We are interested in the existence of infinitely many positive non-radial solutions of a Schrödinger–Poisson system with a positive radial bounded external potential decaying at infinity.

1. Introduction and main results

We outline in this Note the recent results obtained in [1]. More precisely, we consider the following nonlinear Schrödinger–Poisson system

\[
\begin{aligned}
-\Delta u + u + V(x)\phi u &= |u|^{p-1}u, & x \in \mathbb{R}^3, \\
-\Delta \phi &= V(x)u^2, & x \in \mathbb{R}^3,
\end{aligned}
\]

where \( p \in (1, 5) \) and \( V : \mathbb{R}^3 \to \mathbb{R} \) is a positive bounded radial function such that

\((V)\) there are constants \( m > \frac{1}{2}, \theta > 0 \) such that

\[
V(x) = \frac{1}{|x|^m} + O\left(\frac{1}{|x|^{m+\theta}}\right), \quad \text{as } |x| \to +\infty.
\]

This kind of problem has been introduced in [2] and arises in an interesting physical context. Since, for all \( u \in H^1(\mathbb{R}^3) \), the Poisson equation in \((SP)\) admits a unique and positive solution \( \phi_u \in D^{1,2}(\mathbb{R}^3) \), we can reduce ourselves to a single equation which is variational in nature. Hence we look for critical points of the corresponding \( C^2 \) one variable functional

\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + |u|^2) \, dx + \frac{1}{4} \int_{\mathbb{R}^3} V(x)\phi_u u^2 \, dx - \frac{1}{p+1} \int_{\mathbb{R}^3} |u|^{p+1} \, dx.
\]

Our main result can be stated as follows:

Theorem 1.1. If \( V \) satisfies \((V)\), then the problem \((SP)\) has infinitely many non-radial positive solutions.

To prove Theorem 1.1 we follow an idea of Wei and Yan [3] by constructing solutions with large number of bumps near infinity.
2. Sketch of the proof of Theorem 1.1

The proof of Theorem 1.1 relies on a Lyapunov–Schmidt reduction. If \(|x| = r\) and since \(V(r) \to 0\) as \(r \to +\infty\), the solutions of (4.9) can be approximated by using the solution \(U\) of the following limit problem

\[
\begin{aligned}
-\Delta u + u = u^p, & \quad \text{in } \mathbb{R}^3, \\
u > 0, & \quad u(x) \to 0, \quad \text{as } |x| \to +\infty,
\end{aligned}
\]

which decays exponentially at infinity with its derivatives and which is also non-degenerate (see [4]).

For any positive integer \(k\), let us define

\[
P_j = \left( r \cos \frac{2(j-1)\pi}{k}, r \sin \frac{2(j-1)\pi}{k}, 0 \right) \in \mathbb{R}^3, \quad j = 1, \ldots, k,
\]

with \(r \in S_k := \left[ \left( \frac{m}{n} - \beta \right) k \log k, \left( \frac{m}{n} + \beta \right) k \log k \right], \beta > 0\) small and \(z_r(x) = \sum_{j=1}^{k} U_{P_j}(x)\), where \(U_{P_j}(\cdot) := U(\cdot - P_j)\).

If \(x = (x_1, x_2, x_3) \in \mathbb{R}^3\), we set

\[
H_k = \left\{ u \in H^1(\mathbb{R}^3) \mid \begin{array}{l}
u \text{ is even in } x_2, x_3; \\
u(r \cos \theta, r \sin \theta, x_3) = u \left( r \cos \left( \theta + \frac{2\pi j}{k} \right), r \sin \left( \theta + \frac{2\pi j}{k} \right), x_3 \right) \\
j = 1, \ldots, k - 1
\end{array} \right\}.
\]

Let \(Z_{i,j} = \frac{a_{i,j}}{a_{i,j}}, \quad i = 1, \ldots, k\) and \(j = 1, 2, 3\). Let us define

\[
W := \left\{ w \in H_k : \int_{\mathbb{R}^3} U_{P_j}^{p-1} Z_{i,j} w \, dx = 0, \quad i = 1, \ldots, k; \quad j = 1, 2, 3 \right\}
\]

and \(P_W\) the orthogonal projection onto \(W\). Our approach is to find first a solution \(w \in W\) of the auxiliary equation \(P_W l'(z_r + w) = 0\) and then to solve the remaining finite dimensional equation, namely \((I - P_W) l'(z_r + w) = 0\).

The following proposition concerns with the existence of a solution of the auxiliary equation.

Proposition 2.1. There exists an integer \(k_0 > 0\), such that for each \(k \geq k_0\), there is a \(C^1\) map \(w : S_k \to H_k, w = w(r)\), satisfying \(w \in W\) and \(P_W l'(z_r + w) = 0\). Moreover, there is a small \(\sigma > 0\), such that

\[
\|w\| \leq \frac{C}{k^{n-\frac{1}{2} + \sigma}}.
\]

Proof. Let us consider \(J(w) = l(z_r + w), w \in W\) and expand it as follows:

\[
J(w) = l(z_r) + l'(z_r)[w] + \frac{1}{2} l''(z_r)[w, w] + R_{z_r}(w)
\]

\[
= J(0) + l(w) + \frac{1}{2} (Lw, w) + R_{z_r}(w),
\]

where

\[
l(w) = l'(z_r)[w], \quad (Lw, w) = l''(z_r)[w, w],
\]

and

\[
R_{z_r}(w) = \frac{1}{4} \int_{\mathbb{R}^3} V(x) \phi_u w^2 \, dx + \int_{\mathbb{R}^3} V(x) \phi_u z_r \, dx
\]

\[
- \frac{1}{p + 1} \int_{\mathbb{R}^3} \left[ (z_r + w)^{p+1} - z_r^{p+1} - \frac{p(p+1)}{2} z_r^{p-1} w^2 - (p + 1) z_r^p w \right] \, dx.
\]

Since \(l(w)\) is a bounded linear functional in \(W\), by Riesz Theorem there exists an \(l_k \in W\) such that

\[
l(w) = \langle l_k, w \rangle.
\]

Now we want to find a critical point of \(J\), that is a \(w \in W\) such that

\[
0 = J'(w) = l_k + Lw + R_{z_r}'(w).
\]

By standard arguments, we can prove that \(L := P_W l''(z_r) : W \to W\) is invertible with the inverse uniformly bounded and so we rewrite (2.3) in the following way

\[
w = A(w) := -L^{-1} l_k - L^{-1} R_{z_r}'(w).
\]
Thus the problem of finding a critical point of \( J(u) \) is equivalent to find a fixed point of \( A \). To this end, let

\[
B := \left\{ w \in W : \|w\| \leq \frac{C}{k^{m-\frac{1}{2}+\sigma}} \right\},
\]

where \( \sigma > 0 \) is small.

Using the fact that there exists an integer \( k_0 > 0 \), such that for each \( k \geq k_0 \), there is a small \( \sigma > 0 \) such that

\[
\|J'(z_r)\| \leq \frac{C}{k^{m-\frac{1}{2}+\sigma}}
\]

(2.4)

and that \( L \) is invertible with the inverse uniformly bounded, we can prove that, for \( k \) sufficiently large, \( A(B) \subset B \) and \( A \) is a contraction in \( B \) and so we conclude. \( \square \)

Thus we can prove our main result.

**Proof of Theorem 1.1.** Let us define the reduced functional \( F : S_k \to \mathbb{R} \) such that, for all \( r \in S_k \), \( F(r) = I(z_r + w) \), where \( w = w(r, k) \) is the unique solution of the auxiliary equation. By (2.2), (2.4) and since \( I'' \) maps bounded sets onto bounded sets then

\[
F(r) = I(z_r) + I'(z_r)[w] + \frac{1}{2} I''(\xi)[w, w] = I(z_r) + k \cdot O \left( \frac{1}{k^{2m+\sigma}} \right),
\]

where \( \sigma > 0 \) is small. Then, by making some computations, the reduced functional is given by

\[
F(r) = k \left[ C_0 + \frac{B_1}{r^{2m}} + \frac{B_2 k \log k}{r^{2m+1}} - B_3 \sum_{i=2}^k \int_{\mathbb{R}^3} U_{i_1}^p U_{i_1} \, dx + O \left( \frac{1}{k^{2m+\sigma}} \right) \right],
\]

where \( C_0, B_1, B_2, B_3 \) are positive constants. The problem

\[
\max \{ F(r) : r \in S_k \}
\]

(2.5)

has a solution since \( F \) is continuous on a compact set. We have to show that this maximum is an interior point of \( S_k \).

Let us denote with \( \tilde{F} \) the function

\[
\tilde{F}(r) := \frac{B_1}{r^{2m}} + \frac{B_2 k \log k}{r^{2m+1}} - B_3 \sum_{i=2}^k \int_{\mathbb{R}^3} U_{i_1}^p U_{i_1} \, dx.
\]

Since

\[
C_1 \frac{e^{-\frac{2\pi r}{\log k}}}{\log k} \leq \sum_{i=2}^k \int_{\mathbb{R}^3} U_{i_1}^p U_{i_1} \, dx \leq C_2 e^{-\frac{2\pi r}{\log k}},
\]

if we define

\[
F_1(r) := \frac{B_1}{r^{2m}} + \frac{B_2 k \log k}{r^{2m+1}} - B_3 e^{-\frac{2\pi r}{\log k}},
\]

\[
F_2(r) := \frac{B_1}{r^{2m}} + \frac{B_2 k \log k}{r^{2m+1}} - B_4 e^{-\frac{2\pi r}{\log k}},
\]

then, for \( k \) sufficiently large, \( F_1(r) \leq \tilde{F}(r) \leq F_2(r) \) in \( S_k \) and, moreover, we have

\[
\tilde{F} \left( \left( \frac{m}{\pi} + \beta \right) k \log k \right) \geq F_1 \left( \left( \frac{m}{\pi} + \beta \right) k \log k \right) > 0,
\]

\[
\tilde{F} \left( \left( \frac{m}{\pi} - \beta \right) k \log k \right) \leq F_2 \left( \left( \frac{m}{\pi} - \beta \right) k \log k \right) < 0
\]

and

\[
\tilde{F} \left( \left( \frac{m}{\pi} + \frac{\beta}{2} \right) k \log k \right) - \tilde{F} \left( \left( \frac{m}{\pi} + \beta \right) k \log k \right) \geq \frac{C}{k^{2m \log k} k^{2m}} > 0.
\]

Hence \( \tilde{F} \) possesses a critical point (a maximum point) in the interior of \( S_k \).

Finally it is easy to check that (2.5) is achieved by some \( r_k \), which is in the interior of \( S_k \), and so we infer that \( r_k \) is a critical point of \( F(r) \). As a consequence, we can conclude that \( z_{r_k} + w(r_k) \) is a critical point of \( I \). This proves the existence of infinitely many non-trivial non-radial solutions of (3.9). Actually, by standard arguments, we prove that these solutions are also positive. \( \square \)
References

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