A Note on Quantum Hamming Bound

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Proving the quantum Hamming bound for degenerate nonbinary stabilizer codes has been an open problem for a decade. In this note, I prove this bound for double error-correcting degenerate stabilizer codes. Also, I compute the maximum length of single and double error-correcting MDS stabilizer codes over finite fields.

1 Bounds on Quantum Codes

Quantum stabilizer codes are a known class of quantum codes that can protect quantum information against noise and decoherence. Stabilizer codes can be constructed from self-orthogonal or dual-containing classical codes, see for example [3, 8, 11] and references therein. It is desirable to study upper and lower bounds on the minimum distance of classical and quantum codes, so the computer search on the code parameters can be minimized. It is a well-known fact that Singleton and Hamming bounds hold for classical codes [10]. Also, upper and lower bounds on the achievable minimum distance of quantum stabilizer codes are needed. Perhaps the simplest upper bound is the quantum Singleton bound, also known as the Knill-Laflamme bound. The binary version of the quantum Singleton bound was first proved by Knill and Laflamme in [12], see also [1, 2], and later generalized by Rains using weight enumerators in [16].

Theorem 1 (Quantum Singleton Bound). An \((n, K, d)_q\) stabilizer code with \(K > 1\) satisfies

\[ K \leq q^{n-2d+2}. \]  

Codes which meet the quantum Singleton bound with equality are called quantum MDS codes. If we assume that \(K = q^k\), then this bound can be stated as \(k \leq n - 2d + 2\). In [11] It has been shown that these codes cannot be indefinitely long and showed that the maximal length of a \(q\)-ary quantum MDS codes is upper bounded by \(2q^2 - 2\). This could probably be tightened to \(q^2 + 2\). It would be interesting to find quantum MDS codes of length greater than \(q^2 + 2\) since it would disprove the MDS conjecture for classical codes [10]. A related open question is regarding the construction of codes with lengths between \(q\) and \(q^2 - 1\). At the moment there are no analytical methods for constructing a quantum MDS code of arbitrary length in this range (see [9] for some numerical results).

Another important bound for quantum codes is the quantum Hamming bound. The quantum Hamming bound states (see [6, 7]) that:
Theorem 2 (Quantum Hamming Bound). Any pure \((n, K, d)_q\) stabilizer code satisfies

\[
\sum_{i=0}^{\lfloor (d-1)/2 \rfloor} \binom{n}{i} (q^2 - 1)^i \leq q^n / K.
\]  

The previous quantum Hamming bound holds only for nondegenerate (pure) quantum codes. However, the degenerate (impure) quantum codes are particularly interesting class of quantum codes because they can pack more quantum information. In addition, the errors of small weights do not need active error correction strategies.

So far no degenerate quantum code has been found that beats this bound. Gottesman showed that impure single and double error-correcting binary quantum codes cannot beat the quantum Hamming bound \[8\]. It is proved in \[11\] that Hamming bound holds for quantum stabilizer codes with distance \(d = 3\).

In general, does Hamming bound exist for any distance \(d\) in \((n, K, d)_q\) stabilizer codes? This has been an open question for a decade. In this note we prove Hamming bound for double error-correcting stabilizer codes with distance \(d = 5\) and also give a sketch to prove it for general distance \(d\).

2 Quantum Hamming Bound Holds for Distance \(d = 5\)

There have been several approaches to prove bounds on the quantum code parameters. In \[1\] Ashikhmin and Litsyn derived many bounds for quantum codes by extending a novel method originally introduced by Delsarte \[5\] for classical codes. Using this method they proved the binary versions of Theorems 1,2. We use this method to show that the Hamming bound holds for all double error-correcting quantum codes. See \[11\] for a similar result for single error-correcting codes. But first we need Theorem 3 and the Krawtchouk polynomial of degree \(j\) in the variable \(x\),

\[
K_j(x) = \sum_{s=0}^{j} (-1)^s(q^2 - 1)^{j-s} \binom{x}{s} \binom{n-x}{j-s}.
\]

Theorem 3. Let \(Q\) be an \((n, K, d)_q\) stabilizer code of dimension \(K > 1\). Suppose that \(S\) is a nonempty subset of \(\{0, \ldots, d-1\}\) and \(N = \{0, \ldots, n\}\). Let

\[
f(x) = \sum_{i=0}^{n} f_i K_i(x)
\]

be a polynomial satisfying the conditions

i) \(f_x > 0\) for all \(x\) in \(S\), and \(f_x \geq 0\) otherwise;

ii) \(f(x) \leq 0\) for all \(x\) in \(N \setminus S\).

Then

\[
K \leq \frac{1}{q^n} \max_{x \in S} \frac{f(x)}{f_x}.
\]
Proof. See [11].

We demonstrate usefulness of the previous Theorem by showing that quantum Hamming bound holds for impure codes when \( d = 5 \).

**Lemma 4** (Quantum Hamming Bound). An \((n,K,5)\) stabilizer code with \( K > 1 \) satisfies

\[
K \leq q^n / \left( n(n-1)(q^2 - 1)^2/2 + n(q^2 - 1) + 1 \right).
\]

Proof. Let \( f(x) = \sum_{j=0}^{n} f_jK_j(x) \), where \( f_x = (\sum_{j=0}^{n} K_j(x))^2 \), \( S = \{0, 1, \ldots, 4\} \) and \( N = \{0, 1, \ldots, n\} \). Calculating \( f(x) \) and \( f_x \) gives us

\[
\begin{align*}
    f_0 &= (1 + n(q^2 - 1) + n(n-1)(q^2 - 1)^2/2)^2 \\
    f_1 &= \frac{1}{4}(n-1)^2(n-2)^2(q^2-1)^4 \\
    f_2 &= \frac{1}{2}(n-3)(n-2)(q^2-1)^2 - (n-2)(q^2-1)^2 \\
    f_3 &= (1 - 2(n-3)(q^2-1) + \frac{1}{2}(n-4)(n-3)(q^2-1)^2)^2 \\
    f_4 &= (3 - 3(n-4)(q^2-1) + \frac{1}{2}(n-5)(n-4)(q^2-1)^2)^2 \\
\end{align*}
\]

and,

\[
\begin{align*}
    f(0) &= q^{2n}(1 + n(q^2 - 1) + \frac{1}{2}(n-1)n(q^2 - 1)^2) \\
    f(1) &= q^{2n}(q^2 + 2(n-1)(q^2 - 1) + (n-1)(q^2 - 2)(q^2 - 1)) \\
    f(2) &= q^{2n}(4 + 4(q^2 - 2) + (q^2 - 2)^2 + 2(n-2)(q^2 - 1)) \\
    f(3) &= q^{2n}(6 + 6(q^2 - 2)) \\
    f(4) &= 6q^{2n}.
\end{align*}
\]

Clearly \( f_x > 0 \) for all \( x \in S \). Also, \( f(x) \leq 0 \) for all \( x \in N\setminus S \) since the binomial coefficients for the negative values are zero. The Hamming bound is given by

\[
K \leq q^{-n} \max_{x \in S} \frac{f(x)}{f_x} \tag{7}
\]

So, there are four different comparisons where \( f(0)/f_0 \geq f(x)/f_x \), for \( x = 1, 2, 3, 4 \). We find a lower bound for \( n \) that holds for all values of \( q \). From Lemmas 5, 6, 7, and 8 shown below, for \( n \geq 7 \) it follows that

\[
\max\{f(0)/f_0, f(1)/f_1, f(2)/f_2, f(3)/f_3, f(4)/f_4\} = f(0)/f_0 \tag{8}
\]

While the above method is a general method to prove Hamming bound for impure quantum codes, the number of terms increases with a large minimum distance. It becomes difficult to find the true bound using this method. However, one can derive more consequences from Theorem 8, see, for instance, [1, 2, 13, 15].

3
**Lemma 5.** The inequality \( f(0)/f_0 \geq f(1)/f_1 \) holds for \( n \geq 6 \) and \( q \geq 2 \).

**Proof.** Let \( f(0)/f_0 \geq f(1)/f_1 \) then

\[
\frac{1}{1 + n(q^2 - 1) + n(n-1)(q^2 - 1)^2/2} \geq \frac{4q^2((n-1)(q^2 - 1) + 1)}{(n-1)^2(n-2)(q^2 - 1)^4}
\]

\[
(n-1)^2(n-2)^2(q^2 - 1)^4 \geq (1 + n(q^2 - 1) + \frac{n(n-1)}{2}(q^2 - 1)^2)(4q^2((n-1)(q^2 - 1) + 1))
\]

in the left side \((n-1)\) approximates to \((n-2)\). Also, in the right side \((n-2)\) and \((n-1)\) approximate to \((n)\). So,

\[
(n-2)^4(q^2 - 1)^4 \geq 4(1 + n(q^2 - 1) + \frac{n^2}{2}(q^2 - 1)^2)(q^2(q^2 - 1)(n-1) + 1)
\]

divide both sides by \((q^2 - 1)(q^2 - 1)^2\) and approximate \(\frac{1}{q^2 - 1} \leq 1\), we find that

\[
(n-2)^4 \geq 8(1 + n + \frac{n^2}{2})(n-1)
\]

by approximating both sides, the final result is \((n-2) \geq 4\) or \(n \geq 6\).

\[\square\]

**Lemma 6.** The inequality \( f(0)/f_0 \geq f(2)/f_2 \) holds for \( n \geq 7 \) and \( q \geq 2 \).

**Proof.** Let

\[
\frac{q^{2n}}{1 + n(q^2 - 1) + n(n-1)(q^2 - 1)^2/2} \geq \frac{q^{2n}(q^4 + 2(n-2)(q^2 - 1))}{(-(n-2)(q^2 - 1) + (n-3)(n-2)(q^2 - 1)^2/2)^2}
\]

by simplifying both sides

\[
(-(n-2)(q^2 - 1) + (n-3)(n-2)(q^2 - 1)^2/2)^2 \geq (q^4 + 2(n-2)(q^2 - 1))(1 + n(q^2 - 1) + n(n-1)(q^2 - 1)^2/2)
\]

Simplifying L.H.S, \((n-2)\) to \((n-3)\) then

\[
(q^2 - 1)^4((n-3)^2/2 - (n-2))^2 \geq (q^4 + 2(n-2)(q^2 - 1))(1 + n(q^2 - 1) + n(n-1)(q^2 - 1)^2/2)
\]
by simplifying both sides

\[
\frac{(n - 3)^2 / 2 - (n - 2)^2}{2} \geq \frac{2(n - 2)}{q^2 - 1} (1 + n + n(n - 1)/2)
\]

\[
\frac{(n - 3)^2 / 2 - (n - 2)^2}{2} \geq 2(n + 1)(n^2 + 2n + 2)
\]

\[
(n - 3)^2((n - 3)/2 - 1)^2 \geq 2(n + 1)((n + 1)^2 + 1)
\]

\[
(n - 5)^2/4 \geq 2(n + 1)
\]

\[
n \geq 7
\]

Lemma 7. The inequality \( f(0)/f_0 \geq f(3)/f_3 \) holds for \( n \geq 7 \) and \( q \geq 2 \).

Proof. Let

\[
\frac{q^{2n}}{1 + n(q^2 - 1) + n(n - 1)(q^2 - 1)^2/2} \geq \frac{6q^{2n}(q^2 - 1)}{(1 - 2(n - 3)(q^2 - 1) + (n - 3)(n - 4)(q^2 - 1)^2/2)^2}
\]

by simplification

\[
\frac{(1 - 2(n - 3)(q^2 - 1) + (n - 3)(n - 4)(q^2 - 1)^2/2)^2}{6q^2 - 1)(1 + n(q^2 - 1) + n(n - 1)(q^2 - 1)^2/2)} \geq \frac{4(n - 4)^2(q^2 - 1)^2 - 4(n - 4)(q^2 - 1) + 2)^2}{6(q^2 - 1)(1 + n(q^2 - 1) + n(n - 1)(q^2 - 1)^2/2)}
\]

by approximation to \((q^2 - 1)^4\)

\[
\frac{(q^2 - 1)^4}{4} \geq 3(q^2 - 1)^3(2 + 2n + n^2)
\]

\[
(n - 5)^4 \geq 4(2 + 2n + n^2)
\]

\[
(n - 5)^2 \geq 2
\]

\[
n \geq 7
\]

Lemma 8. The inequality \( f(0)/f_0 \geq f(4)/f_4 \) holds for \( n \geq 7 \) and \( q \geq 2 \).

Proof. Let

\[
\frac{q^{2n}}{1 + n(q^2 - 1) + n(n - 1)(q^2 - 1)^2/2} \geq \frac{6q^{2n}}{(3 - 3(n - 4)(q^2 - 1) + (n - 4)(n - 5)(q^2 - 1)^2/2)^2}
\]

divide by \(q^{2n}\) and simplifying

\[
(3 - 3(n - 4)(q^2 - 1) + (n - 4)(n - 5)(q^2 - 1)^2/2)^2 \geq \frac{6(1 + n(q^2 - 1) + n(n - 1)(q^2 - 1)^2/2)}{}
\]
then by approximating \((n - 4)\) to \((n - 5)\) in L.H.S and \((n - 1)\) to \(4n\) in R.H.S, we can find that

\[
\begin{align*}
-3(n - 4)(q^2 - 1) + (n - 5)^2(q^2 - 1)^2/2 & \geq 6(1 + n(q^2 - 1) + n^2(q^2 - 1)^2/2) \\
(n - 5)^2(q^2 - 1) + (n - 5)^2(q^2 - 1)^2/2 & \geq 6(1 + n(q^2 - 1) + n^2(q^2 - 1)^2/2) \\
(q^2 - 1)(n - 5)^2 + (n - 5)^2/2 & \geq 6(1 + n(q^2 - 1) + n^2(q^2 - 1)^2/2)
\end{align*}
\]

dividing both sides by \((q^2 - 1)^4\) and simplifying

\[
(9/4)(n - 5)^4 \geq \frac{6(1 + n(q^2 - 1) + n^2(q^2 - 1)^2/2)}{(q^2 - 1)^4}
\]

\[
\begin{align*}
(9/4)(n - 5)^4 & \geq 6(1 + n + n^2) \\
\frac{n}{n} & \geq \frac{7}{(q^2 - 1)^4}
\end{align*}
\]

Since it is not known if the quantum Hamming bound holds for degenerate nonbinary quantum codes, it would be interesting to find degenerate quantum codes that either meet or beat the quantum Hamming bound. This is obviously a challenging open research problem.

## 3 Maximal Length of MDS Codes

In this section we derive some results on the maximal length of single and double error-correcting quantum MDS codes. These bounds hold for all additive quantum codes.

### 3.1 Maximal Length Single Error-correcting MDS Codes

**Lemma 9.** The maximal length of single error correcting additive quantum MDS codes is given by \(q^2 + 1\).

**Proof.** We know that the quantum Hamming bound holds for \(K > 1\) for \(d = 3\), so

\[
K \leq \frac{q^n}{1 + n(q^2 - 1)} \tag{9}
\]

If the Hamming bound is tighter than the Singleton bound for any \(((n, K, 3))_q\) quantum code, then it means that MDS codes cannot exist for that set of \(n, K\). This occurs when

\[
\begin{align*}
q^{n-2d+2} = q^{n-4} & \geq \frac{q^n}{1 + n(q^2 - 1)} \\
1 + n(q^2 - 1) & \geq q^4 \\
n & \geq q^2 + 1
\end{align*}
\]

Thus there exist no single error correcting quantum MDS codes for \(n > q^2 + 1\). \(\mathbb{Q}\)
3.2 Upper Bound on the Maximal Length of Double Error-correcting MDS Codes

Lemma 10. The maximal length of double error-correcting quantum MDS codes is upper bounded by:

\[ n \leq \frac{(q^2 - 3) + \sqrt{((q^2 - 3) + 8(q^8 - 1))/(q^2 - 1)}}{2} \] (11)

Proof. It is known that the Hamming bound for \( d = 5 \) is given by:

\[ K \leq \frac{q^n}{1 + n(q^2 - 1) + n(n - 1)(q^2 - 1)^2/2} \] (12)

If the Hamming bound is tighter than the Singleton bound for any \((n, K, 5)\) quantum code, then it means that MDS codes cannot exist for that set of code parameters. By simple computation, we find that

\[ q^{n-2d+2} = q^{n-8} \geq \frac{q^n}{1 + n(q^2 - 1) + \frac{n(n-1)}{2}(q^2 - 1)^2/2} \]

\[ n^2(q^2 - 1)^2 - n(q^2 - 1)(q^2 - 3) - 2(q^8 - 1) \geq 0 \] (13)

So, the quadratic equation of \( n \) has two real solutions. This inequality holds for

\[ n \geq \frac{(q^2 - 3) - \sqrt{(q^2 - 3)^2 + 8(q^8 - 1)}}{2(q^2 - 1)} \] (14)
\[ n \leq \frac{(q^2 - 3) + \sqrt{(q^2 - 3)^2 + 8(q^8 - 1)}}{2(q^2 - 1)} \] (15)

Only the positive solution for \( n \) is valid. So, the maximal length of double error-correcting MDS code is upper bounded by

\[ n \leq \frac{(q^2 - 3) + \sqrt{(q^2 - 3)^2 + 8(q^8 - 1)}}{2(q^2 - 1)} \] (16)

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4 Appendix

An Approach (Sketch) to Prove Hamming Bound for Degenerate Nonbinary Stabilizer Codes with Minimum Distance \( d \)

One way to prove the quantum Hamming bound for impure nonbinary stabilizer codes with \( d \leq (n - k + 2)/2 \) is to expand \( f(x)/f_x \) in terms of Krawtchouk polynomials. Let \( f(x) = \sum_{j=0}^{n} f_j K_j(x) \) and \( f_x = (\sum_{i=0}^{e} K_i(x))^2 \). The Krawtchouk polynomial of degree \( e \) in the variables \( x \) and \( q \) is given by

\[
K_e(q, x) = \sum_{j=0}^{e} (-1)^j (q^2 - 1)^{-j} \binom{n-x}{j} \binom{n-x}{e-j} \tag{17}
\]

**Theorem 11.** Let \( Q \) be an \( (n, K, d)_q \) stabilizer code of dimension \( K > 1 \). Suppose that \( S \) is a nonempty subset of \( \{0, 1, \ldots, d-1\} \) and \( N = \{0, 1, \ldots, n\} \). Let

\[
f(x) = \sum_{i=0}^{n} f_i K_i(x)
\]

be a polynomial satisfying the conditions:

i) \( f_x > 0 \) for all \( x \in S \), and \( f_x \geq 0 \) otherwise;

ii) \( f(x) \leq 0 \) for all \( x \in N \backslash S \).

Then

\[
K \leq \frac{1}{q^n} \min_{x \in S} \frac{f(x)}{f_x}.
\]

Notice that \( f(x) = \sum_{i=0}^{n} f_i K_i(x) \) can be written as \( f_i = q^{-2n} \sum_{x=0}^{n} f(x) K_x(i) \).

**Lemma 12 (Sketch).** Let \( Q \) be an \( ((n, K, d))_q \) stabilizer code of dimension \( k \geq 1 \). Suppose that \( S \) is a non-empty subset of \( \{0,1,2,\ldots,2e\} \), where \( e = \left\lfloor \frac{d-1}{2} \right\rfloor \). The Hamming bound is given by

\[
K \leq q^{-n} \max_{x \in S} \frac{f(x)}{f_x} \quad \text{equals to}
\]

\[
K \leq \frac{q^n}{\sum_{i=0}^{e} \binom{n}{i} (q^2 - 1)^i} \tag{18}
\]

If and only if \( f(0)/f_0 \) is the maximum value for \( d \geq 3 \) and \( n \geq n_0 \).

**Proof.** In this proof, we propose \( f_x \) satisfying Theorem [11]. Let \( f_x = \left( \sum_{i=0}^{e} K_i(x) \right)^2 \) and \( f(x) = \sum_{j=0}^{n} f_j K_j(q, x) \).
\[
\frac{f(x)}{f_x} = \sum_{j=0}^{n} f_j K_j(q, x)
\]  

(19)

And our goal is to find \(\max\{f(0)/f_0, f(1)/f_1, \ldots, f(d-1)/f_{d-1}\}\) that may equal to \(f(0)/f_0\).

Now, for \(x = 0\), we find that

\[
\frac{f(0)}{f_0} = \sum_{j=0}^{n} \frac{f_j K_j(0)}{f_0} = K_0(q, 0) + \frac{f_1 K_1(q, 0)}{f_0} + \ldots + \frac{f_n K_n(q, 0)}{f_0}
\]  

(20)

or

\[
\frac{f(0)}{f_0} = \sum_{j=0}^{n} \left( \frac{\sum_{i=0}^{c} K_i(j)}{\sum_{i=0}^{c} K_i(0)} \right)^2 K_j(0)
\]

and for any other value of \(y \in \{1, 2, \ldots, d-1\}\), we find that

\[
\frac{f(y)}{f_y} = \sum_{j=0}^{n} \frac{f_j K_j(y)}{f_y} = \frac{f_0 K_0(q, y)}{f_y} + \frac{f_1 K_1(q, y)}{f_y} + \ldots + \frac{f_n K_n(q, y)}{f_y}
\]  

(21)

or

\[
\frac{f(y)}{f_y} = \sum_{j=0}^{n} \left( \frac{\sum_{i=0}^{c} K_i(j)}{\sum_{i=0}^{c} K_i(0)} \right)^2 K_j(y)
\]

(22)

From (20) and (21) simply we need to show that

\[
\frac{f(0)}{f_0} - \frac{f(y)}{f_y} \geq 0
\]  

(22)

\[
\frac{f(0)}{f_0} - \frac{f(y)}{f_y} = \sum_{j=0}^{n} \left( \sum_{i=0}^{c} K_i(j) \right)^2 K_j(0) - \sum_{j=0}^{n} \left( \sum_{i=0}^{c} K_i(j) \right)^2 K_j(y)
\]

\[
= \sum_{j=0}^{n} \left( \frac{\sum_{i=0}^{c} K_i(j)}{\sum_{i=0}^{c} K_i(0)} \right)^2 \left( \frac{K_j(0)}{\sum_{i=0}^{c} K_i(0)} - \frac{K_j(y)}{\sum_{i=0}^{c} K_i(y)} \right)
\]

\[
= \sum_{j=0}^{n} \left( \frac{f_j K_j(q, 0)}{f_0} - \frac{f_j K_j(q, y)}{f_y} \right)
\]  

(23)
in the previous equation, $f_j > 0$ and $f_y > 0$, so, if we prove that

$$
\frac{f_j K_j(q,0)}{f_0} - \frac{f_j K_j(q,y)}{f_y} \geq 0,
$$

(24)

then the claim holds. As shown in [4], [14], we seek a constant value for the left side in (24) so, multiplying both sides by $K_e(i)$

$$
\frac{K_e(i) K_i(q,0)}{f_0} - \frac{K_e(i) K_i(q,y)}{f_y} \geq 0
$$

(25)

and take $\sum_{i=0}^{n} K_e(i)$

$$
\sum_{i=0}^{n} \left( \frac{K_e(i) K_i(q,0)}{f_0} - \frac{K_e(i) K_i(q,y)}{f_y} \right) \geq 0
$$

$$
\sum_{i=0}^{n} K_e(i) K_i(q,0) - \sum_{i=0}^{n} K_e(i) K_i(q,y) \geq 0
$$

(26)

from [14], given that $\sum_{i=0}^{n} K_e(i) K_i(q,j) = q^n \delta_{ej}$, by substitution,

$$
\frac{q^n \delta_{e0}}{f_0} - \frac{q^n \delta_{ey}}{f_y} \geq 0
$$

$$
\frac{\delta_{e0}}{f_0} - \frac{\delta_{ey}}{f_y} \geq 0
$$

(27)

Now, $\delta_{e0} = 1$, and $\delta_{ey} = 1$ or 0; and obviously $f_y \geq f_0$. So, if $y = e \implies \delta_{e0}/f_0 \geq 0$, and similarly, $\delta_{ey} = 1 \implies f_y - f_0 \geq 0$. \qed