REMARKS CONCERNING THE GEOMETRIES OF GRAVITY AND GAUGE FIELDS

JEEVA ANANDAN

Department of Physics and Astronomy, University of South Carolina, Columbia SC 29208, USA

Abstract

An important limitation is shown in the analogy between the Aharonov-Bohm effect and the parallel transport on a cone. It illustrates a basic difference between gravity and gauge fields due to the existence of the solder form for the space-time geometry. This difference is further shown by the observability of the gravitational phase for open paths. This reinforces a previous suggestion that the fundamental variables for quantizing the gravitational field are the solder form and the connection, and not the metric.

1. INTRODUCTION

I recall with great pleasure the discussions which I had with Charles Misner on fundamental aspects of physics, such as the geometry of gravity, gauge fields, and quantum theory. In particular, I remember the encouragement he gave to my somewhat unorthodox attempts to understand the similarities and differences between gauge fields and gravity from their effects on quantum interference, and their implications to physical geometry. It therefore seems appropriate to present here for his Festschrift some observations which came out of this investigation.

Geometry is a part of mathematics which can be visualized, and is intimately related to symmetries. This may explain the tremendous usefulness of geometry in physics. In section 2, I shall make some basic remarks about the similarities and differences between the geometries of gravity and gauge field. Then I shall illustrate, in section 3, an important difference between them that arises due to the existence of the solder form for gravity, using the Aharonov-Bohm (AB) effect and parallel transport on a cone. In section 4, I shall further illustrate this difference by the fact that the gravitational phase for a spinless particle is observable for an open path, unlike the AB effect. This implies that the translational gauge symmetry of the gravitational...
field is broken by the existence of the solder form. It is then argued that the solder form and the connection are the proper variables for quantizing the gravitational field.

2. LOCALITY OF GRAVITY AND GAUGE FIELDS
Something which Misner emphasized to me during a conversation was the fundamental role assigned to locality by the theory of relativity. Already in special relativity locality is incorporated in the fact that signals cannot travel faster than the speed of light. But in general relativity, locality plays an even more fundamental role: The principle of equivalence states that the laws of physics are locally Minkowskian. Also, because space-time is curved, there is no distant parallelism and vectors at two different points can only be compared by parallel transporting them to a common point with respect to the gravitational connection.

These three aspects of locality are also present in gauge fields which are now being used to describe the three remaining fundamental interactions in physics. The principle of equivalence for gauge fields may be stated as follows: Given any point \( p \) in space-time, a gauge can be chosen so that the corresponding connection coefficients or vector potential vanishes at \( p \) for all fields interacting with the given gauge field. Also, there is no distant parallelism for vectors parallel transported using the gauge field connection if the curvature or the Yang-Mills field strength is non vanishing.

Contrary to what is sometimes said, gravity does not differ fundamentally from gauge fields simply because it is associated with a metric. Because if the gauge group is unitary then it leaves invariant a metric in the vector space at each space-time point that consists of all possible values of any matter field interacting with the gauge field at that point. The essential difference is that the gravitational metric can be used to measure distances along any curve in space-time, unlike the gauge field metric. But I shall show now, by means of physical arguments, that this fundamental difference between gravity and gauge fields exists even prior to introducing the metric.

3. AHARONOV-BOHM EFFECT AND PARALLEL TRANSPORT ON A CONE
It is an interesting fact that the phase shifts in quantum interference due to gravity and gauge fields are obtained in a simple manner from the distance due to the gravitational metric and parallel transport due to gravitational and gauge field connections along the interfering beams. Conversely, the phase shifts in quantum interference can be used to define gauge fields and gravity. This is most easily shown for the simplest gauge field, namely the electromagnetic field, by means of the AB effect.
We recall that the magnetic AB effect is the phase shift in the interference of two coherent electron beams which enclose a cylinder containing a magnetic flux. In the interference region, the wave function may be written as \( \psi_1(r, t) + \psi_2(r, t) \), where \( \psi_1 \) and \( \psi_2 \) are the wave functions corresponding to the two beams. The introduction of the magnetic field inside the cylinder modifies this wave function to

\[
\psi(r, t) = \psi_1(r, t) + F_\gamma \psi_2(r, t),
\]

in an appropriate gauge, where

\[
F_\gamma = \exp\left(-\frac{ie}{\hbar c} \oint_{\gamma} A_\mu dx^\mu \right).
\]  

(1)

Here the integral is along the curve \( \gamma \) going around the cylinder, \( A_\mu \) is the electromagnetic 4-vector potential and \( e \) is the charge of the electron. Therefore the intensity distribution \( |\psi(r, t)|^2 \) in the interference region is modified in an apparently non local way by the magnetic flux via \( F_\gamma \), even though the magnetic and electric field strengths vanish everywhere along the beams.

But this phenomenon is not surprising when we realize the analogy with the geometry of a cone.\(^5\) The cone may be formed by taking a flat sheet of paper bounded by two straight lines making an angle \( \theta \) and identifying the two straight lines (Fig. 1a); we denote this cone by \( C_\theta \). For \( 0 \leq \theta \leq 2\pi \), this is what we do when we make a cone by rolling this flat sheet so that these two lines coincide to form one of the generators of the cone. Since the paper is not stretched or compressed during this process, a cone has no intrinsic curvature except at the apex, which can be smoothed out so that the curvature is finite there. In the multiply connected geometry around the apex, the intrinsic curvature is zero everywhere, same as the flat geometry of the sheet which was rolled up to be the cone. In particular, a vector is parallel transported like on the flat sheet. But a vector \( V \) parallel transported around a closed curve drawn on the curvature free region of the cone so as to enclose the apex undergoes a rotation by the angle \( \theta \), which is the holonomy transformation associated with this curve.

If the curvature at the apex is regarded as analogous to the magnetic field in the cylinder then the zero intrinsic curvature everywhere else corresponds to the vanishing of magnetic field strength outside the cylinder. Then \( V \) moving in a curvature free region is analogous to beams traveling in a field free region. The rotation by the angle \( \theta \) which relates \( V \) and \( V' \) (Fig. 1a) is analogous to the phase difference between the two beams due to \( F_\gamma \). This suggests that the electromagnetic field may be a connection for parallel transporting the value of the wave function and the AB effect arises because a wave function when parallel transported around the closed curve \( \gamma \), gets multiplied by \( F_\gamma \). The electric and magnetic field strengths at each space-time
a) Analogy between the Aharonov-Bohm effect and parallel transport on a cone. The cone may be obtained by identifying the lines OA and OA' on a flat sheet. Therefore, the vector V parallel transported from B around the cone would come back to B (identified with B') as V rotated by the angle $\theta$. This is analogous to the AB phase shift with the magnetic field corresponding to the curvature at the apex of the cone.

b) The limitation of this analogy when $\theta$ is changed to $\theta + 2\pi$ by adding an extra sheet of paper. The vector parallel transported along the closed curve BCDEB' rotates by $\theta + 2\pi$ with respect to the tangent vector to the curve. This enables one to distinguish this cone from the earlier one. This is unlike the AB effect which cannot distinguish between two enclosed magnetic fluxes that differ by one quantum of flux.
point then constitutes the curvature of this connection at this point. Thus the phase factor (1) is the holonomy transformation associated with \( \gamma \) for this connection. The statement that the electromagnetic field is a gauge field is the same as saying that it is a connection as described above.

The above mentioned conical geometry describes the gravitational field in each section normal to a long straight string\(^6\), such as a cosmic string. A gravitational analog of the AB effect is obtained if we interfere two coherent beams of identical particles with intrinsic spin around the string. The resulting phase shift due to the cosmic string is a special case of the phase shift due to an arbitrary gravitational field obtained before\(^2\). Basically, this phase shift consists of two parts, one due to the change in path lengths of the interfering beams, and the other due to the holonomy transformation, which in this case is a rotation undergone by the wave function when it is parallel transported around the interfering beams. This change in path length and holonomy transformation, and consequently the phase shifts, occur even though the space is locally flat.

Now if the AB phase

\[
\phi_\gamma = \frac{e}{\hbar c} \oint_\gamma A_\mu dx^\mu
\]

is changed by \( 2\pi \), which corresponds to changing the magnetic flux inside the cylinder by a “quantum of flux”, then (1) is unchanged. Therefore the AB experiment or for that matter any other experiment outside the cylinder cannot detect the difference between these two magnetic fluxes. Hence, Wu and Yang\(^7\) stated that, because of the AB effect, the electromagnetic field strength \( F_{\mu\nu} \) has too little information, \( \phi_\gamma \) has too much information, and it is the phase factor or the holonomy transformation \( F_\gamma \), for arbitrary closed curves \( \gamma \), which has the right amount of information of the electromagnetic field. This has been generalised to an arbitrary connection by the theorem\(^8\) which states that from the holonomy transformations the connection can be reconstructed and it is then unique up to gauge transformations. A simple physical system to illustrate the Wu-Yang statement is a superconducting ring enclosing a magnetic flux. No experiment performed in the interior of the ring using Cooper pairs can distinguish between a given enclosed flux \( \Phi \) and \( \Phi + n\Phi_0 \), where \( \Phi_0 \) is the quantum of flux for the Cooper pair and \( n \) is an integer. For example, if we measure the flux by inserting a Josephson junction in the ring and observe the Josephson current, we would obtain the same current for both fluxes. Because the AB phases for the two fluxes differ by \( 2\pi n \) and therefore (1) is the same for both fluxes, with \( e \) now being the charge of the Cooper pair.

An important and interesting limitation of the analogy of the AB effect with the cone emerges when we consider the meaning of increasing the flux of the curvature in the apex region of the cone by “one quantum”. The new flux may be regarded
as corresponding to the cone $C_{\theta+2\pi}$ which has one extra sheet of paper compared to $C_{\theta}$. (To embed $C_{\theta+2\pi}$ into a three dimensional Euclidean space it needs to be twisted in some way but it is well defined by the identification stated above.) The holonomy transformations are the same for $C_{\theta}$ and $C_{\theta+2\pi}$ (Fig. 1b). Therefore the above mentioned theorem$^8$ implies that the cones $C_{\theta}$ and $C_{\theta+2\pi}$ are the same as far as their connections are concerned. Here a connection is regarded simply as a rule for parallel transporting abstract vectors attached to points on the cone and not regarded as tangent vectors. Physically, the phase shift arising from spin in an interference experiment which is determined by the holonomy transformation will be the same for both cones for a bosonic particle. For a fermionic particle, there is a difference of $\pi$ between the phase shifts because this phase is acquired by a fermion when it is rotated by $2\pi$ radians. Therefore for fermions, $C_{\theta}$ is not equivalent to $C_{\theta+2\pi}$, but is equivalent to $C_{\theta+4\pi}$, because of the nature of the spinor connection.

A straightforward application of the Gauss-Bonnet theorem shows that the flux or integral of the curvature at the smoothed out apex of the cone $C_{\alpha}$ is $2\pi - \alpha$. Therefore this flux is negative when $\alpha > 2\pi$. In the latter case, it follows via Einstein's field equations that if $C_{\alpha}$ represents the geometry around a cosmic string then the string has negative mass. In particular, $C_{\theta+2\pi}$ and $C_{\theta+4\pi}$ represent geometries around cosmic strings with negative mass.

The two cones, which are the same as far as their linear connections are concerned, are of course, different when we take into account their metrics. This gives rise to the phase shift due to changes in the path lengths of the interfering beams$^2$. But even if we forget their metrics, there is a subtle difference between the two cones. To see this, for each cone, parallel transport a vector around a closed smooth curve that encloses the apex and does not intersect itself. This vector rotates with respect to the tangent vector to the curve by the angle $\theta$ for $C_{\theta}$ and by the angle $\theta + 2\pi$ for $C_{\theta+2\pi}$. This difference, which can be observed by means of local measurements, arises ultimately because we identify the vectors being parallel transported with the tangent vectors to the cone. The mathematical concept used to make this identification in an arbitrary manifold is called the solder form, or the canonical 1-form, or the canonical form$^9$. For the electromagnetic field the vector $\psi(x,t)$ belongs to an internal space and cannot be compared with a tangent vector. Therefore the Wu-Yang statement$^7$ is valid. But the gravitational field connection is for parallel transporting tangent vectors. Hence there is such an identification. This is the most fundamental difference between gravity and gauge fields$^{10}$.

If $C_{\theta}$ and $C_{\theta+2\pi}$ have only the connections or only the solder forms then they are identical. Since the two connections in the the frame bundles over $C_{\theta}$ and $C_{\theta+2\pi}$ have the same holonomy transformations, there exists a fiber bundle isomorphism
between the frame bundles which maps one connection into the other. This induces a unique diffeomorphism between the base manifolds \( C_\theta \) and \( C_{\theta + 2\pi} \) in the obvious way. The differential \( f_* \) is a map between tangent vectors. It determines a fiber bundle isomorphism \( f \) between the two frame bundles that maps one solder form into the other. But \( \tilde{f} \) and \( f \) are topologically different in the sense that one cannot be continuously deformed into the other. This is why we were able to distinguish between \( C_\theta \) and \( C_{\theta + 2\pi} \) when the connections and the solder forms are both present, even when the metric is absent.

4. GRAVITATIONAL PHASE FACTOR

The phase shift in quantum interference due to an arbitrary gauge field, which generalizes the AB effect, is determined by the “phase factor”

\[
F_\gamma = P \exp \left( -\frac{i g}{\hbar c} \oint_\gamma A^k T_k dx^\mu \right),
\]

where \( T_k \) generate the Lie algebra of the gauge group, \( A^k_\mu \) is the Yang-Mills gauge potential, \( P \) denotes path ordering, and \( \gamma \) is a closed curve through the interfering beams. Here, \( F_\gamma \) is an element of the gauge group. Its eigenvalues can be determined by interference experiments. This shows the real significance of (1) as an element of the \( U(1) \) group, which is a special case of the gauge group. When \( \gamma \) is an infinitesimal closed curve spanning a surface element represented by \( d\sigma_{\mu\nu} \),

\[
F_\gamma = 1 + \frac{ig}{2\hbar c} F^k_{\mu\nu} T_k d\sigma_{\mu\nu}
\]

where \( F^k = dA^k - g C^k_{ij} A^i \wedge A^j \) is the Yang-Mills field strength.

The phase shift in quantum interference of a particle due to the gravitational field is determined by

\[
F_\gamma = P \exp \left[ -\frac{i}{\hbar} \oint_\gamma (e^a_\mu P_a + \frac{1}{2} \Gamma^a_{\mu\nu} M_{ab}) dx^{\mu} \right],
\]

which is an element of the Poincare group that may be associated with any path \( \gamma \) in space-time. Here, \( P_a \) and \( M_{ab} \), \( a, b = 0, 1, 2, 3 \) are respectively the energy-momentum and angular momentum operators which generate the representation of the Poincare group corresponding to the given particle, \( e^a_\mu \) is dual to the frame \( e_a^\mu \) used by local observers:

\[
e^a_\mu e^\mu_b = \delta^a_b,
\]

and \( \Gamma^a_{\mu\nu} \) are the connection coefficients with respect to this frame field. If the local observers use orthonormal frames then

\[
e^a_\mu e_b^\nu g_{\mu\nu} = \eta_{ab},
\]
where $g_{\mu\nu}$ is the space-time metric and $\eta_{ab}$ are the coefficients of the Minkowski metric. When the particle has non-zero intrinsic spin then the values of the wave function are what are observed by observers using the frame field $e^a_\mu$. Then (4) implies that the spinor field is parallel transported in addition to a phase that it acquires due to its energy-momentum. This is obtained in the WKB approximation, disregarding here for simplicity a real factor which does not contribute to the phase $^2$.

For the special case of a spinless particle, $M_{ab} = 0$, the gravitational phase acquired by a locally plane wave is, to a good approximation,

$$\phi = \frac{1}{\hbar c} \int_\gamma e^a_\mu p_a,$$

(7)

where $p_a$ are the eigenvalues of the energy-momentum operators $P_a$ and the integral is along the classical trajectory$^{13}$. A remarkable feature of (7) is that it is observable for an open curve $\gamma$ unlike the phase shifts for gauge fields which can be observed only for closed curves. For example, (7) may be observed by the Josephson effect for a path across the Josephson junction$^3$, or by the oscillation of strangeness in the Kaon system for an open time-like path $\gamma$ along the Kaon beam$^{14}$. Both these phases depend on the geometry of space-time as determined by the gravitational field.

To understand this difference between gravity and gauge fields note that the field $e^a_\mu$ plays three roles here: First, comparing (4) with (2) suggests that $e^a_\mu$ is like a connection or gauge potential associated with the translation group. Indeed, $e^a_\mu$ and $\Gamma^{ab}_\mu$ may be regarded as constituting the connection in the affine bundle. The curvature of this connection is obtained by evaluating (4) for an infinitesimal closed curve $\gamma$:

$$F_\gamma = 1 + \frac{i}{2\hbar}(Q^a_\mu P_a + \frac{1}{2}R^{ab}_\mu M_{ab})d\sigma^{\mu\nu},$$

(8)

using the Poincare Lie algebra, where $Q^a = de^a + \Gamma^a_b \wedge e^b$ is the torsion and $R^{ab} = d\Gamma^{ab} + \Gamma^c_b \wedge \Gamma^{cb}$ is the curvature. Eq. (8) is the analog of (3) for gravity. This is the most physical way that I know to regard gravity as the gauge field of the Poincare group. Second, $e^a_\mu$ represents the solder form referred to earlier. In geometrical language, it is the pullback of the solder form with respect to the local section $e^a_\mu$ in the bundle of frames, which follows from (5). The (Lie-algebra valued) 1-form $e^a_\mu P_a$ acts on the tangent vector to $\gamma$ to give an element in the Lie algebra of the translational group, which is also an observable in the Hilbert space. When this observable acts on a WKB wave function it gives as an approximate eigenvalue the rate of change of phase along $\gamma$. When this is integrated along $\gamma$ the phase (7) is obtained. Third, (5) and (6) imply that $e^a_\mu$ is like the square root of the metric:

$$e^a_\mu e^b_\nu \eta_{ab} = g_{\mu\nu}.$$

(9)
In the first role, there is no restriction on the values of $e^a_\mu$ at any given point in spacetime. Indeed $e^a_\mu$ may be made to vanish by an appropriate choice of gauge along any differentiable curve that does not intersect itself. The "gravitational field" then has the full gauge symmetry of the affine group $A(4, R)$, i.e. the group of inhomogeneous linear transformations on a four dimensional real vector space. The holonomy group is a subgroup of the Poincare group which enables only the generators of the Poincare Lie algebra to occur in (4). The corresponding "gravitational phase", like the AB phase, would then be meaningful only for a closed curve $\gamma$.

However, in the second role, the matrix $e^a_\mu$ is restricted to be non singular. The gauge symmetry group is reduced to the general linear group $GL(4, R) \subset A(4, R)$, with $\Gamma^{ab}_\mu$ being the connection or gauge field. It is the breaking of the translational gauge symmetry that enables the phase (7) to be observable. The solder form is the canonical 1-form defined on the frame bundle whose structure group is $GL(4, R)$. The discussion of the parallel transport of a vector around a cone in section 3 shows the important role played by the solder form which makes this theory richer than a gauge theory with $GL(4, R)$ as the internal symmetry. The $e^a_\mu$ now transforms as a tensor, instead of a connection, under local gauge transformations which corresponds to space-time dependent transformations of the frame field. Therefore the phase (7) is invariant under these gauge transformations for an open curve $\gamma$, as it should be because it is observable.

Despite the breaking of the translational gauge invariance, the torsion which appears in (8) as the curvature corresponding to this group nevertheless arises naturally from a physical point of view. Because the motion of the amplitude of a spinor wave function provides an operational definition of the connection which is independent of the Christoffel connection that comes from the metric. Therefore, the connection, in a coordinate basis, can be non symmetric and the torsion is then twice the antisymmetric part of this connection. Hence the burden of proof is on gravitational theories with zero torsion to justify this constraint and not on torsion theories to justify introducing torsion, because kinematically torsion arises naturally whenever there are fields with intrinsic spin as seen above. But it is not necessary to introduce a metric in the first two roles of $e^a_\mu$ discussed here.

In the third role, the specification of the metric, which is the same as specifying the orthonormal frame field $e^\mu_b$, breaks the gauge symmetry further to the Lorentz group $O(3,1,R) \subset GL(4, R)$ that leaves this metric invariant. But from the observed phases (7), the metric may be constructed. Therefore it does not appear to be as fundamental a physical variable as the solder form or connection. From an operational point of view, the motion of a quantum system in a gravitational field is influenced
directly by the solder form and connection, and the metric seems to arise only as a secondary construct. Therefore in the reaction of the quantum system on the gravitational field, which needs to be described by quantum gravity, the solder form and the connection would be the fundamental dynamical variables that are affected.

It was therefore proposed that in quantizing the gravitational field the variables $e^a_{\mu}$ and $\Gamma^{ab}_{\mu}$ should be quantized and not the metric$^{15}$. The arguments in this paper which show further the important role played by the solder form reinforce this view. The important role assigned to the vector potential by the AB effect finds its counterpart in quantum electrodynamics in which it is the vector potential which is quantized. Similarly, the quantum effects discussed above which depend on the gravitational phase factor (4) suggest that the variables $e^a_{\mu}$ and $\Gamma^{ab}_{\mu}$ should be quantized in order to obtain quantum gravitodynamics. It is noteworthy that (4) is an element of the Poincare group, even though the curvature of space-time classically breaks Poincare invariance. This is analogous to the phase factor (2) for gauge fields being an element of the corresponding gauge group. This role of groups in the fundamental interactions, together with the general role of symmetry in quantum physics, which is much more fundamental, substantive and determinative than in classical physics, suggest that the way forward in physics at the present time should perhaps be guided by the precept ‘symmetry is destiny’.

Acknowledgements
I thank P. O. Mazur, R. Penrose and R. Howard for useful remarks. This work was partially supported by NSF grant no. PHY-8807812.

REFERENCES
1. C. N. Yang and R. L. Mills, Phys. Rev. 96 (1954) 191.
2. J. Anandan, Nuov. Cim. 53A (1979) 221.
3. J. Anandan, Phys. Rev. D 33 (1986) 2280; J. Anandan in Topological Properties and Global Structure of Space-Time, eds. P. G. Bergmann and V. De Sabbata (Plenum Press, NY 1985), p. 1-14.
4. Y. Aharonov and D. Bohm, Phys. Rev. 115 (1959) 485.
5. J. S. Dowker, Nuov. Cim. 52B (1967) 129.
6. L. Marder, Proc. Roy. Soc. A 252 (1959) 45; in Recent Developments in General Relativity (Pergamon, New York 1962).
7. T. T. Wu and C. N. Yang, Phys. Rev. D, 33 (1986) 2280.
8. J. Anandan in Conference on Differential Geometric Methods in Physics, edited by G. Denardo and H. D. Doebner (World Scientific, Singapore, 1983) p. 211.
9. S. Kobayashi and K. Nomizu, Foundations of Differential Geometry (John Wiley, New York 1963) p. 118.
10. A. Trautman in *The Physicist’s Conception of Nature*, edited by J. Mehra (Reidel, Holland, 1973).
11. See also, I. Bialynicki-Birula, Bull. Acad. Pol. Sci., Ser. Sci. Math. Astron. Phys. 11 (1963) 135; D. Wisnivesky and Y. Aharonov, Ann. of Phys. 45 (1967) 479.
12. J. Anandan in *Quantum Theory and Gravitation*, edited by A. R. Marlow (Academic Press, New York 1980), p. 157.
13. J. Anandan, Phys. Rev. D, 15 (1977) 1448.
14. L. Stodolsky, J. Gen. Rel. and Grav. 11 (1979) 391.
15. J. Anandan, Found. Phys. 10 (1980) 601.
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/gr-qc/9503055v1