On a $K_{2,3}$ in non-Hamiltonian graphs

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Abstract

In a reduced graph [1], a I cycle-set is the set of cycles only connected by interior points. $|I|$ is the number of I cycle-sets in a given graph. We use a norm graph to denote a reduced graph of $|I|=1$. An operation denotes a procedure to delete removable cycles. We say an operation is rational means that to delete removable cycles if there have no solutions of Grinberg’s Equation of the graph, to delete co-solution cycles if there have solutions of Grinberg’s Equation of the graph, or to delete removable cycles if there is a K (a boundary point of order 4) in the graph. $\mathcal{G}$ is a subgraph derived by rational operations from a graph $G$ such that there have no removable cycles. In this paper, we present a theorem that a graph $G$ is non-Hamiltonian, if and only if, $\mathcal{G}$ and $K_{2,3}$ are homeomorphic.

Keywords Hamilton graphs, I cycle-set, norm graphs, $K_{2,3}$ sets, rationally-removing-cycle operating, a subgraph $\mathcal{G}$

1. Introduction

In a graph $G(V,E)$, m and n are subgraphs of $V(G)$ such that $m \cup n = V(G)$ and $m \cap n = \emptyset$, if two vertices in m and n are adjacent to each other by an edge in $E(G)$, then $G$ is a perfect bipartite graph, marked as $K_{m,n}$. According to the properties of Hamilton graphs, it is clear that $K_{2,3}$ is a non-Hamilton graph with minimum graphic elements (both vertices and edges). A subdivision of a graph $G$ is a graph resulting from the subdivision
of edgees of graph G. The subdivision of an edge \( e \) with endpoints \( \{u, v\} \) yields a graph containing one new vertex \( w \), and with an edge set replacing \( e \) by two new edges, \( \{u, w\}, \{w, v\} \). A graph \( G' \) is called a subdivision of a graph \( G \) if it can be obtained from \( G \) by repeating the operation of edge subdivision several times. See Figure 1. Two graphs \( G' \) and \( G \) are said to be homeomorphic if they have isomorphic subdivisions and write as \( G' \cong G \). A \( K_{2,3} \) set is a cycle set to be homeomorphic to a \( K_{2,3} \). So, we have \( G' \cong G \cong K_{2,3} \).

![Subdivision](image)

**Figure 1** \( G' \), \( G \) and \( K_{2,3} \) are homeomorphic. \( G' \) and \( G \) are \( K_{2,3} \) sets

In a cycle basis of a graph \( G \), let \( R \) be the number of cycles passing by an edge of a graph \( G \). A vertex is boundary if there have only two edges of \( R=1 \) in its incident edges. A boundary edge is an edge of \( R=1 \). A cut point means a vertex that all its incident edges are \( R=1 \). An interior vertex is a point that is neither a boundary nor a cut. \(|P|\) denotes the number of vertices of order 2 in the neighbor of a vertex. In a reduced graph \( G \), a I cycle-set is the set of cycles connected by interior points. \(|I|\) denotes the number of I cycle-sets. A norm graph is a reduced graph of \(|I|=1\). A removable cycle is a cycle that removing this cycle from the given cycle basis of \( G \) remains a subgraph \( G' \) satisfying \( V'=V, E'=E-1 \) and \(|P|<3\). An operation denotes a procedure to delete removable cycles. We say an operation is rational means that to delete removable cycles if there have no solutions of Grinberg’s Equation of the graph, to delete co-solution cycles if there have solutions of Grinberg’s Equation of the graph, or to delete removable cycles if there is a \( K \) (a boundary point of order 4) in the graph. \( g \) is a subgraph derived by rational operations from a graph \( G \).
such that there have no removable cycles. For the sake of the visual effects, all the cycle bases we mentioned are the minimum cycle bases which do not change the results of this paper. For terms not defined in this paper see [3].

In a $|\mathcal{I}|>1$ reduced graph, there have 3 connected ways among $\mathcal{I}$ cycle-sets: by a cut point, by a bridge or by a cycle. Without loss of generality, we consider the case of $|\mathcal{I}|=2$. See Figure 2.

![Figure 2](image)

Figure 2  In graph (1) $I_1$ connects $I_2$ by a cut point, in graph (2) $I_1$ connects $I_2$ by a bridge, and in graph (3) $I_1$ connects $I_2$ by a cycle.

However, it is easy to determine Hamiltonicity of these three connecting cases if we know Hamiltonicity of every $I$ cycle-set. Clearly, the key problem for identifying Hamiltonicity of a graph $G$ is to determine Hamiltonicity of every $I$ cycle-set. Hence, in the following pages, we will discuss the reduced graphs of $|\mathcal{I}|=1$ that are norm graphs.

Let $G$ be a norm graph, $\mathcal{G} \supseteq G$ (By the definition of $\mathcal{G}$, we have $V(\mathcal{G}) = V(G)$, and let $\mathcal{C}_K$ be a $C_k$ neighbor-cycle-set. We have the following results,

**Lemma 1.1**  $\mathcal{G} \approx K_{2,3} \Rightarrow \mathcal{G}$ has only one inducing subgraph $K_{2,3}$.

**Lemma 1.2**  $\mathcal{C}_K \approx K_{2,3}$. 
Lemma 1.3  G is Hamiltonian ⇒ ℓ ≈ C₃.

Theorem 1.1  G is non-Hamiltonian ⇔ ℓ ≈ K₂,₃.

2. Proofs

The proof of Lemma 1.1

Let G be a norm graph, ℓ ⊇ G. Suppose that ℓ ≈ K₂,₃. Considering K₂,₃ can be induced from ℓ, without loss of generality, we need only to discuss the cases of appearing two K₂,₃ subgraphs in operations. There have three cases, see Figure 3.

![Figure 3](image)

**Figure 3**  Three cases of appearing two K₂,₃ subgraphs

In a norm graph, if Case 1 appears in operations, then it does not meet the needs of the definition of ℓ, and so it does not exist. For appearing Case 2, there have only three connections in operations. See the dotted lines in

![Figure 4](image)

**Figure 4**

Figure 4. But, whatever the edge e combines with any edge such as a, b,
or c, there have no Case 2 existing. For Case 3 (bold lines in Figure 5), there only have five connections (subcases). For Subcase 1 and 2, by the solutions of Grinberg’s Equation of the given basis, all the removable cycles in operations must not include the edge a, thus no Case 3 existing in these two subcases. For Subcase 3, we have either a graph of $|I| > 1$ or no removable cycles can be selected in preorder operation, that implies that no Case 3 existing in Subcases 3. For Subcase 4, since one part of the graph exists $|P| \geq 3$ (i.e., right part of Subcases 4 in Figure 5), then removable cycles must not exist in preorder operation, hence, no Case 3 existing in Subcases 4. The inference of Subcase 5 is the same as that of Subcase 1 and 2.

![Figure 5](image_url)
In summary, underling the operations, \( g \approx K_{2,3} \) implies that \( g \) has only one inducing subgraph \( K_{2,3} \). □

**The proof of Lemma 1.2**
Since \( C_K \) is a \( C_k \) neighbor-cycle-set, then by operations it will yield a subgraph \( g \) including \( C_k \). According to **Lemma 3.2** in [1], \( |C_k| \neq 0 \Rightarrow |P| \geq 3 \Rightarrow g \approx K_{2,3} \). □

**The proof of Lemma 1.3**
By the definition of graphic homeomorphism, the lemma holds. □

**The proof of Theorem 1.1**
Let \( G \) be a norm graph. By **Lemma 1.1**, if \( g \approx K_{2,3} \), then, as a \( K_{2,3} \) set, \( g \) has only one subgraph \( K_{2,3} \).

\[ g \approx K_{2,3} \Rightarrow G \text{ is non-Hamiltonian.} \]
Assume \( g \approx K_{2,3} \). We consider the graphs of having solutions and the graphs of no solutions. (i) In a graph of having solutions, since \( g \) is a subgraph without any removable cycles by operations (deleting co-solution cycles in operations), then \( g \) is a subset of cycles of the same order of graph \( G \). Because of \( g \approx K_{2,3} \), there exists the case of \( |P| \geq 3 \) in a graph. By **Rule 1.1** in [1], \( G \) is non-Hamiltonian. Furthermore, we suppose that there exists another \( g' \neq K_{2,3} \) simultaneously in a graph of having solutions. It is clear that the symmetry difference of the cycles in \( g' \) produces a Hamilton cycle. But, in the operations of generating \( g \), there must appear the case of \( |P| \geq 3 \) when deleting a cycle in the given graph of having solutions. So the operations will repeatedly select removable cycles until it yields \( g' \) by deleting the right ones. This implies that eventually we obtain \( g' \) from \( g \). Note that \( g' \) and \( g \) do not exist in the same graph. So the suppose of \( g' \neq K_{2,3} \) is false. Hence, in a graph of having solutions, \( g \approx K_{2,3} \Rightarrow G \) is non-Hamiltonian. (ii) In a graph of no
solutions, by Theorem 1.1 in [2], G is non-Hamiltonian. So we have $g \approx K_{2,3} \Rightarrow G$ is non-Hamiltonian.

$g \not\approx K_{2,3} \Rightarrow G$ is Hamiltonian.  

If $g \not\approx K_{2,3}$ (note that $g$ is a subgraph without any removable cycles by operations), then, by the inverse of Lemma 1.2, $g$ is a common-edge type 2-common-vertex-combination of $|C_k|=0$ and $|P|<3$, that the result of symmetry difference of the cycles in subgraph $g$ is a Hamilton cycle. Thus, $g \not\approx K_{2,3} \Rightarrow G$ is Hamiltonian. \hfill \square

3. Remarks
The main result (Theorem 1.1) of this paper not only provides a theoretical approach to the determination of Hamilton graphs but also a conceptual map for solving graph isomorphism problem. Moreover, combining the results of this paper with that of [1], we find some combinatorial rules between two graphs of specified Hamiltoncity. We will describe these rules in a separate paper.

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