Noncommutative compact manifolds constructed from quivers

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Abstract

The moduli spaces of θ-semistable representations of a finite quiver can be packaged together to form a noncommutative compact manifold.

If noncommutative affine schemes are geometric objects associated to affine associative C-algebras, affine smooth noncommutative varieties ought to correspond to quasi-free (or formally smooth) algebras (having the lifting property for algebra morphisms modulo nilpotent ideals). Indeed, J. Cuntz and D. Quillen have shown that for an algebra to have a rich theory of differential forms allowing natural connections it must be quasi-free [1, Prop. 8.5]. M. Kontsevich and A. Rosenberg introduced noncommutative spaces generalizing the notion of stacks to the noncommutative case [5, §2]. It is hard to construct noncommutative compact manifolds in this framework, due to the scarcity of faithfully flat extensions for quasi-free algebras. An alternative was outlined by M. Kontsevich in [4] and made explicit in [5, §1] (see also [7] and [6]). Here, the geometric object corresponding to the quasi-free algebra A is the collection (rep_n A)_n where rep_n A is the affine GL_n-scheme of n-dimensional representations of A. As A is quasi-free each rep_n A is smooth and endowed with Kapronov's formal noncommutative structure [2]. Moreover, this collection has equivariant sum-maps rep_n A × rep_m A → rep_m+n A.

We define a noncommutative compact manifold to be a collection (Y_n)_n of projective varieties such that Y_n is the quotient-scheme of a smooth GL_n-scheme X_n which is locally isomorphic to rep_n A_α for a fixed set of quasi-free algebras A_α, is endowed with a formal noncommutative structure and there are equivariant sum-maps X_m × X_n → X_{m+n}. In this note we will construct a large class of examples.

An illustrative example: let M_{P_2}(n; 0, n) be the moduli space of semi-stable vectorbundles of rank n over the projective plane P_2 with Chern-numbers c_1 = 0 and c_2 = n, then the collection (M_{P_2}(n; 0, n))_n is a noncommutative compact manifold. In general, let Q be a quiver on k vertices without oriented cycles and let θ = (θ_1, . . . , θ_k) ∈ Z^k. For a finite dimensional representation N of Q with dimension vector α = (a_1, . . . , a_k) we denote θ(N) = Σ_i θ_i a_i and d(α) = Σ_i a_i. A representation M of Q is called θ-semistable if θ(M) = 0 and θ(N) ≥ 0 for every subrepresentation N of M. A. King studied the moduli spaces M_Q(α, θ) of θ-semistable representations of Q of dimension vector α and proved that these are projective varieties [3, Prop 4.3]. We will prove the following result.

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Theorem 1 With notations as above, the collection of projective varieties
\[
\left( \bigsqcup_{d(\alpha)=n} M_Q(\alpha, \theta) \right)_n
\]
is a noncommutative compact manifold.

The claim about moduli spaces of vector bundles on \( \mathbb{P}_2 \) follows by considering the quiver \( \bullet \rightarrow \bullet \rightarrow \bullet \) and \( \theta = (-1, 1) \).

Let \( C \) be a smooth projective curve of genus \( g \) and \( M_C(n, 0) \) the moduli space of semi-stable vector bundles of rank \( n \) and degree 0 over \( C \). We expect the collection \( (M_C(n, 0))_n \) to be a noncommutative compact manifold.

1 The setting.

Let \( Q \) be a quiver on a finite set \( Q_v = \{v_1, \ldots, v_k\} \) of vertices having a finite set \( Q_a \) of arrows. We assume that \( Q \) has no oriented cycles.

The path algebra \( \mathbb{C}Q \) has as underlying \( \mathbb{C} \)-vectorspace basis the set of all oriented paths in \( Q \), including those of length zero which give idempotents corresponding to the vertices \( v_i \). Multiplication in \( \mathbb{C}Q \) is induced by (left) concatenation of paths. \( \mathbb{C}Q \) is a finite dimensional quasi-free algebra.

Let \( \alpha = (a_1, \ldots, a_k) \) be a dimension vector such that \( d(\alpha) = n \). Let \( \text{rep}_Q(\alpha) \) be the affine space of \( \alpha \)-dimensional representations of the quiver \( Q \). That is,
\[
\text{rep}_Q(\alpha) = \bigoplus_j M_{a_j \times a_i}(\mathbb{C})
\]
\( GL(\alpha) = GL_{a_1} \times \cdots \times GL_{a_k} \) acts on this space via basechange in the vertex-spaces. For \( \theta = (\theta_1, \ldots, \theta_k) \in \mathbb{Z}^k \) we denote with \( \text{rep}_Q(\alpha, \theta) \) the open (possibly empty) subvariety of \( \theta \)-semistable representations in \( \text{rep}_Q(\alpha) \). Applying results of A. Schofield \([8]\) there is an algorithm to determine \( (\alpha, \theta) \) such that \( \text{rep}_Q(\alpha, \theta) \neq \emptyset \). Consider the diagonal embedding of \( GL(\alpha) \) in \( GL_n \) and the quotient morphism
\[
X_n = \bigsqcup_{d(\alpha)=n} GL_n \times^{GL(\alpha)} \text{rep}_Q(\alpha, \theta) \overset{\pi_n}{\longrightarrow} Y_n = \bigsqcup_{d(\alpha)=n} M_Q(\alpha, \theta).
\]

Clearly, \( X_n \) is a smooth \( GL_n \)-scheme and the direct sum of representations induces sum-maps \( X_m \times X_n \longrightarrow X_{m+n} \) which are equivariant with respect to \( GL_m \times GL_n \longrightarrow GL_{m+n} \). \( Y_n \) is a projective variety by \([3]\) Prop. 4.3] and its points correspond to isoclasses of \( n \)-dimensional representations of \( CQ \) which are direct sums of \( \theta \)-stable representations by \([3]\) Prop. 3.2]. Recall that a \( \theta \)-semistable representation \( M \) is called \( \theta \)-stable provided the only subrepresentations \( N \) with \( \theta(N) = 0 \) are \( M \) and 0.

2 Universal localizations.

We recall the notion of universal localization and refer to \([3]\) Chp. 4] for full details. Let \( A \) be a \( \mathbb{C} \)-algebra and \( \text{projmod} A \) the category of finitely generated projective left \( A \)-modules. Let \( \Sigma \) be some class of maps in this category. In \([3]\) Chp. 4] it is shown that there exists an algebra map \( A \overset{JS}{\longrightarrow} A_\Sigma \) with the universal property that the maps \( A_\Sigma \otimes_A \sigma \) have an inverse for all \( \sigma \in \Sigma \). \( A_\Sigma \) is called
the universal localization of $A$ with respect to the set of maps $\Sigma$. In the special case when $A$ is the path algebra $CQ$ of a quiver on $k$ vertices, we can identify the isomorphism classes in $\text{projmod} \ CQ$ with $\mathbb{N}^k$. To each vertex $v_i$ corresponds an indecomposable projective left $CQ$-ideal $P_i$ having as $C$-vectorspace basis all paths in $Q$ starting at $v_i$. We can also determine the space of homomorphisms

$$\text{Hom}_{CQ}(P_i, P_j) = \bigoplus_{\sigma} \mathbb{C}p$$

where $p$ is an oriented path in $Q$ starting at $v_j$ and ending at $v_i$. Therefore, any $A$-module morphism $\sigma$ between two projective left modules

$$P_{i_1} \oplus \ldots \oplus P_{i_u} \xrightarrow{\sigma} P_{j_1} \oplus \ldots \oplus P_{j_v}$$

can be represented by an $u \times v$ matrix $M_\sigma$ whose $(p, q)$-entry $m_{pq}$ is a linear combination of oriented paths in $Q$ starting at $v_j$ and ending at $v_i$. Now, form an $v \times u$ matrix $N_\sigma$ of free variables $y_{pq}$ and consider the algebra $CQ_\sigma$ which is the quotient of the free product $CQ \ast \mathbb{C}(y_{11}, \ldots, y_{uv})$ modulo the ideal of relations determined by the matrix equations

$$M_\sigma N_\sigma = \begin{bmatrix} v_{i_1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & v_{i_u} \end{bmatrix}, \quad N_\sigma M_\sigma = \begin{bmatrix} v_{j_1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & v_{j_v} \end{bmatrix}.$$ 

Repeating this procedure for every $\sigma \in \Sigma$ we obtain the universal localization $CQ_\Sigma$. Observe that if $\Sigma$ is a finite set of maps, then the universal localization $CQ_\Sigma$ is an affine algebra.

It is easy to see that $CQ_\Sigma$ is quasi-free and that the representation space $\text{rep}_n \ CQ_\Sigma$ is an open subscheme (but possibly empty) of $\text{rep}_n \ CQ$. Indeed, if $m = (m_{i_k})_{a \in \text{rep}_n CQ_\Sigma}$, then $m$ determines a point in $\text{rep}_n CQ_\Sigma$ if and only if the matrices $M_\sigma(m)$ in which the arrows are all replaced by the matrices $m_{i_k}$ are invertible for all $\sigma \in \Sigma$. In particular, this induces numerical conditions on the dimension vectors $\alpha$ such that $\text{rep}_n CQ_\Sigma \neq \emptyset$. Let $\alpha = (a_1, \ldots, a_k)$ be a dimension vector such that $\sum a_i = n$ then every $\sigma \in \Sigma$ say with

$$P_{i_1}^{c_1} \oplus \ldots \oplus P_{i_u}^{c_u} \xrightarrow{\sigma} P_{j_1}^{f_1} \oplus \ldots \oplus P_{j_v}^{f_v}$$

gives the numerical condition $c_1 a_1 + \ldots + c_u a_u = f_1 a_1 + \ldots + f_v a_v$.

### 3 Local structure.

Fix $\theta = (\theta_1, \ldots, \theta_k) \in \mathbb{Z}^k$ and let $\Sigma = \bigcup_{\alpha \in \mathbb{N}^k} \Sigma_\alpha$ where $\Sigma_\alpha$ is the set of all morphisms $\sigma$

$$P_{i_1}^{\theta_1} \oplus \ldots \oplus P_{i_u}^{\theta_u} \xrightarrow{\sigma} P_{j_1}^{\theta_1} \oplus \ldots \oplus P_{j_v}^{\theta_v}$$

where \{i_1, \ldots, i_u\} (resp. \{j_1, \ldots, j_v\}) is the set of indices $1 \leq i \leq k$ such that $\theta_i > 0$ (resp. $\theta_i < 0$). Fix a dimension vector $\alpha$ with $(\theta, \alpha) = 0$, then $\theta$ determines a character $\chi_\theta$ on $GL(\alpha)$ defined by $\chi_\theta(g_1, \ldots, g_k) = \prod \det(g_i)^{\theta_i}$. With notations as before, the function $d_\sigma(m) = \det(M_\sigma(m))$ for $m \in \text{rep}_Q(\alpha)$ is a semi-invariant of weight $\chi_\theta$ in $\mathbb{C}[\text{rep}_Q(\alpha)]$ if $\sigma \in \Sigma_\alpha$.

The open subset $X_\sigma(\alpha) = \{ m \in \text{rep}_Q(\alpha) \mid d_\sigma(m) \neq 0 \}$ consists of $\theta$-semistable representations which are also $n$-dimensional representations of the universal
localization \( \mathbb{C}Q_\sigma \). Under this correspondence \( \theta \)-stable representations correspond to simple representations of \( \mathbb{C}Q_\sigma \). If we denote

\[
X_{\sigma,n} = \bigsqcup_{d(\alpha)=n} GL_n \times^{GL(\alpha)} X_\sigma(\alpha) \longrightarrow X_n
\]

then \( X_{\sigma,n} = \text{rep}_n \mathbb{C}Q_\sigma \) and the restriction of \( \pi_n \) to \( X_{\sigma,n} \) is the \( GL_n \)-quotient map \( \text{rep}_n \mathbb{C}Q_\sigma \longrightarrow \text{fac}_n \mathbb{C}Q_\sigma \) which sends an \( n \)-dimensional representation to the isomorphism class of the semi-simple \( n \)-dimensional representation of \( \mathbb{C}Q_\sigma \) given by the sum of the Jordan-Hölder components, see [2, 2.3]. As the semi-invariants \( i_\sigma \) for \( \sigma \in \Sigma \) cover the moduli spaces \( M_{\mathbb{C}Q}(\alpha, \theta) \) this proves the local isomorphism condition for the collection \( \{Y_n\}_n \).

A point \( y \in Y_n \) determines a unique closed orbit in \( X_n \) corresponding to a representation

\[
M_y = M_i^{\oplus e_i} \oplus \ldots \oplus M_i^{\oplus e_i}
\]

with the \( M_i \) \( \theta \)-stable representations occurring in \( M_y \) with multiplicity \( e_i \). The local structure of \( Y_n \) near \( y \) is completely determined by a local quiver \( \Gamma_y \) on \( l \) vertices which usually has loops and oriented cycles and a dimension vector \( \beta_y = (e_1, \ldots, e_l) \). The quiver-data \( (\Gamma_y, \beta_y) \) is determined by the canonical \( A_\infty \)-structure on \( \text{Ext}^*_{\mathbb{C}Q}(M_y, M_y) \). As \( \mathbb{C}Q \) is quasi-free, this ext-algebra has only components in degree zero (determining the vertices and the dimension vector \( \beta_y \)) and degree one (giving the arrows in \( \Gamma_y \)).

Using [3, Thm 4.7] and the correspondence between \( \theta \)-stable representations and simples of universal localizations, the local structure is the one outlined in [2, 2.5]. In particular, it can be used to locate the singularities of the projective varieties \( Y_n \).

4 Formal structure.

In [2] M. Kapranov computes the formal neighborhood of commutative manifolds embedded in noncommutative manifolds. Equip a \( \mathbb{C} \)-algebra \( R \) with the commutator filtration having as part of degree \(-d\)

\[
F_{-d} = \sum_m \sum_{i_1 + \ldots + i_m = d} RR_{i_1}^{Lie} R \ldots RR_{i_m}^{Lie} R
\]

where \( R_{i}^{Lie} \) is the subspace spanned by all expressions \([r_1, [r_2, [\ldots, [r_{i-1}, r_i]] \ldots] \) containing \( i-1 \) instances of Lie brackets. We require that for \( R_{ab} = R_{-d} \) affine smooth, the algebras \( \frac{F_{-d}}{F_{-d}} \) have the lifting property modulo nilpotent algebras in the category of \( d \)-nilpotent algebras (that is, those such that \( F_{-d} = 0 \)). The micro-local structuresheaf with respect to the commutator filtration then defines a sheaf of noncommutative algebras on \( \text{Spec} C R_{ab} \), the formal structure. Kapranov shows that in the affine case there exists an essentially unique such structure. For arbitrary manifolds there is an obstruction to the existence of a formal structure and when it exists it is no longer unique. We refer to [2, 4.6] for an operadic interpretation of these obstructions.

We will write down the formal structure on the affine open subscheme \( \text{rep}_n \mathbb{C}Q_\Gamma \) of \( X_n \) where \( \Gamma \) is a finite subset of \( \Sigma \). Functoriality of this construction then implies that one can glue these structures together to define a formal structure on \( X_n \) finishing the proof of theorem 1.

If \( A \) is an affine quasi-free algebra, the formal structure on \( \text{rep}_n A \) is given by the micro-structuresheaf for the commutator filtration on the affine algebra

\[
\nabla A = A * M_n(\mathbb{C})^{M_n(\mathbb{C})} = \{ p \in A * M_n(\mathbb{C}) \mid p.(1 * m) = (1 * m).p \forall m \in M_n(\mathbb{C}) \}
\]
This follows from the fact that $\sqrt{A}$ is again quasi-free by the coproduct theorems, [9, §2]. Specialize to the case when $A = \sqrt{CQ_\Gamma}$. Consider the extended quiver $Q(n)$ by adding one vertex $v_0$ and for every vertex $v_i$ in $Q$ we add $n$ arrows from $v_0$ to $v_i$ denoted $\{x_{i1}, \ldots, x_{in}\}$. Consider the morphism between projective left $CQ(n)$-modules

$$P_1 \oplus P_2 \oplus \ldots \oplus P_k \xrightarrow{\tau} P_0 \oplus \ldots \oplus P_0$$

determined by the matrix

$$M_\tau = \begin{bmatrix} x_{11} & \ldots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{k1} & \ldots & x_{kn} \end{bmatrix}.$$ 

Consider the universal localization $B = C\hat{Q}(n)_{\Gamma \cup \{\tau\}}$. Then, $\sqrt{CQ_\Gamma} = v_0 B v_0$ the algebra of oriented loops based at $v_0$.

5 Odds and ends.

One can build a global combinatorial object from the universal localizations $CQ_\Gamma$ with $\Gamma$ a finite subset of $\Sigma$ and gluings coming from unions of these sets. This example may be useful to modify the Kontsevich-Rosenberg proposal of noncommutative spaces to the quasi-free world.

Finally, allowing oriented cycles in the quiver $Q$ one can repeat the foregoing verbatim and obtain a projective space bundle over the collection $(f\circ C_n CQ)_n$.

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