Solution of the Kadomtsev-Petviashvili equation using an improved homotopy perturbation method

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Abstract. An improved homotopy perturbation method (LH) applied to find approximate solution of KP equation. The results obtained ensure that LH is capable for solving the strongly higher dimension nonlinear partial differential equation such as KP equation. The approximated solution obtained by LH is compared with exact solution.

1. Introduction
The Kadomtsev-Petviashvili (KP) equation is the 2+1-dimensional nonlinear partial differential equation, and considered as the fundamental equation in the context of solitons. This equation describes the weakly dispersive waves [1, 2, 3] and is given by

\[ (-4u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0. \] (1)

Several methods are employed to solve the KP equation such as elementary methods [4], explicit finite difference scheme[5], The Hirota direct method[6] and so on.

Homotopy analysis method [7] and homotopy perturbation method (HPM)[8, 9] are found to be successful in providing solutions to several nonlinear problems. Few attempts are made to apply the aforementioned methods to solve the KP equation of various structures [10, 11]. HPM is an approximation method to find an accurate analytical solution for the nonlinear systems in a simple way. Recently, an improved homotopy perturbation method (LH) [12] introduced by coupling HPM and the Laplace transform with the inclusion of the expandable convergence control parameters and applied nonlinear oscillators. LH approximation produces analytical
solutions in an easy way but with appreciable high accuracy in comparison to those obtained from HPM. The objective of the present work is to extend the application of LH to a higher-dimensional problem such as the KP equation. It is also intended to study the effect of second-order approximation of LH (LH2). This article is arranged in the following way. In Sect. 2, we demonstrate the formulation of the LH method. Application of LH for studying the KP equation with error analysis and benchmarking comparisons with an exact solution in Sect. 3. Finally, we conclude in Sect. 4.

2. Formalism
Let us consider a nonlinear inhomogeneous differential equation as

\[ N[u(x, y, t)] = 0 \] (2)

where \( N \) is the nonlinear differential operator. Let us construct the homotopy of Eq.(2) as,

\[ (1 - p)D[u(x, y, t) - v(x, y, t)] + phN[u(x, y, t)] = 0 \] (3)

where \( p \in [0, 1] \) is the embedding parameter. Here, \( D \) is the linear differential operator \((D[u(x, y, t)] = \frac{\partial^2 u(x, y, t)}{\partial x \partial t})\) and \( v(x, y, t) \) is the initial approximation of the of \( u(x, y, t) \) stratifying the initial condition. The convergence control parameter \( h \) is incorporated in Eq. (3).

Applying, Laplace transform on both sides of the Eq. (3) and using the derivative property, \( L[u'(t)] = sL[u(t)] - u(0) \), of the Laplace transform and then applying initial conditions, we get,

\[ u_x = u_g + p\left(u(x, y, t) - v(x, y, t)\right) - hpL^{-1}\left[\frac{1}{s}L\left[N[u(x, y, t)]\right]\right] = 0 \] (4)

we expand \( u \) and \( h \) in the power series of embedding parameter \((p)\) as follows:

\[ u(x, y, t) = \sum_{n=0}^{\infty} u_n(x, y, t)p^n \] (5)

Substituting Eq. (5) in Eq. (4) and equating same power of the \( p \) \((p^0, p^1, p^2 \cdots)\) gives the contributions to the higher-order approximations to the solutions.

\[ p^0 : u_{0,x} = v_x \] (6)

\[ p^1 : u_{1,x} = -L^{-1}\left[\frac{1}{s}L[A_0(u_0, h)]\right] \] (7)

\[ p^2 : u_{2,x} = u_{1,x} - L^{-1}\left[\frac{1}{s}L[A_1(u_0, u_1, h)]\right] \] (8)

\[ \vdots \]

here the \( A_0 \) and \( A_1 \) are the differential operator. The approximate solution to the original problem is obtained putting \( p = 1 \) in Eq.(5). Thus \( m^{th} \) order approximation as follows

\[ u^{(m)}(x, t) = \sum_{n=0}^{m} u_n(x, t) \] (9)
3. Application

Let us consider the general KP equation as,

\[ (-4u_t + 6u_{xx} + u_{xxx})_x + 3u_{yy} = 0 \]  (10)

or equivalently

\[ -4u_{xt} + 6u_x^2 + 6u_{xx} + u_{xxxx} + 3u_{yy} = 0 \]  (11)

with initial condition

\[ u(x, y, 0) = \frac{2e^{x+y}}{(e^x + e^y)^2} \]  (12)

we construct the homotopy of equation (11) as,

\[ (1 - p)D[u - v] + ph[-4u_{xt} + 6u_x^2 + 6u_{xx} + u_{xxxx} + 3u_{yy}] = 0 \]  (13)

and the initial approximations are as follows:

\[ v(x, y, t) = u(x, y, 0) \]  (14)

Using the Laplace transform (LT) and its inverse (ILT) in eq.(13) and rearranging, one get

\[ u_x = u_y + p(u_x - v_x) - hpL^{-1} \left\{ \frac{1}{s}L \left[ -4u_{0,xt} + 6u_{0,x}^2 + 6u_{0,xx} + u_{0,xxxx} + 3u_{0,yy} \right] \right\} = 0 \]  (15)

Substituting Eq. (5) in Eq. (15) and equating the coefficients \( p^0, p^1, p^2 \ldots \), we get,

\[ p^0 : u_{0,x} = v_x \]  (16)

\[ p^1 : u_{1,x} = -hL^{-1} \left\{ \frac{1}{s}L \left[ -4u_{0,xt} + 6u_{0,x}^2 + 6u_{0,xx} + u_{0,xxxx} + 3u_{0,yy} \right] \right\} \]  (17)

\[ p^2 : u_{2,x} = u_{1,x} - hL^{-1} \left[ \frac{1}{s}L \left[ -4u_{0,xt} + 12u_{0,xx}u_{1,x} + 6u_{1,0,xx} + 6u_{0,1,xx} + u_{1,xxxx} + 3u_{1,yy} \right] \right] \]  (18)

Hence we get the contributions from approximations of different order to the solution as

\[ u_0 = \frac{2e^{x+y}}{(e^x + e^y)^2} \]  (19)

\[ u_1 = \frac{8e^{x+y}(e^x - e^y)th}{(e^x + e^y)^3} \]  (20)

\[ u_2 = \frac{8hxe^{x+y}}{(e^x + e^y)^4} \left( e^{2x}(1 + 2h(2 + t)) - e^{2y}(1 + 2h(2 - t)) - 8e^{x+y}ht \right) \]  (21)

\[ \vdots \]

using Eq.(9), the second-order of Eq.(10) as given below

\[ u(x, y, t) = \frac{2e^{x+y}}{(e^x + e^y)^2} + \frac{8e^{x+y}(e^x - e^y)th}{(e^x + e^y)^3} + \frac{8hxe^{x+y}}{(e^x + e^y)^4} \left( e^{2x}(1 + 2h(2 + t)) - e^{2y}(1 + 2h(2 - t)) - 8e^{x+y}ht \right) \]  (22)
The solutions given by the LH contain convergence control parameters $h$, which helps to control the convergence of the solution. The convergence rate of the approximation series depends upon the value of $h$. The average value of the square residual of the differential equation is calculated for fixing the proper value of $h$. The discrete square residual for the $m^{th}$ order approximate solution is defined as

$$R_m(h) = \frac{1}{(11)^3} \sum_{k_1=0}^{10} \sum_{k_2=0}^{10} \sum_{k_3=0}^{10} \left[ \Delta_m\left(\frac{k_1}{10}, \frac{k_2}{10}, \frac{k_3}{10}, h\right) \right]^2,$$

where,

$$\Delta_m(x_{k_1}, y_{k_2}, t_{k_3}, h) = N[u^{(m)}(x, t, h)];$$

For converged solution equations (23) should be minimum, Then, the condition of minimization would be,

$$\frac{dR_m}{dh} = 0$$

The analytical solution $n^{th}$-line-soliton is constructed by Hirota’s direct method [6] as

$$u(x, y, t) = 2\partial_{xx} \log(f(x, y, t)),$$

where,

$$f(x, y, t) = \sum_{i=1}^{n}(1 + e^{-k_i x + y k_i^2 - t k_i^3})$$

We display both the exact results and the first-order solution obtained from LH are plotted in figure 1 for a different choice of $t$, $x$, and $y$. The first-order result obtained from LH are shown in the left column ($x = 0.1$ (top), and $y = 0.1$ (bottom)) and corresponding exact solution displayed in the right column of figure 1. It is seen that the approximate solution obtained from LH is nicely matching with the exact solution.

To study the accuracy of the approximate method, the error ($u_{\text{exact}} - u_{\text{LH}}$) of the solution obtained from the LH method with respect to the exact solution are displayed in the figure 2. It is noticed that, from the top-left panel ($x = 0.1$), the maximum error occurs about $y = 0$ and the error increase as time ($t$) increases. It is expected from figure 1 as the solution $u(x, y, t)$ is more nonlinear about $y = 0$. The same trend is noticed for the case of $y = 0.1$ (bottom-left). The absolute error in the solution considering the second-order approximation (LH2) is shown in the right panel of figure 2. The maximum absolute error is drastically reduced from the order of $10^{-2}$ to $10^{-5}$ when second-order is considered for the range considered in the article.

To check the accuracy of the method, LH approximate results are compared with the exact solution and absolute error of the LH with respect to the exact solution are presented in the table 1. It is seen that the LH yield a very accurate result for all values of time considered in this article. The absolute error is very small (order of $10^{-5}$) for low values of time and it is increased to order of $10^{-3}$ when $t = 0.1$. The results are found to be improved by two orders of magnitude if the second-order approximation is considered (LH2). It is to conclude that LH is a simple and highly accurate approximate method.

The time of computational for the calculation of solution of KP equation using LH is 18s whereas the time increases to 36s when one considers the second-order. it is to note that computational using LH for the cases considered in this article are fast.
Figure 1. The first-order result obtained from LH are shown in the left panel (x = 0.1 (top), and y = 0.1 (bottom)) and corresponding exact solution displayed in the right panel.

Table 1. Absolute error in the first-order and second-order solution for different values of t are tabulated with x = 10, y = 10

| t   | \( |u_{exact} - u_{LH}| \) | \( |u_{exact} - u_{LH^2}| \) |
|-----|--------------------------|--------------------------|
| 0.01| 1.2500 \times 10^{-5}    | 7.1690 \times 10^{-7}    |
| 0.02| 4.9997 \times 10^{-5}    | 2.8651 \times 10^{-6}    |
| 0.03| 1.1248 \times 10^{-4}    | 6.4371 \times 10^{-6}    |
| 0.04| 1.9995 \times 10^{-4}    | 1.1421 \times 10^{-5}    |
| 0.05| 3.1237 \times 10^{-4}    | 1.7798 \times 10^{-5}    |
| 0.06| 4.4973 \times 10^{-4}    | 2.5564 \times 10^{-5}    |
| 0.07| 6.1200 \times 10^{-4}    | 3.4639 \times 10^{-5}    |
| 0.08| 7.9915 \times 10^{-4}    | 4.5043 \times 10^{-5}    |
| 0.09| 1.0111 \times 10^{-3}    | 5.6721 \times 10^{-5}    |
| 0.10| 1.2479 \times 10^{-3}    | 6.9631 \times 10^{-5}    |

4. Conclusion
Applications of the improved homotopy perturbation method (LH) is extended successfully for solving the Kadomtsev-Petviashvili equation which is the (2+1)-dimensional partial differential
Figure 2. The absolute error ($u_{\text{exact}} - u_{\text{LH}}$) of solution obtained from LH method with respect to the exact solution are displayed in the left panel ($x = 0.1$ (top), and $y = 0.1$ (bottom)) of figure and corresponding absolute error of solution obtained from second-order (LH2) calculation displayed in the right panel.

equation. The absolute error of the LH with respect to the exact solution is presented in the table 1. It is noticed that absolute error in solution is very low (order of $10^{-3}$ – $10^{-5}$). The results obtained from LH are found to be improved by two orders of magnitude if the second-order approximation is considered (LH2). Calculations in LH is very fast (18 – 36s). It is to conclude that LH is a simple and accurate approximate method.

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