Eigenvalue Asymptotics for a Schrödinger Operator with Non-Constant Magnetic Field Along One Direction

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Abstract. We consider the discrete spectrum of the two-dimensional Hamiltonian $H = H_0 + V$, where $H_0$ is a Schrödinger operator with a non-constant magnetic field $B$ that depends only on one of the spatial variables, and $V$ is an electric potential that decays at infinity. We study the accumulation rate of the eigenvalues of $H$ in the gaps of its essential spectrum. First, under some general conditions on $B$ and $V$, we introduce effective Hamiltonians that govern the main asymptotic term of the eigenvalue counting function. Further, we use the effective Hamiltonians to find the asymptotic behavior of the eigenvalues in the case where the potential $V$ is a power-like decaying function and in the case where it is a compactly supported function, showing a semiclassical behavior of the eigenvalues in the first case and a non-semiclassical behavior in the second one. We also provide a criterion for the finiteness of the number of eigenvalues in the gaps of the essential spectrum of $H$

Keywords: magnetic Schrödinger operators, spectral gaps, eigenvalue distribution

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1 Introduction

Let $\mathbb{R}^2 \ni (x, y) \mapsto B(x) \in \mathbb{R}_+$ be a bounded magnetic field and define the Schrödinger operator

$$H_0 := -\frac{\partial^2}{\partial x^2} + \left( -i \frac{\partial}{\partial y} - b(x) \right)^2,$$  \hspace{1cm} (1.1)

where the second component of the magnetic vector potential $\mathbb{R}^2 \ni (x, y) \mapsto (0, b(x)) \in \mathbb{R}^2$ is given by

$$b(x) = \int_0^x B(t) \, dt.$$  \hspace{1cm} (1.2)

Let $V : \mathbb{R}^2 \to [0, \infty)$ be an electric potential that decays at infinity. Set $H = H_0 + V$. It is known that the essential spectrum of $H$, denoted by $\sigma_{\text{ess}}(H)$, satisfies

$$\sigma_{\text{ess}}(H) = \bigcup_{j \in \mathbb{N}} [\mathcal{E}_j^-, \mathcal{E}_j^+],$$  \hspace{1cm} (1.3)

with $\mathcal{E}_j^-, \mathcal{E}_j^+ \in \mathbb{R}_+$. Suppose that there exists a finite gap in the essential spectrum of $H$, which in our context will be equivalent to

$$\mathcal{E}_j^+ < \mathcal{E}_{j+1}^-,$$  \hspace{1cm} (1.4)

for some $j \geq 1$ (see Section 2). Then, it is possible to define

$$N_j(\lambda) := \text{Tr}_1(\mathcal{E}_j^+, \mathcal{E}_{j+1}^-)(H), \quad \text{for } 0 < \lambda < \mathcal{E}_{j+1}^- - \mathcal{E}_j^+,$$  \hspace{1cm} (1.5)
where $\mathbb{1}_\omega(\cdot)$ is the characteristic function of the set $\omega$. The function $N_j$ counts the number of eigenvalues of $H$ on the interval $(\mathcal{E}_j^+ + \lambda, \mathcal{E}_{j+1}^-)$.

Our purpose in this article is to describe the asymptotic behavior of $N_j(\lambda)$ as $\lambda$ goes to zero, for some types of non-constant magnetic fields $B$ and electric potentials $V$.

For constant magnetic fields, the asymptotic behavior of the function $N_j$ was thoroughly described in relation with the decaying regime at infinity of the function $V$. This includes power-like, exponential and compactly supported regimes (see [26], [15], [27], [19], [11], [24]).

A natural extension of these results was to consider the eigenvalue counting function for Schrödinger operators with asymptotically constant magnetic field and decaying electric potential (see [15], [28], [29], and for related problems see [25], [22]). Other natural extensions are the Schrödinger operators with unidirectionally constant magnetic field presented here. This last model was first considered by A. Iwatsuka (with $V \equiv 0$) in order to give examples of magnetic Schrödinger operators with purely absolutely continuous spectrum [16]. The one particle system determined by this Hamiltonian presents some interesting transport and spectral properties which have been studied in the mathematical literature (see [18], [9], [30], [31], [7], [17], [13]), as well as in the physics literature (see e.g. [20], [5], [12], [21]).

The problem of the asymptotic behavior of the counting function (1.5) for the Iwatsuka Hamiltonian was already addressed in [30]. In that article the behavior of $N_j$ was obtained for potentials $V$ that decay at infinity as $(x^2 + y^2)^{-m/2}$ (see (2.16), (2.18) below), supposing that $0 < m < 1$, and under the assumption that $B$ is a monotone bounded function. In Corollary 2.4 we will present a result similar to the semiclassical one given in [30], which completes the description of the first asymptotic term of $N_j$ for power-like decaying potentials, that is we consider the case $m > 1$. Furthermore, using the effective Hamiltonian of Theorem 2.1 we are able to deal with other types of decaying regimes of $V$. Namely, in Corollary 2.2 we give a sufficient condition that guarantees the finiteness of the number of eigenvalues of $H$ in each gap of $\sigma_{\text{ess}}(H)$. This is a geometric condition that depends on the set where $B$ reach its supremum and the support of $V$.

When the condition of Corollary 2.2 does not hold, we can see that $N_j$ is generically unbounded in each gap of $\sigma_{\text{ess}}(H)$, as follows incidentally from Corollary 2.3 where we give asymptotic bounds for $N_j$ if $V$ is of compact support. Contrary to Corollary 2.4, the behavior of $N_j$ is not semiclassical in this situation, since a semiclassical formula would imply the finiteness of the number of eigenvalues. For compact supported potentials $V$, a different non-semiclassical asymptotic behavior of the eigenvalue counting function was obtained in [27], [11], in the constant magnetic fields case. In that context the main asymptotic term is $(\ln |\ln \lambda|)^{-1}|\ln \lambda|$ which is faster than the one obtained here, $|\ln \lambda|^{1/2}$. Similar results to our was previously obtained in [3], [4], for other magnetic Hamiltonians with compact supported electric potentials (see Remark after Theorem 2.1).

For non-positive potentials $V$ we could define the functions

$$N_j^-(\lambda) := \text{Tr} \mathbb{1}_{(-\infty, \mathcal{E}_j^- - \lambda)}(H); \quad N_j^-(\lambda) := \text{Tr} \mathbb{1}_{(\mathcal{E}_j^+, \mathcal{E}_{j+1}^- - \lambda)}(H), \quad j \in \mathbb{N}.$$ 

Although we will present our results only for $N_j$, $j \in \mathbb{N}$, they are still valid, with obvious modifications, for $N_j^-$, $j \in \mathbb{Z}_+ := \{0, 1, 2, \ldots\}$. We omit these in order to simplify the presentation.
2 Main Results

2.1 Effective Hamiltonian

To introduce the effective Hamiltonians that govern the main asymptotic term of $N_j$, we need to estate more specific conditions on the magnetic field $B$ and then recall some well-known properties of the unperturbed operator $H_0$.

Throughout this article we will assume the following:

\begin{enumerate}
  \item $B \in L^\infty(\mathbb{R})$.
  \item $B_- \leq B(x) \leq B_+$ a.e., for some positive constants $B_+ > B_-$. \hfill (2.1)
  \item $\lim_{x \to \infty} B(x) = B_+$, and $\limsup_{x \to -\infty} B(x) < B_+$. \hfill (2.1)
\end{enumerate}

Under condition (2.1) the operator defined by (1.1) is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$ and its spectrum, denoted by $\sigma(H_0)$, is purely absolutely continuous \cite{16}, \cite{17}. Note that the potential $b$ defined by (1.2) is an absolutely continuous strictly increasing function such that

$$B_- |x| \leq |b(x)| \leq B_+ |x|.$$ \hfill (2.2)

Let $\mathcal{F}$ be the partial Fourier transform

$$(F u)(x, k) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{-iky} u(x, y) \, dy, \quad u \in C_0^\infty(\mathbb{R}^2).$$

Then

$$\mathcal{F} H_0 \mathcal{F}^* = \int_{\mathbb{R}} h(k) \, dk,$$ \hfill (2.3)

where $h(k)$ is the self-adjoint operator acting in $L^2(\mathbb{R})$, defined by

$$h(k) = -\frac{d^2}{dx^2} + (b(x) - k)^2, \quad k \in \mathbb{R}.$$ \hfill (2.4)

For any $k \in \mathbb{R}$ the spectrum of the operator $h(k)$ is discrete and simple. We denote the increasing sequence of eigenvalues by $\{E_j(k)\}_{j=1}^\infty$. For any $j \in \mathbb{N}$ the band function $E_j(\cdot)$ is analytic as a function of $k \in \mathbb{R}$.

Set $\mathcal{E}_j^- := \inf_{k \in \mathbb{R}} E_j(k)$; $\mathcal{E}_j^+ := \sup_{k \in \mathbb{R}} E_j(k)$, then

$$\sigma(H_0) = \bigcup_{j=1}^\infty E_j(\mathbb{R}) = \bigcup_{j=1}^\infty [\mathcal{E}_j^-, \mathcal{E}_j^+].$$ \hfill (2.5)

Condition (2.1) b) implies that $B_-(2j - 1) \leq E_j(k) \leq B_+(2j - 1)$ for all $k \in \mathbb{R}$, and (2.1) c) implies that $\lim_{k \to \infty} E_j(k) = B_+(2j - 1) = \mathcal{E}_j^+$, for all $j \in \mathbb{N}$ (see \cite{16}).

Put

$$\varphi_j(x) := \frac{H_{j-1}(x)e^{-x^2/2}}{(\sqrt{\pi}2^{j-1}(j - 1)!)^{1/2}}, \quad x \in \mathbb{R}, \quad j \in \mathbb{N},$$ \hfill (2.6)

where

$$H_q(x) := (-1)^q e^{x^2} \frac{d^q}{dx^q} e^{-x^2}, \quad x \in \mathbb{R}, \quad q \in \mathbb{Z}_+.$$
are the Hermite polynomials (see e.g. [1] Chapter I, Eqs. (8.5), (8.7)), then the real-valued function \( \varphi_j \) satisfies

\[
-\varphi''_j(x) + x^2 \varphi_j(x) = (2j - 1) \varphi_j(x), \quad \| \varphi_j \|_{L^2(\mathbb{R})} = 1.
\]

For \((x, \xi) \in \mathbb{R}^2\) define the function

\[
\Psi_{j;x,\xi}(k) = B_+^{-1/4} e^{-ik\xi} \varphi_j(B_+^{1/2}x - B_+^{1/2}b^{-1}(k)), \quad j \in \mathbb{N}, \quad x \in \mathbb{R}, \quad k \in \mathbb{R}. \quad (2.7)
\]

The system \( \{ \Psi_{j;x,\xi}(x,\xi) \}_{(x,\xi) \in \mathbb{R}^2} \) is overcomplete with respect to the measure \( \frac{B_+}{2\pi} dx d\xi \) (see [1] Subsection 5.2.3 for the definition of an overcomplete system with respect to a given measure). Introduce the orthogonal projection

\[
P_{j;x,\xi} := |\Psi_{j;x,\xi} \rangle \langle \Psi_{j;x,\xi}|, \quad (x, \xi) \in \mathbb{R}^2,
\]

acting in \( L^2(\mathbb{R}) \), and the pseudo-differential operator \( \mathcal{V}_j : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) defined as the weak integral

\[
\mathcal{V}_j := \frac{B_+}{2\pi} \int_{\mathbb{R}^2} V(x, \xi) P_{j;x,\xi} \, dx d\xi, \quad (2.8)
\]

i.e. \( \mathcal{V}_j \) is an operator with contravariant symbol \( V \).

As already mentioned, for the potential \( V \) we will assume the following:

a) \( 0 \leq V \in L^\infty(\mathbb{R}^2) \).

b) \( \lim_{x^2+y^2 \to \infty} V(x, y) = 0. \) (2.9)

The diamagnetic inequality and Weyl’s theorem imply that \( \sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H_0) = \sigma(H_0) \), then (1.3) holds true. Conditions (2.9) also imply that \( \mathcal{V}_j \) is a non-negative and compact operator.

**Theorem 2.1.** Assume that for some \( j \in \mathbb{N} \), (1.4) is true. Assume also that \( B \) satisfies (2.1), and \( V \) satisfies (2.9). Consider \( E_j \) as a multiplication operator in \( L^2(\mathbb{R}) \). Then for each \( \delta \in (0, 1) \)

\[
\text{Tr} \mathbbm{1}_{(\mathcal{E}_j^+ + \lambda, \infty)} (E_j + (1 - \delta) \mathcal{V}_j) + O_\delta(1) 

\leq \mathcal{N}_j(\lambda) \leq \text{Tr} \mathbbm{1}_{(\mathcal{E}_j^+ + \lambda, \infty)} (E_j + (1 + \delta) \mathcal{V}_j) + O_\delta(1), \quad \lambda \downarrow 0. \quad (2.10)
\]

**Remark:** Similar results to Theorem 2.1 appear in [3] and [4]. In [3] the discrete spectrum of operators of the form \( H_1 = H_{\text{Hall}} + V \), is described, where

\[
H_{\text{Landau}} = H_{\text{Landau}} + W(x),
\]

\( H_{\text{Landau}} \) being the two dimensional Schrödinger operator with constant magnetic field, and \( W \) a monotonic function depending only on the first variable \( x \). In the same way, in [4] the operator \( H_2 = H_{\text{Hall},-\text{Plane}} + V \) is considered, where \( H_{\text{Hall},-\text{Plane}} \) is the Schrödinger operator with constant magnetic field defined for a half-plane, with a Dirichlet boundary condition along the edge. In both articles an eigenvalue counting function similar to (1.5) is studied. The effective Hamiltonians obtained in those articles are particular cases of the one given by Theorem 2.1 if we put \( b^{-1}(k) = B_+ k \) in (2.7). All these three models share the particularity that the unperturbed operators \( H_{\text{Hall}}, H_{\text{Landau}} \) and \( H_0 \) admit a direct integral decomposition with fibred operators that converge to shifted harmonic oscillators as \( k \to \infty \). However, despite this similarity, the proof of 2.1 requires the use of some new ideas and presents technical difficulties that do not appear in [3] or [4].
2.2 Asymptotic behavior of $N_j(\lambda)$: Finite number of eigenvalues

In Corollaries 2.2, 2.3 we will see that the finiteness or the infiniteness of the number of eigenvalues of $H$ in the gaps of $\sigma_{\text{ess}}(H)$, depend on a relation between the support of $V$ and the number

$$x^+ := \inf \{ x \in \mathbb{R} ; B(t) = B_+ \text{ for almost all } t \in (x, \infty) \}. \quad (2.11)$$

Note that it is possible to have $x^+ = \infty$.

**Corollary 2.2.** Suppose that (1.4) is true, and that $B$ satisfies (2.1). Assume also that $V$ satisfies (2.9) and $\| \int_{\mathbb{R}} V(x, y) \ dy \|_{L^\infty(\mathbb{R})} < \infty$. Then, if $x^+ > \sup \{ x \in \mathbb{R} ; \text{for some } y \in \mathbb{R}, (x, y) \in \text{ess supp } V \}$,

$$N_j(\lambda) = O(1), \quad \lambda \downarrow 0. \quad (2.12)$$

we have that

$$N_j(\lambda) = O(1), \quad \lambda \downarrow 0. \quad (2.13)$$

2.3 Asymptotic behavior of $N_j(\lambda)$: Infinite number of eigenvalues for $V$ of compact support

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Denote by $c_-(\Omega)$ the maximal length of the vertical segments contained in $\Omega$. Further, let $B_R((x, y)) \subset \mathbb{R}^2$ be a disk of radius $R > 0$ centered at $(x, y) \in \mathbb{R}^2$. For $a \in \mathbb{R}$, set

$$K(\Omega, a) := \{ (\xi, R) \in \mathbb{R} \times \mathbb{R}_+ ; \text{there exists } \eta \in \mathbb{R} \text{ such that } \Omega \subset B_R((\xi + a, \eta)) \},$$

and

$$c_+(\Omega, a) := \inf_{(\xi, R) \in K(\Omega, a)} R \kappa \left( \frac{\xi_+}{eR} \right),$$

where $\xi_+ := \max \{ \xi, 0 \}$, and $\kappa(s) := | \{ t > 0 ; t \ln t < s \} |$, for $s \in [0, \infty)$. Here $| \cdot |$ denotes the Lebesgue measure in $\mathbb{R}$.

Also define

$$\tilde{\Omega} := \{ (x, y) \in \Omega ; x > x^+ \}.$$ 

**Corollary 2.3.** Assume that (1.4) holds true, and that $B$ is a function satisfying (2.1). Further assume that

$$c_- 1_{\Omega_-}(x, y) \leq V(x, y) \leq c_+ 1_{\Omega_+}(x, y), \quad (x, y) \in \mathbb{R}^2, \quad (2.14)$$

where $\Omega_{\pm} \subset \mathbb{R}^2$ are bounded domains with Lipschitz boundaries, and $0 < c_- \leq c_+ < \infty$. Then, if

$$x^+ < \sup \{ x \in \mathbb{R} ; \text{for some } y \in \mathbb{R}, (x, y) \in \Omega_- \},$$

the following asymptotic bounds

$$C_- | \ln \lambda |^{1/2}(1 + o(1)) \leq N_j(\lambda) \leq C_+ | \ln \lambda |^{1/2}(1 + o(1)), \quad \lambda \downarrow 0, \quad (2.15)$$

hold true with $C_- := (2\pi)^{-1/2} \sqrt{bc_-}(\tilde{\Omega}_-) \text{ and } C_+ := e \sqrt{bc_+}(\tilde{\Omega}_+, x^+)$. 

**Remark:** The constants $C_\pm$ already appeared in [3, 4], where it is shown that $C_- < C_+$. 

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2.4 Asymptotic behavior of $N_j(\lambda)$: Infinite number of eigenvalues for power-like decaying $V$

Now we will consider potentials $V$ whose support is not compact. First we will assume that there exists a positive number $m$ such that, for any pair $(\alpha, \beta) \in \mathbb{Z}_+$, there exists a positive constant $C_{\alpha, \beta}$, such that

$$|\partial_\xi^\beta \partial_\xi^\alpha V(x, \xi)| \leq C_{\alpha, \beta} (x, \xi)^{-m-\alpha-\beta} \quad \text{for all } (x, \xi) \in \mathbb{R}^2,$$

(2.16)

where $(x, \xi) = (1 + x^2 + \xi^2)^{1/2}$.

Moreover, let $s \in \mathbb{R}$ and define the volume function

$$N(\lambda, V, s) := \frac{1}{2\pi} \text{vol}\{(x, \xi) \in \mathbb{R}^2; V(x, \xi) > \lambda, x > s\},$$

(2.17)

where $\text{vol}$ denotes the Lebesgue measure in $\mathbb{R}^2$. We will assume that for some $s_0 \in \mathbb{R}$ and positive constants $C$ and $\lambda_0$

$$N(\lambda, V, s_0) \geq C \lambda^{-2/m}, \quad 0 < \lambda < \lambda_0.$$ 

(2.18)

We say that a decreasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the homogeneity condition if

$$\lim_{\epsilon \downarrow 0} \limsup_{\lambda \downarrow 0} \lambda^{2/m} (f(\lambda(1 - \epsilon)) - f(\lambda(1 + \epsilon))) = 0.$$

(2.19)

**Corollary 2.4.** Assume that (1.4) is true. Also suppose that $B$ is a smooth function with all its derivatives bounded and for some $M > m$

$$B_+ - B(x) = O(\langle x \rangle^{-M}), \quad x \to \infty.$$ 

(2.20)

If $V$ satisfies (2.16) with $m > 1$, and for $s_0 \in \mathbb{R}$, $N(\lambda, V, s_0)$ satisfies (2.18) and (2.19), then we have the following asymptotic formula

$$N_j(\lambda) = B_+ N(\lambda, V, s_0)(1 + o(1)), \quad \lambda \downarrow 0.$$ 

(2.21)

**Remarks:**

i) The smoothness condition on $B$ is not essential. For instance, an easy modification of the arguments permits to prove Corollary 2.4 just assuming $x^+ < \infty$.

ii) Condition (2.16) implies that if $N(\lambda, V, s_0)$ satisfies (2.19) for some $s_0 \in \mathbb{R}$, then $N(\lambda, V, s)$ satisfies (2.19) as well, for any $s \in \mathbb{R}$. Moreover, if $N(\lambda, V, s_0)$ satisfies (2.18) then the asymptotic formula (2.21) is true for any $s \in \mathbb{R}$, since

$$\lim_{\lambda \downarrow 0} \frac{N(\lambda, V, s)}{N(\lambda, V, s_0)} = 1.$$ 

iii) Results of the same type of (2.21) were obtained in [30], for non-sign-definite potentials $V$, were the number $m$ in (2.16) is assumed to be $0 < m < 1$, and the function $B$ monotone.

iv) As already mentioned in the Remark after Theorem 2.1 in [3], [4] the eigenvalue counting function for magnetic Schrödinger operators similar to those considered here, was studied. However, in [3], [4], the asymptotic behavior of these counting functions was described only for compactly supported potentials $V$, as in Corollary 2.2 and in a slightly weaker version of Corollary 2.3. Since the effective Hamiltonians obtained in [3], [4] are examples of the one in Theorem 2.1, the conclusions of Corollary 2.4 are also valid for the counting functions of the models considered in the articles [3], [4].


3 Proof of the results

3.1 Proof of Theorem 2.1

Before we begin the proof, let us set some notation and auxiliary results that we will use throughout the text. Let \( r > 0 \) and \( T = T^* \) be a linear compact operator acting in a given Hilbert space. Set

\[
n_{\pm}(r; T) := \text{Tr} \mathbb{1}_{[r, \infty)}(\pm T);
\]

thus the functions \( n_{\pm}(\cdot; T) \) are respectively the counting functions of the positive and negative eigenvalues of the operator \( T \). If \( T \) is compact but not necessarily self-adjoint (in particular, \( T \) could act between two different Hilbert spaces), we will use also the notation

\[
n^*(r; T) := n_{\pm}(r^2; T^*T), \quad r > 0;
\]

thus \( n^*(\cdot; T) \) is the counting function of the singular values of \( T \). Evidently,

\[
n_{\pm}(r; T) = n^*_+(r^2; T^*T), \quad r > 0.
\]

Let us recall also the well-known Weyl inequalities

\[
n_+(r_1 + r_2; T_1 + T_2) \leq n_+(r_1; T_1) + n_+(r_2; T_2) \quad (3.1)
\]

where \( r_j > 0 \) and \( T_j, \ j = 1, 2 \), are linear self-adjoint compact operators (see e.g. [2, Theorem 9.2.9]), as well as the Ky Fan inequalities

\[
n_+(r_1 + r_2; T_1 + T_2) \leq n_+(r_1; T_1) + n_+(r_2; T_2) \quad (3.2)
\]

for compact but not necessarily self-adjoint \( T_j, \ j = 1, 2 \), (see e.g. [2, Subsection 11.1.3]). Further, let \( S_p, \ p \in [1, \infty) \), be the Schatten–von Neumann class of compact operators, equipped with the norm

\[
\|T\|_p := \left( -\int_0^\infty r^p \, dn_+(r; T) \right)^{1/p}.
\]

Then the Chebyshev-type estimate

\[
n_+(r; T) \leq r^{-p}\|T\|_p^p \quad (3.3)
\]

holds true for any \( r > 0 \) and \( p \in [1, \infty) \).

We start the proof by using the Birman-Schwinger principle, which give us

\[
\mathcal{N}_j(\lambda) = n_-(1; V^{1/2}(H_0 - \mathcal{E}_j^+ - \lambda)^{-1}V^{1/2}) + O(1), \quad \lambda \downarrow 0. \quad (3.4)
\]

Note that from [2.3]

\[
(H_0 - \mathcal{E}_j^+ - \lambda)^{-1} = \mathcal{F}^* \int_{\mathbb{R}}^{\oplus} (h(k) - \mathcal{E}_j^+ - \lambda)^{-1} \, dk \mathcal{F}. \quad (3.5)
\]

\(^3\)All Hilbert spaces in this article are supposed to be separable.
Let $\pi_j(k)$ be the orthogonal projection of $h(k)$ corresponding to the eigenvalue $E_j(k)$, and for $\lambda > 0$, $A \in [-\infty, \infty)$ set

$$T_j(\lambda, A) := \mathcal{F}^* \int_{(A, \infty)} (\mathcal{E}_j^+ - E_j(k) + \lambda)^{-1} \pi_j(k) \, dk \mathcal{F}.$$ 

Let $l \in \mathbb{N}$. Suppose that $l < j$, then for all $k \in \mathbb{R}$

$$E_l(k) \leq B_+(2l - 1) < B_+(2j - 1) = \mathcal{E}_j^+.$$ 

Also, (2.5) and (1.4) imply that for all $l > j$

$$E_l(k) \geq \mathcal{E}_l^- \geq \mathcal{E}_{j+1}^- > \mathcal{E}_j^+,$$

for all $k \in \mathbb{R}$. Then, there exists a positive constant $\kappa$ such that for all $l \neq j$ and $k \in \mathbb{R}$

$$|E_l(k) - \mathcal{E}_j^+| > \kappa. \quad (3.6)$$

Therefore, if $I$ is the identity operator in $L^2(\mathbb{R})$, due to (3.6) the limit

$$\lim_{\lambda \downarrow 0} (h(k) - \mathcal{E}_j^+ - \lambda)^{-1} (I - \pi_j(k)) \quad (3.7)$$

exesists in the norm operator topology.

**Lemma 3.1.** For any $j \in \mathbb{N}$, $E_j(k) < \mathcal{E}_j^+$ for all $k \in \mathbb{R}$. Moreover, for any $A \in \mathbb{R}$ there exists $\alpha > 0$ such that $\mathcal{E}_j^+ - E_j(k) > \alpha$, for all $k < A$.

**Proof.** First let us prove that for any $k$ real, $\mathcal{E}_j^+ - E_j(k) > 0$. Let $B_1$ and $B_2$ be two functions satisfying condition (2.1), and let $b_1, b_2$ be the corresponding magnetic potentials as chosen in (1.2). Note that

$$b_s(x) - k = \int_{b_s^{-1}(k)}^x B_s(t) \, dt, \quad s = 1, 2.$$ 

Then it is easy to see that if $B_1(x) \leq B_2(x)$ a.e. in $\mathbb{R}$,

$$(b_1(x) - k)^2 \leq (b_2(x) - b_2(b_1^{-1}(k)))^2, \quad (3.8)$$

for all $k$, and all $x$ in $\mathbb{R}$. For $b_1, b_2$, let $h(k, b_1), h(k, b_2)$ be the operators defined by (2.4), and denote by $E_j(k, b_1), E_j(k, b_2)$ their associated $j$-th eigenvalues. The inequality (3.8) implies that for all $k \in \mathbb{R}$

$$h(k, b_1) \leq h(b_2(b_1^{-1}(k)), b_2), \quad (3.9)$$

and from the min-max principle we obtain that for all $k \in \mathbb{R}$, and all $j \in \mathbb{N}$

$$E_j(k, b_1) \leq E_j(b_2(b_1^{-1}(k)), b_2). \quad (3.10)$$

Now, since $\limsup_{x \to -\infty} B(x) < B_+$, there exists a real number $\beta$ and a non-decreasing smooth function $B_\beta$ such that

$$B_\beta(x) = \begin{cases} 
\limsup_{x \to -\infty} B(x) & \text{if } x \leq \beta \\
B_+ & \text{if } x \geq \beta + 1,
\end{cases}$$
and $B(x) \leq B_\beta(x)$ a.e. in $\mathbb{R}$. From the proof of [18, Theorem 3.2] we know that $E_j(k; b_\beta)$ is a non-decreasing function. Since it is also analytic, (2.5) implies that $E_j(k; b_\beta) < \mathcal{E}_j^+$ for all $k \in \mathbb{R}$. Using (2.10) we obtain that $E_j(k) < \mathcal{E}_j^+$.

To prove the second assertion of the Lemma, just note that $E_j(\cdot; b_\beta)$ satisfies the required condition and use (3.10) again.

Using the Weyl inequalities (3.1) together with (3.5) and (3.7), and together with Lemma 3.1, it can be easily seen that for any $r \in (0, 1)$

$$n_+(1 + r; V^{1/2}T_j(\lambda, A)V^{1/2}) + O(1) \leq n_-(1; V^{1/2}(H_0 - \mathcal{E}_j^+ - \lambda)^{-1}V^{1/2}) \leq n_+(1 - r; V^{1/2}T_j(\lambda, A)V^{1/2}) + O(1),$$

(3.11)
as $\lambda \downarrow 0$.

Next, let $h_\infty(k)$ be the shifted harmonic oscillator

$$h_\infty(k) := -\frac{d^2}{dx^2} + (B_+ x - B_+ b^{-1}(k))^2,$$

self-adjoint in $L^2(\mathbb{R})$, for $k \in \mathbb{R}$. The spectrum of $h_\infty(k)$ coincide with the set of Landau levels \{\(B_+ (2j - 1) = \mathcal{E}_j^+\}_{j=1}^\infty\}. Let $\pi_{j,\infty}(k)$ be the orthogonal projection of $h_\infty(k)$ corresponding to the eigenvalue $\mathcal{E}_j^+$, which can be described explicitly by

$$\pi_{j,\infty}(k) = |\Psi_{j,\infty}(\cdot, k)\rangle \langle \Psi_{j,\infty}(\cdot, k)|,$$ (3.12)

where $\Psi_{j,\infty}(x, k) = B_+^{1/4} \varphi_j(B_+^{1/2} x - B_+^{1/2} b^{-1}(k))$ ($\varphi_j$ defined in (2.6)).

For $\lambda > 0$ and $A \in [-\infty, \infty)$, set

$$T_{j,\infty}(\lambda, A) := \mathcal{F}^* \int_{(A, \infty)}^{\oplus} (\mathcal{E}_j^+ - E_j(k) + \lambda)^{-1} \pi_{j,\infty}(k) \, dk \mathcal{F}.$$ (3.13)

Our next goal is to replace $T_j(\lambda, A)$ by $T_{j,\infty}(\lambda, A)$ in inequality (3.11).

**Theorem 3.2.** For any $j \in \mathbb{N}$

$$\lim_{k \to \infty} \frac{||\pi_j(k) - \pi_{j,\infty}(k)||}{(\mathcal{E}_j^+ - E_j(k))^{1/2}} = 0.$$

The proof of this Theorem follows from the next two lemmas.

**Lemma 3.3.** Define $\Lambda_k := h(k)^{-1} - h_\infty(k)^{-1}$. Then $\Lambda_k \geq 0$ and

$$\lim_{k \to \infty} ||\Lambda_k|| = 0.$$ (3.14)

**Proof.** To see that $\Lambda_k \geq 0$, use (2.11) b) and (3.9). To prove (3.14) we introduce the unitary operators $U_k : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined for any $k \in \mathbb{R}$ by

$$(U_k f)(x) = f(x + b^{-1}(k)),$$
and set
\[ \tilde{h}(k) := U_k h(k) U_k^* = -\frac{d^2}{dx^2} + (b(x + b^{-1}(k)) - k)^2 \]
and
\[ \tilde{h}_\infty := U_k h_\infty(k) U_k^* = -\frac{d^2}{dx^2} + (B_+ x)^2. \]

Instead of (3.14) we will prove the equivalent statement \( \lim_{k \to \infty} ||\tilde{h}(k)^{-1} - \tilde{h}_\infty^{-1}|| = 0. \)

Put \( d_k(x) := \tilde{h}_\infty - \tilde{h}(k) = (B_+ x)^2 - (b(x + b^{-1}(k)) - k)^2. \) Using two times the resolvent identity we get
\[ \tilde{h}(k)^{-1} - \tilde{h}_\infty^{-1} = \tilde{h}_\infty^{-1} d_k \tilde{h}_\infty^{-1} + \tilde{h}(k)^{-1} d_k \tilde{h}_\infty^{-1} d_k \tilde{h}_\infty^{-1}. \] (3.15)

Therefore we need to prove first that \( \tilde{h}_\infty^{-1} d_k \tilde{h}_\infty^{-1} \) converges to zero in norm as \( k \to \infty. \)

Note that due to
\[ |d_k(x)| = |B_+ x + (b(x + b^{-1}(k)) - k)| |B_+ x - (b(x + b^{-1}(k)) - k)| \]

\[ = \left| \int_{b^{-1}(k)+x}^{b^{-1}(k)+x} B_+ + B(t) \ dt \right| \left| \int_{b^{-1}(k)}^{b^{-1}(k)+x} B_+ - B(t) \ dt \right| \] (3.16)

\[ \leq 2B_+ |x| \left| \int_{b^{-1}(k)}^{b^{-1}(k)+x} B_+ - B(t) \ dt \right| , \]

and \( \lim_{x \to \infty} B(x) = B_+ \), the function \( |d_k(x)| \) converges pointwise to zero when \( k \to \infty. \)

Denote by \( D(\tilde{h}(k)), D(\tilde{h}_\infty) \) the domains of \( \tilde{h}(k) \) and \( \tilde{h}_\infty \), respectively. Using (3.8), \( B_- \leq B(x) \leq B_+ \) implies that
\[ (B_- x)^2 \leq (b(x + b^{-1}(k)) - k)^2 \leq (B_+ x)^2, \quad \text{for all } x \in \mathbb{R}. \] (3.17)

Then the domains are equal and coincide with the domain of the harmonic oscillator, i.e. \( D(\tilde{h}(k)) = D(\tilde{h}_\infty) = D(-d^2/dx^2) \cap D(x^2) \). \[ \mathbb{R}. \]

Let \( f \in L^2(\mathbb{R}). \) Since \( \tilde{h}_\infty^{-1} f \in D(x^2) \), for any \( \epsilon > 0 \) one can find \( N > 0 \) such that
\[ \int_{|x|>N} |(b(x + b^{-1}(k)) - k)^2(\tilde{h}_\infty^{-1} f)(x)|^2 \ dx \leq \int_{|x|>N} |(B_+ x)^2(\tilde{h}_\infty^{-1} f)(x)|^2 \ dx < \epsilon. \] (3.18)

Further, \( \tilde{h}_\infty^{-1} f \) is also continuous, then
\[ \int_{-N}^{N} |d_k(x)(\tilde{h}_\infty^{-1} f)(x)|^2 \ dx \leq \sup_{x \in [-N,N]} |(\tilde{h}_\infty^{-1} f)(x)|^2 \int_{-N}^{N} |d_k(x)|^2 \ dx. \] (3.19)

Using (3.18), (3.19) and (3.16) we can conclude that \( d_k \tilde{h}_\infty^{-1} \) converges strongly to zero as \( k \to \infty. \) Consequently, the family \( d_k \tilde{h}_\infty^{-1} \) is uniformly bounded with respect to \( k \), and since \( \tilde{h}_\infty^{-1} \) is compact we get that \( \| \tilde{h}_\infty^{-1} d_k \tilde{h}_\infty^{-1} \| \to 0 \), for \( k \to \infty. \)

To finish the proof of the Lemma it only remains to show that for all \( G \in D(\tilde{h}(k)) = D(\tilde{h}_\infty) \), \( \| \tilde{h}(k)^{-1} d_k G \|_{L^2(\mathbb{R})} \leq C \| G \|_{L^2(\mathbb{R})} \), for some constant \( C \) independent of \( k \) and \( G \).
From [8, Theorem 1] we know that for any \( g \in D(\tilde{h}(k)) \)

\[
\frac{1}{2} \|(b(x + b^{-1}(k)) - k)^2 g\|^2_{L^2(\mathbb{R})} \leq \|\tilde{h}(k)g\|^2_{L^2(\mathbb{R})}.
\]  

(3.20)

Then for \( f \) in \( L^2(\mathbb{R}) \), if \( g = \tilde{h}(k)^{-1}f \) in (3.20)

\[
\frac{1}{2} \|(b(x + b^{-1}(k)) - k)^2 \tilde{h}(k)^{-1}f\|^2_{L^2(\mathbb{R})} \leq \|f\|^2_{L^2(\mathbb{R})}.
\]

Besides, using (3.17) we get \( \|(B_x x)^2 \tilde{h}(k)^{-1}f\|^2_{L^2(\mathbb{R})} \leq 2B_2^2/B_2 \|f\|^2_{L^2(\mathbb{R})} \), which implies the existence of a uniform bound for \( d_k \tilde{h}(k)^{-1} \), from where we can easily get the needed result for \( \tilde{h}(k)^{-1}d_k \).

**Lemma 3.4.** For all \( j \in \mathbb{N} \)

1. There exist a constant \( C_j \) such that for all \( k \) big enough

\[
\|\pi_j(k) - \pi_j,\infty(k)\| \leq C_j\|\Lambda_k \pi_j,\infty(k)\|.
\]

2. It is satisfied the asymptotic formula

\[
E_j^- - E_j(k) = E_j^+ \|\pi_j,\infty(k)\|\Lambda_k \pi_j,\infty(k)\| (1 + o(1)), \quad k \to \infty.
\]

(3.21)

**Proof.** The proof of this Lemma uses Lemma [3.3] and repeats almost word by word the proof of Propositions 3.6 and 3.7 in [4].

Putting together Lemmas [3.1, 3.3] and 3.4 we can prove Theorem 3.2 just by noticing that

\[
\frac{\|\pi_j(k) - \pi_j,\infty(k)\|}{(E_j(k) - E_j^+)^{1/2}} \leq \frac{C_j \|\Lambda_k \pi_j,\infty(k)\|}{E_j^+ \|\Lambda_k^{1/2} \pi_j,\infty(k)\|} (1 + o(1)) \leq \frac{C_j}{E_j^+} \|\Lambda_k^{1/2}(1 + o(1)), \quad k \to \infty.
\]

**Proposition 3.5.** For all \( A \in [-\infty, \infty), r \in \mathbb{R}, \delta \in (0, 1) \) and \( j \in \mathbb{N} \)

\[
\begin{align*}
n_+(r(1 + \delta); V^{1/2}T_j,\lambda, A)V^{1/2} &+ O(1) \\
&\leq n_+(r; V^{1/2}T_j,\lambda, A)V^{1/2} \\
&\leq n_+(r(1 - \delta); V^{1/2}T_j,\lambda, A)V^{1/2} + O(1), \quad \lambda \downarrow 0.
\end{align*}
\]

(3.22)

**Proof.** First note that \( n_+(r; V^{1/2}T_j,\lambda, A)V^{1/2} = n_+(r^{1/2}; V^{1/2}T_j,\lambda, A^{1/2}) \). By Lemma 3.1 for any \( A \in (A, \infty) \)

\[
\begin{align*}
n_+(r; V^{1/2}(T_j,\lambda, A^{1/2} - T_j,\lambda, A^{1/2})) &= O(1) \\
n_+(r; V^{1/2}(T_j,\lambda, A^{1/2} - T_j,\lambda, A^{1/2})) &= O(1), \quad \lambda \downarrow 0,
\end{align*}
\]

since both \( T_j,\lambda, A^{1/2} - T_j,\lambda, A^{1/2} \) and \( T_j,\lambda, A^{1/2} - T_j,\lambda, A^{1/2} \) have a limit in the norm sense when \( \lambda \downarrow 0 \). Thanks to Theorem 3.2 it is possible to choose \( \tilde{A} \) big enough such that

\[
n_+(r; V^{1/2}(T_j,\lambda, \tilde{A}^{1/2} - T_j,\lambda, \tilde{A}^{1/2})) = 0.
\]

Using the Ky-Fan inequalities (3.5) we get (3.22) (see Proposition 3.1 and Theorem 3.2 in [3] for more details of this proof).
Putting together (3.23), (3.24) and (3.25), we obtain that for any \( A \in [-\infty, \infty) \), \( r > 0 \) and \( \delta \in (0,1) \)
\[
n_+(r;V^{1/2}T_{j,\infty}(\lambda,A)V^{1/2}) \leq N_j(\lambda) \leq n_+(r(1-\delta);V^{1/2}T_{j,\infty}(\lambda,A)V^{1/2}) + O(1), \quad \lambda \downarrow 0.
\]
(3.23)

For \( A \in [-\infty, \infty) \) define
\[
P_{j,\infty}(A) := \mathcal{F}^* \int_{(A,\infty)}^{\oplus} \pi_{j,\infty}(k) \, dk \mathcal{F}.
\]

Then, setting \( A = -\infty \) we obtain that for any \( r > 0 \)
\[
n_+(r;V^{1/2}T_{j,\infty}(\lambda,-\infty)V^{1/2})
= n_+(r;T_{j,\infty}(\lambda,-\infty)^{1/2}V^{1/2}T_{j,\infty}(\lambda,-\infty)^{1/2})
= n_+(r;E_j^+ - E_j(\cdot) + \lambda)^{-1/2} F P_{j,\infty}(-\infty) V P_{j,\infty}(-\infty)^* (E_j^+ - E_j(\cdot) + \lambda)^{-1/2}).
\]
(3.24)

Let \( \mathcal{U} : L^2(\mathbb{R}) \rightarrow F P_{j,\infty}(-\infty)^* F L^2(\mathbb{R}^2) \), defined by \( (\mathcal{U}g)(x,k) = B_{+}^{1/4} g(k) \phi_j(B_{+}^{1/2} x - B_{+}^{1/2} b^{-1}(k)) \).
The operator \( \mathcal{U} \) is unitary and
\[
\mathcal{U}^* F P_{j,\infty}(-\infty) V P_{j,\infty}(-\infty)^* F^* \mathcal{U} = V_j,
\]
(3.25)

\[
\mathcal{U}^* (E_j^+ - E_j(\cdot) + \lambda)^{-1} \mathcal{U} = (E_j^+ - E_j(\cdot) + \lambda)^{-1}.
\]

Use (3.23), (3.24) and (3.25) together with the Birman-Schwinger principle to get (2.10).

### 3.2 Proof of Corollary 2.2

From inequality (3.23), we see that to prove this corollary it is enough to show that for some \( A \in [\infty, \infty) \) and \( r > 0 \)
\[
n_+(r^2;V^{1/2}T_{j,\infty}(\lambda,A)V^{1/2}) = n_+(r;T_{j,\infty}(\lambda,A)^{1/2}V^{1/2}) = O(1), \quad \lambda \downarrow 0.
\]
(3.26)

The Chebyshev-type estimate (3.3), with \( p = 2 \), states that
\[
n_+(r;T_{j,\infty}(\lambda,A)^{1/2}V^{1/2}) \leq r^{-2} ||T_{j,\infty}(\lambda,A)^{1/2}V^{1/2}||_2^2
= \frac{1}{2\pi r^2} \int_A^{\infty} \int_{\mathbb{R}^2} (E_j^+ - E_j(\cdot) + \lambda)^{-1} \psi_{j,\infty}(x,k)^2 V(x,y) \, dx \, dy \, dk,
\]
(3.27)

where we have used (3.13) and (3.12). Here and in the sequel we will assume without loss of generality that \( x^+ = 0 \). Indeed, for \( x^+ \) finite, this follows by making a translation along the \( x \)-axis and using the gauge invariance of \( H \). If \( x^+ \) is infinite, thanks to (3.3) we may replace \( B \) by a function \( \tilde{B} \geq B \) such that \( x^+_{\tilde{B}} := \inf \{ x \in \mathbb{R} ; \tilde{B}(t) = B_+ \} \) for almost all \( t \) in \( (x,\infty) \) \( = 0 \), and use (3.10) in (3.30) below.

Put \( X^+ := \sup \{ x \in \mathbb{R} ; \text{for some } y \in \mathbb{R}, (x,y) \in \text{ess supp } V \} \). Take \( \tilde{x} \) such that \( X^+ < \tilde{x} < 0 = x^+ \), and define the step function
\[
W(x) := \begin{cases} 
  b(\tilde{x})^2 - (B_+ \tilde{x})^2 & \text{for } x < \tilde{x} \\
  0 & \text{for } x \geq \tilde{x}.
\end{cases}
\]
(3.28)
Setting $h^W(k)$ as the operator given by

$$-\frac{d^2}{dx^2} + (B_+ x - B_+ b^{-1}(k))^2 + W(x),$$

self-adjoint in $L^2(\mathbb{R})$, it is not difficult to see that for $k > 0$

$$h(k) \leq h^W(k). \quad (3.29)$$

The spectrum of $h^W(k)$ is discrete and simple. Denote by $\{E_j^W(k)\}_{j=1}^{\infty}$ the increasing sequence of eigenvalues of $h^W(k)$. Inequality $(3.29)$ implies that

$$E_j(k) \leq E_j^W(k), \quad (3.30)$$

and then $(\mathcal{E}_j^+ - E_j(k) + \lambda)^{-1} \leq (\mathcal{E}_j^+ - E_j^W(k) + \lambda)^{-1}$ for all $j \in \mathbb{N}$, $k > 0$ and $\lambda > 0$.

By Proposition 4.2 of [3], we know that there exists a positive constant $C_j$ such that for all $k$ big enough

$$\mathcal{E}_j^+ - E_j^W(k) \geq C_j (B_+ b^{-1}(k))^{2j} e^{-B_+ (b^{-1}(k) - \bar{x})^2}.$$

Then for $A > 0$ large

$$\int_A^{\infty} \int_{\mathbb{R}^2} (\mathcal{E}_j^+ - E_j(k) + \lambda)^{-1} \psi_{j, \infty}(x, k)^2 V(x, y) \, dx \, dy \, dk \leq \frac{1}{C_j} \int_A^{\infty} \int_{\mathbb{R}^2} k^{3-2j} e^{2k(x-\bar{x})} e^{B_+(x^2-x^2)} H_j(B_+^{1/2}x - k) V(x, y) \, dy \, dx \, dk,$$

where we have used that $b^{-1}(k) = k/B_+$ for $k > 0$, due to $x^+ = 0$. The last integral can be decomposed into a finite sum of terms of the form

$$C_{l,n} e^{B_+ \bar{x}^2} \int_A^{\infty} \int_{\mathbb{R}^2} k^l x^n e^{2k(x-\bar{x})} e^{-B_+ x^2} V(x, y) \, dy \, dx \, dk \leq \left\| \int_{\mathbb{R}} V(x, y) \, dy \right\|_{L^\infty(\mathbb{R})} |C_{l,n}| e^{B_+ \bar{x}^2} \int_A^{\infty} k^l e^{2k(x-\bar{x})} \, dk \int_{-\infty}^{X_+} |x|^n e^{-B_+ x^2} \, dx,$$

for some constants $C_{l,n}$, and integers $l, n$. Each one of this terms is finite because of our choice of $\bar{x}$.

3.3 Proof of Corollary 2.3

Let us first show how to obtain the upper bound in $(2.15)$. As in the proof of Corollary 2.2 take de function $W$ defined in $(2.28)$, and for $A \in (-\infty, \infty)$, $\lambda > 0$ set

$$T_{j, \infty}^W(\lambda, A) := \mathcal{F}^{\ast} \int_{(A, \infty)}^\oplus (\mathcal{E}_j^+ - E_j^W(k) + \lambda)^{-1} \pi_{j, \infty}(k) \, dk \mathcal{F}. \quad (3.31)$$

From $(3.30)$, $T_{j, \infty}(\lambda, A) \leq T_{j, \infty}^W(\lambda, A)$, thus $(3.28)$ implies that for all $A \in [-\infty, \infty)$ and $r > 0$

$$N_j(\lambda) \leq n_+(r; V^{1/2} T_{j, \infty}^W(\lambda, A) V^{1/2}) + O(1), \quad \lambda \downarrow 0. \quad (3.31)$$
The asymptotic behavior of the function $n_+(r; V^{1/2}T_{j,\infty}^W(\lambda, A)V^{1/2})$ was studied in [3] where it is shown that (Theorems 5.1 and 6.1)

$$\limsup_{\lambda \downarrow 0} \frac{n_+(r; V^{1/2}T_{j,\infty}^W(\lambda, A)V^{1/2})}{|\ln \lambda|^{1/2}} \leq C_+.$$  \hspace{1cm} (3.32)

Putting together (3.31) and (3.32) we get the upper bound in (2.15).

For the lower bound consider the operators $h_{N^+}(k) := -d^2/dx^2 + (B_+ x - k)^2$ and $h_{-}(k) := -d^2/dx^2 + (B_- x - k)^2$ defined in $L^2(\mathbb{R}_+)$ and $L^2(\mathbb{R}_-)$, respectively, both with a Neumann boundary condition at zero. From the monotonicity property with respect to the Neumann conditions, and from (3.8) we obtain that

$$h(k) \geq h_{N^-}(k) \oplus h_{N^+}(k)$$ \hspace{1cm} (3.33)

(recall that $x^+ = 0$, which implies that $b(x) = B_+ x$ for $x \geq 0$). The operators $h_{\pm}(k)$ have discrete and simple spectrum for any $k \in \mathbb{R}$. Denoting by $\{E_j^{N\pm}(k)\}_{j=1}^\infty$ their increasing sequences of eigenvalues, and using that

$$\lim_{k \to \infty} E_1^{N^+}(k) = \infty, \quad \lim_{k \to \infty} E_j^{N^+}(k) = E_j^+,$$

(see e.g. [11]), we can conclude from (3.33) that for any $j \in \mathbb{N}$ there exists a constant $K_j$ such that

$$E_j(k) \geq E_j^{N^+}(k), \quad \text{for } k \geq K_j.$$ \hspace{1cm} (3.34)

Set

$$T_{j,\infty}^N(\lambda, A) := \mathcal{F}^* \int_{(A,\infty)}^{\infty} (\mathcal{E}_j^+ - E_j^{N^+}(k) + \lambda)^{-1} \pi_{j,\infty}(k) \, dk \, \mathcal{F}.$$ \hspace{1cm} (3.35)

Then, (3.23) along with (3.34) imply that for any $r > 0$ and $A \geq K_j$

$$\mathcal{N}_j(\lambda) \geq n_+(r; V^{1/2}T_{j,\infty}^N(\lambda, A)V^{1/2}) + O(1), \quad \lambda \downarrow 0.$$ \hspace{1cm} (3.36)

Besides, it is shown in [23] that for some positive constant $C_j$

$$\mathcal{E}_j^+ - E_j^{N+}(k) = C_j k^{2j-1} e^{-k^2/B_+} (1 + o(1)), \quad k \to \infty.$$ \hspace{1cm} (3.37)

Now we can repeat the proofs of Proposition 3.7 and Corollary 3.9 in [4] in order to obtain

$$\liminf_{\lambda \downarrow 0} \frac{n_+(r; V^{1/2}T_{j,\infty}^N(\lambda, A)V^{1/2})}{|\ln(\lambda)|^{1/2}} \geq C_-.$$ \hspace{1cm} (3.38)

The inequalities (3.35), (3.36) imply the lower bound in (2.15).
3.4 Proof of Corollary 2.4: Upper bound

The starting point of this proof is, as for Corollaries 2.2, 2.3 the inequalities 3.23. We will denote the operator $T_{j;\infty}(\lambda, -\infty)$ simply by $T_{j;\infty}(\lambda)$, and $P_{j;\infty}(-\infty)$ by $P_{j;\infty}$. Also from now on, with any lost of generality, we will take $s = 0$ for the function (2.17). That means, we will prove (2.21) for $N(\lambda, V, 0)$ (see Remark ii after Corollary 2.4).

Let $\varepsilon > 0$ and take a smooth function $\chi_\varepsilon$ with bounded derivatives such that $0 \leq \chi_\varepsilon(x) \leq 1$ for all $x \in \mathbb{R}$, $\chi_\varepsilon(x) = 0$ for $x \leq -2\varepsilon$ and $\chi_\varepsilon(x) = 1$ for $x \geq -\varepsilon$. Define

$$V_\varepsilon(x, \xi) := \chi_\varepsilon(x)V(x, \xi).$$

(3.37)

The Weyl’s inequalities say that for any $r > 0$, $\delta \in (0, 1)$, and $\lambda > 0$

$$n_+(r; T_{j;\infty}(\lambda)^{1/2}VT_{j;\infty}(\lambda)^{1/2}) \leq n_+(r(1 - \delta); T_{j;\infty}(\lambda)^{1/2}V_\varepsilon T_{j;\infty}(\lambda)^{1/2})$$

$$+ n_+(r\delta; T_{j;\infty}(\lambda)^{1/2}(V - V_\varepsilon)T_{j;\infty}(\lambda)^{1/2}).$$

(3.38)

The function $V - V_\varepsilon$ is equal to zero for $x \geq -\varepsilon$. Arguing as in the proof of Corollary 2.2 we can see that for any $r > 0$

$$n_+(r; T_{j;\infty}(\lambda)^{1/2}(V - V_\varepsilon)T_{j;\infty}(\lambda)^{1/2}) = n_+(\sqrt{r}; T_{j;\infty}(\lambda)^{1/2}(V - V_\varepsilon)^{1/2}) = O(1), \ \lambda \downarrow 0. \quad (3.39)$$

Now, since $E_j(k) \leq \mathcal{E}_j^+, T_{j;\infty}(\lambda) \leq \lambda^{-1}P_{j;\infty}$, thus the min-max principle implies that for all $r > 0$ and $\lambda > 0$

$$n_+(r; T_{j;\infty}(\lambda)^{1/2}V_\varepsilon T_{j;\infty}(\lambda)^{1/2}) \leq n_+(r\lambda; P_{j;\infty}V_\varepsilon P_{j;\infty}).$$

(3.40)

Next, let us introduce a class of symbols suitable for our purposes. For $(x, \xi) \in \mathbb{R}^2$ consider the quadratic form in $\mathbb{R}^2$

$$g_{x,\xi}(y, \eta) = |y|^2 + \frac{|\eta|^2}{\langle x, \xi \rangle^2},$$

and for $p, q \in \mathbb{R}$, define the weight $w := \langle x \rangle^p \langle x, \xi \rangle^q$. Then, according to [14, Definition 18.4.6], consider the class of symbols $S^p_q := (w, g)$. A symbol $a$ is in $S^p_q$ if for any $(\alpha, \beta) \in \mathbb{Z}_+^2$, the quantity

$$n_{\alpha,\beta}^{p,q}(a) := \sup_{(x, \xi) \in \mathbb{R}^2} |\langle x \rangle^{-p} \langle x, \xi \rangle^{-q+\alpha} \partial^\alpha_x \partial^\beta_\xi a(x, \xi)|$$

(3.41)

is finite.

For $a \in S^0_p$ we define the operator $Op^W(a)$ according to the Weyl quantization

$$(Op^W(a)u)(x) := \frac{1}{2\pi} \int_{\mathbb{R}^2} a \left( \frac{x + y}{2}, \xi \right) e^{-i(x-y)\xi} u(y) dy d\xi,$$

for $u$ in the Schwartz space $\mathcal{S}(\mathbb{R})$.

Since $V$ satisfies (2.16) it is obvious that $V_\varepsilon$ is in $S_0^{-m}$. Moreover, using (2.17 b), it is also true that the function $\tilde{V}_\varepsilon(x, \xi) := V_\varepsilon(b^{-1}(x), -\xi) \in S_0^{-m}$. Due to $m > 0$, the operator $Op^W(\tilde{V}_\varepsilon)$ is compact in $L^2(\mathbb{R})$.

Using the same notation of Theorem 2.4, write $V_{\varepsilon,j}$ for the pseudodifferential operator with contravariant symbol $V_\varepsilon$ defined by (2.3).

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Lemma 3.6. For any $\varepsilon > 0$ and $j \in \mathbb{N}$

$$\mathcal{V}_{\varepsilon,j} - Op^W(\tilde{V}_\varepsilon) = Op^W(R_1) + Op^W(R_2),$$

(3.42)

where the symbol $R_1 \in S_0^{-m-1}$ and $R_2 \in S_0^{-m}$.

Proof. We give a sketch of the proof which is similar to the proof of [31] Lemma 5.1. Suppose that $V$ is in the Schwartz space $S(\mathbb{R}^2)$. Then, from (2.38) the Weyl symbol $p_V$ of $\mathcal{V}_{\varepsilon,j}$ is given by

$$p_V(\eta, \eta^*) = \frac{B_+}{2\pi} \int_{\mathbb{R}^3} e^{-i\eta \cdot \xi} \Psi_{\varepsilon}(\eta + w/2) \Psi_{\varepsilon}(\eta - w/2) V_\varepsilon(x, \xi) \, dx \, d\xi \, dw,$$

$\Psi_{\varepsilon}^j x, \xi$ being defined in (2.21). We use a first order Taylor expansion of $V_\varepsilon$, noticing that $\partial_1 V_\varepsilon = (\partial_1 V)_{\varepsilon} + V(\partial_1 \varepsilon)$, $\partial_2 V_\varepsilon = (\partial_2 V)_{\varepsilon}$. Because of (2.16), $(\partial_1 V)_{\varepsilon}, (\partial_2 V)_{\varepsilon} \in S_0^{-m-1}$. On the other side, the partial derivative $\partial_1 \varepsilon$ has compact support which implies that $V(\partial_1 \varepsilon) \in S_0^{-m}$ for any $p > 0$, in particular for $p = m$. Now we use the same estimates given in the proof of [31] Lemma 5.1 to conclude that $V_\varepsilon$ is a principal symbol for $\mathcal{V}_{\varepsilon,j}$, and that the remainder terms, coming from the Taylor expansion, satisfy the required conditions.

For a measurable function $a : \mathbb{R}^2 \to \mathbb{R}_+$ define

$$N(\lambda, a) := \frac{1}{2\pi} \text{vol}\{ (x, \xi) \in \mathbb{R}^2 ; a(x, \xi) > \lambda \}.$$

Lemma 3.6 together with [6] Lemma 4.7 imply that there exists a positive $\lambda_0$ such that

$$n_+(\lambda; Op^W(R_1)) = O(N(\lambda, (x, \xi)^{-m-1})) = O(\lambda^{\frac{1}{2m+1}}),$$

$$n_+(\lambda; Op^W(R_2)) = O(N(\lambda, (x, \xi)^{-m} (x)^{-m})) = O(\lambda^{\frac{1}{2m} - \frac{1}{m}}),$$

for $\lambda \in [0, \lambda_0]$. Then, (3.42) and the Weyl inequalities imply that for all $\delta \in (0, 1)$

$$n_+(\lambda; \mathcal{V}_{\varepsilon,j}) \leq n_+((1-\delta)\lambda; Op^W(\tilde{V}_\varepsilon)) + o(\lambda^{-2/m}), \quad \lambda \downarrow 0.$$  (3.43)

Putting together (3.23), (3.38), (3.39), (3.40), (3.25) and (3.43) we obtain that for all $\delta \in (0, 1)$

$$N_j(\lambda) \leq n_+((1-\delta)\lambda; Op^W(\tilde{V}_\varepsilon)) + o(\lambda^{-2/m}), \quad \lambda \downarrow 0.$$  (3.44)

Lemma 3.7. For any $\varepsilon > 0$ the function $N(\lambda, \tilde{V}_\varepsilon)$ satisfies the homogeneity condition (2.19).

Proof. Note that

$$2\pi |N((1-\epsilon)\lambda, \tilde{V}_\varepsilon) - N((1+\epsilon)\lambda, \tilde{V}_\varepsilon)|$$

$$= \text{vol}\{ (x, \xi) \in \mathbb{R}^2 ; (1+\epsilon)\lambda \geq \chi_{\varepsilon}(b^{-1}(x))V(b^{-1}(x), -\xi) > (1-\epsilon)\lambda \}$$

$$= \text{vol}\{ (x, \xi) \in \mathbb{R}^2 ; (1+\epsilon)\lambda \geq V(b^{-1}(x), -\xi) > (1-\epsilon)\lambda, b^{-1}(x) \geq -\varepsilon \}$$

$$\leq \int_{\{ (x', \xi') \in \mathbb{R}^2 ; (1+\epsilon)\lambda \geq V(x', \xi') > (1-\epsilon)\lambda, x', \xi' > -\varepsilon \}} B(x') \, dx' \, d\xi'$$

$$+ \text{vol}\{ (x, \xi) \in \mathbb{R}^2 ; C_{0,0} (b(x)^{-1}, \xi)^{-m} > (1-\epsilon)\lambda, -2\varepsilon < b^{-1}(x) < -\varepsilon \}$$

$$\leq B_2 N((1-\epsilon)\lambda, V, -\varepsilon) - N((1+\epsilon)\lambda, V, -\varepsilon) + O(\lambda^{-1/m}),$$

where in the first inequality we have used the change of variables $b^{-1}(x) = x', -\xi = \xi'$, that $V$ satisfies (2.16) and that $0 \leq \chi_{\varepsilon} \leq 1$. Since $N(\lambda, V, -\varepsilon)$ fulfils (2.19) we obtain the required result.

□
Lemma 3.8. For any $\varepsilon > 0$, $N(\lambda, \tilde{V}_\varepsilon)$ satisfies condition (2.18). Moreover

$$\lim \frac{B_+ N(\lambda, V, 0)}{N(\lambda, \tilde{V}_\varepsilon)} = 1. \quad (3.45)$$

Proof. First let us show that

$$\lim \frac{N(\lambda, V_\varepsilon)}{N(\lambda, V, 0)} = 1. \quad (3.46)$$

To see this we estimate $|N(\lambda, V_\varepsilon) - N(\lambda, V, 0)|$, noticing that $\{(x, \xi) \in \mathbb{R}^2; V_\varepsilon(x, \xi) > \lambda \}$ and $\{(x, \xi) \in \mathbb{R}^2; V(x, \xi) > \lambda, x > 0 \}$ differ in a set contained in

$$\{(x, \xi) \in \mathbb{R}^2; V_\varepsilon(x, \xi) > \lambda, -2\varepsilon < x \leq 0 \}.$$

Then in view of $V_\varepsilon \in S_{-m}^{-}$

$$|N(\lambda, V_\varepsilon) - N(\lambda, V, 0)| = O(\lambda^{-1/m}).$$

Using that $N(\lambda, V, 0)$ satisfies property (2.18) we obtain (3.46).

Now let us prove that

$$\lim \frac{N(\lambda, \tilde{V}_\varepsilon)}{B_+ N(\lambda, V_\varepsilon)} = 1. \quad (3.47)$$

Similarly to the proof of Lemma 3.7 we have

$$2\pi |B_+ N(\lambda, V_\varepsilon) - N(\lambda, \tilde{V}_\varepsilon)| = 2\pi \int_{\{x, \xi); V_\varepsilon(x, \xi) > \lambda \}} B_+ - B(x) \, dx \, d\xi$$

$$\leq 4\pi C_{0,0}^{-1/m} \lambda^{-1/m} \int_{-2\varepsilon}^{\lambda} \langle x \rangle^{-M} \, dx = o(\lambda^{-2/m}), \quad \lambda \downarrow 0. \quad (3.48)$$

Where we have used (2.20) and (2.16). Taking into account (3.46) along with (2.18), we obtain (3.47).

Putting together (3.46) and (3.47) we get (3.45).

Since $N(\lambda, \tilde{V}_\varepsilon)$ satisfies (2.18) and (2.19), it follows that it also satisfies condition $(T')$ of [6]. Then, [6] Theorem 1.3, together with (3.44), (3.45) imply that for all $\delta \in (0, 1)$ and all $\varepsilon > 0$

$$\lim \sup_{\lambda \downarrow 0} \frac{\mathcal{N}_f(\lambda)}{B_+ N((1 - \delta)\lambda; V, 0)} \leq \lim \sup_{\lambda \downarrow 0} \frac{n_+(1 - \delta)\lambda; Op_{-W}(\tilde{V}_\varepsilon))}{N((1 - \delta)\lambda, \tilde{V}_\varepsilon)} = 1. \quad (3.49)$$

To finish the proof of the upper bound in (2.21) it only remains to note that conditions (2.18), (2.19) imply that

$$\lim \lim \sup \frac{N((1 - \delta)\lambda, V, 0)}{N(\lambda, V, 0)} = 1.$$
3.5 Proof of Corollary 2.4: Lower bound

Condition (2.20) implies that there exists a smooth function \(\tilde{B}\) such that \(B(x) \geq \tilde{B}(x) \geq B\) for all \(x \in \mathbb{R}\), and \(B_+ - \tilde{B}(x) = \tilde{C}(x)^{-M}\), for some positive constant \(\tilde{C}\) and \(x\) sufficiently big. Using \(\tilde{B}\) to define \(\tilde{b}\) according to (1.2), we see that (3.10) implies

\[
(\mathcal{E}_j^+ - E_j(k) + \lambda)^{-1} \geq (\mathcal{E}_j^+ - E_j((\tilde{b}^{-1}(k)), \tilde{b}) + \lambda)^{-1}, \quad \text{for all } k \in \mathbb{R}.
\]

Since \(B_+ - \tilde{B}\) is strictly decreasing for \(x\) large, the function \(\mathcal{E}_j^+ - E_j((\tilde{b}^{-1}(k))\) is strictly decreasing for \(k\) big [18, Theorem 3.2]. We denote by \(\rho_j\) its inverse when exists. It is obvious that \(\lim_{w \to 0} \rho(w) = \infty\). Moreover, from Lemma 4.3 in [31], we know that (2.20) implies that \(\mathcal{E}_j^+ - E_j((\tilde{b}^{-1}(k)) = O(k^{-M}), k \to \infty, \) and then

\[
\rho_j(w) = O(w^{-1/M}), \quad w \downarrow 0.
\]

For \(j \in \mathbb{N}, \delta \in (0, 1)\) and \(\lambda > 0\) set \(\varrho = \varphi(\lambda) := \rho_j(\delta \lambda)\). Then (3.50) implies that

\[
(\mathcal{E}_j^+ - E_j(k) + \lambda)^{-1} \geq ((1 + \delta)\lambda)^{-1}, \quad \text{for all } k \geq \varphi(\lambda).
\]

For all \(r \geq 0\) and \(\delta \in (0, 1)\)

\[
n_+ \left( r; V_1^{1/2}T_{j,\infty}(\lambda)V_1^{1/2} \right) \geq n_+ \left( r; V_1^{1/2}T_{j,\infty}(\lambda)V_1^{1/2} \right)
\]

\[
\geq n_+ \left( r; V_1^{1/2}T_{j,\infty}(\lambda, \varrho)V_1^{1/2} \right) \geq n_+ \left( r(1 + \delta)\lambda; V_1^{1/2}P_{j,\infty}(\varrho)V_1^{1/2} \right).
\]

In the first and the second inequality we have used the min-max principle, while for the third inequality we used (3.52).

Next, using the Weyl inequalities, for any \(\lambda > 0\) and \(\delta \in (0, 1)\)

\[
n_+ \left( \lambda; V_1^{1/2}P_{j,\infty}(\varrho)V_1^{1/2} \right)
\]

\[
\geq n_+ (\lambda(1 + \delta); V_1^{1/2}P_{j,\infty}V_1^{1/2}) - n_+ \left( \lambda\delta; V_1^{1/2}(P_{j,\infty} - P_{j,\infty}(\varrho))V_1^{1/2} \right).
\]

The term \(n_+(\lambda; V_1^{1/2}P_{j,\infty}V_1^{1/2}) = n_+(\lambda; P_{j,\infty}V_1^{1/2}P_{j,\infty}) = n_+(r; V_{\epsilon,j})\) was already obtained in (3.40), and its asymptotic behavior can be estimated as in subsection 3.4.

For the second term in (3.52) we have that (3.52) implies

\[
n_+ \left( \lambda; V_1^{1/2}(P_{j,\infty} - P_{j,\infty}(\varrho))V_1^{1/2} \right)
\]

\[
= n_+ \left( \lambda; \int_{\mathbb{R}^\times} -\frac{\pi_{\lambda,\infty}(k)}{\pi_{\lambda,\infty}(\varrho)} dk \mathcal{F}V_1^{\mathcal{F}^*} \int_{\mathbb{R}^\times} -\frac{\pi_{\lambda,\infty}(k)}{\pi_{\lambda,\infty}(\varrho)} dk \right)
\]

\[
= n_+ \left( \lambda; \mathcal{F}_\lambda(\infty)P_{j,\infty}V_1^{\mathcal{F}^*} \mathcal{F}_\lambda(\infty) \right) = n_+ \left( \lambda; \mathcal{F}_\lambda(\infty)P_{j,\infty}V_1^{\mathcal{F}^*} \mathcal{F}_\lambda(\infty) \right) = n_+ \left( \lambda; \mathcal{F}_\lambda(\infty) \mathcal{F}_\lambda(\infty) \right).
\]

Let \(\chi_{\lambda}(x) := \chi_{\epsilon}(x + \rho_j(\delta \lambda))\), the same \(\chi_{\epsilon}\) of the preceding subsection. Then \(\chi_{\lambda}\) is a smooth function with bounded derivatives such that \(0 \leq \chi_{\epsilon} \leq 1\), \(\chi_{\lambda}(x) = 0\) for \(x \geq \varphi(\lambda) + 2\epsilon\) and \(\chi_{\lambda}(x) = 1\) for \(x \leq \varphi(\lambda) + \epsilon\). It is important to note that for all positive \(\lambda, \epsilon,\) and \(\delta \in (0, 1),\)
\( \chi_\lambda \in S^0_0 \) and its semi-norms \( n^{0,0}_{\alpha,\beta}(\chi_\lambda) \) (defined by (3.41)) are independent of \( \delta \) and \( \lambda \) for all \((\alpha,\beta) \in \mathbb{Z}^2_+ \). Indeed,

\[
n^{0,0}_{\alpha,\beta}(\chi_\lambda) = \begin{cases} 
0 & ; \alpha > 0 \\
||\chi_\lambda^{(\beta)}||_{L^\infty(\mathbb{R})} & ; \alpha = 0.
\end{cases}
\]

Write as before \( Op^W(\tilde{V}_\varepsilon) \) for the Pseudo-differential operator with Weyl symbol \( \tilde{V}_\varepsilon \). Then, since for all \( \lambda > 0 \), \( \chi_\lambda \mathbb{I}_{(\infty,\varepsilon]\{\lambda\}} = \mathbb{I}_{(\infty,\varepsilon]\{\lambda\}} \), we have that

\[
\mathbb{I}_{(-\infty,\varepsilon]}\mathbb{I}_{(-\infty,\varepsilon]} = \mathbb{I}_{(-\infty,\varepsilon]} Op^W(\tilde{V}_\varepsilon) \mathbb{I}_{(-\infty,\varepsilon]} + \mathbb{I}_{(-\infty,\varepsilon]} \left( \mathbb{I}_{(-\infty,\varepsilon]} Op^W(\tilde{V}_\varepsilon) \right) \mathbb{I}_{(-\infty,\varepsilon]}.
\]

The symbol \( \tilde{V}_\varepsilon \in S_0^{-m} \) and \( \chi_\lambda \in S_0^0 \), then it is well known that \([14 \text{ Theorem 18.5.4}] \]

\[
Op^W(\tilde{V}_\varepsilon)Op^W(\chi_\lambda) = Op^W(\tilde{V}_\varepsilon\chi_\lambda) + Op^W(R_\lambda),
\]

where \( R_\lambda \in S_0^{-m-1} \), and each one of its semi-norms \( n^{0,-m-1}_{\alpha,\beta}(R_\lambda) \) is polynomially bounded by a finite number of semi-norms of \( \tilde{V}_\varepsilon \) and \( \chi_\lambda \) in \( S_0^{-m} \) and \( S_0^0 \), respectively. Since the semi-norms of \( \chi_\lambda \) are independent of \( \lambda \), \([6 \text{ Lemma 4.7}] \) implies that there exists a positive constants \( \lambda_0 \) such that

\[
n_+(\lambda; Op^W(R_\lambda)) = O(\lambda^{-2/(m+1)}), \quad \text{for } \lambda \in (0, \lambda_0].
\]

**Lemma 3.9.** For every \( \varepsilon > 0 \)

\[
\lim_{\lambda \downarrow 0} n_+(\lambda; Op^W(\tilde{V}_\varepsilon\chi_\lambda)) = 0.
\]

**Proof.** By \([6 \text{ Proposition 4.1}] \), there are positive constants \( C_\lambda \) and \( \zeta \) such that

\[
n_+(\lambda; Op^W(\tilde{V}_\varepsilon\chi_\lambda)) \leq N(\lambda, \tilde{V}_\varepsilon\chi_\lambda) + C_\lambda \lambda^{-2/m} + \zeta.
\]

The constant \( C_\lambda \) depends polynomially on a finite number of semi-norms of the symbol \( \tilde{V}_\varepsilon\chi_\lambda \), but from composition of symbols each one of the semi-norms of \( \tilde{V}_\varepsilon\chi_\lambda \) is polynomially bounded by a finite number of semi-norms of \( V \) in \( S_0^{-m} \) and \( \chi_\lambda \) in \( S_0^0 \). Consequently, the constant \( C_\lambda \) can be taken independent of \( \lambda \).

The proof of (3.59) that appears in \([6] \) is for symbols that do not depend on \( \lambda \). However, it works as well in our case just introducing minor changes.

Now, since \( (\tilde{V}_\varepsilon\chi_\lambda)(x, \xi) = V(b^{-1}(x), -\xi)\chi_\varepsilon(b^{-1}(x))\chi_\lambda(x) \) where the support of \( \chi_\varepsilon(b^{-1}(x))\chi_\lambda(x) \) is contained on the strip \( \{(x, \xi) \in \mathbb{R}^2; b(-2\varepsilon) \leq x \leq g(\lambda) + 2\varepsilon\} \), and \( V \) is in \( S_0^{-m} \), the set \( \{(x, \xi) \in \mathbb{R}^2; \tilde{V}_\varepsilon\chi_\lambda(x, \xi) > \lambda\} \) is contained in

\[
\{(x, \xi) \in \mathbb{R}^2; \langle b^{-1}(x), \xi \rangle^{-m} \geq \lambda, b(-2\varepsilon) \leq x \leq g(\lambda) + 2\varepsilon\}.
\]

Then,

\[
N(\lambda, \tilde{V}_\varepsilon\chi_\lambda) = O(\lambda^{-1/m} ((g(\lambda) + 2\varepsilon) - b(-2\varepsilon))).
\]

Putting together (3.59), (3.60) and (3.51) we finish the proof of the Lemma.\(\square\)
Gathering \((3.56)\), Lemma \(3.6\) \((3.57)\), \((3.58)\) and Lemma \(3.9\) we obtain
\[
\lim_{\lambda \downarrow 0} \frac{n_+(\lambda; \mathbb{1}_{(-\infty,\rho]} V_{\varepsilon}, \mathbb{1}_{(-\infty,\rho]} \lambda^{-2/m})}{n_+(\lambda; 1 + \delta; Op W(\tilde{V}_\varepsilon))} = 0,
\]
(3.61)
thus, for all \(\delta \in (0,1)\) \((3.23)\), \((3.53)\), \((3.54)\), \((3.55)\), \((3.61)\) and Lemma \(3.8\) imply
\[
\liminf_{\lambda \downarrow 0} \frac{N_j(\lambda)}{B_{1/2} N(\lambda(1 + \delta), V, 0)} \geq \liminf_{\lambda \downarrow 0} \frac{n_+(\lambda(1 + \delta); Op W(\tilde{V}_\varepsilon))}{N(\lambda(1 + \delta), V_\varepsilon)}.
\]
Finally, arguing as in the last part of subsection \(3.4\) one can obtain the lower bound in \((2.21)\).

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