THE TOPOLOGY OF SYMPLECTIC CIRCLE BUNDLES

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Abstract. We consider circle bundles over compact three-manifolds with symplectic total spaces. We show that the base of such a space must be irreducible or the product of the two-sphere with the circle. We then deduce that such a bundle admits a symplectic form if and only if it admits one that is invariant under the circle action in three special cases: namely if the base is Seifert fibered, has vanishing Thurston norm, or if the total space admits a Lefschetz fibration.

1. Introduction

A conjecture due to Taubes states that if a closed, compact 4-manifold of the form $M \times S^1$ is symplectic, then $M$ must fiber over $S^1$. A natural extension of this conjecture is to the case where $E \xrightarrow{\pi} M$ is a possibly nontrivial circle bundle. In [4] it was shown that if an $S^1$-bundle admits an $S^1$-invariant symplectic form then the base must fiber over $S^1$ and the Euler class $e(E)$ pairs trivially with the fiber of some fibration. Thus based on the principle that an $S^1$-bundle should admit a symplectic form if and only if it admits an invariant one, one arrives at the following conjecture.

Conjecture 1 (Taubes). If a circle bundle $S^1 \to E \xrightarrow{\pi} M$ over a closed, compact 3-manifold is symplectic, then there is a fibration $\Sigma \to M \xrightarrow{\phi} S^1$ such that $e(E)([\Sigma]) = 0$.

If an oriented 3-manifold fibers over $S^1$ with fiber $\Sigma \neq S^2$, then it follows by the long exact homotopy sequence that $M$ is in fact aspherical. So a necessary condition for Conjecture 1 to hold is that any $M$ that is the base of an $S^1$-bundle, whose total space carries a symplectic form, must in fact be aspherical or $S^2 \times S^1$ in the case $\Sigma = S^2$. This observation provides the motivation for the following theorem, which is the main result of the first part of this paper.

Theorem 2. Let $M$ be an oriented, closed 3-manifold, so that some circle bundle $S^1 \to E \xrightarrow{\pi} M$ admits a symplectic structure, then, either $M$ is diffeomorphic to $S^2 \times S^1$ and the bundle is trivial, or $M$ is irreducible and aspherical.

A similar statement was proved by McCarthy in [18] for the case $E = M \times S^1$. More precisely, McCarthy showed that if $M \times S^1$ admits a symplectic structure then $M$ decomposes as a connected sum $M = A\#B$ where the first Betti number $b_1(A) \geq 1$ and $B$ has no nontrivial connected covering spaces. This can be refined quite substantially following Perelman’s proof of Thurston’s geometrisation conjecture (see [20], [21] or [19]). For one corollary of geometrisation is that the fundamental group of a closed 3-manifold is residually finite (see [10]), meaning that the $B$ in McCarthy’s theorem must have trivial fundamental...
group, and hence by the Poincaré Conjecture is diffeomorphic to $S^3$. Thus in fact $M$ must be prime and hence irreducible and aspherical or $S^2 \times S^1$. Theorem 2 is then a generalisation of this more refined statement to the case of nontrivial $S^1$-bundles. Our argument will rely on a vanishing result of Kronheimer-Mrowka for the Seiberg-Witten invariants of a manifold that splits into two pieces along a copy of $S^2 \times S^1$, which in itself is of independent interest (cf. Proposition 1). One may also prove Theorem 2 by following the argument of [18], see Remark 1 below.

In the remainder of this paper we will show that Conjecture 1 holds in various special cases. Firstly we will verify the conjecture under certain additional assumptions on the topology of the base manifold $M$. In order to be able to do this we will need to understand when a manifold fibers over $S^1$. One gains significant insight into this problem by considering the Thurston norm $\| \cdot \|_T$ on $H^1(M, \mathbb{R})$, which was introduced by Thurston in [25]. The Thurston norm enables one to see which integral classes $\alpha \in H^1(M, \mathbb{Z})$ can be represented by closed, nonvanishing 1-forms, which in turn induce fibrations of $M$ by compact surfaces.

In [5] it was shown that if $E = M \times S^1$ admits a symplectic form and $\| \cdot \|_T \equiv 0$ or $M$ is Seifert fibered, then $M$ must fiber over $S^1$. In Corollary 2 below we will show that in fact Conjecture 1 holds in these two cases. The argument will be based on understanding the Seiberg-Witten invariants of the total space $E$ given that $M$ has vanishing Thurston norm and the Seifert case will be deduced as a corollary of this. Indeed, if $M$ has vanishing Thurston norm and $S^1 \to E \to M$ is symplectic, then the canonical class of $E$ must be trivial. This combined with the restrictions on Seiberg-Witten basic classes of a symplectic manifold as proved by Taubes in [24] means that $K = 0$ is the only Seiberg-Witten basic class and the result then follows by an application of a vanishing result of Lescop (cf. [16] or [26]).

Another special case of the Taubes conjecture is when the total space $E$ admits a Lefschetz fibration, as was considered in [2] and [3] for a trivial bundle. In view of Corollary 2 we will be able to give a comparatively simple proof of the following result.

**Theorem 9.** Let $S^1 \to E \to M$ be a symplectic circle bundle over an irreducible base $M$. If $E$ admits a Lefschetz fibration, then $M$ fibers over $S^1$.

It then follows by considering the Kodaira classification of complex surfaces that Conjecture 1 holds under the assumption that the total space admits a complex structure.

**Outline of paper.** In Section 2 we will state the relevant vanishing result of Kronheimer-Mrowka in order to prove Theorem 2. In Section 3 we recall the definition of the Thurston norm and quote some well known facts about it. In Section 4 we will use our knowledge of the Thurston norm to verify Conjecture 1 under the assumption that the base is Seifert fibered or has vanishing Thurston norm. Finally in Section 5 we will define Lefschetz fibrations and prove that the conjecture is true when one has a Lefschetz fibration on the total space $E$.

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2. ASPHERICITY OF THE BASE $M$

Throughout this article all manifolds will be closed, connected, compact and oriented and $M$ will always denote a manifold of dimension 3. In addition we will make the convention that all (co)homology groups will be taken with integral coefficients unless otherwise stated.
In [18] it was shown that if $M \times S^1$ is symplectic, then $M$ must be irreducible and aspherical or $S^2 \times S^1$. We extend this to the case of a nontrivial $S^1$-bundle. We first collect some relevant lemmas.

**Lemma 1.** Let $M = M_1 \# M_2$ be a nontrivial connect sum decomposition with $b_1(M) \geq 1$, then there is a finite covering $N$ of $M$ that decomposes as a direct sum $N = N_1 \# N_2$ where $b_1(N_i) \geq k$ for any given $k$.

**Proof.** It follows from Mayer-Vietoris that the Betti numbers are additive for a connect sum, hence by assumption we may assume that $b_1(M_1) \geq 1$. By the proof of geometrisation it follows that the fundamental group of a 3-manifold is residually finite (cf. [10]) and hence $M_2$ has a nontrivial $d$-fold cover $\tilde{M}_2$, with $d \geq 2$. By removing a ball from $M_2$ and its disjoint lifts from $\tilde{M}_2$ and then gluing in $d$ copies of $M_1$ we obtain a cover $\tilde{M}$ of $M = M_1 \# M_2$, and by construction $\tilde{M}$ has a connect sum decomposition as $\tilde{M} = \tilde{M}_1 \# \tilde{M}_2$, where $b_1(\tilde{M}_i) \geq 1$. We may now take a $k$-fold cover associated to some surjective homomorphism of $\pi_1(M_1) \to \mathbb{Z}_k$ and glue in copies of $P$ to get a cover of $\tilde{M}$ (and hence of $M$), which decomposes in two pieces one of which has first Betti number at least $k$. One more application of this procedure gives the desired result. □

**Lemma 2.** Let $S^1 \to E \xrightarrow{s} M$ be a circle bundle, whose Euler class we denote by $e(E) \in H^2(M)$, then

1. $b_2(E) = \begin{cases} 2b_1(M) - 2, & \text{if } e(E) \text{ is not torsion} \\ 2b_1(M), & \text{if } e(E) \text{ is torsion} \end{cases}$
2. $b_2^+(E) = b_2^-(E) \geq b_1(M) - 1$.

**Proof.** We consider the Gysin sequence

$$H^0(M) \xrightarrow{\cup e} H^2(M) \xrightarrow{\pi^*} H^2(E) \xrightarrow{\partial} H^1(M) \xrightarrow{\cup e} H^3(M),$$

where here $e \in H^2(M)$ denotes the Euler class of the bundle. By Poincaré duality $H^0(M) = H_3(M) = \mathbb{Z}$ and $b_1(M) = b_2(M)$, so we conclude by exactness that $b_2(E) = 2b_1(M) - 2$ if $e$ is not torsion and $b_2(E) = 2b_1(M)$ if $e$ is torsion. Furthermore since $E$ bounds its associated disc bundle, it has zero signature and hence

$$b_2^+(E) = b_2^-(E) \geq b_1(M) - 1.$$ □

We will need to appeal to a vanishing result for the Seiberg-Witten invariants of manifolds that decompose along $S^2 \times S^1$, which we take from [15]. For this we will need to define a relative notion of $b_2^+$ for an oriented 4-manifold $X$ with boundary. This is done by considering the symmetric form induced on rational cohomology that is obtained as the composition

$$H^2(X, \partial X) \times H^2(X, \partial X) \xrightarrow{i^* \times Id} H^2(X) \times H^2(X, \partial X) \xrightarrow{i} H^4(X) \cup H^4(X, \partial X) \to \mathbb{Q}.$$ 

Here the map $i^*$ is the map coming from the long exact sequence of the pair $(X, \partial X)$ and the second map is non-degenerate by Poincaré duality. This is then a symmetric, possibly degenerate, form on $H^2(X, \partial X)$ and we define $b_2^+(X)$ to be the dimension of a maximal positive definite subspace.
Theorem 1 (Kronheimer-Mrowka, [15]). Let \( X = X_1 \cup_{\partial X_1 = \partial X_2} X_2 \) where \( \partial X_1 = -\partial X_2 = S^2 \times S^1 \) and \( b_2^+ (X_1), b_2^+ (X_2) \geq 1 \). Then for all \( \text{Spin}^c \)-structures \( \xi \)

\[
\sum_{\xi^* - \xi \in \text{Tor}} \text{SW}(\xi^*) = 0.
\]

Although it is not explicitly stated in book [15], Theorem 1 can be deduced as follows: formula 3.27 (p.73) defines the sum of the SW invariants of all \( \text{Spin}^c \)-structures that differ by torsion as given by a pairing of certain Floer groups and these groups are zero for \( S^2 \times S^1 \) by Proposition 3.10.3 in the case of an untwisted coefficient system and by Proposition 3.10.4 in the twisted case.

Theorem 1 then implies certain restrictions on the decomposition of symplectic manifolds along \( S^2 \times S^1 \), which is similar but slightly weaker than the results one obtains for a connected sum.

Proposition 1. A symplectic manifold \( X \) cannot be decomposed as \( X = X_1 \cup_{\partial X_1 = \partial X_2} X_2 \), where \( \partial X_1 = -\partial X_2 = S^2 \times S^1 \) and \( b_2^+ (X_1), b_2^+ (X_2) \geq 1 \).

Proof. By the hypotheses of the proposition, we conclude from Theorem 1 that for every \( \text{Spin}^c \)-structure \( \xi \in \text{Spin}^c (X) \)

\[
\sum_{\xi^* - \xi \in \text{Tor}} \text{SW}(\xi^*) = 0.
\]

However as \( X \) is symplectic and

\[
b_2^+ (X) \geq b_2^+ (X_1) + b_2^+ (X_2) \geq 2\]

the nonvanishing result of Taubes implies \( \text{SW}(\xi_{\text{can}}) = \pm 1 \), where \( \xi_{\text{can}} \) denotes the canonical \( \text{Spin}^c \)-structure associated to the symplectic structure on \( E \) (cf. [23]). Moreover it follows from the constraints on SW basic classes of a symplectic manifold of [24] that if \( \xi^* \) is another \( \text{Spin}^c \)-structure with non-trivial SW invariant and \( \xi_{\text{can}} - \xi^* \in \text{Tor} \) then in fact \( \xi_{\text{can}} = \xi^* \). Hence

\[
\sum_{\xi^* - \xi_{\text{can}} \in \text{Tor}} \text{SW}(\xi^*) = \pm 1
\]

which is a contradiction.

\( \square \)

Theorem 2. Let \( M \) be an oriented, closed 3-manifold, so that some circle bundle \( S^1 \to E \xrightarrow{\pi} M \) admits a symplectic structure, then \( M \) is irreducible and aspherical or \( M = S^2 \times S^1 \) and the bundle is trivial.

Proof. We first show that \( M \) must be prime. Since \( E \) is symplectic it follows from Lemma 2 that \( b_1 (M) \geq 1 \). Assume that \( M = M_1 \# M_2 \) is a nontrivial connected sum, then by taking a suitable covering as in Lemma 1 and pulling back \( E \) and its symplectic form we may assume without loss of generality that \( b_1 (M_i) \geq 2 \). We let \( S \) denote the gluing sphere of the connected sum, then as \( S \) is nullhomologous the bundle restricted to \( S \) is trivial. Thus the connect sum decomposition induces a decomposition \( E = E_1 \cup_{S^2 \times S^1} E_2 \). Since the bundles \( E_i \to M_i / B^3 \) are trivial on the boundary we may extend them to bundles \( \tilde{E}_i \to M_i \) and as \( b_1 (M_i) \geq 2 \), Lemma 2 implies that \( b_2^+ (\tilde{E}_i) \geq 1 \). Further, since \( E_i \simeq \tilde{E}_i / S^1 \times pt \) we have that

\[
b_2^+ (E_i) \geq b_2^+ (\tilde{E}_i) \geq 1.
\]
which then contradicts Proposition 1. Hence $M$ is prime, and thus irreducible or $S^2 \times S^1$.

We assume that $M$ is irreducible, then by the sphere theorem $\pi_2(M) = 0$. Since $b_1(M) \geq 1$, we have that $\pi_1(M)$ is infinite so the universal cover $\tilde{M}$ of $M$ is not compact and has $\pi_i(\tilde{M})$ trivial for $i = 1, 2$. The Hurewicz theorem then implies that the first nontrivial $\pi_i(M)$ is isomorphic to $H_i(\tilde{M})$. But since $\tilde{M}$ is not compact $H_3(\tilde{M}) = 0$ and as $\tilde{M}$ is 3-dimensional $H_i(\tilde{M}) = 0$ for all $i \geq 4$. Hence $\pi_i(\tilde{M}) = 0$ for all $i \geq 1$ and it follows from Whitehead’s Theorem that $\tilde{M}$ is contractible, that is $M$ is aspherical.

In the case where $M = S^2 \times S^1$ any symplectic bundle must be trivial by Lemma 2. □

Remark 1. One may also give a proof of Theorem 2 that uses covering construction in [18]. In order to do this one first takes finite coverings on each of the two pieces in the connect sum decomposition. Then one glues these together to find a covering $\tilde{M}$ where the sphere of the connect sum lifts to a sphere that is nontrivial in real cohomology. This sphere then lifts to the total space of the pullback bundle $\tilde{E}$ over $\tilde{M}$. One may also assume by Lemma 1 that $b_1(\tilde{M})$ is large and hence $b_2^+(\tilde{E})$ is large. Then a standard vanishing theorem for the SW invariants implies that all invariants are zero, which then contradicts Taubes’ result if $E$ and hence $\tilde{E}$ is symplectic.

By considering the long exact homotopy sequence we have the following corollary that was first proved by Kotschick in [13].

**Corollary 1.** Let $S^1 \to E \overset{\pi}{\to} M$ be a symplectic circle bundle over an oriented 3-manifold $M$. Then the map $\pi_1(S^1) \to \pi_1(E)$ induced by the inclusion of the fiber is injective. In particular a fixed point free circle action on a symplectic 4-manifold can never have contractible orbits.

### 3. The Thurston norm

In this section we will define and collect several relevant facts about the Thurston norm. We first define the negative Euler characteristic or complexity of a possibly disconnected, orientable surface $\Sigma = \bigsqcup_i \Sigma_i$ to be

$$\chi_-(\Sigma) = \sum_{\chi(\Sigma_i) \leq 0} -\chi(\Sigma_i)$$

where $\chi$ denotes the Euler characteristic of the surface.

Next we define the Thurston norm $\| \cdot \|_T$ as a map on $H_1(M)$ by

$$\|\sigma\|_T = \min \{ \chi_-(\Sigma) \mid PD(\Sigma) = \sigma \}.$$ 

It is a basic fact that this map extends uniquely to a (semi)norm on $H^1(M, \mathbb{R})$, which we will denote again by $\| \cdot \|_T$. One particularly important property of the Thurston norm is that its unit ball, which we denote by $B_T$, is a (possibly noncompact) convex polytope with finitely many faces. If $B_{T^*}$ denotes the unit ball in the dual space we have the following characterisation of $B_T$.

**Theorem 3** ([25], p. 106). The unit ball $B_{T^*}$ is a polyhedron whose vertices are integral lattice points, $\pm \beta_1, ..., \pm \beta_k$ and the unit ball $B_T$ is defined by the following inequalities

$$B_T = \{ \alpha \mid |\beta_i(\alpha)| \leq 1, \ 1 \leq i \leq k \}.$$
We are interested in understanding how a manifold fibers over $S^1$ and the following theorem says that the Thurston norm determines precisely which cohomology classes can be represented by fibrations.

**Theorem 4** ([25], p. 120). Let $M$ be a compact, oriented 3-manifold. The set $F$ of cohomology classes in $H^1(M, \mathbb{R})$ representable by nonsingular closed 1-forms is the union of the open cones on certain top-dimensional open faces of $B_T$, minus the origin. The set of elements in $H^1(M, \mathbb{Z})$ whose Poincaré dual is represented by the fiber of some fibration consists of the set of lattice points in $F$.

We call a top-dimensional face of the unit ball $B_T$ fibered, if some integral class, and hence all, in the cone over its interior can be represented by a fibration. One also understands how the Thurston norm behaves under finite covers by the following result of Gabai.

**Theorem 5** ([6], Cor. 6.13). Let $\tilde{M} \to M$ be a finite connected $d$-sheeted covering then for $\sigma \in H^2(M, \mathbb{R})$ we have
$$\|\sigma\|_T = \frac{1}{d} \|p^*\sigma\|_T.$$ 

These facts then allow us to completely characterise the Thurston norm of an irreducible Seifert fibered manifold.

**Proposition 2.** Let $M$ be irreducible and Seifert fibered, then either the Thurston norm of $M$ vanishes identically or $M$ fibers over $S^1$ and
$$\|\sigma\|_T = \chi(F) |\sigma(\gamma)|$$
where $\gamma \in H_1(M)$ is a primitive class some multiple of which is homologous to the fiber of a Seifert fibration and $F$ is a connected fiber of a fibration of $M$.

Proof. Since $M$ is irreducible and Seifert fibered either $M$ has a horizontal surface or every surface is isotopic to a vertical surface (cf. [8] Prop 1.11) and is hence a union of tori so the Thurston norm is identically zero. If $M$ has a horizontal surface $F$, which we may assume to be connected, then $M$ is a mapping torus with monodromy $\phi \in Diff^+(F)$ so that $\phi^n = Id$ for some $n$. This means that $M$ is covered by $\tilde{M} = F \times S^1$. If $\tilde{\gamma} = pt \times S^1$, then the Thurston norm of $\tilde{M}$ is given by
$$\|\sigma\|_T = \chi(F) |\sigma(\tilde{\gamma})|$$
and the formula for the norm on $M$ follows from Theorem 5. \qed

**Example 1** (Seifert fibered spaces with horizontal surfaces). We note that in the second case of Proposition 2 the Thurston ball $B_T$ consists of two (noncompact) faces that are both fibered. Thus by [4] any bundle over such an $M$ will admit an $S^1$-invariant symplectic form except possibly in the case where the Euler class $e(E)$ is a multiple of $PD(\gamma) \in \tilde{H}^1(M)$. If a bundle over $M$ with Euler class a multiple of $PD(\gamma)$ is symplectic then by taking the pullback bundle of the cover $\tilde{M} = F \times S^1 \to M$ we may assume that we have a bundle $E$ over $F \times S^1$ that is symplectic and has Euler class that is multiple of $PD(\tilde{\gamma})$. This in turn has a covering $\tilde{E}$ with Euler class equal to $PD(\tilde{\gamma})$. Now if we let $T = \tilde{\gamma} \times S^1$ and $X = \tilde{M} \times S^1$ then the SW polynomial of $X$ can be computed to be
$$SW^4_X = (t_T - t_T^{-1})^{2g-2}.$$
where \( g \) is the genus of \( F \). Then by the formula of Baldridge in [1] it follows that all the SW invariants of \( \bar{E} \) are zero, contradicting Taubes’ non vanishing result. So in fact Conjecture [1] holds for Seifert fibered spaces that have horizontal surfaces.

4. THE CASE OF VANISHING THURSTON NORM

In [2] it was shown that if \( E = M \times S^1 \) admits a symplectic form and \( \| \|_T \equiv 0 \) or \( M \) is Seifert fibered, then \( M \) must fiber over \( S^1 \). In this section we shall extend this to the case of a nontrivial \( S^1 \)-bundle and then show that Conjecture [1] holds in both of these cases. From now on we shall assume that \( M \) is also irreducible, which in view of Theorem 2 only excludes the case where \( M = S^2 \times S^1 \) and the bundle is trivial. Our argument will be based on that of [5] and we begin with the following lemma.

**Lemma 3.** If \( S^1 \to E \xrightarrow{\pi} M \) is a bundle over an \( M \) that has vanishing Thurston norm, then

\[
H^2(E, \mathbb{Z})/\text{Tor} = V \oplus W
\]

where \( V, W \) are isotropic subspaces that admit a basis of embedded tori.

**Proof.** We consider the Gysin sequence

\[
\mathbb{Z} \xrightarrow{s} H^2(M) \xrightarrow{\pi^*} H^2(E) \xrightarrow{\pi_*} H^1(M) \xrightarrow{s} \mathbb{Z}.
\]

Here \( s \) is a section defined on the image of \( \pi_* \) as follows: we represent an element of \( \sigma \in H^1(M) \) by an embedded surface \( \Sigma \). By exactness, \( \sigma \) will be in \( \text{Im}(\pi_*) \) precisely when the bundle is trivial on \( \Sigma \) and in this case we may lift \( \Sigma \) to some \( \tilde{\Sigma} \) in \( E \). As \( H^1(M) \) is free, we define \( s \) on a \( \mathbb{Z} \)-basis \( \{\sigma_i\} \) by \( s(\sigma_i) = \tilde{\Sigma}_i \). We set \( V = \pi^* H^2(M) \) and \( W = s(H^1(M)) \), then \( V \) is clearly spanned by embedded tori and the statement for \( W \) is precisely the assumption on the Thurston norm. \( \square \)

**Proposition 3.** Let \( S^1 \to E \xrightarrow{\pi} M \) be an \( S^1 \)-bundle with torsion Euler class \( e(E) \), then there is a finite cover \( \tilde{M} \xrightarrow{p} M \) such that the pullback bundle \( p^* E \to \tilde{M} \) is trivial.

**Proof.** We choose a splitting of \( H_1(M) = F \oplus T \) where \( T \) is the torsion subgroup and \( F \) is any free complement. We take the cover \( \tilde{M} \xrightarrow{p} M \) associated to the kernel of the composition

\[
\pi_1(M) \to H_1(M) \xrightarrow{\phi} T,
\]

where \( \phi \) is the projection with kernel \( F \). Note that the composition \( H_1(\tilde{M}) \xrightarrow{p} H_1(M) \xrightarrow{\phi} T \) is zero. Then by the Universal Coefficient Theorem we have the following commutative diagram

\[
\begin{array}{ccccccccc}
0 & \xrightarrow{(p_*)^*} & \text{Ext}(H_1(\tilde{M}), \mathbb{Z}) & \xrightarrow{p^*} & H^2(\tilde{M}) & \xrightarrow{(p_*)^*} & \text{Hom}(H_2(\tilde{M}), \mathbb{Z}) & \xrightarrow{(p_*)^*} & 0 \\
&& \downarrow{(p_*)^*} & & \uparrow{p^*} & & \downarrow{(p_*)^*} & & \\
0 & \xrightarrow{(p_*)^*} & \text{Ext}(H_1(M), \mathbb{Z}) & \xrightarrow{p^*} & H^2(M) & \xrightarrow{(p_*)^*} & \text{Hom}(H_2(M), \mathbb{Z}) & \xrightarrow{(p_*)^*} & 0.
\end{array}
\]

This implies that \( p^* \) is zero on torsion in \( H^2(M) \) so the pullback bundle is indeed trivial. \( \square \)
Theorem 6. Let \( S^1 \to E \xrightarrow{\pi} M \) be a symplectic circle bundle over an irreducible manifold for which \(||||_T\) is identically zero, then \( M \) fibers over \( S^1 \).

Proof. Since \( E \) is symplectic it has an associated canonical bundle \( \xi_{\text{can}} \) and canonical class that we denote by \( K \). We claim that our assumption on the Thurston norm of the base implies that \( K \) must be torsion. For by Taubes’ nonvanishing result \( \xi_{\text{can}} \) has nontrivial SW invariant. If \( \alpha \in H^2(E) \), the adjunction inequality (see [14]) and Lemma 3 imply that
\[
|\alpha.K| = 0.
\]
This also holds in the case \( b_2^+ (E) = 1 \) (cf. [17] Theorem E). As \( M \) is irreducible and \( b_2(M) \geq 1 \) the assumption on the vanishing of the Thurston norm implies that \( M \) contains an embedded, incompressible torus \( T \hookrightarrow M \). Then by Proposition 7 of [11] either \( T \) is the fiber of some fibration or there is a finite cover \( \overline{M} \to M \) with large \( b_1 \), say \( b_1(\overline{M}) \geq 4 \). We assume that the latter holds. Then the pullback \( \overline{E} = p^*E \) will be symplectic with canonical class \( \overline{K} = p^*K \), symplectic form \( \overline{\omega} = p^*\omega \) and \( b_2^+ (\overline{E}) \geq 2 \). Then for any \( \text{Spin}^c \)-structure \( \xi_{\text{can}} \otimes F \) that has nontrivial SW invariant we have by [24]
\[
0 \leq F.[\overline{\omega}] \leq \overline{K}.[\overline{\omega}].
\]
Moreover, since \( \overline{K} \) is torsion and equality on the left implies \( F = 0 \), we conclude that in fact \( \overline{K} = 0 \). Thus \( K = 0 \), so \( \xi_{\text{can}} \) is trivial and this is the only \( \text{Spin}^c \)-structure with nonzero SW invariant. We now need to consider two cases. We first assume that \( e(E) \) and hence \( e(\overline{E}) \) is nontorsion. In this case we compute
\[
\pm 1 = \sum_{\xi^* \in \text{Spin}^c(E)} SW^4_E(\xi^*) = \sum_{\xi^* \in \text{Spin}^c(E)} \sum_{\xi^* \equiv \xi \mod \mathbb{E}} SW^3_M(\xi) = \sum_{\xi \in \text{Spin}^c(E)} SW^3_M(\xi),
\]
where the second inequality follows from Theorem 1 in [1]. However as \( b_1(M) \geq 4 \) this sum is zero (cf. [26] p.114) a contradiction. If the Euler class is torsion we may assume by Proposition 3 that it is indeed zero and the above calculation reduces to
\[
\pm 1 = \sum_{\xi \in \text{Spin}^c(E)} SW^4_E(\xi) = \sum_{\xi \in \text{Spin}^c(E)} SW^3_M(\xi) = 0.
\]
In either case we obtain a contradiction and hence \( M \) must fiber over \( S^1 \).

As a consequence of this theorem we conclude that Conjecture 1 holds if \( M \) has vanishing Thurston norm or is Seifert fibered.

Corollary 2. Conjecture 1 holds if \( M \) is Seifert fibered or \(||||_T \equiv 0\).

Proof. If \( M \) has vanishing Thurston norm, then by Theorem 4 we conclude that if one class in \( H^1(M) \) can be represented by a fibration then so can all classes and by the construction of \( 4 \) every bundle over \( M \) admits an \( S^1 \)-invariant symplectic form. If \( M \) is Seifert fibered it either has vanishing Thurston norm by Proposition 2 and we proceed as in the previous case or \( M \) has a horizontal surface and the claim follows by Example 1 above.
5. The case where $E$ admits a Lefschetz fibration

In [2] Chen and Matveyev showed that if $S^1 \times M$ admits a symplectic Lefschetz fibration then $M$ fibers over $S^1$. This was extended by Etněr in [3] to the case where the fibration may or may not be symplectic. In this section we shall show that the same statement holds for arbitrary $S^1$-bundles. Let us begin with some definitions and basic facts concerning Lefschetz fibrations.

**Definition 7.** Let $E$ be a compact, connected, oriented smooth 4-manifold, a Lefschetz fibration is a map $E \xrightarrow{p} B$ to an orientable surface so that any critical point has a chart on which $p(z_1, z_2) = z_1^2 + z_2^2$.

We list some basic properties of Lefschetz fibrations (for proofs see [7]).

(1) There are finitely many critical points, so the generic preimage of a point will be a surface and we may assume that this is connected. To each critical point one associates a vanishing cycle in the fiber.

(2) A Lefschetz fibration admits a symplectic form so that the fiber is a symplectic submanifold if the class $[F]$ of the fiber is nontorsion in $H_2(E)$. Moreover this is always true if $\chi(F) \neq 0$.

(3) We have a formula for the Euler characteristic given by

$$\chi(E) = \chi(B) \cdot \chi(F) + \# \{\text{critical points}\}.$$ 

We will first show that for a symplectic circle bundle any Lefschetz fibration will actually be a proper fibration, i.e. cannot have any critical points. The following lemma is essentially Lemma 3.4 of [2].

**Lemma 4.** Let $S^1 \to E \xrightarrow{\pi} M$ be a circle bundle that admits a Lefschetz fibration $E \xrightarrow{p} B$, then $p$ has no critical points.

**Proof.** We first consider the case where $F = S^2$, then since $E$ is spin and hence has an even intersection form, all critical points are non-separating in $F$. Thus we cannot have any. If $F = T^2$, the equation

$$0 = \chi(E) = \chi(B) \cdot \chi(F) + \# \{\text{critical points}\}.$$ 

implies that $E$ has no critical points.

We now consider the case when $F$ has genus greater than 1. We know that $E$ admits a symplectic Lefschetz fibration. Thus by the adjunction formula for symplectic surfaces we see that

(1) $$K \cdot F = \chi_{-}(F) \neq 0$$

where $K$ is the canonical class on $E$. If $b_2^+ > 1$ then it follows from Taubes’ result that $K$ is a basic class and thus the adjunction inequality holds. In the case where $b^+(E) = 1$ we may apply the adjunction inequality exactly as in the case of $b_2^+ > 1$ by ([17] Theorem E). Now we assume that our fibration has a critical point and hence a vanishing cycle $\gamma$, then we know that this is nonseparating so the fiber $F$ is homologous to a surface obtained by collapsing $\gamma$ to a point and this in turn be thought of as the image of a map $F' \xrightarrow{f} E$ where $\chi_{-}(F') < \chi_{-}(F)$. Hence the image $\pi_*(F)$ may be represented by a surface of complexity at most $\chi_{-}(F')$ (see [6]). We know that any basic class of a circle bundle is a pullback of a
class on the base (see [1]) thus by the adjunction inequality (which still holds for $b_2^+ = 1$) and equation (1)
\[
\chi_-(F) = |K.F| = |K\pi_*F| \leq ||\pi_*F||_T \leq \chi_-(F') < \chi_-(F)
\]
which is a contradiction. □

Our proof of Theorem 9 below, which differs from those of [2] and [3], will rely on a theorem of Stallings that characterises fibered 3-manifolds in terms of their fundamental group.

**Theorem 8 (Stallings).** Let $M$ be a compact, irreducible 3-manifold and suppose there is an extension
\[
1 \to G \to \pi_1(M) \to \mathbb{Z} \to 1
\]
where $G$ is finitely generated and $G \neq \mathbb{Z}_2$, then $M$ fibers over $S^1$.

We now come to the main result of this section.

**Theorem 9.** Let $E \xrightarrow{\pi} M$ be a symplectic circle bundle over an irreducible base $M$. If $E$ admits a Lefschetz fibration, then $M$ fibers over $S^1$.

**Proof.** First of all by Lemma 4 we have that $E$ actually admits a fibration $F \to E \xrightarrow{p} B$. In addition we note that the fiber $\gamma$ of any circle bundle lies in the centre of its total space, $\pi_1(E)$. We shall have to consider two distinct cases according to whether $\gamma$ is in the kernel of $p_*$ or not.

**Case 1:** $p_*(\gamma) \neq 1$.

Since $\gamma$ was central in the fundamental group of $E$ the fact that $p_*(\gamma)$ is nontrivial in $\pi_1(B)$ means that $B$ must be a torus. Hence the long exact homotopy sequence of the fibration gives the following short exact sequence
\[
1 \to \pi_1(F) \to \pi_1(E) \xrightarrow{p_*} \pi_1(T^2) = \mathbb{Z}^2 \to 1.
\]

Since $M$ is assumed to be irreducible and hence aspherical we also have the following exact sequence from the homotopy exact sequence of the fibration $S^1 \to E \xrightarrow{\pi} M$:
\[
1 \to \pi_1(S^1) = \langle \gamma \rangle \to \pi_1(E) \xrightarrow{\pi_*} \pi_1(M) \to 1.
\]

Because $\gamma$ is central in $\pi_1(E)$, the sequence (2) gives the following exact sequence
\[
1 \to \pi_1(F) \to \pi_1(E)/\langle \gamma \rangle \xrightarrow{p_*} \mathbb{Z}^2/\langle p_*\gamma \rangle \to 1.
\]

Moreover since $p_*\gamma \neq 1$ we have that $\mathbb{Z}^2/\langle p_*\gamma \rangle = \mathbb{Z} \oplus \mathbb{Z}_k$ for some $k$. If we let $H = p_*^{-1}(\mathbb{Z}_k)$ we see that $H$ has $\pi_1(F)$ as a finite index subgroup and is thus also finitely generated. Then by taking the projection to $\mathbb{Z}$ in the above sequence we obtain
\[
1 \to H \to \pi_1(E)/\langle \gamma \rangle = \pi_1(M) \xrightarrow{p_*} \mathbb{Z} \to 1.
\]

This is exact and $H \neq \mathbb{Z}_2$ since it contains $\langle \gamma \rangle$. As $M$ is irreducible, the hypotheses of Theorem 8 are satisfied and we conclude that $M$ fibers over $S^1$.

**Case 2:** $p_*(\gamma) = 1$. 

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In this case \( \langle \gamma \rangle \subset \pi_1(F) \) and hence \( F = T^2 \). Thus sequence (2) above yields the following
\[
1 \to \mathbb{Z}^2 \to \pi_1(E) \xrightarrow{p} \pi_1(B) \to 1
\]
and \( \langle \gamma \rangle \subset \mathbb{Z}^2 \). Again by taking the quotient by \( \langle \gamma \rangle \) we obtain the following short exact sequence
\[
1 \to \mathbb{Z} \oplus \mathbb{Z}_k = \mathbb{Z}^2/\langle \gamma \rangle \to \pi_1(E)/\langle \gamma \rangle = \pi_1(M) \xrightarrow{p} \pi_1(B) \to 1.
\]
However since \( M \) is irreducible and hence prime and \( \pi_1(M) \) is infinite it follows from ([9], Corollary 9.9) that \( \pi_1(M) \) is torsion free. Hence \( k = 0 \) and \( \pi_1(M) \) contains an infinite cyclic normal subgroup, thus by ([9], Corollary 12.8) it is in fact Seifert fibered and the result follows from Corollary 2 above.

Theorem 9 then allows us to prove Conjecture 1 under the assumption that the total space is a complex manifold.

**Corollary 3.** Conjecture 1 holds in the case that \( E \) is a complex manifold.

**Proof.** By considering the Kodaira classification and noting that \( E \) is spin, symplectic and has \( \chi(E) = 0 \) one concludes that one of the following must hold (cf. [3] Theorem 5.1)

1. \( E = S^2 \times T^2 \)
2. \( E \) is a \( T^2 \)-bundle over \( T^2 \)
3. \( E \) is a Seifert fibration over a hyperbolic orbifold.

If \( E = S^2 \times T^2 \) then \( M = S^2 \times S^1 \) and one clearly has an \( S^1 \)-invariant symplectic form. In the second case it follows from the argument above that \( M \) is a \( T^2 \)-bundle over \( S^1 \) and hence has vanishing Thurston norm. In the final case \( M \) must be Seifert fibered as in Case 2 in the proof of Theorem 9 and hence the claim holds in the latter two cases by Corollary 2.

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