A Correspondence Principle for the Gowers Norms

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1 Introduction

Informally speaking, the Furstenberg Correspondence \cite{5, 6} shows that the “local behavior” of a dynamical system is controlled by the behavior of sufficiently large finite systems. By the local behavior of a dynamical system \((X, B, \mu, G)\), we mean the properties which can be stated using finitely many actions of \(G\) and the integral given by \(\int f d\mu\). By a finite system, we just mean \((S, \mathcal{P}(S), c, G)\) where \(G\) is an infinite group, \(S\) is a finite quotient of \(G\), and \(c\) is the counting measure \(c(A) := \frac{|A|}{|S|}\).

The most well known example of such a property is the ergodic form of Szemerédi’s Theorem:

For every \(k\), every \(\epsilon > 0\), and every \(L^\infty\) function \(f\), if \(\int f d\mu \geq \epsilon\) then there is some \(n\) such that \(\int \prod_{j=0}^{k-1} T^{-jn} f d\mu > 0\).

The Furstenberg Correspondence shows that this is equivalent to the following statement of Szemerédi’s Theorem:

For every \(k\) and every \(\epsilon > 0\), there is an \(N\) and a \(\delta > 0\) such that if \(m \geq N\) and \(f : [0, m-1] \rightarrow [-1, 1]\) is such that \(\int f dc \geq \epsilon\) then there is some \(n\) such that \(\int \prod_{j=0}^{k-1} f(x + jn) dc(x) \geq \delta\).

In general, the Furstenberg Correspondence states that, given a sequence of functions on increasingly large finite systems, a single function on a single infinite system can be given with the property that suitable calculations are controlled by the limit of the value of analogous calculations in the finite systems.

Recent work \cite{9, 11, 12} has extended this correspondence both to other specific properties and to more general formulations. These methods are not adequate, however, for the study of the uniformity norms introduced for finite systems by Gowers in \cite{7} and for infinite systems by Host and Kra in \cite{8}. While the are strong reasons for believing that functions on finite systems with small Gowers norm should correspond to functions on infinite systems with small Gowers-Host-Kra norm, these norms are

\footnote{A more precise version of this notion would be to say that the local behavior consists of the \(\Pi_2\) formulas in an appropriate extension of the language of arithmetic.}
not local. In particular, the ordinary correspondence may place a sequence of highly 
$k$-uniform functions (that is, functions with $|| \cdot ||_{U^k}$ going to zero) in correspondence 
with a function with large $U^k$ norm.

In [10], Tao and Ziegler give a variant of the correspondence principle which pre-
serves the $U^k$ norms when the group $G$ is vector space over a finite field, $\mathbb{F}_p^\omega$. Their 
argument, however, takes advantage of group theoretic properties of $\mathbb{F}_p$, and does not 
immediately extend to other groups.

In this paper we give a similar correspondence for arbitrary countable Abelian 
groups. While there is no theoretical obstacle to giving the construction explicitly 
in a style similar to [10], the resulting argument would be quite unwieldy. Roughly 
speaking, where Tao and Ziegler can choose representative transformations randomly 
and expect that almost all choices suffice, here we have to choose particular transformations. It is much more convenient to do the work of choosing the correct transformations in an infinitary ergodic setting; the price is that we use an argument from 
nonstandard analysis to give a highly infinitary system acted on by a very large group, 
and then use ergodic methods to reduce the group down to something more manageable. For the sake of readers unfamiliar with nonstandard analysis, we isolate its use in 
a single lemma.

In Section 2 we lay the ergodic-theoretic groundwork for the correspondence, and 
in Section 3 we give the correspondence argument itself.

2 Choosing a Good Subgroup

Because of the nature of the intermediate object which will be produced by the non-
standard argument in Section 3 we want to work with a fairly general notion of a 
dynamical system.

**Definition 2.1.** A dynamical system consists of a probability measure space $(X, B, \mu)$ 
together with an Abelian group $G$, an action of $G$ on $X$ such that for each $g \in G$, 
the action $T_g : X \rightarrow X$ is measurable, and a finitely additive $G$-invariant probability 
measure space $(G, C, \lambda)$.

Usually $G$ is taken to be countable and $\mathcal{C}$ is taken to be the powerset of $G$, which 
is possible since $\lambda$ is only required to be finitely additive. Here, however, we need to 
include the case where $G$ is uncountable and $\mathcal{C}$ is not the full powerset of $G$. We do not 
require that the action of $G$ on $X$ be a measurable function from $G \times X$ to $X$, since we 
need actions where this is not true. Instead we only ask for the weaker condition that 
Fubini’s Theorem holds.

**Definition 2.2.** Given a bounded function $f$ and a group $G$, define $\mathcal{F}(f, G)$ to be the 
collection of functions containing $f$ and the function constantly equal to $1$, and closed 
under pairwise sums, pairwise multiplication, scalar multiplication by a rational, and 
shifts from $G$.

Clearly $\mathcal{F}(f, G)$ is countable as long as $G$ is.
Definition 2.3. We say a bounded, measurable function \( f \) on \( X \) is weakly Fubini for \( (G, C, \lambda) \) if for every \( x \in X \), the function \( g \mapsto f(T_g x) \) is measurable with respect to \( C \) and \( x \mapsto \int f(T_g x) d\lambda \) is measurable with respect to \( B \).

We say \( f \) is Fubini if every function in \( \mathcal{F}(f, G) \) is.

The requirement that the condition hold for every \( x \) could be weakened to almost every \( x \) without much trouble.

In this context, the Mean Ergodic Theorem can be taken to be the following:

Lemma 2.1. If \( f \) is Fubini for \( (G, C, \lambda) \) and \( \mathcal{I}(G) \) is the collection of sets invariant under \( T_g \) for every \( g \in G \) then

\[
\int \left[ E(f \mid \mathcal{I}(G))(x) - \int f(T_g x) d\lambda \right]^2 d\mu = 0.
\]

It will be convenient to extend groups by a single element while preserving the Fubini property.

Definition 2.4. Let \( H \) be a subgroup of \( G \) and suppose \( f \) is Fubini for \( (G, C, \lambda) \). For \( g \in G \), define \( H_g' \) to be the subgroup of \( G \) generated by \( H \cup \{g\} \). Taking \( \pi : H \times \mathbb{Z} \to H_g' \) to be the homomorphism given by \( \pi(h, n) := h \cdot g^n \), any finitely-additive \( H \)-invariant measure \( (H, D, \nu) \) may be extended to a measure on \( H_g' \) by taking the image of \( (H \times g, D \times \mathcal{P} (\mathbb{Z}), \nu \times \sigma) \) where \( \sigma \) is an arbitrary finitely-additive \( \mathbb{Z} \)-invariant measure.

Note that the choice of measure on \( H_g' \) is not canonical, but the Mean Ergodic Theorem tells us that the choice will not matter.

Lemma 2.2. If \( f \) is bounded and Fubini for both \( (G, C, \lambda) \) and \( (H, D, \nu) \) then for any \( g \in G \), \( f \) is Fubini for \( H_g' \).

Proof. Since \( h \mapsto f(T_h x) \) is measurable on \( H \), it is also measurable on \( H_g' \) (since we have taken the product with a discrete set). Measurability of \( x \mapsto \int f(T_h x) d(\nu \times \sigma)(h) \)

follows since

\[
\int f(T_h x) d(\nu \times \sigma)(h) = \lim_{N \to \infty} \frac{1}{|I_N|} \sum_{n \in I_n} \int f(T_h T_g^n x) d\nu(h)
\]

for some Følner sequence \( \{I_N\} \), each \( \frac{1}{|I_N|} \sum_{n \in I_n} \int f(T_h T_g^n x) d\nu(h) \) is measurable, and \( B \) is a \( \sigma \)-algebra. \( \square \)

The following definition and the basic properties of such norms are taken from [8].

Definition 2.5 (Gowers-Host-Kra Norms). Define \( X^{[k]} := X^{2^k} \), \( B^{[k]} := B^{2^k} \), and for any transformation \( T \) on \( (X, B, \mu) \), define \( T^{[k]} := \bigotimes_{\omega \in \{0, 1\}^k} T \).

If \( G \) is an Abelian group acting on \( (X, B, \mu) \), define \( \mu^{[0]}(G) := \mu \), \( \mathcal{T}^{[k]}(G) \) to be the collection of sets in \( B^{[k]} \) invariant under \( T_g^{[k]} \) for each \( g \in G \), and \( \mu^{[k+1]}(G) \) to be the relative joining of \( \mu^{[k]} \) with itself over \( \mathcal{T}^{[k]}(G) \).
For any $L^\infty$ function $f$, define
\[
\|f\|_{U^k(G)} := \left(\int \bigotimes_{\omega \in \{0,1\}^k} f d\mu^{[k]}(G)\right)^{1/2^k}.
\]

**Lemma 2.3.** If $f$ is Fubini for $(G, \mathcal{C}, \lambda)$ then
\[
\|f\|_{U^k(G)}^2 = \iiint \prod_{\omega \in \{0,1\}^k} T_{\tilde{g}\omega} f d\lambda^k(\tilde{g}) d\mu.
\]

**Proof.** We show by induction on $k$ that
\[
\int \bigotimes_{\omega \in \{0,1\}^k} f_\omega d\mu^{[k]} = \iiint \prod_{\omega \in \{0,1\}^k} T_{\tilde{g}\omega} f_\omega d\mu d\lambda^k(\tilde{g}).
\]

For $k = 0$, this is immediate. Assume the claim holds for $k$. Then
\[
\int \bigotimes_{\omega \in \{0,1\}^{k+1}} f_\omega d\mu^{[k+1]}
= \int \mathcal{E}(\bigotimes_{\omega \in \{0,1\}^k} f_{0\omega} | T^{[k]}(G)) E(\bigotimes_{\omega \in \{0,1\}^k} f_{1\omega} | T^{[k]}(G)) d\mu^{[k]}(G)
= \int \int \bigotimes_{\omega \in \{0,1\}^k} f_{0\omega} T_g f_{1\omega} d\mu d\lambda^k(G)
= \int \int \bigotimes_{\omega \in \{0,1\}^k} f_{0\omega} T_g f_{1\omega} d\mu d\lambda^k(G)
= \iiint \prod_{\omega \in \{0,1\}^k} T_{\tilde{g}\omega} T_{\tilde{g}\omega} f_{0\omega} T_g f_{1\omega} d\mu d\lambda^k(\tilde{g}) d\lambda(g)
= \iiint \prod_{\omega \in \{0,1\}^k} T_{\tilde{g}\omega} f_\omega d\mu d\lambda^{k+1}(\tilde{g}).
\]

The following property is easily seen by induction:

**Lemma 2.4.** If $H$ is a subgroup of $G$ then $\|f\|_{U^k(G)} \leq \|f\|_{U^k(H)}$.

It will be convenient to use a slight generalization of the $U^k$ norm, in which a different group is used at the top-most level.

**Definition 2.6.** Let $G, H$ be groups. Then $\mu^{[0]}(G, H) := \mu$, $T^{[k]}(G, H)$ is the space of sets $B \in \mathcal{B}^{[k]}$ such that $\mu^{[k]}(G)(B \triangle (T^{[k]}(G)_h B) = 0$ for each $h \in H$, and $\mu^{[k+1]}(G, H)$ is the relatively independent joining of $\mu^{[k]}(G)$ with itself over $T^{[k]}(G, H)$.

Similarly, $\|f\|_{U^k(G, H)} := \left(\int \bigotimes_{\omega \in \{0,1\}^k} f d\mu^{[k]}(G, H)\right)^{1/2^k}.$

**Lemma 2.5.** If $H$ and $H'$ are subgroups of $G$ and $H$ is a subgroup of $H'$ then $\|f\|_{U^k(G, H')} \leq \|f\|_{U^k(G, H)}$.

**Theorem 2.1.** Let $(X, B, \mu), (\Gamma, \mathcal{C}, \lambda)$ be a dynamical system, and let $H$ be a subgroup of $\Gamma$. Let $f$ be everywhere bounded by 1, let $f$ be Fubini for both $\Gamma$ and $H$, and suppose that $\|f\|_{U^{k+1}(\Gamma, H')}^2 = \|f\|_{U^{k+1}(\Gamma)}^2 + \epsilon$ with $\epsilon > 0$. Then there is a $g \in \Gamma$ such that, $\|f\|_{U^{k+1}(\Gamma, H'_g)} \leq \|f\|_{U^{k+1}(\Gamma)} + 3\epsilon/4.$
Proof. Note that a similar claim for the $U^0$ norm would be trivial, since the premise could never hold (the $U^0$ norm is independent of $\Gamma$). Observe that

$$||f||_{L^k+1(\Gamma)}^{2k+1} = \int E(\otimes f | T^k(\Gamma))^2 d\mu^k(\Gamma).$$

Setting $f' := E(\otimes f | T^k(\Gamma)H)$, this quantity is equal to

$$\int E(f' | T^k(\Gamma))^2 d\mu^k(\Gamma) = ||E(f' | T^k(\Gamma))||^2_{L^2(\mu^k(\Gamma))},$$

Suppose that for every $g \in \Gamma$, $||f' - T^k_g f'||_{L^2(\mu^k(\Gamma))} < \sqrt{\epsilon}$. Then also $||f' - E(f' | T^k(\Gamma))||^2_{L^2(\mu^k(\Gamma))} < \epsilon$, which implies that

$$||f||_{L^k+1(\Gamma)}^{2k+1} - ||f||_{L^k+1(\Gamma)}^{2k+1} = ||E(f' | T^k(\Gamma))||^2_{L^2(\mu^k(\Gamma))} - ||f'||_{L^2} < \epsilon,$$

contradicting the assumption.

So choose $g$ such that $||f' - T^k_g f'|| \geq \sqrt{\epsilon}$. Then for sufficiently large $n$, $||f' - \frac{1}{n} \sum_{i \leq n}(T^k_g)^i f'|| \geq \sqrt{\epsilon}/2$. It follows that $||f' - E(f' | T^k(H'_g))|| \geq \sqrt{\epsilon}/2$, and therefore that

$$||f||_{L^k+1(\Gamma,H'_g)}^{2k+1} = ||E(f' | H'_g)||^2 \leq ||f||_{L^k+1(\Gamma)}^{2k+1} + \frac{3}{4} \epsilon.$$

Lemma 2.6. Let $(X, \mathcal{B}, \mu)$, $(\Gamma, C, \lambda)$ be a dynamical system and, let $G$ be a countable subgroup of $\Gamma$, let $f$ be bounded and Fubini. Then there is a discrete subgroup $H$ of $\Gamma$ containing $G$ such that $||F||_{U^k(\Gamma,H)} = ||F||_{U^k(\Gamma)}$ for every $F \in \mathcal{F}(f, H)$.

Proof. We will construct $H$ so that there is a natural map $\pi : \mathbb{Z}^\omega \to H$. Then we may chose an ordering of order-type $\omega$ of pairs $(\epsilon, F)$ where $F$ is a code for an element of $\mathcal{F}(f, H)$. We set $H_0 := G$, and for each $n$, set $H_{n+1} := (H_n)'_g$ where $g$ is chosen so that $||F_n||_{U^k(H_{n+1})} \leq ||F_n||_{U^k(\Gamma)} + \epsilon_n$ (where $(\epsilon_n, F_n)$ is the $n$-th element in our ordering). Take $H := \bigcup_{n \in \omega} H_n$. Then for every $F \in \mathcal{F}(f, H)$ and every $\epsilon > 0$, for sufficiently large $n$, $||F||_{U^k(H_n)} \leq ||F||_{U^k(\Gamma)} + \epsilon$. $H$ is a subgroup of $\Gamma$, so $||F||_{U^k(\Gamma)} \leq ||F||_{U^k(H)}$, and therefore $||F||_{U^k(\Gamma)} = ||F||_{U^k(H)}$. □

Theorem 2.2. Let $(X, \mathcal{B}, \mu)$, $(\Gamma, C, \lambda)$ be a dynamical system, let $G$ be a countable subgroup of $\Gamma$, and let $f$ be bounded and Fubini. Then there is a discrete subgroup $H$ of $\Gamma$ containing $G$ such that $||F||_{U^k(H)} = ||F||_{U^k(\Gamma)}$ for every $F \in \mathcal{F}(f, H)$.

Proof. Let $H$ be as given in the preceeding lemma and proceed by induction on $k$. For $k = 0$, this is trivial. Assume the result holds for $k$. Then

$$||F||_{U^{k+1}(H)}^{2k+1} = \int E(\otimes F | T^k(\mathbb{Z}^\omega))^2 d\mu^k(H).$$
For any \( \epsilon > 0 \), we may choose \( i \) so that this is within \( \epsilon \) of
\[
\int E(\bigotimes F | I_k(H_i))^2 d\mu^{|k|}(H) = \int \bigotimes FT_h F d\mu^{|k|}(H).
\]
For every \( h \in H_i \),
\[
\int \bigotimes F \cdot T_h F d\mu^{|k|}(H) = ||FT_h F||_{U^k(H)}^{2^k}
\]
and since \( FT_h F \in \mathcal{F}(f, H) \), by IH \( ||FT_h F||_{U^k(H)}^{2^k} = ||FT_h F||_{U^k(\Gamma)}^{2^k} \). It follows that
\[
||F||_{U^{k+1}(H,H_i)}^{2^{k+1}} - ||F||_{U^{k+1}(\Gamma,H_i)}^{2^{k+1}} < \epsilon.
\]
But for sufficiently large \( i \), \( ||F||_{U^{k+1}(\Gamma,H_i)}^{2^{k+1}} \) is arbitrarily close to \( ||F||_{U^{k+1}(\Gamma)}^{2^{k+1}} \). So, taking the limit as \( i \to \infty \), we have
\[
||F||_{U^{k+1}(H)}^{2^{k+1}} = ||F||_{U^{k+1}(\Gamma)}^{2^{k+1}}.
\]

\[\square\]

### 3 A Correspondence Principal

To set up the appropriate analogy between different dynamical systems, we need the notion of a representative of an element of \( \mathcal{F}(\cdot, G) \).

**Definition 3.1.** Let \( \mathcal{F}(G) \) be a set of symbols defined inductively by:

- \( c \in \mathcal{F}(G) \)
- \( 1 \in \mathcal{F}(G) \)
- If \( f, g \in \mathcal{F}(G) \) then \( f + g \) and \( f \cdot g \) belong to \( \mathcal{F}(G) \)
- If \( h \in G \) and \( f \in \mathcal{F}(G) \) then \( \mathcal{T}_h f \in \mathcal{F}(G) \)
- If \( f \in \mathcal{F}(G) \) and \( q \) is a rational then \( q f \in \mathcal{F}(G) \)

If \( f \in \mathcal{F}(G) \) and \( f \) is a bounded Fubini function in a dynamical system, we define \( \mathcal{T}(f) \) recursively by:

- \( c(f) := f \)
- \( 1(f) := 1 \) (the function constantly equal to 1)
- \( (f + g)(f) := f(f) + g(f) \)
- \( (f \cdot g)(f) := f(f) \cdot g(f) \)
- \( (\mathcal{T}_h f)(f) := T_h(f(f)) \)
\[ (qf)(f) := q \cdot (f(f)) \]

It is easy to see that \( g \in \mathcal{F}(f, G) \) iff there is a \( \mathcal{f} \in \mathcal{F}(G) \) such that \( g = \mathcal{f}(f) \).

**Lemma 3.1.** Let \( G \) be a countable Abelian group, let \( \mathcal{N} \) be an infinite set of integers, and for each \( N \in \mathcal{N} \), let \( S_N \) be a finite quotient of \( G \), \( \pi_N : G \to S_N \), with \( |S_N| \to \infty \). Let \( f_N : S_N \to \mathcal{D} \) be given. There is a dynamical system \((X, B, \mu, \Gamma)\), a homomorphism \( \pi : G \to \Gamma \), and a measurable Fubini function \( f : X \to \mathcal{D} \) such that for any \( \mathcal{f} \in \mathcal{F}(G) \),

\[
\lim \inf_{N \in \mathcal{N}} \|f(f_N)\|_{U^k(S_N)} \leq \|f(f)\|_{U^k(\Gamma)} \leq \lim \sup_{N \in \mathcal{N}} \|f(f_N)\|_{U^k(S_N)}
\]

for each \( k \).

Additionally, if for every \( g, h \in G \), \( g \neq h \), \( \pi_N(g) \neq \pi_N(h) \) except for finitely many \( N \in \mathcal{N} \) then \( \pi(g) \neq \pi(h) \).

**Proof.** Fix a non-principle ultrafilter \( U \) and form the nonstandard extension of a universe containing the sequences \( \langle f_N \rangle, \langle S_N \rangle \). The sequence \( \mathcal{N} \) codes a nonstandard integer \( a \) and \( S_a \) is a hyperfinite Abelian group. By the Loeb measure construction, the internal subsets of \( S_a \) may be extended to a \( \sigma \)-algebra on \( S_a \), and the sequence \( \langle f_N \rangle \) represents an internal function \( F : S_a \to \mathcal{D}^* \). The function \( f = st \circ F \) is then a measurable function from \( S_a \) to \( \mathcal{D} \).

This same measure space is also \((\Gamma, C, \lambda)\), with \( \Gamma \) acting on \( S_a = \Gamma \) by the group action. Since \( F \) is internal, for any \( x \in S_a \), the functions \( g \mapsto f(T_gx) \) and \( x \mapsto \int f(T_gx)d\lambda \) are the result of applying the standard part operation to internal functions, and are therefore measurable, and the same applies to any element of \( \mathcal{F}(f, \Gamma) \). So \( f \) is Fubini. The embedding \( \pi : G \to \Gamma \) is simply the embedding represented by the sequence \( \langle \pi_N \rangle \).

The final clause follows from transfer. For instance, if \( \lim \inf_{N \in \mathcal{N}} \|f_N\|_{U^k(S_N)} \geq \alpha \) then for each \( \epsilon > 0 \) and all but finitely many \( N \in \mathcal{N} \),

\[
\frac{1}{|S_N|^{k+1}} \sum_{x \in S_N} \sum_{\bar{g} \in S_N^k} \prod_{\omega \in \{0,1\}^k} f_N(x + \bar{g} \cdot \omega) > \alpha - \epsilon
\]

and therefore

\[
st \left( \frac{1}{|S_a|^{k+1}} \sum_{x \in S_a} \sum_{\bar{g} \in S_N^k} \prod_{\omega \in \{0,1\}^k} F(x + \bar{g} \cdot \omega) \right) \geq \alpha - \epsilon
\]

and therefore

\[
\int \prod_{\omega \in \{0,1\}^k} T_{\bar{g} \cdot \omega} f d\mu d\lambda^k \geq \alpha - \epsilon.
\]

Applying this argument for arbitrary \( \mathcal{f} \in \mathcal{F}(G) \), and the analogous argument for the upper bound, gives the claim. \( \square \)

As shown in the previous section, there is a discrete subgroup \( H \) of \( \Gamma \) containing \( G \) such that for each \( F \in \mathcal{F}(f, H) \), \( \|F\|_{U^k(H)} = \|F\|_{U^k(\Gamma)} \). Putting this together, we obtain the following theorem:
Theorem 3.1. Let $G$ be a countable Abelian group, let $N$ be an infinite set of integers, and for each $N \in N$, let $S_N$ be a finite quotient of $G$, $\pi_N : G \to S_N$, with $|S_N| \to \infty$. Let $f_N : S_N \to D$ be given. Then there is a dynamical system $(X, B, \mu, H)$ and an $L^\infty$ function $f$ such that for any $f \in \mathcal{F}(G)$,

$$\lim\inf_{N \in N} ||f(f_N)||_{U^k(S_N)} \leq ||f||_{U^k(\Gamma)} \leq \lim\sup_{N \in N} ||f(f_N)||_{U^k(S_N)}$$

for each $k$.

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