In this article we give a survey of homology computations for moduli spaces $\mathcal{M}_{g,n}^m$ of Riemann surfaces with genus $g \geq 0$, one boundary curve, and $m \geq 0$ punctures. While rationally and stably this question has a satisfying answer by the Madsen–Weiss theorem, the unstable homology remains notoriously complicated. We discuss calculations with integral, mod-2, and rational coefficients. Furthermore, we determine, in most cases, explicit generators using homology operations.

1. Introduction and overview

Let $\mathcal{M}_{g,n}^m$ denote the moduli space of Riemann surfaces of genus $g \geq 0$ with $n \geq 1$ boundary curves and with $m \geq 0$ permutable interior punctures. The boundary curves are numbered and parametrised; an equivalence is a biholomorphic map between two surfaces that respects the numbering and the parametrisations of the boundary curves; the punctures may be permuted. This space is the quotient of the corresponding Teichmüller space $\mathcal{T}_{g,n}^m$ by the proper action of the mapping class group $\Gamma_{g,n}^m$. Since any self-equivalence must fix the boundary curves pointwise, this action is free, and since the Teichmüller space is homeomorphic to a ball, the quotient $\mathcal{M}_{g,n}^m$ is a manifold and has the homotopy type of the classifying space $B\Gamma_{g,n}^m$.

Some of these moduli spaces have well-understood homotopy types: for $g = 0$, the spaces $\mathcal{M}_{0,1}^m$ are homotopy equivalent to the unordered configuration spaces $C_m(\mathbb{D})$ of $m$ points in the interior of a disc, and thus, the mapping class group $\Gamma_{0,1}^m$ is isomorphic to the braid group $\mathfrak{B}_m$ on $m$ strings. In the case of $g = 1$, we consider the moduli space of bounded 2-tori (or, equivalently, of ‘directed’ elliptic curves). The moduli space $\mathcal{M}_{1,1}$ is equivalent to the complement of the trefoil knot in the 3-sphere and the mapping class group $\Gamma_{1,1}$ is isomorphic to the third braid group $\mathfrak{B}_3$.

We are interested in the homology groups $H_\ast(\mathcal{M}_{g,n}^m)$. While it is a famous result that increasing the genus is homologically stable [Har84], and while the Madsen–Weiss theorem [MW07] gives a complete description of the stable homology with rational
and mod-\( p \) coefficients (for the latter see [Galo4]), the homology outside this stable range remains notoriously complicated. We focus on the case of \( n = 1 \) and consider homology with coefficients in \( \mathbb{Z}, \mathbb{Q}, \) and \( \mathbb{F}_2 \). Many explicit results concerning these homology groups are well-known, and other results have been achieved over the last decades, partially by the authors and in several bachelors’, masters’, and PhD theses. The purpose of this work is two-fold: on the one hand, we give a detailed overview of what is known so far, and on the other hand, we make some new contributions.

1.1. Homology operations

There are various geometric constructions that relate the homology groups \( H_*(\mathcal{M}_{g,1}^m) \) via homology operations. Even though we will study them extensively in § 3, let us mention the most important ones already here.

First of all, it is a classical observation that the boundary connected sum turns the collection \( \bigsqcup_{g,m} \mathcal{M}_{g,1}^m \) into an \( H \)-space, and thus endows its homology with the structure of a (graded) Pontrjagin algebra, whose product we denote by

\[
- \cdot - : H_i(\mathcal{M}_{g_1,1}^m) \otimes H_j(\mathcal{M}_{g_2,1}^m) \to H_{i+j}(\mathcal{M}_{g_1+g_2,1}^{m_1+m_2}).
\]

Even better, this product is part of an \( E_2 \)-structure (this was observed in [Mil86; Böd90]). As a consequence, we additionally have a Browder bracket \([ -, - ]\), and, for homology modulo 2 or for even-dimensional classes, a Dyer–Lashof square \( Q \). There are plenty of relations that involve these operations and which hold for each \( E_2 \)-algebra, see [CLM76, § iii]. We give a short summary of them in § 3.1.

In § 3.4.1, we introduce a further operation \( T : H_i(\mathcal{M}_{g,1}^m) \to H_{i+1}(\mathcal{M}_{g+1,1}^{m-1}) \) that, visually speaking, picks one of the punctures and a tangential direction, and turns it into an extra handle. This \( T \)-operation already appeared in the computations of [Meh11] and has proved useful to describe non-trivial homology classes.

We recall several other homology operations in § 3, for example the Segal–Tillmann map [SeTo8] and various operations involving multiple boundary curves [Kran22].

1.2. Generators and homology tables

In § 4, we describe several explicit homology classes, called \( a, b, c, d, e, f, s, \) and \( v \). Starting with them and using the above homology operations, we can characterise many generators: these calculations are the main result of this paper. (In these tables, an empty entry means that the corresponding homology group is trivial; moreover, all homology groups in degrees higher than the rows shown are trivial.)
Genus 0  We start with the well-known case of genus 0, which means we are looking at unordered configuration spaces of particles in the plane.

**Theorem 6.4.** The homology groups $H_\bullet(\mathcal{M}_m^{0,1}; \mathbb{Z})$ and their generators are, for $m = 0, \ldots, 5$, as shown in the following table:

| $m$ | $\mathcal{M}_0,1$ | $\mathcal{M}_1,1$ | $\mathcal{M}_2,1$ | $\mathcal{M}_3,1$ | $\mathcal{M}_4,1$ |
|-----|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0   | $\mathbb{Z}(1)$| $\mathbb{Z}(a)$| $\mathbb{Z}(a^2)$| $\mathbb{Z}(a^3)$| $\mathbb{Z}(a^5)$|
| 1   | $\mathbb{Z}(b)$| $\mathbb{Z}(ab)$| $\mathbb{Z}(a^2b)$| $\mathbb{Z}(a^3b)$|
| 2   | $\mathbb{Z}_2(b^2)$| $\mathbb{Z}_2(ab^2)$|

Table 1. Homology groups for $g = 0$ and $m = 0, \ldots, 5$

While $a$ is a ground class, $b = Qa$ is a Dyer–Lashof square. In fact, the $\mathbb{F}_2$-homology of these moduli spaces is the free Dyer–Lashof algebra generated by $a$, or, in other words, the polynomial algebra over $\mathbb{F}_2$ generated by all $a_k$ with $a_0 := a$ and $a_{k+1} := Qa_k$.

Genus 1  The table for genus 1 already has an interesting first column, since for $m = 0$ the mapping class group $\Gamma_{1,1}$ is isomorphic to the third braid group.

**Theorem 6.5.** The homology groups $H_\bullet(\mathcal{M}_m^{1,1}; \mathbb{Z})$ and their generators are, for $m = 0, \ldots, 4$, as shown in the following table:

| $m$ | $\mathcal{M}_0,1$ | $\mathcal{M}_1,1$ | $\mathcal{M}_2,1$ | $\mathcal{M}_3,1$ | $\mathcal{M}_4,1$ |
|-----|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0   | $\mathbb{Z}(c)$| $\mathbb{Z}(ac)$| $\mathbb{Z}(a^2c)$| $\mathbb{Z}(a^3c)$| $\mathbb{Z}(a^4c)$|
| 1   | $\mathbb{Z}(d)$| $\mathbb{Z}(ad)$| $\mathbb{Z}(a^2d) \oplus \mathbb{Z}_2(bc)$| $\mathbb{Z}(a^3d) \oplus \mathbb{Z}_2(abc)$| $\mathbb{Z}(a^4d) \oplus \mathbb{Z}_2(a^2bc)$|
| 2   | $\mathbb{Z}_2(e)$| $\mathbb{Z}_2(ae, bd)$| $\mathbb{Z}_2(a^2e, abd)$| $\mathbb{Z}_2(a^3e, a^2bd, b^2c)$|
| 3   | $\mathbb{Z}_2(f)$| $\mathbb{Z} \oplus \mathbb{Z}_2(af, be)$| $\mathbb{Z}^2 \oplus \mathbb{Z}_2(a^2f, abe, b^2d)$|
| 4   | $\mathbb{Z}$| $\mathbb{Z} \oplus \mathbb{Z}_2(bf) \oplus \mathbb{Z}_2$|
| 5   | $\mathbb{Z}$| $\mathbb{Z} \oplus \mathbb{Z}_2$|
| 6   | $\mathbb{Z}$| $\mathbb{Z}$|

Table 2. Homology groups for $g = 1$ and $m = 0, \ldots, 4$

Again, while $c$ is a ground class, the $T$-operation occurs here for the first time in $d = Ta$, which is (regarded as a loop in the classifying space $B\Gamma_{1,1}$) a Dehn twist. Similarly, the class $e$ can be written as $E(a^2)$ where $E$ is the operation from §3.4.2. The class $f$ has a more complicated description, see §4.6. Note that there are homology groups for which we cannot find generators in terms of operations applied to known classes.
Genus 2  The case of $g = 2$ and $m = 0$ was the first problem for which the simplicial model from §2 lead to new computational results [Ehr98; ABe98]. They have also been discovered independently [God07] by graph-theoretical methods.

**Theorem 6.6.** The homology groups $H_\bullet(M_{2,1};\mathbb{Z})$ and their generators are, for $m = 0, 1, 2$, as in following table (where $\lambda = \frac{1}{\mu}$ for some natural number $\mu$):

| $m_{2,1}$ | $m^1_{2,1}$ | $m^2_{2,1}$ |
|----------|--------------|--------------|
| 0 $\mathbb{Z}\langle c^2 \rangle$ | $\mathbb{Z}\langle a^2c^2 \rangle$ | $\mathbb{Z}\langle a^2c^2 \rangle$ |
| 1 $\mathbb{Z}_{10}\langle cd \rangle$ | $\mathbb{Z}_{10}\langle acd \rangle$ | $\mathbb{Z}_{10}(a^2cd) \oplus \mathbb{Z}_2(b^2c)$ |
| 2 $\mathbb{Z}_2(\delta^2)$ | $\mathbb{Z} \oplus \mathbb{Z}_2(\delta^2)$ | $\mathbb{Z} \oplus \mathbb{Z}_2(a^2d^3, bcd)$ |
| 3 $\mathbb{Z}\langle \lambda s \rangle \oplus \mathbb{Z}_2(Te)$ | $\mathbb{Z}\langle \lambda as \rangle \oplus \mathbb{Z} \oplus \mathbb{Z}_2(a \cdot Te) \oplus \mathbb{Z}_2$ | $\mathbb{Z}\langle \lambda a^2s \rangle \oplus \mathbb{Z} \oplus \mathbb{Z}_2(a^2 \cdot Te, bd^2) \oplus \mathbb{Z}_2^2$ |
| 4 $\mathbb{Z}_2 \oplus \mathbb{Z}_3(\nu)$ | $\mathbb{Z}_2^3 \oplus \mathbb{Z}_3(\nu) \oplus \mathbb{Z}_3$ | $\mathbb{Z} \oplus \mathbb{Z}_2(b \cdot Te) \oplus \mathbb{Z}_2^1 \oplus \mathbb{Z}_3(a^2 \nu) \oplus \mathbb{Z}_2^3$ |
| 5 $\mathbb{Z}$ | $\mathbb{Z}^2 \oplus \mathbb{Z}_4^1 \oplus \mathbb{Z}_3$ | $\mathbb{Z} \oplus \mathbb{Z}_2^2$ |
| 6 $\mathbb{Z}$ | $\mathbb{Z} \oplus \mathbb{Z}_2^2$ | $\mathbb{Z} \oplus \mathbb{Z}_2^2$ |
| 7 | | $\mathbb{Z}_2$ |

Table 3. Homology groups for $g = 2$ and $m = 0, 1, 2$

Although nearly all entries in these three tables can easily be exhibited as generators, two classes need a more subtle treatment and hence are separated from the aforementioned theorem; both appear in the PhD thesis of the second author [Boe18]:

**Theorem 6.1.** The class $s$ in $H_3(M_{2,1};\mathbb{Z})$ is a rational generator.

**Theorem 6.2.** The class $v$ generates the $\mathbb{Z}_3$-summand in $H_4(M_{2,1};\mathbb{Z})$.

While the mere computations of the homology groups for $g \leq 2$ were done in [Har91; Ehr98; God07; Abh05; Meh11; Wan11; BoH14], our contribution is the identification of generators. Some of them have already been found in [Meh11; BoH14], others occur here for the first time. In particular, we give generators for the entire integral homology of $M_{2,1}$, except for one $\mathbb{Z}_2$-summand in degree 4. There are similar tables over $\mathbb{F}_2$ which we discuss in Appendix A and which rely on a recent result by Bianchi [Bia20].

**Genus 3**  The case $g = 3$ is already much more complicated. Here we restrict ourselves to $m = 0$, that is: we only care about $H_\bullet(M_{3,1})$.

Let us start by considering the first and the second homology groups: it is a famous result [Pow78, Thm. 1] that for $g \geq 3$, the mapping class group $\Gamma_{g,1}$ is perfect, in other words $H_1(M_{g,1}) = 0$. In [KSo3, Thm. 1.2], it has been shown that $H_2(M_{3,1})$ is either $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}_2$, and it has been shown in [Sak12, Thm. 4.9] that the second of these two cases holds. The $\mathbb{Z}_2$-summand is generated by $cd^2$, see the proof of [GKR19, Lem. 3.6]
for details. We call the \(^1\) generator of the free part \(w\): it plays a prominent role in the study of secondary homological stability, see [GKR19].

Furthermore, we have \(H_i(\mathcal{M}_{3,1}) = 0\) for \(i \geq 10\); this can be seen for example by inspecting the simplicial complex from \(\S\) 2.5. The rational Betti numbers of \(\mathcal{M}_{3,1}\) have been calculated in [BoH14, \S\) 6.5.1.4]; they are given by

\[
\dim_{\mathbb{Q}} H_i(\mathcal{M}_{3,1}; \mathbb{Q}) = 1, 0, 1, 0, 1, 1, 0, 0, 1
\]

In her PhD thesis [Wan11], Wang performed many computer-aided calculations for \(\mathcal{M}_{3,1}\) in prime characteristic. Combining her results with the Betti numbers (1.1), the integral homology of \(\mathcal{M}_{3,1}\) is determined up to possible direct summands of the form \(\mathbb{Z}_p^k\), where \(p \geq 29\) is a prime:

- these possible summands are symbolised by \(\ldots\).

**Summary.** The homology groups \(H_*(\mathcal{M}_{3,1}; \mathbb{Z})\) and some of their generators are as shown in the following table:

| \(\mathcal{M}_{3,1}\) | \(\mathbb{Z}\langle c^3 \rangle\) | \(\mathbb{Z}\langle w \rangle \oplus \langle cd^2 \rangle\) | \(\mathbb{Z}\langle \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_7 \rangle\) | \(\mathbb{Z}\langle \mathbb{Z}_2^2 \oplus \mathbb{Z}_3^2 \rangle\) | \(\mathbb{Z}\langle \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \rangle\) | \(\mathbb{Z}\langle \mathbb{Z} \oplus \mathbb{Z}_2^2 \rangle\) | \(\mathbb{Z}\langle 2 \rangle\) | \(\mathbb{Z}\langle 3 \rangle\) | \(\mathbb{Z}\langle 7 \rangle\) | \(\mathbb{Z}\langle \mathbb{Z}_2 \rangle\) | \(\mathbb{Z}\langle \mathbb{Z} \rangle\) |
|----------------------|------------------------------|---------------------------------|-------------------------------------------------|-------------------------------------------------|-------------------------------------------------|-------------------------------------------------|-------------------------------------------------|-------------------------------------------------|-------------------------------------------------|-------------------------------------------------|-------------------------------------------------|-------------------------------------------------|
| 0                    | \(\mathbb{Z}\langle c^3 \rangle\) | \(\mathbb{Z}\langle w \rangle \oplus \langle cd^2 \rangle\) | \(\mathbb{Z}\langle \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_7 \rangle\) | \(\mathbb{Z}\langle \mathbb{Z}_2^2 \oplus \mathbb{Z}_3^2 \rangle\) | \(\mathbb{Z}\langle \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \rangle\) | \(\mathbb{Z}\langle \mathbb{Z} \oplus \mathbb{Z}_2^2 \rangle\) | \(\mathbb{Z}\langle 2 \rangle\) | \(\mathbb{Z}\langle 3 \rangle\) | \(\mathbb{Z}\langle 7 \rangle\) | \(\mathbb{Z}\langle \mathbb{Z}_2 \rangle\) | \(\mathbb{Z}\langle \mathbb{Z} \rangle\) |

**Table 4.** Homology groups for \(g = 3\) and \(m = 0\)

**Genus 4 and higher**  Rationally, Harer’s stability theorem [Har84; Iva90; Bol12; Ran16] and the Madsen–Weiss theorem [MW07] show that

\[
H_i(\mathcal{M}_{g,1}; \mathbb{Q}) \cong H^i(\mathcal{M}_{g,1}; \mathbb{Q}) \cong \mathbb{Q}[\kappa_1, \kappa_2, \ldots ](\text{degree } i)
\]

\(^1\)The class \(w\) is uniquely determined up to sign; [GKR19] uses the convention that \(\kappa_1(w) = 12\) holds for the first Mumford–Miller–Morita class \(k_1\).

\(^2\)To be precise, Wang states that also summands of the form \(\mathbb{Z}_p^k\) with \(p < 29\) a prime and \(k\) a large exponent can occur. However, she also calculates the \(F_p\)-Betti numbers [Wan11, p. 68] for \(p < 29\), and they are, for each such \(p\), exactly the sum of the rational ones and the ones coming from the \(p\)-torsion summands that we have already found—this excludes these further summands.
for $i \leq \frac{2}{3}g - \frac{2}{3}$. Here $\kappa_i$ is the $i^{th}$ Mumford–Miller–Morita class: it has degree $2i$. In addition to that, we can say something about $H_i$ for $i \leq 3$: we clearly have $H_0(\mathcal{M}_{g,1}) \cong \mathbb{Z}(c^8)$, and $H_1(\mathcal{M}_{g,1}) = 0$ for $g \geq 3$, as already noted. For the homological degrees 2 and 3, we collect the following two results from the literature:

- $H_2(\mathcal{M}_{g,1}) \cong \mathbb{Z}(c^{g-3}w)$ for $g \geq 4$.

  From [KS03, Thm. 3.9], we know that $H_2(\mathcal{M}_{4,1}) \cong \mathbb{Z}$, in particular $c^2d^2 = 0$, as it has order 2. Harer’s stability theorem implies that $c \cdot - : H_2(\mathcal{M}_{3,1}) \to H_2(\mathcal{M}_{4,1})$ is surjective, and hence $c^i$ is a free generator. Using Harer’s stability theorem once again, $c^{g-4} \cdot - : H_2(\mathcal{M}_{4,1}) \to H_2(\mathcal{M}_{g,1})$ is an isomorphism for $g \geq 4$.

- $H_3(\mathcal{M}_{g,1};\mathbb{Q}) = 0$ for $g \geq 4$.

  For $g \geq 6$, we are already in the stable range and can invoke (1.2). The case $g = 4$ is shown in [GKR19, Thm. 6.1], based on [Tom05, Thm. 1.4], and the case $g = 5$ follows from [GKR19, Cor. 5.7]: one shows that $H_4(\mathcal{M}_{6,1};\mathbb{Q}) \to H_4(\mathcal{M}_{6,1},\mathcal{M}_{5,1};\mathbb{Q})$ is epic, hence $H_3(\mathcal{M}_{5,1};\mathbb{Q}) \to H_3(\mathcal{M}_{6,1};\mathbb{Q}) = 0$ is injective.

Summarising these results, we end up with Table 5. Note that for homological degrees at most 2, it is clear what the stabilisation $c \cdot -$ looks like. In degree 3, however, it remains unknown: for example we do not know if $c \cdot s$ is non-trivial or even of infinite order; similarly, we do not know if $c \cdot Te$ is non-trivial.

| $\mathcal{M}_{0,1}$ | $\mathcal{M}_{1,1}$ | $\mathcal{M}_{2,1}$ | $\mathcal{M}_{3,1}$ | $\mathcal{M}_{4,1}$ | $\mathcal{M}_{5,1}$ |
|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| $\mathbb{Z}(c)$     | $\mathbb{Z}(c^2)$   | $\mathbb{Z}(c^4)$   | $\mathbb{Z}(c^4)$   | $\mathbb{Z}(c^5)$   |                     |
| $Z_{10}(cd)$        | $Z_2(d^2)$          | $Z_2(cd^2) \oplus Z(cw)$ | $Z(cw)$          | $Z(c^2w)$          |                     |
|                     | $Z(\lambda s) \oplus Z_2(\text{Te})$ | $Z \oplus [\text{torsion}]$ | [torsion]    | [torsion]           |                     |

Table 5. $H_i(\mathcal{M}_{g,1})$ for $0 \leq i \leq 3$ and $g \leq 5$

1.3. Properties and relations

We also study how the above generators behave when combining them: for example, each class $x \in H_4(\mathcal{M}_{g,1}^{\infty})$ gives rise to a (graded) homology operation by multiplying with it. If $x$ is one of our first generators, we can say the following:

1. multiplication with $a$ is split injective by [BT01, Thm. 1.3];

2. multiplication with $b$ is injective modulo 2, as we see in §4.2 using [Bia20].

3. multiplication with $c$ is the classical genus-stabilisation. Harer’s stability theorem tells us that this map\footnote{To be precise, the optimal slope from [Ran16] is formulated without punctures; however, the punctured case follows from the unpunctured one by a spectral sequence argument [Han09].} is surjective if $\bullet \leq \frac{2}{3}g$ and an isomorphism if $\bullet \leq \frac{2}{3}g - \frac{2}{3}$. 

...
Additionally, we want to describe what relations hold between the above generators and operations. Some of them have already been derived: for example, the relation $Qc = 3 \cdot cd$ can be found [God07, Ex. 6] and appears in [Meh11, § 1.2]. In particular, $[c, c] = 6 \cdot Qc - 9 \cdot cd \neq 0$, showing that the $E_2$-structure on $\mathcal{M}_g \mathcal{M}_g$ cannot be enhanced to an $E_3$-structure [FS96, Thm. 2.5], although the group completion $\Omega B \mathcal{M}_g \mathcal{M}_g$ has the homotopy type of an infinite loop space [Til97, Thm. A]. We contribute to this collection of relations in § 5. For example, we show the following stabilisation property of the Browder bracket:

**Proposition 5.1.** For two classes $x \in H_*(\mathcal{M}_{g,1}^m)$, $x' \in H_*(\mathcal{M}_{g',1}^{m'})$, the Browder bracket vanishes after a single stabilisation step: $c \cdot [x, x'] = 0$.

Let us point out that it follows from abstract considerations that $[x, x']$ vanishes after finitely many stabilisation steps: the group completion $\mathcal{M}_g \mathcal{M}_g \to \mathcal{M}_\infty \mathcal{M}_\infty \times \mathbb{Z}$ is given by iterated stabilisations and respects the $E_2$-structure on both sides, and the right side has the homology of an infinite loop space, so its $E_2$-Browder bracket vanishes. Another result shows that many Browder brackets are trivial:

**Proposition 5.3.** For each $x \in H_*(\mathcal{M}_{g,1}^m)$, the Browder brackets $[c, x]$ and $[d, x]$ are divisible by 2, and $[e, x] = 0$.

**Proposition 5.4.** $[a, c] = 0$ and $[d, d] = 0$.

Finally, we show that the above $T$-operation often behaves like a differential.

**Proposition 5.5.** For each class $x \in H_*(\mathcal{M}_{g,1}^m)$, the class $(T \circ T)(x)$ is divisible by 2 and of order 2, i.e. with coefficients in $F_2$ or $Q$, we have $T \circ T = 0$.

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2. Models for moduli spaces

Many of our results rely on a finite combinatorial model for the moduli space \( \mathcal{M}_{g,n}^m \), namely the space of parallel slit domains \( \mathcal{M}_{g,n}^m \); this is based on an old work of Hilbert [Hil09], has been described in the context of moduli spaces by Bödigheimer [Böd90a], and admits a (relative) multisimplicial description [ABE08; BoH14].

2.1. Moduli spaces of surfaces with a direction

We start with a description of the moduli spaces \( \mathcal{M}_{g,n}^m \) with a non-zero tangent vector \( X \): it is equivalent to the moduli space from the introduction. There is \( \mathcal{Q} \) (purposes: instead of a Riemann surface with itself, has (real) dimension \( 6g - 6 + 4n - 2m \), with one exception: \( \dim(\mathcal{M}_{0,1}) = 0 \).

A conformal equivalence between two directed surfaces \( (F, \mathcal{Q}, \mathcal{P}) \) and \( (F', \mathcal{Q}', \mathcal{P}') \) of type \( (g, n, m) \) is a biholomorphic (conformal) mapping \( F \rightarrow F' \) sending \( Q_i \) to \( Q'_i \), with its derivative at \( Q_i \) sending \( X_i \) to \( X'_i \), and sending \( \mathcal{P} \) to \( \mathcal{P}' \). We denote by \( \mathcal{F} = [F, \mathcal{Q}, \mathcal{P}] \) the conformal equivalence class and let \( \mathcal{M}_{g,n}^m \) be the moduli space of directed surfaces of type \( (g, n, m) \), with the topology induced by the Teichmüller metric of the corresponding Teichmüller space: it is equivalent to the moduli space from the introduction. There is an \( n! \)-sheeted covering map \( \mathcal{M}_{g,n}^m \rightarrow \mathcal{M}_{g,n}^m \) where the set of sinks is ordered.

It is a classical result that the corresponding Teichmüller space, and hence also \( \mathcal{M}_{g,n}^m \) itself, has (real) dimension \( 6g - 6 + 4n - 2m \), with one exception: \( \dim(\mathcal{M}_{0,1}) = 0 \).

2.2. Potential functions

We follow [Böd90a, §3.1]: given a directed surface \( \mathcal{F} = (F, \mathcal{Q}, \mathcal{P}) \) of type \( (g, n, m) \), a potential function is a map \( u: F \rightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \) that satisfies the following conditions:

1. \( u \) is harmonic away from its singularities \( Q_1, \ldots, Q_n \) and \( P_1, \ldots, P_m \).

2. \( u \) has at each \( Q_i \) a dipole in the direction \( X_i \): for a local parameter \( z \) around \( Q_i \) with \( z(Q_i) = 0 \) and \( T_{Q_i} z(X_i) = \partial_x \), there is a real number \( A_i \) and a holomorphic function \( f_i \) satisfying locally

\[
u(z) = \text{Re}(\frac{1}{i}) - A_i \cdot \log |z| + \text{Re}(f_i(z)).\]

3. \( u \) has at each \( P_j \) a logarithmic sink: for a local parameter \( z \) around \( P_j \) with \( z(P_j) = 0 \), there is a positive real number \( B_j \) and a holomorphic function \( g_j \) satisfying locally

\[
u(z) = B_j \cdot \log |z| + \text{Re}(g_j(z)).\]

\[
\]
It then follows from the residue theorem that $\sum A_i = \sum B_i$. For each such collection $A_1, \ldots, A_n, B_1, \ldots, B_m$ of constants, there is, up to a single further additive constant $C \in \mathbb{R}$, exactly one such $u$: while the existence of $u$ follows from the Dirichlet principle, the uniqueness is a consequence of the maximum principle. The space of all potential functions is therefore parametrised by an affine subspace $A_n^m \subset \mathbb{R}^{n+m+1}$, which is contractible and of dimension $n + m$. Furthermore, we have a left action of $S_m$ on $A_n^m$ by permuting the constants $B_1, \ldots, B_m$ (but not the $A_1, \ldots, A_n$) and we obtain a bundle

$$\mathbb{R}^\pi : \mathcal{M}_{g,n}^m := \mathcal{M}_{g,n} \times S_m \rightarrow A_n^m$$

over $\mathcal{M}_{g,n}^m$ with fibre $A_n^m$. Thus, the bundle map $\mathbb{R}^\pi$ is a homotopy equivalence. The elements of the total space $\mathcal{R}\mathcal{M}_{g,n}^m$ are classes $[F, u]$, where $F$ is a directed surface of type $(g, n, m)$ and $u$ is a potential function as above.

2.3. The critical graph

Let $u$ be a potential function on a directed surface $F = (F, Q, \mathcal{P})$ of type $(g, n, m)$. We choose a metric on $F$ that is compatible with the conformal structure, and consider the gradient vector field $\phi := -\nabla u$ of steepest descent: it is defined away from $Q \cup \mathcal{P}$. Then each $Q_i$ is a pole of order 2 and each $P_j$ is a pole of order 1. If we denote the critical points of $\phi$ by $S_1, \ldots, S_l \in F$ and let $l_k$ be the index of $S_k$, then it follows from the Poincaré–Hopf index theorem that $h := h_1 + \cdots + h_l$ coincides with $2g - 2 + 2n + m$. The left side of Figure 1 shows a small part of such a gradient field.

The critical graph $K \subseteq F$ is now declared as in [Böd, §3.2]: it is the embedded graph consisting of all $Q_i$, all $P_j$, and all critical points $S_k$ as vertices; the edges are all flow lines running from some $S_k$ to either another $S_{k'}$ or to some $Q_i$ or some $P_j$. The graph is directed and without loops, and only the parametrisation of the flow lines depends on the above choice of metric. The complement $F \setminus K$ has $n$ contractible components: following the flow of $\phi$ backwards, we obtain a retraction of $F \setminus K$ onto the open set where $u$ is larger than any critical value; this set, however, consists of $n$ contractible open sets, each lying ‘in front of’ some $Q_i$ with respect to the direction $X_i$.

We call the component of $F \setminus K$ that contracts towards $Q_i$ the basin of $Q_i$, denoted $B_i$.

It follows that $u$ is on $F \setminus K$ the real part of a holomorphic function $w = u + iv$, and the imaginary part $v$ is on each component $B_i$ uniquely determined up to an additive constant $D_i \in \mathbb{R}$. Altogether, $w$ is determined by the constants $A_1, \ldots, A_n, B_1, \ldots, B_m, C,$ and $D_1, \ldots, D_n$. Note that, even though $v$ is not globally defined, we can speak of its critical values: they are the limits when approaching a critical point $S_k$ at the boundary of a basin. We can regard $w$ as a map $F \setminus K \rightarrow \mathbb{C} \times \{1, \ldots, n\}$. Its image is a collection of $n$ slitted planes: this will be essential for our reformulation in §2.4.

Keeping track of the additive constants $D_1, \ldots, D_n$, we obtain a trivial $n$-dimensional vector bundle $\mathcal{M}_{g,n}^m \rightarrow \mathbb{R}\mathcal{M}_{g,n}^m$: elements in $\mathcal{M}_{g,n}^m$ are classes $[F, u, w]$ where $F = (F, Q, \mathcal{P})$. 

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is a directed surface of type \((g,n,m)\), \(u\) is a potential function on \(\mathcal{F}\), and \(w\) is a holomorphic extension defined on \(\mathcal{F} \setminus \mathcal{K}\). The composition of bundle projections
\[
\pi : \mathcal{S}_{g,n}^m \rightarrow R\mathcal{S}_{g,n}^m \rightarrow \mathcal{M}_{g,n}^m
\]
is a bundle with fibre \(\mathcal{A}_n^m \times \mathbb{R}^n\), and hence an equivalence. The fibre of the bundle has dimension \(2n + m\), so together with our observation from §2.1 that \(\mathcal{M}_{g,n}^m\) has dimension\(^4\) \(6g - 6 + 4n + 2m\), it follows that \(\mathcal{S}_{g,n}^m\) has dimension \(3h\) for \(h := 2g - 2 + 2n + m\).

### 2.4. Parallel slit domains

In this subsection, we construct a (relative) multisimplicial complex \(\mathcal{P}_{g,n}^m\), which parametrises all possible slitted planes as in Figure 1, together with the extra information how to ‘reglue’ the slits, and which will turn out to be homeomorphic to \(\mathcal{S}_{g,n}^m\). We will be very brief on the combinatorics; more details can be found in [ABE08] for the case \(n = 1\) and in [BoH14, §2.3] for the general case.

**Notation 2.1.** Let \(X\) be a finite set.

1. Each permutation \(\sigma\) on \(X\) can be decomposed into its cycles, which are denoted by \((x_1, \ldots, x_r)\) and read from left to right, i.e. \(\sigma(x_i) = x_{i+1}\) for \(1 \leq i < r\) and \(\sigma(x_r) = x_1\). Let \(C(\sigma)\) be the number of cycles of \(\sigma\), including fixed points.

2. Let \(N(\sigma)\) be the word length norm of \(\sigma\) with respect to the generating set of all transpositions. As an \(r\)-cycle has norm \(r - 1\), we get \(N(\sigma) = \#X - C(\sigma)\).

**Construction 2.2.** For \(p_1, \ldots, p_n \geq 0\), we consider the **tableau**
\[
[p_1, \ldots, p_n] := \{(i,j); 1 \leq i \leq n \text{ and } 0 \leq j \leq p_i\}
\]
as an index set. Let \(\mathfrak{S}[p_1, \ldots, p_n]\) be the group of all permutations of \([p_1, \ldots, p_n]\); it is, up to reindexing, isomorphic to the symmetric group on \(n + p_1 + \cdots + p_n\) elements.

\(^4\)For the exceptional case, we have \(\dim(\mathcal{S}_{0,1}) = 2\).
For each \((i,j) \in [p_1, \ldots, p_n]\) we have a **deletion map**, which is a function of sets,

\[
D^j_i : \Theta[p_1, \ldots, p_n] \to \Theta[p_1, \ldots, p_i - 1, \ldots, p_n],
\]

where \(D^j_i(\sigma)\) is obtained by skipping the cycle representation \((i,j)\) from the cycle representation of \(\sigma\) and shifting down all \(j'\) in \((i,j')\) with \(j < j'\) by one. These functions \(D^j_i\) satisfy the \(n\)-semisimplicial identities, but they are no group homomorphisms.

Without going into detail, let us point out that for \(n = 1\), the collection of deletion maps \(D^j_i : \Theta[p] \to \Theta[p - 1]\), together with adequate degeneracies, is closely related to the notion of a crossed simplicial group in the sense of Krasauskas [Kras87] and Fiedorowicz–Loday [FL91], [Lod92, §6].

**Definition 2.3.** Let \(g, m \geq 0\) and \(n \geq 1\). We put \(h := 2g - 2 + 2n + m\) as before and define a \((1+n)\)-semisimplicial complex \(P^m_{g,n}\) as follows: its \((q, p_1, \ldots, p_n)\)-simplices are given by tuples \(\Sigma = (\sigma_q : \ldots : \sigma_0)\) with \(\sigma_k \in \Theta[p_1, \ldots, p_n]\) such that

1. \(\sum_{k=1}^q N(\sigma_k \sigma_{k-1}) \leq h\) and
2. \(C(\sigma_q) \leq m + n\).

(The notation for the tuple is chosen to be reminiscent of the homogeneous notation for the bar complex of a group.) We denote the \(0^{th}\) face operator by \(\partial_0\), and for \(1 \leq i \leq n\), the \(i^{th}\) face operator by \(\partial_i\): this means we have \(\partial_0, \ldots, \partial_q\) and \(\partial_0, \ldots, \partial_{p_i}\) for each \(i\). For \(\Sigma = (\sigma_q : \ldots : \sigma_0)\), these face operators are given by

\[
\partial'_i \Sigma := (\sigma_q : \ldots : \sigma_{k-i} : \ldots : \sigma_0),
\]

\[
\partial^j_i \Sigma := (D^j_i(\sigma_q) : \ldots : D^j_i(\sigma_0)).
\]

A cell \(\Sigma = (\sigma_q : \ldots : \sigma_0)\) of the complex \(P^m_{g,n}\) is called **non-degenerate** if it satisfies the following properties (see [ABE08, §4.3] and [BoH14, Def. 2.3.3]):

s1. \(\sigma_0 = \prod_{i=1}^n ((i,0), \ldots, (i,p_i))\) for \(1 \leq i \leq n\).

s2. \(\sigma_k(i, p_i) = (i,0)\) for each \(0 \leq k \leq q\) and \(1 \leq i \leq n\).

s3. No cycle of any \(\sigma_k\) contains two different symbols of the form \((i,0)\).

s4. \(C(\sigma_q) = n + m\).

s5. \(\sum_{k=1}^q N(\sigma_k \sigma_{k-1}) = h\).

s6. We have \(\sigma_k \neq \sigma_{k-1}\), and there is no \((i,j)\) such that \(\sigma_k(i, j) = (i, j + 1)\) for all \(k\).

s7. Under the equivalence relation on the set \(\{1, \ldots, n\}\), generated by \(i \sim i'\) if there are \(k, j, j'\) with \(\sigma_k(i, j) = (i', j')\), all elements are equivalent.
The collection $\mathcal{P}^m_{g,n} \subseteq \mathcal{P}^n_{g,n}$ of degenerate cells forms a subcomplex, and we define the space of parallel slit domains as the complement

$$\mathcal{P}^m_{g,n} := |\mathcal{P}^m_{g,n}| \setminus |\mathcal{P}^m_{g,n}|.$$ 

For the top-dimensional non-degenerate cells we have $q = h$, and all $\sigma_i^k\sigma_i^{k-1}$ are disjoint transpositions, implying that $p_1 + p_2 + \cdots + p_n = 2h$; thus, the dimension of $\mathcal{P}^m_{g,n}$ is $3h$.

**Construction 2.4.** Using ordered simplex coordinates, each slit domain in $\mathcal{P}^m_{g,n}$ is represented by a tuple $(\Sigma; a, b_1, \ldots, b_n)$ where:

1. $\Sigma$ is a non-degenerate cell,
2. $a$ is a $q$-tuple of real numbers $-\infty < a_q < \cdots < a_1 < \infty$, and
3. each $b_i$ is a $p_i$-tuple of real numbers $-\infty < b_{i,1} < \cdots < b_{i,p_i} < \infty$.

This data is used in a ‘gluing recipe’ to obtain an element $[\mathcal{F}, u, w] \in \mathcal{E}^m_{g,n}$ as in [Bödgoa, §5.2], see Figure 2: we subdivide $n$ complex planes $C \times \{1, \ldots, n\}$ into rectangles

$$R_{k,i,j} := [a_{k+1}, a_k] \times [b_{i,j}, b_{i,j+1}] \times \{i\}$$

for $0 \leq k \leq q$, $1 \leq i \leq n$ and $0 \leq j \leq p_i$, where we put $a_0 := b_{i,p_i+1} := \infty$ and $a_{q+1} := b_{i,0} := -\infty$, skipping boundary at $\pm \infty$. Then we glue the left edge of $R_{k,i,j}$ to the right edge of $R_{k+1,i,j}$, and the top edge of $R_{k,i,j}$ to the bottom edge of $R_{k+1,i,j}$ (These identified top/bottom edges are called a slit, if $\sigma_k(i,j) \neq (i, j+1)$ for some $k' \geq k$.)

The resulting surface has $n + m$ ends. For each $1 \leq i \leq n$, there is one end that corresponds to the ‘far right’ of the $i^{th}$ plane; we close such an end by adding a point $Q_i$. The remaining ends are closed by adding points $P_1, \ldots, P_m$. Then property s7 ensures that $F$ is connected, while property s5 ensures that $F$ has the correct Euler characteristic.

The complex structure on $F$ is declared by the following atlas: each point in the interior of a rectangle has this rectangle as a coordinate neighbourhood; each point in the interior of an edge of a rectangle needs the two adjacent half-rectangles as a coordinate neighbourhood; each point at a corner of a rectangle uses the $4 \cdot (l + 1)$ quarter-rectangles attached to it, parametrised in such a way to have the point as a branching point of index $l$ when projecting down to a usual union of 4 quarter-rectangles; each $P_i$ has several triangles attached: these are parametrised by using the logarithm function to form a coordinate neighbourhood; and each $Q_i$ has triangles and bigons attached: these are parametrised using the logarithm function and the inversion $z \mapsto \frac{1}{z}$ to form a coordinate neighbourhood as in [Bödgoa, §4.6]: this local parameter $u$ around $Q_i$ also determines the non-zero tangent vector $X_i$ via $T_{Q_i}z(X_i) = \sigma_i$.

The harmonic potential $u : F \to \overline{\mathbb{R}}$ is defined to be the projection of the slit domain to the $x$-axis. Then the critical graph on $F$ is the union of all slits, and on its complement, the holomorphic map $\bar{w}$ is just the reidentification of $F \setminus \mathcal{K}$ with the open subset of our $n$ complex planes with slits removed.
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2.5. Hilbert uniformisation

In the previous subsection, we have constructed a map $\mathcal{P}_{g,n}^m \to \mathcal{S}_{g,n}^m$ by regluing slits according to the combinatorics of the cell. This construction has a very geometric inverse, which we call Hilbert uniformisation: given an element $[\mathcal{F}, \nu, \omega] \in \mathcal{S}_{g,n}^m$, the images of the basins $F_i$ under $\omega$ are slitted complex planes, so the collection $\coprod_i \omega(F_i)$ shows slits on $\mathbb{C} \times \{1, \ldots, n\}$ as in Figure 1. If we additionally track how we have dissected the surface along the edges of the critical graph, then we receive exactly the combinatorial gluing information needed to describe an element $(\Sigma; a, b_1, \ldots, b_n)$ in $\mathcal{P}_{g,n}^m$.

The Hilbert uniformisation and the gluing construction are indeed continuous and inverses of each other, see [Böd90a, Thm. 5.5.1], and thus $\mathcal{P}_{g,n}^m$ and $\mathcal{S}_{g,n}^m$ are homeomorphic. In particular, $(|\mathcal{P}_{g,n}^m|, |\mathcal{P}_{g,n}^m|')$ is a pair such that the quotient is compact and the complement $|\mathcal{P}_{g,n}^m| \smallsetminus |\mathcal{P}_{g,n}^m|$ is a $3h$-dimensional open manifold. By Poincaré–Lefschetz duality, we obtain an isomorphism

$$H_\bullet(\mathcal{P}_{g,n}^m) \cong H_\bullet(\mathcal{P}_{g,n}^m) \cong H^{3h-\bullet}(\mathcal{P}_{g,n}^m, \mathcal{P}_{g,n}^m, \mathcal{O}),$$

where $\mathcal{O}$ is the orientation system, which has a simplicial description as carried out in [Mül96]. We note that $\mathcal{P}_{g,n}^m$ is orientable if the number $m$ of punctures is 0 or 1 (otherwise we have to pass to $\mathcal{P}_{g,n}^m$ where the punctures are ordered), so in these cases, the orientation system is constant.

For $n = 1$, the right side of (2.1) is the homology of a finite double complex with $2h$ columns and $h$ rows and can, in principle, be computed by computer-aided methods.

---

5In the exceptional case $\mathcal{M}_{0,1}$, we obtain (2.1) by noting that both $\mathcal{M}_{0,1}$ and $\mathcal{P}_{0,1}$ are just a point.
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Figure 3. The incidence graph for the double complex of \((P_{1,1}, P'_{1,1})\). The black cells are of bidimension \((2, 4)\), the red ones of bi-dimension \((2, 3)\), the blue one of \((1, 4)\), the green one of \((2, 2)\), and the yellow one of \((1, 3)\). Each edge stands for one direct face relation.

However, the number of cells grows very quickly: for example, the complex \(P_{2,1}\) already has 17,136 non-degenerate cells [ABE08, p. 11]. In his thesis [Vis11], Visy used intricate combinatorics of the symmetric group \(S_p\) to show that the homology of the \(p^{th}\) column of the double complex is concentrated in the top degree. Therefore, the first page of the spectral sequence associated with the column-filtration of the double complex is concentrated in a single row and hence collapses on the second page. This simplifies the chain complex drastically and made many of the cited calculations possible at all. At the same time, Visy’s result gives a new proof that the homological dimension of the moduli space \(M_{g,1}\) is \(4g - 3\): this was first observed by Harer [Har86].

2.6. A computational example

In order to illustrate how the above semisimplicial model can be used for homology calculations, we consider the example of \(M_{1,1}\) as in [ABE08, §6]: the relative complex \((P_{1,1}, P'_{1,1})\) has eight non-degenerate cells and the incidence graph that underlies the simplicial double complex looks as in Figure 3. Taking care of all the signs involved, the total cochain complex of \((P_{1,1}, P'_{1,1})\) is of the form

\[
\begin{array}{c}
\mathbb{Z}^2 \\
\mathbb{Z}^4 \\
\mathbb{Z}^2
\end{array}
\begin{array}{c}
\begin{pmatrix}
0 & -1 & 0 & -1 \\
-1 & 1 & 0 & -1 \\
0 & -1 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
-1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & -1
\end{pmatrix}
\end{array}
\begin{array}{c}
\mathbb{Z}^2
\end{array}
\]

Therefore, we see that \(H_\bullet(M_{1,1}) \cong H^{6-\bullet}(P_{1,1}, P'_{1,1}) \cong (\mathbb{Z}, \mathbb{Z}, 0, \ldots)\), which is not very surprising, as we are calculating the group homology of \(\Gamma_{1,1} \cong \mathbb{B}_3\).

Let us finally give a geometric description of \(\mathcal{M}_{1,1}\); for details see [Dah96]: first of all, we have a proper action of \(\mathbb{R}_{>0} \times \mathbb{R}^2\) on \(|P_{1,1}|\) by translating and scaling the slit picture, which is free on the non-degenerate part \(\mathcal{P}_{1,1}\). The 3-dimensional quotient is a union of two prisms \(\Delta^2 \times \Delta^1\) with two triangles identified to a middle triangle. The remaining top and bottom triangles are degenerate and the six squares are identified in three pairs. Cutting out the degenerate part gives a 3-sphere with a trefoil knot removed.
3. Homology operations

In this section we describe several operations on the collection of homology groups $H_\bullet(M_{g,n})$. We start by recalling the $E_2$-structure on the union $\bigsqcup_{g,m} \mathcal{M}_{g,1}^m$.

3.1. The $E_2$-algebra structure on $\mathcal{P}_{\ast,1}$

It is a well-known result [Mil86; Böd90b] that the collection $\bigsqcup_{g} \mathcal{M}_{g,1}$ admits an action of the little 2-cubes operad $\mathcal{C}_2$ by sewing of surfaces. The same works with punctures, so we obtain a $\mathcal{C}_2$-algebra $\bigsqcup_{g,m} \mathcal{M}_{g,1}^m$, with $\bigsqcup_{g} \mathcal{M}_{g,1}$ as a subalgebra.

This operadic action can be expressed in terms of parallel slit domains (this was actually the model used in [Böd90b, §3] to establish the $E_2$-structure), see Figure 4: given a $k$-ary operation in $\mathcal{C}_2$, i.e. an ordered collection of $k$ disjoint squares $B_1, \ldots, B_k$ in the plane, and, for each $1 \leq i \leq k$, a parallel slit configuration $S_i = (\Sigma_i, a_i, b_i) \in \mathcal{P}_{g_i,1}^{m_i}$, then we can insert $S_i$ into $B_i$. For this purpose, we regard a slit domain as being supported on a square; then the insertion starts with the right-most square and proceeds to the left: here it may happen that we have to ‘weave’ some squares through slits leaving another one: if the squares $B_i$ and $B_j$ have overlapping projections to the $y$-axis, then their projections to the $x$-axis are disjoint, so we can assume that $B_j$ is on the left side of $B_i$. The slits leaving $B_i$ then cut $B_j$ into several horizontal sub-rectangles which are reglued in accordance to the left-most permutation $\sigma_i$ of $\Sigma_i$, and this is the area into which we insert $S_j$, see Figure 4. After applying the gluing recipe from Construction 2.4, the insertion can be visualised as in Figure 5.

![Figure 4](image_url)

**Figure 4.** An instance of $\mathcal{C}_2(2) \times \mathcal{P}_{1,1} \times \mathcal{P}_{1,1} \rightarrow \mathcal{P}_{2,1}$

![Figure 5](image_url)

**Figure 5.** The $E_2$-action after the gluing construction

There is a subtlety if some of the $m_i$ is strictly positive, involving a non-trivial rescaling of the area on the left side of an implanted slit picture; this has been addressed in...
There are several ‘universal’ formulæ that involve these operations and which hold for
them (let $E$ homology operations, $P$ Pontrjagin product, $\lambda$ a bi-graded, i.e. the structure maps are of the form

$$\lambda : C_2(k) \times \prod_{\mathbb{P}^{s_1,1}} \times \cdots \times \prod_{\mathbb{P}^{s_k,1}} \rightarrow \prod_{\mathbb{P}^{s_1+\cdots+s_k,1}}.$$  

### 3.1.1. Operations for $E_2$-algebras

The above $C_2$-action on $P^{s,1}$ gives rise to a unit class $1 \in H_0(\mathbb{M}_{0,1})$ and three (graded) homology operations, which are depicted in Figure 6:

1. The Pontrjagin product is geometrically given by by combining two surfaces by a pair of pants, or, equivalently, joining two slit domains on a single layer. It is a graded-commutative, associative, and unital product

$$- \cdot - : H_i(\mathbb{M}_{s_1,1}) \otimes H_j(\mathbb{M}_{s_2,1}) \rightarrow H_{i+j}(\mathbb{M}_{s_1+s_2,1}).$$

2. The Browder bracket of two homology classes $x$ and $y$ geometrically corresponds to a full twist of two boxes, ‘filled’ with $x$ and $y$, respectively. It is denoted by

$$[-, -] : H_i(\mathbb{M}_{s_1,1}) \otimes H_j(\mathbb{M}_{s_2,1}) \rightarrow H_{i+j+1}(\mathbb{M}_{s_1+s_2,1}).$$

3. If the homological degree of $x$ is even or if we work over $\mathbb{F}_2$, then we additionally have the Dyer–Lashof square $Q(x)$, which geometrically corresponds a half-twist of two boxes, both filled with $x$. Thus, $Q$ is a family of maps

$$Q : H_i(\mathbb{M}_{s,1}) \rightarrow H_{2i+1}(\mathbb{M}_{2s,1}).$$

### 3.1.2. Relations for $E_2$-algebras

There are several ‘universal’ formulæ that involve these operations and which hold for each $E_2$-algebra. They have been derived in [CLM76, §111]; we give a short summary of them (let $|x|$ be the homological degree of $x$, $(-1)^x := (-1)^{|x|}$, and $x' := |x| + 1$):

\[
\begin{align*}
x \cdot (y \cdot z) &= (x \cdot y) \cdot z & \text{(associativity)} \\
x \cdot y &= (-1)^{xy} \cdot y \cdot x & \text{(graded commutativity)} \\
1 \cdot x &= x \cdot 1 = x & \text{(unitality)} \\
[x, y] &= (-1)^{xy} \cdot [y, x] & \text{(graded commutativity)} \\
[x, 1] &= [1, x] = 0 & \text{(annihilation)} \\
0 &= (-1)^{x'y'} \cdot [x, [y, z]] + (-1)^{y'x'} \cdot [y, [z, x]] + (-1)^{x'y'} \cdot [z, [x, y]] & \text{(Jacobi)} \\
[x, y \cdot z] &= [x, y] \cdot z + (-1)^{x'y} \cdot y \cdot [x, z] & \text{(Leibniz)}
\end{align*}
\]

\[\text{Our sign convention for the Browder bracket slightly differs from the one in [CLM76, §111].}\]
The above relations between the Pontrjagin product and the Browder bracket are often summarised in the definition of a Poisson 2-algebra or a Gerstenhaber algebra. If $x$ is an even class, then $Q(x)$ is also defined integrally, and we have the additional relation $[x, x] = 2 \cdot Q(x)$. If we work over $\mathbb{F}_2$, then we additionally have the following relations involving the Dyer–Lashof square:

\[ [x, x] = 2 \cdot Q(x) = 0 \quad \text{(divisibility of the bracket)} \]
\[ Q(0) = Q(1) = 0 \quad \text{(nullification)} \]
\[ Q(x + y) = Q(x) + [x, y] + Q(y) \quad \text{(non-linearity)} \]
\[ Q(x \cdot y) = Q(x) \cdot y^2 + x \cdot [x, y] \cdot y + x^2 \cdot Q(y) \quad \text{(Cartan)} \]
\[ [Q(x), y] = [x, [x, y]] \]

We finally mention the Nishida relations, which involve the dual Steenrod squares in homology $Sq_t : H_\bullet(\mathbb{M}_{g,1}; \mathbb{F}_2) \to H_{\bullet-t}(\mathbb{M}_{g,1}; \mathbb{F}_2)$, namely

\[ Sq_t(x \cdot y) = \sum_{i=0}^t Sq_i x \cdot Sq_{t-i} y \]
\[ Sq_t[x, y] = \sum_{i=0}^t [Sq_i x, Sq_t y], \]

which follow directly from the naturality of the $Sq_t$ and the Cartan formula, and

\[ Sq_{2t}(Q(x)) = Q(Sq_t x) + \sum_{i=0}^{t-1} [Sq_i x, Sq_{2t-i} x], \]
\[ Sq_{2t+1}(Q(x)) = (Sq_t x)^2 + \sum_{i=0}^t [Sq_i x, Sq_{2t+1-i} x]. \]
One particular example of a dual Steenrod square is the Bockstein $\beta$ for the short exact coefficient sequence $0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$, which agrees with the Steenrod square $Sq_1$. In this case, the last Nishida relations is of the form $\beta Q(x) = x^2 + [x, \beta x]$.

### 3.1.3. A splitting modulo 2

Note that $\coprod_{g, m} \mathcal{M}_{g, 1}^m$ has not only $\coprod_{g} \mathcal{M}_{g, 1}$ as a $\mathcal{C}_2$-subalgebra, but also the collection $\coprod_{m} \mathcal{M}_{0, 1}^m$. With coefficients in $\mathbb{F}_2$, its homology is called the Dyer–Lashof algebra, and it has an easy description due to [CLM76, § III]: if $a \in H_0(\mathcal{M}_{0, 1})$ is the ground class, then we have an isomorphism $\bigoplus_m H_*(\mathcal{M}_{0, 1}^m; \mathbb{F}_2) \cong \mathbb{F}_2[Q]^i_{j \geq 0}$. Via the Pontrjagin product, $\bigoplus_{g, m} H_*(\mathcal{M}_{g, 1}^m; \mathbb{F}_2)$ is a module over Dyer–Lashof algebra, and based on [BCT89; BT01], Bianchi [Biazo, Thm. 6.5] recently has established an isomorphism

$$\bigoplus_{m \geq 0} H_*(\mathcal{M}_{g, 1}^m; \mathbb{F}_2) \cong \mathbb{F}_2[Q]^i_{j \geq 0} \otimes H_*(\mathcal{M}_{g, 1}; \text{Sym} \mathcal{H}) \tag{3.1}$$

of modules over the Dyer–Lashof algebra, where $\text{Sym} \mathcal{H} = \bigoplus_k \text{Sym}^k \mathcal{H}$ is the symmetric algebra over the $\Gamma_{g, 1}$-representation $H_1(F; \mathbb{F}_2)$, with $F$ a surface of genus $g$. This isomorphism is bigraded by the number of punctures and by homological degree: here each class in $H_i(\mathcal{M}_{g, 1}; \text{Sym}^k \mathcal{H})$ as $k$ punctures and homological degree $i + k$, and the Dyer–Lashof algebra is bigraded as usual.

### 3.2. Rotating the boundary curve

We have a (generally non-free) action $\rho : S^1 \times \mathcal{M}_{g, 1} \to \mathcal{M}_{g, 1}$ by rotating the parametrisation of the single boundary curve, or, in other words, rotating the non-zero tangent vector $X \in T\mathcal{M}$ of a conformal class $[F, \mathcal{P}, (Q, X)]$. This gives rise to a homology operation

$$R : H_*(\mathcal{M}_{g, 1}^m) \to H_{i+1}(\mathcal{M}_{g, 1}^m), \quad R(x) := \rho_*(|S^1| \times x),$$

called the rotation, where $|S^1|$ is the fundamental class of the circle. Note that $R \circ R = 0$, just because the Pontrjagin product $|S^1| \cdot |S^1|$ lives in $H_2(S^1) = 0$.

This rotation operation is due to the fact that the action of the little 2-discs operad $\mathcal{C}_2$ on $\coprod_{g, m} \mathcal{M}_{g, 1}^m$ can be extended to an action of the framed little 2-discs operad $\mathcal{C}_2^{fr}$ by allowing parametrisations of the boundary curves. As a consequence, we obtain several formulæ, which are induced by the internal structure of $\mathcal{C}_2^{fr}$:

$$R(1) = 0,$$

$$R(x \cdot y) = R(x) \cdot y + [x, y] + (-1)^x \cdot x \cdot R(y).$$

Here the first one follows from the fact that $\mathcal{C}_2^{fr}(0)$ is contractible, and the second one can be checked by comparing the loop in $\mathcal{C}_2^{fr}(2)$ that describes the rotation with the
three loops describing the summands on the right side. From these two formulæ and from $R^2 = 0$, it formally follows that

$$[R(x), y] = R(R(x) \cdot y) + (-1)^x \cdot R(x) \cdot R(y).$$

Let us point out that $R(x)$ is very often trivial: recall that $\mathcal{M}_{g,n}^m$ is a classifying space for the mapping class group $\Gamma_{g,n}^m$, and capping the boundary curve with a disc gives rise to a map $\delta : H_*(\Gamma_{g,n}^m) \to H_*(\Gamma_{g,n}^m)$, which is surjective for $\bullet \leq \frac{2}{3}g + 1$ and injective for $\bullet \leq \frac{2}{3}g$ by Harer’s stability theorem. On the other hand, the above $S^1$-action clearly becomes homotopy trivial after capping with a disc, so the image of $R$ lies in the kernel of $\delta$. It follows that $R(x) = 0$ for $x \in H_*(\mathcal{M}_{g,n}^m)$ with $\bullet \leq \frac{2}{3}g - 1$. We will encounter several non-trivial examples of $R(x)$ in Remark 5.6.

### 3.3. Permuting curves and gluing pairs of pants

If we widen our view for a moment and consider moduli spaces of surfaces with multiple boundary curves, then we obtain, for each $n \geq 1$, an action of the symmetric group $\Sigma_n$ on $\mathcal{M}_{g,n}^m$ by permuting boundary curves, or, in other words, on $\mathcal{P}_{g,n}^m$ by exchanging layers of slit pictures. For a permutation $\sigma \in \Sigma_n$, we denote the induced homology operation by $\sigma_* : H_*(\mathcal{M}_{g,n}^m) \to H_*(\mathcal{M}_{g,n}^m)$. Secondly, we get, for each $1 \leq l \leq n - 1$, a map

$$s_l : \mathcal{M}_{g,n}^m \longrightarrow \mathcal{M}_{g+1,n-1}^m$$

by gluing a pair of pants to the $l$th and the $(l + 1)$st boundary curve. In terms of slit pictures, this is the same as joining the $l$th and the $(l + 1)$st layer of the slit picture on a single new layer, see Figure 7. The codegeneracies constitute one of the three classes of maps in Harer’s stability theorem: the induced maps\(^\text{7}\) $s_l^* : H_*(\mathcal{M}_{g,n}^m) \to H_*(\mathcal{M}_{g+1,n-1}^m)$ are surjective for $\bullet \leq \frac{2}{3}g + \frac{1}{3}$ and isomorphisms for $\bullet \leq \frac{2}{3}g - \frac{2}{3}$.

In [Kran22, Rmk. 4.4.19], it is shown that $s^k s_l^* \simeq s^k s_{l+1}^*$ for $l \leq k$, whence the induced maps in homology behave like codegeneracies, justifying our choice of notation. Moreover, we have $s_l^*(l, l+1) \simeq s_l^*$, where $(l, l+1) \in \Sigma_n$ is the transposition that exchanges $l$ and $l + 1$: this can be seen by comparing $\mathcal{P}$ to different models, see [Kran22, Rmk. 5.2.16]. Altogether, the assignment $n \mapsto H_*(\mathcal{P}^m_{g,n})$ is functorial with respect to surjective maps $\{1, \ldots, n\} \to \{1, \ldots, n'\}$.

### 3.4. Transfer operations

In the following, we construct two operations that, visually speaking, use the homological transfer maps to choose one or several punctures, endow them with tangential directions, and regard them as new boundary curves.

\(^{7}\)Again, the punctured case follows from the unpunctured one, see [Han09].
3.4.1. The $T$-operation

Let $g \geq 0$ and $m \geq 1$, which means that we have at least one puncture. Then there is an $m$-sheeted covering $\alpha : \mathcal{M}_{g,1}^{m-1,1} \to \mathcal{M}_{g,1}^m$ that marks one of the punctures.

We also have a bundle map $\mathcal{M}_{g,1}^{m-1,1} \to \mathcal{M}_{g,1}^{m-1}$ by forgetting the chosen puncture: this is a subbundle of the universal surface bundle over $\mathcal{M}_{g,1}^{m-1}$, where points in the fibre have to avoid the boundary and the given $m-1$ punctures. Over $\mathcal{M}_{g,1}^{m-1,1}$, we consider the vertical unit tangent bundle $p : W^1 := UT \mathcal{M}_{g,1}^{m-1,1} \to \mathcal{M}_{g,1}^{m-1,1}$, which is a 1-dimensional sphere bundle. Note that $W^1$ is oriented, as the structure group of $\mathcal{M}_{g,1}^{m-1,1}$ contains only orientation-preserving diffeomorphisms.

We have an equivalence $\vartheta : W^1 \to \mathcal{M}_{g,2}^{m-1}$ by using that an isolated puncture, together with a unit tangential direction is (by our choice of model) the same as a parametrised boundary curve. Finally, we use the homological transfer and define

$$\hat{T} : H_i(\mathcal{M}_{g,1}^m) \to H_i(\mathcal{M}_{g,1}^{m-1,1}) \to H_{i+1}(W^1) \to H_{i+1}(\mathcal{M}_{g,2}^{m-1}).$$

As we want to focus on moduli spaces of surfaces with a single boundary curve, we glue in a pair of pants and define the $T$-operation by

$$T := s_1 \circ \hat{T} : H_i(\mathcal{M}_{g,1}^m) \to H_{i+1}(\mathcal{M}_{g,1}^{m-1,1}).$$

Under the gluing construction $\mathcal{P}_{g,1}^m \to \mathcal{M}_{g,1}^m$, the map $\vartheta$ can be visualised in terms of slit domains: if $W_{\vartheta}^1$ denotes the pullback of the bundle $W^1$ along the affine bundle $\mathcal{P}_{g,1}^{m-1,1} \to \mathcal{M}_{g,1}^{m-1,1}$, then an element in $W_{\vartheta}^1$ is given by a slit domain, together with the choice of one of the cycles of $\sigma_q$ which does not contain $0$, and, on a circle given by regluing the upper and lower faces of the rectangles inside the chosen cycle, a marked point. We can turn this left face into an honest boundary curve by providing a new layer for it: we draw a small slit at the given position and pair it with a single slit on a new layer. This gives rise to a map $\vartheta_q : W_{\vartheta}^1 \to \mathcal{P}_{g,2}^{m-1}$, covering $\vartheta$ up to homotopy. Finally, we have already seen that gluing in a pair of pants corresponds to joining both layers of the slit picture on a single layer: thus, we end up with Figure 8.
If \( x \) lies in a moduli space of a surface without punctures, then we put \( T(x) := 0 \). It follows immediately from the definition that \( T \) satisfies the graded Leibniz rule

\[
T(x \cdot y) = (-1)^y \cdot T(x) \cdot y + x \cdot T(y)
\]

with respect to the Pontrjagin product. In particular, if \( x \) is a class in \( \mathcal{M}_{g,1} \), then we have \( T(x \cdot y) = x \cdot T(y) \), i.e. the operation \( T \) is \( \bigoplus_h H_\ast(\mathcal{M}_{g,1}) \)-linear.

### 3.4.2. The \( E \)-operation

Let \( g \geq 0 \) and \( m \geq 2 \). Then we consider the \((m)\)-sheeted covering \( \beta : \mathcal{M}^{m-2,2}_{g,1} \to \mathcal{M}^m_{g,1} \) where two punctures are separated from the other ones, but unordered. Over the total space, we consider the torus bundle \( q : W^2 \to \mathcal{M}^{m-2,2}_{g,1} \) given by the symmetric fibre product of the two vertical tangent bundles at the two punctures.\(^8\)

There is a map \( \eta : W^2 \to \mathcal{M}^{m-1}_{g+1,1} \) as follows: via the exponential map, an element in \( W^2 \) is given by a conformal class of a surface \( F \), a subset \( P \subseteq F \) of cardinality \( m - 2 \), two punctures \( x_1, x_2 \in F \) with small disjoint discs \( D_1, D_2 \subseteq F \setminus P \) around them, and points \( x'_i \in \partial D_i \). If we cut along the two straight lines from \( x_1 \) to \( x'_1 \) and from \( x_2 \) to \( x'_2 \) and reglue, then we have identified the two punctures and increased the genus by 1. Again, this construction can be visualised in terms of slit pictures, see Figure 8.

The torus bundle \( q : W^2 \to \mathcal{M}^{m-2,2}_{g,1} \) is not orientable, but if we work over \( \mathbb{F}_2 \), then we still have a homological transfer, which can be used to define

\[
E : H_i(\mathcal{M}^m_{g,1}) \xrightarrow{\beta^!} H_i(\mathcal{M}^{m-2,2}_{g,1}) \xrightarrow{q^!} H_{i+2}(W^2) \xrightarrow{\eta^*} H_{i+2}(\mathcal{M}^{m-1}_{g+1,1}).
\]

Let us point out that the \( E \)-operation can also be applied integrally to ground classes \( x = [\ast] \in H_0(\mathcal{M}^m_{g,n}) \). In this case, \( E(x) \) is just pushforward of some fundamental class of \( S^1 \times S^1 \) along \( S^1 \times S^1 \to W^2 \to \mathcal{M}^{m-1}_{g+1,n} \), where the first map is some fibre inclusion. Choosing a path that exchanges the two punctures, we see that \( 2 \cdot E(x) = 0 \).

\[\begin{array}{c|c|c}
\hline
T & = & \\hline
\hline
\hline
\hline
E & = & \\hline
\hline
\hline
\end{array}\]

**Figure 8.** The two operations \( T : H_i(\mathcal{M}^1_{1,1}) \to H_{i+1}(\mathcal{M}^2_{2,1}) \) and \( E : H_i(\mathcal{M}^2_{1,1}) \to H_{i+2}(\mathcal{M}^1_{2,1}) \), here applied to ground classes.

\(^8\)Formally, we consider the double covering \( \beta := \mathcal{M}^{m-2,1,1}_{g,1} \to \mathcal{M}^{m-2,2}_{g,1} \) where the two points are ordered. There are two unit vertical tangent bundles \( L \) and \( L' \) over \( B \) and the the fibre product \( L \times_B L' \to \mathcal{M}^{m-2,2}_{g,1} \) is \( G_2 \)-equivariant. Then \( q \) is the induced map on quotients.
3.5. Segal–Tillmann maps

In [SeT08], Segal and Tillmann study maps $C_{2g+2}(D^2) \to \mathcal{M}_{g,2}$ from the configuration space of $2g + 2$ unordered points in an open disc to the moduli space of surfaces of genus $g$ and two boundary curves. Because both spaces are aspherical, the map can likewise be described on the level of fundamental groups, where it is a homomorphism $\mathcal{B}r_{2g+2} \to \Gamma_{g,2}$ and has the following description that is due to [SoT07]: the braid group $\mathcal{B}r_{2g+2}$ is generated by $\sigma_1, \ldots, \sigma_{2g+1}$, where $\sigma_i$ interchanges the $i$th and the $(i+1)$st string. We choose $D_1, \ldots, D_{2g+1} \in \Gamma_{g,2}$ to be the Dehn twists about the curves $\alpha_1, \ldots, \alpha_{2g+1}$ as in Figure 9. Then two consecutive curves intersect exactly once while all other pairs of curves do not intersect at all. Therefore, the mapping classes $D_1, \ldots, D_{2g+1}$ satisfy the braid relations and we obtain the desired homomorphism. Its image is contained in the symmetric mapping class group, which was studied in [BiH71].

![Figure 9. The curves $a_1, \ldots, a_7$ defining the images of the elementary braids $\sigma_1, \ldots, \sigma_7$ along the group homomorphism $\mathcal{B}r_8 \to \Gamma_{3,2}$.](image)

For our purposes, it is convenient to cap off the second boundary curves by gluing a disc, i.e. postcomposing with the capping homomorphism $\Gamma_{g,2} \to \Gamma_{g,1}$. We call the composition $\text{ST} : \mathcal{B}r_{2g+2} \to \Gamma_{g,1}$, as well as its topological analogue $\text{ST} : C_{2g+2}(D^2) \to \mathcal{M}_{g,1}$, the Segal–Tillmann map. The main result of [SoT07] tells us that the induced map in homology is trivial in the stable range.

3.6. Vertical Browder brackets

The boundary permutations and codegeneracies from §3.3 are part of a larger operadic action on moduli spaces with multiple boundary curves: in [Kran22], the $\mathcal{C}_2$-action on $\coprod_{g,m} \mathcal{P}^m_{g,1}$ has been generalised to the case of multiple boundary curves, ending up with a coloured operad, called the vertical operad $\mathcal{V}_{1,1}$, acting on the sequence $(\coprod_{g,n} \mathcal{P}^m_{g,n})_{n \geq 1}$.

Both the internal structure of $\mathcal{V}_{1,1}$ and the explicit construction is rather lengthy; we will only use one specific class of operations that arise from it: it generalises the classical Browder bracket and is of the form

$$[-,-] : H_i(\mathcal{M}^m_{g_1,n_1}) \otimes H_j(\mathcal{M}^m_{g_2,n_2}) \longrightarrow H_{i+j+1}(\mathcal{M}^{m_1+m_2}_{g_1+g_2,n_1+n_2-1}).$$

22
We call it the *vertical Browder bracket*, see [Kran22, Def. 4.4.1 & Constr. 5.2.15]. Pictorially, it takes a slit domain $x$ on $n_1$ layers and a slit domain $y$ on $n_2$ layers, places the respective first layers of both arguments on the first layer of a new slit domain, puts the remaining $n_1 - 1$ layers of $x$ on the layers $2, \ldots, n_1$, and the remaining $n_2 - 1$ layers of $y$ on the layers $n_1 + 1, \ldots, n_1 + n_2 - 1$, and lets the respective first layers spin around each other, see Figure 10. If $n_1 = n_2 = 1$, then we recover the classical Browder bracket.

As for $E_2$-algebras, there are several universal formulæ involving the vertical Browder brackets and the boundary permutations and codegeneracies from § 3.3, see [Kran22, Prop. 4.4.16]. We will only be using the following relation\(^9\), with $\tau := (1, 2) \in \mathfrak{S}_{n_1+n_2-1}$:

$$[s_1^* x, y] = s_1^*[x, y] + s_1^*[[\tau^* x, y]].$$

(3.2)

### 4. Generators

In this section, we describe several explicit generators of $H_\bullet(\mathcal{M}^{n,1+1}_\bullet)$, which we call $a$, $b$, $c$, $d$, $e$, $f$, $s$, and $v$. Most of them can easily be visualised in terms of slit pictures. Even though we will show several relations among these generators in § 5, we discuss some immediate properties of these classes directly after having introduced them.

---

\(^9\)For general $\mathcal{T}^{1,1}$-algebras, the second summand in formula (3.2) is $s_1^* \tau_*[[\tau_* x, y]]$. However, for the special algebra $(\bigsqcup_{k,m} \mathcal{T}^{n,m}_k)$, codegeneracies and permutations satisfy all relations to assemble into a functor as in § 3.3, so we can use that $s_1^* \circ \tau$ and $s_1^*$ agree as maps $\{1, 2\} \to \{1\}$. 

---
4.1. Generator a

Our first generator is the integral ground class \( a \in H_0(\mathcal{M}^1_{0,1}) \), which already occurred in §3.1.3 during the description of the Dyer–Lashof algebra. Restricting the Pontrjagin product to \( a \) in one factor defines a homology operation

\[
a \cdot - : H_\bullet(\mathcal{M}^m_{g,1}) \to H_\bullet(\mathcal{M}^{m+1}_{g,1}).
\]

of degree zero. Since adding a puncture \( \mathcal{M}^m_{g,1} \to \mathcal{M}^{m+1}_{g,1} \) admits a stable splitting by [BT01, Thm. 1.3], this homology operation is split injective, even integrally.

4.2. Generator b

Since \( a \) is of even degree, we can consider the Dyer–Lashof square

\[
b = Qa \in H_1(\mathcal{M}^2_{0,1})\]

integrally. Then \( b \) is a fundamental class of \( \mathcal{M}^2_{0,1} \simeq S^1 \), and hence a generator. Multiplying with \( b \) gives rise to a homology operation \( H_\bullet(\mathcal{M}^m_{g,1}) \to H_\bullet(\mathcal{M}^{m+2}_{g,1}) \) of degree 1. In contrast to multiplying with \( a \), this operation is not injective integrally: we have \( 2b^2 = 0 \) by graded commutativity, while \( b \) has infinite order. However, for coefficients in \( \mathbb{F}_2 \), this operation is indeed injective: this follows directly from (3.1).

4.3. Generator c

The next generator is the ground class \( c \in H_0(\mathcal{M}_{1,1}) \). The restriction of the multiplication to \( c \cdot - : H_\bullet(\mathcal{M}^m_{g,1}) \to H_\bullet(\mathcal{M}^{m+1}_{g+1,1}) \) is the classical genus-stabilisation map, which is surjective for \( \bullet \leq \frac{2}{3}g \) and an isomorphism for \( \bullet \leq \frac{2}{3}g - \frac{2}{3} \) by Harer’s stability theorem, as already mentioned in the introduction.

One should point out that \( c \) lifts to a class in \( \mathcal{M}_{0,2} \): if we consider the ground class \( c_2 \in H_0(\mathcal{M}_{0,2}) \), see Figure 12, then we clearly have \( c = s^1 c_2 \).

4.4. Generator d

We consider the class \( d := Ta \in H_1(\mathcal{M}_{1,1}; \mathbb{Z}) \), which can be depicted by the embedded circle shown in Figure 11. Even though the computer-aided calculations in [Meh11, p. 133] exhibit \( d \) as a generator, this is one of the very few cases where it is doable ‘by hand’: recall the incidence graph from Figure 3. Then the embedded circle carrying \( d \) intersects a single 5-cell transversally (and no cells of lower dimension), namely \( \Sigma := \langle 0,2,1,3 \rangle : \langle 0,2,3 \rangle \langle 1 \rangle : \langle 0,1,2,3 \rangle \), and one easily checks that its dual \( \Sigma^* \) generates \( H^5 \) of the small cochain complex discussed in §2.6.

Again, \( d \) lifts to a class \( \Sigma_2 \in H_1(\mathcal{M}_{0,2}) \), which is depicted in Figure 12, i.e. we have \( d = s^1 \Sigma_2 \). Note that the abelianisation of \( \Gamma_{1,1} \) is a free abelian group generated by the Dehn twist along an arbitrary simple closed and non-separating curve; in particular, \( d \) agrees, up to sign, with such a Dehn twist.
4.5. Generator e

We define \( e := Ea^2 \in H_2(\mathcal{M}_{1,1}; \mathbb{Z}) \cong \mathbb{Z}_2 \), which is depicted in Figure 11. Again \( e \) lifts to a class \( e_2 \in H_2(\mathcal{M}_{0,2}) \), which is depicted in Figure 12, i.e. we have \( e = s_1^1e_2 \).

Via Poincaré–Lefschetz duality, \( e \) is represented by a simplicial 7-dimensional cocycle in \((P_{1,1}', P_{1,1}')\) by tracking all transversal intersections. Now a computer-aided calculation [Meh11, p. 133], which is very similar to the one for \( d \), shows that this cocycle is not nullhomologous, i.e. \( e \) is non-trivial and hence a generator.

4.6. Generator f

Using the vertical Browder bracket from § 3.6, we let \( f := s_1^1[a, e_2] \in H_3(\mathcal{M}_{1,1}) \cong \mathbb{Z}_2 \). It is supported on an embedded 3-torus, as one can see in Figure 11.

Again, \( f \) is, via Poincaré–Lefschetz duality, represented by a simplicial 9-cocycle in \((P_{2,1}', P_{1,1}')\) by tracking all transversal intersections (with orientation signs), and a computer-aided calculation [Boe18, Prop. 7.3.1] shows that this cocycle is not nullhomologous, i.e. \( f \) is non-trivial in \( H_3(\mathcal{M}_{2,1}) \cong \mathbb{Z}_2 \).

4.7. Generator s

The homology class \( s \in H_3(\mathcal{M}_{2,1}; \mathbb{Z}) \) is defined in a different way: fixing a closed Riemann surface \( F \) of genus 2, we have a map \( i : UTF \to \mathcal{M}_{2,1} \) from the unit tangent bundle \( UTF \) of \( F \) to the moduli space \( \mathcal{M}_{2,1} \) by assigning to each pair \((Q, X)\) with \( Q \in F \) and \( X \in T_QF \) with norm 1 the conformal class \([F, (Q, X)]\). Note that \( UTF \) is an orientable closed 3-manifold, so we can choose a fundamental class \([UTF] \in H_3(UTF; \mathbb{Z})\) and define \( s := i_*[UTF] \).

In Theorem 6.1, we will give the proof from [Boe18, Prop. 7.1.2] that \( s \) is a rational generator (this is already stated in [Har91, p. 33], but without proof); showing that there is a unique positive natural number \( \mu \) such that for \( \lambda := \frac{1}{\mu} \), the class \( \lambda s \) generates the free part of \( H_3(\mathcal{M}_{2,1}; \mathbb{Z}) \).
4.8. Generator v

Consider the Segal–Tillmann map \( ST \colon C_6(\hat{D}^2) \to \mathcal{M}_{2,1} \). We show in Theorem 6.2 that the induced map on \( H_4(-;\mathbb{Z}) \) is injective. On the other hand, it is a classical calculation \([CLM\,76, \S\,iii]\) that \( H_4(C_6(\mathbb{R}^2);\mathbb{Z}) \) is isomorphic to \( \mathbb{Z}_3 \), and a generator is given by the ptolemaic epicycle \( \tilde{v} \) as in Figure 13. We set \( v := ST^*(\tilde{v}) \in H_4(\mathcal{M}_{2,1};\mathbb{Z}) \).

5. Relations

In this section, we describe several relations that hold between the generators and operations from the previous sections. Some of them have already been found: for example, the relation \( Qc = 3 \cdot cd \) is due to \([God\,07, \text{Ex.} \,6]\) and appears in \([Meh\,11, \S\,1.2]\). In particular, \([c,c] = 2 \cdot Qc = 6 \cdot cd \neq 0 \), showing that the \( E_2 \)-structure on \( \bigvee_g \mathcal{M}_{g,1} \) cannot be enhanced to an \( E_3 \)-structure \([FS\,96, \text{Thm.} \,2.5]\).

In the following, we contribute to this list of relations. Our first observations consider the genus-stabilisation, i.e. multiplication with \( c \). The following proposition claims that a single genus stabilisation step cancels the Browder bracket. Our proof is essentially the same as in \([Kran\,22, \text{Prop.} \,5.3.3]\), using a different model. Let us point out that the very same argument works for each algebra over Tillmann’s surface operad.

**Proposition 5.1.** For each \( x \in H_4(\mathcal{M}_g^{m}) \) and \( x' \in H_4(\mathcal{M}_g^{m'}), \) we have \( c \cdot [x,x'] = 0 \).

**Proof.** We fix a slit picture \( S \in \mathfrak{P}_{1,1} \) and let \( \mathcal{C}_2^S(2) \) be the space of two small numbered cubes inside this slit picture, see Figure 14. By the same implanting procedure as before, we obtain a map \( \lambda^S \colon \mathcal{C}_2^S(2) \times \mathfrak{P}_{g,1}^m \times \mathfrak{P}_{g',1}^{m'} \to \mathfrak{P}_{1+g+g',1}^{m+m'} \).

Now let \( \gamma \colon S^1 \to \mathcal{C}_2^S(2) \) be the loop that is depicted on the left side of Figure 14. Then we clearly have \( c \cdot [x,x'] = \lambda^S([\gamma] \otimes x \otimes x') \), so it suffices to show that \( \gamma \) is a boundary. To do so, let \( I \subseteq S^1 \) be a small closed interval on the standard circle. Then we define a map \( \tilde{\gamma} \colon (S^1 \times S^1) \setminus I^2 \to \mathcal{C}_2^S(2) \) whose source is a 2-torus with a small square removed and which is depicted on the right side of Figure 14: the positions of the boxes on the red and on green circles parametrised by a torus, and the excluded square \( I^2 \) parametrises the situation in which both boxes lie inside the blue region—this is the only case where the disjointness condition can be violated. Now one readily checks that, after identifying \( \partial I^2 \cong S^1 \), the boundary of \( \tilde{\gamma} \) is homotopic to \( \gamma \). \( \square \)

---

Figure 13. The ptolemaic epicycle \( \tilde{v} \) generating \( H_4(C_6(\mathbb{R}^2);\mathbb{Z}) \cong \mathbb{Z}_3 \).
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The very same argument works for $d$
and for $\tau\in\mathcal{G}_2$.

**Figure 14.** Left: The loop $\gamma: S^1 \to \mathcal{G}_2^2(2)$. Right: The map $\tilde{\gamma}: (S^1 \times S^1) \setminus I^2 \to \mathcal{G}_2^2(2)$.

The stabilisation step can also cancel other unstable classes, which do not decompose
into a Browder bracket, as Proposition 5.2 shows.

**Proposition 5.2.** $ce = 0$.

This relation has already been claimed in [Meh11, p. 14], but without a proof.

**Proof.** We have $ce \in H_2(\mathcal{M}_{2,1}^1; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ and $2 \cdot ce = 0$, so it suffices to show that
$ce$ is not the generator of the single $\mathbb{Z}_2$-summand. Here we use that $H_2(\mathcal{M}_{2,1}^1; \mathbb{Z}) \cong \mathbb{Z}_2$,
and we call its generator $x$ (we will show later in Theorem 6.6 that it is $d^2$), but this is not
necessary for the argument. As the Pontrjagin product with $a$ is split injective, the class
$ax$ generates the aforementioned $\mathbb{Z}_2$-summand of $H_2(\mathcal{M}_{2,1}^1; \mathbb{Z})$. We show that $ax$ differs
from $ce$ by inspecting their mod-2 reductions, using the isomorphism (3.1) from [Bia20]:
here $ax$ is a non-trivial element in the direct summand $F_2(a) \otimes H_2(\Gamma_{2,1}; \text{Sym}^0\mathcal{H})$, while
$ce$ lies in the direct summand $F_2(1) \otimes H_2(\Gamma_{2,1}; \text{Sym}^1\mathcal{H})$. \(\square\)

Using the vertical Browder brackets from § 3.6 and exploiting the fact that $c$, $d$, and $e$
can be lifted to classes of $\mathcal{M}_{g,2}$, we can show that certain classical Browder brackets are
divisible by 2 or even vanish. This is essentially the proof from [Kran22, Prop. 5.3.7].

**Proposition 5.3.** For $x \in H_\ast(\mathcal{M}_{g,1}^m)$, $[c, x]$ and $[d, x]$ are divisible by 2, and $[e, x] = 0$.

**Proof.** Let $\tau := (1, 2) \in \mathcal{G}_2$ be the transposition that exchanges 1 and 2. Now we note
that $s_1^\ast c_2 = c$ and $\tau_\ast c_2 = c_2$, so by employing formula (3.2), we get

$$[c, x] = [s_1^\ast c_2, x] = s_1^\ast [c_2, x] + s_1^\ast [\tau_\ast c_2, x] = 2 \cdot s_1^\ast [c_2, x].$$

The very same argument works for $d$, and for $e$, we note that $\tau_\ast e_2 = -e_2$, whence we
get $[e, x] = [s_1^\ast e_2, x] = s_1^\ast [e_2, x] + s_1^\ast [\tau_\ast e_2, x] = 0$. \(\square\)

Note that the proof says a little bit more: the same argument works for each class
$x \in H_\ast(\mathcal{M}_{g,n}^m)$, where the number $n$ of boundary curves is arbitrary and where we use
the vertical Browder bracket from § 3.6. This shows $[c, c] = 2 \cdot s_1^\ast [c_2, c] = 4 \cdot s_1^\ast s_1^\ast [c_2, c_2]$, i.e. $[c, c]$ is divisible by 4, which we already know since $[c, c] = 2 \cdot Qc = 6 \cdot cd = -4 \cdot cd$.

We also conclude that $[c, d] \in H_2(\mathcal{M}_{2,1}) \cong \mathbb{Z}_2$ and $[a, d] \in H_2(\mathcal{M}_{1,1}) \cong \mathbb{Z}_2$ vanish.\(^{10}\)

Even though such an argument cannot be applied to the brackets $[a, c] \in H_2(\mathcal{M}_{1,1}) \cong \mathbb{Z}$
or to $[d, d] \in H_3(\mathcal{M}_{2,1}) \cong \mathbb{Z} \oplus \mathbb{Z}_2$, these Browder brackets vanish as well:

\(^{10}\)In [Meh11, p. 14], one finds $[c, d] = \tilde{d}^2$; this, however, is disproven by the above argument.
Proposition 5.4. \([a, c] = 0\) and \([d, d] = 0\).

**Proof.** The homomorphism \(\delta : \Gamma_{1,1}^1 \to \Gamma_{1,1}^1\) that forgets the single puncture is surjective, as it is part of the Birman exact sequence \(\pi_1(F_{1,1}) \to \Gamma_{1,1}^1 \to \Gamma_{1,1}^1\). Since abelianisation is right exact, the induced map on first homology \(\delta_* : H_1(\mathcal{M}_{1,1}^1) \to H_1(\mathcal{M}_{1,1}^1)\) is epic as well. Since both homology groups are (abstractly) isomorphic to \(\mathbb{Z}\), it follows that \(\delta_*\) is an isomorphism. However, we clearly have \(\delta_* [a, c] = [1, c] = 0\).

We know that \([d, d] = -[d, d]\) by the graded commutativity of the Browder bracket, i.e. \([d, d]\) has to lie in the \(\mathbb{Z}_2\)-summand. However, we also know that \([d, d]\) is divisible by 2, so it has to vanish. \(\square\)

The following proposition shows that the \(T\)-operation often behaves like a differential. More precisely, an expression \((T \circ T)(x) \in H_{i+2}(\mathcal{M}_{g+2,1}^m)\) can only be non-trivial if it lies in a direct summand of the form \(\mathbb{Z}_2^k\) with \(k \geq 2\).

**Proposition 5.5.** \(T \circ T\) is divisible by 2 and of order 2, i.e. over \(\mathbb{F}_2\) or \(\mathbb{Q}\), we have \(T \circ T = 0\).

**Proof.** Recall the \(\binom{m}{2}\)-sheeted covering \(\beta : \mathcal{M}_{g,1}^{m-2,2} \to \mathcal{M}_{g,1}^m\) where two of the punctures are separated from the other ones and indistinguishable, and let \(\gamma : \mathcal{M}_{g,1}^{m-2,1,1} \to \mathcal{M}_{g,1}^{m-2,2}\) be the 2-sheeted covering in which these two punctures can be distinguished.

Let \(p : W_{1,1} \to \mathcal{M}_{g,1}^{m-2,1,1}\) be the fibre product of the two vertical unit tangent bundles: it is an orientable \((S^1 \times S^1)\)-bundle and we have an equivalence \(\theta_{1,1} : W_{1,1} \to \mathcal{M}_{g,3}^{m-2}\) by regarding the two isolated punctures, together with their tangential directions, as boundary curves as in §3.4.1. Gluing pairs of pants, we obtain \(T \circ T = s_1^* s_2^* \theta_{1,1}^! p^! \gamma^! \beta^!\).

Note that both spaces \(W_{1,1}\) and \(\mathcal{M}_{g,1}^{m-2,1,1}\) carry a free \(\mathfrak{S}_2\)-action by interchanging the two isolated punctures (and their tangential directions), and the bundle map \(p\) is \(\mathfrak{S}_2\)-equivariant. Quotienting out this symmetry, we obtain the bundle \(q : W^2 \to \mathcal{M}_{g,1}^{m-2,2}\) from §3.4.2. Moreover, the map \(\theta_{1,1}\) is equivariant with respect to the \(\mathfrak{S}_2\)-action on \(\mathcal{M}_{g,3}^{m-2}\) interchanging the boundary curves 2 and 3. Now recall that \(s_2 : \mathcal{M}_{g,2}^{m-2} \to \mathcal{M}_{g+1,2}^{m-2}\) is homotopic to \(s_2 \circ \langle 2, 3 \rangle\), i.e. \(s_2 \circ \theta_{1,1}\) is homotopy \(\mathfrak{S}_2\)-invariant. Because the projection \(\tilde{\gamma} : W_{1,1} \to W^2\) is a covering and hence a homotopy quotient, we obtain a map \(\theta^\natural : W^2 \to \mathcal{M}_{g+1,2}^{m-2}\) such that \(s_2 \theta_{1,1}\) is homotopic to \(\theta^\natural \tilde{\gamma}\), i.e. we have a diagram (where the right square commutes up to homotopy):

\[
\begin{array}{ccc}
\mathcal{M}_{g,1}^{m-2,2,1,1} & \xleftarrow{p} & W_{1,1}^{1,1,1} & \xrightarrow{\theta_{1,1}} & \mathcal{M}_{g,1}^{m-2} \\
\downarrow{\gamma} & & \downarrow{\tilde{\gamma}} & & \downarrow{s^2} \\
\mathcal{M}_{g,1}^{m-2,2} & \xleftarrow{q} & W^2 & \xrightarrow{\theta^\natural} & \mathcal{M}_{g+1,2}^{m-2} \\
\downarrow{\beta} & & \downarrow{s^1} & & \\
\mathcal{M}_{g,1}^{m} & \xrightarrow{} & \mathcal{M}_{g+2,1}^{m-2} \\
\end{array}
\]
They satisfy the braid relation

\[ T_1 T_2 T_1 = T_2 T_1 T_2 \]

We are left to show that

\[ \tau \cdot \gamma = \gamma \cdot \tau \]

Proof. By Shapiro’s lemma, the Borel construction \( E \times_{\text{Diff}(F)} \text{UTF} \) of the unit tangent bundle of a closed genus-2 surface \( F \), it acts transitively on the unit tangent bundle \( \text{UTF} \).

Recall that \( \text{UTF} \) is the image of a fundamental class under \( i: \text{UTF} \to \text{UTF} \) where \( \text{UTF} \) is the unit tangent bundle of a closed genus-2 surface \( F \).

The map \( i \) is a homotopy fibre: let \( \text{Diff}(F) \) be the group of orientation-preserving diffeomorphisms on \( F \). By a continuous version of Shapiro’s lemma, the Borel construction \( E\text{Diff}(F) \times_{\text{Diff}(F)} \text{UTF} \) acts transitively on the unit tangent bundle \( \text{UTF} \).

In this section, we check that our tables from the introduction are correct. Since the two classes \( s \) and \( v \) need a more subtle treatment, we will deal with them first.

6. Proofs for the tables

In this section, we check that our tables from the introduction are correct. Since the two classes \( s \) and \( v \) need a more subtle treatment, we will deal with them first.

6.1. Proof: Generator \( s \)

Recall that \( s \) is the image of a fundamental class under \( i: \text{UTF} \to \text{UTF} \).

In this section, we check that our tables from the introduction are correct. Since the two classes \( s \) and \( v \) need a more subtle treatment, we will deal with them first.
is a classifying space for the subgroup \( \text{Diff}_{(Q,X)}(F) \) of diffeomorphisms fixing a point \((Q,X) \in \text{UTF}\). Since \( F \) is hyperbolic, both groups \( \text{Diff}_{(Q,X)}(F) \) and \( \text{Diff}(F) \) have contractible components and hence are equivalent to their group of path components, which are the mapping class groups \( \Gamma_{2,1} \) and \( \Gamma_2 \), respectively. Using that \( \mathcal{M}_{2,1} \) is a classifying space for \( \Gamma_{2,1} \), the above Borel construction gives rise to a homotopy fibre sequence \( \text{UTF} \to \mathcal{M}_{2,1} \to B \Gamma_2 \), where \( i \) occurs as the fibre inclusion.

**Theorem 6.1.** The class \( s = i_\ast [\text{UTF}] \) is a generator of \( H_3(\mathcal{M}_{2,1}; \mathbb{Q}) \).

**Proof.** In a first step, we determine the rational homology of \( \text{UTF} \) by inspecting the Serre spectral sequence of the \( S^1 \)-bundle \( \text{UTF} \to F \): note that \( \pi_1(F) \) acts trivially on the homology of the fibre \( H_\ast(S^1; \mathbb{Q}) \) since \( \text{UTF} \) is orientable. Therefore, the second page of the spectral sequence is given as in Figure 15. The only interesting differential is given by multiplication with the Euler number \( \chi(F) = -2 \), and hence is an isomorphism. Therefore, we obtain \( H_\ast(\text{UTF}; \mathbb{Q}) = (\mathbb{Q}, \mathbb{Q}^4, \mathbb{Q}^4, \mathbb{Q}) \). We note that the identifications \( H_1(\text{UTF}; \mathbb{Q}) \cong H_1(F; \mathbb{Q}) \cong H_2(\text{UTF}; \mathbb{Q}) =: A \) are even isomorphisms of \( \Gamma_2 \)-representations: each mapping class \( \varphi \in \Gamma_2 \) induces a bundle automorphism of \( \text{UTF} \to F \) and hence automorphism of spectral sequences converging to the respective automorphism of \( H_\ast(\text{UTF}; \mathbb{Q}) \).

![Figure 15. The second page of the rational Serre spectral sequence for \( S^1 \to \text{UTF} \to F \)](image)

Now we look at the Serre spectral sequence for the fibration \( \text{UTF} \to \mathcal{M}_{2,1} \to B \Gamma_2 \). Here we use that we already know the rational homology of \( \mathcal{M}_{2,1} \): it is \( (\mathbb{Q}, 0, 0, \mathbb{Q}) \), see [Har91, Lem. 1.3]. Moreover, the group \( \Gamma_2 \) is rationally acyclic [Igu60], so if we put \( M_k := H_k(\Gamma_2; A) \), then the second page of the Serre spectral sequence is given as in Figure 16. The map of interest, namely \( i_\ast : \mathbb{Q} = H_3(\text{UTF}; \mathbb{Q}) \to H_3(\mathcal{M}_{2,1}); \mathbb{Q}) \), is given by the edge morphism \( E^2_{0,3} \to E^0_{0,3} \hookrightarrow H_3(\mathcal{M}_{2,1}; \mathbb{Q}) \); hence we only have to show that all differentials that reach \( E^2_{0,3} \) are trivial.

![Figure 16. The second page of the rational Serre spectral sequence for \( \text{UTF} \to \mathcal{M}_{2,1} \to B \Gamma_2 \), together with the only two differentials that could possibly kill \( E^2_{0,3} \)](image)
To this aim, it is enough to show that $E^2_{2,2} = M_2$ and $E^2_{3,1} = M_3$ are trivial; we even show that all $M_k$ are trivial: we start by noting that $M_0 = E^0_{0,1}$ and $M_1 = E^0_{1,1}$ are trivial. Now we use from [Harg91, p. 33] that $H_*(\Gamma^2_2; \mathbb{Q}) \cong (\mathbb{Q}, 0, 0)$. If we consider the Serre spectral sequence $\tilde{E}$ associated with the fibre sequence $F \to \Gamma^2_1 \to \Gamma^2_2$, then $E^2_{k,1} \cong M_k$, and since $\Gamma_2$ is rationally acyclic and $M_0 = M_1 = 0$, the spectral sequence looks as in Figure 17 and converges to $H_*(\Gamma^2_2; \mathbb{Q})$. This already shows that $M_k = E^\infty_{k,1} = 0$ for $k \geq 3$. Finally, since $E^2_{0,2} = \mathbb{Q}$ has to survive, the differential $d_{2,1}$ has to be trivial; and thus, $M_2 = E^\infty_{2,1} = 0$ as desired.

![Figure 17](Image)

**Figure 17.** The second page of the rational Serre spectral sequence $\tilde{E}$ for $F \to \Gamma^2_1 \to \Gamma^2_2$; here the differential $d_{2,1}$ has to be trivial, as the limit $H_2(\Gamma^2_2; \mathbb{Q})$ has dimension 1.

### 6.2. Proof: Generator $v$

In this subsection, we summarise the proof of [Boe18, Prop. 7.2.2] that the homology class $v$ generates the $\mathbb{Z}_3$-summand of $H_4(\mathcal{B}\mathcal{M}_{2,1}; \mathbb{Z}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$. Recall that $v$ was defined using the Segal–Tillmann map $ST: C_6(\mathbb{R}^2) \to \mathcal{M}_{2,1}$, by taking the ptoloeic epicycle $\tilde{v}$, which generates $H_4(\mathcal{B}\mathcal{M}_6; \mathbb{Z}) \cong \mathbb{Z}_3$ and is depicted in Figure 13. It hence suffices to show:

**Theorem 6.2.** The Segal–Tillmann map $ST: C_6(\mathbb{R}^2) \to \mathcal{M}_{2,1}$ induces an injection on the 3-torsion of $H_4$, and hence, $v$ generates the $\mathbb{Z}_3$-summand of $H_4(\mathcal{M}_{2,1}; \mathbb{Z})$.

The proof needs a little preparation: first of all, recall that the Segal–Tillmann map has an algebraic counterpart on the level of mapping class groups, which is a group homomorphism $ST: \mathcal{B}\mathcal{M}_6 \to \Gamma_{2,1}$. Now note that $\mathcal{B}\mathcal{M}_6$ is the same as the mapping class group $\Gamma^6_{0,1}$ of a disc with six punctures. Here we can cap the boundary with a disc, resulting in a homomorphism $\vartheta: \Gamma^6_{0,1} \to \Gamma^6_0$. The main ingredient of the proof of Theorem 6.2 is the following observation from [Boe18, Prop. 7.2.3]:

**Lemma 6.3.** The map $\vartheta: \Gamma^6_{0,1} \to \Gamma^6_0$ induces an isomorphism on the 3-torsion of $H_4$.

The proof of Lemma 6.3 relies on the observation that $\vartheta$ is a composition of two projection maps in short exact sequences: the capping sequence $\mathbb{Z} \to \Gamma^6_{0,1} \to \Gamma^6_0$, whose target is the mapping class group of a sphere with $seven$ punctures, where the first six punctures are allowed to be permuted and the last one is isolated; and the Birman sequence $\pi_1(S^2 \setminus \{p_1, \ldots, p_6\}) \to \Gamma^6_0 \to \Gamma^6_0$ that forgets the isolated puncture. Both
projections induce isomorphisms on the 3-torsion of $H_4$, as Boes shows by considering the respective Lyndon–Hochschild–Serre spectral sequences with coefficients in the ring $\mathbb{Z}_{(3)} \subseteq \mathbb{Q}$ where all primes except for 3 have been inverted. As a composition of these two maps, the same holds for $\vartheta$: this proves the Lemma. Now we are ready to give the main proof of Theorem 6.2:

**Proof of Theorem 6.2.** The Birman–Hilden theorem [BiH71] (see also [FM12, §9.4.2]) gives rise to a map $\gamma: \Gamma_2 \rightarrow \Gamma_0^6$ as follows: let $F$ be a closed surface of genus 2 and consider the hyperelliptic involution $\tau: F \rightarrow F$ as in Figure 18. Let $\mathcal{P} \subseteq F$ be the set of fixed points: it has exactly six elements. Then $\tau^2 = \text{id}$ and $F / \langle \tau \rangle$ is identified with a sphere $S^2$; we call the image of the fixed points $\mathcal{P}' \subseteq S^2$. Now let $\text{SHomeo}(F) \subseteq \text{Homeo}(F)$ be the subgroup of those orientation-preserving homeomorphisms that commute with $\tau$, called symmetric homeomorphisms. Since each symmetric homeomorphism has to fix $\mathcal{P}$ as a set, we obtain a map $\text{SHomeo}(F) \rightarrow \text{Homeo}_{\mathcal{P}'}(S^2)$ to the group of orientation-preserving homeomorphisms on $S^2$ that fix $\mathcal{P}'$ as a set, by quotienting out $\tau$. By passing to $\pi_0$, we obtain a zig-zag of group homomorphisms

$$
\Gamma_2 \xleftarrow{i} \pi_0(\text{SHomeo}(F)) \xrightarrow{j} \Gamma_0^6.
$$

Since all standard generators of $\Gamma_2$ can easily be represented by symmetric homeomorphisms, $i$ is surjective. Secondly, one ingredient for the Birman–Hilden theorem states that if two symmetric homeomorphisms $\varphi$ and $\psi$ are isotopic, then they are even isotopic through symmetric homeomorphisms, see [FM12, Prop. 9.4]. This shows that $i$ is also injective, and hence an isomorphism of groups: we put $\gamma := j \circ i^{-1}$.

If we consider the capping morphism $\beta: \Gamma_{2,1} \rightarrow \Gamma_2$ and the Segal–Tillmann map $\text{ST}: \mathcal{B}t_6 \rightarrow \Gamma_{2,1}$, then the composition $\beta \circ \text{ST}: \mathcal{B}t_6 \rightarrow \Gamma_2$ can, without any choice of isotopy, be described in terms of symmetric homeomorphisms, as the Dehn twists about the curves $\alpha_1, \ldots, \alpha_5$ in Figure 9 (now for genus 2) are already symmetric. Quotienting out $\tau$, we recover the standard homeomorphisms of $\Gamma_0^6 = \mathcal{B}t_6(S^2)$, the sixth braid group of the sphere. This shows that $\gamma \circ \beta \circ \text{ST}$ agrees with the capping map $\vartheta$, so since $\vartheta$ induces an isomorphism on the 3-torsion of $H_4$, it follows that $\text{ST}$ induces an injection on the 3-torsion of $H_4$, which finishes the proof. \hfill \Box
6.3. Proof: Rest of the tables

Now we have everything at hand to verify the remaining entries in our three tables of generators from the introduction. Recall that we already know the isomorphism types of the abelian groups $H_i(\mathcal{M}^m_{g,1})$ for small $g$ and $m$; we only have to argue why our classes are generators.

**Theorem 6.4.** The generators for $H_4(\mathcal{M}^m_{1,1};\mathbb{Z})$ are given as in Table 1.

*Proof.* Both $\mathcal{M}^m_{0,1}$ and $\mathcal{M}^m_{1,0}$ are contractible, with ground classes $1$ and $a$. Moreover, $\mathcal{M}^m_{0,1}$ is a classifying space for the braid group $\mathfrak{B}m$, so $H_1(\mathcal{M}^m_{0,1})$ is generated by a loop corresponding to an elementary braid: this is the class $a^{m-2}b$. Finally, the Pontrjagin product with $b$ is injective over $\mathbb{F}_2$, as discussed in §4.2, and hence $b^2$ is non-trivial in $H_2(\mathcal{M}^m_{0,1};\mathbb{F}_2)$. This shows that $b^2$ cannot vanish integrally, and hence generates $H_2(\mathcal{M}^m_{1,1}) = \mathbb{Z}_2$. Finally, $ab^2 \neq 0$ by the injectivity of $a \cdot \cdot$.

**Theorem 6.5.** The generators for $H_4(\mathcal{M}^m_{1,1};\mathbb{Z})$ are given as in Table 2.

*Proof.* The row for the $0$th homology is obvious, as $a^m c$ is the ground class of $\mathcal{M}^m_{1,1}$. Moreover, we have already seen that $d$ and $e$ generate their respective homology groups. By the injectivity of $a \cdot \cdot$, we conclude that all $a^m d$ are of infinite order, and hence generate the free part of $H_1(\mathcal{M}^m_{1,1})$ for $m \leq 4$. As before, $bc$ is non-trivial modulo $2$ and hence non-trivial integrally. On the other hand, we see $2 \cdot bc = [a,a] \cdot c = 0$ by Proposition 5.1. Thus, $bc$ generates the $\mathbb{Z}_2$-summand of $H_1(\mathcal{M}^m_{1,1})$. Again, by the injectivity of $a \cdot \cdot$, the same holds for $abc$ and $a^2bc$. Having found all generators for $H_0$ and $H_1$, we now consider $H_2$: since $e \neq 0$, we conclude that $a^m e \neq 0$. As before, the mod-2 reduction of $bd$ is non-trivial, and hence, $bd$ itself is non-trivial. Under the isomorphism (3.1), we see that the mod-2 reduction of $ae$ lies in the direct summand $\mathbb{F}_2\langle a \rangle \otimes H_1(\mathcal{M}^m_{1,1};\text{Sym}^1 \mathcal{H})$, while $bd$ lies in the direct summand $\mathbb{F}_2\langle Qa \rangle \otimes H_1(\mathcal{M}^m_{1,1};\text{Sym}^0 \mathcal{H})$, so since they are non-trivial, they cannot agree. By the injectivity of adding a puncture, we get that $a^2 e$ and $abd$, as well as $a^3 e$ and $a^2 bd$ are different generators. Finally, $b^2 c$ is non-trivial and differs from $a^2 e$ and $a^2 bd$ by the very same isomorphism (3.1). For $H_3$, we already know that $f$ is a generator, so the same applies to $af$ and $a^2 f$. Moreover, the mod-2 reduction of $bf$ is non-trivial, and under the isomorphism (3.1), it lies in the summand $\mathbb{F}_2\langle Qa \rangle \otimes H_1(\mathcal{M}^m_{1,1};\text{Sym}^1 \mathcal{H})$, while $af$ lies in the summand $\mathbb{F}_2\langle a \rangle \otimes H_1(\mathcal{M}^m_{1,1};\text{Sym}^2 \mathcal{H})$: this shows that $be$ and $af$ are linearly independent. As both classes are of order $2$, they generate the 2-torsion part of $H_3(\mathcal{M}^m_{1,1};\mathbb{Z})$. By the injectivity of adding a puncture, $a^2 f$ and $abe$ are linearly independent. Moreover, $b^2 d$ differs from $a^2 f$ and $abe$ by the same argument as before, and $bf$ is non-trivial and of order $2$. \[\square\]

**Theorem 6.6.** The generators for $H_4(\mathcal{M}^m_{2,1};\mathbb{Z})$ are given as in Table 3.

*Proof.* Again, $H_0$ is obvious. Since $d$ is represented by a single Dehn twist about a non-separating curve in a surface of genus 1 and one boundary curve, the same applies.
to \( cd \), and we know that each such Dehn twist generates \( H_1(\mathcal{M}_{2,1}) = \Gamma_{2,1}^{ab} \). Using again that adding a puncture is injective, we find that \( acd \) and \( a^2cd \) are generators. Moreover, \( bc^2 \) is non-trivial modulo 2, and hence non-trivial integrally, and it is of order 2, as already \( bc \) is. The non-triviality of \( d^2 \) is due to \[\text{God}07\], Ex. 4, and the non-triviality of the mod-2 reduction of \( Te \) has been shown by a computer-aided calculation in \[\text{Meh}11\], p. 133. Since \( Te \) is of order 2 and \( \lambda s \) is a free generator by Theorem \[\text{6.1}\], the classes \( \lambda s \) and \( T e \) generate \( H_3(\mathcal{M}_{2,1}) \). The very same arguments as before justify the entries \( \lambda as, a \cdot T e, a^2 \cdot T e, \) and \( bd^2 \). By Theorem \[\text{6.2}\], we know that \( v \) generates the \( \mathbb{Z}_3 \)-summand in \( H_4(\mathcal{M}_{2,1}) \). This justifies the entries \( av \) and \( a^2v \), and again, \( b \cdot T e \) is non-trivial and, using the isomorphism \( (3.1) \), different from \( a^2v \).

\[\square\]

**A. Calculations modulo 2**

In this short Appendix, we discuss analogous tables for homology with coefficients in \( \mathbb{F}_2 \). The mere \( \mathbb{F}_2 \)-Betti numbers can easily be computed via the universal coefficient theorem from our integral Tables 1, 2, and 3. Moreover, the mod-2 reductions of our integral generators (for the \( \mathbb{Z} \)- and the \( \mathbb{Z}_2 \)-summands) appear as generators for the \( \mathbb{F}_2 \)-homology. However, several further summands arise from the torsion part in the universal coefficient theorem, and we can identify some further generators of these summands.

First of all, note that Table 6 for \( g = 0 \) only shows the first grading components of the Dyer–Lashof algebra \( \mathbb{F}_2[Q/a]_{j \geq 0} \). For the other two Tables 7 and 8, we extensively use the decomposition \( (3.1) \) from \[\text{Bia}20\]; and for Table 7, we additionally use from \[\text{Meh}11\], p. 13] that \( E b \) is non-trivial in \( H_3(\mathcal{M}_{1,1}) \), so it must generate the Tor-summand. By the naturality of the short exact sequence from the universal coefficient theorem, the same applies to \( a^m \cdot E b \). For Table 8, the last Nishida relation from \( \S 3.1.2 \) tells us that for the mod-2 Bockstein morphism \( \beta \), we have \( \beta Qd = d^2 + [d, \beta d] \). The second summand vanishes by Proposition 5.3, and hence \( \beta Qd = d^2 \neq 0 \). Thus, \( Qd \) is not the mod-2 reduction of an integral class, and hence generates the single Tor-summand of \( H_3(\mathcal{M}_{2,1}; \mathbb{F}_2) \). Moreover, \[\text{Meh}11\], p. 14] shows that \( TEb, c \cdot E b, de, \) and \( d \cdot E b \) are non-trivial and that \( c \cdot E b \) and \( de \) are independent.

| \( m_{0,1} \) | \( m_{1,1} \) | \( m_{2,1} \) | \( m_{3,1} \) | \( m_{4,1} \) | \( m_{5,1} \) |
|---|---|---|---|---|---|
| 0 | a | \( a^2 \) | \( a^3 \) | \( a^4 \) | \( a^5 \) |
| 1 | b | ab | \( a^2b \) | \( a^3b \) |
| 2 | \( b^2 \) | ab^2 |
| 3 | \( Qb \) | a \cdot Qb |

Table 6. Homology groups over \( \mathbb{F}_2 \) for \( g = 0 \) and \( m = 0, \ldots, 5 \)
### Table 7. Homology groups over \( \mathbb{F}_2 \) for \( g = 1 \) and \( m = 0, \ldots, 4 \); here the symbol \( \oplus^k \) means that \( k \) generators have not yet been found.

| \( m_{1,1} \) | \( m_{1,1}^2 \) | \( m_{1,1}^3 \) | \( m_{1,1}^4 \) | \( m_{1,1}^5 \) |
|----------------|----------------|----------------|----------------|----------------|
| 0  \( c \) ac | \( a^2 c \) ac | \( a^3 c \) ac | \( a^4 c \) ac |
| 1  \( d \) ad | \( a^2 d \), bc | \( a^3 d \), abc | \( a^4 d \), \( a^2 b c \) |
| 2  \( e \) ae, bd | \( a^2 e \), abd, | \( a^3 e \), \( a^2 bd \), \( b^2 c \), |
| 3  \( f \), a | \( a f \), be, a^2 E b | \( a f \), a be, b^2 d, Q b c a \ a^3 E b, | \( a^3 E b, | \( a^3 E b, | \( a^3 E b, | \( a^3 E b, |
| 4  ? | \( b \), E b | \( a_3 E b, \oplus^3 \) | \( b f \), Q b d a, b - E b, | \( a_5 \) |
| 5  ? | \( ? \) | \( ? \) | \( ? \) | \( ? \) |
| 6  ? | \( ? \) | \( ? \) | \( ? \) | \( ? \) |

### Table 8. Homology groups over \( \mathbb{F}_2 \) for \( g = 2 \) and \( m = 0, 1, 2 \)

### References

[ABE08] J. Abhau, C.-F. Bödigheimer, and R. Ehrenfried. ‘Homology of the mapping class group \( \Gamma_{2,1} \) for surfaces of genus 2 with a boundary curve’. In: Geom. Topol. Monogr. 14 (2008), pp. 1–25. doi: 10.2140/gtm.2008.14.1.

[Abh05] J. Abhau. ‘Die Homologie von Modulräumen Riemannscher Flächen: Berechnungen für \( g \leq 2 \)’. Diploma thesis. Rheinische Friedrich-Wilhelms-Universität Bonn, 2005.

[BCT89] C.-F. Bödigheimer, F. R. Cohen, and L. R. Taylor. ‘On the homology of configuration spaces’. In: Topol. 28 (1 1989), pp. 111–123. doi: 10.1016/0040-9383(89)90035-9.

[Bia20] A. Bianchi. ‘Splitting of the homology of the punctured mapping class group’. In: J. Topol. 13 (3 2020), pp. 1230–1260. doi: 10.1112/topo.12153.

[BiH71] J. S. Birman and H. M. Hilden. ‘On the mapping class groups of closed surfaces as covering spaces’. In: Annals of Mathematics Studies 66 (1971). Ed. by L. V. Ahlfors, H. M. Farkas, R. C. Gunning, I. Kra, and H. E. Rauch, pp. 81–115. ZBL: 0217.48602.
Unstable homology of moduli spaces

C.-F. Bödigheimer, F. Boes, F. Kranhold

[Böd90a] C.-F. Bödigheimer. ‘On the topology of moduli spaces of Riemann surfaces. Part i: Hilbert Uniformization’. In: Math. Gott. 7+8 (1990). Sfb 170.

[Böd90b] C.-F. Bödigheimer. ‘On the topology of moduli spaces of Riemann surfaces. Part ii: Homology Operations’. In: Math. Gott. 9 (1990). Sfb 170.

[Boe18] F. J. Boes. ‘On moduli spaces of Riemann surfaces: new generators in their unstable homology and the homotopy type of their harmonic compactification’. PhD thesis. Rheinische Friedrich-Wilhelms-Universität Bonn, 2018. hdl: 20.500.11811/7586.

[BoH14] F. J. Boes and A. Hermann. ‘Moduli spaces of Riemann surfaces: homology computations and homology operations’. Master’s thesis. Rheinische Friedrich-Wilhelms-Universität Bonn, 2014.

[Bol12] S. K. Boldsen. ‘Improved homological stability for the mapping class group with integral or twisted coefficients’. In: Math. Z. 270 (2012), pp. 297–329. doi: 10.1007/s00209-010-0798-y.

[BT01] C.-F. Bödigheimer and U. Tillmann. ‘Stripping and splitting decorated mapping class groups’. In: Cohomological Methods in Homotopy Theory. Ed. by J. Aguadé, C. Broto, and C. Casacuberta. Progress in Mathematics 196. Basel: Birkhäuser, 2001, pp. 47–57. doi: 10.1007/978-3-0348-8312-2_6.

[CLM76] F. R. Cohen, T. J. Lada, and J. P. May. The Homology of Iterated Loop Spaces. Lecture Notes in Mathematics 533. Berlin, Heidelberg: Springer, 1976. doi: 10.1007/BFb0080464.

[Dah96] S. Dahlmann. ‘Über den Modulraum gerichteter Tori’. Diploma thesis. Rheinische Friedrich-Wilhelms-Universität Bonn, 1996.

[Ehr98] R. Ehrenfried. Die Homologie der Modulräume berandeter Riemannscher Flächen von kleinem Geschlecht. Bonner Mathematische Schriften 306. PhD thesis. Bonn: Mathematisch-Naturwissenschaftliche Fakultät der Universität Bonn, 1998.

[FL91] Z. Fiedorowicz and J.-L. Loday. ‘Crossed simplicial groups and their associated homology’. In: Trans. Amer. Math. Soc. 326.1 (1991), pp. 57–87. doi: 10.2307/2001855.

[FM12] B. Farb and D. Margalit. A Primer on Mapping Class Groups. Princeton: Princeton University Press, 2012. doi: 10.1515/9781400839049.

[FS96] Z. Fiedorowicz and Y. Song. ‘The braid structure of mapping class groups’. In: Sci. Bull. Josai Univ. 2 (1996), pp. 21–29. zbl: 0891.55012.

[Galo4] S. Galatius. ‘Mod p homology of the stable mapping class group’. In: Topol. 43 (5 2004), pp. 1105–1132. doi: 10.1016/j.top.2004.01.011.

[GKR19] S. Galatius, A. Kupers, and O. Randal-Williams. ‘$E_2$-cells and mapping class groups’. In: Publ. Math. IHÉS 130 (2019), pp. 1–61. doi: 10.1007/s10240-019-00107-8.

[God07] V. Godin. ‘The unstable integral homology of the mapping class groups of a surface with boundary’. In: Math. Ann. 337 (2007), pp. 15–60. doi: 10.1007/s00208-006-0025-7.

[Han09] E. Hanbury. ‘Homological stability of non-orientable mapping class groups with marked points’. In: Proc. Amer. Math. Soc. 137 (1 2009), pp. 385–392. doi: 10.1090/S0002-9939-08-09619-1.
Unstable homology of moduli spaces

C.-F. Bödigheimer, F. Boes, F. Kranhold

[Har84] J. L. Harer. ‘Stability of the homology of the mapping class group of orientable surfaces’. In: Ann. Math. Second series 121.2 (1984), pp. 215–249. doi: 10.2307/1971172.

[Har86] J. L. Harer. ‘The virtual cohomological dimension of the mapping class group of an orientable surface’. In: Invent. Math. 84 (1986), pp. 157–176. doi: 10.1007/BF01388737.

[Har91] J. L. Harer. ‘The third homology group of the moduli space of curves’. In: Duke Math. J. 63.1 (1991), pp. 22–55. doi: 10.1215/S0012-7094-91-06302-7.

[Hil09] D. Hilbert. ‘Zur Theorie der konformen Abbildung’. In: Nachr. Königl. Ges. Wiss. (1909), pp. 314–323.

[Igu60] J.-I. Igusa. ‘Arithmetic Variety of Moduli for Genus Two’. In: Ann. Math. 72.3 (1960), pp. 612–649. doi: 10.2307/1970233.

[Iva90] N. V. Ivanov. ‘On stabilization of the homology of Teichmüller modular groups’. In: Leningr. Math. J. 1.3 (1990), pp. 675–691. zbl: 0727.30036.

[Kran22] F. Kranhold. ‘Coloured topological operads and moduli spaces of surfaces with multiple boundary curves’. PhD thesis. Rheinische Friedrich-Wilhelms-Universität Bonn, 2022. hdl: 20.500.11811/10004.

[Kras87] R. Krasauskas. ‘Skew-simplicial groups’. In: Liet. Mat. Rink. 27 (1987), pp. 47–54. doi: 10.1007/BF00972021.

[KS03] M. Korkmaz and A. I. Stipsicz. ‘The second homology groups of mapping class groups of orientable surfaces’. In: Math. Proc. Cambridge Philos. Soc. 134.3 (2003), pp. 479–489. doi: 10.1017/S0305004102006461.

[Lod92] J.-L. Loday. Cyclic Homology. Grundlehren der mathematischen Wissenschaften 301. Berlin, Heidelberg: Springer, 1992. doi: 10.1007/978-3-662-21739-9.

[Meh11] S. Mehner. ‘Homologieberechnungen von Modulräumen Riemannscher Flächen durch diskrete Morse-Theorie’. Diploma thesis. Rheinische Friedrich-Wilhelms-Universität Bonn, 2011.

[Mil86] E. Y. Miller. ‘The homology of the mapping class group’. In: J. Differ. Geom. 24 (1 1986), pp. 1–14. doi: 10.4310/jdg/121440254.

[Mül96] M. Müller. ‘Orientierbarkeit des Raumes der Parallelschlitzgebiete’. Diploma thesis. Rheinische Friedrich-Wilhelms-Universität Bonn, 1996.

[MWo7] I. Madsen and M. Weiss. ‘The stable moduli space of Riemann surfaces: Mumford’s conjecture’. In: Ann. Math. 165.3 (2007), pp. 843–941. doi: 10.4007/annals.2007.165.843.

[Pow78] J. Powell. ‘Two theorems on the mapping class group of a surface’. In: Proc. Amer. Math. Soc. 68.3 (1978), pp. 347–350. doi: 10.1090/S0002-9939-1978-0494115-8.

[Ran16] O. Randal-Williams. ‘Resolutions of moduli spaces and homological stability’. In: J. Eur. Math. Soc. 18 (1 2016), pp. 1–81. doi: 10.4171/JEMS/583.

[Sak12] T. Sakasai. ‘Lagrangian mapping class groups from a group homological point of view’. In: Algebr. Geom. Topol. 12 (1 2012), pp. 267–291. doi: 10.2140/agt.2012.12.267.
Unstable homology of moduli spaces

C.-F. Bödigheimer, F. Boes, F. Kranhold

[SeTo8] G. B. Segal and U. Tillmann. ‘Mapping configuration spaces to moduli spaces’. In: Adv. Stud. Pure Math. 52 (2008), pp. 469–477. doi: 10.2969/aspm/05210469.

[SoTo7] Y. Song and U. Tillmann. ‘Braids, mapping class groups, and categorical delooping’. In: Math. Ann. 339 (2007), pp. 377–393. doi: 10.1007/s00208-007-0117-z.

[Til97] U. Tillmann. ‘On the homotopy of the stable mapping class group’. In: Invent. Math. 130 (1997), pp. 257–275. doi: 10.1007/s002220050184.

[Tom05] O. Tommasi. ‘Rational cohomology of the moduli space of genus 4 curves’. In: Compos. Math. 141 (2 2005), pp. 359–384. doi: 10.1112/S0010437X0400123X.

[Vis11] B. Visy. ‘Factorable Groups and their Homology’. PhD thesis. Rheinische Friedrich-Wilhelms-Universität Bonn, 2011. HDL: 20.500.11811/4990.

[Wan11] R. Wang. ‘Homology computations for mapping class groups, in particular for Γ_{3,1}^0’. PhD thesis. Rheinische Friedrich-Wilhelms-Universität Bonn, 2011. HDL: 20.500.11811/5018.

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