Complex Surfaces of Locally Conformally Flat Type

Mustafa Kalafat Caner Koca

May 14, 2013

Abstract

We show that if a compact complex surface admits a locally conformally flat metric, then it cannot contain a 2-sphere of non-zero self intersection. In particular, the surface has to be minimal. Then we give a list of possibilities.

1 Introduction

A Riemannian \( n \)-manifold \((M, g)\) is called locally conformally flat (LCF) if \( M \) has an open cover such that for any open set \( U \) of the cover we have a strictly positive function \( f : U \rightarrow \mathbb{R}^+ \) and a diffeomorphism \( h : U \rightarrow \mathbb{R}^n \) such that the pull-back of the Euclidean metric \( g_{\text{Euc}} \) on \( \mathbb{R}^n \) is conformally related to the restriction of \( g \) on \( U \); i.e.

\[ h^* g_{\text{Euc}} = f g. \]

In this paper, we are specifically interested in dimension four. In particular, we would like to see which complex surfaces admit this kind of metric. For this purpose we start with proving the following main result.

Theorem 2.2. If a closed 4-manifold admits a locally conformally flat Riemannian metric, then it cannot contain a sphere of non-zero self-intersection.

Since a reducible (non-minimal) complex surface necessarily contains a rational curve \( \mathbb{CP}_1 \) of self-intersection \(-1\), as a consequence we have the following.

Corollary 1.1. If a compact, complex surface admits a locally conformally flat Riemannian metric, then it has to be minimal.

We apply this corollary to the Enriques–Kodaira classification of surfaces ([BHPV], p.244), and eliminate some of the surfaces appearing on the list (see Theorem 3.1). We also analyze the case of elliptic fibrations separately in Theorem 4.2. As a consequence of these results, we obtain the following list of possibilities:
Theorem 1.2. If a compact, complex surface admits a locally conformally flat Riemannian metric, then it can be either

1. a Hopf surface, or an Inoue surface with vanishing second Betti number,
2. a minimal ruled surface fibered over a curve $\Sigma_g$ of genus $g \geq 2$,
3. a minimal elliptic fibration with no singular, but possibly with multiple fibers over a genus $g \geq 1$ curve,
4. a non-simply-connected minimal surface of general type of Euler characteristic $\chi \geq 4$.

The outline of the paper is as follows: In Section $\S 2$ we recall the developing map construction for LCF manifolds, and prove the main result. In $\S 3$ we obtain a list by analyzing the Kodaira-Enriques classification of compact complex surfaces. In Section $\S 4$ we deal with the elliptic fibration case separately. In $\S 5$ we give information about the converse case. Finally in $\S 6$ we relate our classification to the Hermitian case.

Acknowledgements. We thank Claude LeBrun for his suggestions and encouragement. Thanks to Y. Gürtas and A. Akhmedov for useful discussions.

2 Developing map of a locally conformally flat manifold

Before defining the developing map of a locally conformally flat manifold, let us give some motivation. We would like to modify the definition of local conformal flatness so that one uses charts and transition maps rather than the Riemannian metric directly. The key theorem in this case is due to Liouville and Gehring (see [Ge] p.389 or for a recent survey [Ho]), which states that for $n \geq 3$ and any open set $U \subset \mathbb{R}^n$, any $C^1$ conformal map $\varphi : U \to \mathbb{R}^n$ is the restriction of a Möbius transformation of $S^n$. Möbius transformations $\text{M"{o}b}(S^n) = \text{Conf}(S^n)$ is the group of conformal diffeomorphisms of the round $n$-sphere, and they are generated by inversions in round spheres. So, they constitute a group of real analytic diffeomorphisms of the real analytic manifold $S^n$ by the Liouville-Gehring theorem. Alternatively, they are the restrictions of the full group of isometries of the hyperbolic space $\mathbb{L}^{n+1}$ to its ideal boundary $S^n$ as follows. Consider $\mathbb{R}^{n+2}$ with its Lorentzian metric $g_1 = dx_1^2 + \cdots + dx_{n+1}^2 - dx_{n+2}^2$. Let $O(n+1, 1)$ be the group of linear maps that preserves the Lorentzian metric. We embed the two mentioned spaces into $\mathbb{R}^{n+2}$ as follows.

$$\mathbb{L}^{n+1} = \{ x \in \mathbb{R}^{n+2} : |x|_1^2 = -1 \text{ and } x_{n+2} > 0 \}$$
$$S^n = \{ x \in \mathbb{R}^{n+2} : |x|_1^2 = 0 \text{ and } x_{n+2} = 1 \},$$
i.e. $L$ is the upper part of the hyperboloid asymptotic to the light cone and $S^n$ is the unit sphere in the upper light cone which is the boundary of the Klein model $K$ of the hyperbolic space, see the Figure 5 in [CFKP]. The restriction of the Lorentzian metric on $L^{n+1}$ and $S^n$ gives hyperbolic and round metrics which are positive definite and of constant curvature $-1$ and $1$, respectively. Consider

$$\text{Isom}(L^{n+1}) = O^+(n+1, 1) := \{ A \in O(n+1, 1) : A \text{ preserves } L \}.$$ 

We define an isomorphism,

$$\Psi : \text{Isom}(L^{n+1}) \to \text{M"ob}(S^n), \ a \mapsto \Psi_a$$

by the following procedure. Take $a \in O^+(n+1, 1)$ so that for $y = (y_1 \cdots y_{n+1})$ and $a(y, 1) = (a_1 y, a_2 y) \in \mathbb{R}^{n+1} \times \mathbb{R}^+$ i.e. $a_2 := \pi_{n+2} \circ a \circ \pi_{1 \cdots n+1}$ define

$$\Psi_a : S^n \to S^n \ \text{by} \ \Psi_a (y, 1) := \left( \frac{a_1 y}{a_2 y}, 1 \right)$$

This is a conformal map on the sphere since it is the map $y \mapsto (a_1 y, a_2 y)$, an isometry of the sphere on its image, followed by rescaling via the factor $(a_2 y)^{-1}$. So whenever we define a locally conformally flat structure, instead of local conformal diffeomorphisms into $\mathbb{R}^n$, we map into $S^n$.

**Definition 2.1.** A locally conformally flat structure on a smooth manifold $M$ is a smooth atlas $\{(U_i, h_i)_{i \in I}\}$ where the maps $h_i : U_i \to S^n$ are diffeomorphisms onto their image and the transition maps $h_i \circ h_j^{-1} \in \text{M"ob}(S^n)$ after restriction.

Now start with one of the flattening (or rounding) maps $h_1 : U_1 \to S^n$. Let $\alpha$ be a path in $M$ beginning in $U_1$. We would like to analytically continue $h_1$ along this path. Proceeding inductively, on a component of $\alpha \cap U_i$ the analytic continuation of $h_1$ is a shift away from $h_i$, i.e. of the form $\Gamma \circ h_i$ for some $\Gamma \in \text{M"ob}(S^n)$. This way $h_1$ is analytically continued along every path of $M$ starting at a point in $U_1$. So that there is a global analytic continuation $D$ of $h_1$ defined on the universal cover $\tilde{M}$ since it is defined as a quotient space of paths in $M$. $D$ is called the developing map of the locally conformally flat space.

$$\tilde{M} \xrightarrow{D} S^4$$

If one starts with a different flattening open subset instead of $U_1$, one gets another developing map which differs from $D$ by a composition of a Möbius transformation. So the developing map is defined uniquely up to a composition with an element in $\text{M"ob}(S^n)$. This uniqueness property has the following consequence. Given any covering transformation $T$ of the universal covering, there is a unique element $g \in \text{M"ob}(S^n)$ such that

$$D \circ T = g \circ D.$$
This correspondence defines a homomorphism

\[ \rho : \pi_1(M) \longrightarrow \text{M"ob}(S^n) \]

called the holonomy representation of \( M \). Conversely, starting with a pair \((D, \rho)\) where \( \rho \) is a representation of the fundamental group into Möbius transformations and \( D \) is any \( \rho \)-equivariant local diffeomorphism of \( \tilde{M} \) into \( S^n \), one can construct the corresponding LCF structure on \( M \) by pulling back the standard LCF structure from \( S^n \) to \( \tilde{M} \) via \( D \), and then projecting it down.

The following is the main theorem of this section:

**Theorem 2.2.** If a closed 4-manifold admits a locally conformally flat Riemannian metric, then it cannot contain a sphere of non-zero self-intersection.

**Proof.** Let \( f : S^2 \rightarrow M \) be a smoothly embedded sphere in \( M \). Since the fundamental group of the sphere is trivial we have \( f_*\pi_1(S^2) \subset p_*\pi_1(\tilde{M}) \). So by the general lifting lemma ([Mu] p.478), we can lift the embedding to a continuous map \( \tilde{f} : S^2 \rightarrow \tilde{M} \) at any chosen base-point in a unique way. Since \( p \) is a local diffeomorphism and \( \tilde{f} \) is an embedding, the map \( \tilde{f} \) is also an embedding locally, hence an immersion. We can conclude that the self-intersection numbers in \( M \) and the universal cover

\[ I(f, f) = I(\tilde{f}, \tilde{f}) \]

are the same since there is a local diffeomorphism and the intersection numbers can be computed through the local deformations of the submanifolds. To be precise, self-intersection number is obtained by wiggling a copy of the sphere in a neighborhood to make it transverse to itself and counting the signed number of points according to the orientation. Since the covering map is a local diffeomorphism, it becomes a bijection when restricted to a lifting of the sphere. Since at the same time it is a local diffeomorphism in a neighborhood of a point, by compactness, passing to a finite cover one can introduce a metric and find a uniform \( \epsilon \) neighborhood on which the covering map is a diffeomorphism. If the wiggled sphere goes beyond this neighborhood, then we just push it inside without changing the intersection points.

As the second step, since the developing map is obtained through local flattening conformal diffeomorphisms, it is an immersion. Again, since local diffeomorphisms do not change the intersection numbers, we have

\[ I(\tilde{f}, \tilde{f}) = I(D \circ \tilde{f}, D \circ \tilde{f}). \]

Pick a point \( p \in S^4 - D \circ \tilde{f}(S^2) \) and look at the restriction map

\[ D \circ \tilde{f} : S^2 \rightarrow \mathbb{R}^4 \cong S^4 - \{p\}. \]

Linear translations show that the self-intersection number is zero here. \( \Box \)

In particular, there cannot be a \(-1\) self-intersecting sphere in \( M \), and thus, it is minimal.
3 Kodaira–Enriques classification

In this section we will give a list of complex surfaces of locally conformally flat type. The idea is to go through the classes of surfaces in the Kodaira–Enriques classification. According to the classification ([BHPV], p.244), the following is the complete list of minimal surfaces:

1. Minimal rational surfaces
2. Minimal surfaces of class VII
3. Ruled surfaces of genus $g \geq 1$
4. Enriques surfaces
5. Bi-elliptic surfaces
6. Primary or secondary Kodaira surfaces
7. K3-surfaces
8. Tori
9. Minimal properly elliptic surfaces
10. Minimal surfaces of general type

The assumption that the surface admits a LCF metric helps us to eliminate some of these possibilities by close inspection. First of all, we make the following general remark: A compact complex surface admitting a LCF metric has to be of signature $\tau = 0$ and non-simply-connected. The fact that the signature $\tau$ is zero, follows from the Hirzebruch Signature Theorem, which states that, for any given Riemannian metric $(M, g)$, we have

$$12\pi^2 \tau = \int_M |W^+|^2 - |W^-|^2 d\mu,$$

where $W^\pm$ are the self-dual and anti-self-dual parts of the Weyl tensor. Since $W = 0$ for any LCF metric (see [Bes]), we see that $\tau$ has to be zero.

On the other hand, a celebrated theorem of Kuiper [Kui] states that the 4-sphere $S^4$ is the only compact 4-manifold with a LCF metric. Since $S^4$ is not a complex manifold, a compact complex surface with a LCF metric cannot be simply connected. Now, let us start analyzing the above list.

The first case is a minimal rational surface. A surface is called rational if and only if it is birationally equivalent to the complex projective plane. The possibilities for the minimal models are the complex projective plane $\mathbb{CP}^2$ and the Hirzebruch surfaces $\mathbb{F}_n = \mathbb{P}(O \oplus O_{\mathbb{CP}^1}(n))$ for $n = 0, 2, 3 \ldots$ (see [BHPV]). The
Hirzebruch surfaces fall into two distinct smooth topological types $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{C}P_2 \# \mathbb{C}P_2$ determined by parity of $n$ (see [H]). Both of these types are simply-connected, as is $\mathbb{C}P_2$. Thus, they cannot admit a LCF metric by Kuiper’s theorem.

The second item in the list is the minimal surfaces of class VII. These surfaces are characterized by their Kodaira dimension $\kappa = -\infty$ and Betti number $b_1 = 1$ (therefore, they are not simply-connected). Furthermore their Chern numbers satisfy $c_1^2 \leq 0$ and $c_2 \geq 0$. Combining with the identity

$$c_1^2 = 2\chi + 3\tau = 2\chi$$

for LCF complex surfaces, we reach at the conclusion that $\chi = 0$. Since $b_1 = 1$, we can compute the second Betti number as follows,

$$0 = \chi = 2 - 2b_1 + b_2 = b_2.$$

Class VII minimal surfaces of vanishing second Betti number are classified by Bogomolov in [Bo1, Bo2]: Hopf surfaces and Inoue surfaces are the only two possibilities. A surface is called a Hopf surface if its universal cover is biholomorphic to $\mathbb{C}^2 - 0$. The other possibility are Inoue surfaces with $b_2 = 0$. Their universal cover is biholomorphic to $\mathbb{C} \times \mathbb{H}$, i.e. complex line times the hyperbolic disk.

The third is the case of ruled surface of genus $g \geq 1$. Such a surface admits a ruling, i.e. it admits a locally trivial, holomorphic fibration over a smooth non-rational curve with fiber $\mathbb{C}P_1$ and structural group $\text{PGL}(2, \mathbb{C})$. This can be thought as a projectivization of a complex rank 2-bundle over a Riemann surface. Now, we will see that the base cannot be a torus: Suppose otherwise. Then, topologically, we have the following fibre bundle

$$\mathbb{S}^2 \rightarrow M \rightarrow T^2.$$

The homotopy exact sequence for this bundle involves the following terms

$$\cdots \rightarrow \pi_3 T^2 \rightarrow \pi_2 S^2 \rightarrow \pi_2 M \rightarrow \pi_2 T^2 \rightarrow \cdots$$

Here, the terms at the two ends are zero since the universal cover of torus is contractible. Therefore, we have the isomorphism $\pi_2 M \approx \pi_2 S^2 \approx \mathbb{Z}$. Thus, the second homotopy group of the universal cover $\tilde{M}$ is non-trivial. Taking a look at the remaining terms of the homotopy exact sequence on the right we have

$$\cdots \rightarrow \pi_1 S^2 \rightarrow \pi_1 M \rightarrow \pi_1 T^2 \rightarrow \pi_0 S^2 \rightarrow \cdots$$

Again the end terms vanish, and we have the isomorphism $\pi_1 M \approx \pi_1 T^2 \approx \mathbb{Z} \oplus \mathbb{Z}$. This is an infinite abelian group. However, According to another theorem [Kui2] of Kuiper, universal cover of a compact, LCF 4-space with an infinite abelian fundamental group must be $\mathbb{R}^4$ or $\mathbb{R} \times S^3$. Ours have non-trivial second homotopy group, so it is none of these. Hence the genus $g = 1$ case yields a contradiction.

The fourth and seventh possibilities are eliminated, because the signatures of Enriques and $K_3$ surfaces are nonzero: $\tau(E) = -8$ and $\tau(K_3) = -16$. 

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Finally, let us consider surfaces of general type. We know that the Chern numbers $c_1^2$ and $c_2$ are strictly positive for these surfaces. Recall the formula for the holomorphic Euler characteristic: $12\chi_h = c_1^2 + c_2$. Since this is a non-zero positive integer multiple of 12, we have $c_1^2 + c_2 \geq 12$. Recall the identity $c_1^2 = 2c_2 + 3\tau$ for complex surfaces. Adding $c_2$ to both sides and applying the previous inequality we obtain $c_2 + \tau \geq 4$. Since the signature $\tau$ is zero for LCF surfaces, we get $c_2 = \chi \geq 4$.

Now, we can list the remaining cases as follows.

**Theorem 3.1.** If a compact, complex surface $(M, J)$ admits a locally conformally flat Riemannian metric, then it can be either

1. a Hopf surface, or an Inoue surface with $b_2 = 0$,
2. a ruled surface fibered over a Riemann surface $\Sigma_g$ of genus $g \geq 2$,
3. a bi-elliptic surface,
4. a primary or secondary Kodaira surface,
5. a torus,
6. a minimal properly elliptic surface, or
7. a non-simply-connected minimal surface of general type of Euler characteristic $\chi \geq 4$.

## 4 Elliptic Surfaces

In the case of elliptic surfaces one can actually make a more refined classification. We start with the following classification theorem stated in [GS] p. 314, a summary of research done by different people. See the references therein.

**Theorem 4.1.** A relatively minimal elliptic surface with nonzero Euler characteristic is diffeomorphic to $E(n, g)_{p_1 \ldots p_k}$ for exactly one choice of the integers involved for

$$1 \leq n, 0 \leq g, k, 2 \leq p_1 \cdots \leq p_k \text{ and } k \neq 1 \text{ if } (n, g) = (1, 0).$$

Here, relatively minimal means that the fibers do not contain any sphere of self intersection $-1$. This is a generalization of being minimal. $E(1)$ is defined to be the surface $\mathbb{CP}_2^\# 9 \mathbb{CP}_2$ considered with its elliptic fibration. Then taking its fiber sum with itself $n$-times, one gets $E(n)$. Furthermore taking the fiber sum with the trivial fibration $\Sigma_1 \times \Sigma_g$ over the Riemann surface of genus $g$, one gets the space $E(n, g)$. Finally the subindices $p_i$ denotes the multiplicity of a logarithmic transformation. Using this classification theorem we can prove our result.
Theorem 4.2. If an elliptic surface admits a locally conformally flat metric then it is minimal, also its Euler characteristic and signature vanishes. Moreover it has to be a torus bundle over a surface of genus $g$ outside the multiple fibers.

Proof. The signature of $E(n, g)_{p_1 \ldots p_k}$ is computed to be $\tau = -8n$, see $[GS]$. If we assume that there exists a LCF metric, then signature has to vanish and $n = 0$. Applying the Theorem 4.1 we reach to the conclusion that the Euler characteristic $\chi = 0$. But this is the Euler characteristics of the fiber bundle: $\chi = \chi(\text{fiber}) \times \chi(\text{base}) = 0$. Since a cusp fiber contributes by 2 and a fishtail fiber contributes by 1 to the Euler characteristic, there are no singular fibers.

Logarithmic transformation is a standard way to introduce a multiple fiber. Topologically, picking up a latitute $l$ of a smooth fiber, multiplying with the disc in the base, replacing the solid torus by another solid torus which has multiple Seifert fibered central circle is basically what this operation means. Note that this does not chance the Euler characteristic.

Elliptic surfaces based on a rational curve are classified as a product or a Hopf surface, see $[BHPV]$ p.196. Since the product does not admit LCF metric and Hopf surface is calready counted in the first case, we can assume that the genus $g \geq 1$. In the list we gave in the previous section, the cases between 3-6 are elliptic. These are the cases to which the results of this section apply.

5 Converse

In this section we will mention the surfaces in the list of Theorem 3.1 that are known to admit LCF metrics. Below are the cases. See $[K]$ for a recent survey and $[Bes]$ for references.

Case 1: Among the Hopf surfaces, the primary ones i.e. the ones homeomorphic to $S^1 \times S^3$ admit LCF metrics. The reason is locally it is a product of a line with a constant curvature space. Note that if a complex surface is homeomorphic to $S^1 \times S^3$ then it is diffeomorphic to it by a result of Kodaira $[Ko]$. Among the secondary type Hopf surfaces, the ones obtained by $\mathbb{Z}_p$-action on the second component results in a lens space product $S^1 \times L(p, q)$ hence admits a LCF metric because locally it is as the previous case.

Case 2: Among the ruled surfaces mentioned, the trivial products are LCF. The product metric on $\mathbb{C}P_1 \times \Sigma_g$ admits LCF metric for $g \geq 2$.

Case 5: All tori admit flat metrics.
6 The Hermitian case

In this section we consider the Hermitian locally conformally flat structures on complex surfaces. This means in addition we have the compatibility, i.e. $J$-invariance relation $g(JX,JY) = g(X,Y)$ for all vectors $X,Y$. This case is analyzed by M. Pontecorvo in [Pon], and also stated without proof in [B]. We have one of the following three cases.

1. A Hopf surface, i.e. finitely covered by a complex surface $\mathbb{C}^2*/\mathbb{Z}$ (diffeomorphic to $S^1 \times S^3$) with its standard metric of [Va].

2. A flat $\mathbb{CP}_1$ bundle over a Riemann surface $\Sigma_g$ of genus $g \geq 2$. Its metric is locally the product of constant $\pm 1$ curvature metrics.

3. A complex torus or a hyperelliptic surface with flat metrics.

In this classification the cases 1, 2 and 3 falls into the cases of 1, 2 and 3 of our Theorem respectively.

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