Steering maps and their application to dimension-bounded steering

Tobias Moroder,1 Oleg Gittsovich,2,3,4 Marcus Huber,5,6 Roope Uola,1 and Otfried Gühne1

1Naturwissenschaftlich-Technische Fakultät, Universität Siegen, Walter-Flex-Str. 3, 57068 Siegen, Germany
2Institute of Atomic and Subatomic Physics, TU Wien, Stadionallee 2, 1020 Wien, Austria
3Institute for Theoretical Physics, University of Innsbruck, Technikerstr. 25, 6020 Innsbruck, Austria
4Institute for Quantum Optics and Quantum Information, Austrian Academy of Sciences, Technikerstr. 21a, 6020 Innsbruck, Austria
5Departament de Física, Universitat Autònoma de Barcelona, 08193 Bellaterra, Spain
6ICFO-Institut de Ciències Fotòniques, Mediterranean Technology Park, 08860 Castelldefels (Barcelona), Spain

The existence of quantum correlations that allow one party to steer the quantum state of another party is a counterintuitive quantum effect that has been described already at the beginning of the past century. It has been shown that steering occurs if entanglement can be proven, but with the extra difficulty that the description of the measurements on one party is not known, while the other side is fully characterized. We introduce the concept of steering maps that allow to unlock the sophisticated techniques developed in regular entanglement detection to be used for certifying steerability. As an application we show that this allows to go even beyond the canonical steering scenario, enabling a generalized dimension-bounded steering where one only assumes the Hilbert space dimension on the characterized side, but no description of the measurements. Surprisingly this does not weaken the detection strength of very symmetric scenarios that have recently been carried out in experiments.

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Introduction.—While the term steering was coined already in the early days of quantum mechanics [1], its precise treatment only started alongside modern developments in quantum information theory [2, 3]. The possibility to steer the ensemble in a two-party shared state in quantum mechanics relies on the fact that the two subsystems are entangled. To show steering, entanglement is however not sufficient, since there are even some entangled states that are non-steerable. This highlights the fact that steering can be seen as regular entanglement verification where one relaxes all assumptions about the devices used by one of the parties.

This fundamental fact is also what motivates one of the recent interests in certifying the steerability of quantum states: Any successful steering test constitutes an entanglement test that is completely device independent for one of the parties and can thus be exploited to design more secure quantum protocols in situations where one of the parties may be untrusted. Apart from this it has been observed recently that steering is fundamentally asymmetric [4] and that it is closely connected to joint measurability [5, 6]. Furthermore, steering is known to give an advantage for tasks like subset channel discrimination [7]. Naturally this also spurred the interest in devising strong steering criteria [2, 8–12], to investigate their violation [13] or to develop and to use it quantitatively [14–16]. It has been shown that also bound entangled quantum states exhibit steering [17]. Experimentally, steering has been successfully shown in several recent experiments [18–20], which all demonstrate that steering, taking into account also various loopholes, is already reachable with today’s technology.

In this manuscript we operationally connect steering with regular entanglement verification: We develop a framework that maps the steering certification problem to a regular entanglement detection problem. These steering maps, like we call them, allow us to harness the sophisticated techniques developed in entanglement theory and to go beyond the current state of the art in steering. As an example of the vast possibilities of this framework we introduce a new concept that we call dimension-bounded steering and show that it is accessible with our developed techniques. In this scenario one removes also all assumptions of the usually trusted side, except that all measurements operate in the same Hilbert space of dimension d. In that, this dimension-bounded steering lies between nonlocality and regular steering. Nonetheless we also show that the robustness to experimental noise of dimension-bounded steering can be comparable or even equal to regular steering certification. This implies that recent loophole-free steering experiments could have also shown loophole-free dimension-bounded steering.

The manuscript is organized as follows: We first define steering and set the notation. We continue by demonstrating our approach in a dichotomic setting and then discuss our main technique, the steering maps. With this we then show that deciding steerability of an ensemble is equivalent to a separability problem. In the later part we discuss how our approach can be used to derive criteria for the dimension-bounded case. We end with an explicit example of this criterion for recent experiments and a discussion on its strength.

Steering.—Let us explain the scenario and the notation first. Initially, two parties Alice and Bob share a quantum state ρ. Alice can choose between n different measurements on her side, each having m possible results.
Her choice is denoted by \( x = 1, \ldots, n \) for the setting while the results are labeled by \( a = 1, \ldots, m \). For Bob we assume that he performs full tomography on his reduced state depending on Alice’s measurement and result. So he is able to reconstruct the conditional states \( \rho_{a|x} \) and the data of this experiment is summarized by the ensemble \( \mathcal{E} = \{ \rho_{a|x} \}_{a,x} \) of unnormalized density operators such that Alice’s probability is \( P(a|x) = \text{tr}(\rho_{a|x}) \).

Originally, the question of steering asks whether Alice can convince Bob that she can steer the state at Bob’s side via her measurements. This means that Bob cannot explain the reduced states \( \rho_{a|x} \) as coming from some probability distribution \( p(\lambda) \) of states \( \rho_{\lambda} \), where Alice’s measurements just give additional information about the probability. As shown in Ref. [16] this can be reformulated as follows: An ensemble \( \mathcal{E} \) is non-steerable if and only if there exist unnormalized density operators \( \omega_{i_1 \ldots i_n} \) with \( i_k = 1, \ldots, m \) for each \( k = 1, \ldots, n \) such that

\[
\rho_{a|x} = \sum_{i_1, \ldots, i_n} \delta_{i_x, a} \omega_{i_1 \ldots i_n} \tag{1}
\]

and steerable otherwise. This is the definition from which we start our considerations.

A dichotomic warm up.—Let us first discuss the idea via the most simplest scenario of Alice having two dichotomic measurements, i.e., \( n = m = 2 \), in which case we use labels \( a = \pm \) to provide easier distinguishable formulæ. In this scenario the ensemble \( \mathcal{E} = \{ \rho_{\pm|\pm}, \rho_{\pm|\mp}, \rho_{\mp|\pm}, \rho_{\mp|\mp} \} \) is called non-steerable if and only if there exists positive semidefinite operators \( \omega_{ij} \) with \( i, j = \pm \) such that

\[
\begin{align*}
\rho_{\pm|\pm} &= \omega_{++} + \omega_{--}, & \rho_{\pm|\mp} &= \omega_{++} + \omega_{-+}, \\
\rho_{\mp|\pm} &= \omega_{-+} + \omega_{--}, & \rho_{\mp|\mp} &= \omega_{+-} + \omega_{--},
\end{align*}
\tag{2}
\]

holds. Note that these linear equations are not linearly independent, therefore \( \mathcal{E} \) does not completely determine the unknowns \( \omega_{ij} \). Choosing for instance an arbitrary \( \omega_{++} \) the choices

\[
\begin{align*}
\omega_{++, \omega_{--}} &= \rho_{\pm|\pm} - \omega_{++}, \\
\omega_{+-} &= \rho_{\pm|\mp} - \omega_{+-}, & \omega_{-+} &= \rho_{\mp|\pm} - \omega_{+-},
\end{align*}
\tag{3}
\]

with \( \rho_{\Delta} = \rho - \rho_{\pm|\pm} - \rho_{\pm|\mp} - \rho_{\mp|\mp} \) satisfy the linear constraints.

Recall that steering constitutes semi-device independent entanglement verification, because a non-steerable ensemble can always be reproduced by measurements on a separable state \( \sigma_{AB} \). This works by the using

\[
\sigma_{AB} = \sum_{ij} |i, j \rangle_A \langle i, j| \otimes \omega_{ij}, \tag{4}
\]

where \( |\pm, \pm \rangle_A \) label computational basis states and measurements \( M_{\pm|\pm} = |\pm\rangle \langle \pm| \otimes 1 \), \( M_{\pm|\mp} = 1 \otimes |\pm\rangle \langle \pm| \).

Whether we explicitly search for appropriate \( \omega_{ij} \) satisfying Eq. (2) or for the separable state \( \sigma_{AB} \) in Eq. (4) one could guess there is not much difference. However, looking for a separable state is a task we are well familiar with nowadays, due to extensive research in the past two decades on separability criteria [21, 22]. But there are two things to take into account: Obviously the state \( \sigma_{AB} \) is not completely known to us. Also, \( \sigma_{AB} \) is not just a separable state, because Alice’s states are very special: such states are called classical-quantum [23] or to have zero “quantum discord” [24, 25]. Thus if one naively applies a separability criterion one looses this required extra structure and the criterion will not be very strong. In the following we show how to circumvent these drawbacks.

Steering maps.—To remove the discord zero structure we replace the basis states \( |i, j \rangle \langle i, j| \) by other positive semidefinite operators \( Z_{ij} \) of our choice, so that we get a generic separable structure

\[
\Sigma_{AB} = \sum_{ij} Z_{ij} \otimes \omega_{ij}. \tag{5}
\]

To get a unit trace for \( \Sigma_{AB} \) and to remove the problem that not all \( \omega_{ij} \) are known one enforces certain linear relations on \( Z_{ij} \). Using for instance the solution of Eq. (4) in Eq. (5) one obtains

\[
\Sigma_{AB} = Z_{++} \otimes \rho_{++} + Z_{+-} \otimes \rho_{+-} + Z_{-+} \otimes \rho_{-+} + Z_{--} \otimes \rho_{--},
\]

from which one sees that \( \Sigma_{AB} \) is completely determined if the last term vanishes, i.e., \( Z_{++} = Z_{+-} + Z_{-+} = Z_{--} \). With this identity the normalization of \( \text{tr}(\Sigma_{AB}) = 1 \) is then equal to \( \text{tr}(Z_{++}) \text{tr}(\rho_{++}) + \text{tr}(Z_{+-}) \text{tr}(\rho_{+-}) + \text{tr}(Z_{-+}) \text{tr}(\rho_{-+}) + \text{tr}(Z_{--}) \text{tr}(\rho_{--}) = 1 \). This is exactly what we were looking for and we get the following sufficient criterion for steerability: For any non-steerable ensemble \( \mathcal{E} \) and any choice of positive semidefinite operators \( Z_{ij} \), which satisfy the two just mentioned extra relations, the operator

\[
\Sigma_{AB} = Z_{++} \otimes \rho_{++} + Z_{+-} \otimes \rho_{+-} + Z_{-+} \otimes \rho_{-+} + Z_{--} \otimes \rho_{--} \tag{6}
\]

is a separable quantum state.

If for a given set of \( Z_{ij} \) the state \( \Sigma_{AB} \) is not separable, i.e., entangled or no quantum state at all, then operators \( \omega_{ij} \) with the properties from Eqs. (2, 3) do not exist and the underlying ensemble is steerable. In order to check this we can employ any separability criterion, e.g., partial transposition [26], positive maps [27], entanglement witness [27, 28], computable cross norm or realignment [29, 30], covariance matrices [31], to name only a few. The whole power of this is unlocked by the mapping \( |i, j \rangle \langle i, j| \mapsto Z_{ij} \), which we refer to as steering map from now on.

Next let us generalize this idea. For the most general steering case we know that a non-steerable ensemble can always be obtained by measuring the separable state \( \sigma_{AB} = \sum_{i_1, \ldots, i_n} |i_1, \ldots, i_n \rangle_A \langle i_1, \ldots, i_n| \otimes \omega_{i_1 \ldots i_n} \), with ap-
propriate measurements that only act non-trivially on the respective subsystem for Alice. Each computational basis state is now mapped to a new positive semidefinite operator $Z_{i_1...i_n}$ to obtain

$$\Sigma_{AB} = \sum Z_{i_1...i_n} \otimes \omega_{i_1...i_n}. \quad (7)$$

This operator is uniquely determined by the given ensemble $\mathcal{E}$ if and only if the chosen operators $Z_{i_1...i_n}$ satisfy

$$Z_{i_1...i_n} = Z_{i_1j_2...j_n} + Z_{j_1i_2...j_n} + \ldots + Z_{j_1j_2...i_n} - (n-1)Z_{j_1j_2...j_n} \quad (8)$$

for all possible choices of $i_1,...,i_n$ and $j_1,...,j_n$. With this we are ready to state our first main result, which says that the developed criterion via steering maps is also sufficient. The proof employs the duality of semidefinite programs [32] and is given in the appendix.

**Proposition 1.** For any non-steerable ensemble $\mathcal{E}$ and any set of positive semidefinite operators $Z = \{Z_{i_1...i_n} \}_{i_1...i_n}$ fulfilling (3) the operator given by Eq. (7) has a separable structure.

For any steerable ensemble $\mathcal{E}$ there exists a set of operators $Z$ which uniquely determines $\Sigma_{AB}$ and satisfies $\text{tr}(\Sigma_{AB}) = 1$, but where non-separability of $\Sigma_{AB}$ is detected by the swap entanglement witness. Here, the swap entanglement witness is the flip operator $V = \sum_{ij} |ij\rangle \langle ji|$ where $\text{Tr}(\rho V) < 0$ signals entanglement.

Let us remark that the steering map criterion is strictly stronger than a single steering inequality, which is similarly characterized by $Z$, but where one only checks the swap entanglement witness. Moreover, the proposition also applies to steering scenarios where Bob measures a few observables rather than a tomographic complete set; in this case non-separability of $\Sigma_{AB}$ must be verified via this partial information only. Note that since steering is closely related to joint measurability, Prop. [1] can directly be employed also for this task, and we are using a result from this field [32] to deduce a collection of $Z$ for the case $n = 2, m = d$, cf. appendix.

**Dimension-bounded steering.**—Next let us turn to the dimension-bounded steering case. Contrary to the standard steering setup, where it is essential that the measured observables on the characterized side are fully known, these criteria require only that Bob’s measurements act on a fixed finite dimensional Hilbert space.

To be precise, we assume that Bob can choose between $n_B$ different settings $y$ each yielding one of $m_B$ possible outcomes $b$. Each measurement is described by a POVM, i.e., a set of operators $\{M_{b|y}\}$ which satisfies positivity $M_{b|y} \geq 0$ and normalization $\sum_b M_{b|y} = 1$. As the sole restriction we have to assume that they all act on the same Hilbert space with at most dimension $d_B$. Thus if Bob observes different distributions, $P(b|y, i)$, maybe conditioned onto a separate event $i$ like a measurement result by Alice, then there must exist a collection of different density operators $\{\rho_{b_{\lambda}}\}$ and a single set of appropriate POVMs, both on an $d_B$-dimensional Hilbert space, which reproduce the data, $P(b|y, i) = \text{tr}(M_{b|y}\rho_{b_{\lambda}})$. 1 To complete the description of the problem we assume that $n_A, m_B$ are the subsystem-labeled specifications for Alice, who is the fully uncharacterized side, and refer to it as a $d_B$-dimension-bounded steering scenario with parameters $n_A, m_A, n_B, m_B$.

In order to derive steering criteria for this scenario we employ a fixed steering map to transform the problem to a standard separability question according to Prop. [1]. Afterwards we use the entanglement detection techniques of Ref. [33] which require only a dimension constraint.

The criteria that we derive work best for Bob having dichotomic measurements $n_B = 2$. Before we give the main recipe we like to explain the ideas: As shown in the previous section we know that any steerable ensemble $\mathcal{E}$ can be detected by an appropriate collection $Z$ such that $\Sigma_{AB} = \sum_{ijkl...n} Z_{ijkl...n} \otimes \omega_{ijkl...n}$ is a separable state. Here, $\omega_{ijkl...n}$ should express that the $\omega_{i_1...i_n}$, when using a $Z$ satisfying Eq. (3), is given by a special solution of the linear relations given by Eq. (1), e.g., like in Eq. (9).

To show that $\Sigma_{AB}$ is not separable we can employ the CCNR criterion [24, 30]. This criterion states that the correlation matrix $[\rho_{ijkl}] = [\text{tr}(G_{ij} \otimes G_{kl}) P_{AB}]$ of any separable state $\rho_{AB}^{\text{sep}}$ satisfies $||C(\rho_{AB}^{\text{sep}})||_1 \leq 1$. Here the appearing norm is the trace norm $||C||_1 = \sum_i s_i(C)$ given by the sum of the singular values $s_i(C)$, while the sets $\{G_i\}_i$ are orthonormal Hermitian operators (not necessarily forming a basis) for the respective local side. Thus whenever $||C(\Sigma_{AB})||_1 > 1$ the data $\mathcal{E}$ shows steering.

However, one cannot directly evaluate this for the dimension-bounded scenario, because Bob can neither reconstruct $\rho_{a|x}$ nor compute values $\text{tr}(G^B_k \rho_{a|x})$ because he lacks the precise description of his measurements $M_{b|y}$. Still, we can build a matrix which looks similar to the correlation matrix and for which the dichotomic choice of Bob’s measurements becomes important. For each dichotomic measurement consider the operators given by the difference of the two POVM elements $B_y = M_{+|y} - M_{-|y}$ for $y = 1, \ldots, n_B$ and $B_0 = 1$. Then,

1 Note that we do not “convexify” the set of possible distributions, i.e., we are not assuming the more general form $P(b|y, i) = \sum \lambda P(\lambda) \text{tr}(M_{b|y}\rho_{\lambda})$, with $d_B$-dimensional quantum states and measurements. First, we consider this largely unmotivated for experiments, second, it would considerably weaken the detection strengths of the criteria, and third, since it effectively corresponds to the case of many different $d_B$-dimensional systems it is a strange dimension restriction, except if one distinguishes classical and quantum dimensions [34].
define the matrix \([D(\Sigma_{AB})]_{ky}\) with entries
\[
\text{tr}(G_k^A \otimes B_y \Sigma_{AB}) = \sum_{i_1,...,i_n} \text{tr}(G_k^A Z_{i_1...i_n}) \text{tr}(B_y \omega^{\text{spec}}_{i_1...i_n}).
\]  
(9)

For convenience let us assume that we only pick \(n_B + 1\) different operators \(G_k^A\), such that \(D\) is a square matrix with a determinant. We call this matrix the data matrix \(D\) to further express that \(D\) is completely determined by the observed data \(P(a, b|x, y)\) once having selected \(Z\) and \(\{G_k^A\}_k\).

From the data matrix \(D\) we obtain a correlation matrix \(C = DT\) if \(T\) describes a linear transformation that maps \(B_y\) into an orthonormal set \(\{G_i^B = \sum_y T_y[B_i]\_y\}.\) Though having only the limited information about \(n_B\) being dichotomic measurements on a \(d_B\)-dimensional Hilbert space, this transformation \(T\) is assured to satisfy
\[
|\det(T)| \geq d_B^{\frac{n_B+1}{n_B+1}},
\]  
(10)

which was proven in Ref. [32]. Via this one can then lower bound the trace-norm of \(C\) by
\[
\|C\|_1 = \sum s_i(C) \geq (n_B + 1)|\det(C)|^{\frac{1}{n_B+1}}
\]
\[
= (n_B + 1) \left( |\det(D)| |\det(T)| \right)^{\frac{1}{n_B+1}}
\]
\[
\geq \frac{n_B+1}{\sqrt{d_B}} |\det(D)|^{\frac{1}{n_B+1}}
\]  
(11)

using the inequality of the arithmetic and geometric means in the first step, the determinant rule, and finally Eq. (10). If this lower bound is strictly above 1, we certify that \(\Sigma_{AB}\) is not separable and thus also steerability of the state. This is effective the second condition of the following proposition; the other statement employs a slightly better bounding technique.

**Proposition 2.** Consider a \(d_B\)-dimension-bounded steering scenario with parameters \(n_A, m_A, n_B\) and \(m_B = 2\). From the observed data build up the data matrix
\[
D_{ky} = \sum_{i_1,...,i_n} \text{tr}(G_k^A Z_{i_1...i_n}) \text{tr}(B_y \omega^{\text{spec}}_{i_1...i_n})
\]  
(12)

using \(B_0 = \mathbb{1}\) and \(B_y = M_{+|y} - M_{-|y}\) for \(y = 1, \ldots, n_B\), any set of steering operators \(Z\) with \(n_A, m_A\), and any choice of \(n_B + 1\) orthonormal operators \(G_k^A\).

Let \(d_A\) be the dimension of the chosen \(Z\). If the observed data are non-steerable then the determinant of \(D\) satisfies
\[
|\det(D)| \leq \frac{1}{\sqrt{d_A}} \left( \frac{\sqrt{d_A d_B} - 1}{n_B \sqrt{d_A}} \right)^{n_B}
\]  
(13)

if \(n_B > \sqrt{d_A d_B} - 1\) and \(\mathbb{1} \in \text{span}(\{G_i^A\})\). If this is not the case, non-steerable data give
\[
|\det(D)| \leq \left( \frac{\sqrt{d_B}}{n_B + 1} \right)^{n_B+1}.
\]  
(14)

**Application to experiments.**—In this part we give an explicit example of Prop. 2 to demonstrate its application and also to compare its strength. We pick the scenario that has been implemented in the loophole-free steering experiment performed in Vienna [19]. Alice and Bob have three different dichotomic measurements, \(n_A = n_B = 3\) and \(m_A = m_B = 2\), and we assume that Bob’s measurement act onto a qubit \(d_B = 2\). The settings will be labeled by \(x, y \in \{1, 2, 3\}\) and the outcomes by \(a, b \in \{\pm 1\}\).

According to Prop. 2 let us first pick operators \(Z_{ijk}\) with \(i, j, k \in \{\pm 1\}\) that characterize a steering map with parameters \(n_A = 3\) and \(m_A = 2\). Here we choose \(Z_{ijk} = \frac{\mathbb{1} + i(\sigma_1 + j\sigma_2 + k\sigma_3)/\sqrt{3}}{2}\), which can be interpreted as pure states, whose Bloch vectors point towards the 8 different corners of the cube. It can be checked that these choices satisfy all relations given by Eq. (5), so that, by construction, the operator \(\Sigma_{AB}\) is uniquely determined by the ensemble \(\mathcal{E}\) and furthermore normalized. This operator is given by
\[
\Sigma_{AB} = \frac{1}{2} \left[ \mathbb{1} \otimes \rho + \frac{1}{\sqrt{3}} \sum_{s=1}^3 \sigma_s \otimes (\rho_{+|s} - \rho_{-|s}) \right].
\]  
(15)

In order to get to the data matrix \([D(\Sigma_{AB})]_{ky}\) we still need to fix the operator set \(\{G_k^A\}\) for \(A\), for which the properly normalized Pauli-operators and the identity \(\{\mathbb{1}, \sigma_1, \sigma_2, \sigma_3\}/\sqrt{2}\) are convenient choices since they only act non-trivial on certain terms in Eq. (12). As the final step we rewrite the abstract expectation values \(\text{tr}(B_y \rho_{a|x})\), with \(B_y = M_{+|y} - M_{-|y}\) and \(B_0 = \mathbb{1}\), in terms of the directly observable quantities \(P(a, b|x, y)\). Looking at
\[
\text{tr}(B_y (\rho_{+|x} - \rho_{-|x}))
\]
\[
= \text{tr}[(M_{+|y} - M_{-|y}) (M_{+|x} - M_{-|x})]
\]
\[
= P(\mathbb{1}, + |x, y) - P(-, - |x, y) - P(\mathbb{1}, - |x, y) + P(-, + |x, y)
\]

one sees that correlations \(\langle A_x B_y \rangle\) and respective marginals \(\langle A_z \rangle, \langle B_y \rangle\), which similarly appear in Bell inequalities, give an appropriate formulation. Hence, to sum up one gets the data matrix \(D\)
\[
\frac{1}{\sqrt{2}} \left[ \begin{array}{ccc} 1 & \langle B_1 \rangle & \langle B_2 \rangle & \langle B_3 \rangle \\ \langle A_1 \rangle/\sqrt{3} & \langle A_1 B_1 \rangle/\sqrt{3} & \langle A_1 B_2 \rangle/\sqrt{3} & \langle A_1 B_3 \rangle/\sqrt{3} \\ \langle A_2 \rangle/\sqrt{3} & \langle A_2 B_1 \rangle/\sqrt{3} & \langle A_2 B_2 \rangle/\sqrt{3} & \langle A_2 B_3 \rangle/\sqrt{3} \\ \langle A_3 \rangle/\sqrt{3} & \langle A_3 B_1 \rangle/\sqrt{3} & \langle A_3 B_2 \rangle/\sqrt{3} & \langle A_3 B_3 \rangle/\sqrt{3} \end{array} \right].
\]

Because \(n_B = 3 > \sqrt{d_A d_B} - 1 = 1\) and since the full operator basis for \(A\) includes the identity we can use the
bound given by Eq. (13). Thus if
\[ |\det(D)| > \frac{1}{108}, \]
then the observed data show steering under the sole assumption that Bob’s measurements act onto a qubit.

If one evaluates this criterion for a noisy maximally entangled state \( p |\psi^-\rangle \langle \psi^-| + (1 - p) \mathbb{1}/4 \), measuring along the three spin directions \( \sigma_1, \sigma_2, \sigma_3 \), one verifies steering if \( p > 1/\sqrt{3} \). This is quite surprising, because the visibility to show standard steering, i.e., requiring the knowledge that Bob perfectly measures \( \sigma_1, \sigma_2, \sigma_3 \), is exactly the same. Thus, we learn that for this symmetric case, the only crucial knowledge of the measurements is that they act onto a qubit, but no further characterization is needed. In the appendix we discuss this scenario also under experimentally realistic conditions showing that today’s technology indeed allows (or has already allowed) a loophole-free dimension-bounded steering experiment.

**Conclusion.**—We have introduced a framework that allows to map the steering problem to a standard separability problem. Via this map, we opened the possibility to exploit the sophisticated tools available in entanglement detection, thereby creating strong steering criteria.

Using this we showed dimension-bounded steering, as one particularly further promising application. Considering that many quantum protocols require also a certain level of trust we believe that this dimension-bounded scenario is of high relevance for scenarios where at least one of the parties has some degree of confidence of his or her local device. We have shown that this “nearly” device independent scenario is a lot stronger than the still not attainable full device-independent scenario. It will help to make quantum key distribution more robust [36, 37] and to unify frameworks of resource theories that exist for nonlocality [38] and steering [39] to approach a resource theory of partially device independent entanglement certification.

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**APPENDIX**

**Proof of Eq. (5)**

Let us summarize the statement in the following proposition:

**Proposition 3.** The set \( Z = \{ Z_{i_1 \ldots i_n} \}_{i_1 \ldots i_n} \) uniquely determines \( \Sigma_{AB} \) if and only if Eq. (3) holds for any choices of \( i_1, \ldots, i_n \) and \( j_1, \ldots, j_n \).

Before we prove this proposition let us note a technical lemma, which will be useful in the following. It describes the most general solution of \( \omega_{i_1 \ldots i_n} \) which satisfy the relations demanded for a local hidden state model.

**Lemma 1.** Any collection of hidden states \( \omega_{i_1 \ldots i_n} \) which satisfies the set of linear equations given by Eq. (7) for \( \mathcal{E} \) can be written as \( \omega = \omega^{\text{spec}} + \omega^{\text{homo}} \). A special solution \( \omega^{\text{spec}} \) is given by \( \omega^{\text{spec}}_{i_1 \ldots i_n} = 0 \) for all indices \( i_1, \ldots, i_n \) except

\[ \omega^{\text{spec}}_{\alpha m \ldots m} = \rho_{\alpha 1}, \omega^{\text{spec}}_{\alpha m m \ldots m} = \rho_{\alpha 2}, \ldots, \omega^{\text{spec}}_{m m m m} = \rho_{\alpha n}, \]

for \( a < m \) and

\[ \omega^{\text{spec}}_{m m m} = \rho_{m | x} - (n - 1) \rho. \]

The general solution of the corresponding homogeneous system is given by

\[ \omega^{\text{homo}}_{i_1 \ldots i_n} = \sum_k v^{(k)}_{i_1 \ldots i_n} X_k \]

using arbitrary Hermitian operators \( X_k \). Here \( k = k_1 \ldots k_n \) is an \( n \)-length index similar to the subscripts of \( \omega \), where only the distinct possibilities with at least two \( k_i < m \) are considered. For a fixed \( k \) the vector \( v^{(k)} \) is given by

\[ v^{(k)}_{i_1 \ldots i_n} = \delta_{i_1 \ldots i_n, k_1 \ldots k_n} - \delta_{i_1 \ldots i_n, k_1 m \ldots m} - \ldots - \delta_{i_1 \ldots i_n, m m m} + (n - 1) \delta_{i_1 \ldots i_n, m m m}. \]

**Proof.** Note that Eq. (1) is a standard set of linear equations, except that we have Hermitian operators rather than scalar variables. Therefore all the basic linear algebra results apply.

In total we have \( m^n \) unknowns but only \( n(m + 1) + 1 \) linear independent relations recalling once more that \( \sum_x \rho_{a | x} = \rho \) is independent of the setting. Hence the general solution can be written as a combination of a special solution and the general solution of the homogeneous system \( \sum_{i} \delta_{i, a} \omega_{i_1 \ldots i_n} = 0 \).

That \( \omega^{\text{spec}} \) as given in the Lemma is a special solution can be checked straightforwardly. For the general solution of the homogeneous system \( \omega^{\text{homo}} \) note that via the
Ansatz of Eq. (19) this breaks down to the relation
\[ \sum_{i_1, \ldots, i_n} \delta_{i_x, a} v_{i_1 \ldots i_n}^{(k)} = 0. \] (21)

The dimension of this linear subspace is \( m^n - \left[ n(m - 1) + 1 \right] \), which is precisely the number of the considered \( k \)'s. Now first note that the given \( \{v^{(k)}\}_k \) are linearly independent, since vector \( v^{(k)} \) is the only vector which has a non-zero entry at the position \( i_1 \ldots i_n = k_1 \ldots k_n \). Thus we are left to show that they indeed solve Eq. (21). For the \( x = 1 \) and \( a < m \) this follows for instance by
\[ \sum_{i_2 \ldots i_n} v_{a_{i_2 \ldots i_n}}^{(k)} = \begin{cases} +1 & \text{if } k = a \\ -1 & \text{otherwise} \end{cases} = 0 \] (22)

if \( k_1 = a \), otherwise it holds trivially. The same arguments holds if one picks a different index \( i_x \). At last we still need to check the relation corresponding to reduced state, which is given by
\[ \sum_{i_1 \ldots i_n} v_{i_1 \ldots i_n}^{(k)} = \begin{cases} +1 & \text{if } k = 1 \\ -1 & \text{otherwise} \end{cases} = 0. \] (23)

which finishes the proof. \( \square \)

**Proof of Prop. 1.** Using the general solution \( \omega^{sol} \) as given the Lemma 1 in the operator \( \Sigma_{AB} \) one sees that
\[ \Sigma_{AB} = \sum_{i_1, \ldots, i_n} Z_{i_1, \ldots, i_n} \otimes \omega^{spec}_{i_1, \ldots, i_n} \]
\[ + \sum_k \left( \sum_{i_1, \ldots, i_n} v_{i_1, \ldots, i_n}^{(k)} Z_{i_1, \ldots, i_n} \right) \otimes X_k \] (24)
is uniquely determined by the given ensemble \( \mathcal{E} \) if and only if
\[ \sum_{i_1, \ldots, i_n} v_{i_1, \ldots, i_n}^{(k)} Z_{i_1, \ldots, i_n} = 0 \] (25)
holds for all possibilities \( k \). Using the explicit form of the vectors \( v^{(k)} \) as given in Eq. (20) these constraints can be re-written as
\[ Z_{k_1 \ldots k_n} = Z_{k_1 m \ldots m} + Z_{mk_2 \ldots m} + \ldots + Z_{m \ldots k_n} \]
\[ - (n-1) Z_{m \ldots m} \] (26)
for all admissible \( k_1 \ldots k_n \) with at least two \( k_i < m \). However, this condition also holds also for each \( k_1 \ldots k_n \) without this restriction, because then the vectors \( v^{(k)} \) in Eq. (20) vanish. Thus we have proven Eq. (20) for all \( i_1 \ldots i_n \), but only for the special index set \( j_1 \ldots j_n = m \ldots m \). Still, these conditions already imply the general (more symmetric looking) relation, using an arbitrary \( j_1 \ldots j_n \). This can be inferred more easily directly from the problem formulation by relabeling the individu-
basis elements.

This dual has a strictly feasible point \( Z_{i_1 \ldots i_n} = 1 > 0 \), noting \( \sum_{i_1 \ldots i_n} v_{i_1 \ldots i_n}^{(k)} = 0 \) was already proven in Lemma 1. Therefore we have strong duality, and consequently the statement that, whenever the primal problem is infeasible \((\mathcal{E} \text{ steerable})\) then there exists a sequence of appropriate \( Z_{i_1 \ldots i_n} \) such that \( C = \sum_{i_1 \ldots i_n} \text{tr}(Z_{i_1 \ldots i_n} \omega_{i_1 \ldots i_n}^{\text{spec}}) \) will tend to \(-\infty\), saying that Eq. (28) is unbounded.

Now let us interpret this as the detection statement of the proposition. That we labeled the dual variables by \( Z_{i_1 \ldots i_n} \) as also used in \( \Sigma_{AB} \) is no coincidence. Effectively the solutions \( Z_{i_1 \ldots i_n} \) of the dual program will be the ones used in the operator \( \Sigma_{AB} \) that shows steering. Note that the variables of the dual program already satisfy positivity \( Z_{i_1 \ldots i_n} \geq 0 \) and the linear relations in Eq. (28) uniquely determine \( \Sigma_{AB} = \sum_i Z_{i_1 \ldots i_n} \otimes \omega_{i_1 \ldots i_n}^{\text{spec}} \) as already shown in the proof of Prop. 3. Finally, note here the formal operator connection between \( \Sigma_{AB} \) and the objective function \( C \). Using the swap operator \( V \), i.e., \( \text{tr}(VA \otimes B) = \text{tr}(AB) \), one directly sees that the swap operator evaluated on \( \Sigma_{AB} \) gives the objective value \( \text{tr}(V \Sigma_{AB}) = C \). Since the swap operator \( V \) is an entanglement witness a negative \( \text{tr}(V \Sigma_{AB}) = C < 0 \) signals that the optimal \( \Sigma_{AB} \) has not a separable structure. This finishes the first part of the proof.

Proof. Part 2. It is left to show that we can also find a solution \( \mathcal{Z} \) which satisfies \( \text{tr}(\Sigma_{AB}) = 1 \), since such a condition does not appear in Eq. (28). Note that since the value of an objective function of any steerable ensemble will tend to \(-\infty\), there are for sure parameters \( \mathcal{Z} \) such that \( C < 0 \). Suppose that for these \( Z_{i_1 \ldots i_n} \), the operator \( \Sigma_{AB} \) is not normalized. If \( \text{tr}(\Sigma_{AB}) > 0 \), then one can directly use a rescaled version \( Z_{i_1 \ldots i_n} / \text{tr}(\Sigma_{AB}) \), now also satisfying the trace condition, but still detecting the state. Note that this trick fails if \( \text{tr}(\Sigma_{AB}) \leq 0 \), either due to a division by zero, or due to \( Z_{i_1 \ldots i_n} \) being not positive semidefinite anymore. Thus we are left to prove that \( \text{tr}(\Sigma_{AB}) > 0 \).

To verify \( \text{tr}(\Sigma_{AB}) \geq 0 \) we employ that \( C \geq 0 \) holds for any non-steerable ensemble. From the given ensemble \( \mathcal{E} \) such a non-steerable ensemble is for instance \( \hat{\mathcal{E}} = \{ \hat{\rho}_{a|x} = \text{tr}(\rho_{a|x}) I/d \} \), having a special solution \( \hat{\omega}_{i_1 \ldots i_n} = \text{tr}(\omega_{i_1 \ldots i_n}^{\text{spec}}) I/d \) as can be checked by Eqs. (17, 18). Thus evaluating the objective function of this non-steerable ensemble and the chosen selection \( \mathcal{Z} \) one finds

\[
\sum_{i_1 \ldots i_n} \text{tr}(Z_{i_1 \ldots i_n} \omega_{i_1 \ldots i_n}^{\text{spec}}) = \frac{1}{d} \sum_{i_1 \ldots i_n} \text{tr}(Z_{i_1 \ldots i_n} \omega_{i_1 \ldots i_n}^{\text{spec}}) = \frac{1}{d} \text{tr}(\Sigma_{AB}) \geq 0.
\]

Finally, we show that from \( \mathcal{Z} \) with \( C < 0 \) and \( \text{tr}(\Sigma_{AB}) = 0 \) it is always possible to find a different solution \( \tilde{\mathcal{Z}} \) with \( \tilde{C} > 0 \) but \( \text{tr}(\Sigma_{AB}) > 0 \) such that we can employ the rescaling trick again. Note first that the only negative part in the \( C \) must be due to \( \text{tr}(\Sigma_{m \ldots m} \omega_{m \ldots m}^{\text{spec}}) < 0 \), since all other terms involve only positive semidefinite operators. Now pick any \( \omega_{i_1 \ldots i_n}^{\text{spec}} \) with \( \text{tr}(\omega_{i_1 \ldots i_n}^{\text{spec}}) > 0 \), and assume this is \( \omega_{am \ldots m}^{\text{spec}} \) with \( a < m \). Then define the new set of operator

\[
\tilde{Z}_{am \ldots m} = Z_{am \ldots m} + \epsilon \mathbb{1}, \quad \tilde{Z}_{mam \ldots m}, \ldots, \tilde{Z}_{m \ldots m} = Z_{m \ldots m}
\]

which by Eq. (29) are enough to finally determine the set \( \mathcal{Z} \). This set still contains only positive semidefinite operators because the only operators that change are \( \tilde{Z}_{ai_1 \ldots i_n} = Z_{ai_1 \ldots i_n} + \epsilon \mathbb{1} \). For this new solution \( \mathcal{Z} \) we get \( \text{tr}(\Sigma_{AB}) = \epsilon \text{tr}(\omega_{am \ldots m}^{\text{spec}}) \) and \( \tilde{C} = C + \epsilon \text{tr}(\omega_{am \ldots m}^{\text{spec}}) \), thus choosing \( \epsilon \) small enough one obtains the given statement. This completes the proof.

Proof of Prop. 2

The ideas and bounding techniques are the same as in Ref. 33, which derived similar determinant constraints for the dimension-bounded entanglement verification; here we only need to apply them to a single side.

Proof. Inequality (14) is just a rearrangement of Eq. (11). We remark that the bound of \( T \) as given by Eq. (10) holds only if \( \{ B_i \}_{i=1} \) is linearly independent, which follows from the observation \( |\det(D)| \neq 0 \).

The first and stronger condition in Eq. (13) follows using the extra information of \( C \) that if both sets \( \{ G_k^A \}, \{ G_k^B \}_{i=1} \) have the identity in its linear span, then the largest singular value satisfies \( \sigma_0(C) \geq q = \text{tr}(1/\sqrt{d_A} \otimes 1/\sqrt{d_B} \Sigma_{AB}) = 1/\sqrt{d_B} \).

This follows from the fact that the ordered singular values of \( C \) are lower bounded by the ordered singular values of any submatrix \( C_{\text{sub}} \) of \( C \). While \( \{ G_k^B \}_{i=1} \) satisfies this extra condition automatically since \( B_0 = 1 \), we need this requirement for the choice of \( \{ G_k^A \}_{k} \).

Via this extra condition we can achieve a better bound using the inequality of arithmetic and geometric means only to \( n_B \) singular values and then checking whether the minimal value of \( \sigma_0(C) \) can be reached, more precisely one obtains

\[
\min_{\sigma_0(C) \geq q} \frac{\| C \|_1}{\sigma_0(C)} \geq \min_{\sigma_0(C) \geq q} \left[ \sigma_0(C) + \left( \frac{\| \det(C) \|}{\sigma_0(C)} \right)^{\frac{1}{n_B}} \right] \geq \left\{ \begin{array}{ll} (n_B + 1) \| \det(C) \|^{\frac{1}{n_B}} & \text{if } |\det(C)| \geq q \\ q + n_B \left( \frac{\| \det(C) \|}{q} \right)^{\frac{1}{n_B}} & \text{else} \end{array} \right.,
\]

depending on the determinant of \( C \). Note that both bounds are monotonically increasing functions. By the determinant rule \( |\det(C)| = |\det(D)||\det(T)| \) and the bound of Eq. (10), the possible values are constrained to
satisfy
\[ |\det(C)| \geq |\det(D)|d_B^{-\frac{n_B+1}{2}}. \]
(31)

Thus, depending on the value of $|\det(D)|$ the second bound in Eq. (30) can be used or not. If $|\det(D)|^{1/(n_B+1)} \geq 1/\sqrt{d_A}$ the determinant of $C$ will always satisfy the constraint in Eq. (30) and one obtains
\[
\min_{\sigma_0(C)} \|C\|_1 \geq n_B + \frac{1}{\sqrt{d_B}} |\det(D)|^{\frac{1}{n_B+1}}. \]
(32)

Otherwise one can split the possible region and minimize separately, yielding
\[
\min_{\sigma_0(C)} \|C\|_1 \geq \min \left\{ \frac{1}{\sqrt{d_A d_B}} + n_B \left( \frac{1}{\sqrt{d_A d_B}} |\det(D)|^{\frac{1}{n_B+1}} \right), n_B + \frac{1}{\sqrt{d_A d_B}} \right\}. \]
(33)

At last, if $n_B > \sqrt{d_A d_B} - 1$ note that the bound given by Eq. (32) and the second argument in minimum of Eq. (33) are strictly larger than 1. Thus only the first argument of Eq. (33) must be checked, which is the stated condition. This completes the proof.

**Steering scenario for $n = 2$ and $m = d$**

In this section we exemplify the construction of respective $Z = \{Z_{kl}\}_{kl}$ for the case of two settings but arbitrary number of outcomes. The idea and construction rely on Fourier connected mutually unbiased bases\footnote{33}. Thus we need a couple of definitions first.

Consider a Hilbert space $\mathbb{C}^d$ and suppose that one has a basis $\{|\phi_k\rangle\}_{k \in \mathbb{Z}_d}$ with $\mathbb{Z}_d = \{0, \ldots, d-1\}$, which we also use to label the outcomes. Then one obtains another basis, which is mutually unbiased, by the Fourier transform
\[
|\psi_k\rangle = \mathcal{F}|\phi_k\rangle = \frac{1}{\sqrt{d}} \sum_{i \in \mathbb{Z}_d} q^{i k} |\phi_i\rangle \quad (34)
\]
with $q = e^{2\pi i/d}$.

These two bases even admit further structure which becomes convenient in the following. Consider two representations $U,V$ of the cyclic group $\mathbb{Z}_d$ on $\mathcal{H}$ defined by its action onto the first basis, $U_x|\psi_k\rangle = |\psi_{k+x}\rangle$ and $V_y|\phi_k\rangle = q^{y k} |\phi_k\rangle$ for all $x,y,k$. These two representations further satisfy $U_x V_y = q^{-xy} V_y U_x$ and the Fourier transform is the intertwining map, $U_x \mathcal{F} = \mathcal{F} V_x^\dagger$ and $V_y \mathcal{F} = \mathcal{F} U_y$. Via this one can identify the action on both basis states that we summarize as
\[
U_x |\phi_k\rangle = |\phi_{k+x}\rangle, \quad U_x |\psi_k\rangle = q^{-x k} |\psi_k\rangle, \quad (35)
\]
\[
V_y |\phi_k\rangle = q^{y k} |\phi_k\rangle, \quad V_y |\psi_k\rangle = |\psi_{k+y}\rangle \quad (36)
\]
for all $x,y \in \mathbb{Z}_d$. Then the following set of operators will be our characterization of the steering inequality. The structure can be guessed once one knows the so-called mother observable for the respective joint measurability problem\footnote{33}, from whose result one further knows that the current form is optimal.

**Proposition 4.** Consider the set of operators $Z = \{Z_{kl} = U_k V_l Z_{00} V_l^\dagger U_k^\dagger\}$ with
\[
Z_{00} = \mu_1 |\chi_-\rangle \langle \chi_-| + \mu_2 (1 - |\chi_+\rangle \langle \chi_+| - |\chi_-\rangle \langle \chi_-|), \quad (37)
\]
pure states $|\chi_{\pm}\rangle \propto |\phi_0\rangle \pm |\psi_0\rangle$ and parameters
\[
\mu_1 = \frac{2}{\sqrt{d(\sqrt{d} - 1)(\sqrt{d} + 2)}}, \quad (38)
\]
\[
\mu_2 = \frac{1 + \sqrt{d}}{\sqrt{d(\sqrt{d} - 1)(\sqrt{d} + 2)}}. \quad (39)
\]

Then this set of operators can be used in the steering map, since all operators are positive semidefinite and uniquely determines the operator $\Sigma_{AB}$ and satisfies $\text{tr}(\Sigma_{AB}) = 1$.

**Proof.** Using the form of $Z_{00}$ as given by Eq. (37) one sees that $Z_{00}$ is positive semidefinite, since both $\mu_i$ are strictly positive and $|\chi_-\rangle$ and $|\chi_+\rangle$ are orthogonal, moreover it has unit trace. Since all other $Z_{kl}$ are obtained by a unitary transformation each $Z_{kl}$ is positive semidefinite and satisfies $\text{tr}(Z_{kl}) = 1$, which directly shows that $\Sigma_{AB}$ has unit trace. Thus we are left to show that $Z_{kl}$ uniquely determines $\Sigma_{AB}$, for which we have to show
\[
Z_{kl} = Z_{kt} + Z_{sl} - Z_{st} \quad (40)
\]
for all $k,l,s,t \in \mathbb{Z}_d$ according to Prop. 3. In order to show this we expand the states $|\chi_{\pm}\rangle$ in $Z_{00}$ which results into the structure
\[
Z_{00} = c_1 |\phi_0\rangle \langle \phi_0| + |\psi_0\rangle \langle \psi_0| + c_2 1 \quad (41)
\]
with appropriate coefficients $c_1,c_2$. Note that at this point the very specific choices of $\mu_1$ and $\mu_2$ become important; they are chosen such that cross terms of $|\phi_0\rangle \langle \psi_0|$ or $|\psi_0\rangle \langle \phi_0|$ vanish. Applying now the rules given by Eqs. (35) (36) one gets
\[
Z_{kl} = c_1 (|\phi_k\rangle \langle \phi_k| + |\psi_l\rangle \langle \psi_l|) + c_2 1 \quad (42)
\]
from which the necessary relation given by Eq. (40) can be verified.

In order to obtain a steering criterion one can use the given operators $Z_{kl}$ of the proposition to build up $\Sigma_{AB}$, which is uniquely determined by the given ensemble $\mathcal{E}$ in the $n = 2$ and $m = d$ steering case. Whenever this operator $\Sigma_{AB}$ is then not a separable state the underlying distribution is steerable.
Dimension-bounded steering in a loophole free experiment of Ref. [19]

Next let us explain how the developed criterion can be employed for the real setup used in Vienna [19]. The main difference is that in the actual experiment one additionally observes an inconclusive outcome “inc” due to no click or even double click events. On Bob’s side, the side which is at least partially trusted, this event can safely be discarded assuming that this event is independent of the measurement choice such that it can be viewed as a kind of filter telling whether the final result will be conclusive or not. Only if this filter succeeds one looks at the corresponding state. For those measurements (acting on the conditional state) the measurements are assumed to act on a qubit, respective single photon in two polarization modes. However for Alice, the uncharacterized side, this is not possible. In order to incorporate the inconclusive event for Alice we consider the case that each inconclusive outcome “inc” is randomly assigned to either of the +1 or −1 outcome. This is also the standard for Bell experiments. Then one is left with the dimension-bounded steering scenario considered in the main section.

To finally give an example of the strength of our developed criterion we employ the following model to simulate real data: For the quantum state we assume a noisy maximal criterion we employ the following model to simulate a kind of filter telling whether the final result will be conclusive or not. In order to incorporate the inconclusive outcome “inc” is randomly assigned to either side, this is not possible. In order to incorporate the inconclusive outcome “inc” is randomly assigned to either of the +1 or −1 outcome. This is also the standard for Bell experiments. Then one is left with the dimension-bounded steering scenario considered in the main section.

Putting these observations into the data matrix from the main text one obtains

\[
D = \begin{bmatrix}
\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & -\frac{\rho\lambda}{\sqrt{6}} & 0 & 0 \\
0 & 0 & 0 & -\frac{\rho\lambda}{\sqrt{6}} \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

which shows steering according to Eq. (10) if \( p\lambda > 1/\sqrt{3} \approx 0.577 \). Let us point out that this is also the condition if we would know that the performed measurements are perfect projective measurements in the eigenbasis of \( \sigma_1, \sigma_2, \sigma_3 \). Thus, we see that we have here a scenario where this further characterization is totally redundant and only the knowledge that one measures a qubit is essential.

Assuming the visibility and detection efficiency parameters from Ref. [19], one would obtain the values \{0.74, 0.73, 0.73\} for the respective \( \rho\lambda \), which are all well above the threshold. Assuming that all other correlations and marginals vanish, this would strongly show steering also in the case where one has only the very limited knowledge that the conclusive outcomes were qubit measurements. However, note, that these other observations are essential for the inequality, otherwise one could not gain the required extra knowledge of the uncharacterized qubit measurements. Unfortunately, these experimental data are not available anymore for the experiment of Ref. [19].

\[
(A_x B_y) = -\delta_{x,y} p\lambda, \quad (A_z) = (B_y) = 0. \tag{49}
\]

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