COMPLEX AND REAL RANKS OF REDUCIBLE CUBICS

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Abstract. In this paper, we give a complete description of the complex and the real Waring ranks of reducible cubic forms over $\mathbb{C}$.

1. Introduction

Let $\mathbb{K}$ be a field. Let $V$ be a $(n+1)$-dimensional $\mathbb{K}$-vector space and $F \in \text{Sym}^d V$, a homogeneous polynomial of degree $d$ in $n+1$ variables. The Waring problem for $F$ over $\mathbb{K}$ asks for the least value $s$ such that there exist linear forms $L_1, \ldots, L_s$ over $\mathbb{K}$, for which $F$ can be written as a sum of powers

$$F = L_1^d + \ldots + L_s^d.$$  

This value $s$ is called the Waring rank over $\mathbb{K}$, or the $\mathbb{K}$ rank, of the form $F$, and it is denoted by $\text{rk}_\mathbb{K}(F)$. Note that the rank could be infinite for fields of positive characteristic.

The notion of Waring rank, and its generalization to the case of tensors, are intensively studied because of their many applications which include, but are not limited to, Algebraic Complexity Theory [2], Signal Processing [5], and Quantum Information Theory [9, 10].

Most applications are interested in the real and complex cases, that is the cases in which $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We will call $\text{rk}_\mathbb{R}(F)$, respectively $\text{rk}_\mathbb{C}(F)$, the real, respectively the complex, rank of $F$.

Our knowledge of the Waring rank is very limited even for $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. For example, the complex Waring rank is known for all monomials, see [9], but the real Waring rank is only known in the case of monomials in two variables, see [1, 7].

Since the Waring rank of a quadratic form is always known, it is natural to consider the cubic forms. However, the degree three cases is already too difficult, and a complete description of the complex ranks is only given when at most three variables are involved. In [6], the three variable case is treated using projective changes of coordinates in order to obtain canonical forms of which the complex rank is then computed. In the same paper, a similar idea is used to find the complex rank of some reducible cubic forms in any number of variables.

In this paper we recover, and complete, the description of the complex ranks given in [6] ans we also determine the real ranks of all equivalence classes, over $\mathbb{C}$, of reducible cubics. Our main result is the following:

Theorem (3.5). If $F$ is a complex reducible cubic form in $n + 1$ variables, then one of the following holds:

- $F$ is projectively equivalent to

$$A = x_0(x_0^2 + x_1^2 + \ldots + x_n^2),$$
and \( \text{rk}_F = \text{rk}_C F = 2n. \)

- \( F \) is projectively equivalent to
  \[
  B = x_0(x_1^2 + x_2^2 + \ldots + x_n^2),
  \]
  and \( \text{rk}_F = \text{rk}_C F = 2n. \)

- \( F \) is projectively equivalent to
  \[
  C = x_0(x_0x_1 + x_2x_3 + x_4^2 + \ldots + x_n^2),
  \]
  and \( \text{rk}_F = \text{rk}_C F = 2n + 1. \)

2. Basic results

In this section we work over a field \( \mathbb{K} \) of characteristic zero. First we have the following lemma.

**Lemma 2.1.** Let \( F \) be a form of degree \( d \) in \( n+1 \) variables \( x_0, \ldots, x_n \) and let \( F_k \) denote \( \partial F / \partial x_k \). Let \( r \) be the minimal non-negative integer \( s \), such that there exist \( s \) linear forms \( L_1, \ldots, L_s \), with the property that for each \( 0 \leq k \leq n \), \( F_k \in \langle L_1^{d-1}, \ldots, L_s^{d-1} \rangle \). Then \( \text{rk}_F(F) \geq r. \)

**Proof.** Let \( t = \text{rk}_F(F) \), \( F = \sum_{k=1}^t L_k^d \) and \( F_k = \partial F / \partial x_k \). Hence

\[
F_k = \sum_{k=1}^t \partial L_k^d / \partial x_k = (d-1) \sum_{k=1}^t (\partial L_k / \partial x_k) L_k^{d-1}.
\]

Thus for each \( k, F_k \in \langle L_1^{d-1}, \ldots, L_t^{d-1} \rangle \). By definition of \( r \), we have \( \text{rk}_F(F) \geq r. \)

**Theorem 2.2.** Let \( 1 \leq p \leq n \) be an integer, \( F \) be a form in \( n+1 \) variables \( x_0, \ldots, x_n \) and let \( F_k \) denote \( \partial F / \partial x_k \). If for all \( \lambda_k \in \mathbb{C} \), with \( 1 \leq k \leq p \), we have \( \text{rk}_F(F_0 + \sum_{k=1}^p \lambda_k F_k) \geq m \) and \( F_1, F_2, \ldots, F_p \) are linearly independent, then \( \text{rk}_F(F) \geq m + p. \)

**Proof.** Suppose by contradiction that \( \text{rk}_F(F) < m + p. \) By Lemma 2.1 there exist \( m + p - 1 \) linear forms \( L_1, L_2, \ldots, L_{m+p-1} \), such that for each \( k, F_k \in \langle L_1^{d-1}, \ldots, L_{m+p-1}^{d-1} \rangle \). Hence, there is a \( p \times (m+p-1) \) matrix \( M \) of rank \( p \), such that

\[
\begin{pmatrix}
F_1 \\
F_2 \\
\vdots \\
F_p
\end{pmatrix} = M
\begin{pmatrix}
L_1^{d-1} \\
L_2^{d-1} \\
\vdots \\
L_{m+p-1}^{d-1}
\end{pmatrix}.
\]

Performing Gaussian elimination on \( M \), we can decompose \( M \) as \( M = M_0 M_1 \), where \( M_0 \) is a full rank matrix, which is product of elementary matrices, and \( M_1 \) has the following form

\[
M_1 = \begin{pmatrix}
1 & \cdots & \cdots & \cdots & \cdots \\
0 & 1 & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1
\end{pmatrix}.
\]
Each first non-zero element in each row of $M_1$ is 1 and these 1’s are in different columns. Suppose the 1 in the $k$–th row is in the $h(k)$–th column. We have $1 = h(1) < h(2) < \cdots < h(p) \leq m + p - 1$. Let $H = \{h(1), h(2), \ldots, h(p)\}$. Thus for $1 \leq k \leq p$, there exist $\mu_k, t$, with $h(k) < t \leq p$, such that $L_{h(k)}^{d-1} + m + p - 1 \sum_{t=h(k)+1}^{m+p-1} \mu_{k,t}L_{j}^{d-1} \in \langle F_1, F_2, \ldots, F_p \rangle$.

Suppose $F_0 = \sum_{j=1}^{m+p-1} \nu_j L_j^{d-1}$. Then we have $F_0 = \sum_{j \in H} \nu_j L_j^{d-1} + \sum_{j \notin H} \nu_j L_j^{d-1}$. For $j \notin H$, there exists $\bar{\nu}_j$, such that $\sum_{j \in H} \nu_j L_j^{d-1} + \sum_{j \notin H} \bar{\nu}_j L_j^{d-1} \in \langle F_1, F_2, \ldots, F_p \rangle$.

Therefore

$$F_0 = \sum_{j \in H} \nu_j L_j^{d-1} + \sum_{j \notin H} \nu_j L_j^{d-1} = (\sum_{j \in H} \nu_j L_j^{d-1} + \sum_{j \notin H} \bar{\nu}_j L_j^{d-1}) + \sum_{j \notin H} (\nu_j - \bar{\nu}_j)L_j^{d-1}.$$  

Then

$$\sum_{j \notin H} (\nu_j - \bar{\nu}_j)L_j^{d-1} = F_0 - (\sum_{j \in H} \nu_j L_j^{d-1} + \sum_{j \notin H} \bar{\nu}_j L_j^{d-1}).$$

The rank of left-hand side is at most $m + p - 1 - |H| = m - 1$, while the rank of right-hand side is at least $m$ by assumption and this a contradiction.

Lemma 2.1 and Theorem 2.2 are inspired by the property of quantum states transformation via local operation and classical communication (LOCC) [4, 9, 10].

3. Ranks of Reducible Cubic Forms in $n+1$ variables

In this section we determine the complex and the real ranks of reducible cubic forms in $\mathbb{C}[x_0, \ldots, x_n]$. Since the rank is invariant under projective transformations, we only need to determine the rank for projective equivalence classes of reducible cubic forms. Note that the rank of a reducible cubic form which is projectively equivalent to union of three hyperplanes is an easy computation since it reduces to cubic forms. Note that the rank of a reducible cubic form which is projectively equivalent to union of three hyperplanes is an easy computation since it reduces to cubic forms.

Type A. \{Q = 0\} is not a cone and \{L = 0\} is not tangent to the quadric.

Cubic forms of this type are projectively equivalent to the cubic form

$$A := x_0(x_0^2 + x_1^2 + \ldots + x_n^2).$$

Type B. \{Q = 0\} is a cone and \{L = 0\} does not pass through any vertex of the quadric.

Cubic forms of this type are projectively equivalent to the cubic form

$$B := x_0(x_1^2 + x_2^2 + \ldots + x_n^2).$$

Type C. \{Q = 0\} is not a cone and \{L = 0\} is tangent to the quadric.

Cubic forms of this type are projectively equivalent to the cubic form

$$C := x_0(x_0 x_1 + x_2 x_3 + x_4^2 + \ldots + x_n^2).$$
Remark 3.1. Note that if the quadric \( \{ Q = 0 \} \) is a cone and the linear space contains any vertex of the quadric, then we can project from this vertex. Thus we reduce the number of variables and we end up again with a reducible cubic either of type \( A \), or \( B \) or \( C \) in less than \( n + 1 \) variables.

We give the complete description of the complex and real ranks of these forms. We will prove two propositions giving an upper bound on the real rank and a lower bound on the complex rank. Note that the complex rank for cubics of type \( A \) and \( B \) was given in [5], but we produce here independent proofs. B. Segre proved that the cubic surface in \( \mathbb{P}^3 \) of type \( C \) has rank 7 and that it is the maximal rank among cubic surfaces [5].

Notation 3.2. Let \( \int Fdx_k \) denote a suitable choice of a primitive of \( F \), i.e., a form \( G \) such that \( \partial G/\partial x_k := G_k = F \), which we will specify when needed.

Proposition 3.3. The cubic forms \( A \), \( B \) and \( C \) have real rank at most \( 2n \), \( 2n \) and \( 2n + 1 \) respectively.

Proof. All the statements are proven by induction on \( n \). Let \( A^{(k)} \), \( B^{(k)} \) and \( C^{(k)} \) denote the cubic forms of types \( A \), \( B \) and \( C \) in \( k + 1 \) variables. We will use integrals according to Notation 3.2 in order to easily compute a decomposition, giving an upper-bound.

Let us consider the cubic form \( A \). For \( n = 1 \), the form is
\[
A^{(1)} = x_0(x_0^2 + x_1^2) = \frac{1}{6\sqrt{3}} \left( (\sqrt{3}x_0 - x_1)^3 + (\sqrt{3}x_0 + x_1)^3 \right),
\]
whose real rank is 2. Suppose the statement holds for \( k \leq n - 1 \). Take \( A^{(n)} = x_0(x_0^2 + x_1^2 + \ldots + x_n^2) \). We have \( A_1^{(n)} = 2x_0x_1 = 1/2[(x_0 + x_1)^2 - (x_0 - x_1)^2] \). Set \( A' = \int A_1^{(n)}dx_1 = 1/6[(x_0 + x_1)^3 + (x_0 - x_1)^3] = 1/3x_0^2 + x_0x_1 + A'' = x_0(-1/3x_0^2 + x_1^2 + \ldots + x_n^2) \). Thus \( A^{(n)} = A' + A'' \). The form \( A'' \) is projectively equivalent to a cubic form of type \( A \), and by induction, it has real rank at most \( 2(n-1) \). Hence we have that
\[
\text{rk}_R(A^{(n)}) \leq \text{rk}_R(A') + \text{rk}_R(A'') \leq 2 + 2(n-1).
\]

Let us consider the cubic form \( B \). For \( n = 2 \), we have
\[
B^{(2)} = x_0(x_1^2 + x_2^2) = \frac{1}{6} \left( (x_1 + x_0)^3 - (x_1 - x_0)^3 \right) - \frac{1}{3}x_0^3 + x_0x_2,
\]
where \( \text{rk}_R(1/3x_0^3-x_0x_2^2) = 2 \) as noticed in the discussion of type \( A \). Thus, \( \text{rk}_R(B^{(2)}) \leq 4 \). Suppose the statement holds for \( 3 \leq k \leq n-1 \). Take the form \( B^{(n)} = x_0(x_1^2 + x_2^2 + \ldots + x_k^2) \) and set \( B' = \int B_1^{(n)}dx_1 = 1/3x_0^3 + x_0x_1 \), \( B'' = \int B_2^{(n)}dx_2 = -1/3x_0^3 + x_0x_2 \) and \( B''' = x_0(x_1^2 + \ldots + x_k^2) \). Thus \( B^{(n)} = B' + B'' + B''' \). The form \( B''' \) is projectively equivalent to the cubic form of type \( B \) and hence, by induction, has real rank at most \( 2(n-2) \). On the other hand, the \( \text{rk}_R(B' + B'') \leq 4 \). Hence we have that
\[
\text{rk}_R(B) \leq 4 + 2(n-2) = 2n.
\]

Let us consider the cubic form \( C \). For \( n = 2 \), consider cubic form \( C^{(2)} = x_0(x_0,x_1,x_2) \). Indeed, performing change of variables with the linear transformation
\[
\begin{cases}
x_0 = y_1 \\
x_1 = 1/3y_1 + y_3 \\
x_2 = y_2
\end{cases}
\]
we have $C^{(2)} = 1/3y^3 + y_1^2y_3 + y_1y_2^2$. Let $C' = \int C_2^{(2)} dy_2$ be the primitive given by $C' = 1/6[(y_1 + y_2)^3 + (y_1 - y_2)^3] = 1/3y_1^3 + y_1y_2^2$ and let $C'' = y_1^2y_3 = 1/6[(y_1 + y_3)^3 - (y_1 - y_3)^3 - 2y_3^3]$. Thus $C^{(2)} = C' + C''$. Then $rk_\mathbb{R}(C^{(2)}) \leq 5$. Assume the statement holds for all $k \leq n-1$ and let $C^{(n)} = x_0(x_0x_1 + x_2x_3 + x_4^2 + \ldots + x_n^2)$. Perform the change of variables given by the linear transformation

$$
\begin{cases}
    x_0 = y_1 \\
    x_1 = y_3 \\
    x_2 = y_0 + y_2 \\
    x_3 = y_0 - y_2 \\
    x_4 = y_4 \\
    \vdots \\
    x_n = y_n.
\end{cases}
$$

Then $C^{(n)} = y_0^2y_1 - y_1y_3^2 + y_3y_4 + y_2y_3^2 + \ldots + y_1y_n^2$. Setting $C' = \int C_0^{(n)} dy_0 = y_0^2y_1 + 1/3y_1^3$ and $C'' = -1/3y_1^3 - y_1y_3^2 + y_3y_4 + y_2y_3^2 + \ldots + y_1y_n^2$, we have $C^{(n)} = C' + C''$. Note that $rk_\mathbb{R}(C') = 2$. Since $C''$ is a cubic form of type $C$, by induction, $rk_\mathbb{R}(C'') \leq (n-1) + 1$. Hence we have that

$$
rk_\mathbb{R}(C^{(n)}) \leq rk_\mathbb{R}(C') + rk_\mathbb{R}(C'') \leq 2 + 2(n-1) + 1 = 2n + 1.
$$

The proof is now completed.

\textbf{Proposition 3.4.} The cubic forms $A, B$ and $C$ have complex rank at least $2n$, $2n$ and $2n + 1$ respectively.

\textbf{Proof.} Let us consider the cubic forms $A$ and $B$. The forms $A_1, \ldots, A_n$ are linearly independent and for any $\lambda_k \in \mathbb{C}$, with $1 \leq k \leq n$, we have $rk_\mathbb{C}(A_0 + \sum_{k=1}^n \lambda_k A_k) \geq n$. The last statement follows from considering the matrix and noticing that the rank is at least $n$. By Theorem 2.2 for $m = n$ and $p = n$, we have $rk_\mathbb{C}(A) \geq m + p = 2n$.

Analogously to the cubic form $A$, we have that $rk_\mathbb{C}(B_0 + \sum_{k=1}^n \lambda_k B_k) \geq n$ and hence $rk_\mathbb{C}(B) \geq m + p = 2n$.

We now consider the cubic form $C$. The forms $C_1, C_2, \ldots, C_n$ are linearly independent and for any $\lambda_k \in \mathbb{C}$, with $1 \leq k \leq n$, we have $rk_\mathbb{C}(F_0 + \sum_{k=1}^n \lambda_k C_k) = n + 1$.

The last statement is equivalent to the fact that the following matrix has non-zero determinant

$$
M = \begin{pmatrix}
    \lambda_1 & 1 & \lambda_3/2 & \lambda_2/2 & \lambda_4 & \lambda_5 & \cdots & \lambda_n \\
    1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
    \lambda_3/2 & 0 & 0 & 1/2 & 0 & \cdots & 0 \\
    \lambda_2/2 & 0 & 1/2 & 0 & 0 & \cdots & 0 \\
    \lambda_4 & 0 & 0 & 0 & 1 & \cdots & 0 \\
    \lambda_5 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    \lambda_n & 0 & 0 & 0 & 0 & \cdots & 1
\end{pmatrix}.
$$
A direct computation gives $\det(M) = 1/4$. Hence $rk_C(F_0 + \sum_{k=1}^{n} \lambda_k F_k) = n + 1$ for all $\lambda_k \in \mathbb{C}$. By Theorem 2.2, for $m = n + 1$ and $p = n$ we have $rk_C(C) \geq m + p = 2n + 1$. The proof is now completed.

\begin{theorem}
If $F$ is a complex reducible cubic form in $n + 1$ variables, then one of the following holds:
\begin{itemize}
  \item $F$ is projectively equivalent to
    \[
    A = x_0(x_0^2 + x_1^2 + \ldots + x_n^2),
    \]
    and $rk_R F = rk_C F = 2n$.
  \item $F$ is projectively equivalent to
    \[
    B = x_0(x_1^2 + x_2^2 + \ldots + x_n^2),
    \]
    and $rk_R F = rk_C F = 2n$.
  \item $F$ is projectively equivalent to
    \[
    C = x_0(x_0x_1 + x_2x_3 + x_4^2 + \ldots + x_n^2),
    \]
    and $rk_R F = rk_C F = 2n + 1$.
\end{itemize}
\end{theorem}

\begin{proof}
It is enough to combine Propositions 3.3 and 3.4.
\end{proof}

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