PASCAL TRIANGLE, STIRLING NUMBERS AND THE UNIQUE INVARIANCE OF THE EULER CHARACTERISTIC

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Abstract. We use some basic properties of binomial and Stirling numbers to prove that the Euler characteristic is, essentially, the unique numerical topological invariant for compact polyhedra which can be expressed as a linear combination of the numbers of faces of triangulations. We obtain this result converting it into an eigenvalue problem.

1. Introduction

Following D. Eppstein in [3] and the introduction of N. Levitt in [7], the Euler characteristic $\chi$ is the best known as well as the most ancient topological invariant and the Euler formula $E - V + F = 2$ is one of many theorems in mathematics which are important enough as to be proved repeatedly in surprisingly many different ways.

On this line the unique invariance of the Euler characteristic, among linear combinations on the numbers of faces of triangulations, is known and reproved time after time. Up to our knowledge the first proof appeared in Mayer [12]. More recently in [7], [4] and [13] there are some related results. Very recently in [17] this result is strengthened in the framework of combinatorial manifolds to non-linear functions on the number of faces.

This note was born after the simple observation that the $f$-vectors of $n$-simplices, considered as abstract simplicial complexes, can be placed forming an infinite lower triangular matrix which is almost the Pascal triangle. This allowed us to use the ideas in our previous works about Riordan matrices, see for example [8], [9], [10] and [11]. This kind of matrices and the group structure were introduced in [14] and [15].

Our main idea is to consider the $K$-linear action induced by such a matrix of $f$-vectors on $K[[x]]$, where we consider the natural $K$-linear space structure on the set of formal power series with coefficients on a field $K$ of characteristic zero. Using only this matrix we obtain, in particular, the unique homotopy invariance of the Euler characteristic among all possible linear combinations on components of $f$-vectors.

After that we use the known formula, involving Stirling numbers, of the change of $f$-vectors under barycentric subdivision to prove that the multiples of the Euler characteristic are the unique invariant under barycentric subdivision in the class of all $n$-simplices. Consequently we prove the unique topological invariance of the Euler characteristic among all possible linear combinations on components of $f$-vectors. We obtain this result as a consequence of the description of the eigenspace associated to the eigenvalue 1 in the linear action induced by an infinite matrix describing the variation of $f$-vectors under barycentric subdivisions.
2. The results.

We suppose that the basic definitions of abstract and geometric finite simplicial complexes are known. See the corresponding introductory chapters in [10] or [15]. We call herein the $f$-vector of an $n$-dimensional finite simplicial complex $\mathcal{F}$ to $(f_0, f_1, \cdots, f_n)$, where $f_i$ counts the number of $i$-faces of $\mathcal{F}$. So $f_0$ is the number of vertices, $f_1$ is the number of edges, and so on. We have to note that in other places the $f$-vector includes $f_{-1} = 1$ as its first coordinate which corresponds to the interpretation of the empty set as the unique $(-1)$-dimensional face.

The basic pieces to construct polyhedra in Topology are the geometric $n$-simplices, $\Delta_n$. Topologically they can be described as the convex hull of $n+1$ affinely independent points in a suitable euclidean space.

The abstract description of an $n$-simplex as a simplicial complex is given by considering all the non-empty subsets of a set of vertices $V = \{v_0, \cdots, v_n\}$ with $n+1$ points. So, the corresponding $f$-vectors are easily computed using combinatorial numbers. If we denote by $f_{\Delta_n}$ the $f$-vector of $\Delta_n$ we get

$$f_{\Delta_n} = \left(\begin{array}{c}
\binom{n+1}{1} \\
\binom{n+1}{2} \\
\vdots \\
\binom{n+1}{n+1}
\end{array}\right)$$

We can place these vectors $f_{\Delta_n}$ forming an infinite lower triangular matrix, $F = \begin{bmatrix} (n+1) \\ (k+1) \end{bmatrix}_{n,k \geq 0}$. We note that the non-null part coincides with Pascal’s triangle without the first row and column.

This matrix as well as the corresponding matrix representation of Pascal’s triangle, and some of its generalizations, are elements of a group under the usual product of matrices. This group is known as the Riordan group. We approached this group in [9], see also [8], using the Banach Fixed Point Theorem. To describe the elements $T(\beta | \alpha)$ in this group as in [9], we use a pair of formal power series $\alpha(x) = \sum_{n \geq 0} \alpha_n x^n$ and $\beta(x) = \sum_{n \geq 0} \beta_n x^n$ such that $\alpha_0 \neq 0$ and $\beta_0 \neq 0$. The columns of $T(\beta | \alpha)$ are the coefficients of the elements in the geometric progression, in $\mathbb{K}[\[x]]$, whose first term is the series $\frac{\beta(x)}{\alpha(x)}$ and the ratio is the series $\frac{x}{\alpha(x)}$.

The representations of the product and the inverse in this group are:

$$T(\beta | \alpha)T(\beta | \alpha) = T(\beta | \alpha)$$

where

$$\tilde{\beta}(x) = \beta(x)\tilde{\beta}\left(\frac{x}{\alpha(x)}\right), \quad \tilde{\alpha}(x) = \alpha(x)\tilde{\alpha}\left(\frac{x}{\alpha(x)}\right)$$

$$(T(\beta | \alpha))^{-1} = T \left(\frac{1}{\beta(\omega^{-1})} \frac{1}{\alpha(\omega^{-1})}\right), \quad \omega = \frac{x}{\alpha}, \quad \omega \circ \omega^{-1} = \omega^{-1} \circ \omega = x$$

Besides, we can consider the matrix $T(\beta | \alpha)$, like in Linear Algebra, as the associated matrix to a $\mathbb{K}$-linear isometry for a suitable ultrametric $d$, see [9], defined by:

$$T(\beta | \alpha) : \mathbb{K}[\[x]], d \quad \gamma \quad \rightarrow \quad \mathbb{K}[\[x]], d \quad \gamma \quad \rightarrow \quad T(\beta | \alpha)(\gamma) = \tilde{a}_T(\tilde{\gamma})$$

(1)

In this terms Pascal’s triangle is $T(1 | 1-x)$ and our matrix of $f$-vectors is $F = T \left(\frac{1}{1-x} \right) | 1-x \right)$, because we obtain $F$ from Pascal’s triangle by deleting the first row and column, see page 3614 in [8].

Given a $m$-dimensional simplicial complex $\mathcal{F}$ with $f$-vector $f^\mathcal{F} = (f_0^\mathcal{F}, f_1^\mathcal{F}, \cdots, f_m^\mathcal{F})$ the Euler characteristic is defined by

$$\chi(\mathcal{F}) = \sum_{k=0}^{m} (-1)^k f_k^\mathcal{F}.$$

Consider the geometric realization $|\mathcal{F}|$ of $\mathcal{F}$. It is known that $\chi(\mathcal{F})$ depends on $|\mathcal{F}|$ but not on the triangulation $\mathcal{F}$. Even more, $\chi(\mathcal{F})$ depends only on the homotopy type of $|\mathcal{F}|$ because it can be expressed in terms of the ranks of the homology groups $H_k(|\mathcal{F}|)$ which are homotopy invariants. This previous result is the so called Euler-Poincaré formula. See page 146 in [8].

Note that $\chi(\mathcal{F})$ can be expressed as the following product of infinite matrices:
and the column matrix does not depend on the complex $F$. The generating function of this column matrix is $\frac{1}{1 + x}$ then

$$T \left( \frac{1}{1 - x} \right) \left( \frac{1}{1 + x} \right) = \sum_{n \geq 0} \chi(\Delta_n)x^n$$

because the rows of the matrix $F$ are the vectors $f^{\Delta_n}$.

On the other hand, using (1) we get

$$T \left( \frac{1}{1 - x} \right) \left( \frac{1}{1 + x} \right) = \frac{1}{(1 - x)^2} \frac{1}{1 + \frac{x}{1-x}} = \frac{1}{1 - x}$$

then

$$\sum_{n \geq 0} \chi(\Delta_n)x^n = \frac{1}{1 - x} \quad \text{equivalently} \quad \chi(\Delta_n) = 1 \quad \forall n \geq 0$$

as everybody knows. Note that there is not any topology in the above computation.

The Euler characteristic is, on one hand, a homotopy invariant in the class of finite polyhedra and, on the other hand, it is a linear combination on the number of faces of any triangulation of the polyhedron. A natural way to define linear combinations on the number of faces non-depending on the polyhedron even on its dimension is the following

**Definition 1.** Let $\gamma(x) = \sum_{n \geq 0} \gamma_n x^n$ be any power series with coefficients in $\mathbb{K}$. Suppose that $F$ is a finite simplicial complex with $f$-vector $(f^F_0, f^F_1, \ldots, f^F_m, 0, \ldots)$, we define the linear combination induced by the series $\gamma$, and denote it by $\chi(\gamma, F)$, as

$$\chi(\gamma, F) = \sum_{k=0}^{m} \gamma_k f^F_k.$$

So, the Euler characteristic is

$$\chi \left( \frac{1}{1 + x}, F \right).$$

We want to prove that the Euler characteristic is the unique linear combination which is homotopy invariant. For this propose we define

**Definition 2.** Let $\gamma \in \mathbb{K}[[x]]$, $\gamma(x) = \sum_{n \geq 0} \gamma_n x^n$ be any power series. We say that

(a) $\gamma(x)$ is a homotopy invariant for the class of finite simplicial complexes if given any two of them $F_1$ and $F_2$, then $\chi(\gamma, F_1) = \chi(\gamma, F_2)$ provided $|F_1|$ and $|F_2|$ have the same homotopy type.

(b) $\gamma(x)$ is a topological invariant for the class of finite simplicial complexes if given any two of them $F_1$ and $F_2$, then $\chi(\gamma, F_1) = \chi(\gamma, F_2)$ provided $|F_1|$ and $|F_2|$ are homeomorphic.

Of course, any series which is a homotopy invariant is a topological invariant. We can restrict the above definition to any subclass of finite simplicial complexes. Using, essentially, the Pascal triangle we get

**Theorem 3.** The unique series which are homotopy invariants for the class of all dimensional euclidean closed balls (then for the class of all polyhedra) are $\frac{k}{1 + x}$, $k \in \mathbb{K}$.

In other words, the multiples of the Euler characteristic are the unique linear combinations on the components of the $f$-vectors of finite simplicial complexes which are homotopy invariant.
Proof. Suppose a series $\gamma(x) = \sum_{n \geq 0} \gamma_n x^n$ which is a homotopy invariant for the class of closed euclidean balls. Consider the Riordan matrix $T\left(\frac{1}{1-x} \bigg| 1-x\right)$ whose rows are specific triangulations of the euclidean balls. Then

$$T\left(\frac{1}{1-x} \bigg| 1-x\right) (\gamma(x)) = \frac{k}{1-x}$$

for some $k \in \mathbb{K}$ because $\chi(\gamma, \Delta_n) = k$ for any $n \geq 0$ from the homotopy invariance of $\gamma$. Now, in the Riordan group, $T^{-1}\left(\frac{1}{1-x} \bigg| 1-x\right) = T\left(\frac{1}{1+x} \bigg| 1+x\right)$ consequently

$$\gamma(x) = T\left(\frac{1}{1+x} \bigg| 1+x\right)\left(\frac{k}{1-x}\right) = \frac{k}{1+x}$$

and the proof is finished. \qed

**Remark 4.** (a) The same proof is valid in the more restrictive framework of simple homotopy theory, see [2].

(b) We have really proved that the unique linear combination of the number of faces which assigns the same number to every abstract $n$-simplex is the Euler characteristic. This is related to one of the conditions imposed in [4] because all of them are cones.

In [1] the authors treat $f$-vectors of barycentric subdivision of simplicial complexes to get, in particular, that certain limiting behavior depending on the iteration of the barycentric subdivision of a simplicial complex, does not depend on the complex itself but on the dimension of such complex. In that paper a formula for the variation of the $f$-vector after a barycentric subdivision is given.

Now we reproduce the formula at page 850 in [1] taking into account that we use the $f$-vector $(f_0, f_1, \ldots, f_m)$ and not the extended $f$-vector $(f_{-1}, f_0, f_1, \ldots, f_m)$.

**Proposition 5.** Let $\mathcal{F}$ be a $m$-dimensional simplicial complex. Denote by $sd(\mathcal{F})$ the complex obtained by the barycentric subdivision of $\mathcal{F}$. Then

$$f_j^{sd(\mathcal{F})} = \sum_{i=0}^{m} f_i^\mathcal{F} (j+1)! {i+1 \choose j+1} \quad \text{for} \quad j = 0, \ldots, m$$

In the above result $\{ \frac{k}{l} \}$ represents the corresponding Stirling number of the second kind as denoted in [3] Chapter 6.

Let $B = (b_{i,j})_{i,j \geq 0}$ be the matrix with $b_{i,j} = (j+1)! {i+1 \choose j+1} \ i, j \geq 0$. The formula in the proposition above converts to

$$(f^0_{sd(\mathcal{F})}, f^1_{sd(\mathcal{F})}, \ldots, f^m_{sd(\mathcal{F})}, 0, \ldots) = (f^0_\mathcal{F}, f^1_\mathcal{F}, \ldots, f^m_\mathcal{F}, 0, \ldots) \begin{pmatrix} \{ \frac{1}{1} \} & 2! \{ \frac{2}{1} \} & \cdots \\ \{ \frac{i}{1} \} & \cdots & \cdots \\ \{ \frac{m}{1} \} & 2! \{ \frac{2}{1} \} & \cdots & n! \{ \frac{n}{1} \} \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

By the usual way $B$ induces a $\mathbb{K}$-linear isomorphism $B : \mathbb{K}[[x]] \to \mathbb{K}[[x]]$, $B(\zeta(x)) = \eta(x)$, such that if

$$\zeta(x) = \sum_{n \geq 0} \zeta_n x^n, \quad \text{and} \quad \eta(x) = \sum_{n \geq 0} \eta_n x^n$$

then

$$B(\zeta_n) = (\eta_n)^t$$

For this operator we get

**Proposition 6.** The number 1 is an eigenvalue for the operator $B$. Moreover the eigenspace associated to 1 is $\{ \frac{c}{1+x} \ | \ c \in \mathbb{K} \}$. 

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Proof. If \( D = (d_{i,j}) \) is the diagonal matrix with \( d_{i,i} = (i+1)! \) and \( S = (\binom{i+j}{j+1})_{i,j \in \mathbb{N}} \) is the matrix of the Stirling numbers of second kind, then \( B = SD \). Consider
\[
\delta(x) = D \left( \frac{1}{1 + x} \right) = \sum_{i \geq 0} (i + 1)! (-x)^i
\]
then
\[
B \left( \frac{1}{1 + x} \right) = \frac{1}{1 + x} \iff S(\delta) = \frac{1}{1 + x}
\]
Recall that
\[
S^{-1} = \left( (-1)^{i-j} \binom{i+j}{j+1} \right)_{i,j \in \mathbb{N}}
\]
where \( \binom{i}{j} \) denote the Stirling numbers of the first kind. Since
\[
[\delta] = 0 \quad \forall i \geq 1 \quad \text{and} \quad \sum_{j=0}^{i} [j] = i!
\]
Hence
\[
S^{-1} \left( \frac{1}{1 + x} \right) = \delta(x)
\]
and then
\[
B \left( \frac{1}{1 + x} \right) = \frac{1}{1 + x}
\]
See pages 259-264 in [5] for the properties of Stirling numbers used above.

So we have proved that 1 is an eigenvalue and that \( \frac{1}{1 + x} \) is an associated eigenvector, and then
\[
\left\{ \frac{c}{1 + x}, \ c \in \mathbb{K} \right\}
\]
is contained in the corresponding eigenspace.

We consider the finite central \((m+1) \times (m+1)\) submatrices
\[
B_m = (b_{i,j})_{i,j=0...m}
\]
of the infinite matrix \( B \). Note that, in every \( B_m \), the eigenspace associated to the eigenvalue 1 has always dimension 1, because the entries in the main diagonal are all different. To prove that there is not any other eigenvector for \( B \) associated to 1, we suppose that \( \eta(x) = \sum_{n \geq 0} \eta_n x^n \neq \frac{c}{1 + x}, \ c \in \mathbb{K} \) is one such eigenvector. This means that there is an \( l \in \mathbb{N}, \ l \geq 1 \) such that
\[
\sum_{j=0}^{l} \eta_j x^j \neq c \sum_{j=0}^{l} (-x)^j, \quad \text{for any} \quad c \in \mathbb{K}.
\]
So, \( (\eta_j)_{j=0...l}, \ ((-1)^j)_{j=0...l} \) are eigenvector associated to 1 for the matrix \( B_l \). This is impossible because they are linearly independent.

The first consequence we obtain is that the invariance of the Euler Characteristic by successive barycentric subdivision is not a Topology matter. Our proof is the following: Let \( \mathcal{F} \) be a finite simplicial complex. Denote by \( sd^{(k)}(\mathcal{F}) \) the \( k \)-th barycentric subdivision of \( \mathcal{F} \). Using Proposition 6, we get
\[
f^{sd^{(k)}(\mathcal{F})} = f^\mathcal{F} B^k.
\]
So,
\[
\chi(sd^{(k)}(\mathcal{F})) = \chi \left( \frac{1}{1 + x}, sd^{(k)}(\mathcal{F}) \right) = \chi \left( B^k \left( \frac{1}{1 + x}, \mathcal{F} \right) \right) = \chi \left( \frac{1}{1 + x}, \mathcal{F} \right) = \chi(\mathcal{F})
\]
In the above proof we only used that \( \frac{1}{1 + x} \) is an eigenvector for the matrix \( B \), and then for \( B^k \), associated to the eigenvalue 1. As a consequence of the fact that \( \left\{ \frac{1}{1 + x} \right\} \) is a base for the eigenspace associated to 1 we get
Theorem 7. The unique linear combinations which are invariants under barycentric subdivisions in the class of all dimensional simplices are the multiples of the Euler characteristic. In particular, \( \frac{c}{1+x} \) are the unique series which are topologically invariants in the class of finite simplicial complexes.

Proof. Let \( \gamma(x) = \sum_{n \geq 0} \gamma_n x^n \) be a series which is invariant under barycentric subdivisions in the class of all dimensional simplices. This implies that

\[
T \left( \frac{1}{1-x} \right) (1-x) B(\gamma) = T \left( \frac{1}{1-x} \right) B(\gamma)
\]

this is equivalent to

\[
\gamma = B(\gamma)
\]

Consequently

\[
\gamma = \frac{c}{1+x} \quad \text{for some} \quad c \in \mathbb{K}
\]

The second part follows immediately because the polyhedra \(|F|\) and \(|sd(F)|\) are always homeomorphic.

Remark 8. The above proof can be also applied to the more restrictive framework of PL-Topology.

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