On Homogeneous Landsberg Surfaces

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Abstract

In this paper, we prove that every homogeneous Landsberg surface has isotropic flag curvature. Using this special form of the flag curvature, we prove a rigidity result on homogeneous Landsberg surface. Indeed, we prove that every homogeneous Landsberg surface is Riemannian or locally Minkowskian. This gives a positive answer to the Xu-Deng’s well-known conjecture in 2-dimensional homogeneous Finsler manifolds.

Keywords: Homogeneous Finsler surface, Landsberg metric, Berwald metric, flag curvature.

1 Introduction

Let $(M, F)$ be a Finsler manifold and $c : [a, b] \to M$ be a piecewise $C^\infty$ curve from $c(a) = p$ to $c(b) = q$. For every $u \in T_pM$, let us define $P_c : T_pM \to T_qM$ by $P_c(u) := U(b)$, where $U = U(t)$ is the parallel vector field along $c$ such that $U(a) = u$. $P_c$ is called the parallel translation along $c$. In [11], Ichijyō showed that if $F$ is a Berwald metric, then all tangent spaces $(T_xM, F_x)$ are linearly isometric to each other. Let us consider the Riemannian metric $\hat{g}_x$ on $T_xM$ for $x \in M$ which is defined by $\hat{g}_x := \frac{1}{2}F^2_{ij} \delta y^i \otimes \delta y^j$, where $g_{ij} := \frac{1}{2}F^2_{ij} y^i y^j$ is the fundamental tensor of $F$ and $\{\delta y^i \} = \{dy^i + N^i_j dx^j\}$ is the natural coframe on $T_xM$ associated with the natural basis $\{\partial/\partial x^i\}$ for $T_xM$. If $F$ is a Landsberg metric, then for any $C^\infty$ curve $c$, $P_c$ preserves the induced Riemannian metrics on the tangent spaces, i.e., $P_c : (T_pM, \hat{g}_p) \to (T_qM, \hat{g}_q)$ is an isometry. By definition, every Berwald metric is a Landsberg metric, but the converse may not hold.

In 1996, Matsumoto found a list of rigidity results which almost suggest that such a pure Landsberg metric (non-Berwaldian metric) does not exist [14]. In 2003, Matsumoto emphasized this problem again and looked at it as the most important open problem in Finsler geometry. It is a long-existing open problem in Finsler geometry to find Landsberg metrics which are not Berwaldian. Bao called such metrics unicorns in Finsler geometry, mythical single-horned horse-like creatures that exist in legend but have never been seen by human beings [4]. There are a lot of unsuccessful attempts to find explicit examples of unicorns. In [15], Szabó made an argument to prove that any regular Landsberg metric must be of Berwald type. But unfortunately, there is a little gap in Szabó’s argument. As pointed out in Szabó’s correction to [15], his argument only applies to the so-called dual Landsberg spaces. Hence, the unicorn problem remains open in Finsler geometry. Taking into account of so many unsuccessful efforts of many researchers, one can conclude that unicorn problem is becoming more and more puzzling.

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The unicorn problem in Finsler geometry is well-studied. However, up to now, very little attention has been paid to the subject of homogeneous Finsler metrics. A Finsler manifold \((M, F)\) is said to be homogeneous if its group of isometries acts transitively on \(M\). In [17], the authors consider the unicorn problem in the class of homogeneous \((\alpha, \beta)\)-metric. We proved that every homogeneous \((\alpha, \beta)\)-metric is a stretch metric if and only if it is a Berwald metric. In [21], Xu-Deng introduced a generalization of \((\alpha, \beta)\)-metrics, called \((\alpha_1, \alpha_2)\)-metrics. Let \((M, \alpha)\) be an \(n\)-dimensional Riemannian manifold. Then one can define an \(\alpha\)-orthogonal decomposition of the tangent bundle by \(TM = V_1 \oplus V_2\), where \(V_1\) and \(V_2\) are two linear subbundles with dimensions \(n_1\) and \(n_2\) respectively, and \(\alpha_i = \alpha|_{V_i}\), \(i = 1, 2\) are naturally viewed as functions on \(TM\). An \((\alpha_1, \alpha_2)\)-metric on \(M\) is a Finsler metric \(F\) which can be written as \(F = \sqrt{L(\alpha_1^2, \alpha_2^2)}\). An \((\alpha_1, \alpha_2)\)-metric can also be represented as \(F = \alpha\phi(\alpha_2/\alpha) = \alpha\psi(\alpha_1/\alpha)\), in which \(\phi(s) = \psi(\sqrt{1 - s^2})\). They proved that every Landsberg \((\alpha_1, \alpha_2)\)-metric reduces to a Berwald metric. This result shows that the finding a unicorn cannot be successful even in the very broad class of \((\alpha_1, \alpha_2)\)-metrics. Then, Xu-Deng conjectured the following:

**Conjecture 1.1.** ([21]) A homogeneous Landsberg space must be a Berwald space.

Taking a look at the rigid theorems in Finsler geometry, one can find that this type of result is different for procedures with dimensions greater than three. For example, in [16] Szabó proved that any connected Berwald surface is locally Minkowskian or Riemannian. In [5], Bao-Chern-Shen proved a rigidity result for compact Landsberg surface. They showed that a compact Landsberg surfaces with non-positive flag curvature is locally Minkowskian or Riemannian. Therefore, we preferred to consider the issue of unicorns for homogeneous Finsler surfaces. In this paper, we prove the following rigidity result.

**Theorem 1.1.** Any homogeneous Landsberg surface of is Riemannian or locally Minkowskian.

This result articulates the hunters of unicorns that they do not looking forward to seeing such a creature in the jungle of homogeneous Finsler surfaces.

In order to prove Theorem 1.1, we consider the flag curvature of Landsberg surface and prove the following rigidity result.

**Theorem 1.2.** Every homogeneous Landsberg surface has isotropic flag curvature.

### 2 Preliminaries

Let \((M, F)\) be an \(n\)-dimensional Finsler manifold, and \(TM\) be its tangent space. We denote the slit tangent space of \(M\) by \(TM_0\), i.e., \(T_xM_0 = T_xM - \{0\}\) at every \(x \in M\). The fundamental tensor \(g_y : T_xM \times T_xM \to \mathbb{R}\) of \(F\) is defined by following

\[
g_y(u, v) := \frac{1}{2} \frac{\partial^2}{\partial s \partial t} \left[ F^2(y + su + tv) \right]_{s, t = 0}, \quad u, v \in T_xM.
\]

Let \(x \in M\) and \(F_x := F|_{T_xM}\). To measure the non-Euclidean feature of \(F_x\), define \(C_y : T_xM \times T_xM \times T_xM \to \mathbb{R}\) by

\[
C_y(u, v, w) := \frac{1}{2} \frac{d}{dt} \left[ g_{y+tw}(u, v) \right]_{t = 0}, \quad u, v, w \in T_xM.
\]
The family \( C := \{ C_y \}_{y \in T_x M} \) is called the Cartan torsion. By definition, \( C_y \) is a symmetric trilinear form on \( T_x M \). It is well known that \( C = 0 \) if and only if \( F \) is Riemannian.

Let \( (M, F) \) be a Finsler manifold. For \( y \in T_x M_0 \), define \( I_y : T_x M \to \mathbb{R} \) by

\[
I_y(u) = \sum_{i=1}^n g^{ij}(y) C_y(u, \partial_i, \partial_j),
\]

where \( \{ \partial_i \} \) is a basis for \( T_x M \) at \( x \in M \). The family \( I := \{ I_y \}_{y \in T_x M_0} \) is called the mean Cartan torsion. By definition, \( I_y(u) := I_i(y) u^i \), where \( I_i := g^{jk} C_{ijk} \). By Deicke’s theorem, every positive-definite Finsler metric \( F \) is Riemannian if and only if \( I = 0 \).

Given a Finsler manifold \( (M, F) \), then a global vector field \( G \) is induced by \( F \) on \( TM_0 \), and in a standard coordinate \((x^i, y^j)\) for \( TM_0 \) is given by \( G = y^i \partial/\partial x^i - 2G^i(x, y) \partial/\partial y^i \), where \( G^i = G^i(x, y) \) are scalar functions on \( TM_0 \) given by

\[
G^i := \frac{1}{4} g^{ij} \left\{ \frac{\partial^2 [x^2]}{\partial x^j \partial y^i} y^k - \frac{\partial [x^2]}{\partial x^j} \right\}, \quad y \in T_x M.
\]

The vector field \( G \) is called the spray associated with \((M, F)\).

For \( y \in T_x M_0 \), define \( B_y : T_x M \times T_x M \times T_x M \to T_x M \) by \( B_y(u, v, w) := B_{ijkl}(y) u^j v^k w^l \partial/\partial x^i \) where

\[
B_{ijkl}(y) := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.
\]

The quantity \( B \) is called the Berwald curvature of the Finsler metric \( F \). We call a Finsler metric \( F \) a Berwald metric, if \( B = 0 \).

Define the mean of Berwald curvature by \( E_y : T_x M \times T_x M \to \mathbb{R} \), where

\[
E_y(u, v) := \frac{1}{2} \sum_{i=1}^n g^{ij}(y) g_{ij} \left( B_y(u, v, e_i, e_j) \right).
\]

The family \( E = \{ E_y \}_{y \in T_x M \setminus \{0\}} \) is called the \textit{mean Berwald curvature} or \textit{E-curvature}. In a local coordinates, \( E_y(u, v) := E_{ij}(y) u^i v^j \), where

\[
E_{ij} := \frac{1}{2} B^m_{mij}.
\]

The quantity \( E \) is called the mean Berwald curvature. \( F \) is called a weakly Berwald metric if \( E = 0 \). Also, define \( H_y : T_x M \otimes T_x M \to \mathbb{R} \) by \( H_y(u, v) := H_{ij}(y) u^i v^j \), where

\[
H_{ij} := E_{ij} g^s.
\]

Then \( H_y \) is defined as the covariant derivative of \( E \) along geodesics.

For non-zero vector \( y \in T_x M_0 \), define \( D_y : T_x M \otimes T_x M \otimes T_x M \to T_x M \) by \( D_y(u, v, w) := D_{ijkl}(y) u^j v^k w^l \partial/\partial x^i \), where

\[
D_{ijkl}(y) := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l} \left[ G^i - \frac{2}{n+1} \frac{\partial C^m}{\partial y^m y^i} \right].
\]
\( \textbf{D} \) is called the Douglas curvature. \( F \) is called a Douglas metric if \( \textbf{D} = 0 \). According to the definition, the Douglas tensor can be written as follows

\[
D^i_{\ jkl} = B^i_{\ jkl} - \frac{2}{n+1} \left\{ E_{ijk} \delta^i_l + E_{ikl} \delta^i_j + E_{ijl} \delta^i_k + E_{jkl} y^i \right\}.
\]

For \( y \in T_xM \), define the Landsberg curvature \( \mathbf{L}_y : T_xM \times T_xM \times T_xM \to \mathbb{R} \) by

\[
\mathbf{L}_y(u, v, w) := -\frac{1}{2} g_y(B_y(u, v, w), y).
\]

\( F \) is called a Landsberg metric if \( \mathbf{L}_y = 0 \). By definition, every Berwald metric is a Landsberg metric.

Let \((M, F)\) be a Finsler manifold. For \( y \in T_xM_0 \), define \( \mathbf{J}_y : T_xM \to \mathbb{R} \) by

\[
\mathbf{J}_y(u) = \sum_{i=1}^n g^{ij}(y) \mathbf{L}_y(u, \partial_i, \partial_j).
\]

The quantity \( \mathbf{J} \) is called the mean Landsberg curvature or J-curvature of Finsler metric \( F \). A Finsler metric \( F \) is called a weakly Landsberg metric if \( \mathbf{J}_y = 0 \). By definition, every Landsberg metric is a weakly Landsberg metric. Mean Landsberg curvature can also be defined as following

\[
J_i := y^m \partial I_i \partial x^m - I_m \partial G^m \partial y^i - 2G^m \partial I_i \partial y^m.
\]

By definition, we get

\[
\mathbf{J}_y(u) := \frac{d}{dt} \left[ \mathbf{I}_{\dot{\sigma}(t)}(U(t)) \right]_{t=0},
\]

where \( y \in T_xM \), \( \sigma = \dot{\sigma}(t) \) is the geodesic with \( \sigma(0) = x \), \( \dot{\sigma}(0) = y \), and \( U(t) \) is a linearly parallel vector field along \( \sigma \) with \( U(0) = u \). The mean Landsberg curvature \( \mathbf{J}_y \) is the rate of change of \( \mathbf{I}_y \) along geodesics for any \( y \in T_xM_0 \).

For an arbitrary non-zero vector \( y \in T_xM_0 \), the Riemann curvature is a linear transformation \( \mathbf{R}_y : T_xM \to T_xM \) with homogeneity \( \mathbf{R}_{\lambda y} = \lambda^2 \mathbf{R}_y \), \( \forall \lambda > 0 \), which is defined by \( \mathbf{R}_y(u) := R^i_k(y) u^k \partial / \partial x^i \), where

\[
R^i_k(y) = 2 \frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}.
\]

The family \( \mathbf{R} := \{ \mathbf{R}_y \}_{y \in TM_0} \) is called the Riemann curvature of the Finsler manifold \((M, F)\).

For a flag \( P := \text{span} \{ y, u \} \subset T_xM \) with flagpole \( y \), the flag curvature \( \mathbf{K} = \mathbf{K}(x, y, P) \) is defined by

\[
\mathbf{K}(x, y, P) := \frac{g_y(u, R_y(u))}{g_y(y, y) g_y(u, u) - g_y(y, u)^2}.
\]

The flag curvature \( \mathbf{K}(x, y, P) \) is a function of tangent planes \( P = \text{span} \{ y, v \} \subset T_xM \). This quantity tells us how curved space is at a point. A Finsler metric \( F \) is of scalar flag curvature if \( \mathbf{K} = \mathbf{K}(x, y, P) \) is independent of flags \( P \) containing \( y \in T_xM_0 \).
3 Proof of Theorems

In this section, we are going to prove Theorems 1.1 and 1.2. In order to prove Theorem 1.1, first we consider the flag curvature of homogeneous Landsberg surface. More precisely, we prove Theorem 1.2. For this aim, we need some useful Lemmas as follows.

In [13], Latifi-Razavi proved that every homogeneous Finsler manifold is forward geodesically complete. In [17], Tayebi-Najafi improved their result and proved the following.

Lemma 3.1. ([18]) Every homogeneous Finsler manifold is complete.

By definition, every two points of a homogeneous Finsler manifold \((M, F)\) map to each other under an isometry. This causes the norm of an invariant tensor under the isometries of a homogeneous Finsler manifold is a constant function on \(M\), and consequently, it has a bounded norm. Then, we conclude the following.

Lemma 3.2. ([17]) Let \((M, F)\) be a homogeneous Finsler manifold. Then, every invariant tensor under the isometries of \(F\) has a bounded norm with respect to \(F\).

Proof of Theorem 1.2: We first deal with Finsler surfaces. The special and useful Berwald frame was introduced and developed by Berwald [7]. Let \((M, F)\) be a two-dimensional Finsler manifold. One can define a local field of orthonormal frame \((\ell^i, m^i)\) called the Berwald frame, where \(\ell^i = y^i/F\), \(m^i\) is the unit vector with \(\ell_i m^i = 0\), \(\ell_i = g_{ij} \ell^j\) and \(g_{ij}\) is defined by \(g_{ij} = \ell_i \ell_j + m_i m_j\). In [3], it is proved that the Douglas curvature of the Finsler surface \((M, F)\) is given by following

\[
D_{ijkl} = -\frac{1}{3F^2}(6I_{11} + I_{22} + 2II_2) m_j m_k m_l y^i.
\]

We rewrite it as equivalently

\[
D_y(u, v, w) = T(u, v, w)y, \quad \text{(3.1)}
\]

where \(T(u, v, w) := T_{ijk} u^i v^j w^k\) and \(T_{ijk} := -1/(3F^2)(6I_{11} + I_{22} + 2II_2)m_i m_j m_k\). It is easy to see that \(T\) is a symmetric Finslerian tensor filed and satisfies the following

\[
T(y, v, w) = 0.
\]

Let us denote the Berwald connection of \(F\) by \(D\). The horizontal and vertical derivation with of a Finsler tensor field are denoted by “\(D_u\)” and “\(D_u\)” respectively. Taking a horizontal derivation of (3.1) along Finslerian geodesics implies that

\[
D_0 D_y(u, v, w) = D_0 T(u, v, w)y, \quad \text{(3.2)}
\]

where \(D_0 := D_i y^i\). Let us define \(h_y : T_x M \rightarrow T_x M\) by

\[
h_y(u) = u - \frac{1}{F^2} g_y(u, y)y.
\]

Since \(h_y(y) = 0\), it follows from (3.2) that

\[
h_y(D_0 D_y(u, v, w)) = 0. \quad \text{(3.3)}
\]
On the other hand, the Douglas tensor of $F$ is given by

$$D_y(u, v, w) = B_y(u, v, w) - \frac{2}{3} \left\{ E_y(v, w)u + E_y(w, u)v + E_y(u, v)w + (D_yE_y)(v, w)y \right\}.$$  \hfill (3.4)

Then

$$h_y(D_0D_y(u, v, w)) = h_y(D_0B_y(u, v, w)) - \frac{2}{3} \left\{ H_y(u, v)h_y(w) + H_y(v, w)h_y(u) + H_y(w, u)h_y(v) \right\}.$$ \hfill (3.5)

Let us define

$$\tilde{B}_y := D_0B_y.$$  

Indeed, $\tilde{B}_y$ is the horizontal derivative of Berwald curvature along Finsler geodesics. By (3.3) and (3.4), we get

$$h_y(\tilde{B}_y(u, v, w)) = \frac{2}{3} \left\{ H_y(u, v)h_y(w) + H_y(v, w)h_y(u) + H_y(w, u)h_y(v) \right\}.$$  \hfill (3.6)

Using $D_i h = 0$ yields

$$h_y(D_i \tilde{B}_y(u, v, w)) = \frac{2}{3} \left\{ D_i H_y(u, v)h_y(w) + D_i H_y(v, w)h_y(u) + D_i H_y(w, u)h_y(v) \right\}.$$  \hfill (3.7)

Using $g_y(B_y(u, v, w), y) = -2L_y(u, v, w)$, we get

$$D_i(h_y \tilde{B}_y(u, v, w)) = h_y(D_i \tilde{B}_y(u, v, w)) = D_i D_0(h_y B_y(u, v, w)) = D_i D_0(\tilde{B}_y(u, v, w) - \frac{1}{F^2} g_y(B_y(u, v, w), y)) = D_i \tilde{B}_y(u, v, w) + \frac{2}{F^2} D_i D_0 L_y(u, v, w).$$  \hfill (3.8)

By (3.7), (3.8), and $L = 0$, we obtain

$$D_i \tilde{B}_y(u, v, w) = \frac{2}{3} \left\{ D_i H_y(u, v)h_y(w) + D_i H_y(v, w)h_y(u) + D_i H_y(w, u)h_y(v) \right\}.$$   \hfill (3.9)

The relation (3.9) yields

$$D_y \tilde{B}_y(u, v, \partial_k) - D_k \tilde{B}_y(u, v, \partial_k) = \frac{2}{3} \left\{ D_h H_y(u, v)h_y(\partial_k) - D_k H_y(u, v)h_y(\partial_h) \right\} + \frac{2}{3} \left\{ (D_h H_y(v, \partial_k) - D_k H_y(v, \partial_h))h_y(u) \right\} + \frac{2}{3} \left\{ (D_h H_y(\partial_k, u) - D_k H_y(\partial_k, u))h_y(v) \right\}.$$ \hfill (3.10)

By definition, we have $tr(\tilde{B}) = 2H$ and $tr(h) = 1$. Then, (3.10) implies that

$$D_h H_y(u, \partial_k) - D_k H_y(u, \partial_h) = 2 \left\{ D_h H_y(u, \partial_k) - D_k H_y(u, \partial_h) \right\},$$  \hfill (3.11)

which yields

$$D_h H_y(u, \partial_h) = D_k H_y(u, \partial_h).$$ \hfill (3.12)
Contracting (3.12) with $y^h$ and using $D_k H_y(u, y) = 0$, we get
\[ D_0 H_y(u, w) = 0. \] (3.13)

Take an arbitrary unit vector $y \in T_x M$ and an arbitrary vector $v \in T_x M$. Let $c(t)$ be the geodesic with $\dot{c}(0) = y$ and $U = U(t)$ the parallel vector field along $c$ with $V(0) = v$. In order to avoid clutter, we put
\[ E(t) = E_c(U(t), U(t)), \quad H(t) = H_c(U(t), U(t)). \] (3.14)

From the definition of $H_y$, we have
\[ H(t) = E'(t). \] (3.15)

By (3.13) we have $H'(t) = 0$ which implies that
\[ H(t) = H(0). \] (3.16)

Then by (3.15) and (3.16), we get
\[ E(t) = H(0)t + E(0). \] (3.17)

Since $E(t)$ is a bounded function on $[0, \infty)$, then letting $t \to +\infty$ or $t \to -\infty$ implies that
\[ H_y(v, v) = H(0) = 0. \]

Therefore $H = 0$. According to Akbar-Zadeh’s theorem every Finsler metric $F = F(x, y)$ of scalar flag curvature $K = K(x, y)$ on an $n$-dimensional manifold $M$ has isotropic flag curvature $K = K(x)$ if and only if $H = 0$ [2]. Every Finsler surface has scalar flag curvature $K = K(x, y)$. Then by Akbar-Zadeh theorem, we get $K = K(x)$.

Now, we can prove the Theorem 1.1.

**Proof of Theorem 1.1:** Let $(M, F)$ be a homogeneous Landsberg surface and fix a point $x \in M$. Suppose that $y = y(t)$ is a unit speed parametrization of indicatrix of $M$ at $x$. We know that the curvature along $y(t)$ is completely determined by the Cartan scalar of $F$, i.e.,
\[ K(t) = K(0) e^{\int_0^t I(s)ds}. \]

Thus either $K(t)$ vanishes every where or it is non-zero every where and $K(t)$ has the same sign as the sign of $K(0)$. On the other hand, for homogeneous Finsler surfaces the flag curvature is a bounded scalar function on $SM$. Suppose that $\lambda_1 \leq K(t) \leq \lambda_2$. In this case, we have
\[ e^{\lambda_1 t} \leq C(0) e^{\int_0^t K(s)ds} \leq e^{\lambda_2 t}. \]

Suppose that $C(0) \neq 0$. Then we consider two following cases:

**Case 1:** If $\lambda_1$ and $\lambda_2$ are positive, then letting $t \to \infty$ implies that $C(t)$ is unbounded, which is a contradiction.
Case 2: If \( \lambda_1 \) and \( \lambda_2 \) are negative, then letting \( t \to -\infty \) implies that \( C(t) \) is unbounded, which is a contradiction.

Thus, every homogeneous Landsberg surface is Riemannian or flat. On the other hand, by Akbar-Zadeh’s theorem any positively complete Finsler metric with zero flag curvature must be locally Minkowskian if the first and second Cartan torsions are bounded [2]. For the homogeneous Finsler metrics, the first and second Cartan torsions are bounded. Then in this case, \( F \) reduces to a locally Minkowskian metric. This completes the proof. \( \Box \)

It is worth to mention that, in general, every Landsberg metric of non-zero scalar flag curvature is Riemannian, provided that its dimension is greater than two. Theorem 1.1 is Numata type theorem for homogeneous Finsler surfaces.

**Corollary 3.1.** Let \((M, F)\) be a homogeneous Finsler surface of non-positive flag curvature. Then \( F \) is a Landsberg metric if and only if it has isotropic flag curvature. In this case, \( F \) is Riemannian or locally Minkowskian.

**Proof.** According to Theorem 8.1 of [5], every geodesically complete Finsler surface of non-positive isotropic flag curvature \( K(x) \leq 0 \) and bounded Cartan scalar is a Landsberg metric. Then, by Theorem 1.1, we get the proof. \( \Box \)

In [8], L. Berwald introduced a non-Riemannian curvature so-called stretch curvature and denoted it by \( \Sigma_y \). He showed that this tensor vanishes if and only if the length of a vector remains unchanged under the parallel displacement along an infinitesimal parallelogram.

**Corollary 3.2.** Every homogeneous stretch surface is Riemannian or locally Minkowskian.

**Proof.** Every Landsberg metric is a stretch metric. In [17], it is proved that every homogeneous stretch metric is a Landsberg metric. Then, by Theorem 1.1, we get the proof. \( \Box \)

In [6], Bajancu-Farran introduced a new class of Finsler metrics, called generalized Landsberg metrics. This class of Finsler metrics contains the class of Landsberg metrics as a special case. A Finsler metric \( F \) on a manifold \( M \) is called generalized Landsberg metric the Riemannian curvature tensors of the Berwald and Chern connections coincide.

**Corollary 3.3.** Every homogeneous generalized Landsberg surface is Riemannian or locally Minkowskian.

**Proof.** By definition, we have

\[
L^i_{jkl} - L^i_{jkl} + L^i_{sk}L^s_{jl} - L^i_{sl}L^s_{jk} = 0,
\]

where “\( | \)” denotes the horizontal derivation with respect to the Berwald connection of \( F \). By (3.18), we get

\[
L_{isl}L^s_{jl} - L_{isl}L^s_{jk} = 0,
\]

\[
L_{ijl|k} - L_{ijkl|l} = 0.
\]

The Landsberg curvature of Finsler surface satisfies

\[
L_{jkl} + \mu FC_{jkl} = 0.
\]
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where $\mu := -4I_{i,i}/I$. By (3.19) and (3.21), we get

$$
\mu F \left\{ C_{isk} C_{jl}^{s} - C_{isl} C_{jk}^{s} \right\} = 0. \tag{3.22}
$$

We have two cases: If $C_{isk} C_{jl}^{s} - C_{isl} C_{jk}^{s} = 0$, then the vv-curvature is vanishing. In [19], Schneider proved that vv-curvature is vanishing if and only if $F$ is Riemannian. If $\mu = 0$, then by (3.21) it follows that $F$ is a Landsberg metric. By Theorem 1.1, we get the proof.

Let us define $\tilde{J} = \tilde{J}_{ij} dx^i \otimes dx^j$, by

$$
\tilde{J} := (J_{i,j} + J_{j,i})_{|m} y^m. \tag{3.23}
$$

In [20], Xia proved that every $n$-dimensional compact Finsler manifold with $\tilde{J} = 2\tilde{H}$ is a weakly Landsberg metric. Here, we prove the following.

**Corollary 3.4.** Every homogeneous Finsler surface satisfying $\tilde{J} = 2\tilde{H}$ is Riemannian or locally Minkowskian.

**Proof.** The following Bianchi identity holds

$$
H_{ij} := \frac{1}{2} (J_{i,j} + J_{j,i} - (I_{i,j})_{|p} y^p)_{|m} y^m. \tag{3.24}
$$

See [20]. By (3.23) and (3.24), we get $(I_{i,j})_{|p} y^p = 0$ and contracting it with $y^j$ yields

$$
J_{i,p} y^p = 0. \tag{3.25}
$$

For any geodesic $c = c(t)$ and any parallel vector field $U = U(t)$ along $c$, let us put

$$
I(t) = I_c(U(t), U(t), U(t)), \quad J(t) = J_c(U(t), U(t), U(t)).
$$

Thus, we have

$$
J(t) = I'(t). \tag{3.26}
$$

Integrating (3.25) implies that

$$
I(t) = J(0)t + I(0).
$$

Every homogeneous manifold $M$ is complete and the parameter $t$ takes all the values in $(-\infty, +\infty)$. Letting $t \to +\infty$ or $t \to -\infty$ we have $I(t)$ is unbounded which is a contradiction. Therefore $J(0) = J(t) = 0$. On the other hand, every Finsler surface is $C$-reducible

$$
C_y(u, v, w) = \frac{1}{3} \left\{ I_y(u) h_y(v, w) + I_y(v) h_y(u, w) + I_y(w) h_y(u, v) \right\}. \tag{3.27}
$$

Taking a horizontal derivation of (3.27) yields

$$
L_y(u, v, w) = \frac{1}{3} \left\{ J_y(u) h_y(v, w) + J_y(v) h_y(u, w) + J_y(w) h_y(u, v) \right\}. \tag{3.28}
$$

Putting $J = 0$ in (3.28) implies that $L = 0$. By Theorem 1.1, we get the proof.
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