The Optimal Control Problems of Exponentially Damped System

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Abstract. The optimal control problems of exponentially damped system are considered in this paper. Exponentially damped system involves convolution integrals over exponentially decaying functions, which is used as damping models in viscoelastically damped structures. The dynamic constraint is in the form of a differential equation that includes integer derivatives and the “integral” term. The performance index considered is a function of both the state and the control variables. The traditional state-space approach is extended to the optimal control problems of exponentially damped system, A direct numerical technique is used to solve the resulting equations. Numerical simulations are provided to illustrate the above control design.

1. Introduction
Viscoelastic materials are widely used to control or reduce vibrations and sound radiation in aerospace structures, industrial machines, civil engineering structures, etc. Of many nonviscously damping models, the convolution integral model is possibly the most general model within the scope of linear analysis. The integral constitutive models are derived based on the materials properties of stress relaxation and creep applying Boltzmann’s superposition principle. The stress relaxation functions and creep functions are memory and hereditary kernels in the integral constitutive equations. They can be expressed by a series of exponential functions [1,2], power-law functions [3], Mittage-Leffler functions [3], or other types of functions. The integral constitutive relations can be readily used as damping models in viscoelastically damped structures. Equations of motion of such systems are a set of coupled second-order integro-differential equations. The presence of the “integral” term makes the vibration analysis and control design more complicated than the classical ones. The integral type damping models may be also called nonviscously damping models and the corresponding oscillators are called nonviscously damped oscillators.

Researches on nonviscously damped oscillators are mainly concentrated on two types: one is the exponentially damped oscillators where the damping force are expressed by exponentially fading memory kernel; and the other is the fractional-order oscillators where the viscoelastic relaxation functions are characterized by power-law functions or Mittage-Leffler functions. S. Adhikari and his colleagues have systematically investigated the structural dynamics with exponentially damped models, including dynamics of exponentially damped single degree of freedom and multi-degree-of-freedom systems, identification and quantification of damping in [4-6].

In recent years, Various fractional control techniques have been proposed, such as CRONE control, fractional PID control, fractional sliding control [7], fractional adaptive control [8], fractional optimal control [9], etc. However, control designs for exponentially damped systems are very limited. This
paper is going to propose the optimal control problems of exponentially damped system. Furthermore, this control method can also be applied to N-order exponentially damped system containing multiple integral teams.

2. State-space representation

To describe constitutive relation for viscoelastic material, the following nonviscous damping model that depend on the past history of motion via convolution integral over kernel function is adopted.

\[ \sigma(t) = \int_0^t G(t - \tau) d\varepsilon(\tau) \]  

(1)

Where, \( \sigma(t) \) is the stress, \( \varepsilon(t) \) is the stain, \( G(t) \) is the stress relaxation function. Here, we will use a damping model for which the kernel function is exponentially damped model.

\[ G(t) = \sum_{k=1}^{n} C_k \mu_k e^{-\mu_k t} \]  

(2)

Where, \( \mu_k \in \mathbb{R}^+ \) are known as the relaxation parameters, and \( n \) denotes the number of relaxation parameters used to described the damping behavior. \( C_k \in \mathbb{R}^{N_x N} \) are the damping coefficient matrices. We consider such \( n \)-order exponentially damped system.

\[
\frac{d^n x}{dt^n} + \sum_{i=1}^{n} a_i \frac{d^{n-i} x}{dt^{n-i}} + \cdots + c_n \int_0^t \mu_n e^{-\mu_n (t-\tau)} \dot{x}(\tau) d\tau + \cdots + c_1 \int_0^t \mu_1 e^{-\mu_1 (t-\tau)} \dot{x}(\tau) d\tau + a_0 x(t) = u(t)
\]  

(3)

Where, \( n \) is an integer. Assuming \( x = x_1, \frac{dx}{dt} = x_2, \frac{d^2 x}{dt^2} = x_3, \ldots, \frac{d^{n-1} x}{dt^{n-1}} = x_n \), the above equation can be expressed in the following form

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\vdots \\
\dot{x}_n \\
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
\end{bmatrix}
\int_0^t \mu e^{-\mu (t-\tau)} d\tau + \begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
\end{bmatrix}
\begin{bmatrix}
\int_0^t \mu_1 e^{-\mu_1 (t-\tau)} \dot{x}(\tau) d\tau + \cdots + \int_0^t \mu_n e^{-\mu_n (t-\tau)} \dot{x}(\tau) d\tau \end{bmatrix}
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\vdots \\
\dot{x}_n \\
\end{bmatrix}
\]  

(4)

So we can abbreviate it into the following form

\[
\dot{X} + P_1 \int_0^t \mu e^{-\mu (t-\tau)} \dot{X}(\tau) d\tau + \cdots + P_n \int_0^t \mu e^{-\mu (t-\tau)} \dot{X}(\tau) d\tau
\]  

(5)

\[
+ P_n \int_0^t \mu e^{-\mu (t-\tau)} \dot{X}(\tau) d\tau = AX + bu
\]

It is easy to see that the above system is similar to integer order system, but the existence of integral terms makes the calculation more difficult.

3. The optimal control problem of exponentially damped system

In this section, we consider a single-input multivariable system. The dynamic constraints are expressed in the above state-space formulation, but only one “integral” term is restricted. The performance index function is expressed as follows

\[ J(u) = \frac{1}{2} \int_0^t \left( X^T Q X + ru^2 \right) dt \]  

(6)
Subject to the dynamic constraint

$$\dot{X} + P\int_0^t \mu e^{-\mu(t-\tau)} \dot{X}(\tau) \, d\tau = AX + bu$$

(7)

And the boundary conditions as $X(0) = X_0$, $X(t_f)$ is free and $t_f$ is fixed. Where, $Q$ is a weighted matrix related to the state vector, $r$ is the weighted coefficient associated with the control signal. Here, $X(t)$ and $u(t)$ are $n$-dimensional state vector and a control input respectively. $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^{n \times 1}$.

We assume that the following formula holds

- $Q_{non}$ is a nonnegative matrix.
- $r > 0$
- $P_{non} = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ k & 0 & \ldots & 0 \end{bmatrix}$, where, $k$ is a nonzero real number.

Assuming that $\lambda(t)$ is the Lagrange multiplier. We construct the augmented performance index as

$$J_a(u) = \int_0^t \left[ \frac{1}{2} (X^T Q X + ru^2) + \lambda^T (AX + bu - \dot{X}) - P\int_0^t \mu e^{-\mu(t-\tau)} \dot{X}(\tau) \, d\tau \right] \, dt$$

(8)

Setting the first variation of (8), we get

$$\delta J_a(u) = \int_0^t \left[ (QX + A^T \lambda)^T \delta X - \lambda^T \delta \dot{X} - \lambda^T P\int_0^t \mu e^{-\mu(t-\tau)} \dot{X}(\tau) \, d\tau \right] \, dt$$

$$+ (ru + b^T \lambda) \delta u + (AX + bu - \dot{X} - P\int_0^t \mu e^{-\mu(t-\tau)} \dot{X}(\tau) \, d\tau) \delta \lambda \right] \, dt$$

(9)

The second and third terms on the right side of the upper formula are calculated separately and the following results are obtained.

$$\int_0^t (\lambda^T \delta \dot{X}) \, dt = -\int_0^t \lambda^T \frac{d\delta X}{dt} \, dt = -\int_0^t \lambda^T d\delta X = -\lambda^T \delta X [t_f] + \int_0^t \delta X d\lambda^T$$

(10)

Because $X(0) = X_0$ is specified, so $\delta X(0) = 0$, but $X(t_f)$ is not specified, so we require $\lambda(t_f) = 0$. We can get

$$\int_0^t \lambda^T \delta \dot{X} \, dt = \int_0^t \lambda^T \delta X \, dt$$

(11)

$$= \int_0^t \left[ e^{\mu t} \delta X(t) \, dt - \mu e^{\mu t} \delta X(t) \, dt \right]$$

(12)

$$= \int_0^t \left[ \mu e^{\mu t} \delta X(t) \, dt - \mu \int_0^t e^{\mu \tau} \delta X(\tau) \, d\tau \right]$$

$$= \int_0^t \delta X(t) \, dt - \mu \int_0^t e^{\mu \tau} \delta X(\tau) \, d\tau$$

$$= \int_0^t \left[ \mu e^{\mu t} \int_0^t e^{\mu \tau} \delta X(\tau) \, d\tau - \mu \int_0^t e^{\mu \tau} \delta X(\tau) \, d\tau \right]$$

Substitute the above calculation results into (9)

$$\delta J_a(u) = \int_0^t \left[ (QX + A^T \lambda + \lambda + e^{\mu t} M \int_0^t e^{\mu \tau} \delta X(\tau) \, d\tau - \mu P^T \lambda) \delta X \right]$$

$$+ (ru + b^T \lambda) \delta u + (AX + bu - \dot{X} - P\int_0^t \mu e^{-\mu(t-\tau)} \dot{X}(\tau) \, d\tau) \delta \lambda \right] \, dt$$

(13)
Minimization of $J_a(u)$ (and hence minimization of $J(u)$) requires the coefficients of $\delta \lambda$, $\delta X$ and $\delta u$ in equation (9) be zero. So we can get
\begin{equation}
\dot{X} + P \int_0^t \mu e^{-\mu(t-\tau)} \dot{X}(\tau) d\tau = AX + bu
\end{equation}
(14)
\begin{equation}
\dot{\lambda} + \mu^2 e^{\mu t} P \int_0^t \lambda e^{-\mu(t-\tau)} d\tau - \mu P^T \lambda + QX + A^T \lambda = 0
\end{equation}
(15)
\begin{equation}
r u + b^T \lambda = 0
\end{equation}
(16)
With the boundary conditions as $X(0) = X_0$, $\lambda(t_f) = 0$.
(17)

So, equation (14), (15), (16) with the boundary conditions (17) constitute a boundary value problem of optimal control problems for exponentially damped system. We can use a direct numerical technique to solve. Here we only discuss the optimal control problem with one “integral” term in the dynamical constraint. The same method can be used to solve the optimal control problem with multiple “integrals” in the dynamical constraint. If the single input system is replaced by the multi-input system, this method can also solve the problems.

4. Numerical example
When the integral constitutive equation (1) is applied to model the dynamics of structures incorporated with viscoelastic dampers, the equation of motion is
\begin{equation}
m \ddot{x}(t) + c \int_0^t G(t-\tau) \dot{x}(\tau) d\tau + k x(t) = u(t)
\end{equation}
(18)
\begin{equation}
m \ddot{x}(t) + c \int_0^t \mu e^{\mu(t-\tau)} \dot{x}(\tau) d\tau + k x(t) = u(t)
\end{equation}
(19)
Where, $m$ is the mass, $k$ is the stiffness, $c$ is the damping coefficient, $u(t)$ is the input force acting on the system, $x(t)$ is the displacement of $m$. Defining the following state vectors: $x = x_1$, $\dot{x} = x_2$.

Then the equation of (18) is transformed into the following equation in the state space
\begin{equation}
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
0 & 0 \\
0 & \frac{c}{m}
\end{bmatrix}
\int_0^t \mu e^{-\mu(t-\tau)}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} d\tau
+ \begin{bmatrix}
0 & 1 \\
-\frac{k}{m} & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
+ \begin{bmatrix}
0 \\
1
\end{bmatrix} u
\end{equation}
(20)

For the convenience of the following calculation, we can take $m = 1, c = 1, k = 1$, and then the equation can be abbreviated to the following form
\begin{equation}
\dot{X} + P \int_0^t \mu e^{-\mu(t-\tau)} \dot{X}(\tau) d\tau = AX + bu
\end{equation}
(21)
Where, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $P = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Next, we discuss the optimal control problem with exponentially damped terms in dynamic constraint. The performance index function is expressed as follows
\begin{equation}
J(u) = \frac{1}{2} \int_0^{t_f} (X^T Q X + ru^2) dt
\end{equation}
(22)
Subject to the dynamic constraint
\begin{equation}
\dot{X} + P \int_0^t \mu e^{-\mu(t-\tau)} \dot{X}(\tau) d\tau = AX + bu
\end{equation}
(23)
The given initial condition as $X_0 = [1^T, X(t_f)]$ is free and $t_f = 1$, where $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $r = 1$.

Optimal conditions for this problem are given by
\[ \dot{X} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \int_0^t \mu e^{-\mu(t-\tau)} \dot{X}(\tau) d\tau = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \] (24)

\[ \dot{\lambda} + \mu^2 e^{\mu t} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \int_0^t \lambda e^{-\mu t} d\tau = \mu \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \lambda - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \lambda X - \begin{bmatrix} 1 & -1 \end{bmatrix} \lambda \] (25)

\[ u = -[0 \ 1]^T \lambda \] (26)

\[ X_0 = [1 \ 0]^T, \lambda(1) = [0 \ 0]^T \] (27)

Equation (24), (25), (26), (27) constitute a two point boundary value problem. To solve equations (24) to (26) we divide the entire time domain into \( N \) equal domains. The size of each domain is given as \( h = \frac{1}{N} \). The equation can be expressed as

\[ \frac{X^m - X^{m-1}}{h} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \int_0^t \mu e^{-\mu(t-\tau)} \frac{X^m - X^{m-1}}{h} d\tau = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} X^m - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \lambda^m \] (28)

\[ \frac{\lambda^m - \lambda^{m-1}}{h} + \mu^2 e^{\mu t} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \int_0^t \lambda^m e^{-\mu t} d\tau = \mu \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \lambda^m - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} X^m - \begin{bmatrix} 1 & -1 \end{bmatrix} \lambda^m \] (29)

Where, \( X^i \) and \( \dot{\lambda}^i \) are the numerical approximations of state vector \( X(t) \) and \( \dot{\lambda}(t) \) at node \( i \).

Now let's turn the vector equation above into a scalar form, and assume that \( \mu = 1 \)

\[ \frac{X^m_i - X^{m-1}_i}{h} = x^m_i, \ m = 1, 2, \ldots, N \] (30)

\[ \frac{X^m_i - X^{m-1}_i}{h} + e^t \int_0^t e^{-\mu(t-\tau)} \frac{X^m_i - X^{m-1}_i}{h} d\tau = -x^m_i - \lambda^m_i, \ m = 1, 2, \ldots, N \] (31)

\[ \frac{\dot{\lambda}^m_i - \dot{\lambda}^{m-1}_i}{h} + e^t \int_0^t \lambda^m_i e^{-\mu t} d\tau = 2\lambda^m_i - x^m_i, \ m = N - 1, N - 2, \ldots, 0 \] (32)

\[ \frac{\dot{\lambda}^m_i - \dot{\lambda}^{m+1}_i}{h} = -x^m_i - \dot{\lambda}^m_i, \ m = N - 1, N - 2, \ldots, 0 \] (33)

Equation (30) to (33) form \( 4n \) number of equations in terms of \( 4n \) unknowns. These could be solved directly using any linear equation solver. Once \( \dot{\lambda} = [\dot{\lambda}_1, \dot{\lambda}_2]^T \) is known, control \( u \) can be computed using equation (26).

Figures 1 and 2 show the convergence of the state variables and Figure 3 shows the convergence of control variable as a function of time for \( \mu = 0.2 \). From these figures it is clear to see that solutions converge as \( N \) is increased. However, accuracy of the numerical method can be improved.

5. Conclusion
In this paper, we consider the optimal control problem with exponential damping dynamic. We generalize the traditional state space method to obtain the state space representation of \( n \)-order differential equations with \( m \) integral terms, and use it as dynamic constraints for optimal control design. Through complex calculation, we find the necessary condition to satisfy the minimum performance index, which is the solution of a Two Point Boundary Value problem and can be obtained by numerical simulation directly. Then we take the second-order exponential damper system as an example to design the optimal control, and get the Two Point Boundary Value problem which satisfies the optimal conditions. Then we simulate it by the method of precise integration. The image shows that the analytical solution is consistent with the numerical solution.
Figure 1. Convergence of function $x_1(t)$ ($\cdots N=60, - N=80, \cdot - N=100, \mu=0.2$)

Figure 2. Convergence of function $x_2(t)$ ($\cdots N=60, \cdot - N=80, - N=100, \mu=0.2$)

Figure 3. Convergence of function $u(t)$ ($- N=60, \cdots N=80, \cdot - N=100, \mu=0.2$)

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