HOROCYCLE FLOWS FOR LAMINATIONS BY HYPERBOLIC RIEHMANN SURFACES AND HEDLUND’S THEOREM

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To Étienne Ghys, on the occasion of his 60th birthday

ABSTRACT. We study the dynamics of the geodesic and horocycle flows of the unit tangent bundle \((\hat{M}, T^1F)\) of a compact minimal lamination \((M,F)\) by negatively curved surfaces. We give conditions under which the action of the affine group generated by the joint action of these flows is minimal and examples where this action is not minimal. In the first case, we prove that if \(F\) has a leaf which is not simply connected, the horocycle flow is topologically transitive.

1. INTRODUCTION

The geodesic and horocycle flows over compact hyperbolic surfaces have been studied in great detail since the pioneering work in the 1930’s by E. Hopf and G. Hedlund. Such flows are particular instances of flows on homogeneous spaces induced by one-parameter subgroups, namely, if \(G\) is a Lie group, \(K\) a closed subgroup and \(N\) a one-parameter subgroup of \(G\), then \(N\) acts on the homogeneous space \(K\backslash G\) by right multiplication on left cosets.

One very important case is when \(G = SL(n,\mathbb{R})\), \(K = SL(n,\mathbb{Z})\) and \(N\) is a unipotent one parameter subgroup of \(SL(n,\mathbb{R})\), i.e., all elements of \(N\) consist of matrices having all eigenvalues equal to one. In this case \(SL(n,\mathbb{Z})\backslash SL(n,\mathbb{R})\) is the space of unimodular lattices. By a theorem by Marina Ratner (see [28]), which gives a positive answer to the Raghunathan conjecture, the closure of the orbit under the unipotent flow of a point \(x \in SL(n,\mathbb{Z})\backslash SL(n,\mathbb{R})\) is the orbit of \(x\) under the action of a closed subgroup \(H(x)\). This particular case already has very important applications to number theory; for instance, it was used by G. Margulis and Dani in [10] and Margulis in [18] to give a positive answer to the Oppenheim conjecture.

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When \( n = 2 \) and \( \Gamma \) is a discrete subgroup of \( \text{SL}(2, \mathbb{R}) \) such that \( M = \Gamma \setminus \text{SL}(2, \mathbb{R}) \) is of finite Haar volume, and \( N \) is any unipotent one-parameter subgroup acting on \( M \), Hedlund proved that any orbit of the flow is either a periodic orbit or dense. When \( \Gamma \) is cocompact the flow induced by \( N \) has every orbit dense, so it is a minimal flow. The horocycle flow on a compact hyperbolic surface is a homogeneous flow of the previous type and most of the important dynamic, geometric and ergodic features of the general case are already present in this 3-dimensional case.

On the other hand, the study of Riemann surface laminations has recently played an important role in holomorphic dynamics (see [14] and [17]), polygonal tilings of the Euclidean or hyperbolic plane (see [5], [25]), moduli spaces of Riemann surfaces (see [23]), polygonal billiards (see [21] and [37]), etc. It is natural then to consider compact laminations by surfaces with a Riemannian metric of negative curvature and consider the positive and negative horocycle flows on the unit tangent bundle of the lamination. In the spirit of Raghunathan and Ratner, in this paper we study the closures of the horocycle orbits in this non-transitive \( \text{PSL}(2, \mathbb{R}) \)-space, giving examples of orbit closures which are not algebraic in the sense of Ratner.

Throughout this paper, \((M, \mathcal{F})\) is a compact metric space with a chosen metric and carrying a lamination by hyperbolic surfaces for which all leaves are dense. The unit tangent bundle of the lamination, called \((\hat{M}, T^1 \mathcal{F})\), is the phase space of the geodesic flow and the horocycle flow. The space \(\hat{M}\) is in a natural way a compact metric space since it fibers over \(M\) with fibre the circle. The joint action of these flows is an action of the affine group \(B\). In Section 2 we introduce all the basic definitions and notation, as well as preliminary results.

Section 3 is devoted to the study of the \(B\)-action mentioned above, and it contains the main results of the present paper. In Theorem 9 we prove that this action is topologically transitive, and in Theorem 8 that it has a unique minimal set \(\mathcal{M} \subset \hat{M}\). This set may or may not be the whole of \(\hat{M}\). A family of laminations for which \(\mathcal{M}\) is a proper minimal set is described in Proposition 3; it is the family of hyperbolic surface laminations that come from an action of the affine group. Example 13 shows a lamination which does not come from an affine action and for which \(\mathcal{M}\) is also proper. Two sufficient conditions for the \(B\)-action to be minimal appear in Proposition 4 and in Theorem 14. The last one is the existence of a transverse holonomy-invariant measure for \(\mathcal{F}\).

Section 4 gives an assortment of examples of minimal sets for the horocycle flow. Some of these minimal sets coincide with the minimal set \(\mathcal{M}\) for the affine action, and some do not. We study the dynamics of the geodesic and the horocycle flows, separately, and the main result of this section is Theorem 24, which states that if \(\mathcal{M} = \hat{M}\) and \(\mathcal{F}\) has a non simply connected leaf, then the horocycle flow is topologically transitive.

Finally, we ask the following question: does the minimality of the \(B\)-action imply the minimality of the horocycle flow? There is strong evidence that the answer might be positive, see [1], [2], and [19].
2. Laminations by hyperbolic surfaces

For the definition of a compact lamination or foliated space in general, we refer the reader to Chapter 11 of [8]. In this paper, we only concern ourselves with compact laminations by hyperbolic surfaces.

2.1. Laminations by hyperbolic surfaces and their unit tangent bundles. A compact lamination by hyperbolic surfaces \((M, \mathcal{F})\) (or simply \(\mathcal{F}\)) consists of a compact metrizable space \(M\) together with a family \(\{(U_a, \varphi_a)\}\) such that

- \(\{U_a\}\) is an open covering of \(M\),
- \(\varphi_a : U_a \to D_a \times T_a\) is a homeomorphism, where \(D_a\) is a bounded open disk in the Poincaré upper half plane \(\mathbb{H}\) and \(T_a\) is a topological space, and
- for \((y, t) \in \varphi^{-1}_\beta(U_a \cap U_\beta)\), \(\varphi_a \circ \varphi^{-1}_\beta(y, t) = (g_{a\beta}(t)y, h_{a\beta}(t))\), where \(g_{a\beta}(t)\) is in \(PSL(2, \mathbb{R})\).

Notice that since \(\varphi_a\) is a homeomorphism, the map \(h_{a\beta}\) is a homeomorphism from an open subset of \(T_\beta\) to an open subset of \(T_a\), and \(g_{a\beta}(t)\) is continuous in \(t\).

In the sequel \(M\) will always be compact. Each \(U_a\) is called a foliated chart, a set of the form \(\varphi^{-1}_\alpha([y] \times T_a)\) being its transversal. The sets of the form \(\varphi^{-1}_\alpha(D_a \times \{t\})\), called plaques, glue together to form maximal 2-dimensional connected manifolds called leaves. The lamination \(\mathcal{F}\) is the partition of \(M\) into leaves. The leaves have structures of hyperbolic surfaces with local charts the restriction of \(p_1 \circ \varphi_a\)'s to plaques, where \(p_1 : D_a \times T_a \to D_a\) is the projection onto the first factor.

A lamination is said to be minimal if all the leaves are dense.

A holonomy-invariant measure is a family of Radon measures in all transversals which are invariant under the maps \(h_{a\beta}\) when restricted to their domains and ranges.

For each \(x \in M\), let us denote by \(T^1_x\mathcal{F}\) the set of the unit tangent vectors of the leaf through \(x\), and define the laminated unit tangent bundle \(\hat{M}\) of \((M, \mathcal{F})\) by \(\hat{M} = \bigcup_{x \in M} T^1_x\mathcal{F}\). Points of \(\hat{M}\) are denoted by \((x, v)\), where \(x \in M\) and \(v \in T^1_x\mathcal{F}\). Denote the natural projection by \(\Pi : \hat{M} \to M\). Thus \(\Pi^{-1}(x) = T^1_x\mathcal{F}\). There are homeomorphisms \(\psi_a : \Pi^{-1}(U_a) \to T^1D_a \times T_a\) defined by the leafwise differentials of \(\varphi_a\). The locally defined transition function \(\psi_a \circ \psi^{-1}_\beta : T^1D_\beta \times T_\beta \to T^1D_a \times T_a\) is given by

\[
\psi_a \circ \psi^{-1}_\beta(w, t) = (g_{a\beta}(t)_*(w), h_{a\beta}(t)),
\]

where \(g_{a\beta}(t)_*\) is the differential of \(g_{a\beta}(t)\). Plaques \(\psi^{-1}_a(T^1D_a \times \{t\})\) all together define a lamination which we shall denote by \(T^1\mathcal{F}\).

When we identify \(T^1\mathbb{H}\) with \(PSL(2, \mathbb{R})\) as usual, then \(T^1D_a\) and \(T^1D_\beta\) are open subsets of \(PSL(2, \mathbb{R})\), and the transition function takes the form

\[
\psi_a \circ \psi^{-1}_\beta(g, t) = (g_{a\beta}(t)g, h_{a\beta}(t)),
\]

where \(g \in PSL(2, \mathbb{R})\) and \(g_{a\beta}(t)g\) is the group multiplication. Since the transition functions commute with the locally defined right translation of \(PSL(2, \mathbb{R})\) on the first factor, we obtain a right action of the universal covering group of \(PSL(2, \mathbb{R})\).
on \( \hat{M} \). Since the rotation by \( 2\pi \) is the identity on each foliated chart, we actually obtain a right action of \( PSL(2, \mathbb{R}) \). The orbit foliation of this action is just \( T^1 \mathcal{F} \).

2.2. Laminated geodesic and horocycle flows. We are particularly interested in the right action of the following three subgroups \( D, U \) and \( B \) of \( PSL(2, \mathbb{R}) \), where

\[
D = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} : t \in \mathbb{R} \right\},
\]
\[
U = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\},
\]
\[
B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a > 0, b \in \mathbb{R} \right\}.
\]

The right action of \( D \) (resp. \( U \)) is called the laminated geodesic (resp. horocycle) flow and denoted by \( g^t \) (resp. \( h^t \)). They satisfy

\[
g^t \circ h^s \circ g^{-t} = h^{se^{-t}}.
\]

For each leaf \( L \) of \( \mathcal{F} \), the unit tangent bundle \( T^1 L \) is a leaf of \( T^1 \mathcal{F} \), and the flows \( g^t \) and \( h^t \) restricted to \( T^1 L \) are the usual geodesic and horocycle flows of the hyperbolic surface \( L \).

To study further the right \( B \)-action, we consider the following identification of \( T^1 \mathbb{H} \) with \( \mathbb{H} \times \mathbb{S}^1 \), where \( \mathbb{S}^1 \) is the circle at infinity of \( \mathbb{H} \). For any \( w \in T^1 \mathbb{H} \), denote by \( w(\infty) \) the point in \( \mathbb{S}^1 \) at which the geodesic with initial vector \( w \) terminates, and by \( p : T^1 \mathbb{H} \to \mathbb{H} \) the bundle projection. Now we identify \( w \in T^1 \mathbb{H} \) with \( (p(w), w(\infty)) \in \mathbb{H} \times \mathbb{S}^1 \). For any \( g \in PSL(2, \mathbb{R}) \), \( w \in T^1 \mathbb{H} \), we have \( (p(g_*(w)), (g_*(w))(\infty)) = (gp(w), gw(\infty)) \), where \( gw(\infty) \) is the action of \( PSL(2, \mathbb{R}) \) on \( \mathbb{S}^1 \).

Thus when \( T^1 D_\alpha \times T_\alpha \) and \( T^1 D_\beta \times T_\beta \) are identified with \( D_\alpha \times \mathbb{S}^1 \times T_\alpha \) and \( D_\beta \times \mathbb{S}^1 \times T_\beta \), the transition function \( \psi_\alpha \circ \psi_\beta^{-1} \) takes the form

\[
\psi_\alpha \circ \psi_\beta^{-1}(y, \theta, t) = (g_{\alpha}(t)y, g_{\alpha}(t)\theta, h_{\alpha}(t)).
\]

On the other hand, we have \( w_b(\infty) = w(\infty) \) for any \( w \in T^1 \mathbb{H} \) and \( b \in B \). Therefore the local right \( B \)-translation on a lamination chart \( D_\alpha \times \mathbb{S}^1 \times T_\alpha \) leaves the second and the third factor invariant. In what follows, we shall suppress the map \( \phi_\alpha \) and the subscript \( \alpha \), and write \( D \times \mathbb{S}^1 \times T \) to denote a laminated chart in \( \hat{M} \). The foliation \( \hat{M} \) obtained by the plaques \( D \times |\theta| \times |t| \) is the orbit foliation of the right \( B \)-action.

2.3. Abundance of compact laminations by hyperbolic surfaces. Let \( (M, \mathcal{F}) \) be a compact 2-dimensional lamination equipped with a leafwise Riemannian metric which is continuous and leafwise \( C^2 \). The restriction to a leaf \( L \) of the metric lifts to a Riemannian metric on the universal cover \( \hat{L} \) of \( L \). Let us fix \( x \in \hat{L} \) and denote by \( A(x, r) \) the area of the ball on \( \hat{L} \) centered at \( x \) with radius \( r > 0 \).

Consider the following condition

\[
(1) \quad \lim_{r \to \infty} \frac{1}{r} \log A(x, r) > 0, \quad \forall \ x \in M.
\]
This condition is independent of the Riemannian metric chosen, since all the leafwise metrics on the compact lamination \((M, F)\) are quasi-isometric on leaves. A theorem due to Candel and Verjovsky (see [7] and [34]) guarantees the existence of a Riemannian metric of constant curvature \(-1\) in the same conformal class as the metric we started with. Namely we have the following theorem.

**Theorem 1.** If \((M, F)\) is a compact 2-dimensional lamination which satisfies (1), then \(F\) has a structure of a lamination by hyperbolic surfaces.

Such laminations which satisfy (1) are quite abundant. For example, laminations without holonomy-invariant measures are known to satisfy (1).

2.4. **Harmonic measures.** The study of the laminated Brownian motion was developed by Lucy Garnett in [13]. If \((M, F)\) is a compact lamination by hyperbolic surfaces, the laminated Brownian motion starting at \(x \in M\) is defined simply to be the usual Brownian motion on the leaf \(L\) of \(F\) through \(x\) with respect to the hyperbolic metric on \(L\). A probability measure of \(M\) which is stationary for this process is called harmonic. A harmonic measure \(\mu\) is ergodic if every measurable subset of \(M\) which is a union of leaves of \(F\) has either \(\mu\) measure zero or one.

Lucy Garnett proved the following ergodic theorem for harmonic measures:

**Theorem 2** (Lucy Garnett). Let \(\mu\) be an ergodic harmonic measure on \((M, F)\) and \(f \in L^1(\mu)\). Then for \(\mu\)-almost every \(x \in M\) and almost every Brownian path \(\omega(t)\) on the leaf through \(x\) starting at \(x\)

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T f(\omega(t)) \, dt = \int_M f \, d\mu.
\]

When the lamination \((M, F)\) has a holonomy-invariant measure \(\nu\), we can consider the measure on \(M\) which is locally the product of \(\nu\) and the volume on leaves. This gives a harmonic measure, and harmonic measures of this kind are called completely invariant.

3. **The central stable foliation of the laminated geodesic flow**

As before, let \((M, F)\) be a compact minimal lamination by hyperbolic surfaces. The laminated geodesic flow on \((\tilde{M}, T^1_1 F)\) is an Anosov flow on each leaf \(T^1 L\), whose central stable foliation is the foliation by orbits of the action of the affine group \(B\) restricted to \(T^1 L\). Without any further assumption on the dynamics of the foliation, the central-stable manifolds of the laminated geodesic flow might be bigger than these \(B\)-orbits. Nevertheless, we will refer to the foliation by \(B\)-orbits on \(\tilde{M}\) as the central-stable foliation of the laminated geodesic flow, and to its leaves as the central-stable leaves.

Let \(\Pi : \tilde{M} \to M\) be the canonical projection \(\Pi(x, \nu) = x\); for every \(x \in M\) \(\Pi^{-1}(\{x\}) = T^1_1 F\). Under this projection, each central-stable leaf in \(\tilde{M}\) projects onto a leaf of \(F\).
Let $\mathcal{M} \subset \hat{M}$ be a compact minimal set for the central-stable foliation. Since $\mathcal{F}$ is minimal and central-stable leaves project onto entire $\mathcal{F}$-leaves, $\Pi(\mathcal{M}) = M$. Namely, $\mathcal{M} \cap T^1_1 \mathcal{F}$ is never empty for $x \in M$. We will see that the number of points in this intersection carries crucial information on the dynamics of the $B$-action.

It is possible that $\mathcal{M} \cap T^1_1 \mathcal{F}$ consists of a single point for all $x \in M$. Foliations for which this happens are characterized in the following proposition.

**Proposition 3.** Let $(M, \mathcal{F})$ be a compact minimal lamination by hyperbolic surfaces. Then $(M, \mathcal{F})$ is the lamination by orbits of a continuous, locally free, right action of the group $B$ on $M$ if and only if there is a compact set $\mathcal{M} \subset \hat{M}$ which is minimal for the action of $B$ on $\hat{M}$ such that $T^1_1 \mathcal{F} \cap \mathcal{M}$ consists of a single point for all $x \in M$.

**Proof.** Assume there exists one such $\mathcal{M}$. The fact that it intersects each unit tangent space in only one point means that $\pi : \mathcal{M} \to M$ is a homeomorphism. Since it sends central-stable leaves in $\mathcal{M}$ to leaves of $\mathcal{F}$, via $\Pi$ the right $B$-action on $\mathcal{M}$ defines a right $B$-action on $M$ whose orbits are the leaves of $\mathcal{F}$.

To show the converse, we need to study the geometry associated to a locally free $B$-action. First of all, we fix a left invariant hyperbolic Riemannian metric $g_B$ on $B$. In fact, the group $B$ is isomorphic to $B' = \{(y, x, 1) : x \in \mathbb{R}, y > 0\}$, and the metric $g_{B'} = y^{-2}(dx^2 + dy^2)$ is left invariant. We choose $g_B$ to be the metric on $B$ which corresponds to $g_{B'}$ by the isomorphism. Let $\phi : M \times B \to M$ be a locally free $B$-action. For any $x \in M$, define $\phi_x : B \to M$ by $\phi_x(b) = \phi(x, b)$.

Then there is a leafwise metric $g_\phi$ of the orbit foliation which satisfies $\phi_x^* g_\phi = g_B$, for any $x \in M$.

The metric $g_\phi$ is well defined since $g_B$ is left invariant.

Now assume that the lamination $\mathcal{F}$ on $M$ is defined by a right $B$ action $\phi$. Let $D$ and $U$ be the one-parameter subgroups of $B$ defined in the previous section. On the universal covering space of each $B$-orbit (equipped with the metric $g_\phi$) of a point $x \in M$, there is a point $\xi(x)$ at infinity such that any $D$-orbit is a geodesic tending to $\xi(x)$ and any $U$-orbit is a horocycle at $\xi(x)$. Now since $M$ is compact, the leafwise metric $g_\phi$ is quasi-isometric to the leafwise metric $g$ which is given a priori. Therefore the point $\xi(x)$ at infinity with respect to $g_\phi$ corresponds to a point at infinity with respect to $g$, which we still denote by $\xi(x)$. Notice that $\xi(x)$ is $B$-invariant and continuous in $x$.

Finally for any $x \in M$, define $X(x) \in T^1_x \mathcal{F}$ to be the tangent vector tending to $\xi(x)$. This way, we get a cross section $X : M \to \hat{M}$. The image $\mathcal{M}$ is compact, invariant and minimal for the action of $B$ on $\hat{M}$. \qed
Foliations that arise in this way have been widely studied, see for example [26]. They do not possess holonomy-invariant measures. All of their leaves are homeomorphic to planes or cylinders, and in codimension one they must have countably many cylindrical leaves, and these leaves have non-trivial holonomy.

Although the $C^\infty$ locally free $B$-actions in dimension 3 are found only in the $S^1$-bundles over surfaces ([3]), continuous actions are abundant. The central stable foliation of any topologically transitive Anosov flow is the orbit foliation of a continuous $B$-action, thanks to the existence of the Margulis measure along the stable foliation ([16]). There are many examples of such flows ([11], [4], [12]).

Apart from the situation described in Proposition 3, two possibilities remain: that $T^1_x \mathcal{F} \cap \mathcal{M}$ has more than one point for all $x$, and that $T^1_x \mathcal{F} \cap \mathcal{M}$ has one point for some values of $x$ and more than one point for other values of $x$.

**PROPOSITION 4.** Let $(M, \mathcal{F})$ be a compact minimal lamination by hyperbolic surfaces. If $\mathcal{M} \subset \hat{M}$ is compact and minimal for the action of $B$ on $\hat{M}$ and it intersects each unit tangent space in more than one point, then $\mathcal{M} = \hat{M}$.

To prove this we will use an argument due to Étienne Ghys.

**Proof.** Assume $\mathcal{M}$ is a nontrivial minimal set (that is, that $\mathcal{M}$ is not the whole $\hat{M}$), and that it intersects each unit tangent space in at least two points.

Let us take a point $x \in M$ and call $L_x$ the leaf of $\mathcal{F}$ that passes through $x$. The fact that the lamination $(\hat{M}, T^1 \mathcal{F})$ is minimal and $\mathcal{M}$ is $B$-invariant implies that the unit tangent space to $L_x$ at $x$ is not contained in $\mathcal{M}$, i.e., $T^1_x \mathcal{F} \cap \mathcal{M}$ is a nonempty closed proper subset of $T^1_x \mathcal{F}$. In any case, by identifying in the usual way $T^1_x \mathcal{F}$ with the set of points at infinity of the universal cover $\tilde{L}_x$ of $L_x$, we may think of $T^1_x \mathcal{F} \cap \mathcal{M}$ as a subset $K_x$ of the circle at infinity of $\tilde{L}_x$. Notice that $K_x$ does not depend on $x$ but only on the leaf $L_x$, since $\mathcal{M}$ is $B$-invariant.

For the moment we will also assume that all leaves of $\mathcal{F}$ are simply connected.

For every $x \in M$, let $\hat{K}_x$ be the convex hull of $K_x$ in $\tilde{L}_x$. (Namely, consider all geodesics in $L_x$ joining pairs of points in $K_x$, and take the convex hull of their union. This set is $\hat{K}_x$.) It is possible to do this because $K_x$ has at least two points.

Let $f : M \to [0, +\infty)$ be the function defined by

$$f(y) = d(y, \hat{K}_x),$$

where $d$ is the hyperbolic distance on the leaf passing through $y$.

**REMARK 5.** The function $f$ is measurable.

**Proof of remark.** Let $E \approx D \times T$ be a compact foliated chart of $(M, \mathcal{F})$, where $D$ is a closed disk in $\mathbb{R}^2$ and $T$ is a topological space. In $E$, the $\hat{K}_x$ form a semi-continuous family of compact sets parametrized by $T$, and the function $f$ is the distance on each fiber $D \times \{t\}$ to the corresponding compact set. Therefore $f$ is measurable. 

In [6], the authors prove that the axiom of choice is not needed to prove the existence of minimal sets. This implies that the function $f$ can be defined.
without using the axiom of choice, and this is another reason why we can safely assume that \( f \) is measurable (see [29]).

Let \( \mu \) be an ergodic harmonic measure on \((M, \mathcal{F})\). For every \( n \in \mathbb{N} \), we define
\[
A_n = \{ x \in M : f(x) \leq n \}.
\]
The sequence \( \{A_n\} \) is increasing and \( \mu(\bigcup_n A_n) = 1 \), therefore there exists an \( n \in \mathbb{N} \) for which \( \mu(A_n) > 0 \). The ergodic theorem tells us that for \( \mu \)-almost every \( x \in M \) and almost every continuous path \( \omega \) on \( L_x \) that starts at \( x \)
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \chi_{A_n}(\omega(t)) \, dt > 0,
\]
where \( \chi_{A_n} \) is the characteristic function of the set \( A_n \).

In spite of this, for every \( x \in M \), the set of continuous paths \( \omega(t) \) on \( L_x \) which start at \( x \) and which converge, when \( t \to \infty \), to a point outside \( K_x \) has positive Wiener measure, and for any of these paths
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \chi_{A_n}(\omega(t)) \, dt = 0.
\]
Since this contradicts (3), we have proved the lemma when all leaves of \( \mathcal{F} \) are hyperbolic planes.

If there is a leaf \( L_x \) of \( \mathcal{F} \) which is not simply connected, we can still define \( \hat{K}_x \) on its universal cover. For points on \( L_x \), we define \( f(y) \) as the distance from \( y \) to the projection of \( \hat{K}_x \) and use the same argument as before to complete the proof.

In fact, the previous argument proves the following slightly stronger statement, which will be useful later on:

**Proposition 6.** Let \((M, \mathcal{F})\) be a compact minimal lamination by hyperbolic surfaces, and let \( \mu \) be an ergodic harmonic measure on \( M \). If \( \mathcal{M} \subset M \) is compact and minimal for the action of \( B \) on \( \hat{M} \), then either \( T_x^1 \mathcal{F} \cap \mathcal{M} \) is a singleton for \( \mu \)-almost every \( x \) or \( \mathcal{M} = \hat{M} \).

**Remark 7.** Notice that in the proof of Proposition 4, we do not actually use that \( \mathcal{M} \) is minimal, only that it is a closed \( B \)-invariant set.

We use this fact to prove the uniqueness of \( B \)-minimal sets, even in the case where the \( B \) action is not minimal.

**Theorem 8.** If the action of \( B \) on the unit tangent bundle \( \hat{M} \) is not minimal, then there exists a unique minimal set \( \mathcal{M} \) which is a proper subset of \( \hat{M} \).

**Proof.** If \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) were two minimal sets they have to be disjoint and then \( \mathcal{M}_1 \cup \mathcal{M}_2 \) would be a \( B \)-invariant set which intersects the circle fibers of \( \hat{M} \) in more than one point and therefore, by Proposition 2, \( \mathcal{M}_1 \cup \mathcal{M}_2 = \hat{M} \) which is a contradiction.

Proposition 4, in the slightly stronger version mentioned in Remark 7, has a pair of interesting consequences for the action of the affine group \( B \), as we will now see.
Theorem 9. If $M$ is a compact minimal lamination by hyperbolic surfaces and $\hat{M}$ is the laminated unit tangent bundle then the locally free action of $B$ on $M$ is topologically transitive. In particular the set of $B$-orbits which are dense in $\hat{M}$ is residual (in the sense of Baire).

Lemma 10. Let $\mathcal{U} \subseteq \hat{M}$ be any open set. Let $B\mathcal{U} = \{ b(u) : b \in B, u \in \mathcal{U} \}$ be the $B$-orbit of $\mathcal{U}$. Then $\overline{B\mathcal{U}} = \hat{M}$. In other words, $B\mathcal{U}$ is $B$-invariant open and dense in $\hat{M}$.

Proof. Clearly $B\mathcal{U}$ is open and $B$-invariant. Suppose $\overline{B\mathcal{U}} \neq \hat{M}$. Let $\mathcal{V} \neq \emptyset$ be the exterior in $\hat{M}$ of $B\mathcal{U}$. Then $\mathcal{V}$ is also $B$-invariant and open. Since $\mathcal{F}$ is a minimal hyperbolic lamination it follows that for each $x \in M$ one has that $B\mathcal{U} \cap T^1_x \mathcal{F} =: B\mathcal{U}_x$ and $\mathcal{V} \cap T^1_x \mathcal{F} =: \mathcal{V}_x$ are nonempty disjoint open subsets of the circle fibre $T^1_x \mathcal{F}$, therefore $C_x = T^1_x \mathcal{F} - (B\mathcal{U}_x \cup \mathcal{V}_x)$ is a closed set in $T^1_x \mathcal{F}$ with more than one point. The set $D = \hat{M} - (B\mathcal{U} \cup \mathcal{V})$ is a nonempty compact set which is $B$-invariant and it intersects $T^1_x \mathcal{F}$ in the set $C_x$ with more than one point. Therefore (by Proposition 4) $D = \hat{M}$. This is a contradiction, hence $\mathcal{V} = \emptyset$. Therefore $\mathcal{U}$ is dense.

Proof of Theorem 9. The proof is by the standard Baire category argument. Let $\{ \mathcal{U}_i \}_{i \in \mathbb{N}}$ be a countable open base of the topology of $\hat{M}$. Let $\mathcal{U}_i$ be the set of points $y \in M$ such that the $B$-orbit of $y$ intersects $\mathcal{U}_i$ ($i \in \mathbb{N}$) then, by lemma 1, $\mathcal{U}_i$ is open and dense. By Baire’s category theorem the set $\mathcal{U} = \cap_{i \in \mathbb{N}} \mathcal{U}_i$ is $B$-invariant and residual in $\hat{M}$. The set $\mathcal{U}$ is dense and consists of dense $B$-orbits.

Proposition 11. Let $(M, \mathcal{F})$ be a compact minimal lamination by hyperbolic surfaces. If the $B$-action on $\hat{M}$ is minimal, then the laminated geodesic flow on $\hat{M}$ is topologically transitive.

Proof. Let $U \subseteq \hat{M}$ be an open set. It contains a small segment $I$ of a stable horocycle orbit. When the $B$-action is minimal, $\cup_{t<0} g_t(I)$ is a dense subset of $\hat{M}$. One way to see this is the following: consider the normalized Lebesgue measure $m$ of the horocycle segment $I$, the sequence of probability measures $m_n$ defined by $\int f \, dm_n = \frac{1}{n!} \int_0^1 \frac{d}{dt} g_t^\ast f \, dt \, dm$, and an accumulation point $m_\infty$ of $m_n$. In way of contradiction, assume that $K = \cup_{t<0} g_t(I)$ is not the whole $\hat{M}$. Consider a continuous function $f : \hat{M} \to [0,1]$ taking value 1 on $K$ and value 0 on some point outside $K$. Clearly $\int_M f \, dm_n = 1$ for all $n$, and therefore $\int_M f \, dm_\infty = 1$. But direct computation shows that $m_\infty$ is $B$-invariant. Thus $m_\infty$ has full support on $\hat{M}$ and we must have $\int_M f \, dm_\infty < 1$. A contradiction shows that $\cup_{t<0} g_t(U)$ and hence $\cup_{t<0} g_t(U)$ is dense in $\hat{M}$.

Thus the set of points whose geodesic orbit intersects $U$, namely the set $\cup_{t \in \mathbb{R}} g_t(U)$, is open and dense. Therefore, there is a residual set of points whose geodesic orbit intersects every element of a countable basis for the topology of $\hat{M}$.
The possibility that the $B$-minimal set $\mathcal{M}$ intersects some fibers $T^1_x \mathcal{F}$ in one point and others in more than one point cannot be ruled out. This will be shown in the example at the end of this section. Recall that we assumed that $M$ is a compact metric space and $\hat{M}$ is also a metric space since it is a fiber bundle over $M$ with fiber the circle. Let $S_x = T^1_x \mathcal{F} \cap \mathcal{M}$. Remark that $S_x \neq \emptyset$ for all $x \in M$ and the cardinality of $T^1_x \mathcal{F} \cap \mathcal{M}$ remains constant on leaves. For each $x \in M$ let us provide each unit circle $T^1_x \mathcal{F}$ with its standard Riemannian metric. For each closed subset $C$ of $T^1_x \mathcal{F}$ let $\text{diameter}_x(C)$ denote the diameter of $C \subset T^1_x \mathcal{F}$ with respect to this metric. If $\{x_i\}_{i \in \mathbb{N}}$ is a sequence of points of $M$ which converges to $x$ and such that $S_{x_i}$ converges in the Hausdorff distance (with respect to the metric on $\hat{M}$) to $\hat{S}_x \subset T^1_x \mathcal{F}$ then $\hat{S}_x \subset S_x$ since $\mathcal{M}$ is closed. Hence $\text{diameter}_x(\hat{S}_x) \leq \text{diameter}_x(S_x)$ and therefore the function

$$x \mapsto \text{diameter}_x(S_x)$$

is upper semicontinuous. The set

$$\mathcal{R} = \left\{ x \in M : \#(T^1_x \mathcal{F} \cap \mathcal{M}) = 1 \right\} = \left\{ x \in M : \text{diameter}_x(S_x) = 0 \right\},$$

where $\#$ denotes cardinality, is a countable intersection of open and dense subsets of $M$, unless it is empty, because

$$\mathcal{R} = \bigcap_{n \in \mathbb{N}} R_n$$

where $R_n$ is the open set $R_n = \{ x \in M : \text{diameter}_x(S_x) < 1/n \}$ and $\mathcal{R}$ is saturated by the minimal lamination $\mathcal{F}$ and hence dense if nonempty. Thus we have

**Lemma 12.** The set of points $x \in M$ where the cardinality of $S_x$ is one is residual.

In order to present the aforementioned example, we will make a small digression to recall some facts about hyperbolic three-manifolds.

Let $\Gamma \subset PSL(2, \mathbb{R})$ be a cocompact surface group, and $\Sigma = \Gamma \backslash \mathbb{H}$ the hyperbolic surface it defines. In $\Sigma$, any simple closed curve $[a]$, not homotopic to a constant, is freely homotopic to a unique geodesic $C_a$.

Thus, to a free homotopy class of loops $[a]$ we can associate the hyperbolic length $\ell([a]) = \ell(C_a)$ of its geodesic representative.

Let $f : \Sigma \to \Sigma$ be a diffeomorphism and $f_*$ the induced map on free homotopy classes. Then, $f$ is said to be of **pseudo-Anosov** type if there exist constants $A(|[a]|), B > 0$ such that $\ell(f^n_*([a])) \geq A(|[a]|)e^{Bn\ell([a])}$, where $f^n = f \circ \cdots \circ f$ ($n$ times). Thus pseudo-Anosov diffeomorphisms increase exponentially the length of loops not homotopic to a constant loop. The property of a diffeomorphism being of pseudo-Anosov type depends only on the isotopy class of the diffeomorphism.

Let $f : \Sigma \to \Sigma$ be an orientation-preserving diffeomorphism of pseudo-Anosov type. Then, by a celebrated theorem by Thurston ([15], [20], [24], [32], [31], [33]) the mapping torus of $f$ is a hyperbolic 3-manifold. Its hyperbolic metric is unique by Mostow’s rigidity theorem. Namely, there is a Kleinian group $\Lambda \subset PSL(2, \mathbb{C})$ such that $\Lambda \backslash \mathbb{H}^3$ is the suspension of $f$. 

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The suspension construction implies that \( \Lambda \setminus \mathbb{H}^3 \) fibers over \( S^1 \) with fiber \( \Sigma \). The group \( \Lambda \) is a semidirect product of \( \Gamma \) and \( Z \), that is, there is an exact sequence

\[
1 \to \Gamma \to \Lambda \to Z \to 0.
\]

Therefore, there is an injective homomorphism \( \phi : \Gamma \to \Lambda \) whose image \( \phi(\Gamma) \) is a normal subgroup of \( \Lambda \).

Denote by \( S^1 \) the circle at infinity of \( \mathbb{H}^2 \), and by \( \mathbb{P}^1 \) the sphere at infinity of \( \mathbb{H}^3 \). According to a theorem by Cannon and Thurston (see [9]) there is a \( \phi \)-equivariant continuous and surjective map \( g : S^1 \to \mathbb{P}^1 \) (a sphere filling curve). Let \( \Gamma \) act on \( S^1 \times \mathbb{P}^1 \) by \( \gamma \cdot (\theta, \zeta) = (\gamma \theta, \phi(\gamma) \zeta) \). Then the graph \( \mathcal{X} \) of \( g \) is a minimal set of this action.

**Example 13.** Consider the manifold

\[
M = \Gamma \setminus (\mathbb{H}^2 \times \mathbb{P}^1),
\]

where the action is given by \( \gamma \cdot (x, \zeta) = (\gamma x, \phi(\gamma) \zeta) \), and the horizontal foliation \( \mathcal{F} \) on \( M \). The leafwise unit tangent bundle of \( \mathcal{F} \) is

\[
\hat{M} = \Gamma \setminus (PSL(2, \mathbb{R}) \times \mathbb{P}^1).
\]

Let \( B \) be the affine group. The \( B \)-action on \( \hat{M} \) is dual (i.e., Morita equivalent) to the \( \Gamma \)-action on \( (PSL(2, \mathbb{R})/B) \times \mathbb{P}^1 \) or \( S^1 \times \mathbb{P}^1 \). There is a minimal set \( \mathcal{M} \) of the former \( B \)-action which corresponds to the above minimal set \( \mathcal{X} \) of the latter \( \Gamma \)-action.

Let \( x \in M \) and let \( L \) be the leaf of \( \mathcal{F} \) through \( x \). Take \( \zeta \in \mathbb{P}^1 \) such that \( \mathbb{H}^2 \times [\zeta] \) projects onto \( L \). The intersection \( T^1_x \mathcal{F} \cap \mathcal{M} \) consists of a single point if and only if \( (\Gamma \mathcal{M}) \cap (PSL(2, \mathbb{R}) \times [\zeta]) \) consists of a single \( B \)-orbit. Since \( \Gamma \mathcal{M} = \mathcal{X} \mathcal{B} \), this is equivalent to \( \mathcal{X} \) intersecting \( S^1 \times [\zeta] \) in a single point. This happens if and only if \( \zeta \) has only one preimage under \( g \), which is true for some, but certainly not for any \( \zeta \). The last statement is clear since the map \( g \) cannot be a homeomorphism. To show the first statement, let \( \zeta, \zeta' \) be the fixed points of some \( \phi(\gamma) \in \phi(\Gamma) \sim \{ e \} \), a loxodromic element. Both \( g^{-1}(\zeta) \) and \( g^{-1}(\zeta') \), being closed, nonempty and invariant by \( \gamma \), must contain at least one point from \( \text{Fix}(\gamma) \). The set \( g^{-1}(\zeta) \) cannot contain any point from \( S^1 \setminus \text{Fix}(\gamma) \), since if it does, \( g^{-1}(\zeta) \) would contain the whole \( \text{Fix}(\gamma) \), contradicting the fact that \( g^{-1}(\zeta) \cap g^{-1}(\zeta') = \emptyset \). The same is true for \( g^{-1}(\zeta') \) and one can conclude that \( g^{-1}(\zeta) \), as well as \( g^{-1}(\zeta') \), is a singleton.

Theorem 8 implies that the minimal set \( \mathcal{M} \) in Example 13 is unique.

We will finally give a condition that guarantees the minimality of the \( B \)-action on \( \hat{M} \).

**Theorem 14.** Let \( (M, \mathcal{F}) \) be a compact minimal lamination by hyperbolic surfaces that has a holonomy-invariant measure. Then the \( B \)-action on its unit tangent bundle is minimal.
Proof. Assume for contradiction that there is a proper minimal set \( \mathcal{M} \) for the \( B \)-action. Since \( \mathcal{F} \) is minimal, we have \( \pi(\mathcal{M}) = M \). Consider a foliated chart \( D \times T \) of \( \mathcal{F} \), where \( T \) is a transversal. Its inverse image can be written as
\[
\Pi^{-1}(D \times T) = D \times S^1 \times T,
\]
where \( D \times S^1 \times \{t\} \) is a plaque of \( T^1 \mathcal{F} \) and \( D \times \{\theta\} \times \{t\} \) is a plaque of the foliation by \( B \)-orbits. The intersection \( \mathcal{M} \cap (D \times S^1 \times T) \) has the form
\[
\mathcal{M} \cap (D \times S^1 \times T) = D \times \mathcal{N}
\]
for some closed subset \( \mathcal{N} \) of \( S^1 \times T \). Since \( \mathcal{N} \) and \( T \) are standard Borel spaces and the inverse image of any point in \( T \) under the projection \( \mathcal{N} \to T \) is compact, there is a Borel cross section \( \sigma : T \to \mathcal{N} \). This can also be shown in an elementary way as follows. For each \( t \in T \), the set
\[
\mathcal{N}_t = \{ \theta \in S^1 : (\theta, t) \in \mathcal{N} \}
\]
is a proper nonempty closed subset of \( S^1 \). In fact, if it is \( S^1 \) for some \( t \), \( \mathcal{M} \) would contain at least one leaf of \( T^1 \mathcal{F} \). Since \( \mathcal{F} \) is minimal, this would contradict the properness of \( \mathcal{M} \). Choosing \( T \) smaller if necessary, one may assume that there is an open interval \( I \) of \( S^1 \) such that \( \mathcal{N}_t \) is contained in \( I \). Then there is an upper semicontinuous cross section \( \sigma : T \to S^1 \) defined by \( \sigma(t) = \sup \mathcal{N}_t \).

Let \( \mu \) be an ergodic transverse holonomy-invariant measure. Together with the leafwise hyperbolic volume, it defines an ergodic completely invariant harmonic measure \( \lambda \). Proposition 6 says that \( \mathcal{N}_t \) must be a singleton for \( \mu \)-almost all \( t \). This gives us the following lemma:

**Lemma 15.** The measure \( \sigma_* \mu \) is independent of the Borel cross section \( \sigma \).

Now the family formed by \( \sigma_* \mu \) for each foliation chart yields a transverse invariant measure of the lamination by \( B \)-orbits on \( \mathcal{M} \), and hence a completely invariant measure on \( \mathcal{M} \). But the geodesic flow on \( \mathcal{M} \) preserves the transverse measure on one hand, and contracts the leafwise hyperbolic measure on the other hand\(^1\). A contradiction proves the theorem. \( \square \)

4. Minimal Sets for the Laminated Horocycle Flow

We will begin this section by looking at several examples.

**Example 16.** Let \( A : \mathbb{T}^2 \to \mathbb{T}^2 \) be a hyperbolic linear automorphism of the 2-torus, for example, the one given by the matrix \( \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \). The suspension of \( A \) is a 3-manifold \( \mathbb{T}^3_A \) which fibers over the circle with fiber the torus \( \mathbb{T}^2 \). It is a solvmanifold whose universal cover is a solvable Lie group whose Lie algebra is generated by \( X \), \( Y \) and \( Z \) which satisfy
\[
[X, Y] = 0, \quad [Z, X] = -X, \quad [Z, Y] = Y.
\]
Therefore, \( Z \) and \( Y \) generate a locally free action of the affine group, as well as \( Z \) and \( X \). This is the classical example of an Anosov flow.

\(^1\) In fact one can similarly show that if a lamination comes from a locally free action of any non-unimodular group, it cannot have holonomy-invariant measures.
More precisely, since the matrix $A$ is diagonalizable with positive eigenvalues there exists a $2 \times 2$ matrix $C$ such that $e^C = A$ so that for every integer $n$ one has $e^{nC} = A^n$. Let $\mathbb{R}^3_\delta$ be the solvable group whose elements $((x, y), t)$ belong to $\mathbb{R}^2 \times \mathbb{R}$ and with operation $\star$ defined by

$$(x_1, y_1), t_1 \star (x_2, y_2), t_2 = (e^{nC}(x_2, y_2) + (x_1, y_1), t_1 + t_2)$$

Thus $\mathbb{R}^3_\delta$ is the semidirect product $\mathbb{R}^2 \ltimes_\delta \mathbb{R}$ of $\mathbb{R}$ with $\mathbb{R}^2$ corresponding to the monomorphism $\Phi : \mathbb{R} \rightarrow GL(2, \mathbb{R}) = Aut(\mathbb{R}^2)$ given by $t \mapsto e^{tC}$. The 3-dimensional Lie group $\mathbb{R}^3_\delta$ is isomorphic to the unique 3-dimensional unimodular solvable non-nilpotent Lie group, usually denoted by $Solv_3$. It has as discrete and uniform subgroup the group $\mathbb{Z}_3$ consisting of elements $((n, m), l)$ of $\mathbb{R}^3_\delta$ where $m$, $n$ and $l$ are integers. The group $\mathbb{Z}_3$ is the semidirect product $\mathbb{Z}^2 \ltimes_\phi \mathbb{Z}$ of $\mathbb{Z}^2$ with $\mathbb{Z}$ corresponding to the monomorphism $\phi : \mathbb{Z} \rightarrow SL(2, \mathbb{Z}) = Aut(\mathbb{Z}^2)$ given by $n \mapsto A^n$.

We have $T^3_A = \mathbb{R}^3_\delta / \mathbb{Z}_3$ and the commutative diagram of short exact sequences

$$
\begin{array}{cccc}
0 & \rightarrow & \mathbb{Z}^2 & \rightarrow & \mathbb{Z}_3^3 & \rightarrow & \mathbb{Z} & \rightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \rightarrow & \mathbb{R}^2 & \rightarrow & \mathbb{R}_3^3 & \rightarrow & \mathbb{R} & \rightarrow & 0
\end{array}
$$

From this we obtain the sequence of differentiable maps

$$
0 \rightarrow T^2 \rightarrow T^3_A \xrightarrow{F} S^1 \rightarrow 0
$$

where $F : T^3_A \rightarrow S^1$ is a locally trivial fibration. There are two subgroups $B_1$ and $B_2$ of the simply connected solvable Lie group $\mathbb{R}^3_\delta$

$$
B_1 = \{(sa_1, sa_2), t) \in \mathbb{R}^3_\delta : s, t \in \mathbb{R}\}
$$

$$
B_2 = \{(sb_1, sb_2), t) \in \mathbb{R}^3_\delta : s, t \in \mathbb{R}\}
$$

where $(a_1, a_2)$ and $(b_1, b_2)$ are eigenvectors of $A$ corresponding to the eigenvalues $\lambda = \frac{3 + \sqrt{3}}{2}$ and $\lambda^{-1}$.

Both $B_1$ and $B_2$ are isomorphic to the real affine group $B$ and intersect in the one-parameter subgroup $\{((0, 0), (\log \lambda)^{-1} t) : t \in \mathbb{R}\}$. This one-parameter subgroup generates the vector field $Z$ and the one-parameter subgroups

$$\{((sb_1, sb_2), 0) \in \mathbb{R}^3_\delta : s \in \mathbb{R}\}$$

$$\{((sa_1, sa_2), 0) \in \mathbb{R}^3_\delta : s \in \mathbb{R}\}$$

generate the vector fields $X$ and $Y$. These three vector fields descend to the three vector fields described at the beginning. This is an algebraic description of the suspension of the Anosov diffeomorphism of the 2 torus induced by the matrix $A$. We have two locally free actions of the affine group $B$ on $T^3_A$, so that we have two hyperbolic laminations. We also have a locally free action of $\mathbb{R}^2$ induced by the commuting vector fields $X$ and $Y$. The orbits of this action on $T^3_A$ are the 2-tori which are the fibers of $F$. The orbits of both $X$ and $Y$ are minimal in the fibre which contains them.
Consider one of these laminations whose leaves are the orbits by one of the locally free actions of \( B \). Say the one generated by \( Z \) and \( X \). In the unit tangent bundle of the lamination we also have the action of \( B \). This \( B \)-action is not minimal because the union of the unit vectors tangent to the flow lines of the action of \( Z \) is a closed set \( \mathcal{M} \) invariant by the action. In fact \( \mathcal{M} \) is the unique minimal set of the \( B \)-action. The minimal set \( \mathcal{M} \) is the image of the section \( \sigma : M \to \hat{M} \) which assigns to each point in \( M \) the unit vector tangent to \( Z \) at the point. The periodic orbits of the suspension flow are dense in \( \hat{M} = \mathbb{T}^3_A \). Each orbit admits two lifts as periodic orbits of the laminated geodesic flow on \( \hat{M} \), one with the orientation given by the time parametrization of the suspension flow, and the other opposite. The former is contained in the unique minimal set \( \mathcal{M} \) of the \( B \)-action and, all together, forms a dense subset there. The latter is contained in the unique minimal set \( \mathcal{M}' \) of the central unstable foliation on \( \hat{M} \). The set \( \mathcal{M} \) (resp. \( \mathcal{M}' \)) is a repeller (an attractor) of the laminated geodesic flow. Minimal sets for the horocycle flow are 2-tori.

**Example 17.** This example is related to the previous one. Instead of suspending an Anosov flow one suspends an expanding automorphism of the dyadic solenoid. Let \( \mathcal{S}_2 \) be the dyadic solenoid

\[
\mathcal{S}_2 = \lim \left\{ \mathbb{S}^1 \overset{f}{\leftarrow} \mathbb{S}^1 \overset{f}{\leftarrow} \mathbb{S}^1 \ldots \right\},
\]

where \( f(z) = z^2 \) (\( z \in \mathbb{S}^1 \)). Namely \( \mathcal{S}_2 \) consists of sequences as follows

\[
\mathcal{S}_2 = \left\{ \hat{z} \overset{\text{def}}{=} (z_1, z_2, \ldots) : z_i \in \mathbb{S}^1, z_{i+1}^2 = z_i \quad \forall \ i = 1, 2, \ldots \right\}.
\]

\( \mathcal{S}_2 \) is a solenoidal compact 1-dimensional abelian group, with coordinate wise multiplication. Consider the map

\[
T(\hat{z}) = \hat{z}^2,
\]

that is, \( T(z_1, z_2, \ldots) = (z_1^2, z_2^2, z_3^2, \ldots) = (z_1^2, z_2, z_3, \ldots) \). Note \( T \circ \theta = \theta \circ T = \text{Id} \), where \( \theta \) is the shift map. We refer to [14] for more details. The map \( T \) is an automorphism since obviously \( T(s_1 s_2) = T(s_1) T(s_2) \) and \( T \) is bijective. This solenoidal group is locally the product of a Cantor set and an open interval. It has a vector field \( X \) tangent to the flow \( \eta_t : \mathcal{S}_2 \to \mathcal{S}_2 \) given by the formula

\[
\eta_t(z_1, z_2, \ldots) = (e^{2\pi i t} z_1, e^{\pi i t} z_2, e^{(1/2)\pi i t} z_3, e^{(1/4)\pi i t} z_4, \ldots, e^{(1/2^n - 1)\pi i t} z_n, \ldots)
\]

One has that \( dT(X) = 2X \), so that \( T \) is an expanding automorphism. The flow \( \{\eta_t\}_{t \in \mathbb{R}} \) is minimal by classical results of Vietoris and Van Dantzig ([35], [36]). Therefore \( \mathcal{S}_2 \) admits a minimal lamination by the orbits of \( \{\eta_t\}_{t \in \mathbb{R}} \). The expanding automorphism \( T \) leaves invariant the Haar measure of the compact abelian group \( \mathcal{S}_2 \) and it is ergodic with respect to that measure. The periodic points of \( T \) are dense in \( \mathcal{S}_2 \). The dyadic solenoid \( \mathcal{S}_2 \) gives an example of a one-dimensional expanding attractor known as the Smale-Williams attractor.

We describe the suspension of \( T \) in a slightly different (but completely equivalent) way. Indeed, let \( \mathbb{R}^+ = \{ u \in \mathbb{R} : u > 0 \} \) be the multiplicative group of the positive real numbers. \( \mathbb{R}^+ \times \mathcal{S}_2 \) is a non compact minimal 2-dimensional lamination
with leaves of the form $\mathbb{R}^* \times L$ where $L$ is a leaf of $\mathcal{P}_2$. Let $\mathfrak{F}: \mathbb{R}^* \times \mathcal{P}_2 \to \mathbb{R}^* \times \mathcal{P}_2$ be defined by $\mathfrak{F}(u, z) = (2u, T(z))$.

The group generated by $\mathfrak{F}$ acts freely and properly discontinuously on $\mathbb{R}^* \times \mathcal{P}_2$ and the quotient $M$ is diffeomorphic (as a lamination) to the suspension of $T$. Let 

$$\xi_t: \mathbb{R}^* \times \mathcal{P}_2 \to \mathbb{R}^* \times \mathcal{P}_2, \quad t \in \mathbb{R}$$

be the flow on $\mathbb{R}^* \times \mathcal{P}_2$ given by the formula 

$$\xi_t(u, z) = (2^t u, z)$$

One has $\mathfrak{F} \circ \xi_t = \xi_t \circ \mathfrak{F}$ for all $t \in \mathbb{R}$. This laminated flow is smooth (in the sense of laminations), and it is tangent to the laminated vector field $Z$. Consider the flow 

$$\nu_s = \mathbb{R}^* \times \mathcal{P}_2 \to \mathbb{R}^* \times \mathcal{P}_2, \quad s \in \mathbb{R}$$

given by the formula 

$$\nu_s(u, z) = (u, \eta_{st} z).$$

One has $\mathfrak{F} \circ \nu_s = \nu_s \circ \mathfrak{F}$ for all $s \in \mathbb{R}$. This laminated flow is smooth (in the sense of laminations), and it is tangent to the laminated vector field $X$. The fact that these two flows commute with $\mathfrak{F}$ implies that they descend to two flows, denoted by the same symbols $\{\xi_t\}_{t \in \mathbb{R}}, \{\nu_s\}_{s \in \mathbb{R}}$, on the suspension space $M = \mathbb{R}^* \times \mathcal{P}_2 / \{z^n\}_{n \in \mathbb{N}}$: 

$$\nu_s, \xi_t: M \to M, \quad s \in \mathbb{R}, \ t \in \mathbb{R}$$

The two abelian one-parameter groups of diffeomorphisms $\{\xi_t\}_{t \in \mathbb{R}}, \{\nu_s\}_{s \in \mathbb{R}}$ on $\mathbb{R}^* \times \mathcal{P}_2$ generate a 2-dimensional group $G$ of diffeomorphisms of $\mathbb{R}^* \times \mathcal{P}_2$ and this group is isomorphic to the affine group $B$. Indeed, we have that $\{\nu_s\}_{s \in \mathbb{R}}$ is normal in $G$ because 

$$\xi_{-t} \nu_s \xi_t = \nu_{2^s t}, \quad \forall \ t, s \in \mathbb{R}$$

and any element of $G$ can be written in a unique way as $\xi_t \nu_s$. It also follows from the fact that we have the laminated bracket relation $[Z, X] = (\log 2) X$. The vector field $Z' = (\log 2)^{-1} Z$ satisfies $[Z', X] = X$. Now we identify $G$ with $B$ and one has a free smooth action of the group $B$ on $\mathbb{R}^* \times \mathcal{P}_2$ with leaves $\mathbb{R}^* \times L$ where $L$ is a leaf of $\mathcal{P}_2$, hence we have a minimal lamination by hyperbolic planes on $\mathbb{R}^* \times \mathcal{P}_2$.

The vector fields $Z$ and $X$ descend to $M$ to two vector fields that we still denote by the same letters. Therefore on $M$ we have a minimal lamination by the orbits of the locally free action of $B$. This lamination has a countable number of leaves which are hyperbolic cylinders corresponding to the periodic points of $T$ and all the other leaves are densely embedded copies of the hyperbolic plane.

Let $\mathcal{H} \subset \hat{M}$ be the image of the section $Z': M \to \hat{M}$ of the bundle $\Pi: \hat{M} \to M$ defined by the vector field $Z'$.

Then, as in the proof of Proposition 3, $\mathcal{H}$ is minimal for the action of $B$ on $\hat{M}$. It is not minimal for the laminated horocycle flow alone; in fact, each set of the form $Z'((\mathcal{P}_2 \times \{t\}), t \in [0, 1)$, is minimal for the horocycle flow. The restriction to $\mathcal{H}$ of the geodesic flow is conjugate by $\pi = \Pi_{\mathcal{H}}: \mathcal{H} \to M$ to the suspension flow of $T$, and has dense periodic orbits since $T$ has dense periodic orbits.
EXAMPLE 18. A small twist yields a more general example which is similar in spirit, but without periodic orbits for the geodesic flow. Let \( T : \mathcal{P}_2 \rightarrow \mathcal{P}_2 \) be the expanding map on the dyadic solenoid as in the previous paragraph, and \( F : \mathcal{X} \rightarrow \mathcal{X} \), a minimal continuous dynamical system on a compact space \( \mathcal{X} \). Now let \( M \) be the suspension of the map

\[
\hat{T} : \mathcal{P}_2 \times \mathcal{X} \rightarrow \mathcal{P}_2 \times \mathcal{X}, \quad \hat{T}(x, y) \mapsto (T(x), F(y)).
\]

The suspension is constructed in exactly the same way as in the previous examples: define

\[
\hat{\mathcal{F}} : \mathbb{R}^* \times \mathcal{P}_2 \times \mathcal{X} \rightarrow \mathbb{R}^* \times \mathcal{P}_2 \times \mathcal{X}
\]

by the formula

\[
\hat{\mathcal{F}}(u, \hat{z}, x) = (2u, T(\hat{z}), F(x)) = (2u, (\hat{z})^2, F(x)).
\]

Then the group generated by \( \hat{\mathcal{F}} \) acts freely and properly on \( \mathbb{R}^* \times \mathcal{P}_2 \times \mathcal{X} \) and the quotient space \( M \) is the suspension of \( \hat{\mathcal{F}} \). The space \( \mathbb{R}^* \times \mathcal{P}_2 \times \mathcal{X} \) has a 2-dimensional lamination with leaves \( \mathbb{R}^* \times L \times \{x\} \) \((x \in \mathcal{X})\), which is the orbit lamination of the joint actions

\[
\hat{\nu}_s, \hat{\xi}_t = \mathbb{R}^* \times \mathcal{P}_2 \times \mathcal{X} \rightarrow \mathbb{R}^* \times \mathcal{P}_2 \times \mathcal{X}, \quad s \in \mathbb{R}, \ t \in \mathbb{R}
\]

defined by:

\[
\hat{\xi}_t(u, \hat{z}, x) = (2^t u, \hat{z}, x) \\
\hat{\nu}_s(u, \hat{z}, x) = (u, \eta_{su} \hat{z}, x)
\]

They descend to actions on \( M \) denoted by the same letters

\[
\hat{\xi}_t, \hat{\nu}_s, M \rightarrow M, \quad s \in \mathbb{R}, \ t \in \mathbb{R}.
\]

Again the flows \( \{\hat{\xi}_t\}_{t \in \mathbb{R}} \) and \( \{\hat{\nu}_s\}_{s \in \mathbb{R}} \) generate an action of the affine group \( B \), and the flows are tangent to vector fields \( Z \) and \( X \) satisfying the relation \( [Z, X] = (\log 2) X \). The proof is identical to that of the previous example except that in this case the action is actually free due to the minimal action of \( F \) in \( \mathcal{X} \). As before, it carries a minimal 2-dimensional lamination \( \mathcal{F} \) which comes from the free minimal action of the affine group. The image of the vector field section \( Z' : M \rightarrow \hat{M} \) \((Z' = (\log 2)^{-1} Z)\) is invariant under the central-stable foliation and is homeomorphic, as a foliated space, to \((M, \mathcal{F})\). Minimal sets for the horocycle flow are sets of the form \( Z'((\mathcal{P}_2 \times \{y\}) \times \{t\}) \), with \( y \in \mathcal{X}, \ t \in [0, 1] \). Again \( \mathcal{M} \) is the union of minimal sets for the horocycle flow, and the restriction of the geodesic flow to \( \mathcal{M} \) is a suspension.

In these three examples, the set \( \mathcal{M} \) which is minimal for the action of \( B \) is a union of sets which are minimal for the horocycle flow. Needless to say, there are examples where the horocycle flow is itself minimal in \( \hat{M} \), the main one being when the lamination consists of only one leaf which is a compact hyperbolic surface. We will now give an example where the horocycle flow is minimal in \( \mathcal{M} \) but not in \( \hat{M} \).
**Example 19.** Let $S$ be an orientable compact connected hyperbolic surface. Then its unit tangent bundle $M = T^1S$ has a locally free transitive action of $PSL(2, \mathbb{R})$ and therefore has a locally free action of the affine group determined by two unit vector fields $X, Y : M \to T^1(PSL(2, \mathbb{R}))$ such that $[X, Y] = Y$. Let $\mathcal{F}$ be the foliation given by the orbits of this affine action. Let $(\hat{M}, T^1\mathcal{F})$ be the unit tangent bundle of this foliation. As before, the image of the section $s : M \to \hat{M}$ defined by $s(m) = X(m)$ is a minimal set $\mathcal{M}$ for the $B$-action on $\hat{M}$. This action is obviously differentiably conjugate to the original action of the affine group on $M$. Unlike in the previous examples, the set $\mathcal{M}$ is also minimal for the foliated horocycle flow.

**Remark 20.** When $G = PSL(2, \mathbb{R})$ acts smoothly and transitively on a 3-manifold $M$ (which must therefore be diffeomorphic to a quotient of $G$ under a discrete subgroup), any closed set invariant under the action of the unipotent group $U$ is either a closed orbit or invariant under the normalizer of $U$ which is $B$ (see for example [22]). If $M$ is compact, $U$ can have no closed orbits, so any $U$-invariant compact set is also $D$-invariant. The previous example shows an instance where this sort of Mautner phenomenon does not hold when the $G$-action fails to be transitive.

In the rest of this section $(M, \mathcal{F})$ will be a compact minimal lamination by hyperbolic surfaces such that the action of $B$ on its unit tangent bundle $(\hat{M}, T^1\mathcal{F})$ is minimal.

Let $K \subset \hat{M}$ be a compact invariant set for the laminated horocycle flow $h_t$. It may be that $K = \hat{M}$. In any case, since the action of $B$ on $\hat{M}$ is minimal,

$$\bigcup_{t \in \mathbb{R}} g_t(K),$$

being invariant under both $g_t$ and $h_t$, is dense in $\hat{M}$. For every $t \in \mathbb{R}$ the set $g_t(K)$ is also compact and invariant under $h_t$. Assume that whenever $g_t(K) \cap K \neq \emptyset$, in fact $g_t(K) = K$. This holds, if, for example, $K$ is minimal.

Consider the additive subgroup of $\mathbb{R}$ defined as

$$\mathcal{G} = \{ t \in \mathbb{R} : g_t(K) = K \}.$$

When the laminated horocycle flow is not minimal in $\hat{M}$, $\mathcal{G}$ is either cyclic or trivial. In the first case, let $t_0$ be its generator. The minimality of the affine group action implies that

$$\bigcup_{t \in \mathbb{R}} g_t(K) = \bigcup_{t \in [0, t_0]} g_t(K) = \hat{M},$$

and $K$ is therefore a global transverse section of the geodesic flow $g_t$, which every geodesic orbit intersects exactly at intervals of length $t_0$. We call a closed set having this property a *synchronized global transverse section*. The function

$$p : \hat{M} \to S^1 = \mathbb{R}/\mathcal{G}$$

$$x \mapsto t \mod t_0$$

if $g_{-t}(x) \in K$ is well defined, and it is a locally trivial fibration of $\hat{M}$ over $S^1$. That is, the geodesic flow is a suspension. This was first noticed by Plante in [27].
The main result of this section states that under the assumption that the $B$-action on the foliated unit tangent bundle $\hat{M}$ is minimal, the laminated geodesic flow is never a suspension.

**Proposition 21.** Let $(M, \mathcal{F})$ be a compact minimal lamination by hyperbolic surfaces such that the $B$-action on its unit tangent bundle is minimal. Then the geodesic flow on $(\hat{M}, T^{1} \mathcal{F})$ admits no synchronized global transverse section.

**Proof.** Suppose $K$ is a synchronized global transverse section for the laminated geodesic flow on $(\hat{M}, T^{1} \mathcal{F})$.

Let $T : \hat{M} \to \hat{M}$ be the involution that leaves every unit tangent space $T_{x}^{1} \mathcal{F}$, $x \in M$, invariant and takes a unit tangent vector $v$ to $-v$. We can always assume that $\mathcal{F}$ is oriented; otherwise we take an orientable double covering. Under this assumption, $T$ is homotopic to the identity $Id$ in $\hat{M}$. Let $H_{0}, u \in [0,1]$, be a homotopy taking $T = H_{0}$ to $Id = H_{1}$.

There exists an infinite cyclic covering $\psi : \mathbb{R} \times K \to \hat{M}$ with the property that $\psi(1+s,x) = g_{s}\psi(s,x)$ for all $t, s$. The flow $f_{t}$ in $\mathbb{R} \times K$ defined by $f_{t}(s,x) = (t+s,x)$ is therefore the lifting of $g_{t}$.

Notice that $T \circ g_{t} \circ T^{-1} = T \circ g_{t} \circ T = g_{-t}$, for all $t \in \mathbb{R}$, and in particular the flows $g_{t}$ and $g_{-t}$ are topologically conjugated.

Let us compactify $\mathbb{R} \times K$ by adding two points “to the left” and “to the right.” Namely, the compactification is $X = (\mathbb{R} \times K) \cup \{L,R\}$; a neighborhood of $L$ is a set containing $V_{a} = \{(t,x) : t < a\}$, for some $a \in \mathbb{R}$, and neighborhoods of $R$ are defined analogously.

The flow $f_{t}$ can be continuously extended to a flow $\tilde{f}_{t}$ in $X$ that has $L$ and $R$ as fixed points and that satisfies

$$
\lim_{t \to -\infty} \tilde{f}_{t}(x) = L, \quad \lim_{t \to +\infty} \tilde{f}_{t}(x) = R, \quad \forall \ x \in X \setminus \{L,R\}.
$$

Likewise, the homotopy $H$ can be lifted to a homotopy $\tilde{H} : [0,1] \times X \to X$ such that $\tilde{H}_{1} = \tilde{H}(1,\cdot)$ is the identity in $X$. Then, each map $H_{u} = \tilde{H}(u,\cdot)$ must satisfy $\tilde{H}_{u}(L) = L, \tilde{H}_{u}(R) = R$. Nevertheless, $H_{0}$ conjugates $f_{t}$ to $f_{-t}$, which combined with equation (4) implies that $H_{0}(L) = R$ and $H_{0}(R) = L$.

**Remark 22.** The proof of the previous proposition in fact proves the following: if a flow is conjugate to its inverse by a homeomorphism isotopic to the identity, then it does not admit a global synchronized cross section.

We have the following corollary:

**Corollary 23.** Assume that the $B$-action is minimal on $\hat{M}$. If $K \subset \hat{M}$ is a compact minimal set for the horocycle flow $h_{t}$, then its intersection with any central-stable leaf is one of the following:

(i) the empty set;

(ii) the whole central-stable leaf;

(iii) a single horocycle.

**Proof.** If $o_{1}$ and $o_{2}$ are two horocycle orbits on the same central-stable leaf, there is a time $t$ such that $g_{t}(o_{1}) = o_{2}$. Let $K$ be a minimal set for the horocycle flow
flow. Assume that $K$ intersects a given central-stable leaf $x \cdot B$ on more than one horocycle orbit. If $o_1$ and $o_2$ are two distinct horocycles in $(x \cdot B) \cap K$, there is a $t \neq 0$ such that $g_t(K) \cap K \neq \emptyset$. Proposition 21 then says that $K$ must in fact be invariant under the geodesic flow as well, and therefore $x \cdot B \subset K$.

As Example 18 shows, the family of minimal sets for the flow $h_t$ can in general be very large. Nevertheless, except for the cases when the $B$-action on $(M, \mathcal{F})$ is not minimal, we know of no example of a minimal lamination by hyperbolic surfaces such that the horocycle flow $h_t$ is not minimal. Namely, having ruled out the possibility that the group $\mathcal{F}$ be cyclic, we do not know if it can ever be trivial.

**Question:** Is it true that for any compact minimal lamination $(M, \mathcal{F})$ by hyperbolic surfaces, if the joint action of the laminated geodesic and horocycle flows is minimal, then the laminated horocycle flow $h_t$ is minimal?

A positive answer to this question would constitute a generalization of Hedlund’s theorem to surface laminations.

For the moment, an interesting corollary of Proposition 21 is the following:

**Theorem 24.** If $(M, \mathcal{F})$ is a compact minimal lamination by hyperbolic surfaces for which the $B$-action is minimal and that has a leaf which is not simply connected, then the horocycle flow $h_t$ is topologically transitive on $M$.

**Proof.** A leaf of $\mathcal{F}$ which is not simply connected must have a closed geodesic orbit, since its fundamental group cannot have elliptic or parabolic elements. Let $T$ be the period of this closed geodesic orbit and $x$ one of its points. If the orbit of $x$ under the horocycle flow is dense, we have nothing to prove. Assume it is not dense, and let $K$ be its closure. Consider the set $\mathcal{Z}$ of subsets $K'$ of $\hat{M}$ which are compact and invariant under $h_t$ and such that $g_T(K') \cap K' \neq \emptyset$. Since $K \in \mathcal{Z}$, $\mathcal{Z}$ is nonempty. It follows from Zorn’s Lemma that $\mathcal{Z}$ has a minimal element $K_0$, which is contained in $K$. Clearly $g_T(K_0) = K_0$, and $K_0$ is not invariant under the geodesic flow. It is therefore a global transverse section for $g_t$, which is impossible according to Proposition 21.

**Remark 25.** In fact we have proved that if $\mathcal{F}$ is a compact minimal lamination by hyperbolic surfaces for which the affine action is minimal, then all periodic points for the geodesic flow have dense orbits under the horocycle flow.

We will finish this section by showing that the laminated horocycle flow on Sullivan’s Universal Hyperbolic Solenoid is minimal. The Universal Hyperbolic Solenoid is a compact minimal lamination by hyperbolic surfaces, and its leaves are simply connected.

**Example 26.** Let $\Sigma_0$ be a compact hyperbolic surface and $x_0$ a point in $\Sigma_0$. We consider the family of all marked hyperbolic surfaces $(\Sigma, x)$ which are finite regular covers of $\Sigma_0$ such that the covering map sends $x$ to $x_0$, up to homeomorphisms which send marked points to marked points. Let

$$\mathcal{C} = \{ (\Sigma_\alpha, x_\alpha) : \alpha \in A \}$$
be this family, and \( \pi_a : \Sigma_a \to \Sigma_0 \) be the covering map which corresponds to \( a \in A \). A partial order can be defined on \( \mathcal{C} \) by stating that \((\Sigma_a, x_a) \leq (\Sigma_{\beta}, x_{\beta})\) (or \( \alpha \leq \beta \) for short) if there exists a finite regular cover \( \pi_{a\beta} : \Sigma_{\beta} \to \Sigma_a \) such that \( \pi_{a\beta}(x_{\beta}) = x_a \). The projective limit of \((\mathcal{C}, \leq)\) is

\[
H = \lim_{\longrightarrow}(\Sigma_a, x_a) = \left\{ y = (y_a) \in \prod_{a \in A} \Sigma_a : \pi_{a\beta}(y_{\beta}) = y_a \text{ whenever } \alpha \leq \beta \right\},
\]

seen as a topological subspace of the product \( \prod_{a \in A} \Sigma_a \). It is a compact laminated space whose leaves are dense simply connected hyperbolic surfaces, see [30]. It does not depend on the surface \( \Sigma_0 \), and it is called the Universal Hyperbolic Solenoid.

For \( \alpha, \beta \in A \) such that \( \alpha \leq \beta \), we call \( \hat{\pi}_{a\beta} : T^1\Sigma_{\beta} \to T^1\Sigma_a \) the map naturally defined by \( \pi_{a\beta} \) between the unit tangent bundles of \( \Sigma_{\beta} \) and \( \Sigma_a \), namely the one given by \( \hat{\pi}_{a\beta}(y, v) = (\pi_{a\beta}(y), d_yp_{a\beta}(v)) \).

Let \( H \) be the unit tangent bundle of \( H \). A point \( z \) in \( \hat{H} \) is of the form \( z = (z_a) \in \prod_{a \in A} T^1\Sigma_a \) such that \( \hat{\pi}_{a\beta}(z_{\beta}) = z_{a} \) if \( \alpha \leq \beta \).

The topology on \( \hat{H} \) has a basis composed of open sets of the form

\[
U = \bigcap_{a \in A} U_a,
\]

where \( U_a = T^1\Sigma_a \) except for finitely many values \( a \in A \), which we call \( a_1, \ldots, a_n \), and for each \( i \) the set \( U_{a_i} \) is a connected component of \( \pi_{a_i}^{-1}(U_0) \), for some fixed small open set \( U_0 \subset T^1\Sigma_0 \). (The ‘smallness’ of \( U_0 \) means that each \( \pi_{a_i} \), when restricted to \( U_{a_i} \), is a homeomorphism from \( U_{a_i} \) to \( U_0 \).)

We will show that for any \( z \in \hat{H} \) the horocycle through \( z \) is dense in \( \hat{H} \), that is, it intersects every basic open set \( U \). Using the notation introduced in the previous paragraph, let \( \beta \in A \) be such that \( \alpha_i \leq \beta \) for all \( i = 1, \ldots, n \). Let \( V \) be a connected component of \( \pi^{-1}_{\beta}(U_0) \), chosen in such a way that \( \pi_{a_i\beta}(V) = U_{a_i} \).

Since \( \Sigma_{\beta} \) is a compact hyperbolic surface, Hedlund’s theorem tells us that there is a time \( t \) for which \( h^{-1}_t(\Sigma_{\beta}) \in V \), \( h^t(\beta) \) being the horocycle flow on \( T^1\Sigma_{\beta} \). Therefore, at time \( t \) the horocycle orbit of the point \( z_{a_i} \), on \( T^1\Sigma_{a_i} \), passes through the set \( U_{a_i} \), which means that \( h_{t}(z) \in U \).

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