FRACTIONAL DERIVATIVES AND TIME-FRACTIONAL ORDINARY DIFFERENTIAL EQUATIONS IN $L^p$-SPACE

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ABSTRACT. We define fractional derivatives $\partial_t^\alpha$ in Sobolev spaces based on $L^p(0,T)$ by an operator theory, and characterize the domain of $\partial_t^\alpha$ in subspaces of the Sobolev-Slobodecki spaces $W^{\alpha,p}(0,T)$. Moreover we define $\partial_t^\alpha u$ for $u \in L^p(0,T)$ in a sense of distribution. Then we discuss initial value problems for linear fractional ordinary differential equations by means of such $\partial_t^\alpha$ and establish several results on the unique existence of solutions within specified classes according to the regularity of the coefficients and the non-homogeneous terms in the equations.

Key words. fractional derivative, $L^p$-space, time-fractional ordinary differential equations, fractional Sobolev spaces

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1. Introduction

Recently fractional differential equations have attracted great attention, not only by theoretical interests but also for by the necessity for various applications. For example, time-fractional differential equations are understood as reasonable model equations in various

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anomalous diffusion phenomena which are frequently observed in diffusion in heterogeneous media around us (e.g., Metzler and Klafter [19], Chapter 10 in Podlubny [20]).

For $0 < \alpha < 1$, we can formally define the pointwise Caputo derivative by

$$d_t^\alpha v(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{dv}{ds}(s) \, ds,$$

as long as the right-hand side exists. Here and henceforth, for $\beta > 0$ by $\Gamma(\beta)$ we denote the gamma function: $\Gamma(\beta) := \int_0^\infty e^{-t}t^{\beta-1} \, dt$. As for classical treatments on fractional derivatives, we can refer to monographs Gorenflo, Kilbas, Mainardi and Rogosin [5], Kilbas, Srivastava and Trujillo [13], Podlubny [20] for example.

We can interpret $d_t^\alpha v$ as the $\alpha$-th derivative of $v$, but in (1.1), the first derivative $\frac{dv}{ds}$ appears and so intuitively $d_t^\alpha v$ requires the existence of $\frac{dv}{dt}$ even for defining the lower-order derivative of order $\alpha < 1$. Moreover, a corresponding initial value problem for a time-fractional ordinary differential equation can be formulated as

$$d_t^\alpha u(t) = b(t)u(t) + f(t), \quad 0 < t < T,$$

and

$$u(0) = a.$$  

Since only $\alpha$-th derivative appears with $0 < \alpha < 1$ in (1.2), the initial condition (1.3) is not trivial. This can be understood in a simple case where $b \equiv 0$ and $f(t) = t^{-\gamma}$ with $0 < \gamma < 1$. Then

$$u(t) = \frac{\Gamma(1-\gamma)}{\Gamma(\alpha-\gamma+1)} t^{\alpha-\gamma} + a$$

satisfies (1.2) - (1.3) for $0 < \gamma < \alpha < 1$, but cannot satisfy (1.2) - (1.3) if $0 < \alpha < \gamma < 1$.

This observation suggests us to make adequate formulations for both fractional derivatives and initial value problems, when we consider not smooth functions. There are several studies and here we refer to In Gorenflo, Luchko and Yamamoto [6] and Kubica, Ryszewska and Yamamoto [15], Yamamoto [23], where one constructs the fractional derivative operator in Sobolev spaces based on $L^2(0,T)$ and apply it to fractional differential equations including partial differential equations in spatial variables. Also see Kian [11], Kian and Yamamoto [12], Luchko and Yamamoto [18], Sakamoto and Yamamoto [21], Zacher [24]. The method in those works relies on the structure of Hilbert spaces, that is, $p = 2$. 
The purpose of this article is to establish the corresponding theory based on $L^p(0, T)$ with $1 \leq p < \infty$.

Throughout this article, we assume $1 \leq p < \infty$.

Henceforth let $L^p(0, T) := \{v; \int_0^T |v(t)|^p dt < \infty\}$ with $1 \leq p < \infty$ and $W^{1,p}(0, T) := \{v \in L^p(0, T); \frac{dv}{dt} \in L^p(0, T)\}$. We define the norm by

$$\|v\|_{L^p(0, T)} := \left( \int_0^T |v(t)|^p dt \right)^{\frac{1}{p}}, \quad \|v\|_{W^{1,p}(0, T)} := \|v\|_{L^p(0, T)} + \left\| \frac{dv}{dt} \right\|_{L^p(0, T)}.$$ 

For $\alpha > 0$, we set

$$J^\alpha v(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) ds, \quad 0 < t < T, \quad v \in L^1(0, T), \quad (1.4)$$

which is called the Riemann-Liouville fractional integral operator, and can be interpreted as an $\alpha$-times integral of $v$. By the Young inequality on the convolution (e.g., Lemma A.1 in [15]), we see that

$$J^\alpha L^q(0, T) \subset L^q(0, T) \quad \text{with } q \geq 1. \quad (1.5)$$

Our definition of the fractional derivative relies on the inverse to $J^\alpha$ in $L^p(0, T)$. Henceforth by $\partial_t^\alpha$ we denote such a fractional derivative in order to distinguish from $d_t^\alpha$. The first step is to first clarify the space $J^\alpha L^p(0, T)$ and introduce a suitable norm. After establishing $\partial_t^\alpha$, we formulate an initial value problem and discuss the unique existence of solutions with suitable regularity.

In order to accomplish the task, we organize the article as follows:

- §2. Fractional derivative $\partial_t^\alpha$ in $L^p$-Sobolev-Slobodecki spaces of positive orders
- §3. Fractional derivative $\partial_t^\alpha$ in $L^p(0, T)$
- §4. Initial value problems for time-fractional ordinary differential equations
- §5. Concluding remarks
- §6. Appendix: Proof of Lemma 2.4.
2. Fractional derivative $\partial_t^\alpha$ in $L^p$-Sobolev-Slobodecki spaces of positive orders: characterization of the domain $\mathcal{D}(\partial_t^\alpha)$ and norm estimates

Henceforth by $\mathcal{D}(K)$, we denote the domain of an operator $K$ under consideration, and $C > 0$ denotes generic constants depending on $\alpha, p$ but not on choices of functions $v$.

We recall that $J^\alpha$ is defined by (1.4) and we set $\mathcal{D}(J^\alpha) = L^p(0,T)$. Then by (1.5) we see that

$$\|J^\alpha v\|_{L^p(0,T)} \leq C\|v\|_{L^p(0,T)} \quad \text{for all } v \in L^p(0,T). \quad (2.1)$$

First we prove:

**Lemma 2.1.**

Let $\alpha, \beta > 0$ and $\alpha, \beta \notin \mathbb{N}$.

(i) $J^\alpha J^\beta v = J^{\alpha + \beta} v$ for each $v \in L^1(0,T)$.

(ii) $J^\alpha : L^p(0,T) \to L^p(0,T)$ is injective for $1 \leq p < \infty$.

**Proof of Lemma 2.1.**

(i) Exchanging the order of the integral, we have

$$J^\alpha (J^\beta v)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} J^\beta v(s) ds$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{1}{\Gamma(\beta)} \left( \int_0^s (s-\xi)^{\beta-1} v(\xi) d\xi \right) ds$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \left( \int_\xi^t (t-s)^{\alpha-1} (s-\xi)^{\beta-1} ds \right) v(\xi) d\xi$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} (t-\xi)^{\alpha + \beta - 1} v(\xi) d\xi = (J^{\alpha + \beta} v)(t), \quad 0 < t < T,$$

which completes the proof of (i). □

(ii) We assume that $v \in L^1(0,T)$ satisfies

$$J^\alpha v(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) ds = 0, \quad 0 < t < T.$$

We set $\alpha = (m-1) + \sigma$ with $m \in \mathbb{N}$ and $0 < \sigma < 1$. We operate $J^{1-\sigma}$ to obtain

$$\int_0^t (t-s)^{-\sigma} \left( \int_0^s (s-\xi)^{\alpha-1} v(\xi) d\xi \right) ds = 0, \quad 0 < t < T.$$
Again exchanging the order of the integral and using \( \alpha - \sigma = m - 1 \), we obtain
\[
0 = \int_0^t \left( \int_\xi^t (t - s)^{-\sigma}(s - \xi)^{\alpha - 1} ds \right) v(\xi) d\xi
\]
\[
= \frac{\Gamma(1 - \sigma)\Gamma(\alpha)}{\Gamma(1 - \sigma + \alpha)} \int_0^t (t - \xi)^{m - 1} v(\xi) d\xi, \quad 0 < t < T.
\]
Therefore, the \( m \)-times differentiation yields \( v(t) = 0 \) for \( 0 < t < T \). Thus the proof of Lemma 2.1 is complete. ■

The proof of (ii) is suggested by the classical formula (e.g., Gorenflo and Vessella [7])
\[
D_t^\alpha J^\alpha v = v, \quad v \in L^1(0, T),
\]
where for \( \alpha = (m - 1) + \sigma \) with \( m \in \mathbb{N} \) and \( 0 < \sigma < 1 \), we define the Riemann-Liouville fractional derivative
\[
D_t^\alpha v(t) = \frac{1}{\Gamma(1 - \sigma)} \frac{d^m}{dt^m} \int_0^t (t - s)^{-\sigma} v(s) ds, \quad v \in C^\infty[0, T]. \tag{2.2}
\]

By Lemma 2.1 (ii), there exists an inverse operator \((J^\alpha)^{-1}\) in \( L^p(0, T) \) to \( J^\alpha \) algebraically. We define
\[
W_{\alpha,p}(0, T) := J^\alpha L^p(0, T) \quad \text{as a set} \tag{2.3}
\]
with the norm
\[
\| u \|_{W_{\alpha,p}(0, T)} := \| (J^\alpha)^{-1} u \|_{L^p(0, T)}. \tag{2.4}
\]
We note that
\[
\| J^\alpha w \|_{W_{\alpha,p}(0, T)} = \| w \|_{L^p(0, T)}.
\]
Therefore,

**Lemma 2.2.**

Let \( \alpha > 0 \). If \( w_n \longrightarrow w \) in \( L^p(0, T) \), then \( J^\alpha w_n \longrightarrow J^\alpha w \) in \( W_{\alpha,p}(0, T) \).

Moreover, we prove

**Lemma 2.3.**

The space \( W_{\alpha,p}(0, T) \) is complete with the norm \( \| \cdot \|_{W_{\alpha,p}(0, T)} \), that is, \( W_{\alpha,p}(0, T) \) is a Banach space.

**Proof.**
Let $u_n \in W_{\alpha,p}(0,T)$ and $\lim_{m,n \to \infty} \|u_n - u_m\|_{W_{\alpha,p}(0,T)} = 0$. Then for each $n \in \mathbb{N}$, we can find $w_n \in L^p(0,T)$ such that $u_n = J^\alpha w_n$. Hence,

$$
\|u_m - u_n\|_{W_{\alpha,p}(0,T)} = \|J^\alpha (w_m - w_n)\|_{W_{\alpha,p}(0,T)} = \|w_m - w_n\|_{L^p(0,T)} \to 0
$$

as $m, n \to \infty$. By the completeness of $L^p(0,T)$, there exists $w_0 \in L^p(0,T)$ such that $\lim_{n \to \infty} \|w_n - w_0\|_{L^p(0,T)} = 0$. Set $u_0 := J^\alpha w_0$. Lemma 2.2 yields $J^\alpha w_n \to J^\alpha w_0$ in $W_{\alpha,p}(0,T)$, that is, $u_n \to J^\alpha w_0$ in $W_{\alpha,p}(0,T)$, where $J^\alpha w_0 \in J^\alpha L^p(0,T) = W_{\alpha,p}(0,T)$. Thus the proof of Lemma 2.3 is complete.

We define the fractional derivative.

**Definition 2.1.**

Let $\alpha > 0$ and $\alpha \notin \mathbb{N}$. Then

$$
\partial_t^\alpha := (J^\alpha)^{-1} \mathcal{D}(\partial_t^\alpha) = W_{\alpha,p}(0,T).
$$

At least $\partial_t^\alpha$ is well-defined, but is not useful if we do not specify the domain $\mathcal{D}(\partial_t^\alpha)$. In a special case of $p = 2$, by the structure of the Hilbert space $L^2(0,T)$, we can make the complete characterization in terms of $W^{\alpha,2}(0,T)$ (e.g., [4, 15, 23]). However for $p \neq 2$, to the author's best knowledge, such characterization is not known.

Now we will make characterization of $W_{\alpha,p}(0,T)$ and for it we introduce function spaces. For $m \in \mathbb{N}$, we set $\frac{d^mv}{dt^m}(t) = v(t)$ and

$$
0C^m[0,T] := \left\{ v \in C^m[0,T]; \frac{d^k v}{dt^k}(0) = 0 \quad \text{for} \quad k = 0, 1, ..., m - 1 \right\}.
$$

For $\alpha = m - 1 + \sigma$ with $m \in \mathbb{N}$ and $0 < \sigma < 1$ and $1 \leq p < \infty$, we define the Sobolev-Slobodecki space $W^{\alpha,p}(0,T)$ with the norm $\| \cdot \|_{W^{\alpha,p}(0,T)}$ by

$$
W^{\alpha,p}(0,T) := \{ u \in L^p(0,T); \|u\|_{W^{\alpha,p}(0,T)} < \infty \}
$$

where

$$
\|u\|_{W^{\alpha,p}(0,T)} := \left( \|u\|_{W^{m-1,p}(0,T)}^p + \int_0^T \int_0^T \frac{|d^{m-1}u(t) - d^{m-1}u(s)|^p}{|t - s|^{1+\sigma p}} dt ds \right)^{\frac{1}{p}}
$$
(e.g., Adams [1], Grisvard [8]). By \( \overline{X}^Y \) we denote the closure of \( X \subset Y \) in a normed space \( Y \). Moreover

\[
_0W^{\alpha,p}(0,T) := \overline{C^m[0,T]}^{W^{\alpha,p}(0,T)}.
\]

We set \( W^{0,p}(0,T) = _0W^{\alpha,p}(0,T) = L^p(0,T) \) for uniform notations.

We state the main result in this section.

**Theorem 2.1.**

Let \( \alpha = m - 1 + \sigma \) with \( m \in \mathbb{N} \) and \( 0 < \sigma < 1 \), and \( \varepsilon > 0 \) be arbitrarily given such that \( 0 < \varepsilon < \alpha \). Then

(i) There exists a constant \( C = C(\alpha, \varepsilon) > 0 \) such that

\[
\|J^\alpha v\|_{W^{\alpha-\varepsilon,p}(0,T)} \leq C\|v\|_{L^p(0,T)}
\]

for all \( v \in L^p(0,T) \) and

\[
W_\alpha(0,T) \subset _0W^{\alpha-\varepsilon,p}(0,T).
\]

(ii) We have

\[
_0W^{\alpha+\varepsilon,p}(0,T) \subset W_\alpha(0,T)
\]

and

\[
J^\alpha \left( \frac{d^m}{dt^m} J^{m-\alpha} u \right)(t) = u(t) \quad \text{for each } u \in _0W^{\alpha+\varepsilon,p}(0,T).
\]

(iii) The embedding \( W_{\alpha+\varepsilon,p}(0,T) \rightarrow W_\alpha(0,T) \) is compact.

In terms of the Riemann-Liouville fractional derivative (2.2), we can rewrite (2.6) as

\[
J^\alpha D_t^\alpha u = u \quad \text{for each } u \in _0W^{\alpha+\varepsilon,p}(0,T).
\]

We note that for \( \alpha = m - 1 \) with \( m \in \mathbb{N} \), we can easily prove

\[
J^\alpha L^p(0,T) = \left\{ v \in W^{m,p}(0,T); \frac{d^k v}{dt^k}(0) = 0 \quad \text{for } k = 0, 1, \ldots, m - 1 \right\}.
\]
In the case of \( p = 2 \) and \( 0 < \alpha < 1 \), it is proved in [15] that 
\[ W^\alpha_0(0, T) = C^{1}[0, T]^{\alpha} \]
which means that we can take \( \varepsilon = 0 \) in (i) and (ii). However, in the case \( p \neq 2 \), we do not know the equivalent representation of \( W^\alpha_0(0, T) \), but (i) and (ii) imply
\[ 0 \subset W^\alpha_0(0, T) \subset 0 \]
with arbitrarily small \( \varepsilon \in (0, \alpha) \), which means that \( W^\alpha_0(0, T) \) is interpolated between \( 0 \) and \( 0 \) with arbitrarily small gap \( \varepsilon > 0 \).

**Corollary 2.1.**

Let \( \alpha > 0 \) and \( \beta \geq 0 \). Then

(i) \( J^\alpha : W^\beta_0(0, T) \to W^\beta_0(0, T) \) is a compact operator.

(ii) \( J^\alpha W^\beta_0(0, T) = W^\alpha+\beta_0(0, T) \) and \( \| J^\alpha w \|_{W^\alpha+\beta_0(0, T)} = \| w \|_{W^\beta_0(0, T)} \) for \( w \in W^\beta_0(0, T) \).

We rewrite Theorem 2.1 in terms of \( \partial^\alpha_t = (J^\alpha)^{-1} \).

**Theorem 2.2.**

Let \( p \geq 1 \), \( \alpha = m - 1 + \sigma \) with \( m \in \mathbb{N} \) and \( 0 < \sigma < 1 \), and \( 0 < \varepsilon < \alpha \).

(i) \( W^\alpha_0(0, T) \subset \mathcal{D}(\partial^\alpha_t) \subset W^\alpha_{\alpha+\varepsilon}(0, T) \) topologically and algebraically.

(ii) \( \partial^\alpha_t W^\alpha_{\alpha+\varepsilon}(0, T) = W^\beta_0(0, T) \) for \( \alpha > 0 \) and \( \beta \geq 0 \).

The part (i) of Theorem 2.1 improves Proposition 6 in Carbotti and Comi [3], and see Li and Liu [16], [17] as for results on fractional derivative in Sobolev spaces on \( L^p(0, T) \).

The rest of this section is devoted to the proof of Theorem 2.1, but the arguments in the proof are not used later. The proof of Theorem 2.1 (i) in the case of \( p = 1 \) and \( 0 < \alpha < 1 \) can be found in the proof of Theorem 4.2.2 (pp.73-74) in Gorenflo and Vessella [7], and we adjust their arguments in [7] to the general case \( p > 1 \). Moreover, in König [14], related results are discussed in an operator theoretical setting, while our proof is lengthy but based on direct estimation.

**Proof of Theorem 2.1 (i).**
First Step: $0 < \alpha < 1$.
First we show that for $\delta \in (0, \frac{1}{2})$, there exists a constant $C_\delta > 0$ such that
\[
|1 - \eta^{\alpha - 1}| \leq (1 - \alpha)(1 - \eta) + C_\delta(1 - \eta)^2 \quad \text{if } 1 - \delta \leq \eta \leq 1.
\] (2.7)

Verification of (2.7).
Applying the Taylor theorem to $g(\eta) := \eta^{\alpha - 1}$, we obtain
\[
g(\eta) = g(1) + g'(1)(\eta - 1) + \frac{g''(\zeta)}{2!}(\eta - 1)^2 \quad \text{for } 1 - \delta \leq \eta \leq 1,
\]
where $\zeta \in (1 - \delta, 1)$ is some number. Therefore $|\eta^{\alpha - 1} - (\alpha - 1)(\eta - 1)| \leq C_\delta(\eta - 1)^2$, which implies (2.7). □

We set $\beta := \alpha - \varepsilon$ with $0 < \varepsilon < \alpha$. We recall that
\[
\|v\|_{W^{\beta,p}(0,T)}^p = \|v\|_{L^p(0,T)}^p + \int_0^T \int_0^T \frac{|v(t) - v(s)|^p}{|t - s|^{1 + \beta_p}} dt ds.
\]
We can see that it suffices to estimate for $0 < s < t$. Indeed, dividing $[0,T]^2 = \{(t,s) \in [0,T]^2; 0 \leq s \leq t \leq T\} \cup \{(t,s) \in [0,T]^2; 0 \leq t \leq s \leq T\}$, we have
\[
\int_0^T \int_0^T \frac{|v(t) - v(s)|^p}{|t - s|^{1 + \beta_p}} dt ds = 2 \int_0^T \left( \int_0^t \frac{|v(t) - v(s)|^p}{|t - s|^{1 + \beta_p}} ds \right) dt.
\]
We have
\[
|\Gamma(\alpha)(J^\alpha v(t) - J^\alpha v(s))| = \left| \int_0^t (t - \xi)^{\alpha - 1}v(\xi) d\xi - \int_0^s (s - \xi)^{\alpha - 1}v(\xi) d\xi \right|
\]
\[
= \left| \int_s^t (t - \xi)^{\alpha - 1}v(\xi) d\xi + \int_0^s ((t - \xi)^{\alpha - 1} - (s - \xi)^{\alpha - 1})v(\xi) d\xi \right|
\]
\[
\leq \int_s^t (t - \xi)^{\alpha - 1}|v(\xi)| d\xi + \int_0^s |(t - \xi)^{\alpha - 1} - (s - \xi)^{\alpha - 1}||v(\xi)| d\xi,
\]
and so
\[
|J^\alpha v(t) - J^\alpha v(s)|^p \leq C \left( \int_s^t (t - \xi)^{\alpha - 1}|v(\xi)| d\xi \right)^p + C \left( \int_0^s |(t - \xi)^{\alpha - 1} - (s - \xi)^{\alpha - 1}||v(\xi)| d\xi \right)^p.
\]
Hence
\[
\int_0^T \int_0^t \frac{|J^\alpha v(t) - J^\alpha v(s)|^p}{|t-s|^{1+\beta p}} \, ds \, dt \leq C \int_0^T dt \int_0^t ds (t-s)^{-1-\beta p} \left( \int_s^t (t-\xi)^{\alpha-1} |v(\xi)| \, d\xi \right)^p \\
+ C \int_0^T dt \int_0^t ds (t-s)^{-1-\beta p} \left( \int_0^s |(t-\xi)^{\alpha-1} - (s-\xi)^{\alpha-1}| |v(\xi)| \, d\xi \right)^p \leq CI_1 + CI_2. \quad (2.8)
\]

**Case 1: p = 1.**

Equation (2.8) implies
\[
\int_0^T \int_0^t \frac{|J^\alpha v(t) - J^\alpha v(s)|}{|t-s|^{1+\beta}} \, ds \, dt \leq CI_1 + CI_2,
\]
where
\[
I_1 = \int_0^T \int_0^t (t-s)^{-1-\beta} \left( \int_s^t (t-\xi)^{\alpha-1} |v(\xi)| \, d\xi \right) \, ds \, dt
\]
and
\[
I_2 = \int_0^T \int_0^t (t-s)^{-1-\beta} \left( \int_0^s |(t-\xi)^{\alpha-1} - (s-\xi)^{\alpha-1}| |v(\xi)| \, d\xi \right) \, ds \, dt.
\]

Exchanging the order of the integral, we obtain
\[
I_1 = \int_0^T \int_0^t |v(\xi)|(t-\xi)^{\alpha-1} \left( \int_0^\xi (t-s)^{-1-\beta} \, ds \right) \, d\xi \, dt
\]
\[
= \frac{1}{\beta} \int_0^T \left( \int_0^t |v(\xi)|(t-\xi)^{\alpha-1}((t-\xi)^{-\beta} - t^{-\beta}) \, d\xi \right) \, dt
\]
\[
\leq C \int_0^T \int_0^t |v(\xi)|(t-\xi)^{\alpha-\beta-1} \left( 1 - \left( \frac{t-\xi}{t} \right)^\beta \right) \, d\xi \, dt \leq C \int_0^T \int_0^t |v(\xi)|(t-\xi)^{\alpha-1} \, d\xi \, dt
\]
by \(|\frac{t-\xi}{t}| \leq 1\) and \(\beta > 0\) for \(0 < \xi < t\). Therefore, the Young inequality for the convolution yields
\[
|I_1| \leq C \|s^{\xi-1} * v\|_{L^1(0,T)} \leq C \|v\|_{L^1(0,T)}.
\]

As for the term \(I_2\), by the change \(s := \eta(t-\xi) + \xi\) of the variables, exchanging the order of the integral, we deduce
\[
\int_0^t (t-s)^{-1-\beta} \left( \int_0^s |(t-\xi)^{\alpha-1} - (s-\xi)^{\alpha-1}| |v(\xi)| \, d\xi \right) \, ds
\]
\[
= \int_0^t \left( \int_0^t (t - s)^{-1-\beta} |(s - \xi)^{\alpha-1} - (t - \xi)^{\alpha-1}| \, ds \right) |v(\xi)| \, d\xi \\
\leq C \int_0^t |v(\xi)|(t - \xi)^{\varepsilon-1} \left( \int_0^1 (1 - \eta)^{-1-\beta} |(\eta^{\alpha-1} - 1)| \, d\eta \right) \, d\xi.
\]

Here for arbitrarily fixed constant \( \delta \in (0, \frac{1}{2}) \), we apply (2.7) to estimate
\[
\int_{1-\delta}^1 (1 - \eta)^{-1-\beta} |\eta^{\alpha-1} - 1| \, d\eta \leq \int_{1-\delta}^1 (1 - \eta)^{-1-\beta} ((1 - \alpha)(1 - \eta) + C_\delta (1 - \eta)^2) \, d\eta \\
\leq C_\delta \int_{1-\delta}^1 ((1 - \eta)^{-\beta} + (1 - \eta)^{1-\beta}) \, d\eta < \infty
\]
by \( 0 < \beta < \alpha < 1 \). Furthermore
\[
\int_0^{1-\delta} (1 - \eta)^{-1-\beta} |\eta^{\alpha-1} - 1| \, d\eta \leq C_\delta \int_0^{1-\delta} (|\eta^{\alpha-1}| + 1) \, d\eta < \infty.
\]

Consequently, the Young inequality yields
\[
|I_2| \leq C_\delta \int_0^T \left| \int_0^t |v(\xi)|(t - \xi)^{\varepsilon-1} \, d\xi \right| \, dt = C_\delta \|s^{\varepsilon-1} * |v||_{L^1(0,T)} \leq C_\delta \|v||_{L^1(0,T)}.
\]

Thus the proof of (i) for \( p = 1 \) and \( 0 < \alpha < 1 \) is completed. \( \blacksquare \)

**Case 2:** \( p > 1 \).

First we note that since \( W^{\alpha-\varepsilon,p}(0,T) \subset W^{\alpha-\varepsilon',p}(0,T) \) if \( \alpha > \varepsilon' > \varepsilon > 0 \), it suffices to prove (i) for sufficiently small \( \varepsilon > 0 \), and in particular, we can assume that
\[
\alpha > \varepsilon p. \quad (2.9)
\]

Let \( q \in (1, \infty) \) satisfy \( \frac{1}{p} + \frac{1}{q} = 1 \). By \( 1 < p < \infty \), we have \( q = \frac{p}{p-1} \in (1, \infty) \).

In \( I_1 \) and \( I_2 \), the integrands contain \( |v(\xi)| \), not \( |v(\xi)|^p \), so that we need to change such factors to \( |v(\xi)|^p \). To this end, we resort to the Hölder inequality after factorizing
\[
|t - \xi|^{\alpha-1} |v(\xi)| = |(t - \xi|^{\alpha-1})^{\frac{1}{2}} \{ (|t - \xi|^{\alpha-1})^{1-\frac{1}{q}} |v(\xi)| \}.
\]

Then the Hölder inequality implies
\[
\int_s^t |t - \xi|^{\alpha-1} |v(\xi)| \, d\xi = \int_s^t (|t - \xi|^{\alpha-1})^{\frac{1}{q}} \{ (|t - \xi|^{\alpha-1})^{1-\frac{1}{q}} |v(\xi)| \} \, d\xi
\]
\[
\leq \left( \int_{s}^{t} (t - \xi)^{\alpha-1} d\xi \right)^{\frac{1}{q}} \left( \int_{s}^{t} ((t - \xi)^{\alpha-1})^{p(\frac{1}{q} - 1)} |v(\xi)| \right)^{\frac{1}{p}},
\]
and so
\[
\left( \int_{s}^{t} |t - \xi|^{\alpha-1} |v(\xi)| d\xi \right)^{p} \leq \left( \int_{s}^{t} (t - \xi)^{\alpha-1} d\xi \right)^{\frac{p}{q}} \int_{s}^{t} |t - \xi|^{\alpha-1} |v(\xi)| d\xi
\]
\[
= \alpha^{-\frac{p}{q}} (t - s)^{\frac{p}{q}} \int_{s}^{t} (t - \xi)^{\alpha-1} |v(\xi)| d\xi.
\]
Therefore, since \( \frac{1}{q} = 1 - \frac{1}{p} \) implies
\[
-1 - \beta p + \frac{p}{q} \alpha = -1 - \beta p + \alpha p - \alpha = -1 - \alpha + \varepsilon p,
\]
using \( \int_{0}^{t} \left( \int_{0}^{s} \cdots d\xi \right) ds = \int_{0}^{t} \left( \int_{0}^{\xi} \cdots d\xi \right) d\xi \), we obtain
\[
I_1 \leq C \int_{0}^{T} dt \int_{0}^{t} (t - s)^{-\alpha + \varepsilon p} \left( \int_{s}^{t} (t - \xi)^{\alpha-1} |v(\xi)| d\xi \right) ds
\]
\[
= C \int_{0}^{T} dt \int_{0}^{t} \left( \int_{0}^{\xi} (t - s)^{-1 - \alpha + \varepsilon p} ds \right) (t - \xi)^{\alpha-1} |v(\xi)| d\xi.
\]
Since \( 0 < s < \xi < t \), we have
\[
\int_{0}^{t} (t - s)^{-1 - \alpha + \varepsilon p} ds = \frac{1}{\varepsilon p - \alpha} ((t - \xi)^{-\alpha + \varepsilon p} - t^{-\alpha + \varepsilon p}),
\]
and by (2.9) we deduce
\[
|I_1| \leq C \int_{0}^{T} \left( \int_{0}^{t} |t^{-\alpha + \varepsilon p} - (t - \xi)^{-\alpha + \varepsilon p}||t - \xi|^{\alpha-1} |v(\xi)| d\xi \right) dt.
\]
On the other hand,
\[
|t^{-\alpha + \varepsilon p} - (t - \xi)^{-\alpha + \varepsilon p}| = |t - \xi|^{-\alpha + \varepsilon p} \left| 1 - \frac{t - \xi}{t} \right|^{\alpha-\varepsilon p}.
\]
From (2.9) and \( \left| \frac{t - \xi}{t} \right| \leq 1 \), it follows that \( \left| \frac{t - \xi}{t} \right|^{\alpha-\varepsilon p} \leq 1 \) and
\[
|t^{-\alpha + \varepsilon p} - (t - \xi)^{-\alpha + \varepsilon p}| \leq 2|t - \xi|^{-\alpha + \varepsilon p} \quad \text{for } 0 < \xi < t.
\]
Hence,
\[
|I_1| \leq 2C \int_{0}^{T} \left( \int_{0}^{t} |t - \xi|^{-\alpha + \varepsilon p} |t - \xi|^{\alpha-1} |v(\xi)| d\xi \right) dt
\]
Therefore, the Young inequality implies

\[ |I_1| \leq C \|s^{\alpha p-1}\|_{L^1(0,T)} \|v\|_{L^1(0,T)} \leq C \|v\|_{L^p(0,T)}. \]  

(2.12)

We now proceed to the proof of \( I_2 \). Similarly to \( I_1 \), we factorize

\[ |(t - \xi)^{\alpha - 1} - (s - \xi)^{\alpha - 1}|v(\xi)| \]

\[ = |(t - \xi)^{\alpha - 1} - (s - \xi)^{\alpha - 1}| \left\{ |(t - \xi)^{\alpha - 1} - (s - \xi)^{\alpha - 1}|^{1 - \frac{1}{q}} |v(\xi)| \right\} \]

and apply the Hölder inequality to obtain

\[
\left( \int_0^s |(t - \xi)^{\alpha - 1} - (s - \xi)^{\alpha - 1}|v(\xi)| d\xi \right)^p \\
\leq \left( \int_0^s |(t - \xi)^{\alpha - 1} - (s - \xi)^{\alpha - 1}| d\xi \right)^{\frac{p}{q}} \int_0^s |(t - \xi)^{\alpha - 1} - (s - \xi)^{\alpha - 1}| |v(\xi)|^p d\xi \\
= \left( \frac{(t-s)^\alpha - (t^\alpha - s^\alpha)}{\alpha} \right)^{\frac{q}{p}} \int_0^s |(t - \xi)^{\alpha - 1} - (s - \xi)^{\alpha - 1}| |v(\xi)|^p d\xi \\
\leq C (t-s)^{\frac{p\alpha}{\alpha+1}} \int_0^s |(t - \xi)^{\alpha - 1} - (s - \xi)^{\alpha - 1}| |v(\xi)|^p d\xi.
\]

Therefore,

\[
\int_0^t (t-s)^{1-\beta p} \left( \int_0^s |(t - \xi)^{\alpha - 1} - (s - \xi)^{\alpha - 1}| |v(\xi)| d\xi \right)^p ds \\
\leq C \int_0^t (t-s)^{1-\beta p + \frac{p\alpha}{\alpha+1}} \left( \int_0^s |(t - \xi)^{\alpha - 1} - (s - \xi)^{\alpha - 1}| |v(\xi)|^p d\xi \right) ds.
\]

Exchanging the order of the integral and using (2.10), we have

\[
\int_0^t (t-s)^{1-\beta p} \left( \int_0^s |(t - \xi)^{\alpha - 1} - (s - \xi)^{\alpha - 1}| |v(\xi)|^p d\xi \right) ds \\
\leq C \int_0^t \left( \int_\xi^t (t-s)^{-1+\alpha p} ((s - \xi)^{\alpha - 1} - (t - \xi)^{\alpha - 1}) ds \right) |v(\xi)|^p d\xi.
\]

Changing the variables \( s \mapsto \eta \) by \( s := \eta(t - \xi) + \xi \), we obtain

\[
\int_0^t (t-s)^{-1+\alpha p} \left( \int_0^s |(t - \xi)^{\alpha - 1} - (s - \xi)^{\alpha - 1}| |v(\xi)|^p d\xi \right) ds
\]
\[ I_2 = C \int_0^t |v(\xi)|^p (t - \xi)^{\varepsilon p - 1} \left( \int_0^1 (1 - \eta)^{-1 - \alpha + \varepsilon p} (\eta^{\alpha - 1} - 1) d\eta \right) d\xi. \]

Here
\[ \int_0^1 (1 - \eta)^{-1 - \alpha + \varepsilon p} (\eta^{\alpha - 1} - 1) d\eta < \infty. \]

Indeed, for fixed \( \delta \in (0, \frac{1}{2}) \), we deduce
\[ \int_0^\delta (1 - \eta)^{-1 - \alpha + \varepsilon p} (\eta^{\alpha - 1} - 1) d\eta \leq \int_0^\delta (\eta^{\alpha - 1} - 1) d\eta < \infty \]
and (2.7) yields
\[ \int_{1-\delta}^1 (1 - \eta)^{-1 - \alpha + \varepsilon p} |\eta^{\alpha - 1} - 1| d\eta \]
\[ \leq \int_{1-\delta}^1 (1 - \eta)^{-1 - \alpha + \varepsilon p} ((1 - \alpha)(1 - \eta) + C_\delta (1 - \eta)^2) d\eta \leq C_\delta \int_{1-\delta}^1 (1 - \eta)^{-\alpha + \varepsilon p} d\eta < \infty, \]
because \( -\alpha + \varepsilon p > -1 \) by \( 0 < \alpha < 1 \).

Therefore,
\[ \int_0^t (t - s)^{-1 - \alpha + \varepsilon p} \left( \int_0^s |(t - \xi)^{\alpha - 1} - (s - \xi)^{\alpha - 1}|^p d\xi \right) ds \]
\[ \leq C |(s^{\varepsilon p - 1} * |v|^p)(t)|, \quad 0 < t < T. \]

Then the Young inequality yields
\[ |I_2| \leq C \left\| s^{\varepsilon p - 1} * |v|^p \right\|_{L^1(0,T)} \leq C \left\| s^{\varepsilon p - 1} \right\|_{L^1(0,T)} \left\| |v|^p \right\|_{L^1(0,T)} \leq C \left\| v \right\|_{L^p(0,T)}. \]

(2.13)

Hence, for \( p > 1 \) and \( 0 < \alpha < 1 \), by (2.8), (2.12) and (2.13), we complete the proof of the estimate in Theorem 2.1 (i).

**Second Step: \( \alpha > 1 \).**

Let \( \alpha = \ell + \sigma \) where \( \ell \in \mathbb{N} \) and \( 0 < \sigma < 1 \). Then
\[ J^\alpha v(t) = \frac{1}{\Gamma(\ell + \sigma)} \int_0^t (t - s)^{\ell + \sigma - 1} v(s) ds, \quad 0 < t < T, \ v \in L^p(0, T). \]

Therefore
\[ \frac{d^k}{dt^k} J^\alpha v(t) = \frac{1}{\Gamma(\ell - k + \sigma)} \int_0^t (t - s)^{\ell - k + \sigma - 1} v(s) ds, \quad 0 \leq k \leq \ell, \ 0 < t < T, \ v \in L^p(0, T). \]
In particular,
\[
\frac{d^k}{dt^k} J^\alpha v(t) = \frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} v(s) ds, \quad 0 \leq k \leq \ell, \quad 0 < t < T, \quad v \in L^p(0, T).
\]
Hence,
\[
\left\| \frac{d^k}{dt^k} J^\alpha \right\|_{L^p(0, T)} = \frac{1}{\Gamma(\ell - k + \sigma)} \left\| s^{\ell-k+\alpha-1} v \right\|_{LP(0, T)} \leq C \left\| v \right\|_{LP(0, T)}, \quad 0 \leq k \leq \ell. \tag{2.14}
\]
Since we have already proved (2.5) with \( \alpha \in (0, 1) \), we see
\[
\left\| \frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\alpha-1} v(s) ds \right\|_{W^{\sigma-\varepsilon, p}(0, T)} \leq C \left\| v \right\|_{L^p(0, T)} \tag{2.15}
\]
with small \( \varepsilon \in (0, \sigma) \). Consequently, in terms of (2.14) and (2.15), we deduce
\[
\left\| J^\alpha v \right\|^p_{W^{\alpha-\varepsilon, p}(0, T)} = \sum_{k=0}^\ell \left\| \frac{d^k}{dt^k} J^\alpha v \right\|^p_{L^p(0, T)} + \int_0^T \int_0^T \frac{\left| \frac{d^k}{dt^k} J^\alpha v(t) - \frac{d^k}{dt^k} J^\alpha v(s) \right|^p}{|t-s|^{1+(\sigma-\varepsilon)p}} dt ds \leq C \left\| v \right\|^p_{L^p(0, T)}
\]
for all \( v \in L^p(0, T) \).

Finally, for \( \alpha = m-1+\sigma \) with \( m \in \mathbb{N} \) and \( 0 < \sigma < 1 \), we have to prove that \( W_{\alpha,p}(0, T) \subset_0 W^{\alpha-\varepsilon, p}(0, T) \). Let \( u \in W_{\alpha,p}(0, T) \) be arbitrarily given. By the definition (2.4), there exists \( w \in L^p(0, T) \) such that \( u = J^\alpha w \). By the density, we can find \( w_n \in C^m[0, T], n \in \mathbb{N} \), such that \( w_n \to w \) in \( L^p(0, T) \) as \( n \to \infty \). Setting \( u_n := J^\alpha w_n, n \in \mathbb{N} \), by \( w_n \in C^m[0, T] \), we can directly prove that \( u_n \in C^m[0, T] \) for \( n \in \mathbb{N} \).

Indeed, since we can directly see that \( J^\alpha C[0, T] \subset C[0, T] \) with \( \alpha > 0 \) by estimating \( J^\alpha u(t) - J^\alpha u(s) \) for \( u \in C[0, T] \), this follows by applying (2.16) to \( v := w_n \) and \( \gamma := \alpha \):
\[
\frac{d^k}{dt^k} J^\gamma v = \frac{d^k}{dt^k} J^\alpha \left( \frac{d^k}{dt^k} v \right) \quad \text{for} \quad v \in C^m[0, T] \text{ and } \gamma > 0, k = 0, 1, ..., m. \tag{2.16}
\]

**Verification of (2.16).**

Since \( \frac{d^k}{dt^k}(0) = 0 \) for \( 0 \leq k \leq m-1 \) and \( \int_0^t (t-s)^{\gamma-1} v(s) ds = \int_0^t s^{\gamma-1} v(t-s) ds \), we have
\[
\frac{d}{dt} J^\gamma v(t) = \frac{1}{\Gamma(\gamma)} \left( \int_0^t s^{\gamma-1} \frac{dv}{dt} (t-s) ds + t^{\gamma-1} v(t) \right) = \frac{1}{\Gamma(\gamma)} \int_0^t s^{\gamma-1} \frac{dv}{dt} (t-s) ds
\]
and 
\[
\frac{d^2}{dt^2} J^\gamma v(t) = \frac{1}{\Gamma(\gamma)} \left( \int_0^t \gamma^{-1} \frac{d^2 v}{dt^2}(t-s)ds + \gamma^{-1} \frac{dv}{dt}(0) \right).
\]

We can continue the calculations to finish the proof of (2.16). \[\square\]

Then, by (2.5), we can deduce that 
\[
J^{\alpha} w_n \rightarrow J^{\alpha} w \text{ in } W^{\alpha-\varepsilon,p}(0,T),
\]
that is, 
\[
u_n \rightarrow u \text{ in } W^{\alpha-\varepsilon,p}(0,T) \text{ as } n \rightarrow \infty.
\]
Since \(u_n \in _0C^m[0,T]\), this means that 
\[
u \in _0C^m[0,T] W^{\alpha-\varepsilon,p}(0,T) = _0W^{\alpha-\varepsilon,p}(0,T).
\]

Thus the proof of Theorem 2.1 (i) is completed. \[\square\]

**Proof of Theorem 2.1 (ii).**

**First Step.**

We show Lemma 2.4.

Let \(\alpha = m - 1 + \sigma\) with \(m \in \mathbb{N}\) and \(0 < \varepsilon < 1 - \sigma\), and \(1 \leq p < \infty\). Then there exists a constant \(C = C(p,\alpha,\varepsilon) > 0\) such that

\[
\|J^{\alpha-m}\nu\|_{W^{\alpha,p}(0,T)} \leq C\|\nu\|_{W^{\alpha+p,\varepsilon,p}(0,T)}
\]

for all \(\nu \in _0W^{\alpha+p,\varepsilon,p}(0,T)\).

For \(p = 1\) and \(m = 1\), the proof is found in Theorem 4.2.3 (pp.77-78) in [7]. Also our proof for the general \(m \in \mathbb{N}\) and \(1 < p < \infty\), relies on direct estimation, but is lengthy. Thus the proof of Lemma 2.4 is postponed to Section 6.

**Second Step.**

By (2.16), we can directly prove that formula (2.6) holds for \(\nu \in _0C^m[0,T]\). Indeed, by \(m - \alpha = 1 - \sigma \in (0,1)\), equality (2.16) yields

\[
\frac{d^m}{dt^m} J^{\alpha-m}\nu = J^{\alpha-m} \frac{d^m\nu}{dt^m},
\]

and so

\[
J^{\alpha} \left( \frac{d^m}{dt^m} J^{\alpha-m}\nu \right) = J^{\alpha} J^{\alpha-m} \frac{d^m\nu}{dt^m} = \frac{1}{\Gamma(m)} \int_0^t (t-s)^{m-1} \frac{d^m\nu}{ds^m}(s)ds.
\]

Here we used \(J^{\alpha} J^{\alpha-m} = J^m\) by Lemma 2.1 (i). Repeating the integration by parts and using \(\frac{d^k\nu}{ds^k}(0) = 0\) for \(0 \leq k \leq m - 1\). we see

\[
\frac{1}{\Gamma(m)} \int_0^t (t-s)^{m-1} \frac{d^m\nu}{ds^m}(s)ds = \nu(t), \quad 0 < t < T.
\]
Thus we have verified (2.6) for \( u \in C^m[0,T] \).

Next let \( u \in W^{\alpha+\varepsilon,p}(0,T) \) be arbitrarily given. By the definition, we can choose \( u_n \in C^m[0,T], n \in \mathbb{N} \), such that \( u_n \to u \) in \( W^{\alpha+\varepsilon,p}(0,T) \) as \( n \to \infty \). Applying Lemma 2.4, we see that \( J^{m-\alpha}u_n \to J^{m-\alpha}u \) in \( W^{m,p}(0,T) \), that is, \( \frac{d^m}{dt^m}J^{m-\alpha}u_n \to \frac{d^m}{dt^m}J^{m-\alpha}u \) in \( L^p(0,T) \) as \( n \to \infty \) and \( \frac{d^m}{dt^m}J^{m-\alpha}u \in L^p(0,T) \). Therefore, Lemma 2.2 yields

\[
J^{\alpha}\left(\frac{d^m}{dt^m}J^{m-\alpha}u_n\right) \to J^{\alpha}\left(\frac{d^m}{dt^m}J^{m-\alpha}u\right)
\]

in \( W_{\alpha,p}(0,T) \) as \( n \to \infty \), that is, by (2.6) with \( u_n \in C^m[0,T] \), we obtain

\[
u_n \to J^{\alpha}\left(\frac{d^m}{dt^m}J^{m-\alpha}u\right)
\]

in \( W_{\alpha,p}(0,T) \) as \( n \to \infty \). Consequently, since \( u_n \to u \) in \( W^{\alpha+\varepsilon,p}(0,T) \) as \( n \to \infty \), we verify (2.6) for all \( u \in W^{\alpha+\varepsilon,p}(0,T) \). Thus the proof of Theorem 2.1 (ii) is complete. ■

**Proof of Theorem 2.1 (iii).**

Since the embedding \( W^{\alpha+\varepsilon,p}(0,T) \to L^p(0,T) \) is compact (e.g., [1], [8]), part (i) implies

\[
J^{\alpha} : L^p(0,T) \to L^p(0,T)
\]

is a compact operator. (2.17)

This compactness is known (e.g., [9]) but it follows from (i).

Let \( u_n \in W^{\alpha+\varepsilon,p}(0,T), n \in \mathbb{N} \) satisfy

\[
\sup_{n \in \mathbb{N}}\|u_n\|_{W^{\alpha+\varepsilon,p}(0,T)} < \infty.
\]

(2.18)

Then, for each \( n \in \mathbb{N} \), there exists \( w_n \in L^p(0,T) \) such that \( u_n = J^{\alpha+\varepsilon}w_n \). Moreover the definition of the norm implies \( \|u_n\|_{W^{\alpha+\varepsilon,p}(0,T)} = \|w_n\|_{L^p(0,T)} \) for \( n \in \mathbb{N} \). Hence,

\[
\sup_{n \in \mathbb{N}}\|w_n\|_{L^p(0,T)} < \infty.
\]

(2.19)

Setting \( v_n := J^\varepsilon w_n \), the Young inequality yields

\[
\|v_n\|_{L^p(0,T)} \leq C\|w_n\|_{L^p(0,T)}.
\]

Therefore, in view of (2.17) and (2.19), we conclude that \( \{v_n\}_{n \in \mathbb{N}} \) contains a convergent subsequence \( \{v_{n(k)}\}_{k \in \mathbb{N}} \) in \( L^p(0,T) \). Since \( u_{n(k)} = J^{\alpha}J^\varepsilon w_{n(k)} = J^{\alpha}v_{n(k)} \), by the definition of the norm \( \|\cdot\|_{W^{\alpha,p}(0,T)} \), we see that \( u_{n(k)} \) is convergent in \( W^{\alpha,p}(0,T) \). Thus by (2.18), we
complete the proof of part (iii) and thus the proof of Theorem 2.1 is complete. ■

**Proof of Corollary 2.1.**

(i) We assume that
\[
\sup_{n \in \mathbb{N}} \| u_n \|_{W_{\beta,p}(0,T)} < \infty. \tag{2.20}
\]
By the definition of \( W_{\beta,p}(0,T) \), we can find \( w_n \in L^p(0,T) \) such that \( u_n = J^\beta w_n \) for \( n \in \mathbb{N} \). Moreover \( \| w_n \|_{L^p(0,T)} = \| u_n \|_{W_{\beta,p}(0,T)} \) for \( n \in \mathbb{N} \), so that (2.20) implies
\[
\sup_{n \in \mathbb{N}} \| w_n \|_{L^p(0,T)} < \infty. \tag{2.21}
\]

Since \( J^\alpha u_n = J^\alpha J^\beta w_n \), by the definition of \( \| \cdot \|_{W_{\beta,p}(0,T)} \), we deduce
\[
\| J^\alpha u_n \|_{W_{\beta,p}(0,T)} = \| J^\beta (J^\alpha u_n) \|_{W_{\beta,p}(0,T)} = \| J^\alpha w_n \|_{L^p(0,T)}.
\]
In terms of (2.17) and (2.21), there exists a sequence \( \{ n(k) \}_{k \in \mathbb{N}} \subset \mathbb{N} \) such that \( J^\alpha w_{n(k)} \) is convergent in \( L^p(0,T) \), that is, \( J^\alpha u_{n(k)} \) is convergent in \( W_{\beta,p}(0,T) \). With (2.21), we deduce that \( J^\alpha : W_{\beta,p}(0,T) \rightarrow W_{\beta,p}(0,T) \) is a compact operator. ■

(ii) If \( u \in W_{\beta,p}(0,T) \), then \( u = J^\beta w \) with \( w \in L^p(0,T) \). By Lemma 2.1 (i), we see that \( J^\alpha u = J^\alpha J^\beta w = J^{\alpha+\beta} w \in W_{\alpha+\beta,p}(0,T) \). Therefore, \( J^\alpha W_{\beta,p}(0,T) \subset W_{\alpha+\beta,p}(0,T) \).

Conversely, let \( u \in W_{\alpha+\beta,p}(0,T) \). Then there exists \( w \in L^p(0,T) \) such that \( u = J^{\alpha+\beta} w = J^\alpha (J^\beta w) \). By \( J^\beta w \in W_{\beta,p}(0,T) \), we see that \( u \in J^\alpha W_{\beta,p}(0,T) \), that is, \( W_{\alpha+\beta,p}(0,T) \subset J^\alpha W_{\beta,p}(0,T) \).

Finally we will prove the norm equivalence. Let \( u \in W_{\beta,p}(0,T) \) be arbitrarily given. Then \( u = J^\beta w \) with some \( w \in L^p(0,T) \) and \( \| u \|_{W_{\beta,p}(0,T)} = \| w \|_{L^p(0,T)} \). Moreover the definition of \( \| \cdot \|_{W_{\alpha+\beta,p}(0,T)} \) implies
\[
\| J^\alpha u \|_{W_{\alpha+\beta,p}(0,T)} = \| J^\alpha J^\beta w \|_{W_{\alpha+\beta,p}(0,T)} = \| J^{\alpha+\beta} w \|_{W_{\alpha+\beta,p}(0,T)} = \| w \|_{L^p(0,T)}.
\]
and so \( \| J^\alpha u \|_{W_{\alpha+\beta,p}(0,T)} = \| u \|_{W_{\beta,p}(0,T)} \). Thus the proof of Corollary 2.1 is complete. ■

We close this section with the following proposition which is used in Section 4 and is proved easily.

**Proposition 2.1.**
(i) $W_{\alpha,p}(0,T) \subset W_{\beta,p}(0,T)$ if $\alpha > \beta > 0$ and there exists a constant $C > 0$ such that

$$\|u\|_{W_{\beta,p}(0,T)} \leq C \|u\|_{W_{\alpha,p}(0,T)}$$

for each $u \in W_{\alpha,p}(0,T)$.

(ii) $J^\alpha \partial_t^\beta u = J^{\alpha - \beta}u$ for $\alpha > \beta > 0$ and each $u \in W_{\beta,p}(0,T)$.

Proof of Proposition 2.1.

(i) Let $u \in W_{\alpha,p}(0,T)$. Then $u = J^\alpha w$ with some $w \in L^p(0,T)$. Since $J^\alpha w = J^\beta J^{\alpha - \beta}w$ by Lemma 2.1 (i), we have $u = J^\alpha w = J^\beta(J^{\alpha - \beta}w) \in J^\beta L^p(0,T)$. Moreover, by $u = J^\alpha w = J^\beta(J^{\alpha - \beta}w)$, the definition of the norm implies that $\|u\|_{W_{\alpha,p}(0,T)} = \|w\|_{L^p(0,T)}$ and

$$\|u\|_{W_{\beta,p}(0,T)} = \|J^{\alpha - \beta}w\|_{L^p(0,T)} \leq C \|u\|_{L^p(0,T)}.$$

Therefore, $\|u\|_{W_{\beta,p}(0,T)} \leq C \|u\|_{W_{\alpha,p}(0,T)}$. Thus part (i) is proved.

(ii) Let $u \in W_{\beta,p}(0,T)$. Then $u = J^\beta w$ with some $w \in L^p(0,T)$ and

$$J^\alpha \partial_t^\beta = J^\alpha (J^\beta)^{-1} J^\beta w = J^\alpha w.$$

On the other hand, by $\alpha - \beta, \beta > 0$, Lemma 2.1 (i) implies

$$J^{\alpha - \beta}u = J^{\alpha - \beta} J^\beta w = J^{(\alpha - \beta) + \beta}w = J^\alpha w.$$

Hence $J^\alpha \partial_t^\beta = J^{\alpha - \beta}u$ for $u \in W_{\beta,p}(0,T)$. Thus the proof of Proposition 2.1 is complete.

3. Fractional derivative $\partial_t^\alpha$ in $L^p(0,T)$

We consider an initial value problem (1.2) and (1.3):

$$d_t^\alpha u(t) = b(t)u(t) + f(t), \quad 0 < t < T, \quad u(0) = a,$$

where $f$ is singular in the sense that $f \not\in L^r(0,T)$ for any $r \geq 1$. For example, let $f(t)$ be the Dirac delta function which means an impulsive source at $t = 0$. It is desirable to construct a framework in order to treat such singular terms in fractional differential equations. For the mathematical treatments, we need the formulation of time-fractional derivative $\partial_t^\alpha$ in Sobolev spaces of non-positive orders. In Yamamoto [23], we find such studies in the space $L^2(0,T)$ and the treatment with fixed $p = 2$ is less flexible.

In this section, we define $\partial_t^\alpha$ in $L^p(0,T)$-based Sobolev-Slobodecki spaces of negative orders. Our extension of the domain of $\partial_t^\alpha$ from $W_{\alpha,p}(0,T)$, relies on the adjoint of fractional
differential operator.

We set

\[ J_\alpha v(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (\xi - t)^{\alpha - 1} v(\xi) d\xi, \quad 0 < t < T, \ v \in L^1(0, T). \quad (3.1) \]

Then, similarly to Lemma 2.1, we can prove

**Lemma 3.1.**

Let \( \alpha, \beta > 0 \).

(i) \( J_\alpha J_\beta v = J_{\alpha + \beta} v \) for \( v \in L^1(0, T) \).

(ii) \( J_\alpha : L^p(0, T) \to L^p(0, T) \) is an injective compact operator.

We can prove directly but we can transform the corresponding results for \( J^\alpha \) through the following transformation:

\[ \tau : L^p(0, T) \to L^p(0, T), \quad (\tau v)(t) = v(T - t), \quad 0 < t < T. \quad (3.2) \]

For example, as is readily verified,

\[ J_\alpha v(t) = (\tau J^\alpha (\tau v))(t), \quad 0 < t < T, \ v \in L^p(0, T). \quad (3.3) \]

In view of (3.3), Lemma 3.1 follows directly from Lemma 2.1.

In the same way as in Section 2, for \( 1 \leq p < \infty \) and \( \alpha > 0 \), by Lemma 3.1, we can define

\[ \alpha, p W(0, T) := J_\alpha L^p(0, T), \quad \| u \|_{\alpha, p W(0, T)} := \| (J_\alpha)^{-1} u \|_{L^p(0, T)} \quad \text{for} \ u \in \alpha, p W(0, T). \quad (3.4) \]

Moreover, we can similarly prove

**Lemma 3.2.**

Let \( \alpha > 0 \). If \( w_n \rightharpoonup w \) in \( L^p(0, T) \), then \( J_\alpha w_n \rightharpoonup J_\alpha w \) in \( \alpha, p W(0, T) \).

**Lemma 3.3.**

The space \( \alpha, p W(0, T) \) is a Banach space with the norm \( \| \cdot \|_{\alpha, p W(0, T)} \).

For \( m \in \mathbb{N} \), we set

\[ 0C^m[0, T] := \left\{ v \in C^m[0, T]; \frac{d^k v}{dt^k}(T) = 0 \quad \text{for} \ k = 0, 1, \ldots, m - 1 \right\} \]

and

\[ 0W^{\alpha, p}(0, T) := 0C^m[0, T]^{W^{\alpha, p}(0, T)}. \]
We remark that \( C^m[0, T] = \tau_0 C^m[0, T] \) and \( \tau : W_{\alpha,p}(0, T) \rightarrow \alpha,p W(0, T) \) is an isomorphism.

Then, by means of Theorem 2.1 and the symmetric transform \( \tau \) in \( t \), we can readily prove

**Theorem 3.1.**

Let \( \alpha > 0 \), \( \notin \mathbb{N} \) and \( 0 < \varepsilon < \alpha \).

(i) \( 0 W^{\alpha+\varepsilon,p}(0, T) \subset \alpha,p W(0, T) \subset 0 W^{\alpha-\varepsilon,p}(0, T) \). Moreover there exists a constant \( C = C(\alpha, \varepsilon) > 0 \) such that

\[
\| J_\alpha v \|_{W^{\alpha-\varepsilon,p}(0,T)} \leq C \| v \|_{L^p(0,T)} \quad \text{for all } v \in L^p(0,T).
\]

(ii) The embedding \( \alpha+\varepsilon,p W(0, T) \rightarrow \alpha,p W(0, T) \) is compact.

(iii) For \( \alpha > 0 \) and \( \beta \geq 0 \), the operator \( J_\alpha : \beta,p W(0, T) \rightarrow \beta,p W(0, T) \) is compact.

We will use \( J_\alpha \) in order to define \( \partial_t^\alpha \) in Sobolev spaces of negative orders through the adjoint. As for notations and terminologies, we follow Brezis [2].

Let \( \alpha,p W(0, T)^* \) denote the dual space of \( \alpha,p W(0, T) \), that is, the space of all the bounded linear real-valued functionals defined over \( \alpha,p W(0, T) \).

In place of \( u(\psi) \), by \( \alpha,p W(0,T)^* \subset \alpha,p W(0,T) \), we denote the value of \( u \) at \( \psi \in \alpha,p W(0,T) \). As the norm \( \| \cdot \|_{\alpha,p W(0,T)^*} \) we follow a usual definition:

\[
\| u \|_{\alpha,p W(0,T)^*} := \sup_{\| \psi \|_{\alpha,p W(0,T)} = 1} \left| \alpha,p W(0,T)^* \subset u, \psi > \alpha,p W(0,T) \right|.
\]

Henceforth, throughout this article, for \( 1 \leq p < \infty \), we define \( 1 < q \leq \infty \) by

\[
q = \begin{cases} 
\frac{p}{p-1} & \text{if } p > 1, \\
\infty & \text{if } p = 1,
\end{cases}
\]

and we set

\[
L^q(0,T)(v, u)_{L^p(0,T)} := \int_0^T u(t)v(t)dt, \quad v \in L^q(0,T), \ u \in L^p(0,T) \quad \text{with} \ 1 \leq p < \infty.
\]

We note that \( \frac{1}{p} + \frac{1}{q} = 1 \) and

\[
L^p(0,T)^* = L^q(0,T) \quad \text{if} \ 1 \leq p < \infty.
\]
Example of $\alpha, p W(0, T)^*$. Let $\alpha p > 1$. Choosing $\varepsilon > 0$ sufficiently small such that $(\alpha - \varepsilon)p > 1$, by the Sobolev embedding (e.g., [1], [3]) by Theorem 3.1 (i) we verify that $\alpha, p W(0, T) \subset C[0, T]$. Therefore, for $t_0 \in [0, T]$, we define $\delta_{t_0} \in \alpha, p W(0, T)^*$ by

$$\alpha, p W(0, T)^* \ni \delta_{t_0}, \psi = \psi(t_0)$$

and we can see $\delta_{t_0} \in \alpha, p W(0, T)^*$ if $\alpha p > 1$.

We here sum up useful results.

Lemma 3.4.

(i) $C^{-1} \|J_\alpha u\|_{L^p(0,T)} \leq \|u\|_{L^p(0,T)} \leq C \|J_\alpha u\|_{\alpha, p W(0,T)}$ for each $u \in L^p(0,T)$.

(ii) For arbitrarily chosen $v \in L^q(0,T)$, we define a mapping:

$$F_v : u \mapsto L^q(0,T)(v, u)_{L^p(0,T)} \text{ for each } u \in \alpha, p W(0,T).$$

Then, $F_v$ is a bounded linear functional on $\alpha, p W(0,T)$. Identifying $F_v \in \alpha, p W(0,T)^*$ with $v, in place of $F_v(u)$ we write

$$\alpha, p W(0,T)^* \ni v, \psi = \psi(t_0),$$

so that

$$\|F_v\|_{\alpha, p W(0,T)^*} = \|v\|_{\alpha, p W(0,T)^*} \leq C \|v\|_{L^q(0,T)}.$$

(iii)

$$J_\alpha^* : \alpha + \beta, p W(0, T)^* \rightarrow \beta, p W(0, T)^*$$

is an isomorphism. In particular,

$$\|J_\alpha^* u\|_{\beta, p W(0, T)^*} = \|u\|_{\alpha + \beta, p W(0, T)^*} \text{ for all } u \in \alpha + \beta, p W(0, T)^*.$$

Proof of Lemma 3.4.

(i) By the definition of the norm $\| \cdot \|_{\alpha, p W(0,T)}$, we have $\|J_\alpha u\|_{\alpha, p W(0,T)} = \|u\|_{L^p(0,T)}$ for any $u \in L^p(0,T)$. The Young inequality implies $C \|J_\alpha u\|_{L^p(0,T)} \leq \|u\|_{L^p(0,T)}$. Thus the proof of the lemma (i) is complete. ■
(ii) Immediately we see that $F_v$ is well-defined for $u \in \alpha,p W(0,T)$ and is a linear mapping. Setting $u := J_\alpha w$ with $w \in L^p(0,T)$, we have $\|u\|_{\alpha,p W(0,T)} = \|J_\alpha w\|_{\alpha,p W(0,T)} = \|w\|_{L^p(0,T)}$; by Lemma 3.4 (i) we obtain

$$
\|v\|_{\alpha,p W(0,T)} = \sup_{\|u\|_{\alpha,p W(0,T)} = 1} |L^q(0,T)(v, u)_{L^p(0,T)}| = \sup_{\|w\|_{L^p(0,T)} = 1} |L^q(0,T)(v, J_\alpha w)_{L^p(0,T)}|
$$

$$
\leq \|v\|_{L^q(0,T)} \sup_{\|w\|_{L^p(0,T)} = 1} \|J_\alpha w\|_{L^p(0,T)} \leq C \|v\|_{L^q(0,T)}.
$$

Hence, $\|v\|_{\alpha,p W(0,T)} \leq C \|v\|_{L^q(0,T)}$. Thus the proof of (ii) is complete. ■

(iii) Thanks to the operator $\tau$ defined by (3.2), Corollary 2.1 (ii) yields that

$$
J_\alpha : \beta,p W(0,T) \rightarrow \alpha+\beta,p W(0,T)
$$

is an isomorphism for $\alpha > 0$ and $\beta \geq 0$. Consequently, part (iii) follows directly by the closed range theorem (e.g., Section 7 in Chapter 2 of [2]). Thus the proof of Lemma 3.4 is complete. ■

For arbitrarily chosen $v \in L^q(0,T)$ and $u \in L^p(0,T)$, we apply Lemma 3.4 (ii) by setting $u := J_\alpha u$, we see that

$$
\alpha,p W(0,T) < v, J_\alpha u > \alpha,p W(0,T) := L^q(0,T)(v, J_\alpha u)_{L^p(0,T)}
$$

(3.5)

for all $u \in L^p(0,T)$ and $v \in L^q(0,T)$.

On the other hand, exchanging the order of the integral $\int_0^t (\int_0^s \cdots d\xi) \, ds = \int_0^t (\int_s^t \cdots ds) \, d\xi$, we can readily verify

$$
L^q(0,T)(J_\alpha v, u)_{L^p(0,T)} = L^q(0,T)(v, J_\alpha u)_{L^p(0,T)} \quad \text{for all } u \in L^p(0,T) \text{ and } v \in L^q(0,T).
$$

In terms of (3.5), we can reach

$$
\alpha,p W(0,T) < v, J_\alpha u > \alpha,p W(0,T) = L^q(0,T)(J_\alpha v, u)_{L^p(0,T)}
$$

(3.6)

for all $u \in L^p(0,T)$ and $v \in L^q(0,T)$. We note that since $J_\alpha : L^p(0,T) \rightarrow \alpha,p W(0,T)$ is bounded, the adjoint operator $J_\alpha^* : \alpha,p W(0,T)^* \rightarrow L^p(0,T)^* = L^q(0,T)$ exists and $J_\alpha^*$ is the operator with the maximal domain among operators $J : \alpha,p W(0,T)^* \rightarrow L^q(0,T)$ satisfying

$$
\alpha,p W(0,T)^* < v, J_\alpha^* u > \alpha,p W(0,T) = L^q(0,T)(J^* v, u)_{L^p(0,T)}
$$

(3.7)
for all \( v \in \mathcal{D}(J) \subset \alpha,p W(0,T)^* \) and \( u \in L^p(0,T) \). By taking \( J := J^\alpha \) with \( \mathcal{D}(J) = L^q(0,T) \), from (3.6) it follows that (3.7) holds. Therefore, the maximality of \( J^\alpha \) yields

\[
J^\alpha \subset J^\alpha, \quad \mathcal{D}(J^\alpha) = \alpha,p W(0,T) \supset L^q(0,T).
\] (3.8)

Now we define

Definition 3.1.

We define

\[
\partial^\alpha_t = (J^\alpha)^{-1}, \quad \mathcal{D}(\partial^\alpha_t) = \alpha,p W(0,T)^*.
\]

In view of (3.8), we extend \( \partial^\alpha_t \) defined on \( W_{\alpha,p}(0,T) \) to the domain \( \alpha,p W(0,T)^* \), which means that \( \partial^\alpha_t \) in Definitions 2.1 and 3.1 coincide in \( W_{\alpha,p}(0,T) \):

\[
\partial^\alpha_t u = (J^\alpha)^{-1} u = (J^\alpha)^{-1} u \quad \text{if} \quad u \in W_{\alpha,p}(0,T).
\]

We can generalize the domain of \( \partial^\alpha_t \) to \( \beta,p W(0,T)^* \) with arbitrary \( \beta > 0 \) and refer to Yamamoto [23] for the case of \( p = 2 \). Furthermore we can prove several fundamental formulae in the fractional calculus such as \( \partial^\alpha_t \partial^\beta_t = \partial^{\alpha+\beta} t \) in \( \alpha+\beta,p W(0,T)^* \) and \( W_{\alpha+\beta,p}(0,T) \). However, we here omit the details and proceed to time-fractional ordinary differential equations.

4. Initial value problems for fractional ordinary differential equations

We have constructed \( \partial^\alpha_t \) in \( L^p(0,T) \)-based Sobolev spaces in Sections 2 and 3. Although we should widely pursue the fractional calculus of \( \partial^\alpha_t \), our main purpose is to study initial value problems for time-fractional ordinary differential equations and initial boundary value problems for time-fractional partial differential equations within \( L^p \)-spaces. See [23] as for some fractional calculus of \( \partial^\alpha_t \) for \( p = 2 \).

In this article, we are restricted to initial value problems for simple fractional ordinary differential equations and illustrate the well-posedness of solution within the framework of \( \partial^\alpha_t \) in \( W_{\alpha,p}(0,T) \) or \( \alpha,p W(0,T)^* \).

We can make comprehensive studies for general classes, but we consider only simple types of single linear fractional ordinary differential equations in the case of \( 0 < \alpha < 1 \). See Yamamoto [22] as for similar treatments in the case of \( p = 2 \). One can refer to Diethelm [4],
Jin [10], which treat various topics concerning fractional derivatives and differential equations from different viewpoints.

Throughout this section, we assume that 
\[ 1 \leq p < \infty, \quad 0 < \alpha < 1, \quad a \in \mathbb{R}. \]

**Section 4.1. Single linear fractional differential equation with bounded coefficient**

We consider
\[ \partial_t^\alpha (u(t) - a) = b(t)u + f(t), \quad 0 < t < T, \] (4.1)
and
\[ u - a \in W_{\alpha,p}(0,T). \] (4.2)

By the definition of \( \partial_t^\alpha \) in \( W_{\alpha,p}(0,T) \), we see that the initial value problem (4.1) - (4.2) is equivalent to
\[ u - a = J^\alpha (b(t)u) + J^\alpha f(t), \quad 0 < t < T. \] (4.3)

Similarly to [15] and [23], we interpret the initial condition by (4.2). If \( \alpha p > 1 \), then the Sobolev embedding (e.g., [8]) and Theorem 2.1 (i) yield that \( W_{\alpha,p}(0,T) \subset C[0,T] \), so that we conclude that \( u \in C[0,T] \) and \( \lim_{t \to 0} u(t) = a \), that is, the initial condition (4.2) can be understood in a usual sense. If \( \alpha p \leq 1 \), then such an interpretation is impossible, but we can prove the unique existence of solution to (4.1)-(4.2) for all \( 0 < \alpha < 1 \) and \( 1 \leq p < \infty \).

**Theorem 4.1.**

Let \( b \in L^\infty(0,T) \) and \( f \in L^p(0,T) \). Then there exists a unique solution \( u = u(t) \) to (4.1) - (4.2) and we can find a constant \( C > 0 \) such that
\[ \| u - a \|_{W_{\alpha,p}(0,T)} \leq C(\| f \|_{L^p(0,T)} + |a|) \] (4.4)
for all \( f \in L^p(0,T) \) and \( a \in \mathbb{R} \).

For \( p \neq 2 \), we can not completely characterize \( W_{\alpha,p}(0,T) \), but Theorem 2 (i) and (ii) provide the properties of the domain \( W_{\alpha,p}(0,T) \) of \( \partial_t^\alpha \) intermediated between \( \partial_t^\alpha W^{\alpha+\varepsilon,p}(0,T) \) and \( \partial_t^\alpha W^{\alpha-\varepsilon,p}(0,T) \) with any small gap \( \varepsilon > 0 \), and we can apply \( \partial_t^\alpha \) for the regularity of solutions to fractional equations in a flexible way.
If \( \alpha p < 1 \), then
\[
a = \frac{a}{\Gamma(\alpha)\Gamma(1 - \alpha)} \int_0^t (t - s)^{\alpha - 1}s^{-\alpha}ds \quad \text{for } 0 < t < T \quad \text{and } s^{-\alpha} \in L^p(0, T),
\]
so that \( a \in W_{\alpha,p}(0, T) \). Therefore for \( \alpha p < 1 \), the estimate (4.4) is rewritten as
\[
\|u\|_{W_{\alpha,p}(0, T)} \leq C\left(\|f\|_{L^p(0, T)} + |a|\right).
\]
However, if \( \alpha p \geq 1 \), then we cannot obtain the above estimate.

In the case \( b \in L^\infty(0, T) \), Theorem 4.1 can be found in the existing works (e.g., Theorem 3.2 (p.124) in [20]). We here present Theorem 4.1 in order to illustrate our arguments which can work for more general cases, as are discussed in Sections 4.3 and 4.4 later.

**Proof of Theorem 4.1.**

It suffices to consider (4.3). Setting \( v := u - a \) and defining an operator \( K : L^p(0, T) \rightarrow L^p(0, T) \) by \( K v(t) := J^\alpha(b(t)v) \), we see that (4.3) is equivalent to
\[
v(t) = K v(t) + J^\alpha(b(t)a + f(t)), \quad 0 < t < T. \tag{4.5}
\]
The theorem will be proved if we can show that the operator \( K \) possesses a unique fixed point.

By \( b \in L^\infty(0, T) \), we deduce that \( v \in L^p(0, T) \rightarrow bv \in L^p(0, T) \) is a bounded operator. By (2.17), it follows that \( v \in L^p(0, T) \rightarrow J^\alpha(bv) \in L^p(0, T) \) is a compact operator. Therefore, \( K : L^p(0, T) \rightarrow L^p(0, T) \) is compact.

Assume that \( a = 0 \) and \( f = 0 \) in (4.5), that is, \( v \in L^p(0, T) \) satisfies \( v = K v \) in \((0, T)\). Then
\[
v(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1}v(s)ds, \quad 0 < t < T.
\]
Hence,
\[
|v(t)| \leq C \int_0^t (t - s)^{\alpha - 1}|v(s)|ds, \quad 0 < t < T.
\]
By the generalized Gronwall inequality (e.g., Lemma 7.1.1 (p.188) in Henry [9] or Lemma A.2 in [13]), we obtain that \( v = 0 \) in \((0, T)\). Consequently the Fredholm alternative implies that there exists a unique fixed point to (4.5) and
\[
\|v\|_{L^p(0, T)} \leq C(\|J^\alpha(ba + f)\|_{L^p(0, T)} \leq C(|a| + \|f\|_{L^p(0, T)}))
\]
by $b \in L^\infty(0,T)$. Hence, since $v = J^\alpha(bv + ba + f)$ by (4.5), using the norm of $\| \cdot \|_{W_\alpha,p(0,T)}$ and $b \in L^\infty(0,T)$, we obtain

$$\|v\|_{W_\alpha,p(0,T)} = \|bv + ba + f\|_{L^p(0,T)} \leq C(|a| + \|f\|_{L^p(0,T)}).$$

Thus the proof of Theorem 4.1 is complete. ■

Section 4.2. Single linear multi-term fractional differential equation with bounded coefficient

Let $0 < \alpha_1 < \cdots < \alpha_N < \alpha < 1$.

We consider an initial value problem

$$\begin{cases}
\partial_t^\alpha (u - a) + \sum_{k=1}^N b_k(t) \partial_t^{\alpha_k} (u - a) = b(t)u(t) + f(t), & 0 < t < T, \\
u - a \in W_\alpha,p(0,T).
\end{cases} \quad (4.6)$$

By Proposition 2.1 (i), we note that $W_\alpha,p(0,T) \subset W_{\alpha_k,p}(0,T)$, and so $\partial_t^{\alpha_k} (u - a) \in L^p(0,T)$ with $k = 1, \ldots, N$, are well-defined if $u - a \in W_\alpha,p(0,T)$. Now we can prove

**Theorem 4.2.**

Let $b_1, \ldots, b_N, b \in L^\infty(0,T)$ and $f \in L^p(0,T)$. Then there exists a unique solution $u = u(t)$ to (4.6) and we can find a constant $C >$ such that the estimate (4.4) holds for all $f \in L^p(0,T)$ and $a \in \mathbb{R}$.

**Proof of Theorem 4.2.**

Setting $w := \partial_t^\alpha (u - a) = (J^\alpha)^{-1}(u - a)$, by Lemma 2.1 (i) we have $u - a = J^\alpha w$ and

$$\partial_t^{\alpha_k} (u - a) = \partial_t^{\alpha_k} J^\alpha w = \partial_t^{\alpha_k} (J^{\alpha_k} J^{\alpha - \alpha_k})w = J^{\alpha - \alpha_k} w.$$

Then (4.6) is equivalent to

$$w(t) = -\sum_{k=1}^N b_k(t) J^{\alpha - \alpha_k} w + b(t) J^\alpha w + b(t)a + f(t), \quad 0 < t < T.$$

By (2.17) and $b, b_1, \ldots, b_N \in L^\infty(0,T)$, we can verify that the operator

$$Kw(t) := -\sum_{k=1}^N b_k(t) J^{\alpha - \alpha_k} w + b(t) J^\alpha w$$
is compact from \( L^p(0,T) \) to itself. Similarly to Theorem 4.1, we can verify the unique existence of the fixed point to \( w = Kw + (ba + f) \). Thus we can complete the proof of Theorem 4.2. ■

Section 4.3. Single linear fractional differential equation with unbounded coefficient

We return to a simple equation:

\[
\partial_t^\alpha (u - a) = b(t)u, \quad 0 < t < T, \quad u - a \in W_{\alpha,p}(0,T). \tag{4.7}
\]

Here we study the unique existence and the regularity of the solution to (4.7) for \( b \in L^p(0,T) \) with \( 1 \leq p < \infty \) within our framework.

As is discussed in Section 4.1, the regularity \( b \in L^\infty(0,T) \) makes the total arguments very simple, and for \( b \notin L^\infty(0,T) \), we need more careful discussions.

We remark that the case \( \alpha = 1 \) does not require us any special consideration because a formula of solution

\[
u(t) = a \exp \left( \int_0^t b(s)ds \right), \quad 0 < t < T
\]

implies the unique existence of the solution in \( W^{1,\infty}(0,T) \) for arbitrary \( b \in L^1(0,T) \). However, for \( 0 < \alpha < 1 \), we do not have such an explicit formula of solution to (4.7) and we need proper arguments even though the equation in (4.7) is extremely simple. In order to concentrate on the regularity issue of the coefficient \( b(t) \), we consider (4.7) without the right-hand side of the equation.

In terms of \( \partial_t^\alpha \) with \( \mathcal{D}(\partial_t^\alpha) = W_{\alpha,p}(0,T) \), we prove

Theorem 4.3.

Let \( 0 < \alpha < 1 \). We assume that

\[
b \in L^q(0,T) \quad \text{with} \quad 1 < q < \infty. \tag{4.8}
\]

If

\[
1 < p < \infty, \quad \frac{1}{\alpha} < q < \infty, \quad p - 1 > \frac{1}{q - 1}, \tag{4.9}
\]

then there exists a unique solution to (4.7) such that

\[
u - a \in W_{\alpha, \frac{pq}{p+q}}(0,T), \quad u \in L^p(0,T). \tag{4.10}
\]
Moreover there exists a constant $C > 0$ such that

$$
\|u - a\|_{W^{\alpha, \frac{p}{p}}(0, T)} + \|u\|_{L^p(0, T)} \leq C|a|
$$

for all $a \in \mathbb{R}$.

As is seen by the proof below, if $1 < q \leq \frac{1}{\alpha}$, then our proof does not work, and we do not know the unique existence of $u$.

**Proof of Theorem 4.3.**

Setting $v := u - a$, we see that (4.7) is equivalent to

$$
v = Kv(t) + aJ^\alpha b(t), \quad 0 < t < T,
$$

where we set $Kv(t) := J^\alpha (b(t)v(t)), 0 < t < T$ for $v \in L^p(0, T)$. Theorem 2.1 yields

$$
J^\alpha b \in W_{\alpha, q}(0, T) \subset W^\alpha_{\alpha-\varepsilon, q}(0, T).
$$

Here $\varepsilon > 0$ is a sufficiently small constant.

We note that $p - 1 > \frac{1}{q - 1}$ in (4.9) implies $p > 1$. We set

$$
r = \frac{pq}{p + q}.
$$

Then by the same condition in (4.9), we deduce

$$
1 < r \leq p, q, \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q}. \quad (4.13)
$$

Therefore, for $b \in L^q(0, T)$, the H"older inequality implies $bv \in L^r(0, T)$ for $v \in L^p(0, T)$, Theorem 2.1 (i) implies

$$
J^\alpha (bv) \in W^{\alpha - \varepsilon, r}(0, T) \text{ if } v \in L^p(0, T). \quad (4.14)
$$

We choose a small constant $\varepsilon > 0$ such that $0 < \varepsilon < \frac{q}{r}$. Since $q \geq r$ in (4.13) yields $W^{\alpha - \varepsilon, q}(0, T) \subset W^{\alpha - \varepsilon, r}(0, T)$, so that by (4.12) we have $J^\alpha b \in W^{\alpha - \varepsilon, r}(0, T)$.

Therefore, by (4.14), we see that if

$$
W^{\alpha - 2\varepsilon, r}(0, T) \subset L^p(0, T), \quad (4.15)
$$

then

$$
Kv + aJ^\alpha b \in L^p(0, T) \quad \text{for each } v \in L^p(0, T).
$$
Moreover, Theorem 2.1 (i) and (4.14) yield that \( v \in L^p(0,T) \rightarrow J^{\alpha}(bv) \in W^{\alpha-\varepsilon,r}(0,T) \) is a bounded operator. Since the embedding \( W^{\alpha-\varepsilon,r}(0,T) \rightarrow W^{\alpha-2\varepsilon,r}(0,T) \) is compact, it follows that under (4.15), the operator \( v \in L^p(0,T) \rightarrow Kv \in L^p(0,T) \) is compact.

Thus in view of the Fredholm alternative, similarly to the proof of Theorem 4.1, it is sufficient to verify that (4.9) implies (4.15). We consider the following three cases separately.

**Case 1:** \( \alpha r < 1 \):

Under condition \( q > \frac{1}{\alpha} \), we see that \( \alpha r < 1 \) is equivalent to

\[
p < \frac{q}{\alpha q - 1}. \tag{4.16}
\]

By \((\alpha - 2\varepsilon)r < 1\), the Sobolev embedding implies

\[
W^{\alpha-2\varepsilon,r}(0,T) \subset L^{\frac{r}{1-(\alpha-2\varepsilon)r}}(0,T).
\]

We can directly verify that \( q > \frac{1}{\alpha} \) yields \( \frac{r}{1-\alpha r} > p \). Hence, with sufficiently small \( \varepsilon > 0 \), we have \( \frac{r}{1-(\alpha-2\varepsilon)r} > p \), that is,

\[
L^p(0,T) \supset L^{\frac{r}{1-(\alpha-2\varepsilon)r}}(0,T).
\]

Thus (4.15) holds if (4.9) and (4.16) are satisfied.

**Case 2:** \( \alpha r = 1 \).

We note that \( \alpha r = 1 \) is equivalent to \( p = \frac{q}{\alpha q - 1} \). For sufficiently small \( \varepsilon > 0 \), we have \((\alpha - 2\varepsilon)r < 1\) and

\[
W^{\alpha-2\varepsilon,r}(0,T) \supset L^{\frac{r}{1-(\alpha-2\varepsilon)r}}(0,T)
\]

by the Sobolev embedding. Therefore, in the same was as Case 1, we can verify (4.15) under (4.9) and

\[
p = \frac{q}{\alpha q - 1}. \tag{4.17}
\]

**Case 3:** \( \alpha r > 1 \).

We note that \( \alpha r > 1 \) is equivalent to \( p > \frac{q}{\alpha q - 1} \). Choosing \( \varepsilon > 0 \) sufficiently small, we obtain \((\alpha - 2\varepsilon)r > 1\). Therefore, the Sobolev embedding yields that \( W^{\alpha-2\varepsilon,r}(0,T) \subset L^\infty(0,T) \). Hence, (4.15) holds if (4.9) and

\[
p > \frac{q}{\alpha q - 1}. \tag{4.18}
\]

are satisfied.
Thus, taking the union of the sets of \((p, q)\) satisfying (4.16) - (4.18), we verify that (4.15) holds if (4.9) is satisfied. Thus we complete the proof of Theorem 4.3. ■

Section 4.4. Single linear fractional differential equation with singular non-homogeneous term

Let \(0 < \alpha < 1\), \(\beta \geq 0\) and \(1 < p \leq \infty\). Henceforth we define \(1 \leq q < \infty\) by

\[
p = \begin{cases} 
\frac{q}{q-1} & \text{if } 1 < q < \infty, \\
\infty & \text{if } q = 1.
\end{cases}
\]

We consider

\[
\begin{aligned}
\partial_t^\alpha (u - a) &= b(t)u(t) + f(t), \quad 0 < t < T, \\
u - a &\in L^p(0, T),
\end{aligned}
\]

where \(b \in L^\infty(0, T)\) and \(f \in \alpha,q W(0, T)^*\).

In (4.19), we understand as

\[
\partial_t^\alpha = (J_\alpha^*)^{-1} : L^p(0, T)^* \rightarrow \alpha,q W(0, T)^*,
\]

that is,

\[
\begin{aligned}
\partial_t^\alpha : L^p(0, T) &\rightarrow \alpha,q W(0, T)^* \quad \text{if } 1 < p < \infty, \\
\partial_t^\alpha : L^\infty(0, T) &\rightarrow \alpha,1 W(0, T)^* \quad \text{if } p = \infty.
\end{aligned}
\]

Moreover, by Lemma 3.4 (iii), we see that

\[
J_\alpha^* : \alpha+\beta,q W(0, T)^* \rightarrow \beta,q W(0, T)^*
\]

is an isomorphism.

In terms of (4.20), by setting \(v := u - a\), the initial value problem (4.19) is equivalent to

\[
\begin{aligned}
v &= J_\alpha^* (bv) + J_\alpha^* (ba + f), \\
v &\in L^p(0, T).
\end{aligned}
\]

For simplicity, we assume that \(1 < p < \infty\). Now we can prove

**Theorem 4.4.**

*Let \(b \in L^\infty(0, T)\) be arbitrary. Then there exists a unique solution \(u\) to (4.19) and we can find a constant \(C > 0\) such that*

\[
\|u - a\|_{L^p(0, T)} \leq C(|a| + \|f\|_{\alpha,q W(0, T)^*})
\]
for all $a \in \mathbb{R}$ and $f \in \alpha,q W(0,T)^*$.

Example.

Given constant $t_0 \in [0,T]$, we consider a Dirac delta function $f(t) := \delta_{t_0}$, that is, $C^{(0,T)} < \delta_{t_0}, \psi > C^{[0,T]} := \psi(t_0)$ for all $\psi \in C[0,T]$. Let $\alpha q > 1$ with $q = \frac{p}{p-1}$. Then the Sobolev embedding yields that $\alpha,q W(0,T) \subset C[0,T]$, and so it turns that $\delta_{t_0} \in W_{\alpha,q}(0,T)^*$.

Therefore Theorem 4.4 asserts that there exists a unique solution $u \in L^p(0,T)$ to $\partial_t^\alpha (u-a) = b(t)u(t) + \delta_{t_0}(t)$ in $\alpha,q W(0,T)^*$.

Proof of Theorem 4.4.

By Lemma 3.4 (iii), noting that $\alpha,p W(0,T)^* = L^q(0,T)^* = L^p(0,T)$, we obtain

$$\| J^*_\alpha f \|_{L^p(0,T)} \leq C \| f \|_{\alpha,q W(0,T)^*}.$$ 

By $b \in L^\infty(0,T)$, we apply (3.8) to have $J^*_\alpha b = \alpha b \in L^\infty(0,T)$ by the Young inequality. Therefore,

$$J^*_\alpha (ba + f) \in L^p(0,T), \quad \| J^*_\alpha (ba + f) \|_{L^p(0,T)} \leq C (|a| + \| f \|_{\alpha,q W(0,T)^*}). \quad (4.22)$$

Since $b \in L^\infty(0,T)$ implies $bv \in L^p(0,T)$. Again by (3.8), we deduce that $J^*_\alpha (bv) = \alpha (bv)$ for all $v \in L^p(0,T)$. Thus (4.21) is rewritten as

$$v = J^\alpha (bv) + G, \quad v \in L^p(0,T),$$

where $G := J^\alpha (ba + f) \in L^p(0,T)$. Consequently our argument can be executed within $L^p(0,T)$, and we can repeat the proof of Theorem 4.1. Thus the proof of Theorem 4.4 is complete. ■

5. Concluding remarks

1. In Section 2 of this article, we have established a time-fractional derivative $\partial_t^\alpha$ in $L^p(0,T)$-based Sobolev-Slobodecki spaces for $1 \leq p < \infty$. Theorem 2.2 provides characterization of the domain $\mathcal{D}(\partial_t^\alpha)$ in terms of the Sobolev-Slobodecki spaces, which still admits a gap with any small $\varepsilon > 0$ in Sobolev orders.

We remark that the domain $\mathcal{D}(\partial_t^\alpha)$ in a special case $p = 2$, is completely characterized by the Sobolev-Slobodecki spaces (see [6], [15], [23]).

2. By the duality, in Section 3, for $1 \leq p < \infty$, we extend the domain $\mathcal{D}(\partial_t^\alpha)$ to $L^p(0,T)$ with the range in the dual space $\alpha,p W(0,T)^*$. 


3. In Section 4, we established the unique existence of solutions to initial value problems for fractional ordinary differential equations on the basis of \( \frac{\partial}{\partial t^\alpha} \). We can develop more comprehensive treatments for wider classes of equations, but we postpone them and are restricted to single fractional equations. Even for such simple equations, there are no works for example in the case where a coefficient \( b(t) \) is not bounded or non-homogeneous term \( f(t) \) does not belong to \( L^p(0, T) \), and we established the unique existence of solution for \( b \in L^p(0, T) \) with \( p \neq \infty \) or \( f \in \alpha,q W(0, T)^* \) with \( \alpha > 0 \).

Furthermore we here considered \( \frac{\partial}{\partial t^\alpha} \) only in the case of \( 0 < \alpha < 1 \) although in Sections 2 and 3 the fractional derivative \( \frac{\partial}{\partial t^\alpha} \) is defined for all \( \alpha > 0, \notin \mathbb{N} \), and more general treatments will be provided in a future work.

We recall that Theorems 2.1 and 2.2 give characterization of the domain of \( \frac{\partial}{\partial t^\alpha} \) which is not the best possible because of regularity loss \( \varepsilon > 0 \), even though \( \varepsilon > 0 \) can be chosen arbitrarily small. However, for studying fractional differential equations, such characterization of the domain by the Sobolev spaces is used mainly for applying the Sobolev embedding, and, as is illustrated in Section 4, we emphasize that the \( \varepsilon \)-gap is not a serious disadvantage.

4. We do not discuss time-fractional partial differential equations at all. The related topics should be studied also in future works for \( p \geq 1 \). We remark that the case \( p = 2 \) is relatively satisfactorily argued already in e.g., [6, 15].

6. Appendix: Proof of Lemma 2.4.

For \( m = 1 \) and \( p = 1 \), Lemma 2.4 is proved as Theorem 4.2.3 (p.77) in [7], and our proof is based on the expression (6.1) (which is used also on p.77 of [7]).

First Step: \( 0 < \alpha < 1 \).

We note that \( gC^1[0, T] \subset W^{\alpha + \varepsilon, p}(0, T) \) if \( 0 < \alpha + \varepsilon < 1 \). Let \( u \in gC^1[0, T] \). Then

\[
J^{1-\alpha}u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u(s) ds
\]

\[
= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} (u(s) - u(t)) ds + \frac{u(t)}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} ds,
\]

that is,

\[
J^{1-\alpha}u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} (u(s) - u(t)) ds + \frac{u(t)t^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}, \quad 0 < t < T.
\]
Since \( u \in \mathcal{O}C^1[0,T] \), we see
\[ |(t-s)^{-\alpha}(u(s) - u(t))| \leq C|t-s|^{1-\alpha} \]
and
\[ \left| \frac{\partial}{\partial t}((t-s)^{-\alpha}(u(s) - u(t))) \right| = \left| -\alpha(t-s)^{-\alpha-1}(u(s) - u(t)) - (t-s)^{-\alpha}\frac{du}{dt}(t) \right| \leq C|t-s|^{-\alpha} \]
so that \((t-s)^{-\alpha}(u(s) - u(t))\) is in \( W^{1,1}(0,T) \) as a function in \( s \in (0,t) \) for arbitrarily fixed \( t \in (0,T) \). Therefore,
\[
\frac{\partial}{\partial t} \int_0^t (t-s)^{-\alpha}(u(s) - u(t))ds = -\alpha \int_0^t (t-s)^{-\alpha-1}(u(s) - u(t))ds + \int_0^t (t-s)^{-\alpha} \left( -\frac{du}{dt}(t) \right) ds
\]
\[
= -\alpha \int_0^t (t-s)^{-\alpha-1}(u(s) - u(t))ds - \frac{t^{1-\alpha}}{1-\alpha} \frac{du}{dt}(t).
\]
Hence,
\[
\frac{d}{dt} J^{1-\alpha}u(t) = -\alpha \int_0^t (t-s)^{-\alpha-1}(u(s) - u(t))ds + \frac{t^{-\alpha}}{\Gamma(1-\alpha)} u(t)
\]
\[
=: S_1(t) + S_2(t), \quad 0 < t < T \quad \text{for} \ u \in \mathcal{O}C^1[0,T]. \tag{6.1}
\]

**Second Step: Estimation of \( S_1(t) \) for \( 0 < \alpha < 1 \) and \( u \in \mathcal{O}C^1[0,T] \).**

**Case 1: \( p = 1 \).**

We have
\[
\|S_1\|_{L^1(0,T)} \leq \int_0^T \left| \int_0^t (t-s)^{-\alpha-1}(u(s) - u(t))ds \right| dt
\]
\[
\leq \int_0^T \int_0^T \frac{|u(t) - u(s)|}{|t-s|^\alpha dsdt} \leq \|u\|_{W^{\alpha,1}(0,T)}, \tag{6.2}
\]

**Case 2: \( 1 < p < \infty \).**

We have
\[
\frac{|u(t) - u(s)|}{|t-s|^\alpha} = \frac{|u(t) - u(s)|}{|t-s|^{\frac{1}{p}+\alpha+\varepsilon}} \frac{1}{|t-s|^{1-\frac{1}{p}-\varepsilon}}.
\]
Set \( \frac{1}{q} := 1 - \frac{1}{p} \), that is, \( q = \frac{p}{p-1} > 1 \). Then

\[
\int_0^T |t - s|^{q(\frac{1}{p} + \varepsilon - 1)} ds = \int_0^T |t - s|^{-1 + \frac{p\varepsilon}{p-1}} ds \leq (T^\frac{p}{p-1} + (T - t)^\frac{p}{p-1}) \frac{p-1}{p\varepsilon} < \infty.
\]

Consequently, the Hölder inequality yields

\[
\int_0^T \frac{|u(t) - u(s)|}{|t - s|^\alpha} ds = \int_0^T \frac{|u(t) - u(s)|}{|t - s|^\alpha + \varepsilon} \frac{1}{|t - s|^\frac{1}{p} - \varepsilon} ds 
\leq \left( \int_0^T \frac{|u(t) - u(s)|^p}{|t - s|^{1+p(\alpha+\varepsilon)}} ds \right)^{\frac{1}{p}} \left( \int_0^T |t - s|^{q(\frac{1}{p} + \varepsilon - 1)} ds \right)^{\frac{1}{q}} \leq C \left( \int_0^T \frac{|u(t) - u(s)|^p}{|t - s|^{1+p(\alpha+\varepsilon)}} ds \right)^{\frac{1}{p}}.
\]

Therefore,

\[
\int_0^T |S_1(t)|^p dt \leq C \int_0^T \left( \int_0^t \frac{|u(t) - u(s)|}{|t - s|^\alpha} ds \right)^p dt 
\leq C \int_0^T \int_0^t \frac{|u(t) - u(s)|^p}{|t - s|^{1+p(\alpha+\varepsilon)}} ds dt \leq C \|u\|_{W^{\alpha+\varepsilon,p}(0,T)}.
\]

By (6.2) and (6.3), for \( p \geq 1 \), we obtain

\[
\|S_1\|_{L^p(0,T)} \leq C \|u\|_{W^{\alpha+\varepsilon,p}(0,T)}.
\]

**Third Step: Estimation of \( S_2(t) \) for \( 0 < \alpha < 1 \) and \( u \in C^1[0,T] \).**

We have

\[
\|S_2\|_{L^p(0,T)} \leq C \int_0^T t^{-p\alpha} |u(t)|^p dt.
\]

**Case 1: \( p(\alpha + \varepsilon) < 1 \).**

We can have the following Sobolev embedding:

\[
W^{\alpha+\varepsilon,p}(0,T) \subset L^{\frac{p(1-\delta_0)}{p(\alpha+\varepsilon)}}(0,T),
\]

where we fix a sufficiently small constant \( \delta_0 > 0 \). The inclusion (6.5) is proved in Adams [1] or Grisvard [8] for \( p > 1 \) and as Lemma 4.2.1 (pp.72-73) in Gorenflo and Vessella [7] for \( p = 1 \).
We set \( q := \frac{1-\delta_0}{1-p(\alpha+\varepsilon)} \) and \( r := \frac{1-\delta_0}{p(\alpha+\varepsilon)-\delta_0} \). By \( p(\alpha+\varepsilon) < 1 \), choosing sufficiently small \( \delta_0 > 0 \), see that \( q, r > 1 \) and \( \frac{1}{q} + \frac{1}{r} = 1 \). Hence the Hölder inequality yields
\[
\int_0^T t^{-\alpha p} |u(t)|^p dt \leq \left( \int_0^T |u(t)|^{pq} dt \right)^{\frac{1}{q}} \left( \int_0^T t^{-\alpha pr} \frac{1-\delta_0}{p(\alpha+\varepsilon)-\delta_0} dt \right)^{\frac{1}{r}}.
\]
In view of \( \alpha p < p(\alpha+\varepsilon) < 1 \), choosing \( \delta_0 > 0 \) smaller, we can assume that \( \delta_0 < \frac{p\varepsilon}{1-\alpha p} \). Then direct calculations imply
\[
\alpha p \frac{1-\delta_0}{p(\alpha+\varepsilon)-\delta_0} < 1,
\]
and so
\[
\left( \int_0^T t^{-\alpha p} \frac{1-\delta_0}{p(\alpha+\varepsilon)-\delta_0} dt \right)^{\frac{1}{r}} < \infty.
\]
With (6.5), we obtain
\[
\|S_2\|_{L^p(0, T)}^p \leq C \left( \|u\|_{W^{\alpha+\varepsilon,p}(0, T)}^{p(1-\delta_0)} \right)^{\frac{1}{q}} = C \|u\|_{W^{\alpha+\varepsilon,p}(0, T)}^p.
\]  (6.6)

**Case 2:** \( p(\alpha+\varepsilon) = 1 \).

By \( \alpha + \varepsilon < 1 \), we have \( p > 1 \). Then the Sobolev embedding (e.g., [1], [2]) implies
\[
W^{\alpha+\varepsilon,p}(0, T) \subset L^{pq}(0, T)
\]  (6.7)
for any \( q \geq 1 \). We set \( r := \frac{q}{q-1} \) with \( q > 1 \). Choosing \( q > 1 \) sufficiently large, since \( \lim_{q \to \infty} r = 1 \) and \( \alpha p < \alpha(p+\varepsilon) = 1 \), we can obtain \( \alpha pr < 1 \). Therefore, (6.7) and the Hölder inequality imply
\[
\int_0^T t^{-\alpha p} |u(t)|^p dt \leq \left( \int_0^T |u(t)|^{pq} dt \right)^{\frac{1}{q}} \left( \int_0^T t^{-\alpha pr} dt \right)^{\frac{1}{r}} \leq C \|u\|_{L^{pq}(0, T)}^p \leq C \|u\|_{W^{\alpha+\varepsilon,p}(0, T)}^p,
\]
that is, we reach (6.6) in Case 2.

**Case 3:** \( p(\alpha+\varepsilon) > 1 \).

Then we note that \( p > 1 \). The Sobolev embedding yields
\[
W^{\alpha+\varepsilon,p}(0, T) \subset C^{q}[0, T],
\]  (6.8)

where
\[ 0 < \theta < \alpha + \varepsilon - \frac{1}{p} \]
(e.g., [8]). Here we set
\[ C^\theta[0, T] := \{ v \in C[0, T]; \text{there exists a constant } C = C_v > 0 \text{ such that} \]
\[ |v(t) - v(s)| \leq C_v |t - s|^\theta \}
and
\[ \|v\|_{C^\theta[0, T]} := \|v\|_{C[0, T]} + \sup_{t, s \in [0, T], t \neq s} \frac{|u(t) - u(s)|}{|t - s|^\theta}. \]

By \( u \in 0C^1[0, T] \), we have \( u(0) = 0 \) and
\[ |u(t)| = |u(t) - u(0)| \leq \|u\|_{C^\theta[0, T]} t^\theta. \]

Therefore,
\[ \|S_2\|^p_{L^p(0, T)} \leq C \int_0^T \frac{|u(t)|^p}{t^\alpha} dt \leq C \int_0^T t^{p(\theta - \alpha)} \|u\|^p_{C^\theta[0, T]} dt. \]
Since \( 0 < \theta < \alpha + \varepsilon - \frac{1}{p} \), we choose \( 0 < \delta_1 < \varepsilon \) and set \( \theta := \alpha + \varepsilon - \frac{1}{p} - \delta_1 \). Then \( p(\theta - \alpha) = -1 + p(\varepsilon - \delta_1) > -1 \), and \( \int_0^T t^{p(\theta - \alpha)} dt < \infty \). Consequently, by (6.8) we reach
\[ \|u\|^p_{L^p(0, T)} \leq C\|u\|^p_{W^{\alpha + \varepsilon, p}(0, T)} \]
and (6.6) is proved for Case 3.

Thus, in view of (6.1), (6.4) and (6.6), we finished the proof of
\[ \|J^{1-\alpha}u\|_{W^{1, p}(0, T)} \leq C\|u\|_{W^{\alpha + \varepsilon, p}(0, T)} \quad \text{for } u \in 0C^1[0, T] \]  
(6.9)
with \( 0 < \alpha < 1 \), where \( p \geq 1 \) and the constant \( C > 0 \) depends only on \( T, p, \alpha, \) and \( \varepsilon \).

**Fourth Step: Completion of the proof for** \( 0 < \alpha < 1 \).

Let \( u \in W^{\alpha + \varepsilon, p}(0, T) \) be arbitrary. By the definition, we can choose \( u_n \in 0C^1[0, T], n \in \mathbb{N} \) such that \( u_n \rightarrow u \) in \( W^{\alpha + \varepsilon, p}(0, T) \) as \( n \rightarrow \infty \). In view of (6.9), we deduce that
\[ \|J^{1-\alpha}u_n - J^{1-\alpha}u_m\|_{W^{1, p}(0, T)} \leq C\|u_n - u_m\|_{W^{\alpha + \varepsilon, p}(0, T)} \rightarrow 0 \]
as \( n, m \rightarrow \infty \). Therefore, again by (6.9) and \( J^{1-\alpha}u_n \in 0C^1[0, T] \) for \( n \in \mathbb{N} \), there exists \( \tilde{w} \in W^{\alpha + \varepsilon, p}(0, T) \) such that
\[ J^{1-\alpha}u_n \rightarrow \tilde{w} \quad \text{in } W^{1, p}(0, T) \text{ as } n \rightarrow \infty, \]  
(6.10)
and
\[
\|\widetilde{w}\|_{W^{1,p}(0,T)} = \lim_{n \to \infty} \|J^{1-\alpha}u_n\|_{W^{1,p}(0,T)} \leq C \lim_{n \to \infty} \|u_n\|_{W^{\alpha+\varepsilon,p}(0,T)}.
\] (6.11)

Here we applied the part of the lemma already proved for $0 < \alpha + \varepsilon < 1$ and $u_n \in \theta C^1[0,T]$.

Since $u_n \to u$ in $W^{\alpha+\varepsilon,p}(0,T)$, implies that $u_n \to u$ in $L^p(0,T)$, by the Young inequality, we obtain that $J^{1-\alpha}u_n \to J^{1-\alpha}u$ in $L^p(0,T)$ as $n \to \infty$.

Combining with (6.10), we see that $J^{1-\alpha}u = \widetilde{w}$. Therefore, in terms of (6.11), we reach the conclusion of Lemma 2.4 for $0 < \alpha < 1$ and $u \in \theta W^{\alpha+\varepsilon,p}(0,T)$. □

**Fifth Step: Completion of the proof of Lemma 2.4.**

Let $\alpha = m - 1 + \sigma$ with $m = 2, 3, 4, \ldots$ and $0 < \sigma < 1$. By the density argument, similarly to Fourth Step, it suffices to prove for $u \in \theta C^m[0,T]$.

Let $u \in \theta C^m[0,T]$ be arbitrarily given. Then
\[
\frac{d^{m-1}u}{dt^{m-1}} \in \theta C^1[0,T]
\]
by the definition of $\theta W^{\alpha,p}(0,T)$.

Since the conclusion of Lemma 2.4 with $0 < \alpha < 1$ and $u \in \theta C^1[0,T]$ was already proved, we replace $m := 1$ and $u := \frac{d^{m-1}u}{dt^{m-1}} \in \theta C^1[0,T]$ to apply, so that we obtain
\[
\left\| J^{1-\sigma}\frac{d^{m-1}u}{dt^{m-1}} \right\|_{W^{1,p}(0,T)} \leq C \left\| \frac{d^{m-1}u}{dt^{m-1}} \right\|_{W^{\alpha+\varepsilon,p}(0,T)}.
\] (6.12)

Similarly to (2.16), we can see
\[
\frac{d^{m-1}}{dt^{m-1}}J^{1-\sigma}u = J^{1-\sigma}\frac{d^{m-1}u}{dt^{m-1}} \quad \text{for} \quad u \in \theta C^m[0,T].
\]

Consequently, (6.12) yields
\[
\left\| \frac{d^m}{dt^m}J^{1-\sigma}u \right\|_{L^p(0,T)} \leq \left\| \frac{d^{m-1}}{dt^{m-1}}J^{1-\sigma}u \right\|_{W^{1,p}(0,T)} = \left\| J^{1-\sigma}\left(\frac{d^{m-1}u}{dt^{m-1}}\right) \right\|_{W^{1,p}(0,T)} \leq C \left\| \frac{d^{m-1}u}{dt^{m-1}} \right\|_{W^{\alpha+\varepsilon,p}(0,T)}.
\]

Hence, the definition of the norm $\| \cdot \|_{W^{\alpha+\varepsilon,p}(0,T)}$ yields
\[
\left\| \frac{d^m}{dt^m}J^{1-\sigma}u \right\|_{L^p(0,T)} \leq C \|u\|_{W^{m-1+\sigma+\varepsilon,p}(0,T)}.
\] (6.13)
Since \( \frac{d^k}{dt^k}(J^{1-\sigma}u)(0) = 0 \) by \( \frac{du}{dt}(0) = 0 \) for \( 0 \leq k \leq m - 1 \), we have

\[
\frac{d^{m-k}}{dt^{m-k}}(J^{1-\sigma}u)(t) = \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} \frac{d^m}{dt^m}(J^{1-\sigma}u)(s)ds, \quad k = 1, \ldots, m
\]

and so

\[
\|J^{1-\sigma}u\|_{W^{m-1,p}(0,T)} \leq C \left\| \frac{d^m}{dt^m}J^{1-\sigma}u \right\|_{L^p(0,T)}.
\]

In view of (6.13) and \( m - \alpha = 1 - \sigma \), we reach

\[
\|J^{m-\alpha}u\|_{W^m,p(0,T)} = \|J^{1-\sigma}u\|_{W^{m-\alpha,p}(0,T)} \leq C \|u\|_{W^{\alpha+\epsilon,p}(0,T)}
\]

for \( u \in 0C^m[0, T] \). Thus the proof of Lemma 2.4 is complete. \( \blacksquare \)

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