ON VECTOR BUNDLES OVER MODULI SPACES TRIVIAL ON HECKE CURVES

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Abstract. Let $M_X(r, \xi)$ be the moduli space of stable vector bundles, on a smooth complex projective curve $X$, of rank $r$ and fixed determinant $\xi$ such that $\deg(\xi)$ is coprime to $r$. If $E$ is a vector bundle $M_X(r, \xi)$ whose restriction to every Hecke curve in $M_X(r, \xi)$ is trivial, we prove that $E$ is trivial.

1. Introduction

Moduli spaces of vector bundles on a complex projective curve have a long history. Apart form algebraic geometry, the context in which these moduli spaces were introduced, they also arise in symplectic geometry, geometric representation theory, differential geometry and mathematical physics. Line bundles and higher rank vector bundles on these moduli spaces play central role in their study. On the other hand, these moduli spaces contain a distinguished class of rational curves known as Hecke lines. They can be characterized as minimal degree rational curves on the moduli spaces [Ty] (also proved in [Su]). These Hecke curves play important role in the geometric representation theoretic aspect of the moduli spaces and also in the computation of cohomology of coherent sheaves on the moduli spaces.

Here we study restriction of vector bundles on moduli spaces to the Hecke lines. To describe the result proved here, fix a smooth complex projective curve $X$ of genus at least two. Let $\xi$ be line bundle on $X$ and $r \geq 2$ an integer coprime to $\deg(\xi)$. Let $M_X(r, \xi)$ be the moduli space of stable vector bundles on $X$ of rank $r$ and determinant $\xi$. We prove the following (see Theorem 5.4):

Theorem 1.1. Let $E$ be a vector bundle on $M_X(r, \xi)$ such that its restriction to every Hecke curve on $M_X(r, \xi)$ is trivial. Then $E$ is trivial.

Our motivation to study this problem comes from the result that says that a vector bundle on a projective space $\mathbb{P}^N$ is trivial when the restriction of it to every line is trivial (in fact, it is enough to check it for lines through a fixed point, cf. [OSS, p. 51, Theorem 3.2.1]). In this article we are replacing $\mathbb{P}^N$ by $M_X(r, \xi)$, and lines in $\mathbb{P}^N$ by Hecke curves in $M_X(r, \xi)$ (which are also rational curves of minimal degree).

To prove Theorem 1.1 we crucially use a theorem of Simpson which says that a semistable vector bundle $W$ on a smooth complex projective variety admits a flat holomorphic connection if $c_1(W) = 0 = c_2(W)$. It is relatively straightforward to deduce that the vector bundle $E$ in Theorem 1.1 is semistable and $c_1(E) = 0$. Almost all of our work is devoted in proving that $c_2(E) = 0$.

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2. COHOMOLOGY OF MODULI SPACE

Let $X$ be a smooth complex projective curve of genus $g$, with $g \geq 2$. Fix an integer $r \geq 2$ and also fix a line bundle $\xi$ on $X$ such that $\deg(\xi)$ is coprime to $r$. Let

$$M = M_X(r, \xi)$$

be the moduli space of stable bundles on $X$ of rank $r$ and degree $\deg(\xi)$. This moduli space $M$ is a smooth projective variety of dimension $(r^2 - 1)(g - 1)$. There is a Poincaré bundle $P$ on $X \times M$; two different Poincaré bundles on $X \times M$ differ by tensoring with a line bundle pulled back from $M$. It is known that $\text{Pic}(M) = \mathbb{Z}$ [Ra, p. 69], [Ra, p. 78, Proposition 3.4(ii)]. The ample generator of $\text{Pic}(M)$ will be denoted by $O_M(1)$. The degrees of any torsionfree coherent sheaf $F$ on $M$ is defined to be

$$\deg(F) := (c_1(F) \cup c_1(\mathcal{O}_M(1))(r^2-1)(g-1)-1) \cap [M] \in \mathbb{Z}.$$  

Let $U$ be a rank $r$ vector bundle on $X \times T$ such that for every point $t \in T$, the restriction $U_t := U|_{X \times t}$ is stable and has determinant $\xi$. Let

$$\phi : T \longrightarrow M = M_X(r, \xi)$$

be the corresponding classifying morphism. Define

$$\text{Det } U := (\det(Rp_T^*U))^{-1} := (\det(R^0p_T^*U))^{-1} \otimes (\det(R^1p_T^*U))^{-1} \longrightarrow T,$$

where $p_T : X \times T \longrightarrow T$ is the natural projection. Then, by [Na, Proposition 2.1],

$$\phi^*\mathcal{O}_M(1) = (\text{Det } U)^r \otimes (\bigwedge^r U_p)^{d+r(1-g)} \quad (2.1)$$

where $p \in X$ is any point, and $U_p = U|_{p \times M}$. Applying this to the Poincaré bundle $U = P$, it follows that

$$(\deg P_p) \cdot d \equiv 1 \mod r \quad (2.2)$$

(see [Ra, p. 75, Remark 2.9] and [Ra, p. 75, Definition 2.10]). Using the slant product operation, construct the integral cohomology classes

$$f_2 := c_2(P)/[X] \in H^2(M, \mathbb{Z}), \quad a_2 := c_2(P)/[p] \in H^4(M, \mathbb{Z}) \quad (2.3)$$

and

$$f_3 := c_3(P)/[X] \in H^4(M, \mathbb{Z}),$$

where $[X] \in H_2(X, \mathbb{Z})$ and $[p] \in H_0(X, \mathbb{Z})$ are the positive generators.

The following result is standard.

**Proposition 2.1.**

- The integral cohomology of $M$ has no torsion.
- The rank of $H^2(M, \mathbb{Z})$ is 1. The cohomology class $f_2$ in (2.3) generates the $\mathbb{Q}$–vector space $H^2(M, \mathbb{Q})$.
- For $r \geq 3$, the rank of $H^4(M, \mathbb{Z})$ is 3, while rank($H^4(M, \mathbb{Z})$) = 2 for $r = 2$. The $\mathbb{Q}$–vector space $H^4(M, \mathbb{Q})$ is generated by

$$f_2^2, \quad a_2, \quad \text{and } f_3, \quad (2.4)$$

where $a_2$ and $f_3$ are defined in (2.3). (Note that $f_3 = 0$ if $r = 2$.)
In [AB, p. 578, Theorem 9.9] it is proved that $H^*(M, \mathbb{Z})$ is torsionfree. See [AB, p. 582, Proposition 9.13] for the second statement. For the third statement, see [AB, p. 543, Proposition 2.20], [JK, p. 114, Section 2], [BR, p. 2, Theorem 1.5].

The cohomology class $a_2$ in Proposition 2.1(3) depends on the choice of Poincaré bundle $\mathcal{P}$. In the following lemma we show that $c_2(\mathcal{P})/[p]$ in (2.3) can be replaced by $c_2(\text{End}(\mathcal{P}))/[p]$, which does not depend on the choice of Poincaré bundle. This would simplify our later calculations.

**Lemma 2.2.** The cohomology classes

$$(f_2)^2, \ b_2 := c_2(\text{End}(\mathcal{P}))/[p] \text{ and } f_3$$

also generate $H^4(M, \mathbb{Q})$.

**Proof.** In view of Proposition 2.1(3), it suffices to prove that $a_2$ in (2.4) can be expressed as a function of the classes (2.4). The slant product

$$H^k(X \times M, \mathbb{Z}) \otimes H_{\ell}(X, \mathbb{Z}) \to H^{k-\ell}(M, \mathbb{Z}) \quad (\eta, c) \mapsto \eta/c$$

satisfies the following natural condition [GH, p. 264, (29.23)]: For morphisms $f : X' \to X$ and $g : M' \to M$,

$$(f \times g)^*(\eta) = g^*(\eta)$$

In particular, if $i : x \to X$ is a point and $\eta \in H^k(X \times M, \mathbb{Z})$, then

$$\xi([i(x)]) = (i \times \text{id}_M)^*\xi \in H^k(M, \mathbb{Z}). \quad (2.5)$$

Now consider $c_2(\mathcal{P})/[p]$ in (2.3). We have

$$c_2(\text{End}(\mathcal{P}))/[p] = -2rc_2(\mathcal{P})/[p] + (c_1(\mathcal{P})^2)/[p].$$

Using (2.5) it follows that $(c_1(\mathcal{P})^2)/[p] = c_1(P_p)^2$. Note that Proposition 2.1(2) says that $c_1(P_p) = kf_2$ for some $k \in \mathbb{Q}$. Consequently, we have

$$a_2 = \frac{-b_2 + k^2(f_2)^2}{2r},$$

which proves the lemma. \qed

## 3. HECKE TRANSFORMATION ON TWO POINTS OF THE CURVE

**Definition 3.1.** ([NR, p. 306, Definition 5.1 and Remark 5.2]). Let $l, m$ be integers. A vector bundle $F$ over $X$ is $(l, m)$–stable if, for every proper subbundle $G$ of $F$,

$$\frac{\deg(G) + l}{\text{rk } G} < \frac{\deg(F) + l - m}{\text{rk } F}.$$

Let $F$ be a $(0, 2)$–stable bundle with rank $r$ and determinant $\xi(x_1 + x_2) = \xi \otimes \mathcal{O}_X(x_1 + x_2)$ for fixed points $x_1, x_2 \in X$. We are going to perform Hecke transformations on $F$ over these two points $x_1, x_2$. The parameter space will be

$$\mathbb{P}_1 \times \mathbb{P}_2 := \mathbb{P}(E^\vee_{x_1}) \times \mathbb{P}(E^\vee_{x_2}) \cong \mathbb{P}^{r-1} \times \mathbb{P}^{r-1}$$

For a point $x \in X$ let

$$i_x : \mathbb{P}_1 \times \mathbb{P}_2 \to X \times \mathbb{P}_1 \times \mathbb{P}_2, \quad (y, z) \mapsto (x, y, z)$$

be the inclusion map. Let $p_{\mathbb{P}_1 \times \mathbb{P}_2} : X \times \mathbb{P}_1 \times \mathbb{P}_2 \to \mathbb{P}_1 \times \mathbb{P}_2$ be the natural projection.
For integers $a$, $b$, the line bundle $\mathcal{O}_{\mathbb{P}(E')}(a)\boxtimes\mathcal{O}_{\mathbb{P}(E')}(b)$ on $\mathbb{P}(E'_1)\times\mathbb{P}(E'_2)$ will be denoted by $\mathcal{O}(a,b)$. Consider the vector bundle $U$ on $X\times\mathbb{P}_1\times\mathbb{P}_2$ defined by the short exact sequence

$$0 \to U \to \mathcal{O}_X^2 \to (i_x)_*\mathcal{O}(1,0)^2 \to 0.$$  

(3.1)

Using the fact that $F$ is $(0,2)$–stable it can be shown that $U$ is a family of stable bundles on $X$. Indeed, for $(p_1, p_2) \in \mathbb{P}_1\times\mathbb{P}_2$, if a subbundle $G$ of the vector bundle $U|_{X\times\mathbb{P}_1\times\mathbb{P}_2}$ on $X$ contradicts the stability condition, then the subbundle of $F$ generated by $G$ contradicts the $(0,2)$–stability of $F$ (see [NR, p. 307, Lemma 5.5]).

From (3.1) it follows that

$$\bigwedge^r U_{(p_1,p_2)} \otimes \mathcal{O}_X(x_1 + x_2) = \bigwedge^r F = \xi \otimes \mathcal{O}_X(x_1 + x_2).$$

This implies that $\bigwedge^r U_{(p_1,p_2)} = \xi$. Let

$$\psi : \mathbb{P}_1\times\mathbb{P}_2 \to M = M_X(r, \xi)$$

be the corresponding classifying morphism.

If the point $p$ in (2.1) is different from $x_1$ and $x_2$, then $U_p$ (as in (2.1)) for the family $U$ in (3.1) is evidently trivial. Therefore, from (2.1) it follows that

$$\psi^* \mathcal{O}_M(1) \cong \mathcal{O}(r,r).$$

(3.3)

We assume that the Poincaré bundle is normalized by imposing the condition (see (2.2))

$$0 < d' := \deg(P_p) < r.$$  

(3.4)

By the universal property of the Poincaré bundle, there exist integers $a_1$, $a_2$ such that

$$(\text{id}_X \times \psi)^* \mathcal{P} = U \otimes p_{\mathbb{P}_1\times\mathbb{P}_2}^* \mathcal{O}(a_1, a_2).$$

(3.5)

Once we restrict the isomorphism in (3.5) to $p \times M$, it follows from (3.3) and (3.4) that $a_1 = a_2 = d'$. Hence, denoting $L := p^*_{\mathbb{P}_1\times\mathbb{P}_2} \mathcal{O}(d', d')$,

$$(\text{id}_X \times \phi)^* \mathcal{P} = U \otimes L.$$ (3.6)

We now calculate the Chern classes

$$c_1(U \otimes L) = dP + rd' (D_1 + D_2)$$

$$c_2(U \otimes L) = (1 + (r - 1)d'd') P(D_1 + D_2) + d'^2 (r - 1)2 (D_1 + D_2)^2$$

$$c_3(U \otimes L) = -P(D_1^2 + D_2^2) + (d'(r - 2) + d'^2 (r - 1)(r - 2)2 (D_1 + D_2)^2 +$$

$$+ (d'^2 (r - 1)(r - 2)6) (D_1 + D_2)^3$$

$$c_2(\text{End}(U)) = -2r P(D_1 + D_2),$$

where $D_1 \in H^2(X \times \mathbb{P}_1 \times \mathbb{P}_2, \mathbb{Z})$ (respectively, $D_2 \in H^2(X \times \mathbb{P}_1 \times \mathbb{P}_2, \mathbb{Z})$) is the pullback of the first Chern class $H_1 \in H^2(\mathbb{P}_1 \times \mathbb{P}_2, \mathbb{Z})$ (respectively, $H_2 \in H^2(\mathbb{P}_1 \times \mathbb{P}_2, \mathbb{Q})$) of the line bundle $\mathcal{O}(1,0)$ (respectively, $\mathcal{O}(0,1)$), and $P$ is the pullback of the class of a point in $X$.

We calculate the pullback of the generators using the pullback formula for the slant product:

$$\psi^* f_2 = \psi^* (c_2(\mathcal{P})/\!/[X]) = ((\text{id}_X \times \psi)^* c_2(\mathcal{P}))/\!/[X]$$

$$= c_2(U \otimes L)/\!/[X] = (1 + dd'(r - 1))(H_1 + H_2).$$
Analogously, we get that
\[ \psi^* f_2^2 = (1 + dd'(r - 1))^2 (H_1 + H_2)^2, \]
\[ \psi^* b_2 = 0, \]  
\[ \psi^* f_3 = - (H_1^2 + H_2^2) + (d'(r - 2) + d^2 d \frac{(r - 1)(r - 2)}{2}) (H_1 + H_2)^2. \]  

(3.6)

4. Hecke transformation on moving point

It this section we shall construct a family of vector bundles parametrized by Hecke lines with a moving point.

Let \( W \) be a \((0, 1)\)-stable bundle on \( X \) with determinant
\[ \xi(x_0) = \xi \otimes \mathcal{O}_X(x_0), \]  
for a fixed point \( x_0 \in X \), and let
\[ W \to Q \]  
be a rank 2 torsionfree quotient. Let \( X_1 \) be a copy of \( X \), i.e., \( X_1 \) is a curve with a fixed isomorphism with \( X \); the parameter space that we are going to construct involves several copies, so we shall employ this notation to distinguish between them. The points of \( X_1 \) will parametrize the points used to perform a Hecke transformation. Consider the projective bundle
\[ \mathbb{P}(Q^\vee) \]  
\[ \xrightarrow{\pi} X_1 \to X \]  
\[ X \]  
A point in \( y \in \mathbb{P}(Q^\vee) \) over \( x_1 = \pi(y) \in X_1 \) gives to a 1-dimensional quotient
\[ W_{x_1} \to Q_{x_1} \to \mathbb{C} \]  
of the fiber over \( x_1 \), so \( \mathbb{P}(Q^\vee) \) is, in a natural way, the parameter space of a family of Hecke transformations with respect to a moving point. We shall write this family explicitly. Consider the Cartesian diagram:
\[ P_\Delta \xrightarrow{i} X \times \mathbb{P}(Q^\vee) \]  
\[ \xrightarrow{\text{id}_X \times \pi} X_1 \to X \times X_1 \]  
\[ \Delta \]  
where the morphism at the bottom is the diagonal embedding \( \Delta = X \to X \times X_1 \), \( t \mapsto (t, t) \). Note that \( P_\Delta \) is a \( \mathbb{P}^1 \)-bundle over \( \Delta \). In fact it is canonically identified with \( \mathbb{P}(Q^\vee) \) once we invoke the natural isomorphism between the diagonal \( \Delta \) and \( X_1 = X \). From this identification between \( P_\Delta \) and \( \mathbb{P}(Q^\vee) \), let \( \mathcal{O}_{P_\Delta} (1) \to P_\Delta \) be the line bundle corresponding to the tautological line bundle \( \mathcal{O}_{\mathbb{P}(Q^\vee)}(1) \).

There is a canonical short exact sequence of sheaves on \( X \times \mathbb{P}(Q^\vee) \):
\[ 0 \to F \to p_X^* W \to i_* \mathcal{O}_{P_\Delta} (1) \to 0; \]  
recall from (4.2) that \( i \) is the inclusion of \( P \) in \( X \times \mathbb{P}(Q^\vee) \); here we consider \( F \) as a family of vector bundles on \( X \) parametrized by \( \mathbb{P}(Q^\vee) \). Using the condition that \( W \) is \((0, 1)\)-stable it can be shown that the vector bundle \( F_y := F |_{X \times y} \) on \( X \) is stable for every point \( y \in \mathbb{P}(Q^\vee) \).
Indeed, if a subbundle $S$ of $F_y$ contradicts the stability condition, then the subbundle of $W$ generated by $S$ contradicts that $(0,1)$–stability condition for $W$ [NR] p. 307, Lemma 5.5.

We are going to calculate the Chern character of the vector bundle $F$ in (4.3). The following notation for the Chow classes on $X \times \mathbb{P}(Q^\vee)$ will be used:

- $P_1$ (respectively, $P$) is the pullback of the class of a point in $X_1$ (respectively, $X$).
- $P_{01}$ is the pullback of the class of a point in $X \times X_1$.
- $\delta$ is the pullback of the class of the diagonal in $X \times X_1$.
- $D$ is the pullback of the divisor $\mathcal{O}_{\mathbb{P}(Q^\vee)}(1)$ on $\mathbb{P}(Q^\vee)$.

We can now calculate:

$$\text{ch}(W) = r + [(d+1)P].$$

Let $i(P_\Delta) \subset X \times \mathbb{P}(Q^\vee)$ be the image of the closed inclusion $i$ in (4.2). To identify $\text{ch}(\mathcal{O}_{i(P_\Delta)})$, we first do the following calculations on $X \times X_1$:

$$\text{ch}(\mathcal{O}_{X \times X_1}(-\Delta)) = 1 - \Delta + \frac{\Delta^2}{2},$$

$$\text{ch}(\mathcal{O}_\Delta) = \Delta - \frac{\Delta^2}{2} = \Delta - \frac{2g(X) - 2p}{2},$$

where $p \in H^4(X \times X_1, \mathbb{Z})$ is the class of a point in $X \times X_1$. It follows that

$$\text{ch}(\mathcal{O}_{i(P_\Delta)}) = p^*_X \times X_1 \text{ch}(\mathcal{O}_\Delta) = \delta - (g(X) - 1)P_{01}.$$

Now,

$$\text{ch}(i_* \mathcal{O}_{P_\Delta}(1)) = \text{ch}(\mathcal{O}_{i(P_\Delta)} \otimes \mathcal{O}_{\mathbb{P}(Q^\vee)}(1)) = \left(\delta - (g(X) - 1)P_{01}\right) \left(1 + D + \frac{D^2}{2}\right)$$

$$= \delta + \left[\delta \tilde{D} - (g(X) - 1)P_{01}\right] + \left[\frac{\delta D^2}{2} + (g(X) - 1)P_{01}D\right].$$

Finally we obtain the Chern character of $F$:

$$\text{ch}(F) = r + [(d+1)P_2 - \delta] + [-\delta D + (g(X) - 1)P_{01}] + \left[-\frac{\delta D^2}{2} - (g(X) - 1)P_{01}D\right]. \quad (4.4)$$

It may be clarified that $F$ in (4.3) is a family of stable vector bundles of degree $d$ parametrized by $\mathbb{P}(Q^\vee)$, but the determinant is not fixed. Indeed, if $y \in \mathbb{P}(Q^\vee)$ and $x_1 = \pi(y)$, then the determinant of the vector bundle corresponding to the point $y$ is $(\bigwedge^r W) \otimes \mathcal{O}_X(-x_1) = \xi \otimes \mathcal{O}_X(x_0 - x_1)$ (see (4.1)). In particular, the family $F$ induces a morphism from $\mathbb{P}(Q^\vee)$ to the moduli space $M_X(r, d)$ of stable vector bundles on $X$ of rank $r$ and degree $d$. But we want a morphism to the fixed determinant moduli space, so we shall tensor this family with an $r$-th root of $\mathcal{O}_X(x_1 - x_0)$. Since $x_1 \in X_1$ is a moving point, to have a family of $r$-th roots we need to pass to a Galois cover of the parameter space $X_1$.

Let

$$f : X_1 \longrightarrow J(X), \quad x_1 \longmapsto \mathcal{O}_X(x_1 - x_0)$$

be the Abel-Jacobi map for $X_1$, where $x_0$ is the point in (4.1) (recall that $X_1 = X$ is a copy of the same curve, but we make this distinction in notation because of the different roles they will play in the construction). This morphism $f$ corresponds to a family of line bundles on $X$ of degree zero parametrized by $X_1$, i.e., a line bundle $\mathcal{L}$ on $X \times X_1$ such that $\mathcal{L}|_{X \times x_1} \cong \mathcal{O}_X(x_1 - x_0)$. Let

$$w_r : J(X) \longrightarrow J(X), \quad L \longmapsto L^\otimes r$$
be the morphism that sends a line bundle to its \( r \)-th tensor power. Consider the Cartesian diagram

\[
\begin{array}{ccc}
T & \xrightarrow{f_r} & J(X) \\
\downarrow t & & \downarrow w_r \\
X_1 & \xrightarrow{f} & J(X)
\end{array}
\] (4.5)

We note that \( T \) is a connected Galois covering of \( X_1 \), because \( w_r \) is a Galois covering and the homomorphism \( f_* : \pi_1(X_1) \to \pi_1(J(X)) \) induced by \( f \) is surjective. It is easy to check that, if the morphism \( f \) in (4.5) corresponds to a line bundle \( L \) on \( X \times X_1 \), then the morphism \( f_r \) corresponds to a line bundle \( M \) on \( X \times T \) such that \( M \otimes r \cong (\text{id}_X \times t)^*L \), where \( t \) is the map in (4.5). In other words, after pulling back from \( X_1 \) to \( T \), the family \( L \) admits an \( r \)-th root namely \( M \).

Let \( Z \) be defined by the Cartesian diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{q} & \mathbb{P}(Q^\vee) \\
\downarrow \pi_T & & \downarrow \pi \\
T & \xrightarrow{t} & X_1
\end{array}
\] (4.6)

Finally, we define the vector bundle

\[
U = (\text{id}_X \times q)^*F \otimes (\text{id}_X \times \pi_T)^*M^{-1}
\]

on \( X \times Z \), where \( F \) is the vector bundle in (4.3). From the construction of \( U \) it is evident that \( U \) is a vector bundle on \( X \times T \) which represents a family of vector bundles on \( X \) of fixed determinant \( \xi \). Also, this is a family of stable vector bundles, because \( F \) is a family of stable vector bundles. Consequently, we have a classifying morphism

\[
\varphi : Z \to M.
\] (4.7)

Our objective now is to calculate the class \( \varphi^*b_2 = \varphi^*(c_2(\text{End}(\mathcal{P}))/[p] \). Note that the advantage of working with \( \text{End}(\mathcal{P}) \) instead of \( \mathcal{P} \) is that we do not have to worry about normalization of the Poincaré bundle, and also the tensorization by the line bundle \( M \) will not appear in the calculation. We have

\[
\text{End}(U) = (\text{id}_X \times q)^*\text{End}(F),
\]

and

\[
c_2(\text{End}(F)) = -\text{ch}_2(\text{End}(F)) - \frac{1}{2}\text{c}_1(\text{End}(F))^2 = \text{ch}_2(\text{End}(F))
\]

\[
= -[\text{ch}(F^\vee) \otimes \text{ch}(F)]_2 = 2r \text{ch}_2(F) - \text{ch}_1(F)^2
\]

\[
= 2r \delta D + (2(r - 1) - 2(d + 1) + (2 - 2g)) P_{01}
\]

(see (4.4)). Recall that “slanting with the class of a point is the same thing as restriction to the slice” (formula (2.3)). It follows that, if \( p \) is a point in \( X \) and \( [p] \in H_0(X, \mathbb{Z}) \) is its homology class, then \( P_{01}/[p] = 0 \) and \( \delta D/[p] = [\varpi] \in H^4(\mathbb{P}(Q^\vee), \mathbb{Q}) \), where \([\varpi] \in H^4(\mathbb{P}(Q^\vee), \mathbb{Z}) \) is the positive generator. So, we have

\[
c_2(\text{End}(F))/[p] = 2r[\varpi] \in H^4(\mathbb{P}(Q^\vee), \mathbb{Q}).
\]
Also, \( c_2(\text{End}(F))/[p] = c_2(\text{End}(F_p)) \), where \( F_p \) is the restriction of \( F \) to the slice \( p \times \mathbb{P}(Q') \) and then

\[
\varphi^* b_2 = \varphi^* c_2(\text{End}(\mathcal{P}_p)) = c_2(\text{End}(U_p)) = q^* c_2(\text{End}(F_p)) = \deg(q)2r = r^2g2r, \tag{4.8}
\]

where \( \mathcal{P}_p \) and \( U_p \) respectively are the restrictions of \( \mathcal{P} \) and \( U \) to the slice \( p \times M \).

## 5. Vanishing of Chern classes

Let \( E \) be a vector bundle on \( M = M_X(r, \xi) \) such that the restriction of \( E \) to every Hecke curve is trivial. Throughout this section, \( E \) would satisfy this condition.

From the above condition it can be deduced that

\[
c_1(E) = 0. \tag{5.1}
\]

Indeed, \( H^2(M, \mathbb{Z}) = \mathbb{Z} \), and a Hecke curve \( f : \mathbb{P}^1 \to M \) induces an injection on the second cohomology

\[
f^* : H^2(M, \mathbb{Z}) \to H^2(\mathbb{P}, \mathbb{Z}) \cong \mathbb{Z}.
\]

Now, \( f^* c_1(E) = c_1(f^* E) = 0 \), and hence (5.1) holds. Recall that the rank of \( H^4(M, \mathbb{Z}) \) is 3 when \( r \geq 3 \), and it is 2 when \( r = 2 \), and the generators are given by Lemma 2.2.

**Lemma 5.1.** Let \( \psi : \mathbb{P}_1 \times \mathbb{P}_2 \to M \) be the morphism (3.2). The pullback \( E' := \psi^*(E) \) is a trivial vector bundle on \( \mathbb{P}_1 \times \mathbb{P}_2 \).

**Proof.** For every line \( l_1 \subset \mathbb{P}_1 \) and every point \( p_2 \in \mathbb{P}_2 \), the restriction of \( \psi \) to \( l_1 \times p_2 \cong \mathbb{P}^1 \) is a Hecke curve, so \( E'|_{l_1 \times p_2} \) is trivial by the hypothesis. A vector bundle on a projective space is trivial if it is trivial when restricted to every line on the projective space ([OSS, p. 51, Theorem 3.2.1]). Consequently, \( E'|_{\mathbb{P}_1 \times \mathbb{P}_2} \) is trivial, and this is true for every point \( p_2 \in \mathbb{P}_2 \).

Therefore \( E' \) descends to a vector bundle \( F \) on \( \mathbb{P}_2 \), i.e., there is a vector bundle \( F \) on \( \mathbb{P}_2 \) such that \( q^* F \cong E' \), where \( q \) is the projection of \( \mathbb{P}_1 \times \mathbb{P}_2 \) to \( \mathbb{P}_2 \). In fact \( F = q_2 E' \).

Note that, for any \( p_1 \in \mathbb{P}_1 \), the restriction \( E'|_{p_1 \times \mathbb{P}_2} \) is isomorphic to \( F \). As before, for every line \( l_2 \) in \( \mathbb{P}_2 \), the restriction of \( \psi \) to \( p_1 \times l_2 \) is a Hecke curve, so \( F \) is trivial on \( p_1 \times l_2 \). Hence \( F \) is trivial by the above argument. Consequently, \( E' = q^* F \) is trivial. \( \square \)

**Lemma 5.2.** Let \( \varphi : Z \to M \) be the morphism in (4.7). The pullback \( E_Z := \varphi^*(E) \) has Chern classes

\[
c_1(E_Z) = 0 \in H^2(Z, \mathbb{Z}) \quad \text{and} \quad c_2(E_Z) = 0 \in H^4(Z, \mathbb{Z}).
\]

**Proof.** The vector bundle \( E \) on \( M \) has \( c_1(E) = 0 \) (see (5.1)), and hence \( c_1(E_Z) = 0 \).

The scheme \( Z \) fibers over a curve \( \pi_T : Z \to T \), and the restriction of \( \varphi \) to any fiber is a Hecke line (see (4.6)). Hence, the vector bundle \( E_Z \) is trivial on the fibers of \( \pi_T \). This implies that \( E_Z \) descends to \( T \), i.e., there exists a vector bundle \( F \) on \( T \) such that \( E_Z = \pi_T^* F \). In fact, \( F = \pi_T E_Z \).

Since \( F \) is a vector bundle on a curve, namely \( Z \), it follows that \( c_2(F) = 0 \) (as \( H^4(Z, \mathbb{Z}) = 0 \)). Therefore, we have \( c_2(E_Z) = \pi_T^* c_2(F) = 0 \). \( \square \)

**Proposition 5.3.** The second Chern class \( c_2(E) \in H^4(M, \mathbb{Q}) \) of the vector bundle \( E \) on \( M \) is zero.
Proof. Using Lemma 2.2 we write $c_2(E)$ as a combination of the generators with coefficients in $\mathbb{Q}$:

$$c_2(E) = \alpha(f_2)^2 + \beta b_2 + \gamma f_3.$$ 

In the case rank $r = 2$, we have $f_3 = 0$, so we set $\gamma = 0$.

Consider the pullback of $c_2(E)$ by the morphism $\psi$ in (3.2). We have $\psi^*c_2(E) = 0$ by Lemma 5.1, and hence using (3.6) it follows that

$$\psi^*c_2(E) = 0 = \alpha((1 + dd'(r-1))^2(H_1 + H_2)^2) - \gamma(H_1^2 + H_2^2).$$  \tag{5.2}$$

It is easy to check, using (2.2), that $1 + dd'(r-1) \neq 0$.

We have $\mathbb{P}_1 \times \mathbb{P}_2 \cong \mathbb{P}^{r-1} \times \mathbb{P}^{r-1}$, so if rank $r = 2$, then

$$H_1^2 = 0 = H_2^2,$$

and hence $\alpha = 0$.

On the other hand, if $r > 2$, then the classes $H_1^2 + H_2^2$ and $(H_1 + H_2)^2$ are linearly independent in $H^4(\mathbb{P}_1 \times \mathbb{P}_2, \mathbb{Z}) \cong H^4(\mathbb{P}^{r-1} \times \mathbb{P}^{r-1}, \mathbb{Z}) \cong \mathbb{Z}^2$, so (5.2) implies $\alpha = \gamma = 0$.

Summing up, for any rank $r \geq 2$ we have

$$c_2(E) = \beta b_2.$$  \tag{5.3}$$

Pulling back (5.3) by the map $\varphi$ in (4.7), and using (4.8) we get that

$$0 = \beta r^{2g-2}.$$

Therefore, $\beta = 0$, and hence $c_2(E) = 0$ by (5.3).

**Theorem 5.4.** Let $E$ be a vector bundle on $M = M_X(r, \xi)$ satisfying the condition that the restriction of $E$ to every Hecke curve is trivial. Then $E$ is trivial.

**Proof.** The restriction of $E$ to a Hecke curve on $M$ is trivial. From this it can be deduced that $E$ is semistable. Indeed, if $E$ is not semistable, there is a coherent subsheaf $V \subset E$ such that $E/V$ is torsionfree, and

$$\frac{\deg(V)}{\rank(V)} > \frac{\deg(E)}{\rank(E)} = 0$$  \tag{5.4}$$

(see (5.1)). Now, for a general Hecke curve $\mathbb{P} \subset M$, the restriction $V|_\mathbb{P}$ is torsionfree. Moreover for any smooth closed curve $C \subset M$, and any torsionfree coherent sheaf $W$ on $M$ which is locally free on $C$, we have $\deg(W|_C) = \deg(C) \cdot \deg(W)$. Consequently, from (5.4) it is deduced that

$$\deg(V|_\mathbb{P}) > 0$$

for any Hecke curve $\mathbb{P} \subset M$ that is contained in the open subset where $V$ is locally free. But the trivial bundle $E|_\mathbb{P}$ on $\mathbb{P}$ does not contain any subsheaf of positive degree. From this contradiction we conclude that $E$ is semistable.

Since

- $E$ is semistable,
- $c_1(E) = 0$ (5.1) and
- $c_2(E) = 0$ (Proposition 5.3),
the vector bundle $E$ admits a filtration of subbundles

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{\ell-1} \subset E_\ell = E$$

such that for every $1 \leq i \leq \ell$, the quotient $E_i/E_{i-1}$ is a stable vector bundle with $c_1(E_i/E_{i-1}) = 0 = c_2(E_i/E_{i-1})$ [Si, p. 39, Theorem 2] (note that $\ell = 1$ is allowed). This implies that $E$ admits a flat holomorphic connection [Si, p. 40, Corollary 3.10] (set the Higgs field to be zero in [Si, Corollary 3.10]); see [BS, p. 4015, Proposition 3.10] for an extension of this result.

Since $E$ admits a flat holomorphic connection it is given by a representation of $\pi_1(M)$ in $GL(r, \mathbb{C})$, where $r$ is the rank of $E$. On the other hand, $M$ is simply connected [AB, p. 581, Theorem 9.12]. Therefore, the vector bundle $E$ is trivial. \hfill \Box

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