On linear independence of values of certain $q$-series

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Abstract. We obtain qualitative and quantitative results on the linear independence of the values of functions in a fairly wide class generalizing $q$-hypergeometric series and of their derivatives at algebraic points. The results are proved in both the Archimedean and $p$-adic cases.

Keywords: algebraic number field, absolute height of an algebraic number, $q$-series, $q$-exponential function, $q$-logarithm, linear independence, linear independence measure, Hankel determinant, cyclotomic polynomial.

§ 1. Notation and main results

Let $K$ be a finite extension of the field $\mathbb{Q}$ of degree $\kappa = [K : \mathbb{Q}]$ and let $M$ be the set of all valuations on $K$. For $v \in M$ we normalize the absolute value $\cdot |_v$ as follows:

1) $|p|_v = p^{-1}$ if $v|p$,
2) $|x|_v = |x|$ for $x \in \mathbb{Q}$ if $v|\infty$,

where $|x|$ is the modulus of a number $x$. Then for any $\alpha \in K^* = K \setminus \{0\}$ we have the product formula

$$\prod_v |\alpha|_v^{\kappa_v} = 1,$$

where the $\kappa_v = [K_v : \mathbb{Q}_v]$ are the corresponding local degrees.

For $\alpha \in K$ we let $H(\alpha)$ denote the absolute (multiplicative) height of the number $\alpha$:

$$H(\alpha) = \prod_v \max\{|\alpha|_v^{\kappa_v/\kappa}; 1\}.$$ 

We note the equation $H(\alpha) = (M(\alpha))^{1/\deg \alpha}$, where $M(\alpha)$ is the Mahler measure of a number $\alpha$. In particular, if $\alpha = a/b \in \mathbb{Q}$, where $a, b \in \mathbb{Z}$, $(a, b) = 1$, then $H(\alpha) = \max\{|a|/|b|\}$.

If $\alpha \in K^*$, then for any $v \in M$ we have the fundamental inequality

$$(H(\alpha))^{-1} \leq |\alpha|_v^{\kappa_v/\kappa} \leq H(\alpha).$$

Next, for an arbitrary vector $\alpha = (\alpha_0, \ldots, \alpha_k) \in K^{1+k}$ we write

$$|\alpha|_v = \max\{|\alpha_0|_v; \ldots; |\alpha_k|_v\}, \quad v \in M, \quad H(\alpha) = \prod_v |\alpha|_v^{\kappa_v/\kappa},$$

(in particular, $H((1, \alpha)) = H(\alpha)$).

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Suppose that \( q \in K \) and \( w \in M \) are such that \(|q|_w > 1\). We set
\[
\lambda = \frac{\kappa \log H(q)}{\kappa_w \log |q|_w},
\]
(1.1)
Note that \( \lambda \geq 1 \), and \( \lambda = 1 \) if and only if the inequality \(|q|_v \leq 1\) holds for all \( v \in M \setminus \{w\} \).

Suppose that polynomials \( P(x, y) \in K[x, y] \) and \( Q(x) \in K[x] \) satisfy the conditions \( d := \deg_y P \geq 1 \) and \( P(n, q^n)Q(n) \neq 0 \) for \( n = 1, 2, 3, \ldots \). We set
\[
\Pi_n(z) = \prod_{k=1}^{n} \frac{P(k, z^k)}{Q(k)}, \quad n \in \mathbb{Z}_{\geq 0},
\]
(1.2)
and consider the function
\[
f(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Pi_n(q)}, \quad z \in C_w,
\]
(1.3)
where \( C_w \) is the completion of the algebraic closure of \( K_w \).

The function \( f(z) \) is entire. Indeed, in the notation
\[
P(x, y) = \sum_{\nu=0}^{d} p_\nu(x) y^\nu,
\]
(1.4)
for all sufficiently large \( n \in \mathbb{Z}_{>0} \) we have
\[
|p_d(n)|_w \geq H(p_d(n))^{-\kappa/\kappa_w} \geq n^{-c}
\]
with some constant \( c \), and so for large \( n \) we have
\[
|P(n, q^n)|_w \geq 0.5 n^{-c} |q|_w^{dn},
\]
whence we obtain what is required.

There are many papers devoted to the study of the arithmetic properties of the values of functions of the form (1.3). For example, by setting \( Q(x) = 1 \) and \( P(x, y) = P(y) \in K[y] \) we obtain a fairly wide class of so-called \( q \)-hypergeometric series (see [1]), special cases of which are the \( q \)-exponential function
\[
E_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{\prod_{k=1}^{n}(q^k - 1)} = \prod_{n=1}^{\infty} \left( 1 + \frac{z}{q^n} \right),
\]
(1.5)
and the Chakalov function
\[
T_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{q^{n(n+1)/2}} = \sum_{n=0}^{\infty} \frac{z^n}{\prod_{k=1}^{n} q^k}.
\]
(1.6)
If \( P(x, y) = P(x)y \), then the function (1.3) is a special case of the so-called generalized Chakalov series (see [2]).
In 1988 Bézivin [3] proposed a new method for proving the linear independence of the values of a very wide class of functions. Although this approach has not as yet made it possible to obtain estimates for the linear independence measure, it has been used successfully to obtain a number of results that were inaccessible using other methods. For example, Bézivin [4] used a certain modification of his original method to prove that for \( q \in \mathbb{Z} \setminus \{0; \pm 1\} \) and \( \alpha \in \mathbb{Q}^* \) the number \( T_q(\alpha) \) is neither a rational number nor a quadratic irrational (a more general assertion is proved in [4]). In 2001 Choulet [5] used analytic considerations to extend Bézivin’s method to the case of the \( q \)-exponential function \( E_q(z) \) (note that Choulet’s method can also be used for functions of the more general form (1.3)). The recent paper [6] contains an elementary analogue of Choulet’s method that is close in spirit to the original paper [4] but involves considerations absent from Bézivin’s and Choulet’s papers. In particular, Bézivin’s result on non-quadraticity was extended to the \( q \)-exponential function (see also Corollary 1.1 below). In the present paper we give a quantitative generalization of the method of [6] and prove a number of assertions on the linear independence over \( \mathbb{K} \) of the values of the function (1.3) and its derivatives at points of \( \mathbb{K} \) (including an estimate for the linear independence measure) under various restrictions on \( q \), \( P(x, y) \), and \( Q(x) \). As applications we refine a number of known results.

We introduce the notation

\[
Z(a, b) = \pi^{-2} \sum_{n \geq 0} [an + b]^{-2}, \tag{1.7}
\]

where \([x]\) is the integer part of a number \( x \) (in particular, if \( a, b \in \mathbb{Z}_{>0} \), then \( Z(a, b) = (a\pi)^{-2}\zeta(2, b/a) \), where \( \zeta(s, a) = \sum_{n \geq 0} (n + a)^{-s} \) is the Hurwitz zeta-function).

**Theorem 1.1.** Suppose that the polynomials \( P(x, y) = P(y) \) and \( Q(x) = 1 \) are independent of \( x \).

Suppose that numbers \( \alpha_1, \ldots, \alpha_m \in \mathbb{K}^* \) satisfy the conditions

(i) \( \alpha_j \alpha_k^{-1} \notin q\mathbb{Z} \) for \( 1 \leq j, k \leq m, \ j \neq k \),

(ii) \( \alpha_j \notin P(0)q^{Z>0} \) for \( 1 \leq j \leq m \).

Let \( s_1, \ldots, s_m \in \mathbb{Z}_{>0} \). We set

\[
s = \sum_{j=1}^{m} s_j, \tag{1.8}
\]

\[
\alpha = \begin{cases} 
2s + 1/d & \text{if } P(y) = py^d, \ p \in \mathbb{K}^*, \\
1/6s^2 & \text{otherwise}, \\
3s & \text{if } P(0) \neq 0,
\end{cases} \tag{1.9}
\]

\[
\beta = \begin{cases} 
1/3(s + 1) & \text{if } \delta_0 := \text{ord}_{y=0} P(y) > 0, \\
2s + 3/\delta_0 & \text{if } \delta_0 := \text{ord}_{y=0} P(y) > 0.
\end{cases} \tag{1.10}
\]
Next, if $P(1) \neq 0$, we write

$$\gamma = \beta + \begin{cases} 
\frac{5}{32} - \frac{289}{900 \pi^2} - Z(4, 1) & \text{for } m = 2, s_1 = s_2 = 2, \\
Z(m + 2, s + 1) + Z(m + 2, s + 2) & \text{otherwise.}
\end{cases} \tag{1.11}$$

But if $\delta_1 := \text{ord}_{y=1} P(y) > 0$, we set

$$\gamma = \beta + \begin{cases} 
\frac{1}{6} - \frac{41}{36 \pi^2} & \text{for } \delta_1 = 1, m = 1, s_1 = 2, \\
2Z \left( m + \frac{1}{\delta_1}, s + 1 \right) & \text{otherwise.}
\end{cases} \tag{1.12}$$

If

$$\lambda < \frac{2d/3 + \alpha}{2d/3 - \gamma}, \tag{1.13}$$

where $\lambda$ is defined in (1.1), then the numbers $1$, $f^{(\sigma)}(\alpha_j q^k)$, $1 \leq j \leq m$, $0 \leq k < d$, $0 \leq \sigma < s_j$, are linearly independent over $\mathbb{K}$. Moreover, for any $\varepsilon > 0$ there exists a constant $H_0 = H_0(P, q, \lambda, m, \alpha_j, s_j, \varepsilon) > 0$ such that for any vector $\overline{\eta} = (\eta_0, \eta_{j,k,\sigma}) \in \mathbb{K}^{1+ds} \setminus \{0\}$ we have

$$\left| \eta_0 + \sum_{j=1}^{m} \sum_{k=0}^{d-1} \sum_{\sigma=0}^{s_j-1} \eta_{j,k,\sigma} f^{(\sigma)}(\alpha_j q^k) \right|_w \geq |\overline{\eta}|_w \exp \left( -(C_0 + \varepsilon)(\log H)^{3/2} \right),$$

where $H = \max \{H(\overline{\eta}); H_0\}$ and

$$C_0 = \frac{d^3/2 (2d/3 + \alpha + (d^2 - 1)(2d/3 - \gamma) \lambda) \log |q|_w}{((2d/3 + \alpha - (2d/3 - \gamma) \lambda) \log H(q))^{3/2}}.$$

In the case when $\lambda = 1$ and the numbers $\alpha_j$ satisfy the stronger (than (ii)) condition $\alpha_j \notin P(0)q^Z$, the qualitative part of Theorem 1.1 (without an estimate for the linear independence measure) is contained in [1], Theorem 3. In the case when $\lambda > 1$, such a general result on linear independence is apparently new, and so is the quantitative result of that generality (even in the case $\mathbb{K} = \mathbb{Q}$ and $q \in \mathbb{Z}$). Stronger estimates for linear forms in various special cases can be found in [7]–[9].

**Remark 1.1.** Our method of proof of Theorem 1.1 often makes it possible to weaken condition (1.13) a little, for example, if $P(y)$ has roots that are roots of unity different from 1 (see §3.3). Furthermore, if $P(y) = R(y^t)$ for $R(y) \in \mathbb{K}[y]$ and some $t \in \mathbb{Z}_{>1}$, we can apply Theorem 1.1 to the polynomial $R(y)$, the number $q^t$, and the points $\alpha_j q^r$, $1 \leq j \leq m$, $0 \leq r < t$, instead of to $P(y)$, $q$, and $\alpha_j$, taking into account the fact that the equation $\lambda(q^t) = \lambda(q)$ holds for the quantity $\lambda = \lambda(q)$ defined in (1.1).

**Remark 1.2.** Condition (1.13) can also be weakened in the case when the ratio of two of the numbers $\alpha_j q^k$, $1 \leq j \leq m$, $k \in \mathbb{Z}$, is a root of unity. For example, one can show (see Remark 3.1 below) that if $\lambda < 270/163$, then for any $\alpha \in \mathbb{K}^*$, $\alpha \notin \pm q^{Z>0}$,
the numbers $1, E_q(\alpha)$, and $E_q(-\alpha)$ (where $E_q(z)$ is the $q$-exponential function (1.5)) are linearly independent over $\mathbb{K}$ (in this case the condition of Theorem 1.1 has the form $\lambda < 45/28$).

By applying Theorem 1.1 to $E_q(z)$ we obtain the following assertion, which refines the main result of [6] (note that a similar refinement can also be given, for example, for the function $F_q(z; -1)$ in the notation of [6]; see Remark 1.1).

**Corollary 1.1.** Let $q = \rho/\sigma \in \mathbb{Q}$, where $\rho, \sigma \in \mathbb{Z} \setminus \{0\}$, $(\rho, \sigma) = 1$, $|\rho| > |\sigma|$, and let $\alpha \in \mathbb{Q}^*$, $\alpha \notin q^{\mathbb{Z} > 0}$. Let $\gamma = \log |\sigma|/\log |\rho|$. If $\gamma < 7/12$, then the number $E_q(\alpha)$ is irrational. Moreover, for any $\varepsilon > 0$ there exists a positive constant $S_0 = S_0(q, \alpha, \varepsilon)$ such that for any rational number $r/s$, where $r \in \mathbb{Z}$, $s \in \mathbb{Z}_{>0}$, the inequality

$$|E_q(\alpha) - r/s| \geq \exp(-(C_1 + \varepsilon)(\log S)^{3/2})$$

holds, where $S = \max\{s; S_0\}$ and

$$C_1 = \frac{24\sqrt{3}(1-\gamma)}{(7-12\gamma)^{3/2}(\log |\rho|)^{1/2}}.$$

Furthermore, if $\gamma < 1/6$, then $E_q(\alpha)$ is not a quadratic irrational, and for any $\varepsilon > 0$ there exists a positive constant $S_0 = S_0(q, \alpha, \varepsilon)$ such that for any quadratic polynomial $A(z) \in \mathbb{Z}[z]$ the inequality

$$|A(E_q(\alpha))| \geq \exp(-(C_2 + \varepsilon)(\log L)^{3/2})$$

holds, where $L = \max\{L(A); L_0\}$, $L(A)$ is the length of the polynomial $A$ (the sum of the moduli of its coefficients), and

$$C_2 = \frac{6\sqrt{6}(1-\gamma)}{(1-6\gamma)^{3/2}(\log |\rho|)^{1/2}}.$$

Corollary 1.1 improves the conditions $\gamma < 0.55301134 \ldots$ and $\gamma < 0.10602269 \ldots$ in the results on irrationality and non-quadraticity, respectively, obtained in [6].

Another corollary of Theorem 1.1 for $E_q(z)$ is the following result for the $q$-logarithm (see [10]):

$$L_q(z) = \sum_{n=1}^{\infty} \frac{z}{q^n - z} = \frac{E_q'(z)}{E_q(z)}.$$

**Corollary 1.2.** Suppose that

$$\lambda < \frac{30\pi^2}{14\pi^2 + 41} = 1.65251302 \ldots,$$

where $\lambda$ is defined in (1.1). Then for any $\alpha \in \mathbb{K}^*$, $\alpha \notin q^{\mathbb{Z} > 0}$, we have $L_q(\alpha) \notin \mathbb{K}$. Moreover, for any $\varepsilon > 0$ there exists a positive constant $H_0 = H_0(q, \lambda, \alpha, \varepsilon)$ such that for any number $\theta \in \mathbb{K}$ the inequality

$$|L_q(\alpha) - \theta| \geq \exp(-(C_0 + \varepsilon)(\log H)^{3/2})$$
holds, where $H = \max\{H(\theta); H_0\}$ and
\[
C_0 = \frac{180\pi^3 \lambda^{3/2} \log |q|_w}{(30\pi^2 - (14\pi^2 + 41)\lambda) \log H(q))^{3/2}}.
\]

Corollary 1.2 is a qualitative strengthening of Theorem 3 in [10], where the corresponding result was proved under the condition $\lambda < 1.33943985\ldots$ (this condition was not stated explicitly, but it defines the domain of the function $m(-\lambda)$ in [10]) and the additional requirement that $|q|_v \neq 1$ for all $v|\infty$. However, Corollary 1.2 is quantitatively weaker than Theorem 3 in [10], where the estimate for $|L_q(\alpha) - \theta|_w$ is polynomial in $H$.

Corollary 1.2 yields the following strengthening of Theorem 5 in [10].

**Corollary 1.3.** Let $r, s \in \mathbb{Z} \setminus \{0\}$ be such that $D = r^2 + 4s > 0$. Suppose that a sequence $u_n$ is a solution of the recurrence relation
\[
u_{n+2} = ru_{n+1} + su_n
\]
with the initial conditions $u_0 = 0$, $u_1 = u \in \mathbb{Q}(\sqrt{D})^*$. We set
\[
d = \begin{cases} (r^2, s) & \text{if } \sqrt{D} \notin \mathbb{Z}, \\ \left(\frac{|r| + \sqrt{D})^2}{4}, s\right) & \text{if } \sqrt{D} \in \mathbb{Z}, \end{cases}
\]
where $(a, b)$ is the greatest common divisor of numbers $a, b \in \mathbb{Z}$. If $r' := |r|/\sqrt{d}$ and $s' := s/d$ satisfy the inequality
\[
r' > |s'|^a - \frac{s'}{|s'|^a}, \quad \text{where } a := \frac{15\pi^2}{16\pi^2 - 41} = 1.26626822\ldots,
\]
then for any $k \in \mathbb{Z}_{>0}$ and $b \in \mathbb{Q}(\sqrt{D})^*$ such that $|b| < ((|r| + \sqrt{D})/2)^k$ we have
\[
\sum_{n=1}^{\infty} \frac{b^n}{u_{kn}} \notin \mathbb{Q}(\sqrt{D}).
\]
Moreover, for any $\varepsilon > 0$ there exist positive constants $C_0 = C_0(r, s)$ and $H_0 = H_0(r, s, k, u, b, \varepsilon)$ such that for any number $\theta \in \mathbb{Q}(\sqrt{D})$ the inequality
\[
\left| \sum_{n=1}^{\infty} \frac{b^n}{u_{kn}} - \theta \right| \geq \exp\left(-(C_0k^{-1/2} + \varepsilon)(\log H)^{3/2}\right)
\]
holds, where $H = \max\{H(\theta); H_0\}$.

The corresponding qualitative result in [10] was proved under the condition that
\[
|r| > |s|^{a_1} - \frac{s'}{|s'|^{a_1}}, \quad \text{where } a_1 := \frac{1}{3 - \sqrt{5 + 12/\pi^2}} = 1.97301502\ldots, \quad (1.14)
\]
$\sqrt{D} \notin \mathbb{Z}$, and $b = 1$ (see the proof of Theorem 5 in [10]).
Remark 1.3. In the case when the inequality (1.14) holds for the quantities \( r' \) and \( s' \) defined in Corollary 1.3 instead of \( r \) and \( s \), respectively, we can use Corollary 1 in [10] to obtain an estimate for \( \left| \sum_{n=1}^{\infty} b^n/u_{kn} - \theta \right| \) that is polynomial in \( H \). In particular, if \( s|r^2 \) (in which case it is easy to show that \( \sqrt{D} \notin \mathbb{Z} \)), then there exists a positive constant \( H_0 = H_0(r, k, u, b) \) such that

\[
\left| \sum_{n=1}^{\infty} \frac{b^n}{u_{kn}} - \theta \right| > H^{-7.8921}
\]

for all \( \theta \in \mathbb{Q}(\sqrt{D}) \), where \( H = \max\{H(\theta); H_0\} \) (see [10], Theorem 4 for the cases \( s = \pm 1, \pm 2 \) and \( b = 1 \)).

The statement of the following theorem uses the representation (1.4). In addition we shall assume that \( \max\{\deg x P(x, y); \deg Q(x)\} > 0 \). Note that the author is unaware of any general results concerning similar functions, except for the generalized Chakalov series mentioned above.

We write

\[
h = \deg Q(x),
\]

\[
g_1 = \max \left\{ \max_{1 \leq \nu \leq d} \frac{\deg p_{d-\nu}(x)}{\nu}; \frac{h}{d} \right\},
\]

\[
g_2 = \max \left\{ \max_{1 \leq \nu \leq d} \frac{\deg p_{\nu}(x)}{\nu} \right\},
\]

\[
\varepsilon_0 = \begin{cases} 0 & \text{if } p_0(x) = 0, \\ 1 & \text{if } p_0(x) \neq 0, \end{cases}
\]

\[
\mathcal{D} = d + \max\{h; \deg p_0(x)\} + \sum_{\nu=1}^{d} \deg p_{\nu}(x)
\]

(The degree of the zero polynomial is assumed to be equal to zero).

**Theorem 1.2.** Suppose that \( \lambda = 1 \), where \( \lambda \) is defined in (1.1), and the polynomials \( P(x, y) \) and \( Q(x) \) satisfy (at least) one of the two conditions

(a) \( p_d(x) \) is independent of \( x \),

(b) \( Q(x) \) and \( p_0(x) \) are independent of \( x \).

Let \( m \in \mathbb{Z}_{>0} \) and \( d_0 \in \mathbb{Z}_{>d} \) and suppose that numbers \( \alpha_1, \ldots, \alpha_m \in \mathbb{K}^* \) and \( s_{j,k} \in \mathbb{Z}_{>0} \), \( 1 \leq j \leq m, \ 0 \leq k < d_0 \), satisfy the conditions

(i) \( \alpha_j \alpha_{k}^{-1} \notin \mathbb{Z} \) for \( 1 \leq j, k \leq m, \ j \neq k \),

(ii) the inequality \( s_{j,k} \leq \deg p_d(x) \) holds for \( 1 \leq j \leq m \) and \( d \leq k < d_0 \),

(iii) if \( \deg p_0(x) = \deg Q(x) \), then \( \alpha_j \notin (a/b)q^{Z \alpha} \) for \( 1 \leq j \leq m \), where \( a \) and \( b \) are the leading coefficients of the polynomials \( p_0(x) \) and \( Q(x) \), respectively.
Then the numbers \(1, f^{(\sigma)}(\alpha_j q^k), 1 \leq j \leq m, 0 \leq k < d_0, 0 \leq \sigma < s_{j,k},\) are linearly independent over \(\mathbb{K}\). Moreover, if we write

\[
c_1 = \begin{cases} 
\frac{8}{15} \sqrt{\frac{2}{mg_1}} & \text{if } p_0(x) \text{ is independent of } x, \\
0 & \text{otherwise},
\end{cases}
(1.20)
\]

\[
c_2 = \begin{cases} 
\frac{8}{15} \sqrt{\frac{2}{(m + \varepsilon_0)g_2}} & \text{if } p_0(x) \text{ and } Q(x) \text{ are independent of } x, \\
0 & \text{otherwise},
\end{cases}
(1.21)
\]

\[
C_0 = \frac{2dD^3 \log |q|_w}{3((c_1 + c_2) \log H(q))^2},
\]

then for any \(\varepsilon > 0\) there exists a positive constant \(H_0 = H_0(P, Q, q, m, d_0, \alpha_j, s_{j,k}, \varepsilon)\) such that for any vector

\[
\eta = (\eta_0, \eta_{j,k,\sigma}) \in \mathbb{K}^{1+\sum_{j=1}^m \sum_{k=0}^{d_0-1} s_{j,k}} \setminus \{0\},
\]

the inequality

\[
\left| \eta_0 + \sum_{j=1}^m \sum_{k=0}^{s_{j,k}} \sum_{\sigma=0}^{s_{j,k}-1} \eta_{j,k,\sigma} f^{(\sigma)}(\alpha_j q^k) \right| \geq |\eta|_w \exp\left(-\left( C_0 + \varepsilon \right) (\log H)^2 \right)
\]

holds, where \(H = \max\{H(\eta); H_0\}\).

In particular, we obtain the following result for the function

\[
H_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{n! q^{n(n+1)/2}}
\]

(see also [2], [11]).

**Corollary 1.4.** Let \(q\) be an integer in an imaginary quadratic field \(\mathbb{K}\), \(|q| > 1\), let \(d_0, s_0 \in \mathbb{Z}_{>0}\), and suppose that numbers \(\alpha_1, \ldots, \alpha_m \in \mathbb{K}^*\) satisfy the condition \(\alpha_j \alpha_k^{-1} \notin q^\mathbb{Z}\) for \(1 \leq j, k \leq m, j \neq k\). Then the numbers \(1, H_q^{(\sigma)}(\alpha_j), 1 \leq j \leq m, 0 \leq \sigma < s_0,\) and \(H_q(\alpha_j q^k), 1 \leq j \leq m, 1 \leq k < d_0,\) are linearly independent over \(\mathbb{K}\). Moreover, for any \(\varepsilon > 0\) there exists a positive constant \(H_0 = H_0(q, m, \alpha_j, d_0, s_0, \varepsilon)\) such that for any vector \(\eta = (\eta_0, \eta_{j,k,\sigma}) \in \mathbb{Z}_{\mathbb{K}}^{1+(s_0 + d_0-1)m} \setminus \{0\}\) the inequality

\[
\left| \eta_0 + \sum_{j=1}^m \left( \sum_{\sigma=0}^{s_{j,0}} \eta_{j,0,\sigma} H_q^{(\sigma)}(\alpha_j) + \sum_{k=1}^{d_0-1} \eta_{j,k,0} H_q(\alpha_j q^k) \right) \right| \geq \exp\left(-\left( C_0 + \varepsilon \right) (\log H)^2 \right)
\]

holds, where \(H = \max\{H(\eta); H_0\}\) and \(C_0 = 75 m/(8 \log |q|)\).

Corollary 1.4 refines the corollary in [2] for the function \(H_q(z)\), where the corresponding qualitative result was proved for the field \(\mathbb{K} = \mathbb{Q}\) of rational numbers under the condition \(|\alpha_j \alpha_k^{-1}| \notin |q|^\mathbb{Z}, j \neq k\). Note that in Corollary in [2], the values
of the antiderivatives $H_q^{(−k)}(α_j)$ are used instead of the values $H_q(α_j q^k)$. However, using the functional equation

$$qH_q'(qz) = H_q(z)$$

for $H_q(z)$, they can easily be expressed in terms of one another (for example, $H_q^{(−1)}(z) = H_q(qz) − 1$). The same equation can be used to reduce the general case of Corollary 1.4 to the case $d_0 = 1$ (with the numbers $α_j$ replaced by $α_j q^{d_0 − 1}$). In this case a rather stronger quantitative result was proved in [11] for $K = Q$, but under certain additional restrictions of an arithmetic nature on the numbers $α_j$.

Remark 1.4. Theorem 1.2 makes it possible to give a complete description of all linear relations (over $K$) between the values of $f(z)$ and its derivatives at points of $K$ (in the case when $P(x, y), Q(x)$, and $q$ satisfy the conditions of the theorem).

Indeed, the function $f(z)$ satisfies the equation

$$\sum_{ν=0}^{d} p_ν \left(z \frac{d}{dz} (f(q^ν z))\right) = P(0, 1) + Q \left(z \frac{d}{dz} (zf(z))\right),$$

(1.22)

which implies that for any $α ∈ K^*$ and $s ≥ \text{deg} p_d(x)$ the number $f^{(s)}(α)$ is linearly expressible in terms of $1, f^{(σ)}(α), 0 ≤ σ < \text{deg} p_d(x)$, and $f^{(σ)}(q^{−ν}α), 1 ≤ ν ≤ d, σ ≥ 0$. Consequently, whatever the numbers $β_1, \ldots, β_l ∈ K^*, s ∈ Z_{>0}$, there exist numbers $α_j$ and $s_{j,k}$ for which the conditions of Theorem 1.2 hold, so that $f^{(σ)}(β_j), 1 ≤ j ≤ l, 0 ≤ σ ≤ s$, are linearly expressible in terms of $1, f^{(σ)}(α_j q^k), 1 ≤ j ≤ m, 0 ≤ k < d_0, 0 ≤ σ < s_{j,k}$. Then the relation

$$\eta_0 + \sum_{j=1}^{l} \sum_{σ=0}^{s} η_{j,σ} f^{(σ)}(β_j) = 0$$

can be rewritten in the form

$$\tilde{η}_0 + \sum_{j=1}^{m} \sum_{k=0}^{d_0−1} \sum_{σ=0}^{s_{j,k}−1} \tilde{η}_{j,k,σ} f^{(σ)}(α_j q^k) = 0,$$

where $\tilde{η}_0, \tilde{η}_{j,k,σ}$ are certain linear combinations of $η_0, η_{j,σ}$. Therefore Theorem 1.2 implies that the coefficients $η_0, η_{j,σ}$ must satisfy the system of linear equations $\tilde{η}_0 = \tilde{η}_{j,k,σ} = 0$. In other words, all the non-trivial linear relations between the values of $f(z)$ and its derivatives at points of $K$ are consequences of equation (1.22).

Finally, the following theorem generalizes Theorem 1 of [12] to the case of an arbitrary algebraic number field. To do this, we consider the entire function

$$h(z) = \prod_{n=1}^{∞} \left(1 − \frac{1}{q^n}\right) \left(1 + \frac{z}{q^n} + \frac{1}{q^{2n}}\right), \quad z ∈ \mathbb{C}_w.$$
Theorem 1.3. Let \( a_1, \ldots, a_m \in \mathbb{K} \) be distinct numbers satisfying the conditions

(i) \( a_j \neq \pm 2 \) for \( 1 \leq j \leq m \),
(ii) \( q^n(a_jq^n - a_k)(a_kq^n - a_j) \neq (q^{2n} - 1)^2 \) for \( 1 \leq j, k \leq m \) and \( n \in \mathbb{Z} \setminus \{0\} \).

Let \( s_1, \ldots, s_m \in \mathbb{Z}_{>0} \). We set

\[
\alpha = \frac{4s + 1}{24s^2},
\]

(1.23)

\[
\gamma = \begin{cases} 
\frac{551}{2400} - \frac{289}{900\pi^2} - Z(4,1) & \text{if } m = 1, s_1 = 2, \\
\frac{4s + 3}{6(2s + 1)^2} + Z(2m + 2, 2s + 1) + Z(2m + 2, 2s + 2) & \text{otherwise},
\end{cases}
\]

(1.24)

where \( s = \sum_{j=1}^m s_j \). If

\[
\lambda < \frac{2/3 + \alpha}{2/3 - \gamma},
\]

where \( \lambda \) is defined in (1.1), then the numbers \( 1, h(\sigma)(a_j), 1 \leq j \leq m, 0 \leq \sigma < s_j \), are linearly independent over \( \mathbb{K} \). Moreover, for any \( \varepsilon > 0 \) there exists a positive constant \( H_0 = H_0(q, \lambda, m, a_j, s_j, \varepsilon) \) such that for any vector \( \eta = (\eta_0, \eta_{\sigma}) \in \mathbb{K}^{1+s} \setminus \{0\} \) the inequality

\[
\left| \eta_0 + \sum_{j=1}^m s_j^{-1} \sum_{\sigma=0}^{s_j-1} \eta_{\sigma}h(\sigma)(a_j) \right|_w \geq |\eta|_w \exp\left(-C_0 + \varepsilon\right)\left(\log H\right)^{3/2}
\]

holds, where \( H = \max\{H(\eta); H_0\} \) and

\[
C_0 = \frac{(2/3 + \alpha)\lambda^{3/2} \log q}{((2/3 + \alpha - (2/3 - \gamma)\lambda) \log H(q))^{3/2}}.
\]

In the case \( \mathbb{K} = \mathbb{Q} \) and \( q \in \mathbb{Z} \), the qualitative part of Theorem 1.3 was proved by Bézivin [12]. However, since he considered the function \( h(z)/h(0) \) instead of \( h(z) \), the additional restriction that the \( a_j \neq 0 \) was needed in that paper.

In particular, for the meromorphic function

\[
S(z) = \frac{h'(z)}{h(z)} = \sum_{n=1}^{\infty} \frac{q^n}{q^{2n} + q^n z + 1}
\]

we obtain the following result, which partially generalizes Corollary 1 of [12] (where the validity of the inequality (1.26) was required only for \( n \neq 0 \)).

Corollary 1.5. Suppose that the inequality

\[
\lambda < \frac{5475\pi^2}{3147\pi^2 + 2312 + 7200\pi^2 Z(4,1)} = 1.31441259\ldots
\]

(1.25)

holds, where \( \lambda \) is defined in (1.1). Then for any \( a \in \mathbb{K} \) such that

\[
q^n a^2 \neq (q^n + 1)^2
\]

(1.26)
for \( n \in \mathbb{Z} \), we have \( S(a) \notin \mathbb{K} \). Moreover, for any \( \varepsilon > 0 \) there exists a positive constant \( H_0 = H_0(q, \lambda, a, \varepsilon) \) such that for any number \( \theta \in \mathbb{K} \) the inequality

\[
|S(a) - \theta|_w \geq \exp\left(-(C_0 + \varepsilon)(\log H)^{3/2}\right)
\]

holds, where \( H = \max\{H(\theta); H_0\} \) and

\[
C_0 = \frac{1314(50\pi^2)^{3/2} \log |q|_w}{((5475\pi^2 - (3147\pi^2 + 2312 + 7200\pi^2 Z(4, 1))\lambda) \log H(q))^{3/2}}.
\]

Remark 1.5. The function \( S(z) \) is expressed in terms of the \( q \)-logarithm \( L_q(z) \) by the formula

\[
S\left(-\left(z + \frac{1}{z}\right)\right) = \frac{z}{z^2 - 1}\left(L_q(z) - L_q\left(\frac{1}{z}\right)\right).
\]

Therefore if \( q \) satisfies condition (1.25), then for any \( \alpha \in \mathbb{K}^* \) such that \( \alpha^2 \notin q^\mathbb{Z} \) we have

\[
L_q(\alpha) - L_q\left(\frac{1}{\alpha}\right) \notin \mathbb{K}.
\]

This paper has the following structure. In §2 we prove a certain identity for difference operators which will be used later. In §3, following Bézivin’s construction, we define auxiliary polynomials and establish a number of their properties which will be used for proving Theorems 1.1–1.3. In §4 we prove a result on the non-vanishing of the auxiliary polynomials which, along with the approximation lemma proved in §5, will make it possible to obtain quantitative results (see [6], Remark 8). Since all the theorems are proved using the same scheme, we carry out the arguments for arbitrary polynomials \( P(x, y) \) and \( Q(x) \). In the case when special properties of these polynomials are used, this is indicated explicitly (in particular, for convenience these conditions are included in the statements of the main intermediate results; see Propositions 3.1, 3.2, 3.6). Finally, in §6 we prove Theorems 1.1–1.3, as well as Corollaries 1.1, 1.3 (the other corollaries follow immediately from the corresponding theorems).

§2. Difference operators

Let \( \mathcal{B} \) denote the shift operator with respect to \( n \) acting on an arbitrary function \( \xi(n) \) of the integer argument \( n \) by the rule

\[
\mathcal{B}(\xi(n)) = \xi(n - 1).
\]

Thus, if \( \xi(n) \) is defined for \( n \geq n_0 \), then \( \mathcal{B}(\xi(n)) \) is defined for \( n \geq n_0 + 1 \). Note that we reserve \( n \) as the name of the argument of a function: if \( \xi(n) \) also depends on other arguments, then they are regarded as parameters.

For \( a \in \mathbb{F} \), where \( \mathbb{F} \) is an arbitrary field, we define the difference operator

\[
D_a = I - a\mathcal{B},
\]
where $\mathcal{I}$ denotes the identity operator, $\mathcal{I}(\xi(n)) = \xi(n)$. Note that the operators (2.2) commute with one another as well as with the operator $B$, so that, for example, we have the identity
\[
B(D_a(\xi(n))) = D_a(\xi(n - 1)).
\]
In what follows we make repeated use of this fact without special reference.

It is known that for any $a \neq 0$ and an arbitrary polynomial $p(x) \in \mathbb{F}[x]$ of degree at most $t \in \mathbb{Z}_{\geq 0}$, we have the identity
\[
D_{a}^{t+1}(p(n)a^n) = 0, \quad n \in \mathbb{Z}.
\]  
(2.3)

It is easy to see that the identity
\[
D_{a}(b^n \xi(n)) = b^n D_{ab}^{-1}(\xi(n))
\]  
(2.4)
holds for arbitrary $a$ and $b \neq 0$. The following property of the operators $D_a$ is a discrete analogue of the Leibniz product rule for differentiation.

**Lemma 2.1.** For $k \in \mathbb{Z}_{\geq 0}$ we have the identity
\[
D_{a}^k(\eta(n)\xi(n)) = \sum_{\nu=0}^{k} \binom{k}{\nu} a^\nu D_{1}^{\nu}(\eta(n)) D_{a}^{k-\nu}(\xi(n-\nu)).
\]

**Proof.** For $k = 0$ the assertion is trivial. For $k = 1$ it has the form
\[
D_{a}(\eta(n)\xi(n)) = \eta(n)D_{a}(\xi(n)) + aD_{1}(\eta(n))\xi(n-1)
\]
and can be verified directly.

If the identity is true for $k - 1$, then we have
\[
D_{a}^k(\eta(n)\xi(n)) = D_{a} \left( \sum_{\nu=0}^{k-1} \binom{k-1}{\nu} a^\nu D_{1}^{\nu}(\eta(n)) D_{a}^{k-1-\nu}(\xi(n-\nu)) \right)
\]
\[
= \sum_{\nu=0}^{k-1} \binom{k-1}{\nu} a^\nu D_{a} \left( D_{1}^{\nu}(\eta(n)) D_{a}^{k-1-\nu}(\xi(n-\nu)) \right)
\]
\[
= \sum_{\nu=0}^{k-1} \binom{k-1}{\nu} a^\nu \left( D_{1}^{\nu}(\eta(n)) D_{a}^{k-\nu}(\xi(n-\nu)) + aD_{1}^{\nu+1}(\eta(n)) D_{a}^{k-1-\nu}(\xi(n-1-\nu)) \right)
\]
\[
= \sum_{\nu=0}^{k} \binom{k}{\nu} a^\nu D_{1}^{\nu}(\eta(n)) D_{a}^{k-\nu}(\xi(n-\nu)).
\]

The lemma is proved.

We obtain the following assertion from Lemma 2.1 by induction on $\ell$. 


Lemma 2.2. For \(k_1, \ldots, k_\ell \in \mathbb{Z}_{\geq 0}\) we have the identity

\[
D_{a_1}^{k_1} \cdots D_{a_\ell}^{k_\ell}(\eta(n)\xi(n)) = \sum_{\nu_1=0}^{k_1} \cdots \sum_{\nu_\ell=0}^{k_\ell} \prod_{j=1}^{\ell} (k_j a_j)^{\nu_j} \cdot D_{1+\cdots+\nu_\ell}^{\nu_1+\cdots+\nu_\ell}(\eta(n))
\]

\[
\times D_{a_1}^{k_1-\nu_1} \cdots D_{a_\ell}^{k_\ell-\nu_\ell}(\xi(n-\nu_1-\cdots-\nu_\ell)).
\]

In particular, if \(\eta(n)\) is a polynomial of degree at most \(t\), it is sufficient to carry out the summation only over those \(\nu_j\) that satisfy the inequality \(\nu_1 + \cdots + \nu_\ell \leq t\).

§ 3. Construction of the auxiliary polynomials

We fix numbers \(m \in \mathbb{Z}_{\geq 0}, d_0 \in \mathbb{Z}_{\geq d}, \alpha_j \in \mathbb{C}_w^*, 1 \leq j \leq m\), and \(s_{j,k} \in \mathbb{Z}_{\geq 0}, 1 \leq j \leq m, 0 \leq k < d_0\). Furthermore, for \(1 \leq j \leq m\) we set

\[
s_j = \max_{0 \leq k < d_0} s_{j,k}.
\]

Let \(\overline{x}\) denote the vector of variables \((x_0, \overline{x}) = (x_0, x_{j,k,\sigma})\), where the subscripts vary between the limits \(1 \leq j \leq m, 0 \leq k < d_0, 0 \leq \sigma < s_{j,k}\).

We define sequences of polynomials as follows:

\[
u_{n} = u_{n}(z, \overline{x}) = \sum_{j=1}^{m} \sum_{k=0}^{d_0-1} s_{j,k}^{-1} \sum_{\sigma=0}^{n} \sigma! \binom{n}{\sigma} (\alpha_j z)^{n-\sigma} x_{j,k,\sigma} \in \mathbb{C}_w[z, \overline{x}], \quad n \in \mathbb{Z}_{\geq 0}, \tag{3.1}
\]

\[
u_{n+1} = v_{n}(z, \overline{x}) = \Pi_{n}(z) \left(x_0 + \sum_{k=0}^{n} \frac{u_k(z, \overline{x})}{\Pi_k(z)}\right) \in \mathbb{C}_w[z, \overline{x}], \quad n \in \mathbb{Z}_{\geq 0}, \tag{3.2}
\]

where the \(\Pi_{n}(z)\) are defined in (1.2). It is easy to see that the \(v_{n}\) satisfy the recurrence relation

\[
u_{0}(z, \overline{x}) = x_0 + u_0(z, \overline{x}), \tag{3.3}
\]

\[Q(n)\nu_{n}(z, \overline{x}) = P(n, z^\nu)\nu_{n-1}(z, \overline{x}) + Q(n)u_{n}(z, \overline{x}), \quad n \geq 1.
\]

Finally, for \(n \in \mathbb{Z}_{\geq 0}\) we consider the Hankel determinant

\[
u_{n} = V_{n}(z, \overline{x}) = \det(v_{i+j})_{i,j=0}^{n-1} \in \mathbb{C}_w[z, \overline{x}]. \tag{3.4}
\]

The present section is devoted to proving some properties of the \(V_{n}\).

3.1. Estimating \(|V_{n}(q, \overline{w})|_w\). In this subsection we assume that the polynomial \(P(x, y)\) satisfies condition (a) of Theorem 1.2, that is,

\[
P(x, y) = p_d y^d + \sum_{\nu=0}^{d-1} p_\nu(x) y^\nu, \quad p_d \in \mathbb{K}^*.
\]

We set

\[
s' = \begin{cases} s - 1/d & \text{if } P(x, y) = p_d y^d, \\ s & \text{otherwise}, \end{cases}
\]

where \(s\) is defined in (1.8).
For $1 \leq j \leq m$ and $\nu \in \mathbb{Z}$ we introduce the difference operator

$$\mathcal{A}_{\nu,j} = D_{\alpha,j} q^\nu,$$

where the operators $D_a$ are defined in (2.2). By (2.4) for $\sigma \in \mathbb{Z}$ we have the identity

$$\mathcal{A}_{\nu,j} (q^{\sigma n} \xi(n)) = q^{\sigma n} \mathcal{A}_{\nu-\sigma,j} (\xi(n)). \tag{3.5}$$

For $l \in \mathbb{Z}_{\geq 0}$ and $n \geq (s + mh)l + m \sum_{k=0}^{l-1} |kg_1|$, where $h$ and $g_1$ are defined by equations (1.15) and (1.16), we set

$$v_{l,n}(\bar{x}) = \prod_{k=0}^{l-1} \prod_{j=1}^{m} \mathcal{A}_{d_0-1-k,j} (v_{n}(q, \bar{x})).$$

Henceforth, an expression of the form $\prod_{j=1}^{\ell} \mathcal{R}_j \mathcal{A} (\xi(n))$, where the $\mathcal{R}_j$ are difference operators, means $\left( \prod_{j=1}^{\ell} \mathcal{R}_j \right) \mathcal{A} (\xi(n))$.

**Lemma 3.1.** Let $l > d_0$ and $\bar{\omega} = (\omega_0, \omega_{j,k,\nu}) \in \mathcal{C}_w^{1+1+\sum_{j=1}^{m} \sum_{k=0}^{d_0-1} s_{j,k}}$. Suppose that for $0 \leq \nu < l$ and $n \geq (s + mh)\nu + m \sum_{k=0}^{\nu-1} |kg_1|$ the inequality

$$|v_{\nu,n}(\bar{\omega})|_w \leq (n + 1)^{\nu g_1} |q|^{-\nu n + s' \nu^2 / 2 + m g_1 \nu^2 / 6 + a n + b}$$

holds with positive constants $a, b$ independent of $\nu, n$. Then for $n \geq (s + mh)l + m \sum_{k=0}^{l-1} |kg_1|$, the inequality

$$|v_{l,n}(\bar{\omega})|_w \leq (n + 1)^{t g_1} |q|^{-ln + s'l^2 / 2 + m g_1 l^2 / 6 + a n + b + c' + c' g_1 l}$$

holds with a positive constant $c'$ depending only on the numbers $q, m, \alpha_j, d_0, s_{j,k}$ and the polynomials $P(x,y), Q(x)$.

**Proof.** In the proof, $c_1, c_2, \ldots$ denote positive constants depending only on the numbers $q, m, \alpha_j, d_0, s_{j,k}$ and the polynomials $P(x,y), Q(x)$. Furthermore, for brevity we write $v_n$ and $v_{\nu,n}$ instead of $v_{\nu,n}(q, \bar{\omega})$ and $v_{\nu,n}(\bar{\omega})$.

Assuming that the $u_n$ are defined by (3.1) for all integers $n$, equations (2.3), (3.1) and the inequality $l \geq d_0$ imply the relation

$$\prod_{k=0}^{l-1} \prod_{j=1}^{m} \mathcal{A}_{d_0-1-k,j} (Q(n) u_{n}(q, \bar{\omega})) = 0, \quad n \in \mathbb{Z}.$$

Therefore it follows from (3.3) that for $n \geq (s + mh)l + m \sum_{k=0}^{l-1} |kg_1|$ we have

$$\prod_{k=0}^{l-1} \prod_{j=1}^{m} \mathcal{A}_{d_0-1-k,j} (Q(n+1) v_{n+1} - P(n+1, q^{n+1}) v_n) = 0.$$
In view of (3.5), we can rewrite the last relation in the form

\[ p_d v_{l,n} = q^{-d(n+1)} \prod_{k=0}^{l-1} A_{d_0-1-k,j}^{s_j+h+\lfloor kg_1 \rfloor} (Q(n+1) v_{n+1}) \]

\[ - \sum_{\nu=1}^{d} q^{-\nu(n+1)} \prod_{k=0}^{l-1} A_{d_0-1+\nu-d-k,j}^{s_j+h+\lfloor kg_1 \rfloor} (p_d - \nu(n+1) v_n). \]  

(3.6)

We now estimate the terms on the right-hand side.

Let \( 1 \leq \nu \leq d \). First we estimate

\[ \left| \prod_{k=\nu}^{l-1} \prod_{j=1}^{m} A_{d_0-1+\nu-d-k,j}^{s_j+h+\lfloor kg_1 \rfloor} (p_d - \nu(n+1) v_n) \right|_w \]

\[ = \left| \prod_{k=0}^{l-\nu-1} \prod_{j=1}^{m} A_{d_0-1-d-k,j}^{s_j+h+\lfloor (k+\nu)g_1 \rfloor} (p_d - \nu(n+1) v_n) \right|_w \]

for \( n \geq (s + mh)(l - \nu) + m \sum_{k=\nu}^{l-1} \lfloor kg_1 \rfloor \) when \( p_d - \nu(x) \neq 0 \) (in this case \( s' = s \)). Since \( \deg p_d - \nu(x) \leq \lfloor \nu g_1 \rfloor \), we obtain from Lemma 2.2 that

\[ \prod_{k=0}^{l-\nu-1} \prod_{j=1}^{m} A_{d_0-1-d-k,j}^{s_j+h+\lfloor (k+\nu)g_1 \rfloor} (p_d - \nu(n+1) v_n) \]

\[ = \sum_{\nu_{k,j}} \prod_{k=\nu}^{l-1} \prod_{j=1}^{m} \left( s_j + h + \lfloor (k + \nu)g_1 \rfloor \right) (\alpha_j q^{d_0-1-d-k} v_{k,j}) . D_1^N (p_d - \nu(n+1)) \]

\[ \times \prod_{k=0}^{l-\nu-1} \prod_{j=1}^{m} A_{d_0-1-d-k,j}^{\lfloor (k+\nu)g_1 \rfloor - \lfloor kg_1 \rfloor} (v_{l-n,n-N}), \]  

(3.7)

where the summation is carried out over all sets of numbers \( \nu_{k,j} \in \mathbb{Z}_{\geq 0}, 0 \leq k \leq l - \nu - 1, 1 \leq j \leq m \), satisfying the inequality \( N := \sum_{k,j} \nu_{k,j} \leq \lfloor \nu g_1 \rfloor \).

We observe that if for a sequence \( \xi(n) \in \mathbb{C}_w \) the inequality \( |\xi(n)|_w \leq \eta(n) \) holds for \( n \geq n_0 \), then for an arbitrary \( a \in \mathbb{C}_w \) the inequality

\[ |D_a (\xi(n))|_w \leq D_{-|a|_w} (\eta(n)) \leq (1 + |a|_w) \max \{ \eta(n); \eta(n-1) \} \]  

(3.8)

holds for \( n \geq n_0 + 1 \).

By taking into account the hypotheses of the lemma and the relations (2.4), (3.8), we have

\[ \left| \prod_{k=0}^{l-\nu-1} \prod_{j=1}^{m} A_{d_0-1-d-k,j}^{\lfloor (k+\nu)g_1 \rfloor - \lfloor kg_1 \rfloor - \nu_{k,j}} (v_{l-n,n-N}) \right|_w \]

\[ \leq \prod_{k=0}^{l-\nu-1} \prod_{j=1}^{m} D_{-|\alpha_j q^{d_0-1-d-k}|_w} (n + 1)^{l-\nu} g_1 \]

\[ \times |q|_{-w}^{(l-\nu)(n-N) + s(l-\nu)^2/2 + mg_1 (l-\nu)^3/6 + an+b} \]
\[
= \left| q \right|_{w}^{-(l-\nu)(n-N)+s(l-\nu)^2/2+mg_1(l-\nu)^3/6+an+b} \times \prod_{k=0}^{l-\nu-1} \prod_{j=1}^{m} \mathcal{D}_{l-\nu-a+\nu-k,j}^{(k+\nu)g_1 - [kg_1] - \nu_k,j} \left( n+1 \right)^{(l-\nu)g_1} \\
\leq (n+1)^{(l-\nu)g_1} \left| q \right|_{w}^{-(l-\nu)(n-N)+s(l-\nu)^2/2+mg_1(l-\nu)^3/6+an+b} \times \prod_{k=0}^{l-\nu-1} \prod_{j=1}^{m} \left( 1 + |\alpha_j|_w \left| q \right|_{w}^{l-\nu-a+\nu-k,j} \right)^{(k+\nu)g_1 - [kg_1]}.
\]

Since
\[
\sum_{k=0}^{l-\nu-1} (l-\nu-k)([(k+\nu)g_1] - [kg_1]) = \sum_{k=\nu}^{l-1} (l-k)[kg_1] - \sum_{k=0}^{l-\nu-1} (l-\nu-k)[kg_1] = \nu \sum_{k=0}^{l-\nu-1} [kg_1] + \sum_{k=\nu}^{l-1} (l-k)[kg_1] - \sum_{k=0}^{l-\nu-1} (l-\nu-k)[kg_1] \leq \nu g_1 \sum_{k=0}^{l-1} k \leq \frac{\nu g_1 l^2}{2},
\]

we have the inequality
\[
\prod_{k=0}^{l-\nu-1} \prod_{j=1}^{m} \left( 1 + |\alpha_j|_w \left| q \right|_{w}^{l-\nu-a+\nu-k,j} \right)^{(k+\nu)g_1 - [kg_1]} \leq \left| q \right|_{w}^{\nu g_1 l^2/2 + c_1 g_1 l}.
\]

Therefore we have
\[
\left| \prod_{k=0}^{l-\nu-1} \prod_{j=1}^{m} \mathcal{A}_{d_0-1-d-k,j}^{(k+\nu)g_1 - [kg_1] - \nu_k,j} \left( v_{l-\nu,n-N} \right) \right|_{w} \leq (n+1)^{(l-\nu)g_1} \left| q \right|_{w}^{-(l-\nu)n+s(l-\nu)^2/2+mg_1 l^3/6+an+b+c_2 g_1 l}.
\]

In view of the inequality
\[
\left| \mathcal{D}_1^N \left( p_{d-\nu}(n+1) \right) \right|_{w} \leq 2^N c_3(n+1)^{\nu g_1},
\]

it follows from the relation (3.7) that
\[
\left| \prod_{k=0}^{l-\nu-1} \prod_{j=1}^{m} \mathcal{A}_{d_0-1-d-k,j}^{s_j+h+[(k+\nu)g_1]} \left( p_{d-\nu}(n+1)v_n \right) \right|_{w} \leq c_3(n+1)^{g_1} \left| q \right|_{w}^{-(l-\nu)n+s(l-\nu)^2/2+mg_1 l^3/6+an+b+c_2 g_1 l} \times \prod_{k=0}^{l-\nu-1} \prod_{j=1}^{m} \left( 1 + 2 |\alpha_j|_w \left| q \right|_{w}^{d_0-1-d-k} s_j+h+[(k+\nu)g_1] \right) \leq (n+1)^{g_1} \left| q \right|_{w}^{-(l-\nu)n+s(l-\nu)^2/2+mg_1 l^3/6+an+b+c_2 g_1 l+c_4}.
\]
Therefore, it follows by (2.4) and (3.8) that for \( n \geq (s + mh)l + m \sum_{k=0}^{l-1} |kg_1| \), we finally have

\[
|q^{-\nu(n+1)} \prod_{k=0}^{l-1} \prod_{j=1}^{m} A_{d_0-1+\nu-d-k,j}^{s_j+h+[kg_1]} \left( P_{d-\nu}(n+1)u_n \right)_w | \leq |q|^{-\nu(n+1)} \prod_{k=0}^{l-1} \prod_{j=1}^{m} D_{s_j+h+[kg_1]} \left( (n+1)^{l_1} \right) |q|^{-l(l-\nu)n+s(l-\nu)^2/2+mg_1l^3/6+an+b+c_2g_1l+c_4} \prod_{k=0}^{l-1} \prod_{j=1}^{m} D_{s_j+h+[kg_1]} \left( (n+1)^{l_1} \right) \leq (n+1)^{l_1} |q|^{-l(n+1)+s(l-\nu)^2/2+mg_1l^3/6+(n+1)+b+c_6g_1l+c_6} \prod_{k=0}^{l-1} \prod_{j=1}^{m} D_{s_j+h+[kg_1]} \left( (n+2)^{l_1} \right) \leq (n+1)^{l_1} |q|^{-l(n+1)+s(l-\nu)^2/2+mg_1l^3/6+(n+1)+b+c_7g_1l+c_7} \prod_{k=0}^{l-1} \prod_{j=1}^{m} D_{s_j+h+[kg_1]} \left( (n+1)^{l_1} \right) \leq (n+1)^{l_1} |q|^{-l(n+1)+s(l-\nu)^2/2+mg_1l^3/6+an+b+a+c_8g_1l+c_8}.
\]

Taking into account (3.6), we obtain what is required.

**Lemma 3.2.** Suppose that \( \varpi = (\omega_0, \varpi_1) = (\omega_0, \omega_{j,k,\sigma}) \in \mathbb{C}_w^{1+\sum_{j=1}^{m} \sum_{k=0}^{d_0-1} s_{j,k}} \) satisfies the condition

\[
\omega_0 + \sum_{j=1}^{m} \sum_{k=0}^{d_0-1} s_{j,k} \sum_{\sigma=0}^{1} \omega_{j,k,\sigma} f^{(\sigma)}(\alpha_j q^k) = 0.
\]

Then for \( l \geq 0 \) and \( n \geq (s + mh)l + m \sum_{k=0}^{l-1} |kg_1| \), the inequality

\[
|u_{l,n}(\varpi)|_w \leq |\varpi_1|_w |q|^{-l(n+1)+s(l-\nu)^2/2+mg_1l^3/6+c(n+1)}
\]

holds with a positive constant \( c \) depending only on the numbers \( q, m, \alpha_j, d_0, s_{j,k} \) and the polynomials \( P(x,y), Q(x) \).

**Proof.** It follows from (3.1) that

\[
\sum_{j=1}^{m} \sum_{k=0}^{d_0-1} \sum_{\sigma=0}^{1} \omega_{j,k,\sigma} f^{(\sigma)}(\alpha_j q^k) = \sum_{n=0}^{\infty} \frac{u_n(q, \varpi)}{\Pi_n(q)}.\]
Therefore it follows from (3.2) that

\[ v_n(q, \bar{\omega}) = - \sum_{k=n+1}^{\infty} \frac{u_k(q, \bar{\omega})}{\prod_{j=n+1}^{k} P(j, q^j)/Q(j)}. \] (3.9)

By (3.1), the inequality \(|u_n(q, \bar{\omega})|_w \leq |\bar{\omega}_1|_w c_0^n\) holds for all \(n \in \mathbb{Z}_{\geq 0}\) with some constant \(c_0 > 1\). Therefore it follows from (3.9) that

\[ |v_n(q, \bar{\omega})|_w \leq \sum_{k=n+1}^{\infty} \frac{c_1 |\bar{\omega}_1|_w c_0^k}{(2c_0)^{k-n}} = c_1 |\bar{\omega}_1|_w c_0^n \]

for \(n \geq 0\) with some constant \(c_1 > 0\).

Using (3.8), we obtain that the inequality

\[ |v_{\nu,n}(\bar{\omega})|_w \leq |\bar{\omega}_1|_w (n+1)^\nu g_1 |q|_w^{-\nu n + s' \nu^2/2 + m g_1 \nu^3/6 + an + a} \]

holds for \(0 \leq \nu \leq d_0\) and \(n \geq (s + mh)\nu + m \sum_{k=0}^{\nu-1} |kg_1|\) with some positive constant \(a\). It follows from Lemma 3.1 by induction on \(l\) that

\[ |v_{l,n}(\bar{\omega})|_w \leq |\bar{\omega}_1|_w (n+1)^{l g_1} |q|_w^{-ln + s' l^2/2 + m g_1 l^3/6 + an + a + \sum_{k=1}^{l} (a' + c' g_1 k)}, \]

where \(c' > 0\) is the constant in Lemma 3.1. We obtain what is required by taking into account the inequality \((s + mh)l + m \sum_{k=0}^{l-1} |kg_1| \leq n\).

**Proposition 3.1.** Suppose that \(\bar{\omega} = (\omega_0, \bar{\omega}_1) = (\omega_0, \omega_{j,k}, \sigma) \in C_{w}^{1+\sum_{j=1}^{m} \sum_{k=0}^{d_0-1} s_{j,k}}\) satisfies the condition

\[ \omega_0 + \sum_{j=1}^{m} \sum_{k=0}^{d_0-1} \sum_{\sigma=0}^{s_{j,k}-1} \omega_{j,k,\sigma} f^{(\sigma)}(\alpha_j q^k) = 0. \]

If \(pd(x) = \delta d \in \mathbb{K}^\times\), then we have the following asymptotic estimate (uniform in \(\bar{\omega}\)) for the Hankel determinant (3.4) as \(n \to \infty:\)

\[ |V_n(q, \bar{\omega})|_w \leq \begin{cases} |\bar{\omega}_1|_w |q|_w^{-\alpha n^3 + O(n^2)} & \text{if } g_1 = 0, \\ |\bar{\omega}_1|_w |q|_w^{-c_1 n^{5/2} + O(n^2)} & \text{if } g_1 > 0, \end{cases} \]

where \(\alpha, g_1,\) and \(c_1\) are defined in (1.9), (1.16), and (1.20), respectively.

**Proof.** For \(0 \leq i \leq n - 1\) we set

\[ l_i = \max \left\{ l \in \mathbb{Z}_{\geq 0}: (s + mh)l + m \sum_{k=0}^{l-1} |kg_1| \leq i \right\}. \]

Then we have

\[ V_n(q, \bar{\omega}) = \det(v_{i,i+j}(\bar{\omega}))_{i,j=0}^{n-1}, \]
since the last determinant is obtained from $V_n(q, \omega)$ by elementary row transformations. By taking Lemma 3.2 into account, we obtain

$$|V_n(q, \omega)|_w \leq n! \max_{\sigma \in S_n} \prod_{i=0}^{n-1} |v_{l_i, i+\sigma(i)}(\omega)|_w$$

$$\leq \|\omega\|_w^n |q|_w^{\max_{\sigma \in S_n} \left(-l_i(i+\sigma(i)) + s'i^2/2 + mg_1l_i^3/6\right)} + O(n^2),$$  

(3.10)

where $S_n$ is the group of permutations of the set $\{0, 1, \ldots, n-1\}$.

Since the sequence $l_i$ is non-decreasing, the rearrangement inequality (see [13], Theorem 368) implies that the maximum on the right-hand side of (3.10) is attained at the permutation $\sigma(i) = n - 1 - i$. If $g_1 = 0$, then $l_i = i/s + O(1)$ and

$$\sum_{i=0}^{n-1} \left(-(n-1)l_i + s'i^2/2 \right) = -\alpha n^3 + O(n^2).$$

If $g_1 > 0$, then $l_i = \sqrt{2i/mg_1} + O(1)$ and

$$\sum_{i=0}^{n-1} \left(-(n-1)l_i + s'i^2/2 + mg_1l_i^3/6 \right) = -8/15 \sqrt{2/mg_1} n^{5/2} + O(n^2).$$

In view of (3.10) we obtain what is required.

### 3.2. Factorization of $V_n$. Part I.

In this subsection we assume that the polynomials $p_0(x) = p_0$ and $Q(x)$ are independent of $x$ and, without loss of generality, that $Q(x) = 1$. We set

$$\delta_0 = \min\{1 \leq \nu \leq d \mid p_\nu(x) \neq 0\}, \quad s'' = s + \varepsilon_0 + \frac{1 - \varepsilon_0}{\delta_0},$$

where $s$ and $\varepsilon_0$ are defined in (1.8) and (1.18), respectively.

For $\nu \in \mathbb{Z}$ and $0 \leq j \leq m$ we define the difference operator

$$A_{\nu, j} = \begin{cases} D_{p_0z^\nu} & \text{for } j = 0, \\ D_{z^\nu} & \text{for } 1 \leq j \leq m, \end{cases}$$

where the operators $D_a$ are defined in (2.2). We observe that if $p_0 = 0$, then $A_{\nu, 0} = I$. By the identity (2.4), we have

$$A_{\nu, j}(z^{\sigma_n}x(n)) = z^{\sigma_n} A_{\nu-\sigma, j}(x(n))$$  

(3.11)

for $\sigma \in \mathbb{Z}$. We set

$$v_{l, n} = v_{l, n}(z, \omega) = \prod_{k=0}^{l-1} \prod_{j=0}^{m} A_{k, j}^{s_{k, j} + \lfloor kg_2 \rfloor}(v_n) \in \mathbb{C}_w[\omega][z], \quad s_0 := 1,$$

for $l \in \mathbb{Z}_{\geq 0}$ and $n \geq (s + \varepsilon_0)l + (m + \varepsilon_0) \sum_{k=0}^{l-1} \lfloor kg_2 \rfloor$, where $g_2$ is defined by equation (1.17).
Let $\text{ord}_z \xi$ denote the order (with respect to $z$) of the formal Laurent series $\xi(z) = \sum_{\nu \in \mathbb{Z}} a_{\nu} z^{\nu}$, $a_{\nu} \neq 0$:

$$\text{ord}_z \xi = \min\{\nu \in \mathbb{Z} | a_{\nu} \neq 0\},$$

and take $\text{ord}_z 0 = +\infty$.

**Lemma 3.3.** For $l \in \mathbb{Z}_{\geq 0}$ and $n \geq (s + \varepsilon_0) l + (m + \varepsilon_0) \sum_{k=0}^{l-1} [kg_2]$, we have

$$\text{ord}_z v_{l,n} \geq ln - s'' \left(\frac{l+1}{2}\right) - (m + \varepsilon_0) \sum_{k=0}^{l-1} (l-k)[kg_2] - c$$

with some constant $c > 0$ depending only on the numbers $m$, $d_0$, $s_{j,k}$ and the polynomial $P(x,y)$.

**Proof.** We use induction on $l$. For $l = 0$ the assertion is trivial.

Let $l \geq 1$. First suppose that $n \geq (s + \varepsilon_0) l + (m + \varepsilon_0) \sum_{k=0}^{l-1} [kg_2] + 1 - \varepsilon_0$. The relation (3.3) can be rewritten in the form

$$A_{0,0}(v_n) = \sum_{\nu = \delta_0}^{d} p_{\nu}(n) z^{\nu n} v_{n-1} + u_n, \quad n \geq 1,$$

whence we obtain (for $n \geq (s + \varepsilon_0) l + (m + \varepsilon_0) \sum_{k=0}^{l-1} [kg_2] + 1 - \varepsilon_0$)

$$v_{l,n} = \sum_{\nu = \delta_0}^{d} \prod_{k=0}^{l-1} \prod_{j=0}^{m} A_{k,j}^{s_{j,k}+[kg_2]} (p_{\nu}(n) z^{\nu n} v_{n-1}) + \prod_{k=0}^{l-1} \prod_{j=0}^{m} A_{k,j}^{s_{j,k}+[kg_2]} (u_n),$$

where the prime means that for $k = 0$ we set $s_0 = 0$. Consequently,

$$\text{ord}_z v_{l,n} \geq \min\left\{ \min_{\delta_0 \leq \nu \leq d} \text{ord}_z \prod_{k=0}^{l-1} \prod_{j=0}^{m} A_{k,j}^{s_{j,k}+[kg_2]} (p_{\nu}(n) z^{\nu n} v_{n-1}); \right.$$

$$\left. \text{ord}_z \prod_{k=0}^{l-1} \prod_{j=0}^{m} A_{k,j}^{s_{j,k}+[kg_2]} (u_n) \right\}. \quad (3.12)$$

In the case $p_0 = 0$ it is sufficient to consider the product over $j$ between the limits 1 and $m$.

We observe that if for a sequence $\xi_n(z) \in \mathbb{F}((z))$, where $\mathbb{F}$ is some field, the inequality $\text{ord}_z \xi_n \geq an + b$ holds for $n \geq n_0$ with some $a, b \in \mathbb{R}$ (independent of $n$), then for any $c \in \mathbb{F}$ and $\nu \in \mathbb{Z}$, $\nu \leq a$, we have for $n \geq n_0 + 1$ that

$$\text{ord}_z D_{cz^\nu}(\xi_n) \geq \min\{an + b; a(n - 1) + b + \nu\} = a(n - 1) + b + \nu. \quad (3.13)$$

We write

$$u_{l,n} = \sum_{j=1}^{m} \sum_{l \leq k < d_0} ^{s_{j,k-1}} \sum_{\sigma = 0}^{\sigma!} \binom{n}{\sigma} (\alpha_j z^k)^{n-\sigma} x_{j,k,\sigma}. $$
If \( l \geq d_{0} \), then \( u_{l,n} = 0 \), while if \( l < d_{0} \), then \( \text{ord}_{z} u_{l,n} \geq ln + O(1) \). Therefore in view of (2.3), (3.1), and (3.13) we obtain

\[
\text{ord}_{z} \prod_{k=0}^{l-1} \prod_{j=0}^{m} A_{k,j}^{s_{j}+[kg_{2}]}(u_{n}) = \text{ord}_{z} \prod_{k=0}^{l-1} \prod_{j=0}^{m} A_{k,j}^{s_{j}+[kg_{2}]}(u_{l,n}) \geq ln - c.
\]

Let \( l \leq \nu \leq d \). Since \( \text{ord}_{z}(p_{\nu}(n)z^{\nu n}v_{n-1}) \geq \nu n \geq ln \) for \( n \geq 1 \), in view of (3.13) and the inequality \( l \leq d \) we have

\[
\text{ord}_{z} \prod_{k=0}^{l-1} \prod_{j=0}^{m} A_{k,j}^{s_{j}+[kg_{2}]}(p_{\nu}(n)z^{\nu n}v_{n-1}) \geq \ln - c.
\]

Let \( \delta_{0} \leq \nu < l \). First we estimate

\[
\text{ord}_{z} \prod_{k=0}^{l-1} \prod_{j=0}^{m} A_{k,j}^{s_{j}+[kg_{2}]}(p_{\nu}(n)z^{\nu n}v_{n-1}) = \nu n + \text{ord}_{z} \prod_{k=0}^{l-\nu-1} \prod_{j=0}^{m} A_{k,j}^{s_{j}+[\nu g_{2}]}(p_{\nu}(n)v_{n-1}) \quad (3.14)
\]

for \( n \geq 1 + (s + \varepsilon_{0})(l - \nu) + (m + \varepsilon_{0}) \sum_{k=\nu}^{l-1} \nu g_{2} \) (we have used equation (3.11)). In view of Lemma 2.2 we obtain

\[
\text{ord}_{z} \prod_{k=0}^{l-\nu-1} \prod_{j=0}^{m} A_{k,j}^{s_{j}+[(\nu+1)g_{2}]}(p_{\nu}(n)v_{n-1}) \geq \min_{\nu_{k,j}} \left( \sum_{k,j} k
\nu_{k,j} + \text{ord}_{z} \prod_{k,j} A_{k,j}^{(\nu+1)g_{2}} - g_{2} - \nu_{k,j}(v_{l-\nu,n-1-N}) \right), \quad (3.15)
\]

where the minimum is taken over all sets of integers \( \nu_{k,j} \geq 0, 0 \leq k \leq l - \nu - 1, 0 \leq j \leq m \), satisfying the inequality \( N := \sum_{k,j} v_{k,j} \leq [\nu g_{2}] \). If \( p_{0} = 0 \), then \( j \) varies between the limits \( 1 \leq j \leq m \).

Taking into account the induction hypothesis, repeated application of (3.13) yields that

\[
\sum_{k,j} k
\nu_{k,j} + \text{ord}_{z} \prod_{k,j} A_{k,j}^{(\nu+1)g_{2}} - g_{2} - \nu_{k,j}(v_{l-\nu,n-1-N})
\]

\[
\geq \sum_{k,j} k
\nu_{k,j} + (l - \nu) \left( n - 1 - (m + \varepsilon_{0}) \sum_{k=0}^{l-\nu-1} ([k + \nu]g_{2} - [kg_{2}]) \right)
\]

\[
- s^{'2} \left( \frac{l - \nu + 1}{2} \right) - (m + \varepsilon_{0}) \sum_{k=0}^{l-\nu-1} (l - \nu - k)[kg_{2}] - c
\]

\[
+ \sum_{k=0}^{l-\nu-1} \left( (m + \varepsilon_{0})([k + \nu]g_{2} - [kg_{2}]) - \sum_{j} \nu_{k,j} \right) k
\]

\[
= (l - \nu)n - s^{'2} \left( \frac{l - \nu + 1}{2} \right) - (l - \nu) - (m + \varepsilon_{0}) \sum_{k=0}^{l-1} (l - k)[kg_{2}] - c.
\]
Therefore, in view of (3.14) and (3.15), for
\[ n \geq (s + \varepsilon_0)l + (m + \varepsilon_0) \sum_{k=0}^{l-1} |kg_2| + 1 - \varepsilon_0 \]
we obtain
\[
\text{ord}_z \prod_{k=0}^{l-1} \prod_{j=0}^{m} A_{k,j}^{s_j + |kg_2|} (p_\nu(n) z^{\nu} v_{n-1})
\]
\[
= \text{ord}_z \prod_{k=\nu}^{\nu-1} \prod_{j=0}^{m} A_{k,j}^{s_j + |kg_2|} \left( \prod_{k=\nu}^{l-1} \prod_{j=0}^{m} A_{k,j}^{s_j + |kg_2|} (p_\nu(n) z^{\nu} v_{n-1}) \right)
\]
\[
\geq l \left( n + \varepsilon_0 - (s + \varepsilon_0) \nu - (m + \varepsilon_0) \sum_{k=0}^{\nu-1} |kg_2| \right) - s'' \left( \frac{l - \nu + 1}{2} \right) - (l - \nu)
\]
\[
- (m + \varepsilon_0) \sum_{k=\nu}^{l-1} (l - k) |kg_2| - c + \sum_{k=0}^{\nu-1} (s + \varepsilon_0 + (m + \varepsilon_0) |kg_2|) k
\]
\[
= ln - s'' \left( \frac{l + 1}{2} \right) - (m + \varepsilon_0) \sum_{k=0}^{l-1} (l - k) |kg_2| - c
\]
\[
+ \frac{(1 - \varepsilon_0)(2l - \nu)(\nu - \delta_0) + (\delta_0 + 1 + (\delta_0 - 1)\varepsilon_0)\nu}{2\delta_0}
\]
\[
\geq ln - s'' \left( \frac{l + 1}{2} \right) - (m + \varepsilon_0) \sum_{k=0}^{l-1} (l - k) |kg_2| - c.
\]
Thus, for \( n \geq (s + \varepsilon_0)l + (m + \varepsilon_0) \sum_{k=0}^{l-1} |kg_2| + 1 - \varepsilon_0 \) (where \( l \) is fixed) the required inequality is proved. If \( \varepsilon_0 = 0 \) and \( n = sl + m \sum_{k=0}^{l-1} |kg_2| \), then in view of (3.13) and the induction hypothesis we obtain
\[
\text{ord}_z v_{l,n} = \text{ord}_z \prod_{j=1}^{m} A_{l-1,j}^{s_j + (l-1)g_2} (v_{l-1,n})
\]
\[
\geq (l - 1)n - s'' \left( \frac{l}{2} \right) - m \sum_{k=0}^{l-2} (l - 1 - k) |kg_2| - c
\]
\[
\geq ln - s'' \left( \frac{l + 1}{2} \right) - m \sum_{k=0}^{l-1} (l - k) |kg_2| - c.
\]
The lemma is proved.

**Proposition 3.2.** If the polynomials \( p_0(x) \) and \( Q(x) \) are independent of \( x \), then as \( n \to \infty \), we have the following asymptotic estimate for the order of the Hankel determinant (3.4):
\[
\text{ord}_z V_n \geq \begin{cases} 
\beta n^3 + O(n^2) & \text{if } g_2 = 0, \\
\epsilon_2 n^{5/2} + O(n^2) & \text{if } g_2 > 0,
\end{cases}
\]
where \( \beta, g_2, \) and \( \epsilon_2 \) are defined in (1.10), (1.17), and (1.21), respectively.
Proof. For $0 \leq i \leq n - 1$ we set

$$l_i = \begin{cases} 
\min \left\{ \left\lfloor \frac{i}{s + \varepsilon_0} \right\rfloor; \left\lfloor \frac{n}{s''} \right\rfloor \right\} & \text{if } g_2 = 0, \\
\max \left\{ l \in \mathbb{Z}_{\geq 0}: (s + \varepsilon_0)l + (m + \varepsilon_0) \sum_{k=0}^{l-1} \lfloor kg_2 \rfloor \leq i \right\} & \text{if } g_2 > 0.
\end{cases}$$

Then we have

$$V_n = \det(v_{l_i,i+j})_{i,j=0}^{n-1},$$

since the last determinant is obtained from $V_n$ by elementary row transformations. Taking into account Lemma 3.3 and the rearrangement inequality, we obtain

$$\text{ord}_z V_n \geq \min_{\sigma \in S_n} \sum_{i=0}^{n-1} \text{ord}_z v_{l_i,i+\sigma(i)}$$

$$\geq \min_{\sigma \in S_n} \sum_{i=0}^{n-1} \left( l_i(i+\sigma(i)) - \frac{s''l_i^2}{2} - (m + \varepsilon_0) \frac{g_2l_i^3}{6} \right) + O(n^2)$$

$$= \sum_{i=0}^{n-1} \left( l_i(n-1) - \frac{s''l_i^2}{2} - (m + \varepsilon_0) \frac{g_2l_i^3}{6} \right) + O(n^2)$$

$$= \begin{cases} 
\beta n^3 + O(n^2) & \text{if } g_2 = 0, \\
c_2n^{5/2} + O(n^2) & \text{if } g_2 > 0.
\end{cases}$$

The proposition is proved.

3.3. Factorization of $V_n$. Part II. In this subsection we assume that $P(x, y) = P(y) \in \mathbb{K}[y]$ and $Q(x) = 1$.

We fix $l \in \mathbb{Z}_{>0}$ and an arbitrary primitive $l$th root $\zeta$ of unity. We set

$$\delta = \delta_l = \text{ord}_y \prod_{k=0}^{l-1} P(\zeta^k y), \quad \text{(3.16)}$$

$$A = A_l = \prod_{k=0}^{l-1} P(\zeta^k). \quad \text{(3.17)}$$

$\delta$ can also be defined as the number of zeros (taking account of multiplicities) of the polynomial $P(y)$ that are $l$th roots of unity. $A$ and $\delta$ are obviously independent of the choice of the root $\zeta$.

For $0 \leq j \leq m$ we define the difference operator

$$\mathcal{F}_j = \begin{cases} 
\mathcal{I} - AB^l & \text{for } j = 0, \\
\mathcal{I} - \alpha_j^l B^l & \text{for } 1 \leq j \leq m,
\end{cases}$$

where $\mathcal{I}$ and $B$ have the previous meaning. We observe that if $\delta > 0$ (that is, $A = 0$), then $\mathcal{F}_0 = \mathcal{I}$.
For \( t, n \in \mathbb{Z}_{\geq 0} \) we set
\[
v_n^{(t)}(x) = \frac{\partial^t v_n}{\partial z^t} \bigg|_{z=\zeta} \in \mathbb{C}_w[\bar{x}].
\]

**Lemma 3.4.** Let \( t, t_0, \ldots, t_m \in \mathbb{Z}_{\geq 0} \) and suppose that
\[
t_0 \geq 2t + 1,
\]
\[
t_j \geq s_j + t, \quad 1 \leq j \leq m.
\]
Then for \( n \geq n_0l \) we have
\[
\prod_{j=0}^m \mathcal{F}_{t_j}(v_n^{(t)}) = 0,
\]
where
\[
n_0 = \begin{cases} \sum_{j=0}^m t_j & \text{if } \delta = 0, \\ \sum_{j=1}^m t_j + \lfloor t/\delta \rfloor + 1 & \text{if } \delta > 0. \end{cases}
\]

**Proof.** For \( j \in \mathbb{Z}_{\geq 0} \) and \( n \in \mathbb{Z} \) we set \( P_j(n, z) = \prod_{k=0}^{j-1} P(z^{n-k}) \). We observe that
\[
\text{ord}_{z=\zeta} P_l(n, z) = \delta
\]
for any \( n \geq l \), where \( \delta \) is defined in (3.16). It follows from (3.3) that
\[
v_n = P_l(n, z)v_{n-l} + \sum_{j=0}^{l-1} P_j(n, z)u_{n-j}
\]
for \( n \geq l \).

Next, for \( j, t \in \mathbb{Z}_{\geq 0} \) and \( n \in \mathbb{Z} \) we set
\[
u_n^{(t)}(x) = \frac{\partial^t v_n}{\partial z^t} \bigg|_{z=\zeta}, \quad P_j^{(t)}(n) = \frac{d^t}{dz^t} P_j(n, z) \bigg|_{z=\zeta}.
\]

As shown in the proof of Proposition 4 in [6], for arbitrary \( j, t \in \mathbb{Z}_{\geq 0} \) the sequence \( \{P_j^{(t)}(n)\}_{n \in \mathbb{Z}} \) is a quasi-polynomial of degree at most \( t \) with quasi-period \( l \) (see [14], §4.4), that is, it has the form \( \sum_{\nu=0}^t a_\nu(n)n^\nu \), where the sequence \( \{a_\nu(n)\}_{n \in \mathbb{Z}} \) is \( l \)-periodic for every \( \nu \).

It follows from (2.3) that for an arbitrary quasi-polynomial \( R(n) \) of degree at most \( t \in \mathbb{Z}_{\geq 0} \) with quasi-period \( l \), we have
\[
(I - B^l)^{t+1}(R(n)) = 0, \quad n \in \mathbb{Z}.
\]

Furthermore, in view of (3.1), for \( \tau \in \mathbb{Z}_{\geq 0} \) we have
\[
\prod_{j=1}^m \mathcal{F}_{t_j}^{s_j+t+\tau}(R(n)u_n^{(\tau)}) = 0, \quad n \in \mathbb{Z}.
\]
Note that the sequence \( \{P_l^{(0)}(n)\}_{n \in \mathbb{Z}} \) is constant: namely,

\[
P_l^{(0)}(n) = A, \tag{3.22}
\]

where \( A \) is defined in (3.17). By differentiating \( t \) times with respect to \( z \) in (3.19) and substituting \( z = \zeta \), we obtain using (3.18) and (3.22) that

\[
\mathcal{F}_0(v_n^{(t)}) = \sum_{\max\{1; \delta\} \leq \tau \leq t} \left( \frac{t}{\tau} \right) P_l^{(\tau)}(n)v_n^{(t-\tau)} + \sum_{j=0}^{l-1} \sum_{\tau=0}^{t} \left( \frac{t}{\tau} \right) P_j^{(\tau)}(n)v_{n-j}^{(t-\tau)}, \quad n \geq l.
\tag{3.23}
\]

We now pass to proving the lemma. We use induction on \( t \).

For \( t < \max\{1; \delta\} \) the assertion follows directly from (3.21) and (3.23).

Let \( t \geq \max\{1; \delta\} \). Then we obtain from (3.21) and (3.23) that for \( n \geq n_0l \),

\[
\prod_{j=0}^{m} \mathcal{F}_j^{t_j}(v_n^{(t)}) = \sum_{\max\{1; \delta\} \leq \tau \leq t} \left( \frac{t}{\tau} \right) \mathcal{F}_0^{t_0-1} \prod_{j=1}^{m} \mathcal{F}_j^{t_j}(P_l^{(\tau)}(n)v_n^{(t-\tau)}).
\]

Therefore it suffices to prove that

\[
\mathcal{F}_0^{t_0-1} \prod_{j=1}^{m} \mathcal{F}_j^{t_j}(P_l^{(\tau)}(n)v_n^{(t-\tau)}) = 0
\]

for \( \max\{1; \delta\} \leq \tau \leq t \) and \( n \geq n_0l \).

As in the proof of the identity of Lemma 2.2, we obtain

\[
\mathcal{F}_0^{t_0-1} \prod_{j=1}^{m} \mathcal{F}_j^{t_j}(P_l^{(\tau)}(n)v_n^{(t-\tau)})
\]

\[
= \sum_{\nu_0=0}^{t_0-1} \sum_{\nu_1=0}^{t_1} \cdots \sum_{\nu_m=0}^{t_m} \left( \frac{t_0-1}{\nu_0} \right) A^{\nu_0} \prod_{j=1}^{m} \left( \frac{t_j}{\nu_j} \right) \alpha_j^{l_{\nu_j}} \cdot (I - B^l)^{N}(P_l^{(\tau)}(n))
\]

\[
\times \mathcal{F}_0^{t_0-1-\nu_0} \prod_{j=1}^{m} \mathcal{F}_j^{t_j-\nu_j}(v_n^{(t-\tau)}),
\]

where \( N := \sum_{j=0}^{m} \nu_j \). In the case \( \delta > 0 \), in the sum over \( \nu_0 \) we keep only the term with \( \nu_0 = 0 \). It follows from (3.20) that the terms for which \( N > \tau \) are equal to zero. Since the inequalities

\[
t_0 - 1 - \nu_0 \geq 2t - \tau \geq 2(t - \tau) + 1,
\]

\[
t_j - \nu_j \geq s_j + t - \tau, \quad 1 \leq j \leq m,
\]

hold for \( N \leq \tau \), it follows from the induction hypothesis that all the terms with \( N \leq \tau \) are also equal to zero (because in the case \( \delta > 0 \) we have the equation \( \nu_0 = 0 \) and the inequality \( \tau \geq \delta \)). The lemma is proved.
Remark 3.1. If $\alpha_{j_1}^l = \alpha_{j_2}^l$ for some $j_1 \neq j_2$, then the order of the recurrence relation for $v_{n}^{(t)}$ can be lowered. Indeed if, say, $s_{j_1} \geq s_{j_2}$, then in the product (3.21) we can drop the factor $\mathcal{F}_{j_2}^{s_{j_2} + l + \tau}$, and the statement of Lemma 3.4 is also modified accordingly. Therefore in the following proposition, in this case we can take $m - 1$ and $s - s_{j_2}$ instead of $m$ and $s$, respectively (see also Propositions 3.4 and 3.5).

The more general case when $(\alpha_{j_1} q^{k_1})^l = (\alpha_{j_2} q^{k_2})^l$ can be reduced to the one considered above by ‘touching up’ the definition of the polynomials (3.1). Namely, in the notation $\beta_{\nu} = \alpha_{j_\nu} q^{k_{\nu} - k_0}$, $\nu = 1, 2$, for $k_0 = \max\{k_1; k_2\}$, it is sufficient to replace $\alpha_{j_\nu}$ by $\beta_{j_\nu} z^{k_{0} - k_{\nu}}$ in Definition (3.1). In particular, if $m = 2$, $s_1 = s_2 = 1$, and $\alpha_2 = -\alpha_1$, then for the polynomial $P(y) = y - 1$ we can take

$$e_l(n) = \begin{cases} 2 \sum_{i=0}^{n-1} \lfloor \frac{i}{2l} \rfloor & \text{if } l \equiv 0 \pmod{2}, \\ 2 \sum_{i=0}^{n-1} \lfloor \frac{i}{3l} \rfloor & \text{if } l \equiv 1 \pmod{2}. \end{cases}$$

We can use the asymptotic formulae

$$\sum_{\substack{l \leq x \leq \ell \equiv a \pmod{2} \leq x}} \varphi(l) = \frac{cx^2}{\pi^2} + O(x \log x), \quad c = \begin{cases} 1 & \text{if } a = 0, \\ 2 & \text{if } a = 1, \end{cases}$$

to show that we then have the asymptotic relation

$$\sum_{l \geq 1} e_l(n) \varphi(l) \sim \frac{17n^3}{324} \quad \text{as } n \to \infty$$

(see Lemma 3.5 below), which implies the validity of Remark 1.2 (see below Proposition 3.6 and §6.1).

For $t_0, \ldots, t_m \in \mathbb{Z}_{\geq 0}$ we set

$$v_{t_0, \ldots, t_m, n}(z, \overline{x}) = v_{t_0, \ldots, t_m, n}(z, \overline{x}) = \prod_{j=0}^{m} \mathcal{F}_j^{t_j}(v_{n}), \quad n \geq \begin{cases} \sum_{j=0}^{m} t_j l & \text{if } \delta = 0, \\ \left(\sum_{j=1}^{m} t_j + \left\lfloor \frac{t - 1}{\delta} \right\rfloor + 1\right) l & \text{if } \delta > 0. \end{cases}$$

If the numbers $t, t_0, \ldots, t_m, n \in \mathbb{Z}_{\geq 0}$ satisfy the conditions

$$t_0 \geq 2t - 1,$$

$$t_j \geq s_j + t - 1, \quad 1 \leq j \leq m,$$

$$n \geq \begin{cases} \sum_{j=0}^{m} t_j l & \text{if } \delta = 0, \\ \left(\sum_{j=1}^{m} t_j + \left\lfloor \frac{t - 1}{\delta} \right\rfloor + 1\right) l & \text{if } \delta > 0, \end{cases}$$
then we have

\[ \frac{\partial \tau}{\partial z} v_{t_0, \ldots, t_m, n} \bigg|_{z=\zeta} = \prod_{j=0}^{m} F_{t_j}^j(v_n(\tau)) = 0, \quad 0 \leq \tau < t, \]

by Lemma 3.4 (if \( \delta > 0 \), the conditions imposed on \( t_0 \) are not needed). Since this holds for any choice of the root \( \zeta \), we obtain under these conditions that

\[ \Phi_l(z)^t|v_{t_0, \ldots, t_m, n}, \]

where

\[ \Phi_l(z) = \prod_{\zeta}(z - \zeta) = \prod_{n|l}(z^n - 1)^\mu(l/n) \in \mathbb{Z}[z] \]

is the cyclotomic polynomial. Furthermore, (3.24) holds trivially for \( t = 0 \), arbitrary \( t_0, \ldots, t_m \in \mathbb{Z}_{\geq 0} \), and admissible \( n \).

**Proposition 3.3.** For real numbers \( a > 0, b, x \) we write

\[ L(a, b, x) = \max\{\max\{n \in \mathbb{Z} \mid \lfloor a(n-1)+b \rfloor \leq x\}; 0\}. \quad (3.25) \]

Then for \( n \geq 1 \), we have \( \Phi_l(z)^{e_l(n)}|V_n \) for the Hankel determinant (3.4), where

\[ e_l(n) = \begin{cases} \sum_{i=0}^{n-1} \left( L\left(m+2, s+1, \frac{i}{l}\right) + L\left(m+2, s+2, \frac{i}{l}\right) \right) & \text{if } \delta = 0, \\
2 \sum_{i=0}^{n-1} L\left(m+\frac{1}{\delta}, s+1, \frac{i}{l}\right) & \text{if } \delta > 0,
\end{cases} \quad (3.26) \]

and \( \delta = \delta_l \) is defined in (3.16).

**Proof.** For \( 0 \leq i \leq n-1 \) we set

\[ l_i = \begin{cases} L\left(m+2, s+1, \frac{i}{l}\right) & \text{if } \delta = 0, \\
L\left(m+\frac{1}{\delta}, s+1, \frac{i}{l}\right) & \text{if } \delta > 0,
\end{cases} \]

Next, for \( 1 \leq j \leq m \) we write

\[ l_{i,j} = \begin{cases} 0 & \text{if } l_i = 0, \\
s_j + l_i - 1 & \text{if } l_i > 0,
\end{cases} \]

\[ \ell_{i,j} = \begin{cases} 0 & \text{if } \ell_i = 0, \\
s_j + \ell_i - 1 & \text{if } \ell_i > 0.
\end{cases} \]
Furthermore, we write
\[ l_{i,0} = \begin{cases} \max\{2l_i - 1; 0\} & \text{if } \delta = 0, \\ 0 & \text{if } \delta > 0, \end{cases} \]
\[ \ell_{i,0} = \begin{cases} 2\ell_i & \text{if } \delta = 0, \\ 0 & \text{if } \delta > 0. \end{cases} \]

It is easy to verify that the inequalities
\[ \sum_{j=0}^{m} l_{i,j} \leq \frac{i}{l}, \quad \sum_{j=0}^{m} \ell_{i,j} \leq \frac{i}{l} \]
hold for any \( i \). Therefore the determinant \( V_n \) can be reduced to the form
\[ V_n = \det(a_{i,j})_{i,j=0}^{n-1} = \sum_{\sigma \in S_n} \text{sign } \sigma \prod_{i=0}^{n-1} a_{i,\sigma(i)} \tag{3.27} \]
using elementary transformations of rows and columns, where
\[ a_{i,j} = v_{l_{i,0} + \ell_{j,0}, \ldots, l_{i,m} + \ell_{j,m} + i + j}. \]

Next, if \( \delta > 0 \) (so that \( l_i = \ell_i \)) and \( l_i + \ell_j > 0 \), then we have the inequality
\[ \sum_{k=1}^{m} (l_{i,k} + \ell_{j,k}) + \left[ \frac{l_i + \ell_j - 1}{\delta} \right] + 1 \leq \sum_{k=1}^{m} (l_{i,k} + \ell_{j,k}) + \left[ \frac{l_i - 1}{\delta} \right] + \left[ \frac{\ell_j - 1}{\delta} \right] + 2 \leq \frac{i + j}{l}. \]

Therefore we obtain by (3.24) that \( \Phi_l(z)^{\ell_i + \ell_j} |a_{i,j}| \) for any \( i, j \). By taking into account (3.27), we obtain what is required.

Remark 3.2. Proposition 3.3 implies the validity of the conjecture stated in [6], Remark 7, since it is easy to show that the equation
\[ \prod_{l \geq 1} \Phi_l(z)^{e_l} = \prod_{l \geq 1} (z^l - 1)^{\tilde{e}_l} \]
holds for any \( m, n \in \mathbb{Z}_{>0} \), where
\[ e_l = \sum_{i=0}^{n-1} \left\lfloor \frac{i}{ml} \right\rfloor, \quad \tilde{e}_l = \max\{n - ml; 0\}. \]

Remark 3.3. In the case when \( \max\{\delta_l; \max_{1 \leq j \leq m} s_j\} > 1 \), the quantities \( e_l(n) \) defined in (3.26) can be increased (by means of a better choice of the parameters in the proof of Proposition 3.3). We consider two special cases that will be used for proving Corollaries 1.2, 1.5.

**Proposition 3.4.** If \( \delta_l = 1, m = 1, s_1 = 2 \), then in the condition of Proposition 3.3 we can set
\[ e_l(n) = \begin{cases} 0 & \text{if } n \leq 3l, \\ \sum_{i=0}^{n-1} \left\lfloor \frac{i}{l} \right\rfloor - n & \text{if } n \geq 3l. \end{cases} \tag{3.28} \]
Proof. We can assume that \( n > 3l \). For \( 0 \leq i < n \) we set
\[
l_i = \left\lfloor \frac{i}{l} \right\rfloor, \quad l_{i,1} = \left\lfloor \frac{l_i + 1}{2} \right\rfloor.
\]

Since \( l_{i,1} \leq i/l \) for any \( i \), we can use elementary transformations of rows and columns to reduce the determinant \( V_n \) to the form
\[
V_n = \det(a_{i,j})_{i,j=0}^{n-1}
\] (3.29)
with the elements \( a_{i,j} = v_{0,l_{i,1}+l_{j,1},i+j} \). Next, for \( 0 \leq i, j < n \) we set
\[
t_{i,j} = \max\left\{\left\lfloor \frac{l_i + l_j - 1}{2} \right\rfloor; 0\right\}.
\]
It is easy to verify that if \( t_{i,j} > 0 \), then the inequalities
\[
l_{i,1} + l_{j,1} \geq t_{i,j} + 1,
\]
\[
l_{i,1} + l_{j,1} + t_{i,j} \leq l_i + l_j \leq \frac{i+j}{l}
\]
hold, whence it follows from (3.24) that
\[
\Phi_l(z)^{t_{i,j}} |a_{i,j}
\]
for any \( i, j \). Since
\[
\sum_{i=0}^{n-1} t_{i,\sigma(i)} \geq \sum_{i=0}^{n-1} \frac{l_i + l_{\sigma(i)} - 2}{2} = \sum_{i=0}^{n-1} \left\lfloor \frac{i}{l} \right\rfloor - n
\]
for any permutation \( \sigma \in \mathfrak{S}_n \), we obtain what is required from (3.29).

**Proposition 3.5.** If \( \delta_l = 0, m = 2, s_1 = s_2 = 2 \), then in the condition of Proposition 3.3 we can set
\[
e_l(n) = \sum_{i=0}^{n-1} \left( \left\lfloor \frac{i}{4l} \right\rfloor + \max\left\{ \left\lfloor \frac{i-l}{4l} \right\rfloor; 0\right\} \right)
\] (3.30)
\[
\begin{cases}
0 & \text{if } n \leq 4l, \\
4l - n & \text{if } 4l \leq n \leq 5l, \\
4kl - n & \text{if } (4k+1)l \leq n \leq (4k+2)l,
\end{cases}
\]
\[
\begin{cases}
-2l & \text{if } (4k+2)l \leq n \leq (4k+3)l, \\
n - (4k+5)l & \text{if } (4k+3)l \leq n \leq (4k+4)l, \\
-l & \text{if } (4k+4)l \leq n \leq (4k+5)l,
\end{cases}
\]
where \( k \in \mathbb{Z}_{>0} \).
Proof. We can assume that \( n > 5l \). For \( 0 \leq i < n \) we set

\[
\ell_i = \left\lfloor \frac{i}{l} \right\rfloor,
\]

\[
l_{i,0} = 2\left\lfloor \frac{l_i}{4} \right\rfloor + \begin{cases} 
    -1 & \text{if } l_i \in 4\mathbb{Z}_{\geq 0} \cup (1 + 4\mathbb{Z}_{>0}), \\
    1 & \text{if } l_i \equiv 3 \pmod{4}, \\
    0 & \text{otherwise}, 
\end{cases}
\]

\[
l_{i,1} = \begin{cases} 
    0 & \text{if } l_i = 0, \\
    \left\lfloor \frac{l_i}{4} \right\rfloor + 1 & \text{if } l_i > 0, 
\end{cases}
\]

\[
l_{i,2} = \left\lfloor \frac{l_i + 3}{4} \right\rfloor + \begin{cases} 
    -1 & \text{if } l_i = 1, \\
    0 & \text{otherwise}, 
\end{cases}
\]

\[
\ell_{i,0} = 2\left\lfloor \frac{l_i}{4} \right\rfloor + \begin{cases} 
    1 & \text{if } l_i \equiv 3 \pmod{4}, \\
    0 & \text{otherwise}, 
\end{cases}
\]

\[
\ell_{i,1} = \left\lfloor \frac{l_i + 2}{4} \right\rfloor, \quad \ell_{i,2} = \left\lfloor \frac{l_i + 3}{4} \right\rfloor.
\]

It is easy to verify that \( l_{i,0} + l_{i,1} + l_{i,2} = \ell_{i,0} + \ell_{i,1} + \ell_{i,2} = l_i \leq i/l \) for any \( i \). Therefore we can use elementary transformations of rows and columns to reduce the determinant \( V_n \) to the form

\[
V_n = \det(a_{i,j})_{i,j=0}^{n-1} \tag{3.31}
\]

with the elements \( a_{i,j} = v_{i,0} + \ell_{j,0} + l_{i,1} + l_{i,2} + \ell_{j,2,i+j} \).

Next, for \( 0 \leq i, j < n \) we set

\[
t_{i,j} = \left\lfloor \frac{l_i}{4} \right\rfloor + \max\left\{ \left\lfloor \frac{l_j - 1}{4} \right\rfloor ; 0 \right\}
\]

\[
+ \begin{cases} 
    -1 & \text{if } \left( l_i, l_j \right) \in \{ \{0\} \times (1 + 4\mathbb{Z}_{>0}) \} \cup (4\mathbb{Z}_{>0} \times \{0\}), \\
    1 & \text{if } \left( l_i, l_j \right) \in (3 + 4\mathbb{Z}_{>0}) \times (\mathbb{Z}_{>0} \setminus (1 + 4\mathbb{Z}_{>0})), \\
    0 & \text{otherwise}. 
\end{cases}
\]

It can be verified that if \( t_{i,j} > 0 \), then the inequalities

\[
l_{i,0} + \ell_{j,0} \geq 2t_{i,j} - 1, \quad l_{i,1} + \ell_{j,1} \geq t_{i,j} + 1, \quad l_{i,2} + \ell_{j,2} \geq t_{i,j} + 1
\]

hold, whence it follows from (3.24) that

\[
\Phi_l(z)^{t_{i,j}} \mid_{a_{i,j}}
\]

for any \( i, j \).

For any permutation \( \sigma \in S_n \) we have

\[
\sum_{i=0}^{n-1} t_{i,\sigma(i)} - \sum_{i=0}^{n-1} \left( \left\lfloor \frac{i}{4l} \right\rfloor + \max\left\{ \left\lfloor \frac{i-l}{4l} \right\rfloor ; 0 \right\} \right) = \#\{ i \mid l_i \equiv 3 \pmod{4} \}
\]

\[
- \#\{ i \mid l_i \equiv 3 \pmod{4}, \ l_{\sigma(i)} = 0 \}
\]

\[
- \#\{ i \mid l_i \equiv 3 \pmod{4}, \ l_{\sigma(i)} \equiv 1 \pmod{4} \}
\]

\[
- \#\{ i \mid l_i = 0, \ l_{\sigma(i)} \in 1 + 4\mathbb{Z}_{>0} \} - \#\{ i \mid l_i \in 4\mathbb{Z}_{>0}, \ l_{\sigma(i)} = 0 \}.
\]
\[
\geq \# \{ i \mid l_i \equiv 3 \, (\text{mod} \, 4) \} - \# \{ i \mid l_{\sigma(i)} \equiv 1 \, (\text{mod} \, 4) \} - \# \{ i \mid l_{\sigma(i)} = 0 \}
\]
\[
= \begin{cases}
4kl - n & \text{if } (4k + 1)l < n \leq (4k + 2)l, \\
-2l & \text{if } (4k + 2)l < n \leq (4k + 3)l, \\
n - (4k + 5)l & \text{if } (4k + 3)l < n \leq (4k + 4)l, \\
-l & \text{if } (4k + 4)l < n \leq (4k + 5)l.
\end{cases}
\]

Therefore we obtain what is required from (3.31).

The following assertion will be needed later.

**Lemma 3.5.** Let \( a > 0 \) and \( b \geq 1 \). Then as \( n \to \infty \) we have

\[
\sum_{l \geq 1} \varphi(l) \sum_{i=0}^{n-1} L \left( a, b, \frac{i}{7} \right) \sim Z(a, b) n^3,
\]

where \( \varphi(l) = \deg \Phi_l(z) \) is the Euler function and \( Z(a, b) \) and \( L(a, b, x) \) are defined by formulae (1.7) and (3.25). Furthermore, the asymptotic relations

\[
\sum_{l \geq 1} e_l(n) \varphi(l) \sim \left( \frac{1}{6} - \frac{41}{36\pi^2} \right) n^3
\]

hold if the \( e_l(n) \) are defined by (3.28), and

\[
\sum_{l \geq 1} e_l(n) \varphi(l) \sim \left( \frac{5}{32} - \frac{289}{900\pi^2} - Z(4, 1) \right) n^3
\]

if the \( e_l(n) \) are defined by (3.30).

**Proof.** For \( a \geq -1 \) we have

\[
\sum_{l \leq x} ^a \varphi(l) = \sum_{l \leq x} \sum_{d \mid l} \frac{\mu(d) l^{a+1}}{d} = \sum_{d \leq x} \mu(d) \sum_{n \leq x/d} n^{a+1}
\]

\[
= \sum_{d \leq x} \left( \frac{\mu(d) x^{a+2}}{(a + 2)d^2} + O \left( \frac{x^{a+1}}{d} \right) \right) = \frac{x^{a+2}}{(a + 2)} \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} + O(x^{a+1} \log x)
\]

\[
= \frac{6x^{a+2}}{(a + 2)^2 \pi^2} + O(x^{a+1} \log x), \quad x \to \infty.
\]

Next, if \( S \) is some statement, we denote its truth value by \([S]\) (that is, \([S] = 1\) if \( S \) is true, and \([S] = 0\) otherwise). Then \( L(a, b, x) \) can be written in the form

\[
L(a, b, x) = \sum_{m=0}^{\infty} \left[ [am + b] \leq x \right].
\]
Consequently,
\[ \sum_{l \geq 1} \varphi(l) \sum_{i=0}^{n-1} L\left(a, b, \frac{i}{l}\right) = \sum_{l \geq 1} \sum_{i=0}^{n-1} \sum_{m \geq 0} \varphi(l) [am + b] l \leq i \]
\[ = \sum_{[b] \leq i \leq n-1} \sum_{0 \leq m \leq i/a} \sum_{1 \leq l \leq i/[am+b]} \varphi(l) \]
\[ = \sum_{[b] \leq i \leq n-1} \sum_{0 \leq m \leq i/a} \left( \frac{3i^2}{\pi^2 [am+b]^2} + O \left( \frac{i \log n}{m+1} \right) \right) \]
\[ = \sum_{[b] \leq i \leq n-1} \left( \frac{3i^2}{\pi^2} \sum_{m \geq 0} [am+b]^{-2} + O(i \log^2 n) \right) \]
\[ = Z(a, b) n^3 + O(n^2 \log^2 n). \]

Next, formula (3.28) can be rewritten in the form
\[ e_l(n) = \sum_{i=0}^{n-1} \left| \frac{i}{l} \right| + \begin{cases} 0 & \text{if } n \leq l, \\
-l + \frac{n}{2} & \text{if } l \leq n \leq 2l, \\
3l - 2n & \text{if } 2l \leq n \leq 3l, \\
-n & \text{if } n \geq 3l. \end{cases} \]

Taking into account the equation \([i/l] = L(1, 1, i/l)\), we obtain in this case that
\[ \sum_{l \geq 1} e_l(n) \varphi(l) = \frac{n^3}{6} + O(n^2 \log^2 n) - \sum_{i \leq n/3} n \varphi(l) + \sum_{n/3 < l \leq n/2} (3l - 2n) \varphi(l) \]
\[ + \sum_{n/2 < l \leq n} (l - n) \varphi(l) \]
\[ = \left( \frac{1}{6} - \frac{1}{3\pi^2} \right) + \frac{6}{\pi^2} \left( \frac{1}{8} - \frac{1}{27} \right) - \frac{6}{\pi^2} \left( \frac{1}{4} - \frac{1}{9} \right) \]
\[ + \frac{2}{\pi^2} \left( \frac{3}{8} \right) - \frac{3}{\pi^2} \left( \frac{1}{4} \right) \]
\[ = \left( \frac{1}{6} - \frac{41}{36\pi^2} \right) n^3 + O(n^2 \log^2 n). \]

Similarly, for the \(e_l(n)\) defined in (3.30) we have
\[ \sum_{l \geq 1} e_l(n) \varphi(l) = \left( \frac{1}{96} + Z(4, 5) + \frac{8}{\pi^2} \left( \frac{1}{64} - \frac{1}{125} \right) - \frac{3}{\pi^2} \left( \frac{1}{16} - \frac{1}{25} \right) \right) n^3 \]
\[ + O(n^2 \log^2 n) + \sum_{1 \leq k \leq n/4} \left( C(k) n^3 + O\left( \frac{n^2 \log n}{k} \right) \right) \]
\[ = \left( \frac{1}{96} - \frac{2013}{2000\pi^2} + Z(4, 1) + \sum_{k=1}^{\infty} C(k) \right) n^3 + O(n^2 \log^2 n), \]
Values of certain $q$-series

where

$$C(k) = -\frac{2}{\pi^2} \left( \frac{1}{(4k+4)^3} - \frac{1}{(4k+5)^3} \right) + \frac{3}{\pi^2} \left( \frac{1}{(4k+3)^2} - \frac{1}{(4k+4)^2} \right) - \frac{2(4k+5)}{\pi^2} \left( \frac{1}{(4k+3)^3} - \frac{1}{(4k+4)^3} \right) - \frac{4}{\pi^2} \left( \frac{1}{(4k+2)^3} - \frac{1}{(4k+3)^3} \right) + \frac{8k}{\pi^2} \left( \frac{1}{(4k+1)^3} - \frac{1}{(4k+2)^3} \right) - \frac{3}{\pi^2} \left( \frac{1}{(4k+1)^2} - \frac{1}{(4k+2)^2} \right) - \frac{2}{\pi^2} \left( \frac{2}{(4k+5)^3} - \frac{2}{(4k+1)^3} \right) - \frac{1}{(4k+4)^2} + \frac{1}{(4k+3)^2} + \frac{1}{(4k+2)^2} - \frac{1}{(4k+1)^2} \right).$$

We have

$$\sum_{k=1}^{\infty} C(k) = \frac{7}{48} + \frac{12337}{18000\pi^2} - 2Z(4, 1),$$

whence we obtain what is required.

Taking into account the fact that $\delta_l \geq \delta_1$ for any $l \geq 1$, Propositions 3.2–3.5 and Lemma 3.5 yield the following result.

**Proposition 3.6.** Suppose that the polynomials $P(x, y) = P(y)$ and $Q(x)$ are independent of $x$. For $n \geq 1$ let

$$\Delta_n = \Delta_n(z) = z^{e_0(n)} \prod_{l \geq 1} (\Phi_l(z))^{e_l(n)} \in \mathbb{Z}[z],$$

(3.32)

where $e_0(n) := \text{ord}_z V_n$ and the $e_l(n)$ are defined in (3.26), (3.28), and (3.30). Then $\Delta_n(z)|V_n(z, \pi)$. Furthermore, as $n \to \infty$ we have

$$\deg \Delta_n \geq (\gamma + o(1))n^3,$$

where $\gamma$ is defined in (1.11), (1.12).

§ 4. The non-vanishing lemma

We retain the notation of the preceding section. In the following assertion we refine Kronecker’s rationality criterion (see [15], Corollary 5.2.3) for the sequence (3.2).

**Lemma 4.1.** Let $\overline{\omega} = (\omega_0, \omega_{j,k}) \in \mathbb{C}_w^{1+\sum_{j=1}^{m} \sum_{k=0}^{d_0-1} s_{j,k}}$. Suppose that for some $n_0 \in \mathbb{Z}_{>0}$ the equations

$$V_{n_0}(q, \overline{\omega}) = V_{n_0+1}(q, \overline{\omega}) = \cdots = V_{n_0+N(n_0)}(q, \overline{\omega}) = 0$$

(4.1)

hold, where

$$N(n) = (\mathfrak{d} - 1)(n - 1) + \sum_{j=1}^{m} \sum_{k=0}^{d_0-1} (\deg Q(x) + s_{j,k})$$

(4.2)
and \( \mathcal{D} \) is defined in (1.19). Then the generating function

\[ F(z) = F_{\mathcal{D}}(z) = \sum_{n=0}^{\infty} v_n(q, \omega) z^n \]

of the sequence \( v_n(q, \omega) \) is rational.

Proof. For brevity we write \( v_n \) instead of \( v_n(q, \omega) \). For \( N, n \in \mathbb{Z}_{\geq 0} \) we write

\[ V_{N,n} = \det(v_{N+i+j}^{n-1})_{i,j=0}^{n} \]

(taking the empty determinant to be equal to 1). It is known that the \( V_{N,n} \) satisfy the following recurrence relations (see [15], §5.2):

\[ V_{N-1,n+1}V_{N+1,n-1} = V_{N-1,n}V_{N+1,n} - (V_{N,n})^2, \quad N, n \geq 1. \quad (4.3) \]

Hence, if \( V_{N-1,n} = V_{N-1,n+1} = 0 \), then \( V_{N,n} = 0 \). Therefore by using induction on \( k \), we obtain from (4.1) that the equations

\[ V_{k,n_0} = V_{k,n_0+1} = \cdots = V_{k,n_0+N(n_0)-k} = 0 \]

hold for \( 0 \leq k \leq N(n_0) \).

In particular, we have

\[ V_{0,n_0} = V_{1,n_0} = \cdots = V_{N(n_0),n_0} = 0. \]

Let \( n_1 \in \mathbb{Z}_{\geq 0} \) be the smallest number such that the equations

\[ V_{N_1,n_1} = V_{N_1+1,n_1} = \cdots = V_{N_1+N(n_1),n_1} = 0 \quad (4.4) \]

hold for some \( N_1 \in \mathbb{Z}_{\geq 0} \). Using (4.3), it is easy to show that then, the determinants

\[ V_{N_1+1,n_1-1}, V_{N_1+2,n_1-1}, \ldots, V_{N_1+N(n_1)+1,n_1-1} \quad (4.5) \]

either are all equal to zero or are all non-zero (see [16], part VII, problem 20). By the choice of \( n_1 \), the second case holds.

It follows from (4.4) that for every integer \( k \) in the interval \( 0 \leq k \leq N(n_1) \) there exists a non-zero set of numbers \( a_{k,j} \in \mathbb{C}_w, \ 0 \leq j \leq n_1 - 1 \), such that

\[ \sum_{j=0}^{n_1-1} a_{k,j} v_{N_1+k+l+j} = 0, \quad 0 \leq l \leq n_1 - 1. \quad (4.6) \]

Since the determinants (4.5) are non-zero, \( a_{k,n_1-1} \neq 0 \) for every \( k \), whence we can assume that \( a_{k,n_1-1} = 1 \). But then we obtain from (4.6) that \( a_{k,j} = a_j \) is independent of \( k \) (since \( a_{k,j} \) and \( a_{k+1,j} \), \( 0 \leq j \leq n_1 - 2 \), satisfy the same system of linear equations with non-zero determinant).

Consider the sequence \( \{w_n\}_{n \geq 0} \) defined by the formula

\[ w_n = v_{N_1+n}, \quad 0 \leq n < n_1 - 1, \quad \sum_{j=0}^{n_1-1} a_j w_{n+j} = 0, \quad n \geq 0. \]

Then we obtain from (4.6) that
\[ w_n = v_{N_1+n} \quad \text{for } 0 \leq n \leq N(n_1) + 2n_1 - 2. \] (4.7)

We observe that if a sequence \( \{r_n\}_{n \geq 0} \subset \mathbb{F} \), where \( \mathbb{F} \) is some field, satisfies a recurrence relation of the form
\[ R(\mathcal{B})(r_n) = 0, \quad n \geq h, \]
where \( R(x) \in \mathbb{F}[x] \) is a polynomial of degree at most \( h \in \mathbb{Z}_{\geq 0} \), \( R(0) \neq 0 \), and the operator \( \mathcal{B} \) is defined in (2.1), then for any \( a \in \mathbb{F}^{*} \) and any polynomial \( p(x) \in \mathbb{F}[x] \) of degree at most \( t \in \mathbb{Z}_{\geq 0} \) we have
\[ R(a\mathcal{B})^{t+1}(p(n)a^{n}r_n) = 0, \quad n \geq (t + 1)h. \]

For the proof it is sufficient to decompose \( R(x) \) into linear factors (in the algebraic closure of \( \mathbb{F} \)) and use the relation (2.4) and Lemma 2.2.

Consequently, the sequence
\[ t_n = Q(N_1 + n + 1)w_{n+1} - P(N_1 + n + 1, q^{N_1+n+1})w_n - Q(N_1 + n + 1)u_{N_1+n+1}(q, \overline{w}), \quad n \geq 0, \]
for \( n \geq N(n_1) + 2n_1 - 2 \) satisfies the following linear recurrence relation with constant coefficients of order \( N(n_1) + 2n_1 - 2 \):
\[ A(\mathcal{B})^{\max\{\deg Q(x); \deg p_0(x)\}+1} \prod_{\nu=1}^{d} A(q^{\nu}\mathcal{B})^{\deg p_{\nu}(x)+1} \prod_{j=1}^{m} \prod_{k=0}^{d_0-1} \mathcal{D}_{\alpha_j q^k}^{\deg Q(x)+s_j,k}(t_n) = 0, \]
where
\[ A(x) = \sum_{j=0}^{n_1-1} a_j x^{n_1-1-j} \]
and the operators \( \mathcal{D}_a \) are defined in (2.2).

On the other hand, it follows from (3.3) and (4.7) that \( t_n = 0 \) for \( 0 \leq n < N(n_1) + 2n_1 - 2 \), whence \( t_n = 0 \) for all \( n \geq 0 \). But this means that \( v_{N_1+n} = w_n \) for \( n \geq 0 \), and so the function
\[ F(z) = \sum_{n \geq 0} v_{n} z^{n} \]
is rational. The lemma is proved.

**Lemma 4.2.** Suppose that the polynomials \( P(x, y) \), \( Q(x) \) and the numbers \( \alpha_j \) and \( s_{j,k} \) satisfy the hypotheses of Theorem 1.2 and \( \overline{w} \in \mathbb{C}^{1+\sum_{j=1}^{m} \sum_{k=0}^{d_0-1} s_{j,k}} \setminus \{0\} \). Then the function
\[ F(z) = F_{\overline{w}}(z) = \sum_{n=0}^{\infty} v_{n}(q, \overline{w}) z^{n} \]
is not rational.
Proof. Suppose that for some $\omega \neq 0$ the function $F(z)$ belongs to $\mathbb{C}_w(z)$. Then the equation $|v_n(q, \omega)|_w = O(C^n)$ holds for some constant $C > 1$, whence we obtain from (3.2) that

$$\omega_0 + \sum_{j=1}^m \sum_{k=0}^{d_0-1} \sum_{\sigma=0}^{s_{j,k}-1} \omega_{j,k,\sigma} f^{(\sigma)}(\alpha_j q^k) = \omega_0 + \sum_{n=0}^{\infty} \frac{u_n(q, \omega)}{\Pi_n(q)} = 0.$$

In particular, at least one of the numbers $\omega_{j,k,\sigma}$ is non-zero.

By (3.3) the function $F(z)$ satisfies the equation

$$Q \left( z \frac{d}{dz} \right) (F(z)) - p_0 \left( z \frac{d}{dz} \right) (z F(z))$$

$$= \sum_{\nu=1}^d p_{\nu} \left( z \frac{d}{dz} \right) (q^{\nu} z F(q^\nu z)) + Q \left( z \frac{d}{dz} \right) (R(z)), \quad (4.8)$$

where

$$R(z) = \omega_0 + \sum_{n \geq 0} u_n(q, \omega) z^n = \omega_0 + \sum_{j=1}^m \sum_{k=0}^{d_0-1} \sum_{\sigma=0}^{s_{j,k}-1} \frac{\omega_{j,k,\sigma} \sigma! z^\sigma}{(1 - \alpha_j q^k z)^{\sigma+1}} \in \mathbb{C}_w(z). \quad (4.9)$$

It follows from condition (i) of Theorem 1.2 that all the numbers $\alpha_j q^k$ are distinct. Since not all the numbers $\omega_{j,k,\sigma}$ are equal to zero, the function $R(z)$ has at least one pole, whence we obtain from (4.8) that the function $F(z)$ also has a pole in $\mathbb{C}_w \setminus \{0\}$.

We claim that every pole of $F(z)$ has the form $\alpha_j^{-1} q^n$, where $1 \leq j \leq m$, $n \in \mathbb{Z}_{>0}$. Suppose the opposite. Let $\beta$ be a pole of $F(z)$ that cannot be represented in such a form, with the smallest value of $|\beta|_w$. Then it follows from (4.9) and condition (ii) of Theorem 1.2 that the order of the pole of the function $Q \left( z \frac{d}{dz} \right) (R(z))$ at the point $q^{-d} \beta$ is at most $\deg p_d(x)$, for otherwise $\beta$ would have the required form. (This uses the assumption about the polynomials $P(x, y)$ and $Q(x)$ that if $\deg Q(x) > 0$, then $\deg p_d(x) = 0$, whence it follows from condition (ii) of Theorem 1.2 that in this case $d_0 = d$.) Since $q^{-d} \beta$ is a pole of the function $p_d \left( z \frac{d}{dz} \right) (q^d z F(q^d z))$ of order at least $\deg p_d(x) + 1$, we obtain from (4.8) that $q^{-d} \beta$ is a pole of one of the functions $F(q^\nu z)$, $0 \leq \nu < d$. Consequently, $\beta = q^{d-\nu} \beta'$, where $\beta'$ is also a pole of $F(z)$. Since $|\beta'|_w < |\beta|_w$, it follows that $\beta'$ can be represented in the required form. But then $\beta$ is also of that form and this contradiction proves the assertion. In particular, it follows from (4.9) and condition (i) of Theorem 1.2 that the functions $F(z)$ and $R(z)$ have no common poles.

Let $\beta$ be a pole of $F(z)$ with the greatest value of $|\beta|_w$. Then in view of what was proved above, it follows from (4.8) that the function $Q \left( z \frac{d}{dz} \right) (F(z)) - p_0 \left( z \frac{d}{dz} \right) (z F(z))$ has no pole at the point $\beta$. This is possible only in the case $\deg p_0(x) = \deg Q(x)$.

Let

$$p_0(x) = \sum_{\nu=0}^h a_{\nu} x^\nu, \quad Q(x) = \sum_{\nu=0}^h b_{\nu} x^\nu.$$
It follows from condition (iii) of Theorem 1.2 and from what was proved above that \(b_h - a_h \beta \neq 0\). Therefore we obtain that the function
\[
Q \left( z \frac{d}{dz} \right) (F(z)) - p_0 \left( z \frac{d}{dz} \right) (zF(z)) = \sum_{\nu=0}^{h} \left( z \frac{d}{dz} \right)^{\nu} \left( (b_\nu - a_\nu z)F(z) \right)
\]
has a pole of order \(o + h\) at the point \(\beta\), where \(o\) is the order of the pole of the function \(F(z)\) at the same point. But this contradicts what was proved above and thus completes the proof of the lemma.

Lemmas 4.1, 4.2 yield the following assertion.

**Lemma 4.3.** Suppose that the polynomials \(P(x, y), Q(x)\) and the numbers \(\alpha_j, s_{j,k}\) satisfy the conditions of Theorem 1.2 and \(\varpi \in C^1_{\text{w}} \sum_{j=1}^{m} \sum_{k=0}^{n-1} s_{j,k} \setminus \{0\}\). Then for any \(n_0 \in \mathbb{Z}_{>0}\) there exists an integer \(n\) in the interval \(n_0 \leq n \leq n_0 + N(n_0)\) for which \(V_n(q, \varpi) \neq 0\), where \(N(n)\) is defined in (4.2).

### § 5. The approximation lemma

For a polynomial \(A(\overline{x}) = A(x_0, \ldots, x_k) = \sum_{\overline{\nu}} A_{\overline{\nu}} \overline{x}^{\overline{\nu}} \in \mathbb{K}[\overline{x}]\) and a valuation \(v \in \mathcal{M}\) we write
\[
|A|_v = \begin{cases} 
\sum_{\overline{\nu}} |A_{\overline{\nu}}|_v & \text{for } v | \infty, \\
\max_{\overline{\nu}} |A_{\overline{\nu}}|_v & \text{for } v \nmid \infty
\end{cases}
\]

\[
H(A) = \prod_{\overline{\nu}} |A_{\overline{\nu}}|^{|\overline{\nu}|_{\overline{x}}}, \quad H_w(A) = \prod_{\overline{\nu} \neq w} |A_{\overline{\nu}}|_{w / \overline{x}}.
\]

**Lemma 5.1.** Let \(k, n \in \mathbb{Z}_{>0}\), let \(W(\overline{x}) = W(x_0, \ldots, x_k) \in \mathbb{K}[\overline{x}]\) be a homogeneous polynomial of degree \(n\), and let \(\overline{\eta} = (\eta_0, \overline{\eta}') \in \mathbb{K}^{1+k}, W(\overline{\eta}) \neq 0, \omega_0 \in \mathbb{C}_w\). Suppose that the following inequalities hold:
\[
|W(\omega_0, \overline{\eta}')|_w \leq \frac{1}{2} |\overline{\eta}'_w|^n (H_w(W)(H(\overline{\eta}))^n)^{-|\overline{x}|_{w}},
\]
\[
|\omega_0|_w \leq C|\overline{\eta}|_w,
\]
where \(C \geq 1\). Then
\[
|\eta_0 - \omega_0|_w \geq |\overline{\eta}|_w (2nC^{n-1})^{-1} (H(W)(H(\overline{\eta}))^n)^{-|\overline{x}|_{w}}.
\]

**Proof.** Since \(W(\overline{\eta}) \neq 0\), we have
\[
|W(\overline{\eta})|_{w / \overline{x}} = \prod_{\overline{\nu} \neq w} |W(\overline{\eta})|_{v / \overline{x}} \geq \prod_{\overline{\nu} \neq w} (|W|_v |\overline{\eta}|_v)^n_{v / \overline{x}} = |\overline{\eta}|_{w / \overline{x}}^n (H_w(W)(H(\overline{\eta}))^n)^{-1}.
\]

On the other hand, by (5.1) we have
\[
|W(\overline{\eta})|_w \leq |W(\omega_0, \overline{\eta}')|_w + |W(\overline{\eta}) - W(\omega_0, \overline{\eta}')|_w
\]
\[
\leq \frac{1}{2} |\overline{\eta}'_w|^n (H_w(W)(H(\overline{\eta}))^n)^{-|\overline{x}|_{w}} + |W(\overline{\eta}) - W(\omega_0, \overline{\eta}')|_w.
\]
whence
\[ |W(\eta) - W(\omega_0, \eta')|_w \geq \frac{1}{2} |\eta|^n_w (H_w(W)(H(\eta))^n)^{-\omega/\omega_w}. \]

Writing
\[ W(x) = \sum_{l=0}^{n} A_l(x)x_0^l, \]
we have
\[ \frac{W(\eta) - W(\omega_0, \eta')}{\eta_0 - \omega_0} = \sum_{l=1}^{n} \sum_{j=0}^{l-1} A_l(\eta') \eta_0^{l-1-j}. \]

By taking (5.2) into account, we obtain
\[ |W(\eta) - W(\omega_0, \eta')|_w \leq n |W|_w (C|\eta|_w)^{n-1} |\eta_0 - \omega_0|_w, \]
whence the assertion follows.

§ 6. Proofs of the main results

6.1. Proof of Theorems 1.1, 1.2. Suppose that numbers \( \alpha_j \) and \( s_{j,k} \) satisfy the conditions of Theorem 1.2. Furthermore, under the hypotheses of Theorem 1.1 we assume that \( d_0 = d \) and \( s_{j,k} = s_j \). We consider the polynomials \( V_n \) defined in § 3.

Under the hypotheses of Theorem 1.1, we define polynomials \( \Delta_n(z) \) by the relation (3.32). Furthermore, under the hypotheses of Theorem 1.2 we set \( \Delta_n(z) = _{\omega_0}^{\omega_0} V_n \).

We define polynomials \( W_n(\bar{x}) \in \mathbb{K}[\bar{x}] \) by the formulae
\[ W_n(\bar{x}) = V_n(q, \bar{x})(\Delta_n(q))^{-1}. \]

Proposition 6.1. The polynomial \( W_n(\bar{x}) \) either is equal to zero or is a homogeneous polynomial of degree \( n \), and as \( n \to \infty \) the following asymptotic estimates hold:
\[ H(W_n) \leq (H(q))(2d/3 - \gamma + o(1))n^3, \]
\[ H_w(W_n) \leq \begin{cases} (H(q))(1 - 1/\lambda)(2d/3 - \gamma + o(1))n^3 & \text{under the hypotheses of Theorem 1.1,} \\ \exp(o(n^{5/2})) & \text{under the hypotheses of Theorem 1.2,} \end{cases} \]
where \( \lambda \) is defined in (1.1), \( \gamma \) is defined in (1.11), (1.12) under the hypotheses of Theorem 1.1, and \( \gamma = 0 \) otherwise. Next, given \( \bar{x} = (\omega_0, \bar{x}_1) = (\omega_0, \omega_{j,k,\sigma}) \in \mathbb{C}_w^{1 + \sum_{j=1}^{m} \sum_{k=0}^{d_j - 1} s_{j,k}} \) satisfying the condition
\[ \omega_0 + \sum_{j=1}^{m} \sum_{k=0}^{d_j - 1} s_{j,k} - 1 \sum_{\sigma=0}^{s_{j,k}} \omega_{j,k,\sigma} f^{(\sigma)}(\alpha_j q^k) = 0, \]
the inequality
\[ |W_n(\bar{x})|_w \leq \begin{cases} |\bar{x}_1|^{n} q^{-(\alpha + \gamma + o(1))n^3} & \text{under the hypotheses of Theorem 1.1,} \\ |\bar{x}_1|^{n} q^{-(\alpha + \gamma + o(1))n^{5/2}} & \text{under the hypotheses of Theorem 1.2} \end{cases} \]
holds uniformly in $\varpi$, where $\alpha$, $c_1$, and $c_2$ are defined in (1.9), (1.20), and (1.21). Finally, for any $\varpi \neq 0$ and $n_0 \in \mathbb{Z}_{>0}$ there exists an integer $n$ in the interval $n_0 \leq n \leq \mathcal{D}n_0 + O(1)$ for which $W_n(\varpi) \neq 0$, where $\mathcal{D}$ is defined in (1.19).

Proof. We can assume without loss of generality that $Q(x) \in \mathbb{Z}_K[x]$. Furthermore, we let $D \in \mathbb{Z}_{>0}$ denote the least common denominator of the numbers $\alpha_j$ and the coefficients of the polynomial $P(x, y)$.

It follows from (3.1), (3.2) that for $n \in \mathbb{Z}_{>0}$ the polynomial $v_n$ is homogeneous in $\varpi$, $\deg_{\varpi} v_n = 1$, $\deg_x v_n \leq dn^2/2 + O(n + 1)$, and

$$D^n \prod_{k=1}^{n} Q(k) \cdot v_n \in \mathbb{Z}_K[z, \varpi].$$

We obtain from (3.4) that for $n \in \mathbb{Z}_{>0}$ the polynomial $V_n$ is homogeneous in $\varpi$ of degree $n$ (or is equal to zero). Furthermore, in view of the rearrangement inequality we have

$$\deg_z V_n \leq \max_{\sigma \in \mathbb{C}_n} \frac{d(i + \sigma(i))^2}{2} + O(n^2) = \sum_{i=0}^{n-1} \frac{d(2i)^2}{2} + O(n^2) = \frac{2dn^3}{3} + O(n^2).$$

Next, in view of Proposition 3.6 we obtain that

$$U_n V_n(z, \varpi)(\Delta_n(z))^{-1} \in \mathbb{Z}_K[z, \varpi],$$

where

$$U_n = D^{2n^2} \prod_{k=1}^{2n} (Q(k))^n,$$

and $\deg_z(V_n \Delta_n^{-1}) \leq (2d/3 - \gamma + o(1))n^3$.

We obtain from (6.1) that the inequality

$$|W_n|_v \leq |U_n|_v^{-1} \max\{1; |q|_v\}^{(2d/3-\gamma+o(1))n^3}$$

holds (uniformly in $v$) for a non-Archimedean valuation $v$.

Given a polynomial $A(z, \varpi) = \sum_{k,l} A_{k,l} z^k \varpi^l \in K[z, \varpi]$, we write

$$\mathcal{L}(A) = \sum_{k,l} |A_{k,l}|,$$

where $|\xi| := \max_{v | \infty} |\xi|_v$ (that is, $|\xi|$ is the maximum of the moduli of the conjugates of the number $\xi$).

It follows from (3.1), (3.2) that $\log \mathcal{L}(v_n) = O(n \log n)$, whence we obtain from (3.4) that

$$\mathcal{L}(V_n) \leq \exp(O(n^2 \log n)).$$

Suppose that $v \in M$ is an Archimedean valuation, $z_0 \in \mathbb{C}_v$, and $|z_0|_v \geq 1 + 1/n$. Then for $l \geq 1$ we have

$$|\log |\Phi_l(z_0)|_v - \varphi(l) \log |z_0|_v| = \left| \sum_{t|l} \mu\left(\frac{l}{t}\right) \log |1 - z_0^{-t}|_v \right|$$

$$\leq -\tau(l) \log(1 - |z_0|_v^{-1}) = O(\tau(l) \log n),$$
where \( \tau(l) = \sum_{t|l} 1 \). Therefore for \( \Delta_n(z) \) (defined in (3.32)) we obtain

\[
|\log |\Delta_n(z_0)|_v - \deg \Delta_n \cdot \log |z_0|_v| = O\left(\frac{\sum_{l \leq n} \tau(l)n^2 \log n}{l}\right) = O(n^2 \log^3 n).
\]

Consequently, if we define polynomials \( A_{n,k}(z) \) by putting

\[
V_n(z, \overline{x})(\Delta_n(z))^{-1} = \sum_k A_{n,k}(z)\overline{x}^k,
\]

we obtain

\[
\sum_k |A_{n,k}(z_0)|_v \leq \mathcal{L}(V_n)|z_0|^{\deg_x V_n}\frac{1}{|\Delta_n(z_0)|_v} \leq \exp \left( \left( \frac{2d}{3} - \gamma + o(1) \right) n^3 \log |z_0|_v + O(n^2 \log^3 n) \right).
\]

Furthermore, it follows from the maximum principle that for \( z_0 \in \mathbb{C}_v \) with \( |z_0|_v \leq 1 + 1/n \), we have

\[
\sum_k |A_{n,k}(z_0)|_v \leq \sum_k \max_{|z|_v = 1 + n^{-1}} |A_{n,k}(z)|_v \leq (n + 1)^{1 + \sum_{j=1}^m \sum_{k=0}^{d_0-1} s_{j,k}} \exp\left( O(n^2 \log^3 n) \right) \leq \exp\left( O(n^2 \log^3 n) \right).
\]

Consequently,

\[
|W_n|_v \leq \max\{1; |q|_v\}^{(2d/3 - \gamma + o(1))n^3} \cdot \exp\left( O(n^2 \log^3 n) \right).
\]

From the estimates obtained and the relation

\[
\prod_{v \nmid \infty} |U_n|_v^{-\kappa_v/\kappa} = \prod_{v \nmid \infty} |U_n|_v^{\kappa_v/\kappa} \leq |U_n| \leq \exp\left( O(n^2 \log n) \right),
\]

we obtain the first part of the proposition.

The second part follows from Propositions 3.1, 3.2 using the estimate

\[
|v_n(q, \overline{x})|_w \leq |\overline{x}|_w \exp\left( O(n) \right)
\]

in the case \( \deg p_d(x) > 0 \) (see the proof of Lemma 3.2), whence we obtain that

\[
|V_n(q, \overline{x})|_w \leq |\overline{x}|_w^n \exp\left( O(n^2) \right).
\]

The last assertion of the proposition follows from Lemma 4.3.

Let \( \eta = (\eta_0, \eta_1) = (\eta_0, \eta_{j,k,\sigma}) \in \mathbb{R}^{1 + \sum_{j=1}^m \sum_{k=0}^{d_0-1} s_{j,k}} \setminus \{0\} \). We define \( \omega_0 \) by the equation

\[
\omega_0 = -\sum_{j=1}^m \sum_{k=0}^{d_0-1} \sum_{\sigma=0}^{s_{j,k}-1} \eta_{j,k,\sigma} f^{(\sigma)}(\alpha_j q^k).
\]

(6.2)
Then
\[ |\omega_0|_w \leq |\eta|_w \sum_{j=1}^{m} \sum_{k=0}^{d_n-1} \sum_{\sigma=0}^{s_{j,k}-1} |f^{(\sigma)}(\alpha_j q^k)|_w. \]  \tag{6.3}

Taking into account equation (1.1), we obtain from Proposition 6.1 that
\[ 2|W_n(\omega_0, \eta_1)|_w (H_w(W)(H(\eta))^n)^{\infty/\kappa_w} \leq |\eta|_w^n (H(\eta))^{\infty / \kappa_w} \times \begin{cases} |q|_w^{-2d/3+\alpha-(2d/3-\gamma)\lambda+o(1)n^3} & \text{under the hypotheses of Theorem 1.1,} \\ |q|_w^{-(c_1+c_2+o(1))n^{5/2}} & \text{under the hypotheses of Theorem 1.2.} \end{cases} \]

Taking into account the inequality (1.13) and the last part of Proposition 6.1, we obtain that for any \( \varepsilon > 0 \) there exists \( n \) satisfying the inequalities
\[ n \leq \begin{cases} (d+\varepsilon) \left( \frac{\lambda \log H(\eta)}{(2d/3+\alpha-(2d/3-\gamma)\lambda) \log H(q)} \right)^{1/2} + O(1) & \text{under the hypotheses of Theorem 1.1,} \\ (D+\varepsilon) \left( \frac{\log H(\eta)}{(c_1+c_2) \log H(q)} \right)^{2/3} + O(1) & \text{otherwise} \end{cases} \]

such that \( W_n(\eta) \neq 0 \) and
\[ 2|W_n(\omega_0, \eta_1)|_w (H_w(W)H(\eta))^n)^{\infty/\kappa_w} \leq |\eta|_w^n. \]

Applying Lemma 5.1 and taking into account (6.2), (6.3), we obtain what is required.

6.2. Proof of Corollary 1.1. To prove the irrationality and non-quadraticity (in the qualitative form) it is sufficient to observe that in the case under consideration the quantity (1.1) has the form
\[ \lambda = \frac{a \log |\rho|}{\log |\rho| - \log |\sigma|} = \frac{a}{1-\gamma}, \]
where \( a = 1 \) if \( \mathbb{K} \) is \( \mathbb{Q} \) or an imaginary quadratic field, and \( a = 2 \) if \( \mathbb{K} \) is a real quadratic field, while condition (1.13) has the form
\[ \lambda < \frac{12}{5}. \]

The estimate for \( |E_q(\alpha) - r/s| \) follows directly from Theorem 1.1.

Suppose that \( \gamma < 1/6 \). From Theorem 1.1 we obtain that for any \( \varepsilon > 0 \) there exists a positive constant \( c = c(q, \alpha, \varepsilon) \) such that for any number \( \theta \) belonging to some quadratic extension of the field \( \mathbb{Q} \) we have the inequality
\[ |E_q(\alpha) - \theta| \geq \begin{cases} c \exp(-(C_1+\varepsilon)(\log H(\theta))^{3/2}) & \text{if } \theta \in \mathbb{Q}, \\ c \exp(-(2\sqrt{2}C_2+\varepsilon)(\log H(\theta))^{3/2}) & \text{otherwise}. \end{cases} \]
Let
\[ A(z) = a(z - \theta_1)(z - \theta_2) \in \mathbb{Z}[z] \setminus \{0\}. \]
We can assume without loss of generality that \( A \) is primitive (that is, its coefficients are coprime in toto).

If \( \theta_{1,2} \in \mathbb{Q} \), then in view of the fact that
\[ H(\theta_1)H(\theta_2) = M(A) \leq L(A) \]
(where \( M(A) \) is the Mahler measure of the polynomial \( A \)), we have
\[
|A(E_q(\alpha))| \geq |(E_q(\alpha) - \theta_1)(E_q(\alpha) - \theta_2)|
\geq c^2 \exp\left(- (C_1 + \varepsilon)(\log H(\theta_1))^{3/2} - (C_1 + \varepsilon)(\log H(\theta_2))^{3/2}\right)
\geq c^2 \exp\left(- (C_1 + \varepsilon)(\log H(\theta_1)H(\theta_2))^{3/2}\right)
\geq c^2 \exp\left(- (C_2 + \varepsilon)(\log L(A))^{3/2}\right).
\]

If the polynomial \( A \) is irreducible over \( \mathbb{Q} \), then
\[ H(\theta_1) = H(\theta_2) = (M(A))^{1/2}; \]
\[
\max\{|E_q(\alpha) - \theta_1|; |E_q(\alpha) - \theta_2|\} \geq \frac{1}{2}|\theta_1 - \theta_2| \geq \frac{1}{2}|a|,
\]
whence
\[
|A(E_q(\alpha))| \geq \frac{1}{2}c \exp\left(- (2\sqrt{2} C_2 + \varepsilon) \log^{3/2}(L(A))^{1/2}\right)
\geq \frac{1}{2}c \exp\left(- (C_2 + \varepsilon)(\log L(A))^{3/2}\right).
\]

6.3. Proof of Corollary 1.3. It follows from the hypotheses of the corollary that
\[
u_n = \frac{u}{\sqrt{D}} \left(\left(\frac{r + \sqrt{D}}{2}\right)^n - \left(\frac{r - \sqrt{D}}{2}\right)^n\right).
\]
Let \( \varepsilon_0 = \text{sign } r \). Then
\[
\sum_{n=1}^\infty \frac{b^n}{u_{kn}} = \frac{\varepsilon_0 \sqrt{D}}{u} L_{q^k}(\alpha),
\]
where
\[
\alpha = b \left(\frac{2\varepsilon_0}{|r| - \sqrt{D}}\right)^k, \quad q = \frac{|r| + \sqrt{D}}{|r| - \sqrt{D}},
\]
since for \( |z| < |q| \) we have the equation
\[ L_q(z) = \sum_{n=1}^\infty \frac{z^n}{q^n - 1}. \]

If \( \sqrt{D} \notin \mathbb{Z} \), then the minimal polynomial of the number \( q \) is equal to \( z^2 + (r^2/s + 2)z + 1 \), whence
\[ H(q) = (M(q))^{1/2} = |s'q|^{1/2}. \]
If $\sqrt{D} \in \mathbb{Z}$, then
\[
H(q) = \frac{|r| + \sqrt{D}}{(|r| + \sqrt{D}, |r| - \sqrt{D})} = \frac{(|r| + \sqrt{D})^2}{4d} = |s'|.
\]

We apply Corollary 1.2 with $\mathbb{K} = \mathbb{Q}(\sqrt{D})$ and $| \cdot |_w = | \cdot |$. In view of the fact that
\[
\lambda(q^k) = \lambda(q) = 1 + \frac{\log |s'|}{\log |q|} = 1 + \frac{\log |s'|}{2 \log(r' + \sqrt{(r')^2 + 4s'}) - \log(4|s'|)},
\]
we obtain what is required.

6.4. Proof of Theorem 1.3. As shown in [12], Lemma 2, the equation
\[
h^{(l)}(z + \frac{1}{z}) = \sum_{k=0}^{l} \left( A_{l,k}(z)T_q^{(k)}(z) + A_{l,k}(\frac{1}{z})T_q^{(k)}(\frac{1}{z}) \right)
\]
holds for $z \neq 0, \pm 1$ and $l \in \mathbb{Z}_{\geq 0}$, where $T_q(z)$ is Chakalov’s function (1.6), the $A_{l,k}(z)$ are rational functions with coefficients in $\mathbb{Q}$ that have no poles other than $\pm 1$, and
\[
A_{l,l}(z) = \frac{z^{2l+1}}{(z^2 - 1)^l(z + 1)}.
\]

For $1 \leq j \leq m$ let $\alpha_{j,1}, \alpha_{j,2} \in C^\ast_w$ be the roots of the equation $z^2 - a_j z + 1 = 0$ (so that $\alpha_{j,2} = 1/\alpha_{j,1}$). We observe that $\alpha_{j,1} \neq \pm 1$ by condition (i) of Theorem 1.3. Then for $\overline{\omega} = (\omega_0, \omega_j, \sigma) \in C^{1+s}$ we have the equation
\[
\omega_0 + \sum_{j=1}^{m} \sum_{\sigma=0}^{s_j-1} \omega_j, \sigma h^{(\sigma)}(a_j) = \omega_0 + \sum_{j=1}^{m} \sum_{\nu=1}^{2} \sum_{\sigma=0}^{s_j-1} C_{j,\sigma}(\alpha_{j,\nu}, \overline{\omega}) T_q^{(\sigma)}(\alpha_{j,\nu}),
\]
where
\[
C_{j,\sigma}(z, x) = \sum_{\tau=\sigma}^{s_j-1} A_{\tau,\sigma}(z)x_{j,\tau}.
\]

If not all the $\omega_j, \sigma$ are equal to zero, then the same is true for $C_{j,\sigma}(\alpha_{j,\nu}, \overline{\omega})$.

Consider the sequences of polynomials
\[
u_n = u_n(\overline{y}) = \sum_{j=1}^{m} \sum_{\nu=1}^{2} \sum_{\sigma=0}^{s_j-1} \sigma! \binom{n}{\sigma} \alpha_{j,\nu}^{n-\sigma} y_{j,\nu,\sigma},
\]
\[
v_n = v_n(z, \overline{y}) = z^{n(n+1)/2} y_0 + \sum_{k=0}^{n} z^{n(n+1)/2 - k(k+1)/2} u_k(\overline{y}),
\]
\[
V_n = V_n(z, \overline{y}) = \det(v_{i+j})_{i,j=0}^{n-1},
\]
where $\overline{y} = (y_0, y_{j,\nu,\sigma})$. The results in § 3 are valid for $V_n$ (with $P(x, y) = y$, $Q(x) = 1$ and the corresponding values of the parameters).
In order to use the results obtained in §4, it suffices to verify that \(\alpha_{j,\nu}^{-1} \notin q^Z\) for \((j, \nu) \neq (k, \mu)\). But if \(\alpha_{j,\nu} = \alpha_{k,\mu}\), then \(a_j = a_k\), so that \(j = k\), whence also \(\nu = \mu\) (since \(\alpha_{j,1} \neq \pm 1\)). If, however, \(\alpha_{j,\nu} = \alpha_{k,\mu}q^n\) for \(n \neq 0\), then the polynomials

\[
x^2 - a_k x + 1, \quad q^{2n}x^2 - a_jq^nx + 1
\]

have a common root. Therefore their resultant is equal to zero, which contradicts condition (ii) of Theorem 1.3.

We define polynomials \(W_n\) by the equation

\[
W_n(\xi) = W_n(x_0, x_{j,\sigma}) = V_n(q, \eta(\xi))(\Delta_n(q))^{-1},
\]

where \(\Delta_n(z)\) is defined in (3.32) (for the corresponding values of the parameters),

\[
y_0(\xi) = x_0, \quad y_{j,\nu,\sigma}(\xi) = C_{j,\sigma}(\alpha_{j,\nu}, \xi),
\]

while the \(C_{j,\sigma}\) are defined in (6.5).

We observe that \(\tilde{u}_n := u_n(\eta(\xi)) \in K[\xi]\) since \(\tilde{u}_n^\sigma = \tilde{u}_n\) for any element \(\sigma\) of the Galois group \(\text{Gal}(L/K)\), where \(L = K(\alpha_{j,\nu})_{1 \leq j \leq m}\), and so \(W_n \in K[\xi]\).

As in the proof of Proposition 6.1, we obtain the following assertion in view of equation (6.4).

**Proposition 6.2.** The polynomial \(W_n\) either is equal to zero or is a homogeneous polynomial of degree \(n\), and as \(n \to \infty\) the asymptotic estimates

\[
H(W_n) \leq (H(q))^{(2/3-\gamma+o(1))n^3}, \quad H_w(W_n) \leq (H(q))^{((1-1/\lambda)(2/3-\gamma)+o(1))n^3}
\]

hold, where \(\lambda\) and \(\gamma\) are defined in (1.1) and (1.24). Next, given \(\varpi = (\omega_0, \omega_{j,\sigma}) = (\omega_0, \omega_{j,\sigma}) \in C^{1+s}_w\) satisfying the condition

\[
\omega_0 + \sum_{j=1}^m \sum_{\sigma=0}^{s_j-1} \omega_{j,\sigma}h^{(\sigma)}(a_j) = 0,
\]

we have

\[
|W_n(\varpi)|_w \leq |\omega_1|_w^n |q|_w^{-(\alpha+\gamma+o(1))n^3}
\]

uniformly in \(\varpi\) as \(n \to \infty\), where \(\alpha\) is defined in (1.23). Furthermore, for any \(\varpi \in C^{1+s}_w\setminus \{0\}\) and \(n_0 \in Z_{>0}\), there exists \(n \in Z_{>0}\) in the interval \(n_0 \leq n \leq n_0 + 2s\) such that \(W_n(\varpi) \neq 0\).

Using Lemma 5.1, we obtain the assertion of Theorem 1.3 from Proposition 6.2 (see the concluding part of §6.1).

**Remark 6.1.** Small changes in the proof make it possible to improve the estimates for the linear independence measure in our main theorems and their corollaries. The author is currently preparing the corresponding results for publication (see [17]).

**Remark 6.2.** In the case when \(K = \mathbb{Q}\) and \(|\cdot|_w = |\cdot|\), the qualitative part of Theorem 1.2 follows in essence from the results of [18].

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