1. Introduction

In this paper I study the intrinsic geometry of convex polyhedra in three-dimensional hyperbolic space $\mathbb{H}^3$, with all vertices on the sphere at infinity $S^2_{\infty}$. Such a polyhedron $P$ is homeomorphic to the sphere $S^2$ with a number of punctures (corresponding to the vertices of $P$). It is not hard to see that $P$ is a complete hyperbolic surface of finite area.

In this paper I prove the following converse:

**Theorem 1.1. Characterization of ideal polyhedra** Let $M_N$ be a complete hyperbolic surface of finite area, homeomorphic to the $N$ times punctured sphere. Then $M_N$ can be isometrically embedded in $H^3$ as a convex polyhedron $P_N$ with all vertices on the sphere at infinity.

Furthermore,

**Theorem 1.2. Uniqueness of realization** The polyhedron $P_N$ promised by Theorem 1.1 is unique, up to congruence.

In fact, Theorem 1.2 is a special case of the following more general result:

**Theorem 1.3. Uniqueness of generalized polyhedra** A generalized polyhedron $P_N$ in $H^3$ is determined by its intrinsic metric, up to congruence.

A generalized polyhedron is a polyhedron some of whose vertices are inside $\mathbb{H}^3$, some are on the ideal boundary of $\mathbb{H}^3$ and some are beyond the ideal boundary. This is best visualized in the projective model of $\mathbb{H}^3$ as a (Euclidean) polyhedron, all of whose edges intersect the unit ball $\mathbb{H}^3$. The derivation of Theorem 1.3 from Theorem 1.4 is essentially the same as that of Theorem 1.2.
The proof of Theorem 1.1 uses the Invariance of Domain Principle of A. D. Aleksandrov. This is as follows:

Given a map \( f : A \to B \) between topological spaces \( A \) and \( B \), then \( f \) is onto, provided the following criteria are satisfied:

1. The image of \( f \) is non-empty.
2. \( f \) is continuous.
3. \( f \) maps open sets in \( A \) to open sets in \( B \).
4. \( f \) maps closed sets in \( A \) to closed sets in \( B \).
5. \( B \) is connected.

of Theorem 1.1 (Outline). Let \( M_\mu \) be a surface homeomorphic to the \( N \)-times punctured 2-sphere, with a marking \( \mu \), that is, a labelling of the punctures. Let \( \mathcal{P}^N \) be the space of convex ideal polyhedra in \( \mathbb{H}^3 \), parametrized by the positions of their vertices on the sphere at infinity of \( \mathbb{H}^3 \) (interpreted as the Riemann sphere \( \mathbb{C} \)). Three of the vertices are fixed at 0, 1 and \( \infty \), which eliminates the action of the isometry group of \( \mathbb{H}^3 \). \( \mathcal{P}^N \) is easily seen to be a \( 2N - 6 \) dimensional manifold. \( \mathcal{P}^N \) plays the role of \( A \) in the invariance of domain principle. We will abuse notation and view a polyhedron \( P \) both as a geometric object and as a polyhedral isometric embedding of \( M \) into \( \mathbb{H}^3 \). \( P \) will inherit the labelling of vertices from \( \mu \).

Let \( \mathcal{T}^N \) be the set of complete, finite volume hyperbolic structures on \( M_\mu \).

This set is parametrized by shears along the edges of a geodesic triangulation. This parametrization is explained in detail in section 2. It will also be shown (Theorem 2.10) that \( \mathcal{T}^N \) is a \( 2N - 6 \) dimensional contractible manifold. Although many of the results of section 2 are known to Teichmüller theorists, they are so elementary in this particular setting that it was impossible to resist including a full exposition.

\( \mathcal{T}^N \) will play the role of \( B \) in the invariance of domain principle.

The role of the map \( f \) will be played by the map \( g : \mathcal{P}^N \to \mathcal{T}^N \). \( g(P) \) is \( P \) viewed as an abstract Riemannian manifold. The continuity of \( g \) with respect to the chosen coordinate systems on \( \mathcal{P}^N \) and \( \mathcal{T}^N \) is the content of Theorem 3.7.

Since \( \mathcal{P}^N \) and \( \mathcal{T}^N \) are manifolds of the same dimension and \( g \) is continuous, Theorem 1.2 shows that \( g \) is an open map. That \( g \) is closed is the content of Theorem 3.9.

The theory developed in Section 3 is of independent interest. In particular, it leads to a very simple derivation of a set of conditions satisfied by dihedral angles of an ideal polyhedron (Theorem 3.12).

2. Hyperbolic geometry of the \( N \)-times punctured sphere

2.1. Geometry of triangles. Let \( ABC \) be an ideal triangle in \( \mathbb{H}^2 \). Pick a point \( p \) on the geodesic \( AB \). How far is \( p \) from \( A \)? This question turns out to make sense in \( \mathbb{H}^2 \):
Consider the unique horocycle $h_A$ centered at $A$ and passing through $p$. $h_A$ will intersect $AC$ in a point $q$. Define $D_{ABC}(p)$ to be the distance along $h_A$ between $p$ and $q$. $D_{ABC}$ has the following important property:

**Lemma 2.1.** Let $p_1$ and $p_2$ be two points on $AB$. The hyperbolic distance between $p_1$ and $p_2$ is equal to $|\log(D_{ABC}(p_1)/D_{ABC}(p_2))|$. 

**Proof.** Use the upper half-space model of $H^2$. Recall that the hyperbolic metric is related to the Euclidean metric on the upper half-space by $ds_h = |dz|/\Im(z)$. This means, in particular, that the hyperbolic distance between $z_1 = x + iy_1$ and $z_2 = x + iy_2$ is $|\log(y_1/y_2)|$.

By a hyperbolic isometry $A$ can be sent to $\infty$, $B$ to 0 and $C$ to 1. Horocycles centered on $A$ are then simply horizontal lines, and if $p = x + iy$, then $D(p) = 1/y$, since the metric on the horocycle around infinity through $p$ is then simply the standard metric on the real line, rescaled by $1/y$. The assertion of the Lemma now follows. \qed

Lemma 2.1 gives a way to quantify the ways in which two ideal triangles can be joined together along a side to form an ideal quadrilateral. Intuitively, ideal triangles $ABC$ and $ADC$ can slide with respect to each other along the common side $AC$. Pick a point $p$ on $AC$. If $D_{ACB}(p) = D_{ACD}(p)$ ($D$ is taken with respect to the vertex $A$), then we say that $ABC$ and $ADC$ are joined without a shear (and it is easy to see that reflection in $AC$ will send $B$ to $D$ and vice versa). Otherwise, $ABC$ and $ADC$ are joined with shear $\log(D_{ACB}(p)/D_{ACD}(p))$. It is clear that shear doesn’t depend on which of the vertices $A$ or $C$ is taken as the center of the horocycles. Henceforth the shear between triangles $t_1$ and $t_2$ will be denoted by $s(t_1, t_2)$. By abuse of notation $s(ABCD) = s(ABC, ADC)$.

All of the above is quite easily seen in the upper half-space model of $H^2$ – see Figure 1. Figure 1 also demonstrates Lemma 2.2.

**Lemma 2.2.** If the shear is $\alpha$, and the triangle $ABC$ is positioned so that $A = \infty$, $B = 1$, $C = 0$, then $\log |D| = \alpha$.

This also shows that the following fundamental lemma:

**Lemma 2.3.** $s(ABCD)$ is equal to the log of the absolute value of the cross ratio $[C, B, D, A]$.

Recall that cross-ratio of $z_1, z_2, z_3, z_4$ is defined to be

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_2)(z_3 - z_4)}$$

**Proof.** Both $s(ABCD)$ and $[C, B, D, A]$ are invariant under the Möbius group, and since any three points can be transformed to 0, 1, and $\infty$ by a Möbius transformation, the Lemma follows from the obvious special case where $A = \infty$, $B = 1$, and $C = 0$. \qed
2.2. Geometry of triangulations. Recall that $M_{\mu}$ is a surface homeomorphic to the 2-sphere with $N$ punctures, together with its marking.

Let $T$ be a triangulation of $M_{\mu}$. Then the above discussion serves to parametrize all the complete hyperbolic structures on $(M, \mu)$ where the faces of $T$ are ideal triangles — to each edge of $T$ we associate the shear of the two abutting faces of $T$. This information specifies the geometry completely. On the other hand, it is not quite true that any assignment of real numbers to edges of $T$ corresponds to a complete hyperbolic structure on $M_{\mu}$ with those numbers as shears – it is necessary and sufficient that the sum of shears around any cusp add up to zero (pick a horocycle $h$ centered at a vertex $v$ of $T$. In order for a hyperbolic structure to be complete, $h$ must close up). Thus if there are $V$ vertices and $E$ edges of $T$, the set of hyperbolic structures on $M_{\mu}$ such that $T$ is an ideal triangulation is naturally parametrized as $R^{E-V}$. Note that an Euler’s formula computation yields $E - V = 2V - 6$, so the dimension of the space of hyperbolic structures depends only on the number of cusps.

The following lemma shows that the reliance on the particular triangulation $T$ in the above discussion is not critical — any other topological ideal triangulation will do as well:

Lemma 2.4. straightening Any topological triangulation $\tilde{T}$ with all vertices at cusps of $M_{\mu}$ can be straightened to a geodesic triangulation.

Proof. All that is necessary to show is that if $v_1, v_2, v_3$ and $v_4$ are cusps and there are non-intersecting curves $\gamma_1$ connecting $v_1$ and $v_2$ and $\gamma_2$ connecting $v_3$ and $v_4$, then the corresponding geodesics also don’t intersect. That
this last statement is true can be observed by examining the geometry of universal cover of $S$. The postulated curves $\gamma_1$ and $\gamma_2$ exist if and only if the lifts of $v_1$ and $v_2$ do not separate the lifts of $v_3$ and $v_4$ on the circle at infinity of $H^2$. In that case, however, it is seen that the corresponding geodesic segments do not intersect either.$\square$

Thus, every topological triangulation of $M_\mu$ with vertices at the cusps (topological ideal triangulation) corresponds to a coordinate system on the space of hyperbolic metrics on $S$ with cusps at the prescribed vertices.

**Definition 2.5.** The space of complete hyperbolic structures on $M_\mu$ shall be denoted by $T^N$. A shear coordinate system corresponding to a triangulation $T$ on $T^N$ is the map $C_T : T^N \to \mathbb{R}^{2N-6}$ associating to a particular metric its shears along the straightened edges of $T$.

**Theorem 2.6.** For any two triangulations $T_1$ and $T_2$, the map $c_{T_1T_2} = C_{T_2} \circ C_{T_1}^{-1}$ is a continuous function from $\mathbb{R}^{2N-6}$ to itself.

To prove Theorem 2.6 it will first be necessary to understand triangulation graph of the sphere with $N$ vertices.

**Definition 2.7.** Let $T$ be a triangulation, and $ABC$ and $ADC$ be two of the triangles of $T$ sharing the edge $AC$. Then the Whitehead move $w_{ABCD}$ transforms $T$ into a triangulation $T'$, where the triangles $ABC$ and $ADC$ are replaced by triangles $BAD$ and $BCD$ (in other words the diagonal of the quadrilateral $ABCD$ is “flipped”).

![Figure 2. A Whitehead move.](image-url)
Definition 2.8. The triangulation graph $T_N$ is a graph whose vertices are isotopy classes of triangulations of $S^2$ on $N$ vertices, and there is an edge joining nodes corresponding to $T_1$ and $T_2$ if and only if there exists a Whitehead move transforming $T_1$ and $T_2$.

Theorem 2.9. The graph $T_N$ is connected.

Proof. Pick two distinguished vertices $v_1$ and $v_2$. For any starting triangulation $T$ there is a sequence of Whitehead moves transforming $T$ into a triangulation $T_{v_1}$ where $v_1$ is connected to every other vertex. Now consider the complement in $T_{v_1}$ of $v_1$ and all the edges incident to it. This will be a triangulation of an $(N - 1)$-gon, all of whose vertices are those of the $(N - 1)$-gon. By a similar argument, this can be transformed by a sequence of Whitehead moves into a triangulation where every vertex is connected to $v_2$. Thus, it is seen that by a sequence of Whitehead moves, every triangulation can be transformed to a particular triangulation (where both $v_1$ and $v_2$ have valence $N - 1$), and thus $T_N$ is connected.

We summarize the results of this section for convenience:

Theorem 2.10. $T^N$ is a contractible $2N - 6$ dimensional manifold, as evidenced by coordinate systems coming from ideal triangulations of $S^2_N$. Any two such coordinate systems are analytically equivalent.

3. Geometry of ideal tessellations

First, let us review briefly the geometry of the upper half-space model of $\text{H}^3$. We will think of the ideal boundary $S^2_{\infty}$ of $\text{H}^3$ as the Riemann sphere $\overline{C}$. Hyperbolic planes are represented by hemispheres whose equatorial circles are in $S^2_{\infty}$. In the present context we think of straight lines as circles passing through $\infty$. The corresponding hemispheres are vertical planes rising above the lines.

Let $P$ be a convex polyhedron with all vertices on the ideal boundary of $\text{H}^3$. $P$ is the intersection of the half-spaces defined by its faces. By an isometry of $\text{H}^3$ and relabelling we can transform $P$ so that the face $f_1$ of $P$ lies in the plane rising above the real axis in $S^2_{\infty}$, and the vertices $v_1$, $v_2$ and $v_3$ are 0, 1 and $\infty$ respectively. Furthermore, without loss of generality we assume that $P$ lies above the half-plane $\Im(z) \geq 0$.

The rest of the faces of $P$ are then oriented in such a way that the interior of the corresponding hemispheres lie outside of $P$. 
$P$ defines a Euclidean tessellation of $\mathbb{C}$ in the natural way: $P$ casts a shadow on the ideal boundary of $\mathbb{H}^3$ under the orthogonal projection. The edges of $P$ are then mapped to straight-line segments, and the faces of $P$ to convex polygons. Denote the resulting tessellation of $\mathbb{C}$ by $T_P$. This tessellation has the following properties:

**Condition 1.** Every face $F$ of $T_P$ is inscribed in the circle $C_F$.

**Condition 2.** No vertices of $T_P$ are contained in the interior of $C_F$.

**Condition 3.** $T_P$ is contained in the upper half-plane of $\mathbb{C}$.

In the sequel we will assume for simplicity that $T_P$ is a triangulation (unless otherwise indicated). Any more general tessellation can be subdivided until it is a triangulation. First we note the following:

**Lemma 3.1.** Condition 2 of is equivalent to the following:

**Condition 2'.** for any two abutting triangles $ABC$ and $ADC$ of $T_P$, $D$ is not in the interior of $C_{ABC}$.

**Proof.** Consider $P$. Lemma 3.1 is equivalent to the observation that the polyhedron $P$ is convex (Condition 2) if and only if all of its edges are convexly bent (Condition 2').

**Note.** A simple direct Euclidean proof of Lemma 3.1 is also possible. This will be left as an exercise for the reader.

**Corollary 3.2.** Given an arbitrary triangulation $T'$ on the same vertex set as $T_P$, $T'$ can be transformed into $T_P$ by a finite sequence of Whitehead moves of the following kind: whenever $ABC$ and $ADC$ are abutting triangles of $T'$ such that $D$ lies inside $C_{ABC}$, we change $ABC$ and $ADC$ into $ABD$ and $CBD$.

**Proof.** A Whitehead move of the described type corresponds to filling in a missing tetrahedron $ABCD$ of a polyhedron lying above $T'$. Every time a move as above happens, the edge $AC$ is buried, never to be seen again. Since the number of possible edges is finite, the result follows.

The following facts from elementary Euclidean geometry will be needed in the sequel.

**Lemma 3.3.** Let $C$ be a circle with center $O$ and let $ABC$ be a triangle inscribed in $C$. Then the $\angle ACB = \frac{1}{2} \angle AOB$ if $C$ and $O$ are on the same side of $AB$ and $\angle ACB = \pi - \frac{1}{2} \angle AOB$ otherwise.

**Lemma 3.4.** Let $ABCD$ be a quadrilateral.

- $D$ is outside $C_{ABC}$ if $\angle D + \angle B < \pi$.
- $D$ is inside $C_{ABC}$ if $\angle D + \angle B > \pi$.
- $D$ is on $C_{ABC}$ if $\angle D + \angle B = \pi$.

The following important fact follows from Lemma 3.3.
Theorem 3.5. The dihedral angle between the faces $ABC$ and $ADC$ of $P$ is equal to the sum of angles $\angle ABC$ and $\angle ADC$.

Proof. First, observe that the angle between the the faces $ABC$ and $ADC$ is equal to the angle between the circles $C_{ABC}$ and $C_{ADC}$. The rest of the proof is contained in Figure 4.

The sum of the angles $\angle ABC$ and $\angle ADC$ is easily seen to equal the argument of the cross-ratio $c(A,B,C,D) = [B,C,A,D] = \frac{(B-A)(C-D)}{(B-C)(A-D)}$.

If $A$, $B$, $C$, and $D$ are transformed by a hyperbolic isometry in such a way that $A = \infty$, $B = 1$, $C = 0$, and $D = z$, then $c(A,B,C,D) = z$, then Theorem 3.6 below follows from the discussion of section 2.

Theorem 3.6. With notation as above $s(ABC, ADC) = \log |c(A, B, C, D)|$.

Proof. This is just a “bent” three-dimensional version of Lemma 2.3.

The above observations allow us to prove Theorems 3.7 and 3.9, which are two of the steps of the proof of Theorem 1.1.

Theorem 3.7. The map $g$ is continuous.

Proof. The simpler case is one where $T_P$ is a genuine triangulation. Then, it is clear that a small perturbation of the vertices of $T_P$ doesn’t change the combinatorics of $T_P$, and so continuity follows from Theorem 3.6 and the continuity of the cross-ratio. Things are very slightly more complicated
when $T_P$ has non-triangular faces. Then, $T_P$ is combinatorially unstable: a small perturbation in the vertices changes the combinatorial structure, but only in the following simple fashion:

**Lemma 3.8.** For a sufficiently small $\epsilon$, a perturbation $T_P^\epsilon$ of $T_P$ is combinatorially equivalent to $T_P$ with some diagonals added to the non-triangular faces.

Proof of Lemma. There exists an $\epsilon > 0$, such that if a point $D$ is closer than $\epsilon$ to $C_{ABC}$ then $A$, $B$, $C$, and $D$ are co-circular, for any triangle $ABC$ and vertex $D$ of $T_P$. 

Every way of adding diagonals to $T_P$ until we get a triangulation corresponds to a different coordinate system of $T_N$ (where $N$ is the number of vertices of $P$), and Lemma 3.8 shows that every sufficiently small perturbation of $T_P$ is close to $T_P$ in at least one of the coordinate systems. Since the transition maps between the various coordinate systems are continuous (Theorem 2.6), Theorem 3.7 follows.

**Theorem 3.9.** The image of $g$ is closed.

Proof. Let $\mathcal{H}_1, \ldots, \mathcal{H}_k, \ldots$ be a sequence of metrics on $S^2_{\lambda}$ converging to a metric $\mathcal{H}$. Let $P_1$ be such that $g(P_1) = \mathcal{H}_1$. We will show that there exists a $P_\infty$ such that $g(P_\infty) = \mathcal{H}$. First choose a subsequence $P_1', \ldots, P_i', \ldots$ such that all of the $P_i'$ have the same combinatorics. This is possible since the number of possible combinatorial structures is finite. As before, the vertices and faces of $P_i'$ are labelled in such a way that $v_1(P_i') = 0$, $v_2(P_i') = 1$, 

![Figure 4. Dihedral angle](image-url)
$v_3(P'_i) = \infty$, and $f_1(P'_i)$ lies above the real axis. By compactness of the sphere $\mathbb{C}$, there exists a limiting tessellation $T_{P'\nu}$. If $T_{P'\nu}$ is non-degenerate (that is, no two vertices of a triangle have coalesced into one), then by Theorem 3.7 it follows that $g(P') = \mathcal{F}$.

We will show that $T_{P'\nu}$ is always non-degenerate. If this is not the case, let $t_i$ be a collapsing face of $T_{P'\nu}$ which is abutting a non-collapsing face $t_j$. Such a pair of faces must exist, since at least one face ($f_1$) is not collapsing. By relabelling and a hyperbolic isometry, send $t_j$ to the triangle $0, 1, \infty$, and $t_i$ to $0, \infty, z$. Since $s(t_j, t_i) = \log |z|$, and $\mathcal{F}$ is a non-degenerate metric it follows that $z$ stays away from 0 and $\infty$, and so $t_i$ is not collapsing after all.

Remark 3.10. Theorem 3.5 and subsequent discussion is easily seen to lead to the following pleasing “hyperbolic” interpretation of planar triangulations. Consider a (not necessarily convex) polygon $Q$ in the plane, such that the interior of $Q$ is triangulated in such a way that edges of $Q$ are edges of the triangulation $T(Q)$. Then $T(Q)$ is the projection of an ideal polyhedron $\tilde{Q}$ onto the plane $\mathbb{C}$ at infinity of $H^3$, such that:

1. $\tilde{Q}$ has vertices at the vertices of $T(Q)$, plus one vertex at the point $\infty$ of $\mathbb{C}$.
2. $Q$ is similar to the link of the vertex $v_\infty$ of $\tilde{Q}$ at $\infty$.
3. $\tilde{Q}$ is star-shaped with respect to $v_\infty$.
4. If $v_1, \ldots, v_k$ are the vertices of $Q$, then the dihedral angle of $\tilde{Q}$ corresponding to the edge $v_kv_\infty$ is the Euclidean angle of $Q$ at $v_k$.
5. The dihedral angle of $\tilde{Q}$ corresponding to the boundary edge $v_iv_{i+1}$ is equal to the euclidean angle at the third vertex $w$ of the (unique) triangle $wv_iv_{i+1}$ of $T(Q)$ containing the edge $v_iv_{i+1}$.
6. The dihedral angle of $\tilde{Q}$ corresponding to a non-boundary edge $AB$ of $T(Q)$ is equal to the sum of the angles at $C$ and $D$ of the two triangles $ACB$ and $ADB$ abutting along the edge $AB$.

The triangulation $T(Q)$ could also be taken to be immersed, in which case all of the above statements still hold, with the obvious changes in interpretation.

Definition 3.11. A set of edges $C = \{e_1, \ldots, e_k\}$ in a graph $G$ is called a cutset, if the removal of those edges disconnects $G$. A cutset $C$ is called minimal, if no subset of $C$ is a cutset.

The simplest example of a cutset is the set of edges incident to a single vertex of $G$.

The correspondence above can be used to prove the following result:

Theorem 3.12. Let $e_1, \ldots, e_k$ be a minimal cutset of the 1-skeleton $\tilde{Q}$. Then the sum $\Sigma$ of dihedral angles at $e_1, \ldots, e_k$ is strictly smaller than $(k - 2)\pi$ if $e_1, \ldots, e_k$ are not all incident to one vertex. If $e_1, \ldots, e_k$ are all incident to one vertex then $\Sigma$ is exactly $(k - 2)\pi$. 
Proof. (also see figure 5) It is easy to see that any minimal cutset as above is actually the set of internal edges of a triangulation $T(A)$ of an annulus $A$ (possibly with one boundary component collapsed to a point $v$, if all of the $e_i$ are incident to $v$.) From now on all references will be to quantities in $T(Q)$ Let the inner and outer boundary components of $A$ be $A_1$ and $A_2$, respectively. The edges of $T(A)$ naturally fall into three categories – outer boundary edges, inner boundary edges and internal edges. Similarly, divide the angles of the triangles of $T(A)$ into the three sets – $A$ (angles opposite outer boundary), $B$ (angles opposite inner boundary) and $\Gamma$ (angles opposite inner edges). Obviously, 
\[ \sum A + \sum B + \sum \Gamma = k\pi \] (where $k$ is the number of triangles and the cardinality of the cutset). Furthermore, by Theorem 3.5, 
\[ \sum \Gamma = \Sigma. \] Now, note that the sum of the angles incident (not opposite) to $A_2$ is $(\text{card } A_2 - 2)\pi$, and further note that this sum is equal to $\sum B + (\text{card } A_2)2\pi - \sum A$. That is true since if $\alpha$ is opposite to $A_2$, then the other angles of the triangle containing $\alpha$ are incident to $A_2$. Now, since 
\[ (\text{card } A_2 - 2)\pi = \sum B + \pi \text{ card } A_2 - \sum A, \] it follows that $\sum A - \sum B = 2\pi$. Since $\sum B$ is greater than zero precisely when the inner boundary of $A$ is non-degenerate, it follows that $\sum A + \sum B > 2\pi$ whenever the inner boundary of $A$ is non-degenerate and $\sum A + \sum B = 2\pi$ otherwise. The statement of the theorem then follows from equations 1, 2, and 3. 

The above theorem is slightly stronger than Theorem 1 of [2], which is stated only for convex polyhedra. It turns out that the conditions of Theorem 3.12 together with the convexity conditions (dihedral angles are between 0 and $\pi$) completely characterize the sets of dihedral angles of convex polyhedra. Proof of sufficiency is given in an upcoming paper of the author.

4. IDEAL POLYHEDRA ARE DETERMINED BY THEIR METRIC

The purpose of this section is to prove Theorem 1.2. First, a couple of definitions:

**Definition 4.1. (Generalized polyhedra and polygons)** A generalized convex hyperbolic polyhedron is represented in the projective model of $\mathbb{H}^3$ by a Euclidean convex polyhedron which may have some vertices on or outside the sphere at infinity (called “infinite” and “hyperinfinite” vertices respectively). However, each edge must contain some points inside hyperbolic space. We will usually only be concerned with the part of a generalized
polyhedron lying within $\mathbb{H}^3$. Generalized convex polygons in $\mathbb{H}^2$ are defined similarly.

**Definition 4.2.** *(Links of vertices)* A generalized hyperbolic polyhedron has vertices of three types: finite, hyperinfinite and infinite vertices.

The “link” of a finite vertex of a polyhedron is the spherical polygon obtained by intersecting a small sphere centered at the vertex with the polyhedron, and rescaling so the sphere has radius 1. So the edge lengths in the link are precisely the face angles at the vertex.

For each hyperinfinite vertex there is a unique hyperbolic plane orthogonal to the faces meeting at the vertex. The intersection of this plane with these faces is a hyperbolic polygon which we will call the “link” of the vertex. The edge lengths in the link are precisely the lengths of common perpendiculars to adjacent sides meeting at the hyperinfinite vertex.

For each infinite vertex there a 1-parameter family of horospheres centered at the vertex. Each small horosphere intersects the polyhedron in a Euclidean polygon, which we will call the “link” of the vertex. In this case the link is only well defined up to Euclidean similarities.

**Remark.** The link of an infinite or hyperinfinite vertex then determines the corresponding *end* of the polyhedron up to congruence.

**Remark 4.3.** The link of an ideal vertex of $P$ is a Euclidean convex polygon. Theorem 3.6 shows that if all vertices of $P$ are ideal, then the logarithm of the ratio of two adjacent sides of the link of a vertex $v$ is equal to the shear between the two corresponding faces.
The following result is obtained in [1] (see [3], Theorem 4.10):

**Theorem 4.4.** A generalized convex polyhedron $P$ in hyperbolic 3-space is determined up to congruence by the type of its vertices and the edge lengths of the links of its vertices.

**Note 4.5.** The edges of $P$ are not required to be non-degenerate, so some of the dihedral angles may be $\pi$.

This theorem means that two combinatorially equivalent polyhedra $P_1$ and $P_2$ such that the corresponding sides of corresponding links of $P_1$ and $P_2$ are equal are congruent.

of Theorem 4.4. Let $M$ be a complete finite-volume hyperbolic surface homeomorphic to $S^2 \times \mathbb{R}$. Let $P_1$ and $P_2$ be two different embeddings of $M$ into $\mathbb{H}^3$ as convex polyhedra. If $P_1(M)$ and $P_2(M)$ are combinatorially equivalent, Theorem 4.4 implies that $P_1(M)$ and $P_2(M)$ are congruent.

Assume that $P_1(M)$ and $P_2(M)$ are not combinatorially equivalent. Then $P_1$ and $P_2$ induce two different cell decompositions $Q_1$ and $Q_2$ of $M$, where the edges of $Q_i$ are preimages of corresponding edges of $P_i(M)$. Produce a new cell decomposition $Q$ of $M$ by superimposing $Q_1$ and $Q_2$. The vertex set of $Q$ is the union of $V$ (the cusps of $M$) with the set $V'$ of intersections of edges of $Q_1$ with those of $Q_2$. The image of $Q$ under $P_i$ will be $P_i(M)$ with some extra edges and vertices drawn on it. Then we can treat $P_1(M)$ and $P_2(M)$ as being of the same type (that of $Q$), and Theorem 4.4 may be applied. Thus $P_1(M)$ and $P_2(M)$ are congruent.

5. Directions for further research

To the author, the most painful shortcoming of the results presented in this paper is the lack of any constructive method of producing an embedding of a hyperbolic $N$-punctured sphere $S$ into $\mathbb{H}^3$ as a convex ideal polyhedron. In particular, the tesselation of $S$ induced by such an embedding is clearly canonical (in view of Theorem 4.2), and yet there seems no known method of producing it.

The simple Theorem 3.3 turns out to be very useful. A number of consequences are given in the author’s paper [4]. An efficient algorithm for producing a convex ideal polyhedron with prescribed dihedral angles is contained in an upcoming joint paper of the author and Warren D. Smith.

Acknowledgements. The author would like to thank Craig Hodgson and Warren D. Smith for valuable comments on earlier drafts of this paper.

**References**

[1] Igor Rivin. *On geometry of convex polyhedra in hyperbolic 3-space*. PhD thesis, Princeton, June 1986.

[2] Igor Rivin. On geometry of convex ideal polyhedra in Hyperbolic 3-space. To appear in *Topology*, January 1993.
[3] Igor Rivin and C. D. Hodgson. A characterization of compact convex polyhedra in hyperbolic 3-space. *Inventiones Mathematicae*, 111(1), January 1993.

[4] Igor Rivin Euclidean Structures on simplicial surfaces and hyperbolic volume. *Annals of Math. (2)*, **139** (1994), no. 3, pp. 553-580.

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