The R-operator for a modular double

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Abstract

We construct the R-operator—the solution of the Yang–Baxter equation acting in the tensor product \(\pi_1 \otimes \pi_2\) of two infinite-dimensional representations of Faddeev’s modular double. This R-operator intertwines the product of two L-operators associated with the modular double and it is built from three basic operators generating the permutation group of four parameters \(S_4\).

Keywords: modular double, Yang–Baxter equation, intertwining operators

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1. Introduction

We construct the solution of the Yang–Baxter equation

\[
R_{12}(u - v) R_{13}(u) R_{23}(v) = R_{23}(v) R_{13}(u) R_{12}(u - v)
\]

(1.1)

where the operators \(R_{ik}(u)\) are acting in the tensor product of two infinite-dimensional representations \(\pi_i \otimes \pi_{-i}\) of the modular double of \(U_q(s\ell_2)\).

Firstly, we solve the defining RLL-relation [1]

\[
R_{12}(u - v) L_1(u) L_2(v) = L_2(v) L_1(u) R_{12}(u - v)
\]

(1.2)

using as the main building blocks two operators. The first building block is an intertwining operator for the equivalent representations \(\pi_i\) and \(\pi_{-i}\), and the second building block is obtained from the intertwining operator by some duality transformation.

The proof that the R-operator obtained obeys the general Yang–Baxter equation is based on the Coxeter or star–triangle relation for two main building blocks. The operator \(R_{12}(u)\) was constructed in the work of Bytsko and Teschner [9] as a function of the operator argument. We hope that our construction is simpler and allows generalization to a higher rank [28, 29]. It shows that the R-operator can be represented in two forms: as a product of four simple and explicit operators and as an integral operator.
In the last section we consider some reductions of the operator obtained, \( \mathbb{R}_{12}(u) \), and reproduce the universal R-operator for the modular double from [2, 5].

2. The modular double and the intertwining operator

2.1. The modular double

We consider the modular double of \( U_q(sl_2) \) introduced by Faddeev in [2]. This algebra is formed from two sets of generators \( E, K, F \) and \( \tilde{E}, \tilde{K}, \tilde{F} \). The usual relations for \( E, K, F, \tilde{E}, \tilde{K}, \tilde{F} \)

\[
[E, F] = \frac{K^2 - K^{-2}}{q - q^{-1}}, \quad KE = qEK, \quad KF = q^{-1}FK,
\]

(2.1)

where \( q = e^{i\pi \tau} \) (\( \tau \in \mathbb{C} \) and it is not a rational number), are supplemented by similar relations for \( \tilde{E}, \tilde{K}, \tilde{F} \) with parameter \( \tilde{q} = e^{i\pi /\tau} \). The generators \( E, \tilde{E} \) commute with \( \tilde{E}, \tilde{F} \). The generators \( K \) and \( \tilde{K} \) commute, \( K \) anti-commutes with \( \tilde{E}, \tilde{F} \) and \( \tilde{K} \) anti-commutes with \( E, F, \tilde{E} \). This algebra possesses two central elements, which are Casimir operators. One of them has the form

\[
C = -(q - q^{-1})^2 FE - qK^2 - q^{-1}K^{-2} + 2,
\]

(2.2)

and the second is constructed from \( \tilde{E}, \tilde{K}, \tilde{F} \) and \( \tilde{q} \) and its explicit expression is similar to (2.2).

We shall use the parametrization \( \tau = \frac{\omega}{\omega'} \) where the complex numbers \( \omega \) and \( \omega' \) with positive imaginary parts are restricted by the relation \( \omega \omega' = -\frac{1}{4} \). Then

\[
q = \exp(i\pi \omega'/\omega), \quad \tilde{q} = \exp(i\pi \omega/\omega'),
\]

and the change \( q \rightleftharpoons \tilde{q} \) is equivalent to \( \omega \rightleftharpoons \omega' \). There exists another standard parametrization

\[
\omega = \frac{i}{2b}; \quad \omega' = \frac{ib}{2},
\]

where the change \( q \rightleftharpoons \tilde{q} \) is equivalent to \( b \rightleftharpoons b^{-1} \) [3, 9–11]. The representations of the modular double, introduced in [2], were investigated in [3–5]. In the following we deal with representation \( \pi_s \) of the modular double generators by finite-difference operators \( K_s = \pi_s(K), E_s = \pi_s(E), F_s = \pi_s(F) \) in the space of entire functions rapidly decaying at infinity along contours parallel to the real line. It is parametrized by one parameter \( s \) which we refer to as a spin, and generators are given by the explicit formulae [5, 9]

\[
(q - q^{-1})E_s = e^{-i\frac{s}{2b}} \left[ e^{-i\frac{s}{2b}(p + s - \omega')} - e^{i\frac{s}{2b}(p - s - \omega')} \right],
\]

(2.3)

\[
(q - q^{-1})F_s = e^{-i\frac{s}{2b}} \left[ e^{i\frac{s}{2b}(p + s + \omega')} - e^{-i\frac{s}{2b}(p + s + \omega')} \right],
\]

where \( p \) denotes the momentum operator in the coordinate representation: \( p = \frac{1}{2\pi} \partial_x \) and \( \omega'' = \omega + \omega' \). The formulae for generators \( \tilde{K}_s, \tilde{E}_s, \tilde{F}_s \) are obtained by the simple interchange \( \omega \rightleftharpoons \omega' \). In the representation \( \pi_s \), Casimir operators (2.2) take the form

\[
C_s = 4 \cos^2 \left( \frac{\pi s}{2b} \right), \quad \tilde{C}_s = 4 \cos^2 \left( \frac{\pi s}{2b'} \right).
\]

(2.4)

The formula (2.4) implies that representations \( \pi_s \) and \( \pi_{-s} \) are equivalent. In order to show this explicitly, we construct the corresponding intertwining operator \( W \) as a solution of the defining equations

\[
WK_s = K_{-s}W, \quad WE_s = E_{-s}W, \quad WF_s = F_{-s}W
\]

(2.5)

and similar equations with generators \( \tilde{K}_s, \tilde{E}_s, \tilde{F}_s \). The explicit construction of an intertwining operator is given in [3] and we rederive it here for convenience. Intertwining operator \( W \) will be a basic building block in the construction of the general R-operator which solves the Yang–Baxter relation (1.1).
The defining system (2.5) is equivalent to a set of functional relations which fix the intertwining operator \( W \) unambiguously. The relations \( W K_x = K_{-x} W \), \( W \bar{K}_x = \bar{K}_{-x} W \) imply that the intertwining operator is a function of the momentum operator, \( W = W(p) \). Then relations \( WE_x = E_{-x} W \), \( WF_x = F_{-x} W \) and their dual lead to the finite-difference equations
\[
W(p - \omega') = \frac{\cos \frac{\pi}{2s} (p + s)}{\cos \frac{\pi}{2s} (p - s)}, \quad W(p - \omega) = \frac{\cos \frac{\pi}{2s} (p + s)}{\cos \frac{\pi}{2s} (p - s)}.
\]
(2.6)
The solution of these finite-difference equations is given in terms of some special functions.

2.2. Special functions

We shall use two basic special functions. The first one is the non-compact quantum dilogarithm which has the following integral representation:
\[
\gamma(z) = \exp \left( -\frac{1}{4} \int_{-\infty}^{+\infty} dt \frac{e^{iz}}{t \sin(\omega t) \sin(\omega' t)} \right),
\]
(2.7)
where the contour goes above the singularity at \( t = 0 \). This function is closely related to double-sine function of Barnes [6], and in the context of quantum integrable systems Faddeev pointed out, in [7], its remarkable properties. The key formulae for \( \gamma(z) \) are given in [10–12].

We shall use the notation
\[
\omega'' = \omega + \omega', \quad \beta = \frac{\pi}{12} \left( \frac{\omega}{\omega'} + \frac{\omega'}{\omega} \right).
\]
(2.8)
The function \( \gamma(z) \) respects a pair of finite-difference equations
\[
\frac{\gamma(z + \omega')}{\gamma(z - \omega')} = 1 + e^{-\pi z}; \quad \frac{\gamma(z + \omega)}{\gamma(z - \omega)} = 1 + e^{-\pi z}
\]
and a reflection relation
\[
\gamma(z)\gamma(-z) = e^{i\beta} e^{i\pi z}.
\]
(2.9)
The asymptotic behaviour is \( \gamma(z) \to 1 \) for \( \Re(z) \to +\infty \) and a reflection relation can be used to get the asymptotic behaviour for \( \Re(z) \to -\infty \). The second function is
\[
D_a(z) = e^{-2\pi ia} \gamma(z + a) / \gamma(z - a).
\]
(2.10)
In fact it coincides with the Faddeev–Volkov R-matrix [8, 26], and it is extensively used in the paper [9] where another parametrization and basic special function are chosen:
\[
\omega = \frac{i}{2b}; \quad \omega' = \frac{ib}{2}; \quad \gamma(z) = e^{\frac{\pi}{2} + iz} w_b(z).
\]
The function \( D_a(z) \) is even and it obeys a simple reflection relation:
\[
D_a(z) = D_a(-z); \quad D_a(z)D_{-a}(z) = 1
\]
(2.11)
and a pair of finite-difference equations:
\[
\frac{D_a(z - \omega')}{D_a(z + \omega')} = \frac{\cos \frac{\pi}{2s} (z - b)}{\cos \frac{\pi}{2s} (z + b)}, \quad \frac{D_a(z - \omega)}{D_a(z + \omega)} = \frac{\cos \frac{\pi}{2s} (z - b)}{\cos \frac{\pi}{2s} (z + b)}.
\]
(2.12)
Note that the functions \( \gamma(z) \) and \( D_a(z) \) are invariant at \( \omega \leftrightarrow \omega' \).

Now we return to the equations (2.6) for the intertwining operator. Comparing (2.6) with (2.12) one concludes that
\[
W(p) = D_x(p).
\]
(2.13)
Below we shall use the notation $L$. Note that in the notation $L$ it is possible to represent $W$ as an integral operator. Indeed, due to (2.13) and (2.14) one has
\begin{align}
W \Phi(x) &= A(s - \omega) \int_{-\infty}^{+\infty} dx' D_{s-\omega}(-x') e^{2\pi i / p} \Phi(x) = A(s - \omega) \\
&\times \int_{-\infty}^{+\infty} dx' D_{s-\omega}(x-x') \Phi(x').
\end{align}

3. The $L$-operator and its factorization

The $L$-operator is constructed from generators of the modular double taken in the representation $\pi_1$ (2.3) and has the form [9]
\begin{equation}
L(u) = \begin{pmatrix}
e^{\pi i s K_s} - e^{-\pi i s K_s^{-1}} & f_p \\
e^{\pi i s K_s^{-1}} - e^{-\pi i s K_s}
\end{pmatrix}.
\end{equation}

The $L$-operator respects the standard intertwining relation with the $4 \times 4$ trigonometric $R$-matrix which is equivalent to the set of commutation relations (2.1). The second $L$-operator is obtained from $L(u)$ through the interchange $\omega \leftrightarrow \omega'$. $\tilde{L}(u) = L(u)|_{\omega = \omega'}$. In the following we indicate formulae only for the $L$-operator (3.1), and all relations for the $\tilde{L}$-operator have the same form with $\omega \leftrightarrow \omega'$.

The $L$-operator (3.1) can be represented in the factorized form
\begin{align}
L(u_1, u_2) &= \begin{pmatrix} U_2 & \omega - \omega' \\
-U_2^{-1} e^{\frac{\pi i s}{2}} & U_2 e^{\frac{\pi i s}{2}} \end{pmatrix}\begin{pmatrix} e^{\frac{\pi i s}{2}} (p-\omega) & 0 \\
0 & e^{\frac{\pi i s}{2}} (p-\omega') \end{pmatrix}\begin{pmatrix} -U_1 & U^{-1}_1 e^{\frac{-\pi i s}{2}} \\
-U_1^{-1} & U e^{\frac{-\pi i s}{2}} \end{pmatrix} \\
U_1 &= e^{\frac{\pi i s}{2}}, \quad U_2 = e^{\frac{\pi i s}{2}}
\end{align}

where we have introduced parameters $u_1$ and $u_2$ instead of $u$ and $s$:
\begin{equation}
u_1 = u + \frac{s + \omega - \omega'}{2}; \quad u_2 = u - \frac{s + \omega - \omega'}{2}.
\end{equation}

Note that in the notation $L(u)$ we omit for simplicity the dependence on the spin parameter $s$. Below we shall use the notation $L(u_1, u_2)$ to show all parameters explicitly and the shorthand notation for the building blocks in the factorized representation
\begin{equation}
L(u_1, u_2) = M_{u_2}(x) H(p) N_{u_1}(x)
\end{equation}

where $H(p) = \text{diag}(e^{\frac{\pi i s}{2} (p-\omega)}, e^{\frac{\pi i s}{2} (p-\omega')})$ and
\begin{align}
M_{u_2}(x) &= \begin{pmatrix} U & -U^{-1} \\
-U^{-1} e^{\frac{\pi i s}{2}} & U e^{\frac{\pi i s}{2}} \end{pmatrix}, \quad N_{u_1}(x) = \begin{pmatrix} -U & U^{-1} e^{\frac{-\pi i s}{2}} \\
-U^{-1} & U e^{\frac{-\pi i s}{2}} \end{pmatrix}.
\end{align}

3.1. $L^\pm$-operators

Taking one of the parameters to infinity we obtain reduced $L$-operators
\begin{align}
e^{\frac{\pi i s}{2}} L(u_1, u_2) &\rightarrow L^+ (u_1) \equiv \begin{pmatrix} 1 & 0 \\
0 & e^{\frac{\pi i s}{2}} \end{pmatrix} H(p) N_{u_1}(x) \quad \text{at} \quad u_2 \rightarrow +\infty, \\
e^{-\frac{\pi i s}{2}} L(u_1, u_2) &\rightarrow L^- (u_2) \equiv M_{u_2}(x) H(p) \begin{pmatrix} 0 & 1 \\
0 & e^{-\frac{\pi i s}{2}} \end{pmatrix} \quad \text{at} \quad u_1 \rightarrow +\infty.
\end{align}
or explicitly
\[
\begin{align*}
L^+(u) &= \begin{pmatrix} e^{-\frac{\pi}{2} (p-\omega')} & 0 \\ e^{\frac{\pi}{2} (p-\omega')} & 0 \end{pmatrix} \begin{pmatrix} -U & U^{-1} e^{-\frac{\pi}{2} \lambda} \\ -U^{-1} e^{\frac{\pi}{2} \lambda} & U \end{pmatrix}, \\
L^-(u) &= \begin{pmatrix} U & -U^{-1} e^{\frac{\pi}{2} \lambda} \\ -U^{-1} e^{\frac{\pi}{2} \lambda} & U e^{-\frac{\pi}{2} \lambda} \end{pmatrix} \begin{pmatrix} -e^{-\frac{\pi}{2} (p-\omega')} & 0 \\ 0 & e^{\frac{\pi}{2} (p-\omega')} \end{pmatrix}.
\end{align*}
\]

The initial operator \(L(u)\) and operators \(L^\pm(u)\) respect the same intertwining relations with the trigonometric \(R\)-matrix. Using formulae
\[
e^{-i\pi \hat{P}_x} e^{i\pi \hat{P}_x} = e^{i\pi \hat{P}_x} e^{-i\pi \hat{P}_x},
\]
one can check that \(L^- (u)\) and \(L^+(u)\) are unitarily equivalent:
\[
e^{-i\pi \hat{P}_x} L^+(u) e^{i\pi \hat{P}_x} = L^-(u). \tag{3.4}
\]

It is possible to reconstruct the initial \(L\)-operator in terms of \(L^\pm\)-operators:
\[
- e^{\frac{\pi}{2} \omega''} L_1(u, v) = e^{-i\pi (p_2-x_12)} \hat{L}_1^-(v) L_2^+(u) e^{i\pi (p_2-x_12)} \begin{pmatrix} e^{\frac{\pi}{2} \hat{p}_1} & 0 \\ 0 & e^{-\frac{\pi}{2} \hat{p}_1} \end{pmatrix}, \tag{3.5}
\]
\[
- e^{\frac{\pi}{2} \omega''} L_2(u, v) = e^{i\pi (p_1-x_12)} \hat{L}_1^+(v) L_2^+(u) e^{-i\pi (p_1-x_12)} \begin{pmatrix} e^{\frac{\pi}{2} \hat{p}_1} & 0 \\ 0 & e^{-\frac{\pi}{2} \hat{p}_1} \end{pmatrix}, \tag{3.6}
\]

where the operators \(x, p\) entering \(L_1^\pm, L_2^\pm\) are replaced by \(x_k, p_k\) and \(x_{12} = x_1 - x_2\). The proof is straightforward and it is reduced to a simple direct check using (3.3). This variant of factorization for the Lax operator is an analogue of the factorization obtained in the context of the chiral Potts model [16–20], and it expresses the equivalence of the corresponding representations of the RLL-algebra [21].

Due to relation (3.4) it is possible to use just one reduced operator as a building block. This leads to yet another variant of factorization [22]. We rewrite (3.5), taking into account (3.4):
\[
- e^{\frac{\pi}{2} \omega''} \hat{L}_1(u, v) = L_1^+(v) L_2^+(u) \begin{pmatrix} e^{\frac{\pi}{2} (p_1+x_12)} & 0 \\ 0 & e^{-\frac{\pi}{2} (p_1+x_12)} \end{pmatrix}
\]

where \(\hat{L}_1(u, v)\) is obtained from \(L_1(u, v)\) by a canonical transformation:
\[
\begin{align*}
x_1 &\rightarrow X_1 \equiv p_1 + x_1 \\
p_1 &\rightarrow P_1 \equiv -x_1 + p_2 + x_2.
\end{align*}
\]

Under this canonical transformation, \(P_2 \equiv p_1 + x_{12}\) and it commutes with \(X_1\) and \(P_1\). We can impose the constraint \(p_1 + x_{12} = 0\) or, equivalently, take the zero eigenspace of the operator \(P_2\), which reproduces the result of [22].

Let us note that the reduced operators \(L^\pm\) are rather well known [14–16]. In order to render them in the form used in the literature,
\[
L^+(u) = -e^{\frac{\pi}{2} (\omega + \omega')} \begin{pmatrix} \hat{u} & -e^{\frac{i}{2} \lambda^2} \hat{v}^{-1} \\ e^{\frac{i}{2} \lambda^2} \hat{u} \hat{v}^{-1} \end{pmatrix} \begin{pmatrix} -e^{\frac{i}{2} \lambda^2} & 0 \\ 0 & e^{\frac{i}{2} \lambda^2} \end{pmatrix}.
\]

we need a Weyl pair \(\hat{u}, \hat{v}\):
\[
\hat{u} = e^{\frac{i}{2} \lambda}, \quad \hat{v} = e^{\frac{i}{2} \lambda} e^{\frac{\pi}{2} (p-\omega')}, \quad \hat{u} \hat{v} = q \hat{v} \hat{u}.
\]
4. Basic intertwining relations

Note that interchange \( u_i \leftrightarrow u_2 \) is equivalent to the substitution \( s \rightarrow -s \). L-operators (3.1) are linear in generators \( E_i \), \( K_i \), \( F_i \) and \( \bar{E}_i \), \( \bar{F}_i \), \( \bar{K}_i \), so the set of intertwining relations (2.5) is equivalent to a single matrix relation

\[
D_{u_2 \rightarrow u_1}(p)L(u_1, u_2) = L(u_2, u_1)D_{u_2 \rightarrow u_1}(p)
\]

or, more explicitly, taking into account factorization relation (3.3),

\[
D_{u_2 \rightarrow u_1}(p)M_{u_2}(x)H(p)N_{u_2}(x) = M_{u_1}(x)H(p)N_{u_1}(x)D_{u_2 \rightarrow u_1}(p).
\]

In the following we indicate formulae only for the L-operator (3.1) and all relations for the L-operator have the same form with \( \omega \rightarrow \omega' \).

4.1. The duality transformation

To proceed further let us note that the transformation \( p \rightarrow -x \) and \( x \rightarrow p \) (Fourier transformation) preserves the canonical commutation relation. Consequently, applying this transformation to generators in the representation \( \pi_s (2.3) \) one finds that they continue to fulfill the commutation relations of the modular double (2.1). We refer to the corresponding L-operator as the dual. The intertwining relation (4.2) subjected to this duality transformation takes the form

\[
D_{u_2 \rightarrow u_1}(x) \cdot M_{u_2}(p)H(-x)N_{u_2}(p) = M_{u_1}(p)H(-x)N_{u_1}(p) \cdot D_{u_2 \rightarrow u_1}(x)
\]

where we used that \( D_{u_2 \rightarrow u_1}(-x) = D_{u_2 \rightarrow u_1}(x) \) (see (2.11)). This observation allows us to prove the second intertwining relation

\[
D_{u_1 \rightarrow u_2}(v_1, v_2) \cdot L_1(u_1, u_2) = L_1(v_2, u_2) \cdot D_{u_1 \rightarrow u_2}(v_1, v_2)
\]

which will be used in the next section to construct the general R-operator.

Indeed, like for (4.3), the transformation

\[
p \rightarrow x_2 \equiv x_2 - x_1, \quad x \rightarrow p_1
\]

of the relation (4.2) leads to the intertwining relation

\[
D_{u_1 \rightarrow u_2}(x_1, x_2)M_{u_1}(p_1)H(x_1)N_{u_2}(p_1) = M_{u_1}(p_1)H(x_2)N_{u_2}(p_1)D_{u_1 \rightarrow u_2}(x_2).
\]

Further we note that the dual L-operator in the previous formula can be factorized as follows:

\[
M_{u_1}(p_1)H(x_1)N_{u_2}(p_1) = -i e^{\frac{\pi \omega}{\omega'}} H(p_1) N_{u_1}(x_1) M_{u_2}(x_2) H^{-1}(p_1) \begin{pmatrix} -e^{\frac{\pi \omega'}{\omega}} & 0 \\ 0 & e^{-\frac{\pi \omega'}{\omega}} \end{pmatrix}.
\]

Substituting the latter formula in (4.6), and taking into account that \( D_{u_1 \rightarrow u_2}(x_1, x_2) \) commutes with \( p_1 + p_2 \) and so

\[
H(p_2)D_{u_1 \rightarrow u_2}(x_1, x_2)H^{-1}(p_2) = H^{-1}(p_1)D_{u_1 \rightarrow u_2}(x_1, x_2)H(p_1),
\]

one obtains

\[
D_{u_1 \rightarrow u_2}(x_1, x_2)H(p_1)N_{u_1}(x_1)M_{u_2}(x_2)H(p_2) = H(p_1)N_{u_1}(x_1)M_{u_2}(x_2)H(p_2)D_{u_1 \rightarrow u_2}(x_1, x_2).
\]

This relation is equivalent to (4.4) in view of factorization relation (3.3).
4.2. Basic intertwining relations for $L^\pm$-operators

We take into account local factorization (3.5) and rewrite intertwining relation (4.1) using the formula $e^{i\pi(p_2-s_1)^2}p_1 e^{-i\pi(p_2-s_1)^2} = p_1 + p_2 - x_{12}$ in the following form:

$$D_{v-u}(p_1 + p_2 - x_{12})L_1^{-}(v)L_2^{+}(u) = L_1^{-}(u)L_2^{+}(v)D_{v-u}(p_1 + p_2 - x_{12}). \quad (4.7)$$

The relation between two reduced L-operators (3.4) allows us to rewrite it in another form:

$$D_{v-u}(x_{12})L_1^{+}(v)L_2^{-}(u) = L_1^{+}(u)L_2^{-}(v)D_{v-u}(x_{12}), \quad (4.8)$$

and also

$$D_{v-u}(p_2 - x_{12})L_1^{-}(v)L_2^{+}(u) = L_1^{-}(u)L_2^{+}(v)D_{v-u}(p_2 - x_{12}),$$

$$D_{v-u}(p_1 - x_{12})L_1^{-}(v)L_2^{-}(u) = L_1^{-}(u)L_2^{-}(v)D_{v-u}(p_1 - x_{12}).$$

We should note that the intertwining relations considered for $L^\pm$ and the expressions for the corresponding intertwining operators are direct counterparts of Volkov’s solution of the same problem given in [13].

5. The R-operator

Now we proceed to construct the general R-operator acting in the tensor product of two representations $\pi_u \otimes \pi_v$ (2.3). To do this we solve the RLL-relation (1.2). It is convenient to extract from the R-matrix the permutation operator $P_{12}(u) = P_{12}R_{12}(u)$, where the permutation operator interchanges arguments: $P_{12}(x_1, x_2) = \Phi(x_2, x_1)$. Then we obtain the RLL-relation with the L-operator (3.1):

$$R(u) L_1(u_1, u_2) L_2(v_1, v_2) = L_1(v_1, v_2) L_2(u_1, u_2) R(u) \quad (5.1)$$

and the analogous one for the second L-operator:

$$R(u) \tilde{L}_1(u_1, u_2) \tilde{L}_2(v_1, v_2) = \tilde{L}_1(v_1, v_2) \tilde{L}_2(u_1, u_2) R(u). \quad (5.2)$$

Here subscripts 1, 2 in L-operators denote distinct quantum spaces in $\pi_u \otimes \pi_v$, where corresponding operators act nontrivially. The spin parameters $s_1, s_2$ and spectral parameters $u, v$ are related to four parameters appearing in the RLL-relation according to (3.2). We combine these parameters into one set in the following order: $u \equiv (u_2, u_1, v_2, v_1)$ and use the notation $R(u)$ to show explicitly the dependence of the R-operator on all parameters.

Let us emphasize that the general R-operator is the same in (5.1) and (5.2). As we will see shortly, it is invariant under $\omega \rightarrow \omega'$. Further, we propose two constructions of the R-operator.

5.1. The first construction

Equation (5.1) admits a natural interpretation: the R-operator interchanges the set of parameters $(u_1, u_2)$ in the first L-operator with the set of parameters $(v_1, v_2)$ in the second L-operator. The operator $R(u)$ corresponds to a particular permutation $r$ in the group of permutations of four parameters $S_4$:

$$r \rightarrow R(u); \quad ru \equiv r(u_2, u_1, v_2, v_1) = (v_2, v_1, u_2, u_1).$$

Any permutation from the group $S_4$ can be composed from the elementary transpositions $s_1$, $s_2$, and $s_3$:

$$s_1 u = (u_1, u_2, v_2, v_1), \quad s_2 u = (u_2, v_2, u_1, v_1), \quad s_3 u = (u_2, u_1, v_1, v_2).$$
which interchange only two nearest neighbouring elements:

\[
\begin{align*}
  s_1 \quad s_2 \quad s_3 \\
  (u_2, u_1, v_2, v_1); \
  (u_2, u_1, v_2, v_1).
\end{align*}
\]

It is natural to search for the operators representing these elementary transpositions in L-operators. Namely, we demand that the \(S_i(u)\) obey the following defining relations:

\[
\begin{align*}
  S_1(u) L_1(u_1, u_2) &= L_1(u_2, u_1) S_1(u); \\
  S_3(u) L_2(v_1, v_2) &= L_2(v_2, v_1) S_3(u), \quad (5.3) \\
  S_2(u) L_1(u_1, u_2) L_2(v_1, v_2) &= L_1(v_2, u_2) L_2(v_1, u_1) S_2(u). \quad (5.4)
\end{align*}
\]

In the previous section we have already seen these relations. Indeed (4.1) leads to

\[
\begin{align*}
  D_{u_2 \rightarrow u_1}(p_1) L_1(u_1, u_2) &= L_1(u_2, u_1) D_{u_2 \rightarrow u_1} (p_1); \\
  D_{v_2 \rightarrow v_1}(p_2) L_2(v_1, v_2) &= L_2(v_2, v_1) D_{v_2 \rightarrow v_1} (p_2)
\end{align*}
\]

and (4.4) can be reformulated as

\[
D_{u_1 \rightarrow u_2}(x_{12}) L_1(u_1, u_2) L_2(v_1, v_2) = L_1(v_2, u_2) L_2(v_1, u_1) D_{u_1 \rightarrow u_2}(x_{12}),
\]

so we have the following identification:

\[
S_1(u) = D_{u_2 \rightarrow u_1}(p_1); \quad S_2(u) = D_{v_2 \rightarrow v_1}(p_2); \quad S_3(u) = D_{v_2 \rightarrow v_1}(p_2).
\]

The operator \(R(u)\) corresponds to a particular permutation \(r = s_3 s_1 s_3 s_2\) in the group \(S_4\). We have the following correspondence between permutations and operators:

\[
s_i \rightarrow S_i(u); \quad s_j s_k \rightarrow S_i(s_j s_k u) S_j(s_j s_k u) S_k(s_j s_k u) \ldots, \quad (5.5)
\]

and it is easy to see that the composite operator

\[
R(u) = S_2(s_1 s_3 s_2 u) S_1(s_3 s_2 u) S_3(s_2 u) S_2(u)
\]

satisfies equation (5.1). We have, in explicit form,

\[
R(u) = D_{u_2 \rightarrow u_1}(x_{12}) D_{u_2 \rightarrow u_1}(p_1) D_{u_2 \rightarrow u_1}(p_2) D_{u_2 \rightarrow u_1}(x_{12}). \quad (5.6)
\]

This operator is invariant under \(\omega \rightarrow \omega'\) so it solves both RLL-relations (5.1), (5.2).

Let us rewrite this expression using the spectral parameter and initial spin parameters. The R-operator depends only on the difference of spectral parameters \(u - v\) so, taking into account (3.2) and making the substitution \(u - v \rightarrow u\), one obtains

\[
R_{12}(u) = D_{u_2 \rightarrow u_1}(x_{12}) D_{u_2 \rightarrow u_1}(p_2) D_{u_2 \rightarrow u_1}(p_1) D_{u_2 \rightarrow u_1}(x_{12}). \quad (5.7)
\]

In view of (2.15) the R-operator can also be rewritten as an integral operator:

\[
R_{12}(u) \Phi(x_1, x_2) = A(-\omega'' - u + (s_1 - s_2)/2) A(-\omega'' - u + (s_1 - s_2)/2)
\]

\[
\times D_{u_2 \rightarrow u_1}(x_{12}) \cdot \int dx'_1 dx'_2 D_{u_2 \rightarrow u_1}(x_{12}) D_{u_2 \rightarrow u_1}(p_2) D_{u_2 \rightarrow u_1}(p_1) D_{u_2 \rightarrow u_1}(x_{12})
\]

\[
\times \Phi(x'_1, x'_2). \quad (5.8)
\]

Note that we use a common notation \(R_{12}(u)\) for the R-operator. Subscripts 1, 2 denote quantum spaces where the operator acts nontrivially, and we indicate only the dependence on the spectral parameter \(u\), skipping all other parameters for simplicity. The R-operator can be represented in two forms which are complementary to each other: as a function of operator argument (5.6) and as an integral operator (5.7).

We have constructed operator \(R_{12}(u)\) which solves RLL-relations (5.1), (5.2) and now we have to check that the corresponding operator with permutation \(R_{12}(u) = P_{12} R_{12}(u)\) respects the Yang–Baxter equation

\[
R_{12}(u - v) R_{13}(u) R_{23}(v) = R_{23}(v) R_{13}(u) R_{12}(u - v).
\]
in the tensor product $\pi_1 \otimes \pi_2 \otimes \pi_3$. We state that if following two Coxeter relations hold:

\[ s_1 s_2 s_1 = s_2 s_1 s_2 \rightarrow S_1(s_2 s_1 u) S_2(s_1 u) S_1(u) = S_2(s_1 s_2 u) S_1(s_2 u) S_2(u), \tag{5.9} \]
\[ s_2 s_3 s_2 = s_3 s_2 s_3 \rightarrow S_2(s_3 s_2 u) S_3(s_2 u) S_2(u) = S_3(s_2 s_3 u) S_2(s_3 u) S_3(u), \tag{5.10} \]
then the Yang–Baxter equation is satisfied. For more details see [23–25].

Relations (5.8) and (5.9) are equivalent to the relations

\[ D_u(p_1)D_{a+v}(x_1)D_v(p_1) = D_v(x_1)D_{a+v}(p_1)D_u(x_1), \tag{5.11} \]
\[ D_u(p_2)D_{a+v}(x_1)D_v(p_2) = D_v(x_1)D_{a+v}(p_2)D_u(x_1) \tag{5.12} \]
which are reduced to the known star–triangle relation [8, 12]

\[ D_u(p) D_{a+v}(x) D_v(p) = D_v(x) D_{a+v}(p) D_u(x). \tag{5.13} \]

This relation can be rewritten in an integral form due to (2.15):

\[ \frac{A(u + v)}{A(u)A(v)} \int_{-\infty}^{+\infty} dx'' D_{-a'-v}(x - x'')D_{a'}(x'')D_{-a-v}(x'' - x') = D_v(x)D_{-a'-v}(x - x')D_u(x') \]

and it is a special example of the integral relation [9, 26] for the $D$-functions

\[ A(a)A(b)A(c) \int_{-\infty}^{+\infty} D_u(z - z_1)D_v(z - z_2)D_v(z - z_3) = D_{-a'-v}(z_2 - z_3)D_{-a-v}(z_3 - z_1)D_{-a'-v}(z_1 - z_2) \]
\[ \text{at } a + b + c = -2a''. \tag{5.14} \]

The generalization of the Faddeev–Volkov type of solution of the star–triangle relation is discussed in [27].

### 5.2. The second construction

In this section we shall reproduce the solution of the RLL-relation using a different approach based on the factorization of the $L$-operator in terms of $L^\pm$-operators (3.5), (3.6).

The initial RLL-relation for the operator $R(u)$

\[ R(u) L_1(u_1, u_2) L_2(v_1, v_2) = L_1(v_1, v_2) L_2(u_1, u_2) R(u) \]

is equivalent to the relation

\[ R^{ab}(u) L_1^a(u_1) L_2^a(v_1) = L_1^b(v_2) L_2^b(u_2) R^{ab}(u), \tag{5.15} \]

for the operator $R^{ab}(u)$ which is obtained from $R(u)$ by the similarity transformation

\[ R^{ab}(u) = e^{i\pi(p_b - s_a)^2} e^{-i\pi(p_a - s_b)^2} R(u) e^{i\pi(p_b - s_a)^2} e^{-i\pi(p_a - s_b)^2}. \]

Here $x_{ab} = x_a - x_1, x_{ba} = x_2 - x_a$ and the operators $x, p$ entering $L_1^\pm$ and $L_2^\pm$ are replaced by $x_a, p_a$ and $x_b, p_b$, respectively.

To transform the initial RLL-relation to the form (5.14) we multiply it by a simple diagonal matrix: $\text{diag}(e^{-\frac{i\pi}{2} p_a}, e^{\frac{i\pi}{2} p_a})$ from the left-hand side and by $\text{diag}(e^{-\frac{i\pi}{2} p_b}, e^{\frac{i\pi}{2} p_b})$ from the right-hand side, and take into account local factorization (3.5)–(3.6).

Using the interchange relations (4.7) and (4.8) it is easy to check that the operator

\[ R^{ab}(u) = D_{a_2-a_1}(x_1) D_{a_1-a_2}(p_a + p_1 - x_{a_1}) D_{a_1-a_2}(p_2 + p_b - x_{a_2}) D_{a_2-a_1}(x_1) \]

solves the relation (5.14). Then after the similarity transformation needed we reproduce the expression for the operator $R(u)$ from the previous section:

\[ R(u) = e^{i\pi(x_0 - s_0)^2} e^{-i\pi(x_0 - s_0)^2} R^{ab}(u) e^{i\pi(x_0 - s_0)^2} e^{-i\pi(x_0 - s_0)^2} \]
\[ = D_{a_2-a_1}(x_1) D_{a_1-a_2}(p_1) D_{a_1-a_2}(p_2) D_{a_2-a_1}(x_1). \]
This construction is based on two local factorization formulae (3.5), (3.6), the connection between two reduced L-operators (3.4) and the expression for the intertwining operator (2.13).

Note that the first construction of the R-operator is the same as for the elliptic modular double [25]. In principle all formulae needed can be obtained by some reduction from the elliptic case, but the direct approach is much simpler.

We do not know the analogue of the second construction in the elliptic situation. In the appendix we show for completeness how it works for the group \( \text{SL}(2, \mathbb{C}) \).

6. Reductions of the general R-operator

In this section we show that the universal R-matrix \( \mathcal{R}_{12} \) and its Yang–Baxterized version \( \mathcal{R}_{12}(u) \) can be extracted from the R-operator constructed. Let us perform certain reductions of the R-operator with the permutation \( \mathbb{P}_{12}(u) = \mathbb{P}_{12}(u) \) where \( \mathbb{P}_{12}(u) \) is given by (5.6).

Using asymptotic behaviour: \( \gamma(z) \to 1 \) for \( \Re(z) \to +\infty \), and the reflection formula (2.11), we obtain

\[
e^{2\pi i u v} \mathcal{R}_{12}(u) \mathcal{R}_{12}(u + v) e^{-2\pi i u p_1} \to \mathcal{R}_{12}(u) \quad \text{at} \quad v \to +\infty
\]

where

\[
\mathcal{R}_{12}(u) = e^{2\pi i u p_1} \mathcal{R}_{12}(u) e^{2\pi i u p_1},
\]

where

\[
\mathcal{R}_{12} = \mathbb{P}_{12} \frac{e^{-\pi i (s_1 + s_2)s_{12}} e^{-\pi i (s_1 - s_2)s_{12}}}{\gamma(-x_{12} + \frac{3i}{2} - u) \gamma(p_2 - \frac{i}{2}) \gamma(-x_{12} - \frac{3i}{2} - u)} \frac{e^{-\pi i (s_1 + s_2)s_{12}}}{\gamma(-p_1 + \frac{3i}{2}) \gamma(x_{12} - \frac{i}{2})}.
\]

The operator \( \mathcal{R}_{12} \) respects the Yang–Baxter equation without spectral parameters:

\[
\mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12} = \mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23}.
\]

As a direct consequence of this relation, we immediately obtain the Yang–Baxter equation

\[
\mathcal{R}_{23}(v) \mathcal{R}_{13}(u) \mathcal{R}_{12}(u - v) = \mathcal{R}_{12}(u - v) \mathcal{R}_{13}(u) \mathcal{R}_{23}(v)
\]

for the operator \( \mathcal{R}_{12}(u) \) (6.1).

The operator (6.2) plays the role of the universal R-matrix for the modular double. In this form it first appeared in the paper [5] where it was obtained after some nontrivial calculations starting with the expression for the universal R-matrix in terms of quantum algebra generators [2].

The simplest way to prove (6.3) is to start from the Yang–Baxter equation

\[
\mathbb{R}_{23}(v) \mathbb{R}_{13}(u) \mathbb{R}_{12}(u - v) = \mathbb{R}_{12}(u - v) \mathbb{R}_{13}(u) \mathbb{R}_{23}(v)
\]

and to perform the appropriate reductions. We have

\[
\mathbb{R}_{23}(v) e^{2\pi i u p_1} \mathbb{R}_{13}(u) e^{-2\pi i u p_1} = e^{2\pi i u p_1} \mathbb{R}_{13}(u) e^{-2\pi i u p_1} \mathbb{R}_{23}(v).
\]
so it is possible to take the limit $u \to +\infty$ in the underlined factors by means of (5.15). In view of the translation invariance of the operator (6.2): $[\mathcal{R}_{12}, p_1 + p_2] = 0$, one obtains

$$e^{2\pi i p_1} \mathcal{R}_{23}(v) e^{-2\pi i p_1} \mathcal{R}_{13} \mathcal{R}_{12} = \mathcal{R}_{12} \mathcal{R}_{13} e^{2\pi i p_2} \mathcal{R}_{23}(v) e^{-2\pi i p_1},$$

and taking the limit $v \to \infty$ in the underlined factors we obtain the Yang–Baxter relation (6.3).

Note that there exists the additional three-term relation

$$\mathbb{R}_{23}(v) \mathcal{R}_{13}(u) \mathcal{R}_{12}(u - v) = \mathcal{R}_{12}(u - v) \mathcal{R}_{13}(u) \mathbb{R}_{23}(v),$$

and the corresponding analogue of the RLL-relation

$$\mathcal{R}_{12}(u - v) \ell_1(\ell_2(v) \ell_1(\mathcal{R}_{12}(u - v),$$

where operators $\ell$, $\overline{\ell}$ are obtained by some reduction from the standard L-operator:

$$e^{-\frac{2\pi i}{\omega} v} e^{2\pi i p_1} L(u + v) e^{-2\pi i p_1} \to \ell(u) \equiv \begin{pmatrix} e^{2\pi i p_1} & 0 \\ e^{-2\pi i p_1} & e^{-2\pi i p_1} \ell \end{pmatrix}, \text{ for } v \to +\infty,$$

$$e^{-\frac{2\pi i}{\omega} v} e^{2\pi i p_1} L(u + v) e^{-2\pi i p_1} \to \overline{\ell}(u) \equiv \begin{pmatrix} e^{-2\pi i p_1} \ell^{-1} & -\ell \\ 0 & e^{-2\pi i p_1} \ell \end{pmatrix}, \text{ for } v \to -\infty.$$ (9)

To derive (6.5) we again start with the Yang–Baxter equation (6.4) and transform it as follows:

$$\mathbb{R}_{23}(v) e^{2\pi i p_1} \mathcal{R}_{13}(u + w) e^{-2\pi i p_1} \mathcal{R}_{12}(u - v + w) e^{-2\pi i p_1},$$

and taking the limit $w \to \infty$ in the underlined factors and obtain (6.5).

The formulae (6.6) and (6.7) are consequences of the standard RLL-relation

$$\mathcal{R}_{12}(u - v) \ell_1(\ell_2(v) \ell_1(\mathcal{R}_{12}(u - v).$$

For example, to derive (6.6) we transform it to the form

$$e^{2\pi i p_1} \mathcal{R}_{12}(u - v + w) e^{-2\pi i p_1} e^{2\pi i p_1} \ell_1(\ell_2(v) e^{-2\pi i p_1},$$

and take the limit $w \to +\infty$ using obvious formulae for the similarity transformation of the generators in representation (2.3):

$$e^{2\pi i p_1} \ell_1 = e^{2\pi i p_1} \ell_1, \quad e^{2\pi i p_1} \ell_1 = e^{2\pi i p_1} \ell_1.$$

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Appendix A. The $R$-operator with SL(2, $\mathbb{C}$)-symmetry

In this appendix we adopt the R-operator construction of section 5.2 for the SL(2, $\mathbb{C}$) group. We start with the holomorphic L-operator

$$L(u_1, u_2) = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \begin{pmatrix} u_1 & -\partial \\ 0 & u_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix},$$

and use this factorization to derive what happens with the L-operator in the limit $u_2 \to \infty$:

$$\left( \begin{array}{cc} 1 & 0 \\ 0 & u_2 \end{array} \right)^{-1} L(u_1, u_2) = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \cdot L^+(u_1) \longrightarrow L^+(u_1), \quad (A.1)$$

and similarly in the limit $u_1 \to \infty$:

$$L_+(u_1) = \left( \begin{array}{cc} u_1 & -\partial \\ 0 & 1 \end{array} \right) \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix}, \quad L_-(u_2) = \left( \begin{array}{cc} 1 & 0 \\ 0 & u_2 \end{array} \right) \cdot L^+(u_1). \quad (A.2)$$

where

Asymptotically we have

$$L(u_1, u_2) \overset{u_1 \to \infty}{\longrightarrow} L^-(u_2) \cdot \left( \begin{array}{cc} u_1 & 0 \\ 0 & 1 \end{array} \right); \quad L(u_1, u_2) \overset{u_2 \to \infty}{\longrightarrow} \left( \begin{array}{cc} 1 & 0 \\ 0 & u_2 \end{array} \right) \cdot L^+(u_1). \quad (A.3)$$

In the following we omit the antiholomorphic sector (for more details see [24]). It can be checked that local factorizations take place (cf (3.5) and (3.6)):

$$L^-(v) L^+_z(u) = e^{-z\partial} L_1(u, v) \left( \begin{array}{cc} 1 & 0 \\ -z & 1 \end{array} \right) e^{z\partial}, \quad (A.4)$$

$$L^-(v) L^+_z(u) = e^{-z\partial} \left( \begin{array}{cc} 1 & 0 \\ z & 1 \end{array} \right) L_2(u, v) e^{z\partial}. \quad (A.5)$$

Then we take into account the intertwining relation (cf (4.1))

$$\partial^{v-u} L_1(u, v) = L_1(v, u) \partial^{v-u},$$

multiply it by the matrix $\left( \begin{array}{cc} 1 & 0 \\ z & 1 \end{array} \right)$ on the right, and apply (A.3) and $e^{-z\partial} \partial^{v-u} e^{z\partial} = (\partial_t + \partial_z)^{v-u}$ which gives the intertwining relation for $L^+_1 L^+_z$ (cf (4.7)):

$$(\partial_t + \partial_z)^{v-u} L^+_1(v) L^+_z(u) = L^+_1(u) L^+_z(v) (\partial_t + \partial_z)^{v-u}. \quad (A.6)$$

To obtain a similar intertwining relation for $L^+_1 L^-_z$ we note that the composition of the matrix similarity transformation and the canonical transformation connects $L^+$ and $L^-$ (cf 3.4):

$$(0 1) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = L^+(u), \quad (0 1) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = L^-(u),$$

which leads to (cf (4.8))

$$z_1^{v-u} L^+_1(v) L^+_z(u) = L^+_1(u) L^-_z(v) z_1^{v-u}. \quad (A.7)$$

Our aim is to construct the operator $R(u)$ which solves the RLL-relation

$$R(u) L_1(u_1, u_2) L_2(v_1, v_2) = L_1(v_1, v_2) L_2(u_1, u_2) R(u).$$

We multiply it by $\left( \begin{array}{cc} 1 & 0 \\ z_2 & 1 \end{array} \right)$ on the left and by $\left( \begin{array}{cc} 1 & 0 \\ -z_1 & 1 \end{array} \right)$ on the right, and substitute in (A.3) and (A.4), which leads to

$$R^{ab}(u) L^a_2(v_2) L^+_1(u_1) L^+_z(v_1) = L^a_2(v_2) L^+_1(u_1) L^+_z(v_1) R^{ab}(u), \quad (A.8)$$

12
$$R^{ab}(u) = e^{- zi_{ab} \hbar} e^{- zi_{ba} \hbar} R(u) e^{zi_{ab} \hbar} e^{zi_{ba} \hbar}.$$  

We can easily solve (A.7) and find $R^{ab}$ due to (A.5) and (A.6):

$$R^{ab}(u) = z_{12}^{v_{a} - v_{b}} (d_{a} + d_{1})^{v_{a} - v_{b}} (d_{2} + d_{b})^{v_{1} - v_{2}} z_{12}^{v_{b} - v_{a}}.$$  

Applying the relation between $R^{ab}(u)$ and $R(u)$, we obtain (cf (5.5))

$$R_{12}(u) = z_{12}^{v_{a} - v_{b}} o_{1}^{v_{a} - v_{b}} o_{2}^{v_{1} - v_{2}} o_{1}^{v_{1} - v_{2}} z_{12}^{v_{b} - v_{a}}.$$  

Taking into account the antiholomorphic sector, the modification of the latter $R$-operator that arises can then be rewritten in an integral form which is analogous to (5.7) [24].

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