A Study on the Fundamental Unit of Certain Real Quadratic Number Fields

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Abstract In this paper, we consider the certain types of real quadratic fields \( \mathbb{Q}(\sqrt{d}) \) where \( d \) is a square free positive integer. We obtain new parametric representation of the fundamental unit \( \epsilon_d \) for such types of fields. Also, we get a fix on Yokoi’s invariants as well as class numbers and support all results with tables.

Keywords: quadratic fields, continued fraction expansions, class numbers, fundamental units, Yokoi’s invariants

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1. Introduction

Quadratic fields have many applications to different fields of mathematics which contain algebraic number theory, algebraic geometry, algebra, cryptography, and also other scientific fields like computer science. It is also well known that the fundamental units play an important role in studying the class number problem, unit group, Pell equations, cryptography, network security and even computer science.

Recently, in [1], Benamar and his co-authors worked on a type of special monic and non square free polynomials related with fixed period continued fraction expansion of square root of rational integers. In [2], Clemens with collaborators proved explicit continued fractions with almost periodic or almost symmetric patterns in their partial quotients, and infinite series whose terms satisfy certain recurrence relations using Newton’s method.

Tomita and Kawamoto [5] constructed an infinite family of real quadratic fields with large even period of minimal fundamental unit with positive norm as well as several invariants using Newton’s method. Halter-Koch [4] studied on a construction of infinite families of real quadratic number fields where \( d \) is purely periodic in the continued fraction expansion and the denominator of its modular automorphism is equal to fundamental unit \( \epsilon_d \) of \( \mathbb{Q}(\sqrt{d}) \).

Also, Yokoi’s invariants, which were defined by H. Yokoi, are determined by the coefficient of fundamental unit \( \epsilon_d \) in the case of \( d \equiv 1(\text{mod}4) \) or \( d \equiv 2,3(\text{mod}4) \), \( \ell(d) \) is the period length of continued fraction expansion.

For the set \( I(d) \) of all quadratic irrational numbers in \( \mathbb{Q}(\sqrt{d}) \), we say that \( \alpha \) in \( I(d) \) is reduced if \( \alpha > 1, -1 < \alpha’ < 0 \) ( \( \alpha’ \) is the conjugate of \( \alpha \) with respect to \( \mathbb{Q} \)), and denote by \( R(d) \) the set of all reduced quadratic irrational numbers in \( I(d) \). Then, it is well known that any number \( \alpha \) in \( R(d) \) is purely periodic in the continued fraction expansion and the denominator of its modular automorphism is equal to fundamental unit \( \epsilon_d \) of \( \mathbb{Q}(\sqrt{d}) \).

The fundamental unit \( \epsilon_d = (t_d + u_d \sqrt{d})/2 > 1 \) of the ring of algebraic integers in a real quadratic number field \( \mathbb{Q}(\sqrt{d}) \) is a generator of the group of units. Furthermore, integral basis element of algebraic integer’s ring in real quadratic fields is determined by either

\[
w_d = \sqrt{d} = [a_0; a_1, a_2, \ldots, a_{\ell(d)-1}, 2a_0]
\]

in the case of \( d \equiv 2,3(\text{mod}4) \) or

\[
w_d = \frac{1+\sqrt{d}}{2} = [a_0; a_1, a_2, \ldots, a_{\ell(d)-1}, 2a_0-1]
\]

in the case of \( d \equiv 1(\text{mod}4) \), \( \ell(d) \) is the period length of continued fraction expansion.

The aim of this paper is to classify some types of \( \mathbb{Q}(\sqrt{d}) \) real quadratic number fields where \( d \) is a positive integer. We obtain new parametric representation of the fundamental unit \( \epsilon_d \) for such types of fields. Also, we get a fix on Yokoi’s invariants as well as class numbers and support all results with tables.

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square free integer. Such real quadratic fields include the continued fraction expansion of the integral basis element $w_d$ which has got partial constant elements are same and written as nines (except the last digit of the period). The representation of fundamental unit is determined for such types of real quadratic fields using the parametrization of positive square free integers $d$ (not only in the case of $d \equiv 2,3 (\text{mod} 4)$ but also in the case of $d \equiv 1 (\text{mod} 4)$).

Also, the present paper deals with computing Yokoi’s invariants $m_d$ and $n_d$ as well as class numbers. Using the practical way, the results obtained in this paper are supported by numerical tables.

2. Prelimineries

In this section we also give some fundamental concepts for the proof of our main theorems defined in the next section.

**Definition 2.1.** $[K_i]$ is called as a sequence defined by the recurrence relation

$$K_i = 9K_{i-1} + K_{i-2}$$

with the initial conditions $K_0 = 0$ and $K_1 = 1$ for $i \geq 2$.

**Lemma 2.2.** Let $d$ be a square free positive integer such that $d$ congruent to $1$ modulo $4$. If we put $w_d = \frac{1 + \sqrt{d}}{2}$, $a_0 = \lfloor w_d \rfloor$ into the $w_R = (a_0 - 1) + w_d$, then $w_d \notin R(d)$ but $w_R \in R(d)$ holds. Moreover, for the period $l = \ell(d)$ of $w_R$, we get

$$w_R = \left[2a_0-1,a_1,\ldots,a_{l-1}\right]$$

and $w_d = \left[a_0,a_1,\ldots,a_{l-1}-2a_0-1\right]$.

Let

$$w_R = \frac{(P_l w_R + P_{l-1})}{(Q_l w_R + Q_{l-1})} = \left[\frac{2a_0-1,a_1,\ldots,a_{l-1},w_R}{a_d} \right]$$

be a modular automorphism of $w_R$, then the fundamental unit $\varepsilon_d$ of $\mathbb{Q}(\sqrt{d})$ is given by the formulae:

$$\varepsilon_d = \frac{t_d + u_d \sqrt{d}}{2},$$

$$t_d = (2a_0 - 1)Q_{l-1} + 2Q_l, u_d = Q_l$$

where $Q_i$ is determined by $Q_0 = 0$, $Q_1 = 1$ and $Q_{i+1} = a_iQ_i + Q_{i-1}$, $i \geq 1$.

**Proof.** Proof is omitted in [15].

**Lemma 2.3.** Let $d$ be a square free positive integer congruent to $2,3$ modulo $4$. If we put $w_d = \sqrt{d}$, $a_0 = \lfloor \sqrt{d} \rfloor$ into the $w_R = a_0 + w_d$, then we get $w_d \notin R(d)$, but $w_R \in R(d)$ holds.

Furthermore, for the period $l = l(d)$ of $w_R$, we have $w_R = \left[2a_0,a_1,a_2,\ldots,a_{l(d)}\right]$ and $w_d = \left[a_0,a_1,a_2,\ldots,a_{l(d)-1},a_{l(d)}\right]$.

Besides, let

$$w_R = \frac{w_R P_l + P_{l-1}}{w_R Q_l + Q_{l-1}} = \left[2a_0,a_1,a_2,\ldots,a_{l(d)-1},w_R\right]$$

be a modular automorphism of $w_R$. Then the fundamental unit $\varepsilon_d$ of $\mathbb{Q}(\sqrt{d})$ is given by the following formula:

$$\varepsilon_d = \frac{t_d + u_d \sqrt{d}}{2} = \left(1 + \frac{\sqrt{d}}{2}\right)Q_{l(d)} + Q_{l(d)-1}$$

$$t_d = 2a_0Q_{l(d)} + 2Q_{l(d)-1} \text{ and } u_d = 2Q_{l(d)}$$

where $Q_i$ is determined by $Q_0 = 0$, $Q_1 = 1$ and $Q_{i+1} = a_iQ_i + Q_{i-1}$ for $i \geq 1$.

**Proof.** Proof can be obtained in a similar way as the proof of Lemma 2.2.

3. Theorems and Results

**Theorem 3.1.** Let $d$ be square free positive integer and $\ell \geq 2$ be a positive integer.

(1) We suppose that $d = (2\xi K_\ell + 9)^2 + 8\xi K_{\ell-1} + 4$ where $\xi > 0$ is a positive integer. In this case, we obtain that $d \equiv 1 (\text{mod} 4)$ and

$$w_d = \left[5 + \frac{\xi K_\ell}{2}, \frac{9}{2}, \ldots, \frac{9}{2}, \xi K_\ell + \frac{9}{2}\right]$$

with $\ell = \ell(d)$. Moreover, we get

$$t_d = 2\xi K_\ell^2 + 9K_{\ell+1} + 2K_{\ell-1} \text{ and } u_d = K_{\ell+1}$$

for $\varepsilon_d = \frac{t_d + u_d \sqrt{d}}{2}$.

(2) In the case of $d \equiv 0 (\text{mod} 3)$, if we assume that $d = (9 + \xi K_\ell)^2 + 4\xi K_{\ell-1} + 4$ for $\xi > 0$ odd positive integer, then $d \equiv 1 (\text{mod} 4)$ and

$$w_d = \left[5 + \frac{\xi K_\ell}{2}, \frac{9}{2}, \ldots, \frac{9}{2}, \xi K_\ell + \frac{9}{2}\right].$$

Also, in this case

$$t_d = \xi K_\ell^2 + 9K_{\ell+1} + 2K_{\ell-1} \text{ and } u_d = K_{\ell+1}$$

hold for $\varepsilon_d = \frac{t_d + u_d \sqrt{d}}{2}$.

**Remark 1.** It is clear that $K_\ell$ is odd number if $\ell \equiv 0 (\text{mod} 3)$. In the case of (2), $\frac{\xi K_\ell}{2}$ is not integer if we substitute $\xi$ odd positive numbers into the parametrization of $d$ for $\ell \equiv 0 (\text{mod} 3)$. So, we have to put a condition as $\ell$ is divided by 3 in the case of (2). Also, if we choose $\xi$ is even integer, the parametrization of $d$ coincides with the case of (1). That’s why, we have to consider $\ell \equiv 0 (\text{mod} 3)$ and $\xi > 0$ odd positive integer in the case of (2).

**Proof.** (1) For any $\ell \geq 2$ and $\xi > 0$ positive integer, $d \equiv 1 (\text{mod} 4)$ holds since $(2\xi K_\ell + 9)$ is odd integer.

From Lemma 2.2, we know that $w_d = \frac{1 + \sqrt{d}}{2}$, $a_0 = \lfloor w_d \rfloor$ and $w_R = (a_0 - 1) + w_d$.

By using these equations, we obtain
holds for

By using Lemma 2.2 and the parametrization of $d$

By substituting this equivalence into the parametrization of $d$, we have

Using Definition 2.1 and rearranging the above equality, we obtain

This implies that $w_R = \left(9 + 2\xi \ell \right) + \frac{\sqrt{d}}{2}$ since $w_R > 0$. If we consider Lemma 2.2, we get

and $\ell = \ell(d)$.

Now, we have to determine $e_d$, $t_d$ and $u_d$ using Lemma 2.2 again. It is clear that $Q_1 = K_{\ell}$ by induction for all $i \geq 0$.

If we substitute the values of sequence into the coefficients of fundamental unit

holds for $e_d = \frac{t_d + u_d\sqrt{d}}{2}$.

(2) In the case of $\ell \equiv 0 \pmod{3}$, we get $K_{\ell} \equiv 0 \pmod{2}$. By substituting this equivalence into the parametrization of $d$, we have $d \equiv 1 \pmod{2}$ for $\ell > 0$ positive odd integer. By using Lemma 2.2 and the parametrization of $d = (9 + \xi \ell)^2 + 4\xi K_{\ell-1} + 4$, we have $w_R = (d_0 - 1) + w_d$. Then, we have

Then, we have

Rearranging and using Definition 2.1 into the above equality, we obtain

This implies that $w_R = \left(9 + \xi \ell \right) + \frac{\sqrt{d}}{2}$ since $w_R > 0$. If we consider Lemma 2.2, we get

and $\ell = \ell(d)$.

Using $Q_1 = K_{\ell}$ for all $i \geq 0$, we obtain the coefficients of fundamental unit $t_d = \xi K_{\ell} + 9K_{\ell} + 2K_{\ell-1}$ and $u_d = K_{\ell}$ for $e_d = \frac{t_d + u_d\sqrt{d}}{2}$.

Proof. $m_d$ is defined as $m_d = \left[\frac{u_d}{t_d}\right]$ by Yokoi. In the case of (1) if we substitute $t_d$ and $u_d$ into the $m_d$, then we obtain,

So, we get $m_d = 0$ since $t_d > u_d^2$ for $\xi > 0$ positive odd integer.

In a similar way, we obtain

since $t_d > u_d^2$ for $\xi > 0$ positive odd integer in the case of (2).

Corollary 3.3. Let $d$ be the square free positive integer corresponding to $\mathbb{Q}\left(\sqrt{d}\right)$ holding (1) in the Theorem 3.1. We tabulate the Table 3.1, where fundamental unit is $w_d$ and Yokoi’s invariant is $n_d$ for $\xi = 1.2$ and $2 \leq \ell(d) \leq 10$.

Proof. This Corollary is obtained from Theorem 3.1 by taking $\xi = 1$ or 2 in the case of (1). $n_d$ is defined as $n_d = \left[\frac{t_d}{u_d}\right]$. If we substitute $t_d$ and $u_d$ into the $n_d$, then we get

for $\xi = 1$. Also, we get $n_d = 3$ for $\ell = 2$. Since $K_{\ell}$ is increasing sequence, we obtain

for $\ell \geq 3$. So, we have $n_d = 2$ for $\ell \geq 3$. Besides, in the case of $\xi = 2$, we get $n_d = 5$ for $\ell = 2$ as well as $n_d = 4$ for $\ell \geq 3$ by using similar way. The proof of Corollary 3.3 is completed.
Table 3.1. Square-free positive integers d with $2 \leq \ell(d) \leq 10$ (In this table, we rule out $\ell(d) = 4$ for $\xi = 2$ since d is not a square free positive integer.)

| d     | $\xi$ | $\ell(d)$ | n_d  | w_d            | $\varepsilon_d$ |
|-------|-------|-----------|------|----------------|-----------------|
| 741   | 1     | 2         | 3    | [14, 9, 27]    | (245 + 9v741)/2 |
| 30005 | 1     | 3         | 2    | [97, 9, 173]   | (14204 + 8320005)/2 |
| 2259669 | 1  | 4         | 2    | [752, 9, 9, 1503] | (1122905 + 747v2259669)/2 |
| 185483141 | 1     | 5         | 2    | [6810, 9, 132679] | (92678789 + 6805v185483141)/2 |
| 15374318493 | 1     | 6         | 2    | [61997, 9, 123993] | (7606587666 + 61992v15374318493)/2 |
| 1275714271565 | 1     | 7         | 2    | [564738, 9, 112945] | (57286192159 + 564738v1275714271565)/2 |
| 10586737368837 | 1     | 8         | 2    | [5144594, 9, 10289187] | (5923639386979 + 10586737368837)/2 |
| 8785702299880645 | 1     | 9         | 2    | [46866039, 9, 9, 93732077] | (493250718679 + 46866039v8785702299880645)/2 |
| 729107296000012677 | 1     | 10        | 2    | [426938900, 9, 9, 853877799] | (426398095v729107296000012677)/2 |
| 2045  | 2     | 2         | 5    | [23, 9, 45]    | (407 + 9v2045)/2  |
| 113717 | 2     | 3         | 4    | [169, 9, 337]  | (27652 + 8320005)/2  |
| 741430397 | 2     | 5         | 4    | [13615, 9, 9, 27299] | (185294839 + 6805v741430397)/2 |
| 6149204113 | 2     | 6         | 4    | [1123989, 9, 9, 247977] | (15372603794 + 701v6149204113)/2 |
| 5102815433537 | 2     | 7         | 4    | [1192471, 9, 9, 2258941] | (1275968515737 + 564733v5102815433537)/2 |
| 423469115108957 | 2     | 8         | 4    | [10289183, 9, 9, 20578365] | (105483271346451 + 564733v423469115108957)/2 |
| 35142805742894453 | 2     | 9         | 4    | [93732073, 9, 9, 187464145] | (878570103640108 + 46866034v35142805742894453)/2 |
| 2916429152510593469 | 2     | 10        | 4    | [853877799, 9, 9, 1707755589] | (729107296191464623 + 426938900v2916429152510593469)/2 |

Remark 2. In the Table 3.1, using the classical Dirichlet class number formula, we calculate class number $h_d = 2$ for both the real quadratic field $\mathbb{Q}(\sqrt{741})$ and $\mathbb{Q}(\sqrt{2045})$. These fields are also obtained in the Table 2.1 of reference [8]. Additionally, using the classical Dirichlet class number formula and computer calculations, we can see the other class numbers for several real quadratic fields in the Table 3.1 as follows:

| d     | $h_d$ |
|-------|-------|
| 741   | 2     |
| 30005 | 4     |
| 2259669 | 36    |
| 185483141 | 294  |
| 2045  | 2     |
| 113717 | 7     |
| 741430397 | 383  |

Corollary 3.4. Let d be the square free positive integer positive integer corresponding to $\mathbb{Q}(v_d)$ holding (2) in Theorem 3.1. We state the Table 3.2 where fundamental unit is $\varepsilon_d$ , integral basis element is $w_d$ and Yokoi’s invariant is $n_d$ for $\xi = 1,3$ and $3 \leq \ell(d) \leq 12$.

Table 3.2. Square-free positive integers d with $3 \leq \ell(d) \leq 12$ (In this table, we rule out $\ell(d) = 6$ for $\xi = 1$ since d is not a square free positive integer.)

| d     | $\ell$ | $\ell(d)$ | n_d  | w_d            | $\varepsilon_d$ |
|-------|-------|-----------|------|----------------|-----------------|
| 8321  | 1     | 3         | 1    | [46, 9, 9, 91]  | (3748 + 8321)/2  |
| 2196426007056209 | 1     | 9         | 1    | [23433022, 9, 4, 4866643] | (219642557492640 + 46666034v2196426007056209)/2 |
| 1255340433966050791385 | 1     | 12        | 1    | [17715391853, 9, 9, 35407083705] | (12766890309650105858 + 354070837065v1255340433966050791385)/2 |
| 6513 | 3     | 3         | 3    | [128, 9, 9, 255]  | (20928 + 8320005)/2  |
| 34590501889 | 3     | 6         | 3    | [92993, 9, 9, 1859985] | (11529595730 + 6192v34590501889)/2 |
| 19767828878503393 | 3     | 9         | 3    | [70299056, 9, 9, 140598111] | (46866034v19767828878503393)/2  |
| 11298063901774588896481 | 3     | 12        | 3    | [53146175549, 9, 9, 106292351097] | (375682130284679346690v11298063901774588896481)/2 |
Remark 3. We obtain class number as \( h_d = 10 \) for the real quadratic field \( \mathbb{Q}(\sqrt{8321}) \) and \( h_d = 32 \) for the real quadratic field \( \mathbb{Q}(\sqrt{65137}) \) in the Table 3.2 using the classical Dirichlet class number formula and computer calculations. Besides, we can see that other class numbers are too bigger than class number two by using Proposition 4.1 of reference [8].

Theorem 3.5. Let \( d \) be a square free positive integer and \( \ell \geq 2 \) be a positive integer such that \( 3 \nmid \ell \). We assume that the parametrization of \( d \) is
\[
d = \left( \frac{9 + (2\xi + 1)K_\ell}{2} \right)^2 + (2\xi + 1)K_{\ell-1} + 1
\]
where \( \xi \geq 0 \) is a positive integer. Then, following conditions hold:
1. If \( \ell \equiv 1, 2, 4 \pmod{6} \) and \( \xi \) is even positive integer then \( d \equiv 2, 3 \pmod{6} \).
2. If \( \ell \equiv 5 \pmod{6} \) and \( \xi \) is odd positive integer then \( d \equiv 2 \pmod{6} \).

Moreover, in \( \mathbb{Q}(\sqrt{d}) \) real quadratic fields, we obtain
\[
w_d = \left( \frac{9 + (2\xi + 1)K_\ell}{2} \right)^{\ell-1} + 9, 9, \ldots, 9, \left( \frac{2\xi + 1)K_\ell}{2} \right) + 9
\]
with \( \ell = \ell(d) \) for \( d \equiv 2, 3 \pmod{6} \).

Furthermore, we have the fundamental unit \( \varepsilon_d \) and coefficients of fundamental unit \( t_d, u_d \) as follows:
\[
\varepsilon_d = \left( \frac{9 + (2\xi + 1)K_\ell}{2} \right) K_\ell + K_{\ell-1} + 1
\]
\[
t_d = \left( \frac{9 + (2\xi + 1)K_\ell}{2} \right) K_\ell + 2K_{\ell-1}
\]

Proof. If we choose \( \ell \equiv 0, 3 \pmod{6} \) we obtain that \( d \) is not integer because of the parametrization of
\[
d = \left( \frac{9 + (2\xi + 1)K_\ell}{2} \right)^2 + (2\xi + 1)K_{\ell-1} + 1
\]
So, we have to consider that \( 3 \nmid \ell \) and \( \ell \geq 2 \) in order to get \( d \in \mathbb{Z}_+ \).

1. If we suppose that \( \ell \equiv 1, 2 \pmod{6} \), then \( K_\ell \equiv 1 \pmod{4} \). Also, either \( K_{\ell-1} \equiv 0 \pmod{4} \) or \( K_{\ell-1} \equiv 1 \pmod{4} \) hold. By substituting these values into parametrization of \( d \) by considering \( \xi \) is even positive integer, we obtain \( d \equiv 2, 3 \pmod{6} \). Moreover, if \( \ell \equiv 0 \pmod{6} \) and \( \xi \) is even positive integer, then \( K_\ell \equiv 3 \pmod{6} \) and \( K_{\ell-1} \equiv 2 \pmod{4} \). By substituting these values into parametrization of \( d \), then we get \( d \equiv 3 \pmod{4} \).

2. If \( \ell \equiv 5 \pmod{6} \) and \( \xi \) is odd positive integer then we get \( K_\ell \equiv 1 \pmod{4} \) and \( K_{\ell-1} \equiv 3 \pmod{6} \). Substituting these values into parametrization of \( d \) and rearranging, we have \( d \equiv 2 \pmod{4} \).

By Lemma 2.3 we get
\[
w_d = \left( \frac{9 + (2\xi + 1)K_\ell}{2} \right)^{\ell-1} + 9, 9, \ldots, 9, (2\xi + 1)K_{\ell} + 9
\]
\[
\Rightarrow w_d = (9 + (2\xi + 1)K_\ell) + \frac{1}{9} + \frac{1}{9 + \frac{1}{w_d}}
\]
So, we have
\[
w_d = (9 + (2\xi + 1)K_\ell) + \frac{1}{9} + \ldots + \frac{1}{9 + \frac{1}{w_d}}
\]
By Lemma 2.3 we obtain
\[
w_d = (9 + (2\xi + 1)K_\ell) + K_{\ell-1}w_d + K_{\ell-2}
\]
Using Definition 2.1 and put \( 9K_\ell + K_{\ell-1} = K_{\ell+1} \) equation into the above equality, we have
\[
w_d^2 - (9 + (2\xi + 1)K_\ell)w_d - (1 + (2\xi + 1)K_{\ell-1}) = 0.
\]
This implies that \( w_d = \left( \frac{9 + (2\xi + 1)K_\ell}{2} \right) + \sqrt{d} \) since \( w_d > 0 \).

Let us consider Lemma 2.3, then we obtain
\[
w_d = \sqrt{d} \left( \frac{9 + (2\xi + 1)K_\ell}{2} \right) + 9, 9, \ldots, 9, (2\xi + 1)K_{\ell}
\]
and \( \ell = \ell(d) \) hold.

Now, we can determine \( \varepsilon_d, t_d, u_d \) using Lemma 2.3 as follows:
\[
\varepsilon_d \equiv \left( \frac{9 + (2\xi + 1)K_\ell}{2} \right) K_\ell + K_{\ell-1} + K_{\ell+1}
\]
\[
t_d = (2\xi + 1)K_\ell + 9K_\ell + 2K_{\ell-1}
\]
and \( u_d = 2K_{\ell} \) using the \( Q_\ell = K_\ell \) by induction for \( \forall \ell \geq 0 \).

Remark 4. We should say that the present paper has got the most general results for such type real quadratic fields. Moreover, we can obtain infinitely many values of \( d \) which correspond to new real quadratic fields \( \mathbb{Q}(\sqrt{d}) \) by using the results.

Corollary 3.6. Let \( d \) be square free positive integer and \( \ell \geq 2 \) be a positive integer satisfying that \( \ell \not\equiv 5 \pmod{6} \). Suppose that the parametrization of \( d \) is
\[
d = \left( \frac{9 + K_\ell}{2} \right)^2 + K_{\ell-1} + 1
\]
Then ,we get \( d \equiv 2, 3 \pmod{6} \) and
\[
w_d = \left( \frac{9 + K_\ell}{2} \right) + 9, 9, \ldots, 9, \frac{K_{\ell}}{2}
\]
with \( \ell = \ell(d) \). Additionally, we get the fundamental unit \( \varepsilon_d \), coefficients of fundamental unit \( t_d, u_d \) and Yokoishi’s invariant \( m_d \) as follows:
\[
\varepsilon_d \equiv \left( \frac{9 + K_\ell}{2} + \sqrt{d} \right) K_\ell + K_{\ell-1},
\]
\[
t_d = K_\ell^2 + 9K_\ell + K_{\ell-1} + 2K_{\ell}
\]
\[
m_d = \left\{ \begin{array}{ll} 1 & : \ell = 2 \\ 3 & : \ell \geq 4 \end{array} \right.
\]
Proof. If we put \( \xi = 0 \) into Theorem 3.5, then we get
\[ \varepsilon_d = \left( \frac{9 + K_\ell}{2} + \sqrt{d} \right) K_\ell + \varepsilon_{\ell-1}, \]
\[ t_d = K_\ell^2 + 9K_\ell + K_{\ell-1} \text{ and } u_d = 2K_\ell. \]

We have to calculate \( m_d = \left\lfloor \frac{u_d^2}{t_d} \right\rfloor \) defined in the H. Yokoi’s references. If we substitute \( t_d \) and \( u_d \) into the \( m_d \), then we get
\[ m_d = \left\lfloor \frac{u_d^2}{t_d} \right\rfloor = \left\lfloor \frac{4K_\ell^2}{K_\ell^2 + 9K_\ell + K_{\ell-1}} \right\rfloor. \]

We obtain \( m_d = 1 \) for \( \ell = 2 \). For \( \ell \geq 4 \), we get
\[ 4 > \frac{4K_\ell^2}{K_\ell^2 + 9K_\ell + K_{\ell-1}} > 3.951 \]
since \( K_\ell \) is increasing sequence as well as the assumption. Therefore, we obtain \( m_d = \left\lfloor \frac{4K_\ell^2}{K_\ell^2 + 9K_\ell + K_{\ell-1}} \right\rfloor = 3 \) for \( \ell \geq 4 \).

**Remark 5.** The class number is obtained \( h_\ell = 1 \) for the real quadratic field \( \mathbb{Q}(\sqrt{83}) \) and this field was got with same class number by Mollin in the reference [7] too. The real quadratic field \( \mathbb{Q}(\sqrt{142967}) \) has got class number \( h_\ell = 12 \) in the Table 3.3 using the classical Dirichlet class number formula and computer calculations. Furthermore, we cannot calculate easily class numbers for the other real quadratic fields since they are too bigger than class number two by using Proposition 4.1 of reference [8].

**Corollary 3.7.** Let \( d \) be a square free positive integer and \( \ell > 1 \) be a positive integer satisfying that \( \ell \equiv 5(\text{mod}6) \). Suppose that the parametrization of \( d \) is
\[ d = \left( \frac{9 + 3K_\ell}{2} \right)^2 + 3K_{\ell-1} + 1. \]

Then, we have \( d \equiv 2(\text{mod}4) \) and
\[ w_d = \left( \frac{9 + 3K_\ell}{2} ; 9, 9, \ldots, 9 + 3K_\ell \right) \]
\( \ell = \ell(d) \). Furthermore, we obtain following equalities for \( \varepsilon_d, t_d, u_d \) and Yokoi’s invariant \( m_d \).
\[ \varepsilon_d = \left( \frac{9 + 3K_\ell}{2} \right) K_\ell + K_{\ell-1} + K_\ell \sqrt{d} \]
\( t_d = 3K_\ell^2 + 9K_\ell + 2K_{\ell-1} \) and \( u_d = 2K_\ell \).

**Proof.** This Corollary is got by substituting \( \xi = 1 \) into the Theorem 3.5.

We assume that \( \ell \equiv 5(\text{mod}6) \) and \( \ell > 1 \), so we have
\[ \varepsilon_d = \left( \frac{9 + 3K_\ell}{2} \right) K_\ell + K_{\ell-1} + K_\ell \sqrt{d} \]
\( t_d = 3K_\ell^2 + 9K_\ell + 2K_{\ell-1} \) and \( u_d = 2K_\ell \).

If we substitute \( t_d \) and \( u_d \) into the \( m_d \) and rearranged, then we get
\[ m_d = \left\lfloor \frac{u_d^2}{t_d} \right\rfloor = \left\lfloor \frac{4K_\ell^2}{3K_\ell^2 + 9K_\ell + 2K_{\ell-1}} \right\rfloor. \]

So, we have
\[ 2 > \frac{K_\ell^2}{3K_\ell^2 + 9K_\ell + 2K_{\ell-1}} > 1.332 \]
since \( K_\ell \) is increasing sequence. Therefore, we obtain
\[ m_d = \left\lfloor \frac{4K_\ell^2}{3K_\ell^2 + 9K_\ell + 2K_{\ell-1}} \right\rfloor = 1 \] for \( \ell \equiv 5(\text{mod}6) \).

To illustrate, let us consider the following Table 3.4 where fundamental unit is \( \varepsilon_d \), integral basis element is \( w_d \) and Yokoi’s invariant is \( m_d \) for \( 1 < \ell(d) \leq 17 \).

| \( d \)   | \( \ell(d) \) | \( m_d \)     | \( w_d \)     | \( \varepsilon_d \)  |
|----------|---------------|--------------|--------------|----------------------|
| 83       | 2             | 1            | [9, 9, 18]    | 82 + 9\sqrt{83}     |
| 142967   | 4             | 3            | [378;9,9,9,756] | 282448 + 747\sqrt{142967} |
| 79733443634 | 7         | 3            | [282371;9,9,...,9,564742] | 159464283935 + 546733\sqrt{79733443634} |
| 6616722710135 | 8     | 3            | [2572299;9,9,...,9,5144598] | 13233421708484 + 6414589\sqrt{6616722710135} |
| 45569206984046339 | 10  | 3            | [213469452;9,9,...,9,426938904] | 91138412000001574 + 42693890454683648046339 |
| 26044532297817183855458 | 13  | 3            | [1613839184601;9,9,...,9,3227663639362] | 520890645941468265089 + 32276636393526044532297817183855458 |

| \( d \)   | \( \ell(d) \) | \( m_d \)     | \( w_d \)     | \( \varepsilon_d \)  |
|----------|---------------|--------------|--------------|----------------------|
| 104287186 | 5             | 1            | [10212;9,9,9,260424] | 138986814 + 6805\sqrt{104287186} |
| 340352524244133659730 | 11 | 1            | [58339741389;9,9,...,9,11667948276] | 22606914978160245177 + 388031609804703525244133659730 |
| 11117809286740557533656603036090 | 17 | 1            | [333433790830120;9,9,9,6668675816604240] | 74118728378208108745541593003 + 2228891938868077 \sqrt{11117809286740557533656603036090} |
Corollary 3.8 Let \( d \) be square free positive integer and \( \ell \geq 2 \) be a positive integer satisfying that \( \ell \not\equiv 5 \pmod{6}, 3 \not\mid \ell \). We assume that the parametrization of \( d \) is

\[
d = \left( \frac{5K_\ell + 9}{2} \right)^2 + 5K_{\ell - 1} + 1.
\]

Then, we get \( d \equiv 2, 3 \pmod{4} \) and

\[
w_d = \left[ \frac{5K_\ell + 9}{2} ; 9, 9, \ldots, 9, + 5K_\ell \right] \]

with \( \ell = \ell(d) \). Besides, we obtain the fundamental unit \( \varepsilon_d \), coefficients of fundamental unit \( t_d, u_d \) and Yokoi’s invariant \( n_d \) as follows:

\[
\varepsilon_d = \left( \frac{5K_\ell + 9}{2} + \sqrt{d} \right) K_\ell + K_{\ell - 1},
\]

\( t_d = 5K_\ell^2 + 9K_\ell + 2K_{\ell - 1} \) and \( u_d = 2K_\ell, \ n_d = 1 \).

Proof. We have this corollary by using Theorem 3.5 for \( \xi = 2 \). It is just enough to calculate \( n_d \) defined as:

\[
n_d = \left[ \frac{t_d}{u_d} \right].
\]

If we substitute \( t_d \) and \( u_d \) into the \( n_d \), then we get

\[
n_d = \left[ \frac{5K_\ell^2 + 9K_\ell + 2K_{\ell - 1}}{4K_\ell^2} \right] = 1 \quad \text{for} \quad \ell \geq 2.
\]

For numerical example, let us consider the following table where the fundamental unit is \( w_d \) and and Yokoi’s invariant is \( n_d \) for \( 2 \leq \ell(d) \leq 13 \).

### Table 3.5. Numerical example for Corollary 3.7 (In the following table, we rule out \( \ell(d) = 2, 10, 13 \) since \( d \) is not a square free positive integer in these periods)

| \( d \)     | \( \ell(d) \) | \( n_d \) | \( w_d \) | \( \varepsilon_d \)     |
|------------|--------------|----------|---------|-------------------------|
| 3504795    | 4            | 1        | [1872 ; 99, 9, 3744] | 1398466 +747√3504795 |
| 1993284024530 | 7       | 1        | [1411837 ; 99, ..., 9, 2823674] | 797311006513 +564733√1993284024530 |
| 165417593445195 | 8       | 1        | [12861477 ; 99, ..., 9, 25722954] | 66167013662686 + 5144589√165417593445195 |

Remark 6. The class number is \( h_d = 128 \) for the real quadratic field \( \mathbb{Q}(\sqrt{3504795}) \) in the Table 3.5 using the classical Dirichlet class number formula and computer calculations. Furthermore, we can not calculate easily other class numbers since they are too bigger than class number two by using Proposition 4.1 of reference [8].

4. Conclusion

In this paper, we introduced the notion of real quadratic field structures such as continued fraction expansions, fundamental unit and Yokoi invariants where \( d \) is square free positive integer. We established general interesting and significant results for that. Results obtained in this paper provide us a useful and practical method so as to rapidly determine continued fraction expansion of \( w_d \) fundamental unit \( \varepsilon_d \) and and Yokoi invariants \( n_d \) for such real quadratic number fields. There are some authors work on structure of the real quadratic number fields, but the results in this paper are new and more general for such types of real quadratic fields.

Findings in this paper will help the researchers to enhance and promote their studies on quadratic fields to carry out a general framework for their applications in life.

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