Weak controllability of second order evolution systems and applications

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Abstract. Controllability and observability are important properties of a distributed parameter systems. The equivalence between the notion of exact observability and exact controllability holds in general. In this work, we define a new notion of controllability say weak which is related to some weak observability inequality and we give the equivalence between.

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1 Introduction

Problems of control and observations of waves arises in many different context and for various models. Hence controllability refers to the possibility of driving the system under consideration to prescribed final state at a given final time using a control function. This question is very interesting when the control function doesn’t act everywhere but is rather located in some part of the domain or in its boundary through suitable actuators. On the other hand, observability refers to the possibility of measuring the whole energy of the solutions of the free trajectories (i.e., without control) through partial measurements. It turns out that these two properties are equivalent and dual one from another. This is the basis of the so-called Hilbert Uniqueness method [12].

Our starting point is the following. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an open bounded domain with a sufficiently smooth boundary $\partial \Omega = \Gamma_0 \cup \Gamma_1$ such that $\Gamma_0, \Gamma_1$ are disjoints parts of the boundary relatively open in $\partial \Omega$, $\text{int}(\Gamma_0) \neq \emptyset$.

We consider the following homogenous wave equation

$$\frac{\partial^2 \phi}{\partial t^2} - \Delta \phi = 0, \quad \Omega \times (0, +\infty),$$

$$\phi = 0, \quad \partial \Omega \times (0, +\infty),$$

$$\phi(x, 0) = \phi^0(x), \quad \frac{\partial \phi}{\partial t}(x, 0) = \phi^1(x),$$

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then, using theorem of Hille-Yoshida, one can easily check that problem (1.1)-(1.3) is well-posed, i.e., for all \((\phi^0,\phi^1) \in H^1_0(\Omega) \times L^2(\Omega)\), equations (1.1)-(1.2) admits unique solution

\[
\phi \in C([0, +\infty); H^1_0(\Omega)) \cap C^1([0, +\infty); L^2(\Omega)).
\]

It’s well known that the problem of controllability, that’s., there exists a constant \(C_0 > 0\) such that for all \((z^0, z^1) \in L^2(\Omega) \times H^{-1}(\Omega)\) there exist a control \(g \in L^2([0, T], L^2(\Gamma))\) such that

\[
\|g\|_{L^2([0, T], L^2(\Gamma_0))} \leq C_0(\|z^0\|_{L^2(\Omega)} + \|z^1\|_{H^{-1}(\Omega)})
\]

such that the solution of

\[
\frac{\partial^2 z}{\partial t^2} - \Delta z = 0, \quad \Omega \times (0, +\infty),
\]

\[
z = g, \quad \Gamma_0 \times (0, +\infty),
\]

\[
z = 0, \quad \Gamma_1 \times (0, +\infty),
\]

\[
z(x, 0) = z^0(x), \quad \frac{\partial z}{\partial t}(x, 0) = z^1(x), \quad \Omega,
\]

satisfy

\[
z(x, t) = 0, \quad \forall t \geq T.
\]

is equivalent to the following observability inequality

\[
\int_0^T \int_{\Gamma_0} \left| \frac{\partial \phi}{\partial \nu} \right|^2 d\Gamma_0 dt \geq C(\|(\phi^0, \phi^1)\|_{H^1_0(\Omega) \times L^2(\Omega)}^2).
\]

Due to [5],[6] this last inequality is equivalent to some geometric conditions\(\text{CGC}\)\(\text{\cite{5}}\) satisfied by the part of the boundary \(\Gamma_0\) and the time of control \(T > 0\).

As a first example of this paper, we consider \(\Omega = (0, 1) \times (0, 1)\) and we prove that the solution of the homogenous system (1.1)-(1.3) satisfy the following weakly observability inequality. For the proof, see appendix.

**Proposition 1.1.** \(\text{\cite{3}}\) Let \(\Gamma_0 = \{(0, x_2); \ x_2 \in (0, 1)\} = \{0\} \times (0, 1)\). There exist \(T_0 > 0, C_{T_0} > 0\) such that for all \(T > T_0\) and for all \((\phi_0, \phi_1) \in H^1_0(\Omega) \times L^2(\Omega)\) we have

\[
\|(\phi_0, \phi_1)\|_{L^2(\Omega) \times H^{-1}(\Omega)}^2 \leq C_{T_0} \int_0^T \int_{\Gamma_0} |\partial \nu \phi(x, t)|^2 d\Gamma_0(x) dt.
\]
We prove that.

**Theorem 1.2.** System \((1.1)-(1.3)\) is weakly observable in time \(T > 0\), that’s \((1.12)\) holds true if and only if there exists a control \(g \in L^2(0,T;L^2(\Gamma_0))\) such that
\[
\|g\|_{L^2(0,T;L^2(\Gamma_0))} \leq C_0(\|\phi^0\|_{H^1_0(\Omega)} + \|\phi^1\|_{L^2(\Omega)}),
\]
and that the solution of
\[
\frac{\partial^2 z}{\partial t^2}(x,t) - \Delta z(x,t) = 0, \quad (x,t) \in \Omega \times (0, +\infty),
\]
\[
z(x,t) = g, \quad (x,t) \in \Gamma_0 \times (0, +\infty),
\]
\[
z(x,t) = 0, \quad (x,t) \in \partial\Omega \setminus \Gamma_0 \times (0, +\infty),
\]
\[
z(x,0) = z^0(x), \quad x \in \Omega.
\]
satisfy
\[
z(x,t) = 0, \quad \forall t \geq T.
\]

**Definition 1.3.** With \(g\) and \(z\) as above satisfy respectively \((1.13)\) and \((1.19)\), we said that the system \((1.6)-(1.9)\) is weakly controllable in time \(T > 0\).

The outline of this paper is as follows. In the second section we give some background on HUM method needed here, section 3 contains the proof of the main result and abstract framework. The last section is devoted to some applications.

### 2 Survey on HUM method

Before starting the proof of Theorem 1.2 we recall the HUM method in the classical case. For more details, see [12].

In fact, J. L. Lions gave a systematic method to reduce the study of exact controllability problem of system \((1.6)-(1.9)\) to obtain some inequality, say observability inequality or inverse inequality of the adjoint problem \((1.1)-(1.3)\). He called this method Hilbert Uniqueness method which can be found in [12] and we briefly describe below.

Let \((\phi^0, \phi^1) \in H^1_0(\Omega) \times L^2(\Omega),\) and we solve the problem \((1.1)-(1.3)\) then, \((1.1)-(1.3)\) admits a unique solution \(\phi \in C(0,T;H^1_0(\Omega)) \cap C^1(0,T;L^2(\Omega)).\)

In addition, we have the following regularity result: \(\frac{\partial \phi}{\partial t} \in L^2(0,T;L^2(\Gamma_0))\) and there exist a constant \(C_0 > 0\) such that
\[
\forall (\phi^0, \phi^1) \in D(\Omega) \times D(\Omega), \quad \left\| \frac{\partial \phi}{\partial t} \right\|_{L^2(0,T;L^2(\Gamma_0))} \leq C_0(\|\phi^0\|_{H^1_0}^2 + \|\phi^1\|_{L^2}^2).
\]

This inequality express that the application \((\phi^0, \phi^1) \mapsto \frac{\partial \phi}{\partial t}\) extends to a continuous linear application from \(H^1_0(\Omega) \times L^2(\Omega)\) to \(L^2(0,T;L^2(\Gamma_0)).\) Now, we consider the following backward problem associated to \((1.1)-(1.3)\)
\[
\frac{\partial^2 \psi}{\partial t^2} - \Delta \psi = 0, \quad \Omega \times (0, +\infty),
\]
\[
(2.20)
\]
\[
\psi = -\frac{\partial \phi}{\partial \nu}, \quad \Gamma_0 \times (0, +\infty), \quad (2.21)
\]
\[
\psi = 0, \quad \Gamma_1 \times (0, +\infty), \quad (2.22)
\]
\[
\psi(x, T) = \frac{\partial \psi}{\partial t}(x, T) = 0, \quad \Omega, \quad (2.23)
\]
then, (2.20)-(2.23) admits unique solution
\[
\psi \in C(0, T; L^2(\Omega)) \cap C^1(0, T; H^{-1}(\Omega)),
\]
thus, \(\psi(x, 0)\) and \(\frac{\partial \psi}{\partial t}(x, 0)\) are well defined in \(L^2(\Omega)\) and \(H^{-1}(\Omega)\) respectively. In fact, the density of \(D(\Omega)\) in \(H^1_0(\Omega)\) and \(L^2(\Omega)\) allows us to do all the above steps for \((\phi^0, \phi^1)\) \(\in H^1_0(\Omega) \times L^2(\Omega)\). If we can find \((\phi^0, \phi^1)\) \(\in H^1_0(\Omega) \times L^2(\Omega)\) such that
\[
\left\{
\begin{array}{l}
\psi(x, 0) = z^0(x) \\
\frac{\partial \psi}{\partial t}(x, 0) = z^1(x)
\end{array}
\right.
\]
we resolve the control problem (1.15)-(1.19) with \(g = -\frac{\partial \phi}{\partial \nu}\) and \(z = \psi\). Hence, for \((\phi^0, \phi^1)\) \(\in H^1_0(\Omega) \times L^2(\Omega)\), we define the following operator
\[
\Lambda(\phi^0, \phi^1) = \left(\frac{\partial \psi}{\partial t}(0), -\psi(0)\right) \in L^2(\Omega) \times H^{-1}(\Omega).
\]
It’s easy to check that \(\Lambda\) is a linear continuous operator from \(F = H^1_0(\Omega) \times L^2(\Omega)\) onto its dual \(F' = L^2(\Omega) \times H^{-1}(\Omega)\).
In fact, let \(\phi = (\phi^0, \phi^1) \in F\), \(\tilde{\phi} = (\tilde{\phi}^0, \tilde{\phi}^1) \in F\), then multiplying (2.20) by \(\tilde{\phi}\) and integrating by parts, we obtain
\[
\left\langle \Lambda(\phi^0, \phi^1), (\tilde{\phi}^0, \tilde{\phi}^1) \right\rangle_{F', F} = \int_0^T \int_{\Gamma_0} \frac{\partial \phi}{\partial \nu} \frac{\partial \tilde{\phi}}{\partial \nu} d\Gamma_0 dt,
\]
in particular
\[
\left\langle \Lambda(\phi^0, \phi^1), (\phi^0, \phi^1) \right\rangle_{F', F} = \int_0^T \int_{\Gamma_0} \left|\frac{\partial \phi}{\partial \nu}\right|^2 d\Gamma_0 dt.
\]
If we show that the continuous bilinear form defined on \(F \times F\) by
\[
((\phi^0, \phi^1), (\tilde{\phi}^0, \tilde{\phi}^1)) \rightarrow \left\langle \Lambda(\phi^0, \phi^1), (\tilde{\phi}^0, \tilde{\phi}^1) \right\rangle_{F', F}
\]
is coercive, then according to the Lax-Milgram lemma, we have: for all \((z^0, z^1)\) \(\in L^2(\Omega) \times H^{-1}(\Omega)\), there exist \((\phi^0, \phi^1)\) \(\in H^1_0(\Omega) \times L^2(\Omega)\) such that
\[
\Lambda(\phi^0, \phi^1) = (z^1, -z^0).
\]
That’s to say that we have solved the problem of exact controllability of (1.15)-(1.19). The coercivity is equivalent to: there exists a constant \(C > 0\) such that \(\forall (\phi^0, \phi^1) \in H^1_0(\Omega) \times L^2(\Omega)\)
\[
\int_0^T \int_{\Gamma_0} \left|\frac{\partial \phi}{\partial \nu}\right|^2 d\Gamma_0 dt \geq C \|(\phi^0, \phi^1)\|_{H^1_0(\Omega) \times L^2(\Omega)}^2.
\]
(2.24)
Thus, obtaining (2.24) is sufficient condition for exact controllability of (1.6)-(1.9). More precisely, we show that (1.6)-(1.9) is exactly controllable in time \( T > 0 \) if and only if (2.24) holds. Hence, HUM can reduce the problem of exact controllability to the obtention of such inequality (2.24) for (1.1)-(1.3). But we cannot hope to get (2.24) without any conditions, in fact several types of conditions have been considered and in [5], Bardos, Lebeau and Rauch also Burq and Gérard [6] gave a necessary and sufficient condition in the case of "very regular" geometrical domain.

### 3 Proof of main result and abstract setting

Before starting the proof of our result, we shall make the following hypothesis:

\[
\Sigma_0 \text{ allows the application of the Holmgren’s Uniqueness Theorem, (3.25)}
\]

where

\[
\Sigma_0 = \Gamma_0 \times [0, T].
\]

Let \( F \) be the completion of \( H^1_0(\Omega) \times L^2(\Omega) \) with respect to the norm

\[
\|(\phi^0, \phi^1)\|_F^2 = \int_0^T \int_{\Gamma_0} \left| \frac{\partial \phi}{\partial \nu} \right|^2 d\Gamma_0 dt,
\]

and such that

\[
F \subset L^2(\Omega) \times H^{-1}(\Omega).
\]

In fact, since we assume (3.25) holds true, the functional \( \|\cdot\|_F \) is a norm. Let us recall the Holmgren’s Uniqueness Theorem (see Hörmander [8]).

**Theorem 3.1.** If \( u \in D'(\Omega) \) is a solution of a differential equation

\[
P(x, D)u = 0
\]

with analytic coefficients, then \( u = 0 \) in a neighborhood of non-characteristic \( C^1 \) if this true on one side.

We recall that a \( C^1 \) surface \( S \subset \mathbb{R}^n \) with normal \( \xi \) at \( x \) is said to be non-characteristic at \( x \) for \( P(x, D) \) if

\[
P_m(x, \xi) \neq 0,
\]

where \( P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha \) is a differential operator with principal symbol

\[
P_m(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha.
\]

Since we suppose that (1.12) holds true, then \( \Sigma_0 \) doesn’t necessarily satisfy the geometrical control condition, but satisfying the condition for the Holmgren’s Uniqueness Theorem to apply, then

\[
\|(\phi^0, \phi^1)\|_F^2 = \int_0^T \int_{\Gamma_0} \left| \frac{\partial \phi}{\partial \nu} \right|^2 d\Gamma_0 dt
\]
is a norm strictly weaker than the $H^1_0(\Omega) \times L^2(\Omega)$.

Define the operator $\Lambda$ by

$$\Lambda : F \rightarrow F', \quad \Lambda(\phi^0, \phi^1) = (-\frac{\partial z}{\partial t}(x, 0), z(x, 0)).$$

An important ingredient of the proof of Proposition 1.2 is the following technical result.

**Lemma 3.2.** $\Lambda$ is a norm isomorphism of $F$ onto $F'$.

**Proof.** Clearly $\Lambda$ is a bounded linear operator. The backward problem associated to (1.1)-(1.3) is $\partial^2 \psi / \partial t^2 - \Delta \psi = 0$, $\psi = -\partial \phi / \partial \nu$, $\psi = 0$, $\partial \Omega \setminus \Gamma_0 \times (0, +\infty)$, $\psi(x, T) = \partial \psi / \partial t(x, T) = 0$, $\Omega$.

$F$ and $F'$ are in duality by: $\forall \phi = (\phi^0, \phi^1) \in F'$, $\bar{\phi} = (\bar{\phi}^0, \bar{\phi}^1) \in F$

$$\langle (\phi^0, \phi^1), (\bar{\phi}^0, \bar{\phi}^1) \rangle_{F', F} = \int_\Omega \phi^1 \bar{\phi}^0 - \phi^0 \bar{\phi}^1 \, dx,$$

we prove that $\Lambda$ is coercive. In fact, applying the Lax-Milgram theorem, it suffices to show the existence of a constant $c > 0$ such that

$$\langle \Lambda(\phi^0, \phi^1), (\phi^0, \phi^1) \rangle_{F', F} \geq c \| (\phi^0, \phi^1) \|_F^2, \quad \forall (\phi^0, \phi^1) \in F.$$

In fact, multiplying (2.20) by $\phi$ and integrating by parts, we obtain

$$0 = \int_0^T \int_\Omega \phi(\psi'' - \Delta \psi) \, dx \, dt = \left[ \int_\Omega (\phi \psi' - \phi' \psi) \, dx \right]_0^T$$

$$+ \int_0^T \int_\Omega \phi'' \psi \, dx \, dt - \int_0^T \int_\Omega (\Delta \psi) \phi \, dx \, dt$$

and therefore

$$\int_\Omega [\phi^0 \psi'(0) - \phi^1 \psi(0)] \, dx = \int_0^T \int_{\Gamma_0} \left| \frac{\partial \phi}{\partial \nu} \right|^2 \, d\Gamma_0 \, dt.$$

Consequently,

$$\langle \Lambda(\phi^0, \phi^1), (\phi^0, \phi^1) \rangle_{F', F} = \int_0^T \int_{\Gamma_0} \left| \frac{\partial \phi}{\partial \nu} \right|^2 \, d\Gamma_0 \, dt \geq c \| (\phi^0, \phi^1) \|_F^2, \quad \forall (\phi^0, \phi^1) \in H^1_0(\Omega) \times L^2(\Omega)$$

and by density argument, we get

$$\langle \Lambda(\phi^0, \phi^1), (\phi^0, \phi^1) \rangle_{F', F} = \int_0^T \int_{\Gamma_0} \left| \frac{\partial \phi}{\partial \nu} \right|^2 \, d\Gamma_0 \, dt \geq c \| (\phi^0, \phi^1) \|_F^2, \quad \forall (\phi^0, \phi^1) \in F.$$

\[\square\]
Proof. (of Theorem 1.2). As we have seen in the previous lemma, we have the following inequality

\[
\langle \Lambda(\phi^0, \phi^1), (\phi^0, \phi^1) \rangle_{F', F} = \int_0^T \int_{\Gamma_0} \left| \frac{\partial \phi}{\partial \nu} \right|^2 d\Gamma_0 dt \geq c\|\phi^0, \phi^1\|_F^2, \quad \forall (\phi^0, \phi^1) \in F,
\]
on the other hand

\[
\|\phi^0, \phi^1\|_F^2 \leq C \int_0^T \int_{\Gamma_0} \left| \frac{\partial \phi}{\partial \nu} \right|^2 d\Gamma_0 dt
\]

\[
= C \langle \Lambda(\phi^0, \phi^1), (\phi^0, \phi^1) \rangle_{F', F}
\]

\[
= C \langle (\psi'(0), -\psi(0)), (\phi^0, \phi^1) \rangle_{F', F}
\]

\[
\leq C\|(z^1, z^0)\|_{F'}\|(\phi^0, \phi^1)\|_F
\]
hence

\[
\|\phi^0, \phi^1\|_F \leq C\|(z^1, z^0)\|_{F'}.
\] (3.26)

Let \( g = -\frac{\partial \phi}{\partial \nu} \), then

\[
\|g\|^2_{L^2(0,T;L^2(\Gamma_0))} = \int_0^T \int_{\Gamma_0} \left| \frac{\partial \phi}{\partial \nu} \right|^2 d\Gamma_0 dt
\]

\[
= \langle (z^1, -z^0), (\phi^0, \phi^1) \rangle_{F', F}
\]

\[
\leq \|(z^1, z^0)\|_{F'}\|(\phi^0, \phi^1)\|_F
\]

by (3.26), we get

\[
\|g\|_{L^2(0,T;L^2(\Gamma_0))} \leq C\|(z^1, z^0)\|_{F'} \leq C\|(z^0, z^1)\|_{H^0_0(\Omega) \times L^2(\Omega)},
\]

and \( g \) derives the system (1.6)-(1.9) to zero.

Conversely, suppose that there exists a control \( g \in L^2([0, T], L^2(\Gamma_0)) \) satisfying (1.13) and that the solution of (1.6)-(1.9) verify (1.19), therefore, by the previous lemma we have in particular that \( \Lambda^{-1} \) is continuous, in particular it verifies

\[
\|\Lambda(\phi^0, \phi^1)\|_{F'} \geq C\|(\phi^0, \phi^1)\|_F, \quad \text{for some } C > 0,
\]

and then by continuous imbedding of \( F \) in \( L^2(\Omega) \times H^{-1}(\Omega) \), we get (1.12).
\[\square\]

3.1 Variational approach

In this section we will see how the weak controllability property of system (1.6)-(1.9) is a consequence of (1.12) by a minimization method which yields the control of minimal \( L^2(0, T; L^2(\Gamma)) \)-norm. Spaces \( L^2(\Omega) \times H^{-1}(\Omega) \) and \( H^0_0(\Omega) \times L^2(\Omega) \) are in duality by

\[
\langle (\phi^0, \phi^1), (z^0, z^1) \rangle = \langle \phi^0, z^1 \rangle_{H^{-1}, H^0_0} - \langle \phi^1, z^0 \rangle_{L^2},
\]
for all \((\phi^0, \phi^1) \in L^2(\Omega) \times H^{-1}(\Omega)\).

Let us consider the functional \(J : L^2(\Omega) \times H^{-1}(\Omega) \rightarrow \mathbb{R}\) defined by

\[
J(\phi^0, \phi^1) = \frac{1}{2} \int_0^T \int_{\Gamma_0} \left| \frac{\partial \phi}{\partial \nu} \right|^2 d\Gamma_0 dt + \langle (\phi^0, \phi^1), (z^0, z^1) \rangle.
\] (3.27)

where \(\phi\) is the solution of the homogenous system \((1.1)-(1.3)\) with inital data \((\phi^0, \phi^1) \in L^2(\Omega) \times H^{-1}(\Omega)\).

Thus we have:

**Theorem 3.3.** Let \((\phi^0, \phi^1) \in L^2(\Omega) \times H^{-1}(\Omega)\) and suppose that \((\tilde{\phi}^0, \tilde{\phi}^1) \in L^2(\Omega) \times H^{-1}(\Omega)\) is a minimizer of \(J\). If \(\tilde{\phi}\) is the corresponding solution of \((1.1)-(1.3)\) with initial data \((\tilde{\phi}^0, \tilde{\phi}^1)\) then

\[
g = -\frac{\partial \tilde{\phi}}{\partial \nu} \big|_{\Gamma_0} \tag{3.28}
\]

is a control which leads \((z^0, z^1)\) to zero in time \(T\).

Let us give sufficient conditions ensuring the existence of a minimizer of \(J\). For that we recall the following fundamental result in the calculus of variations which is a consequence of the so called Direct method for the calculus of variations. For a proof, see [7].

**Proposition 3.4.** Let \(H\) be a reflexive Banach space, \(K\) a closed convex subset of \(H\) and \(\phi : K \rightarrow \mathbb{R}\) is a function with the following properties:

1. \(\phi\) is convex
2. \(\phi\) is lower semi-continuous
3. If \(K\) is unbounded then \(\phi\) is coercive, i.e.,

\[
\lim_{\|x\| \to +\infty} \phi(x) = +\infty.
\]

Then \(\phi\) attains its minimum in \(K\), i.e., there exists \(x_0 \in K\) such that

\[
\phi(x_0) = \min_{x \in K} \phi(x). \tag{3.29}
\]

As a consequence, we get the following.

**Theorem 3.5.** Let \((\phi^0, \phi^1) \in H^1_0(\Omega) \times L^2(\Omega)\) and suppose that \((1.12)\) holds, that’s system \((1.1)-(1.3)\) is weakly observable in time \(T\). Then the functional \(J\) defined by \((3.27)\) has a unique minimizer \((\tilde{\phi}^0, \tilde{\phi}^1) \in L^2(\Omega) \times H^{-1}(\Omega)\).
Continuity and convexity are easy to prove. The existence of minimum of $J$ is ensured is also coercive, that’s
\[
\lim_{\|(\phi^0, \phi^1)\|_{E^{-1}} \to +\infty} J(\phi^0, \phi^1) = +\infty.
\]

In fact, coercivity of $J$ follows from (1.12), indeed
\[
J(\phi^0, \phi^1) \geq \frac{1}{2} \left( \int_0^T \int_{\Gamma_0} \left| \frac{\partial \phi}{\partial \nu} \right|^2 \, d\Gamma_0 dt - \|(z^0, z^1)\|_{E^{-1}} \|(\phi^0, \phi^1)\|_{E_0} \right)
\]
\[
\geq \frac{C}{2} \|(\phi^0, \phi^1)\|_{E^{-1}}^2 - \frac{1}{2} \|(z^0, z^1)\|_{E^{-1}} \|(\phi^0, \phi^1)\|_{E_0}.
\]

Hence, we conclude from the previous theorem that $J$ has a minimizer $(\tilde{\phi}^0, \tilde{\phi}^1) \in E^{-1}$. Next we prove that $J$ is strictly convex and hence the minimizer is unique. In fact, let $(\phi^0, \phi^1), (\psi^0, \psi^1) \in E^{-1}$ and $\lambda \in [0, 1]$. We have
\[
J(\lambda(\phi^0, \phi^1) + (1 - \lambda)(\psi^0, \psi^1)) = \lambda J(\phi^0, \phi^1) + (1 - \lambda)J(\psi^0, \psi^1) - \frac{\lambda(1 - \lambda)}{2} \int_0^T \int_{\Gamma_0} \left| \frac{\partial \phi}{\partial \nu} - \frac{\partial \psi}{\partial \nu} \right|^2 \, d\Gamma_0 dt.
\]

From (1.12) we deduce that
\[
\int_0^T \int_{\Gamma_0} \left| \frac{\partial \phi}{\partial \nu} - \frac{\partial \psi}{\partial \nu} \right|^2 \, d\Gamma_0 dt \geq \|(\phi^0, \phi^1) - (\psi^0, \psi^1)\|_{E^{-1}}^2,
\]
and hence, for any $(\phi^0, \phi^1) \neq (\psi^0, \psi^1),
\[
J(\lambda(\phi^0, \phi^1) + (1 - \lambda)(\psi^0, \psi^1)) < \lambda J(\phi^0, \phi^1) + (1 - \lambda)J(\psi^0, \psi^1).
\]

\[\square\]

**Remark 3.6.** According to the previous theorem and under hypothesis (1.12), system (1.6)-(1.9) is controllable, and this control may be obtained as in (3.28) from the solution of the homogenous system (1.11)-(1.3) with initial data minimizing the functional $J$.

### 3.2 Abstract framework

Let $H$ be a Hilbert space, and let $A_1 : D(A_1) \to H$ be a self-adjoint, positive, and boundedly invertible operator. We introduce the scale of Hilbert spaces $H_\alpha$, $\alpha \in \mathbb{R}$, as follows: for every $\alpha \geq 0$, $H_\alpha = D(A_1^\alpha)$, with the norm $\|z\|_\alpha = \|A_1^\alpha z\|_H$. The space $H_{-\alpha}$ is defined by duality with respect to the pivot space $H$ as follows: $H_{-\alpha} = H_\alpha^*$ for $\alpha > 0$. Equivalently, $H_{-\alpha}$ is the completion of $H$ with respect to the norm $\|z\|_{-\alpha} = \|A_1^{-\alpha} z\|_H$. 

The operator $A_1$ can be extended (or restricted) to each $H_\alpha$ such that it becomes a bounded operator

$$A_1 : H_\alpha \rightarrow H_{\alpha-1} \quad \forall \alpha \in \mathbb{R}. $$

Let $B_1$ be an unbounded linear operator from $U$ to $H_{-\frac{1}{2}}$, where $U$ is another Hilbert space. We identify $U$ with its dual, so that $U = U^*$. The systems we consider are described by

$$\ddot{z}(t) + A_1 z(t) = B_1 v(t), \quad (3.30)$$

$$z(0) = z^0, \quad \dot{z}(0) = z^1. \quad (3.31)$$

where $t \in [0, +\infty)$ is the time and $v \in L^2([0, +\infty); U)$. The equation (3.30) is understood as an equation in $H_{-\frac{1}{2}}$, i.e., all the terms are in $H_{-\frac{1}{2}}$.

Let us now consider the initial value problem

$$\ddot{\phi}(t) + A_1 \phi(t) = 0, \quad (3.32)$$

$$\phi(0) = z^0, \quad \dot{\phi}(0) = z^1, \quad (3.33)$$

It’s well known that (3.32)-(3.33) is well-posed in $D(A_1) \times D(A_1^2)$ and in $D(A_1^2) \times H$. System (3.30)-(3.31) is well-posed, in fact

**Proposition 3.7.** Suppose that $v \in L^2([0,T]; U)$ and that the solutions $\phi$ of (3.32)-(3.33) are such that $B_1^* \phi(\cdot) \in H^1([0,T]; U)$ and there exists a constant $C > 0$ such that

$$\|B_1^* \phi(\cdot)\|_{L^2(0,T; U)} \leq C \|z^0\|_{H^1} + \|z^1\|_{H_{-\alpha - \frac{1}{2}}}, \quad \forall (z^0, z^1) \in H_{\frac{1}{2}} \times H.$$

Then the system (3.30)-(3.31) admits a unique solution having the regularity

$$z \in C([0,T]; H_{\frac{1}{2}}) \cap C^1([0,T]; H).$$

Next, we give the definition of $\alpha$-weak observability of (3.30)-(3.31).

**Definition 3.8.** The system (3.32)-(3.33) is $\alpha$-weakly observable if there exist a time $T > 0$ and a constant $C_T > 0$ such that

$$\int_0^T \|B_1^* \phi(t)\|^2 dt \geq C_T \|z^0\|^2_{H_{\alpha}} + \|z^1\|^2_{H_{\alpha - \frac{1}{2}}}, \quad \forall (z^0, z^1) \in H_{\frac{1}{2}} \times H, \quad (3.34)$$

where $\alpha > -\frac{1}{2}$.

The system (3.32)-(3.33) is $\alpha$-weakly observable if it’s $\alpha$-weakly observable in some $T > 0$.

**Theorem 3.9.** (3.34) holds if and only if there exist a control $v \in L^2([0,T]; U)$ satisfying

$$\|v\|^2_{L^2(0,T)} \leq (\|z^0\|^2_{H_{\alpha + \frac{1}{2}}} + \|z^1\|^2_{H_{\alpha}}), \quad \forall (z^0, z^1) \in H_{1} \times H_{\frac{1}{2}}, \quad (3.35)$$

such that the solution of (3.30)-(3.31) satisfy

$$z \equiv 0, \quad t \geq T.$$
The previous theorem allows us to introduce a new notion of controllability said \( \alpha \)-weak controllability as follows.

**Definition 3.10.** With \( v \) as above in Theorem 3.9, system (3.30)-(3.31) is said to be \( \alpha \)-weakly controllable in time \( T > 0 \), i.e., \( v \) verify (3.35) and derives the system (3.30)-(3.31) to zero in time \( T > 0 \).

**Proof.** (of Theorem 3.9). Let \( F \) be the completion of \( H^{1/2} \times H \) with respect to the norm

\[
\| (z^0, z^1) \|_F = \int_0^T \| B_1^* \phi(t) \|_U^2 dt
\]

where \( \phi \) is a solution of (3.32)-(3.33) and such that

\[
F \subset H^{-\alpha} \times H^{-\alpha - 1/2}.
\]

Since (3.34) holds true, then for all \( T > 0 \) there exist \( C_T > 0 \) such that

\[
\int_0^T \| B_1^* \phi(t) \|_U^2 dt \geq C_T (\| (z^0, z^1) \|_F^2), \quad \forall (z^0, z^1) \in H_1 \times H_{1/2}.
\]

Thus \( F \) and \( F' \) are in duality by

\[
\langle (u^0, u^1), (v^0, v^1) \rangle_{F', F} = \int_\Omega u^0 v^1 - u^1 v^0 dx, \quad \forall (u^0, u^1) \in F', (v^0, v^1) \in F.
\]

and the backward problem associated to (3.32)-(3.33) is

\[
\begin{align*}
\ddot{\psi}(t) + A_1 \psi(t) &= B_1 v(t), \\
\psi(0) &= z^0, \quad \dot{\psi}(0) = z^1.
\end{align*}
\]

Define the following operator \( \Lambda : F \to F' \) by

\[
\Lambda(u^0, u^1) = (\frac{\partial}{\partial t} \psi(x, 0), -\psi(x, 0)).
\]

Hence we have

\[
\langle \Lambda(u^0, u^1), (v^0, v^1) \rangle_{F', F} = \int_\Omega u^0 v^1 - u^1 v^0 dx, \quad \forall (u^0, u^1) \in F, (v^0, v^1) \in F.
\]

It’s easy to check that \( \Lambda \) is linear and continuous operator, in particular

\[
\langle \Lambda(u^0, u^1), (u^0, u^1) \rangle_{F', F} = \int_0^T \| B_1^* \phi(t) \|_U^2 dt \geq C_T (\| (z^0, z^1) \|_F^2)
\]

and hence \( \Lambda \) is coercive, which imply by the Lax-Milgram lemma that \( \Lambda \) is an isomorphism between \( F \) and \( F' \).

Let \( v = B_1^* \phi \). One can easily verify that \( v \) is in \( L^2([0, T]; U) \) (it’s simply the closed-loop admissibility hypothesis or the direct inequality) and for

\[
\begin{align*}
\psi(0) &= z^0, \\
\dot{\psi}(0) &= z^1
\end{align*}
\]
the solution of (3.30)-(3.31) satisfy
\[ z \equiv 0, \quad \forall t \geq T, \]
and the control \( v \) satisfy (3.35).

Conversely, from (3.38) and the embedding of \( F \) in \( H_{-\alpha} \times H_{-\frac{\alpha}{2}} \), we easily get (3.34) and therefore system (3.32)-(3.33) is \( \alpha \)-weakly observable.

**Corollary 3.11.** Suppose that system (3.32)-(3.33) is \( \alpha \)-weakly observable (that’s equivalent to system (3.30)-(3.31) is \( \alpha \)-weakly controllable) and we consider the following observation
\[ y(t) = B^*z(t). \] (3.39)

If for fixed \( \delta > 0 \), the function defined by
\[ H(\lambda) = \lambda B^*_1(\lambda^2 I + A_1)^{-1}B_1 \in \mathcal{L}(U), \quad \forall \lambda \in \mathbb{C}_0 \]
is uniformly bounded on \( \mathbb{C}_\delta = \{ s \in \mathbb{C} | \text{Re } s = \delta > 0 \} \), then system (3.32)-(3.33) is weakly stable. See [2] for more details.

### 4 Application

We consider the following initial and boundary problem:

\[ \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = v(t)\delta_\xi, \quad (x,t) \in (0,1) \times (0, +\infty), \] (4.40)

\[ u(0, t) = u(1, t) = 0, \quad t \in (0, +\infty), \] (4.41)

\[ u(x, 0) = u^0(x), \quad \frac{\partial u}{\partial t}(x, 0) = u^1(x), \quad x \in (0, 1) \] (4.42)

where \( \xi \in \mathcal{S} \), \( \delta_\xi \) is the Dirac mass concentrated in the point \( \xi \in (0, 1) \). Let

\[ H = L^2(0,1), \quad U = \mathbb{R}, \quad H_{1/2} = H^1_0(0,1), \]

and

\[ A = -\frac{d^2}{dx^2}, \quad H_1 = H^2(0,1) \cap H^1_0(0,1), \quad Bk = k\delta_\xi, \quad \forall k \in \mathbb{R}. \]

The homogenous problem associated to (4.40)-(4.42) is

\[ \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = 0, \quad (0,1) \times (0, +\infty), \] (4.43)

\[ \phi(0, t) = \phi(1, t) = 0, \quad (0, +\infty), \] (4.44)

\[ \phi(x, 0) = u^0(x), \quad \frac{\partial \phi}{\partial t}(x, 0) = u^1(x), \quad x \in (0, 1) \] (4.45)

---

\[ ^\dagger \text{ We denote by } \mathcal{S} \text{ the set of all numbers } \rho \in (0, 1) \text{ such that } \rho \in \mathbb{Q} \text{ and if } [0,a_1,...,a_n,...] \text{ is the expansion of } \rho \text{ as a continued fraction, then } (a_n) \text{ is bounded. Let us notice that } \mathcal{S} \text{ is obviously uncountable and, by classical results on diophantine approximation (cf. [7], p. 120), its Lebesgue measure is equal to zero.} \]
According to Proposition 5.4 of Ammari-Tucsnak [2], see also [4], the observability inequality concerning the trace at the point \( x = \xi \) of the solutions of (4.43)-(4.45) reads as: For all \( T > 0 \) and \( \xi \in S \), there exists a constant \( C_{\xi,T} \) such that
\[
\int_0^T \phi^2(\xi,t) dt \geq C_{\xi,T}(\|u^0\|^2_{H^{-1}(0,1)} + \|u^1\|^2_{(H^2(0,1) \cap H^1_0(0,1))'}), \quad \forall (u^0, u^1) \in H^{1/2}_2 \times H.
\]

If we consider \( F \) as the completion of \( H^{1/2}_2 \times H \) with respect to the norm
\[
\|(u^0, u^1)\|^2_F = \int_0^T \phi^2(\xi,t) dt.
\]
and such
\[ F \subset H^{-1}(0,1) \times (H^2(0,1) \cap H^1_0(0,1))'. \]
If we put
\[
u^0(x) = \sum_{n \geq 1} a_n \sin(n\pi x), \quad u^1(x) = \sum_{n \geq 1} b_n \sin(n\pi x),
\]
with
\[ (na_n), \quad (b_n) \subset l^2(\mathbb{R}), \]
then the dual space of \( F \) with respect to the pivot space \( L^2(0,1) \) can be characterized by
\[
(u^0, u^1) \in F' \iff \sum_{n \geq 1} \frac{n^2 a_n^2 + b_n^2}{\sin^2(n\pi \xi)} < \infty,
\]
with \( u^0, u^1 \) as in (4.47). Therefore, inequality (4.46) and Theorem 3.9 gives the following corollary.

**Corollary 4.1.** For a given time \( T > 0 \) and \( \xi \in S \), there exists a control \( v \in L^2([0,T];\mathbb{R}) \) such that
\[
\|v\|^2_{L^2([0,T];\mathbb{R})} \leq C(\|u^0\|^2_{H^2(0,1) \cap H^1_0(0,1)} + \|u^1\|^2_{H^1_0(0,1)}),
\]
and such that the solution of (4.40)-(4.42) satisfy
\[ u \equiv 0, \quad \forall t \geq T. \]

**5 Comments**

More generally, the problem of observability refers to dominate the solution defined in \( \Omega \) of some PDE’s to the restriction on portion of the boundary in appropriate norms. For a large class of PDE’s such estimates are false without constraints on the Cauchy data or without geometric hypothesis. F. John [9] introduced estimates of Hölder and logarithmic dependency type that reads in our model case for wave equation with control acting in a portion of the boundary in the following way:
The logarithmic dependency type is the existence of a constant $C > 0$ such that for all $(u^0, u^1) \neq 0$, we have
\[
\|(u^0, u^1)\|_{H^1_0(\Omega) \times L^2(\Omega)} \leq \exp \left( C \frac{\|(u^0, u^1)\|_{H^1_0(\Omega) \times L^2(\Omega)}}{\|(u^0, u^1)\|_{L^2(\Omega) \times H^{-1}(\Omega)}} \right) \left\| \frac{\partial u}{\partial t} \right\|_{L^2(0,T \times \Gamma_0)}
\]
(5.48)
where $\beta \in (0,1)$. These estimates can be viewed as observability inequalities with low frequency where the quantity
\[
\frac{\|(u^0, u^1)\|_{H^1_0(\Omega) \times L^2(\Omega)}}{\|(u^0, u^1)\|_{L^2(\Omega) \times H^{-1}(\Omega)}}
\]
is a natural measure of the frequency of the wave.
By the same way, we can study the controllability concept associated to the weakly observability inequality (5.48).

5.1 Appendix

Proof. (of Proposition 1.1) [3]. The solution of (1.1)-(1.3) is explicitly given by:
\[
\phi(x, t) = \sum_{k \in \mathbb{N}^*} (\alpha_k e^{i\omega_k t} + \alpha_{-k} e^{-i\omega_k t}) \sin(k_1 \pi x_1) \sin(k_2 \pi x_2).
\]
(5.49)
with suitable coefficients $\alpha_k$, and where $k = (k_1, k_2)$, $\omega_k = \pi \sqrt{k_1^2 + k_2^2}$.
Now by using [14, Theorem 1], we first express the inequality (1.12) in terms of the Fourier series. We have
\[
\int_0^T \int_{\Gamma_0} |\partial_\nu \phi(x, t)|^2 d\Gamma_0(x) dt = \int_0^1 \int_0^1 |\partial_{x_1} \phi(0, x_2, t)|^2 dx_2 dt =
\]
\[
\int_0^1 \int_0^1 \left| \sum_{k \in \mathbb{N}^*} k_1 \pi (\alpha_k e^{i\omega_k t} + \alpha_{-k} e^{-i\omega_k t}) \sin(k_2 \pi x_2) \right|^2 dx_2 dt.
\]
By using the orthogonality of the family $(\sin(k_2 \pi x_2))_{k \in \mathbb{N}^*}$ in $L^2(0,1)$, we get
\[
\int_0^T \int_{\Gamma_0} |\partial_\nu \phi(x, t)|^2 d\Gamma_0(x) dt \asymp \sum_{k_2 \in \mathbb{N}^*} \int_0^T \left| \sum_{k_1 \in \mathbb{N}^*} k_1 (\alpha_k e^{i\omega_k t} + \alpha_{-k} e^{-i\omega_k t}) \right|^2 dt.
\]
On the other hand,
\[
\sum_{k \in \mathbb{N}^*} k_1^2 (|\alpha_k|^2 + |\alpha_{-k}|^2) \geq \sum_{k \in \mathbb{N}^*} (|\alpha_k|^2 + |\alpha_{-k}|^2) = \|(\phi^0, \phi^1)\|^2_{L^2(\Omega) \times H^{-1}(\Omega)}.
\]
Then, we apply again [14, Theorem 1], we take
\[
d = N = 2, \ \lambda_k = \omega_k = \pi \sqrt{k_1^2 + k_2^2}, \ \forall k \in (\mathbb{N}^*)^2, \ p_l = l, \ \forall l \in \mathbb{N}^*, \ \gamma_1 = \gamma_2 = \frac{\pi}{2\sqrt{2}}.
\]
We finally get (1.12) with $T > 8$, ie., for $T_0 = 8$. \qed
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