Extension of nested array for large aperture and high degree of freedom

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Abstract: This paper presents a novel array configuration for high-resolution direction of arrival (DOA) estimation by extending the two-level nested array based on the concept of the Khatri-Rao product. We found that the proposed array configuration had larger aperture and higher degree of freedom in comparison with those of the two-level nested array, and also had a closed-form sensor position representation. Performance of the proposed array is evaluated through some computer simulation of DOA estimation.

Keywords: direction of arrival estimation, array signal processing

Classification: Antennas and Propagation

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1 Introduction

Direction of arrival (DOA) estimation plays an important role in radar, sonar, and indoor and outdoor wireless communications. High resolution DOA estimation methods have been studied in the last three decades and have attracted much attention [1, 2]. The well-known methods, MUSIC, Root-MUSIC, ESPRIT, and Unitary-ESPRIT are based on the eigenvalue decomposition of a sample covariance matrix of an array input.

Recently the nested array [3] has been proposed as an efficient array configuration with a closed-form sensor location representation. The nested array creates
a virtual uniform linear array (ULA) from a given physical nonuniform array, and estimates more number of DOAs than the number of physical array elements using the concept of Khatri-Rao (KR) product [4]. The aperture of the virtual ULA indeed becomes larger than that of the physical ULA, however the aperture is still smaller than that of the minimum redundancy array (MRA) [5] which could be regarded as a kind of optimum array configuration but does not have a closed-form location representation.

In this paper, we try to find a novel array configuration with a closed-form location representation which can be regarded as an extension of two-level nested array. The proposed array has larger aperture and higher degree of freedom (DOF) in comparison with those of the two-level nested array. Performance of the proposed array is evaluated through computer simulation of DOA estimation.

2 Preliminaries

2.1 Signal model
Assume that \( L \) far-field incident signals are received by an \( M \)-element linear array in an additive white Gaussian noise (AWGN) environment, where signals and noises are statistically independent. The array input vector \( x(t) \) can be written as

\[
x(t) = As(t) + n(t),
\]

where \( A, s(t), \) and \( n(t) \) denote the array steering matrix, incident signal vector, and noise vector, respectively. The covariance matrix of the array input vector \( x(t) \) can be written as

\[
R_{xx} = E[x(n)x^H(n)] = ASA^H + R_N,
\]

where \( E[-], S, \) and \( R_N \) denote the expectation, source autocorrelation matrix, and noise covariance matrix, respectively.

2.2 DOA estimation using the concept of KR product [4]
We assume uncorrelated sources which make the matrix \( S \) diagonal. The covariance matrix \( R_{xx} \) is then vectorized as

\[
z = \text{vec}(R_{xx}) = (A^* \odot A)p + \text{vec}(R_N),
\]

where \( *, \odot \) and \( \text{vec}(\cdot) \) respectively denote the complex conjugate, KR product and the vectorization operator, and \( p \) consists of the diagonal element of \( S \) [4].

The distinct rows of \( A^* \odot A \) behave like the manifold of a longer array whose sensor locations are given by the distinct values in the set \( \{d_i - d_j \mid i, j = 0, 1, \ldots, M - 1 \} \) where \( d_i \) denotes the position vector of the \( i \)-th sensor of the original array. Hence the vector \( z \) has the same manner as the original signal vector \( x(t) \), and we can apply DOA estimation to the vector \( z \).

2.3 Two-level nested array [3]
The two-level nested array is basically a concatenation of two ULAs, where the first (level-1) ULA has \( M_1 \) elements with the interval \( \Delta d_1 \), and the second (level-2) ULA
has $M_2$ elements with the interval $\Delta d_2 = (M_1 + 1)\Delta d_1$ [3]. More precisely, the two-level nested array can be regarded as a nonuniform linear array with the sensor locations given by the union of the sets $S_1 = \{(m_1 - 1)\Delta d_1 | m_1 = 1, 2, \ldots, M_1\}$ and $S_2 = \{m_2\Delta d_2 - \Delta d_1 | m_2 = 1, 2, \ldots, M_2\}$ as shown in Fig. 1(a). Thanks to the KR product, the two-level nested array can create a virtual ULA from $-\{M_2(M_1 + 1) - 1\}\Delta d_1$ to $\{M_2(M_1 + 1) - 1\}\Delta d_1$.

### 3 Proposed array configuration

The two-level nested array has higher DOF than that of the ULA with a same number of elements, which will lead more accurate DOA estimation, and holds a closed-form representation of array geometry even the MRA does not hold it [5].

We now try to extend the concept of two-level nested array by shifting all the level-2 ULA elements to have $\Delta d_1$-more interval as shown in Fig. 1(b), where $M'_1, M'_2$ denote the number of elements in each level in the proposed array. The location of the proposed array is represented as

$$
\Delta d_2 = M'_1\Delta d_1, 
$$

$$
S'_1 = \{(m_1 - 1)\Delta d_1 | m_1 = 1, 2, \ldots, M'_1\},
$$

$$
S'_2 = \{(m_2 + 1)\Delta d_2 | m_2 = 1, 2, \ldots, M'_2\} = \{(m_2 + 1)M'_1\Delta d_1 | m_2 = 1, 2, \ldots, M'_2\},
$$

where $M'_1 = M_1 + 1$ and $M'_2 = M_2 - 1$. Then the proposed array can create a virtual ULA from $-M'_1(M'_2 + 1)\Delta d_1$ to $M'_1(M'_2 + 1)\Delta d_1$ after the KR-product extension, as proven in the following proposition.

**Proposition 1:**

The difference co-array of the proposed configuration becomes a ULA from $-M'_1(M'_2 + 1)\Delta d_1$ to $M'_1(M'_2 + 1)\Delta d_1$.

**Proof:**

Hereafter we assume $\Delta d_1 = 1$ for simplicity. Let $S'$ denote the set of sensor positions in the proposed array, i.e.,
We consider another subset
Case 2:
Then the difference set of the proposed array can be written as
\[ S'_\text{aug} = \{ u_1 - u_2 \}, \quad u_1, u_2 \in S'. \] (8)
The set \( S'_\text{aug} \) in (8) can be divided as a union of the following three cases.
Case 1:
Consider a subset \( S'_{\text{aug}1} = \{ u_1 - u_2 \} \) of the set \( S'_\text{aug} \), where \( u_1 = M'_1 - 1 \) and \( u_2 \in S'_1 \). Then we have:
\[
S'_{\text{aug}1} = \{ u_1 - u_2 \} \\
= \{ M'_1 - 1 - u_2 \} \\
= \{ 0, 1, 2, \ldots, M'_1 - 1 \}. \] (9)
Case 2:
We consider another subset \( S'_{\text{aug}2} = \{ u_1 - u_2 \} \), where \( u_1 \in S'_2 \) and \( u_2 \in S'_2 \). Here we have:
\[
S'_{\text{aug}2} = \{ u_1 - u_2 \} \\
= \{ (m_2 + 1)M'_1 - (m_2 + 1 + 1)M'_1 \} \\
= \{ M'_1 \}. \] (10)
Case 3:
Consider the subsets \( S'_{\text{aug}3,m} = \{ u_1 - u_2 \} \), where \( u_1 = (m + 1)M'_1 \in S'_2, \ u_2 \in S'_1 \), and \( m = 1, \ldots, M'_2 \). Then we have
\[
S'_{\text{aug}3,m} = \{ u_1 - u_2 \} \\
= \{ (m + 1)M'_1 - u_2 \} \\
= \{ mM'_1 + 1, mM'_1 + 2, \ldots, (m + 1)M'_1 \}, \] (11)
\[
S'_{\text{aug}3,m+1} = \{ (m + 2)M'_1 - u_2 \} \\
= \{ (m + 1)M'_1 + 1, (m + 1)M'_1 + 2, \ldots, (m + 2)M'_1 \}. \] (12)
Then the sum \( S'_{\text{aug}3} \) of the subsets \( S'_{\text{aug}3,m} \) also becomes a subset of \( S' \), and is written as
\[
S'_{\text{aug}3} = \sum_{m=1}^{M'_2} S'_{\text{aug}3,m} \\
= \{ M'_1 + 1, M'_1 + 2, \ldots, (M'_2 + 1)M'_1 \}. \] (13)
From the above three cases, the sum of the difference co-arrays makes a difference co-array on the positive (right) side in the proposed configuration, i.e.,
\[
S'_+ = S'_{\text{aug}1} \cup S'_{\text{aug}2} \cup S'_{\text{aug}3} \\
= \{ 0, 1, 2, \ldots, (M'_2 + 1)M'_1 \}. \] (14)
In a similar manner, we can make the other co-array on the negative (left) side by the same approach which is given by
\[ S'_+ = \{- (M'_1 + 1)M'_1, \ldots, 2, 1, 0\}. \]  

(15)

As a whole, we have the sum of the positive and negative difference co-arrays:

\[ S'_{\text{aug}} = S'_+ \cup S'_- \]

\[ = \{- (M'_2 + 1)M'_1, \ldots, -1, 0, 1, \ldots, (M'_2 + 1)M'_1\}, \quad (16) \]

which becomes a ULA from \(-M'_1(M'_2 + 1)\) to \(M'_1(M'_2 + 1)\). Therefore the Proposition 1 holds.

In [3], the \(K\)-level nested array is defined as a union of \(K\) ULAs with \(M_i\) sensors in the \(i\)-th level of nesting. We can apply the same manner of extension for \(K\)-level nested array and can make larger array aperture.

### 4 Simulation

The DOA estimation accuracy of the proposed array is evaluated through computer simulation and compared with the accuracy of the method of [3]. We try \(N = 1,000\) times Monte Carlo simulation and evaluate the RMSE given by

\[ \text{RMSE} = \sqrt{\frac{1}{LN} \sum_{i=1}^{N} \sum_{\ell=1}^{L} (\theta_i(\ell) - \hat{\theta}_i(\ell))^2}. \]  

(17)

First we consider the case of \(M = 6\) elements ULA and \(L = 6\) uncorrelated incident signals. The numbers of each level of arrays are given as \(M_1 = 3, M_2 = 3, M'_1 = 4, M'_2 = 2\), hence the sensor locations are \(\{0, 1, 2, 3, 7, 11\}\) for the two-level nested array, and \(\{0, 1, 2, 3, 8, 12\}\) for the proposed array. The six DOAs are randomly given between \([-60, 60]\) where the DOAs have difference at least 5 degree.

Fig. 2 show the behavior of RMSE as a function of SNR and the number of snapshots in the case of 6 element array. The number of snapshots is set to 200 in Figs. 2(a), while the SNR is set to 10 dB in Figs. 2(b). We can see from Fig. 2 that the proposed configuration achieves better performance than the two-level nested array due of the larger array aperture.

Fig. 3 shows the behavior of RMSE in another scenario: we consider the case of \(M = 8\) array elements and \(L = 10\) uncorrelated incident signals. The numbers
of each level of arrays are $M_1 = 4$, $M_2 = 4$, $M'_1 = 5$, $M'_2 = 3$, so the sensor locations are \{0, 1, 2, 3, 4, 9, 14, 19\} for the two-level nested array, and \{0, 1, 2, 3, 4, 10, 15, 20\} for the proposed array. Ten DOAs in this case are given by \{-60, -52, -45, -30, -15, 0, 15, 30, 45, 60\}, and the other specifications follow the case of Fig. 2. We can see from Fig. 3 that the proposed method achieves better performance than the two-level nested array in various scenarios.

Also we confirmed that the proposed array have more DOF; the proposed array could estimate DOAs of up to 12 sources while the two-level nested array could do up to 11 sources in the case of 6 element array.

5 Concluding remarks

This paper proposed an extended version of two-level nested array. The performance of the proposed method was evaluated through computer simulation, and we confirmed that the proposed array worked effectively. The proposed method achieved higher DOA estimation accuracy and larger DOF.

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