STEADY RICCI SOLITONS WITH HORIZONTALLY
ε-PINCHED RICCI CURVATURE

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ABSTRACT. In this paper, we prove that any κ-noncollapsed gradient steady Ricci soliton with nonnegative curvature operator and horizontally ε-pinched Ricci curvature must be rotationally symmetric. As an application, we show that any κ-noncollapsed gradient steady Ricci soliton $(M^n, g, f)$ with nonnegative curvature operator must be rotationally symmetric if it admits a unique equilibrium point and its scalar curvature $R(x)$ satisfies $\lim_{r(x)\to\infty} R(x)f(x) = C_0 \sup_{x \in M} R(x)$ with $C_0 > \frac{n-2}{2}$.

1. INTRODUCTION

As one of singular model solutions of Ricci flow, it is important to classify steady Ricci solitons under a suitable curvature condition [14]. In his celebrated paper [20], Perelman conjectured that all 3-dimensional κ-noncollapsed steady (gradient) Ricci solitons must be rotationally symmetric. The conjecture is solved by Brendle in 2012 [3]. For higher dimensions, under an extra condition that the soliton is asymptotically cylindrical, Brendle also proves that any κ-noncollapsed steady Ricci soliton with positive sectional curvature must be rotationally symmetric in [4]. In general, it is still open whether an $n$-dimensional κ-noncollapsed steady Ricci soliton with positive curvature operator is rotationally symmetric for $n \geq 4$. For κ-noncollapsed steady Kähler-Ricci solitons with nonnegative bisectional curvature, the authors have recently proved that they must be flat [10], [11].

Recall from [4],

Definition 1.1. An $n$-dimensional steady Ricci soliton $(M, g, f)$ is called asymptotically cylindrical if the following holds:

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1It is proved by Chen that any 3-dimensional ancient solution has nonnegative sectional curvature [6].
(i) Scalar curvature $R(x)$ of $g$ satisfies

$$\frac{C_1}{\rho(x)} \leq R(x) \leq \frac{C_2}{\rho(x)}, \quad \forall \rho(x) \geq r_0,$$

where $C_1, C_2$ are two positive constants and $\rho(x)$ denotes the distance of $x$ from a fixed point $x_0$.

(ii) Let $p_m$ be an arbitrary sequence of marked points going to infinity. Consider rescaled metrics $g_m(t) = r_m^{-1} \phi_t^* r_m g$, where $r_m R(p_m) = \frac{m-1}{2} + o(1)$ and $\phi_t$ is a one-parameter subgroup generated by $X = -\nabla f$. As $m \to \infty$, flows $(M, g_m(t), p_m)$ converge in the Cheeger-Gromov sense to a family of shrinking cylinders $(\mathbb{R} \times S^{n-1}(1), \tilde{g}(t)), t \in (0, 1)$. The metric $\tilde{g}(t)$ is given by

$$\tilde{g}(t) = dr^2 + (n-2)(2-2t)g_{S^{n-1}(1)},$$

where $S^{n-1}(1)$ is the unit sphere of euclidean space.

The purpose of present paper is to give an approach to verify whether a $\kappa$-noncollapsed steady Ricci soliton with positive curvature operator is asymptotically cylindrical. As a generalization of $\epsilon$-pinched Ricci curvature (cf. [13], [8]), we introduce a notion of horizontally $\epsilon$-pinched Ricci curvature as follows.

**Definition 1.2.** A steady soliton $(M, g, f)$ with positive Ricci curvature is called horizontal Ricci curvature $\epsilon$-pinched if $(M, g, f)$ admits a unique equilibrium point and there exist $r_0 > 0$ and $\epsilon > 0$ such that on each level set

$$\Sigma_r = \{x \in M| f(x) = r\}, \quad \forall r \geq r_0,$$

Ricci curvature is $\epsilon$-pinched, i.e.,

$$\overline{\text{Ric}}(v, v) \geq \epsilon R(x) \overline{g}(v, v), \quad \forall \, v \in T_x \Sigma_r,$$

where $\overline{\text{Ric}}$ is Ricci curvature of $(\Sigma_r, \overline{\mathcal{g}})$ with the induced metric $\overline{\mathcal{g}}$ on $\Sigma_r$ as a hypersurface of $(M, g)$.

A point $o$ in $(M, g, f)$ is called an equilibrium one if $\nabla f(o) = 0$. Such a point is unique if Ricci curvature of $(M, g, f)$ is positive. Moreover, each $\Sigma_r$ is smooth as long as $r$ is sufficiently large (cf. Lemma 2.1 below).

The following is our main result in this paper.

**Theorem 1.3.** Any $\kappa$-noncollapsed steady Ricci soliton $(M, g, f)$ with non-negative curvature operator and horizontally $\epsilon$-pinched Ricci curvature is asymptotically cylindrical in sense of Brendle. As a consequence, $(M, g, f)$ must be rotationally symmetric.
Steady Ricci solitons with \( \epsilon \)-pinched Ricci curvature have been studied by many people [15], [8], [19], [9], etc.. For example, Ni proves that any steady Ricci soliton with \( \epsilon \)-pinched Ricci curvature and nonnegative sectional curvature must be flat [19]. For steady Kähler-Ricci solitons, the authors prove that Ni’s result is still true even without nonnegative sectional curvature condition [9]. In order to study asymptotic behavior of steady Ricci solitons with horizontally \( \epsilon \)-pinched Ricci curvature, we will give a classification of \( \kappa \)-solutions with \( \epsilon \)-pinched Ricci curvature in Section 3.

In the proof of Theorem 1.3, we essentially show that for any sequence \( p_i \to \infty \), there exists a subsequence \( p_{i_k} \to \infty \) such that

\[
(M, g_{p_{i_k}}(t), p_{i_k}) \to (\mathbb{R} \times S^{n-1}, \tilde{g}(t), p_\infty), \text{ for } t \in (-\infty, 1),
\]

where \( g_{p_{i_k}}(t) = R(p_{i_k})g(R^{-1}(p_{i_k})t) \) and \( (\mathbb{R} \times S^{n-1}, \tilde{g}(t)) \) is a shrinking cylinders flow, i.e.

\[
\tilde{g}(t) = dr^2 + (n - 2)((n - 1) - 2t)g_{S^{n-1}}.
\]

It is interesting to mention that as one of steps in the proof of Theorem 1.3, we prove that scalar curvature of \( g \) decays uniformly (cf. Corollary 3.6). We say that scalar curvature \( R(x) \) of a Riemannian manifold \( (M, g) \) decays uniformly if

\[
|R(x)| \to 0, \text{ as } \rho(x) \to \infty.
\]

(1.3)

By the way, we also point that the Ricci curvature pinching condition of \( \tilde{g} \) in (1.2) in Theorem 1.3 can be replaced by (3.29) for the ambient metric \( g \) if (1.3) is true (cf. Corollary 3.11).

As an application of Theorem 1.3, we study \( \kappa \)-noncollapsed and positively curved steady Ricci solitons with a linear curvature decay,

\[
\lim_{\rho(x) \to \infty} R(x)f(x) = C_0 \sup_{x \in M} R(x),
\]

(1.4)

where \( C_0 > 0 \) is a constant. It is known that there are constants \( c_1, c_2 > 0 \) such that

\[
c_1 \rho(x) \leq f(x) \leq c_2 \rho(x), \quad \rho(x) \geq r_0,
\]

(1.5)

if a steady Ricci soliton has positive Ricci curvature and an equilibrium point (cf. [5]). By check of pinching condition of horizontal Ricci curvature, we are able to prove

**Theorem 1.4.** Any \( n \)-dimensional \( \kappa \)-noncollapsed steady Ricci soliton \( (M, g, f) \) with nonnegative curvature operator and a unique equilibrium point must be rotationally symmetric if scalar curvature \( R(x) \) of \( g \) satisfies (1.4) with \( C_0 > \frac{n - 2}{2} \).
The paper is organized as follows. In Section 2, we give a classification of \( \kappa \)-solutions with \( \epsilon \)-pinched Ricci curvature. Theorem 1.3 and Theorem 1.4 will be proved in Section 3, 4 respectively. Some related estimates for steady Ricci solitons are used from [10].

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2. **Classification of Ricci pinched \( \kappa \)-solutions**

Throughout the paper, we say that \((M, g, f)\) is a (gradient) steady Ricci soliton if
\[
R_{ij} = \nabla_i \nabla_j f.
\]
The following lemma shows that each level set \( \Sigma_r \) of \((M, g, f)\) is diffeomorphic to a sphere if it admits an equilibrium point and has positive Ricci curvature.

**Lemma 2.1.** Let \((M, g, f)\) be an \(n\)-dimensional noncompact steady Ricci soliton with positive Ricci curvature and an equilibrium point \(o\). Then any \( \Sigma_r \) is diffeomorphic to \( S^{n-1} \), where \( r > f(o) \).

**Proof.** By soliton equation and positivity of Ricci curvature, \( f \) is strictly convex. Moreover, the equilibrium point is unique. Thus by the Morse Lemma, there exists a coordinates system \((x_1, \ldots, x_n)\) in a neighbourhood \(U\) near \(o\) such that
\[
f(x) = f(o) + x_1^2 + \cdots + x_n^2.
\]
It follows that \( \Sigma_{f(o)+\epsilon} = \{ x \in M | f(x) = f(o) + \epsilon \} \) is diffeomorphic to \( S^{n-1} \) for any small \( \epsilon > 0 \). Since \( \Sigma_r \) is evolved along the gradient flow of \( -\nabla f \), each \( \Sigma_r \) is diffeomorphic to \( \Sigma_{f(o)+\epsilon} \), where \( r > f(o) + \epsilon \). The lemma is proved. \( \square \)

This lemma will play an important role in the analysis of the asymptotic behavior of steady Ricci solitons below. In this section, we give a classification of steady Ricci solitons with \( \epsilon \)-pinched Ricci curvature. Recall

**Definition 2.2.** A Riemannian manifold \((M, g)\) is called Ricci curvature \( \epsilon \)-pinched, if it has nonnegative Ricci curvature and its Ricci curvature satisfies
\[
\text{Ric}(x) \geq \epsilon R(x)g(x),
\]
where \( \epsilon \) is a positive constant independent of \( x \in M \). Similarly, a Ricci flow \((M, g(t))\) on \( t \in (a, b) \) is called Ricci curvature \( \epsilon \)-pinched, if it has nonnegative Ricci curvature along the flow and its Ricci curvature satisfies
\[
\text{Ric}(x, t) \geq \epsilon R(x, t)g(x, t), \forall t \in (a, b).
\]
where $\epsilon$ is a positive constant independent of $x \in M$ and $t \in (a, b)$.

we prove

**Theorem 2.3.** Let $(M, g(t))$ be a simply connected and Ricci curvature $\epsilon$-pinched $\kappa$-solution. Then $(M, g(t)) = (M_1, g_1(t)) \times (M_2, g_2(t)) \times \cdots \times (M_k, g_k(t))$, and each $(M_i, g_i(t))$ is an Einstein metrics flow on a simply connected and compact symmetric space. More precisely, each $(M_i, g_i(t))$ is one of the following three types:

(i) $(S^n_i, g(t))$ is a Ricci flow with positive constant sectional curvature;
(ii) $(\mathbb{C}P^n_i, g(t))$ is a Ricci flow with constant positive bisectional curvature;
(iii) $(M_i, g(t))$ is an Einstein metrics flow on an irreducible symmetric space except (i) and (ii).

To prove Theorem 2.3 we need the following classification result.

**Lemma 2.4.**

i) A simply connected Einstein manifold with positive curvature operator must be $S^n$ with constant sectional curvature.
ii) A simply connected shrinking soliton with positive curvature operator must be $S^n$ with constant sectional curvature.
iii) A simply connected Kähler-Einstein manifold with positive bisectional curvature must be $\mathbb{C}P^n$ with constant bisectional curvature.
iv) A simply connected Kähler-Ricci soliton with positive bisectional curvature must be $\mathbb{C}P^n$ with constant bisectional curvature.

**Proof.** i) It is clear that the manifold is compact. Let $g(t) = (1 - 2\lambda t)g$, where $n\lambda$ is scalar curvature of $g$. Then $g(t)$ satisfies Ricci flow equation,

(2.3) \[ \frac{\partial g(t)}{\partial t} = -2\text{Ric}(g(t)), \]

By Wilking’s result [22], there exists $r(t)$ such that

(2.4) \[ r(t)g(t) \to g_{S^n}, \text{ as } t \to \frac{1}{2\lambda}, \]

where $g_{S^n}$ is the metric with constant sectional curvature $1$ on $S^n$. Thus $g$ must be a metric with constant sectional curvature on $S^n$.

ii) By Munteanu and Wang’s result [18], any shrinking soliton with non-negative curvature and positive Ricci curvature must be compact. Then by the same argument as in i), the manifold should be isometric to one with constant sectional curvature on $S^n$.
iii) and iv) are known, since the manifold is compact and so it is biholomorphic to \( \mathbb{CP}^n \) by Frankel’s conjecture. By the uniqueness of Kähler-Einstein metrics [1], \((M, g)\) has constant bisectional curvature. Another proof can come from [7] by using the Ricci flow.

\[ \Box \]

**Proof of Theorem 2.3.** By Theorem 11.3 for \( \kappa \)-solutions in [20], there exists a sequence \((M, \tau_i^{-1}g(t - \tau_i), p_{\tau_i}) (\tau_i \to \infty)\) converging to a limit \((M_\infty, g_\infty(t), p_\infty)\), which is a non-flat shrinking solitons solution with nonnegative curvature operator. Since the pinching condition is preserved under rescaling, \((M_\infty, g_\infty(t))\) is also \(\epsilon\)-Ricci pinched. We claim that Ricci curvature of \(g_\infty(t)\) is strictly positive. If the claim is not true, Ricci curvature of \((M_\infty, g_\infty(t))\) vanishes along some direction somewhere. Then scalar curvature of the soliton is zero at some point \(p \in M_\infty\) and some time \(t_0\) by pinched \(\epsilon\)-Ricci curvature property. We may assume that \((M_\infty, g_\infty((0)))\) is a Ricci flow generated by the non-flat shrinking Ricci soliton \((M_\infty, g_\infty(t_0))\) which satisfies

\[
\frac{\partial g'_\infty(\tau)}{\partial \tau} = -2\text{Ric}'(\infty)(\tau), \ \tau \in (-\infty, 1),
\]

and scalar curvature \(R'\infty(\tau)\) is zero at \((p, t'_0)\). It turns that \(R'\infty(\tau)\) satisfies evolution equation

\[
\frac{\partial R'\infty(\tau)}{\partial \tau} = \Delta R'\infty(\tau) + 2|\text{Ric}'(\infty)(\tau)|^2.
\]

Note that \(R'\infty(\tau)\) attains its minimum at \((p, t'_0)\) for some \(t'_0 < 1\). Thus \(R'\infty(t) \equiv 0\) by the maximum principle. Namely, \((M_\infty, g_\infty(t_0))\) is flat. This is impossible!

Applying Munteanu and Wang’s result for shrinking solitons with nonnegative sectional curvature and positive Ricci curvature [18], we know that \((M_\infty, g_\infty(t))\) is compact. Next we show that \(M\) is also compact and it is diffeomorphic to \(M_\infty\). Let \(g_\tau = \tau^{-1}g(t_0 - \tau)\), for \(\tau \in (1, \infty)\). By the convergence, there exists a sequence of isometries \(\Phi_{\tau_i} : B(p_{\tau_i}, r_{\tau_i}; g_{\tau_i}) \to \Phi_{\tau_i}(B(p_{\tau_i}, r_{\tau_i}; g_{\tau_i}); g_{\tau_i}) (\subset M_\infty)\) such that \(\Phi_{\tau_i}(B(p_{\tau_i}, r_{\tau_i}; g_{\tau_i}))\) exhaust \(M_\infty\) as \(r_{\tau_i} \to \infty\). Since \(M_\infty\) is compact, \(\Phi_{\tau_i}(B(p_{\tau_i}, r_{\tau_i}; g_{\tau_i})) = M_\infty\), for \(\tau\) large enough. Hence, \(B(p_{\tau_i}, r_{\tau_i}; g_{\tau_i})\) is diffeomorphic to \(M_\infty\) and it is compact without boundary. As a consequence, \(B(p_{\tau_i}, r_{\tau_i}; g_{\tau_i})\) is a complete manifold and it must cover \(M\). Hence \(M\) is compact and diffeomorphic to \(M_\infty\). By a classification theorem of compact manifolds with nonnegative curvature operator [22], \((M, g(t))\) is isometric to flow \((M_1, g_1(t)) \times (M_2, g_2(t)) \times \cdots \times (M_k, g_k(t))\), where each \((M_i, g_i(t))\) is one of the following three types:

(i) Ricci flow on \(S^n\) with positive curvature operator;
(ii) Ricci flow on $\mathbb{C} \mathbb{P}^n$ with positive sectional curvature;
(iii) Einstein metrics flow on a symmetric space except (i) and (ii).

It remains to deal with case (i) and (ii). For case (i), we may assume that $M = S^n$. Then $(M, g_\infty(t))$ is a round spheres flow by Lemma 2.4. Thus $(M, g_\tau, p_\tau)$ converge to a round sphere as $\tau \to \infty$. On the other hand, by Theorem 3.1 in [16] (also see [2]), the curvature pinching property is preserved along the flow $(M, g(t))$. Hence $(M, g(t))$ is getting more and more round from largely negative $t$. Therefore, $(M, g(t))$ must be a round sphere for all $t$.

For case (ii), we may assume that $M = \mathbb{C} \mathbb{P}^n$. Then $g(t)$ are all Kähler metrics. Moreover, there is some $C_0 > 0$ such that Kähler classes of $\hat{g}(t) = C_0^{-1}e^{-t}g(C_0(1 - e^{-t}))$ are all $2\pi c_1(\mathbb{C} \mathbb{P}^n)$. It follows that $\hat{g}(t)$ satisfies the normalized Kähler-Ricci flow,

$$\frac{\partial \hat{g}}{\partial t} = -\text{Ric}(\hat{g}) + \hat{g}, \quad t \in (-\infty, \infty).$$

By the convergence of $g_\tau$ and Lemma 2.4, it is easy to see that there exists a sequence of $\hat{g}(t_i)$ which converges to the Fubini-Study metric of $\mathbb{C} \mathbb{P}^n$ as $t_i \to -\infty$. Now we can apply the stability result for Kähler-Ricci flow near a Kähler-Einstein metric in [23] to conclude that $\hat{g}(t)$ is the Fubini-Study metric for any $t$. Hence, $g(t)$ are all Kähler metrics with positive constant bisectional curvature.

3. Proof of Theorem 1.3

In this section, we prove Theorem 1.3 under the condition (1.2). First, we recall a result of asymptotic behavior for $\kappa$-noncollapsed steady Ricci solitons with nonnegative curvature operator proved in Theorem 1.6, [10].

**Theorem 3.1.** Let $(M, g, f)$ be an $n$-dimensional noncompact $\kappa$-noncollapsed steady Ricci soliton with a unique equilibrium point. Suppose that $M$ has nonnegative curvature operator and positive Ricci curvature. Then, for any $p_i \to \infty$, the sequence of rescaled flows $(M, R(p_i)g(R^{-1}(p_i)t), p_i)$ converges subsequently to a Ricci flow $(\mathbb{R} \times N, \bar{g}(t))$ ($t \in (-\infty, 0]$) in the Cheeger-Gromov topology, where

$$\bar{g}(t) = ds \otimes ds + g_N(t),$$

and $(N, g_N(t))$ is a $\kappa$-noncollapsed Ricci flow with nonnegative curvature operator on $N$ with dimension $n - 1$.

**Remark 3.2.** The proof of Theorem 3.1 is based on an estimate

$$\frac{C}{\rho(x)} \leq R(x), \text{ if } \rho(x) \geq r_0 >> 1,$$

where $C > 0$. This estimate is crucial in establishing the required curvature pinching properties along the flow. The details of this estimate are given in Section 3.2.
where $C > 0$ is a uniform constant. (3.1) is proved for $\kappa$-noncollapsed Kähler-Ricci solitons with nonnegative bisectional curvature in Proposition 4.3 in [10]. For steady Ricci solitons in Theorem 3.1 it is still true since we assume the existence of equilibrium points. (3.1) will be frequently used below (cf. (3.9)).

Under horizontally $\epsilon$-pinched Ricci curvature, we can also control the local structure of steady Ricci solitons.

**Lemma 3.3.** Let $(M, g, f)$ be a $\kappa$-noncollapsed steady Ricci soliton as in Theorem 1.3. Let $p_r \in \Sigma_r$ such that $R(p_r) = \inf_{x \in \Sigma_r} R(x)$. Then, for any $k \in \mathbb{N}$, there exists $r(k, \epsilon)$ such that for any $r \geq r(k, \epsilon)$

\[
 B(p_r, \frac{k}{\sqrt{R_{\max}}}; g_r) \subset M'_{r,k} \subset B(p_r, 2\pi \sqrt{n - 2} \epsilon + \frac{2k}{\sqrt{C_0}}; g_r),
\]

(3.2)

where $g_r = R(p_r)g$, $R_{\max} = \max_{x \in M} R(x)$, $C_0 = R_{\max} - \sup_{x \in \Sigma f(o) + 1} R(x) > 0$, and $M'_{r,k}$ is a subset of $M$ defined by

\[
 M'_{r,k} = \{ x \in M | r - \frac{k}{\sqrt{R(p_r)}} \leq f(x) \leq r + \frac{k}{\sqrt{R(p_r)}}, R(p_r) = \inf_{x \in \Sigma_r} R(x) \}.
\]

Proof. Let $\overline{g} = g|_{\Sigma_r}$ be an induced metric of hypersurface $\Sigma_r$ of $(M, g)$ and $\overline{\text{Ric}}(\overline{g})$ Ricci curvature of $\overline{g}$ with components $\overline{R}_{i'j'}$, where indices $i', j'$ are corresponding to a basis of vector fields on $\Sigma_r$. By (1.2), we have

\[
 \overline{R}_{i'j'} \geq \epsilon R_{i'j'}, \quad \overline{R}_{i'j'} \geq \epsilon R(p_r) \overline{g}_{i'j'}, \quad \forall x \in \Sigma_r,
\]

as long as $r$ is large enough. By the Myer’s theorem, the diameter of $\Sigma_r$ is bounded by

\[
 \text{diam}(\Sigma_r, g) \leq \text{diam}(\Sigma_r, \overline{g}_r) \leq 2\pi \sqrt{n - 2} \epsilon R(p_r).
\]

(3.3)

It follows

\[
 \Sigma_r \subset B(p_r, A(\epsilon); g_r),
\]

(3.4)

where $A(\epsilon) = 2\pi \sqrt{\frac{n-2}{\epsilon}}$.

Let $\phi_t$ be a one parameter subgroup generated by $-\nabla f$. Then there exists a largely negative $t_0$ such that (cf. Lemma 4.1, [10]),

\[
 c_1 |t| \leq r(\phi_t(p), o) \leq c_2 |t|, \quad \forall t \leq t_0,
\]

(3.5)
where the uniform constant $c_1$ and $c_2 > 0$ independent of $p \in \Sigma_{f(o)+1}$. Moreover, by the identity,

$$\left|\nabla f\right|^2 + R = R_{\text{max}},$$

(3.6)

where $R_{\text{max}} = R(o)$, we have

$$\frac{dR(\phi_t(q))}{dt} = 2 \text{Ric}(\nabla f, \nabla f) > 0, \ \forall \ q \in M \setminus \{o\}.$$  

(3.7)

(3.7) means that scalar curvature is decreasing along the integral curves $\phi_t(q)$. Thus by (3.5), there exists a large $r_0$ such that for any $x$ with $\rho(x, o) \geq r_0$

$$0 \leq R(x) \leq \sup_{z \in \Sigma_{f(o)+1}} R(z) < R_{\text{max}},$$

(3.8)

On the other hand, by Remark 3.2 together with (1.5), there exists $C > 0$ such that

$$R(p_r) \cdot r \geq C > 0.$$  

(3.9)

Then for a fixed $k$, we have

$$\frac{r - k}{\sqrt{R(p_r)}} \to 1, \ \text{as} \ r \to \infty.$$  

Since

$$C_1 \rho(x, o) \leq f(x) \leq C_2 \rho(x, o),$$

we see

$$\rho(x, o) \to \infty, \ \text{as} \ r \to \infty, \ \forall \ x \in M'_{r,k}.$$  

Hence by (3.8), we get

$$0 \leq R(x) \leq \sup_{z \in \Sigma_{f(o)+1}} R(z) < R_{\text{max}}, \ x \in M'_{r,k}$$

as long as $r$ is large enough. This implies

$$\sqrt{C_0} \leq |\nabla f|(x) \leq \sqrt{R_{\text{max}}}, \ x \in M'_{r,k},$$

(3.10)

where $C_0 = R_{\text{max}} - \sup_{x \in \Sigma_{f(o)+1}} R(x) > 0.$
For any $q \in M_{r,k}'$, there exists $q' \in \Sigma_r$ such that $\phi_s(q) = q'$ for some $s \in \mathbb{R}$.

Then by (3.3) and (3.10), we have

$$d(q, p_r) \leq d(q', p_r) + d(q, q')$$

$$\leq \text{diam}(\Sigma_r, g) + L(\phi_{\tau}|_{[0, s]})$$

$$\leq 2\pi \sqrt{\frac{n - 2}{\epsilon R(p_r)}} + \left| \int_0^s \frac{d\phi_{\tau}(q)}{d\tau} |d\tau| \right|$$

$$= 2\pi \sqrt{\frac{n - 2}{\epsilon R(p_r)}} + \int_0^s |\nabla f(\phi_{\tau}(q))| |d\tau|$$

$$\leq 2\pi \sqrt{\frac{n - 2}{\epsilon R(p_r)}} + \frac{1}{\sqrt{C_0}} |f(q) - f(p_r)|$$

$$\leq \left( 2\pi \sqrt{\frac{n - 2}{\epsilon}} + \frac{2k}{\sqrt{C_0}} \right) \cdot \frac{1}{\sqrt{R(p_r)}}$$

Thus

$$M_{r,k}' \subset B(p_r, A(\epsilon) + \frac{2k}{\sqrt{C_0}} : g_r).$$

This proves the second relation in (3.2).

For any $q \in M$, let $\gamma(s)$ be any curve connecting $p_r$ and $q$ such that $\gamma(s_1) = q$ and $\gamma(s_2) = p_r$. Then,

$$\mathcal{L}(q, p_r) = \int_{s_1}^{s_2} \sqrt{\langle \gamma'(s), \gamma'(s) \rangle} ds$$

$$\geq \int_{s_1}^{s_2} \frac{\langle \gamma'(s), \nabla f \rangle}{|\nabla f|} ds$$

$$\geq \frac{1}{\sqrt{R_{\max}}} \int_{s_1}^{s_2} \langle \gamma'(s), \nabla f \rangle ds$$

$$= \frac{1}{\sqrt{R_{\max}}} |f(p_r) - f(q)|.$$
In particular, for \( q \in M \setminus M'_{r,k} \), we get
\[
d(q, p_r) \geq \frac{1}{\sqrt{R_{\text{max}}}} : \frac{k}{\sqrt{R(p_r)}}.
\]
Hence
\[
B_{d}(p_r, \frac{k}{\sqrt{R_{\text{max}}}}) \subset M'_{r,k}.
\]

(3.11)

The first relation in (3.2) is also true. \( \square \)

**Lemma 3.4.** Under the condition of Theorem 1.3, there exists a constant \( C(\epsilon) > 0 \) independent of \( x, y \) and \( r \) such that
\[
\sup_{x, y \in \Sigma_r} \frac{R(x)}{R(y)} \leq C(\epsilon), \text{ for } r > f(o).
\]

(3.12)

**Proof.** Let \( p_r \) be chosen as same as in Lemma 3.3 for \( r > f(o) \). We consider rescaled \( \kappa \)-solutions \( (M, R(p_r)g(R(p_r)^{-1}t), p_r) \). As in the proof of Theorem 3.3 in [10], we see that for any \( d > 0 \) there exists a constant \( C(d) > 0 \) such that
\[
R_{R(p_r)}(x) \leq C(d), \ \forall \ x \in B(p_r, d; R(p_r)g).
\]

(3.13)

On the other hand, by (3.4), there exists a constant \( r(\epsilon) \) such that for any \( r \geq r(\epsilon) \) it holds
\[
\Sigma_r \subset B(p_r, 2\pi \sqrt{\frac{n-2}{\epsilon}}; R(p_r)g).
\]

(3.14)

Then there exists a \( C(\epsilon) \) such that
\[
\frac{R(x)}{R(y)} \leq \frac{R(x)}{R(p_r)} \leq C(\epsilon), \ \forall \ x, y \in \Sigma_r, \ \forall \ r \geq r(\epsilon).
\]

Since \( \{x \in M | f(x) \leq r(\epsilon)\} \) is compact, the lemma is true. \( \square \)

**Lemma 3.5.** Under the condition of Theorem 1.3, for any \( x \in M \setminus \{o\} \),
\[
R(\phi_t(x)) \to 0, \text{ as } t \to -\infty.
\]

**Proof.** If the lemma is not true, by (3.7), there exists \( x_0 \) such that
\[
\lim_{t \to -\infty} R(x_0, t) = \lim_{t \to -\infty} R(\phi_t(x_0)) = C_0 > 0.
\]

Let \( r_t = f(\phi_t(x_0)) \) and \( p_{r_t} \) be chosen as in Lemma 3.3. By Lemma 3.4, \( R(p_{r_t}) \geq C_0/C(\epsilon) \). On the other hand, for any \( x \in \{y \in M | f(y) \geq f(\phi_t(x_0))\} \), there exists \( t_x \leq -1 \) such that \( f(x) = f(\phi_{t_x}(x_0)) \). Thus
\[
R(x) \geq R(p_{r_{t_x}}) \geq \frac{C_0}{C(\epsilon)}.
\]
This implies that there exists a uniform constant $C'_0$ such that
\begin{equation}
R(x) \geq C'_0, \quad \forall \, x \in M.
\end{equation}

However, by Lemma 4.3 in [12], we know
\[ \frac{1}{\text{Vol}(B(o, r))} \int_{B(o, r)} R(x) \, dv \leq \frac{C}{r}, \quad \forall \, r > 0, \]
for some uniform $C$ independent of $r$. This is a contradiction of (3.15). The lemma is proved.

As a corollary of Lemma 3.4 and Lemma 3.5, we prove

**Corollary 3.6.** Let $(M, g, f)$ be a $\kappa$-noncollapsed steady Ricci soliton with nonnegative curvature operator. If $g$ has horizontally $\epsilon$-pinched Ricci curvature, scalar curvature $R(x)$ of $g$ decays uniformly. Namely, $R(x)$ satisfies (1.3).

**Proof.** We suffice to prove that $R(y_i) \to 0$ for any sequence of $\{y_i\}$ with $f(y_i) \to \infty$. Let $r_i = f(y_i)$ and $p_{r_i}$ be defined as in Lemma 3.3. By Lemma 3.4
\begin{equation}
\frac{R(y_i)}{R(p_{r_i})} \leq C(\epsilon).
\end{equation}

On the other hand, for a fixed $x_0$, there is a $t_i$ such that $f(\phi_{t_i}(x_0)) = r_i = f(p_{r_i})$. Thus by Lemma 3.5 we have
\[ R(p_{r_i}) \leq R(\phi_{t_i}(x_0)) \to 0, \quad \text{as} \, r_i \to \infty.\]

Hence $R(y_i) \to 0$ as $i \to \infty$. \hfill \Box

**Remark 3.7.** By Corollary 3.6, the constant $C_0$ at relation (3.2) in Lemma 3.3 can be chosen by $R_{\max}$ (cf. Lemma 3.10 below).

Combining Theorem 3.1, Lemma 3.3 and Corollary 3.6, we prove

**Lemma 3.8.** Let $(M, g, f)$ be a $\kappa$-noncollapsed steady Ricci soliton as in Theorem 1.3 and $p_r \in \Sigma_r$ chosen as in Lemma 3.3. Then for any sequence of $r \to \infty$, there exists a subsequence $r_i \to \infty$ such that
\[ (M, g_{r_i}(t), p_{r_i}) \to (\mathbb{R} \times S^{n-1}, \tilde{g}(t), p_{\infty}), \quad \text{for} \, t \in (-\infty, 0], \]
where $g_{r_i}(t) = R(p_{r_i})g(R^{-1}(p_{r_i})t)$ and $(\mathbb{R} \times S^{n-1}, \tilde{g}(t))$ is a shrinking cylinders flow, namely,
\[ \tilde{g}(t) = ds \otimes ds + (n - 2)(n - 1 - 2t)g_{S^{n-1}}. \]
Proof. By Theorem 3.1 for any sequence of \( r \to \infty \), there exists a subsequence of \( r_i \to \infty \) such that

\[
(M, g_{r_i}(t), p_{r_i}) \to (M_{\infty}, \bar{g}(t), p_{\infty}), \text{ for } t \in (-\infty, 0],
\]

where \((M_\infty, \bar{g}(t)) = (\mathbb{R} \times N, ds^2 + g_N(t))\) and \(g_N(t)\) satisfies the Ricci flow equation for \( t \in (-\infty, 0) \]. In the following, we first need to show that \( N \) is diffeomorphic to \( S^{n-1} \). Fix a point \( x_0 \in M \setminus \{ o \} \) and let \( x_r = \phi_r(x_0) \) such that \( f(x_r) = r_i \). We are going to construct diffeomorphism \( \Phi_{r_i} : M'_{r_i,k} \to \Phi_{r_i}(M'_{r_i,k}) \subset \mathbb{R} \times S^{n-1} \) and show that \((\Phi_{r_i}(M'_{r_i,k}), (\Phi_{r_i}^{-1})^* g_{r_i}(0), \Phi_{r_i}(p_{r_i}))\) subsequently converge to \((\mathbb{R} \times S^{n-1}, \bar{g}(0), p_{\infty})\) in \( C^\infty_{\text{loc}} \) sense, where \( M'_{r_i,k} \) is a subset of \( M \) as defined in Lemma 3.3.

By Lemma 2.1 we know that the level set \( \Sigma_{f(o)+1} \) is diffeomorphic to \( S^{n-1} \). For any \( x \in M \), there exists a unique \( t_x \) and \( \Phi \in \Sigma_{f(o)+1} \) such that \( x = \phi_{t_x}(\Phi) \). We define \( \Phi_{r_i}(x) = ((f(x) - f(x_r)) \sqrt{R(p_{r_i})}, \Phi) \in \mathbb{R} \times \Sigma_{f(o)+1} \) for \( r \to \infty \). Since \( |\nabla f| > 0 \) on \( M \setminus \{ o \} \), \( \Phi_{r_i} \) is essentially a diffeomorphism from \( M'_{r_i,k} \) to \( \mathbb{R} \times S^{n-1} \). Note that \( \Phi_{r_i}(M'_{r_i,k}) \) exhausts \( \mathbb{R} \times S^{n-1} \) as \( k \to \infty \) by Lemma 3.3. On the other hand, by Lemma 3.6 there exist constants \( C_1 \) and \( C_2 \) depends only on \( \epsilon \) and \( n \) such that

\[
B(p_{r_i}, C_1^{-1}k; g_{r_i}(0)) \subset M'_{r_i,k} \subset B(p_{r_i}, C_2k; g_{r_i}(0)).
\]

Thus \( \Phi_{r_i}(B(p_{r_i}, k; g_{r_i}(0))) \) also exhausts \( \mathbb{R} \times S^{n-1} \). As a consequence, \((\Phi_{r_i}(M'_{r_i,k}), (\Phi_{r_i}^{-1})^* g_{r_i}(0), \Phi_{r_i}(p_{r_i}))\) subsequently converges to \((\mathbb{R} \times S^{n-1}, \bar{g}(0), p_{\infty})\) in \( C^\infty_{\text{loc}} \) sense. Hence \( \mathbb{R} \times S^{n-1} \) is diffeomorphic to \( M_{\infty} \). Therefore, to prove that \( N \) is diffeomorphic to \( S^{n-1} \), it reduces to show that the limit flow \( \bar{g}(t) \) splits along the direction of \( \mathbb{R} \) in \( \mathbb{R} \times S^{n-1} \).

Fix any \( d_0 > 0 \). Let \( g_{r_i} = g_{r_i}(0) \) and \( X_{(i)} = R(p_{r_i})^{-\frac{1}{2}} \nabla f \). Then as in (3.13), it holds

\[
|\text{Ric}_{g_{r_i}}(x)|_{g_{r_i}}(x) \leq C(d_0), \forall x \in B(p_{r_i}, d_0; g_{r_i}).
\]

Namely,

\[
|\text{Ric}|(x) \leq C(d_0) R(p_{r_i}), \forall x \in B(p_{r_i}, d_0; g_{r_i}).
\]

By Corollary 3.6 it follows

\[
\sup_{B(p_{r_i}, d_0; g_{r_i})} |\nabla_{(g_{r_i})} X_{(i)}|_{g_{r_i}} = \sup_{B(p_{r_i}, d_0; g_{r_i})} \frac{|\text{Ric}|}{\sqrt{R(p_{r_i})}} \leq C(d_0) \sqrt{R(p_{r_i})} \to 0.
\]

On the other hand, by Shi’s higher order estimate, we have

\[
\sup_{B(p_{r_i}, d_0; g_{r_i})} |\nabla^m_{(g_{r_i})} X_{(i)}|_{g_{r_i}} \leq C(n) \sup_{B(p_{r_i}, d_0; g_{r_i})} |\nabla^{m-1}_{(g_{r_i})} \text{Ric}(g_{r_i})|_{g_{r_i}} \leq C_3.
\]


Thus
\[
|\nabla (\tilde{g}(0)) X(\infty)|_{\tilde{g}(0)} = \lim_{k \to \infty} |\nabla (g_{r_k}) X(i)|_{g_{r_k}} = 0,
\]
where the convergence is uniform on \(B(p_{r_k}, d_0; g_{r_k})\). This means that \(X(\infty)\) is parallel. It remains to show that \(X(\infty)\) is tangent to \(\mathbb{R}\) in \(\mathbb{R} \times S^{n-1}\).

Let \(\frac{\partial}{\partial w}\) be the vector field tangent to \(\mathbb{R}\) in \(\mathbb{R} \times S^{n-1}\). By the construction of \(\Phi_{r_k}\), we have
\[
(\Phi_{r_k})_* (X(i)) = \frac{\partial}{\partial w}.
\]
Then
\[
X(\infty) = \lim_{i \to \infty} (\Phi_{r_k})_* (X(i)) = \frac{\partial}{\partial w}.
\]
Moreover, by (3.6) and Corollary 3.6,
\[
|X(i)|_{g_{r_k}} (x) = |\nabla f| (p_{r_k}) = \sqrt{R_{\max}} + o(1) > 0, \quad \forall \ x \in B(p_{r_k}, d_0; g_{r_k}),
\]
as long as \(r_i\) is large enough. Thus, \(X(\infty)\) is nonzero and is tangent to \(\mathbb{R}\) in \(\mathbb{R} \times S^{n-1}\). Since we have already proved that \(X(\infty)\) is parallel, \((\mathbb{R} \times S^{n-1}, \tilde{g}(0))\) must split off a line along \(\mathbb{R}\) in \(\mathbb{R} \times S^{n-1}\), and so does the limit flow \(\tilde{g}(t)\). This proves that \(N\) is diffeomorphic to \(S^{n-1}\).

Secondly, we prove that \((S^{n-1}, g_{S^{n-1}}(t))\) has \(\epsilon\)-pinched Ricci curvature. In other words, Ricci curvature of \((M_\infty, \tilde{g}(t))\) is \(\epsilon\)-pinched along the vectors vertical to \(X(\infty)\). By Gauss formula, we have
\[
R(X, Y, Z, W) = \overline{R}(X, Y, Z, W) + \langle B(X, Z), B(Y, W) \rangle - \langle B(X, W), B(Y, Z) \rangle,
\]
where \(X, Y, Z, W \in TS_r\) and \(B(X, Y) = (\nabla_X Y)^\perp\). Note that
\[
B(X, Y) = \langle \nabla_X Y, \nabla f \rangle \cdot \frac{\nabla f}{|\nabla f|^2} = [\nabla_X (Y, \nabla f) - \langle Y, \nabla_X \nabla f \rangle] \cdot \frac{\nabla f}{|\nabla f|^2} = - \text{Ric}(X, Y) \cdot \frac{\nabla f}{|\nabla f|^2},
\]
Then
\[
R_{i'j'} = \overline{R}_{i'j'},
\]
\[
+ R(\frac{\nabla f}{|\nabla f|}, e_i', e_{j'}, \frac{\nabla f}{|\nabla f|}) - \frac{1}{|\nabla f|^2} \sum_k (R_{i'j'} R_{k'k'} - R_{i'k'} R_{k'j'}),
\]
where indices \(i', j', k'\) are corresponding to a basis of vector fields on \(TS_r\). Since \(g_\infty(0)\) splits along \(X(\infty)\), by the convergence of the rescaled metrics, we have
\[
R(\frac{\nabla f}{|\nabla f|}, e_i', e_{j'}, \frac{\nabla f}{|\nabla f|}) = o(1) R_{i'j'}, \quad \forall \ x \in \ B(p_i, d_0; g_{r_k}).
\]
On the other hand, by (3.19), it is easy to see
\[
0 \leq \frac{1}{|\nabla f|^2} \sum_k (R_{i'j'}R_{k'k'} - R_{i'k'}R_{k'j'}) \leq \frac{3}{2} R_{i'j'}.
\]
Thus by (3.20) and (1.2), we get
\[
R_{i'j'} \geq \frac{1}{3} R_{i'j'} \geq \frac{\epsilon}{3} R_{i'j'}, \text{ on } B(p_i, d_0; g_{r_i}),
\]
when \( r_i \) are large enough. Since the above relation is independent of sequence of \( \{p_i\} \) by Theorem 3.1, we prove in fact,
\[
R_{i'j'} \geq \frac{1}{3} R_{i'j'} \geq \frac{\epsilon}{3} R_{i'j'}, \text{ on } B(p_r, d_0; g_{r}), \forall r > r_1 > 1.
\]

We show that (3.21) holds along the flow \( g_{\tau}(t) \). By (3.4), it is easy to see
\[
\{ x \in M \mid f(x) > r_1 \} \subseteq \bigcup_{r > r_1} B(p_r, d_0; g_r),
\]
where \( d_0 \geq 2\pi \sqrt{\frac{n-2}{\epsilon}} \). Then by (3.21),
\[
R_{i'j'} \geq \frac{\epsilon}{3} R_{i'j'}, \text{ on } \{ x \in M \mid f(x) > r_1 \}.\]
Since \( \{ x \in M \mid f(x) \leq r_1 \} \) is compact and Ricci curvature is positive, we obtain
\[
(3.22) \quad R_{i'j'}(x) \geq \epsilon_0 R_{i'j'}(x), \forall x \in M,
\]
for some \( \epsilon_0 > 0 \). Note that (3.22) is preserved under the metric scaling. Thus (3.22) holds along the flow \( g_{\tau}(t) \) for any \( \tau > 0 \). By the convergence of rescaled metrics, the Ricci curvature of \( (M_\infty, \bar{g}(t)) \) is \( \epsilon_0 \)-pinched along the vector fields vertical to \( X(\infty) \). Therefore, Ricci curvature of \( (N = S^{n-1}, g_N(t)) \) is \( \epsilon_0 \)-pinched.

By the above two steps, we see that \( (N, g_N(t)) \) is a \( \kappa \)-solution with \( \epsilon \)-pinched Ricci curvature. By Theorem 2.3 \( (N, g_N(t)) \) must be a shrinking sphere flow. Note that \( R^{(\infty)}(p_\infty, 0) = 1 \). Then it is easy to see that
\[
(N, g_N(t)) = g_{S^{n-1}}(t) = (n-2) [(n-1) - 2t] g_{S^{n-1}}.\]
The lemma is proved.

In Lemma 3.8, we prove the asymptotically cylindrical behavior of the steady soliton for a special rescaling sequence with base points \( p_{r_i} \). In the following, we show that the same result holds for an arbitrary sequence.

**Lemma 3.9.** Under the condition and notations of Lemma 3.8, let \( p_i \to \infty \) be any sequence. Then by taking a subsequence of \( p_i \) if necessary, we have
\[
(M, g_{p_i}(t), p_i) \to (\mathbb{R} \times S^{n-1}, \bar{g}(t), p_\infty), \text{ for } t \in (-\infty, 0],
\]
where \( g_{p_i}(t) = R(p_i)g(R^{-1}(p_i)t), \ (\mathbb{R} \times S^{n-1}, \tilde{g}(t)) \) is a shrinking cylinders flow defined as in Lemma 3.8.

**Proof.** By Corollary 3.6 there exists a \( C(\epsilon) \) such that \( R(x) \leq C(\epsilon), \ \forall \ x, p \in \Sigma_r. \) By (1.2), it follows

\[
\overline{R}_i' j'(x) \geq \epsilon R(x) g_{i' j'} \geq \epsilon C(\epsilon)^{-1} R(p) g, \ \forall \ x, p \in \Sigma_r.
\]

Thus, by the Myer’s theorem, we get as in (3.3),

\[
diam(\Sigma_r, g) \leq 2\pi \sqrt{(n-2)C(\epsilon)} \frac{\epsilon R(p)}{\epsilon R(p)}, \ \forall \ x, p \in \Sigma_r.
\]

Following the argument in Lemma 3.3 (also see Remark 3.7), we see that for any \( k \in \mathbb{N} \), there exists \( r'_{k, \epsilon} \) such that for any \( r \geq r'_{k, \epsilon} \) and \( p \in \Sigma_r \) it holds

\[
B(p, \frac{k}{\sqrt{R_{\max}}}; g_{p_i}) \subset M'_{p, k} \subset B(p, 2\pi \sqrt{(n-2)C(\epsilon)} \frac{\epsilon R(p)}{\epsilon R(p)} + \frac{2k}{\sqrt{R_{\max}}}; g_{p_i}),
\]

where \( g_{p_i} = R(p_i)g \) and \( M'_{p, k} \) is defined as

\[
M'_{p, k} = \{x \in M | f(p) - \frac{k}{\sqrt{R(p)}} \leq f(x) \leq f(p) + \frac{k}{\sqrt{R(p)}}\}.
\]

Once (3.23) is true, we can use the argument in the proof of Lemma 3.8 to the sequence \((M, g_{p_i}(t), p_i)\) to prove Lemma 3.9.

As a corollary of Lemma 3.9 we get

**Corollary 3.10.** Under the condition of Theorem 1.3 we have

\[
\lim_{r \to \infty} \sup_{x, y \in \Sigma_r} |\frac{R(x)}{R(y)} - 1| = 0.
\]

**Proof.** Suppose that (3.24) is not true. Then, we can find \( \delta > 0 \) and two sequences \( \{x_i\}, \{y_i\} \to \infty \) such that \( f(x_i) = f(y_i) \) and

\[
|\frac{R(x_i)}{R(y_i)} - 1| > \delta, \ \text{as} \ i \to \infty.
\]

Applying Lemma 3.9 to the sequence \( \{x_i\} \), we get

\[
(M, g_{x_i}(t), x_i) \to (\mathbb{R} \times S^{n-1}, g(t), x_\infty),
\]

where \( g_{x_i}(t) = R(x_i)g(R^{-1}(x_i)t) \) and \( g(t) \) is defined as in Lemma 3.9. By (3.23), we have

\[
y_i \in \Sigma_{f(x_i)} \subset B(x_i, 2\pi \sqrt{(n-2)C(\epsilon)} \frac{\epsilon}{\epsilon}; g_{x_i}),\]

where \( g_{x_i} = R(x_i)g \).
where $g_{x_i} = g_{x_i}(0)$. According to the convergence of $(M, g_{x_i}(t), x_i)$, we can find a sequence of diffeomorphism $\Phi_i$ such that $\Phi_i^*(g_{x_i}(t))$ converges to $\tilde{g}(t)$ in $C^\infty_{loc}$ topology and $\Phi_i(y_i) \in B(x, 3\sqrt{\frac{(n-2)C(\epsilon)}{\epsilon}}; \tilde{g}(0))$. As a consequence, there is a subsequence $y_{i_k}$ such that $\Phi_{i_k}(y_{i_k}) \to y_\infty$ in $B(x, 3\sqrt{\frac{(n-2)C(\epsilon)}{\epsilon}}; \tilde{g}(0))$. Note

$$\tilde{R}(q, 0) \equiv 1, \quad \forall q \in \mathbb{R} \times S^{n-1}.$$ 

Thus

$$\frac{\tilde{R}(y_{i_k}, 0)}{\tilde{R}(x_{i_k}, 0)} \to \tilde{R}(y_\infty, 0) = 1, \quad as \ i_k \to \infty.$$ 

This contradicts to (3.25). Hence, the corollary is true. $\square$

Since the sequence in Lemma 3.9 and the sequence in Definition 1.1 are the same up to a scale, to finish the proof of Theorem 1.3, we need to derive the curvature decay property (i) in Definition 1.1.

**Proof of Theorem 1.3.** Let $\phi_t$ be the one parameter subgroup generated by $-\nabla f$ and $g(t) = \phi_t^* g$ as before. Let any $p \in M$ such that $\nabla f(p) \neq 0$ and $p_i = \phi_{t_i}(p)$, where $t_i \to -\infty$. Then by Lemma 3.9 we may assume that sequence $(M, g_{r_i}(t), p_i)$ converges to a shrinking cylinders flow $(\mathbb{R} \times S^{n-1}, \tilde{g}(t))$, where

$$\tilde{g}(t) = ds \otimes ds + (n-2)(n-1) - 2t \delta_{i_{n-1}}.$$ 

Thus scalar curvature $\tilde{R}(-, t)$ of $\tilde{g}(t)$ is given by

$$\tilde{R}(-, t) = \frac{n-1}{(n-1)-2t}.$$ 

It follows

$$\frac{\partial}{\partial t} \tilde{R}(p_\infty, 0) = \frac{2}{n-1}.$$ 

On the other hand, by the convergence of $(M, g_{r_i}(t), p_i)$, we have

$$\frac{\partial}{\partial t} \tilde{R}(p_\infty, 0) = \lim_{i \to \infty} \frac{1}{R^2(p_i, 0)} \frac{\partial}{\partial t} R(p_i, 0) = \lim_{i_k \to \infty} \frac{1}{R^2(p_i, -t_{i_k})} \frac{\partial}{\partial t} R(p_i, -t_{i_k}).$$ 

Hence we get

$$\lim_{i \to \infty} F'(t_i) = \frac{n-1}{2},$$ 

where $F(t) = R^{-1}(p, -t)$. Since $t_i$ is an arbitrary sequence,

$$\lim_{t \to -\infty} F'(t) = \frac{n-1}{2},$$ 

This implies

$$\lim_{t \to -\infty} \frac{1}{tR(p, -t)} = \lim_{t \to -\infty} F'(t) = \frac{n-1}{2}.$$
Therefore, we derive

\[ \lim_{t \to -\infty} R(\phi_t(p)) |t| = \frac{n-1}{2}, \text{ as } t \to -\infty. \]

By the identity (3.6) and Corollary 3.6, we have

\[ \lim_{t \to -\infty} \frac{f(\phi_{-t}(p))}{t} = \lim_{t \to -\infty} \frac{df(\phi_{-t}(p))}{dt} = \lim_{t \to -\infty} |\nabla f|^2 = R_{\text{max}}. \]

On the other hand, by Corollary 3.10, it holds

\[ \lim_{r \to \infty} \sup_{x, y \in \Sigma_r} \left| \frac{R(x)}{R(y)} - 1 \right| = 0. \]

Since \( f(\phi_t(p)) \) goes to \( \infty \) as \( t \to -\infty \), we see that for any \( x \in M \setminus \{o\} \) with \( f(x) \geq r_0 \), there exists a \( t_x < 0 \) such that \( f(x) = f(\phi_{t_x}(p)) \). Thus

\[ \lim_{f(x) \to \infty} \left| \frac{R(x)}{R(\phi_{t_x}(p))} - 1 \right| = 0. \]

Combining this with (3.27) and (3.28), we deduce

\[ \lim_{f(x) \to \infty} R(x)f(x) = \lim_{t_x \to -\infty} R(\phi_{t_x}(p))f(\phi_{t_x}(p)) = \frac{n-1}{2}R_{\text{max}}. \]

By (1.5), we finally get the property (i) in Definition 1.1.

If (1.2) is replaced for the ambient metric \( g \) by

\[ \text{Ric}(v, v) \geq \epsilon R(x)g(v, v), \forall v \in T_x \Sigma_r, \ r > r_0, \]

we give another version of Theorem 1.3 as follows.

**Corollary 3.11.** Any \( \kappa \)-noncollapsed steady Ricci soliton \( (M, g, f) \) with nonnegative curvature operator, positive Ricci curvature and a uniform scalar curvature decay must be rotationally symmetric, if \( (M, g, f) \) admits a unique equilibrium point and there exist \( r_0 > 0 \) and \( \epsilon > 0 \) such that the pinching condition (3.29) holds.

**Proof.** By Theorem 1.3, it suffices to verify that \( (M, g, f) \) satisfies (1.2). Let \( e_{i'}(i' = 1, 2, \cdots, n-1) \) be normal eigenvector fields of \( \text{Ric}(\nabla f) \). Then we have Gauss equation

\[ \text{Ric}(\nabla f, e_{i'}, e_{i'}, \nabla f) - \frac{1}{|\nabla f|^2} \sum_{k'} (R_{i'i'k'k'} - R_{i'i'k'k'}) \]

Note that \( R(x) \) decays uniformly. We have

\[ |\nabla f|(x) \geq \frac{\sqrt{R_{\text{max}}}}{2}, \forall r(x) > r_0, \]
for some \( r_0 > 0 \). Thus by (3.29), it follows
\[
|\sum_{k'} (R_{i'i'} R_{k'k'} - R_{i'k'} R_{k'i'})| \leq R_{i'i'} R + \sum_{k'} R_{i'k'} R_{k'i'}
\]
\[
(3.31)
\]

Also we have
\[
|R(\nabla f, e_i', e_{i'}, \nabla f)| \leq \sum_{i'=1}^{n-1} R(\nabla f, e_i', e_{i'}, \nabla f)|
\]
\[
= |\text{Ric}(\nabla f, \nabla f)|
\]
\[
= |\langle \nabla R, \nabla f \rangle| - \frac{2|\nabla f|^2}{2|\nabla f|^2}
\]
\[
= \frac{|\Delta R + 2|\text{Ric}|^2|}{2|\nabla f|^2}.
\]
\[
(3.32)
\]

On the other hand, by Proposition 3.8 in [10], we see that there exists a constant \( C > 0 \) such that
\[
|\Delta R(x)| \leq C, \ \forall \ x \in M.
\]

Then by (3.32) and (3.29), we get
\[
|R(\nabla f, e_i', e_{i'}, \nabla f)| \leq \frac{(C + 2)R^2}{|\nabla f|^2} \leq \frac{(C + 2)R}{|\nabla f|^2} R_{i'i'} = o(1) R_{i'i'}.
\]
Combining this with (3.30) and (3.31), we obtain
\[
R_{i'i'}(x) \geq \frac{1}{2} R_{i'i'}(x) \geq \frac{\epsilon}{2} R(x), \ \text{as} \ r(x) > r_0.
\]
This implies (1.2).

4. Proof of Theorem 1.4

In this section, we apply Theorem 1.3 to prove Theorem 1.4 by verifying (1.2) for steady Ricci solitons under the curvature decay condition (1.4).

We begin with

**Lemma 4.1.** Let \( p_i \in M \to \infty \) and \((M, g_i(t), p_i)\) be a sequence of rescaling Ricci flows, where \( g_i(t) = R(p_i) g(R^{-1}(p_i)t) \). Suppose that \((M, g_i(t), p_i)\) converges to a limit flow \((M_\infty, g_\infty(t), p_\infty)\) in the Cheeger-Gromov topology. Then scalar curvature \( R^{(\infty)}(x,t) \) of \( g_\infty(t) \) satisfies
\[
(4.1) \quad (1 - \frac{t}{C_0}) R^{(\infty)}(x,t) \equiv 1, \ \forall \ x \in M_\infty, \ t \in (-\infty, 0].
\]
if \((M, g, f)\) admits a unique equilibrium point and it satisfies (1.4).

**Proof.** First we note that \(R(g)\) decays uniformly by (1.4) and (1.5). For any \(x \in B(p_\infty, g_\infty(0); r_0) \subset M_\infty\), we choose a sequence of \(x_i \in B(p_i, 2r_0; g_i)\) which converges to \(x\) in Cheeger-Gromov topology, where \(g_i = R(p_i)g_i\). Since the proof of relation (3.11) is still true for any \(k > 0\) when \(p_r\) is replaced by any \(p_i\), we have

\[
B(p_i, 2r_0; g_i) \subset \{y \in M| f(p_i) - \frac{2r_0\sqrt{R_{\text{max}}}}{\sqrt{R(p_i)}} \leq f(y) \leq f(p_i) + \frac{2r_0\sqrt{R_{\text{max}}}}{\sqrt{R(p_i)}}\}.
\]

By (1.4), it follows

\[
\lim_{i \to \infty} \frac{f(x_i)}{f(p_i)} = 1.
\]

Thus

\[
R^{(\infty)}(x, t) = \lim_{i \to \infty} \frac{R(x_i, R^{-1}(p_i)t)}{R(p_i)} = \lim_{i \to \infty} \frac{R(x_i, R^{-1}(p_i)t)}{R(x_i)}.
\]

Fix a point \(p \in M \setminus \{o\}\). Then (3.28) holds. Choose \(\tau_i < 0\) such that \(f(\varphi_{\tau_i}(p)) = f(x_i)\). We claim

\[
\lim_{i \to \infty} \frac{f(\varphi_{R^{-1}(p_i)_t}(x_i))}{f(\varphi_{\tau_i, R^{-1}(p_i)_t}(p))} = 1, \ \forall t \leq 0.
\]

It suffices to consider the case \(t < 0\). By (1.4) and (1.2), we have

\[
\frac{f(\varphi_{R^{-1}(p_i)_t}(x_i))}{|R^{-1}(p_i)t|} = \frac{f(\varphi_{R^{-1}(p_i)_t}(p)) - f(\varphi_{\tau_i}(p))}{|R^{-1}(p_i)t|} + \frac{f(x_i)}{|R^{-1}(p_i)t|}
= \frac{\|\int_{R^{-1}(p_i)_t} \nabla f|^2 ds\|}{|R^{-1}(p_i)t|} + \frac{f(x_i)}{|R^{-1}(p_i)t|}
\to R_{\text{max}}(1 + \frac{C_0}{|t|}), \ \text{as } i \to \infty.
\]

Similarly,

\[
\frac{f(\varphi_{R^{-1}(p_i)_t}(x_i))}{|R^{-1}(p_i)t|} = \frac{f(\varphi_{R^{-1}(p_i)_t}(x_i)) - f(x_i)}{|R^{-1}(p_i)t|} + \frac{f(x_i)}{|R^{-1}(p_i)t|}
\to R_{\text{max}}(1 + \frac{C_0}{|t|}), \ \text{as } i \to \infty.
\]

Thus (1.4) comes from the above two relations.

By (1.4) and the identity (3.6), we have

\[
\lim_{t \to \infty} \frac{f(\varphi_{-t}(p))}{t} = \lim_{t \to \infty} \frac{df(\varphi_{-t}(p))}{dt} = \lim_{t \to \infty} |\nabla f|^2 = R_{\text{max}}.
\]
Combining this with (1.4), (4.4) and (4.2), we obtain from (4.3),
\[
R^{(\infty)}(x, t) = \lim_{i \to \infty} \frac{f(x_i)}{f(\phi_{R^{-1}(p_i)t}(x_i))}
\]
\[
= \lim_{i \to \infty} \frac{f(\phi_{t_i}(p))}{\tau_i} \cdot \frac{\tau_i}{\tau_i + R^{-1}(p_i)t}
\]
\[
= \lim_{i \to \infty} \frac{\tau_i + f(x_i)t/(C_0 \cdot R_{\max})}{\tau_i}
\]
\[
= \lim_{i \to \infty} \frac{\tau_i + f(\phi_{t_i}(p))t/(C_0 \cdot R_{\max})}{\tau_i}
\]
\[
= \frac{C_0}{C_0 - t}.
\]
This proves (4.1).

\[\square\]

**Proof of Theorem 4.4.** It suffices to verify that horizontally Ricci curvature of \((M, g)\) is \(\epsilon\)-pinched for some \(\epsilon > 0\). We use the contradiction argument. Then there exist a sequence of points \(p_i \to \infty\) and vectors \(v^{(i)} \in T_{p_i} \Sigma f(p_i)\) such that
\[
\text{Ric}(v^{(i)}, v^{(i)}) \to 0, \text{ as } i \to \infty.
\]
By Theorem 3.1, we may assume that \((M, g_i(t), p_i)\) converges to \((M_{\infty}, \tilde{g}(t), p_{\infty})\), where \(g_i(t) = R(p_i)g(R^{-1}(p_i)t), (M_{\infty}, g_{\infty}(t)) = (\mathbb{R} \times N, dr^2 + g_N(t))\).

Let \(X^{(i)}(i) = R(p_i)^{-1/2} \nabla f\). Then, for any fixed \(r > 0\), we have
\[
\lim_{i \to \infty} \sup_{B(p_i, r; g_i)} |X^{(i)}(i)|_{g_i} = \lim_{i \to \infty} \sup_{B(p_i, r; g_i)} \sqrt{R_{\max} - R(x)} = \sqrt{R_{\max}},
\]
where \(g_i = R(p_i)g\). It follows
\[
\sup_{B(p_i, r; g_i)} |\nabla^k_{(g_i)} X^{(i)}(i)|_{g_i} \leq C_0 \sup_{B(p_i, r; g_i)} |\nabla^{k-1}_{(g_i)} \text{Ric}|_{g_i} \leq C.
\]
Thus we may assume that \(X^{(i)}(i)\) converges to \(X^{(\infty)}\). On the other hand, by Lemma 4.1 we have
\[
R^{(i)}(p, 0) \to 1, \forall p \in B(p_i, r; g_i),
\]
where \(R^{(i)}\) are scalar curvatures of \(g_i\). Hence similar to (3.18), we have
\[
|\nabla_{(g_i)} X^{(\infty)}(i)|_{g_{\infty}}(0) = \lim_{i \to \infty} |\nabla_{(g_i)} X^{(i)}(i)|_{g_i} = \lim_{i \to \infty} \frac{|\text{Ric}|_{(g_i)}}{R^{(i)}(p_i)} = 0.
\]
This implies that the limit manifold will split off a real line along \(X^{(\infty)}\). As a consequence, \(X^{(\infty)}\) is tangent to \(\mathbb{R}\) in \(\mathbb{R} \times N\).
Now, we prove that \( N \) is compact. By Lemma 4.4, we have
\[
\Delta_{\tilde{g}(t)} R^{(\infty)}(q, t) \equiv 0,
\]
\[
\frac{\partial R^{(\infty)}(q, t)}{\partial t} = \frac{1}{C_0} (R^{(\infty)}(q, t))^2.
\]
By flow equation
\[
\frac{\partial R^{(\infty)}(q, t)}{\partial t} = \Delta^{(\infty)} R^{(\infty)}(q, t) + 2|\text{Ric}^{(\infty)}|^2(q, t),
\]
It follows
\[(4.6) \quad (R^{(\infty)}(q, t))^2 = 2C_0 |\text{Ric}^{(\infty)}|^2(q, t).
\]
Let \( \lambda_1(q, t) \leq \lambda_2(q, t) \leq \cdots \leq \lambda_{n-1}(q, t) \) be the eigenvalues of \( \text{Ric}(g_N(t)) \). By the condition \( C_0 > \frac{n-2}{2} \), we see that there exists a constant \( \delta(n) > 0 \) such that
\[(4.7) \quad \frac{\lambda_1(q, t)}{\lambda_{n-1}(q, t)} \geq \delta(n).
\]
This implies that \( N \) is compact.

At last, we check the pinching condition. Choose an orthonormal basis \( \{e_1, \ldots, e_n\} \) w.r.t \( g_\infty(0) \) of \( T_{p_\infty} M_\infty \) such that \( e_n = \frac{X^{(\infty)} (q, t)}{|X^{(\infty)} (q, t)|} \). By the convergence of \( X(i) \), we can choose a sequence of orthonormal bases \( \{e_1^{(i)}, \ldots, e_n^{(i)}\} \) w.r.t \( g_i \) of \( T_{p_i} M \) such that \( e_n^{(i)} = \frac{X^{(i)} (q, t)}{|X^{(i)} (q, t)|} \),
\[\text{span}\{e_1^{(i)}, \ldots, e_{n-1}^{(i)}\} = T_{p_i}\Sigma_f(p_i)\]
and each \( e_k^{(i)} \) converge to \( e_k \) for \( 1 \leq k \leq n-1 \). Then by (4.7), we have
\[
\text{Ric}^{(\infty)}(e_k, e_k) \geq \frac{\delta(n)}{n} R^{(\infty)}(p_\infty), \quad \forall 1 \leq k \leq n-1,
\]
\[
\text{Ric}^{(\infty)}(e_k, e_l) = 0, \quad \text{for } k \neq l.
\]
By the convergence, it follows
\[
\text{Ric}^{(i)}(e_k^{(i)}, e_k^{(i)}) \geq (1 - \varepsilon_i) \frac{\delta(n)}{n} R^{(i)}(p_i), \quad \forall 1 \leq k \leq n-1,
\]
\[
\text{Ric}^{(i)}(e_k^{(i)}, e_l^{(i)}) \to 0, \quad \text{as } i \to \infty, \quad \forall k \neq l,
\]
where \( \varepsilon_i \to 0 \) as \( i \to \infty \). Since \( v^{(i)} \in T_{p_i} \Sigma_f(p_i) \), we get
\[
\frac{\text{Ric}^{(i)}(v^{(i)}, v^{(i)})}{g(v^{(i)}, v^{(i)})} \geq \frac{\delta(n)}{2n} R^{(i)}(p_i), \quad \text{as } i \to \infty.
\]
On the other hand, since the limit manifold splits along \( X(\infty) \), we also have
\[
R^{(i)}(X(i), e_k^{(i)}, e_l^{(i)}, X(i)) = o(1) R^{(i)}_{kl}, \quad \text{as } i \to \infty.
\]
By (3.30) and (1.4), it is easy to see
\[ R_{i}^{(i)}(kl) \geq \frac{1}{2} R_{i}^{(i)}(kl). \]
Thus
\[ \frac{\text{Ric}^{(i)}(v^{(i)}, v^{(i)})}{g(v^{(i)}, v^{(i)})} \geq \frac{\text{Ric}^{(i)}(v^{(i)}, v^{(i)})}{2g(v^{(i)}, v^{(i)})} \geq \frac{\delta(n)}{4n} R^{(i)}(p_{i}), \text{ as } i \to \infty. \]
This is a contradiction to (4.5). Hence we complete the proof. \[\square\]

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