Distribution of Small Values of Bohr Almost Periodic Functions with Bounded Spectrum

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For f a nonzero Bohr almost periodic function on \( \mathbb{R} \) with a bounded spectrum we proved there exist \( C_f > 0 \) and integer \( n > 0 \) such that for every \( u > 0 \) the mean measure of the set \( \{ x : |f(x)| < u \} \) is less than \( C_f u^{-1/n} \). For trigonometric polynomials with \( \leq n + 1 \) frequencies we showed that \( C_f \) can be chosen to depend only on \( n \) and the modulus of the largest coefficient of \( f \): We showed this bound implies that the Mahler measure \( M(h) \) of the lift \( h \) of \( f \) to a compactification \( G \) of \( \mathbb{R} \), is positive and discussed the relationship of Mahler measure to the Riemann Hypothesis.

Keywords: almost periodic function, entire function, Beurling factorization, Mahler measure, Riemann hypothesis.

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1. Distribution of small values

\( \mathbb{N} := \{1, 2, \ldots\}, \mathbb{Z}, \mathbb{R}, \mathbb{C}, \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\} \) are the natural, integer, real, complex and circle group numbers, \( C_0(\mathbb{R}) \) is the \( C^* \)-algebra of bounded continuous functions and \( \chi_\omega : \mathbb{R} \to \mathbb{T}, \omega \in \mathbb{R} \) the homomorphisms \( \chi_\omega(x) := e^{i\omega x}, \omega \in \mathbb{R} \). A finite sum \( f = \sum a_\omega \chi_\omega \) with distinct \( \omega \) is called a trigonometric polynomial with height \( H_f := \max |a_\omega| \) and they comprise the algebra \( T(\mathbb{R}) \) of trigonometric polynomials. Bohr [9] defined the \( C^* \)-algebra \( U(\mathbb{R}) \) of uniformly almost periodic functions to be the closure of \( T(\mathbb{R}) \) in \( C_0(\mathbb{R}) \) and proved that their means \( m(f) := \lim_{L \to \infty} (2L)^{-1} \int_{-L}^L f(t)dt \) exist. The Fourier transform \( \hat{f} : \mathbb{R} \to \mathbb{C} \) of \( f \in U(\mathbb{R}) \) is \( \hat{f}(\omega) := m(f \overline{\chi_\omega}) \) and its spectrum \( \Omega(f) := \text{support } \hat{f} \). If \( f \) is nonzero then \( \Omega(f) \) is nonempty and countable and we say \( f \) has bounded spectrum if its bandwidth \( b(f) \in [0, \infty) \), defined by \( b(f) := \sup \Omega(f) - \inf \Omega(f) \), is finite. We observe that if \( S \subseteq \mathbb{R} \) is defined by a finite number of inequalities involving functions in \( U(\mathbb{R}) \) then \( m(S) := \lim_{L \to \infty} (2L)^{-1} \text{measure } [-L, L] \cap S \) exists and define \( J_f : (0, \infty) \to [0, 1] \) by

\[
J_f(u) := m(\{ x \in \mathbb{R} : |f(x)| < u \})
\]

Theorem 1.1. If \( f \in U(\mathbb{R}) \) is nonzero and has a bounded spectrum then there exist \( C_f > 0 \) and \( n \in \mathbb{N} \) such that:

\[
J_f(u) \leq C_f u^{1/n}, \quad u > 0.
\]

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There exists a sequence $C_n$ such that if $f \in T(\mathbb{R})$ has $n + 1$ frequencies then

$$J_f(u) \leq C_n H_f^{\frac{1}{2}} u^\frac{2}{3}, \quad u > 0.$$  \hfill (3)

Proof. For $f \in U(\mathbb{R}), \omega \in \mathbb{R}, k \in \mathbb{N}, u > 0$ define $\Xi_{f, \omega, k, u}, K_f : (0, \infty) \to [0, 1]$ by

$$\Xi_{f, \omega, k, u}(v) := m \{ x \in \mathbb{R} : |f(x)| < u, |(\chi_\omega f)^{(j)}(x)| < v, j = 1, \ldots, k \},$$

$$K_f(u) := \inf_{\omega \in \mathbb{R}} \inf_{k \in \mathbb{N}} \inf_{v > 0} \left[ 3 \sqrt{2} \pi^{-1} b(f) v^{-1} u^\frac{2}{3} + \Xi_{f, \omega, k, u}(v) \right].$$

We first prove Theorem 1 assuming the following result which we prove latter.

**Lemma 1.1.** Every nonzero $f \in U(\mathbb{R})$ with bounded spectrum satisfies $J_f \leq K_f$.

We observe that for every $\omega \in \mathbb{R}$ and every $a \in \mathbb{R} \setminus \{0\}$, if $h(x) = \chi_\omega(x) f(ax)$ then $J_h = J_f$ and $K_h = K_f$. Without loss of generality we can assume that $\Omega(f) \subseteq \left[ -\frac{b(f)}{2}, \frac{b(f)}{2} \right]$. If $b(f) = 0$ then $f = c$ and $J_f(u) \leq |c|^{-1} u$. If $b(f) > 0$ then Bohr [10] proved that $f$ extends to an entire function $F$ of exponential type $\frac{b(f)}{2}$, and Boas [6], ([7], p. 11, Equation 2.2.12) proved that

$$\limsup_{k \to \infty} |f^{(k)}(x)|^\frac{2}{k} = \frac{b(f)}{2}$$

uniformly in $x$. Therefore for any $v_0 > \frac{b(f)}{2}$ there exists $k \in \mathbb{N}$ such that $\Xi_{f, 0, k, u}(v_0) = 0$ so Lemma 1.1 implies $J_f$ satisfies (2) with $C_f = 3 \sqrt{2} \pi^{-1} b(f) v_0^{-1} u$ and $n = k$. This proves the first assertion. To prove the second we assume, without loss of generality, that $b(f) = 1$, $\Omega(f) \subseteq [0, 1]$ and

$$f(x) = \sum_{j=1}^{n+1} a_j e^{i \omega_j x}, \quad 0 = \omega_1 < \cdots < \omega_{n+1} = 1, \quad H_f = \max \{|a_j| : j = 2, \ldots, n+1\}.$$

Define $C_1 := \frac{1}{2}$. If $n = 1$ and $f$ has $n + 1 = 2$ terms and $f = a_0 + a_1 \chi_1$ with $|a_1| = H_f$ and $h = H_f (1 - \chi_1)$, then $J_f(u) \leq J_h(u) = (2/\pi) \sin^{-1}(u/(2H_f)) \leq C_1 H_f^{-1} u$ therefore (3) holds for $n = 1$. For $n \geq 2$ we assume by induction that (3) holds for $n - 1$ and therefore, since $f^{(1)}$ has $n$ terms and $H_{f^{(1)}} = H_f$, it follows that for all $v > 0$,

$$J_{f^{(1)}}(v) \leq C_{n-1} H_f^{\frac{1}{2}} v^{\frac{2}{3}},$$

$$\Xi_{f, 0, 1, u}(v) \leq C_{n-1} H_f^{\frac{1}{2}} v^{\frac{2}{3}}.$$  \hfill (8)

Therefore Lemma 1 with $\omega = 0$, $b(f) = k = 1$ gives

$$J_f(u) \leq \inf_{v > 0} \left[ 3 \sqrt{2} \pi^{-1} v^{-1} u + C_{n-1} H_f^{\frac{1}{2}} v^{\frac{2}{3}} \right] = C_n H_f^{\frac{1}{2}} u^{\frac{2}{3}}$$

where $C_n := C_{n-1}^{\frac{1}{2}} (3 \sqrt{2} \pi^{-1} (n-1))^{\frac{2}{3}} n(n-1)^{-1}$. \hfill (9)

**Remark 1.1.** Computation of 200 million terms shows that $n^{-1} C_n \to 0.900316322$

**Conjecture 1.1.** In (3) $C_n$ can be replaced by a bounded sequence.
Lemma 1.2. If \( \phi : [a, b] \to \mathbb{C} \) is differentiable and \( \phi'(a, b) \) is contained in a quadrant then

\[
b - a \leq 2\sqrt{2} \max \frac{|\phi'|([a, b])}{\min |\phi'|([a, b])}.
\]

Proof of Lemma 1.2. We first proved this result in ([18], Lemma 1) where we used it to give a proof of a conjecture of Boyd [11] about monic polynomials related to Lehmer’s conjecture [20], which was reviewed in ([13], Section 3.5) and extended to monic trigonometric polynomials in ([19], Lemma 2). The triangle inequality \( |\phi'| \leq |\Re \phi'| + |\Im \phi'| \) gives

\[
(b - a) \min |\phi'|([a, b]) \leq \int_a^b |\phi'(y)| \, dy \leq \int_a^b (|\Re \phi'(y)| + |\Im \phi'(y)|) \, dy.
\]

Since \( \phi'(a, b) \) is contained in a quadrant of \( \mathbb{C} \) there exist \( c, d \in \{1, -1\} \) such that \( |\Re \phi'(y)| = c \Re \phi'(y) \) and \( |\Im \phi'(y)| = d \Im \phi'(y) \) for all \( y \in [a, b] \). Therefore

\[
\int_a^b (|\Re \phi'(y)| + |\Im \phi'(y)|) \, dy = (c \Re \phi(b) + d \Im \phi(b)) - (c \Re \phi(a) + d \Im \phi(a)).
\]

The result follows since the right side is bounded above by \( 2 \sqrt{2} \max |\phi'(a, b)| \).

Proof of Lemma 1.1. Assume that \( f \in \mathcal{U}(\mathbb{R}) \) is nonzero. We may assume without loss of generality that \( \Omega(f) \subset [-\frac{b(f)}{2}, \frac{b(f)}{2}] \). For \( k \in \mathbb{N}, u > 0, v > 0 \) we define the set

\[
S_{f, k, u, v} := \{ x \in \mathbb{R} : |f(u)| < u, \max_{j \in \{1, \ldots, k\}} |f^{(j)}(x)|^{\frac{1}{v}} \geq v \}.
\]

We observe that the set of functions in \( \mathcal{U}(\mathbb{R}) \) whose spectrums are in \([\frac{-b(f)}{2}, \frac{b(f)}{2}]\) is closed under differentiation, and define \( s(f, k, u, v) := m(S_{f, k, u, v}) \).

It suffices to prove that \( s(f, k, u, v) \leq 3 \sqrt{2} \pi^{-1} b(f) k v^{-1} u^{\frac{v}{u}} \).

Define \( \gamma_j := u^{\frac{k-j}{k}} v^j, j \in \{0, \ldots, k\} \), and \( \mathcal{I} := \) set of closed intervals \( I \) satisfying, for some \( j \in \{0, 1, \ldots, k - 1\} \), the following three properties:

1. \( f^{(j+1)}(I) \) is a subset of a closed quadrant,
2. \( \max |f^{(j)}(I)| \leq \gamma_j \) and \( \min |f^{(j+1)}(I)| \geq \gamma_{j+1} \),
3. \( I \) is maximum with respect to properties 1 and 2.

Define \( \mathcal{E} := \) set of endpoints of intervals in \( \mathcal{I} \), and

\[
\psi := \prod_{j=0}^{k-1} (\Re f^{(j+1)}(I) \Im f^{(j+1)}(I)) |f^{(j+1)}(I)|^{\frac{2}{v}} - \gamma_j^2 (|f^{(j+1)}(I)|^2 - \gamma_j^2).
\]

Lemma 1.2 implies that \( \text{length } (I) \leq 2 \sqrt{2} \frac{\gamma_k}{\gamma_{k+1}} = 2 \sqrt{2} v^{-1} u^{\frac{v}{u}}, I \in \mathcal{I} \),

\[
\text{and (12) and Property 3 implies that } S_{f, k, u, v} \subset \bigcup_{I \in \mathcal{I}} I.
\]
by $3\pi^{-1}b(f)k$. Property 3 implies that all points in $E$ are zeros of $\Psi$ so the upper density of intervals in $I$ is bounded by $\frac{3}{2}\pi^{-1}b(f)k$. Combining these facts gives $s(f, k, u, v) \leq (\frac{3}{2}\pi^{-1}b(f)k) (2\sqrt{2}\pi^{-1}u^\frac{1}{p}) = 3\sqrt{2}\pi^{-1}b(kk^{-1}u^\frac{1}{p})$ which proves (13) and concludes the proof of Lemma 1. □

For $p \in [1, \infty]$ Besicovitch [4] proved that the completion $B^p(\mathbb{R})$ of $U(\mathbb{R})$ with norm $(m(|f|^p))^{\frac{1}{p}}$ is a subset of $L^p_{loc}(\mathbb{R})$. For $x \geq 0$ we define $\log^+(x) := \log(\max\{1, x\}) \in [0, \infty)$, $\log^-(x) := \log(\min\{1, x\}) \in [-\infty, 0)$, and $|x| := \max\{|x|, \frac{1}{x}\}$ for $j \in \mathbb{N}$.

**Corollary 1.1.** If $f \in U(\mathbb{R})$ satisfies (2), then $\log^- \circ |f| \in B^p(\mathbb{R})$,

$$m(|\log^- \circ |f||^p) \leq \int_0^1 |\log(u)|^p \, dC_f u^\frac{1}{p} = C_f n^p \Gamma(p), \quad (17)$$

and $\log \circ |f| \in B^p(\mathbb{R})$.

**Proof of Corollary 1.1.** Since the means of the functions $\log^- \circ |f|_j$ are nondecreasing and bounded by the right side of (17), the sequence $\log^- \circ |f|_j$ is a Cauchy sequence in $B^p(\mathbb{R})$ so it converges to a function $\eta \in B^p(\mathbb{R})$. Therefore $\log^- \circ |f| = \eta$ since it is the pointwise limit of $\log^- \circ |f|_j$ and $\eta \in L^p_{loc}(\mathbb{R})$. The last fact follows since $\log = \log^+ + \log^-$. □

2. **Compactifications and Hardy Spaces**

**Definition 2.1.** A compactification of $\mathbb{R}$ is a pair $(G, \theta)$ where $G$ is a compact abelian group and $\theta : \mathbb{R} \to G$ is a continuous homomorphism with a dense image.

$C(G)$ is the set of continuous functions on $G$ and $L^p(G), p \in [1, \infty)$ are Banach spaces. If $h \in C(G)$ then $f := h \circ \theta \in U(\mathbb{R})$ since by a theorem of Bochner [8] every sequence of translates of $f$ has a subsequence that converges uniformly. We call $h$ the lift of $f$ to $G$. The Pontryagin dual $\hat{G}$ of a compact abelian group $G$ is the discrete group of continuous homomorphisms $\chi : G \to \mathbb{T}$ under pointwise multiplication. Bohr proved the existence of a compactification $(\mathbb{B}, \theta)$ such that $U(\mathbb{R}) = \{ h \circ \theta : h \in C(\mathbb{B}) \}$. The group $\mathbb{B}$ is nonseparable and $\hat{\mathbb{B}}$ is isomorphic to $\mathbb{R}_d$ := real numbers with the discrete topology.

**Lemma 2.1.** For every $f \in U(\mathbb{R})$ there exists a compactification $(G, \theta)$, with $G(f)$ separable, and $h \in C(G)$ such that $f = h \circ \theta$.

**Proof of Lemma 2.1.** If $f \in U(\mathbb{R})$ is nonzero its spectrum $\Omega(f)$ is nonempty and countable so the product group $\mathbb{T}^{\Omega(f)}$ is compact and separable. The function $\theta : \mathbb{R} \to \mathbb{T}^{\Omega(f)}$ defined by $\theta(x)(\omega) := \chi_\omega(x)$ is a continuous homomorphism. Define $G(f) := \overline{\theta(f)}$. Then $(G(f), \theta)$ is a compactification. The function $\tilde{h} : \theta(\mathbb{R}) \to \mathbb{C}$ defined by $\tilde{h}(\theta(x)) := f(x)$ is uniformly continuous so extends to a unique function $h : G \to \mathbb{C}$ and $f = h \circ \theta$. □

**Lemma 2.2.** If $(G, \theta)$ is a compactification, $h \in C(G)$, $f = h \circ \theta$, and $\log \circ |f| \in B^p(\mathbb{R})$, then $\log \circ |h| \in L^p(G)$ and $\int_G |\log \circ |h||^p = m(|\log \circ |f||^p$ for all $p \in [1, \infty)$.

**Proof of Lemma 2.2.** The theorem of averages ([3], p. 286) implies that

$$\int_G |\log^- \circ |h||^p = m(|\log^- \circ |f||^p) \leq m(|\log^- \circ |f||^p). \quad (18)$$

The result follows from Lebesgue’s monotone convergence theorem since the sequence $|\log \circ |h||^p$ is nondecreasing, converges pointwise to $|\log \circ |h||^p$ pointwise and by (18) their integrals are uniformly bounded. □
Definition 2.2. The Fourier transform $\mathcal{F} : L^1(G) \to \ell^\infty(\hat{G})$ is defined by $\mathcal{F}(h)(\chi) := \int_G h \, \chi$.

We define the spectrum $\Omega(h) := \text{support } \mathcal{F}(h)$. The Hausdorff-Young theorem [15, 26] implies that the restrictions give bounded operators $\mathcal{F} : L^p(G) \to \ell^q(\hat{G})$ for $p \in [1, \infty)$ and $p^{-1} + q^{-1} = 1$.

Definition 2.3. A compactification $(G, \theta)$ induces an injective homomorphism $\xi : \hat{G} \to \mathbb{R}$, $\xi(\chi) := \omega$ where $\chi \circ \theta = \chi_{\omega}$, by which we will identify $\hat{G}$ as a subset of $\mathbb{R}$ with the same archimedian order. Therefore if $\xi(h) \in \mathcal{O}(G)$ is the lift of $f \in U(\mathbb{R})$, then $\Omega(h) = \Omega(f)$. The compactification gives Hardy spaces $H^p(G, \theta) := \{h \in L^p(G) : \Omega(h) \subset [0, \infty), p \in [1, \infty]\}$.

Definition 2.4. A function $h \in H^p(G, \theta)$ is outer if $\int_G \log |h| = \log \int_G h$. (19)

A function $h \in H^p(G, \theta)$ is inner if $|h| = 1$.

A polynomial $h$ is outer iff it has no zeros in the open unit disk since a formula of Jensen [16] gives $\int_G \log |h| = \log |h(0)| - \sum_{h(\lambda) = 0} \log^{-1}(|\lambda|)$. Beurling [5] proved that a function $h \in H^2(\mathbb{T})$ admits a factorization $h = h_o h_i$, with $h_o$ outer and $h_i$ inner, iff $\log |h| \in L^1(\mathbb{T})$.

Let $(G, \theta)$ be a compactification. If $h \in \mathcal{O}(G)$ has a bounded spectrum $\Omega(h) \subset [0, \infty)$ and $\int h \, d\sigma > 0$ then $f = h \circ \theta$ extends to an entire function $F$ bounded in the upper half plane. We observe that if $F$ has no zeros in the upper half plane, then $\chi_{-\log F/2} \in \mathcal{O}$ is the Ahiezer spectral factor [1] of the entire function $F(z)\overline{F(\overline{z})}$.

Conjecture 2.1. $h$ above is outer iff $F$ has no zeros in the open upper half plane.

3. Mahler Measure and the Riemann Hypothesis

Definition 3.1. For $G$ a compact abelian group the Mahler measure [22, 23] of $h \in L^1(G)$ is $M(h) := \exp \left( \int_G \log |h| \right) \in [0, \infty)$. We also define $M^\pm(h) := \exp \left( \int_G \log^\pm |h| \right)$.

Since $M(h) = M^+(h)M^-(h)$ and $M^+(h) \in [1, \text{max}\{1, ||h||_\infty\}]$, it follows that $M(h) > 0$ iff $\log^+ |h| \in L^1(G)$ and then $M^-(h) = \exp \left( -|| \log^+ |h| ||_1 \right)$. Lemma 2.2 implies that this condition holds whenever $h \in \mathcal{O}(G)$ is nonzero and $\Omega(h)$ is bounded.

Definition 3.2. For $N \in \mathbb{N}$, $\Phi_N := \text{product of the first } N \text{ cyclotomic polynomials}$.

Amoroso ([2], Theorem 1.3) proved that the Riemann Hypothesis is equivalent to

$$\log M^+(\Phi_N) \ll_\epsilon N^{1/2+\epsilon}, \quad \epsilon > 0. \quad (20)$$

Define $f_N := \Phi_N \circ \chi_1 \in U(\mathbb{R})$ and define $J_{f_N} : (0, \infty) \to [0, 1]$ by (1). Jensen’s formula implies that $M(\Phi_N) = 1$ therefore

$$\log M^+(\Phi_N) = - \int_0^1 \log(u) \, dJ_{f_N}(u). \quad (21)$$

The bounds that we obtained for $J_f$ in (2) and (3) were exceptionally crude and totally inadequate to obtain (20). When deriving (3) for general polynomials we used the bound $\Xi_{f,0,1,\alpha}(v) =$
Distribution of Small Values of Bohr Almost Periodic Functions

Let \( m \{ x : |f(x)| < u, |f^{(1)}(x)| < v \} \leq m \{ x : |f^{(1)}(x)| < v \} \). Conjecture (1.1) was based on our intuition that a smaller upper bound holds. We suspect that much smaller upper bounds hold for specific sequences of polynomials as illustrated by the following examples. Construct sequences of height 1 polynomials

\[
P_n(z) := 1 + z + \cdots + z^n; \quad Q_n(z) := \left( \frac{n}{[n/2]} \right)^{-1} (1 + z)^n
\]

and \( p_n := P_n \circ \chi_1, q_n := Q_n \circ \chi_1 \). Both polynomials have maxima at \( z = 1 \), \( ||P_n||_{\infty} = n + 1 \), Stirling’s approximation gives \( ||Q_n||_{\infty} \approx \sqrt{n/2} \) for large \( n \), and for \( u \in (0, 1] \)

\[
J_{p_n}(u) \leq \frac{2}{\pi} \sin^{-1} \left( \min \{ 1, u \} \right) \leq u \Rightarrow \log(M^-(P_n)) > -1,
\]

\[
J_{q_n}(u) = \frac{2}{\pi} \sin^{-1} \left( \min \left\{ 1, \frac{1}{2} \left( \frac{n}{[n/2]} \right)^\frac{1}{n} \frac{1}{u^{\frac{1}{n}}} \right\} \right) \geq \frac{2}{\pi} u^{\frac{1}{n}} \Rightarrow \log(M^-(Q_n)) < -\frac{2n}{\pi}.
\]

Differences between these polynomials arise from their root discrepancy. Those of \( P_n \) are nearly evenly spaced. Those of \( Q_n \), all at \( z = 1 \), have maximally discrepancy.

**Conjecture 3.1.** If \( R_n \) is a polynomial with \( n + 1 \) terms and height \( H(R_n) = 1 \) then \( M^-(Q_n) \leq M^-(P_n) \).

The roots of \( \Phi_N \) have the form \( \exp(2\pi i a_k), k = 1, \ldots, \deg \Phi_N \) where \( a_k \) are the Farey series consisting of rational numbers in \( [0, 1) \) whose denominators are \( \leq N \). Bounds on the discrepancy of the Farey series were shown by Franel [14] and by Landau [17] to imply the Riemann Hypothesis. The relationship between the discrepancy of roots of a polynomial and its coefficients, and the distributions of roots of entire functions have been extensively studied since the seminal paper by Erdős and Turán [12] and the extensive work by Levin and his school [21]. We suggest that investigation of the functions \( \Xi_{f, \omega, k, v, u} \) in (4) and derived functions \( K_f \) in (5) may further elucidate how the distribution of small values of polynomials and entire functions depend on their coefficients and roots.

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Распределение малых значений почти периодических функций Бора с ограниченным спектром

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Для не нулевой почти периодической функции Бора на $\mathbb{R}$ с ограниченным спектром мы доказали, что существуют $C_f > 0$ и целое число $n > 0$ такие что для каждого $u > 0$ средняя мера установить $\{ x : |f(x)| < u \}$ меньше $C_f u^{1/n}$. Для тригонометрических полиномов с частотами $\leq n + 1$ мы показали, что $C_f$ можно выбрать так, чтобы он зависел только от $n$ и модуль наибольшего коэффициента $f$. Из этой оценки следует, что мера Малера $M(h)$, подъема $h$ из $f$ к компактификации $G$ из $\mathbb{R}$ положительна и обсуждена связь меры Малера с гипотезой Римана.

Ключевые слова: почти периодическая функция, целая функция, факторизация Берлинга, мера Малера, гипотеза Римана.