Constant of Motion for One-Dimensional Non Autonomous Linear Systems and Harmonic Oscillator

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ABSTRACT

For a one-dimensional motion, a constant of motion for non autonomous an linear system (position and velocity) is given from the constant of motion associated to its autonomous system. This approach is used in the study of the harmonic oscillator with an additional time depending force.
I. Introduction

The importance of the constant of motion of an autonomous system (the forces are time independent) and its relation with the Lagrangian and the Hamiltonian is well known (López, 1996). For autonomous systems, the constant of motion, Lagrangian and Hamiltonian do not need to depend explicitly on time (Goldstein, 1950). If one looks for these quantities to be time explicitly depending, one could get ambiguities in the formulation (López, 1998). On the other hand, constant of motion for nonautonomous systems (the forces depend explicitly on time) is very little known, and the usual approach to study these systems is looking for a Hamiltonian without worry about whether or not it is a constant of motion (Zel’dovich, 1967; Dekker, 1981, and references therein). A constant of motion of nonautonomous system must depend explicitly on time and has its own importance by its own, independently on a possible relation with the Lagrangian and Hamiltonian of the system (López and Hernández, 1987). In this paper, a constant of motion for a nonautonomous system which has a linear dependence on coordinate and velocity is deduced. This approach is applied to the harmonic oscillator with dissipation and with a time depending force.

II. Constant of Motion

Consider the following nonautonomous dynamical system

\[
\frac{dx}{dt} = v \tag{1a}
\]

and

\[
\frac{dv}{dt} = ax + bv + f(t) , \tag{1b}
\]

where \(a, b\) are constants and \(f(t)\) is an arbitrary function. For a constant of motion associated to this system, one understands a function \(K = K(x,v,t)\) such that its total derivation with respect the time is zero \((dK/dt = 0)\), that is, \(K\) must be solution of the following partial differential equation

\[
v \frac{\partial K}{\partial x} + (ax + bv + f(t)) \frac{\partial K}{\partial v} + \frac{\partial K}{\partial t} = 0 . \tag{2}
\]

Assume that \(K_o = K_o(x,v)\) is the constant of motion associated to (1) but for \(f = 0\) (associated autonomous system), that is, \(K_o\) is solution of th equation

\[
v \frac{\partial K_o}{\partial x} + (ax + bv) \frac{\partial K_o}{\partial v} = 0 . \tag{3}
\]
Thus, it follows that

\[ K(x, v, t) = K_o \left( x - \alpha(t), v - \frac{d\alpha}{dt} \right) \]  \hspace{1cm} (4)

is solution of Eq. (2), where \( \alpha(t) \) is the particular solution of the equation

\[ \frac{d^2\alpha}{dt^2} = a\alpha + b \frac{d\alpha}{dt} + f(t) \]  \hspace{1cm} (5)

To see this, defining the variables \( \xi_1 \) and \( \xi_2 \) as

\[ \xi_1 = x - \alpha(t), \quad \xi_2 = v - \frac{d\alpha}{dt} \]  \hspace{1cm} (6)

one gets the following derivations

\[ v \frac{\partial K}{\partial x} = v \frac{\partial K_o}{\partial \xi_1}, \]  \hspace{1cm} (7a)

\[ (ax + bv + f) \frac{\partial K}{\partial v} = (ax + bv + f) \frac{\partial K_o}{\partial \xi_2}, \]  \hspace{1cm} (7b)

and

\[ \frac{\partial K}{\partial t} = -\frac{d\alpha}{dt} \frac{\partial K_o}{\partial \xi_1} - \frac{d^2\alpha}{dt^2} \frac{\partial K_o}{\partial \xi_2}. \]  \hspace{1cm} (7c)

Substituting Eqs. (7a), (7b), and (7c) in Eq. (2), rearranging terms, and using Eqs. (5) and (6), one has

\[ \frac{dK}{dt} = \xi_2 \frac{\partial K_o}{\partial \xi_1} + \left[ a\xi_1 + b\xi_2 \right] \frac{\partial K_o}{\partial \xi_2} = 0 \]  \hspace{1cm} (8)

since by definition (3), \( K_o \) satisfies this type of equation. Therefore, (4) is solution of Eq. (2) and is a constant of motion of the nonautonomous system (1).

III. Harmonic Oscillator with Periodic Force

This dynamical system is defined by the equations

\[ \frac{dx}{dt} = v \]  \hspace{1cm} (9a)

and

\[ \frac{dv}{dt} = -\omega^2 + \frac{A}{m} \sin(\Omega t) \]  \hspace{1cm} (9b)
where $\omega$ is the natural frequency of oscillations, $m$ is the mass of the particle, $\Omega$ and $A$ are the frequency and amplitude of the external periodic force. According to Eqs. (4) and (5), the function $\alpha$ is the particular solution of Eq. (9) which is given by

$$\alpha(t) = -\frac{A \sin(\Omega t)}{m(\Omega^2 - \omega^2)}.$$  

(10)

Therefore, a constant of motion of system (9) can be written for $\Omega \neq \omega$ (out of resonance) as

$$K(x, v, t) = \frac{1}{2}mv^2 + \frac{1}{2}m\omega^2x^2$$

$$+ \frac{A}{\Omega^2 - \omega^2} \left( \Omega v \cos(\Omega t) + \omega^2x \sin(\Omega t) \right)$$

$$- \frac{A^2}{2m(\Omega^2 - \omega^2)} \sin^2(\Omega t),$$

(11)

where a term of the form $A^2\Omega^2/2m(\Omega^2 - \omega^2)$ has been ignored since it represents a constant term.

For the resonant case ($\Omega = \omega$), one gets

$$\alpha(t) = \frac{A}{4m\omega^2} \sin(\omega t) - \frac{A}{2m\omega} t \cos(\omega t),$$

(12)

and the constant of motion can be written as

$$K(x, v, t) = \frac{1}{2}mv^2 + \frac{1}{2}m\omega^2x^2$$

$$+ \frac{A}{4\omega} \left[ (v + x\omega^2t) \cos(\omega t) - (x\omega + 2v\omega t) \sin(\omega t) \right]$$

$$+ \frac{A^2t}{8m\omega} [\omega t - \sin(2\omega t)],$$

(13)

where a term of the form $A^2/32m\omega^2$ has been ignored. It is not difficult to see that expression (12) satisfies indeed the following equation

$$v \frac{\partial K}{\partial x} + \left[ -\omega^2x + \frac{A}{m} \sin(\Omega t) \right] \frac{\partial K}{\partial v} + \frac{\partial K}{\partial t} = 0.$$  

(14)
IV. Dissipative Harmonic Oscillator with Periodic Force

This dynamical system is defined by the equations

\[
\frac{dx}{dt} = v \tag{15a}
\]

and

\[
\frac{dv}{dt} = -\omega^2 x - \frac{\lambda}{m} v + \frac{A}{m} \sin(\Omega t) \tag{15b}
\]

where \( \lambda \) is the parameter which characterizes the dissipation. The particular solution of this system is given by

\[
\alpha(t) = \frac{A/m}{\left(\frac{\lambda \Omega}{m}\right)^2 + (\omega^2 - \Omega^2)^2} \left[ (\omega^2 - \Omega^2) \sin(\Omega t) - \frac{\lambda \Omega}{m} \cos(\Omega t) \right]. \tag{16}
\]

System (15) is also of the form (1), and the constant of motion for \( A = 0 \) (associated autonomous system) was given somewhere else (López, 1996). Thus, according to this reference and Eq. (4), a constant of motion of the system (15) can be given by

\[
K(x, v, t) = \frac{m}{2} \left[ (v - \beta(t))^2 + \frac{\lambda}{m} (x - \alpha(t))(v - \beta(t)) + \omega^2(x - \alpha(t))^2 \right]
\times \exp \left( -\frac{\lambda}{m} G \left( \frac{v - \beta(t)}{x - \alpha(t)}, w, \lambda \right) \right), \tag{18a}
\]

where \( G = G(\xi, w, \lambda) \) is the function defined as

\[
G = \begin{cases} 
\frac{1}{2\sqrt{(\lambda/2m)^2 - \omega^2}} \log \frac{\lambda/2m + \xi - \sqrt{(\lambda/2m)^2 - \omega^2}}{\lambda/2m + \xi + \sqrt{(\lambda/2m)^2 - \omega^2}}, & \text{if } \omega^2 < (\lambda/2m)^2 \\
\frac{1}{\lambda/2m + \xi}, & \text{if } \omega^2 = (\lambda/2m)^2 \tag{18c}
\end{cases}
\]

and the function \( \beta(t) \) is defined as

\[
\beta(t) = \frac{A\Omega/m}{\left(\frac{\lambda \Omega}{m}\right)^2 + (\omega^2 - \Omega^2)^2} \left[ (\omega^2 - \Omega^2) \cos(\Omega t) + \frac{\lambda \Omega}{m} \sin(\Omega t) \right]. \tag{18c}
\]

For the particular case of very weak dissipation \((\lambda/2m \ll \omega)\), one gets the expression

\[
K(x, v, t) = \frac{m}{2} \left[ (v - \beta(t))^2 + \omega^2(x - \alpha(t))^2 \right] + \frac{\lambda}{2} (x - \alpha(t))(v - \beta(t))
- \frac{\lambda}{2\omega} \left[ (v - \beta(t))^2 + \omega^2(x - \alpha(t))^2 \right] \arctan \frac{v - \beta(t)}{\omega(x - \alpha(t))}. \tag{19}
\]
V. CONCLUSION

For a particle moving in one dimension where forces are linear in position and velocity, to know a constant of motion for this system (autonomous) allows one to get a constant of motion when a time-depending force is added. This approach was applied to the harmonic oscillator with external force and dissipation.
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