One rate function to explain them all: large deviations for bulk and extreme eigenvalues of the Gaussian ensemble

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We show that a rate function $\Psi(c,x)$ of two variables explains: (i) the large deviations of bulk eigenvalues; (ii) the large deviations of extreme eigenvalues (both, left and right large deviations); (iii) the statistics of the fraction $c$ of eigenvalues to the left of a position $x$. Thus, $\Psi(c,x)$ captures the full order statistics of the eigenvalues of random Gaussian matrices as well as the statistics of the shifted index number. All our analytical findings are compared with Monte Carlo simulations, obtaining excellent agreement. A summary of preliminary results was already presented in [60] in the context of one-dimensional trapped spinless fermions in an harmonic potential.

I. INTRODUCTION

We can state somewhat in confidence that physicists have not only been more than a hundred years using the theory of large deviations but, it appears, we were the first ones in establishing the very first result on large deviations [1–2]. However, when discussing about it, the theory of large deviations rather evokes the mathematical treatises of Cramér, Donsker and Varadhan, and Freidlin and Wentzell [11]. Nowadays, this trend seems to continue: to the question What is the theory of large deviations?, physicists and mathematicians reply relatively differently, even though we are doing the same. We could say, perhaps provokingly, that the theory of large deviations is nothing else but the mathematics of statistical physics [11]. This confusion is also perpetuated by the unfortunate name of large deviations, as it evokes a theory that only cares about rare events. This could not be further from the truth since, for instance, such a theory may be contemplated by physicists as a generalization of Einstein’s fluctuation theory and it can be used to analyse phase transitions that typically occur when the law of large numbers breaks down (see, for instance, treatment of the Curie-Weiss model in [2]). In fact, the central object in this theory, what mathematicians called rate function, is nothing else but our revered entropy or free energy.

While some mathematicians were busy developing the intricacies of this theory, others, around the late 1920s, were getting hold to the Extreme Value Statistics (EVS), that is the statistics of the maximum and minimum of a set independent and identically distributed (IID) random variables [6–7]. During the following 80 years, its importance was rapidly recognised, first in engineering [8–10], finding later applications in other branches of science like environmental [9], finance [10–11], or within the theory of complex and disordered systems [12].

Most natural processes [13–15] are, however, characterised by large variations that result from long-time correlations. Due to this, and due to its ubiquity in many branches of sciences, there has been a frenetic activity to extend EVS from IID to correlated random variables. Evidently, from a solely mathematical point of view, it is worth to understand whether the limiting distributions characterising the IID case (these are Gumbel, Weibull, and Fréchet distributions) still hold in the correlated case or whether new limiting distributions emerge. Studies about the appearance of a new type of emergence can be found in fluctuating interfaces [19–25, 57–65], in quantum chaotic states [61], in fragmentation [27], in directed polymers on Cayley trees [28], in random binary tree searches [29], in whether records [30, 31], in $1/f^\alpha$ signals [32] (on the latter, natural processes with this type of fluctuations abound: in voltage fluctuations in resistors [41], in temperature fluctuations in oceans [42], in records on climatological temperatures [43], in daily market transactions [44]). Other relevant work has been carried out in one-dimensional random landscapes [33], in Gibbs measures of a particle on random Gaussian potentials in $d$ dimensions [34–35], in percolation [36], in the Brownian motion [37–40], in cosmology [61, 62] and, last but not least, all the work done on strongly correlated eigenvalues in the theory of random matrices [45–50, 74].

Considering the abundance of literature here presented, it should be obvious that the understanding of EVS is of vital importance. One needs, however, to recognise as well that by focusing only on the extremes, one is analysing a rather small fraction of the information available. Are the extreme values isolated or are there crowding effects? There are studies trying to understand what happens close to extreme values by, for instance, deriving a density close to them for IID variables [50] or for uncorrelated and correlated time series [67]. Other studies can be found on the statistics of return and first passage times of fluctuating interfaces [68–69], in persistent effects of the Brownian motion [71], in record statistics [72–73] (for record daily temperatures), and for the biased Brownian motion and applications to financial data [79] (see also the review on record statistics [83]), as well as some mathematical studies of record for InID (independent but not identically distributed) variables, IID variables whose distribution changes over time, inverse problems [75–76, 80–81], and other mathematical studies with applications in climatology and evolutionary biology [77–78].

A more general and profound vision of this type of statis-
tistics is given by the so called order statistics. This has been studied in detail in the mathematical literature both for IID and weakly correlated random variables \[70, 83, 85\]. In Physics, we find ourselves in a somewhat poorer position, being able to find some results of order statistics for IID variables in cosmology \[59, 87\]. More recent work on correlated random variables can be found in branching random walks \[88\], in \(1/f^\alpha\) signals \[91\], and the Brownian motion \[59, 90\].

Since not much is known about order statistics for strongly correlated random variables, it is appropriate to explore this topic in systems which are amenable for mathematical analysis. Random matrix theory offers precisely an ideal mathematical laboratory of strongly correlated random variables, that of the eigenvalues of random matrices, and we will exploit it in this work to dig a bit deeper in the complexities of EVS and order statistics.

As our working framework is that of random matrices, it is worth to devote a paragraph to see in which way the work presented here fits the already abundant literature \[4, 5, 14, 41, 52, 94\]. First of all, apart from the rather formal mathematical works of \[92, 93\], the statistics of the \(k\)-th eigenvalue has not been studied in detail. Moreover, these works only offer the typical case and not large deviations. In this work, we show that our new results go further from those presented in \[92, 93\], as we are able to describe typical as well as atypical fluctuations in the asymptotic limit of very large matrices. Secondly, and with respect to the statistics of extreme values, we offer a unifying way to obtain large deviations to the left and to the right of the typical value of an extreme eigenvalue. In the past, the large deviations to the left and to the right of, for instance, the maximum eigenvalue had to be analysed separately, using the Coulomb fluid picture to expose as well the various definitions that will be used throughout the paper and we also point out the connection of the full order statistics with the statistics of the shifted index number; in Sect. \[IV\] we describe the mathematical derivations using the Coulomb fluid picture, exposing as well the various ways to evaluate the free energy associated to the Coulomb fluid, how to derive exact expressions of the constrained spectral density resulting from fixing a fraction of eigenvalues to the left a point \(x\), and how to obtain an exact expression of the rate function using complete and incomplete elliptic integrals; in Sect. \[V\] the tail cumulative distribution for the shifted index number is presented together with its connection to the statistics of the \(k\)-th eigenvalue; in Sect. \[V\] we focus on recovering the large deviations for the extreme eigenvalues while in Sect. \[VI\] we deal with the typical fluctuations of the bulk eigenvalues; in Sect. \[VII\] we describe the various Monte Carlo simulations performed in this work. Finally, Sect. \[VIII\] contains some concluding remarks and describes some possibly interesting future research lines.

II. MODEL DEFINITIONS

Let us start with some definitions. Given a random variable \(X\) taking values \(x\) on a given set \(x \in \Omega\), we denote as \(F_X(x) = \text{Prob}[X \leq x]\) its cumulative density function (CDF) and \(F_X(x) = 1 - F_X(x)\) its tail CDF. Similarly, we denote the probability density function (PDF) as \(f_X(x) = \text{Prob}[X = x]\).

We are interested in studying the statistical properties related to the joint probability density function (jPDF) of eigenvalues \(y = (y_1, \ldots, y_N)\) of the Gaussian ensemble

\[
P(y) = \frac{1}{A_0} e^{-\frac{1}{2} \sum_{i=1}^{N} y_i^2} \prod_{i < j} |y_i - y_j|^\beta,
\]

where \(\beta\) is Dyson’s index, \(A_0\) is a normalising factor for \(P(y)\) such that \(-\infty < y_i < \infty\) for \(i = 1, \ldots, N\). In particular, we focus on what we call the shifted index number (SIN), as the random variable \(N_x = \sum_{i=1}^{N} \Theta(x - y_i)\). This is the number of eigenvalues to the left of \(x\) and, due to the stochastic character of the eigenvalues, \(N_x\) can take values from the set \(n_x \in \{0, \ldots, N\}\). Thus, its PDF is

\[
f_{N_x}(n_x) = \int_{n_x}^{\infty} dy P(y) \delta \left( n_x - \sum_{i=1}^{N} \Theta(x - y_i) \right).
\]

Notice that, even though the SIN takes discrete values, it is mathematically harmless to consider Dirac deltas instead of Kronecker deltas, at least in the thermodynamic limit. Its tail CDF is obviously given as

\[
F_{N_x}(n_x) = \int_{n_x}^{\infty} dy f_{N_x}(y).
\]

Next, we observe trivially that the probability that the \(k\)-th eigenvalue \(y_k\) is smaller than \(x\) is precisely the probability that at least \(N_k\) is greater than \(k\), that is

\[
F_{N_k}(k) = F_{y_k}(x).
\]

Alternatively, we have that \(F_{N_k}(k) = F_{y_k}(x)\). Thus, by studying \(F_{N_k}\) (or alternatively \(f_{N_k}\)) we have access not only to the statistical properties of SIN, but also to
the ones of the $k$-th eigenvalue. This (what some might consider a modest) generalisation of the index distribution problem [23], when analysed using the Coulomb fluid method, turns out to capture a rich fauna of statistical cases. More surprisingly, we even recover the full statistical properties (that is, for large $N$, we obtain left and right large deviation functions) for extreme eigenvalues.

III. COULOMB FLUID APPROACH

The analysis in the thermodynamic limit of $f_{N^c}$ is obtained by using the Coulomb fluid approach (see, for instance, [23] for the treatment of the standard index number). We must first write the jPDF of eigenvalues as $P(\lambda) = \frac{1}{Z} e^{-\beta N^2 E(\lambda)}$ with $E(\lambda) = \frac{1}{2N} \sum_{i=1}^{N} \lambda_i^2 - \frac{1}{2N^2} \sum_{i\neq j} \log |\lambda_i - \lambda_j|$. Here we have scaled the eigenvalues as $\gamma_i = \sqrt{N} \lambda_i$ and have kept only the relevant terms. We obviously need to scale $x$ accordingly: $x \to \sqrt{N} x$. Similarly, in our quest to work with intensive variables, we define the random variable $C_x = N^x / N$, which takes values $c \in [0, 1]$ for large $N$. This allows us to write $\varrho(c) \equiv f_{C_x}(c) = \Pr \{ C_x = c \}$ as follows:

$$\varrho(c) = \frac{1}{Z_0} \int_{-\infty}^{\infty} d\lambda e^{-\beta N^2 E[\rho(\lambda, \lambda)]} \times \delta \left( c - \int \rho(\lambda, \lambda) \Theta(x - \lambda) \right) ,$$

with $\rho(\lambda, \lambda) = (1/N) \sum_{i=1}^{N} \delta(\lambda - \lambda_i)$ and

$$E[\rho(\lambda, \lambda)] = -\frac{1}{2} \int d\lambda d\lambda' \rho(\lambda) \rho(\lambda') \log |\lambda - \lambda'| + \frac{1}{2} \int d\lambda \rho(\lambda) \lambda^2 + \text{irrelevant terms} .$$

To push towards a continuous theory, we introduce a functional Dirac delta to enforce the definition of the density

$$\varrho(c) = \frac{1}{Z_0} \int D[\rho] e^{-\beta N^2 E[\rho(\lambda, \lambda)]} \delta \left( c - \int \rho(\lambda, \lambda) \Theta(x - \lambda) \right)$$

$$\int_{-\infty}^{\infty} d\lambda \delta(\rho(\lambda) - \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i))$$

$$= \frac{1}{Z_0} \int D[\rho, \gamma] \exp \left[ -\beta N^2 E[\rho(\lambda, \lambda)] \right] + iN \int d\lambda \gamma(\lambda) \rho(\lambda) + N \log \int_{-\infty}^{\infty} d\lambda e^{-i\gamma(\lambda)}$$

$$\times \delta \left( c - \int d\lambda \rho(\lambda) \Theta(x - \lambda) \right) .$$

Here $Z_0$ is a normalisation constant. Since other constant terms may appear during the derivation, we take the approach of hiding then in $Z_0$ without the necessity of keep changing its name. The saddle-point equation with respect to $\gamma(\lambda)$ gives us the following equation

$$\rho(\lambda) = \frac{e^{-i\gamma(\lambda)}}{\int_{-\infty}^{\infty} d\lambda' e^{-i\gamma(\lambda')}} ,$$

that is, $\rho(\lambda)$ is normalised. Plugging back this solution will only give an $\mathcal{O}(N)$ term plus the normalisation constraint. Thus, after some standard manipulations, one is able to write the PDF of the fraction $c$ of eigenvalues to the left of $x$ as follows:

$$\varrho(c) = \frac{1}{Z_0} \int D[\rho, A_1, A_2] e^{-\frac{2}{N} \beta N^2 S[\rho(\lambda, \lambda, \lambda)]} ,$$

$$S[p, A_1, A_2] = \int d\lambda d\lambda' \rho(\lambda) \rho(\lambda') \log |\lambda - \lambda'|$$

$$+ A_1 \left( \int_{-\infty}^{\infty} d\lambda \rho(\lambda) \Theta(x - \epsilon) \right)$$

$$+ A_2 \left( \int_{-\infty}^{\infty} d\lambda \rho(\lambda) - 1 \right),$$

(1)

where $D[\rho, A_1, A_2]$ stands for integrating over the set of functions $\rho(\lambda)$ and variables $A_1$ and $A_2$. For large $N$, $\varrho(c)$ can be evaluated by the saddle-point method, yielding:

$$\varrho(c) = \exp \left[ -\beta N^2 \Psi(c, x) \right] ,$$

where we have defined the rate function

$$\Psi(c, x) = \frac{1}{2} \left( S_0(c, x) - \Omega_0 \right)$$

(2)

with $\Omega_0 = \frac{3 + 2 \log 2}{4}$. Here $S_0(c, x)$ refers to the action $S$ evaluated at the saddle point obtained by seeking stationarity with respect to $\rho$, $A_1$, and $A_2$, while $\Omega_0$ refers to the action related to the normalising constant $Z_0$. The saddle-point equations

$$\frac{\delta S[p, A_1, A_2]}{\delta \rho(\lambda)} = \frac{\delta S[p, A_1, A_2]}{\delta A_1} = \frac{\delta S[p, A_1, A_2]}{\delta A_2} = 0$$

provide the following:

$$\lambda^2 + A_1 \Theta(x - \epsilon) + A_2 = 2 \int d\lambda' \rho(\lambda') \log |\lambda - \lambda'|$$

$$c = \int_{-\infty}^{\infty} d\lambda \rho(\lambda) \Theta(x - \epsilon) , \quad 1 = \int_{-\infty}^{\infty} d\lambda \rho(\lambda) .$$

(3)

Let us pause here and comment briefly on what we expect to obtain by solving the set of equations (3). They obviously reflect that we are looking for the stationary solution in the thermodynamic limit of the charge density of a Coulomb fluid with external harmonic potential in which a fraction $c$ of charges (read eigenvalues) must be to the left of a value $x$. We already know that if the latter constraint were absence we would obtain the celebrated Wigner’s semi-circle law $\rho_{wc}(\lambda) = \frac{1}{\pi} \sqrt{2 - \lambda^2}$. There is, of course, another possibility of obtaining the semi-circle
law even in the presence of such constraint. This corresponds to the case when the constraint does not add anything to the problem, or in other words, when given $x$, the fraction $c$ is precisely the fraction of eigenvalues to the left $x$ in Wigner’s law. Anticipating the mathematics, let us denote such a value as $c_\star(x)$. Then it is intuitively clear that we will have two distinct regimes: when $c > c_\star(x)$ and $c < c_\star(x)$.

Let us remind ourselves that our main goal is to obtain an expression for the rate function $\Psi(c, x)$. In order to do so, we must unavoidably follow a rather torturous mathematical path which, for clarity, we have divided into: (A) rewriting appropriately the action in terms of the second moment of the constrained density and two constants; (B) solving the saddle-point equations; (C) obtaining expressions for the second moment and those constants, viz.

\[
\int d\lambda d\lambda' \rho(\lambda)\rho(\lambda') \log |\lambda - \lambda'| = \frac{1}{2} \int d\lambda \rho(\lambda) \lambda^2 + \frac{A_1}{2} c + \frac{A_2}{2}.
\]

We can use this to replace the double integral appearing in the action by a single integral and the two constants $A_1$ and $A_2$ constants, viz.

\[
S_0(c, x) = \frac{1}{2} \int d\lambda \rho(\lambda) \lambda^2 - \frac{A_1}{2} c - \frac{A_2}{2}.
\]  

Alternatively, and as it was reported in [60], one can recall that the variation with respect to external parameters at the saddle point enters only through its explicit derivatives, that is

\[
\frac{\partial S_0(c, x)}{\partial c} = \frac{\partial S[p, A_1, A_2]}{\partial c} + \frac{\partial S[p, A_1, A_2]}{\partial \rho} \frac{\partial \rho}{\partial c}
+ \frac{\partial S[p, A_1, A_2]}{\partial A_1} \frac{\partial A_1}{\partial c} + \frac{\partial S[p, A_1, A_2]}{\partial A_2} \frac{\partial A_2}{\partial c}
= \frac{\partial S[p, A_1, A_2]}{\partial c},
\]

which yields

\[
\frac{\partial S_0(c, x)}{\partial c} = -A_1(c, x)
\]

and, therefore, integrating over $c$ we have

\[
S_0(c, x) - \Omega_0 = \int_{c_\star(x)}^c \frac{d\rho}{dc} \frac{\partial S_0(c', x)}{\partial c'} = -\int_{c_\star(x)}^c \frac{d\rho}{dc} A_1(c', x).
\]

Here, recalling our prior comment on $c_\star(x)$, we have used that $S_0(c_\star(x), x) = \Omega_0$ or, alternatively, $\Psi(c_\star(x), x) = 0$. Thus, we see from (4) that, in order to evaluate the action at the saddle point, we must evaluate the second moment of $\rho(\lambda)$ and determine the constants $A_1$ and $A_2$.

**B. Solving the saddle-point equations.** A complete picture of the constrained spectral density

Mathematically, the most comfortable way to solve the saddle-point equations (3) is by introducing the Stieltjes transform of the density $\rho(\lambda)$, $S(z) = \int d\lambda \frac{\rho(\lambda)}{z - \lambda}$ (Nb. $S(z)$ is usually called resolvent) with $z \in \mathbb{C}$. This helps to write the derivative of the first the saddle-point equation (3) as a simple algebraic equation for $S(z)$. Indeed, differentiating with respect to $\lambda$ on the first saddle-point eq. in (3) we obtain the Tricomi equation:

\[
\lambda + B_1 \delta(x - \lambda) = \int d\lambda' \rho(\lambda') \frac{\lambda'}{\lambda - \lambda'},
\]

with $B_1 = -A_1/2$. To solve eq. (5), we first multiply it by $\rho(\lambda)/(z - \lambda)$ and integrate over $\lambda$, viz.

\[
\int d\lambda \frac{\rho(\lambda)}{z - \lambda} \lambda + B_1 \int d\lambda \frac{\rho(\lambda)}{z - \lambda} \delta(x - \lambda)
= \int d\lambda \int d\lambda' \frac{\rho(\lambda)\rho(\lambda')}{(z - \lambda')(z - \lambda)}.
\]

Next, we must work out the three terms in (7) to write them in terms of $S(z)$. For the first term we write

\[
\int d\lambda \frac{\rho(\lambda)}{z - \lambda} = \int d\lambda \frac{\rho(\lambda)}{z - \lambda}(\lambda - z) + \int d\lambda \frac{\rho(\lambda)}{z - \lambda}
= -1 + zS(z).
\]

For the second term we have:

\[
B_1 \int d\lambda \frac{\rho(\lambda)}{z - \lambda} \delta(x - \lambda) = B_1 \frac{\rho(x)}{z - x}.
\]

Finally, for the third term we obtain

\[
T_3 = \int d\lambda \int d\lambda' \frac{\rho(\lambda)\rho(\lambda')}{(z - \lambda')(z - \lambda)},
\]

\[
= \int d\lambda \int d\lambda' \rho(\lambda)\rho(\lambda') \left[ \frac{1}{z - \lambda} + \frac{1}{\lambda - \lambda'} \right] \frac{1}{z - \lambda'}
= S^2(z) - T_3.
\]
which implies that \( T_3 = \frac{1}{2} S^2(z) \). All in all, we obtain the following algebraic equation for \( S(z) \)
\[
\frac{1}{2} S^2(z) - z S(z) + 1 + \frac{\alpha}{z - x} = 0,
\]
where we have denoted \( \alpha = -B_1 \rho(x) \) and whose solution is
\[
S_{\pm}(z) = z \pm \sqrt{\frac{P_3(z)}{z - x}},
\]
with \( P_3(z) = z^3 - xz^2 - 2z + 2(x - \alpha) \), and where \( \alpha \) is a constant to be determined by the 2nd and 3rd saddle-point equations (3), that is \( \alpha = \alpha(c, x) \). In anticipation of the detailed mathematical analysis below we can intuitively discuss which role this unphysical parameter \( \alpha \) plays: for \( \alpha = 0 \) notice from the expression of \( P_3(z) \) that we should recover the semicircle law. Hence \( \alpha(c, x) = 0 \) and, as we will see below, whenever \( \alpha \) is negative or negative we are either in the regime \( c > c_0 \) or \( c < c_0 \), respectively.

To extract the density of eigenvalues \( \rho(x) \) from \( S(z) \), we recall that the resolvent \( S(z) \) has the following properties: (i) it is analytic everywhere on \( \mathbb{C} \) but not on the cuts on the real line where the density is defined; (ii) it must behave as \( \frac{1}{z} \) as \( |z| \to \infty \). This is easy to see by doing and expansion in powers of \( 1/z \) in the definition; (iii) it is real for real \( z \) outside of the domain of the density; (iv) from distribution theory we have that if we approach a point \( \lambda \) belong to the domain of \( \rho(x) \), that is \( S(\lambda \pm i\epsilon) = g(\lambda) \mp i\epsilon \rho(\lambda) \), from which we have \( \rho(x) = -\lim_{\epsilon \to 0} \frac{1}{2\pi i} \text{Im}[S(\lambda + i\epsilon)] \). Property (ii) implies that we must take the negative sign solution of \( S_{\pm}(z) \).

To elucidate the spectral density \( \rho(x) \) one first needs to understand the behaviour of the three roots of \( P_3(z) \). These are given by the formulas:

\[
\lambda_+ = \frac{1}{3} \left( \Delta(\alpha, x) + \frac{x^2}{2} + 6 \frac{\alpha}{\Delta(\alpha, x)} + x \right),
\]
\[
\lambda_0 = \frac{1}{3} \left( -1 + \frac{i\sqrt{3}}{2} \Delta(\alpha, x) - \frac{1}{2} \frac{\alpha}{\Delta(\alpha, x)} + x \right),
\]
\[
\lambda_- = \frac{1}{3} \left( 1 - \frac{i\sqrt{3}}{2} \Delta(\alpha, x) - \frac{1}{2} \frac{\alpha}{\Delta(\alpha, x)} + x \right),
\]
where we have defined
\[
\Delta(\alpha, x) = \left( 27 \sqrt{(\alpha_-(x))} (\alpha - \alpha_+(x)) \right)^{1/3},
\]
\[
\alpha_{\pm}(x) = \frac{1}{27} \left( (18 - x^2) \pm (6 + x^2)^{3/2} \right). 
\]

Given a value of \( x \), for all values of \( \alpha \in [\alpha_-(x), \alpha_+(x)] \) the three roots are real. However, not all possible points in this subregion of the \((x, \alpha)\)-plane are physical. This can be understood by looking at plots of the roots as a function of \( \alpha \) for fixed values of \( x \), like the ones presented in Fig. 1.

Here we can see (as well as algebraically from the form of the polynomial \( P_3(z) \)) that when \( \alpha = 0 \), we recover Wigner’s semi-circle law. Then we have two cases: either the position of the barrier is within the natural support of Wigner’s law, that is \( x \in [-\sqrt{2}, \sqrt{2}] \) (corresponding to the middle plot in Fig. 1), or it is not \( x \notin [-\sqrt{2}, \sqrt{2}] \) (corresponding to the left and right plots in Fig. 1). In the first case \((x \in [-\sqrt{2}, \sqrt{2}]\) we have that \( \alpha \in [\alpha_-, \alpha_+] \).

The value of \( \alpha \) controls the fraction of eigenvalues to the left of \( x \): when varying \( \alpha \), denoted as a black vertical line in Fig. 1, we can go from all eigenvalues within the interval \([\lambda_-, x] \) (left-most position and corresponding to \( c = 1 \)), to all eigenvalues within the interval \([x, \lambda_+] \) (right-most position corresponding to \( c = 0 \)). At any other position of \( \alpha \), the eigenvalues are found within two blobs: either \([\lambda_-, x] \cup [0, \lambda_+] \) (corresponding to \( \alpha \in [\alpha_-, 0] \)) or \([\lambda_-, \lambda_0] \cup [x, \lambda_+] \) (corresponding to \( \alpha \in [0, \alpha_+] \)).

On the other side, when \( x \notin [-\sqrt{2}, \sqrt{2}] \) (left and right plots in Fig. 1) then either \( \alpha \in [\alpha_-, 0] \) (corresponding to \( x < -\sqrt{2} \), left figure) or \( \alpha \in [0, \alpha_+] \) (corresponding to \( x > \sqrt{2} \), right figure). Consider the first case (the second one is thought similarly). When \( \alpha \) is in its left-most position, then all eigenvalues are within the interval \([\lambda_-, x] \), which corresponds to \( c = 1 \). As \( \alpha \) moves to the right, the eigenvalues can be found within the two blobs \([\lambda_0, \lambda_+] \cup [0, \lambda_+] \) until, finally for \( \alpha = 0 \) we recover Wigner’s law. The latter corresponds, for this position of the barrier, to having a zero fraction of eigenvalues \( c = 0 \).

These restrictions are summarised in Fig. 2. Here we have redefined the solutions of \( \alpha_{\pm}(x) \) coming from the mathematical analysis but adding in the previous the physical interpretation:
\[
\alpha_-(x) = \begin{cases} 
\frac{1}{9} \left( (18 - x^2) - (6 + x^2)^{3/2} \right) & \text{if } x \leq \sqrt{2}, \\
0 & \text{if } x > \sqrt{2}.
\end{cases}
\]
and
\[
\alpha_+(x) = \begin{cases} 
\frac{1}{9} \left( (18 - x^2) + (6 + x^2)^{3/2} \right) & \text{if } x < -\sqrt{2}, \\
0 & \text{if } x \geq -\sqrt{2}.
\end{cases}
\]

In Fig. 2 we can see the allowed region of physical solutions in the \((x, \alpha)\)-plane, which is enclosed by the lines \( \alpha = \alpha_+(x) \) (solid blue line in Fig. 2) and \( \alpha = \alpha_-(x) \) (solid green line in Fig. 2). Recalling that \( \alpha = 0 \) (solid red line in Fig. 2) corresponds to the Wigner law, if we follow this line we have that the fraction of eigenvalues to the left of \( x \) should be \( c = c_0(x) \). From \( \alpha = 0 \) and by increasing \( \alpha \), we are exploring those constrained spectral densities with \( c < c_0(x) \) (corresponding to the blue filled region in Fig. 2). Upon further increasing \( \alpha \) we will eventually arrive to the line \( \alpha = \alpha_+(x) \), which corresponds precisely to \( c = 0 \) or, in order words, a constrained spectral density with all the eigenvalues to the right of \( x \). Alternatively, starting from \( \alpha = 0 \) and by decreasing \( \alpha \) we are considering constrained spectral densities with
c > c_*(x) (green filled region in Fig. 2) until we eventually arrive to the boundary line \( \alpha = \alpha_-(x) \). The latter corresponds to \( c = 1 \) or to a constrained spectral density with all eigenvalues to the left of \( x \).

This analysis relies on the admitting annoying parameter \( \alpha \), which lacks a simple physical interpretation and it is unnecessarily constrained. It is better to switch the whole discussion to the \((x, c)\)-plane, as both are free parameters. For this we need to define first \( c_*(x) \) which, as we recall is the fraction of eigenvalues to the left of \( x \) when they form precisely the the semi-circle law 
\[
\int_{-\sqrt{2}}^{x} dy \rho_{sc}(y),
\]
that is
\[
c_*(x) = \begin{cases} 
0 & x < -\sqrt{2} \\
\frac{x + \sqrt{x^2 - 4\arcsin\left(\frac{x}{2}\right)}}{2\pi} & x \in [-\sqrt{2}, \sqrt{2}] \\
1 & x > \sqrt{2}
\end{cases}
\]
Then the allowed region appearing in Fig. 2 is transformed into that reported in Fig. 3. All this detailed analysis allows us to write the constrained spectral density for both regimes. Indeed, for \( c > c_*(x) \) we have that
\[
\rho(\lambda) = \frac{1}{\pi} \sqrt{\frac{(\lambda_+ - \lambda)(\lambda - \lambda_0)(\lambda - \lambda_-)}{x - \lambda}} \mathbb{1}_{\lambda \in [\lambda_-]} + \frac{1}{\pi} \sqrt{\frac{(\lambda_+ - \lambda)(\lambda - \lambda_0)(\lambda - \lambda_-)}{\lambda - x}} \mathbb{1}_{\lambda \in [\lambda_0, \lambda_+]},
\]
while for \( c < c_*(x) \) we write instead
\[
\rho(\lambda) = \frac{1}{\pi} \sqrt{\frac{(\lambda_+ - \lambda)(\lambda - \lambda_0)(\lambda - \lambda_-)}{x - \lambda}} \mathbb{1}_{\lambda \in [\lambda_-]} + \frac{1}{\pi} \sqrt{\frac{(\lambda_+ - \lambda)(\lambda - \lambda_0)(\lambda - \lambda_-)}{\lambda - x}} \mathbb{1}_{\lambda \in [\lambda, \lambda_+]}.
\]
Here $\mathbb{I}_{x \in [a, b]}$ is an indicator function equal to the unity if $x \in [a, b]$ or zero otherwise. In particular notice that we recover the well-known expressions of the spectral density for all eigenvalues to the left of the barrier ($c = 1$) and all eigenvalues to the right of the barrier ($c = 0$), which we denote as $\rho_{\ell}(\lambda)$ and $\rho_{R}(\lambda)$, respectively. This would correspond to following the lines $\alpha_{-}(x)$ and $\alpha_{+}(x)$ in the $(x, \alpha)$-plane, respectively. This gives

$$\rho_{\ell}(\lambda) = \begin{cases} \frac{1}{2} \sqrt{\frac{\lambda - a_{-}(x)}{x - a_{-}}} |\lambda - b_{-}(x)| \mathbb{I}_{\lambda \in [a_{-}(x), x]} & x \leq \sqrt{2} \\ \rho_{\ell}(\lambda) & x > \sqrt{2} \end{cases},$$

and

$$\rho_{R}(\lambda) = \begin{cases} \rho_{R}(\lambda) & x < -\sqrt{2} \\ \frac{1}{2} \sqrt{\frac{\lambda - a_{+}(x)}{x - a_{+}}} |\lambda - b_{+}(x)| \mathbb{I}_{\lambda \in [a_{+}(x), x]} & x \geq -\sqrt{2} \end{cases},$$

with $a_{+}(x) = \frac{1}{2}(x \pm 2\sqrt{6 + x^2})$ and $b_{+}(x) = \frac{1}{2}(x \pm \sqrt{6 + x^2})$.

Unfortunately, to get rid of the parameter $\alpha$ we still need to derive its dependence on the pair of parameters $(x, c)$, that is, we are seeking for the function $\alpha(c, x)$. This can only be given implicitly via the function $c(\alpha, x)$ which is obtained from the condition $c = \int_{-\infty}^{x} d\lambda \rho(\lambda)$. Using complete elliptic integrals we find that for $c > c_{*}(x)$, the function $c(\alpha, x)$ takes the following form

$$c(\alpha, x) = \frac{1}{2\pi \sqrt{(a - c)(b - d)}} \left[ 4(a - d) \Pi \left( \frac{d - c}{a - c} \right) + (a - c) \{c(b - d)E - (a - d)(a - b + c)K\} \right],$$

where $a = \lambda_{+}$, $b = \lambda_{0}$, $c = x$, and $d = \lambda_{-}$. For $c < c_{*}(x)$, the function $c(\alpha, x)$ reads

$$c(\alpha, x) = \frac{1}{2\pi \sqrt{(a - c)(b - d)}} \left[ 4(b - c) \Pi \left( \frac{c - d}{b - d} \right) + (b - d) \{b(a - c)E - (b - c)(a - 2c - 2d)K\} \right],$$

where $a = \lambda_{+}$, $b = x$, $c = \lambda_{0}$, and $d = \lambda_{-}$ (see appendix A). Here $K \equiv K(k)$, $E \equiv E(k)$, and $\Pi(n) \equiv \Pi(n, k)$ are the complete elliptic integrals of the first, second, and third kind, respectively, with elliptic modulus $k = \sqrt{(a - b)(c - d)/(a - c)(b - d)}$. In Fig. 4 we have plotted the function $c(\alpha, x)$ versus $\alpha$ for a fixed value of $x$. We have also compared the analytical results with the numerical evaluation of $c = \int_{-\infty}^{x} d\lambda \rho(\lambda)$. Once we have $\alpha(c, x)$ (given of course implicitly) we can plot the constrained spectral density $\rho(\lambda)$ for varying values of $c$ by fixing a value of $x$. This is done in Fig. 5 where we have plotted the $\rho(\lambda)$ in the region $c > c_{*}(x)$ (corresponding to the green filled region in Fig. 3). In these plots, we have fixed the value of $x = -2$ and plot $\rho(\lambda)$ along this vertical line in Fig. 3 for three values of $c = 1, 2/3, 1/6$. Notice that the first value $c = 1$ corresponds precisely to being on the top of the horizontal dark green line in Fig. 3 (or, equivalently, on top of $\alpha = \alpha_{+}(x)$ of Fig. 2), which implies that that constrained spectral density is precisely given by $\rho_{\ell}(\lambda)$.

C. Determining the expressions for the second moment of the spectral density and constants $A_{1}$ and $A_{2}$

Once the behaviour of the constrained density has been fully determined and hopefully clearly understood, we need to evaluate the second moment of the constrained density and determine the constants $A_{1}$ and $A_{2}$ appearing in (3), which will allow us to obtain a final expression for the rate function (2). To obtain a neat result for the second moment, we recall the expression for the resolvent and do an expansion in powers of $\lambda$, viz.

$$S_{-}(z) = \int d\lambda \frac{\rho(\lambda)}{z - \lambda} = \frac{1}{z} \int d\lambda \frac{\rho(\lambda)}{1 - \lambda} = \frac{1}{z} \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} \int d\lambda \lambda^{n} \rho(\lambda)$$

or $S_{-}(z) = \sum_{n=0}^{\infty} \frac{\mu_{n}}{z^{n+1}}$ where we have defined the moments of $\rho(\lambda)$ as $\mu_{n} = \int \rho(\lambda)\lambda^{n}$. In our particular case, we have that the resolvent reads $S_{-}(z) = z - \frac{z^{2}(z - x) - 2(z - x + \alpha)}{z - x}$. Its expansion in inverse powers of $z$ automatically yields

$$\int d\lambda \rho(\lambda)\lambda^{2} = \frac{1}{2} + \alpha x.$$
To find expressions for the constants $A_1$ and $A_2$ we need to treat the cases $c > c_\epsilon(x)$ and $c < c_\epsilon(x)$ separately. For $c > c_\epsilon(x)$, the order of the roots and the barrier, that is the domain of the density, is $[\lambda_-, x] \cup [\lambda_0, \lambda_+]$. From the first saddle-point equation (9), we take the following three points within the domain of $\rho(\lambda)$: $\lambda_+ - \epsilon, \lambda_0 + \epsilon$ and $x - \epsilon$ for $\epsilon \to 0^+$. They yield the following 3 equations

\[
\begin{align*}
\lambda_+^2 + A_2 &= 2 \int d\lambda' \rho(\lambda') \log |\lambda_+ - \lambda'|, \\
\lambda_0^2 + A_2 &= 2 \int d\lambda' \rho(\lambda') \log |\lambda_0 - \lambda'|, \\
x^2 + A_1 + A_2 &= 2 \int d\lambda' \rho(\lambda') \log |x - \lambda'|, \\
\end{align*}
\]

which can be combined to obtain the following two identities for $A_1$ and $A_2$:

\[
\begin{align*}
A_1 &= \lambda_0^2 - x^2 + 2 \int d\lambda' \rho(\lambda') \log \left| \frac{x - \lambda'}{\lambda_0 - \lambda'} \right|, \\
A_2 &= -\lambda_+^2 + 2 \int d\lambda' \rho(\lambda') \log |\lambda_+ - \lambda'|. \\
\end{align*}
\]

This, in turn, can be written in terms of the resolvent as follows

\[
\begin{align*}
A_1 &= \lambda_0^2 - x^2 - 2 \int_{\lambda_0}^\lambda dz F_-(z) \\
A_2 &= -\lambda_+^2 + 2 \left[ \log(\lambda_+) - \int_{\lambda_+}^\infty dz \left( S_+(z) - \frac{1}{z} \right) \right].
\end{align*}
\]

For $c < c_\epsilon(x)$, we have that the domain of the spectral density is instead $[\lambda_-, x] \cup [\lambda_0, \lambda_+]$. Again, we evaluate the first saddle-point equation (9) at the points $\lambda_+ - \epsilon, x + \epsilon, \lambda_0 - \epsilon$, for $\epsilon \to 0^+$:

\[
\begin{align*}
2 \int d\lambda' \rho(\lambda') \log |\lambda_+ - \lambda'| &= \lambda_+^2 + A_2, \\
2 \int d\lambda' \rho(\lambda') \log |x - \lambda'| &= x^2 + A_2, \\
2 \int d\lambda' \rho(\lambda') \log |\lambda_0 - \lambda'| &= \lambda_0^2 + A_1 + A_2,
\end{align*}
\]

which, after combining, provide the following:

\[
\begin{align*}
A_2 &= 2 \int d\lambda' \rho(\lambda') \log |\lambda_+ - \lambda'| - \lambda_+^2, \\
A_1 &= x^2 - \lambda_0^2 + 2 \int d\lambda' \rho(\lambda') \log \left| \frac{\lambda_0 - \lambda'}{x - \lambda'} \right|. \\
\end{align*}
\]

These again can be written in terms of the resolvent, viz.

\[
\begin{align*}
A_1 &= x^2 - \lambda_0^2 - 2 \int_{\lambda_0}^x dz S_-(z) \\
A_2 &= -\lambda_+^2 + 2 \int_{\lambda_+}^\infty dz \left( S_+(z) - \frac{1}{z} \right). \\
\end{align*}
\]

From here we see that the constant $A_1$ is actually the same for both cases, while $A_2$ is trivially the same. All in all, this allows us to write the following compact expression for the rate function $\Psi(c, x)$:

\[
\begin{align*}
\Psi(c, x) &= \frac{1}{2} \left( \frac{1}{2} + c x + \lambda_+ \right) - \log(\lambda_+) \\
&\quad + \int_{\lambda_+}^\infty dz \left( S_+(z) - \frac{1}{z} \right) - 3 + 2 \log(2) \frac{c^2}{4} \\
&\quad + \frac{c}{2} \left( 2 \int_{\lambda_0}^x dz S_-(z) + \lambda_0^2 - x^2 \right). \\
\end{align*}
\]

As we will discuss later, the expression (11) is useful to analyse deviations around the typical line $(x, c, x)$ in the $(x, c)$-plane. Still, for general values of $(x, c)$ we need to evaluate the integrals appearing in (11). After a lengthy derivations one eventually finds (see appendix B) that:

\[
\begin{align*}
\Psi(c, x) &= \frac{1}{2} \left( \frac{3}{2} + c x - A_1 c - A_2 - \Omega_0 \right), \\
\end{align*}
\]

where the constants $A_1$ and $A_2$ can be written in terms of incomplete elliptic integrals. Indeed, for $c > c_\epsilon(x)$ we
have
\[ A_1 = -2I_1(\lambda_+, \lambda_0, x, \lambda_-), \]
\[ A_2 = -\lambda_+^2 + 2(\log \lambda_+ + J_2(\lambda_+, \lambda_0, x, \lambda_-)), \]
\[ I_1 = \frac{1}{2\sqrt{(a-c)(b-d)}} \left( 4(a-b)\Pi' \left( \frac{b-c}{a-c} \right) \right. \]
\[ + (a-c)[(b-a)(a+c-d)K' + c(d-b)E'] \left. \right) \],
\[ I_2 = \frac{1}{2\sqrt{(a-c)(b-d)}} \left( 4(a-b)\Pi' \left( \frac{\theta}{a-c} \right) \right. \]
\[ - (b-d)[(c-d)(a-b-d)K' - b(a-c)E'] \left. \right) \],
\[ J_1 = \frac{1}{2\sqrt{(a-c)(b-d)}} \left( 4(c-d)\Pi' \left( \frac{c-d}{a-d} \right) \right. \]
\[ - (b-d)[(c-d)(a-b-d)K' - b(a-c)E'] \left. \right) \],
\[ J_2 = \frac{1}{2\sqrt{(a-c)(b-d)}} \left( 4(a-b)\Pi' \left( \frac{a-c}{b-c} \right) \right. \]
\[ - (a-c)[(a-b)(a+b-d)E'(\theta) + b(d-b)E'(\theta)] \left. \right) \]
\[ + \frac{1}{2} \left[ 1 - \log(4) + (b-c)(b-d) \right] \right). \]
(12)

with definitions \( a = \lambda_+ \), \( b = \lambda_0 \), \( c = x \), and \( d = \lambda_- \). For \( c < c_\ast(x) \) we arrive instead at
\[ A_1 = 2J_1(\lambda_+, x, \lambda_0, \lambda_-), \]
\[ A_2 = -\lambda_+^2 + 2(\log(\lambda_+ + J_2(\lambda_+, \lambda_0, x, \lambda_-)), \]
\[ J_1 = \frac{1}{2\sqrt{(a-c)(b-d)}} \left( 4(c-d)\Pi' \left( \frac{c-d}{a-d} \right) \right. \]
\[ - (b-d)[(c-d)(a-b-d)K' - b(a-c)E'] \left. \right) \],
\[ J_2 = \frac{1}{2\sqrt{(a-c)(b-d)}} \left( 4(a-b)\Pi' \left( \frac{a-c}{b-c} \right) \right. \]
\[ - (a-c)[(a-b)(a+b-d)E'(\theta) + b(d-b)E'(\theta)] \left. \right) \]
\[ + \frac{1}{2} \left[ 1 - \log(4) + (b-c)(b-d) \right] \right) \right). \]
(13)

with definitions \( a = \lambda_+ \), \( b = x \), \( c = \lambda_0 \), and \( d = \lambda_- \). In both cases \( F'(\theta) \equiv F'(\theta, k') \), \( E'(\theta) \equiv E(\theta, k') \), and \( \Pi'(\theta, n) \equiv \Pi(\theta, n, k') \) are the incomplete elliptic integrals of the first, second, and third kind, respectively, with \( k' = \sqrt{1-k^2} \), elliptic modulus \( k = \sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}} \), and argument \( \theta = \sin^{-1} \sqrt{\frac{(b-d)}{(a-b)(c-d)}} \).

Fig. 6 shows some plots of the rate function \( \Psi(c, x) \). Looking at them, a few points are in order. The first one is to notice that the rate function is zero along the line \( c_\ast(x), \) that is \( \Psi(c_\ast(x), x) = 0 \). This is to be expected as it corresponds to the unperturbed Coulomb fluid configuration of Wigner’s semi-circle law. Any perturbation of this configuration will increase the energy of the Coulomb fluid, increasing the value of the rate function. This is shown in Fig. 6 where we present two cuts of the rate function. In the middle plot of Fig. 6 we have fixed the position \( x \) of the barrier and vary the fraction \( c \). Here the rate function is zero for \( c_\ast(x), \) that is, the value of \( c \) corresponding to Wigner’s law. In this setup, and if we want to keep the position of the barrier constant, there are only two ways to perturb Wigner’s law: either we pull eigenvalues to the left of the barrier (\( c > c_\ast(x) \)) or we pull eigenvalues to the right of the barrier (\( c < c_\ast(x) \)). Similarly, in the right plot in Fig. 6 we have instead fixed the fraction \( c \) and vary the position of the barrier. Here the rate function is zero at \( x_\ast(c) \), that is, the position of the barrier that will give precisely the fraction \( c \) in the semi-circle law (i.e. \( x_\ast(c) \) is the inverse function of \( c_\ast(x) \)). While keeping \( c \) constant, there are two ways to perturb Wigner’s law: either by pushing the barrier to the left (\( x < x_\ast(c) \)) or to the right (\( x > x_\ast(c) \)).

IV. TAIL CUMULATIVE DISTRIBUTION FUNCTION OF \( c \)

Recall that for large \( N \), \( g(c) = e^{-\beta N^2 \Psi(c,x)} \). From here, its tail CDF reads
\[ \mathcal{F}_{C}(c) = \text{Prob}[C_x > c] = \int_c^\infty dc' g(c'). \]

Since the pdf \( g(c) \) is peaked at \( c = c_\ast(x) \) (nb. if the rate function is one-branched the same reasoning obviously applies), then we have that:
\[ \mathcal{F}_{C}(c) = e^{-\beta N^2 \Psi(c,x)} = c > c_\ast(x) \]
\[ \mathcal{F}_{C}(c) = e^{-\beta N^2 \Psi(c,x)} = c < c_\ast(x). \]

Once that the CDF of the SIN is found, we automatically obtain the corresponding CDF for the \( k \)-th eigenvalue, viz.
\[ \mathcal{F}_{\lambda_k}(x) = e^{-\beta N^2 \Psi(k/N,x)} = x < x_\ast(k/N) \]
\[ \mathcal{F}_{\lambda_k}(x) = e^{-\beta N^2 \Psi(k/N,x)} = x > x_\ast(k/N). \]

Thus, the rate function \( \Psi(c,x) \) has a two-fold meaning: as a function of \( c \) (for \( x \) fixed) gives information about the large deviations of the SIN; as a function of \( x \) (for \( c = k/N \) fixed) gives the large deviations of the \( k \)-th eigenvalue. The latter case is even more surprising when, as we will see later, we find that for extreme eigenvalues, that is for \( k = 1 \) (smallest) or \( k = N \) (largest), one is able to derive their left and right large deviation functions within the Coulomb fluid picture. This goes against common knowledge that one was required different approaches to obtain both tail distributions.

V. LARGE DEVIATION FUNCTIONS FOR EXTREME EIGENVALUES

We move on and consider in this section the statistics for the extreme eigenvalues. We focus on the smallest eigenvalues \( k = 1 \) (the largest one has identical properties since we are dealing here with the Gaussian ensemble). This corresponds to study the behavior of the rate
function $\Psi(c, x)$ along the horizontal line $c = 0$ in the left plot in Fig. [4]. We have clearly two distinct cases corresponding to whether either $x > -\sqrt{2}$ (right rate function, which we will denote as $\Psi_+(x)$) or $x < -\sqrt{2}$ (left rate function, which we will denote as $\Psi_-(x)$). The right rate function is simply obtained by taking the limit $\Psi_+(x) = \lim_{\lambda \to 0^+} \Psi(c, x)$ so that $F_{\lambda}(x) = e^{-N^2 \Psi_+(x)}$ for $x > -\sqrt{2}$. In this case, which corresponds to $0 < c < c_0(x)$, we have that $\lambda_+ > x > \lambda_0 = \lambda$, with $\lambda_+ = a_+(x)$ and $\lambda_0 = \lambda_0 = b_-(x)$. Looking at the corresponding expressions for $c < c_0(x)$, we notice we only have to worry about the constant $A_2$ or, equivalently the function $J_2(\lambda_-, \lambda_0, \lambda_-)$. A simple eye inspection of them reveals that since $\lambda_0 = \lambda_-$, the elliptic modulus is zero (or the complementary elliptic modulus is one), so that the incomplete elliptic integrals appearing in $J_2$ take the following forms:

$$F(\theta, 1) = \log(\tan \theta + \sec \theta), \quad E(\theta, 1) = \sin \theta,$$

$$\Pi(\theta, \alpha^2, 1) = \frac{\log(\tan \theta + \sec \theta) - \alpha \log \sqrt{\frac{1 + \alpha \sin \theta}{1 - \alpha \sin \theta}}}{1 - \alpha^2}.$$

Plugging this result back into $J_2$, and after some torturous derivations, we finally obtain:

$$J_2(a, b, c, d) = \frac{1}{18} \left( x \left( \sqrt{x^2 + 6} - x \right) 
- 18 \log \left( x \left( \sqrt{x^2 + 6} + x \right) + 4 \right) + 15 \right).$$

This helps us to arrive to the following expression for the $A_2$, viz.

$$A_2 = -\frac{1}{9} \left( x + 2\sqrt{6 + x^2} \right)^2 + 2 \log \left[ \frac{1}{3} \left( x + 2\sqrt{6 + x^2} \right) \right] + \frac{1}{9} \left( 15 + x \left( \sqrt{x^2 + 6} - x \right) \right) - 18 \log \left( x \left( \sqrt{x^2 + 6} + x \right) + 4 \right),$$

while the action $S_0(c, x)$ takes the eventual form

$$S_0(c, x) = \frac{3}{4} + \frac{1}{54} \left[ x^2(36 - x^2) + x(15 + x^2)\sqrt{6 + x^2} \right.
+ 54 \log(3) - 54 \log \left( x + 2\sqrt{6 + x^2} \right)
+ 54 \log \left( x \left( \sqrt{x^2 + 6} + 4 \right) \right).$$

After some final standard manipulations with the logarithms we arrive at

$$\Psi_+(x) = \frac{1}{108} \left[ -x^4 + 36x^2 + \sqrt{x^2 + 6} \left( x^3 + 15x \right) 
+ 27 \left( \log(18) - 2 \log \left( \sqrt{x^2 + 6} - x \right) \right) \right],$$

as reported in [46].

To derive the left rate function we first realise that $\lim_{\lambda \to 0^+} \Psi(c, x) = 0$ for $x \leq -\sqrt{2}$. Hence it becomes relevant to see how the rate function vanishes with $c$ in this limit. Before doing any derivation let us imagine what would happen if $\Psi(c, x) = c \Psi_-(x) + \cdots$. Then since $c = 1/N$ we would have that $F_{\lambda_0}(x) = e^{-N^2 \Psi(k/N,x)} = e^{-\beta N \Psi_-(x)}$, which is precisely the correct scaling in $N$ we expect for the deviation of the the smallest eigenvalue to the left of its typical value $-\sqrt{2}$.

This is precisely what happens mathematically and what allows us to recover the rate function within the Coulomb fluid picture. To obtain the precise expression for $\Psi_-(x)$ we need to do an expansion around $c = 0^+$ for $x < -\sqrt{2}$. It is helpful in this case to realize that this is equivalent of doing the expansion around $a = 0$, since we are close to the typical value of $c_0(x)$ for $x < -\sqrt{2}$. There are two ways to tackle the expansion: either directly in the exact expressions or using the expression [11] as a starting point. The latter turns out to be more straightforward. The first thing to do is to find an expression of the roots of $P_3(x)$ for small $a$. Starting from the exact expression of the roots and doing an expansion

![Graph showing density plot of $\Psi(c, x)$ along with rate functions for various $c$ values.](image)
for small $\alpha$ is actually hopeless. It is easier to go back to the equation $P_\lambda(z) = 0$ and to treat it perturbatively. This readily yields:

$$\begin{align*}
\lambda_+ &= \sqrt{2} - \frac{2\alpha}{-4 + 2\sqrt{2}x} + \cdots , \\
\lambda_0 &= -\sqrt{2} - \frac{2\alpha}{-4 - 2\sqrt{2}x} + \cdots , \\
\lambda_- &= x - \frac{2\alpha}{2 - x^2} + \cdots .
\end{align*}$$

Armed with this result, we can go to back to eq. (11) to obtain the following results for each term. We have that

$$\begin{align*}
\lambda_+^2/2 - \log(\lambda_+) + \int_{\lambda_+}^\infty dz \left( S_-(z) - \frac{1}{z} \right) \\
= 1 + \log(2) + \frac{\alpha}{\sqrt{x^2 - 2} - x} \log \left( \frac{\sqrt{x^2 - 2} - x}{\sqrt{x^2 - 2} + x} \right) + O(\alpha^{3/2}),
\end{align*}$$

where $\lambda_+ = \sqrt{2} - \frac{2\alpha}{-4 + 2\sqrt{2}x}$ and $\lambda_0 = -\sqrt{2} - \frac{2\alpha}{-4 - 2\sqrt{2}x}$. Gathering all results we obtain

$$\begin{align*}
\Psi(c, x) &= \frac{1}{2} \left[ \frac{1}{2} \left( \frac{1}{2} + \alpha x \right) + \frac{c}{2} \left( 2 \int_{\lambda_0}^x dz S_-(z) + \lambda_0^2 - x^2 \right) + \frac{\lambda_+^2}{2} - \log(\lambda_+) + \int_{\lambda_+}^\infty dz \left( S_-(z) - \frac{1}{z} \right) - \frac{3 + 2\log(2)}{4} \right] \\
&= \frac{1}{2} \left[ \frac{1}{2} \alpha x + \frac{c}{2} \left( -x \sqrt{x^2 - 2} - 2 \log \left( \sqrt{x^2 - 2} - x \right) + \log(2) \right) + \frac{\alpha}{\sqrt{x^2 - 2} - x} \log \left( \frac{\sqrt{x^2 - 2} - x}{\sqrt{x^2 - 2} + x} \right) \right] + \cdots \\
&= \frac{c}{2} \left[ -x \sqrt{x^2 - 2} - 2 \log \left( \sqrt{x^2 - 2} - x \right) + \log(2) \right] + \cdots ,
\end{align*}$$

from which we can directly read the left rate function for the smallest eigenvalue, viz

$$\begin{align*}
\Psi_-(x) &= \frac{1}{2} \left[ \log(2) - x \sqrt{x^2 - 2} - 2 \log \left( \sqrt{x^2 - 2} - x \right) \right].
\end{align*}$$

In agreement with [74].

VI. TYPICAL FLUCTUATIONS AROUND THE POINT $(x, c_\star(x))$ FOR BULK EIGENVALUES

In the previous section, we have convincingly established that the rate function $\Psi(c, x)$ contains the left and right rate functions for the extreme eigenvalues. Here, we focus on the typical statistics for bulk eigenvalues, that is, those eigenvalues within the region $x \in (-\sqrt{2}, \sqrt{2})$. From a mathematical viewpoint, this calls for doing an expansion of the rate function around the line $c_\star(x)$. Or strictly speaking, if we denote as $(x_\star, c_\star)$ a particular point along this line, since $\Psi(c, x)$ rests on the $(x, c)$-plane, we are seeking for the change of the rate function to a new point $(x, c)$ close to $(x_\star, c_\star)$. As we are close to $c_\star(x)$ we can use again the trick of expanding around $\alpha = 0$ the expression (11). Firstly, we need an expansion of the roots, viz.

$$\begin{align*}
\lambda_- &= -\sqrt{2} - \frac{2\alpha}{-4 - 2\sqrt{2}x} + \cdots , \\
\lambda_0 &= x_\star + (x - x_\star) - \frac{2\alpha}{2 - x_\star^2} + \cdots , \\
\lambda_+ &= \sqrt{2} - \frac{2\alpha}{-4 + 2\sqrt{2}x_\star} + \cdots .
\end{align*}$$

We then proceed to perform this expansion to the various factors appearing in the rate function.

A. Asymptotic analysis of $c$

Let us start with the expression for the fraction $c$. We do the derivation as follows: we expand the integrand in power of $\alpha$ and $dx = x_\star - x$ and then we integrate term by term, viz

$$c = \frac{1}{\pi} \int_{\lambda_-}^{\lambda_0} d\lambda \left[ \sqrt{2 - \lambda^2} + \frac{\alpha}{(-x_\star + \lambda)\sqrt{2 - \lambda^2}} + O(\alpha^2) \right].$$
The first integral gives:
\[
\frac{1}{\pi} \int_{-\infty}^{\lambda_0} d\lambda \sqrt{2 - \lambda^2} = c_\ast(x_\ast) + \frac{\sqrt{2 - x_\ast^2}}{\pi}(x - x_\ast)
\]
while for the second integral we have instead:
\[
\frac{\alpha}{\pi} \int_{\lambda_0}^{\lambda_\ast(\alpha)} \frac{d\lambda}{(-x + \lambda)\sqrt{2 - \lambda^2}} = \frac{\alpha}{\pi \sqrt{2 - x_\ast^2}} \log \left| \frac{1}{\sqrt{2} - x_\ast^2} \left( \delta x + \frac{2\alpha}{2 - x_\ast^2} \right) \right| + \cdots.
\]
Putting together these two results we finally write the following expansion for \(c\):
\[
c = c_\ast(x_\ast) - \frac{\sqrt{2 - x_\ast^2}}{\pi} \delta x - \frac{2\alpha}{\pi \sqrt{2 - x_\ast^2}}
+ \frac{\alpha}{\pi \sqrt{2 - x_\ast^2}} \log \left| \frac{1}{\sqrt{2} - x_\ast^2} \left( \delta x + \frac{2\alpha}{2 - x_\ast^2} \right) \right| + \cdots.
\] (14)

Eq. (14) allows us to obtain the expansion of \(\alpha(c, x)\) around \((x_\ast, c_\ast)\). This is given implicitly, but for small perturbations one can iterate (14) to obtain:
\[
\alpha = -\frac{\pi \sqrt{2 - x_\ast^2} \delta c + (2 - x_\ast^2) \delta x}{2 - \log \left| \frac{1}{\sqrt{2} - x_\ast^2} \left( \delta x + \frac{2\alpha}{2 - x_\ast^2} \right) \right|},
\] (15)
where we have defined \(\delta c = c - c_\ast\).

B. Asymptotic analysis of \(2 \int_{x_0}^{x} dz S_-(z) + \lambda_0^2 - x^2\)

This expansion is done similarly to the previous case. We simply write the final result:
\[
2 \int_{x_0}^{x} dz S_-(z) + \lambda_0^2 - x^2 = -\frac{2\pi\alpha}{\sqrt{2 - x_\ast^2}} + \cdots.
\]

C. Asymptotic analysis of \(\int_{x_0}^{\infty} dz \left( S_-(z) - \frac{1}{z} \right)\)

This is similar to the result before: again there is no linear term in the expansion of the resolvent in \(\delta x\). So the final result is:
\[
\frac{\lambda_0^2}{2} - \log(\lambda_\ast) + \int_{x_0}^{\infty} dz \left( S_-(z) - \frac{1}{z} \right)
= 1 + \log(2) + \frac{\alpha}{\sqrt{2 - x_\ast^2}} \left( \pi - \arccos \left( \frac{x_\ast}{\sqrt{2}} \right) \right) + \cdots.
\]

D. Gathering results: asymptotic analysis for the rate function

By gathering all these results into the expression (11) we obtain the following expression:
\[
\Psi(c, x) = \frac{\alpha}{2} \left[ -\frac{1}{2} \frac{\delta x}{\sqrt{2 - x_\ast^2}} + \frac{1}{\sqrt{2} - x_\ast^2} \left( -\pi c + \frac{1}{2} x_\ast \sqrt{2 - x_\ast^2} \right) \right]
\]
\[
+ \pi - \arccos \left( \frac{x_\ast}{\sqrt{2}} \right) + \cdots,
\]
and recalling that \(2\pi c_\ast = 2\pi + x_\ast \sqrt{2 - x_\ast^2} - 2\arccos \left( \frac{x_\ast}{\sqrt{2}} \right)\) we finally have:
\[
\Psi(c_\ast + \delta c, x_\ast - \delta x) = -\frac{\alpha}{2} \left[ \frac{1}{2} \frac{\delta x}{\sqrt{2 - x_\ast^2}} + \pi \delta c + \cdots \right],
\] (16)

where \(\alpha = \alpha(\delta c, \delta x)\) is given by (15).
Suppose now we sit at a point \((x_\ast, c_\ast)\) and we want to look at the typical fluctuations along the \(x\)-direction and the typical fluctuations along the \(c\)-direction. This means that we must take \(\delta c = 0\) and \(\delta x = 0\), respectively, which produces the following two expansions of the rate function:
\[
\Psi(c_\ast, x_\ast - \delta x) = \frac{2 - x_\ast^2}{4} \left( \frac{\delta x}{2 - x_\ast^2} \right)^2,
\]
\[
\Psi(c_\ast + \delta c, x_\ast) = \frac{\pi^2}{2} \frac{(\delta c)^2}{2 - \log \left| \frac{\sqrt{2}\delta c}{(2 - x_\ast^2)} \right|}.
\]

Recalling that for large \(N\) we have that \(F_X(x) \sim f_X(x)\) and assuming, moreover, that that the fluctuations are symmetric around \((c_\ast, x_\ast)\) we obtain the following results for the Gaussian fluctuations of \(y_k\) and \(N'\):
\[
f_{y_k}(y_k) \sim \exp \left[ \frac{(y_k - x_\ast \sqrt{N})^2}{2\Delta_1} \right],
\]
\[
f_{N'}(n_k) \sim \exp \left[ \frac{(n_k - c_\ast N)^2}{2\Delta_2} \right],
\]
with variances \(\Delta_1 = \log((2 - x_\ast^2)^3/2N)\) and \(\Delta_2 = \log(N/(2 - x_\ast^2))\).
Here we have used that \(\delta c = c - c_\ast = (n_k - c_\ast N)/N\) and \(\delta x = x_\ast - x = (x_\ast \sqrt{N} - y_k)/\sqrt{N}\). The expressions for \(\Delta_1\) and \(\Delta_2\) are in agreement with those found in [92, 93].
In Fig. 7 we compare the analytical results of \(\Delta_1\) and \(\Delta_2\) with standard Monte Carlo simulations of the Coulomb fluid.

VII. COMPARISON WITH MONTE CARLO SIMULATIONS

We move to explain how we have performed the two types of Monte Carlo simulations to check our analytical results. Let us first recall that given the jPDF of
FIG. 7: Comparison between Monte Carlo simulations and Gustavsson’s formulas. For $\Delta_2(x)$ we have taken $N = 500$ eigenvalues and $\beta = 1$. We have relaxed the Coulomb fluid using Metropolis for $5 \cdot 10^5$ Monte Carlo steps. After that we have use a window of $1 \cdot 10^6$ Monte Carlo setps to sample the mean value and variance of each eigenvalue $y_i$ with $y_1 < \cdots < y_N$ each 100 steps, resulting in the reported figure. For $\Delta_1(x)$ we have taken $N = 100$ eigenvalues and $\beta = 1$. We have relaxed the Coulomb gas for $10^6$ Monte Carlo steps and we have used a window of $10^7$ to take samples of $N_c$ every 100 Monte Carlo steps. We repeat this process for various values of $x$. In both cases we have added a small constant to have the theoretical results on top of the numerical ones.

The standard Metropolis algorithm consists in the following steps: i) pick a $k \in \{1, \ldots, N\}$ at random; ii) propose an update $y_k \rightarrow y_k';$ iii) calculate the energy difference $\Delta F$ due to the change; iv) update with probability $p = \min\left\{1, e^{-\frac{2}{\beta} \Delta F}\right\}$. The energy difference due to this move is

$$\Delta F = (y_k')^2 - y_k^2 - 2 \sum_{j(\neq k)} y_j^2 \log \frac{|y_k - y_j|}{|y_k' - y_j|}.$$ 

In what follows, we explain how we modify the standard Metropolis algorithm to improve the performance when checking the exposed analytical results.

A. Monte Carlo simulations for the constrained density $\rho(\lambda)$ and for the action $S_0(c, x)$

As already noted in Figs. [3] and [6] we have compared our analytical findings with Monte Carlo Metropolis simulations using a slightly modified metropolis algorithm. Given a position $c$ of the barrier, to simulate a given fraction $c$, we choose a total number of eigenvalues $N$ and divide them into those to the left of the barrier, $N_{\text{left}}$, and those to the right of the barrier, $N_{\text{right}}$. Initial conditions are chosen so that $c = N_{\text{left}}/N$. Then one performs the standard Metropolis algorithm, but the updates are wisely chosen so that $N_{\text{left}}$ remains constant throughout the Monte Carlo updating. This can obviously be done without adding extra rejection to the Metropolis algorithm. After letting the system thermalise, one performs averages over the Monte Carlo Markov chain generated by the algorithm.

The estimation of the density $\rho(\lambda)$ by Monte Carlo, which appears in Fig. [5] is done straightforwardly and does not required much thought. The estimation of the action (or equivalently the rate function $\Psi(c, x)$ as shown in Fig. [6]) requires a bit of care, though. Here we must remember that when doing the analytics we have gone from $y_i \rightarrow \lambda_i$, and we have ignored constant terms which depend on $N$. Tracing our steps back and correcting for those factors, we find that the action is to be estimated by the formula

$$S_0(c, x) = \frac{1}{N^2} \left[\langle F(y)\rangle_{\text{thermal}} + \frac{1}{2} N(N - 1) \log N \right],$$

with $F(y)$ defined in (17) and where $\langle \cdots \rangle_{\text{thermal}}$ stands for the thermal averaging, which is estimated by averaging over the Monte Carlo Markov chain generated by the algorithm.

B. Modified Metropolis algorithm for the statistics of $\lambda_k$

We finish this part by briefly explaining how we estimate the rate function for bulk eigenvalues. As it has been explained somewhere else, if we were to use the standard Metropolis algorithm to construct an histogram for $\text{Prob}[\lambda_k < x]$ it would require a prohibitively large number of samples to obtain a reliable statistics for $x$ away from its typical value $x_*$. However, this can be written as $\text{Prob}[\lambda_k < x|x \leq y]K_y$, for a constant $K_y$. Essentially, one is putting a barrier at $y$ so that $\lambda_k$ would be concentrated around $x \in [y - \delta, \delta]$. This gives directly the
The derivative of the rate function with respect to $x$,

$$\Psi(x, x) = -\frac{1}{\beta N^2} \frac{d}{dx} \log \text{Prob}[\lambda_k < x | x \leq y].$$

This nice little trick gives the branch of $\Psi(c, x)$ for $x < x_*(c)$. A similar argument gives the branch corresponding to $x > x_*(c)$.

The result of this algorithm is presented in Fig. 8. Here we have taken $N = 20$ with $N_{\text{left}} = 14$, which means that we are estimating the statistics of the 14-th eigenvalue or, for very large systems, that eigenvalue with typical value $x_*(c)$ corresponding to $c = 7/10$. As we can see the results are strikingly good.

VIII. SUMMARY AND FUTURE WORK

In this work we have presented a method to obtain the full order statistics of the eigenvalues of the Gaussian ensemble. This has been achieved by introducing a rate function $\Psi(c, x)$ depending on two parameters $c$ and $x$. When $c = k/N$ the rate function $\Psi(k/N, x)$ gives the large deviations of the $k$-th eigenvalue as a function of $x$. Importantly, when $k = 1$ (or $k = N$) the rate function $\Psi(k/N, x)$, when analysed carefully, provides both the fluctuations to the left and to the right of the typical value $-\sqrt{2}$ of $\lambda_1$. If we fix $x$, then $\Psi(c, x)$ provides the large deviations of the SIN, that is, the large deviations of the fraction $c$ of eigenvalues to the left of $x$, hence generalising the work on the index number problem.

It would be interesting to see how easily the methods presented here can be extended to other ensembles (e.g. Wishart, Jacobi, Cauchy), as it might be that for some ensembles it is not possible to obtain exact expressions for the integrals involving the resolvent in terms of e.g. elliptic integrals. However, perturbations around the typical behaviour would be certainly possible. The latter will help to see how the variances of typical fluctuations are affected when considering eigenvalues in a restricted subset of $\mathbb{R}$. An analysis of the Coulomb fluid on a general external polynomial potential $V(x)$ would be of some interest too.

It will be interesting and challenging to see the connection with the known results using determinantal methods (and its relation to Painlevé systems) applied to the $k$-th eigenvalue and valid for finite $N$.

Finally, the approach of this work can also be extended to study the order statistics of a pair of eigenvalues $\lambda_k$ and $\lambda_\ell$, with applications, for instance, to the study the statistics of the gap between consecutive eigenvalues and to study the condition number. These are current lines of research.

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APPENDICES

Appendix A: Deriving $c(\alpha, x)$

In this section we derive an exact expression for the fraction $c$ as a function of $\alpha$ and $x$ in terms of complete elliptic integrals. To do so, we need to distinguish between the cases $c > c_*(x)$ and $c < c_*(x)$.

1. Case $c > c_*$

Here the fraction $c$ of eigenvalues to the left of the barrier is given by

$$c(\alpha, x) = \frac{1}{\pi} \int_{\lambda_-}^{\lambda_+} d\lambda \sqrt{\frac{(\lambda_+ - \lambda)(\lambda_0 - \lambda)(\lambda - \lambda_-)}{x - \lambda}}, \quad (A1)$$

which is related to the the following integral

$$I = \int_{d}^{y} \frac{(a - t)(b - t)(t - d)}{(c - t)},$$

with $\lambda_+ = a$, $\lambda_0 = b$, $x = c$, $\lambda_- = d$, and $y = c$. This can be related to elliptic integrals (given by [100], formula 252.36):

$$I = (a - d)^2(d - b)\tilde{\alpha}^2g\int_{0}^{w_1} \frac{sn^2(u)dn^2(u)du}{(1 - \tilde{\alpha}^2sn^2(u))^{3}}, \quad (A2)$$

$$g = \frac{2}{\sqrt{(a - c)(b - d)}}, \quad k^2 = \frac{(a - b)(c - d)}{(a - c)(b - d)},$$

$$\tilde{\alpha}^2 = \frac{d - c}{a - c} < 0, \quad u_1 = K(k),$$

where the following definition of the $V_m = \int_{a}^{w_1} \frac{du}{(1 - \tilde{\alpha}^2sn^2(u))^{m}}$ can also be found at [100] (formulas 336). Gathering terms together and after massaging a bit we obtain the final expression:
\[
c(\alpha, x) = \frac{1}{\pi} \frac{1}{4\sqrt{(a-c)(b-d)}} \left[(a - d) \left(a^2 - 2d(a + b - c) - 2ab + 2ac + b^2 + 2bc - 3c^2 + d^2\right) \Pi \left(\frac{d - c}{a - c}, k\right) \right. \\
- (a - d)(a - c)(a - 3b + 3c - d)K(k) - (a - c)(b - d)(a + b - 3c + d)E(k), \quad k^2 = \frac{(a - b)(c - d)}{(a - c)(b - d)}. \tag{A2}
\]

This is still valid for general values of \(a, b, c,\) and \(d\) (as soon as they are ordered as our roots). In our particular case we have, however, that \(a = \lambda_+, \ b = \lambda_0, \ c = x\) and \(d = \lambda_-\). This implies that, the coefficients \(a, b, c,\) and \(d\) obey certain equalities, which allows us to simplify further this finding, as presented in the text.

2. Case \(c < c_*(x)\)

In this case the fraction \(c\) of eigenvalues reads:

\[
c(\alpha, x) = \frac{1}{\pi} \int_{\lambda_-}^{\lambda_+} d\lambda \sqrt{\frac{(\lambda_+ - \lambda)(\lambda_0 - \lambda)(\lambda - \lambda_-)}{x - \lambda}}, \tag{A3}
\]

which is related to the elliptic integral

\[
I = \int_{y}^{c} \sqrt{(a - t)(c - t)(t - d)} \frac{b - t}{d - t}
\]

with \(a = \lambda_+, \ b = x, \ c = \lambda_0, \ d = \lambda_-\), and with \(y = d\). This corresponds to the elliptic integral 253.34 page 111 in [100], viz.

\[
I = (a - c)(b - c)(c - d)a^{2}g \int_{0}^{u_{1}} \frac{\sin^{2}(u) \sin^{2}(a^{2})}{(1 - a^{2} \sin^{2}(u))^{3/2}} du, \\
k^{2} = \frac{(a - b)(c - d)}{(a - c)(b - d)}, \quad g = \frac{2}{\sqrt{(a - c)(b - d)}}, \\
\tilde{a}^{2} = \frac{c - d}{b - d}, \quad u_{1} = K(k).
\]

The final expression for fraction \(c\) of eigenvalues becomes

\[
c(\alpha, x) = \frac{1}{4\pi \sqrt{(a-c)(b-d)}} \left[(b - c) \left(a^{2} + 2a(b - c - d) - 3b^{2} + 2b(c + d) + (c - d)^{2}\right) \Pi \left(\frac{c - d}{b - d}, k\right) \right. \\
+ (b - c)(b - d)(-5a + 3b + c - d)K(k) - (a - c)(b - d)(a - 3b + c - d)E(k), \quad k^2 = \frac{(a - b)(c - d)}{(a - c)(b - d)}, 
\]

which again can be simplified, as given in the main text recalling that the roots and teh barrier \(x\) obey certain equalities.

### Appendix B: Exact expressions for the integrals in Eq. [11]

Let us find exact expressions for the integrals appearing in the expressions of the constants \(A_1\) and \(A_2\). Again we have to make a distinction between cases \(c > c_*(x)\) and \(c < c_*(x)\).

1. Case \(c > c_*(x)\)

a. Integral \(\int_{x}^{\lambda_0} dz S_+(z)\)

To evaluate the integral \(\int_{x}^{\lambda_0} dz S_+(z)\), we note that it is related to the following elliptic integral (255.39 page 120 from [100])

\[
I_1 = \int_{c}^{d} dt \sqrt{\frac{(t - a)(t - b)(t - d)}{t - c}} = (a - b)(b - d) g \frac{2}{\tilde{a}^{2}} \Pi(\tilde{a}^{2}, k) + \left(2k^{2} - \tilde{a}^{2}\right)V_2 + \left(\tilde{a}^{2} - k^{2}\right)V_3, \\
k^{2} = \frac{(b - c)(a - d)}{(a - c)(b - d)}, \quad g = \frac{2}{\sqrt{(a - c)(b - d)}}, \quad \tilde{a}^{2} = \frac{b - c}{a - c},
\]
and with the already given the definitions for the $V$ functions. Gathering all these results and simplifying, it comes:

$$I_1(a,b,c,d) = \frac{1}{4\sqrt{(a-c)(b-d)}} \left[ (a-b)(a^2 - 2d(a+b-c) - 2ab + 2ac + b^2 + 2bc - 3c^2 + d^2) \Pi \left( \frac{b-c}{a-c}, k \right) - (a-b)(a-c)(a-b+3c-3d)K(k) + (a-c)(b-d)(a+b-3c+d)E(k) \right].$$

Apply to our particular case we obtain:

$$\int d\lambda \rho(\lambda') \log \frac{\lambda_0 - \lambda'}{x - \lambda'} = \frac{\lambda_0^2 - x^2}{2} + I_1(\lambda_+, \lambda_0, x, \lambda_-).$$

As mentioned before, due to the roots obeying certain equalities, $I_1$ is simplified further as reported in the main text.

**b. Evaluation of $\int_{\lambda_+}^{\infty} dz \left[ S_- (z) - \frac{1}{z} \right]$**

Next, we move to evaluate $\int_{\lambda_+}^{\infty} dz \left[ S_- (z) - \frac{1}{z} \right]$. Firstly, we notice that since the resolvent behaves at infinity as

$$I_2(a,b,c,d) = \lim_{y \to \infty} \int_y^y dt \left[ \sqrt{\frac{(t-a)(t-b)(t-d)}{(t-c)}} - \frac{1}{2}(-a-b+c-d) - t \right.\left. - \frac{a^2 + 2a(b-c+d) - b^2 + 2b(d-c) + (c-d)(3c+d)}{8t} \right],$$

where we have subtracted the divergences at infinity. Or in other words: we must evaluate the integral and study its asymptotic behaviour for $y \to \infty$, viz.

$$\int_a^y dt \sqrt{\frac{(t-a)(t-b)(t-d)}{(t-c)}} \sim Ay^2 + By \\ \text{log}(y) + C + O(y^{-1}).$$

From formula 258.36 page 132 in [100] that (Nb. there seems to be a typo in [100]. The one reported here is the correct one) $S_-(z) = \frac{1}{z} + O(z^{-2})$, the integral is clearly convergent. Here, to obtain an exact expression in terms of elliptic integrals we must work a bit harder. Essentially, we want to do the following integral and then evaluate the limit

$$I_3 = \int_a^y dt \sqrt{\frac{(t-a)(t-b)(t-d)}{(t-c)}} = (a-b)^2(a-d) \frac{g}{\alpha^2} \left[ -\Pi(u_1, \alpha^2, k) + (2 - \alpha^2)V_2(u_1) + (\alpha^2 - 1)V_3(u_1) \right],$$

$$k^2 = \frac{(b-c)(a-d)}{(a-c)(b-d)}, \quad g = \frac{2}{\sqrt{(a-c)(b-d)}}, \quad sn(u_1) = \sin \varphi, \quad \varphi = \sin^{-1} \sqrt{\frac{(b-d)(y-a)}{(a-d)(y-b)}}.$$
with the following definitions for the $V$ functions:

\[ V_0 = u = F(\varphi, k), \]
\[ V_1 = \Pi(\varphi, \tilde{\alpha}^2, k), \]
\[ V_2 = \frac{1}{2(\tilde{\alpha}^2 - 1)(k^2 - \tilde{\alpha}^2)} \left[ \tilde{\alpha}^2 E(u_1, k) + (k^2 - \tilde{\alpha}^2)u_1 + (2\tilde{\alpha}^2k^2 + 2\tilde{\alpha}^2 - \tilde{\alpha}^4 - 3k^2)\Pi(\varphi, \tilde{\alpha}^2, k) - \frac{\tilde{\alpha}^4 sn(u_1)cn(u_1)dn(u_1)}{1 - \tilde{\alpha}^2 sn^2(u_1)} \right], \]
\[ V_3 = \frac{1}{4(1 - \tilde{\alpha}^2)(k^2 - \tilde{\alpha}^2)} \left[ k^2V_0 + 2(\tilde{\alpha}^2k^2 + \tilde{\alpha}^2 - 3k^2)V_1 + 3(\tilde{\alpha}^4 - 2\tilde{\alpha}^2k^2 - 2\tilde{\alpha}^2 + 3k^2)V_2 + \frac{\tilde{\alpha}^4 sn(u_1)cn(u_1)dn(u_1)}{(1 - \tilde{\alpha}^2 sn^2(u_1))^2} \right]. \]

In the limit $y \to \infty$ it is possible to see that the divergences are coming from the following terms:

\[ \Pi \left( \sin^{-1} \sqrt{\frac{(b - d)(y - a)}{(a - d)(y - b)}}, \frac{a - d}{b - d}, k \right) = -\Pi \left( \sin^{-1} \sqrt{\frac{(b - d)}{(a - d)}}, \frac{b - c}{a - c}, k \right) + F \left( \sin^{-1} \sqrt{\frac{(b - d)}{(a - d)}}, k \right) + E_1(a, b, c, d) + A \log(y) + \cdots, \]
\[ \frac{\tilde{\alpha}^4 sn(u_1)cn(u_1)dn(u_1)}{1 - \tilde{\alpha}^2 sn^2(u_1)} = E_2(a, b, c, d) + Ay + B \frac{1}{y} + O(y^{-2}), \]
\[ \frac{\tilde{\alpha}^4 sn(u_1)cn(u_1)dn(u_1)}{(1 - \tilde{\alpha}^2 sn^2(u_1))^2} = E_3(a, b, c, d) + Ay^2 + By + Cy^{-1} + O(y^{-2}), \]

where we have defined

\[ E_1(a, b, c, d) = \frac{\sqrt{(a - c)(b - d)}}{2(a - b)} \log \left( \frac{4}{a + b - c - d} \right), \]
\[ E_2(a, b, c, d) = -\frac{(a - d)(a - b + c + d)}{2\sqrt{a - c(b - d)^{3/2}}}, \]
\[ E_3(a, b, c, d) = \frac{(a - d)(a^2 - 2a(b + c + d) + b^2 - 2bc + 2cd + d^2)}{8(b - a)\sqrt{a - c(b - d)^{3/2}}} , \]
\[ I_2(a, b, c, d) = \frac{(a - b)(a^2 - 2a(b - c + d) + b^2 + 2bc - 2bd - 3c^2 + 2cd + d^2) \Pi \left( \frac{\theta}{\pi}, k \right)}{4\sqrt{(a - c)(b - d)}}, \]

This allows us to obtain the following expression for $I_2$

Recalling again that the coefficients are related to the roots, the final expression takes the form
Thus we have that
\[ I_2(a, b, c, d) = -\frac{(a - b)(a - c)(a + c - d)}{2\sqrt{(a - c)(b - d)}} F(\theta, k) - \frac{c}{2} \sqrt{(a - c)(b - d)} E(\theta, k) \]
\[ + \frac{2(a - b)}{\sqrt{(a - c)(b - d)}} \Pi \left( \frac{b - c}{a - c}, k \right) + \frac{1}{2} \left( d(b - c) + 1 - 2 \log \left( \frac{2a}{d} \right) \right) \]
\[ a = \lambda_+, \quad b = \lambda_0, \quad c = x, \quad d = \lambda_-, \quad k^2 = \frac{(b - c)(a - d)}{(a - c)(b - d)}, \quad \theta = \sin^{-1} \sqrt{\frac{b - d}{a - d}}. \]

Thus we have
\[ I_2(\lambda_+, \lambda_0, x, \lambda_-) + \log \lambda_+ = \int d\lambda \rho(\lambda) \log |\lambda_+ - \lambda|. \]

2. Case \( c < c_i(x) \)

a. Evaluation of \( \int_{z_0}^z dz S_-(z) \)

As before we can exploit the known results on elliptic integrals to obtain exact expressions of the integrals ap-

Thus we have that
\[ \int_{z_0}^z dz S_-(z) = \int d\lambda \rho(\lambda) \log \left| \frac{x - \lambda'}{\lambda_0 - \lambda'} \right| \]
\[ = \frac{x^2 - \lambda_0^2}{2} - J_1(\lambda_+, \lambda_0, \lambda_-). \]
b. evaluation of \( \int_{\lambda_+}^{\infty} dz [S_-(z) - \frac{1}{z}] \)

The evaluation of \( \int_{\lambda_+}^{\infty} dz [S_-(z) - \frac{1}{z}] \) is straightforward. As the parameters seem to be the same as before we can readily say that the asymptotic solution for \( J_2 \) is

\[
J_2(a, b, c, d) = \frac{(a - b) (a^2 + 2a(b - c - d) - 3b^2 + 2b(c + d) + (c - d)^2)}{4(a - c)(b - d)} \Pi \left( \frac{b - c}{a - c}, k \right) \\
- \frac{(a - b)(a - c)(a + 3b - c - 3d)}{4(a - c)(b - d)} F(\theta, k) + \frac{1}{4} \sqrt{(a - c)(b - d)}(a - 3b + c + d) E(\theta, k) \\
+ \frac{1}{16} \left( 2(a^2 + 2ab - c - d) - 3b^2 + 2b(c + d) + (c - d)^2 \right) \log \left( \frac{a + b - c - d}{4a} \right) \\
+ a^2 - 2a(-b + c + d) + 5b^2 - 6b(c + d) + c^2 + 6cd + d^2 )
\]

with further simplifications due to the relations of the coefficients, viz.

\[
J_2(a, b, c, d) = \frac{(a - b)(a - c)(a + b - d)}{2(a - c)(b - d)} F(\theta, k) - \frac{1}{2} b\sqrt{(a - c)(b - d)} E(\theta, k) \\
+ \frac{2(a - b)}{\sqrt{(a - c)(b - d)}} \Pi \left( \frac{b - c}{a - c}, k \right) + \frac{1}{2} \left( -2 \log (2) + (b - c)(b - d) + 1 \right)
\]

\[a = \lambda_+, \quad b = x, \quad c = \lambda_0, \quad d = \lambda_-, \quad k^2 = \frac{(b - c)(a - d)}{(a - c)(b - d)}, \quad \theta = \sin^{-1} \sqrt{\frac{(b - d)}{(a - d)}}.\]

We can then write:

\[
\int d\lambda \rho(\lambda) \log |\lambda_+ - \lambda| = \log(\lambda_+) - \int_{\lambda_+}^{\infty} dz \left( S_-(z) - \frac{1}{z} \right) \\
= J_2(\lambda_+, x, \lambda_0, \lambda_-) + \log(\lambda_+).
\]