaces are non-degenerate is generic, and the proof is the same in a smooth neighborhood of an arbitrary pair $(g, h)$ when $h \in S(g)$, hence the result follows as the set of $g$ where $h \in S(g)$ is open and dense. In [45] Theorem 33, White further proved self-transverse property for a generic set of metrics. Our almost embedded hypersurfaces are simple immersed. So for such generic metrics, almost embedded PMC’s are properly embedded.

To finish the proof, we take the intersection of the two generic sets of metrics, which is still generic in the Baire sense. □

The following theorem is a counterpart of [27] Theorem 6.1, and the proof follows closely. We remark that by Theorem 2.6(iv), if $(g, h)$ is a good pair, then there are only finitely many elements in $\mathcal{P}^h(A, I)$.

Theorem 3.6. Assume that $(g, h)$ is a good pair and let $\{\Phi_i\}_{i \in \mathbb{N}}$ be a min-max sequence of $\Pi$ such that $L^h(\{\Phi_i\}_{i \in \mathbb{N}}) = L^h(\Pi) = L^h$ and (3.1) is satisfied.

There exists a smooth, closed, properly embedded hypersurface $\Sigma = \partial \Omega \in C(\{\Phi_i\}_{i \in \mathbb{N}})$ such that $\Sigma \in \mathcal{P}^h$ with

$$L^h(\Pi) = A^h(\Omega), \text{ and } \text{index}(\Sigma) \leq k.$$  

Proof. By the finiteness remark above, it suffices to show that, for every $r > 0$, there is a $\tilde{\Sigma} = \partial \tilde{\Omega} \in \mathcal{P}^h$ such that $F(\Sigma, C(\{\Phi_i\}_{i \in \mathbb{N}})) < r$,

$$L^h(\Pi) = A^h(\tilde{\Omega}), \text{ and } \text{index}(\tilde{\Sigma}) \leq k.$$  

Denote by $\mathcal{W}$ the set of all $\tilde{\Sigma} = \partial \tilde{\Omega} \in \mathcal{P}^h$ with $A^h(\tilde{\Omega}) = L^h$ and by $\mathcal{W}(r)$ the set

$$\{\Sigma \in \mathcal{W} : F(\Sigma, C(\{\Phi_i\}_{i \in \mathbb{N}})) \geq r\}.$$  

Lemma 3.7. There exist $i_0 \in \mathbb{N}$ and $\tilde{\epsilon}_0 > 0$ such that $F(\Phi_i(X), \mathcal{W}(r)) > \tilde{\epsilon}_0$ for all $i \geq i_0$.

Proof. Suppose by contradiction for some subsequence $\{j\} \subset \{i\}$, $x_j \in X$, $\tilde{\Sigma}_j = \partial \tilde{\Omega}_j \in \mathcal{W}(r)$ so that

$$\lim_{j \to \infty} F(\Phi_j(x_j), \tilde{\Sigma}_j) = 0.$$  

Since $A^h(\tilde{\Omega}_j) \equiv L^h$, we have $\lim_{j \to \infty} A^h(\Phi_j(x_j)) = L^h$. Hence a subsequence $|\partial \Phi_j(x_j)|$ will converge as varifolds to some $V \in C(\{\Phi_i\}_{i \in \mathbb{N}})$, which is a contradiction to $F(|\partial \tilde{\Omega}_j|, C(\{\Phi_i\}_{i \in \mathbb{N}})) \geq r$. □

Denote $\mathcal{W}^{k+1}$ as the collection of elements in $\mathcal{W}$ with index greater than or equal to $(k + 1)$. As $(g, h)$ is a good pair, this set is countable by the remark above the theorem, and we can write
\[ \mathcal{W}^{k+1} \backslash \overline{B}_{\varepsilon_0}^{F}(\mathcal{W}(r)) = \{ \Sigma_1, \Sigma_2, \ldots \}, \text{ where } \Sigma_i = \partial \Omega_i. \] Note that by possibly perturbing \( \varepsilon_0 \), we can make sure \( \mathcal{W}^{k+1} \cap \partial \overline{B}_{\varepsilon_0}^{F}(\mathcal{W}(r)) = \emptyset \).

Using Theorem 3.4 (we can take \( X_{i,\sigma} \) to be \( X \)) with \( K = \overline{B}_{\varepsilon_0}^{F}(\mathcal{W}(r)) \) and \( \Sigma = \Sigma_1 \), we find \( \varepsilon_1 > 0 \), \( i_1 \in \mathbb{N} \), and \( \{ \Phi_{i_1}^1 \}_{i_1 \in \mathbb{N}} \) so that

- \( \Phi_i^1 \) is homotopic to \( \Phi_i \) in the \( F \)-topology for all \( i \in \mathbb{N} \) and \( \Phi_i^1|_Z = \Phi_i|_Z \) for \( i \geq i_1 \);
- \( L^h(\{ \Phi_i^1 \}_{i_1 \in \mathbb{N}}) \leq L^h; \)
- \( F(\Phi_i^1(X), \overline{B}_{\varepsilon_1}^{F}(\Omega_1(1)) \cup \overline{B}_{\varepsilon_0}^{F}(\mathcal{W}(r))) > 0 \) for \( i \geq i_1 \).
- no \( \Omega_j \) belongs to \( \partial \overline{B}_{\varepsilon_1}^{F}(\Omega_1) \).

We consider \( \Sigma_2 \) now. If \( \Omega_2 \not\in \overline{B}_{\varepsilon_1}^{F}(\Omega_1) \), we apply Theorem 3.4 with \( K = \overline{B}_{\varepsilon_1}^{F}(\Omega_1(1)) \cup \overline{B}_{\varepsilon_0}^{F}(\mathcal{W}(r)) \), \( \Sigma = \Sigma_2 \), and find \( \varepsilon_2 > 0 \), \( i_2 \in \mathbb{N} \), and \( \{ \Phi_{i_2}^2 \}_{i_2 \in \mathbb{N}} \) so that

- \( \Phi_i^2 \) is homotopic to \( \Phi_i \) in the \( F \)-topology for all \( i \in \mathbb{N} \) and \( \Phi_i^2|_Z = \Phi_i|_Z \) for \( i \geq i_2 \);
- \( L^h(\{ \Phi_i^2 \}_{i_2 \in \mathbb{N}}) \leq L^h; \)
- \( F(\Phi_i^2(X), \overline{B}_{\varepsilon_2}^{F}(\Omega_2(1)) \cup \overline{B}_{\varepsilon_0}^{F}(\mathcal{W}(r))) > 0 \) for \( i \geq i_2 \);
- no \( \Omega_j \) belongs to \( \partial \overline{B}_{\varepsilon_1}^{F}(\Omega_1(1)) \cup \partial \overline{B}_{\varepsilon_2}^{F}(\Omega_2) \).

If \( F(\Omega_2, \Omega_1) < \varepsilon_1 \), we skip it and repeat the construction with \( \Sigma_3 \).

By induction there are two possibilities. We can find for all \( l \in \mathbb{N} \) a sequence \( \{ \Phi_i^l \}_{i_l \in \mathbb{N}}, \varepsilon_l > 0, i_l \in \mathbb{N} \), and \( \Sigma_{j_l} \in \mathcal{W}^{k+1} \backslash \overline{B}_{\varepsilon_0}^{F}(\mathcal{W}(r)) \) for some subsequences \( \{ j_l \} \subset \mathbb{N} \) so that

- \( \Phi_i^l \) is homotopic to \( \Phi_i \) in the \( F \)-topology for all \( i \in \mathbb{N} \) and \( \Phi_i^l|_Z = \Phi_i|_Z \) for \( i \geq i_l \);
- \( L^h(\{ \Phi_i^l \}_{i_l \in \mathbb{N}}) \leq L^h; \)
- \( F(\Phi_i^l(X), \overline{B}_{\varepsilon}^{F}(\Omega_{j_l}(1)) \cup \overline{B}_{\varepsilon_0}^{F}(\mathcal{W}(r))) > 0 \) for \( i \geq i_l \);
- \( \Omega_1, \ldots, \Omega_l \) \( \cup \Omega_{l+1, \ldots, \Omega_l} \subset \overline{B}_{\varepsilon}^{F}(\Omega_{j_l}); \)
- no \( \Omega_j \) belongs to \( \partial \overline{B}_{\varepsilon_l}^{F}(\Omega_{j_l}) \) for all \( q = 1, \ldots, l \).

Or the process ends in finitely many steps. That means we can find some \( m \in \mathbb{N} \), a sequence \( \{ \Phi_i^m \}_{i_m \in \mathbb{N}}, \varepsilon_1, \ldots, \varepsilon_m > 0, i_m \in \mathbb{N} \), and \( \Sigma_{j_1}, \ldots, \Sigma_{j_m} \in \mathcal{W}^{k+1} \backslash \overline{B}_{\varepsilon_0}^{F}(\mathcal{W}(r)) \) so that

- \( \Phi_i^m \) is homotopic to \( \Phi_i \) in the \( F \)-topology for all \( i \in \mathbb{N} \) and \( \Phi_i^m|_Z = \Phi_i|_Z \) for \( i \geq i_m \);
- \( L^h(\{ \Phi_i^m \}_{i_m \in \mathbb{N}}) \leq L^h; \)
- \( F(\Phi_i^m(X), \overline{B}_{\varepsilon}^{F}(\Omega_{j_m}(1)) \cup \overline{B}_{\varepsilon_0}^{F}(\mathcal{W}(r))) > 0 \) for \( i \geq i_m \);
- \( \Omega_{j_q} : j_q \geq 1 \) \( \cup \Omega_{q=1, \ldots, q_m} \overline{B}_{\varepsilon_l}^{F}(\Omega_{j_q}); \)

In the first case we choose an increasing sequence \( p_l \geq i_l \) so that

\[ \sup_{x \in X} A^h(\Phi_{p_l}^l) \leq L^h + \frac{1}{l}, \]

and set \( \Psi_l = \Phi_{p_l}^l \). In the second case we set \( p_l = l \) and \( \Psi_l = \Phi_l^m \). The sequence \( \{ \Psi_l \}_{l \in \mathbb{N}} \) satisfies that

(i) \( \Psi_l \) is homotopic to \( \Phi_{p_l} \) in the \( F \)-topology, and \( \Psi_l|_Z = \Phi_{p_l}|_Z \) for all \( l \);
(ii) \( L^h(\{ \Psi_l \}_{l \in \mathbb{N}}) \leq L^h; \)
(iii) given any subsequence \( \{ l_j \} \subset \{ l \} \), \( x_j \in X \), if \( \lim_{j \to \infty} A^h(\Psi_{l_j}(x_j)) = L^h \), then \( \{ \Psi_{l_j}(x_j) \}_{j \in \mathbb{N}} \) does not converge in \( F \)-topology to any element in \( \mathcal{W}^{k+1} \cup \mathcal{W}(r) \).

The Min-max Theorem applied to \( \{ \Psi_l \}_{l \in \mathbb{N}} \) implies that \( \mathcal{W} \backslash (\mathcal{W}^{k+1} \cup \mathcal{W}(r)) \) is not empty and this proves the theorem. \( \Box \)
Now we can use the previous theorem and the Compactness Theorem 2.8 to prove Theorem 3.1.

**Proof of Theorem 3.1.** Given \((g,h)\) as in the theorem, then \(h \in S(g) \subset S_0\). By Lemma 3.5, there exists a sequence of metrics \(\{g_j\}_{j \in \mathbb{N}}\) converging smoothly to \(g\) such that \((g_j,h)\) is a good pair for all \(j \in \mathbb{N}\). If \(L^h_j = L^h_j(\Pi, g_j)\) is the \(h\)-width of \(\Pi\) with respect to \(g_j\), then the sequence \(\{L^h_j\}_{j \in \mathbb{N}}\) tends to the \(h\)-width \(L^h(\Pi, g)\) with respect to \(g\), and for \(j\) large enough (3.1) is satisfied with \(g_j\) in place of \(g\). For each \(j\) large enough, the previous theorem gives a properly embedded closed hypersurface \(\Sigma_j = \partial \Omega_j \in \mathcal{P}^h\) with \(\mathcal{A}^h(\Omega_j) = L^h_j\) and \(\text{index}(\Sigma_j) \leq k\) (with respect to \(g_j\)). Let \(\Sigma_\infty = \partial \Omega_\infty\) be the limit of \(\{\Sigma_j\}_{j \in \mathbb{N}}\) given in Theorem 2.8 then the locally smooth convergence implies that \(\mathcal{A}^h(\Omega_\infty) = L^h(\Pi, g)\) and \(\text{index}(\Sigma_\infty) \leq k\).

4. **Min-max hypersurfaces associated with sweepouts of boundaries have multiplicity one in a bumpy metric**

We present our first multiplicity one result. In particular, we will prove that the min-max minimal hypersurfaces associated with sweepouts of boundaries of Caccioppoli sets are two-sided and have multiplicity one, and we will prove that the limit minimal hypersurfaces are two-sided with multiplicity one, and we will prove that the limit minimal hypersurfaces are two-sided with multiplicity one by choosing the right pre-

Recall that a Riemannian metric \(g\) is said to be **bumpy** if every smooth closed immersed minimal hypersurface is non-degenerate. White proved that the set of bumpy metrics is generic in the Baire sense [42, 44].

**Theorem 4.1** (Multiplicity one theorem for sweepouts of boundaries). Let \((M^{n+1}, g)\) be a closed Riemannian manifold of dimension \(3 \leq (n + 1) \leq 7\). Let \(X\) be a \(k\)-dimensional cubical complex and \(Z \subset X\) be a subcomplex, and \(\Phi_0 : X \to \mathcal{C}(M)\) be a map continuous in the \(\mathcal{F}\)-topology. Let \(\Pi\) be the associated \((X, Z)\)-homotopy class of \(\Phi_0\). Assume that

\[
L(\Pi) > \max_{x \in Z} M(\partial \Phi_0(x)),
\]

where we let \(h \equiv 0\) in Section 1.7.

If \(g\) is a bumpy metric, then there exists a disjoint collection of smooth, connected, closed, embedded, two-sided, minimal hypersurfaces \(\Sigma = \bigcup_{i=1}^N \Sigma_i\), such that

\[
L(\Pi) = \sum_{i=1}^N \text{Area}(\Sigma_i), \quad \text{and} \quad \text{index}(\Sigma) = \sum_{i=1}^N \text{index}(\Sigma_i) \leq k.
\]

In particular, each component of \(\Sigma\) is two-sided and has exactly multiplicity one.

**Proof.** Pick a \(h \in S(g)\) with \(\int_M h \geq 0\) (to be fixed at the end), and \(\epsilon > 0\) small enough so that

\[
L(\Pi) - \max_{x \in Z} M(\partial \Phi_0(x)) > 2\epsilon \sup_M |h| \cdot \text{Vol}(M).
\]

Note that we have for each \(\Omega \in \mathcal{C}(M)\)

\[
M(\partial \Omega) - \epsilon \sup_M |h| \cdot \text{Vol}(M) \leq \mathcal{A}^h(\Omega) \leq M(\partial \Omega) + \epsilon \sup_M |h| \cdot \text{Vol}(M).
\]
The above two inequalities imply that if we consider the $\mathcal{A}^{\epsilon h}$-functional in place of the mass $\mathcal{M}$-functional for the $(X, Z)$-homotopy class $\Pi$, we have
\[
L^{\epsilon h}(\Pi) > \max_{x \in Z} \mathcal{A}^{\epsilon h}(\Phi_0(x)).
\]
Note that when $h \in \mathcal{S}(g)$, $\epsilon h$ also belongs to $\mathcal{S}(g)$. Therefore Theorem 3.1 applies to $\Pi$ and produces a nontrivial, smooth, closed, almost embedded hypersurface $\Sigma$, such that
- $\Sigma$ is the boundary for some $\Omega \in \mathcal{C}(M)$ where its mean curvature with respect to the unit outer normal $\nu$ (of $\Omega$) is $\epsilon \cdot h$, i.e.
\[
H_{\Sigma} = \epsilon \cdot h|_{\Sigma};
\]
- $\mathcal{A}^{\epsilon h}(\Omega) = L^{\epsilon h}(\Pi)$;
- $\text{index}(\Sigma) \leq k$.
We denote $L = L(\Pi)$ and $L^\epsilon = L^{\epsilon h}(\Pi)$. In the following, we proceed the proof by parts.

**Part 1:** $L^\epsilon \to L$ when $\epsilon \to 0$.

*Proof:* From (4.2), it is easy to see
\[
L - \epsilon \sup_M |h| \text{Vol}(M) \leq L^\epsilon \leq L + \epsilon \sup_M |h| \text{Vol}(M).
\]

**Part 2:** By Theorem 2.6, there exists a subsequence $\{\epsilon_k\} \to 0$, such that $\Sigma_k = \Sigma_{\epsilon_k}$ converges to some smooth, closed, embedded, minimal hypersurface $\Sigma_\infty$ (with integer multiplicity) in the sense of Theorem 2.6(i)(ii). We denote $\mathcal{Y}$ as the set of points where the convergence fails to be smooth. In particular, by (4.2) and Part 1 and Theorem 2.6(v), we have
\[
\mathcal{M}(\Sigma_\infty) = L, \quad \text{and} \quad \text{index}(\Sigma_\infty) \leq k.
\]
That is to say that $\Sigma_\infty$ is a min-max minimal hypersurface associated with $\Pi$.

Without loss of generality, we assume from Part 3 to Part 8 that $\Sigma_\infty$ has only one connected component. If $\Sigma_\infty$ is 2-sided with the multiplicity equal to one, then we are done; otherwise we may assume that the multiplicity $m > 1$ or $\Sigma_\infty$ is 1-sided.

**Part 3:** We first assume that $\Sigma_\infty$ is 2-sided. We will implicitly use exponential normal coordinates of $\Sigma_\infty$ with respect to one fixed unit normal of $\Sigma_\infty$. By the local, smooth graphical convergence $\Sigma_k \to \Sigma_\infty$ away from $\mathcal{Y}$, we know that there exists an exhaustion by compact domains $\{U_k \subset \Sigma_\infty \setminus \mathcal{Y}\}$ and some small $\delta > 0$, so that for $k$ large enough, $\Sigma_k \cap (U_k \times (-\delta, \delta))$ can be written as a set of $m$-normal graphs $\{u^1_k, \ldots, u^m_k : u^i_k \in C^\infty(U_k)\}$ over $U_k$, and such that
\[
u^1_k \leq \nu^2_k \leq \cdots \leq \nu^m_k, \quad \text{and} \quad u^i_k \to 0, \quad \text{in smooth topology as} \quad k \to \infty.
\]
Since $\Sigma_k$ is the boundary of some set $\Omega_k$, by the Constancy Theorem (applied to $\Omega_k$ in $U_k \times (-\delta, \delta)$), we know that the unit outer normal $\nu_k$ of $\Omega_k$ will alternate orientations along these graphs. In particular, if $u_k$ restricted to the graph of $u^i_k$ points upward (or downward), then $\nu_k$ restricted to the graph of $u^{i+1}_k$ will point downward (or upward).

**Part 4:** We first deal with an easier case: $m$ is an odd number. Hence $m \geq 3$. In this case $\nu_k$ restricted to the bottom $(u^1_k)$ and top $(u^m_k)$ sheets point to the same side of $\Sigma_\infty$, and without loss of
generality we may assume that $\nu_k$ points upward therein. That means:

$$H|_{\text{Graph}(u_k^m)}(x) = \epsilon_k h(x, u_k^m(x)), \quad H|_{\text{Graph}(u_k^1)}(x) = \epsilon_k h(x, u_k^1(x)), \quad \text{for } x \in U_k.$$ 

Here and in the following the sign convention is made so that $H|_{\text{Graph}(u)}$ is defined with respect to the upward pointing normal of $\text{Graph}(u)$, and hence the linearized operator is positively definite.

Note that since $\epsilon h \in \mathcal{S}(g)$, by the Strong Maximum Principle \cite{52} Lemma 3.12] (applied to two sheets of the same orientation), we know

$$u_k^m(x) - u_k^1(x) > 0, \quad \text{for all } x \in U_k.$$ 

Now by subtracting the above two equations, and using the fact $H|_{\text{Graph}(u_k^m)} - H|_{\text{Graph}(u_k^1)} = L_{\Sigma_\infty}(u_k^m - u_k^1) + o(u_k^m - u_k^1)$ (see \cite{55}, page 331] and part 3 in the proof of Theorem \cite{26}, we have

$$(4.3) \quad L_{\Sigma_\infty}(u_k^m - u_k^1) + o(u_k^m - u_k^1) = \epsilon_k \cdot \partial_n h(x, v_k(x)) \cdot (v_k^m(x) - v_k^1(x)),$$

where $v_k(x) = t(x)u_k^m(x) + (1 - t(x))u_k^1(x)$ for some $t(x) \in [0, 1]$.

Now it is a standard argument to produce a nontrivial positive Jacobi field on $\Sigma_\infty \setminus \mathcal{Y}$. Let us present the details for completeness. Write $h_k = u_k^m - u_k^1$, and pick a fixed point $p \in U_1$. Let $\tilde{h}_k = h_k/h_k(p)$, then $\tilde{h}_k(p) = 1$. By standard Harnack and elliptic estimates, $\tilde{h}_k$ will converge locally smoothly to a positive function $\varphi$ on any fixed $U \subset U_k$, and by a diagonalization process, we can extend $\varphi$ to $\Sigma_\infty \setminus \mathcal{Y}$, and such that

$$L_{\Sigma_\infty} \varphi = 0, \quad \text{outside } \mathcal{Y}.$$ 

**Part 5:** Next we use White’s local foliation argument \cite{41} to prove that $\varphi$ extends smoothly across $\mathcal{Y}$, and this will contradict the bumpy assumption of $g$.

Fix $y \in \mathcal{Y}$. We use the exponential normal coordinates $(x, z) \in \Sigma_\infty \times [-\delta, \delta]$. Let $\epsilon > 0$ be as given in Proposition \cite{D1}. Fix a small radius $0 < \eta < \epsilon$, and choose $k$ large enough such that $\|u_k^1\|_{2, \alpha}, \|u_k^m\|_{2, \alpha} \ll \epsilon \eta$ near $\partial B_\eta^0(y)$ so that some extensions of them to the whole $B_\eta^0(y)$ have $C^{2, \alpha}$-norms bounded by $\epsilon \eta$. Let $v_{k,t}^1, v_{k,t}^m : B_\eta^0(y) \to \mathbb{R}, \ t \in [-\eta, \eta]$, be the PMC local foliations associated with $\epsilon h_k$,

$$H_{\text{Graph}(v_{k,t}^1)}(x) = \epsilon_k h(x, v_{k,t}^1(x)), \quad i = 1, m, \ x \in B_\eta^0(y),$$

and

$$v_{k,t}^1(x) = u_k^1(x) + t, \quad i = 1, m, \ x \in \partial B_\eta^0(y).$$

By the Hausdorff convergence of $\Sigma_k \to \Sigma_\infty$ and the Strong Maximum Principle \cite{52} Lemma 3.12] (applied to $\text{Graph}(u_k^1)$ and $\{\text{Graph}(v_{k,t}^1)\}$, $\text{Graph}(u_k^m)$ and $\{\text{Graph}(v_{k,t}^m)\}$), we have

$$v_k^m(x) - u_k^1(x) \leq v_k^m(x) - v_k^1(x), \quad \text{when } x \in U_k \cap B_\eta^0(y).$$

By subtracting the mean curvature equations for $\text{Graph}(v_{k,t}^1)$, $i = 1, m$, we get an equation similar to $$(4.3),$$

$$L_{\Sigma_\infty}(v_{k,t}^m - v_{k,t}^1) + o(v_{k,t}^m - v_{k,t}^1) = \epsilon_k \cdot \partial_n h(x, v_k(x)) \cdot (v_{k,t}^m(x) - v_{k,t}^1(x)).$$

Note that the two graphs $\text{Graph}(v_{k,t}^i), i = 1, m$ must be disjoint by the Strong Maximum Principle. By elliptic estimates via the weak maximum principle \cite{13} Theorem 3.7], we have for $\eta$ small enough and $k$ sufficiently large and a uniform $C > 0$ so that,

$$\max_{\partial B_\eta^0} (v_{k,0}^m - v_{k,0}^1) \leq C \max_{\partial B_\eta^0} (v_{k,0}^m - v_{k,0}^1).$$
This implies 
\[ \max_{U_k \cap B^n_\eta} (u_k^m(x) - u_k^1(x)) \leq C \max_{\partial B^n_\eta} (u_k^m(x) - u_k^1(x)). \]

Hence \( \max_{U_k \cap B^n_\eta} \bar{h}_k \leq C \max_{\partial B^n_\eta} \bar{h}_k \), so \( \varphi \) is uniformly bounded and hence extends smoothly across \( y \).

Part 6: We now take care the more interesting case: \( m \) is an even number. Hence \( m \geq 2 \). In this case \( u_k \) restricted to the bottom \( (u_k^1) \) and top \( (u_k^m) \) sheets point to different side of \( \Sigma^\infty \), and without loss of generality we may assume that \( \nu_k \) points downward on top sheet, and upward on bottom sheet. That means:
\[ H|_{\Gamma(u^m_k)}(x) = -\epsilon_k h(x, u_k^m(x)), \quad H|_{\Gamma(u^1_k)}(x) = \epsilon_k h(x, u_k^1(x)), \quad \text{for } x \in U_k. \]

Note that
\[ u_k^m(x) - u_k^1(x) \geq 0, \quad \text{for all } x \in U_k, \]
but it may take zeros in a co-dimension 1 subset by \([52] \text{ Proposition 3.17}\).

Again by subtracting the above two equations, and using the fact \( H|_{\Gamma(u^m_k)} - H|_{\Gamma(u^1_k)} = L_{\Sigma^\infty}(u_k^m - u_k^1) + o(u_k^m - u_k^1) \), we have
\[ L_{\Sigma^\infty}(u_k^m - u_k^1) + o(u_k^m - u_k^1) = -\epsilon_k \cdot (h(x, u_k^1(x)) + h(x, u_k^m(x))). \]

Fix a point \( p \in U_1 \), and we discuss the renormalization in two cases. Again write \( h_k = u_k^m - u_k^1 \).

**Case 1:** \( \limsup_{k \to \infty} \frac{h_k(p)}{\epsilon_k} = +\infty \). Consider renormalizations \( \tilde{h}_k(x) = h_k(x)/h_k(p) \). Then by the same reasoning as Part 4, \( \tilde{h}_k \) converges locally smoothly to a nontrivial function \( \varphi \geq 0 \) on \( \Sigma^\infty \setminus \mathcal{Y} \), and such that
\[ L_{\Sigma^\infty} \varphi = 0, \quad \text{outside } \mathcal{Y}. \]

**Case 2:** \( \limsup_{k \to \infty} \frac{h_k(p)}{\epsilon_k} < +\infty \). Consider renormalizations \( \tilde{h}_k(x) = h_k(x)/\epsilon_k \). Then again by the same reasoning, \( \tilde{h}_k \) converges locally smoothly to a nonnegative \( \varphi \geq 0 \) on \( \Sigma^\infty \setminus \mathcal{Y} \), and such that
\[ L_{\Sigma^\infty} \varphi = -2h|_{\Sigma^\infty}, \quad \text{outside } \mathcal{Y}. \]

Part 7: We will follow a slightly different local foliation argument to prove removable singularity for \( \varphi \). We inherit all notations in Part 5. Without loss of generality, we may assume \( \sup_M |h| = 1 \). Let \( v_{i,t}^1, v_{i,t}^m : B^n_\eta \to \mathbb{R}, \ t \in [-\eta, \eta] \), be the CMC local foliations associated with \( -\epsilon_k \) and \( \epsilon_k \) respectively,
\[ H_{\Gamma(v_{i,t}^m)}(x) = \epsilon_k, \quad \text{and } H_{\Gamma(v_{i,t}^1)}(x) = -\epsilon_k, \quad x \in B^n_\eta(y), \]
and
\[ v_{i,t}^1(x) = u_k^1(x) + t, \ i = 1, m, \ x \in \partial B^n_\eta(y). \]

By the same reasoning as Part 5 using the Strong Maximum Principle for varifolds by White \([43]\), we get
\[ \max_{U_k \cap B^n_\eta} (u_k^m(x) - u_k^1(x)) \leq \max_{B^n_\eta} (u_{k,0}^m(x) - u_{k,0}^1(x)). \]

Note that slightly different with Part 5, we have
\[ L_{\Sigma^\infty} (v_{k,0}^m - v_{k,0}^1) + o(v_{k,0}^m - v_{k,0}^1) = 2\epsilon_k. \]

By \([13] \text{ Theorem 3.7}\), we have for \( \eta \) small enough, \( k \) large enough and for some uniform \( C > 0 \)
\[ \max_{U_k \cap B^n_\eta} (u_k^m(x) - u_k^1(x)) \leq C \left( \max_{\partial B^n_\eta} (u_k^m(x) - u_k^1(x)) + \epsilon_k \right). \]
Then for both Case 1 and Case 2, this implies that $\varphi$ is uniformly bounded and hence extends smoothly across $\mathcal{Y}$.

Note that if we flip the orientations of the top and bottom sheets, then in Case 2 the limit of renormalizations of heights will converge to a solution of $L_{\Sigma_\infty} \varphi = 2h|_{\Sigma_\infty}$, where $\varphi \geq 0$. Note that in the previous case, we can just flip the sign of $\varphi$, and obtain

$$L_{\Sigma_\infty} \varphi = 2h|_{\Sigma_\infty}, \text{ where } \varphi \leq 0.\$$

**Part 8:** Now we briefly record the case when $\Sigma_\infty$ is only one-sided. Then the convergence of $\Sigma_k$ must have multiplicity at least 2; otherwise the convergence will be smooth by the Allard regularity theorem [10], and hence all $\Sigma_k$ will be 1-sided for $k$ sufficiently large, which is a contradiction. Denote $\pi : \tilde{\Sigma}_\infty \to \Sigma_\infty$ as the 2-sided double cover of $\Sigma_\infty$, and $\tau : \Sigma_\infty \to \tilde{\Sigma}_\infty$ the deck transformation map. By the same argument for the 2-sided case applied to the double cover $\tilde{\Sigma}_\infty$, we can either construct a non-trivial Jacobi field $\varphi$ on $\tilde{\Sigma}_\infty$ with $\varphi \circ \tau = \varphi$ and

$$L_{\tilde{\Sigma}_\infty} \varphi = 0;$$

or a smooth function $\varphi$ on $\tilde{\Sigma}_\infty$ with $\varphi \circ \tau = \varphi$, such that $\varphi$ does not change sign, and

$$L_{\Sigma_\infty} \varphi = 2h|_{\Sigma_\infty} \circ \pi.$$

By [44], the first case cannot happen in a bumpy metric.

Summarizing the discussion, we proved that if $g$ is bumpy, then each connected 2-sided component $\Sigma_o$ of $\Sigma_\infty$ with multiplicity bigger than one must carry a smooth solution $\varphi$ to the equation

(4.5) $$L_{\Sigma_o} \varphi = 2h|_{\Sigma_o};$$

and the double cover $\tilde{\Sigma}_u$ of each 1-sided component $\Sigma_u$ of $\Sigma_\infty$ must carry a smooth solution $\varphi$

(4.6) $$L_{\Sigma_u} \varphi = 2h|_{\Sigma_u} \circ \pi, \text{ and } \varphi \circ \tau = \varphi.$$

Moreover, in both cases $\varphi$ does not change sign.

**Part 9:** We will show that for a nicely chosen $h \in S(g)$, the (unique) solutions to (4.5) and (4.6) must change sign. Thus there is no 1-sided component, and the multiplicity for 2-sided component must be one.

**Lemma 4.2 (Key Lemma).** Assume that $g$ is bumpy. Given $L > 0$ and $k \in \mathbb{N}$, there exists $h \in S(g)$, such that if $\Sigma$ is a smooth, connected, closed, embedded minimal hypersurface with

$$\text{Area}(\Sigma) \leq L, \text{ and } \text{index}(\Sigma) \leq k,$$

then the solution of (4.5) (when $\Sigma$ is 2-sided) or (4.6) (when $\Sigma$ is 1-sided) must change sign.

**Proof.** As $g$ is bumpy, by the compactness analysis of Sharp [35], there are only finitely many such $\Sigma$ with $\text{Area}(\Sigma) \leq L$ and $\text{index}(\Sigma) \leq k$, and we can denote them as $\{\Sigma_1, \cdots, \Sigma_L\}$. If $\Sigma_i$ is 1-sided, we use $\pi_i : \tilde{\Sigma}_i \to \Sigma_i$ to denote the 2-sided double cover, and $\tau_i : \tilde{\Sigma}_i \to \tilde{\Sigma}_i$ to denote the deck transformation map.

On each $\Sigma_i$, we can choose two disjoint open subsets $U_i^+$ and $U_i^- \subset \Sigma_i$, so that the collection of subsets $\{U_i^\pm\}_{i=1, \cdots, L}$ are pairwise disjoint. Moreover, by possibly changing $U_i^\pm$, we can make
sure that the pre-image $\pi_i^{-1}(U_i^+)$, $\pi_i^{-1}(U_i^-)$ are diffeomorphic to two disjoint copies of $U_i^+$, $U_i^-$ respectively. In that case, we will denote the two copies as $\tilde{U}_{i,1}^+ \cup \tilde{U}_{i,2}^+$, and $\tilde{U}_{i,1}^- \cup \tilde{U}_{i,2}^-$. That is $\pi_i^{-1}(U_i^+) = \tilde{U}_{i,1}^+ \cup \tilde{U}_{i,2}^+$, and $\pi_i^{-1}(U_i^-) = \tilde{U}_{i,1}^- \cup \tilde{U}_{i,2}^-$. 

For each $i \in \{1, \ldots, L\}$ such that $\Sigma_i$ is 2-sided, we can choose an arbitrary pair of nontrivial smooth functions $f_i^+ \in C_c^\infty(U_i^+)$, $f_i^- \in C_c^\infty(U_i^-)$ such that $f_i^+ \geq 0$, and $f_i^+(p_i^+) > 0$ at some $p_i^+ \in U_i^+$, and $f_i^- \leq 0$, and $f_i^-(p_i^-) < 0$ at some $p_i^- \in U_i^-$. 

Let $h_i^+ \in C_c^\infty(U_i^+)$ and $h_i^- \in C_c^\infty(U_i^-)$ be defined by:

$h_i^+ = L\Sigma_i f_i^+$, \quad $h_i^- = L\Sigma_i f_i^-$. 

If $\Sigma_i$ is 1-sided, we choose $\tilde{f}_{i,1}^+ \in C_c^\infty(\tilde{U}_{i,1}^+)$, $\tilde{f}_{i,2}^+ \in C_c^\infty(\tilde{U}_{i,2}^+)$ in the same way, and we can make sure they are the same under deck transformation: $\tilde{f}_{i,1}^+ \circ \tau = \tilde{f}_{i,2}^+$. In particular,

$\tilde{f}_{i,1}^+ \geq 0$, \quad and $\tilde{f}_{i,1}^+ > 0$ somewhere in $\tilde{U}_{i,1}^+$,

and

$\tilde{f}_{i,1}^- \leq 0$, \quad and $\tilde{f}_{i,1}^- < 0$ somewhere in $\tilde{U}_{i,1}^-$. 

Then we define $h_{i,1}^+, h_{i,2}^+$ in the same manner, so obviously $h_{i,1}^+ \circ \tau = h_{i,2}^+$, and they pass to two functions

$h_i^+ \in C_c^\infty(U_i^+)$, \quad and $h_i^- \in C_c^\infty(U_i^-)$. 

We can extend each $h_i^\pm$ to a function defined on $\Sigma_i$ by letting it be zero outside $U_i^\pm$. Using the fact that the set of smooth functions $\mathcal{S}(g)$ is open and dense in $C^\infty(M)$, we can choose a $h \in \mathcal{S}(g)$ so that $h$ is as close to $h_i^\pm$ as we want in any $C^{k,\alpha}$-norm when restricted to $\Sigma_i$.

We may need to flip the sign of $h$ to make $\int_M h \geq 0$, but the following argument proceeds the same way. Since all $\{\Sigma_i : i = 1, \cdots, L\}$ are non-degenerate (the Jacobi operator is an isomorphism), we know that if

$L\Sigma_i \varphi = 2h|_{\Sigma_i}$ when $\Sigma_i$ is 2-sided, or

$L\tilde{\Sigma}_i \varphi = 2h|_{\Sigma_i} \circ \pi_i$ when $\Sigma$ is 1-sided,

then

$\varphi$ is as close to $f_i^\pm$ or $\tilde{f}_{i,j}^\pm$ ($j = 1, 2$) as we want in $C^{k+2,\alpha}$-norm when restricted to $\Sigma_i$ or $\tilde{\Sigma}_i$.

Then $\varphi$ must change sign, and this is what we want to prove. \hfill \Box

Note that by Part 2, all connected components of a min-max minimal hypersurface must satisfy the area and index bound in Lemma 4.2 So we finish the proof of the theorem. \hfill \Box

**Remark 4.3.** Indeed, we can obtain more information. Since $\Sigma_\infty$ has multiplicity one, the Allard regularity theorem [1] implies that the convergence $\Sigma_k \to \Sigma_\infty$ is smooth everywhere, and hence $\Sigma_k$ is properly embedded for $k$ large.
Remark 4.4. Without assuming that \( g \) is bumpy, our proof says that if the multiplicity of a 2-sided component is greater than 2, or if the multiplicity for a 1-sided component is greater than 1, then there exists a nontrivial, nonnegative Jacobi field. Let us point out the necessary details for 2-sided case, and the 1-sided case follows the same way. Indeed, we only need to focus on the case when the multiplicity \( m \) is even and \( m \geq 4 \); and moreover, we can focus on Case 2 in Part 6. Using notations in Part 6 and 7, we consider the height difference between the two pairs \((u_k^1, u_k^{m-1})\) and \((u_k^2, u_k^m)\),

\[
\bar{h}_k^a = u_k^{m-1} - u_k^1, \quad \bar{h}_k^b = u_k^m - u_k^2.
\]

Then both \( \bar{h}_k^a, \bar{h}_k^b > 0 \) and satisfy equations of type (4.3) since the graphs of the two pairs have outer normals pointing to the same side. Consider the renormalizations: \( \bar{h}_k^a = h_k^a/\epsilon_k \) and \( \bar{h}_k^b = h_k^b/\epsilon_k \). Then

\[
\bar{h}_k^a, \bar{h}_k^b \leq \bar{h}_k, \quad \text{and} \quad \bar{h}_k^a + \bar{h}_k^b \geq \bar{h}_k.
\]

Note that the limit of \( \bar{h}_k \) can not be identically zero, as then \( h|_{\Sigma_{\infty}} \equiv 0 \), violating the assumption \( h \in \mathcal{S}(g) \). Then the above two inequalities and standard elliptic estimates imply that at least one limit of the two sequences \( \{h_k^a\}_{k \in \mathbb{N}} \) and \( \{h_k^b\}_{k \in \mathbb{N}} \) must be a smooth, nontrivial, nonnegative Jacobi field.

Part of the proof of the theorem can be summarized as the following multiplicity one convergence result, which we believe has its independent interests.

**Theorem 4.5 (Multiplicity one convergence).** Let \((M^{n+1}, g)\) be a closed manifold of dimension \( 3 \leq (n + 1) \leq 7 \) with a bumpy metric \( g \). Given \( \mathbf{L} > 0, I \in \mathbb{N} \), then there exists a smooth function \( h : M \to \mathbb{R} \), \( h \in \mathcal{S}(g) \), such that:

Let \( \{\Sigma_k\}_{k \in \mathbb{N}} \) be a sequence of smooth, closed, almost embedded hypersurfaces, and \( \{\epsilon_k\}_{k \in \mathbb{N}} \to 0 \), such that

- \( \Sigma_k \) is the boundary of some open set \( \Omega_k \), and the mean curvature of \( \Sigma_k \) with respect to the outer normal of \( \Omega_k \) is prescribed by \( \epsilon_k h \);
- \( \text{Area}(\Sigma_k) \leq \mathbf{L} \), and \( \text{index}(\Sigma_k) \leq I \).

Then up to a subsequence \( \{\Sigma_k\}_{k \in \mathbb{N}} \) converges smoothly to a smooth, closed, embedded, two-sided, minimal hypersurface \( \Sigma_{\infty} \) with multiplicity one.

### 5. Application to Volume Spectrum

In this part, we will show how to apply the result in Section 4 to study volume spectrum introduced by Gromov, Guth, and Marques-Neves. In particular, we will prove that in a bumpy metric, the volume spectrum can be realized by the area of min-max minimal hypersurfaces produced by Theorem 4.1.

To do this, we will carefully pick a sequence of sweepouts of mod 2 cycles, and open the parameter space so as to produce sweepouts of boundaries of Caccioppoli sets, whose relative homotopy classes satisfy (4.1). As the space of Caccioppoli sets forms a double cover of the space of mod 2 cycles, the parameter-space-opening process is achieved by lifting to the double cover.

We first recall the definition of volume spectrum following [28, Section 4]. Let \((M^{n+1}, g)\) be a closed Riemannian manifold. Let \( X \) be a cubical subcomplex of \( I^m = [0, 1]^m \) for some \( m \in \mathbb{N} \). Given \( k \in \mathbb{N} \), a continuous map \( \Phi : X \to \mathcal{Z}_n(M, \mathbb{Z}_2) \) is a \( k \)-sweepout if

\[
\Phi^*(\lambda^k) \neq 0 \in H^k(X, \mathbb{Z}_2),
\]

where \( \lambda \in H^1(\mathcal{Z}_n(M, \mathbb{Z}_2), \mathbb{Z}_2) = \mathbb{Z}_2 \) is the generator. \( \Phi \) is said to be admissible if it has no concentration of mass. Denote by \( \mathcal{P}_k \) as the set of all admissible \( k \)-sweepouts. Then
Definition 5.1. The \( k \)-width of \((M, g)\) is
\[
\omega_k(M, g) = \inf_{\Phi \in \mathcal{P}_k} \sup \{ \text{Area}(\Sigma_i^k) : x \in \text{dmn}(\Phi) \},
\]
where \( \text{dmn}(\Phi) \) is the domain of \( \Phi \).

It was proved in [28] Theorem 5.1 and 8.1] that there exists some constant \( C = C(M, g) \), such that
\[
\frac{1}{C^{k+1}} \leq \omega_k(M, g) \leq C^{k+1}.
\]

Assume from now on that the dimension satisfies \( 3 \leq (n + 1) \leq 7 \). It was later observed by Marques-Neves in [27] that one can restrict to a subclass of \( \mathcal{P}_k \) in the definition of \( \omega_k(M, g) \). In particular, let \( \mathcal{P}_{\kappa} \) denote those elements \( \Phi \in \mathcal{P}_k \) which is continuous under the \( \mathbf{F} \)-topology, and whose domain \( X = \text{dmn}(\Phi) \) has dimension \( k \) (and is identical to its \( k \)-skeleton). Then
\[
\omega_k(M, g) = \inf_{\Phi \in \mathcal{P}_k} \{ \text{Area}(\Sigma_i^k) : x \in \text{dmn}(\Phi) \}.
\]

They also proved in [27] that for each \( \kappa \in \mathbb{N} \) there exists a disjoint collection of smooth, connected, closed, embedded minimal hypersurfaces \( \{\Sigma_i^k : i = 1, \cdots, l_k\} \) with integer multiplicities \( \{m_i^k : i = 1, \cdots, l_k\} \in \mathbb{N} \), such that
\[
\omega_k(M, g) = \sum_{i=1}^{l_k} m_i^k \cdot \text{Area}(\Sigma_i^k), \quad \text{and} \quad \sum_{i=1}^{l_k} \text{index}(\Sigma_i^k) \leq k.
\]

Now we are going to state and prove our main theorem.

Theorem 5.2 (Theorem A). If \( g \) is a bumpy metric and \( 3 \leq (n + 1) \leq 7 \), then for each \( \kappa \in \mathbb{N} \), there exists a disjoint collection of smooth, connected, closed, embedded, two-sided minimal hypersurfaces \( \{\Sigma_i^k : i = 1, \cdots, l_k\} \) such that
\[
\omega_k(M, g) = \sum_{i=1}^{l_k} \text{Area}(\Sigma_i^k), \quad \text{and} \quad \sum_{i=1}^{l_k} \text{index}(\Sigma_i^k) \leq k.
\]

That is to say, the min-max minimal hypersurfaces are all two-sided and have multiplicity one.

Proof. If \( g \) is bumpy, then there are only finitely many closed, embedded, minimal hypersurfaces with \( \text{Area} \leq \Lambda \) and \( \text{index} \leq I \) for given \( \Lambda > 0, I \in \mathbb{N} \) by Sharp’s result [35]. Using the Morse index upper bound estimates for min-max theory by Marques-Neves [27], we have

Lemma 5.3. Suppose \( g \) is bumpy, then for each \( \kappa \in \mathbb{N} \), there exists a \( k \)-dimensional cubical complex \( X_k \) and a map \( \Phi_{0,k} : X_k \to \mathcal{Z}_n(M, \mathbf{F}, \mathbb{Z}_2) \) continuous in the \( \mathbf{F} \)-topology with \( \Phi_{0,k} \in \mathcal{P}_k \), such that
\[
\text{L}(\Pi_k) = \omega_k(M, g),
\]
where \( \Pi_k = \Pi(\Phi_{0,k}) \) is the class of all maps \( \Phi : X_k \to \mathcal{Z}_n(M, \mathbf{F}, \mathbb{Z}_2) \) continuous in the \( \mathbf{F} \)-topology that are homotopic to \( \Phi_{0,k} \) in flat topology.

Proof. From definition we know that
\[
\omega_k(M, g) = \inf \{ \text{L}(\Pi(\Phi)) : \Phi \in \mathcal{P}_k \}.
\]

By area and index upper bounds and the finiteness result, the infimum is achieved. \( \square \)

Now we fix \( k \in \mathbb{N} \) and omit the sub-index \( k \) in the following. Take \( \Pi = [\Phi_0 : X \to \mathcal{Z}_n(M, \mathbf{F}, \mathbb{Z}_2)] \) with \( \text{L}(\Pi) = \omega_k \). The following result is an outcome of the proof of [27] Theorem 6.1].
Lemma 5.4. Suppose $g$ is bumpy. Then there exists a pull-tight (see [27, 3.7]) min-max sequence \( \{\Phi_i\}_{i\in\mathbb{N}} \) of \( \Pi \) such that if \( \Sigma \subset C(\{\Phi_i\}_{i\in\mathbb{N}}) \) has support a smooth, closed, embedded minimal hypersurface, then
\[
\|\Sigma\|(M) = \omega_k(M, g), \quad \text{and index}(\text{support of } \Sigma) \leq k.
\]

We proceed the proof by the following four steps.

**Step 1:** In this and the next step, we show how to find another min-max sequence, still denoted as \( \{\Phi_i\}_{i\in\mathbb{N}} \), such that for \( i \) sufficiently large, either \( |\Phi_i(x)| \) is close to a regular min-max minimal hypersurface, or the mass \( M(\Phi_i(x)) \) is strictly less than \( \omega_k(M, g) \).

We recall the following observation by [28, Claim 6.2]. Let \( S \) be the set of all stationary integral varifolds with \( \text{Area}(\Phi_i) \leq \omega_k \) whose support is a smooth closed embedded minimal hypersurface with \( \text{index}(\text{support}) \leq k \). Consider the set \( T \) of all mod 2 flat cycles \( T \in \mathcal{Z}_n(M, \mathbb{Z}_2) \) with \( M(T) \leq \omega_k \) and such that either \( T = 0 \) or the support of \( T \) is a smooth closed embedded minimal hypersurface with index \( \leq k \). By the bumpy assumption, both sets \( S \) and \( T \) are finite. Moreover,

**Lemma 5.5** (Claim 6.2 in [28]). For every \( \bar{\epsilon} > 0 \), there exists \( \epsilon > 0 \) such that
\[
T \in \mathcal{Z}_n(M, \mathbb{Z}_2) \text{ with } F(|T|, S) \leq 2\epsilon \implies \mathcal{F}(T, T) < \bar{\epsilon}.
\]

We also need another observation by [28, Corollary 3.6]. Denote \( S^1 \) by the unit circle.

**Lemma 5.6** (Corollary 3.6 in [28]). If \( \bar{\epsilon} \) is sufficiently small, depending on \( T \), then every map \( \Phi : S^1 \to \mathcal{Z}_n(M, \mathbb{Z}_2) \) with
\[
\Phi(S^1) \subset B^F_\epsilon(T) = \{T \in \mathcal{Z}_n(M, \mathbb{Z}_2) : \mathcal{F}(T, T) < \bar{\epsilon}\}
\]
is homotopically trivial.

Let \( \{\Phi_i\}_{i\in\mathbb{N}} \) be chosen as in Lemma 5.4. We choose \( \bar{\epsilon} \) as Lemma 5.6 and \( \epsilon \) by Lemma 5.5. Take a sequence \( \{k_i\}_{i\in\mathbb{N}} \to \infty \), such that
\[
\sup \{F(\Phi_i(x), \Phi_i(y)) : \alpha \in X(k_i), x, y \in \alpha\} \leq \epsilon/2.
\]

Consider \( Z_i \) to be the cubical subcomplex of \( X(k_i) \) consisting of all cells \( \alpha \in X(k_i) \) so that
\[
F(|\Phi_i(x)|, S) \geq \epsilon \quad \text{for every vertex } x \text{ in } \alpha.
\]

Hence \( F(|\Phi_i(x)|, S) \geq \epsilon/2 \) for all \( x \in Z_i \).

Consider this sub-coordinating sequence \( \{\Phi_i |_{Z_i}\}_{i\in\mathbb{N}} \). \( L(\{\Phi_i |_{Z_i}\}) \) and \( C(\{\Phi_i |_{Z_i}\}) \) are defined in the same way as in Section 1.1 with \( A^h \) replaced by \( M \).

**Lemma 5.7.** We have the following dichotomy:
- no element \( V \in C(\{\Phi_i |_{Z_i}\}_{i\in\mathbb{N}}) \) is \( \mathbb{Z}_2 \)-almost minimizing in small annuli (see [28, 2.10]),
- or
\[
L(\{\Phi_i |_{Z_i}\}_{i\in\mathbb{N}}) < L(\Pi) = \omega_k.
\]

**Proof.** Suppose that (5.1) does not hold, then \( L(\{\Phi_i |_{Z_i}\}_{i\in\mathbb{N}}) = L(\Pi) \). As \( \{\Phi_i\}_{i\in\mathbb{N}} \) is pull-tight, we know that every \( V \in C(\{\Phi_i |_{Z_i}\}_{i\in\mathbb{N}}) \) is stationary. If \( V \) is also \( \mathbb{Z}_2 \)-almost minimizing in small annuli, then \( V \) is regular by the regularity of Pitts [31, Theorem 7.11] and Schoen-Simon [33, Theorem 4]; (see also [28, Theorem 2.11] for the adaption to \( \mathbb{Z}_2 \)-coefficients). By Lemma 5.4, \( V \in S \), which is a contradiction. \( \square \)
Let $Y_i = X \setminus Z_i$. It then follows that

$$F(|\Phi(x)|, S) \leq \frac{3}{2} \epsilon, \text{ for all } x \in Y_i.$$  

(5.2)

We also denote $B_i = Y_i \cap Z_i$. In fact, $B_i$ is the topological boundary of $Y_i$ and $Z_i$. For later purpose, we also consider the set

$$B_i = \text{ the union of all cells } \alpha \in Z_i \text{ such that } \alpha \cap B_i \neq \emptyset.$$  

$B_i$ can be thought of the “thickening” of $B_i$ inside $Z_i$.

Let $\lambda = \Phi^* (\lambda) \in H^1(X, \mathbb{Z}_2)$. Consider the inclusion maps $i_1 : Y_i \to X$ and $i_2 : Z_i \to X$. It then follows from (5.2), Lemma 5.5 and Lemma 5.6 that

$$i^*_i(\lambda) = 0 \in H^1(Y_i, \mathbb{Z}_2).$$  

Then by [28, Claim 6.3], $(\Phi_i)|_{Z_i}$ is a $(k-1)$-sweepout, i.e.

$$i^*_2(\lambda^{k-1}) \neq 0 \in H^{k-1}(Z_i, \mathbb{Z}_2).$$

Now we let $Y'_i = Y_i \cup B_i$ and $Z'_i = Z_i \setminus B_i$, and $i'_1 : Y'_i \to X$ and $i'_2 : Z'_i \to X$ be the inclusion maps. Note that (5.2) is satisfied with $Y_i, \frac{2}{\epsilon}$ replaced by $Y'_i, 2\epsilon$ respectively, so by similar reasoning we have

$$(i'_1)^*(\lambda) = 0 \in H^1(Y'_i, \mathbb{Z}_2), \text{ and } (i'_2)^*(\lambda^{k-1}) \neq 0 \in H^{k-1}(Z'_i, \mathbb{Z}_2).$$

**Step 2:** The strategy is to follow the idea in the proof of Theorem 1.7 and apply [28, Theorem 2.13] (see also Theorem 1.16 to deform $\{\Phi_i\}_{i \in \mathbb{N}}$ so as to decrease $L(\{\Phi_i\}_{i \in \mathbb{N}})$ and make (5.1) be satisfied.

If (5.1) holds true, then we are done for this step. So let us assume that

$$L(\{\Phi_i|_{Z_i}\}_{i \in \mathbb{N}}) = L(\Pi) = \omega_k.$$  

By Lemma 5.7 and our assumption 5.3, we know that no element $V \in C(\{\Phi_i|_{Z_i}\}_{i \in \mathbb{N}})$ is $\mathbb{Z}_2$-almost minimizing in small annuli.

Since $\Phi_i : X \to Z_n(M, F, \mathbb{Z}_2)$ has no concentration of mass as it is continuous in $F$-topology, we can apply [28, Theorem 3.9] (the counterpart of Theorem 1.11 for maps to $Z_n(M, \mathbb{Z}_2)$) to produce a sequence of maps

$$\phi^j_i : X(k_i + k^j_i) \to Z_n(M, \mathbb{Z}_2),$$

with $k_i \in \mathbb{N}$ and $k^j_i < k^{j+1}_i$ for all $j \in \mathbb{N}$ and a sequence of positive $\{\delta^j_i\}_{j \in \mathbb{N}} \to 0$, such that

(i) the fineness $f(\phi^j_i) \leq \delta^j_i$;

(ii) $\sup\{F(\phi^j_i(x) - \Phi_i(x)) : x \in X(k_i + k^j_i)\} \leq \delta^j_i$;

(iii) for some sequence $l^j_i \to \infty$ with $l^j_i < k^j_i$,

$$M(\phi^j_i(x)) \leq \sup\{M(\Phi_i(y)) : x, y \in \alpha, \text{ for some } \alpha \in X(k_i + l^j_i)\} + \delta^j_i.$$  

As $\Phi_i$ is continuous in $F$-topology, we get from property (iii) that for all $x \in X(k_i + k^j_i)$, $\alpha$,

$$M(\phi^j_i(x)) \leq M(\Phi_i(x)) + \eta^j_i$$

with $\eta^j_i \to 0$ as $j \to \infty$. Applying [26, Lemma 4.1] with $S = \Phi_i(X)$, we get by (ii) that

(iv) $\sup\{F(\phi^j_i(x), \Phi_i(x)) : x \in X(k_i + k^j_i)\} \to 0$, as $j \to \infty$. 


We can choose $j(i) \to \infty$ as $i \to \infty$ (then $k_i^{j(i)} \to \infty$) such that $\varphi_i = \phi_i^{j(i)} : X(k_i + k_i^{j(i)})_0 \to Z_n(M, \mathbb{Z}_2)$ satisfies:

- $\sup \{F(\varphi_i(x), \Phi_i(x)) : x \in X(k_i + k_i^{j(i)})_0 \} \leq a_i$ with $a_i \to 0$ as $i \to \infty$;
- $\sup \{F(\Phi_i(x), \Phi_i(y)) : x, y \in \alpha, \alpha \in X(k_i + k_i^{j(i)}) \} \leq a_i$;
- the fineness $F(\varphi_i) \to 0$ as $i \to \infty$;
- the Almgren extension $\Phi_i^{j(i)} : X \to Z_n(M, M, \mathbb{Z}_2)$ (see [28] 3.10 for definition, and it is continuous in the $M$-topology) is homotopic to $\varphi_i$ in the flat topology (by [28] Corollary 3.12), and $\sup \{F(\Phi_i^{j(i)}(x), \Phi_i(x)) : x \in X \} \to 0$ as $i \to \infty$ (by [28] 3.10).

If we let $S = \{\varphi_i\}_{i \in \mathbb{N}}$ be a discrete swee, then we have $L(S) = L(\{\Phi_i\}_{i \in \mathbb{N}})$ and $C(S) = C(\{\Phi_i\}_{i \in \mathbb{N}})$. Moreover, consider the restrictions of $\varphi_i$ to $Z_i(k_i^{j(i)})_0$:

$$S_Z = \{\varphi_i : Z_i(k_i^{j(i)})_0 \to Z_n(M, \mathbb{Z}_2)\}.$$

Similarly we have

$$L(S_Z) = L(\{\Phi_i|Z_i\}_{i \in \mathbb{N}}) = L(\Pi),$$
and $C(S_Z) = C(\{\Phi_i|Z_i\}_{i \in \mathbb{N}})$.

As no $V \in C(S_Z)$ is $\mathbb{Z}_2$-almost minimizing in small annuli, by [28] Theorem 2.13 (which is a reformulation of Almgren-Pitts combinatorial argument [51] Theorem 4.10], we can find a sequence $S_{\tilde{Z}} = \{\tilde{\varphi}_i\}$ of maps:

$$\tilde{\varphi}_i : Z_i(k_i^{j(i)} + l_i)_0 \to Z_n(M, \mathbb{Z}_2),$$

and a sequence of homotopies

$$\psi_i : I(l_i)_0 \times Z_i(k_i^{j(i)} + l_i)_0 \to Z_n(M, \mathbb{Z}_2),$$

such that

- $\psi_i([0], x) = \varphi_i \circ n(k_i^{j(i)} + l_i, k_i^{j(i)})(x)$ and $\psi_i([1], x) = \tilde{\varphi}_i(x)$;
- the fineness $\psi_i$ tends to zero as $i \to \infty$;
- $\limsup_{i \to \infty} \sup \{M(\psi_i(t, x)) : (t, x) \in I(l_i)_0 \times Z_i(k_i^{j(i)} + l_i)_0 \} = L(S_{\tilde{Z}})$;

(note that this property was not explicitly listed in [28] Theorem 2.13, but it follows from the construction in [51] Theorem 4.10].)
- $L(S_{\tilde{Z}}) < L(S_{Z})$.

Now we construct a new sequence $S^* = \{\varphi^*_i\}_{i \in \mathbb{N}}$ with

$$\varphi^*_i : X(k_i + k_i^{j(i)} + l_i)_0 \to Z_n(M, \mathbb{Z}_2),$$

defined as

- $\varphi^*_i(x) = \varphi_i \circ n(k_i^{j(i)} + l_i, k_i^{j(i)})(x)$, when $x \in Y_i(k_i^{j(i)} + l_i)_0$;
- $\varphi^*_i(x) = \psi_i(t(x), x)$, where $x \in B_i(l_i)_0$ and $t(x) = \min \{3^{-l_i} \cdot d(x, B_i \cap Y_i), 1\} \in I(l_i)_0$ ;

(Here $d$ is the distance function restricted to $B_i(l_i)_0$; see Appendix A);
- $\varphi^*_i(x) = \tilde{\varphi}_i(x)$, when $x \in Z_i'(k_i^{j(i)} + l_i)_0$; (note that $t(x) \geq 1$ when $x \in Z_i' \cap B_i$).

By the construction, we see that

- $\varphi^*_i$ is homotopic to $\varphi_i$ with fineness tending to zero as $i \to \infty$;
- $L(S^*) = L(\Pi)$;
- $\limsup_{i \to \infty} \sup \{M(\varphi^*_i(x)) : x \in Z_i'(k_i^{j(i)} + l_i)_0 \} \leq L(S_{\tilde{Z}}) < L(\Pi)$. 

Consider the Almgren’s extension of \( \varphi_i^* \):

\[
\Phi_i^* : X \to \mathcal{Z}_n(M, \mathcal{M}, \mathbb{Z}_2).
\]

Then

(a) \( \Phi_i^* \) is homotopic to \( \Phi_i^{(i)} \) and hence to \( \Phi_i \) in the flat topology by [28, 3.11]; and by [28, 3.10] (b) \( \sup \{ \mathbf{F}(\Phi_i^{(i)}(x), \Phi_i(x)) : x \in Y_i \} \to 0 \);

(c) \( \mathbf{L}(\{ \Phi_i^* \}) = \mathbf{L}(S^*) = \mathbf{L}(\Pi) \);

(d) \[
\lim_{i \to \infty} \sup \{ \mathbf{M}(\Phi_i^*(x)) : x \in Z_i \} \leq \mathbf{L}(S_Z) < \mathbf{L}(\Pi).
\]

By summarizing what we have done (and abusing the notation \( Y_i = Y_i' \) and \( Z_i = Z_i' \)), we produced another min-max sequence \( \{ \Phi_i^* \}_{i \in \mathbb{N}} \subseteq \Pi \) such that

1. \( X \) can be decomposed to \( Y_i \) and \( Z_i \) with \( Z_i = X \setminus Y_i \), and for \( i \) large enough,
   \[
i_i^*(\lambda) = 0 \in H^1(Y_i, \mathbb{Z}_2), \quad \text{and} \quad i_i^*(\lambda^{-1}) \neq 0 \in H^{k-1}(Z_i, \mathbb{Z}_2).
   \]

2. \( \mathbf{L}(\{ \Phi_i^* \}) = \mathbf{L}(\{ \Phi_i \}) = \mathbf{L}(\Pi) \);

3. \[
\lim_{i \to \infty} \sup \{ \mathbf{M}(\Phi_i^*(x)) : x \in Z_i \} < \mathbf{L}(\Pi).
\]

Note that both \( Y_i \) and \( Z_i \) are nonempty for \( i \) large enough by (1)(3).

**Step 3:** Now we want to produce sweepouts in \( \mathcal{C}(M) \) by lifting to the double cover \( \partial : \mathcal{C}(M) \to \mathcal{Z}_n(M, \mathbb{Z}_2) \) so as to produce sweepouts satisfying the assumption of Theorem 4.1.

We abuse notation and still write \( \{ \Phi_i^* \} \) as \( \{ \Phi_i \} \). Since \( (\Phi_i)^*(\lambda) \neq 0 \in H^1(X, \mathbb{Z}_2) = \mathbb{Z}_2 \), there exist a double cover \( \pi : \tilde{X} \to X \) with deck transformation map \( \tau : \tilde{X} \to \tilde{X} \), and the lifting maps:

\[
\tilde{\Phi}_i : \tilde{X} \to (\mathcal{C}(M), \mathbf{F}),
\]

satisfying \( \partial \tilde{\Phi}_i = \Phi_i \circ \pi \). Indeed, the cohomological condition implies that the induced maps \( (\Phi_i)^* : \pi_1(X) \to \pi_1(\mathcal{Z}_n(M, \mathbb{Z}_2)) = \mathbb{Z}_2 \) are surjective; see [28, Definition 4.1 (i)]. So the kernel of \( (\Phi_i)^* \) is a subgroup of \( \pi_1(X) \) with index 2. Then the existence of such liftings follows from [20, Proposition 1.36 and Proposition 1.33].

Note that \( i_i^*(\lambda) = 0 \in H^1(Y_i, \mathbb{Z}_2) \), so the pre-image of \( Y_i \) is disconnected, and is a disjoint union of two copies of \( Y_i \):

\[
\tilde{Y}_i = \pi^{-1}(Y_i) = Y_i^+ \cup Y_i^-,
\]

where both \( Y_i^+ \) and \( Y_i^- \) are homeomorphic to \( Y_i \). In fact, the cohomological condition implies that every closed curve \( \gamma : S^1 \to Y_i \) lies in the kernel of \( (\Phi_i)^* \), so the lift \( \tilde{\gamma} \) of \( \gamma \) to \( \tilde{X} \) is still a closed curve. This means that \( \tilde{Y}_i \) is disconnected as we want.

Denote \( \tilde{Z}_i, \tilde{B}_i \) and \( \tilde{B}_i \) the pre-images of \( Z_i, B_i, B_i \) under \( \pi \) respectively. Then \( \tilde{B}_i = B_i^+ \cup B_i^- \) is also a disjoint union of two copies of \( B_i \).

**Lemma 5.8.** For \( i \) large enough, if \( \tilde{\Pi}_i \) is the \( (\tilde{X}, \tilde{Y}_i) \)-homotopy class associated with \( \tilde{\Phi}_i \), then we have

\[
\mathbf{L}(\tilde{\Pi}_i) \geq \mathbf{L}(\Pi) > \max_{x \in \tilde{Z}_i} \mathbf{M}(\partial \tilde{\Phi}_i(x)).
\]

**Proof.** Fix \( i \) large, so that

\[
\sup_{x \in \tilde{Z}_i} \mathbf{M}(\Phi_i(x)) < \mathbf{L}(\Pi),
\]

and we will omit the sub-index in the following proof.
If the conclusion were not true, then we can find a sequence of maps \( \tilde{\Psi}_j : \tilde{X} \to (\mathcal{C}(M), F) \) for \( j \in \mathbb{N} \), such that
\[
\limsup_{j \to \infty} \sup_{x \in X} \{ M(\partial \tilde{\Psi}_j(x)) : x \in X \} < L(\Pi),
\]
and homotopy maps \( \{ H_j : [0, 1] \times \tilde{X} \to \mathcal{C}(M) \} \) which are continuous in the flat topology, \( H_j(0, \cdot) = \tilde{\Psi}_j, H_j(1, \cdot) = \Phi \), and
\[
\limsup_{j \to \infty} \sup_{x \in \tilde{Z}} \{ F(H_j(t, x), \Phi(x)) : t \in [0, 1], x \in \tilde{Z} \} = 0.
\]

We construct a new sequence of maps \( \tilde{\Psi}_j^* \) defined as
- \( \tilde{\Psi}_j^*(x) = \tilde{\Psi}_j(x) \), if \( x \in Y^+ \), and \( \tilde{\Psi}_j^*(x) = \tilde{\Psi}_j \circ \tau(x) \), if \( x \in Y^- \);
- \( \tilde{\Psi}_j^*(x) = \tilde{H}_j(t(x), x) \), where \( t(x) = \min \{ \text{dist}(x, \mathcal{B}^+ \cap Y^+), 1 \} \) if \( x \in \mathcal{B}^+ \), and \( \tilde{\Psi}_j^*(x) = \tilde{H}_j \circ \tau(x) \), if \( x \in \mathcal{B}^- \); (here dist is the distance function by viewing \( \mathcal{B} \) as a cube complex in some \( \mathcal{I}(m, l) \);
- \( \tilde{\Psi}_j^*(x) = \Phi(x) \), if \( x \in \tilde{Z}' \); (note that \( t(x) \geq 1 \) for \( x \in \tilde{Z}' \cap (\mathcal{B}^+ \cap \mathcal{B}^-) \)).

Note that though \( \tilde{\Psi}_j^* \) themselves may not be continuous as maps to \( \mathcal{C}(M) \), \( \tilde{\Psi}_j^* \) can be passed to quotient as continuous maps from \( X \) to \( \mathcal{Z}_n(M, \mathbb{Z}_2) \). This is essentially where we used the structures of \( \tilde{Y} \) and \( \tilde{B} \), that is, \( (Y^+, Y^-) \) and \( (\mathcal{B}^+, \mathcal{B}^-) \) are pairwise disjoint.

Denote the quotient maps of \( \{ \tilde{\Psi}_j^* \} \) for \( j \in \mathbb{N} \) by \( \{ \tilde{\psi}_j^* = \partial \circ \tilde{\Psi}_j^* : X \to \mathcal{Z}_n(M, \mathbb{Z}_2) \} \). We have
- \( \tilde{\psi}_j^* \) is homotopic to \( \Phi \) in the flat topology;
- \( \limsup_{j \to \infty} \sup_{x \in X} \{ M(\tilde{\psi}_j^*(x)) : x \in X \} < L(\Pi) = \omega_k(M, g) \) (by the three above inequalities).

This will lead to a contradiction with the definition of \( k \)-width once we prove that \( \tilde{\psi}_j^* \) is an admissible \( k \)-sweepout when \( j \) is sufficiently large. Indeed, the only thing left is to show that \( \tilde{\psi}_j^* \) has no concentration of mass. This follows from the third inequality above. So we finish the proof. \( \square \)

**Step 4:** We are ready to finish the proof of Theorem 5.2

For \( i \) large enough as in Lemma 5.8, Theorem 4.1 applied to \( \tilde{\Pi}_i \) gives a disjoint collection of smooth, connected, closed, embedded, 2-sided, minimal hypersurfaces \( \Sigma_i = \bigcup_{j=1}^{N_i} \Sigma_{i,j} \), such that
\[
L(\tilde{\Pi}_i) = \sum_{j=1}^{N_i} \text{Area}(\Sigma_{i,j}), \text{ and } \text{index}(\Sigma_i) \leq k.
\]

Note also that \( L(\tilde{\Pi}_i) \leq L(\Phi_i) \to L(\Pi) = \omega_k \). Counting the fact that there are only finitely many smooth, closed, embedded minimal hypersurfaces with Area \( \leq \omega_k+1 \) and index \( \leq k \), for \( i \) sufficiently large we have
\[
L(\tilde{\Pi}_i) = L(\tilde{\Pi}_{i+1}) = \cdots = \omega_k.
\]
Hence we finish the proof of Theorem 5.2. \( \square \)

**Remark 5.9.** By the course of the above proof, in a bumpy metric, the min-max minimal hypersurfaces associated with any homotopically nontrivial sweepouts of mod-2 cycles are always two-sided and have multiplicity one. In fact, if \( \Phi : X \to \mathcal{Z}_n(M, \mathbb{Z}_2) \) is homotopically nontrivial, then the induced map \( \Phi_* : \pi_1(X) \to \pi_1(\mathcal{Z}_n(M, \mathbb{Z}_2)) = \mathbb{Z}_2 \) must be surjective. Otherwise by [20, Proposition 1.33] \( \Phi \) can be lifted to a map \( \tilde{\Phi} : X \to \mathcal{C}(M) \) which is then homotopically trivial as \( \mathcal{C}(M) \) is contractible.
With these topological information, the above proof works the same way and implies the two-sidedness and multiplicity one for min-max minimal hypersurfaces associated with $\Pi(\Phi)$.

**Appendix A. Cubical complex structures**

Here we recall several cubical complex structures in [28, 2.1].

For each $k \in \mathbb{N}$, $I(1, k)$ denotes the cubical complex on the unit interval $I = [0, 1]$ whose 1-cells and 0-cells (which are also called vertices) are, respectively,

$$[0, 3^{-k}], [3^{-k}, 2 \cdot 3^{-k}], \ldots, [1 - 3^{-k}, 1] \text{ and } [0], [3^{-k}], \ldots, [1 - 3^{-k}], [1].$$

We then denote by $I(m, k)$ the cell complex on $I^m$:

$$I(m, k) = I(1, k) \otimes \cdots \otimes I(1, k) \quad m \text{ times.}$$

Then $\alpha = \alpha_1 \otimes \cdots \otimes \alpha_m$ is a $q$-cell of $I(m, k)$ if and only if $\alpha_i$ is a cell of $I(1, k)$ for each $i$, and $\sum_{i=1}^m \dim(\alpha_i) = q$. We often identify a $q$-cell $\alpha$ with its support $\alpha_1 \times \cdots \times \alpha_m \subset I^m$. The distance function $d$ on $I(m, k)_0$ is defined as $d(x, y) = 3^k \sum_{i=1}^m |x_i - y_i|, x, y \in I(m, k)_0, [31]$ 4.1(e).

Let $X \subset I^m$ be a cubical subcomplex. The cubical complex $X(k)$ is the union of all cells of $I(m, k)$ whose support is contained in some cell of $X$. We use the notation $X(k)_q$ to denote the set of all $q$-cells in $X(k)$, and particularly $X(k)_0$ to denote the set of vertices in $X(k)$. Two vertices $x, y \in X(k)_0$ are adjacent if they belong to a common cell in $X(k)_1$.

Let $Y \subset I(m, k)$ be a cubical subcomplex. Similarly, the cubical complex $Y(l)$ is the union of all cells of $I(m, k + l)$ whose support is contained in some cell of $Y$. $Y(k)_q$ is defined in the same way.

Given $k, l \in \mathbb{N}$, we define $n(k, l) : X(k)_0 \to X(l)_0$ so that $n(i, j)(x)$ is the element in $X(l)_0$ that is closest to $x$; (see [31] page 141).

**Appendix B. Removing singularity for weakly stable PMC**

We record the following standard removable singularity result.

**Theorem B.1.** Let $(M^{n+1}, g)$ be a closed Riemannian manifold of dimension $3 \leq (n + 1) \leq 7$. Given $h \in C^\infty(M)$ and $\Sigma \subset B_r(p) \setminus \{p\}$ an almost embedded hypersurface with $\partial \Sigma \cap B_r(p) \setminus \{p\} = \emptyset$, assume that $\Sigma$ has prescribing mean curvature $h$, and $\Sigma$ is weakly stable in $B_r(p) \setminus \{p\}$ as in Theorem 2.6. Part 1 of proof. If $\Sigma$ represents a varifold of bounded first variation in $B_r(p)$, then $\Sigma$ extends smoothly across $p$ as an almost embedded hypersurface in $B_r(p)$.

**Proof.** Given any sequence of positive $\lambda_i \to 0$, consider the blowups $\{\mu_{p, \lambda_i}(\Sigma) \subset \mu_{p, \lambda_i}(M)\}$, where $\mu_{p, \lambda_i}(x) = \frac{x - p}{\lambda_i}$. Since $\Sigma$ has bounded first variation, $\mu_{p, \lambda_i}(\Sigma)$ converges (up to a subsequence) to a stationary integral rectifiable cone $C \subset \mathbb{R}^{n+1} = T_pM$. By weakly stability and Theorem 2.5 (which works well for the notion of weak stability of $\Sigma_\infty$), the convergence is locally smooth and graphical away from the origin, so $C$ is an integer multiple of some embedded minimal hypercone; moreover, $C$ is weakly stable, and hence is stable as an embedded minimal hypersurface away from $0$. Therefore $C$ is an integer multiple of some $n$-plane $P$ by Simons’s classification [38], i.e. $C = m \cdot P$ where $m = \Theta^n(\Sigma, p)$. Note that a priori $C$ may not be unique.

By the locally smooth and graphical convergence, there exists $\sigma_0 > 0$ small enough, such that for any $0 < \sigma \leq \sigma_0$, $\Sigma$ has an $m$-sheeted, ordered (in the sense of 52 Definition 3.2)), graphical decomposition in the annulus $A_{\sigma/2, \sigma}(p) = B_{\sigma}(p) \setminus \overline{B}_{\sigma/2}(p)$:

$$\Sigma \cap A_{\sigma/2, \sigma}(p) = \bigcup_{i=1}^m \Sigma_i(\sigma).$$

Here each $\Sigma_i(\sigma)$ is a graph over $A_{\sigma/2, \sigma}(p) \cap P$ for some $n$-plane $P \subset T_pM$. 
We can continue each $\Sigma_i(\sigma)$ all the way to $B_{\sigma_0}(p)\setminus\{p\}$, and we denote the continuation by $\Sigma_i$. Each $\Sigma_i$ can be extended as a varifold across $p$ with uniformly bounded first variation (since $\Sigma_i \subset \Sigma$ satisfies the area decay estimates, $\text{area}(\Sigma_i \cap B_{\sigma}(p)) \leq C\sigma^n$). We claim that the density satisfies $\Theta^n(\Sigma_i, p) = 1$ for each $i$. In fact, $\Theta^n(\Sigma_i, p) \geq 1$ as any blowups of $\Sigma_i$ converges to an $n$-plane, but $m = \Theta^n(\Sigma, p) = \sum_{i=1}^n \Theta^n(\Sigma_i, p)$. Now applying the Allard regularity theorem \[1\] to each $\Sigma_i$, we get that $\Sigma_i$ extends as a $C^{1,\alpha}$ hypersurface across $p$. Higher regularity of $\Sigma_i$ follows from the prescribing mean curvature equation and elliptic regularity. \hfill \qed

**Appendix C. Proof of Lemma 3.3**

[27] Lemma 4.5] is purely a result in finite dimensional multi-variable calculus. Let us translate the problem as follows: let $B$ be some compact topological space with $0 \in B$, and $\{f^\omega \in C^\infty(B^k) : \omega \in B\}$ be a family of smooth functions defined on $\overline{B^k}$, such that $\omega \to f^\omega$ is a continuous map in the smooth topology on $C^\infty(B^k)$. Moreover we assume

- $f^\omega$ has a unique maximum $m(\omega) \in B^k_{c_0/\sqrt{10}}$ and $m(0) = 0$;
- $-\frac{1}{c_0} \Id \leq D^2 f^\omega(u) \leq -c_0 \Id$, for all $u \in B^k$ and some $c_0 \in (0, 1)$.

So for each $\omega \in B$, we have

\[
(C.1) \quad f^\omega(m(\omega)) - \frac{1}{2c_0}|u - m(\omega)|^2 \leq f^\omega(u) \leq f^\omega(m(\omega)) - \frac{c_0}{2}|u - m(\omega)|^2
\]

for all $u \in B^k$.

For each $f^\omega$, consider the one-parameter flow $\{\phi^\omega(\cdot, t) : t \geq 0\} \subset \text{Diff}(B^k)$ generated by the vector field

\[ u \to -(1 - |u|^2)\nabla f^\omega(u), \quad u \in B^k. \]

For fixed $u \in B^k$, the function $t \to f^\omega(\phi^\omega(u, t))$ is non-increasing.

The prototype of [27] Lemma 4.5 is the following lemma, and the proof is essentially the same as therein so we omit it.

**Lemma C.1.** For any $\delta < \frac{1}{10}$, there exists $T = T(\delta, B, \{f^\omega\}, c_0) \geq 0$ such that for any $\omega \in B$ and $v \in B^k$ with $|v - m(\omega)| \geq \delta$, we have

\[ f^\omega(\phi^\omega(v, T)) < f^\omega(0) - \frac{c_0}{10} \quad \text{and} \quad |\phi^\omega(v, T)| > \frac{c_0}{4}. \]

Now we are ready to prove Lemma 3.3. Note that the ball $\overline{B}_{2R}(\Omega_0)$ is not compact under the $F$-topology, so to apply Lemma C.1, we need to introduce a compactification of $\overline{B}_{2R}(\Omega_0)$.

**Proof of Lemma 3.3** Given a $F$-Cauchy sequence $\{\Omega_i\} \subset \overline{B}_{2R}(\Omega_0)$, we denote $(V_\infty, \Omega_\infty) \in \mathcal{V}_u(M) \times C(M)$ as the limit such that $V_\infty = \lim_{i \to \infty} |\partial \Omega_i|$ as varifolds and $\Omega_\infty = \lim_{i \to \infty} \Omega_i$ as Caccioppoli sets. If we define

\[
A^h_\Omega(v) = \|(F_v)_\# V_\infty\|(M) - \int_{F_v(\Omega_\infty)} h \, d\mathcal{H}^{n+1}, \quad \text{for each } v \in B^k,
\]

Then $A^h_\Omega$ converges smoothly to $A^{h,\infty}$ as functions in $C^\infty(\overline{B}_{2R})$.

Now take $B$ as the union of $\overline{B}_{2R}(\Omega_0)$ with the limits of the form $(V_\infty, \Omega_\infty)$, $f^{\Omega} = A^h_\Omega$ and $f(V_{\infty}, \Omega_\infty) = A^{h,\infty}$, then Lemma 3.3 follows from Lemma C.1. \hfill \qed
APPENDIX D. EXISTENCE OF LOCAL PMC FOLIATIONS

We recall the following classical result of White [41, Appendix and Remark 2]. Note that the $A^h$-functional can be locally expressed as the integration of an elliptic integrand.

**Proposition D.1.** Given a Riemannian metric $g$ in a neighborhood $U$ of $0 \in \mathbb{R}^{n+1}$, there exists an $\epsilon > 0$, such that if $h : U \rightarrow \mathbb{R}$ is a smooth function with $\|h\|_{4, \alpha} < \epsilon$, $r < \epsilon$, and if

$$w : B^r_0 \subset \mathbb{R}^n \rightarrow \mathbb{R} \text{ satisfies } \|w\|_{2, \alpha} < \epsilon r,$$

then for each $t \in [-r, r]$, there exists a $C^{2, \alpha}$ function $v_t : B^r_0 \rightarrow \mathbb{R}$ whose graph $G_t$ satisfies:

$$H_{G_t} = h|_{G_t},$$

(where $H_{G_t}$ is evaluated with respect to the upward pointing normal of $G_t$), and

$$v_t(x) = w(x) + t, \text{ if } x \in \partial B^r_0.$$

Furthermore, $v_t$ depends on $r, t, h, w$ in $C^1$ and the graphs $\{G_t : t \in [-r, r]\}$ forms a foliation.

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