The Support and Resistance Line Method:
An Analysis via Optimal Stopping

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Abstract
We study a mathematical model capturing the support/resistance line method (a technique in technical analysis) where the underlying stock price transitions between two states of nature in a path-dependent manner. For optimal stopping problems with respect to a general class of reward functions and dynamics, using probabilistic methods, we show that the value function is $C^1$ and solves a general free boundary problem. Moreover, for a wide range of utilities, we prove that the best time to buy and sell the stock is obtained by solving free boundary problems corresponding to two linked optimal stopping problems. We use this to numerically compute optimal trading strategies for several types of dynamics and varying degrees of relative risk aversion. We then compare the strategies with the standard trading rule to investigate the viability of this form of technical analysis.

Keywords: Optimal stopping, technical analysis, resistance level, support line.

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1 Introduction

Technical analysis (TA) is a method to identify trading opportunities by analysing historical market data and price patterns. Some traders believe that by observing key market indicators and charts (i.e. graphs of price data) they can predict future price movement, and construct profitable trading strategies. TA is extremely popular among investors. In a survey of 678 fund

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managers Menkhoff [21] found that 86% of fund managers rely on TA as one of their investment tools. Hoffmann and Shefrin [12] analyze survey responses from individual investors and report that 32% use TA.

Through the continuous development of TA, numerous trading rules have been introduced. For example, traders may generate buy/sell signals by comparisons of short and long-term moving-averages; from breakthroughs of market support and resistance levels; from so-called Bollinger bands, and from directional indicators.

In this paper we study the support/resistance line method. Under this method, traders usually buy (sell) an asset if its price goes below (above) a support (resistance) level, or simply, “buy at low” (BL) and “sell at high” (SH). This is the so-called standard trading rule. The support (resistance) line is viewed as a local minimum (maximum) of the asset price over a period of time. However, when the price goes substantially below (above) the support (resistance) line, it is said to have broken-through the line and it is widely accepted that the support (resistance) line will become the new resistance (support) line because of the negative (positive) outlook for the asset resulting from such a price movement.

Despite the richness of technical trading strategies, many of them have been criticised for being subjective and lacking mathematical justification. Furthermore, TA is also contentious due to the perception of conflict between its claimed predictive power and the Efficient Markets Hypothesis (see Section 2 of Park and Irwin [23] for more details).

The vast majority of studies of TA devote their efforts to finding empirical evidence for the profitability of technical trading rules by examining historical data. For example, Brock, Lakonishok, and LeBaron [4] tested moving-average-type trading rules and the support/resistance line method on the Dow Jones Industrial Average on a time scale of 90 years. This study suggested that the technical trading strategies considered there were significantly profitable. Based on a similar approach but with the data taken from Asian markets, Bessembinder and Chan [1] further confirmed the forecasting power of trading rules based on TA. Lo, Mamaysky, and Wang [19] implemented an automatic trading algorithm based on more sophisticated pattern-based trading rules (such as triangle, rectangle, and head-and-shoulders) by using kernel regressions, and a significant profit was observed. Park and Irwin [23] provided a comprehensive review of the literature on the profitability of TA and concluded that more than half showed positive evidence, though many of them had imperfections in their test procedures (for example, some ignored transaction costs). Ebert and Hilpert [8] demonstrated that the market timing of technical trading rules induced skewed trading profits. Popular rules were studied by a combination of simple models, simulations and analysis of empirical data. They argued that investors’ preference for positive skewness partially explained the popularity of TA.

Tremendous effort has also been made to build algorithms which implement technical analysis-based trading strategies fast and accurately. For instance, Sezer, Ozbayoglu, and Dogdu [31] designed a trading system based on a neural network constructed by using technical trading rules (based on the simple moving average and the relative strength index), and they showed the optimised system did outperform a buy-and-hold strategy.
In contrast, very little research has been done on the mathematical modelling side. Blanchet-Scalliet et al. [2] derived the optimal expected portfolio wealth at some terminal time $T$ where the underlying price process was assumed to have a mis-specified drift from time 0 to an exponentially distributed random time $\tau$, and (using Monte Carlo methods) they numerically compared it with the expected portfolio wealth resulted from a simple moving-average trading strategy. Lorig, Zhou, and Bin [20] studied a logarithmic utility maximization problem when trading strategies are based on exponential moving averages of the price of an underlying risky asset. De Angelis and Peskir [7] determined the optimal stopping time that minimised the expected absolute distance between the stock price and the unknown support/resistance line which was assumed to be a random variable independent of the price. Furthermore, under an unrealistic constraint and with only linear utility, Jacka and Maeda [14] solved two linked optimal stopping problems with a novel stochastic process model for the stock price inspired by the support/resistance line method.

Nevertheless, this literature either focused on particular dynamics (e.g. De Angelis and Peskir [7]) or a specific utility function (e.g. Blanchet-Scalliet et al. [2] and Lorig, Zhou, and Bin [20], Jacka and Maeda [14]).

Our aim in this paper is to study the modelling of the support/resistance line method and provide very general results for a wide class of dynamics and utility functions. The underlying asset price process is that introduced by Jacka and Maeda [14], and we will rigorously study its mathematical properties and substantially extend their work to a much broader range of reward functions with the aid of elegant probabilistic arguments. We will show, under mild assumptions of reward functions and dynamics, the $C^1$ smoothness of the value function. Hence, we will prove the value function is the solution to a generalized free boundary problem. Using these results, we show how solutions of two relevant linked optimal stopping problems are found by solving two free boundary problems whose solutions are easily (numerically) computed. Then, the resulting optimal trading strategy derived from various plausible choices for price dynamics and the trader’s utility function will be compared with the standard trading rule. To the best of our knowledge, no literature to date attempts to address these issues.

The stock price process is as follows. We assume there are two regimes for the stock price process, termed the positive and negative regime respectively. The dynamics of the stock price process are dependent on its current regime. We further assume that there is an unobserved fixed price level located in some known interval $[L, H]$, and this price level is the support line if the stock is in the positive regime and the resistance line if it is in the negative regime. The regime changes from the negative (positive) to the positive (negative) regime if the stock price crosses $H(L)$ from below (above). So, when a transition of regimes occurs, there is a reversal of the role of the resistance and support level in line with what traders would expect. Note that the stock price process can be in either regime on the interval $(L, H)$, which provides the flexibility to move around the support/resistance line without changing regimes.

We emphasise that, in contrast to standard regime-switching models, the regime transition in our model is path-dependent and not specified by an exogenous Markov chain. The path-dependent regime-changing can be viewed as a novel method of introducing a market signalling
effect into the price process (see Lehalle and Neuman [18] for a different approach).

By making mild assumptions on dynamics, no-arbitrage can be ensured (for a market consisting of a bond paying the risk-free rate and the stock). A potential criticism of our class of models might be that the exogenous stock price process reduces the economic credibility of our results and that a multiple-agent-based model with an endogenous price for the stock would be more desirable. We argue that, since endogenous specification of prices in a dynamic context usually requires that agents, although they may differ as to their utilities, agree on the probabilistic specification of the world, modelling a world where only some agents believe in TA renders this approach non-viable and the best that can be achieved is a price specification which is arbitrage-free and hence in dynamic equilibrium.

We stress that the optimal stopping problems presented here are not standard since the stock price process is not a diffusion and the regime process on its own is not Markovian.

The rest of this paper proceeds as follows. In Section 2 we provide definitions for key ingredients of the model and establish important mathematical properties. In Section 3 we give some general results regarding the optimal stopping problem. In Section 4 we describe and solve the seller’s problem and obtain the optimal selling boundaries. In Section 5 we define and solve the buyer’s problem, which provides the optimal buying boundaries. Finally, in Section 6 we analyse numerically the influence of the degree of relative risk aversion to optimal trading strategies under different types of dynamics, and we will discuss the optimality of the standard trading rule within the context of our modelling.

2 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t \in \mathbb{R}_+ \cup \{\infty\}}, P)$ be a filtered probability space satisfying the usual conditions, i.e., $\mathbb{F}$ is right-continuous and $P$-complete. By convention, we set $\mathbb{R}_+ = [0, \infty)$ and $\mathcal{F}_\infty = \sigma(\cup_{0 \leq t < \infty} \mathcal{F}_t)$. Assume $(\Omega, \mathcal{F}, \mathbb{F}, P)$ supports a one dimensional Brownian motion denoted by $W$. Suppose for the stock price, there are two price levels $L$ and $H$ such that $0 < L < H$ and two regimes: positive (denoted $+$) and negative (denoted $-$). Further assume there is a continuous stock process $S_t$ and a càdlàg flag-process $F_t$, $t \in \mathbb{R}_+ \cup \{\infty\}$, living on this filtered probability space (adapted to $\mathbb{F}$) and taking values in $\mathbb{R}_+ \cup \{\infty\}$ and $\{+, -\}$ respectively, with $S$ satisfying

$$dS_t = \mu_{F_{t^-}}(S_t)dt + \sigma_{F_{t^-}}(S_t)dW_t, \quad (2.1)$$

and $F$ satisfying

$$F_t = \begin{cases} + & \text{if } F_{t^-} = - \text{, and } S_t = H \\ - & \text{if } F_{t^-} = + \text{, and } S_t = L, \end{cases} \quad (2.2)$$

where $\mu_+, \mu_-, \sigma_+$, and $\sigma_-$ are locally bounded Borel measurable functions. Note that, to obtain the existence and uniqueness (in law) of the weak solution of the corresponding Itô SDEs (given in (2.6) below), by Theorem 4.53 of Engelbert and Schmidt [9], it is sufficient to assume:
Assumption 2.1.

\[ \sigma^2_{\pm}(x) > 0 \forall x \in \mathbb{R}_+, \quad (2.3) \]

\[ \int_{x-\epsilon}^{x+\epsilon} \frac{1 + \mu_{\pm}(y)}{\sigma^2_{\pm}(y)} \, dy < +\infty, \text{ for some } \epsilon > 0 \text{ at every } x > 0. \quad (2.4) \]

By our construction, \( S \) transitions from the positive to the negative regime when it hits \( L \) from above and from the negative to the positive regime when it hits \( H \) from below. Clearly, by (2.1), \( S \) is a continuous semimartingale. Furthermore, the resulting joint process \((S, F)\) is strong Markov with state space \((E, \mathcal{B})\), where \( E \) equals to \([0, H) \times \{-\} \cup [L, \infty) \times \{+\} \) and \( \mathcal{B} = \mathcal{B}(\mathbb{R}_+ \times \{+, -\}) \). Define a metric on \( E \) by \( d((x_1, f_1), (x_2, f_2)) := |x_1 - x_2| + \mathbb{1}_{f_1 \neq f_2} \) for \((x_1, f_1), (x_2, f_2) \in E\), where \( \mathbb{1}_{f_1 \neq f_2} = 1 \) if \( f_1 \neq f_2 \) and 0 otherwise. It is obvious that \( E \) is LCCB with respect to the induced topology. The initial position of the process \((S, F)\) (i.e. \((S_0, F_0)\)) is denoted by \((x, f)\). In addition, denoting the scale function and speed measure in the positive/negative regime by \( s_{\pm} \) and \( m_{\pm} \) respectively, we assume:

Assumption 2.2. Khasminskii's condition holds in the positive regime:

\[ \int_1^{\infty} s'(x)dx \int_1^{x} m'_+(y)dy = \infty. \quad (2.5) \]

This implies the process \( S \) does not explode in finite time (see Rogers and Williams [29] p.297). Since Assumption 2.1 and 2.2 are standard, we will assume them in the rest of this paper and use them without mentioning explicitly.

For any \( A \in \mathcal{B} \), we define the hitting time of \( A \) by \( \tau_A := \inf \{ t \geq 0 : (S_t, F_t) \in A \} \). In the case of \( A \in \mathcal{B}(\mathbb{R}_+) \), we denote by \( \tau^+_A \) and \( \tau^-_A \) the first time \( S \) enters \( A \) while in the positive regime and negative regime respectively, e.g., \( \tau^+_A := \inf \{ t \geq 0 : S_t \in A, F_t = + \} \). If \( A = \{a\} \) for some \( a \in \mathbb{R}_+ \), we simply use \( \tau^+_a \) to denote \( \tau^+_A \), \( f \in \{+,-\} \). If \( A \in \mathcal{B}(\mathbb{R}_+) \), we set \( \tau_A := \tau^+_A \land \tau^-_A \). According to Kallenberg [16], since \((S, F)\) is a right-continuous adapted process, it is progressively measurable (i.e., \((S, F)\) restricted to \( \Omega \times [0, t) \) is \( F_t \otimes \mathcal{B}[0, t] \)-measurable for every \( t \geq 0 \)) and Theorem 7.7 in [16] ensures \( \tau_A \) is a Markov time (a stopping time if \( \tau_A \) is finite a.s.) for any \( A \in \mathcal{B} \).

From (2.1), we may observe that \( S \) behaves like a diffusion process between the times when it changes regime. Thus, it is often useful to work with the diffusion processes \( S^+ \) and \( S^- \) which are defined by the following SDEs,

\[
\begin{align*}
\,dS^+_t &= \mu_+(S^+_t)\,dt + \sigma_+(S^+_t)\,dW_t, \\
\,dS^-_t &= \mu_-(S^-_t)\,dt + \sigma_-(S^-_t)\,dW_t,
\end{align*}
\]

where \( \sigma_+, \sigma_-, \mu_+, \mu_- \) are defined as above. Let \( \sigma^+_A \) and \( \sigma^-_A \) denote the first hitting time by \( S^+ \) and \( S^- \) of a Borel measurable set \( A \) respectively. By (2.6), for every \( t \geq 0 \), it is obvious that \((S_{t \land \tau^+_A}, F_{t \land \tau^+_A}) = (S^+_t, +)\) for \( A \in \mathcal{B}(L, \infty) \) if \( x \in A, f = + \), and \((S_{t \land \tau^-_A}, F_{t \land \tau^-_A}) = (S^-_t, -)\) for \( A \in \mathcal{B}[0, H) \) if \( x \in A, f = - \). As a result, \( S \) has the same law as \( S^f \) until the first time that the regime changes, and hence
Lemma 2.3. For \((x, f) \in E \setminus \{(0, -)\}, P_{x, f}(\exists \epsilon > 0, \forall t < \epsilon, F_t = f) = 1.\)

Proof. WLOG, take \(x > L\) and \(f = +\). By path continuity of \(S^+\) (where we set \(S^+_0 = x\), \(\exists \epsilon > 0\), such that \(|S^+_t - S^+_0| < x - L\) for all \(t < \epsilon\) a.s.. Hence, \(\exists \epsilon > 0\), \(S^+\) hit \(L\) after time \(\epsilon\) a.s.. Thus, \(P_{x, +}(\exists \epsilon > 0, \forall t < \epsilon, F_t = +) = P_{x, +}(\exists \epsilon > 0, \tau^+_L > \epsilon) = P_x(\exists \epsilon > 0, \sigma^+_L > \epsilon) = 1.\) ♦

From Assumption 2.1 the Itô diffusions \(S^+\) and \(S^-\) are regular except at 0 (i.e. \(P_x(S^f)\) hits \(y\) > 0, for all \(x > 0\) and \(y \geq 0\), for each \(f \in \{+,-\}\).

Lemma 2.4. For each \((x, f) \in E \setminus \{(0, -)\}\) and \((y, g) \in \mathbb{R}^+ \times \{+,-\}\), \(P_{x, f}((S, F)\) hits \((y, g)) > 0, and hence \((S, F)\) is regular except at \((0,-)\).

See Appendix A.2 for the proof.

Let

\[A_x = \{\omega : \forall t > 0 \exists s \in (0, t) \text{ with } S_s(\omega) > x\}.\]

Analogously, we define

\[B_x = \{\omega : \forall t > 0 \exists s \in (0, t) \text{ with } S_s(\omega) < x\}.\]

With the aid of Lemma 2.4 we establish the following lemma. The proof can be found in Appendix A.2.

Lemma 2.5. \(P_{x, f}(A_x) = 1\) and \(P_{x, f}(B_x) = 1, \text{ for each } (x, f) \in E \setminus \{(0, -)\}.\)

Let \((P_t)\) denote the semigroup of \((S_t, F_t)\) so that \(P_t h(x, f) = \mathbb{E}^{x,f}[h(S_t, F_t)]\) for \(h\) a bounded \(\mathcal{B}\)-measurable real-valued function on the domain \(E\) with \(||h|| := \sup_E |h(x, f)|\). In line with Shiryaev \[32\], we call \((P_t)\) a Feller semigroup (and the corresponding process \((S,F)\) is a Feller process) if \(P_t\) maps \(C^b(E)\), the space of bounded continuous function on \(E\), to itself.

Lemma 2.6. The process \((S, F)\) is a Feller process.

See Appendix A.2 for the proof.

That \((S, F)\) is a Feller process is a key ingredient in the proof of Lemma 3.6.

For \((S,F)\), the infinitesimal generator, \(\mathcal{L}\), is given by \(\mathcal{L}h(x, f) := \lim_{t \downarrow 0} \frac{\mathbb{E}^{x,f}[h(S_t, F_t)] - h(x)}{t}\) for a measurable function \(h : E \rightarrow \mathbb{R}\), and say \(h \in \mathcal{D}(\mathcal{L})\), the domain of \(\mathcal{L}\), if the limit exists. Analogously, we have \(\mathcal{L}^f\) for diffusion processes \(S^f, f \in \{+,-\}\). If \(h \in C^2_0(\mathbb{R}_+)\), by Itô’s Lemma, we see

\[\mathcal{L}^f h = \mu_f h' + \frac{1}{2} \sigma_f^2 h''.\] (2.7)

More generally, let \(L\) denote the martingale generator of \((S,F)\), i.e. for a measurable function \(h\), if there is a measurable function \(g\) such that, \(\int_0^t |g(S_s, F_s)| ds < \infty\) a.s. and

\[M_t := h(S_t, F_t) - h(x, f) + \int_0^t g(S_s, F_s) ds\] (2.8)
is a local martingale, then $L = g$, and we say $h \in D(L)$. Similarly, for $S^f$, we denote its martingale generator by $L^f$. If $h \in C^1$ with absolutely continuous first derivative, then $Lh = \mathcal{L}h$.

Note that $D(\mathcal{L}) \subset D(L)$. Furthermore, we can show the following equivalence.

**Lemma 2.7.** Restrict $S^+$ on $(L, \infty)$ by stopping $S^+$ at $L$ and $S^-$ on $[0, H)$ by stopping $S^-$ at $H$ and $0$. If $h : E \to \mathbb{R}$ is in $D(\mathcal{L})$, then $h(\cdot, f) \in D(\mathcal{L}^f)$ and $\mathcal{L}h(\cdot, f) = \mathcal{L}^f h(\cdot, f)$. Moreover, if $h \in D(L)$, then $h(\cdot, f) \in D(D(\mathcal{L}^f))$ and $\mathcal{L}h(\cdot, f) = \mathcal{L}^f h(\cdot, f)$.

**Proof.** The case for $(0, -)$ is obvious. Now fix an initial position $(x, f) \in E \setminus \{(0, -)\}$. Fix a small $\epsilon > 0$ such that $x > L + \epsilon$ or $x < H - \epsilon$ and set $\tau := \tau_{(L+\epsilon, \infty)}^+ \wedge \tau_{(0, H-\epsilon)}^-$. We have $h(S_{0\wedge\tau}, F_{0\wedge\tau}) = h(S^f_{0\wedge\tau}, f)$. Thus, if $h \in D(\mathcal{L})$, then

$$
\mathcal{L}h(x, f) = \lim_{t \downarrow 0} \frac{\mathbb{E}^x[h(S_{0\wedge\tau}, F_{0\wedge\tau}) - h(x, f)]}{t} = \lim_{t \downarrow 0} \frac{\mathbb{E}^x[h(S^f_{0\wedge\tau}, f) - h(x, f)]}{t} = \mathcal{L}^f h(x, f),
$$

where the first and third equality follows from Dykin’s formula.$^\dagger$

Now, let $h \in D(L)$ and $\tau$ be the first time that the regime changes, then there is $g$ such that

$$
M_{0\wedge\tau} := h(S_{0\wedge\tau}, F_{0\wedge\tau}) - h(x, f) + \int_0^{\tau_{0\wedge\tau}} g(S_s, F_s)ds
$$

is a local martingale. Since $S^f$ is stopped at the boundary where the regime switches, we see that

$$
M_{0\wedge\tau} = N_{\tau} := h(S^f_{\tau}, f) - h(x, f) + \int_0^{\tau} g(S^f_s, f)ds,
$$

which shows $N$ is also a local martingale. Finally, $\int_0^t |g(S^f_s, +)|ds = \int_0^{\tau_{0\wedge\tau}} |g(S_s, F_s)|ds < \infty$, a.s.

$^\dagger$

Fix a positive constant $r$, which shall be understood as the “interest rate” in later sections. Then, we define fundamental solutions, which we denote by $\phi^+$ and $\psi^+$ for (2.13), and $\phi^-$ and $\psi^-$ for (2.14) as follows:

$$
\psi^f(x) = \begin{cases} 
\mathbb{E}^x [e^{-r \sigma f}] & \text{if } x \leq c^f \\
\frac{1}{\mathbb{E}^x [e^{-r \sigma f}]} & \text{if } x > c^f
\end{cases}, \quad \phi^f(x) = \begin{cases} 
\frac{1}{\mathbb{E}^x [e^{-r \sigma f}]} & \text{if } x \leq c^f \\
\mathbb{E}^x [e^{-r \sigma f}] & \text{if } x > c^f
\end{cases},
$$

where $c^+ = H$ and $c^- = L$. Note that $\psi^f$ is increasing and $\phi^f$ is decreasing. Since both the speed measures and scale functions are absolutely continuous with respect to Lebesgue measure, fundamental solutions are solutions to the following ODEs:

$$
\begin{align*}
\mathcal{L}^+ v - rv &= 0, \\
\mathcal{L}^- v - rv &= 0.
\end{align*}
$$

Finally, it is important to note that, for one dimensional Itô diffusions, fundamental solutions are $C^1$ since both speed measures and scale functions are $C^1$.

$^\dagger$So $\mathbb{E}^x[h(S_{0\wedge\tau}, F_{0\wedge\tau}) - h(x, f)] = \frac{1}{t} \int_0^t \mathbb{E}^x \mathbb{E}^x[h(S_u, F_u) \mathbb{1}_{u \leq \tau}]du \to \mathcal{L}h(x, f)$ as $t \to 0$. 

7
3 The optimal stopping problem

We would like to study two problems in this paper. The first one is called the seller’s problem. In this problem, a trader holds the stock initially and seeks the best selling time to obtain the maximum gains (utility in this paper). Likewise, the second one is called the buyer’s problem: here a trader wants to maximise expected utility (gain) by purchasing a stock first and then selling it later. Both problems are formulated as optimal stopping problems, and we will present some general results here.

We assume we have a gains function \( h : E \to \mathbb{R}_+ \) and a discount factor \( r \geq 0 \). We introduce the following assumptions.

Assumption 3.1. \( h \in \mathcal{D}(\mathbb{L}) \), i.e. \( h \) is in the domain of the martingale generator.

Assumption 3.2. \( \mathbb{E}^x,f \left[ \sup_{t \geq 0} e^{-rt} h(S_t, F_t) \right] < \infty \).

Remark 3.3. Assumption (3.1) holds for any \( h \in C^2 \), and even \( h \in C^1 \) whose first derivative is absolutely continuous following an extended version of Itô’s formula (e.g. (45.9) Lemma in [29] p.105).

The optimal stopping problem is defined by

\[
V(x, f) := \sup_{\tau} \mathbb{E}^x,f[e^{-r\tau} h(S_\tau, F_\tau)],
\]

where the supremum is taken over all stopping times, and we call \( V(x, f) \) the value function. We also look for the optimal stopping time \( \tau^* \) making

\[
V(x, f) = \mathbb{E}^x,f[e^{-r\tau^*} h(S_{\tau^*}, F_{\tau^*})].
\]

Remark 3.4. In the seller’s problem, we take \( h \) to be a utility function denoted by \( u \). In this case, the economic interpretation of \( r \) is the time value of utility. We have assumed that \( h \) is non-negative since otherwise there will be a conflict with the natural time preference.

The following lemmas are required for the proof of Theorem 3.7.

Lemma 3.5. Under Assumption 3.2 \( V(x, f) < \infty \) for any \( (x, f) \in E \).

Proof. By definition,

\[
V(x, f) = \sup_{\tau} \mathbb{E}^x,f[e^{-r\tau} h(S_\tau, F_\tau)] \leq \mathbb{E}^x,f \left[ \sup_{\tau} e^{-r\tau} h(S_\tau, F_\tau) \right] = \mathbb{E}^x,f \left[ \sup_t e^{-rt} h(S_t, F_t) \right]
\]

Hence, by Assumption 3.2 \( V(x, f) < \infty \).

Lemma 3.6. Under Assumption 3.2 the value function \( V(x, f) \) is lower semicontinuous (i.e. \( \liminf_{y \to x} V(y, f) \geq V(x, f) \)).

Proof. By Assumption 3.2 we can apply Theorem 1 in Chapter 3 of [32] to see \( V \) is the smallest excessive majorant of the gains function \( h \). Then by Lemma 4 in Chapter 3 of [32], since \( (S, F) \) is a Feller process and \( h \) is bounded below by 0, \( V \) is lower semicontinuous.
Define the stopping set $D$ and continuation set $C$ by
\begin{align}
D &= \{(x, f) \in [0, \infty) \times \{+, -\} | V(x, f) = h(x, f)\}, \\
C &= \{(x, f) \in [0, \infty) \times \{+, -\} | V(x, f) > h(x, f)\}.
\end{align}

(3.3)  
(3.4)

As $V$ is lower semicontinuous, $D$ is closed and $C$ is open. The following theorem follows immediately from Shiryaev [32] Chapter 3 Theorem 3.

**Theorem 3.7.** For any non-negative gains function $h$ satisfying Assumptions 3.1 and 3.2, if $\tau_D < \infty$ a.s. for every $(x, f) \in E$, the stopping time $\tau_D$ is optimal in the sense that equation (3.1) holds.

By a well-known result (e.g. see Jacka and Norgilas [15] Theorem 2.10), $e^{-rt}V(S_t, F_t)$ is the Snell envelope of $e^{-rt}h(S_t, F_t)$ under Assumption 3.2 i.e. $e^{-rt}V(S_t, F_t) = \operatorname{ess} \sup_{s \geq t} \mathbb{E}[e^{-r\tau}h(S_{\tau}, F_{\tau}) | F_t]$, a.s. Moreover, standard theory in optimal stopping (e.g. Theorem 2.2 of [24]) tells us that $e^{-rt}V(S_t, F_t)$ is a supermartingale and the stopped process $e^{-rt\wedge \tau_D}V(S_{\tau_D}, F_{\tau_D})$ is a martingale. The theorem below is a direct consequence of Theorem 3.11 in Jacka and Norgilas [15].

**Theorem 3.8.** Let $\mathbb{L}$ denote the martingale generator of $(S, F)$. Then, under Assumption 3.1 and 3.2 $\mathbb{L}V(x, f) - rV(x, f) = 0$ on $C$ almost everywhere.

**Proof.** Fix $(x, f) \in C$. Theorem 3.11 in Jacka and Norgilas [15] states that, under mild assumptions, $h \in D(\mathbb{L})$ implies that $V \in D(\mathbb{L})$. If so, we get
\begin{equation}
V(x, f) + \int_0^{t \wedge \tau_D} e^{-rs}dM_s + \int_0^{t \wedge \tau_D} e^{-rs}(L-r)\mathbb{V}(S_s, F_s)ds.
\end{equation}

(3.5)

Since $e^{-rt\wedge \tau_D}V(S_{\tau_D}, F_{\tau_D})$ is a martingale, we must have, for all $t \geq 0$,
\begin{equation}
\int_0^t e^{-rs}(L-r)\mathbb{V}(S_s, F_s)1_{(S_s, F_s) \in C}ds = 0 \text{ a.s.}
\end{equation}

(3.6)

Moreover, by the Doob-Meyer decomposition (e.g. Theorem 2.4 of [15]), $\int_0^t e^{-rs}(L-r)\mathbb{V}(S_s, F_s)ds$ is the unique decreasing integrable variation process in the decomposition of $e^{-rt}V(S_t, F_t)$. This implies $\mathbb{L}V(x, f) - rV(x, f) \leq 0$ a.e. on $E$, because otherwise the compensator would be increasing on a set with positive probability. If $\mathbb{L}V(x, f) - rV(x, f) < 0$ on a subset of $C$ with positive measure, then with positive probability, there is a $t$ such that
\begin{equation}
\int_0^t e^{-rs}(L-r)\mathbb{V}(S_s, F_s)1_{(S_s, F_s) \in C}ds < 0.
\end{equation}

(3.7)

This leads to a contradiction. Thus, we have $\mathbb{L}V(x, f) - rV(x, f) = 0$ on $C$ almost everywhere. We are left to verify the remaining assumption of Theorem 3.11 in Jacka and Norgilas [15] — that $(S, F)$ is a right process, which is satisfied since $(S, F)$ is Feller. 

\Diamond
If a process $X$ starts at $x \in \partial C$ and enters $\text{int}(D)$ immediately, then the smooth pasting principle is often valid at $x$ (see Section 9 in Peskir and Shiryaev [24]). The smooth pasting principle is well established for Itô diffusion processes (e.g., Jacka and Norgilas [15]), but not in greater generality. Nevertheless, since the process $S$ is an Itô diffusion before regime transitions, the smooth pasting principle indeed holds.

**Theorem 3.9.** Under Assumption 3.1 and 3.2, $V(\cdot,f) \in C^1$. In particular, $V$ is continuously differentiable at the boundary $\partial C$.

**Proof.** The strategy is similar to the proof of Theorem 4.9 of Jacka and Norgilas [15].

Fix $f = +$ and $x > L$. Let $\tilde{s}(x) := \psi_+(x)/\phi_+(x)$ and $\tilde{s}$ is continuous and increasing. Pick an arbitrary interval $x \in [a,b]$ and $a > L$. Set $\tau := \inf\{t \geq 0 : S = a \text{ or } S = b\}$.

By Jacka and Norgilas [15], $J(x) := V(x,+)/\phi_+(x)$ is $\tilde{s}$-concave. Let $K : [\tilde{s}(a), \tilde{s}(b)] \to \mathbb{R}_+$ be the function defined by $K(x) := J \circ \tilde{s}^{-1}(x)$. Then, $K$ is concave and $K(\tilde{s}(x))\phi_+(x) = V(x,+)$. Further define $Y_t = \tilde{s}(S_t)$, and we have $e^{-rt}\varphi(V(S_{t\wedge \tau},+)) = e^{-rt}\varphi(S_t)k(V(Y_t))$. Set $N_t := e^{-rt}\phi_+(S_t)$, which implies that $N_{t\wedge \tau}$ is a local martingale by Itô and McKean [13]. Applying the generalised Itô formula for convex/concave functions (see Revuz and Yor [27]), we have

$$K(Y_{t\wedge \tau}) = K(Y_0) + \int_0^{t\wedge \tau} K'_-(Y_s)dY_s - \int_{\tilde{s}(a)}^{\tilde{s}(b)} L^Z_{t\wedge \tau}\nu(dz),$$

(3.8)

where $L^Z_t$ is the local time of $Y_t$ at $z$, and $\nu$ is the measure corresponding to the second derivative of $-K$ in the sense of distribution. Therefore,

$$e^{-rt}\varphi(V(S_{t\wedge \tau},+)) = N_0K(Y_0) + \int_0^{t\wedge \tau} K(Y_s)dN_s + \int_0^{t\wedge \tau} K'_-(Y_s)(N_s\nu Y_s + d[N,Y]_s) + \int_0^{t\wedge \tau} N_s\nu dA_s,$$

(3.9)

where $A_t := \int_{\tilde{s}(a)}^{\tilde{s}(b)} L^Z_t\nu(dz)$ is a continuous and increasing process. Note that $N_{t\wedge \tau}Y_{t\wedge \tau} = e^{-rt}\varphi(S_{t\wedge \tau})$ is a local martingale. Hence,

$$\int_0^{t\wedge \tau} N_s\nu Y_s + [N,Y]_{t\wedge \tau} = e^{-rt}\varphi(S_{t\wedge \tau}) - \int_0^{t\wedge \tau} Y_sN_sdz,$$

(3.10)

which again is a local martingale. Furthermore,

$$\int_0^t N_s\nu dA_s = N_tA_t - \int_0^t A_sN_sdz.$$

(3.11)

So, we get

$$e^{-rt}\varphi(V(S_{t\wedge \tau},+)) = V(x,+) + M_{t\wedge \tau} - \int_{\tilde{s}(a)}^{\tilde{s}(b)} N_{t\wedge \tau}L^Z_{t\wedge \tau}\nu(dz)$$

(3.12)

for some local martingale $M$.

On the other hand, by assumptions, we know $V \in D(L)$. Thus,

$$e^{-rt}\varphi(V(S_{t\wedge \tau},+)) = V(x,+) + M_{t\wedge \tau} + \int_0^{t\wedge \tau} e^{-rs}(L^+ - r)V(S_s,+)ds,$$

(3.13)

10
where \( M^V \) is a local martingale. By the uniqueness of Doob-Meyer decomposition, the finite variation term of (3.12) and (3.13) must agree. Hence,

\[
\int_0^{t\wedge \tau} e^{-rs}(L^+ - r)V(S_s, +)ds = -\int_{\tilde{s}(a)} e^{-rt\wedge \tau} \phi_+(S_{t\wedge \tau})L^z_{t\wedge \tau} \nu(dz) \tag{3.14}
\]

Suppose \( \nu(\{\tilde{s}(x)\}) > 0 \) (i.e. \( K \) is not differentiable at \( \tilde{s}(x) \)). Then, (3.14) becomes

\[
\int_0^{t\wedge \tau} e^{-rs}(L^+ - r)V(S_s, +)ds = -e^{-rt\wedge \tau} \phi_+(S_{t\wedge \tau})(L^z_{t\wedge \tau} \nu(\tilde{s}(x))) + \int_{\tilde{s}(a)} \mathbb{1}_{z \neq \tilde{s}(x)} L^z_{t\wedge \tau} \nu(dz) \tag{3.15}
\]

We see the left hand side of (3.15) is absolutely continuous with respect to Lebesgue measure. However, the right hand side is not absolutely continuous since \( L^{\tilde{s}(x)} \) is singular with respect to Lebesgue measure. Hence, we find a contradiction, which implies \( V(x, +) \) is differentiable at \( x \) since \( \tilde{s} \in C^1 \). Therefore, as \( x \) is arbitrary and the left and right derivative of \( V(x, +) \) exist, we conclude \( V(x, +) \in C^1 \). The proof for \( V(x, -) \in C^1 \) is symmetric.

\[\Diamond\]

Remark 3.10. We might extend the dynamics of \( S \) to have multiple regimes on overlapping intervals in \( \mathbb{R}^+ \). The proof of Theorem 3.9 can be easily extended to show that the \( C^1 \) smoothness of the value function holds in this more general set-up.

We look for a measurable function \( v : E \to \mathbb{R} \) and a set \( \tilde{D} \) such that \( v \in D(L) \) and

\[
\mathbb{L}v - rv = 0 \text{ in } \tilde{C}, \tag{3.16}
\]

\[
v|_{\tilde{D}} = h|_{\tilde{D}}, \tag{3.17}
\]

\[
\frac{\partial v}{\partial x}|_{\partial \tilde{C}} = \frac{\partial h}{\partial x}|_{\partial \tilde{C}}, \tag{3.18}
\]

where \( \tilde{C} := \tilde{D}^c \). By Theorem 3.8 and 3.9, the value function \( V \) and stopping set \( D \) is a solution to the free boundary problem (under some assumptions). Moreover, if the drifts and volatilities are sufficiently smooth (so that fundamental solutions are \( C^2 \)), the usual argument (e.g. Section 7.1 in Peskir and Shiryaev [24]) allows us to replace condition (3.16) by ODEs (2.13) and (2.14).

We want to show that conversely, any solution to the free boundary problem gives the value function. This is done in Section 4.3 for the seller’s problem and in Section 5.2 for the buyer’s problem.

### 4 The seller’s problem

Let \( u : [0, \infty) \to \mathbb{R}_+ \) denote an increasing concave utility function. We assume \( u \) is \( C^2 \) in \((0, \infty)\), which meets Assumption 3.1. Moreover, we further assume \( u \) also satisfies Assumption 3.2. Replacing the gains function \( h \) in \( (P) \) by \( u \), the seller’s problem is defined by \( (SP) \).

\[
V(x, f) := \sup_{\tau} \mathbb{E}^{x,f}[e^{-r\tau}u(S_{\tau})]. \tag{SP}
\]

\[\text{For simplicity, from now on, when we refer to Assumption 3.1, we mean } u \in C^2(0, \infty).\]
By Theorem 3.7, the arg max of (SP) is given by $\tau_D$. So, our next step is to consider what additional assumptions are needed on $u$ to determine the shape of the stopping set $D$.

### 4.1 Assumptions and their motivations

Jacka and Maeda [14] fix the gains function to be $h : x \mapsto x$, and in this case, by assuming $\mu_- < r < \mu_+$, it is clear that the sign of $\mathcal{L}^h - rh$ is $f$. In a similar manner, we discuss here some reasonable assumptions to make on the sign of $\mathcal{L}^u - ru$. Consonant with the description ‘negative regime’, we assume that the sign of $\mathcal{L}^- u - ru$ is $-$. Moreover, because $u$ is an increasing concave function, it is reasonable to posit that, for large $x$, the sign of $\mathcal{L}^+ u - ru$ is $-$. So, we assume there is a constant $A$ such that, the sign of $\mathcal{L}^+ u - ru$ is $+$ for $x < A$ and $-$ for $x > A$. In short, we have:

**Assumption 4.1.**

\[
\begin{align*}
\mathcal{L}^- u - ru &< 0 \quad \text{in } [0, H]. \\
\mathcal{L}^+ u - ru &> 0 \quad \text{in } [L, A). \\
\mathcal{L}^+ u - ru &< 0 \quad \text{in } (A, \infty).
\end{align*}
\]

We will solve the seller’s and buyer’s problem under these three assumptions on the seller’s utility.

**Remark 4.2.** For some common choices of dynamics and utility functions (e.g., a geometric Brownian motion with a power utility function), $\mathcal{L}^+ u - ru$ can only have one sign. In this case, the optimal stopping time (in the positive regime) can be proven to be either $0$ or $\infty$, which is neither very interesting nor realistic. We observe that our assumptions are satisfied for a wide class of realistic dynamics and utility functions.

**Remark 4.3.** Note that neither does $u$ being a utility function imply Assumption 4.1 nor the reverse. For a reward function $u$ (not necessarily a utility function) satisfying Assumption 4.1, the results in the rest of this paper remain valid.

### 4.2 The shape of the stopping set, $D$

We define $D^+, C^+, D^-$ and $C^-$ by

\[
\begin{align*}
D^+ &= \{x \in (L, \infty) | V(x, +) = u(x)\}, \\
C^+ &= \{x \in (L, \infty) | V(x, +) > u(x)\}, \\
D^- &= \{x \in [0, H) | V(x, -) = u(x)\}, \\
C^- &= \{x \in [0, H) | V(x, -) > u(x)\}.
\end{align*}
\]

It is clear that $D = D^+ \times \{+\} \cup D^- \times \{-\}$ and $C = C^+ \times \{+\} \cup C^- \times \{-\}$.

**Theorem 4.4.** Under Assumptions 3.1, 3.2, and 4.1, $D^+ = [B, \infty)$ for some $B \in [A, \infty)$, and $D^- = [0, m]$ for some $m \in [0, H)$. 

12
Proof. Before we start, recall $D^+$ and $D^-$ are both closed as $V$ is lower semicontinuous. If $\exists y \in (L, A)$ such that $y \in D^+$, there is $\epsilon > 0$ making $L + \epsilon < y < A - \epsilon$. Define $\tau = \tau_{L+\epsilon}^+ \wedge \tau_{A-\epsilon}^-$. So $S_{\tau}^y$ is an Itô diffusion staring at $y$ with absorbing states $L + \epsilon$ and $A - \epsilon$. Then, by Dynkin’s formula,

$$\mathbb{E}^{(y, +)}[e^{-r\tau}u(S_{\tau})] = u(y) + \mathbb{E}^{(y, +)}\left[ \int_0^\tau e^{-r\tau}(\mathcal{L}^+ u(S_t) - ru(S_t))dt \right] > u(y) = V(y, +),$$

where the inequality follows from Assumption 4.1 (more specifically 4.2). But this contradicts the definition of $V$. Thus, $D^+ \cap (L, A) = \emptyset$.

Now, we need to show there are no gaps in $D^+$. Since $C^+$ is open, it can be written as countably unions of open intervals. Let $(y_1, y_2)$ be one of the intervals and assume $y_1, y_2 \in D^+$, and hence $y_2 > y_1 \geq A$. Take any $y \in (y_1, y_2)$ and define $\tau = \tau_{y_1}^+ \wedge \tau_{y_2}^-$. It is obvious that

$$V(y, +) = \mathbb{E}^{(y, +)}[e^{-r\tau}u(S_{\tau})] = u(y) + \mathbb{E}^{(y, +)}\left[ \int_0^\tau e^{-r\tau}(\mathcal{L}^+ u(S_t) - ru(S_t))dt \right] \leq u(y),$$

which contradicts the assumption that $y \in (y_1, y_2) \subset C^+$. This implies that $D^+$ must be an interval. Since $D^+$ is closed, we complete our claim apart from the special case where $D^+$ is empty.

To prove $D^-$ is an interval, suppose there are $y_1, y_2 \in D^-$ such that $H > y_2 > y_1 > 0$ and $(y_1, y_2) \subset C^-$. Take any $y \in (y_1, y_2)$ and define $\tau = \tau_{y_1}^- \wedge \tau_{y_2}^-$. It is obvious that

$$V(y, -) = \mathbb{E}^{(y, -)}[e^{-r\tau}u(S_{\tau})] = u(y) + \mathbb{E}^{(y, -)}\left[ \int_0^\tau e^{-r\tau}(\mathcal{L}^- u(S_t) - ru(S_t))dt \right] \leq u(y),$$

but this contradicts the inequality $V(y, -) > u(y)$ because $y \in C^-$. Therefore, $D^-$ is a closed interval.

Moreover, since $(0, -)$ is assumed to be an absorbing state, starting the process at $(0, -)$, we must have $e^{-r\tau}u(S_{\tau}) = e^{-r\tau}u(0) \leq u(0)$ for all stopping time $\tau$. Hence, $\mathbb{E}^{(0, -)}[\sup_\tau e^{-r\tau}u(S_{\tau})] \leq u(0)$, which implies $0 \in D^-$. So either $D^- = [0, m]$ for some $m \in [0, H]$ or $D^- = [0, H)$. To rule out the latter possibility, assume that it is true. Then $\forall \epsilon > 0, V(H - \epsilon, -) = u(H - \epsilon)$, which implies $\lim_{\epsilon \to 0} V(H - \epsilon, -) = u(H)$. However, because

$$V(H - \epsilon, -) \geq V(H, +)E^{(H-\epsilon,-)}[e^{-r\tau_H}1_{\tau_H<\tau_{H/2}}] + u(H/2)E^{(H-\epsilon,-)}[e^{-r\tau_{H/2}}1_{\tau_{H/2}<\tau_H}],$$

taking $\epsilon$ to 0, we can see that $\lim_{\epsilon \to 0} V(H - \epsilon, -) \geq V(H, +) > u(H)$ as $\mathbb{E}^{(H-\epsilon,-)}[e^{-r\tau_H}1_{\tau_H<\tau_{H/2}}]$ converges to 1 and $\mathbb{E}^{(H-\epsilon,-)}[e^{-r\tau_{H/2}}1_{\tau_{H/2}<\tau_H}]$ converges to 0 by continuity of $\phi_-$ and $\psi_-$. Therefore, by contradiction, $D^- \neq [0, H]$.

Finally, we shall rule out the case where $D^+$ is empty. Suppose $D^+ = \emptyset$. Denote $D_\epsilon := \{(x, f) \in E; V(x, f) \leq u(x + \epsilon)\}$. Since $D^-$ takes the form $[0, m]$, we know $D_\epsilon$ decreases to $D^- \times \{-\}$ as $\epsilon$ tends to 0. Hence, $\tau_{D_\epsilon}$ converges to $\tau_{D^-}$ a.s. Therefore, by the Dominated Convergence Theorem,

$$\lim_{\epsilon \to 0} \mathbb{E}^{x, f}_{\tau_{D_\epsilon}}[e^{-r\tau_{D_\epsilon}}u(S_{\tau_{D_\epsilon}})] = \mathbb{E}^{x, f}[e^{-r\tau_{D^-}}u(S_{\tau_{D^-}})] \leq u(m). \quad (4.4)$$
On the other hand,
\[ \mathbb{E}^{x,f}[e^{-rT_D}u(S_{T_{D^x}})] \geq \mathbb{E}^{x,f}[e^{-rT_D}(V(S_{T_{D^x}}, F_{T_{D^x}}) - \epsilon)] \geq \mathbb{E}^{x,f}[e^{-rT_D}V(S_{T_{D^x}}, F_{T_{D^x}})] - \epsilon = V(x, f) - \epsilon, \]
where the first inequality follows from the definition of \( D_\epsilon \) and the last equality follows from Theorem 2 in Chapter 3 of Shiryaev [32]. Letting \( \epsilon \) go to 0, we find
\[ \lim_{\epsilon \downarrow 0} \mathbb{E}^{x,f}[e^{-rT_D}u(S_{T_{D^x}})] \geq V(x, f). \] (4.5)

Therefore, as the left hand side of (4.4) and (4.5) are the same, we have \( u(m) \geq V(x, f) \) for any \((x, f) \in E\). However, since \( u \) is increasing, \( V(x, f) \geq u(x) \) for any \((x, f) \in E \setminus ([0, m] \times \{-\})\), which results in a contradiction.

\section{4.3 The value function}

Recalling the definition of the free boundary problem (3.16)-(3.18), we have shown that \( V \) is a solution to the free boundary problem (suitably modified when \( m = 0 \), where the smooth pasting condition (3.18) only holds at \( B \)).

Now we want to show the converse, namely that \( V \) is the unique solution of the free boundary problem based on \( L \) (instead of \( \mathbb{L} \)). So, we seek a pair of functions \( v(x, f) \in C^2 \), constants \( \tilde{B} \geq A \) and \( \tilde{m} < H \), such that
\[
\begin{align*}
L^+ v(\cdot, +) - rv(\cdot, +) &= 0, \text{ in } (L, \tilde{B}) \quad \text{(4.6)} \\
L^- v(\cdot, -) - rv(\cdot, -) &= 0, \text{ in } (\tilde{m}, H) \quad \text{(4.7)} \\
v(\tilde{B}, +) &= u(\tilde{B}), \quad v(L, +) = v(L, -)1_{\tilde{m} < L} + u(L)1_{\tilde{m} \geq L}, \quad \text{(4.8)} \\
v(H, -) &= v(H, +), \quad v(\tilde{m}, -) = u(\tilde{m}), \quad \text{(4.9)} \\
\frac{\partial v}{\partial x}(x, +; \tilde{B}, \tilde{m}) \bigg|_{x = \tilde{B}} &= u'(\tilde{B}), \quad \text{(4.10)} \\
\frac{\partial v}{\partial x}(x, -; \tilde{B}, \tilde{m}) \bigg|_{x = \tilde{m}} &= u'(\tilde{m}) \quad \text{if } \tilde{m} > 0. \quad \text{(4.11)}
\end{align*}
\]

The following theorem provides an approach to derive the value function, which is the main result in this section.

\textbf{Theorem 4.6.} Under Assumptions 3.1, 3.2, 4.1, and 4.5, a (classical) solution to the free boundary problem (4.6)-(4.11) exists. Let \((B^*, m^*)\) and \(v(x, f; B^*, m^*)\) denote the solution. Define \( V : E \to \mathbb{R} \) by:
\[
V(x, f) = \begin{cases} 
  v(x, f; B^*, m^*) & \text{if } x \in (L, B^*), \ f = + \ or \ x \in (m^*, H), \ f = -; \\
  u(x) & \text{if } x \in [B^*, \infty), \ f = + \ or \ x \in [0, m^*], \ f = -.
\end{cases} \quad \text{(4.12)}
\]

Then, \( V = V \) on \( E \) and \((B^*, m^*) = (B, m)\).
The following lemma provides sufficient conditions for a function to be the value function.

**Lemma 4.7.** Let \( V(x, f) \) denote a function on \( E \). Define \( N_t := e^{-rt}V(S_t, F_t) \). If \( N \) satisfies P1, P2 and P3 defined as follows:

(P1) \( N_t \) is a class D supermartingale,

(P2) \( \exists \tau < \infty \) a.s. such that \( N_0 = \mathbb{E}^{x,f}[e^{-r\tau}u(S_\tau)] \),

(P3) \( N_t \geq e^{-rt}u(S_t) \) for all \( t \geq 0 \)

then, \( V(x, f) = V(x, f) \).

**Proof.** By the Optional Sampling Theorem for class D supermartingales (see Rogers and Williams [28] pp.189), for any stopping time \( \tau \),

\[
V(x, f) = N_0 \geq \mathbb{E}^{x,f}[e^{-r\tau}V(S_\tau, F_\tau)] \geq \mathbb{E}^{x,f}[e^{-r\tau}u(S_\tau)],
\]

where the last inequality follows from P3. Since (4.13) holds for any \( \tau \), we get \( V(x, f) \geq V(x, f) \).

On the other hand, by P2, for some \( \tau \),

\[
V(x, f) = N_0 = \mathbb{E}^{x,f}[e^{-r\tau}u(S_\tau)],
\]

and hence \( V(x, f) \leq V(x, f) \). \( \Box \)

We can now prove Theorem 4.6.

**Proof of Theorem 4.6**. Since the dynamics are Hölder-continuous and \( L^\pm \) are strictly elliptic, the fundamental solutions \( \phi^\pm \) and \( \psi^\pm \) are \( C^2 \) in \((0, \infty)\). Thus, by Theorem 3.8, 3.9, and 4.4, we can see \( V \) is a solution to the free boundary problem defined via conditions (4.6) to (4.11). Conversely, let \( v \) denote a solution to the free boundary problem with boundaries \( B^* \) and \( m^* \). Define \( N_t := e^{-rt}V(S_t, F_t) \) where \( V \) is defined by (4.12). According to Lemma 4.7 to show \( V \) is the value function, it is sufficient to prove \( N \) satisfies P1-P3.

Firstly, it is obvious that \( V(0, -) = u(0) = V(0, -) \). Now, take an arbitrary initial position \((x, f) \in E \setminus \{(0, -)\}\).

(P1) As \( v(x, +) \) and \( v(x, -) \) are \( C^1 \) on compact domains, they are bounded by some finite constant \( M \). So,

\[
|N_t| = e^{-rt}|V(S_t, F_t)| \leq e^{-rt}(|M| \vee |u(S_t)|) \leq |M| \vee e^{-rt}|u(S_t)|,
\]

Hence,

\[
\mathbb{E}^{x,f}[\sup_\tau |N_\tau|] \leq \mathbb{E}^{x,f}[|M| \vee \sup_\tau e^{-r\tau}|u(S_\tau)|] \leq |M| + \mathbb{E}^{x,f}[\sup_\tau e^{-r\tau}|u(S_\tau)|] < \infty,
\]

which implies that \( N_t \) is of class D.

Now, we are going to show \( N_t \) is a supermartingale. Using Peskir’s change-of-variable formula
with local time (see [25]), as $S_t$ is a continuous semimartingale and $V(x,f)$ is a piecewise $C^2$ function of $x$ given $f$, it follows

$$dN_t = e^{-rt} \left[ -rV(S_t, +) + \mathcal{L}^+ V(S_t, +) \mathbb{1}_{(F_t = +, S_t \neq B^*)} dt + \mathcal{L}^- V(S_t, -) \mathbb{1}_{(F_t = -, S_t \leq m^*)} dt \right. \\
+ \left. \frac{\partial V}{\partial x}(S_t, F_t) \sigma F_t(S_t) \mathbb{1}_{(S_t, F_t) \notin \{(B^*, +), (m^*, -)\}} dW_t \right. \\
+ \left. \frac{1}{2} \left( \frac{\partial V}{\partial x}(m^+, -; m^*, B^*) - \frac{\partial V}{\partial x}(m^-, -; m^*, B^*) \right) \mathbb{1}_{(F_t = -, m^* \neq 0)} dW^*_t(S) \right]$$

$$dN_t = e^{-rt} \left[ -ru(S_t) \mathbb{1}_{(F_t = -, S_t \leq m^*)} dt + \mathcal{L}^+ u(S_t) \mathbb{1}_{(F_t = +, S_t > B^*)} dt \right. \\
+ \left. \frac{\partial V}{\partial x}(S_t, F_t) \sigma F_t(S_t) \mathbb{1}_{(S_t, F_t) \notin \{(B^*, +), (m^*, -)\}} dW_t \right].$$

(4.16)

Since $\mathcal{L}^- u - ru \leq 0$ on $[0, m^*]$ and $\mathcal{L}^+ u - ru \leq 0$ on $[B^*, \infty)$, we can conclude that the drift terms are non-positive. Moreover, we can find a localising sequence for the $dW_t$ term such that the stopped process is a martingale. Thus, we conclude that $N_t$ is a local supermartingale. Since it is also class D, $N_t$ is therefore a supermartingale.

(P2) Define $\tau_{B^*}^{m*} := \inf\{t \geq 0; S_t \leq m^* \text{ if } F_t = - \text{ or } S_t \geq B^* \text{ if } F_t = +\}$. Suppose for the starting position $(x,f)$ we have that $x \geq m^*$ if $f = -$ and $x \leq B^*$ if $f = +$. Applying Itô’s formula to $N_{t \wedge \tau_{B^*}^{m*}}$, we obtain:

$$dN_{t \wedge \tau_{B^*}^{m*}} = \mathbb{1}_{t \wedge \tau_{B^*}^{m*}} e^{-rt} \left[ -rv(S_t, -) \mathbb{1}_{(F_t = -, S_t \leq m^*)} dt + \mathcal{L}^+ v(S_t, +) - rv(S_t, +) \mathbb{1}_{(F_t = +)} dt \right.$$

$$+ \left. \frac{\partial v}{\partial x}(S_t, F_t) \sigma F_t(S_t) dW_t \right].$$

Again, by (4.6) and (4.7), the drift terms vanish, which implies $\mathbb{E}^{x,f}[N_{t \wedge \tau_{B^*}^{m*}}] = N_0$. Thus, $\mathbb{E}^{x,f}[N_{t \wedge \tau_{B^*}^{m*}}] = N_0$ by the Dominated Convergence Theorem.

(P3) It is sufficient to show $v(x, +) \geq u(x)$ on $[L, B^*]$ and $v(x, -) \geq u(x)$ on $[m^*, H]$.

Let’s start by proving the second inequality. Define $g(x) := v(x, -) - u(x)$. By Assumption 4.1, $L^- g - rg \geq 0$ on $[m^*, H]$. Now we further define $g_\epsilon(x) := g(x) + \epsilon \psi_-(x)$ where $\psi_-(x)$ is the strictly increasing fundamental solution for the diffusion process $S^-$ as before (which solves (2.14)). Hence,

$$L^- g_\epsilon - rg_\epsilon = ru - L^- u \geq 0.$$
\( g \geq 0 \). Therefore, \( v(x, -) \geq u(x) \).

To show \( v(x, +) \geq u(x) \) on \([L, B^*]\), we can now define \( g \) by \( g(x) := v(x, +) - u(x) \) so \( g \) satisfies \( \mathcal{L}^+ g - rg \leq 0 \) on \((L, A)\) and \( \mathcal{L}^+ g - rg \geq 0 \) on \([A, B^*]\).

For \( x \in [A, B^*] \), as before, define \( g_\epsilon(x) = g(x) + \epsilon \phi_+(x) \) where \( \phi_+(x) \). Hence,

\[
\mathcal{L}^+ g_\epsilon - rg_\epsilon = ru - \mathcal{L}^+ u \geq 0.
\]

Therefore, \( \mathcal{L}^+ g_\epsilon - rg_\epsilon \) is non-negative on \([A, B^*]\). Moreover, we have \( g_\epsilon(B^*) = \epsilon \phi_+(B^*) > 0 \), and \( \frac{\partial g_\epsilon}{\partial x} \bigg| _{x=B^*} = \phi'_-(B^*) < 0 \). By the strong maximum principle, there is no positive maximum of \( g \) on \((A, B^*)\). So \( g_\epsilon \) must be strictly decreasing and hence positive. Let \( \epsilon \) go to 0 to see \( g \geq 0 \) on \([A, B^*]\).

Finally, for \( x \in [L, A] \), we know \( \mathcal{L}^+ g - rg \leq 0 \) and \( g(A) \geq 0 \). We further notice that \( v(L, +) = u(L)\mathbb{1}_{m^\star \geq L} + v(L, -)\mathbb{1}_{m^\star < L} \geq u(L) \) since we have shown that \( v(x, -) \geq u(x) \). Thus, by the strong minimum principle, there is no negative minimum of \( g \) on \((L, A)\), which ensures \( g \geq 0 \). So we conclude that \( v(x, +) \geq u(x) \) on \([L, A]\).

\[\Box\]

### 4.4 An example

Here we present a simple example where we calculate all quantities in closed form. Section 6 will present a numerical approach to treat more realistic examples.

Let \( u(x) = x^\gamma, \gamma \in (0, 1) \). We set \( \mu_- = \mu_-, \sigma_- = \sigma_-, \mu_+ = \mu_+(x + 1), \sigma_+ = \sigma_+ x \), where \( \mu_-, \sigma_-, r, \sigma_- \) and \( \sigma_+ \) are all positive constants. It is not hard to see that Assumption 3.1, 3.2, 4.1 and 4.5 all hold. To have a closed form solution, we further assume \( \mu_+ = \sigma_+^2 = r = c \) (later in Section 6 we will relax this assumption to obtain numerical solutions).

Thus, we can find the value function by solving the free boundary problem.

The ODE

\[
\mathcal{L}^- v - rv = \frac{1}{2} \sigma^2 x^2 v''(x) + \mu_- x v'(x) - rv(x) = 0,
\]

admits a general solution of the form \( v(x, -) = C_3 x^\alpha + C_4 x^\beta \) where \( \alpha \) and \( \beta \) are the roots of \( \frac{1}{2} \sigma^2 - (\mu_- - \frac{1}{2} \sigma^2) - r = 0 \). The ODE

\[
\mathcal{L}^+ v - rv = \frac{1}{2} r x^2 v''(x) + r(x + 1) v'(x) - rv(x) = 0
\]

has general solution \( v(x, +) = C_1(x - 1)e^{\frac{2}{r}} + C_2(x + 1) \). Now, set \( \gamma = 0.8, \mu_- = 1/30, \sigma^2 = 1/30, r = 0.1, L = 1, \) and \( H = 2 \). After some calculations, one obtains \( A = 20/7 \) and \( v(x, -) = C_3 x^{-3} + C_4 x^2 \). Assume \( m^\star \geq 1 \), we compute the value of \( B^* \) by condition (4.8) and (4.10). Numerical approximation gives \( B^* = 3.839282 \) and \( v_1(x, +) = 0.1075171(x - 1)e^{\frac{2}{r}} + 0.5(x + 1) \). Therefore by (4.9) and (4.11), we compute \( m^\star = 1.775502 \) and \( v_1(x, -) = 2.126333x^{-3} + 0.3816175x^2 \). Thus, by Theorem 4.6 we derive the value function \( V(x, f) \):

\[
V(x, f) = \begin{cases} 
0.1075171(x - 1)e^{\frac{2}{r}} + 0.5(x + 1) & \text{if } x \in (1, 3.839282), f = + \\
2.126333x^{-3} + 0.3816175x^2 & \text{if } x \in (1.775502, 2), f = - \\
x^{0.8} & \text{otherwise}.
\end{cases}
\]
The optimal strategy is to sell the stock when its price is higher than 3.839282 in the positive regime or lower than 1.775502 in the negative regime.

5 The buyer’s problem

If traders want to find the best time to purchase a stock and sell it later to maximise their incremental expected utility, they will try to solve the following optimal stopping problem:

$$V_p(x, f) := \sup_{\tau} E_x[f e^{-r\tau} \{V(S_\tau, F_\tau) - u(S_\tau)\}]$$

where $V$ is the value function from the seller’s problem (SP). This defines the buyer’s problem.

Let $g(x, f) := V(x, f) - u(x)$. Hence $g(x, f)$ is the gains function and for fixed $f$, it is a $C^1$ and piecewise $C^2$ function on $\mathbb{R}_+$, which has value 0 on $[0, m] \times \{-\} \cup [B, \infty) \times \{+\}$ and is positive elsewhere. We can see $g$ still satisfies Assumptions 3.1, 3.2 (the proof can be found in Lemma A.1). Moreover, by Assumptions 4.1, $g$ satisfies the following conditions

$$\mathcal{L}^+ g(x, +) - rg(x, +) < 0 \quad \text{for } x \in (L, A). \quad (5.1)$$

$$\mathcal{L}^+ g(x, +) - rg(x, +) > 0 \quad \text{for } x \in (A, B]. \quad (5.2)$$

$$\mathcal{L}^- g(x, -) - rg(x, -) > 0 \quad \text{for } x \in [m, H). \quad (5.3)$$

Let $D_p$ and $C_p$ denote the stopping set and continuation set for the buyer’s problem. By Theorem 3.7, since Assumption 3.1 and 3.2 hold, $\tau_{D_p}$ is the optimal stopping time.

5.1 The shape of the stopping set $D_p$

Similarly to the seller’s problem, we define analogously the stopping sets $D_p^+$ and $D_p^-$.  

**Theorem 5.1.** Under Assumptions 3.1, 3.2, and 4.1, $D_p^+ = [a, b]$ where $L < a < b \leq A$ and $D_p^- = \emptyset$.

**Proof.** If $S_0 = 0$, then $g(S_t, F_t) = 0$ for any $t$ and hence $V_p(0, -) = 0 = g(0, -)$, which implies $0 \in D_p^-$. If $x \in (0, m]$, set $\tau := \tau_{mH}^- \land \tau_0^-$. Then, $\mathbb{E}[e^{-r\tau} g(S_\tau, F_\tau)] > 0 = g(x, -)$. So $D_p^- \cap (0, m] = \emptyset$.

Moreover, for $x \in (m, H)$, we can define $\tau := \tau_m^- \land \tau_H^-$. Then by (5.3),

$$\mathbb{E}^{x-} [e^{-r\tau} g(S_\tau, F_\tau)] = g(x, -) + \mathbb{E}^x \left[ \int_0^\tau \mathcal{L}^- g(S_{\tau}^-, -) - rg(S_{\tau}^-, -)dt \right] > g(x, -). \quad (5.4)$$

Thus, $D_p^- \cap (m, H) = \emptyset$, and this shows $D_p^- = \emptyset$.

For $D_p^+$, suppose it is empty. Then, $\tau_{D_p} = \tau_0$, which implies $V = 0$ on $E$. However, this contradicts the definition of $C_p^+$ because $g(x, +) > 0 = V(x, +)$ for $x < B$. Thus, $D_p^+ \neq \emptyset$.

Since $g(x, +) = 0$ in $[B, \infty)$, it is clear that $D_p^+ \cap [B, \infty) = \emptyset$. By (5.2), we can make a very similar argument to show $D_p^+ \cap (A, B) = \emptyset$. Now we claim $D^+$ is an interval. If it is not, then
there exist $L < y_1 < y < y_2 < A$ such that $y_1, y_2 \in D_p^+$ and $y \in (y_1, y_2) \subset C_p^+$. Let $\tau := \tau_{y_1}^+ \wedge \tau_{y_2}^+$. By strong Markov property, 

$$V_p(y, +) = E^{y, +}[e^{-r\tau} g(S_{\tau}, F_{\tau})] = g(y, -) + E^y [\int_0^{\tau} \mathcal{L}^+ g(S_t^+, +) - r g(S_t^+, +) dt] < g(y, -). \quad (5.5)$$

By contradiction, $D_p^+$ must be an interval. Now we claim that $L \notin \partial D_p^+$ (where $\partial$ denotes the boundary of a set). Suppose $L \in \partial D_p^+$, i.e. $D_p^+$ is of the form $(L, b)$ for some $b \in (L, A]$ (we know $D_p^+$ is closed). Then $\lim_{x \to L} V_p(x, +) = \lim_{x \to L} g(x, +) = g(L, -)$. However we also have

$$V_p(x, +) = V_p(L, -) E^{x, +}[e^{-r\tau} \mathbf{1}_{\tau_A^+ < t_A^+}] + g(A, +) E^x [e^{-r\tau} \mathbf{1}_{\tau_A^+ < t_A^+}]. \quad (5.6)$$

Taking $x$ down to $L$ in (5.6), we can conclude $\lim_{x \to L} V_p(x, +) \geq V(L, -) > g(L, -)$ (since $L \notin D_p^-$), which gives a contradiction. Therefore, putting every pieces together, $D_p^+$ has to be of the form $[a, b]$ where $L < a < b \leq A$.

Thus, the trader should only buy the stock when it is in the positive regime and its price falls into the interval $[a, b]$.

Note that $b > L$. If the initial position $x$ is greater than $b$ and the stock is in the positive regime, then we will stop before $S$ hit $L$, which means no regime transition at all. Hence, the value of $V_p$ on $[b, \infty)$ is easy to compute.

**Lemma 5.2.** $V_p(x, +) = \frac{g(b, +)}{\phi+(b)} \phi_+(x)$ for $x \in [b, \infty)$.

**Proof.** Let $x \in [b, \infty)$. Then, $V_p(x, +) = E^{x, +}[e^{-r\tau} g(S_{\tau}, F_{\tau})] = E^x [e^{-r\tau} g(S_{\tau}^+, +)] = g(b, +) E^x [e^{-r\tau}].$ By strong Markov property, $\phi_+(x) = E^x [e^{-r\tau}] \phi_+(b).$ Thus, $V_p(x, +) = \frac{g(b, +)}{\phi+(b)} \phi_+(x).$ \hfill \qed

### 5.2 The value function

Similarly to the seller’s problem, the free boundary problem corresponding to the buyer’s problem is as follows. We look for a pair of functions $v(x, f) \in C^2$, constants $\hat{a} \geq L$ and $\hat{b} \in (\hat{a}, A]$, such that

$$\mathcal{L}^+ v_p(\cdot, +) - r v_p(\cdot, +) = 0, \text{ in } (L, \hat{a}) \quad (5.7)$$

$$\mathcal{L}^- v_p(\cdot, -) - r v_p(\cdot, -) = 0, \text{ in } (0, H) \quad (5.8)$$

$$v_p(\hat{a}, +) = g(\hat{a}, +), \quad v_p(L, +) = v_p(L, -) \quad (5.9)$$

$$v_p(H, -) = k(\hat{a}, \hat{b}), \quad v_p(0, -) = 0 \quad (5.10)$$

$$\frac{\partial v_p}{\partial x}(x, +) \bigg|_{x=\hat{a}} = g'(\hat{a}, +) \quad (5.11)$$

$$\frac{d}{dx} \frac{g(\hat{b}, +)}{\phi_+(\hat{b})} \phi_+(x) \bigg|_{x=b} = g'(\hat{b}, +) \quad (5.12)$$

where $k(\hat{a}, \hat{b}) := v_p(H, +) \mathbf{1}_{\{H \geq \hat{a}\}} + g(H, +) \mathbf{1}_{\{\hat{a} < H \leq \hat{b}\}} + \frac{g(b, +)}{\phi_+(b)} \phi_+(H) \mathbf{1}_{\{b < H\}}.$
We are now ready to present the main result in this section: the solution to the buyer's problem.

**Theorem 5.3.** Under Assumption 3.1, 3.2, 4.1, and 4.5, the free boundary problem defined via conditions (5.7) to (5.12) admits a solution \( v_p, a^*, \) and \( b^* \). Moreover, \( V_p(x, f) \) defined by (5.13) is equal to the value function \( V_p(x, f) \) and \( \tau := \tau_{[a^*, b^*]}^+ \) is the optimal stopping time in (BP).

\[
V_p(x, f) = \begin{cases} 
  v_p(x, f; a^*, b^*) & \text{if } x \in (L, a^*], \\ 
g(x, +) & \text{if } x \in (a^*, b^*], \\ 
g(b^*, +) \phi_+(x) & \text{if } x \in (b^*, \infty), 
  \end{cases} 
\tag{5.13}
\]

**Proof.** By Theorem 3.8, 3.9, and 5.1, \( V_p \) is indeed a solution to the free boundary problem. Conversely, let \( (x, f) \in E \setminus \{(0, -)\} \) and assume we have a solution denote by \( v_p, a^*, \) and \( b^* \) to the free boundary problem. Define \( N_t := e^{-rt}V_p(S_t, F_t) \). According to Lemma 4.7, to show \( V_p \) is the value function, it is sufficient to prove \( N_t \) satisfies (P1-P3).

(P1) \( |V_p| \) is bounded by some constant \( K \). So \( \mathbb{E}^{x,f}[\sup_t N_t] \leq K \), which implies class D. Using Peskir’s change-of-variable formula with local time (see [25]), as \( S_t \) is a continuous semimartingale and \( V_p(x, f) \) is a piecewise \( C^2 \) function of \( x \) given \( f \), it follows

\[
dN_t = e^{-rt} \left[ \left( -rV_p(S_t, +) + \mathcal{L}^+ V_p(S_t, +) \right) 1_{\{F_t=+, S_t \neq a^*\}} 1_{\{F_t=+, S_t \neq b^*\}} dt + \left( -rV_p(S_t, -) + \mathcal{L}^- V_p(S_t, -) \right) 1_{\{F_t=-\}} dt + \frac{\partial V_p}{\partial x}(S_t, F_t) \sigma F_t(S_t) 1_{\{F_t=\}, \{S_t \neq a^*\}, \{S_t \neq b^*\}} dW_t \right. \\
+ \frac{1}{2} \left( \frac{\partial^2 V_p}{\partial x^2}(a^*, +; a^*, b^*) - \frac{\partial^2 V_p}{\partial x^2}(a^-, +; a^*, b^*) \right) 1_{\{F_t=-\}} dt + \\
+ \frac{1}{2} \left( \frac{\partial^2 V_p}{\partial x^2}(b^*, +; a^*, b^*) - \frac{\partial^2 V_p}{\partial x^2}(b^-, +; a^*, b^*) \right) 1_{\{F_t=+\}} dt + dW_t(S) \right].
\tag{5.14}
\]

By smooth pasting principles (5.11) and (5.12), the local time terms in (5.14) disappears. Recall \( v_p \) satisfies (5.7) and (5.12). By the construction of \( V_p \), equation (5.14) becomes

\[
dN_t = e^{-rt} \left[ \left( \mathcal{L}^+ g(S_t, +) - rg(S_t, +) \right) 1_{\{F_t=+, a^* < S_t < b^*\}} dt + \frac{\partial V_p}{\partial x}(S_t, F_t) 1_{\{F_t=\}, \{S_t \neq a^*\}, \{S_t \neq b^*\}} dW_t \right].
\]

Since \( \mathcal{L}^+ g(x, +) - rg(x, +) < 0 \) on \( [a^*, b^*] \), we can conclude the drift terms are non-positive. Moreover, by smoothness of \( v_p \) and \( g \), we have \( \frac{\partial V_p}{\partial x}(S_t, F_t) 1_{\{F_t=\}, \{S_t \neq a^*\}, \{S_t \neq b^*\}} \) is locally bounded. We also have \( \sigma \) is bounded locally. These together imply that the \( dW_t \) term is a local martingale. Thus, we conclude that \( N_t \) is a local supermartingale. Since \( N \) is class D, it is also a supermartingale.

(P2) Define \( \tau_{b^*}^a := \inf\{t \geq 0; S_t \leq a^* \text{ if } F_t = + \text{ or } S_t \geq b^* \text{ if } F_t = +\} \). Suppose for the initial position \((x, f)\) we have that \( x \leq a^* \) or \( x \geq b^* \) when \( f = + \). Applying Itô's formula to \( N_{t \wedge \tau_{b^*}^a} \), we
By the strong maximum principle, there is no positive maximum on \((b^*, A)\). Hence, \(dW_t = 0\), which is the expectation of the \(dW_t\) term should be 0. Let \(t\) go to infinity to see that \(E^x f[N_{t\wedge \tau_{b^*}}] = 0\) by dominated convergence theorem. For other initial positions \((x, f)\), it is trivial that \(N_{t\wedge \tau_{b^*}} = 0\), which also leads to \(E^x f[N_{t\wedge \tau_{b^*}}] = 0\).

(P3) It is sufficient to show \(v_p(x, -) \geq g(x, -)\) on \([m, H]\), \(v_p(x, +) \geq g(x, +)\) on \([L, a^*]\), and \(\frac{g(b^*, +)}{\phi_+(b^*)} \phi_+(x) \geq g(x, +)\) on \([b^*, B]\). Let’s start with proving the third inequality. Define \(h(x) := \frac{g(b^*, +)}{\phi_+(b^*)} \phi_+(x) - g(x, +)\), and we have \(\mathcal{L}^+ h - rh \geq 0\) on \([b^*, A]\) and \(\mathcal{L}^+ h - rh \leq 0\) on \([A, B]\). Now we further define \(h_\epsilon(x) := h(x) + \epsilon \phi_+(x)\). Hence

\[\mathcal{L}^+ h_\epsilon - r h_\epsilon = \mathcal{L}^+ h - rh \geq 0\] on \([b^*, A]\).

By the strong maximum principle, there is no positive maximum on \((b^*, A)\). Because \(h_\epsilon(b) > 0\) and \(h_\epsilon'(b) > 0\), \(h_\epsilon\) must be strictly increasing and hence positive. Let \(\epsilon\) go to 0 to see \(h \geq 0\). Therefore, \(\frac{g(b^*, +)}{\phi_+(b^*)} \phi_+(x) \geq g(x)\). Moreover, as \(\mathcal{L}^+ h - rh \leq 0\) on \([A, B]\), by strong minimum principle, there is no negative minimum on \((A, B)\). As \(h(A) \geq 0\) and \(h(B) > 0\), we can see \(h(x) \geq 0\) on \([A, B]\).

To show \(v_p(x, +) \geq g(x, +)\) on \([L, a^*]\), we first define \(h(x) := v_p(x, +) - g(x, +)\). Then, further define \(h_\epsilon(x) := h(x) + \epsilon \phi_+(x)\). Hence we have

\[\mathcal{L}^+ h_\epsilon - r h_\epsilon = \mathcal{L}^+ h - rh \geq 0\] on \([L, a^*]\)

By the strong maximum principle, there is no positive maximum on \((b^*, A)\). Moreover, \(h_\epsilon(a) > 0\) and \(h_\epsilon'(a) < 0\), we must have \(h_\epsilon\) being strictly increasing and hence positive. Let \(\epsilon\) go to 0 to see \(h \geq 0\).

Finally, we can define \(h(x) := v_p(x, -) - g(x, -)\). Then, \(\mathcal{L}^+ h_\epsilon - r h_\epsilon \leq 0\) on \([m, H]\). By strong minimum principle, there is no negative minimum on \((m, H)\). However, since \(h(H) = v_p(H, +) - g(H, +) \geq 0\) and \(h(m) \geq v_p(m, -) > 0\), \(h\) must stay non-negative, i.e. \(h(x) \geq 0\) on \([m, H]\).
5.3 Example revisited

Recall the example studied in Section 4.4. We now solve the purchase problem. By Theorem 5.3, we can compute \( a^* = 1.1632 \) and \( b^* = 2.1686 \). Thus, the value function is given below.

\[
V_p(x, f) = \begin{cases} 
0.0408(x - 1)e^{\frac{x}{2}} + 0.0138(x + 1) & \text{if } x \in (1, 1.1632), f = + \\
0.1075(x - 1)e^{\frac{x}{2}} + 0.5(x + 1) - x^{0.8} & \text{if } x \in [1.1632, 2.1686], f = + \\
-0.1858(x - 1)e^{\frac{x}{2}} + 0.1858(x + 1) & \text{if } x \in (2.1686, \infty), f = + \\
0.0277x^2 & \text{if } x \in (0, 2), f = - 
\end{cases}
\]

(5.15)

6 Optimal trading strategies and degrees of relative risk aversion

In previous sections, we studied the seller’s and buyer’s problems. In summary, we identified four price levels, namely \( B, m, b, a \), which together determine the optimal trading strategies. In this section, we are going to explore the relation between these price levels and degrees of relative risk aversion. To do this, we first need to implement an algorithm which allows us to estimate these stopping boundaries numerically.

6.1 The numerical algorithm

The algorithm needs to numerically solve two free boundary problems defined via (4.6)-(4.11) and (5.7)-(5.12). Essentially, for each free boundary problem, we need to solve two linear second order ODEs that are linked via boundary conditions where the boundaries are estimated simultaneously. The numerical methods for solving an ODE are well established. For each iteration with a different boundary value, we numerically solve the corresponding boundary value problem (BVP) and check the smooth pasting condition. In the case where the boundary condition is given by the solution of the other BVP (e.g. \( v(L, +) = v(L, -) \)), we have to loop through different values of the boundary, and in every iteration, we numerically solve the linked BVPs with the aid of the smooth pasting conditions and check the boundary condition.

Figure 1 provides more details about the algorithm implemented for the seller’s problem. Essentially, we begin with the assumption that \( m \geq L \) so that the BVP for the positive regime can be solved in isolation, which gives \( v(x, +) \). Then, we compute \( v(x, -) \) and check whether \( m \geq L \). If not, then we have to set initially the boundary condition \( v(L, \pm) = u(L) \) and continue the computation as Figure 1 indicates until the boundary condition \( v(H, +) = v(H, -) \) is (approximately) satisfied. Whenever “compute” appears in Figure 1, we mean numerically solve the related BVPs.

In the following sections, we take the utility function to be a power function of the form \( u(x) = x^\gamma \) and the dynamics in the negative regime are of the form \( \mu_-(x) = \mu_- x \) and \( \sigma_-(x) = \sigma_- x \).


6.2 Affine drift with linear volatility

Recall the example studied in Section 4.4 and 5.3 where we have parameters $\mu_\pm, \sigma_\pm$ and $c$. Table 1 summarises the values of all parameters. Note the assumption $\mu_+ = \sigma_+^2 = r = c$ is relaxed. We vary $\gamma$ between 0.7 and 0.95. Then, we can compute $B, m, b, a$ accordingly. The results are plotted in Figure 2 where the solid horizontal line is the reference line representing $L = 1$.

Table 1: Parameter values.

| $\mu_+$ | $\mu_-$ | $\sigma_+^2$ | $\sigma_-^2$ | $c$ | $r$ | $L$ | $H$ |
|---------|---------|--------------|--------------|-----|-----|-----|-----|
| 0.15    | 1/30    | 0.1          | 1/30         | 0.16| 0.15| 1   | 2   |

We observe some patterns in Figure 2. For the seller’s problem, as $\gamma$ increases, $B$ increases exponentially fast, and $m$ decreases roughly linearly and never drops below $L$. Moreover, for the buyer’s problem, as $\gamma$ increases, $b$ increases exponentially fast and $a$ increases approximately linearly. These patterns can be explained qualitatively. Since $1 - \gamma$ is equal to the relative
risk aversion for the power utility, it is obvious that as $\gamma$ increases, the degree of risk aversion decreases. Therefore, as $\gamma$ increases, traders are increasingly happier to afford more risk, which implies optimally traders shall sell at a higher profit-taking boundary $B$ or a lower stop-loss boundary $m$. Similarly, the increase of $b$ and decrease of $a$ are easily understood.

From Figure 2, we can clearly see how the magnitude of the trader’s risk preference influences the optimal trading strategies. When a trader is risk-neutral, as Jacka and Maeda [14] observe, the profit-taking boundary $B$ is equal to infinity. Then, as the trader becomes more risk-averse, $B$ quickly decreases to a comparably small level. On the other hand, the stop-loss boundary $m$ is relatively stable with regard to the change of risk aversion.

We are now at a good point to explain the relation between the standard trading rule (recall trading strategies BL and SH from Section 1) and the optimal solution derived in our model. If traders follow BL (SH), they should buy (sell) at a price between 1 and 2 in the positive (negative) regime. Assuming that the trader’s $\gamma$ is below 0.75, we can see that, since $a$ is close to 1 and $b$ is close to 2, BL would result in the trader behaving approximately optimally. However, in the negative regime, the optimal trading strategy looks very different from SH, because the trader optimally sells at any price levels below $m$ (instead of only at somewhere between 1 and 2) and not to sell at price levels above $m$. Hence, we find that there is some inconsistency between the standard trading rule and our optimal trading strategy.

How should we understand this? When technical analysts trade under the assumption of a resistance line, they think of it as a true reflecting boundary, i.e. the price must go down once
Table 2: Parameter values.

| $\mu_+$ | $\mu_-$ | $\sigma_+^2$ | $\sigma_-^2$ | $c$ | $r$ | $L$ | $H$ |
|----------|----------|---------------|---------------|-----|-----|-----|-----|
| 0.1      | 1/30     | 0.1           | 1/30          | 0.7 | 0.1 | 1   | 2   |

it reaches this price level. This naturally leads to arbitrage opportunities (e.g. short sell at the resistance line) — there is no EMM for such a model. We can see, under this scenario, it would be clearly optimal to sell the stock at the resistance line. However, our model does not permit arbitrage. So, fundamentally, we are in a different situation comparing to where the SH is optimal. Instead of merely modelling the concept of a resistance line, our stock dynamics model the idea of the resistance line being *broken-through* (the old resistance line becomes the new support after it gets broken from below), which is compatible with technical analysis. This explains why sell-at-high is not the best trading strategy in our model. Indeed, no matter how strong the downwards drift around the resistance line, Theorem 4.6 tells us that it is optimal not to sell when the price is near $H$ in the negative regime.

We can provide another viewpoint on the inconsistency raised from the negative regime. Since traders know the possibility of break-through in our model, the optimal strategy takes this into account. When the stock price goes above $m$ in the negative regime, it is likely that there will be a break-through. Hence, the current resistance can be thought as the future support, which means “not selling” by SH because the trader now believes the stock is going to enter the positive regime very soon. This provides another explanation to the inconsistency.

Our analysis provides us some evidence that, under the possibility of a break-through, the standard trading rule is not optimal. So, when examining the profitability of technical analysis, the standard trading rule should be used with caution.

### 6.3 Mean-reverting models: Vasicek and CIR

We would like to investigate how our findings in the previous section change when the drift becomes mean-reverting. In this section we consider the Vasicek [33] and Cox, Ingersoll and Ross [6] (CIR) models. The drifts of Vasicek and CIR are both of the form $c - \mu_+ x$, and the volatilities are of the form $\sigma_+^2$ and $\sigma_+^2 \sqrt{x}$ respectively, for some positive constants $\mu_+, \sigma_+$ and $c$. We can check that (5.1)-(5.3) have all been met for our choice of parameters listed in Table 2. We will allow the value of $\gamma$ to be greater than 1 (i.e. traders become risk-seeking). We can do this because concavity is not needed for Theorem 4.6 and 5.3 to be valid (cf. Remark 4.3). Figure 3 and 4 present the results for the Vasicek and CIR models, respectively, and vary $\gamma$ between 0.5 and 1.5.

Comparing to Figure 2, the main pattern (e.g. the signs of the slopes of boundaries) is preserved, which makes our previous analysis regarding the standard trading rule more evident. It is also worth noticing that Figure 3 and 4 from the two mean-reverting models are very similar.

On the other hand, it is clear that there are also a few distinctions. Firstly, the slopes of $B$ and
Figure 3: Values of $B, m, b, a$ against $\gamma$ for the Vasicek [33] model.

Figure 4: Values of $B, m, b, a$ against $\gamma$ for the CIR [6] model.
$b$ decrease as $\gamma$ increases. Intuitively, since the mean-reverting drift would push the stock price down with an increasing force as the stock price increases, which means the risk associated with waiting for a higher selling boundary $B$ (less chance of getting there) or buying at a higher price $b$ (greater chance of making a loss) is much greater than in the model with affine drift. This makes the traders become increasingly less willing to increase $B$ or $b$ for each smaller (and eventually negative) degree of relative risk aversion, which results in the concavity seen in Figures 3 and 4.

Secondly, in contrast to the affine drift model, the values of $B$ or $b$ do not explode, even for $\gamma$ close to, or greater than one. Qualitatively, this can be explained by the same reasoning as above. Mathematically, this is also obvious, because Theorem 4.4 and 5.1 together show, under the existence of $A$, we must have $b \leq A \leq B < \infty$ (the finiteness of $B$ is proved in Theorem 4.4). Since the value of $A$ is determined once we are given dynamics and utilities, the finiteness of $B$ and $b$ comes from the finiteness of $A$. We need to stress the importance of $A < \infty$, because in the previous affine drift case where $\gamma = 1$ (which leads to $A = \infty$), Jacka and Meada [14] have proved $B = \infty$. In addition, we see that the existence of $A$ is sufficient to prevent the value function from exploding.

Finally, the value of $m$ drops below reference line $L = 1$ at $\gamma \geq 1.25$. This is not observed in Figure 2 for the affine drift model. Moreover, there is a kink for $m$ for $\gamma$ around 1.25. This is because the boundary condition changes substantially for $m < L$ as Figure 1 shows. Essentially, when $m < L$, the trader would continue to hold the stock when the price process transitions from the positive to the negative regime. From Figure 3 and 4, this happens only in the case where traders are risk-seeking (i.e. $\gamma > 1$), which suggests (at least under our modelling and specifications) waiting for a break-through from the negative to the positive regime is a very risky strategy and would be avoided by risk-averse traders.

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A Proofs and additional results

A.1 Additional lemmas

Lemma A.1. \( g(x, f) := V(x, f) - u(x) \) satisfies Assumptions 3.1, 3.2.

Proof. \( u \) is of \( C^2 \) on \((0, \infty)\). Moreover, by Theorem 4.6 \( V \) is \( C^2 \) on \( C \cup \text{int}(D) \) and \( C^1 \) on \( E \setminus (0, -) \). Therefore, Assumption 3.1 holds for \( g \).

Since \( V - u \) is continuous with a compact support, \( g \) is bounded by some constant \( K \). Hence,

\[
0 \leq \mathbb{E}^{x,f} \left[ \sup_{t \geq 0} e^{-r t} g(S_t, F_t) \right] \leq \mathbb{E}^{x,f} \left[ \sup_{t \geq 0} e^{-r t} K \right] \leq K < \infty. \tag{A.1}
\]

\( \diamond \)
A.2 Proofs of results in Section 2

Proof of Lemma 2.4. WLOG, take \((x,+)\) \(\in E \setminus (0,-)\) and \(y \in \mathbb{R}^+\). Then,

\[
P_{x,+}(S \text{ hits } y) = P_{x,+}(S \text{ hits } y \text{ before regime changes}) + P_{x,+}(S \text{ hits } y \text{ after regime changes})
\]

\[
= P_x(S^+ \text{ hits } y \text{ before } S^+ \text{ hits } L) + P_{L,-}(S \text{ hits } y)P_x(S^+ \text{ hits } L \text{ before } S^+ \text{ hits } y)
\]

\[
= P_x(\sigma_y^- \leq \sigma_L^+) + \left( P_{L,-}(S \text{ hits } y \text{ before regime changes}) \right) P_x(\sigma_L^+ < \sigma_y^-)
\]

\[
+ P_{L,-}(S \text{ hits } y \text{ after regime changes})\left( P_x(\sigma_y^- < \sigma_L^-) + P_{L,-}(\tau_H < \tau_y, \tau_y < \infty) P_x(\sigma_L^- < \sigma_y^-) \right).
\]

(A.2)

If \(y \geq L\), then \(P_x(\sigma_y^+ \leq \sigma_L^+) > 0\) by the regularity of \(S^+\). If \(y < L\), then \(P_x(\sigma_L^- < \sigma_y^-) > 0\) and \(P_{L,-}(\sigma_y^- < \sigma_H^-) > 0\) by the regularity of \(S^+\) and \(S^-\). Therefore, \(P_{x,f}(S \text{ hits } y) > 0\) for any \(y\).

Proof of Lemma 2.5. The strategy is similar to the proof of (6) in Chapter 2 of Freedman [10]. By Blumenthal’s 0-1 Law, \(P_{x,f}(\mathbb{A}_x)\) is either 1 or 0. Suppose it is 0. We set \(\tau := \inf\{t \geq 0 : S_t \not> x\}\). Then, \(P_{x,f}(\tau = 0) = 0\) since \(\{\tau = 0\} = \mathbb{A}_x\). Moreover, it is not hard to check \(\{\tau = 0\} = \{S_t \not> x \text{ for all } t\} = S_t = x \text{ on } \{\tau < \infty\}\), and \(\tau = 0\) on \(\{\tau < \infty\}\) given starting position \(S_t\). Therefore, by strong Markov property,

\[
P_{x,f}(\tau < \infty) = P_{x,f}(\tau < \infty, \tau = 0 \text{ given starting position } S_t)
\]

\[
= P_{x,f}(\tau = 0 \text{ given starting position } S_t \mid \tau < \infty)P_{x,f}(\tau < \infty)
\]

\[
= P_{x,f}(\tau = 0)P_{x,f}(\tau < \infty) = 0.
\]

Thus, \(P_{x,f}(\tau = \infty) = 1\), and hence \(P_{x,f}(S_t \not> x \text{ for all } t) = 1\), which contradicts Lemma 2.4. Therefore, \(P_{x,f}(\mathbb{A}_x) = 1\). The proof for the event \(\mathbb{B}_x\) is symmetric.

Proof of Lemma 2.6. The strategy is similar to the proof of (30) in Chapter 2 of Freedman [10]. Fix \(h \in C^b(E), t \in \mathbb{R}_+\) and \((x,f) \in E \setminus (0,-)\). Consider a sequence \((x_n, f_n)_{n \in \mathbb{N}} \subset E\) converges to \((x,f)\). Since we put discrete metric on \([+, -]\), it must follow that \(\exists N \in \mathbb{N}\), such that for any \(n \geq N\), \(f_n = f\). WLOG, we can assume \(f_n = f\) for any \(n\). Let \(\tau_n := \tau_{x_n}^f\). We further define

\[
L_t := \min\{S_s : s \in [0, t]\} \quad \text{and} \quad U_t := \max\{S_s : s \in [0, t]\}.
\]

(A.4)

By Lemma 2.5 \(P_{x,f}(L_t < x < U_t \text{ for all } t > 0) = 1\). Pick \(\omega \in \{L_t < x < U_t \text{ for all } t > 0\}\) and fix \(\epsilon > 0\). Then there is an integer \(N_\epsilon\) such that for any \(n > N_\epsilon\), \(L_\epsilon(\omega) < x_n < U_\epsilon(\omega)\), which implies \(S\) hits \(x_n\) before time \(\epsilon\) by path continuity. Therefore, \(\omega \in \cap_{\epsilon > 0} \bigcup_{n > N_\epsilon} \{S \text{ hits } x_n \text{ before } \epsilon\}\), and hence

\[
P_{x,f}(\tau_n \to 0) = P_{x,f}\left( \bigcap_{\epsilon > 0} \bigcup_{n > N_\epsilon} \{\tau_n < \epsilon\} \right) = P_{x,f}\left( \bigcap_{\epsilon > 0} \bigcup_{n > N_\epsilon} \{S \text{ hits } x_n \text{ before } \epsilon\} \right) = 1.
\]

(A.5)

Thus, \(\tau_n\) converges to 0 \(P\)-a.s. Consequently, \(1_{\tau_n \to 0} \to 1\) \(P\)-a.s. By dominated convergence,

\[
\mathbb{E}^{x,f}[h(S_{\tau_n+t}, F_{\tau_n+t}) 1_{\tau_n \to 0}] \to \mathbb{E}^{x,f}[h(S_t, F_t)].
\]

(A.6)
By strong Markov property,

$$
\mathbb{E}^{x,f}[h(S_{\tau_n+t}, F_{\tau_n+t})1_{\tau_n<\infty}] = \mathbb{E}^{x,f}[\mathbb{E}^{x,f}[h(S_{\tau_n+t}, F_{\tau_n+t})1_{\tau_n<\infty}|F_{\tau_n}]]
$$

$$
= \mathbb{E}^{x,f}[1_{\tau_n<\infty}\mathbb{E}^{x,f}[h(S_t, F_t)]]
$$

$$
= \mathbb{E}^{x,f}[1_{\tau_n<\infty}\mathbb{E}^{x,f}[h(S_t, F_t)]].
$$

(A.7)

Therefore,

$$
\mathbb{E}^{x,f}[h(S_t, F_t)] \rightarrow \mathbb{E}^{x,f}[h(S_t, F_t)].
$$

(A.8)

We now consider the case where \((x, f) = (0, -)\). Since it is absorbing, the previous argument cannot be applied. However, choosing \(\epsilon > 0\) sufficiently small, there is \(0 < \delta < L\) such that \(|x| \leq \delta\) implies \(|h(x, -) - h(0, -)| \leq \epsilon/2\). Pick a sequence \(x_n \rightarrow 0\). Then \(x_n \leq \delta\) for \(n\) large enough. Moreover, \(P_{x_n, -}(\tau^-_\delta < \tau^-_0) \rightarrow 0\). So if we choose \(n\) sufficiently large that \(P_{x_n, -}(\tau^-_\delta < \tau^-_0) \leq \frac{\epsilon}{2\|h\|}\), we can see

$$
|\mathbb{E}^{x_n, -}[h(S_t, F_t)] - \mathbb{E}^{0, -}[h(S_t, F_t)]| \\
\leq \mathbb{E}^{x_n, -}[|h(S_t, -) - h(0, -)|] + \mathbb{E}^{x_n, -}[|h(S_t, -) - h(0, -)||1_{t<\tau^-_\delta}|] \\
\leq \frac{\epsilon}{2} + \mathbb{E}^{x_n, -}[|h(S_t, -) - h(0, -)||1_{t<\tau^-_\delta}|] \\
\leq \frac{\epsilon}{2} + \mathbb{E}^{x_n, -}[|h(S_t, -) - h(0, -)||1_{t<\tau^-_0}|] \\
\leq \frac{\epsilon}{2} + ||h||P_{x_n, -}(\tau^-_\delta < \tau^-_0) \leq \epsilon,
$$

(A.9)

which completes the proof. ♦