ALGEBRAIC INDEPENDENCE OF THE CARLITZ PERIOD
AND THE POSITIVE CHARACTERISTIC MULTIZETA VALUES
AT \( n \) AND \((n,n)\)

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Abstract. Let \( k \) be the rational function field over the finite field of \( q \) elements
and \( \bar{k} \) its fixed algebraic closure. In this paper, we study algebraic relations
over \( \bar{k} \) among the fundamental period \( \tilde{\pi} \) of the Carlitz module and the positive
characteristic multizeta values \( \zeta(n) \) and \( \zeta(n, n) \) for an “odd” integer \( n \),
where we say that \( n \) is “odd” if \( q - 1 \) does not divide \( n \). We prove that these three
elements are either algebraically independent over \( \bar{k} \) or satisfy some simple
relation over \( k \). We also prove that if \( 2n \) is “odd”, then these three elements
are algebraically independent over \( \bar{k} \).

1. Introduction

Let \( k := \mathbb{F}_q(\theta) \) be the rational function field over the finite field of \( q \) elements
with variable \( \theta \), \( p \) the characteristic of \( k \), \( k_\infty := \mathbb{F}_q((\theta^{-1})) \) the \( \infty \)-adic completion
of \( k \), \( k_\infty \) a fixed algebraic closure of \( k_\infty \), and \( \bar{k} \) the algebraic closure of \( k \) in \( k_\infty \).
Thakur ([T1]) defined the positive characteristic multizeta values (MZV) by
\[
\zeta(n_1, \ldots, n_d) := \sum_{a_1, \ldots, a_d} \frac{1}{a_1^{n_1} \cdots a_d^{n_d}} \in k_\infty
\]
for integers \( n_1, \ldots, n_d \geq 1 \), where the sum is over all monic polynomials \( a_i \) in \( \mathbb{F}_q[\theta] \)
such that \( \deg a_1 > \cdots > \deg a_d \geq 0 \). Throughout this paper, we use the notation
\( \zeta \) for the above sums, and we use the notation \( \zeta_Q \) for the classical multiple zeta
values in characteristic zero. The sum \( \sum_i n_i \) is called the weight and \( d \) is called
the depth of the MZV \( \zeta(n_1, \ldots, n_d) \). We fix a \( (q - 1) \)-st root of \(-\theta\) and set
\[
\tilde{\pi} := (-\theta)^{\frac{q}{1-q^r}} \prod_{i=1}^{\infty} \left( 1 - \theta^{1-q^i} \right)^{-1} \in (-\theta)^{\frac{1}{1-q^r}} \cdot k_\infty^x,
\]
the fundamental period of the Carlitz module. Since \( \#\mathbb{F}_q[\theta]^x = q - 1 \), we say that
an integer \( n \) is “odd” if \( q - 1 \) does not divide \( n \) and “even” if \( q - 1 \) divides \( n \).

It is clear that \( \zeta(pn) = \zeta(n)^p \) for all \( n \geq 1 \). Carlitz ([Ca]) showed that \( \zeta(n)/\tilde{\pi}^n \in \bar{k}^x \)
for each “even” integer \( n \geq 1 \). Chang and Yu ([CY]) proved that all the algebraic relations over \( \bar{k} \)
among \( \tilde{\pi} \) and MZVs of depth one come from the above types. Thus we are interested in the case where the depth is greater than 1. In [Ch], Chang showed that all the algebraic relations over \( \bar{k} \) among MZVs are \( k \)-homogeneous. However he did not treat linear relations over \( \bar{k} \) among MZVs of
same weights. Linear relations of same weight MZVs are complicated and not completely understood. See for example [T2]. In this paper, we prove the following theorem:

**Theorem 1.1.** Let \( n \geq 1 \) be an “odd” integer. Then \( \bar{\pi}, \zeta(n) \) and \( \zeta(n, n) \) are algebraically independent over \( \overline{k} \) or \( \zeta(n)^2 - 2\zeta(n, n) \in \pi^{2n} \cdot k^\times \). If \( 2n \) is “odd”, then we have the former case.

**Remark 1.2.** If \( p = 2 \), then \( 2n \) is “odd” if and only if \( n \) is “odd”. Thus \( \bar{\pi}, \zeta(n) \) and \( \zeta(n, n) \) are algebraically independent over \( \overline{k} \) for each “odd” integer \( n \).

On the other hand, in characteristic zero, \( 2n \) is always even. Thus the second part of Theorem 1.1 does not occur in this case. In fact, we have the relation \( \zeta_Q(n)^2 - 2\zeta_Q(n, n) = \zeta_Q(2n) \in \pi^{2n} \cdot \mathbb{Q}^\times \).

**Remark 1.3.** If \( p^e \) divides \( n_1 \) and \( n_2 \) and \( n_1/p^e + n_2/p^e \leq q \) for some \( e \geq 0 \), then we have the harmonic shuffle product \( \zeta(n_1)\zeta(n_2) = \zeta(n_1, n_2) + \zeta(n_2, n_1) + \zeta(n_1 + n_2) \) ([T2] Theorem 1) or see Remark 1.1. In particular, if \( 2n = p^e(q - 1) \), then we have the relation \( \zeta(n)^2 - 2\zeta(n, n) = \zeta(2n) \in \pi^{2n} \cdot k^\times \) (when \( p = 2 \), this follows directly, but in this case \( n \) is “even”). Thus, the latter case of the first part of Theorem 1.1 actually occurs when \( p \geq 3 \). We do not know what happens in the case where \( 2n = m(q - 1) \) for general \( m \) (including the case where \( n \) is “even”).

Since \( \bar{\pi} \) and \( \zeta(n) \) are algebraically independent over \( \overline{k} \) for each “odd” integer \( n \) ([CY]), we have the following corollary:

**Corollary 1.4.** Let \( n \geq 1 \) be an “odd” integer. Then \( \bar{\pi} \) and \( \zeta(n, n) \) are algebraically independent over \( \overline{k} \).

Corollary 1.4 also follows from the “Eulerian” criterion ([CPY]) and the fact that if a multizeta value is not “Eulerian”, then it is algebraically independent from \( \bar{\pi} \) over \( \overline{k} \) ([Ch]).

**Corollary 1.5.** Let \( \overline{Z}_2 \) be the \( \overline{k} \)-vector space spanned by \( \zeta(1, 1) \) and \( \zeta(2) \) (the weight two MZVs). We have \( \dim_{\overline{k}} \overline{Z}_2 = 2 \) if \( q \geq 2 \), and \( \dim_{\overline{k}} \overline{Z}_2 = 1 \) if \( q = 2 \).

**Proof.** Note that by Remark 1.3 we have \( \zeta(1)^2 = 2\zeta(1, 1) + \zeta(2) \in \overline{Z}_2 \) for each \( q \). If \( q \geq 4 \), then 2 is “odd”. Thus \( \zeta(1) \) and \( \zeta(1, 1) \) (and \( \bar{\pi} \)) are algebraically independent over \( \overline{k} \) by Theorem 1.1. Thus \( \zeta(1)^2 \) and \( \zeta(1, 1) \) form a basis of \( \overline{Z}_2 \). If \( q = 3 \), then 2 is “even”, and hence we have \( \zeta(2) \in \pi^2 \cdot k^\times \). However \( \bar{\pi} \) and \( \zeta(1) \) are algebraically independent over \( \overline{k} \) ([CY]). Thus \( \zeta(1)^2 \) and \( \zeta(2) \) form a basis of \( \overline{Z}_2 \). When \( q = 2 \), we have the relation \( \zeta(1, 1) = \zeta(2)/(\theta^2 + \theta) \) ([T2] Theorem 5.10.13)).

**Remark 1.6.** If \( p \neq 2 \), then \( \zeta(1) \) and \( \zeta(2) \) are algebraically independent over \( \overline{k} \) ([CY]), and hence in this case we already know that \( \zeta(1)^2 \) and \( \zeta(2) \) form a basis of \( \overline{Z}_2 \). Thus a new result for determining the dimension of \( \overline{Z}_2 \) in Corollary 1.5 is the characteristic 2 case with \( q \neq 2 \).

In Section 2 we review Papanikolas’ theory for \( t \)-motives briefly. Theorem 1.1 is proved in Section 3. The outline of the proof is as follows: Following Anderson-Thakur ([AT2]), we construct a rigid analytically trivial pre-\( t \)-motive \( M \) such that the MZVs we want to know appear in its “period matrix” \( \Psi \). Then we obtain two group schemes, \( \Gamma_M \) and \( \Gamma_\Psi \), where \( \Gamma_M \) is the fundamental group of the Tannakian category generated by \( M \), and \( \Gamma_\Psi \) is defined from \( \Psi \) more explicitly than \( \Gamma_M \). By
Papanikolas’ theory, these two group schemes are isomorphic and their dimensions are the same as the transcendence degree which we want, in this case, the transcendence degree of \( \overline{k}(\pi, \zeta(n), \zeta(n, n)) \) over \( \overline{k} \). By the Tannakian duality, we have a natural projection \( \Gamma_M \to \mathbb{G}_m \). Set \( V \) to be its kernel. Since \( \pi \) and \( \zeta(n) \) are algebraically independent over \( \overline{k} \), the dimension of \( \Gamma_M \) is greater than or equal to 2. We suppose that the dimension of \( \Gamma_M \) is 2. Then we can determine \( V \) explicitly. By using the expression of \( V \), we can also compute \( \Gamma_M \) explicitly. Then we obtain a non-trivial defining polynomial of \( \Gamma_M \). By returning to the definition of \( \Gamma_\varphi \), the MZVs in question must satisfy some relation coming from the defining polynomial. If \( 2n \) is “odd”, we can check easily that the MZVs do not satisfy this relation, and so we have a contradiction, showing that \( \operatorname{tr.deg}_k \overline{k}(\pi, \zeta(n), \zeta(n, n)) \geq 3 \).

2. Review of \( t \)-motives

In this section, we review the notions of pre-\( t \)-motives. For more details, see [P]. We continue to use the notation of the previous section. Let \( \mathbb{C}_\infty \) be the \( \infty \)-adic completion of \( \overline{k}_\infty \) and \( |·|_\infty \) its multiplicative valuation. For any subset \( T \) of \( \mathbb{C}_\infty \), we denote by \( \langle T \rangle_k \) the \( k \)-vector subspace of \( \mathbb{C}_\infty \) spanned by \( T \). Let \( t \) be a variable independent of \( \theta \) and \( \mathbb{C}_\infty((t)) \) the field of formal Laurent series over \( \mathbb{C}_\infty \). Let \( T := \{ f \in \mathbb{C}_\infty[[t]] | f \text{ converges on } |t|_\infty \leq 1 \} \) be the Tate algebra and \( \mathbb{L} \) the fractional field of \( T \). We set

\[
E := \left\{ \sum_i a_i t^i \in \mathbb{C}_\infty[[t]] | \lim_{i \to \infty} \sqrt[n]{a_i} | \mathbb{C}_\infty = 0, \ [k_\infty(a_0, a_1, \ldots) : k_\infty] < \infty \right\}.
\]

For any integer \( n \in \mathbb{Z} \) and any formal Laurent series \( f = \sum_i a_i t^i \in \mathbb{C}_\infty((t)) \), we set

\[
f^{(n)} := \sum_i a_i^{(n)} t^i,
\]

the \( n \)-fold twist of \( f \), and \( \sigma(f) := f^{(-1)} \). The operation \( f \mapsto f^{(n)} \) stabilizes the subfields \( \mathbb{L} \) and \( \overline{k}(t) \) of \( \mathbb{C}_\infty((t)) \). For a ring or a module \( R \), we denote by \( \text{Mat}_{r \times s}(R) \) the set of \( r \times s \) matrices with coefficients in \( R \).

A pre-\( t \)-motive is an étale \( \varphi \)-module over \( (\overline{k}(t), \sigma) \), meaning a finite-dimensional \( \overline{k}(t) \)-vector space \( M \) equipped with a \( \sigma \)-semilinear bijective map \( \varphi : M \to M \). A morphism of pre-\( t \)-motives is a \( \overline{k}(t) \)-linear map which is compatible with their structure maps \( \varphi \). There is a tensor product of two pre-\( t \)-motives. For any pre-\( t \)-motive \( M \), the Betti realization of \( M \) is an \( \mathbb{F}_q(t) \)-vector space defined by

\[
M^B := \left( \mathbb{L} \otimes_{\overline{k}(t)} M \right)^{\sigma \otimes \varphi = 1},
\]

where \((-)_{\sigma \otimes \varphi = 1}\) is the \( \sigma \otimes \varphi \)-fixed part. Then we have a natural injection

\[
\mathbb{L} \otimes_{\mathbb{F}_q(t)} M^B \hookrightarrow \mathbb{L} \otimes_{\overline{k}(t)} M.
\]

A pre-\( t \)-motive \( M \) is called rigid analytically trivial if the above injection is an isomorphism. The category of rigid analytically trivial pre-\( t \)-motives forms a neutral Tannakian category over \( \mathbb{F}_q(t) \) with fiber functor \( M \mapsto M^B \). For any such \( M \), we set \( \Gamma_M \) to be the fundamental group of the Tannakian subcategory generated by \( M \) with respect to the Betti realization. By definition, \( \Gamma_M \) is an \( \mathbb{F}_q(t) \)-subgroup scheme of \( \text{GL}(M^B) \).

Let \( M \) be a pre-\( t \)-motive and \( r \) the dimension of \( M \) over \( \overline{k}(t) \). For a \( \overline{k}(t) \)-basis \( \mathbf{m} \in \text{Mat}_{r \times 1}(M) \) of \( M \), there exists a unique matrix \( \Phi \in \text{GL}_r(\overline{k}(t)) \) such that \( \varphi \mathbf{m} = \Phi \mathbf{m} \). Conversely when a matrix \( \Phi \in \text{GL}_r(\overline{k}(t)) \) is given, we can construct a
pre-t-motive $M$ by $M := \overline{k}(t)^r$ with $\varphi(x_1, \ldots, x_r) := (x_1^{(-1)}, \ldots, x_r^{(-1)})\Phi$. Clearly the pre-t-motive $M$ is determined by the matrix $\Phi$, and we say that $M$ is the pre-t-motive defined by $\Phi$. We can show that $M$ is rigid analytically trivial if and only if there exists a matrix $\Psi \in \text{GL}_r(L)$ such that $\Psi^{(-1)} = \Phi\Psi$. The matrix $\Psi$ is called a rigid analytic trivialization of $\Phi$. Let $\Psi_1, \Psi_2 \in \text{GL}_r(L \otimes \overline{\mathbb{T}(t)} L)$ be the matrices such that $(\Psi_1)_{ij} = \Psi_{ij} \otimes 1$ and $(\Psi_2)_{ij} = 1 \otimes \Psi_{ij}$ and set $\widetilde{\Psi} := \Psi_1^{-1}\Psi_2$.

Let $X := (X_{ij})_{ij}$ be the $r \times r$ matrix of independent variables $X_{ij}$. We define an $\mathbb{F}_q(t)$-algebra homomorphism $\nu$ by

$$\nu: \mathbb{F}_q(t)[X_{11}, X_{12}, \ldots, X_{rr}, 1/\det X] \to L \otimes \overline{\mathbb{T}(t)} L; \ X_{ij} \mapsto \widetilde{\Psi}_{ij}$$

and set

$$\Gamma_\Psi := \text{Spec}(\mathbb{F}_q(t)[X_{11}, X_{12}, \ldots, X_{rr}, 1/\det X]/\text{Ker} \nu) \subset \text{GL}_r.$$

We can easily check that $\Psi^{-1}m$ is an $\mathbb{F}_q(t)$-basis of $M^B$. For each $\mathbb{F}_q(t)$-algebra $R$, we have a map

$$\Gamma_\Psi(R) \to \Gamma_M(R); \ \gamma \mapsto (f \cdot \Psi^{-1}m \mapsto f\gamma^{-1} \cdot \Psi^{-1}m),$$

where $f$ runs over all elements of $\text{Mat}_{1 \times r}(R)$.

**Theorem 2.1** ([P, Theorems 4.3.1, 4.5.10, 5.2.2]). Let $M$ be a rigid analytically trivial pre-t-motive equipped with $\Phi$ and $\Psi$ as above.

(1) The scheme $\Gamma_\Psi$ is a smooth subgroup scheme of $\text{GL}_r$, and the above map $\Gamma_\Psi \to \Gamma_M$ is an isomorphism of group schemes over $\mathbb{F}_q(t)$.

(2) We have

$$\dim \Gamma_\Psi = \text{tr.deg}_{\overline{k}(t)} \overline{k}(t)(\Psi_{11}, \Psi_{12}, \ldots, \Psi_{rr}).$$

(3) Assume that $\Phi \in \text{Mat}_{r \times r}(\overline{k}[t])$, $\Psi \in \text{GL}_r(\overline{T}) \cap \text{Mat}_{r \times r}(E)$, and $\det \Phi = c(t - \theta)^d$ for some $c \in \overline{k}^\times$ and $d \geq 0$. Then we have

$$\text{tr.deg}_{\overline{k}(t)} \overline{k}(t)(\Psi_{11}, \Psi_{12}, \ldots, \Psi_{rr}) = \text{tr.deg}_{\overline{k}} \overline{k}(\Psi_{11}(\theta), \Psi_{12}(\theta), \ldots, \Psi_{rr}(\theta)).$$

**Remark 2.2.** The result (3) of Theorem 2.1 is proved by using [ABP, Theorem 3.1.1], which is a very deep result and is called the ABP-criterion. All transcendence results in this paper have their genesis in the paper [ABP]. Also, the assumption $\Psi \in \text{Mat}_{r \times r}(E)$ in (3) can be dropped by [ABP, Proposition 3.1.3].

When a $t$-motive $M$ has some special form, we can describe $\dim \Gamma_M$ in terms of the dimension of some $k$-linear space. We set

$$\Omega(t) := (-\theta)^{-\frac{\text{deg} \Phi}{\text{dim} \Phi}} \prod_{i=1}^{\text{dim} \Phi} \left(1 - \frac{t}{\theta^i}\right) \in \overline{k}_t[t],$$

which is an element of $E$. The function $\Omega(t)$ is essentially due to Anderson and Thakur from [AT1, Section 2.5], though they work with the reciprocal. Since $\Omega$ has infinitely many zeros, it is transcendental over $\overline{k}(t)$. It satisfies the equations

$$\Omega^{(-1)} = (t - \theta)\Omega \quad \text{and} \quad \Omega(\theta) = \frac{1}{\pi}.$$
For each \( \alpha = \sum_i a_i t^i \in \bar{k}[t] \), we set \( |\alpha|_\infty := \max_i \{ |a_i|_\infty \} \). When \( \alpha = (\alpha_1, \ldots, \alpha_d) \) and \( n = (n_1, \ldots, n_d) \) (\( \alpha_j \in \bar{k}[t] \sim \{0\}, n_j \geq 1 \)) satisfy \( |\alpha_j|_\infty < |\theta|_\infty^{-\frac{1}{n_j}} \), we set

\[
L_{\alpha, n}(t) := \sum_{i_1 > \cdots > i_d \geq 0} \alpha_{(i_1)}^{(1)} \cdots \alpha_{(i_d)}^{(d)} \frac{((t - \theta^1) \cdots (t - \theta^{n_1}))^{n_1} \cdots ((t - \theta^d) \cdots (t - \theta^{n_d}))^{n_d}}{(t - \theta)^{n_1 + \cdots + n_d}}.
\]

Note that \( L_{\alpha, n}(t) \) converges on \( |t|_\infty < |\theta|_\infty^q \) and satisfies

\[
L_{\alpha, n}^{(-1)} = \frac{\alpha_{(1)}^{(-1)} (t - \theta)^n}{(t - \theta)^{n_1 + \cdots + n_d}} L_{\alpha_{(1)}, n_1 \ldots n_d} + \frac{1}{(t - \theta)^{n_1 + \cdots + n_d}} L_{\alpha_{(1)}, n_1 \ldots n_d},
\]

where we set \( L_{\emptyset, 0} := 1 \) for the cases \( d = 1 \). Note that when \( \alpha_i \in \bar{k}^\times \) for each \( i \), \( L_{\alpha, n}(\theta) \) is the Carlitz multiple polylogarithm at the algebraic point \( \alpha \), which is defined by Chang ([Ch]).

Let \( n \geq 1 \) be an integer and \( \alpha_1, \ldots, \alpha_r \in \bar{k}[t] \sim \{0\} \) with \( |\alpha_i|_\infty < |\theta|_\infty^{-\frac{n_q}{n_i}} \) for each \( i \). We set

\[
\Phi := \begin{bmatrix}
(t - \theta)^n & 1 \\
\alpha_{1}^{(-1)} (t - \theta)^n & 1 \\
\vdots & \ddots \\
\alpha_{r}^{(-1)} (t - \theta)^n & 1
\end{bmatrix} \in \text{GL}_{r+1}(\bar{k}(t)),
\]

\[
\Psi := \begin{bmatrix}
\Omega^n & 1 \\
\Omega^n L_{\alpha_1, n} & 1 \\
\vdots & \ddots \\
\Omega^n L_{\alpha_r, n} & 1
\end{bmatrix} \in \text{GL}_{r+1}(\mathbb{L}).
\]

Then we have

\[
\Psi^{(-1)} = \Phi \Psi.
\]

**Theorem 2.3** ([P] Theorem 6.3.2, [CY] Theorem 3.1). Let \( M \) be the \( t \)-motive defined by the matrix \( \Phi \) as above. Then we have

\[
\dim \Gamma_M = \dim_k \langle \pi^n, L_{\alpha_1, n}(\theta), \ldots, L_{\alpha_r, n}(\theta) \rangle_k.
\]

**Remark 2.4.** In [P] and [CY], the authors discuss only the case where \( \alpha_i \in \bar{k}^\times \). However their proofs work also for any \( \alpha_i \in \bar{k}[t] \sim \{0\} \).

3. **Proof of Theorem 1.1**

We define two square matrices of size \( d + 1 \):

\[
\Phi := \begin{bmatrix}
(t - \theta)^{n_1 + \cdots + n_d} & (t - \theta)^{n_2 + \cdots + n_d} & \cdots & (t - \theta)^{n_d} \\
\alpha_{1}^{(-1)} (t - \theta)^{n_1 + \cdots + n_d} & \alpha_{2}^{(-1)} (t - \theta)^{n_2 + \cdots + n_d} & \cdots & \alpha_{d}^{(-1)} (t - \theta)^{n_d}
\end{bmatrix}
\]

and \( \Psi := (\Psi_{ij}) \), where

\[
\Psi_{ij} := \Omega^{n_j + \cdots + n_d} L_{\alpha_{j}, \alpha_{j-1}, n_{j-1}, \ldots, n_{i-1}} \quad \text{if} \quad 1 \leq j \leq i \leq d + 1,
\]

and \( \Psi_{ij} := 0 \) if \( i < j \). Then we have \( \Psi^{(-1)} = \Phi \Psi \). In particular each component \( \Psi_{ij} \) of \( \Psi \) is an element of \( \mathbb{E} \) by Theorem 2.1.
We set \( D_i := \prod_{j=0}^{i-1} (\theta^q - \theta^j) \) for \( i \geq 0 \). For each integer \( n \geq 0 \) with \( q \)-adic expansion \( n = \sum n_i q^i \) (\( 0 \leq n_i < q \)), the Carlitz factorial is defined by
\[
\Gamma_{n+1} := \prod_i D_i^{n_i}.
\]
We consider the power sum \( S_i(n) := \sum \frac{1}{a^n} \), where the sum is over all monic polynomials \( a \) in \( \mathbb{F}_q[\theta] \) with \( \deg a = i \). For each \( n \geq 1 \), Anderson and Thakur ([AT1], [AT2]) defined a polynomial \( H_{n-1} \in \mathbb{F}_q[\theta, t] \) such that
\[
(H_{n-1} \Omega^n)^{(i)}(\theta) = \frac{\Gamma_{n} S_i(n)}{\pi^n}
\]
for each \( i \geq 0 \) and \( |H_{n-1}|_\infty < \frac{nq^{|n|}}{|q|} \). Chang ([Ch]) showed that
\[
(\Omega^{n_1+\cdots+n_d} L_{H(n)}^{\alpha,n})^{(\theta^q N)} = \left( \frac{\Gamma_{n_1} \cdots \Gamma_{n_d} \zeta(n_1, \ldots, n_d)}{\pi^{n_1+\cdots+n_d}} \right)^q
\]
for each \( N \geq 0 \), where \( n := (n_1, \ldots, n_d) \) and \( H(n) := (H_{n_1-1}, \ldots, H_{n_d-1}) \). Thus to prove Theorem 1.1, we shall consider the algebraic relations between the special values of \( \Omega, L_{\alpha,n} \) and \( L_{\alpha,\alpha,n,n} \).

**Remark 3.1.** We can easily show that
\[
L_{\alpha_1,n_1} L_{\alpha_2,n_2} = L_{\alpha_1,\alpha_2,n_1,n_2} + L_{\alpha_2,\alpha_1,n_2,n_1} + L_{\alpha_1,\alpha_2,n_1,n_2}
\]
for each \( \alpha_i \) and \( n_i \) (for more general cases, see [Ch], Section 6.2). He treated \( L_{\alpha,\alpha}(\theta) \), but the arguments are the same). By definition, \( \Gamma_n = 1 \) for each \( 1 \leq n \leq q \), and by the construction in [AT1], we know that \( H_{n-1} = 1 \) for \( 1 \leq n \leq q \). Thus if \( n_1 + n_2 \leq q \), we have
\[
L_{H_{n-1} H_{n-1} n_1+n_2} = L_{1,n_1+n_2} = L_{H_{n-1} n_1+n_2}.
\]
Therefore, we obtain the harmonic shuffle product formula in Remark 1.3.

We fix \( \alpha \in \overline{k}[t] \setminus \{0\} \) and \( n \geq 1 \) such that \(|\alpha|_\infty < \frac{nq^{|n|}}{|q|} \), and set matrices
\[
\Phi := \begin{bmatrix}
(t - \theta)^{2n} & (t - \theta)^n \\
\alpha^{-1} (t - \theta)^{2n} & \alpha^{-1} (t - \theta)^n
\end{bmatrix} \in \text{GL}_3(\overline{k}(t))
\]
and
\[
\Psi := \begin{bmatrix}
\Omega^{2n} & \Omega^n \\
\Omega^n L_{\alpha,\alpha,n,n} & \Omega^n L_{\alpha,n}
\end{bmatrix} \in \text{GL}_3(\mathbb{L}).
\]
These matrices satisfy \( \Psi^{-1} = \Phi \Psi \). Let \( M \) be the rigid analytically trivial \( \tau \)-motive defined by \( \Phi \) and set \( \Gamma := \Gamma_M \) as its fundamental group. By Theorem 2.1, we have
\[
\dim \Gamma = \text{tr.deg}_{\overline{k}(t)}(\overline{k}(t), L_{\alpha,n}, L_{\alpha,\alpha,n,n}) = \text{tr.deg}_{\overline{k}}(\overline{k}(t), L_{\alpha,n}(\theta), L_{\alpha,\alpha,n,n}(\theta)).
\]

**Theorem 3.2.** Let \( \alpha \) and \( n \) be as above. Assume that \( \overline{\pi}^n \) and \( L_{\alpha,n}(\theta) \) are linearly independent over \( k \). If \( p > 2 \), assume further that \( (\Omega^n L_{\alpha,n})^2 - 2 \Omega^{2n} L_{\alpha,\alpha,n,n} - c \) and \( \Omega^{2n} \) are linearly independent over \( \overline{k}(t) \) for each \( c \in \mathbb{F}_q(t) \). Then we have
\[
\text{tr.deg}_{\overline{k}}(\overline{k}(\tau), L_{\alpha,n}(\theta), L_{\alpha,\alpha,n,n}(\theta)) = 3.
\]
Theorem 2.1.}

We set \( \tilde{\Psi} = (\tilde{\Psi}_{ij})_{ij} \), we have the relations \( \tilde{\Psi}_{11} = \tilde{\Psi}_{22} = \tilde{\Psi}_{13} = 1 \),
\( \tilde{\Psi}_{21} = \tilde{\Psi}_{23} \), \( \tilde{\Psi}_{32} = 1 \), \( \tilde{\Psi}_{31} = (L^2 - L_{a,n} \otimes \Omega^n) \).
Thus we have the inclusion
\[
\Gamma \cong \Gamma_{\psi} \subset G := \left\{ \begin{bmatrix} a^2 & a \cr ax & y \cr x & 1 \end{bmatrix} \right\},
\]
where we use the letters \( a, x, y \) as coordinate variables. Similarly, we obtain
\[
\Gamma' \cong \Gamma_{\psi'} \subset G' := \left\{ \begin{bmatrix} a \\ x \cr 1 \end{bmatrix} \right\}.
\]

Since \( \dim \Gamma_{\psi'} = \dim_k (\pi^n L_{a,n}(0) \otimes \Omega^n) = 2 = \dim G' \) and \( G' \) is irreducible and reduced, we have the equality \( \Gamma_{\psi'} = G' \). By using the above identifications, we can write
\[
\psi : \Gamma \to \Gamma', \quad \begin{bmatrix} a^2 & a \\ ax & y \\ x & 1 \end{bmatrix} \mapsto \begin{bmatrix} a \\ x \\ 1 \end{bmatrix}
\]
and
\[
\pi' : \Gamma' \to \mathbb{G}_m, \quad \begin{bmatrix} a \\ x \\ 1 \end{bmatrix} \mapsto a.
\]

We set \( \pi := \pi' \circ \psi \), \( V := \text{Ker} \pi \) and \( V' := \text{Ker} \pi' \). Then we have
\[
V \subset W := \left\{ \begin{bmatrix} 1 & x & 1 \cr y & x & 1 \end{bmatrix} \right\}, \quad V' \approx \mathbb{G}_a.
\]
and obtain the following diagram:

\[
\begin{array}{cccccc}
1 & \rightarrow & W & \rightarrow & G & \rightarrow \mathbb{G}_m & \rightarrow 1 \\
1 & \rightarrow & V & \rightarrow & \Gamma & \pi & \rightarrow \mathbb{G}_m & \rightarrow 1 \\
& & \psi |_V & \downarrow & \psi & & & \\
1 & \rightarrow & V' & \rightarrow & \Gamma' & \pi' & \rightarrow \mathbb{G}_m & \rightarrow 1 \\
\end{array}
\]

which is commutative and whose rows are exact. Clearly \(\psi|_V\) is surjective. The group scheme \(V\) is smooth over \(F\). Indeed, take \(a_0 \in \mathbb{G}_m(F) \setminus \mathbb{G}_m(F_q)\) and its lift \(\tilde{a}_0 \in \Gamma(F)\). Let \(T \subset \Gamma F\) be the Zariski closure of the group generated by \(\tilde{a}_0\). Then \(T\) is a rank one torus and isomorphic to \(\mathbb{G}_m, F\) via \(\pi\). In particular, \(\text{Lie}(\pi F) : \text{Lie}(\Gamma F) \rightarrow \text{Lie}(\mathbb{G}_m, F)\) is non-zero. Since \(\Gamma F\) and \(\mathbb{G}_m, F\) are smooth over \(F\), we have \(\text{dim} \text{Lie}(\pi F) = \text{dim} \text{Lie}(\Gamma F)\). If \(\text{dim} \Gamma = 3\), then there is nothing to prove. Thus we suppose that \(\text{dim} \Gamma = 2\) and hence \(\text{dim} V = 1\). Then \(\text{Ker}(\psi|_V : V(F) \rightarrow V'(F)) = V(F) \cap W_1(F)\) has dimension zero, where we set

\[
W_1 := \left\{ \begin{bmatrix} 1 & 1 \\ y & 1 \end{bmatrix} \right\}.
\]

In view of the above short exact sequence, we let \(\mathbb{G}_m(F)\) act on \(V(F)\) by \(a.X := \tilde{a}^{-1}X\tilde{a}\) for \(a \in \mathbb{G}_m(F)\) and \(X \in V(F)\), where \(\tilde{a} \in \Gamma(F)\) is a lift of \(a\). In terms of matrices, this action is given by

\[
\begin{bmatrix} 1 \\ x & 1 \\ y & x & 1 \end{bmatrix} = \begin{bmatrix} 1 & ax & 1 \\ axy & ax & 1 \end{bmatrix}.
\]

Since \(V(F) \cap W_1(F)\) is closed under this action, we conclude that \(V(F) \cap W_1(F) = 1\). For a matrix

\[
X = \begin{bmatrix} 1 \\ x & 1 \\ y & x & 1 \end{bmatrix} \in V(F),
\]

we have

\[
X^r = \begin{bmatrix} 1 \\ rx & 1 \\ (r^{-1}r)x^2 + ry & rx & 1 \end{bmatrix} \in V(F)
\]

for each integer \(r\). Thus if \(r\) is not divisible by \(p\), we obtain

\[
V(F) \ni (r.X)X^{-r} = \begin{bmatrix} 1 \\ 0 \\ r((r-1)y - \frac{r-1}{2}x^2) & 1 \\ 0 & 1 \end{bmatrix} \in W_1(F).
\]

Since \(V(F) \cap W_1(F) = 1\), we have the relation

\[
\frac{r-1}{2}x^2 = (r-1)y.
\]

When \(p = 2\), if we take \(r \equiv 3 \mod 4\), then we have \(x^2 = 2y = 0\), and thus \(x = 0\). This means \(X \in V(F) \cap W_1(F) = 1\), and therefore \(V(F) = 1\). This is
a contradiction. Thus when \( p = 2 \), we always have \( \dim \Gamma = 3 \). In the following, we assume that \( p \geq 3 \). In this case, if we take \( r \not\equiv 0, 1 \mod p \), then we have the relation \( y = \frac{x^2}{2} \). Since \( \dim V = 1 \), we conclude that

\[
V = \left\{ \begin{bmatrix} 1 \\ x \\ \frac{x^2}{2} \\ x \\ 1 \end{bmatrix} \right\}.
\]

Next, we determine the group scheme \( \Gamma \). Fix an element \( a_0 \in \mathbb{G}_m(F) \) which has infinite order and set \( \tilde{a}_0 \in \mathbb{G}_m(F) \) to be the geometric point above \( a_0 \). Since the geometric fiber \( \Gamma_{\tilde{a}_0} \) of \( \pi \) over \( \tilde{a}_0 \) is a \( V_T \)-torsor, it is isomorphic to \( V_T \) which is smooth over \( T \). Thus the fiber \( \Gamma_{\tilde{a}_0} \) is smooth over \( F \). By [L, Chapter 3, Proposition 2.20], we have \( \Gamma_{\tilde{a}_0}(F_{\text{sep}}) \neq \emptyset \), and hence we can take a lift \( \tilde{a}_0 \) of \( a_0 \) in \( \Gamma(F_{\text{sep}}) \). Since \( \Gamma(F_{\text{sep}}) \) contains \( V(F_{\text{sep}}) \), we can eliminate the \( x \)-coordinate of \( \tilde{a}_0 \). Thus we may assume that

\[
\tilde{a}_0 = \begin{bmatrix} a_0^2 \\ a_0 \\ y_0 \\ 1 \end{bmatrix} \in \Gamma(F_{\text{sep}}).
\]

Then for each \( r \in \mathbb{Z} \), we have

\[
\tilde{a}_0^r = \begin{bmatrix} a_0^{2r} \\ a_0^r \\ y_0 \\ a_0^{-r} - 1 \end{bmatrix} \in \Gamma(F_{\text{sep}}).
\]

Since \( a_0 \) has infinite order, we have

\[
\left\{ \begin{bmatrix} a^2 \\ y_0 \\ a^2 - 1 \\ 1 \end{bmatrix} \right\} \subset \Gamma_T,
\]

where \( c_0 := \frac{y_0}{a_0^{-r} - 1} \in F_{\text{sep}} \). Since \( \Gamma_T \) is a two-dimensional irreducible reduced group scheme which also contains \( V_T \), we conclude that

\[
\Gamma_T = \left\{ \begin{bmatrix} a^2 \\ ax \\ c_0 (a^2 - 1) + \frac{x^2}{2} \\ x \\ 1 \end{bmatrix} \right\}.
\]

We set a polynomial

\[
Q := 2X_3 + c_0 X_{33} - X_{22} - 1 - X_{22} X_{33} \in F_{\text{sep}}[X_{22}, X_{33}, X_{33}] \subset F_{\text{sep}}[X_{11}, \ldots, X_{32}, 1/\det X].
\]

Then \( \Gamma_T = \text{Spec} \mathcal{F}[X_{22}, X_{33}, X_{33}, X_{22}^{-1}] / (Q) \). Since \( \Gamma \) is defined over \( F \), the ideal \( (Q) \) is stable under the action of \( \text{Gal}(F_{\text{sep}}/F) = \text{Aut}(\mathcal{F}/F) \). Therefore for each \( \sigma \in \text{Gal}(F_{\text{sep}}/F) \), we can write \( \sigma(Q) = P_\sigma Q \) for some \( P_\sigma \in \mathcal{F}[X_{22}, X_{33}, X_{33}, X_{22}^{-1}] \). By comparing the degree of each variable, \( P_\sigma \) must be a constant. Comparing both sides again, we have \( P_\sigma = 1 \) and \( \sigma(c_0) = c_0 \) for each \( \sigma \in \text{Gal}(F_{\text{sep}}/F) \). Hence we have \( c_0 \in F \) and \( Q \in F[X_{11}, \ldots, X_{33}, 1/\det X] \). Since \( Q \equiv 0 \) on the reduced scheme \( \Gamma_T \), we have \( Q(\bar{\Psi}) = 0 \). By the definition of \( \bar{\Psi} \), this is equivalent to the equality

\[
((\Omega^n L_{\alpha,n})^2 - 2 \Omega^{2n} L_{\alpha,n,n} - c_0) \otimes \Omega^{2n} = \Omega^{2n} \otimes ((\Omega^n L_{\alpha,n})^2 - 2 \Omega^{2n} L_{\alpha,n,n} - c_0)
\]

in \( \mathcal{L} \otimes \mathcal{F}(t) \). However, this is a contradiction to the assumption about linear independence. \( \square \)
Proof of Theorem 1.1. We fix an “odd” integer $n \geq 1$ and set $\alpha := H_{n-1}$. Since $\tilde{\pi}^n \not\in k_\infty$ and $L_{\alpha,n}(\theta) = \Gamma_n \zeta(n) \in k_\infty$, they are linearly independent over $k$. If $(\Omega^n L_{\alpha,n})^2 - 2\Omega^2 L_{\alpha,a,n,n} - c$ and $\Omega^{2n}$ are linearly independent over $\bar{k}(t)$ for each $c \in \mathbb{F}_q(t)$, we have that $\tilde{\pi}$, $\zeta(n)$ and $\zeta(n,n)$ are algebraically independent over $\bar{k}$ by Theorem 3.2. Suppose that $(\Omega^n L_{\alpha,n})^2 - 2\Omega^2 L_{\alpha,a,n,n} - c$ and $\Omega^{2n}$ are linearly dependent over $\bar{k}(t)$ for some $c \in \mathbb{F}_q(t)$. Then there exists an element $f \in \bar{k}(t)$ such that $(\Omega^n L_{\alpha,n})^2 - 2\Omega^2 L_{\alpha,a,n,n} - c = f\Omega^{2n}$. Since $c$ and $f$ are rational functions, they have no poles or zeros at $t = \theta^{\eta^n}$ for some $N \geq 1$. Then we have

$$
\left(\frac{\Gamma_n \zeta(n)}{\pi^n}\right)^{2N} - 2\left(\frac{\Gamma_n \zeta(n,n)}{\pi^{2n}}\right)^{n} - c(\theta^{\eta^n}) = 0.
$$

Thus we obtain the relation

$$
\zeta(n)^2/\pi^{2n} - 2\zeta(n,n)/\pi^{2n} = c(\theta)/\Gamma_n^2 \in k^\times.
$$

Now we assume that $2n$ is “odd”. Then the left hand side of the above equation is contained in $\overline{\pi}^{-2n} \cdot k_\infty$. However $\overline{\pi}^{2n} \not\in k_\infty$ by the definition of $\tilde{\pi}$. This is a contradiction. Thus the required linear independence holds in this case.

Remark 3.3. In the proof of Theorem 1.1, we used Theorem 2.3 to show the algebraic independence of $\tilde{\pi}$ and $\zeta(n)$ over $\bar{k}$ for an “odd” integer $n \geq 1$ (this is equivalent to $\dim \Gamma' = 2$ by Theorem 3.2). However, by using similar arguments to determine the dimension of $\Gamma$, we can show that $\dim \Gamma' = 2$ directly. This gives a simple proof of the algebraic independence of $\tilde{\pi}$ and $\zeta(n)$ over $\bar{k}$, which is proved in [CY]. The proof is as follows: Assume that $\dim \Gamma' = 1$. Then $\dim V' = 0$, and hence $V' = 1$ since $\mathbb{G}_m$ acts on $V'$. As before, we obtain

$$
\Gamma' = \left\{ \begin{bmatrix} a \\ c_0(a-1) \\ 1 \end{bmatrix} \right\}
$$

for some $c_0 \in \mathbb{F}_q(t)$. This implies the equation

$$(\Omega^n L_{\alpha,n} + c_0) \otimes \Omega^n = \Omega^n \otimes (\Omega^n L_{\alpha,n} + c_0)$$

in $\mathbb{L} \otimes_{\bar{k}(t)} \mathbb{L}$. Therefore if $\Omega^n L_{\alpha,n} + c$ and $\Omega^n$ are linearly independent over $\bar{k}(t)$ for each $c \in \mathbb{F}_q(t)$, we have a contradiction, and hence $\dim \Gamma' = 2$. When $q - 1$ does not divide $n$ and $\alpha = H_{n-1}$, this condition is satisfied.

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