BINARY REPRESENTATION OF COORDINATE AND MOMENTUM IN QUANTUM MECHANICS

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To simulate a quantum system with continuous degrees of freedom on a quantum computer based on qubits, it is necessary to reduce continuous observables (primarily coordinates and momenta) to binary observables. We consider this problem based on expanding quantum observables in series in powers of two, analogous to the binary representation of real numbers. The coefficients of the series (“digits”) are therefore orthogonal projectors. We investigate the corresponding quantum mechanical operators and the relations between them and show that the binary expansion of quantum observables automatically leads to renormalization of some divergent integrals and series (giving them finite values).

Keywords: quantum computing, qubit, binary expansion, renormalization

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1. Introduction

When future applications of quantum computers are discussed, attention is often focused primarily on cryptanalysis tasks. Nevertheless, by the time quantum computers become sufficiently powerful, post-quantum cryptography, which is resistant to cryptanalysis on a quantum computer, will be universally introduced [1]. In this regard, we can expect that the main use of quantum computers will turn out to be “peaceful” (unrelated to breaking of ciphers), in particular, much attention will be paid to modeling quantum systems in accordance with Feynman’s original idea [2].

Modeling quantum systems is a very relevant task from the standpoint of practical applications such as quantum chemistry, the creation of new materials, quantum biophysics, the development of new drugs, nuclear physics, and elementary particle physics. Many of these applications require discretizing continuous quantum observables, coordinates and momenta, which should be described in a quantum computer using a set of discrete quantum cells (qubits and/or qutrits and other qudits). Of course, momentum operators are differential operators, and we can use well-known difference schemes from computational mathematics to discretize them (see, e.g., [3], [4]). But such a direct approach disregards the specifics of quantum mechanics, in which the momentum operator is a generator of shifts along the corresponding coordinate.

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Here, we construct a representation of continuous quantum observables based on an expansion in powers of two, analogous to the binary representation of real numbers. An individual binary digit of the expansion of the observable is itself observable and is described by a single quantum bit. We construct the operators of binary digits of the coordinate and momentum on a lattice and on the line in an explicit form, for which we obtain commutation relations. We construct a binary integral representation of real numbers and quantum observables. In the representation of observables in the form of binary series and integrals, we naturally assign finite values to some formally divergent expressions, i.e., we introduce a renormalization. This renormalization method is applicable to finite quantities (which is related to the ambiguity in choosing representatives of the space $\mathbb{Z}_N$), and this allows calculating the renormalized quantities on the lattice.

In Sec. 2, we introduce coordinates and momenta on a finite one-dimensional lattice of size $2^n$ and the momentum operator as a coordinate shift generator. In Sec. 3, we represent the coordinate and momentum operators (on both the lattice and line) as series in powers of two and construct the expansion coefficients (the operators of binary digits) explicitly. For a number on the line, we construct an integral representation by analogy with a binary decomposition and study the resulting renormalization procedures (assignment of a finite value to series and integrals). In Sec. 4, we investigate the commutation relations between the digits of the coordinate and the momentum. Section 5 contains the conclusion. At the end of the paper, three appendices contain the details of calculating the explicit form of the digits of the coordinates and momentum, the calculations by which the integral binary representation of numbers on a line can be verified, and examples of matrices of digits of the coordinates and momentum for lattices of size $2^1$, $2^2$, and $2^3$.

2. Coordinates and momenta on a finite lattice

Here and hereafter, we use the coordinate representation (unless otherwise stated) and assume $\hbar = 1$ and $\hbar = 1/2\pi$, $1/\hbar = 2\pi$ for the Planck constant.

2.1. Coordinates and momenta on a finite lattice. We assume that the coordinate is described by $n$ digit-qubits and the coordinate lattice consists of $N = 2^n$ nodes, which we assume to be cyclic (after the last one comes the first). If we suppose that the coordinate lattice constant is $\Delta x = 2^{-n-}$, then the lattice period is equal to $\Xi = N\Delta x = 2^{n+}$, $n_+ = n - n_-$. We assume that the values $x$ range from 0 to $\Delta x \cdot (N - 1)$.

On the coordinate lattice, a natural addition operation is induced from $\mathbb{Z}_N$, for which $x = x + \Xi$. It is possible to use other representations of the lattice $\Delta x \cdot \mathbb{Z}_N$ by real numbers. For example, taking the equivalence of $x$ and $x + \Xi$ into account, we further need a representation in which $x$ ranges from $-\Delta x \cdot N/2$ to $+\Delta x \cdot (N/2 - 1)$. The series for the coordinate on a finite lattice is finite:

$$x = \sum_{s=-n_-}^{n_+-1} x_s 2^s = \sum_{s=-n_-}^{n_+-1} c(s, x) 2^s. \quad (1)$$

Here, $x_s = c(s, x)$ is the $s$th digit in the binary expansion of $x$. We sometimes specify a range of powers of two that defines a lattice and write $x_s = c_{n-n_+}(s, x)$.

We introduce the coordinate basis $\{|x\rangle\}_{x \in \Delta x \cdot \mathbb{Z}_N}$ for functions defined on the lattice:

$$\hat{x}|x\rangle = x|x\rangle, \quad \langle x'|x''\rangle = \delta_{x',x''}, \quad \psi(x) = \langle x|\psi\rangle, \quad x \in \Delta x \cdot \mathbb{Z}_N. \quad (2)$$

We represent wave functions (ket vectors) in the form of columns whose rows are ordered in decreasing
order of $x$. Thus, if $x$ varies from 0 to $(N - 1)\Delta x$, then

$$\psi(x) = \begin{pmatrix} \psi((N - 1)\Delta x) \\ \psi((N - 2)\Delta x) \\ \vdots \\ \psi(\Delta x) \\ \psi(0) \end{pmatrix}. \quad (3)$$

### 2.2. Momentum lattice

We define the momentum operator $\hat{p}$ as the generator of the shift $\hat{T}_A$ along the coordinate lattice:

$$\hat{T}_A \psi(x) = \psi(x + A), \quad \hat{T}_A = e^{2\pi i A\hat{p}}, \quad A \in \Delta x \cdot \mathbb{Z}. \quad (4)$$

Such operators were considered in Weyl's classic book [5] and in more detail later by Schwinger [6].

Because the coordinate lattice is periodic, the shift by the period $\Xi$ must be the identity transformation, i.e., for eigenvalues of the operator $\hat{p}$, we have $\Xi \cdot p \in \mathbb{Z}$. This gives the momentum step $\Delta p$,

$$\Xi \cdot \Delta p = 1, \quad \Delta p = 2^{-n+1}, \quad \Delta p \cdot \Delta x = \frac{1}{N} = 2^{-n}. \quad (5)$$

The number of points in the spectrum of the momentum is the same as for the coordinate, i.e., for momentum, we have a periodic lattice with the same number of nodes but a different period $\Pi = \Delta p \cdot N = 2^n$, $\Pi \Xi = N$. The momentum lattice is denoted by $\Delta p \cdot \mathbb{Z}_N$. The series for the momentum is also finite:

$$p = \sum_{r=-n+1}^{n-1} p_r 2^r = \sum_{r=-n+1}^{n-1} c(r, p)2^r. \quad (6)$$

Here, $p_r = c(r, p)$ is the $r$th digit in the binary expansion of $p$. We sometimes specify a range of powers of two that defines a lattice and write $p_r = c_{n+}(r, p)$.

### 2.3. Minimum shift

The minimum shift $\hat{T}_{\Delta x}$ is a shift by the lattice step $\Delta x$; any other shift on a given lattice is a power $\hat{T}_A = (\hat{T}_{\Delta x})^{A/\Delta x}$, where $A/\Delta x \in \mathbb{Z}_N$:

$$\hat{T}_A \psi(x) = \psi(x + A), \quad \hat{T}_A |x\rangle = |x - A\rangle, \quad \langle x' | \hat{T}_A | x'' \rangle = \delta_{x' + A, x''} = \delta_{x', x'' - A}. \quad (7)$$

Moreover,

$$\hat{T}_{\Delta x} \psi(x) = \hat{T}_{\Delta x} \begin{pmatrix} \psi((N - 1)\Delta x) \\ \psi((N - 2)\Delta x) \\ \vdots \\ \psi(\Delta x) \\ \psi(0) \end{pmatrix} = \begin{pmatrix} \psi(0) \\ \psi((N - 1)\Delta x) \\ \vdots \\ \psi(\Delta x) \end{pmatrix} = \psi(x + \Delta x). \quad (8)$$

The sum $x + \Delta x$ is taken in the sense $x \in \Delta x \cdot \mathbb{Z}_N$, i.e., this is a cyclic shift of the function on the lattice down one position.

The eigenvalues of the minimum shift operator are $N$th roots of unity and are related to the eigenvalues of the momentum operator (which has not yet been introduced explicitly):

$$\lambda^N = 1, \quad \lambda_p = e^{2\pi i \Delta x \cdot p} = e^{2\pi i \Delta x \Delta p / \Delta p} = (\lambda_{\Delta p})^{p / \Delta p}, \quad (9)$$
where we take $\Delta x \Delta p = 1/N$ and $p/\Delta p \in \mathbb{Z}_N$ into account and

$$\Lambda = \lambda_{\Delta p} = e^{2\pi i \Delta x \Delta p} = e^{2\pi i N}.$$  \hfill (10)

The corresponding eigenvectors are obtained from the relation $\psi(x) = \hat{T}_{\Delta} \psi(0)$. The normalized eigenvectors have the forms

$$\psi_{\lambda_p}(x') = \langle x' | \psi_{\lambda_p} \rangle = \frac{\lambda_p^{x'/\Delta x} \sqrt{N}}{\sqrt{N}},$$

$$\langle \psi_{\lambda_p} | x'' \rangle = \langle x'' | \psi_{\lambda_p} \rangle^* = \frac{\lambda_p^{-x''/\Delta x} \sqrt{N}}{\sqrt{N}} = e^{-2\pi i x''/p}.$$  \hfill (11)

We can write the projector on the (one-dimensional) eigensubspace of the operator $\hat{T}_{\Delta x}$ as

$$\hat{P}_{\lambda_p} = |\psi_{\lambda_p}\rangle \langle \psi_{\lambda_p}|,$$

$$\langle x' | \hat{P}_{\lambda_p} | x'' \rangle = \left( \frac{\lambda_p^{x'-x''/\Delta x}}{\sqrt{N}} \right) \left( \frac{\lambda_p^{d/\Delta x}}{\sqrt{N}} \right) = e^{2\pi ipd}, \quad d = x' - x''.$$  \hfill (12)

In this notation $x'$ labels rows of a matrix, and $x''$ labels columns.

The eigenstates of the minimum shift operator are also eigenstates of the momentum operator and can be hence written differently:

$$|\psi_{\lambda_p}\rangle = |\psi_p\rangle = |p\rangle, \quad \langle x | p \rangle = \frac{e^{2\pi i px}}{\sqrt{N}}.$$  \hfill (13)

2.4. Group of shifts. We make a trivial remark that might nevertheless be of some interest. We constructed the momentum operator such that it generates a symmetry group with respect to the shifts of the coordinate lattice by an integer number of nodes, i.e., a group isomorphic to the group (with respect to addition) of the residues modulo division by $N$: $\Delta x \cdot \mathbb{Z}_N = \mathbb{Z}_N$. But we can consider unitary operators of the form $\hat{T}_A = e^{2\pi i A \hat{p}}$, $A \in \mathbb{R}$. Such operators correspond to cyclic shifts by an arbitrary value (not necessarily a multiple of $\Delta x$). The corresponding group is isomorphic to the group $\mathbb{R}/\Xi \approx SO(1) \approx U(1)$ of rotations of a circle by an arbitrary angle. Addition is again understood modulo $\Xi$, $A = A + \Xi$. In the case $\Xi = \infty$, the group of symmetries coincides with the group $\mathbb{R}$ of real numbers with respect to addition.

We see that if the Hamiltonian on the lattice is expressed in terms of the operator $\hat{p}$, then the presence of the lattice does not violate translation invariance under arbitrary translations (not necessarily by an integer number of lattice sites), but the operator $\hat{p}$ (as we see below) turns out to be nonlocal, i.e., matrix elements $\langle x' | \hat{p} | x'' \rangle$ can be nonzero for arbitrarily large values $x' - x''$ (in the lattice).

We can specify a state $|x_0\rangle = \hat{T}_{x_0} |0\rangle$ with an arbitrary value $x_0 \notin \Delta x \cdot \mathbb{Z}_N$, but such a state is not a state with a certain value of the coordinate, because it decomposes into several basic states $\{|x\rangle\}_{x \in \Delta x \cdot \mathbb{Z}_N}$.

3. Operators of digits and their decomposition by shifts

3.1. Operators of digits on the lattice. We defined the momentum operator such that the Fourier harmonic of the momentum is given by the operator $\hat{T}_A = e^{2\pi i A \hat{p}}$ of the coordinate shift. Therefore, if we take the Fourier transform for the momentum digits

$$c_{n+} (r) = \sum_{A \in \Delta x \cdot \mathbb{Z}_N} \tilde{c}_{n+} (r, A) e^{2\pi i A r},$$

then we obtain the decomposition of the momentum digit by the coordinate shifts:

$$c_{n+} (r, \hat{p}) = \sum_{A \in \Delta x \cdot \mathbb{Z}_N} \tilde{c}_{n+} (r, A) \hat{T}_A.$$  \hfill (15)
Fig. 1. A plot of the value of the binary digit before the binary point (multiplier with $2^0$) for a conventional binary form (the sign is specified by a separate bit) is not periodic.

After simple computations (see the appendix), we obtain the expansion of the operator $\hat{p}_r$ of the momentum digit of the particle on the lattice over the shift operators $\hat{T}_A$:

$$\hat{p}_r = c_{n+n-}(r, \hat{p}) = \frac{1}{2} - \Delta p \cdot 2^{-r} \sum_{D \in \mathbb{Z}_2} \frac{\hat{T}_{-2^{-r}(D+1/2)}}{1 - e^{2\pi i \Delta p \cdot 2^{-r}(D+1/2)}}. \quad (16)$$

Similarly, the operator of the coordinate digit can be expanded in momentum shifts $\hat{S}_B = e^{-2\pi i \hat{x} B}$:

$$\hat{x}_s = c_{n+n+}(s, \hat{x}) = \frac{1}{2} - \Delta x \cdot 2^{-s} \sum_{D \in \mathbb{Z}_2} \frac{\hat{S}_{2^{-s}(D+1/2)}}{1 - e^{2\pi i \Delta x \cdot 2^{-s}(D+1/2)}}. \quad (17)$$

3.2. Binary expansion on the line. We have considered the case where the coordinate and momentum are given on a finite lattice. We consider the transition to a continuous limit. As we see below, this transition is associated with a nontrivial generalization of the sum of the binary series.

The digit of the binary expansion with the number $s \in \mathbb{Z}$ of the coordinate $x \in \mathbb{R}$ is defined by the value $x$, i.e., for each $s$, we have a function $c(s, x)$. The usual binary decomposition

$$x = \sum_{s=-\infty}^{+\infty} c_o(s, x) \cdot 2^s, \quad c_o(s, x) \in \{0, 1\}, \quad (18)$$

in which the sign is given by an individual bit, is not a generalization of the formalism described above for the lattice. The inconvenience of such a representation is that the digit $c_o(s, x)$ is not a periodic function of $x$, which is clearly seen in the plot in Fig. 1. The periodicity fails because the continuation of $c_o(s, x)$ to the negative values of $x$ is even.

If we continue $c(s, x)$ to the negative values of $x$ periodically, then we obtain another variant of the binary representation of real numbers. This version generalizes the binary representation of numbers on the lattice $\Delta x \cdot \mathbb{Z}_{2^n}$, which is easy to see if we apply the representation of the ring $\mathbb{Z}_{2^n}$ by integers from $-2^{n-1}$ to $2^{n-1} - 1$.

From the plot in Fig. 2, we see that when the sign of a number changes, the value of the digit changes by inversion $0 \leftrightarrow 1$. Such a representation is called the ones’ complement representation of negative binary numbers. It is used in computers.

The function $c(s, x)$ can be expressed in terms of a function of one variable $C(x) = c(0, x)$:

$$c(s, x) = c(0, 2^{-s} x) = C(2^{-s} x). \quad (19)$$

It is convenient to introduce the functions $\lfloor \cdot \rfloor$, the integer part (floor function) and $\{ \cdot \}$, the fractional part of a number:

$$x = \lfloor x \rfloor + \{ x \}, \quad \lfloor x \rfloor = \max\{ n \in \mathbb{Z} \mid n \leq x \}. \quad (20)$$
In this case,

\[ C(x) = \frac{1 - (-1)^{|x|}}{2}. \]  

(21)

A real number is determined by the formal binary series

\[ x = \sum_{s=-\infty}^{+\infty} x_s \cdot 2^s = \sum_{s=-\infty}^{+\infty} c(s, x) \cdot 2^s. \]  

(22)

For positive numbers, both representations yield the same expansions. For any number \( x > 0 \) there exists \( s_0 \in \mathbb{Z} \) such that \( 2^{s_0} < x < 2^{s_0+1} \). The highest digits of the binary expansion are zero, \( x_S = 0 \) for all \( S > s_0 \). A binary expansion series

\[ x = \sum_{s=-\infty}^{+\infty} x_s \cdot 2^s = \sum_{s=-\infty}^{s_0} x_s \cdot 2^s \]  

(23)

converges because the series is truncated as \( s \to +\infty \) and is majorized by a decreasing geometric progression as \( s \to -\infty \).

For any number \( x < 0 \), there exists \( s_0 \in \mathbb{Z} \) such that \( 2^{s_0} < |x| < 2^{s_0+1} \). The highest digits of the binary expansion are equal to unity: \( x_S = 1 \) for all \( S > s_0 \). A binary expansion series

\[ x = \sum_{s=-\infty}^{+\infty} x_s \cdot 2^s = \sum_{s=-\infty}^{s_0} x_s \cdot 2^s + \sum_{s=s_0+1}^{+\infty} 2^s \]  

(24)

diverges because the convergent series

\[ \sum_{s=-\infty}^{s_0} x_s \cdot 2^s = \sum_{s=-\infty}^{s_0} (1 - c(s, |x|)) \cdot 2^s = 2^{s_0+1} + x, \quad x = -|x|, \]  

(25)

is added to the sum of the increasing geometric progression

\[ \sum_{s=s_0+1}^{+\infty} 2^s = 2^{s_0+1} \sum_{s=0}^{+\infty} 2^s \to +\infty. \]  

(26)

### 3.3. Renormalization of binary sums.

To assign a finite value to the binary expansion of a negative number \( x < 0 \), we must renormalize it, which reduces to assigning a finite value to the sum of a divergent geometric progression using the formula \( \sum_{s=0}^{+\infty} q^s = 1/(1 - q) \) beyond its applicability limits: for \( q = 2 \). The renormalized sum is denoted by the sum symbol with a prime:

\[ \sum_{s=0}^{+\infty} 2^s = \frac{1}{1 - 2} = -1. \]  

(27)
The same rule can be reduced to the form
\[ \sum_{s=-\infty}^{+\infty} 2^s = 0. \] (28)

We could consider the same series in the 2-adic sense [7], but we then lose the convergence of fractions with an infinite number of nonzero digits after the binary point and obtain the convergence of fractions with an infinite number of digits before the binary point. Hence, 2-adic convergence proves inconvenient for real coordinates because it must be applied selectively (only for the integer part of negative numbers). Nevertheless, we can consider 2-adic observables, also decomposing them in binary digits.

To apply the binary expansion to the operators \( \hat{x} \) and \( \hat{p} \), a general formula for negative and positive numbers is useful. It is easily obtained using the formal computations
\[ x = 2x - x = \sum_{s=-\infty}^{+\infty} x_{s-1} \cdot 2^s - \sum_{s=-\infty}^{+\infty} x_s \cdot 2^s = \sum_{s=-\infty}^{+\infty} (x_{s-1} - x_s) \cdot 2^s. \] (29)

It is easy to see that such a series converges to \( x \) regardless of the sign because in both cases for the higher powers \( x_{s-1} - x_s = 0 \). The same rule ensures that the series is terminated in negative powers if all digits following the binary point are the same after a certain position. Hence, the renormalized sum (with a prime) has the form
\[ x = \sum_{s=-\infty}^{+\infty} x_s 2^s = \sum_{s=-\infty}^{+\infty} (x_{s-1} - x_s) \cdot 2^s. \] (30)

This representation is known in computer science as the signed-digit representation [8], [9]. It was introduced to increase computational speed by reducing the number of carries of digits.

For the operator \( \hat{x} \) and its digits, we similarly have
\[ \hat{x} = \sum_{s=-\infty}^{+\infty} \hat{x}_s 2^s = \sum_{s=-\infty}^{+\infty} (\hat{x}_{s-1} - \hat{x}_s) \cdot 2^s. \] (31)

A digit of the operator \( \hat{x}_s = c(s, \hat{x}) \) is a projector (self-adjoint operator with the eigenvalues 0 and 1).

We can also generalize the reduced renormalization method to the case where the coordinate is defined on the lattice:
\[ x' = \sum_{s=-n_-}^{n_+ - 1} c(s, x) 2^s = \sum_{s=-n_-}^{n_+ - 1} (c(s - 1, x) - c(s, x)) 2^s, \quad c(-n_- - 1, x) = 0. \] (32)

The same rule is reducible to the form
\[ \sum_{s=-n_-}^{n_+ - 1} 2^s = -2^{-n_-} = -\Delta x. \] (33)

Then
\[ x' = \sum_{s=-n_-}^{n_+ - 1} c(s, x) 2^s = 2 \sum_{s=-n_-}^{n_+ - 2} c(s, x) 2^s - \sum_{s=-n_-}^{n_+ - 1} c(s, x) 2^s = \]
\[ = 2(x - c(n_+ - 1, x) 2^{n_- - 1}) - x = x - c(n_+ - 1, x) 2^{n_- - 1}. \] (34)
Hence, if the highest digit \( x_{n+1} = c(n+1, x) \) is equal to zero, then the renormalized coordinate \( x' \) is positive and coincides with \( x \), and if the highest digit is unity, then the renormalized coordinate \( x' \) is negative and differs from \( x \) for the lattice period \( \Xi \), \( x' = x - x_{n+1} \Xi \). The original expression for \( x \) corresponds to the representation of the lattice \( \Delta x \cdot \mathbb{Z}_N \) by real numbers from 0 to \( \Delta x \cdot (N - 1) \) in steps of \( \Delta x \). The renormalized coordinate \( x' \) corresponds to the representation of the same lattice by real numbers from \(-\Delta x \cdot N/2\) to \(+\Delta x \cdot (N/2 - 1)\) in steps of \( \Delta x \). In computer science, such a renormalization corresponds to a transition from a positive integer to an integer with a sign. For a renormalized number, the highest digit specifies the sign (0 corresponds to a plus sign and 1 corresponds to a minus sign).

3.4. Integral binary representation. The function \( c(s, x) \) is expressed in terms of a function of one variable \( C(x) = c(0, x) \) by formula (19). We can consider noninteger values of \( s \), which can be useful, for example, when changing the scale of a unit interval. Then \( x \to 2^n x \) and \( c(s, x) \to c(s - a, x) \). For positive numbers \( c(s, x) = c(s - \log_2 x, 1) \), \( x > 0 \). For negative numbers \( c(s, x) = 1 - c(s, |x|) = 1 - c(s - \log_2 |x|, 1) \), \( x < 0 \).

By analogy with the binary series, we consider the formal binary integral

\[
\int_{-\infty}^{+\infty} c(s, x)2^s \, ds.
\]

For any \( x > 0 \), the highest digits are zero, \( c(s, x) = 0 \) for all \( s > \log_2 x \). The binary integral

\[
\int_{-\infty}^{+\infty} c(s, x)2^s \, ds = \int_{-\infty}^{\log_2 x} c(s, x)2^s \, ds
\]

converges because the integrand tends to zero as \( s \to +\infty \) and is majorized by a convergent integral of the exponential function as \( s \to -\infty \). For positive numbers, it is easy to prove by a direct calculation (see the appendix) that the binary integral, like the binary sum, gives \( x \):

\[
x = \sum_{s=-\infty}^{+\infty} c(s, x)2^s = \int_{-\infty}^{+\infty} c(s, x)2^s \, ds.
\]

For any \( x < 0 \), the highest digits are equal to unity, \( c(s, x) = 1 \) for all \( s \geq \log_2 |x| \). The binary integral

\[
x = \int_{-\infty}^{+\infty} c(s, x)2^s \, ds = \int_{-\infty}^{+\infty} 2^s \, ds - \int_{-\infty}^{+\infty} c(s, |x|)2^s \, ds = \int_{-\infty}^{+\infty} 2^s \, ds + x
\]

diverges because a divergent summand is added to the convergent integral.

To assign a finite value to the binary integral representation of a negative number \( x < 0 \), we should renormalize it. Similarly to the binary sum, the renormalization corresponds to applying the formula \( \int_{0}^{\infty} q^s \, ds = -1/\log q \), which holds for \( 0 < q < 1 \), to the case \( q > 1 \):

\[
\sum_{s=-\infty}^{+\infty} 2^s = \int_{-\infty}^{+\infty} 2^s \, ds = 0 \iff \sum_{s=0}^{+\infty} 2^s = \log 2 \int_{0}^{+\infty} 2^s \, ds = -1.
\]

As for the binary sum, we can obtain a general formula that converges regardless of the sign of \( x \):

\[
x = 2x - x = \int_{-\infty}^{+\infty} (c(s - 1, x) - c(s, x)) \cdot 2^s \, ds.
\]
The integral representation is overdetermined because to specify a number, it suffices to specify its digits with integer numbers or digits with a unit step, which corresponds to the choice of \(2^{\Delta s}\) as a scale segment: for any \(\Delta s \in \mathbb{R}\),

\[
x = \sum_{s=-\infty}^{+\infty} c(s + \Delta s, x) 2^{s+\Delta s} = 2^{\Delta s} \sum_{s=-\infty}^{+\infty} c(s, x \cdot 2^{-\Delta s}) 2^s.
\]

(41)

Such a representation has its advantages despite being overdetermined. In particular, the integral representation is scale invariant: it does not distinguish scales of the form \(2^s, s \in \mathbb{Z}\).

### 3.5. Operators of digits on the line.

The function \(c(r, p)\) is periodic with the period \(2^{r+1}\). Expanding it in a Fourier series, we obtain an expansion in terms of the shift operators:

\[
\hat{p}_r = \sum_{A \in 2^{-r-1}\mathbb{Z}} \tilde{c}(r, A)e^{2\pi i A \hat{p}} = \tilde{T}_A
\]

\[
\tilde{c}(r, A) = 2^{-r-1} \int_0^{2^{r+1}} c(r, p) e^{2\pi i Ap} dp.
\]

Moreover,

\[
\hat{p}_r = \frac{\hat{1}}{2} + \sum_{D \in \mathbb{Z}} \tilde{T}_{2^{-r}(D+1/2)} - \frac{1}{2} e^{2\pi i Ap}.
\]

(43)

The same expression can be obtained if we take the limit \(n \to +\infty\) for \(\Delta p \to 0\) in expansion (16) of the momentum digits on a lattice. This corresponds to an infinite number of digits before the binary point for the coordinate and an infinite number of digits after the binary point for the momentum, i.e., to a lattice in the coordinate and a circle in the momentum (if also \(\Delta x \to 0\), then both the coordinate and momentum yield a line). Specifically, we note that in the limit we obtain a sum over all integers including negative integers. Before passing to the limit for the lattice of finite size closed to a circle, separating numbers into negative and positive did not make sense.

### 4. Commutation relations

#### 4.1. Digit–digit commutator.

The operator of the momentum digits is expanded in terms of shift operators:

\[
\hat{p}_r = \frac{\hat{1}}{2} - \Delta p \cdot 2^{-r} \sum_{D \in \mathbb{Z}/\Delta p} \tilde{T}_{-2^{-r}(D+1/2)} - e^{2\pi i Ap}2^{-r}(D+1/2).
\]

(44)

It is easy to derive the commutation relation between an arbitrary function of the coordinate \(f(\hat{x})\) and the shift operator \(\tilde{T}_A\):

\[
[f(\hat{x}), \tilde{T}_A] \psi(x) = f(\hat{x})\tilde{T}_A \psi(x) - \tilde{T}_A f(\hat{x}) \psi(x) = f(x)\psi(x + A) - f(x + A)\psi(x + A) = (f(x) - f(x + A))\psi(x + A) = (f(\hat{x}) - f(\hat{x} + A))\tilde{T}_A \psi(x).
\]

Hence,

\[
[f(\hat{x}), \tilde{T}_A] = (f(\hat{x}) - f(\hat{x} + A))\tilde{T}_A.
\]

(46)

Because \(\hat{x}_s = c(s, \hat{x})\), the commutator of the digits of the coordinate and of the momentum on the lattice has the form

\[
[\hat{x}_s, \hat{p}_r] = -\Delta p \cdot 2^{-r} \sum_{D \in \mathbb{Z}/\Delta p} \frac{c(s, \hat{x}) - c(s, \hat{x} - 2^{-r}(D + 1/2))}{1 - e^{2\pi i Ap}2^{-r}(D+1/2)} \tilde{T}_{-2^{-r}(D+1/2)}.
\]

(47)
On the line, the commutator of the coordinate and momentum digits has the form

\[ [\hat{x}_s, \hat{p}_r] = \sum_{D \in \mathbb{Z}} \frac{c(s, \hat{x}) - c(s, \hat{x} - 2^{-r}(D + 1/2))}{2\pi i (D + 1/2)} \hat{T}_{-2^{-r}(D+1/2)}. \]  

(48)

The digit \( \hat{x}_s \) in the coordinate representation is given by \( c(s, x) \), which has the period \( 2^{s+1} \). The shift value \( 2^{-r-1}(2D + 1) \) is the period of \( c(s, x) \) if

\[ \frac{2^{-r}(D + 1/2)}{2^{s+1}} = 2^{-r-s-2}(2D + 1) \in \mathbb{Z}, \]  

(49)
i.e., for \( -r - s - 2 \geq 0 \). Therefore, \( [\hat{x}_s, \hat{p}_r] = 0 \) for \( s + r \leq -2 \). In particular, the fractional part of the momentum commutes with the fractional part of the coordinate,\(^1\) the lowest digit of the momentum does not commute only with the highest digit of the coordinate, and the lowest coordinate digit does not commute only with the highest momentum digit.

4.2. Coordinate–digit commutator. Because \( \hat{x} = \sum_{s=0}^{n} 2^s c(s, \hat{x}) \), we obtain the commutator of the coordinate and momentum digit on the lattice:

\[ [\hat{x}, \hat{p}_r] = -\Delta p \cdot 2^{-r} \sum_{D \in \mathbb{Z}^{2r}/\Delta_p} \frac{\hat{x} - (\hat{x} - 2^{-r}(D + 1/2))}{1 - e^{2\pi i \Delta_p 2^{-r}(D+1/2)}} \hat{T}_{-2^{-r}(D+1/2)} = \]

\[ = \frac{1}{2^r} \sum_{D \in \mathbb{Z}^{2r}/\Delta_p} \frac{-\Delta p \cdot 2^{-r}(D + 1/2)}{1 - e^{2\pi i \Delta_p 2^{-r}(D+1/2)}} \hat{T}_{-2^{-r}(D+1/2)}. \]  

(50)

On the line, the commutator of the coordinate and the momentum digit has the form

\[ [\hat{x}, \hat{p}_r] = \frac{1}{2^r} \sum_{D \in \mathbb{Z}} \hat{T}_{-2^{-r}(D+1/2)}. \]  

(51)

4.3. Coordinate–momentum commutator. Because \( \hat{p} = \sum_{r=0}^{n} 2^r \hat{p}_r \), we obtain the commutator of the coordinate and momentum on the lattice:

\[ [\hat{x}, \hat{p}] = \sum_{r=0}^{n} \sum_{D \in \mathbb{Z}^{2r}/\Delta_p} \frac{-\Delta p \cdot 2^{-r}(D + 1/2)}{1 - e^{2\pi i \Delta_p 2^{-r}(D+1/2)}} \hat{T}_{-2^{-r}(D+1/2)}. \]  

(52)

On the line, we obtain the \textit{formal} relations

\[ [\hat{x}, \hat{p}_r] = \frac{1}{2^r} \sum_{D \in \mathbb{Z}} \hat{T}_{-2^{-r}(D+1/2)}, \]

\[ [\hat{x}, \hat{p}] = \frac{1}{2^r} \sum_{D \in \mathbb{Z}} \hat{T}_{-2^{-r}(D+1/2)} = -i\hbar \sum_{r \in \mathbb{Z}} \sum_{D \in \mathbb{Z}} \hat{T}_{-2^{-r}(D+1/2)}. \]  

(53)

We recall that \( h = 1 \) and \( \hbar = 1/2\pi \) in the chosen system of units.

\(^1\)The fractional parts of the coordinate and momentum can be taken as the complete set of observables for one-dimensional motion. The area of such a phase cell is \( h = 2\pi \hbar \) (1 in our system of units), which corresponds to the fact that parallel transfer along the cell boundary should give the identity transformation \([10]\).
4.4. Renormalization of the commutator on the line. We obtain a formal decomposition of the commutator in the sum of the shift operators:

\[ [\hat{x}, \hat{p}] = -i\hbar \sum_{r \in \mathbb{Z}} \sum_{D \in \mathbb{Z}} \hat{T}_{2^{-r}(D+1/2)}. \]  

(54)

The values of the shifts have the form \(-2^{-r}(D + 1/2) = -2^{-r-1}(2D + 1)\), where \(r, D \in \mathbb{Z}\).

Let \(\Lambda\) be a set of numbers whose binary expansion contains a finite number of nonzero factors with negative powers of two (a finite number of significant binary digits after the binary point); \(\Lambda\) is a group under the summation operation. Then the set of values of the shifts along which summation occurs has the form \(\Lambda \setminus \{0\}\). Given that \(\hat{T}_0 = 1\), we obtain

\[ [\hat{x}, \hat{p}] = -i\hbar \sum_{\Lambda \in \Lambda \setminus \{0\}} \hat{T}_\Lambda = -i\hbar \left( \sum_{\Lambda \in \Lambda} \hat{T}_\Lambda - 1 \right) = i\hbar \hat{1} - i\hbar \sum_{\Lambda \in \Lambda} \hat{T}_\Lambda. \]  

(55)

We know that for the particle coordinate and momentum on the line, there is the canonical commutation relation \([\hat{x}, \hat{p}] = i\hbar 1\). We thus obtain the renormalization

\[ \sum_{\Lambda \in \Lambda} \hat{T}_\Lambda = \sum_{\Lambda \in \Lambda} e^{2\pi i \hat{p} \Lambda} = 0. \]  

(56)

This renormalization is similar to the formal equality \(\int_{\mathbb{R}} e^{2\pi i px} dx = 0\) for all \(p \neq 0\) arising in the Fourier transforms.

5. Conclusion

The binary expansion of the coordinate and momentum operators that we have constructed here are applicable not only in the field of quantum computations but also in numerically solving partial differential equations on a classical computer. The natural appearance of renormalizations is particularly interesting. Renormalizing not only infinite but also finite quantities (on the lattice) allows finding the renormalization numerically by passing from the lattice to the limit of a continuous quantity. Time and energy can also be considered as a coordinate and momentum \([11]\), which allows applying the same renormalization methods to them. Renormalizations in this context are probably related to the quantum theory of measurements (see \([12]\), \([13]\) and the references therein for the quantum theory of measurements). The representation of the coordinate and momentum in the form of a binary expansion assumes that the coordinate and momentum are not themselves observed in the experiment. Instead, individual digits of the coordinate and momentum are directly observed. The measurement of a binary digit of the spatial coordinate corresponds to the passage/nonpassage of a particle through a diffraction grating.

Appendix A: Calculating the digits of the coordinate and momentum

We calculate the Fourier amplitudes of the momentum digits explicitly:

\[ \hat{c}_{n+n-}(r, A) = \frac{1}{N} \sum_{p \in \Delta p \subset \mathbb{Z}} e^{-2\pi i p A} c_{n+n-}(r, p). \]

It is obvious that \(\hat{c}_{n+n-}(0, 0) = 1/2\). The period of \(c_{n+n-}(r, p)\) is equal to \(2^{r+1}\). The number of harmonic periods \(e^{2\pi i p A}\) on an interval of length \(2^{r+1}\) is \(2^{r+1} \cdot A\). For the amplitude \(\hat{c}_{n+n-}(r, A)\) to be nonzero, this number must be an integer. The function \(c_{n+n-}(r, p)\) is equal to zero in the first half of the period and
equal to unity on the second half. At half the period of \( c_{n+} \), there are \( 2^r \cdot A \) harmonic periods. For the amplitude \( \tilde{c}_{n+} \) to be nonzero, this number should be a noninteger.

Therefore, for nonzero amplitudes, the number \( 2^r \cdot A \) should be half-integer, \( 2^r \cdot A = D + 1/2, \ D \in \mathbb{Z} \). Then

\[
\tilde{c}_{n+} (r, A) = \frac{1}{N} \cdot \frac{N \cdot \Delta p}{2^r + 1} \sum_{p \in \Delta p} e^{-2\pi i p A} c_{n+} (r, p), \quad A = 2^{-r} \left( D + \frac{1}{2} \right).
\]

Because \( c_{n+} (r, p) \) is equal to zero in the first half of the period and becomes unity by the second, taking \( \Delta p = 2^{-n_+} \) into account, we obtain a geometric series:

\[
\tilde{c}_{n+} (r, A) = \frac{1}{2^{r+n_+} + 1} \sum_{p \in \Delta p} e^{-2\pi i p A} c_{n+} (r, p) = \frac{-1}{2^{r+n_+} + 1} \sum_{p \in \Delta p} e^{-2\pi i p A} \left( c_{n+} (r, p) - 1 \right) = \frac{-1}{2^{r+n_+} + 1} \sum_{p \in \Delta p} e^{-2\pi i p A}.
\]

The sum of the geometric series is

\[
\sum_{j=0}^{J-1} \lambda^j = \begin{cases} \frac{1 - \lambda^J}{1 - \lambda}, & \lambda \neq 1, \\ J, & \lambda = 1. \end{cases}
\]

In this case,

\[
J = 2^r + n_+, \quad \lambda = e^{-2\pi i A \Delta p} = e^{-2\pi i 2^{-r} \cdot (D+1/2)} \neq 1,
\]

\[
\lambda^J = e^{-2\pi i (D+1/2)} = e^{\pi i} = -1.
\]

Hence, the nonzero components have the form

\[
\tilde{c}_{n+} (r, A) = \frac{-1}{2^{r+n_+} + 1} \frac{1}{1 - e^{-2\pi i 2^{-r} \cdot (D+1/2)}}, \quad A = 2^{-r} \left( D + \frac{1}{2} \right).
\]

The number \( D \) is an integer, and we need \( r + n_+ \) binary digits to write it. We can therefore assume that \( D \in \mathbb{Z}_{2^r+n_+} = \mathbb{Z}_{2^r}/\Delta p \). We hence obtain the expansion of the momentum digits in the shift operators:

\[
\hat{p}_r = c_{n+} (r, \hat{p}) = \sum_{A \in \Delta x / \mathbb{Z}} \tilde{c}_{n+} (r, A) e^{2\pi i A \hat{p}} = \\
= \frac{1}{2} - \Delta p \cdot 2^{-r} \sum_{D \in \mathbb{Z}_{2^r}/\Delta x} \frac{\hat{T}_{-2^{-r}(D+1/2)}}{1 - e^{2\pi i \Delta p \cdot 2^{-r}(D+1/2)}}.
\]

Similarly, after replacing \( n_+ \leftrightarrow n_-, \Delta x \leftrightarrow \Delta p, \ r \leftrightarrow s, \text{ and } A \rightarrow B \), we obtain the operator of the digit of the coordinate in terms of the shifts in the momentum:

\[
\hat{x}_s = c_{n-} (s, \hat{x}) = \sum_{B \in \Delta p / \mathbb{Z}} \tilde{c}_{n-} (s, B) e^{2\pi i B \hat{x}} = \\
= c_{n-} (s, \hat{x}) = \frac{1}{2} - \Delta x \cdot 2^{-s} \sum_{D \in \mathbb{Z}_{2^s}/\Delta x} \frac{\hat{S}_{-2^{-s}(D+1/2)}}{1 - e^{2\pi i \Delta x \cdot 2^{-s}(D+1/2)}}.
\]
The matrix elements of the momentum digits and the coordinates have the forms

\[
\langle x' | \hat{p}_r | x'' \rangle = \frac{1}{2} \delta_{x',x''} - \frac{1}{2^{n_++r}} \sum_{D \in \mathbb{Z}_{+}^{n_++r}} \frac{\delta_{x',x''+2^{r-1}(2D+1)}}{1 - e^{2\pi i 2^{n_++r-(2D+1)}}},
\]

\[
\langle p' | \hat{x}_s | p'' \rangle = \frac{1}{2} \delta_{p',p''} - \frac{1}{2^{n_++s}} \sum_{D \in \mathbb{Z}_{+}^{n_++s}} \frac{\delta_{p',p''-2^{r-1}(2D+1)}}{1 - e^{2\pi i 2^{n_++r-(2D+1)}}}.
\]

We can convert the coefficient as

\[
\frac{1}{1 - e^{i 2 \alpha}} = \frac{ie^{-i \alpha}}{2 \sin \alpha} = \frac{i}{2} \cot \alpha + \frac{1}{2}.
\]

Appendix B: Checking the integral binary representation for \( x > 0 \)

We prove the validity of (37) for \( x > 0 \) by direct calculation. We have

\[
c(s, x) = \begin{cases} 
  0, & s > \log_2 x, \\
  1, & s \in [\log_2 \frac{x}{2}, \log_2 \frac{x}{2^{k-1}}], \\
  0, & s \in [\log_2 \frac{x}{2^{k-1}}, \log_2 \frac{x}{2^{k}}], \\
  \vdots & \vdots \\
  1, & s \in [\log_2 \frac{x}{2^{k-1}}, \log_2 \frac{x}{2^{k}}], \\
  0, & s \in [\log_2 \frac{x}{2^{k}}, \log_2 \frac{x}{2^{k+1}}], \\
  \vdots & \vdots 
\end{cases}
\]

Hence,

\[
\int_{-\infty}^{+\infty} c(s, x) 2^s \, ds = \int_{-\infty}^{+\log_2 x} c(s, x) 2^s \, ds = \sum_{k=1}^{+\infty} \left( \frac{1}{2k} - \frac{1}{2^{k-1}} \right) = \frac{x}{\log_2 2} \left( \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots + \frac{(-1)^{n+1}}{n} \cdots \right) = x.
\]

Appendix C: Examples

Everywhere in this section, \( \Delta x = 1 \), \( x \in \mathbb{Z}_N = \{0, 1, \ldots, N-1\} \), coordinates and momenta are numbered by binary numbers, which are marked with a lower index 2, and \( \Delta p = 2^{-n} = 1/N \).

C.1. The case \( n = 1 \) and \( N = 2^1 = 2 \). In this case, we have

\[
\dot{x} = \dot{x}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1 + \sigma_+}{2} \equiv \dot{c}, \quad \hat{T}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2\hat{h},
\]

\[
\lambda_0 = 1 = e^{2\pi i \cdot 0}, \quad \psi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]
C.2. The case $n = 2$ and $N = 2^2 = 4$. In this case, we have

$$\hat{P}_0 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{i + \sigma_x}{2} \equiv \hat{a},$$

$$\lambda_{0,12} = -1 = e^{2\pi i \cdot 0.12}, \quad \psi_{0,12} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

$$\hat{P}_{0,12} = \hat{p} = \hat{p}_{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{i - \sigma_x}{2} \equiv \hat{b},$$

$$[\hat{x}, \hat{p}] = \frac{1}{2} [\hat{x}_0, \hat{p}_{-1}] = \frac{1}{4} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\frac{i}{4} \sigma_y \equiv \frac{1}{2} \hat{f}.$$
\[ [\hat{x}_0, \hat{p}_{-1}] = \hat{a} \otimes \hat{f} - i \hat{b} \otimes \hat{h}, \quad [\hat{x}_1, \hat{p}_{-1}] = \hat{f} \otimes (\hat{h} + i \hat{f}), \]
\[ [\hat{x}_0, \hat{p}_{-2}] = 0, \quad [\hat{x}_1, \hat{p}_{-2}] = \hat{f} \otimes \hat{1}_2, \]
\[ \hat{p} = 0.12 \cdot \hat{p}_{-1} + 0.012 \cdot \hat{p}_{-2} = \frac{1}{8} \begin{pmatrix}
3 & -1-i & -1 & -1+i \\
-1+i & 3 & -1-i & -1 \\
-1 & -1+i & 3 & -1-i \\
-1-i & -1 & -1+i & 3
\end{pmatrix}. \]

C.3. The case \( n = 3 \) and \( N = 2^3 = 8 \). In this case, we have

\[ \dot{x} = \hat{x}_0 + 2\hat{x}_1 + 4\hat{x}_2 = \begin{pmatrix}
7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 9 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad \hat{T}_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \]

\[ \hat{x}_0 = \hat{1}_2 \otimes \hat{1}_2 \otimes \hat{c} = \text{diag}(1, 0, 1, 0, 1, 0, 1, 0, 0), \]
\[ \hat{x}_1 = \hat{1}_2 \otimes \hat{c} \otimes \hat{1}_2 = \text{diag}(1, 1, 0, 0, 1, 1, 0, 0, 0), \]
\[ \hat{x}_2 = \hat{c} \otimes \hat{1}_2 \otimes \hat{1}_2 = \text{diag}(1, 1, 1, 1, 0, 0, 0, 0, 0), \]
\[ \hat{p}_{-3} = \frac{\hat{T}_2}{2} - \frac{\hat{T}_3}{2} = \frac{1}{2} \begin{pmatrix}
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0
\end{pmatrix} = \hat{b} \otimes \hat{1}_2 \otimes \hat{1}_2, \]
\[ \hat{p}_{-2} = \frac{1}{2} - \frac{1+i}{4} \hat{T}_6 - \frac{1-i}{4} \hat{T}_2 = \begin{pmatrix}
1 & 0 & \frac{-1-i}{2} & 0 & 0 & 0 & \frac{-1+i}{2} & 0 \\
0 & 1 & 0 & \frac{-1+i}{2} & 0 & 0 & 0 & \frac{-1+i}{2} \\
\frac{-1+i}{2} & 0 & 1 & 0 & \frac{-1-i}{2} & 0 & 0 & 0 \\
0 & \frac{-1+i}{2} & 0 & 1 & 0 & \frac{-1-i}{2} & 0 & 0 \\
0 & 0 & \frac{-1+i}{2} & 0 & 1 & 0 & \frac{-1-i}{2} & 0 \\
0 & 0 & 0 & \frac{-1+i}{2} & 0 & 1 & 0 & \frac{-1-i}{2} \\
\frac{-1-i}{2} & 0 & 0 & 0 & \frac{-1+i}{2} & 0 & 1 & 0 \\
0 & \frac{-1-i}{2} & 0 & 0 & 0 & \frac{-1+i}{2} & 0 & 1
\end{pmatrix}, \]
\[ \hat{p}_{-1} = \frac{1}{4} \left( \frac{\hat{T}_7}{1-e^{\pi i/4}} + \frac{\hat{T}_5}{1-e^{2\pi i/4}} + \frac{\hat{T}_3}{1-e^{3\pi i/4}} + \frac{\hat{T}_1}{1-e^{4\pi i/4}} \right) = \begin{pmatrix}
2 & -E_1 & 0 & -E_3 & 0 & -E_5 & 0 & -E_7 \\
-E_7 & 2 & -E_1 & 0 & -E_3 & 0 & -E_5 & 0 \\
0 & -E_7 & 2 & -E_1 & 0 & -E_3 & 0 & -E_5 \\
-E_5 & 0 & -E_7 & 2 & -E_1 & 0 & -E_3 & 0 \\
0 & -E_5 & 0 & -E_7 & 2 & -E_1 & 0 & -E_3 \\
-E_3 & 0 & -E_5 & 0 & -E_7 & 2 & -E_1 & 0 \\
0 & -E_3 & 0 & -E_5 & 0 & -E_7 & 2 & -E_1 \\
-E_1 & 0 & -E_3 & 0 & -E_5 & 0 & -E_7 & 2
\end{pmatrix}. \]
where we use the notation $E_k = 1/(1 - e^{k\pi i/4})$, $k = 1, 3, 5, 7$, for brevity.

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