An Estimating the Local Deformation Parameters in Grained Structures via Stereological Methods

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Abstract. A lot of sophisticated methods have been developed to study grained structures by 2D imaging such as image correlation method, triple points technique, electron backscatter diffraction (EBSD), computer tomography (CT), etc. With applying 3D volume data mentioned methods can significantly help in precise investigation of grained structures. However these methods have several limitations. They are neither always economically efficient nor can be applied for different situations. In this paper we present relationship between change of grain’s shape caused by deformation and deformation parameters. Next we investigate a ratio ξ between total surface area of images of all grains in structure displayed in orthogonal 2D projection and their total surface area in the 3D (that means entire surface of grain boundaries). During an applied mechanical loading the grained structure is deformed and this ratio ξ varies. One specific conclusion of our work is that in cases of giant random grained structures mentioned ratio is a definite smooth function of deformation parameters. This result suggests that deformation of grained structure can be intuitively estimated on the basis of grains surface distortion in given area of bulk sample which is visible on metallographic cut. Unfortunately 2D images of real grains in orthogonal projection overlap each other and stereological methods must be used in precise deformation analysis.

1. Introduction
If the loading in grained structure spatially varies it is difficult to determine the value of the local strain experimentally. Local strain distribution can be only intuitively estimated on the basis of changes of grains spatial orientation caused by structure deformation. Grains orientation can be easily evaluated experimentally by stereological methods.

Stereology provides the three-dimensional interpretation of two-dimensional cross sections of materials. Stereological techniques allow the extracting of quantitative information about a three-dimensional material from measurements made on its two-dimensional planar sections. Despite the fact that stereological methods are quite old indeed nothing new and better has been developed in this area so far. These methods are widely used today especially in the investigation of the character of grained and foamed structures.

Spatial orientation of grained structure can be experimentally evaluated by statistical Saltykov method of oriented lines [1-2] directly from metallographic cut of bulk sample. If the correlation between the change of grains spatial orientation and deformation parameters would be known a detailed
deformation map in the structure could be designed. Our work is motivated by the fact that exact
developmental basis for assessment of deformation by means of the change of orientation is still missing.
We suggest that results of such measurement, i.e. degree of grains orientation in volume of plastically
deformed structure, can be converted to values of local deformation parameters.

Presented paper contains a mathematical model for describing the deformation of a fictional,
simplified grained structure. No real material is specified, nor is the actual deformation process
described. We want to contribute to the theoretical work on this topic published in the past [3-6]. Purely
theoretical approach is presented. However it has been shown that such a deformation mechanism, i.e.
elongation (or the shortening) of grains under applied load, exists in a real-world material [7-8].

2. Grain-surface distortion at linear deformation

During an applied mechanical loading, according to the different orientations and the influences from
neighboring grains, most of grains are submitted to a deformation. The deformation of grains results in a
change in orientation and amount of grain surface area. This is evident in the case if the typical grain
prolongation in direction of loading force is visible (see Figure 1. and Figure 2.).

Consider isotropic, linear elastic material with the grained structure. Generally the parametrization of
any single grain surface in 3D can be accomplished by spatial coordinates of each points on the surface:

$$X_i = X_i(u,v), \text{ where } i = 1, 2, 3 \quad (1.1)$$

and $u, v$ are parameters that changes in appropriate intervals. Therefore the position vectors of each
points on the surface of grain with any possible shape in undeformed configurations can be written as:

$$\bar{R}(u,v) = [X_1(u,v), X_2(u,v), X_3(u,v)], \text{ where } u \in \{u_1, u_2\} \quad v \in \{v_1, v_2\} \quad (1.2)$$

Let the position vectors of each points on the surface of the same grain in deformed configurations are

$$\bar{r}(u,v) = [x_1(u,v), x_2(u,v), x_3(u,v)], \text{ where } u \in \{u'_1, u'_2\} \quad v \in \{v'_1, v'_2\} \quad (1.3)$$

Figure 1. Scheme illustrating a view on an
undeformed grained structure cross-section
(for example on metallographic cut). Scheme
shows cross-section of some individual grains.

Figure 2. Scheme illustrating a view on the same
grained structure cross-section in deformed
configuration. Scheme demonstrates grain surface
area distortion and typical prolongation of individual
grains.

In general case, any 3D deformation can be described by a 3x3 deformation tensor

$$\bar{e} = \begin{pmatrix}
\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\
\varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\
\varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33}
\end{pmatrix} \quad (1.4)$$

Using notation mentioned above the grain linear deformation (for small strain) can be expressed as
anisomorphic linear transformation

$$\bar{r} = \bar{e} \cdot \bar{R}, \quad \text{or} \quad x_i = e_{ij} X_j \quad (1.5)$$
where \(i, j = 1, 2, 3\) and \(\varepsilon_{ij}\) are coefficients of deformation tensor (1.4). It is no problem to verify that \(\varepsilon_{ij} = \delta_{ij}\) in undeformed case (where \(\delta_{ij}\) is Kronecker symbol). We used typical Einstein’s summation convention in the second formula in (1.5) whereby when an index variable appears twice in a single term it implies summation of that term over all the values of the index. For the simplicity this convention is used in all next text.

Vector of a surface element of undeformed grain is determined as follows

\[
\mathbf{S}_{\text{undeform}} = -\varepsilon_{ij} \mathbf{X}_{ijk} \quad \text{(1.6)}
\]

An example of this vector is shown in the Figure 3. Using transformation (1.5) the vector of same surface element of the grain in deformed configuration can be written in the form

\[
\mathbf{S}_{\text{defor}} = \varepsilon_{ijk} \mathbf{X}_{ijk} \quad \text{(1.7)}
\]

where \(i, j, k = 1, 2, 3\). This vector is illustrated in the Figure 4. After suitable treatment and re-indexing of sums in formula (1.7) the change of mentioned vector of surface element during deformation can be expressed as

\[
\mathbf{S}_{\text{defor}} - \mathbf{S}_{\text{undeform}} = \mathbf{\Sigma} \quad \text{(1.8)}
\]

where \(\mathbf{\Sigma}\) can be defined as a tensor of grain surface deformation. Coefficients of this tensor are related to coefficients of deformation tensor.

\[
\sigma_{mn} = (-1)^{m+n} \text{det} \left( \mathbf{\Sigma}_{mn} \right)
\]

where \(\text{det} \left( \mathbf{\Sigma}_{mn} \right)\) is the minor (or subdeterminant) of matrix of the deformation tensor \(\mathbf{\Sigma}\). This minor is obtained from the matrix of tensor \(\mathbf{\Sigma}\) by the deleting of \(m\)-th row and \(w\)-th column. So it is easily to show that coefficients of tensor of grain surface deformation can be found as

\[
\sigma_{mn} = \varepsilon_{ik} \varepsilon_{jn} - \varepsilon_{in} \varepsilon_{jk}
\]

where indexes

\[
i = \frac{9 - 2w - (-1)^w}{4}, \quad j = \frac{15 - 2w + (-1)^w}{4},
\]

\[
k = \frac{12 - 2m - 3(-1)^w}{4}, \quad n = \frac{12 - 2m + 3(-1)^w}{4}, \quad w, m = 1, 2, 3.
\]
Then the magnitude of vector (1.8), i.e. size of the deformed surface element is given by

$$
\left( d^2 S_{\text{defor}} \right)^2 = \sum_{jnk} \Gamma_{jnks} \frac{\partial X_j}{\partial u} \frac{\partial X_n}{\partial u} \frac{\partial X_k}{\partial v} \frac{\partial X_s}{\partial v}, \tag{1.12}
$$

where coefficients

$$
\Gamma_{jnks} = \sum_{p,q} \epsilon_{jp} \epsilon_{kq} \left( \epsilon_{pq} \epsilon_{qs} - \epsilon_{qs} \epsilon_{pq} \right). \tag{1.13}
$$

Notation under the summation symbol in (1.13) means sum over all variations of indexes $p, q$ while $p \neq q$. The entire surface area of the deformed grains can be determined by formula

$$
S_{\text{defor}} = \int \int dS \Gamma_{jklm} \frac{\partial X_j}{\partial u} \frac{\partial X_k}{\partial v} \frac{\partial X_l}{\partial u} \frac{\partial X_m}{\partial v}. \tag{1.14}
$$

where $j, k, l, m = 1, 2, 3$. We note, that we permanently use Einstein’s summation convention and the sum of 81 members is under the square root in formula (1.14).

3. Grain surface 2D projection

Orientation of each grain-surface elements in 3D is determined by vector $d\tilde{S}$ which is perpendicular on the surface. Planar character of infinitesimal surface area enables clearly define orientation of surface element in 3D just by means of the vector $d\tilde{S}$ . Hence it is possible to analyze change of surface element spatial orientation during grain deformation. Parameter $\xi$ defined by means of orthogonal 2D projection of the element onto the plane characterized by unit vector $\tau$ ca be used:

$$
d\xi = \frac{\tilde{\tau} \cdot d\tilde{S}}{S}, \tag{2.1}
$$

where $S$ is surface area size of whole grain. Absolute value in the numerator determines the size of the orthogonal projection of the surface element $dS$ onto the plane perpendicular to the vector $\tilde{\tau}$. By integrating of (2.1) over the entire grain surface we get:

$$
\xi(\tilde{\tau}) = \frac{1}{S} \int \int \tilde{\tau} \cdot d\tilde{S}. \tag{2.2}
$$

Parameter $\xi$ in both deformed and undeformed configurations can be quantified by (2.2):

$$
\xi_{\text{undeformed}} = \frac{\int \int \tilde{\tau} \cdot dS_{\text{undeformed}}}{\int \int dS_{\text{undeformed}}}, \quad \xi_{\text{deformed}} = \frac{\int \int \tilde{\tau} \cdot dS_{\text{defor}}}{\int \int dS_{\text{defor}}}. \tag{2.3a,b}
$$

Let $\rho$ is unit vector paralel to vector of grain-surface element of undeformed grain, i.e.

$$
dS_{\text{undeformed}} = \rho dS_{\text{undeformed}} \text{ where } |\rho| = 1.
$$

Taking into account the formula (1.8) and considering the fact:

$$
\tilde{\tau} \cdot d^2 S_{\text{defor}} = \tilde{\tau} \left( \tilde{\sigma} \cdot d^2 S_{\text{undeformed}} \right) = \left( \tilde{\sigma}^T \cdot \tilde{\tau} \right) \cdot d^2 S_{\text{undeformed}}, \tag{2.4}
$$

where $\tilde{\sigma}^T$ is transposed tensor to the grain surface deformation tensor $\tilde{\sigma}$ determined by (1.9), formula (2.2) in deformed configuration can be rewritten to the next form
\[
\xi_{\text{defor}} = \frac{\int_{(\text{Surface})} \left\langle \bar{\sigma} \cdot \bar{\tau} \right\rangle dS_{\text{undeform}}} {\int_{(\text{Surface})} |\bar{\sigma}| dS_{\text{undeform}}}. \tag{2.5}
\]

As can be concluded from comparison of above equations (2.3a) and (2.5), transformation of parameter \(\xi\) during grain deformation can be realized as follows:

\[
\bar{\tau} \rightarrow \bar{\sigma} \cdot \bar{\tau} \quad \text{and} \quad \bar{\rho} \rightarrow \bar{\sigma} \cdot \bar{\rho}. \tag{2.6}
\]

It can be seen from (2.5) that in general the transformation of \(\xi\) during grain deformation appears to be relatively complicated and we will not examine its mathematical properties in detail. However, it should be noted that we integrate over the surface of undeformed grain in both formulas (2.3a) and (2.5). So to determine the value of parameter \(\xi\) in deformed configuration we need to know only the shape of undeformed grain and coefficients of deformation tensor \(\varepsilon_{ij}\). Therefore we believe, that \(\xi\) is suitable for the quantitative evaluation of grain-surface distortion during local deformation of grained structure. (1.14).

4. Random matrix approximation

Integrals in formula (2.5) can be modeled by summation over infinitely small areas placed on grains boundaries (see Figure 5.). So we get

\[
\xi_{\text{defor}} = \frac{\sum_{i=1}^{N} \left( \bar{\tau} \cdot \bar{\rho} \right) S_{\text{undeform}}^{(i)}} {\sum_{i=1}^{N} \left| \bar{\rho} \right| S_{\text{undeform}}^{(i)}} \tag{3.1}
\]

where \(\bar{\tau} = [\tau_1, \tau_2, \tau_3]\) is unit vector perpendicular to the orthogonal projection plane (\(\tau_1^2 + \tau_2^2 + \tau_3^2 = 1\)) and \(\bar{\rho}^{(i)}\) is unit vector perpendicular to the \(i\)-th grain-surface element of undeformed grain

\[
S_{\text{undeform}}^{(i)} = \bar{\rho}^{(i)} S^{(i)}. \tag{3.2}
\]

Vector \(\bar{\rho}^{(i)}\) can be written in the form

\[
\bar{\rho}^{(i)} = \sin(\eta^{(i)}) \cos(\chi^{(i)}) \sin(\eta^{(i)}) \sin(\chi^{(i)}) \cos(\eta^{(i))}]
\]

while angles \(\eta^{(i)}\) and \(\chi^{(i)}\) determines orientation of the \(i\)-th grain-surface element in 3D and then

\[
S_{\text{undeform}}^{(i)} = \begin{bmatrix} S^{(i)}(\eta^{(i)}) \cos(\chi^{(i)}) \sin(\eta^{(i)}) & S^{(i)}(\eta^{(i)}) \sin(\eta^{(i)}) \sin(\chi^{(i)}) \end{bmatrix} \tag{3.3}
\]

If we use result (1.9) and notations (3.3), the formula (3.1) can be calculated as

\[
\xi_{\text{defor}} = \frac{\sum_{i=1}^{N} S^{(i)} \sum_{n=1}^{N} \tau_n \det \bar{A}_n^{(i)}} {\sum_{i=1}^{N} S^{(i)} \left( \sum_{n=1}^{N} \left( \det \bar{A}_n^{(i)} \right)^2 \right)} \quad \text{...} \quad n = 1, 2, 3 \tag{3.4}
\]

where \(\bar{A}_n^{(i)}\) are following matrix-valued variables.
Parameter $\xi$ of whole deformed grained structure is calculated by means of the formula (3.4) while sums in the numerator and denominator on the right side of this formula can be determined using (3.5), (3.6) and (3.7).

Any randomly oriented surface of grain boundary $S(i)$ in the given bulk sample with grained structure can be included into the calculation of $\xi$ if corresponding parameters $\eta(i)$ and $\chi(i)$ will be applied. Therefore random values of angles $\eta(i)$ and $\chi(i)$ should be substituted step by step into formula (3.4) when the calculation progress. The problem of parameter $\xi$ calculation can be formulated as a random matrix problems because matrices (3.5), (3.6) and (3.7) can be smartly used in the formal mathematical notation as well as numerical implementation of mentioned approximation approach.

5. The case of plastic deformation

Fundamental assumption used to establish the theory of plasticity is that plastic deformation is isochoric or volume preserving. Next we apply results obtained in previous section 3 in the case of plastic grain deformation in which the change in volume of grain is zero.

Grain volume conservation during plastic deformation is given by formula

$$V_{\text{undefor}} = V_{\text{defor}}. \quad (4.1)$$

Gauss-Ostrogradski integral theorem can be used for determination of grain volume in both undeformed ($V_{\text{undefor}}$) and deformed ($V_{\text{defor}}$) configurations

$$\int_{S_{\text{undefor}}} \mathbf{R} \cdot d\mathbf{S}_{\text{undefor}} = \int_{V_{\text{undefor}}} \text{div} (\mathbf{R}) \, dV = 3V_{\text{undefor}}, \quad \int_{S_{\text{defor}}} \mathbf{r} \cdot d\mathbf{S}_{\text{defor}} = \int_{V_{\text{defor}}} \text{div} (\mathbf{r}) \, dV' = 3V_{\text{defor}}. \quad (4.2)$$

Next formula results from the comparison of equations (4.2)

$$\int_{S_{\text{undefor}}} \mathbf{R} \cdot d\mathbf{S}_{\text{ undefor}} = \int_{S_{\text{defor}}} \mathbf{r} \cdot d\mathbf{S}_{\text{defor}} = \int_{S_{\text{undefor}}} \left\{ \mathbf{\tilde{\sigma}}^T \left( \mathbf{\tilde{\epsilon}} \cdot \mathbf{R} \right) \right\} d\mathbf{S}_{\text{undefor}}, \quad (4.3)$$

where we used (1.5) and (1.8). We integrate over the same (but any) area on the both sides of previous equation (4.3). Therefore for plastic deformation apply

$$\mathbf{R} = \mathbf{\tilde{\sigma}}^T \left( \mathbf{\tilde{\epsilon}} \cdot \mathbf{R} \right), \quad \mathbf{\tilde{\sigma}}^T = \mathbf{\tilde{\epsilon}}^{-1} \quad (4.4)$$

Transposed tensor to the grain surface deformation tensor is inverse to tensor of deformation in this case. If we consider (4.3), next equation can be written in coordinate form
\[
\left\{ (e_{1z}E_{2z}E_{3x} - e_{2x}E_{1z}E_{3x} + e_{3x}E_{1z}E_{2z}) - \left( \delta_{1z}\delta_{2z}\delta_{3x} - \delta_{2x}\delta_{1z}\delta_{3x} + \delta_{3x}\delta_{1z}\delta_{2z} \right) \right\} \left( \frac{\partial X_i}{\partial u} \frac{\partial X_j}{\partial v} - \frac{\partial X_i}{\partial v} \frac{\partial X_j}{\partial u} \right) X_n = 0 . \tag{4.5}
\]

Equation (4.5) must be satisfied for any function \( X_i = X_i(u,v) \) when the grain is plastically deformed. If (4.5) is expanded without further assumptions, it will lead to the following 27 equations

\[
e_{1u} \left( E_{2z}E_{3x} - E_{2x}E_{3z} \right) - e_{2u} \left( E_{1z}E_{3x} - E_{1x}E_{3z} \right) + e_{3u} \left( E_{1z}E_{2x} - E_{1x}E_{2z} \right) -
\]

\[
- \delta_{1u} \left( \delta_{2z}\delta_{3x} - \delta_{2x}\delta_{3z} \right) + \delta_{2u} \left( \delta_{1z}\delta_{3x} - \delta_{1x}\delta_{3z} \right) - \delta_{3u} \left( \delta_{1z}\delta_{2x} - \delta_{1x}\delta_{2z} \right) = 0 \quad \forall \; n,j,k \tag{4.6}
\]

i.e. next equation must be satisfied in case of plastic deformation

\[
det(\vec{e}) = 1 . \tag{4.7}
\]

If the homogeneous and isotropic grained structure is considered, following conditions for the deformation tensor can be applied

\[
e_{ij} = 0 \quad \text{for all} \; i \neq j \neq k \quad \land \quad det(\vec{e}) = e_{xx}e_{yy}e_{zz} = 1. \tag{4.8}
\]

In the case of plastic deformation of isotropic and homogeneous grained structure tensor elements are related to grain prolongation (contraction) \( \delta (-1 < \delta < \infty) \). If the loading force is applied in direction of \( x \)-axis it holds

\[
e_{11} = \delta + 1 , \quad e_{22} = e_{33} = \frac{1}{\sqrt{\delta + 1}} . \tag{4.9}
\]

Application of (4.8) and (4.9) in formulas (2.4) and (4.4) will simplify integration in (2.5). Nevertheless detail information about the shape of grain is necessary for the practical application because we integrate over entire surface of the grain in this formula. However it is absolutely impossible to determine real shape of each grain in material structure exactly. Therefore deformation of modeled grained structure consisting from grains with various idealized shapes are investigated usually \([9]\).

6. Monte-Carlo simulation of \( \vec{e}, \vec{\delta} \) for plastically deformed grained structures

In the frame of random matrix approximation presented in the previous section, grain boundaries are considered to be a large set of planar surfaces and therefore all derivatives with respect to parameters \( u \) and \( v \) in formulas (1.7) and (1.14) are constants. These constants are different for each planar surface and integrals can be replaced by sums. Random variables \( \eta^{(i)} \), \( \chi^{(i)} \) and \( S^{(i)} \) actually represent orientation and size of \( i \)-th planar surface in the structure.

We consider random homogeneous grained structure in which the single grain may be of any shape and can have any orientation. The size and orientation of mentioned small flat surfaces in the structure in the approximation, they are not limited by particular shape of the grain.

In the case of homogeneous and isotropic plastically deformed grained structure formulas (4.8) and (4.9) must be considered and numerator and denominator on the right side of formula (3.1) can be simplified to the following form

\[
\sum_{i=1}^{N} \left| \vec{\varphi} \cdot \vec{t} \right| S^{(i)} = \sum_{i=1}^{N} S^{(i)} \left| \frac{1}{(\delta + 1)} \sin(\eta^{(i)}) \cos(\chi^{(i)}) \tau_1 + \sqrt{\delta + 1} \sin(\eta^{(i)}) \sin(\chi^{(i)}) \tau_2 + \sqrt{\delta + 1} \cos(\eta^{(i)}) \tau_3 \right| \tag{5.1}
\]

\[
\sum_{i=1}^{N} \left| \vec{\varphi} \right| \bar{S}^{(i)} = \sum_{i=1}^{N} S^{(i)} \left| \frac{1}{(\delta + 1)} \cos(\chi^{(i)}) \sin(\eta^{(i)}) \right|^2 + \left| \frac{1}{(\delta + 1)} \sin(\chi^{(i)}) \sin(\eta^{(i)}) \right|^2 + \left| \frac{1}{(\delta + 1)} \cos(\eta^{(i)}) \right|^2 . \tag{5.2}
\]

where \( N \) is number of random variables (i.e. number of considered planar grain boundaries in the model of grained structure).
a) The case of „out of plane“ deformation

If the vector \( \tau = [1, 0, 0] \) (out of plane deformation), we get next formula by means of substituting (5.1) and (5.2) into (3.1)

\[
\xi_{\text{defor}} = \sum_{i=1}^{N} S^{(i)} \left| \sin(\eta^{(i)}) \cos(\chi^{(i)}) \right| \sum_{i=1}^{N} S^{(i)} \left[ \cos^{2}(\chi^{(i)}) \sin^{2}(\eta^{(i)}) + (\delta + 1)^{2} \left[ \sin^{2}(\chi^{(i)}) \sin^{2}(\eta^{(i)}) \right] + \cos^{2}(\eta^{(i)}) \right].
\] (5.3)

Parameter \( \xi \) calculated via the vector oriented parallel to the direction of deformation force can be evaluated in this simple case by means of formula (5.3). Values \( \eta^{(i)}, \chi^{(i)} \) and \( S^{(i)} \) must be randomly generated and substituted to formula (5.3) during the numerical simulation of the function \( \xi_{\text{defor}}(\delta) \). However it is also necessary to take into account the value \( \xi_{\text{undefor}} \) for the grained structure in the undeformed configuration during the simulation. Prolongation \( \delta = 0 \) in the mentioned undeformed case and then from (5.3) it holds

\[
\xi_{\text{undefor}} = \xi_{\text{defor}}(\delta = 0) = \sum_{i=1}^{N} S^{(i)} \left| \sin(\eta^{(i)}) \cos(\chi^{(i)}) \right| \sum_{i=1}^{N} S^{(i)} \left[ \cos^{2}(\chi^{(i)}) \sin^{2}(\eta^{(i)}) + (\delta + 1)^{2} \left[ \sin^{2}(\chi^{(i)}) \sin^{2}(\eta^{(i)}) \right] + \cos^{2}(\eta^{(i)}) \right].
\] (5.4)

Random values \( \eta^{(i)}, \chi^{(i)} \) and \( S^{(i)} \) must be generated in a such way to meet an initial condition (5.4). Considering (5.3) and (5.4) we get

\[
\xi_{\text{defor}} = \sum_{i=1}^{N} S^{(i)} \left| \sin(\eta^{(i)}) \cos(\chi^{(i)}) \right| \sum_{i=1}^{N} S^{(i)} \left[ \cos^{2}(\chi^{(i)}) \sin^{2}(\eta^{(i)}) + (\delta + 1)^{2} \left[ \sin^{2}(\chi^{(i)}) \sin^{2}(\eta^{(i)}) \right] + \cos^{2}(\eta^{(i)}) \right].
\] (5.5)

We propose to evaluate the value \( \xi \) for the grained structure by the formula (5.5) in cases when an extension of this stucture caused by the deformation force, i.e if \( 0 < \delta \) is observed.

b) The case of „in plane“ deformation

In the case if there is contraction of the structure under the deformation force, i.e. \(-1 < \delta < 0 \) (in plane deformation), it is appropriate to evaluate \( \xi \) via any vector oriented perpendicular to deformation force. Vector

\[
\tilde{\tau} = [0, \cos(\theta), \sin(\theta)]
\] (5.6)

where \( 0 < \theta < 2\pi \) must be considered in this case of structure squeezing and numerator and denominator on the right side of formula (3.1) take the form:

\[
\sum_{i=1}^{N} \left| \tilde{\tau}^{T} \cdot \beta^{(i)} S^{(i)} \right| = \sum_{i=1}^{N} S^{(i)} \left[ \sqrt{\delta + 1} \cos(\eta^{(i)}) \sin(\eta^{(i)}) \cos(\theta) + \sqrt{\delta + 1} \sin(\eta^{(i)}) \sin(\theta) \right]
\] (5.7)

\[
\sum_{i=1}^{N} \left| \tilde{\tau} \cdot \beta^{(i)} S^{(i)} \right| = \sum_{i=1}^{N} S^{(i)} \left[ \sqrt{\delta + 1} \cos^{2}(\eta^{(i)}) \right] + \left[ \sqrt{\delta + 1} \sin^{2}(\eta^{(i)}) \right] + \left[ \sqrt{\delta + 1} \cos(\eta^{(i)}) \right]^{2}
\] (5.8)

Applying (5.6), (5.7) and (5.8) in formula (3.1) we get
\[\xi_{\text{defor}} = \frac{\sum_{i=1}^{N} S^{(i)} \left[ \sqrt{\delta + 1} \sin(\eta^{(i)}) \sin(\chi^{(i)}) \cos(\theta^{(i)}) + \sqrt{\delta + 1} \cos(\eta^{(i)}) \sin(\theta^{(i)}) \right]}{\sum_{i=1}^{N} S^{(i)} \left[ \frac{1}{\sqrt{\delta + 1}} \cos(\chi^{(i)}) \sin(\eta^{(i)}) \right]^2 + \left( \sqrt{\delta + 1} \sin(\chi^{(i)}) \sin(\eta^{(i)}) \right)^2 + \left( \sqrt{\delta + 1} \cos(\eta^{(i)}) \right)^2}.\]  

(5.9)

Therefore in undeformed configuration:

\[\xi'_{\text{undeform}} = \xi_{\text{defor}}(\delta = 0) = \frac{\sum_{i=1}^{N} s^{(i)} \left[ \sin(\eta^{(i)}) \sin(\chi^{(i)}) \cos(\theta) + \cos(\eta^{(i)}) \sin(\theta) \right]}{\sum_{i=1}^{N} s^{(i)}}\]  

(5.10)

and we get next formula for the calculation of the \(\xi\) in the same way as before:

\[\xi_{\text{defor}} = \sqrt{\delta + 1} (\delta + 1) \xi'_{\text{undeform}} \sum_{i=1}^{N} s^{(i)} \left[ \cos^2(\chi^{(i)}) \sin^2(\eta^{(i)}) + (\delta + 1) \left( \sin^2(\chi^{(i)}) \sin^2(\eta^{(i)}) + \cos^2(\eta^{(i)}) \right) \right].\]  

(5.11)

Functions \(\xi_{\text{defor}}(\delta)\) were found using formulas (5.5) and (5.11) numerically by Monte Carlo simulation. Procedure of this simulation is illustrated by means of graphs shown in the Figure 6. (for zero boundary conditions \(1-2\xi_{\text{undeform}} = 0\)). Mentioned graphs were obtain by successive substitution of \(N\) randomly chosen triples of variables \(\{\eta^{(i)}, \chi^{(i)}, S^{(i)}\}\) into formulas (5.5) and (5.11). Values \(1-2\xi\) are displayed on vertical axes in mentioned graphs as these values are assumed to be corresponding to parameters describing spatial orientation of grains from Saltykov method. With an increasing of \(N\) the calculated values \(\xi_{\text{defor}}\) converge to some (real) value for every \(\delta\). Computational stability of Monte Carlo algorithm is sufficiently clear from Figure 6.

![Graphs showing out of plane and in plane deformation](image-url)
7. Conclusion
In this work we present brief mathematical treatment of quantities resulting from standard stereological procedures. According to our findings, these quantities can be applicable for the modelling of the local plastic deformation in grained structures. Our approach is based on the assumption that the grain, as it appears on a two-dimensional cross section, will change its shape and spatial orientation during plastic deformation. Obtained results are applied to a homogeneous and isotropic grained structure.

We note that formulas (5.5) and (5.11) can be actually used for the conversion of grained structure orientation represented by quantity $\xi$ into local plastic deformation of the structure characterized by $\delta$. Graphs of the calculated dependence $\xi_{\text{defor}}(\delta)$ obtained using these formulas by means of Monte Carlo simulation demonstrate the relevance of stereological methods for the local plastic deformation modelling in grained structures.

Figure 6. Graphs illustrating Monte Carlo simulation of values (1-2$\xi_{\text{defor}}$) as a function of prolongation ($\delta > 0$) and contraction ($\delta < 0$) for plastically deformed random grained structure.
Results expressed in (5.5) and (5.11) are not a general solution. It can only be applied to the particular case for which there is no rigid solid rotation of grains in structure and the axes chosen are principal axes of strain (therefore, there are no shear components present). In addition another limitation is imposed, which is that plastic deformation must be isotropic. However it does not seem that a grain with anisotropic behavior would suppose complications in practical applications.

In conclusion, we reported the problem of applicability of stereological methods to determine the local plastic deformation in grained structure. One can easily understand that there is a correlation between the change in the grain orientation and grain deformation. This correlation was mathematically demonstrated in our work using parameter \( \xi \) reflecting orthogonal 2D projection of the grained structure onto the oriented plane. Indeed, the expression (3.4) allows one to investigate this correlation.

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