Efficient Signal-Time Coding Design and its Application in Wireless Gaussian Relay Networks

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Abstract

Signal-time coding, which combines the traditional encoding/modulation mode in the signal domain with signal pulse phase modulation in the time domain, was proposed to improve the information flow rate in relay networks. In this paper, we mainly focus on the efficient signal-time coding design. We first derive an explicit iterative algorithm to estimate the maximum number of available codes given the code length of signal-time coding, and then present an iterative construction method of codebooks. It is shown that compared with conventional computer search, the proposed iterative construction method can reduce the complexity greatly. Numerical results will also indicate that the new constructed codebook is optimal in terms of coding rate. To minimize the buffer size needed to store the codebook while keeping a relatively high efficiency, we shall propose a combinatorial construction method. We will then consider applications in wireless Gaussian relay networks. It will be shown that in the three node network model, the mixed transmission by using two-hop and direct transmissions is not always a good option.

Index Terms

Signal-time coding, network information flow, network information theory, max-flow min-cut, rate splitting, Gaussian relay network.

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I. INTRODUCTION

Relay network information transmission, as an important and necessary technique, has been receiving significant attention in recent years. However, only few results on the network capacities and its efficient implementations were obtained for networks with relatively simple topologies and special applications [1] [2][3]. In the literature, there are two different strategies to investigate relay network information transmission. The first one is based on considering the networks with relatively simple topologies, such as two-hop networks, three-node networks, diamond network, linear networks and butterfly networks [1] [5] [6] [7] [8]. The second one considers the networks composed of a lot of nodes and links between them. It then employs random geometry and random graph theory to find some statistical characteristics [9] [10]. In this paper, we mainly investigate the three-node networks.

It is well known that the first Gaussian relay channel model was introduced by [5], and was thoroughly analyzed by Cover and El Gamal in [2] [6] where they presented upper and lower bounds for the general relay channel. In [4], the authors discussed different node cooperation strategies and presented their corresponding capacity bounds. The power allocation and capacity bounds for wireless relay networks were discussed in [3]. On the capacity approaching implementation, [11] investigated the performance limits using distributed space time coding in wireless relay networks. Other works, e.g., [12][13] [14] considered the coding design and its implementation strategy for half-duplex relay channels. Network coding was also included in the discussion for relay networks [15][16]. Other discussions on interference relay channels can be found in [17] [18].

Recently, we considered the simple three-node network in a systematic way and found that it is possible to get a higher information flow rate beyond the known min-cut upper bound using conventional transmission techniques [19]. This observation led us to propose a new coding method, signal-time coding, which combines the traditional encoding/modulation mode in the signal domain with the signal pulse phase modulation in the time domain. Such a hybrid signal-time coding approach can be considered as an integrated codec/modem processing in the two dimensional combinatorial space: Signal domain and temporal domain. The main feature of the proposed signal-time coding is that one can separately design the codec/modem in the signal domain and in the time domain. Therefore, the well known efficient codec/modems, such as LDPC, Turbo coding, and TCM in the signal domain can be employed. By doing so, the coding design problem becomes equivalent to that of designing an efficient signal-time coding in the time domain. In [19], we provided an iterative construction method of codebooks that
can reach a derived lower bound of the coding rate per pulse of signal-time coding. In this paper, we will further investigate the efficient coding design. The main contributions are as follows: 1) Explicit formulas are derived to find the maximum available signal-time code number for an arbitrary code length, which can then be used to reduce the code search time by exhaustive computer search; 2) An iterative construction algorithm of codebooks, which can reach the upper bounds in terms of coding efficiency, is given; 3) A combinatorial codebook generation strategy, which can keep a relatively high coding efficiency and save the storing buffer size greatly, is developed; and 4) The application of our proposed signal-time coding in wireless Gaussian relay networks is considered and investigated.

The rest of this paper is organized as follows. In Section II, we shall revisit the three-node network model and introduce the signal-time coding proposed in [19]. We derive explicit formulas to calculate the maximum available signal-time codes in Section III and then present an iterative construction algorithm of codebooks in Section IV. A combinatorial codebook generation strategy and its coding efficiency are presented in Section V. Some applications of signal-time coding in wireless Gaussian relay networks are discussed in Section VI. Finally, we present our conclusion in Section VII.

II. SYSTEM MODEL AND SIGNAL-TIME CODING

Consider a three-node network as shown in Fig. 1. The system consists of three nodes, one source $S$, one relay $R$ and one destination $D$. For simplicity, assume that all the source information must be forwarded by the relay node $R$ to its destination $D$ and node $R$ can not transmit and receive signals simultaneously. That is, the relay is in half-duplex mode and there does not exist a direct link between $R$ and $D$. Such kind of model usually appears in wireline communications or satellite communications, where the source node can not transmit information to the destination node directly. In Section V, we shall discuss the general case.

Let $C_1$ and $C_2$ denote the channel capacities of the links $S - R$ and $R - D$, respectively. For simplicity, we assume that the link capacities of $S - R$ and $R - D$ are time invariant. In this case, one wants to transmit some information from the source node to the destination node by relay forwarding. It is easy to see that the maximum value of information flow $R_{\text{max}}$ satisfies

$$R_{\text{max}} \leq \min\{C_1, C_2\}$$

(1)

by using the max-flow min-cut theorem.
Eqn. (1) is the result obtained for a relay operating in full duplex mode. It also seems to be an upper of the achievable information flow rate for relays in half duplex mode.

Now let us consider the case of a relay working in half duplex mode. For a given time period $T$, using the conventional signal transmission method, the maximum achievable amount of information bits and its corresponding maximum average information rate are given by the following lemmas.

**Lemma 1:** Consider a simple relay network with three nodes, the source $S$, relay $R$ and the destination $D$. Also assume that the direct link between the source and destination does not exist and that all the information from the source to the destination need to be forwarded by the relay. If the relay node $R$ is allowed to operate in half duplex mode, then the maximum amount of information bits per period of $T$ received by the destination is $C_1C_2T$ and the maximum average value of information flow is $\frac{C_1C_2}{C_1+C_2}$, where $C_1$ and $C_2$ represent the channel capacities of the links $S-R$ and $R-D$, respectively.

The outline of the proof of Lemma 1 can be given as follows. For each time period $T$, a fraction of it is used to transmit signals for the source to the relay, denoted as $T_1$. The remaining period of time duration, denoted as $T_2$ is used for the relay to forward its received signals to the destination. Suppose that all the links can be employed to transmit information at their capacities, then we have

$$T_1 + T_2 = T$$  \hspace{1cm} (2)

$$C_1T_1 = C_2T_2$$  \hspace{1cm} (3)

By solving the above two equations, we get

$$C_1T_1 = C_2T_2 = \frac{C_1C_2}{C_1+C_2}T$$  \hspace{1cm} (4)

$$T_1 = \frac{C_2}{C_1+C_2}T$$  \hspace{1cm} (5)

$$T_2 = \frac{C_1}{C_1+C_2}T$$  \hspace{1cm} (6)

This indicates that the maximum amount of information bits per period of $T$ received by the destination is $\frac{C_1C_2}{C_1+C_2}T$ and the maximum average value of information flow is $\frac{C_1C_2}{C_1+C_2}$.

By using the results in Lemma 1, one can easily get Lemma 2.

**Lemma 2:** Consider a simple relay network model with three nodes as described in Lemma 1, then the maximum
The average value of information flow is bounded by the following inequality

\[
\frac{1}{2} \min\{C_1, C_2\} \leq \frac{C_1 C_2}{C_1 + C_2} \leq \min\{\frac{1}{2} \max\{C_1, C_2\}, \min\{C_1, C_2\}\}.
\] (7)

The results in Lemma 2 indicate that using conventional transmission techniques, the achievable rate will not beyond the value of the minimum one between \(\frac{1}{2} \max\{C_1, C_2\}\) and \(\min\{C_1, C_2\}\). In the sequel, we shall refer to the right-hand side of Eqn. (7) as two-hop upper bound.

By observing the results in Lemma 1, we find that in each period of time \(T\), only a part of the period with duration \(T_1 = \frac{C_2}{C_1 + C_2} T\) is used to transmit the signal from the source \(S\) to the relay \(R\), and the remaining period of time duration \(T_2 = T - T_1\) is used for the relay to forward the received signals from the source \(S\) to the destination \(D\). One usually would like to divide each period of time \(T\) into two subintervals. The first one with length \(T_1\) is employed for the source transmitting signals to the relay, and the remaining part is used for the relay to forward its received signals previously to the destination. Obviously, this time division scheduling approach is an efficient way to reach the so called maximum average value of information flow for this simple relay network. However, it neglects an important transmission characteristic of information flow. For instance, consider the following scenario.

Suppose that \(C_1 = C_2\) and the period of time \(T\) is divided into four time slots. Based on Lemma 1, we know that the two slots are used for transmitting signals from the source \(S\) to the relay \(R\), and the other two slots for forwarding signals from \(R\) to the destination \(D\). It is easy to see that there are four different arrangements as shown in Fig. 2 where the blank and filled blocks represent the time slots for \(S-R\) transmission and \(R-D\) transmission, respectively. According to the information flow rule, \(R\) will be permitted to transmit signals if it receives some signals that it did not forward previously. We find that only cases (3) and (4) in Fig. 2 are the available time scheduling modes. Nevertheless, these two different available time scheduling modes can bring 1 bit information and the corresponding information rate with signal-time coding is at least \(1/T\).

The above example shows that if we adjust the signal transmission time intervals for the source to the relay and the relay to the destination while still keeping the relay operating in half duplex mode, we will find that there are various different scheduling modes that can be employed to reach the so called maximum average value of information flow. In addition, if we observe the information flow over the two hop routing at the destination, it is not hard to see that the variation of time scheduling modes is able to carry some information and such time scheduling modes can be detected by the destination if each time slot is long enough. Thus, the variation of time scheduling
modes can be employed to carry the information at the source node if all the nodes have perfect knowledge of the whole network. As a result, we can employ a hybrid way to encode and modulate the source information. In other words, if the relay and destination employ carrier sensing, then one can use rate splitting at the source node by transmitting a part of the information using the time scheduling mode and the remaining part using conventional signal transmission way. By doing so, it is easy to see that the obtained average value of information flow is given by the sum of the two different kinds of information flows as shown in Fig. 3 and is larger than that given in Lemma 1.

The above hybrid signal coding mode was called signal-time coding, because it combines the traditional encoding/modulation mode in the signal domain with the signal pulse phase modulation in the time domain. One important point to be mentioned is that the main feature of the signal-time coding is that one can separately design the codec/modem in the signal domain and in the time domain. Therefore, the well known efficient codec/modems, such as LDPC, Turbo coding, and TCM in the signal domain can be employed here. The remaining problem is that how to design an efficient coding mode in the time domain.

Now let us consider the general case of \( C_1 \neq C_2 \). In each period time \( T \), \( T_1 \) and \( T_2 \) are the total time durations of employing links \( S \rightarrow R \) and \( R \rightarrow D \), respectively. By decomposing them into \( N \) time slots with different lengths, one can get

\[
T_1 = \frac{C_2 T}{C_1 + C_2} = N \Delta t_1
\]

\[
T_2 = \frac{C_1 T}{C_1 + C_2} = N \Delta t_2.
\]

That is,

\[
\Delta t_1 = \frac{C_2 T}{C_1 + C_2} \frac{T}{N}
\]

\[
\Delta t_2 = \frac{C_1 T}{C_1 + C_2} \frac{T}{N}.
\]

Note that \( C_1 \Delta t_1 = C_2 \Delta t_2 \). This implies that the network transmits the same amount of information every active time slot, which is independent of the links transmission status.

For simplicity, we assume that both the relay and the destination employ the same carrier sensing method so that they have the same minimum resolution of signal detection in the time domain, which we shall refer to \( \Delta T \). Thus,
we have \( \min\{\Delta t_1, \Delta t_2\} \geq \Delta T \). After some manipulations, we get

\[ \frac{C_1C_2}{C_1 + C_2} \geq \max\{C_1, C_2\} \frac{\Delta T}{N} \]  \hspace{1cm} (12)

Let \( S_N \) denote the maximum number of available time slot order sequences with length \( 2N \). Then, the information rate carried by the signal-time coding in the time domain is given by \( f_{ct} = \frac{\log_2(S_N)}{N} = A(N) \frac{N}{T} \), where \( \log_2(S_N) = A(N)N \) and \( A(N) \) is a weighting factor being dependent on \( N \). By using Eqn. (12), we have

\[ f_{ct} \leq \frac{C_1C_2}{C_1 + C_2} \frac{A(N)}{\max\{C_1, C_2\}} \frac{\Delta T}{N} \]  \hspace{1cm} (13)

In fact, the upper bound in the right-hand side of Eqn. (13) can be reached when both the relay and destination employ the minimum time resolution \( \Delta T \). As a result, the total achievable information rate using signal-time coding is given by

\[ f_{ST} = \frac{C_1C_2}{C_1 + C_2} \left(1 + \frac{A(N)}{\max\{C_1, C_2\}} \frac{\Delta T}{N}\right) \]  \hspace{1cm} (14)

This clearly shows that using signal-time coding can get a higher information rate than that given in Lemma 1 and may be beyond the two-hop bound given in Lemma 2 in some scenarios.

In this paper, we shall try to answer the following three questions: 1) What is the maximum available code number of signal time coding for an arbitrary code length?; 2) How to construct a codebook with relatively high efficiency and low memory units?; and 3) What is the coding gain in different scenarios.

### III. Optimal Signal-Time Code Design

In Section II, we introduced signal-time coding. In this section, we shall discuss the optimal signal-time coding design in terms of coding efficiency. In fact, there exist two basic problems: 1) What is the maximum number of the available signal-time codes with an arbitrary length \( 2n \)?; and 2) How to design it in a constructed form? To answer these two questions, we shall discuss them separately in the following part.

#### A. Maximum number estimation of signal-time codes

We begin by introducing a lemma [19].

**Lemma 3:** Consider a relay network with three-nodes \( S, R \) and \( D \) as shown in Fig. 1 but without a direct link between \( S \) and \( D \). Assume that the links \( S - R \) and \( R - D \) are with equal capacity \( C \). If \( R \) is only permitted to operate in half duplex, then an available time scheduling mode, which can achieve the average information flow
rate \( C/2 \), must satisfy the following condition: For an arbitrary number \( k \) \((0 < k \leq 2n)\), the appearance times of 
'1' is larger than or equal to that of the number '0' in the first \( k \) time slots, where '1' and '0' represent the indicator of a time slot being used by the links \( S - R \) and \( R - D \), respectively.

Note that this lemma can be directly obtained with the information flow rule. That is, each available signal-time code must satisfy the assignment rule given in Lemma 1, where '1' and '0' represent two kinds of time slots with different lengths if the link capacities of \( S - R \) and \( R - D \) are different. Therefore, the code design can follow the rule in Lemma 3. To do so, we need to introduce a definition.

**Definition 1:** Consider a '0' and '1' sequence with length \( 2n \). If it satisfies the following two conditions

1) It has an equal number of '0' and '1'. That is, the number of '0' is equal to that of '1' in the sequence;
2) In any sub-sequence composed of the first \( k \) \((k \leq 2n)\) symbols, the number of '1' is not less than that of '0';
then, we we shall call the sequence as the available signal sequence.

Based on this definition, the estimation of the maximum number of available signal-time codes with length \( 2n \) can be obtained by solving the following problem: How many available signal sequences are there for a given length \( 2n \)?

To answer this question, we shall define some symbols. let \( S_n \) denote the total number of available signal sequences with length \( 2n \), and let \( F_{n}^{k} \) denote the total number of available signal sequences with length \( 2n \), and whose first \( k \) symbols are '1'. Obviously, we have \( S_n = F_{n}^{1} \). In addition, by observing the first two symbols in the available signal sequence, we get the following

\[
S_n = S_{n-1} + F_{n}^{2} \tag{15}
\]

Eqn. (15) implies that the first symbol should be assigned as '1'. If the second symbol is assigned as '0', the following part can then follow for the available signal sequence with length \( 2(n-1) \). Otherwise, the second symbol will be assigned as '1'. In this case, the total number of available signal sequences is \( F_{n}^{2} \). Thus, the total number of the available signal sequences is given in Eqn. (15).

Next we shall estimate \( F_{n}^{k} \). By observing the property of \( F_{n}^{k} \), we have \( F_{n}^{k} = 0 \) if \( k > n \) and \( F_{n}^{k} = 1 \) when \( k = n \). Consider the case when \( k < n \). In this case, the available codes are in the following form,
\[
\begin{array}{cccccccc}
1 & 1 & \cdots & 1 & \ast & \cdots & \ast & 0 \\
\end{array}
\]
\[2n-1-k\]
(16)

where \(\ast\) denotes one symbol position, which may be filled by ‘0’ or ‘1’.

Let us decompose the marked part with \(\ast\) into two different subsections with the form

\[
\begin{array}{cccccccc}
1 & 1 & \cdots & 1 & \square & \cdots & \square & \ast & \cdots & \ast & 0 \\
\end{array}
\]
\[k \quad k \quad 2n-1-2k\]
(17)

where the \(k\) symbols in the positions marked by \(\square\) can be selected as ‘0’ or ‘1’ randomly without any constraint, while the selection of the remaining \(2n-1-2k\) symbols marked by \(\ast\) is only dependent on the numbers of ‘0’ and ‘1’ in the positions marked by \(\square\), and is independent of their exact sequence order.

Without loss of generality, let \(i\) \((0 \leq i \leq k)\) denote the number of ‘1’ in the positions marked by \(\square\), then the number of ‘0’ in the positions marked by \(\square\) is \(k-i\), which implies that the possible combination number is \(C_{k}^{i} = \frac{k!}{i!(k-i)!}\). In this case, the possible selection number of the remaining \(2n-1-2k\) symbols marked by \(\ast\) is equal to the possible number of the available sequence in the following form,

\[
\begin{array}{cccccccc}
1 & 1 & \cdots & 1 & 0 & \cdots & 0 & \ast & \cdots & \ast & 0 \\
\end{array}
\]
\[k+i \quad k-i \quad 2n-1-2k\]
(18)

By removing the first \(k-i\) ‘0’ and ‘1’ pairs, we have

\[
\begin{array}{cccccccc}
1 & 1 & \cdots & 1 & \ast & \cdots & \ast & 0 \\
\end{array}
\]
\[2i \quad 2n-1-2k\]
(19)

Obviously, the above sequence length is \(2n-1-2k+2i+1 = 2(n-k+i)\), which implies that the possible number of the available sequence is \(F_{n-k+i}^{2i}\). By using the multiplier principle, we get the possible number of the available signal sequences in the form of Eqn. (17) with ‘1’ and \(k-i\) ‘0’ filled in the \(\square\) positions as \(C_{k}^{i}F_{n-k+i}^{2i}\).

By summing all the cases with different \(i\), we have

\[
F_{n}^{k} = \sum_{i=0}^{k} C_{k}^{i}F_{n-k+i}^{2i}
\]
(20)

Note that in Eqn. (20), the equality \(F_{k}^{0} = F_{k}^{1} = S_{k}\) is true for all positive integer \(k\). In particular, when \(k = 1\), we have

\[
F_{n}^{1} = F_{n-1}^{0} + F_{n}^{2} = F_{n-1}^{1} + F_{n}^{2}.
\]
(21)

This is equivalent to Eqn. (15).
By using Eqns. (15) and (20), we get

\[ S_n = S_{n-1} + F^2_n; \]
\[ F^2_n = C^0_2 F^0_{n-2} + C^1_2 F^2_{n-1} + C^2_2 F^4_n, \]
\[ F^4_n = C^0_4 F^0_{n-4} + C^1_4 F^2_{n-3} + C^2_4 F^4_{n-2} + C^3_4 F^6_{n-1} + C^4_4 F^8_n, \]
\[ \ldots \]
\[ F^{2p}_n = \sum_{i=0}^{k} C^i_{2p-1} F^{2i}_{n-2p+1} + C^{2p}_{2p} F^{2p+1}_n. \]  

(22)

From Eqn. (22), it is not hard to see that the calculation of \( S_n \) is dependent on \( S_{n-1} (or F^0_{n-1}) \) and \( F^2_n \), where \( F^2_n \) is dependent on \( F^{2i}_k, (1 < k < n) \) and \( F^4_n \). Likewise, \( F^4_n \) is dependent on \( F^{2i}_k, (1 < k < n) \) and \( F^8_n \). This iterative procedure can then follow. That is, \( F^{2p}_n \) is dependent on \( F^{2i}_k, (1 < k < n) \) and \( F^{2p+1}_n \). If \( n \leq 2^q \) for some integer \( q \), then the iterative procedure will be stopped until \( F^{2q}_n \) due to \( F^{2q+1}_n = 0 \). That is, \( F^{2q}_n \) is only dependent on \( F^{2i}_k, (1 < k < n) \). As a result, we can use an iterative algorithm to calculate \( S_n \). The iterative procedure is given as follows.

\[ Initialization \quad F^0_1 = 1, \quad F^0_2 = 1, \quad F^2_2 = 2; \]

\[ Using \ Eqn. (20) \ \text{iteratively to calculate the following terms} \]
\[ F^2_3 \rightarrow F^0_3 \ (F^{2k}_3 = 0, k > 1) \]
\[ F^4_4 \rightarrow F^2_4 \rightarrow F^0_4 \ (F^{2k}_4 = 0, k > 1) \]
\[ \ldots \]
\[ F^{2p}_n \rightarrow F^{2p-1}_n \rightarrow \ldots \rightarrow F^2_n \rightarrow F^0_n \ (2^p \geq n, \ F^{2k}_n = 0, k \geq 2^{p+1}) \]

It is easy to see that using the above algorithm, we can save much time to estimate the maximum number of the available codes for signal-time coding compared to using exhaustive computer search.

B. Codebook construction

Consider the derivation of Eqns. (15), (17) and (20), we find that for a given code-length \( 2n \), the construction of codebook with the highest coding efficiency can be iteratively built up by using the codebooks with relatively shorter lengths. Table 1 shows the constructing process, in which \( CBF^k_n \) represents the sub-codebook with length

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code-length $n$ and code number $F_n^k$. Obviously, any pair of codes in the constructed codebook with code-length $2n$ are different. This means that the constructed codebook are uniquely decodable. We shall state this more formally in a theorem.

**Theorem 1:** For any positive integer $n$, the constructed code book with length $2n$ using the method as shown in Table I is uniquely decodable.

This theorem indicates that the codebook constructed as shown in Table I can be used to perform signal-time coding. Therefore, the optimal information bit numbers per pulse with different code-length are presented in Fig. 4.

It is known that a loose upper bound of the coding rate on the signal-time coding is 1 bit/per pulse. The numerical results are very close to the upper bound when the code lengths are large enough. i.e. $2n = 300$, the bit rate per pulse is 0.9611. However, due to the limited capability of the computer search, we are not able to provide what is the exact upper bound of the coding rate per pulse for signal-time coding. Fortunately, by various numerical tests, we found that $S_n = \frac{C_n}{n+1}$ is true for all $n < 150$! If this is true for all $n$, then using the Stirling formula,

$$n! \approx n^{n+1/2}e^{-n}\sqrt{2\pi},$$

we can get

$$\frac{C_{2n}}{n+1} \approx 2^{2n} \frac{1}{(n+1)\sqrt{\pi n}}$$

and

$$\lim_{n \to \infty} \frac{\log \frac{C_n}{n+1}}{2n} = 1.$$

As a result, we can conjecture that 1 bit/per pulse is a tight upper bound of the coding rate per pulse for signal time coding. However, this still needs to be proved.

**IV. Efficient Signal-Time Coding Design**

In Section III, we presented the optimal code design. However, as the code length increases, the codebook storage will become a serious problem because the number of codes will increase exponentially with respect to the code-length. In this section, we shall propose a combinatorial method, which combine a set of codebooks with relatively shorter code-lengths to compose a codebook with an arbitrarily required code-length. This combinatorial method can be considered as an universal combinatorial method.
The construction steps are described as follows.

1) Select a relatively large positive integer $L$;

2) Construct all the codebooks with code length not greater than $L$ by using the iterative method in Section III, and let $CBST_k$ denote the codebook with code-length $k$, ($1 \leq k \leq L$);

3) Construct a codebook with code-length greater than $L$.

The basic idea is that we shall construct the codebook with code-length $2n$ by concatenating some component codes. Here we give a simple construction method.

Suppose that the required code-length is $2n$, then we get $2n = mL + r$, where $m$ and $r$ are two positive even integers and $0 \leq r < L$. Based on this result, the codebook with code-length $L$ and that with code-length $r$ obtained in Step 2) are selected as the two component codes. Each code with code-length $2n$ is setup in the following form.

$$C_1(L), C_2(L), \ldots, C_m(L) C_{m+1}(r)$$

where $C_k(L)$ and $C_k(r)$ denote the selected codes with length $L$ and $r$ arranged in a fixed order of $k$, respectively. By collecting all the codes in the form Eqn. (26), we can obtain the codebook with code-length $2n$.

Using the concatenated coding, it is easy to evaluate the average coding rates and its storage sizes for an arbitrarily length codebook. The average coding rate per pulse $R_c(n)$ is given by

$$R_c(n) = \frac{mLR_L}{2n} + \frac{rR_r}{2n}$$

where $R_L$ and $R_r$ denote the coding rates per pulse for the optimal codebooks obtained in Section III with length $L$ and that with length $r$, respectively.

In order to clearly characterize the coding rate and its buffer size, we introduce two measures.

Definition 2: The coding rate loss coefficient of one codebook with code-length $2n$ is defined by

$$\rho_R = 1 - \frac{R(n)}{R_n}$$

where $R(n)$ and $R_n$ represent the coding rate of one codebook with code-length $2n$ and that of the optimal codebook with the same code-length obtained in Section III.
Definition 3: The code storage ratio of one codebook with code-length $2n$ is defined by

$$\rho_M = \frac{S(n)}{S_n}$$

where $S(n)$ and $S_n$ represent the required storing code number of one codebook with code-length $2n$ and that of the optimal codebook with the same code-length obtained in Section III.

Figs. 4-6 present some sample results on the coding rate, coding rate loss coefficients and code storage ratios with different values of $L$. From the results of Figs. 5 and 6, we get the following observations: 1) As $L$ increases, the code rate loss ratios will decrease. In particular, when $L$ is 60, the code rate loss ratio is less than 6 percent for the code-lengths in the interval of 2 to 300; and 2) The combinatorial codebook storage will reduce greatly compared to that of the optimal codebook with the same length. For example, when $L = 60$ and $n = 100$, the combinatorial codebooks storing only requires about $1/10^{23}$ times memory units of the optimal codebooks with the same length while the corresponding coding rate loss is about 4 percent. This indicates that the universal combinatorial coding method will bring a good tradeoff between the coding rate loss and the codebook storing.

V. APPLICATION IN WIRELESS GAUSSIAN RELAY NETWORKS

In Sections III and IV, we mainly discussed the signal-time coding design under the assumption that the relay network topology adopts a two-hop model. In this section, we shall consider the general three-node model as shown in Fig. 3. The main difference from the two-hop three-node model is that there exist a direct link between $R$ and $D$. By using the general model, we shall discuss the benefit of signal-time coding in wireless Gaussian relay networks.

In [3], an upper bound on the average information flow rate in time division relaying was given. Later, the achievable rate for the general half-duplex Gaussian relay channel with the decode-and-forward protocol at the relay $R$ were presented in [23]. To observe the coding gain of signal-time coding over the conventional signal transmission technique, we employed the achievable rate for the general half-duplex Gaussian relay channel with the decode-and-forward protocol at relay $R$ given in [23] as the baseline. The achievable rate for the general half-duplex Gaussian relay channel with the decode-and-forward protocol is introduced here as a lemma.

Lemma 4: For the general half-duplex Gaussian relay channel, the decode-and forward protocol at relay $R$
achieves the following rate,

\[
R_{GC} = \sup_{0 \leq t, r \leq 1} \min \{ tC(P_{SR}) + (1 - t)C((1 - r^2)P_{SD_2}) , \\
tC(P_{SD_1}) + (1 - t)C(P_{SD_2} + P_{RD} + 2r\sqrt{P_{SD_1}P_{RD}}) \}
\]

(30)

where the power spectral density of Gaussian additive noises at the relay and destination are both normalized as 1, and \( P_{SR} \) and \( P_{RD} \) denote the received powers at relay \( R \) from link \( S - R \) and at destination \( D \) from link \( R - D \), respectively; \( P_{SD_1} \) and \( P_{SD_2} \) represent the received power at destination \( D \) from link \( S - D \) directly in the time intervals of \( R \) in the receiving and transmitting modes, respectively; \( r \) is the correlation between the source and relay signals in multiple access mode and \( C(x) = \frac{1}{2} \log(1 + x) \) is the capacity of a Gaussian link.

Note that the results in Lemma 4 are based on the use of the normalized bandwidth of the transmission. In the following comparison, we will consider the effect of transmission bandwidth.

Assume that we employ the following transmission strategy: 1) At the beginning of each transmission period \( T \), the information at source \( R \) will be decomposed into three parts by rate splitting. The first part will be transmitted by the direct link of \( R - D \) using the conventional signal transmission technique. The second part will be transmitted over links \( S - R \) and \( R - D \) in a time division mode, and the third part will be transmitted by signal-time coding in the time domain. 2) The time division mode is adopted here. We first divide the time period \( T \) into \( 2n \) time sub-slots. In each receiving sub-slot of relay \( R \), the source transmit two different information steams to \( R \) and \( D \) over links \( S - R \) and \( S - D \), respectively, with different transmission powers, and in each transmitting sub-slot of relay \( R \), the source \( S \) will stop its signal transmission, then the destination \( D \) only receives the forwarded information from \( R \), while it always detects the signal pulses in the time domain and receives the carried information by signal-time coding in the time domain regardless of the time sub-slot being used by relay for receiving or transmitting. In this way, we use three information sub-flows to transmit signals. The total information amount in time period \( T \) is given by

\[
I_A = I_{A1} + I_{A2} + I_{A3}
\]

(31)
where

\[
I_{A1} = \frac{C_1 C_2}{C_1 + C_2} T \quad (32)
\]

\[
I_{A2} = \frac{C_1 C_2}{C_1 + C_2} \frac{1.9222T}{\max\{C_1, C_2\} \Delta T} \quad (33)
\]

\[
I_{A3} = \frac{C_2 C_3}{C_1 + C_2} T \quad (34)
\]

and \(C_1, C_2\) and \(C_3\) represent the link capacities of \(S - R\), \(R - D\) and \(S - D\), respectively; and \(\Delta T\) is the minimum required time resolution for signal detections at relay \(R\) and destination \(D\). In fact, \(I_{A1}\) is the information amount transmitted over a two-hop routing of \(S - R\) and \(R - D\) in the time period \(T\). \(I_{A2}\) is the information amount transmitted by signal-time coding in the time domain in the duration of \(T\) where we use the obtained result of the bit rate per pulse 0.9611 is employed and \(I_{A3}\) is the information amount transmitted over the direct link \(S - D\) in the time period \(T\).

The corresponding average achievable information rate by using the proposed signal-time coding is given by

\[
R_{ST} = \frac{C_1 C_2}{C_1 + C_2} (1 + \frac{1.9222}{\max\{C_1, C_2\} \Delta T}) + \frac{C_2 C_3}{C_1 + C_2} \quad (35)
\]

In order to clearly characterize the coding gain of signal-time coding over Gaussian relay channels, we make the following assumptions.

1) In each time slot, the total transmission powers are limited by a constant \(P\).

2) The distance between \(R - D\) is denoted by \(d\) while the sum of the distances from \(S\) to \(R\) and that from \(R\) to \(D\) is equal to \(ad\) where \(a > 1\). Obviously, the relay is at a point of an ellipse with \(S\) and \(D\) as its two focuses. This will help us observe the effect of the distance variation between \(S\) to \(R\) (or the ratio between the other two sides of the triangle) on the information flow rate.

3) The propagation path loss exponential factor \(\alpha\) is set up as 2.

4) The source and relay employ the same transmission frequency band, and the bandwidth is denoted as \(B\).

5) The normalized power of the additive Gaussian white noises (AWGN) are set as 1. That is, \(B N_0 = 1\) where \(N_0\) is the power spectral density of AWGN.
Based on the above assumptions, the parameters in Eqn. (30) are given by

\[ P_{SR} = \frac{P}{d_1^3}, \quad P_{SD_1} = \frac{P}{d_3^3}; \]  

\[ P_{SD_2} = \frac{(1-\beta)P}{d_3^3}, \quad P_{RD_2} = \frac{\beta P}{d_2^3}. \]  

while the parameters in Eqns. (35), and (7) are given by, respectively,

\[ C_1 = \log_2(1 + \frac{\zeta P}{d_1^3}); \]  

\[ C_2 = \log_2(1 + \frac{P}{d_2^3}); \]  

\[ C_3 = \log_2(1 + \frac{(1-\zeta)P}{d_3^3}); \]  

where \( B \) is the transmission bandwidth, \( d_3 = ad, d_1 = \kappa d_3, d_2 = (1-\kappa)d_3 \), and \( 0 \leq \beta, \zeta \leq 1 \).

In this case, the maximum achievable information rate using signal-time coding for a given normalized time resolution \( B\Delta T \) is given by

\[ R_{ST}^{opt}(B\Delta T) = \sup_{0 \leq \zeta \leq 1} \{ R_{ST}(\zeta, B\Delta T) \} \]  

and the two hop upper bound is given by

\[ U_{two} = \min\{U_1, U_2\} \]  

where

\[ U_1 = \min\{\log_2(1 + \frac{P}{d_1^3}), \log_2(1 + \frac{P}{d_2^3})\} \]  

\[ U_2 = \frac{1}{2} \max\{\log_2(1 + \frac{P}{d_1^3}), \log_2(1 + \frac{P}{d_2^3})\}. \]  

To guarantee fairness, \( R_{ST}^{opt}(B\Delta T) \) and \( U_{two} \) are required to be normalized by \( 2B \). For simplicity, we denote them as \( R_{N-ST}^{opt}(B\Delta T) \) and \( U_{N-two} \), respectively. Here we also need to introduce a new concept, the coding gain of signal-time coding, which is defined as the ratio of the maximum achievable information rate using signal-time coding to the two-hop upper bound.
coding to that using the conventional transmission technique. It is given by

\[
\gamma = \frac{R_{N-ST}^{\text{opt}}}{R_{GC}}
\]  

(47)

Various numerical investigations have shown that using signal time coding will bring more benefits in terms of information flow rate compared with that only using the conventional transmission techniques in the signal domain. That is, the coding gain is always larger than 1. Due to space limitation, we do not include the result of these investigations here. One can observe some of these from the results in Figs. 7-10.

We shall now use some examples to illustrate the increment of information flow rate by using signal time coding. Let \( d = 10, \kappa = 0.35 \) and 0.75, and \( a = 1.5 \) and 2. The value range of \( P \) is selected from 1 to 50 dB, which indicates the received SNR variation over the direct link S-D is from \(-19\) to \(30\) dB. Two normalized time resolutions are considered here, which are selected as 6 and 12. Figs. 7-10 show some comparison results with time division achievable upper bound and the two hop upper bound, where the SNR used is that over the direct link \( S-D \). Fig. 11 shows the optimal power allocation of signal-time coding over different links. From these results, we get the following observations.

1) When the normalized time resolution is relatively small, the signal-time coding will provide more information rate gains. Thus, reducing the required normalized time resolution will get more benefits in terms of information flow rate.

2) When SNR is relatively low and the relay is relatively close to the source, the relative coding gain of signal-time coding is bigger and as the SNR increases, the relative coding gain will become smaller. This suggests that when SNR is relatively small, we need to adopt signal time coding to get more coding gain in terms of information flow rate. Comparing the results in Figs. 7 and 9, we can see that if the sum distances of \( d_1 \) and \( d_2 \) is relatively small, the relative coding gain of signal-time coding decreases more slowly as SNR increases.

3) As the relay is relatively far away from the source, if SNR is relatively low, then using signal time coding will get a smaller information rate gain compared to that case when the relay is relatively close to the source. In contrast, when SNR is high enough, using signal time coding will get more information rate gains. And the gain will become bigger along with the increase of SNR. This suggests that we need to employ signal-time coding when the relay is relatively far away from the source. A more important point is that the valid range of SNR using
signal-time coding is from a few dB to infinite when $\kappa = 0.75$.

All the above observations can be explained from the results in Fig. 11. When relay $R$ is relatively close to the source, it needs to adjust its transmission power allocation to the different links $S - R$ and $S - D$ in the receiving time-slots of relay $R$ to get its maximum information rate. Fig. 11 indicates that when $a = 2$, about 29 percent power will be allocated to the link $S - R$, so that its capacity can have a good match with that one over link $R - D$, and other 71 percent will be allocated to the $S - D$ link when the normalized time resolution, NRT, is equal to 6 and SNRs are from -19 to -1 dB. As the normalized time resolution, NRT, increases, the corresponding valid SNR range will expand. Due to space limitation, we do not show this result. As the SNR is relatively high, the optimal power allocation strategy is to allocate all the power to the direct link $S - D$ and not to the link $S - R$. This implies that the transmission system will not use the relay again to transmit information.

In contrast, when the relay $R$ is relatively far from the source, the analysis will become more complicated. It is dependent not only on the sum of the distances of $S - R$ and $R - D$, but also on their relative ratio $\kappa$.

Let us observe the case $a = 2$ first. In this case, it needs to allocate the transmission power to different links $S - R$ and $S - D$ in the receiving time slots of relay $R$. Fig. 11 indicates that when SNR is very low, all the power will be allocated to link $S - R$ and not to the direct link of $S - D$. This implies that the transmission system would like to select two hop transmission, which further confirms that our considered simple three-node model with direct link of $S - D$ in Section II is reasonable in some scenarios even it may have a direct link between the source and destination. Fig. 11 also indicates that when SNR is relatively low, it will have a balance point on the power allocation to the link $S - R$ and $S - D$, (see SNR = -11 dB, NRT=6, and $a = 2$ in Fig. 11). After that, from $-10$ to $2$ dB, all the power will be allocated to the direct link $S - D$ and without using the two-hop transmission again. Hence, there will be no opportunity to use the relay to forward the information. As a result, there will be no opportunity to use the signal time coding in the time domain. After $3$ dB, all the transmission power will be allocated to the direct link $S - D$ and $S - R$ again. Hence, there will have an opportunity to use the relay to forward the information again. As a result, there will have an opportunity to use the signal time coding in the time domain and the information flow rate gain by using signal time coding in relatively high SNR will become greater.

Observe the case $a = 1.5$, when SNR is very low, all the power will be allocated to link $S - R$ and not to the direct link of $S - D$. This is similar to that of $a = 2$. When SNR is higher than -9 dB, the optimal power allocation
will be balanced between the link $S - R$ and $S - D$. This indicates that there will have an opportunity to use the relay to forward the information. As a result, there will have an opportunity to use the signal time coding in the time domain to get a higher information rate.

The results in Fig. 11 also show that when SNR is very high, the optimal power allocation will have the same trends which mainly be dependent on the relative distance ratio $\kappa$.

We conclude this section by noting that in the three node network model, the mixed transmission by using two-hop and direct transmissions is not always a good option. That is, only two-hop transmission, only direct link transmission and the mixed transmission may occur, which is dependent on the practical scenarios.

VI. CONCLUSION

In this paper, we first derived explicit formulas to accurately estimate the maximum number of available codes by using an iterative algorithm for an arbitrary code length of signal-time coding, and then presented an iterative construction method of codebooks. Compared to exhaustive computer search, the proposed method can reduce the complexity greatly. In addition and to avoid having the codebook storing complexity increasing exponentially with code-length as well as reduce the buffer size while keeping a relatively high efficiency, we proposed a combinatorial universal construction method. Furthermore, we considered the application of signal-time coding in wireless Gaussian relay networks. It was shown that it can get a higher information rate than the upper bound using conventional signal transmission technique in some scenarios. An optimal power allocation analysis was presented and it was found that in the three node network model, the mixed transmission by using two-hop and direct transmissions is not always a good option.

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Fig. 1. Three node relay model

Fig. 2. Four time slots for signal time coding

Fig. 3. Information transmission using signal time coding
| Code Length | Code Book Offset | Code Word |
|-------------|------------------|------------|
| $n = 1$     | $CBF_1^0$        | 10         |
| $n = 2$     | $CBF_0^0$        | 110CBF_0^1 |
|             | $CBF_0^2$        | 110CBF_0^2 |
| $n = 3$     | $CBF_0^3$        | 1110CBF_2^5 |
|             | $CBF_2^0$        | 1110CBF_2^3 |
|             | $CBF_0^4$        | 1111CBF_4^6 |
| $n = 4$     | $CBF_0^5$        | 11110CBF_4^8 |
|             | $CBF_0^8$        | 1111100CBF_8^9 |
|             | $CBF_0^{16}$     | 111110100CBF_8^9 |
|             | $CBF_0^{20}$     | 111110101CBF_8^9 |
|             | $CBF_0^{22}$     | 111110102CBF_8^9 |

For $n = 5$, the table continues similarly with increasing code length and offset values, maintaining the pattern of constructing codebooks with the highest coding efficiency.
Fig. 4. Coding rate per pulse for different codebooks

Fig. 5. Coding rate loss ratio of Concatenated codes
Fig. 6. Code storage ratio of Concatenated codes

Fig. 7. Information Rate comparison, where $\kappa = 0.35$, $a = 2$ and the normalized time resolution (NTR) $B\Delta T = 6$ and 12
Fig. 8. Information Rate comparison, where $\kappa = 0.75$, $\alpha = 2$ and the normalized time resolution (NTR) $B\Delta T = 6$ and 12

Fig. 9. Information Rate comparison, where $\kappa = 0.35$, $\alpha = 1.5$ and the normalized time resolution (NTR) $B\Delta T = 6$ and 12
Fig. 10. Information Rate comparison, where $\kappa = 0.75$, $a = 1.5$ and the normalized time resolution (NTR) $B\Delta T = 6$ and 12

Fig. 11. Optimal power allocation in different scenarios
Normalized $BT=3$, $disbeta=0.75$, HD Bound via ST coding
Normalized $BT=3$, $disbeta=0.75$, HD Bound via ST coding
Normalized BT=3, disbeta=0.45, HD Bound via ST coding
Normalized $BT=3$, $disbeta=0.35$, HD Bound via ST coding
Normalized BT=6, disbeta=0.35

- HD bound Aazhang
- Two-Hop HD bound
- Signal-Time Coding
| Combinatorial Code | L=20 | L=40 | L=60 | Optimal Code |
|--------------------|------|------|------|--------------|
| Code Length        |      |      |      |              |
| Information Bit Number Per Code |      |      |      |              |
Optimal Power Allocation

NTR=3, d1/d3=0.35
NTR=6, d1/d3=0.35
NTR=3, d1/d3=0.75
NTR=6, d1/d3=0.75

SNR (dB) vs. Optimal Power Allocation
Normal Transmission in Signal Domain

Pulse Phase Transmission in Time Domain