On Supersymmetry at Finite Temperature

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Abstract

We consider the effective theories governing the sensitivity to the plasma of certain high-energy observables in supersymmetric plasmas, and point out that they preserve supersymmetry. Our findings generalize previous observations on asymptotic thermal masses in weakly coupled plasmas, to both the real and imaginary parts of self-energies, on the light cone and away from it, in weakly and strongly interacting theories. All observed supersymmetry violations due to thermal effects turn out to vanish faster than $E^{-2}$ in the high energy limit.
1 Introduction

Supersymmetry is usually considered as being broken at finite temperature (see e.g. [1]). A simple reason for this is the different statistics and population functions assumed by Bose and Fermi fields, making it hard to see how a Bose-Fermi symmetry could be preserved. In Euclidean space, Bose-Fermi symmetry is explicitly broken by boundary conditions along the periodic time direction (with period $1/T$), which are respectively periodic for bosonic fields and anti-periodic for fermionic fields. Nevertheless, one can still ask whether the supersymmetry of the underlying equations of motion leaves any trace in physical observables.

One implication of supersymmetry at finite temperature along these lines was described in [2]: due to the existence of a conserved supercurrent, the effective hydrodynamics theory which describes the long-wavelength modes of the plasma must contain fermionic degrees of freedom. These fermionic modes enter the discussion of non-linear effects within this effective theory (e.g. loop corrections), but, of course, are not allowed to take on classical expectation values.

In this letter we propose to look at a different sector of the theory: that of high-energy observables. In the strict high-energy limit the plasma decouples and supersymmetry is recovered (provided it is present in the vacuum theory). Nontrivial results may be obtained by looking at the leading thermal corrections to this limit, such as thermal-induced dispersion relations and decay rates. We see no obvious reason why these should preserve supersymmetry. Nevertheless, in this letter we wish to report an intriguing fact: for a wide class of high-energy observables, supersymmetry is preserved.

Our original motivation for this work was to investigate whether the well-known observation of supersymmetry for asymptotic thermal masses, in weakly coupled plasmas, did extend to other quantities. We will answer this question in the affirmative, and will in fact extend it to all high-energy correlators we could study. More precisely, if high-energy supersymmetry violations are characterized by the power of the energy $E^{-n}$ by which they are suppressed (relative to the vacuum correlators), our finding is that $n > 2$ with strict inequality in all considered cases. This is a nonempty statement, since the leading thermal effects all have $n \leq 2$.

We will discuss in turn the effective theories we have considered. These include: the effective theory for particle masses at weak coupling, in section 2; the effective theories for the imaginary part of self-energies (including collinear bremsstrahlung processes and $2 \rightarrow 2$ collisions), at weak coupling, in section 3; the self-energies of uncharged particles in strongly interacting plasmas with a gravity dual, in section 4; and finally, the operator product expansion for deeply virtual correlators, in section 5.

We use the phrase “effective theory” to emphasize that the details of the plasma are only probed through a restricted set of low-energy correlators, which provide param-
eters for medium-independent high-energy effective theories. Supersymmetry should be understood as an intrinsic property of these effective theories — we do not believe that the thermal nature of the underlying medium plays any role.

2 Thermal masses at weak coupling

Thermal dispersion relations (of massless particles) are known to approach the form \( E^2 = p^2 + m_{\infty}^2 \) at large momenta \( p \gg gT \) \[^{[3]}\], at the leading order in perturbation theory. In applications to supersymmetric theories, it has been repeatedly observed that the asymptotic masses \( m_{\infty} \) are the same among particles within a supersymmetry multiplet\[^{[1]}\]. Compiling results from the literature \[^{[4]}\] \[^{[7]}\], or by directly evaluating one-loop diagrams such as those shown in fig.\[^{[1]}\] one arrives at the formulae:

\[
m^2_{\infty,g} = m^2_{\infty,\lambda} = g^2 C_A \left( Z_g + Z_f^\lambda \right) + g^2 N_{\text{matter}} T_M \left( Z_f^\psi + Z_S \right),
\]

\[
m^2_{\infty,\psi} = m^2_{\infty,\phi} = g^2 C_M \left( Z_g + Z_f^\lambda + Z_f^\psi + Z_S \right) + y^2 \left( Z_f^\phi + Z_S \right),
\]

(1)

where the \( Z_i \) are certain (non-local) dimension-two condensates that we give shortly. Terms in eq. (1) are in one-to-one correspondence with particles and interactions of renormalizable supersymmetric gauge theories. \( C_A, C_M \) and \( T_M \) are quadratic Casimirs and Dynkin indices for the adjoint and matter representations, respectively, and \( N_{\text{matter}} \) is the number of chiral superfields, with \( g \) the ordinary Yang-Mills coupling. For simplicity the Yukawa contribution in eq. (1) is normalized to correspond to a term \( \sim y^2 \sqrt{2} \phi \psi \psi \) in the Lagrangian of a single-field Wess-Zumino model. We expect supersymmetry to be preserved for more general (e.g. nonrenormalizable) superpotentials, though we have not checked this explicitly. Nonzero expectation values for the \( D \) or \( F \) auxiliary fields, not considered in eq. (1), could break the supersymmetry by lifting the bosonic masses\[^{[2]}\]. However, we do not view such effects as being specifically thermal sources of supersymmetry violations, since they would do the same thing in vacuum.

The condensates in eq. (1), each normalized to give the contribution from two degrees of freedom, admit the following gauge-invariant definitions and thermal expectation

\[^{[1]}\] Although this observation has been described to me as “well known” in the course of several private conversations, I did not succeed in finding a reference to it in the early literature. A recent appearance is in \[^{[6]}\].

\[^{[2]}\] Is is perfectly consistent to turn on a temperature without generating expectation values for these fields. For instance, at the leading order in perturbation theory, \( F \sim \phi^\dagger \phi^\dagger \) vanishes (it involves two antiholomorphic fields), and so does \( D^a \sim \phi^\dagger t^a \phi \) for \( t^a \) traceless.
Figure 1: One-loop fermion self-energy of a fermion $\psi$ due to gauge interaction, at large energy $E$. The plasma is probed by soft propagators (with all components $\sim T$ in Minkowski spacetime), which give rise to the low-energy correlators (2) and are denoted here by the crosses. The (hard) remainder of the diagram is expanded in powers of $T/E$.

values:

$$Z_g \equiv \frac{1}{d_A} \langle v_\sigma F^{\sigma \mu} \psi v_\sigma' F^{\sigma' \mu} \rangle = 2 \int \frac{d^3 q}{(2\pi)^3} \frac{n_B(q)}{q} = \frac{T^2}{6},$$

$$Z_S \equiv \frac{2}{d_M} \langle \phi^* \phi \rangle = 2 \int \frac{d^3 q}{(2\pi)^3} \frac{n_B(q)}{q} = \frac{T^2}{6},$$

$$Z_f^\psi \equiv \frac{1}{2d_M} \langle \bar{\psi} \frac{\psi}{v \cdot D} \psi \rangle = 2 \int \frac{d^3 q}{(2\pi)^3} \frac{n_F(q)}{q} = \frac{T^2}{12},$$

with $v^\mu = (1, v)$ the four-velocity of the hard particle, $n_{B,F}$ the standard Bose-Einstein and Fermi-Dirac distribution functions, and $d_{A,M}$ the dimensions of the adjoint and matter representations, respectively. Useful examples include the thermal masses in $\mathcal{N} = 4$ SYM, which are all equal to $m_\infty^2 = g^2 N_c T^2$, and the gluon and gluino masses in pure glue SQCD, $m_\infty^2, g(\lambda) = \frac{1}{4} g^2 N_c T^2$.

The structures in eq. (2) are identical to those entering the hard thermal loop effective action [9, 10]: these are in fact the unique dimension-two gauge-invariant operators that can be built out of a light-like four-vector $v^\mu$ [9]. Given that next-to-leading order ($O(g)$) corrections are associated with hard thermal loop physics [8] ($gT$ scale physics), but only $g^2$ corrections arise from the hard scale $\sim E$, one concludes that at $O(g)$ only the matrix elements (2) get corrected but not their coefficients (1). Thus, $O(g)$ corrections preserve supersymmetry. For completeness, we record their values here. They are most readily obtained by working within the dimensionally reduced Euclidean theory [11], in which the corrections to $Z_g$ only arise from the contribution $Z_g \rightarrow \frac{1}{d_A} 2 A_4^a A_4^a$ of longitudinal gluons with zero Matsubara frequency. Next-to-leading order thermal masses were also obtained in [12] by means of an indirect thermodynamic argument, by relating them to the well-known $\sim g^3 T^3$ corrections to the QCD entropy.

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3 The authors of [12] also expressed worries that actual thermal dispersion relations of hard particles could grow like $m^2 \sim ET$ at high energies, which would spoil the physical interpretation of the
Either way one obtains ($Z_{f}^{\text{NLO}} = Z_{f}^{\text{LO}} + \mathcal{O}(g^{2}T^{2})$):

$$Z_{g}^{\text{NLO}} = \frac{T^{2}}{6} - \frac{Tm_{\infty,g}}{\pi\sqrt{2}} + \mathcal{O}(g^{2}T^{2}), \quad Z_{S}^{\text{NLO}} = \frac{T^{2}}{6} - \frac{Tm_{\infty,S}}{2\pi} + \mathcal{O}(g^{2}T^{2}).$$  (3)

What the factorization formula (1) (and the fact that it preserves supersymmetry) becomes when $\mathcal{O}(g^{2})$ effects are accounted for, if it retains any meaning at all, is unknown. The derivation outlined in fig. 1 shows that the supersymmetry of the thermal masses at leading order may be interpreted as a statement about the couplings of soft particles with various spins to hard propagators: these turn out to be universal in supersymmetric theories.

3 Imaginary parts of self-energies at weak coupling

The imaginary parts of self-energies at weak coupling arise from $2 \rightarrow 2$ scattering against plasma particles, as well as from induced collinear splitting processes (bremsstrahlung or pair production). For charged particles in gauge theories, the dominant contribution to $\text{Im} \Pi$ is $\sim g^{2}TE$ due to elastic small angle Coulomb scattering, though the dominant inelastic contribution $\sim g^{4}T^{3/2}E^{1/2}$ (barring logarithms) is due to induced collinear processes, which we will discuss first. These processes also dominate the self-energies of neutral particles in gauge theories, provided these particles are allowed to split into charged ones. In non-gauge theories, self-energies begin at $\sim g^{4}T^{2}E^{0}$ due to ordinary $2 \rightarrow 2$ scattering, which we will discuss in subsection 3.2.

3.1 Collinear pair production

The key aspects of collinear pair production may be briefly summarized as follows. This process is only relevant in gauge theories, in which it is initiated by the very frequent small-angle (Coulomb) scatterings suffered by either the parent or the daughter particles. At high-energies $E \gg gT$, its long formation time (associated with its
collinearity) requires that multiple thermal scatterings occurring during it be summed coherently. This causes a parametrically significant destructive interference, the so-called LPM effect [15], which is responsible for the non-analytic behavior $\Pi \propto E^{1/2}$. For relativistic plasmas, a complete leading-order treatment was given (for photons) in [16] (see also earlier discussions [18, 17], in which different approximations are made). Somewhat schematically, the result may be written in the form:

$$-2\text{Im } \Pi_a(E) = \sum_{bc} \int_0^1 dz P_{a\rightarrow bc}(z) F_{a\rightarrow bc}(E,z)$$

(4)

where $P_{a\rightarrow bc}$ are ordinary DGLAP kernels [20], governing collinear physics, $bc$ indexes final states and $z = E_b/E_a$ is the longitudinal momentum fraction. In eq. (4) we have omitted final state Bose-enhancement or Pauli-blocking factors which are not needed unless $z$ or $(1 - z)$ are very small, $\sim T/E$. The functions $F(E,z)$ depend in a complicated way on $E$ and $z$ and are to be obtained by solving an effective inhomogeneous Schrödinger equation governing the evolution of the pair in the transverse plane [16]. This equation depends on the details of the plasma through a collision kernel $C(q_\perp)$, which is a function of the transverse momentum transfer. Its only property that we need is that it involves only eikonal physics: it does not depend on the spins of the particles. For our purposes $F(E,z)$ in eq. (4) is thus just some universal function, the same for all final states among a given supersymmetry multiplet. In the leading logarithmic approximation [16], $F(E,z) \sim g^4 N_c^2 T^{5/2} E^{1/2} z^{-1/2} (1 - z)^{-1/2} (\log \left( \frac{g T}{2 z (1 - z)} \right))^{1/2}$.

The only ingredients in eq. (4) which could possibly break supersymmetry are the DGLAP splitting kernels $P_{a\rightarrow bc}(z)$. Such kernels are listed in table 1 for various supermultiplets of initial and final states. As shown in the table, when complete supermultiplets of final states are summed over (thereby enforcing the symmetry under $z \rightarrow (1 - z)$), supersymmetry with respect to the initial particle is restored. Not shown in the table (it is related to the first three entries by a crossing symmetry [21]), but which also preserves supersymmetry, is the process of bremsstrahlung of a gauge particle off a chiral multiplet. Thus, all in-medium splitting rates preserve supersymmetry.

Observations of supersymmetry in DGLAP kernels were made long ago by Dokshitzer [21], and subsequently given an explanation (at the one-loop level, we believe) by Lipatov and collaborators [22]. Here we are merely reporting their implications at finite temperature.

We expect coupling constant corrections to eq. (4) to first arise at $O(g)$. In thermal perturbation theory such $\sim g$ factors arise from ordinary loop factors $g^2$ multiplied by large bosonic occupation numbers $n_B \sim T/p^0 \sim T/gT$, and are associated with hard hard thermal loop physics [8] at the $gT$ scale. Such soft physics can only interfere with processes having a sufficiently long duration, such as the soft scatterings contributing to
Table 1: DGLAP splitting kernels. The first three entries govern the splitting of photons and photinos to a chiral multiplet, the next three pertain to the Yang-Mills splitting of a nonabelian gauge multiplet, and the last two govern the Yukawa splitting of a chiral multiplet. Supersymmetry is restored when complete supermultiplets of final states are summed over.

| Process | DGLAP kernel $P(z)$ | Sum |
|---------|---------------------|-----|
| $\gamma \to \psi^\dagger \psi$ | $e^2 [z^2 + (1 - z)^2]$ | $e^2$ |
| $\gamma \to \phi^\dagger \phi$ | $e^2 [2z(1 - z)]$ | $e^2$ |
| $\tilde{\gamma} \to \phi^\dagger \psi$ | $e^2 [2z]$ | $e^2$ |
| $g \to gg$ | $g^2 C_A [2(1 - z)/z + 2z/(1 - z) + 2z(1 - z)]$ | $g^2 C_A [2/z + 2/(1 - z) - 3]$ |
| $g \to \lambda^\dagger \lambda$ | $g^2 C_A [z^2 + (1 - z)^2]$ | $g^2 C_A [2/z + 2/(1 - z) - 3]$ |
| $\lambda \to g \lambda$ | $g^2 C_A [4z/(1 - z) + 2(1 - z)]$ | $g^2 C_A [2/z + 2/(1 - z) - 3]$ |
| $\phi \to \psi^\dagger \psi^\dagger$ | $y^2 [1]$ | $y^2$ |
| $\psi \to \phi^\dagger \psi^\dagger$ | $y^2 [2z]$ | $y^2$ |

$C(q_\perp)$ with $q_\perp \sim gT$, so we believe that this is the only ingredient suffering from $O(g)$ corrections. These soft collisions have a purely diffusive effect, so we could equivalently say that all $O(g)$ corrections pertain to the so-called transverse momentum diffusion coefficient “$\hat{q}$”. These corrections should be calculable using techniques similar to those used for heavy quark diffusion in $[23]$, but this has not yet been done. Since only eikonal physics is involved, these corrections trivially preserve supersymmetry.

The $O(g^2)$ corrections to eq. (4) are expected to possess a much more interesting and richer structure. For instance, they will most certainly require dealing with the scale dependence of the partonic constituents of the plasma, which should ultimately lead to “saturation” effects $[24]$ at very high energies upon summation of large logarithms $\alpha_s \log(E/T)$ and $\alpha_s \log(q_\perp^2/T^2)$ with $q_\perp^2 \sim E^{1/2}T^{3/2}$. The scale evolution of the constituents of the probe, which has to be treated in the presence of the LPM effect, should also enter at this order. Other interesting (though manifestly supersymmetry-preserving) effects may include sensitivity to nonperturbative $g^2 T$-scale magnetic physics, which we believe contributes to $\hat{q}$ at $O(g^2)$. We leave to future work a detailed analysis of these effects and of the question of whether they preserve supersymmetry.

As for the subleading corrections in $T/E$, we expect supersymmetry-breaking effects in $\Pi$ not to be larger than $\sim T^{5/2}E^{-1/2}$. These could arise from various $\sim T/E$ or $\sim q_\perp^2/E^2 \sim (T/E)^{3/2}$ corrections to ingredients entering $F(E,z)$, such as the eikonal vertices.
Table 2: Left panel: scattering amplitudes $|\mathcal{M}|^2$ in Wess-Zumino model, with amplitudes related by crossing symmetry not shown. Right panel: amplitudes summed over final states, for which supersymmetry is manifest as a function of particle 1 with particle 2 held fixed.

3.2 $2 \rightarrow 2$ scattering at weak coupling

Ordinary $2 \rightarrow 2$ collisions dominate self-energies in non-gauge models, which we will now discuss; their total rate is found to preserve supersymmetry. We first recall the general formula for the total collision rate ($-\text{Im } \Pi = -\Gamma E$):

$$-2\text{Im } \Pi(p_1) = \int \frac{d^3p_2d^3p_3d^3p_4}{(2\pi)^{12}E_2E_3E_4^2} (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)$$

$$\times \sum_{s_2s_3s_4} |\mathcal{M}_{1s_2\rightarrow s_3s_4}|^2 n_b(E_2)(1 \pm n_c(E_3))(1 \pm n_d(E_4)). \quad (6)$$

Here the particle labels are as defined as in fig. 2, the $s_i$ label the corresponding particle species, and $n_i$ are the corresponding distribution functions.

Let us first assume, for a moment, that the distribution functions can be omitted in the final state ("Bose-enhancement" and "Pauli-blocking") factors $(1 \pm n_i)$, which is justified for generic final state energies $E_3 \sim E_4 \sim \sqrt{E_1E_2} \sim \sqrt{E_T}$. The integrand then depends only on the sum $\sum_{s_2s_3s_4} |\mathcal{M}|^2_{1s_2\rightarrow s_3s_4}$. Such matrix elements summed over final states turn out to obey supersymmetry identities, with respect to the particle 1, for fixed identities of particle 2. This is exemplified in table 2 for single-field Wess-Zumino model with cubic superpotential, and the generalization to other models will be given shortly. Therefore, the contribution to eq. (6) from the region $E_3, E_4 \gg T$ preserves supersymmetry.

It is easy to convince oneself that for bounded amplitudes $|\mathcal{M}|^2$, the regions $E_3 \sim T$ and $E_4 \sim T$ suffer from $\sim T/E$ phase-space suppressions, and so the neglect of the final state distributions in eq. (3) is justified. However, $s/t \sim ET/T^2$ singularities in squared matrix elements when $t \lesssim T^2$ can overcome this suppression, and a separate discussion is required for these singular terms.\footnote{The total integral of such $\sim 1/t$ singularities is logarithmically divergent at $t \rightarrow 0$. This is cured by resumming hard thermal loop self-energies \cite{5} to the soft exchanged fermion propagator.}
or $E_3 \sim T$, can be treated similarly). To establish the supersymmetry of this region, for which the distribution function $n(E_4)$ must be kept, we need another ingredient: universality of the $1/t$ singularities. Indeed, the coefficient of $1/t$ at $t \to 0$, which is due to soft fermion exchange, is left unchanged when the hard particle 1 is replaced by its superpartner (e.g. if particles 1 and 3 are exchanged in fig. 2). This shows that the complete $\sim T^2 E^0$ self-energies in the Wess-Zumino model preserve supersymmetry, up to $\sim T^3 E^{-1}$ corrections.

It is interesting to analyze the $t \to 0$ singularities by means of the effective theory language used for thermal masses in section 2. Indeed, the region $E_4 \sim T$, $t \sim T^2$ in fig. 2 is characterized by soft fields coupled to a hard line and is thus governed by the gradient expansion depicted in fig. 1. This means that the $\sim T^2 E^0$ contribution to eq. (6) from soft fermion exchange may be understood as a contribution to the imaginary part of the fermion condensate in eq. (2), at one-loop in thermal perturbation theory. This way we see that the universality of $1/t$ singularities as a function of particle 1 is closely linked to the supersymmetry of thermal masses.

We now show, as promised, that supersymmetry of scattering amplitudes summed over final states holds in any supersymmetric theory, as a property of the S-matrix. Introducing the notation $P_{i_1\ldots i_n}$ for projection operators which perform the sum over complete supermultiplets of scattering states with $n$ particles, this follows from considering the following trace (over scattering states):

$$
\text{Tr} \left[ S^\dagger P_{34} S \left( |2\rangle\langle 2| \otimes [Q, |1\rangle\langle \bar{1}|] \right) \right],
$$

with $S$ the S-matrix and $\bar{1}$ denotes the superpartner of particle 1. For any supersymmetry generator $Q$ which does not annihilate particle 1, the commutator $[Q, |1\rangle\langle \bar{1}|] \propto (|1\rangle\langle 1| - |\bar{1}\rangle\langle \bar{1}|)$, so eq. (7) computes the difference:

$$
\sum_{s_3,s_4} \left( |M_{12\to s_3 s_4}|^2 - |M_{\bar{1}2\to s_3 s_4}|^2 \right).
$$

For scalar exchange, this argument relates the locality of the scalar condensate $\phi^*\phi$ in eq. (2) (which implies that it is purely real) to the absence of $1/t$ singularities due to scalar exchange.
For a massless particle 2 one can always choose $Q$ so as to annihilate particle 2; such a $Q$ commutes with $|2\rangle \langle 2|$, with the S-matrix as well as with the projectors $P_{i_1...i_n}$ (by construction), showing that eq. (7) (and thus eq. (8)) vanishes, being the trace of a commutator. This establishes the supersymmetry of the contributions from $E_3, E_4 \gg T$ to eq. (6) in any theory.

Combining the results of the preceding sections, we have reached a simple conclusion: the full thermal self-energies of gauge-neutral particles preserve supersymmetry, at leading order in the coupling, up to corrections suppressed by a at least $T^{5/2}E^{-1/2}$. Although we believe the analysis can be generalized to charged particles (for which the analysis is made more complicated by the stronger singularities $M \sim 1/t$ associated with gluon exchange and various sources of infrared divergences which make these self-energies less cleanly defined), here we will refrain from doing so: we are content with a robust result for gauge-invariant self-energies.

4 Strong coupling

Maldacena’s conjectured gauge/gravity correspondence renders possible, among other things, the calculation of correlators of currents in certain strongly coupled large $N_c$ gauge theories. In theories which have a continuous R-symmetry, such as the SU(4) of $\mathcal{N} = 4$ super Yang-Mills, “photons” and “photinos” can be introduced by weakly gauging a U(1) subgroup of the R-symmetry. Their self-energies are then given by suitable two-point functions of currents and their superpartners, which we now evaluate by means of the correspondence.

In the case of the on-shell photon self-energy in $\mathcal{N} = 4$ SYM, it was argued by means of a WKB approximation (in appendix) that at high energy the calculation localizes itself near the boundary of the AdS space. Here we generalize this phenomenon to other backgrounds, and establish two properties of the resulting effective theory of high-energy photon/photino propagation. First, it only probes the underlying low-energy medium through the expectation value of its energy-momentum tensor (actually, only through one component $\propto p_\mu p_\nu T^{\mu\nu}$). Second, it preserves supersymmetry: the absorption rates and dispersion relations of a photon and of a photino are identical.

6 In an effective theory language this implies the existence of certain pure imaginary operators which are not probed by the thermal masses, such as the dimension-1 operator described in footnote. At dimension-2 I find operators like $v \cdot A \delta(−iv\cdot D)D$ (representing e.g. an interference term between t-channel gluon exchange and $D$-term scalar self-interaction in $\phi\phi \rightarrow \phi\phi$ scattering), as well as its superpartner involving $\lambda$. At least for vanishing chemical potential I could check that such operators take on no expectation value.
We will be considering five-dimensional metrics of the general form

$$ds^2 = R^2 g(z) dz^2 + h_{\mu\nu}(z) dx^\mu dx^\nu$$

for which, near the boundary $z = 0$, the metric approaches that of AdS$_5$ with radius $R$ (for which $g(z) = 1$ and $h_{\mu\nu}(z) = \eta_{\mu\nu}$). The metric eq. (9) should be sufficiently general to cover any system invariant under space-time translation that admits a gravity dual. For the AdS$_5$ black hole, relevant for $N = 4$ SYM at finite temperature $T$, $$-h_{00} = 1 - (\pi T z)^4, h_{ij} = \delta_{ij}, h_{i0} = 0$$ and $$g(z) = (-h_{00})^{-1}$$. At certain steps below rotational invariance will be assumed; these steps will be highlighted.

### 4.1 Bulk equations

The bulk dual of the spin-1 current which couples to the photon is a five-dimensional gauge field, whose field strength tensor obeys Maxwell’s equations:

$$0 = \frac{z}{\sqrt{g(z) \det(-h(z))}} \partial_z \left( \frac{h^{\mu\sigma}}{z} \sqrt{\frac{\det(-h(z))}{g(z)}} F_{\sigma}^\mu \right) + h^{\mu\sigma} h^{\rho \rho} \partial_\mu F_{\rho \sigma}, \quad (10)$$

$$\partial_\alpha F_{\mu\nu} = \partial_\mu F_{\alpha\nu} - \partial_\nu F_{\alpha\mu}. \quad (11)$$

Here $\mu, \nu, \sigma, \rho$ are space-time indices but $\alpha$ may cover all five coordinates. We will restrict our attention to space-time momentum eigenstates $\partial_\mu = i p_\mu$. A closed equation for the transverse electric field $F_{0, \perp}$, for $\nu = \perp$ a component perpendicular to $p_\mu$, may be obtained by acting on the first equation with a partial time derivative $\partial_0$, and using the second equation. Specifically, one uses relations such as $\partial_0 F_{z, \perp} = \partial_z F_{0, \perp}$, which follow from dropping perpendicular derivatives $\partial_\perp$ in the latter. To fully exploit such simplifications, rotational invariance must be assumed, so that upstairs derivative $h^{\perp\sigma} \partial_\sigma$ also vanish. This yields the closed equation:

$$\frac{zh_{\perp\perp}}{\sqrt{g(z) \det(-h(z))}} \partial_z \left( \frac{h^{\perp\perp}}{z} \sqrt{\frac{\det(-h(z))}{g(z)}} \partial_z F_{0, \perp} \right) = h^{\mu\nu} p_\mu p_\nu F_{0, \perp}, \quad (12)$$

in which no summation over $\perp$-indices is implied.

The bulk dual of the spin-$\frac{1}{2}$ operator coupling to the photino is a five-dimensional Dirac fermion with mass $m = \frac{1}{2} [28]$ (in units with $R = 1$). It possesses as many components as two four-dimensional Weyl spinors, but it is dual to only one such spinor: the sign of $m$ breaks the symmetry between the two Weyl components. The bulk Dirac equation reads:

$$[\slashed{D} + m] \psi = 0 \equiv \left[ \gamma^a e_a^\alpha \left( \partial_\alpha + \frac{1}{4} \omega_{\alpha}^{ab} \gamma_a \gamma_b \right) + m \right] \psi, \quad (13)$$
with \( \alpha, a = 0 \ldots 4 \) and \( e^a_\alpha \) the orthogonal basis. Under the assumption of rotational invariance, the term involving the spin connection \( \omega \) must be proportional to the single matrix \( \gamma_z \), and it can be removed by a \( z \)-dependent field rescaling. We choose the rescaling \( \psi = z^2 (\det(-h))^{-1/4} e^{-m} \int dz \sqrt{g(z)/z} \phi \), which leads to the following equations for the Weyl components of \( \psi_L,R \) of \( \psi \):

\[
\begin{align*}
\partial_z \psi_L &= \sqrt{g(z)} \hat{\psi}_R \psi_R, \\
\left[ \frac{1}{\sqrt{g(z)}} \partial_z - \frac{2m}{z} \right] \psi_R &= \hat{\psi}_L \psi_L.
\end{align*}
\]

(14)

(15)

Here \( \hat{\psi}_L,R \) are Weyl operators associated with the four-dimensional metric \( h_{\mu\nu}(z) \). With \( m = \pm \frac{1}{2} \) the component relevant near the \( z = 0 \) boundary is \( \psi_L \) and we are calculating the self-energy of a left-handed photino. Eq. (15) implies a closed equation for \( \psi_L \):

\[
\hat{\psi}_R \left[ \frac{1}{\sqrt{g(z)}} \partial_z - \frac{2m}{z} \right] \frac{1}{\hat{\psi}_R \sqrt{g(z)}} \partial_z \psi_L = h^{\mu\nu} p_\mu p_\nu \psi_L.
\]

(16)

4.2 WKB solution and supersymmetry

We are now in position to discuss the WKB approximation. By means of a change of variable \( y \equiv y(z) \), Maxwell’s equation eq. [12] may be cast in a Schrödinger form with potential proportional to the squared energy \( p^2_0 \), provided

\[
\frac{dy}{dz} = 2zh_{\perp\perp} \sqrt{\frac{g(z)}{\det(-h(z))}}.
\]

(17)

The factor of 2 is convenient: near the boundary \( y \sim z^2 \). For black holes (like the AdS\(_5 \) black hole metric given above) the function \( g(z) \) has a pole at a finite value of \( z \) (the location of the horizon), while the function \( \det(-h) \) vanishes there. In this limit \( y \) is mapped logarithmically to infinity. The rescaled potential remains finite there, though, due to the blowing up of \( h_{00} \); it then depends only on the energy \( E = p_0^2 \).

The qualitative features of the Schrödinger potential in the equation \( [\partial_y^2 - V(y)] F_{0\perp} = 0 \) are sketched in fig. 3. The shape of the potential depends on the geometry but not on the energy, which only determines its overall normalization. At large \( y \) the potential becomes constant, while for \( y \to 0 \) the leading term becomes, for on-shell and off-shell momenta respectively,

\[
V(y) \to \begin{cases} 
\frac{1}{4y} p^2, & \quad p^2 \neq 0, \\
\frac{y}{4} p_\mu p_\nu \frac{d h^{\mu\nu}(z)}{dz^2} = -\frac{\pi^2 T^{\mu\nu} p_\mu p_\nu}{2N_c^2}, & \quad p^2 = 0.
\end{cases}
\]

(18)
Figure 3: Schematic features of the Schrödinger potential $V(y)/p^2$, when $p^2 = 0$. It approaches the universal linear behavior eq. (18) near the boundary and tends to a constant at the horizon $y \to \infty$, with a transition regime that may depend on the details of the theory and on possible intrinsic mass scales $m$.

Here we have used that the leading corrections to the metric near the boundary are proportional to $z^4$ and are related to the expectation value of the stress-energy tensor $T_{\mu\nu}$; its trace part, if nonzero, does not contribute when $p^2 = 0$. The normalization in eq. (18) is appropriate to the $N = 4$ SYM theory.

At the horizon $y \to \infty$ the solutions are oscillatory and in-falling boundary conditions $F_0 \propto e^{i\omega}$ must be imposed for calculating retarded correlators [25], with $\omega = p^0 \pi T/2$ the natural frequency near the horizon. To obtain correlators of currents, as described shortly, this solution must be continued to the AdS$_5$ boundary $z = 0$. For energies high relative to all intrinsic scales in the metric, a WKB approximation can be used. This is applicable for $y$ down to $y \sim 1/p^2$ (resp. $y \sim (T^4 E^2)^{-1/3}$) for $p^2 \neq 0$ (resp. $p^2 = 0$), at which it breaks down due to the redshift factors.\[7 The transition between the two regimes $p^2 \neq 0$ and $p^2 = 0$ takes place smoothly around $|p^2_s| \sim E^{2/3} T^{4/3}$, for which value of $p^2$ the two estimates for $y$ cross each other. This natural scale $p^2_s(E)$ is exactly the “saturation scale” $p_s \sim T/x_s$ discussed in [29] at strong coupling, viewed as a function of $E$ with $x_s \equiv p^2_s/2ET$.\]

For the on-shell component $\psi^-_L$ of a left-handed photino in a rotationally-invariant background, the operator $\hat{p}_R$ is nonsingular with eigenvalue $E(\sqrt{|h^{00}|} + \sqrt{|h^{33}|})$. Here $h^{33}$ gives the metric component along the longitudinal direction. Just as for eq. (17),

$$\frac{dy}{dz} = 2z\sqrt{g(z)} \frac{\hat{p}_R(z)}{\hat{p}_R(z = 0)}. \quad (19)$$

For the Dirac equation eq. (16) one needs only modify the change of variable eq. (17), to:

\[7 The transition between the two regimes $p^2 \neq 0$ and $p^2 = 0$ takes place smoothly around $|p^2_s| \sim E^{2/3} T^{4/3}$, for which value of $p^2$ the two estimates for $y$ cross each other. This natural scale $p^2_s(E)$ is exactly the “saturation scale” $p_s \sim T/x_s$ discussed in [29] at strong coupling, viewed as a function of $E$ with $x_s \equiv p^2_s/2ET$.\]
near the boundary $y \sim z^2$ and the horizon is mapped logarithmically to $y = \infty$, so the same WKB approximation applies. More significantly, one readily sees from eq. (16) that the approximate potentials near the boundary are identical to the photon case, eqs. (18).

Correlation functions are obtained by prescribing the limiting values of the fields $F_{0,\perp}$ and $\psi_L$ near the boundary and evaluating boundary terms $\propto \partial_y F_{0,\perp}$ (see e.g. [28]), or proportional to $\bar{\psi}\psi \sim \psi_R / z \sim 1 / \rho_R$, or more precisely,

$$\Pi_\gamma = -\frac{N_c^2 T^2}{8 \pi^2} \lim_{y \to 0} \frac{\partial_y F_{0,\perp}(y)}{F_{0,\perp}(y)}, \quad \Pi_\gamma = -\frac{N_c^2 T^2}{8 \pi^2} \lim_{y \to 0} \frac{\partial_y \psi_L(y)}{\psi_L(y)}. \quad (20)$$

Here $\Pi_\gamma \equiv \bar{\pi} \Sigma u$ is the photino self-energy sandwiched between on-shell polarization spinors $u$, whose real part yields the thermal mass squared. The normalizations eq. (20) are fixed by the well-known supersymmetry-preserving vacuum result, $\Pi_\gamma = \Pi_\gamma = -\frac{N_c^2 T^2}{32 \pi^2} \log(p^2 / \mu^2)$, $p^2 = p^0 - p_0^2$. On the light-cone, Schrödinger’s equation with the approximate potential (18) is solved in terms of Bessel (Hankel) functions,

$$F_{0,\perp}(y) = \psi_L^- = y^{1/2} \left( J_{1/3} \left( \frac{2}{3} \omega y^{3/2} \right) + i Y_{1/3} \left( \frac{2}{3} \omega y^{3/2} \right) \right), \quad (21)$$

with $\omega^2 = \pi^2 T^2 \rho^0 / 2 N_c^2$, yielding with eq. (20) the result:

$$\Pi_\gamma(p) = \Pi_\gamma(p) = \frac{N_c^2 T^2}{16 \pi^2 \Gamma(\frac{2}{3})} \left( \frac{3}{2} - i \frac{3}{2} \right) \omega^{\frac{2}{3}}, \quad p^0 = p \text{ large.} \quad (22)$$

The imaginary part of this result reproduces that given in [28], with $\omega = p^0 T^2 \pi^2 / 2$ in thermal $\mathcal{N} = 4$ SYM. Corrections in $T/E$ to this result may be found by expanding the potential eq. (18) to higher orders near the boundary; for the AdS$_5$ black hole this expansion proceeds in powers of $y^2 \sim \omega^{-4/3}$, so the first subleading corrections to $\Pi$ are $\sim \omega^{-2/3}$.

It is remarkable that photon self-energies at strong coupling, and high energies, depend only on one property of the plasma: its stress-energy tensor. An heuristic picture of strongly coupled plasmas, based on the idea of parton saturation, has been proposed recently [29] in which such a universality comes out naturally.

## 5 Operator Product Expansion

Supersymmetry violations in deeply virtual correlators may be analyzed by means of the operator product expansion (OPE) [30]. The OPE is a means of separating short-distance and long-distance physics, so that the thermal corrections to deeply virtual (short-distance) correlators with $E \gg T$ are expressed in terms of the expectation
value of local operators. Thermal corrections are thus suppressed by powers \( \sim E^{-\Delta} \), with the powers \( \Delta \) set by the scaling dimensions of local operators.

The difference between a correlator of operators and of their superpartners may be written as a supersymmetry variation (which corresponds to the fact that it vanishes in supersymmetry-preserving vacua). For instance, for correlators of transverse currents \( \epsilon \cdot J \) and of their superpartners \( \lambda_{\alpha} \) \cite{32}, one schematically has:

\[
\epsilon_1 \cdot J(p) \epsilon_2 \cdot J - \frac{1}{2} \lambda^\dagger(p) \bar{\phi}_1 \bar{\phi}_2 \lambda \propto \epsilon_1^{\dot{\alpha}} Q_{\alpha} \left[ \lambda^\dagger_{\dot{\alpha}}(p) \epsilon_2 \right]
\]

with \( p_{\mu} \epsilon_1^{\mu} = 0 \) and \( \alpha, \dot{\alpha} \) spinor indices. As an operator equation the OPE must commute with the supersymmetries, so from the OPE of the right-hand side of eq. (23) one concludes that the operators on its left-hand side are supersymmetry variations. Thus, in order to see supersymmetry violations at order \( E^{-2} \) or stronger, one must find local gauge-invariant fermionic operators of dimension \( \frac{3}{2} \) or less.

In a wide class of theories, an accidental symmetry comes into play: such operator do not exist. These theories certainly include all weakly coupled gauge theories containing no U(1) vector multiplets and no gauge-neutral chiral superfields. In these theories, gauge-invariant dimension-2 bosonic operators (such as \( \text{Tr} \phi^* \phi \) or \( \text{Tr} \phi^* \phi^* \)) do not correspond to any supersymmetry variations, and must thus enter in the same way in the OPEs of fields and of their superpartners; this is easily verified at leading order for the OPE of currents in weakly coupled gauge theories. The lowest-dimensional fermionic operators are dimension-\( \frac{5}{2} \) supercurrents, from which we conclude that thermal supersymmetry breaking can only be seen through dimension-3 operators, \( \sim E^{-3} \).

When neutral chiral superfields or U(1) vector multiplets are present, nonzero expectation values for \( \bar{D} \sim \phi^* \phi \) or \( \bar{F} \sim \phi^* \phi^* \) auxiliary fields (which enter the supersymmetry transformations of the gaugino and fermionic matter fields, respectively) could produce supersymmetry violations at dimension 2. A similar possibility was observed for thermal masses, in section eq. (2) but, as we discussed there, we do not view it as being specifically related to thermal effects. Thus, we conclude that in weakly coupled theories, the absence of thermal supersymmetry breaking below dimension 3 is generic.

It is not possible to analyze general theories at finite values of the coupling constants, because finite anomalous dimensions can alter the power-counting. Nevertheless, for certain strongly coupled theories accessible to the AdS/CFT correspondence, it is easy to be more quantitative. For instance, it is known \cite{33} that in \( \mathcal{N} = 4 \) SYM at large \( \text{'t} \) Hooft coupling \( \lambda \gg 1 \), only protected (chiral) operators have finite dimensions \( \Delta \ll \lambda^{1/4} \) and that the lowest-dimensional fermionic operator has dimension \( 2 + \frac{1}{2} = \frac{5}{2} \) (being

\footnote{It is worth noting that the OPE, being most rigorously formulated in Euclidean signature, does not enjoy at the moment the same rigorous status in Minkowski space-time with respect to nonperturbative physics (see e.g. \cite{31}).}
the supersymmetry variation of a dimension-2 primary). Similarly, the $\mathcal{N} = 1$ theory dual to IIB string theory on $\text{AdS}_5 \times T^{11}$ \cite{33} is known to contain no fermionic operator of dimension less than 2 \cite{35}. Thus, in these theories, supersymmetry violations can only be seen at the level of $\sim E^{-3}$ or $\sim E^{-\frac{7}{2}}$ corrections, respectively. A discussion of more general strongly coupled theories will not be attempted here.

6 Discussion

In this letter we have shown that supersymmetry is a generic property of the effective theories which describe high-energy correlators in supersymmetric plasmas. These correlators included self-energies at high energies on the light-cone as well as far away from it (large virtuality).

For all correlators (except for the more tractable deeply virtual correlators, treated in section 5) our analysis has been limited to the leading nontrivial order in the relevant coupling constant expansion (at both weak and strong coupling). Unfortunately, this seems like a major limitation of this work: it makes it hard to decide whether our findings highlight general properties of supersymmetric theories, or whether they are mere artefacts of these extreme limits. In this respect it would be interesting to know the structure of higher order corrections, at both weak and strong coupling.

We have found that thermal supersymmetry violations in all correlators are suppressed by a power of the energy $E^{-n}$ (relative to the vacuum correlators), with $n$ strictly greater than 2. (Violations with $n = 2$ were observed in sections 2 and 5 due to nonvanishing $D$-term of $F$-term expectation values, but we do not regard these as specifically thermal effects. These would have identical supersymmetry-breaking effects in vacuum correlators.)

We find pleasing that such a simple uniform bound holds. This makes one wonder whether our findings could be elevated to some sort of theorem, valid independently of perturbation theory, though we have no concrete proposal to make along these lines.

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