Abstract: We give to the categorical theory $\text{PR}$ of Primitive Recursion a logically simple, algebraic presentation, via equations between maps, plus one genuine Horner type schema, namely Freyd’s uniqueness of the initialised iterated. Free Variables are introduced – formally – as another names for projections. Predicates $\chi : A \to 2$ admit interpretation as (formal) Objects $\{A \mid \chi\}$ of a surrounding Theory $\text{PR}_A = \text{PR} + (\text{abstr}) : \text{schema (abstr)}$ formalises this predicate abstraction into additional Objects. Categorical Theory $\text{P} \hat{\text{R}}_A \sqsubseteq \text{PR}_A \sqsubseteq \text{PR}$ then is the Theory of formally partial $\text{PR}$-maps, having Theory $\text{PR}_A$ embedded. This Theory $\text{P} \hat{\text{R}}_A$ bears the structure of a (still) diagonal monoidal category. It is equivalent to “the” categorical theory of $\mu$-recursion (and of while loops), viewed as partial $\text{PR}$ maps. So the present approach to partial maps sheds new light on Church’s Thesis, “embedded” into a Free-Variables, formally variable-free (categorical) framework.
1 Introduction

We develop here, from scratch, a formally variable-free, categorical Theory of $\mu$-recursion, without use of formal quantification: This Theory is formalised on the basis of a theory $\mathbf{PR}_A$ of partial PR maps which in turn is introduced as a definitional, conservative extension of Theory $\mathbf{PR}_A = \mathbf{PR} + \text{(abstr)}$, the latter obtained from fundamental categorical Theory $\mathbf{PR}$ of Primitive Recursion: we formally interpret PR predicates $\chi : A \to 2$ as additional, defined Objects of emerging Theory $\mathbf{PR}_A$: schema of abstraction already “hidden” in fundamental Theory $\mathbf{PR}$ of Primitive Recursion. The latter is given as Cartesian Hull over data and axioms of a Natural Numbers Object $\mathbb{N}$, this in the sense of Lawvere, Eilenberg & Elgot, and Freyd.

Central in present approach is the notion of a partial PR map: Such a (formally) partial map $f : A \to B$ is given, in categorical, variable-free terms, by:

- an enumeration source Object $D_f$ in $\mathbf{PR}_A$ (e.g.: $D_f = \mathbb{N}$),
- a $\mathbf{PR}_A$-map $d_f : D_f \to A$, meaning for $\mathbf{PR}_A$-enumeration of defined arguments of $f$,
- and a $\mathbf{PR}_A$-rule-map $\hat{f} : D_f \to B$ for $f$,
- these data with (intuitive) meaning: for $a \in A$ defined argument, of form $a = d_f(\hat{a}) \in A$ for suitable $\hat{a} \in D_f$:

$$a = d_f(\hat{a}) \in A \text{ for suitable } \hat{a} \in D_f : d_f(\hat{a}) = a \xrightarrow{f} \hat{f}(\hat{a}) \in B.$$ 

We prove a Structure Theorem for partial-map extensions $\hat{S} \sqsubseteq S$, where $S$ is a Cartesian PR theory with schema of predicate abstraction – mostly $S := \mathbf{PR}_A$ or one of its strengthenings – which establishes these extensions (via embedding) as diagonal monoidal PR theories: Cartesian structure is lost in part, since the (still present) projection- and terminal-map-families do not preserve their character as natural transformations in the extension. These partial-map extensions turn out to be Closures: $\hat{S} \cong \hat{S} :$ Partial partial maps have a representation as just partial maps.

Within this Free-Variables (formally: variable free) categorical framework $\mathbf{PR}_A$ for partial PR map theories, we discuss (Free-Variables) category based $\mu$-recursion as well as content driven loops such as while loops.

This prepares in particular discussion of termination for suitable special such loops, namely those given as Complexity Controlled Iterations, for which iteration step decreases a complexity measure within a suitably given (constructive) ordinal $O$, “until” minimum $0$ of $O$ is reached. Complexity Controlled Iteration is basic for the following second part of investigation on Recursive Categorical Foundations, RCF 2, entitled Evaluation and Consistency.
Evaluation of PR map codes is there resolved into such an iteration with descending complexity values. “Hence” we can “hope” this formally partial evaluation to always terminate, by reaching complexity 0. In that case within ordinal $O$ taken the lexicographically ordered set of polynomials over $\mathbb{N}$, in one indeterminate.

2 Notions, Axioms, Results for Theories PR and PR$_A$

Fundamental Theory PR of Primitive Recursion is the minimal, “initial” Cartesian Theory with (universal) Natural Numbers Object:

As Objects it has $1, N, \ldots, A, B, \ldots, (A \times B)$, i.e. all (binary bracketed) finite powers of Natural Numbers Object (“NNO”) $\mathbb{N}$.

It comes with associative composition “$\circ$”, functorial cylindrification $(A \times g) : A \times B \to A \times B'$ – and from this with bifunctorial Cartesian product “$\times$” – and is generated over basic map constants $0 : 1 \to \mathbb{N}$ (zero), successor $s : \mathbb{N} \times \mathbb{N}$, as well as terminal map $! : A \to 1$, diagonal $\Delta : A \to A \times A$, and projections $\ell : A \times B \to A$ and $r : A \times B \to B$.

For given $f : C \to A$ and $g : C \to B$, induced map $(f, g) : C \to A \times B$ into Cartesian Product is defined as

$$(f, g) =_{\text{def}} (f \times g) \circ \Delta_C : C \to C^2 =_{\text{by def}} C \times C \to A \times B,$$

with $(f \times g) : C^2 \to A \times B$ defined by either – equal(!) – sequence of cylindrified above, see diamond sub-DIAGRAM in

Binary Cartesian Map Product DIAGRAM

In present context we take Godement’s equations for Cartesian (!) Prod-
uct, as axioms:

Uniqueness of the induced \((f, g) =: \text{by def } (f \times g) \circ \Delta_C : C \to C^2 \to A \times B\) is forced – axiomatically – by the following Fourman’s equation (equational schema):

\[
\begin{align*}
    \text{(Four!)} \quad h : C &\to A \times B \text{ map} \\
    (\ell_{A,B} \circ h, \ r_{A,B} \circ h) &= h : C \to A \times B.
\end{align*}
\]

The NNO \( \mathbb{1} \xrightarrow{0} \mathbb{N} \xrightarrow{\delta} \mathbb{N} \) is given – by axiom – the following property, expressed as iteration schema:

As Uniqueness schema we need the following one – of Freyd, formally stronger than uniqueness of the iterated \( f^\delta : A \times \mathbb{N} \to A \) satisfying the equations of Basic Iteration diagram above – namely Uniqueness of initialised iterated:

\[
\begin{align*}
    f : A &\to B, \ g : B \to B, \ h : A \times \mathbb{N} \to B, \text{ all in } \text{PR,} \\
    h \circ (\text{id}_A, 0 \circ !_A) &= f : A \to B, \text{ (init)} \\
    h \circ (A \times s) &= g \circ h : A \times \mathbb{N} \to B, \text{ (step)} \\
    h = g^\delta \circ (f \times \mathbb{N}) : A \times \mathbb{N} \to B \times \mathbb{N} \to B,
\end{align*}
\]

in terms of Freyd’s pentagonal diagram:
Schemata (it) and (FR!) give in fact the well-known full schema (pr) of Primitive Recursion:

\[
g = g(a) : A \rightarrow B \quad \text{PR (init map)}
\]

\[
h = h((a, n), b) : (A \times \mathbb{N}) \times B \rightarrow B \quad \text{(step map)}
\]

\[
f = f(a, n) = \text{pr}[g, h](a, n) : A \times \mathbb{N} \rightarrow B \quad \text{in PR such that}
\]

\[
f(a, 0) = g(a) : A \rightarrow B \quad \text{(init), and}
\]

\[
f(a, s n) = h((a, n), f(a, n)) : (A \times \mathbb{N}) \rightarrow B \quad \text{(step)}
\]

as well as

\[
\text{(pr!)} : f \text{ is unique with these properties.}
\]

This full schema of PR has as consequence in particular the elegant and powerful Uniqueness schemata of Goodstein, U₁ - U₄, whith passive parameter, usually \( a \in \mathbb{N} \) here made explicit.

Using these schemata, Goodstein proves central equations for addition, truncated subtraction, and multiplication on the “NNO” \( \mathbb{N} \):

As usual, the basic structure of a unitary commutative semiring on NNO \( \mathbb{N} \) – (plus truncated subtraction and exponentiation) – is defined – and characterised, map theoretically – with interpretation of Free Variables \( a \) and \( n \) as
projections $\ell : A \times \mathbb{N} \to A$ and $r : A \times \mathbb{N} \to \mathbb{N}$ — as follows:

- $0, 1 =_{\text{by def}} s 0 : 1 \to \mathbb{N}$: zero resp. one,
- addition $a + n = s^{\ell}(a, n) : \mathbb{N}^2 \to \mathbb{N}$ by
  
  $a + 0 = a : \mathbb{N} \to \mathbb{N}$, $a + s n = s(a + n) : \mathbb{N}^2 \to \mathbb{N}$,
- truncated subtraction $a \div n : \mathbb{N}^2 \to \mathbb{N}$ by:
  
  $a \div 0 = a$, $a \div s n = \text{pre}(a \div n)$, with predecessor $\text{pre} : \mathbb{N} \to \mathbb{N}$
- PR defined (and characterised) by $\text{pre} 0 = 0$, $\text{pre} s n = n$, so for example $5 \div 2 = 3$, but $2 \div 5 = 0$;
- multiplication $a \cdot n : \mathbb{N}^2 \to \mathbb{N}$ by
  
  $a \cdot 0 = 0$, $a \cdot s n = a \cdot n + a$,
- and exponentiation $a^n : \mathbb{N}^2 \to \mathbb{N}$ by
  
  $a^0 = 0$, $a^{s n} = a^n \cdot a$.

**Remark** on use of free variables: Interpretation of Free Variables as (identities) resp. projections, (possibly nested) is subject to formal rules extending this interpretation in the above example. Vice versa, projections can be seen as free variables, by (re)naming them with names $a, b, \ldots, x, y, \ldots$ usually standing for free “individual” variables. From now on, we will make extensive use of Free Variables, anchored in the Cartesian structure of basic theory $\text{PR}_A = \text{PR} + (\text{abstr})$ and its strengthenings.

**Continuation** of elementary map equations for NNO $\mathbb{N} : \text{Multiplication}$ on $\mathbb{N}$ distributes not only over addition but over truncated subtraction as well, almost (again) by definition.

(Boolean) Logic and Order then are defined, and characterised as follows, the latter essentially via truncated subtraction:

- $\text{sign} = \text{sign}(n) : \mathbb{N} \to \mathbb{N}$ by $\text{sign}(0) = 0$, $\text{sign}(s n) = 1 = s 0 : \mathbb{N} \to 2$, $\neg = \neg(n) : \mathbb{N} \to \mathbb{N}$ by $\neg(0) = 1$, $\neg(s n) = 0$.
- A map $\chi : A \to \mathbb{N}$, here such a map within $\text{PR}$, is called a predicate (on Object $A$) if

  $\text{sign} \circ \chi = \neg \circ \neg \circ \chi = \text{chi} : \mathbb{N} \to \mathbb{N}$.

By obvious reason, we write for this

$\chi : A \to 2 \subset \mathbb{N}$.

For the moment, this is just notation. Later, in theory $\text{PR}_A$ to come, Object $2 = \{n \in \mathbb{N} | n < s s 0\} \subset \mathbb{N}$ will become an Object on its own right.\(^1\)

\(^1\) The idea to introduce Boolean (!) Free-Variables predicate Calculus into the Theory of Primitive Recursion this way, without an explicit basic (“undefined”) Object $2 = \{\text{false}, \text{true}\} \equiv \{0, 1\}$ goes back to GOODSTEIN 1971 and REITER 1982.
Using this notation, we get the Boolean Operations, and basic binary predicates on \( \mathbb{N} \) as
\[
\land = [a \land b] = \text{def} \quad \text{sign}(a \cdot b) : \mathbb{N}^2 \to \mathbb{N},
\]
\[
[m \leq n] = \text{def} \quad \neg [m \div n] : \mathbb{N}^2 \to 2.
\]
\[
[m < n] = [m \leq n] \land [n \leq m] : \mathbb{N}^2 \to 2,
\]

and directly from “\( \leq \)”:
\[
[m = n] = [m \leq n] \land [n \leq m] : \mathbb{N}^2 \to 2.
\]
The latter predicate is \textit{predicative equality}. It can be extended to all \textit{fundamental Objects} (finite bracketed powers of \( \mathbb{N} \)), by componentwise conjunctive definition. And furthermore to (formal, virtual) predicate \textit{extensions} \( \{A | \chi : A \to 2\} \) of \( \text{PR} \)-predicates \( \chi \), see below.

[Addition as well as multiplication – the latter with arguments greater zero – are strictly \textit{monotonous} with respect to the (linear) order introduced above via truncated subtraction.]

This equality “\textit{of individuals}” \( [a \doteq a'] : A^2 \to 2 \) is \textit{reflexive, symmetric, and transitive}. On \( \mathbb{N} \) it satisfies – with regard to strict order \( < : \mathbb{N}^2 \to \mathbb{N} \) – the \textit{law of trichotomy}.

Furthermore it is – as we expect it from a “sound” notion of \textit{equality} – \textit{substitutive} (in \textsc{Leibniz}’ sense), with respect to the “basic meta-operations” \textit{composition, Cartesian Product}, as well as \textit{Iteration,} and therefore in particular with respect to \textit{all} operations (above), introduced on \( \mathbb{N} \) via these basic \textit{meta-operations}.

The key relationship between this \textit{predicate} and the – already given – notion of \textit{equality} between maps:
\[
f = \text{PR} \quad g : A \to B, \text{ also written } \text{PR} \vdash f = g : A \to B :
\]

is the following \textit{Equality Definability Schema}, which is \textit{derivable} in theory \( \text{PR} \), and also in Theory \( \text{PR}_A = \text{PR}+(\text{abstr}) \) to come, as well as in all strengthenings of these:

\textbf{Equality Definability Theorem:} An Arithmetical Theory \( T \), i.e. \( T \) an extension of \( \text{PR} \), admits the following schema (EquDef):
\[
\begin{align*}
\text{(EquDef)} & \quad f, g : A \to \mathbb{N} \text{ in } T, \\
T \vdash f \equiv \circ (f, g) = \text{true}_2 : \mathbb{N}^2 \to 2 & \quad T \vdash f = g : A \to \mathbb{N}, \text{ algebraically: } f =^T g : A \to \mathbb{N}.
\end{align*}
\]

\textit{Equality Definability} extends to \( T \)-map pairs \( f, g : A \to B \) with (common) codomain a (finite) Cartesian Product \( B \) of Objects \( \mathbb{N} \), a \textit{fundamental Object}
– in PR – or even B a predicative extension $B = \{ C \mid \gamma \}$ of such an Object, an Object of Theory PR$_A$ for short, see below.

**Remark:** For proof of the laws of multiplication, and also for proof of logically all important Equality Definability above, GOODSTEIN proves commutativity of the maximum function, namely

$$\max(a, b) =_{\text{def}} a + (b - a) = b + (a - b) =_{\text{by def}} \max(b, a) : \mathbb{N}^2 \rightarrow \mathbb{N}.$$

We now “realise” straight forward the schema of predicate abstraction: We start with a (PR) predicate $\chi : A \rightarrow 2$ on a fundamental Object $A$ is a (binary bracketed) Cartesian product out of $\mathbb{N}$ and $\mathbb{1}$. Such a predicate $\chi : A \rightarrow 2$, formally: $\chi : \mathbb{N} \rightarrow \mathbb{N}$ (see above), can be turned into a virtual, “new” Object easily: just take as Objects of new frame, extended Theory PR$_A$ the fundamental maps of form $\chi : A \rightarrow 2$ (predicates) within fundamental Theory PR above, and as maps between such Objects $\{ A \mid \chi \}$ and $\{ B \mid \psi \}$ those PR-maps $f : A \rightarrow B$ which transform $\chi$ into $\psi$. Two such maps $f, g : A \rightarrow B$ are identified, declared equal, if they agree “on” $\chi$. This definitional, conservative extension is called here PR$_A = \text{PR} + (\text{abstr}) \sqcup \text{PR}$, the basic Theory of Primitive Recursion.

**Structure Theorem for PR$_A$ : Basic Theory PR$_A$ becomes a Cartesian Category (Theory), has all Extensions of its predicates and therefore all equalisers, and contains fundamental Theory PR embedded via**

$$\langle f : A \rightarrow B \rangle \mapsto \langle f : \{ A \mid \text{true}_A \} \rightarrow \{ B \mid \text{true}_B \} \rangle.$$  

Furthermore, it has all equality predicates – by restriction – and admits the schema of Equality Definability.

**Proof** (Reiter 1980): A preliminary version of PR$_A$ is constructed as canonical extension of PR’s Class of Objects into its predicates and then admitting as maps between these “new” Objects those of PR which are compatible with the given, defining predicates for these “Objects”. This way one gets a Cartesian PR theory. Equality of PR$_A$ maps, given by their equality on the predicate of their PR$_A$ Domain Object, is compatible with this Cartesian PR structure and thereby gives the Cartesian PR structure of PR$_A$, defined as Quotient theory by this notion of equality. Forming the iterated is likewise compatible with this canonical notion of equality in the extended world, and therefore PR$_A$ becomes this way a PR category. By construction, PR$_A$ gets in addition the wanted predicate-extension and equaliser-by-extension structure: it has finite limits, in particular pullbacks (and multiple pullbacks) q.e.d.

**Remarks:**

(i) A PR$_A$-map $f : \{ A \mid \chi \} \rightarrow \{ B \mid \psi \}$ can be viewed as a defined partial PR map from A to B with values in $\psi$: Set of defined arguments, namely $\{ a \in A \mid \chi(a) \}$ is PR decidable. By definin of PR$_A$’s equality, PR-map $f : A \rightarrow B$ “doesn’t care” about arguments a in the complement $\{ a \in A \mid \neg \chi(a) \}$.  

8
So wouldn’t it be easier to realise this view to *defined partial maps* just by throwing the *undefined arguments* into a *waste basket* \{⊥\} say?

But where to place this waste basket, this for each Codomain Object \(B\)? The fundamental Objects have a zero-vector as a candidate. For example we could interprete truncated subtraction as a *defined partial map*

\[
a - b : \{(m, n) \in \mathbb{N} \times \mathbb{N} \mid m \geq n\} \rightarrow \mathbb{N},
\]

and throw the complement \{\((m, n) \in \mathbb{N} \times \mathbb{N} \mid m < n\}\} into waste basket \{0\} \subset \mathbb{N}.

But this is not a good interpretation of *truncated (!) subtraction*: Value 0 is not waste, it has an important meaning as zero.

“The” waste basket \{⊥\} should be an entity with a *natural* extra representation, and we should have only one such entity in a later theory of defined partial PR maps to come. This theory, to be called \(\text{PR}^X_A\), will be constructed with the help of Universal Object \(X\) which is to contain *codes* of all singletons and (nested) pairs of natural numbers, and “below” these codes it has room for code of *undefined value* symbol \(_\bot\), in a “Hilbert’s hotel”. All this will be carried through within present theory \(\text{PR}_A\).

(ii) A PR-map \(f : A \rightarrow B\) such that \(f\) “is” a \(\text{PR}_A\)-map

\[
\begin{align*}
f : \{A \mid \chi \lor \chi' : A \rightarrow 2\} & \rightarrow \{B \mid \psi\}, \text{ also “works” as a } \text{PR}_A\text{-map} \\
f : \{A \mid \chi\} & \rightarrow \{B \mid \psi\}, \text{ and a } \text{PR}_A\text{-map} \\
g : \{A \mid \chi\} & \rightarrow \{B \mid \psi \land \psi'\} \text{ also “works” as a } \text{PR}_A\text{-map} \\
g : \{A \mid \chi\} & \rightarrow \{B \mid \psi\}.
\end{align*}
\]

Since map-properties of *injectivity*, *epi-property* of PR-maps viewed as \(\text{PR}_A\)-maps, *depend* on choice of hosting (predicative) \(\text{PR}_A\) Objects – examples above – *specification* of a \(\text{PR}_A\) map \(f : \{A \mid \chi\} \rightarrow \{B \mid \psi\}\) must contain, besides PR-map \(f : A \rightarrow B\), Domain and Codomain Objects \(\chi : A \rightarrow 2\) and \(\psi : B \rightarrow 2\) as well. This way the members of Map set family \(\text{PR}_A(A, B) : A, B\ \text{PR}_A\text{-Objects, become mutually disjoint}. Inclusions \(i : A' \subseteq A''\) are realised in \(\text{PR}_A\) as restricted PR-identities \(\text{id}_A : \{A \mid \chi'\} \xrightarrow{\subseteq} \{A \mid \chi''\}, \chi' \Rightarrow \chi''\).

3 Goodstein Arithmetic GA

In “Development of Mathematical Logic” (Logos Press 1971) R. L. Goodstein gives four basic uniqueness–rules for Free–Variable Arithmetics. We show here that these four rules are sufficient for proving the commutative and associative laws for multiplication and the distributive law, for addition as well as for truncated subtraction.

We include\(^2\) into Goodstein’s uniqueness–rules a “passive parameter” \(a\). These extended rules are provable using Freyd’s uniqueness Theorem (pr!), part

\(^2\)Sandra Andrasek and the author
of full schema (pr) of Primitive Recursion which he deduces from his uniqueness (FR!) of the initialised iterated. FREYD deduces the latter from availability of a Natural Numbers Object \( N \) in LAWVERE’s sense, and (!) axiomatic availability of “higher order” internal hom objects with, again axiomatic, evaluation map family for these objects, of form \( \epsilon_{A,B} : B^A \times A \to B \) within (!) the category considered.

**Goodstein’s rules with passive parameter:** Let \( f, g : A \times N \to N \) be primitive recursive maps, \( s : N \to N \) the successor map \( n \mapsto n + 1 \) and \( \text{pre} : N \to N \) the predecessor map, usually written as \( n \mapsto n - 1 \).

Then Goodstein’s rules read:

\[
\begin{align*}
\text{U}_1 & \quad f(a, sn) = f(a, n) : A \times N \to B \\
& \quad f(a, n) = f(a, 0) : A \times N \to B \\
\text{U}_2 & \quad f(a, s n) = s f(a, n) : A \times N \to N \\
& \quad f(a, n) = f(a, 0) + n : A \times N \to N \\
\text{U}_3 & \quad f(a, sn) = \text{pre} f(a, n) : A \times N \to N \\
& \quad f(a, n) = f(a, 0) \downarrow n : A \times N \to N \\
\text{U}_4 & \quad f(a, sn) = g(a, sn) : A \times N \to N \\
& \quad f(a, n) = g(a, n) : A \times N \to N.
\end{align*}
\]

**Comment:** Theories PR and PR\(_A\) allow, within rules U\(_1\) and U\(_4\) above, for replacing \( N \) as a Codomain Object, by an arbitrary object \( B \) of PR resp. PR\(_A\).

Rule U\(_4\), of uniqueness of maps defined by case distinction, is nothing else than uniqueness of the induced map out of the sum \( A \times N \cong (A \times 1) + (A \times N) \), this sum canonically realised via injections \( \iota = (\text{id}_A \times 0) : A \times 1 \to A \times N \) as well as, right injection: \( \kappa = \text{id}_A \times s : A \times N \to A \times N \):

This uniqueness combined with LEIBNIZ’ compatibility of induced-map-out-of-a-sum with map (term) equality, compatibility available in Theories PR, PR\(_A\), and their strengthenings.

**Proof** of these four rules is straight forward for theories PR, PR\(_A\) (and strengthenings), using FREYD’s uniqueness (FR!) and uniqueness clause (pr!) of the full schema of Primitive Recursion respectively.

U\(_1\)-U\(_4\) give, by means of a derived schema \( V_4 \), the wanted

**Structure Theorem** for NNO \( N : N \) admits the structure of a commutative semiring with zero, truncated subtraction \( \downarrow = m \downarrow n \mapsto N \), over which multiplication distributes, a linear order \( m < n : N^2 \to 2 \), and equality predicate \( \doteq = [m \doteq n] : N^2 \to 2 \), both of the latter defined via truncated subtraction.
Order and equality (predicates) satisfy the law of trichotomy; addition is strictly monotoneous in both arguments; truncated subtraction is weakly monotoneous in first, and weakly antitoneous in second argument, whereas multiplication is strongly monotoneous in both arguments, on \( \mathbb{N}_{>0}^2 = \{ n \in \mathbb{N} | n > 0 \}^2 \).

\( \mathbb{N} \) and \( \{ n \in \mathbb{N} | n < 2 \} \) admit (2-valued) **Boolean Logic** sign, \( \neg, \land, \lor, \implies \).

Last but not least, the maximum \( \max(a,b) : \mathbb{N}^2 \to \mathbb{N} \) commutes:

\[
\text{PR} \vdash \max(a,b) =_{\text{def}} a + (b - a) = b + (a - b) =_{\text{by def}} \max(b,a) : \mathbb{N}^2 \to \mathbb{N}.
\]

This goes into **Proof of Equality Definability**, for PR, \( \text{PR}_A \), and strengthenings.

**Proof**, by \( U_1 - U_4 \), straightforward but tedious.

### 4 Theories of Partial PR Maps

We now turn to Extension of PR theories – here Theory \( \text{PR}_A \) or a strengthening – into Theories of **partial** PR maps (**map terms**). This extension is understood best, wenn looking at the following **DIAGRAM** which shows **composition** of two such partial maps, \( g : B \to C \) with \( f : A \to B \):

![Composition Diagram](attachment:composition_diagram.png)

**Composition diagram for \( \tilde{S} \)**

A partial map \( f = \{ (d_f : \tilde{f}) : D_f \to A \times B \} : A \to B \) consists of a (PR) enumeration \( d_f = d_f(\hat{a}) : D_f \to A \) of **defined-arguments** for \( f \), and a **rule** \( \tilde{f} = \tilde{f}(\hat{a}) : D_f \to B \), fixing the values \( f(a) =_{\text{def}} \tilde{f}(\hat{a}) \) for **defined** arguments \( a \in A \), i.e. \( a \) of form \( a = d_f(\hat{a}) \), \( \hat{a} \in D_f \).

Up to here, \( f \) defines just a **relation**, in the sense of Brinkmann/Puppe 1969. What is lacking is **right uniqueness**, see the following
Partial-Map Schema:

\[\gamma f = \gamma f(\hat{a}) : D_f \to A \times B\]  
S-map,
called graph (of \(f : A \to B\) to be introduced),
\[d_f = d_f(\hat{a}) = \ell \circ \gamma f : D_f \to A\]  
defined arguments enumeration
\[\hat{f} = \hat{f}(\hat{a}) : D_f \to B\]  
rule
\[S \vdash d_f(\hat{a}) \cong d_f(\hat{a}') \implies \hat{f}(\hat{a}) \cong \hat{f}(\hat{a}') : D_f^2 \to 2\]  
(right uniqueness)
\[\hat{S}\]-map, partial S-map.

\[\hat{f} \triangleq \langle \gamma f = (d_f, \hat{f}) : D_f \to A \times B \rangle : A \to B\]

\(\hat{S}\)-morphism, partial S-map.

Diagram above then shows – Brinkmann & Puppe type – partial map composition \(g \circ f : A \to B \to C\) via pullback.

Equality \(f \cong g : A \to B\) of partial maps – over \(S\) – is established by availability of a pair \(D_f \xrightarrow{i} D_{f'}\) of defined-arguments comparison maps (in \(S\)) which are compatible as such with \(d_f, d_{f'}\) as well as with \(\hat{f}, \hat{f}'\). Availability of just one of these \(S\) maps, of \(i : D_f \to D_{f'}\) say, defines “graph” inclusion, here \(f \subseteq f' : A \to B\).

Basic compatibility of (partial) composition “\(\circ\)”, with graph inclusion \(\subseteq\) – and hence with partial equality “\(\cong\)” then is given by the universal properties of (composed) pullback in the following diagram:\[3\]

Compatibility Diagram of \(\circ\) with \(\subseteq\)

Furthermore, composition via pullback above then is associative, by natural equivalence of the (finite) limits defining compositions

\[h \circ (g \circ f), (h \circ g) \circ f : A \to B \to C \to D.\]

\[^3\text{F. Hermann}\]
Cylindrification is componentwise, and gives the Cartesian Product for $\text{PR}_A$ as a monoidal – again bifunctorial one – within extended Theory $\text{PR}_A \sqsubset \text{PR}_A$. But this extended Product does not have anymore (Godement’s) universal properties of a Cartesian Product, within $\text{PR}_A$.

Iteration in $\text{PR}_A$ works analogously to composition, using in this case pullback iteration.

**Equality Definability:** There is such a Theorem also for partial map Theory $\text{PR}_A$ and its strengthenings.

**Structure Theorem for $\hat{\text{S}}$:**

(i) $\hat{\text{S}}$ carries a – canonical – structure of a diagonal symmetric monoidal category, with partial composition $\circ$ and identities introduced above, (monoidal) product $\times$ extending $\times$ of $\text{S}$, association $\text{ass} : (A \times B) \times C \overset{\simeq}{\to} A \times (B \times C)$, symmetry ("transposition") $\Theta : A \times B \overset{\simeq}{\to} B \times A$, and diagonal $\Delta : A \to A \times A$ inherited from $\text{S}$; cf. BUDACH & HOEHNKE 1975 and – later – PFENDER 1974 for an axiomatic approach to categories with a given type of substitution transformations. Our present theory $\hat{\text{S}}$, a theory of partial PR maps, is a monoidal category, which has – in addition to natural transformations $\text{ass}$, $\Theta$, and $\Delta$ above –, so-called half-terminal maps, and the former projections as half-projective ones, in the terminology of BUDACH & HOEHNKE, “half” since the latter natural families of Theory $\text{S}$, are no longer natural transformations for Theory $\hat{\text{S}}$. All of this substitutive structure is obviously preserved by the embedding $\text{S} \sqsubseteq \hat{\text{S}}$.

(ii) The defining diagram for an $\hat{\text{S}}$-map – namely

\[
\begin{array}{ccc}
D_f & \xrightarrow{d_f} & \hat{f} \\
\downarrow{d_f} & & \downarrow{\hat{f}} \\
A & \xrightarrow{f} & B
\end{array}
\]

**Partial Map Diagram**

– constitutes in fact a commuting $\hat{\text{S}}$ diagram.

Conversely – with same notation as above – define the minimised opposite to $d_f$, beginning with formally partial, $\hat{\text{S}}$ map

\[
d_f' = \langle (d_f, \; [\hat{f}]) : D_f \to A \times D_f \rangle : A \to D_f,
\]

opposite (graph) to given $\text{S}$ map $d_f : D_f \to A$. This opposite is made right-unique by selecting $D_f$-minimal $\hat{f}$ equivalence representant

\[
[\hat{f}] = \{ \hat{a} \} = \min_{d_f} \{ \hat{a}' \leq \hat{a} \mid \hat{f}(\hat{a}') \vdash_{D_f} \hat{f}(\hat{a}) \} : D_f \to D_f:
\]

---

4there is an earlier preprint of BUDACH & HOEHNKE
Minimal with respect to here “canonical”, CANTOR-ordering on $S$ Object $D_f = \{D \mid \delta : D \to 2\}$. This order is inherited from $D_f$’s “mother” fundamental Object, $D$, say. This object in turn is (well) ordered via canonical counting

$$\text{cantor}_D = \text{cantor}_D(n) : \mathbb{N} \xrightarrow{\cong} D,$$

(see general schema above of PR dominated minimum), and get the commuting $\hat{S}$-DIAGRAM

\[
\begin{array}{c}
\text{Basic Partial Map DIAGRAM} \\
\begin{array}{c}
D_f \\
\downarrow \hat{f}
\end{array}
\begin{array}{c}
\hat{f}
\end{array}
\begin{array}{c}
\downarrow \\
B
\end{array}
\begin{array}{c}
A \\
\hat{f} \equiv f
\end{array}
\end{array}
\]

(iii) The first factor $f : A \to B$ in a $\hat{P}R_A$-composition $h = g \circ f : A \to B \to C$, when giving an (embedded) $PR_A$ map $h : A \to C$, is itself an (embedded) $PR_A$ map: first factor of a total map is total.

So each section (“coretraction”) of $\hat{P}R_A$ is a $PR_A$ map, in particular a $\hat{P}R_A$ section of a $PR_A$ map is in $PR_A$.

(iv) $\hat{S}$ clearly inherits from $S$ FOURMAN’s uniqueness equation:

For $h : C \to A \times B$ in $\hat{S}$: $h \cong (h \circ \ell, h \circ r) : C \to A \times B$,

where for $f : C \to A$, $g : C \to B$,

$$(f, g) =_{\text{def}} (f \times g) \circ \Delta_C : C \to C \times C \to A \times B,$$

with diagonal $\Delta_C : C \to C \times C$ of $S$.

This equation guarantees uniqueness of the “induced” $(f, g) : C \to A \times B$, but $(f, g)$ does not satisfy (both of) the Cartesian equations

$$\ell \circ (f, g) \cong f \quad \text{and} \quad r \circ (f, g) \cong g,$$

except $f$ and $g$ have equal domains of definition, i.e. if $i : D_f \to D_g$, $j : D_g \to D_f$ are available such that $d_g \circ i = d_f : D_f \to A$ as well as $d_f \circ j = d_g : D_g \to A$.

(v) Iteration $f^\$ : $A \times \mathbb{N} \to A$ of $\hat{S}$-endo is available in $\hat{S}$, satisfying the characteristic $\hat{S}$-equations

$$f^\$ \circ (\text{id}_A, 0) =_{\text{by def}} f^\$ \circ (A \times 0) \circ \Delta_A \cong \text{id}_A : A \to A, \quad \text{and}$$

$$f^\$ \circ (A \times s) \cong f \circ f^\$ : A \times \mathbb{N} \to A.$$
(vi) Freyd’s uniqueness of the initialised iterated holds in $\hat{S}$:

\[
\begin{align*}
\text{(FR!)}_{\hat{S}} & \quad f : A \rightarrow B, \ g : B \rightarrow B, \ h : A \times \mathbb{N} \rightarrow B \text{ in } \hat{S} \text{ such that} \\
& \quad h \circ (\text{id}_A, 0) \sim f : A \rightarrow B \text{ and} \\
& \quad h \circ (A \times s) \equiv g \circ h : A \times \mathbb{N} \rightarrow B \\
& \quad h \equiv g^s \circ (f \times \mathbb{N}) : A \times \mathbb{N} \rightarrow B \times \mathbb{N} \rightarrow B.
\end{align*}
\]

[The latter two statements are not so easy to prove: PR construction of comparison maps is needed, for comparing the respective enumerations of defined arguments in the postcedent, proceeding from the comparison maps given by the antecedents.]

(vii) For extension $\hat{S}$ of $S$ again, we get – by the general Freyd’s argument – the corresponding full schema of primitive recursion, namely

\[
\begin{align*}
\text{(pr)}_{\hat{S}} & \quad g : A \rightarrow B \text{ in } \hat{S} \text{ (initialisation),} \\
& \quad h : (A \times \mathbb{N}) \times B \rightarrow B \text{ (step map)} \\
& \quad f = \text{pr}[g, h] : A \times \mathbb{N} \rightarrow B \text{ is available in } \hat{S}, \\
& \quad \text{characterised (up to equality } \equiv \text{ ) in } \hat{S} \text{ by} \\
& \quad f \circ (\text{id}_A, 0) \equiv g : A \rightarrow B \text{ and} \\
& \quad f \circ (A \times s) \equiv h \circ (\text{id}_{A \times \mathbb{N}}, f) \\
& \quad = \text{by def } h \circ ((A \times \mathbb{N}) \times f) \circ \Delta_{A \times \mathbb{N}} : \\
& \quad A \times \mathbb{N} \rightarrow (A \times \mathbb{N})^2 \rightarrow (A \times \mathbb{N}) \times B \rightarrow B.
\end{align*}
\]

The Proof of this Structure Theorem is long, already since we have to show that many assertions; but mainly since assertion (vi) needs some auxiliary arguments.

Nevertheless, all of these assertions look plausible: they are “straightforward” extrapolations from the case of finite partial maps, by means we have at our disposition for the potentially infinite, primitive recursive case as basic theory\footnote{full Proof is ready within detailed version, as a pdf file}.

5 Partial-Map Extension as a Closure Operator

Closure: Theory $\hat{S}$ of partial maps over $\hat{S}$, of partial partial maps over $S$, is (category) equivalent to Theory $\hat{S}$. Theory $S$ is a strengthening of $PR_A$ as always here.

Mutatis mutandis, construction of Partiality Hull $\hat{S} \triangleleft S$ above of a Cartesian PR theory $S$ can be applied again to diagonal monoidal Theory $\hat{S}$ “again”.

\footnote{full Proof is ready within detailed version, as a pdf file}
Because of lack of Cartesianness this is more involved, and so is verification of the properties of this Double Closure $\mathcal{S}$. In particular it is more difficult to define composition: If you want to go into this detail, look at next diagram:

For defining composition of such $\mathcal{S}$-morphisms, composition of say $f : A \to B$ and $g : B \to C$, consider the following $\mathcal{S}/\mathcal{S}/\mathcal{S}$-diagram which displays the $\mathcal{S}/\mathcal{S}$ data of $f$ and $g$ to be composed into an $\mathcal{S}$-morphism $g \circ f : A \to B \to C$:

Composition diagram for $\mathcal{S}$

Composition $g \circ f : A \to C$ then is defined to have as graph $\gamma_g \circ f$ the map “induced” by the left and right frame morphisms of the diagram, namely:

$$\gamma_g \circ f = \text{def} \ (\ell \circ \gamma_f \circ d_f \circ \pi_\ell, r \circ \gamma_g \circ d_g \circ \pi_r) : D_{g \circ f} \to A \times C.$$

The next assertion (really) to be proved is idempotence of our Closure operator, namely that each $\mathcal{S}$ map $f : A \to B$ is represented – with respect to notion of equality $\simeq$ of $\mathcal{S}$, by a suitable $\mathcal{S}$-map $h : A \to B$.

For a Proof look at the following diagram, for given $\mathcal{S}$-map

$$f = \langle \gamma f = (\ell \circ \gamma f, r \circ \gamma f) = : (d_f, \hat{f}) : D_f \to A \times B \rangle : A \to B.$$
Closure diagram for Extension by partial maps

In this diagram, \( \gamma f : D_f \rightarrow A \times B \) is the graph of \( \hat{S} \)-morphism \( f : A \rightarrow B \) to be considered. The \( S \)-maps \( d_{\gamma f} : D_{\gamma f} \rightarrow D_f \) (defined-arguments enumeration) and \( \hat{\gamma} f : D_{\gamma f} \rightarrow A \times B \) (rule) are to define \( \gamma f : D_f \rightarrow A \times B \) as a partial \( S \)-map, an \( \hat{S} \) morphism.

This diagram shows the way of Proof for

**Closure Theorem for Extension of Theory \( S \) by Partial Maps:**

*Closure by Partial Maps is idempotent:* Partial map Closure of theory \( \hat{S} \) is again a diagonal monoidal category \( \hat{S} \) which is in fact equivalent – as such a category – to theory \( \hat{S} \):

\[ \hat{S} \cong \hat{S}. \]

[Both inherit – from \( S \) – Object \( \mathbb{N} \) as NNO in the sense of the (full) schema of PR for \( \mathbb{N} \).]

6 \( \mu \)-Recursion without Quantifiers

**Church type “Inclusion”:** For given \( PR_A \)-predicate \( \varphi : A \times \mathbb{N} \rightarrow 2 \), partially defined “map”

\[ \mu \varphi = \mu \varphi(a) : A \rightarrow \mathbb{N}, \]

classically defined by

\[
\mu \varphi(a) = \begin{cases} 
\min \{n \in \mathbb{N} \mid \varphi(a,n)\} & \text{if } (\exists n \in \mathbb{N}) \varphi(a,n) \\
\text{undefined} & \text{if } (\forall n \in \mathbb{N}) \neg \varphi(a,n),
\end{cases}
\]
has a \textit{classically correct representation} within (strengthenings of) Theory \( \mathbf{PR\_A} \) as

\[
D_{\mu\phi} \overset{\text{def}}{=} \{ A \times \mathbb{N} \mid \varphi \} = \{(a, n) \mid \varphi(a, n)\}
\]

\[
d_{\mu\phi} = \ell \circ \subseteq
\]

Here \textit{defined-arguments} (PR) enumeration is

\[
d_{\mu\phi} = d_{\mu\phi}(a, n) = \text{def} \ a : \{ A \times \mathbb{N} \mid \varphi \} \subseteq A \times \mathbb{N} \xrightarrow{\ell} A,
\]

and rule \( \tilde{\mu}\phi \) : \( \{(a, n) \mid \varphi(a, n)\} \to \mathbb{N} \) is (totally) PR defined by

\[
\tilde{\mu}\phi = \tilde{\mu}\phi(a, n) = \min \{ m \leq n \mid \varphi(a, m) \} : \{(a, n) \mid \varphi(a, n)\} \to \mathbb{N}.
\]

\textbf{\( \mu\)-Lemma:} \( \hat{S} \) admits the following (Free-Variables) schema \((\mu)\), operator \( \mu \)'s “property”, combined with \textit{uniqueness} schema \((\mu!)\), as a \textit{characterisation} of the \( \mu \)-operator \( \langle \varphi : A \times \mathbb{N} \to 2 \rangle \leftrightarrow \langle \mu\phi : A \nrightarrow \mathbb{N} \rangle \) above:

\[
(\mu) \quad \varphi = \varphi(a, n) : A \times \mathbb{N} \to 2 \text{ S-map ("predicate"), }
\]

\[
\begin{align*}
[a \in A \text{ (free) is the passive parameter,} & \\
n \in \mathbb{N} \text{ free the recursion parameter,} & \\
\mu\phi = \langle (d_{\mu\phi}, \tilde{\mu}\phi) : D_{\mu\phi} \to A \times \mathbb{N} \rangle : A \nrightarrow \mathbb{N} & \\
\text{is an } \hat{S} \text{-map such that} & \\
S \vdash \varphi(d_{\mu\phi}(\hat{a}), \tilde{\mu}\phi(\hat{a})) = \text{true}_{D_{\mu\phi}} : D_{\mu\phi} \to 2, & \\
[\hat{a} \in D_{\mu\phi} \text{ free, so just } a \in A \text{ of form } a : = d_{\mu\phi}(\hat{a}) & \\
\text{counts as } \text{- is enumerated as } \text{- "defined argument" for } \mu\phi & \\
\text{]} & \\
\text{+ "argumentwise" minimality:} & \\
S \vdash [\varphi(d_{\mu\phi}(\hat{a}), n) \implies \tilde{\mu}\phi(\hat{a}) \leq n] : D_{\mu\phi} \times \mathbb{N} \to 2 & \\
\text{as well as uniqueness, by maximal extension:} & \\
\text{by definition of } \mu\phi = \mu\phi(a) : A \nrightarrow \mathbb{N}. & \\
\text{Proof idea for uniqueness schema } (\mu!) \text{ is "displayed" as the following DIAGRAM:} & \\
\]

Proof of schema \((\mu)\) is by \textit{definition} of \( \mu\phi = \mu\phi(a) : A \nrightarrow \mathbb{N} \). Proof idea for uniqueness schema \((\mu!)\) is “displayed” as the following DIAGRAM:
Remark: Within Peano-Arithmétique $\mathbf{PA}$, and hence also within set theory, our $\mu \varphi : A \to N$ equals

$$\mu \varphi = \{ (\subseteq , \tilde{\mu} \varphi) : \hat{A} \to A \times N \} : A \supset \hat{A} \to N,$$

with $\hat{A} = \{ \hat{a} \in A \mid \exists n \varphi(\hat{a}, n) \}$, and $\tilde{\mu} \varphi(\hat{a}) = \min\{ m \in N \mid \varphi(\hat{a}, m) \} : \hat{A} \to N$, i.e. it is given there by the classical partial minimum definition. But this definition lacks constructivity, since $\hat{A} \subseteq A$ is in general not PR decidable.

Conversely, consider a partial PR map,

$$f = \langle (d_f , \hat{f}) : D_f \to A \times B \rangle : A \to B$$

out of $\hat{\mathbf{PR}}_A$.

Standard, pointed case: $f$ is defined at least at one point, say at $a_0 = d_f(\hat{a}_0) : 1 \to D_f \to A$.

Such $f$ is represented easily, within Theory $\hat{\mathbf{PR}}_A$, by a $\mu$-recursive $\hat{\mathbf{PR}}_A$-map (followed by a $\mathbf{PR}_A$ map), namely by

$$\mu[f] \equiv (\hat{f} \circ \text{count}_{D_f}) \circ \mu \varphi : A \to N \to D_f \to N.$$

$\mathbf{PR}_A$-predicate $\varphi_f = \varphi_f(a, n) : A \times N \to 2$ is given as

$$\varphi_f = \varphi_f(a, n) \equiv [ a \triangleq A d_f \circ \text{count}_{D_f}(n) ] : A \times N \to A \times D_f \to 2 \text{ PR}.$$

Here $\text{count}_{D_f} : N \to D_f$ designates a (retractive) (PR) count of $D_f$. For disposing on this count of $D_f$, we had to assume that $D_f$ comes with a $\mathbf{PR}_A$-point, $\hat{a}_0 : 1 \to D_f$ above.

Partial map $\mu \varphi_f : A \to N$ is the genuine $\mu$-recursive kernel of $\mu$-representation $\mu[f] : A \to B$ of (pointed) partial map $f : A \to B$. We count composition of $\mu$-recursive maps with PR maps equally under the $\mu$-recursive ones. So in this sense, $\mu[f] \equiv f : A \to B$ is a $\mu$-recursive representant of $f$ within $\hat{\mathbf{PR}}_A$ and its strengthenings, a (partial) map in $\mu \mathbf{R}$, and in $\mu \mathbf{S}$ for strengthenings $\mathbf{S}$ of $\mathbf{PR}_A$.

The case that $D_f$ has no point, and is nevertheless not $\mathbf{S}$-derivably empty, causes a problem, formally. We “solve” this problem by modifying (extending) the definition of $\mu[f] : A \to N$ as follows:
**PR**-Object $D_f$ is predicative restriction $D_f : D \to 2$ of a *fundamental* Object $D$, which comes as such with a (componentwise) zero $0 : \mathbb{1} \to D$ as (privilegded) *point*. This Object admits a CANTOR count $\text{count}_D : \mathbb{N} \cong D$. (Trivial exception: for $D = \mathbb{1}$, $\text{count}_D = ! : \mathbb{N} \to \mathbb{1}$ is still a retraction.)

In present general case, replace – on the way – Object $A \subset X$ by sum, (disjoint) union $\mathbb{1} \oplus A \subset X$, and define $\hat{\varphi}_f = \hat{\varphi}_f(a,n) : A \times \mathbb{N} \to 2$ by the following **PR**-DIAGRAM:

$$
\begin{array}{c}
A \times \mathbb{N} \\
\downarrow \iota \times \mathbb{N} \subset \\
(\mathbb{1} \oplus A) \times \mathbb{N} =_{\text{def}} (\mathbb{1} \oplus A) \times (\mathbb{1} \oplus A) \\
\downarrow (\mathbb{1} \oplus A) \times \text{count}_D \\
(\mathbb{1} \oplus A) \times D =_{\text{id}} ((D \setminus D_f) \oplus D_f) \\
\end{array}
$$

$\hat{\varphi}_f = \hat{\varphi}_f(a,n) : A \times \mathbb{N} \to 2$ in Free-Variables notation:

$$
A \times \mathbb{N} \ni (a,n) \mapsto (a,n) \mapsto (a, \text{count}_D(n)) \\
\to \begin{cases} 
\text{false if } \text{count}_D(n) \notin D_f, \text{ (“outside” case),} \\
[ a \equiv_A d_f(\text{count}_D(n)) ] [ \in 2 ] \text{ if } \text{count}_D(n) \in D_f. 
\end{cases}
$$

This given, we *define* in this general case

$$
\mu[f] =_{\text{def}} \hat{f} \circ \text{count}_D \circ \mu \hat{\varphi}_f : A \to \mathbb{N} \to D \to B.
$$

Note first that $\hat{f} : D_f \to B$ comes by (formal) Domain restriction of a genuine **PR** map $\hat{f} : D \to B' : D_f = \{ D \mid D_f \}, D$ (and $B'$) fundamental: This by definition of *maps* of Theory $\text{PR}_A = \text{PR} + (\text{abstr})$.

Second: wider count $\text{count}_D$, available in particular for $D$ as a fundamental Object, Codomain-restricts here nicely, gives “again” $\mu$-representation of $f$, here via $D \supset D_f = \{ D \mid D_f : D \to 2 \}$.

This taken together gives $\mu$-representation of general partial **PR**-map $f : A \to B$ as

$$
f \equiv \mu[f] =_{\text{by def}} ( \hat{f} \circ \text{count}_D ) \circ \mu \hat{\varphi}_f : A \xrightarrow{\mu \hat{\varphi}_f} \mathbb{N} \xrightarrow{\text{count}_D} D \xrightarrow{\hat{f}} B.
$$

So we have reached

**Another Proved Instance of Church’s Thesis:**

- The notion of a $\mu$-recursive (partial) map is *equivalent* to that of a *Partial* **PR** map, over “all” Theories of Primitive Recursion.
- Theories $\text{PR}_A$ and $\mu R$ are equivalent, and the Closure Theorem $\hat{S} \cong \hat{S}$ above then shows $\mu \mu R \cong \mu R$.

- Level-one $\mu$-recursion is enough for getting all $\mu$-recursive maps. By the above, this gives the well known corresponding result for while programs: one while loop is enough: Any such program can be equivalently transformed into a while loop program without nesting of while loops.

- All this works as well for strengthenings $S$, $\hat{S}$ of $\text{PR}_A$ and $\text{PR}_A$ respectively. We would name the corresponding Theory of $\mu$-recursion $\mu S \cong \hat{S}$.

**Conclusion** so far:

- We can eliminate formal existential quantification – as well as (individual, formal) variables – from the theory of $\mu$-recursion: we interpret application of $\mu$-operator to predicates of theories $S$ strengthening PR Theory $\text{PR}_A = \text{PR} + (\text{abstr})$ as suitable partial maps, maps in Theory $\hat{S}$.

- The $\mu$-operator canonically extends to all partial predicates $\varphi : A \times \mathbb{N} \rightarrow 2$, and associates to them just partial maps $\mu \varphi : A \rightarrow \mathbb{N}$, within $\hat{S}$ itself. So, “once again”, we see, that theories $\hat{S}$ of partial PR maps are closed, this time under the $\mu$-operator, “in parallel” to Closure of $\hat{S}$ under forming partial maps: partial partial PR maps “are” partial PR maps.

- We have the following chain of categorical equivalences of theories considered so far:

$$S \sqsubseteq \mu S \cong \mu \mu S \cong \hat{S} \cong \hat{S} \sqsubseteq S,$$

the inclusions being diagonal-monoidal PR compatible with the equivalences.

[A partial PR map $f : A \rightarrow B$ which is, “by hazard”, a total map – discussion of overall termination = total definedness in part RCF 2 ($\varepsilon$&C), is in general not itself PR: only its graph $(d_f, f) : D_f \rightarrow A \times B$ is PR. Ackermann type maps, in particular evaluation of all PR-map-codes, are formally partial maps. In well defined cases, they can be forced – by plausible additional axiom – to become on-terminating, i.e. to get defined-argument enumeration epimorphic.]

- Conversely, application of the $\mu$-operator, already just to $\text{PR}_A$-predicates $\varphi = \varphi(a, n) : A \times \mathbb{N} \rightarrow 2$, generates all partial $\text{PR}_A$-maps $f : A \rightarrow B$.

- As important special cases of basic PR theories $S$ we have at the moment Theory $\text{PR}_A = \text{PR} + (\text{abstr})$ itself as well as the PR trace $\text{PA} \upharpoonright \text{PR}_A$ of $\text{PA}$: All $\text{PR}_A$-maps (map terms) with all those equations in between, which are derivable by $\text{PA}$ : Our theories, notions, and results have a structure-preserving interpretation into (within) Peano-Arithmetic $\text{PA}$. 

21
Same for set theories in place of PA, in particular ZF, ZFC and their first order subsystems 1ZF, 1ZFC = 1ZF + ACC. ACC is the Axiom of Countable Choice.

7 Content Driven Loops, in particular while Loops

Classically, with variables, a while loop \( w = w[\chi \mid f] \) is “defined” in pseudocode by

\[
wh(a) := [a' := a; \\
\text{while } \chi(a') \text{ do } a' := f(a') \text{ od}; \\
wh(a) := a'].
\]

The formal version of this – within a classical, element based setting – is the following partial-(PEANO)-map characterisation:

\[
wh(a) = wh[\chi \mid f](a) = \begin{cases} a \text{ if } \neg \chi(a) \\ wh(f(a)) \text{ if } \chi(a) \end{cases} : A \rightarrow A.
\]

It is possible to give a static Definition of \( w = w[\chi \mid f] : A \rightarrow A \), within \( \hat{\text{PR}}_A \supseteq \text{PR}_A \) (and strengthenings) as follows:

With \( \varphi = \varphi[\chi \mid f](a, n) =_{\text{def}} \neg \chi f^n(a) \\
=_{\text{by def}} \neg \chi f^\sharp(a, n) : A \times \mathbb{N} \rightarrow A \rightarrow 2 \rightarrow 2, \\
the \text{while loop} \text{wh} = wh[\chi \mid f] : A \rightarrow A \) is given as

\[
wh =_{\text{def}} f^\mu \varphi[\chi \mid f](a) \\
=_{\text{by def}} f^\sharp \circ (id_A, \mu \varphi[\chi \mid f]) \\
=_{\text{by def}} f^\sharp \circ (A \times \mu \varphi[\chi \mid f]) \circ A : \\
A \rightarrow A \times A \rightarrow A \times \mathbb{N} \rightarrow A.
\]

In this generalised categorical sense, we have within theories \( \hat{S} \supseteq S \) (S strengthening \( \text{PR}_A \), all while loops.

Characterisation Theorem for while loops over S, within Theory \( \hat{S} \) : For \( \chi : A \rightarrow 2 \) (control) and \( f : A \rightarrow A \) (step), both – for the time being – S-maps, while loop \( w = w[\chi \mid f] : A \rightarrow A \) (as defined above), is characterised by the following implications within \( \hat{S} \):

\[
\hat{S} \vdash \neg \chi \circ a \Rightarrow wh \circ a \equiv a : A \rightarrow 2, \text{ uland} \\
\hat{S} \vdash \chi \circ a \Rightarrow wh \circ a \equiv wh \circ f \circ a.
\]
Dominated Termination of while loop

\[ wh = wh[\chi | f] = wh[\chi | f](a) : A \rightarrow A, \]

at argument \( a \in A \), is expressed by

\[ [m \text{ def } wh[\chi | f](a)] =_{\text{def}} \neg \chi f^n(a) : \mathbb{N} \times A \rightarrow 2 : \]

In words: \textit{iteration of endo }\( f : A \rightarrow A \), \textit{applied to argument }\( a \), \textit{reaches stop condition }\( \neg \chi \) \textit{after (at most) }\( m \) \textit{steps.} \ Here both: \textit{argument }\( a \in A \) \textit{as well as iteration counter }\( m \in \mathbb{N} \), \textit{are free variables (categorically: projections).} \[ [m \text{ def } wh[\chi | f](a)] \equiv_A \bar{a} \]

\[ \text{by def } f_\chi^{\text{min}\{n \leq m | \neg \chi f^n(a)\}}(a) \]

\[ \mathbb{N} \times A^2 \rightarrow 2, ~ m \in \mathbb{N}, ~ a, \bar{a} \in A \text{ free.} \]

[“Variable” \( n \in \mathbb{N} \) used in the min PR Operator is auxiliary, bounded by \( m \). It does not count as a (free) variable.]

From a \textit{logical} point of view, there are – at least – the following open Questions, in

Arithmetics Complexity Problem:

(i) Does Theory \( \text{PR} \) admit \textit{strict, consistent} strengthenings, or is it a \textit{simple theory}, will say that it admits its given notion of equality and the indiscr e (inconsistency) equality of its maps as only “congruences?”, cf. a simple \textit{group} which has as normal subgroups only itself and \( \{1\} \). Because of reasons to be explained in later work, my guess here is: \( \text{PR} \) \textit{admits} non-trivial strengthenings, in particular I suppose that the PR \textit{trace} of \( \text{PA} \), explained above, is a strict strengthening of \( \text{PR} \textit{resp. PRA} = \text{PR} + \text{(abstr)} \).

We cannot exclude at present that all these strengthenings of \( \text{PR} \textit{make up a whole lattice} of (Free-Variables) Arithmetical Theories, each of them giving particular, “new” features to Primitive Recursive Arithmetics.

(ii) Already at start we possibly have such a strengthening: If Free-Variables ("Free Variables" in the classical sense) \textit{Primitive Recursive Arithmetic PRA} is \textit{defined} to have as its terms all map terms obtainable by the (full) schema of Primitive Recursion, and as formulae just the \textit{defining equations} for the maps introduced by that schema, then I see no way to \textit{prove} all of the usual semiring equations for \( \mathbb{N} \):

We need Freyd’s \textit{uniqueness} (FR!) – section 1 above – of the \textit{initialised iterated}: From this \textit{HORN} clause we can show (!) in particular \textit{Goodstein’s uniqueness rules }\( U_1 \text{ to } U_4 \) on which \textit{his} Proof of the semiring properties of \( \mathbb{N} \) is based. He takes these rules as \textit{axioms}.

23
My guess is here – if I have understood right the definition of \textit{PRA} – that $\text{PR} = \text{PRA} + (\text{FR!})$ is a strict strengthening of \textit{PRA}, at least if there is no “underground” connection to the set theoretic view of maps as (possibly infinite) \textit{argument-value tables}.

(iii) So again, Arithmetic would \textit{become} simpler, if Theory $\text{PR}$ would turn out to be \textit{simple}. If not, we have a diversity of Arithmetics, a diversity intuitively far below such issues as Independence of the Axiom of Choice or of the Continuum Hypothesis. At least the latter is open in the context of a \textbf{Constructive Analysis} based on map theoretic, Free-Variables (variable-free) Primitive Recursive and $\mu$-recursive Arithmetics.
References

J. Barwise ed. 1977: *Handbook of Mathematical Logic*. North Holland.

H.-B. Brinkmann, D. Puppe 1969: *Abelsche und exakte Kategorien, Korrespondenzen*. L.N. in Math. 96. Springer.

L. Budach, H.-J. Hoehnke 1975: *Automaten und Funktoren*. Akademie-Verlag Berlin.

C. Ehresmann 1965: *Catégories et Structures*. Dunod Paris.

H. Ehrig, W. Künnel, M. Pfender 1975: Diagram Characterization of Recursion. LN in Comp. Sc. 25, 137-143.

S. Eilenberg, C. C. Elgot 1970: *Recursiveness*. Academic Press.

S. Eilenberg, S. Mac Lane 1945: General Theory of Natural Equivalences. *Trans. AMS* 58, 231-294.

P. J. Freyd 1972: Aspects of Topoi. *Bull. Australian Math. Soc.* 7, 1-76.

R. L. Goodstein 1971: *Development of Mathematical Logic*, ch. 7: Free-Variable Arithmetics. Logos Press.

F. Hausdorff 1908: Grundzüge einer Theorie der geordneten Mengen. *Math. Ann.* 65, 435-505.

D. Hilbert: Mathematische Probleme. Vortrag Paris 1900. *Gesammelte Abhandlungen*. Springer 1970.

P. T. Johnstone 1977: *Topos Theory*. Academic Press

A. Joyal 1973: Arithmetical Universes. Talk at Oberwolfach.

J. Lambek, P. J. Scott 1986: *Introduction to higher order categorical logic*. Cambridge University Press.

F. W. Lawvere 1964: An Elementary Theory of the Category of Sets. *Proc. Nat. Acad. Sc. USA* 51, 1506-1510.

F. W. Lawvere, S. H. Schanuel 1997: *Conceptual Mathematics, A first introduction to categories*. Cambridge University Press.

S. Mac Lane 1972: *Categories for the working mathematician*. Springer.

B. Pareigis 1969: *Kategorien und Funktoren*. Teubner.

R. Péter 1967: *Recursive Functions*. Academic Press.

M. Pfender 1974: Universal Algebra in S-Monoidal Categories. Algebra-Berichte Nr. 20, Mathematisches Institut der Universität München. Verlag Uni-Druck München.

M. Pfender 2008 RCF1d: Theories of PR Maps and Partial PR Maps, detailed version. pdf file.

M. Pfender, M. Kröplin, D. Pape 1994: Primitive Recursion, Equality, and a Universal Set. *Math. Struct. in Comp. Sc.* 4, 295-313.

H. Reichel 1987: *Initial Computability, Algebraic Specifications, and Partial Algebras*. Oxford Science Publications.

W. Rautenberg 1995/2006: *A Concise Introduction to Mathematical Logic*. Universitext Springer 2006.
R. Reiter 1980: Mengentheoretische Konstruktionen in arithmetischen Universen. Diploma Thesis. Techn. Univ. Berlin.

R. Reiter 1982: Ein algebraisch-konstruktiver Abbildungskalkül zur Fundierung der elementaren Arithmetik. Dissertation, rejected by the Mathematics Department of TU Berlin.

L. Roman 1989: Cartesian categories with natural numbers object. J. Pure and Appl. Alg. 58, 267-278.

W. W. Tait 1996: Frege versus Cantor and Dedekind: on the concept of number. Frege, Russell, Wittgenstein: Essays in Early Analytic Philosophy (in honor of Leonhard Linsky) (ed. W. W. Tait). Lasalle: Open Court Press (1996): 213-248. Reprinted in Frege: Importance and Legacy (ed. M. Schirn). Berlin: Walter de Gruyter (1996): 70-113.

U. Thiel 1982: Der Körper der rationalen Zahlen im arithmetischen Universum. Diploma Thesis. Techn. Univ. Berlin.

A. Tarski, S. Givant 1987: A formalization of set theory without variables. AMS Coll. Publ. vol. 41.

Address of the author:
M. Pfender
Institut für Mathematik
Technische Universität Berlin
D-10623 Berlin
pfender@math.TU-Berlin.DE