Cabling the Vassiliev Invariants

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Abstract

We characterise the cabling operations on the weight systems of finite type knot invariants. The eigenvectors and eigenvalues of this family of operations are described. The canonical deframing projection for these knot invariants is described over the cable eigenbasis. The action of immanent weight systems on general Feynman diagrams is considered, and the highest eigenvalue cabling eigenvectors are shown to be dual to the immanent weight systems. Using these results, we prove a recent conjecture of Bar-Natan and Garoufalidis on cblings of weight systems.

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1 Introduction.

The theory of knot invariants of finite order (or Vassiliev invariants [Vas]) is today well established. In particular, the essential combinatorial structure of the constructions has been elucidated by Birman and Lin [BL], and Bar-Natan [BN]. Kontsevich showed that, up to knot invariants of lesser order, these combinatorial representations are faithful [Kon]. Several important questions remain. First amongst these is the question of whether all knot invariants of finite order are realisable as finite linear combinations of the coefficients of Lie algebraic (quantum group) knot invariants. Another question is whether a better understanding of Vassiliev knot invariants would follow from a more traditional algebraic-topological methodology.

In a 1994 paper [BNG], Bar-Natan and Garoufalidis presented the following theorem, originally conjectured by Melvin and Morton [MM]. Note that their proof followed a path-integral demonstration by Rozansky [Roz].

We denote by $\hat{J}_{sl(2),\lambda}(K)(\bar{h})$ the $U_q(sl(2))$ invariant evaluated in a representation of dimension $\lambda + 1$ at $q = e^{\bar{h}}$ and we will denote the Alexander polynomial as $A(K)(z)$.

Theorem 1 ([BNG]) Expanding $\hat{J}_{sl(2),\lambda}/(\lambda + 1)$ in powers of $\lambda$ and $\bar{h}$,

$$\frac{\hat{J}_{sl(2),\lambda}(K)(\bar{h})}{\lambda + 1} = \sum_{j,m \geq 0} b_{jm}(K) \lambda^j \bar{h}^m, \quad (1)$$

we have,

1. $b_{jm}(K) = 0$ if $j > m$.

2. Define

$$JJ(K)(\bar{h}) = \sum_{m=0}^{\infty} b_{mm}(K) \bar{h}^m.$$

Then,

$$JJ(K)(\bar{h}) \frac{\bar{h}}{e^{\frac{\bar{h}}{2}} - e^{-\frac{\bar{h}}{2}}} A(K)(e^{\bar{h}}) = 1. \quad (2)$$

$\square$
Thus Bar-Natan and Garoufalidis showed that one can reconstruct the Alexander-Conway polynomial from the highest order terms in the extended Jones polynomial, when that polynomial is expanded in the dimension of the representation. They also extended this result to any semi-simple Lie algebra.

There is a certain philosophy motivated by extensive numerical calculations that the Lie algebraic invariants should span the set of finite order invariants. Thus their result led Bar-Natan and Garoufalidis to conjecture that there was in some natural sense a highest-order term in an arbitrary Vassiliev invariant expressible over the algebra of coefficients of the Alexander-Conway polynomial. They conjectured that the order of the natural cabling operation would be the intrinsic variable in which one could expand weight systems, replacing the dimension of the representation.

We will prove the following formulation of this conjecture.

**Theorem 2** Let $W_m$ be a weight system of order $m$ (a weight system on $m$-chorded diagrams, $A_m$). Denote for the deframed $n$-th cable of this weight system,

$$
\hat{\psi}_n^*W_m = W_m \circ \psi_m^n \circ \phi_m.
$$

In the above $\phi_m$ is a deframing projector (which we shall discuss below).

1. As a function of $n$, $\psi^*_nW_m$ is a polynomial in $n$, of highest order $n^m$.

2. The coefficient of $n^m$ in the polynomial is equal to a linear combination of immanent weight systems.

$\Box$

Immanent weight systems were written down in [BNG]. They are a means of calculating invariants of chord diagrams from their intersection matrices. These authors also showed that the algebra of immanent weight systems is the algebra of the weight systems for the coefficients of the Alexander polynomial. That is to say, the subspace of weight systems coming from sums of products of coefficients of the Alexander-Conway polynomial is the same as the subspace coming from immanent weight systems. Details of this can be found in the discussion in Section 6 of [BNG].

In the following, we will begin in section 2 by reviewing briefly some essential concepts - the notions of the chord diagram algebra, weight systems
and Feynman diagrams. In section 3, we introduce cabling, and construct eigenvectors of the cabling operation (our results in this section partially answer Bar-Natan’s Problem 7.4 [BN], providing a topological understanding of the alternative grading on the chord diagram algebra).

These eigenvectors have in fact already been written down in another context, relating to the enumeration of primitive vectors for this algebra. In section 4, we consider the deframing operation on the chord diagram algebra. The action of deframing finds a simple expression on our collection of eigenvectors. We use this to prove the first part of Theorem 2, enumerating the eigenvectors that are in the deframing invariant subspace. In section 5, we recall immanent weight systems, and their relation with the Alexander-Conway weight system. In this section we show that they form an orthogonal dual basis on precisely the eigenvectors within the highest eigenvalue subspace, and are zero on all others. This concludes the proof of Theorem 2. Section 6 contains the analysis of the cycle decomposition sums of FDs: this facilitates the proof of many statements made in section 5. We finish by discussing some further directions of research suggested by these results.

2 The Chord Diagram Algebra.

In what follows we employ a representative field of characteristic 0, the complex numbers \( \mathbb{C} \) (unless stated otherwise). We shall consider framed knots: when we say knot invariant, it may be framing dependent. We begin with knot invariants of finite order. To facilitate this definition we first extend any knot invariant \( V \) to an invariant of knots with self-intersections (with blackboard framing):

\[
V(\begin{array}{c}
\bullet
\end{array}) = V(\begin{array}{c}
\bullet
\\downarrow
\\downarrow
\end{array}) - V(\begin{array}{c}
\bullet
\\uparrow
\\uparrow
\end{array}).
\] (4)

**Definition 2.1** A knot invariant is of finite order \( n \), if it vanishes on knots with more than \( n \) self-intersections.

\( \square \)
Denote by $\hat{A}_m$ the finite-dimensional vector space spanned by $\mathbb{C}$-linear sums of chord diagrams with $m$ chords. The loop is oriented; in diagrams we shall assume counterclockwise. For example,

$$
\begin{align*}
\begin{array}{c}
\includegraphics[width=2cm]{diagram1} \\
\includegraphics[width=2cm]{diagram2}
\end{array}
- 2
\begin{array}{c}
\includegraphics[width=2cm]{diagram3} \\
\includegraphics[width=2cm]{diagram4}
\end{array}
+ \pi
\begin{array}{c}
\includegraphics[width=2cm]{diagram5} \\
\includegraphics[width=2cm]{diagram6}
\end{array}
\in \mathcal{D}_3.
\end{align*}
$$

(5)

Chord diagrams, in a natural way, represent the order in which intersections are encountered in a singular knot. An invariant of finite type $n$, provides a well-defined dual vector to $\hat{A}_n$. Namely, such an invariant will only observe the order in which self-intersections are encountered, and not extra “knotting” information. The invariant is then linearly extended to linear combinations of diagrams.

There are certain relationships that the resulting functional satisfies if it is obtained in this way: the 4-T relations, which can be understood most easily from a three-dimensional picture \[BN\]. Generalising:

**Definition 2.2** A $\mathbb{C}$-valued weight system of degree $m$, is a linear functional $W : \mathcal{D}_m \rightarrow \mathbb{C}$ satisfying the 4-T relations.

**4-T.**

$$
W\left(\begin{array}{c}
\includegraphics[width=2cm]{diagram1} \\
\includegraphics[width=2cm]{diagram2}
\end{array}\right) - W\left(\begin{array}{c}
\includegraphics[width=2cm]{diagram3} \\
\includegraphics[width=2cm]{diagram4}
\end{array}\right) + W\left(\begin{array}{c}
\includegraphics[width=2cm]{diagram5} \\
\includegraphics[width=2cm]{diagram6}
\end{array}\right) - W\left(\begin{array}{c}
\includegraphics[width=2cm]{diagram7} \\
\includegraphics[width=2cm]{diagram8}
\end{array}\right) = 0.
$$

(6)

With regard to pictures of chord diagrams in this paper, sections of the outer circle of a chord diagram which are dotted denote parts of the outer circle where further chords or lines not shown in the diagram may end. Sections of the outer circle which are full lines show regions where all allowed terminal points of chords or lines are shown. In what follows we shall refer to the outer loop as the Wilson loop of the diagram.

We denote the degree $m$ weight system that one constructs from a given knot invariant of finite order $m$, $V$, by $W_m[V]$. The vector space dual to the space of weight systems at degree $m$ is denoted by $\mathcal{A}_m$ (i.e. $\mathcal{D}_m$ quotiented by 4-T expressions).
It is convenient to express frequently recurring linear combinations of diagrams in $A_m$ using another notation. The more general diagrams are referred to as **Feynman Diagrams** (FDs), and allow internal trivalent vertices, where the incoming edges to the vertex are assigned a cyclic ordering.

The following definitions will be useful.

**Definition 2.3** The graph of a FD is the graph that remains when the Wilson loop is removed. It has univalent and trivalent vertices, and the incoming edges to a trivalent vertex are cyclically ordered. A **leg** is a neighbourhood in the graph of some univalent vertex. Denote by $D^t_m$ the vector space whose basis is formed by FDs whose graphs contain $2m$ vertices.

Let the STU vectors of $D^t_m$ be the following (where only a part of the graph is shown):

\[ r - r + \]

(7)

**Definition 2.4** Denote by $A^t_m$ the quotient of $D^t_m$ by the subspace spanned by the STU vectors. According to [BN], $A_m \equiv A^t_m$. Hereafter we shall only refer to $A_m$.

A FD with $2m$ vertices (internal and external) resolves to an $m$-chorded diagram. For example, the following diagram in $A_3$ can be expanded as

\[ = - + - \]

(8)
We equip $\mathcal{A}_m$ with a product, being the connect-sum of Wilson loops at some choice of point, 4-T relations being required in order that this product be well-defined. For example:

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram1.png} \\
\end{array}
\end{array}
\cdot 
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram2.png}
\end{array}
\end{array} = 
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram3.png}
\end{array}
\end{array} 
\epsilon \mathcal{A}_4.
\] (9)

This forms a graded algebra with identity element the empty Wilson loop. Furthermore, there is a co-product homomorphism $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$. It is defined as a sum over all ways of partitioning the chords of a diagram into two sets, and separating them accordingly. For example:

\[
\Delta \left( \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram4.png}
\end{array}
\end{array} \right) =
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram5.png}
\end{array}
\end{array} \otimes 
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram6.png}
\end{array}
\end{array} + 2 
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram7.png}
\end{array}
\end{array} \otimes 
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram8.png}
\end{array}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram9.png}
\end{array}
\end{array} + 
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram10.png}
\end{array}
\end{array} \otimes 
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram11.png}
\end{array}
\end{array} + 
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram12.png}
\end{array}
\end{array} \otimes 
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram13.png}
\end{array}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram14.png}
\end{array}
\end{array} + 2 \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram15.png}
\end{array}
\end{array} \otimes 
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram16.png}
\end{array}
\end{array} + 
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram17.png}
\end{array}
\end{array} \otimes 
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram18.png}
\end{array}
\end{array}
\end{array}
\]

(10)

These operations are well-defined and satisfy the axioms of a (commutative and co-commutative) Hopf algebra. By the structure theory of Hopf algebras, such an algebra is generated by its primitive elements, namely those $v \in \mathcal{A}_*$ such that

\[
\Delta(v) = v \otimes 1 + 1 \otimes v.
\] (11)

In the next section we shall describe the primitive subspace in terms of eigenvectors of certain cabling operations.

3 Cabling

Cabling is a natural topological operation on framed knots. Moreover, when composed with knot invariants of finite order, it leads to an interesting collec-
tion of linear transformations on the weight systems that characterise those knot invariants.

**Definition 3.1** Consider a knot $z^\mu : (0, 2\pi] \rightarrow \mathbb{R}^3$. Observe that the normal plane sufficiently close to a point on the knot can be parameterised as a domain in the complex plane, with the knot intersecting at the origin. With this, take a framing of the knot $\epsilon : (0, 2\pi] \rightarrow \mathbb{C}$, and the embedding of the normal plane in $\mathbb{R}^3$, $e^\mu : \mathbb{C} \times (0, 2\pi] \rightarrow \mathbb{R}^3$.

The nth-connected cabling of $z^\mu$ is the knot given by

$$\psi^n z^\mu(t) = z^\mu(nt) + e^\mu(nt, e^{2\pi it}\epsilon(nt)),$$  \hspace{1cm} (12)

(with the understanding that angles are identified mod $2\pi$).

\[\blacksquare\]

Operating inside $V$, a (framing independent) knot invariant of finite order $m$, this yields $V \circ \psi^n$, a framing dependent invariant also of finite order $m$. We wish to describe an operator $\psi^m_n$ satisfying

$$W_m[V \circ \psi^n] = W_m[V] \circ \psi^m_n.$$  \hspace{1cm} (13)

**Definition 3.2** Denote by $\psi^m_n$ the linear transformation, $\psi^m_n : A_m \rightarrow A_m$ defined as follows. Take the nth cyclic cover of $S^1$, which is also $S^1$. Sum over all ways of “lifting” the ends of chords to the different covers. For example,

$$\psi^2_2 \left(\begin{array}{c} \bigcirc \bigotimes \bigcirc \end{array}\right) = \begin{array}{c} \bigcirc \bigotimes \bigcirc + \bigcirc \bigotimes \bigcirc + \bigcirc \bigotimes \bigcirc + \bigcirc \bigotimes \bigcirc + \ldots \end{array}$$

$$= 8 \begin{array}{c} \bigcirc \bigotimes \bigcirc \end{array} + 8 \begin{array}{c} \bigcirc \bigotimes \bigcirc \end{array}$$  \hspace{1cm} (14)

This operation satisfies equation (13) [BN].

\[\blacksquare\]
The cabling transformation is well defined on $A_*$, with the 4-T relations being mapped into themselves. To see this, we describe cabling on FDs with internal trivalent vertices.

Consider first the effect of cabling on a single trivalent vertex:

$$
\psi^2 \left( \begin{array}{c}
\bullet
\end{array} \right) = \psi^2 \left( \begin{array}{c}
\bullet
\end{array} \right)
$$

$$
= \begin{array}{c}
\bullet
\bullet
\bullet
\bullet
\end{array} + \begin{array}{c}
\bullet
\bullet
\bullet
\bullet
\end{array} + \begin{array}{c}
\bullet
\bullet
\bullet
\bullet
\end{array} + \begin{array}{c}
\bullet
\bullet
\bullet
\bullet
\end{array}
$$

$$
= \begin{array}{c}
\bullet
\bullet
\bullet
\bullet
\end{array} - \begin{array}{c}
\bullet
\bullet
\bullet
\bullet
\end{array} - \begin{array}{c}
\bullet
\bullet
\bullet
\bullet
\end{array} - \begin{array}{c}
\bullet
\bullet
\bullet
\bullet
\end{array}
$$

(15)

It is clear from this picture that in order to cable a diagram whose graph has a single trivalent vertex one ‘sums over lifts’ of the legs of $v$. By induction this understanding extends to diagrams with an arbitrary number of internal vertices.

Denote by $S_p$ the group of permutations of $p$ letters. There exists a natural action of $S_p$ on ‘line’ chord diagrams, with $p$ legs (this action depends on the point where one breaks the Wilson loop). For example, we may break the following three chord diagram.

![Three chord diagram]

Elements of $S_p$ then act by permuting the order of the location of the univalent vertices (the legs) on the Wilson loop. To construct a cabling eigenvector, let us start with some choice of a Feynman diagram, $v \in A_n$, with $p$ legs. Construct the vector $Sym_v \in A_n$ as follows.

**Definition 3.3**

$$
Sym_v = \sum_{\sigma \in S_p} \sigma(v).
$$

(16)
(Note that this is independent of the choice of break because we have summed over all permutations.) This vector is the same for initial FDs related by a permutation of their legs. The graphs of two such FDs have the same set of connected components when the Wilson loop is removed. We record these vectors in this way: the collection of graphs with ‘internal’, trivalent vertices and ‘external’, univalent vertices. In fact, they have been written down previously, in another context, by Kontsevich \cite{Kon}. There they were used to describe the primitive vectors of the chord diagram algebra, and were dubbed ‘Chinese Characters’ by Bar-Natan, \cite{BN}. Here we shall refer to them as symmetrised Feynman diagrams (SFDs). Examples are

\[
\begin{array}{c}
\includegraphics{example1.png} \\
\includegraphics{example2.png}
\end{array}
\]

Keeping in mind their representation as sums over ways of ordering the univalent vertices of the graph on a Wilson loop, and the STU relations, there are further identities of importance in this space. In the diagrams below, we have extracted one part of a SFD, further connections from the univalent vertices to other parts of the diagram are possible.

\textbf{Antisymmetry.}

\[
\includegraphics{antisymmetry.png} = - \includegraphics{antisymmetry.png}
\] (17)

\textbf{IHX relation.}

\[
\includegraphics{ihx.png} = \includegraphics{ihx.png} - \includegraphics{ihx.png}
\] (18)

\textbf{Theorem 3} If the graph of some diagram $v$ has $p$ univalent vertices, then

\[
\psi_m^n \text{Sym}_v = n^p \text{Sym}_v.
\]
Proof.

The action of cabling on any Feynman diagram is expressible as a sum over actions by elements of $S_p$, once a labelling of the legs has been chosen. Recall that $Sym_v$ is a sum over all possible orderings of the external edges of a FD. Thus the action of any $\sigma \in S_p$ just returns $Sym_v$. To decide the eigenvalue it remains to count the number of permutations by which a cabling is expressed: every leg can be raised to one of $n$ possible covers, and there are $p$ legs, so $\psi_m^n$ is expressed on a FD with $p$ legs as a sum of $n^p$ permutation actions. Thus $n^p$ is the eigenvalue.

Hence we have a useful collection of eigenvectors of the cabling operation. In fact, we have a diagonalisation. In [BN] it is shown how the difference between any FD and a permutation of its legs is expressible as a sum of FDs with fewer legs. It follows that any FD can be expressed as a sum of these totally symmetrised vectors.

4 Deframing

We have seen that cabling is an operation on framed knots. We require a framing to choose a particular cabling, and so cabling a knot invariant, even if it previously was framing independent, introduces a framing dependence.

Framing independence of a Vassiliev knot invariant translates into an additional set of relations on the associated weight system, the 1-T relations. Diagrammatically,

$$W[V] \left( \begin{array}{c} r \\ \circ \end{array} \right) = V \left( \begin{array}{c} \circ \end{array} \right) = V \left( \begin{array}{c} \circ \end{array} \right) - V \left( \begin{array}{c} \circ \end{array} \right) = 0.$$

Recall that one of our goals is to seek an intrinsic explanation for the result of Bar-Natan and Garoufalidis. In this result, the invariant employed was always the writhe-normalised quantum group invariant. If we are seeking to generalise the scaling of the dimension of the chosen representation with a cabling operation, then we must conceive of a way to remove the framing dependence once cabling has introduced it.
What we require is an operator which projects out the subspace of chord diagrams with isolated chords, whilst preserving 4-T relations. We will investigate the properties of such a projector here, [Wil]. First, some technical maps:

**Definition 4.1** Write \( s : A_n \to A_{n-1} \), for the map which acts on chord diagrams by summing over ways of deleting a single chord, extended linearly. For example

\[
s\left(\begin{array}{c}
\circlearrowleft \\
\circlearrowleft \\
\circlearrowleft \\
\circlearrowleft \\
\end{array}\right) = 3 \left(\begin{array}{c}
\circlearrowleft \\
\circlearrowleft \\
\circlearrowleft \\
\circlearrowleft \\
\end{array}\right) + \left(\begin{array}{c}
\circlearrowleft \\
\circlearrowleft \\
\circlearrowleft \\
\circlearrowleft \\
\end{array}\right).
\]

(20)

Write \( \theta : A_n \to A_{n+1} \) for the map that connect-sums in the chord diagram with a single chord.

These maps are well-defined, preserving 4-T relations. If we take a 4-T relation, then we can see that the image is a sum of two terms, being terms where a chord is removed which is either active or not in the 4-T relation. The latter obviously still involves a 4-T relation at lesser degree, whilst the former vanish as terms where an “active” chord is removed always pair up and cancel.

It is also worth noting that with respect to the product in the natural algebra, the \( s \) operation satisfies a Leibniz rule \( s(a \cdot b) = s(a) \cdot b + a \cdot s(b) \). With this operation we can re-express the deframing operation in a form that will suit our analyses. The definition follows.

**Definition 4.2** Define \( \phi : A_n \to A_n \) to be the following operation.

\[
\phi = 1d - \theta \circ s + \frac{\theta^2 \circ s^2}{2!} - \ldots + \frac{(-1)^n \theta^n \circ s^n}{n!}.
\]

(21)

**Lemma 4.3**

\[
s \circ \phi = 0.
\]

(22)

**Proof.**

We have from the Leibniz rule that \( s \circ \theta^m \circ s^m = m \theta^{m-1} \circ s^m + \theta^m \circ s^{m+1} \).
Consider any two successive terms in the expansion of \( s \circ \phi \), and use the fact that
\[
\begin{align*}
s \circ \frac{1}{m!} \theta^m \circ s^m - \frac{1}{(m+1)!} \theta^{m+1} \circ s^{m+1} \\
= \frac{1}{m!} m \theta^{m-1} \circ s^m + \frac{1}{m!} \theta^m \circ s^{m+1} - \frac{1}{(m+1)!} \theta^{m+1} \circ s^{m+2}
\end{align*}
\]
\[= \frac{1}{(m-1)!} \theta^{m-1} \circ s^m - \frac{1}{(m+1)!} \theta^{m+1} \circ s^{m+2}. \tag{23}\]

Adding terms, and using the fact that \( s^{n+1} = 0 \) on diagrams with \( n \) chords, the lemma follows.

\(\square\)

**Corollary 4.4** \( \phi \) is a projection operator.

**Proof.**

\[
\phi^2 = (Id - \theta \circ s + \frac{\theta^2 \circ s^2}{2!} - \ldots + \frac{(-1)^n \theta^n \circ s^n}{n!}) \circ \phi
\]
\[= \phi \tag{24}\]

\(\square\)

**Lemma 4.5**

\[
\phi \left( \frac{\circ}{\circ} \right) = 0.
\]

**Proof.**

Consider a chord diagram with an isolated chord. \( \theta^m \circ s^m \) is the operation of summing over the “trivialisation” of all choices of \( m \) chords. This produces two terms, one where the already trivial chord is included in the choice, and one where it is not. The term where it is not cancels with the term where it is included, at the \((m + 1)\)-th term in the expansion of \( \phi \). This is because they are the same diagram and there are \( m + 1 \) more such terms from \( \theta^{m+1} \circ s^{m+1} \). This terminates at \( \theta^n \circ s^n \), as there is no term without trivial chord included in the set to be trivialised.
Thus φ is our desired ‘deframing’ operator. There is a nice description of the invariant subspace of deframing over the eigenvectors of cabling. The following criterion is useful for this task.

**Lemma 4.6** For v ∈ A_n, φ(v) = v if and only if s(v) = 0.

**Proof.**
It is clear that if s(v) = 0, then φ(v) = v, by construction. Further, we have shown that s ◦ φ = 0, so that if φ(v) = v, then s(v) = s(φ(v)) = 0.

How does the operator s act on the eigenvectors of cabling? First consider its action on Feynman diagrams.

**Lemma 4.7** The operator s acts on FDs by summing over all ways of removing a chord. If there are no chords, then it takes the value zero.

**Proof.**
Take a Feynman diagram. Resolve all but one of the trivalent vertices. In the sum over removals of chords, the terms where we remove the chords resulting from the trivalent vertex cancel. Thus with one vertex, the action of s is equivalent to a sum over removals of the non-participating chords. This understanding proceeds by induction.

The above procedure translates simply to our previously constructed cabling eigenvectors – the operator s acts on a SFD by striking out a single isolated chord in all possible ways, if they exist, otherwise s maps the SFD to zero. For example,

\[
s \left( \begin{array}{c} | \hline \end{array} \right) = \begin{array}{c} | \hline \end{array}, \quad s \left( \begin{array}{c} | \hline | \hline \end{array} \right) = 2, \quad s \left( \begin{array}{c} | \hline | \hline | \hline \end{array} \right) = 0. \tag{25}\]

It is immediate that the set of SFDs with no isolated chords is in the kernel of s, and hence in the deframing invariant subspace. With the unique SFD at level one, these generate the full kernel of s.
Note that on the space of SFDs, the primitive subspace of the Hopf algebra is spanned by the connected diagrams.

The first part of Theorem 2 now follows easily. Recall that the eigenvalue of a SFD with \( m \) legs under \( \psi^n \) is \( n^m \). The eigenvectors in the deframing invariant subspace whose eigenvalues are of leading order are those that have no isolated chords, and the maximum number of legs.

Note first that two legs cannot join in a trivalent vertex. This follows immediately from the antisymmetry of trivalent vertices. (Note that the edges are not oriented, the arrows here point to the rest of the diagram.)

\[
\begin{align*}
\raisebox{-0.5cm}{\includegraphics{triangle}} &= - \quad \raisebox{-0.5cm}{\includegraphics{doubletriangle}} &= - \quad \raisebox{-0.5cm}{\includegraphics{tripletriangle}}
\end{align*}
\]  

(26)

The maximum admissible number of legs then, at level \( n \), is \( n \), and the eigenvectors spanning the highest weight, deframing invariant subspace are all the ways of connecting the \( n \) separate trivalent vertices that each leg joins, with extra edges. This corresponds with the different ways of connecting \( n \) points with closed loops. These ways are enumerated by the partitions of \( n \).

**Definition 4.8** Denote by \( P \) some partition of a positive integer \( n \) (say \( P = \{P_1, \ldots, P_{#P}\} \)). We construct the SFD \( \tau_P \). It has \( #P \) components. The \( i \)th component is a loop of \( P_i \) edges with legs attached radially at every vertex.

Some examples –

\[
\tau_{\{2\}} = \quad \tau_{\{4,2\}} = \quad \tau_{\{6\}} =
\]

Observe that if such a vector is built from an odd partition (i.e. has an odd-legged component) then it is zero. This again comes from the antisymmetry condition. We can ‘flip’ such a loop over a given external chord, yielding the same SFD, multiplied by a minus sign -

\[
\begin{align*}
\raisebox{-0.5cm}{\includegraphics{flip1}} &= - \quad \raisebox{-0.5cm}{\includegraphics{flip2}} &= - \quad \raisebox{-0.5cm}{\includegraphics{flip3}} = - \quad \raisebox{-0.5cm}{\includegraphics{flip4}} = - \quad \raisebox{-0.5cm}{\includegraphics{flip5}} &= -
\end{align*}
\]
It is not hard to see that moreover, the $\tau_P$ are linearly independent (for even partitions $P$.)

We finish this section by noting that, as well as the operator $s$ defined above, there are further operations one can perform with interesting properties. For example, define an operator $d$ to act upon a chord diagram by summing the diagrams obtained by replacing each chord in turn firstly by two parallel chords, and then subtracting the diagrams obtained by replacing each chord in turn by two intersecting chords. For example,

$$d \left( \begin{array}{c} ✫✪ \\ ✬✩ \end{array} \right) = \begin{array}{c} ✫✪ \\ ✬✩ \end{array} - \begin{array}{c} ✫✪ \\ ✬✩ \end{array} \right) , \quad (27)$$

$$d \left( \begin{array}{c} ✫✪ \\ ✬✩ \\ ✬✩ \end{array} \right) = 2 \begin{array}{c} ✫✪ \\ ✬✩ \end{array} - 2 \begin{array}{c} ✫✪ \\ ✬✩ \end{array} + \begin{array}{c} ✫✪ \\ ✬✩ \end{array} - \begin{array}{c} ✫✪ \\ ✬✩ \end{array} \right) . \quad (28)$$

It is straightforward (if tedious) to show that this operator preserves the 4-T relations. One can also show that the operators $d$ and $s$ map between cabling eigenvectors, and that moreover $d \circ s - s \circ d = 0$. There are also interesting generalisations of these operators. Simple realisations of these operators exist for some of the Lie algebraic weight systems. For example, in [FKV] it is shown how the Alexander-Conway weight system arises using the superalgebra $gl(1|1)$. Before applying deframing, this weight system assigns a function of two variables $c$ and $y$ to each chord diagram. One can show by induction, using the recursion relation of [FKV], that the operators $d$ and $s$ are realised as the differential operators $\frac{\partial}{\partial c}$ and $\frac{\partial}{\partial c}$, respectively. Thus the action of deframing can be interpreted as the specification $c = 0$.

## 5 Immanents and cabling eigenvectors.

There is another way of representing the information in a chord diagram – by its *labelled intersection graph* (LIG) [BNG].

Consider a chord diagram $D \in \mathcal{A}_m$. Construct a labelled graph as follows. The $m$ vertices of this graph correspond to the $m$ chords of the diagram, and there is an edge connecting two vertices when the corresponding chords
intersect once in \( D \). Number the vertices according to the order of appearance going anti-clockwise from some arbitrarily chosen point on the external loop of the chord diagram. For example,

\[
v_0 = \begin{array}{ccc}
1 & 2 & 3 \\
4 & 3 & 1
\end{array}
\]

This information can be coded in the intersection matrix of the LIG. Construct this matrix via the following prescription.

**Definition 5.1** *The Intersection Matrix (IM) of a LIG is defined as follows.*

\[
(IM)_{ij} = \begin{cases} 
sign(i - j) & \text{if the vertices labelled } i \text{ and } j \text{ are linked}, \\
0 & \text{otherwise}. 
\end{cases}
\]  

Write \( IM : \hat{A}_n \to Gl(n, \mathbb{Z}) \).

The IM for the above example follows.

\[
\begin{bmatrix}
0 & 0 & -1 & -1 \\
0 & 0 & -1 & -1 \\
1 & 1 & 0 & -1 \\
1 & 1 & 1 & 0
\end{bmatrix}
\]

Up to 4-T relations, the IM contains sufficient information to reconstruct the original chord diagram \([\text{BNG}]\). To build a weight system from the IM of the LIG, one needs to generate numbers from the IM in a way which does not depend on the choice of break defining the numbering in addition to satisfying 4T relations. The determinant of the IM proves a well-defined choice \([\text{BNG}]\). The following result provides our connection with the AC polynomial.

**Fact 5.2** \([\text{BNG}]\) *The Alexander-Conway polynomial \( C(h) \) is a series in powers of \( h \), \( C(h) = \sum c_n h^n \). It is not difficult to see from the skein relation that \( c_n \) is in fact of finite type of order \( n \). Choose \( v \in \hat{A}_n \). Then*

\[
W_n[c_n](v) = Det(IM(v)).
\]  

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Immanents are alternative matrix invariants which yield well-defined weight systems in this fashion \([\text{BNG}]\). Denote by \(\mathcal{ZS}_n\) the integer module generated by the conjugacy classes of \(S_n\) (this is not formally the group ring - we maintain the notational conventions established in \([\text{BNG}]\)). Denote by \([\sigma]\) the conjugacy class of \(\sigma \in S_n\).

**Definition 5.3** The Universal Immanent Map of an \(n \times n\) matrix \((M)_{ij}\), \(\text{Imm} : \text{Gl}_n(\mathbb{Z}) \rightarrow \mathcal{ZS}_n\), is defined by

\[
\text{Imm}(M) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} (M)_{i\sigma(i)}[\sigma].
\]

We define the universal immanent weight system \(I : \mathcal{A}_n \rightarrow \mathcal{ZS}_n\) by \(I = \text{Imm} \circ IM\).

The conjugacy classes of \(S_n\) are bijective with the partitions of \(n\). To see this, construct a graph from \(\sigma \in S_n\) with \(n\) vertices, and a link from \(i\) to \(\sigma(i)\). The connected components of the resulting graph represent the corresponding partition.

To project to a \(\mathbb{C}\)-valued weight system, we compose some vector \(W \in \mathcal{ZS}_n^*\) (i.e. \(W \in \text{Hom}(\mathcal{ZS}_n, \mathbb{C})\)) with \(I\). There are some distinguished elements in \(\mathcal{ZS}_n^*\). Namely, any representation of \(S_n\) will furnish a well-defined functional on conjugacy classes by taking the trace of a representative element. With the alternating representation of \(S_n\) one obtains the usual matrix determinant. Taking the trivial representation, one gets the permanent of the matrix, for example.

We can understand the universal immanent weight system differently. First we note some graph theoretic terminology.

**Definition 5.4** A Hamiltonian cycle on a graph is a directed and non-repeating cycle of at least two vertices, where consecutive vertices are linked in the graph.
Definition 5.5 A Hamiltonian cycle decomposition (HCD) of a graph is a collection of disjoint Hamiltonian cycles such that every vertex in the graph appears in exactly one. The descent of a HCD of a labelled graph is the number of instances in a cycle decomposition where consecutive vertices in a cycle decrease in label value.

Fact 5.6 ([BNG]) To every cycle decomposition of an \(n\)-verticed graph we can associate a partition of \(n\). The universal immanent invariant of a labelled graph is precisely a sum over the partitions corresponding to the different cycle decompositions of the graph, with each decomposition weighted by \((-1)^d\), where \(d\) is the descent of the decomposition.

We illustrate this calculus with our previous example – the LIG above Definition (5.1), \(v_0\). The cycle decompositions and descents here are

\[
1 \rightarrow 4 \rightarrow 2 \rightarrow 3 \rightarrow 1 \quad d = 2 \\
3 \rightarrow 2 \rightarrow 4 \rightarrow 1 \rightarrow 3 \quad d = 2 \\
1 \rightarrow 4 \rightarrow 1, \quad 2 \rightarrow 3 \rightarrow 2 \quad d = 2 \\
2 \rightarrow 4 \rightarrow 2, \quad 1 \rightarrow 3 \rightarrow 1 \quad d = 2
\]

Thus

\[
I(v_0) = 2[4] + 2[2, 2].
\]

The weighting of the decomposition by \((-1)^d\) allows us to ignore decompositions which include cycles of odd length: reversing the direction of the odd-lengthed cycle produces the same decomposition with opposite sign, which cancels in the summation.

We have an important connection between the immanent weight systems at level \(n\) and the highest weight deframing invariant subspace of \(A_n\). For each \(n\), they have precisely the same dimension, the number of possible even partitions of \(n\). This motivates the following theorem.
**Theorem 4** Consider \( v \in A_m \).

1. If \( \phi(v) = 0 \), then \( I(v) = 0 \).

2. If \( \phi(v) = v \), and \( \psi_m^n(v) = n^p v \) for \( p < m \), then
   \[
   I(v) = 0. \tag{32}
   \]

3. \[
   I(\tau_{[\sigma]}) = 2^{#[\sigma]} m! [\sigma]. \tag{33}
   \]

In the above \( #[\sigma] \) denotes the number of components of \( |\sigma| \).

\( \square \)

Before presenting the proof of this proposition, we will explain how the main theorem follows from it.

**Definition 5.7** Take \( \sigma, \rho \in S_n \). Define \( \delta_{[\sigma]} : ZS_n \to C \), defined on the basis of \( ZS_n \) by
   \[
   \delta_{[\sigma]}([\rho]) = \begin{cases} 1 & \text{if } [\sigma] = [\rho], \\ 0 & \text{otherwise} \end{cases}
   \tag{34}
   \]
and extend linearly. This is the canonical dual basis.

Define \( \alpha_{[\sigma]} : A_n \to C \), by
   \[
   \alpha_{[\sigma]} = \delta_{[\sigma]} \circ I. \tag{35}
   \]

\( \square \)

The \( \alpha_{[\sigma]} \) span the set of immanent weight systems. Bar-Natan and Garoufalidis showed that this subspace was equivalent to the subspace of weight systems coming from sums of products of the coefficients of the AC polynomial \[BNG\].

**Proof of theorem 2.**

Consider the equation, for \( v \in A_m \),
   \[
   \psi^m W_m(v) = \phi^*(\psi^m W_m)(v),
   \]
   \[
   = W_m(\psi_m^\alpha(\phi(v))). \tag{36}
   \]

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Expand \( \phi(v) \) over the \( \tau_{[\sigma]} \) and SFDs with fewer univalent vertices, as
\[
\phi(v) = \sum_{[\sigma]} b_{[\sigma]} \tau_{[\sigma]} + \text{Rem},
\]
where \( \text{Rem} \) denotes terms with fewer than \( m \) univalent vertices. From Theorem 4 it then follows that
\[
\alpha_{[\sigma]}(v) = \alpha_{[\sigma]}(\phi(v) + (\text{Id} - \phi)(v))
\]
\[
= \alpha_{[\sigma]}(\sum_{[\sigma]} b_{[\sigma]} \tau_{[\sigma]} + \text{Rem})
\]
\[
= b_{[\sigma]} 2^{\# [\sigma] m}!.
\]

Applying the cabling operator and using Theorem 3, this implies that
\[
\psi^m W_m(v) = W_m \left( \psi_m(\sum_{[\sigma]} b_{[\sigma]} \tau_{[\sigma]} + \text{Rem}) \right),
\]
\[
= n^m \sum_{[\sigma]} b_{[\sigma]} W_m(\tau_{[\sigma]}) + \left( \text{lower powers in } n. \right),
\]
\[
= n^m \sum_{[\sigma]} k_{[\sigma]} W_m(\alpha_{[\sigma]}(v)) + \left( \text{lower powers in } n. \right),
\]
setting \( k_{[\sigma]} W_m = 1/(2^{\# [\sigma] m}) W_m(\tau_{[\sigma]}) \). Thus we see that the \( n \)-th cabling of a weight system of order \( m \) is a polynomial in \( n \) of highest order \( n^m \), and that the coefficient of \( n^m \) in this polynomial is a linear combination of immanent weight systems. These are statements 1 and 2 of Theorem 2.

Now we consider the proof of Theorem 4.

**Proof of part (1) of Theorem 4**

Consider \( v \in \mathcal{A}_n \) such that \( \phi(v) = 0 \). This implies that
\[
(Id - \theta \circ s + \frac{1}{2!} \theta^2 \circ s^2 + \ldots + \frac{(-1)^n}{n!} \theta^n \circ s^n)(v) = 0.
\]
Now \( I(w) = 0 \) if \( w \) has an isolated chord – as the LIG has an isolated vertex, there can be no cycle decompositions. Operating on both sides of \((43)\) with \( I \) we get
\[
I(v) = 0.
\]
\[
\square
\]
6 The universal immanent weight system and FDs.

Recall that the genus of a connected graph is calculated as $1 - \#\text{vertices} + \#\text{edges}$. Take some FD $v$ whose graph has $n$ components. Define $G(v)$ to be the unordered $n$-tuple of the genera of the connected components of the graph of $v$. We introduce the notation $\hat{v}$ for the graph that represents the SFD $v$.

Our principal technical tool is:

**Lemma 6.1** Take some Feynman Diagram $v \in \mathcal{A}_n$. If the graph of $v$ has a genus 1 component not equal to $\hat{\tau}_{\{p\}}$ for some even integer $p$ then $I(v) = 0$.

This will be proved in a later section. Part (2) of Theorem 4 is a statement about the values $I$ takes on SFDs at grade $n$ whose graphs have less than $n$ univalent vertices. We characterise these vectors:

**Lemma 6.2** Take a SFD $v \in \mathcal{A}_n$ with less than $n$ univalent vertices. Then $\hat{v}$ has a component of genus at least two.

This follows from a straightforward Euler characteristic calculation. Namely: there will be at least one component of $\hat{v}$ with more trivalent than univalent vertices, $t > u$. That component will have genus

$$G = 1 - (u + t) + \left(\frac{3t + u}{2}\right)$$

$$= 1 + \left(\frac{t - u}{2}\right) > 1.$$ 

With this understanding, Part (2) of Theorem 4 follows from the following lemma.

**Lemma 6.3** Take a FD $v \in \mathcal{A}_n$. If the graph of $v$ has a component of at least genus 2 then $I(v) = 0$.

**Proof.**

An STU resolution decreases the number of trivalent vertices on the graph of $v$. $v$ is then expressed as a linear combination of FDs whose graphs are the
same. On account of this identification it makes sense to speak of resolving the trivalent vertices of the graph of \( v \) in a particular order.

At each resolution, the genus of the graph of the FDs in the sum can alter in two ways. If the number of components increases by one as a result of the resolution then \( \{g_1, g_2, \ldots, g_n\} \to \{g'_1, g_1 - g'_1, g_2, \ldots, g_n\} \). Otherwise (when the number of components of the graph is unchanged) \( \{g_1, g_2, \ldots, g_n\} \to \{g_1 - 1, g_2, \ldots, g_n\} \).

It is always possible to resolve a choice of trivalent vertices such that \( v \) is equal to a sum over FDs with genus \( \{1, \ldots\} \). If the genus 1 component is not \( \hat{\tau}_{\{p\}} \) for some even \( p \), then \( I(v) = 0 \) from Lemma 6.1.

Assume then, that we have expressed (by some sequence of STU resolutions of \( v \)) \( v \) as a sum over FDs whose graphs have a genus 1 component \( \tau_{\{p\}} \) for some even \( p \). The step which led to this was either \( \{g_1, g_2, \ldots, g_n\} \to \{1, g_1 - 1, g_2, \ldots, g_n\} \) or \( \{2, g_2, \ldots, g_n\} \to \{1, g_2, \ldots, g_n\} \). We show here that in both these cases we can always choose a different sequence of vertex resolutions so that \( v \) is expressible as a sum of FDs with genus 1 components not some \( \hat{\tau}_{\{p\}} \) (and hence \( I(v) \) vanishes by Lemma 6.1).

Take the first case then, where some genus \( g > 1 \) component “splits” into a genus 1 and a genus \( g - 1 \) component when some joining vertex is resolved. For example:

\[
 a = \begin{array}{c}
 \begin{array}{c}
 \bullet
 \end{array}
 \end{array} \quad = \begin{array}{c}
 \begin{array}{c}
 \bullet
 \end{array}
 \end{array} - \begin{array}{c}
 \begin{array}{c}
 \bullet
 \end{array}
 \end{array} \quad \equiv b - c.
\]

\[
 G(a) = \{0, 2\} \quad G(b) = G(c) = \{0, 1, 1\}.
\]

We can always choose to resolve all the vertices that make up the genus \( g - 1 \) subgraph instead of the “joining” vertex. The genus 1 component is always then \( \hat{\tau}_{\{p\}} \) with some tree adjoined. Consider our example:

\[
 a = \begin{array}{c}
 \begin{array}{c}
 \bullet
 \end{array}
 \end{array} \quad = 2 \begin{array}{c}
 \begin{array}{c}
 \bullet
 \end{array}
 \end{array} \quad 22
\]
The second possibility is that the genus 1 component $\hat{\tau}_{\{p\}}$ is obtained by resolving the vertex on some genus 2 component. For example:

\[
\begin{align*}
    a &= \begin{array}{c}
        \text{Diagram 1}
    \end{array} \\
    &= \begin{array}{c}
        \text{Diagram 2} - \text{Diagram 3}
    \end{array} \\
    &\equiv b - c,
\end{align*}
\]

\[
\begin{align*}
    G(a) &= \{0,0,2\} \\
    G(b) &= G(c) = \{0,0,1\}.
\end{align*}
\]

Possible genus 2 components of this sort are $\hat{\tau}_{\{p\}}$ with two of the legs joined in an extra trivalent vertex. The case $p = 2$ is resolved from a null vector:

\[
\begin{align*}
    \begin{array}{c}
        \text{Diagram 4}
    \end{array} &= - \begin{array}{c}
        \text{Diagram 5}
    \end{array} = - \begin{array}{c}
        \text{Diagram 6}
    \end{array}
\end{align*}
\]

Thus we can assume $p \geq 4$. Joining the two legs in this fashion partitions the remaining legs around the loop into two sets, according to how they appear on the internal loop. If one set has $q$ legs, the other will have $p - q - 2$ legs. If either of these sets has more than one leg, then choosing instead to resolve a trivalent vertex along some leg from that set yields a genus 1 component not some $\hat{\tau}_{\{p\}}$. Taking our example:

\[
\begin{align*}
    \begin{array}{c}
        \text{Diagram 7}
    \end{array} &= - \begin{array}{c}
        \text{Diagram 8}
    \end{array} = - \begin{array}{c}
        \text{Diagram 9}
    \end{array}
\end{align*}
\]

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In this case we see that the extra join divides the other legs of \( \hat{\tau}_{\{4\}} \) into sets \( \{x, y\} \) and \( \phi \) (the joined legs are adjacent on the loop). As \( \{x, y\} \) has two legs, resolving along the leg \( x \) yields \( \hat{\tau}_3 \) with a tree adjoined.

There is one exceptional case. Joining opposite legs of \( \hat{\tau}_{\{4\}} \) in an extra vertex gives two sets of one leg each (resolving along any vertex always leads to \( \tau_{\{4\}} \) which we cannot say \( I \) vanishes on). Happily, this possibility is the zero vector.

\[
\begin{align*}
\text{We turn to the proof of Part 3 of Theorem 4. Write} & \\
v &= a \ b \\
&= a \ b \\
&= v_L - v_R \quad (46)
\end{align*}
\]

**Lemma 6.4** The cycle decomposition sum of the labelled intersection graph of \( v \) (above) is a linear combination of cycle decompositions, each of which includes a step from the vertex corresponding to \( a \) to the vertex corresponding to \( b \).

**Proof.**
On the level of labelled intersection graphs, $v_L - v_R$ looks like:

\[
\begin{array}{c}
\rightarrow \\
\cdot \\
\leftarrow \\
\end{array}
\] (47)

(where we have drawn in possible further links). Note that the labels on the vertices corresponding to $a$ and $b$ are either unchanged between $v_L$ and $v_R$, or they swap (in which case the labels are consecutive). Any cycle decomposition of $v_R$ which does not include a step from the vertex corresponding to chord $a$ to the vertex corresponding to chord $b$ decomposes $v_L$ with the same descent, and hence cancels.

Define $\tau'_P$ to be the FD that corresponds to the planar embedding of the graph $\hat{\tau}_P$ in a Wilson loop. For example:

\[
\tau'_{\{4\}} = \begin{array}{c}
\bullet \\
\oplus \\
\bullet \\
\end{array}
\] (48)

**Lemma 6.5**

\[
I(\tau'_{\{p\}}) = 2\{p\}.
\] (49)

**Proof.** Take the example $\tau'_{\{4\}}$. Resolve every trivalent vertex through the leg it joins:

\[
\tau'_{\{4\}} = \begin{array}{c}
\bullet \\
\oplus \\
\bullet \\
\end{array}
\] (50)

Lemma 6.4 indicates that the cycle decomposition sum is a linear combination of cycle decompositions which cycle around the four chords in the order in which they meet. The cycle decompositions are:

\[
\begin{array}{c}
1 \rightarrow 3 \rightarrow 4 \rightarrow 2 \rightarrow 1, \\
1 \rightarrow 2 \rightarrow 4 \rightarrow 3 \rightarrow 1.
\end{array}
\] (51)
Both these cycles have descent two. Thus \( I(\tau_{(4)}') = 2\{4\} \). The only case that is treated slightly differently to this example is for \( \tau_{(2)}' \). Here:

\[
\tau_{(2)}' = 2 \left( \begin{array}{c}
\end{array} \right). 
\]

(52)

It is easy to see in this case that \( I(\tau_{(2)}') = 2\{2\} \).

\(\square\)

Recall that \( ZS_n \) has a generator for every partition of \( n \). We define a multiplication \( \cdot : ZS_n \times ZS_m \rightarrow ZS_{n+m} \) defined on the generators by juxtaposition of partitions (i.e. \( \{p_1, \ldots, p_i\} \cdot \{q_1, \ldots, q_j\} = \{p_1, \ldots, p_i, q_1, \ldots, q_j\} \)), extended linearly. The following property is manifest from the definition of \( I \).

**Lemma 6.6** Take \( v \in A_n \), \( w \in A_m \). Then

\[
I(v.w) = I(v) \cdot I(w).
\]

(53)

Take some SFD \( \tau_{\{p_1, \ldots, p_i\}} \). Recall that this is a sum over the FDs that correspond to all different orderings of the legs of the graph \( \hat{\tau}_{\{p_1, \ldots, p_i\}} \) on a Wilson loop. Recall that on account of the STU relations, the FDs corresponding to different orderings of the legs on the Wilson loop differ by FDs with more internal vertices. In fact here we observe:

\[
\tau_{\{p_1, \ldots, p_i\}} = (p_1 + \ldots + p_i)! \tau'_{\{p_1\}} \cdot \ldots \tau'_{\{p_i\}} + \left\{ \text{FDs of genus } \geq 2 \right\}.
\]

(54)

This observation, together with Lemma 6.3, Lemma 6.5 and Lemma 6.6 yields Part 3 of Theorem 4.

7 Proof of Lemma 6.1
Lemma 6.1 details conditions under which $I$ vanishes. We separate the proof into two parts.

**Lemma 6.1 A**

If the graph of a FD $v$ has a component of its graph $\hat{\tau}_{\{p\}}$ for some odd integer $p$, then $I(v) = 0$.

**Proof.**

Such a FD has a presentation, taking the example $p = 3$,

\begin{equation}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{example.png}
\end{array}
\end{equation}

By Lemma 6.4 the cycle decomposition sum is a linear combination of cycle decompositions which include a 3-cycle around these chords. However, all decompositions with odd cycles cancel on account of the weighting by descent.

\[ \square \]

**Lemma 6.1 B.**

If the graph of a FD $v$ has a genus 1 component not some $\hat{\tau}_{\{p\}}$ then $I(v) = 0$.

**Proof.**

A connected genus one trivalent graph $v$ has a single cycle with a number of 'trees' attached (with the obvious meaning):

\begin{equation}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{example.png}
\end{array}
\end{equation}

Via a sequence of STU resolutions any FD $v$ containing a genus 1 component not some $\hat{\tau}_{\{p\}}$ may be expressed as a linear combination of FDs whose
graphs contain a component from the following list (i.e. some $\tilde{\tau}_v$ with one extra trivalent vertex):

\[
\begin{array}{c}
\includegraphics{image1} \\
\includegraphics{image2} \\
\includegraphics{image3} \\
\end{array}, \ldots
\]

Thus we need to show that $I(v) = 0$ when $v$ has such a component in its graph. There is no loss of generality if we assume there are no trivalent vertices in the other graph components (i.e. they are all chords). There are two cases to address: when the cycle has two edges, and when the cycle has more than two edges.

Assume first that the cycle has more than two edges. Such a FD has a presentation as follows:

\[
= v_L - v_R.
\]  \tag{56}

From Lemma 6.4 we see that the cycle decomposition sum of $v_L$ (or $v_R$) is a linear combination of cycle decompositions which include a cycle around the vertices corresponding to the chords $a, b, c$ and $d$.

However note further, that any cycle decomposition which does not include a step from the vertex corresponding to $*$ to at least one of $a$ or $d$ appears equally signed in the cycle decomposition sums of $v_L$ and $v_R$ and hence cancels in the sum (one must be careful to check that the descents are the same, the point being that when labels swap, they are consecutive).

Thus, the cycle decomposition sum is zero. $I(v) = 0$.

The logic for when the genus 1 component has a two cycle is almost identical. In this case $v$ has a presentation:
8 Conclusions

The result we have proved in Theorem 2 is a statement about weight systems. What this says about the actual knot invariants is more subtle. Recall that the Kontsevich integral inverts weight systems \([\text{Kon}, \text{BN}]\). That is, it is a knot invariant taking values in the algebra of chord diagrams \(Z_K : \{\text{knots}\} \rightarrow A_*\). In general we know that for a knot invariant of finite order \(m\)

\[
W_m(V) \circ Z_K = V + (\text{invariants of order } < m).
\]  

(58)

This expression can be iterated to construct a weight system:

\[
\hat{W}^V = \hat{W}^V_m + \hat{W}^V_{m-1} + \ldots + \hat{W}^V_1 \neq W[V]
\]

(59)

Consider the invariant \(V^n = \hat{W}^V \circ \psi^n \circ \phi \circ Z_K\). Obviously \(V^1 = V\). In this work we have shown that \(V^n\) is a finite polynomial in \(n\) of highest order \(n^m\) and that the coefficient of \(n^m\) is a linear sum of the knot invariants \(\alpha_{[\sigma]} \circ Z_K\). The weight systems \(\alpha_{[\sigma]}\) are in the algebra of the (normalised) Alexander-Conway weight systems \([\text{BNG}]\). As these weight systems are canonical (the remainder in eqn. (58) vanishes) we have shown that the highest order term of an arbitrary Vassiliev knot invariant is in the algebra of coefficients of the Conway polynomial.

This is an intriguing result: every Vassiliev knot invariant has a term which can be calculated from traditional methods of algebraic topology. An immediate question is whether such an understanding extends to the lower powers in \(n\) of an invariant. Alternatively, how must the Alexander construction be perturbed to account for the next-to-highest orders?
There are some obvious generalisations, in that it is not difficult to generate sequences of weight systems of which the immanent variety form the simplest example. For instance, one may count the number of graph morphisms from some more sophisticated graph into the LIG, appropriately weighted. Such generalisations would presumably filter FDs according to genus. The difficult and interesting task is to seek topological candidates for these generalisations.

The Melvin-Morton-Rozansky conjecture follows naturally from our results here (as was certainly anticipated by Bar-Natan and Garoufalidis when they formulated their conjecture) and is the subject of a paper in preparation. It would be interesting to more fully incorporate the Lie algebraic weight systems into this picture: such an incorporation might lend some insight into the role of Lie algebras in the space of weight systems. In [BNG] an unusual generating formula was provided for the \(sl(2)\) weight system. This formula indicates that the lesser powers in \(\lambda\) arise by considering cycle decompositions with a certain number of “self-intersections” in generalised intersection graphs, with an extra singular point for each reduction in \(\lambda\).

In the context of Lie algebras and cabling operators, it is appropriate to point the reader towards [AT]. In this work, Atiyah and Tall thoroughly investigate the “Adams operations” on lambda rings: in our case the act of cabling descends to such an operation on the representation ring of the Lie algebra.

We finish by noting that this work relates to the BF topological field theories investigated by Cattaneo et al. In this reference the authors recovered the Alexander-Conway polynomial from the BF theory without cosmological constant [CCM], and the Jones polynomial from the BF theory with cosmological constant [CFM]. Our work suggests that the correlators yielding the Alexander-Conway polynomial in the theory without cosmological constant can be related to the correlators for the theory with cosmological constant evaluated along a cabled Wilson loop.

**Acknowledgements**

AK was supported by an Australian Government Postgraduate Award and would like to express his thanks to the strings group at Queen Mary & Westfield College London, where he was a guest whilst this work was begun. BS is supported by the Engineering and Physical Sciences Research Council of the UK. We would like to thank Paul Martin and Jose Figueroa-O’Farrell for helpful conversations, and the latter also for kindly allowing us to use his
routines for drawing chord diagrams and for showing us his work prior to publication.

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