Integral operator approach over octonions to solution of nonlinear PDE.

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Abstract

Integration of nonlinear partial differential equations with the help of the non-commutative integration over octonions is studied. An apparatus permitting to take into account symmetry properties of PDOs is developed. For this purpose formulas for calculations of commutators of integral and partial differential operators are deduced. Transformations of partial differential operators and solutions of partial differential equations are investigated. Theorems providing solutions of nonlinear PDEs are proved. Examples are given. Applications to PDEs of hydrodynamics and other types PDEs are described.

\[\text{key words and phrases: hypercomplex, octonion algebra, nonlinear partial differential equation, non-commutative integration, integral operator}\]

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1 Introduction.

Analysis over hypercomplex numbers develops fast and has important applications in geometry and partial differential equations including that of nonlinear (see [5] - [11], [27]-[33] and references therein). As a consequence it gives new opportunities for integration of different types of partial differential equations (PDEs). It is worth to mention that the quaternion skew field $H = A_2$, the octonion algebra $O = A_3$ and Cayley-Dickson algebras $A_r$ have found a lot of applications not only in mathematics, but also in theoretical physics (see [5] - [10] and references therein).

Mixed type PDEs play very important role not only in mathematics, but also in physics. For example, they describe two-dimensional motions of a stratified rotating liquid, electromagnetic fields in crystals, internal gravitational waves, non stationary filtration process of liquid in a fissure porous medium, dissipation process, cold plasma, two temperature plasma in an external magnetic field, etc. (see [46] and references therein). They were studied with the real time and two spatial variables. But there are needs to integrate more general PDE of such type with larger number of variables or their systems.

Frequently types of considered PDE in modeling different processes and in studying their solutions are restricted by available tools of mathematical analysis. Otherwise numerical methods and computer calculations are used. But in many cases it is also necessary to analyze properties of solutions. Therefore analytic approaches apart from that of numerical provide in this respect many advantages. On the other hand, ranges of mathematical analysis strongly depend on used number systems. Classically in mathematical analysis and PDEs real and complex fields are used. But hypercomplex analysis enlarges its scopes.

This article is devoted to analytic approaches to solution of PDEs and taking into account their symmetry properties. For this purpose the octonion algebra is used. This is actual especially in recent period because of increasing interest to non-commutative analysis and its applications. It is worth to mention that each problem of PDE can be reformulated using the octonion
algebra. The approach over octonions enlarges a class of PDEs which can be analytically integrated in comparison with approaches over the real field and the complex field.

There is stimulus for investigations caused by needs to integrate known PDEs (see, for example, [13, 14, 18, 46] and references therein) and by the progress of algebra [2, 19, 42, 43]. Certainly algebras are widely used in PDEs (see also, for example, [6, 12, 39]-[41] and references therein). But in previous works mainly associative algebras were used for such purposes.

On the other hand nonlinear PDEs frequently are more complicated and demand specific approaches to get their solutions [14, 17, 37, 46] in comparison with that of linear.

We exploit a new approach based on the non-commutative integration over non-associative Cayley-Dickson algebras that to integrate definite types of nonlinear PDEs. This work develops further results of the previous article [27]. The obtained below results open new perspectives and permit to integrate nonlinear PDEs with variable coefficients and analyze symmetries of solutions as well.

In the following sections integration of nonlinear PDEs with the help of the non-commutative integration over quaternions, octonions and Cayley-Dickson algebras is studied. For this purpose formulas for calculations of commutators of integral and partial differential operators are deduced. Transformations of partial differential operators and solutions of partial differential equations are investigated. An apparatus permitting to take into account symmetry properties of PDOs is developed. Theorems providing solutions of nonlinear PDEs are proved. Examples are given. Applications to PDEs used in hydrodynamics and other types PDEs are described. The results of this paper can be applied to integration of some kinds of nonlinear Sobolev type PDEs as well.

All main results of this paper are obtained for the first time. They can be used for further investigations of PDEs and properties of their solutions. For example, generalized PDEs including terms such as $\Delta^p$ or $\nabla^p$ for $p > 0$ or even complex $p$ can be investigated.
2 Integral operators over octonions.

To avoid misunderstandings we first present our definitions and notations.

1. Notations and Definitions. By \( \mathcal{A}_r \) we denote the Cayley-Dickson algebra over the real field \( \mathbb{R} \) with generators \( i_0, ..., i_{2r-1} \) so that \( i_0 = 1, \)
\( i_j^2 = -1 \) for each \( j \geq 1, \) \( i_j i_k = -i_k i_j \) for each \( j \neq k \geq 1, 2 \leq r \in \mathbb{N} \).

Henceforward PDEs are considered on a domain \( U \) in \( \mathcal{A}_r \) such that
\[
\text{(D1) each projection } p_j(U) =: U_j \text{ is } (2^r - 1)-\text{connected};
\]
\[
\text{(D2) } \pi_{s,p,t}(U_j) \text{ is simply connected in } \mathbb{C} \text{ for each } k = 0, 1, ..., 2^r - 1, s = i_{2k}, p = i_{2k+1}, t \in \mathcal{A}_{r,s,p} \text{ and } u \in \mathcal{C}_{s,p}, \text{ for which a Cayley-Dickson number } z \text{ exists satisfying the condition } z = u + t \in U_j,
\]
where \( e_j = (0, ..., 0, 1, 0, ..., 0) \in \mathcal{A}_m \) is the vector with 1 on the \( j \)-th place, \( p_j(z) = i_jz \) for each \( z \in \mathcal{A}_m \), where the decomposition is used \( z = \sum_{j=1}^{m} i_j e_j \), \( i_j \in \mathcal{A}_r \) for each \( j = 1, ..., m, m \in \mathbb{N} := \{1, 2, 3, \ldots \} \), the projections are the following
\[
\pi_{s,p,t}(V) := \{ u : z \in V, z = \sum_{v \in b} w_v v, u = w_s s + w_p p \} \text{ for a domain } V \text{ in } \mathcal{A}_r \text{ for each } s \neq p \in b, \text{ where } t := \sum_{v \in b \setminus \{s,p\}} w_v v \in \mathcal{A}_{r,s,p} := \{ z \in \mathcal{A}_r : z = \sum_{v \in b} w_v v, w_s = w_p = 0, w_v \in \mathbb{R} \forall v \in b \}, \text{ where } b := \{i_0, i_1, ..., i_{2r-1}\} \text{ is the family of standard generators of the Cayley-Dickson algebra } \mathcal{A}_r. \text{ Frequently we take } m = 1. \text{ Henceforth, we consider a domain } U \text{ satisfying Conditions (D1, D2) if something other is not outlined.}

2. Operators. An \( \mathbb{R} \) linear space \( X \) which is also left and right \( \mathcal{A}_r \) module will be called an \( \mathcal{A}_r \) vector space. It is supposed that \( X \) can be presented as the direct sum \( X = \sum_{0 \leq i_0 \leq \ldots \leq i_{2r-1}} X_{i_0 \ldots i_{2r-1}} \), where \( X_{0 \ldots i_{2r-1}} \) are pairwise isomorphic real linear spaces. Particularly, for \( r = 2 \) this module is associative: \( (xa)b = x(ab) \) and \( (ab)x = a(bx) \) for all \( x \in X \) and \( a, b \in \mathbb{H} \), since the quaternion skew field \( \mathcal{A}_2 = \mathbb{H} \) is associative. This module is alternative for \( r = 3 \): \( (xa)a = x(a^2) \) and \( (a^2)x = a(ax) \) for all \( x \in X \) and \( a \in \mathcal{O} \), since the octonion algebra \( \mathcal{O} = \mathcal{A}_3 \) is alternative.

Let \( X \) and \( Y \) be two \( \mathbb{R} \) linear normed spaces which are also left and right \( \mathcal{A}_r \) modules, where \( 2 \leq r, \) such that
\[
(1) \ 0 \leq \|ax\|_X \leq \|a\|_X \|x\|_X \text{ and } \|xa\|_X \leq \|a\|_X \|x\|_X \text{ for all } x \in X \text{ and } a \in \mathcal{A}_r,
\]
and
\[
(2) \ \|x + y\|_X \leq \|x\|_X + \|y\|_X \text{ for all } x, y \in X \text{ and and}
\]
(3) $\|bx\|_X = |b|\|x\|_X = \|xb\|_X$ for each $b \in \mathbb{R}$ and $x \in X$, where for $r = 2$ and $r = 3$ Condition (1) takes the form

$$(1') 0 \leq \|ax\|_X = |a|\|x\|_X = \|xa\|_X$$

for all $x \in X$ and $a \in A_r$.

Such spaces $X$ and $Y$ will be called $A_r$ normed spaces.

An $A_r$ normed space complete relative to its norm will be called an $A_r$ Banach space.

Put $X^\otimes k := X \otimes_R \ldots \otimes_R X$ to be the $k$ times ordered tensor product over $\mathbb{R}$ of $X$. By $L_{q,k}(X^\otimes k, Y)$ we denote a family of all continuous $k$ times $\mathbb{R}$ poly-linear and $A_r$ additive operators from $X^\otimes k$ into $Y$. If $X$ and $Y$ are normed $A_r$ spaces and $Y$ is complete relative to its norm, then $L_{q,k}(X^\otimes k, Y)$ is also a normed $\mathbb{R}$ linear and left and right $A_r$ module complete relative to its norm. In particular, $L_{q,1}(X, Y)$ is denoted by $L_q(X, Y)$ as well.

If $A \in L_q(X, Y)$ and $A(xb) = (Ax)b$ or $A(bx) = b(Ax)$ for each $x \in X_0$ and $b \in A_r$, then an operator $A$ we call right or left $A_r$-linear respectively.

An $\mathbb{R}$ linear space of left (or right) $k$ times $A_r$ poly-linear operators is denoted by $L_{l,k}(X^\otimes k, Y)$ (or $L_{r,k}(X^\otimes k, Y)$ respectively).

An $\mathbb{R}$-linear operator $A : X \to X$ will be called right or left strongly $A_r$ linear if

$$(x)(xb) = (Ax)b \text{ or } A(bx) = b(Ax) \text{ for each } x \in X \text{ and } b \in A_r \text{ correspondingly.}$$

An $\mathbb{R}$ linear $A_r$ additive operator $A$ is called invertible if it is densely defined and one-to-one and has a dense range $\mathcal{R}(A)$.

Henceforward, if an expression of the form

$$(4) \sum_k (I - A_x)_{\otimes k}f(x, y) = g(y)$$

will appear on a domain $U$, which need to be inverted we consider the case when

$$(RS) (I - A_x)$$

is either right strongly $A_r$ linear, or right $A_r$ linear and $\otimes_k f \in X_0$ for each $k$, or $\mathbb{R}$ linear and $\otimes_k g(y) \in \mathbb{R}$ for each $k$ and every $y \in U$, at each point $x \in U$, since $\mathbb{R}$ is the center of the Cayley-Dickson algebra $A_r$, where $2 \leq r$.

3. First order PDOs. We consider an arbitrary first order partial differential operator $\sigma$ given by the formula

$$(1) \sigma f = \sum_{j=0}^{2r-1} i^*_j (\partial f / \partial z_{\xi(j)}) \psi_j,$$
where \( f \) is a differentiable \( \mathcal{A}_r \)-valued function on the domain \( U \) satisfying Conditions 1(\( D_1, D_2 \)), \( 2 \leq r, i_0, ..., i_{2r-1} \) are the standard generators of the Cayley-Dickson algebra \( \mathcal{A}_r \), \( a^* = \tilde{a} := a_0i_0 - a_1i_1 - ... - a_{2r-1}i_{2r-1} \) for each \( a = a_0i_0 + a_1i_1 + ... + a_{2r-1}i_{2r-1} \) in \( \mathcal{A}_r \) with \( a_0, ..., a_{2r-1} \in \mathbb{R} \); \( \psi_j \) are real constants so that \( \sum_j \psi_j^2 > 0 \), \( \xi : \{0, 1, ..., 2^r - 1\} \to \{0, 1, ..., 2^r - 1\} \) is a surjective bijective mapping, i.e. \( \xi \) belongs to the symmetric group \( S_{2^r} \) (see also §2 in [26]).

For an ordered product \( \{1f...kf\}_{q(k)} \) of differentiable functions \( s f \) we put

\[
(2) \sigma \{1f...kf\}_{q(k)} = \sum_{j=0}^n i_j^* \{1f...(\partial s f/\partial z_{\xi(j)})...kf\}_{q(k)} \psi_j,
\]

where a vector \( q(k) \) indicates on an order of the multiplication in the curled brackets (see also §2 [21, 20]), so that

\[
(3) \sigma \{1f...kf\}_{q(k)} = \sum_{s=1}^k \sigma \{1f...k\}_{q(k)}.
\]

Symmetrically other operators

\[
(4) \hat{\sigma} f = \sum_{j=0}^{2^r-1} (\partial f/\partial z_{\xi(j)})i_j \psi_j,
\]

are defined. Therefore, these operators are related by the formula:

\[
(5) (\sigma f)^* = \hat{\sigma} (f^*).
\]

Operators \( \sigma \) given by (1) are right \( \mathcal{A}_r \) linear.

4. Integral operators. We consider integral operators of the form:

\[
(1) K(x, y) = F(x, y) + p_\sigma \int_x^\infty F(z, y) N(x, z, y) dz,
\]

where \( \sigma \) is an \( \mathbb{R} \)-linear partial differential operator as in §3 and \( \sigma f \) is the non-commutative line integral (anti-derivative operator) over the Cayley-Dickson algebra \( \mathcal{A}_r \) from [26] or §4.2.5 [22], where \( F \) and \( K \) are continuous functions with values in the Cayley-Dickson algebra \( \mathcal{A}_r \) or more generally in the real algebra \( Mat_{n\times n}(\mathcal{A}_r) \) of \( n \times n \) matrices with entries in \( \mathcal{A}_r \), \( p \) is a nonzero real parameter. For definiteness we take the right \( \mathcal{A}_r \) linear anti-derivative operator \( \sigma \int g(z) dz \).

Let a domain \( U \) be provided with a foliation by locally rectifiable paths \( \{\gamma^\alpha : \alpha \in \Lambda\} \) (see also [26] or [22]), where \( \Lambda \) is a set described below. We take for definiteness a canonical closed domain \( U \) in \( \hat{\mathcal{A}}_r \) satisfying Conditions 1(\( D_1, D_2 \)) so that \( \infty \in U \), where \( \hat{\mathcal{A}}_r = \mathcal{A}_r \cup \{\infty\} \) denotes the one-point compactification of \( \mathcal{A}_r \), \( 2 \leq r < \infty \).
A domain $U$ is called foliated by locally rectifiable paths $\{\gamma^\alpha : \alpha \in \Lambda\}$ if

$$\gamma : \langle a_\alpha, b_\alpha \rangle \rightarrow U$$

for each $\alpha$ and it satisfies the following three conditions:

1. For $\alpha \in \Lambda$, $\cup_{\alpha \in \Lambda} \gamma^\alpha(\langle a_\alpha, b_\alpha \rangle) = U$ and
2. $\gamma^\alpha(\langle a_\alpha, b_\alpha \rangle) \cap \gamma^\beta(\langle a_\beta, b_\beta \rangle) = \emptyset$ for each $\alpha \neq \beta \in \Lambda$.

Moreover, if the boundary $\partial U = cl(U) \setminus Int(U)$ of the domain $U$ is non-void then

3. $\partial U = (\cup_{\alpha \in \Lambda_1} \gamma^\alpha(a_\alpha)) \cup (\cup_{\beta \in \Lambda_2} \gamma^\beta(b_\beta))$,

where $\Lambda_1 = \{\alpha \in \Lambda : a_\alpha \leq b_\alpha \}$ and $\Lambda_2 = \{\alpha \in \Lambda : a_\alpha \geq b_\alpha \}$.

For the canonical closed subset $U$ we have $cl(U) = U = cl(Int(U))$, where $cl(U)$ denotes the closure of $U$ in $\mathcal{A}_v$ and $Int(U)$ denotes the interior of $U$ in $\mathcal{A}_v$. For convenience one can choose a $C^m$ foliation, that is each $\gamma^\alpha$ is of class $C^m$, where $n \in \mathbb{N}$. When $U$ is with non-void boundary we choose a foliation family such that $\cup_{\alpha \in \Lambda} \gamma(a_\alpha) = \partial U_1$, where a set $\partial U_1$ is open in the boundary $\partial U$ and so that $w|_{\partial U_1}$ would be a sufficient initial condition to characterize a unique branch of an anti-derivative $w(x) = \mathcal{I}_\sigma f(x) = \sigma \int_{(0,x)} f(z)dz$, where $0_x \in \partial U_1$, $x \in U$, $\gamma^\alpha(t_0) = 0_x$, $\gamma^\alpha(t) = x$ for some $\alpha \in \Lambda$, $t_0$ and $t \in \langle a_\alpha, b_\alpha \rangle$.

$$\sigma \int_{0_x}^x f(z)dz = \sigma \int_{\gamma^\alpha|_{[0,t]}} f(z)dz.$$

In accordance with Theorems 2.4.1 and 2.5.2 [26] or 4.2.5 and 4.2.23 [22] the equality

$$\sigma \int_{0_x}^x g(z)dz = g(x)$$

is satisfied for a continuous function $g$ on a domain $U$ as in §1 and a foliation as above.

Particularly in the class of $\mathcal{A}_v$ holomorphic functions in the domain satisfying Conditions 1($D1, D2$) this line integral depends only on initial and final points due to the homotopy theorem [21, 20].

We denote by $P = P(U)$ the family of all locally rectifiable paths $\gamma : \langle a_\gamma, b_\gamma \rangle \rightarrow U$ supplied with the family of pseudo-metrics

$$\rho^{a,b,c,d}(\gamma, \omega) := |\gamma(a) - \omega(c)| + \inf_{\phi} V_a^b(\gamma(t) - \omega(\phi(t)))$$

where the infimum is taken by all diffeomorphisms $\phi : [a,b] \rightarrow [c,d]$ so that $\phi(a) = c$ and $\phi(b) = d$, $a < b$, $c < d$, $[a,b] \subset \langle a_\gamma, b_\gamma \rangle$, $[c,d] \subset \langle a_\omega, b_\omega \rangle$.

We take a foliation such that $\Lambda$ is a uniform space and the limit

$$\lim_{\beta \rightarrow \alpha} \rho^{a,b,a,b}(\gamma^\beta, \gamma^\alpha) = 0$$
is zero for each \([a, b] < < a_\alpha, b_\alpha >\).

For example, we can take \(\Lambda = \mathbb{R}^{n-1}\) for a foliation of the entire Cayley-Dickson algebra \(\mathcal{A}_r\), where \(n = 2^r, t \in \mathbb{R}, \alpha \in \Lambda\), so that \(\bigcup_{\alpha \in \Lambda} \gamma^\alpha([0, \infty)) = \{z \in \mathbb{A}_r : \text{Re}((z - y)v^*) \geq 0\}\) and \(\bigcup_{\alpha \in \Lambda} \gamma^\alpha((-\infty, 0]) = \{z \in \mathbb{A}_r : \text{Re}((z - y)v^*) \leq 0\}\) are two real half-spaces, where \(v, y \in \mathbb{A}_r\) are marked Cayley-Dickson numbers and \(v \neq 0\). Particularly, we can choose the foliation such that \(\gamma^\alpha(0) = y + \alpha_1 v_1 + \ldots + \alpha_{2^r-1} v_{2^r-1}\) and \(\gamma^\alpha(t) = tv_0 + \gamma^\alpha(0)\) for each \(t \in \mathbb{R}\), where \(v_0, \ldots, v_{2^r-1}\) are \(\mathbb{R}\)-linearly independent vectors in \(\mathcal{A}_r\).

Therefore, the expression

\[
\sigma \int_x^\infty g(z)dz := \sigma \int_{\gamma^\alpha|_{[x,b_\alpha]}} g(z)dz
\]
denotes a non-commutative line integral over \(\mathcal{A}_r\) along a path \(\gamma^\alpha\) so that \(\gamma^\alpha(t_x) = x\) and \(\lim_{b_\alpha \to b_\alpha} \gamma^\alpha(t) = \infty\) for an integrable function \(g\), where \(t_x < a_\alpha, b_\alpha >\), \(\alpha \in \Lambda, a_\alpha = a_{\gamma^\alpha}, b_\alpha = b_{\gamma^\alpha}\). It is sometimes convenient to use the line integral

\[
\sigma \int_x^{-\infty} g(z)dz := \sigma \int_{\gamma^\alpha|_{[a_\alpha,t_x]}} g(z)dz,
\]
when \(\lim_{a_\alpha \to a_\alpha} \gamma^\alpha(t) = \infty\).

Put for convenience \(\sigma^0 = I\), where \(I\) denotes the unit operator, \(\sigma^m\) denotes the \(m\)-th power of \(\sigma\) for each non-negative integer \(0 \leq m \in \mathbb{Z}\).

5. Proposition. Let \(F \in \mathcal{C}^m(U^2, \text{Mat}_{n \times n}(\mathcal{A}_r))\) and \(N \in \mathcal{C}^m(U^3, \text{Mat}_{n \times n}(\mathcal{A}_r))\) and let

\[
(1) \quad \lim_{z \to \infty} \sigma_z^k \sigma_z^s \sigma_z^l F(z, y)N(x, z, y) = 0
\]
for each \(x, y\) in a domain \(U\) satisfying Conditions 1(D1, D2) with \(\infty \in U\) and every non-negative integers \(0 \leq k, s, l \in \mathbb{Z}\) such that \(k + s + l \leq m\). Suppose also that \(\sigma \int_x^\infty \partial_1^\alpha \partial_2^\beta \partial_3^\gamma [F(z, y)N(x, z, y)]dz\) converges uniformly by parameters \(x, y\) on each compact subset \(W \subset U \subset \mathcal{A}_r^2\) for each \(|\alpha| + |\beta| + |\gamma| \leq m\), where \(\alpha = (\alpha_0, \ldots, \alpha_{2^r-1}), |\alpha| = \alpha_0 + \ldots + \alpha_{2^r-1}, \partial_1^\alpha = \partial_1^{\alpha_1}/\partial x_0^{\alpha_1} \ldots \partial x_{2^r-1}^{\alpha_{2^r-1}}\). Then the non-commutative line integral \(\sigma \int_x^\infty F(z, y)N(x, z, y)dz\) from \(\S_4\) satisfies the identities:

\[
(2) \quad \sigma_x^m \sigma \int_x^\infty F(z, y)N(x, z, y)dz = 2\sigma_x^m \sigma \int_x^\infty F(z, y)N(x, z, y)dz + A_m(F, N)(x, y),
\]

\[
(3) \quad \sigma_z^m \sigma \int_x^\infty F(z, y)N(x, z, y)dz = (-1)^m 2\sigma_z^m \sigma \int_x^\infty F(z, y)N(x, z, y)dz + B_m(F, N)(x, y),
\]
where

\[ A_m(F, N)(x, y) = -2\sigma_x^{m-1}[F(x, y)N(x, z, y)]|_{z=x} + \sigma_x A_{m-1}(F, N)(x, y) \]

for \( m \geq 2 \),

\[ B_m(F(z, y), N(x, z, y)) = (-1)^m 2\sigma_x^{m-1}F(x, y)N(x, z, y) + \sigma_x B_{m-1}(F(z, y), N(x, z, y)) \]

for \( m \geq 2 \), \( B_m(F, N)(x, y) = B_m(F(z, y), N(x, z, y))|_{z=x} \);

\( A_1(F, N)(x, y) = -F(x, y)N(x, x, y) \),

\( B_1(F(z, y), N(x, z, y)) = -F(z, y)N(x, z, y) \),

\( \sigma_x \) is an operator \( \sigma \) acting by the variable \( x \in U \subset A_r \).

**Proof.** Using the conditions of this proposition and the theorem about differentiability of improper integrals by parameters (see, for example, Part IV, Chapter 2, §4 in [15]) we get the equality

\[ \sigma \int_x^\infty \partial_x^\alpha \partial_y^\beta \partial_z^\gamma [F(z, y)N(x, z, y)]dz = \partial_x^\alpha \partial_y^\beta \sigma \int_x^\infty \partial_x^\gamma [F(z, y)N(x, z, y)]dz \]

for each \( |\alpha| + |\beta| + |\gamma| \leq m \).

In virtue of Theorems 2.4.1 and 2.5.2 [20] or 4.2.5 and 4.2.23 and Corollary 4.2.6 [22] there are satisfied the equalities

\[ \sigma \int_x^\infty g(z)dz = -g(x) \] and

\[ \sigma \int_{0x}^x [\sigma_z f(z)]dz = f(x) - f(0x) \]

for each continuous function \( g \) and a continuously differentiable function \( f \),

where \( 0x \) is a marked point in \( U \),

\[ 1\sigma_x \int_x^\infty F(z, y)N(x, z, y)dz := \sum_{j=0}^{2^r-1} \sigma \int_x^\infty \{i_j^*[(\partial F(z, y)/\partial z_{(j)})N(x, z, y)]\psi_j\}dz \] and

\[ 2\sigma_x \int_{0x}^x F(z, y)N(x, z, y)dz := \sum_{j=0}^{2^r-1} \sigma \int_{0x}^x \{i_j^*[F(z, y)(\partial N(x, z, y)/\partial z_{(j)})]\psi_j\}dz \] and

\[ 2\sigma_x \int_x^\infty F(z, y)N(x, z, y)dz := \sum_{j=0}^{2^r-1} \sigma \int_x^\infty \{i_j^*[F(z, y)(\partial N(x, z, y)/\partial x_{(j)})]\psi_j\}dz \].

Therefore, from Equalities (8,9), 3(3) and 4(5) and Condition (1) we infer that:

\[ \sigma_x \int_x^\infty F(z, y)N(x, z, y)dz = 2\sigma_x \int_x^\infty F(z, y)N(x, z, y)dz - F(x, y)N(x, x, y), \]
since \(F(z, y)N(x, z, y)|_x^\infty = -F(x, y)N(x, x, y)\), that demonstrates Formula (2) for \(m = 1\) and \(A_1 = -F(x, y)N(x, x, y)\). Proceeding by induction for \(p = 2, \ldots, m\) leads to the identities:

\[
(14) \quad \sigma_x^p \int_x^\infty F(z, y)N(x, z, y)dz = \\
\sigma_x \left( [\sigma_x^{p-1} \int_x^\infty F(z, y)N(x, z, y)dz] + \sigma_x A_{p-1}(F, N)(x, z, y) \right) \\
= 2\sigma_x^p \int_x^\infty F(z, y)N(x, z, y)dz \\
- \left[ 2\sigma_x^{p-1}F(z, y)N(x, z, y) \right]|_{z=x} + \sigma_x A_{p-1}(F, N)(x, y).
\]

Thus (14) implies Formulas (2, 4, 6). Then with the help of Formulas (8, 9) and Condition (1) we infer also that

\[
(15) \quad 1\sigma_z \int_x^\infty F(z, y)N(x, z, y)dz = -2\sigma_z \int_x^\infty F(z, y)N(x, z, y)dz + F(z, y)N(x, z, y)|_x^\infty \\
= -F(x, y)N(x, x, y) - 2\sigma_z \int_x^\infty F(z, y)N(x, z, y)dz.
\]

Thus Formulas (3) for \(m = 1\) and (7) are valid. Then we deduce Formulas (3, 5) by induction on \(p = 2, \ldots, m\):

\[
(16) \quad 1\sigma_z^p \int_x^\infty F(z, y)N(x, z, y)dz = \\
1\sigma_z^{p-1} \left( 1\sigma_z \int_x^\infty F(z, y)N(x, z, y)dz \right) \\
= 1\sigma_z^{p-1} \left[ -2\sigma_z \int_x^\infty F(z, y)N(x, z, y)dz \right] \\
- \left[ 1\sigma_z^{p-1}F(z, y)N(x, z, y) \right]|_{z=x} \\
= 1\sigma_z^{p-2} \left\{ 1\sigma_z \left[ -2\sigma_z \int_x^\infty F(z, y)N(x, z, y)dz \right] \right\} \\
- \left[ 1\sigma_z^{p-1}F(z, y)N(x, z, y) \right]|_{z=x} \\
= 1\sigma_z^{p-2} \left\{ (-2\sigma_z)^2 \int_x^\infty F(z, y)N(x, z, y)dz \right\} \\
+ \left[ 1\sigma_z^{p-2}(2\sigma_z F(z, y)N(x, z, y)) \right]|_{z=x} - \left[ 1\sigma_z^{p-1}F(z, y)N(x, z, y) \right]|_{z=x} \\
= \ldots = (-2\sigma_z)^p \int_x^\infty F(z, y)N(x, z, y)dz + B_p(F(z, y), N(x, z, y))|_{z=x}
\]

and

\[
B_p(F(z, y), N(x, z, y)) = -(-2\sigma_z)^p F(z, y)N(x, z, y) + 1\sigma_z B_{p-1}(F(z, y), N(x, z, y)).
\]

6. Corollary. If suppositions of Proposition 5 are satisfied, then

(1) \(A_2(F, N)(x, y) = -\sigma_x[F(x, y)N(x, x, y)] - 2\sigma_x[F(z, y)N(x, z, y)]|_{z=x}\),

(2) \(A_3(F, N)(x, y) = -\sigma_x^2[F(x, y)N(x, x, y)] - \sigma_x(2\sigma_x[F(x, y)N(x, z, y)]|_{z=x}) - 2\sigma_x^2[F(x, y)N(x, z, y)]|_{z=x}\),
(3) \( A_m(F, N)(x, y) = - \sum_{j=0}^{m-1} \sigma_x^j \{ 2^{m-j} F(z, y)N(x, z, y) \}_{z=x} \),

(4) \( B_2(F(z, y), N(x, z, y)) = -1 \sigma_z [F(z, y)N(x, z, y)] + 2 \sigma_z [F(z, y)N(x, z, y)] \),

(5) \( B_3(F(z, y), N(x, z, y)) = -1 \sigma_x^2 [F(z, y)N(x, z, y)] + \sigma_z [2^2 \sigma_x^2 F(z, y)N(x, z, y)] \),

\( + 1 \sigma_z [2^2 \sigma_x^2 F(z, y)N(x, z, y)] - 2^2 \sigma_x^2 [F(z, y)N(x, z, y)] \),

\( B_m(F(z, y), N(x, z, y)) = [\sum_{k=0}^{m-1} (-1)^{k+1} 1 \sigma_z^{m-1-k} 2 \sigma_x^k]F(z, y)N(x, z, y) \),

(7) \( A_2(F, N)(x, y) - B_2(F, N)(x, y) = -2^2 \sigma_x [F(x, y)N(x, x, y)] \),

where \( \sigma_x N(x, x, y) = [\sigma_x N(x, x, y) + \sigma_z N(x, x, y)]_{z=x} \),

(8) \( A_3(F, N)(x, y) - B_3(F, N)(x, y) = -3 \sigma_z^2 + 2 \sigma_x^2 \sigma_z + 2^2 \sigma_x^2 \sigma_z [F(x, y)N(x, x, y)]_{z=x} \)

\(- (2^2 \sigma_x^2 \sigma_z + 2^2 \sigma_x^2 \sigma_z) [F(x, y)N(x, x, y)] \).

Particularly, if either \( p \) is even and \( \psi_0 = 0 \) or \( F \in Mat_{m \times n}(R) \) and \( N \in Mat_{m \times n}(A_r) \), then

\( 2^2 \sigma_x^p [F(x, y)N(x, x, y)] = F(z, y) \sigma_x^p N(x, x, y) \) and \( 2^2 \sigma_x^p [F(x, y)N(x, x, y)] = F(z, y) \sigma_x^p N(x, x, y) \).

\textbf{Proof.} From Formulas (13–16) Identities (1–8) follow by induction, since

\( A_m(F, N)(x, y) = - \sum_{j=0}^{m-1} \sigma_x^j \{ 2^{m-j} F(z, y)N(x, z, y) \}_{z=x} + \sigma_x A_{m-1}(F, N)(x, y) = \)

\( ... = - \sum_{j=0}^{m-1} \sigma_x^j \{ 2^{m-j} F(z, y)N(x, z, y) \}_{z=x} - \sigma_x \{ [2^{m-2} \sigma_x F(z, y)N(x, z, y)]_{z=x} \} \)

\(- \sigma_x^2 \{ [2^{m-3} \sigma_x F(z, y)N(x, z, y)]_{z=x} \} - \sigma_x^{m-2} \{ [2^{m-1} \sigma_x F(z, y)N(x, z, y)]_{z=x} \} \)

\( \sigma_x^{m-1} F(x, y)N(x, x, y) \) and

\( B_m(F(z, y), N(x, z, y)) = - (2^2 \sigma_x^m F(z, y)N(x, z, y) + \)

\( 1 \sigma_z B_{m-1}(F(z, y), N(x, z, y)) = ... = \)

\(- (2^2 \sigma_x^m F(z, y)N(x, z, y) + 1 \sigma_z^2 [2^2 \sigma_x F(z, y)N(x, z, y)] \)

\(- 1 \sigma_z^{m-2} (2^2 \sigma_x^2 F(z, y)N(x, z, y)] + ... + (-1)^m (2^2 \sigma_x^m F(z, y)N(x, z, y)] \).

Particularly when \( p \) is even and \( \psi_0 = 0 \), \( p = 2k \), \( k \in N \) we get that

\( \sigma_x^p f(x) = A^k f(x) \)

for \( p \) times differentiable function \( f : U \to A_r \), where

\( Af = \sum_j b_j \partial f(x)/\partial x_j \), \( b_j = i_{-1(j)}^2 \in R \) according to §2.2 [26] or Formulas 4.2.4(7–9) [22].

On the other hand the operators \( 2^2 \sigma_x^p \) and \( 2^2 \sigma_x^p \) commute with the left multiplication on \( F(z, y) \in Mat_{m \times n}(R) \), that is \( 2^2 \sigma_x^p [F(z, y)K(x, z)] = F(z, y) \sigma_x^p K(x, z) \)

and \( 2^2 \sigma_x^p [F(z, y)K(x, z)] = F(z, y) \sigma_x^p K(x, z) \) for \( p = 2k \), since \( R \) is the center of the Cayley-Dickson algebra \( A_r \).
3 Some types of integrable nonlinear PDE.

1. Partial differential operators $L_j$ are considered on domains $\mathcal{D}(L_j)$ contained in suitable spaces of differentiable functions, for example, in the space $C^\infty(U, Mat_{n \times n}(\mathbb{A}_r))$ of infinitely differentiable by real variables functions on an open domain $U$ in $\mathbb{A}_r$ and with values in $Mat_{n \times n}(\mathbb{A}_r)$, because $U$ has the real shadow $U_{\mathbb{R}}$, where $n \in \mathbb{N}$. Or it is possible to use the Sobolev space $H^m(U, Mat_{n \times n}(\mathbb{A}_r))$, where $m \geq \text{ord}(L_j)$, $\text{ord}(L_j)$ denotes the order of a PDE $L_j$, while on $U$ the Lebesgue measure is provided. The spaces $C^m(U, Mat_{n \times n}(\mathbb{A}_r))$ and $H^m(U, Mat_{n \times n}(\mathbb{A}_r))$ with $m \leq \infty$ are linear over the real field $\mathbb{R}$, also they have the structure of the left and the right modules over the Cayley-Dickson algebra $\mathbb{A}_r$, $r \geq 2$. To each Cayley-Dickson number $z = z_0i_0 + \ldots + z_{2r-1}i_{2r-1} \in \mathbb{A}_r$ there corresponds a vector $[z] = (z_0, \ldots, z_{2r-1})$ in its real shadow $\mathbb{R}^{2r}$, where $z_j \in \mathbb{R}$ for each $j$. For functions $f([z])$ of $[z]$ we shall write for short $f(z)$ also.

Henceforth, if something other will not be specified, we shall take a function $N$ may be depending on $F$, $K$ and satisfying the following conditions:

(1) $N(x, y) = EK(x, y)$ with an operator $E$ in the form

(2) $E = BST_g$,

(3) $[L_j, E] = 0$ for each $j$,

where $B$ is a nonzero bounded right $\mathbb{A}_r$ linear (or strongly right $\mathbb{A}_r$ linear) operator, $S = S(x, y) \in \text{Aut}(Mat_{n \times n}(\mathbb{A}_r))$, so that $B$ is independent of $x, y \in U$, $g \in \text{Diff}^\infty(U_{\mathbb{R}}^2)$, $g = (g_1, g_2)$, $g_l(U_{\mathbb{R}}^2) = U_{\mathbb{R}}$ for $l = 1$ and $l = 2$, $U_{\mathbb{R}}$ denotes the real shadow of the domain $U$, $\text{Aut}(Mat_{n \times n}(\mathbb{A}_r))$ notates the automorphism group of the algebra $Mat_{n \times n}(\mathbb{A}_r)$,

(4) $T_gK(x, y) := K(g_1(x, y), g_2(x, y))$.

Condition (3) is implied by the following:

(5) $[L_j, B] = 0$ for each $j$,

(6) $L_{j,x,y}(T_gK(x, y)) = T_g(L_{j,x,y}K(x, y))$ and

(7) $L_{j,x,y}(S(x, y)K(x, y)) = S(x, y)(L_{j,x,y}K(x, y))$ for each $j$ and each $x, y \in U$, where $L_{j,x,y}$ are PDOs considered below.

Evidently Conditions (6, 7) are fulfilled, when $L_{j,x,y}$ are polynomials of $\sigma^k_x$ and $\sigma^h_y$, all coefficients of $L_j$ are real and the following stronger conditions
are imposed:

(8) $\sigma_x^k(T_gK(x, y)) = T_g(\sigma_x^kK(x, y))$, $\sigma_y^k(T_gK(x, y)) = T_g(\sigma_y^kK(x, y))$ and

(9) $\sigma_x^k(S(x, y)K(x, y)) = S(x, y)(\sigma_x^kK(x, y))$, $\sigma_y^k(S(x, y)K(x, y)) = S(x, y)(\sigma_y^kK(x, y))$,

since $S|_{\text{iso}_{\mathbb{R}}} = I$.

If coefficients of $L_j$ may be Cayley-Dickson numbers, $S = I$, then Condition (8) will suffice as well.

Particularly there may be $E = B$, $B \in SL_n(\mathbb{R})$, or $E = I$. It will also be indicated, when $E$ or $K$ and hence $N$ depend on some parameter or a variable.

2. General approach to solutions of nonlinear vector partial differential equations with the help of non-commutative integration over Cayley-Dickson algebras. We consider an equation over the Cayley-Dickson algebra $\mathcal{A}_r$ which is presented in the non-commutative line integral form:

\[ K(x, y) = F(x, y) + p \sigma \int_x^\infty F(z, y)N(x, z, y)dz, \]

where $K$, $F$ and $N$ are continuous integrable functions of $\mathcal{A}_r$ variables $x, y, z \in U$ so that $F$, $K$ and $N$ have values in $Mat_{n \times n}(\mathcal{A}_r)$, where $n \geq 1$, $r \geq 2$. $N$ and $K$ are related by $1(1, 2)$, $p \in \mathbb{R} \setminus \{0\}$ is a non-zero real constant. These functions $F$, $K$ and $N$ may depend on additional parameters $t, \tau, \ldots$.

At first it is necessary to specify the function $N$ and its expression throughout $F$ and $K$. It is supposed that an operator

\[ (I - A_xE)K(x, y) = F(x, y) \]

is invertible, when $N(x, z, y) = E_xK(x, z)$ for each $x, y, z \in U$, so that $(I - A_xE)^{-1}$ is continuous, where $I$ denotes the unit operator,

\[ A_xK(x, y) := p \sigma \int_x^\infty F(z, y)K(x, z)dz \]

is an operator acting by variables $x$.

Then $\mathbb{R}$-linear partial differential operators $L_k$ over the Cayley-Dickson algebra $\mathcal{A}_r$ are provided for $k = 1, \ldots, k_0$, where $k_0 \in \mathbb{N}$. It is frequently helpful to consider their decompositions:

\[ L_kf = \sum_j \iota_j^*(L_{k,j}f), \]

where $f$ is a differentiable function in the domain of each operator $L_k$, $L_{k,j}$
are components of the operators $L_k$ so that each $L_{k,j}$ is a PDO written in real variables with real coefficients. That is $L_{k,j}g$ is a real-valued function for each $ord(L_{k,j})$ times differentiable real-valued function $g$ in the domain of $L_{k,j}$ for every $j$, where $ord(L_{k,j})$ denotes the order of the PDO $L_{k,j}$. Next the conditions are imposed on the function $F$:

$$L_k F = 0$$

for $k = 1, \ldots, k_0$.

It may be necessary to consider in some problems stronger conditions:

$$\sum_{j \in \Psi_l} i_j^n c_{k,j}(L_{k,0}F) + L_{k,j}F = 0$$

for each $k$ and $1 \leq l \leq m$, where $c_{k,j}$ are constants $c_{k,j} \in A_r$, $\Psi_l \subset \{0, 1, \ldots, 2^r - 1\}$ for each $l$, $\bigcup_l \Psi_l = \{0, 1, \ldots, 2^r - 1\}$, $\Psi_n \cap \Psi_l = \emptyset$ for each $n \neq l$, $1 \leq m \leq 2^r$. There is not excluded that the coefficients $c_{k,j}$ or the operators $L_{k,j}$ may be zero for some $(k, j)$.

After this a function $K$ is determined from Equation (2).

This function $K$ may be satisfying some PDEs, when suitable PDOs $L_s$ and the operator $E$ are chosen (see also §1). Indeed acting by the operator $L_k$ from the left on both sides of (2) one may get with the help of Conditions either (5) or (6) the PDEs either

$$L_s[(I - A_x E_y)K] = 0$$

or

$$\sum_{j \in \Psi_k} i_j^n c_{k,j}(L_{s,0}F) + L_{s,j}[(I - A_x E_y)K] = 0$$

for each $k = 1, \ldots, m$ respectively for $s = 1, \ldots, k_1$, where $k_1 \leq k_0$.

Therefore this leads to the equalities

$$(I - A_x E_y)(L_s K) = R_s(K)$$

for $s = 1, \ldots, k_1$,

where each operator of the form

$$R_s(f) = (I - A_x E_y)(L_s f) - L_s[(I - A_x E_y)f]$$

is obtained by calculations of appearing commutators $[A, B] = AB - BA$ and anti-commutators $\{A, B\} = AB + BA$ of operators $(I - (A_x E_y)_0)$, $(A_x E_y)_j$, $L_{s,j}$, $j = 0, \ldots, 2^r - 1$. The latter can be realized when the function $N$ and the PDOs $L_j$ are chosen such that

$$R_s(K) = (I - A_x E_y)M_s(K)$$

for $s = 1, \ldots, k_1$,

where $M_s(K)$ are operators or functionals acting on $K$. Generally the operators $M_s$ may be non-$R$-linear and besides terms of a partial differential operator it may contain terms containing the integral operator $A$. Therefore
due to Condition (2) the function \( K \) must satisfy the PDEs or the partial integro-differential equations (PIDEs)

\[
L_s K - M_s(K) = 0 \quad \text{for} \quad s = 1, \ldots, k_1,
\]

which generally may be non-\( \mathbb{R} \)-linear. Thus each solution \( K \) of the \( \mathbb{R} \)-linear integral equation (1) should also be the solution of the aforementioned PDEs or PIDEs (12).

It is worthwhile to choose the \( \mathcal{A}_r \) vector independent PDOs \( L_s \) for \( s = 1, \ldots, k_0 \) and so that \( c_{k,j} \in \mathbb{R} i_{\xi(k,j)} \) and \( \xi(k,j) \in \{0, 1, \ldots, 2^r - 1\} \) for each \( k, j \).

Henceforward, if something other will not be outlined, we consider the variants:

\[
F, K, N \in \text{Mat}_{n \times n}(\mathcal{A}_r) \quad \text{with} \quad 2 \leq r \leq 3 \quad \text{and} \quad B \text{ is the strongly right } \mathcal{A}_r\text{-linear operator}; \quad \text{or}
\]

\[
F \in \text{Mat}_{n \times n}(\mathbb{R}) \quad \text{and} \quad K, N \in \text{Mat}_{n \times n}(\mathcal{A}_r) \quad \text{with} \quad 2 \leq r \quad \text{and} \quad B \text{ is the right } \mathcal{A}_r\text{-linear operator} \quad \text{(see also §1)}, \quad \text{where} \quad 1 \leq n \in \mathbb{N}.
\]

3. Theorem. Suppose that conditions of Proposition 2.5 and 2.2(RS) are fulfilled over the Cayley-Dickson algebra \( \mathcal{A}_r \) with \( 2 \leq r \) and on a domain \( U \) satisfying Conditions 2.1(D1, D2) for the corresponding terms of operators \( L_s \) for all \( s = 1, \ldots, k_0 \) so that

\[
\begin{align*}
(1) & \quad \text{the appearing in the terms } M_s(K) \text{ integrals uniformly converge by parameters on compact sub-domains in } U \quad \text{and} \\
(2) & \quad \lim_{z \to \infty} \partial_x^\alpha \partial_y^\beta \partial_z^\omega (F(z,y)N(x,z,y)) = 0
\end{align*}
\]

the limit converges uniformly by \( x, y \in U \setminus V \) for some compact subset \( V \) in \( U \) and for each \( |\alpha| + |\beta| + |\omega| \leq m \), where \( 1 \leq m = \max\{\deg(L_s) : s = 1, \ldots, k_0\} \) and

\[
(3) \quad \text{the operator } (I - A_x E_y) \text{ is invertible, where } F \text{ is in the domain of PDOs } L_1, \ldots, L_{k_0}, \quad F(x,y) \in \text{Mat}_{n \times n}(\mathcal{A}_r) \quad \text{and} \quad K(x,y) \in \text{Mat}_{n \times n}(\mathcal{A}_r), \quad n \in \mathbb{N}.
\]

Then there exists a solution \( K \) of PDEs or PIDEs 2(12) such that \( K \) is given by Formulas 1(1, 2), 2(1) and either 2(5) or 2(6).

Proof. The anti-derivative operator \( g \mapsto \sigma \int_{a x}^z g(z)dz \) is compact from \( C^0(V, \mathcal{A}_r) \) into \( C^0(V, \mathcal{A}_r) \) for a compact domain \( V \) in \( \mathcal{A}_r \), where \( C^0(V, \mathcal{A}_r) \) is the Banach space over \( \mathcal{A}_r \) of all continuous functions \( g : V \to \mathcal{A}_r \) supplied with the supremum norm \( \|g\| := \sup_{x \in V} |g(x)| \), \( a x \) is a marked point in \( V \), \( x \in V \). A function \( F \) satisfying the system of \( \mathbb{R} \) linear PDEs 2(5) or 2(6) is
continuous.

Therefore, due to Conditions (1–3) the anti-derivative operator $\sigma \int_2^{\infty} F(z, y) N(x, z, y) \, dz$ is compact. Hence there exists $\delta > 0$ such that the operator $I - A_x E_y$ is invertible when $|p| < \delta$, where $p \in \mathbb{R} \setminus \{0\}$. Mention that the operator $T_g$ is strongly left and right $A_r$-linear (see §1), while $S$ is the automorphism of the Cayley-Dickson algebra, that is $S[ab] = S[a]S[b]$ and $S[a + b] = S[a] + S[b]$ for each $a, b \in A_r$.

Since the operator $(I - A_x E_y)$ is invertible and Conditions 2.2(RS) and either 2(13) or 2(14) are satisfied, then Equation 2(12) can be resolved:

$$(4) \sum_k k f(x, y) g(y) = (I - A_x E_y)^{-1} u(x, y),$$

since if $A : X \to X$ is a bounded $\mathcal{R}$ linear operator on a Banach space $X$ with the norm $\|A\| < 1$, then the inverse of $I - A$ exists:

$$(I - A)^{-1} = \sum_{n=0}^{\infty} A^n.$$ Applying Proposition 2.5 and §2 we get the statement of this theorem.

4. **Remark.** If Condition 2.2(RS) is not fulfilled, the corresponding system of PDEs in real components $(A_x E_y)_{j,s}$, $k_f s$ and $k_g s$ can be considered.

5. **Lemma.** Let suppositions of Proposition 2.5 be satisfied and the operator $A_x$ be given by Formula 2(3), let also $E = E_y$ may be depending on the parameter $y \in U$ and let $N(x, z, y) = E_y K(x, z)$ (see Formulas 1(1–4)). Suppose that $F(x, y) \in Mat_{n \times n}(\mathbb{R})$ and $K(x, z) \in Mat_{n \times n}(A_r)$ for each $x, y, z \in U$, where $r \geq 2$. Then

$$(1) A_m(F, E_y K)(x, y) = (I - A_x E_y) \hat{A}_m(K, E_y K)(x, y) + P_m(K, E_y K)(x, y),$$

$$(2) B_m(F, E_y K)(x, y) = (I - A_x E_y) \hat{B}_m(K, E_y K)(x, y) + Q_m(K, E_y K)(x, y),$$

where

$$(3) \hat{A}_{m,x,y}(K(z, y), E_y K(x, z))\big|_{z=x} = \hat{A}_m(K, E_y K)(x, y) = - \sum_{j=0}^{m-1} \sigma_j^1 K_{1,m-j-1}(x, y)$$

$$+ p \sum_{j=1}^{m-1} \sum_{j_1=0}^{j-1} \sigma_j^2 K_{2,m-j-1,j-j_1-1}(x, y)$$

$$+ p^2 \sum_{j=1}^{m-1} \sum_{j_1=1}^{j-1} \sum_{j_2=0}^{j_1-1} \sigma_j^{j_1} K_{3,m-j-1,j-j_1-1,j-j_2-1}(x, y) + \ldots + p^{m-2} K_{m-1,0,\ldots,0}(x, y),$$

$$(4) \hat{B}_m(K(z, y), E_y K(x, z)) = \sum_{j=1}^{m} (-1)^j \{ \frac{1}{2} \sigma_j^2 [K(z, y) \sigma_j^{j-1} (E_y K(w, v))]$$

$$+ p \hat{A}_{m-j,z,y}(K(z, y), E_y K(z, x)(\sigma_j^{j-1} E_y K(w, v)))) \big|_{v=z,w=x},$$
(5) \(\hat{B}_m(K, E_y K)(x, y) := \hat{B}_m(K(z, y), E_y K(x, z))|_{z=x}\),
(6) \(\hat{A}_1(K, E_y K)(x, y) = -K(z, y)(E_y K(x, z))|_{z=x}\),
(7) \(\hat{B}_1(K(z, y), E_y K(x, z)) = -K(z, y)(E_y K(x, z))\)

for each \(m \geq 2\) in (3, 4), where \(\sum_{j=1}^m a_j := 0\) for all \(l > m\),
(8) \(K_{1,j}(x, y|z) := K(x, y)\sigma_x^{j-1}(E_y K(x, z))\),
(9) \(K_{m,1,...,m}(x, y|z) := K(x, y)\sigma_x^{m-1}E_y K_{m-1,1,...,1}(x, z)\),
(10) \(K_{m,1,...,m}(x, y) := K_{m,1,...,m}(x, y|z)|_{z=x}\),
(11) \(P_m(K(z, y), E_y K(x, z))|_{z=x} = P_m(K, E_y K)(x, y) := A_x\{\sum_{j=1}^{m-1}[\sigma_x^{j-1}, E_y]K_{1,m-j-1}(x, y) + p\sum_{j=1}^{m-1}\sum_{j=1}^{j-1}[\sigma_x^{j-1}, E_y]K_{2,m-j-1,j-1-1}(x, y) + ... + p^{m-3}\sum_{j=1}^{m-1}\sum_{j=1}^{j-1}\sum_{j=1}^{j-1}[\sigma_x^{j-1}, E_y]K_{m-2,m-j-1,j-1-1,...,m-j-3-1}(x, y)\}\),
(12) \(Q_m(K, E K)(x, y|z) := \sum_{j=1}^{m-1}(-1)^jP_m-j(K, E_y K(x, z), E_y)(\sigma_x^{j-1}E_y K(w, v))|_{w=v,z=z, w=x}\),
(13) \(Q_m(K, E K)(x, y) := Q_m(K, E K)(x, y|z)|_{z=x}\).

**Proof.** Formulas (6) and (7) follow immediately from that of 2.2(2, 3) and 2.5(6, 7). Write \(A_m\) for each \(m \geq 2\) in the form:

\[
(14) \quad A_m(F, E_y K)(x, y) = -\sum_{j=1}^{m} \sigma_x^{j-1}\{[2\sigma_x^{m-j} F(z, y)(E_y K(x, z))]|_{z=x}\},
\]

where \(\sigma^0 = I\). Using that \(F(z, y) = (I - A_x E_y)K(z, y)\) and \(F(z, y) \in Mat_{n \times n}(R)\) we get from (14):

\[
(15) \quad A_m(F, E_y K)(x, y) = -\sum_{j=1}^{m} \sigma_x^{j-1}\{(I - A_x E_y)K(x, y)(\sigma_x^{m-j} E_y K(x, z))]|_{z=x}\}.
\]

In virtue of Proposition 2.5 we deduce from (12) that

\[
(16) \quad A_m(F, E_y K)(x, y) = -(I - A_x E_y)\{\sum_{j=1}^{m} \sigma_x^{j-1}K_{1,m-j}(x, y)\} + p\sum_{j=2}^{m} A_{j-1}(F(z, y), E_y K_{1,m-j}(z, z))|_{z=x} + A_x\sum_{j=1}^{m-1} [\sigma_x^{j}, E_y]K_{1,m-j-1}(x, y)
\]

\[
= -(I - A_x E_y)\{\sum_{j=0}^{m-1} \sigma_x^{j}K_{1,m-j-1}(x, y) + p\sum_{j=1}^{m-1}\sum_{j=1}^{j-1} \sigma_x^{j}K_{2,m-j-1,j-j-1}(x, y)\}\)
\]

\[
+ A_x\sum_{j=1}^{m-1} [\sigma_x^{j}, E_y]K_{1,m-j-1}(x, y) + pA_x\sum_{j=1}^{m-1}\sum_{j=1}^{j-1} [\sigma_x^{j}, E_y]K_{2,m-j-1,j-j-1}(x, y) + p^2\sum_{j=1}^{m-1}\sum_{j=1}^{j-1} A_{j-1}(F(z, y), E_y K_{2,m-j-1,j-j-1}(z, x))|_{z=x} = ...,
\]

since \([\sigma_x^{j}, E_y] + E_y \sigma_x^{j} = \sigma_x^{j} E_y\) for \(j \geq 1\),
\[ A_x E_y K(x, y) = p \sigma \int_z^\infty F(z, y) E_y K(x, z) dz. \] Iterating relations (13) we infer by induction Formulas (1, 3, 11). Then we have

(17) \[ B_m(F(z, y), E_y K(x, z)) = \sum_{j=1}^{m} (-1)^j 1 \sigma_z^{m-j} 2 \sigma_z^{j-1} F(z, y)(E_y K(x, z)) \]

and for \( F(z, y) \in Mat_{n \times n}(R) \) for each \( z, y \in U \) this reduces to:

(18) \[ B_m(F(z, y), E_y K(x, z)) = \sum_{j=1}^{m} (-1)^j 1 \sigma_z^{m-j}(I - A_x E_y) K(z, y)(\sigma_z^{j-1}(E_y K(w, v)))|_{v=z, w=x}. \]

Therefore, in view of Proposition 2.5 and Formula (1) the identity

(19) \[ B_m(F(z, y), E_y K(x, z)) = (I - A_x E_y) \{ \sum_{j=1}^{m} \{ 1 \sigma_z^{m-j} K(z, y)(\sigma_z^{j-1}(E_y K(w, v))) \} \}|_{z=v=x, w=x} \]

\[ + \sum_{j=1}^{m} (-1)^j P_{m-j}(K(z, y), E_y K(z, y)(\sigma_z^{j-1}(E_y K(w, v)))|_{v=z, w=x} \]

is valid, since

\[ A_x E_y K(z, y)\{ \sigma_z^{j-1}(E_y K(w, v)) \}|_{v=z, w=x} = \]

\[ p \sigma \int_z^\infty F(\eta, y)\{ E_y K(z, y)\{ \sigma_z^{j-1}(E_y K(w, v)) \} \}d\eta|_{v=z, w=x}. \]

6. Proposition. Suppose that

(1) a PDO \( L_j \) is a polynomial \( \Omega_j(\sigma_x, \sigma_y) \) of \( \sigma_x \) and \( \sigma_y \) for each \( j = 1, \ldots, k_0 \),
coefficients of \( \Omega_j \) are real and Condition 1(3) is fulfilled for all \( j \);

(2) \( L_{s,x,y} K(x, y) - (A_x E_y L_{s,x,y}) K(x, y) =: R_s(K)(x, y) \) for each \( s = 1, \ldots, k_1 \);

\( F(x, y) \) is in \( Mat_{n \times n}(R) \) and \( K(x, y) \in Mat_{n \times n}(A_r) \) for each \( x, y \in U \) (see 2(3)), \( \sigma \) and \( A \) are over the Cayley-Dickson algebra \( A_r \), \( 2 \leq r \), \( n \in N \),
\( 1 \leq k_1 \leq k_0 \);

(3) \( L_{j,x,y} F(x, y) = 0 \) for every \( x, y \in U \) and \( j = 1, \ldots, k_0 \).

Then there exists a polynomial \( M_s \) of \( K, E, \sigma \) and \( A \) such that

(4) \( R_s(K)(x, y) = (I - A_x E_y) M_s(K)(x, y) \) for each \( s = 1, \ldots, k_1 \).

Proof. Proposition 2.5 and Corollary 2.6 imply that \( R_s(K)(x, y) \) can be expressed as a polynomial of \( A_m(F, EK), B_m(F, EK), \sigma \) and \( AEK \), where \( m \in N \).
Take an algebra $\mathcal{B}$ over the real field generated by the operators $\sigma, A, E$ and $I$:

$$\mathcal{B} = \text{alg}_R(\sigma_x, \sigma_y, \sigma_z, A_x, A_y, A_z, E_y(x, z) \forall x, y, z \in U; I),$$

where $I$ denotes the unit operator. In view of Proposition 2.1.1 [3], the algebra $\mathcal{B}$ is associative, since $F(z, y)$ is in $\text{Mat}_{n \times n}(\mathbb{R})$ for each $z, y \in U$, the algebra $\text{Mat}_{n \times n}(\mathbb{R})$ is associative, also $E$ is given by $1(2, 4)$. Therefore, there exists the Lie algebra $L(\mathcal{B})$ generated from $\mathcal{B}$ with the help of commutators $[H, G] := HG - GH$ of elements $H, G \in \mathcal{B}$ (see also about abstract algebras of operators and their Lie algebras in [43]). Then $Y_s := [L_s, E]L(\mathcal{B})$ is the (two-sided) ideal in $L(\mathcal{B})$ and hence there exist the quotient algebra $L_s := L(\mathcal{B})/Y_s$ and the quotient morphism $\pi_s : L(\mathcal{B}) \to L_s$.

Next consider the universal enveloping algebra $\mathcal{U}$ of the Lie algebra $L(\mathcal{B})$. In virtue of Proposition 2.1.1 [3] there exists a unique homomorphism $\tau$ from $\mathcal{U}$ into $\mathcal{B}$. The algebra $C^\infty(U, \text{Mat}_{n \times n}(\mathcal{A}_r))$ over the real field also has the structure of the left module of the operator ring $\mathcal{B}$ and hence of $L(\mathcal{B})$ and $\mathcal{U}$ as well, where $C^\infty(U, \text{Mat}_{n \times n}(\mathcal{A}_r))$ denotes the algebra of all infinitely differentiable functions from $U_R$ into $\text{Mat}_{n \times n}(\mathcal{A}_r)$ (see §1). Since $C^\infty(U, \text{Mat}_{n \times n}(\mathcal{A}_r))$ is dense in $C^l(U, \text{Mat}_{n \times n}(\mathcal{A}_r))$, then it is sufficient to consider $C^\infty(U, \text{Mat}_{n \times n}(\mathcal{A}_r))$, where $l = \max_{s=1,...,k_0} \text{ord}(L_s)$. On the other hand, $[L_s, E]\mathcal{U} =: \mathcal{U}_s$ is the (two-sided) ideal in $\mathcal{U}$.

Let $\mathcal{P}(x, y)$ denote the $\mathbb{R}$-linear algebra generated by sums and products of all terms $QP$ so that $P$ are polynomials of functions $K \in C^\infty(U, \text{Mat}_{n \times n}(\mathcal{A}_r))$ and $Q$ are acting on them polynomials of operators $\sigma, A, E$ (or $B, S, T_g$ instead of $E$, since $E = BST_g$), where coefficients of $P$ and $Q$ are chosen to be real, since coefficients of each polynomial $\Omega_j$ are real. Certainly the equality $\alpha T = T \alpha$ is valid for each $T \in \text{Mat}_{n \times n}(\mathbb{R})$ and $\alpha \in \mathcal{A}_r$, since $\mathbb{R}$ is the center of the Cayley-Dickson algebra $\mathcal{A}_r$, $2 \leq r$ and $(\alpha T)_{i,j} = \alpha T_{i,j} = T_{i,j} \alpha = (T \alpha)_{i,j}$ for each $(i, j)$ matrix element $\alpha T_j$ of $\alpha T$.

The polynomial $R_s(K)(x, y)$ is calculated with the help of Conditions (3), where $s = 1, ..., k_1$ (see also §2). From Formula (2) it follows that the polynomial $R_s(K)(x, y)$ belongs to $(1 - A_x E_y)\mathcal{P}(x, y) + \sum_{j=1}^{k_0} \lbrace \mathcal{U}_j C^\infty(U, \text{Mat}_{n \times n}(\mathcal{A}_r)) \rbrace$ for each $s = 1, ..., k_0$. Applying the quotient mapping $\pi_j$ for all $j = 1, ..., k_0$ and using Proposition 2.3.3 [3] we get Formula (4), since $\pi_j(\mathcal{U}_j) = 0$ and
7. **Example.** Take two partial differential operators

1. \( L_1 = L_{1,x,y} := \sum_l a_l ((-\sigma_x)^l - \sigma_y^l) \) and
2. \( L_2 = L_{2,x,y} := \sum_l a_l (\sigma_x^l - (-\sigma_y)^l) , \)

where \( a_l \in \mathbb{R} \) for each \( l \) when \( \sigma \) is over \( \mathcal{A}_r \) with \( r \geq 2 \), the sum is finite or infinite, \( l \in \mathbb{N} \). The functions \( F(x,y) \) and \( K(x,y) \) of \( \mathcal{A}_r \) variables \( x,y \in U \) have values in \( \text{Mat}_{n \times n}(\mathbb{R}) \) and \( \text{Mat}_{n \times n}(\mathcal{A}_r) \) respectively, where \( n \geq 1, r \geq 2 \). A domain \( U \) in \( \mathcal{A}_r \) satisfies conditions 2.1(\( D_1, D_2 \)) with \( \infty \in U \). On a function \( F(x,y) \) are imposed two conditions:

1. \( L_{1,x,y} F(x,y) = 0 \) and
2. \( L_{2,x,y} F(x,y) = 0. \)

Suppose that conditions of Proposition 2.5 are fulfilled and

1. \( K(x,y) = F(x,y) + p_\sigma \int_x^\infty F(z,y)N(x,z)dz \),

where \( p \) is a non-zero real parameter, \( N(x,z) = EK(x,z) \) for each \( x,z \in U \), while \( E \) is a bounded right \( \mathcal{A}_r \)-linear operator satisfying Conditions 1(\( 2-4 \)) and either 2(13) or 2(14).

Condition (3) is equivalent to

1. \( \sum_l a_l (-\sigma_x)^l F(x,y) = \sum_l a_l \sigma_y^l F(x,y) \) and (4) to
2. \( \sum_l a_l \sigma_x^l F(x,y) = \sum_l a_l (-1)^l \sigma_y^l F(x,y) = 0 \) correspondingly. Acting on both sides of the equality (5) by the operator \( L_1 \) and using (6) and Proposition 2.5 we get

3. \( L_{1,x,y} K(x,y) = p \sum_l a_l ((-\sigma_x)^l - (-1)^l \sigma_y^l) \int_x^\infty F(z,y)N(x,z)dz \)

4. \( = p \sum_l a_l ((-2\sigma_x)^l - 2\sigma_y^l) \int_x^\infty F(z,y)N(x,z)dz \)

5. \( + p \sum_l a_l (-1)^l (A_l(F;N)(x,y) - B_l(F;N)(x,y)) \) and hence
6. \( L_{1,x,y} K(x,y) = p^2 L_{1,x,z} \int_x^\infty F(z,y)N(x,z)dz \)

7. \( + p \sum_l (-1)^l a_l (A_l(F;N)(x,y) - B_l(F;N)(x,y)). \)

Then from (5,7) we infer that

8. \( L_{2,x,y} K(x,y) = p \sum_l a_l (\sigma_x^l - (-1)^l \sigma_y^l) \int_x^\infty F(z,y)N(x,z)dz \)

9. \( = p \sum_l a_l (2\sigma_x^l - (-2\sigma_y^l)) \int_x^\infty F(z,y)N(x,z)dz \)

10. \( + p \sum_l a_l (A_l(F;N)(x,y) - B_l(F;N)(x,y)) \) and consequently,
11. \( L_{2,x,y} K(x,y) = p^2 L_{2,x,z} \int_x^\infty F(z,y)N(x,z)dz \)

12. \( + p \sum_l a_l (A_l(F;N)(x,y) - B_l(F;N)(x,y)). \)

Then Equalities (9,11) imply that
We take into account sufficiently small values of the parameter \( p \), when the operator \( I - A_x E \) is invertible, for example, \( \|A_x E\| < 1 \), where

\[
A_x K(x, y) = p_\sigma \int_x^\infty F(z, y) K(x, z) dz.
\]

In the case \( F \in \text{Mat}_{n \times n}(\mathbb{R}) \) and \( K \in \text{Mat}_{n \times n}(A_r) \) with \( \sigma \) over \( A_r \), from (5, 12), Lemma 5 and Proposition 6 it follows that \( K \) satisfies the nonlinear PDE

\[
(L^\pm_{x,y} K(x, y) - p \sum_l (\pm 1 + (-1)^l) a_l \hat{A}_l(K; EK)(x, y) - B_l(K; EK)(x, y)) = 0,
\]

where \( \hat{A}_l \) and \( \hat{B}_l \) are given by Formulas (3–7), since \( < a, b, c > = 0 \) when particularly \( a \in \text{Mat}_{n \times n}(\mathbb{R}) \), where \( < e, b, c > = (eb)c - e(bc) \) denotes the associator of the Cayley-Dickson matrices \( e, b, c \in \text{Mat}_{n \times n}(A_r) \), also since \( aa = a^2 \) for each \( \alpha \in A_r \). A solution of (13) reduces to linear PDEs and is prescribed by (3–5). Equivalently the function \( K \) satisfies also the PDEs

\[
(L^\pm_{s,x,y} K(x, y) - p \sum_l (-1)^l a_l \hat{A}_l(K; EK)(x, y) - \hat{B}_l(K; EK)(x, y)) = 0
\]

for \( s = 1 \) and \( s = 2 \). Instead of this system it is possible also to consider separately PDOs \( L_1 \) and \( L_2 \) and the corresponding PDEs for \( F \) and \( K \) as well. Thus with the help of Theorem 3 we get the following.

**7.1. Theorem.** Suppose that conditions of Theorem 3 and Example 7 are fulfilled, then a solution of PDE (15) is given by (3, 5), where \( L_1 \) is prescribed by Formula (1), \( s = 1 \).

**8. Example.** Let PDOs be

\[
(1) \quad L_1 = L_{1;x,y} = \sigma_x - \sigma_y,
\]

\[
(2) \quad L_{2,j} = L_{2;j;x,y} = \sum_l (a_l \sigma_x^l + (-1)^l b_l \sigma_y^l),
\]

where \( a_l, b_l \in \mathbb{R} \) for each \( l \) when \( \sigma \) is over \( A_m \) with \( m \geq 2 \), the sum is finite or infinite, \( j = 1 \) or \( j = 2 \). It is also supposed that the functions \( F(x, y) \) and \( K(x, y) \) of \( A_m \) variables \( x, y \in U \) have values in \( \text{Mat}_{n \times n}(\mathbb{R}) \) and \( \text{Mat}_{n \times n}(A_m) \) correspondingly, where \( n \geq 1 \), \( m \geq 2 \). A domain \( U \) in \( A_m \) satisfies Conditions 2.1(D1, D2) with \( \infty \in U \). Suppose that

\[
(3) \quad L_{1;x,y} F(x, y) = 0 \quad \text{and consider the integral relation:}
\]

\[
(4) \quad K(x, y) = F(x, y) + p_\sigma \int_x^\infty F(z, y) N(x, z) dz,
\]

where \( N \) and \( K \) are related by Formulas 1(1, 2).
Using condition (3) we can write \(F(x, y) = F\left(\frac{x+y}{2}\right)\). Therefore we deduce that

\begin{align}
(5) \quad & L_{2,j; x,y}K(x, y) = L_{2,j; x,y}F\left(\frac{x+y}{2}\right) + pL_{2,j; x,y} \int_x^\infty F\left(\frac{z+y}{2}\right)N(x, z)dz \\
& = L_{2,j; x,y}F\left(\frac{x+y}{2}\right) + p \sum_l (a_l \sigma_x^l + (-1)^{j+1}b_l \sigma_y^l) \int_x^\infty F\left(\frac{z+y}{2}\right)N(x, z)dz \\
& = L_{2,j; x,y}F\left(\frac{x+y}{2}\right) + p \sum_l \{\left[ (a_l \sigma_x^l + (-1)^{j+1}b_l \sigma_y^l) \int_x^\infty F\left(\frac{z+y}{2}\right)N(x, z)dz \right] \\
& + a_l A_l(F; N)(x, y) + (-1)^{j+1}b_l \hat{B}_l(F; N)(x, y)\}. \\
\end{align}

Imposing the condition

\( (6) \quad L_{2,x,y}F(x, y) = 0, \)

we get the nonlinear PDE with the help of Lemma 5 and Proposition 6

\( (7) \quad L_{2,1; x,y} = L_{2,1; x,y} + L_{2,2; x,y} = \sum_l (2a_l \sigma_x^l + (1 + (-1)^j)b_l \sigma_y^l), \)

where PDOs \(\sigma_x^l + \sigma_y^l\) instead of \(\sigma_x - \sigma_y\) and the corresponding changes in the PDO \(L_2\). Then Theorem 3 implies the following.

8.1. Theorem. Let conditions of Theorem 3 and Example 8 be fulfilled, then a solution of PDE (8) is described by Formulas (3, 4, 6), where PDOs \(L_1\) and \(L_2\) are provided by expressions (1, 7).

9. Example. Consider now the generalization of PDOs from §5 with \(k \geq 2\):

\( (1) \quad L_1 = L_{1; x,y} = \sigma_x^k - \sigma_y^k, \)

\( (2) \quad L_{2,j} = L_{2,j; x,y} = \sum_l (a_l \sigma_x^{kl} + (-1)^{j+1}b_l \sigma_y^{kl}), \)

where \(k\) is a natural number, \(a_l, b_l \in \mathbb{R}\) for each \(l\) when the Dirac type operator \(\sigma\) is over \(\mathcal{A}_m\) with \(m \geq 2\), the sum is finite or infinite, \(j = 1\) or \(j = 2\). Other suppositions are as in §8. Let
(3) \( L_{1;x,y}F(x, y) = 0 \) and 
(4) \( K(x, y) = F(x, y) + p \sigma \int_{x}^{\infty} F(z, y) N(x, z) dz \),
where \( N \) is expressed through \( K \) by 1(1, 2).

Then we deduce the identities:

(5) \[ L_{2;j;x,y}K(x, y) = L_{2;j;x,y}F(x, y) + pL_{2;j;x,y}\sigma \int_{x}^{\infty} F(z, y) N(x, z) dz \]

\[ = L_{2;j;x,y}F(x, y) + p \sum_{l} (a_l \sigma^{kl}_{x} + (-1)^{jkl} b_l \sigma^{kl}_{y}) \sigma \int_{x}^{\infty} F(z, y) N(x, z) dz \]

\[ = L_{2;j;x,y}F(x, y) + p \sum_{l} \left\{ (a_l \sigma^{kl}_{x} + (-1)^{(j+1)kl} b_l \sigma^{kl}_{y}) \sigma \int_{x}^{\infty} F(z, y) N(x, z) dz \right\} \]

\[ + a_l \hat{A}_{kl}(F; N)(x, y) + (-1)^{jkl} b_l \hat{B}_{kl}(F; N)(x, y) \}.

From the condition

(6) \( L_{2;x,y}F(x, y) = 0 \) with the PDO
(7) \( L_{2;x,y} = L_{2;1;x,y} + L_{2;2;x,y} = \sum_{l} (2a_l \sigma^{kl}_{x} + (1 + (-1)^{kl}) b_l \sigma^{kl}_{y}) \)

we infer that a function \( K \) is a solution of the nonlinear PDE of the form:

(8) \[ L_{2;x,y}K(x, y) - p \sum_{l} \left\{ 2a_l \hat{A}_{kl}(K; N)(x, y) + (1 + (-1)^{kl}) b_l \hat{B}_{kl}(K; N)(x, y) \right\} = 0. \]

PDE (8) can be resolved with the help of the linear problem (1, 3, 6, 7) and the
integral operator (4), where terms \( \hat{A}_{l} \) and \( \hat{B}_{l} \) are given by Formulas 5(3 – 7),
a function \( N \) satisfies conditions 1(1 – 4) and either 2(13) or 2(14). Thus
due to Theorem 3 we have proved the following.

9.1. Theorem. Let conditions of Theorem 3 and Example 9 be satisfied,
then a solution of PDE (8) is provided by Formulas (3, 4, 6), where PDOs \( L_{1} \)
and \( L_{2} \) are given by (1, 7).

10. Example. Let now the pair of PDOs be

(1) \( L_{1} = L_{1;x,y} = \sigma^{k}_{x} + \sigma^{l}_{y} \);

(2) \( L_{2;j} = L_{2;j;x,y} = \sum_{l} (a_l \sigma^{kl}_{x} + (-1)^{j(k+1)} b_l \sigma^{kl}_{y}) \),

where \( k \) is a natural number, \( k \geq 2, K, F, U \) and \( \sigma \) have the same meaning
as in §1 and 2, \( a_l, b_l \in \mathbb{R} \) for each \( l \) when \( \sigma \) are over \( A_{m} \) with \( m \geq 2, \)
\( F \in Mat_{n \times n}(\mathbb{R}) \) and \( K \in Mat_{n \times n}(\mathbb{A}_{r}) \), the sum is finite or infinite, \( j = 1 \) or
\( j = 2 \). Imposing the conditions

(3) \( L_{1;x,y}F(x, y) = 0 \) and 
(4) \( L_{2;x,y}F(x, y) = 0 \) with
(5) \( L_{2;x,y} = L_{2;1;x,y} + L_{2;2;x,y} = \sum_l (2a_l \sigma_x^{kl} + (1 + (-1)^l(k+1))b_l \sigma_y^{kl}) \) and considering the integral transform

(6) \( K(x, y) = F(x, y) + p_\sigma \int_x^\infty F(z, y)N(x, z)dz, \)

where \( N \) is related with \( K \) by expressions (1, 2), we infer that

(7) \( L_{2;j;x,y}K(x, y) = L_{2;j;x,y}F(x, y) + pL_{2;j;x,y}\sigma \int_x^\infty F(z, y)N(x, z)dz \)

\( = L_{2;j;x,y}F(x, y) + p \sum_l ((a_l 2\sigma_x^{kl} + (1 + (-1)^l(l+1)+jkl)B_l)F; N)(x, y) + (-1)^l(j+1)+jklb_lB_{kl}(F; N)(x, y). \)

Thus in virtue of Lemma 5 and Proposition 6 a function \( K \) satisfies the nonlinear PDE:

(8) \( L_{2;x,y}K(x, y) - p \sum_l (2a_l \hat{A}_{kl}(K; N)(x, y) + ((-1)^l + (-1)^l)b_l \hat{B}_{kl}(K; N)(x, y)) = 0. \)

The solution of the latter PDE reduces to the linear problem (1, 3 – 5) and using the integral operator (6), where terms \( \hat{A}_l \) and \( \hat{B}_l \) are given by Formulas 5(3 – 7), a function \( N \) is of the form 1(1 – 4) and either 2(13) or 2(14) is fulfilled also. It is also possible to change the notation \( b_l \mapsto (-1)^lb_l \) in this example or \( b_l \mapsto -b_l \) in §5. Examples 7–10 correspond to different types of PDEs such as elliptic, hyperbolic and mixed types. In view of Theorem 3 this implies the following.

10.1. **Theorem.** If conditions of Theorem 3 and Example 10 are satisfied, then a solution of PDE (8) is given by Formulas (3, 4, 6), where PDOs \( L_1 \) and \( L_2 \) are as in (1, 5).

10.2. **Remark.** Transformation groups related with the quaternion skew field are described in [38]. Automorphisms and derivations of the quaternion skew field and the octonion algebra are contained in [43], that of Lie algebras and groups in [3].

11. **Example.** Consider now the term \( N \) in the integral operator

(1) \( f(y)(g(x)K(x, y)) = F(x, y) + p_\sigma \int_x^\infty F(z, y)[f(z)(g(x)E K(x, z))]dz \)

with multiplier functions \( f(z) \) and \( g(x) \) satisfying definite conditions (see
below, where \( F, K \) and \( N(x, z) = f(z)(g(x)EK(x, z)) \), \( p \) have the meaning of the preceding paragraphs, \( E \) is an operator fulfilling Conditions 1(2, 3) and either 2(13) or 2(14) also. Suppose that

\[
(2) \quad \sigma_z f(z) = \sum_j i_j \psi_j \partial f(z)/\partial z(j) = \lambda f(z) \quad \text{and} \\
(3) \quad \sigma_z g(x) = \sum_j i_j \psi_j \partial g(x)/\partial x(z(j)) = \mu g(x), \quad \text{where} \\
\lambda = \sum_j i_j \psi_j \lambda_j \quad \text{and} \\
\mu = \sum_j i_j \psi_j \mu_j \quad \text{with} \quad \lambda_j, \mu_j \in \mathbb{R} \quad \text{for each} \quad j. \quad \text{We choose the functions} \\
f(z) = C_1 \exp(\sum_j z_j \lambda_j) \quad \text{and} \quad g(x) = C_2 \exp(\sum_j x_j \mu_j) \quad \text{satisfying PDEs} \ (2) \quad \text{and} \quad (3) \ \text{correspondingly, where} \quad C_1 \quad \text{and} \quad C_2 \quad \text{are real non-zero constants,} \ x, z_j \in \mathbb{R},
\]
\( x = \sum_j i_j x_j, \ x, z \in U. \) The first PDO we take as

\[
(4) \quad L_1 = L_{1,x,y} = \sigma_x^k + s \sigma_y^k,
\]
where \( k \geq 1, \) either \( s = 1 \) or \( s = -1. \) Then the condition

\[
(5) \quad L_{1,x,y} F(x, y) = 0 \quad \text{is equivalent to} \\
(6) \quad \sigma_x^k F(x, y) = -s \sigma_y^k F(x, y).
\]
Therefore we get from Proposition 2.5 with \( N(x, z) = f(z)(g(x)EK(x, z)) \) that

\[
(7) \quad \sigma_y^{kl} \int_x^\infty F(z, y)[f(z)(g(x)EK(x, z))]dz = (-s)^1 1 \sigma_z^k \int_x^\infty F(z, y)[f(z)(g(x)EK(x, z))]dz
\]
\[
= s'(-1)^{(k+1)}[4 \sigma_z + 4 \lambda]^{kl} \int_x^\infty F(z, y)[f(z)(g(x)EK(x, z))]dz
\]
\[
+ (-s)^l B_{kl}(F(z, y); [f(z)(g(x)EK(x, z))]|_{z=x},
\]
where \( F \) stands on the first place, \( f \) on the second, \( g \) on the third and \((EK)\) on the fourth place. Then from (2) and (7) it follows that

\[
(8) \quad \sigma_y^{kl} \int_x^\infty F(z, y)[f(z)(g(x)EK(x, z))]dz =
\]
\[
s'(-1)^{(k+1)}[4 \sigma_z + \lambda]^{kl} \int_x^\infty F(z, y)[f(z)(g(x)EK(x, z))]dz
\]
\[
+ (-s)^l B_{kl}(F(z, y); [f(z)(g(x)EK(x, z))]|_{z=x}.
\]
Evaluation of the other integral with the help of Proposition 2.5 and Formula (3) leads to:

\[
(9) \quad \sigma_x^{kl} \int_x^\infty F(z, y)[f(z)(g(x)EK(x, z))]dz =
\]
\[
[3 \sigma_x + 4 \lambda]^{kl} \int_x^\infty F(z, y)[f(z)(g(x)EK(x, z))]dz
\]
Thus in this particular case PDEs of Examples 7-10 change. For example, PDE 10(8) takes the form:

\[
\sum_l (2a_l (\sigma_x + \mu)^{kl} + (1 + (-1)^{(l+1)})b_l (\sigma_y + \lambda)^{kl}) K(x,y)
\]

\[- \frac{p}{f(y)g(x)} \sum_l \{2a_l \hat{A}_kl([f(y)(g(z)K(z,y))]; [f(z)(g(x)EK(x,z))]|_{z=x})
\]

\[+ (1 + (-1)^{(l+1)})b_l \hat{B}_kl([f(y)(g(z)K(z,y))]; [f(z)(g(x)EK(x,z))]|_{z=x})\} = 0,
\]

when \(F \in Mat_{n \times n}(\mathbb{R})\) and \(K \in Mat_{n \times n}(\mathcal{A}_r)\) with \(2 \leq r\) and \(E\) is the right linear operator over \(\mathcal{A}_r\), since the operator \(E\) satisfies Conditions 1(2, 3) and either 2(13) or 2(14); the functions \(f(y)\) and \(g(x)\) have values in \(\mathbb{R} \setminus \{0\}\) for each \(x, y \in U\), whilst \(\mathbb{R}\) is the center of the Cayley-Dickson algebra.

Analogous changes will be in Examples 7-9.

**12. Example.** Let the non-commutative integral operator be

(1) \(K(x, y) = F(x, y) + B_y K(x, y)\) with

(2) \(B_y K(x, y) = p \int_x^\infty F(z, y) N(x, z, y) dz\),

where \(F, K, N(x, z, y)\) are as in Proposition 2.5 and Theorem 3.3, \(F \in Mat_{n \times n}(\mathbb{R})\), \(K\) and \(N\) are in \(Mat_{n \times n}(\mathcal{A}_m)\), while \(p\) is a sufficiently small non-zero real parameter, \(N\) is an operator function right linear in \(K\) as in §1. Put

(3) \(N(x, z, y) = E_y K(x, z)\) for every \(x, y\) and \(z\) in \(U\),

where \([E_y, L_j] = 0\) for each \(j = 1, ..., k_0\) and \(y \in U\), \(E = E_y\) may depend on the variable \(y \in U\) also, \(E\) is an operator satisfying Conditions 1(2, 3) and either 2(13) or 2(14), \(m \geq 2\). Choose two PDO

(4) \(L_1 = L_{1,x,y} = \sigma_x + \sigma_y\),

(5) \(L_2 = L_{2,x,y} = (\sum_l a_l \sigma_x^l) + s \sigma_y\),

where \(s \in \mathbb{R}\), \(s\) is a non-zero real constant, \(a_l \in \mathbb{R}\) for each \(l\) when the Dirac type operator \(\sigma\) is over \(\mathcal{A}_m\) with \(m \geq 2\). We impose the conditions:
(6) \( L_{j,x,y}F(x,y) = 0 \) for \( j = 1 \) and \( j = 2 \), for all \( x,y \in U \). Then it is possible to write \( F(z,y) = F\left(\frac{x-y}{2}\right) \). Applying the PDO \( L_2 \) to both sides of (1) and using (2), Proposition 2.5 and Conditions (3–6) we deduce that

\[
L_{2,x,y}K(x,y) = p\left\{ \sum_l a_l \sigma_x^l - s^1 \sigma_z + s^2 \sigma_y \right\} \int_x^\infty F(z,y)N(x,z,y)dz
\]

\[
= [p\left\{ \sum_l a_l \sigma_x^l + s^2 \sigma_z + s^2 \sigma_y \right\} \int_x^\infty F(z,y)N(x,z,y)dz]
\]

\[
+ [p \sum_l a_l A_l(F;N)(x,y)] - psB_1(F;N)(x,y).
\]

For sufficiently small non-zero real values of \( p \) the operator \( I - B_x \) is invertible and hence Equality (7), Lemma 5 and Proposition 6 imply that \( K \) satisfies the nonlinear partial integro-differential equation:

\[
L_{2,x,y}K(x,y) - \left[ p \sum_l a_l A_l(K;N)(x,y) \right] - psK(x,y)N(x,x,y)
\]

\[
- ps \int_x^\infty K(z,y)\sigma_y N(x,z,y)dz = 0.
\]

Using Theorem 3 we deduce the following.

12.1. Theorem. A solution of PIDE (8) is described by (1, 2, 3, 6), where PDOs \( L_1 \) and \( L_2 \) are given by (4, 5), provided that conditions of Theorem 3 and Example 12 are satisfied.

13. Example. Suppose that functions \( K \) and \( F \) are related by Equations 12(1, 2) and take two PDOs

\[
L_{1,x,y} = \sigma_x - \sigma_y \quad \text{and} \quad L_{2,x,y} = \Delta_x + s\Delta_y,
\]

where the coefficient \( \psi_0 \) is null in \( \sigma \) and hence the Laplace operator is expressed as \( \Delta = -\sigma^2 \), while \( s \in \mathbb{R} \setminus \{0\} \). Now we take a function \( N \) in the form

\[
N(x, z, y) = EK(x, ay + bz),
\]

where \( a \) and \( b \) real parameters to be calculated below such that \( a^2 + b^2 > 0 \), \( b \) is non-zero. Then from the conditions

\[
L_{j,x,y}F(x,y) = 0 \quad \text{for} \ j = 1 \ \text{and} \ j = 2,
\]

Proposition 2.5 and Corollary 2.6 it follows that

\[
L_{2,x,y}K(x,y) = -p(\sigma_x^2 + s\sigma_y^2) \int_x^\infty F(z,y)EK(x,ay+bz)dz
\]
\[ p(2\Delta_x - s(1^2 + 2^{2y}))\frac{\partial}{\partial z} \int_x^\infty F(z, y)EK(x, ay+zb)dz - pA_2(F(z, y), EK(x, ay+zb)) \]  
and
\[ (1^2 + 2^{2y})\frac{\partial}{\partial z} \int_x^\infty F(z, y)EK(x, ay+zb)dz = \]
\[ 1^2 + 2^{2y} \int_x^\infty F(z, y)EK(x, ay+zb)dz \]
\[ = [ab^{-1}(1^2 + 2^{2y} + (1-ab^{-1})1^2 + (a^2b^2 - ab^{-1})2^{2y})\frac{\partial}{\partial z} \int_x^\infty F(z, y)EK(x, ay+zb)dz = \]
\[ = (1-ab^{-1})2^{2y}\int_x^\infty F(z, y)EK(x, ay+zb)dz \]
\[ -ab^{-1}[\sigma_z(F(z, y)EK(x, ay+zb))]_{z=x} + (1-ab^{-1})B_2(F, EK)(x, y) \]
\[ = p^{-1}(1-ab^{-1})2^{2y}\int_x^\infty F(z, y)EK(x, ay+zb)dz \]
\[ -ab^{-1}[\sigma_z(F(z, y)EK(x, ay+zb))]_{z=x} + (1-ab^{-1})B_2(F, EK)(x, y), \]

since
\[ \sigma_z^{2y} \int_x^\infty F(z, y)N(x, z, y)dz = \sigma \int_x^\infty \sigma_z^{2y}F(z, y)N(x, z, y)dz \]
\[ = -\sigma_z[\sigma_z(F(z, y)EK(x, ay+zb))]_{z=x}. \]

Then Identities (5, 6) imply that
\[ L_{2,x,y}K(x, y) = B_x[L_{2,x,y}K(x, y)] - pA_2(F(z, y), EK(x, ay+zb)) |_{z=x} \]
\[ +psab^{-1}[\sigma_z(F(z, y)EK(x, ay+zb))]_{z=x} - ps(1-ab^{-1})B_2(F, EK)(x, y) \]
when \((b-a)^2 = 1\) and \(b\) is non-zero, that is either \(a = b + 1\) or \(a = b - 1\). In virtue of Lemma 5 and Proposition 6 this gives the nonlinear PDE for \(K\):
\[ L_{2,x,y}K(x, y) + p\hat{A}_2(K(z, y), EK(x, ay+zb)) |_{z=x} \]
\[ -ab^{-1}ps[\sigma_z(K(z, y)EK(x, ay+zb))]_{z=x} + (1-ab^{-1})ps\hat{B}_2(K(z, y), EK(x, ay+zb)) |_{z=x} = 0. \]

From Theorem 3 we infer the following.

13.1. Theorem. A solution of PDE (8) is given by (3, 4) and 12(1, 2), where PDOs \(L_1\) and \(L_2\) are prescribed by (1, 2), whenever conditions of Theorem 3 and Example 13 are satisfied.

14. Nonlinear PDE with parabolic terms. Let
\[ \partial_t := \sum_{k=1}^v \partial/\partial t_k \]
be the first order PDO, where \(t_1, ..., t_v\) are real variables independent of other
variables \( x, y, z \in U, \ t = (t_1, ..., t_n) \in W, \ W := \{ t \in \mathbb{R}^n : \forall k = 1, ..., n \ 0 \leq t_k < T_k \} \), where \( T_k \) is a constant, \( 0 < T_k \leq \infty \) for each \( k \).

Suppose that

(2) \( F \) and \( K \) are continuously differentiable functions by \( t_k \) for each \( k \) so that \( \sigma \int_x^\infty F(z, y)N(x, z, y)dz \) converges for some \( t \in W \) and

(3) the integrals \( \sigma \int_x^\infty (\partial_t F(z, y))N(x, z, y)dz \) and \( \sigma \int_x^\infty F(z, y)(\partial_t N(x, z, y))dz \)

converge uniformly on \( W \) in the parameter \( t \).

In virtue of the theorem about differentiation of an improper integral by a parameter the equality is valid:

\[
(4) \quad \partial_t \sigma \int_x^\infty F(z, y)N(x, z, y)dz = \sigma \int_x^\infty (\partial_t F(z, y))N(x, z, y)dz + \sigma \int_x^\infty F(z, y)(\partial_t N(x, z, y))dz.
\]

Using (4) the commutator \((I - A_x E_y)((\partial_t + L_s)f) - (\partial_t + L_s)[(I - A_x E_y)f] \) can be calculated, when there is possible to evaluate the commutator \((I - A_x E_y)(L_s f) - L_s[(I - A_x E_y)f] = R_s(f) \) for suitable functions \( f \) and a PDO \( L_s = L_{s, x, y} \) (see also §2).

For solution of nonlinear PDEs or PIDEs also the following will be useful for integral operators of the form \( \sigma \int_x^\infty N(x, z, y)K(x, z)dz \).

14.1. Example. Let a PDO be

(1) \( L_1 = \partial_t + \sum_l a_l(\sigma_x^l + (-1)^{l+1}\sigma_y^l) \) and let

\[
(2) \quad N(x, z, y) = E_y K(x, z),
\]

where \( a_l \in \mathbb{R} \) for all \( l = 0, 1, 2, ..., \) so that conditions 1(2, 3) and 2(1, 5) are fulfilled, \( F \in Mat_{n \times n}(\mathbb{R}), K \in Mat_{n \times n}(\mathcal{A}_r), 2 \leq r \), the first order PDO \( \sigma \) is over the Cayley-Dickson algebra \( \mathcal{A}_r \) (see §§1, 2 and 14). Then

\[
(\partial_t + \sum_l a_l(-1)^{l+1}\sigma_y^l)F(x, y) = -(\sum_l a_l \sigma_x^l)F(x, y)
\]

for all \( x, y \in U \). Therefore we infer from Proposition 2.5 that

\[
(3) \quad L_1 K(x, y) = p(2\partial_t + \sum_l a_l(\sigma_x^l - 1^l\sigma_z^l)) \sigma \int_x^\infty F(z, y)E_y N(x, z)dz
\]

\[
= p(2\partial_t + \sum_l a_l(2\sigma_x^l + (-1)^{l+1}2\sigma_z^l)) \sigma \int_x^\infty F(z, y)E_y N(x, z)dz
\]

\[
+ p \sum_l a_l(A_l(F; N)(x, y) - B_l(F; N)(x, y)).
\]
Hence we deduce a nonlinear PDE

\[(4) \quad L_1K(x, y) - p \sum_l a_l(\hat{A}_l(K;EK)(x, y) - \hat{B}_l(K;EK)(x, y)) = 0\]

according to Lemma 5 and Proposition 6. Its solution reduces to the linear problem 2(2,5). We mention that PDE (4) corresponds to some kinds of Sobolev type nonlinear PDEs.

14.2. Remark. Suppose that \(L\) and \(S\) are PDOs and functions \(f : (a, b) \times U^m \rightarrow A_r\) and \(g : (a, b) \times U^m \rightarrow A_r\) are in the domains of operators \(\exp(tL)\) and \(S\) correspondingly, where \(t\) is a real parameter, \(t \in (a, b), a < b, 2 \leq r\), where PDOs \(L\) and \(S\) are by variables in \(U^m, m \in \mathbb{N}\). If they satisfy the PDE

\[(1) \quad \exp(tL)f(t, x_1, ..., x_m) = Sg(t, x_1, ..., x_m)\]

for all \(t \in (a, b)\) and \(x_1, ..., x_m \in U\), then

\[\frac{\partial \exp(tL)f(t, x_1, ..., x_m)}{\partial t} = \exp(tL)(\frac{\partial}{\partial t} + L)f(t, x_1, ..., x_m)\]

\[= \exp(tL)(\frac{\partial}{\partial t} + L)\exp(-tL)Sg(t, x_1, ..., x_m) = \frac{\partial Sg(t, x_1, ..., x_m)}{\partial t},\]

consequently,

\[(2) \quad \left(\frac{\partial}{\partial t} + L\right)f(t, x_1, ..., x_m) = \exp(-tL)\frac{\partial Sg(t, x_1, ..., x_m)}{\partial t}.\]

The latter also may be helpful for solutions of nonlinear PDEs with parabolic terms.

14.3. Generalized approach. Let \(L_1, ..., L_k\) and \(S_1, ..., S_k\) be PDOs which are polynomials or series of \(\sigma_x\) and \(\sigma_y\) so that

\[(1) \quad [L_j, S_j] = 0\]

for each \(j = 1, ..., k\), where \(x\) and \(y\) are in a domain \(U\) in the Cayley-Dickson algebra \(A_r, 2 \leq r\) (see §2.3). Instead of the conditions \(L_jF = 0\) it is possible to consider more generally

\[(2) \quad L_jF = G_j,\]

where \(G_j\) are some functions known or defined by some relations, while functions \(F, G_j\) and \(K\) may also depend on a parameter \(t \in W\) (see §14) so that \(F \in Mat_{n \times n}(\mathbb{R}), G_j\) for all \(j\) and \(K\) have values in \(Mat_{n \times n}(A_r)\). It is also supposed that \(F\) and \(K\) are related by the integral equation 2(1) and \(N(x, y, z) = E_yK(x, z)\) and Conditions 1(2, 3) are satisfied. In particular, if
(3) $G_j = L_j(I + S_j)K$, then a solution of the linear system of PIDEs

(4) $L_jF(x, y) = L_j(I + S_j)K(x, y)$ and

(5) $(I - A_xE)K(x, y) = F(x, y)$

would also be a solution of nonlinear PIDEs

(6) $S_j L_j K(x, y) + M_j(K) = 0$,

where $M_j$ corresponds to $L_j$ for each $j = 1, ..., k_0$ with $1 \leq k_0 \leq k$ as in §2. Thus this generalizes PIDEs 2(12). Particularly, taking $S_j = \partial_t$ we get that Condition (1) is valid. Therefore, the technique presented in §7-14.3 encompasses some kinds of nonlinear Sobolev type PDEs as well.

15. Proposition. Let

\[
\lim_{z \to \infty} \int_{x}^{\infty} \sigma^{2} \sigma^{1} N(x, z, y)K(x, z) = 0
\]

for each $x, y$ in a domain $U$ satisfying Conditions 2.1(D1, D2) with $\infty \in U$ and every non-negative integers $0 \leq k, s, n \in \mathbb{Z}$ such that $k + s + n \leq m$. Suppose also that $\sigma \int_{x}^{\infty} \sigma^{1} \sigma^{0} [N(x, z, y)K(x, z)]dz$ converges uniformly by parameters $x, y$ on each compact subset $W \subset U \subset \mathbb{A}^{2}$ for each $|\alpha| + |\beta| + |\omega| \leq m$, where $\alpha = (\alpha_0, ..., \alpha_{2r-1})$, $|\alpha| = \alpha_0 + ... + \alpha_{2r-1}$, $\sigma^{\alpha} = \partial^{\alpha}|/\partial x_{0}^{\alpha_0} ... \partial x_{2r-1}^{\alpha_{2r-1}}$, where $N \in C^{m}(U^3, Mat_{n \times n}(A_r))$ and $K \in C^{m}(U^2, Mat_{n \times n}(A_r))$. Then

\[
\int_{x}^{\infty} N(x, z, y)K(x, z)dz = \int_{x}^{\infty} N(x, z, y)K(x, z)dz + \tilde{A}_m(N; K)(x, y),
\]

where

\[
\tilde{A}_m(N; K)(x, y) = - \sum_{j=0}^{m-1} \sigma^{j}_{x} \{[\sigma^{m-j-1} N(x, z, y)K(x, z)]\}_{z=x}
\]

for each $m \geq 1$, $\sigma^{0}_{x} = I$. Moreover,

\[
\int_{x}^{\infty} N(x, z, y)K(x, z)dz = \int_{x}^{\infty} N(x, z, y)K(x, z)dz + \tilde{B}_m(N; K)(x, y),
\]

where $\tilde{B}_m(N; K)(x, y) = \tilde{B}_m(N(x, z, y); K(x, z))|_{z=x}$,

\[
\tilde{B}_m(N(x, z, y); K(x, z)) = \sum_{k=0}^{m-1} (-1)^{k+1} \sigma^{m-k-1}_{x} \sigma^{k}_{z} N(x, z, y)K(x, z).
\]
Proof. For $m = 1$ we infer that

\begin{equation}
(6) \quad \sigma_x \int_x^\infty N(x, z, y)K(x, z)dz =
\end{equation}

\[ (1 \sigma_x + 2 \sigma_x) \sigma \int_x^\infty N(x, z, y)K(x, z)dz - N(x, x, y)K(x, x) \]

and put $\tilde{A}_1(N; K)(x, y) = \tilde{A}_1(N(x, z, y); K(x, z))|_{z=x}$, where

\begin{equation}
(7) \quad \tilde{A}_1(N(x, z, y); K(x, z)) = -N(x, z, y)K(x, z).
\end{equation}

Then we deduce by induction that

\begin{equation}
(8) \quad \sigma_x^m \sigma \int_x^\infty N(x, z, y)K(x, z)dz =
\end{equation}

\[ \sigma_x[(1 \sigma_x + 2 \sigma_x)^m \sigma \int_x^\infty N(x, z, y)K(x, z)dz + \tilde{A}_{m-1}(N(x, z, y); K(x, z))|_{z=x}] =
\end{equation}

\[ (1 \sigma_x + 2 \sigma_x)^m \sigma \int_x^\infty N(x, z, y)K(x, z)dz + \tilde{A}_m(N; K)(x, y) \]

with the convention that the operator $\sigma \int_x^\infty$ stands in the zero position, $N$ in the first and $K$ in the second positions correspondingly, where

\begin{equation}
(9) \quad \tilde{A}_m(N; K)(x, y) = \sigma_x \tilde{A}_{m-1}(N; K)(x, y) - [\sigma_x^{m-1}N(x, z, y)K(x, z)]|_{z=x}
\end{equation}

for each $m \geq 2$. Therefore, by induction we deduce that

\begin{equation}
(10) \quad \tilde{A}_m(N; K)(x, y) = -[\sigma_x^{m-1}N(x, z, y)K(x, z)]|_{z=x} - \sigma_x\{[\sigma_x^{m-2}N(x, z, y)K(x, z)]|_{z=x}\} - ... - \sigma_x^{m-2}\{[\sigma_xN(x, z, y)K(x, z)]|_{z=x}\} - \sigma_x^{m-1}N(x, x, y)K(x, x).
\end{equation}

Then we infer:

\begin{equation}
(11) \quad 1 \sigma_x \int_x^\infty N(x, z, y)K(x, z)dz =
\end{equation}

\[ -2 \sigma_z \int_x^\infty N(x, z, y)K(x, z)dz + \tilde{B}_1(N; K)(x, y), \]

where $\tilde{B}_1(N; K)(x, y) = \tilde{B}_1(N(x, z, y); K(x, z))|_{z=x}$,

\[ \tilde{B}_1(N(x, z, y); K(x, z)) = -N(x, z, y)K(x, z). \]

Therefore applying the operator $1 \sigma_z$ by induction we get the formulas

\begin{equation}
(12) \quad 1 \sigma_z^m \sigma \int_x^\infty N(x, z, y)K(x, z)dz =
\end{equation}

\[ 1 \sigma_z^{m-1}[-2 \sigma_z \int_x^\infty N(x, z, y)K(x, z)dz] - [1 \sigma_z^{m-1}N(x, z, y)K(x, z)]|_{z=x}
\end{equation}

\[ = ... = (-2 \sigma_z)^m \sigma \int_x^\infty N(x, z, y)K(x, z)dz + \tilde{B}_m(N; K)(x, y), \]

32
where \( \tilde{B}_m(N; K)(x, y) = \tilde{B}_m(N(x, z, y); K(x, z)) \) |\( z = x \),

\[
(13) \quad \tilde{B}_m(N(x, z, y); K(x, z)) = \left[ \sum_{k=0}^{m-1} (-1)^{k+1} 1^{m-k-1} 2^{k} \sigma_z^k \right] N(x, z, y) K(x, z) \\
= -(-2\sigma_z)^{m-1} N(x, z, y) K(x, z) + 1\sigma_z \tilde{B}_{m-1}(N(x, z, y); K(x, z)).
\]

16. **Theorem.** Let \( \{ L_s : s = 1, ..., k_0 \} \) be a set of PDOs which are polynomials \( \Omega_s(1\sigma_x, 2\sigma_y) \) over \( A_r \) or \( R \). Let also \( G \) be the family of all operators \( E = BST_g \) satisfying the condition \( [L_s, E] = 0 \) for each \( s = 1, ..., k_0 \), where \( B \in SL_n(R) \), \( S \in Aut(Mat_{n \times n}(A_r)) \), \( g \in Diff^\infty(U) \), \( T_g \) is prescribed by Formula 1(4), \( 1\sigma_x \) and \( 2\sigma_y \) are over the Cayley-Dickson algebra \( A_r \), \( r \geq 2 \). Then the family \( G \) forms the group and there exists an embedding of \( G \) into \( SL_n(R) \times Aut(Mat_{n \times n}(A_r)) \times Diff^\infty(U) \).

**Proof.** The composition (set theoretic) in the family \( G \) of the aforementioned operators is associative. Then the inverse \( E^{-1} = T_g^{-1} S^{-1} B^{-1} \) of \( E = BST_g \) exists, since \( B, S \) and \( T_g \) are invertible for every \( B \in SL_n(R) \), \( S \in Aut(Mat_{n \times n}(A_r)) \) and \( g \in Diff^\infty(U) \) so that \( T_g^{-1} = T_{g^{-1}} \). On the other hand, the identity \( E^{-1}[L_s, E]E^{-1} = -[L_s, E^{-1}] \) is valid. Thus the equality \( [L_s, E] = 0 \) implies that \( L_s \) and \( E^{-1} \) commute, \( [L_s, E^{-1}] = 0 \), as well. Therefore, from \( E \in G \) the inclusion \( E^{-1} \in G \) follows. The identity \( [L_s, E_1 E_2] = [L_s, E_1] E_2 + E_1 [L_s, E_2] \) implies that \( E_1 E_2 \in G \) whenever \( E_1 \in G \) and \( E_2 \in G \). Thus the family \( G \) has the group structure. There exists the bijective correspondence between diffeomorphisms \( g \in Diff^\infty(U) \) and operators \( T_g \) acting on functions defined on \( U \) with values in \( Mat_{n \times n}(A_r) \) according to Formula 1(4). Each element \( E \) in \( G \) is of the form \( E = BST_g \), where \( B \in SL_n(R) \), \( S \in Aut(Mat_{n \times n}(A_r)) \), \( g \in Diff^\infty(U) \), consequently, an embedding \( \omega : G \hookrightarrow SL_n(R) \times Aut(Mat_{n \times n}(A_r)) \times Diff^\infty(U) \) exists.

4 Nonlinear PDEs used in hydrodynamics.

1. **Remark.** In the previous article [27] vector hydrodynamical PDEs were investigated. Using results of Sections 2 and 3 we generalize the approach using transformations of functions by operators \( E \) of the form 3.1(2). It permits to consider other PDEs and study the symmetry of solutions.
2. Example. Generalized Korteweg-de-Vries’ type PDE. Let

(1) \( N(x, z, y) = EK(x, z) \) as in 3.1(1) and let \( A_x \) be given by 3.2(3), where \( E \) satisfies conditions 3.1(2, 4). Foliations of a domain \( U \), operators \( L_s \) and the cases of \( F \) and \( K \) are the same as in [27]:

(2) \( L_1 = 1 \sigma_x^2 - 2 \sigma_y^2 \)
(3) \( L_2 = 3 \sigma_t + 1 \sigma_x^3 + 3 \sigma_y \sigma_x^2 + 3 \sigma_y^2 \sigma_x + 2 \sigma_y^3 \),
where \( 1 \psi_0 = 2 \psi_0 = 0, \)

(4) \( L_1 F = 0 \) and \( L_{2,j} F = 0 \) for each \( j = 0, ..., 2^{r-1} \). Then equations from example 4.2 [27] take the following form. In view of 2.6(7) and Proposition 2.5 above PDE (4.19) [27] transforms into:

(5) \( (1 \sigma_x^2 - 2 \sigma_y^2) K(x, y) + 2pK(x, y)[1 \sigma_x EK(x, x)] = 0. \)

Putting

(6) \( u(x) = 2 \ 1 \sigma_x EK(x, x) \)
over the quaternion skew field \( H = A_2 \) and substituting \( K(x, y) = \Phi(x, k) \exp(JRe(ky)) \) into (4), we get Schrödinger’s equation:

(7) \( 1 \sigma_x^2 \Phi(x, k) + \Phi(x, k)(pu - \sum_j k_j^2 2 \psi_j^2) = 0, \)
where \( k \in H \), the generator \( J \) commutes with \( i_0, ..., i_{2^r-1} \). There is supposed that functions \( F \) and \( K \) may depend on \( t \). Then in formulas (4.24 – 28) [27] \( K \) changes into \( EK \), while (4.30) [27] due to (5), 3.1(2 – 5) and 3.2(1, 2) transforms into:

(8) \(- 2p \ 1 \sigma \int_{- \infty}^{x} F(z, y)[K(x, y)(1 \sigma_x EK(x, x))]dz = [K(x, y) - F(x, y)]ST_g u(x), \)

since \( E(Ku) = B((ST_g K)u) = B(ST_g K)(ST_g u) = (EK)(ST_g u) \) (see \( u \) in (6)). Therefore (4.29) [27] changes into:

(9) \( I_2 = -3 \ 2 \sigma_y[K(x, y) - F(x, y)](ST_g u(x)) \)
\[ + 3p \ 2 \sigma_y[1A_2(F, EK)(x, y) - 1B_2(F, EK)(x, y)]. \]

Hence from Formulas (5, 8, 9) it follows that

(10) \( (3 \sigma_t + 1 \sigma_x^3 + 3 \sigma_y \sigma_x^2 + 3 \sigma_y^2 \sigma_x + 2 \sigma_y^3) K(x, y) + 3 \sigma_y [K(x, y)] ST_g u(x) = \)
\[ p(3 \sigma_t + 3 \sigma_x^2 + 3 \sigma_y \sigma_x^2 + 3 \sigma_y^2 \sigma_x + 3 \sigma_y^3) \ 1 \sigma_x EK(x, z)dz \]
\[ + 3p \ 1 \sigma \int_1 \ 2 \sigma_y [F(x, y) ST_g u(x)] dz + T, \]

where

\( T = p \ 1A_3(F, EK)(x, y) - p \ 1B_3(F, EK)(x, y) \)
\[ + 3p \ 2 \sigma_y[1A_2(F, EK)(x, y) - 1B_2(F, EK)(x, y)] + \]
\[ 3p \ 1 \sigma_y[F(x, y) ST_g u(x)] + 3p \ 1 \sigma_y^2 + 2 \sigma_x^2 - 1 \sigma_x \sigma_y - 1 \sigma_x \sigma_y \sigma_z)[F(x, y) EK(x, z)]_{z=x}. \]

Then using (5, 8 – 10) we infer that
(11) \[ T = -p(3 2 \sigma_x^2 + 3 \sigma_x 2 \sigma_z + 2 2 \sigma_z 2 \sigma_x) [F(x, y) EK(x, z)]_{z=x} \]
\[ -p(2 2 \sigma_x^2 + 2 \sigma_x 2 \sigma_z) [F(x, y) EK(x, x)] \]
\[ +3(1-p) 2 \sigma_y [F(x, y) ST_y u(x)] + 3p (2 2 \sigma_x 2 \sigma_z + 2 \sigma_x 2 \sigma_z 2 \sigma_x) [F(x, y) EK(x, z)]_{z=x} \]
\[ = -3p 2 \sigma_x [F(x, y) u(x)] - 3p (2 2 \sigma_x 2 \sigma_z) [F(x, y) EK(x, z)]_{z=x} \]
\[ +3(1-p) 2 \sigma_y [F(x, y) ST_y u(x)] \]
\[ +p(2 2 \sigma_x 2 \sigma_z) [F(x, y) EK(x, z)]_{z=x} + p(2 2 \sigma_x 2 \sigma_z) [F(x, y) EK(x, x)] \]
\[ = -3p 2 \sigma_x [F(x, y) u(x)] + 3p^2 2 \sigma_x (2 \sigma_x F(z, y) EK(x, z) d z) ST_y u(\eta)]_{\eta=x} + 3p F(x, y) [E(K(x, x) u(x))] + 3(1-p) 2 \sigma_y [F(x, y) ST_y u(x)] \]
\[ +p(2 2 \sigma_x 2 \sigma_z) [F(x, y) EK(x, z)]_{z=x} + p(2 2 \sigma_x 2 \sigma_z) [F(x, y) EK(x, x)]. \]

Let \( p = 1 \). Therefore, in accordance with Formulas (10,11) and 3.2(1,2)
the equality

(12) \[ 3 \sigma_t + 1 \sigma_x^3 + 3 2 \sigma_y 2 \sigma_x + 3 2 \sigma_y 1 \sigma_x + 2 \sigma_y^3 K(x, y) \]
\[ +6(2 \sigma_x + 1 \sigma_y) (K(x, y) (1 \sigma_x EK(x, x)) - K(x, y) ([1 \sigma_x, 1 \sigma_x] EK(x, z))]_{z=x} \]
\[ -([1 \sigma_x, 1 \sigma_x] EK(x, x)] = 0 \]
follows, when the operator \((I - A_x E)\) is invertible.

In view of Theorem 3.3 this implies:

2.1. **Theorem.** If suppositions of Theorem 3.3 and Example 2 are satisfied. Then a solution of PDE 2(12) with \( 1 \psi_0 = 2 \psi_0 = 0 \) over the Cayley-Dickson algebra \( \mathcal{A}_r \) with \( 2 \leq r \leq 3 \) is given by Formulas (2 - 4) and 2(1), when \( p = 1 \).

2.2. **Example.** Korteweg-de-Vries’ type PDE. Continuing Example 2 mention that on the diagonal \( x = y \) the operators are: \( L_{1,x,x} = 0, L_{2,x,x} = \partial/\partial t + 8 1 \sigma_x^3 \). Therefore, \( [L_{1,x,x}, E] = 0 \) is valid. Let \( E \) be independent of the parameter \( t \), then \( [\partial/\partial t, E] = 0 \), since \( t \in \mathbb{R} \) and \( \mathbb{R} \) is the center of the Cayley-Dickson algebra \( \mathcal{A}_r \). To the term \( 1 \sigma_x^3 \) the cubic form \( (Im w)^3 = -|w|^2 w \)
corresponds, since \( 1 \psi_0 = 0 \), where \( Im w = (w - w^*)/2, w = i_1 x_1 \psi_1 + ... + i_{2^r-1} x_{2^r-1} \psi_{2^r-1}, \ x_j \in \mathbb{R} \) for each \( j \). That is for \( E = ST_g \) the restriction is \([|w|^2 w, E] = 0, \) where \( n = 1 \) and \( B = 1 \). Geometrically in the real shadow of \( Im(\mathcal{A}_r) \) such \( E = E(x) \) permits any rotations along the axis \( J_w \) parallel to \( w \) such that \( J_w \) crosses the origin of the coordinate system. Evidently \([w^3, E] = 0 \) is satisfied if \([w, E] = 0 \), that is \([1 \sigma_x, E(x)] = 0 \). In the latter
case and when \( n = 1, \) \( 1\sigma = 2\sigma, \) \( 1\psi_0 = 0 \) and \( 3\sigma_i = \partial / \partial t_0 \) the differentiation of \( 2(12) \) with the operator \( 1\sigma_x \) and the restriction on the diagonal \( x = y \) provides the PDE

\[
(1) \ v_t(t, x) + 6 \ 1\sigma_x [v(t, x)Ev(t, x)] + 1\sigma_x^3 v(t, x) = 0
\]
of Korteweg-de-Vries’ type, where \( v(t, x) = 2 \ 1\sigma_x K(x, x) \). Particularly there are solutions of PDE \( (1) \) which have the symmetry property \( Ev(t, x) = v(t, x) \).

3. Example. Non-isothermal flow of a non-compressible Newtonian liquid with a dissipative heating. Take the pair of PDOs

(1) \( L_1 = \sigma_x + \sigma_y \) and

(2) \( L_2 = 1\sigma_t + \sigma_x^2 + q\sigma_y\sigma_x + \sigma_y^2, \)

where \( q \in \mathbb{R} \) is a real constant, and consider the integral equation \( 3.2(1) \) with \( N \) of the form \( 3.1(1) \), so that

(3) \( L_1 F(x, y) = 0 \) and

(4) \( L_{2,j} F(x, y) = 0 \) for each \( j \), (see also \( 4.81 \) and \( 4.82 \) in [27]). Therefore, in \( (4.83) \) [27] the term \( K \) changes into \( EK \) and due to Proposition 2.5 and Corollary 2.6 we deduce the formula:

(5) \( L_2 K(x, y) = I_1 + I_2, \)

where

(6) \( I_1 = p(2\sigma_t + \sigma_x^2 + q \ 1\sigma_y \ 1\sigma_x) \sigma \int_x^\infty F(z, y) EK(x, z) dz \)

\( = p(2\sigma_x^2 + q \ 1\sigma_y \ 2\sigma_x) \sigma \int_x^\infty F(z, y) EK(x, z) dz \)

\( - p\sigma_x [F(x, y) EK(x, x)] - p \ 2\sigma_x [F(x, y) EK(x, z)] \big|_{z=x} - q p\sigma_y [F(x, y) EK(x, x)] \)

and

(7) \( I_2 = p(\sigma_x^2 + q \ 1\sigma_y \ 1\sigma_z) \sigma \int_x^\infty F(z, y) EK(x, z) dz \)

\( = p(2\sigma_x^2 - q \ 1\sigma_y \ 2\sigma_z) \sigma \int_x^\infty F(z, y) EK(x, z) dz \)

\( - p \ 1\sigma_x [F(x, y) EK(x, x)] + p \ 2\sigma_z [F(x, y) EK(x, z)] \big|_{z=x} - q p\sigma_y [F(x, y) EK(x, x)] \).

Then in \( (4.88, 4.89) \) [27] \( K \) changes into \( EK \) as well and \( (4.90) \) takes the form:

(8) \( L_2 K(x, y) = p(\sigma_t + \sigma_x^2 + q \ 2\sigma_x \ 2\sigma_z + (q-1) \ 2\sigma_z^2) \sigma \int_x^\infty F(z, y) EK(x, z) dz \)

\( - (q+2)p \ 2\sigma_x [F(x, y) EK(x, x)] + q p \ 2\sigma_x [F(x, y) EK(x, z)] \big|_{z=x} + q p \ 2\sigma_z [F(x, y) EK(x, z)] \big|_{z=x}. \)

Take \( q = 2 \). If the cases of \( F \) and \( K \) are the same as in Example 4.6 [27], \([L_1, E] = 0 \) and \([L_2, E] = 0 \) and when conditions of Theorem 3.3 are fulfilled, the equality follows:
(9) \((1\sigma_t + \sigma_x^2 + 2\sigma_y^2\sigma_x + \sigma_y^2)K(x,y) = -2pK(x,y)[\sigma_x EK(x,x)]\)
where \(K\) depends on the parameter \(t\).

Let \(g(x,t) = K(x,x)\), then on the diagonal \(x = y\) this implies the PDE:

\[(10) \((1\sigma_t + \sigma_x^2)g(x,t) = -2pg(x,t)[\sigma_x Eg(x,t)]\).

Then Equality (3) and Proposition 2.5 imply that \(K(x,y) = K(\frac{\psi,x-y}{2})\)
and the condition \([L_1, E]K = 0\) is fulfilled, when \(E(x,y) = E(\frac{\psi,x-y}{2})\).

Mention that \(L_{2,x,x} = 1\sigma_t + 4\sigma_x^2\) on the diagonal \(x = y\). Taking \(E\) independent of \(t\), the condition \([L_{2,x,x}, E(x,x)] = 0\) means that \([L_{2,x,x}, E(0)] = 0\).

Thus in the real shadow of \(Im(\mathbb{A}_r)\) this \(E(0)\) induces any element of the orthogonal group \(O(2^r - 1)\). Then more general PDE (10) can be applied to non-isothermal flow of a non-compressible Newtonian liquid with a dissipative heating as in [27]. This also provides symmetry properties of \(g(t,x)\), particularly, when a solution satisfies the condition \(Eg = g\).

3.1 Theorem. Suppose that conditions of Theorem 3.3 and Example 3 are satisfied, then PDE (9) over the Cayley-Dickson algebra \(\mathbb{A}_r\) with \(2 \leq r \leq 3\) has a solution given by Formulas (3, 4), 3.1(1) and 3.2(1), where PDOs \(L_1\) and \(L_2\) are given by 2.1(1, 2), \(F \in Mat_{n \times n}(\mathbb{R})\) and \(K \in Mat_{n \times n}(\mathbb{A}_r)\), \(n \in \mathbb{N}\) for \(r = 2\), \(n = 1\) for \(r = 3\).

Conclusion. In the paper new integrable PDEs were found with the help of non-commutative integration over octonions and Cayley-Dickson algebras. It enlarges possibilities of previous approaches based on real and complex numbers, because each PDE over them can be reformulated over octonions and new types of PDEs can be encompassed. There is the vast general research theme on integrability of differential equations and PDEs over real and complex numbers basing on Lie groups and algebras. It is interesting to develop this theme further and investigate integrable PDEs using nonassociative analogs of Lie groups and algebras over octonions and Cayley-Dickson algebras. It is planned to be continued in a next paper.

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