A note on non-coercive first order Mean Field Games with analytic data

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Abstract

We study first order evolutive Mean Field Games whose operators are non-coercive. This situation occurs, for instance, when some directions are “forbidden” to the generic player at some points. Under some regularity assumptions, we establish existence of a weak solution of the system. Mainly, we shall describe the evolution of the population’s distribution as the push-forward of the initial distribution through a flow, suitably defined in terms of the underlying optimal control problem.

Keywords: Mean Field Games, non-coercive first order Hamilton-Jacobi equations, continuity equation.

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1 Introduction

In this paper we study the following Mean Field Game (briefly, MFG)

\[
\begin{align*}
(i) & \quad -\partial_t u + H(x, Du) = F(x, t, m) \quad \text{in } \mathbb{R}^d \times (0, T) \\
(ii) & \quad \partial_t m - \text{div}(m \partial_p H(x, Du)) = 0 \quad \text{in } \mathbb{R}^d \times (0, T) \\
(iii) & \quad m(x, 0) = m_0(x), u(x, T) = G(x, m(T)) \quad \text{on } \mathbb{R}^d,
\end{align*}
\]

where, if \( p = (p_1, \ldots, p_d) \) and \( x = (x_1, \ldots, x_d) \), the functions \( H(x, p) \) is

\[
H(x, p) = \frac{1}{2} |pB(x)|^2
\]

with \( B(x) = B(x_1, \ldots, x_d) \) equal to the \( d \times d \) matrix

\[
\begin{pmatrix}
h_{11} & 0 & 0 & 0 & \cdots & 0 \\
h_{21}(x_1) & h_{22}(x_1) & 0 & 0 & \cdots & 0 \\
h_{31}(x_1, x_2) & h_{32}(x_1, x_2) & h_{33}(x_1, x_2) & 0 & \cdots & 0 \\
& \cdots & \cdots & \cdots & \cdots & \cdots \\
h_{d1}(x_1, \ldots, x_{d-1}) & h_{d2}(x_1, \ldots, x_{d-1}) & h_{d3}(x_1, \ldots, x_{d-1}) & \cdots & \cdots & h_{dd}(x_1, \ldots, x_{d-1})
\end{pmatrix}
\]

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where \( h_{ij} = h_{ij}(x_1, x_2, \ldots, x_{i-1}) \) and \( h_{11} \) is a non-null constant. From (1.2) and (1.3) we have \( \partial_p H(x, p) = p B(x) B^T(x) \).

We shall assume that the functions \( h_{ij}(x) \) are analytic, bounded, possibly vanishing and that \( F \) is a nonlocal strongly regularizing coupling (see assumptions below). It is our intention to study much less regular \( B \) and \( F \) in a forthcoming paper [18].

These MFG systems arise when the dynamics of the generic player are deterministic and, in some points, may have a “forbidden” direction (for instance, when the \( h_{ij} \) vanish); actually, if the evolution of the whole population’s distribution \( m \) is given, each agent wants to choose the control \( \alpha = (\alpha_1, \ldots, \alpha_d) \) in the set

\[
A = \{ \alpha : [t, T] \rightarrow \mathbb{R}^d, \alpha \in L^2([t, T]) \}
\]

in order to minimize the cost

\[
\int_t^T \left[ \frac{1}{2} |\alpha(\tau)|^2 + f(x(\tau), \tau, m) \right] d\tau + g(x(T), m(T))
\]

where, in \([t, T]\), its dynamics \( x(\cdot) \) are governed by

\[
\begin{cases}
x'(s) = \alpha(s) B^T(x) \\
x(t) = x.
\end{cases}
\]

We study a problem with dynamics as in (1.6) because the structure of the matrix \( B \) in (1.3) allows us to simplify the calculations in Section 2. Moreover this class of problems encompasses the Grushin and the Carnot type model (as the Heisenberg one, see Examples 1.2 and 1.3 below).

Let us recall that the MFG theory studies Nash equilibria in games with a huge number of (“infinitely many”) rational and indistinguishable agents. This theory started with the pioneering papers by Lasry and Lions [14, 15, 16] and by Huang, Malhamé and Caines [13]. A detailed description of the achievements obtained in these years goes beyond the scope of this paper; we just refer the reader to the monographs [1, 6, 3, 11, 12]. As far as we know, degenerate MFG systems have been poorly investigated up to now. Dragoni and Feleqi [10] studied a second order (stationary) system where the principal part of the operator fulfills the Hörmander condition; moreover, Cardaliaguet, Graber, Porretta and Tonon [7] tackled degenerate second order systems with coercive (and convex as well) first order operators. Hence, these results cannot be directly applied to the non-coercive problem (1.1). On the other hand, the existence of a solution to (1.1) can be obtained by the vanishing viscosity approach as in [6, Sect. 4.4] (see also [16, Sect. 2.5]). Unfortunately, the vanishing viscosity method seems to give no interpretation for the solution to the system.

Therefore, we shall pursue a different approach in order to obtain more detailed information on the evolution of the population’s density. In particular, following the arguments in [6, Sect. 4.3], we are able to describe this evolution as the push-forward of the distribution at the initial time through a flow which is suitably defined in terms of the optimal control problem underlying (1.1). As a matter of facts, the non-coercivity of \( H \) prevents from applying directly the arguments of [6, Sect. 4.3]. The well-posedness and several properties of this flow may be considered among the main novelties of this paper. In particular, we prove that the optimal trajectories of the control problem associated to the Hamilton-Jacobi equation (1.1)-(i) are unique after the starting time (see Proposition 2.1 below); as far as we know this property has never been tackled before for
degenerate dynamics as \((1.6)\). The proof of this property is the only point where our very strong regularity assumptions play a role.

We now list our notations and the assumptions that will hold throughout the whole paper, we give the definition of (weak) solution to system \((1.1)\) and we state the existence result for system \((1.1)\).

**Notations and Assumptions.** We denote by \(P^1\) the space of Borel probability measures on \(\mathbb{R}^d\) with finite first order moment, endowed with the Kantorovich-Rubinstein distance \(d_1\). Throughout this paper, we shall require the following hypotheses:

1. \((H1)\) for every \(m \in C([0, T], P^1)\), the function \(F(\cdot, \cdot, m)\) is analytic; \(G\) is a real-valued function, continuous on \(\mathbb{R}^d \times P^1\);

2. \((H2)\) there exists \(C\) such that \(\|F(\cdot, t, m_1)\|_{C^2}, \|G(\cdot, m_2)\|_{C^2} \leq C, \forall m_1 \in C([0, T], P^1), m_2 \in P^1;\)

3. \((H3)\) the functions of the matrix \(B\), \(h_{ij} : \mathbb{R}^{i-1} \to \mathbb{R}\) are analytic, with \(\|h_{ij}\|_{C^2} \leq C;\)

4. \((H4)\) the initial distribution \(m_0\) is absolutely continuous with a density (that, with a slight abuse of notation, we still denote by \(m_0\)) bounded and with compact support.

**Example 1.1** A coupling \(F\) satisfying the above assumptions is of the form

\[
F(x, t, m) = V(x, t, (\rho \ast m)(x, t)),
\]

where \(\rho\) and \(V\) are analytic functions with \(\|V\|_{C^2}\) bounded. This assumption is very restrictive and it plays a role only in the proof of the uniqueness of optimal trajectories; we refer the reader to a forthcoming paper \([18]\) for much weaker hypotheses.

We now introduce our definition of solution of the MFG system \((1.1)\) and state the main result concerning its existence.

**Definition 1.1** The pair \((u, m)\) is a solution of Problem \((1.1)\) if:

1) \((u, m) \in W^{1,\infty}(\mathbb{R}^d \times [0, T]) \times L^1(\mathbb{R}^d \times [0, T]);\)

2) Equation \((1.1)-(i)\) is satisfied by \(u\) in the viscosity sense;

3) Equation \((1.1)-(ii)\) is satisfied by \(m\) in the sense of distributions.

Here below we state the main result of this paper.

**Theorem 1.1** Under the above assumptions, system \((1.1)\) has a solution \((u, m)\) as in Definition \((1.1)\).

For the sake of simplicity, in the following sections we prove Theorem \((1.1)\) in the particular case \(d = 2, h_{11} = 1\) and \(h_{21}(x_1) \equiv 0\). Denoting \(h(x_1) := h_{22}(x_1)\) the matrix \(B\) is

\[
(1.7) \quad B(x) = \begin{pmatrix} 1 & 0 \\ 0 & h(x_1) \end{pmatrix}
\]
and the dynamic system (1.6) becomes:

\begin{align*}
(1.8) & \quad \left\{ \begin{array}{ll}
  x'_1(s) = \alpha_1(s) & x_1(t) = x_1 \\
  x'_2(s) = h(x_1(s))\alpha_2(s) & x_2(t) = x_2.
\end{array} \right.
\end{align*}

In this case the Mean Field Game (1.1) is

\begin{align*}
(1.9) & \quad \left\{ \begin{array}{ll}
  \hspace{2cm} (i) \ -\partial_t u + \frac{1}{2} |Du|^2 = F(x, t, m) & \text{in } \mathbb{R}^2 \times (0, T) \\
  \hspace{2cm} (ii) \ \partial_m m - \text{div}_B(mDu) = 0 & \text{in } \mathbb{R}^2 \times (0, T) \\
  \hspace{2cm} (iii) \ m(x, 0) = m_0(x), u(x, T) = G(x, m(T)) & \text{on } \mathbb{R}^2,
\end{array} \right.
\end{align*}

where, for \( x = (x_1, x_2) \in \mathbb{R}^2 \), \( \phi : \mathbb{R}^2 \to \mathbb{R} \) and \( \Phi : \mathbb{R}^2 \to \mathbb{R} \) differentiable, we set

\[ D_B \phi(x) := (\partial_{x_1} \phi(x), h(x_1)\partial_{x_2} \phi(x)), \quad \text{div}_B \Phi(x) := \partial_{x_1} \Phi_1(x) + h(x_1)\partial_{x_2} \Phi_2(x). \]

In this case we easily see that the direction along \( x_2 \) is forbidden when \( h(x_1) \) has zero value.

**Example 1.2** Examples of metric defined by (1.7), are the Grushin type problems, with analytic and bounded \( h \), as \( h(x_1) = \sin(x_1) \) or \( h(x_1) = \frac{x_1}{\sqrt{1+x_1^2}} \) (see [17]).

**Example 1.3** For \( d = 3 \), the matrix \( B(x) \) can be of Heisenberg type (see [3])

\[ B(x) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
h_{31}(x_1, x_2) & h_{32}(x_1, x_2) & h_{33}(x_1, x_2)
\end{pmatrix} \]

and the corresponding dynamics are
\[
\left\{ \begin{array}{ll}
  x'_1(s) = \alpha_1(s) \\
  x'_2(s) = \alpha_2(s) \\
  x'_3(s) = h_{31}(x_1(s), x_2(s))\alpha_1(s) + h_{32}(x_1(s), x_2(s))\alpha_2(s) + h_{33}(x_1(s), x_2(s))\alpha_3(s).
\end{array} \right.
\]

This paper is organized as follows: in section 2 we will find some properties of the solution of the Hamilton-Jacobi equation (1.9)-(i) with fixed \( m \). In section 3 we study the continuity equation (1.9)-(ii) where \( u \) is the solution of a Hamilton-Jacobi equation found in the previous section. In section 4 we give the proof of the existence of the solution to system (1.9). Finally, in the Appendix, we state the problem and the results for the general d-dimensional case (1.1) and we introduce the notion of \( B \)-differentiability with the main properties of \( B \)-differentiable functions.

## 2 The Hamilton-Jacobi equation

The aim of this section is to study the Hamilton-Jacobi equation (1.9)-(i) with \( m \) fixed, namely

\begin{align*}
(2.1) & \quad \left\{ \begin{array}{ll}
  -\partial_t u + \frac{1}{2}((\partial_{x_1} u)^2 + h(x_1)^2(\partial_{x_2} u)^2) = f(x, t) & \text{in } \mathbb{R}^2 \times (0, T), \\
  u(x, T) = g(x) & \text{on } \mathbb{R}^2,
\end{array} \right.
\end{align*}

where \( f(x, t) := F(x, t, m) \) and \( g(x) := G(x, m(T)) \); hence, our assumptions (H1)-(H3) read

\begin{align*}
(2.2) & \quad h \text{ and } f \text{ are analytic with } \|h\|_{C^2} + \|f\|_{C^2} + \|g\|_{C^2} < C.
\end{align*}
In particular, we shall prove several regularity properties of the solution (Lipschitz continuity and semiconcavity) and mainly the uniqueness of optimal trajectories. In our opinion this uniqueness result has its own interest because, as far as we know, this property has never been tackled before for non-coercive dynamics as \[ (1.6) \]

The solution \( u \) of \( (2.1) \) can be represented as the value function of the following control problem. Let the set of the admissible controls \( \mathcal{A} \) be defined as in \[ (1.4) \] with \( d = 2 \) and, for each control \( \alpha \in \mathcal{A} \), let \( x(\cdot) \) be the trajectory given by \[ (1.8) \]; we define the cost

\[
J_t(\alpha(\cdot), x(\cdot)) := \int_t^T \frac{1}{2}|\alpha(\tau)|^2 + f(x(\tau), \tau) d\tau + g(x(T)).
\]

**Definition 2.1** The value function for the cost \( J_t \) in \( (2.3) \) is

\[
u(x, t) := \inf \left\{ J_t(\alpha(\cdot), x(\cdot)) : \alpha(\cdot) \in \mathcal{A}, (x(\cdot), \alpha(\cdot)) \text{ satisfying } (1.8) \right\}.
\]

**Lemma 2.1** Under assumptions \( (2.2) \), the value function \( u(x, t) \) is the unique bounded uniformly continuous viscosity solution to problem \( (2.1) \).

**Proof.** Arguing as in [2, Proposition III.2.5], one can easily prove the Dynamical Programming Principle: for every \((x, t) \in \mathbb{R}^2 \times [0, T]\) and \( s \in [t, T] \), there holds

\[
u(x, t) = \inf_{\alpha(\cdot) \in \mathcal{A}} \left\{ u(x_{\alpha,x,t}(s), s) + \int_t^s \frac{1}{2}|\alpha(\tau)|^2 + f(x_{\alpha,x,t}(\tau), \tau) d\tau \right\}
\]

where \( x_{\alpha,x,t}(\cdot) \) is the solution in \([t, T]\) of \( (1.8) \). By this property, applying classical results uniqueness (see, for example, [5, eq. (7.40) and Thm. 7.4.14]) for the Hamiltonian

\[
H(x, p) := \max_{a = (a_1, a_2) \in \mathbb{R}^2} \left\{ a_1p_1 + h(x_1)a_2p_2 - \frac{|a|^2}{2} - f(x, t) - \frac{(p_1)^2 + (h(x_1)p_2)^2}{2} - f(x, t), \right\}
\]

we obtain the statement. \( \square \)

**Lemma 2.2** Under assumptions \( (2.2) \), there hold:

1. The function \( u \) defined in \( (2.4) \) is bounded in \( \mathbb{R}^2 \times [0, T] \),
2. \( u(x, t) \) is Lipschitz continuous with respect to the spatial variable \( x \),
3. \( u(x, t) \) is Lipschitz continuous with respect to the time variable \( t \).

**Proof.** In this proof, \( C_T \) will denote a constant which may change from line to line but it always depends only on the constants in the assumptions (especially the Lipschitz constants of \( f \) and \( g \)) and on \( T \).

1. Taking as admissible control the law \( \alpha = 0 \), from the representation formula \( (2.4) \), using the boundedness of \( f \) and \( g \) we have \(|u(x, t)| \leq C_T\).
2. Let \( t \) be fixed. We follow the proof of [6, Lemma 4.7]. Let \( \alpha^\varepsilon \) be an \( \varepsilon \)-optimal control for \( u(x_1, x_2, t) \) i.e.,

\[
u(x_1, x_2, t) + \varepsilon \geq \int_t^T \frac{1}{2}|\alpha^\varepsilon(s)|^2 + f(x(s), s) ds + g(x(T))
\]
where \( x(\cdot) \) satisfies the following dynamics

\[
\forall s \in [t, T] \quad \begin{cases} 
  x_1'(s) = \alpha_1^x(s), \\
  x_2'(s) = h(x_1'(s))\alpha_2^x(s), \\
  x_1(t) = x_1, \\
  x_2(t) = x_2.
\end{cases}
\]

From the boundedness of \( u \) (established in Point 1) and our assumptions, there exists a constant \( C_T \) such that \( \|\alpha^x\|_{L^2((t, T))} \leq C_T \).

We consider the path \( x^*(s) \) starting from \( y = (y_1, y_2) \), with the velocity as in (2.6). Hence

\[
x_1^*(s) = y_1 + \int_t^s \alpha_1^x(\tau) \, d\tau = y_1 - x_1 + x_1(s)
\]

\[
x_2^*(s) = y_2 + \int_t^s h(y_1 - x_1 + x_1(\tau))\alpha_2^x(\tau) \, d\tau
\]

\[
= y_2 - x_2 + x_2(s) + \int_t^s h(y_1 - x_1 + x_1(\tau))\alpha_2^x(\tau) - h(x_1(\tau))\alpha_2^x(\tau) \, d\tau.
\]

Using the Lipschitz continuity of \( f \) and \( h \) and the boundedness of \( h \) we get

\[
f(x^*(s), s) =
\]

\[
f\left(y_1 - x_1 + x_1(s), y_2 - x_2 + x_2(s) + \int_t^s h(y_1 - x_1 + x_1(\tau))\alpha_2^x(\tau) - h(x_1(\tau))\alpha_2^x(\tau) \, d\tau, s\right) \leq
\]

\[
\leq f(x_1(s), x_2(s), s) + L|y_1 - x_1| + L|y_2 - x_2| + L'|y_1 - x_1| \int_t^s |\alpha_2^x(\tau)| \, d\tau
\]

\[
\leq f(x_1(s), x_2(s), s) + L|y_1 - x_1| + L|y_2 - x_2| + L'|y_1 - x_1| T^\frac{1}{2} \left(\int_t^s (\alpha_2^x(s))^2 \, ds\right)^{\frac{1}{2}}.
\]

By the same calculations for \( g \) and substituting inequality (2.5) in

\[
u(y_1, y_2, t) \leq \int_t^T \frac{1}{2} |\alpha^x(s)|^2 + f(x^*(s), s) \, ds + g(x^*(T)),
\]

we get

\[
u(y_1, y_2, t) \leq u(x_1, x_2, t) + C_T(|y_2 - x_2| + |y_1 - x_1|).
\]

Reversing the role of \( x \) and \( y \) we get the result.

3. We follow the same arguments as those in the proof of [6, Lemma 4.7]; to this end, we recall that \( h \) is bounded and we observe that there holds

\[
|x(s) - x| \leq C(s - t)\|\alpha\|_{\infty}
\]

since \( \alpha \) is bounded (see (2.11)–(2.12) in Lemma 2.4 below).

\[
\square
\]

**Lemma 2.3** Under assumptions (2.2), the value function \( u(x, t) \), defined in (2.1), is semi-concave with respect to the variable \( x \).
Proof. For any \(x, y \in \mathbb{R}^2\) and \(\lambda \in [0, 1]\), consider \(x_\lambda := \lambda x + (1-\lambda)y\). Let \(\alpha\) be an \(\varepsilon\)-optimal control for \(u(x_\lambda, t)\); we set

\[
x_\lambda(s) = (x_{\lambda,1}(s), x_{\lambda,2}(s)) := \left( x_{\lambda,1} + \int_t^s \alpha_1(\tau) \, d\tau, \ x_{\lambda,2} + \int_t^s h(x_{\lambda,1}(\tau))\alpha_2(\tau) \, d\tau \right).
\]

Let \(x(s)\) and \(y(s)\) satisfy (1.8) with initial condition respectively \(x\) and \(y\) still with the control \(\alpha\), \(\varepsilon\)-optimal for \(u(x_\lambda, t)\). We have to estimate \(\lambda u(x, s) + (1-\lambda)u(y, s)\) in terms of \(u(x_\lambda, t)\). To this end, arguing as in the proof of [6] Lemma 4.7, we have to estimate the terms \(\lambda f(x(s), s) + (1-\lambda)f(y(s), s)\) and \(\lambda g(x(T)) + (1-\lambda)g(y(T))\).

We explicitly provide the calculations for the second component \(x_2(s)\) since the calculations for \(x_1(s)\) are the same as in [6]. We have

\[
x_2(s) = x_2 + \int_t^s h(x_1(\tau))\alpha_2(\tau) \, d\tau
\]

and analogously for \(y_2(s)\). For the sake of brevity we provide the explicit calculations only for \(f\) and we omit the analogous ones for \(g\); we write \(f(x_1, x_2) := f(x_1, x_2, s)\). We have

\[
\lambda f(x(s)) + (1-\lambda)f(y(s)) = \lambda f(x_1(s), x_2, s) + x_2 - x_{\lambda,2} + \int_t^s (h(x_1(\tau)) - h(x_{\lambda,1}(\tau)))\alpha_2(\tau) \, d\tau
\]

\[
+ (1-\lambda)f(y_1(s), x_{\lambda,2}(s) + y_2 - x_{\lambda,2} + \int_t^s (h(y_1(\tau)) - h(x_{\lambda,1}(\tau)))\alpha_2(\tau) \, d\tau).
\]

In the Taylor expansion of \(f\) centered in \(x_\lambda(s)\) the contribution of the first variable can be dealt with as in [6]. Assuming without any loss of generality \(x_1 = y_1\), the contribution of the second variable gives

\[
\lambda f(x(s)) + (1-\lambda)f(y(s)) = f(x_\lambda(s)) + \partial_{x_2} f(x_\lambda(s)) \left( \lambda(x_2 - x_{\lambda,2}) + (1-\lambda)(y_2 - x_{\lambda,2})
\right.
\]

\[
+ \lambda \int_t^s (h(x_1(\tau)) - h(x_{\lambda,1}(\tau)))\alpha_2(\tau) \, d\tau
\]

\[
+ (1-\lambda) \int_t^s (h(y_1(\tau)) - h(x_{\lambda,1}(\tau)))\alpha_2(\tau) \, d\tau + R,
\]

where \(R\) is the error term of the expansion, namely

\[
(2.7) \quad R = \lambda \frac{\partial^2_{x_2} f(\xi_1)}{2} \left( x_2 - x_{\lambda,2} + \int_t^s (h(x_1(\tau)) - h(x_{\lambda,1}(\tau)))\alpha_2(\tau) \, d\tau \right)^2
\]

\[
+ (1-\lambda) \frac{\partial^2_{x_2} f(\xi_2)}{2} \left( y_2 - x_{\lambda,2} + \int_t^s (h(y_1(\tau)) - h(x_{\lambda,1}(\tau)))\alpha_2(\tau) \, d\tau \right)^2,
\]

for suitable \(\xi_1, \xi_2 \in \mathbb{R}^2\).

Since \(\lambda(x_2 - x_{\lambda,2}) + (1-\lambda)(y_2 - x_{\lambda,2}) = 0\), we get

\[
(2.8) \quad \lambda f(x(s)) + (1-\lambda)f(y(s)) = f(x_\lambda(s)) + \partial_{x_2} f(x_\lambda(s)) \int_t^s I(\tau)\alpha_2(\tau) \, d\tau + R,
\]

\[
I(t) = \frac{1}{2} \left( \lambda^2 \frac{\partial^2_{x_2} f(\xi_1)}{2} \left( x_2 - x_{\lambda,2} + \int_t^s (h(x_1(\tau)) - h(x_{\lambda,1}(\tau)))\alpha_2(\tau) \, d\tau \right)^2
\]

\[
+ (1-\lambda)^2 \frac{\partial^2_{x_2} f(\xi_2)}{2} \left( y_2 - x_{\lambda,2} + \int_t^s (h(y_1(\tau)) - h(x_{\lambda,1}(\tau)))\alpha_2(\tau) \, d\tau \right)^2.
\]

\[
\]
with \( I(\tau) := -h(x_{\lambda,1}(\tau)) + \lambda h(x_1(\tau)) + (1 - \lambda)h(y_1(\tau)) \). Now, our aim is to estimate \( I(\tau) \).

Since \( x_{\lambda,1}(\tau) = \lambda x_1(\tau) + (1 - \lambda)y_1(\tau) \), \( x_1(\tau) - x_{\lambda,1}(\tau) = (1 - \lambda)(x_1 - y_1) \) and \( y_1(\tau) - x_{\lambda,1}(\tau) = \lambda(y_1 - x_1) \), the Taylor expansion for \( h \) centered in \( x_{\lambda,1}(\tau) \) yields

\[
I(\tau) = \frac{1}{2} (1 - \lambda)\lambda (y_1 - x_1)^2 \left[ (1 - \lambda)h''(\xi) + \lambda h''(\tilde{\xi}) \right],
\]

for suitable \( \xi, \tilde{\xi} \in \mathbb{R} \). Our assumption (2.2) entails

\[
|I(\tau)| \leq (1 - \lambda)\lambda (y_1 - x_1)^2.
\]

Replacing the inequality above in (2.8), we obtain

(2.9) \quad \lambda f(x_2(s)) + (1 - \lambda)f(y_2(s)) \leq f(x_{\lambda,2}(s)) + C^2 T (1 - \lambda) (y_1 - x_1)^2 + R.

Let us now estimate the error term \( R \) in (2.7). We have

\[
\left( x_2 - x_{\lambda,2} + \int_t^s (h(x_1(\tau)) - h(x_{\lambda,1}(\tau))) \alpha_2(\tau) d\tau \right)^2
\]

\[
\leq 2(x_2 - x_{\lambda,2})^2 + 2 \left( \int_t^s (h(x_1(\tau)) - h(x_{\lambda,1}(\tau))) \alpha_2(\tau) d\tau \right)^2
\]

\[
\leq 2(1 - \lambda)^2 (x_2 - y_2)^2 + 2C(1 - \lambda)^2 (x_1 - y_1)^2 \leq C(1 - \lambda)^2 |x - y|^2
\]

and, analogously

\[
\left( y_2 - x_{\lambda,2} + \int_t^s (h(y_1(\tau)) - h(x_{\lambda,1}(\tau))) \alpha_2(\tau) d\tau \right)^2 \leq C \lambda^2 |x - y|^2
\]

Then, replacing these two inequalities in (2.7), we infer

(2.10) \quad R \leq C(1 - \lambda)\lambda |x - y|^2.

Taking into account (2.10) and (2.9), we get the semiconcavity of \( u \). \( \square \)

For any \((x, t) \in \mathbb{R}^2 \times [0, T]\), we denote by \( A(x, t) \) the set of optimal controls of the minimization problem (2.4) whose trajectories are governed by (1.8). As in [6], we easily see that if \((t_n, x_n) \to (x, t)\) and \( \alpha_n \in A(x_n, t_n) \), then, possibly passing to some subsequence, \( \alpha_n \) weakly converges in \( L^2 \) to some \( \alpha \in A(x, t) \).

The following result gives the optimality condition:

Lemma 2.4 Let \( \alpha := (\alpha_1, \alpha_2) \in A(x, t) \). Let \( x(\cdot) := (x_1(\cdot), x_2(\cdot)) \) be the associated optimal trajectory governed by (1.8) under assumptions (2.2); then the following properties hold:

1. The optimal control \( \alpha \) satisfies

   (2.11) \quad \forall s \in [t, T], \quad \begin{cases} 
   \alpha_1(s) = p_1(s) \\
   \alpha_2(s) = h(x_1(s))p_2(s)
   \end{cases}

   where \( p := (p_1, p_2) : [t, T] \to \mathbb{R}^2 \) satisfies the following implicit equations:

   (2.12) \quad \forall s \in [t, T], \quad \begin{cases} 
   p_1(s) := -g_{x_1}(x(T)) - \int_s^T f_{x_1}(x, \tau) - p_2^2 h'(x_1(h(x_1))) d\tau, \\
   p_2(s) := -g_{x_2}(x(T)) - \int_s^T f_{x_2}(x, \tau) d\tau.
   \end{cases}
2. The control \( \alpha \) is of class \( C^1 \), as well as the pair \((x, p) = ((x_1, x_2), (p_1, p_2))\), and the latter satisfies the system of differential equations:

\[
\begin{align*}
(1) & \quad x_1' = p_1 \\
(2) & \quad x_2' = h^2(x_1)p_2 \\
(3) & \quad p_1' = -p_2^2h'(x_1)h(x_1) + f_{x_1}(x, s) \\
(4) & \quad p_2' = f_{x_2}(x, s),
\end{align*}
\]

with the mixed boundary conditions \( x(t) = x, p(T) = -\nabla g(x(T)) \).

Proof. If \((x, \alpha)\) is an optimal solution of \((2.4)\), then from the Maximum Principle (see, for example, the monograph by Clarke [3, Theorem 22.17] and [9, Corollary 22.3]) there exists an absolutely continuous arc in \([t, T]\), \( p(s) = (p_1(s), p_2(s)) \), such that the pair \((\alpha, p)\) satisfies the adjoint equations for a.e. \( s \in [t, T] \)

\[
\begin{align*}
(2.14) & \quad \begin{cases}
p_1' = -p_2^2h'(x_1)\alpha_2 + f_{x_1}(x, s), \\
p_2' = f_{x_2}(x, s),
\end{cases}
\end{align*}
\]

and the transversality condition

\[-p(T) = \nabla g(x(T))\]

together with the maximum condition

\[
\max_{a = (a_1, a_2) \in \mathbb{R}^2} p_1(s)a_1 + p_2(s)h(x_1(s))a_2 - \frac{|a|^2}{2} = p_1(s)\alpha_1(s) + p_2(s)h(x_1(s))\alpha_2(s) - \frac{|\alpha(s)|^2}{2}, \text{ a.e. } s \in [t, T].
\]

This condition implies that

\[
\nabla_a \left( p_1(s)a_1 + p_2(s)h(x_1(s))a_2 - \frac{|a|^2}{2} \right)_{a = \alpha} = 0, \quad s \in [t, T],
\]

from which we get \((2.11)\). Assumptions \((2.2)\) and \((2.11)\) imply the continuity of \( \alpha \).

Conditions \((1)\) and \((2)\) of \((2.13)\) follow directly from the dynamics \((1.8)\) replacing \( \alpha_1, \alpha_2 \) by \((2.11)\). Condition \((3)\) and \((4)\) of \((2.13)\) follow similarly from \((2.14)\).

Now \((2.12)\) shows that \( p \) is actually of class \( C^1 \), therefore \((2.13)\) implies that the same holds for the optimal trajectory \( x \). The regularity of \( \alpha \) is a consequence of \((2.11)\). Finally, the regularity of \( (x, p, \alpha) \) imply that the equalities in \((2.11)-(2.13)\) actually hold for every \( s \) in \([t, T]\) and not just up to a negligible set. \( \square \)

By standard arguments, one can prove the following Lemma so we omit the proof.

Lemma 2.5 Let \( \alpha_* \) be optimal control in \( \mathcal{A}(x, t) \) and \( x_*(\cdot) \) be the corresponding optimal trajectory for \( J_t \). Let \( \bar{\alpha}(\cdot) \in \mathcal{A}(x_*(s), s) \). The control law

\[
\bar{\alpha}(\tau) := \begin{cases} 
\alpha_*(\tau) & \text{if } \tau \in [t, s] \\
\bar{\alpha}(\tau) & \text{if } \tau \in [s, T]
\end{cases}
\]

is optimal for \( u(x, t) \), i.e., \( \bar{\alpha} \in \mathcal{A}(x, t) \).
In the following Lemma 2.6 we show that the optimal trajectories are analytic; this property will play a crucial role in the proof of uniqueness of optimal trajectories (after the starting time) which is established in the next proposition.

Lemma 2.6 Under assumptions (2.2), the solutions \((x, p)\) of the system (2.13) are analytic on \([t, T]\).

Proof. The system (2.13) is of the form
\[
(x_1, x_2, p_1, p_2)' = F(s, x_1, x_2, p_1, p_2),
\]
where \(F : [t, T] \times \mathbb{R}^4 \to \mathbb{R}^4\) is the analytic function
\[
F(s, x_1, x_2, p_1, p_2) := (p_1, h^2(x_1)p_2, -p_2^2h'(x_1)h(x_1) + f_{x_1}(x_1, x_2, s), f_{x_2}(x_1, x_2, s)).
\]
The Cauchy-Kovalevskaya Theorem [19, Theorem 2.2.21] and the Cauchy-Lipschitz theorem yield the conclusion. \(\Box\)

Proposition 2.1 Under assumptions (2.2), for any \(\alpha_* \in A(x, t)\), let \(x_*(\cdot)\) be the corresponding optimal trajectory.

1. For every \(s \in (t, T]\), there are no optimal trajectories for \(J_s(\alpha(\cdot), x(\cdot))\) starting from \(x_*(s)\) at time \(s\) other than \(x_*(\cdot)\), restricted to \((s, T]\).

2. For every \(s \in (t, T]\), \(DBu(x(s), s)\) exists if and only if \(A(x(s), s) = \{\alpha\}\) is a singleton and \(DBu(x(s), s) = -\alpha(s)\) (i.e., \(u_{x_1}(x(s), s) = -\alpha_1(s), h(x_1(s))u_{x_2}(x(s), s) = -\alpha_2(s)\)).

Proof. 1. Let us suppose that there exists another optimal trajectory \(\tilde{x}(\cdot)\) for \([s, T]\). From Lemma 2.5 the trajectory \(\tilde{x}(\cdot)\) obtained linking \(x(\cdot)\) in \([t, s]\) with \(\tilde{x}(\cdot)\) on \([s, T]\) is optimal whence, from Lemma 2.6 analytic on \([t, T]\) as well as \(x_*(\cdot)\). Since \(x_*(\cdot)\) and \(\tilde{x}(\cdot)\) coincide on the non trivial interval \([t, s]\), it follows from the Identity Theorem for analytic functions that \(x_*(\cdot) = \tilde{x}(\cdot)\) on \([t, T]\). Therefore, their restrictions \(x_*(\cdot)\) and \(\tilde{x}(\cdot)\) to \([s, T]\) are equal.

2. From Point 1, arguing as in [1] and using Proposition 4.1 of the Appendix, we get that \(u(\cdot, s)\) is always \(B\)-differentiable (see the Definition 4.1 below) at \(x(s)\) for any \(s \in (t, T]\) and \(DBu(x(s), s) = -\alpha(s)\). \(\Box\)

Lemma 2.7 Let \(x(\cdot) := (x_1(\cdot), x_2(\cdot))\) be an absolutely continuous solution of the problem
\[
\begin{align*}
\begin{cases}
x_1'(s) = -u_{x_1}(x(s), s), & x_1(t) = x_1, \\
x_2'(s) = -h^2(x_1(s))u_{x_2}(x(s), s), & x_2(t) = x_2,
\end{cases}
\end{align*}
(2.15)
\]
then the control \(\alpha = (\alpha_1, \alpha_2)\), with
\[
\alpha_1(s) = -u_{x_1}(x(s), s), \quad \alpha_2(s) = -h(x_1(s))u_{x_2}(x(s), s)
\]
is optimal for \(u(x, t)\). In particular if \(u(\cdot, t)\) is differentiable at \(x\) then problem (2.15) has a unique solution corresponding to the optimal trajectory.
Proof. Deriving formally \( u \) with respect to \( s \) we obtain
\[
\frac{du}{ds}(x(s), s) = -\frac{1}{2} |\alpha|^2 - f
\]
and, integrating in \([t, T]\), we get that \( \alpha \) is optimal since
\[
u(x(t), t) = \int_t^T \frac{1}{2} |\alpha(s)|^2 + f(x(s), s)\, ds + g(x(T)).
\]
Hence, one can easily accomplish the proof arguing as in the proof of [6, Lemma 4.11]. □

By the same arguments of [6, page 25], we infer that the multivalued function \( A(x, t) \) has
a Borel measurable selection \( \alpha(x, t) \in A(x, t) \). We fix a Borel measurable selection \( \alpha \) and
we define the flow \( \Phi(x, t, s) \) as follows:
\[
\Phi := (\Phi_1, \Phi_2),
\]
(2.16) \[
\Phi_1(x, t, s) := x_1 + \int_t^s \alpha_1(\tau)\, d\tau, \quad \Phi_2(x, t, s) := x_2 + \int_t^s h(x_1(\tau))\alpha_1(\tau)\, d\tau
\]
where \( x = (x_1, x_2) \). Thus \( \Phi(x, t, s) \) is the value, at time \( s \), of the optimal trajectory starting
at point \( x \) at time \( t \) associated to the optimal control \( \alpha(x, t) \).

Lemma 2.8 Under assumptions (H3) and (2.2), the flow \( \Phi \) defined in (2.16) has the
following properties: for any \( x, y \in \mathbb{R}^2 \), \( t \leq s \leq s' \leq T \), there hold
1. the semigroup property: \( \Phi(x, t, s') = \Phi(\Phi(x, t, s), s, s') \);
2. there holds
\[
\begin{aligned}
&\partial_s \Phi_1(x, t, s) = -\partial x_1 u(\Phi(x, t, s), s), \\
&\partial_s \Phi_2(x, t, s) = -h(\Phi_1(x, t, s))\partial x_2 u(\Phi(x, t, s), s);
\end{aligned}
\]
3. \( |\Phi(x, t, s') - \Phi(x, t, s)| \leq \|D u\|_\infty |s' - s| \);
4. \( |x - y| \leq C|\Phi(x, t, s) - \Phi(y, t, s)| \).

Proof. We follow the same arguments of [6] taking advantage of the boundedness of the
function \( h \), of the uniqueness result in Proposition 2.1 and of the fact that the semiconvexity of \( u \) gives \( D^2 u \leq C I \). □

Remark 2.1 From Lemma 2.8 (4), the map \( x \to \Phi(x, t, s) \) has an inverse function which is Lipschitz continuous. The push-forward of a measure \( m \) will be defined in the next section in terms of this inverse function.

3 The continuity equation

In this section we want to study the well posedness of the problem
\[
\begin{aligned}
\text{div} \partial_t m - \text{div} B(m D_B u) &= 0 \quad \text{in } \mathbb{R}^2 \times (0, T), \\
m(x, 0) &= m_0(x) \quad \text{on } \mathbb{R}^2,
\end{aligned}
\]
where \( u \) is a solution to the Hamilton-Jacobi problem (2.1) while \( m_0 \) fulfills assumption (H4). The aim of this Section is to prove that problem (3.1) has a unique solution which, moreover, can be characterized as the push-forward of the measure \( m_0 \) by the map \( \Phi(\cdot, 0, s) \), namely: \( \mu(s) = \Phi(\cdot, 0, s) \sharp m_0 \) where \( \Phi(\cdot, 0, s) \sharp m_0(A) := m_0(\Phi(\cdot, 0, s)^{-1}(A)) \) for any \( A \subset \mathbb{R}^2 \).
Proposition 3.1 Under assumption (2.2) and (H4), problem (3.1) has a unique solution given by \( \mu(s) := \Phi(\cdot, 0, s) \# m_0 \) where \( \Phi \) is defined by (2.16).

Proof. The steps to achieve the statement are similar to that of [6, Section 4.2] so we just sketch them. Let \( \mu \) be the function defined in the statement. By the same arguments as in [6, Lemma 4.14] with \( D_z u \) replaced by \( D_B u \), we obtain that \( \mu(s) \) is a bounded, absolutely continuous function with compact support and satisfies:

\[
d_1(\mu(s'), \mu(s)) \leq \|D_B u\|_\infty |s' - s| \quad \forall 0 \leq s' \leq s \leq T.
\]

Moreover, by the same arguments as those in the proof of [6, Lemma 4.15], the function \( \mu \) is a weak solution to problem (3.1). Now we want to approximate the function \( \mu \) by regular solutions to problems similar to (3.1). To this end, consider a positive smooth kernel \( \rho_\varepsilon \) and set

\[
\mu_\varepsilon(x, t) := \mu(x, t) \ast \rho_\varepsilon(x) \quad \text{and} \quad b_\varepsilon(x, t) := \frac{1}{\mu_\varepsilon}(u_{x_1} \mu, h^2(x_1) u_{x_2} \mu) \ast \rho_\varepsilon.
\]

We note that, by Lemma 2.2, \( b_\varepsilon \) is bounded and locally Lipschitz continuous in the space variable while \( \mu_\varepsilon \) satisfies the (standard) continuity equation

\[
\partial_t \mu_\varepsilon - \text{div}(b_\varepsilon \mu_\varepsilon) = 0 \quad \text{in } \mathbb{R}^2 \times (0, T).
\]

Invoking [6, Lemma 4.16], we infer that \( \mu_\varepsilon(s) = \Phi_\varepsilon(\cdot, 0, s) \# m_0 \) where \( \Phi_\varepsilon \) is the flow associated to \( b_\varepsilon \), namely:

\[
\partial_t \Phi_\varepsilon(x, 0, t) = b_\varepsilon(\Phi_\varepsilon(x, 0, t), t), \quad \Phi_\varepsilon(x, 0, 0) = x.
\]

Letting \( \varepsilon \to 0 \), by the same arguments as in [6, Thm 4.18], we accomplish the proof. \( \Box \)

4 Proof of the main Theorem

This section is devoted to the proof of our main Theorem 1.1. To this end, it is expedient to recall the stability result (as [6, Lemma 4.19]) which still holds in our case.

Lemma 4.1 Let \( m_n \in C([0, T], P_1) \) be uniformly convergent to \( m \in C([0, T], P_1) \). Then the solution \( u_n \) of (2.1) with \( f_n(x, t) = F(x, t, m_n) \) and \( g(x) = G(x, m_n(T)) \) converges locally uniformly to the solution \( u \) to (2.1).

Moreover, denote \( \mu_n(s) := \Phi_n(\cdot, 0, s) \# m_0 \) and \( \mu(s) := \Phi(\cdot, 0, s) \# m_0 \) where \( \Phi_n \) (respectively, \( \Phi \)) is the flow associated to \( u_n \) (resp., \( u \)). Then, \( \mu_n \to \mu \) in \( C([0, T], P_1) \).

Proof of Theorem 1.1 We shall argue following the proof of [6, Theorem 4.1] (see also [14, 15, 19]). Consider the set \( \mathcal{C} := \{ m \in C([0, T], P_1) \mid m(0) = m_0 \} \) and observe that it is convex. We also introduce a map \( T : \mathcal{C} \to \mathcal{C} \) as follows: to any \( m \in \mathcal{C} \) we associate the solution \( u \) to problem (2.1) with \( f(x, t) = F(x, t, m) \) and \( g(x) = G(x, m(T)) \) and to this \( u \) we associate the solution \( \mu := T(m) \) to problem (3.1). By the stability result of Lemma 4.1, the map \( T \) is continuous. Moreover, estimate (3.2) (note that the constant \( C \) is independent of \( m \)) implies that the map \( s \to T(m)(s) \) is uniformly Lipschitz continuous with value in the compact set of measures on a compact set (still independent of \( m \)); hence, the map \( T \) is compact. Invoking Schauder fix point Theorem, we accomplish the proof. \( \Box \)
Appendix

4.1 The general case

In this subsection we collect the ingredients to prove Theorem 1.1 in the general case as stated in the Introduction. We find the system satisfied by \((x, p)\) analogous to (2.13), we deduce the analyticity of the solution and then the uniqueness of the optimal trajectory after the starting time as in Proposition 2.1. The proofs rely on the same arguments of the model problem studied in the sections above hence we only emphasize the main differences.

Following the same procedure as in Section 2, we can prove that the solution \(u\) of the Hamilton-Jacobi equation (1.1)-(i) with \(m\) fixed is bounded in \(\mathbb{R}^d \times [0, T]\), Lipschitz continuous with respect to \(x\) and \(t\) and it is semiconcave with respect to \(x\). Moreover if \(\alpha\) and \(x\) are respectively the optimal control and the trajectory of the control problem (1.5)-(1.6) then, as in Lemma 2.4, we can prove:

1. The pair \((x, p)\) is of class \(C^1\) and satisfies the system of differential equations:

\[
\begin{align*}
(1) & \quad x' = p B(x) B^T(x), \\
(2) & \quad p' = -\frac{D_x[pB(x)]^2}{2} + D_x f(x, s) \\
(3) & \quad x(t) = x, \quad p(T) = -\nabla g(x(T)).
\end{align*}
\]

Since the functions \(h_{ij}\) in the matrix \(B\) do not depend on the last variable \(x_d\), then, in our case, the last coordinate of \(D_x|pB(x)|^2\) is 0, i.e. in (2) of (4.1) the last equation for \(p\) is \(p_d'(s) = f_{x_d}(x, s)\).

2. The optimal control \(\alpha\) is of class \(C^1\) and is given by

\[
\forall s \in [t, T] \quad \alpha(s) = p(s)B(x(s)).
\]

As in Lemma 2.6 we can prove that, under our assumptions, the solutions of (4.1) are analytic and we can use the same argument to prove the statement of Proposition 2.1 where the equalities in Point 2 are replaced by

\[
\forall s \in [t, T] \quad \alpha(s) = -D_x u(x(s), s) B(x(s)),
\]

and from (1.6), equation (4.1)-(1) becomes

\[
x'(s) = -D_x u(x(s), s) B(x(s)) B^T(x(s)).
\]

By the uniqueness of the optimal trajectory after the starting time, we can fix a Borel measurable selection \(\alpha\) of \(\mathcal{A}(x, t)\); we define the flow \(\Phi(x, t, s)\) as in (2.16):

\[
\Phi(x, t, s) = x + \int_t^s \alpha(\tau)B^T(x(\tau)) \, d\tau,
\]

where \(\alpha\) is the selected element in \(\mathcal{A}(x, t)\). Equation (2.16) in the general case becomes

\[
\partial_s \Phi(x, t, s) = -D_x u(x(s), s) B(x(s)) B^T(x(s)).
\]

The main issue for extending the results of Section 3 is the definition of the approximating smooth sequence \(b_\varepsilon\). In this case taking

\[
b_\varepsilon(x, t) := -\frac{1}{\mu^\varepsilon} (\mu \partial_p H(x, D_x u)) \ast \rho_\varepsilon
\]

we can obtain the same results as in Proposition 3.1.

Collecting all the results recalled here we prove also in this general case Theorem 1.1.
4.2 B-differentiability

In this subsection we introduce the notion of $B$-differentiability and the main properties for semiconcave functions which have been used in Proposition 2.1.

**Definition 4.1** A function $u : \mathbb{R}^d \to \mathbb{R}$ is $B$-differentiable in $x \in \mathbb{R}^d$ if there exists $\rho_B \in \mathbb{R}^d$ such that

$$\lim_{v \to 0} \frac{u(\tilde{x}) - u(x) - (\rho_B, v)}{|v|} = 0;$$

where, for $v \in \mathbb{R}^d$ we iteratively define $\tilde{x}_1 = x_1 + h_{11} v_1$, and $\tilde{x}_i = x_i + \sum_{j=1}^{i-1} h_{ij}(\tilde{x}_1, \cdots, \tilde{x}_{i-1}) v_j$, where $h_{ij}$ are defined in (1.3).

In the case treated in the previous sections definition 4.1 becomes

$$\lim_{v \to 0} \frac{u(x_1 + v_1, x_2 + h(x_1 + v_1) v_2) - u(x_1, x_2) - (\rho_B, v)}{|v|} = 0.$$

**Remark 4.1** If $u$ is differentiable then $\rho_B =: D_B u = Du B$, where $B$ is defined in (1.3).

The next proposition is used in Proposition 2.1 will be proved in details in [18] using techniques introduced by [5] by sub- and superdifferential notion.

**Proposition 4.1** Let $u$ be a semiconcave function. Then there hold

1. if $D_B u(x)$ is not empty, then $u$ is $B$-differentiable.

2. if $D_B u(x) = \{p\}$ (i.e., it is a singleton), then $u$ is $B$-differentiable at $x$.

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