SC-REG: TRAINING OVERPARAMETERIZED NEURAL NETWORKS UNDER SELF-CONCORDANT REGULARIZATION

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ABSTRACT

In this paper we propose the SC-Reg (self-concordant regularization) framework for learning overparameterized feedforward neural networks by incorporating second-order information in the Newton decrement framework for convex problems. We propose the generalized Gauss-Newton with Self-Concordant Regularization (SCoRe-GGN) algorithm that updates the network parameters each time it receives a new input batch. The proposed algorithm exploits the structure of the second-order information in the Hessian matrix, thereby reducing the training computational overhead. Although our current analysis considers only the convex case, numerical experiments show the efficiency of our method and its fast convergence under both convex and non-convex settings, which compare favorably against baseline first-order methods and a quasi-Newton method.

1 Introduction

The results presented in this paper apply to the pseudo-online training of feedforward neural networks based on solving a regularized nonlinear least-squares problem under the assumption of strong convexity and Lipschitz continuity. Unlike first-order methods such as stochastic gradient descent (SGD) [2] and its variants [3, 4, 5, 6] that only make use of first-order information through the function gradients, second-order methods [7, 8, 9, 10, 11, 12, 13] use second-order information obtained from the Hessian matrix or the Fisher information matrix (FIM). It is well known that this generally provides second-order methods with better (quadratic) convergence than a typical first-order method which only converges linearly in the neighbourhood of the solution [14].

Contrary to the classical theory of data interpolation, overparameterization in learning (deep) neural networks is provably found necessary if we must achieve a (smooth) interpolation [15, 16, 17, 18]. However, compared to first-order methods for learning such neural networks, and despite their convergence advantage, learning the parameters of a neural network in the overparameterized setting with second-order methods result into highly prohibitive computations, namely, inverting an \(n_w \times n_w\) matrix at each training iteration, where \(n_w\) is the number of trainable parameters in the neural network. In most commonly used second-order methods, the natural gradient, Gauss-Newton, and the sub-sampled Newton [19] – or its regularized version [20] – used for incorporating second-order information, while maintaining desirable convergence properties, compute the Hessian matrix (or FIM) \(H\) by using the approximation \(H \approx J^T J\), where \(J\) is the Jacobian matrix. This approximation is still an \(n_w \times n_w\) matrix and remains computationally demanding for overparameterized deep neural networks. In recent works [21, 22, 23, 24], second-order methods for overparameterized models are made to bypass this difficulty by applying a matrix identity, and instead only compute the matrix \(JJ^T\) which is a \(d \times N \times d \times N\) matrix, where \(d \times N\) is typically much smaller than \(n_w\) when training an overparameterized neural network. This approach significantly reduces computational overhead for overparameterized models and helps to accelerate convergence [21]. Nevertheless, for an objective function with a differentiable (up to two times) convex regularizer, this simplification requires a closer attention and special modifications.

The idea of exploiting desirable properties of regularization for improving the convergence of the (Gauss-)Newton scheme has been around for decades and most of the published works on the topic combine in different ways the idea of Levenberg-Marquardt (LM) regularization, line search, and the trust region methods [25]. For example, the recent work [26] combines the idea of cubic regularization (originally proposed by [24]) and a particular variant of the adaptive LM penalty that uses the Euclidean gradient norm of the output-fit loss function (see [27] for a comprehensive list

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and complexity bounds). Their proposed scheme achieves a global $O(k^{-2})$ rate, where $k$ is the iteration number. [28] follows a similar idea using the Bregman distances, and extends the idea to developing an accelerated variant of the scheme that achieves a $O(k^{-3})$ rate.

In this paper, we propose a new self-concordant regularization (SC-Reg) scheme for efficiently training an overparameterized feedforward neural network involving smooth, strongly convex optimization objectives, where one of the objective functions regularizes the network’s parameter vector and hence avoid overfitting, ultimately improving the neural network’s ability to generalize well and make accurate predictions on new, previously unseen data [29]. By an extra assumption that the regularization function is self-concordant, we propose SCoRe-GGN algorithm (see Algorithm [1]) that updates the neural network parameters in the framework of the local norm $\| \cdot \|_x$ (a.k.a., Newton decrement) of a self-concordant function $f(x)$ such as seen in [30]. Our proposed scheme does not require that the output-fit loss function is self-concordant, which in many applications does not hold [20]. Instead, we exploit the greedy descent provision of self-concordant functions to achieve a fast convergence rate while maintaining feasible assumptions on the combined objective function (from an application point of view).

This paper is organized as follows: First, we introduce some notations and formulate the learning problem in §2. In §3, we derive a generalized method for reducing computational overhead when training an overparameterized network by the second-order method. The idea of SC-Reg is introduced in §4 and our SCoRe-GGN algorithm is presented thereafter. Experimental results that show the efficiency and fast convergence of the proposed method are presented in §5.

## 2 Notation and preliminaries

Let $\{(x_n, y_n)\}_{n=1}^N$ be a sequence of input and output sample pairs, $x_n \in \mathbb{R}^{n_x}$, $y_n \in \mathbb{R}^d$, where $n_x$ is the number of features and $d$ is the number of targets. Let the neural network model $f(\theta; x_n)$, defined by $f: \mathbb{R}^{n_w} \times \mathbb{R}^{n_x} \rightarrow \mathbb{Y}$, be parameterized by $\theta \in \mathbb{R}^{n_w}$ which is the vectorization of the set $\{W_1, b_1, \ldots, W_L, b_L\}$ of trainable parameters of the network, with $W_l \in \mathbb{R}^{d_l \times d_{l-1}}$, $b_l \in \mathbb{R}^{d_l}$, $l = 1, 2, \ldots, L$, and $L - 1$ is the number of hidden layers (or “depth”) of the network. If $z_l \in \mathbb{R}^{d_l}$ is the hidden state of layer $l$ computed through the recursive formula

$$z_0 = x_n; \quad z_l = \sigma(W_l z_{l-1} + b_l), \quad (1)$$

then the value $y_n \in \mathbb{R}^d$ of the prediction $f(\theta; x_n)$ of the neural network is given by the composition

$$f(\theta; x_n) \rightarrow f(\theta; x_n),$$

$$f(\theta; x_n) := \sigma_L(T_L \sigma_{L-1}(T_{L-1} \cdots \sigma_2(T_2 \sigma_1(T_1(x_n))))), \quad (2)$$

where $T_l = W_l z_{l-1} + b_l$, $l = 1, 2, \ldots, L$. The activation functions $\sigma_l(\cdot)$, $l = 1, 2, \ldots, L - 1$ determine the nonlinearity of the network and are applied element-wise, and $\sigma_L(\cdot)$ is a “squashing” activation function of the output layer $L$. We denote by $\partial_a b$ the gradient (or first derivative) of $b$ (with respect to $a$) and $\partial_{aa} b$ the second derivative of $b$ with respect to $a$. Wherever the subscripts do not appear, we assume derivatives with respect to the function variables. We write $\| \cdot \|$ to denote the 2-norm. Throughout the paper, bold face letters denote vectors and matrices. Any new notations that arise within the text will be clearly defined.

By training the neural network, we attempt to solve the regularized nonlinear least-squares problem

$$\min_{\theta} \mathcal{L}(\theta) := \sum_{n=1}^{N} \ell(y_n, \hat{y}_n) + \lambda \sum_{j=1}^{n_w} r_j(\theta_j), \quad (3)$$

where $\ell: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ is a (strongly) convex twice-differentiable output-fit loss function, $r_j: \mathbb{R} \rightarrow \mathbb{R}$, $j = 1, \ldots, n_w$, define a separable regularization term on $\theta$, $g(\theta): \mathbb{R}^{n_w} \rightarrow \mathbb{R}$, $h(\theta): \mathbb{R}^{n_w} \rightarrow \mathbb{R}$. We assume that the regularization function $h(\theta)$, scaled by the parameter $\lambda > 0$, is twice differentiable and strongly convex. Preliminary assumptions defining the regularity of the Hessian are as follows:

**Assumption 1.** The functions $g$ and $h$ are twice-differentiable with respect to $\theta$ and are respectively $\gamma_1$- and $\gamma_2$-strongly convex, that is, given the identity matrix $I_{n_w} \in \mathbb{R}^{n_w \times n_w}$, there exists some $\gamma_1, \gamma_2$ satisfying $0 \leq \gamma_1 \leq \gamma_2 < \infty$, $0 < \gamma_0 \leq \gamma_2 < \infty$ such that

$$\gamma_1 I_{n_w} \preceq \partial_{\theta\theta} g(\theta) \preceq \gamma_2 I_{n_w}, \quad \gamma_1 I_{n_w} \preceq \partial_{\theta\theta} h(\theta) \preceq \gamma_2 I_{n_w}, \quad (4a)$$

$$\gamma_0 I_{n_w} \preceq \partial_\theta g(\theta) \preceq \gamma_1 I_{n_w}, \quad (4b)$$

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and hence
\[(\gamma_l + \lambda \gamma_u) I_{nw} \preceq \partial^2_{\theta \theta} \mathcal{L}(\theta) \preceq (\gamma_u + \lambda \gamma_l) I_{nw}, \quad (5)\]
holds almost everywhere. This implies that \(\forall x \in \mathbb{R}^{n_r}, \forall y \in \mathbb{R}^d\), the gradient \(\partial_\theta \mathcal{L}(\theta) = \partial_\theta g(\theta) + \lambda \partial_\theta h(\theta)\) satisfies
\[\left\| \partial_\theta \mathcal{L}(y, f(\theta_1; x)) - \partial_\theta \mathcal{L}(y, f(\theta_2; x)) \right\| \leq (\gamma_u + \lambda \gamma_l) \left\| \theta_1 - \theta_2 \right\|, \quad (6)\]
for any \(\theta_1, \theta_2 \in \mathbb{R}^{nw}\). In other words, the gradient of \(\mathcal{L}(\theta)\) is \((\gamma_u + \lambda \gamma_l)\)-Lipschitz continuous \(\forall x \in \mathbb{R}^{n_r}, y \in \mathbb{R}^d\).

Assumption 2. \(\exists \gamma_g, \gamma_h \) with \(0 < \gamma_g, \gamma_h < \infty\) such that \(\forall x \in \mathbb{R}^{n_r}, \forall y \in \mathbb{R}^d\), the second derivatives of \(g(\theta)\) and \(h(\theta)\) respectively satisfy
\[\left\| \partial^2_{\theta \theta} g(y, f(\theta_1; x)) - \partial^2_{\theta \theta} g(y, f(\theta_2; x)) \right\| \leq \gamma_g \left\| \theta_1 - \theta_2 \right\|, \quad (7a)\]
\[\left\| \partial^2_{\theta \theta} h(y, f(\theta_1; x)) - \partial^2_{\theta \theta} h(y, f(\theta_2; x)) \right\| \leq \gamma_h \left\| \theta_1 - \theta_2 \right\|, \quad (7b)\]
for any \(\theta_1, \theta_2 \in \mathbb{R}^{nw}\). This implies
\[\left\| \partial^2_{\theta \theta} \mathcal{L}(y, f(\theta_1; x)) - \partial^2_{\theta \theta} \mathcal{L}(y, f(\theta_2; x)) \right\| \leq (\gamma_g + \lambda \gamma_h) \left\| \theta_1 - \theta_2 \right\|. \quad (8)\]

In other words, the second derivative of \(\mathcal{L}(\theta)\) is \((\gamma_g + \lambda \gamma_h)\)-Lipschitz continuous \(\forall x \in \mathbb{R}^{n_r}, y \in \mathbb{R}^d\).

Although the loss landscapes of neural networks are generally non-convex, one can build convex loss functions from neural networks under certain conditions, namely, by using the non-decreasing ReLU activation function \(\sigma(\cdot) = \max\{0, \cdot\}\) in the hidden layers of the network and “squashing” the outputs by a convex (in the parameters of the network) function in the output layer, so that the loss function formed from the network outputs may be convex \([31, 32, 33]\). Hence, the assumptions above imply \(\sigma_l(\cdot)\) defined in \([2]\) is a (strongly) convex function, and each \(\sigma_l(\cdot), l = 1, \ldots, L - 1\), is a non-decreasing convex function. Commonly used loss functions such as the squared loss, and the cross-entropy loss (with a sigmoidal output \(\hat{y}_n\)) are twice differentiable and (strongly) convex in the network parameters. Certain smoothed sparsity-inducing penalties such as the (pseudo-)Huber function – presented later in this paper – constitute the class of functions that may be well-suited for \(h(\theta)\) defined above.

The assumptions of strong convexity and smoothness about the objective \(\mathcal{L}(\theta)\) are standard conventions in many optimization problems as they help to characterize the convergence properties of the underlying solution method \([34, 35]\). However, the smoothness assumption about the objective \(\mathcal{L}(\theta)\) is sometimes not feasible for some multi-objective (or regularized) problems where a non-smooth (penalty-inducing) function \(h(\theta)\) is used, such as is presented in the recent work \([36]\). In such a case, and when the need to incorporate second-order information in their solution techniques arise, a well-known approach in the optimization literature is generally either to approximate the non-smooth objectives by a smooth quadratic function (when such an approximation is available) or use a “proximal splitting” method and replace the 2-norm in this setting with the \(Q\)-norm, where \(Q\) is the Hessian matrix or its approximation \([37]\). In \([37]\), the authors propose two techniques that help to avoid the complexity that is often introduced in subproblems when the latter approach is used. While proposing new approaches, \([38]\) highlights some popularly used techniques to handle non-differentiability. Each of these works highlight the importance of incorporating second-order information in the solution techniques of optimization problems. By conveniently solving the optimization problem \([2]\) where the assumptions made above are satisfied, our method ensures the full curvature information is captured while reducing computational overhead.

2.1 Approximate Hessian

**Definition 2.1** (Generalized Gauss-Newton Hessian). Let \(q_n \in \mathbb{R}^d\) be the second derivative of the loss function \(\ell(y_n, \hat{y}_n)\) with respect to the predictor \(y_n\), \(q_n = \partial^2_{y_n y_n} \ell(y_n, \hat{y}_n)\) for \(n = 1, 2, \ldots, N\), and let \(Q_g \in \mathbb{R}^{dN \times dN}\) be a block diagonal matrix with \(q_n\) being the \(n\)-th diagonal block. Let \(J_n \in \mathbb{R}^{d \times nw}\) denote the Jacobian of \(\hat{y}_n\) with respect to \(\theta\) for \(n = 1, 2, \ldots, N\), and let \(J_g \in \mathbb{R}^{dN \times nw}\) be the vertical concatenation of all \(J_n\)'s. Then, the generalized Gauss-Newton (GGN) approximation of the Hessian matrix \(H_g \in \mathbb{R}^{nw \times nw}\) associated with the fit loss \(\ell(y_n, \hat{y}_n)\) with respect to \(\theta\) is defined by
\[H_g \approx J_g^T Q_g J_g = \sum_{n=1}^{N} J_n^T \text{diag}(q_n) J_n. \quad (9)\]
Let \( e_n \in \mathbb{R}^d \) be the Jacobian of the fit loss defined by \( e_n = \partial y_n \ell(y_n, \hat{y}_n) \) for \( n = 1, 2, \ldots, N \). For example, in case of squared loss \( \ell(y_n, \hat{y}_n) = \frac{1}{2}(y_n - \hat{y}_n)^2 \) we get that \( e_n \) is the residual \( e_n = y_n - \hat{y}_n \). Let \( e_g \in \mathbb{R}^{dN} \) be the vertical concatenation of all \( e_n \)'s. Then using the chain rule, we write

\[
J_n^T e_n^T = \begin{bmatrix}
\partial_{\theta_1} \hat{y}^{(1)} & \partial_{\theta_1} \hat{y}^{(2)} & \cdots & \partial_{\theta_1} \hat{y}^{(d)} \\
\partial_{\theta_2} \hat{y}^{(1)} & \partial_{\theta_2} \hat{y}^{(2)} & \cdots & \partial_{\theta_2} \hat{y}^{(d)} \\
\vdots & \vdots & \ddots & \vdots \\
\partial_{\theta_{nw}} \hat{y}^{(1)} & \partial_{\theta_{nw}} \hat{y}^{(2)} & \cdots & \partial_{\theta_{nw}} \hat{y}^{(d)} 
\end{bmatrix}
\begin{bmatrix}
\partial y^{(1)} \\
\partial y^{(2)} \\
\vdots \\
\partial y^{(d)} 
\end{bmatrix}
= \begin{bmatrix}
\partial_{\theta_1} \ell^{(1)} + \partial_{\theta_1} \ell^{(2)} + \cdots + \partial_{\theta_1} \ell^{(d)} \\
\partial_{\theta_2} \ell^{(1)} + \partial_{\theta_2} \ell^{(2)} + \cdots + \partial_{\theta_2} \ell^{(d)} \\
\vdots \\
\partial_{\theta_{nw}} \ell^{(1)} + \partial_{\theta_{nw}} \ell^{(2)} + \cdots + \partial_{\theta_{nw}} \ell^{(d)} 
\end{bmatrix}^T,
\]

and

\[
g_g(\theta) = J_g^T e_g = \sum_{n=1}^N J_n^T e_n.\]

### 3 Pseudo-online Learning of Overparameterized Neural Networks

Suppose that at each iteration \( k \) of the neural network training we uniformly select a random index \( I_k \subset \{1, 2, \ldots, N\} \) and access a mini-batch of \( m \) samples from the training set. The loss derivatives used for the recursive update of the training parameters \( \theta \) in this way is computed at each iteration \( k \) of the training, and are estimated as running averages over the batch-wise computations. This leads to a stochastic approximation of the true derivatives at each iteration for which we assume unbiased estimations.

The problem of finding the optimal adjustment \( \delta \theta_{mk} := \theta_{k+1} - \theta_k \) that solves (3) results in solving either an overdetermined or an underdetermined linear system depending on whether \( dm \geq n_w \) or \( dm < n_w \), respectively. Consider, for example, the squared fit loss and the penalty-inducing square norm as the scalar-valued functions \( g(\theta) \) and \( h(\theta) \), respectively in (3). Then, \( Q_g \) will be the identity matrix, and the LM solution \( \delta \theta \) [39, 40] is estimated at each iteration \( k \) according to the rule

\[
\delta \theta = -(H_g + \lambda I)^{-1}g_g = -(J_g^T J_g + \lambda I)^{-1}J_g^T e_g.
\]

If the model is overparameterized and \( dm \ll n_w \), then by using the “Searle” identity \((AB + \lambda I)A = A(BA + \lambda I)\) [41], we can conveniently update the adjustment \( \delta \theta \) by

\[
\delta \theta = -J_g^T (J_g J_g^T + \lambda I)^{-1} e_g.
\]

Clearly, this provides a more computationally efficient way of solving for \( \delta \theta \). In what follows, we formulate a generalized solution method for the regularized problem [3] which similarly exploits the Hessian matrix structure when training an overparameterized neural network, thereby conveniently and efficiently computing the adjustment \( \delta \theta \).

Taking the second-order approximation of \( \mathcal{L}(\theta) \), we have

\[
\mathcal{L}(\theta + \delta \theta) \approx \mathcal{L}(\theta) + g^T \delta \theta + \frac{1}{2} \delta \theta^T H \delta \theta,
\]

where \( H \in \mathbb{R}^{n_w \times n_w} \) is the Hessian of \( \mathcal{L}(\theta) \) and \( g \in \mathbb{R}^{n_w} \) is its gradient. Let \( M = d(m + 1) \) and define the Jacobian \( J \in \mathbb{R}^{M \times n_w} \):

\[
J = \begin{bmatrix}
\partial_{\theta_1} \hat{y}_1 & \partial_{\theta_1} \hat{y}_2 & \cdots & \partial_{\theta_1} \hat{y}_m & \lambda \partial_{\theta_1} r_1 \\
\partial_{\theta_2} \hat{y}_1 & \partial_{\theta_2} \hat{y}_2 & \cdots & \partial_{\theta_2} \hat{y}_m & \lambda \partial_{\theta_2} r_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\partial_{\theta_{nw}} \hat{y}_1 & \partial_{\theta_{nw}} \hat{y}_2 & \cdots & \partial_{\theta_{nw}} \hat{y}_m & \lambda \partial_{\theta_{nw}} r_{nw}
\end{bmatrix}.
\]

\[^3\text{For simplicity of notation, and unless where the full notations are explicitly required, we shall subsequently drop the subscripts } m \text{ and } k, \text{ and assume that each expression represents stochastic approximations performed at iteration } k \text{ using randomly selected data batches each of size } m.\]
Let $e_M = 1$ and denote by $e \in \mathbb{R}^M$ the vertical concatenation of all $e_n$’s, $n = 1, 2, \ldots, m + 1$. Then by using the chain rule as in (10) and (11), we obtain
\[
g(\theta) = \partial_\theta \mathcal{L}(\theta) = J^T e. \tag{16}
\]
Let $q_n = \partial_{y_n}^2 \ell(y_n, \hat{y}_n)$, $q_n \in \mathbb{R}^d$ for $n = 1, 2, \ldots, m$ (clearly $q_n = 1$ in case of squared fit loss terms) and let $q_{m+1} = 0$. Define $Q \in \mathbb{R}^{M \times M}$ as the diagonal matrix with diagonal elements $q_n$ for $n = 1, 2, \ldots, m + 1$, where $Q = Q^T \succeq 0$ by convexity of $\ell$. Consider the following slightly modified GGN approximation of the Hessian $H \in \mathbb{R}^{n_w \times n_w}$ associated with $\mathcal{L}(\theta)$:
\[
H \approx J^T Q J + \lambda H_h, \tag{17}
\]
where $H_h$ is the Hessian of the regularization term $h(\theta)$, $H_h \in \mathbb{R}^{n_w \times n_w}$, and is a diagonal matrix whose diagonal terms are
\[
H_{h_{jj}} = \frac{d^2 r_j(\theta_j)}{d \theta_j^2}, \ j = 1, \ldots, n_w.
\]
We hold on to the notation $H$ to represent the modified GGN approximation of the full Hessian matrix $\partial^2 \mathcal{L}$. By differentiating (14) with respect to $\delta \theta$ and equating to zero, we obtain the optimal adjustment
\[
\delta \theta = -(J^T Q J + \lambda H_h)^{-1} J^T e. \tag{18}
\]

Remark 1. The inverse matrix in (18) exists due to the strong convexity assumption on the regularization function $h$ which makes $H_h \succeq (\min_j H_{h_{jj}}) I$ and therefore matrix $J^T Q J + \lambda H_h$ is symmetric positive definite and hence invertible.

Let $U = J^T Q$. Using the identity \[D - VA^{-1}B \succeq D^{-1}V(A - BD^{-1}V)^{-1} \]
with $D = \lambda H_h$, $V = -J^T$, $A = I_M$, and $B = U^T$, and recalling that $Q$ is symmetric, from (18) we get
\[
\delta \theta = \left(\lambda H_h - (-J^T) I_M U^T \right)^{-1} \left(-J^T\right) I_M^{-1} e = -\frac{1}{\lambda} H_h^{-1} J^T \left(I_M + U^T \frac{1}{\lambda} H_h^{-1} J^T \right)^{-1} e
\]
\[
= -H_h^{-1} J^T \left(\lambda I_M + Q J H_h^{-1} J^T \right)^{-1} e. \tag{19}
\]

Compared to (18), the clear advantage of the form (19) is that it requires the factorization of an $M \times M$ matrix rather than an $n_w \times n_w$ matrix. Given these modifications, we proceed by making an assumption that define the residual $e$ and the Jacobian $J$ in the region of convergence where we assume the starting point $\theta_0$ of the process lies.

Let $\theta^*$ be a nondegenerate minimizer of $\mathcal{L}$, and define $\mathcal{E}(\theta^*) := \{ \theta_k \in \mathbb{R}^{n_w} : \| \theta_k - \theta^* \| \leq \epsilon \}$, a closed ball of a sufficiently small radius $\epsilon > 0$ about $\theta^*$. We denote by $\mathcal{N}_i(\theta^*)$ an open neighbourhood of the sublevel set $\Gamma(\mathcal{L}) := \{ \theta_k : \mathcal{L}(\theta_k) \leq \mathcal{L}(\theta_0) \}$, so that $\mathcal{E}(\theta^*) = \text{cl}(\mathcal{N}_1(\theta^*))$. We then have $\mathcal{N}_i(\theta^*) := \{ \theta_k \in \mathbb{R}^{n_w} : \| \theta_k - \theta^* \| < \epsilon \}$.

Assumption 3. (i) Each $e_n(\theta_k)$ and each $q_n(\theta_k)$ is Lipschitz smooth, and for all $\theta_k \in \mathcal{N}_i(\theta^*)$ there exists $\nu > 0$ such that $\| J(\theta_k) z \| \geq \nu \| z \|$.\footnote{Subsequently, we shall omit the notation for the Hessian and gradient estimates as we assume unbiasedness.}

(ii) $\forall \theta_k \in \mathcal{N}_i(\theta^*)$, $\lim_{k \to \infty} \mathbb{E}_m[|H(\theta_k)| - \partial^2 \mathcal{L}(\theta_k)|] = 0$ almost surely whenever $\lim_{k \to \infty} \mathbb{E}_m[|g(\theta_k)|] = 0$, where $\mathbb{E}_m[\cdot]$ denotes the expectation with respect to $m$.

Remark 2. Assumption (3(ii)) implies the singular values of $J$ are uniformly bounded away from zero and $\exists \beta, \beta > 0$ such that $\| e \| \leq \beta \| J \| = \| J^T \| \leq \beta$, then as $Q \to 0$, we have $\exists K_1$ such that $Q \leq K_1 I$, and hence $\| I + Q J H_h^{-1} J^T \| \leq \lambda + (K/\gamma_0)$, where $K = K_1 \beta^2$. Note that although we use limits in Assumption (3(ii)), the assumption similarly holds in expectation by unbiasedness. Also, a sufficient sample size $m$ may be required for Assumption (3(ii)) to hold, by law of large numbers, see e.g. [44, Lemma 1, Lemma 2].

Remark 3. $H_g(\theta)$ and $H_h(\theta)$ respectively satisfy
\[
\begin{align*}
\gamma_i H_{i,w} & \preceq H_g(\theta_k) \preceq \gamma_i H_{i,w}, \quad \| H_g(y, f(\theta_1; x)) - H_g(y, f(\theta_2; x)) \| \leq \gamma_g \| \theta_1 - \theta_2 \|, \tag{20a} \\
\gamma_w H_{w,w} & \preceq H_h(\theta_k) \preceq \gamma_w H_{i,w}, \quad \| H_h(y, f(\theta_1; x)) - H_h(y, f(\theta_2; x)) \| \leq \gamma_h \| \theta_1 - \theta_2 \|, \tag{20b}
\end{align*}
\]
at any point $\theta_k \in \mathbb{R}^{n_w}$, for any $x \in \mathbb{R}^{n_f}$, $y \in \mathbb{R}^d$, and for any $\theta_1, \theta_2 \in \mathcal{N}_i(\theta^*)$ where Assumption 3 holds.
We now state a convergence result for the update type (18) (and hence (19)). First, we define the second-order optimality condition and state two useful lemmas.

**Lemma 3.1 (Second-order sufficiency condition (SOSC)).** Let \( \theta^* \) be a local minimum of a twice-differentiable function \( \mathcal{L}(\cdot) \). Then second-order sufficiency condition (SOSC) states:

\[
\partial \mathcal{L}(\theta^*) = 0, \quad \partial^2 \mathcal{L}(\theta^*) \succ 0. \tag{SOSC}
\]

**Theorem 3.3.** Suppose that Assumptions 1, 2, and 3 hold, and that \( \mathcal{L}(\theta) \) is twice-continuously differentiable. Let \( S \subset \mathbb{W} \) be an open interval containing some \( \theta^* \), and suppose that \( \mathcal{L}(\theta^*) \) satisfies SOSC. Then, there exists \( \mathcal{L}^* = \mathcal{L}(\theta^*) \) satisfying

\[
\mathcal{L}(\theta_k) > \mathcal{L}^* \quad \forall \theta_k \in S. \tag{21}
\]

**Remark 4.** Lemma 3.2 and the first part of SOSC ensure that the second part of Assumption 3 (ii) always hold. In essence, it holds at every point \( \theta_k \) of the sequence \( \{\theta_k\} \) generated by the process (19) as long as we choose a starting point \( \theta_0 \in \mathcal{B}_r(\theta^*) \).

**Theorem 3.3.** Suppose that Assumptions 1, 2, and 3 hold, and that \( \theta^* \) is a local minimizer of \( \mathcal{L}(\theta) \) for which the assumptions in Lemma 3.1 hold. Let \( \{\theta_k\} \) be the sequence generated by the process (19). Then starting from a point \( \theta_0 \in \mathcal{N}_r(\theta^*) \), \( \{\theta_k\} \) converges at a \( Q \)-quadratic rate. Namely:

\[
\|\theta_{k+1} - \theta^*\| \leq \xi_k \|\theta_k - \theta^*\|^2,
\]

where

\[
\xi_k = \frac{1}{2} \left( \gamma_g + b \gamma_h \right) \frac{1}{\gamma_t + a \gamma_a - (\gamma_g + b \gamma_h) \|\theta_k - \theta^*\|}.
\]

The proofs of these results are provided in Appendix A and Appendix B.

4 **Self-concordant Regularization (SC-Reg)**

In the overparameterized setting, one could deduce by mere visual inspection that the matrix \( H_h^{-1} \) in (19) plays an important role in the perturbation of \( \theta \) within the region of convergence due to its “dominating” structure in the equation. This may be true as it appears. But beyond this “naive” view, let us derive a more technical intuition about the update step (19) by using a similar analogy as that given in [14, Chapter 5]. We have

\[
\theta_{k+1} = \theta_k - H_h^{-1} J^T \left( \lambda I + Q J H_h^{-1} J^T \right)^{-1} e, \tag{22}
\]

where \( I \) is the identity matrix of suitable dimension. By some simple algebra (see e.g., [14, Lemma 5.1.1]), one can show that indeed the update method (22) is affine-invariant. In other words, the region of convergence \( \mathcal{N}_r(\theta^*) \) does not depend on the problem scaling but on the local topological structure of the minimization functions \( g(\theta) \) and \( h(\theta) \) [14].

Consider the non-augmented form of (22): \( \theta_{k+1} = \theta_k - Q_g J_g^T G_g^{-1} e_g \), where \( G_g = J_g J_g^T \) is the so-called Gram matrix. It has been shown that indeed when training an overparameterized neural network, and as long as we start from an initial point \( \theta_0 \) close enough to the minimizer \( \theta^* \) of \( \mathcal{L}(\theta) \) (assuming that such a minimizer exists), both \( J_g \) and \( G_g \) remain stable throughout the training process (see e.g., [21, 45, 46]). The term \( Q_g \) may have little to no effect on this notion, for example, in case of the squared fit loss, \( Q_g \) is just an identity term. However, the original Equation (22) involves the Hessian matrix \( H_h \) which, together with its bounds (namely, \( \gamma_a, \gamma_b \) characterizing the region of convergence), changes rapidly when we scale the problem [14]. It therefore suffices to impose an additional assumption on the function \( h(\theta) \) that will help to control the rate at which its second-derivative \( H_h \) changes, thereby enabling us to characterize an affine-invariant structure for the region of convergence. Namely:

**Assumption 4 (SC-Reg).** The scaled regularization function \( \lambda h(\theta) \) has a third derivative and is \( M_h \)-self-concordant. That is, the inequality

\[
\left| \langle u, \partial^3 h(\theta) | u \rangle u \right| \leq 2 M_h \left( \langle u, \partial^2 h(\theta) u \rangle \right)^{3/2}, \tag{23}
\]

holds for any \( \theta \) in the closed convex set \( \mathbb{W} \subseteq \mathbb{R}^{n_w} \) and \( u \in \mathbb{R}^{n_w} \).
Then starting from a point $\theta_0$, data $\{(x_n, y_n)\}_{n=1}^m, H, Q, J$ (see (17)), $e$ (see (16)), parameters $\alpha, M_\lambda, \gamma$.

**Algorithm 1 SCoRe-GGN**

1: **Input:** weights vector $\theta_k$, data $\{(x_n, y_n)\}_{n=1}^m, H, Q, J$ (see (17)), $e$ (see (16)), parameters $\alpha, M_\lambda, \gamma$.
2: **Output:** weights vector $\theta_{k+1}$
3: Compute $g_h = \partial h(\theta_k)$
4: Choose $\eta_k = \langle g_h, H^{-1}g_h \rangle^{1/2}$
5: Set $\rho_k = \frac{\alpha_k}{1 + M_\lambda \eta_k}$
6: Set $G = H^{-1}J^T \left( \lambda I + QJH^{-1}J^T \right)^{-1} e$
7: Compute $\theta_{k+1} = \theta_k - \rho_k G$

Here, $\partial^3 h(\theta)[u] \in \mathbb{R}^{n \times n \times n}$ denotes the limit

$$\partial^3 h(\theta)[u] := \lim_{t \to 0} \frac{1}{t} \left( \partial^2(\theta + tu) - \partial^2(\theta) \right), \quad t \in \mathbb{R}.$$ 

For a detailed analysis of self-concordant functions, we refer the interested reader to [30] [14].

Given this extra assumption, and following the idea of Newton decrement in [30], we propose to update $\theta$ by

$$\delta(\theta) = -\frac{\alpha_k}{1 + M_\lambda \eta_k} H^{-1}J^T \left( \lambda I + QJH^{-1}J^T \right)^{-1} e,$$

(24)

where $\alpha_k > 0, \eta_k = \langle g_h, H^{-1}g_h \rangle^{1/2}$ and $g_h$ is the gradient of $h(\theta)$. The proposed method which we call SCoRe-GGN is summarized, for one oracle call, in Algorithm 1.

Typical examples of penalty-inducing functions that satisfy Assumption 3 either in their scaled form [14] Corollary 5.1.3 or in their original form are the (pseudo-)Huber function [47] parameterized by $\mu > 0$ [48, 49]:

$$h_\mu(\theta) := \sqrt{\mu^2 + \theta^2} - \mu,$$

and the $l^2$-norm function [50]

$$h_2(\theta) = \|\theta\|_2 := \sum_{i=1}^{n_w} |\theta_i|^2.$$  

(25)

(26)

For the sake of completeness, we state here that the functions $h_\mu(\theta)$ and $h_2(\theta)$ satisfy all the assumptions on $h(\theta)$ in Assumptions 1, 2, 3 and 4. For details on the convexity and smoothness results, and the bounds of the Hessian of the pseudo-Huber function, we refer the interested reader to [51, 52].

In the following, we state a local convergence result for the process (24). We introduce the notation $\|\Delta\|_q$, representing the local norm of a direction $\Delta$ taken with respect to the local Hessian of a self-concordant function $h$, $\|\Delta\|_q := \langle \Delta, \partial^2 h(\theta) \Delta \rangle^{1/2} = \left\| \partial^2 h(\theta) \right\|^{1/2} \Delta$.

**Theorem 4.1.** Let Assumptions 1, 2, 3, and 4 hold, and let $\theta^*$ be a local minimizer of $\mathcal{L}(\theta)$ for which the assumptions in Lemma 3.1 hold. Let $\{\theta_k\}$ be the sequence generated by Algorithm 1 with $\alpha_k = \alpha = \frac{\gamma_2}{M_\lambda} (K + \lambda \gamma_a)$, $\beta_1 = \beta \beta_2$. Then starting from a point $\theta_0 \in N_{M_\lambda, \gamma}(\theta^*)$, $\{\theta_k\}$ converges to $\theta^*$ according to the following descent properties:

$$E[\mathcal{L}(\theta_{k+1})] \leq \mathcal{L}(\theta_k) - \left( \frac{\lambda}{M_\lambda^2} \omega(\zeta_k) + \frac{\gamma_2}{2\gamma_a} \omega''(\zeta_k) - \xi \right),$$

$$E\|\theta_{k+1} - \theta^*\|_{\theta_{k+1}} \leq \psi(\|\theta_k - \theta^*\|_{\theta_k} + \frac{\gamma_2}{\beta_1} \|\theta_k - \theta^*\| + \frac{\gamma_2}{2} \|\theta_k - \theta^*\|^2,.$$

where $\omega(\cdot)$ is an auxiliary univariate function defined by $\omega(t) := t - \ln(1 + t)$ and has a second derivative $\omega''(t) = 1/(1 + t)^2$, and

$$\zeta_k := \frac{M_\lambda}{1 + M_\lambda \eta_k}, \quad \tilde{\zeta}_k := M_\lambda \eta_k, \quad \xi := \frac{2(\gamma_2 + \lambda \gamma_b)}{\sqrt{\gamma_a}},$$

$$\psi := 1 + \frac{\lambda}{\sqrt{\gamma_a \beta_1} (1 - M_\lambda \|\theta_k - \theta^*\|_{\theta_k})}.$$
The proof is provided in Appendix B. The results in Theorem 4.1 combine strong convexity and smoothness properties of both \( g(\theta) \) and \( h(\theta) \), and requires that only \( h(\theta) \) is self-concordant.

5 Experiments

In this section, we validate the efficiency of SCoRe-GGN (Algorithm 1) on a binary classification task. We use a real dataset: \( ijcnn \) from the LIBSVM repository \([53]\), with an 80:20 train:test split. We fit the nonlinear, convex model using the ReLU activation function in the network hidden layers and the cross-entropy fit loss function \( \ell(y_n, \hat{y}_n) = \frac{1}{2} \sum_{n=1}^{N} y_n \log \left( \frac{1}{\hat{y}_n} \right) + (1 - y_n) \log \left( \frac{1}{1-\hat{y}_n} \right) \) with a sigmoidal output \( \hat{y}_n \), and the “deviance” residual \([54]\) \( e_n = (-1)^{y_n+1} \sqrt{-2 \left[ y_n \log(\hat{y}_n) + (1 - y_n) \log(1 - \hat{y}_n) \right]} \), \( y_n, \hat{y}_n \in \{0, 1\} \). We perform a feature mapping using the radial basis function (RBF) kernel \( K(x_n, x'_n) = \exp(-\gamma \|x_n - x'_n\|^2) \) with \( \gamma = 50.0 \). The model built with the resulting input space contains a total of \( n_w = 28,005 \) trainable weights, and we train with a batch size of 4096. In all experiments, we choose \( \lambda = 0.1 \). For the regularization function \( h(\theta) \), we experiment with both \( h_{\mu}(\theta) \) and \( h_2(\theta) \) defined respectively in (25) and (26) with \( \mu = 1.0 \). We choose \( \alpha_k = \alpha = 1.0 \) and assume a scaled self-concordant regularization so that \( M_h = 1 \) (see [14] Corollary 5.1.3). We compare results, under the same (regularized) settings, from solving \([5] \) using the SGD, Adam \([5] \), and the quasi-Newton Limited-memory Broyden-Fletcher-Goldfarb-Shanno (LBFGS) \([55] \) algorithms with a fixed learning rate of 0.01.

Figure 1: Training curves

(a) Convex model

(b) Non-convex model
Table 1: Classification accuracies (on test dataset).

| REGULARIZER | SC-REG (SGD) | SC-REG (ADAM) | SC-REG (LBFGS) | SC-REG (SCoRe-GGN) | SCoRe-GGN generalizes best? |
|-------------|--------------|---------------|----------------|---------------------|----------------------------|
| CONVEX      | PSEUDO-HUBER | 90.27%        | 90.27%         | 90.27%              | 93.06%                     | ✓                          |
|             | ℓ²           | 90.27%        | 90.27%         | 90.27%              | 93.06%                     | ✓                          |
| NON-CONVEX  | PSEUDO-HUBER | 90.27%        | 90.27%         | 90.27%              | 93.10%                     | ✓                          |
|             | ℓ²           | 90.27%        | 90.27%         | 90.27%              | 93.14%                     | ✓                          |

To investigate how well the trained model generalizes, we use the binary accuracy metric which measures how often the model predictions match the true labels when presented with new, previously unseen data:

\[
\text{Accuracy} = \frac{1}{N} \sum_{n=1}^{N} \left( 2y_n\hat{y}_n - y_n\hat{y}_n + 1 \right).
\]

The LBFGS is implemented with PyTorch [56] (v. 1.9.0) in the full-batch mode. The rest of the methods are implemented using the open-source Keras API with TensorFlow [57] backend (v. 2.5.0). All experiments are performed on a laptop with dual (1.90GHz + 2.11GHz) Intel Core i7-8665U CPU and 16GB RAM.

As displayed in Figure [1] and Table [1], SCoRe-GGN converges way faster (in terms of number of epochs) and generalizes better than all three SGD, Adam, and the LBFGS algorithms. Although SCoRe-GGN comes with a higher computational cost (in terms of wall-clock time per epoch) than the first-order methods, in the presence of a GPU to speed up computations, and if training time is not provided as a bottleneck, this may not pose an extreme issue as we only need to pass the trainer on the dataset three times (epochs) to obtain an optimal solution in our experiments.

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Appendices

A Useful Results

Lemma A.1. Let Assumptions 1 and 2 hold. Let \( \theta_1, \theta_2 \) be any two points in \( \mathbb{R}^n \). Then we have

\[
\|g(\theta_1) - g(\theta_2) - \langle H(\theta_2), \theta_1 - \theta_2 \rangle \| \leq \frac{\gamma_g + \lambda \gamma_h}{2} \| \theta_1 - \theta_2 \|^2, \tag{27}
\]

\[
\left| \mathcal{L}(\theta_1) - \mathcal{L}(\theta_2) - \langle g(\theta_2), \theta_1 - \theta_2 \rangle - \frac{1}{2} \langle H(\theta_2)(\theta_1 - \theta_2), \theta_1 - \theta_2 \rangle \right| \leq \frac{\gamma_g + \lambda \gamma_h}{6} \| \theta_1 - \theta_2 \|^3. \tag{28}
\]

Proof. Fix \( \theta_1, \theta_2 \in \mathbb{R}^n \). Then

\[
g(\theta_1) - g(\theta_2) = \int_0^1 \partial^2 \mathcal{L}(\theta_2 + \tau(\theta_1 - \theta_2))(\theta_1 - \theta_2) d\tau.
\]

As \( \tau \in [0, 1] \), we have \( \theta_2 + \tau(\theta_1 - \theta_2) \in \mathbb{R}^n \). Hence writing \( \partial^2 \mathcal{L} = H \), we have

\[
g(\theta_1) - g(\theta_2) = \int_0^1 H(\theta_2 + \tau(\theta_1 - \theta_2))(\theta_1 - \theta_2) d\tau
\]

\[
g(\theta_1) - g(\theta_2) - H(\theta_2)(\theta_1 - \theta_2) = \int_0^1 (H(\theta_2 + \tau(\theta_1 - \theta_2)) - H(\theta_2))(\theta_1 - \theta_2) d\tau
\]

\[
\|g(\theta_1) - g(\theta_2) - H(\theta_2)(\theta_1 - \theta_2)\| = \int_0^1 \| (H(\theta_2 + \tau(\theta_1 - \theta_2)) - H(\theta_2))(\theta_1 - \theta_2) d\tau
\]

\[
\leq \int_0^1 \| (H(\theta_2 + \tau(\theta_1 - \theta_2)) - H(\theta_2))(\theta_1 - \theta_2) \| d\tau
\]

\[
\leq \int_0^1 \| H(\theta_2 + \tau(\theta_1 - \theta_2)) - H(\theta_2) \| \cdot \| \theta_1 - \theta_2 \| d\tau
\]

\[
\leq \int_0^1 \tau (\gamma_g + \lambda \gamma_h) \| \theta_1 - \theta_2 \|^2 d\tau
\]

\[
= \frac{\gamma_g + \lambda \gamma_h}{2} \| \theta_1 - \theta_2 \|^2.
\]

The proof of (28) follows immediately using a similar procedure (see e.g., [58]).

Corollary A.1.1. Let Assumptions 1 and 2 hold. Let \( \theta_1, \theta_2 \) be any two points in \( \mathbb{R}^n \). Then by writing \( \partial^2 \mathcal{L} = H \),

\[
H(\theta_2) - (\gamma_g + \lambda \gamma_h) \| \theta_1 - \theta_2 \| \leq H(\theta_1) \leq H(\theta_2) + (\gamma_g + \lambda \gamma_h) \| \theta_1 - \theta_2 \|. \tag{29}
\]

Proof. The proof follows immediately by recalling for any \( \theta \in \mathbb{R}^n \), \( H(\theta) \) is positive definite, and hence the eigenvalues \( s_j \) of the difference \( H(\theta_1) - H(\theta_2) \) satisfy

\[
|s_j| \leq (\gamma_g + \lambda \gamma_h) \| \theta_1 - \theta_2 \|, \quad j = 1, 2, \ldots, n_w,
\]

so that we have

\[
-(\gamma_g + \lambda \gamma_h) \| \theta_1 - \theta_2 \| \leq H(\theta_1) - H(\theta_2) \leq (\gamma_g + \lambda \gamma_h) \| \theta_1 - \theta_2 \|.
\]

Lemma A.2 ([14], Theorem 2.1.5]). Let the first derivative of a function \( \Phi(\cdot) \) be \( L \)-Lipschitz on \( \text{dom}(\Phi) \). Then for any \( \theta_1, \theta_2 \in \text{dom}(\Phi) \), we have

\[
0 \leq \Phi(\theta_1) - \Phi(\theta_2) - \langle \partial \Phi(\theta_2), \theta_1 - \theta_2 \rangle \leq \frac{L}{2} \| \theta_2 - \theta_1 \|^2.
\]
We remark that if the first derivative of the function \( \psi \) is \( \gamma_{sc} \)-strongly convex on \( \text{dom}(\psi) \), we have
\[
\psi(\theta_1) \geq \psi(\theta_2) + \langle \partial \psi(\theta_2), \theta_1 - \theta_2 \rangle + \frac{\gamma_{sc}}{2} \| \theta_1 - \theta_2 \|^2, \quad \gamma_{sc} > 0.
\]

We remark that if the first derivative of the function \( \psi \) in the above definition is \( L \)-Lipschitz continuous, then by construction, for any \( \theta_1, \theta_2 \in \text{dom}(\psi) \), \( \psi \) satisfies the Lipschitz constraint
\[
|\psi(\theta_1) - \psi(\theta_2)| \leq L\| \theta_1 - \theta_2 \|.
\]

**Lemma A.3** (Theorem 5.1.8)]. Let the function \( \phi \) be \( M_\phi \)-self-concordant. Then, for any \( \theta_1, \theta_2 \in \text{dom}(\phi) \), we have
\[
\phi(\theta_1) \geq \phi(\theta_2) + \langle \partial \phi(\theta_2), \theta_1 - \theta_2 \rangle + \frac{1}{M_\phi} \omega(\| \theta_1 - \theta_2 \|),
\]
where \( \omega(\cdot) \) is an auxiliary univariate function defined by \( \omega(t) := t - \ln(1 + t) \).

**Lemma A.4** (Corollary 5.1.5)]. Let the function \( \phi \) be \( M_\phi \)-self-concordant. Let \( \theta_1, \theta_2 \in \text{dom}(\phi) \) and \( r = \| \theta_1 - \theta_2 \|_{\theta_1} < \frac{1}{\lambda_{\phi}} \). Then
\[
\left(1 - M_\phi r + \frac{1}{3} M_\phi^2 r^2\right) \partial^2 \phi(\theta_1) \leq \int_0^1 \partial^2 \phi(\theta_2 + t(\theta_1 - \theta_2)) dt \leq \frac{1}{1 - M_\phi r} \partial^2 \phi(\theta_1).
\]

### B Missing Proofs

**Proof of Lemma 3.2** By the arguments of Remark 1, we have that the matrix \( (J(\theta_k)^T QQ J(\theta_k) + \lambda H_k(\theta_k))^{-1} \) is positive definite. Hence, with \( g(\theta_k) \neq 0 \), we have \( gH(\theta_k)^{-1} g(\theta_k) = -\delta \theta^T g(\theta_k) > 0 \) and \( \delta \theta^T g(\theta_k) < 0 \).

**Proof of Theorem 3.3** The process formulated in (19) performs the update
\[
\theta_{k+1} = \theta_k - H(\theta_k)^{-1} g(\theta_k).
\]
As \( g(\theta^*) = 0 \) by mean value theorem and the first part of (SOSC), we have
\[
\theta_{k+1} - \theta^* = \theta_k - \theta^* - H(\theta_k)^{-1} \int_0^1 \partial^2 L(\theta^* + \tau(\theta_k - \theta^*)) d\tau
\]
\[
= \left[I - H(\theta_k)^{-1} \int_0^1 \partial^2 L(\theta^* + \tau(\theta_k - \theta^*)) d\tau\right] (\theta_k - \theta^*)
\]
\[
= H(\theta_k)^{-1} \int_0^1 \left( H(\theta_k) - \partial^2 L(\theta^* + \tau(\theta_k - \theta^*)) \right) d\tau (\theta_k - \theta^*)
\]
\[
\| \theta_{k+1} - \theta^* \| = \left\| H(\theta_k)^{-1} \int_0^1 \left( H(\theta_k) - \partial^2 L(\theta^* + \tau(\theta_k - \theta^*)) \right) d\tau (\theta_k - \theta^*) \right\|
\]
\[
\leq \left\| H(\theta_k)^{-1} \right\| \int_0^1 \left\| H(\theta_k) - \partial^2 L(\theta^* + \tau(\theta_k - \theta^*)) \right\| d\tau \| \theta_k - \theta^* \|
\]
Also, as \( \tau \in [0, 1] \) we have that \( \theta^* + \tau(\theta_k - \theta^*) \in \mathcal{B}_{s}(\theta^*) \subseteq \mathbb{R}^n \). By taking the limit \( \lim_{k \to \infty} \| \theta_{k+1} - \theta^* \| \) implicitly, we have by the first part of (SOSC) and Assumption 3(ii)
\[
\| \theta_{k+1} - \theta^* \| \leq \left\| H(\theta_k)^{-1} \right\| \int_0^1 \left\| H(\theta_k) - H(\theta^* + \tau(\theta_k - \theta^*)) \right\| d\tau \| \theta_k - \theta^* \|
\]
\[
\leq \left\| H(\theta_k)^{-1} \right\| \int_0^1 (1 - \tau)(\gamma_g + \lambda \gamma_h) \| \theta_k - \theta^* \| d\tau \| \theta_k - \theta^* \|
\]
\[
= \frac{\gamma_g + \lambda \gamma_h}{2} \left\| H(\theta_k)^{-1} \right\| \| \theta_k - \theta^* \|^2.
\]
By combining the claims in Corollary [A.1] and the bounds of $H(\theta_k)$ in (??), we obtain the relation

$$\gamma_l + a\gamma_a - (\gamma_g + \lambda\gamma_h) \|\theta_k - \theta^*\| \leq H(\theta^*) - (\gamma_g + \lambda\gamma_h) \|\theta_k - \theta^*\| \leq H(\theta_k).$$

Recall that $H(\theta_k)$ is positive definite, and hence invertible. We deduce that, indeed for all $\theta_k$ satisfying $\|\theta_k - \theta^*\| \leq \epsilon$, $\epsilon$ small enough, we have

$$\left\| H(\theta_k)^{-1} \right\| \leq (\gamma_l - \gamma_g - \lambda(\gamma_h - \gamma_a) \|\theta_k - \theta^*\|)^{-1}.$$  

Therefore,

$$\|\theta_{k+1} - \theta^*\| \leq \xi_k \|\theta_k - \theta^*\|^2,$$

where

$$\xi_k = \frac{1}{2} \frac{\gamma_l + \lambda\gamma_h}{(\gamma_l - \gamma_g - \lambda(\gamma_h - \gamma_a) \|\theta_k - \theta^*\|)}.$$ 

**Proof of Theorem 4.1** First, we upper bound the norm $\|\delta\theta|_{\theta_k} := \|\theta_{k+1} - \theta_k\|_{\theta_k} = \|H^{1/2}_h(\theta_{k+1} - \theta_k)\|$. From Remark 3 we have

$$\|J(\theta_k)^T(\lambda I + QJH^{-1}_hJ^T(\theta_k))^{-1}\| \leq \frac{\gamma_a \|J(\theta_k)^T\|}{K + \lambda\gamma_a} \leq \frac{\tilde{\beta}\gamma_a}{K + \lambda\gamma_a}.$$ 

Hence,

$$\|\theta_{k+1} - \theta_k\|_{\theta_k} \leq \frac{\alpha_k}{1 + M_h\eta_k} \left\| H^{1/2}_h H^{-1}_h J^T(\theta_k)(\lambda I + QJH^{-1}_hJ^T(\theta_k))^{-1} e(\theta_k) \right\| \leq \frac{\alpha_k}{1 + M_h\eta_k} \left\| H_h(\theta_k) \right\| \left\| H^{-1}_h(\theta_k) \right\| \left\| J(\theta_k)^T(\lambda I + QJH^{-1}_hJ^T(\theta_k))^{-1} \right\| e(\theta_k) \right\| \leq \frac{\alpha_k\beta\tilde{\beta}\gamma_a^{-1/2}}{(K + \lambda\gamma_a)(1 + M_h\eta_k)}.$$ 

Similarly, we have

$$\|\theta_{k+1} - \theta_k\| \leq \frac{\alpha_k\beta\tilde{\beta}}{(K + \lambda\gamma_a)(1 + M_h\eta_k)}.$$ 

By Lemma [A.2] the function $L(\theta)$ satisfies

$$L(\theta_{k+1}) \leq L(\theta_k) + \left\langle \delta L(\theta_k), \theta_{k+1} - \theta_k \right\rangle + \frac{\gamma_u + \lambda\gamma_b}{2} \|\theta_{k+1} - \theta_k\|^2.$$ 

By convexity of $g$ and $h$, and self-concordance of $h$, we have (using Definition [A.1] Lemma [A.3] and (30))

$$L(\theta_{k+1}) \leq L(\theta_k) + (\frac{\gamma_u + \lambda\gamma_b}{2} \|\theta_{k+1} - \theta_k\|^2 - \frac{\lambda M_h^2\omega(M_h\|\theta_{k+1} - \theta_k\|)}{2} \left( \frac{\alpha_k\beta\tilde{\beta}\gamma_a^{-1/2}M_h}{(K + \lambda\gamma_a)(1 + M_h\eta_k)} \right))^2.$$ 

Substituting the choice $\alpha_k = (\beta\tilde{\beta})^{-1}(\gamma_a)^{1/2}(K + \lambda\gamma_a)$, we have

$$L(\theta_{k+1}) \leq L(\theta_k) + (\frac{\gamma_u + \lambda\gamma_b}{2} \|\theta_{k+1} - \theta_k\|^2 - \frac{\lambda M_h^2\omega(M_h\|\theta_{k+1} - \theta_k\|)}{2} \left( \frac{M_h}{1 + M_h\eta_k} \right))^2.$$ 

Taking expectation on both sides with respect to $m$ conditioned on $\theta_k$, we get

$$\mathbb{E}[L(\theta_{k+1})] \leq L(\theta_k) + (\frac{\gamma_u + \lambda\gamma_b}{2} \|\theta_{k+1} - \theta_k\|^2 - \mathbb{E} \left[ \frac{\lambda M_h^2\omega}{2} \left( \frac{M_h}{1 + M_h\eta_k} \right) \right],$$
Note the second derivative $\omega''$ of $\omega$: $\omega''(t) = 1/(1 + t)^2$. By the convexity of $\omega$ and using Jensen’s inequality, also recalling unbiasedness of the derivatives,

$$
\mathbb{E}[\mathcal{L}(\theta_{k+1})] \leq \mathcal{L}(\theta_k) + \frac{\gamma_u + \lambda \gamma_b}{\sqrt{\gamma_a}(1 + M_h \eta_k)} + \frac{\gamma_u + \lambda \gamma_b}{2 \gamma_a(1 + M_h \eta_k)^2} - \frac{\gamma_u}{2 \gamma_a} \omega''(M_h \eta_k) - \frac{\lambda}{M^2_h} \omega \left( \frac{M_h}{1 + M_h \eta_k} \right)
$$

$$
\leq \mathcal{L}(\theta_k) - \left[ \frac{\lambda}{M^2_h} \omega \left( \frac{M_h}{1 + M_h \eta_k} \right) + \frac{\gamma_u}{2 \gamma_a} \omega''(M_h \eta_k) - \frac{2(\gamma_u + \lambda \eta)}{\sqrt{\gamma_a}} \right].
$$

In the above, we used the bounds of the norm in (32), and again used the choice $\alpha_k = (\beta \tilde{\beta})^{-1}(\gamma_a)^{1/2}(K + \lambda \gamma_a)$.

To proceed, let us make a simple remark that is not explicitly stated in Remark 3: For any $\theta_k, \theta_{k+1} \in \mathcal{N}(\theta^*)$, we have

$$
\left\| H_g(\theta_{k+1}) - H_g(\theta_k) \right\| \leq \gamma_g \left\| \theta_1 - \theta_2 \right\| \to 0 \text{ as } k \to \infty,
$$

$$
\left\| H_h(\theta_{k+1}) - H_h(\theta_k) \right\| \leq \gamma_h \left\| \theta_1 - \theta_2 \right\| \to 0 \text{ as } k \to \infty.
$$

Now, recall the proposed update step (24):

$$
\theta_{k+1} = \theta_k - \frac{\alpha_k}{1 + M_h \eta_k} H^{-1} J^T \left( \lambda I + Q J H^{-1} J^T \right)^{-1} e.
$$

Then the above remark allows us to perform the following operation: Subtract $\theta^*$ from both sides and multiply by $H_h^{1/2}(\theta_{k+1}) \approx H_h^{1/2}(\theta_k) = H_h^{1/2}$. We get the recursion

$$
H_h^{1/2}(\theta_{k+1} - \theta^*) = H_h^{1/2}(\theta_k - \theta^*) - \frac{\alpha_k}{1 + M_h \eta_k} H_h^{1/2} H^{-1} J^T \left( \lambda I + Q J H^{-1} J^T \right)^{-1} e(\theta_k)
$$

$$
\left\| H_h^{1/2}(\theta_{k+1} - \theta^*) \right\| = \left\| H_h^{1/2}(\theta_k - \theta^*) - \frac{\alpha_k}{1 + M_h \eta_k} H_h^{1/2} H^{-1} J^T \left( \lambda I + Q J H^{-1} J^T \right)^{-1} e(\theta_k) \right\|
$$

$$
\leq \left\| H_h^{1/2}(\theta_k - \theta^*) \right\| + \frac{\alpha_k}{1 + M_h \eta_k} \left\| H_h^{1/2} H^{-1} J^T \left( \lambda I + Q J H^{-1} J^T \right)^{-1} e(\theta_k) \right\|.
$$

Take expectation with respect to $m$ on both sides conditioned on $\theta_k$ and again consider unbiasedness of the derivatives. Further, recall the definition of the local norm $\| \cdot \|_{\theta, \phi}$, and the bounds of $H_h$, then

$$
\mathbb{E} \left\| \theta_{k+1} - \theta^* \|_{\theta_{k+1}} \right\|_{\theta_{k+1}} \leq \left\| \theta_k - \theta^* \|_{\theta_k} \right\|_{\theta_k} + \frac{\alpha_k}{\sqrt{\gamma_a}(1 + M_h \eta_k)} \left\| \left( \lambda I + Q J H^{-1} J^T \right)^{-1} \right\|_{H_h^{-1} J^T e(\theta_k)}
$$

$$
\leq \left\| \theta_k - \theta^* \|_{\theta_k} \right\|_{\theta_k} + \frac{\alpha_k \gamma_a}{\sqrt{\gamma_a}(1 + M_h \eta_k)(K + \lambda \gamma_a)} \left\| H_h^{-1} g(\theta_k) \right\|
$$

$$
= \left\| \theta_k - \theta^* \|_{\theta_k} \right\|_{\theta_k} + \frac{\gamma_a}{\beta \gamma_a} \left\| H_h^{-1} g(\theta_k) + \lambda H_h^{-1} h(\theta_k) \right\|
$$

$$
\leq \left\| \theta_k - \theta^* \|_{\theta_k} \right\|_{\theta_k} + \frac{\gamma_a}{\beta \gamma_a} \left\| H_h^{-1} g(\theta_k) \right\| + \frac{\lambda \gamma a}{\beta \gamma_a} \left\| H_h^{-1} h(\theta_k) \right\|.
$$

By the mean value theorem and the first part of (SOSC),

$$
g_g(\theta_k) = \int_0^1 H_g(\theta^* + \tau(\theta_k - \theta^*))((\theta_k - \theta^*)d\tau
$$

$$
= H_g(\theta^*)(\theta_k - \theta^*) + \int_0^1 (H_g(\theta^* + \tau(\theta_k - \theta^*)) - H_g(\theta^*))((\theta_k - \theta^*)d\tau,
$$

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where $H_g$ is the second derivative of $g$.

\[
\|g_g(\theta_k)\| = \left\| H_g(\theta^*)(\theta_k - \theta^*) + \int_0^1 (H_g(\theta^* + \tau(\theta_k - \theta^*)) - H_g(\theta^*)) (\theta_k - \theta^*) d\tau \right\|
\]

\[
\leq \|H_g(\theta^*)\| \|\theta_k - \theta^*\| + \int_0^1 \|H_g(\theta^* + \tau(\theta_k - \theta^*)) - H_g(\theta^*)\| \|\theta_k - \theta^*\| d\tau
\]

\[
= \|H_g(\theta^*)\| \|\theta_k - \theta^*\| + \int_0^1 \tau \gamma_g \|\theta_k - \theta^*\|^2 d\tau
\]

\[
= \|H_g(\theta^*)\| \|\theta_k - \theta^*\| + \frac{\gamma_g}{2} \|\theta_k - \theta^*\|^2 d\tau
\]

\[
\leq \gamma_u \|\theta_k - \theta^*\| + \frac{\gamma_g}{2} \|\theta_k - \theta^*\|^2.
\]

In the above steps, we have used Assumption 3 and the remarks that follow it. Further,

\[
\|H_h^{-1} g_h(\theta_k)\| = \|H_h^{-1} g_h(\theta_k)\|
\]

\[
\leq \frac{1}{\gamma_a} \left( \gamma_u \|\theta_k - \theta^*\| + \frac{\gamma_g}{2} \|\theta_k - \theta^*\|^2 \right)
\]

\[
= \frac{\gamma_u}{\gamma_a} \|\theta_k - \theta^*\| + \frac{\gamma_g}{2\gamma_a} \|\theta_k - \theta^*\|^2.
\]

Next, we analyze $\|H_h^{-1} g_h(\theta_k)\|$. We have

\[
\|H_h^{-1} g_h(\theta_k)\| = \|H_h^{-3/2} H_h^{1/2} g_h(\theta_k)\|
\]

\[
\leq \frac{1}{\gamma_a} \left( \gamma_u \|\theta_k - \theta^*\| + \frac{\gamma_g}{2} \|\theta_k - \theta^*\|^2 \right)
\]

\[
= \frac{\gamma_u}{\gamma_a} \|\theta_k - \theta^*\| + \frac{\gamma_g}{2\gamma_a} \|\theta_k - \theta^*\|^2.
\]

and by the mean value theorem,

\[
\|H_h^{-1} g_h(\theta_k)\| = \frac{1}{\gamma_a} \left( \int_0^1 H_h^{1/2}(\theta_k) H_h(\theta^* + \tau(\theta_k - \theta^*)) (\theta_k - \theta^*) d\tau \right)
\]

\[
\leq \frac{1}{\gamma_a} \left( \int_0^1 H_h^{1/2}(\theta_k) (\theta_k - \theta^*) \right) \left( \int_0^1 H_h(\theta^* + \tau(\theta_k - \theta^*)) d\tau \right)
\]

\[
= \frac{1}{\gamma_a} \left( \|\theta_k - \theta^*\| \theta_k \right) \left( \int_0^1 H_h(\theta^* + \tau(\theta_k - \theta^*)) d\tau \right)
\]

\[
\leq \frac{1}{\gamma_a} \left( \|\theta_k - \theta^*\| \theta_k \right) \left( \int_0^1 H_h^{1/2}(\theta_k) H_h(\theta^* + \tau(\theta_k - \theta^*)) H_h^{1/2}(\theta_k) d\tau \right)
\]

\[
\leq \frac{1}{\gamma_a} \left( \|\theta_k - \theta^*\| \theta_k \right) \left( \int_0^1 \gamma_a \|\theta_k - \theta^*\|^2 d\tau \right)
\]

\[
\leq \frac{\gamma_a}{1 - M_h \|\theta_k - \theta^*\| \theta_k}
\]

In the above, we have used the fact $\theta_0 \in \mathcal{N}_{M_h^{-1}}(\theta^*) \implies \theta_k \in \mathcal{N}_{M_h^{-1}}(\theta^*)$ for all $\theta_k$ generated by the process (24).

Combining the above results, we have

\[
\mathbb{E} \|\theta_{k+1} - \theta^*\|_{\theta_k} \leq \left[ 1 + \frac{\lambda_a^{-1/2}}{\beta \beta} \left( 1 - M_h \|\theta_k - \theta^*\| \theta_k \right) \right] \|\theta_k - \theta^*\|_{\theta_k} + \frac{\gamma_u}{\beta \beta} \|\theta_k - \theta^*\| + \frac{\gamma_g}{2} \|\theta_k - \theta^*\|^2.
\]