Monodromy groups of Lagrangian tori in $\mathbb{R}^4$

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Abstract

We determine the Lagrangian monodromy group $L(T)$ and the smooth monodromy group $S(T)$ of a Clifford torus $T$ in $\mathbb{R}^4$. We show that $L(T)$ is isomorphic to the infinite dihedral group, and $S(T)$ is generated by three reflections. We give explicit formulas for both groups. We also show that if a Lagrangian torus is smoothly isotopic to a Clifford torus then the smooth isotopy can be chosen to be Lagrangian outside of a disc.

1 Introduction

In this note we work in the standard symplectic 4-space $(\mathbb{R}^4, \omega = \sum_{j=1}^{2} dx_j \wedge dy_j)$ unless otherwise mentioned. Let $L \hookrightarrow (\mathbb{R}^4, \omega)$ be an embedded Lagrangian torus with respect to the standard symplectic 2-form $\omega$. The Lagrangian condition means that the pull-back 2-form $\iota^* \omega = 0 \in \Omega^2(L)$ vanishes on $L$. Gromov [7] proved that $L$ is not exact, i.e., the pull-back 1-form $\iota^* \lambda$ of a primitive $\lambda$ of $\omega = d\lambda$ represents a nontrivial class in the cohomology group $H^1(L, \mathbb{R})$.

Let $\text{Diff}_c^0(\mathbb{R}^4)$ denote the group of orientation preserving diffeomorphisms with compact support on $\mathbb{R}^4$ that are isotopic to the identity map. We are interested in studying various types of self-isotopies of $L$. It is well-known that for a smooth isotopy $L_s$, $s \in [0, 1]$, between two embedded tori $L_0, L_1$, there associates a family of maps $\phi_s \in \text{Diff}_c^0(\mathbb{R}^4)$ with $\phi_0 = \text{id}$ such that $\phi_s(L) = L_s$. We will make no distinction between $L_s$ and the associated maps $\phi_s$ from now on.

Starting from $L$, a path $\phi_s \in \text{Diff}_c^0(\mathbb{R}^4)$ with $0 \leq s \leq 1$ and $\phi_0 = \text{id}$ associates a family of tori $L_s : \phi_s(L)$ in $\mathbb{R}^4$. The family of maps $\phi_1 \in \text{Diff}_c^0(\mathbb{R}^4)$ is called a smooth self-isotopy of $L$ if $\phi_1(L) = L$. Moreover, if all $L_s$ are Lagrangian with respect to $\omega$ ($\omega$-Lagrangian) then $\phi_s$ is called a Lagrangian self-isotopy of $L$. This is equivalent to say that $L$ is $\phi_s^* \omega$-Lagrangian. Suppose in addition that the cohomology class of $\iota^* \phi_s^* \lambda$ is independent of $s$, then $\phi_s$ is called a Hamiltonian self-isotopy of $L$. Equivalently, $\phi_s$ is Hamiltonian if it is generated by a Hamiltonian vector field. Each self-isotopy $\phi_s$ of $L$ associates an isomorphism

$$(\phi_1)_* : H_1(L, \mathbb{Z}) \rightarrow H_1(L, \mathbb{Z}),$$

called a smooth (resp. Lagrangian, Hamiltonian) monodromy of $L$ if $\phi_1$ is smooth (resp. Lagrangian, Hamiltonian). The group of all smooth monodromies of $L$
is called the smooth monodromy group (SMG) of $L$, and is denoted by $\mathcal{S}(L)$. Likewise, $\mathcal{L}(L)$ and $\mathcal{H}(L)$ denote the Lagrangian monodromy group (LMG) and the Hamiltonian monodromy group (HMG) of $L$ respectively. It is easy to see that $\mathcal{H}(L) \subset \mathcal{L}(L) \subset \mathcal{S}(L)$.

The interest in such monodromy groups is to study the Lagrangian knot problem [6] from a different perspective. Clearly if $L$ and $L'$ are smoothly isotopic, then their smooth monodromy groups are isomorphic. Similar conclusion holds for the Lagrangian and the Hamiltonian cases as well. In [17] we studied $\mathcal{H}(L)$ for $L$ being either a monotone Clifford torus or a Chekanov torus. The latter was constructed (and called a special torus) by Chekanov in [3]. We proved that these two tori are distinguished by their spectrums associated to their Hamiltonian monodromy groups [17]. In this note we will focus on $\mathcal{L}(L)$ instead.

Recall from [12] that the Maslov class $\mu = \mu_L \in H^1(L, \mathbb{Z})$ of a Lagrangian torus $L \subset \mathbb{R}^4$ is nonzero with divisibility 2. An element $h \in \mathcal{L}(L)$ clearly has to satisfy $\mu \circ h = \mu$. Note that in general symplectic manifolds, $h \in \mathcal{L}(L)$ also has to preserve the linking class $\ell_L \in H_1(L, \mathbb{Z})$ (see [5] and Section 2) whenever defined. However, since $\ell_L = 0$ for any embedded $L \subset \mathbb{R}^4$ [5], it imposes no further restriction on $\mathcal{L}(L)$. Let $G_\mu$ denote the formal subgroup of all group isomorphisms $g : H_1(L, \mathbb{Z}) \to H_1(L, \mathbb{Z})$ such that $\mu \circ g = \mu$. Clearly $\mathcal{L}(L)$ is a subgroup of $G_\mu$. Our first result is the following:

**Theorem 1.1.** Assume that $T$ is a Clifford torus. Then $\mathcal{L}(T) = G_\mu$.

The group $G_\mu$ is freely generated by two generalized reflections $f_0, f_1$ (see [1], [2], [3] in Section 4) with $f_1(\gamma_0) = -\gamma_0$, where $\gamma_0 \in H_1(T, \mathbb{Z})$ is a primitive class with $\mu_T(\gamma_0) = 0$, hence $G_\mu$ is isomorphic to the infinite dihedral group $D_\infty$ [8].

For the smooth counterpart, we obtain the following result due to the vanishing of $\ell_L$:

**Theorem 1.2.** Let $L_s = \phi_s(L_0), 0 \leq s \leq 1, \phi_0 = id$, be a smooth isotopy between two Lagrangian tori $L_0, L_1 \subset \mathbb{R}^4$. Then for any $\gamma \in H_1(L_0, \mathbb{Z})$,

$$\mu(\phi_{1*}(\gamma)) - \mu(\gamma) \in 4\mathbb{Z}.$$  

I.e.,

$$\phi_1^* \mu - \mu \in H^1(L_0, \mathbb{Z}) \text{ has divisibility 4.}$$

Thus $\mathcal{S}(L)$ is a subgroup of

$$\mathcal{X} = \mathcal{X}_L := \{g \in \text{Isom}(H_1(L, \mathbb{Z})) \mid \mu_L \circ g - \mu_L \in 4 \cdot H^1(L, \mathbb{Z})\}.$$  

We determine $\mathcal{L}(L)$ for the case of a Clifford torus:

**Theorem 1.3.** Assume that $T$ is a Clifford torus, then $\mathcal{S}(T) = \mathcal{X}_T$. In particular, $\mathcal{S}(T)$ is generated by $\mathcal{L}(T)$ and a reflection along a class $\gamma \in H_1(T, \mathbb{Z})$ with $\mu_T(\gamma) = 2$.  

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As it turns out, since the group $\mathcal{S}(T)$ of a Clifford torus is large enough, any smooth isotopy between a Lagrangian torus and a Clifford torus can be modified at either end by a self-isotopy to match the Maslov classes at both ends. We have the following:

**Proposition 1.4.** Let $L \subset \mathbb{R}^4$ be an embedded Lagrangian torus smoothly isotopic to a Clifford torus $T$. Then there exists a smooth isotopy $\phi_s \in \text{Diff}_0^c(\mathbb{R}^4)$, $s \in [0, 1]$ with $\phi_0 = \text{id}$ and $\phi_1(T) = L$, such that $\phi_s(T \setminus D)$ is Lagrangian for $s \in [0, 1]$, in particular, $\phi_s^* \mu_L = \mu_T$.

However, at the present stage we do not know how to improve $\phi_s(T)$ to a genuine Lagrangian isotopy between $T$ and $L$. To achieve the goal, it seems necessary (and perhaps enough) to have a better understanding on the isotopy of Lagrangian discs with prescribed boundary conditions.

We remark here that in [11] K. Mohnke showed that all embedded Lagrangian tori in $\mathbb{R}^4$ are smoothly isotopic to a Clifford torus. Also in his thesis [9] A. Ivrii showed that any embedded Lagrangian torus in $\mathbb{R}^4$ is Lagrangian isotopic to a Clifford torus. Both of them used pseudoholomorphic curve techniques [7] and methods of symplectic field theory [4, 1].

This article is organized as follows. In Section 2 we review necessary background on Maslov class and the linking class. In Section 3 we discuss the framings of the symplectic normal bundle of a loop in $\mathbb{R}^4$, and the change of framings under diffeomorphisms. Theorem 1.1 is proved in Section 4. Theorem 1.2 is proved in the beginning of Section 5, followed by the proof of Theorem 1.3 which consists of Propositions 5.3-5.5. Proposition 1.4 is proved in Section 6. We will use the convention $S^1 \cong \mathbb{R}/2\pi \mathbb{Z}$ throughout the paper.

2 Maslov class and linking class

As we are concerned with monodromies of self-isotopies of a Lagrangian torus, we should discuss at first two relevant classes in $H^1(L, \mathbb{Z})$: the *Maslov class* $\mu = \mu_L$ (see [10] for more detail) and the *linking class* $l = l_L$. The latter is defined (and denoted by $\sigma$) in [5].

**Maslov class.** The Maslov class $\mu$ is defined as follows. Given $\gamma \in H_1(L, \mathbb{Z})$ and let $C \subset L$ be an immersed curve representing $\gamma$, the tangent bundle $T_C L$ over $C$ is a closed path of Lagrangian planes and hence a cycle in the Grassmanian of Lagrangian planes in the symplectic vector space $\mathbb{R}^4$. Then $\mu(\gamma)$ is defined to be the *Maslov index* of the cycle $T_C L$.

**Theorem 2.1** ([13]). The Maslov class $\mu$ of a Lagrangian torus $L \subset \mathbb{R}^4$ is nontrivial and is of divisibility two.

**Example 2.2.** Consider a Clifford torus

$$T = T_{a, b} := \{(ae^{it_1}, be^{it_2}) \in \mathbb{C}^2 \mid t_1, t_2 \in S^1 \cong \mathbb{R}/2\pi \mathbb{Z}\}.$$
Let $\gamma_1 \in H_1(T, \mathbb{Z})$ be the class represented by the curve $\{(ae^{i t_1}, b) \in \mathbb{C}^2 \mid t_1 \in \mathbb{R}/2\pi \mathbb{Z}\}$, and $\gamma_2 \in H_1(T, \mathbb{Z})$ the class represented by $\{(a, be^{i t_2}) \in \mathbb{C}^2 \mid t_2 \in \mathbb{R}/2\pi \mathbb{Z}\}$. Then $\mu_T(\gamma_1) = 2 = \mu_T(\gamma_2)$.

That $\mu_L \neq 0$ implies that the Lagrangian monodromy group $\mathcal{L}(L)$ can only be a proper subgroup of $\text{Isom}(H_1(L, \mathbb{Z})) \cong GL(2, \mathbb{Z})$.

**Linking class.** The linking class $\ell = \ell_L \in H^1(L, \mathbb{Z})$ is defined as follows. Take $v$ to be any non-vanishing vector field on $L$ which is homotopically trivial, i.e., $v$ is homotopic to some $v'$ in the space of non-vanishing vector fields on $L$, such that $v'$ generates the kernel of a non-vanishing closed 1-form on $L$. Any two of such vector fields are homotopic. Let $J$ be an $\omega$-compatible almost complex structure on $\mathbb{R}^4$. For example, we can take $J$ to be the standard complex structure $J_0$ on $\mathbb{R}^4 \cong \mathbb{C}^2$. Then $\ell(\gamma) := lk(C + \epsilon Jv, L)$ is defined to be the linking number with $L$ of the push-off of $C$ in the direction of $Jv$, where $C \subset L$ is an immersed curve representing the class $\gamma$. The class $\ell$ is independent of the choices involved.

**Example 2.3.** Let $C \subset L$ be an embedded closed curve representing a nontrivial class $\gamma \in H_1(L, \mathbb{Z})$. Parameterize $C$ by $t \in S^1 \cong \mathbb{R}/2\pi \mathbb{Z}$ so that its tangent vector field $\dot{C}(t)$ is non-vanishing. Then $\dot{C}(t)$ extends to a homotopically trivial vector field $v$ on $L$. For example, we can view $L$ as an $S^1$ bundle over $S^1$ with fibers representing the class $[C] \in H_1(L, \mathbb{Z})$, and $C$ is one of the fibers. then take $v$ to be a non-vanishing vector field tangent to the fibers.

**Theorem 2.4** ([5]). The linking class $\ell_L = 0$ for any embedded Lagrangian torus $L \subset \mathbb{R}^4$.

**Remark 2.5.** Later we will show that the vanishing of $\ell$ imposes some restriction on the smooth monodromy group $S(L)$.

### 3 Loops in $\mathbb{R}^4$ and their framings

Before moving on to Lagrangian tori in $\mathbb{R}^4$, it helps to have a closer look at loops in $\mathbb{R}^4$ at first.

A loop in $\mathbb{R}^4$ is an embedded 1-dimensional submanifold diffeomorphic to $S^1$. The pull-back of $\omega$ on a loop vanishes, hence a loop is an isotropic submanifold. Take a loop $C \subset \mathbb{R}^4$. We fix an orientation of $C$ and fix a trivialization of $C \cong S^1 = \mathbb{R}/2\pi \mathbb{Z}$, and write $\dot{C}(t)$ for the tangent vector of $C$ at $C(t)$.

**Symplectic normal bundle.** Let us recall some basic properties of the normal bundle $N$ of $C$. The bundle $N$ splits as

$$N = (T^*C) \oplus N^\omega \cong S^1 \times \mathbb{R} \times \mathbb{R}^2,$$
where $N^\omega$, called the symplectic normal bundle of $C$, is the trivial $\mathbb{R}^2$-bundle over $C$ defined by

$$N^\omega := \{(C(t), v) \mid t \in S^1, \ v \in N|_{C(t)}, \ \omega(\dot{C}(t), v) = 0\}.$$ 

By Weinstein’s isotropic neighborhood theorem (see [14, 15, 10]), there exists a tubular neighborhood $U \subset \mathbb{R}^4$ of $C$, a tubular neighborhood $V \subset N$ of the zero section of the normal bundle $C \subset \mathbb{R}^4$, and a symplectomorphism with $C \subset U$ identified with the zero section of $N$:

$$(U \subset \mathbb{R}^4, \omega) \rightarrow (V \subset N = T^*C \times \mathbb{R}^2, \omega_C \times \omega_{\text{can}}).$$

Here $\omega_{\text{can}} = dx \wedge dy$ is the standard symplectic 2-form on $\mathbb{R}^2$, $\omega_C = dt \wedge dt^*$ is the canonical symplectic 2-form on $T^*C$, and $t^*$ is the fiber coordinate of $T^*C$ dual to $t$. The symplectic normal bundle $N^\omega$ is identified with $\{(t, 0, x, y) \in S^1 \times \mathbb{R} \times \mathbb{R}^2\}$.

Below we explore some properties of $N^\omega$ that will be applied in later sections.

**Lagrangian tori associated to a loop.** Let

$$D^\omega \subset N^\omega$$

denote the associated symplectic normal disc bundle with fiber an open disc $\{(x, y) \in \mathbb{R}^2, \ x^2 + y^2 < \epsilon\}$ with some positive radius $\epsilon$. With the symplectomorphism near $C$ described as above, the boundary $L = L_C := \partial D^\omega$ is an embedded Lagrangian torus in $\mathbb{R}^4$ provided that $\epsilon > 0$ is small enough. Note that for each $\epsilon$ small enough, $L_C$ is unique up to a Hamiltonian isotopy.

It is well known that any two loops in $\mathbb{R}^4$ are smoothly isotopic. The following proposition can be easily verified.

**Proposition 3.1.** Let $C_s, \ 0 \leq s \leq 1$, be a smooth isotopy of loops. Let $D^\omega_s$ denote the symplectic normal disc bundle of $C_s$ with fiber radius $\epsilon_s > 0$. Let $L_s := \partial D^\omega_s$. Then there exists an $\epsilon > 0$ such that $L_s$ is a Lagrangian isotopy of embedded Lagrangian tori provided that $0 < \epsilon_s < \epsilon$. In particular, if $C_0 = C_1$ as a set and $\epsilon_0 = \epsilon_1$, then $D^\omega_0 = D^\omega_1$ and we get a Lagrangian self-isotopy of $L_0 = \partial D^\omega_0$.

In Section 4 we will use this observation to construct Lagrangian self-isotopies of a Clifford torus.

**Framings of $N^\omega$.**

**Definition 3.2.** To a non-vanishing section (i.e., a framing) $\sigma$ of $N^\omega$ one can associate an $S^1$-family of Lagrangian planes

$$\dot{C}(t) \wedge \sigma(t), \ t \in S^1.$$
We denote the corresponding Maslov index by

\[ \mu_C(\sigma) := \mu(\hat{C}(t) \cap \sigma(t)) \in 2\mathbb{Z}. \]

Note that \( \mu_C(\sigma) \) depends only on the orientation of \( C \) and the homotopy class of \( \sigma \) among framings of \( N^\omega \).

If we fix a trivialization \( \Phi : N^\omega \to C \times \mathbb{R}^2 = C \times \mathbb{C}^1 \) then the homotopy classes of framings of \( N^\omega \) can be identified with \( [S^1, \mathbb{R}^2 \setminus \{0\}] = [S^1, S^1] = \mathbb{Z} \). Then for a map \( \theta : S^1 \to S^1 \) of degree \( m \), the Maslov index associated to the section \( \sigma'(t) := e^{i\theta(t)}\sigma(t) \) is \( \mu_C(\sigma') = \mu_C(\sigma) + 2m \). In particular, there is a framing \( \sigma^0 \) of \( N^\omega \) such that \( \mu_C(\sigma^0) = 0 \). We call \( \sigma^0 \) a 0-framing of \( C \), it is unique up to homotopy. Likewise, for each \( m \in \mathbb{Z} \) there is a framing \( \sigma^m \) of \( N^\omega \), \( \sigma^m \) is unique up to homotopy, such that \( \mu_C(\sigma^m) = 2m \).

**Definition 3.3.** We call \( \sigma^m \) an \( m \)-framing of \( N^\omega \) or, an \( m \)-framing of \( C \).

Note that the homotopy classes of framings of \( N^\omega \) is classified by the framing number \( \mu_C(\sigma)/2 \).

**Example 3.4.** Let \( C \subset L \) be a simple closed curve representing the class \( \gamma \in H_1(L, \mathbb{Z}) \) of a Lagrangian torus. Let \( v \) be a non-vanishing section of \( N_C^\omega \cap T_CL \). Then \( v \) is a \( \mu(\gamma)/2 \)-framing of \( N_C^\omega \).

**Proposition 3.5.** Let \( C_s, s \in [0,1] \) be a smooth isotopy between loops \( C_0 \) and \( C_1 \). Write \( C_s = \phi_s(C_0) \) where \( \phi_s \in \text{Diff}_0(\mathbb{R}^4) \) with \( \phi_0 = \text{id} \). Let \( N_s^\omega \) and \( \sigma_s^m \) denote the symplectic normal bundle and the \( m \)-framing of \( C_s \) respectively.

(i). Assume that \((\phi_1)_*N_0^\omega = N_1^\omega \). Then

\[ \mu_{C_1}((\phi_1)_*\sigma^m_0) - \mu_{C_1}(\sigma^m_1) = \mu_{C_1}((\phi_1)_*\sigma^0_0) - \mu_{C_1}(\sigma^0_1) \in 4\mathbb{Z}. \]

(ii). If \( \mu_{C_1}((\phi_1)_*\sigma^m_0) = \mu_{C_1}(\sigma^m_1) = 2m \) then up to a perturbation of \( \phi_s \) we may assume that \((\phi_s)_*N_s^\omega = N_s^\omega \) and \((\phi_s)_*\sigma^m_s = \sigma^m_s \).

**Proof.** (i). First consider the case \( m = 0 \). Fix a trivialization \( S^1 \cong \mathbb{R}/2\pi \mathbb{Z} \to C_0 \) for \( C_0 \). This trivialization composed with \( \phi_s \) becomes a trivialization of \( C_s \). Applying Weinstein's isotropic neighborhood theorem we may symplectically identify a neighborhood of \( C_s \in \mathbb{R}^4 \) with a neighborhood of the zero section of the normal bundle \( N_s \) of \( C_s \). We can trivialize \( N_s = S^1 \times \mathbb{R} \times \mathbb{R}^2 \) with coordinates \((t, t^*, x, y)\) so that

- \( C_s = S^1 \times \{0\} \times \{0\} \),
- \( N_s^\omega = S^1 \times \{0\} \times \mathbb{R}^2 \), and
- \( \sigma^0_s(t) = (t, 0, \epsilon, 0) \), for some \( \epsilon > 0 \).
Then for each \( s \), the differential \( (\phi_\ast)_\ast(t) \) at \( C_0(t) \) with \( t \in S^1 \cong \mathbb{R}/2\pi \mathbb{Z} \) can be thought of as a smooth loop in \( GL^+(4, \mathbb{R}) \):

\[
(\phi_\ast)_\ast(t) \in \begin{bmatrix}
0 & * \\
0 & GL^+(3, \mathbb{R})
\end{bmatrix}, \quad (\phi_0)_\ast(t) = Id, \quad (\phi_1)_\ast(t) \in \begin{bmatrix}
1 & * \\
0 & GL(2, \mathbb{R})
\end{bmatrix}.
\]

Note that \( c(t) \neq 0 \) for \( t \in S^1 \). Note also that \( (\phi_1)_\ast(t) \) is free homotopic to the trivial class of \( \pi_1(GL^+(4, \mathbb{R})) \cong \pi_1(GL^+(3, \mathbb{R})) = \mathbb{Z}_2 \).

We can perturb \( \phi_s \) by composing it with some suitable family of maps in \( \text{Diff}_0(\mathbb{R}^4) \), each of them fixing \( C_s \) pointwise, so that the perturbed \( \phi_s \) satisfy

\[
(\phi_\ast)_\ast(t) \in \begin{bmatrix}
1 & 0 \\
0 & GL^+(3, \mathbb{R})
\end{bmatrix}, \quad \text{with} \quad (\phi_0)_\ast(t) = Id,
\]

\[
(\phi_1)_\ast(t) \in \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & c = \pm 1 & 0 & 0 \\
0 & 0 & \cos kt & -\sin kt \\
0 & 0 & \sin kt & \cos kt
\end{bmatrix}, \quad c k \in \mathbb{N} \cup \{0\}.
\]

That \( [(\phi_1)_\ast(t)] = 0 \) in \( \pi_1(GL^+(4, \mathbb{R})) \) implies that \( k \in 2\mathbb{Z} \). Hence \( \mu_{C_1}((\phi_1)_\ast(\sigma^0)) = 2k + \mu(\sigma^0) = 2k \in 4\mathbb{Z} \).

That \( \mu_{C_1}((\phi_1)_\ast(\sigma^m)) = \mu_{C_1}((\phi_1)_\ast(\sigma^0)) \) follows from the property that \( \sigma^m(t) = e^{*\mu t} \phi_s(t) \) up to homotopy.

(ii). Follow from the perturbation of \( \phi_s \) constructed in (i).

\[\Box\]

4 \ Lagrangian monodromy group (LMG) of a Clifford torus

In general, the LMG \( L(L) \) has to preserve both the Maslov class \( \mu_L \) and the linking class \( \ell_L \) whenever defined. However, for \( L \subset \mathbb{R}^4 \) the class \( \ell_L = 0 \) is automatically preserved. In this section we will determine the LMG of a Clifford torus in \( \mathbb{R}^4 \).

Identify \( \mathbb{R}^4 \cong \mathbb{C}^2 \). For \( a, b > 0 \) the Clifford torus \( T_{a,b} \) is defined to be

\[ T = T_{a,b} := \{(z_1, z_2) \mid |z_1| = a, \ |z_2| = b\}. \]

We fix a basis \( \{\gamma_1, \gamma_2\} \) of \( H_1(T, \mathbb{Z}) \) so that

- \( \gamma_1 \) is represented by the cycle \( \{(ae^{it}, b) \mid t \in \mathbb{R}/2\pi \mathbb{Z}\} \),
- \( \gamma_2 \) is represented by the cycle \( \{(a, be^{it}) \mid t \in \mathbb{R}/2\pi \mathbb{Z}\} \).

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Then $\gamma_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\gamma_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ when expressed as column vectors. We also denote $\gamma_0 := -\gamma_1 + \gamma_2$. Then $\mu(\gamma_0) = 0$ and $\gamma_0 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ as a column vector. Likewise, the Maslov class $\mu \in H^1(T, \mathbb{Z})$ is expressed as a row vector $\mu = (2 - 2)$.

The mapping class group of $T$ is then isomorphic to $GL(2, \mathbb{Z})$, the group of $2 \times 2$ matrices with integral coefficients and with determinant $\pm 1$. Let

$$G_\mu := \{ g \in GL(2, \mathbb{Z}) \mid \mu \circ g = \mu \}.$$ 

A direct computation showed that $G_\mu = G_\mu^+ \cup G_\mu^-$, where

1. $G_\mu^+ = \{ g_n := \begin{pmatrix} 1-n \\ n \\ n \end{pmatrix} \mid n \in \mathbb{Z} \}$
2. $G_\mu^- = \{ f_n := \begin{pmatrix} 1-n \\ n \\ -1+n \end{pmatrix} \mid n \in \mathbb{Z} \}$

Elements of $G_\mu^+$ are of determinant 1, and elements of $G_\mu^-$ are of determinant -1. Also $g_n = (g_1)^n$, where $g_1$ is a generator of $G_\mu \cong \mathbb{Z}$. On the other hand, $G_\mu^-$ consists elements of order 2 in $G_\mu$. Geometrically $g_n = (g_1)^n$ is the $(-n)$-Dehn twist along $\gamma_0$ while each of $f_n$ is a generalized reflection with $f_n(\gamma_0) = -\gamma_0$.

Note that $f_0^2 = e = f_1^2$, $(f_1f_0)^n = g_n$, $(f_0f_1)^n = g_{-n} = (g_n)^{-1}$, $g_nf_m = f_{n+m}$.

Here $e$ denotes the identity element of $G_\mu$. Hence

$$G_\mu = \langle f_0, f_1 \mid f_0^2 = e = f_1^2 \rangle \cong D_\infty$$

is freely generated by the two elements $f_0, f_1$ of order 2, hence is isomorphic to the infinite dihedral group $D_\infty$ [8].

Note that if $L_s = \phi_s(T)$, $s \in [0, 1]$, is a Lagrangian self-isotopy of $T$ so that $L_0 = L_1 = T$, $\phi_0 = id$, then the induced isomorphism $(\phi_1)_* : H_1(T, \mathbb{Z}) \to H_1(T, \mathbb{Z})$ is an element of $G_\mu$. I.e., the LMG $\mathcal{L}(T)$ is a subgroup of $G_\mu$.

**Proposition 4.1.** The LMGs of $T_{a,b}$ and $T_{a',b'}$ are isomorphic.

**Proof.** Identity the ordered pairs $(a, b), (a', b')$ with the coordinates of two points in the first quadrant of the $\mathbb{R}^2$ plane. Take a smooth path $c(s) = (c_1(s), c_2(s))$, $s \in [0, 1]$, in the first quadrant so that $c(0) = (a, b), c(1) = (a', b')$, then $T_c(s)$ is a Lagrangian isotopy of Clifford tori between $T_{a,b}$ and $T_{a',b'}$. Note if we take $\{ \gamma_1, \gamma_2 \}$ as the basis of $H_1(T_{a,b}, \mathbb{Z})$ for any $a, b > 0$, then this isotopy induces an isomorphism $H_1(T_{a,b}, \mathbb{Z}) \to H_1(T_{a',b'}, \mathbb{Z})$ which is represented by the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. \square

**Theorem 4.2.** The LMG of a Clifford torus $T$ is $\mathcal{L}(T) = G_\mu$. 

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**Proof.** We will explicitly construct Lagrangian self-isotopies of $T$ with monodromies $f_0$ and $f_1$ respectively. Then $\mathcal{L}(T) = G_\mu$ following (3).

**Case 1:** The monodromy $f_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Recall that in [17] we have constructed a Lagrangian self-isotopy for $T_{b,b}$ with monodromy $f_1$ (denoted by $f_1$ in [17]). For completeness we repeat the construction here. First let us consider the path in the unitary group $U(2)$ defined by

$$A_s := \begin{pmatrix} \cos \frac{\pi s}{2} & -\sin \frac{\pi s}{2} \\ \sin \frac{\pi s}{2} & \cos \frac{\pi s}{2} \end{pmatrix}, \quad 0 \leq s \leq 1.$$ 

$A_s$ acts on $\mathbb{C}^2$, is the time $s$ map of the Hamiltonian vector field $X = \frac{\pi}{2}(x_1 \partial_{x_2} - x_2 \partial_{x_1} + y_1 \partial_{y_2} - y_2 \partial_{y_1})$, $\omega(X, \cdot) = -dH$, $H = \frac{\pi}{2}(x_2 y_1 - x_1 y_2)$. Observe that $A_1(T_{a,b}) = T_{b,a}$, $(A_1)_* = f_1$ on $H_1(T_{b,b}, \mathbb{Z})$. Fix $b > 0$ and modify $H$ to get a $C^\infty$ function $\tilde{H}$ with compact support such that $\tilde{H} = H$ on $\{|z_1| \leq 2b, |z_2| \leq 2b\}$. Let $\phi_s$ be the time $s$ map of the flow of the Hamiltonian vector field associated to $\tilde{H}$. Then $\phi_1(T_{b,b}) = (T_{b,b})$, and $(\phi_1)_* = (A_1)_* = f_1$ on $H_1(T_{b,b}, \mathbb{Z})$. Now extend this self-isotopy of $T_{b,b}$ by conjugating it smoothly by a Lagrangian isotopy between $T_{a,b}$ and $T_{b,b}$ as described in Proposition 4.1. We may assume that the basis $\{\gamma_1, \gamma_2\}$ of $T_{b,b}$ is transported to the basis $\{\gamma_1, \gamma_2\}$ of $T_{a,b}$ along the latter isotopy. Readers can check now that the extended isotopy induces a Lagrangian self-isotopy of $T_{a,b}$ with monodromy $f_1$.

**Case 2:** The monodromy $f_0 = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$. For $s \in [0, 1]$ consider the family of diffeomorphisms $\Psi_s : \mathbb{R}^4 \to \mathbb{R}^4$,

$$\Psi_s(x_1, y_1, x_2, y_2) := (x_1 \cos \pi s - y_2 \sin \pi s, y_1, x_2, y_2 \cos \pi s + x_1 \sin \pi s).$$

Note that $\Psi_s \in SO(4, \mathbb{R})$ are indeed rotations on the $x_1 y_2$-plane, with the $y_1 x_2$-plane fixed. Consider the simple closed curve $C_0$ defined by

$$\{(x_1 = 0, y_1 = 0, x_2 = b \cos t, y_2 = b \sin t) \in \mathbb{R}^4 \mid t \in [0, 2\pi]\}.$$ 

Define $C_s(t) := \Psi_s(C_0)(t)$. $C_s, s \in [0, 1]$ is a smooth family of curves. Note that $C_1$ equals $C_0$ but with the reversed orientation. Recall from Proposition [17] that for $\epsilon > 0$ small enough, the Lagrangian torus boundary $L_s$ of the symplectic normal disc bundle $D^\omega_\epsilon$ of radius $\epsilon$ of $C_s$ are embedded in $\mathbb{R}^4$, with core curve $C_s$. Note that $L_0 = T_{\epsilon,b} = L_1$ as sets, so we obtain a Lagrangian self-isotopy of $T_{\epsilon,b}$ for $\epsilon > 0$ small enough. This self-isotopy of $T_{\epsilon,b}$ reverses the orientation of $T_{\epsilon,b}$, so the corresponding monodromy $f$ is an element of $G_\mu^c$, with determinant $-1$.
the fiber of \( N \) for \( N \) and an (\( n \)t \( C \)the symplectic normal bundle of \( C \) ≤ \( 0 \), so that \( J \)core curve \( C \)Remark 4.3. If we take \(-1 \) when expressed as a matrix. Note that \( \Psi_1 \) reverses the orientation of the fiber of \( D_0^+ \). Since \( \gamma_2 \subset \partial D_0^+ = T_{r,b} \) is longitudinal, this implies that \( f \) sends \( \gamma_2 \) to \(-\gamma_2 + m\gamma_1 \) for some \( m \in \mathbb{Z} \). Then by comparing with the formula of \( f_n \) in 2 one finds that \( f = f_0 = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \) and \( m = 2 \).

Now, similar to what is done in Case 1, extend the Lagrangian self-isotopy of \( T_{r,b} \) into an Lagrangian self-isotopy of \( T_{a,b} \) through Clifford tori. The corresponding monodromy is \( f_0 \). This completes the proof.

**Remark 4.3.** If we take \( C_0 \) to be the curve

\[
\{(x_1 = a \cos t, y_1 = a \sin t, x_2 = 0, y_2 = 0) \in \mathbb{R}^4 \mid t \in [0, 2\pi]\}
\]

instead, then \( \Psi_s \) will induce a Lagrangian self-isotopy of \( T_{a,e} \) with monodromy \( f_2 = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix} \). The reader can check that \( G_\mu = \langle f_1, f_2 \mid f_1^2 = e = f_2^2 \rangle \). Hence \( \mathcal{L}(T) = G_\mu \) again.

### 5 Smooth Monodromy Group (SMG) of a Clifford torus

We start with the proof of Theorem 1.2

**Proof of Theorem 1.2** By the linearity of \((\phi_1)_s\) and \(\mu\) it is enough to prove for the case when \(\gamma \in H_1(L_0, \mathbb{Z})\) is primitive.

Fix a positive basis \(\{\gamma_1, \gamma_2\}\) for \(H_1(L_1, \mathbb{Z})\) with \(\mu(\gamma_1) = 2 = \mu(\gamma_2)\). Given a primitive class \(\gamma \in H_1(L_0, \mathbb{Z})\) we have \((\phi_1)_s(\gamma) = n_1\gamma_1 + n_2\gamma_2\) for some \(n_1, n_2 \in \mathbb{Z}\). Let \(C_0 \subset L_0\) be an embedded curve representing the class \(\gamma\). Let \(C_s := \phi_s(C_0)\). We denote by \(N_s\) and \(N_s^\omega\) respectively the normal bundle and the symplectic normal bundle of \(C_s\). By assumption \(C_1\) represents the class \(n_1\gamma_1 + n_2\gamma_2\).

Let \(\sigma_0\) denote a non-vanishing section of the \(\mathbb{R}^1\)-bundle \((T_{C_0}T) \cap N_0^\omega\) over \(C_0\). Then \(\sigma_0\) is a \(\mu(\gamma)/2\)-framing of \(N_0^\omega\). Extend \(\sigma_0\) to a smooth family \(\sigma_s\) with \(0 \leq t \leq 1\), so that \(\sigma_s\) is a \(\mu(\gamma)/2\)-framing of \(N_s^\omega\). Let \(m := \mu(\gamma)/2\).

Recall \(J_0\) the standard complex structure over \(\mathbb{R}^4 \cong \mathbb{C}^2\). Fix a trivialization for \(N_s \cong S^1 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}\) by taking \(\{J_0 \hat{C}_s(t), \sigma_s(t), J_0 \sigma_s(t)\}\) as the basis of the fiber of \(N_s\) at \(C_s(t)\), so that the coordinate \((t,t^*,x,y)\) represents the fiber \(t^* J_0 \hat{C}_s(t) + x \sigma_s(t) + y J_0 \sigma_s(t)\).

Now let \(\eta_s := \phi_s(\sigma_0)\). Note that \(\eta_1\) is a non-vanishing section of \(N_1^\omega \cap T_{C_1} L_1\), and an \((n_1 + n_2)\)-framing of \(N_1^\omega\). Let \(k := n_1 + n_2\).
Recall that $\sigma_1$ is an $m$-framing of $N^\tau_C$. Up to a homotopy of $\sigma_s$ if necessary, we may assume the followings:

- For each $s$, $\eta_s = \sigma_s$ at $t = 0$.
- For $t \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$, $\eta_1(t) = \sigma_1(t) \cos(k - m)t + J_0 \sigma_1(t) \sin(k - m)t$.

Then for each $s$, $\phi_s$ associates a smooth map $\Phi_s : S^1 \to GL^+(4, \mathbb{R})$,

$$
\Phi_s(t) := (\phi_s)_*(t) \in \begin{pmatrix} 1 & * & 0 & * \\ 0 & GL^+(3, \mathbb{R}) \\ \end{pmatrix},
$$

$$
\Phi_0(t) = Id, \quad \Phi_1(t) = \begin{pmatrix} 1 & * & 0 & * \\ 0 & * & 0 & * \\ 0 & * \cos(k - m)t & * \\ 0 & * \sin(k - m)t & * \\ \end{pmatrix}.
$$

Then second and fourth columns of $\Phi_1$ represent $(\phi_1)_*(J_0 \tilde{C}_0)$ and $(\phi_1)_*(J_0 \sigma_0)$ respectively.

Extend $\tilde{C}_0$ to a homotopically trivial non-vanishing vector field $u_0$ on $L_0$.

Let $u_s := (\phi_s)_* u_0$. Then $u_1|_{C_1} = \tilde{C}_1$. Then by continuity and $\ell_{L_0} = 0$ we have

$$
lk(C_1 + \epsilon \cdot (\phi_1)_* J_0 u_0, L_1) = lk(C_0 + \epsilon J_0 u_0, L_0) = 0.
$$

Similarly, since $\ell_{L_1} = 0$,

$$
lk(C_1 + \epsilon J_0 (\phi_1)_* u_0, L_1) = lk(C_1 + \epsilon J_0 u_1, L_1) = 0.
$$

Note that (4) and (5) holds true for any class $[C_0]$ and hence $[C_1] = (\phi_1)_*[C_0]$. Hence $(\phi_1)_* J_0 u_0$ is homotopic to $J_0 u_1$ as non-vanishing sections of the normal bundle $N_{L_1}$ of $L_1 \subset \mathbb{R}^4$.

We can extend $\eta_0 = \sigma_0$ over $L_0$ as a homotopically trivial vector field transversal to $u_0$. Then the extended $\eta_1 := (\phi_1)_* \sigma_0$ is a homotopically trivial vector field transversal to $u_1$ on $L_1$ as well. Now $N_{L_0} = J_0 u_0 \wedge J_0 \eta_0$, and $(\phi_1)_* N_{L_0} = N_{L_1} = J_0 u_1 \wedge J_0 \eta_1$. Recall that $u_0|_{C_0} = \tilde{C}_0$, $u_1|_{C_1} = \tilde{C}_1$. That $(\phi_1)_* J_0 u_0$ is homotopic to $J_0 u_1$ as non-vanishing sections of the normal bundle $N_{L_1}$ implies that, along $C_1$, the projection $v$ of $(\phi_1)_* J_0 u_0|_{C_1} = (\phi_1)_* (J_0 \tilde{C}_0)$ into $N_{L_1}|_{C_1} = J_0 \tilde{C}_1 \wedge J_0 \eta_1$ is homotopic to $J_0 \tilde{C}_1$ as framings of $N_{L_1}|_{C_1}$.

This implies that, up to an $L_1$-fixing isotopy, $\Phi_1 = (\phi_1)_*$ satisfies

$$
\Phi_1(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(k - m)t & 0 \\ 0 & 0 & \sin(k - m)t & 0 \\ \end{pmatrix} \in GL^+(4, \mathbb{R}).
$$
Then apply another $L_1$-fixing isotopy if necessary, we have

$\Phi_1(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(k-m)t & -\sin(k-m)t \\ 0 & 0 & \sin(k-m)t & \cos(k-m)t \end{pmatrix} \in GL^+(4, \mathbb{R})$.

that $\Phi_0 = \text{Id}$ implies that the cycle $\Phi_1 \subset GL^+(4, \mathbb{R})$ represents the trivial element of $\pi_1(GL^+(4, \mathbb{R})) \cong \mathbb{Z}_2$. Hence $(k-m)$ is even, and $\mu((\phi_1)_*\gamma) - \mu(\gamma) = 2(k-m) \in 4\mathbb{Z}$. This completes the proof.

**Corollary 5.1.** The SMG $S(L)$ of an embedded Lagrangian torus $L \subset \mathbb{R}^4$ is contained in the subset $\mathcal{X} \subset \text{Isom}(H^1(L, \mathbb{Z}))$ defined by

$\mathcal{X} := \{g \in \text{Isom}(H_1(L, \mathbb{Z})) \mid \mu_L \circ g - \mu_L \in 4 \cdot H_1(L, \mathbb{Z})\}$.

**Corollary 5.2.** Let $L \subset \mathbb{R}^4$ be an embedded Lagrangian torus. Fix a positive basis $\{\gamma_1, \gamma_2\}$ for $H_1(L, \mathbb{Z})$ with $\mu(\gamma_1) = 2 = \mu(\gamma_2)$. Then with respect to $\{\gamma_1, \gamma_2\}$, $\mathcal{X}$ is represented as

$\mathcal{X} = \mathcal{X}^o \sqcup \mathcal{X}^e \subset GL(2, \mathbb{Z})$,

(6) $\mathcal{X}^o := \{\begin{pmatrix} 1 + 2p & 2s \\ 2r & 1 + 2q \end{pmatrix} \in GL(2, \mathbb{Z}) \mid p, q, r, s \in \mathbb{Z}\}$,

(7) $\mathcal{X}^e := \{\begin{pmatrix} 2r & 1 + 2q \\ 1 + 2p & 2s \end{pmatrix} \in GL(2, \mathbb{Z}) \mid p, q, r, s \in \mathbb{Z}\}$.

This in particular implies that $\mathcal{X}$ is a group.

**Proof.** Recall that $\mu = \mu_L$ has divisibility 2. Express $\gamma_1, \gamma_2$ as column vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively. For $g = (g_{ij}) \in \mathcal{X}$, that $\mu(g_{ij}) = \mu_{ij} \in 4\mathbb{Z}$ implies that both $2(g_{11} + g_{21}) - 2$ and $2(g_{12} + g_{22}) - 2$ are divisible by 4. Hence (i) $g_{11}$ and $g_{21}$ have different parity, and (ii) $g_{12}$ and $g_{22}$ have different parity. Since $\det g = \pm 1$, the two even valued entries of $g$ cannot lie in the same column nor the same row of $g$, hence either $g \in \mathcal{X}^o$ or $g \in \mathcal{X}^e$. With the formula for $\mathcal{X}$ the reader can easily verify that $\mathcal{X}$ is a group.

We now move on to determine the group $S(T)$ of a Clifford torus $T$. The proof is divided into the following three propositions.

**Proposition 5.3.** Recall the basis $\{\gamma_1, \gamma_2\}$ for $H_1(T_{a,b}, \mathbb{Z})$. Each of the following four types of elements of $GL(2, \mathbb{Z}) \cong \text{Isom}(H_1(T_{a,b}, \mathbb{Z}))$ can be realized as the monodromy of some smooth self isotopy of $T_{a,b}$:
(i). a $k$-Dehn twist $\tau_k^1 := \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ along $\gamma_1$ with $k \in \mathbb{Z} \setminus \{0\}$.

(ii). a $k$-Dehn twist $\tau_k^2 := \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix}$ along $\gamma_1$ with $k \in \mathbb{Z} \setminus \{0\}$.

(iii). the $\gamma_1$-reflection $\bar{r}_1 := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

(iv). the $\gamma_2$-reflection $\bar{r}_2 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Proof. Since the specific values of $a, b > 0$ are immaterial, we may take values of $a, b$ that are convenient for the construction of a smooth self-isotopy. In the following we will denote a Clifford torus as $T$. Also, since the Lagrangian monodromy $f_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ swaps elements in (i)(iii) with elements in (ii)(iv), we only have to prove the two cases: (i) and (iii).

Let $C := \{(0, be^{it} | t \in [0, 2\pi]) \subset \mathbb{R}^4\}$.

Case (i): $\tau_k^1, k \neq 0$ is even.

Let $U$ be a tubular neighborhood of $C$, $U \cong B^3 \times S^1$. Parameterize $U$ by $(\rho, \varphi, \theta, t)$ where $(\rho, \varphi, \theta) \in [0, \rho_0] \times S^2$ are the spherical coordinates of the 3-ball $B^3$ with $\rho$ being the radial coordinate, $(\varphi, \theta)$ being spherical coordinates on $S^2$ and $(\rho_0, \pi/2, \theta, t)$ parameterizes the equator of the $S^2$-fiber over $t$. We also assume that $(\rho_0, \pi/2, \theta, t) \in S^1 \times S^1$ parameterize $T$ so that $\tau_k^1$ is represented by the map $\phi(\theta, t) = (\theta + kt, t)$. Extend $\phi$ over $U$ to get $\tilde{\phi} : U \to U, \tilde{\phi}(\rho, \varphi, \theta, t) = (\rho, \psi_t(\varphi, \theta), t) := (\rho, (\varphi, \theta + kt), t)$.

As a loop in $SO(3)$ parameterized by $t$, the maps $\psi_t$ represents the trivial class of $\pi_1(SO(3))$, following the assumption that $k$ is even. Then there exists between $\psi_t$ and the constant loop $Id$ a smooth homotopy $\psi_{s,t} \in SO(3), s, t \in [0, 1] \times S^1$, such that $\psi_{0,t} = Id = \psi_{s,0}, \psi_{1,t} = \psi_t$. This induces a smooth homotopy $\tilde{\phi}_s, s \in [0, 1]$, between $\tilde{\phi}_1 = \tilde{\phi}$ and $\tilde{\phi}_0 = id_U$ with $\tilde{\phi}_s(\rho, (\varphi, \theta), t) := (\rho, \psi_{s,t}(\varphi, \theta), t)$.

Let $X_s$ be the time dependent vector field on $U$ that generates the isotopy $\tilde{\phi}_s$, i.e., $\frac{d\tilde{\phi}_s}{ds} = X_s \circ \tilde{\phi}_s, \tilde{\phi}_0 = id$. Note that $X_s$ is tangent to $\partial U$. Extend $X_s$ over $\mathbb{R}^4$ smoothly with compact support. Denote the time 1 map of the extended $X_s$ as $\phi'$. Then $\phi' \in \text{Diff}_0(\mathbb{R}^4)$ is isotopic to the identity map, and $\phi'|_L = \phi$. 

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Case (iii): $\bar{r}_1$.

Parameterize $B^3$ by Cartesian coordinates $(x_1, y_1, x_2)$ with $x_1^2 + y_1^2 + x_2^2 \leq 1$ so that $T \subset U = B^3 \times S^1$ is parameterized by $\{(x_1, y_1, 0, t) \mid x_1^2 + y_1^2 = 1\}$. Without loss of generality we may assume that $\bar{r}_1$ is represented by the map $\phi(x_1, y_1, 0, t) = (-x_1, y_1, 0, t)$ for $(x_1, y_1, 0, t) \in T$. Extend $\phi$ over $U$ to get

$$\tilde{\phi} : U \to U, \quad \tilde{\phi}(x_1, y_1, x_2, t) = (\psi(x_1, y_1, x_2, t) := ((-x_1, y_1, -x_2), t).$$

The map $\psi = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in SO(3)$ is isotopic to the identity map. Let $\psi_s$ be a smooth path in $SO(3)$ with $s \in [0, 1], \psi_0 = Id$ and $\psi_1 = \psi$. This induces an isotopy $\tilde{\phi}_s : U \to U, s \in [0, 1],$

$$\tilde{\phi}_s((x_1, y_1, x_2), t) = (\psi_s(x_1, y_1, x_2), t).$$

Now we extend $\tilde{\phi}_s$ over $\mathbb{R}^4$ with compact support just as in (i) to get $\phi' \in Diff^c_0(\mathbb{R}^4)$ which is isotopic to the identity map, and $\phi'|_L = \phi$. This completes the proof.

Let $R \subset GL(2, \mathbb{Z})$

be the subgroup generated by elements of $\mathcal{L}(T) = G_\mu$ and by $\tau_j^2, \bar{r}_j$ for $j = 1, 2$. Clearly we have the following inclusions as subgroups:

$$R \subset S(T) \subset X.$$

Below we will show that $X \subset R$, then $R = S(T) = X$. To begin with, let us consider the subgroup $\mathcal{E} \subset GL(2, \mathbb{Z})$ generated by $\tau_1^2$ and $\tau_2^2$. It is shown by Sanov [13] that $\mathcal{E}$ is free (see also [2]) and

$$\mathcal{E} = \left\{ \begin{pmatrix} 1 + 4p \\ 2r \\ 2s \\ 1 + 4q \end{pmatrix} \in GL(2, \mathbb{Z}) \mid p, q, r, s \in \mathbb{Z} \right\}.$$

**Proposition 5.4.** The group $X$ is contained in $R$. Hence $R = S(T) = X$.

**Proof.** Since $X^c = f_1X^c$ and $f_1 \in R$, it suffices to show that if $h \in X^c$ then $h \in R$. Our strategy here is to show that for $h \in X^0$ there exists a suitable element $g \in R$ such that $gh \in \mathcal{E}$. Then $h = g^{-1}(gh) \in R$.

Write $h = \begin{pmatrix} 1 + 2p \\ 2r \\ 2s \\ 1 + 2q \end{pmatrix}$. We divide the proof into four cases according to the parity of $p$ and $q$:

(i). If both $p$ and $q$ are even, then already $h \in \mathcal{E} \subset R$. 

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exists a smooth self-isotopy \( \psi \) with every element of \( S \). Indeed, \( \psi \) and \( S \).

Recall that Proposition 5.5.

Proof. Let \( \phi \) is an isotopy that \( \phi \) is generated by \( \phi_{1} = L \) such that \( \phi_{1} \mu_{L} = \mu_{T} \). In Step 2 we modify \( \phi_{s} \) so that \( \phi_{s}(T \setminus D) \) is Lagrangian for all \( t \).

**Step 1:** Let \( \psi_{s} \in \text{Diff}_{0}^{1}(\mathbb{R}^{4}) \), \( s \in [0, 1] \), be a smooth isotopy with \( \psi_{0} = \text{id} \) and \( \psi_{1}(L) = T \). Then \( \psi_{1} \mu_{L} - \mu_{T} \in 4 \cdot H^{1}(T, \mathbb{Z}) \) by Theorem 1.2 and hence \( \psi_{1} \mu_{L} = \mu_{T} \circ \text{g} \) for some \( \text{g} \in X_{T} \). Since \( X_{T} = S(T) \) by Proposition 5.4 there exists a smooth self-isotopy \( \psi_{1} \) of \( T \) with \( (\psi_{1})_{s} = g^{-1} \) and hence \( (\psi_{1})^{*}(\psi_{1}^{*} \mu_{L}) = (\psi_{1}^{*})^{*}(\psi_{1}^{*} \mu_{L}) = \mu_{T} \).

Now define \( \phi_{s} = \begin{cases} \psi_{s}^{2} & \text{for } 0 \leq s \leq 1/2, \\ \psi_{2s-1} \circ \psi_{1} & \text{for } 1/2 \leq s \leq 1. \end{cases} \)

Thus we have proved that \( X \subset R \) and hence \( S(T) = X = R \).

**Proposition 5.5.** The group \( S(T) \subset GL(2, \mathbb{Z}) \) is generated by \( f_{1}, f_{2} \) and \( \bar{r}_{1} \).

Proof. Recall that \( S(T) = R \) is generated by \( \bar{r}_{j} \) and \( \tau_{j}^{2} \) with \( j = 1, 2 \), and by elements of \( G_{\mu} \). The group \( G_{\mu} \) is generated by \( f_{1} \) and \( f_{0} \). Observe that

\[
\bar{r}_{1}^{2} = f_{2}f_{0}, \quad \bar{r}_{2} = f_{2}\bar{r}_{1} = f_{1}f_{0}f_{1}\bar{r}_{1}, \quad \bar{r}_{2} = f_{1}\bar{r}_{1}f_{1}.
\]

So indeed \( S(T) \) is generated by the three elements \( f_{0}, f_{1}, \bar{r}_{1} \) of order 2. Note that

\[
(\bar{r}_{1}f_{1})^{-1} = f_{1}\bar{r}_{1} = -\bar{r}_{1}f_{1}, \quad (\bar{r}_{1}f_{1})^{2} = (f_{1}\bar{r}_{1})^{2} = -e.
\]

The element \(-e\) commutes with every element of \( S(T) \).

This concludes the proof of Theorem 1.3.

**6 Proof of Proposition 1.4**

We divide the proof into two steps. In Step 1 we show that there exists a smooth isotopy \( \phi_{s} \) with \( \phi_{1}(T) = L \) such that \( \phi_{1}^{*} \mu_{L} = \mu_{T} \). In Step 2 we modify \( \phi_{s} \) so that \( \phi_{s}(T \setminus D) \) is Lagrangian for all \( t \).
Then $\phi_s \in \text{Diff}_0^\infty(\mathbb{R}^4)$, $\phi_0 = \text{id}$, $\phi_1(T) = L$, and $\phi_s^*\mu_L = (\psi_1 \circ \psi_1')^*\mu_L = (\psi_1')^*\psi_1^*\mu_L = \mu_T$.

Let $L_s := \phi_s(T)$ for $s \in [0, 1]$. Then $L_0 = T$ and $L_1 = L$.

**Step 2:** We can improve the smooth isotopy $L_s$ so that it is indeed a Lagrangian isotopy outside a disc:

**Lemma 6.1.** Let $L_s = \phi_s(L_0)$, $s \in [0, 1]$, be a smooth isotopy between a Clifford torus $T = L_0$ and a Lagrangian torus $L = L_1$ with $\phi_s \in \text{Diff}_0^\infty(\mathbb{R}^4)$, $\phi_0 = \text{id}$, and $\phi_1^*\mu_L = \mu_T$. Then there exists an smooth isotopy $L'_s = \phi_s'(L'_0)$ between $T = L_0$ and $L = L_1'$ and a disc $D \subset T$ such that $L'_s \setminus \phi'_s(D)$ is Lagrangian for all $s \in [0, 1]$.

**Proof.** Take two simple curves $\gamma, \gamma' \subset T$ which generate $H_1(T, \mathbb{Z})$, and $\gamma$ intersects with $\gamma'$ exactly at one point $p \in T$. Fix an orientation of $T$. We orient $\gamma$ and $\gamma'$ so that the homological intersection $\gamma \cdot \gamma'$ is 1. Denote $\gamma_s := \phi_s(\gamma)$ and $\gamma'_s := \phi_s(\gamma')$ with induced orientations. Also let $p_s := \phi_s(p)$.

Let us start with $\gamma_s$. Let $2m = \mu_T(\gamma_0) = \mu_L(\gamma_1)$. Let $\sigma_s^m \subset N_s^\omega$ denote the $m$-framing of the symplectic normal bundle $N_s^\omega$ of $\gamma_s$, so $\mu_{\gamma_s}(\sigma_s^m) = 2m$. Clearly we may take $\sigma_s^m$ to be a non-vanishing section of the normal bundle $N_{\gamma_s/T}$ of $\gamma = \gamma_0 \subset T$. Likewise we may take $\sigma_1^m = (\phi_1)_*(\sigma_0^m)$ since $\phi_1^*\mu_L = \mu_T$.

Trivialize the normal bundle $N_s$ of $\gamma_s$ as $N_s = S^1 \times \mathbb{R} \times \mathbb{R}^2$ with coordinates $(t, t^*, x, y)$ so that (i) $\gamma_s = S^1 \times \{0\} \times \{0\}$, (ii) $N_s^\omega = S^1 \times \{0\} \times \mathbb{R}^2$, and (iii) $\sigma_0^m(t) = (t, 0, 0, 0)$ for some $\epsilon > 0$. This is exactly the same setup used in the proof of Proposition 6.5(ii) except that $\sigma_s^0$ is replaced by $\sigma_s^m$ here. With respect to the trivialization of $N_s$ the differential of $\phi_s$ along $\gamma_s$ defines a loop with base point $Id$ in the subgroup $A \subset GL^+(3, \mathbb{R})$ consisting of matrices of the form $\begin{bmatrix} 1 & * \\ 0 & GL^+(3, \mathbb{R}) \end{bmatrix}$. Note that $\phi_0$ and $\phi_1$ correspond to the constant loop. Thus the total of the family $\phi_s$ corresponds to a smooth map $\Phi : I^2/\partial I \cong S^2 \to A$, with $I^2 = [0, 1]_s \times [0, 2\pi]_t$, $\Phi(s, t) := (\phi_s)_*(t)$. Since $\pi_2(A, Id) \cong \pi_2(SO(3, \mathbb{R}), Id) = 0$ there exists a smooth homotopy $\Xi : (I^2/\partial I^2) \times [0, 1] \to A$ such that $\Xi(\cdot, 0) = \Phi$, $\Xi(\cdot, 1) = Id$, and $\Xi(p, u) = Id$ for $p \in \partial I^2$, $\forall u \in [0, 1]$.

This implies that for each $s$ there is a tubular neighborhood $U_s \subset \mathbb{R}^3$ of $\gamma_s$, a smooth family of maps $\phi_{s,u} \in \text{Diff}_0^\infty(\mathbb{R}^4)$ with $\phi_{s,0} = \phi_s$, $\phi_{s,u} = \phi_s$ on $\gamma_s$ and $\mathbb{R}^4 \setminus U_s$, and $\phi_{s,u} = \phi_t$ for $i = 0, 1$, such that $\phi_{s,1}(T)$ is Lagrangian along $\gamma_s$, i.e., $T_{\gamma_s,\phi_{s,1}(T)}$ is Lagrangian. By a further perturbation if necessary, we may assume that there exists a tubular neighborhood $V \subset T$ of $\gamma_0$ such that $\phi_{s,1}(V)$ is Lagrangian.

Now apply the same argument to $\gamma'_s$ and $\phi_{s,1}$, just like what we have done for $\gamma_s$ and $\phi_s$. We then get an open neighborhood $Q \subset T$ of $\gamma \cup \gamma'$ with $D := T \setminus Q$.
diffeomorphic to a 2-disc, and a new isotopy $L'_s = \phi'_s(T)$ of $T = L_0$ and $L = L_1$ with $\phi'_s \in \text{Diff}_0^c(\mathbb{R}^4)$, $\phi'_0 = \text{id}$, such that $Q_s := \phi'_s(Q) \subset L'_s$ is Lagrangian for $s \in [0, 1]$. We may assume that $C_s := \partial Q_s$ are smooth for all $s$. Take $D = T \setminus Q$.

This completes the proof of Proposition 1.4.

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