An area law for one-dimensional quantum systems

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Abstract. We prove an area law for the entanglement entropy in gapped one-dimensional quantum systems. The bound on the entropy grows surprisingly rapidly with the correlation length; we discuss this in terms of properties of quantum expanders and present a conjecture on matrix product states which may provide an alternate way of arriving at an area law. We also show that, for gapped, local systems, the bound on Von Neumann entropy implies a bound on Rényi entropy for sufficiently large $\alpha < 1$ and implies the ability to approximate the ground state by a matrix product state.

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There are many reasons to believe that the entanglement entropy of a quantum system with a gap obeys an area law: that the entanglement entropy of a given region scales as the boundary area, rather than as the volume. In one dimension, conformal field theory calculations show that away from the critical point the entanglement entropy is bounded, diverging proportionally to the correlation length as a critical point is approached [1]. In higher dimensions, systems represented by matrix product states [2, 3] obey an area law.

However, despite this, there is no general proof of an area law. This is somewhat surprising, since it has been proven that correlation functions in a gapped system decay exponentially [4], and one might guess that the decay of correlation functions implies that only degrees of freedom near the boundary of the region may entangle with those outside. However, the existence of data hiding states [5] shows that one can have states on bipartite systems with small correlations and large entanglement. Further, the existence of quantum expanders [6, 7] shows that one may have matrix product states in one dimension that have all correlation functions decaying exponentially in the distance between the operators, but still have large entanglement. This indicates some of the difficulty in proving an area law.

At the same time, for a gapped system to violate an area law would require some very strange properties. For one thing [8], the thermal density matrix can be well approximated by a matrix product operator. This implies that unless a plausible assumption [6] on the density of low energy states is violated, the ground state can be well approximated by a matrix product state.

In this paper, we succeed in providing a proof of an area law for one-dimensional systems under the assumption of a gap. The result, however, bounds the entanglement entropy by a quantity that grows exponentially in the correlation length. This is much faster than the linear growth one might have expected. We will comment later on why this bound might in fact be reasonably tight. In the process of deriving this result, we will derive bounds for gapped local systems which interrelate three quantities: the Von Neumann entropy, the Rényi entropy, and the error involved in approximating the ground state by a matrix product state.

We begin by defining the lattice and Hamiltonian. We consider finite range Hamiltonians for simplicity. It is likely that the results can be extended to exponentially

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decaying interactions, but for simplicity we do not consider this here. In fact, having decided to consider only finite range interactions, we may group several sites into a single site, and thus simplify to a problem with only nearest neighbor interactions.

Specifically, we consider a finite volume one-dimensional lattice, with sites labeled \( i = 1, 2, \ldots, N \), with a \( D \)-dimensional Hilbert space on each site. We consider finite range Hamiltonians of the form \( H = \sum_{i=1}^{N} H_{i,i+1} \). The finite range condition is that \( H_{i,i+1} \) has support on the set of sites \( i \) and \( i+1 \). We additionally impose a finite interaction strength condition bounding the operator norm, that the operator norm \( \| H_{i,i+1} \| \leq J \) for some \( J \).

The properties imply a Lieb–Robinson bound [9]–[11]: there exists a velocity \( v \) and length scale \( \xi_C \) such that for any two operators \( A, B \) with support on sets \( X, Y \) respectively,

\[
\| [A(t), B] \| \leq c \times |X| \| A \| \| B \| \exp[−\xi_C \text{ dist}(X,Y)] \tag{1}
\]

for \( |t| \leq l/v \), where \( A(t) = \exp[iHt]A\exp[-iHt] \), where \( c \) is a numeric constant of order unity, and where the distance \( \text{dist}(X,Y) \) between sets \( X, Y \) is defined by \( \text{dist}(X,Y) = \min_{i\in X, j\in Y} |i−j| \). The velocity \( v \) will be of order \( J \), while \( \xi_C \) will be of order unity.

We now introduce some notation. We let \( X_{j,k} \) denote the set of sites \( i \) with \( j \leq i \leq k \). We let \( \Psi_0 \) denote the ground state of the Hamiltonian \( H \) and we let \( \rho_{1,N}^0 = \Psi_0 \langle \Psi_0 | \) be the ground state density matrix. We let \( \rho_{j,k}^0 \) denote the reduced ground state density matrix on the interval \( X_{j,k} \). That is, \( \rho_{j,k}^0 = \text{tr}_{i\in X_{j,k}} (\rho_{1,N}^0) \), where the partial trace is over sites not in \( X_{j,k} \). We define the entropy of any density matrix \( \rho_{j,k} \) by \( S(\rho_{j,k}) = −\text{tr}_{i\in X_{j,k}} (\rho_{j,k} \ln(\rho_{j,k})) \).

**Theorem 1.** Consider a Hamiltonian on a finite lattice as above satisfying the finite range and finite interaction strength conditions above. Suppose \( H \) has a unique ground state with a gap \( \Delta E \) to the first excited state. Then, for any \( i \),

\[
S(\rho_{1,i}^0) \leq S_{\text{max}} \tag{2}
\]

where we define \( S_{\text{max}} = c_0 \xi' \ln(\xi') \ln(D)2^{\xi' \ln(D)} \),

\[
c_0 = 4 \xi \ln(D), \tag{3}
\]

for some numerical constant \( c_0 \) of order unity, and where we define

\[
\xi = \min(2v/\Delta E, \xi_C),
\]

\[
\xi' = 6 \xi. \tag{4}
\]

We do not consider the case of a degenerate ground state, but we expect that the theorem can be strengthened to include this case also.

1. **Proof of main theorem**

To prove the theorem, we assume that it is false, so for some \( i_0 \), we have \( S(\rho_{1,i}^0) > S_{\text{max}} \). Then, for any \( k > i \) we have \( S(\rho_{1,k}^0) > S_{\text{max}} - (k - i) \ln(D) \), and therefore, for all \( k \) with \( i \leq k \leq i + l_0 \) where

\[
l_0 \equiv S_{\text{max}}/3 \ln(D), \tag{5}
\]

we have

\[
S(\rho_{1,k}^0) \geq 2S_{\text{max}}/3 \equiv S_{\text{cut}}. \tag{6}
\]

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Then, define $S_l$, for each $l \leq l_0$, to be the maximum entropy of an interval of length $l$ contained in $X_{j,i+l_0}$. That is, $S_l$ is the maximum of $S(\rho^0_{j-i+1,j+l})$ over $j$ such that $X_{j+1,j+l} \subseteq X_{i,l+l_0}$. Clearly,

$$S_1 \leq \ln(D).$$

(7)

Further, for any $l$, we have

$$S_{2l} \leq 2S_l.$$

(8)

However, it should be apparent that it is not possible for equation (8) to be saturated. If $S_{2l} = S_l$, then we have for some $j$ that $S(\rho^0_{j-l+1,j+l}) = S(\rho^0_{j-l+1,j}) + S(\rho^0_{j+1,j+l})$. In this case, the density matrix $\rho^0_{j-l+1,j+l}$ is equal to the product of density matrices $\rho^0_{j-l+1,j} \otimes \rho^0_{j+1,j+l}$. This implies that $\text{tr}(H \rho^0_{j,j} \rho^0_{j+1,j,N}) = \text{tr}(H \rho^0_{j,j})$ since $H$ is a sum of terms $H_{j,j+1}$ acting on nearest neighbors. However, given a unique ground state, this implies that $\rho^0_{j,N} = \rho^0_{j,j} \otimes \rho^0_{j+1,N}$, contradicting the assumption of a non-vanishing entanglement entropy $S(\rho^0_{j,j})$. The key idea of the proof will be to improve on equation (8) even further, and use the fact that $S(\rho^0_{j,j})$ is not only non-vanishing, but bounded below by $S_{\text{cut}}$ to show that the following stronger claim holds for all $l \leq l_0$:

$$S_{2l} \leq 2S_l - (1 - 2C_1(\xi) \exp(-l/\xi'))l/\xi' + \ln(C_1(\xi)) + C_2,$$

(9)

for some function $C_1(\xi)$ which is bounded by a polynomial in $\xi$, and some numerical constant $C_2$ of order unity.

Once equation (9) is shown, we can iterate it. The dominant terms on the right-hand side of equation (9) at large $l$ are $2S_l - l/\xi'$, so that one can guess from the behavior of these dominant terms that we will find $S_l \approx l \ln(D) - \log_2(l/\xi_0)l/\xi'$, up to subleading terms. Indeed this is the case. Set $\xi_0 = \xi'/2 \ln(2C_1(\xi))$. We use $S_l \leq l \ln(D)$ for $l = \xi_l$ as an initial bound and iterate to $l = 2\xi_0, l = 4\xi_0$, and so on. Note that $2C_1(\xi) \exp[-\xi_0/\xi']\xi_0/\xi' \leq 1$, and $\sum_{n=0}^{\infty} 2C_1(\xi) \exp[-\xi_0/\xi']\xi_0/\xi' \leq 1 + e^{-2} + e^{-4} + \cdots \leq 2$. It follows that for $l > \xi_0$,

$$S_l \leq \ln(D)l - l \lfloor \log_2(l/\xi_0) \rfloor/\xi' + (2 + \ln(C_1(\xi)) + C_2)l/\xi_0.$$  

(10)

However, since $S_l$ must be positive, a contradiction will arise when $\lfloor \log_2(l/\xi_0) \rfloor \geq \ln(D)l/\xi' + (2 + C_2)\xi'/\xi_0 + 1/2$ and so

$$l_0 \leq \xi' \ln(2C_1(\xi))2^{\xi' \ln(D)/\xi' + (2 + C_2)\xi'/\xi_0 + 1/2}.$$  

(11)

Combined with equation (5) this gives the main result for some constant $c_0$.

Thus, we must now show equation (9). To show this, it suffices to show that for all $j,l$ with $X_{j+1,l+l} \subseteq X_{i,l+l_0}$ that,

$$S(\rho^0_{j-l+1,j+l}) \leq S(\rho^0_{j-l+1,j}) + S(\rho^0_{j+1,j+l}) - (1 - 2C_1(\xi) \exp(-l/\xi'))l/\xi' + \ln(C_1(\xi)) + C_2.$$  

(12)

At this point, we need two lemmas that rely on the Lieb–Robinson bound and the existence of a gap.
Lemma 1. Let $H$ be a Hamiltonian that satisfies the finite range, finite interaction strength, and gap conditions above. Then for any $j$ and any $l$ there exist Hermitian, positive definite, operators $O_B(j,l), O_L(j,l), O_R(j,l)$ with the following properties. First, $\|O_B(j,l)\| \leq 1, \|O_L(j,l)\| \leq 1,$ and $\|O_R(j,l)\| \leq 1.$

Second,

$$\|O_B(j,l)O_L(j,l)O_R(j,l) - P_0\| \leq C_1(\xi) \exp[-l/\xi'] \equiv \epsilon(l),$$

where $P_0 = \langle \Psi_0 | \Psi_0 \rangle$ is the projection operator onto the ground state and the function $C_1(\xi)$ is bounded by a polynomial in $\xi$. Finally, $O_L(j,l)$ is supported on $X_{1,j}$, $O_R(j,l)$ is supported on $X_{j+1,N}$, and $O_B(j,l)$ is supported on $X_{j-l+1,j+l}$.

We prove this lemma in the appendix.

The next lemma provides a way of bounding the entanglement entropy of a given region, given a certain assumption: that one can approximate the ground state to a certain accuracy in Hilbert space norm by a state with given Schmidt rank. In general, if one knows that a matrix product state provides a good approximation to the ground state, this lemma can be used to ‘bootstrap’ that result into a bound on the entanglement entropy of the ground state.

Lemma 2. Let

$$\rho = \sum_{\gamma} P(\gamma) \sum_{\alpha=1}^{k} A(\alpha, \gamma) \Psi_L(\alpha, \gamma) \otimes \Psi_R(\alpha, \gamma) \middle| \sum_{\beta=1}^{k} A(\beta, \gamma) \Psi_L(\beta, \gamma) \otimes \Psi_R(\beta, \gamma)$$

be some density matrix with unit trace. Here, $\Psi_L(\alpha, \gamma)$ are states on $X_{1,j}$ and $\Psi_R(\alpha, \gamma)$ are states on $X_{j+1,N}$. Then, we say that $\rho$ is a mixture of pure states with Schmidt rank at most $k$. Suppose that $\langle \Psi_0, \rho \Psi_0 \rangle = P > 0$, where $\Psi_0$ is the ground state of a Hamiltonian that satisfies the finite range, finite interaction length, and gap conditions above. Then,

$$S(\rho_{1,j}) \leq \ln(k) + \xi' \ln(2C_1(\xi)^2/P) \ln(D) + F(\xi', D),$$

where

$$F(\xi', D) = (\xi' + 4) \ln(D) + 1 + \ln(D^2 - 1) + \ln(\xi'/2 + 1),$$

and a similar bound holds for Rényi entropies with sufficiently large $\alpha$ as discussed below (here, $\alpha$ refers to the order of the Rényi entropy, not to a particular state; the particular use of $\alpha$ should be clear in context).

Proof. To prove this, note that for any $m$ the positive definite operator $\rho(m) = O_B(j,m)O_L(j,m)O_R(j,m)\rho O_R(j,m)O_L(j,m)O_B(j,m)$ has the following properties. First, $\rho(m)$ is a mixture of pure states with Schmidt rank at most $kD^2m$. Second, from equation (13), $\text{tr}(P_0 \rho(m)) \geq (\sqrt{P} - \epsilon(l))^2$ and $\text{tr}((1 - P_0) \rho(m)) \leq \epsilon^2$ and thus

$$\frac{\langle \Psi_0, \rho(m) \rangle \Psi_0}{\text{tr}(\rho(m))} \geq 1 - 2C_1(\xi)^2 \exp[-2m/\xi']/P.$$

Let the ground state $\Psi_0$ be equal to $\sum_{\alpha=1} A_0(\alpha) \Psi_L(0,\alpha) \otimes \Psi_R(0,\alpha)$, where the states $\Psi_L(0,\alpha)$ and $\Psi_R(0,\alpha)$ are again states on $X_{1,j}$ and $X_{j+1,N}$ respectively, with $\langle \Psi_L(0,\alpha), \Psi_L(0,\beta) \rangle = \langle \Psi_R(0,\alpha), \Psi_R(0,\beta) \rangle = \delta_{\alpha,\beta}$. We order the different states such that if
\( \alpha < \beta \) then \( |A_0(\alpha)| \geq |A_0(\beta)| \) and we normalize so that \( \sum_\alpha |A_0(\alpha)|^2 = 1 \). It follows from equation (17) that for all integer \( m \)

\[
\sum_{\alpha \geq kD^{2m}+1} |A_0(\alpha)|^2 \leq 2C_1(\xi)^2 \exp[-2m/\xi]/P. \tag{18}
\]

Let \( m' \) be the smallest integer \( m \) such that \( 2 \exp[-2m/\xi]C_1(\xi)^2/P \leq 1 \). Thus, for \( m > m' \)

\[
\sum_{\alpha \geq kD^{2m}+1} |A_0(\alpha)|^2 \leq \exp[-2(m - m')/\xi]. \tag{19}
\]

We now maximize the entropy \( S(\rho_{1,j+l}^0) = -\sum_{\alpha=1} |A_0(\alpha)|^2 \ln(|A_0(\alpha)|^2) \) subject to the constraint (19). The maximum occurs when \( \sum_{kD^{2m'-2}} |A_0(\alpha)|^2 = (1 - \exp[-2/\xi']) \) and for \( m > m' \), \( \sum_{\alpha=kD^{2m+2}} |A_0(\alpha)|^2 = (1 - \exp[-2/\xi']) \exp[-2(m - m')/\xi'], \) giving an entropy bounded by

\[
\ln(k) + (2m' + 2) \ln(D) \sum_{n=1}^\infty (2(n - 1) \ln(D) + \ln(D^2 - 1) + 2n/\xi')
\]

\[
= \ln(k) + \left( 2m' + 2 + \frac{2\exp[-4/\xi']}{1 - \exp[-2/\xi']} \right) \ln(D) + (2\xi'/1 - \exp[-2/\xi']) \ln(D^2 - 1) + \ln(\xi'/2 + 1), \tag{20}
\]

where we used the inequalities \( 2 \exp[-4/\xi']/(1 - \exp[-2/\xi']) \leq \xi', (2/\xi') \exp[-2/\xi']/(1 - \exp[-2/\xi']) \leq 1, \) and \( 1/(1 - \exp[-2/\xi']) \leq \xi'/2 + 1 \) for \( \xi' > 0 \) giving equation (15) as claimed.

We note that this proof can be extended to Rényi entropies \( S_{\alpha}(\rho_{1,j}^0) \) defined by \( S_{\alpha}(\rho_{1,j}^0) = (1 - \alpha)^{-1} \ln(\text{tr}(\rho_{1,j}^0)) \) for sufficiently large \( \alpha \). Maximizing the Rényi entropy subject to the constraint (18) gives \( S_{\alpha}(\rho_{1,j}^0) \leq (1 - \alpha)^{-1} \ln\{\sum_{n=0}^{\infty}(kD^{2m'+2}D^{2n})^{1-\alpha}(1 - \exp[-2/\xi'])^\alpha \exp[-2n/\xi'\alpha]\}. \) The sum converges so long as

\[
\exp[-2\alpha/\xi'](D^2)^{(1-\alpha)} < 1, \tag{21}
\]

in which case we have a bound on the Rényi entropy which differs from the bound (15) on the von Neumann entropy only in that the function \( F(\xi', D) \) is replaced by an \( \alpha \)-dependent function.

We now return to proving the main theorem. From equation (13) it follows that \( (\Psi_0, O_B(j,l)O_L(j,l)O_R(j,l)\Psi_0) \geq 1 - \epsilon(l) \) and hence \( (\Psi_0, O_B(j,l)\Psi_0) \times (\Psi_0, O_L(j,l)O_R(j,l)\Psi_0) \geq (1 - \epsilon(l))^2 \). Therefore, \( (\Psi_0, O_B(j,l)\Psi_0) \geq 1 - 2\epsilon(l) \) and \( (\Psi_0, O_L(j,l)\Psi_0) \geq 1 - 2\epsilon(l) \). Thus,

\[
\text{tr}(\rho_{j-1,j+1,N}) \geq 1 - 2\epsilon(l). \tag{22}
\]

Let \( P = \text{tr}(P_0\rho_{0,j}^0 \otimes \rho_{0,j+1,N}^0) \). From equations (6) and (15), we find that \( \xi' \ln(2C_1(\xi)^2/P) \ln(D) + F(\xi', D) \geq S_{\text{cut}} \). Therefore, \( P \leq 2C_1(\xi)^2 \exp[-(S_{\text{cut}} - \)
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\[ F(\xi', D) / \ln(D) \xi' \]

Let

\[ x = \text{tr}(O_B(j, l)\rho^0_{l,j} \otimes \rho^0_{j+1,N}) \]
\[ = \text{tr}(O_B(j, l)\rho^0_{j-l+1,j} \otimes \rho^0_{j+1,j+l}), \quad (23) \]

and

\[ y = \text{tr}(O_L(j, l)O_R(j, l)\rho^0_{l,j} \otimes \rho^0_{j+1,N}) \geq 1 - 2\epsilon(l). \]

Then, \( P \geq \text{tr}(O_B(j, l)O_L(j, l)O_R(j, l, l)\rho^0_{l,j} \otimes \rho^0_{j+1,N}) - \epsilon(l) \geq xy - \sqrt{x - x^2} \sqrt{y - y^2} - \epsilon(l) \) as follows from a Cauchy–Schwarz inequality for the expectation value of a product of operators. Thus, \( P \geq x(1 - 2\epsilon(l)) - \sqrt{x - x^2} \sqrt{2\epsilon(l)} - \epsilon(l) \geq x(1 - 2\epsilon(l)) - \sqrt{x} \sqrt{2\epsilon(l)} - \epsilon(l) \). Thus,

\[ x \leq \{2C_2(\xi)^2 \exp[-(S_{\text{cut}} - F(\xi', D))/\ln(D)\xi']} + \sqrt{\sqrt{2\epsilon(l) + 2\epsilon(l)}/(1 - 2\epsilon(l))}. \quad (24) \]

Thus, equations (22) and (23) imply that the operator \( O_B(j, l) \) has a large expectation value for the state \( \rho^0_{j-l+1,j+l} \) and a small expectation value for the state \( \rho^0_{l,j} \otimes \rho^0_{j+1,j+l} \). Then, the Lindblad–Uhlmann theorem [12] provides a lower bound on the relative entropy \( S(\rho^0_{l,j} \otimes \rho^0_{j+1,j+l}) \) giving

\[ S(\rho^0_{j-l+1,j}) + S(\rho^0_{j+1,j+l}) - S(\rho^0_{l,j} \otimes \rho^0_{j+1,j+l}) \geq (1 - 2\epsilon(l)) \ln((1 - 2\epsilon(l))/x) \]
\[ + 2\epsilon(l) \ln(2\epsilon(l)/(1 - x)). \quad (25) \]

Equation (25) is the key step. Everything that follows consists of picking the constant \( C_2 \) correctly. We first assume that \( l_0 \leq (S_{\text{cut}} - F(\xi', D))/\ln(D) - \xi' \ln(C_1(\xi)) \). If this assumption fails, then since \( l_0 = S_{\text{cut}}/2 \ln(D) \), we have \( S_{\text{cut}}/2 \ln(D) \leq F(\xi', D)/\ln(D) + \xi' \ln(C_1(\xi)) \), and thus by picking the constant \( c_0 \) large enough equation (3) will still hold. Then, since \( l \leq l_0 \) it follows from this assumption that \( 2C_2(\xi)^2 \exp[-(S_{\text{cut}} - F(\xi', D))/\ln(D)\xi'] \leq 2\epsilon(l) \). Then from equation (24), \( x \leq (4\epsilon(l) + \sqrt{2\epsilon(l)}x)/(1 - 2\epsilon(x)) \).

Thus, as \( l \) becomes large, we find that \( x \) and \( \epsilon(l) \) approach zero exponentially. Thus, for large enough \( l \), we have

\[ S(\rho^0_{j-l+1,j}) + S(\rho^0_{j+1,j+l}) - S(\rho^0_{l,j} \otimes \rho^0_{j+1,j+l}) \geq (1 - 2\epsilon(l)) \ln(1/x) + \text{some constant of order unity and hence for all } l \text{ we have} \]

\[ S(\rho^0_{j-l+1,j}) + S(\rho^0_{j+1,j+l}) - S(\rho^0_{l,j} \otimes \rho^0_{j+1,j+l}) \geq (1 - 2\epsilon(l)) \ln(1/\epsilon(l)) - C_2, \quad (26) \]

where \( C_2 \) is a numeric constant of order unity. This shows equation (12) and completes the proof.

2. Matrix product states

Our main result is the bound (2). We now use the existence of a bound on entropy to construct an approximation to the ground state by a matrix product state. At first, this might seem difficult, given that previous such constructions [13] relied on the existence of a bound on the Rényi entropy, and we have a bound on the von Neumann entropy. However ideas similar to those used in the bootstrap lemma 2 will let us avoid these difficulties.

The construction applies to any system given the finite interaction range, finite interaction strength, and gap conditions, and given the existence of a bound \( S(\rho^0_{j,l}) \leq S_{\text{max}} \) for all \( j \). Consider a given \( j \) and write the ground state as in lemma 2 by \( \Psi_0 = \sum_{\alpha=1} A_0(\alpha)\Psi_{L,0}(\alpha) \otimes \Psi_{R,0}(\alpha) \), with \( \Psi_{L,0}(\alpha) \), \( \Psi_{R,0}(\alpha) \) being states on \( X_{1,j} \) and \( X_{j+1,0} \) respectively and the states ordered so that \( |A_0(\alpha)| \) is a non-increasing function of \( \alpha \).
Suppose for some \( k' \) we have \( \sum_{\alpha=0}^{\infty} |A_0(\alpha)|^2 > 1/2 \). Then, \( |A_0(k' + 1)|^2 < 1/2k \). Thus, \( S(\rho_{1,j}) > (1/2) \ln(2k') \). So, \( k' > \exp(2S(\rho_{1,j}))/2 \). Hence, for

\[
 k_0 = \exp(2S(\rho_{1,j}))/2
\]

we have \( \sum_{\alpha=1}^{k_0} |A_0(\alpha)|^2 \geq 1/2 \). At this point we could directly apply lemma 2 to bound the Rényi entropies, and thus get a matrix product form following results in [13], but we prefer to proceed more directly. We apply equation (18) with \( P = 1/2 \), getting

\[
\sum_{\alpha \geq k_0 D^{2m+1}} |A_0(\alpha)|^2 \leq 4C_1(\xi)^2 \exp[-2m/\xi'].
\]

Therefore,

\[
\sum_{\alpha \geq k'} |A_0(\alpha)|^2 \leq 4C_1(\xi)^2 \exp(-[\log_D(k'/k_0)]/\xi') \leq 4C_1(\xi)^2 \exp(1/\xi')(k'/k_0)^{1/\xi' \ln(D)}.
\]

This provides an estimate on how rapidly the Schmidt coefficients decay and hence how accurately the ground state may be approximated by a matrix product state. In particular, for a chain of length \( N \), the error in approximating the ground state by a matrix product state of bond dimension \( k' \) scales as \( N(k'/k_0)^{1/\xi' \ln(D)} \).

### 3. Discussion and a conjecture on matrix product states

We now further explore the relationship between a gap, exponentially decaying correlations, and an area law. Consider a one-dimensional system with a gap. We know that this state must have exponentially decaying correlations. However, this in itself does not imply an area law. For example, there exist matrix product states

\[
\Psi(s_1, s_2, \ldots s_N) = \sum_{\alpha, \beta, \gamma, \delta} A_{\alpha, \beta}(s_1) A_{\beta, \gamma}(s_2) A_{\gamma, \delta}(s_3) \ldots
\]

with \( s_i = 1 \ldots D \) in which the associated completely positive map forms a quantum expander so that a system with a low Hilbert space dimension \( D \) on each site and a short correlation length of order unity may have a very large entropy [6]. Consider, however, the following model of a quantum expander modeled on that in [6]: we have a graph with coordination number \( D \) (which is a classical expander graph), where each node of the graph corresponds to a value of the bond variable \( \alpha \), each link from one node to another gets labeled with one particular value of \( s \), and \( A_{\alpha, \beta}(s) \) is non-vanishing only if the link from \( \alpha \) to \( \beta \) is labeled with the given \( s \). The total number of nodes in the graph is equal to the range of the bond variables, and we will denote this number by \( k \). For a given value of \( \alpha \), the bond variable \( \beta \) can assume \( D \) different values, \( \gamma \) can assume \( D(D-1) \) different values, and so on, so that any correlations between a bond variable \( \alpha \) and another far away bond variable which connects a distant pair of sites become small as required. This behavior is shown in figure 1 for a system with \( D = 3 \) and an arbitrary possible choice of \( \alpha, \beta, \gamma, \delta, \epsilon \).

However, as we have seen in this paper, if there is entropy across a bond variable in a gapped, local system, then there must be mutual information of order \( l/\xi' \) between the set of sites within some distance \( l \) to the left of the bond and the set of sites within some distance \( l \) to the right of the bond. However, in the given expander map, there is

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Figure 1. Illustration of an expander graph, with a particular choice of \(\alpha, \beta, \ldots\) marked. Note that each Greek index marks a site which is one site away from the previous index.

no such mutual information. Consider a given set of sites \(i = i_{\text{min}}, i_{\text{min}} + 1, \ldots, i_{\text{max}}\), with \(i_{\text{max}} - i_{\text{min}} \ll N\) and \(D_{i_{\text{max}} - i_{\text{min}} + 1}\) much smaller than the range \(k\) of the bond variable \(\alpha\). Trace out all sites outside to construct a reduced density matrix on the given sites: the result is proportional to the identity matrix and any of the \(D_{i_{\text{max}} - i_{\text{min}} + 1}\) states is possible with equal probability. Thus, this expander map is not the ground state of a Hamiltonian with an energy gap of order unity and an interaction range of order unity, even though it is the ground state of a gapped Hamiltonian with an interaction range of order \(\log_{N} k\). In a sense, this expander map is mixing too quickly to be the ground state of a gapped Hamiltonian.

There is, however, an interesting alternative way of seeing that this expander map cannot be the ground state of a gapped local Hamiltonian, which will lead to a conjecture we have on matrix product states. We begin with a result on the decay of a certain kind of correlation function.

Lemma 3. Let \(H\) a Hamiltonian \(H\) which satisfies the finite range, finite interaction strength, and gap conditions. Let \(\Psi_{0}\) be the ground state of \(H\). Suppose \(\Psi_{0} = \sum_{\alpha=1} A_{0}(\alpha)\Psi_{L,0}(\alpha) \otimes \Psi_{R,0}(\alpha)\), where \(\Psi_{L,0}(\alpha)\) are orthonormal states on \(X_{1,j}\) and \(\Psi_{R,0}(\alpha)\) are orthonormal states on \(X_{j_{+1},N}\). Let \(B_{L} = \sum_{\alpha=1} O(\alpha)\Psi_{L,0}(\alpha)\rangle\langle \Psi_{L,0} \otimes \mathbb{I}_{R}\), where \(\mathbb{I}_{R}\) is the unit operator on \(X_{j_{+1},N}\), for some function \(O(\alpha)\). Similarly, let \(B_{R} = \mathbb{I}_{L} \otimes \sum_{\alpha=1} O(\alpha)\Psi_{R,0}(\alpha)\rangle\langle \Psi_{R,0}\). Suppose \(\|B_{L}\| \leq 1\), so that \(\|O(\alpha)\| \leq 1\) for all \(\alpha\). Finally, let \(A\) be an operator with support on \(X_{1,j-1} \cup X_{j_{+1},N}\) and with \(\|A\| \leq 1\). Then

\[
\langle \Psi_{0}, AB_{L}\Psi_{0}\rangle - \langle \Psi_{0}, A\Psi_{0}\rangle\langle \Psi_{0}, B_{L}\Psi_{0}\rangle \leq 3\sqrt{2\epsilon(l)} + \epsilon(l),
\]

where \(\epsilon(l)\) is given as before by \(C_{1}(\xi) \exp[-l/\xi]\).

**Proof.** To prove this, define \(O_{B}(j,l), O_{L}(j,l), O_{R}(j,l)\) as in Lemma 1. Recall that \(\langle \Psi_{0}, O_{B}(j,l)\Psi_{0}\rangle \geq 1 - 2\epsilon(l)\) and \(\langle \Psi_{0}, O_{R}(j,l)\Psi_{0}\rangle \geq 1 - 2\epsilon(l)\). Hence, \(|(O_{B}(j,l) - 1)\Psi_{0}| \leq \ldots\)

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An area law for one-dimensional quantum systems

\[\sqrt{2\epsilon(l)} \text{ and } |(O_R(j, l) - 1)\Psi_0| \leq \sqrt{2\epsilon(l)}. \] We now use a series of triangle inequalities. First,

\[|\langle \Psi_0, AB_L\Psi_0 \rangle - \langle \Psi_0, AB_LO_R(j, l)\Psi_0 \rangle| \leq \sqrt{2\epsilon(l)}. \] (32)

Next,

\[|\langle \Psi_0, AB_LO_R(j, l)\Psi_0 \rangle - \langle \Psi_0, O_B(j, l)AB_LO_R(j, l)\Psi_0 \rangle| \leq \sqrt{2\epsilon(l)}. \] (33)

However, \([O_R(j, l), B_L] = 0 \) and \([O_B(j, l), A] = 0\). Also, \(B_L\Psi_0 = B_R\Psi_0\). Thus,

\[\langle \Psi_0, O_B(j, l)AB_LO_R(j, l)\Psi_0 \rangle = \langle \Psi_0, AO_B(j, l)O_R(j, l)B_R\Psi_0 \rangle. \] (34)

Next,

\[|\langle \Psi_0, AO_B(j, l)O_R(j, l)B_R\Psi_0 \rangle - \langle \Psi_0, AO_B(j, l)O_R(j, l)B_RO_L(j, l)\Psi_0 \rangle| \leq \sqrt{2\epsilon(l)}. \] (35)

Using \([B_R, O_L(j, l)] = 0\) we find that

\[|\langle \Psi_0, AB_L\Psi_0 \rangle - \langle \Psi_0, AO_B(j, l)O_R(j, l)O_L(j, l)B_R\Psi_0 \rangle| \leq 3\sqrt{2\epsilon(l)}. \] (36)

Thus, \(|\langle \Psi_0, AB_L\Psi_0 \rangle - \langle AP_OB_L\Psi_0 \rangle| \leq 3\sqrt{2\epsilon(l)} + \epsilon(l), \) completing the proof.

Now, this lemma also implies that the matrix product state arising from the given expander map cannot be the ground state of a Hamiltonian with gap and Lieb–Robinson velocity of order unity. In a slight abuse of notation, let \(\alpha\) be the bond variable which joins site \(j - l\) to \(j - l + 1\), let \(\gamma\) be the bond variable which joins site \(j\) to \(j + 1\), and let \(\epsilon\) be the bond variable which joins site \(j + l\) to \(j + l + 1\). This is shown in figure 1 for a system with \(l = 2\).

For any given value of the bond variable \(\alpha\), the bond variable \(\gamma\) can have a wide range of possible values, of order \((D - 1)^l\). However, for given values of both \(\alpha\) and \(\epsilon\), the possible range of values of \(\gamma\) is much more restricted, as \(\gamma\) will tend to lie on the shortest path joining \(\alpha\) to \(\epsilon\) as shown in the figure. In this manner, it is possible to construct operators \(A, B_L\) such that equation (31) is violated for this state, although we omit the detailed construction.

In fact, we have not been able to find any matrix product states which obey equation (31) while still having a large entanglement entropy. We thus make the following conjecture on matrix product states:

**Conjecture 1.** For any matrix product state \(\Psi_0\) as in equation (30), define \(\xi'(\Psi_0)\) to be the infimum of the set of all \(\xi'\) such that equation (31) is satisfied for all \(j, l, A, B_L\), where the operators \(A, B_L\) obey the same conditions as in lemma 3. Then, there exists a function \(f\) such that for any \(N\) and \(D\), for any \(\Psi_0\) as in equation (30) with \(\xi'(\Psi_0) < \infty\), the entropy bound \(S(\rho^0_{l,k}) \leq f(D^{\xi'(\Psi_0)}) < \infty\) holds for all \(k\).

If this conjecture were shown, then it would give a different way to prove an area law.
4. Conclusion

In conclusion we have given a proof of an area law, although the bound is quite weak. We note that the bootstrap lemma 2 gives some intuitive idea as to why it is difficult to prove bounds on the entanglement entropy. Given a good approximation to the ground state with a state of given Schmidt rank, we have a bound on the ground state entropy. Also, given a bound on the ground state entropy, we can estimate how well we can approximate the ground state with a state of given Schmidt rank as in equations (27) and (29). However, this kind of argument leads to circular reasoning and thus does not help provide an area law. These arguments, do however, interrelate the Von Neumann entropy, the Rényi entropy, and the error in approximating the ground state by a matrix product state in gapped system, as in equations (15), (27) and (29).

We finally consider the implications for numerical simulation of one-dimensional quantum systems. If such a system has a gap, then the ground state is close to a matrix product state. However, finding the best matrix product state may be hard problem [14]. Imagine, though, that we consider a family of Hamiltonians which start from a Hamiltonian with a known ground state and keep the gap open. Then, by following a quasi-adiabatic evolution [15] along this path and truncating the matrix product state to keep the bond dimension bounded, we can well approximate the ground state at the end of the evolution [16].

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Appendix. Approximating the projection operator

In this appendix we show how to approximate the projection operator by a product $O_B(j, l)O_L(j, l)O_R(j, l)$ as described in lemma 1. For simplicity, let us add a constant to $H$ so that the ground state energy is equal to zero. We start by recalling the result in [15], that it is possible, given a gapped Hamiltonian, to write the Hamiltonian as a sum of terms such each term approximately annihilates the ground state. We perform the derivation slightly differently to get a tighter bound ((A.3) and (A.4)) which depends only on the boundary of the terms. Let

\begin{align}
H_L &= \sum_{i,i \leq j-l/3} H_{i,i+1}, \\
H_B &= \sum_{i,j-l/3+1 \leq i \leq j+l/3} H_{i,i+1}, \\
H_R &= \sum_{i,i \geq j+1+l/3} H_{i,i+1},
\end{align}  

(A.1)
so that $H = H_L + H_B + H_R$. We then choose to add constants to $H_L, H_B, H_R$ so that $\langle H_L \rangle = \langle H_B \rangle = \langle H_R \rangle = 0$. Define

$$\tilde{H}^0_L = \frac{\Delta E}{\sqrt{2\pi q}} \int_{-\infty}^{\infty} dt \, H_L(t) \exp[-(t\Delta E)^2/2q],$$

$$\tilde{H}^0_B = \frac{\Delta E}{\sqrt{2\pi q}} \int_{-\infty}^{\infty} dt \, H_B(t) \exp[-(t\Delta E)^2/2q],$$

$$\tilde{H}^0_R = \frac{\Delta E}{\sqrt{2\pi q}} \int_{-\infty}^{\infty} dt \, H_R(t) \exp[-(t\Delta E)^2/2q],$$

where we pick $q = (l/3)\Delta E/(2v)$. Note that $\tilde{H}^0_L + \tilde{H}^0_B + \tilde{H}^0_R = H$. Let $b_L = [H, H_L]$. Note that $\|b_L\| \leq J^2$. Then

$$|\tilde{H}^0_L \Psi_0| \leq \Delta E^{-1} |H \tilde{H}^0_L \Psi_0| = \Delta E^{-1} |[H, \tilde{H}^0_L] \Psi_0| = \Delta E^{-1} \left| \frac{\Delta E}{\sqrt{2\pi q}} \int_{-\infty}^{\infty} dt \, b_L(t) \Psi_0 \right| \leq \Delta E^{-1} J^2 \mathcal{O}(\exp[-l/3\xi]),$$

where on the last inequality we used the assumption of a gap and $\mathcal{O}(\cdots)$ is used to denote a bound up to a numeric constant of order unity. Similarly,

$$|\tilde{H}^0_B \Psi_0| \leq \mathcal{O}(\Delta E^{-1} J^2 \exp[-l/3\xi]),$$

$$|\tilde{H}^0_R \Psi_0| \leq \mathcal{O}(\Delta E^{-1} J^2 \exp[-l/3\xi]).$$

Using the Lieb–Robinson bound, for the given value of $q$ it is possible to approximate $\tilde{H}^0_L, \tilde{H}^0_B, \tilde{H}^0_R$ by operators $M_L, M_B, M_R$ respectively such that $M_L - H_L$ is supported on $X_{j-2l/3,j}$, $M_B - H_B$ is supported on $X_{j-2l/3,j+1+2l/3}$, and $M_R - H_R$ is supported on $X_{j+1j+1+2l/3}$ and such that $\|M_L - H_L\| \leq \mathcal{O}(\Delta E^{-1} J^2 \exp[-l/3\xi]), \|M_B - H_B\| \leq \mathcal{O}(\Delta E^{-1} J^2 \exp[-l/3\xi]), \|M_R - H_R\| \leq \mathcal{O}(\Delta E^{-1} J^2 \exp[-l/3\xi])$.

Thus, we have $|M_L \Psi_0| \leq \mathcal{O}(\Delta E^{-1} J^2 \exp[-l/6\xi])$, and similarly for $M_B$ and $M_R$. We now define $O_L(j,l)$ to project onto eigenvectors of $M_L$ with eigenvalue less than or equal to $\Delta E^{-1} J^2 \exp[-l/6\xi]$, and define $O_R(j,l)$ to project onto eigenvectors of $M_R$ with eigenvalue less than or equal to $\Delta E^{-1} J^2 \exp[-l/6\xi]$. Thus,

$$|(O_L(j,l) - 1) \Psi_0| \leq \mathcal{O}(\exp[-l/6\xi]),$$

$$|(O_R(j,l) - 1) \Psi_0| \leq \mathcal{O}(\exp[-l/6\xi]).$$

Clearly, $O_L(j,l)$ and $O_R(j,l)$ are supported as required by lemma 1.

We now define an approximation to the projection operator

$$P_q \equiv \frac{\Delta E}{\sqrt{2\pi q}} \int dt \, \exp[iHt] \exp[-(t\Delta E)^2/2q].$$
Using the spectral gap we have \( \|P_q - P_0\| \leq \mathcal{O}(\exp[-t/3\xi]) \). Thus, using \( \|M_L + M_R + M_R - H\| \leq \mathcal{O}(\Delta E^{-1}J^2\exp[-t/3\xi]) \), we have
\[
\left\| \frac{\Delta E}{\sqrt{2\pi q}} \int dt \exp[i(M_L + M_R + M_R)t] \exp[-(t\Delta E)^2/2q] - P_0 \right\|
\]
\[
= \left\| \frac{\Delta E}{\sqrt{2\pi q}} \int dt \exp \left[ i \int_0^t dt' \exp(i(M_L + M_R)t') M_B \exp(-i(M_L + M_R)t') \right] \times \exp[i(M_L + M_R)t] \exp[-(t\Delta E)^2/2q] - P_0 \right\|
\]
\[
\leq \mathcal{O}(\Delta E^{-2}J^2\sqrt{q}\exp[-t/3\xi]), \tag{A.7}
\]
where the exponential of the integral over \( t' \) is \( t' \)-ordered. Thus from equation (A.5),
\[
\left\| \frac{\Delta E}{\sqrt{2\pi q}} \int dt \exp \left[ i \int_0^t dt' \exp(i(M_L + M_R)t') M_B \exp(-i(M_L + M_R)t') \right] \times \exp[i(M_L + M_R)t] \exp[-(t\Delta E)^2/2q]O_L(j,l)O_R(j,l) - P_0 \right\|
\]
\[
\leq \mathcal{O}(\Delta E^{-2}J^2\sqrt{q}\exp[-t/3\xi] + \exp[-t/6\xi]). \tag{A.8}
\]
However,
\[
\| \exp[i(M_L + M_R)t]O_L(j,l)O_R(j,l) - O_L(j,l)O_R(j,l) \| \leq 2|t|\Delta E^{-1}J^2\mathcal{O}(\exp[-t/6\xi]).
\]
Combining this with equation (A.8) we find that
\[
\left\| \frac{\Delta E}{\sqrt{2\pi q}} \int dt \exp \left[ i \int_0^t dt' \exp(i(M_L + M_R)t') M_B \exp(-i(M_L + M_R)t') \right] \times \exp[-(t\Delta E)^2/2q]O_L(j,l)O_R(j,l) - P_0 \right\|
\]
\[
\leq \mathcal{O}(\Delta E^{-2}J^2\sqrt{q}\exp[-t/6\xi]). \tag{A.9}
\]
Consider the operator
\[
P_B \equiv \frac{\Delta E}{\sqrt{2\pi q}} \int dt \exp \left[ i \int_0^t dt' \exp(i(M_L + M_R)t') M_B \exp(-i(M_L + M_R)t') \right] \times \exp[-(t\Delta E)^2/2q]. \tag{A.10}
\]
Using a Lieb–Robinson bound for \( M_L, M_R \) (and noting that the difference \( M_L + M_R - H_L - H_R \) has support on \( X_j-2l/3,j+1+2l/3 \)), we can approximate \( P_B \) by an operator \( O_B(j,l) \) with support on \( X_{j-l+1,j+l} \) such that \( \|P_B - O_B(j,l)\| \leq \mathcal{O}(\Delta E^{-1}J\sqrt{q}\exp[-t/6\xi]) \). Thus,
\[
\|O_B(j,l)O_L(j,l)O_R(j,l) - P_0\| \leq \mathcal{O} \left( \Delta E^{-2}J^2\sqrt{t/\xi^2}\exp[-t/6\xi] \right).
\]
This completes the result.

References

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To see that the Lindblad–Uhlmann theorem implies a bound on the relative entropy between two states when an operator, such as $O_B(j, l)$, has a different expectation value in the two states, see, for example, the discussion of ‘a posteriori relative entropy’ in Petz D, 2003 Rev. Math. Phys. 15 79.