On the Number of Representation of Integers into Quadratic Cubic and Quintic Forms

N.D. Bagis and M.L. Glasser

Abstract

We give formulas for the number of representations of non negative integers into diagonal quadratic forms. A proof of the asymptotic behavior of the function $r_2(x)$ in the case of two squares is also given. Lastly we consider the case cubic and quintic forms.

keywords: Diophantine Equations; Sums of Squares; Asymptotics; Quadratic Forms; Cubic Form; Quintic Form; Special Functions

1 Introduction.

Let $K(x)$ be the complete elliptic integral of the first kind. Some expressions of $K(x)$ are

$$K(x) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - x^2 \sin^2(\theta)}} = \frac{\pi}{2} 2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; x^2 \right),$$

(1)

where $2F_1$ is Gauss hypergeometric function.

The elliptic singular modulus $k = k_r, \ r > 0$ can defined as a restriction of $\lambda(\tau)$-modular function which using Weber functions is (see [3],[4])

$$\lambda(\tau) = 16q \prod_{n=1}^{\infty} \left( \frac{1 + q^{2n}}{1 + q^{2n-1}} \right)^8,$$

(2)

where $q = e^{i\pi \tau}, \ Im(\tau) > 0$.

Hence if $r > 0$ and $\tau = \sqrt{-r}$, the singular modulus is

$$k_r^2 = \lambda(\tau) = \left( \frac{\theta_2(q)}{\theta_3(q)} \right)^4, \ q = e^{i\pi \tau}$$

(3)

with

$$\theta_2(q) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2} \text{ and } \theta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \ |q| < 1$$

(4)
Also \( k = k_r \), \( 0 < k < 1 \) is the solution of the equation

\[
\frac{K(\sqrt{1-k^2})}{K(k_r)} = \sqrt{r}
\]  

(5)

Continuing, traditionally set \( K \) to be \( K = K(k_r) \) (the complete elliptic integral at singular values) and \( K' = K(k'_r) \), where \( k'_r = \sqrt{1-k_r^2} \) is the complementary singular modulus. Then hold the next known Fourier expansion for the Jacobi elliptic function \( \text{dn} \) (see [3] p.51-53)

\[
\text{dn}(q, u) = \frac{\pi}{2K} + \frac{2\pi}{K} \sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}} \cos(2nz) 
\]

(6)

where \( z = \left( \frac{\pi}{2K} \right) u \), in the strip \( |\text{Im}(z)| < \frac{\pi}{2} \text{Im}(\tau) \), \( \tau = i\frac{2K'}{K} \).

A very interesting connection between number theory and the theory of elliptic functions rises from the famous Jacobi theorem

**Theorem 1.** (Jacobi [3])

If \( q = e^{-\pi\sqrt{r}} \), \( r > 0 \), then

\[
\theta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \sqrt{\frac{2K}{\pi}}
\]

(7)

It is very easy to see someone by setting \( u = 0 \) in (5) and using \( \text{dn}(q, 0) = 1 \) and then multiply both sides of (2) by \( 2K/\pi \) that

\[
\frac{2K}{\pi} = 1 + 4 \sum_{m=1}^{\infty} \frac{q^m}{1+q^{2m}} = 1 + 4 \sum_{m=1}^{\infty} q^m \sum_{l=0}^{\infty} (-1)^l q^{2ml} = \]

\[
= 1 + 4 \sum_{m=1}^{\infty} \sum_{l=0}^{\infty} (-1)^l q^{(2l+1)m}.
\]

Writing \( n = (2l+1)m, d = 2l+1 \), whence \( l = (d-1)/2 \). Hence if \( d \) runs through the odd divisors of \( n \) we have

\[
\frac{2K}{\pi} = 1 + 4 \sum_{n=1}^{\infty} \left[ \sum_{d-odd, d|n} (-1)^{\frac{d+1}{2}} \right] q^n 
\]

(8)

Now define \( \delta_0(n) = 1 \), if \( n = 0 \) and \( \delta_0(n) = 4 \sum_{d-odd, d|n} (-1)^{\frac{d+1}{2}} \), if \( n \geq 1 \). Denote by \( r(n) \) the number of representations of \( n \) into the form

\[ n = x^2 + y^2, \ (x, y \in \mathbb{Z}) \]
Then if we consider the fact that
\[
\theta_3(q)^2 = \sum_{n=-\infty}^{\infty} q^n \sum_{m=-\infty}^{\infty} q^{n^2} = \sum_{n,m=-\infty}^{\infty} q^{n^2+m^2} = \sum_{n=0}^{\infty} r(n)q^n
\]
and Jacobi’s Theorem 1 we get \( r(n) = \delta_1(n) \), i.e.

**Theorem 2.** (Jacobi [8])

For \( n = 1, 2, \ldots \) we have
\[
r(n) = 4 \sum_{d \text{ odd, } d|n} (-1)^{\frac{d+1}{2}} \tag{9}
\]
and \( r(0) = 1 \).

The above theorem plays a key role not only to the theory of elliptic functions with the form of (6) but we can use it here in our investigation of quadratic forms of general type.

# 2 Generalizations of Jacobi’s two-square theorem

Theory of quadratic forms has been treated by many great mathematicians such Euler, Gauss, Dirichlet, Liouville, Eisenstein, Glaisher, Ramanujan and many others. Its applications are wide. Some of them are Gauss circle problem in higher dimensions, class number theory, algebraic geometry, evaluations of elliptic functions, Fermat-Wiles theorem, evaluations of series (such of Eisenstein) and many other subjects (see [2-10]).

As many problems in number theory is very elementary in his expression but very hard to handle with. In this article using simple arguments we try to address the problem.

Suppose we have two positive integers \( A, B \) with \( \gcd(A, B) = 1 \), then set \( r_{A,B}(n) \) to be the number of representations of \( n \) into the quadratic form
\[
n = Ax^2 + By^2 \tag{10}
\]
Then
\[
\theta_3(q^A)^2 \theta_3(q^B)^2 = \left( \sum_{n,m=-\infty}^{\infty} q^{4n^2+4m^2} \right)^2 = \left( \sum_{n=0}^{\infty} r_{A,B}(n)q^n \right)^2
\]
But also
\[
\theta_3(q^A)^2 \theta_3(q^B)^2 = \left( \sum_{n=0}^{\infty} r(n)q^{nA} \right) \left( \sum_{m=0}^{\infty} r(m)q^{mB} \right)
\]
\[ = \sum_{n=0}^{\infty} \left( \sum_{kA+lB=n} r(k)r(l) \right) q^n \]

The linear Diophantine equation \( kA + lB = n \) has solutions for all \( n \) since \( \gcd(A,B) = 1 | n \).

Hence if we assume the transformation \( T \), which assigns the Taylor coefficient \( f_n \) of every function \( f(q) \) to the Taylor coefficient \( (\sqrt{T})_n \) of the function \( \sqrt{T(q)} \), then

\[ \left( \sqrt{T} \right)_n = T(f_n) \]

The \( T \) transform can be evaluated using Faa Di Bruno’s Formula (see [1] p.823), which in this case is

\[ T(f_n) = \sum_{m=0}^{n} h_m(f_0) \sum_{j=1}^{r} \prod_{j=1}^{a_j} \frac{f_j^{a_j}}{a_j!} \]  \hspace{1cm} (11)

where the prime on the sum means that we take all the values of numbers (non-negative integers) \( a_j \) such that \( a_1 + 2a_2 + 3a_3 + \ldots + na_n = n \) and \( a_1 + a_2 + a_3 + \ldots + a_n = m \). The function \( h_m \) is \( h_m(x) = (-1)^m x^{1/2-m} \left( \frac{-1}{2} \right)_m \), \( (a)_m = \frac{\Gamma(a+m)}{\Gamma(a)} \), \( m = 1, 2, \ldots \).

Hence with the above notation we can proceed to

**Proposition 1.**
Given two positive integers \( A, B \) with \( \gcd(A,B) = 1 \) the number of the representations of \( n = 1, 2, \ldots \) into the form \( Ax^2 + By^2 \) is exactly

\[ r_{A,B}(n) = T \left( \sum_{kA+lB=n} r(k)r(l) \right) \]  \hspace{1cm} (12)

Note that \( r_{A,B}(0) \) is obviously 1. Also the action of \( T \) transform is directly using where \( f_n \) the sum \( \sum_{kA+lB=n} r(k)r(l) \).

**Proposition 2.**
Given two positive integers \( A, B \) with \( \gcd(A,B) = 1 \), the number of the representations of \( n = 1, 2, \ldots \) into the form \( Ax^2 + By^2 \) is exactly

\[ r_{A,B}(n) = \left[ \frac{1}{n!} \frac{d^n}{dq^n} \left( \sum_{t=0}^{n} \left( \sum_{kA+lB=t} r(k)r(l) \right) q^t \right) \right]_{q=0} \]  \hspace{1cm} (13)

In the same way as above we can prove

**Theorem 3.**
If $A_1, A_2, \ldots, A_N$ are positive integers such that $\gcd(A_1, A_2, \ldots, A_N) = 1$, the number of the representations of $n = 1, 2, \ldots$ into the form $\sum_{k=1}^{N} A_k x_k^2$ is exactly

$$r_2(N, n) = T \left( \sum_{k_1 A_1 + k_2 A_2 + \ldots + k_N A_N = n} r(k_1) r(k_2) \ldots r(k_N) \right)$$

(14)

and

$$r_2(N, n) = \frac{1}{n! d^n} \left( \sum_{t=0}^{n} \left( \sum_{k_1 A_1 + k_2 A_2 + \ldots + k_N A_N = t} r(k_1) r(k_2) \ldots r(k_N) \right) q^t \right)_{q=0}$$

(15)

**Proposition 3.**
Assume the non-homogeneous quadratic form

$$Ax^2 + By^2 + Cx + Dy + E$$

(16)

with $A, B$ positive integers, $C, D, E$ general integers $\gcd(A, B) = 1$ and $C \equiv 0 \mod(2A), D \equiv 0 \mod(2B)$. Then $n$ have exactly

$$r_{A,B} \left( n + \frac{C^2}{4A} + \frac{D^2}{4B} - E \right)$$

representations over (15).

**Proof.**
Write $C = -2L_1 A$ and $D = -2L_2 B$. Then $n = Ax^2 + By^2 + Cx + Dy + E$ is equivalent to $n = A(x - L_1)^2 + B(y - L_2)^2 - AL_1^2 - BL_2^2 + E$ and the number of representations of $n$ over (15) is equivalent to the number of representation of $n + AL_1^2 + BL_2^2 - E = n + \frac{C^2}{4A} + \frac{D^2}{4B} - E$, over $Ax^2 + By^2$. qed

**Application 1.**
Let $A, B, C, D$ be as in Proposition 3, then

$$\sum_{n=-\infty}^{\infty} q^{An^2 + Cn} \cdot \sum_{n=-\infty}^{\infty} q^{Bn^2 + Dn} =$$

$$= 2\pi^{-1} q^{-n_0} K(k_r) \sqrt{m_{A,r} m_{B,r}} + q^{-n_0} \sum_{n=0}^{n_0-1} r_{A,B}(n) q^n$$

(17)

where $n_0 = \frac{C^2}{4A} + \frac{D^2}{4B}$ and $q = e^{-\pi \sqrt{r}}$. The function $m_{n,r} = \frac{K(k_n, r)}{K(k_r)}$ is called multiplier (see [4] pg.136) and takes algebraic values when $n$ is positive integer and $r$ rational.

**Proof.**
From Proposition 3 we have

$$\sum_{n=-\infty}^{\infty} q^{An^2 + Cn} \cdot \sum_{n=-\infty}^{\infty} q^{Bn^2 + Dn} = \sum_{n,m=-\infty}^{\infty} q^{A_2 + Bm^2 + Cn + Dm} = \sum_{n=0}^{\infty} r_{A,B}(n+n_0) q^n =$$


\[
q^{-n_0} \left( \sum_{n=0}^{\infty} r_{A,B}(n + n_0)q^{n+n_0} + \sum_{n=0}^{n_0-1} r_{A,B}(n)q^n - \sum_{n=0}^{n_0-1} r_{A,B}(n)q^n \right) = \\
q^{-n_0} \left( \sum_{n=0}^{\infty} r_{A,B}(n)q^n - \sum_{n=0}^{n_0-1} r_{A,B}(n)q^n \right) = \\
q^{-n_0} \left( \vartheta(q^A)\vartheta(q^B) - \sum_{n=0}^{n_0-1} r_{A,B}(n)q^n \right) = \\
q^{-n_0} \left( \vartheta(q^A)\vartheta(q^B) \right) \sqrt{\sum_{n=0}^{\infty} r_{A,B}(n)q^n}. \text{qed}
\]

**Application 2.**

The equation

\[
k(Ax^2 + By^2 + Cx + Dy + E) + l = n
\]

has \(r = r_{A,B} \left( \frac{n_0}{k} + \frac{C^2}{4A} + \frac{D^2}{4B} - E \right)\) solutions.

In general if \(P_N(x) = \sum_{k=0}^{N} a_k x^k\) and there exists exactly one integer \(n'\) such that \(P_N(n') = n\) then

\[
P_N \left( Ax^2 + By^2 + Cx + Dy + E \right) = n
\]

has

\[
r_{A,B} \left( n' + \frac{C^2}{4A} + \frac{D^2}{4B} - E \right)
\]

integer solutions, (including 0).

Further if the equation \(P_N(n') = n\) have solutions \(n' = n'_1, n'_2, \ldots, n'_s\) all integers with \(s \leq N\), then the number of representations of \(n\) in (19) will be

\[
r = \sum_{i=1}^{s} r_{A,B} \left( n_i' + \frac{C^2}{4A} + \frac{D^2}{4B} - E \right).
\]

In case we have non-integer solution \(n'\) of \(P_N(n') = n\) then we have no contribution of these \(n'\) in the representation (19) and hence in sum (20).

Consider now the function \(\sum_{n=0}^{\infty} q^\nu, \nu \in \mathbb{N} \text{ and } \nu > 2\). It holds

\[
\sum_{n=0}^{\infty} q^{\nu n} = \left( \sum_{t=0}^{\infty} \left( \sum_{\nu'} \frac{1}{\nu' + \nu'' = t} \right) q^t \right)
\]
or

\[ T \left( \sum_{a^\nu+b^\nu=t} 1 \right) = 1_\nu(t) \]  

(23)

where \( P_\nu(n) := 1_\nu(n) = 1 \), when \( n \) is if the form \( m^\nu \), \( m \) positive integer) and 0 else.

**Theorem 4.**
The number of representations of \( n \) into \( x^\nu + y^\nu \), \( x, y \) non-negative integers, is

\[ r_\nu(n) = \sum_{k=0}^{n} P_\nu(k)P_\nu(n-k) \]  

(24)

**Proof.**
From (22) we get that

\[ \sum_{a^\nu+b^\nu=n} 1 = T^{(-1)}(1_\nu(n)) \]  

(25)

where \( T^{(-1)}(f_n) \) is the \( n \)-th Taylor coefficient of \( f^2 \). From Leibnitz formula is

\[ T^{(-1)}(f_n) = \sum_{k+l=n} f_kf_l \]  

(26)

Also, in the case which \( n \) is \( \nu \)-th power of positive integer, then formula (23) reduces to a trivial identity. qed

In general if \( A(n) \) is a polynomial with positive integer coefficients, then assuming the general equation \( A(a) + A(b) = n \), where \( a, b, n \) are non negative integers, this equation have

\[ \sum_{A(a)+A(b)=n} 1 = \sum_{k=0}^{n} G_A(k)G_A(n-k) \]  

(27)

solutions. The function \( G_A(n) \) is such that \( G_A(n) = 1 \) if exists \( m \) positive integer such that \( n = A(m) \) and 0 else.

3 The cubic and quintic representation

In this section we give two formulas similar to that of Jacobi(Theorem 2), for the representation of a positive integer in cubic and quintic forms. For this we make use of the square function i.e. \( S(n) = 1 \), if is perfect square, 0 else. The results of section 2 can generalized to higher order of terms under certain conditions. Historical there are some known results regarding the case of the cubic representation. For example it is known that Diophantine equation

\[ ax^3 - by^3 = n \]  

(28)
for $a, b, n$ integers, has finite number of solutions (see [6]).

From Fermat-Wiles theorem it is known that

$$x^3 + y^3 = z^3$$  \hspace{1cm} (29)

have only trivial solutions i.e. \{x, 0, x\} and \{0, x, x\}.

Also a result of Euler is that equation

$$x^3 + y^3 = z^2$$  \hspace{1cm} (30)

admits parametric solution in integers (see [5] p.578-579).

We proceed by stating and prove our first theorem

**Theorem 5.**

The representations of $n$ into $x^3 + y^3$ form $(x, y)$ non negative integers) is given from

$$r_3(n) = \sum_{d|n} 1 + 2 \cdot \sum_{d|n \atop d^3 - 4n = 0} S \left( \frac{-d^2 + 4n}{3} \right)$$  \hspace{1cm} (31)

where $S(n) = 1$ if $n$ is perfect square and 0 else.

**Proof.**

It holds $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$ hence if we set $u = x + y$ and $v = x^2 - xy + y^2$ the $x, y$ are given from

$$x = \frac{1}{6} \left( 3u - \sqrt{-3u^2 + 12v} \right), \quad y = \frac{1}{6} \left( 3u + \sqrt{-3u^2 + 12v} \right)$$

Hence we get the necessary and sufficient conditions for $u, v$ to determine $n = uv$ that can be expanded into two cubes.

The quintic representation is the same as the cubic. We have

**Theorem 6.**

The representations of $n$ into $x^5 + y^5$ form $(x, y)$ non negative integers) is given from $r_5(0) = 1$ and if $n$ positive integer

$$r_5(n) = - \sum_{d|n \atop d^5 - 16n = 0} 1 + 2 \cdot \sum_{d|n \atop d^5 - 16n \neq 0} x_N \left( \frac{5d^5 - \sqrt{-25d^2 + 10\sqrt{5d^4 + 20\frac{n}{d}}}}{10} \right) S \left( \frac{5d^4 + 20\frac{n}{d}}{d} \right) \times S \left( -25d^2 + 10\sqrt{5d^4 + 20\frac{n}{d}} \right)$$  \hspace{1cm} (32)

where $x_N$ is the characteristic function on positive integers.
4 Representation of Generalized Triangular Numbers

We call $m$-triangular number

$$t_m(n) = \frac{n^2 + mn}{2}, \text{ with } m = 0, 1, 2, \ldots$$  (33)

We interested about the number of representations of a certain non-negative integer $n$ as

$$n = \sum_{k=1}^{N} t_m(x_k) = \sum_{k=1}^{N} \frac{x_k^2 + mx_k^2}{2}, \text{ where } x_k \in \mathbb{Z} \text{ and } m = 0, 1, 2, \ldots$$  (34)

The case of $m = 1$, $N = 2, 3, 4, \ldots$ is the well known representation of $n$ into simple triangular numbers (1-triangular numbers) and have been treated by many various scientists (see[11]). The case $m = 0$ is similar to Jacobi’s two-square theorem. Dropping the notation of $r_{A,B}(n)$ in the above sections, we denote the number of representations of $n$ in (34) by $r_{m,N}(n)$. Also we denote $r(n)$ of (8) as $r_2(n)$ and the symbol $r_N(n)$ is left as in previous sections.

Also for our purpose we recall the definition of certain theta functions studied by Ramanujan (see [14] pg.36):

**Definition 1.**

If $|q| < 1$, then

$$\phi(q) := \sum_{n=-\infty}^{\infty} q^n^2 = \frac{(-q, q^2)_{\infty}(q^2; q^2)_{\infty}}{(q, q^2)_{\infty}(-q^2; q^2)_{\infty}}$$  (35)

$$\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}}$$  (36)

$$f(-q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_{\infty}$$  (37)

Consider the Jacobi triple product formula (see [15] pg.169-172):

$$\sum_{n=-\infty}^{\infty} q^{n^2+zn} = \prod_{n=0}^{\infty} (1 - q^{2n+2})(1 + q^{2n+1-z})(1 + q^{2n+1+z})$$  (38)

where $|q| < 1$.

In case of $z = 2p + 1$, with $p$ non-negative integer we get

$$\sum_{n=-\infty}^{\infty} q^{n^2+(2p+1)n} = f(-q^2) \prod_{n=0}^{\infty} (1 + q^{2(n-p)})(1 + q^{2(n+p)+2}) = $$
\[
= f(-q^2) \prod_{n=0}^{\infty} (1 + q^{2(n-p)})(1 + q^{2(n+p)+2}) = \\
= f(-q^2) \prod_{n=0}^{\infty} (1 + q^{2n}) \prod_{n=0}^{p-1} (1 + q^{2(n-p)}) \prod_{n=0}^{\infty} (1 + q^{2n+2}) \prod_{n=0}^{\infty} (1 + q^{2n+2}) = \\
= 2q^{-p(p+1)} f(-q^2)(-q^2; q^2)_{\infty}
\]

since

\[
\prod_{n=0}^{p-1} \frac{1 + q^{2(n-p)}}{1 + q^{2n+2}} = q^{-p(p+1)}
\]

and

\[
(-q^2; q^2)_{\infty} = \frac{(q^4; q^4)_{\infty}}{(q^2; q^2)_{\infty}}
\]

we get

**Proposition 4.**

If \(|q| < 1\) and \(p = 0, 1, 2, \ldots,\) then

\[
\sum_{n=-\infty}^{\infty} q^{t_{2p+1}(n)} = 2q^{-p(p+1)/2} \frac{f(-q^2)^2}{f(-q)} = 2q^{-p(p+1)/2} \psi(q) \quad (39)
\]

**Proof.**

The first equality follows from above discussion. For the second equality we have

\[
\frac{f(-q^2)^2}{f(-q)} = \frac{(q^2; q^2)_{\infty}}{(q; q)} = \frac{(q^2; q^2)}{(q; q^2)} = \psi(q)
\]

since

\[
(q; q^2)_{\infty} \cdot (-q; q)_{\infty} = 1 \quad (40)
\]

**Proposition 5.**

If \(|q| < 1\) and \(p \in \mathbb{Z}\), then

\[
\sum_{n=-\infty}^{\infty} q^{t_{2p}(n)} = q^{-p^2/2} \phi(q^{1/2}) \quad (41)
\]

**Proof.**

The proof is elementary since

\[
\sum_{n=-\infty}^{\infty} q^{(n+p)^2} = \phi(q)
\]

when \(|q| < 1\) and \(p \in \mathbb{Z}\).
Using the above Propositions we can generalize all results in [11]. We start from the $2p + 1$-triangular numbers and Proposition 1. We immediately have

**Theorem 7.**
If $N$ integer $N \geq 2$ and $\delta_N(n)$ are the representations of $n$ into $N$ 1-triangular numbers

$$n = \sum_{k=1}^{N} \frac{x_k^2 + x_k}{2}$$

(42)

and $r_{2p+1,N}(n)$ is the number of representations of the positive integer $n$ into $N 2p + 1$-triangular numbers

$$n = \sum_{k=1}^{N} \frac{x_k^2 + (2p + 1)x_k}{2}$$

(43)

then

$$r_{2p+1,N}(n) = \delta_N\left(n + \frac{Np(p+1)}{2}\right)$$

(44)

For example we have

**Example 1.**
i) The number of representations of $n$ into the form

$$n = \frac{x^2 + (2p + 1)x}{2} + \frac{y^2 + (2p + 1)y}{2}$$

(45)

is

$$r_{2p+1,2}(n) = 4d_1(8(n + p^2 + p) + 2) - 4d_3(8(n + p^2 + p) + 2)$$

(46)

where

$$d_a(n) = \sum_{d|n, d = a(4)} \frac{1}{d}, \text{ where } a = 1, 3$$

(47)

ii) The number of representations of $n$ into the form

$$n = \frac{x^2 + (2p + 1)x}{2} + \frac{y^2 + (2p + 1)y}{2} + \frac{z^2 + (2p + 1)z}{2}$$

(48)

is

$$r_{2p+1,3}(n) = \sum_{d^2|(n + p(p+1))} R_3\left(\frac{n + p(p+1)}{d}\right)$$

(49)

where

$$R_3(n) = 24 \sum_{r=1}^{[\frac{n}{d}]} \left(\frac{r}{n}\right) \text{ if } n = 1(4)$$

(50)
and

\[ R_3(n) = 8 \sum_{r=1}^{\lceil \frac{n}{r} \rceil} \left( \frac{r}{n} \right) \text{ if } n = 3(4) \]  

(51)

The function \([x] \) is the greatest integer that is smaller or equal to \(x\) and \(\left( \frac{r}{n} \right) \) is the usual Jacobi symbol.

iii)

\[ r_{2p + 1, 4}(n) = \sigma_1(2n + 4p(p + 1) + 1) \]  

(52)

where \(\sigma_\nu(n) = \sum_{d|n} d^\nu\).

Continuing from Section 2 we get evaluations of \(r_{2p, N}(n)\).

Since we know

\[ r_N(n) = T \left( \sum_{k_1 + k_2 + \ldots + k_N = n} r(k_1)r(k_2) \ldots r(k_N) \right) \]  

(53)

then from Proposition 2 we get

**Theorem 8.**

It is known that \(r_N(2n)\) is the number of representations of the positive integer \(n\) into \(N\) 0-triangular numbers

\[ n = \sum_{k=1}^{N} \frac{x_k^2}{2} \]  

(54)

If \(r_{2p, N}(n)\) are the representations of \(n\) into

\[ n = \sum_{k=1}^{N} \frac{x_k^2 + 2px_k}{2} \]  

(55)

then

\[ r_{2p, N}(n) = r_N(2n + Np^2) \]  

(56)

From Jacobi two-square theorem we know that

\[ r_2(n) = \sum_{d=\text{odd, } d|n} (-1)^{\frac{n}{d}}, \]

if \(n = 1, 2, \ldots\) and \(r_2(0) = 1\). Combining the above results we obtain
Proposition 6.
The number of representations \( s_m(n) \) of \( n \) into
\[
x^2 + mx + \frac{y^2 + my}{2}
\]
form is
i) If \( m \) is even
\[
s_m(n) = \sum_{d \mid (n + \frac{m^2}{4})} (-1)^{\frac{d-1}{2}} \quad (58)
\]
ii) If \( m \) is odd
\[
s_m(n) = 4 \sum_{d \mid (m^2 + 4n)} 1 - 4 \sum_{d \mid (m^2 + 4n)} 1 \quad (59)
\]

Proposition 7.
The number of representations of \( n \) as a four \( m \)-triangular numbers
\[
x^2 + mx + \frac{y^2 + my}{2} + \frac{z^2 + mz}{2} + \frac{w^2 + mw}{2}
\]
is
i) If \( m = 2p, p = 0, 1, 2, \ldots \),
\[
r_{2p,4}(n) = r_4(2n + 4p^2) \quad (61)
\]
where (see [12])
\[
r_4(n) = 8 \sum_{d \mid n} d, \text{ if } n \text{ is odd and } 24 \sum_{d \mid n} d, \text{ if } n \text{ is even} \quad (62)
\]
ii) If \( m = 2p + 1, p = 0, 1, 2, \ldots \),
\[
r_{2p+1,4}(n) = \sigma_1 (2n + 4p(p + 1) + 1) \quad (63)
\]
From Proposition 8 we get the next
Corollary.
Every non negative integer \( n \) can represented into sum of four \( m \)-triangular numbers.

Proposition 8.
The number of representation of \( n \) into
\[
n = \frac{x^2 + 2px}{2} + \frac{y^2 + 2py}{2} + \frac{z^2 + 2pz}{2}
\]
(64)
is
\[ r_{2p,3}(n) = r_3(2n + 3p^2) \quad (65) \]
where
\[ r_3(n) = \begin{cases} 
24h(-n), & n \equiv 3(8) \\
12h(-4n), & n \equiv 1, 2, 5, 6(8) \\
0, & n \equiv 7(8)
\end{cases} \quad (66) \]
where \( h(n) \) is the class number of \( n \).

5 Asymptotic Expansion of \( \sum_{n \leq x} r_2(n) \)

In this section we provide asymptotic formulas for the mean value of \( r_2(n) \), using a formula of Hardy (see [16]).

\[ \sum_{n \leq x} r_2(n) = \pi x + x^{1/2} \sum_{n=1}^{\infty} \frac{r_2(n)}{\sqrt{n}} J_1(2\pi \sqrt{nx}) \quad (67) \]

Moreover if \( a, b \in \mathbb{R} \), then we define
\[ M_s(a, b) = \sum_{k-odd, k=1}^{\infty} (-1)^{k+1} \frac{\cos(a + b\sqrt{k})}{k^s} \quad (68) \]
\[ N_s(a, b) = \sum_{k-odd, k=1}^{\infty} (-1)^{k+1} \frac{\sin(a + b\sqrt{k})}{k^s} \quad (69) \]
and
\[ P_s(a, b) = \sum_{n=1}^{\infty} \frac{M_s(a, b\sqrt{n})}{n^s}, \quad Q_s(a, b) = \sum_{n=1}^{\infty} \frac{N_s(a, b\sqrt{n})}{n^s} \quad (70) \]
we prove that

**Theorem 9.**

\[ R(x) = \sum_{n \leq x} r_2(n) - x\pi = \frac{x^{1/4}}{\pi} P_{3/4} \left( \frac{x^{1/4}}{4}, 2\pi \sqrt{x} \right) + \sum_{s=1}^{N} (-1)^s c_1(2s) P_{s+3/4} \left( \frac{x^{1/4}}{4}, 2\pi \sqrt{x} \right) \]
\[ - \sum_{s=0}^{N} (-1)^s c_1(2s+1) Q_{s+5/4} \left( \frac{x^{1/4}}{4}, 2\pi \sqrt{x} \right) + \mathcal{O} \left( c_1(2N) x^{-N-1/2} \right) \quad (71) \]
where \( c_1(m) = (-1)^m \frac{(\pm 1)^m}{m!} \).

**Proof.**

From (67) and (9) we have
\[ \sqrt{x} \sum_{n=1}^{\infty} \frac{r_2(n)}{\sqrt{n}} J_1(2\pi \sqrt{nx}) = \sqrt{x} \sum_{n=1}^{\infty} \left( \sum_{d-odd, d|n} (-1)^{d-1} \right) \frac{1}{\sqrt{n}} J_1(2\pi \sqrt{nx}) = \]

\[ \sum_{n=1}^{\infty} \frac{r_2(n)}{\sqrt{n}} J_1(2\pi \sqrt{nx}) = \]
\[
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^m}{\sqrt{n(2m-1)}} J_1 \left( 2\pi \sqrt{n(2m-1)x} \right) = \\
= \sqrt{x} \sum_{p-\text{odd}, n, p=1}^{\infty} \frac{(-1)^{p+1}}{\sqrt{np}} J_1 \left( 2\pi \sqrt{np}x \right)
\]

But for the function \( J_1(x) \) holds the following asymptotic expansion as \( x \to \infty \)

\[
J_1(x) = \sqrt{\frac{2}{\pi x}} \cos \left( x - \frac{3\pi}{4} \right) \sum_{n=0}^{\infty} \frac{(-1)^n c_1(2n)}{(2x)^{2n}} - \\
- \sin \left( x - \frac{3\pi}{4} \right) \sum_{n=0}^{\infty} \frac{(-1)^n c_1(2n+1)}{(2x)^{2n+1}} \quad (73)
\]

The error due to stopping the summation at any term is the order of the magnitude of that term multiplied by \( 1/x \). Hence using (73) in (72) we get (71).

Setting \( N = 1 \) in (71), we get

\[
R(x) = - \sum_{n, p=1}^{\infty} \frac{1}{n p} \left[ 105(-1)^{p+1} \sin \left( 2\pi \sqrt{np}x + \frac{\pi}{4} \right) - 15(-1)^{p+1} \cos \left( 2\pi \sqrt{np}x + \frac{\pi}{4} \right) \right] + \\
+ \frac{3(-1)^{p+1}}{8\pi (np)^{3/4} \sqrt{x}} \sin \left( 2\pi \sqrt{np}x + \frac{\pi}{4} \right) - \frac{2(-1)^{p+1} \sqrt{x} \cos \left( 2\pi \sqrt{np}x + \frac{\pi}{4} \right)}{(np)^{3/4}} + O \left( x^{-3/4} \right)
\]

For to prove \( R(x) = O \left( x^{1/4} \right) \) is equivalent to show that

\[
S(x) = \sum_{n=1}^{\infty} \frac{r_2(n) \cos \left( 2\pi \sqrt{n}x + \frac{\pi}{4} \right)}{n^{3/4}} = \sum_{n, l=1}^{\infty} \frac{(-1)^{l-1} \cos \left( 2\pi \sqrt{n(2l-1)}x + \frac{\pi}{4} \right)}{(n(2l-1))^{3/4}}
\]

is convergent and bounded in \( x \).

We can write

\[
S(x) = \sum_{n, l=1}^{\infty} \frac{1}{n^{3/4}} \left( \frac{\cos \left( 2\pi \sqrt{n(4l+1)}x + \frac{\pi}{4} \right)}{(4l+1)^{3/4}} - \frac{\cos \left( 2\pi \sqrt{n(4l-1)}x + \frac{\pi}{4} \right)}{(4l-1)^{3/4}} \right)
\]

Also if we set

\[
\theta(n, l, x) = \cos \left( 2\pi \sqrt{n(4l-1)}x + \frac{\pi}{4} \right) - \cos \left( 2\pi \sqrt{n(4l+1)}x + \frac{\pi}{4} \right)
\]

Then because

\[
\lim_{l \to \infty} \frac{1}{l^{3/4+1}} \left( \frac{1}{(4l+1)^{3/4}} - \frac{1}{(4l-1)^{3/4}} \right) = C_0 = \text{const},
\]

we have
exist $C > 0$ and $l_0 > 0$ such that for every $l > l_0$:

$$\left| \frac{1}{(4l+1)^{3/4}} - \frac{1}{(4l-1)^{3/4}} \right| \leq \frac{C}{(4l+1)^{3/4+1}}$$

Hence

$$\frac{1}{(4l+1)^{3/4}} - \frac{1}{(4l-1)^{3/4}} = O \left( \frac{1}{(4l+1)^{3/4+1}} \right) \quad (76)$$

and

$$S(x) = \sum_{n,l=1}^{\infty} \frac{\cos \left( 2\pi \sqrt{n(4l+1)x + \frac{x}{4}} \right)}{n^{3/4}} \left[ \frac{1}{(4l+1)^{3/4}} - \frac{1}{(4l-1)^{3/4}} \right] +$$

$$+ \sum_{n,l=1}^{\infty} \frac{\theta(n,l,x)}{n^{3/4}(4l+1)^{3/4}}.$$

But if we set $\sigma'_\nu(n) = \sum_{d \equiv 1(4), d|n} d^\nu$, then (since all sequences are bounded above (see [9] pg.135-136)):

$$\left| \sum_{n,l=1}^{\infty} \frac{\cos \left( 2\pi \sqrt{n(4l+1)x + \frac{x}{4}} \right)}{n^{3/4}} \left[ \frac{1}{(4l+1)^{3/4}} - \frac{1}{(4l-1)^{3/4}} \right] \right| =$$

$$= O \left( \sum_{n,l=1}^{\infty} \frac{\cos \left( 2\pi \sqrt{n(4l+1)x + \frac{x}{4}} \right)}{n^{3/4}(4l+1)^{3/4+1}} \right) =$$

$$= O \left( \sum_{n,l=1}^{\infty} \frac{n \cos \left( 2\pi \sqrt{n(4l+1)x + \frac{x}{4}} \right)}{(n(4l+1))^{3/4+1}} \right) =$$

$$= O \left( \sum_{n=1}^{\infty} \sigma'_1(n) \cos \left( 2\pi \sqrt{n(x + \frac{x}{4})} \right) \right) =$$

$$= O \left( \sum_{n=1}^{\infty} \frac{n \log \log(n) \cos \left( 2\pi \sqrt{n(x + \frac{x}{4})} \right)}{n^{1+\delta}} \right)$$

Using Dirichlet’s test (see [13] pg.347) the last series is uniformly convergent, when

$$\sum_{n=1}^{N} \frac{\cos \left( 2\pi \sqrt{nx + \frac{x}{4}} \right)}{n^{3/4-\delta}} \quad (77)$$

are uniformly bounded for every $\delta > 0$ small enough.

Also

$$\sum_{n,l=1}^{N} \frac{\theta(n,l,x)}{n^{3/4}(4l+1)^{3/4}} = O \left( \sum_{n,l=1}^{N} \frac{\cos \left( 2\pi \sqrt{n(4l+1)x + \frac{x}{4}} \right)}{(n(4l+1))^{3/4}} \right) =$$
\[ \sum_{n=1}^{N} \frac{\sigma_0(n) \cos \left( \frac{2\pi \sqrt{nx}}{n^{3/4}} + \frac{\pi}{4} \right)}{n^{3/4}} = O \left( \sum_{n=1}^{N} \frac{1}{n^{3/4}} \cos \left( \frac{2\pi \sqrt{nx}}{n^{3/4}} + \frac{\pi}{4} \right) \right) \]

Since it is known that exist \( C_0 \) such that \( \sigma_0(n) \leq C_0 n^\epsilon \), for \( 0 < \epsilon < \delta \). Hence \( S(x) = O(1) \) and holds the following

**Theorem 10.**

When \( x \) is large then

\[ \sum_{n \leq x} r_2(n) = x \pi + O \left( x^{1/4} \right) \quad (78) \]

**References**

[1]: M. Abramowitz and I.A. Stegun. 'Handbook of Mathematical Functions'. Dover Publications, New York. (1972)

[2]: T. Apostol. 'Introduction to Analytic Number Theory'. Springer Verlang, New York, Berlin, Heidelberg, Tokyo, (1974)

[3]: J.V. Armitage W.F. Eberlein. 'Elliptic Functions'. Cambridge University Press. (2006)

[4]: J.M. Borwein and P.B. Borwein. 'Pi and the AGM'. John Wiley and Sons, Inc. New York, Chichester, Brisbane, Toronto, Singapore. (1987)

[5]: L.E. Dickson. 'History of the Theory of Numbers, Vol2: Diophantine Analysis'. Dover, New York, (2005)

[6]: G.H. Hardy. 'Ramanujan Twelve Lectures on Subjects Suggested by his Life and Work, 3rd ed.' Chelsea. New York, (1999)

[7]: M.D. Hirschhor. 'Three classical results on representations of a number'. Seminaire Lotharingien de Combinatoire, (42) (1999)

[8]: C.G.J. Jacobi. 'Fundamenta Nova Functionum Ellipticarum'. Werke I, 49-239. (1829)

[9]: William J. LeVeque. 'Fundamentals of Number Theory'. Dover Publiccations. New York. (1996)

[10]: E.T. Whittaker and G.N. Watson. 'A course on Modern Analysis'. Cambridge U.P. (1927)

[11]: Ken Ono. 'Representations of Integers as Sums of Squares'.Journal of Number Theory. (95), 253-258. (2002)

[12]: Ila Varma. 'Sums of Squares, Modular Forms, and Hecke Characters'. Master thesis. Mathematisch Institut, Universiteit Leiden. June 18 (2010).

[13]: Konrad Knopp. 'Theory and Applications of Infinite Series'. Dover Publications, Inc. New York. (1990).

[14]: Bruce C. Berndt. 'Ramanujan’s Notebooks Part III'. Springer Verlag, New York (1991)

[15]: G.E. Andrews, Number Theory. Dover Publiccations, New York, 1994.
[16] G.H. Hardy. 'On the expression of a number as the sum of two squares'. Quart. J. Math. (Oxford) 46 (1915), 263283.