DISTANCE GRAPHS AND SETS OF POSITIVE UPPER DENSITY IN $\mathbb{R}^d$

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Abstract. We present a sharp extension of a result of Bourgain on finding configurations of $k+1$ points in general position in measurable subset of $\mathbb{R}^d$ of positive upper density whenever $d \geq k+1$ to all proper $k$-degenerate distance graphs.

1. Introduction

1.1. Background. A result of Katznelson and Weiss [4] states that if $A \subseteq \mathbb{R}^2$ has positive upper Banach density, then its distance set $\{ |x-x'| : x, x' \in A \}$ contains all large numbers. Recall that the upper Banach density of a measurable set $A \subseteq \mathbb{R}^d$ is defined by

$$\delta^*(A) = \lim_{N \to \infty} \sup_{t \in \mathbb{R}^d} \frac{|A \cap (t + Q_N)|}{|Q_N|},$$

where $| \cdot |$ denotes Lebesgue measure on $\mathbb{R}^d$ and $Q_N$ denotes the cube $[-N/2, N/2]^d$.

Note that the distance set any set of positive Lebesgue measure in $\mathbb{R}^d$ automatically contains all sufficiently small numbers (by the Lebesgue density theorem) and that it is easy to construct a set of positive upper density which does not contain a fixed distance by placing small balls centered on an appropriate square grid.

This result was later reproved using Fourier analytic techniques by Bourgain in [2] where he established the following more general result for finite point configurations in general position.

Theorem 1 (Bourgain [2]). Let $\Delta_k \subseteq \mathbb{R}^d$ be a fixed collection of $(k+1)$ points in general position.

If $A \subseteq \mathbb{R}^d$ has positive upper Banach density and $d \geq k+1$, then there exists a threshold $\lambda_0 = \lambda_0(A, \Delta_k)$ such that $A$ contains an isometric copy of $\lambda \cdot \Delta_k$ for all $\lambda \geq \lambda_0$.

Recall that a point configuration $\Delta_k$ is said to be an isometric copy of $\lambda \cdot \Delta_k$ if there exists a bijection $\phi : \Delta_k \to \Delta_k'$ such that $|\phi(v) - \phi(w)| = \lambda |v - w|$ for all $v, w \in \Delta_k$.

These results may be viewed as initial results in geometric Ramsey theory where, roughly speaking, one shows that “large” but otherwise arbitrary sets necessarily contain certain geometric configurations. Recently there has been a number of results in this direction in various contexts, see [3], [1], and [5]. One of the aims of this article is to present a common extension for measurable subsets of Euclidean spaces of positive upper density. At the same time we present a new approach, based on a simple notion of uniform distribution attached to an appropriate scale. For another instance of this new approach see [6] where configurations of points that form the vertices of a rigid geometric square, and more generally the direct product of any two finite point configurations in general position, are addressed.

1.2. Distance Graphs and Main Result. A distance graph $\Gamma = \Gamma(V,E)$ is a connected finite graph with vertex set $V$ contained in $\mathbb{R}^d$ for some $d \geq 1$. We say that $\Gamma$ is $k$-degenerate if each of its subgraphs contain a vertex with degree at most $k$ and refer to the smallest $k$ such that it is $k$-degenerate as the degeneracy of $\Gamma$. It is easy to see that if a given graph is $k$-degenerate, then there exists an ordering of its vertex set $V = \{v_0, v_1, \ldots, v_n\}$ in such a way that $|V_j| \leq k$ for all $1 \leq j \leq n$, where

$$V_j := \{v_i : (v_i, v_j) \in E \text{ with } 0 \leq i < j\}$$

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Remark on the Sharpness of the dimension condition in Theorem 2 when $\Gamma$ is a finite path. Very recently, parallel to our work, embedding of bounded degree distance graphs Theorem 2. Let the given distance graph and not on the number of its vertices which could in fact be arbitrarily large. It is important to further observe that the length of the interval of dilations guaranteed by Part (ii) of Theorem larger than $d$ around the vertices, and an isometric embedding of a distance graph into a set dense subsets of the integer lattice. In [1] it was shown that measurable subsets contexts. Indeed, in [3] the embedding of large copies of trees (1-degenerate distance graphs) was shown for dimension larger than $d$.

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Remark on the Sharpness of the dimension condition in Theorem 2

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Let \( e_1, \ldots, e_k \) be the standard basis vectors of \( \mathbb{R}^k \) and \( \Delta_+ \) and \( \Delta_- \) denote the complete graphs with vertices \( \{0, e_1, e_2, \ldots, e_k\} \) and \( \{0, -e_1, e_2, \ldots, e_k\} \) respectively. It is clear that \( \Gamma = \Delta_+ \cup \Delta_- \) then defines a proper \( k \)-degenerate distance graph with the property that any isometric copy of \( \lambda \cdot \Gamma \) in \( \mathbb{R}^k \) must contain three collinear points, i.e., a copy of \( \{-\lambda e_1, 0, \lambda e_1\} \) obtained by a translation and a rotation. It was shown in \([2]\) that there are sets of positive upper density \( A \subseteq \mathbb{R}^k \), for any \( k \), which do not contain such configurations for all large \( \lambda \). This example shows the sharpness of the dimension condition \( d \geq k + 1 \) in Theorem \([2]\).

2. A Counting Function and Generalized von-Neumann Inequality

Let \( \Gamma = \Gamma(V, E) \) be a fixed proper \( k \)-degenerate distance graph with vertex set \( V = \{v_0, v_1, \ldots, v_n\} \) with \( v_0 = 0 \) in \( \mathbb{R}^d \) with \( d \geq k + 1 \). As our arguments are analytic, we need to define a measure on the configuration space of all isometric copies of \( \Gamma \). For each \( (v_i, v_j) \in E \) let \( t_{ij} = |v_i - v_j|^2 \). The configuration space of all isometric copies of \( \Gamma \), with the vertex \( v_0 \) remaining fixed at 0, namely

\[
S_\Gamma := \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{d(n+1)} : x_0 = 0 \text{ and } |x_i - x_j|^2 = t_{ij} \text{ for all } i, j \text{ for which } (v_i, v_j) \in E\}
\]

is clearly a real subvariety. Since \( \Gamma \) is proper, there exists a points \( (x_0, x_1, \ldots, x_n) \in S_\Gamma \) such that for all \( 1 \leq j \leq n \), the sets \( \overline{X}_j := x_j \cup X_j \) where \( X_j := \{x_i : v_i \in V_j\} \) are in general position and each of the spheres

\[
S_j := \{x \in \mathbb{R}^d : |x - x_i|^2 = t_{ij} \text{ for all } x_i \in X_j\}
\]

have dimension \( d - |X_j| \), which is at least 1 if our distance graph \( \Gamma \) is \( k \)-degenerate and \( d \geq k + 1 \).

It is easy to see that the radius \( r_j \) of \( S_j \) is positive and equals the distance from \( x_j \) to the affine subspace \( \text{span } X_j \). If \( X_j = \{x_{i_1}, \ldots, x_{i_\ell}\} \), then the fact that \( \overline{X}_j = \{x_{i_1}, \ldots, x_{i_\ell}, x_j\} \) is in general position ensures that the volume of the \( \ell \)-dimensional fundamental parallelepiped determined by the vectors \( x_j - x_{i_1}, \ldots, x_j - x_{i_\ell} \) is non-zero. Since this volume is equal to the square root of the determinant of the inner product matrix

\[
\det \{(x_j - x_{im_1}) \cdot (x_j - x_{im_2})\}_{1 \leq m_1, m_2 \leq \ell}
\]

it follows that

\[
r_j = \sqrt{\frac{\det \{(x_j - x_{im_1}) \cdot (x_j - x_{im_2})\}_{1 \leq m_1, m_2 \leq \ell}}{\det \{(x_{i_1} - x_{im_1}) \cdot (x_{i_\ell} - x_{im_2})\}_{1 \leq m_1, m_2 \leq \ell-1}}}
\]

and hence that \( r_j \) in fact depends continuously on \( X_j \).

An important consequence of the discussion above that there exists a constant \( r_\Gamma > 0 \) and compactly supported functions \( \eta_j \in C^\infty(\mathbb{R}^d) \) with \( 0 \leq \eta_j \leq 1 \) such that for all \( (x_0, x_1, \ldots, x_n) \in S_\Gamma \) with \( x_j \in \text{supp } \eta_j \), the corresponding spheres \( S_j \) will all have radius \( r_j \geq r_\Gamma \).

**Definition 2.1** (Localized Counting Function). For any \( 0 < \lambda \ll 1 \) and functions

\[
f_0, f_1, \ldots, f_n : [0, 1]^d \to \mathbb{R}
\]

with \( d \geq k + 1 \) we define

\[
T_r(f_0, f_1, \ldots, f_n)(\lambda) = \int \cdots \int f_0(x_0) f_1(x - \lambda x_1) \cdots f_n(x - \lambda x_n) \, d\mu_n(x_0) \cdots d\mu_1(x_1) \, dx
\]

where \( d\mu_j(x_j) = \eta_j(x_j) \, d\sigma_j(x_j) \) and \( \sigma_j \) denotes the normalized surface measure on \( S_j \).

Note that if \( A \subseteq [0, 1]^d \) with \( d \geq k + 1 \) and \( T_r(1_A, 1_A, \ldots, 1_A)(\lambda) > 0 \), then \( A \) must contain a point configuration \( \Gamma' = \{x, x + \lambda x_1, \ldots, x + \lambda x_n\} \) with \( (0, x_1, \ldots, x_n) \in S_\Gamma \) and hence an isometric copy of \( \lambda \cdot \Gamma \).

The key to showing that \( T_r(1_A, 1_A, \ldots, 1_A)(\lambda) \) is positive for certain sets \( A \) is to estimate \([6]\) in terms of a suitable uniformity norm localized to a scale \( L \) (related to \( \lambda \))

**Definition 2.2** (\( U^1(L) \)-norm). For \( 0 < L \ll 1 \) and functions \( f : [0, 1]^d \to \mathbb{R} \) we define

\[
\|f\|_{U^1(L)} = \|f \ast \varphi_L\|_2
\]

where \( \varphi_L(x) = L^{-d} \varphi(L^{-1}x) \) with \( \varphi = 1_{[-1/2,1/2]^d} \).
Note that if $A \subseteq [0,1]^d$ with $\alpha = |A| > 0$ and we define $f_A := 1_A - \alpha 1_{[0,1]^d}$, then

\begin{equation}
\|f_A\|_{U^1(L)}^2 = \int_{\mathbb{R}^d} \left| \frac{|A \cap (t + Q_L)|}{|Q_L|} - |A| \right|^2 dt,
\end{equation}

where $Q_L = [-L/2, L/2]^d$.

Evidently the $U^1(L)$-norm is measuring the mean-square uniform distribution of $A$ on scale $L$. The engine that drives our approach to Theorem 2 is the following

**Proposition 1** (Generalized von-Neumann). Let $0 < \varepsilon, \lambda \ll 1$. For any $L \leq \varepsilon \lambda$, $0 \leq m \leq n$ and functions $f_0, f_1, \ldots, f_m : [0,1]^d \to [-1,1]$ we have that

\[ |T_\Gamma(f_0, f_1, \ldots, f_m, 1, \ldots, 1)(\lambda)| \leq \|f_m\|_{U^1(L)} + O_\Gamma(\varepsilon). \]

Here $1$ stands for the indicator function of the unit cube $[0,1]^d$ and $O_\Gamma(\varepsilon)$ means a quantity bounded by $C_\Gamma \varepsilon$ with a constant $C_\Gamma$ depending only on $\Gamma$. We will also use the notation $f \ll g$ to indicate that $|f| \leq cg$ with a constant $c > 0$ sufficiently small for our purposes.

The above proposition immediately implies the following result for uniformly distributed sets from which we will deduce both parts of Theorem 2 in Section 3 below.

**Corollary 1.** Let $\Gamma$ be a proper $k$-degenerate distance graph with $n + 1$ vertices in $\mathbb{R}^d$ with $d \geq k + 1$.

Let $\alpha \in (0,1)$ and $0 < \lambda \ll \alpha \alpha^{n+1}$. If $A \subseteq [0,1]^d$ with $|A| = \alpha$ satisfies $\|f_A\|_{U^1(\varepsilon \lambda)} \ll \varepsilon$, then

\[ T_\Gamma(1_A, 1_A, \ldots, 1_A)(\lambda) \geq \frac{c_0}{2} \alpha^{n+1} \]

where

\[ c_0 = \int \cdots \int d\mu_n(x_n) \cdots d\mu_1(x_1) dx \]

**Proof.** The result follows immediately from Proposition 1 since

\[ T_\Gamma(1_A, 1_A, \ldots, 1_A)(\lambda) = c_0 \alpha^{n+1} + \sum_{m=0}^{n} \alpha^{n-m} T_\Gamma(1_A, \ldots, 1_A, f_A, 1, \ldots, 1)(\lambda) \]

where $f_A = 1_A - \alpha 1_{[0,1]^d}$.

We conclude this section with the proof of Proposition 1.

**Proof of Proposition 1.** Fix $0 \leq m \leq n$. We have

\[ |T_\Gamma(f_0, f_1, \ldots, f_m, 1, \ldots, 1)(\lambda)| \]

\[ \leq \int \cdots \int \left( \int |f_m(x - \lambda x_m) c_{m+1}(x_1, \ldots, x_m) d\mu_m(x_m)| dx \right) d\mu_{m-1}(x_{m-1}) \cdots d\mu_1(x_1) \]

where

\begin{equation}
\begin{aligned}
c_{m+1}(x_1, \ldots, x_m) &= \int \cdots \int d\mu_n(x_n) \cdots d\mu_{m+1}(x_{m+1}) \\
f_0 &= 1_A - \alpha 1_{[0,1]^d},
\end{aligned}
\end{equation}

if $0 \leq m \leq n - 1$ and $c_{n+1} = 1$. It follows from an application of Cauchy-Schwarz and Plancherel that

\begin{equation}
|T_\Gamma(f_0, f_1, \ldots, f_m, 1, \ldots, 1)(\lambda)|^2 \leq \int |\widehat{f_m}(\xi)|^2 I_m(\lambda \xi) d\xi
\end{equation}

where

\begin{equation}
I_m(\xi) = \int \cdots \int |\widehat{c_{m+1}}(\xi)|^2 d\mu_{m-1}(x_{m-1}) \cdots d\mu_1(x_1)
\end{equation}
with

\[ c_{m+1}\mu_m(\xi) = \int c_{m+1}(x_1, \ldots, x_m) \eta_m(x_m) e^{-2\pi i x_m \cdot \xi} \, d\sigma_m(x_m) \]

if \( 2 \leq m \leq n \) and \( I_1 = |c_{2\mu_1}|^2 \). In light of the trivial uniform bound \( 0 \leq I_m(\xi) \leq 1 \) and the fact that

\[ \|f_m\|^2_{U^1(L)} = \int |\hat{f}_m(\xi)|^2 |\hat{\varphi}(L\xi)|^2 \, d\xi \]

it suffices to establish that

\[ I_m(\lambda \xi)(1 - \hat{\varphi}(L\xi)^2) = O_T(\varepsilon^2). \tag{11} \]

Since \( 0 \leq \hat{\varphi}(\xi)^2 \leq 1 \) for all \( \xi \in \mathbb{R}^d \) and \( \hat{\varphi}(0) = 1 \) it follows that \( 0 \leq 1 - \hat{\varphi}(L\xi)^2 \leq \min\{1, 4\pi L|\xi|\} \). The uniform bound \((11)\) thus reduces to establishing the decay estimate

\[ I_m(\xi) \leq \min\{1, C_T |\xi|^{-1/2}\} \tag{12} \]

since this would in turn imply that

\[ I_m(\lambda \xi)(1 - \hat{\varphi}(L\xi)^2) \leq C_T \min\{(\lambda|\xi|)^{-1/2}, \varepsilon^6 \lambda|\xi|\} \leq C_T \varepsilon^2 \]

whenever \( L \leq \varepsilon^6 \lambda \).

To establish \((12)\) we will use the fact that in addition to being trivially bounded by 1, the Fourier transform of \( c_{m+1}\mu_m \) also decays for large \( \xi \) in certain directions, specifically

\[ |c_{m+1}\mu_m(\xi)| \leq \min\{1, (\gamma_1 \cdot (\text{dist}(\xi, \text{span } X_m))^{-1/2}\} \tag{13} \]

uniformly over all \( x_1, \ldots, x_{m-1} \) with \( x_j \in \text{supp} \eta_j \). This estimate is an easy consequence of the well-known asymptotic behavior of the Fourier transform of the measure on the unit sphere \( S^{d-|X_m|} \subseteq \mathbb{R}^{d-|X_m|+1} \) induced by Lebesgue measure, see for example \([7]\).

Using the fact that the measure \( d\sigma_{m-1}(x_m) \cdots d\sigma_1(x_1) \) is clearly invariant under the rotations

\[ (x_1, \ldots, x_m) \rightarrow (U x_1, \ldots, U x_m), \]

for any \( U \in SO(d) \), together with \((13)\) and the fact that \( 0 \leq \eta_j \leq 1 \) for \( 1 \leq j \leq m \), then gives

\[
I_m(\xi) \leq C \int \cdots \int (1 + \gamma_1 \cdot \text{dist}(\xi, \text{span } X_m))^{-1} d\sigma_{m-1}(x_m) \cdots d\sigma_1(x_1)
= C \int \cdots \int \int_{SO(d)} (1 + \gamma_1 \cdot \text{dist}(\xi, \text{span } U X_m))^{-1} d\mu(U) d\sigma_{m-1}(x_m) \cdots d\sigma_1(x_1)
= C \int \cdots \int \int_{S^{d-1}} (1 + \gamma_1 |\xi| \cdot \text{dist}(y, \text{span } X_m))^{-1} d\sigma(y) d\sigma_{m-1}(x_m) \cdots d\sigma_1(x_1)
\]

where \( \sigma \) denote normalized measure on the unit sphere \( S^{d-1} \) in \( \mathbb{R}^d \) induced by Lebesgue measure. Estimate \((12)\) then follows from the easy observation that the inner integral above satisfies the uniform estimate

\[
\int_{S^{d-1}} (1 + \gamma_1 |\xi| \cdot \text{dist}(y, \text{span } X_m))^{-1} d\sigma(y) = O((1 + \gamma_1 |\xi|)^{-1/2}). \tag{14} \]

\[ \square \]

### 3. Proof of Theorem 2

We will deduce Theorem 2 from Corollary 1 by localizing to cubes on which our set is suitably uniformly distributed. In the case of Part (i) this is achieved as a direct consequence of the definition of upper Banach density, while for Part (ii) this is achieved via an energy increment argument.
3.1. Direct Proof of Part (i) of Theorem 2

Let \( \varepsilon > 0 \) and \( A \subseteq \mathbb{R}^d \) with \( \delta^*(A) > 0 \).

The following two facts follow immediately from the definition of upper Banach density, see (1):

(i) There exist \( M_0 = M_0(A, \varepsilon) \) such that for all \( M \geq M_0 \) and all \( t \in \mathbb{R}^d \)

\[
\frac{|A \cap (t + Q_M)|}{|Q_M|} \leq (1 + \varepsilon^4/3) \delta^*(A).
\]

(ii) There exist arbitrarily large \( N \in \mathbb{R} \) such that

\[
\frac{|A \cap (t_0 + Q_N)|}{|Q_N|} \geq (1 - \varepsilon^4/3) \delta^*(A)
\]

for some \( t_0 \in \mathbb{R}^d \).

Combining (i) and (ii) above we see that for any \( \lambda \geq \lambda_0 := \varepsilon^{-6}M_0 \), there exist \( N \geq \varepsilon^{-6}\lambda \) and \( t_0 \in \mathbb{R}^d \) such that

\[
\frac{|A \cap (t + Q_{\varepsilon^6\lambda})|}{|Q_{\varepsilon^6\lambda}|} \leq (1 + \varepsilon^4)|A|
\]

for all \( t \in \mathbb{R}^d \). Consequently, Theorem 2 reduces, via a rescaling of \( A \cap (t_0 + Q_N) \) to a subset of \([0,1]^d\), to establishing that if \( \Gamma \) is a proper \( k \)-degenerate distance graph, \( 0 < \lambda \leq \varepsilon \ll 1 \) and \( A \subseteq [0,1]^d \) is measurable with \( |A| > 0 \) and the property that

\[
\frac{|A \cap (t + Q_{\varepsilon^6\lambda})|}{|Q_{\varepsilon^6\lambda}|} \leq (1 + \varepsilon^4)|A|
\]

for all \( t \in \mathbb{R}^d \), then \( A \) contains an isometric copy of \( \lambda \cdot \Gamma \).

Now since \( A \cap (t + Q_{\varepsilon^6\lambda}) \) is only supported in \([-\varepsilon^6\lambda, 1 + \varepsilon^6\lambda]^d\) and

\[
|A| = \int_{\mathbb{R}^d} \frac{|A \cap (t + Q_{\varepsilon^6\lambda})|}{|Q_{\varepsilon^6\lambda}|} \, dt
\]

it easily follows that

\[
\left| \left\{ t \in \mathbb{R}^d : 0 < \frac{|A \cap (t + Q_{\varepsilon^6\lambda})|}{|Q_{\varepsilon^6\lambda}|} \leq (1 - \varepsilon^2)|A| \right\} \right| = O(\varepsilon^2)
\]

and hence that

\[
\|f_A\|_{L^1(\varepsilon^6\lambda)}^2 = \int_{\mathbb{R}^d} \left| \frac{|A \cap (t + Q_{\varepsilon^6\lambda})|}{|Q_{\varepsilon^6\lambda}|} - |A| \right|^2 \, dt = O(\varepsilon^2).
\]

The result thus follows from Corollary 1 above provided \( \varepsilon \ll \delta^*(A)^{n+1} \). \( \square \)

3.2. Proof of Part (ii) of Theorem 2

Lemma 1 (Localization Principle). Let \( A \subseteq [0,1]^d \) with \( d \geq k+1 \) and \( |A| = \alpha > 0 \).

Let \( \varepsilon > 0 \) and \( \varepsilon^7 \gg L_1 \gg L_2 \gg \cdots \) be any decreasing sequence with \( L_1^{-1} \in \mathbb{N} \) and \( L_{j+1} \leq \varepsilon^7L_j \) for all \( j \geq 1 \). If we let \( G_j \) denote the partition of \([0,1]^d\) into cubes of side length \( L_j \), then there exists \( 1 \leq j \leq C\varepsilon^{-2} \) such that for all but at most \( \varepsilon L_j^{-d} \) of the cubes \( Q \in G_j \) the set \( A \) will be uniformly distributed on the smaller scale \( L_{j+1} \) inside \( Q \) in the sense that

\[
\frac{1}{|Q|} \int_Q \left| \frac{|A \cap Q \cap (t + Q_{L_{j+1}})|}{|Q_{L_{j+1}}|} - \frac{|A \cap Q|}{|Q|} \right|^2 \, dt \leq \varepsilon.
\]

Before proving Lemma 1 we first show that it, together with Corollary 1 (after rescaling), is sufficient to establish Part (ii) of Theorem 2. Let \( \varepsilon \ll \alpha \) \( \delta^*(A)^{n+1} \) and \( \{Q_i\} \) denote the cubes of side length \( L_j \) in the partition \( G_j \) of \([0,1]^d\) that we obtain from Lemma 1. If we then let \( A_i = A \cap Q_i \) and set \( \alpha_i = |A \cap Q_i|/|Q_i| \) it follows from Corollary 1 (after rescaling) and Hölder’s inequality that for any \( \lambda \in (\varepsilon^{-6}L_{j+1}, \varepsilon L_j) \) we have

\[
T_{\lambda}(1, \ldots, 1_A)(\lambda) \geq \sum_{i=1}^{L_j^{-d}} T_{\lambda}(1, \ldots, 1_A)(\lambda) \geq \frac{c_0}{4} L_j^{-d} \sum_{i=1}^{L_j^{-d}} \alpha_i \geq \frac{c_0}{4} \left( \frac{L_j^{-d}}{L_j^{d-1}} \sum_{i=1}^{L_j^{-d}} \alpha_i \right)^{n+1} = \frac{c_0}{4} |A|^n.
\]
**Proof of Lemma 1** Let \( \{Q_i\} \) denote the cubes of sidelength \( L_j \) in the partition \( \mathcal{G}_j \) of \([0, 1]^d\) and

\[
g_j = 1_A - \mathbb{E}(1_A | \mathcal{G}_j)
\]

where

\[
\mathbb{E}(1_A | \mathcal{G}_j)(x) = \frac{|A \cap Q_i|}{|Q_i|}
\]

for each \( x \in Q_i \). If \( \|g_j\|_{U^2(L_{j+1})} \geq \varepsilon \), then by definition

\[
\int \left| \frac{1}{|Q_{L_{j+1}}|} \int_{x+Q_{L_{j+1}}} g_j(y) \, dy \right|^2 \, dx \geq c \varepsilon^2.
\]

It follows that there must exist a \( x_0 \in [0, 1]^d \) for which the shifted grid \( x_0 + \mathcal{G}_{j+1} \) satisfies

\[
\int \left| \mathbb{E}(g_j | x_0 + \mathcal{G}_{j+1}) \right|^2 \, dx \geq c \varepsilon^2
\]

from which one can easily conclude that the (unshifted) refined grid \( \mathcal{G}_{j+2} \) satisfies

(17) \[
\int \left| \mathbb{E}(g_j | \mathcal{G}_{j+2}) \right|^2 \, dx \geq c \varepsilon^2
\]

provided \( L_{j+2} \ll \varepsilon^2 L_{j+1} \). By orthogonality, it follows immediately from (17) and the definition of \( g_j \) that

(18) \[
\|\mathbb{E}(1_A | \mathcal{G}_{j+2})\|^2 \geq \|\mathbb{E}(1_A | \mathcal{G}_j)\|^2 + c \varepsilon^2
\]

and hence that there must exist \( 1 \leq j \leq C \varepsilon^{-2} \) such that \( \|g_j\|_{U^2(L_{j+1})} \leq \varepsilon \) from which it follows that

\[
\sum_{i=1}^{L_j} \int \left| \frac{1}{|Q_{L_{j+1}}|} \int_{x+Q_{L_{j+1}}} (1_A - \alpha_i 1_{Q_i})(y) \, dy \right|^2 \, dx \leq C \varepsilon^2
\]

provided \( L_{j+1} \ll \varepsilon^2 L_j \). \( \square \)

4. A SECOND PROOF OF THEOREM 2

We conclude by presenting a second proof of Theorem 2 which is closer in spirit to Bourgain’s original proof of Theorem 1. We include this in order to highlight the simplicity and directness of our approach to Part (i) of Theorem 2 above, but also to emphasize that our approach to Part (ii) of Theorem 2 is in essence a physical space reinterpretation of Bourgain original approach.

4.1. Reducing Theorem 2 to a Dichotomy between Randomness and Structure. Let \( \Gamma \) be a proper \( k \)-degenerate distance graph in \([0, 1]^d\) with \( d \geq k+1 \). As we shall see, Theorem 2 is an immediate consequence of the following proposition which reveals that if \( A \subseteq [0, 1]^d \) has positive measure but does not contain an isometric copy of \( \lambda \cdot \Gamma \) for all \( \lambda \) in a given interval, then this “non-random” behavior is detected by the Fourier transform of the characteristic function of \( A \) and results in “structural information”, specifically a concentration of its \( L^2 \)-mass on appropriate annuli.

**Proposition 2** (Dichotomy). Let \( \Gamma = \Gamma(V, E) \) be a proper \( k \)-degenerate distance graph in \([0, 1]^d\) with \( d \geq k+1 \).

If \( A \subseteq [0, 1]^d \) with \( |A| > 0 \), \( 0 < a \leq b \ll \varepsilon^4 \) with \( 0 < \varepsilon \ll |A|^{n+1} \), and \( A \) does not contain an isometric copy of \( \lambda \cdot \Gamma \) for some \( \lambda \) in \([a, b]\), then

(19) \[
\int_{\varepsilon^2/b \leq |\xi| \leq 1/\varepsilon^2 a} |\hat{1}_A(\xi)|^2 \, d\xi \gg |A|^{2n+2}
\]

with the implied constant above independent of \( a \), \( b \), and \( \varepsilon \).
Proof that Proposition 2 implies Theorem 2 We shall first establish Part (ii) of Theorem 2 so we start by letting $A \subseteq [0,1]^d$ with $|A| > 0$. For any fixed $0 < \varepsilon \ll \Gamma |A|^{n+1}$, let $\{I_j\}_{j=1}^J$ denote a sequence of intervals with $I_j := [a_j, b_j]$ satisfying
\begin{equation}
J\varepsilon^2 \leq \sum_{j=1}^J \int_{\varepsilon^2 b_j \leq \xi \leq \varepsilon^2 a_j} |\hat{I}_A(\xi)|^2 d\xi \leq \int |\hat{I}_A(\xi)|^2 d\xi
\end{equation}
a contradiction if $J(\varepsilon) \gg \varepsilon^{-2}$ since by Plancherel we know that $\int |\hat{I}_A(\xi)|^2 d\xi = |A| \leq 1$.

To establish Part (i) of Theorem 2 with this approach we will argue indirectly and thus suppose that $A \subseteq \mathbb{R}^d$ is a set with $\delta^*(A) > 0$ for which the conclusion of Part (i) of Theorem 2 fails to hold, namely that there exist arbitrarily large $\lambda \in \mathbb{R}$ for which $A$ does not contain an isometric copy of $\lambda \cdot \Gamma$.

We now let $0 < \alpha < \delta^*(A)$, $0 < \varepsilon \ll \Gamma \alpha^{n+1}$, and fix $J \gg \varepsilon^{-2}$ as above. By our indirect assumption we can choose a sequence $\{\lambda_j\}_{j=1}^J$ with the property that $\lambda_{j+1} \ll \varepsilon^4 \lambda_j$ for all $1 \leq j \leq J - 1$ and $A$ does not contain an isometric copy of $\lambda_j \cdot \Gamma$ for each $1 \leq j \leq J$. It follows from the definition of upper Banach density that exist $N \in \mathbb{R}$ with $N \gg \lambda_1$ and $t_0 \in \mathbb{R}^d$ for which
\begin{equation}
\frac{|A \cap (t_0 + Q_N)|}{|Q_N|} \geq \alpha.
\end{equation}
Rescaling $A \cap (t_0 + Q_N)$ to a subset of $[0,1]^d$ and arguing as in the proof of Part (ii) above but this time with $b_j = \lambda_j/N$ again leads to a contradiction.

4.2. Proof of Proposition 2 Let $f = 1_A$ and $\Gamma = \{0, v_1, \ldots, v_n\}$ be a fixed proper $k$-degenerate distance graph. We will utilize the existence of a suitably smoothed version of $f$ with the certain properties, specifically

**Lemma 2.** For any $\varepsilon > 0$ there exists a function $g : \mathbb{R}^d \to (0,1]$, an appropriate smoothing of $f$, such that
\begin{equation}
|g(x - \lambda z) - g(x)| \ll \varepsilon
\end{equation}
uniformly in $x \in [0,1]^d$ and $|z| \leq 1$. Moreover, if $\varepsilon \ll |A|^{n+1}$, then
\begin{equation}
\int f(x)g(x)^n\,dx \gg |A|^{n+1}.
\end{equation}

The proof of Lemma 2 is straightforward and presented in Section 4.3 below. Assuming for now the existence of a function $g$ with property (23) it follows that
\begin{equation}
T_{\Gamma}(f,f,\ldots,f)(\lambda) = \int f(x)g(x)^n\,dx + \sum_{m=1}^n T_{\Gamma}(fg^{n-m},f,\ldots,f,f-g,1,\ldots,1)(\lambda) + O(n\varepsilon)
\end{equation}
where, as in (6) in Section 2 we define
\begin{equation}
T_{\Gamma}(f_0,f_1,\ldots,f_n)(\lambda) = \int \cdots \int f_0(x)f_1(x-\lambda x_1)\cdots f_n(x-\lambda x_n)\,\mu_n(x_n)\cdots d\mu_1(x_1)\,dx
\end{equation}
with $d\mu_j(x_j) = \eta_j(x_j)\,d\sigma_j(x_j)$ and $\sigma_j$ denotes the normalized surface measure on $S_j$.

If $A$ does not contain an isometric copy of $\lambda \cdot \Gamma$ for some $\lambda \in [a,b]$, then it clearly follows that
\begin{equation}
T_{\Gamma}(f,f,\ldots,f)(\lambda) = 0.
\end{equation}
In light of (21) and (25) it follows that if $\varepsilon \ll |A|^{n+1}/n$ then there must exist $1 \leq m \leq n$ such that
\begin{equation}
\int \cdots \int \left( \int |f - g|(x - \lambda x_m)\,c_{m+1}(x_1,\ldots,x_m)\,d\mu_m(x_m)\,dx \right) d\mu_{m-1}(x_{m-1})\cdots d\mu_1(x_1) \gg |A|^{n+1}
\end{equation}
with $c_{m+1}$ defined as before in equation (3) above. It then follows from an application of Cauchy-Schwarz and Plancherel that
\begin{equation}
\int |\hat{f}(\xi) - \hat{g}(\xi)|^2 I_m(\lambda \xi) \, d\xi \gg |A|^{2n+2}
\end{equation}
with $I_m$ again defined as before in equation (11) above. The fact that $g$ will be taken to be a sufficient smoothing of $f$ ensures that its Fourier transform satisfies
\begin{equation}
|\hat{f}(\xi)| \leq |\hat{g}(\xi)| \leq \varepsilon |\hat{f}(\xi)|
\end{equation}
provided $|\xi| \leq \varepsilon^2 b^{-1}$, see Section 4.3 below. This, together with the fact that $I_m(\xi)$ is bounded by 1 uniformly in $\xi$, and Plancherel, ensures that (27) implies
\begin{equation}
\int_{|\xi| \leq \varepsilon^2 b^{-1}} |\hat{f}(\xi)|^2 I_m(\lambda \xi) \, d\xi \gg |A|^{2n+2}.
\end{equation}

Estimate (19), and hence Proposition 2, then follows easily from estimate (29) and our previously established estimates for $I_m$, namely (12).

### 4.3. A smooth cutoff function and Proof of Lemma 2

#### 4.3.1. A smooth cutoff function

Let $\psi : \mathbb{R}^d \to (0, \infty)$ be a Schwartz function that satisfies
\begin{equation}
1 = \hat{\psi}(0) \geq \hat{\psi}(\xi) \geq 0 \quad \text{and} \quad \hat{\psi}(\xi) = 0 \quad \text{for} \quad |\xi| > 1.
\end{equation}
As usual, for any given $t > 0$, we define
\begin{equation}
\psi_t(x) = t^{-d} \psi(t^{-1} x).
\end{equation}

First we record the trivial observation that
\begin{equation}
\int \psi_t(x) \, dx = \int \psi(x) \, dx = \hat{\psi}(0) = 1
\end{equation}
as well as the simple, but important, observation that $\psi$ may be chosen so that
\begin{equation}
|1 - \hat{\psi}_t(\xi)| = |1 - \hat{\psi}(t\xi)| \ll \min\{1, t|\xi|\}.
\end{equation}

Finally we record a formulation, appropriate to our needs, of the fact that for any given small parameter $\varepsilon$, our cutoff function $\psi_t(x)$ will be essentially supported where $|x| \leq \varepsilon^{-1} t$ and is approximately constant on smaller scales. More precisely,

**Lemma 3.** Let $\varepsilon > 0$ and $t > 0$, then
\begin{equation}
\int_{|y| \geq \varepsilon^{-1} t} \psi_t(y) \, dy \ll \varepsilon.
\end{equation}

and
\begin{equation}
\int |\psi_t(y - \lambda z) - \psi_t(y)| \, dy \ll \varepsilon
\end{equation}
uniformly for $|z| \leq 1$, provided $t \gg \varepsilon^{-1} \lambda$.

**Proof of Lemma 3** Estimate (33) is easily verified using the fact that $\psi$ is a Schwartz function on $\mathbb{R}^d$ as
\begin{equation}
\int_{|y| \geq \varepsilon^{-1} t} \psi_t(y) \, dy = \int_{|y| \geq \varepsilon^{-1}} \psi(y) \, dy \ll \int_{|y| \geq \varepsilon^{-1}} (1 + |y|)^{-d-1} \, dy \ll \varepsilon.
\end{equation}

To verify estimate (34) we make use of the fact that both $\psi$ and its derivative are rapidly decreasing, specifically
\begin{equation}
\int |\psi_t(y - \lambda z) - \psi_t(x)| \, dy \leq \int |\psi(y - \lambda z/t) - \psi(y)| \, dy \ll \frac{\lambda}{t} \int (1 + |y|)^{-d-1} \, dy \ll \frac{\lambda}{t}.
\end{equation}
4.3.2. Proof of Lemma 2. Let \( g = f * \psi_{\varepsilon^{-1}b} \).

We first note that estimates (28) and (23) follow immediately from (32) and (34) respectively. In order to establishing the remaining “main term” estimate (24), we need only establish that if \( \varepsilon \ll |A|^{n+1} \), then

\[
\int f(x)g(x)\,dx \geq (1 - C\varepsilon)|A|^2
\]

for some constant \( C > 0 \), since by Hölder we would then obtain

\[
(1 - C\varepsilon)^n|A|^{2n} \leq \left( \int f(x)g(x)\,dx \right)^n \leq |A|^{n-1} \int f(x)g(x)^n\,dx
\]

from which (24) clearly follows for sufficiently small \( \varepsilon > 0 \).

To establish (35) we first note that Parseval, the fact that \( 0 \leq \hat{\psi} \leq 1 \), and a final application of Cauchy-Schwarz gives

\[
\int f(x)g(x)\,dx = \left| \int \hat{f}(\xi)^2\hat{\psi}(\varepsilon^{-1}b\xi)\,d\xi \right| \geq \int |\hat{f}(\xi)|^2|\hat{\psi}(\varepsilon^{-1}b\xi)|^2\,d\xi = \int g(x)^2\,dx \geq \left( \int_{[0,1]^d} g(x)\,dx \right)^2.
\]

Establishing (35) therefore reduces to showing that if \( \varepsilon \ll |A|^3 \), then

\[
\int_{[0,1]^d} g(x)\,dx \geq (1 - C\varepsilon)|A|
\]

for some constant \( C > 0 \). To establish (37) we use (31) and write

\[
|A| = \int_{\mathbb{R}^d} g(x)\,dx = \int_{[0,1]^d} g(x)\,dx + \int_{\{x \in \mathbb{R}^d : \text{dist}(x,[0,1]^d) \geq \varepsilon^{-1}b\}} g(x)\,dx + \int_{\{x \in \mathbb{R}^d : 0 < \text{dist}(x,[0,1]^d) < \varepsilon^{-2}b\}} g(x)\,dx.
\]

The fact that \( b \leq \varepsilon^4 \) ensures that

\[
\left| \{ x \in \mathbb{R}^d : 0 < \text{dist}(x,[0,1]^d) < \varepsilon^{-2}b\} \right| \ll \varepsilon^2
\]

and hence, since \( \varepsilon \ll |A| \) and \( 0 \leq g \leq 1 \), that

\[
\int_{\{x \in \mathbb{R}^d : 0 < \text{dist}(x,[0,1]^d) < \varepsilon^{-2}b\}} g(x)\,dx \ll \varepsilon^2 \leq \varepsilon|A|
\]

while (34) ensures that

\[
\int_{\{x \in \mathbb{R}^d : \text{dist}(x,[0,1]^d) \geq \varepsilon^{-2}b\}} g(x)\,dx \leq |A| \int_{|y| > \varepsilon^{-2}b} \psi_{\varepsilon^{-1}b}(y)\,dy \ll \varepsilon|A|
\]

which completes the proof. \( \square \)

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