SERIES OF RECIPROCAL POWERS OF $k$–ALMOST PRIMES

RICHARD J. MATHAR

Abstract. Sums over inverse $s$-th powers of semiprimes and $k$-almost primes are transformed into sums over products of powers of ordinary prime zeta functions. Multinomial coefficients known from the cycle decomposition of permutation groups play the role of expansion coefficients. Founded on a known convergence acceleration for the ordinary prime zeta functions, the sums and first derivatives are tabulated with high precision for indices $k = 2, \ldots, 6$ and integer powers $s = 2, \ldots, 8$.

1. Overview

Series over rational polynomials evaluated at integer arguments contain subseries summing over integers classified by the count of their prime factors. The core example is the Riemann zeta function $\zeta$ which accumulates the prime zeta function $\P_1$ plus what we shall define the almost-prime zeta functions $\P_k$ (Section 2). The central observation of this manuscript is that the almost-prime zeta functions are combinatorial sums over the prime zeta function (Section 3). Since earlier work by Cohen, Sebah and Gourdon has pointed at efficient numerical algorithms to compute $\P_1$, series over reciprocal almost-primes—which may suffer from slow convergence in their defining format—may be computed efficiently by reference to the $\P_1$.

In consequence, any converging series over the positive integers which has a Taylor expansion in reciprocal powers of these integers splits into $k$-almost prime components. Section 4 illustrates this for the most basic formats.

Number theory as such will not be advanced. The meromorphic landscape of the prime zeta functions as a function of their main variable appears to be more complicated than what is known for their host, the Riemann zeta function; so only some remarks on the calculation of first derivatives are dropped.

2. Prime Zeta Function

Definition 1. The prime zeta function $P(s)$ is the sum over the reciprocal $s$-th powers of the prime numbers $p$ [16] [15] [27]

$$P(s) \equiv \sum_{p} \frac{1}{p^s}; \quad \Re s > 1.$$  (1)

Remark 1. The primes are represented by sequence A000040 in the Online-Encyclopedia of Integer Sequences [32], and we will adopt the nomenclature that a letter A followed by a 6-digit number points at a sequence in this database. Accuracy values
for $s \leq 9$ are provided by the sequences A085548, A085541, and A085964–A085969.

Table 1 complements these by using [1] 44 [2] 32

$$P(s) = \sum_{p \leq M} \frac{1}{p^s} + \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log P(M, sn)$$

for a suitably large prime $M$, where $\mu$ is the Möbius function [1] (24.3.1), where $\zeta$ is the Riemann zeta function [11, 22] [18, (9.5)], and

$$P(M, s) \equiv \zeta(s) \prod_{p \leq M} \left(1 - p^{-s}\right)$$

an associated definition of a partial product.

Moments are listed in Table 2. The first line,

$$\sum_{s=1}^{\infty} \frac{1}{s} P(s) = -\sum_{p} \left[ \log \left(1 - \frac{1}{p}\right) + \frac{1}{p} \right] = \lim_{s \to 1} \left[ \zeta(s) - \sum_{p} \frac{1}{p^s} \right],$$

is A143524 [32, 13]. The total of all values in Table 2 is

$$\sum_{s=1}^{\infty} \sum_{u=1}^{\infty} \frac{1}{su} P(s) = -\sum_{s=2}^{\infty} \sum_{p} \frac{1}{(s-1)p^s} = -\sum_{p} \frac{\log(1 - 1/p)}{p},$$

(5)

The first derivatives $dP(s)/ds$ of (1) are evaluated as the first derivatives of (2) [1, (3.3.6)],

$$P'(s) = -\sum_{p} \frac{\log p}{p^s} = -\sum_{p \leq M} \frac{\log p}{p^s} + \sum_{n=1}^{\infty} \frac{\mu(n)}{n} P'(M, sn)$$

(6)

$$P'(M, s) \equiv \zeta'(s) \sum_{p \leq M} \frac{\log p}{p^s(1 - p^{-s})}.$$

Here, primes denote derivatives with respect to the main argument, which is the second argument for the case of $P(\ldots)$. Table 3 shows some of them for selected small integer $s$.

Remark 2. The relation $-\zeta'(s)/\zeta(s) = \sum \log(p)/(p^s - 1)$ is a gateway to acceleration of series involving logarithmic numerators. Examples are growth rate coefficients of unitary square-free divisors [33],

$$\sum_{\text{square-free divisors}} \frac{(2p + 1) \log p}{(p^2 - 1)(p + 1)(p^2 + p - 1)} = \sum_{p} \log p \left[ \frac{2}{p^2 - 1} - \frac{3}{p^3 - 1} + \frac{4}{p^4 - 1} - \frac{10}{p^5 - 1} + \frac{18}{p^6 - 1} \right]$$

$$-\frac{28}{p^7 - 1} + \frac{40}{p^8 - 1} - \frac{72}{p^9 - 1} + \cdots = 0.748372333429674\ldots$$

(8)

or of unitary cube-free divisors,

$$\sum_{\text{cube-free divisors}} \frac{(4p^2 - 2p) \log p}{(p^2 - 1)(p^2 + p - 1)} = \sum_{p} \log p \left[ \frac{4}{p^2 - 1} - \frac{5}{p^3 - 1} + \frac{7}{p^4 - 1} - \frac{17}{p^5 - 1} + \frac{31}{p^6 - 1} \right]$$

$$-\frac{48}{p^7 - 1} + \frac{69}{p^8 - 1} - \frac{124}{p^9 - 1} + \cdots = 1.647948081159756\ldots$$

(9)
Table 1. The Prime Zeta Function of some integer arguments. In a style adopted from [1], optional trailing parentheses contain an additional power of 10. Example: The number 3.45 \times 10^{-3} may appear as 0.00345 or as 3.45(-3) or .345(-2). Trailing dots indicate that more digits are chopped off, not rounded, at the rightmost places.

| s | \( P(s) \) |
|---|---|
| 10 | .993603574436980721875585070147739416301872545285203320553566(-3) ... |
| 11 | .493947269104654976591621768334398712155939700960495218166074(-3) ... |
| 12 | .2460267400345456795266485921650809279793226794732191741459(-3) ... |
| 13 | .12269367527869279905488792431403323914728525577690135256528(-3) ... |
| 14 | .61244396725444783775083429987454197282126872378013541885303(-4) ... |
| 15 | .305873047575262931371262800130114525389809330765981(-4) ... |
| 16 | .15282021939341481080192641180559746612698767641310788064(-5) ... |
| 17 | .76371393706489725090455604339762017569839042162662520251345(-5) ... |
| 18 | .38172870317496631225153163109136162494263682614195748077(-5) ... |
| 19 | .1908290769228257218617998769776618145616195066986381165765(-5) ... |
| 20 | .9539611241036233263528834939770057955700555885822134364992986(-6) ... |
| 21 | .476932793684272505083726618887610604190810253478311286107(-6) ... |
| 22 | .23845040458767019282631168520159505067787325069914076736138961(-6) ... |
| 23 | .1192199175188258616024645338339857710830411659141349570467(-6) ... |
| 24 | .596016950483345227113066655008612013146568248011715598729492(-7) ... |
| 25 | .290835026264065867662659401781459495278827831139923166297(-7) ... |
| 26 | .14901554606314570543459077394423733840265740940037170949826319(-7) ... |
| 27 | .74507117343230007801645461240936933495593461489271525177775(-8) ... |
| 28 | .372533401091056351833912287693071753007133176180958755544001(-8) ... |
| 29 | .186265072004357490752214511335360117288334716131657109067707(-8) ... |
| 30 | .93132431552601296776645896544775995113591735984504142578(-9) ... |
| 31 | .4656629062856732188024756168244570478371109046830714604848803(-9) ... |
| 32 | .23283118331344031491367214292013438395601283935372695549588(-9) ... |
| 33 | .1164155017134526546960071628601971730190064295175902585555278(-9) ... |
| 34 | .58207208756338736129611032927989146113554437146187077805157(-10) ... |
| 35 | .291038504441239633403052831321248180971854383563509582681388(-10) ... |
| 36 | .1455192819803022590216132905087468529073564054445529854681582(-10) ... |
| 37 | .7275959835044149315848417671131286009680602799652884873652(-11) ... |
| 38 | .36379795473654162974339172591421915260411920813259213301657(-11) ... |
| 39 | .18189865030375722468576390305856620025233007531508321634193(-11) ... |

3. Zeta Functions of Almost-primes

3.1. Nomenclature. We generalize the notation, and define the \( k \)-almost prime zeta functions by summation over inverse powers of \( k \)-almost primes \( q_j \),

\[
P_k(s) \equiv \sum_{j=1}^{\infty} \frac{1}{q_j^s}.
\]
Table 2. Moments of the Prime Zeta function. The difference between the first two lines has been quoted by Chan [10].

| \( u \) | \( \sum_{s=2}^{\infty} P(s)/s^u = \sum_{s=2}^{\infty} \sum_{p} 1/(s^u p^s) \) |
|---|---|
| 1 | 0.3151845205389007685105825147360571990592687872439261370... |
| 2 | 0.1394706396113086818030776729373945942859632164855482900601... |
| 3 | 0.064081378884568610840324258154627347930040089339678692107... |
| 4 | 0.3079906011277953305364576891280831527791590439776419610652576(-1)... |
| 5 | 0.194140748444275035892118810037703316021054538295740449(-1)... |
| 6 | 0.73276376441199457647510781673927492110198196876194280213(-2)... |

Table 3. The first derivative of the Prime Zeta function at some integer arguments \( s \). The value in the first line differs from Cohen’s value [13] after 42 digits.

| \( s \) | \( P'(s) \) |
|---|---|
| 2 | -4.930911093687644621978282050564912580555881263464682907133271(-1)... |
| 3 | -1.50755554395042217983636516363542919575011615306893318187970(-1)... |
| 4 | -6.06076335077006392309837097133784063828776412598439911276(-2)... |
| 5 | -2.683860127679835742218751329245015994333014955355822812481980(-2)... |
| 6 | -1.24590807279999152702772774689970040911350415758785740993(-2)... |
| 7 | -5.904690391481961425505928290160901936818950592951075166813(-3)... |
| 8 | -2.8795247087824739134602438573340469989837616758684107618(-3)... |
| 9 | -1.410491921245312915514965630819997901657311693496192836500(-3)... |
| 10 | -6.956784474346208402007019778041591384486370332983954712256(-4)... |
| 11 | -3.446842563054190162052798303143722105514850939872073205298024(-4)... |
| 12 | -1.71299352462175657553249312138275372004981118241302326420951(-5)... |
| 13 | -8.5303109167110566532088760172156919726173261505421474299073(-5)... |
| 14 | -4.253630557412291035547547415368175167208935443884363104558(-5)... |
| 15 | -2.12297056274934569384621302375720453112762226994727510845(-5)... |
| 16 | -1.060211861676127903320578231686279299858288732873256230264968(-5)... |
| 17 | -5.2968025756348407499697331902062921354548700744729695167083(-6)... |
| 18 | -2.6498287878099735252632618541821018065965865359403974219570106(-6)... |
| 19 | -1.3230186485122927354434206851957658773372595613019420828763900(-6)... |
| 20 | -6.6135175941726002108914570295021050064035779681754585968634604(-7)... |
| 21 | -3.3062336148252086577302308859128631398893730346735089329(-7)... |
| 22 | -1.65294175342597266938254323706724250523775460957895302949(-7)... |
| 23 | -8.2642526736573812777916086294362294534901824290993527749559(-8)... |
| 24 | -4.1318681364650687420545459801639580884643026496428380702300(-8)... |
| 25 | -2.0658623636712237908526789676180100331701376239519947296098(-9)... |
| 26 | -1.03291300766983384061006103497396886706968131430320354135307(-9)... |
| 27 | -5.1649300351952183949907124602884025560927707601929381567114(-9)... |
| 28 | -2.5822224901930837368041277870336236263231350157972853671708(-9)... |
| 29 | -1.291103241249637065884459649285079993129747526229207669548247(-9)... |
In slight violation of the almost-terminology, the Prime Zeta Function is incorporated as just one special case,

\[ P_1(s) \equiv P(s). \]

**Remark 3.** The sequence \( q_j \) is given by the primes A000040 if \( k = 1 \), by the semiprimes A001358 if \( k = 2 \), by the 3-almost primes A014612 if \( k = 3 \), by the 4-almost primes A014613 if \( k = 4 \), by the 5-almost primes A014614 if \( k = 5 \), by the 6-almost primes A046306 if \( k = 6 \), by the 7-almost primes A046308 if \( k = 7 \) etc. \( P_2(2) \) is A117543, and \( P_3(2) \) is A131653.

**Remark 4.** One step further defines prime multi-zeta functions of the form

\[ P(s_1, s_2, \ldots, s_k) \equiv \sum_{p_1, p_2, \ldots, p_k = 2, 3, 5, \ldots} \frac{1}{p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k}}. \]

If the exponents are restricted to a common number \( s \), they retrieve the information of the \( P_k(s) \) in slightly entangled form, for example \( P(s, s) = 2 P_2(s) - P(2s) \) [8].

Each integer \( n > 0 \) is either a member of the set \( \{1\} \), or of the set of primes, or of the set of semi-primes, etc. These disjoint sets are labeled by the sum of the exponents of the prime number factorizations of their members,

\[ \Omega(n) = k; \quad \Omega(1) \equiv 0. \]

The Riemann zeta function may be partitioned into sums over the almost-prime zeta functions,

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \sum_{k=1}^{\infty} \sum_{\Omega(n)=k} \frac{1}{n^s} = 1 + \sum_{k=1}^{\infty} P_k(s). \]

**Remark 5.** \( \zeta(s) \) may be taken from [1] (Table 23.3) for \( s \leq 42 \), or from A013661, A002117, A013662 – A013678 while \( s \leq 20 \), or from the link to “Recent additions of tables” in Plouffe’s database [28] while \( s \leq 99 \). \( \Omega(n) \) is tabulated in A001222.

### 3.2. Numerical Scheme

Table 4 of \( P_k(s) \) is deduced from

**Theorem 1.** [21]

\[ P_2(s) = \frac{P(2s) + P^2(s)}{2!}; \]

\[ P_3(s) = \frac{2P(3s) + 3P(2s)P(s) + P^3(s)}{3!}; \]

\[ P_4(s) = \frac{6P(4s) + 8P(3s)P(s) + 3P^2(2s) + 6P(2s)P^2(s) + P^4(s)}{4!}; \]

\[ P_k(s) = \frac{1}{k!} \sum_{k_1 + 2k_2 + 3k_3 + \cdots + kk_k = k} (k; k_1k_2 \ldots k_k)^* P^{k_1}(s) P^{k_2}(2s) \cdots P^{k_k}(ks), \]

utilizing values of the prime zeta function as discussed above and summing over the partitions \( \pi(k) \) of \( k \) with weight coefficients

\[ (k; k_1k_2 \ldots k_k)^* = k! \prod_{m=1}^{k} (m^{k_m} k_m!); \]

of Table 24.2 in [1] (multinomials \( M_2 \)) and A036039 or A102189.
Proof. (18) is the main result of the paper. For small \( k \), explicit verification can be done along Price’s [29] construction, where the \( k \)-almost primes fill triangular, \((k = 2)\), tetrahedral \((k = 3)\) etc. sections of a \( k \)-dimensional Euclidean lattice labeled by the prime numbers along its Cartesian axes [5]. The case \( P_1(s) = P(s) \) just repeats the definition (10). The case (15) accumulates in \( P(2s) \) the sum over the squares, and in \( P^2(s) \)—with the binomial expansion—again the sum over the squares and twice the sum over products of distinct primes. After division through \( 2! \), each semiprime is effectively represented once.

The generic proof follows through induction: the terms of the right hand side of (18) contain the factor \( (k; k_1 k_2, \ldots k_k)^* \), which is the number of distinct permutations with \( k_m \) cycles of length \( m \) for \( m = 1, 2 \ldots k \) [6, (p. 123)] [1, (24.1.2)]. The right hand side is the cycle index

\[
Z(S_k) = \frac{1}{k!} \sum_{k_1 + 2k_2 + 3k_3 + \cdots + kk_k = k} (k; k_1 k_2 \ldots k_k)^* P^{k_1}(s) P^{k_2}(2s) \cdots P^{k_k}(k s)
\]

of the symmetric group \( S_k \) [20, (2.2.5)] with \( P(ms) \) substituted for the indeterminates of cycle length \( m \). Skipping any interpretation within a Redfield-Pólya symmetry, its recurrence

\[
Z(S_k) = \frac{1}{k} \sum_{j=1}^{k} P(js) Z(S_{k-j})
\]

is already established [20, (2.2.9)]. This matches precisely the recurrence on the left hand side which generates \( P_k \) by a combination of products of lower-indexed almost-primes,

\[
P_k(s) = \frac{1}{k} \sum_{j=1}^{k} P(js) P_{k-j}(s).
\]

This recurrence is valid because each \( k \)-almost prime which appears on the left hand side of this equation can be generated in \( k \) ways by a product of the form \( P(js) P_{k-j}(s) \): in \( \omega(k) \) ways by splitting off a prime number and multiplication with a number of the sum in \( P_{k-1}(s) \), for each divisor of the \( k \)-almost prime which is a square of some prime in addition by multiplication of the square with a term in the sum in \( P_{k-2}(s) \), and so on for divisors that are cubes of some prime etc. □

The simple pole of \( \zeta(s) \) at \( s - 1 \) with Stieltjes constants \( \gamma_j \) [12, 3, 23],

\[
\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \gamma_j (s-1)^j,
\]

is associated with a logarithmic singularity of \( P(s) \) at \( s = 1 \) and singularities of \( P(s) \) on the real line between \( s = 0 \) and \( s = 1 \) where \( s \) is the inverse of a square-free integer [16]. If \( k > 1 \), the \( k \)-almost zeta functions inherit these and add more by the mechanism evident from the multipliers in Theorem [1] \( P_2(s) \) in [15], for example, inherits singularities at 1/2, 1/3, 1/5, 1/6, 1/7, 1/10 etc. from the term \( P^2(s) \), and singularities at 1/2, 1/4, 1/6, 1/10, 1/12 etc. from the term \( P(2s) \), illustrated in Fig. [1]
Table 4. Almost-prime zeta functions $P_k(s)$ at small integer arguments $s$.

| $k$ | $s$ | $P_k(s)$ |
|-----|-----|----------|
| 2   | 2   | $1.407604343490233882227509254138772537741972760048802639241489(-1)$ |
| 2   | 3   | $2.3860334377271959687695955852836200628932178484034845684562765(-2)$ |
| 2   | 4   | $4.9946744686373963527668740495792893225020578482380667228500909(-3)$ |
| 2   | 5   | $1.136012248563535756551607037577266369348055063108551651424(-3)$ |
| 2   | 6   | $2.6870176561409632421738798614087553579878747125719642936101(-4)$ |
| 2   | 7   | $6.4933411756911455788540615078367145199897516715283327216508(-5)$ |
| 2   | 8   | $1.5888519885295888585373209535187923452758597132723300748108(-6)$ |
| 3   | 2   | $3.8516192982696446091283792262806039543890016747838157193719155(-2)$ |
| 3   | 3   | $3.0436208233431294674809884707930299984694548619577993637287(-3)$ |
| 3   | 4   | $3.1442249683294147218212466614097192073070176953574340102524244(-4)$ |
| 3   | 5   | $3.5577533706826911188017622799305930206716602282958084700573(-5)$ |
| 3   | 6   | $4.20127553396671214387834295923202794509520951879928447828323(-6)$ |
| 3   | 7   | $5.07368799451597922712787865492065044179712489921349789148696(-7)$ |
| 3   | 8   | $6.2068136216146994555196445893296566913542547740137362544719901(-8)$ |
| 4   | 2   | $1.000943620148325082041084351808525466652473851036634849174401(-2)$ |
| 4   | 3   | $3.83904543615729607462800842516248330089079010633110559279434(-4)$ |
| 4   | 4   | $1.9679636328119146794085596157341095509879595042823958199339(-5)$ |
| 4   | 5   | $1.112054938414704206541643932409614810339288829649464410277(-6)$ |
| 4   | 6   | $6.514686966273645939263094291756536419279606893245430136952(-8)$ |
| 4   | 7   | $3.964020938138935586748245375642870705531500802782383854435(-9)$ |
| 4   | 8   | $2.42454206719812971921522140682757858198415539059019871808441(-10)$ |
| 5   | 2   | $2.54076168069302058221776985605516223094313334404435645812102(-3)$ |
| 5   | 3   | $4.08940110832567973019045453662867670709263774628437825310148(-5)$ |
| 5   | 4   | $1.23032174727824954320836389049979176133316153015200078253(-6)$ |
| 5   | 5   | $3.47545960029756789328750518499380973103878236550655202548(-8)$ |
| 5   | 6   | $1.02576593049305828602017782544418055295894431706212794650(-9)$ |
| 5   | 7   | $3.96892760398295200747746359132814874356383401665599175776(-11)$ |
| 5   | 8   | $9.47086828755709953170729241488657640110148079145995132172493(-13)$ |
| 6   | 2   | $6.41033582604280712627076320846767912178985251438428485882042(-4)$ |
| 6   | 3   | $6.014928701790894186295286615522838575739776334332316566701(-6)$ |
| 6   | 4   | $7.689364141651761724094522188631876613018667310055486761046(-8)$ |
| 6   | 5   | $1.08680663383684175561530146294503063068695695783050109331817(-9)$ |
| 6   | 6   | $1.6027594427579381648667907110112243763271754888808749039669(-11)$ |
| 6   | 7   | $2.41944756334425756885672198950160973207458203936956720368419(-13)$ |
| 6   | 8   | $3.6995579375927960791942236715804939304459496592590589532(-15)$ |

Remark 6. Sums over odd indices are [31, 2]:

\[
\sum_{k=1}^{\infty} P_{2k-1}(2) = \frac{\pi^2}{20}; \quad \sum_{k=1}^{\infty} P_{2k-1}(2s) = \frac{\zeta^2(2s) - \zeta(4s)}{2\zeta(2s)}.
\]

Sums over even indices follow from there as the complement in [14].
Corollary 1. The product rule (3.3.3) applied to (18) yields the first derivative

\[ P_k'(s) = \frac{1}{k!} \sum_{k_1 + 2k_2 + 3k_3 + \ldots + kk_k = k} (k; k_1 k_2 \ldots k_k)^* P^{k_1}(s) P^{k_2}(2s) \cdots P^{k_k}(ks) \]

\[ \times \left[ \frac{k_1 P'(s)}{P(s)} + \frac{2k_2 P'(2s)}{P(2s)} + \cdots + \frac{kk_k P'(ks)}{P(ks)} \right] . \]

Numerical evaluation yields Table 5. The univariate \( \int_1^\infty \frac{1}{P_k(s) \log p} \) has been evaluated numerically for \( k = 1 \) by Cohen [13, 25].

3.3. Möbius (Square-free) Variant. Reduction of the summation to \( k \)-almost primes with \( k \) distinct prime factors defines a signed variant of the prime zeta functions:

Definition 3.

\[ P_k^{(\mu)}(s) = \sum_{\Omega(n) = k} \frac{1}{n^s} \frac{\mu(n)}{n^s} = (-1)^k \sum_{\Omega(n) = \omega(n) = k} \frac{1}{n^s} ; \quad P_1^{(\mu)}(s) = -P(s) , \]

where \( \omega(.) \) is the number of distinct prime factors of its argument.

Remark 7. The criterion \( \Omega(n) = \omega(n) = k \) selects the prime number products (square-free \( k \)-almost primes) of A000040 (\( k = 1 \)), A006881 (\( k = 2 \)), A007304 (\( k = 3 \)), A046386 (\( k = 4 \)), A046387 (\( k = 5 \)), A067885 (\( k = 6 \)), A123321 (\( k = 7 \)), A123322 (\( k = 8 \)), A115343 (\( k = 9 \)), etc.
The sum
\[ \frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \sum_{k=1}^{\infty} P_k^{(\mu)}(s) \]
converges for \( s > \frac{1}{2} \) if the Riemann hypothesis holds \[14\]. Application of the multinomial expansion \[1, (24.1.2)\] to the powers \( P_k(s) \) leads to the recurrences

**Theorem 2.**

\[ P_2^{(\mu)}(s) = \frac{P^2(s) - P(2s)}{2!}; \]
\[ -P_3^{(\mu)}(s) = \frac{P^3(s) - 3P(s)P(2s) + 2P(3s)}{3!}; \]
\[ P_4^{(\mu)}(s) = \frac{P^4(s) - 6P^2(s)P(2s) + 3P^2(2s) + 8P(s)P(3s) - 6P(4s)}{4!}; \]
\[ P_k^{(\mu)}(s) = \frac{1}{k!} \sum_{k_1+2k_2+3k_3+\cdots+kk_k=k, k_\alpha \geq 0} (-1)^m (k; k_1 k_2 \ldots k_k)^* \times P_{k_1}(s)^{k_2}(2s) \cdots P_{k_k}(ks), \]

where \( m \equiv k_1 + k_2 + k_3 + \cdots + k_k \).
Table 5. First derivatives $P'_k(s)$. $P'_1(s)$ is Table 3.

| $k$ | $s$ | $P'_k(s)$ |
|-----|-----|-----------|
| 2   | 2   | -2.8360681514057980652242582225482783360793505782378140131411118(-1) ... |
| 3   | 2   | -3.8805699293325206692460172856731896741264975825135409952156(-2) ... |
| 4   | 7.54889669408531250260697097667196350193021639416359677908501448(-3) ... |
| 5   | 1.6532931502460167402506098879514629396865668818280047120(-3) ... |
| 6   | -3.839796424635045625755737180011945613921162583983225237893084(-4) ... |
| 7   | -9.1747980624914395264462602702318824641281810420282594760147(-4) ... |
| 8   | -2.229357521814937325224031302167294498658575057693036222970(-6) ... |

3

| $k$ | $s$ | $P'_k(s)$ |
|-----|-----|-----------|
| 2   | -1.092764452688696718239350444063723187427760242890148910943(-1) ... |
| 3   | 7.1761163513348341438718965718681375688595376202629921788618(-3) ... |
| 4   | -6.957183997016348677998754615917611673903874611362524278817441(-4) ... |
| 5   | 7.659271206549039063747431108080791018988690026393975133120885(-5) ... |
| 6   | 8.918921902960271370285096585954009850472582686458231428562033(-6) ... |
| 7   | -1.068734610887186385367349143263696994248021103907624353475(-6) ... |
| 8   | -3.30129258649541517522122901848855066736701369507851049403307(-7) ... |

4

| $k$ | $s$ | $P'_k(s)$ |
|-----|-----|-----------|
| 2   | -3.6037260943517984850662665618111112411308366479695596293855(-2) ... |
| 3   | -1.1741163095729879467781601016882720464081644182323562864446(-3) ... |
| 4   | -5.7229988598192580006173046090236901289250874897789672642039(-5) ... |
| 5   | -3.165566397960624496652503472091496425474993776593819875806(-6) ... |
| 6   | -1.8487630226155200614707971908610805456608541345894494226(-7) ... |
| 7   | -1.10973061941958343230779385594975325149430785124555161690094(-8) ... |
| 8   | -6.76377111705292290998115786756787319416415761969510610649579(-10) ... |

5

| $k$ | $s$ | $P'_k(s)$ |
|-----|-----|-----------|
| 2   | -1.1021620985050701831043192005537942165829991652167921097474(-2) ... |
| 3   | -1.8061349293876311798921697170772329704116896588483316621559(-4) ... |
| 4   | -4.4316644276805933899208960923370954681107567501701221273078(-6) ... |
| 5   | -1.2302362375879194296629288412459227277403432769621159402(-7) ... |
| 6   | -3.59972345372087197841453883752982298321080426510863532161742(-9) ... |
| 7   | -1.0816382882901610115099616768322567562515889929760513749543586(-10) ... |
| 8   | -3.29856907616376826327472012152104849391167675473850381292530352(-12) ... |

6

| $k$ | $s$ | $P'_k(s)$ |
|-----|-----|-----------|
| 2   | -3.2372630121052318130409824396954130354284174135060291495318(-3) ... |
| 3   | -2.6769115386443168037448713352769040439823804026861339614141(-5) ... |
| 4   | -3.302848626555066781698851083367477685502382798238084421381(-7) ... |
| 5   | -4.597234953883457819527050674957592810761129310502451431327254(-9) ... |
| 6   | -6.735520381050295876809240060087628109974302091955919905410423(-11) ... |
| 7   | -6.101732492072411533139207362754178556047001434410873508589(-12) ... |
| 8   | -1.54493736829491827569615909580294393406555439166236656388540(-14) ... |

Redistributing the sign with

$$(-1)^m P^{k_1}(s)P^{k_2}(2s)\cdots P^{k_s}(ks) = P^{(\mu)k_1}(s)P^{(\mu)k_2}(2s)\cdots P^{(\mu)k_s}(ks),$$

shows that a recurrence equivalent to (21) is applicable.

**Remark 8.** Sums over odd indices are [31]:

$$\sum_{k=1}^{\infty} P^{(\mu)k-1}(2) = -\frac{9}{2\pi^2}; \quad \sum_{k=1}^{\infty} P^{(\mu)k-1}(2s) = -\frac{\zeta(2s) - \zeta(4s)}{2\zeta(2s)\zeta(4s)}.$$
Table 6. Almost-prime zeta functions $P_k^{(µ)}(s)$ at small integer arguments $s$ computed from (31).

| $k$ | $s$ | $P_k^{(µ)}(s)$ |
|-----|-----|----------------|
| 2   | 2   | $6.3767294584776543280131629480719383612878216290037073692109(-2)$ ... |
| 3   | 2   | $6.735966221354467245622858677680141943623660580421211246428(-3)$ ... |
| 4   | 9.326910211590747534349069208799524913363159126727467407(-4) ... |
| 5   | 1.424885041931714574695705495880323493187551074090523559785(-4) ... |
| 6   | 2.26806752686395289509147440066255999464767652487721194642(-5) ... |
| 7   | 3.8887450314469741103258077849263227078482497419695331204(-6) ... |
| 8   | 6.0649366592024470492179454162832357061731983656840189960047(-3) ... |
| 9   | 3.6962441634523839553463239468115591397130304272497472(-3) ... |
| 10  | 3.6148651246349939521729289639111156410218947240410606989(-5) ... |
| 11  | 1.72714580937233046302125882710417325726718418066397876940(-6) ... |
| 12  | 5.0940194598826356852050108113237778072664477868404106866961(-8) ... |
| 13  | 2.18522329154492938697063292501476827895728535727841472395(-9) ... |
| 14  | 7.0453603114670674519320455267424860581256194921499261430(-11) ... |
| 15  | 1.6329431236938215954403416838730541572919535205263255(-12) ... |
| 4   | 2   | $-3.6962441634523839553463239468115591397130304272497472(-3)$ ... |
| 5   | 2   | $-6.6148651246349939521729289639111156410218947240410606989(-5)$ ... |
| 6   | 2   | $-1.72714580937233046302125882710417325726718418066397876940(-6)$ ... |
| 7   | 2   | $-5.0940194598826356852050108113237778072664477868404106866961(-8)$ ... |
| 8   | 2   | $-1.582322154942938697063292501476827895728535727841472395(-9)$ ... |

4. Applications

4.1. Geometric Series. By integrating [18, (1.511)]

\[
\log(1 - x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}
\]
Table 7. Series of the form \((39)\).

| \(s\) | \(\sum_{n=2}^{\infty} 1/[n^s(n-1)]\) |
|---|---|
| 1 | 1. |
| 2 | 3.55065933157735635275848333539748107810500987932015622(-1) \ldots |
| 3 | 1.530092999217927812678466718425248200160638064527026804(-1) \ldots |
| 4 | 3.55065933157735635275848333539748107810500987932015622(-1) \ldots |
| 5 | 3.3758041137611602804774888432274918423193503206296081(-2) \ldots |
| 6 | 1.6414979153222056595955905340222128241444499920939897(-2) \ldots |
| 7 | 0.8657017712991937261620092036054616825508843397069255(-3) \ldots |
| 8 | 3.988345573354854347476770694952964359093784120672226(-3) \ldots |
| 9 | 1.97995274727263992962400146254093593784242389231915677(-3) \ldots |
| 10 | 9.853776194545545924878042562221981939647108247543984198(-4) \ldots |

we obtain the generating function [18, (1.513.5)],
\[
x + (1-x) \log(1-x) = \sum_{n=2}^{\infty} \frac{x^n}{n(n-1)}.
\]
In the limit \(x \to 1\) we find [18, (0.141)]
\[
\sum_{n=2}^{\infty} \frac{1}{n(n-1)} = 1.
\]
Larger powers of the first factor of the denominator display partial sums of zeta functions through partial fraction decomposition,
\[
\sum_{n=2}^{\infty} \frac{1}{n^2(n-1)} = \sum_{n=2}^{\infty} \left( \frac{1}{n(n-1)} - \frac{1}{n^2} \right) = 2 - \frac{\pi^2}{6};
\]
\[
\sum_{n=2}^{\infty} \frac{1}{n^3(n-1)} = \sum_{n=2}^{\infty} \left( \frac{1}{n(n-1)} - \frac{1}{n^2} - \frac{1}{n^3} \right) = 3 - \frac{\pi^2}{6} - \zeta(3);
\]
\[
\sum_{n=2}^{\infty} \frac{1}{n^s(n-1)} = s - \sum_{l=2}^{s} \zeta(l); \quad s \geq 1.
\]
Examples of these numbers are collected in Table 7, A152416 and A152419. Their sum is
\[
\sum_{s=1}^{\infty} \sum_{n=2}^{\infty} \frac{1}{n^s(n-1)} = \sum_{n=2}^{\infty} \frac{1}{(n-1)^2} = \zeta(2) = \frac{\pi^2}{6}.
\]

**Definition 4.** Restriction of the summation in \((39)\) to \(k\)-almost primes defines a set of constants \(B_{k,s}\) [19].
\[
B_{k,s} = \sum_{\Omega(n)=k} \frac{1}{n^s(n-1)} = \sum_{n=2}^{\infty} \sum_{l=0}^{\infty} \frac{1}{n^{s+l+1}} = \sum_{l=0}^{\infty} P_k(s+1+l).
\]
This reduction to a geometric series and sum over the $P_k$ has been used to calculate Table 8. This definition introduces an analog to (14).

\[ \sum_{n=2}^{\infty} \frac{1}{n^s(n-1)} = \sum_{k=1}^{\infty} B_{k,s}. \]

**Remark 9.** $B_{1,1}$ is A136141, calculated by Cohen [13]. $B_{1,2}$ is A152441. $B_{2,1}$ is A152447.

Projection of (39) onto the $n$ of a fixed $\Omega(n)$ yields

\[ B_{k,s} = B_{k,1} - \sum_{l=2}^{s} P_k(l). \]

The benefit of this formula is that the $B_{k,s}$ can be derived from $B_{k,1}$ without accumulating the individual terms of the geometric series proposed in (11), reaching back to Tables 1 and 4 instead. Consider for example $B_{3,2} \approx 0.003404 = B_{3,1} - P_3(2) \approx 0.041920 - 0.038516$ in Table 9.

The square-free variant of (11) is...
Table 9. Some values of \( B_{k,s} \). The series limits of the partial sums \( \sum_k B_{k,s} \) are in Table 7.

| \( k \) | \( s \) | \( B_{k,s} = \sum_{\Omega(n)=k} \frac{1}{n^s(n-1)} \) |
|---|---|---|
| 1 | 2 | 3.2090924900872962935782490950236946144614455699284329362657458713... |
| 2 | 2 | 0.32914586309732483994561638958496960216327336030620335566930993(-1) ... |
| 3 | 2 | 0.34041462909583149671090654107628833843579424470347730472318(-2) ... |
| 4 | 2 | 0.40762148179053260981037088630456765933440285294839259784(-3) ... |
| 1 | 3 | 1.4616609702869029347107803579877604700978709714433668591512 ... |
| 2 | 3 | 0.68542558201288720755683105660759573951575502718767213033361(-2) ... |
| 3 | 3 | 0.357842167525436821901075833145988403566339382745677940994463(-3) ... |
| 4 | 3 | 0.208610202178413235709572750020237570255452209283788001114444(-4) ... |
| 5 | 3 | 1.26613405850629820433773990228912341234438356263326013819(-5) ... |
| 1 | 4 | 0.6915344094503924799296148342428298903085668112023011462097711(-1) ... |
| 2 | 4 | 0.149073568956394085487880905601814702648930997020410089927942399(-2) ... |
| 3 | 4 | 0.403567199023023063086094140740676728492698470022769157503043...(-4) ... |
| 4 | 4 | 0.1181976739964855630069754286128019256652750014484191210481(-5) ... |
| 5 | 4 | 0.358123103275388385013901827921059901276826250860625278844402(-7) ... |
| 1 | 5 | 0.333984524611419908592732418859717917635953447564981743084436(-1) ... |
| 2 | 5 | 0.3547826407594318636252856354697599919935164601490043643281503(-3) ... |
| 3 | 5 | 0.37946653162093491800591971274761742642553424719318834097486(-5) ... |
| 4 | 5 | 0.6987107560235151423555111035371840444631341617179854710770233(-7) ... |
| 5 | 5 | 0.1057711729199790417355375910626719164172948014247187268165467(-8) ... |
| 1 | 6 | 0.163283656104787497051395686199147799667735926011056939972721067(-1) ... |
| 2 | 6 | 0.860310971461846654151425731604561947290130232847234920489(-4) ... |
| 3 | 6 | 0.578190997602637036180682444163146581149531325986267368068(-5) ... |
| 4 | 6 | 0.422240539627862956248090245275108212062010286509607400760(-8) ... |
| 5 | 6 | 0.319461361560404131327358128082301108877053582540365987872046674(-10) ... |

Definition 5.

(44) \[ B_{k,s}^{(\mu)} = \sum_{n=2}^{\infty} \frac{\mu(n)}{n^s(n-1)} \]

As in (41), there is a representation generated by expansion as a geometric series, and another one from the decomposition in partial fractions:

(45) \[ B_{k,s}^{(\mu)} = \sum_{l=0}^{\infty} P_k^{(\mu)}(s + 1 + l) = B_{k,1}^{(\mu)} - \sum_{l=2}^{s} P_k^{(\mu)}(l). \]

Explicit values follow in Tables 10 and 11. The special values of

(46) \[ B_{1,s}^{(\mu)} = -B_{1,s} \]

can be read off Table 8 and 9. Because the squared primes are those 2-almost primes which are not square-free, constants like

(47) \[ \sum_p \frac{1}{p^{2s}(p^2-1)} = \sum_{l=0}^{\infty} P(2(1+s+l)) = B_{2,s} - B_{2,s}^{(\mu)} \]

can be extracted subtracting values of Tables 9 and 11.
Table 10. Some values of (44) at $s = 1$.

| $k$ | $(-1)^k B_{k,1}^{(p)} = \sum_{\Omega(n)=k, n \sim k} 1/[n(n-1)]$ |
|-----|---------------------------------------------------------------|
| 1   | 0.77315666904979512786436745985594239561874133608318604831100606... |
| 2   | 0.7160615364062956890149523327857003297757749676476966831566... |
| 3   | 0.376417253617739897891446442934388451317167828473395361749... |
| 4   | 0.105324142637030907318956105761580620676017573617479796070137... |
| 5   | 0.166234636913663863135981299997075804303019325500048549975... |
| 6   | 0.161096519050745263581951060263809976151843704660230861574529... |
| 7   | 0.1037654958138210450127184359674418404044704893780297641014... |
| 8   | 0.105324142637030907318956105761580620676017573617479796070137... |
| 9   | 0.0466816435033902366579272500035285659530910663817211356715781... |
| 10  | 0.252791650837407417918191699527156261287684908198212066313664... |
| 11  | 0.0376417253617739897891446442934388451317167828473395361749... |
| 12  | 0.0724160310435881305896567637397207142589348985613246864573289... |
| 13  | 0.10966184934062961737951091947363648217302666195170... |
| 14  | 0.13491272590180301878068618207218104997991413625692668361826582... |
| 15  | 0.136500391426215659399117854245799710072414674625983750046... |
| 16  | 0.1154502993058919020363207484833721347361648310620287413349173... |
| 17  | 0.128461663669460638199906712482357029072394743693281076128206927... |
| 18  | 0.5033138770096510735720427167107691393238682251603074192908100... |
| 19  | 0.2649852985653062334484766800012725887780001122254618728853331610... |
| 20  | 0.12135579687976106037893591360857756602310726382140603104260606... |
| 21  | 0.05871343094024372739476025792024304066514972560359369318110118... |
| 22  | 0.1725587625660721457243856607674119990525918130947542550053715... |
| 23  | 0.5427013725578710178835908998482929048581333704208758782973105... |
| 24  | 0.15238881562580862127942235619175415202306455024167076901840995... |
| 25  | 0.3839755144980947896292097416496478946908646039053220138237362... |
| 26  | 0.872214203479708566977716463365058358980257342379119911641791... |

4.2. Hurwitz Zeta Decompositions.

**Definition 6.** The Hurwitz Zeta Function is

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} = \sum_{n=1}^{\infty} \frac{1}{(n+a-1)^s}; \quad \Re s > 1, \quad \Re a > 1.$$  

On the trot, we project this sum onto the subspaces of $k$-almost primes, too,

**Definition 7.** (Hurwitz Prime and Almost-Prime Zeta Functions)

$$P_k(s,a) = \sum_{\Omega(n)=k} \frac{1}{(a-1+n)^s}; \quad P_k(s,1) = P_k(s).$$

$$P_k^{(p)}(s,a) = \sum_{\Omega(n)=k} \frac{\mu(n)}{(a-1+n)^s}; \quad P_k^{(p)}(s,1) = P_k^{(p)}(s).$$

This partitions (44) into

$$\zeta(s,a) = \frac{1}{a^s} + \sum_{k} P_k(s,a),$$

as opposed to

$$\zeta(s,a) = \sum_{k} P_k(s,a).$$
Table 11. Some values of \((-1)^kB_{k,s}^{(\rho)}\).

| \(k\) | \(s\) | \((-1)^kB_{k,s}^{(\rho)}\) |
|-----|-----|------------------|
| 2   | 2   | \(.78387207792864074088832755752559186420093612067276933376705990(-2)\) |
| 3   | 2   | \(.679283717149368203941798318982537285977454798046059789812729(-4)\) |
| 4   | 2   | \(.214907713932382436626011100595357020867171177897016336529944(-6)\) |
| 5   | 2   | \(.26416111685102616016738596172379283737929209557704345022153(-9)\) |
| 2   | 3   | \(.11027741571506068432099316578238500075312460092348122252063101(-2)\) |
| 3   | 3   | \(.177920460858868087245002259712457218755560070641953719082922(-5)\) |
| 4   | 3   | \(.7325781797354076886116612880679583412683957083696886124052352(-9)\) |
| 5   | 3   | \(.7632343049784011106574781119488120313414403098406009257503235(-13)\) |
| 2   | 4   | \(.2\.679283717149386203941798318982537285977454798046059789812729(-4)\) |
| 3   | 4   | \(.525746592164504942874343261008396148837588975555842142889903(-7)\) |
| 4   | 4   | \(.29750108591388430703115241596312267861630644973967306755420261(-11)\) |
| 5   | 4   | \(.266849846873132284278576178941457041398915987179742043245376(-16)\) |
| 2   | 5   | \(.2\.2149057713932382436626011100595357020867171177897016336529944(-6)\) |
| 3   | 5   | \(.2\.26416111685102616016738596172379283737929209557704345022153(-9)\) |
| 4   | 5   | \(.2\.266849846873132284278576178941457041398915987179742043245376(-16)\) |
| 5   | 5   | \(.2\.11027741571506068432099316578238500075312460092348122252063101(-2)\) |
| 2   | 6   | \(.4415507085213108012762620849136326236382772582004900125801602(-5)\) |
| 3   | 6   | \(.52141702169123237807057183141513675412654987329380358363271(-10)\) |
| 4   | 6   | \(.59376215895380392526466226214719270223297866728632203157065385(-16)\) |
| 5   | 6   | \(.4010546162489087900384620760650048636932117449331297950429083(-23)\) |

generalizing (14).

Remark 10. By series expansion \(18\) (1.112.1), the sum rule associated with (41) is

\[
B_{k,s} = \sum_{n=2}^{\infty} \left( \frac{1}{n-1} \right)^{1+s} = \sum_{l=0}^{\infty} (-1)^l P_k(1 + s + l, 0).
\]

The reduction of (49) to the Prime Zeta Functions is obtained by the binomial expansion \([7\text{, Entry 22}]\)

\[
P_k(s,a) = \sum_{n=2}^{\infty} \frac{1}{n^s} \left( \frac{1}{1-a} \right)_n = \sum_{l=0}^{\infty} \left( \frac{s}{l} \right)_l \left( 1-a \right)_l P_k(s + l).
\]

Pochhammer’s symbol \((s)_l = \Gamma(s + l)/\Gamma(s)\) is introduced to simplify the notation \([1\text{, (6.1.22)]}\). Brute-force accumulation of partial sums over \(l\) for the case \(a = 0\)
Table 12. Some values of $P_k(s,0)$. Cohen has reported $P_1(2,0)$ \cite{13}, which is A086242 \cite{32}.

| $k$ | $s$ | $P_k(s,0) = \sum_{\Omega(n)=k} 1/(n-1)^s$ |
|-----|-----|--------------------------------------------------|
| 1   | 2   | 1.3750649474863528701725313052243969917950996017531745870918933... |
| 2   | 2   | 0.2097832394001949275536860246918236268613932921851343752817089... |
| 3   | 2   | 0.457250649473356179590942895789867622396635450843043104898823(-1)...
| 4   | 2   | 1.0814086447646394130511466215222764287414954775183405819497437(-1)...
| 5   | 2   | 2.26509027543661792197346769155986673457680151009430906200960165(-2)...
| 1   | 3   | 1.147529077585804696323830628213040164476473552511225527582412... |
| 2   | 3   | 0.4976105113266659818759508637705240584130477547563781478393517(-1)...
| 3   | 3   | 1.3257287912996225107947200519463804641452546738693718720601966(-2)...
| 4   | 3   | 4.503375616016164969705044013832457739796302018827147548523156327(-3)...
| 5   | 3   | 5.1999636872926720244682115275828429690875495318025444066596364(-4)...
| 1   | 4   | 1.0673601122715176911512740206525870839352859304550836811508241... |
| 2   | 4   | 0.1442518670501253472063217138552577465573291687146611337540069(-1)...
| 3   | 4   | 0.31154573343901127268316650380324470399731764156523619083247750(-3)...
| 4   | 4   | 0.24873203818407033068602411510121468875276416146626659372052(-4)...
| 5   | 4   | 1.1380520309846403690931448223425176600445470887305178065(-6)...
| 1   | 5   | 1.032371005978341965851775792963868294504382496931767985434695475... |
| 2   | 5   | 0.448814553317860858004525242214145722419706205813943952361742(-2)...
| 3   | 5   | 0.670525710706644507999917906002173405129522585443401784675937527(-4)...
| 4   | 5   | 1.509417183618483834163159714104371580744915253875014126944996(-5)...
| 5   | 5   | 0.40382718323494954952893153228228432613799229837706544183828(-7)...
| 1   | 6   | 1.01589201139972411006657659184573525510134375805731168636932262... |
| 2   | 6   | 0.144118068391275520265154654521805154418089925454726070129138(-2)...
| 3   | 6   | 0.31523107817504087489510416351034274165014034086162005143925(-5)...
| 4   | 6   | 0.954942045918423081706205835423838952509548418059401298115858(-7)...
| 5   | 6   | 1.233223955626350727396638246189276451856567079235951708582994(-8)...

yields Table 12. Resummation in \cite{19} imposes the sum rule

(55) \[ \sum_{k=1}^{\infty} P_k(s,0) = \zeta(s). \]

The derivative of (53) with respect to $s$ is

\[ \frac{\partial}{\partial s} P_k(s,a) = P_k'(s,a) = - \sum_{\Omega(n)=k} \log(a - 1 + n) \frac{(a - 1 + n)^s}{(a - 1 + n)^s} \]

(56) \[ \sum_{l=0}^{\infty} \frac{(s)_l (1-a)^l}{l!} \left[ (\psi(s + l) - \psi(s)) P_k(s + l) + P_k'(s + l) \right] \]

in terms of digamma functions $\psi$ \cite{1, (6.3)]. Cases with $a = 0$ are illustrated by Table 13.

4.3. Logarithmic Functions. Repeated integration of (53) with respect to $x$ yields a ladder of functions with increasing order $l + 1$ of the polynomial of $n$
### Table 13. Some absolute values of (56) at $a = 0.$

| $k$ | $s$ | $|P_k(s,0)| = \sum_{\Omega(n)=k} \log(n-1)/(n-1)^s$ |
|-----|-----|--------------------------------------------------|
| 1   | 2   | $4,120,362694845359298953672788691959310869395993272284789334253\ldots$ |
| 2   | 2   | $4,494,064202832790491021858840523284117552689458536744645947573\ldots$ |
| 3   | 2   | $12,205,38775570122922528860135234470821144298661274431502394539860\ldots$ |
| 4   | 2   | $3,813,24604512840984062050578361697402655973888834901273296193385(1-1)\ldots$ |
| 5   | 2   | $11,453,50145901906536851293804267575046096249239456318219727158(1-1)\ldots$ |
| 1   | 3   | $1,224,91944696114944181066412654630614807170036457469067227278\ldots$ |
| 2   | 3   | $6,471,4774935307071820333326031334769038262596303945154513521204(1-1)\ldots$ |
| 3   | 3   | $9,351,18221996044867853448492950310390608212991090028893869639531(2-1)\ldots$ |
| 4   | 3   | $1,342,6815050192511286561753670177005489358798054445521080127694(2-1)\ldots$ |
| 5   | 3   | $19,332,293801196743055978270801199939526414775611810214336768401(3-1)\ldots$ |
| 1   | 4   | $5,057,1030112820461135804493661138529079953089942673024892230106(1-1)\ldots$ |
| 2   | 4   | $1,720,9705239404148940955906998207933054016728793461(2-1)\ldots$ |
| 3   | 4   | $1,055,4371770636664676702388193888381593405142250008490197320(2-1)\ldots$ |
| 4   | 4   | $7,052,55128958924189079712608249243671781890722901833430(2-1)\ldots$ |
| 5   | 4   | $4,921,74845158517304655381936518534428557535927933728449269(5-1)\ldots$ |
| 1   | 5   | $2,282,3372972589090974754746653849346646934243019653066544(1-1)\ldots$ |
| 2   | 5   | $5,151,18463816847340424656578480839865650309296489970884531366991(2-1)\ldots$ |
| 3   | 5   | $1,465,4514948629680558875457457686959728794768723723639017(3-1)\ldots$ |
| 4   | 5   | $4,190,8173714005228515335452727687681983243734757815669907193790(4-1)\ldots$ |
| 5   | 5   | $1,455,006129099320290909037191919984238407760119056451514240612(6-1)\ldots$ |
| 1   | 6   | $1,121,07016694258479319488347840155835256000252535562240625898(1-1)\ldots$ |
| 2   | 6   | $1,623,09029366737494046263910398904379017023031727725964622063(2-1)\ldots$ |
| 3   | 6   | $1,810,9924875729486622670211139219188523262792500530173877649790(4-1)\ldots$ |
| 4   | 6   | $2,623,9352241109293153260491098675091309689211371790669768498(5-1)\ldots$ |
| 5   | 6   | $4,826,7884903977448189220166642396011211315801456016154892551835(-8)\ldots$ |

In the denominator $[18, (1.513.7)] [24],$

\[(57) \quad x + (1 - x)^2 \ln(1 - x) = \frac{3x^2}{2} - 2 \sum_{n=1}^{\infty} \frac{x^{n+2}}{n(n+1)(n+2)};\]

\[(58) \quad x + (1 - x)^3 \ln(1 - x) = \frac{5x^3}{2} - \frac{11x^3}{6} + 6 \sum_{n=1}^{\infty} \frac{x^{n+3}}{n(n+1)(n+2)(n+3)};\]

\[(59) \quad x + (1 - x)^4 \ln(1 - x) = \frac{7x^4}{2} - \frac{13x^4}{3} + \frac{25x^4}{12} - 24 \sum_{n=1}^{\infty} \frac{x^{n+4}}{n(n+1)(n+2)(n+3)(n+4)};\]

\[(60) \quad x + (1 - x)^l \ln(1 - x) = \sum_{i=2}^{l} \tau_{i,l} x^i - (-1)^l l! \sum_{n=1}^{\infty} \frac{x^{n+l}}{n(n+1)\cdots(n+l)};\]

The rational coefficients of the auxiliary polynomials obey the recurrence

\[(61) \quad \tau_{2,l} = l - \frac{1}{2}, \quad l \geq 2; \quad i \tau_{i,l} = (-1)^l \left(\frac{l-1}{i-1}\right) l \tau_{i-1,l-1}, \quad i \geq 3.\]

$i! \tau_{i,l}$ are tabulated in A067176 and A093905.
Table 14. Logarithmic sums of $k$-almost primes defined in (62).

| $k$ | $l$ | $L_{k,l}$ |
|-----|-----|-----------|
| 1   | 1   | 0.263400411760216467317382459506461492528060270047150992644... |
| 1   | 2   | 0.61209609608164688109735009030749177445368176853791322041... |
| 1   | 3   | 0.83162016227013645858608159976581294069621021299936900143624... |
| 1   | 4   | 0.978145894979949499502571065792051537852172306539544173835... |
| 2   | 1   | 0.0748317675355275088816314288402705712300662738030777281173... |
| 2   | 2   | 0.202746006799052856234785763142299711085906727618973818502... |
| 2   | 3   | 0.3095671685644667043848886087141852332384931207915051554876... |
| 2   | 4   | 0.3996707719920095241763105708301979615263486781675913515832... |
| 3   | 1   | 0.0197944580087094218491696866714839521363828266446266352... |
| 3   | 2   | 0.056730371990503161507401093742116599202154721447174575277... |
| 3   | 3   | 0.09078044492621077365939579101033447804508695808347... |
| 3   | 4   | 0.122240568076260947821379734978191934659685467095976821370... |

Substitution of inverse $k$-almost primes for $x$ in (60) followed by summation defines constants $L_{k,l}$ which generalize (4),

$$L_{k,l} = \sum_{\Omega(n)=k} \left[ \frac{1}{n} + \left(1 - \frac{1}{n}\right) \ln \left(1 - \frac{1}{n}\right) \right]$$

(62)

placed in Table 14. At $l = 1$, partial fraction decomposition of $1/[s(s+1)]$ in (62) implies that $L_{1,1}$ plus the value at $u = 1$ in Table 2 equals the value in (5). Series of the form \( \sum_{s=1}^{\infty} P_s(s+l)/\prod_{i=0}^{l} (s+i) \) are quickly accessible by subtracting the $\tau$-dependent terms of (62) with the aid of Table 4. Varieties with non-contiguous factors in the denominator can be reduced to the $L_{k,l}$ basis by partial fraction synthesis: examples of this approach with one missing factor $s+2$ or two missing factors $s$ and $s+2$ on the left hand side are

$$\frac{1}{s(s+1)(s+3)} = \frac{1}{s(s+1)(s+2)} - \frac{1}{s(s+1)(s+2)(s+3)}$$

(63)

and

$$\frac{1}{(s+1)(s+3)} = \frac{1}{s(s+1)} - \frac{3}{s(s+1)(s+2)} + \frac{3}{s(s+1)(s+2)(s+3)}$$

(64)

Remark 11. The case of $x = 1/2$ in (54) is decomposable in $k$-almost primes as well [17, 9].

$$\log 2 = \sum_{n=1}^{\infty} \frac{1}{n2^n} = \frac{1}{2} + \sum_{k=1}^{\infty} \sum_{\Omega(n)=k} \frac{1}{n2^n}.$$
The dominating components are

\[
\sum_{n=1}^{\infty} \frac{1}{n^{2k}} 
\]

\[\quad \Omega(n) = k \implies 0 \quad (k = 1)\]
\[\quad = 0.17408707176979362471993316621554442658749500081033068401 \ldots (k = 1)\]
\[\quad = 0.185502662799497056892654852882047774301689318692751270328(-1) \ldots (k = 2)\]
\[\quad = 0.5088863534699730953864506326222795951839970876499567910(-3) \ldots (k = 3)\]
\[\quad = 0.956158270230416532955864299570858933417801804520522813509(-6) \ldots (k = 4)\]

These sums generated from \(|x| < 1\) converge rapidly on their own and do not need the acceleration strategies outlined in the first sections.

**Remark 12.** The limits \(x \to 1\) in (57)–(60) generalize [34, (p. 42)]

\[
\frac{1}{k \cdot k!} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2) \ldots (n+k)}
\]

Restricted summation over \(k\)-almost primes in the spirit of (41) would split these into another basic type of constants.

5. **Summary**

Almost-prime zeta functions have been defined by restriction of the summation of the standard definition of zeta functions to \(k\)-almost primes. Their values can be bootstrapped from a multinomial overlay of the values of the ordinary prime zeta functions. Efficient schemes to compute the latter have been employed to calculate series summed over \(k\)-almost primes of some basic inverse polynomials to high accuracy.

**References**

1. Milton Abramowitz and Irene A. Stegun (eds.), *Handbook of mathematical functions*, 9th ed., Dover Publications, New York, 1972. MR 0167642 (29 #4914)
2. Takashi Aoki and Yasuo Ohno, *Multiple zeta values and connection formulas of Gauss’s hypergeometric functions*, Recent Trends in Microlocal Analysis 1412 (2005), 144–150. MR 2138027 (2005m:11165)
3. Tom M. Apostol, *Formulas for higher derivatives of the Riemann Zeta Function*, Math. Comput. 44 (1985), no. 169, 223–232. MR 0771044 (86c:11063)
4. Eric Bach, *The complexity of number-theoretic constants*, Inform. Proc. Lett. 62 (1997), no. 3, 145–152. MR 1453698 (98g:11148)
5. M. K. Bennett and G. Birkhoff, *Two families of Newman lattices*, Alg. Univ. 32 (1994), no. 1, 115–144. MR 1287019 (95m:06022)
6. C. Berge, *Principles of combinatorics*, Mathematics in Science and Engineering, vol. 72, Academic Press, New York, London, 1971. MR 0270922 (42 #5805)
7. Bruce C. Berndt and Ronald J. Evans, *Chapter 7 of Ramanujan’s second notebook*, Proc. Indian Acad. Sci. (Math. Sci.) 92 (1983), no. 2, 67–96. MR 0755124 (86d:11020)
8. David M. Bradley, *Partition identities for the multiple zeta function*, Zeta Functions, Topology and Quantum Physics (T. Aoki, S. Kanemitsu, M. Nakahara, and Y. Ohno, eds.), Springer Series: Developments in Mathematics, vol. 14, Springer, 2005, pp. 19–29. MR 2179270 (2006f:11105)
9. Marc Chamberland, *Binary BBP-formulae for logarithms and generalized Gaussian-Mersenne primes*, J. Int. Seq. 6 (2003), 03.3.7. MR 2044407 (2005a:11201)
10. Tsz Ho Chan, *Lower order terms of the second moment of \(S(t)\)*, Acta Arith. 234 (2006), no. 4, 313–333. MR 2202247 (2007f:11004)
11. Howard Cheng, Guillaume Hanrot, Emmanuel Thomé, Eugene Zima, and Paul Zimmermann, Time- and space-efficient evaluation of some hypergeometric constants, arXiv:cs/0701151 (2007).
12. Mark W. Coffey, Series of zeta values, the Stieltjes constants, and a sum $S_{n}(n)$, arXiv:math-ph/0706.0345 (2007).
13. Henri Cohen, High precision computation of Hardy-Littlewood constants, 1998, http://www.math.u-bordeaux.fr/~cohen/hardyLw.dvi.
14. Arthur Erdélyi, Wilhelm Magnus, Fritz Oberhettinger, and Francesco G. Tricomi (eds.), Higher transcendental functions, vol. 3, McGraw-Hill, New York, London, 1953. MR 0066496 (16,586c)
15. Steven Finch and Pascal Sebah, Squares and cubes modulo $n$, arXiv:math/0604465 (2006).
16. Carl-Erik Fröberg, On the prime zeta function, BIT 8 (1968), no. 3, 187–202. MR 0236123 (38 #4421)
17. Xavier Gourdon and Pascal Sebah, The logarithmic constant: log 2, 2008, http://numbers.computation.free.fr/Constants/Log2/log2.html.
18. I. Gradstein and I. Ryzhik, Summen-, Produkt- und Integraltafeln, 1st ed., Harri Deutsch, Thun, 1981. MR 0671418 (83i:00012)
19. Jacques Grah, Comportement moyen du cardinal de certains ensembles de facteurs premiers, Monatsh. Math. 118 (1994), no. 1–2, 91–109. MR 1289851 (95e:11102)
20. Frank Harary and Edgar M. Palmer, Graphical enumeration, Academic Press, New York, London, 1973. MR 0357214 (50 #9682)
21. A. Knopfmacher and M. E. Mays, Dirichlet generating functions and factorization identities, Congr. Numer. 173 (2005), 117–127. MR 2193939 (2006m:11005)
22. K. S. Kölbig, Table errata 617, Math. Comp. 64 (1995), no. 209, 449–460. MR 1270626
23. Rick Kreminski, Newton-Cotes integration for approximating Stieltjes (generalized Euler) constants, Math. Comp. 72 (2003), no. 243, 1379–1397. MR 1972742 (2004a:11140)
24. Derrick Henry Lehmer, Interesting series involving the central binomial coefficient, Amer. Math. Monthly 92 (1985), no. 7, 449–457, E: [26]. MR 0801217 (87c:40002)
25. Richard J. Mathar, Twenty digits of some integrals of the prime zeta function, arXiv:0811.4739 [math.NT] (2008).
26. , Corrigenda to “interesting series involving the central binomial coefficient” [Am. Math. Monthly vol 92 (1985)], arXiv:0905.0215 [math.CA] (2009).
27. C. W. Merrifield, The sums of the series of the reciprocals of the prime numbers and of their powers, Proc. Roy. Soc. Lond. 33 (1881), 4–10.
28. Simon Plouffe, Plouffe’s inverter, 1999, http://pi.lacim.uqam.ca/eng/.
29. G. B. Price, Distributions derived from the multinomial expansion, Amer. Math. Monthly 53 (1946), no. 2, 59–74. MR 0014607 (7,309i)
30. Pascal Sebah and Xavier Gourdon, Constants from number theory, 2001, http://numbers.computation.free.fr/Constants/const.html.
31. , Collection of series for $\pi$, 2004, http://numbers.computation.free.fr/Constants/Pi/piSeries.html.
32. Neil J. A. Sloane, The On-Line Encyclopedia Of Integer Sequences, Notices Am. Math. Soc. 50 (2003), no. 8, 912–915, http://www.research.att.com/~njas/sequences/. MR 1992789 (2004f:11151)
33. D. Suryanarayana and V. Siva Rama Prasad, The number of $k$-ary, $(k+1)$-free divisors of an integer, J. Reine Angew. Math. 276 (1975), 15–35. MR 0376574 (51 #12749)
34. Daniel Zwillinger (ed.), CRC standard mathematical tables and formulae, 31 ed., Chapman & Hall/CRC, Boca Raton, FL, 2003, E: the lower limit in eqs. (11)–(14) on page 42 ought be $k = 1$, not $k = 0$. URL: http://www.strw.leidenuniv.nl/~mathar
E-mail address: mathar@strw.leidenuniv.nl

LEIDEN OBSERVATORY, LEIDEN UNIVERSITY, P.O. BOX 9513, 2300 RA LEIDEN, THE NETHERLANDS.