STABILITY IN THE INSTANTANEOUS BETHE–SALPETER FORMALISM: REDUCED EXACT-PROPAGATOR BOUND-STATE EQUATION WITH HARMONIC INTERACTION

Zhi-Feng LI
Faculty of Physics, University of Vienna,
Boltzmanngasse 5, A-1090 Vienna, Austria

Wolfgang LUCHA*
Institute for High Energy Physics,
Austrian Academy of Sciences,
Nikolsdorfgasse 18, A-1050 Vienna, Austria

Franz F. SCHÖBERL†
Faculty of Physics, University of Vienna,
Boltzmanngasse 5, A-1090 Vienna, Austria

Abstract

Several numerical investigations of the Salpeter equation with static confining interactions of Lorentz-scalar type revealed that its solutions are plagued by instabilities of presumably Klein-paradox nature. By proving rigorously that the energies of all predicted bound states are part of real, entirely discrete spectra bounded from below, these instabilities are shown, for confining interactions of harmonic-oscillator shape, to be absent for a “reduced” version of an instantaneous Bethe–Salpeter formalism designed to generalize the Salpeter equation towards an approximate inclusion of the exact propagators of all bound-state constituents.

PACS numbers: 11.10.St, 03.65.Ge, 03.65.Pm

* E-mail address: wolfgang.lucha@oeaw.ac.at
† E-mail address: franz.schoeberl@univie.ac.at
1 Introduction

The presumably most well-known (and in elementary particle physics most widely applied) three-dimensional reduction of the (four-dimensional) Bethe–Salpeter formalism [1] for the description of bound states within quantum field theories, such as, for example, in quantum electrodynamics or quantum chromodynamics, is its instantaneous approximation, derived by assuming any interaction between bound-state constituents as static in their rest frame. The additional assumption of free propagation of all bound-state constituents then leads to Salpeter’s equation, an integral equation determining the bound-state’s Salpeter amplitude (which encodes the distribution of the bound-state constituents’ relative momenta) and its mass eigenvalue [2]. This equation can be solved by, for instance, reduction to a set of radial relations [3–5] and subsequent conversion to an equivalent matrix eigenvalue problem [6–9].

However, one would, of course, like to incorporate into the formalism also effects such as dynamical breakdown of chiral symmetry, allowing one to interpret the lowest pseudoscalar quark–antiquark bound states as (pseudo-)Goldstone bosons. This necessitates to take into account the exact propagators of the bound-state constituents, a rather ambitious goal but certainly missed by the free-propagator assumption on which the Salpeter equation relies.

One recent attempt in this direction has been undertaken, by two of the present authors (W. L. and F. F. S.), in Ref. [10], with the implications of this improvement for both energy levels and Salpeter amplitudes of the bound states being (tentatively) explored in Ref. [11]. Unfortunately, numerical treatments of the Salpeter equation [2] with (in configuration space linearly rising) confining interaction observed, for a specific class of Lorentz nature of this interaction, nasty instabilities of its solutions, likely related to Klein’s paradox [12–14].

In view of this clearly unsatisfactory state of the art, we scrutinized, for various popular Lorentz structures, including one suggested by Böhm, Joos, and Krammer (BJK hereafter) [15,16], a reduced form [17–21] of Salpeter’s equation with harmonic-oscillator interactions, which allowed for an analytical investigation of the stability problem [22–24]. By a rigorous stability analysis, we managed to prove all bound states to be stable by demonstrating that their masses form real, purely discrete spectra bounded from below [22–24]. Here, we extend this earlier analysis, with precisely the same findings, to the generalized Salpeter formalism of Ref. [10], where the behaviour of the full propagators slightly complicates the discussion.

The outline of the paper is as follows. In Sec. 2 we briefly recall the full exact-propagator bound-state equation previously derived within the framework of the specific instantaneous Bethe–Salpeter formalism introduced in Ref. [10] and perform the (standard) truncation of this bound-state equation to an exact-propagator version of the reduced Salpeter equation. Assuming the integral kernel encoding the interaction to be of convolution type, we reduce, in Sec. 3, our truncated equation to a radial eigenvalue equation for any Salpeter amplitude describing bound states with spin-parity-charge conjugation quantum numbers $J^{PC} = 0^{-+}$ (which is the environment where all instabilities we are concerned about should arise first). For interactions of harmonic-oscillator form in configuration space, any such radial integral equation simplifies to an ordinary differential equation, given, for various kernels, in Sec. 4. All these differential equations are then transformed, along the lines sketched in Sec. 5, into eigenvalue equations for Schrödinger operators, which can be analyzed by standard means. A systematic rigorous analytical inspection of the spectral properties of all these operators, briefly sketched in Sec. 6, then leads us to conclude, in Sec. 7, that for reasonable behaviour of the exact propagators of the bound-state constituents all bound states are indeed stable.
2 Instantaneous Bethe–Salpeter formalism for nearly exact propagators of the bound-state constituents

Within instantaneous formulations of the Bethe–Salpeter framework, a bound state $|B(P)\rangle$ of momentum $P$ and mass $M$, composed of a fermion of mass $m_1$ and momentum $p_1$ and an antifermion of mass $m_2$ and momentum $p_2$, represented by Dirac field operators $\psi_1(x_1)$ and $\psi_2(x_2)$, respectively, is described in momentum space by the equal-time Salpeter amplitude

$$\Phi(p) = \int d^3x \exp(-i p \cdot x) \langle 0|\psi_1(0, \zeta x) \bar{\psi}_2(0, -\eta x)|B(P)\rangle ,$$

involving the total momentum, $P \equiv p_1 + p_2$, the relative momentum $p \equiv \zeta p_1 - \eta p_2$, and the relative coordinate $x \equiv x_1 - x_2$ of this two-particle system, with $\eta$ and $\zeta$ satisfying $\eta + \zeta = 1$. (We suppress all spinor and internal indices and all dependence on the total momentum $P$.)

Instantaneous approximations to the Bethe–Salpeter equation are found by integrating the latter over the zero component, $p_0$, of $p$. The four-dimensional Bethe–Salpeter equation involves two dynamical ingredients: the exact propagators of both bound-state constituents and a Bethe–Salpeter kernel representing all their interactions. Let us discuss these in turn.

**Interaction kernel:** The instantaneous approximation assumes that the Bethe–Salpeter kernel $K(p, q)$ depends, in the center-of-momentum frame of the bound state studied, exclusively on the spatial components, $p, q$, of the two relative momenta $p, q$ involved:

$$K(p, q) = K(p, q) .$$

This regards all interactions as instantaneous and thus ignores all retardation effects.

**Exact propagators:** By Lorentz covariance (if preserved by the gauge-fixing procedure), the exact fermion propagator $S_i(p)$ is fully determined, in parity-conserving theories, by two real $p$-dependent Lorentz-scalar functions; the latter can be interpreted as the mass function $m_i(p^2)$ and wave-function renormalization factor $Z_i(p^2)$ of the fermion:

$$S_i(p) = \frac{i Z_i(p^2)}{\not{p} - m_i(p^2) + i \varepsilon} , \quad \not{p} \equiv p^\mu \gamma^\mu , \quad \varepsilon \downarrow 0 , \quad i = 1, 2 .$$

The exact propagator $S_i(p)$ can be found as solution of the fermion Dyson–Schwinger equation or from lattice gauge theory. The integration of the Bethe–Salpeter equation over $p_0$ requires, of course, the knowledge of the explicit functional dependence of the propagator functions $m_i(p^2)$ and $Z_i(p^2)$ on $p_0$. In view of the lack of such information, in general cases, these propagator functions have been assumed in Ref. [10] to depend *approximately* just on $p$ by replacing them by $m_i(p^2) \to m_i(p^2)$ and $Z_i(p^2) \to Z_i(p^2)$. Moreover, we impose as reasonable constraints $0 < m_i(p^2) < \infty$ and $0 < Z_i(p^2) \leq 1$.

In the free-propagator limit, $m_i(p^2) \to m_i$, $Z_i(p^2) \to 1$, Salpeter’s equation [2] is recovered.

Defining, for particle $i = 1, 2$, free-particle energy $E_i(p)$, generalized Dirac Hamiltonian $H_i(p)$, and energy projection operators $\Lambda_i^\pm(p)$ for positive or negative energies according to

$$E_i(p) \equiv \sqrt{p^2 + m_i^2(p^2)} , \quad i = 1, 2 ,$$

$$H_i(p) \equiv \gamma_0 [\gamma \cdot p + m_i(p^2)] , \quad i = 1, 2 ,$$

$$\Lambda_i^\pm(p) \equiv \frac{E_i(p) \pm H_i(p)}{2 E_i(p)} , \quad i = 1, 2 ,$$
our full-propagator instantaneous Bethe–Salpeter equation for fermion–antifermion bound states, proposed in Ref. [10] as generalization of Salpeter’s equation [2], then takes the form

\[ \Phi(p) = Z_1(p_1^2) Z_2(p_2^2) \int \frac{d^3q}{(2\pi)^3} \left( \frac{\Lambda^+(p_1) \gamma_0 [K(p,q) \Phi(q)] \gamma_0 \Lambda_-(p_2)}{P_0 - E_1(p_1) - E_2(p_2)} - \frac{\Lambda^-(p_1) \gamma_0 [K(p,q) \Phi(q)] \gamma_0 \Lambda^+(p_2)}{P_0 + E_1(p_1) + E_2(p_2)} \right). \quad (1) \]

Every solution satisfies the constraint \( \Lambda^+(p_1) \Phi(p) \Lambda^-(p_2) = \Lambda^-(p_1) \Phi(p) \Lambda^+(p_2) = 0 \) [10]; this entails its projector decomposition \( \Phi(p) = \Lambda^+(p_1) \Phi(p) \Lambda^-(p_2) + \Lambda^-(p_1) \Phi(p) \Lambda^+(p_2) \).

Any Bethe–Salpeter interaction kernel \( K(p,q) \) can be represented as sum of terms each of which is the product of a Lorentz-scalar potential function with a tensor product of some Dirac matrices. If in each of these terms the couplings of the bound fermions to the effective interaction involves the same generic Dirac matrix \( \Gamma \), and if \( V\Gamma(p,q) \) denotes the associated potential function, the action of the kernel \( K(p,q) \) on Salpeter amplitudes \( \Phi(p) \) thus reads

\[ [K(p,q) \Phi(q)] = \sum_{\Gamma} V_{\Gamma}(p,q) \Gamma \Phi(q) \Gamma. \quad (2) \]

First attempts to explore the consequences of introducing the exact propagators arising in quantum chromodynamics have been undertaken in Ref. [11]: within the rainbow–ladder truncation scheme the Dyson–Schwinger equation suggests for light-quark propagators [25]

\[ m(p^2) = \frac{a}{1 + p^4/b} + m_0, \quad Z(p^2) = 1 - \frac{c}{1 + p^2/d}, \]

with \( a = 0.745 \text{ GeV}, \ b = (0.744 \text{ GeV})^4, \ m_0 = 0.0055 \text{ GeV}, \ c = 0.545, \ d = (1.85508 \text{ GeV})^2 \).

Subjecting \( \Phi(p) \) to either of the (equivalent) additional constraints \( \Lambda^+(p_1) \Phi(p) = 0 \) or \( \Phi(p) \Lambda^-(p_2) = 0 \) yields the exact-propagator counterpart of the reduced Salpeter equation

\[ [P_0 - E_1(p_1) - E_2(p_2)] \Phi(p) \]

\[ = Z_1(p_1^2) Z_2(p_2^2) \int \frac{d^3q}{(2\pi)^3} \Lambda^+(p_1) [K(p,q) \Phi(q)] \gamma_0 \Lambda_-(p_2). \quad (3) \]

For the study of the spectrum of bound-state mass eigenvalues \( M \) it is sufficient to consider the center-of-momentum frame of the two-particle system defined by \( P = 0 \), which implies \( p = p_1 = -p_2 \). There the time component, \( P_0 \), of the total momentum \( P \) reduces to \( M \), i.e., \( P_0 = M \). Accordingly, we will perform our spectral analysis in the bound state’s rest frame.

For kernels of the form (2), by suitable generalization of Eq. (18) of Ref. [5], all solutions of our exact-propagator reduced instantaneous Bethe–Salpeter equation (3) have to satisfy

\[ M \int \frac{d^3p}{(2\pi)^3} \text{Tr} \left[ \Phi^\dagger(p) \Phi(p) \right] = \int \frac{d^3p}{(2\pi)^3} \left[ E_1(p) + E_2(p) \right] \text{Tr} \left[ \Phi^\dagger(p) \Phi(p) \right] \]

\[ + \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} Z_1(p^2) Z_2(p^2) \sum_{\Gamma} V_{\Gamma}(p,q) \text{Tr} \left[ \Phi^\dagger(p) \gamma_0 \Gamma \Phi(q) \Gamma \gamma_0 \right]. \]

Recalling our line of argument given in Sec. 7 of Ref. [22] (see also Ref. [23]), in this relation both the integral on its left-hand side and the first term on its right-hand side are obviously nonvanishing and real while the second term on its right-hand side is real if \( Z_1(p^2) Z_2(p^2) \) is real, the potential functions \( V_{\Gamma}(p,q) \) satisfy \( V_{\Gamma}^\dagger(q,p) = V_{\Gamma}(p,q) \), and the Dirac couplings \( \Gamma \) satisfy \( \gamma_0 \Gamma^\dagger \gamma_0 = \pm \Gamma \). If this holds, all bound-state mass eigenvalues \( M \) are necessarily real.
3 Radial eigenvalue equations for pseudoscalar states

Following, or mimicking, the path paved in Refs. [3–5,22–24], as first step of our analysis we simplify the bound-state equation (3), for given Dirac structures $\Gamma \otimes \Gamma$ of the interaction, to radial eigenvalue equations by factorizing off all dependence on angular variables, which for Bethe–Salpeter interaction kernels of convolution type, $K(p, q) = K(p-q)$, is a trivial one.

For notational brevity we restrict the presentation of our considerations to bound states built up by fermion and associated antifermion. This entails, with $p \equiv |p|$, for the masses of both bound-state constituents $m_1(p_1^2) = m_2(p_2^2) =: m(p)$, for their renormalization factors $Z_1(p_1^2) = Z_2(p_2^2) =: Z(p)$ and for their energies $E_1(p_1) = E_2(p_2) =: E(p) \equiv \sqrt{p^2 + m^2(p)}$.

On simple and purely energetic grounds, instabilities of the kind we worry about should manifest themselves first for pseudoscalar bound states [13]. Consequently, we will consider fermion–antifermion bound states with total spin $J$, parity $P = (-1)^{J+1}$ and (well-defined) charge-conjugation quantum number $C = (-1)^J$. The particular projector structure of the bound-state equation (3) entails, for all its solutions $\Phi(p)$, the unique component structure $\Phi(p) = \Lambda^+_1(p_1) \Phi(p) \Lambda^+_2(p_2)$ [22]. As consequence of this, for the states under consideration any solution $\Phi(p)$ of Eq. (3) involves only one independent component, $\phi(p)$. Dropping the indices $i = 1, 2$ in the definitions of Sec. 2, any generic solution of Eq. (3) is thus of the form

$$\Phi(p) = \phi(p) \frac{H(p) + E(p)}{E(p)} \gamma_5 \equiv 2 \phi(p) \Lambda^+(p) \gamma_5 .$$

The bound states in the focus of our interest, i.e., the pseudoscalar states, are characterized by total spin $J = 0$ and thus by the spin-parity-charge conjugation assignment $J^{PC} = 0^-$. Stripping off all spherical harmonics reduces Eq. (3) to an equation for the radial factor, $\phi(p)$, in the independent amplitude $\phi(p)$. Therein the interaction between the bound-state constituents defined, in configuration space, by some spherically symmetric static potential $V(r)$, $r \equiv |\mathbf{x}|$, enters in form of a set of Fourier–Bessel transforms $V_L(p, q)$ ($L = 0, 1, 2, \ldots$):

$$V_L(p, q) \equiv 8\pi \int_0^\infty dr r^2 j_L(p r) j_L(q r) V(r) , \quad L = 0, 1, 2, \ldots ,$$

where $j_n(z)$, for $n = 0, \pm 1, \pm 2, \ldots$, label the spherical Bessel functions of the first kind [26]. Specifying the Lorentz behaviour of the Bethe–Salpeter kernel $K(p-q)$, we thus obtain the radial eigenvalue equations, for interactions of Lorentz-scalar Dirac structure, $\Gamma \otimes \Gamma = 1 \otimes 1$,

$$2 E(p) \phi(p) - \frac{1}{2} Z^2(p) \int dq q^2 (2\pi)^2 \left[ \left( 1 + m(p) m(q) \right) \frac{V_0(p, q)}{E(p) E(q)} - \frac{p q V_1(p, q)}{E(p) E(q)} \right] \phi(q) = M \phi(p) ,$$

for interactions of time-component Lorentz-vector Dirac structure, $\Gamma \otimes \Gamma = \gamma^0 \otimes \gamma^0$,

$$2 E(p) \phi(p) + \frac{1}{2} Z^2(p) \int dq q^2 (2\pi)^2 \left[ \left( 1 + m(p) m(q) \right) \frac{V_0(p, q)}{E(p) E(q)} + \frac{p q V_1(p, q)}{E(p) E(q)} \right] \phi(q) = M \phi(p) ,$$

for interactions of Lorentz-vector Dirac structure, $\Gamma \otimes \Gamma = \gamma_\mu \otimes \gamma^\mu$,

$$2 E(p) \phi(p) + Z^2(p) \int dq q^2 (2\pi)^2 \left( 2 - \frac{m(p) m(q)}{E(p) E(q)} \right) V_0(p, q) \phi(q) = M \phi(p) ,$$

$$2 \int dq q^2 (2\pi)^2 \left[ \left( 1 + m(p) m(q) \right) \frac{V_0(p, q)}{E(p) E(q)} + \frac{p q V_1(p, q)}{E(p) E(q)} \right] \phi(q) = M \phi(p) ,$$

for interactions of Lorentz-vector Dirac structure, $\Gamma \otimes \Gamma = \gamma^\mu \otimes \gamma^\nu$. 


for interactions of Lorentz-pseudoscalar Dirac structure, $\Gamma \otimes \Gamma = \gamma_5 \otimes \gamma_5$,

$$2 E(p) \phi(p) - \frac{1}{2} Z^2(p) \int_0^\infty \frac{dq\ q^2}{(2\pi)^2} \left[ \left(1 - \frac{m(p)\ m(q)}{E(p)\ E(q)}\right) V_0(p, q) - \frac{p\ q\ V_1(p, q)\ E(q)}{E(p)\ E(q)}\right] \phi(q) = M \phi(p) ,$$

and, for interactions of BJK [15, 16] Dirac structure, $\Gamma \otimes \Gamma = \frac{1}{2} (\gamma_\mu \otimes \gamma^\mu + \gamma_5 \otimes \gamma_5 - 1 \otimes 1)$,

$$2 E(p) \phi(p) + Z^2(p) \int_0^\infty \frac{dq\ q^2}{(2\pi)^2} V_0(p, q) \phi(q) = M \phi(p) .$$

4 Confining interactions of harmonic-oscillator type

For pure harmonic-oscillator interactions, represented by the configuration-space potential

$$V(r) = a\ r^2 , \quad a = a^* \neq 0 , \quad r \equiv |\mathbf{x}| ,$$

upon trading the harmonic-oscillator interaction for the second-order differential operators

$$D_{\mu}^{(L)} = \frac{d^2}{dp^2} + 2 \frac{d}{p\ \frac{d}{dp}} - \frac{L\ (L+1)}{p^2} , \quad L = 0, 1, 2, \ldots ,$$

(which are nothing but the Laplacian $\Delta \equiv \nabla \cdot \nabla$ acting on states of angular momentum $L$), the Fourier–Bessel integral transforms $V_L(p, q)$ encoding all interactions explicitly read [22]

$$V_L(p, q) = -\frac{(2\pi)^2 a}{q^2} D_{\mu}^{(L)} \delta(p - q) , \quad L = 0, 1, 2, \ldots .$$

In this case all integral equations representing our reduced exact-propagator instantaneous bound-state equation (3) simplify to second-order homogeneous linear ordinary differential equations; the latter read, for interactions of Lorentz-scalar Dirac structure, $\Gamma \otimes \Gamma = 1 \otimes 1$,

$$2 E(p) + \frac{Z^2(p)\ a}{2} \left( D_{\mu}^{(0)} + \frac{m(p)\ m(p)}{E(p)\ E(p)}\right)\ D_{\mu}^{(1)}\ p\ E(p) - \frac{E(p)}{E(p)}\ D_{\mu}^{(1)}\ \frac{p}{E(p)}\right)\right)\phi(p) = M \phi(p) ,$$

for interactions of time-component Lorentz-vector Dirac structure, $\Gamma \otimes \Gamma = \gamma_0 \otimes \gamma_0$,

$$2 E(p) - \frac{Z^2(p)\ a}{2} \left( D_{\mu}^{(0)} + \frac{m(p)\ m(p)}{E(p)\ E(p)}\right)\ D_{\mu}^{(1)}\ p\ E(p) + \frac{E(p)}{E(p)}\ D_{\mu}^{(1)}\ \frac{p}{E(p)}\right)\right)\phi(p) = M \phi(p) ,$$

for interactions of Lorentz-vector Dirac structure, $\Gamma \otimes \Gamma = \gamma_\mu \otimes \gamma^\mu$,

$$2 E(p) - Z^2(p)\ a \left( 2 D_{\mu}^{(0)} - \frac{m(p)\ m(p)}{E(p)\ E(p)}\right)\ D_{\mu}^{(1)}\ p\ E(p)\right)\right)\phi(p) = M \phi(p) ,$$

for interactions of Lorentz-pseudoscalar Dirac structure, $\Gamma \otimes \Gamma = \gamma_5 \otimes \gamma_5$,

$$2 E(p) + \frac{Z^2(p)\ a}{2} \left( D_{\mu}^{(0)} - \frac{m(p)\ m(p)}{E(p)\ E(p)}\right)\ D_{\mu}^{(1)}\ p\ E(p) - \frac{E(p)}{E(p)}\ D_{\mu}^{(1)}\ \frac{p}{E(p)}\right)\right)\phi(p) = M \phi(p) ,$$
and, for interactions of BJK \[15,16\] Dirac structure, \( \Gamma \otimes \Gamma = \frac{1}{2} (\gamma_\mu \otimes \gamma^\mu + \gamma_5 \otimes \gamma_5 - 1 \otimes 1) \),

\[
[2 \, E(p) - Z^2(p) \, a \, D_p^{(0)}] \, \phi(p) = M \, \phi(p) .
\] (9)

Apart from the Lorentz-pseudoscalar case all differential operators on the left-hand sides of these eigenvalue equations are not self-adjoint, their spectra, therefore, not necessarily real. Nevertheless, by our arguments of Sec. 2 and the fact that our potential function \( V_\Gamma(p, q) \) is the Fourier transform \( V_\Gamma(p, q) = V_\Gamma(p - q) \equiv \int d^3p \exp[-i \, (p - q) \cdot x] \, V_\Gamma(r) \) of a real central configuration-space potential \( V_\Gamma(r) = V_\Gamma^*(r) \) we can be sure that all eigenvalues \( M \) are real.

The above (differential) equations may be further simplified by application of the identities

\[
D_p^{(0)} \frac{m(p)}{E(p)} = \frac{m(p)}{E(p)} D_p^{(0)} + \frac{2}{E(p)} \left[ \frac{d m(p)}{dp} \frac{m(p)}{E(p)} \frac{d E(p)}{dp} \right] \frac{d}{dp} + \left[ D_p^{(0)} \frac{m(p)}{E(p)} \right] ,
\]

\[
D_p^{(0)} \frac{p}{E(p)} = \frac{p}{E(p)} D_p^{(0)} + \frac{2}{E(p)} \left[ \frac{1 - \frac{p}{E(p)} \frac{d E(p)}{dp}}{\frac{d}{dp} + \left[ D_p^{(0)} \frac{p}{E(p)} \right]} .
\] (10)

First of all, by adopting these identities, the definition \( E^2(p) \equiv p^2 + m^2(p) \), and the relation

\[
p + m(p) \frac{d m(p)}{dp} = E(p) \frac{d E(p)}{dp} ,
\]

it is very straightforward to convince oneself that, in spite of its appearance, the differential operators cancel in Eq. (8). Thus, as was the case already for the reduced Salpeter equation [22, Sec. 6], for all harmonic-oscillator interactions of Lorentz-pseudoscalar Dirac structure \( \Gamma \otimes \Gamma = \gamma_5 \otimes \gamma_5 \) the eigenvalue problem of Eq. (8) is posed by a pure multiplication operator,

\[
\left\{ 2 \, E(p) + \frac{Z^2(p) \, a}{2 \, E^4(p)} \left[ 2 \, E^2(p) + \left( m(p) - p \frac{d m(p)}{dp} \right)^2 \right] \right\} \phi(p) = M \, \phi(p) ,
\]

which has a continuous spectrum but no eigenvalue. Accordingly, the Lorentz-pseudoscalar interaction kernel cannot describe bound states and does not need to be considered further.

5 Transformation to Schrödinger eigenvalue equation

Because of the momentum dependence of \( Z(p) \), \( m(p) \), and \( E(p) \), all our genuine differential equations (5), (6), (7), and (9) are not standard Schrödinger equations. However, they may be easily reformulated [22–24] as usual Schrödinger eigenvalue equation for zero eigenvalue,

\[
[-D_p^{(0)} + U(p)] \psi(p) \equiv \left[ -\frac{d^2}{dp^2} - \frac{2}{p} \frac{d}{dp} + U(p) \right] \psi(p) = 0 ,
\] (11)

where \( U(p) \) is an auxiliary potential to be found case by case and the Laplacian expected in a Schrödinger equation is assumed to act on states of vanishing orbital angular momentum: First, dividing by \( [Z^2(p) \, a] \neq 0 \) (which is nonvanishing by assumption) and working out the derivatives with the aid of Eqs. (10) simplifies all differential equations to the common form

\[
\left[ -\frac{d^2}{dp^2} - 2 \, g(p) \frac{d}{dp} + h(p) \right] \phi(p) = 0 ,
\] (12)
where in each case the two functions $g(p)$ and $h(p)$ may be easily read off from Eqs. (5), (6), (7), or (9). Here, merely $h(p)$ involves $M$ as a parameter, whereas $g(p)$ is independent of $M$; this observation will considerably facilitate our analysis. Then, performing the substitution $\phi(p) = f(p) \psi(p)$ of the bound-state amplitude leads to the desired Schrödinger shape (11), provided the transforming function $f(p)$ is found as the solution of the differential equation

$$\left[ \frac{d}{dp} + g(p) \right] f(p) = \frac{f(p)}{p},$$

(13)

which may be easily integrated, yielding the (formal) solution, up to an irrelevant constant,

$$f(p) = p \exp \left[ - \int dp \, g(p) \right].$$

(14)

Our auxiliary potential $U(p)$ is, of course, fully determined by the quantities $g(p)$ and $h(p)$:

$$U(p) \equiv h(p) + \frac{dg}{dp}(p) + g^2(p).$$

(15)

Clearly, if $g(p) = 1/p$, as is the case for the time-component Lorentz-vector Dirac structure $\Gamma \otimes \Gamma = \gamma^0 \otimes \gamma^0$ and the BJK [15,16] Lorentz structure $\Gamma \otimes \Gamma = \frac{1}{2} (\gamma_\mu \otimes \gamma^\mu + \gamma_5 \otimes \gamma_5 - 1 \otimes 1)$, the differential equation (12) is already of the desired Schrödinger form (11). In these cases, the integration (14) of our definition (13) of $f(p)$ trivially yields $f(p) = 1$, that is, $\psi(p) = \phi(p)$, and the effective potential (15) becomes just the $M$-dependent function $h(p)$: $U(p) = h(p)$. In both our nontrivial cases, $g(p)$ reads, for the Lorentz-scalar Dirac structure $\Gamma \otimes \Gamma = 1 \otimes 1$,

$$g(p) = \frac{1}{p} - \frac{m(p)}{m(p) \, E^2(p)} \left( m(p) - p \frac{dm}{dp}(p) \right) = \frac{1}{pm(p) \, E^2(p)} \left( m^3(p) + p^3 \frac{dm}{dp}(p) \right)$$

and, for the Lorentz-vector Dirac structure, $\Gamma \otimes \Gamma = \gamma_\mu \otimes \gamma^\mu$,

$$g(p) = \frac{1}{p} + \frac{pm(p)}{E^2(p) \, (E^2(p) + p^2)} \left( m(p) - p \frac{dm}{dp}(p) \right).$$

The effective potentials $U(p)$ are, for kernels of Lorentz-scalar Dirac structure $\Gamma \otimes \Gamma = 1 \otimes 1$,

$$U(p) = -\frac{2 \, E(p) - M \, E^2(p)}{2 \, m^2(p) \, a} - \frac{1}{2 \, m^2(p) \, E^2(p)} \left[ 2 \, E^2(p) + \left( m(p) - p \frac{dm}{dp}(p) \right)^2 \right],$$

for kernels of time-component Lorentz-vector Dirac structure $\Gamma \otimes \Gamma = \gamma^0 \otimes \gamma^0$,

$$U(p) = \frac{2 \, E(p) - M \, E^2(p)}{2 \, m^2(p) \, a} + \frac{1}{2 \, E^4(p)} \left[ 2 \, E^2(p) + \left( m(p) - p \frac{dm}{dp}(p) \right)^2 \right],$$

for kernels of Lorentz-vector Dirac structure $\Gamma \otimes \Gamma = \gamma_\mu \otimes \gamma^\mu$,

$$U(p) = \frac{2 \, E(p) - M \, E^2(p)}{2 \, m^2(p) \, a} + \frac{E^2(p)}{E^2(p) + p^2} - \frac{2 \, p^2}{E^2(p) \, (E^2(p) + p^2)^2} \left( m(p) - p \frac{dm}{dp}(p) \right)^2,$$
and, for kernels of the BJK [15, 16] Dirac structure $\Gamma \otimes \Gamma = \frac{1}{2} (\gamma_\mu \otimes \gamma^\mu + \gamma_5 \otimes \gamma_5 - 1 \otimes 1),$

$$U(p) = \frac{2 E(p) - M}{Z^2(p) a}.$$  

Interestingly, precisely a linear dependence $m(p) \propto p$ of the mass function $m(p)$ on $p$ entails

$$m(p) - p \frac{dm}{dp}(p) = 0.$$  

In this latter case, our transformation becomes trivial also for Lorentz-scalar ($\Gamma \otimes \Gamma = 1 \otimes 1$) and Lorentz-vector ($\Gamma \otimes \Gamma = \gamma_\mu \otimes \gamma^\mu$) kernels: $g(p) = 1/p$ implies $f(p) = 1$ and $\psi(p) = \phi(p).$

In the “Salpeter limit” of free propagators involving constant constituent masses, that is, if $Z(p) \equiv 1,$ $m(p) = \text{const} \iff \frac{dm}{dp}(p) = 0,$

all these potentials $U(p)$ must reduce to the corresponding expressions in Sec. 6 of Ref. [22].

6 Spectra: discreteness, semiboundedness, stability

In the preceding section, we succeeded to rewrite the differential equations (5), (6), (7), and (9) as eigenvalue equations for eigenvalue zero of Schrödinger Hamiltonian operators of the form $\mathcal{H} \equiv -\Delta + U$ (acting only on states of vanishing orbital angular momentum). In order to proceed with our spectral analysis of the bound-state masses $M,$ we recall a fundamental theorem about the spectra of Hamiltonians with potentials increasing beyond bound [27]: a Schrödinger operator $H \equiv -\Delta + V,$ defined as sum of quadratic forms, with positive, locally bounded, infinitely rising potential $V(x) \to \infty$ for $|x| \to \infty$ has a purely discrete spectrum. This theorem may be trivially generalized to all potentials $V$ that are bounded from below. Thus, if the effective potential $U(p)$ satisfies all requirements of this theorem, the spectrum of the corresponding $M$-dependent Hamiltonian $\mathcal{H}$ will be, for any value of the bound-state mass $M,$ entirely discrete: it will consist exclusively of isolated eigenvalues $\mathcal{E}_i(M)$ ($i \in \mathbb{Z}$) of finite multiplicity, depending, of course, on one parameter, $M.$ By construction, the zeros of these eigenvalue functions $\mathcal{E}_i(M)$ define the wanted set of bound-state mass eigenvalues $M.$

A closer inspection reveals that for sufficiently well-behaved propagator functions $m(p)$ and $Z(p)$ [in particular, if $m(p)$ is an element of the space of differentiable functions on $\mathbb{R}^+$] and for an “appropriate” choice of the sign of the harmonic-oscillator interaction strength $a$ all auxiliary potentials $U(p)$ resulting, by means of Eq. (15), from the differential equations (5), (6), (7), and (9) satisfy all the assumptions of the “infinitely-rising-potential theorem:”

1. The behaviour of all potentials $U(p)$ for large relative momenta $p$ is dominated by the contribution of the kinetic part $2 E(p)$ of our exact-propagator reduced instantaneous Bethe–Salpeter equation (3). This contribution is necessarily proportional to $1/a$: for the choice $a < 0$ in the case of a kernel of Lorentz-scalar Dirac structure, $\Gamma \otimes \Gamma = 1 \otimes 1,$ and for the choice $a > 0$ in the case of interactions of time-component Lorentz-vector Dirac structure, $\Gamma \otimes \Gamma = \gamma^0 \otimes \gamma^0,$ or Lorentz-vector Dirac structure, $\Gamma \otimes \Gamma = \gamma_\mu \otimes \gamma^\mu,$ or BJK [15,16] Dirac structure, $\Gamma \otimes \Gamma = \frac{1}{2} (\gamma_\mu \otimes \gamma^\mu + \gamma_5 \otimes \gamma_5 - 1 \otimes 1),$ all our potentials $U(p)$ exhibit in the large-$p$ limit the rise to positive infinity required by the above theorem.
2. In order to be on the safe side, we avoid the vanishing of denominators by relying on a strict positivity $m(p) > 0$ of the mass functions $m(p)$ of the bound-state constituents: $m(p) \neq 0$ for all $p \in \mathbb{R}^+ \equiv [0, \infty)$ should, together with $Z(p) \neq 0$, suffice to guarantee the absence of singularities in all our potentials $U(p)$ and, as immediate consequence, for all potentials $U(p)$ both their local boundedness and their boundedness from below. In particular instances, this requirement of strict positivity of the mass function $m(p)$ can be loosened to an extent which depends on the Dirac structure $\Gamma \otimes \Gamma$ of the kernel: for the BJK [15, 16] structure $\Gamma \otimes \Gamma = \frac{1}{2} (\gamma_\mu \otimes \gamma^\mu + \gamma_5 \otimes \gamma_5 - 1 \otimes 1)$, all prerequisites of the above theorem are satisfied automatically, without imposing any constraint on $m(p)$; in the case of the Lorentz-vector structure $\Gamma \otimes \Gamma = \gamma^0 \otimes \gamma^0$, we find as sufficient to require that $m(p)$ is nonvanishing at the origin, i.e., to demand $m(0) \neq 0$; in the case of the Lorentz-vector structure $\Gamma \otimes \Gamma = \gamma_\mu \otimes \gamma^\mu$ the mass $m(p)$ even may be allowed to approach zero in the limit $p \to 0$, without doing any harm, if it behaves for small $p$ like $m(p) \propto p^d$ with an exponent $d \in (0, \frac{1}{2}) \cup \{1\} \cup [2, \infty)$, i.e., $d \notin \left(\frac{1}{2}, 1\right) \cup (1, 2)$.

Hence, we know that all eigenvalues $E_i(M)$ of all our auxiliary Hamiltonians $H$ are discrete. The discreteness of all auxiliary eigenvalues $E_i(M)$ for all $M$ guarantees the discreteness of the spectrum of bound-state masses $M$ [22, 23] if for each eigenvalue $E_i(M)$ the derivative of $E_i(M)$ with respect to $M$ can be shown to be strictly definite, that is, if, for every $i$, either

$$\frac{dE_i}{dM}(M) > 0 \quad \forall \ M$$

or

$$\frac{dE_i}{dM}(M) < 0 \quad \forall \ M$$

holds, because in this case every zero of $E_i(M)$ is also an isolated point of finite multiplicity. By the Hellmann–Feynman theorem [28], the derivative of a given $E_i(M)$ with respect to $M$ is identical to the expectation value over the associated eigenstate $|i\rangle$ (taken as normalized, $\langle i|i\rangle = 1$, for brevity of notation) of the derivative of the Hamiltonian $\mathcal{H}$ with respect to $M$:

$$\frac{d\mathcal{E}_i}{dM}(M) = \left\langle i \bigg| \frac{\partial \mathcal{H}}{\partial M} \bigg| i \right\rangle.$$ (16)

By construction, all the functions $h(p)$ in the differential equation (12), and thus all our auxiliary potentials $U(p)$, and, consequently, all our Hamiltonians $\mathcal{H}$ exhibit a very simple, that is, a linear dependence on the bound-state mass $M$. The derivatives with respect to $M$

$$\frac{\partial \mathcal{H}}{\partial M} = \frac{\partial U}{\partial M} = \frac{\partial h}{\partial M}$$ (17)

are summarized, for the Lorentz structures of interaction kernels still of interest, in Table 1. According to this, for precisely those choices of the sign of the harmonic-oscillator coupling $a$ for which the above “infinitely-rising-potential theorem” was found to be applicable [viz., for $a < 0$ in the case of kernels of Lorentz-scalar structure, $\Gamma \otimes \Gamma = 1 \otimes 1$, and for $a > 0$ in the case of kernels of time-component Lorentz-vector structure, $\Gamma \otimes \Gamma = \gamma^0 \otimes \gamma^0$, Lorentz-vector structure, $\Gamma \otimes \Gamma = \gamma_\mu \otimes \gamma^\mu$, and BJK [15, 16] structure, $\Gamma \otimes \Gamma = \frac{1}{2} (\gamma_\mu \otimes \gamma^\mu + \gamma_5 \otimes \gamma_5 - 1 \otimes 1)$, the derivatives (17), and thus the derivatives with respect to $M$ of all eigenvalues $E_i(M)$, are all negative definite: The associated spectra of bound-state masses $M$ will be entirely discrete.
Table 1: Derivative of our auxiliary Hamiltonian operators $\mathcal{H} \equiv -\Delta + U$ with respect to the bound-state mass $M$ entering in the effective potential $U(p; M)$ as a parameter, for Lorentz structures $\Gamma \otimes \Gamma$ entailing differential operators in the reduced instantaneous Bethe–Salpeter equation (3) with configuration-space harmonic-oscillator interactions $V(r) = a r^2$ ($a \neq 0$).

| Lorentz structure | $\Gamma \otimes \Gamma$ | $\frac{\partial \mathcal{H}}{\partial M} = \frac{\partial U}{\partial M} = \frac{\partial h}{\partial M}$ |
|------------------|------------------------|-------------------------------------------------|
| Lorentz scalar   | $1 \otimes 1$          | $\frac{E^2(p)}{Z^2(p)} \frac{a}{m^2(p)}$       |
| time-component   | $\gamma^0 \otimes \gamma^0$ | $-\frac{1}{Z^2(p)} \frac{a}{E^2(p)}$       |
| Lorentz vector   | $\gamma^\mu \otimes \gamma^\mu$ | $-\frac{1}{Z^2(p)} \frac{a}{E^2(p) + p^2}$ |
| BJK [15, 16]     | $\frac{1}{2} (\gamma^\mu \otimes \gamma^\mu + \gamma_5 \otimes \gamma_5 - 1 \otimes 1)$ | $-\frac{1}{Z^2(p)} \frac{a}{E^2(p)}$ |

Under the conditions discussed above, all effective potentials $U(p)$ derived in Sec. 5 and thus all associated Schrödinger Hamiltonians $\mathcal{H}$ are for any given value of the parameter $M$ bounded from below. Accordingly, for each Lorentz structure $\Gamma \otimes \Gamma$ under consideration the spectrum of auxiliary eigenvalues $\mathcal{E}_i(M)$ is bounded from below. The derivatives (16) of the functions $\mathcal{E}_i(M)$ proved to be negative definite. Therefore, the zero of the “lowest” of all the trajectories $\mathcal{E}_i(M)$ defines a lower bound on the spectrum of bound-state mass eigenvalues.

Altogether, the sequence of findings of this section forms the basis of the firm conviction that all bound states encountered in any actual evaluation of the reduced exact-propagator instantaneous Bethe–Salpeter equation (3) with harmonic-oscillator interaction are stable.

7 Summary, Conclusions, and Outlook

Motivated by instabilities observed [4,12–14] for the solutions of the Salpeter equation with confining interaction, we investigated the stability of the solutions of the three-dimensional reduction of the Bethe–Salpeter equation proposed in Ref. [10] to retain exact propagators. Summarizing our findings, for each Lorentz structure analyzed the solutions of our reduced exact-propagator instantaneous Bethe–Salpeter equation (3) with pure harmonic-oscillator interactions, which enter by means of Eq. (4), exhibit very desirable characteristic features:

- The main result is that, indeed, all bound-state masses are discrete.
- For their spectra, we could show: they are bounded from below.
- For all bound states, we were able thus to prove that they are stable.

It goes without saying that, in spite of some mainly technical complications to be overcome [22–24], a similar stability discussion may be envisaged for the (full) Salpeter equation [29].
Acknowledgements

W. L. would like to sincerely thank Bernhard Baumgartner for an enlightening discussion.

References

[1] E. E. Salpeter and H. A. Bethe, Phys. Rev. 84 (1951) 1232.
[2] E. E. Salpeter, Phys. Rev. 87 (1952) 328.
[3] J.-F. Lagaë, Phys. Rev. D 45 (1992) 305.
[4] M. G. Olsson, S. Veseli, and K. Williams, Phys. Rev. D 52 (1995) 5141, hep-ph/9503477.
[5] M. G. Olsson, S. Veseli, and K. Williams, Phys. Rev. D 53 (1996) 504, hep-ph/9504221.
[6] W. Lucha, K. Maung Maung, and F. F. Schöberl, Phys. Rev. D 63 (2001) 056002, hep-ph/0009185.
[7] W. Lucha, K. Maung Maung, and F. F. Schöberl, in: Proceedings of the International Conference on Quark Confinement and the Hadron Spectrum IV, edited by W. Lucha and K. Maung Maung (World Scientific, New Jersey/London/Singapore/Hong Kong, 2002), p. 340, hep-ph/0010078.
[8] W. Lucha, K. Maung Maung, and F. F. Schöberl, Phys. Rev. D 64 (2001) 036007, hep-ph/0011235.
[9] W. Lucha and F. F. Schöberl, Int. J. Mod. Phys. A 17 (2002) 2233, hep-ph/0109165.
[10] W. Lucha and F. F. Schöberl, J. Phys. G: Nucl. Part. Phys. 31 (2005) 1133, hep-th/0507281.
[11] Li Z.-F., W. Lucha, and F. F. Schöberl, Mod. Phys. Lett. A 21 (2006) 1657, hep-ph/0510372.
[12] J. Parramore and J. Piekarewicz, Nucl. Phys. A 585 (1995) 705, nucl-th/9402019.
[13] J. Parramore, H.-C. Jean, and J. Piekarewicz, Phys. Rev. C 53 (1996) 2449, nucl-th/9510024.
[14] M. Uzzo and F. Gross, Phys. Rev. C 59 (1999) 1009, nucl-th/9808041.
[15] M. Böhm, H. Joos, and M. Krammer, Nucl. Phys. B 51 (1973) 397.
[16] F. Gross and J. Milana, Phys. Rev. D 43 (1991) 2401.
[17] A. B. Henriques, B. H. Kellett, and R. G. Moorhouse, Phys. Lett. B 64 (1976) 85.
[18] S. Jacobs, M. G. Olsson, and C. J. Suchyta III, Phys. Rev. D 35 (1987) 2448.
[19] A. Gara, B. Durand, L. Durand, and L. J. Nickisch, Phys. Rev. D 40 (1989) 843.
[20] A. Gara, B. Durand, and L. Durand, Phys. Rev. D 42 (1990) 1651; *ibid.* 43 (1991) 2447 (erratum).
[21] W. Lucha, H. Rupprecht, and F. F. Schöberl, Phys. Rev. D 45 (1992) 385.
[22] Z.-F. Li, W. Lucha, and F. F. Schöberl, Phys. Rev. D (in press), arXiv:0707.3202 [hep-ph].
[23] W. Lucha and F. F. Schöberl, in: Proceedings of QCD@Work 2007: International Workshop on Quantum Chromodynamics: Theory and Experiment, edited by P. Colangelo, D. Creanza, F. De Fazio, R. A. Fini, E. Nappi, and G. Nardulli, AIP Conf. Proc. (AIP, New York, 2007), Vol. 964, p. 318, arXiv:0707.1440 [hep-ph].

[24] W. Lucha and F. F. Schöberl, preprint HEPHY-PUB 856/07 (2007), arXiv:0711.1736 [hep-ph], to appear in the Proceedings of Hadron 07, XIIth International Conference on Hadron Spectroscopy, Frascati (Rome), Italy, October 8–13, 2007.

[25] P. Maris and C. D. Roberts, Phys. Rev. C 56 (1997) 3369, nucl-th/9708029.

[26] Handbook of Mathematical Functions, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1964).

[27] M. Reed and B. Simon, Methods of Modern Mathematical Physics IV: Analysis of Operators (Academic Press, New York, 1978).

[28] H. Hellmann, Acta Physicochim. URSS 1 (1934/1935) 913; ibid. 4 (1936) 225; Einführung in die Quantenchemie (F. Deuticke, Leipzig/Wien, 1937), p. 285; R. P. Feynman, Phys. Rev. 56 (1939) 340.

[29] W. Lucha and F. F. Schöberl (in preparation).