Asymptotics of the convolution of the Airy function and a function of the power-like behavior

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Abstract. The asymptotic behavior of the convolution-integral of a special form of the Airy function and a function of the power-like behavior at infinity is obtained. The integral under consideration is the solution of the Cauchy problem for an evolutionary third-order partial differential equation used in the theory of wave propagation in physical media with dispersion. The obtained result can be applied to studying asymptotics of solutions of the KdV equation by the matching method.

Keywords: Airy function, convolution, Cauchy problem, asymptotics.
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1 Introduction

In the present paper, we study the behavior as \(|x| + t \to \infty\) of the convolution

\[
 u(x, t) = \frac{1}{\sqrt{3t}} \int_{-\infty}^{+\infty} f(y) \operatorname{Ai}\left(\frac{x - y}{\sqrt{3t}}\right) dy
\]

of the Airy function

\[
 \operatorname{Ai}(x) = \frac{1}{\pi} \int_{0}^{+\infty} \cos\left(\frac{\theta^3}{3} + x\theta\right) d\theta
\]

and a locally Lebesgue integrable function \(f\), which satisfies the following asymptotic (in the sense of Poincaré) relations:

\[
 f(x) = \sum_{n=0}^{\infty} f_n^\pm x^{-n}, \quad x \to \pm \infty.
\]

Integral (1.1), which is understood in this case in the sense of the limit \(\lim_{R \to +\infty} \int_{-\infty}^{R}\) of a standard Lebesgue integral, is of interest as the solution of the Cauchy problem for a third-order equation with the initial function \(f\):

\[
 \frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} = 0, \quad u(x, 0) = f(x), \quad t \geq 0, \quad x \in \mathbb{R}.
\]

This equation is used in the theory of wave propagation in physical media with dispersion \(\Pi\) and it is sometimes called the linearized Korteweg–de Vries (KdV) equation. Notice that the Airy function and the Airy transform (convolution of a general form) have applications
in wave optics, quantum mechanics, and in modern laser technologies [2, 3]. In addition, the investigation of the behavior of integral (1.1) is of independent interest for asymptotic analysis, for example, the obtained result can be applied to studying asymptotics of solutions of the nonlinear KdV equation by the matching method [4] in the same way as the large-time asymptotics of solutions of the heat equation [5] is applied to the Cauchy problem for a nonlinear parabolic equation [6, 7].

An asymptotics can be easily found for large values of \( t \) and bounded values of \( x \), if \( f(x) \) exponentially tends to zero as \( x \to \infty \). In this case, it suffices to expand the Airy function into a series in \( t^{-1/3} \). However, for more general conditions (1.3) the problem becomes much more difficult.

\section{Investigation integral}

Following the approach used in paper [5], we represent function (1.1) in the form

\[ u(x, t) = U_1^-(x, t) + U_0^-(x, t) + U_0^+(x, t) + U_1^+(x, t), \quad (2.1) \]

where

\[ U_1^-(x, t) = \int_{-\infty}^{-\sigma} \ldots, \quad U_0^-(x, t) = \int_{-\sigma}^{0} \ldots, \quad U_0^+(x, t) = \int_{0}^{\sigma} \ldots, \quad U_1^+(x, t) = \int_{\sigma}^{+\infty} \ldots, \]

\[ \sigma = (|x|^3 + t)^{p/3}, \quad 0 < p < 1, \]

and dots denote the integrand from formula (1.1) together with the factor \((3t)^{-1/3}dy\).

In the integral \( U^-(x, t) \), we make the change \( y = \theta \sqrt[3]{3t} \), setting

\[ \mu = \frac{\sigma}{\sqrt[3]{3t}}, \quad \eta = \frac{x}{\sqrt[3]{3t}}. \]

Using condition (1.3) as \( x \to -\infty \), we obtain

\[ U_1^-(x, t) = \int_{-\infty}^{-\mu} f(\theta \sqrt[3]{3t}) \text{Ai}(\eta - \theta)d\theta = \]

\[ = \sum_{n=0}^{N-1} f_n(3t)^{-n/3} \int_{-\infty}^{-\mu} \theta^{-n} \text{Ai}(\eta - \theta)d\theta + \int_{-\infty}^{-\mu} R_N(\theta \sqrt[3]{3t}) \text{Ai}(\eta - \theta)d\theta, \quad (2.2) \]

where \( R_N(s) = O(|s|^{-N}) \). From the last relation, we have an estimate of the remainder:

\[ \int_{-\infty}^{-\mu} R_N(\theta \sqrt[3]{3t}) \text{Ai}(\eta - \theta)d\theta = O(\sigma^{-N+1}). \quad (2.3) \]

Let us find the dependence of the integral

\[ \int_{-\infty}^{-\mu} \theta^{-n} \text{Ai}(\eta - \theta)d\theta \]
on value of $\mu$. For $n = 0$ and $t \geq |x|^\alpha$, $2 + p < \alpha < 3$, we have

$$
\int_{-\infty}^{-\mu} \tilde{\theta}^{-n} \Psi_n(\theta, \eta) d\theta = \int_{-\infty}^{-1} \tilde{\theta}^{-n} \Psi_n(\tilde{\theta}, \eta) d\tilde{\theta} + \sum_{r=0}^{n-1} \frac{(-1)^r}{r!} \frac{\ln \mu}{(r-\nu+1)!} \ln \mu + \sum_{r=0}^{n-2} \frac{\ln \mu}{r!(r-\nu+1)!} + \int_{-\mu}^{0} \Psi_n(\theta, \eta) d\theta + B_n^{-}(\eta),
$$

(2.7)

where

$$
B_n^{-}(\eta) = \sum_{r=0}^{n-2} \frac{\ln \mu}{r!(r-\nu+1)!} + \int_{-\mu}^{0} \Psi_n(\theta, \eta) d\theta.
$$

(2.8)

From formulas (2.6) we conclude that $\Psi_n(\theta, \eta)$ has no singularities as $\theta \to -0$; whence we get

$$
\int_{-\mu}^{0} \Psi_n(\theta, \eta) d\theta = \sum_{r=1}^{N-1} \mu^r q_r \frac{\ln \mu}{r!(r-\nu+1)!} + O(\sigma^{-\gamma N}), \quad \eta \to \infty.
$$

(2.9)
Substituting relations (2.4) and (2.7) in formula (2.2), and using estimate (2.3), we find

\[ U^{-1}(x,t) = \int_{-\infty}^{\infty} \text{Ai}(\theta) d\theta + \sum_{n=1}^{N-1} \int_{-\infty}^{1} \theta^{-n} \text{Ai}(\eta - \theta) d\theta + B^{-1}_{n}(\eta) + V^{-1}(\mu, \eta, t) + O(\sigma^{-\gamma N}), \]

where

\[ V^{-1}(\mu, \eta, t) = \sum_{r+q\neq 0} v_{r,s}^{-1} \text{Ai}^{(rs)}(\eta) t^{l_s} \mu^{r_s} \ln^{q_s} \mu, \]

\[ v_{r,s}^{-1} \text{ are constant coefficients.} \]

From the explicit form the Airy function (1.2), it is easy to see that

\[ \int_{-\infty}^{\infty} \text{Ai}(\theta) d\theta = -\int_{0}^{+\infty} \frac{1}{\pi \theta} \sin \left( \frac{\theta^3}{3} + \eta \theta \right) d\theta + \text{const.} \]

In addition, for \( \eta > 0 \)

\[ \int_{0}^{+\infty} \frac{1}{\theta} \sin \left( \frac{\theta^3}{3} + \eta \theta \right) d\theta = \int_{0}^{\eta^{-1/2}} + \int_{\eta^{-1/2}}^{+\infty} = \]

\[ = \int_{0}^{\eta^{-1/2}} \frac{1}{z} \sin \left( \frac{z^3}{3\eta^3} + z \right) dz - \int_{\eta^{-1/2}}^{+\infty} \frac{1}{\theta(\theta^2 + \eta)} d\cos \left( \frac{\theta^3}{3} + \eta \theta \right). \]

The first integral on the right-hand side of this relation as \( \eta \to +\infty \) tends to the limit value of the sine integral

\[ \text{Si}(+\infty) = \int_{0}^{+\infty} \frac{\sin z}{z} dz = \frac{\pi}{2}, \]

since \( z^3/\eta^3 \leq \eta^{-3/2} \) on the whole integration interval. Integrating by parts, we see that

\[ \int_{\eta^{-1/2}}^{+\infty} \frac{1}{\theta(\theta^2 + \eta)} d\cos \left( \frac{\theta^3}{3} + \eta \theta \right) = O(\eta^{-1/2}). \]

Thus, we obtain

\[ \int_{\eta}^{+\infty} \text{Ai}(\theta) d\theta = -\int_{0}^{+\infty} \frac{1}{\pi \theta} \sin \left( \frac{\theta^3}{3} + \eta \theta \right) d\theta + \frac{1}{2}. \]

(2.11)

Now, let us study the integral

\[ U^{-1}_{0}(x,t) = \frac{1}{\sqrt{3t}} \int_{-\infty}^{0} f(y) \text{Ai} \left( \frac{x-y}{\sqrt{3}t} \right) dy. \]

Since \( 2t \geq \sigma^{\alpha/p} \) on the set \( T_{\alpha} \) and there holds the estimate

\[ \frac{y^{\gamma}}{\sqrt{3t}} = O(\sigma^{-\gamma}), \quad \gamma = \frac{\alpha}{3p} - 1 > 0, \]
for $|y| \leq \sigma$, we can use Taylor's formula. Then,

$$U_0^-(x, t) = \sum_{n=0}^{N-1} t^{-(n+1)/3} \frac{(-1)^n 3^{-(n+1)/3}}{n!} \text{Ai}^{(n)}(\eta) \int_{-\sigma}^{0} y^n f(y) dy + O(\sigma^{-\gamma N}).$$

Let us transform the integral $\int_{-\sigma}^{0}$ as follows:

$$\int_{-\sigma}^{0} y^n f(y) dy = \int_{-1}^{0} y^n f(y) dy + \int_{-\sigma}^{0} \left( f_0^- y^n + \cdots + f_{n+1}^- \right) dy +$$

$$+ \int_{-\sigma}^{-1} y^n \left[ f(y) - f_0^- - \cdots - \frac{f_n^-}{y^n} \frac{f_{n+1}^-}{y^{n+1}} \right] dy =$$

$$= \int_{-1}^{0} y^n f(y) dy + \int_{-\infty}^{-1} \Phi_n^-(y) dy - \int_{-\infty}^{-\sigma} \Phi_n^-(y) dy - f_{n+1}^- \ln \sigma + P_n(\sigma), \quad (2.12)$$

where

$$\Phi_n^-(y) = y^n \left[ f(y) - \sum_{m=0}^{n+1} \frac{f_m^-}{y^m} \right].$$

From relations (1.3) we obtain estimates

$$|\Phi_n^-(y)| \leq C_n y^{-2}, \quad y \leq -1,$$

$$\int_{-\sigma}^{-\infty} \Phi_n^-(y) dy = \sum_{m=1}^{N-1} \varphi_{n,m} \sigma^{-m} + O(\sigma^{-N}), \quad \sigma \to \infty,$$

where $\varphi_{n,m}$ are some constants. Taking into account these estimates and substituting $\sigma = \mu^{3/3} t$ in (2.12), we obtain

$$\int_{-\sigma}^{0} y^n f(y) dy = I_n - \frac{f_{n+1}^-}{3} \ln t - f_{n+1}^- \ln \mu + \sum_{r \neq 0} a_{n,r} \mu^r \ln \mu + O(\sigma^{-N}),$$

where $I_n$ and $a_{n,r}$ are constants. Then, we have

$$U_0^-(x, t) = \sum_{n=0}^{N-1} t^{-(n+1)/3} \text{Ai}^{(n)}(\eta)[b_n^- + \tilde{b}_n^- \ln t] + V_0^-(\mu, \eta, t) + O(\sigma^{-\gamma N}), \quad (2.13)$$

$$V_0^-(\mu, \eta, t) = \sum_{r \neq 0} v_{0,s}^- \text{Ai}^{(m_s)}(\eta) t^r \mu^r \ln q^s \mu,$$

where $b_n^-$, $\tilde{b}_n^-$, and $v_{0,s}^-$ are constant coefficients. From the asymptotics of the Airy function

$$\text{Ai}(x) = \frac{x^{-1/4}}{2\pi} \exp \left( -\frac{2}{3} x^{3/2} \right) \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma \left( \frac{3n+1}{2} \right)}{3^{2n} (2n)!} x^{-3n/2}, \quad x \to +\infty, \quad (2.14)$$
it is seen that estimates of remainders in formulas (2.10) and (2.13) are also valid on the set

\[ X_\alpha^+ = \{(x, t) : x > 0, \; 0 < t < |x|^\alpha \}, \]

where

\[ \mu = o(\eta), \quad |\eta| \geq |x|^{1-\alpha/3} \to \infty \quad \text{as} \quad \sigma \to \infty. \]

To find the dependence on \( \mu \) of integrals

\[ \int_0^\mu Ai(\eta + \theta')d\theta', \quad \int_{-\mu}^0 \Psi_n(\theta, \eta)d\theta, \quad \int_0^\mu \Psi_n(\theta, \eta)d\theta \]

on the set

\[ X_\alpha^- = \{(x, t) : x < 0, \; 0 < t < |x|^\alpha \}, \]

we use the expansion

\[ Ai(x) = \frac{|x|^{-1/4}}{\pi} \sum_{n=0}^\infty (-1)^{n/2} \frac{\Gamma\left(\frac{3n+1}{2}\right)}{3^{3n}(2n)!} \sin \left(\frac{2}{3}|x|^{3/2} + \frac{\pi}{4}(-1)^n\right) |x|^{-3n/2} \quad (2.15) \]

as \( x \to -\infty \). Then we obtain expressions of the form

\[ \sum_{r^2+q^2\neq 0} b_s \mu^r(\ln \mu)^q t^l \eta^m t^l \mu^r \frac{\ln \mu}{t^l}, \quad \sin \left(\frac{2}{3}|\eta|^{3/2} \pm \frac{\pi}{4}\right), \]

where \( b_s \) are constant coefficients.

Analogously to formulas (2.10) and (2.13), we find

\[ U_0^+(x, t) = \sum_{n=0}^{N-1} t^{-\alpha_n} \int_{-\alpha_n}^\alpha \Psi_n(\theta, \eta)d\theta + \int_0^\alpha \Psi_n(\theta, \eta)d\theta + B_n^+(\eta) \]

\[ V_0^+(\mu, \eta, t) + O(\sigma^{-\gamma N}), \quad (2.18) \]

are constructed similarly to functions (2.8),
\( b_n^+ \) and \( \tilde{b}_n^+ \) are some constants,

\[
V_0^+ (\mu, \eta, t) = \sum_{r_1^2 + q_1^2 \neq 0} G_{0,s}(\eta) t^{r_*} \mu^{r_*} \ln^{q_*} \mu,
\]

\( G_{0,s}(\eta) \) and \( G_{1,s}(\eta) \) are smooth functions.

Since \( \mu \) contains an arbitrary parameter \( p \) and \( \mu \to 0 \) for \( t > |x|^h (3p < h < 3) \), from A.R. Danilin’s lemma [8, Lemma 4.4], it follows that

\[
V_0^- (\mu, \eta, t) + V_1^- (\mu, \eta, t) + V_0^+ (\mu, \eta, t) + V_1^+ (\mu, \eta, t) = O(\sigma^{-\gamma N}). \tag{2.19}
\]

Then substituting expressions (2.10), (2.13), (2.16), and (2.18) into (2.1) and using formulas (2.11) and (2.19), we obtain

\[
u(x, t) = (f_0^+ - f_0^-) \int_0^{+\infty} \frac{1}{\pi \theta} \sin \left( \frac{\theta^3}{3} + \frac{x \theta}{\sqrt{3t}} \right) d\theta + \frac{f_0^+ + f_0^-}{2} + \\
+ \sum_{n=1}^{N-1} t^{-n/3} \left( F_n(\eta) + \tilde{F}_n(\eta) \ln t \right) + O(\sigma^{-\gamma N}). \tag{2.20}
\]

The coefficients \( F_n(\eta) \) and \( \tilde{F}_n(\eta) \) are linear combinations of smooth bounded functions, derivatives of the Airy function up to the \((n - 1)\)th order, inclusively, and integrals

\[
\int_{-1}^{0} \Psi_n(\theta, \eta) d\theta, \quad \int_{0}^{1} \Psi_n(\theta, \eta) d\theta.
\]

Then from asymptotics (2.14), (2.15), (2.9), and (2.17), it follows that functions \( F_n(\eta) \) and \( \tilde{F}_n(\eta) \) cannot grow faster than \(|\eta|^{-1/4 + n/2}\) as \( \eta \to \infty \). Consequently, the \( n \)th term in expansion (2.20) cannot grow faster than

\[
\ln t |\eta|^{-1/4} \left| \frac{x}{t} \right|^{n/2}.
\]

Therefore, the series

\[
\sum_{n=1}^{\infty} t^{-n/3} \left( F_n(\eta) + \tilde{F}_n(\eta) \ln t \right)
\]

keeps its asymptotic character for \( t > |x|^{1+\delta} \) with any \( \delta > 0 \).

Thus, we arrive at the following statement.

**Theorem.** If for a locally Lebesgue integrable function \( f \) the conditions

\[
f(x) = \sum_{n=0}^{\infty} f_n^\pm x^{-n}, \quad x \to \pm \infty,
\]

are fulfilled, then for any \( \delta > 0 \) and \( t > |x|^{1+\delta} \) there holds the asymptotic formula

\[
\frac{1}{\sqrt{3t}} \int_{-\infty}^{+\infty} f(y) \text{Ai} \left( \frac{x - y}{\sqrt{3t}} \right) dy = 
\]

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\[= (f^+_0 - f^-_0) \int_{0}^{\infty} \frac{1}{\pi \theta} \sin \left( \frac{\theta^3}{3} + \frac{x\theta}{\sqrt{3}t} \right) d\theta + \frac{f^+_0 + f^-_0}{2} + \sum_{n=1}^{\infty} t^{-n/3} \left( F_n(\eta) + \tilde{F}_n(\eta) \ln t \right), \]

as \(|x| + t \to \infty\), where \(F_n(\eta)\) and \(\tilde{F}_n(\eta)\) are \(C^\infty\)-smooth functions of the self-similar variable

\[\eta = \frac{x}{\sqrt{3}t}.\]

According to formula (2.20), the asymptotic series in the theorem should be understood in the sense of Erdelyi with the asymptotic sequence

\[\{(|x|^3 + t)^{-\gamma}n\}_{n=1}^{\infty},\]

where \(\gamma' > 0\).

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