On the EIT problem for nonorientable surfaces

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Abstract

Let \((\Omega, g)\) be a smooth compact two-dimensional Riemannian manifold with boundary, \(\Lambda_g : f \mapsto \partial\nu u|_{\partial\Omega}\) its DN map, where \(u\) obeys \(\Delta_g u = 0\) in \(\Omega\) and \(u|_{\partial\Omega} = f\). The Electric Impedance Tomography problem is to determine \(\Omega\) from \(\Lambda_g\).

A criterion is proposed that enables one to detect (via \(\Lambda_g\)) whether \(\Omega\) is orientable or not.

The algebraic version of the BC-method is applied to solve the EIT problem for the Moebius band. The main instrument is the algebra of holomorphic functions on the double covering \(\mathbb{M}\) of \(M\), which is determined by \(\Lambda_g\) up to an isometric isomorphism. Its Gelfand spectrum (the set of characters) plays the role of the material for constructing a relevant copy \((M', g')\) of \((M, g)\). This copy is conformally equivalent to the original, provides \(\partial M' = \partial M\), \(\Lambda_{g'} = \Lambda_g\), and thus solves the problem.

Key words: 2d Riemannian manifold with boundary, determination of manifold from DN map, criterion of orientability via DN map.

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1 Introduction

About the paper

The fact that the Dirichlet-to-Neumann map does determine the Riemannian surface with boundary up to conformal equivalence, is well known [9, 10, 8]. It was first established in [9]. In [8] the explicit complex analysis formulas for determination of the (oriented) Riemannian surface are provided.

In [10] this fact is proved for the (oriented) Riemannian surfaces by the use of the connection between the EIT problem and Banach algebras of analytic functions. Our prospective goal is to extend the approach [10] to the nonorientable surfaces. The present paper is the first step in this direction.

Results

• Let $(\Omega, g)$ be a two-dimensional smooth compact Riemannian manifold endowed with the smooth metric tensor $g$, $\Delta_g$ the Beltrami-Laplace operator on $M$. Let $u = u^f(x)$ be a solution to the elliptic Dirichlet boundary value problem

$$\begin{align*}
\Delta_g u &= 0 & \text{in } \text{int } \Omega, \\
u f &= f & \text{on } \Gamma,
\end{align*}$$

where $\text{int } \Omega := \Omega \setminus \partial \Omega$ and $\partial \Omega =: \Gamma = \Gamma_1 \cup \cdots \cup \Gamma_N$. The Dirichlet-to-Neumann map of $\Omega$ is the operator $\Lambda_g : f \mapsto \partial_\nu u^f|\Gamma$, where $\nu$ is the outward normal to $\Gamma$.

Our main result is Theorem [10] that provides a criterion, which enables one to detect orientability of the surface $\Omega$ via its DN map $\Lambda_g$. We prove that orientability is equivalent to the solvability of a Hamilton-type system on $\Gamma$ with the ‘Hamiltonian’ $\Lambda_g$. If the boundary consists of the single component, the criterion is simplified: $\Omega$ is orientable iff $\text{Ker } [I + (\Lambda J)^2] \neq \{0\}$, where $J$ is the integration along $\Gamma$. It is noteworthy that the operator $I + (\Lambda J)^2$ introduced in [10] has multidimensional analogs [2, 5].

• The possibilities of the algebraic approach [10] for nonorientable surfaces are demonstrated by the example of EIT problem for the Moebius band $(M, g)$. The main tool for solving is the algebra of the boundary values of the holomorphic functions on the (orientable) double covering $\mathbb{M}$ of $M$.

\footnote{everywhere in the paper, \textit{smooth} means $C^\infty$-smooth}
This algebra is determined by $\Lambda_g$ up to isometry. Its Gelfand spectrum (the set of characters) plays the role of the material for constructing a relevant copy $(M', g')$ of $(M, g)$. This copy is conformally equivalent to the original, provides $\partial M' = \partial M$, $\Lambda_{g'} = \Lambda_g$, and thus solves the problem. Note that to construct such a copy is the only relevant understanding of ‘to solve the EIT problem for the unknown manifold’ [1, 3, 4].

Comments

• According to the physical meaning of EIT problem, an external observer must reconstruct the shape of the conducting shell $\Omega$ from measurements taken at its border $\Gamma$. The observer prospects the shell with electric current $\nabla_g u^f$ initiated by potential $f$ applied to the border, and registers the current $\partial_{\nu} u^f = \Lambda_g f$ flowing across the border. The above mentioned criterion enables him to determine whether the shell is orientable or not, without solving the inverse problem.

• In the most general case, the shell has a multicomponent boundary and is homeomorphic to a sphere with handles and Mobius bands glued into it. To visualize such a complex structure from the boundary is a worthy and challenging task.

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2 Orientability via DN-map

Harmonic functions

• In the sequel, $(\Omega, g)$ is a 2d smooth compact Riemannian manifold with the boundary $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_N$, where $\Gamma_j$ are diffeomorphic to the circle in $\mathbb{R}^2$. For $A \subset \Omega$, we denote

$$B_r[A] := \{ x \in \Omega \mid \text{dist} \ (x, A) < r \}.$$


The inner product of functions \( u, v \in L_2(\Omega) \) is
\[
(u, v) := \int_\Omega u(x)v(x) \, dx
\]
where \( dx \) is the area element. By \( \langle a, b \rangle \) we denote the inner product of the vectors \( a, b \in T\Omega_x \) and put
\[
(a, b) := \int_\Omega \langle a(x), b(x) \rangle \, dx
\]
for the vector fields \( a, b \in \vec{L}_2(\Omega) \).

Let \( \Delta \) be the Beltrami-Laplace operator, \( \nabla \) the gradient in \( \Omega \). The divergence is defined by
\[
(\text{div} \, a, \varphi) = -(a, \nabla \varphi), \quad \varphi \in C^\infty_0(\Omega).
\]
The relation \( \Delta = \text{div} \, \nabla \) holds. In some places, emphasizing the correspondence to the given metric, we write \( \Delta_g, \nabla_g, \text{div}_g \), and so on.

Unless otherwise specified, we deal with \textit{real} functions and fields. However, later on the \( \mathbb{C} \)-valued functions are also in the use.

- A function \( u \) obeying \( \Delta u = 0 \) is \textit{harmonic}. Harmonic functions are smooth in \( \text{int} \, \Omega \).

Let \( \omega \subset \Omega \). Two smooth functions \( u \) and \( v \) are called \textit{conjugate} (we write \( u \sim \omega v \)), if \( \langle \nabla u, \nabla v \rangle = 0 \) and \( |\nabla u| = |\nabla v| \) holds everywhere in \( \omega \). Note that \( u \sim \omega v \) and \( u \sim \omega \pm v + \text{const} \) are equivalent. The fields \( \nabla u \) and \( \nabla v \) are also called conjugate, and we write \( \nabla u \sim \omega \nabla v \)). We also agree in the case \( \omega = \text{int} \, \Omega \) to write just \( u \sim v \) and \( \nabla u \sim \nabla v \).

As is well known, if \( \Omega \) is orientable then it supports the pairs of conjugate functions. The orientation can be fixed by the choice of the continuous family of isometries \textit{(rotations)} \( \{ \Phi(x) \in \text{End} \, T\Omega_x \mid x \in \omega \} \) such that
\[
\Phi^* = \Phi^{-1}, \quad \Phi^2 = -I, \quad \text{and} \quad \nabla v = \Phi \nabla u \quad \text{in} \, \Omega
\]
for a pair \( u, v \) provided \( \text{const} \neq u \sim v \). The third relation is the Cauchy-Riemann conditions on \( u \) and \( v \). By (3) one has \( \Delta v = \text{div} \, \Phi \nabla u \equiv 0 \) and \( \Delta u = -\text{div} \, \Phi \nabla v \equiv 0 \), so that conjugacy implies harmonicity.

- Also, the local conjugacy implies orientability.
Lemma 1. Let the functions $u \not\equiv \text{const}$ and $v$ be harmonic in $\Omega$ and $u \overset{\Gamma'}{\sim} v$ hold for a segment $\Gamma' \subset \Gamma$ of positive length. Then $u \sim v$ in $\Omega$, whereas $\Omega$ is orientable.

Proof. 1. Let $\omega$ be a (small enough) neighborhood of $\Gamma'$ diffeomorphic to a disk in $\mathbb{R}^2$ and such that $\partial \omega \supset \Gamma'$. By the Poincare Theorem, there is a harmonic $v'$ provided $v' \overset{\omega}{\sim} u$ and, in particular, $v' \overset{\Gamma'}{\sim} u$. By the uniqueness of harmonic continuation, the latter implies $v' = \pm v + c$, because $v'$ and $v$ have the same Cauchy data at $\Gamma'$ for the properly chosen sign and constant. Hence, we have $u \overset{\omega}{\sim} v$.

Fix the points $x \in \text{int } \Omega$ and $x' \in \omega$. Choose a smooth simple curve $l$ connecting $x$ and $x'$ and its (small enough) neighborhood $\omega'$ diffeomorphic to a disk. Let $v''$ be conjugate to $u$ in $\omega'$. Since the conjugate function is determined uniquely up to the sign and constant summand, one can take $v'' = v' = v$ in $\omega \cap \omega'$ that obviously leads to $v \overset{\omega}{\sim} u$. Since $x$ is arbitrary, we conclude that $v \sim u$ in $\Omega$. Thus, $u \overset{\Gamma'}{\sim} v$ leads to $u \sim v$ and $\nabla u \sim \nabla v$.

2. The conjugated fields $\nabla u$ and $\nabla v$ may have (the same) zeros $x_1, x_2, \ldots$ in $\omega$. By harmonicity, these zeros are isolated in $\text{int } \Omega$, whereas the open set $\Omega_0 := \text{int } \Omega \setminus \{x_1, x_2, \ldots\}$ is a chart with the coordinates $u, v$ oriented by the Cauchy-Riemann conditions (3).

Choose a sequence of (small enough) positive $r_1, r_2, \ldots$ such that each $B_{r_j}[x_j] \subset \Omega$ is diffeomorphic to a disk, and endow it with the orientation consistent with the orientation of $\Omega_0$. Thus, $\text{int } \Omega$ admits the oriented atlas $\{\Omega_0, B_{r_1}[x_1], B_{r_2}[x_2], \ldots\}$, i.e., is orientable. Hence, $u \overset{\Gamma'}{\sim} v$ yields orientability of $\Omega$. \hfill $\square$

Harmonic fields

- A vector field $h$ is said to be harmonic in $\Omega$ if for every $x \in \text{int } \Omega$ there is a disk $D := B_r[x]$ oriented by a family of rotations $\Phi_D$ such that
  \[
  \text{div } h = \text{div } \Phi_D h = 0 \quad \text{in } D.
  \]

By the Poincare Theorem, harmonic fields are locally potential: there is a harmonic function $u$ such that $h = \nabla u$ in $D$. The potential $u$ has the local conjugate $v$, so that $u \overset{D}{\sim} v$ and $h = -\Phi_D \nabla v$ holds.

Let
\[
\mathcal{H} := \{h \in L_2(\Omega) \mid h \text{ is harmonic in } \Omega\}, \quad \mathcal{E} := \{h \in \mathcal{H} \mid h = \nabla u \text{ in } \Omega\}
\]
be the space of harmonic fields and its subspace of potential harmonic fields. In the second definition, the potential $u$ belongs to the Sobolev class $H^1(\Omega)$. The smooth fields are dense in $\mathcal{H}$ and $\mathcal{E}$.

- Depending on the topology of $\Omega$, the subspace $\mathcal{N} := \mathcal{H} \cap \mathcal{E}$ may be nontrivial. In the latter case, it consists of the so-called Neumann fields, which are tangent on $\Gamma$. Indeed, for a smooth $f$ in $\mathcal{N}$ and $n \in \mathcal{N}$ one has $\nabla u^f \in \mathcal{E}$ and

$$0 = (\nabla u^f, n) = \int_{\Omega} \langle \nabla u^f, n \rangle \, dx = \int_{\Gamma} f \langle n, \nu \rangle \, ds - \int_{\Omega} u^f \, \text{div} \, n \, dx = \int_{\Gamma} f \langle n, \nu \rangle \, ds$$

($ds$ is the length element of the metric $g$ on $\Gamma$), which leads to $\langle n, \nu \rangle = 0$ on $\Gamma$ by arbitrariness of $f$. Also, $\dim \mathcal{N}$ is finite and determined by topology of $\Omega$: see, e.g., [12].

Let $\Omega$ be orientable and (globally) oriented by a family of rotations $\Phi$. The family determines the unitary operator in $\mathcal{H}$ that acts point-wise by $(\Phi h)(x) = \Phi(x)h(x)$; we denote it by the same symbol $\Phi$. The subspace

$$\mathcal{E}^c := \mathcal{E} \cap \Phi \mathcal{E}$$

is invariant with respect to $\Phi$. A field $h \in \mathcal{H}$ belongs to $\mathcal{E}^c$ iff it has the conjugate $h^c \sim h$ in $\Omega$. In such a case, the relations $h = \nabla u$, $h^c = \nabla v = \Phi \nabla u$ (or $h^c = \nabla v = -\Phi \nabla u$), $u \sim v$ hold. As is well known, for any orientable $\Omega$ the subspace $\mathcal{E}^c$ is nontrivial and, moreover, $\dim \mathcal{E}^c = \infty$ holds.

**Criterion of orientability**

- Denote $\tilde{L}_2(\Gamma) := \{ f \in L_2(\Gamma) \mid \int_{\Gamma} f \, ds = 0 \}$ and recall that $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_N$. For a smooth $f$, the Green formula implies

$$0 = \int_{\Omega} \Delta u^f \, dx = \int_{\Gamma} \partial_n u^f \, ds = \int_{\Gamma} \Lambda f \, ds = \int_{\Gamma_1} \Lambda f \, ds + \cdots + \int_{\Gamma_N} \Lambda f \, ds,$$

so that $\Lambda f \in \tilde{L}_2(\Gamma)$. Moreover, the following is valid.

Recall that $\Lambda$ is a positive selfadjoint 1-st order pseudo-differential operator in $L_2(\Gamma)$, $\text{Dom} \, \Lambda = H^1(\Gamma)$, whereas $\text{Ker} \, \Lambda = \{ \text{const} \}$ and $\text{Ran} \, \Lambda = \tilde{L}_2(\Gamma)$ holds. Note in addition that the length element $ds$ (the metric on $\Gamma$) is determined by the principal symbol of $\Lambda$ [13].
Lemma 2. Let $\Omega$ be orientable and $N \geq 2$. Let a smooth $f$ be such that $\nabla u^f \in \mathcal{E}^c$. Then

$$(\Lambda f)|_{\Gamma_j} \in \hat{L}_2(\Gamma_j), \quad j = 1, \ldots, N$$

holds.

Proof. The harmonic potential fields

$$d_j := \nabla u^{\psi_j}, \quad \psi_j|_{\Gamma_k} = \delta_{jk}, \quad j, k = 1, \ldots, N$$

are normal to $\Gamma$ and, hence, $\Phi d_j$ are tangent. Therefore $\Phi d_j$ is a harmonic field tangent to $\Gamma$, i.e., $\Phi d_j \in \mathcal{H} \cap \mathcal{E}$.

Since $\nabla u^f \in \mathcal{E}^c$, one has $\Phi \nabla u^f \in \mathcal{E}^c$, so that $\Phi \nabla u^f \perp \mathcal{N}$. The latter implies

$$0 = (\Phi \nabla u^f, \Phi d_j) = (\nabla u^f, d_j) = \int_{\Gamma} f \langle d_j, \nu \rangle \, ds = \int_{\Gamma} f \langle \nabla u^{\psi_j}, \nu \rangle \, ds =$$

$$= \int_{\Gamma} f \Lambda \psi_j \, ds = \int_{\Gamma} \Lambda f \psi_j \, ds = \int_{\Gamma_j} \Lambda f \, ds$$

and we arrive at (4).

$\square$

• Let $\gamma$ be a continuous tangent field of unit vectors on $\Gamma$. For functions on the boundary, by $\dot{f} := \partial_s f$ we denote the derivative with respect to the length $s$ in direction $\gamma$, so that $\nabla_{\Gamma} f = \dot{f} \gamma$. Also, note the evident relation $\dot{f}|_{\Gamma_j} \in \hat{L}_2(\Gamma_j)$. For the solution to (1) and (2) one has

$$\Lambda p = \langle \nu, \nabla u^p \rangle = \langle \nu, \Phi \nabla u^f \rangle = -\langle \Phi \nu, \nabla u^f \rangle = -\langle \gamma, \nabla u^f \rangle = -\dot{f},$$

$$\Lambda f = \langle \nu, \nabla u^f \rangle = -\langle \nu, \Phi \nabla u^p \rangle = \langle \Phi \nu, \nabla u^p \rangle = \langle \gamma, \nabla u^p \rangle = \dot{p}$$

hold and lead to a ‘Hamiltonian’ system of the form

$$\dot{f} = -\Lambda p, \quad \dot{p} = \Lambda f \quad \text{on } \Gamma.$$ (5)
In addition, note that these relations are consistent with (4) on each $\Gamma_j$.

- Assume that the smooth functions $f$ and $p$ are such that (5) is satisfied at least on one component $\Gamma_j$ of the boundary. Then $u^f \sim u^p$ holds, whereas $\Omega$ is orientable.

  Indeed, we have

  \[
  \langle \nabla u^f, \nabla u^p \rangle = \langle \dot{f}\gamma + (\Lambda f)\nu, \dot{p}\gamma + (\Lambda p)\nu \rangle = \dot{f}\dot{p} + \Lambda f\Lambda p \text{ see } (3) = 0, \\
  |\nabla u^f|^2 = \dot{f}^2 + (\Lambda f)^2 \text{ see } (5) = (\Lambda p)^2 + \dot{p}^2 = |\nabla u^p|^2 \text{ on } \Gamma_j,
  \]

so that $u^f \Gamma_j \sim u^p$. The latter, by the Lemma 1, implies $u^f \Omega \sim u^p$ and follows to orientability of $\Omega$.

Summarizing, we arrive at the following criterion.

**Theorem 1.** The manifold $\Omega$ is orientable if and only if there are a tangent field $\gamma$ and a pair of smooth functions $f \not\equiv \text{const}$ and $p$ on $\Gamma$ such that (5) holds at least on one of the components $\Gamma_j$ of the boundary.

By Lemma 1 this statement remains true if we replace ‘one of the components $\Gamma_j$’ with ‘any segment $\Gamma'$ on one of the components $\Gamma_j$’.

- Let $\Gamma$ consist of the single component. Introduce the integration $J : \mathcal{L}_2(\Gamma) \to \mathcal{L}_2(\Gamma)$ by $\partial_\gamma J = \text{id}$. In this case, one has

  \[
  \dot{f} = -\Lambda p = -\Lambda J \dot{p} = -(\Lambda J)^2 \dot{f}
  \]

that leads to $[I + (\Lambda J)^2] \dot{f} = 0$ and $\dot{p} = \Lambda J \dot{f}$. In the case of the orientable $\Omega$, the latter relations enable one to find the traces of the conjugated functions $u^f$ and $u^p$ at the boundary via $\Lambda$: see 1. The above established criterion can be formulated as follows.

**Corollary 1.** Let $\Gamma$ consist of a single component. Then $\Omega$ is orientable iff $\text{Ker } [I + (\Lambda J)^2] \neq \{0\}$.

So, given the DN-map, one can determine whether the manifold is orientable or not.
3 Moebius band

Attributes

- Let $\Gamma$ be diffeomorphic to a circle in $\mathbb{R}^2$, $ds$ the length element on $\Gamma$. For $m, m' \in \Gamma$, we put $m' := -m$ if $\text{dist}_\Gamma(m, m') = \frac{\text{mes}_\Gamma}{2}$ holds.

  Let $\mathbb{M} := \Gamma \times [-1, 1]$ be the cylinder. For $x = \{m, \alpha\} \in \mathbb{M}$ we put $-x := \{-m, -\alpha\}$ and denote by $\tau$ the involution $\tau : x \mapsto -x$. The relation $x \sim x' \iff \tau(x) = x'$ is an equivalence on $M$.

  The Moebius band is $M := \mathbb{M}/\tau$, so that $M$ is the double covering of $\mathbb{M}$. By $\pi : \mathbb{M} \to M$ we denote the natural projection, which is a local diffeomorphism. The boundary $\partial M$ consists of two components $\Gamma_\pm = \{x = \{m, \pm 1\} \mid m \in \Gamma\}$. The boundary $\partial M = \pi(\partial \mathbb{M}) = \pi(\Gamma_+) = \pi(\Gamma_-)$ is identified with $\Gamma$ by $\pi(\{m, \pm 1\}) \equiv m$.

- Let $M$ be endowed with a metric $g$, $ds$ be the length element of $g$ on $\Gamma$. By $\Delta_g, \nabla_g, \Lambda_g, \ldots$ we denote the corresponding operations in $M$.

  The metric on $M$ induces the metric $g = \pi_* g$ on $M$; recall that $g(a, b)_{|\pi(x)} := g(D_\pi a, D_\pi b)_{|\pi(x)}$ for the tangent vectors $a, b \in T\mathbb{M}_x$, where $D_\pi$ is the differential of the projection. Also, $\pi$ is a local isometry, i.e., $\text{dist}_M(x, y) = \text{dist}_M(\pi(x), \pi(y))$ holds for the close enough $x$ and $y$. As is easy to see, the induced metric obeys

$$\tau_* g = g. \quad (6)$$

Simplifying the notation, we denote $\Delta := \Delta_g, \nabla := \nabla_g, \Lambda := \Lambda_g$ and so on.

In contrast to $M$, its covering $\mathbb{M}$ is orientable, and in the subsequent we assume $\mathbb{M}$ to be oriented by a rotation $\Phi$. Its boundary is also oriented by the tangent field $\gamma = \Phi \nu$, where $\nu$ is the outward normal to $\partial \mathbb{M}$. There are two orientations of the boundary $\Gamma = \partial M$. For definiteness, we put it to be oriented by the tangent field $D_\pi[\gamma_{|\Gamma_+}]$, and denote this field by the same $\gamma$.

- A function $u$ on $\mathbb{M}$ is said to be even (odd) if $u = u \circ \tau$ ($u = -u \circ \tau$) holds. If $u$ is even, there is a function $\tilde{u}$ on $M$ such that $u = \tilde{u} \circ \pi$. If $f$ is even (odd), then $\tilde{f} := \partial_\gamma f$ is odd (even), and the following relations can be easily derived and will be used later:

$$\nabla (f \circ \pi) = \sigma (\partial_\gamma f) \circ \pi \quad \text{on } \partial \mathbb{M}, \quad (7)$$

where $f$ is a function on $\Gamma$ and $\sigma_{|\Gamma_\pm} := \pm 1$.

As is easy to verify, the relation

$$\Delta (u \circ \pi) = (\Delta_g u) \circ \pi \quad \text{in int } \mathbb{M}$$
holds and $\Delta$ preserves the parity. As a consequence, if $u = u^f(x)$ satisfies
\[ \Delta u = 0 \text{ in } \text{int } M, \quad u = f \text{ on } \partial M \] (8)
and $f = f \circ \pi$ then one has the relations
\[ u^{f \circ \pi} = u^f \circ \pi, \quad \Lambda(f \circ \pi) = (\Lambda g \circ f) \circ \pi \text{ on } \partial M, \] (9)
where $u^f$ solves (1), (2) on $M$. So, $\Lambda$ also preserves the parity.

- The plan of solving the EIT problem for $M$ is, loosely speaking, as follows. First, we show that $\Lambda g$ determines (up to isometry) the analytic function algebra on the cylinder $\tilde{M}$, which does exist owing to its orientability. Then, by the use of the technique [1], we construct a homeomorphic copy $M'$ of $M$ and endow it with a relevant metric $g'$, which obeys (6). At last, we determine a copy $M'$ of $M$ and supply it with the metric $g' = \pi_*^{-1}g'$. As a result, the manifold $(M', g')$ turns out to be isometric to the (unknown) original $(M, g)$ and, thus, provides the solution of the problem.

### Harmonicity in $(M, g)$

- Let $\phi := u^f$ be the solution of (8) for $f = \pm 1$ on $\Gamma_{\pm}$. The harmonic potential field $\nabla \phi \in E$ in $\tilde{M}$ is normal on $\Gamma_{\pm}$. For harmonic fields in $\tilde{M}$, we have $H = E \oplus N$ and, as is known, $\dim N = 1$ and $N = \{ c\Phi \nabla \phi \mid c = \text{const} \}$ holds.

**Lemma 3.** For any smooth $f$, there is a smooth $p$ the a constant $c$ such that the equality
\[ \Phi \nabla u^f = \nabla u^p + c \Phi \nabla \phi \] (10)
holds, where
\[ \dot{p} = \Lambda f - c \Lambda \phi \text{ on } \partial \tilde{M}, \quad c = \frac{\int_{\Gamma_+} \Lambda f \, ds - \int_{\Gamma_-} \Lambda f \, ds}{\| \nabla \phi \|^2}. \] (11)

**Proof.** The equality (10) follows from $\Phi \nabla u^f \in \mathcal{H}$ and $\mathcal{H} = E \oplus N$. Multiplying it by $\gamma$, one has
\[ \langle \gamma, \Phi \nabla u^f \rangle = -\langle \Phi \gamma, \nabla u^f \rangle = \langle \nu, u^f \rangle = \Lambda f \quad \langle \gamma, \nabla u^p \rangle + c \langle \gamma, \Phi \nabla \phi \rangle = \dot{p} + c \langle \nu, \nabla \Phi \rangle = \dot{p} + c \Lambda \phi, \]
so that \(\dot{p} = \Lambda f - c\Lambda \phi\) does hold. Multiplying by \(\Phi \nabla \phi\) and integrating over \(\mathbb{M}\), one gets

\[
(\Phi \nabla \phi, \Phi \nabla u^f) = (\nabla \phi, \nabla u^f) = \int_{\partial M} f \Lambda \phi \, ds = \int_{\partial M} \Lambda f \, ds =
\]

\[
eq \int_{\Gamma_+} \Lambda f \, ds - \int_{\Gamma_-} \Lambda f \, ds = (\Phi \nabla \phi, \nabla u^p) + c \|\nabla \phi\|^2 = c \|\nabla \phi\|^2
\]

since \(\Phi \nabla \phi \in \mathcal{N}\), whereas \(\nabla u^p \perp \mathcal{N}\). Thus, (11) is valid. \(\square\)

- As a consequence of (10), we have the following.

**Corollary 2.** The relations

\[
\Phi \nabla u^{f \phi} = \nabla u^p \in \mathcal{E}, \quad \dot{p} = (\Lambda_g f) \circ \pi, \quad p \circ \tau = -p \quad \text{in} \quad \mathbb{M}
\]

(12)

hold for any \(f\) smooth on \(\Gamma = \partial M\).

Indeed, the function \(f = f \circ \pi\) is even on \(\mathbb{M}\). Hence, by (9) the function \(\Lambda f\) is also even and, as a result, we have \(c = 0\) in (11). In the meanwhile, the function \(p\) satisfies \(\dot{p} = \Lambda f \equiv (\Lambda_g f) \circ \pi\) and, hence, one can choose it to be odd. In what follows we accept such a choice.

Let \(J : \tilde{L}_2(\Gamma) \to \tilde{L}_2(\Gamma), \quad \partial \gamma J = \text{id}\) be the corresponding integration. Then, in addition to (12) one has

\[
p = \sigma [(J \Lambda_g f) \circ \pi + b] = \sigma [(J \Lambda_g f) \circ \pi] + b \phi \quad \text{on} \quad \partial \mathbb{M},
\]

(13)

where \(\sigma|_{\Gamma_+} = \pm 1\) and \(b\) is a constant. Respectively, one gets

\[
\nabla u^p = \nabla u^{\sigma[(J \Lambda_g f) \circ \pi]} + b \nabla \phi \quad \text{in} \quad \mathbb{M}.
\]

(14)

To find \(b\) we use the orthogonality \(\nabla u^{f \phi} \perp \Phi \nabla \phi\) in \(\mathcal{E}\): the relations

\[
0 = -(\nabla u^{f \phi}, \Phi \nabla \phi) = (\Phi \nabla u^{f \phi}, \nabla \phi) \overset{(12)}{=} (\nabla u^p, \nabla \phi) + b (\nabla \phi, \nabla \phi) \overset{\text{int. by parts}}{=} \int_{\partial \mathbb{M}} \sigma [(J \Lambda_g f) \circ \pi] \langle \nu, \nabla \phi \rangle \, ds + b \int_{\partial \mathbb{M}} \phi \langle \nu, \nabla \phi \rangle \, ds =
\]

\[
= 2 \int_{\Gamma_+} [(J \Lambda_g f) \circ \pi] \Lambda \phi \, ds + 2b \int_{\Gamma_+} \Lambda \phi \, ds
\]
hold and imply

\[ b = - \frac{\int_{\Gamma_+} [(J\Lambda g f) \circ \pi] \Lambda \phi \, ds}{\int_{\Gamma_+} \Lambda \phi \, ds}. \]  

(15)

• The first relation in (12) shows that \( u_f^{\text{for}} \) and \( u^p \) are conjugate by Cauchy-Riemann and, hence, the \( \mathbb{C} \)-valued function \( w = u_f^{\text{for}} + iu^p \) is holomorphic in \( \mathbb{M} \). Its boundary value (trace) \( \text{Tr} w := w \big|_{\partial \mathbb{M}} \) is represented by (13) and (15):

\[ \text{Tr} w = \text{Tr} u_f^{\text{for}} + \text{Tr} u^p = f \circ \pi + ip = f \circ \pi + i\sigma [(J\Lambda g f) \circ \pi + b]. \]

However, there is a disadvantage of this representation. When solving the EIT problem, the observer possesses the operator \( \Lambda_g \) but not \( \Lambda \), which enters in (15). To eliminate it, we use the following artificial trick.

To simplify the notation, denote \( q := J\Lambda g f \). The function \( w^2 \) is holomorphic in \( \mathbb{M} \) and, hence, \( \Re w^2 = (u_f^{\text{for}})^2 - (u^p)^2 \) is harmonic in \( \mathbb{M} \). Therefore, for \( v = (u_f^{\text{for}})^2 - (u^p)^2 \) we have

\[ \partial_v v = \Lambda\nu \big|_{\partial \mathbb{M}} = \Lambda[(f \circ \pi)^2 - p^2] \quad \text{[7], [9], [13]} \]

\[ = (\Lambda_g f^2) \circ \pi - \Lambda[(q \circ \pi)^2 + 2b(q \circ \pi)\sigma \phi + b^2 \phi^2] = \]

\[ = (\Lambda_g f^2) \circ \pi - (\Lambda_g q^2) \circ \pi - 2b\Lambda(q \circ \pi) = [\Lambda_g f^2 - \Lambda_g q^2 - 2b\Lambda q] \circ \pi, \]  

(16)

where \( \phi^2 = \sigma \phi = 1 \) and \( \Lambda b^2 = b^2 \Lambda 1 = 0 \) were used. On the other hand, we have

\[ \partial_v v = \partial_v (u_f^{\text{for}})^2 - \partial_v (u^p)^2 = 2u_f^{\text{for}} \partial_r u_f^{\text{for}} - 2u^p \partial_r u^p = \]

\[ = 2(f \circ \pi)\Lambda(f \circ \pi) - 2p\Lambda p = 2(f\Lambda g f) \circ \pi - 2p\Lambda p. \]  

(17)

In the meanwhile, multiplying the first relation in (12) by \( \nu \) at \( \partial \mathbb{M} \), we have

\[ \Lambda p = \langle \nu, \nabla u^p \rangle = \langle \nu, \Phi \nabla u_f^{\text{for}} \rangle = -\langle \gamma, \nabla u_f^{\text{for}} \rangle = \]

\[ = -\partial_\gamma (f \circ \pi) \quad \text{[7]} \]

\[ = -\sigma \partial_\gamma f \circ \pi, \]

which follows to

\[ p\Lambda p = -p \sigma (\partial_\gamma f) \circ \pi = -[\sigma q \circ \pi + b\phi] \sigma (\partial_\gamma f) \circ \pi = -[q \circ \pi] [(\partial_\gamma f) \circ \pi] - b[(\partial_\gamma f) \circ \pi] = -[q \partial_\gamma f + b \partial_\gamma f] \circ \pi. \]
Substituting to (17), we get
\[ \partial_v v = 2 \left[ f \Lambda g f + q \partial_v f + b \partial_v f \right] \circ \pi. \]  
(18)

At last, equating the results in (18) and (16) one easily arrives at
\[ b = \frac{1}{2} \left[ \Lambda_g f^2 - \Lambda_g q^2 \right] - f \Lambda_g f - q \dot{f}, \quad \text{where} \quad q = J \Lambda_g f, \quad \dot{f} = \partial_v f. \]  
(19)

The remarkable feature of this representation is that, first, it contains \( \Lambda_g \) only (does not contain \( \Lambda \)) and, second, the terms entering in the right hand side are the functions of \( x \in \Gamma \) but the ratio is constant. Also, the denominator
\[ \dot{f} + \Lambda_g q = \dot{f} + \Lambda_g J \Lambda_g f = \dot{f} + \Lambda_g J \Lambda_g J \dot{f} = [I + (\Lambda_g J)^2] \dot{f} \]
can not have too many zeros on \( \Gamma \) since, by Corollary 1, one has \( \text{Ker} [I + (\Lambda_g J)^2] = \{0\} \) for the nonorientable \( M \).

**Algebra \( \mathfrak{A}(M) \)**

- Let
\[ \mathfrak{A}(M) := \{ w = u + iv \mid u, v \in C(M), \nabla v = \Phi \nabla u \text{ in int } M \} \]

be the Banach algebra of holomorphic continuous functions with the norm \( ||w|| = \sup_{M} |w| \). Its smooth elements \( \mathfrak{A}^\infty(M) := \mathfrak{A}(M) \cap C^\infty(M; \mathbb{C}) \) are dense in \( \mathfrak{A}(M) \). A special feature of the algebra \( \mathfrak{A}(M) \) is the presence of the involution
\[ w \mapsto w^* := w \circ \tau. \]

By
\[ \mathfrak{A}_s(M) := \{ v \in \mathfrak{A}(M) \mid w^* = w \}, \quad \mathfrak{A}^\infty_s(M) := \mathfrak{A}_s(M) \cap C^\infty(M; \mathbb{C}) \]
we denote the sets of the Hermitian elements. For any element of the algebra, one represents
\[ w = y + iz, \quad y = \frac{v + v^*}{2}, \quad z = \frac{v - v^*}{2i}, \]  
(20)

with the Hermitian \( y \) and \( z \).
In accordance with the maximal principle, one has \( \sup_{M} |w| = \sup_{\partial M} |w| \) and, hence, the map

\[
\mathfrak{A}(M) \ni w \mapsto w|_{\partial M} \in C(\partial M; \mathbb{C})
\]

is an isometry on its image. In the meantime, obviously, \( \text{Tr} \) is an isomorphism of algebras. The (sub)algebra

\[
\text{Tr} \mathfrak{A}(M) \subset C(\partial M; \mathbb{C})
\]

contains the dense set \( \text{Tr} \mathfrak{A}^\infty(M) \) and is isometrically isomorphic to \( \mathfrak{A}(M) \) via the map \( \text{Tr} \).

The functions \( \Re w \) and \( \Im w \) of \( w \in \mathfrak{A}(M) \) can be used as the local (isothermal) coordinates consistent with the smooth structure of \( M \).

- Let us show that the algebra \( \mathfrak{A}(M) \) is determined by the DN map \( \Lambda_g \) of the Moebius band, which is the key fact for the EIT problem.

Each element of the form \( w = u^f + iu^p \) obeying (12) and (19), is Hermitian. This is a simple consequence of the fact that \( \Re w \) and \( \Im w \) are even and odd respectively. It is easy to check that the converse is also valid. As a result, passing to the traces on \( \partial M \), we have the representation

\[
\text{Tr} \mathfrak{A}^\infty(M) = \{ w = f + i\sigma [(J\Lambda_g f) + b] \mid f \in C^\infty(\Gamma), \ b \ \text{obeys (19)} \}.
\]  

(21)

By (20) and (21), for any \( w \in \mathfrak{A}^\infty(M) \) one has

\[
\text{Tr} \mathfrak{A}^\infty(M) = \{ w = y + iz \mid y = f + i\sigma [J\Lambda_g f + b], \ z = f' + i\sigma [J\Lambda_g f' + b'] ; \\
\quad \ b \text{ and } b' \text{ obey (19) for } f \text{ and } f' \text{ respectively} \}.
\]

(22)

Summarizing and denoting by \( \mathfrak{A} \cong \mathfrak{B} \) the isomorphic isometry of algebras, we arrive at the following scheme of determination of \( \mathfrak{A}(M) \) via the DN-map:

\[
\Lambda_g \xrightarrow{(22)} \text{Tr} \mathfrak{A}^\infty(M) \Rightarrow \text{clos}_{C(\Gamma; \mathbb{C})} \text{Tr} \mathfrak{A}^\infty(M) = \text{Tr} \mathfrak{A}(M) \cong \mathfrak{A}(M).
\]

(23)

Thus, \( \Lambda_g \) determines the algebra \( \mathfrak{A}(M) \) up to an isometric isomorphism.
Determination of \((M, g)\)

• Recall the well-known notions and facts (see, e.g., [7, 10])

A character of the complex commutative Banach algebra is a nonzero homomorphism \(\chi : \mathfrak{A} \to \mathbb{C}\). The set of characters (spectrum of \(\mathfrak{A}\)) is denoted by \(\hat{\mathfrak{A}}\) and endowed with the canonical Gelfand \((\ast\)-weak) topology. The Gelfand transform \(\mathfrak{A} \to C(\hat{\mathfrak{A}}; \mathbb{C})\) maps \(a \in \mathfrak{A}\) to the function \(\hat{a} \in C(\hat{\mathfrak{A}}; \mathbb{C})\) by \(\hat{a}(\chi) := \chi(a)\).

We write \(S \cong T\) if the topological spaces \(S\) and \(T\) are homeomorphic. If the algebras are isometrically isomorphic \((\mathfrak{A} \cong \mathfrak{B})\) then their spectra are homeomorphic: \(\hat{\mathfrak{A}} \cong \hat{\mathfrak{B}}\).

For a function algebra \(F \subset C(T; \mathbb{C})\), the set \(D\) of the Dirac measures \(\delta_t : a \mapsto a(t), t \in T\) is a subset of \(\hat{\mathfrak{F}}\). This algebra is called generic if \(\hat{\mathfrak{F}} = D\) holds, which is equivalent to \(\hat{\mathfrak{F}} \cong T\). In this case, the algebra \(\hat{\mathfrak{F}}\) is isometrically isomorphic to \(C(\hat{\mathfrak{F}}; \mathbb{C})\), the isometry being realized by the Gelfand transform \(a \mapsto \hat{a}\).

The algebra \(\mathfrak{A}(M)\) is generic [10, 11]. Therefore, \(\mathfrak{A}(M) \cong \text{Tr } \mathfrak{A}(M)\) implies \(M \cong \mathfrak{A}(M) \cong \text{Tr } \mathfrak{A}(M) =: M'\). By the latter and in accordance with the scheme (23), the DN-map \(\Lambda_g\) determines the spectrum \(M'\).

• Recall that ‘to solve the EIT problem’ is: given the DN-map \(\Lambda_g\) of the Möbius band \((M, g)\), to provide a manifold \((M', g')\) such that \(\partial M' = \partial M\) and \(\Lambda_g = \Lambda_{g'}\) holds. Such a copy \((M', g')\) of the original \((M, g)\) is considered to be the solution [1, 3, 4].

The copy is constructed by means of the following procedure. We describe it briefly, referring the reader to the papers [1, 3] for details. Note that, starting the procedure, we have nothing but the operator \(\Lambda_g\) on \(\Gamma\). However, we know a priori that \(\Lambda_g\) is the DN-map of some unknown Möbius band.

Step 1. Take \(\hat{M} = \Gamma \times [-1, 1], \partial \hat{M} = \Gamma_+ \cup \Gamma_-\) and identify \(\Gamma_+ \equiv \Gamma\). Given \(\Lambda_g\), determine the algebra \(\text{Tr } \mathfrak{A}(\hat{M})\) by (23) and find its spectrum \(\hat{M}'\). Applying the Gelfand transform

\[
\text{Tr } \mathfrak{A}(\hat{M}) \ni w \mapsto \hat{w} \in C(\hat{M}'; \mathbb{C}),
\]

we get the relevant copies \(\hat{w}\) of the (unknown) holomorphic functions \(w\) in \(\hat{M}\).

Step 2. Using \(\hat{\mathfrak{N}}\) and \(\hat{\mathfrak{F}}\) in capacity of the local coordinates on \(\hat{M}'\), we supply the spectrum with the structure of a smooth 2d manifold.
Step 3. By Silov \[7, 1\], the boundary \(\partial \mathcal{M}'\) is identified as the subset of \(\mathcal{M}'\), at which the functions \(|\hat{w}|\) attain the maximum. The boundary is disconnected and consists of two connected components \(\Gamma'_\pm\). Also, we identify the boundary points by

\[
\partial \mathcal{M} \ni x \equiv \delta \in \partial \mathcal{M}' \iff \hat{w}(x) = \hat{w}(\delta) \quad \text{for all smooth } w
\]

and, thus, attach \(\partial \mathcal{M}'\) to \(\partial \mathcal{M}\).

Step 4. The involution \(\tau'\) in \(\mathcal{M}'\), which is the copy of \(\tau\) in \(\mathcal{M}\), is determined as follows. Let \(w = w^f \in \text{Tr} \mathcal{A}^\infty(\mathcal{M})\) be a function on \(\partial \mathcal{M}\) specified by the conditions in (21), \(\hat{w}^f\) its Gelfand transform (a function on \(\mathcal{M}'\)). For a character \(\chi \in \mathcal{M}'\), we define \(\chi' := \tau'(\chi)\) if

\[
\hat{w}^f(\chi') = \overline{\hat{w}^f(\chi)} \quad \text{holds for all } f \in C^\infty(\Gamma).
\]

As is easy to recognize, such a definition is motivated by the relation \(w(\tau(x)) = w(x)\) on \(\mathcal{M}\) for the functions of the form \(w = u^{f_0} + i u^p\) with the even \(\Re w\) and odd \(\Im w\).

Step 5. At the moment, there is no metric on the spectrum \(\mathcal{M}'\). In the meantime, it supports the reserve of functions \(\Re \hat{w}^f \Im \hat{w}^f\), which are the relevant copies of the (unknown) harmonic functions \(u^{f_0} + i u^p\) on \(\mathcal{M}\). As is known, this reserve determines a metric \(\tilde{g}\) on \(\mathcal{M}'\), which provides \(\Delta_{\tilde{g}} \Re \hat{w}^f = \Delta_{\tilde{g}} \Im \hat{w}^f = 0\), such a metric being determined up to a conformal deformation. In \[9, 1\] the reader can find concrete tricks for determination of \(\tilde{g}\) (see also \[6\], page 16). One of them is to write the equation \(\Delta_{\tilde{g}} \Re \hat{w}^f = 0\) for a rich enough set \(f = f_1, \ldots, f_n\) in coordinates and then use these equations as a system for finding \(\tilde{g}_{ij}\) (up to a smooth functional factor). Also, it is easily seen that one can choose the metric \(\tilde{g}\) to be \(\tau'\)-invariant, i.e., obeying \(\tau'_* \tilde{g} = \tilde{g}\).

Let such a metric \(\tilde{g}\) be chosen. Find a smooth positive function \(\rho\) on \(\mathcal{M}'\) provided \(\rho = \rho \circ \tau'\) and such that the length element \(ds''\) of the metric \(g'' = \rho \tilde{g}\) at \(\Gamma' = \Gamma\) coincides with the (known) element \(ds\).

Step 6. Passing to the factor-space \(M' := \mathcal{M}'/\tau'\), we get a homeomorphic copy of the unknown original \(M\). The projection \(\tau' : M' \to M'\) (a copy of the unknown \(\pi\)) is a local homeomorphism by construction. Endowing \(M'\) with the metric \(g' = \pi'^{-1} \tilde{g}\), we obtain the manifold \((M', g')\), which satisfies \(\partial M' = \Gamma\) and \(\Lambda_{g'} = \Lambda_g\) by construction, and thus solves the EIT problem.

It is worth noting the following. In the course of solving the problem by this procedure, the observer operates not with the attributes of \(M\) themselves
(which is impossible in principle!), but creates their copies, and recovers not the original $M$, but its relevant copy $M'$. This view of what is happening is fully consistent with the philosophy of the BC-method in inverse problems [1, 4]: the only reasonable understanding of ‘to restore unreachable object’ is to construct its (preferably, isomorphic) copy.

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