A NEW FAMILY OF ELLIPTIC CURVES WITH POSITIVE RANKS ARISING FROM THE HERON TRIANGLES

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Abstract. The aim of this paper is to introduce a new family of elliptic curves with positive ranks. These elliptic curves have been constructed with certain rational numbers, namely a, b, and c as sides of Heron triangles having rational areas \( k \). It turned out that the torsion groups of this family are of the form \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) and also the rank is positive.

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1. Introduction

As is well-known, an elliptic curve \( E \) over a field \( \mathbb{K} \) can be explicitly expressed by the generalized Weierstrass equation of the form

\[
E : y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6
\]

where, \( a_1, a_2, a_3, a_4, a_6 \in \mathbb{K} \). In this paper we are interested in the case of \( \mathbb{K} = \mathbb{Q} \).

By the Mordell-Weil theorem [6], every elliptic curve over \( \mathbb{Q} \) has a commutative group \( E(\mathbb{Q}) \) which is finitely generated, i.e., \( E(\mathbb{Q}) \cong \mathbb{Z}^r \oplus E(\mathbb{Q})_{\text{tors}}, \) where \( r \) is a nonnegative integer and \( E(\mathbb{Q})_{\text{tors}} \) is the subgroup of elements of finite order in \( E(\mathbb{Q}) \). This subgroup is called the torsion subgroup of \( E(\mathbb{Q}) \) and the integer \( r \) is called the rank of \( E \) and is denoted by the rank(\( E \)).

By Mazur’s theorem, the torsion subgroup \( E(\mathbb{Q})_{\text{tors}} \) is one of the following 15 forms: \( \mathbb{Z}/n\mathbb{Z} \) with \( 1 \leq n \leq 10 \) or \( n = 12 \), \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2m\mathbb{Z} \) with \( 1 \leq m \leq 4 \). Besides, it is not known which values of rank \( r \) are possible. The folklore conjecture is that a rank can be arbitrarily large, but it seems to be very hard to find examples with large ranks. The current record is an example of elliptic curve over \( \mathbb{Q} \) with rank \( \geq 28 \), found by Elkies in May 2006 (see [2]). Having classified the torsion part, one interested in seeing whether or not the rank is unbounded among all the elliptic curves. There is no known guaranteed algorithm to determine the rank and it is not known which integers can occur as ranks.

2. NEW FAMILY OF ELLIPTIC CURVES

In our work, we are going to study the rank of a new family of elliptic curves based on some triangles with rational areas \( k \) and rational sides. Recall that any triangle with rational sides and rational area is simply called
a Heron triangle due to Heron of Alexandria (C. 10 A.D. - C. 75 A.D.) and that Heron’s formula for the area, say $S$, of a triangle with sides $(a, b, c)$ is given by

$$S = \sqrt{P(P-a)(P-b)(P-c)}$$

Where, $P = \frac{(a+b+c)}{2}$ is the semi perimeter.

Take $S = v$ and $P = u$, so we have $v^2 = u(u-a)(u-b)(u-c)$, where $a$, $b$, and $c$ are the sides of a Heron triangle.

With the change of coordinates, $(u, v) \rightarrow (\frac{1}{\zeta}, \frac{\eta}{\zeta^2})$, we obtain

$$\eta^2 = (1 - a\zeta)(1 - b\zeta)(1 - c\zeta)$$

with one more change of coordinates, $(\zeta, \eta) \rightarrow (\frac{x}{abc}, \frac{y}{abc})$, we get

$$y^2 = (x + ab)(x + bc)(x + ac).$$

We will show that

$$E(\mathbb{Q})_{\text{tors}} = \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} = \{(-ab, 0), (-bc, 0), (-ac, 0), \infty\}.$$  

However from $x = 0$ we get $y = \pm abc$. We will show that this two points $(0, \pm abc)$ are belong to the free torsion part of $E(\mathbb{Q})$.

Now we define the elliptic curve associated to the triple \{a(k), b(k), c(k)\} arising from a Heron triangle with rational area $k$ and sides

$$\left\{\begin{array}{l}
a(k) = 5k^2 - 4k + 4, \\
b(k) = \frac{1}{2}k(k^2 - 4k + 20), \\
c(k) = \frac{1}{2}(k + 2)(k^2 - 4),
\end{array}\right.$$  

given by N. J. Fine [3],

i.e.,

$$y^2 = (x + a(k)b(k))(x + b(k)c(k))(x + a(k)c(k)).$$

Then the curve $E$ has three rational points of order two :

$$\left\{\begin{array}{l}
P_1 = [\frac{1}{2}(5k^2 - 4k + 4)(k^2 - 4k + 20), 0], \\
P_2 = [\frac{1}{2}(5k^2 - 4k + 4)(k + 2)(k^2 - 4), 0], \\
P_3 = [\frac{1}{2}(k^2 - 4k + 20)(k + 2)(k^2 - 4), 0].
\end{array}\right.$$  

We can make a change of coordinates, $(x, y) \rightarrow (x - a(k)c(k), y)$ which sends $(0, 0) \rightarrow (a(k)c(k), 0)$. Obviously such a change won’t affect the structure of the group $E(\mathbb{Q})$. Thus, given the restriction that we are considering curves with a rational two-torsion point, we can assume that $E$ is given by
For the third equation it’s trivial that the solutions are $k = 2$. Also for the second equation, we see that there is no solution.

By the Eisenstein theorem if $A$ and $B$:

$$A = \frac{1}{4}k^6 - 16k^4 - 3k^5 + 96k^3 - 44k^2 - 16k + 32,$$

$$B = -\frac{1}{4}(k^6 - 4k^4 - 12k^5 + 96k^3 - 16k^2 - 192k + 64) \times (15k^4 - 72k^3 + 40k^2 - 32k - 16).$$

(2.1)

The corresponding value of the discriminant is

$$
\Delta = 16(A^2 - 4B)B^2 = \frac{1}{16}k^2(k^2 - 4k + 20)^2(k^3 - 8k^2 + 4k - 16)^2 \times (k^2 - 12k + 4)^2(k - 2)^4(k + 2)^4(5k^2 - 4k + 4)^2(3k^2 - 12k - 4)^2.
$$

From the discriminant of the curve we see that for the values $k = 0, 2$, and $-2$ curve reduces to singular form.

**Theorem 2.1.** Let $a(k), b(k)$ and $c(k)$ be defined as above. then the elliptic curve

$$E : y^2 = (x + a(k)b(k))(x + b(k)c(k))(x + a(k)c(k))$$

over $\mathbb{Q}(k)$ has the torsion group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

**Proof.** With coordinate transformation $x \rightarrow x - a(k)c(k)$ our curve leads to

$$y^2 = x\left(x - a(k)b(k) + a(k)c(k)\right)\left(x - b(k)c(k) + a(k)c(k)\right).$$

Kubert [4] showed that if $y^2 = x(x + r)(x + s)$, with $r, s \neq 0$ and $s \neq r$, then the torsion subgroup is $\mathbb{Z}_2 \times \mathbb{Z}$. Therefore for our curve it is sufficient to prove that $-a(k)b(k) + a(k)c(k) \neq 0$, $-b(k)c(k) + a(k)c(k) \neq 0$, and $-a(k)b(k) + a(k)c(k) \neq -b(k)c(k) + a(k)c(k)$ for all $k$ in $\mathbb{Q}(k)$. Firstly we try to find all $k$ in $\mathbb{Q}(k)$ that satisfy in the equalities, $a(k) = b(k)$, $a(k) = c(k)$, and $b(k) = c(k)$. These equalities give rise to the following equalities:

$$\begin{cases}
  k^3 - 14k^2 + 28k - 8 = 0, \\
  k^3 - 8k^2 + 4k + 16 = 0, \\
  6k^2 - 24k - 8 = 0.
\end{cases}$$

By the Eisenstein theorem if $k = \frac{p}{q} \in \mathbb{Q}$ for relatively primes $p$ and $q$, be a solution to the first equation, the possibilities for $p$ are $\{\pm 1, \pm 2, \pm 4, \pm 8\}$ which shows that $k = 2$ satisfies this cubic equation, but we have already assume that $k \neq 0, 2, -2$.

Also for the second equation, we see that there is no solution.

For the third equation it’s trivial that the solutions are $k = 2 \pm \frac{2}{3} \sqrt{3}$.

Therefore, for every $k$ in $\mathbb{Q}$, $a(k) \neq b(k)$, $c(k) \neq b(k)$, $c(k) \neq a(k)$ and the torsion group for main curve is $\mathbb{Z}_2 \times \mathbb{Z}_2$. \qed
Corollary 2.2. \( \text{rank}(E(\mathbb{Q})) \geq 1. \)

Proof. By Theorem 2.1, the point

\[ P_k = (0, abc) = (0, 1/4(5k^2 - 4k + 4)k(k^2 - 4k + 20)(k + 2)(k^2 - 4)) \]

on \( E(\mathbb{Q}) \) has infinite order, which shows that \( \text{rank}(E(\mathbb{Q})) \geq 1. \) \( \square \)

Lemma 2.3. For each \( 1 \leq r \leq 6 \) there exists a \( k \) such that the elliptic curve

\[ y^2 = (x + ab)(x + bc)(x + ac) \]

has a torsion group isomorphic to \( \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \) and its rank equals to \( r. \)

Proof. It is trivial by the results given in the table above. \( \square \)

Remark 2.4. If \( a, b \) and \( c \) are the sides of right triangle then, the relations

\[ a^2 + b^2 = c^2, \quad a^2 + c^2 = b^2 \quad \text{and} \quad b^2 + c^2 = a^2 \]

lead to \( k = 6, 2/3 \) that has the curves with ranks equal 1. It is remarkable that in our computation we only meet this two curves with rank =1.

3. Our Computation

In this stage we want to find curves having large ranks possible. The main idea here is that a curve is more likely to have large rank if \( |E(\mathbb{Q})| \) is relatively large for many primes \( p. \) We will use the following realization of this idea. For a prime \( p \) we put \( a_p = a_p(E) = p + 1 - |E(\mathbb{F}_p)| \) and

\[ S(N, E) = \sum_{p \leq N,p \text{prime}} (1 - \frac{p - 1}{|E(\mathbb{F}_p)|}) \log(p) = \sum_{p \leq N,p \text{prime}} (-a_p + 2 \frac{p}{p + 1 - a_p}) \log(p). \]

It is experimentally known that one may expect the high rank curves have large \( S(N, E). \) Some arguments show that the Birch and Swinnerton-Dyer conjecture gives support to this observation. The sum \( S(N, E) \) can be very efficiently computed (e.g., using PARI )for \( N < 10000. \) After this sieving method, we may continue to investigate the best, let us say, 1 percent of the curves. Since, we are working with curves with torsion points of order 2, we may compute the Selmer rank which is the best known upper bound for the actual rank of the curves. In our computations, we used the sage software [5] and Cremona’s MWRANK program [1] for computing the Mordell-Weil rank of the curves. Following is a table of elliptic curves with corresponding values of \( a, b, c \) and ranks =1, 2, 3, 4, 5 and 6. Also we have calculated the rank of this family for 100000 curves in the range of \( 1 \leq k \leq 100 \) using the MWRANK. The results appear in the following table.
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| $k$ | Curve | Rank |
|------|--------|------|
| 6/25 | $y^2 = x^3 + \frac{-385968810117979136}{59604644775890625} x^2 + \frac{-302381690231490275394654830592}{1168888777216160297393798828125} x$ | 6 |
|      | $+ \frac{720840680923379929171885236706081749399532498944}{2168404349710088680149056017398834228515625}$ | |
| 11   | $y^2 = x^3 + 4547347x^2 + 6818384095380x + 3363133863125708100$ | 5 |
| 19   | $y^2 = x^3 + 89515187x^2 + 25234525031220x + 22674567869155130588100$ | 4 |
| 3    | $y^2 = x^3 + 6899x^2 + 14152500x + 8901922500$ | 3 |
| 4    | $y^2 = x^3 + 26432x^2 + 225607680x + 613652889600$ | 2 |
| 6    | $y^2 = x^3 + 192512x^2 + 12079595520x + 247390116249600$ | 1 |

A table of elliptic curves with ranks $= 1,2,3,4,5,6$.

| Rank | Percent |
|------|---------|
| 6    | 0.5%    |
| 5    | 3.7%    |
| 4    | 23.4%   |
| 3    | 36.7%   |
| 2    | 16.2%   |
| undetermined | 19.5% |

The table for percents.

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