Probability of consensus in the multivariate Deffuant model on finite connected graphs

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Abstract The Deffuant model is a spatial stochastic model for the dynamics of opinions in which individuals are located on a connected graph representing a social network and characterized by a number in the unit interval representing their opinion. The system evolves according to the following averaging procedure: pairs of neighbors interact independently at rate one if and only if the distance between their opinions does not exceed a certain confidence threshold, with each interaction resulting in the neighbors’ opinions getting closer to each other. All the mathematical results collected so far about this model assume that the individuals are located on the integers. In contrast, we study the more realistic case where the social network can be any finite connected graph. In addition, we extend the opinion space to any bounded convex subset of a normed vector space where the norm is used to measure the level of disagreement or distance between the opinions. Our main result gives a lower bound for the probability of consensus. Interestingly, our proof leads to a universal lower bound that depends on the confidence threshold, the opinion space (convex subset and norm) and the initial distribution, but not on the size or the topology of the social network.

1. Introduction

This paper is concerned with opinion dynamics on connected graphs. The first and most popular stochastic model in this topic is the voter model, introduced independently in [5, 13]. The main mechanism in the voter model is social influence, the tendency of individuals to become more similar when they interact. More precisely, individuals located on the vertex set of a connected graph (traditionally the $d$-dimensional integer lattice) are characterized by one of two competing opinions, and update their opinion at rate one by simply mimicking one of their neighbors chosen uniformly at random. Using a duality relationship between the voter model and a system of coalescing random walks, it can be proved that the process on the infinite square lattice clusters in one and two dimensions whereas opinions coexist at equilibrium in higher dimensions [13]. While mathematicians studied analytically various aspects of the model such as the asymptotics for the cluster size in one and two dimensions [3, 7], the spatial correlations at equilibrium in higher dimensions [2], and the occupation time of the process [6], social scientists and statistical physicists developed and studied numerically more realistic models of opinion dynamics. We refer to [10, 23] for reviews of the main results about the voter model, and to [4] for a review of more recent stochastic models of opinion dynamics introduced by applied scientists.

Apart from social influence, an important component of opinion dynamics is homophily, the tendency to interact more frequently with individuals who are more similar. The most popular spatial model that includes social influence and homophily is probably the Axelrod model [1] where individuals are now characterized by a vector of cultural features, and interact with their neighbors at a rate proportional to the number of features they share (homophily), which results in the two neighbors having one more feature in common (social influence). For a mathematical treatment

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of the Axelrod model, we refer to \cite{13, 17, 18, 21, 22}. Other spatial stochastic models of opinion dynamics include homophily in the form of a confidence threshold: individuals interact with their neighbors on the graph if and only if the level of disagreement between the two individuals before the interaction does not exceed a certain threshold. The simplest such model is the constrained voter model \cite{25}, the voter model with three opinions (leftist, centrist and rightist) where leftists and rightists do not interact. Extensions of this model where the opinion space takes the form of a finite connected graph and the level of disagreement is measured using the geodesic distance on this graph were introduced and studied analytically in \cite{20, 24}. The Deffuant model \cite{8} and the Hegselmann-Krause model \cite{10} are two other important spatial stochastic models that include social influence and homophily in the form of a confidence threshold.

In the original version of the Deffuant model \cite{8}, individuals are located on a general finite connected graph representing a social network and characterized by opinions that are initially chosen independently and uniformly at random in the unit interval. Pairs of neighbors interact at rate one if and only if the distance between their opinions before the interaction does not exceed a confidence threshold $\tau$ (homophily), which results in the two neighbors’ opinions getting closer to each other after the interaction (social influence). Because \cite{8} is purely based on numerical simulations, the authors only considered specific social networks: the complete graph and the two-dimensional torus. Their simulations on large graphs suggest the following conjecture for the infinite system: the process exhibits a phase transition at the critical threshold one-half in that a consensus is reached when $\tau > 1/2$ whereas disagreements persist in the long long when $\tau < 1/2$. This conjecture was first established in \cite{15} for the process on the integers using a combination of probabilistic and geometric techniques while a slightly stronger result was proved shortly after in \cite{9} using a different approach for part of the proof. The existence of a phase transition along with lower and upper bounds for the critical threshold were also proved for two variants of the model: a multivariate version where the opinion space is a finite-dimensional vector space and certain metrics are used to quantify the disagreement between individuals \cite{11, 12}, and a discrete version called the vectorial Deffuant model also introduced in \cite{8} where the opinion space is the hypercube and the disagreement between individuals is quantified using the Hamming distance \cite{19}.

In this paper, we study a version of the model where both the opinion space and the social network are fairly general. The opinion space is a bounded convex subset of a finite-dimensional normed vector space (where the norm is used to measure the disagreements). Convexity is in fact a necessary assumption following from the model’s evolution rules because future opinions must be on the segment connecting past opinions. More importantly, while \cite{9, 11, 12, 15, 19} assume that the individuals are located on the integers, we follow \cite{8} by assuming more realistically that the individuals are located on the vertex set of a general finite connected graph, meaning any possible real-world social networks. But unlike \cite{8} that relies on numerical simulations and therefore can only look at a few specific graphs, our results apply to all possible finite connected graphs. Due to the finiteness of the graph, the existence of a phase transition at a specific critical threshold no longer holds, and we instead derive a general lower bound for the probability of consensus. Interestingly, while our bound depends on the choice of the opinion space (convex subset and norm), it is uniform in all possible choices of the social network.

2. Model description and main results

The two key components of the model studied in this paper are the social network on which the individuals are located and the opinion space. To define these two components,
we let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a finite connected graph and

- we let $\Delta \subset \mathbb{R}^d$ be a bounded convex subset and $\| \cdot \|$ be a norm on $\mathbb{R}^d$.

The multivariate Deffuant model is a continuous-time Markov chain whose state at time $t$ is a configuration of opinions on the graph:

$$\xi_t : \mathcal{V} \to \Delta \quad \text{where} \quad \xi_t(x) = \text{opinion at vertex } x \text{ at time } t.$$

Following all the previous works in this topic, we assume that the process starts from a constant product measure, meaning that the initial opinions $\xi_0(x), x \in \mathcal{V}$, are independent and identically distributed, and we let $X$ be the random variable with distribution $P(X \in B) = P(\xi_0(x) \in B)$ for all $x \in \mathcal{V}$ and all Borel subsets $B \subset \mathcal{V}$.

The evolution rules are based on two parameters: the confidence threshold $\tau > 0$ and the convergence parameter $\mu \in (0, 1/2)$. The edges become independently active at rate one, which results in a potential update of the system at the two vertices connected by the active edge. More precisely, assuming that edge $(x, y) \in \mathcal{E}$ is active at time $t$, we let

$$\xi_t(x) = \xi_{t-}(x) + \mu (\xi_{t-}(y) - \xi_{t-}(x)) \mathbf{1}\{|\xi_{t-}(x) - \xi_{t-}(y)| \leq \tau\}$$

$$\xi_t(y) = \xi_{t-}(y) + \mu (\xi_{t-}(x) - \xi_{t-}(y)) \mathbf{1}\{|\xi_{t-}(x) - \xi_{t-}(y)| \leq \tau\}$$

while the opinions at the other vertices remain unchanged. In words, neighbors interact at rate one if and only if their opinion distance or level of disagreement before the interaction does not exceed the confidence threshold $\tau$, which results in a partial averaging of their opinions by a factor $\mu$. Note that the model is well-defined because the probability that different edges become active simultaneously is equal to zero.

Our main result gives a lower bound for the probability of consensus that applies to any finite connected graph, any opinion space (convex set and norm), and any initial distribution with value in the opinion space. To state this result, we let

$$d = \sup_{a, b \in \Delta} \|a - b\| \quad \text{and} \quad c \in \Delta \quad \text{such that} \quad \sup_{a \in \Delta} \|a - c\| = d/2$$

be the diameter and the center of the convex set $\Delta$, respectively.

**Theorem 1 (probability of consensus)** – For all $\tau > d/2$,

$$P(\mathcal{C}) \geq 1 - \frac{E \|X - c\|}{\tau - d/2} \quad \text{where} \quad \mathcal{C} = \left\{ \lim_{t \to \infty} \sup_{x, y \in \mathcal{V}} \|\xi_t(x) - \xi_t(y)\| = 0 \right\}.$$

The key to proving the theorem is to study a collection of auxiliary processes (see (2) below) that keep track of the cumulative disagreement between a fixed opinion $c \in \Delta$ and the opinions at each of the vertices at time $t$. Using a triangle-type inequality (Lemma 2), we first prove that all these auxiliary processes are almost surely nonincreasing, meaning that, for all $c$, the averaging procedure can only decrease the overall level of disagreement between an observer with fixed opinion $c$ and the population (Lemma 3). Almost sure monotonicity implies two important results:

1. The opinion model converges almost surely to a (random) limiting configuration.
In addition, due to the evolution rules, each limiting configuration is characterized by a partition of the graph into connected components such that all the individuals in the same component share the same opinion and the distance between opinions in two adjacent components exceeds the confidence threshold $\tau$ (Lemma 6).

2. All the auxiliary processes are bounded supermartingales.

In particular, one may apply the optional stopping theorem to these supermartingales and a certain stopping time (Lemma 7) to obtain a lower bound for the probability that the random partition above consists of only one set, meaning that all the individuals in the limiting configuration share the same opinion and consensus occurs.

Interestingly, our proof leads to a lower bound that depends on the confidence threshold, the opinion space (convex set and norm) and the initial distribution, but not on the size or the topology of the social network. The probability of consensus, however, depends on the choice of the network so our lower bound is a universal bound that is uniform over all possible choices of the network.

To illustrate our result, we now give two numerical examples where the lower bound in the theorem can be computed explicitly. In both examples, we let $\| \cdot \|$ be any norm on the vector space $\mathbb{R}^d$ and assume that the set of opinions is the ball

$$\Delta = B(c, r) = \{ a \in \mathbb{R}^d : \| a - c \| < r \} \quad \text{where} \quad c \in \mathbb{R}^d \quad \text{and} \quad r > 0.$$ 

In particular, the norm used to define the set of opinions is the same as the norm used to measure the distance between the opinions. In our first example, we assume that the opinions are initially uniformly distributed over the opinion set ($X = \text{Uniform}(\Delta)$), while in our second example, we assume that the initial distribution is of the form

$$P(\xi_0(x) \in B) = \frac{\int_B (r - \| a - c \|) \, d\lambda(a)}{\int_\Delta (r - \| a - c \|) \, d\lambda(a)}$$

(1)

for all vertices $x \in \mathcal{V}$ and all Borel sets $B \subset \Delta$. In both examples, $c$ can be viewed as the centrist opinion. The initial opinions are closer to this centrist opinion in the second example than in the first example. Using the theorem, we get the following explicit lower bounds.

Example 1 – Assume $X = \text{Uniform}(\Delta)$. Then, $P(\mathcal{C}) \geq 1 - dr/(d + 1)(\tau - r)$.

Example 2 – Assume (1). Then, $P(\mathcal{C}) \geq 1 - dr/(d + 2)(\tau - r)$.

The rest of the paper is devoted to proofs. In the next section, we show that the opinion model converges almost surely to a (random) limiting configuration in which neighbors either share the same opinion or disagree too much to interact. Then, we use the optional stopping theorem for supermartingales to derive the universal lower bound for the probability of consensus. Finally, we compute the lower bound explicitly for our two examples.

3. Limiting configurations

The objective of this section is to prove that, regardless of the initial configuration, the process converges almost surely to a limiting configuration in which any two neighbors either share the same opinion or disagree too much to interact, i.e.,

(P1) $\lim_{t \to \infty} \xi_t(x) = \xi_\infty(x)$ exists for all $x \in \mathcal{V}$

(P2) $\| \xi_\infty(x) - \xi_\infty(y) \| \notin (0, \tau]$ for all edges $(x, y) \in \mathcal{E}$. 
From now on, we let \( (X^c_t) \) be the process defined by

\[
X^c_t = \sum_{x \in V} \|\xi_t(x) - c\| \quad \text{for all} \quad c \in \mathbb{R}^d.
\] (2)

That is, the process keeps track of the cumulative disagreement between a fixed opinion \( c \) possibly outside \( \Delta \) and the opinions at each of the vertices. To shorten the notation, we also let

\[
\phi : \Delta \times \Delta \rightarrow \Delta \quad \text{defined as} \quad \phi(a, b) = (1 - \mu)a + \mu b = a + \mu(b - a).
\]

In particular, whenever a vertex \( x \) that has opinion \( a \) interacts with a vertex \( y \) that has a compatible opinion \( b \in B(a, \tau) \), meaning that the distance between the two opinions does not exceed \( \tau \), the opinion at \( x \) becomes \( \phi(a, b) \) and the opinion at \( y \) becomes \( \phi(b, a) \). Although the details are somewhat more complicated, the basic idea to prove the two properties above is to show that the processes \( (X^c_t) \) converge almost surely. To begin with, we prove the following lemma which is illustrated in Figure 1 and gives two variations of the triangle inequality.

**Lemma 2 (triangle inequalities)** – For all \( a, b \in \Delta \) and all \( c \in \mathbb{R}^d \),

\[
\|\phi(a, b) - c\| + \|\phi(b, a) - c\| \leq |a - c| + |b - c|
\]

\[
\|\phi(a, b) - c\| + \|\phi(b, a) - c\| \leq |a - c| + |b - c| - 2\|\phi(a, b) - a\| + \|a + b - 2c\|.
\]

**Proof.** Using the triangle inequality and absolute homogeneity, we get
the jumps at each vertex get smaller and smaller as time goes to infinity.

to see the change of opinions so it is not clear how to deduce convergence of the system. To prove
this order. For some norms, the lack of alignment is not even a sufficient condition for the process
be used later with the optional stopping theorem to derive our universal lower bound for the prob-
ability of consensus. By the martingale convergence theorem, each of these processes converges
almost surely to a finite random variable, which suggests almost sure convergence of the interacting
particle system. The main difficulty to prove this result is that whenever two vertices with compat-
able opinions interact, the process \((X^c_t)\) does not “see the update” when \(a, b, c\) are aligned in
this order. For some norms, the lack of alignment is not even a sufficient condition for the process
to see the change of opinions so it is not clear how to deduce convergence of the system. To prove
this result rigorously, we now use Lemma \(2\) and the second inequality in Lemma \(2\) to show that
the jumps at each vertex get smaller and smaller as time goes to infinity.

\[
\|\phi(a, b) - c\| + \|\phi(b, a) - c\| = \| (1 - \mu) a + \mu b - c \| + \| (1 - \mu) b + \mu a - c \|
\]
\[
= \| (1 - \mu) (a - c) + \mu (b - c) \| + \| (1 - \mu) (b - c) + \mu (a - c) \|
\]
\[
\leq \| (1 - \mu) (a - c) \| + \| \mu (b - c) \| + \| (1 - \mu) (b - c) \| + \| \mu (a - c) \|
\]
\[
= \| a - c \| + \| b - c \|
\]

which proves the first inequality. Now, because \(0 < \mu \leq 1/2\),

\[
a, \quad \phi(a, b), \quad c_0 = (a + b)/2, \quad \phi(b, a), \quad b
\]

are aligned in this order, so using again the triangle inequality, we get

\[
\|\phi(a, b) - c\| + \|\phi(b, a) - c\| \leq \|\phi(a, b) - c_0\| + \|\phi(b, a) - c_0\| + \|c_0 - c\|
\]
\[
= \|\phi(a, b) - \phi(b, a)\| + 2\|c_0 - c\|
\]
\[
= \|a - b\| - |\phi(a, b) - a| - |\phi(b, a) - b| + 2\|c_0 - c\|
\]
\[
\leq |a - c| + |b - c| - 2(|\phi(a, b) - a| + |a + b - 2c|)
\]

which proves the second inequality. This completes the proof. \(\Box\)

In the next lemma, we use the first inequality in Lemma \(2\) to prove that, for all \(c \in \Delta\), the
processes \((X^c_t)\) are almost surely nonincreasing.

**Lemma 3 (monotonicity)** — For all \(c \in \Delta\),

\[
0 \leq X^c_t \leq X^c_s \leq d \cdot \text{card}(\mathcal{V}) \quad \text{for all} \quad s \leq t.
\]

**Proof.** At each update of the processes, say at time \(s\),

\[
\xi_s(x) = \phi(\xi_{s-}(x), \xi_{s-}(y)) \quad \text{and} \quad \xi_s(y) = \phi(\xi_{s-}(y), \xi_{s-}(x)) \quad \text{for some} \quad (x, y) \in \mathcal{E}.
\]

In particular, applying Lemma \(2\) with \(a = \xi_{s-}(x)\) and \(b = \xi_{s-}(y)\), we get

\[
X^c_s - X^c_{s-} = \|\xi_s(x) - c\| + \|\xi_s(y) - c\| - \|\xi_{s-}(x) - c\| - \|\xi_{s-}(y) - c\|
\]
\[
= \|\phi(a, b) - c\| + \|\phi(b, a) - c\| - |a - c| - |b - c| \leq 0.
\]

In addition, because \(c \in \Delta\), we have

\[
0 \leq X^c_t = \sum_{x \in \mathcal{V}} |\xi_t(x) - c| \leq \text{card}(\mathcal{V}) \cdot \sup_{a \in \Delta} |a - c| \leq d \cdot \text{card}(\mathcal{V}) < \infty.
\]

This completes the proof. \(\Box\)

Note that Lemma \(3\) implies that the processes \((X^c_t)\) are bounded supermartingales, which will
be used later with the optional stopping theorem to derive our universal lower bound for the prob-
bility of consensus. By the martingale convergence theorem, each of these processes converges
almost surely to a finite random variable, which suggests almost sure convergence of the interacting
particle system. The main difficulty to prove this result is that whenever two vertices with compat-
able opinions \(a\) and \(b\) interact, the process \((X^c_t)\) does not “see the update” when \(a, b, c\) are aligned in
this order. For some norms, the lack of alignment is not even a sufficient condition for the process
to see the change of opinions so it is not clear how to deduce convergence of the system. To prove
this result rigorously, we now use Lemma \(3\) and the second inequality in Lemma \(2\) to show that
the jumps at each vertex get smaller and smaller as time goes to infinity.
Lemma 4 – For all $\epsilon > 0$, there exists $S = S(\epsilon)$ almost surely finite such that

$$\|\xi_s(x) - \xi_{s-}(x)\| < \epsilon \quad \text{for all} \quad s \geq S \text{ and } x \in \mathcal{V}.$$ 

Proof. Assume by contradiction that there exist $\epsilon > 0$ and $x \in \mathcal{V}$ such that the opinion at $x$ jumps by more than $\epsilon$ infinitely often, and let $(s_i)$ be the times of these updates:

$$\|\xi_{s_i}(x) - \xi_{s_i-}(x)\| \geq \epsilon \quad \text{for all} \quad i > 0.$$ 

Letting $y_i \in \mathcal{V}$ be the vertex that interacts with $x$ at time $s_i$, setting

$$a_i = \xi_{s_i-}(x), \quad b_i = \xi_{s_i-}(y) \quad \text{and} \quad c_i = (a_i + b_i)/2,$$

and applying the second inequality in Lemma 2 with $a = a_i$ and $b = b_i$, we get

$$X_{s_i}^c - X_{s_i-}^c = \|\xi_{s_i}(x) - c\| + \|\xi_{s_i}(y) - c\| - \|\xi_{s_i-}(x) - c\| - \|\xi_{s_i-}(y) - c\|
= |\phi(a_i, b_i) - c| + |\phi(b_i, a_i) - c| - |a_i - c| - |b_i - c|
\leq -2|\phi(a_i, b_i) - a_i| + |a_i + b_i - 2c| = -2|\xi_{s_i}(x) - \xi_{s_i-}(x)| + 2|c_i - c|
\leq -2\epsilon + 2|c_i - c| \leq -\epsilon$$

for all $c \in B(c_i, \epsilon/2)$. Now, observe that there exists $\epsilon' > 0$ such that

$$B(c, \epsilon/2) \cap \Delta(\epsilon') \neq \emptyset \quad \text{for all} \quad c \in \Delta \quad \text{where} \quad \Delta(\epsilon') = \Delta \cap (\epsilon'\mathbb{Z})^d. \quad (4)$$

For the Euclidean norm, (4) holds for $\epsilon' = \epsilon/2$. This and the equivalence of the norms in finite dimensions imply that, for each norm, there indeed exists $\epsilon' > 0$ such that (4) holds. In addition, because the opinion space $\Delta$ is bounded, and again the dimension is finite,

$$\text{card}(\Delta(\epsilon')) < \infty \quad \text{for all} \quad \epsilon' > 0. \quad (5)$$

Combining (4) and (5), we deduce that

$$\Delta'(\epsilon') = \{c \in \Delta(\epsilon') : \text{card}\{i : c \in B(c_i, \epsilon/2)\} = \infty\} \neq \emptyset.$$

In particular, there exists

$$c' \in \Delta'(\epsilon') \quad \text{such that} \quad I = \{i \in \mathbb{N} : c \in B(c_i, \epsilon/2)\} \text{ is infinite.}$$

This, together with (4) and Lemma 3, implies that

$$\lim_{t \to \infty} X_t^{c'} \leq X_0^{c'} + \sum_{i \in I} (X_{s_i}^{c'} - X_{s_i-}^{c'}) = X_0^{c'} + \sum_{i \in I} (-\epsilon) = -\infty,$$

which contradicts the fact that $(X_t^{c'})$ is positive. \qed

The next lemma shows that the jumps getting smaller and smaller implies that, for large times, neighbors must either be incompatible or have almost the same opinion.

Lemma 5 – For all $0 < \epsilon < \tau$, there exists $T = T(\epsilon)$ almost surely finite such that

$$\|\xi_s(x) - \xi_s(y)\| \notin [\epsilon, \tau] \quad \text{for all} \quad s \geq T \text{ and } (x, y) \in \mathcal{E}.$$
In particular, by the triangle inequality,
and we write
\[ x \leftrightarrow y \]
Let \( x \leftrightarrow y \) if there exist \( x_0 = x, x_1, \ldots, x_j = y \) all distinct such that
\( (x_i, x_{i+1}) \in E \) and \( \| \xi_s(x_i) - \xi_s(x_{i+1}) \| < \epsilon/N \) for all \( 0 \leq i < j \) and \( s \geq T \). In particular, by the triangle inequality,
\[
\| \xi_T(x) - \xi_T(y) \| \leq \sum_{i=0}^{j-1} \| \xi_T(x_i) - \xi_T(x_{i+1}) \| < \frac{j\epsilon}{N} \leq \epsilon.
\] (6)

The relationship \( \leftrightarrow \) defines an equivalence relationship so it induces a partition of the vertex set into equivalence classes \( \mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_k \) that correspond to connected components of the graph. In addition, by \( \Box \) and the definition of \( \leftrightarrow \), there exist \( c_1, c_2, \ldots, c_k \in \Delta \) such that
(a) for all \( i = 1, 2, \ldots, k \), we have \( \xi_T(x) \in B(c_i, \epsilon) \) for all \( x \in \mathcal{Y}_i \) and
(b) whenever \( \mathcal{Y}_i \) and \( \mathcal{Y}_j \) are connected by \( (x, y) \in E \), we have \( \| \xi_T(x) - \xi_T(y) \| > \tau \).

Assume that properties (a) and (b) hold from time \( T \) to time \( s > T \) and that two neighbors \( x \) and \( y \) interact at time \( s \). Then, either \( x \leftrightarrow y \), say \( x, y \in \mathcal{Y}_i \), in which case
\[
[\xi_s(x), \xi_s(y)] = [(1 - \mu) \xi_{s-}(x) + \mu \xi_{s-}(y), (1 - \mu) \xi_{s-}(y) + \mu \xi_{s-}(x)] \\
\subset [\xi_{s-}(x), \xi_{s-}(y)] \subset B(c_i, \epsilon)
\]
by convexity of $B(c, \epsilon)$, or edge $(x, y)$ connects two different classes in which case
\[
\|\xi_s(x) - \xi_s(y)\| > \tau \quad \text{therefore} \quad \xi_s(x) = \xi_s(y) \text{ and } \xi_s(y) = \xi_s(y).
\]
In either case, properties (a) and (b) remain true after the interaction. Because $\epsilon > 0$ can be chosen arbitrarily small, this proves that properties (P1) and (P2) hold. □

4. Stopping time and consensus event

This section is devoted to the proof of Theorem 1. As mentioned after the proof of Lemma 3, the processes $(X^c_t)$ are bounded supermartingales so the idea is to apply the optional stopping theorem. Before proving the theorem, we define a suitable stopping time and show how the consensus event relates to the configuration of the system at this stopping time. Let
\[
T^* = \inf \{ t : |\xi_t(x) - \xi_t(y)| \notin [\tau/2, \tau] \text{ for all } x, y \in V \}.
\]
Note that time $T^*$ is a stopping time for the natural filtration of the process. Time $T^*$ is also almost surely finite according to Lemma 5, so we have the following lemma.

Lemma 7 – Time $T^*$ is an almost surely finite stopping time.

We now identify a collection of configurations at the stopping time $T^*$ that always lead the population to consensus eventually. More precisely, letting
\[
\mathcal{A} = \bigcup_{x \in V} \left\{ \sup_{c \in \Delta} |\xi_{T^*}(x) - c| < \tau \right\},
\]
we have the following inclusion.

Lemma 8 – We have the inclusion $\mathcal{A} \subset \mathcal{C}$. 

Proof. The definition of $T^*$ implies that
\[
\xi_{T^*}(y) \in B(\xi_{T^*}(x), \tau) \Rightarrow \xi_{T^*}(y) \in B(\xi_{T^*}(x), \tau/2). \quad (7)
\]
In addition, by the proof of Lemma 6 (convexity argument),
\[
\xi_{T^*}(y) \in B(c, \tau/2) \text{ for all } y \in V^c \Rightarrow \xi_{T^*}(y) \in B(c, \tau/2) \text{ for all } y \in V \text{ and } s > T^*. \quad (8)
\]
This, together with Lemma 6 itself, gives the implications
\[
\sup_{c \in \Delta} |\xi_{T^*}(x) - c| < \tau \text{ for some } x \in V^c \\
\Rightarrow (\xi_{T^*}(y) \in B(\xi_{T^*}(x), \tau) \text{ for all } y \in V^c \text{ for some } x \in V^c \\
\Rightarrow (\xi_{T^*}(y) \in B(\xi_{T^*}(x), \tau/2) \text{ for all } y \in V^c \text{ for some } x \in V^c \text{ (by (7))} \\
\Rightarrow (\xi_{T^*}(y) \in B(c, \tau/2) \text{ for all } y \in V \text{ for some } c \in \Delta \\
\Rightarrow (\xi_s(y) \in B(c, \tau/2) \text{ for all } y \in V \text{ and } s > T^* \text{ for some } c \in \Delta \text{ (by (5))} \\
\Rightarrow (\lim_{s \to \infty} \xi_s(y) \in B(c, \tau/2) \text{ for all } y \in V \text{ for some } c \in \Delta \\
\Rightarrow \lim_{s \to \infty} |\xi_s(y) - \xi_s(z)| = 0 \text{ for all } y, z \in V \text{ (by (P2) and choice of } \tau/2).}
\]
This completes the proof. □

Using Lemmas 3, 7 and 8 we can now deduce the theorem.

**Proof of Theorem** According to Lemma 3 for all $c \in \Delta$, the processes $(X^c_t)$ is bounded and almost surely nonincreasing. In particular, the process is a bounded supermartingale with respect to the natural filtration of the opinion model. According to Lemma 7 we also have that the random time $T_*$ is an almost surely finite stopping time with respect to the same filtration. In particular, it follows from the optional stopping theorem that, for all $c \in \Delta$,

$$E(X^c_{T_*}) \leq E(X^c_0) = E\left(\sum_{x \in \mathcal{V}} |\xi_0(x) - c|\right) = \text{card}(\mathcal{V}) \cdot E\|X - c\|. \quad (9)$$

Now, on the complement of $\mathcal{A}$,

for all $x \in \mathcal{V}$, there exists $c_x \in \Delta$ such that $|\xi_{T_*}(x) - c_x| \geq \tau$.

This and the triangle inequality imply that

$$|\xi_{T_*}(x) - c| \geq |\xi_{T_*}(x) - c_x| - |c_x - c| \geq \tau - d/2 \quad \text{for all } x \in \Delta.$$

This gives the following bound for the conditional expectation:

$$E(X^c_{T_*} | \mathcal{A}^c) = E\left(\sum_{x \in \mathcal{V}} |\xi_{T_*}(x) - c| \bigg| \mathcal{A}^c\right) \geq \text{card}(\mathcal{V}) \cdot (\tau - d/2). \quad (10)$$

Combining (9) with $c = c$ and (10), we deduce that

$$\left(\tau - \frac{d}{2}\right) (1 - P(\mathcal{A})) \leq \frac{E(X^c_{T_*} | \mathcal{A}^c) P(\mathcal{A}^c)}{\text{card}(\mathcal{V})} \leq \frac{E(X^c_{T_*})}{\text{card}(\mathcal{V})} \leq E\|X - c\|$$

which, together with Lemma 8, implies that

$$P(\mathcal{A}^c) \geq P(\mathcal{A}) \geq 1 - \frac{E\|X - c\|}{\tau - d/2} \quad \text{for all } \tau > d/2.$$

This completes the proof of the theorem. □

5. Numerical examples

In this section, we use Theorem 1 to prove Examples 1 and 2.

**Proof of Example** To deal with the uniform case, we first observe that

$$\lambda(B(c, s)) = \lambda(B(0, s)) = s^d \lambda(B(0, 1)). \quad (11)$$

In particular, letting $X = \text{Uniform}(\Delta)$, we get

$$P(\|X - c\| < s) = P(X \in B(c, s)) = \frac{\lambda(B(c, s))}{\lambda(B(c, r))} = \left(\frac{s}{r}\right)^d$$
for all $s \leq r$, from which it follows that

$$E |X - c| = \int_0^\infty P(|X - c| > s) \, ds = \int_0^r \left(1 - \left(\frac{s}{r}\right)^d\right) \, ds = \frac{dr}{d+1}.$$ 

Observing also that $d = 2r$ and $c = c$, and applying Theorem 1, we get

$$P(\mathcal{C}) \geq 1 - \frac{E |X - c|}{\tau - d/2} = 1 - \frac{E |X - c|}{\tau - r} = 1 - \frac{dr}{(d+1)(\tau - r)}.$$

This completes the proof. \qed

**Proof of Example 2.** To begin with, we observe that

$$\int_\Delta (r - |a - c|) \, d\lambda(a) = \frac{r \lambda(B(c, r))}{d+1}.$$ 

Using also (11) and thinking of the probability that $X \in B(c, s)$ as the volume of a cone plus the volume of a cylinder in dimension $d + 1$, we deduce that

$$P(|X - c| < s) = P(X \in B(c, s)) = \frac{s \lambda(B(c, s))/d + 1 + (r - s) \lambda(B(c, s))}{r \lambda(B(c, r))/d+1}$$

$$= \left(\frac{s}{r}\right)^d \left(\frac{s + (d+1)(r - s)}{r}\right) = \left(\frac{s}{r}\right)^d \left(1 + d\left(1 - \frac{s}{r}\right)\right)$$

for all $s \leq r$, from which it follows that

$$E |X - c| = \int_0^\infty P(|X - c| > s) \, ds$$

$$= \int_0^r \left(1 - \left(\frac{s}{r}\right)^d \left(1 + d\left(1 - \frac{s}{r}\right)\right)\right) \, ds = \frac{dr}{d+2}.$$ 

As previously, $d = 2r$ and $c = c$, so, according to Theorem 1, we have

$$P(\mathcal{C}) \geq 1 - \frac{E |X - c|}{\tau - d/2} = 1 - \frac{E |X - c|}{\tau - r} = 1 - \frac{dr}{(d+2)(\tau - r)}.$$ 

This completes the proof. \qed

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