Remarks on the probability of finding a particle in the interaction zone of a scattering process

Alfonso Jaimes-Nájera
Physics Department, Cinvestav, A.P. 14-740, México DF 07000, Mexico
E-mail: ajaimes@fis.cinvestav.mx

Abstract. The probability of finding a particle in the interaction zone of a scattering process is analyzed in one dimension. The quantum state of the particle is described by a Gaussian wave packet that is narrow in the momentum distribution. We find the intervals of time that are required in order to get a maximum probability in the interaction zone as a lower bound of the duration of the scattering process. As an immediate application, times involved in the scattering associated with symmetric potentials (as the rectangular barriers and wells) are calculated.

1. Introduction

The intervals of time involved in a decaying or scattering process are relevant in contemporary quantum mechanics [1, 2]. Sound examples are the tunneling time [3–8], the delay produced in the crossover of particles in a given region of space [9–12], and the lifetimes associated to the resonance phenomenon [13, 14]. In this contribution we are interested in determining lower bounds to the duration of scattering processes in one dimension. Our motivation is mainly due to the fact that the reflection and transmission times associated to scattering depend on the profile of the scatterer, and the non-locality of the wave functions introduces a lot of difficulties in the determination of such times. That is, no matter what is the definition used to calculate these times, it is not clear when the scattering process has been concluded since the tails of the incoming and outgoing waves are formally different from zero in all the real line for any time. One option is to say that the scattering process is finished whenever the profile of the transmitted and reflected waves is well defined. However, the values of time required to fulfill such a condition might be extremely large, depending on the scattering potential we are interested in. Therefore, we propose to calculate the interval of time \( (0, t^*_s) \) required to get a probability of finding the particle in the interaction zone that is maximum for a given scatterer. At that times the scattering process is still in progress, so that the transmitted and reflected times should not be shorter than \( t^*_s \).

The organization of the paper is as follows. In Section 2 we revisit the generalities of the one-dimensional scattering of a Gaussian wave packet. Then, we calculate the probability of finding the particle in the interaction zone and make the assumption that the initial wave packet is very sharp in the momenta distribution. Conditions for the extremal probability are also found. In Section 3 we apply our approach to the case of symmetric scattering potentials. Final remarks are given in Section 4.
2. Scattering of Gaussian wave packets in one dimension

Consider a one dimensional scattering process defined by a function of compact support $V(x)$. That is, the potential $V(x)$ is different from zero in $(a,b) \in \mathbb{R}$ and null everywhere in $\mathbb{R} \setminus (a,b)$, see e.g. [15]. Let us assume that the particle is injected from the right at $t = 0$ with a momentum $k_0$. Thus, we can represent the quantum state of the particle by a wave packet of Gaussian profile in the momentum space $\Lambda(k)$ that is centered at $k = k_0$ in the initial spatial coordinate $x = x_0$. Using dimensionless variables we write

$$\Psi(x,t) = \int_{-\infty}^{\infty} dk \Lambda(k) \psi_k(x)e^{i(kx_0-\omega t)},$$

(1)

where $\psi_k(x)$ is solution of the stationary Schrödinger equation with eigenvalue $\omega = k^2$, and the coefficient of the superposition

$$\Lambda(k) = \frac{1}{\sqrt{\pi \sigma \Delta}} e^{-(k-k_0)^2/2\sigma^2},$$

(2)

is a normalized Gaussian distribution of width $\sigma$ and normalization constant $\Delta$. The center of this packet behaves like a classical particle of velocity equal to the packet’s group velocity:

$$v_g(k_0) = \frac{d\omega}{dk} \bigg|_{k=k_0} = 2k_0.$$

(3)

We are interested in conservative processes where the initial Gaussian profile in momentum space is preserved (neither sources nor sinks of probability are allowed). Then, to the right of the scatterer ($x > b$) one should get a superposition of incident $\Psi_{inc}(x,t)$ and reflected $\Psi_r(x,t)$ wave packets of the form

$$\Psi_{inc}(x,t) = \int_{-\infty}^{\infty} dk \Lambda(k)e^{-i(k(x-x_0)+\omega t)},$$

(4)

and

$$\Psi_r(x,t) = \int_{-\infty}^{\infty} dk \Lambda(k)S_R(k)e^{i(k(x+x_0)-\omega t - \zeta)},$$

(5)

where $S_R(k)e^{-i\zeta(k)}$ is the reflection amplitude in polar form and $\zeta(k)$ is the reflection phase shift. To the left of the scatterer only a transmitted wave packet is expected

$$\Psi_t(x,t) = \int_{-\infty}^{\infty} dk \Lambda(k)S_T(k)e^{-i(k(x-x_0)+\omega t + \gamma)},$$

(6)

with $S_T(k)e^{-i\gamma(k)}$ the transmission amplitude in polar form and $\gamma(k)$ the transmission phase shift.

2.1. Finding the particle in the interaction zone

The idea of working with compact support functions $V(x)$ is to represent short range potentials in the scattering process. We define the support $(a,b)$ of $V(x)$ as the interaction zone of the scattering. Our principal interest is to calculate the time at which the probability $P_{(a,b)}$ of finding the particle in the interaction zone is maximum. This information will be used to get intervals of time, bounded from below, for the complete scattering process. That is, if at $t = t_*$ the probability of finding the particle in $(a,b)$ is maximum, then the scattering is still in progress.
As the initial wave packet $\Psi(x,t)$ is normalized, the probability of finding the particle in the interaction zone $(a,b)$ is

$$P_{(a,b)}(t) = \int_a^b dx \left| \Psi(x,t) \right|^2 = 1 - \int_{-\infty}^a dx \left| \Psi(x,t) \right|^2 - \int_b^\infty dx \left| \Psi(x,t) \right|^2. \quad (7)$$

Considering the composition of the wave packet in the different regions, functions (4), (5) and (6), we get

$$P_{(a,b)}(t) = 1 - \int_{-\infty}^a dx \left| \Psi_t(x,t) \right|^2 - \int_b^\infty dx \left[ \left| \Psi_t(x,t) \right|^2 + \left| \Psi_{inc}(x,t) \right|^2 \right]
- 2 \Re \left[ \int_b^\infty dx \Psi_{inc}^*(x,t) \Psi_t(x,t) \right]. \quad (8)$$

To make the calculations we shall assume that the initial wave packet is (almost) monochromatic so that the width $\sigma$ in (2) is arbitrarily narrow; that is, $\sigma < < 1$. In this form, we hope small variations of the values of $S_R$ and $S_T$ in the vicinity of $k_0$. Thus, in order of removing the amplitudes from the integrals (5) and (6), we respectively make $S_R(k) \approx S_R(k_0) = S_R$ and $S_T(k) \approx S_T(k_0) = S_T$ in the interval $k_0 \pm \frac{1}{2}\sigma$, see e.g. [16]. Moreover, the sharpness of the initial wave packet also allows to expand the phase shifts around $k = k_0$,

$$\gamma(k) = \gamma_0 + \frac{\partial \gamma}{\partial k} \Big|_{k=k_0} (k - k_0) + \cdots, \quad \zeta(k) = \zeta_0 + \frac{\partial \zeta}{\partial k} \Big|_{k=k_0} (k - k_0) + \cdots, \quad (9)$$

and to take into account the linear terms in $k$ only. Therefore, the transmitted and reflected wave packets read as

$$\Psi_t(x,t) = \frac{S_T}{\sqrt{\Delta^2 + \pi^2}} e^{i(k_0 \frac{\sigma^2}{\Delta}(k_0) - \gamma_0)} \int_{-\infty}^\infty dk e^{-(k-k_0)^2/2\sigma^2 - i k (x-x_0 + \frac{\sigma^2}{\Delta}(k_0)) + i k t}, \quad (10)$$

$$\Psi_r(x,t) = \frac{S_R}{\sqrt{\Delta^2 + \pi^2}} e^{i(k_0 \frac{\sigma^2}{\Delta}(k_0) - \zeta_0)} \int_{-\infty}^\infty dk e^{-(k-k_0)^2/2\sigma^2 + i k (x-x_0 + \frac{\sigma^2}{\Delta}(k_0)) - i k t}, \quad (11)$$

where the explicit form of $\Lambda(k)$ given in (2) has been introduced. The probability (8) reads now as follows

$$P_{(a,b)}(t) = \frac{1}{2\Delta} \left\{ \text{erf} \left[ \frac{1}{\sigma_x} (2k_0 t - (x_0 - b)) \right] - \left( S_T \right)^2 \text{erf} \left[ \frac{1}{\sigma_x} \left( 2k_0 t - \left( x_0 - b + \frac{\partial \phi_t}{\partial k} \Big|_{k=k_0} \right) \right) \right] \right. \right.
- \left( S_R \right)^2 \text{erf} \left[ \frac{1}{\sigma_x} \left( 2k_0 t - \left( x_0 - b + \frac{\partial \phi_r}{\partial k} \Big|_{k=k_0} \right) \right) \right]
- S_R e^{-c} \left[ e^{i\varphi} \left( \text{erf} \left[ \frac{d}{\sigma_x} \right] + 1 \right) + \text{c.c.} \right] + 2(\Delta - 1) \right\}, \quad (12)$$

where erf($z$) stands for the error function of $z$ [17] and c.c. means complex conjugate; other expressions involved in (12) are the phase shifts

$$\phi_t = k(b - a) - \gamma, \quad \phi_r = 2kb - \zeta, \quad \varphi = k_0 \frac{\partial \zeta}{\partial k} \Big|_{k=k_0} - \zeta_0, \quad (13)$$

the spatial shift

$$\sigma_x = \sqrt{\frac{1}{\sigma^2} + 4\sigma^2 t^2}, \quad (14)$$

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt e^{-t^2}.$$
the interference argument
\[ d = i \left( \frac{k_0}{\sigma^2} + 2\sigma^2 t \left( x_0 - b + \frac{1}{2} \frac{\partial \phi_t}{\partial k} \bigg|_{k=k_0} \right) \right) - \frac{1}{2} \frac{\partial \phi_t}{\partial k} \bigg|_{k=k_0} \] (15)
and the constants
\[ c = \frac{k_0^2}{\sigma^2} + \sigma^2 \left( x_0 - b + \frac{1}{2} \frac{\partial \phi_t}{\partial k} \bigg|_{k=k_0} \right)^2, \]
\[ \Delta = 1 + e^{-c} S_R \left( \cos \varphi - \text{erfi} \left[ \sigma \left( x_0 - b + \frac{1}{2} \frac{\partial \phi_t}{\partial k} \bigg|_{k=k_0} \right) \right] \sin \varphi \right). \] (16)
In the sequel, it will be useful to rewrite the derivative with respect to the momentum \( k \) as a derivative with respect to the energy
\[ \frac{\partial}{\partial E} = \frac{1}{2k} \frac{\partial}{\partial k}. \] (17)
In this form the expressions
\[ \tau_t = \frac{\partial \phi_t}{\partial E} \bigg|_{E=k_0^2}, \quad \tau_r = \frac{\partial \phi_t}{\partial E} \bigg|_{E=k_0^2}, \] (18)
correspond to the transmission and reflection phase times, respectively (for details see [10–12,18] and references quoted therein). Using the phase times (18) in Eqs. (12)–(16), the notation is notably simplified.

Figure 1. Probability in the interaction zone \( P_{(a,b)} \) as a function of time for a rectangular barrier (left) and a rectangular well (right). In both cases \( k_0 = 1.0724, x_0 = 30, b = -a = 3\pi/2 \), and the time \( t_b \) spent by the particle in crossing the distance between \( x_0 \) and \( b \) is \( t_b = (x_0 - b)/(2k_0) = 11.79 \). The height \( V_0 = 1 \) of the barrier implies \( S_T^2 = 0.744 \) and \( \tau_t = \tau_r = 13.0671 \), while the depth \( V_0 = -1 \) of the well leads to \( S_T^2 = 0.916 \) and \( \tau_t = \tau_r = 3.073 \). The curves correspond to the width \( \sigma = 0.55 \) (black), \( \sigma = 0.43 \) (red), and \( \sigma = 0.1 \) (blue).

Our expression for the probability \( P_{(a,b)} \) in (12) encodes the information of the scatterer in the amplitudes \( S_R, S_T \), and in the phase (times) shifts \( \phi_t \) and \( \phi_t \) (\( \tau_t \) and \( \tau_r \)). In Fig. 1 we have depicted the probability \( P_{(a,b)} \) associated with a rectangular barrier (left) and a rectangular well (right) as a function of time. In both cases the values of \( S_R, S_T, \tau_t \) and \( \tau_r \) have been calculated for \( (a,b) = (-3\pi/2,3\pi/3) \). The well has a depth \( V_0 = -1 \) and the barrier a height \( V_0 = 1 \). The time scale goes from \( t = 0 \), the moment in which the particle crosses the point \( x_0 \), till an
arbitrary value \( t \) at which the probability goes to zero. Initially, \( P_{(a,b)} \) is negligible and reaches its maximum at a time \( t = t_* \) that is smaller than \( \min \{ \tau_r + t_b, \tau_t + t_b \} \), with \( t_b = (x_0 - b)/(2k_0) \) the time spent by the particle in crossing the distance between \( x_0 \) and \( b \). As we can see, the value of the maximum increases as the initial width \( \sigma \) decreases. Thus, the sharpness of the momenta distribution \( \Lambda(k) \) in the initial packet defines the height of \( P_{(a,b)} \): the probability of finding the particle in the interaction zone \((a, b)\) is as great as peaked is the distribution \( \Lambda(k) \) in the value \( k = k_0 \). Another profile of \( P_{(a,b)} \) that can be appreciated is that the time \( t_* \) at which this probability is maximum increases as the width \( \sigma \) of \( \Lambda(k) \) decreases.

2.2. Extreme values of the probability

We are interested in calculating the time \( t = t_* \) at which the probability (12) is maximum. As we have discussed in the previous section, such a time depends on the width \( \sigma \) of the initial wave packet: \( t_* \) is greater as \( \sigma \) is narrower. To get an idea of the influence of the sharpness of \( \Lambda(k) \) on \( t_* \) let us consider the spatial width \( \sigma_x \) defined in (14). Given \( \sigma \), the spatial distributions are broader as the time goes pass. In this context we realize that the approximation

\[
\sigma_x = \frac{1}{\sigma} \sqrt{1 + 4\sigma^4 t^2} \approx \frac{1}{\sigma} (1 + 2\sigma^4 t^2) \tag{19}
\]

holds whenever \( t << \frac{1}{2\sigma} \). That is, we can identify an interval of time \((0, t_0)\) in which the spatial width \( \sigma_x \) is approximately constant \( \sigma_x \approx \sigma^{-1} \); it will be enough to take \( t_0 << \frac{1}{2\sigma} \). Note that this interval is larger as \( \sigma \) is narrower. Within this approximation, one can identify the extreme values of the probability (12) by calculating the zeros of its time-derivative:

\[
\frac{dP_{(a,b)}(t)}{dt} = \frac{4k_0 \sigma}{\sqrt{\pi} (1 + 4t^2 \sigma^4)^{3/2}} \left\{ \left( 1 + 4\sigma^4 t_b t \right) \exp\left[ -\frac{y_0^2}{1 + 4\sigma^4 t} \right] \right. \\
- \left. (S_T)^2 \left[ 1 + 4\sigma^4 t \left( t_b + \tau_t \right) \right] \exp\left[ -\frac{y_0^2}{1 + 4\sigma^4 t} \right] - (S_R)^2 \left[ 1 + 4\sigma^4 t \left( t_b + \tau_t \right) \right] \exp\left[ -\frac{y_0^2}{1 + 4\sigma^4 t} \right] \right. \\
- \left. 4S_R \left( \sigma^4 t \tau_t \cos \Theta + \sigma^2 \left( t - \left( t_b + \frac{1}{2} \tau_t \right) \right) \sin \Theta \right) \exp\left[ -\frac{y_0^2 (y_0 - 2\sigma k_0 \tau_t)}{1 + 4\sigma^4 t} \right] \right\} = 0, \tag{20}
\]

where

\[
\Theta = \frac{1}{1 + 4\sigma^4 t^2} \left\{ 2k_0^2 \tau_t \left[ 1 + 4\sigma^4 t \left( \frac{t_b}{2} + \frac{1}{2} \tau_t \right) \right] \right\} + \varphi, \tag{21}
\]

\[
y_0 = \sigma (2k_0 t - (x_0 - b)), \quad y_t = y_0 - 2\sigma k_0 \tau_t, \quad y_0 = y_0 - 2\sigma k_0 \tau_t. \tag{22}
\]

Thus, (20) is reduced to the expression

\[
\exp\left[ -(a_t t - b_t) \sigma^2 \right] - (S_R)^2 \exp\left[ (a_t t - b_t) \sigma^2 \right] - (S_T)^2 \exp\left[ (a_t t - b_t) \sigma^2 \right] \\
- 4\sigma^2 S_R \left( t - \left( t_b + \frac{1}{2} \tau_t \right) \right) \sin \varphi = 0, \tag{23}
\]

with

\[
a_t = 4k_0^2 \tau_t, \quad b_t = 2k_0 \tau_t (x_0 - b + k_0 \tau_t), \quad a_t = 4k_0^2 (2\tau_t - \tau_r), \quad b_t = 4k_0 \tau_t (x_0 - b + k_0 \tau_t) - 2k_0 \tau_t (x_0 - b + k_0 \tau_t). \tag{24}
\]

Note that (23) is a transcendental equation of time. To solve it, remember that \( \sigma \) is arbitrarily narrow and \( t \in (0, t_0) \), with \( t_0 << \frac{1}{2\sigma} \). As the argument in the first three exponentials of (23) includes \( \sigma^2 \) as a global factor we look for the simultaneous solution of the inequalities

\[
t < \frac{1}{a_t} \left( \frac{1}{\sigma^2} + b_t \right), \quad t < \frac{1}{a_t} \left( \frac{1}{\sigma^2} + b_t \right),
\]

\[
t < \frac{1}{a_t} \left( \frac{1}{\sigma^2} + b_t \right), \quad t < \frac{1}{a_t} \left( \frac{1}{\sigma^2} + b_t \right),
\]
this last allows to approximate the exponentials by their first terms in a Taylor series. Therefore, if \( t \) is in \( (0, t_0) \) and
\[
b_t, b_t << \frac{1}{\sigma^2}
\]
the above inequalities are fulfilled. Within this approximations the equation (23) is reduced to

\[
1 - (a_t t - b_t)(2k_0 \sigma)^2 - (S_{\text{ext}R})^2 [1 + (a_t t - b_t)(2k_0 \sigma)^2]
\]
\[
- (S_T)^2 [1 + (a_t t - b_t)(2k_0 \sigma)^2] - 4\sigma^2 S_R \left( t - \left( t_b + \frac{1}{2} \tau_t \right) \right) \sin \phi_t = 0. \tag{25}
\]

The root \( t_m \) of this last equation is given by the weighted average
\[
t_m = \frac{(S_T)^2 \tau_t (t_b + \frac{1}{2} \tau_t) + \left( (S_R)^2 \tau_t + \frac{1}{2k_0^2} S_R \sin \phi_t \right) (t_b + \frac{1}{2} \tau_t)}{(S_T)^2 \tau_t + (S_R)^2 \tau_t + \frac{1}{2k_0^2} S_R \sin \phi_t}. \tag{26}
\]

The weight factor is the sum of three terms: \( (S_T)^2 \tau_t \) and \( (S_R)^2 \tau_t \) are the contribution of the transmission and reflection processes, and
\[
\tau_t = - \frac{S_R \sin \phi_t}{2k_0^2} \tag{27}
\]
gives account of the contribution due to the interference between the incident and the reflected wave packets. This last becomes important as \( k_0 \) goes to zero, i.e., for scattering at low energies. Therefore, due to the interference between the incident and the reflected wave packets, slower particles will spend larger intervals of time outside the interaction zone. The contribution to \( t_m \) associated to the interference is less important for faster particles (see the discussion on the matter in [18]).

3. Symmetric potentials

For symmetric potentials the transmission and reflection times coincide \( \tau_t = \tau_r = \tau \), consequently (26) is reduced to its simplest expression
\[
t_m = t_b + \frac{1}{2} \tau. \tag{28}
\]

This extreme value gives the maximum of \( P_{(a,b)} \) just at a half the time \( \tau \) required to complete the scattering process (remember: the approach includes Gaussian wave packets that are peaked in the momentum \( k_0 \)). That is, \( t_m = t_s \). The probabilities depicted in Figure 1 correspond to symmetric barriers and wells for which \( t_s \approx 11.79 + 6.53 = 18.32 \) and \( t_s \approx 11.79 + 1.53 = 13.32 \), respectively.

Now, let us evaluate \( P_{(a,b)} \) at the critical time defined in (28). We get
\[
P_{(a,b)} \big|_{t=t_m} = \frac{1}{\Delta} \text{erf}(z - ik_0/\sigma) + e^{-\frac{k_0^2}{2\sigma^2}} S_R \left[ \cos \varphi \left( \text{Re} \left[ \text{erf} (z) \right] - 1 \right) + \sin \varphi \left( \text{Im} \left[ \text{erf} (z) \right] \right) \right], \tag{29}
\]
with
\[
z = \sigma k_0 \tau + i \frac{k_0}{\sigma}.
\]
The first term in (29) is a monotonically increasing function of \( \tau, k_0, \) and \( \sigma \) with limit \( 1/\Delta \) at \( t \to \infty \); this function corresponds to the individual contributions of the incident, transmitted
and reflected wave packets. The second term, including $S_R$ as a factor, is the contribution of the interference between the incident and reflected packets in the region $x > b$. This term oscillates with $\tau$ and goes to zero as $\tau$ increases (see Figure 2). Therefore, at $t = t_0$, the probability $P_{(a,b)}$ will oscillate around the values of $\text{erf}(\sigma k_0 \tau)$ from zero to $1/\Delta$ (within our approximation, $\sigma \ll 1$, one has $\Delta \approx 1$). We have depicted the behavior of $P_{(a,b)}$ as a function of $\tau$ in Figure 2. There, we can appreciate that the interference contributes mainly for small values of $\tau$ while the probability $P_{(a,b)}$ takes its smaller values. As $\tau$ depends on the analytic form of the function of compact support $V(x)$, among other variables, we realize that short times of scattering $\tau < 1$ imply small probabilities $P_{(a,b)}$ even for sharp momenta distributions in the wave packets. For the same packets and properly chosen scatterers $V(x)$, the scattering times $\tau$ are larger and the probabilities $P_{(a,b)}$ are bigger.

![Figure 2. The probability $P_{(a,b)}$ defined in (29) as a function of the scattering time $\tau$ (left) and the contribution to $P_{(a,b)}$ due to the interference of the incident and reflected wave packets. In both cases $\sigma = 0.05$, $x_0 = 200$, $b = 1$, $S_R = 0.5$, $k_0 = 1$, $\varphi = 2.664$.](image)

4. Concluding remarks

We have studied the probability of finding a particle, represented by a Gaussian wave packet, in the interaction zone of a scattering process that is characterized by a function of compact support $V(x)$. Our approach considers a series of approximations addressed to solve the transcendental equations that rule the times in which the probability is maximum. The main approximation is to consider (almost) monochromatic wave packets; that is, we assume that sharp momentum distributions can be used to construct the Gaussian wave packets. Within this approximation we have determined the extreme points of the probability we are interested in, and we have shown that these points can be used to determine the maximum of the probability. As an immediate application of our results, we have analyzed the case of symmetric potentials $V(x)$. Our results indicate that the interference between the incident and reflected waves plays a relevant role in the determination of the probabilities for scattering processes of short duration. That is, even for peaked distributions of momentum the analytic form of the potential $V(x)$ would produce small probabilities in the interaction zone. The times discussed in this work can be used to bound from below the duration of a scattering process, a condition that is crucial in the determination of dwell times and the related phenomena in quantum mechanics (see, e.g., [1, 2]).

Acknowledgments

The support of CONACyT and Cinvestav is acknowledged. The author appreciates the enlightening discussions with Prof. Oscar Rosas-Ortiz.
References

[1] Muga J G, Sala Mayato R, and Egusquiza I L 2008 *Time in Quantum Mechanics* (Berlin: Springer)
[2] Muga J G, Ruschhaupt A and del Campo A 2010 *Time in Quantum Mechanics* vol 2 (Berlin: Springer)
[3] Wigner E P 1955 Lower limit for the energy derivative of the scattering phase shift *Phys. Rev.* 98 145
[4] Smith F T 1960 Lifetime matrix in collision theory *Phys. Rev.* 118 349
[5] Hartman T E 1962 Tunneling of a wave packet *J. Appl. Phys.* 33 3427
[6] Büttiker M 1983 Larmor precession and the traversal time for tunneling *Phys. Rev. E* 27 6178
[7] Golub R, Felber S, Gähler R and Gutsniel E 1990 A modest proposal concerning tunneling times *Phys. Lett.* A 148 27-30
[8] Tsuchiya M, Matsusue T and Sakaki H 1987 Tunneling escape rate of electrons from quantum well in double-barrier heterostructures *Phys. Rev. Lett.* 59 2356
[9] Steinberg A M, Kwiat P G and Chiao R Y 1993 Measurement of the single-photon tunneling time *Phys. Rev. Lett.* 71 708
[10] Fernández-García N and Rosas-Ortiz O 2011 Rectangular potentials in a semi-harmonic background: spectrum, resonances and dwell time SIGMA 7 044
[11] Rosas-Ortiz O, Cruz y Cruz S, and Fernández-García N 2012 Negative time delay for wave reflection from a one-dimensional semi-harmonic well *Proc. of the XXX Workshop on Geom. Meth. Phys.* ed P Kielanowski et al (Poland: Springer Basel) 275-281
[12] Rosas-Ortiz O, Cruz y Cruz S, and Fernández-García N 2012 Time delay in the reflection of particles by semi-harmonic wells *J. Phys.: Conf. Ser.* 380 012018
[13] Rosas-Ortiz O, Fernández-García N and Cruz y Cruz S 2008 A primer on resonances in quantum mechanics *AIP Conf. Proc.* ed L M Montaño Zetina et al (Mexico city: American Institute of Physics) 31
[14] Fernández-García N and Rosas-Ortiz O 2008 Gamow-Siegert functions and Darboux-deformed short range potentials *Ann. Phys.* 323 1397-1414
[15] Naylor A.W. and Sell G.R., *Linear Operator Theory in Engineering and Science*, Springer, New York, 1982
[16] Cohen-Tannoudji C, Diu B and Lalöe F 1977 *Quantum Mechanics* vol 1 (New York: John Wiley) 83-84
[17] Abramowitz M and Stegun I A 1972 *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (AMS-55) (Washington D.C.: National Bureau of Standars)
[18] Winful H G 2003 Delay time and the Hartman effect in quantum tunneling *Phys. Rev. Lett.* 91 260401