Infinite-Dimensional Estabrook-Wahlquist Prolongations for the sine-Gordon Equation

J. D. Finley, III and John K. McIver
Department of Physics and Astronomy
University of New Mexico
Albuquerque, N.M. 87131

We are looking for the universal covering algebra for all symmetries of a given pde, using the sine-Gordon equation as a typical example for a non-evolution equation. For non-evolution equations, Estabrook-Wahlquist prolongation structures for non-local symmetries depend on the choice of a specific sub-ideal, of the contact module, to define the pde. For each inequivalent such choice we determine the most general solution of the prolongation equations, as sub-algebras of the (infinite-dimensional) algebra of all vector fields over the space of non-local variables associated with the pde, in the style of Vinogradov covering spaces. We show explicitly how previously-known prolongation structures, known to lie within the Kac-Moody algebra, $A_{1}^{(1)}$, are special cases of these general solutions, although we are unable to identify the most general solutions with previously-studied algebras.

We show the existence of gauge transformations between prolongation structures, viewed as determining connections over the solution space, and use these to relate (otherwise) distinct algebras. Faithful realizations of the universal algebra allow integral representations of the prolongation structure, opening up interesting connections with algebras of Toeplitz operators over Banach spaces, an area that has only begun to be explored.

I. Universal Algebras for Symmetries

The sine-Gordon equation has been applied to physically interesting problems for over a hundred years; quite a large list of applications is given by Rogers and Shadwick. The generalized form $u_{xy} = f(u)$ has been the subject of numerous investigations into questions of symmetries. Geometrical questions concerning it occasioned the creation of the first Bäcklund transformation, which we view as involving a non-local symmetry, since it relates the dependent variables in two distinct copies of the original equation. A particular approach to determining non-local symmetries was created by Estabrook and Wahlquist, based on Cartan’s method of describing pde’s via the use of differential forms. They refer to the new variables allowed by their non-local symmetries as pseudopotentials, and the associated algebras as prolongation structures. This method is quite popular, and has, of course, been used to study our form of the sine-Gordon equation. Since we believe these prolongation structures are an important tool in the general classification of “equivalent” pde’s, or systems of pde’s, and in the general determination of their solutions, we, along with many
others,\textsuperscript{24−29} are quite interested in the determination of the most general such structures, and have found the sine-Gordon equation useful in one aspect of this.

A k-th order pde may be realized as a subvariety, \( Y \), of a finite jet bundle, \( J^{(k)}(M,N) \), where \( M \) is intended as the space of independent variables and \( N \) the space of dependent variables in the original partial differential equation. The search for generalized symmetries of a pde is most easily performed on the infinite prolongation of that pde, to \( Y_\infty \subset J_\infty \), as described, for instance, by Vinogradov.\textsuperscript{30}, while the further prolongation to \textit{coverings} of \( J_\infty \), by Vinogradov and Krasil’shchik (VK),\textsuperscript{31−32} allows the determination of non-local symmetries as well. Beginning with \( \partial_a \equiv \partial/\partial x^a \) as a choice of basis for tangent vectors over the base manifold, \( M \), the total derivative operators, \( D_a \), are the (standard) lifts of these vector fields to the infinite jet bundle, which still commute: \([D_a, D_b] = 0\). The covering space being searched for will have fibers, \( W \) over \( J_\infty \), where a choice of coordinates, \( \{w^A\} \), may be thought of as allowed pseudopotentials, or non-local variables, for the pde in question. The further prolongation of the total derivative operators into these fibers is accomplished by the addition of some vector fields vertical with respect to the fibers, i.e., \( X_a = \sum X_a^A (\partial/\partial w^A) \), where the coefficients depend on the jet variables as well as the \( \{w^A\} \) themselves. In general these vector fields, \( D_a + X_a \), would no longer commute with each other; however, the requirement that they commute when restricted to the subspace defined by the original pde is exactly the requirement needed in order to ensure that these additional coordinates can in fact act as pseudopotentials for that pde:\textsuperscript{31}

\[
0 = [D_a + X_a, D_b + X_b]_{Y_\infty \times W} \equiv [\overline{D}_a + X_a, \overline{D}_b + X_b] \\
= \{\overline{D}_a(X_b^C) - \overline{D}_b(X_a^C)\} \frac{\partial}{\partial w^C} + [X_a, X_b], \tag{1.1}
\]

where the overbar on the total derivative operators indicates that they have been restricted to \( Y_\infty \). The general solution of these equations for the \( X_a \) will describe all possible coverings associated with this pde. This requirement is of course reminiscent of the usual sort of “zero-curvature” requirements,\textsuperscript{33,34} the distinctions being that we specify first the pde, rather than
the particular algebra within which the $X_a$’s reside, and that we do not specify in advance the dimension of the covering fibers.

As this equation is an identity in the coordinates of the jet bundle, several independent equations will be determined. Some of these equations will explicate some (or all) of the dependence on the jet variables of the $X_a$, but none of their dependence on the fiber variables, $\{w^A\}$, so that each $X_a$ is now given in terms of linear combinations of vector fields $W_\alpha$ with coefficients that depend only on the $\{w^A\}$. The constraints on the $w^A$-dependence is then encoded in the commutators of these $\{W_\alpha\}$ among themselves. The solution to Eq. (1.1) will already have made specific requirements concerning the values for some of the commutators of the $\{W_\alpha\}$. Since these vector fields lie within the entire algebra of vector fields over the fiber, $W$, the universal algebra in which we are interested is the smallest subalgebra that maintains the linear independence of all of the $\{W_\alpha\}$ and faithfully reproduces the values of those of their commutators required by Eq. (1.1). This subalgebra will usually be infinite-dimensional. It determines the general solution to the covering problem, and all faithful realizations of it, by making particular choices for the number of pseudopotentials. We believe this algebra is a universal object for the given pde and other pde’s related to it via some (local or nonlocal) symmetry transformation; i.e., it may be used to characterize related classes of pde’s. The isolation and identification of such algebras is an important part of the process of determining and understanding all the solutions of nonlinear pde’s. Vector-field realizations of this algebra can be used to generate Bäcklund transformations, inverse scattering problems, etc., although such complete realizations of the entire algebra are not usually necessary to obtain these transformations. Infinite-dimensional algebras were originally not considered a profitable direction for study, because of the difficulty with their identification. In their work on the generalized sine-Gordon equation, Dodd and Gibbon already noted that they were left with an infinite-dimensional algebra; however, since they could map it homomorphically to $\mathfrak{sl}(2, \mathbb{C})$, they did not “need” to consider the entire algebra. Beginning with the work by van Eck, and
Estabrook,\textsuperscript{28} on identification of these universal algebras for the KdV equation, such an approach was extended considerably by Hoenselaers and co-workers\textsuperscript{17,20,24}, by Omote\textsuperscript{19}, and by the group at Twente, who seem to have made this a studied art-form.\textsuperscript{15,26}

Two general statements have yet to be made concerning the determination of such algebras. Firstly, the dependence of the $X_a$ on the various jet variables is determined by algebraic equations if the original pde is an evolution equation, i.e., of only first order with respect to one of the variables. On the other hand, for equations of higher order, the determining equations are (linear) partial differential equations, thus substantially increasing the difficulty of finding the general solutions. In the most general cases, these equations have not been solved; nonetheless, a small set of additional assumptions concerning the dependencies on the jet variables, will in fact allow general solutions, as we intend to demonstrate.

The second distinct property associated with non-evolution equations is that they allow distinct subideals of the entire ideal of total derivative operators to remain effective\textsuperscript{2,13} as complete descriptions of the original pde. That is, each such subideal retains sufficient information to completely re-construct the pde; the existence of such subideals correlates with the existence of characteristic vectors for the pde, and, at least when there are only two independent variables, requires that the pde be higher than first order in both variables. These questions were already discussed by Pirani and Shadwick,\textsuperscript{13,27} who showed that the existence of various, distinct subideals of the contact module causes different prolongation algebras to be generated. It is exactly this situation which shows up the significant differences between the methods of EW, relative to those of VK. As described above, VK use the entire, restricted ideal of total derivative operators, $\mathcal{D}_a$, while the essential essence of the EW approach, via differential forms, is to use a (differentially closed) subideal of the entire contact module. The smaller size of this subideal reduces the generality allowed to the $X_a$, and therefore reduces the number of independent vector fields, $\{W_\alpha\}$, without losing any generality in the pde.

A major purpose of this article is to explicate more fully how these differences can lead to major differences in the prolongation algebra generated, and to outline how those algebras are
related. In addition, we will show that distinct choices of the coordinates in the covering fibers, $W$, can obscure the differences in these algebras, especially when one does not investigate the maximal algebras created by the different subideals. These differences in coordinate choices may be viewed as related by gauge transformations between different solutions of Eqs. (1.1). This viewpoint is not as easily seen in the EW approach, as it is in the VK approach, via total derivatives, where the $X_a$ may be thought of as connections over the covering manifold, and gauge transformations are a very natural thing.$^{34,35}$

We have found the generalized form of the sine-Gordon equation to be an ideal vehicle for the understanding of these differences. It is true that one might easily think that all possible interesting questions had been already been answered for this equation. Nonetheless, research on new properties of this equation, and new ways of looking at it, seem to still be surfacing today.$^{36}$ In our case, it is actually the detailed knowledge of the properties of this equation and its solutions that allows us to use it to describe and discuss these previously-unanswered questions concerning the meaning, behavior and use of the EW procedure. Therefore, we will consider the equation,

$$u_{xy} = f(u),$$

(1.2)

where we do not allow the function $f$ to depend on the independent variables, $\{x, y\}$. (There are in fact some interesting aspects that occur when such dependence is allowed; a separate report on a typical and important equation of that type is being made.$^{37}$ ) We will consider all the non-isomorphic ideals of 2-forms that can be used to generate this equation, and will in each case describe the general solution to the determining differential equations and detail those commutation relations for the associated vector fields $W_\alpha$ that the original pde requires. We give sufficiently complete descriptions of these algebras to show why the identification question is particularly difficult for them. We also show how each of them has homomorphisms onto quite large subalgebras which are gauge equivalent.

Having already described a $k$-th order pde as a submanifold, $Y \subset J^k(M, N)$, a solution to the pde is then locally a map $u : U \subseteq M \to N$ such that, $\forall x \in U$, the section $j_x^k u$ lies entirely
within $Y$. From this point of view, Cartan’s approach to pde’s may be said to begin with the ‘contact module,’ $\Omega^k(M, N) \subseteq [J^k(M, N)]_*$, which is used to determine whether a given section of $J^k(M, N)$ is the lift of a function, $u$, on the base manifold. It is generated by the following set of 1-forms:

$$
\begin{align*}
\Omega^k(M, N) : \\
\theta^\mu &= dz^\mu - z_\alpha^\mu dx^\alpha, \\
\theta^\mu_a &= dz_\alpha^\mu - z_\alpha^\mu dx^b, \\
\vdots \\
\theta^\mu_{a_1 a_2 \ldots a_{k-1}} &= dz_{a_1 a_2 \ldots a_{k-1}}^\mu - z_{a_1 a_2 \ldots a_{k-1} a_k}^\mu dx^{a_k},
\end{align*}
$$

(1.3)

where the summation convention has been used with respect to repeated (once upper, once lower) indices, and a choice for a local coordinate chart for a neighborhood in $J^k(M, N)$ is given by $\{x^a, z^\mu, z_\alpha^\mu, z_{a_1 a_2}^\mu, \ldots, z_{a_1 \ldots a_q}^\mu\}$. (The $z_{a_1 \ldots a_q}^\mu$ are symmetric in their subscripts.)

The contact module ‘remembers’ the relation that the various coordinates of the jet bundle are ‘supposed to have’ when one is dealing with an actual function, i.e., when a section has been chosen. Therefore the contact module vanishes when pulled back to $M$ by a function $u : U \subseteq M \rightarrow N$, i.e., $(J^k u)^*(\Omega^k) = 0$. The ideal, $\mathcal{I}$, is the differential closure of the pullback of the contact module to $Y$. It constitutes the Cartan description of the original pde, and is completely equivalent (and dual) to the description via the variety $Y$ and the restricted total derivative operators.

The essence of the EW procedure, for 2 independent variables, is to first determine an effective, proper subideal, $\mathcal{K} \subset \mathcal{I}$, generated by a set of 2-forms over $J^{k-1}$, to describe the pde of interest. Such smaller ideals have long been used to search for potentials for pde’s, and are characterized in Refs. 2 and 16 as “effective.” EW then append to this ideal a set of contact 1-forms for the to-be-determined pseudopotentials, $w^A$:

$$
\omega^A = -dw^A + F^A dx + G^A dy, \quad A = 1, \ldots, N, 
$$

(1.4)

where we will, however, now begin using EW’s symbol $F$ for VK’s $X_x$ and EW’s symbol $G$ for VK’s $X_y$, with the $F^A$ and $G^A$ as labels for their respective coefficients. In the more classical
approach, such a potential would only depend on the jet variables. When one potential is found it may be appended to the original variables, and the enlarged system again searched for potentials. In principle this process generates a (possibly infinite) sequence of potentials. The original contribution of Estabrook and Wahlquist was to notice that one could simply suppose the existence of an as-yet-undiscovered space $W$—the fibers of VK’s covering space—of such potentials and look for them all at once, in which case they were called pseudopotentials.

The EW approach allows the $F^A$ and $G^A$ to depend upon themselves; therefore proper phrasing of the closure question now requires that the $d\omega^A$ be contained within the extended ideal, $\mathcal{K} \oplus \{\omega^1\}$. Labelling the (2-form) generators of $\mathcal{K}$ by $\{\alpha^r | r = 1, \ldots, p\}$, this means we are searching for pseudopotentials $w^A$ that allow the existence of some functions $f^A_r$ and 1-forms $\eta^A_B$ such that

$$dF^A \wedge dx + dG^A \wedge dy = f^A_r \alpha^r + \eta^A_B \wedge \omega^B .$$

(1.5)

Allowing dependence of $F^A$ and $G^B$ on all of the variables of $J^{k-1} \times W$, and comparing coefficients of the various independent 2-forms on both sides of Eq. (1.5), three sorts of information are acquired. Firstly, we determine on which jet variables they are allowed to depend. Secondly, we are able to relate the values of the previously-unknown Lagrange multipliers, $f^A_r$ and $\eta^A_B$, to various derivatives of the $F^A$ and the $G^B$. While the continuation of the prolongation process, toward eventually finding new solutions of the original pde, does not explicitly require that we know the values of these Lagrange multipliers, it is nonetheless quite important that the final choices of $F^A$ and $G^B$ should be such as to maintain non-zero the multipliers, $f^A_r$. After all they retain the information needed by the procedure to “remember” the original ideal, $\mathcal{K}$, i.e., to remember the pde with which the process began. Lastly, but importantly, the coefficients of $dx \wedge dy$ in Eq. (1.5) generate a commutator equation for these vertical vector fields which always has the following general form:

$$[F + \partial_x, G + \partial_y] = \sum_{\sigma} a_{\sigma} \frac{\partial}{\partial z_{\sigma}} F + \sum_{\sigma} b_{\sigma} \frac{\partial}{\partial z_{\sigma}} G ,$$

(1.6)

where the sum is over all coordinates of $Y$ and the coefficients $a_{\sigma}$ and $b_{\sigma}$ depend on the specific system being considered. This equation is in fact identical to Eq. (1.1), and makes explicit the connection between the EW approach to the calculations and the VK approach, that begins with zero-curvature equations similar to those of Zakharov and Shabat.33
II. The Prolongation Problem for the (Generalized) sine-Gordon Equation

We now begin explicit applications of these procedures to our Eq. (1.2), choosing coordinates on the solution space $Y \subset J^2(M, N)$, by eliminating all the “mixed derivatives” of $u$, so that $\mathcal{I}$, the contact module restricted to $Y$, is generated by

$$\bar{\theta}_u \equiv du - p\,dx - q\,dy$$
$$\bar{\theta}_p \equiv dp - r\,dx - f(u)\,dy$$
$$\bar{\theta}_q \equiv dq - f(u)\,dx - t\,dy$$

(2.1)

We recognize 3 inequivalent subideals, $\mathcal{K}_i$, each being effective and differentially closed, with generators as follows:

$$\begin{array}{ccc}
\mathcal{K}_1 & \mathcal{K}_2 & \mathcal{K}_3 \\
\bar{\theta}_u \wedge dy & \bar{\theta}_u \wedge dx & \bar{\theta}_u \wedge dx \\
\bar{\theta}_p \wedge dx & \bar{\theta}_u \wedge dy & \bar{\theta}_u \wedge dy \\
\bar{\theta}_p \wedge dx - \bar{\theta}_q \wedge dy & \bar{\theta}_p \wedge dx & \bar{\theta}_q \wedge dy.
\end{array}$$

(2.2)

In Ref. 2, for example, the authors also distinguish 3 effective subideals, equivalent to our $\mathcal{K}_1$ and $\mathcal{K}_2$. Their third subideal, as they indicate, is simply diffeomorphic to $\mathcal{K}_1$, so that they limit their discussion to these two. They omit our ideal $\mathcal{K}_3$ since it generates all of $\mathcal{I}$ restricted to $Y$, except for the 1-forms. It might also be characterized as generating all 2-forms annihilated on $Y$; one might then be justified in calling it “minimally complete” relative to the EW procedure.38

From the point of view of the “classical” symmetries, over $J^2(M, N)$, or the higher symmetries over $J^\infty$, each ideal has exactly the same content, as determined by the standard methods.39 Nonetheless, they are properly described as inequivalent since they engender distinct E-W coverings, as already noted by Pirani et al.40, although they did not pursue the consequences of that inequivalence very far.
Application of the E-W procedure to any of our $K_i$ gives the E-W commutator equation, (1.6), in the form

$$[F, G] = -p G_u + q F_u + f(u)(F_p - G_q), \quad (2.3a)$$

with $F = F(u, p; w^A)$ and $G = G(u, q; w^A), \quad (2.3b)$

where, as is usual, we have ignored any possible dependence on the independent variables, $\{x, y\}$. Notice explicitly that Eq. (2.3b) indicates that $F_q = 0 = G_p$. In addition to these requirements on $F$ and $G$, the smaller ideals described in Eq. (2.2) cause additional constraints on the functional dependencies of $F$ and $G$:

$$K_1 \quad K_2 \quad K_3 \quad (2.3c)$$

$$F_u = 0 = G_q \quad F_p + G_q = 0 \quad \text{no additional constraints}.$$

As expected, the ideal $K_3$ generates a set of covering equations identical to those of Krasil’shchik. An expansion of $F$ in a Taylor series in the variable $p$, and $G$ in the variable $q$, leads to his equations (1.25). These equations are so general that we have been unable to find the general solution, except when the covering space is restricted to be only one-dimensional.

On the other hand, the additional constraints required by the use of either of the two smaller subideals causes the set of equations to become manageable. The equations we obtain, when solving the covering equations for the set of generators, $K_2$, were first obtained by Shadwick. He gave the general solution for the case of 1-dimensional coverings, and then wrote down the general form of the solution for multi-dimensional coverings. He referred to the algebra so generated as $\alpha_f$, and was apparently only interested in cases where it had a non-trivial, homomorphic mapping to a finite-dimensional algebra, $\alpha$. He considered 4 fairly simple cases for $\alpha$, showing that the set of functions $f(u)$ for which this gave Bäcklund transformations depended on one’s choice of $\alpha$. We will generalize this, determining the form of the general solution to those equations, for arbitrary dimension.

The simplest ideal, that generated by $K_1$, has occurred many times in the literature, as might be expected. Pirani et. al. distinguish it by noting that it possesses a characteristic
vector, \( \partial_q \). As discussed in some detail in Ref. 2, those authors add on the additional constraints that the solution for \( F \) should be linear in \( p \), and indicate that “a solution” is obtained where the Lie algebra describing \( F \) and \( G \) is simply \( \mathfrak{sl}(2, \mathbb{C}) \). On the other hand Hoenselaers\textsuperscript{20} also requires that the solution for \( F \) should be linear in \( p \), but finds a rather more general solution involving the infinite-dimensional Lie algebra \( \mathfrak{sl}(2, \mathbb{C}) \otimes \mathbb{C}[\lambda^{-1}, \lambda] \), the loop algebra over \( \mathfrak{sl}(2, \mathbb{C}) \).

Along the way to determining this solution, Hoenselaers requires, additionally, of the equations that \( G_{uu} + G = 0 \), and that a particular central element should be ignored. As we will in fact obtain the most general solution to the equations, we will indicate how these special cases are contained in them, in Section III. The requirement of linearity was avoided by Dodd and Gibbon, in the Appendix of Ref. 12. They looked for a solution for \( F \) in the form of a finite polynomial in \( p \), and indicated that their structure yielded an infinite-dimensional Lie algebra. However, for reasons unclear to us they only slightly generalized Hoenselaers requirement, to the requirement \( G_{uu} + G = Z \). Our general solution, with infinitely many terms in the description of both \( F \) and \( G \), will also include their case.

Since the requirements of \( \mathcal{K}_1 \) are stronger than those of \( \mathcal{K}_2 \), surely one is embedded within the other. The explicit embedding is not immediate since \( \mathcal{K}_1 \) is quite asymmetric, while \( \mathcal{K}_2 \) is symmetric, from the point of view of the first derivatives. However, after having obtained both general solutions, for \( \mathcal{K}_2 \) and \( \mathcal{K}_1 \), we will describe the complete set of defining relations for those prolongation structures, and show how there is a gauge transformation that transforms a large subset of the \( \mathcal{K}_2 \) structure into a large subset of the one for \( \mathcal{K}_1 \). We will also show why both universal algebras must be infinite-dimensional, even though we cannot explicitly identify them in terms of already-known algebras. From Hoenselaers’ work,\textsuperscript{20,24} we do know that the universal algebras of these three incomplete Lie algebras (of vector fields over the covering fibers) form a sequence: \( \mathfrak{sl}(2, \mathbb{C}) \otimes \mathbb{C}[\lambda^{-1}, \lambda] \equiv \mathfrak{A}_1 \otimes \mathbb{C}[\lambda^{-1}, \lambda] \subseteq \mathfrak{A}_2 \subseteq \mathfrak{A}_3 \).

One last, quite interesting ideal used for the sine-Gordon equation should be mentioned, even though it is in fact not a subideal of the restricted contact module. This is the CC ideal used by Estabrook\textsuperscript{43}, by Harrison\textsuperscript{14}, and also espoused by Hoenselaers\textsuperscript{24}. Such ideals
were chosen by them because of the ease with which they can be manipulated, if they can be found. A CC ideal is composed of two distinct sets of 2-forms, made from some original set of 1-forms \( \{\xi^a\} \) using only constant coefficients, thereby creating the name CC ideal. The first set expresses the closure conditions for the \( \{d\xi^a\} \) in terms of wedge products of the \( \{\xi^a\} \) themselves, while the second set is simply sums of wedge products of the \( \{\xi^a\} \), which may be thought of as expressing various linear dependence conditions between them. (It is of course this last property that causes such ideals not to be subideals of the restricted contact module, since all the generators there are linearly independent.) Estabrook came upon the study of such ideals by realizing that they may be created by beginning with a set of Maurer-Cartan relations for some group and annihilating some of the members “by hand.” On the other hand, if one begins with a specific pde, such as our sine-Gordon equation, one begins with some particular coordinate representations of the 1-forms, choosing them so that the closure conditions may be written with constant coefficients. Once such an ideal is created, these authors then “forget” the original coordinate formulation of the 1-forms \( \xi^a \), and treat the set of closure conditions as a defining ideal for their problem. Typically such an ideal need no longer be a subideal of the contact module restricted back to \( Y \), nor any prolongation of it. For the sine-Gordon equation they choose four 1-forms, and set up their defining relations along the lines followed earlier by Chern and Terng, who interpreted the original geometric visualizations, involving two distinct surfaces of constant negative curvature embedded in \( \mathbb{R}^3 \), in terms of differential geometry.

From our point of view Estabrook’s ideal appears rather like two distinct copies of our subideal \( K_1 \), multiplied with coefficients \( \sin u \) and \( \cos u \), respectively. In addition, since they use the ideal itself to determine suitable coordinates on the manifold, this actually does generate two copies of the sine-Gordon equation, in distinct dependent variables; i.e., this ideal comes pre-disposed toward an auto-Bäcklund transformation. Taking the notion that \( F^A \) and \( G^A \), from Eq. (1.4), should be replaced by some set of \( F^A_a(w) \) such that \( \omega^A = -dw^A + F^A_a \xi^a \), one may again find equations to determine the Lie algebraic structure generated by these \( F^A_a \),
also usually infinite-dimensional. In Ref. 43, Estabrook writes explicitly a realization of the generators and known structure constants for this algebra; however, no one has yet been able to identify that entire Lie algebra or place it in a universal context. (Of course it does have a homomorphism to $\mathfrak{sl}(2, \mathbb{C})$, which generates the usual Bäcklund transform.)

III. The Algebra $\mathfrak{A}_1$ for the sine-Gordon Equation:

Specializing directly to the explicit sine-Gordon equation, we first consider our smallest ideal, described by the $K_1$ part of Eq. (2.3c), which reduces the commutator equation to the form:

$$\mathfrak{A}_1 : \begin{cases} [F, G] = -p G_u + F_p \sin u , \\ F_u = 0 = F_q , G_p = 0 = G_q . \end{cases} \quad (3.1)$$

Taking the second derivative with respect to $u$, of the first of Eqs. (3.1), we acquire $[F, G_{uu}] = -p G_{uuu} - F_p \sin u$. Summing this one with the original eliminates the term containing $\sin u$, so that we may first solve

$$[F, Z] = -p Z_u , \quad \text{with} \quad Z \equiv G + G_{uu} . \quad (3.2)$$

As usual, assuming that $F$ is analytic about the origin, we write

$$F(p) = \sum_{m=0}^{\infty} \frac{p^n}{n!} F_n \implies \begin{cases} [F_n, Z] = 0 , \quad n \neq 1 , \\ [F_1, Z] = -Z_u , \quad n = 1 . \end{cases} \quad (3.3)$$

Since $F_u = 0$, this vector-field valued pde simply describes the ‘flow’ of one vector field along the direction described by another. The adjoint presentation of such a flow is well-known\textsuperscript{44,45}; phrased in notation suitable for our current purposes, we described it in more detail in Ref. 23. From that we infer the existence of a vector field, $Q \in W_s$, and some constraints, such that

$$Z = e^{-u \text{ad} F_1} Q , \quad [F_n, (\text{ad} F_1)^m Q] = 0 , \quad n \neq 1 , m = 0, 1, 2, \ldots \quad (3.4)$$

where the $\text{ad}$-operator is the usual mapping $\text{ad} : \mathcal{A} \to \mathcal{A}$ so that $\{\text{ad} X\}(Y) \equiv [X, Y]$. 

Having this form for \( Z \), we may solve the differential equation for \( G \). The homogeneous portion is straightforward; however, the inhomogeneous term that \( Z \) represents requires writing \( G \) also as a series in \( u \), comparing coefficients of \( u^n \) on both sides of the inhomogeneous equation, and then re-summing the series. The comparison gives the recursion relation

\[
G_{n+2} = -G_n + (-\text{ad} F_1)^n Q,
\]

which has as solution the form

\[
G = G_0 \cos u - G_1 \sin u + \sum_{n=0}^{\infty} \left( \cos u \right)^{(n-2)} (-\text{ad} F_1)^n Q,
\]

where the negative superscripts in parentheses, on the cosine function, are meant to indicate integrals, from 0 to an upper limit of \( u \):

\[
\left( \cos u \right)^{(p-2)} \equiv \int_0^u du_1 \cdots \int_0^{u_{p+1}} du_{i_{p+2}} \cos u_{i_{p+2}} = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n+p+2}}{(2n+p+2)!}.
\]

Integral forms like this are also obtained by Dodd and Gibbon,22 although they appear to have ignored the contributions from a possible lower limit.

Inserting this form back into the original pde, we acquire the additional constraints

\[
[F_{pp}, G_0] = 0, \quad [F_{pp}, G_1] = -F_{ppp}, \quad [F_{pp}, e^{-w \text{ad} F_1} Q] = 0,
\]

from which we obtain an explicit equation for \( F \), giving us the general solution in terms of some six generating vector fields, \( \{F_0, F_1, K, G_0, G_1, Q\} \in W_* \), which elaborate the forms of \( F \) as follows:

\[
F = F_0 + pF_1 + \sum_{n=0}^{\infty} \frac{p^{n+2}}{(n+2)!} (\text{ad} G_1)^n K
\]

\[
= F_0 + pF_1 + \int_0^p ds (p - s) e^{s \text{ad} G_1} K,
\]

(3.7)
in addition to the set of requirements that the pde makes on the commutators of these fields. Defining new quantities

\[
K_n \equiv (\text{ad} G_1)^n K, \quad Q_m \equiv (-\text{ad} F_1)^m Q, \quad (3.8)
\]

we may write the requirements of the pde, on the commutators, in the following display, where the entry inside the table is the commutator of the element labelling the row with the element labelling the column, and the space in the matrix is left blank if the pde makes no requirement on that particular commutator:

\[
\begin{pmatrix}
F_0 & F_1 & K_n & G_0 & G_1 & Q_n \\
F_0 & 0 & 0 & -F_1 & 0 & \\
F_1 & 0 & G_1 & -G_0 + Q - K & -Q_{n+1} & \\
K_m & 0 & 0 & -K_{n+1} & 0 & \\
G_0 & 0 & -G_1 & 0 & 0 & \\
G_1 & F_1 & G_0 + K - Q & K_{n+1} & 0 & \\
Q_m & 0 & Q_{n+1} & 0 & \\
\end{pmatrix}.
\quad (3.9)
\]

One obtains Dodd and Gibbon’s form by insisting that our \(Z\) should be the same as theirs, which requires insisting that our commutator, \(Q_1 = [Q_0, F_1]\), and all higher such commutators, should vanish. On the other hand, their approach allows for the existence of our infinite sequence of vector fields, \(K_n\). Hoenselaers’ approach makes the requirement that \(F\) should be linear in \(p\), so that one must set to zero \(K_0\), and all commutators of it with \(F_1\). Additionally Hoenselaers’ requirements on \(G\), i.e., that \(G_{uu} + G = 0\), require that we set to zero our \(Q_0\) and all repeated commutators with \(F_1\). In our notation, Hoenselaers’ mapping, \(\Psi: \mathfrak{A}_1 \rightarrow A_1 \otimes \mathbb{C}[\lambda^{-1}, \lambda] \subset A_1^{(1)}\) is given by

\[
\Psi: \left\{ \begin{array}{l}
F_0 \rightarrow J_1^{(1)}, F_1 \rightarrow J_3^{(0)}, \\
G_0 \rightarrow J_1^{(-1)}, G_1 \rightarrow J_2^{(-1)},
\end{array} \right.
\quad (3.10)
\]

where the three \(J_i\)’s at level zero, i.e., \(\{J_i^{(0)} \equiv J_i \mid i = 1, 2, 3\}\), form a basis for \(\mathfrak{sl}(2, \mathbb{C})\), with commutators \([J_1, J_2] = -J_3, [J_2, J_3] = J_1, [J_3, J_1] = J_2\). In order for these requirements to be
consistent, we must impose two extra restrictions on our algebra: \( 0 = N = [F_1, [F_1, F_0]] + [F_1, F_0] \), and \([G_0, [G_1, [G_1, G_0]]] = 0 \). The first of these is the statement that the central quantity, \( N \), is to be ignored. This mapping is also pointed out by Omote,\(^{19}\) and is equivalent to that specified by the use of powers of \( \lambda \) in §6.2 of Leznov and Saveliev.\(^{34,46}\)

IV. The algebra \( \mathfrak{A}_2 \)

For \( K_2 \) generated by the elements in Eq. (2.3), we easily see that \( F \) and \( G \) must have the following forms where \( B \) and \( C \) are as-yet-undetermined functions of \( u \), which must, however, satisfy the equations indicated:

\[
\mathfrak{A}_2 : \begin{cases} 
F = \frac{1}{2} p R + B , & G = -\frac{1}{2} q R + C , & R_u = 0 , \\
\left[ \frac{1}{2} R , B \right] = B_u , & \left[ \frac{1}{2} R , C \right] = -C_u , & [B , C] = R f(u) .
\end{cases}
\] (4.1)

We will usually simply refer to the arbitrary function \( f(u) \) for our equation; however, we will shift to the special case \( f(u) = \sin u \) when it becomes necessary. Following other cases, such as, for instance, the Liouville equation, or the Tzitzeica-Dodd-Bullough equation\(^{47}\) would also be possible from our general equations, but we have not carried it through in any detail.

Eqs. (4.1) are the same as those found by Shadwick in Ref. 15, under the notational transformation (from our notation to his), \( B \rightarrow A , R \rightarrow 2B , \) and \( C \rightarrow C \). The vector-field valued pde’s in Eqs. (4.1) are the same sort as discussed in the previous section. Therefore, integration of the two differential equations for \( B \) and \( C \) puts into evidence two new, vertical vector fields, \( E , J \in W_* \), such that

\[
B = e^{+\frac{1}{2} u(ad R) E} , \quad C = e^{-\frac{1}{2} u(ad R) J} .
\] (4.2)

Insertion of these forms into the last of Eqs. (4.1) gives the following commutator equation, which we will solve by treating it as an identity in formal series in the variable \( u \):

\[
[e^{+\frac{1}{2} u(ad R) E} , e^{-\frac{1}{2} u(ad R) J}] = R f(u) .
\] (4.3)
Comparing coefficients of \( u \) gives a countable list of commutation relations to be satisfied, namely

\[
\sum_{m=0}^{k} \left( \frac{1}{2} \right)^k \binom{k}{m} [E_{k-m}, J_m] = c_k R, \quad \forall k = 0, 1, 2, \ldots ,
\]

(4.4a)

where

\[
E_m \equiv (+\text{ad } R)^m E, \quad J_n \equiv (-\text{ad } R)^n J, \quad \forall m, n = 0, 1, 2, \ldots ,
\]

(4.4b)

and the \( c_n \) are simply the constants given by the coefficients in the power series for \( f(u) \):

\[
\text{define } f(u) \equiv \sum_{n=0}^{\infty} \frac{c_n}{n!} u^n, \quad \text{or } c_n \equiv (f(u))^{(n)} \bigg|_{u=0}.
\]

(4.5)

The sum in Eqs. (4.4) simplifies enormously. To see this, we first use the definitions in Eqs. (4.4), the equation for \( k = 0 \), and the Jacobi identity to tell us that

\[
[J, E_1] = [J, [R, E]] = [R, [J, E]] + [E, [R, J]] = -c_0 [R, R] - [E, J_1] = [J_1, E].
\]

At the next level, the same sorts of manipulations yield \([J, E_2] = \cdots = +[J_1, E_1] = \cdots = [J_2, E]\). An induction hypothesis then shows that the value of \([E_{k-m}, J_m]\) is independent of \( m \), when \( 0 \leq m \leq k \):

\[
[E_{k-m}, J_m] = c_k R, \quad \forall m \geq 0, m \leq k.
\]

(4.6)

The original free algebra, generated by \( \{ J, R, E \} \), when divided by the countably infinite set of relations in Eq. (4.6), is still enormous. We have so far been unable to resolve it into an already-studied algebra. Since, algebraically, there are really only three generators—so that it can be said to be finitely generated—it might well be that there are interesting realizations to be found, for instance, among the Kac-Moody algebras. If, however, we desire to maintain distinct the realizations of all the \( \{ E_m, J_n \} \) satisfying Eq. (4.6), which does not immediately appear to be consistent with the usual gradings of such algebras, it may in fact require some other approach to infinite-dimensional algebras. Surely it is true that the restriction to only
one pseudopotential reduces the problem to the well-known homomorphism, with a parameter, \( \lambda \), onto its smallest interesting subalgebra, \( \mathfrak{sl}(2, \mathbb{R}) \), taken with Chevalley basis, \( \{h, e, f\} \) such that \( [h, e] = 2e \), \( [h, f] = -2f \), and \( [e, f] = h \):

\[
\begin{align*}
R & \rightarrow \frac{1}{2}(f - e), \\
J_n & \rightarrow \lambda^{-2}E_n, \\
E_n & \rightarrow \frac{1}{2}\lambda \left\{ \begin{array}{ll}
(-1)^{\frac{n}{2}}h, & n \text{ even}, \\
(-1)^{\frac{n+1}{2}}(e + f), & n \text{ odd}.
\end{array} \right.
\end{align*}
\]

(4.7)

V. Gauge Transformations of the Algebras

Because there appear to be several different avenues to follow at this point, we first turn to the questions of gauge equivalence of these algebras, seen as generated by connections over the covering with fibers \( W \). The (usual) language describing Eq. (1.1) as a \textit{zero-curvature equation} has behind it the notion that the total derivative over \( J^\infty \), namely \( D_a \), when prolonged into the covering space \( J^\infty \otimes W \), should be treated as a covariant derivative,\(^{34,48}\) with the additional quantities \( X_a \) generating the associated connection, since they prevent the commutator of the prolonged total derivatives from vanishing, at least until it is restricted to the variety that describes the (infinite) prolongation of the pde being studied. The role the various \( X_a \) play in these investigations requires that they be elements of the Lie algebra of all vector fields over \( J^\infty \otimes W \), preserving the fibered structure. This makes it clear that \( \Gamma \equiv X_a \, dx^a \) is in fact a Lie-algebra-valued connection 1-form for this covariant derivative, \( \nabla \equiv dx^a \{D_a + X_a\} \); therefore our \( \Gamma \) should transform in the usual way for connections.

The transformations we consider correspond to flows of the covering space generated by particular tangent vector fields, so that we are simply moving along a congruence of curves from one point on the manifold to another. By restricting our attention to vertical vector fields, i.e., those of the form of the \( X_a = X^A_a \partial_{w^A} \), any individual curve, within the congruence, simply runs within a single fiber over its base point, in \( J^\infty \), at which it began. Therefore, different positions along such a congruence of curves, i.e., different values of a parameter along the curves, just correspond to different values of fiber coordinates \( \{w^A\} \) over the same base point. We surely want the structure of our theory to be independent of distinctions such as
this; therefore, we refer to such transformations as **gauge** transformations, i.e., transformations which leave invariant the meaning of the equations presented, only mapping different explicit presentations of the underlying geometry into one another.

We now describe explicitly such a gauge transformation, generated by $R$, say, a vertical vector field defined over some (local) portion of our manifold. The flow of that vector field is a (local) mapping of the manifold into itself, that can be presented via a congruence of curves, described by $\Phi_t \equiv e^{tR} : U \subseteq M \rightarrow M$. As the base point is the same for all points on this curve, the actual coordinates along the curve can be specified by first giving the coordinates, $z_0 \in J^\infty$, for that base point and then integrating the flow equations for the $w^A(z_0, t)$:

$$\frac{dw^A}{dt} = \frac{d}{dt} \Phi_t = \frac{d}{dt} \{tR^A(z_0, w^B)\} \Phi_0 .$$

The induced mapping of the tangent bundle, $(\Phi_t)_* : U_* \rightarrow M_*$, may be presented in the form $X_P \rightarrow e^{t(adR)}X_{\Phi_t(P)}$, for $P \in U \subseteq M$, and $X_P$ an arbitrary tangent vector at that point. Under this flow a covariant derivative operator, $\nabla \in M_* \otimes M^*$, with a Lie-algebra-valued connection 1-form $\Gamma$, would have a transformation law usual for connections:

$$\Gamma_t \equiv e^{t(adR)}\Gamma - d(tR) .$$

In our case, the derivation fills in the steps in the following reasoning:

$$(D'_a + X'_a)_{\Phi_t(P)} = (\nabla'_a)_{\Phi_t(P)} \equiv e^{t(adR)}\{\nabla_aP\} = e^{t(adR)}\{(D_a + X_a)P\}$$

$$=\{D_a - (R)_a\}e^{tR}p + \{e^{t(adR)}X_a\}_{\Phi_t(P)} = \{D_a - (tR)_a + e^{t(adR)}X_a\}_{\Phi_t(P)} ,$$

where the primes indicate the transformed objects, and the subscript $a$ on $(tR)_a$ is only a shorthand for $\{D_a(tR)\}$. As well we have used explicitly the fact that the invariance of the base manifold, $J^\infty$, under these transformations leaves the underlying total derivatives, $D_a$, invariant. Suppressing the manifold points, the result may then be re-stated as simply

$$(\Phi_t)_*(X_a) \equiv X'_a = e^{t(adR)}X_a - \{D_a(tR)\} ,$$

(5.2')
with the same content as Eq. (5.2).

Applying these notions to our current considerations, we must modify our notation slightly, so that we can discuss both prolongations at once. We do this by first changing from the symbols \( \{F, G\} \) to new ones \( \{\mathcal{F}, \mathcal{G}\} \), for this section only, so that we may then further label separately the connection forms \( \{F_2, G_2\} \), as the ones determined by \( \mathfrak{A}_2 \), via Eqs. (4.1-2), and the other connection forms \( \{F_1, G_1\} \), determined by \( \mathfrak{A}_1 \) via Eqs. (3.5,7):

\[
\mathfrak{A}_2 : \begin{cases} 
F_2 &= \frac{1}{2} p \mathbf{R} + e^{+\frac{1}{2} u(\text{ad} \mathbf{R})} \mathbf{E}, \\
G_2 &= -\frac{1}{2} q \mathbf{R} + e^{-\frac{1}{2} u(\text{ad} \mathbf{R})} \mathbf{J}, 
\end{cases}
\]

\[
\mathfrak{A}_1 : \begin{cases} 
F_1 &= F_0 + pF_1 + \sum_{n=0}^{\infty} \frac{p^{n+2}}{(n+2)!} (\text{ad} \mathbf{G}_1)^n \mathbf{K}, \\
G_1 &= G_0 \cos u - G_1 \sin u + \sum_{n=0}^{\infty} (\cos u)^{-n-2} (\text{ad} \mathbf{F}_1)^n \mathbf{Q}. 
\end{cases}
\]

The form of \( F_2 \) immediately suggests a gauge transformation with the group element \( e^{-\frac{1}{2} u \mathbf{R}} \). The result, via Eq. (5.2), is

\[
(\Phi_{-\frac{1}{2} u})_* \{F_2, G_2\} = \left\{ F'_2 = p \mathbf{R} + \mathbf{E}, \ G'_2 = e^{-u(\text{ad} \mathbf{R})} \mathbf{J} \right\}, \tag{5.4}
\]

which resembles the Hoenselaers’ form of the prolongation achieved via our algebra \( \mathfrak{A}_1 \), as discussed in Section III. Comparing this equation for \( F'_2 \) with our general equation for \( F_1 \), we see that the equality of \( F_2 \) and \( F_1 \) requires \( \mathbf{K}_0 = 0 \), along with its repeated commutators. Although one must still also check that the required commutation relations are the same, it turns out that they indeed are. On the other hand, the two forms for \( G \) appear different; however, by inserting the quantities in Eq. (5.4) into Eqs. (3.1), we find that they do constitute a solution of the general equations obtained from \( \mathfrak{A}_1 \). Since \( F'_2 \) and \( G'_2 \) do satisfy the original \( \mathcal{K}_1 \) equations, they must also constitute a special case of the general solution, \( \{F, G\} \) as given in Eq. (5.1) The truth of this may be seen by comparing the fairly-simply-displayed infinite series for \( G'_2 \) in Eq. (5.4) with the infinite sum for \( G_1 \) in Eq. (3.5). Since all the repeated
cosine integrals, \((\cos u)^{(-n-2)}\), discussed near Eq. (3.5), may be written as either \(\sin u\) or \(\cos u\), plus finite polynomials in \(u\), we may expand the trigonometric functions in infinite series and re-collect the expression for \(G_1\) in Eq. (5.3) in the form

\[
G_1 = G_0 - u(G_1) + \frac{u^2}{2!} (Q_0 - G_0) - \frac{u^3}{3!} (Q_1 - G_1) + \frac{u^4}{4!} (Q_2 - Q_0 + G_0) - \frac{u^5}{5!} (Q_3 - Q_1 + G_1) + \cdots
\]

Since \(G'_2\), in Eq. (5.4), is already of the form of an infinite power series, we may identify completely the general forms of \(G'_2\) and \(G_1\), allowing us to write a full presentation for an identification mapping of one to the other:

\[
\Xi : (\Phi_{-\frac{1}{2}u})_* (\mathfrak{A}_2) \rightarrow (\mathfrak{A}_1) \bigg|_{K=0} \quad \begin{cases} 
R \rightarrow F_1, \quad E_0 \rightarrow F_0, \\
J_0 \rightarrow G_0, \quad J_1 \rightarrow -G_1, \quad J_2 \rightarrow Q_0 - G_0, \\
J_3 \rightarrow Q_1 + G_1, \quad \ldots, \quad J_j \rightarrow Q_{j-2} - J_{j-2}.
\end{cases}
\]

After also checking that all the required commutators, from Eqs. (3.9) and Eq. (4.4), do indeed map into each other, these comparisons tell us that the general algebraic solution for the prolongation structure defined by \(K_2\), i.e., \(\mathfrak{A}_2\), is actually gauge equivalent to a subalgebra of the one defined by \(K_1\), which we believe to be unexpected. (The subalgebra in question is obtained from \(\mathfrak{A}_1\) by setting the generator \(K_0\), and therefore all the other \(K_n\)'s, to zero.)

An alternative approach to gauge transformations of this sort is to view them as simply effecting a different set of choices for the fiber coordinates \(w^A\). This is a reasonable viewpoint since the vector fields generating the flows are always vertical. Unlike the previous calculations, this approach is not as abstract; however, it will require an explicit realization of the vector field \(R\), in order to integrate Eqs. (5.1). Equations (1.4) tells us that an acceptable solution to the problem allows us to interpret the components of \(F_2\) and \(G_2\) as first derivatives of the pseudopotentials. In order to distinguish the two, in-principle-distinct sets of pseudopotentials, we use \(\{w^A\}\) for those from \(\mathfrak{A}_2\) and \(\{v^A\}\) for those from \(\mathfrak{A}_1\):

\[
F_2^A = (w^A)_x, \quad G_2^A = (w^A)_y, \quad F_1^A = (v^A)_x, \quad G_1^A = (v^A)_y.
\]
Beginning with any particular solution to the prolongation problem for $K_2$, a plausible (and usual) realization for $R$ is simply $2\partial_{w^1}$, as done, for instance, by Shadwick. This choice realizes our gauge transformation as the flow presented as $e^{-p\partial_{w^1}}$, making the solutions to the flow equations very simple. The curves of the congruence for the gauge transformation simply go along the direction of increase of $w^1$, from the (arbitrary) original value, by an amount $-p$, the jet variable for $-u_x$; i.e., we have

$$w^{1'} = w^1 - p,$$

all other coordinates stay the same. (5.7)

Using this realization for $R$ the $\mathfrak{A}_2$ portion of Eqs. (5.3) can be re-written as

$$(w^A)_x = u_x \delta^A_1 + \{e^{u\partial_1}E\}^A = u_x \delta^A_1 + \sum_{n=0}^{\infty} \frac{u^n}{n!} (\partial_{w^1})^n E^A(w),$$

$$(w^A)_y = -u_y \delta^A_1 + \{e^{u\partial_1}J\}^A = -u_y \delta^A_1 + \sum_{n=0}^{\infty} \frac{(-u)^n}{n!} (p_{w^1})^n J^A(w).$$

Writing everything at the transformed point, Eqs. (5.7) give us the gauge-transformed realization:

$$(w^A')_x = (1 + \delta^A_1) u_x + E^A,$$

$$(w^A')_y = -u_y \delta^A_1 + \sum_{n=0}^{\infty} \frac{(-2u)^n}{n!} (\partial_{w^1})^n J^A.$$  

On the other hand, using the identification of $F_1$ with $R$ and the realization of $R$ already agreed to, we re-write the $\mathfrak{A}_1$-equations, with $K = 0$:

$$(v^A)_x = 2u_x \delta^A_1 + F^A_0,$$

$$(v^A)_y = G^A_0 \cos u - G^A_1 \sin u + \sum_{n=0}^{\infty} (\cos u)(-n-2)(-2\partial_{w^1})^n Q^A.$$  

Comparing Eqs. (5.9) and (5.10), and recalling that we have already shown that the two forms of $G$ are equivalent, we see that the two equations show that $w^{1'}$ is the same as $v^1$, although the other fiber coordinates differ substantially in the two presentations. This difficulty lies only with the particular choice for a realization for $R$, which works well for the 1-dimensional situation, but must be generalized for a higher-dimensional case. For instance,
a realization of the form $R^A = 2$, for all allowed values of $A$, would arrange it so that each of the transformed variables $\{w^A\}$ would simply be a translation of the un-transformed ones, and would then correspond to the variables $\{v^A\}$, appropriate to the other algebra. This would realize the gauge equivalence of $K_2$ with that subalgebra of $K_1$ that has set to zero $K_0$ and its repeated commutators with $F_1$, with the meaning that the origin in $\{w^A\}$-space must be chosen differently for different presentations of the prolongation structure. This result might easily be expected for objects like potentials!

Returning to the 1-dimensional case, it is well known in that case\textsuperscript{21} that the general solution will not permit non-zero values for either $Q$ or $K$. Therefore, when restricted to 1-dimensional fiber spaces, the two algebras are completely equivalent. In a less formal mode of expression, this fact has been well-known for quite some time. This gauge equivalence, when phrased simply as a change in the names of the variables, so that one replaces $w$ by some $w' + u$, has long been known\textsuperscript{49} as a simple way of beginning, say, with the simplest ideal of 2-forms, our $K_1$, and nonetheless recovering exactly the 19th century version of the Bäcklund auto-transformation that comes out immediately if, instead, one uses the ideal $K_2$ and restricts oneself to 1-dimensional fibers.\textsuperscript{21} This last transformation only works, of course, provided the same (1-dimensional) realization of $\mathfrak{sl}(2, \mathbb{C})$ has been chosen for the generators of the prolongations coming from the two ideals.

Our understanding of gauge transformations as relevant to EW prolongations allows us to also return to our work on Burgers’ equation and the KdV equation, in Ref. 23, where we determined the continuation of those prolongation structures to higher-level jets. We noted then that the geometric meaning was still unresolved. Now it is clear that the exponential operators in Eqs. (4.10-14), for Burgers’ equation, may all be removed. Beginning with Eq. (4.10) there, the extra generator $A$ is really simply a gauge freedom for the origin of coordinates in the fiber space $W$. A gauge transformation of the form $e^{z(ad A)}$ will remove the exponential in front of the expression, and also the additional term $z_1 A$, so that this new choice of $w^B$'s, over $J^2$, retains the same expression for the $X_a$ as they had over $J^1$. This process
may be continued as one allows the EW prolongation to be defined on higher and higher jets. Bringing the connection from, say, $J^m$ to $J^{m+1}$, a gauge transformation $e^{z_m \text{ad} A}$ will retain invariant the connection, with this amounting only to the transformation of fiber coordinates, $w^B \rightarrow w^B + z_{m-1} A^B(w) + \ldots$, so that no interesting new structure is in fact added. On the other hand, when the EW prolongation for the KdV equation, in Eqs. (4.28) and (4.34), there, is gauge transformed in the same way, they take on a much simpler appearance, but it is clear that they nonetheless retain the important new structures that are identified there as generators for the higher (local) symmetries of the KdV equation. The process of defining the nonlocal EW prolongation on a higher jet adds to it explicitly new generators which carry the information concerning the higher local symmetries that are defined on the corresponding higher jet.

VI. Further Attempts to Identify $\mathcal{A}_2$

We have been quite involved in attempts to seriously identify the entirety of the algebra $\mathcal{A}_2$. While some partial results are presented below, the algebra has so far resisted its identification in terms of objects with which we are familiar. Therefore, we present these results in the hope that other researchers interested in such results may be able to utilize them to proceed further. As a beginning, we may utilize the gauge transformation from the previous section to transform our previous homomorphism, $\Psi : \mathcal{A}_1 \rightarrow A_1 \otimes \mathbb{C}[\lambda^{-1}, \lambda]$, given by Eqs. (3.10), into a homomorphism for $\mathcal{A}_2$, namely $\Psi' \equiv \Psi \circ (\Phi_{-\frac{1}{2}u})_* : \mathcal{A}_2 \rightarrow A_1 \otimes \mathbb{C}[\lambda^{-1}, \lambda]$, which may lead to some additional insight concerning more general mappings and identifications for that algebra. The first step in a complete description for $\mathcal{A}_2$ lies in a determination of the commutators of the $E_i$ with themselves and the commutators of the $J_k$ with themselves. Beginning with $\Xi$, from Eqs. (5.5), we may construct:

$$E_m \equiv (\text{ad} R)^m E \xrightarrow{\Xi} (\text{ad} F_1)^m F_0 \xrightarrow{\Psi} \begin{cases} (-1)^{\frac{m-2}{2}} J_1^{(1)}, & \text{for } m \text{ even,} \\ (-1)^{\frac{m-1}{2}} J_2^{(1)}, & \text{for } m \text{ odd,} \end{cases}$$

(6.1)
and then continue by attempting to construct the double commutators \([E_m, E_n]\). Under the action of \(\Xi\) itself, these undetermined commutators remain undetermined. However, \(\Psi'\) does indeed cause these to be determined:

\[
[E_m, E_n] \rightarrow \begin{cases} 0, & m + n \text{ even,} \\ (-1)^{m+n} J^{(2)}_3, & m + n \text{ odd.} \end{cases}
\] (6.2)

It is clear that triple commutators of the \(E_m\)'s among themselves will generate elements of \(A_1^{(1)}\) at the third level, quadruple commutators will generate elements at the fourth level, etc. The situation for the original \(J_k\)'s is slightly different in the sense that the ‘extra’ elements, \(Q_j\), are also involved, as described in Eqs. (5.5). However, none of the commutators involving only elements from the set \(\{G_0, G_1, \{Q_j\}\}\) are determined within the general form of \(\mathfrak{A}_1\). Therefore, the final mapping, \(\Psi'\) that sends them into \(A_1^{(1)}\) sends them in exactly the same way as it does the \(E_m\)’s, except that the result lies at negative levels of the structure of \(A_1^{(1)}\), rather than positive levels. In this way, the entire structure of \(\mathfrak{A}_2\) fits very nicely into the loop algebra that is \(A_1^{(1)}\) without its center. This homomorphism does, however, lose quite a lot of information that might have been contained within the larger algebra. For example, we see that \(\Psi'(E_{m+2}) = -\Psi'(E_m)\), \(\Psi'(J_{k+2}) = -\Psi'(J_k)\), so that those particular (countably-) infinite strings of generators are reduced to only 4. As well, the mapping loses all the information that might be in the original generators \(Q_i\) and \(K_j\), since it also relates \(\Psi'(E_m)\) and \(\Psi'(J_m)\) simply by the change of sign of the index \(m\); when the loop-algebra elements are realized by powers of a “spectral,” or loop parameter, \(\lambda\), this simply maps \(\lambda^m\) into \(\lambda^{-m}\).

As described in Section I, we still feel that a better understanding of these algebras would benefit our understanding of the general solutions of the pde’s involved. There we insisted that (at least) those generators involved explicitly in the description of \(F\) and \(G\) should remain linearly independent, which requirement is violated by the mappings \(\Psi\) and \(\Psi'\), that map the infinite sequence of, for instance, the \(E_m\)'s into just two unique ones, \(\Psi'(E_0)\) and \(\Psi'(E_1)\), the rest just being \(\pm 1\) multiplied by these two. Therefore, we have searched for other realizations that would be faithful for at least the entire set \(\{J_n, R, E_m \mid m, n = 0, 1, 2, \ldots\}\).
Unfortunately, we have been unable to find such a faithful realization. It seems necessary to make some additional requirements on the algebra, still of course preserving the requirement of faithfulness. The most reasonable alternative we know was suggested by Alice Fialowski, namely that we should restrict ourselves to the subalgebra which is spanned, in the sense of a vector space, on the countable list of generators presented just above. This is again quite distinct from the homomorphisms, $\Psi$ and $\Psi'$, since they send multiple commutators of the $E_m$‘s with themselves into new quantities, corresponding to larger powers of the loop parameter $\lambda$, which are not within the span of the original generators. Nonetheless, we do believe that there is some merit in continuing this path for awhile. As we will show, it leads to interesting connections with Banach algebras of Toeplitz-type operators, a different direction than has so far been explored by researchers studying symmetry groups for pde’s.

To describe the algebraic restrictions on $\mathfrak{A}_2$ generated by insisting that the generators should be linearly independent and should span the (vector) space, we could require that the double commutators, $[E_m, E_n]$ and $[J_m, J_n]$ be given as linear combinations of the original set of generators. The general form of the requirements on all these coefficients, as required by the satisfaction of the Jacobi relations, is rather large and complicated. However, one property may be quickly seen. If all the coefficients in the directions $J_k$ of the commutators $[E_m, E_n]$ are zero, or all the coefficients in the directions $E_m$ of the commutators $[J_j, J_k]$ are zero, then it is impossible to maintain the linear independence of all the original generators. Therefore, once again we make yet one more ‘extra’ assumption, going toward some notion of linear independence which is clearly not the most general, but which did at least arrive at some unexpected places.

We therefore now discuss, in considerable detail, the situation where one assumes that the commutators are indeed spanned only by those coefficients essential for linear independence. More precisely, we consider the particular special subalgebra of $\mathfrak{A}_2$, which we characterize as $\mathfrak{B}_2$, which contains as a vector space basis the generators

$$\{J_n, R, E_m \mid m, n = 0, 1, 2, \ldots\}, \text{ a vector space basis for } \mathfrak{B}_2.$$  

(6.3)
As this is the most general case we will consider in any detail, it seems reasonable to recapitulate here some of the earlier relations as well so that we may collect together the entire set of relations that define this subalgebra:

\[
\mathfrak{B}_2 : \begin{cases}
\mathbf{E}_m \equiv (\text{ad } \mathbf{R})^m \mathbf{E} , & \mathbf{J}_n \equiv (-\text{ad } \mathbf{R})^n \mathbf{J} , & \forall m, n = 0, 1, 2, \ldots , \\
[\mathbf{E}_{k-m}, \mathbf{J}_m] = c_k \mathbf{R} , & \forall m \geq 0 \leq m \leq k , \\
[\mathbf{E}_i, \mathbf{E}_j] = A_{ij} \mathbf{J}_m , & [\mathbf{J}_m, \mathbf{J}_n] = B_{mn} \mathbf{E}_i , 
\end{cases}
\] (6.4)

where the scalar quantities, \( A_{ij} \) and \( B_{mn} \), are the needed coefficients already mentioned.

Applying the Jacobi relations to this set of definitions determines a complete set of constraints on them which we present below. In addition, we find that they offer an unexpected expression as the defining equations for an algebra of Toeplitz operators, although we do not yet have explicit realizations.

To describe the constraints on these coefficients, it is more useful to consider each of them as being the elements of a single countable sequence of (semi-infinite by semi-infinite) matrices, that of course may be taken as the adjoint representation of our algebra. We collect these coefficients into matrices, \( \mathcal{A}_i \) and \( \mathcal{B}_m \), by the rules that

\[
(\mathcal{A}_i)^{m}_{j} \equiv A_{ij}^m , & (\mathcal{B}_m)^{i}_{j} \equiv B_{mj}^i , & i, j, m = 0, 1, 2, \ldots , \] (6.5)

To manipulate the elements, it is useful to have the usual shift, or ‘creation’ matrix, \( \Lambda \), with each element +1 in its first “super-diagonal,” and all other elements zero, together with its transpose, the ‘destruction’ matrix, \( \Lambda^T \):

\[
\Lambda^j_i = \delta^j_{i+1} , & (\Lambda^T)^j_i = \delta^j_i , \quad \Lambda \Lambda^T = I , \quad \Lambda^T \Lambda = I - (\vec{e}_0)^0 , \quad \vec{e}_{i+1} = \Lambda^T \vec{e}_i = (\Lambda^T)^{i+1} \vec{e}_0 , \]

(6.6)

where \( \{ \vec{e}_i \mid i = 0, 1, \ldots \} \) are the usual basis vectors which have the entry +1 at the i-th row and zeros elsewhere. We also define a sequence of vectors, \( \vec{C}_j \), with elements determined by the constants \( c_i \), from the derivatives of \( f(u) \), at \( u = 0 \), as at Eq. (4.5), which we take to be \( \sin u \) from now on:

\[
\vec{C}_j \equiv (\Lambda)^j C_0 , \quad (\vec{C}_i)^j = (\vec{C}_0)^{i+j} \equiv c_{i+j} .
\] (6.7)
We may now detail the requirements which are placed on valid matrices $A_i$ and $B_j$ by virtue of the Jacobi identity. We list them in some detail, and order, since their solutions will require that order. The **zero-th requirement** is simply that of skew-symmetry, which is required by their definition in terms of commutators:

\[
(A_i)^j_k + (A_k)^j_i = 0 \quad , \quad (B_i)^j_k + (B_k)^j_i = 0 \quad .
\]

(6.8)

Assuming the satisfaction of Eqs. (6.6), the **first requirement** may be formed as a first-order recursion relation for the matrices $A_i$:

\[
A_{i+1} = -\Lambda^T A_i - A_i \Lambda^T \equiv -\{\Lambda^T, A_i\} \quad , \quad B_{i+1} = -\Lambda^T B_i - B_i \Lambda^T \equiv -\{\Lambda^T, B_i\} \quad ,
\]

(6.9a)

where the braces are used to indicate an anti-commutator of the matrices. This relation can immediately be iterated to give

\[
A_i = (-1)^i \sum_{p=0}^{i} \binom{i}{p} (\Lambda^T)^{i-p} A_0 (\Lambda^T)^p \quad , \quad (A_i)^k_\ell = (-1)^i \sum_{m=0}^{i} \binom{i}{m} (A_0)^{k-i+m}_\ell+m \quad ,
\]

(6.9b)

with the same form for the $B_i$, so that only the two matrices $A_0$ and $B_0$ are needed.

The **second requirement** is an “inverse” eigenvector equation, since in this case we are given the eigenvectors and eigenvalues and must find the associated matrices. It is also reminiscent of group representations on vector-valued operators:

\[
\bar{C}^T [j \, A_i] = \sum_{k=0}^{\infty} (A_i)^k_j \bar{C}^T_k \quad , \quad \bar{C}^T [j \, B_i] = \sum_{k=0}^{\infty} (B_i)^k_j \bar{C}^T_k \quad ,
\]

(6.10)

where the brackets around indices indicate a “commutator” on the indices, i.e., $X_i (Y_j) \equiv X_i Y_j - X_j Y_i$. At this point the requirements on the $A_i$’s and the $B_m$’s have been of identical type. However, the **third requirement** establishes quadratic relations among them, which should serve to distinguish the two sets:

\[
A_n B_p = B_n A_p = \bar{c}_{p+1} \bar{C}^T - c_{n+p} \Lambda^T = \Lambda^T \{Q_{np} - c_{n+p} I\} \quad ;
\]

(6.11)

\[
Q_{np} \equiv \bar{c}_p \bar{C}^T_n \quad , \quad \text{trace} (Q_{np}) = c_{n+p} \quad .
\]
These matrix requirements, except for the one concerning skew-symmetry, may also be very naturally expressed via generating functions, with the recursion relation having an especially simple form:

\[
\sum_{i=0}^{\infty} \frac{t^i}{i!} A_i = e^{-t\Lambda^T} A_0 e^{-t\Lambda^T}, \quad \tag{6.9'}
\]

\[
\sum_{j,k=0}^{\infty} \frac{s^j t^k}{j!k!} (\tilde{C}_j^T A_i - \tilde{C}_i^T A_j) = C_0^T \left\{ e^{-(s-t)\Lambda^T} A_0 e^{-s\Lambda^T} - e^{+(s-t)\Lambda^T} A_0 e^{-t\Lambda^T} \right\}, \quad \tag{6.10'}
\]

\[
e^{-t\Lambda^T} A_0 e^{-(s+t)\Lambda^T} B_0 e^{-s\Lambda^T} = \{ e^{s\Lambda} \tilde{C}_0^T (\epsilon_0^T) e^{t\Lambda} - \sin(s+t)I \} \Lambda. \quad \tag{6.11'}
\]

These relations suggest that the structures are ‘tightly’ defined and that, perhaps, determination of the correct form for those at level 0 would in fact determine correctly all the others.

The general solution to requirements zero and one puts non-trivial constraints on the matrices, with the complete derivation of these constraints being relegated to the Appendix. The general form is determined by the fact that skew symmetry requires that \((A_i)^k_i\) must vanish. Iteration of this requirement provides the relationships

\[
(A_0)^k_{2i} = - \sum_{m=0}^{i-1} \binom{i}{m} (A_0)^{k-i+m}_{i+m} = - \sum_{n=1}^{i} \binom{i}{n} (A_0)^{k-m}_{2i-n}, \quad \tag{6.12}
\]

constituting an \(i\)-term recursion relation for the elements of the \(2i\)-th column. These relations determine the even-numbered columns in terms of the odd-numbered ones, which are still arbitrary. To describe this result, we first conceive of the columns as vectors, introducing two new symbols \(\tilde{a}_j\) and \(\tilde{A}_{2m}\) to describe the odd- and even-numbered columns of \(A_0\), respectively, with like objects for \(B_0\):

\[
(\tilde{a}_j)^k \equiv (A_0)^{k}_{2j+1}, \quad (\tilde{A}_{2m})^k \equiv (A_0)^{k}_{2m}, \quad \tag{6.13}
\]

which allows the explicit description

\[
\tilde{A}_0 = 0, \quad \tilde{A}_2 = -\Lambda^T \tilde{a}_0, \quad \tilde{A}_4 = -2\Lambda^T \tilde{a}_1 + (\Lambda^T)^3 \tilde{a}_0, \quad \tilde{A}_6 = -3\Lambda^T \tilde{a}_2 + 5(\Lambda^T)^3 \tilde{a}_1 - 3(\Lambda^T)^5 \tilde{a}_0, \ldots, \quad \tilde{A}_{2i} = \sum_{n=0}^{i-1} \mathcal{P}_{i,n} \{ (\Lambda^T)^{2n+1} \tilde{a}_{i-n-1} \}, \quad \tag{6.14}
\]

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with completely similar representations for the columns of $B_0$, and the (integer) coefficients $P_{i,n}$ are of alternating sign, and are determined as finite sums of binomial coefficients, given explicitly in the Appendix.

The second set of requirements may be explicitly solved as well. To demonstrate the solution, we must first define the alternating sums of even and of odd elements for the odd columns, $\vec{a}_i$:

$$R_j \equiv \sum_{m=0}^{\infty} (-1)^{j+m}(\vec{a}_j)^{2m} = \vec{C}_1 \cdot \vec{a}_j \quad , \quad S_j \equiv \sum_{m=0}^{\infty} (-1)^{j+m}(\vec{a}_j)^{2m+1} = \vec{C}_0 \cdot \vec{a}_j \ , \quad (6.15)$$

with entirely similar quantities, $U_j$ and $V_j$, respectively, being defined for the vectors $\vec{b}_k$. The recursion relations induce $(n+2)$-term recursion relations for these sums, which may be iteratively solved, in terms of $R_0$, $R_1$, and $S_0$:

$$R_m = w_m(R_0 + R_1) + R_1 \ , \ \forall \ m \geq 2 \ , \quad w_m = 4, 20, 84, 340, 1,364, 5,460, 21,844, \ldots ,$$

$$S_j = q_jS_0 \ , \ \forall \ j \geq 1 \ , \quad q_j = 3, 11, 43, 171, 2,393, 2,731, 10,923, 43,691, \ldots , \quad (6.16)$$

with, again, identical relations for $U_m$ and $V_n$. The calculated coefficients, $w_m$ and $q_j$, increase very rapidly; therefore, we have made the additional hypothesis that all of $R_0 + R_1$, $S_0$, $U_0 + U_1$ and $V_0$ must vanish, which satisfies the equations without the need of allowing these divergent sequences of constants, with the result that all of these sums are explicitly required to have the following values:

$$\vec{C}_1 \cdot \vec{a}_i \equiv R_i = (-1)^iR_0 \equiv (-1)^i\vec{C}_1 \cdot \vec{a}_0 \ , \quad \vec{C}_0 \cdot \vec{a}_i \equiv S_i = 0 \ ;$$

$$\vec{C}_1 \cdot \vec{b}_j \equiv U_j = (-1)^jU_0 \equiv (-1)^j\vec{C}_1 \cdot \vec{a}_0 \ , \quad \vec{C}_0 \cdot \vec{b}_j \equiv V_j = 0 \ , \quad (6.17)$$

quite analogous to the fact that for each value of $p$, we always have $\sum_{j=0}^{\infty}(\vec{C}_p)^j = 0$.

With the even columns of $A_0$ and $B_0$ determined by Eqs. (6.12) and the odd columns restrained (slightly) so that their sums satisfy Eqs. (6.15), all the independent conditions have been satisfied, and now only the quadratic equations, Eqs. (6.9), need to be satisfied. As a first step, one easily sees that the product $A_nB_p$ must have the same sort of periodicity in $n$ as
does \( c_n \), i.e., \( c_{n+2} = -c_n \); therefore, we need only resolve the products \( A_0 B_p \) and \( A_1 B_p \). The complete set of requirements that this generates, on the (almost) arbitrary vectors \( \vec{a}_j \) and \( \vec{b}_k \), are written out explicitly in the Appendix, in Eqs. (A11-14).

Once again, these requirements have been found too complicated for us to completely resolve. However, some useful comments may nonetheless be made. For example, the requirements include the statement that

\[
\sum_{k=0}^{\infty} (A_0)^{1 \ k} (\vec{b}_m)^k \equiv (A_0 \vec{b}_m)^1 = c_{2m+1} = (-1)^m \neq 0 . \tag{6.18}
\]

From this we see that insisting that only finitely many of the arbitrary (odd) columns are non-zero is simply not allowed. More precisely, if we set \( \vec{b}_m \equiv 0 \) for all \( m \) greater than some integer, requirement Eq. (6.16) would be impossible to satisfy. By switching, and considering the column vector \( (B_0 \vec{a}_m)^1 \), we can prove the same requirement for \( A_0 \); therefore, it is necessary that our matrices have infinitely many columns. This lack of finite-dimensional representations is already mirrored in the literature by results of Shadwick, although he came to it from a rather different direction. An alternative plausible mode for simplification would involve \( \vec{C}_0 \) more intimately. If one tries the ansatz that \( \vec{a}_k = \alpha_k \vec{C}_0 \), with \( \alpha_k \) a scalar, one easily finds contradictions to this within the equations.

Attempting, nonetheless, to find some route to follow further, we notice that the requirements for, say \( A_0 (\Lambda^T)^n \vec{b}_r \), in Eqs. (A11-A13) in the Appendix, insist that half of all the entries must explicitly be zero. These entries do suggest, and indeed allow, yet an additional specialization. One may simply set to zero all the even-numbered elements of all of our previously arbitrary vectors, which of course simplifies greatly the equations, the general versions of which are listed in the Appendix. It is in fact this particular specialization that seems to offer the intriguing connections with Toeplitz operators on Banach spaces. Therefore, we add this new assumption to the list, give new names to the resulting arbitrary vectors that now determine our matrices, and display the resulting equations in a compact form, involving (semi-infinite) Toeplitz matrices.
In this special case, where \((\vec{a}_j)^{2k} = 0 = (\vec{b}_m)^{2n}\), we put an underline on the vectors to denote a symbol that contains only the remaining elements:

\[
(\vec{a}_j)^k \equiv (\vec{a}_j)^{2k+1}, \quad (\vec{b}_m)^n \equiv (\vec{b}_m)^{2n+1}.
\] (6.19)

The entire set of equations still to be satisfied are specified in the Appendix, in component form, in Eqs. (A15-7). It is however much easier to write and comprehend the equations when they are written in the form of Toeplitz matrices. For each \(\vec{a}_j\) we define the matrices \(A_j\) and \(\vec{A}_j\), where the last is conceived of as a column vector with elements which are each row vectors:

\[
A_j \equiv \sum_{k=0}^{\infty} (\vec{a}_j)^k \Lambda^k = \begin{pmatrix}
(a_j^0 & a_j^1 & a_j^2 & \cdots \\
0 & a_j^0 & a_j^1 & \cdots \\
0 & 0 & a_j^0 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \quad \vec{A}_j \equiv \begin{pmatrix}
\vec{a}_j^T \\
\vec{a}_j^{T+1} \\
\vec{a}_j^{T+2} \\
\vdots
\end{pmatrix} = \begin{pmatrix}
a_0^0 & a_0^1 & a_0^2 & \cdots \\
a_1^0 & a_1^1 & a_1^2 & \cdots \\
a_2^0 & a_2^1 & a_2^2 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\] (6.20a)

and identical objects made from the \(\vec{b}_j\), and use the binomial coefficients to define a matrix of the same type as \(\vec{A}_j\):

\[
\vec{M}_j \equiv \begin{pmatrix}
\vec{M}_{0j}^T \\
\vec{M}_{1j}^T \\
\vec{M}_{2j}^T \\
\vdots
\end{pmatrix} = \begin{pmatrix}
\mathcal{M}_0^0 & \mathcal{M}_0^1 & \mathcal{M}_0^2 & \cdots \\
\mathcal{M}_1^0 & \mathcal{M}_1^1 & \mathcal{M}_1^2 & \cdots \\
\mathcal{M}_2^0 & \mathcal{M}_2^1 & \mathcal{M}_2^2 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\] (6.20b)

\[
(\vec{M}_{mj})^j \equiv (-1)^{m+\ell+j} \left\{ \binom{2m}{2j} - \binom{2m}{2j-2\ell-1} \right\}.
\]

Using these constructions, we may re-write the set of equations from the Appendix in the very simple form

\[
\vec{A}_j \vec{B} = \vec{M}_j = \mathbb{B}_j \vec{A}_j.
\] (6.21)

Since every finite-dimensional sub-matrix of \(\vec{A}_j\) has a straight-forwardly-defined determinant, one could suppose that it also has one when the limit is taken, which would allow the formulation

\[
\vec{A}_j = (\mathbb{B}_j)^{-1} \vec{M}_j, \quad \forall j = 0, 1, 2, \ldots
\] (6.22)
This gives a multitude of definitions of $\vec{A}$, all of which must of course hold simultaneously. In addition, the existence of an inverse for $\vec{M}_0$ is sufficient to allow a description of all the rest of the $\vec{B}_j$ from just the first one, $\vec{B}_0$:

$$\vec{B}_j = \vec{M}_j (\vec{M}_0)^{-1} \vec{B}_0, \quad \vec{A}_j = \vec{M}_j (\vec{M}_0)^{-1} \vec{A}_0. \quad (6.23)$$

In order to justify the above manipulations with infinite-dimensional matrices, we now show that all this may be put into the general theory of Toeplitz operators acting on Banach spaces. We attempt to give a brief introduction to these operators, but surely refer interested parties to the literature mentioned in Ref. 50 for details, justifications, and assorted special cases. We describe these Toeplitz operators by drawing the (usual) comparisons to simple multiplication operators in, for instance, $L^p$, as viewed via a point-wise definition or their Fourier coefficients. Let $\{a, b, m, n, p, \ldots\}$ be arbitrary functions on the circle, and introduce Fourier coefficients for each of them according to the following (usual) pattern:

$$a(\theta) = \sum_{k=-\infty}^{\infty} a_k e^{ik\theta} \quad \iff \quad a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta a(\theta) e^{-ik\theta}. \quad (6.24)$$

Thinking of these functions as defined in either space, their (pointwise) product, as functions on the circle, takes quite a different form when mapped into the space of Fourier coefficients. We describe these two operations as simply being two different representations of a multiplication operator associated with $a$:

$$(M_a b)(\theta) \equiv n(\theta) \equiv a(\theta)b(\theta) \quad \iff \quad (M_a b)_k \equiv n_k = \sum_{n=-\infty}^{\infty} a_{k-n} b_n. \quad (6.25)$$

With this as background, the Toeplitz operator associated with the function $a$, $T_a$, is created by this sort of a product, but where both $b$ and the product are restricted to have only non-negative Fourier coefficients. (We say that they are elements of the Hardy space, $\mathcal{H}^+$, the
space of functions defined on the circle which have only non-negative Fourier coefficients.) The corresponding analogues to Eqs. (6.24-5) are now, with \( a, b \in \mathcal{H}^+ \):

\[
b(\theta) = \sum_{k=0}^{\infty} b_k e^{ik\theta} \quad \iff \quad b_k = \begin{cases} \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \, b(\theta) e^{-ik\theta}, & k \geq 0, \\ 0, & k < 0. \end{cases} \tag{6.26}
\]

\[
p(\theta) \equiv (T_a b)(\theta) \quad \iff \quad p_k = \begin{cases} \sum_{n=0}^{\infty} a_{k-n} b_n, & k \geq 0, \\ 0, & k < 0. \end{cases} \tag{6.27}
\]

For our purposes, we will also need related functions that lie within the other Hardy space, \( \mathcal{H}^- \), the space of functions defined on the circle which have only non-positive Fourier coefficients. The (conjugate)\(^{51} \) function \( \tilde{a}(\theta) \): associated with \( a(\theta) \), and the Toeplitz operator associated with \( \tilde{a} \), are defined as follows:

\[
a(\theta) = \sum_{k=-\infty}^{\infty} a_k e^{ik\theta} \quad \iff \quad \tilde{a}(\theta) = \sum_{k=-\infty}^{\infty} a_k e^{-ik\theta}, \quad \tag{6.28}
\]

\[
m(\theta) \equiv (T_{\tilde{a}} b)(\theta), \quad \iff \quad m_k = \begin{cases} \sum_{n=k}^{\infty} a_{n-k} b_n = \sum_{p=0}^{\infty} a_p b_{p+k}, & k \geq 0, \\ 0, & k < 0. \end{cases} \tag{6.29}
\]

The Toeplitz operator associated with a conjugate function, \( T_{\tilde{a}} : \mathcal{H}^- \times \mathcal{H}^+ \rightarrow \mathcal{H}^+ \), has exactly the same structure as the problem we want to solve. In this language, we may re-write the equations from the Appendix in a mode that has somewhat firmer foundation than the infinite matrices in Eqs. (6.21) and following. In order to do this, we require the existence of five (related) sequences of functions on the circle:

\[
\{ \mathcal{A}^i, \mathcal{B}^k, \mathcal{M}^i \mid i, j, k = 0, 1, \ldots \} \subseteq \mathcal{H}^+, \quad \{ \mathcal{\tilde{A}}_\ell, \mathcal{\tilde{B}}_m \mid \ell, m = 0, 1, \ldots \} \subseteq \mathcal{H}^- . \tag{6.30}
\]

Using these quantities, our constraints take the following form, involving the Toeplitz action of the conjugate functions on the other functions:

\[
(T_{\mathcal{\tilde{B}}^i} \mathcal{A}^j)(\theta) = \mathcal{M}^i(\theta) \equiv \sum_{m=0}^{\infty} \mathcal{M}^i_m e^{im\theta} = (T_{\mathcal{\tilde{A}}_\ell} \mathcal{B}^j)(\theta) , \tag{6.31}
\]

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The functions in question are related to our earlier ones, by considering them as Fourier coefficients of functions of two variables, \( \mathcal{A}(\theta, \zeta), \mathcal{B}(\theta, \zeta) \), as follows:

\[
\mathcal{A}(\theta, \zeta) = \sum_{\ell=0}^{\infty} \sum_{j=0}^{\infty} (\vec{a}_\ell) e^{-ij\theta} e^{i\ell\zeta}
\]

\[
\sum_{s=0}^{\infty} (\vec{a}_s)^s e^{-is\theta} = \tilde{\mathcal{A}}_\ell(\theta) = \begin{cases} 
\frac{1}{2\pi} \int_{-\pi}^{\pi} d\zeta \mathcal{A}(\theta, \zeta) e^{-i\ell\zeta} & , \ell \geq 0, \\
0 & , \ell < 0,
\end{cases} \tag{6.32}
\]

\[
\sum_{\ell=0}^{\infty} (\vec{a}_\ell)^\ell e^{i\ell\zeta} = \mathcal{A}^j(\zeta) = \begin{cases} 
\frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \mathcal{A}(\theta, \zeta) e^{ij\theta} & , j \geq 0, \\
0 & , j < 0,
\end{cases}
\]

In terms of functions of one variable only, two pair of the functions originate from the same pair of sources, our original set of matrix elements. Therefore one may write a transformation between them:

\[
\mathcal{A}^j(\theta) = \frac{1}{2\pi} \sum_{\ell=0}^{\infty} e^{i\ell\theta} \int_{-\pi}^{\pi} d\eta \tilde{\mathcal{A}}_\ell(\eta) e^{ij\eta}, \text{ or } \tilde{\mathcal{A}}_\ell(\eta) = \frac{1}{2\pi} \sum_{n=0}^{\infty} e^{-ijn} \int_{-\pi}^{\pi} d\zeta \mathcal{A}^n(\zeta) e^{-i\ell\zeta}. \tag{6.33}
\]

With \( \mathcal{M}^j(\theta) \) being given, these are equations to determine the unknown functions \( \mathcal{A}(\theta, \zeta) \) and \( \mathcal{B}(\theta, \zeta) \). The sum determining \( \mathcal{M}^j(\theta) \), from the coefficients given in Eq. (6.18c), does not determine a function that exists in the usual Hilbert space; nonetheless, there are indeed Banach spaces in which it exists and makes good sense. What is the optimum approach to considering these equations, and making sense of the set? Unfortunately, we do not know, but do hope that some interest can be generated among the considerable community of experts in the area of Toeplitz problems. We have considered an approach via formal series, which may not be any more rigorous than our earlier semi-infinite matrices; on the other hand, re-writing them in terms of integral equations may well allow them to be re-considered from within some more appropriate Banach space. To do this, we begin with a \textit{projection operator} that projects general functions over the circle into \( \mathcal{H}^+ \):

\[
b(\theta) = \frac{1}{2\pi} \sum_{k=0}^{\infty} e^{ik\theta} \int_{-\pi}^{\pi} d\phi b(\phi) e^{-ik\phi} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \frac{b(\phi)}{1 - e^{i(\theta-\phi)}}, \tag{6.34}
\]
which allows one to give an integral representation, valid at least in terms of formal power series, of the Toeplitz operator:

\[ m(\theta) \equiv (T_{ab})(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \frac{a(\phi)b(\phi)}{1 - e^{i(\theta - \phi)}} , \]  

(6.35)

Using these projection operators, we may produce formal expressions that re-write Eqs. (6.31) explicitly in terms of the 4 functions of one variable:

\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \frac{A^j(\phi)B_\ell(\phi)}{1 - e^{i(\theta - \phi)}} = \mathcal{M}^j(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \frac{B^j(\phi)A_\ell(\phi)}{1 - e^{i(\theta - \eta)}} , \]  

(6.36)

or a form with triple integrals can be written so that only the two unknown functions of two variables need to be considered:

\[ \frac{1}{(2\pi)^3} \int_{-\pi}^{\pi} d\eta \int_{-\pi}^{\pi} d\alpha \int_{-\pi}^{\pi} d\beta \frac{B(\alpha, \eta)A(\eta, \beta)}{1 - e^{i(\theta - \eta)}} = \mathcal{M}^j(\theta) \]

\[ = \frac{1}{(2\pi)^3} \int_{-\pi}^{\pi} d\eta \int_{-\pi}^{\pi} d\alpha \int_{-\pi}^{\pi} d\beta \frac{B(\eta, \beta)A(\alpha, \eta)}{1 - e^{i(\theta - \eta)}} \]  

(6.37)

Multiplication of the above equation by \( e^{-ij\eta} \) and summing on all non-negative values of \( j \) reduces the complexity of the equations, by reducing the problem formulation to double integrals:

\[ \int_{-\pi}^{\pi} d\phi \int_{-\pi}^{\pi} d\beta \frac{B(\eta, \phi)A(\phi, \beta)}{1 - e^{i(\theta - \phi)}} = (2\pi)^2 \mathcal{M}(\eta, \theta) = \int_{-\pi}^{\pi} d\phi \int_{-\pi}^{\pi} d\beta \frac{B(\eta, \phi)A(\phi, \beta)}{1 - e^{i(\theta - \phi)}} \]  

(6.38)

It is plausible that one or the other of the last two sets of equations, taken \textit{ab initio} would be a place from which the construction would seem more reasonable and calculable. From this point of view, of course, one should consider how this would affect the original starting point, which is simply an attempt to determine the completely general form of the prolongation functions \( \mathbf{F} \) and \( \mathbf{G} \). The original indexed scalars, \( A^i_{mn} \) were introduced because we needed to express the commutator \( [\mathbf{E}_m, \mathbf{E}_n] \). Since we are now mapping the Hardy space into itself, from the index (Fourier) approach to the representation involving functions of \( \theta \), this necessitates the existence of (vector-field-valued) functions on the circle, \( \mathbf{E}(\theta) \) and \( \mathbf{J}(\theta) \), such that

\[ E_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta e^{-in\theta} E(\theta) = -\frac{1}{2\pi i} \int \frac{dt}{t} t^n E(\theta) , \]  

(6.39)

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where the contour integral is taken around the unit circle in the complex plane, along with a completely similar equation for $J_m$. This alternative representation for $E_m$ also explains why functions of two circle variables appeared in our equations, since the two such variables correspond to the two distinct occurrences of $E_n$ in the commutators under consideration. Recalling the definitions of $F$ and $G$, from Eqs. (4.1-4), this approach gives a complex-integral representation of them, involving operators that might be from a Banach algebra, which is surely a plausible alternative to a series expansion when that expansion is not truncated:

$$
F = \frac{1}{2} R - \frac{1}{2\pi i} \oint \frac{dt}{t} E(t) e^{\frac{i}{2} ut}, \quad G = -\frac{1}{2} R - \frac{1}{2\pi i} \oint \frac{dt}{t} J(t) e^{-\frac{i}{2} ut}
$$

(6.40)

**Acknowledgments:** We would like to express our appreciation for the very considerable assistance with symbolic-calculation computer codes given to us by Michael Wester. Our understanding of gauge transformations was greatly increased by discussions with Mikhail Saveliev. As well one of us (JDF) has benefited from extensive discussions with Mark Hickman, Cornelius Hoenselaers and Alice Fialowski concerning these problems.
Appendix: Some Technical Details of the Resolution of $\mathfrak{B}_2$

Requirement One:

The skew-symmetry condition, Eqs. (6.8), and the recursion relation, Eqs. (6.9), impose non-trivial constraints on the semi-infinite matrices, $A_0$. To determine a form that satisfies these constraints, we begin with an important part of the skew symmetry constraint, that the diagonal elements must vanish: $(A_0)^{k_0} = 0, (A_1)^{k_1} = 0, \ldots , (A_i)^{k_i} = 0$. When these simple statements are mixed with the recursion relation for the full matrices, we obtain series that allow us to determine other elements of $A_0$. The first of these statements simply says that the first column of the matrix is zero. The lowest non-trivial examples are

$$0 = (A_1)^{k_1} = - (A_0)^{k_{-1}} - (A_0)^{k_2} , \quad \Rightarrow (A_0)^{k_2} = - (A_0)^{k_{-1}} ,$$

$$0 = (A_2)^{k_2} = (A_0)^{k_{-2}} + 2(A_0)^{k_{-1}} + (A_0)^{k_4} , \quad \Rightarrow (A_0)^{k_4} = - 2(A_0)^{k_{-1}} - (A_0)^{k_{-2}} .$$

The general term of this relationship takes the form

$$0 = (A_i)^{k_i} = (-1)^i \sum_{m=0}^{i} \binom{i}{m} (A_0)^{k_{-i+m}}$$

$$\Rightarrow (A_0)^{2i} = - \sum_{m=0}^{i-1} \binom{i}{m} (A_0)^{k_{-i+m}} = - \sum_{n=1}^{i} \binom{i}{n} (A_0)^{k_{-2i-n}} .$$

By looking in detail at the procedure in Eqs. (A1), one can use the equation for $i = 1$ to determine $(A_0)^{k_2}$, then set $i = 2$ and insert into it the just-determined value for $(A_0)^{k_2}$ so that $(A_0)^{k_4}$ can be determined. After that, both values are inserted into the $i = 3$ equation to determine $(A_0)^{k_6}$. Therefore, operating in this doubly-recursive manner, we may sequentially solve an $i$-term recursion relation for each element of the $2i$-th column. The result is given in Eqs. (6.14), where the process described below determines the values of the coefficients, generated by sums of products of binomial coefficients, needed for this.

To display the solutions for arbitrary values of $i$, we first divide up the sum, in Eqs. (A2), into those terms involving elements from even columns and those involving elements from odd columns:

$$(A_0)^{2i} = - \sum_{p=0}^{[\frac{i-1}{2}]} \binom{i}{2p+1} (A_0)^{k_{-2p-1}} - \sum_{q=1}^{[\frac{i}{2}]} \binom{i}{2q} (A_0)^{k_{-2q}} .$$
Within the expansion of some arbitrary even column, the highest-appearing odd column occurs only once, with coefficient \( \binom{i}{1} \); however, the next-highest odd column occurs twice, so that its contribution to the final sum must be determined by summing those two coefficients. The next columns appears three times, etc. We must re-sum the series so as to determine the coefficient that multiplies entries from \((A_0)_{2j-2r}\) in the expansion to determine \((A_0)_{2j}\). These coefficients we denote by \(\{Q_{j,r} \mid j = 1, 2, 3, \ldots; r = 0, 1, 2, \ldots, r-1\}\), and specify the following recursive method of obtaining them:

\[
Q_{j,0} = 1, \quad Q_{j,r} = -\sum_{k=0}^{r-1} \left( \frac{j-k}{2r-2k} \right) Q_{j,k}, \quad r = 0, 1, \ldots, r-1 , \quad (A4)
\]

\[
Q_{1,r} : [1] , \\
Q_{2,r} : [1, -1] , \\
Q_{3,r} : [1, -3, 3] , \\
Q_{4,r} : [1, -6, 17, -17] , \\
Q_{5,r} : [1, -10, 55, -155, 155] , \\
Q_{6,r} : [1, -15, 135, -736, 2073, -2073] .
\]

\[
\ldots
\]

Having those coefficients for each \((A_0)_{2j-2r}\), we may now sum these quantities multiplied by each appropriate weight factor, to determine the coefficients actually desired, which we will refer to by the notation \(\{P_{i,n} \mid i = 1, 2, \ldots; n = 0, 1, \ldots i-1\}\):

\[
P_{i,w} \equiv -\sum_{r=0}^{w} \left( \frac{i-r}{2w+1-2r} \right) Q_{i,r} , \quad (A5)
\]

\[
P_{1,r} : [-1] , \\
P_{2,r} : [-2, 1] , \\
P_{3,r} : [-3, 5, -3] , \\
P_{4,r} : [-4, 14, -28, 17] , \\
P_{5,r} : [-5, 30, -126, 255, -155] , \\
\ldots
\]

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and we always take the binomial coefficients as zero when negative factorials appear in the denominator. To display more clearly the general form of a $A_0$ that satisfies the first 2 of our requirements, we write out the elements nearest the upper left corner:

\[
\begin{pmatrix}
0 & (a_0)^0 & 0 & (a_1)^0 & 0 & (a_2)^0 & 0 & \cdots \\
0 & (a_0)^1 & -(a_0)^0 & (a_1)^1 & -2(a_1)^0 & (a_2)^0 & 0 & \cdots \\
0 & (a_0)^2 & -(a_0)^1 & (a_1)^2 & -2(a_1)^1 & (a_2)^2 & 0 & \cdots \\
0 & (a_0)^3 & -(a_0)^2 & (a_1)^3 & -2(a_1)^2 + (a_0)^0 & (a_2)^3 & -3(a_2)^0 & \cdots \\
0 & (a_0)^4 & -(a_0)^3 & (a_1)^4 & -2(a_1)^3 + (a_0)^1 & (a_2)^4 & -3(a_2)^1 & \cdots \\
0 & (a_0)^5 & -(a_0)^4 & (a_1)^5 & -2(a_1)^4 + (a_0)^2 & (a_2)^5 & -3(a_2)^2 + 5(a_1)^0 & \cdots \\
0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
\]

(A6)

Of course there is an identical expression for $B_0$, made from arbitrary vectors $\vec{b}_k$.

**Requirement Two:**

The second set of requirements, imposed by the eigenvalue-type equations, Eqs. (6.10), may also be explicitly resolved. Both sides of the equations involve an infinite sum of elements of $A_i$ with elements of $C_k$. However, since the $c_{j+k} = (C_k)^j$ take on only the values 0, 1, 0, −1, and then repeat, such sums only generate alternating-sign sums of alternating elements of the matrices $A_i$. Defining (subscripted) vectors $\vec{u}_{ji}$,

\[
(\vec{u}_{ji})_\ell \equiv \begin{cases} 
(-1)^{j/2} \sum_{m=0}^{\infty} (-1)^m (A_i)^{2m+1}_\ell, & j \text{ even,} \\
(-1)^{(j-1)/2} \sum_{m=0}^{\infty} (-1)^m (A_i)^{2m}_\ell, & j \text{ odd,} 
\end{cases} \tag{A7}
\]

allows us to re-write Eqs. (6.10) and display the periodicity they possess:

\[
(\vec{u}_{ji})_\ell - (\vec{u}_{ij})_\ell - (\vec{u}_{i\ell})_j = 0 \quad . \tag{A8}
\]

The general recursion relation for these matrices allows to restrict our attention to $A_0$. This resolution does of course involve entries from both the (arbitrary) odd columns, and the even columns, which are given already as sums of the odd columns, via Eqs. (6.14). Therefore the results will involve the coefficients $P_{i,r}$ just determined, at Eqs. (A6).
We give explicit names to these sums for the matrix $A_0$:

$$R_j \equiv \sum_{m=0}^{\infty} (-1)^{j+m} (\vec{a}_j)^{2m} = \vec{C}_1 \cdot \vec{a}_j \ , \quad S_j \equiv \sum_{m=0}^{\infty} (-1)^{j+m} (\vec{a}_j)^{2m+1} = \vec{C}_0 \cdot \vec{a}_j \ .$$  \hspace{1cm} (A9)

When the requirements are written in terms of these quantities, they induce recursion relations for them. For example, values of \{0, 1, 2\} for the indices gives us $3S_0 + S_1 = 0$. As the indices grow, the number of terms in the relations grows. After some little algebra, we acquire the relations in the form of $(n+2)$-term recursion relations for the desired quantities, $R_n$ and $S_n$:

$$(n-1)R_n = -R_0 + \sum_{h=1}^{n} \mathcal{P}_{n+1,h} R_{n-h} \ , \quad S_n = S_0 - 2 \sum_{h=1}^{n} \mathcal{P}_{n,h-1} S_{n-h} \ ,$$

along with identical relations for $U_n$ and $V_n$, made analogously with the sums made from the elements of $\mathcal{B}_0$. The equations require an initial input of values for $R_0$, $R_1$, and $S_0$, but then allow explicit, recursive calculation of the values of all the others. This calculation results in the numerical sequences, for $w_m$ and $q_j$, given in Eqs. (6.16).

**Requirement Three:**

Taking the form already determined for each of the matrices $\mathcal{A}_n$ and $\mathcal{B}_m$, the quadratic requirements may be written in terms of the various quantities $\mathcal{A}_0 (\Lambda^T)^n \vec{b}_j$ and $\mathcal{B}_0 (\Lambda^T)^n \vec{a}_j$:

$$\mathcal{A}_0 (\Lambda^T)^n \vec{b}_r = \vec{K}_n \ell = \mathcal{B}_0 (\Lambda^T)^n \vec{a}_r \ ,$$

where the 2-index vectors $\vec{K}_n \ell$ have elements that are always integers. The equations may be written out explicitly in terms of the vectors themselves, or in terms of the components, with the latter actually being simpler-appearing set of expressions. The numerical quantities have the form

$$\vec{K}_n \ell \equiv \sum_{p=0}^{[n/2]} (-1)^{n+p+\ell} \binom{n}{2p} (\Lambda^T)^{n-2p} \ell - \sum_{q=0}^{[(n-1)/2]} (-1)^{n+q} \binom{n}{2q+1} (\Lambda^T)^{n+2\ell-2q} \ell \ ,$$

$$\left(\vec{K}_n \ell^i\right) = \begin{cases} (-1)^{m+r+j} \left( \binom{2m+1}{2j-1} - \binom{2m+1}{2(j-r-1)} \right), & \text{for } i \equiv 2j \text{ and odd } n \equiv 2m+1, \\ 0, & \text{for odd } i \text{ and odd } n \\ 0, & \text{for even } i \text{ and even } n \\ (-1)^{m+r+j} \left( \binom{2m}{2j} - \binom{2m}{2(j-r)-1} \right), & \text{for odd } i \equiv 2j+1 \text{ and even } n \equiv 2m, \end{cases}$$

(A13)
where the vector, $\vec{\ell}$ is just $(0, 1, 0, 0, 0, \ldots)^T$, the symbol $[n/2]$ means the greatest integer in $n/2$. The left- and right-hand sides of the equations must also be decomposed:

$$A_0(\Lambda^T)^n\vec{b}_r = \begin{cases} \sum_{k=0}^{\infty} \left\{ (\vec{b}_\ell)^{2k} \vec{a}_{m+k} + \sum_{u=0}^{m+k} P_{m+k+1,u}(\vec{b}_\ell)^{2k+1}(\Lambda^T)^{2u+1}\vec{a}_{m+k-u} \right\}, & \text{for odd } n \equiv 2m + 1, \\ \sum_{k=0}^{\infty} \left\{ (\vec{b}_\ell)^{2k+1} \vec{a}_{m+k} + \sum_{u=0}^{m+k-1} P_{m+k,u}(\vec{b}_\ell)^{2k}(\Lambda^T)^{2u+1}\vec{a}_{m+k-1-u} \right\}, & \text{for even } n \equiv 2m, \end{cases} \quad (A14)$$

and a similar expression for $B_0(\Lambda^T)^n\vec{a}_r$, with the roles of $\vec{b}_j$ and $\vec{a}_i$ reversed. When inserted into Eqs. (A11), these constitute the most general form of the quadratic conditions.

As discussed in Section VI, the equations just displayed seem to be too difficult to fully digest. On the other hand, as noted there, the fact that half of the quantities in Eqs. (A13) are in fact simply zero causes one to think of proposing that a simpler solution might still be obtained in half of the elements of our arbitrary vectors were set directly to zero. When this is done, the remaining half of the equations, still to be satisfied, take the following form:

$$\sum_{k=0}^{\infty} (\vec{b}_\ell)^k \vec{a}_{m+k} = \vec{M}_m = \sum_{k=0}^{\infty} (\vec{a}_\ell)^k \vec{b}_{m+k}, \quad (A15)$$

$$\sum_{k=0}^{\infty} \sum_{u=0}^{m+k} P_{m+k+1,u}(\vec{b}_\ell)^k(\vec{a}_{k+m-u})^{j-u-1} = (-1)^{m+\ell+j} \left\{ \begin{pmatrix} 2m + 1 \\ 2j - 1 \end{pmatrix} - \begin{pmatrix} 2m + 1 \\ 2j - 2\ell - 2 \end{pmatrix} \right\}$$

$$= \sum_{k=0}^{\infty} \sum_{u=0}^{m+k} P_{m+k+1,u}(\vec{a}_\ell)^k(\vec{b}_{k+m-u})^{j-u-1}, \quad (A16)$$

where the elements of the vectors $\vec{M}_m$ are simply combinations of binomial coefficients:

$$(\vec{M}_m)^j \equiv (-1)^{m+\ell+j} \left\{ \begin{pmatrix} 2m \\ 2j \end{pmatrix} - \begin{pmatrix} 2m \\ 2j - 2\ell - 1 \end{pmatrix} \right\} . \quad (A17)$$
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37. J. D. Finley, III, “The Robinson-Trautman Type III Prolongation Structure Contains \(K_2\),” to be published.

38. Private communication from Mark Hickman.

39. B. Kent Harrison and Frank B. Estabrook, “Geometric Approach to Invariance Groups and Solution of Partial Differential Systems,” J. Math. Phys. 12, 653-65 (1971); Peter J. Olver, Applications of Lie Groups to Differential Equations (Springer-Verlag, New York, 1986).

40. See p. 114 ff of Ref. 13 for a description of effective subideals, and some examples for this equation.

41. P. Molino, J. Math. Phys. 25, 2222-5 (1984), discusses why this can be done for the KdV equation in a way which easily generalizes to the case when the independent variables do not appear explicitly in the equation, and that equation is quasilinear.

42. Mark Hickman and JDF spent a few weeks attempting to find general integrals for this problem. Krasil’shchik, in Ref. 32, gives only such one-dimensional coverings.

43. Frank B. Estabrook, “Moving frames and prolongation algebras,” J. Math. Phys. 23, 2071-6 (1982).

44. Shlomo Sternberg, Lectures on Differential Geometry (Chelsea Publ. Co., New York, 1983). See Ch. 5. Our form is simply a coordinate presentation of the action on a Lie algebra of the flow of a vector field generated by the adjoint action of another element of the Lie algebra.

45. R. Hermann, Differential Geometry and the Calculus of Variations, 2nd Ed., Interdisciplinary Mathematics XVII (Math. Sci. Press, Brookline, Mass., 1977). See Ch. 18, Accessibility Problems for Path Systems.

46. In Ref. 34, §3.1.2, those authors begin from a set of potential equations which has the sine-Gordon equation as an integrability condition. This allows them to generate the center of
$A_1^{(1)}$ itself, which distinguishes it from $A_1 \otimes \mathbb{C}[^{\lambda^{-1}}, \lambda]$. Our approach seems unable to do this.

47. Although re-discovered by Dodd and Bullough, in Ref. 8, this equation was first studied by M. Tzitzéica, “Sur une nouvelle classe de surfaces,” Comptes Rendu Acad. Sci. 150, 955-6 (1910), as I learned from Graeme A. Guthrie.

48. L. O’Raifeartaigh, *Group structure of gauge theories* (Cambridge Univ. Press, Cambridge, UK, 1986).

49. Cornelius Hoenselaers, private communication.

50. Ulf Grenander and Gabor Szegö, *Toeplitz Forms and Their Applications* (University of California Press, Berkeley, Calif., 1958), describes earlier work in this area. R. G. Douglas, *Banach algebra techniques in the theory of Toeplitz operators*, Conference Board of the Mathematical Sciences, No. 15 (American Mathematical Society, Providence, RI, 1973), and Ref. 51 are modern presentations of what seems to us to be near the “state of the art.”

51. Albrecht Böttcher and Bernd Silbermann, *Analysis of Toeplitz Operators* (Springer-Verlag, Berlin, 1990). See p. 57 for this particular notion of conjugate functions.