Towards a Fair Allocation of Rewards in Multi-Level Marketing

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An increasing number of businesses and organisations rely on existing users for finding new users or spreading a message. One of the widely used “refer-a-friend” mechanisms offers an equal reward to both the referrer and the invitee. This mechanism provides incentives for direct referrals and is fair to the invitee. On the other hand, multi-level marketing and recent social mobilisation experiments focus on mechanisms that incentivise both direct and indirect referrals. Such mechanisms share the reward for inviting a new member among the ancestors, usually in geometrically decreasing shares. A new member receives nothing at the time of joining. We study fairness in multi-level marketing mechanisms. We show how characteristic function games can be used to model referral marketing, show how the canonical fairness concept of the Shapley value can be applied to this setting, and establish the complexity of finding the Shapley value in each class, and provide a comparison of the Shapley value-based mechanism to existing referral mechanisms.

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1. INTRODUCTION

Social networks and email made it easy to share information. These tools provide powerful ways to spread a message without using mainstream media. Compelling examples of their use can be found in politics (e.g., twitter revolutions), humanitarian relief (e.g., help-maps such as Ushahidi in post-earthquake Haiti), businesses (e.g., promoting products through Facebook), and entertainment (e.g., music videos such as Gangnam Style). Many famous cases of messages going viral are organic: users share because they like the message, or identify with it. However, it is hard to make a message viral by design, and in fact very few messages achieve viral stardom. A more realistic goal is a moderate level of social reach achieved by providing incentives for sharing information.

The marketing literature acknowledges the importance of referral incentives [Buttle 1998] but provides limited guidance on how to design them (see, e.g., [Bivalogorsky et al. 2001]). Multi-level marketing mechanisms—mechanisms that offer not only direct but also indirect referrals—have recently received some attention in computer science. Some of the interest has been fuelled by a social mobilisation experiment known as the Red Balloon Challenge [Defense Advanced Research Projects Agency 2010], which provided validation for using multi-level mechanisms for referral incentivisation in a non-commercial context. The task posed by the challenge was to locate ten red weather balloons moored at undisclosed locations throughout the US. The winning incentive mechanism [Pickard et al. 2011] shared the reward associated with each balloon in a recursive manner: the referrer received half of the finder’s prize, the referrer’s referrer received a quarter, etc. In the context of multi-level marketing this is an example of a geometric reward mechanism.

1For example, an absurd and funny Old Spice commercial collected nearly 50 million views http://www.youtube.com/watch?v=owGykVbfgUE
An axiomatic justification of geometric mechanisms appears in Emek et al. [2011]. An important property for such mechanisms is Sybil-proofness: i.e., the users should have no incentive to create fake identities of themselves. Sybil-proof mechanisms have been studied in a number of papers including Babaioff et al. [2012] Drucker and Fleischer [2012], Chen et al. [2013], Lv and Moscibroda [2013]. Work on referral incentives has also been carried out in the model of Query Incentive Networks (QIN) [Kleinberg and Raghavan 2005], where the root of a tree needs to incentivise the nodes to propagate the query until a node holding the answer is reached. Performance of geometric mechanisms in QINs was analyzed in Cebrian et al. [2012].

A number of desirable properties that a referral incentive mechanism should satisfy are described in Douceur and Moscibroda [2007]. One of the properties relevant to fairness is “value proportional to contribution”. This property ensures that a user recovers a fraction of the effort he has put in. The effort, however, refers to performing non-recruitment activities such as sharing files on a p2p network or classifying galaxies on GalaxyZoo.

Unlike the above-cited work, our interest is in fairness when applied to referrals as opposed to other types of contributions. In this work, the contribution of a node is measured by the descendents he brings. We formalise a class of cooperative games that enables us to model such contributions. This class of games, which we call tree games,[2] allows us to formally study fairness in referral mechanisms by applying the best-known game-theoretic fairness model—the Shapley value [Shapley 1953].

The work is motivated not only by an academic interest in applying a standard fairness concept to a new and popular domain, but also by the fact that fairness of referral mechanisms is important in practice. In particular, many popular mechanisms compensate not only the referrer but also the invitee with an equal reward. Examples of such fair “refer-a-friend” mechanisms include Dropbox and GiffGaff where both the referrer and the invitee receive 500Mb of extra space or £5, respectively, for each successful referral of a new user. As we will show, compensation based on the Shapley value combines features of both fair refer-a-friend mechanisms and geometric mechanisms. Specifically, the Shapley value of an invitee is non-zero, and all of the ancestors are compensated.

The following example illustrates our work.

**Example 1.1.** The Dropbox referral program offers 500MB of extra free space to the referrer and the invitee. Since a fixed amount is awarded for each referral, we can say that Dropbox values each referral at 1GB per user. Consider the referral tree on the right of Figure 1: 1 invites 3 who invites 6 and 7.

There are 3 referrals, so Dropbox will distribute 3GB of free space. Under the Dropbox’s refer-a-friend scheme, user 1 receives 500MB when he invites 3. User 3 receives 500MB when accepting the referral from 1. Inviting 6 and 7 gives 3 an extra 500MB more for each invite. Users 6 and 7 receive 500MB at the time of joining. Distributing 3GB according to the geometric mechanism that passes 1/2 of referral reward to the referrer, results in user 1 receiving 1/2 for 3 and 1/3 for each of 6 and 7. User 3 receives 1/3 for each of 6 and 7. The total shares for splitting 3GB are (1, 1, 0, 0) giving 1.5GB to nodes 1 and 3 and nothing to the leaf nodes 6 and 7. The resulting compensations are shown in Table I.

The current Dropbox mechanism is fair to the invitee: the invitee gets the same reward as the referrer. However it is not fair to nodes who bring many descendants, as a node is compensated only for direct referrals but not for the nodes his children

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1Not to be confused with game trees.
bring. On the other hand, the geometric mechanism acknowledges indirect referrals (node 1 is compensated for nodes 6 and 7) but does not provide any compensation to the invitees at the time of joining. This can be viewed as unfair by the invited node. Our interest is in defining a mechanism that is fair to everyone. Every node joining through a referral brings value to Dropbox, and each should acknowledged. However, nodes whose referral activity brings more new customers should receive more. The mechanism that we derive using the game-theoretic concept of Shapley value satisfied both of these properties.

The derivation of the Shapley value for a general class of referral marketing games follows in Section 3, but for the simple example given here, the Shapley-value mechanism has a simple form: the value of a referral is distributed in equal shares among the invitee and all of his ancestors. Thus, when node 6 joins, nodes 1, 3, and 6 receive 333MB each. The final distribution of the reward appears in Table I.

The example illustrates how referral marketing can be modelled by referral trees. Specifically, there is one special node, the root, that initiates referral activities. This node can be viewed as a customer who joined Dropbox independently. The root node can invite its friends, and each successful referral has a value associated with it that can be shared among the joining nodes. However, if an invitation is not made, that is if an edge is removed, then the entire bottom subtree does not join. This modelling choice means that there is a single chance for each node to get invited. This assumption is consistent with most models in the referral literature [Babaioff et al. 2012; Cebrian et al. 2012; Chen et al. 2013; Douceur and Moscibroda 2007; Drucker and Fleischer 2012; Emek et al. 2011; Kleinberg and Raghavan 2005; Lv and Moscibroda 2013]. This corresponds to modelling the referral process as trees where a node can be referred by only one other node, as opposed to graphs allowing multiple referrals. Considering graphs in general is an interesting open question, but given that they occur naturally in many settings, trees seem an obvious place to start an analysis.

A model related to ours was studied in [Myerson 1977]. Myerson defined games on graphs where the graph encodes which of the players can cooperate. The value of a coalition is then defined as the sum of the values of connected components in it. For the graph model, Myerson proved that there is one value that meets certain natural fairness axioms, and this value equals the Shapley value. We take the graph games to the referral marketing domain. Our model is a special case where the graph is a tree and the value of any connected component not containing the root is zero.

Interestingly, since our model is a special case of [Myerson 1977], the fairness axioms defined there carry over to our marketing domain. In particular, this means that any two players on both sides of the edge profit from the edge equally. This property provides justification for using the Shapley value in the referral marketing context. “Refer-a-friend” mechanisms such as Dropbox equally compensate the referral and the invitee. The Shapley value on referral trees generalises this fairness property to indirect referrals. Thus, the Shapley value compensates all types of contributions—it is not only the direct referrals that matter but also the indirect ones—and it provides a fair way of compensating the referrer and the invitee in equal shares.

Our main contribution is a fair mechanism for referral marketing. To this end, we introduce tree games that allow us to model rewards in multi-level marketing.
present a simple technique for computing the Shapley value in the subclass where each referral has a fixed value, and a simplified computation for the general tree games. In the example above, each referral carried the same value of 1GB. Our work extends to cases where the contribution of a referral depends on the total number of nodes that join or, in the most general case on the identities of all of the nodes that join.

The remainder of this paper is organised as follows. Section 2 is intended to familiarise the reader with the necessary concepts from cooperative game theory. In Section 3, we introduce a class of games defined over trees, and show how the Shapley value can be computed for these games, thus providing a fair compensation for referral activities. Section 3.3 focuses on a natural subclass of games where we show that the Shapley value is computable in linear time. Section 4 concludes the paper and presents some potential future extensions. Appendix A provides a summary of the main notation used throughout the paper.

2. GAME THEORETIC BACKGROUND

In this section, we will introduce some necessary definitions and concepts from cooperative game theory.

A characteristic function game is a pair \((A, v)\), where \(A = \{1, \ldots, n\}\) is a set of players (or agents), and \(v : 2^A \rightarrow \mathbb{R}\) is a characteristic function that maps each subset (or coalition) of agents, \(C \subseteq A\), to a real number, \(v(C)\), which represents the payoff of \(C\). This number is called the value of coalition \(C\).

A coalition structure, \(CS \subseteq 2^A\), is a partition of \(A\) into disjoint and exhaustive coalitions. The set of possible coalition structures is denoted as \(CS^A\).

An outcome of a game is a pair, \((CS, x)\), where \(CS \in CS^A\) is a coalition structure, and \(x = (x_1, \ldots, x_n)\) is a payoff vector, which specifies a payoff, \(x_i\), for every agent \(i \in A\), such that: \(x_i \geq 0\) for all \(i = 1, \ldots, m\), and \(\sum_{i \in C} x_i = v(C)\) for all \(C \in CS\). Intuitively, an outcome specifies which coalitions to form, and how the payoff of each coalition is divided among its members.

A solution concept specifies the set of outcomes that meet certain criteria. In this context, one desirable criterion is stability; an outcome, \((CS, x)\), is said to be stable if no group of agents can receive a payoff greater than what was allocated to them in that outcome, i.e., if \(\sum_{i \in C} x_i \geq v(C), \forall C \subseteq A\). The set of all stable outcomes in a game is called the core of that game [Gillies, 1959]. In general, the problem of determining whether there exists a stable outcome is NP-complete [Conitzer and Sandholm, 2003].

Fairness is another desirable criterion when dealing with outcomes; it is typically evaluated based on the degree to which every agent’s payoff reflects its contribution. In this vein, the best-known solution concept is the Shapley value [Shapley, 1953]. To formally present this concept, we need to first introduce the notion of marginal contribution.

**Definition 2.1.** [Marginal Contribution] Given a characteristic function game, \((A, v)\), the marginal contribution of an agent \(i \in A\) to a coalition \(C \subseteq A \setminus \{i\}\) is denoted by \(mc(i, C)\), and defined as the difference in value that is caused when \(i\) joins \(C\). Formally:

\[
mc(i, C) = v(C \cup \{i\}) - v(C).
\]  

(1)

Finally, let \(\Pi^A\) denote the set of all permutations of \(A\). For any arbitrary permutation, \(\pi \in \Pi^A\), let \(\pi(i)\) be the location of \(i\) in \(\pi\), and let \(C_\pi^i\) be the coalition consisting of the agents that precede \(i\) in \(\pi\). Formally,

\[
C_\pi^i = \{ j \in A : \pi(j) < \pi(i) \}.
\]
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Now, we are ready to define the Shapley value—a solution concept that specifies how to divide the value of the grand coalition, i.e., the coalition containing all agents. According to the Shapley value, the payoff of an agent, \( i \in A \), equals its average marginal contribution to the agents that precede it in an arbitrary permutation. This payoff is referred to as the Shapley value of \( i \). Formally:

**Definition 2.2. [Shapley Value]** Given a characteristic function game \((A, v)\), the Shapley value of an agent \( i \in A \) is denoted by \( Sh_i \) and is given by

\[
Sh_i = \frac{1}{n!} \sum_{\pi \in \Pi^A} mc(i, C^\pi). \tag{2}
\]

The outcome \( (\{A\}, (Sh_1, \ldots, Sh_n)) \) is called the Shapley value of \((A, v)\).

Due to its various desirable properties [Shapley 1953], the Shapley value is arguably the fairest solution concept known to date in cooperative game theory.

A special class of coalitional games are convex games [Shapley 1971]. A characteristic function game \((A, v)\) is said to be convex if \( v \) is supermodular, i.e., if

\[
v(C \cup C') + v(C \cap C') \geq v(C) + v(C'), \quad \forall C, C' \subseteq A.
\]

Some of the desirable properties of convex games include the fact that the Shapley value is always in the core, which implies that the core is always non-empty [Shapley 1971].

3. TREE GAMES

This section is divided into three subsections. The first introduces the main notation that will be used for graphs and trees. Section 3.2 introduces a class of characteristic function games, which we call tree games, while Section 3.3 focuses on an intuitive subclass, namely basic tree games.

### 3.1. Main Notation for Graphs and Trees

Throughout this paper, a graph will be denoted by \( G \), with \( N(G) \) and \( E(G) \) being the set of nodes and the set of edges in \( G \), respectively. In all the graphs that will be considered in this paper, each node will represent a unique agent. As such, a node will be denoted by a number \( i \in \{1, \ldots, n\} \) (just as we denote an agent), and a subset of nodes will be denoted by \( C \) (just as we denote a coalition of agents). For every \( C \subseteq N(G) \), we will write \( \text{adj}(C, G) \) to denote the nodes in \( N(G) \setminus C \) that are each adjacent to one or more of the nodes in \( C \). More formally,

\[
\text{adj}(C, G) = \{ i \in N(G) \setminus C : \exists j \in C, (i, j) \in E(G) \}. \tag{3}
\]

Furthermore, we will write \( G^C \) to denote the subgraph of \( G \) that is induced by \( C \). In other words, we have:

\[
N(G^C) = C, \tag{4}
\]

\[
E(G^C) = \{ (i, j) \in E(G) : i, j \in C \}. \tag{5}
\]

Next, we introduce our notation for trees. We use the notation \( T \) instead of \( G \) if the graph happens to be a rooted tree. Since every tree considered in this paper is rooted, we simply write “tree” instead of “rooted tree”, and write \( r(T) \) to denote the root of \( T \). For every node \( i \in N(T) \), let \( \text{par}(i, T) \) be the parent of \( i \) in \( T \), and let \( \text{chi}(i, T), \text{des}(i, T) \) and \( \text{anc}(i, T) \) be the set of children, descendants, and ancestors of \( i \) in \( T \), respectively. \footnote{Observe that \( \text{anc}(r(T), T) = \emptyset \).}
Fig. 1. An example of $T$, $T^C$, and $T^C_{\text{trim}}$, where $C = \{1, 3, 4, 6, 7, 8, 9\}$.

Just as we denote by $G_C$ the subgraph of $G$ induced by $C$, we denote by $T^C$ the subgraph of $T$ induced by $C$. Furthermore, for every $i \in N(T)$, we write $T_i$ to denote the subset of $T$ rooted at $i$. More formally, $T_i = T(i) \cup \text{des}(i, T)$. We denote by $\text{depth}(i, T)$ the depth of $i$ in $T$, and by $\text{Level}_j(T)$ the set of nodes in $N(T)$ whose depth is $j$. The height of $T$ is denoted by $\text{height}(T)$.

Now, let us introduce the concept of “trimming”. Basically, for every $C \subseteq N(T)$, by trimming $T^C$ we obtain another subgraph $T^C_{\text{trim}}$, which is the connected component in $T^C$ that contains the root of $T$; if no such component exists then $T^C_{\text{trim}}$ is the null graph. An example is illustrated in Figure 1. More formally:

\begin{align*}
N(T^C_{\text{trim}}) &= \{i \in C : \text{anc}(i, T) \subseteq C\}, \quad \text{(6)} \\
E(T^C_{\text{trim}}) &= \{(i, j) \in E(T) : i, j \in N(T^C_{\text{trim}})\}. \quad \text{(7)}
\end{align*}

For every $C \subseteq N(T)$, the subgraph $T^C$ is said to be “trimmed” if and only if $T^C = T^C_{\text{trim}}$. In Figure 1, for example, the subgraphs $T^{\{1,3,6,7\}}$ and $T^{\{1,2,5\}}$ are trimmed, while the subgraph $T^{\{1,2,8,9\}}$ is not. We will denote by $\text{Trimmed}(T)$ the set of coalitions that induce trimmed subgraphs. More formally:

$$\text{Trimmed}(T) = \{C \subseteq N(T) : T^C = T^C_{\text{trim}}\}.$$ \quad (8)

Finally, for every $C \subseteq N(T)$, we will denote by $\text{SameTrim}(C, T)$ the set consisting of every coalition $C' \subseteq N(C)$ such that by trimming $T^{C'}$ we obtain the same subgraph as the one obtained by trimming $T^C$. More formally:

$$\text{SameTrim}(C, T) = \{C' \subseteq N(C) : T^{C'}_{\text{trim}} = T^C_{\text{trim}}\}. \quad \text{(9)}$$

An example of $\text{Trimmed}(T)$ and $\text{SameTrim}(C, T)$ is illustrated in Figure 2. Appendix A provides a summary all notation introduced thus far.

3.2. Tree Games

We now show how trees induce a class of characteristic function games, which we call tree games, and analyze some properties of these games.

**Definition 3.1.** [Tree Game] A tree game is a pair $(T, f)$, where $T$ is a tree, and $f : \text{Trimmed}(T) \to \mathbb{R}$. The set of agents in the tree game $(T, f)$ is $N(T)$, and the value

\[ \text{Recall that the depth of a node in a tree is the number of edges in the path between that node and the root of the tree. As such, the depth of the root itself is zero.} \]
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of every coalition \( C \subseteq N(T) \) is given by:

\[
v(C) = f \left( N(T_{trim}^C) \right).
\]

Intuitively, a tree game \((T, f)\) is a characteristic function game, \((A, v)\), where \(A = N(T)\) and the value of every \( C \subseteq N(T) \) depends solely on the agents in \( C \) that are connected to the root of \( T \) via other members of \( C \).

We will assume throughout this paper that \( f(\emptyset) = 0 \). As such, the following holds since \( N(T_{trim}^C) = \emptyset \) for all \( C \subseteq N(T) \setminus \{r(T)\} \):

\[
\forall C \subseteq N(T) \setminus \{r(T)\}, \quad v(C) = 0.
\]

The following lemmas will prove useful in subsequent proofs.

**Lemma 3.2.** Let \((T, f)\) be a tree game. For every agent \( i \in N(T) \), and every coalition \( C \subseteq N(T) : i \in C \), the following holds:

\[
i \notin N(T_{trim}^C) \quad \Rightarrow \quad mc(i, C \setminus \{i\}) = 0.
\]

**Lemma 3.3.** Let \((T, f)\) be a tree game. For every agent \( i \in N(T) \), and every pair of coalitions \( C_1, C_2 \subseteq N(T) : i \in C_1 \cap C_2 \), the following holds:

\[
T_{trim}^{C_1} = T_{trim}^{C_2} \quad \Rightarrow \quad mc(i, C_1 \setminus \{i\}) = mc(i, C_2 \setminus \{i\}).
\]

**Lemma 3.4.** Let \((T, f)\) be a tree game. For every pair of coalitions \( C_1, C_2 \in \text{Trimmed}(T) : C_1 \neq C_2 \), the following holds:

\[
\text{SameTrim}(C_1, T) \cap \text{SameTrim}(C_2, T) = \emptyset.
\]

**Lemma 3.5.** Let \((T, f)\) be a tree game. The following holds, where \( 2^{N(T)} \) is the set of possible coalitions, i.e., subsets of \( N(T) \):

\[
\bigcup_{C \in \text{Trimmed}(T)} \text{SameTrim}(C, T) = 2^{N(T)}.
\]

**Lemma 3.6.** Let \((T, f)\) be a tree game. For every \( i \in N(T) \), the following holds:

\[
\bigcup_{C \in \text{Trimmed}(T) : i \in C} \text{SameTrim}(C, T) = \{ C \subseteq N(T) : i \in N(T_{trim}^C) \}.
\]

**Lemma 3.7.** Let \((T, f)\) be a tree game, and let \( C \in \text{Trimmed}(T) \). For every coalition \( C' \subseteq N(T) \), the following holds:

\[
C' \in \text{SameTrim}(C, T) \quad \Leftrightarrow \quad (C \subseteq C') \land (C' \cap \text{adj}(T_{trim}^C, T) = \emptyset)
\]

Having outlined the necessary lemmas, we now establish a key result with respect to the Shapley value in tree games.
Lemma 3.6, it is possible to iterate over every permutation \( \pi \) for a given coalition \( C \). Moreover, Lemma 3.3 implies that:

\[
\text{Based on this, we have:}
\]

\[
\text{This can be written differently as follows (because } i \in C \text{ and } i \notin C^*_i:\]

\[
\text{Next, based on Lemma 3.7, we will compute } |\{ \pi \in \Pi^{N(T)} : C^*_i \cup \{i\} \in \text{SameTrim}(C, T) \}| \text{ for a given coalition } C \in \text{Trimmed}(T) : i \in C. \text{ To this end, for every } \pi \in \Pi^{N(T)}, \text{ Lemma 3.7 implies that } C^*_i \cup \{i\} \in \text{SameTrim}(C, T) \text{ if and only if } C^*_i \cup \{i\} \text{ contains...}
\]
all the agents in $C$, and none of the agents in $\text{adj}(T_{\text{trim}}^C, T)$. It is easy to see how the following steps cover all possible ways to construct such a permutation:

— **Step 1:** Place the members of $N(T) \setminus (C \cup \text{adj}(T_{\text{trim}}^C, T))$ in any slots in $\pi$, without any restrictions.

— **Step 2:** Out of all remaining slots, place the members of $\text{adj}(T_{\text{trim}}^C, T)$ in the last slots, without any restriction on the order in which those members are placed in the slots.

— **Step 3:** Out of all remaining slots, place $i$ in the last slot. Steps 2 and 3 ensure that $C_i^\pi$ does not contain any of the agents in $\text{adj}(T_{\text{trim}}^C, T)$.

— **Step 4:** Place the members of $C \setminus \{i\}$ in the remaining slots in any order, without any restrictions. Steps 3 and 4 ensure that $C_i^\pi$ contains all the agents in $C$.

— **Step 4:** Place the remaining members of $C$ (other than $i$) in the remaining slots in any order, without any restrictions. Steps 3 and 4 ensure that $C_i^\pi \cup \{i\}$ contains all the agents in $C$.

Let us count the number of possible ways in which each step can be performed:

— For step 1, the number is: $(n)(n-1)(n-2)\ldots(n-|N(T) \setminus (C \cup \text{adj}(T_{\text{trim}}^C, T))|+1)$.

— For step 2, the number is: $|\text{adj}(T_{\text{trim}}^C, T)|!$.

— For step 3, the number is: 1.

— For step 4, the number is: $(|C|-1)!$.

By multiplying those numbers, we obtain the size of the set $\{\pi \in \Pi^{N(T)} : C_i^\pi \cup \{i\} \in \text{SameTrim}(C(T))\}$, where $C$ is a coalition in $\text{Trimmed}(T)$ that contains $i$. Based on this, Equation (15) can be written differently as follows:

\[
S_{h_i} = \frac{1}{n!} \sum_{C \in \text{Trimmed}(T) : i \in C} mc(i, C \setminus \{i\}) \left( \binom{n}{|N(T) \setminus (C \cup \text{adj}(T_{\text{trim}}^C, T))|+1} \times |\text{adj}(T_{\text{trim}}^C, T)| \times (|C|-1)! \right)
\]

where $mc(i, C \setminus \{i\}) = \frac{|\text{adj}(T_{\text{trim}}^C, T)|! \times (|C|-1)!}{|N(T) \setminus (C \cup \text{adj}(T_{\text{trim}}^C, T))|!}$.

Based on this, in order to complete the proof of Theorem 3.8, it remains to show that:

\[
mc(i, C \setminus \{i\}) = f(C) - f(C \setminus N(T_i)).
\]  

(16)

To this end, based on the definition of marginal contribution and the definition of tree games, we know that for every tree game $(T, f)$ and every $C \subseteq N(T) : i \in C$, the following holds:

\[
mc(i, C \setminus \{i\}) = f(N(T_{\text{trim}}^C)) - f(N(T_{\text{trim}}^{C \setminus \{i\}})).
\]  

(17)

We also know that $C \in \text{Trimmed}(T)$. Thus, based on Equation (8)—the definition of $\text{Trimmed}(T)$—as well as Equation (17), we have:

\[
mc(i, C \setminus \{i\}) = f(C) - f(N(T_{\text{trim}}^{C \setminus \{i\}})).
\]  

(18)

Based on this, to prove the correctness of Equation (16), it suffices to show that:

\[
C \setminus N(T_i) = N(T_{\text{trim}}^{C \setminus \{i\}}).
\]  

(18)
From the definition of Trimmed$(T)$, for every $C \in$ Trimmed$(T)$ : $i \in C$, we know that
\[ C = N(T^C) = N(T^C_{\text{trim}}), \] (19)
and it is easy to see that
\[ N(T^C_{\text{trim}}) \setminus N(T_i) = N(T^C_{\text{trim}} \setminus \{i\}). \] (20)
Equations (19) and (20) imply the correctness of Equation (18) and thus conclude the proof of Theorem 3.8.

Now that we have defined tree games, and analyzed some of their relevant properties, in the following subsections we will focus on a subclass of tree games that is potentially relevant in some multi-level-marketing settings.

3.3. Basic Tree Games

Intuitively, basic tree games are those where the value of a coalition, $C$, equals the number of agents in $C$ that are connected to the root via other members of $C$. More formally:

**Definition 3.9.** [Basic Tree Game] A tree game, $(T, f)$, is said to be basic if and only if: $f(C) = |C|, \forall C \in \text{Trimmed}(T)$.

Our interest in this subclass is driven by our focus on multi-level marketing. To see how this is relevant, recall that one of main properties of the Shapley value is “Additivity”—for every pair of games, $(N, v)$ and $(N, w)$, and every agent $i \in N$, we have: $sh_i(N, v) + sh_i(N, w) = sh_i(N, v + w)$, where the game $(N, v + w)$ is defined by $(v + w)(C) = v(C) + w(C)$ for every $C \subseteq N$. While this property is admittedly not very intuitive, it implies the linearity of the Shapley value. That is, if we scale a game—i.e., multiply all coalition values by some constant—the Shapley values will simply be multiplied by that same constant. To see how this relates to basic tree games, consider a website like [www.888casino.com](http://www.888casino.com), which claims to offer the same amount of reward, £88 to be precise, for every subscriber. This can be interpreted as follows. For every subset of agents $C \subseteq N(T)$, the reward is £88 multiplied by the number of agents in $C$ who reach the root via other members of $C$ (other members of $C$ receive no reward simply because they did not reach the root). In other word, in the tree game $(T, f)$ which represents this website’s preferences, we have:
\[ f(C) = £88 \times |C|, \forall C \in \text{Trimmed}(T). \]
Now instead of computing the Shapley values in this game, it is possible—based on the aforementioned linearity property—to first compute the Shapley values in an alternative basic tree game $(N, f')$, where:
\[ f'(C) = £1 \times |C|, \forall C \in \text{Trimmed}(T), \]
and then simply multiply the resulting Shapley values by 88. This would return the Shapley values of the original game, $(T, f)$. More generally, for every tree game $(T, f)$ where there is a constant reward $x$ for every player that reaches the root, it is possible to first compute the Shapley values in a basic tree game which has the same tree, $T$, and then multiply the resulting Shapley values by $x$.

Now that we have elaborated on the intuition behind tree games, let us analyze some of their properties.

**Lemma 3.10.** In a basic tree game, $(T, f)$, for every agent $i \in N(T)$, and every coalition $C \subseteq N(T) \setminus \{i\}$, the following holds:
\[ mc(i, C) = \left| N(T^C_{\text{trim}} \setminus \{i\}) \cap N(T_i) \right|. \]
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PROOF. Following the definitions of marginal contribution and basic tree games:

\[ mc(i, C) = \left| N(T_{trim}^{C \cup \{i\}}) \right| - \left| N(T_{trim}^{C}) \right|. \] (21)

Furthermore, we know from Equation (5)—the definition of \( N(T_{trim}^{C}) \)—that the following holds, simply because \( C \) is a subset of \( C \cup \{i\} \):

\[ N(T_{trim}^{C}) \subseteq N(T_{trim}^{C \cup \{i\}}). \] (22)

Based on (21) and (22), we have:

\[ mc(i, C) = \left| N(T_{trim}^{C \cup \{i\}}) \setminus N(T_{trim}^{C}) \right|. \] (23)

We also know from Equation (6) that an agent \( j \in N(T_{trim}^{C \cup \{i\}}) \) does not belong to \( N(T_{trim}^{C}) \) if and only if \( i = j \) or \( i \in \text{anc}(j, T) \); in either case we have \( j \in N(T_i) \). This, as well as Equation (23), imply the correctness of the lemma. \( \Box \)

One of the desirable properties of a basic tree game is that it is convex, as stated in the following theorem.

THEOREM 3.11. A basic tree game is a convex game.

PROOF. It is known (see, e.g., [Elkind et al. 2013]) that a characteristic function game is convex if and only if for every pair of coalitions \( C, C' \) such that \( C \subset C' \), and for every agent \( i \) not belonging to \( C' \), the following holds:

\[ mc(i, C) \leq mc(i, C'). \] (24)

Now since \( C \subset C' \), then based on Equation (6), we have:

\[ N(T_{trim}^{C \cup \{i\}}) \subseteq N(T_{trim}^{C'}) \].

This, as well as Lemma 3.10, imply the inequality in (24) holds, which implies the correctness of Theorem 3.11. \( \Box \)

Theorem 3.11 immediately implies that the Shapley value of a basic tree game is in the core of that game. This makes the Shapley value-based division scheme even more attractive, since it is not only fair but also stable (see Section 2 for more details).

THEOREM 3.12. Let \((T, f)\) be a basic tree game. For every agent \( i \in N(T) \), we have:

\[ Sh_i = \sum_{j=0}^{\text{height}(T_i)} \frac{|\text{Level}_j(T_i)|}{\text{depth}(i, T) + j + 1}. \] (25)

PROOF. Since \((T, f)\) is a basic tree game, Equation (2) and Lemma 3.10 imply that:

\[ Sh_i = \frac{1}{n!} \sum_{\pi \in \Pi^N(T)} \left| N(T_{trim}^{C \cup \{i\}} \cap N(T_i) \right|. \]

This can be written differently as follows:

\[ Sh_i = \frac{1}{n!} \sum_{j \in N(T_i)} \left| \left\{ \pi \in \Pi^N(T) : j \in N(T_{trim}^{C \cup \{i\}} \right\} \right|. \]

Based on this, as well as (6)—the equation that defines \( N(T_{trim}^{C}) \)—we have:

\[ Sh_i = \sum_{j \in N(T_i)} \frac{\Pi^U_j}{n!}. \] (26)
where
\[ \Pi^j = \left\{ \pi \in \Pi^{N(T)} : \text{anc}(j, T) \subseteq C^\pi \cup \{i\} \right\}. \] (27)
Let us compute \(|\Pi^j|/n!\). Based on Equation (27), for every \(\pi \in \Pi^{N(T)}\), we have:
\[ \pi \in \Pi^j \iff \text{anc}(j, T) \backslash \{i\} \subseteq C^\pi_i. \]
Thus, the number of permutations in \(\Pi^j\) equals the number of permutations in which all the agents in \(\text{anc}(j, T) \backslash \{i\}\) appear before \(i\) in \(\pi\). It is easy to see how the following steps cover all possible ways to construct such a permutation:

— **Step 1:** Place the members of \(N(T) \setminus (\text{anc}(j, T) \cup \{i\})\) in any slots in \(\pi\), without any restrictions.
— **Step 2:** Out of all remaining slots, place \(i\) in the last one.
— **Step 3:** Place the members of \(\text{anc}(j, T)\) in the remaining slots (from the previous step, all these slots are before \(i\)) in any order, without any restrictions.

Based on this, we have:
\[ |\Pi^j| = \left((n)(n-1)\ldots(n-|N(T) \setminus (\text{anc}(j, T) \cup \{i\})|+1)\right) \cdot \left(|\text{anc}(j, T)|!\right). \]
This, as well as the fact that \(|\text{anc}(j, T)| = \text{depth}(j, T)\), imply that:
\[ |\Pi^j| = (n)\ldots(\text{depth}(j, T) + 2)\left(\text{depth}(j, T)\right)\ldots(1). \]
Based on this, we have:
\[ \frac{|\Pi^j|}{n!} = \frac{(n)\ldots(\text{depth}(j, T) + 2)\left(\text{depth}(j, T)\right)\ldots(1)}{(n)\ldots(1)} \]
This can be simplified as follows:
\[ \frac{|\Pi^j|}{n!} = \frac{1}{\text{depth}(j, T) + 1} \] (28)
Finally, since \(j \in N(T_i)\), we have:
\[ \text{depth}(j, T) = \text{depth}(i, T) + \text{depth}(j, T_i). \] (29)
From (26), (28) and (29), we find that:
\[ Sh_i = \sum_{j \in N(T_i)} \frac{1}{\text{depth}(j, T) + 1} \]
\[ = \sum_{j \in N(T_i)} \frac{1}{\text{depth}(i, T) + \text{depth}(j, T) + 1} \]
\[ = \sum_{j=0}^{\text{height}(T_i)} \sum_{j \in \text{Level}_{j}(T_i)} \frac{1}{\text{depth}(i, T) + j + 1} \]
\[ = \sum_{j=0}^{\text{height}(T_i)} \frac{|\text{Level}_j(T_i)|}{\text{depth}(i, T) + j + 1} \]
\[ \Box \]
Based on Theorem 3.12, as well as the fact that \(\text{height}(T_i) \leq n\) for all \(i \in N(T)\), the Shapley value of any agent in a basic tree game is computable in \(O(n)\) time.
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Table II. Complexity of the Shapley Value.

|                | CFG | Tree Game                  | Basic Tree Game          |
|----------------|-----|----------------------------|--------------------------|
| Chain          | $2^{n-1}$ | $h - d + 1$ | $h - d + 1$ |
| Star           | $2^{n-1}$ | $2^{h-2}$ | $2 - d$ |
| Binary Tree    | $2^{h-1}$ | $h(h, d)$ | $h - d + 1$ |
| Arbitrary tree | $2^{n-1}$ | $|\{C \in \text{Trimmed}(T) : i \in C\}|$ | $\text{height}(T_i) + 1$ |

3.4. Computational Complexity of Shapley Value

We have shown that the Shapley value for basic tree games is linear-time computable. In this section, we compare the computational complexity of the Shapley in characteristic function games (CFG), tree games and basic tree games. We consider three standard tree structures: chain, star, and compete binary tree. Table II shows the number of coalitions that need to be considered when computing the Shapley value of agent $i$ at depth $d$ of a tree with height $h = \text{height}(T)$ when the total number of agents is $n$. The computation for the case of a complete binary tree in tree games appears below:

$$b(h, d) = (y_{h-d} + 1)^2 \times \prod_{j=h-d+1}^{h} (y_j + 1),$$

where $y_1 = 1$ and $y_j = (y_{j-1} + 1)^2$ for $j > 1$.

4. DISCUSSION AND CONCLUSIONS

We now revisit Example 1.1. The fact that Dropbox values a coalition of extra users at 1GB per user, can be encoded as $f(C) = |C|$ satisfying the definition of basic tree games (see Definition 3.9). Referring to Table I, the Dropbox mechanism give the highest reward to the user who made the most direct referrals: 3 gets three times as much as any of the other user. Users 6 and 7 who did not invite anyone receive 500MB each. In contrast, the geometric mechanism rewards both direct an indirect referrals but does not give anything to non-referring nodes. The Shapley value provides a middle ground: it acknowledges indirect referrals and also compensates non-referring nodes. The Shapley rewards are fair in the sense that extra value is assigned to 1 relative to 3 because without the referral of 1, agent 3 would have never joined. The values for the Shapley rewards appearing in Table I can be computed using (25). The rewards can also be computed dynamically as each new user joins. Should node 7 bring a friend, the compensation of each node would change by adding 250MB to compensations of the new node and of nodes 1, 3, 7.

We derived a mechanism based on a fairness concept. However, it also provides participation incentives for the agents. The property of the Shapley value that in compensates the “right” types of contribution (compensation is based on the entire subtree and not just on direct referrals) makes it more appealing for successful incentivisation from the intuitive standpoint. Observe that without introducing some costs into the model, any mechanism that offers a non-zero reward to the referrer and the invitee provides rational agents with incentives to participate and to refer all of their friends. To differentiate among Shapley and other mechanisms a future study may consider models where costs (e.g., of making a referral) are present.

General tree games allow applying the Shapley value to scenarios where the firm’s value for a new user depends on the identity of the user and the identities of other users who join (this allows expressing combinatorial preferences). This is relevant,
for example, when the firm has a particularly high value for certain celebrities who shape the image of the firm. This also allows modelling domains like the Red Balloon Challenge and QINs, where the value is obtained only if a node that holds the answer joins.

The Shapley value has a strong axiomatic justification, but its use in practice is limited. The referral marketing domain may be a compelling setting for applying the Shapley value. Empirical evaluation of its merits is an interesting direction for future investigation.

Implicit in our results is the assumption that each user has only one chance to receive a referral, or in other words, that a node has a single parent through which he can be referred. Relaxation of this assumption is a possible direction for further theoretical work.

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A. SUMMARY OF NOTATION

$A$ the set of agents in the game.
$n$ the number of agents in the game.
$C$ a coalition, i.e., a subset of $A$.
$v(C)$ the value of coalition $C$.
$CS$ a coalition structure, i.e., a partition of $A$.
$CS^A$ the set of coalition structures over $A$.
$\pi$ a permutation of $A$.
$\Pi^A$ the set of all possible permutations of $A$.
$C_i^\pi$ the coalition consisting of the agents that precede $i$ in $\pi$.
$x$ a payoff vector, where $x_i \in x$ is the payoff of agent $i \in A$.
$mc(i,C)$ the marginal contribution of $i$ to $C$, i.e., $v(C \cup \{i\}) - v(C)$.
$Sh_i$ the Shapley value of $i$.
$G$ a graph.
$G^C$ the subgraph of $G$ that is induced by $C$.
$N(G)$ the set of nodes in $G$.
$E(G)$ the set of edges in $G$.
$adj(G',G)$ the set of nodes in $G$ that are adjacent to $G'$, where $G'$ is an induced subgraph of $G$.
$T$ a tree.
$T_i$ the subset of $T$ rooted at $i$.
$r(T)$ the root of $T$.
$chi(i,T)$ the set of children of $i$ in $T$.
$des(i,T)$ the set of descendants of $i$ in $T$.
$anc(i,T)$ the set of ancestors of $i$ in $T$.
$depth(i,T)$ the depth of $i$ in $T$.
$Level_j(T)$ the set of nodes in $N(T)$ whose depth is $j$.
$height(T)$ the height of $T$.
$T^C$ the subgraph of $T$ that is induced by $C$.
$T^C_{trim}$ the subgraph obtained by “trimming” $T^C$. In other words, it is the connected component in $T^C$ that contains $r(T)$. See Figure 1.
$Trimmed(T)$ the set of coalitions that induce “trimmed” subgraphs. In other words, it contains every $C \subseteq N(T)$ such that $T^C = T^C_{trim}$.
$SameTrim(C,T)$ the set of coalitions $C' \subseteq N(C)$ such that by trimming $T^C$ we obtain the same subgraph as the one obtained by trimming $T^C$. 