The World Function in Robertson-Walker Spacetime.

Mark D. Roberts,

Department of Mathematics and Applied Mathematics,
University of Cape Town,
Rondbosch 7701,
South Africa

roberts@gmunu.mth.uct.ac.za

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Abstract

A method for finding the world function of Robertson-Walker space-times is presented. It is applied to find the world function for the $k = 0$, $\gamma = 2$, solution. The close point approximation for the Robertson-Walker world function is calculated up to fourth order.

1 Introduction

A fundamental invariant of a spacetime is the world function, which is given by

$$\sigma = \pm \frac{1}{2} \tau^2, \quad (1.1)$$

where $\tau$ is the geodesic distance. $\sigma$ is positive for timelike geodesics, zero for null geodesics, and negative for spacelike geodesics. This invariant contains a lot of information about an arbitrary Riemannian manifold. Once it is known the metric can be calculated from it using the coincidence limit of one point on the geodesic approaching the other, thus $\lim_{x' \to x} \tau_{ab} = g_{ab}$, and then the Riemann tensor can be calculated from the metric. Defining the Van Vleck determinant $\Delta \equiv g^{1/2}(x) \det(\sigma_{ab}) g^{1/2}(x')$ a Hadmard Green function is

$$G(x, x') = i(8\pi^2)^{-1} \Delta^{1/2} (\sigma + i\epsilon)^{-1},$$

thus the world function, or an approximation to it, is needed when this Green function is required. These Green functions are used a lot in the theory of quantum fields in curved spaces, see for example Brown and Ottewill (1983) [1]; and also they are used in the study of the motion of charged particles in curved spacetimes, Roberts (1989) [2].

There are very few metrics for which the exact world function is known. It has been found for: DeSitter spacetime by Ruse (1930) [3], for $pp$ waves by Günther (1965) [4], for Gödel’s solution by Buchdahl and Warner (1980) [5], and for the Bertotti-Robinson solution by John (1984,1989) [6,7]. Buchdahl (1972) [8] has studied the production of the world function in Robertson-Walker spacetimes. In section 3 one new exact world function is found for the $k = 0$, $\gamma = 2$ Robertson-walker solution, and previously known exact world functions for Robertson-Walker solutions collected together. For the majority of Robertson-Walker metrics it is impossible to find the world function because intractable problems with elliptical integrals occur. For many applications it is only necessary to know the world function in the close point approximation and this is given in section 4. The conventions used
are: signature $+--$; $a, b, c = 0, 1, 2, 3$; $i, j, k = 1, 2, 3$;

\[
R_{bcd}^a = \partial_c \Gamma_{db}^a - \partial_d \Gamma_{cb}^a + \Gamma_{cf}^a \Gamma_{db}^f - \gamma_{df} \Gamma_{cb}^f \tag{1.2}
\]

\[
R_{bd} = R_{bda}.
\]

The Robertson-Walker line element is

\[
ds^2 = N(t)^2 dt^2 - R(t)^2 (d\chi^2 + f(\chi)^2 (d\theta^2 + \sin^2(\theta) d\phi^2)) \tag{1.3}
\]

where $R$ is the scale factor and

\[
f(\chi) = \begin{cases} 
  \sin(\chi) & \text{if } k = +1 \\
  \chi & \text{if } k = 0 \\
  \sinh(\chi) & \text{if } k = -1
\end{cases} \tag{1.4}
\]

in the conventions for $k$ of Hawking and Ellis (1973) [5]p.136.

The lapse function $N$ is arbitrary, and it can be absorbed into the line element; two common choices for $N$ are $N = 1$, and $N = R$ in which form the line element is conformal to the Einstein static universe. The Ricci tensor is given by

\[
R_{tt} = \frac{3 R_{tt}}{R} - \frac{3 N_t R_t}{N R},
\]

\[
R_{ij} = \frac{g_{ij}}{R^2} \left( -2k - \frac{R R_{tt}}{N^2} - \frac{2 R_t^2}{N^2} + \frac{R R_{tt} N_t}{N^3} \right). \tag{1.5}
\]

Using Einstein’s field equations the density $\mu$ and the pressure $p$ for a perfect fluid are given by

\[
8\pi G \mu = \frac{3k}{R^2} + \frac{3 R_t^2}{N^2 R^2},
\]

\[
8\pi G p = -\frac{k}{R^2} - \frac{R^2}{N^2 R^2} + \frac{2 N_t R_t}{N^3 R} + \frac{2 R_{tt} R_t}{N^2 R^3}. \tag{1.6}
\]

solutions to these equations with the equation of state

\[
p = (\gamma - 1) \mu, \tag{1.7}
\]

have been given by Vajk (1969) [12]. The speed of sound in a fluid is given by $c \, dp/d\mu$, Hawking and Ellis (1973) [5]p.91, thus the speed of sound for the equation of state [1.7] is

\[
c(\gamma - 1). \tag{1.8}
\]
The Geodesics of Robertson-Walker Spacetime

The geodesic equations can be most elegantly derived by the Euler method, the Lagrangian is

\[ 2\mathcal{L} = N^2\dot{t}^2 - R^2 g^{(3)}_{ij} \dot{x}^i \dot{x}^j, \]  

(2.1)

where:

(i) \( \dot{x}^a = \frac{dx^a}{d\tau} \)  

(2.2)

(ii) \( \tau \) is the geodesic distance,

(iii) \( 2\mathcal{L} = 1, 0, \text{or} -1 \) for timelike, null, or spacelike geodescs respectively,

(iv) \( g^{(3)}_{ij} \) is the metric of the three space of constant curvature, and \( i, j = 1, 2, 3 \).

The momenta are

\[ p^t = \frac{\partial \mathcal{L}}{\partial \dot{t}} = N^2 \dot{t} \]  

(2.3)

\[ p_i = -\frac{\partial \mathcal{L}}{\partial \dot{x}^i} = R^2 \dot{x}_i, \]

which give

\[ \frac{dp_t}{d\tau} = 2NN_t\dot{t}^2 + N^2 \dot{t} \]

\[ \frac{dp_i}{d\tau} = 2RR_t\dot{t} \dot{x}_i + R^2 \dot{x}_i. \]  

(2.4)

The Lagrangian gives

\[ \frac{\partial \mathcal{L}}{\partial t} = N N_t \dot{t}^2 - R R_t g^{(3)}_{ij} \dot{x}^i \dot{x}^j \]

\[ \frac{\partial \mathcal{L}}{\partial x^i} = -R^2 g^{(3)}_{kj,i} \dot{x}^k \dot{x}^j \dot{x}^i. \]  

(2.5)

The Euler equation

\[ \frac{\partial \mathcal{L}}{\partial x^i} = \frac{dp^i}{d\tau}, \]  

(2.6)

after dividing by \( N^2 \) becomes

\[ \dot{t} + \frac{N_t}{N} \dot{t}^2 + \frac{RR_t}{R^2} g^{(3)}_{ij} \dot{x}^i \dot{x}^j = 0. \]  

(2.7)
The Euler equation
\[- \frac{\partial L}{\partial x^i} = \frac{dp_i}{d\tau}, \quad (2.8)\]
after dividing by $R^2$ becomes
\[\ddot{x}_i + \frac{2R}{R} \dot{t} \dot{x}_i - g_{kj,i} \dot{x}_k \dot{x}_j \dot{x}_i = 0. \quad (2.9)\]

The geodesic distance $\omega$ on the three space of constant curvature is given by the well known expression
\[
\begin{align*}
\cos(\omega) &= \cos(\chi) \cos(\chi') + \sin(\chi) \sin(\chi') \cos(\Theta) \quad \text{if } k = +1 \\
\omega^2 &= \chi^2 + \chi'^2 - 2\chi \chi' \cos(\Theta) \quad \text{if } k = 0 \quad (2.10) \\
cosh(\omega) &= \cosh(\chi) \cosh(\chi') + \sinh(\chi) \sinh(\chi') \cos(\Theta) \quad \text{if } k = -1,
\end{align*}
\]
where
\[\cos(\Theta) = \cos(\theta) \cos(\theta') + \sin(\theta) \sin(\theta') \cos(\phi - \phi'). \quad (2.11)\]

Now
\[g^{(3)}_{ij} \dot{x}_i \dot{x}_j = \left( \frac{d\omega}{d\tau} \right)^2 \frac{dx^i}{d\omega} \frac{dx^j}{d\omega} = \dot{\omega}^2 \quad (2.12)\]

Equations 2.1 and 2.12 give
\[2L = N^2 t^2 - R^2 \dot{\omega}^2, \quad (2.13)\]
and Equations 2.7 and 2.8 give
\[\ddot{t} + \frac{N}{N} \dot{t}^2 + \frac{RR}{N^2} \dot{\omega}^2 = 0. \quad (2.14)\]

Noting that
\[(g^{(3)}_{kj} \dot{x}_k \dot{x}_j)_i = (\dot{\omega})_i = 0, \quad (2.15)\]
Equation 2.9 becomes
\[\dot{\omega} + 2RRd \dot{t} \dot{\omega} = 0. \quad (2.16)\]

The geodesics are determined by 2.13, 2.14, and 2.16; 2.14 can be obtained by differentiating 2.13 and substituting 2.16, therefore only equations 2.13 and 2.16 are used. Equation 2.16 can be integrated to give
\[\dot{\omega} = \frac{\alpha}{R^2} \quad (2.17)\]
where $\alpha$ is a constant independent of $\tau$. Rearranging 2.13 and substituting 2.17 gives

$$N\dot{t} = \pm \sqrt{R^2 \dot{\omega}^2 + 2L} = \pm \sqrt{\frac{\alpha^2}{R^2} + 2L}$$

thus the dynamics of the geodesics are determined by 2.17 and 2.18. The integral form of these equations are

$$\tau = \pm \int \frac{RN \, dt}{\sqrt{\alpha^2 + 2lgR^2}}, \quad (2.19)$$

$$\omega = \pm \int \frac{\alpha N \, dt}{R\sqrt{\alpha^2 + 2LR^2}}, \quad (2.20)$$

taking $2L = +1$ for timelike geodesics, these are just the equations describing geodesics given by Robertson (1933) [10]. $\tau$ is evaluated from 2.19 with the positive sign and the limits of integration from $T'$ to $t$ and with $\alpha$ replaced by its value evaluated from 2.20. The world function $\sigma$ is given from $\tau$ by 1.4.

3 Examples of exact World Functions for Robertson-Walker Spacetime.

*Example 1* The Einstein Static Universe. The metric is given by 1.3 with

$$N = 1, \quad R = R_0,$$

where $R_0$ is a constant. The Ricci tensor is given by

$$R_{tt} = 0$$

$$R_{ij} = -\frac{2k g_{ij}}{R_0^2}, \quad (3.2)$$

thus for $f = 0$ the spacetime is flat. The pressure and the density are given by

$$8\pi G \mu = -24\pi G p = \frac{3k}{R_0^2}, \quad (3.3)$$

therefore the equation of state is 1.7 with $\gamma = \frac{2}{3}$. The world function is given by

$$2\sigma = (t - t') - R_0^2 \omega^2 \quad (3.4)$$
Example 2 The Generalized Milne Universe. For small time variation the metric can be expanded as a Taylor series in $t$

$$
N = 1
R = R_0[1 + H_0 t - \frac{1}{2} q_0 H_0^2 t^2 = O(t^3)]
$$

(3.5)

where $H_0$ is the Hubble constant and $q_0$ is the deceleration parameter. The $q_0 \neq 0$ case proves to be intractable, so that only terms up to first order are considered

$$
N = 1
R = R_0[1 + H_0 t].
$$

(3.6)

The Ricci tensor is

$$
R_{tt} = 0
R_{ij} = -2 g_{ij} \left(\frac{R_0^2 H_0^2}{1 + H_0 t} \right),
$$

(3.7)

thus when $k = -1$ and $R_0 H_0 = 1$ the spacetime is flat, and called the Milne Universe, which is just Minkowski spacetime with a point removed at the origin. The pressure and density are given by

$$
8 \pi G \mu = -24 \pi G p = \frac{3(k + R_0^2 H_0^2)}{R_0^2(1 + H_0 t)^2}
$$

(3.8)

therefore the equation of state is $\frac{\mu}{p} = \frac{3}{2}$ with $\gamma = \frac{2}{3}$. Defining the new time coordinate

$$
\bar{t} = \frac{1}{H_0} + t,
$$

(3.9)

the metric is given by

$$
N = 1,
R = R_0 H_0 \bar{t}.
$$

(3.10)

The world function is given by

$$
2 \sigma = \bar{t}^2 + \bar{t}^2 - 2 \bar{t} t \cosh(R_0 H_0 \omega).
$$

(3.11)
Transforming to the original coordinate system this is

\[ 2\sigma = (t + t')^2 - 4 \left( \frac{1}{H_0} + t \right) \left( \frac{1}{H_0} + t' \right) \sinh^2 \left( \frac{1}{2} R_0 H_0 \omega \right) \] (3.12)

**Example 3** DeSitter and Anti DeSitter Spacetimes. These are maximally symmetric solutions to

\[ R_{ab} = -\Lambda g_{ab} \] (3.13)

with \( \Lambda > 0 \) for DeSitter spacetime and \( \Lambda < 0 \) for Anti DeSitter spacetime. For DeSitter spacetime the metric pressure and density are given by

\[ k = 0, \quad N = R = \frac{1}{\lambda t}, \quad \lambda = \sqrt{\frac{\Lambda}{3}}, \] (3.14)

\[ 8\pi G\mu = -8\pi Gp = \Lambda, \]

and therefore the equation of state is given by with \( \gamma = 0 \). For DeSitter spacetime the world function has been found by Ruse (1930) and is

\[ \tau = \sqrt{2\sigma} = \frac{1}{\lambda} \arccos(Q), \] (3.15)

where

\[ Q = \frac{1}{2tt'}(t + t'\omega^2) \] (3.16)

is the geometric distance in the five dimensional embedding space, and obeys the differential equation

\[ Q_a Q^a = \lambda^2(1 - Q^2). \] (3.17)

Transferring coordinates the result for DeSitter spacetime can be expressed as

\[ k = +1, \quad N = 1, \quad R = \frac{1}{\lambda} \cosh(\lambda t), \quad \lambda = \sqrt{-\frac{\Lambda}{3}}, \] (3.18)

\[ \cosh(\tau) = \sinh(\lambda t) \sinh(\lambda t') + \cosh(\lambda t) \cosh(\lambda t') \cos(\omega/\lambda), \]

footnote added 1999, a more complete treatment of the world function in DeSitter and Anti DeSitter spacetimes can be found in my work “Two Point Gravitational Energy” in preparation.
by symmetry the world function for Anti DeSitter spacetime can be found to be

\begin{align}
  k = -1, \quad N = 1, \quad R = \frac{1}{\lambda} \cos(\lambda t), \quad \lambda = \sqrt{\frac{\Lambda}{3}},
\end{align}

\begin{align}
  \cos(\tau) = \sin(\lambda t) \sin(\lambda t') + \cos(\lambda t) \cos(\lambda t') \cosh(\omega/\lambda).
\end{align}

**Example 4.** Robertson-Walker Spacetime with a Massless Scalar Field as Source. The world function for this spacetime is new and is derived here in detail, thus illustrating the method of Section 2. The metric is given by

\begin{align}
  N = R = bt^{1/2},
\end{align}

where \( b \) is a constant. The Ricci tensor is given by

\begin{align}
  R_{tt} &= -\frac{3}{2t^2},
  \\
  R_{ij} &= -\frac{2}{b^2t} g_{ij}.
\end{align}

The pressure and density are

\begin{align}
  8\pi G\mu &= \frac{3}{b^2t} \left( k + \frac{1}{4t^2} \right), \\
  8\pi Gp &= \frac{1}{b^2t} \left( k + \frac{3}{4t^2} \right),
\end{align}

thus when \( k = 0 \) the equation of state is just \( \frac{\gamma}{\gamma - 1} \) with \( \gamma = 2 \), for which the speed of sound in the fluid equals the speed of light in a vacuum.

Such a fluid is equivalent to a massless scalar field, see for example Roberts (1989), with a Ricci tensor

\begin{align}
  R_{ab} = -2\phi_a \phi_b,
\end{align}

for \( \phi \) is given by

\begin{align}
  \phi = \frac{1}{2} \sqrt{3} \ln(t).
\end{align}

The integrals 2.19 and 2.20 become

\begin{align}
  \tau &= \int_{t'}^{t} \frac{b^2t \ dt}{\sqrt{\alpha^2 + b^2t}} = \frac{2}{3} \sqrt{\alpha^2 + b^2t} \left( t - \frac{2\alpha^2}{b} \right) \bigg|_{t'},
  \\
  \omega &= \int_{t'}^{t} \frac{\alpha \ dt}{\sqrt{\alpha^2 + b^2t}} = \frac{2\alpha}{b^2} \sqrt{\alpha^2 + b^2t} \bigg|_{t'}
\end{align}
Squaring 3.26 gives
\[
\frac{b^4 \omega^2}{4 \alpha^2} - 2 \alpha^2 - b^2 (t + t') = -2 \sqrt{\alpha^2 + b^2 t} \sqrt{\alpha^2 + b^2 t'}, \tag{3.27}
\]
Squaring 3.27 and multiplying by \( \alpha^4 b^{-4} \) gives the quadratic in \( \alpha^2 \)
\[
\frac{\omega^4 b^4}{16} - \frac{\alpha^2 \omega^2 b^2}{4}(t + t') + \alpha^4 ((t - t')^2 - \omega^2) = 0, \tag{3.28}
\]
which has solution
\[
\alpha^2 = \frac{1}{8} \sigma_e^{-1} b^2 \omega^2 (t + t' \pm \sqrt{4tt' + \omega^2}), \tag{3.29}
\]
where
\[
2 \sigma_e = (t - t')^2 - \omega^2 \tag{3.30}
\]
then from 3.25 the world function is given by
\[
\frac{3}{2b} \sqrt{2 \sigma} = \frac{3}{2b} \tau = \left( \frac{1}{8} \sigma^{-1}_e \omega^2 (t + t' \pm \sqrt{4tt' + \omega^2}) + t \right)^{1/2}
\times (t - \frac{1}{4} \sigma^{-1}_e \omega^2 (t + t' \pm \sqrt{4tt' + \omega^2}))
\]
\[-(\frac{1}{4} \sigma^{-1}_e \omega^2 (t + t' \pm \sqrt{4tt' + \omega^2} + t')^{1/2} \times (t' - \frac{1}{4} \sigma^{-1}_e \omega^2 (t + t' \pm \sqrt{4tt' + \omega^2})) \tag{3.31}
\]

**Example 5** The Tolman Universe. For the Tolman radiation Universe
the metric is given by 1.3 with
\[
R = N = bt, \tag{3.32}
\]
\[
k = o,
\]
\[
\gamma = \frac{4}{3}.
\]
The integrals 2.19 and 2.20 can be evaluated to give
\[
\tau = \left[ \frac{\alpha t}{2} \sqrt{1 - \frac{b^2 t}{\alpha^2}} - \frac{\alpha^2}{2b} \cdot \arcsinh \left( \frac{b t}{\alpha} \right) \right]_\nu^t \tag{3.33}
\]
\[ \omega = \frac{\alpha}{b} \arcsinh \left( \frac{b t}{\alpha} \right) \Bigg|_{t'}, \quad (3.34) \]

and (3.34) cannot be inverted to give \( \alpha \). For the majority of solutions with equation of state \( [1.7] \), the integrals \( 2.19 \) and \( 3.20 \) are elliptical and the problem becomes intractable. The Tolman universe illustrates that the occurrence of elliptical integrals is not the only problem that might arise; because in this case the integrals can be evaluated to give simple functions for \( \tau \) and \( \omega \), however an expression for \( \alpha \) cannot be extracted from these functions.

### 4 The Close Point Approximation

Introducing the notation

\[ \pi^{ab...c} = (x^a - x'^a)(x^b - x'^b) \cdots (x^c - x'^c), \quad (4.1) \]

and

\[ \pi^n = \pi^a_1 \cdots \pi^a_n, \quad (4.2) \]

the world function can be expressed in the form

\[ 2\sigma = g_{ab}\pi^{ab} + h_{abc}\pi^{abc} + O(\pi^4). \quad (4.3) \]

Differentiating

\[ 2\sigma_e = 2g_{ae}\pi^{a} + \pi^{ab}(g_{ab,e} + h_{abe} + h_{eab}) + O(\pi^3), \quad (4.4) \]

thus

\[ \sigma_e\sigma^e = g_{ab}\pi^{ab} + \frac{1}{2}(g_{ab,c} + 3h_{abc})\pi^{abc} + O(\pi^4), \quad (4.5) \]

now because

\[ 2\sigma = \sigma_e\sigma^e, \quad (4.6) \]

Equations 4.3 and 4.5 can be equated to give

\[ h_{abc} = -\frac{1}{2}g_{ab,c}. \quad (4.7) \]

Carrying out the above approximation scheme to fourth order gives

\[ 2\sigma = g_{ab}\pi^{ab} - \frac{1}{2}g_{ab,c}\pi^{abc} + \]

\[ + \frac{1}{12} [2g_{ab,cd} - g^{ef}\left( \frac{1}{2}g_{ab,e} - g_{ae,b}\right)(\frac{1}{2}g_{cd,f} - g_{cf,d})] \pi^{abcd} + O(\pi^5). (4.8) \]
For the line element this gives

\[ 2\sigma = N^2 \delta t^2 - R^2 \omega^2 - NN_t \delta t^3 + RR_t \delta t \omega^2 \]

\[ \frac{1}{12} (3N_{t'}^2 + 4NN_{tt}) \delta t^4 - \frac{1}{3} \left( RR_{tt} + \frac{RN_t R_t}{N} \right) \delta t^2 \omega^2 \]

\[ -\frac{1}{12} \left( \frac{RR_t}{N} \right)^2 \omega^4 + O(\pi^5), \]

where

\[ \delta t = t - t', \]

and \( \omega \) is given by 2.11.

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References

[1] Brown,M.R. and Ottewill,A.C.(1983) *Proc.R.Soc.Lond. A* **389**,379.
[2] Buchdahl,H.A.(1972) *Gen.Rel.Grav.* **3**,35.
[3] Buchdahl,H.A. and Warner,N.P.(1980) *J.Phys.A* **13**,509.
[4] Gúnter,P.(1965) *Arch.Rat.Mech.Analys.* **18**,103.
[5] Hawking,S.W. and Ellis,G.F.R.(1973) “The Large Scale Structure of Space-Time” (Cambridge University Press).
[6] John,R.W.(1984) *Ann,Phys.Leipzig* **41**,67.
[7] John,R.W.(1989) *Trans.Inst.Phys.Estonian Acad.Sci.* **65**,58.
[8] Roberts,M.D.(1989a) *Class.Q.Grav.* **6**,419.
[9] Roberts,M.D.(1989b) *Hadronic J.* **12**,93.
[10] Robertson,H.P.(1933) *Rev.Mod.Phys.* **5**,68.
[11] Ruse,H.S.(1930) *Quart.J.Math. Ser.1 Oxford* **1**,146.
[12] Vajk,P.J.(1969) *J.Math.Phys.* **10**,1145.