Isgur–Wise function for heavy–light mesons in the \(D\)-dimensional potential model

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Abstract

We report the results of a wave function for mesons in \(D\) space–time dimension developed by considering the quark–antiquark potential of Nambu–Goto strings. With this wave function, we have studied the Isgur–Wise function for heavy–light mesons and its derivatives such as slope and curvature. The dimensional dependence of our results and a comparative study with the results of three-dimensional QCD are also reported.

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1. Introduction

The two-quark composite system, with one heavy quark and one light quark, has been the focus of interest for many years [1]. In the semileptonic transitions for mesons [2], in the infinite quark mass limit, all the mesonic form factors can be expressed in terms of a single universal function, the Isgur–Wise function (IWF) [3], which is a function involving the wave function for mesons. Thus, the wave function of mesons is important for the detailed study of IWF.

In the low-energy limit, QCD is non-perturbative in nature. Construction of phenomenological models by employing basic properties of QCD is very useful in predicting properties of hadrons such as its mass, form factors, decay widths, etc. In this context, potential models for mesons involving potential between heavy and light quarks have been very successful in the study of hadrons and their properties [4, 5].

It is worth mentioning here that, for obtaining the mesonic wave function in this potential model approach, the choice of potential is of utmost importance. Here, the Cornell potential [6], with a linear plus Coulombic form, has been very popular and useful in such a phenomenological study. The wave function for heavy–light mesons has been calculated earlier with such a potential by applying the quantum mechanical perturbation technique [7]. This has been deduced both with Coulombic term in potential as the parent [8–10] and with linear confinement term as the parent [11, 12]. The parameters such as slope (charge radii) and curvature (convexity parameter) of IWF have been reported in both the cases with certain limitations. Inspired by the higher dimensional potential model of classical string model for hadrons as proposed by Nambu and Goto [13–15], in this paper we develop wave function for heavy–light mesons by solving the non-relativistic Schrödinger wave equation in higher dimension with such a potential. In the recent past, the higher dimensional Schrödinger equation has been solved exactly with different types of potentials such as the Pöschl–Teller potential, Möbius square potential, hypercentral potential, etc [16–21]. However, the generally accepted exact solution of the \(D\)-dimensional Schrödinger equation with linear plus Coulombic-type potential is yet to be achieved. In our case, the quark–antiquark potential, as predicted by the Nambu–Goto action of bosonic strings [22], is of the form

\[ V(r) = -\frac{\gamma}{r} + \sigma r + \mu_0, \tag{1} \]

which is analogous to the linear plus Coulombic-type Cornell potential of three-dimensional QCD. Here, in equation (1), the coefficient \(\gamma = \frac{\pi (d-2)}{2}\) is the universal Lüscher coefficient [23] involved in the Lüscher term \(-\frac{\gamma}{r}\). \(d\) is the space–time dimension, with \(d = D + 1\), with \(D\) being the ordinary spatial
dimension. $\sigma$ is the string tension whose value is 0.178 GeV$^2$ and $\mu_0$ is a regularization-dependent constant.

As mentioned earlier, due to the lack of an exact solution for the $D$-dimensional Schrödinger equation with such a potential, we opt for applying the perturbation method. As a generalization of our previous works on three-dimensional QCD [9, 10, 27] to higher dimension, here we choose Lüscher term in the potential as the parent and linear term as perturbation. We expect that the choice of $1/r$ part of potential in the parent Hamiltonian would be justified when this part dominates the potential in the perturbation method. It is observed that Lüscher term, being dimension dependent, becomes more and more dominant as $D$ goes on increasing, thus justifying such a consideration.

With Lüscher term as the parent, we employ the Schrödinger wave equation in higher dimension [35] for getting unperturbed wave function. Next, we consider linear confinement term in potential as perturbation and find the total wave function for mesons using Dalgarno’s method of perturbation [24–26]. This can be looked at as a generalization of our earlier work [27], when Lüscher term is considered to be analogous to the Coulombic term of Cornell potential in QCD.

With this wave function, we then study IWF and its derivatives (slope and curvature). As stated above, the same studies have been carried out in three-dimensional QCD with the Cornell potential. In this work, we also make a comparative study of our results in higher dimension with those obtained in standard QCD.

Section 2 contains the essential formalism, section 3 contains the results, and the concluding remarks are given in section 4.

2. Formalism

2.1. Potential model

From equation (1), we take $-\mu$ as the parent and $\sigma r + \mu_0$ as perturbation. Our unperturbed Hamiltonian is then

$$H_0 = -\frac{\hbar^2}{2\mu} \frac{\nabla_D^2}{r} + V_0(r)$$  \hspace{1cm} (2)

with

$$H' = \sigma r + \mu_0$$  \hspace{1cm} (3)

as perturbation. $\mu$ is the reduced mass of the meson.

2.2. Wave function with Lüscher term as the parent

In $D$ dimension, the Schrödinger equation is [28–32]

$$\left[ -\frac{\hbar^2}{2\mu} \nabla_D^2 + V_0(r) \right] \Psi(r, \Omega_D) = E \Psi(r, \Omega_D)$$  \hspace{1cm} (4)

with

$$\nabla_D^2 = \frac{1}{r^{D-1}} \frac{d}{dr} \left( r^{D-1} \frac{d}{dr} \right) - \frac{\Lambda_D^2(\Omega_D)}{r^2}$$  \hspace{1cm} (5)

and

$$\Psi(r, \Omega_D) = R(r) Y(\Omega_D).$$  \hspace{1cm} (6)

Here, $\Lambda_D^2(\Omega_D)$ is a generalization of the centrifugal barrier [33] for the case of $D$-dimensional space–time.

The eigenvalues of $\Lambda^2_D(\Omega_D)$ are given by

$$\Lambda^2_D(\Omega_D) Y(\Omega_D) = l(l+D-2) Y(\Omega_D).$$  \hspace{1cm} (7)

Here $Y(\Omega_D)$ and $R(r)$ are the spherical harmonics and radial wave function; $l$ is the angular momentum quantum number; and $E$ is the energy eigenvalue for the unperturbed wave function.

This gives equation (4) in terms of the radial part as

$$\left[ \frac{d^2}{dr^2} + \frac{D-1}{r} \frac{d}{dr} - \frac{l(l+D-2) + 2\mu}{\hbar^2}(E-V_0) \right] R(r) = 0.$$  \hspace{1cm} (8)

For $l = 0$ and taking $\hbar = 1$, equation (8) yields

$$R''(r) + \frac{D-1}{r} R'(r) + 2\mu \left( \mu + \frac{\gamma}{r} \right) R(r) = 0.$$  \hspace{1cm} (9)

We look for the solution of equation (9) by taking the radial wave function of the form [34, 35]

$$R(r) = F(r)e^{-\mu r \gamma}.$$  \hspace{1cm} (10)

Here, it is to be noted that the negative sign in the exponent ensures the square integrability at the origin and infinity.

Using equation (10) in (9), we obtain

$$F''(r) + \mu^2 \gamma^2 - \frac{D-1}{r} \mu \gamma + 2\mu E + \frac{2\mu^2}{r} F(r) = 0.$$  \hspace{1cm} (11)

Now, we consider the series expansion of $F(r)$ as

$$F(r) = \sum_{n=0}^{\infty} a_n r^n f(r, D),$$  \hspace{1cm} (12)

such that $f(r, D) = 1$ at $D = 3$. Here, we take $f(r, D) = r^{\frac{D-2}{2}}$, which satisfies this condition. The radial solution can now be expressed as

$$R(r) = \sum_{n=0}^{\infty} a_n r^{n+\frac{D-1}{2}} e^{-\mu r \gamma}.$$  \hspace{1cm} (13)

With this radial wave function we construct the unperturbed wave function $\Psi^0(r)$ for ground state ($n = 0$) as

$$\Psi^0(r) = N r^{\frac{D-1}{2}} e^{-\mu r \gamma},$$  \hspace{1cm} (14)

where $N$ is the normalization constant, which we obtain from the condition

$$\int_0^{\infty} 4\pi r^2 |\Psi^0(r)|^2 \, dr = 1.$$  \hspace{1cm} (15)

Applying equation (14) in (15), we obtain

$$N = \left[ \frac{(2\mu)^D}{4\pi \Gamma(D)} \right]^{1/2}.$$  \hspace{1cm} (16)

At $D = 3$, this gives $N = \left( \frac{4\mu}{\pi} \right)^{1/2}$, which is similar to the value obtained earlier [27] for the corresponding case with QCD potential. With this unperturbed wave function, we measure the eigenenergy $E$ as

$$E = W^0 = -\int_0^{\infty} \frac{\gamma}{r} 4\pi r^2 |\Psi^0(r)|^2 \, dr = -\frac{\mu \gamma^2}{D-1}.$$  \hspace{1cm} (17)
2.3. Wave function with linear term as perturbation

The first-order perturbed eigenfunction \( \Psi'(r) \) can be calculated using the relation \[ H_0 \Psi'(r) + H' \Psi^0(r) = W^0 \Psi'(r) + W' \Psi^0(r), \] (18)

where \[ W' = \int_0^\infty 4\pi r^2 H' | \Psi^0(r) |^2 dr = \frac{\sigma D}{2\mu r} + \mu_0. \] (19)

Then, from equation (18) we obtain

\[
\left[ \frac{\hbar^2}{2\mu} \nabla_D^2 - \frac{\gamma}{r} - W^0 \right] \Psi'(r) = (W' - \sigma r - \mu_0) \Psi^0(r). \] (20)

Taking \( h = 1 \) and expanding,

\[
\left[ \frac{d^2}{dr^2} + \frac{D - 1}{r} \frac{d}{dr} + \frac{2\mu \gamma}{r} - 2\mu W^0 \right] \Psi'(r) = 2\mu (\sigma r + \mu_0 - W') \Psi^0(r). \] (21)

From equation (21), we obtain the following Dalgarno’s method of perturbation (see the appendix):

\[
\Psi'(r) = -\frac{\sigma D}{6\gamma} r^3 \psi_{\text{total}} e^{-\mu y r}. \] (22)

With this perturbed wave function, we construct the total wave function as

\[
\Psi_{\text{total}}(r) = N'_1 [\Psi^0(r) + \Psi'(r)]. \] (23)

Using equations (14) and (22) into (23):

\[
\psi_{\text{total}}(r) = N_1 \left[ 1 - \frac{\sigma D}{6\gamma} r^3 \right] e^{-\mu y r}, \] (24)

where \( N_1 \) is the normalization constant for the total wave function and is obtained from

\[
\int_0^\infty 4\pi r^2 | \psi_{\text{total}}(r) |^2 dr = 1 \] (25)

as

\[
N_1 = \frac{1}{2\sqrt{\pi} \Gamma(D) \Gamma(2D+2) \Gamma(2D+4)} \left[ \frac{\Gamma(D)}{\Gamma(D)^2} - \frac{2\sigma D}{\mu r} \frac{\Gamma(D+2)}{\mu_0 r} \right]^{1/2}. \] (26)

For \( D = 3 \), \( N_1 \) gives the value of equation (12) of [27] when \( \gamma \) is replaced by \( \frac{\hbar^2}{\mu} \) and \( \sigma \) by \( h \).

Now, if there is only Coulombic term as parent, with no perturbation, then \( \sigma \to 0 \). In this case, equation (27) becomes

\[
N_1 = \frac{1}{2\sqrt{\pi} \Gamma(D)} \left[ \frac{2\mu y}{\Gamma(D)} \right]^{1/2} = \left[ \frac{2\mu y}{4\pi \Gamma(D)} \right]^{1/2}. \] (27)

This is our normalization constant \( N \) for the unperturbed wave function \( \Psi^0(r) \) as obtained in equation (16).

2.4. The Isgur–Wise function

Under heavy quark symmetry (HQS), the strong interactions of heavy quarks are independent of its spin and mass [37] and all the form factors are completely determined at all momentum transfers in terms of the universal IWF. It is useful to parameterize IWF in terms of its derivatives at zero recoil \( (y = 1) \) [38]. In explicit form, for small non-zero recoil, IWF can be expressed as

\[
\xi(y) = 1 - \rho^2 (y - 1) + C (y - 1)^2 + \cdots. \] (28)

Thus, HQS provides us with a prediction for the normalization of the IWF at zero recoil point \( (y = 1) \). Here \( \rho^2 \) is the slope (charge radii) and \( C \) is the curvature (convexity parameter) of IWF, which are measured at zero recoil point as

\[
\rho^2 = -\frac{\delta \xi(y)}{\delta y} \bigg|_{y=1}, \quad C = \frac{\delta^2 \xi(y)}{\delta y^2} \bigg|_{y=1}. \] (29)

It should be mentioned here that for a reliable analysis of the IWF, the first two terms in the expansion of IWF (equation (28)) are required to be taken into consideration, thus making it necessary to calculate both slope and curvature parameters. The calculation of \( \rho^2 \) and \( C \) provides a measure of the validity of HQS in the infinite mass limit along with a valid test for the confirmation of our wave function. There have been several attempts to calculate \( \rho^2 \) and \( C \) from theory and models [39–46]. The corresponding results are shown in table 2. On general ground, the slope parameter should have a value around unity and the curvature of IWF is expected to have a small positive value for all \( y > 1 \) [38].

The calculation of this IWF is non-perturbative in principle and is performed for different phenomenological wave functions of mesons [47]. This function depends upon the meson wave function and some kinematic factor, as given below:

\[
\xi(y) = \int_0^\infty 4\pi r^2 | \Psi(r) |^2 \cos(pr) dr, \] (30)

where \( \cos(pr) = 1 - \frac{\rho^2}{2} y + \frac{\rho^4}{4} y^2 + \cdots \) with \( p^2 = 2\mu^2 (y - 1) \). Taking \( \cos(pr) \) up to \( O(r^4) \), we obtain

\[
\xi(y) = \int_0^\infty 4\pi r^2 | \Psi(r) |^2 dr - \left[ \frac{4\pi \mu^2}{2} \int_0^\infty r^4 | \Psi(r) |^2 dr \right] \] (31)

\[
\times (y - 1) + \left[ \frac{2}{3} \pi \mu^4 \int_0^\infty r^6 | \Psi(r) |^2 dr \right] (y - 1)^2. \] (31)

Equations (28) and (31) give us:

\[
\rho^2 = 4\pi \mu^2 \int_0^\infty r^4 | \Psi(r) |^2 dr, \] (32)

\[
C = \frac{2}{3} \pi \mu^4 \int_0^\infty r^6 | \Psi(r) |^2 dr, \] (33)

\[
\int_0^\infty 4\pi r^2 | \Psi(r) |^2 dr = 1. \] (34)

Equation (34) gives the normalization constants \( N \) and \( N' \) for \( \Psi^0(r) \) and \( \Psi_{\text{total}}(r) \) as obtained earlier in equations (16) and (27).
2.5. Derivatives of IWF

Taking the unperturbed wave function \( \Psi_0(r) \) from equation (14), we have calculated the slope and curvature of IWF, \( \xi(y) \), as given below:

\[
\rho^2 = \frac{D(D+1)}{4y^2},
\]

\[
C = \frac{D(D+1)(D+2)(D+3)}{96y^4}.
\]

Thus, for \( D = 3 \), we obtain \( \rho^2 = \frac{3}{y^2} \) and \( C = \frac{15}{4y^4} \), which are expressions (25) and (26) of [27], if \( y \) and \( \sigma \) are replaced by \( \frac{4\alpha_c}{\pi} \) and \( b \) of QCD potential as given in

\[
V(r) = -\frac{4\alpha_c}{3r} + br + c.
\]

Also, with the total wave function \( \Psi_{\text{total}}(r) \), we have calculated the slope and curvature:

\[
\rho^2 = \mu^2 \frac{\Gamma(D+2) \Gamma(D+4) - 2\pi D \Gamma(D+4)}{\Gamma(D)^2 - 2\pi D \Gamma(D+2) + \pi^2 D^2 \Gamma(D+2)^2 + \pi^2 D^2 \Gamma(D+4)^2}
\]

\[
C = \mu^4 \frac{\Gamma(D+2) \Gamma(D+4) - 2\pi D \Gamma(D+4)}{6 \Gamma(D)^2 - 2\pi D \Gamma(D+2) + \pi^2 D^2 \Gamma(D+2)^2 + \pi^2 D^2 \Gamma(D+4)^2}
\]

These at \( D = 3 \) become

\[
\rho^2 = \mu^2 \frac{3.4 - 3.45.6 + \frac{\sigma^2}{4} 3.45.6.7.8}{1 - \frac{3.4}{4y^2} + \frac{\sigma^2}{4} 3.45.6}
\]

\[
C = \mu^4 \frac{3.45.6 - 3.45.6.7.8 + \frac{\sigma^2}{4} 3.45.6.7.8.9.10}{6 \left[ 1 - \frac{3.4}{4y^2} + \frac{\sigma^2}{4} 3.45.6 \right]}
\]

\[
\rho^2 \quad \text{and} \quad C \quad \text{expressions in equations (38) and (39) also give back equations (35) and (36) when} \quad \sigma \rightarrow 0.
\]

Again, at \( D = 3 \), equations (40) and (41) transform to equations (36) and (38) of [27] on substituting \( \frac{\sigma}{y} \) for \( \gamma \) and \( b \) for \( \sigma \), if we neglect the relativistic effect of [27].

3. Results

We have studied the variation of IWF with \( y \) for different space–time dimensions \( D \) for \( B \) meson for both unperturbed and total wave functions (figure 1) taking string tension \( \sigma = 0.178 \text{ GeV}^2 \). The boundary condition \( \xi(1) = 1 \) is found to be satisfied everywhere.

The variation of derivatives of IWF \( (\rho^2 \text{ and } C) \) with space–time dimension \( D \) for \( B \) meson taking the unperturbed wave function \( \Psi_0(r) \) and the total wave function \( \Psi_{\text{total}}(r) \) is given in table 1.

We find that with an increase in \( D \) value, the slope and curvature decrease. With the total wave function, the variation of \( \rho^2 \text{ and } C \) with space–time dimension \( D \) for \( B \) meson is shown in figure 2.

As \( D \rightarrow \infty \) we have also obtained the asymptotic forms of \( \rho^2 \text{ and } C \) from equations (38) and (39):

\[
\rho^2_{\text{asym}} = \mu^2 \frac{1}{12\pi^2 y^4} - \frac{2\gamma}{6\pi^2 y^4} + \frac{1}{12\pi^2 y^4} + \frac{\sigma^2}{4\pi^2 y^4} - \frac{1}{12\pi^2 y^4},
\]

\[
C_{\text{asym}} = \mu^4 \frac{1}{12\pi^2 y^4} - \frac{2\gamma}{6\pi^2 y^4} + \frac{1}{12\pi^2 y^4} + \frac{\sigma^2}{4\pi^2 y^4} - \frac{1}{12\pi^2 y^4}.
\]
and figure (A.1). 10 (A.2)

previous work with the QCD Cornell potential (see [49]) and results for \( \rho^2 \) and \( C \) may also change (improve) accordingly.

Lastly, we conclude by commenting on our opting for ground state studies. It is true that higher (excited) state analysis ushers in a wider scope for exploring hadron spectra and distribution of hadron characteristic properties among its QCD constituents. Such a study of excited hadron spectra using lattice QCD is currently evolving [50]. Nevertheless, the ground state consideration also provides necessary information for the confirmation of the model and the study of hadron properties. Similar to our earlier works [9, 10, 27] with a three-dimensional potential model, here also we have opted for calculating only ground state (\( l = 0 \)) wave function by solving the \( D \)-dimensional Schrödinger equation for the string potential. We look at the present work as an extension of our previous works, to higher dimension. Development of formalism for higher state wave function and studies of meson excited states are currently under consideration.

4. Conclusion and remarks

From table 1 and figure 2, we find that, with increasing \( D \), the \( \rho^2 \) and \( C \) values go on decreasing, eventually reaching the asymptotic limit. We have shown our results for \( B \) meson only as a representative case. A similar pattern of results should also follow for other \( B \) and \( D \) sector heavy–light mesons such as \( D_i, D_{i*}, B_i, B_{i*} \).

Although expressions for \( \rho^2 \) and \( C \) give back the corresponding expressions obtained with standard QCD potential for \( D = 3 \) (see [27]) by replacing \( \gamma \) with \( \frac{4\mu^2}{r} \) and \( \sigma \) with \( h \), the values of \( \rho^2 \) and \( C \) are higher in the present case for \( D = 3 \) than the QCD results of \( \rho^2 \) and \( C \) as shown in table 2. This is due to the much lower value of \( \gamma \) for lower \( D \), as compared to that of the corresponding term \( \frac{4h^2}{r} \) in the previous work with the QCD Cornell potential (see [27]).

However, at higher \( D \) when \( \gamma \) becomes more and more dominant, our \( \rho^2 \) and \( C \) values go on decreasing and approaching their asymptotic values. This higher dimensional approach also, in turn, supports our consideration of treating Lüscher term as parent and confinement term as perturbation in the present formalism. However, our asymptotic values of \( \rho^2 \) and \( C \) are still higher than the standard QCD expectations of table 2. Here, we put forward our observation. We have taken the standard value of the string tension \( \sigma \) to be 0.178 GeV$^2$ [48], which is lower than the corresponding confinement parameter in the QCD Cornell potential. This \( \sigma \) value can vary [49] and results for \( \rho^2 \) and \( C \) may also change (improve) accordingly.

The asymptotic values of \( \rho^2 \) and \( C \) are found to be 14.5865 and 35.4608, respectively, for \( B \) meson.

4. Conclusion and remarks

From figure 2, we find that, with increasing \( D \), the \( \rho^2 \) and \( C \) values go on decreasing, eventually reaching the asymptotic limit. We have shown our results for \( B \) meson only as a representative case. A similar pattern of results should also follow for other \( B \) and \( D \) sector heavy–light mesons such as \( D_i, D_{i*}, B_i, B_{i*} \).

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Lastly, we conclude by commenting on our opting for ground state studies. It is true that higher (excited) state analysis ushers in a wider scope for exploring hadron spectra and distribution of hadron characteristic properties among its QCD constituents. Such a study of excited hadron spectra using lattice QCD is currently evolving [50]. Nevertheless, the ground state consideration also provides necessary information for the confirmation of the model and the study of hadron properties. Similar to our earlier works [9, 10, 27] with a three-dimensional potential model, here also we have opted for calculating only ground state (\( l = 0 \)) wave function by solving the \( D \)-dimensional Schrödinger equation for the string potential. We look at the present work as an extension of our previous works, to higher dimension. Development of formalism for higher state wave function and studies of meson excited states are currently under consideration.

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Appendix

From equation (21), the Schrödinger equation for first-order perturbation is

\[
\left[ \frac{d^2}{dr^2} + \frac{D-1}{r} \frac{d}{dr} + \frac{2\mu \gamma}{r} - 2\mu W^0 \right] \Psi'(r) = 2\mu (\sigma r + \mu_0 - W') \Psi^0(r). \tag{A.1}
\]

Following Dalgarno’s method [24–26] of perturbation, we obtain

\[
\Psi'(r) = (\sigma r + \mu_0) R(r). \tag{A.2}
\]

Taking

\[
R(r) = G(r) e^{-\mu r} \tag{A.3}
\]
and then expanding $G(r)$ in terms of power series,

$$G(r) = \sum_{n=0}^{\infty} A_{n} r^{\nu_{n} + \frac{n+2}{2}}. \tag{A.4}$$

we ultimately obtain

$$\Psi'(r) = (\sigma r + \mu_{0}) \sum_{n=0}^{\infty} A_{n} r^{\nu_{n} + \frac{n+2}{2}} e^{-\mu_{n} r}. \tag{A.5}$$

This upon expansion gives

$$\Psi'(r) = [\mu_{0} A_{0} r^{0} + (\mu_{0} A_{1} + \sigma A_{0}) r^{1} + (\mu_{0} A_{2} + \sigma A_{1}) r^{2} + \cdots] e^{-\mu_{n} r}, \tag{A.6}$$

and

$$\Psi'(r) = \sum_{n=0}^{\infty} C_{n} r^{n} r^{\frac{n+2}{2}} e^{-\mu_{n} r}, \tag{A.7}$$

where

$$C_{0} = \mu_{0} A_{0}, \tag{A.8}$$

$$C_{k+1} = \mu_{0} A_{k+1} + \sigma A_{k} \tag{A.9}$$

with $k = 0, 1, 2, 3, \ldots$

Using (A.7), (A.8) and (A.9) in (A.1), we obtain

$$C_{k+1} = 0 \quad \text{for} \quad k \neq 1, \tag{A.10}$$

$$C_{2} = \frac{\mu}{3} (\mu_{0} - W'). \tag{A.11}$$

Equation (A.7) then reduces to

$$\Psi'(r) = \left[ \mu_{0} A_{0} + \frac{\mu}{3} (\mu_{0} - W') r^{2} \right] r^{\frac{n+2}{2}} e^{-\mu_{n} r}. \tag{A.12}$$

Now, using equation (19), namely $\mu_{0} - W' = -\frac{\sigma D}{2\mu_{r}}$, equation (A.12) becomes

$$\Psi'(r) = \left[ \mu_{0} A_{0} - \frac{\mu}{3} \frac{\sigma D}{2\mu_{r}} r^{2} \right] r^{\frac{n+2}{2}} e^{-\mu_{n} r}. \tag{A.13}$$

As $A_{0}$ is undetermined, we set $\mu_{0} A_{0}$ to be equal to zero. This results in

$$\Psi'(r) = -\frac{\sigma D}{6\mu_{r}} r^{2} \frac{n+2}{2} r^{-\mu_{n} r}. \tag{A.14}$$

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