Two-Dimensional Legendre Wavelets and Their Applications to Integral Equations

Masood Roodaki*, Zahra JafariBehbahani
Department of Mathematics, Marvdasht Branch, Islamic Azad University, Marvdasht, Iran;
roodaki_1436@yahoo.com

Abstract
The main focus of this paper is to present an effective numerical method for solving two-dimensional Fredholm integral equations, which appear in many phenomena in physics and engineering. For this purpose, a new set of two-dimensional wavelets is constructed by Legendre polynomials. The properties of the novelty wavelets are studied. The suggested method reduces a two-dimensional Fredholm integral equation to an algebraic equations system. Furthermore, to illustrate uniform convergence and accuracy of these wavelets, some theorems are proved. The method is applied on some examples to confirm its accuracy and computational efficiency.

Keywords: Direct Method, Legendre Polynomials, Legendre Wavelets, Two-Dimensional Fredholm Integral Equations, Operational Matrix

1. Introduction
Two-dimensional Fredholm integral equations are appeared in mathematical modeling of various phenomena in physics and engineering. For example, the transport equation, that arises in mathematical modeling of fluid flows, biology, etc., can be formulated as a two-dimensional Fredholm integral equation. This type of integral equations are also developed and used for electromagnetic analysis, specifically for antennas and radar scattering.

This paper considers a linear case of these equations of the form

\[ f(s,t) = g(s,t) + \int_0^{T_2} \int_0^{T_1} k(s,t,x,y) f(x,y) dx dy, \quad (x,y) \in D, \]  

(1)

where \( f(s,t) \) is an unknown function defined on \( D=[0,T_1] \times [0,T_2] \), and the functions \( k(s,t,x,y) \) and \( g(s,t) \) are given functions defined on \( S=\{(s,t,x,y) : 0 \leq x \leq s \leq T_1, 0 \leq y \leq t \leq T_2 \} \), and \( D \), respectively. Moreover, without any loss of generality, we suppose that \( [0,T_1]=[0,T_2]=[0,1] \) because any finite interval \( [0,T] \) can be transformed to \( [0,1] \) by a linear map.

Usually, evaluation of the exact solution of integral equations by analytical methods may be difficult, so the numerical methods has a great appeal for mathematicians.

In comparison of one-dimensional integral equations, few numerical methods are known for approximating solution of equation (1). Hanson et al. proposed numerical solution of two-dimensional integral equations using linear elements. Guoqiang et al. used extrapolation for computing Nyström solution of two-dimensional nonlinear Fredholm integral equations. Gaussian radial basis functions was used for (1) by Alipanah et al. Some numerical method based on piecewise polynomial interpolation was presented. Recently, the direct approaches for estimating numerical solution of two-dimensional Volterra-Fredholm and mixed integral equations were proposed using triangular orthogonal functions.

In this paper, a new set of two-dimensional wavelets are constructed by Legendre polynomials, that we call them two-dimensional Legendre Wavelets (2D-LWs). Using 2D-LWs in a direct approach, allows us to purpose a computational method for solving 2D-Fredholm integral equations, numerically. Our method reduces equation (1) to a system of mere algebraic equations in a
The function \( f(s) \in L^2[0, 1] \) can be approximated by Legendre wavelets as
\[
 f(s) = \sum_{n=1}^{2^k-1} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(s) = C^T \Psi(s),
\]
in which \( C \) and \( \Psi(s) \) are \( 2^k \times M \) vectors of the form
\[
 C^T = [c_{1,0}, c_{1,1}, \ldots, c_{1,M-1}, c_{2,0}, \ldots, c_{2,M-1}, \ldots, c_{k-1,0}, \ldots, c_{k-1,M-1}],
\]
and
\[
 \Psi(s) = \left[ \psi_{1,0}(s), \psi_{1,1}(s), \ldots, \psi_{1,M-1}(s), \psi_{2,0}(s), \ldots, \psi_{2,M-1}(s), \ldots, \psi_{k-1,0}(s), \psi_{k-1,M-1}(s) \right]^T.
\]

### 3. Two-Dimensional Legendre Wavelets

The two-dimensional Legendre wavelets over the region \([0, 1) \times [0, 1)\) can be defined as follows
\[
 \psi_{n,m,s}(t) = \left| a_0 \right|^{\frac{k}{2}} \psi(a_0^n t - n b_0).
\]

The one-dimensional Legendre wavelets over the interval \([0, 1]\) are defined as
\[
 \psi_{n,m}(s) = \begin{cases} 
 (m+\frac{1}{2})^2 L_m(2^k s - 2n+1), & \text{if } \frac{n-1}{2^{k-1}} \leq s < \frac{n}{2^{k-1}}, \\
 0, & \text{otherwise},
\end{cases}
\]
where \( n = 1, 2, \ldots, 2^k - 1 \) and \( m = 0, 1, 2, \ldots, M - 1 \). In above definition, the polynomials \( L_m \) are Legendre polynomials of degree \( m \) over the interval \([-1, 1]\), which can be defined as follows
\[
 L_0(s) = 1, \\
 L_1(s) = s, \\
 L_{m+1}(s) = \frac{2m+1}{m+1} L_m(s) - \frac{m}{m+1} L_{m-1}(s), \quad m = 1, 2, 3, \ldots.
\]

The set of \( \{ L_m(s) : m = 0, 1, \ldots \} \) in the Hilbert space \( L^2[-1, 1] \) is a complete orthogonal set. Orthogonality of Legendre polynomials on the interval \([-1, 1]\) implies that
\[
 < L_m(s), L_{m'}(s) > = \int_{-1}^{1} L_m(s) L_{m'}(s) ds = \begin{cases} 
 \frac{2}{2m+1}, & m = m', \\
 0, & m \neq m',
\end{cases}
\]
for \( m, m' = 0, 1, \ldots, 14 \).

Furthermore, the set of wavelets \( \psi_{n,m}(s) \) makes an orthonormal basis set in \( L^2[0, 1] \), that is
\[
 \int_{0}^{1} \psi_{n,m}(s) \psi_{n',m'}(s) ds = \delta_{n,n'} \delta_{m,m'},
\]
in which \( \delta \) denotes the Kronecker delta function.

By above definition, the region \([0, 1) \times [0, 1)\) is divided to \( (2^k-1) \times (2^k-1) \) subregions. The parameters \( M \) and \( M' \) denote the number of Legendre polynomials considered for variables \( s \) and \( t \), respectively. So, \( M \times M' \) wavelets constructed on each of subregions.
Paraphrase, by considering \{\psi_{n,s}(s)\} and \{\psi_{n',t}(t)\} as two sets of one-dimensional Legendre wavelets over variables \(s\) and \(t\), respectively, the two-dimensional Legendre wavelets over the region \([0,1) \times [0,1)\), may be written as

\[
\psi_{n,m,n',m'}(s,t) = \psi_{n,s}(s) \cdot \psi_{n',t}(t).
\]

If

\[
\Psi_{n',n}(s,t) = \begin{bmatrix}
\psi_{n,0,n',0} & \psi_{n,0,n',1} & \cdots & \psi_{n,0,n',m' - 1} \\
\psi_{n,1,n',0} & \psi_{n,1,n',1} & \cdots & \psi_{n,1,n',m' - 1} \\
\vdots & \vdots & \ddots & \vdots \\
\psi_{n,m-1,n',0} & \psi_{n,m-1,n',1} & \cdots & \psi_{n,m-1,n',m' - 1}
\end{bmatrix},
\]

(2)

be an \(MM'\)-vector of 2D-LWs defined on \((m')\)-th sub-region, then

\[
\Psi(s,t) = \begin{bmatrix}
\Psi^T_{1,1} & \Psi^T_{1,2} & \cdots & \Psi^T_{1,n'} \\
\Psi^T_{2,1} & \Psi^T_{2,2} & \cdots & \Psi^T_{2,n'} \\
\vdots & \vdots & \ddots & \vdots \\
\Psi^T_{n,1} & \Psi^T_{n,2} & \cdots & \Psi^T_{n,n'}
\end{bmatrix},
\]

(3)

is a \(2^{k-1}2^{k-1}\)-vector concluding whole 2D-LWs. In equations (2) and (3), the term \((s, t)\) is canceled for convenience. It is simple to verify that the function \(\psi_{n,m,n',m'}(s,t)\) is attached in \(k\)-th component of vector \(\Psi\) where

\[
k = \left\lfloor \left( (n-1)2^{k-1} + (n' - 1) \right) M + m \right\rfloor M' + (m' + 1).
\]

The set of 2D-LWs is an orthonormal set over the region \([0,1) \times [0,1)\), that is

\[
\int_0^1 \int_0^1 \psi_{n,m,n',m'}(s,t) \psi_{n,m,n',m'}(s,t) ds dt = \delta_{n,n'} \delta_{m,m'} \delta_{n,n'} \delta_{m,m'}.
\]

### 3.1 Function Expansion

Any function \(f(s,t)\) in \(L^2([0,1) \times [0,1)\)) has an truncated expansion with respect to 2D-LWs as

\[
f(s,t) = \sum_{n=1}^{\infty} \sum_{n'=1}^{\infty} \sum_{n,m,m' = 0}^{M-1} \sum_{n,m,m' = 0}^{M'-1} \epsilon_{n,m,n',m'} \psi_{n,m,n',m'}(s,t)
\]

\[
= C^T \cdot \Psi(s,t),
\]

where \(\Psi(s,t)\) defined in (3) and the \(2^{k-1}2^{k-1}\)-vector \(C\) contains the coefficients \(\epsilon_{n,m,n',m'}\) that is defined as

\[
C_k = \epsilon_{n,m,n',m'} = \frac{< \psi_{n,m,n',m'}(s), < f(s,t), \psi_{n',m'}(t) > >}{< \psi_{n,m,n',m'}(s), \psi_{n,m,n',m'}(s) >}.
\]

in which \(<.,.>\) denotes the inner product. Since

\[
< \psi_{n,m,n',m'}(s), \psi_{n,m,n',m'}(s) > = \int_0^1 \int_0^1 \psi_{n,m,n',m'}(s) \psi_{n,m,n',m'}(s) ds dt = 1,
\]

\[
< \psi_{n,m,n',m'}(s), \psi_{n',m',m',m'}(t) > = \int_0^1 \int_0^1 \psi_{n,m,n',m'}(s) \psi_{n',m',m',m'}(t) dt ds = 1,
\]

we have

\[
C_k = \frac{1}{2^{k-1}} \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 f(s,t) \psi_{n',m',m',m'}(t) dt ds,
\]

(5)

for \(k = 1,2,\cdots,2^{k-1}2^{k-1}\).

Furthermore, a function \(k(s,t,x,y)\) can be similarly expanded with respect to 2D-LWs so that

\[
k(s,t,x,y) = \Psi^T(s,t)K \cdot \Psi(x,y),
\]

where \(K\) is the \((2^{k-1}2^{k-1}) MM' \times (2^{k-1}2^{k-1}) MM'\) coefficient matrix with components as

\[
K_{ij} = \frac{1}{2^{k-1}} \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 k(s,t,x,y) \psi_{n,m}(x) \psi_{n',m'}(y) dx dy ds dt,
\]

in which

\[
i = \left\lfloor \left( (n-1)2^{k-1} + (n' - 1) \right) M + m \right\rfloor M' + (m' + 1),
\]

\[
j = \left\lfloor \left( (n-1)2^{k-1} + (n' - 1) \right) M + m \right\rfloor M' + (m' + 1),
\]

for \(n, n' = 1,2,\cdots,2^{k-1}, m, m' = 0,1,\cdots, M - 1\) and \(m', m' = 0,1,\cdots, M' - 1\).
3.2 Integration of Product Vectors

It can be clearly concluded from equations (2) and (3) and disjointness property of \( \psi_{n,m,n',m'}(s,t) \) that

\[
\Psi(s,t) \cdot \Psi^T(s,t) = \begin{bmatrix}
\Psi_{1,1} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \Psi_{k-1,k-1} \\
\end{bmatrix}
\]

where \( \Psi_{1,1}, \Psi_{1,2}, \ldots, \Psi_{k-1,k-1} \) are \( 2^{k-1} \times 2^{k-1} \) matrices.

So,

\[
\int_0^1 \int_0^1 \Psi(s,t) \cdot \Psi^T(s,t) \, ds \, dt = I,
\]

where \( I \) is a \( (2^{k-1} \times 2^{k-1}) \times (2^{k-1} \times 2^{k-1}) \)-identity matrix.

4. Convergence Analysis

In this section, we demonstrate that the expansion of any continuous function \( f(s,t) \in [0,1] \times [0,1] \) with respect to 2D-LWs, as presented in equation (4), converges uniformly to \( f \).

4.1 Theorem 1

Let \( f(s,t) \) be a continuous function on \([0,1] \times [0,1]\) and

\[
\frac{\partial^4 f(s,t)}{\partial s^2 \partial t^2} \leq W,
\]

then the coefficients of function expansion in equation (4) satisfy the following relation

\[
|c_{n,m,n',m'}| \leq 3W \frac{2^{k+k'}}{2m+1} \frac{2m'}{2m' - 2}.
\]

**Proof:**

Put \( C_k = |c_{n,m,n',m'}| \), for convenience. From equation (5) we have

\[
C_k = 2^2 \left\lfloor \frac{k'}{2} \right\rfloor \frac{1}{2m+1} \frac{1}{2m'-1} \frac{2m' + 1}{2m' + 3} \frac{2^{k+k'}}{2m+1} \frac{2m'}{2m' - 2} \int_{\frac{n_2}{2^{k-1}}}^{\frac{n_2}{2^{k-1}}} f(s,t) L_{m'}(2^{k'} t - 2n' + 1) dt ds.
\]

Now, let

\[
A(s) = \int_{\frac{n_2}{2^{k-1}}}^{\frac{n_2}{2^{k-1}}} f(s,t) L_{m'}(2^{k'} t - 2n' + 1) dt.
\]

If \( n_2 = 2n' - 1 \) and \( \eta = 2^{k'} t - n_2 \), then

\[
A(s) = 2^{k'} \int_{\frac{n_2}{2^{k-1}}}^{\frac{n_2}{2^{k-1}}} f(s,n) L_{m'}(\eta) d\eta.
\]

The Legendre polynomials satisfy in

\[
L_0(t) = \frac{d}{dt} L_1(t),
L_m(t) = \frac{1}{2m+1} \frac{d}{dt} (L_{m+1}(t) - L_{m-1}(t)), \quad m \geq 1.
\]

Applying two times integration by parts in equation (9), we get

\[
A(s) = \frac{-2^{k'}}{2m'+1} \int_{\frac{n_2}{2^{k-1}}}^{\frac{n_2}{2^{k-1}}} \frac{\partial^2 f(s,n)}{\partial \eta^2} (\frac{n_2}{2^{k-1}} - L_{m+1}(\eta) - L_{m-1}(\eta)) d\eta
\]

\[
= \frac{2^{k'}}{2m'+1} \int_{\frac{n_2}{2^{k-1}}}^{\frac{n_2}{2^{k-1}}} \frac{\partial^2 f(s,n)}{\partial \eta^2} (\frac{2^{k'} \eta}{2^{k-1}} - L_{m+1}(\eta) - L_{m-1}(\eta)) d\eta
\]

\[
= \frac{2^{k'}}{2m'+1} \int_{\frac{n_2}{2^{k-1}}}^{\frac{n_2}{2^{k-1}}} \frac{\partial^2 f(s,n)}{\partial \eta^2} (\frac{2^{k'} \eta}{2^{k-1}} - 2m'+3)(2m'+1)(2m'-1) d\eta
\]

in which \( P_{m'}(\eta) = (2m'-1)L_{m+2}(\eta) - (4m'+2)L_m(\eta) + (2m'+3)L_{m'-2}(\eta) \).

We can use the above procedure for integration with respect to \( s \) in equation (8), similarly.

Let \( n = 2n - 1 \) and \( s = \frac{2^{k'} \eta}{2^{k-1}} - \frac{n_2}{2^{k-1}} \), then

\[
C_k = \frac{1}{2m+1} \frac{1}{2m'-1} \frac{1}{2m'+3} \frac{1}{2m'+1} \frac{2m'}{2m' - 2} \int_{\frac{n_2}{2^{k-1}}}^{\frac{n_2}{2^{k-1}}} \frac{\partial^2 f(s,n)}{\partial \eta^2} (\frac{2^{k'} \eta}{2^{k-1}} - \frac{n_2}{2^{k-1}}) P_{m'}(\eta) d\eta ds.
\]
in which \( P_m(\xi) = (2m-1)L_{m+2}(\xi) - (4m+2)L_m(\xi) + (2m+3)L_{m-2}(\xi) \). On the other hand, orthogonality of Legendre polynomials implies that
\[
\int_{-1}^{1} |P_m(\xi)| d\xi \leq \frac{\sqrt{12}(2m+3)}{\sqrt{2m-3}},
\]
and
\[
\int_{-1}^{1} |P_m'(\eta)| d\eta \leq \frac{\sqrt{12}(2m'+3)}{\sqrt{2m'-3}}.
\]

Therefore we obtain
\[
|C_k| \leq 3W \frac{2^{-k}}{(2m-1)\sqrt{2m+1}\sqrt{2m-3}} \frac{2^{-k'}}{(2m'-1)\sqrt{2m'+1}\sqrt{2m'-3}} \leq 3W \frac{2^{-k+k'}}{(2m-3)^2(2m'-3)^2}.
\]

Now, we prove the uniform convergence and accuracy estimation of the function expansion with respect to 2D-LWs.

### 4.2 Theorem 2

Let \( f(s, t) \) be a continuous function defined on \([0, 1) \times [0, 1)\), with \( \frac{\partial^3 f(s,t)}{ds^3 dt^3} \leq M \), and \( \tilde{f}(s, t) \) be its truncated expansion of the form
\[
\tilde{f}(s,t) = \sum_{n=1}^{\infty} \sum_{n'=1}^{\infty} \sum_{m=0}^{n-1} \sum_{m'=0}^{n'-1} c_{n,m,n',m'} \psi_{n,m,n',m'}(s,t).
\]

Then \( \tilde{f}(s,t) \) converges uniformly to \( f(s, t) \), and
\[
\left\| f(s,t) - \tilde{f}(s,t) \right\| \leq 3W \frac{2^{-k+k'}}{(2m-3)^2(2m'-3)^2} \left( \frac{1}{(2m-3)^4(2m'-3)^4} \right)^{1/2}.
\]

**Proof:**

For convenience, let \( C_k = c_{n,m,n',m'} \psi_{k}(s, t) = \psi_{n,m,n',m'}(s,t) \), and \( \delta_k = \left\| f(s,t) - \tilde{f}(s,t) \right\| \). Then
\[
\delta_k = \int_0^1 \left( f(s,t) - \sum_{n=1}^{\infty} \sum_{n'=1}^{\infty} \sum_{m=0}^{n-1} \sum_{m'=0}^{n'-1} C_k \psi_k(s,t) \right)^2 ds dt.
\]

Orthogonal property of 2D-LWs implies that
\[
\delta_k^2 = \int_0^1 \int_0^1 \left( f(s,t) - \sum_{n=1}^{\infty} \sum_{n'=1}^{\infty} \sum_{m=0}^{n-1} \sum_{m'=0}^{n'-1} C_k \psi_k(s,t) \right)^2 ds dt.
\]

It can be concluded from equation (7) that
\[
\delta_k^2 \leq \frac{9W^2}{2^{k+k'}} \sum_{n=2^{k+1}+1}^{\infty} \sum_{n'=2^{k+1}+1}^{\infty} \sum_{m=0}^{n-1} \sum_{m'=0}^{n'-1} \frac{1}{(2m-3)^4(2m'-3)^4}.
\]

Hence
\[
\delta_k \leq \frac{3W}{\sqrt{2^{k+k'}}} \left( \sum_{n=2^{k+1}+1}^{\infty} \sum_{n'=2^{k+1}+1}^{\infty} \sum_{m=0}^{n-1} \sum_{m'=0}^{n'-1} \frac{1}{(2m-3)^4(2m'-3)^4} \right)^{1/2}.
\]

So \( \tilde{f}(s,t) \) converges to \( f(s, t) \) uniformly.

### 5. Problem Statement

The results obtained in section 3 are applied to present an effective method to solve two-dimensional Fredholm integral equations (2D-FIE), numerically.

Consider the following 2D-FIE
\[
f(s,t) = g(s,t) + \int_0^1 k(s,t,x,y) f(x,y) dx dy,
\]
where \( k \) and \( g \) are known but \( f \) is not. Approximating functions \( f, g \) and \( k \) with respect to 2D-LWs, gives
\[
f(s,t) = C^T \cdot \Psi(s,t) = \Psi^T (s,t) \cdot C,
\]
\[
g(s,t) = G^T \cdot \Psi(s,t) = \Psi^T (s,t) \cdot G,
\]
\[
k(s,t,x,y) = \Psi^T (s,t) \cdot K \cdot \Psi(x,y),
\]
where \( \Psi(s,t) \) is defined in (3), and \( 2^{k+1}1^{k+1}MM' \)-vectors \( C \) and \( G \) are 2D-LWs coefficients of \( f(s,t) \) and \( g(s,t) \),
Two-Dimensional Legendre Wavelets and Their Applications to Integral Equations

respectively. Also the \((2^{k-1}2^{k-1} MM' \times 2^{k-1}2^{k-1} MM')\)-matrix \(K\) is 2D-LWs coefficients of \(k(s, t, x, y)\).

The integral part of equation (10), using (11) and (6), can be approximated as

\[
\int_0^1 \int_0^1 k(s, t, x, y)f(x, y)dxdy = \int_0^1 \int_0^1 \Psi^T(s, t) \cdot K \cdot \Psi(x, y) \cdot \Psi^T(x, y) \cdot C dxdy
\]

\[
= \Psi^T(s, t) \cdot K \cdot \int_0^1 \int_0^1 \Psi(x, y) \cdot \Psi^T(x, y) dxdy \cdot C
\]

\[
= \Psi^T(s, t) \cdot K \cdot C.
\]

Substituting (11) and (12) in (10), and replacing \(\approx\) with \(=\), follows

\[
\Psi^T(s, t) \cdot C = \Psi^T(s, t) \cdot G + \Psi^T(s, t) \cdot K \cdot C,
\]

or

\[
(I - K)C = G. \quad (13)
\]

Equation (13) is a linear system of \(2^{k-1}2^{k-1} MM'\) algebraic equations. The \(2^{k-1}2^{k-1} MM'\) components of \(C\) are unknown and can be obtained by solving this system. Hence, an approximate solution

\[
f(s, t) = C^T \cdot \Psi(s, t),
\]

can be computed for equation (10).

6. Illustrative Examples

In this section we implemented the proposed method on some examples. In first example, the matrix \(K\) and vector \(G\) are computed in details. The accuracy of our method is studied in second example by computing \(e_{\text{total}}\) of

\[
e_{\text{total}} = \left( \sum_{n=1}^{M} \sum_{n'=1}^{M'} \left( f_{n,n'}(s, t) - \widetilde{f}_{n,n'}(s, t) \right)^2 \right)^{\frac{1}{2}},
\]

and listed in this table.

Furthermore the results are compared with the exact solutions by calculating the following error function

\[
e_{\text{grid}}(s, t) = |f(s, t) - \widetilde{f}_{n,n'}(s, t)|, \quad \frac{n-1}{2^{k-1}} \leq s < \frac{n}{2^{k-1}}, \quad \frac{n'-1}{2^{k-1}} \leq t < \frac{n'}{2^{k-1}}.
\]

The values of \(e_{\text{grid}}(s, t)\) over the set

\[
D_{\text{grid}} = \{(0,0,0,0),(0,1,0,1),(0,2,0,2),\ldots,(1,0,1,0)\},
\]

are computed for different values of \(M, M', k\) and \(k'\), and demonstrated in Table 2 and Figure 1.

The computations associated with the examples were performed using Matlab 7.0 software on a personal computer.

| \((n, n')\) | \(e_{n,n'}\) |
|------------|-------------|
| \((1,1)\) | 1.74924 e -03 |
| \((1,2)\) | 2.21404 e -04 |

Table 1. The numerical results for example 2 with \(M = M' = 3\)
The computations associated with the examples were performed using Matlab 7.0 software on a personal computer.

Figure 1. The error function graph for examples 2 (up) and 3 (down) with $M = M' = 3$ and $k = k' = 3$.

**Example 1.**

Consider the linear Fredholm integral equation as follows, 

$$ f(s,t) = g(s,t) + \int_{0}^{t} \left[ 4(s+2st-4xy+3y) f(x,y) \right] dy dx, \quad 0 \leq s, t \leq 1, \quad (14) $$

where 

$$ g(s,t) = \begin{cases} 
\frac{1}{12} s^2 + \frac{1}{4} st - \frac{1}{4} s - \frac{34}{9}, & 0 \leq s \leq \frac{1}{2}, \quad 0 \leq t \leq 1, \\
\frac{3}{2} s^2 + 2st - \frac{9}{4} s - \frac{43}{9}, & \frac{1}{2} \leq s \leq 1, \quad 0 \leq t \leq 1, \\
0, & \text{otherwise,} 
\end{cases} $$

with the exact solution

$$ f(s,t) = \begin{cases} 
-3st + 4t + s - 1, & 0 \leq s < \frac{1}{2}, \quad 0 \leq t \leq 1, \\
4st + 2t - s - 2, & \frac{1}{2} \leq s \leq 1, \quad 0 \leq t \leq 1, \\
0, & \text{otherwise.} 
\end{cases} $$

Choosing $k = 2$, $k' = 1$ and $M = M' = 2$, we have

$$ K = \begin{bmatrix}
3 & 1 & -1 & -1 & 3 & 0 & -1 & -1 & 3 \\
\frac{1}{12} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\frac{1}{6} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
5 & -1 & -1 & -1 & 5 & 0 & -1 & -1 & 5 \\
\frac{1}{4} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\frac{1}{6} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\end{bmatrix}, $$

and

$$ G = \begin{bmatrix}
-91\sqrt{2} & 7\sqrt{6} & -\sqrt{6} & -11\sqrt{2} & -35\sqrt{2} & 25\sqrt{6} & -\sqrt{6} & \sqrt{2} \\
72 & 32 & 8 & 96 & 144 & 96 & 16 & 32 \\
\end{bmatrix}^T. $$

Solving linear system (13), the unknown vector $C$ is obtained as

$$ C = \begin{bmatrix}
7\sqrt{2} & 13\sqrt{6} & -\sqrt{6} & -\sqrt{2} & -\sqrt{2} & 10\sqrt{6} & \sqrt{6} & \sqrt{2} \\
16 & 48 & 48 & 16 & 24 & 24 & 12 & 12 \\
\end{bmatrix}^T, $$

which confirms that the proposed method gives the analytical solution of equation (14).

**Example 2.**

For the following Fredholm integral equation.

$$ f(s,t) = g(s,t) + 4\int_{0}^{t} e^{(s+st+x+y)} f(x,y) dy dx, \quad 0 \leq s, t \leq 1, $$

where $g(s,t) = 2e^{s+2t} - e^{s+t}$, with the exact solution $f(s,t) = e^{s+t}$, the values of $e_{total}(s,t)$ and $e_{num}(s,t)$ for $M = M' = 3$ and various $k$ and $k'$ are shown in table 1.
Example 3.
Consider the following Fredholm integral equation.

\[ f(s,t) = g(s,t) - \int_{0}^{1} \int_{0}^{1} \frac{8}{3} \cos(s + t + x + y) f(x,y) \, dy \, dx, \quad 0 \leq s, t \leq 1, \]

where \((s, t) \in [0,1] \times [0,1]\) and \(g(s,t) = \frac{1}{3} \cos(s + t + 4) - \frac{2}{3} \cos(s + t + 2)\), with the exact solution \(f(s,t) = \cos(s + t)\).

The values of \(e_{\text{grid}}(s,t)\) over the set \(D_{\text{grid}}\) are shown in table 2 for three cases.

7. Comments on the Results
In this approach, applying the Legendre wavelets, a 2D Fredholm integral equation can be reduced to a system of algebraic equations. Since equation (13) is set up in a simple manner, the suggested method can be used easily in practical cases.

The accuracy and applicability of method is checked on some examples and the following advantages are obtained:

Example 1 shows that the exact solution of the integral equation can be computed by the method with suitable choice of \(M\) and \(M'\), when the kernel and the known term is selected by polynomials.

In example 3, three choices of \(M, M', k\) and \(k'\) are considered in cases (I), (II) and (III). In comparison of the results, it is believed that increasing order of sub-regions is better than increasing order of polynomials. Notice to the fact that the dimensions of algebraic systems in case (II) and (III) are equal, illustrates the efficiency of our method.

8. Acknowledgement
This research was supported, in part, by a grant from Marvdasht branch, Islamic Azad University.

9. References
1. Kadem A, Baleanu D. Two-dimensional transport equation as Fredholm integral equation. Comm Nonlinear Sci Numer Simulat. 2012; 17(2):530–5.
2. Volakis JL, Sertel K. Integral equation methods for electromagnetic. Scitech Publishing; 2012.
3. Atkinson KE. The Numerical Solutions of Integral Equations of the Second Kind. Cambridge: Cambridge University Press; 1997.
4. Delves LM, Mohammed JL. Computational Methods for Integral Equations. Cambridge: Cambridge University Press; 1985.
5. Hanson R, Phillips J. Numerical solution of two-dimensional integral equations using linear elements. SIAM J Numer Anal. 1978; 15(1):113–21.
6. Guoqiang H, Jiong W. Extrapolation of Nystrom solution for two dimensional nonlinear Fredholm integral equations. J Comput Appl Math. 2001; 134(1–2):259–68.
7. Alipanah A, Esmaeili Sh. Numerical solution of the two-dimensional Fredholm integral equations using Gaussian radial basis function. J Comput Appl Math. 2011; 235(18):5342–7.
8. Xie WJ, Lin FR. A fast numerical solution method for two dimensional Fredholm integral equations of the second kind. Appl Numer Math. 2009; 59(7):1709–19.
9. Liang F, Lin FR. A fast numerical solution method for two dimensional Fredholm integral equations of the second kind based on piecewise polynomial interpolation. Appl Numer Math. 2010; 216(10):3073–88.
10. Babolian E, Maleknejad K, Roodaki M, Almasieh H. Two-dimensional triangular functions and its applications to nonlinear 2D Volterra-Fredholm integral equations. Comput Math Appl. 2010; 60 (6):1711–22.
11. Maleknejad K, Jafaribehbahani Z. Applications of two-dimensional triangular functions for solving nonlinear class of mixed Volterra-Fredholm integral equations. Math Comput Model. 2012; 55(5–6):1833–44.
12. Gu JS, Jiang WS. The Haar wavelets operational matrix of integration. Int J Syst Sci. 1996; 27 (7):623–8.
13. Razzaghi M, Yousefi S. Legendre wavelets direct method for variational problems. Math Comput Simulat. 2000; 53 (3):185–92.
14. Chihara TS. An introduction to orthogonal polynomials. New York: Gordon and Breach Science Publisher Inc.; 1978.
15. Sohrabi S. Study on convergence of hybrid functions method for solution of nonlinear integral equations. Appl Anal. 2011; iFirst:1–13.