EINSTEIN HERMITIAN METRICS OF NON NEGATIVE SECTIONAL CURVATURE

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Abstract

In this paper we will prove that if $M$ is a compact simply connected Hermitian Einstein 4-manifold with non negative sectional curvature then $M$ is isometric to complex projective space $\mathbb{CP}^2$ with the Fubini-Study metric or $M$ is isometric to a product of two two-spheres $S^2 \times S^2$, with theirs canonical metrics.

1. Introduction

Let $M = M^4$ be a 4-manifold. A Riemannian metric $g$ on $M$ is called Einstein if $M$ has constant Ricci curvature and called Hermitian if $g(J\cdot, J\cdot) = g(\cdot, \cdot)$ for a complex structure $J$ on $M$. In [5], C. LeBrun proved the following

**Theorem 1. (LeBrun)**- Let $(M = M^4, g, J)$ be a compact connected complex surface with metric $g$ and complex structure $J$. If $g$ is Einstein and Hermitian with respect to $J$ then only one of the following holds:

1. $g$ is Kaehler-Einstein with positive Ricci curvature.
2. $M$ is isometric to $\mathbb{CP}^2 \# \mathbb{CP}^2$ and $g$ is the Page metric.
3. $M$ is isometric to $\mathbb{CP}^2 \# \mathbb{CP}^2$ and $g$ is the Chen-LeBrun-Weber metric.

Using the previous theorem, C. Koca proved in [4] the following:

**Theorem 2. (Koca)**-Let $(M = M^4, g, J)$ be a compact complex surface with metric $g$ and complex structure $J$. If $g$ is Einstein and Hermitian with respect to $J$ and $g$ has positive sectional curvature then $M$ is isometric to complex projective space $\mathbb{CP}^2$ with the Fubini-Study metric metric.

Now, consider $M$ a compact simply connected Kaehler-Einstein 4-manifold with non negative sectional curvature. In this case, M. Berger proved in [1], that $M$ is isometric to complex projective space $\mathbb{CP}^2$ with the Fubini-Study metric or isometric to a product of two spheres $S^2 \times S^2$, with theirs canonical metrics.

In the next two sections we will prove that the Page metric and the Chen-LeBrun-Weber metrics no has non negative sectional curvature. This will conclude the proof our main result:

**Theorem 3.** Let $(M = M^4, g, J)$ be a compact simply connected complex surface with metric $g$ and complex structure $J$. If $g$ is Einstein and Hermitian with respect to $J$ and $g$ has non negative sectional curvature then $M$ is isometric to complex projective space $\mathbb{CP}^2$ with the Fubini-Study metric or $M$ is isometric to a product of two spheres $S^2 \times S^2$, with theirs canonical metrics.
2. Page metric

The Page metric (see [6]) lives in connected sum $\mathbb{CP}^2 \# \mathbb{CP}^2$, where $\mathbb{CP}^2$ is the complex projective space $\mathbb{CP}^2$ with opposite orientation. In [4], C. Koca showed that the Page metric has non negative sectional curvature using a computer program like Maple. In this section we will prove this result using a different argument. For this consider the page metric $g$ as in Koca [4]:

$$g = W^2(x)dx^2 + g^2(x)(\sigma^2_1 + \sigma^2_2) + \frac{D^2}{W(x)}\sigma^2_3$$

where $x \in (-1, 1)$,

$$W(x) = \sqrt{\frac{1 - a^2x^2}{3 - a^2 - a^2(1 + a^2)x^2}(1 - x^2)},$$

$$g(x) = \frac{2}{\sqrt{3 + 6a^2 - a^4}}\sqrt{1 - a^2x^2},$$

$$D = \frac{2}{3+a^2}$$

and $a$ is the unique positive root of equation $f(x) = x^4 + 4x^3 - 6x^2 + 12x - 3 = 0$. Notice that $a < 1$, since that $f(0) = -3$ and $f(1) = 8$.

In accord with Koca, there exists a two-plane where the sectional curvature satisfies

$$K_{01} = 2\left[\frac{gW'}{gW^3} - \frac{g''W}{gW^3}\right].$$

Then we have

$$K_{01} = -\frac{2}{gW}F',$$

where

$$F = \frac{g'}{W}.$$

**Claim:** There exist $c \in [0, 1)$ such that $F'(c) > 0$.

**Proof of Claim:**

Notice that $g' = -Ax(1 - a^2x^2)^{-1/2}$, where $A = \frac{2a}{\sqrt{3+6a^2-a^4}} > 0$. Moreover,

$$F = -Ax\sqrt{3 - a^2 - a^2(1 + a^2)x^2}(1 - x^2)(1 - a^2x^2)^{-1},$$

where $1 - a^2x^2 > 0$. Assumes that $F'(x) \leq 0$ for all $x \in [0, 1)$.

Then $F$ is a increasing function in $[0,1)$ and follows of this that $F(y) \leq F(x) \leq F(0) = 0$, for all $y > x \in [0, 1)$. Since that $F$ is continuo in $[0,1]$, we have that $0 = F(1) \leq F(x) \leq 0$, for all $x \in [0, 1)$. So $F = 0$ in $[0,1)$ (contradiction)

This proves that there exists points where $K_{01} < 0$. 
3. Chen-LeBrun-Weber metric

In [2], Chen, LeBrun and Weber proved that \( M = \mathbb{CP}^2 \# 2 \mathbb{CP}^2 \) admits an Hermitian non-Kaehler Einstein metric \( g \). In particular, there exists a Kaehler metric \( h \) on \( M \) of positive scalar curvature \( s \) such that \( g = s^2 h \). Now consider \( W^+_h \) and \( W^+_g \) the self-dual Weyl part of the Weyl tensor \( W \) of the respective metrics \( h \) and \( g \). Since that \( h \) and \( g \) are conformally related we have \( W^+_g = \frac{1}{s^2} W^+_h \). On the other hand, the self-dual tensor \( W^+_h \) of the Kaehler metric \( h \) has exactly two different eigenvalues and so \( W^+_g \) has also two different eigenvalues. By Proposition 4 of Derdzinski [3], \( M \) admits an non trivial Killing vector field with respect to metric \( g \). Assumes that \( M \) has non negative sectional curvature with respect to metric \( g \). Then \( M \) is a compact simply connected 4-manifold with non negative sectional curvature and with a non trivial Killing vector field. By Theorem 1 of Searle and Yang in [7], we have that the Euler characteristic of \( M \) satisfies \( \chi(M) \leq 4 \) which contradicts the fact of that \( \chi(\mathbb{CP}^2 \# 2 \mathbb{CP}^2) = 5 \).

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