Elliptic deformed superalgebra $U_{q,p}(\hat{\mathfrak{sl}}(M|N))$

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Abstract

We introduce the elliptic superalgebra $U_{q,p}(\hat{\mathfrak{sl}}(M|N))$ as one parameter deformation of the quantum superalgebra $U_q(\hat{\mathfrak{sl}}(M|N))$. For an arbitrary level $k \neq 1$, we give the bosonization of the elliptic superalgebra $U_{q,p}(\hat{\mathfrak{sl}}(1|2))$ and the screening currents that commute with $U_{q,p}(\hat{\mathfrak{sl}}(1|2))$ modulo total difference.

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1. Introduction

Infinite-dimensional symmetry has been an impressive success in conformal field theory (CFT) [1]. A solvable lattice model is an off-critical extension of CFT, and infinite-dimensional symmetry plays an important role in algebraic analysis of the solvable lattice model [2]. The lattice counterpart of minimal unitary CFT is the Andrews–Baxter–Forrester (ABF) model [3], whose Boltzmann weights are elliptic solutions of the Yang–Baxter equation (YBE) of the face type. Among the solvable models based on the YBE, those related to elliptic solutions occupy a fundamental place. Elliptic algebras are certain algebraic structures introduced to investigate these elliptic models. In study of $k$-fusion hierarchy of the ABF model, Konno [4] introduced the elliptic algebra $U_{q,g}(\hat{\mathfrak{gl}}(2))$ and constructed bosonization of the vertex operator by using this algebra. Jimbo–Konno–Odake–Shiraishi [5] constructed the elliptic algebra $U_{q,p}(g)$ by dressing the usual Drinfeld currents [20] of the quantum group $U_q(g)$ for the non-twisted affine Lie algebra $g$. In this paper, we introduce the elliptic deformed superalgebra $U_{q,p}(\hat{\mathfrak{sl}}(M|N))$ as one parameter deformation of the quantum superalgebra $U_q(\hat{\mathfrak{sl}}(M|N))$. We give the bosonization of the elliptic superalgebra $U_{q,p}(\hat{\mathfrak{sl}}(1|2))$ and $U_{q,p}(\hat{\mathfrak{sl}}(2|1))$ for a generic level $k$, and we give the screening currents that commute with $U_{q,p}(\hat{\mathfrak{sl}}(1|2))$ and $U_{q,p}(\hat{\mathfrak{sl}}(2|1))$ modulo total difference.

In this paper, we aim to contribute mathematical tools for the study of super-$\hat{\mathfrak{sl}}(M|N)$ family of the ABF model [19]. Mathematical tools are the elliptic algebra $U_{q,p}(\hat{\mathfrak{sl}}(M|N))$ and bosonizations. We comment on $\hat{\mathfrak{sl}}(N)$ family of the ABF model, where such mathematical tools
have been used previously in an analogous, but simpler case. Andrews, Baxter and Forrester [3] introduced the ABF model, which gives an extension of the hard hexagon model, and derived local height probabilities by Baxter’s corner transfer matrix method (CTM) [6]. The \( k \)-fusion and higher rank generalization, that we call \( \hat{sl}(N) \)-family of the ABF model, have been studied in [7, 8, 10, 11]. Inspired by the vertex operator approach to the six-vertex model [2, 12–14], that originated from CTM, Lukyanov and Pugai [15] studied the vertex operator approach to the ABF model and derived integral representations of multi-point local height probabilities. In the study of \( k \)-fusion hierarchy of the ABF model, Konno [4] introduced the elliptic algebra \( U_{q,p}(\hat{sl}(2)) \) and constructed the bosonization of the vertex operator by using this algebra. The vertex operator approach to the higher rank generalization of the ABF model has been studied in [16–18]. In the vertex operator approach to the \( \hat{sl}(N) \) family of the ABF model, the bosonization of the vertex operator played an important role. In the construction of the vertex operator, the current of the elliptic algebra \( U_{q,p}(\hat{sl}(N)) \) and its bosonization played important roles. In order to derive the integral representation of multi-point local height probabilities of super-\( \hat{sl}(M|N) \) family of the ABF model, we have to construct bosonizations of the vertex operators by using the current of the elliptic algebra \( U_{q,p}(\hat{sl}(M|N)) \), and understand the structure of the space of state of the model by CTM [6] that has been an open problem for the superalgebra \( \hat{sl}(M|N) \).

Next, we comment on pure mathematical aspects. Through an attempt to understand solvable models based on elliptic solutions of the YBE, various versions of elliptic algebras have been introduced [4, 5, 21–30]. It is important to understand not only them but also relations between them. Here, we summarize some basic facts on the elliptic quantum group \( B_{q,g}(\hat{g}) \) and the elliptic algebra \( U_{q,g}(\hat{g}) \). The elliptic quantum group \( B_{q,g}(\hat{g}) \) was introduced by twisting the standard quantum group \( U_q(g) \) [25–29], where \( g \) is the symmetrizable Kac–Moody algebra. The elliptic quantum group \( B_{q,g}(\hat{g}) \) has the quasi-Hopf structure and the elliptic algebra \( U_{q,g}(\hat{sl}(2)) \) has the \( H \)-Hopf algebroid structure [31, 32]. The realizations of the \( L \)-operators of the elliptic quantum group \( B_{q,g}(\hat{g}) \) were constructed in [5, 21, 22] by using the currents of the elliptic algebra \( U_{q,g}(\hat{g}) \) for \( g = \hat{sl}(N), A^{(2)}_2 \). This suggests that the currents of \( U_{q,g}(\hat{g}) \) give the Drinfeld currents [20] of the elliptic quantum group \( B_{q,g}(\hat{g}) \). The construction of the elliptic quantum group \( B_{q,g}(\hat{g}) \) has been extended to the superalgebra \( g = \hat{sl}(M|N) \) [30]. In this paper, we introduce the elliptic algebra \( U_{q,p}(\hat{sl}(M|N)) \). We conjecture that the \( L \)-operator of \( B_{q,g}(\hat{sl}(M|N)) \) is constructed by using the currents of \( U_{q,p}(\hat{sl}(M|N)) \) and that there exists the \( H \)-Hopf algebroid structure for \( U_{q,p}(\hat{sl}(M|N)) \). The bosonizations of the vertex operators give useful information for the construction of the \( L \)-operator, so-called Miki’s construction [33] of the \( L \)-operator. The above mentioned is background mathematical theory of the vertex operator. Next, we give a comment on the mathematical phenomenon of the space of state. Date–Jimbo–Kuniba–Miwa–Okado [8–11] found that local height probabilities of the \( \hat{sl}(N) \) family of the ABF model were expressed in terms of the branching coefficients appearing in the irreducible decomposition of the character of \( \hat{sl}(N) \) [34, 35]. In order to extend this to the super-\( \hat{sl}(M|N) \) family of the ABF model, we have to know the character formulæ of the superalgebra \( \hat{sl}(M|N) \) that gives the affine generalization of the formulæ [36, 37].

The paper is organized as follows. In section 2, after preparing the notation and giving the definition of the quantum group \( U_q(\hat{sl}(M|N)) \), we introduce the elliptic deformed superalgebra \( U_{q,p}(\hat{sl}(M|N)) \). Our approach is based on the dressing procedure of the Drinfeld current of the quantum group. In section 3, we give bosonizations of the superalgebra \( U_q(g), U_{q,p}(g) \) (\( g = \hat{sl}(1|2), \hat{sl}(2|1) \)) for an arbitrary level \( k \). We give the screening currents that commute with \( U_q(g), U_{q,p}(g) (g = \hat{sl}(1|2), \hat{sl}(2|1)) \) modulo total difference. In the appendix, we summarize some useful formulæ of bosonizations and screening currents.
2. Elliptic deformed superalgebra $U_{q,p}(\hat{\mathfrak{s}\mathfrak{l}}(M|N))$

In this section, we introduce the elliptic superalgebra $U_{q,p}(\hat{\mathfrak{s}\mathfrak{l}}(M|N))$. Kac [38] introduced the contragredient Lie algebra. Van de Leur [39] classified the contragredient superalgebra $\mathfrak{g}$ of finite growth. Yamane [40] introduced the quantum affine superalgebra $U_q(\mathfrak{g})$ and constructed the Drinfeld currents. We give the elliptic deformation of the quantum affine superalgebra by developing the dressing procedure [5].

2.1. Quantum superalgebra $U_q(\hat{\mathfrak{s}\mathfrak{l}}(M|N))$

In this section, we review the Drinfeld realization of the quantum superalgebra $U_q(\hat{\mathfrak{s}\mathfrak{l}}(M|N))$ for $M, N = 1, 2, 3, \ldots$ [40]. We restrict our consideration to $M \neq N$. The quantum superalgebra $U_q(\hat{\mathfrak{s}\mathfrak{l}}(M|N))$ in [40] is a $q$-deformation of the universal enveloping algebra of $\hat{\mathfrak{s}\mathfrak{l}}(M|N)$ [39].

Hereafter, we fix a complex number $q \neq 0$, $|q| < 1$. Let us set

$$[x, y] = xy - yx, \quad \{x, y\} = xy + yx, \quad [a]_q = \frac{q^a - q^{-a}}{q - q^{-1}}. \quad (2.1)$$

The Cartan matrix of the Lie superalgebra $\hat{\mathfrak{s}\mathfrak{l}}(M|N)$ is given by

$$(A_{i,j})_{0 \leq i, j \leq M+N-1} = \begin{pmatrix}
0 & -1 & 0 & \cdots & 0 & 1 \\
-1 & 2 & -1 & \cdots & 0 & 0 \\
0 & -1 & 2 & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & \cdots & 1 & -2 \ \ 1 \\
1 & 0 & \cdots & \cdots & 0 & 1 & -2
\end{pmatrix}, \quad (2.2)$$

where the diagonal part is $(A_{i,i})_{0 \leq i \leq M+N-1} = (0, 2, \ldots, 2, 0, -2, \ldots, -2)$.

**Definition 2.1.** The generators of the quantum superalgebra $U_q(\hat{\mathfrak{s}\mathfrak{l}}(M|N))$, which we call the Drinfeld generators, are given by [40]

$$x_{i,m}^\pm, a_{i,n}, h_i, c, \quad (1 \leq i \leq M+N-1, m \in \mathbb{Z}, n \in \mathbb{Z}_{\neq 0}). \quad (2.3)$$

Defining relations are

$c$: central, \hspace{1cm} $[h_i, a_{j,m}] = 0, \quad (2.4)$

$$[a_{i,m}, a_{j,n}] = \frac{[A_{i,j}]_d [c m]_q}{m} q^{-c|m|} \delta_{m+n,0}, \quad (2.5)$$

$$[h_i, x_j^+(z)] = \pm A_{i,j} x_j^+(z), \quad (2.6)$$

$$[a_{i,m}, x_j^+(z)] = \frac{[A_{i,j}]_d}{m} q^{-c|m|} z^{m+n} x_j^+(z), \quad (2.7)$$
where we have used $\in \mathcal{Z}$. (2.8)

\[
(z_1 - q^{\pm A_0/2})x_+^+(z_1)x_+^+(z_2) = (q^{\pm A_0/2}z_1 - z_2)x_+^+(z_2)x_+^+(z_1), \quad \text{for } |A_{i,j}| \neq 0,
\]

(2.9)

\[
x_+^+(z_1)x_+^+(z_2) = x_+^+(z_2)x_+^+(z_1), \quad \text{for } |A_{i,j}| = 0, (i, j) \neq (M, M),
\]

(2.10)

\[
\{x_+^+(z_1), x_+^+(z_2)\} = 0,
\]

(2.11)

\[
[x_+^+(z_1), x_+^+(z_2)] = \frac{\delta_{i,j}}{(q - q^{-1})z_1z_2}(\delta(q^{-z_1/z_2})\psi_i^+(q^{z}z_2) - \delta(q^{z_1/z_2})\psi_i^+(q^{-z}z_2)),
\]

(2.12)

\[
\{x_+^+(z_1), x_+^-(z_2)\} = \frac{1}{(q - q^{-1})z_1z_2}(\delta(q^{-z_1/z_2})\psi_i^+(q^{z}z_2) - \delta(q^{z_1/z_2})\psi_i^+(q^{-z}z_2)),
\]

(2.13)

\[
(x_+^+(z_1)x_+^+(z_2) - (q + q^{-1})x_+^+(z_1)x_+^-(z_2) + x_+^+(z_2)x_+^+(z_1)) + (z_1 \leftrightarrow z_2) = 0, \quad \text{for } |A_{i,j}| = 1, i \neq M,
\]

(2.14)

\[
(x_+^+(z_1)x_+^+(z_2)x_+^+(z_3) - q^{-1}x_+^+(z_1)x_+^+(z_2)x_+^+(z_3) - q^{-1}x_+^+(z_1)x_+^+(z_2)x_+^+(z_3) - q^{-1}x_+^+(z_1)x_+^+(z_2)x_+^+(z_3) - q^{-1}x_+^+(z_1)x_+^+(z_2)x_+^+(z_3) + x_+^+(z_1)x_+^+(z_2)x_+^+(z_3)) + (z_1 \leftrightarrow z_2) = 0,
\]

(2.15)

where we have used $\delta(z) = \sum_{m \in \mathbb{Z}} z^m$. Here, we have set the generating functions

\[
x_+^+(z) = \sum_{m \in \mathbb{Z}} x_+^+(z)^m,
\]

(2.16)

\[
\psi_i^+(q^{z}) = q^{\delta_i} \exp \left( (q - q^{-1}) \sum_{m > 0} a_{i,m} z^{-m} \right),
\]

(2.17)

\[
\psi_i^-(q^{z}) = q^{-\delta_i} \exp \left( -(q - q^{-1}) \sum_{m > 0} a_{i,-m} z^{m} \right).
\]

(2.18)

We changed the gauge of boson $a_{i,m}$ from those of [40]. In what follows, we assume $c \in \mathbb{C}$.
2.2. Elliptic deformed superalgebra $U_{q,p}(\hat{sl}(M|N))$

In this section, we introduce the elliptic superalgebra $U_{q,p}(\hat{sl}(M|N))$ for $M, N = 1, 2, 3, \ldots, (M \neq N)$. Let us introduce a deformation parameter $r$ such that

$$r, r^{*} = r - c > 0.$$  \hfill (2.19)

We often use the parameterization

$$p = q^{2r} = e^{-2 \pi i}, \quad p^{*} = q^{-2r} = e^{2 \pi i}, \quad z = q^{2u}, \quad w = q^{2v}.$$  \hfill (2.20)

We have $r \tau = r^{*} \tau^{*}$. Let us set the Jacobi theta functions $[u], [u]^{*}$ by

$$[u] = q^{2z - u} \Theta_{p}(q^{2u}) \Theta_{p}(q^{2u}); \quad [u]^{*} = q^{2z - u} \Theta_{p}(q^{2u}) \Theta_{p}(q^{2u}).$$  \hfill (2.21)

Here, we have used the standard symbols

$$\Theta_{p}(z) = (p, p)_{\infty} (z; p)_{\infty} (p^{2}; p)_{\infty}.$$  \hfill (2.22)

$$(z; t_{1}, \ldots, t_{n})_{\infty} = \prod_{n_{1}, \ldots, n_{n} \geq 0} (1 - zt_{1}^{n_{1}} \cdots t_{n}^{n_{n}}).$$  \hfill (2.23)

**Definition 2.2.** The elliptic superalgebra $U_{q,p}(\hat{sl}(M|N))$ is generated by the currents (operator valued function) and elements

$$E_{j}(z), F_{j}(z), B_{j,n}, h_{j}, \quad c(1 \leq j \leq M + N - 1, n \in \mathbb{Z}_{\neq 0}).$$  \hfill (2.24)

The defining relations are given as follows.

*For $1 \leq i, j \leq M + N - 1$, the relations are*

$$c: \text{central}, \quad [h_{i}, B_{j,m}] = 0,$$  \hfill (2.25)

$$[B_{i,m}, B_{j,n}] = \frac{[A_{i,j}m]_{q}[cm]_{q} [r^{*}m]_{q} \delta_{m+n,0}}{m}$$  \hfill (2.26)

$$[h_{i}, E_{j}(z)] = A_{i,j}E_{j}(z), \quad [h_{i}, F_{j}(z)] = -A_{i,j}F_{j}(z),$$  \hfill (2.27)

$$[B_{i,m}, E_{j}(z)] = \frac{[A_{i,j}m]_{q} z^{m} E_{j}(z), \quad [B_{i,m}, F_{j}(z)] = -\frac{[A_{i,j}m]_{q} [r^{*}m]_{q} z^{m} F_{j}(z)}{m}.$$  \hfill (2.28)

*For $1 \leq i, j \leq M + N - 1$ such that $(i, j) \neq (M, M)$, the relations are*

$$[u_{1} - u_{2} - \frac{A_{i,j}}{2}] E_{j}(z_{1})E_{j}(z_{2}) = [u_{1} - u_{2} + \frac{A_{i,j}}{2}]^{*} E_{j}(z_{2})E_{j}(z_{1}),$$  \hfill (2.29)

$$[u_{1} - u_{2} + \frac{A_{i,j}}{2}] F_{j}(z_{1})F_{j}(z_{2}) = [u_{1} - u_{2} - \frac{A_{i,j}}{2}] F_{j}(z_{2})F_{j}(z_{1}),$$  \hfill (2.30)

$$[E_{j}(z_{1}), F_{j}(z_{2})] = \frac{\delta_{i,j}}{(q^{-1} - q^{-1}) z_{1}z_{2}} \left( \delta(q^{-1}z_{1}/z_{2})H_{i}(q^{2}z_{2}) - \delta(q^{2}z_{1}/z_{2})H_{i}(q^{-2}z_{2}) \right).$$  \hfill (2.31)

$$[E_{M}(z_{1}), E_{M}(z_{2})] = 0, \quad [F_{M}(z_{1}), F_{M}(z_{2})] = 0,$$  \hfill (2.32)

$$[E_{M}(z_{1}), F_{M}(z_{2})] = \frac{1}{(q^{-1} - q^{-1}) z_{1}z_{2}} \left( \delta(q^{-1}z_{1}/z_{2})H_{M}(q^{2}z_{2}) - \delta(q^{2}z_{1}/z_{2})H_{M}(q^{-2}z_{2}) \right).$$  \hfill (2.33)
For $1 \leq i, j \leq M + N - 1$, the relations are

$$H_i(z_1)H_j(z_2) = \left[ \frac{u_2 - u_1 - \frac{\lambda_i}{z_1}}{u_2 - u_1 + \frac{\lambda_i}{z_1}} \right]^* \left[ \frac{u_2 - u_1 + \frac{\lambda_i}{z_1}}{u_2 - u_1 - \frac{\lambda_i}{z_1}} \right]^* H_j(z_2)H_i(z_1),$$

(2.34)

$$H_i(z_1)E_j(z_2) = \left[ \frac{u_1 - u_2 + \frac{\lambda_j}{z_1}}{u_1 - u_2 - \frac{\lambda_j}{z_1}} \right]^* E_j(z_2)H_i(z_1),$$

(2.35)

$$H_i(z_1)F_j(z_2) = \left[ \frac{u_1 - u_2 + \frac{\lambda_j}{z_1}}{u_1 - u_2 - \frac{\lambda_j}{z_1}} \right]^* F_j(z_2)H_i(z_1).$$

(2.36)

For $1 \leq i, j \leq M + N - 1$ (i $\neq$ M) such that $|\lambda_i| = 1$, they satisfy the Serre relations

$$\left( E_i(z_1)E_j(z_2)E_j(z_1) \right) \left\{ \frac{q^{\frac{\lambda_i}{z_1}}}{q^{\frac{-\lambda_i}{z_1}}} \right\} \left[ \frac{q^{\frac{\lambda_j}{z_2}}}{q^{\frac{-\lambda_j}{z_2}}} \right]^* \left( \frac{z_1}{z_2} \right) = \delta_{\lambda_i, \lambda_j},$$

$$- (q + q^{-1})E_i(z_1)E_j(z_1)E_j(z_2) \left\{ \frac{q^{\frac{\lambda_i}{z_1}}}{q^{\frac{-\lambda_i}{z_1}}} \right\} \left[ \frac{q^{\frac{\lambda_j}{z_2}}}{q^{\frac{-\lambda_j}{z_2}}} \right]^* \left( \frac{z_1}{z_2} \right) \left[ \frac{q^{\frac{\lambda_j}{z_2}}}{q^{\frac{-\lambda_j}{z_2}}} \right]^* \left( \frac{z_2}{z_1} \right) = \delta_{\lambda_i, \lambda_j},$$

$$+ E_j(z_1)E_i(z_1)E_j(z_2) \left\{ \frac{q^{\frac{\lambda_i}{z_1}}}{q^{\frac{-\lambda_i}{z_1}}} \right\} \left[ \frac{q^{\frac{\lambda_j}{z_2}}}{q^{\frac{-\lambda_j}{z_2}}} \right]^* \left( \frac{z_1}{z_2} \right) \left[ \frac{q^{\frac{\lambda_j}{z_2}}}{q^{\frac{-\lambda_j}{z_2}}} \right]^* \left( \frac{z_2}{z_1} \right) = \delta_{\lambda_i, \lambda_j},$$

(2.37)

$$\left( F_i(z_1)F_j(z_2)F_j(z_1) \right) \left\{ \frac{q^{-\frac{\lambda_i}{z_1}}}{q^{-\frac{\lambda_i}{z_1}}} \right\} \left[ \frac{q^{-\frac{\lambda_j}{z_2}}}{q^{-\frac{\lambda_j}{z_2}}} \right]^* \left( \frac{z_2}{z} \right) = \delta_{\lambda_i, \lambda_j},$$

$$- (q + q^{-1})F_i(z_1)F_j(z_1)F_j(z_2) \left\{ \frac{q^{-\frac{\lambda_i}{z_1}}}{q^{-\frac{\lambda_i}{z_1}}} \right\} \left[ \frac{q^{-\frac{\lambda_j}{z_2}}}{q^{-\frac{\lambda_j}{z_2}}} \right]^* \left( \frac{z_1}{z_2} \right) \left[ \frac{q^{-\frac{\lambda_j}{z_2}}}{q^{-\frac{\lambda_j}{z_2}}} \right]^* \left( \frac{z_2}{z_1} \right) = \delta_{\lambda_i, \lambda_j},$$

$$+ F_j(z_1)F_i(z_1)F_j(z_2) \left\{ \frac{q^{-\frac{\lambda_i}{z_1}}}{q^{-\frac{\lambda_i}{z_1}}} \right\} \left[ \frac{q^{-\frac{\lambda_j}{z_2}}}{q^{-\frac{\lambda_j}{z_2}}} \right]^* \left( \frac{z_1}{z_2} \right) \left[ \frac{q^{-\frac{\lambda_j}{z_2}}}{q^{-\frac{\lambda_j}{z_2}}} \right]^* \left( \frac{z_2}{z_1} \right) = \delta_{\lambda_i, \lambda_j},$$

(2.38)

and

$$\left( E_M(z_1)E_{M+1}(w_1)E_M(z_2)E_{M-1}(w_2) \right) \left\{ \frac{q^{\frac{\lambda_i}{z_1}}}{q^{\frac{\lambda_i}{z_1}}} \right\} \left[ \frac{q^{\frac{\lambda_j}{z_2}}}{q^{\frac{\lambda_j}{z_2}}} \right]^* \left[ \frac{q^{\frac{\lambda_j}{w_1}}}{q^{\frac{\lambda_j}{w_1}}} \right]^* \left[ \frac{q^{\frac{\lambda_i}{w_2}}}{q^{\frac{\lambda_i}{w_2}}} \right]^* \left( \frac{w_2}{z_2} \right) \left( \frac{w_2}{z_2} \right) = \delta_{\lambda_i, \lambda_j},$$

$$- q^{-1}E_M(z_1)E_{M+1}(w_1)E_{M-1}(w_2)E_M(z_2) \left\{ \frac{q^{\frac{\lambda_i}{z_1}}}{q^{\frac{\lambda_i}{z_1}}} \right\} \left[ \frac{q^{\frac{\lambda_j}{z_2}}}{q^{\frac{\lambda_j}{z_2}}} \right]^* \left[ \frac{q^{\frac{\lambda_j}{w_1}}}{q^{\frac{\lambda_j}{w_1}}} \right]^* \left[ \frac{q^{\frac{\lambda_i}{w_2}}}{q^{\frac{\lambda_i}{w_2}}} \right]^* \left( \frac{w_2}{z_2} \right) \left( \frac{w_2}{z_2} \right) = \delta_{\lambda_i, \lambda_j},$$

$$- qE_M(z_1)E_M(z_2)E_{M-1}(w_2)E_{M+1}(w_1) \left\{ \frac{q^{\frac{\lambda_i}{z_1}}}{q^{\frac{\lambda_i}{z_1}}} \right\} \left[ \frac{q^{\frac{\lambda_j}{z_2}}}{q^{\frac{\lambda_j}{z_2}}} \right]^* \left[ \frac{q^{\frac{\lambda_j}{w_1}}}{q^{\frac{\lambda_j}{w_1}}} \right]^* \left[ \frac{q^{\lambda_i}}{q^{\lambda_i}} \right]^* \left( \frac{w_2}{w_1} \right) \left( \frac{w_2}{w_1} \right) = \delta_{\lambda_i, \lambda_j}.$$
Here, we have used the abbreviations
\[ \{z\}^* = (p^* z; p^*)_\infty, \quad \{z\} = (p z; p)_\infty. \] (2.41)

2.3. Dressing construction

In this section, we construct \( U_{\theta \rho}(\hat{sl}(M|N)) \) from \( U_{\theta}(\hat{sl}(M|N)) \) by developing the dressing procedure [5].
**Definition 2.3.** Let us introduce the dressing operators \( u_j^\pm (z, p) \) (\( 1 \leq j \leq M + N - 1 \)) by

\[
u_j^+ (z, p) = \exp \left( \sum_{m>0} \frac{1}{[r^m]_q} a_{j,-m}(q^r z)^m \right),
\]

\[
u_j^- (z, p) = \exp \left( - \sum_{m>0} \frac{1}{[r^m]_q} a_{j,m}(q^{-r} z)^{-m} \right).
\]

**Proposition 2.4.** For \( 1 \leq i, j \leq M + N - 1 \), we have

\[
u_i^+ (z_1, p)\nu_j^+ (z_2) = \left( \frac{p^r q^{h_i} z_1/2 : p^r}{p^r q^{h_i} + z_1/2 : p^r} \right)_\infty \nu_i^+ (z_2)u_j^+ (z_1, p),
\]

\[
u_i^+ (z_1, p)\nu_j^- (z_2) = \left( \frac{p^r q^{-h_i} z_1/2 : p^r}{p^r q^{-h_i} + z_1/2 : p^r} \right)_\infty \nu_j^- (z_2)u_i^+ (z_1, p),
\]

\[
u_i^- (z_1, p)\nu_j^+ (z_2) = \left( \frac{pq^{-A_{i,j} - c} z_1/2 : p}{pq^{-A_{i,j} - c} z_1/2 : p} \right)_\infty \nu_j^+ (z_2)u_i^- (z_1, p),
\]

\[
u_i^- (z_1, p)\nu_j^- (z_2) = \left( \frac{pq^{A_{i,j} - c} z_1/2 : p}{pq^{A_{i,j} - c} z_1/2 : p} \right)_\infty \nu_j^- (z_2)u_i^- (z_1, p).
\]

**Definition 2.5.** We define the dressing currents \( e_j(z, p), f_j(z, p), \psi_j^\pm (z, p) \) (\( 1 \leq j \leq M + N - 1 \)) by

\[
e_j(z, p) = u_j^+ (z, p)x_j^+ (z),
\]

\[
f_j(z, p) = x_j^- (z)u_j^- (z, p),
\]

\[
\psi_j^+ (z, p) = u_j^+ (q^{-} z, p)\psi_j^+ (z)u_j^+ (q^{-} z, p),
\]

\[
\psi_j^- (z, p) = u_j^+ (q^{-} z, p)\psi_j^- (z)u_j^+ (q^{-} z, p).
\]

**Proposition 2.6.** The currents \( e_i(z, p), f_i(z, p) \) and \( a_i, c, h_i, a_{i,n} \) (\( 1 \leq i \leq M + N - 1, n \in \mathbb{Z} \)) satisfy the following relations:

\[c: \text{central}, \quad [h_i, a_{i,m}] = 0,\]

\[
[a_{i,m}, a_{i,n}] = \frac{[A_{i,j} ]_q [cm]_q}{m} q^{-cm} \delta_{m+n,0},
\]

\[
[h_i, e_j(z, p)] = A_{i,j} e_j(z, p), \quad [h_i, f_j(z, p)] = -A_{i,j} f_j(z, p).
\]
\[
\begin{align*}
[a_{i,m}, e_j(z, p)] &= \frac{[A_{i,j}]_m}{m} z_m e_j(z, p) \times \left\{ \begin{array}{cc}
\frac{[r_{m}]_q}{[r^* m]_q} & (m > 0), \\
q^m & (m < 0),
\end{array} \right. \\
\{a_{i,m}, f_j(z, p)\} &= -\frac{[A_{i,j}]_m}{m} z_m f_j(z, p) \times \left\{ \begin{array}{cc}
\frac{1}{[r^* m]_q} & (m > 0), \\
q^m & (m < 0),
\end{array} \right.
\end{align*}
\]

\[z_1 \Theta_{p}(q^{A_{i,j}/2z_1}) e_j(z_1, p) e_j(z_2, p) = -z_2 \Theta_{p}(q^{A_{i,j}/2z_1}) e_j(z_2, p) e_j(z_1, p), \quad \text{for} \quad |A_{i,j}| \neq 0,
\]

\[e_i(z_1, p), e_j(z_2, p) = 0, \quad \text{for} \quad |A_{i,j}| = 0, (i, j) \neq (M, M),
\]

\[e_M(z_1, p), e_M(z_2, p) = 0,
\]

\[z_1 \Theta_{p}(q^{-A_{i,j}/2z_1}) f_i(z_1, p) f_j(z_2, p) = -z_2 \Theta_{p}(q^{-A_{i,j}/2z_1}) f_j(z_2, p) f_i(z_1, p), \quad \text{for} \quad |A_{i,j}| \neq 0,
\]

\[f_i(z_1, p), f_j(z_2, p) = 0, \quad \text{for} \quad |A_{i,j}| = 0, (i, j) \neq (M, M),
\]

\[f_M(z_1, p), f_M(z_2, p) = 0,
\]

\[e_i(z_1, p), f_j(z_2, p) = \frac{\delta_{i,j}}{(q - q^{-1})z_1 z_2} \left( \delta(q^{-c_{i,j} z_1 z_2}) \psi_{i}^+(q^{p^2 z_1 z_2}, p) - \delta(q^{\psi_{i}^- z_1 z_2}) \psi_{i}^-(q^{p^2 z_1 z_2}, p) \right), \quad \text{for} \quad (i, j) \neq (M, M),
\]

\[e_M(z_1, p), f_M(z_2, p) = \frac{1}{(q - q^{-1})z_1 z_2} \times \left( \delta(q^{-c_{i,j} z_1 z_2}) \psi_{i}^+(q^{p^2 z_1 z_2}, p) - \delta(q^{\psi_{i}^- z_1 z_2}) \psi_{i}^-(q^{p^2 z_1 z_2}, p) \right),
\]

\[
\begin{align*}
&\left( e_i(z_1, p) e_i(z_2, p) e_j(z, p) \right) \left\{ \frac{[q^{A_{i,j}/2}]^{*}}{[q^{-A_{i,j}/2}]^{*}} \right\}^{\frac{p^2}{2}} \left\{ \frac{[q^{A_{i,j}/2}]^{*}}{[q^{-A_{i,j}/2}]^{*}} \right\}^{\frac{p^2}{2}} \\
&- (q + q^{-1}) e_i(z_1, p) e_j(z, p) e_i(z_2, p) \left\{ \frac{[q^{A_{i,j}/2}]^{*}}{[q^{-A_{i,j}/2}]^{*}} \right\}^{\frac{p^2}{2}} \\
&+ e_j(z, p) e_i(z_1, p) e_i(z_2, p) \left\{ \frac{[q^{A_{i,j}/2}]^{*}}{[q^{-A_{i,j}/2}]^{*}} \right\}^{\frac{p^2}{2}} \\
&+ (z_1 \leftrightarrow z_2) = 0, \quad \text{for} \quad |A_{i,j}| = 1, i \neq M,
\end{align*}
\]
\begin{equation}
\left(f_i(z_1, p)f_i(z_2, p)f_j(z, p) \left[\frac{q^{-\Lambda_{ij}} z}{z} \right] \frac{\left[\frac{q^{-\Lambda_{ij}} z}{z} \right]}{\left[\frac{q^{-\Lambda_{ij}} z}{z} \right]}
\right)
- (q + q^{-1}) f_i(z_1, p)f_j(z, p)f_j(z_2, p) \left[\frac{q^{-\Lambda_{ij}} z}{z} \right] \frac{\left[\frac{q^{-\Lambda_{ij}} z}{z} \right]}{\left[\frac{q^{-\Lambda_{ij}} z}{z} \right]}
+ f_i(z, p)f_i(z_1, p)f_i(z_2, p) \left[\frac{q^{-\Lambda_{ij}} z}{z} \right] \frac{\left[\frac{q^{-\Lambda_{ij}} z}{z} \right]}{\left[\frac{q^{-\Lambda_{ij}} z}{z} \right]}
+ (z_1 \leftrightarrow z_2) = 0, \hspace{1em} \text{for} \hspace{1em} |A_{i,j}| = 1, i \neq M,
\end{equation}

\begin{equation}
\left(e_M(z_1, p)e_{M+1}(w_1, p)e_M(z_2, p)e_{M-1}(w_2, p) \left[\frac{w_1}{w_2} \right] \frac{\left[\frac{w_1}{w_2} \right]}{\left[\frac{w_1}{w_2} \right]}
\right)
- q^{-1} e_M(z_1, p)e_{M+1}(w_1, p)e_{M-1}(w_2, p)e_M(z_2, p) \left[\frac{w_1}{w_2} \right] \frac{\left[\frac{w_1}{w_2} \right]}{\left[\frac{w_1}{w_2} \right]}
- q e_M(z_1, p)e_M(z_2, p)e_{M-1}(w_2, p)e_{M+1}(w_1, p) \left[\frac{w_1}{w_2} \right] \frac{\left[\frac{w_1}{w_2} \right]}{\left[\frac{w_1}{w_2} \right]}
+ e_M(z_1, p)e_{M-1}(w_2, p)e_M(z_2, p)e_{M+1}(w_1, p) \left[\frac{w_1}{w_2} \right] \frac{\left[\frac{w_1}{w_2} \right]}{\left[\frac{w_1}{w_2} \right]}
+ e_{M+1}(w_1, p)e_M(z_2, p)e_{M-1}(w_2, p)e_M(z_1, p) \left[\frac{w_1}{w_2} \right] \frac{\left[\frac{w_1}{w_2} \right]}{\left[\frac{w_1}{w_2} \right]}
- q^{-1} e_{M+1}(w_1, p)e_{M-1}(w_2, p)e_M(z_2, p)e_M(z_1, p) \left[\frac{w_1}{w_2} \right] \frac{\left[\frac{w_1}{w_2} \right]}{\left[\frac{w_1}{w_2} \right]}
- q e_M(z_2, p)e_{M-1}(w_2, p)e_{M+1}(w_1, p)e_M(z_1, p) \left[\frac{w_1}{w_2} \right] \frac{\left[\frac{w_1}{w_2} \right]}{\left[\frac{w_1}{w_2} \right]}
+ e_{M-1}(w_2, p)e_M(z_2, p)e_{M+1}(w_1, p)e_M(z_1, p) \left[\frac{w_1}{w_2} \right] \frac{\left[\frac{w_1}{w_2} \right]}{\left[\frac{w_1}{w_2} \right]}
+ (z_1 \leftrightarrow z_2) = 0,
\end{equation}

\begin{equation}
\left(f_M(z_1, p)f_{M+1}(w_1, p)f_M(z_2, p)f_{M-1}(w_2, p) \left[\frac{w_1}{w_2} \right] \frac{\left[\frac{w_1}{w_2} \right]}{\left[\frac{w_1}{w_2} \right]}
\right)
- q^{-1} f_M(z_1, p)f_{M+1}(w_1, p)f_{M-1}(w_2, p)f_M(z_2, p) \left[\frac{w_1}{w_2} \right] \frac{\left[\frac{w_1}{w_2} \right]}{\left[\frac{w_1}{w_2} \right]}
- q f_M(z_1, p)f_M(z_2, p)f_{M-1}(w_2, p)f_{M+1}(w_1, p) \left[\frac{w_1}{w_2} \right] \frac{\left[\frac{w_1}{w_2} \right]}{\left[\frac{w_1}{w_2} \right]}
+ f_M(z_1, p)f_{M-1}(w_2, p)f_M(z_2, p)f_{M+1}(w_1, p) \left[\frac{w_1}{w_2} \right] \frac{\left[\frac{w_1}{w_2} \right]}{\left[\frac{w_1}{w_2} \right]}
\end{equation}
relations (2.37), (2.38) and (2.39), (2.40) for J. Phys. A: Math. Theor.

We define elliptic currents \( E_j \), symbol:.

We have used the abbreviations (2.41). In what follows we use the standard normal ordering symbol.

Proposition 2.9. We use the zero-mode operators \( P_j \)
satisfy the defining relations of the elliptic superalgebra \( U_q^{\text{sl}}(M|N) \) (2.25), (2.26), (2.27), (2.28), (2.29), (2.30), (2.31), (2.32), (2.33), (2.34), (2.35) and (2.36). They satisfy the Serre relations (2.27), (2.28) and (2.29), (2.30) for \( 1 \leq i, j \leq M + N - 1 \) (i \( \neq M \)) such that \( |A_{i,j}| = 1 \).

We have constructed the elliptic deformed superalgebra \( U_{q,p}(\widehat{\text{sl}}(M|N)) \) from the quantum superalgebra \( U_q(\widehat{\text{sl}}(M|N)) \).
3. Bosonization

In this section, we give a new bosonization of the superalgebra $U_q(\hat{\mathfrak{sl}}(1|2))$, $U_{q,p}(\hat{\mathfrak{sl}}(1|2))$ for an arbitrary level $k$ and their screening currents. Wakimoto [41] constructed bosonization of the affine algebra $\hat{\mathfrak{sl}}(2)$ for an arbitrary level $k$. We call this type of bosonization based on the flag manifold [43] the Wakimoto realization. Feigin–Frenkel [42] generalized the Wakimoto realization to the higher rank affine algebra $\hat{\mathfrak{sl}}(N)$. Shiraiishi [44] constructed the Wakimoto realization of the quantum algebra $U_q(\hat{\mathfrak{sl}}(2))$ and its screening currents. Awata–Odake–Shiraishi constructed the Wakimoto realization for the quantum algebra $U_q(\hat{\mathfrak{sl}}(N))$ and its screening currents [45]. In the case of $U_q(\hat{\mathfrak{sl}}(2|1))$, Awata–Odake–Shiraishi [46] constructed the Wakimoto realization and Zhang–Gould [47] constructed the screening currents.

3.1. $U_q(\hat{\mathfrak{sl}}(1|2))$, $U_{q,p}(\hat{\mathfrak{sl}}(1|2))$, screening

In this section, we give new bosonizations of $U_q(\hat{\mathfrak{sl}}(1|2))$, $U_{q,p}(\hat{\mathfrak{sl}}(1|2))$ and their screening currents. In this section, we assume the central element $c = k \neq 1$. The Cartan matrix $(A_{i,j})_{0 \leq i, j \leq 2}$ of $\hat{\mathfrak{sl}}(1|2)$ is given by

$$
(A_{i,j})_{0 \leq i, j \leq 2} = \begin{pmatrix}
0 & -1 & 1 \\
-1 & 0 & 1 \\
1 & 1 & -2
\end{pmatrix}.
$$

(3.1)

The Cartan matrix of the classical part $\mathfrak{sl}(1|2)$ is written as

$$(A_{i,j})_{1 \leq i, j \leq 2} = ((v_1 + v_{i+1})\delta_{i,j} - v_i\delta_{i,j+1} - v_{i+1}\delta_{i,j+1})_{1 \leq i, j \leq 2},$$

where we have set $v_1 = +$, $v_2 = v_3 = -$. Let us introduce the bosons and the zero-mode operators $a_i^m$, $Q_i^m$ ($m \in \mathbb{Z}$, $j = 1, 2$) $b^{i,j}_m$, $c^{i,j}_m$, $Q^{i,j}_n$ ($m \in \mathbb{Z}$, $1 \leq i < j \leq 3$) by

$$
[a^m_i, a^n_k] = \frac{[(k-1)m]_q[A_{i,j}m]_q}{m} \delta_{m+n,0}, \quad [a^m_i, Q^k_n] = (k-1)A_{i,j}\delta_{m,0}.
$$

(3.2)

$$
[b^{i,j}_m, b^{k,j}_n] = -v_jv_i \frac{[m]_q^2}{m} \delta_{i,j}\delta_{j,0} \delta_{m+n,0}, \quad [b^{i,j}_m, Q^{i,j}_n] = -v_jv_i \delta_{i,j}\delta_{j,0} \delta_{m,0}.
$$

(3.3)

$$
[c^{i,j}_m, c^{k,j}_n] = v_jv_i \frac{[m]_q^2}{m} \delta_{i,j}\delta_{j,0} \delta_{m+n,0}, \quad [c^{i,j}_m, Q^{i,j}_n] = v_jv_i \delta_{i,j}\delta_{j,0} \delta_{m,0}.
$$

(3.4)

Let us set the bosonic fields $a(z)$, $a_\pm(z)$ and $\left(\frac{1}{\beta} a\right) (z|\alpha)$ as follows:

$$a(z) = -\sum_{m \neq 0} \frac{a_m}{[m]_q} z^{-m} + Q_a + a_0 \log z,
$$

(3.5)

$$a_\pm(z) = \pm(q-q^{-1}) \sum_{m>0} a_{\pm m} z^m \pm a_0 \log q,
$$

(3.6)

$$\left(\frac{1}{\beta} a\right) (z|\alpha) = -\sum_{m \neq 0} \frac{a_m}{[\beta m]} q^{-|m|} z^{-m} + \frac{1}{\beta} (Q_a + a_0 \log z).
$$

(3.7)

We impose the co-cycle condition on the zero-mode operator:

$$e^{Q_a^2} e^{Q_a^3} = -e^{Q_a^3} e^{Q_a^2}, \quad e^{Q_a^2} e^{Q_a^2} = e^{Q_a^3} e^{Q_a^2}, \quad e^{Q_a^2} e^{Q_a^1} = e^{Q_a^3} e^{Q_a^1}.
$$

(3.8)

Straightforward OPE calculations show the following propositions.
Proposition 3.1. Bosonization of the quantum superalgebra $U_{q,p}(\widehat{sl}(1/2))$ is given as follows:

\[ c = k, \quad h_1 = a_0^2 - b_0^2 - b_0^1, \quad h_2 = a_0^2 + 2b_0^2 + b_0^2 - b_0^1. \]  

\[ a_{1,m} = a_m^1 q^m \frac{k^1}{q - 1} - b_m^2 q^{(k-1)\mid m\rangle} - b_m^1 q^{(k-1)\mid m\rangle}, \]  

\[ a_{2,m} = a_m^1 q^m \frac{k^1}{q - 1} + b_m^2 q^{(k-2)\mid m\rangle} (q^m + q^{-m}) + b_m^1 q^{(k-2)\mid m\rangle} - b_m^1 q^{(k-1)\mid m\rangle}, \]  

\[ x_i^+ (z) = c_{1,1}^+ x_{1,1}^+ (z) + c_{1,2}^+ x_{1,2}^+ (z), \]  

\[ x_i^- (z) = c_{1,1}^- x_{1,1}^- (z) - c_{1,2}^- x_{1,2}^- (z), \]  

\[ x_i^+ (z) = \frac{1}{(q - q^{-1})z} (c_{2,1}^+ x_{2,1}^+ (z) - c_{2,2}^+ x_{2,2}^+ (z)), \]  

\[ x_i^- (z) = \frac{1}{(q - q^{-1})z} (c_{2,1}^- x_{2,1}^- (z) - c_{2,2}^- x_{2,2}^- (z) + c_{2,3}^- x_{2,3}^- (z)), \]  

where we have set

\[ x_{1,1}^+ (z) := e^{-(b^2 + b^2 \beta + b^2 \gamma)} - b^2 (q^{-1} - q) ; \]  

\[ x_{1,2}^+ (z) := e^{-(b^2 + b^2 \beta + b^2 \gamma)} - b^2 (q^{-1} - q) ; \]  

\[ x_{1,1}^- (z) := e^{b^2 (q^{-1} - q)} + b^2 (q^{-1} - q) ; \]  

\[ x_{1,2}^- (z) := e^{b^2 (q^{-1} - q)} + b^2 (q^{-1} - q) ; \]  

\[ x_{1,3}^- (z) := e^{b^2 (q^{-1} - q) - b^2 \beta (q^{-1} + q) + b^2 \gamma (q^{-1} + q)} ; \]  

\[ x_{1,4}^- (z) := e^{b^2 (q^{-1} - q) - b^2 \beta (q^{-1} + q) - b^2 \gamma (q^{-1} + q)} + b^2 \beta (q^{-1} + q) - b^2 \gamma (q^{-1} + q) ; \]  

\[ x_{1,1}^- (z) := e^{b^2 (q^{-1} - q) + b^2 \beta (q^{-1} + q) - b^2 \gamma (q^{-1} + q) - b^2 \beta (q^{-1} + q)} ; \]  

\[ x_{1,2}^- (z) := e^{b^2 (q^{-1} - q) - b^2 \beta (q^{-1} + q) + b^2 \gamma (q^{-1} + q) - b^2 \beta (q^{-1} + q)} ; \]  

\[ x_{1,3}^- (z) := e^{b^2 (q^{-1} - q) - b^2 \beta (q^{-1} + q) + b^2 \gamma (q^{-1} + q) - b^2 \beta (q^{-1} + q)} - b^2 \beta (q^{-1} + q) + b^2 \gamma (q^{-1} + q) ; \]  

\[ x_{1,4}^- (z) := e^{b^2 (q^{-1} - q) - b^2 \beta (q^{-1} + q) - b^2 \gamma (q^{-1} + q) - b^2 \beta (q^{-1} + q)} - b^2 \beta (q^{-1} + q) + b^2 \gamma (q^{-1} + q) ; \]  

\[ x_{1,3}^- (z) := e^{b^2 (q^{-1} - q) - b^2 \beta (q^{-1} + q) - b^2 \gamma (q^{-1} + q) - b^2 \beta (q^{-1} + q)} ; \]  

\[ x_{1,4}^- (z) := e^{b^2 (q^{-1} - q) + b^2 \beta (q^{-1} + q) - b^2 \gamma (q^{-1} + q) - b^2 \beta (q^{-1} + q)} ; \]  

Here, we have set the coefficients as follows:

\[ (c_{1,1}^+, c_{1,2}^+, c_{1,3}^+, c_{1,4}^+) = (\alpha, \beta, \gamma, \gamma), \]  

\[ (c_{1,1}^-, c_{1,2}^-, c_{1,3}^-, c_{1,4}^-) = \left( \frac{1}{qa}, \frac{1}{qa}, \frac{1}{\beta \gamma}, \frac{1}{\beta \gamma}, \frac{q^{k-1} \alpha}{\beta \gamma} \right). \]  

Here $\alpha, \beta, \gamma \neq 0$ are arbitrary parameters.
Next, we give bosonization of the elliptic superalgebra $U_{q,p}(\hat{sl}(1|2))$. Our construction is based on the dressing procedure of the quantum algebra developed in this paper.

**Proposition 3.2.** Bosonization of the elliptic superalgebra $U_{q,p}(\hat{sl}(1|2))$ is given as follows:

\[ c = k, \quad h_1 = a_0^1 - b_0^{2,3} - b_0^{1,2}, \quad h_2 = a_0^2 + 2b_0^{2,3} + b_0^{1,3} - b_0^{1,2}, \quad (3.29) \]

\[ B_{j,m} = \begin{cases} \frac{[r^m]_q}{[r]_q} a_{j,m} & \text{(m > 0)} \\ \frac{q^{m|j}}{[r^m]_q} a_{j,m} & \text{(m < 0)} \end{cases} \quad (j = 1, 2), \quad (3.30) \]

\[ a_{1,m} = a_m^1 q^{-m-1|j|} - b_m^{2,3} q^{-|j|-1|j|} - b_m^{1,3} q^{-|j|-1|j|}, \quad (3.31) \]

\[ a_{2,m} = a_m^2 q^{-m-1|j|} + b_m^{2,3} q^{-|j|-1|j|} (q^m + q^{-m}) + b_m^{1,3} q^{-|j|-1|j|} - b_m^{1,2} q^{-|j|-1|j|}, \quad (3.32) \]

\[ E_j(z) = u_j^{\uparrow}(z, p)x_j^{\uparrow}(z) \exp \left( \frac{j}{2} \rho_j \right) \quad (j = 1, 2), \quad (3.33) \]

\[ F_j(z) = x_j^{\downarrow}(z)u_j^{\downarrow}(z, p)z^{1/2(\rho_j + h_j)} \quad (j = 1, 2), \quad (3.34) \]

\[ H_j^{\pm}(z) = H_j(q^{\pm (r-1)}z) \quad (j = 1, 2), \quad (3.35) \]

where we have used (3.12), (3.13), (3.14), (3.15) and

\[ u_j^{\uparrow}(z, p) = \exp \left( \sum_{m=0}^{\infty} \frac{q^m}{[r^m]_q} B_{j,m} a^{m} \right) \quad (j = 1, 2), \quad (3.36) \]

\[ u_j^{\downarrow}(z, p) = \exp \left( -\sum_{m=0}^{\infty} \frac{q^m}{[r^m]_q} B_{j,m} a^{-m} \right) \quad (j = 1, 2), \quad (3.37) \]

\[ H_j(z) = \exp \left( -\sum_{m=0}^{\infty} \frac{B_{j,m}}{[r^m]_q} z^{-m} \right) : \exp \left( \frac{j}{2} \rho_j + h_j \right) : \quad (j = 1, 2). \quad (3.38) \]

Here, we have used the zero-mode operators

\[ [P_i, Q_j] = -\frac{A_{i,j}}{2} \quad (1 \leq i, j \leq 2). \quad (3.39) \]

**Proposition 3.3.** The bosonic operators $s_j(z) (j = 1, 2)$ given below are the screening currents that commute with the quantum superalgebra $U_{q,p}(\hat{sl}(1|2))$ modulo total difference:

\[ s_j(z) = : \exp \left( -\left( \hat{\mu}_{\pm} \right)^{\pm \frac{1}{2}} \right) : \quad (j = 1, 2). \quad (3.40) \]

Here, we have set

\[ s_1(z) = -c_{1,2} \hat{s}_1(z), \quad (3.41) \]

\[ s_2(z) = \frac{1}{(q-q^{-1})z} (-c_{2,3} \hat{s}_2(z) + c_{2,4} \hat{s}_2(z)) + c_{2,5} \hat{s}_2(z). \quad (3.42) \]
where
\[
\tilde{s}_{1,5}(z) := e^{-b_2^2(z)} : \; ,
\]
\[
\tilde{s}_{2,3}(z) := e^{-b_2^2(qz) + (b + c)^2(z) - b_1^2(qz) + b_2^2(z)} : ,
\]
\[
\tilde{s}_{2,4}(z) := e^{-b_2^2(qz) + (b + c)^2(z) - b_1^2(qz) + b_1^2(z)} : ,
\]
\[
\tilde{s}_{2,5}(z) := e^{-b_2^2(z) + b_1^2(z) + b_2^2(q^{-1}z)} : .
\]

Here, we have set the coefficients as follows:
\[
(c_{1,5}, c_{2,3}, c_{2,4}, c_{2,5}) = \left( q \alpha, \gamma, \gamma, \frac{b \gamma}{q \alpha} \right) ,
\]
where parameters \( \alpha, \beta, \gamma \neq 0 \) have been introduced in (3.27), (3.28) for the bosonization of \( U_q(\hat{sl}(1|2)) \). Explicitly, the bosonic operators \( s_1(z), s_2(z) \) and \( x_1^\pm(z), x_2^\pm(z) \) satisfy the following relations:
\[
[a_{i,m}, s_j(z_2)] = 0 ,
\]
\[
[x_j^+(z_1), s_j(z_2)] = 0 ,
\]
\[
[x_j^-(z_1), s_j(z_2)] = \frac{\delta_{i,j}}{(q - q^{-1})z_1z_2} (\delta(q^{k-1}z_2/z_1) - \delta(q^{k+1}z_2/z_1)) : e^{-\frac{1}{(1 + \rho)(1 - \rho)}} u_i^{-1}(z_1, p) z_2^2(q^{\pm 1}) : .
\]

and
\[
[\tilde{s}_1(z_1), \tilde{s}_1(z_2)] = 0 ,
\]
\[
(z_1 - q^{-A_1}z_2) \tilde{s}_1(z_1) \tilde{s}_2(z_2) = (q^{-A_1}z_1 - z_2) \tilde{s}_2(z_2) \tilde{s}_1(z_1) ,
\]
\[
(z_1 - q^{-A_1}z_2) \tilde{s}_2(z_1) \tilde{s}_2(z_2) = (q^{-A_1}z_1 - z_2) \tilde{s}_2(z_2) \tilde{s}_2(z_1) .
\]

By the commutation relation \([a_{i,m}, s_j(z_2)] = 0\), we conclude the following.

**Proposition 3.4.** The bosonic operators \( s_j(z) \ (j = 1, 2) \) given in (3.40) become the screening currents that commute with the elliptic algebra \( U_{q,p}(\hat{sl}(1|2)) \) modulo total difference. Explicitly, the bosonic operators \( s_1(z), s_2(z) \) and \( E_1(z), E_2(z), F_1(z), F_2(z) \) satisfy the following relations:
\[
[B_{i,m}, s_j(z_2)] = 0 ,
\]
\[
[E_i(z_1), s_j(z_2)] = 0 ,
\]
\[
[F_i(z_1), s_j(z_2)] = \frac{\delta_{i,j}}{(q - q^{-1})z_1z_2} (\delta(q^{k-1}z_2/z_1) - \delta(q^{k+1}z_2/z_1)) \times : e^{-\frac{1}{(1 + \rho)(1 - \rho)}} u_i^{-1}(z_1, p) z_2^2(q^{\pm 1}) : .
\]


The Jackson integral with parameters $p$ and $s \neq 0$ is defined by
\[
\int_0^{\infty} f(z)dz = s(1-p) \sum_{n \in \mathbb{Z}} f(sp^n) p^n.
\] (3.57)

From the above proposition we have
\[
\left[ \int_0^{\infty} s_j(z)d\phi_{j=1,z}, U_q(\widehat{sl}(1|2)) \right] = 0.
\] (3.58)

### 3.2. $U_q(\widehat{sl}(2|1)), U_q(\widehat{sl}(2|1)), \text{screening}$

In this section, we review the known results on the bosonization of $U_q(\widehat{sl}(2|1))$ \[46\] and its screening currents \[47\]. We give bosonizations of $U_q(\widehat{sl}(2|1))$ and its screenings. In this section, we assume the central element $c = k \neq -1$. The Cartan matrix $(A_{i,j})_{0 \leq i, j \leq 2}$ of $\widehat{sl}(2|1)$ is given by
\[
(A_{i,j})_{0 \leq i, j \leq 2} = \begin{pmatrix}
0 & -1 & -1 \\
-1 & 2 & 1 \\
-1 & 1 & 0
\end{pmatrix}.
\] (3.59)

The Cartan matrix of the classical part $sl(2|1)$ is written by
\[
(A_{i,j})_{1 \leq i, j \leq 2} = \left((-v_1 + v_{i+1})\delta_{i,j} - v_i\delta_{i,j+1} - v_{i+1}\delta_{i,j+1}\right)_{1 \leq i, j \leq 2},
\]
where we have set $v_1 = v_2 = +, v_3 = -$. Let us introduce the bosons and the zero-mode operators $a_m, \bar{Q}_m (m \in \mathbb{Z}, j = 1, 2) b_m^i, \bar{Q}_m^i, c_m^i, \bar{Q}_m^i (m \in \mathbb{Z}, 1 \leq i < j \leq 3)$ by
\[
[a_m, a_n^i] = \frac{(k+1)m}{m} \delta_{m+n,0}, \quad [a_m^i, \bar{Q}_n^i] = (k+1)A_{i,m}\delta_{m,0}.
\] (3.60)

\[
[b_m^i, b_n^{i'}] = -v_jv_{j'} \frac{[m]_2^2}{m} \delta_{m+n,0}, \quad [b_m^i, \bar{Q}_n^{i'}] = -v_jv_{i'} \delta_{m+n,0},
\] (3.61)

\[
[c_m^i, c_n^{i'}] = v_jv_{j'} \frac{[m]_2^2}{m} \delta_{m+n,0}, \quad [c_m^i, \bar{Q}_n^{i'}] = v_jv_{i'} \delta_{m+n,0}.
\] (3.62)

We impose the co-cycle condition on the zero-mode operators:
\[
e^{0}_s^3 e^{0}_{s'}^3 = e^{0}_{s'} e^{0}_{s}^3, \quad e^{0}_{s}^3 e^{0}_{s'}^3 = e^{0}_{s'}^3 e^{0}_{s}^3, \quad e^{0}_{s}^3 e^{0}_{s'}^3 = -e^{0}_{s'}^3 e^{0}_{s}^3.
\] (3.63)

**Proposition 3.5. Bosonization of the quantum superalgebra $U_q(\widehat{sl}(2|1))$ is given as follows \[46\]:**

\[
c = k, \quad h_1 = a_0^2 + 2b_0^{1,2} + b_0^{1,3} - b_0^{2,3}, \quad h_2 = a_0^2 - b_0^{1,2} - b_0^{1,3},
\] (3.64)

\[
a_{1,m} = a_m^2 q^{-\frac{[m]_2}{2}} + b_m^{1,2} q^{-(k+1)|m|} (q^m + q^{-m}) + b_m^{1,3} q^{-(k+2)|m|} - b_m^{2,3} q^{-(k+1)|m|},
\] (3.65)

\[
a_{2,m} = a_m^2 q^{-\frac{[m]_2}{2}} - b_m^{1,2} q^{-(k+1)|m|} - b_m^{1,3} q^{-(k+1)|m|},
\] (3.66)

\[
x_1^+(z) = \frac{1}{(q - q^{-1})_z} \left( c_{1,1}^+ e_{1,1}^+(z) - c_{1,2}^+ e_{1,2}^+(z) \right).
\] (3.67)
Here, we have set the coefficients as follows:
\[
\begin{align*}
(c_1, c_2, c_3, c_4) &= (\alpha, \beta, \gamma), \\
(c_1', c_2', c_3', c_4') &= \left( \frac{1}{\alpha}, \frac{1}{\beta}, \frac{q^{k+1}}{\alpha' \cdot \beta' \cdot \gamma}, \frac{q}{\alpha' \cdot \beta' \cdot \gamma} \right).
\end{align*}
\]

Here, \(\alpha, \beta, \gamma \neq 0\) are arbitrary parameters.

Note: the coefficients of the currents \(x_i^+(z)\) have four free parameters in [46]. In this paper, we have only three free parameters \(\alpha, \beta\) and \(\gamma\), because we assume the commutation relations (3.102), (3.103) and (3.104) with the screening currents.

Proposition 3.6. Bosonization of the elliptic superalgebra \(U_{q,p}(\mathfrak{s}\mathfrak{l}(2|1))\) is given as follows:
\[
c = k, \quad h_1 = a_0^1 + 2b_0^{1,2} + b_0^{1,3} - b_0^{2,3}, \quad h_2 = a_0^{1,2} - b_0^{1,2} - b_0^{1,3},
\]

\[
B_{j,m} = \begin{cases} 
[r^m q^{j+1} a_j] & (m > 0) \\
[r^m a_j] & (j = 1, 2) \\
[q^{km} a_j] & (m < 0)
\end{cases}
\]

\[
a_{1,m} = a_m q^{-|m|} + b_m^{1,2} q^{-(k+1)|m|} (q^m + q^{-m}) + b_m^{1,3} q^{-(k+2)|m|} - b_m^{2,3} q^{-(k+1)|m|}.
\]
\[ a_{2,m} = a_m^{-1} q^{-\frac{1}{2}+2|m|} - b_m^{-1} q^{-\frac{1}{2}+2|m|} - b_m^{1/2} q^{-\frac{1}{2}+2|m|}, \] (3.77)

\[ E_j(z) = u_j^+(z, p) x_j^+(z) e^{2Q_j z^{-\frac{1}{2}} \hat{P}_j} \quad (j = 1, 2), \] (3.78)

\[ F_j(z) = x_j^-(z) u_j^-(z, p) z^{-\frac{1}{2}} (\hat{P}_j + \hat{h}_j) \quad (j = 1, 2). \] (3.79)

\[ H_j^\pm(z) = H_j(q^{\pm|\ell-\ell'|} z) \quad (j = 1, 2), \] (3.80)

where we have used (3.67), (3.68), (3.69), (3.70) and

\[ u_j^+(z, p) = \exp \left( \sum_{m>0} \frac{q^m}{|r^m| q} B_{j, m} - m \right) \quad (j = 1, 2), \] (3.81)

\[ u_j^-(z, p) = \exp \left( - \sum_{m>0} \frac{q^m}{|r^m| q} B_{j, m} - m \right) \quad (j = 1, 2), \] (3.82)

\[ H_j(z) = \exp \left( - \sum_{m \neq 0} \frac{B_{j, m}}{|r^m| q} z^{-m} \right) : e^{2Q_j z^{-\frac{1}{2}} \hat{P}_j + \frac{1}{2} \hat{h}_j} : \quad (j = 1, 2). \] (3.83)

Here, we have used the zero-mode operators

\[ [P_i, Q_j] = -\frac{A_{ij}}{2} \quad (1 \leq i, j \leq 2). \] (3.84)

**Proposition 3.7.** The bosonic operators \( s_1(z) \) and \( s_2(z) \) given below are the screening currents that commute with the quantum superalgebra \( U_q(sl(2|1)) \) modulo total difference [47]:

\[ s_j(z) = e^{-\left( \frac{1}{2} \hat{a}^o \right)(z)(\frac{\hat{a}^{+o}}{2})} s_j(z) \quad (j = 1, 2). \] (3.85)

Here, we have set

\[ \tilde{s}_1(z) = \frac{1}{(q - q^{-1})z} (-c_{1,3} \tilde{s}_{1,3}(z) + c_{1,4} \tilde{s}_{1,4}(z)) + c_{1,5} \tilde{s}_{1,5}(z), \] (3.86)

\[ \tilde{s}_2(z) = -c_{2,5} \tilde{s}_{2,5}(z), \] (3.87)

where

\[ \tilde{s}_{1,5}(z) := e^{b_{1,5} z_{(1)}} \exp(b_{1,5} z_{(2)} - b_{1,5} z_{(3)} + b_{1,5} z_{(4)} - b_{1,5} z_{(5)} + b_{1,5} z_{(6)} - b_{1,5} z_{(7)}); \] (3.88)

\[ \tilde{s}_{1,4}(z) := e^{b_{1,4} z_{(1)}} \exp(b_{1,4} z_{(2)} - b_{1,4} z_{(3)} + b_{1,4} z_{(4)} - b_{1,4} z_{(5)} + b_{1,4} z_{(6)} - b_{1,4} z_{(7)}); \] (3.89)

\[ \tilde{s}_{1,3}(z) := e^{b_{1,3} z_{(1)}} \exp(b_{1,3} z_{(2)} - b_{1,3} z_{(3)} + b_{1,3} z_{(4)} - b_{1,3} z_{(5)} + b_{1,3} z_{(6)} - b_{1,3} z_{(7)}); \] (3.90)

\[ \tilde{s}_{2,5}(z) := e^{b_{2,5} z_{(1)}} \exp(b_{2,5} z_{(2)} - b_{2,5} z_{(3)} + b_{2,5} z_{(4)} - b_{2,5} z_{(5)} + b_{2,5} z_{(6)} - b_{2,5} z_{(7)}); \] (3.91)

Here, we have set the coefficients as follows:

\[ (c_{1,3}, c_{1,4}, c_{1,5}, c_{2,5}) = \left( \alpha, \alpha, \frac{\alpha \beta}{q}, \frac{\beta}{q} \right), \] (3.92)
where parameters $\alpha, \beta, \gamma \neq 0$ have been introduced in (3.82) and (3.83) for the bosonization of $U_q(\widehat{\mathfrak{sl}}(2|1))$. Explicitly, the bosonic operators $s_1(z), s_2(z)$ and $x_1^+ (z), x_2^+ (z)$ satisfy the following relations:

$$\left[ a_{i,m}, s_j(z_2) \right] = 0, \quad (3.102)$$

$$\left[ x_1^+ (z_1), s_j(z_2) \right] = 0, \quad (3.103)$$

$$\left[ x_2^- (z_1), s_j(z_2) \right] = \frac{\delta_{i,j}}{(q - q^{-1})z_1 z_2} \left( \delta (q^{k+1} z_2/z_1) - \delta (q^{-k-1} z_2/z_1) \right) : e^{-\left( \frac{i}{1!} d \left| z_2 - z_1 \right| \right)} :. \quad (3.104)$$

and

$$(z_1 - q^{-A_{1,1}} z_2) \tilde{s}_1 (z_1) \tilde{s}_1 (z_2) = (q^{-A_{1,1}} z_1 - z_2) \tilde{s}_1 (z_2) \tilde{s}_1 (z_1), \quad (3.105)$$

$$(z_1 - q^{-A_{1,2}} z_2) \tilde{s}_1 (z_1) \tilde{s}_2 (z_2) = (q^{-A_{1,1}} z_1 - z_2) \tilde{s}_2 (z_2) \tilde{s}_1 (z_1), \quad (3.106)$$

$$\left[ \tilde{s}_2 (z_1), \tilde{s}_2 (z_2) \right] = 0. \quad (3.107)$$

By the commutation relation $\left[ a_{i,m}, s_j(z_2) \right] = 0$, we conclude the following.

**Proposition 3.8.** The bosonic operators $s_j(z) (j = 1, 2)$ given in (3.95) become the screening currents that commute with the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}(2|1))$ modulo total difference. Explicitly, the bosonic operators $s_1(z), s_2(z)$ and $E_1(z), E_2(z), F_1(z), F_2(z)$ satisfy the following relations:

$$\left[ B_{i,m}, s_j(z_2) \right] = 0, \quad (3.108)$$

$$\left[ E_i(z_1), s_j(z_2) \right] = 0, \quad (3.109)$$

$$\left[ F_i(z_1), s_j(z_2) \right] = \frac{\delta_{i,j}}{(q - q^{-1})z_1 z_2} \left( \delta (q^{k+1} z_2/z_1) - \delta (q^{-k-1} z_2/z_1) \right) \times : e^{-\left( \frac{i}{1!} d \left| z_2 - z_1 \right| \right)} \left( P_{i,h} \right) :. \quad (3.110)$$

From the above proposition we have

$$\left[ \int_{0}^{2\pi} s_j(z) \, d_{q,p,z}, U_{q,p}(\widehat{\mathfrak{sl}}(2|1)) \right] = 0. \quad (3.111)$$

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Appendix. Bosonization

In the appendix, we summarize relations of bosonization for $U_q(\widehat{sl}(1|2))$ and its screening currents relating to the delta function $\delta(z) = \sum_{m \in \mathbb{Z}} z^m$:

\[ [x_{1,1}^+, (q^\pm_{22}/z_1)] = \frac{q}{z_1} \delta(q^\pm_{22}/z_1) e^{a_1(z_1 \pm z_2) - b_1^\pm_{(q^\pm_{22}-1)} z_2}, \quad (A.1) \]

\[ [x_{1,1}^+, (q^\pm_{22}/z_1)] = \frac{q}{z_1} \delta(q^{-k-2}_{22}/z_1) e^{a_1(z_1 \pm z_2) - b_1^\pm_{(q^{k-2}+1)} z_2}, \quad (A.2) \]

\[ [x_{1,1}^+, (q^\pm_{22}/z_1)] = -(q - q^{-1}) \delta(q^{-k-2}_{22}/z_1) \times :e^{a_1(z_1 \pm z_2) - b_1^\pm_{(q^{k-2})} z_2}:, \quad (A.3) \]

\[ [x_{1,2}^+, (q^\pm_{22}/z_1)] = \frac{1}{z_1} \delta(q^{-k-2}_{22}/z_1) e^{a_1(z_1 \pm z_2) - b_1^\pm_{(q^{k+1}+2)} z_2}, \quad (A.4) \]

\[ [x_{1,2}^+, (q^\pm_{22}/z_1)] = \frac{1}{z_1} \delta(q^{-k-2}_{22}/z_1) e^{a_1(z_1 \pm z_2) - b_1^\pm_{(q^{k-1})} z_2}, \quad (A.5) \]

\[ [x_{1,2}^+, (q^\pm_{22}/z_1)] = -(q - q^{-1}) \delta(q^{-k-2}_{22}/z_1) \times :e^{a_1(z_1 \pm z_2) - b_1^\pm_{(q^{k-2})} z_2}:, \quad (A.6) \]

\[ [x_{1,1}^+, (q^\pm_{22}/z_1)] = -(q - q^{-1}) \delta(q^{-k+1}_{22}/z_1) \times :e^{a_1(z_1 \pm z_2) - b_1^\pm_{(q^{k-2})} z_2}:, \quad (A.7) \]

\[ [x_{1,2}^+, (q^\pm_{22}/z_1)] = (q - q^{-1}) \delta(q^2_{22}/z_1) e^{a_1(z_1 \pm z_2) + b_1^\pm_{(q^{k-2}+2)} z_2}, \quad (A.8) \]

\[ [x_{1,2}^+, (q^\pm_{22}/z_1)] = (q - q^{-1}) \delta(q^{-k+1}_{22}/z_1) \times :e^{a_1(z_1 \pm z_2) - b_1^\pm_{(q^{k-2})} z_2}:, \quad (A.9) \]

\[ [x_{1,1}^+, (q^\pm_{22}/z_1)] = -(q - q^{-1}) \delta(q^{-k}_{22}/z_1) \times e^{a_1(z_1 \pm z_2) + b_1^\pm_{(q^{k-2})} z_2}, \quad (A.10) \]

\[ [x_{1,2}^+, (q^\pm_{22}/z_1)] = -(q - q^{-1}) \delta(q^2_{22}/z_1) \times :e^{a_1(z_1 \pm z_2) - b_1^\pm_{(q^{k-2})} z_2}:, \quad (A.11) \]

\[ [x_{1,2}^+, (q^\pm_{22}/z_1)] = (q - q^{-1}) \delta(q^2_{22}/z_1) \times :e^{a_1(z_1 \pm z_2) - b_1^\pm_{(q^{k-2})} z_2}:, \quad (A.12) \]
\[ [x^+_1(z_1), x^+_2(z_2)] = \frac{1}{z_2} \delta(q^2 z_2 / z_1) : e^{-\left(\frac{q z}{x^+_2(z_2)} - \frac{q z}{x^+_2(z_2)} + b^+_1(z_2) - b^+_1(z_2) - b^+_1(z_2) - b^+_1(z_2) \right)} : \] (A.13)

\[ [x^+_1(z_1), x^+_3(z_2)] = q^{-1} (q - q^{-1}) \delta(q^2 z_2 / z_1) \times : e^{-\left(\frac{q z}{x^+_2(z_2)} - \frac{q z}{x^+_2(z_2)} + b^+_2(z_2) - b^+_2(z_2) - b^+_2(z_2) - b^+_2(z_2) \right)} : \] (A.14)

\[ [x^+_1(z_1), x^+_4(z_2)] = \frac{1}{z_2} \delta(q^{-1} z_2 / z_1) : e^{-\left(\frac{q z}{x^+_2(z_2)} - \frac{q z}{x^+_2(z_2)} \right)} : \] (A.15)

\[ [x^+_2(z_1), x^+_1(z_2)] = \frac{1}{z_2} \delta(q^{-1} z_2 / z_1) : e^{-\left(\frac{q z}{x^+_2(z_2)} - \frac{q z}{x^+_2(z_2)} \right)} : \] (A.16)

\[ [x^+_2(z_1), x^+_3(z_2)] = \delta(q^{-1} z_2 / z_1) \times : e^{\left(\frac{q z}{x^+_2(z_2)} - \frac{q z}{x^+_2(z_2)} + b^+_3(z_2) - b^+_3(z_2) - b^+_3(z_2) - b^+_3(z_2) + b^+_3(z_2) - b^+_3(z_2) - b^+_3(z_2) - b^+_3(z_2) \right)} : \] (A.16)

\[ [x^+_2(z_1), x^+_4(z_2)] = -(q - q^{-1}) \delta(q^{-1} z_2 / z_1) : e^{-\left(\frac{q z}{x^+_2(z_2)} - \frac{q z}{x^+_2(z_2)} \right)} : \] (A.18)

\[ [x^+_3(z_1), x^+_1(z_2)] = \frac{-1}{(q - q^{-1}) q^{-1} z_2} \delta(q^{-1} z_2 / z_1) : e^{-\left(\frac{q z}{x^+_2(z_2)} - \frac{q z}{x^+_2(z_2)} \right)} : \]
\[ -\delta(q^{-1} z_2 / z_1) : e^{\left(\frac{q z}{x^+_2(z_2)} - \frac{q z}{x^+_2(z_2)} + b^+_1(z_2) - b^+_1(z_2) - b^+_1(z_2) - b^+_1(z_2) + b^+_1(z_2) - b^+_1(z_2) - b^+_1(z_2) - b^+_1(z_2) \right)} : \] (A.18)

\[ [x^+_3(z_1), x^+_2(z_2)] = \delta(q z_2 / z_1) \times : e^{\left(\frac{q z}{x^+_2(z_2)} - \frac{q z}{x^+_2(z_2)} + b^+_2(z_2) + b^+_2(z_2) + b^+_2(z_2) + b^+_2(z_2) + b^+_2(z_2) + b^+_2(z_2) + b^+_2(z_2) + b^+_2(z_2) \right)} : \] (A.20)

\[ [x^+_3(z_1), x^+_4(z_2)] = \delta(q z_2 / z_1) \times : e^{\left(\frac{q z}{x^+_2(z_2)} - \frac{q z}{x^+_2(z_2)} + b^+_3(z_2) + b^+_3(z_2) + b^+_3(z_2) + b^+_3(z_2) + b^+_3(z_2) + b^+_3(z_2) + b^+_3(z_2) + b^+_3(z_2) \right)} : \] (A.21)

\[ [x^+_4(z_1), x^+_1(z_2)] = -\delta(q z_2 / z_1) \times : e^{\left(\frac{q z}{x^+_2(z_2)} - \frac{q z}{x^+_2(z_2)} + b^+_1(z_2) + 2b^+_1(z_2) + b^+_1(z_2) + b^+_1(z_2) + b^+_1(z_2) + b^+_1(z_2) + b^+_1(z_2) + b^+_1(z_2) \right)} : \] (A.22)

\[ [x^+_4(z_1), x^+_2(z_2)] = \frac{1}{z_2} \delta(q z_2 / z_1) \times : e^{\left(\frac{q z}{x^+_2(z_2)} - \frac{q z}{x^+_2(z_2)} + b^+_2(z_2) + b^+_2(z_2) + b^+_2(z_2) + b^+_2(z_2) + b^+_2(z_2) + b^+_2(z_2) + b^+_2(z_2) + b^+_2(z_2) \right)} : \] (A.23)

\[ [x^+_4(z_1), x^+_3(z_2)] = \delta(q z_2 / z_1) \times : e^{\left(\frac{q z}{x^+_2(z_2)} - \frac{q z}{x^+_2(z_2)} + b^+_3(z_2) + b^+_3(z_2) + b^+_3(z_2) + b^+_3(z_2) + b^+_3(z_2) + b^+_3(z_2) + b^+_3(z_2) + b^+_3(z_2) \right)} : \] (A.24)
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\[
\frac{[x^1_{1,4}(z_1), \xi_2, x^2_5(z_2)]}{z^2_2} = \frac{1}{z^2_2} \delta(q^2 z_2/z_1) \times e^{D_2(q^2 z_2)} \left( \frac{1}{(q^2 z_2)} \right) + b_{-2}^2(q_2) + b_{-3}^2(q_2) - b_{-1}^2(q_2) + (b + c) \gamma^3(q^2 z_2) \right).
\]

(A.25)

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