Scaling Laws of Stress and Strain in Brittle Fracture

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A numerical realization of an elastic beam lattice is used to obtain scaling exponents relevant to the extent of damage within the controlled, catastrophic and total regimes of mode-I brittle fracture. The relative fraction of damage at the onset of catastrophic rupture approaches a fixed value in the continuum limit. This enables disorder in a real material to be quantified through its relationship with random samples generated on the computer.

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Besides having wide practical relevance, breakdown in complex media touches on a range of fundamental issues involving structural disorder. However, it is only within the past two decades or so that adequate tools have become available which mimic how a truly complex material breaks. These tools, known as lattice models, have their origin in statistical physics and are especially well suited to describe the interplay between a continually evolving non-uniform stress-field and a random meso-structure.

Deviation from a perfect structure will usually affect the way a material breaks under strain. This is especially so where fibrous, porous or granular media are concerned. Moreover, in comparisons between components made from the same non-perfect material, sample-to-sample variations obtained in the response of stress to strain are typically large when the disorder is high. In this paper we calculate the average values of stress versus strain for a very large number of samples to obtain the breakdown characteristics of materials with a given type, or strength, of disorder.

Since the breaking characteristics of brittle materials depend crucially on the disorder, such information can be very useful in quality control. The size of the stable regime of fracture, for instance, within which an increasing amount of strain may be applied before catastrophic rupture sets in, increases with the disorder. Indeed, structural disorder on the microscopic level is often a desired trait in many materials. Consequently, if a relationship can be established between disorder and the breaking characteristics, preliminary conclusions may be drawn about average strength and its expected variability based on knowledge about the disorder in the material. This is helpful in cases where experiments are either costly or difficult to set up. With the disorder known, ultrasound or other non-destructive probing techniques can be used to determine how close a given component under stress is to complete failure. Vice versa, with a properly calibrated lattice model detailed experimental knowledge of the average stress-strain response should help to quantify the disorder and thereby predict, through simulations, other disorder-dependent properties which might not be easy to access experimentally. In alloys and composite materials such information can be used as an aid to optimize the mixture used, for instance, with respect to a desired strength specification.

Earlier results indicate that the scaling with system size of the maximum force obtained in quasi-static brittle fracture is universal with respect to the disorder. This has been reported in calculations with both the random fuse model and the beam model. Those results, however, were obtained at a time when numerical resources were far more limited than today. Our calculations present evidence which is contrary to this. Specifically, in large-scale numerical simulations involving a great many samples and spanning a wide range of disorders, the exponents which characterize scaling in the controlled, catastrophic and total regimes of mode-I brittle fracture are all found to be non-universal. Whereas for weak disorder the exponents are strongly dependent on the disorder, there is a slow convergence on the screened percolation value in the limit of infinite disorder.

A universal exponent, on the other hand, would be very useful for predicting strength properties since the scaling behaviour of stress and strain would then be the same for all materials, regardless of the details of the disorder. An additional relationship is required for the exponents to be useful in the case of non-universal scaling behaviour. Specifically, it should be possible for the disorder in a real material to be quantified in terms of the same parameters as those that are used to generate random samples on the computer. Presently we show that, in the continuum limit, the fraction of damage which occurs prior to catastrophic breakdown approaches a fixed value which is specific to the disorder.

It is indeed the ease with which disorder can be included which makes stochastic lattice models practical. The model used in our calculations is the elastic beam model used by Herrmann et al. in Ref. 1 and Skjetne et al. in Ref. 2. The disorder is imposed on the break-
existence of scaling laws [6]. Hence, for a range of disorders with $D > 0$. On the left are shown (a) $D = 0.1$, (b) $D = 0.17$, (c) $D = 0.25$ and (d) $D = 0.33$. On the right are shown (e) $D = 0.5$, (f) $D = 1$, (g) $D = 2$ and (h) $D = 4$. Whereas in (a)–(d) the scale on $\lambda_L$ is $\times 1.33 f_L$, it varies on the right as (e) $\times 2.5 f_L$, (f) $\times 5 f_L$, (g) $\times 13 f_L$ and (h) $\times 100 f_L$. The arrow in (a)–(d) shows the onset of damage for the $L = 100$ system.

where $0 \leq t \leq 1$ is distributed with a tail towards weak beams. For $D < 0$, we have

$$P(t) = 1 - t^{1/D},$$

now with $1 \leq t < \infty$ being distributed with a tail towards strong beams. The important thing is to include zero or infinity within the range of thresholds, otherwise the distribution is asymptotically equivalent to no disorder [6]. Both fundamental types of disorder are included in the present calculations. Fracture may then be fully explored as a function of disorder simply by varying the magnitude of $D$, with small or large values of $|D|$ corresponding to weak or strong disorders, respectively.

The lattice is broken by applying a uniform displacement to the nodes defining the top row, and the first beam to break is that which has the lowest axial strength. After this, the breaking sequence depends on how local stresses interact with the quenched disorder. Each time a beam is removed from the lattice, the new mechanical equilibrium is obtained by minimizing the elastic energy. In practice this is done via relaxation, i.e., using conjugate gradients [7]. The process by which mechanical equilibrium is attained is assumed to be much more rapid than the breaking of the beams, hence the simulation emulates quasi-static fracture.

Results obtained for $D > 0$ are shown in Fig. 1, where the data points represent the average values of stress and strain calculated for each broken beam. The exponents $\alpha$, $\beta$ and $\gamma$, obtained for the scaling of broken beams in the controlled (stable), catastrophic (unstable) and total regimes of fracture, i.e., $N_S$, $N_U$ and $N_T$, respectively, have been obtained from

$$N_S \sim L^\alpha, \quad N_U \sim L^\beta, \quad N_T \sim L^\gamma,$$

and are shown in Fig. 2.

For weak disorders most of the thresholds are to be found in the vicinity of the upper bound. On the application of external force fracture therefore proceeds in a controlled manner only for the first few breaks. During this phase, any small crack caused by the removal of a

FIG. 1: Force $f_L$ versus displacement $\lambda_L$ obtained for system sizes $L = 10, 14, 17, 20, 23, 27, 32, 40, 50, 63, 80$ and $100$, for a range of disorders with $D > 0$. On the left are shown (a) $D = 0.1$, (b) $D = 0.17$, (c) $D = 0.25$ and (d) $D = 0.33$. On the right are shown (e) $D = 0.5$, (f) $D = 1$, (g) $D = 2$ and (h) $D = 4$. Whereas in (a)–(d) the scale on $\lambda_L$ is $\times 1.33 f_L$, it varies on the right as (e) $\times 2.5 f_L$, (f) $\times 5 f_L$, (g) $\times 13 f_L$ and (h) $\times 100 f_L$. The arrow in (a)–(d) shows the onset of damage for the $L = 100$ system.

FIG. 2: The exponents $\alpha$ and $\beta$ for the scaling with system size, $L$, of damage in the controlled (solid lines) and catastrophic (dotted lines) regimes, respectively, as a function of the disorder $D$. Also shown is the exponent $\gamma$ for the scaling of total damage (●) with $L$. For comparison, the data from Ref. 4 (×), relevant to the total damage for $D = 0.5$, $D = 1$ and $D = 2$, have also been included.
weak beam is prevented from further opening up by the stronger thresholds in the immediate neighbourhood of the beam just broken. Since, at this stage, fracture is dominated by quenched disorder rather than stress, the external force must now be increased to break the next beam. A new crack will then most likely appear away from the neighbourhood of the previously broken beam, especially in the case of a large lattice. For small lattices the statistics of extremes dictates that the limited number of beams present should reduce the probability of weak thresholds occurring. Hence, fracture is unstable from the outset.

This cross-over in disorder between, on the one hand, systems for which there is always a regime of stable crack growth (regardless of system size) and, on the other hand, systems for which crack growth is unstable, or conditionally stable, from the very beginning, can, in fact, be identified. Using general arguments Roux et al. [8] showed that, for a fracture criterion of the type relevant to the fuse model, such a cross-over can be expected to occur at $D \sim 0.5$. Hence, for lower values of $D$ one should not expect scaling laws to exist.

The exact value of the disorder beyond which a controlled regime appears might be slightly different in the beam model, due to a different breaking formula. Nonetheless, Fig. 2 shows that a transition indeed occurs in the region $0.5 \lesssim D \lesssim 1$. Although the dependence of the exponents upon the disorder is less pronounced beyond $D = 1$, especially in the case of $\alpha$, the scaling is clearly non-universal, with the $D > 1$ exponents slowly approaching the value 2 in the limit of infinite disorder. The exponent which governs the scaling behaviour of the total damage, however, is very nearly $\alpha \approx 1.9$ within a wide range of disorders.

In Fig. 3 the onset of damage for the largest system included, $L = 100$, is shown by the small arrow in the cases of (a) $D = 0.1$, (b) $D = 0.17$, (c) $D = 0.25$ and (d) $D = 0.33$. In (a) there is a very small controlled regime before catastrophic rupture sets in. Stress and strain then backtracks along the original straight-line response towards the origin before encountering a section where decreasing values of force correspond to the same displacement. The situation here is one of conditional stability, where a small perturbation is sufficient to propagate the crack further.

Exponents relevant to fracture in the $D < 0$ regime are shown in Fig. 4 where stress-strain curves for disorders up to $|D| = 15$ have been analyzed. Stress-strain curves between $D = -0.5$ and $D = -4$ are shown in Fig. 4. Here the initial response, a straight line from the origin to the position of the first broken beam, is not shown. Fig. 5 shows that for small $D$ there exists a regime of finite extent, i.e., $-2 \leq D \leq 0$, within which $\alpha = 1$. The scaling behaviour here is trivial, being simply proportional to system size, and the first beam to break triggers catastrophic rupture. Within this regime the tail towards strong beams is not very pronounced and consequently the next beam in the path of the crack will most likely be comparable in strength with that just broken. Since the loading on neighbouring beams is rendered higher by its removal, crack growth is now localized. The case of
For large $|D|$ the essential features are similar to those for $D > 0$, with the scaling being governed by non-universal exponents. As before, the scaling of total damage shows the weakest variation, with $\alpha \approx 1.9$ within a very large region of disorders.

At the onset of catastrophic rupture the relative amount of damage is $\tilde{C} = N_C/N_T$, where $N_C$ is the number of beams broken in the controlled phase and $N_T$ is the total number of beams broken. In Fig. 5 this quantity is shown as a function of inverse system size for $D > 0$. For the sizes included in the present calculations, a straight-line relationship is obtained for disorders larger than $D = 1$. With the $L = 125$ data having been left out due to low statistical significance, a line has also been fit to the data for $D = 0.5$, although the relationship is probably not linear here. For disorders below $D = 0.5$ the relationship is clearly non-linear, as can be expected based on the arguments of Ref. 8.

The same data are shown in Fig. 6 now however as functions of the disorder. The result extrapolated for the continuum limit, corresponding to $L = \infty$, is the intersection of the straight-line fit in Fig. 5 with the vertical axis $1/L = 0$. This establishes a link between the disorder, as generated on the computer (presently using a power-law distribution), and the stress-strain characteristic obtained in the breaking of real materials. Such a relationship is valuable where comparisons are sought between simulation and experiment. As such, the present quasi-static result for brittle materials is mainly relevant to breakdown associated with fatigue, i.e., slow fracture processes.

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