Blow-up criterion for the chemotaxis-fluid equations in a 3D unbounded domain with mixed boundary conditions

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Abstract. In this paper, we consider a coupled chemotaxis-fluid system in a 3D unbounded domain with mixed boundary conditions. A blow-up criterion for such a system is established by using the proper elliptic estimates and Stokes estimates under some assumptions on the chemotactic sensitivity function.

1. Introduction
It is well known that the chemotaxis phenomena of organisms have been attracted the attention of a large number of scholars recently. Massive properties of the solutions have been obtained, e.g., see [1-3] and the references therein.

In recent decades, since a lot of organisms live in a viscous fluid with chemical stimulation in it, the coupled chemotaxis-fluid systems which is firstly proposed by Tuval et al. [4] have been intensively studied in bounded or unbounded domain. For instance, Duan et al. [5] firstly established the global strong solutions of the chemotaxis-Navier-Stokes system near a constant equilibrium and obtained some temporal decay estimates of the resulting solutions in \( \mathbb{R}^3 \), in the same paper, they also proved the global weak well-posedness of the chemotaxis-Stokes system in \( \mathbb{R}^2 \). When considering the domain to be bounded with smooth boundary, Wang et al. [6] investigated the small-convection limit of chemotaxis-(Navier-)Stokes system. More results on the coupled chemotaxis-fluid system, we refer to [7-10] for the Cauchy problems in whole spaces and [11-14] for initial-boundary problems in bounded domains.

In this paper, we consider the following coupled chemotaxis-fluid equations (see [15,16])

\[
\begin{align*}
\partial_t n + u \cdot \nabla n &= \Delta n - \nabla \cdot (\chi c) n \nabla c \\
\partial_t c + u \cdot \nabla c &= \Delta c - f(c) n n' \\
\partial_t u + u \cdot \nabla u + \nabla P &= \Delta u - n \nabla \phi \\
\nabla \cdot u &= 0
\end{align*}
\]

in \( \Omega \times (0, \infty) \) with \( c \in \mathbb{R}_+ \) and \( \Omega := \{ x = (x', x_3) \in \mathbb{R}^3 \mid x' = (x_1, x_2) \in \mathbb{R}^2, 0 \leq x_3 \leq 1 \} \) around the equilibrium state \( (0, c_{\text{sat}}, 0) \). Here, the unknowns are \( n, c, u \) and \( P \), denoting the cell density, the oxygen concentration, the velocity of fluid and the associated pressure, respectively. \( \chi, f, \phi \) are given functions depicting the chemotactic sensitivity, the consumption rate of the oxygen by cells, and the potential produced by some physical mechanisms, respectively. For simplicity, we denote the
boundary as \( \partial \Omega = \{(x', x_3) \in \mathbb{R}^3 \mid x' \in \mathbb{R}^2, x_3 = 1\} \cup \{(x', x_3) \in \mathbb{R}^3 \mid x' \in \mathbb{R}^2, x_3 = 0\} : = \Gamma_T \cup \Gamma_B \). In order to match the experiment and the numerical analysis [4, 17-19], we impose the following mixed boundary conditions [16]:

\[
(\nabla n - \chi(c_{\text{satur}}) n \nabla c) \cdot v = 0, \quad c = c_{\text{satur}} \quad \text{and} \quad u \cdot v = (\nabla \times u) \cdot r = 0 \quad \text{on} \quad \Gamma_T
\]

and

\[
\nabla n \cdot v = \nabla c \cdot v = 0 \quad \text{and} \quad u = 0 \quad \text{on} \quad \Gamma_B,
\]

where \( c_{\text{satur}} \geq 0 \) stands for the oxygen concentration at the water-air interface. The initial conditions are set as \( n(x, 0) = n_0(x) \), \( \phi(x, 0) = \phi_0(x) \), and \( \mathbf{u}(x, 0) = \mathbf{u}_0(x) \) with \( x \in \Omega \).

The authors in [15] proved that the chemotaxis-Navier-Stokes system is strongly, globally and uniquely solvable near a constant equilibrium, while the chemotaxis-Stokes system is weakly solvable when the boundary conditions are all assumed to be like (1.3) on both \( \Gamma_T \) and \( \Gamma_B \). For the problem of mixed boundary condition (1.1)-(1.3), the authors [16] proved the global existence of strong solutions near the equilibrium \((0, c_{\text{satur}}, 0)\) and obtained some convergence rate. In this paper, the author will consider the blow-up criterion for system (1.1)-(1.3) with \( \chi \equiv 0 \). As far as I know, this is the first work considering the additional influence of mixed boundary conditions to the coupled chemotaxis-fluid system of blow-up mechanisms. The main difficulty is the appearance of the mixed boundary conditions. Throughout this paper, we assume that \( \chi, f \) and \( \phi \) are smooth functions satisfying

\[
f(0) = 0, \quad f'(s) \geq 0 \quad \text{for all} \quad s \in \mathbb{R}, \quad \text{and} \quad \sup_{x \in \Omega} \phi(x) \left( |\nabla \phi(x)| + |\nabla^2 \phi(x)| \right) < \infty \tag{1.4}
\]

with \( \phi(x) = (1 + |x'|)(1 + \ln(1 + |x'|)) \).

The rest of this paper is organized as follows. In section 2, some basic lemmas are given. Section 3 is devoted to proving the blow-up criterion which is the main result.

2. Preliminaries

In this section, we will recall the local-in-time existence result of system (1.1)-(1.3). Its detailed proof can be found in [16].

**Lemma 2.1.** Let \( \chi \geq 2 \), \( c_{\text{satur}} \geq 0 \) and (1.4) hold. Assume that the initial data satisfies \( n_0 \geq 0 \), \( c_0 \in [0, c_{\text{satur}}] \), \( n_0 \in L^1(\Omega) \) and \((n_0, c_0 - c_{\text{satur}}, u_0) \in (H^2(\Omega))^3 \). There exist positive constants \( \varepsilon_1, T_0 \) and \( K \) such that if \( \|n_0, c_0 - c_{\text{satur}}, u_0\|_{H^1} \leq \varepsilon_1 \), then the solution \((n, c - c_{\text{satur}}, \mathbf{u})\) of system (1.1)-(1.3) is well-defined in \((C([0, T_0]; H^2(\Omega))^3)\) and satisfies the uniform estimates

\[
\sup_{0 \leq t \leq T} \left\| (n, c - c_{\text{satur}}, \mathbf{u})(\cdot, t) \right\|_{H^2}^2 + \int_0^T \left\| \nabla n, \nabla (c - c_{\text{satur}}), \nabla \mathbf{u}(\cdot, t) \right\|_{H^1}^2 \, dt \leq K. \tag{2.1}
\]

**Remark 2.1.** We can infer from the local existence result Lemma 2.1 that if \( T_0 \) is the maximal time of existence with \( T_0 < \infty \), then

\[
\limsup_{t \nearrow T_0} \int_0^T \left\| (n, c - c_{\text{satur}}, \mathbf{u})(\cdot, t) \right\|_{H^2}^2 + \int_0^T \left\| \nabla n, \nabla (c - c_{\text{satur}}), \nabla \mathbf{u}(\cdot, t) \right\|_{H^1}^2 \, dt \, ds = \infty.
\]

3. Blow-up criterion

In this section, the blow-up criterion for system (1.1)-(1.3) with \( \chi \equiv 0 \) is established. The main theorem of this paper is stated as follows.

**Theorem 3.1.** Let \( \chi \equiv 0 \) and all conditions in Lemma 2.1 hold. If \( T_0 \), the maximal time existence in Lemma 2.1, is finite, then

\[
\int_0^{T_0} \left( \|\mathbf{u}(\cdot, t)\|_{L^p}^p + \|\nabla (c - c_{\text{satur}})(\cdot, t)\|_{L^p}^p \right) dt = \infty \quad p \geq 3.
\]
Remark 3.1. The blow-up criterion in Theorem 3.1 is coincide with the one in [10]:
\[ \int_0^T \left[ \|u\|_{L^p(\mathbb{R}^3)}^p + \|\nabla n\|_{L^{p'}(\mathbb{R}^3)}^{p'} \right] \, dt = \infty, \quad \frac{3}{\beta} + \frac{2}{\gamma} = 1, \quad 3 < \beta \leq \infty. \]
The slightly difference between the above two blow-up criterion is somehow due to the appearance of the mixed boundary conditions.

Proof.
For simplicity, we firstly take change of variables \( \sigma = c_{\text{sat}} - c \). Thus, we can rewrite the system (1.1)-(1.3) as following since the assumption \( \chi(s) = 0 \) for all \( s \in \mathbb{R} \):

\[
\begin{aligned}
\partial_t n + u \cdot \nabla n &= \Delta n \\
\partial_t \sigma + u \cdot \nabla \sigma &= \Delta \sigma + f(c_{\text{sat}} - \sigma)n' \\
\partial_t u + u \cdot \nabla u + \nabla P &= \Delta u - n \nabla \phi \\
\nabla \cdot u &= 0
\end{aligned}
\]

with the corresponding initial-boundary conditions

\[
\begin{aligned}
\partial_t n &= \sigma = 0, \quad \partial_t u_1 = \partial_t u_2 = u_3 = 0 \quad \text{on } \Gamma_T \\
\partial_t \sigma &= \partial_t \sigma = 0 \quad \text{on } \Gamma_B \\
n(x,0) = n_0(x), \quad \sigma(x,0) = \sigma_0(x), \quad u(x,0) = u_0(x) \quad \text{in } \Omega
\end{aligned}
\]

Now, we will get the \( L^\infty \)-estimate of \( n \). Multiplying (3.1) by \( p n^{p-1} \) \( (p \geq 2) \) and integrating the resulting equation over \( \Omega \), one has

\[
\frac{d}{dt} \|n\|_{L^p}^p + \frac{4(p-1)}{p} \|\nabla n^{p/2}\|_{L^p}^2 = 0,
\]

which implies that

\[
\|n\|_{L^p} \leq \|n_0\|_{L^p} \quad \text{and} \quad \|n\|_{L^p} \leq \|n_0\|_{L^p}
\]

by letting \( p \to \infty \) for the later inequality.

Next, we are going to obtain the \( L^2 \)-estimate of \( (n,\sigma,u) \). Taking the \( L^2 \)-inner product of (3.1i)-(3.1ii) with \( n, \sigma, \text{ and } u \), respectively, one has

\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|n\|_{L^2}^2 + \|\nabla n\|_{L^2}^2 &= 0, \\
\frac{1}{2} \frac{d}{dt} \|\sigma\|_{L^2}^2 + \|\nabla \sigma\|_{L^2}^2 &= \int_\Omega f(c_{\text{sat}} - \sigma) \sigma n' \, dx \leq C \|\sigma\|_{L^{p}} \|\nabla \sigma\|_{L^2} \|n\|_{L^{p}} \leq C \|\sigma\|_{L^{p}} \left( \|\sigma\|_{L^2}^2 + \|n\|_{L^2}^2 \right), \\
\frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 &= -\int_\Omega u \cdot n \nabla \phi \, dx \leq C \left( \|n\|_{L^2}^2 + \|u\|_{L^2}^2 \right).
\end{aligned}
\]

Combining (3.3)-(3.5) , one has

\[
\begin{aligned}
\frac{d}{dt} \|(n,\sigma,u)\|_{L^2}^2 + 2 \|(\nabla n, \nabla \sigma, \nabla u)\|_{L^2}^2 &\leq C \left( 1 + \|n\|_{L^p}^{p-1} \right) \|(n,\sigma,u)\|_{L^2}^2.
\end{aligned}
\]

Then, we deal with the \( L^2 \)-estimate of \( (\nabla n, \nabla \sigma, \nabla u) \). Firstly, taking the \( L^2 \)-inner product of (3.1i) with \( \partial_t n \) and integrating by parts, one has

\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\nabla n\|_{L^2}^2 + \|\partial_t n\|_{L^2}^2 &= -\int_\Omega \partial_t n \, u \cdot \nabla n \, dx \leq \frac{1}{2} \|\partial_t n\|_{L^2}^2 + \frac{1}{2} \|u\|_{L^2}^2 \|\nabla n\|_{L^2}^2,
\end{aligned}
\]

which implies that

\[
\frac{1}{2} \frac{d}{dt} \|\nabla n\|_{L^2}^2 + \frac{1}{2} \|\partial_t n\|_{L^2}^2 \leq \frac{1}{2} \|u\|_{L^2}^2 \|\nabla n\|_{L^2}^2.
\]

Taking the \( L^2 \)-inner product of (3.1ii) with \( \partial_t \sigma \) and using the Hölder inequality, one has
\[
\frac{1}{2} \frac{d}{dt} \|\nabla \sigma\|_{L^2}^2 + \|\partial_t \sigma\|_{L^2}^2 = -\int_\Omega \partial_t \sigma \cdot \nabla \sigma \, dx + \int_\Omega f(c_{sat} - \sigma) \partial_t \sigma \, n^i \, dx
\]
\[
\leq \frac{1}{2} \|\nabla \sigma\|_{L^2}^2 + \frac{1}{2} \|\partial_t \sigma\|_{L^2}^2 + \int_\Omega \nabla \sigma \cdot (\nabla \sigma + \sigma u) \, dx
\]
which implies that
\[
\frac{1}{2} \frac{d}{dt} \|\nabla \sigma\|_{L^2}^2 + \frac{1}{2} \|\partial_t \sigma\|_{L^2}^2 \leq C \left( \|\nabla \sigma\|_{L^2}^2 + \|\sigma \|_{L^\infty}^2 \right) \left( \|\nabla \sigma\|_{L^2}^2 + \|\partial_t \sigma\|_{L^2}^2 \right).
\] (3.8)

Similarly for \( u \), we obtain that
\[
\frac{1}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \frac{1}{2} \|\partial_t u\|_{L^2}^2 = -\int_\Omega \partial_t u \cdot (u \cdot \nabla u) \, dx - \int_\Omega \partial_t u \cdot n \nabla \phi \, dx \leq \frac{1}{2} \|\partial_t u\|_{L^2}^2 + \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2 + C \|n\|_{L^2}^2
\]
which implies that
\[
\frac{1}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \frac{1}{2} \|\partial_t u\|_{L^2}^2 \leq \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2 + C \|n\|_{L^2}^2.
\] (3.9)

Combining (3.7)-(3.9), one can conclude that
\[
\frac{1}{2} \frac{d}{dt} \|\nabla n, \nabla \sigma, \nabla u\|_{L^2}^2 + \|\partial_t \sigma, \sigma, \partial_t \sigma, \sigma, \partial_t u\|_{L^2}^2
\]
\[
\leq C \left( 1 + \|u\|_{L^2}^2 + \|n\|_{L^\infty}^2 \right) \left( \|n\|_{L^2}^2 + \|\nabla n, \nabla \sigma, \nabla u\|_{L^2}^2 + \|\partial_t \sigma\|_{L^2}^2 \right).
\] (3.10)

For the \( L^2 \)-estimate of \( \partial_t n, \partial_t \sigma, \partial_t \sigma, \partial_t u \). Firstly, applying \( \partial_t \) to (3.1), taking the \( L^2 \)-inner product of the resulting equation with \( \partial_t n \), and integrating by parts, one has
\[
\frac{1}{2} \frac{d}{dt} \|\partial_t n\|_{L^2}^2 + \|\nabla \partial_t n\|_{L^2}^2 = -\int_\Omega \partial_t n \partial_t (u \cdot \nabla n) \, dx = \int_\Omega n \nabla \partial_t n \cdot \partial_t u \, dx \leq \frac{1}{2} \|\nabla \partial_t n\|_{L^2}^2 + \frac{1}{2} \|u\|_{L^2}^2 \|\partial_t u\|_{L^2}^2,
\]
which implies that
\[
\frac{1}{2} \frac{d}{dt} \|\partial_t n\|_{L^2}^2 + \frac{1}{2} \|\nabla \partial_t n\|_{L^2}^2 \leq \frac{1}{2} \|u\|_{L^2}^2 \|\partial_t u\|_{L^2}^2.
\] (3.11)

Then, applying \( \partial_t \) to (3.1), taking the \( L^2 \)-inner product of the resulting equation with \( \partial_t \sigma \), and using the Sobolev embedding inequality, one obtain that
\[
\frac{1}{2} \frac{d}{dt} \|\partial_t \sigma\|_{L^2}^2 + \|\nabla \partial_t \sigma\|_{L^2}^2
\]
\[
= -\int_\Omega \partial_t \sigma \partial_t \sigma \cdot \nabla \sigma \, dx - \int_\Omega f(c_{sat} - \sigma) \partial_t \sigma \partial_t \sigma^i \, dx + \gamma \int_\Omega f(c_{sat} - \sigma) \partial_t \sigma \partial_t \sigma^i \partial_t n \, dx
\]
\[
\leq \|\partial_t \sigma\|_{L^2} \|\partial_t \sigma\|_{L^2} \|\nabla \sigma\|_{L^2} + C \|\partial_t \sigma\|_{L^2} \|n\|_{L^\infty} \|\partial_t n\|_{L^2}
\]
\[
\leq \mu_t \|\partial_t \sigma\|_{L^2} + C \|\nabla \sigma\|_{L^2} \|\partial_t \sigma\|_{L^2} + C \|n\|_{L^\infty} \|\partial_t \sigma\|_{L^2} \|\partial_t n\|_{L^2},
\]
which implies by choosing \( \mu_t \) to be small enough that
\[
\frac{1}{2} \frac{d}{dt} \|\partial_t \sigma\|_{L^2}^2 + \frac{1}{2} \|\nabla \partial_t \sigma\|_{L^2}^2 \leq \mu_t \|\partial_t \sigma\|_{L^2}^2 + C \|\nabla \sigma\|_{L^2} \|\partial_t \sigma\|_{L^2} + C \|n\|_{L^\infty} \|\partial_t \sigma\|_{L^2} \|\partial_t n\|_{L^2}^2,
\] (3.12)

where \( \frac{1}{p} + \frac{1}{q} = \frac{1}{2} \) and \( 1 \leq q \leq 6 \) in the above inequalities. For the estimate of \( \|\nabla \partial_t u\|_{L^2} \), we deduce that
\[
\frac{1}{2} \frac{d}{dt} \|\nabla \partial_t u\|_{L^2}^2 + \|\nabla \partial_t u\|_{L^2}^2 = -\int_\Omega \partial_t u \cdot (\partial_t u \cdot \nabla \partial_t u) \, dx - \int_\Omega \partial_t u \cdot \partial_t u \cdot n \nabla \phi \, dx
\]
\[
\leq \frac{1}{2} \|\nabla \partial_t u\|_{L^2}^2 + C \|\partial_t u\|_{L^2} \|\partial_t \sigma\|_{L^2} + C \|\partial_t \sigma\|_{L^2} \|\partial_t u\|_{L^2}^2,
\]
which implies that
\[ \frac{1}{2} \frac{d}{dt} \| \sigma_n \|_{L^2}^2 + \frac{1}{2} \| \nabla \sigma_n \|_{L^2}^2 \leq C \| \sigma_n \|_{L^2}^2 + C \| (\partial_n, \sigma, n) \|_{L^2}^2. \]  

(3.13)

Combing (3.11)-(3.13), one can conclude that
\[ \frac{d}{dt} \left( \| (n, \sigma, u) \|_{L^2}^2 + \| (\partial_n, \sigma, \nabla \sigma, \partial_u) \|_{L^2}^2 \right) \]
\[ \leq C \left( \| n \|_{L^2}^2 + \| \sigma \|_{L^2}^2 + \| u \|_{L^2}^2 \right) \left( \| \partial_n, \sigma, \partial_u \|_{L^2}^2 \right). \]

(3.14)

Now, we are in the position to get the $H^2$-estimate of $(n, \sigma, u)$. Firstly, according to the above estimates, we can claim by combing (3.6), (3.10), and (3.14) that
\[ \frac{d}{dt} \left( \| (n, \sigma, u) \|_{L^2}^2 + \| (\partial_n, \sigma, \nabla \sigma, \partial_u) \|_{L^2}^2 \right) \]
\[ \leq C \left( \| n \|_{L^2}^2 + \| \sigma \|_{L^2}^2 + \| u \|_{L^2}^2 \right) \left( \| \partial_n, \sigma, \partial_u \|_{L^2}^2 \right). \]

(3.15)

By Gronwall’s inequality, we note that $\| (n, \sigma, u) \|_{L^2(0, T, L^2)}$ and $\| (\partial_n, \sigma, \partial_u) \|_{L^2(0, T, L^2)}$ are uniformly bounded if $\int_0^T \| \nabla \sigma \|_{L^2}^2 + \| \sigma \|_{L^2}^2 \ dt$ $(3 \leq p \leq \infty)$ is bounded.

Next, we will close the energy in the framework of $H^2$-norm and obtain the boundedness of $\| (n, \sigma, u) \|_{L^2(0, T, L^2)}$ and $\| \nabla n, \nabla \sigma, \nabla u \|_{L^2(0, T, L^2)}$ by applying the elliptic estimates. To this end, we can testify by recalling the boundary condition for $n$ in (3.2) and using integration by parts and the Hölder inequality that
\[ \| \nabla n \|_{L^2}^2 = \sum_{i,j=1}^3 \int_{\Omega} \partial_i n \partial_j n \ dx - \sum_{i,j=1}^3 \int_{\partial \Omega} \partial_i n \partial_j n \nu_i \ ds + \sum_{i,j=1}^3 \int_{\Omega} \partial_i n \partial_j n \ dx = \sum_{i,j=1}^3 \int_{\Omega} \partial_i n \partial_j n \ dx - \sum_{i,j=1}^3 \int_{\partial \Omega} \partial_i n \partial_j n \nu_i \ dx = \| \Delta n \|_{L^2}^2. \]

and
\[ \| u \cdot \nabla n \|_{L^2}^2 = \int_{\Omega} (u \cdot \nabla n) (u \cdot \nabla n) \ dx \leq \| u \|_{L^2}^2 \| \nabla n \|_{L^2} \| \nabla n \|_{L^2} \leq C \| u \|_{L^2}^2 \| \nabla n \|_{L^2} \| \nabla n \|_{L^2}. \]

Thus, we deduce from (3.1), that
\[ \| \nabla n \|_{L^2}^2 \leq \| u \|_{L^2}^2 \| \nabla n \|_{L^2}^2 + \| \sigma \|_{L^2}^2 + \frac{1}{2} \| \nabla n \|_{L^2}^2 + C \left( \| \sigma \|_{L^2}^2 + \| \sigma \|_{L^2}^2 \right) \| \nabla n \|_{L^2}^2, \]

which implies that $\| \nabla n \|_{L^2}^2 \leq C \| \sigma \|_{L^2}^2 + C \left( \| \sigma \|_{L^2}^2 + \| \sigma \|_{L^2}^2 \right) \| \nabla n \|_{L^2}^2$, and thus we have $\| \nabla n \|_{L^2(0, T, L^2)}$ is bounded. Due to the mixed boundary conditions in (3.2) for $\sigma$, we rewrite (3.1) as
\[ -\Delta \sigma = -\partial_i \sigma - u \cdot \nabla \sigma + f (c_{\text{sat}} - \sigma) n'. \]

(3.16)

and compute that
\[ \| \nabla \sigma \|_{L^2}^2 = \int_{\Omega} \Delta \sigma \sigma + u \cdot \nabla \sigma \ dx - \int_{\Omega} \partial_i \sigma f (c_{\text{sat}} - \sigma) n' \ dx \]
\[ \leq \frac{1}{2} \| \nabla \sigma \|_{L^2}^2 + \| \sigma \|_{L^2}^2 + \mu_2 \| \nabla \sigma \|_{L^2}^2 + C \left( \| \sigma \|_{H^1}^2 + \| \sigma \|_{L^2}^2 \right) \| \nabla \sigma \|_{L^2}^2 + C \| \sigma \|_{L^2}^2 \| \nabla \sigma \|_{L^2}^2, \]

which implies that
\[ \| \nabla \sigma \|_{L^2}^2 \leq 2 \| \sigma \|_{L^2}^2 + \mu_2 \| \nabla \sigma \|_{L^2}^2 + C \left( \| \sigma \|_{H^1}^2 + \| \sigma \|_{L^2}^2 \right) \| \nabla \sigma \|_{L^2}^2 + C \| \sigma \|_{L^2}^2 \| \sigma \|_{L^2}^2. \]

(3.17)

Noticing that (3.16) also can be rewritten as
\[-\partial_t^2 \sigma = \Delta_x \sigma - \partial_x \sigma - \mathbf{u} \cdot \nabla \sigma + f (\sigma_{\text{can}} - \sigma) n', \tag{3.18}\]

Thus, we further deduce that
\[
\|\partial_t^2 \sigma\|_{L^2} \leq C \|\Delta_x \sigma\|_{L^2} + C \|\partial_x \sigma\|_{L^2} + \|\mathbf{u} \cdot \nabla \sigma\|_{L^2} + C \|\n\|_{L^2} \|\sigma\|^{(y-1)}_{L^2}.
\]

Taking a linear combination of (3.17) and (3.19), rearranging the resulting coefficients carefully, and choosing $\mu_2$ to be suitably small, one has
\[
\|\n\|_{L^2} \leq C \|\partial_t \sigma\|_{L^2} + C \left(\|u\|_{H^1} + \|u\|_{H^1}\right) \|\n\|_{L^2} + C \|\sigma\|_{L^2} + C \|\mu_2\|_{L^2} \|\sigma\|^{(y-1)}_{L^2},
\]

which implies that $\|\n\|_{L^2(t_0,T,t')}$ (1). Analogously, for the velocity $\mathbf{u}$, we employ the Stokes estimates (refer to Lemma 3.1 in [1] for details) to obtain that
\[
\|\n\|_{L^2(t_0,T,t')} \leq C \|\partial_t \vo\|_{L^2} + C \left(\|u\|_{H^1} + \|u\|_{H^1}\right) \|\n\|_{L^2} + C \|\vo\|_{L^2} + C \|\mu_2\|_{L^2} \|\vo\|^{(y-1)}_{L^2},
\]

which implies that
\[
\|\n\|_{L^2(t_0,T,t')} \leq C \|\partial_t \vo\|_{L^2} + C \left(\|u\|_{H^1} + \|u\|_{H^1}\right) \|\n\|_{L^2} + C \|\vo\|_{L^2} + C \|\mu_2\|_{L^2} \|\vo\|^{(y-1)}_{L^2},
\]

and thus $\|\n\|_{L^2(t_0,T,t')}$. Now we will prove that $\|\n\|_{L^2(t_0,T,t')}$. Firstly we deduce from integration by parts that
\[
\|\n\|_{L^2(t_0,T,t')} = \|\n\|_{L^2(t_0,T,t')} \leq C \|\partial_t \n\|_{L^2} + C \|\sigma\|_{L^2} + C \|\mu_2\|_{L^2} \|\sigma\|^{(y-1)}_{L^2},
\]

which entails that $\|\n\|_{L^2(t_0,T,t')}$. For the equation of $\sigma$ in (3.16), integration by parts also leads to
\[
\|\partial_t \n\|_{L^2} = \|\n\|_{L^2(t_0,T,t')} \leq C \|\partial_t \sigma\|_{L^2} + C \|\sigma\|_{L^2} + C \|\n\|_{L^2} + C \|\n\|_{L^2} \|\sigma\|^{(y-1)}_{L^2}. \tag{3.20}\]

Then, we apply $\partial_t$ to (3.18) and similarly compute that
\[
\|\partial_t \sigma\|_{L^2} \leq C \|\partial_t \sigma\|_{L^2} + C \|\sigma\|_{L^2} + C \|\n\|_{L^2} + C \|\n\|_{L^2} \|\sigma\|^{(y-1)}_{L^2}, \tag{3.21}\]

Taking a linear combination of (3.20) and (3.21) and rearranging the resulting coefficients carefully, one obtains that
\[
\|\n\|_{L^2(t_0,T,t')} \leq C \|\partial_t \sigma\|_{L^2} + C \|\sigma\|_{L^2} + C \|\n\|_{L^2} + C \|\n\|_{L^2} \|\sigma\|^{(y-1)}_{L^2}, \tag{3.22}\]

which shows that $\|\n\|_{L^2(t_0,T,t')}$. Recall the detailed Stokes estimates in the proof of Lemma 3.1 in [1], one can further obtain that
\[
\|\n\|_{L^2(t_0,T,t')} \leq C \|\partial_t \sigma\|_{L^2} + C \|\sigma\|_{L^2} + C \|\n\|_{L^2} + C \|\sigma\|^{(y-1)}_{L^2} + C \|\mu_2\|_{L^2} \|\sigma\|^{(y-1)}_{L^2},
\]

and this suggests that $\|\n\|_{L^2(t_0,T,t')}$. Consequently, by the above estimates, we can conclude that $(n,\sigma,\mathbf{u}) \in L^2(0,T;H^1) \cap L^2(0,T;H^3)$. This completes the proof of the Theorem.

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