Offline Estimation of Controlled Markov Chains: Minimax Nonparametric Estimators and Sample Efficiency

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Abstract

Controlled Markov chains (CMCs) form the bedrock for model-based reinforcement learning. In this work, we consider the estimation of the transition probability matrices of a finite-state finite-control CMC using a fixed dataset, collected using a so-called logging policy, and develop minimax sample complexity bounds for nonparametric estimation of these transition probability matrices. Our results are general, and the statistical bounds depend on the logging policy through a natural mixing coefficient. We demonstrate an interesting trade-off between stronger assumptions on mixing versus requiring more samples to achieve a particular PAC-bound. We demonstrate the validity of our results under various examples, such as ergodic Markov chains, weakly ergodic inhomogeneous Markov chains, and controlled Markov chains with non-stationary Markov, episodic, and greedy controls. Lastly, we use these sample complexity bounds to establish concomitant ones for offline evaluation of stationary, Markov policies.

1 Introduction

This paper presents probably approximately correct (PAC)-style minimax sample complexity results for statistical estimation of transition kernels of discrete-time, finite controlled Markov chains (CMC) \(^7\), motivated by the need for rigorous statistical inference in offline reinforcement learning (RL) and system identification problems. CMCs represent a natural model setting for stochastic optimal control problems, but also reinforcement learning problems, including pure or reward-free exploration and/or learning a value function or optimal policy when endowed with a reward structure. In fact, a wide range of now standard RL methods have been developed to manage the exploration-exploitation tradeoff (see \([54]\)) inherent in RL, assuming that a reward signal is available in real time that simultaneously allows for estimating the optimal value function while exploring the policy space.

In contrast to this more well-known online RL setting, offline RL uses a fixed dataset, pre-collected under some fixed ‘logging’ policy or control sequence, to learn an optimal value function or a policy function \([30]\). Offline RL problems are fundamentally different, and significantly harder to solve, in comparison with online RL problems, owing to the fact that the agent cannot interact with the environment repeatedly (implying that globally suboptimal decisions can be corrected). However, the offline RL framework is quite natural in the setting of learning to manage operations and service systems, and decision-making in mission-critical settings, where it is either too risky and/or uneconomical to use online RL. To fix these ideas, consider the following two vignettes from clinical and operational decision-making.
A Training an RL agent to aid clinical decision-making in medicine has received substantial interest recently [52, 33, 69, 11]. Data sets on prognosis, diagnosis, and treatment decisions made by physicians have been proposed to be used to train RL agents to potentially identify new (superior) paths to achieving the same (better) outcomes for patients. A crucial fact, as noted in [33], is that “Unlike other domains, such as game playing...the RL agent has to learn from a limited set of data and intervention[s]...collected offline.”

B Analogous learning problems also arise in a variety of manufacturing and service operations management settings. Recent literature has considered the problem of learning to route jobs in a multi-class stochastic processing network, using online RL [13]. However, in many application settings, such as patient flow modelling in a hospital network, as observed in [2], data collected using pre-existing flow control and routing policies can be mined to discover new/better protocols and policies. Offline RL is a natural learning framework to achieve this.

We observe that offline RL methods can be categorized as either model-free or model-based; these notions parallel the definitions in the online RL setting. Model-free methods aim at learning an (optimal) value function without directly learning a CMC model of the underlying stochastic dynamics; in contrast, learning in model-based methods involves two steps – system identification to estimate a CMC model of the system dynamics, followed by value function estimation and policy evaluation using the estimated system dynamics model. Clearly, the estimation errors in the latter step are a function of those in system identification.

The desideratum for sample efficiency arises from the fact that in many application settings datasets are limited in size and collecting new data may be prohibitively expensive, as observed in [33, 2] above. Furthermore, it is often the case that the datasets consist of a single, finite length sample path of a stochastic system under observation; see [2] in the patient flow context, for example. Concomitant statistical inferential questions are further complicated by the fact that the logging policy used to collect data is unknown or structurally complex; again, see the setting of medical decision-making [33] above. Of course, assuming that the policy is stationary Markovian (i.e., for instance, solely dependent on the current state of a patient) is technically appealing as it simplifies the inference problems, often reducing it to the setting of an ergodic Markov chain (and opening the door to a plethora of results, including frequentist PAC bounds [62], Bayesian PAC bounds [4], asymptotic convergence [65] etc.). On the other hand, if the logging policy is non-Markovian, the corresponding stochastic dynamics are not Markovian, and statistical inference results are quite sparse for this setting. Highlighting the importance of non-Markovian policies, a recent paper by Mutti et al. [42] argues that Markovianity is actually insufficient for finite sample exploration of the state-control space and one must consider non-Markovian policies for optimal reward-free exploration in online RL settings.

Given these requirements, an understanding of the sample complexity of system identification and offline policy evaluation is an important open problem. This paper precisely addresses this question in the context of finite-state CMCs. We adopt a broader view of CMCs following [7] and model the sequence of controls generated by the logging policy as an adapted and ‘mixing’ (in a sense defined in Assumptions 3 and 4) stochastic sequence. We focus specifically on the robustness of a simple non-parametric estimator of CMC transition kernels. Although non-parametric estimators of transition kernels have been used in various model-based RL studies, for instance [31, 38], there are no extant statistical guarantees on the specific estimator we study. This paper fills this obvious gap in the literature. The main contributions of this paper can be summarised as follows.

1. We show that the non-parametric estimator is minimax optimal in Theorem 2 if the number of samples is large enough and identify an explicit lower bound on the sample size to achieve a high probability estimate. Informally, we prove that the sample complexity of estimating the transition matrices in a CMC with $d$ states and $k$ controls is $\Theta(d^2 k)$ if the CMC is geometrically fast mixing. As we argue in section 4, a geometrically ergodic Markov chain with $d k$ states can be thought of as a special
case of a $d$-state, $k$-control CMC where the controls are so-called “stationary”. Therefore, our result (Theorem 2) recovers the optimal sample complexity of estimating Markov chains from [62] as a special case.

2. We tease out a subtle trade-off between assuming stronger mixing conditions and obtaining tighter sample complexity. In particular, we prove in Theorem 1 that the transition probabilities can be estimated even under a weaker mixing assumption than is required for minimaxity (Theorem 2). However, there is a trade-off in requiring more samples (roughly $\Omega(d^2k^2)$ in place of $\Omega(d^2k)$) to achieve the same level of estimation error and, thus, losing guarantees of minimax optimality.

3. Our sample complexity results on the transition kernel estimator immediately yield error bounds on the offline policy evaluation (OPE). OPE in offline model-based RL involves estimating the average reward obtained from a given policy (different from the logging policy) using the estimated transition kernel as a plug-in. In particular, Theorem 4 focuses on evaluating stationary Markov policies on a system model estimated using data logged under a mixing logging control sequence that may not be Markovian. The sample complexity results recover minimax optimal rates obtained in the literature where the logging policy is universally assumed to be Markovian.

From a methodological perspective, our analysis reveals two principles that are broadly useful in establishing sample efficiency results for learning CMCs and other controlled stochastic models. First, a crucial fact underlying the results in this paper is a ‘Goldilocks principle’ that no state-control pair must be visited too many or too few times in a single observed sample path. This can be achieved by ensuring that the control sequence is such that no part of the state-control space is visited deterministically (i.e., the controlled process should neither visit a particular pair with probability one, nor not visit a pair with probability one), and that the time-to-return to a particular pair is uniformly bounded over the state-control space (see Assumptions 1 and 2).

Second, the effect of history on the probability of under- or over-visiting any part of the state-control space is controlled by the mixing properties of the control sequence, as defined in Assumption 5 and Assumption 7. Broadly speaking, weaker mixing properties imply looser bounds on these probabilities, in turn implying that estimators are possibly sample inefficient. The bulk of the existing literature on offline estimation of CMCs focuses on the setting where the control sequence forms a stationary ergodic Markov sequence and, under this condition, the nonparametric estimator is minimax optimal, as implied by Theorem 2 and Proposition 13. However, if the control sequence mixes comparatively slowly (say, polynomially), then Theorem 1 yields a loose sample complexity bound. This point is of particular interest in the presence of model misspecification. As we prove in Sections 4.2 and 4.4, it is relatively straightforward to verify the geometric mixing properties of the control sequences when the controls are Markovian. However, when the controls are non-Markovian, there is no general result to demonstrate geometric mixing. Thus, a practitioner must be cautious of erroneously assuming the logging policy to be Markovian when it is not. If the controls are not Markovian, then $\Omega(d^2k^2)$ samples are no longer sufficient to control the probability of large estimation errors on the transition probabilities. Instead, one needs $\Omega(d^2k^2)$ samples to guarantee a low probability of large estimation errors.

As a final note on the methodological implications, while we focus on finite state-control spaces, we believe that these principles, and our analysis, yield a broad framework for proving sample efficiency results for offline estimation of CMCs, and (potentially) other controlled stochastic models, under much broader modelling conditions. For instance, if one uses a histogram or a density estimator of a transition kernel on continuous state spaces and compact control spaces, then our results are directly applicable, although the optimal sample complexity would depend upon the smoothness properties of the transition function.

From an operational and applications perspective, we observe that online RL is increasingly seen as a vital tool in the modellers toolbox for discovering new control policies and treatment paths, particularly in
settings that are analytically hard to analyse; for instance, see [13] in the context of multi-class queueing networks. Offline RL, although apparently not as popular, is a natural learning framework for solving many data-driven operations management and decision-making problems, wherein a fixed dataset of system performance and inputs has been precollected. As noted before, this is a fundamentally different setting from online RL, and yet a very natural one for the broad audience of this journal. The sharp sample complexity results in this paper provide confidence in the conditions under which a natural non-parametric estimator of the system dynamics (and hence, value function estimators) is sample efficient. The methodological principles outlined above should provide guidelines for statistical inference in specific operational modelling problems.

1.1 Related Literature

We divide the literature review into two parts. In the first part, we place our results in the context of the existing literature and on non-parametric estimation for stochastic processes. In the second part, we relate our sample complexity results to existing relevant ones in the literature on offline RL, and system identification.

Non-parametric estimation. The foundations of non-parametric estimation [57] of finite ergodic Markov chains were laid by Billingsley [6]. Subsequently, Yakowitz [66] presented an important extension to infinite state spaces, with follow-up work on applications to regression [65]. There is also extensive literature establishing laws of large numbers (LLNs) [18] and central limit theorems (CLTs) [23] for a range of time-homogeneous Markov chains. However, somewhat surprisingly, Minimax sample complexity bounds for finite ergodic Markov chains were only established recently in [62]. However, barring some results on Laws of Large Numbers, and CLTs [47, 48, 14, 15], results on statistical inference for time-inhomogenous Markov chains remain sparse. Furthermore, such properties when the controls are stochastic in nature are even less understood. A crucial complication in our setting is that the state-control pair process need not be Markovian, complicating the application of existing results. Nonetheless, we recover rates similar to those of Wolfer and Kontorovich [62] as a special case in section 4.2, which demonstrates the generality of our results.

Offline reinforcement learning. Model-based offline reinforcement learning methods [25, 72, 51] have recently received significant attention in the literature. Controlled Markov chains are most relevant in offline reinforcement learning as a tool towards maximising a value function. It involves constructing a “model” for the transition probability matrix and then using it to solve the expected Bellman equation (a detailed discussion of the problem can be found in Levine et al. [30, Section 2.1].

Although there is a plethora of articles based on model-free learning the value or Q functions [68, 44, 51], few articles discuss the optimality of their results. Notable works in this regard are the trio of papers [31, 32, 67], which prove optimality of their results by proving minimaxity of their recovered sample complexities. Crucially, their work involved the more limited scope of discounted or finite-horizon problems under “Markovian” policies.

Of particular interest is the fact that, in many optimal model-based RL problems (for example, [31, 32, 67]), our proposed empirical estimator reflects their model. Since we discard the assumption of Markovianity, our study, conclusively provide evidence that not only is the empirical estimator a correct model to choose for model-based offline RL, it is also the best model to choose for many important statistical questions. Furthermore, it continues to be an optimal estimator in the non-Markovian regime under amenable mixing conditions of the stochastic process.
System identification. The problem setting in this paper is also contiguous with system identification in optimal control theory [60, 35, 56]. System identification revolves around being able to identify the transition dynamics of a dynamical system, which for us culminates in the estimation of the transition matrices. Most of the work on system identification revolves around the so-called “active learning” [37, 12, 45] which aims to learn the system dynamics in a fashion similar to online RL. However, due to the obvious cost benefits of being able to use offline data, there has been a growing interest in the question of offline learning of the system [55, 27, 36]. We stress that the estimation of the transition matrix is congruent to the problem of system identification. Furthermore, we solve the problem in greater generality and without strong linear dynamics assumptions that are prevalent in that literature.

Outline. The rough outline of the paper is as follows. In Section 2, we introduce some notations and concepts of uniform mixing and weak mixing. In Section 3, we construct the empirical estimator $\hat{M}^{(l)}$ for the transition matrix $M^{(l)}$ for any control $l$ and formally introduce our assumptions. We then illustrate the trade-off discussed previously by producing weaker PAC bounds for the estimation error $\sup_C \| \hat{M}^{(l)} - M^{(l)} \|_{\infty}$ under weaker mixing assumptions, and a stronger minimax PAC bound under stronger mixing assumptions. In Section 4, we apply our main result to derive statistical guarantees for various reward-free offline RL tasks under a range of settings, such as stationary controls, Markov controls, and episodic controls. Finally, in Section 5, we use our estimator to obtain estimation guarantees for learning the value function. We end with a summary and discussion of the open questions in Section 6.

2 Preliminaries

Notations. Let $\mathbb{N}$ and $\mathbb{R}$, respectively, denote the set of natural and real numbers. The symbol $[\cdot]$ represents the floor function. All random variables in this paper will be defined with respect to a filtered probability sample space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{P})$, where $\mathcal{F}$ is a $\sigma$-algebra and $\mathbb{P} := \{ \mathcal{F}_i \}$, with $\mathcal{F}_i \subset \mathcal{F}$, is a given filtration. Let $\{ X_i \}$ represent a discrete-time stochastic process adapted to filtration $\mathbb{P}$, with finite state space $\chi$. $\mathcal{L}(X)$ denotes the law associated with the random variable $X$, $\mathbb{E}[X]$ the expectation and $\sigma(X)$ the $\sigma$-algebra induced by $X$. A $d \times d$ matrix $Q$ is a stochastic matrix if the rows of $Q$ add up to 1 and $Q_{s,t}$ denotes the $(s,t)$th entry of $Q$. Let $I$ be a finite set representing the control space, and $\{ a_i \}$ represent the sequence of controls where $a_i \in I \forall i$.

Definitions. Following [7], we define a controlled Markov chain (CMC) as an $\mathbb{F}$-adapted pair process $\{(X_i, a_i)\}$, where the process $\{X_i\}$ satisfies ‘controlled’ Markovian dynamics,

$\mathbb{P} \left( X_{i+1} = s_{i+1} | \mathcal{F}_i \right) = \mathbb{P} \left( X_{i+1} = s_{i+1} | X_i = s_i, a_i = l \right) =: M^{(l)}_{s_i, s_{i+1}}$.

Let $\mathcal{M} := \{ M^{(1)}, \ldots, M^{(k)} \}$ represent the set of transition probability matrices where $M^{(l)} = \left[ M_{s_i, s_{i+1}}^{(l)} \right]$ for all $s, s' \in \chi$ and $l \in I$. The control sequence $\{ a_i \}$ is assumed to satisfy $a_i \in \mathcal{F}_i$ for each $i \geq 0$ (i.e., $\{ a_i \}$ is an adapted sequence of controls). In the case, $\{ a_i \}$ is deterministic, $\{ X_i \}$ forms a time inhomogeneous Markov chain, where the transition kernel changes at time step $i$ according to the control $a_i$. Note that there are a finite number of possible transition kernels (since $I$ is assumed to be finite). Observe, in particular, that if $a_i = f(X_i)$, for some given function $f : S \rightarrow I$, then $\{ a_i \}$ is a Markov control sequence.

For two time points $i < j$, we define the history $\mathcal{H}^j_i$ to be $\mathcal{H}^j_i := \sigma(X_j, a_j, \ldots, X_i, a_i) \subset \mathcal{F}_j$ and sample history $h^j_i \in (\chi \times I)^{(j-i)}$ to be a fixed sequence of states and controls $h^j_i := (s_j, l_j, \ldots, s_i, a_i)$.
Unless stated otherwise, the initial states are assumed to be sampled from some distribution $D_0$ over $\chi \times \mathbb{I}$.

Next, we define the ‘time to return’ for every pair of states and controls $(X_i, a_i)$.

**Definition 1.** When $i = 1$ the first hitting time $(s, l)$ is defined as

$$
\tau_{s,l}^{(1)} := \min \{n : (X_n = s, a_n = l), (X_j \neq s, a_n \neq l) \text{ } \forall \text{ } 0 < j < n \}. 
$$

When $i \geq 2$ the $i$-th time to return of the state-control pair $(s, l)$ is recursively defined as

$$
\tau_{s,l}^{(i)} := \min \{n : (X_{\sum_{k=1}^{i-1} \tau_{s,l}^{(k)} + n} = s, a_{\sum_{k=1}^{i-1} \tau_{s,l}^{(k)} + n} = l), (X_j \neq s \cup a_j \neq l) \text{ } \forall \text{ } i-1 \leq j < \sum_{k=1}^{i-1} \tau_{s,l}^{(k)} + n \}. 
$$

**Remark.** In the ensuing sections, we will use the phrases time to return and return time interchangeably.

### 2.1 Mixing Coefficients

In this subsection, we define the weak and uniform mixing coefficients, and some fundamental lemmas concerning them. Let $\{X_i, a_i\}$ be an CMC.

For any $j \leq m \in \mathbb{N}$, let $T \in (\chi \times \mathbb{I})^{m-j}$, $s_1, s_2 \in \chi$, and $l_1, l_2 \in \mathbb{I}$. Finally, let $h_0^{i-1}$ be an element of $(\chi \times \mathbb{I})^i$. Define the map $(T, s_1, s_2, l_1, l_2, h_0^{i-1}) \rightarrow \eta_{i,j}(T, s_1, s_2, l_1, l_2, h_0^{i-1})$ as

$$
\eta_{i,j}(T, s_1, s_2, l_1, l_2, h_0^{i-1}) := |\mathbb{P}((X_m, a_m, \ldots, X_j, a_j) \in T | X_i = s_1, a_i = l_1, h_0^{i-1} = h_0^{i-1}) - \mathbb{P}((X_m, a_m, \ldots, X_j, a_j) \in T | X_i = s_2, a_i = l_2, h_0^{i-1} = h_0^{i-1})|.
$$

Then the weak-mixing coefficient $\bar{\eta}_{i,j}$ is

$$
\bar{\eta}_{i,j} := \sup_{T, s_1, s_2, l_1, l_2, h_0^{i-1}, \mathbb{P}(X_i = s_1, a_i = l_1, h_0^{i-1} = h_0^{i-1}) > 0, \mathbb{P}(X_i = s_2, a_i = l_2, h_0^{i-1} = h_0^{i-1}) > 0} \eta_{i,j}(T, s_1, s_2, l_1, l_2, h_0^{i-1}). \tag{2.1}
$$

Next, we define the uniform mixing coefficient. As before, let $T \in (\chi \times \mathbb{I})^{m-j}$, and $h_0^i$ be an element of $(\chi \times \mathbb{I})^{i+1}$. Then, the uniform-mixing coefficient is defined as

$$
\sup_{T, h_0^i} \phi_{i,j} := \sup_{T, h_0^i} |\mathbb{P}((X_m, a_m, \ldots, X_j, a_j) \in T | h_0^i = h_0^i) - \mathbb{P}((X_m, a_m, \ldots, X_j, a_j) \in T)|. \tag{2.2}
$$

**Lemma 1.** The uniform and the weak mixing coefficients defined in equations 2.1 and 2.2 (respectively) satisfy

$$
\phi_{i,j} \leq \bar{\eta}_{i,j} \leq 2\phi_{i,j}.
$$

Next, consider the following version of Hölder’s inequality for uniformly mixing sequences from Hall and Heyde [22, Theorem A.6], by setting $p = 1$ and $q = \infty$ in that result.

**Lemma 2.** For any two real functions $f$, and $g$,

$$
|\text{Cov}(f(X_j, a_j), g(X_i, a_i))| \leq \phi_{i,j} \mathbb{E}|f(X_j, a_j) - \mathbb{E}[f(X_j, a_j)]| \text{ ess sup }_{X_i, a_i} |g(X_i, a_i)|.
$$

It follows from Lemma 1 that,

$$
|\text{Cov}(f(X_j, a_j), g(X_i, a_i))| \leq \bar{\eta}_{i,j} \mathbb{E}|f(X_j, a_j) - \mathbb{E}[f(X_j, a_j)]| \text{ ess sup }_{X_i, a_i} |g(X_i, a_i)|.
$$
3 Empirical Estimation of Transition Probability Matrices

As stated before, our objective is to estimate the transition kernels of the Markov chain from a single, large but finite sample path of length $m \gg 1$. Consequently, in this section, we formally describe the estimation problem and our proposed empirical estimator $\hat{M}$ of the CMC. Recall that the sequence of controls $\{a_0, a_1, \ldots\}$ is an adapted stochastic sequence of controls taking values $\mathbb{I}$. It can be readily seen that $\{X_i, a_i\}$ need not be Markovian or ergodic. In particular, we focus on the ‘natural’ empirical estimator of the transition kernels for any given state-control-state (SCS) tuple, $(s, l, t)$. Define the number of visits to a state control pair $(s, l)$, $N_{s,l}^{(l)}$, and the number of transitions from state $s$ to state $t$ under $l$: $N_{s,t}^{(l)}$ as

$$N_{s}^{(l)} := \sum_{i=1}^{m} \mathbb{1}[X_i = s, a_i = l], \quad N_{s,t}^{(l)} := \sum_{i=1}^{m} \mathbb{1}[X_i = s, X_{i+1} = t, a_i = l].$$ (3.1)

Correspondingly, define the empirical estimator of $M_{s,t}^{(l)}$ as

$$\hat{M}_{s,t}^{(l)} := \frac{N_{s,t}^{(l)}}{N_{s}^{(l)}},$$ (3.2)

and construct the estimated matrix $\hat{M}^{(l)}$ as $[\hat{M}_{s,t}^{(l)}]$. The following proposition are obtained as a consequence of the previous definitions.

Proposition 3. Consider a sample $\{(X_0, a_0), \ldots, (X_m, a_m)\}$ from a controlled Markov chain.

Define

$$\hat{M}^{(l)}(s, \cdot) := \left(\hat{M}_{s,1}^{(l)}, \hat{M}_{s,2}^{(l)}, \ldots, \hat{M}_{s,d}^{(l)}\right), \quad \text{and} \quad M^{(l)}(s, \cdot) := \left(M_{s,1}^{(l)}, M_{s,2}^{(l)}, \ldots, M_{s,d}^{(l)}\right),$$ (3.3)

and let $0 < n_{low,s} < n_{high,s} < m$ be any two integers. Then we have

$$\mathbb{P}\left(\left\|\hat{M}^{(l)}(s, \cdot) - M^{(l)}(s, \cdot)\right\|_1 > \varepsilon, \; n_{low,s} \leq N_{s}^{(l)} \leq n_{high,s}\right) \leq m \exp\left(-\frac{n_{low,s}}{2}\max\left\{0, \varepsilon - \sqrt{\frac{d}{n_{high,s}}}\right\}^2\right).$$ (3.4)

It is apparent from Proposition 3 that one needs to control the number of visits to a state-control pair $N_{s}^{(l)}$ to find theoretical guarantees for $\hat{M}_{s,t}^{(l)}$. Here, we return to the 3 challenges of transitioning from a Markov chain to a controlled Markov chain.

1. Question of “Aperiodicity”. As mentioned previously, the notion of aperiodicity is fundamental to a Markov chain but ill-defined for a controlled Markov chain. As an example, consider the following three transition probability matrices

$$M^{(1)} = \begin{bmatrix} 0 & 1 & 0 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}, \quad M^{(2)} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}, \quad M^{(3)} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1 & 0 & 0 \end{bmatrix}.$$

It can be verified easily that each of the transition probability matrices are aperiodic (and, in fact, ergodic). However, consider a time-inhomogenous Markov chain on state-space $\{1, 2, 3\}$ where the transition matrices arrive in a sequence “$M^{(1)}, M^{(2)}, M^{(3)}, M^{(1)}, M^{(2)}, M^{(3)}, \ldots$”. Not only is it periodic if the initial state is 1, it is guaranteed to eventually become periodic!
2. **Question of “Irreducibility”**. As mentioned previously, irreducibility [40, page 85] is a fundamental concept to any Markov chain. Furthermore, observe from [40, Theorem 13.0.1] that an aperiodic and irreducible Markov chain admits a stationary distribution (defined as in [40, Chapter 13]), a key consequence of which is KAC’s theorem [40, Theorem 10.2.2] establishing the finiteness of the return times of every state in a Markov chain. However, such notions do not translate to a controlled Markov chain.

3. **Question of “Mixing”**. We already mentioned that an ergodic Markov chain on a finite state space is uniformly mixing geometrically fast. However, there exist no such equivalent result for controlled Markov chain. Furthermore, a glaring question remains about the speed of mixing of controlled Markov chains. Although geometrically fast mixing might be sufficient, it might not be necessary for the estimation of transition probability matrices. Finally, one may wonder how the overall mixing of the controlled Markov chain affects the mixing of the states and the controls.

Consequently, we make simplifying assumptions that allow us to mitigate the challenges. Our first two assumptions address the notions of aperiodicity and irreducibility. Informally, we need to ensure that no part of the chain is deterministic in nature, and every state-control pair \((s, l)\) is visited sufficiently often. Our first assumption follows.

**Assumption 1.** For all time steps \(i\), there exists two constants \(\zeta_1\) and \(\zeta_2\) and a set \(S_i \subset \{1, \ldots, d\} \times I\) such that

\[
0 < \zeta_2 \leq P[(X_i, a_i) \in S_i] \leq \zeta_1 < 1.
\]

**Remark.** Since we can always discard finitely many samples, our results continue to hold even when the controlled Markov chain satisfies the previous assumptions on all but a finite number of time points. However, it would lead to more cumbersome (but very similar) calculations.

**Assumption 2 (Return Time).** There exists an integer \(T > 0\) such that,

\[
\sup_{s,l,i} E\left[\tau_{s,l}^{(i)} | F_{\sum_{p=0}^{i-1} \tau_{s,l}^{(p)}}\right] < T \text{ almost everywhere}.
\]

The following is an immediate consequence of the previous assumptions.

**Lemma 4.** For any controlled Markov chain that satisfies Assumption 1 and Assumption 2,

\[
m \frac{T}{2} - 1 < E\left[N_s^{(l)}\right] \leq m \max\{\zeta_1, 1 - \zeta_2\},
\]

where \(\zeta_1, \zeta_2 \in (0,1)\) are defined in Assumption 1. In particular, if \(m \geq 2T\), then

\[
\frac{m}{2T} < E\left[N_s^{(l)}\right] \leq m \max\{\zeta_1, 1 - \zeta_2\}.
\]

**Remark.** Observe a parallel between this lemma and the minorization property described in various texts related to Markov chains such as Meyn and Tweedie [40, Chapter 5.1.1], Rosenthal [49], etc. In particular, when \(m = 1\) and \(k = 1\), and \(D_0\) is uniform over \(\chi\), we notice that this lemma recovers the minorization condition described in Meyn and Tweedie [40, Equation 5.3] for an uniform measure. Furthermore, taking summation over all \(l\) and \(s\) in the lower bound. Consequently,

\[
\sum_{s,l} \frac{m}{2T} < \sum_{s,l} E\left[N_s^{(l)}\right].
\]
Obviously, $\sum_{s,l} E \left[ N_{s,l}^{(i)} \right] = E \left[ \sum_{s,l} N_{s,l}^{(i)} \right] = m$. Therefore,
\[
dk \frac{m}{2T} < m,
\]
which in turn implies that,
\[
T > \frac{dk}{2}.
\]

Next, we make the following two assumptions on the $\bar{\eta}$-mixing coefficients of the stochastic process $\{X_i, a_i\}$:

**Assumption 3** ($\eta$-mixing). There exists a constant $C_\Delta > 1$ independent of $m$ such that,
\[
\|\Delta_m\| := \max_{1 \leq i \leq m} (1 + \bar{\eta}_{i,i+1} + \bar{\eta}_{i,i+2} + \ldots \bar{\eta}_{i,m}) \leq C_\Delta.
\]

**Assumption 4** (Exponential $\eta$-mixing). There exists an constant $C_\Delta > 1$ independent of $m$ such that,
\[
\bar{\eta}_{i,j} \leq \exp\left(- (j-i) \log \left( \frac{C_\Delta}{C_\Delta-1} \right) \right).
\]

**Remark.** It is obvious that if Assumption 4 is satisfied, so is Assumption 3 with the same constant. It also follows from Lemma 1 that if the $\bar{\eta}$-mixing coefficients satisfy the previous assumptions, so does the $\phi$-mixing coefficients with appropriately adjusted constants.

We have the following concentration inequalities as an immediate consequence of the previous assumptions.

**Proposition 5.** Consider a sample $\{(X_0, a_0), \ldots, (X_m, a_m)\}$ from a controlled Markov chain that satisfies Assumptions 1, 2, and 3. Let $N_{s,l}^{(i)}$ be the number of visits to state-control pair $(s, l)$ as defined in eq. (3.1). Then for all integers $n_{high,s} > \mathbb{E}[N_{s,l}^{(i)}] > n_{low,s}$, we have
\[
P(N_{s,l}^{(i)} \notin [n_{low,s}, n_{high,s}]) \leq 2 \exp\left( - \frac{(n_{low,s} - \mathbb{E}[N_{s,l}^{(i)}])^2}{2m (C_\Delta)^2} \right) + 2 \exp\left( - \frac{(n_{high,s} - \mathbb{E}[N_{s,l}^{(i)}])^2}{2m (C_\Delta)^2} \right).
\]

**Proof.** We observe that,
\[
P(N_{s,l}^{(i)} \notin [n_{low,s}, n_{high,s}]) = P(N_{s,l}^{(i)} - \mathbb{E}[N_{s,l}^{(i)}] < n_{low,s} - \mathbb{E}[N_{s,l}^{(i)}]) + P(N_{s,l}^{(i)} - \mathbb{E}[N_{s,l}^{(i)}] > n_{high,s} - \mathbb{E}[N_{s,l}^{(i)}]).
\]

The rest of the proof follows readily by an application of Lemmas 16 and 7.

Next, define $\rho_{s,l}^{(i)} := \sup_{1 \leq i \leq m} P(X_i = s, a_i = l)$. Then, we get the following version of Chernoff/Bernstein inequality for tail probability of $N_{s,l}^{(i)}$.

**Proposition 6.** Consider a sample $\{(X_0, a_0), \ldots, (X_m, a_m)\}$ from a controlled Markov chain that satisfies Assumptions 1, 2, and 4. Let $N_{s,l}^{(i)}$ be the number of visits to state-control pair $(s, l)$ as defined in eq. (3.1). Then there exists a positive constant $C_{pel}$ depending only upon $C_\Delta$ such that for all integers $n_{low,s} < \mathbb{E}[N_{s,l}^{(i)}] < n_{high,s}$, we have
\[
P(N_{s,l}^{(i)} \notin [n_{low,s}, n_{high,s}]) \leq 2 \exp\left( - \frac{C_{pel} (n_{low,s} - \mathbb{E}[N_{s,l}^{(i)}])^2}{4m (C_\Delta)^2 \rho_{s,l}^{(i)} + 1 + (\frac{m}{2T} - n_{low,s}) (\log m)^2} \right) + 2 \exp\left( - \frac{C_{pel} (n_{high,s} - \mathbb{E}[N_{s,l}^{(i)}])^2}{4m (C_\Delta)^2 \rho_{s,l}^{(i)} + 1 + (n_{high,s} - \mathbb{E}[N_{s,l}^{(i)}]) (\log m)^2} \right).
\]
Proposition 6 is obtained as a consequence of Merlevède, Peligrad, Rio, et al. [39, Theorem 2] adapted in this current work as Lemma 18. In many practical examples, $C_{pel}$ is a universal constant. We discuss this in greater detail in the remark ensuing Lemma 21 and produce one particular example in Section B. We can now state our first theorem regarding the sample complexity of estimating the transition probability matrices of controlled Markov chain.

**Theorem 1.** Consider a sample $\{(X_0, a_0), \ldots, (X_m, a_m)\}$ from a controlled Markov chain that satisfies Assumptions 1, 2, and 3. Let, $\{\hat{M}^{(l)}_{s,d} : l \in \mathbb{I}\}$ be the empirical estimators as defined in eq. (3.2) with $\hat{M}^{(l)}$ being the corresponding estimated transition matrix. There exists a universal constant $c > 1$, such that for any $\varepsilon > 0$, and $\delta \in (0, 1)$, if

$$m > c \max \left\{ \frac{T}{\varepsilon^2} \log \left( \frac{dkT}{\varepsilon^2 \delta} \right), (\mathbb{C}_\Delta)^2 \max \left\{ T^2, \frac{1}{(1 - \max(\zeta_1, 1 - \zeta_2))^2} \right\} \log \left( \frac{dk}{\delta} \right) \right\},$$

then the empirical estimator satisfies,

$$\mathbb{P} \left( \sup_{l \in \mathbb{I}} \left\| \hat{M}^{(l)} - M^{(l)} \right\|_{\infty} > \varepsilon \right) < \delta,$$

where $d = |\chi|$ and $k = |\mathbb{I}|$.

As we see in Theorem 1, assuming that the mixing coefficients are summable (Assumption 3) allows us to find the sample complexity. However, in Theorem 2 we will see that if we further assume the mixing coefficients to be geometrically decaying (Assumption 4), then we will have a reduced sample complexity, and it would also be minimax. So a tradeoff exists in the form of requiring a stronger assumption versus requiring more samples. The tradeoff appears in terms of having a Hoeffding’s inequality (Proposition 5) versus having a more powerful Chernoff’s inequality (Proposition 6) for the tail probabilities. It is widely known that Chernoff/Bernstein type inequalities are tighter than Azuma/Hoeffding type inequalities when the variances are small. One instance of such a discussion might be found in the introduction of Section 2.3 in Vershynin [59].

### 3.1 Sketch of Proof of Theorem 1

**STEP 1.** As in eq. (3.3) let,

$$\hat{M}^{(l)}(s, \cdot) = \left( \hat{M}^{(l)}_{s,1}, \hat{M}^{(l)}_{s,2}, \ldots, \hat{M}^{(l)}_{s,d} \right), \text{ and } M^{(l)}(s, \cdot) = \left( M^{(l)}_{s,1}, M^{(l)}_{s,2}, \ldots, M^{(l)}_{s,d} \right).$$

By an application of the union bound, we get

$$\mathbb{P} \left( \sup_{l \in \mathbb{I}} \left\| \hat{M}^{(l)} - M^{(l)} \right\|_{\infty} > \varepsilon \right) \leq \sum_{l \in \mathbb{I}} \sum_{s \in \chi} \mathbb{P} \left( \left\| \hat{M}^{(l)}(s, \cdot) - M^{(l)}(s, \cdot) \right\|_1 > \varepsilon \right)$$

**STEP 2.** For each $s \in \chi$ and $l \in \mathbb{I}$, we use law of total probability [20, Proposition 4.1] to decompose $\mathbb{P} \left( \left\| \hat{M}^{(l)}(s, \cdot) - M^{(l)}(s, \cdot) \right\|_1 > \varepsilon \right)$ into a high probability region and a low probability region. To be precise, for two integers $n_{\text{high},s}$ and $n_{\text{low},s}$ chosen appropriately by Lemma 4, we write

$$\sum_{s} \sum_{l} \mathbb{P} \left( \left\| \hat{M}^{(l)}(s, \cdot) - M^{(l)}(s, \cdot) \right\|_1 > \varepsilon \right)$$

$$\leq \sum_{s} \sum_{n = n_{\text{low},s}}^{n_{\text{high},s}} \sum_{l} \mathbb{P} \left( \left\| \hat{M}^{(l)}(s, \cdot) - M^{(l)}(s, \cdot) \right\|_1 > \varepsilon, N^{(l)}_s = n \right)$$

$$+ \sum_{s} \sum_{l} \mathbb{P} \left( N^{(l)}_s \notin [n_{\text{low},s}, n_{\text{high},s}] \right).$$

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STEP 3. In this step we observe that if \( m > \max \left\{ \frac{d}{\varepsilon^2 (1 + \max \{\zeta_1, 1 - \zeta_2\})}, \frac{64T^2 \log \left( \frac{6d}{\delta} \right)}{\varepsilon} \right\} \), Proposition 3 gives us the following upper bound, for an appropriate choice of \( m \):

\[
\sum_{s, l} \sum_{n=n_{\text{low}, s}}^{n_{\text{high}, s}} \mathbb{P}\left( \left\| \hat{M}^{(l)}(s, \cdot) - M^{(l)}(s, \cdot) \right\|_1 > \varepsilon, N^{(l)}_s = n \right) \leq \delta/3.
\]

STEP 4. In this step, we use Proposition 5 to upper bound \( \sum_{s, l} \mathbb{P}(N^{(l)}_s \notin [n_{\text{low}, s}, n_{\text{high}, s}]) \). Using Proposition 5 we get,

\[
\sum_{s, l} \mathbb{P}(N^{(l)}_s \notin [n_{\text{low}, s}, n_{\text{high}, s}]) \leq dk \left( 2 \exp \left( -\frac{(n_{\text{low}, s} - m)^2}{2m (C_\Delta)^2} \right) + 2 \exp \left( -\frac{(n_{\text{high}, s} - m \max \{\zeta_1, 1 - \zeta_2\})^2}{2m (C_\Delta)^2} \right) \right).
\]

Hence, as long as,

\[
m > c \max \left\{ C_\Delta^2 \log \left( \frac{dk}{\delta} \right) \max \left\{ T^2, \frac{1}{1 - \max \{\zeta_1, 1 - \zeta_2\}} \right\} \right\},
\]

\[
\sum_{s, l} \mathbb{P}(N^{(l)}_s \notin [n_{\text{low}, s}, n_{\text{high}, s}]) \leq 2\delta/3.
\]

This completes the sketch.

Before proceeding to the next theorem, we introduce some notation. Consider a sample \( \{(X_0, a_0), \ldots, (X_m, a_m)\} \) from a controlled Markov chain that satisfies Assumptions 1, 2, and 4. Let \( \rho_* := \sup_{s, l} \rho^{(l)}_s \) where \( \rho^{(l)}_s \) is defined in Proposition 6 and define,

\[
C_T := \frac{64 (C_\Delta \rho_* T^2 + 2T)}{C_{\text{pel}}},
\]

(3.10)

\[
C_\zeta := \frac{8 \left( \frac{2C_\Delta \rho_* (1 - \max \{\zeta_1, 1 - \zeta_2\})^{-2} + (1 - \max \{\zeta_1, 1 - \zeta_2\})^{-1} \right)}{C_{\text{pel}}},
\]

(3.11)

\[
C_{T, \delta} := C_T \log \left( \frac{6d}{\delta} \right),
\]

(3.12)

\[
C_{\zeta, \delta} := C_\zeta \log \left( \frac{6d}{\delta} \right),
\]

(3.13)

where \( C_{\text{pel}} \) is as in Proposition 6.

**Theorem 2.** In the setting of Theorem 1 suppose that Assumptions 1, 2, and 4 are satisfied, and let \( \rho_* = \max_{s, l} \rho^{(l)}_s \). Then, there exists an universal constant \( c > 1 \) such that if

\[
m > c \max \left\{ \frac{8d}{\varepsilon^2 (1 + \max \{\zeta_1, 1 - \zeta_2\})}, 2 \frac{C_{T, \delta} \log C_{T, \delta}^2, 2 \frac{C_{\zeta, \delta} \log C_{\zeta, \delta}^2} \right\}
\]

then the empirical estimator satisfies

\[
\mathbb{P}(\sup_{l \in I} \left\| \hat{M}^{(l)} - M^{(l)} \right\|_\infty > \varepsilon) < \delta,
\]

(3.14)

for all \( \varepsilon, \delta > 0 \) and is minimax up to \( \log \log \) and \( \log \log \) terms whenever \( 0 < \varepsilon < 1/32 \).
**Remark.** It is straightforward to see that \( \rho_{\ell}(l) \leq \max \{ \zeta_1, 1 - \zeta_2 \} \). However, this upper bound is loose to the degree that the resulting sample complexity is no longer provably Minimax.

We want to point out that this result is tighter than the previous one by a factor of \( \frac{C_\Delta \rho_*/C_{pel}}{C_{pel}} \). \( \rho_* \) can be \( O(1/T) \) with \( C_{pel} \) being independent of model parameters, and we present such an example in Section B. Therefore, this result is significantly tighter than the previous one. The presence of this difference is significant in the context of model misspecification. If a model is erroneously assumed to be exponentially mixing when it is not, \( O(T) \) samples are no longer sufficient to guarantee precision while estimating the transition matrices.

In Sections 4.2 and 4.4 we prove that if the controls are Markovian, then the process is exponentially mixing under mild conditions. This provides yet another justification of why it is tempting to assume Markovianity. However, if the assumption is erroneous, then there is no guarantee that the estimated probabilities would be accurate. As such, our prescription is to follow pessimism in the face of uncertainty and collect \( O(T^2) \) samples whenever there is uncertainty about the logging policy.

### 3.2 Sketch of Proof of Theorem 2

The proof of sample complexity proceeds similarly to the proof of Theorem 1. The key difference is in STEP 4, where instead of using Proposition 5, we use Proposition 6 instead. The details can be found in Appendix D.3. For this sketch, we focus on the minimaxity. Let \( \mathcal{M}_{\chi, \bar{I}} \) be the class of all Markov decision processes on state space \( \chi \) with control space \( \bar{I} \). Observe that the minimax risk,

\[
\mathcal{R}_m = \min_{\hat{M}} \sup_{(M, P) \in \mathcal{M}_{\chi, \bar{I}}} \mathbb{P} \left( \| \hat{M} - M \|_\infty^* > \varepsilon \right)
\]

for any subclass of controlled Markov chains \( \mathcal{M}' \subset \mathcal{M}_{\chi, \bar{I}} \) and any estimation procedure, \( \hat{M} \) dependent on the data (informally). The rest of the proof proceeds through 2 cases by constructing appropriate subclasses \( \mathcal{M}' \).

**CASE I:** \( m < \frac{8d}{\varepsilon^2 (1 + \max(\zeta_1, 1 - \zeta_2))} \)

For this case, we choose a class of controlled Markov chains with controls distributed uniformly over \( \bar{I} \) and transition matrices

\[
M_{\sigma} = \begin{pmatrix}
\frac{1-p_\star}{d} & \ldots & \frac{1-p_\star}{d} & p_\star \\
\vdots & \ddots & \vdots & \vdots \\
\frac{1-p_\star}{d} & \ldots & \frac{1-p_\star}{d} & p_\star \\
\frac{1-p_\star + 16 \sigma_1 \varepsilon}{d} & \ldots & \frac{1-p_\star - 16 \sigma_1 \varepsilon}{d} & p_\star
\end{pmatrix}.
\]

for vectors \( \sigma = (\sigma_1, \ldots, \sigma_d) \in \{-1, 1\}^d \). We then use Tasbykov’s reduction method to lower bound

\[
\min_{\hat{M}} \sup_{(M, P) \in \mathcal{M}} \mathbb{P} \left( \| \hat{M} - M \|_\infty^* > \varepsilon \right)
\]

for our chosen subclass of controlled Markov chains.

**CASE II:** \( m < \left( 2C_{T, \delta} (\log C_{T, \delta})^2, 2C_{\zeta, \delta} (\log C_{\zeta, \delta})^2 \right) \)

**STEP 1.** For this case, we set \( \mathcal{M}' \) to be a class of controlled Markov chains with controls and transition probability matrices described in Section B.
STEP 2. We then use Tsybakov’s reduction method [57, Chapter 2.2] to observe that for any random variable $T$,
\[
\mathcal{R}_m \geq \inf_{\hat{M}} \sup_{(\hat{M}, P) \in \mathcal{M}'} \mathbb{P} \left( \left\| \hat{M} - M^* \right\|_\infty > \varepsilon \mid T > m \right) \mathbb{P} \left( T > m \right).
\]
$T$ is chosen to be an appropriate “touring time” (the time to visit sufficiently many state-control pairs).

STEP 3. We then prove that as long as $m < 2 C_{T, \delta} (\log C_{T, \delta})$, \(\mathbb{P} \left( T > m \right)\) is bounded away from zero.

STEP 4. We then argue that whenever $T > m$, there exists a state-control pair $s_0, l_0$ such that $N^{(l_0)}_{s_0} = 0$.

STEP 5. If $N^{(l_0)}_{s_0} = 0$, so is $N^{(l_0)}_{s_0, t} = 0$ for all $t \in \chi$. This proves that there is an uniform error to estimate $M^{(l_0)}_{s_0, t}$, which proves our claim.

4 Applications

Before we move on to the main content of the applications section, we make a brief discussion about how Assumptions 2, 3, 4, and can be reduced to simpler assumptions for the purpose of analysis.

4.1 Reduction of Assumptions

Reduction of Return Times. First consider the assumption on return times introduced in Assumption 2. Observe that if $a_i$ is $k$-connected Markovian, then
\[
\sup_{s, l, i} E \left[ \tau_i^{(l) \tau_i} \mid \mathcal{F}_{\tau_i} \right] = \sup_{s, l, i} E \left[ \tau_i^{(l) \tau_i} \mid \mathcal{H}_{\tau_i - k} \right] \text{ almost everywhere},
\]
where we have defined $\tau_i := \sum_{p=0}^{i-1} \tau_i^{(p)}$ for notational convenience. Moreover, if $a_i$ is independent of time point $i$ (also called “stationary”), then we further have
\[
\sup_{s, l, i} E \left[ \tau_i^{(l) \tau_i} \mid \mathcal{H}_{\tau_i - k} \right] = \sup_{s, l} E \left[ \tau_i^{(1)} \mid X_0, a_0 \right] \text{ almost everywhere.} \tag{4.1}
\]

Reduction of Mixing Coefficients. Next, we decompose the $\eta$-mixing coefficients of the paired process \{\(X_i, a_i\)\} into mixing coefficients over states and controls. We motivate this decomposition using two facts:

- The controls of a controlled Markov chain are often chosen with well-behaved properties by the user.
- In practice, it is easier to analyse the mixing coefficients of the individual processes than the paired processes.

We begin by defining the $\gamma$-mixing coefficients $\gamma_{p,j,i}$ for controls as the following total variation distance
\[
\gamma_{p,j,i} := \sup_{s_p, h_{p,j}^{i-1}, h_0^i} \left\| \mathcal{L} \left( a_p \mid X_p = s_p, \mathcal{H}_{p,j}^{i-1} = h_{p,j}^{i-1}, \mathcal{H}_0^i = h_0^i \right) - \mathcal{L} \left( a_p \mid X_p = s_p, \mathcal{H}_{p,j}^{i-1} = h_{p,j}^{i-1} \right) \right\|_{TV}, \tag{4.2}
\]
where $\mathbb{P} \left( X_p = s_p, \mathcal{H}_{p,j}^{i-1} = h_{p,j}^{i-1}, \mathcal{H}_0^i = h_0^i \right) > 0$. 

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**Assumption 5** (Mixing of controls). There exists a constant $C \geq 0$,

$$\sup_{1 \leq i \leq \infty} \sum_{j=1}^{\infty} \sum_{p=i+j+1}^{\infty} \gamma_{p,j,i} \leq \frac{C}{2}.$$

**Remark.** In the Markovian settings, when the sequence of controls $a_i$ depend only upon $X_i$, it readily follows that

$$\gamma_{p,j,i} = 0$$

for all $p, j, i$. In such case, Assumption 5 is satisfied with $C = 0$. This phenomenon extends to the case where $a_i$ depends upon $j$ many past time points. If $a_i$ depend only upon $X_i, a_{i-1}, X_{i-1}, \ldots, a_{i-1}, X_{i-1}$, then Assumption 5 is satisfied with $C = k - 1$.

We can now make the final technical assumption on the mixing of the underlying Markov chain. Let $\{(X_0, a_0), \ldots, (X_m, a_m)\}$ be a collected sample. To be specific, for all integers $j \geq i$ define the mixing coefficient $\bar{\theta}_{i,j}$

$$\bar{\theta}_{i,j} := \sup_{s_1, s_2 \in \chi, l_1, l_2 \in I, \ P(X_i = s_1, a_i = l_1) > 0, \ P(X_i = s_2, a_i = l_2) > 0} \| \mathcal{L}(X_j | X_i = s_1, a_i = l_1) - \mathcal{L}(X_j | X_i = s_2, a_i = l_2) \|_{TV},$$

(4.3)

such that $(s_1, l_1) \neq (s_2, l_2)$. Our final technical assumption is following.

**Assumption 6** (Mixing of States). There exists a constant $C_\theta \geq 0$,

$$\sup_{1 \leq i \leq \infty} \sum_{j=i+1}^{\infty} \bar{\theta}_{i,j} \leq C_\theta.$$

We make note that neither of Assumptions 5 and 6 imply the other. The following counter-examples illustrate this point.

1. Let $(X_i, a_i)$ be an inhomogenous Markov chain for which

$$\sup_{1 \leq i \leq \infty} \sum_{j=i+1}^{\infty} \bar{\theta}_{i,j} = \infty.$$

One example of such an inhomogenous Markov chain can be found in Lemma 23. However, since the controls are deterministic, every inhomogenous Markov chain satisfies Assumption 5. We prove this fact formally in Proposition 11. Therefore, this chain satisfies Assumption 5 but not Assumption 6.

2. For the second counter-example consider a controlled Markov chain $(X_i, a_i)$ where the $a_i$’s do not satisfy Assumption 5. Let $X_i$ be independent draws from a uniform distribution over $\chi$. It is easily seen that $\bar{\theta}_{i,j} = 0$ for this example. Therefore, this chain satisfies Assumption 6 but not Assumption 5.

Observe that the previous assumptions on the states and controls imply the summability of the weak mixing coefficients. We formalise it in the following Lemma.

**Lemma 7.** For any controlled Markov chain that satisfies Assumptions 5 and 6,

$$\| \Delta_m \| \leq C + C_\theta + 1,$$

where $\| \Delta_m \| = \max_{1 \leq i \leq m} (1 + \bar{\eta}_{i,i+1} + \bar{\eta}_{i,j+2} + \ldots \bar{\eta}_{i,m})$, and $\bar{\eta}_{i,j}$ is as defined in eq. (2.1)
Remark. As before, observe that Assumptions 5 and 6 provide a sufficient condition for $\|\Delta_m\|$ to be bounded uniformly over $m$. We remark that Theorem 1 continues to hold under this weaker condition. However, in practice, it will often be easier to prove Assumptions 5 and 6. In Section 4, we demonstrate a number of examples where we prove that some commonly found CMC’s like inhomogenous Markov chains, controlled Markov chains with stationary Markov controls etc. satisfy our assumptions.

Next, we make the following two assumptions as stronger versions of Assumptions 5 and 6.

**Assumption 7** (Geometric mixing of controls). There exists a constant $C_* > 0$ independent of $m$ such that $\forall$ integers $j \geq i$,
\[
\sum_{p=i+j+1}^{\infty} \gamma_p,j,i \leq e^{-C_*(j-i)}.
\]

**Assumption 8** (Geometric mixing of States). There exists a constant $C_{\theta,*} > 0$ independent of $m$ such that $\forall$ integers $j \geq i$,
\[
\bar{\theta}_{i,j} \leq e^{-C_{\theta,*}(j-i)}.
\]

We then get the following lemma as a counterpart to Lemma 7.

**Lemma 8.** For any controlled Markov chain that satisfies Assumptions 7 and 8, there exists a positive constant $c_{\text{cof}}$ independent of $m$ such that $\forall$ integers $j \geq i$,
\[
\bar{n}_{i,j} \leq e^{-c_{\text{cof}}(j-i)}.
\]

Remark. It is easy to observe that if Assumptions 7 and 8 are satisfied, then so are Assumptions 5 and 6 with constants $1/(1-e^{-C_*})$ and $1/(1-e^{-C_{\theta,*}})$ respectively. To simplify notations, we will not use these explicit forms and will denote $1/(1-e^{-C_*})$ by $C$ and $1/(1-e^{-C_{\theta,*}})$ by $C_{\theta}$ respectively. Finally, observe that Assumptions 7 and 8 provide a sufficient condition for $\bar{n}_{i,j}$ to be geometrically decaying uniformly over $m$.

### 4.2 Controlled Markov chains with stationary randomised controls and time homogenous ergodic Markov chains

A Markov decision process is said to have stationary randomised controls if for any time period $i$, state $s_i$, and sample history $h_{0}^{i-1}$,
\[
\mathcal{L}(a_i|X_i = s_i, h_{0}^{i-1} = h_0^{i-1}) = \mathcal{L}(a_i|X_i = s_i) = \mathcal{L}(a_1|X_1 = s_i).
\]

In this section we show that assumptions 1, 2, 5, and 6 hold for a Markov decision process with stationary controls.

We observe that the transition probability of the joint state control pair is
\[
P(X_i = t, a_i = l|X_{i-1} = s, a_{i-1} = l') = P(X_i = t|X_{i-1} = s, a_{i-1} = l) P(a_i = l|X_i = t) = M_{s,t}^{(l)} \times P_{t}^{(l)},
\]
where $P_{s}^{(l)}$, denotes $P(a_1 = l|X_1 = s$). It is straightforward to see that the state-control pair is a time homogeneous Markov chain with transition probabilities given by $M_{s,t}^{(l)} \times P_{t}^{(l)}$. We have the following consequences of the previous definition.

Our goal is to estimate the transition probabilities $P(X_1 = t|X_0 = s, a_0 = l)$. Observe that this estimation problem only makes sense in the case when $P(X_0 = s, a_0 = l) > 0$ for any $s$ and $l$. This follows
from the fact \( \mathbb{P}(X_0 = s, a_0 = l) = P_s^{(l)} \times \mathbb{P}(X_0 = s) \) and, \( P_s^{(l)} > 0 \) for all \( s \in \chi \) and \( l \in \mathbb{I} \). By \( P_{\min} \), denote \( \inf_{s,l} P_s^{(l)} \). Assume that \( M^{(l)} \) is an aperiodic and irreducible (ergodic) transition probability matrix for all \( l \in \mathbb{I} \). Then, we have the following proposition.

**Proposition 9.** The paired process \( \{(X_0, a_0), \ldots, (X_m, a_m)\} \) is an uniformly ergodic Markov chain. □

By verifying the aperiodicity and irreducibility of the paired process, the proof of Proposition 9 is follows readily from [40, Theorem 16.0.2] and is therefore omitted.

Let \( \pi \) be the invariant distribution of this Markov chain with \( \pi_{(s,l)} \) being stationary probability corresponding to \( (s,l) \). For simplicity, assume that \( D_0 = \pi \).

**Proposition 10.** Let \( \{(X_0, a_0), \ldots, (X_m, a_m)\} \) be a sample from a controlled Markov chain with \( d = |\chi| \), \( k = |\mathbb{I}| \), and stationary randomised controls. Fix \( \varepsilon > 0 \), and \( \delta \in (0,1) \). Then there exists a universal constant \( c > 0 \) and a constant \( C_{\theta} > 0 \) such Theorem 1 is satisfied with \( T = \sup_{s,t} 1/\pi_{s,t} \), \( \zeta_2 = P_{\min} \), \( \zeta_1 = 1 - (k-1)P_{\min} \) and \( C_{\theta} \). Moreover, if \( D_0 = \pi \), then \( \zeta_1 = \zeta_2 = 1/T \) satisfied Assumption 1.

**Proof.** The proof of this result can be found in Appendix E.1. □

### 4.3 Inhomogenous Markov chains

A controlled Markov chain is said to be an inhomogenous Markov chain if there exists a sequence of constants \( l_0, l_1, \ldots \) such that for any non-negative integer \( i \), history \( h_{0}^{i-1} \in (\chi \times \mathbb{I})^i \), and state \( s \in \chi \)

\[
\mathbb{P}(a_i = l_i | h_{0}^{i-1} = h_{0}^{i-1}, X_i = s) = 1,
\]

We make the following assumptions on the controls and the transition probabilities

Let \( T \in \mathbb{N} \) be a known integer. Then, we assume that

\[
\sum_{i=j}^{T+j} \mathbbm{1}[a_i = l] > 1, \tag{4.5}
\]

for all \( 1 \leq j \leq m - T \) and \( l \in \mathbb{I} \). Moreover we assume that the transition matrices \( M^{(l)} \) are positive stochastic matrices where every entry in the matrix is strictly positive. It follows from the fact that

\[
\mathbb{P}(X_{i+1} = t | X_i = s, a_i = l) \geq M_{\min} \coloneqq \min_{s,t,l} M_{s,t}^{(l)}, \tag{I1}
\]

it further follows that

\[
\mathbb{P}(X_{i+1} \neq t | X_i = s, a_i = l) < 1 - M_{\min}. \tag{I2}
\]

**Proposition 11.** Let \( \{(X_0, a_0), \ldots, (X_m, a_m)\} \) be a sample from an inhomogenous Markov chain satisfying \( M_{\min} > 0 \) and eq. (4.5). Fix \( \varepsilon > 0 \), and \( \delta \in (0,1) \). Then Theorem 1 holds with

\[
T = \left( 1 - M_{\min} \right)^{1-1/T} / \left( 1 - (1 - M_{\min})^{\frac{1}{T}} \right), \quad \zeta_2 = M_{\min}, \quad \zeta_1 = M_{\max} \coloneqq \max_{s,t,l} M_{s,t}^{(l)}, \text{ and } C_{\theta} = e/(e-1).
\]

**Proof.** The proof of this result can be found in Appendix E.2. □

**Remark.** If \( \{X_n\} \) is Markov, then it is well known that \( Y_n = (X_n, \ldots, X_{n-j+1}) \) is a Markov process on state space \( \chi^j \). Marginalising on the first component, conclusions can then be obtained about the distribution of \( X_n \) given \( X_{n-1}, \ldots, X_{n-j+1} \). It follows that our conclusions also hold for \( j \)-connected inhomogenous Markov chains.
4.4 Controlled Markov chains with non-stationary Markov controls

An controlled Markov chain is said to have non-stationary Markov controls if for any time period $i$, state $s_i$, and sample history $h^{i-1}_0$,

$$L(a_i|X_i = s_i, h^{i-1}_0 = h^{i-1}_0) = L(a_i|X_i = s_i).$$

For convenience, we refer to it as simply Markov controls. Observe that we allow the law of the control sequence to depend upon the time step $i$. Observe that we can write the transition probability of the joint state control pair as

$$P(X_i = t, a_i = l'|X_{i-1} = s, a_{i-1} = l) = M^{(l)}_{s,t} \times P^{(l)}_{t,l'},$$

It is straightforward to see that the state-control pair is a time inhomogeneous Markov chain with transition probabilities given by $M^{(l)}_{s,t} \times P^{(l)}_{t,l'}$. Our goal is to estimate the transition probabilities $P(X_i = t|X_{i-1} = s, a_{i-1} = l)$. We proceed by making assumptions on the return times of the controls.

**Definition 2.** Define $\tau_{s,l}^{(i,*,j)}$ to be the time between the $j - 1$-th and $j$-th visit to control $l$ after visiting state-control pair $s, l$ for the $i$-th time. For ease of notation, denote $\sum_{k=1}^{i} \tau_{s,l}^{(k)} + \sum_{j=1}^{i-1} \tau_{s,l}^{(i,*,k)} = \tau_*$. Then $\tau_{s,l}^{(i,*,j)}$ is recursively defined as

$$\tau_{s,l}^{(i,*,j)} := \min\{n : (a_{\tau_* + n} = l), a_j \neq l \forall \tau_* < j < \tau_* + n\}.$$

Next we make some simplifying assumptions on $\tau_{s,l}^{(i,*,j)}$ and $M^{(l)}$

**Assumption 9.**

1. For some constant $T_* > 0$,

$$\sup_{i \geq 0} \mathbb{E}[\tau_{s,l}^{(i,*,j)}|F_{\sum_{p=1}^{i-1} s^{(p)} + \sum_{j=1}^{i-1} s^{(i,*,p)}}] < T_* \text{ almost everywhere.}$$

2. For all $i \in \mathbb{N}$ the stochastic matrix $M^{(l)}$ is positive. To be precise,

$$0 < M_{min} \leq M_{max} < 1,$$

where $M_{min}$ and $M_{max}$ are defined as,

$$M_{max} := \max_{s,t} M^{(l)}_{s,t}, \text{ and } M_{min} := \min_{s,t} M^{(l)}_{s,t}.$$

Then, we have the following lemma on the return times.

**Lemma 12.** Under the conditions of Assumption 9,

$$\mathbb{E}[\tau_{s,l}^{(i)}|F_{\sum_{p=1}^{i-1} s^{(p)}}] < \frac{T_* M_{max}}{\max\{M_{max}, 1 - M_{min}\}(1 - \max\{M_{max}, 1 - M_{min}\})}$$

almost everywhere for all $(i, s, l) \in \mathbb{N} \times \chi \times \mathbb{I}$.

We can now state our main result about the sample complexity of a controlled Markov chain with a non-stationary Markov controls.

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Proposition 13. Let \( \{(X_0, a_0), \ldots, (X_m, a_m)\} \) be a sample from an controlled Markov chain with non-stationary Markovian controls satisfying assumption 9. Fix \( \varepsilon > 0 \) and \( \delta \in (0,1) \). Then Theorem 1 holds with \( \zeta_1 = M_{\max}, \zeta_2 = M_{\min}, T = \max\{M_{\max}, 1 - M_{\min}\}(1 - \max\{M_{\max}, 1 - M_{\min}\}) \) and \( C_{\Theta} := \frac{1}{1 - d_{\text{min}}} \).

Proof. The proof of this result can be found in Appendix E.3.

Remark. Observe that

\[
\frac{1}{1 - (1 - M_{\min})^\frac{1}{T}} \leq \frac{T}{M_{\min}} \leq \frac{T M_{\max}}{\max\{M_{\max}, 1 - M_{\min}\}(1 - \max\{M_{\max}, 1 - M_{\min}\})}
\]

where the first inequality follows from Bernoulli’s inequality [10]. Comparing this to eq. (4.7) consequently implies that tighter upper bounds to expected return times can be derived when the controls corresponding to a controlled Markov chain are deterministic. In other words, a CMC with Markov non-deterministic controls require more samples to guarantee PAC-bounds than an inhomogenous Markov chain.

4.5 Controlled Markov chains with episodic controls

Controlled Markov chains with episodic controls appear frequently in offline reinforcement learning [44, 31, 32, 68, 51] and constitute a fixed horizon \( H \) after which the algorithm restarts. To formalise, let \( H \) be a fixed positive integer. For any positive integer \( i > H \), let \( H^{(i)} \) denote the greatest multiple of \( H \) less than or equal to \( i \). To be precise

\[
H^{(i)} := H \lfloor \frac{i}{H} \rfloor.
\]

A sample \( \{(X_0, a_0), \ldots, (X_m, a_m)\} \) is said to be from a controlled Markov chain with episodic controls and horizon length \( H \) if for any \( s \in \chi \) and \( h^{i-1}_0 \in (\chi \times \mathbb{I})^i \),

\[
\mathcal{L}(a_i|X_i = s, H^{i-1}_0 = h^{i-1}_0) = \mathcal{L}(a_i|X_i = s, H^{(i)}_H = h^{(i)}_H),
\]

We assume that for all state-control pair \( (t, l') \), there exists a time step \( i^{(t,l')} < H \) such that

\[
\min_{i^{(t,l')} \in \chi \times \mathbb{I}} \mathbb{P}(X_{i^{(t,l')}} = t, a_{i^{(t,l')}} = l'|X_0 = s, a_0 = l) > 0.
\]

for some state control pair \( (s, a) \). Since this infimum is taken over finitely many objects there exists a probability \( M_{\min} \) such that

\[
\mathbb{P}(X_{i(s,l)} = s, a_{i(s,l)} = l|X_0 = s', a_0 = l') > M_{\min}.
\]

We observe that this assumption is similar to the so called persistence of excitation (Assumption 1 in [5]) and is close to the infinite updates assumption in [70] and [71].

Our second technical assumption is that

\[
(X_i, a_i) \overset{i \sim d}{\sim} \text{Uniform}\{(1,1), \ldots, (d,k)\},
\]

whenever \( i \) is a multiple of \( H \). In other words, multiples of \( H \) denote the start of a new episode.

Remark. We would like to point out that assuming eq. (4.9) simplifies our calculations. However, our results also hold when the distribution is not uniform. The corresponding calculations are very similar (yet more tedious and notationally challenging).
Let $\zeta_1 = M_{\text{max}}$, $\zeta_2 = M_{\text{min}}$, $T = dkH - 1$, $C = H^2$, and $C_\theta = H$. We can now state our main result about the sample complexity of a controlled Markov chain with episodic controls.

**Proposition 14.** Let $\{(X_0, a_0), \ldots, (X_m, a_m)\}$ be a sample from a controlled Markov chain with episodic controls. Fix $\varepsilon > 0$, and $\delta \in (0, 1)$. Then Theorem 1 holds with $\zeta_1 = M_{\text{max}}$, $\zeta_2 = M_{\text{min}}$, $T = dkH - 1$, $C = H^2$, and $C_\theta = H$.

**Proof.** The proof of this result can be found in Appendix E.4.

### 4.6 Controlled Markov chains with greedy controls

Greedy algorithms are an important facet of reinforcement learning appearing in numerous exploration-exploitation like multi-armed bandit [28], Q-learning [64], Levy-flights [34] among others. In the current context, we define a controlled Markov chain with greedy controls as follows.

**Definition 3.** Let $\{\omega_i\}_{i \geq 0}$ be a sequence of i.i.d. Bernoulli variables with $\mathbb{P}(\omega_i = 1) = \nu$ for some fixed $\nu \in (0, 1)$. A controlled Markov chain with greedy controls is defined to be a triplet $(X_i, \omega_i, a_i)$ taking values in $\chi \times \{0, 1\} \times \mathbb{I}$, where $X_i$ represents the state and $a_i$ represents the control at time point $i$ such that

$$a_i = (1 - \omega_i)\alpha_i + \omega_i D_i^{(1)}$$

where $\alpha_i$ are random variables on $\mathbb{I}$ adapted to the history and $D_i^{(1)}$ are i.i.d. uniform random variables supported on $\mathbb{I}$.

Moreover for some probability $M_{s,t}^{(l)}$ depending only upon $s, t, l$, the transition probabilities satisfy,

$$\mathbb{P}(X_i = t|X_{i-1} = s, a_{i-1} = l, \omega_i = \xi) = M_{s,t}^{(l)}$$

for any $\xi \in \{0, 1\}$.

This completes our formalisation of a controlled Markov chains with greedy controls. Intuitively, the definition suggests that greedy controls keep exploring the control space irrespective of the history. We make the mild assumption that $M^{(l)}$ is an aperiodic and irreducible (ergodic) transition probability matrix for all $l \in \mathbb{I}$ and observe that the $l$-th transition matrix of $(X_i, \omega_i)$ is a block matrix $\mathbf{M}^{(l)}$ where

$$\mathbf{M}^{(l)} = \begin{bmatrix} (1 - \nu)M^{(l)} & \nu M^{(l)} \\ (1 - \nu)M^{(l)} & \nu M^{(l)} \end{bmatrix}.$$ 

$\nu$ is fixed, so we only need to accurately estimate $M^{(l)}$. The proof proceeds by making a suitable transformation on data.

**Transformation.** Our objective in this transformation is to carefully isolate all those time points $i$ where $\omega_i = \omega_{i-1} = 1$ and use it to create our empirical estimator. Let $D_i^{(2)}$ be a sequence of i.i.d. uniform random variables distributed on $\chi$ and $D_i^{(3)}$ be the same on $\mathbb{I}$. We construct a sequence of random variables $\tilde{X}_i, \tilde{a}_i$ as follows,

$$\tilde{X}_i = (1 - \omega_i)D_i^{(2)} + \omega_i X_i \quad (G1)$$

$$\tilde{a}_i = (1 - \omega_i)D_i^{(3)} + \omega_i a_i. \quad (G2)$$
From (G1) and (G2) we get,
\[
\mathbb{P}\left(\tilde{X}_i = t|\tilde{a}_{i-1} = l, \omega_i = \omega_{i-1} = 1, \tilde{X}_{i-1} = s\right) = \frac{\mathbb{P}\left(\tilde{X}_i = t, \tilde{X}_{i-1} = s, \tilde{a}_{i-1} = l|\omega_i = \omega_{i-1} = 1\right)}{\mathbb{P}\left(\tilde{X}_{i-1} = s, \tilde{a}_{i-1} = l|\omega_i = \omega_{i-1} = 1\right)} = \frac{\mathbb{P}\left(X_i = t, X_{i-1} = s, a_{i-1} = l|\omega_i = \omega_{i-1} = 1\right)}{\mathbb{P}\left( X_{i-1} = s, a_{i-1} = l|\omega_i = \omega_{i-1} = 1\right)} = \mathbb{P}\left(X_i = t|a_{i-1} = l, \omega_i = \omega_{i-1} = 1, X_{i-1} = s\right) = M^{(l)}_{s,t}.
\] (4.11)

Therefore, the \(l\)-th transition probability matrix associated with \((\tilde{X}_{i+1}, \omega_{i+1})\) is
\[
\tilde{M}^{(l)} = \begin{bmatrix} (1 - \nu)J & \nu M^{(l)} \\ (1 - \nu)J & \nu M^{(l)} \end{bmatrix}
\] (4.12)

where \(J\) is a \(d \times d\) matrix with each element \(1/d\). The following lemma holds

**Lemma 15.** The controlled Markov chain \((Y_i, \tilde{a}_i)\), where \(Y_i := (\tilde{X}_i, \omega_i)\) denotes the state at time point \(i\), and \(\tilde{a}_i\) denotes the control, is a CMC over \(\chi \times \{0, 1\} \times \bar{I}\) with ergodic transition probability matrix \(\tilde{M}^{(l)}\) for all \(l \in \bar{I}\) and stationary controls \(\tilde{a}_i\).

As an immediate consequence, we apply Proposition 9 to obtain that the \((\tilde{X}_i, \omega_i, \tilde{a}_i)\) is a uniformly ergodic Markov chain. Let \(\pi\) be its stationary distribution on \(\chi \times \{0, 1\} \times \bar{I}\) and for \((s, \xi, l) \in \chi \times \{0, 1\} \times \bar{I}\), let \(\pi_* := \inf_{s,\xi,l} \pi_{s,\xi,l}\). Furthermore, without losing generality let \(\nu < (1 - \nu)\). Next, let \(C = 0\), \(C_\theta = 1/\nu\), \(T = 1/\pi_*\), \(\zeta_2 = 1/k\), \(\zeta_1 = 1/k\), and by \(\tilde{M}^{(l)}\) denote the empirical estimator for \(\bar{M}^{(l)}\). Then we have the following theorem,

**Theorem 3.** There exists a universal constant \(c > 1\), such that for any \(\varepsilon > 0\), and \(\delta \in (0, 1)\), if
\[
m > c \max \left\{ \frac{T}{\varepsilon^2} \log \left( \frac{dkT}{\varepsilon^2 \delta} \right), (1 + C_\theta)^2 \log \left( \frac{dk}{\delta} \right) \max \left\{ T^2, \frac{1}{(1 - \max\{\zeta_1, 1 - \zeta_2\})^2} \right\} \right\},
\]
then the empirical estimator of \(\bar{M}^{(l)}\) satisfies,
\[
\mathbb{P}\left( \sup_{l \in \bar{I}} \left\| \tilde{M}^{(l)} - \tilde{M}^{(l)} \right\|_\infty > \varepsilon \right) < \delta.
\] (4.13)

Applying Proposition 10, the proof is straightforward. The immediate corollary follows.

**Corollary 1.** Let \(\bar{M}^{(l)}\) be the \(d \times d\) block given by rows and columns \(\{d + 1, \ldots, 2d\}\) of \(\tilde{M}^{(l)}\). Then,
\[
\mathbb{P}\left( \sup_{l \in \bar{I}} \left\| \frac{1}{\nu} \bar{M}^{(l)} - M^{(l)} \right\|_\infty > \varepsilon \right) < \delta,
\] (4.14)

whenever
\[
m > c \max \left\{ \frac{\nu^2 T}{\varepsilon^2} \log \left( \frac{\nu^2 dkT}{\varepsilon^2 \delta} \right), (1 + C_\theta)^2 \log \left( \frac{dk}{\delta} \right) \max \left\{ T^2, \frac{1}{(1 - \max\{\zeta_1, 1 - \zeta_2\})^2} \right\} \right\}.
\]

**Proof.** The proof of this follows readily from the fact that
\[
\mathbb{P}\left( \sup_{l \in \bar{I}} \left\| \bar{M}^{(l)} - \nu M^{(l)} \right\|_\infty > \frac{\varepsilon}{\nu} \right) > \mathbb{P}\left( \sup_{l \in \bar{I}} \left\| \tilde{M}^{(l)} - \tilde{M}^{(l)} \right\|_\infty > \frac{\varepsilon}{\nu} \right).
\]
\(\square\)
5 Sample Complexity of Offline Policy Evaluation

Offline policy evaluation (OPE) is a significant issue in offline RL settings, wherein candidate policies are comparatively ‘evaluated’ by computing their value functions (note that the policies need not be optimal). The sample complexity results in Theorem 1 and Theorem 2 immediately yield bounds for OPE. To be precise, we use the estimated transition probability matrices $\hat{M}$ as a plugin to evaluate the value of a given stationary Markov policy $\pi$. As observed in Section 4.2, a controlled Markov chain with stationary controls is equivalent to a geometrically ergodic Markov chain. Therefore, we assume without losing generality that $k = 1$, and drop the superscript from $M^{(l)}$. As in Section 4.2, we assume the Markov chain satisfies Assumption 2 and Assumption 6, with constants $T$ and $C_\theta$ respectively.

We start in an abstract setting. Let $V := (V(x) : x \in \mathcal{X}) \in \mathbb{R}^d$ be a value function to be computed, $0 < \alpha_{dis} < 1$ a known discount factor, and $g := (g(x) : x \in \mathcal{X}) \in \mathbb{R}^d$ is a known ‘per-stage’ cost function. Recall that the value function $V$ is the solution of the value function or Bellman equation $\mathcal{B}_{eq. 10}$,

$$V = g + \alpha_{dis} MV,$$

implying that

$$\hat{V} = (I - \alpha_{dis} \hat{M})^{-1} g.$$  \hfill (5.2)

Observe that, for a finite state space, the cost function should satisfy $\|g\|_1 < \infty$, and define

$$T_\alpha := \frac{\|g\|^2_1 d \alpha_{dis}^2 T}{(1 - \alpha_{dis})^4}$$

The next theorem provides a sample complexity bound on estimating the value function $V$.

**Theorem 4.** Let $X_0, \ldots, X_m$ be a sample from a controlled Markov chain with stationary controls and $V$ and $\hat{V}$ be as defined in equations 5.1 and 5.2 respectively. Then, there exists a universal constant $c > 1$ such that if,

$$m > c \max \left\{ \frac{T_\alpha}{\varepsilon^2} \log \left( \frac{dkT_\alpha}{\varepsilon^2 \delta} \right), C_\theta^2 \max \left\{ T_\alpha^2, \frac{1}{(1 - \max\{\zeta_1, 1 - \zeta_2\})^2} \right\} \log \left( \frac{dk}{\delta} \right) \right\},$$

then

$$\mathbb{P} \left( \|\hat{V} - V\| > \varepsilon \right) < \delta.$$  

**Proof.** The proof of this theorem can be found in Appendix D.4

Now, suppose $\pi : \mathcal{X} \to \Delta(\mathcal{I})$, where $\Delta$ represents the probability simplex on the control space, is a given stochastic policy; similar comments hold when $\pi$ is a deterministic policy. Let $\Pi = [\pi(x, a)]$ represent the $d \times k$ ‘policy’ matrix. Let $\tilde{g} := (\tilde{g}(x, a) : (x, a) \in \mathcal{X} \times \mathcal{I})$ be a given cost function on the state-control space $\mathcal{X} \times \mathcal{I}$. Then, $g$ is the per-stage expected cost function defined as $g(x) := \sum_{a \in \mathcal{I}} \pi(x, a) \tilde{g}(x, a)$, and the value function is given by

$$V_\pi := \Pi (\tilde{g} + \alpha_{dis} MV_\pi),$$

$$= g + \alpha_{dis} \Pi M V_\pi.$$  

Similarly, define $\hat{V}_\pi := g + \alpha_{dis} + \Pi \hat{M} V_\pi$. Then, since $\Pi M$ is a stochastic matrix, Theorem 4 can be straightforwardly applied yielding a sample complexity result for policy evaluation.
6 Conclusions

In this paper, we derive exact rates of convergence for the empirical estimator of the transition probability matrix of a controlled Markov chain, and used that to derive the sample complexity of achieving a particular estimation error. We further provide optimality guarantees of the recovered sample complexity by proving minimaxity. We also tease out the exact interplay between the mixing coefficients of the states and the controls on the sample complexity. Finally, we demonstrate how the estimation error bound for the transition matrix translates into an estimation error bound for the value function and corresponding applications to policy evaluation. There remains a plethora of unanswered questions which are related to this paper. We highlight three of these below.

Countable state spaces. As an obvious extension, consider the interesting problem of countably infinite state and control spaces. Some work in this regard can be found in [62] where state spaces are countably infinite, but there are no reasonable extensions to the setting with countably infinite control space.

Uncountably infinite state space and finite controls/continuous time. We have also found no result which derives the minimax sample complexity of estimating the transition probability distribution of a Markov chain on an uncountably infinite state space. The histogram estimator is the obvious counterpart to the proportion based estimator discussed in this work. Indeed, a recent study [50] demonstrates promising properties of the former. But to the best of our knowledge, the techniques in [50] do not translate to uncountable state spaces, and optimally estimating the transition probability distribution remains an open question. In the same note, the extensions to a continuous time finite state controlled Markov chain is unclear as well.

Learning in presence of weaker mixing or adversarial controls. Although strong mixing properties of the controls is a sufficient condition for the ‘estimability’ of the transition matrices, it may not be a realistic assumption when the system dynamics are weakly mixing or adversarial [43]. Indeed, system identification under the presence of an adversary remains an interesting question, which was addressed in a recent paper [53] using strong linearization assumptions and exponential computation times. However, this is well beyond the scope of the current work and is a direction for future study.

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References

[1] Tom M Apostol. “An elementary view of Euler’s summation formula”. In: The American Mathematical Monthly 106.5 (1999), pp. 409–418.

[2] Mor Armony et al. “On patient flow in hospitals: A data-based queueing-science perspective”. In: Stochastic systems 5.1 (2015), pp. 146–194.

[3] Kazuoki Azuma. “Weighted sums of certain dependent random variables”. In: Tohoku Mathematical Journal 19.3 (1967), pp. 357–367.
[4] Imon Banerjee, Vinayak A Rao, and Harsha Honnappa. “PAC-Bayes Bounds on Variational Tempered Posteriors for Markov Models”. In: Entropy 23.3 (2021), p. 313.

[5] Carolyn L. Beck and Rayadurgam Srikant. “Error bounds for constant step-size Q-learning”. In: Systems & Control Letters 61.12 (2012), pp. 1203–1208.

[6] Patrick Billingsley. “Statistical methods in Markov chains”. In: The Annals of Mathematical Statistics (1961), pp. 12–40.

[7] Vivek S Borkar. Topics in controlled Markov chains. 1991.

[8] Richard C. Bradley. “Basic Properties of Strong Mixing Conditions. A Survey and Some Open Questions”. In: Probab. Surveys 2 (2005), pp. 107–144. DOI: 10.1214/154957805100000104. URL: https://doi.org/10.1214/154957805100000104.

[9] Ann Cohen Brandwein and William E Strawderman. “Minimax estimation of location parameters for spherically symmetric distributions with concave loss”. In: The Annals of Statistics 8.2 (1980), pp. 279–284.

[10] David Alexander Brannan. A first course in mathematical analysis. Cambridge University Press, 2006.

[11] Irene Y. Chen et al. “Probabilistic machine learning for healthcare”. In: Annual Review of Biomedical Data Science 4 (2021), pp. 393–415.

[12] Robert Chin et al. “Active learning for linear parameter-varying system identification”. In: IFAC-PapersOnLine 53.2 (2020), pp. 989–994.

[13] Jim G. Dai and Mark Gluzman. “Queueing network controls via deep reinforcement learning”. In: Stochastic Systems 12.1 (2022), pp. 30–67.

[14] Roland L. Dobrushin. “Central limit theorem for nonstationary Markov chains. I”. In: Theory of Probability & Its Applications 1.1 (1956), pp. 65–80.

[15] Roland L’vovich Dobrushin. “Central limit theorem for nonstationary Markov chains. II”. In: Theory of Probability & Its Applications 1.4 (1956), pp. 329–383.

[16] Constance van Eeden. “Minimax estimators and their admissibility”. In: Restricted Parameter Space Estimation Problems: Admissibility and Minimaxity Properties (2006), pp. 33–67.

[17] Dominique Fourdrinier, Fatiha Mezoued, and William E Strawderman. “Bayes minimax estimation under power priors of location parameters for a wide class of spherically symmetric distributions”. In: Electronic Journal of Statistics 7 (2013), pp. 717–741.

[18] Charles J. Geyer. “Markov chain Monte Carlo lecture notes”. In: Course notes, Spring Quarter 80 (1998).

[19] BK Ghosh. “Probability inequalities related to Markov’s theorem”. In: The American Statistician 56.3 (2002), pp. 186–190.

[20] Allan Gut and Allan Gut. Probability: a graduate course. Vol. 5. Springer, 2005.

[21] John Hajnal and Maurice S. Bartlett. “Weak ergodicity in non-homogeneous Markov chains”. In: Mathematical Proceedings of the Cambridge Philosophical Society. Vol. 54. Cambridge University Press. 1958, pp. 233–246.

[22] Peter Hall and Christopher C. Heyde. Martingale limit theory and its application. Academic Press, 2014.
[23] Galin L Jones. “On the Markov chain central limit theorem”. In: Probability surveys 1 (2004), pp. 299–320.

[24] Sanyam Kapoor. “Multi-agent reinforcement learning: A report on challenges and approaches”. In: arXiv preprint arXiv:1807.09427 (2018).

[25] Rahul Kidambi et al. “Morel: Model-based offline reinforcement learning”. In: Advances in neural information processing systems 33 (2020), pp. 21810–21823.

[26] Leonid Aryeh Kontorovich, Kavita Ramanan, et al. “Concentration inequalities for dependent random variables via the martingale method”. In: Annals of Probability 36.6 (2008), pp. 2126–2158.

[27] Alexander Krolicki and Pierre-Yves Lavertu. “Supervised DKRC with Images for Offline System Identification”. In: arXiv preprint arXiv:2109.02241 (2021).

[28] Volodymyr Kuleshov and Doina Precup. “Algorithms for multi-armed bandit problems”. In: arXiv preprint arXiv:1402.6028 (2014).

[29] Erich L Lehmann and George Casella. Theory of point estimation. Springer Science & Business Media, 2006.

[30] Sergey Levine et al. “Offline reinforcement learning: Tutorial, review, and perspectives on open problems”. In: arXiv preprint arXiv:2005.01643 (2020).

[31] Gen Li et al. “Settling the sample complexity of model-based offline reinforcement learning”. In: arXiv preprint arXiv:2204.05275 (2022).

[32] Yuanzhi Li, Ruosong Wang, and Lin F Yang. “Settling the horizon-dependence of sample complexity in reinforcement learning”. In: 2021 IEEE 62nd Annual Symposium on Foundations of Computer Science (FOCS). IEEE. 2022, pp. 965–976.

[33] Siqi Liu et al. “Reinforcement learning for clinical decision support in critical care: comprehensive review”. In: Journal of medical Internet research 22.7 (2020), e18477.

[34] Yahui Liu, Buyang Cao, and Hehua Li. “Improving ant colony optimization algorithm with epsilon greedy and Levy flight”. In: Complex & Intelligent Systems 7.4 (2021), pp. 1711–1722.

[35] Lennart Ljung. “Perspectives on system identification”. In: Annual Reviews in Control 34.1 (2010), pp. 1–12.

[36] Kristof Maes et al. “Offline synchronization of data acquisition systems using system identification”. In: Journal of Sound and Vibration 381 (2016), pp. 264–272.

[37] Horia Mania, Michael I Jordan, and Benjamin Recht. “Active learning for nonlinear system identification with guarantees”. In: arXiv preprint arXiv:2006.10277 (2020).

[38] Shie Mannor and John N Tsitsiklis. “On the empirical state-action frequencies in Markov decision processes under general policies”. In: Mathematics of Operations Research 30.3 (2005), pp. 545–561.

[39] Florence Merlevède, Magda Peligrad, Emmanuel Rio, et al. “Bernstein inequality and moderate deviations under strong mixing conditions”. In: High dimensional probability V: the Luminy volume 5 (2009), pp. 273–292.

[40] Sean P Meyn and Richard L Tweedie. Markov chains and stochastic stability. Springer Science & Business Media, 2012.

[41] Farrukh Mukhamedov. “The Dobrushin ergodicity coefficient and the ergodicity of noncommutative Markov chains”. In: Journal of Mathematical Analysis and Applications 408.1 (2013), pp. 364–373.
[42] Mirco Mutti, Riccardo De Santi, and Marcello Restelli. “The Importance of Non-Markovianity in Maximum State Entropy Exploration”. In: arXiv preprint arXiv:2202.03060 (2022).

[43] Lerrel Pinto et al. “Robust adversarial reinforcement learning”. In: International Conference on Machine Learning, PMLR. 2017, pp. 2817–2826.

[44] Paria Rashidinejad et al. “Bridging offline reinforcement learning and imitation learning: A tale of pessimism”. In: Advances in Neural Information Processing Systems 34 (2021), pp. 11702–11716.

[45] Tirthankar RayChaudhuri and Leonard GC Hamey. “Active learning for nonlinear system identification and control”. In: IFAC Proceedings Volumes 29.1 (1996), pp. 2592–2596.

[46] Emmanuel Rio. “On McDiarmid’s concentration inequality”. In: Electronic Communications in Probability 18 (2013), pp. 1–11.

[47] M Rosenblatt-Roth. “Some theorems concerning the law of large numbers for non-homogeneous Markoff chains”. In: Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete 1.5 (1963), pp. 433–445.

[48] M Rosenblatt-Roth. “Some theorems concerning the strong law of large numbers for non-homogeneous Markov chains”. In: The Annals of Mathematical Statistics 35.2 (1964), pp. 566–576.

[49] Jeffrey S Rosenthal. “Minorization conditions and convergence rates for Markov chain Monte Carlo”. In: Journal of the American Statistical Association 90.430 (1995), pp. 558–566.

[50] Mathieu Sart. “Estimation of the transition density of a Markov chain”. In: Annales de l’IHP Probabilités et statistiques. Vol. 50. 3. 2014, pp. 1028–1068.

[51] Laixi Shi et al. “Pessimistic q-learning for offline reinforcement learning: Towards optimal sample complexity”. In: arXiv preprint arXiv:2202.13890 (2022).

[52] Susan M Shortreed et al. “Informing sequential clinical decision-making through reinforcement learning: an empirical study”. In: Machine learning 84.1 (2011), pp. 109–136.

[53] Mehrdad Showkatbakhsh, Paulo Tabuada, and Suhas Diggavi. “System identification in the presence of adversarial outputs”. In: 2016 IEEE 55th Conference on Decision and Control (CDC). IEEE. 2016, pp. 7177–7182.

[54] Richard S Sutton and Andrew G Barto. Reinforcement learning: An introduction. MIT press, 2018.

[55] Serge Suruto and C Guedes Soares. “Offline system identification of ship manoeuvring mathematical models with a global optimization algorithm”. In: MARSIM 2015 (2015), pp. 8–11.

[56] Arun K Tangirala. Principles of system identification: theory and practice. Crc Press, 2018.

[57] Alexandre B Tsybakov. Introduction to Nonparametric Estimation. Springer, 2009, pp. I–XII.

[58] Jacobus Hendricus Van Lint. Introduction to coding theory. Vol. 86. Springer Science & Business Media, 2012.

[59] Roman Vershynin. High-dimensional probability: An introduction with applications in data science. Vol. 47. Cambridge university press, 2018.

[60] Mathukumalli Vidyasagar and Rajeeva L Karandikar. “A learning theory approach to system identification and stochastic adaptive control”. In: Probabilistic and randomized methods for design under uncertainty (2006), pp. 265–302.

[61] Yimin Wei, Yanhua Cao, and Hua Xiang. “A note on the componentwise perturbation bounds of matrix inverse and linear systems”. In: Applied mathematics and computation 169.2 (2005), pp. 1221–1236.
[62] Geoffrey Wolfer, Aryeh Kontorovich, et al. “Statistical estimation of ergodic Markov chain kernel over discrete state space”. In: Bernoulli 27.1 (2021), pp. 532–553.

[63] Jacob Wolfowitz. “Products of indecomposable, aperiodic, stochastic matrices”. In: Proceedings of the American Mathematical Society 14.5 (1963), pp. 733–737.

[64] Michael Wunder, Michael L Littman, and Monica Babes. “Classes of multiagent q-learning dynamics with epsilon-greedy exploration”. In: ICML. 2010.

[65] Sidney Yakowitz. “Nonparametric density and regression estimation for Markov sequences without mixing assumptions”. In: Journal of Multivariate Analysis 30.1 (1989), pp. 124–136.

[66] Sidney Yakowitz. “Nonparametric estimation of Markov transition functions”. In: The Annals of Statistics (1979), pp. 671–679.

[67] Yuling Yan et al. “Model-Based Reinforcement Learning Is Minimax-Optimal for Offline Zero-Sum Markov Games”. In: arXiv preprint arXiv:2206.04044 (2022).

[68] Ming Yin and Yu-Xiang Wang. “Towards instance-optimal offline reinforcement learning with pessimism”. In: Advances in neural information processing systems 34 (2021), pp. 4065–4078.

[69] Chao Yu et al. “Reinforcement learning in healthcare: A survey”. In: ACM Computing Surveys (CSUR) 55.1 (2021), pp. 1–36.

[70] Huizhen Yu and Dimitri P Bertsekas. “On boundedness of Q-learning iterates for stochastic shortest path problems”. In: Mathematics of Operations Research 38.2 (2013), pp. 209–227.

[71] Huizhen Yu and Dimitri P Bertsekas. “Q-learning and policy iteration algorithms for stochastic shortest path problems”. In: Annals of Operations Research 208.1 (2013), pp. 95–132.

[72] Tianhe Yu et al. “Mopo: Model-based offline policy optimization”. In: Advances in Neural Information Processing Systems 33 (2020), pp. 14129–14142.

[73] Fuzhen Zhang. Matrix theory: basic results and techniques. Springer, 2011.
A Technical Desiderata

A.1 Concentration Inequalities

Hoeffding’s concentration inequality for mixing sequences. For two sequences of real numbers \( \bar{x} = \{x_0, \ldots, x_m\} \) and \( \bar{y} = \{y_0, \ldots, y_m\} \), define the Hamming metric \( d \) between them as \( d(\bar{x}, \bar{y}) := \sum_i 1[x_i \neq y_i] \). Observe that for any two sequences \( \bar{x} \) and \( \bar{y} \) we have,

\[
|N_s(\bar{x}) - N_s(\bar{y})| = \left| \sum_{i=1}^m 1[x_i = s] - \sum_{i=1}^m 1[y_i = s] \right| \\
\leq \sum_{i=1}^m |1[x_i = s] - 1[y_i = s]| \\
\leq \sum_{i=1}^m 1[x_i \neq y_i] = d(\bar{x}, \bar{y}).
\]

Therefore, the function \( N_s(\bar{x}) := \sum_{i=1}^m 1[x_i = s] \) is 1-Lipschitz in Hamming metric. This allows us to specialise Theorem 1.1 from [26] to our current setting.

Lemma 16. Let \( \{(X_0, a_0), \ldots, (X_m, a_m)\} \) be a sequence of stochastic random variables on a finite state space \( \chi \times \mathbb{I} \). Then, for any \( t > 0 \) we have,

\[
\mathbb{P}(|N_s^{(l)}(\mathbb{I}) - \mathbb{E}[N_s^{(l)}(\mathbb{I})]| > t) \leq 2 \exp\left( -\frac{t^2}{2m||\Delta_{m}||^2} \right),
\]

where \( ||\Delta_{m}|| = \max_{1 \leq i \leq m} (1 + \eta_{i,i+1} + \eta_{i,i+2} + \ldots \eta_{i,m}) \), and \( \eta_{i,j} \) is as defined in eq. (2.1).

Remark. It is clear that analysing the tail properties of \( N_s^{(l)} \) requires an appropriate concentration inequality. The result from Kontorovich, Ramanan, et al. [26], as recalled in Lemma 16 is a refinement of the celebrated Azuma-Hoeffding’s inequality [3] where the martingale differences are linked to the underlying mixing coefficients of the stochastic process. Observe that the inequality is guaranteed to be as tight as Azuma-Hoeffding’s inequality [26, Theorem 7.5]. However, as we discuss is Remark 4.1, it is not tight enough to achieve minimaxity.

Chernoff’s concentration inequality for geometrically mixing sequences. We begin this subsection with the following lemma.

Lemma 17. For some \( i, j \), let \( T_* = \{0, 1\}^{m-j} \) and \( h_* \in \{0, 1\}^{i+1} \). For the convenience of notation, denote \( (\mathbb{I}[X_i, a_i], \ldots, \mathbb{I}[X_0, a_0]) \) by \( 1(\mathcal{H}_0^0) \). Observe that

\[
\phi_{i,j} = \sup_{T, h_*} \left[ \mathbb{P}\left( (X_m, a_m, \ldots, X_j, a_j) \in T|\mathcal{H}_0^0 = h_0^0 \right) - \mathbb{P}\left( (X_m, a_m, \ldots, X_j, a_j) \in T_*|\mathcal{H}_0^0 = h_* \right) \right] \\
\geq \sup_{T, h_*} \left[ \mathbb{P}\left( \mathbb{I}[X_m = s, a_m = l], \ldots, \mathbb{I}[X_j = s, a_j = l] \in T_*|\mathcal{H}_0^0 = h_* \right) \right] \\
- \mathbb{P}\left( \mathbb{I}[X_m = s, a_m = l], \ldots, \mathbb{I}[X_j = s, a_j = l] \in T_* \right).
\]

Remark. The uniform mixing coefficients of random variables form an upper bound to the uniform mixing sequences of the indicator functions of the same random variables.

Using the aforementioned fact, we specialise Theorem 2 from [39]. As in the proof of Proposition 6, we drop \( s \) and \( l \), and denote \( \mathbb{I}[X_i = s, a_i = l] \) by \( I_i \). Then we have the following lemma.
Lemma 18. Let \((X_m, a_m, \ldots, X_0, a_0)\) be a sequence of stochastic random variables on a finite state space \(\chi \times \mathbb{I}\). Assume that there exists a positive constant \(c_{cof} > 0\) such that the uniform mixing coefficient \(\phi_{i,j}\) satisfies
\[
\sup_{j-i} \phi_{i,j} \leq e^{-c_{cof}(j-1)}.
\]
Then, there exists a constant \(C_{pel}\) which depends only upon \(c_{cof}\) such that for all \(m \geq 2\)
\[
\mathbb{P}\left(\left|N_s^{(l)} - \mathbb{E}[N_s^{(l)}]\right| \geq t\right) \leq \exp\left(-\frac{C_{pel}t^2}{m \sup_{i \geq 0}\left(\text{Var}(I_i) + 2 \sum_{j \geq 1} |\text{Cov}(I_i, I_j)|\right) + 1 + t(log m)^2}\right) \quad (A.2)
\]

**Proof.** The proof follows from [39, eqn. 2.1] by observing from Bradley [8, eqn. 1.12] that uniform mixing coefficients form a natural upper bound to strong mixing coefficients (defined as in Bradley [8, eqn. 1.1]), and then observing that for indicator variables \(\bar{M} = 1\).

**Remark.** This gives us a cleaner version of the original inequality appropriated to our current setup.

Lemma 19. Let \(X, Y,\) and \(Z\) be 3 discrete random variables in the sample space \(\Omega\). Then
\[
\|\mathcal{L}(X, Y|Z = z_1) - \mathcal{L}(X, Y|Z = z_2)\|_{TV} \leq \|\mathcal{L}(Y|z = z_1) - \mathcal{L}(Y|Z = z_2)\|_{TV} + \sup_y \|\mathcal{L}(X|Y = y, Z = z_1) - \mathcal{L}(X|Y = y, Z = z_2)\|_{TV}.
\]

Two finite-state Markov chains are of the same type if their transition matrices have zeros in the same entries. The following lemma is found in Wolfowitz [63, Theorem 1].

Lemma 20. Suppose that \(\{(X_0, a_0), \ldots, (X_m, a_m)\}\) is a sample from a time-inhomogeneous Markov chain. Then \(\bar{\theta}_{i,j} \leq e^{-\nu|j-i|}\), where \(\nu\) is a constant depending only on the number of types of matrices in the set \(\mathcal{M}\).

As a follow-up remark, we note from Wolfowitz [63] that whether a transition matrix belongs to an ergodic and irreducible Markov chain depends solely on the type. The following lemma extends the previous one to the case of controlled Markov chains with non-stationary Markov controls.

Lemma 21. Let \(\{(X_0, a_0), \ldots, (X_m, a_m)\}\) be a sample from a controlled Markov chain with transition matrices \(M^{(l)}\) and Markov controls such that for any \(s \in \chi, l \in \mathbb{I}, h_i^{l-1} \in (\chi \times \mathbb{I})^{i-1}\) and \(i \in \mathbb{N} \cup \{0\}\),
\[
P^{(i)}_{s,l} := \mathbb{P}\left[a_i = l \mid X_i = s, \mathcal{H}_0^{l-1} = h_0^{l-1}\right] = \mathbb{P}\left[a_i = l \mid X_i = s\right].
\]

If there exists a \(\chi_0 \subseteq \chi\) and \(M_{min} > 0\) such that,
\[
\min_{s \in \chi, l \in \mathbb{I}} M^{(l)}_{s,t} \geq M_{min}, \quad \forall \ t \in \chi_0,
\]
then, for this controlled Markov chain,
\[
\bar{\theta}_{i,j} \leq (1 - |\chi_0|M_{min})^{j-i-1}.
\]

As a consequence, this CMC satisfies Assumption 6 with \(C_{\theta} = 1/(1 - |\chi_0|M_{min})\).
Remark. If the number of types of matrices is independent of \( k \), observe it follows from Lemma 20 that there exists a class of Inhomogenous Markov chains for which \( c_{\text{cof}} \) is independent of the parameters of the Markov chain, where the definition of \( c_{\text{cof}} \) is as in Lemma 18. This naturally implies that \( C_{\text{pel}} \) is a universal constant for that class.

Furthermore, it can be derived from Lemma 21 that whenever \( \| \chi_0 \| M_{\text{min}} \) is independent of \( d, k \) there exists a CMC with non-stationary Markov controls, which satisfy a similar property.

Lemma 22. Let \( \{(X_0, a_0), \ldots, (X_m, a_m)\} \) be a sample from a controlled Markov chain belonging to the class of CMC’s as defined in eq. (D.20). Then, \( \{(X_0, a_0), \ldots, (X_m, a_m)\} \) satisfies Assumptions 1,2,5, and 6.

Proof. This controlled Markov chain has only 1 control with the corresponding transition matrix positive. Therefore, it is irreducible and aperiodic [40, Proposition 8.1.1] and therefore, corresponds to a geometrically ergodic Markov chain [40, Theorem 15.0.1]. The rest of the proof follows similarly to the proof of Proposition 10. \( \square \)

Lemma 23. Let \( \{X_i, a_i\} \) be an inhomogenous Markov chain with \( \chi = \{1, 2, 4, 5\} \) and \( \mathbb{I} = \{1, 2\} \) such that \( P(a_0 = 1) = 1, P(a_1 = 2) = 1, P(a_2 = 1) = 1 \) and so on. Moreover, \( a_i \) depends only upon time point \( i \) and is independent of \( (X_i, \mathcal{H}_i) \). The transition matrices are given by

\[
M^{(1)} = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix} \quad \text{and} \quad M^{(2)} = \begin{bmatrix} 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix}.
\]

\( X_0 \) is drawn uniformly from \( \{1, 2, 3, 4\} \). Then, for this inhomogenous Markov chain \( \bar{\theta}_{i,j} = 1 \) for any time points \( j > i \).

Proof. We observe that

\[
\bar{\theta}_{i,i+1} = \sup_{s_1, s_2 \in \chi, l_1, l_2 \in \mathbb{I}, P(X_i = s_1, a_i = l_1) > 0, P(X_i = s_2, a_i = l_2) > 0} \| \mathcal{L}(X_j | X_i = s_1, a_i = l_1) - \mathcal{L}(X_j | X_i = s_2, a_i = l_2) \|_{TV}
\]

is only well-defined if \( l_1 = l_2 \). Let \( s_1 = 1, \) and \( s_2 = 3 \). Then,

\[
\bar{\theta}_{i,i+1} \geq |P(X_j \in \{1, 2\} | X_i = 1, a_i = l_1) - P(X_j \in \{1, 2\} | X_i = 3, a_i = l_1)| = 1.
\]

Thus, \( \bar{\theta}_{i,i+1} = 1 \). Similarly, \( \bar{\theta}_{i,i+1} = 1 \) for any \( j > i \) and

\[
\sup_{1 \leq i \leq \infty} \sum_{j=i+1}^{\infty} \bar{\theta}_{i,j} = \infty.
\]

\( \square \)
B Minimality Example

That uniform distribution is the least favourable choice for estimation purposes are well studied both in frequentist and in Bayesian statistics, and can be found in Brandwein and Strawderman [9, Section 3.2], Eeden [16], Lehmann and Casella [29, page 340-342, example 3.4], and Fourdrinier, Mezoued, and Strawderman [17]. Consequently, the controlled Markov chain which we use to prove minimality of the empirical estimator closely mirrors an uniform distribution.

Let \( \ell \) be a fixed real number between 0 and 31/64 and furthermore, let \( d \) be an integer divisible by 3. Consider a class of Markov decision processes with \( \chi = \{1, \ldots, d\} \) and \( I = \{1, \ldots, k + 1\} \) such that the sequence of controls \( a_i \) satisfies,

\[
a_i = l \text{ with probability } \frac{1}{k} \quad \forall \ l \in I
\]  
(B.1)

Our next step is to construct the transition probability matrices.

**Transition Matrices.** Let \( \ell \) be a fixed real number between 0 and 1 and for each \( l \in \{1, \ldots, k\} \), let \( \xi^{(l)} = (\xi^{(l)}_1, \ldots, \xi^{(l)}_{d/3}) \) be some vector in \( \{0,1\}^{d/3} \). For convenience, assume that \( \xi^{(l)} \neq (0, \ldots, 0) \) for at least some \( l \in I \). Then the \( l \)-th transition probability matrix \( M_{l,\xi^{(l)}}^{(l)} \) is a block matrix such that

\[
M_{l,\xi^{(l)}}^{(l)} = \begin{bmatrix}
C_{\ell} & R_{\xi^{(l)}} \\
J_{\ell} & L_{\ell}
\end{bmatrix},
\]  
(B.2)

where the blocks \( C_{\ell} \in \mathbb{R}^{d/3 \times d/3} \), \( L_{\ell} \in \mathbb{R}^{2d/3 \times 2d/3} \), \( J_{\ell} \in \mathbb{R}^{2d/3 \times d/3} \), and \( R_{\xi^{(l)}} \in \mathbb{R}^{d/3 \times 2d/3} \) are given by

\[
L_{\ell} = \text{diag} \left(1 - \ell, 1 - \ell, \ldots, 1 - \ell\right),
\]

\[
R_{\xi^{(l)}} = \frac{1}{2}
\begin{bmatrix}
3\ell - 1 & \frac{3\ell}{d-3} & 0 & \cdots & 0 \\
\frac{3\ell}{d-3} & \frac{3\ell}{d-3} & \frac{3\ell}{d-3} & \cdots & \frac{3\ell}{d-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & 3\ell - 1 \\
\end{bmatrix},
\]

and, \( C_{\ell} \) and \( J_{\ell} \) are matrices with every element equal to \( 3\ell/d \). It can be verified by summing over the rows that for each \( l \), \( M_{l,\xi^{(l)}}^{(l)} \) is a valid transition probability matrix. We get the following lemma:

**Lemma 24.** The stationary distribution of a Markov chain with transition probability matrix \( M_{l,\xi^{(l)}}^{(l)} \) is a block vector \( \Pi^{(l,\ell,\xi^{(l)})} := \left[ \Pi_1^{(l)}, \Pi_2^{(l,\ell,\xi^{(l)})} \right] \) where \( \Pi_1^{(l)} \) is a row vector of length \( d/3 \) and every element \( 3\ell/d \). Furthermore,

\[
\Pi_2^{(l,\ell,\xi^{(l)})} = \left( \frac{3(1 + \xi^{(l)}_1 \ell - \ell)}{2d}, \frac{3(1 - \xi^{(l)}_1 \ell - \ell)}{2d}, \cdots, \frac{3(1 - \xi^{(l)}_{d/3} \ell - \ell)}{2d} \right).
\]

The proof follows by verifying \( \Pi^{(l,\ell,\xi^{(l)})} M_{l,\xi^{(l)}}^{(l)} = \Pi^{(l,\ell,\xi^{(l)})} \) and is straightforward. Therefore, we omit it.

**Proposition 25.** Let \( \{(X_0, a_0), \ldots, (X_m, a_m)\} \) be a sample from a controlled Markov chain with controls given by eq. (B.1), transition probability matrices given by eq. (B.2), and initial distribution \( \Pi = \frac{1}{k} \left( \Pi^{(1,\ell,\xi^{(l)})}, \ldots, \Pi^{(k,\ell,\xi^{(l)})} \right). \) Then, it satisfies Assumptions 1 with \( \zeta_1 = \zeta_2 = \ell \), 2 with \( T = 2dk/3\ell \), and 7-8 with \( C_* \) and \( C_{d,*} \) independent of \( d, k \) or \( \xi^{(l)} \) for any \( l \in I \). Furthermore, for this controlled Markov chain, \( \rho_* = 3(1 - \ell)/2dk \) where \( \rho_* \) is defined as in Theorem 2.
Proof. It is obvious that the state-control pair \((X_i, a_i)\) is a Markov chain with transition probability matrix given by the block matrix
\[
\frac{1}{k} \begin{bmatrix}
M_{i,\xi}^{(1)} & \cdots & M_{i,\xi}^{(k)} \\
\vdots & \ddots & \vdots \\
M_{i,\xi}^{(1)} & \cdots & M_{i,\xi}^{(k)}
\end{bmatrix}.
\]

Taking \(S_t = \{(1, 1), \ldots, (d/3, 1), (2, 1), \ldots, (d/3, k)\}\), we can immediately see that Assumption 1 is satisfied with \(\zeta_1 = \zeta_2 = \iota\). Next, let \(\Pi\) be the stationary distribution of this Markov chain. Recall from the proof of Proposition 10 that any controlled Markov chain with stationary Markov controls satisfies Assumption 2 with \(T\) to be the supremum of the inverse of its stationary probabilities. In other words,
\[
T = \sup_{s, l} \frac{1}{\Pi_{s, l}}.
\]

We simply verify that \(\Pi_{s, l} > 3\iota/2dk\) for any \((s, l) \in \chi \times \mathbb{I}\). It is known from Lemma 24 that \(\Pi^{(d, i, \xi(l))}\) is the stationary distribution of \(M_{i,\xi(l)}^{(l)}\). Using this fact, it can be easily verified that the stationary distribution of the paired process \((X_i, a_i)\) is given by
\[
\Pi = \frac{1}{k} \left(\Pi^{(1, i, \xi(1))}, \ldots, \Pi^{(k, i, \xi(k))}\right).
\]

Recall from hypothesis that \(\varepsilon < 1/32\). This implies that, for any \(\xi \in \{0, 1\}\)
\[1 - \xi\varepsilon - \iota > 31/32 - \iota > \iota\]
whenever \(\iota < 31/64\). Thus,
\[
\frac{3(1 - \xi\varepsilon - \iota)}{2dk} > \frac{3\iota}{2dk}
\]
Obviously, \(3\iota/2dk > 3\iota/2dk\). Thus, \(\Pi_{s, l} > 3\iota/2dk\) for any pair \((s, l) \in \chi \times \mathbb{I}\). As a consequence, since \(\{X_i, a_i\}\) is a stationary Markov chain, it follows that the marginal probabilities
\[
P(X_i = s, a_i = l) = \Pi_{s, l} < \frac{3(1 - \xi\varepsilon - \iota)}{2dk} < \frac{3(1 - \iota)}{2dk}.
\]

This establishes that \(\rho_* = 3(1 - \iota)/2dk\). Next, because \(a_i\)'s are distributed uniformly over \(\mathbb{I}\), it is obvious that
\[
\gamma_{p, j, i} = \sup_{s_p, h_{i+j}^p, h_0^i} \left\|L \left( a_p | X_p = s_p, h_{i+j}^p = h_{i+j}^0, h_0^i = h_0^i \right) - L \left( a_p | X_p = s_p, h_{i+j}^{p-1} = h_{i+j}^{p-1} \right) \right\|_{TV}
= 0.
\]

Consequently, we get that \(C_*\) is independent of \(d, k, \xi(l)\) and \(C = 0\), where \(l \in \mathbb{I}\). Finally, observe that every controlled Markov chain with stationary controls is also trivially a CMC with non-stationary Markov controls. Therefore, we can use Lemma 21 with \(x_0 = (1, \ldots, d/3)\), and \(M_{min} = 3\iota/d\) to see that
\[
\tilde{\beta}_{i, j} \leq \left(1 - \frac{d}{3} \frac{3\iota}{d} \right)^{j-i-1} = (1 - \iota)^{j-i-1},
\]
Consequently, we get that \(C_*\) is independent of \(d, k, \xi(l)\) and \(C_\theta = 1 - \iota\), where \(l \in \mathbb{I}\). This proves our claims. \(\square\)
C The controlled Markov chain Sampling Scheme

For each \( l \in \mathbb{I} \) and \( M(l) \in \mathbb{M} \), create the following infinite array of i.i.d random variables which are also independent of the data \( \{(X_0, a_0), \ldots, (X_m, a_m)\} \).

\[
\begin{array}{cccc}
X_{1,1}^{(l)} & X_{1,2}^{(l)} & \cdots & X_{1,\tau}^{(l)} \\
X_{2,1}^{(l)} & X_{2,2}^{(l)} & \cdots & X_{2,\tau}^{(l)} \\
\vdots & \vdots & \ddots & \vdots \\
X_{d,1}^{(l)} & X_{d,2}^{(l)} & \cdots & X_{d,\tau}^{(l)}
\end{array}
\]

(C.1)

where, \( \forall (s, t, \tau) \in \{1, \ldots, d\} \times \{1, \ldots, d\} \times \mathbb{N} \), the random variables \( X_{s,\tau}^{(l)} \) follow the mass function given by \( P(X_{s,\tau}^{(l)} = t) = M_{s,t}^{(l)} \). Moreover, for every time point \( i \geq 1 \), and \( (s_0, l_0, \ldots, s_{i-1}, l_{i-1}, s_i) \in (\chi \times \mathbb{I})^{i-1} \times \chi \), let, \( \alpha_i^{(s_0,l_0,\ldots,s_{i-1},l_{i-1},s_i)} \) be independent random variables with support \( \mathbb{I} \) and mass function given by,

\[
P(\alpha_i^{(s_0,l_0,\ldots,s_{i-1},l_{i-1},s_i)} = l) = P(a_i = l | X_i = s_i, H_i^{i-1} = s_0, l_0, \ldots, s_{i-1}, l_{i-1})
\]

The sampling scheme runs as follows: sample \( \tilde{X}_0 \sim D_0 \) and set \( \tilde{a}_0 \overset{d}{=} a_0 \). Recursively sample \( \tilde{X}_{i+1} = X_{\tilde{a}_i,\tilde{X}_{\tilde{a}_i}}^{(\tilde{a}_i)} \) from the array \( \tilde{X}_{\tilde{a}_i} \) and \( \tilde{a}_{i+1} = \alpha_{i+1}^{(\tilde{X}_0, \tilde{a}_0, \ldots, \tilde{X}_{i+1})} \), where each \( i \geq 0 \), define \( \tilde{N}_s^{(i,l)} := \sum_{j \leq i} [\tilde{X}_j = s, \tilde{a}_j = l] \) and \( \tilde{N}_s^{(m,l_m)} = \tilde{N}_s^{(l_m)} \). This completes the sampling scheme.

**Proposition 26.** \( (X_0, a_0, \ldots, X_m, a_m) \) is identically distributed to \( (\tilde{X}_0, \tilde{a}_0, \ldots, \tilde{X}_m, \tilde{a}_m) \).

**Proof.** Using induction, the proof is straightforward and can be found in Appendix F.12. \( \square \)
D Proofs of Theorems

D.1 Proof of Theorem 1 (Sample Complexity)

Proof. We start by analysing the event \( \left\{ \sup_{l \in \mathbb{I}} \| \hat{M}^{(l)} - M^{(l)} \|_{\infty} > \varepsilon \right\} \). We note that if \( \sup_{l \in \mathbb{I}} \| \hat{M}^{(l)} - M^{(l)} \|_{\infty} > \varepsilon \), then it must be that for at least some \( l_0 \in \mathbb{I} \), \( \| \hat{M}^{(l_0)} - M^{(l_0)} \|_{\infty} > \varepsilon \) and vice versa. Therefore, it follows that

\[
\left\{ \sup_{l \in \mathbb{I}} \| \hat{M}^{(l)} - M^{(l)} \|_{\infty} > \varepsilon \right\} = \bigcup_{l=1}^{k} \left\{ \| \hat{M}^{(l)} - M^{(l)} \|_{\infty} > \varepsilon \right\}.
\]

Hence, applying the union bound,

\[
P \left( \sup_{l \in \mathbb{I}} \| \hat{M}^{(l)} - M^{(l)} \|_{\infty} > \varepsilon \right) = P \left( \bigcup_{l \in \mathbb{I}} \left\{ \| \hat{M}^{(l)} - M^{(l)} \|_{\infty} > \varepsilon \right\} \right) \leq \sum_{l \in \mathbb{I}} P \left( \| \hat{M}^{(l)} - M^{(l)} \|_{\infty} > \varepsilon \right). \tag{D.1}
\]

Fix \( l \in \mathbb{I} \). Recall the definition of \( \hat{M}^{(l)}(s, \cdot) \) and \( M^{(l)}(s, \cdot) \) from eq. (3.9).

Using the fact that \( \| \cdot \|_{\infty} < \| \cdot \|_1 \), we get the following.

\[
\left\{ \| \hat{M}^{(l)} - M^{(l)} \|_{\infty} > \varepsilon \right\} \subseteq \bigcup_{s \in \chi} \left\{ \| \hat{M}^{(l)}(s, \cdot) - M^{(l)}(s, \cdot) \|_1 > \varepsilon \right\}.
\]

It follows from the union bound that

\[
P \left( \| \hat{M}^{(l)} - M^{(l)} \|_{\infty} > \varepsilon \right) \leq \sum_{s \in \chi} P \left( \| \hat{M}^{(l)}(s, \cdot) - M^{(l)}(s, \cdot) \|_1 > \varepsilon \right). \tag{D.2}
\]

Our next objective is to find an upper bound for the probability \( P \left( \| \hat{M}^{(l)}(s, \cdot) - M^{(l)}(s, \cdot) \|_1 > \varepsilon \right) \) that is independent of \( s \). Fix an \( s \in \chi \) and recall the definition of \( N^{(l)}_s \) from Section 3. Using the law of total probability [20, Proposition 4.1], it follows that

\[
P \left( \| \hat{M}^{(l)}(s, \cdot) - M^{(l)}(s, \cdot) \|_1 > \varepsilon \right) = \sum_{n=1}^{m} P \left( \| \hat{M}^{(l)}(s, \cdot) - M^{(l)}(s, \cdot) \|_1 > \varepsilon, N^{(l)}_s = n \right). \tag{D.3}
\]

For any two integers \( 0 \leq n_{low,s} < n_{high,s} \leq m \), we can decompose the right hand side of eq. (D.3) into two parts,

\[
\sum_{n=1}^{m} P \left( \| \hat{M}^{(l)}(s, \cdot) - M^{(l)}(s, \cdot) \|_1 > \varepsilon, N^{(l)}_s = n \right) = \sum_{n=n_{low,s}}^{n_{high,s}} P \left( \| \hat{M}^{(l)}(s, \cdot) - M^{(l)}(s, \cdot) \|_1 > \varepsilon, N^{(l)}_s = n \right) + \sum_{n \notin [n_{low,s}, n_{high,s}]} P \left( \| \hat{M}^{(l)}(s, \cdot) - M^{(l)}(s, \cdot) \|_1 > \varepsilon, N^{(l)}_s = n \right). \tag{D.4}
\]
We can further decompose the second summation by noting that for each \( n \), every summand,
\[
\mathbb{P} \left( \left\| \hat{M}^{(l)}(s, \cdot) - M^{(l)}(s, \cdot) \right\|_1 > \varepsilon, N_s^{(l)} = n \right) \leq \mathbb{P} \left( N_s^{(l)} = n \right).
\]
Therefore, it follows that
\[
\sum_{n \notin [n_{low,s}, n_{high,s}]} \mathbb{P} \left( \left\| \hat{M}^{(l)}(s, \cdot) - M^{(l)}(s, \cdot) \right\|_1 > \varepsilon, N_s^{(l)} = n \right) \leq \mathbb{P} \left( N_s^{(l)} \notin [n_{low,s}, n_{high,s}] \right).
\]
Hence, the right hand side of Equation (D.4) is upper bounded by
\[
\sum_{n = n_{low,s}}^{n_{high,s}} \mathbb{P} \left( \left\| \hat{M}^{(l)}(s, \cdot) - M^{(l)}(s, \cdot) \right\|_1 > \varepsilon, N_s^{(l)} = n \right) + \mathbb{P} \left( N_s^{(l)} \notin [n_{low,s}, n_{high,s}] \right) \tag{D.5}
\]
\[= \text{Term 1} + \text{Term 2},\]
We deal with the two terms on the right hand side (RHS) separately.

**Term 1.**
The analysis of the first term follows directly via Proposition 3. We get
\[
\sum_{n = n_{low,s}}^{n_{high,s}} \mathbb{P} \left( \left\| \hat{M}^{(l)}(s, \cdot) - M^{(l)}(s, \cdot) \right\|_1 > \varepsilon, N_s^{(l)} = n \right) \leq m \exp \left( -\frac{n_{low,s}}{2} \max \left\{ 0, \varepsilon - \sqrt{\frac{d}{n_{high,s}}} \right\}^2 \right).
\]

**Term 2.**
We begin the analysis of the second term by observing that
\[
\mathbb{P} \left( N_s^{(l)} \notin [n_{low,s}, n_{high,s}] \right) = \mathbb{P} \left[ N_s^{(l)} - E[N_s^{(l)}] < n_{low,s} - E[N_s^{(l)}] \right] + \mathbb{P} \left[ N_s^{(l)} - E[N_s^{(l)}] > n_{high,s} - E[N_s^{(l)}] \right]. \tag{D.6}
\]
As long as \( n_{high,s} - E[N_s^{(l)}] > 0 \), directly applying the upper bound in Lemma 16 gives us,
\[
\mathbb{P} \left( N_s^{(l)} - E[N_s^{(l)}] > n_{high,s} - E[N_s^{(l)}] \right) \leq 2 \exp \left( -\frac{\left( n_{high,s} - E[N_s^{(l)}] \right)^2}{2m \| \Delta_m \|^2} \right). \tag{D.7}
\]
Our next step is to select an \( n_{high,s} \) such that
\[
\left( n_{high,s} - E[N_s^{(l)}] \right) > 0. \tag{D.8}
\]
Recall from Lemma 4 that under our hypothesis,
\[
E \left[ N_s^{(l)} \right] \leq m \max \{ \zeta_1, 1 - \zeta_2 \}.
\]
Therefore, by setting \( n_{high,s} = m \left( \frac{1 + \max \{ \zeta_1, 1 - \zeta_2 \}}{2} \right) \), we can ensure that
\[
\left( n_{high,s} - E[N_s^{(l)}] \right) = m \left( \frac{1 + \max \{ \zeta_1, 1 - \zeta_2 \}}{2} \right) - E \left[ N_s^{(l)} \right] > m \left( \frac{1 - \max \{ \zeta_1, 1 - \zeta_2 \}}{2} \right) > 0.
\]
Similarly, by choosing \( n_{low,s} = \frac{m}{2T} \) we ensure \( n_{low,s} - E[N_s^{(t)}] < -\frac{m}{2T} \). Using Lemma 16 again we obtain,

\[
P \left( N_s^{(t)} - E[N_s^{(t)}] < n_{low,s} - E[N_s^{(t)}] \right) \leq 2 \exp \left( -\frac{(n_{low,s} - E[N_s^{(t)}])^2}{2m\|\Delta_m\|^2} \right).
\] (D.9)

This completes the analysis of term 2. Combining the results from equations F.5, D.7 and D.9, we arrive at the following upper bound.

\[
P \left( \left\| \hat{M}^{(t)}(s, \cdot) - M^{(t)}(s, \cdot) \right\|_1 > \varepsilon \right) \leq m \exp \left( -\frac{n_{low,s}}{2} \max \left\{ 0, \varepsilon - \sqrt{\frac{d}{n_{high,s}}} \right\}^2 \right) + 2 \exp \left( -\frac{(n_{high,s} - E[N_s^{(t)}])^2}{2m\|\Delta_m\|^2} \right) + 2 \exp \left( -\frac{(n_{low,s} - E[N_s^{(t)}])^2}{2m\|\Delta_m\|^2} \right)
\]  

= A+B+C.  

(D.10)

Substituting the values of \( n_{high,s} \) and \( n_{low,s} \) in A we get,

\[
m \exp \left( -\frac{n_{low,s}}{2} \max \left\{ 0, \varepsilon - \sqrt{\frac{d}{n_{high,s}}} \right\}^2 \right) = m \exp \left( -\frac{m}{16T} \max \left\{ 0, \varepsilon - \sqrt{\frac{d}{m (1 + \max \{\zeta_1, 1 - \zeta_2\})}} \right\}^2 \right).
\]

Recall that by hypothesis, \( m > \frac{8d}{\varepsilon^2 (1 + \max \{\zeta_1, 1 - \zeta_2\})} \).

This implies that,

\[
\left( \varepsilon - \sqrt{\frac{d}{m (1 + \max \{\zeta_1, 1 - \zeta_2\})}} \right)^2 > \varepsilon^2 \left( 1 - \frac{1}{2} \right)^2 = \frac{\varepsilon^2}{4}.
\]

Thus,

\[
\exp \left( -\frac{n_{low,s}}{2} \max \left\{ 0, \varepsilon - \sqrt{\frac{d}{n_{high,s}}} \right\}^2 \right) \leq \exp \left( -\frac{m\varepsilon^2}{64T} \right).
\]

This gives us an upper bound for A.

Recall that, we have chosen \( n_{high,s} \) such that \( n_{high,s} - E[N_s^{(t)}] > m \left( \frac{1 - \max \{\zeta_1, 1 - \zeta_2\}}{2} \right) \). Consequently, \( (n_{high,s} - E[N_s^{(t)}])^2 > m^2 \left( \frac{1 - \max \{\zeta_1, 1 - \zeta_2\}}{2} \right)^2 \), and

\[
2 \exp \left( -\frac{(n_{high,s} - E[N_s^{(t)}])^2}{2m\|\Delta_m\|^2} \right) \leq 2 \exp \left( -\frac{m (1 - \max \{\zeta_1, 1 - \zeta_2\})^2}{8\|\Delta_m\|^2} \right),
\]

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which provides us an upper bound for B. Recall that $n_{\text{low},s} - \mathbb{E}[N_s^{(l)}] < \frac{m}{8T}$. Therefore, $(n_{\text{low},s} - \mathbb{E}[N_s^{(l)}])^2 > (\frac{m}{8T})^2$, and

$$2\exp \left( -\frac{(n_{\text{low},s} - \mathbb{E}[N_s^{(l)}])^2}{2m\|\Delta_m\|^2} \right) \leq 2\exp \left( -\frac{m}{128T^2\|\Delta_m\|^2} \right).$$

This gives us an upper bound for C.

Returning to eq. (D.10), we substitute in the upper bounds for A, B, and C. Consequently,

$$\mathbb{P}\left( \left\| \hat{M}^{(l)}(s, \cdot) - M^{(l)}(s, \cdot) \right\|_1 > \varepsilon \right) \leq m\exp \left( -\frac{m\varepsilon^2}{64T} \right) + \frac{2m}{128T^2\|\Delta_m\|^2} + \frac{2m(1 - \max\{\zeta_1, 1 - \zeta_2\})^2}{8\|\Delta_m\|^2} \leq \frac{64T}{\varepsilon^2} \exp \left( -\frac{m\varepsilon^2}{128T} \right) + \frac{2m}{128T^2\|\Delta_m\|^2} + \frac{2m(1 - \max\{\zeta_1, 1 - \zeta_2\})^2}{8\|\Delta_m\|^2},$$

where the inequality follows from the fact that $xe^{-x} \leq e^{-x/2}$. Recall that the control/control space satisfies $|\Pi| = k$ and the state space satisfies $|\chi| = d$. Using a union bound on $l$, we see,

$$\mathbb{P}\left( \sup_{l \in \Pi} \left\| \hat{M}^{(l)} - M^{(l)} \right\|_\infty > \varepsilon \right) \leq \sum_l \sum_s \mathbb{P}\left( \left\| \hat{M}^{(l)}(s, \cdot) - M^{(l)}(s, \cdot) \right\|_1 > \varepsilon \right) \leq \sum_l \sum_s \left( \frac{64T}{\varepsilon^2} \exp \left( -\frac{m\varepsilon^2}{128T} \right) + \frac{2m}{128T^2\|\Delta_m\|^2} + \frac{2m(1 - \max\{\zeta_1, 1 - \zeta_2\})^2}{8\|\Delta_m\|^2} \right) = dk \left( \frac{64T}{\varepsilon^2} \exp \left( -\frac{m\varepsilon^2}{128T} \right) + \frac{2m}{128T^2\|\Delta_m\|^2} + \frac{2m(1 - \max\{\zeta_1, 1 - \zeta_2\})^2}{8\|\Delta_m\|^2} \right), \quad \text{(D.11)}$$
By Assumption 3, \( \| \Delta_m \| \leq C \). Let \( \gamma_1, \gamma_2, \gamma_3, \gamma_4 \) be four constants such that,

\[
\begin{align*}
\gamma_1 &= C^2 T^2 \log \left( \frac{6dk}{\delta} \right) \\
\gamma_2 &= C^2 \Delta \frac{1}{(1 - \max(\zeta_1, 1 - \zeta_2))^2} \log \left( \frac{6dk}{\delta} \right) \\
\gamma_3 &= \frac{128T}{\varepsilon^2} \log \left( \frac{192dkT}{\varepsilon^2\delta} \right) \\
\gamma_4 &= \frac{d}{\varepsilon^2 (1 + \max(\zeta_1, 1 - \zeta_2))}.
\end{align*}
\]

Hence, there exists a universal constant \( c \) large enough such that if,

\[ m > c \{ \gamma_1, \gamma_2, \gamma_3, \gamma_4 \}, \]

then

\[
\mathbb{P} \left( \sup_{l \in I} \| \hat{M}(l) - M(l) \|_\infty > \varepsilon \right) < \delta. \tag{D.12}
\]

However, recall from eq. (3.7) that \( T > \frac{dk}{\varepsilon^2} \). Therefore, it follows that \( \gamma_3 > \gamma_4 \). Hence, there exists an universal constant \( c \) such that as long as

\[ m > c \max \left\{ T^2, \frac{1}{(1 - \max(\zeta_1, 1 - \zeta_2))^2} \right\}, \]

\[
\mathbb{P} \left( \sup_{l \in I} \| \hat{M}(l) - M(l) \|_\infty > \varepsilon \right) < \delta. \tag{D.13}
\]

\[ \square \]

### D.2 Proof of Sample Complexity in Theorem 2

**Proof:** The first part of this proof follows similarly to that of Theorem 1. We proceed until eq. (D.5), and analyse Term 1 similarly as before to get

\[
\sum_{n=n_{low,s}}^{n_{high,s}} \mathbb{P} \left( \left\| \hat{M}^{(l)}(s, \cdot) - M^{(l)}(s, \cdot) \right\|_1 > \varepsilon, N_s^{(l)} = n \right) \leq m \exp \left( - \frac{n_{low,s}}{2} \max \left\{ 0, \varepsilon - \sqrt{\frac{d}{n_{high,s}}} \right\} \right). \tag{D.14}
\]

The difference arises in the analysis of Term 2, where, instead of using Proposition 5, we use Proposition 6 to obtain

\[
\text{Term 2} = \mathbb{P} \left( N_s^{(l)} \notin [n_{low,s}, n_{high,s}] \right) \leq 2 \exp \left( - \frac{C_{pel} (n_{low,s} - \frac{m}{2T})^2}{4mC_{\Delta} \rho_s^{(l)} + 1 + (\frac{m}{2T} - n_{low,s}) (\log m)^2} \right) \\
+ 2 \exp \left( - \frac{C_{pel} (n_{high,s} - m \max(\zeta_1, 1 - \zeta_2))^2}{4mC_{\Delta} \rho_s^{(l)} + 1 + (n_{high,s} - m \max(\zeta_1, 1 - \zeta_2)) (\log m)^2} \right).
\]

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Observe that the right-hand side of the previous equation is increasing in $\rho_{s}^{(l)}$. Furthermore, we also have $\rho_{s}^{(l)} \leq \rho_{*}$. Thus, we can replace $\rho_{s}^{(l)}$ by $\rho_{*}$ in the upper bound to get

$$
\text{Term 2} \leq 2 \exp \left( - \frac{C_{pel} \left( n_{low,s} - \frac{m}{2T} \right)^2}{4m C_{\Delta \rho_{*}} + 1 + \left( \frac{m}{2T} - n_{low,s} \right) (\log m)^2} \right) + 2 \exp \left( - \frac{C_{pel} \left( n_{high,s} - m \max\{\zeta_{1},1-\zeta_{2}\}\right)^2}{4m C_{\Delta \rho_{*}} + 1 + \left(n_{high,s} - m \max\{\zeta_{1},1-\zeta_{2}\}\right) (\log m)^2} \right).
$$

Next, we substitute the values $n_{high,s} = m \left(1 + \max\{\zeta_{1},1-\zeta_{2}\}\right)/2$ and $n_{low,s} = m/4T$ into the previous term to get

$$
\text{Term 2} \leq 2 \exp \left( - \frac{C_{pel} m^2}{16T^2} \frac{\left(1 + \max\{\zeta_{1},1-\zeta_{2}\}\right)^2}{4m C_{\Delta \rho_{*}} + 1 + m \left(1 + \max\{\zeta_{1},1-\zeta_{2}\}\right) \left(\log m\right)^2} \right). \tag{D.15}
$$

We only analyse the first term. The calculations for the second term follow in a similar way. Recall from hypothesis that $m > C_{T}$. Obviously, $m > 4T$. In other words, $1 < m/T$. Substituting 1 for $m/T$ into the denominator, we get

$$
2 \exp \left( - \frac{1}{16T^2} \frac{C_{pel} m^2}{4m C_{\Delta \rho_{*}} + 1 + m \left(\log m\right)^2} \right)
\leq 2 \exp \left( - \frac{1}{16T^2} \frac{C_{pel} m^2}{4m C_{\Delta \rho_{*}} + \frac{m}{1T} + m \left(\log m\right)^2} \right) = 2 \exp \left( - \frac{1}{4} \frac{C_{pel} m}{16C_{\Delta \rho_{*}} T^2 + T + T \left(\log m\right)^2} \right) \leq 2 \exp \left( - \frac{1}{64} \frac{C_{pel} m}{\left(\log m\right)^2} \right).
$$

The last inequality follows by taking $16(\log m)^2$ common from the denominator and trivially upper bounding $(\log m)^{-2}$ by 1. Observe that $C_{pel}/64 \left(C_{\Delta \rho_{*}} T^2 + 2T\right)$ is less than 1 and constant in $m$. Recall from eq. (3.10) that the inverse of this term was defined to be $C_{T}$. It is clear that $2dk \exp \left( - \frac{m}{C_{T}(\log m)^2} \right)$ is decreasing in $m$. Our objective is to find an $m$ such that

$$
2dk \exp \left( - \frac{m}{C_{T}(\log m)^2} \right) \leq \frac{\delta}{3},
$$

which is equivalent to finding an $m$ such that

$$
\frac{m}{(\log m)^2} > C_{T} \log \left( \frac{6dk}{\delta} \right) = C_{T,\delta}. \tag{D.16}
$$
Let \( m = 2C_{T,\delta} (\log C_{T,\delta})^2 \). The denominator of the term on the left-hand side of the previous equation decomposes into

\[
(\log m)^2 = \left( \log \left( 2C_{T,\delta} (\log C_{T,\delta})^2 \right) \right)^2
= (\log 2 + \log C_{T,\delta} + 2 \log \log C_{T,\delta})^2
= (\log C_{T,\delta})^2 \left( 1 + \frac{\log 2}{\log C_{T,\delta}} + 2 \left( \frac{\log \log C_{T,\delta}}{\log C_{T,\delta}} \right)^2 \right).
\]

Consider the function,

\[
f(x) = \left( 1 + \frac{\log 2}{\log x} + 2 \frac{\log \log x}{\log x} \right)^2.
\]

This is obviously decreasing in \( x \). It can be easily verified that there exists a universal constant ‘\( c \)’ such that \( f(x) < 2 \) if \( x > c \). Using this fact and substituting \( m \) in eq. (D.16), we get

\[
\frac{m}{(\log m)^2} = 2C_{T,\delta} \frac{(\log C_{T,\delta})^2}{(\log C_{T,\delta})^2 f(C_{T,\delta})} \geq C_{T,\delta}.
\]

Therefore, the first term on the left-hand side of eq. (D.15) is upper bounded by \( \delta/3 \). We can similarly show that, whenever \( m > 2C_{\zeta,\delta} (\log C_{\zeta,\delta})^2 \) the second term on the right-hand side of eq. (D.15) can be upper bounded by \( \delta/3 \). We now proceed similarly to eq. (D.11). This gives us that, under our current hypothesis,

\[
P \left( \sup_{l \in I} \| \hat{M}(l) - M(l) \|_\infty > \varepsilon \right) < \delta.
\]

This completes the proof of the sample complexity. We can now proceed to the proof of Minimaxity. \( \square \)

### D.3 Proof of Minimaxity in Theorem 2

**Proof.** As before, let \( \mathcal{M}_{\chi,\I} \) be the class of all Markov decision processes on state space \( \chi \) with control space \( \I \). We will view an element of \( \mathcal{M}_{\chi,\I} \) as doublet \((\tilde{M}, P)\), where \( \tilde{M} := (M^{(1)}, \ldots, M^{(k)}) \) is a collection of distinct \( d \)-state Markov transition matrix, and \( P := (P_1, P_2, \ldots) \) is the distribution of control sequences, with each \( P_i \) being a probability measure on \( \I \) that depends on the history until time point \( i \). Let \( \mathcal{M}_{\chi,1} \) be the set of all \( \tilde{M} \) and \( P \), respectively. For \( \tilde{M}_1, \tilde{M}_2 \in \mathcal{M}_{\chi,1} \), define

\[
\| \tilde{M}_1 - \tilde{M}_2 \|_\infty^* := \sup_{i \in I} \| M_1^{(i)} - M_2^{(i)} \|_{\infty}.
\]

For \( \{(X_0, a_0), \ldots, (X_m, a_m)\} \in (\chi \times \I)^m \), a sample of length \( m \) from some MDP belonging to \( \mathcal{M}_{\chi,1} \), define an estimation procedure \( \hat{\tilde{M}} \) as the mapping \( \hat{\tilde{M}} : (\chi \times \I)^m \mapsto \mathcal{M}_{\chi} \). We seek to provide a lower bound for the minimax risk over all estimation procedures:

\[
\mathcal{R}_m = \inf_{\tilde{M}} \sup_{(\tilde{M}, P) \in \mathcal{M}_{\chi,1}} \mathbb{P} \left( \| \hat{\tilde{M}} - \tilde{M} \|_\infty^* > \varepsilon \right).
\]

We note that if \( \mathcal{M}' \subset \mathcal{M}_{\chi,1} \) is a subclass of MDP’s, then

\[
\mathcal{R}_m \geq \inf_{\tilde{M}} \sup_{(\tilde{M}, P) \in \mathcal{M}'} \mathbb{P} \left( \| \hat{\tilde{M}} - \tilde{M} \|_\infty^* > \varepsilon \right).
\]

The rest of the proof proceeds by constructing an appropriate subclass \( \mathcal{M}' \).
Part 1 \((m < cT/\varepsilon^2)\):

Our example below considers an MDP with stationary Markovian controls. Any such MDP with \(k\) transition matrices and \(d'\) states can be viewed as a single Markov chain with \(d = d'k\) states. For convenience, we use the latter representation. By \(M\) denote its transition matrix, and by \(\hat{M}\) denote the estimate. Without losing generality, let there be \(d + 1\) states where \(d\) is even; the odd case is handled similarly. Let \(0 < p_\ast < \frac{1}{d+1}\) and for a vector \(\sigma = (\sigma_1, \ldots, \sigma_d) \in \{-1, 1\}^d\), define \(\eta(\sigma)\) as the following perturbation of \((\frac{1-p_\ast}{d}, \frac{1-p_\ast}{d}, \ldots, p_\ast)\):

\[
\eta(\sigma) = \left(\frac{1-p_\ast + 16\sigma_1 \varepsilon}{d}, \frac{1-p_\ast - 16\sigma_1 \varepsilon}{d}, \ldots, \frac{1-p_\ast + 16\sigma_d \varepsilon}{d}, \frac{1-p_\ast - 16\sigma_d \varepsilon}{d}, p_\ast\right).
\]

Since \(\varepsilon < \frac{1}{16d}\) and \(d > 2\) by hypothesis, it follows that \(\eta(\sigma)\) is a valid probability mass function on \(\{1, \ldots, d + 1\}\). Let \(\mathcal{M}_\sigma\) be a class of transition matrices indexed by \(\sigma\), taking the form

\[
M_\sigma = \begin{pmatrix}
\frac{1-p_\ast}{d} & \cdots & \frac{1-p_\ast}{d} & p_\ast \\
\vdots & \ddots & \vdots & \vdots \\
\frac{1-p_\ast}{d} & \cdots & \frac{1-p_\ast}{d} & p_\ast \\
\frac{1-p_\ast + 16\sigma_1 \varepsilon}{d} & \cdots & \frac{1-p_\ast - 16\sigma_d \varepsilon}{d} & p_\ast
\end{pmatrix}.
\]

(D.19)

From the Varshamov-Gilbert lemma [58, Theorem 5.1.7], there exists \(\Sigma \subset \{-1, 1\}^{d/2}, |\Sigma| = 2^{d/16}\), such that for \((\sigma, \sigma') \in \Sigma\) with \(\sigma \neq \sigma'\), we have

\[
\sum_{i=1}^{d/2} 1[\sigma_i \neq \sigma'_i] \geq \frac{d}{16},
\]

Define the subclass \(\mathcal{M}'\) as

\[
\mathcal{M}' := \{M_\sigma : \sigma \in \Sigma\}.
\]

(D.20)

Recall that by eq. (D.18), that it is enough to find a lower bound on

\[
\inf_{M} \sup_{M \in \mathcal{M}'} \mathbb{P}\left(\left\|M - \hat{M}\right\|_\infty > \varepsilon\right).
\]

By applying Tsybakov’s reduction method [57, Theorem 2.5] to our problem, we obtain the following lower bound,

\[
\inf_{M} \sup_{M \in \mathcal{M}'} \mathbb{P}\left(\left\|M - \hat{M}\right\|_\infty > \varepsilon\right) \geq \frac{1}{2} \left(1 - 2^{2-d/16} \sum_{\sigma \in \Sigma} D_{\sigma, m}\right),
\]

(D.21)

where \(D_{\sigma, m}\) is the KL-divergence between \(M_0\) and \(M_\sigma\) (for some \(\sigma \in \Sigma\)), both viewed as distributions over sequences of length \(m\). Recall the following chain rule for KL-divergence from Wolfer, Kontorovich, et al. [62, Lemma 6.2]

\[
D_{\sigma, m} \leq p_\ast m D_{\sigma},
\]

(D.22)

where \(D_{\sigma}\) is the KL-divergence between \(\eta(\sigma)\) and \((\frac{1-p_\ast}{d}, \ldots, \frac{1-p_\ast}{d}, p_\ast)\). A direct computation of \(D_{\sigma}\) yields

\[
D_{\sigma} = \sum_{i=1}^{d/2} \frac{1-p_\ast + 16\sigma_i \varepsilon}{d} \log \left(\frac{\frac{1-p_\ast + 16\sigma_i \varepsilon}{d}}{\frac{1-p_\ast}{d}}\right) + \frac{1-p_\ast - 16\sigma_i \varepsilon}{d} \log \left(\frac{\frac{1-p_\ast - 16\sigma_i \varepsilon}{d}}{\frac{1-p_\ast}{d}}\right)
\]

\[
= \frac{d}{2} \left(\frac{1-p_\ast + 16\varepsilon}{d} \log \left(\frac{1-p_\ast + 16\varepsilon}{1-p_\ast}\right) + \frac{1-p_\ast - 16\varepsilon}{d} \log \left(\frac{1-p_\ast - 16\varepsilon}{1-p_\ast}\right)\right).
\]

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Denoting $1 - p_*$ by $A$ and $16\varepsilon$ by $B$ allows us to rewrite the previous equation as

$$D_\sigma = \frac{1}{2} \left( (A + B) \log \left( 1 + \frac{B}{A} \right) + (A - B) \log \left( 1 - \frac{B}{A} \right) \right).$$

Observe that $B = 16\varepsilon < \frac{1}{2}$ and $A = 1 - p_* > 1 - \frac{1}{d+1} > \frac{1}{2}$. This implies that $\frac{B}{A} < 1$. Since $\log (1 + x) \leq x$ whenever $x > -1$, it follows that

$$D_\sigma \leq \frac{1}{2} \left( (A + B) \frac{B}{A} - (A - B) \frac{B^2}{A} \right) = \frac{B^2}{A} = \frac{256\varepsilon^2}{1 - p_*} \leq 512\varepsilon^2.$$

Substituting this value in eq. (D.22), and further substituting in that value into eq. (D.21), we obtain

$$\inf_M \sup_{M' \in M'} \mathbb{P} \left( \left\| M - \hat{M} \right\|_\infty > \varepsilon \right) \geq \frac{1}{2} \left( 1 - \frac{2^{2 - \frac{d}{16}} \sum_{\sigma \in \Sigma} 512 p_* \varepsilon^2}{\log 2^{\frac{d}{16}}} \right) = \frac{1}{2} \left( 1 - \frac{2^{2 - \frac{d}{16}} |\Sigma| 512 p_* \varepsilon^2}{\log 2^{\frac{d}{16}}} \right).$$

Recall that our choice of $\Sigma$ satisfies $|\Sigma| = 2^\frac{d}{16}$. Thus,

$$\frac{1}{2} \left( 1 - \frac{2^{2 - \frac{d}{16}} |\Sigma| 512 p_* \varepsilon^2}{\log 2^{\frac{d}{16}}} \right) = \frac{1}{2} \left( 1 - \frac{32768 p_* \varepsilon^2}{d \log 2} \right).$$

Therefore, whenever $m \leq \frac{d(1 - 2\delta) \log 2}{32768 p_* \varepsilon^2}$,

$$\inf_M \sup_{M' \in M'} \mathbb{P} \left( \left\| \hat{M} - \hat{M} \right\|_\infty > \varepsilon \right) > \delta.$$

To complete the proof, we need to relate the quantities $d$ and $p_*$ to the expected return time $T$ of the process. From the statement of the theorem, we need to show that there exists an universal constant $c_1$ such that $T$ satisfies

$$T \leq c_1 \frac{d}{p_*},$$

where $T$ is the return time as defined in Assumption 2. It is easily verifiable that for any $\sigma \in \{-1, 1\}^{d/2}$ the vector

$$\left( \frac{(1 - p_*)^2 + (1 - p_* + 16\sigma_1 \varepsilon) p_*}{d}, \ldots, \frac{(1 - p_*)^2 + (1 - p_* - 16\sigma_{d/2} \varepsilon) p_*}{d}, p_* \right) \quad \text{(D.23)}$$

represents the stationary distribution corresponding to the transition probability matrix $M_\sigma$ as defined in D.19. We know from Kac’s theorem [40, Theorem 10.2.2] that the expected return time to any state is the inverse of its stationary probability. It follows from eq. (D.23) that the expected return times are

$$\left( \frac{d}{(1 - p_*)^2 + (1 - p_* + 16\sigma_1 \varepsilon) p_*}, \ldots, \frac{d}{(1 - p_*)^2 + (1 - p_* - 16\sigma_{d/2} \varepsilon) p_*}, \frac{1}{p_*} \right).$$

Since $\frac{d}{(1 - p_*)^2 + (1 - p_* + 16\sigma_1 \varepsilon) p_*} \leq \frac{d}{(1 - p_*)^2}$ for any value of $\sigma_1 \in \{-1, 1\}$ it follows that

$$\max \left\{ \frac{d}{(1 - p_*)^2}, 1 \right\} \leq \max \left\{ \frac{d}{(1 - p_*)^2}, \frac{1}{p_*} \right\}.$$
Our next objective is to produce a subclass $M$ function on controls as in eq. (B.1). Let Lemma 27.

This completes the proof of the first part.

**Part 2** ($m < \left\{ 2 C_{T,\delta} (\log C_{T,\delta})^2, 2 C_{\zeta,\delta} (\log C_{\zeta,\delta})^2 \right\}$):

**Case 1:** $m < 2 C_{T,\delta} (\log C_{T,\delta})^2$. In this part, we prove that there exists a subclass $\mathcal{M}' \subset \mathcal{M}_{x,1}$ and an universal constant $c > 0$ for which $\mathcal{R}_m \geq 1/(2 + 2\pi^2)$ whenever

\[
 m < c (1 + C + C_\delta)^2 \max \left\{ T^2, \frac{1}{(1 - \max\{\zeta_1, 1 - \zeta_2\})^2} \right\}.
\]

For an MDP with $d$ states and $k$ transition matrices, define the random variable $T$ to be the first time all of the states $1, \ldots, d/3$ were visited in all of the $k$ transition matrices. That is,

\[
 T = \min \left\{ n \geq 0 : \bigcap_{s \in \{1, \ldots, d/3\}} \bigcup_{j \in \{1, \ldots, k\}} \left\{ \bigcup_{k=0}^n \{ X_k = s, a_k = l \} \right\} \neq \emptyset \right\}. \tag{D.24}
\]

Then, we can further lower bound $\mathcal{R}_m$ as

\[
 \mathcal{R}_m \geq \inf_{\hat{M}} \sup_{(M, P) \in \mathcal{M}'} \mathbb{P} \left( \|\hat{M} - \hat{M}\|_\infty > \varepsilon \mid T > m \right) \mathbb{P} \left( T > m \right).
\]

Our next objective is to produce a subclass $\mathcal{M}'$. Define $P^{(0)}$ to be the sequences of probability mass function on controls as in eq. (B.1). Let $\mathbb{H}_i$ be the set of all $k + 1$ tuples $(M_{l,\xi(1)}, \ldots, M_{l,\xi(k)})$ where $M_{l,\xi(1)}, \ldots, M_{l,\xi(k)}$ are matrices as defined in eq. (B.2). To be precise,

\[
 \mathbb{H}_i := \left\{ \{ M_{l,\xi(l)} : l \in \{1, \ldots, k\} \} : (\xi(1), \ldots, \xi(k)) \in \{0, 1\}^{d/3 \times k} \right\}. \tag{D.25}
\]

Set $\mathcal{M}' = \mathbb{H}_i \times \{ P^{(0)} \}$. As a consequence, we get the following lemma.

**Lemma 27.** Let $T$ be the time to visit the state-control pairs $\{(1, 1), \ldots, (d/3, 1), (2, 1), \ldots, (d/3, k)\}$ of an MDP belonging to class $\mathbb{H}_i \times \{ P^{(0)} \}$ as defined in eq. (D.25). If $n < \frac{d}{6} \log \left( \frac{d}{3}\right)$, then

\[
 \mathbb{P} (T > n) \geq \frac{1}{1 + \pi^2}.
\]

Substituting it in the lower bound to $\mathcal{R}_m$ gives

\[
 \mathcal{R}_m \geq \inf_{\hat{M}} \sup_{(M, P) \in \mathbb{H}_i \times \{ P^{(0)} \}} \mathbb{P} \left( \|\hat{M} - \hat{M}\|_\infty > \varepsilon \mid T > m \right) \mathbb{P} \left( T > m \right).
\]
An application of Lemma 27 and Proposition 25 implies that whenever \( m \leq C_T \rho T^2 \log T \) for some universal constant \( C_T \)

\[
\mathbb{P}(T > m) \geq \frac{1}{1 + \pi^2}.
\]

Consequently, we get,

\[
\mathcal{R}_m \geq \frac{1}{1 + \pi^2} \inf_{\mathcal{M}} \sup_{(M,P) \in \mathcal{H} \times \{P^{(0)}\}} \mathbb{P} \left( \left\| \hat{M} - \hat{M} \right\|_\infty > \varepsilon \mid T > m \right).
\]

Next, let \( l_0 \) be any control. By definition of \( \left\| \cdot \right\|_\infty^* \), it holds that

\[
\left\| \hat{M} - \hat{M} \right\|_\infty^* \geq \left\| \hat{M}^{(l_0)} - M_{l_0} \right\|_\infty^*.
\]

We recall from the construction in eq. (B.2) that \( \imath \) is known. Therefore, to correctly estimate the transition matrix \( M^{(l_0)}_{l_0, \xi} \), we only need to correctly estimate \( \xi^{(l_0)} \). Combining these facts we get,

\[
\inf_{\mathcal{M}} \sup_{(M,P) \in \mathcal{H} \times \{P^{(0)}\}} \mathbb{P} \left( \left\| \hat{M} - \hat{M} \right\|_\infty > \varepsilon \mid T > m \right) \geq \inf_{\mathcal{M}^{(l_0)}_{l_0, \xi}} \sup_{\{0,1\}^{d/3}} \mathbb{P} \left( \left\| \hat{M}^{(l_0)}_{l_0, \xi} - M^{(l_0)}_{l_0, \xi} \right\|_\infty > \varepsilon \mid T > m \right).
\]

We note that whenever \( \xi_1^{(l_0)} \neq \xi_2^{(l_0)} \in \{0,1\}^{d/3} \), we have \( M^{(l_0)}_{l_0, \xi_1} - M^{(l_0)}_{l_0, \xi_2} \geq 2 \varepsilon \). For any estimate \( \hat{M}^{(l_0)} \) define \( \xi^* = \arg\min_{\xi} \left\| M^{(l_0)}_{l_0, \xi} - \hat{M}^{(l_0)}_{l_0, \xi} \right\|_\infty^* \). Then for \( \xi^{(l_0)} \neq \xi^* \) we have

\[
2\varepsilon = \left\| M^{(l_0)}_{l_0, \xi} - \hat{M}^{(l_0)}_{l_0, \xi^*} \right\|_\infty^* \leq \left\| M^{(l_0)}_{l_0, \xi} - \hat{M}^{(l_0)}_{l_0, \xi} \right\|_\infty^* + \left\| \hat{M}^{(l_0)} - M^{(l_0)}_{l_0, \xi^*} \right\|_\infty^* \leq 2 \left\| M^{(l_0)}_{l_0, \xi} - \hat{M}^{(l_0)} \right\|_\infty^*.
\]

Therefore, \( \{ l_0 : \xi^* \neq \xi^{(l_0)} \} \subset \{ l_0 : \left\| M^{(l_0)}_{l_0, \xi} - \hat{M}^{(l_0)} \right\|_\infty^* > \varepsilon \} \) and \( \mathcal{R}_m \) can be further lower bounded by

\[
\mathcal{R}_m \geq \frac{1}{1 + \pi^2} \inf_{\mathcal{M}^{(l_0)}_{l_0, \xi}} \max_{\{0,1\}^{d/3}} \mathbb{P} \left( \xi^* \neq \xi^{(l_0)} \mid T > m \right) = \frac{1}{1 + \pi^2} \inf_{\xi} \max_{\{0,1\}^{d/3}} \mathbb{P} \left( \xi^* \neq \xi^{(l_0)} \mid T > m \right),
\]

where \( \hat{\xi} \) any estimate of \( \xi^* \) \( (X_0, a_0, \ldots, X_m, a_m) \to \{0,1\}^{d/3} \). We now observe that that the events \( \{ N_s^{(l_0)} = 0 \text{ for some } l_0 \in \mathcal{I} \text{ and some } s \in \chi \} \) and \( \{ T > m \} \) are equivalent. Therefore,

\[
\mathbb{P}(\hat{\xi} \neq \xi^{(l_0)} \mid T > m) = \mathbb{P}(\hat{\xi} \neq \xi^{(l_0)} \mid N_s^{(l_0)} = 0).
\]

When \( N_s^{(l_0)} = 0 \), the estimate \( \hat{\xi} \) is equivalent to choosing uniformly over all possible \( \xi^{(l_0)} \). Since there are \( 2^{d/3} \) many possible choices for \( \xi^{(l_0)} \), the probability of choosing incorrectly is \( 1 - \frac{1}{2^{d/3}} \). We get as a consequence that,

\[
\inf_{\xi} \max_{\{0,1\}^{d/3}} \mathbb{P} \left( \xi^* \neq \xi^{(l_0)} \mid T > m \right) \geq 1 - \frac{1}{2^{d/3}} > \frac{1}{2}.
\]

In conclusion, whenever \( m \leq C_T T \log T \),

\[
\inf_{\mathcal{M}} \sup_{(M,P) \in \mathcal{M} \times \{P^{(0)}\}} \mathbb{P} \left( \left\| \hat{M} - \hat{M} \right\|_\infty^* > \varepsilon \right) \geq \frac{1}{2}.
\]

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Case II: $m < 2 \mathbb{C}_{\zeta, \delta} (\log \mathbb{C}_{\zeta, \delta})^2$. For the final case, we must now show that for our chosen class of MDP’s, $1/(1 - \max\{\zeta_1, 1 - \zeta_2\})^2$ must lie in a fixed interval for appropriate choices of $\zeta_1$ and $\zeta_2$. Recall that by Proposition 25 $\zeta_1 = 2 = \epsilon$, which is a known constant. The rest of the argument follows.

Next, recall from Proposition 25 that $C_\epsilon$ and $C_{\alpha, \epsilon}$ are independent of $d$ and $k$. Since $\epsilon$ is a known constant, this implies that $C_{pel}$ is an universal constant. This completes the proof.

D.4 Proof of Theorem 4

Proof: The proof proceeds by using known perturbation equalities of matrices. Observe the following variation of the Woodbury matrix identity from eq. 2.3 [61]

$$B^{-1} - A^{-1} = A^{-1}(A - B)B^{-1}$$

where $A$ and $B$ are matrices of appropriate dimension. Let $\|\cdot\|_\text{op}$ be the operator norm on $d \times d$ matrices as defined in Section 4.2 [73]. Using the facts that operator norm is sub-multiplicative and $\|\cdot\|_\infty \leq \|\cdot\|_\text{op} \leq \sqrt{d} \|\cdot\|_\infty$, we have that

$$\|B^{-1} - A^{-1}\|_\infty \leq \|B^{-1} - A^{-1}\|_\text{op} = \|A^{-1}(A - B)B^{-1}\|_\text{op} \leq \|A^{-1}\|_\text{op} \|B^{-1}\|_\text{op} \|A - B\|_\text{op} \leq \|A^{-1}\|_\text{op} \|B^{-1}\|_\text{op} \sqrt{d} \|A - B\|_\infty.$$ 

Substitute $A = \left(I - \alpha_{\text{dis}} \bar{M}\right)$, $B = \left(I - \alpha_{\text{dis}} M\right)$. It is well known that the eigenvalues of stochastic matrices lie between $[-1, 1]$. As a consequence, the eigenvalues of $A$ and $B$ are at least $1 - \alpha_{\text{dis}}$ and therefore, $\|A^{-1}\|_\text{op}, \|B^{-1}\|_\text{op} \leq (1 - \alpha_{\text{dis}})^{-1}$. Therefore,

$$\|B^{-1} - A^{-1}\|_\infty \leq (1 - \alpha_{\text{dis}})^2 \sqrt{d} \|A - B\|_\infty.$$ 

Next, observe that

$$\|V - \hat{V}\|_\infty = \left\|\left(I - \alpha_{\text{dis}} \bar{M}\right)^{-1} - \left(I - \alpha_{\text{dis}} M\right)^{-1}\right\|_\infty \leq \left\|\left(I - \alpha_{\text{dis}} \bar{M}\right)^{-1} - \left(I - \alpha_{\text{dis}} M\right)^{-1}\right\|_\infty \|g\|_1 \leq (1 - \alpha_{\text{dis}})^{-2} \sqrt{d} \left\|\left(I - \alpha_{\text{dis}} \bar{M}\right) - \left(I - \alpha_{\text{dis}} M\right)\right\|_\infty \|g\|_1 \leq \frac{\alpha_{\text{dis}}}{(1 - \alpha_{\text{dis}})^2} \sqrt{d} \left\|\bar{M} - M\right\|_\infty \|g\|_1.$$ 

It follows from theorem 1 that,

$$\mathbb{P}\left(\|V - \hat{V}\|_\infty > \epsilon\right) \leq \mathbb{P}\left(\frac{\alpha_{\text{dis}}}{(1 - \alpha_{\text{dis}})^2} \sqrt{d} \left\|\bar{M} - M\right\|_\infty \|g\|_1 > \epsilon\right) = \mathbb{P}\left(\left\|\left(M - M\right)\right\|_\infty > \frac{(1 - \alpha_{\text{dis}})^2}{\|g\|_1 \sqrt{d} \alpha_{\text{dis}}} \epsilon\right) \leq \delta$$ 

whenever

$$m > c \max\left\{\frac{T_p}{\epsilon^2} \log\left(\frac{dkT_p}{\epsilon^2 \delta}\right), \mathbb{C}_{\alpha} \max\left\{T^2, \frac{1}{(1 - \max\{\zeta_1, 1 - \zeta_2\})^2}\right\}\log\left(\frac{dk}{\delta}\right)\right\}$$ 

□

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E  Proofs of Applications

E.1 Proof of Proposition 10

Proof. The proof proceeds by verifying Assumptions 1, 2, 5, and 6.

Assumption 1: To verify Assumption 1, we recall that $P_s^{(l)} > 0$ and $P_{\min} = \min_{s,l} P_s^{(l)}$. Since the minimisation is over finitely many positive quantities, it must follow that $P_{\min} > 0$. For any $l_0 \in I$, it follows that

$$P_s^{(l_0)} = 1 - \sum_{l \neq l_0} P_s^{(l)} < 1 - (k - 1)P_{\min}.$$ 

Recall that to satisfy Assumption 1, it is enough to produce sets $S_i$ such that $\zeta_2 < \mathbb{P}((X_i, a_i) \in S_i) < \zeta_1$ for some probabilities $\zeta_2$ and $\zeta_1$. We observe by setting $S_i = \arg\sup\mathbb{P}((X_i, a_i) \in S_i) = \mathbb{P}(a_i = 1)$. It then follows,

$$P_{\min} < \mathbb{P}((X_i, a_i) \in S_i) < 1 - (k - 1)P_{\min}.$$ 

and Assumption 1 is satisfied for $\zeta_2 = P_{\min}$ and $\zeta_1 = 1 - (k - 1)P_{\min}$.

Assumption 2: Next, we proceed to the verification of Assumption 2. As discussed in eq. (4.1), we only need to show that for some $T > 0$,

$$\sup_{s,l} \mathbb{E}[\tau_{s,l}^{(1)} | X_0, a_0] < T \text{ almost everywhere.}$$ 

Recall from Proposition 9 that $(X_i, a_i)$ is a time homogenous uniformly ergodic Markov chain. It follows from KAC’s theorem [40, Theorem 10.2.2] that for any positive integer $i$,

$$\mathbb{E}[\tau_{s,l}^{(i)}] = \frac{1}{\pi_{s,l}^{(x,a)}}.$$ 

Setting $T = \sup_{s,l} 1/\pi_{s,l}^{(x,a)}$ then completes the verification of Assumption 2.

Assumption 5: Next, we verify that an MDP with stationary controls satisfy Assumption 5. Recall from eq. (4.4) the definition of stationary controls.

$$\mathcal{L} (a_i | X_i = s_i, \mathcal{H}_0^{i-1} = h_0^{i-1}) = \mathcal{L} (a_i | X_i = s_i) = \mathcal{L} (a_1 | X_1 = s_i).$$ 

Which consequently implies that

$$\mathcal{L} (a_p | X_p = s_p, \mathcal{H}_{i+j}^{p-1} = h_{i+j}^{p-1}, \mathcal{H}_0 = h_0^i) - \mathcal{L} (a_p | X_p = s_p, \mathcal{H}_{i+j}^{p-1} = h_{i+j}^{p-1})$$

$$= \mathcal{L} (a_1 | X_1 = s_i) - \mathcal{L} (a_1 | X_1 = s_i) = 0.$$ 

Therefore $\gamma_{p,j,i} = 0$ and setting $\mathbb{C} = 0$ completes the verification of Assumption 5.

Assumption 6: Next, we verify that a MDP with stationary controls satisfy Assumption 6. It follows from Meyn and Tweedie [40, Theorem 16.0.2] that any aperiodic time homogenous Markov chain on a finite state space with a single communicating class is uniformly ergodic. Mukhamedov [41, Theorem 3.4] establishes the equivalence of weak and uniform ergodicity for time homogenous Markov chains with the ergodic coefficient between 0 and 1. This completes the verification of Assumption 6.
Lastly, let \( (s_0, l_0) := \arg \sup_{s, l} 1/\pi_{s,l} \) and observe that if \( D_0 = \pi \), then

\[
\mathbb{P}((X_i, a_i) = (s_0, l_0)) = \frac{1}{\pi_{s_0,l_0}} = \frac{1}{T} = \zeta_1 = \zeta_2.
\]

This completes the proof. \( \square \)

### E.2 Proof of Proposition 11

**Proof.** The proof proceeds by verifying Assumptions 1, 2, 5, and 6.

**Assumption 1:** Set \( S_i = (1, I) \) and observe that

\[
\mathbb{P}((X_i, a_i) \in S_i) = \mathbb{P}(X_i = 1) = E[\mathbb{P}(X_i = 1|X_{i-1}, a_{i-1})] = E[M_{X_{i-1}, 1}].
\]

We observe that

\[
M_{\min} \leq \mathbb{P}(S_i) \leq M_{\max}.
\]

Thus, setting \( \zeta_1 \) as \( M_{\min} \) and \( \zeta_2 \) as \( M_{\max} \), completes the verification of Assumption 1.

**Assumption 2:** A consequence of Equation (4.5) is that for each \( l \in I \), the transition matrix \( M_{l} \) is visited at least \( \lfloor n/T \rfloor \) times for any time interval of length \( n \). For some time point \( i \) let \( a_i = l \).

For any integer \( p > i \), (I2) implies

\[
\mathbb{P}(\tau_{s,l}^{(i)} > p) = \mathbb{P}\left( \left\{ X_j \neq s \bigcup a_j \neq l \right\} \forall j \in \{i + 1 \ldots, p + i\}|X_i = s, a_i = l \right) \\
\leq (1 - M_{\min})^{\lfloor \frac{p}{T} \rfloor} \\
\leq (1 - M_{\min})^{\frac{p}{T} - 1}.
\]

Thus,

\[
E[\tau_{s,l}^{(i)}] = \sum_{p \geq 1} \mathbb{P}(\tau_{s,l}^{(i)} > p) \leq \frac{1}{(1 - M_{\min})^{1 - \frac{1}{T}} (1 - (1 - M_{\min})^{\frac{1}{T}})}.
\]

This completes the verification of Assumption 2.

**Assumption 5:** Recall that \( a_i \) is a deterministic sequence of indices. It follows that,

\[
\gamma_{p,j,i} = \sup_{s_p, h_{i+j}^{p-1}, h_0^i} \left\| \mathcal{L} \left( a_p|X_p = s_p, \mathcal{H}_{i+j}^{p-1} = h_{i+j}^{p-1}, \mathcal{H}_0^i = h_0^i \right) - \mathcal{L} \left( a_p|X_p = s_p, \mathcal{H}_{i+j}^{p-1} = h_{i+j}^{p-1} \right) \right\|_{TV}
\]

\[
= 0.
\]

Hence,

\[
\sup_{1 \leq i \leq m} \sum_{j=i}^{m} \sum_{p=i+j+1}^{m} \gamma_{p,j,i} = 0,
\]

and Assumption 5 is verified with \( C = 0 \).

**Assumption 6:** Observe from the definition of \( \bar{\theta}_{i,j} \) in eq. (4.3) that,

\[
\bar{\theta}_{i,j} = \sup_{l,l', s_1, s_2} \left\| \mathcal{L} \left( X_j|X_i = s_1, a_i = l \right) - \mathcal{L} \left( X_j|X_i = s_2, a_i = l' \right) \right\|_{TV}.
\]
Observe from the definition of an inhomogenous Markov chains that
\[ \sup_{s_1, s_2} \| \mathcal{L}(X_j | X_i = s_1, a_i = l_1) - \mathcal{L}(X_j | X_i = s_2, a_i = l_2) \|_{TV} \]
\[ = \sup_{s_1, s_2} \| \mathcal{L}(X_j | X_i = s_1, (a_{j-1}, \ldots, a_i) = (l_{j-1}, \ldots, l_i)) - \mathcal{L}(X_j | X_i = s_2, (a_{j-1}, \ldots, a_i) = (l_{j-1}, \ldots, l_i)) \|_{TV} , \]
where the last line follows since all the controls are deterministic. Since all of the transition matrices are positive, an application of Wolfowitz \cite[Theorem 1]{Wolfowitz} implies that, there exists an integer \( C \) for which,
\[ \bar{\theta}_{i,j} \leq e^{-C(j-i)} . \]
Since \( e^{-C(j-i)} < e^{-(j-i)} \) for any integer \( C \), it consequently implies that,
\[ \sup_{i \geq 1} \sum_{j > l} \bar{\theta}_{i,j} \leq \frac{e^{-1}}{1 - e^{-1}} \leq \frac{1}{1 - e^{-1}} . \]
This completes the proof. \( \square \)

E.3 Proof of Proposition 13

Proof. The proof proceeds by verifying Assumptions 1, 2, 5, and 6. Assumptions 1 and 5 are verified similarly to that in the proof of Proposition 11. Assumption 2 is verified by Lemma 12. Finally, Assumption 6 follows from Lemma 21 by setting \( \chi_0 = \chi \). \( \square \)

E.4 Proof of Proposition 14

Proof. The proof proceeds by verifying Assumptions 1, 2, 5, and 6.
Assumption 1: Assumption 1 is verified similarly to that in the proof of Proposition 11.
Assumption 2: Our objective is to provide an upper bound for \( \sup_{i \geq 0} \mathbb{E}[	au_{s,l}^{(j)}] \). We prove only when \( i = 0 \). All other cases follow similarly. To compute the expectation, it is sufficient to bound from above the survival function \( \mathbb{P}\left( \tau_{s,l}^{(0)} > p \right) \) for all \( p \geq 1 \). When \( p \in \{1, \ldots, H - 1\} \), we trivially upper bound this probability by 1. When \( p = H \), writing the expression for \( \mathbb{P}\left( \tau_{s,l}^{(0)} > H \right) \) we get
\[
\mathbb{P}\left( \left\{ X_j \neq s \bigcup a_i \neq l \right\} \forall j \in \{1, \ldots, H\} | X_0 = s, a_0 = l \right) \\
= \mathbb{P}\left( \left\{ X_H \neq s \bigcup a_H \neq l \right\} \right) \\
\times \mathbb{P}\left( \left\{ X_j \neq s \bigcup a_i \neq l \right\} \forall j \in \{1, \ldots, H - 1\} | X_0 = s, a_0 = l \right) \\
\leq \left( 1 - \frac{1}{dk} \right) . \tag{E1}
\]
(\( E1 \)) follows from eq. (4.9) since for all \( j \neq H \), \( X_H, a_H \) is independent of \( (X_j, a_j) \). (\( E2 \)) follows by substituting the appropriate probability in \( \mathbb{P}\left( \left\{ X_H \neq s \bigcup a_H \neq l \right\} \right) \) and trivially upper bounding \( \mathbb{P}\left( \left\{ X_j \neq s \bigcup a_i \neq l \right\} \forall j \in \{1, \ldots, H - 1\} | X_0 = s, a_0 = l \right) \) by 1. We can proceed similarly for \( p \in \)
\{H + 1, \ldots, 2H - 1\}. For \( p = 2H \) we can similarly decompose \( \mathbb{P}\left( \tau_{s,l}^{(i)} > 2H \right) \) as,

\[
\mathbb{P}\left( \tau_{s,l}^{(i)} > 2H \right) = \mathbb{P}\left( \left\{ X_{2H} \neq s \bigcup a_{2H} \neq l \right\} \right)
\times \mathbb{P}\left( \left\{ X_{H} \neq s \bigcup a_{H} \neq l \right\} \right)
\times \mathbb{P}\left( \left\{ X_{j} \neq s \bigcup a_{i} \neq l \right\} \forall j \in \{1, \ldots, 2H - 1\}\setminus\{H\}|X_0 = s, a_0 = l \right) 
\leq \left( 1 - \frac{1}{dk} \right)^2 .
\]

Proceeding similarly, we bound from above \( \mathbb{P}\left( \tau_{s,l}^{(0)} > p \right) \) for each \( p \). Substituting these bounds in the expression for \( \mathbb{E}[\tau_{s,l}^{(0)}] \) we get

\[
\mathbb{E}[\tau_{s,l}^{(0)}] = \sum_{p=1}^{H-1} \mathbb{P}\left( \tau_{s,l}^{(0)} > p \right) + \sum_{p=H}^{2H-1} \mathbb{P}\left( \tau_{s,l}^{(0)} > p \right) + \ldots
\leq H - 1 + \left( 1 - \frac{1}{dk} \right) H + \left( 1 - \frac{1}{dk} \right)^2 H + \ldots
= \left( H + \left( 1 - \frac{1}{dk} \right) H + \left( 1 - \frac{1}{dk} \right)^2 H + \ldots \right) - 1
= dkH - 1.
\]

This completes the verification of Assumption 2.

Assumption 5: To verify Assumption 5 let \( (p, i, j) \) be any triplet in \( \mathbb{N}^3 \) such that \( j > i \). We first consider the case when \( j \leq i + H \). Observe from eq. (4.8) that whenever \( p > i + j + H \),

\[
\mathcal{L}\left( a_p|X_p = s_p, \mathcal{H}_{i+j}^{p-1} = \mathcal{H}_{i+j}^{p-1}, \mathcal{H}_0 = \mathcal{H}_0 \right) - \mathcal{L}\left( a_p|X_p = s_p, \mathcal{H}_{i+j}^{p-1} = \mathcal{H}_{i+j}^{p-1} \right)
= \mathcal{L}\left( a_p|X_p = s_p, \mathcal{H}_{H(p)}^{p-1} = \mathcal{H}_{H(p)}^{p-1} \right) - \mathcal{L}\left( a_p|X_p = s_p, \mathcal{H}_{H(p)}^{p-1} = \mathcal{H}_{H(p)}^{p-1} \right)
= 0.
\]

Whenever \( p \leq i + j + H \), it follows trivially that

\[-1 \leq \mathcal{L}\left( a_p|X_p = s_p, \mathcal{H}_{i+j}^{p-1} = \mathcal{H}_{i+j}^{p-1}, \mathcal{H}_0 = \mathcal{H}_0 \right) - \mathcal{L}\left( a_p|X_p = s_p, \mathcal{H}_{i+j}^{p-1} = \mathcal{H}_{i+j}^{p-1} \right) \leq 1.
\]

Now we consider the case when \( j > i + H \). Using the law of iterated expectations we get

\[
\mathcal{L}\left( a_p|X_p = s_p, \mathcal{H}_{i+j}^{p-1} = \mathcal{H}_{i+j}^{p-1}, \mathcal{H}_0 = \mathcal{H}_0 \right) = \mathbb{E} \left[ \mathcal{L}\left( a_p|X_p = s_p, \mathcal{H}_{i+j}^{p-1} = \mathcal{H}_{i+j}^{p-1}, \mathcal{H}_0 = \mathcal{H}_0 \right) \right]
= \mathbb{E} \left[ \mathcal{L}\left( a_p|X_p = s_p, \mathcal{H}_{i+j}^{p-1} = \mathcal{H}_{i+j}^{p-1}, \mathcal{H}_0 = \mathcal{H}_0 \right) \right] ,
\]

where the second equality follows from eq. (4.9). Substituting this expression in eq. (E.2) we get for all \( j > i + H \)

\[
\mathcal{L}\left( a_p|X_p = s_p, \mathcal{H}_{i+j}^{p-1} = \mathcal{H}_{i+j}^{p-1}, \mathcal{H}_0 = \mathcal{H}_0 \right) - \mathcal{L}\left( a_p|X_p = s_p, \mathcal{H}_{i+j}^{p-1} = \mathcal{H}_{i+j}^{p-1} \right)
= \mathbb{E} \left[ \mathcal{L}\left( a_p|X_p = s_p, \mathcal{H}_{i+j}^{p-1} = \mathcal{H}_{i+j}^{p-1}, \mathcal{H}_0 = \mathcal{H}_0 \right) \right] - \mathbb{E} \left[ \mathcal{L}\left( a_p|X_p = s_p, \mathcal{H}_{i+j}^{p-1} = \mathcal{H}_{i+j}^{p-1}, \mathcal{H}_0 = \mathcal{H}_0 \right) \right]
= 0.
\]
Substituting the previous upper bounds into the expression for $\gamma_{p,j,i}$, we get that

$$
\sum_{j=1}^{m} \sum_{p=i+j+1}^{m} \gamma_{p,j,i} = \sum_{j=1}^{i+H} \sum_{p=i+j+1}^{i+j+H} \gamma_{p,j,i} + \sum_{j=1}^{i+H} \sum_{p=i+j+H+1}^{m} \gamma_{p,j,i} + \sum_{j=H(p)}^{m} \sum_{p=i+j+1}^{m} \gamma_{p,j,i} \\
= \sum_{j=1}^{i+H} \sum_{p=i+j+1}^{i+j+H} \gamma_{p,j,i} \\
< H^2.
$$

This completes the verification of Assumption 5.

Assumption 6: Applying law of iterated expectation and decomposing $\mathcal{L}(X_j|X_i = s_1, a_i = l_1)$ similar to eq. (E.3), we obtain that whenever $j > i + H \bar{\theta}_{i,j} = 1$. It follows that $\sum_{j>i} \bar{\theta}_{i,j} < H$. This completes the verification of assumption 6. \hfill $\square$
F Proofs of Propositions and Lemmas

F.1 Proof of Lemma 1

Proof. The upper bound follows easily by adding and subtracting \( \mathbb{P} \left( (X_m, a_m, \ldots, X_j, a_j) \in T \right) \), and using triangle inequality. To prove the lower bound, let \( (h_0^{(i)})' \in (\chi \times 1)^{(i+1)} \). We can write

\[
\mathbb{P} \left( (X_m, a_m, \ldots, X_j, a_j) \in T \mid h_0' = h_0' \right) - \mathbb{P} \left( (X_m, a_m, \ldots, X_j, a_j) \in T \right)
\]

\[
= \sum_{(h_0^{(i)})' \in (\chi \times 1)^{(i+1)}} \left( \mathbb{P} \left( (X_m, a_m, \ldots, X_j, a_j) \in T \mid h_0' = (h_0')' \right) - \mathbb{P} \left( (X_m, a_m, \ldots, X_j, a_j) \in T \right) \right)
\]

\[
\leq \sum_{(h_0^{(i)})' \in (\chi \times 1)^{(i+1)}} \left( \mathbb{P} \left( (X_m, a_m, \ldots, X_j, a_j) \in T \mid h_0' = (h_0')' \right) \mathbb{P}(h_0' = (h_0')') \right)
\]

\[
\leq \bar{\eta}_{i,j} \sum_{(h_0^{(i)})' \in (\chi \times 1)^{(i+1)}} \mathbb{P}(h_0' = (h_0')') = \bar{\eta}_{i,j}.
\]

The first inequality follows using the triangle inequality, and the second inequality follows from equation 2.1. This completes the proof.

F.2 Proof of Proposition 3

Proof. The analysis of the proposition term is done via the sampling scheme introduced in Section C. To begin, use the sampling scheme to get \( \{ \hat{X}_0, \hat{a}_0, \ldots, \hat{X}_m, \hat{a}_m \} \). We construct the estimators \( \tilde{N}_s^{(l)} := \sum_i 1[\hat{X}_i = s, \hat{a}_i = l] \) and \( \tilde{N}_{s,t}^{(l)} := \sum_i 1[\hat{X}_i = s, \hat{X}_{i+1} = t, \hat{a}_i = l] \). Consequently, we define \( \tilde{M}_s^{(l)} := \frac{\tilde{N}_s^{(l)}}{N_s^{(l)}} \) and \( \tilde{M}_{s,t}^{(l)} := (\tilde{M}_s^{(l)}, \tilde{M}_{s,2}^{(l)}, \ldots, \tilde{M}_{s,d}^{(l)}) \). We observe that \( (\tilde{M}_s^{(l)}(s, \cdot), N_s^{(l)}(s, \cdot)) \overset{d}{=} (\tilde{M}_s^{(l)}(s, \cdot), \tilde{N}_s^{(l)}) \) by construction we have,

\[
\mathbb{P} \left( \left\| \tilde{M}_s^{(l)}(s, \cdot) - M_s^{(l)}(s, \cdot) \right\|_1 > \varepsilon, N_s^{(l)} = n \right) = \mathbb{P} \left( \left\| \tilde{M}_s^{(l)}(s, \cdot) - M_s^{(l)}(s, \cdot) \right\|_1 > \varepsilon, \tilde{N}_s^{(l)} = n \right) \quad \text{(F.1)}
\]

Next, we observe that given \( \tilde{N}_s^{(l)} = n \leq m \), and for any \( t \in \chi \), one can reduce \( \tilde{M}_{s,t}^{(l)} \) into sum of independent random variables. Recall from Appendix C that \( \tilde{X}_{i+1} = X^{(\hat{a}_i)}_{\hat{X}_i, \hat{X}_i^{(\hat{a}_i)} + 1} \). Therefore, we can write

\[
\tilde{M}_{s,t}^{(l)} = \frac{1}{n} \sum_{i=1}^{m} 1 \left[ \tilde{X}_i = s, X^{(\hat{a}_i)}_{\hat{X}_i, \hat{X}_i^{(\hat{a}_i)} + 1} = t, \hat{a}_i = l \right] = \frac{1}{n} \sum_{i=1}^{n} 1[X_{s,t}^{(l)} = t]. \quad \text{(F.2)}
\]
Let \( \tilde{M}_n^{(l)}(s, \cdot) \) be defined as the \( d \) dimensional vector whose \( t \)-th coordinate is \( \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}[X_{s,i}^{(l)} = t] \). Therefore,
\[
\mathbb{P}\left( \left\| \tilde{M}^{(l)}(s, \cdot) - M^{(l)}(s, \cdot) \right\|_1 > \varepsilon, \tilde{N}_s^{(l)} = n \right) = \mathbb{P}\left( \sum_{t \in \mathcal{X}} \left| \tilde{M}^{(l)}_{s,t} - M^{(l)}_{s,t} \right| > \varepsilon, \tilde{N}_s^{(l)} = n \right)
\leq \mathbb{P} \left( \left\| \tilde{M}^{(l)}(s, \cdot) - M^{(l)}(s, \cdot) \right\|_1 > \varepsilon \right) ,
\]
We obtain the following facts as a consequence of equations F.2 and F.3.

First, \( \sum_{i=1}^{n} \mathbb{1}[X_{s,i}^{(l)} = t] \) is a sum of independent Bernoulli random variables and \( \mathbb{E}[n\tilde{M}_n^{(l)}(s, \cdot)] = nM^{(l)}(s, \cdot) \). Furthermore,
\[
\text{Var} \left( \sum_{i=1}^{n} \mathbb{1}[X_{s,i}^{(l)} = t] \right) = nM^{(l)}_{s,t}(1 - M^{(l)}_{s,t}) \leq nM^{(l)}_{s,t}.
\]
Second, the mean absolute deviation satisfies
\[
\mathbb{E} \left[ \left\| \tilde{M}_n^{(l)}(s, \cdot) - M^{(l)}(s, \cdot) \right\|_1 \right] = \sum_{t \in \mathcal{X}} \mathbb{E} \left[ \left| \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}[X_{s,i}^{(l)} = t] - M^{(l)}_{s,t} \right| \right]
\leq \sum_{t \in \mathcal{X}} \sqrt{\text{Var} \left( \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}[X_{s,i}^{(l)} = t] \right)}
\leq \sum_{t \in \mathcal{X}} \sqrt{M^{(l)}_{s,t}}
\leq \sqrt{\frac{d}{n}},
\]
where the first and the last inequalities follows by Cauchy-Schwarz inequality.

Let \( \Phi \) be a function such that
\[
\Phi(X_{s,1}^{(l)}, \ldots, X_{s,n}^{(l)}) = \begin{bmatrix}
\frac{1}{n} \sum_{i=1}^{n} \mathbb{1}[X_{s,i}^{(l)} = 1] \\
\vdots \\
\frac{1}{n} \sum_{i=1}^{n} \mathbb{1}[X_{s,i}^{(l)} = d]
\end{bmatrix} - \begin{bmatrix}
M_{s,1}^{(l)} \\
\vdots \\
M_{s,d}^{(l)}
\end{bmatrix}.
\]

For a fixed \( j \in \{1, \ldots, n\} \), let \( (X_{s,1}^{(l)}, \ldots, X_{s,j}^{(l)}, \ldots, X_{s,n}^{(l)}) \) and \( (X_{s,1}^{(l)}, \ldots, (X_{s,j}^{(l)})', \ldots, X_{s,n}^{(l)}) \) be two random vectors which differ only in the \( j \)-th coordinate. Thus, \( X_{s,j}^{(l)} = t_1 \) and \( (X_{s,j}^{(l)})' = t_2 \), with \( t_1 \neq t_2 \). As a consequence, all but the \( t_1 \) and \( t_2 \)-th coordinates of the vector
\[
\Phi(X_{s,1}^{(l)}, \ldots, X_{s,j}^{(l)}, \ldots, X_{s,n}^{(l)}) - \Phi \left( X_{s,1}^{(l)}, \ldots, (X_{s,j}^{(l)})', \ldots, X_{s,n}^{(l)} \right)
\]
are 0, while the \( t_1 \)-th and \( t_2 \)-th coordinates are \( \frac{1}{n} \) and \( -\frac{1}{n} \) respectively. Thus,
\[
\left\| \Phi(X_{s,1}^{(l)}, \ldots, X_{s,j}^{(l)}, \ldots, X_{s,n}^{(l)}) - \Phi \left( X_{s,1}^{(l)}, \ldots, (X_{s,j}^{(l)})', \ldots, X_{s,n}^{(l)} \right) \right\|_1 \leq \frac{2}{n}.
\]
The reverse triangle inequality gives us,
\[
\left\| \Phi(X_{s,1}^{(l)}, \ldots, X_{s,j}^{(l)}, \ldots, X_{s,n}^{(l)}) \right\|_{1} - \left\| \Phi(X_{s,1}^{(l)}, \ldots, (X_{s,j}^{(l)})', \ldots, X_{s,n}^{(l)}) \right\|_{1} \\
\leq \left\| \Phi(X_{s,1}^{(l)}, \ldots, X_{s,j}^{(l)}, \ldots, X_{s,n}^{(l)}) - \Phi(X_{s,1}^{(l)}, \ldots, (X_{s,j}^{(l)})', \ldots, X_{s,n}^{(l)}) \right\|_{1} \\
\leq \frac{2}{n}.
\]

We can now apply McDiarmid’s inequality [46, Equation 1.3] to the probability in eq. (F.3), which, combined with the previous facts implies,
\[
P\left( \left\| \tilde{M}^{(l)}(s, \cdot) - M^{(l)}(s, \cdot) \right\|_{1} > \varepsilon, N_{s}^{(l)} = n \right) \leq P\left( \left\| \tilde{M}^{(l)}(s, \cdot) - M^{(l)}(s, \cdot) \right\|_{1} > \varepsilon \right) \\
\leq \exp \left( -\frac{n}{2} \max \left\{ 0, \varepsilon - \sqrt{\frac{d}{n}} \right\} \right).
\]

Now it follows from eq. (D.5),
\[
\sum_{n=n_{\text{low},s}}^{n_{\text{high},s}} P\left( \left\| \tilde{M}^{(l)}(s, \cdot) - M^{(l)}(s, \cdot) \right\|_{1} > \varepsilon, N_{s}^{(l)} = n \right) \\
\leq \sum_{n=n_{\text{low},s}}^{n_{\text{high},s}} \exp \left( -\frac{n}{2} \max \left\{ 0, \varepsilon - \sqrt{\frac{d}{n}} \right\} \right) \\
\leq \sum_{n=n_{\text{low},s}}^{n_{\text{high},s}} \exp \left( -\frac{n_{\text{low},s}}{2} \max \left\{ 0, \varepsilon - \sqrt{\frac{d}{n_{\text{high},s}}} \right\} \right).
\]

Since the term in the exponent does not depend upon \( n \), it can be taken out of the summation. This yields the following upper bound,
\[
(n_{\text{high},s} - n_{\text{low},s}) \exp \left( -\frac{n_{\text{low},s}}{2} \max \left\{ 0, \varepsilon - \sqrt{\frac{d}{n_{\text{high},s}}} \right\} \right) \\
\leq m \exp \left( -\frac{n_{\text{low},s}}{2} \max \left\{ 0, \varepsilon - \sqrt{\frac{d}{n_{\text{high},s}}} \right\} \right) \quad (F.5)
\]

This completes the analysis.

\section*{F.3 Proof of Lemma 4}

To find the upper bound, observe that,
\[
\mathbb{E}[N_{s}^{(l)}] = \sum_{i=1}^{m} \mathbb{E}[1[X_{i} = s, a_{i} = l]] = \sum_{i=1}^{m} \mathbb{P}(X_{i} = s, a_{i} = l).
\]
From Assumption 1 it follows that if \((s, l) \in S_i\) then, 
\[
P(X_i = s, a_i = l) \leq \zeta_1.
\]
Furthermore, if \((s, l) \notin S_i\), then \((s, l) \in S_i^c\), where \(S_i^c\) is the complement of the set \(S_i\). In that case,
\[
P(X_i = s, a_i = l) = 1 - P((X_i, a_i) \in S_i)
\]  
\[
\leq 1 - \zeta_2.
\]
Thus, we have proved that \(P(X_i = s, a_i = l) \leq \max \{\zeta_1, 1 - \zeta_2\}\). It follows as a consequence that,
\[
\mathbb{E}[N_{s}] = \sum_{i=1}^{m} P(X_i = s, a_i = l)
\]
\[
\leq m \max \{\zeta_1, 1 - \zeta_2\}.
\]
For the lower bound, define the random variable \(\{Z^{(p)}_{s,l}\}\) and the filtration \(\mathcal{F}^p\) as,
\[
Z^{(0)}_{s,l} := 0
\]
\[
Z^{(p)}_{s,l} := \sum_{i=1}^{p} \tau_{s,l}^{(i)} T - p
\]
\[
\mathcal{F}^p := \mathcal{F}_{\sum_{i=1}^{p} \tau_{s,l}^{(i)}}.
\]
Observe that
\[
\mathbb{E}[Z^{(p)}_{s,l} | \mathcal{F}^{p-1}] = \mathbb{E}[\sum_{i=1}^{p} \tau_{s,l}^{(i)} | \mathcal{F}^{p-1}] T - p
\]
\[
= \mathbb{E}[\sum_{i=1}^{p-1} \tau_{s,l}^{(i)} | \mathcal{F}^{p-1}] T - (p - 1) + \mathbb{E}[\tau_{s,l}^{(p)} | \mathcal{F}^{p-1}] - 1
\]
\[
\leq \mathbb{E}[Z^{(p-1)}_{s,l} | \mathcal{F}^{p-1}] + \frac{T}{T - 1}
\]
\[
= Z^{(p-1)}_{s,l},
\]
where the last inequality follow from Assumption 2 and the last equality follow from the fact that \(Z^{(p-1)}_{s,l}\) is \(\mathcal{F}^{p-1}\) measurable. It follows that, \(\{Z^{(p)}_{s,l}\}\) is a supermartingale. Now, define
\[
N := \min\{n : \sum_{i=1}^{p} \tau_{s,l}^{(i)} > m\}.
\]
It can be seen easily that \(N\) is a valid stopping time. Moreover, since the return times \(\tau_{s,l}^{(i)} \geq 1\) almost everywhere, it easily follows that \(P(N \leq m + 1) = 1\). Therefore, it follows from Doob’s Optional Stopping Theorem for supermartingales [20, Theorem 7.1, page 495] that,
\[
\mathbb{E}[Z_N] \leq \mathbb{E}[Z_0].
\]
Since \(Z_0 = 0\), we can write
\[
\mathbb{E}\left[\frac{\sum_{i=1}^{N} \tau_{s,l}^{(i)} T}{N} - N\right] \leq 0.
\]
This in turn implies
\[ E \left[ \sum_{i=1}^{N} \frac{\tau_{s,l}^{(i)}}{T} \right] \leq E[N]. \]

Next, we observe that \( N_s^{(l)} \) can be written as
\[ N_s^{(l)} = \max \{ n : \sum_{i=1}^{n} \tau_{s,l}^{(i)} \leq m \}. \]

In other words, \( N_s^{(l)} = N + 1 \) almost everywhere. It follows that,
\[ E \left[ \sum_{i=1}^{N} \frac{\tau_{s,l}^{(i)}}{T} \right] \leq E[N_s^{(l)}] + 1. \]

This in turn implies
\[ E \left[ \sum_{i=1}^{N} \frac{\tau_{s,l}^{(i)}}{T} \right] - 1 \leq E[N_s^{(l)}]. \]

Finally, observe that by the definition of \( N_s^{(l)} \), \( \sum_{i=1}^{N} \tau_{s,l}^{(i)} > m \) almost everywhere. Therefore,
\[ \frac{m}{T} - 1 < E[N_s^{(l)}]. \]

Finally, if \( m \geq 2T \)
\[ \frac{m}{T} - 1 \geq \frac{m}{2T}. \]

This completes the proof of the lower bound and consequently proves our lemma.

F.4 Proof of Proposition 6

\textit{Proof.} For ease of notation, we drop \( s \), and \( l \), and denote \( 1[X_i = s, a_i = l] \) by \( I_i \). Our first step is to prove that for any integer \( i \),
\[ \left( \text{Var}(I_i) + 2 \sum_{j \geq i} |\text{Cov}(I_i, I_j)| \right) \leq 4C_\Delta \rho_s^{(l)}. \]

By an application of Lemma 2, observe that
\[ |\text{Cov}(I_i, I_j)| \leq \tilde{\eta}_{i,j} \mathbb{E}[|I_j - \mathbb{E}[I_j]| \text{ess sup} |I_i|] \leq \tilde{\eta}_{i,j} \mathbb{E}[|I_j - \mathbb{E}[I_j]|] = \tilde{\eta}_{i,j} (\mathbb{P}(I_j = 1) (1 - \mathbb{P}(I_j = 1)) + |0 - \mathbb{P}(I_j = 1)|(1 - \mathbb{P}(I_j = 1))) = 2\tilde{\eta}_{i,j} \mathbb{P}(I_j = 1) (1 - \mathbb{P}(I_j = 1)) \leq 2\tilde{\eta}_{i,j} \rho_s^{(l)}. \]
Moreover, \[ \text{Var}(I_i) = \mathbb{P}(I_j = 1) (1 - \mathbb{P}(I_j = 1)) \leq \mathbb{P}(I_j = 1) \leq 4\rho_s^{(l)}. \]

Combining these facts we get,
\[
\left( \text{Var}(I_i) + 2 \sum_{j \geq i} |\text{Cov}(I_i, I_j)| \right) \leq 4 \left( 1 + \sum_{j \geq i} \bar{\eta}_{i,j} \right) \rho_s^{(l)} \\
\leq 4C_{\Delta} \rho_s^{(l)},
\]

where the final line follows from Assumption 4. Next, we observe that,
\[
\mathbb{P}(N_s^{(l)} \notin [n_{low,s}, n_{high,s}]) = \mathbb{P}(N_s^{(l)} - \mathbb{E}[N_s^{(l)}] < n_{low,s} - \mathbb{E}[N_s^{(l)}]) + \mathbb{P}(N_s^{(l)} - \mathbb{E}[N_s^{(l)}] > n_{high,s} - \mathbb{E}[N_s^{(l)}]).
\]

(F.6)

The rest of the proof follows from an application of Lemma 18 and eq. (F.6). Since $4C_{\Delta} \rho_s^{(l)}$ is independent of $i$, our proof is complete. \qed

**F.5 Proof of Lemma 7**

*Proof.* Recall from Lemma 16 the definition of $\|\Delta_m\|$.
\[
\|\Delta_m\| := \max_{1 \leq i \leq m} (1 + \bar{\eta}_{i,i+1} + \bar{\eta}_{i,i+2} + \ldots + \bar{\eta}_{i,m}),
\]

where,
\[
\bar{\eta}_{i,j} := \sup_{T, s_1, s_2, l_1, l_2, h_0^{-1}} \eta_{i,j},
\]

and,
\[
\eta_{i,j} := \mathbb{P}\left( (X_m, a_m, \ldots, X_j, a_j) \in T | X_i = s_1, a_i = l_1, H_0^{i-1} = h_0^{i-1} \right) - \mathbb{P}\left( (X_m, a_m, \ldots, X_j, a_j) \in T | X_i = s_2, a_i = l_2, H_0^{i-1} = h_0^{i-1} \right).
\]

For any fixed $s_1, s_2 \in \chi, a_1, a_2 \in \mathbb{I}$, and $h_0^{i-1} \in (\chi \times \mathbb{I})^i$, define the total variation distance $\eta_{i,j}^{(m)}$ as,
\[
\eta_{i,j}^{(m)} := \|\mathcal{L}( (X_m, a_m, \ldots, X_j, a_j) | X_i = s_1, a_i = l_1, H_1^{i-1} = h_1^{i-1}) - \mathcal{L}( (X_m, a_m, \ldots, X_j, a_j) | X_i = s_2, a_i = l_2, H_1^{i-1} = h_1^{i-1}) \|_{TV}.
\]

We note that for each fixed $m$, $\bar{\eta}_{i,j}$ can be recovered from the corresponding $\eta_{i,j}^{(m)}$ as
\[
\bar{\eta}_{i,j} = \sup_{s_1, s_2, l_1, l_2, h_0^{-1}} \eta_{i,j}^{(m)}.
\]

(F.7)

Thus is is enough to find an upper bound for $\eta_{i,j}^{(m)}$. For the sake of simplicity, consider the case when $m = 2$. Then,
\[
\eta_{1,2}^{(2)} = \|\mathcal{L}( (X_2, a_2) | X_1 = s_1, a_1 = l_1, X_0 = s_0, a_0 = l_0) - \mathcal{L}( (X_2, a_2) | X_1 = s_2, a_1 = l_2, X_0 = s_0, a_0 = l_0) \|_{TV}.
\]
Use Lemma 19 by substituting $X = a_2$, $Y = X_2$ and $Z = (X_1, a_1, X_0, a_0)$ with $z_1$ and $z_2$ defined accordingly. This reduces the $\eta$ mixing coefficient into a sum of the $\gamma$ mixing coefficients and $\theta$ mixing coefficients. To be precise,

$$
\eta^{(2)}_{1,2} \leq \sup_{s''} \| \mathcal{L} (a_2|X_2 = s'', X_1 = s_1, a_1 = l_1, X_0 = s_0, a_0 = l_0) - \mathcal{L} (a_2|X_2 = s'') \|_{TV}
$$

(\text{F.8})

where the last step follows from taking supremum over $s_1, l_1, s_2, l_2, s_0,$ and $a_0$ on both sides of the inequality.

It follows that, when $m = 2$,

$$
\bar{\eta}_{1,2} \leq 2\gamma_{2,1,1},
$$

(\text{F.9})

We can decompose the first term in eq. (F.8) as,

$$
\sup_{s''} \| \mathcal{L} (a_2|X_2 = s'', X_1 = s_1, a_1 = l_1, X_0 = s_0, a_0 = l_0) - \mathcal{L} (a_2|X_2 = s'') \|_{TV}
$$

$$
\leq \sup_{s''} \| \mathcal{L} (a_2|X_2 = s''|X_1 = s_1, a_1 = l_1, X_0 = s_0, a_0 = l_0) - \mathcal{L} (a_2|X_2 = s'') \|_{TV}
$$

$$
+ \sup_{s''} \| \mathcal{L} (a_2|X_2 = s'', X_1 = s_2, a_1 = l_2, X_0 = s_0, a_0 = l_0) - \mathcal{L} (a_2|X_2 = s'') \|_{TV}
$$

$$
\leq 2\gamma_{2,1,1},
$$

We now establish the following recursion,

$$
\eta^{(m+1)}_{i,j} \leq \eta^{(m)}_{i,j} + 2\gamma_{m+1,j,i}.
$$

(\text{F.10})

Fix an arbitrary $m \geq 3$ and fix $i, j \in 1, 2, \ldots, m$ such that $i < j$ and decompose $\eta^{(m+1)}_{i,j}$ using Lemma 19 by taking $X = (X_{m+1}, a_{m+1})$, and defining $Y$ and $Z$ accordingly to get,

$$
\eta^{(m+1)}_{i,j} \leq \eta^{(m)}_{i,j} + \| \mathcal{L}(X_{m+1}, a_{m+1} | X_m = x_m, a_m = l_m, X_i = s_1, a_i = l_1, \ldots )
$$

$$
- \mathcal{L}(X_{m+1}, a_{m+1} | X_m = x_m, a_m = l_m, X_i = s_2, a_i = l_2, \ldots ) \|_{TV}
$$

$$
\leq \| \mathcal{L}(X_{m+1} | X_m = x_m, a_m = l_m, X_i = s_1, a_i = l_1, \ldots )
$$

$$
- \mathcal{L}(X_{m+1} | X_m = x_m, a_m = l_m, X_i = s_2, a_i = l_2, \ldots ) \|_{TV}
$$

where we have replaced the terms common to both the conditionals by \ldots for convenience of notation.

Using Lemma 19 again on the second term by taking $X = a_{m+1}, Y = X_{m+1}$, and $Z$ as all the random variables in the conditional, we get the following upper bound,

$$
\eta^{(m+1)}_{i,j} \leq \eta^{(m)}_{i,j} + \sup_{x_{m+1} \in \mathcal{X}} \| \mathcal{L}(a_{m+1} | X_{m+1} = x_{m+1}, X_m = x_m, a_m = l_m, X_i = s_1, a_i = l_1, \ldots )
$$

$$
- \mathcal{L}(a_{m+1} | X_{m+1} = x_{m+1}, X_m = x_m, a_m = l_m, X_i = s_2, a_i = l_2, \ldots ) \|_{TV}
$$

$$
+ \| \mathcal{L}(X_{m+1} | X_m = x_m, a_m = l_m, X_i = s_1, a_i = l_1, \ldots )
$$

$$
- \mathcal{L}(X_{m+1} | X_m = x_m, a_m = l_m, X_i = s_2, a_i = l_2, \ldots ) \|_{TV}
$$

(\text{F.11})

Since $X_{m+1}$ is Markovian conditional on $X_m$ and $a_m$, the third term vanishes. Proceeding similarly to eq. (F.9), the second term can be upper bounded by $\gamma_{m+1,j,i}$. This proves the recursion in eq. (F.10).
Applying the recursion multiple times we are left with,

$$\eta_{i,j}^{(m)} \leq \sum_{p=i+j+1}^{m} 2\gamma_{p,j,i} + \eta_{i,j}^{(j)}.$$ 

Proceeding similarly to eq. (F.8), we can decompose \(\eta_{i,j}^{(j)}\) to get,

$$\eta_{i,j}^{(j)} \leq 2\gamma_{j,i,i} + \hat{\theta}_{i,j}.$$ 

The terms in the right hand side of the previous equation is a constant. Following the arguments of eq. (F.7), we can now take an appropriate supremum on the left hand side to recover \(\bar{\eta}_{i,j}\). Therefore,

$$\bar{\eta}_{i,j} \leq 2 \sum_{p=i+j}^{m} \gamma_{p,j,i} + \bar{\theta}_{i,j}. \text{(F.12)}$$

By substituting the upper bound into the expression of \(\|\Delta_m\|\), it follows that,

$$\|\Delta_m\| \leq \sup_{1 \leq i \leq m} \left\{ 1 + 2 \sum_{j>i}^{m} \gamma_{p,j,i} + \sum_{j>i} \bar{\theta}_{i,j} \right\},$$

From Assumptions 5 and 6 it now follows that,

$$\|\Delta_m\| \leq 1 + C + C\theta,$$

which completes the proof. \(\Box\)

F.6 Proof of Lemma 8

Proof: The proof of this Lemma follows similarly to that of Lemma 7. We begin by observing from eq. (F.12) that

$$\bar{\eta}_{i,j} \leq 2 \sum_{p=i+j}^{m} \gamma_{p,j,i} + \bar{\theta}_{i,j}.$$ 

The rest of the proof follows from a direct application of Assumptions 7 and 8. \(\Box\)

F.7 Proof of Lemma 12

Proof: As before, let \(\tau_{s,l}^{(i,*)}\) as the time between the \(j-1\)-th and \(j\)-th visit to control \(l\) after visiting state-control pair \(s,l\) for the \(i\)-th time. Our next step is to represent \(\tau_{s,l}^{(i)}\) in terms of \(\tau_{s,l}^{(i,*)}\)'s. Observe that

$$\tau_{s,l}^{(i+1)} = \begin{cases} \tau_{s,l}^{(i,*1)} & \text{if } \left\{ X_{\sum_{p=1}^{i} s_{l}(p) + s_{s,l}^{(i,*1)}} = s \right\} \\ \tau_{s,l}^{(i,*2)} & \text{if } \left\{ X_{\sum_{p=1}^{i} s_{l}(p) + s_{s,l}^{(i,*1)}} \neq s \text{ and } X_{\sum_{p=1}^{i} s_{l}(p) + s_{s,l}^{(i,*1)} + s_{s,l}^{(i,*2)}} = s \right\} \\ \vdots & \vdots \end{cases}$$
Informally, \( \tau_{s,d}^{(i+1)} \) is: \( \tau_{s,d}^{(i,1)} \) if the state at the corresponding time is \( s \); it is \( \tau_{s,d}^{(i,1)} + \tau_{s,d}^{(i,2)} \) if the state was not \( s \) after \( \tau_{s,d}^{(i,1)} \) time points and \( s \) after \( \tau_{s,d}^{(i,1)} + \tau_{s,d}^{(i,2)} \) time points, and so on. In other words,

\[
\tau_{s,d}^{(i+1)} = \tau_{s,d}^{(i,1)} \mathbb{1} \left[ X_{\sum_{p=1}^{i} \tau_{s,d}^{(p)} + \tau_{s,d}^{(i,1)}} = s \right] \\
+ \left( \tau_{s,d}^{(i,1)} + \tau_{s,d}^{(i,2)} \right) \mathbb{1} \left[ X_{\sum_{p=1}^{i} \tau_{s,d}^{(p)} + \tau_{s,d}^{(i,1)} + \tau_{s,d}^{(i,2)}} = s \right] \\
+ \ldots
\]

which in turn implies

\[
\mathbb{E} \left[ \tau_{s,d}^{(i+1)} | \mathcal{F}_{\sum_{p=1}^{i-1} \tau_{s,d}^{(p)}} \right] = \mathbb{E} \left[ \tau_{s,d}^{(i,1)} | \mathcal{F}_{\sum_{p=1}^{i-1} \tau_{s,d}^{(p)}} \right] \mathbb{1} \left[ X_{\sum_{p=1}^{i-1} \tau_{s,d}^{(p)} + \tau_{s,d}^{(i,1)}} = s \right] \\
+ \mathbb{E} \left[ \left( \tau_{s,d}^{(i,1)} + \tau_{s,d}^{(i,2)} \right) \mathbb{1} \left[ X_{\sum_{p=1}^{i-1} \tau_{s,d}^{(p)} + \tau_{s,d}^{(i,1)} + \tau_{s,d}^{(i,2)}} = s \right] \mathcal{F}_{\sum_{p=1}^{i-1} \tau_{s,d}^{(p)}} \right] \\
+ \ldots
\]

To compute an upper bound to \( \mathbb{E} \left[ \tau_{s,d}^{(i)} \right] \), it is thus sufficient to individually find an upper bound to each term of the summation in the right-hand side of the previous equation by a careful bookkeeping of the conditional expectations.

**Term 1:** Applying the law of conditional expectation to the first term we get

\[
\mathbb{E} \left[ \tau_{s,d}^{(i,1)} | \mathcal{F}_{\sum_{p=1}^{i-1} \tau_{s,d}^{(p)}} \right] = \mathbb{E} \left[ \tau_{s,d}^{(i,1)} | \mathcal{F}_{\sum_{p=1}^{i-1} \tau_{s,d}^{(p)}} \right] \mathbb{1} \left[ X_{\sum_{p=1}^{i-1} \tau_{s,d}^{(p)} + \tau_{s,d}^{(i,1)}} = s \right] \\
+ \mathbb{E} \left[ \left( \tau_{s,d}^{(i,1)} + \tau_{s,d}^{(i,2)} \right) \mathbb{1} \left[ X_{\sum_{p=1}^{i-1} \tau_{s,d}^{(p)} + \tau_{s,d}^{(i,1)} + \tau_{s,d}^{(i,2)}} = s \right] \mathcal{F}_{\sum_{p=1}^{i-1} \tau_{s,d}^{(p)}} \right]
\]

Recall from eq. (4.6) our assumption that

\[
\max_{s,t,l} M_{s,t,l}^{(l)} = M_{\max}, \text{ and } \min_{s,t,l} M_{s,t,l}^{(l)} = M_{\min}
\]

for two numbers \( 0 < M_{\min}, M_{\max} < 1 \). It follows that for any time \( p \), state \( s \), and history \( H_{0}^{p-1} \),

\[
M_{\min} \leq P \left( X_{p} = s \mid H_{0}^{p-1} = H_{0}^{p-1} \right) \leq M_{\max}, \quad \text{and}
\]

\[
P \left( X_{p} \neq s \mid H_{0}^{p-1} = H_{0}^{p-1} \right) \leq \max \{ M_{\max}, 1 - M_{\min} \} =: M_{\opt}. \quad (E1)
\]

It follows from (E1) that, \( P \left( X_{p} = s \mid H_{0}^{p-1} = H_{0}^{p-1} \right) \leq M_{\max} \). Substituting this value in the right hand side of eq. (F.14), we get the following upper bound to Term 1

\[
\mathbb{E} \left[ \tau_{s,d}^{(i,1)} | \mathcal{F}_{\sum_{p=1}^{i-1} \tau_{s,d}^{(p)}} \right] \leq \mathbb{E} \left[ \tau_{s,d}^{(i,1)} \mathcal{F}_{\sum_{p=1}^{i-1} \tau_{s,d}^{(p)}} \right] \leq T_{\tau} M_{\max},
\]

where the last inequality follows from tower property since

\[
\mathcal{F}_{\sum_{p=1}^{i-1} \tau_{s,d}^{(p)}} \subseteq \mathcal{F}_{\sum_{p=1}^{i-1} \tau_{s,d}^{(p)} + \sum_{p=1}^{i-1} \tau_{s,d}^{(i,1,p)}}.
\]
Term 2: For ease of notation, we define

\[ E^* \{ \} = E\{ |F_{s,l}^{i-1} T(p) \}_s \}

and proceed similarly as before to get

\[
E^* \left[ \left( T_{s,l}^{(i,1)} \right)^1 \right] \left[ X_{\sum_{p=1}^s \tau_{s,l}^{(p)} + \tau_{s,l}^{(i,1)} \neq s}, \ X_{\sum_{p=1}^s \tau_{s,l}^{(p)} + \tau_{s,l}^{(i,1)} + \tau_{s,l}^{(i,2)}} = s \right] \\
\quad = E^* \left[ \left( T_{s,l}^{(i,1)} \right)^1 \right] \left[ X_{\sum_{p=1}^s \tau_{s,l}^{(p)} + \tau_{s,l}^{(i,1)} + \tau_{s,l}^{(i,2)}} = s | T_{s,l}^{(i,1)}, T_{s,l}^{(i,2)} \right] . \tag{F.15}
\]

We decompose \( \mathbb{P} \left( X_{\sum_{p=1}^s \tau_{s,l}^{(p)} + \tau_{s,l}^{(i,1)} \neq s}, X_{\sum_{p=1}^s \tau_{s,l}^{(p)} + \tau_{s,l}^{(i,1)} + \tau_{s,l}^{(i,2)}} = s | T_{s,l}^{(i,1)}, T_{s,l}^{(i,2)} \right) \) into

\[
\mathbb{P} \left( X_{\sum_{p=1}^s \tau_{s,l}^{(p)} + \tau_{s,l}^{(i,1)} + \tau_{s,l}^{(i,2)}} = s | T_{s,l}^{(i,1)}, T_{s,l}^{(i,2)} \right) \times \mathbb{P} \left( X_{\sum_{p=1}^s \tau_{s,l}^{(p)} + \tau_{s,l}^{(i,1)} \neq s} \right)
\]

We use (E1) to bound the first term from above and (E2) to bound the second term from above in the previous equation. This gives us,

\[
\mathbb{P} \left( X_{\sum_{p=1}^s \tau_{s,l}^{(p)} + \tau_{s,l}^{(i,1)} \neq s}, X_{\sum_{p=1}^s \tau_{s,l}^{(p)} + \tau_{s,l}^{(i,1)} + \tau_{s,l}^{(i,2)}} = s | T_{s,l}^{(i,1)}, T_{s,l}^{(i,2)} \right) \leq M_{max} \cdot M_{opt},
\]

Substituting this value in the right hand side of eq. (F.15) we get

\[
E^* \left[ \left( T_{s,l}^{(i,1)} \right)^1 \right] \left[ X_{\sum_{p=1}^s \tau_{s,l}^{(p)} + \tau_{s,l}^{(i,1)} \neq s}, X_{\sum_{p=1}^s \tau_{s,l}^{(p)} + \tau_{s,l}^{(i,1)} + \tau_{s,l}^{(i,2)}} = s \right] \leq E^* \left[ T_{s,l}^{(i,1)} \right] M_{max} \cdot M_{opt}
\]

\[
\leq T_s M_{max} \cdot M_{opt},
\]

where the last inequality follows from an application of tower property. We similarly have,

\[
E^* \left[ \left( T_{s,l}^{(i,2)} \right)^1 \right] \left[ X_{\sum_{p=1}^s \tau_{s,l}^{(p)} + \tau_{s,l}^{(i,1)} \neq s}, X_{\sum_{p=1}^s \tau_{s,l}^{(p)} + \tau_{s,l}^{(i,1)} + \tau_{s,l}^{(i,2)}} = s \right] \leq T_s M_{max} \cdot M_{opt}.
\]

Thus \( 2T_s M_{max} \cdot M_{opt} \) is an upper bound to Term 2. Proceeding similarly, we can find an upper bound to each term. Substituting these upper bounds back into eq. (F.13) we get

\[
E \left[ \left( T_{s,l}^{(i)} \right) \left| \sum_{p=1}^{s} \tau_{s,l}^{(p)} \right| \right] \leq T_s M_{max} + 2T_s M_{max} \cdot M_{opt} + \ldots
\]

\[
= \sum_{j=1}^{\infty} jT_s M_{max} M_{opt}^{j-1}
\]

\[
= T_s \frac{M_{max}}{1 - M_{opt}} \sum_{j=1}^{\infty} j(1 - M_{opt}) M_{opt}^{j-1}
\]

\[
= T_s \frac{M_{max}}{M_{opt}(1 - M_{opt})}.
\]

\[ \square \]
where the first inequality follows because there exists at least one

However, if \( \xi \)

Since \( P \)

that it is a controlled Markov chain is obvious. We only need to verify ergodicity of \( \mathfrak{M}(l) \), and stationarity of controls \( a_i \). We first show that \( \mathfrak{M}(l) \) is irreducible. Since \( M(l) \) is irreducible and aperiodic, there exists an integer \( n_{ap} \) such that \( (M(l))^{n_{ap}} \) is a positive stochastic matrix. Observe that \( J \times J = J \), \( J \times M(l) = J \). Then,

\[
\left( \mathfrak{M}(l) \right)^2 = \mathfrak{M}(l) \times \mathfrak{M}(l) = \begin{bmatrix}
(1 - v)^2 J & (1 - v) v J \\
(1 - v) v J & v^2 (M(l))^2
\end{bmatrix}.
\]

Similarly, it can be shown that \( (\mathfrak{M}(l))^{n_{ap}} \) depends only upon \( v, (M(l))^{n_{ap}} \) and \( J \). Thus, \( (\mathfrak{M}(l))^{n_{ap}} \) is positive and it is irreducible.

\[
P(X_2 = 1, \omega_2 = 0 | X_1 = 1, \omega_1 = 0) = \frac{1 - v}{d} P(\omega_2 = \xi) > 0
\]

Since \( P(\omega_2 = \xi) = \min\{v, 1 - v\} > 0 \), this proves that there exists a one-step path of positive probability from \( (1, 0) \) to \( (1, 0) \). Thus, state \( (1, 0) \) is aperiodic. Since the matrix is also irreducible, all the states are aperiodic, and the transition matrix is ergodic. Next, we verify the stationarity of the controls \( a_i \). Let \( h_0^{-1} \in (\chi \times \{0, 1\} \times I)^i \) be any history and \( (s_i, \xi_i) \in \chi \times \{0, 1\} \). Then,

\[
\mathcal{L}(\tilde{a}_i | X_i = s_i, \omega_i = \xi_i, \mathcal{H}_0^{i-1} = h_0^{i-1}) = \begin{cases}
\mathcal{L}(a_i | X_i = s_i, \omega_i = \xi_i, \mathcal{H}_0^{i-1} = h_0^{i-1}) & \text{if } \xi_i = 1 \\
\mathcal{L}(D_i^{(3)} | X_i = s_i, \omega_i = \xi_i, \mathcal{H}_0^{i-1} = h_0^{i-1}) & \text{if } \xi_i = 0
\end{cases}
\]

However, if \( \xi_i = 1 \), then \( a_i \) is drawn uniformly over \( I \) independent of the history. And if \( \xi_i = 0 \), then we draw from \( D_i^{(3)} \), which is distributed uniformly over \( I \) independent of the history. Thus the controls are trivially stationary. This completes the proof. \( \square \)

F.9 Proof of Lemma 17

Proof: As before, for some \( i, j \) let \( T_s \in \{0, 1\}^{m-j} \) and \( h_s \in \{0, 1\}^{i+1} \), and denote by \( (\mathbb{1} [X_i = s, a_i = l], \ldots, \mathbb{1} [X_0 = s, a_0 = l]) \) by \( \mathbb{1} (\mathcal{H}_0^i) \). Define \( S_h^* := \{h_0^i \in (\chi \times \mathbb{I})^{i+1} : \mathbb{1} (h_0^i) = h_s\} \)

Next observe that,

\[
|P(\mathbb{1} [X_i = s, a_i = l], \ldots, \mathbb{1} [X_j = s, a_j = l]) \in T_s | \mathbb{1} (\mathcal{H}_0^i) = h_s|,
\]

\[
- P(\mathbb{1} [X_i = s, a_i = l], \ldots, \mathbb{1} [X_j = s, a_j = l]) \in T_s | \mathbb{1} (\mathcal{H}_0^i) = h_0^i 
\]

\[
\leq \sup_{h_0^i \in S_h^*} \big| P(\mathbb{1} [X_i = s, a_i = l], \ldots, \mathbb{1} [X_j = s, a_j = l]) \in T_s | \mathbb{1} (\mathcal{H}_0^i) = h_0^i 
\]

\[
- P(\mathbb{1} [X_i = s, a_i = l], \ldots, \mathbb{1} [X_j = s, a_j = l]) \in T_s | \mathbb{1} (\mathcal{H}_0^i) = h_0^i 
\]

\[
\leq \sup_{h_0^i \in (\chi \times \mathbb{I})^{i+1}} \big| P(\mathbb{1} [X_i = s, a_i = l], \ldots, \mathbb{1} [X_j = s, a_j = l]) \in T_s | \mathbb{1} (\mathcal{H}_0^i) = h_0^i 
\]

\[
- P(\mathbb{1} [X_i = s, a_i = l], \ldots, \mathbb{1} [X_j = s, a_j = l]) \in T_s | \mathbb{1} (\mathcal{H}_0^i) = h_0^i 
\]

\[
\leq \sup_{T_s, h_0^i} \big| P((X_i, a_i, \ldots, X_j, a_j) \in T | \mathbb{1} (\mathcal{H}_0^i) = h_0^i) - P(X_i, a_i, \ldots, X_j, a_j \in T) \big|,
\]

where the first inequality follows because there exists at least one \( h_0^i \) such that \( \mathbb{1} (h_0^i) = h_s \), and the last inequality follows naturally by taking inversion of \( \mathbb{1} (\cdot) \) and the appropriate supremum. Now taking an
appropriate supremum over $T_*, h_*$ we get,
\[
\sup_{T_*, h_*} |\mathbb{P} (\mathbb{1}[X_m = s, a_m = l], \ldots, \mathbb{1}[X_j = s, a_j = l]) \in T_* |\mathbb{1}(H_0^T) = h_*) 
- \mathbb{P} (\mathbb{1}[X_m = s, a_m = l], \ldots, \mathbb{1}[X_j = s, a_j = l]) \in T_* |\mathbb{1}(H_0^{T}) = h_0) \leq \sup_{T, H_0^{-1}} |\mathbb{P} ((X_m, a_m, \ldots, X_j, a_j) \in T|H_0^T = h_0^T) - \mathbb{P} (X_m, a_m, \ldots, X_j, a_j \in T)|
= \phi_{i,j}.
\]
This proves our lemma. \qed

F.10 Proof of Lemma 19

Proof. Let $\mathcal{X}$ and $\mathcal{Y}$ be two events. Consider the following term,
\[
|\mathbb{P}(X \in \mathcal{X}, Y \in \mathcal{Y}|Z = z_1) - \mathbb{P}(X \in \mathcal{X}, Y \in \mathcal{Y}|Z = z_2)|
\]
\[
= \sum_{y \in \mathcal{Y}} \left| \mathbb{P}(X \in \mathcal{X}, Y = y|Z = z_1) - \mathbb{P}(X \in \mathcal{X}, Y = y|Z = z_2) \right|
\]
\[
= \sum_{y \in \mathcal{Y}} \left| \mathbb{P}(X \in \mathcal{X}|Y = y, Z = z_1)\mathbb{P}(Y = y|Z = z_1) - \mathbb{P}(X \in \mathcal{X}|Y = y, Z = z_2)\mathbb{P}(Y = y|Z = z_2) \right|
\]
Adding and subtracting $\mathbb{P}(X \in \mathcal{X}|Y = y, Z = z_1)\mathbb{P}(Y = y|Z = z_2)$ inside the summation and applying triangle inequality we get,
\[
|\mathbb{P}(X \in \mathcal{X}, Y \in \mathcal{Y}|Z = z_1) - \mathbb{P}(X \in \mathcal{X}, Y \in \mathcal{Y}|Z = z_2)|
\]
\[
\leq \sum_{y \in \mathcal{Y}} \left| \mathbb{P}(X \in \mathcal{X}|Y = y, Z = z_1)\mathbb{P}(Y = y|Z = z_2) - \mathbb{P}(X \in \mathcal{X}|Y = y, Z = z_2)\mathbb{P}(Y = y|Z = z_2) \right|
+ \sum_{y \in \mathcal{Y}} \left| \mathbb{P}(X \in \mathcal{X}|Y = y, Z = z_1)\mathbb{P}(Y = y|Z = z_2) - \mathbb{P}(X \in \mathcal{X}|Y = y, Z = z_1)\mathbb{P}(Y = y|Z = z_1) \right|
= \text{TERM 1} + \text{TERM 2}
\]
In the first term in the previous equation, $\mathbb{P}(Y = y|Z = z_2)$ is common. So, the first term can be rewritten as,
\[
\text{TERM 1} = \sum_{y \in \mathcal{Y}} \left( |\mathbb{P}(X \in \mathcal{X}|Y = y, Z = z_1) - \mathbb{P}(X \in \mathcal{X}|Y = y, Z = z_2) | \times \mathbb{P}(Y = y|Z = z_2) \right)
\]
We take the modulus inside the summation to get,
\[
\text{TERM 1} \leq \sum_{y \in \mathcal{Y}} \left( |\mathbb{P}(X \in \mathcal{X}|Y = y, Z = z_1) - \mathbb{P}(X \in \mathcal{X}|Y = y, Z = z_2) | \times \mathbb{P}(Y = y|Z = z_2) \right)
\]
\[
\leq \sup_{y_0 \in \mathcal{Y}} |\mathbb{P}(X \in \mathcal{X}|Y = y_0, Z = z_1) - \mathbb{P}(X \in \mathcal{X}|Y = y_0, Z = z_2) | \sum_{y \in \mathcal{Y}} \mathbb{P}(Y = y|Z = z_2)
= \sup_{y_0 \in \mathcal{Y}} |\mathbb{P}(X \in \mathcal{X}|Y = y_0, Z = z_1) - \mathbb{P}(X \in \mathcal{X}|Y = y_0, Z = z_2) | \mathbb{P}(Y = \mathcal{Y}|Z = z_2)
\]
\[
\leq \sup_{y_0 \in \mathcal{Y}} |\mathbb{P}(X \in \mathcal{X}|Y = y_0, Z = z_1) - \mathbb{P}(X \in \mathcal{X}|Y = y_0, Z = z_2) |
\]
We upper bound \( |\mathbb{P}(X \in \mathcal{X}|Y = y_0, Z = z_1) - \mathbb{P}(X \in \mathcal{X}|Y = y_0, Z = z_2)| \) by its corresponding total variation distance as,

\[
\sup_{y_0 \in \mathcal{Y}} \|\mathcal{L}(X|Y = y_0, Z = z_1) - \mathcal{L}(X|Y = y_0, Z = z_2)\|_{TV}.
\]

Since \( \mathcal{Y} \subseteq \Omega \), we can further upper bound the previous term by

\[
\sup_{y_0 \in \Omega} \|\mathcal{L}(X|Y = y_0, Z = z_1) - \mathcal{L}(X|Y = y_0, Z = z_2)\|_{TV}.
\]

This gives us the following upper bound to the first term,

\[
\text{TERM 1} \leq \sup_{y_0 \in \Omega} \|\mathcal{L}(X|Y = y_0, Z = z_1) - \mathcal{L}(X|Y = y_0, Z = z_2)\|_{TV}.
\]

(F.16)

Now focusing on the second term, let \( \mathcal{Y}_1 \subseteq \mathcal{Y} \) be the largest set such that the probability difference \( \mathbb{P}(Y = y|Z = z_1) - \mathbb{P}(Y = y|Z = z_2) > 0 \) for any \( y \in \mathcal{Y}_1 \). Let \( \mathcal{Y}_2 \) be the set difference \( \mathcal{Y} \setminus \mathcal{Y}_1 \). This yields us the following decomposition of the second term,

\[
\text{TERM 2} = \sum_{y \in \mathcal{Y}} \mathbb{P}(X \in \mathcal{X}|Y = y, Z = z_1)\left[\mathbb{P}(Y = y|Z = z_1) - \mathbb{P}(Y = y|Z = z_2)\right]
\]

\[
= \sum_{y \in \mathcal{Y}_1} \mathbb{P}(X \in \mathcal{X}|Y = y, Z = z_1)\left[\mathbb{P}(Y = y|Z = z_1) - \mathbb{P}(Y = y|Z = z_2)\right]
\]

\[
+ \sum_{y \in \mathcal{Y}_2} \mathbb{P}(X \in \mathcal{X}|Y = y, Z = z_1)\left[\mathbb{P}(Y = y|Z = z_1) - \mathbb{P}(Y = y|Z = z_2)\right],
\]

which can be further upper bounded by,

\[
\left| \left( \sup_{y_1 \in \mathcal{Y}_1} \mathbb{P}(X = \mathcal{X}|Y = y_1, Z = z_1) \right) \sum_{y_1 \in \mathcal{Y}_1} (\mathbb{P}(Y = y_1|Z = z_1) - \mathbb{P}(Y = y_1|Z = z_2)) \right| + \\
\left| \left( \inf_{y_2 \in \mathcal{Y}_2} \mathbb{P}(X = \mathcal{X}|Y = y_2, Z = z_1) \right) \sum_{y_2 \in \mathcal{Y}_2} (\mathbb{P}(Y = y_2|Z = z_1) - \mathbb{P}(Y = y_2|Z = z_2)) \right|
\]

\[
= \left| \left( \sup_{y_1 \in \mathcal{Y}_1} \mathbb{P}(X = \mathcal{X}|Y = y_1, Z = z_1) \right) (\mathbb{P}(Y = \mathcal{Y}_1|Z = z_1) - \mathbb{P}(Y = \mathcal{Y}_1|Z = z_2)) \right| + \\
\left| \left( \inf_{y_2 \in \mathcal{Y}_2} \mathbb{P}(X = \mathcal{X}|Y = y_2, Z = z_1) \right) (\mathbb{P}(Y = \mathcal{Y}_2|Z = z_1) - \mathbb{P}(Y = \mathcal{Y}_2|Z = z_2)) \right|
\]

Recall that if \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) are real numbers with different signs, \( |\mathcal{P}_1 + \mathcal{P}_2| \leq \max\{|\mathcal{P}_1|,|\mathcal{P}_2|\} \). Using this fact, we can get the following upper bound to the previous equation.

\[
\max \left\{ \left| \left( \sup_{y_1 \in \mathcal{Y}_1} \mathbb{P}(X = \mathcal{X}|Y = y_1, Z = z_1) \right) (\mathbb{P}(Y \in \mathcal{Y}_1|Z = z_1) - \mathbb{P}(Y \in \mathcal{Y}_1|Z = z_2)) \right|, \\
\left| \left( \inf_{y_2 \in \mathcal{Y}_2} \mathbb{P}(X = \mathcal{X}|Y = y_2, Z = z_1) \right) (\mathbb{P}(Y \in \mathcal{Y}_2|Z = z_1) - \mathbb{P}(Y \in \mathcal{Y}_2|Z = z_2)) \right| \right\}.
\]
We can upper bound the common probabilities in the previous terms by 1. This gives us the following upper bound to the previous term.

\[
\max \left\{ \| \mathbb{P}(Y = Y_1|Z = z_1) - \mathbb{P}(Y = Y_2|Z = z_2) \|, \| \mathbb{P}(Y = Y_1|Z = z_1) - \mathbb{P}(Y = Y_2|Z = z_2) \| \right\},
\]

Since \( \|\mathcal{L}(Y|Z = z_1) - \mathcal{L}(Y|Z = z_2)\|_{TV} \) is an upper bound to both of the terms inside the maximum, we find the following upper bound to the second term.

\[
\text{TERM 2} \leq \|\mathcal{L}(Y|Z = z_1) - \mathcal{L}(Y|Z = z_2)\|_{TV}.
\]

Combining equations F.16 and F.17 we find

\[
\| \mathbb{P}(X \in \mathcal{X}, Y \in \mathcal{Y}|Z = z_1) - \mathbb{P}(X \in \mathcal{X}, Y \in \mathcal{Y}|Z = z_2) \| \leq \| \mathcal{L}(Y|Z = z_1) - \mathcal{L}(Y|Z = z_2)\|_{TV}.
\]

Taking supremum over \( \mathcal{X} \) and \( \mathcal{Y} \),

\[
\| \mathcal{L}(X,Y|Z = z_1) - \mathcal{L}(X,Y|Z = z_2)\|_{TV} \leq \| \mathcal{L}(Y|Z = z_1) - \mathcal{L}(Y|Z = z_2)\|_{TV}.
\]

This completes the proof. \( \square \)

### F.11 Proof of Lemma 21

**Proof.** Define the paired process \( Y_i := (X_i, a_i) \) on the paired state space \( \chi \times \mathbb{I} \). It follows from the definition of \( \theta_{i,j} \) in eq. (4.3) that,

\[
\hat{\theta}_{i,j} = \sup_{s_1, s_2 \in \chi, l_1, l_2 \in \mathbb{I}} \| \mathcal{L}(X_j|X_i = s_1, a_i = l_1) - \mathcal{L}(X_j|X_i = s_2, a_i = l_2) \|_{TV}
\]

\[
\leq \sup_{s_1, s_2 \in \chi, l_1, l_2 \in \mathbb{I}} \| \mathcal{L}(X_j, a_j|X_i = s_1, a_i = l_1) - \mathcal{L}(X_j, a_j|X_i = s_2, a_i = l_2) \|_{TV}
\]

As seen in section 4.4, \((X_i, a_i)\) forms an inhomogenous Markov chain with the probability of transition from \((s, l)\) to \((t, l')\) at time point \(i\) is \( M_{s,t}^{(i)} \times P_{t,t'}^{(i)} \). It follows from Hajnal and Bartlett [21, Theorem 2] that

\[
\sup_{s_1, s_2 \in \chi, l_1, l_2 \in \mathbb{I}} \| \mathcal{L}(X_j, a_j|X_i = s_1, a_i = l_1) - \mathcal{L}(X_j, a_j|X_i = s_2, a_i = l_2) \|_{TV}
\]

\[
\leq \prod_{p=i}^{j-1} \left( 1 - \min_{(s_1, l_1), (s_2, l_2) \in \chi \times \mathbb{I}} \sum_{(t, t') \in \chi \times \mathbb{I}} \min \left\{ \left( M_{s_1,t}^{(i)} \times P_{t,t'}^{(i)} \right), \left( M_{s_2,t}^{(i)} \times P_{t,t'}^{(i)} \right) \right\} \right).
\]

Recall that by hypothesis

\[
\min_{s \in \chi, t \in \mathbb{I}} M_{s,t}^{(i)} > M_{\min},
\]

for any \( t \in \chi_0 \). This implies that for all \( t \in \chi_0 \),

\[
\min \left\{ \left( M_{s_1,t}^{(i)} \times P_{t,t'}^{(i)} \right), \left( M_{s_2,t}^{(i)} \times P_{t,t'}^{(i)} \right) \right\} \geq M_{\min} P_{t,t'}^{(i)}.
\]

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Decomposing the summation over \((t, l) \in \chi \times \Pi\) in eq. (F.18) into a summation over \((t, l) \in (\chi \setminus \chi_0) \times \Pi\) and \((t, l) \in \chi_0 \times \Pi\) and substituting \(M_{\min, P_{t,l}}^{(i)}\) as the appropriate lower bound we get,

\[
\sum_{(t', l') \in \chi \times \Pi} \min \left\{ \left( M_{s_1, t}^{(l_1)} \times P_{t', l'}^{(i)} \right), \left( M_{s_2, t}^{(l_2)} \times P_{t', l'}^{(i)} \right) \right\}
\]

\[
\geq \sum_{t \in \chi \setminus \chi_0} \sum_{l' \in \Pi} \min \left\{ \left( M_{s_1, t}^{(l_1)} \times P_{t', l'}^{(i)} \right), \left( M_{s_2, t}^{(l_2)} \times P_{t', l'}^{(i)} \right) \right\} + \sum_{(t', l') \in \chi_0 \times \Pi} M_{\min, P_{t,l}}^{(i)}
\]

\[
= \sum_{t \in \chi_0} \sum_{l' \in \Pi} \min \left\{ \left( M_{s_1, t}^{(l_1)} \times P_{t', l'}^{(i)} \right), \left( M_{s_2, t}^{(l_2)} \times P_{t', l'}^{(i)} \right) \right\}
\]

\[
= |\chi_0|M_{\min}.
\]

It follows that

\[
\prod_{p=i}^{j-1} \left( 1 - \min_{(s_1, l_1), (s_2, l_2) \in \chi \times \Pi} \sum_{(t', l') \in \chi \times \Pi} \min \left\{ \left( M_{s_1, t}^{(l_1)} \times P_{t', l'}^{(i)} \right), \left( M_{s_2, t}^{(l_2)} \times P_{t', l'}^{(i)} \right) \right\} \right)
\]

\[
\leq \prod_{p=i}^{j-1} (1 - |\chi_0|M_{\min})
\]

\[
= (1 - |\chi_0|M_{\min})^{j-i-1}.
\]

Thus we prove that,

\[
\bar{\theta}_{i,j} \leq (1 - |\chi_0|M_{\min})^{j-i-1}.
\]

Therefore,

\[
\sum_{j=i+1}^{\infty} \bar{\theta}_{i,j} \leq \frac{1}{1 - |\chi_0|M_{\min}}.
\]

Since \(|\chi_0|M_{\min} < \sum_{t \in \chi_0} M_{s,t}^{(l)}\) for any \(s\) and \(l\), we observe that \(|\chi_0|M_{\min} < \sum_{t \in \chi_0} M_{s,t}^{(l)} \leq 1\). Therefore, \(1 - |\chi_0|M_{\min} > 0\). Choosing \(C_0 = 1/(1 - |\chi_0|M_{\min})\) now completes the proof.

### F.12 Proof of Proposition 26

**Proof.** We prove this fact by induction. Obviously, \((X_0, \alpha_0) \overset{d}{=} (\tilde{X}_0, \tilde{\alpha}_0)\). Now, for some \(i \geq 1\), let
(X_0, a_0, \ldots, X_i, a_i) be identically distributed to \( \tilde{X}_0, \tilde{a}_0, \ldots, \tilde{X}_i, \tilde{a}_i \). Then, for \( i + 1 \), we note that,

\[
\begin{align*}
\mathbb{P}(\tilde{X}_{i+1} = s_{i+1}, \tilde{a}_{i+1} = l_{i+1}, \ldots, \tilde{X}_0 = s_0, \tilde{a}_0 = l_0) \\
= \mathbb{P}(\tilde{a}_{i+1} = l_{i+1}|\tilde{X}_{i+1} = s_{i+1}, \ldots, \tilde{X}_0 = s_0) \\
\times \mathbb{P}(\tilde{X}_{i+1} = s_{i+1}|\tilde{X}_i = s_i, \tilde{a}_i = l_i, \ldots, \tilde{a}_0 = l_0, \tilde{X}_0 = s_0) \\
\times \mathbb{P}(\tilde{X}_i = s_i, \tilde{a}_i = l_i, \ldots, \tilde{X}_0 = s_0, \tilde{a}_0 = l_0) \\
= \mathbb{P}(\alpha_i^{(s_0, l_0, \ldots, s_{i+1})} = l_{i+1}|\tilde{X}_{i+1} = s_{i+1}, \ldots, \tilde{X}_0 = s_0) \\
\times \mathbb{P}(X_{s_i}^{(l_i)} = s_{i+1}|\tilde{X}_i = s_i, \tilde{a}_i = l_i, \ldots, \tilde{a}_0 = l_0, \tilde{X}_0 = s_0) \\
\times \mathbb{P}(\tilde{X}_i = s_i, \tilde{a}_i = l_i, \ldots, \tilde{X}_0 = s_0, \tilde{a}_0 = l_0), \tag{F.19}
\end{align*}
\]

where the equalities follow by substituting in the corresponding value of each quantity. Observe that under the given conditional, such that \( \tilde{N}_{s_i}^{(l_i)} + 1 \) is some fixed integer \( n \). Then, the right-hand side of the previous equation can be further decomposed into,

\[
\begin{align*}
\mathbb{P}(\alpha_i^{(s_0, l_0, \ldots, s_{i+1})} = l_{i+1}|\tilde{X}_{i+1} = s_{i+1}, \ldots, \tilde{X}_0 = s_0) \\
\times \mathbb{P}(X_{s_i}^{(l_i)} = s_{i+1}|\tilde{X}_i = s_i, \tilde{a}_i = l_i, \ldots, \tilde{a}_0 = l_0, \tilde{X}_0 = s_0) \\
\times \mathbb{P}(\tilde{X}_i = s_i, \tilde{a}_i = l_i, \ldots, \tilde{X}_0 = s_0, \tilde{a}_0 = l_0) \\
= \mathbb{P}(\alpha_i^{(s_0, l_0, \ldots, s_{i+1})} = l_{i+1}|\tilde{X}_{i+1} = s_{i+1}, \ldots, \tilde{X}_0 = s_0) \\
\times \mathbb{P}(X_{s_i}^{(l_i)} = s_{i+1}|\tilde{X}_i = s_i, \tilde{a}_i = l_i, \ldots, \tilde{a}_0 = l_0, \tilde{X}_0 = s_0) \\
\times \mathbb{P}(\tilde{X}_i = s_i, \tilde{a}_i = l_i, \ldots, \tilde{X}_0 = s_0, \tilde{a}_0 = l_0) \\
= \mathbb{P}(\alpha_i^{(s_0, l_0, \ldots, s_{i+1})} = l_{i+1}) \times \mathbb{P}(X_{s_i}^{(l_i)} = s_{i+1}) \\
\times \mathbb{P}(\tilde{X}_i = s_i, \tilde{a}_i = l_i, \ldots, \tilde{X}_0 = s_0, \tilde{a}_0 = l_0) \\
= P_{t}^{(s_i, l_i, \ldots, s_0)} M_{s,t}^{(l_i)} \times \mathbb{P}(\tilde{X}_i = s_i, \tilde{a}_i = l_i, \ldots, \tilde{X}_0 = s_0, \tilde{a}_0 = l_0) \\
= P_{t}^{(s_i, l_i, \ldots, s_0)} M_{s,t}^{(l_i)} \times \mathbb{P}(X_i = s_i, a_i = l_i, \ldots, X_0 = s_0, a_0 = l_0),
\end{align*}
\]

where the last equality follows by induction hypothesis. It follows easily from the last equality that

\[
\begin{align*}
\mathbb{P}(\tilde{X}_{i+1} = s_{i+1}, \tilde{a}_{i+1} = l_{i+1}, \ldots, \tilde{X}_0 = s_0, \tilde{a}_0 = l_0) \\
= \mathbb{P}(X_{i+1} = s_{i+1}, a_{i+1} = l_{i+1}, \ldots, X_0 = s_0, a_0 = l_0).
\end{align*}
\]

This completes the proof. \( \square \)
F.13 Proof of Lemma 27

Proof. We introduce the notation $\chi'$ to denote \{(1, 1), \ldots, (d/3, 1), (2, 1), \ldots, (d/3, k)\}. Observe that $T$ can be written as,

$$T := \sum_{\Upsilon=0}^{dk/3-1} U_{\Upsilon} \quad (F.20)$$

where $U_{\Upsilon}$ is the time spent between the $\Upsilon$-th and the $\Upsilon + 1$-th unique state-control pair visited in $\chi'$. Next, we observe two facts. Firstly, observe that for any element $(t, l')$ belonging to $\chi'$ we have

$$P \left( X_i = t, a_i = l' \mid X_{i-1} = s, a_{i-1} = l \right) = P \left( X_i = t, a_i = l' \right) = \frac{3t}{dk}$$

independent of any $(s, l) \in \chi \times \mathbb{I}$. Secondly, observe that the probability of visiting a new state-control pair in $\chi_I$ when $\Upsilon$ unique states have already been visited is $\frac{3t}{dk} \left( \frac{dk}{3} - \Upsilon \right) / dk$. Together, these facts imply that

$$U_{\Upsilon} \overset{d}{=} X_{\Upsilon} \text{ where } X_{\Upsilon} \sim \text{Geometric} \left( \left( \frac{dk}{3} - \Upsilon \right) \frac{3t}{dk} \right). \quad (F.21)$$

It follows from eq. (F.21) that,

$$E[T] = \left( \frac{dk}{3t} \sum_{\Upsilon=0}^{dk/3-1} \frac{1}{\frac{dk}{3} - \Upsilon} \right)$$

where we have dropped the superscript $l$ from $T^{(l)}$ for convenience. Rewriting the previous equation we get,

$$E[T] = \frac{dk}{3t} \sum_{\Upsilon=1}^{dk/3} \frac{1}{\Upsilon} > \frac{dk}{3t} \log \left( \frac{dk}{3} + 1 \right). \quad (F.22)$$

where the last inequality follows from the Euler-Maclaurin (see for example, Apostol [1]) approximation of a sum by its integral. We also observe that,

$$\text{Var}(U_{\Upsilon}) = \frac{d^2k^2}{9t^2} \left( \frac{dk}{3} - \Upsilon \right)^{-2} \left[ 1 - \left( \frac{dk}{3} - \Upsilon \right) \frac{3t}{dk} \right].$$

The term inside the square brackets is a probability, and can be upper bounded by 1. Observe that when $\Upsilon \leq \frac{dk}{3} - 1$ we can upper bound $\text{Var}(T)$ as

$$\text{Var}(T) \leq \sum_{\Upsilon=0}^{dk/3-1} \frac{d^2k^2}{9t^2} \left( \frac{dk}{3} - \Upsilon \right)^{-2} = \sum_{\Upsilon=1}^{dk/3} \frac{d^2k^2}{9t^2} \frac{1}{\Upsilon^2} < \frac{d^2k^2}{9t^2} \frac{\pi^2}{6} < \frac{d^2k^2}{9t^2} \frac{\pi^2}{4}. \quad (F.23)$$
where the second inequality follows from the fact that $\sum_{Y \geq 1} 1/Y^2 = \pi^2/6$. Using Cantelli’s inequality [19, Equation 5], we obtain, for all $0 < \theta < \mathbb{E}[T]/\sqrt{\text{Var}(T)}$,

$$
\mathbb{P}\left(T > \frac{dk}{3\ell} \log\left(\frac{dk}{3} + 1\right) - \theta \frac{dk \pi}{3\ell} \frac{1}{2}\right) \geq \frac{\theta^2}{1 + \theta^2}.
$$

From the equations F.22 and F.23, we get that $\mathbb{E}[T]/\left(\sqrt{\text{Var}(T)}\right) > (\log(dk/3) + 1)/\pi$. Substituting $\theta = (\log(dk/3) + 1)/\pi$ we get

$$
\mathbb{P}\left(T > \frac{dk}{6\ell} \left(\log\left(\frac{dk}{3}\right) + 1\right)\right) \geq \frac{1}{1 + \left(\frac{\pi}{\log(dk/3) + 1}\right)^2} > \frac{1}{1 + \pi^2}.
$$

This proves the lemma. \Box