ISOMETRIC ACTIONS OF SIMPLE GROUPS AND TRANSVERSE STRUCTURES: THE INTEGRABLE NORMAL CASE

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With gratitude to Robert J. Zimmer on the occasion of his 60th birthday.

Abstract. For actions with a dense orbit of a connected noncompact simple Lie group $G$, we obtain some global rigidity results when the actions preserve certain geometric structures. In particular, we prove that for a $G$-action to be equivalent to one on a space of the form $(G \times K\backslash H)/\Gamma$, it is necessary and sufficient for the $G$-action to preserve a pseudo-Riemannian metric and a transverse Riemannian metric to the orbits. A similar result proves that the $G$-actions on spaces of the form $(G \times H)/\Gamma$ are characterized by preserving transverse parallelisms. By relating our techniques to the notion of the algebraic hull of an action, we obtain infinitesimal Lie algebra structures on certain geometric manifolds acted upon by $G$.

1. Introduction

In the rest of this work, we let $G$ be a connected noncompact simple Lie group with Lie algebra $\mathfrak{g}$ and $M$ a smooth connected manifold acted upon smoothly by $G$. There are several examples of such actions that preserve a finite volume. Some of the most interesting are obtained from Lie group homomorphisms $G \rightarrow L$ where $L$ is a connected Lie group that admits a lattice $\Gamma$. The relevant $G$-action is then given on the double coset $K\backslash L/\Gamma$, where $K$ is some compact subgroup that centralizes $G$. Robert Zimmer proposed in [15] to study the finite measure preserving $G$-actions on $M$ and determine to what extent these are given by such double cosets.

To understand how to tackle Zimmer’s program, it has been proved very useful to consider $G$-actions preserving a suitable geometric structure (see [2] and [18]). This is a natural condition since the above double coset examples carry a $G$-invariant pseudo-Riemannian metric when $L$ is semisimple. Such metric comes from the bi-invariant metric on $L$ obtained from the Killing form of its Lie algebra.

The properties of the $G$-orbits of finite measure preserving $G$-actions are reasonably well known. For example, such $G$-actions are known to be everywhere locally free when they preserve suitable pseudo-Riemannian metrics (see [2], [13] and [14]). Also in such case, the metric restricted to the orbits can be described precisely (see Lemma 2.6). However, there is still a lack of knowledge of the properties of a manifold, acted upon by $G$, in the transverse direction to the orbits.

In this work, we propose to study the $G$-actions on $M$ by emphasizing the need to understand the properties of the transverse to the $G$-orbits. For this, we will be

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dealing with $G$-actions that have a dense orbit and preserve a finite volume pseudo-Riemannian metric. As observed above, in this case the results from [13] show that the $G$-orbits define a foliation, which from now on we will denote with $O$. By assuming that the orbits are nondegenerate for the pseudo-Riemannian metric on $M$, we can consider the normal bundle $TO^\perp$ to the orbits as realizing the transverse direction in $M$. With this respect, we obtain the following structure theorem for $G$-actions on $M$ when this normal bundle is integrable.

**Theorem 1.1.** Suppose that the $G$-action on $M$ has a dense orbit and preserves a finite volume complete pseudo-Riemannian metric. If the $G$-orbits in $M$ are nondegenerate and the normal bundle to the orbits $TO^\perp$ is integrable, then there exist:

1. an isometric finite covering map $\hat{M} \to M$ to which the $G$-action lifts,
2. a simply connected complete pseudo-Riemannian manifold $\tilde{N}$, and
3. a discrete subgroup $\Gamma \subset G \times \text{Iso}(\tilde{N})$,

such that $\hat{M}$ is $G$-equivariantly isometric to $(G \times \tilde{N})/\Gamma$.

We observe that in the double coset examples $K\setminus L/\Gamma$ as above with $L$ semisimple and with the metric coming from the Killing form, the $G$-orbits are always nondegenerate. Besides that, we prove that for a general $G$-action on $M$, the orbits are always nondegenerate when $\dim(M) < 2\dim(G)$ (see Lemma [27]). Also, by developing criteria for the normal bundle $TO^\perp$ to be integrable, we obtain some results where the conclusion of Theorem 1.1 holds. For example, Corollary 3.1 ensures such conclusion when $G$ has high enough real rank. Also, Corollary 3.2 does the same for manifolds $M$ whose dimensions have a suitable bound in terms of $G$. In this last result we can even dispense with the assumption of having nondegenerate orbits by applying Lemma [27].

Without assuming in Theorem 1.1 that the $G$-action has a dense orbit it is still possible to draw some conclusions. More specifically, if we assume the rest of the hypotheses, a description of the universal covering space of the manifold as a warped product is obtained. This sort of result already appears in [3].

For a $G$-action on $M$ preserving a pseudo-Riemannian metric, in [10] we considered a certain geometric condition between the metrics on $G$ and $M$ (the former for a bi-invariant metric) from which we concluded that $M$ is, up to a finite covering space, a double coset of the form $(G \times K\setminus H)/\Gamma$. In other words, a double coset as above where $G$ appears as a factor in $L$. One of the steps used in [10] to achieve this was to prove that the normal bundle $TO^\perp$ is Riemannian. Considering the relevance we are giving to the transverse to the orbits, it is natural to determine the properties of the $G$-actions that preserve a transverse Riemannian structure. In this context, we obtain the following result which proves that, up to a finite covering, the double cosets of the form $(G \times K\setminus H)/\Gamma$ are characterized as those isometric $G$-actions that preserve a transverse Riemannian structure on the foliation $O$. We recall that a semisimple Lie group is isotypic if the complexification of its Lie algebra is isomorphic to a sum of identical simple complex ideals.

**Theorem 1.2.** If $G$ is a connected noncompact simple Lie group acting faithfully on a compact manifold $M$, then the following conditions are equivalent:

1. There is a $G$-equivariant finite covering map $(G \times K\setminus H)/\Gamma \to M$ where $H$ is a connected Lie group with a compact subgroup $K$ and $\Gamma \subset G \times H$ is a discrete cocompact subgroup such that $G\Gamma$ is dense in $G \times H$. 

There is a finite covering map \( \hat{M} \to M \) for which the \( G \)-action on \( M \) lifts to a \( G \)-action on \( \hat{M} \) with a dense orbit and preserving:

- a pseudo-Riemannian metric for which the orbits are nondegenerate,
- a transverse Riemannian structure for the foliation \( O \) by \( G \)-orbits.

Furthermore, if \( G \) has finite center and real rank at least 2, then we can assume in (1) of the above equivalence that \( G \times H \) is semisimple isotypic with finite center and that \( \Gamma \) is an irreducible lattice.

Based on the previous result, we prove in Theorem 5.1 that the \( G \)-actions on \( M \) preserving a Lorentzian metric are, up to a finite covering, those given by double cosets \((G \times K)\backslash H/\Gamma\) with \( G \) locally isomorphic to \( \text{SL}(2, \mathbb{R}) \). We observe that this result improves a similar one found in [2] (see also the Introduction of [3]).

Continuing with our study of transverse structures, we next consider isometric \( G \)-actions preserving a transverse parallelism for the foliation \( O \). We prove that, up to a finite covering, such actions characterize the double cosets of the form \((G \times H)\backslash \Gamma\).

**Theorem 1.3.** If \( G \) is a connected noncompact simple Lie group acting faithfully on a compact manifold \( M \), then the following conditions are equivalent:

1. There is a \( G \)-equivariant finite covering map \((G \times H)\backslash \Gamma \to M\) where \( H \) is a connected Lie group and \( \Gamma \subset G \times H \) is a discrete cocompact subgroup such that \( G\Gamma \) is dense in \( G \times H \).
2. There is a finite covering map \( \hat{M} \to M \) for which the \( G \)-action on \( M \) lifts to a \( G \)-action on \( \hat{M} \) with a dense orbit and preserving:
   - a pseudo-Riemannian metric for which the orbits are nondegenerate,
   - a transverse parallelism for the foliation \( O \) by \( G \)-orbits.

Furthermore, if \( G \) has finite center and real rank at least 2, then we can assume in (1) of the above equivalence that \( G \times H \) is semisimple isotypic with finite center and that \( \Gamma \) is an irreducible lattice.

The notion of the algebraic hull of an action on a bundle, introduced by Zimmer, is a fundamental tool to understand \( G \)-actions as they relate to geometric structures. We recall that the algebraic hull is the smallest algebraic subgroup for which there is a measurable \( G \)-invariant reduction of the bundle being acted upon (see [17]). For our setup, where we are interested in the transverse to the orbits, it is then natural to consider the algebraic hull of the \( G \)-action on the bundle \( L(TO^\perp) \) for an isometric \( G \)-action with nondegenerate orbits. Since a \( G \)-action of this sort preserves a pseudo-Riemannian metric on \( TO^\perp \), the algebraic hull for \( L(TO^\perp) \) is in this case a subgroup of \( \text{O}(p, q) \), for some \( (p, q) \). The next result shows that for weakly irreducible manifolds and when the algebraic hull for \( L(TO^\perp) \) is the largest possible for this setup, the manifold being acted upon has infinitesimally at some point the structure of a specific Lie algebra that contains \( g \). By the last claim in the statement, such Lie algebra structure is nontrivially linked to the geometry of the manifold. For the explicit description of the Lie algebra structure obtained in this result we refer to Lemma 7.1 and Theorem 7.2.

We recall that a pseudo-Riemannian manifold is weakly irreducible if the tangent space at some (and hence any) point has no proper nondegenerate subspaces invariant under the restricted holonomy group at that point. Also recall that every simple Lie group with a bi-invariant metric is weakly irreducible.
Theorem 1.4. Suppose that $G$ has finite center and real rank at least 2, and that the $G$-action on $M$ preserves a finite volume complete pseudo-Riemannian metric. Also assume that $G$ acts ergodically on $M$ and that the foliation by $G$-orbits is nondegenerate. Denote with $L$ the algebraic hull for the $G$-action on the bundle $L(TO^\perp)$ and with $l$ its Lie algebra. In particular, there is an embedding of Lie algebras $l \hookrightarrow so(p,q)$, where $(p,q)$ is the signature of the metric of $M$ restricted to $TO^\perp$. If this embedding is surjective and $M$ is weakly irreducible, then the following holds:

1. $G$ is locally isomorphic to $SO_0(p,q)$ and $\dim(M) = (p+q)(p+q+1)/2$,
2. for some $x \in M$ the tangent space $T_xM$ admits a Lie algebra structure isomorphic to either $so(p,q+1)$ or $so(p+1,q)$ such that $T_x\mathcal{O}$ is a Lie subalgebra isomorphic to $g$.

Furthermore, with respect to the representation of Lemma 2.1, there is a Lie algebra of local Killing fields vanishing at $x$ which is isomorphic to $g$ and acts nontrivially on $T_xM$ by derivations of the Lie algebra structure given in (2).

With the results developed in this work, we try to show the importance of considering transverse geometric structures to understand the actions of noncompact simple Lie groups. In fact, we believe that some form of the following conjecture could provide a geometric characterization of the double coset examples of $G$-actions mentioned above.

Conjecture 1.5. Consider the double coset $G$-spaces of the form $K\backslash L/\Gamma$, where $L$ is a semisimple Lie group with an irreducible lattice $\Gamma$ and a compact subgroup $K$; the $G$-action is then induced by a nontrivial homomorphism $G \to L$ whose image centralizes $K$. Then, a $G$-action on a manifold $M$ is equivalent to such a double coset $G$-action for some $L, \Gamma, K$ if and only if the $G$-action on $M$:

- has a dense orbit,
- preserves a pseudo-Riemannian metric, and
- preserves a transverse geometric structure to the orbits suitably related to the geometry of $GK\backslash L$.

Note that for $L = G \times H$ and $K$ a compact subgroup of $H$, we obtain quotients $GK\backslash L = K\backslash H$ and $G\backslash L = H$ which naturally carry a Riemannian metric and a parallelism, respectively. Thus Theorems 1.2 and 1.3 verify the conjecture in these cases.

One of the main tools used to obtain our results is Gromov’s machinery on geometric $G$-actions (see [1], [2], [10] and [18]). Such machinery ensures the existence of large families of local Killing fields. We develop these techniques in Section 2 making emphasis on the fact that the Killing fields thus obtained yield $g$-module structures on the tangent spaces to $M$; with this respect, our main result is Proposition 2.5 which is due to Gromov [2] in the analytic/compact case (see also [15]) and was extended to the smooth/finite volume case in [1] and [10]. In Section 3 we prove that the latter impose restrictions strong enough to guarantee the integrability of the normal bundle under suitable conditions (Corollaries 3.1 to 3.3). We observe that Theorem 1.1 and its consequences obtained in Section 3 are in fact extensions of results obtained in [2]: Theorem 5.3.F in page 129 of [2] states, under the assumptions of our Corollary 3.1, that $M$ has a covering $G$-equivariantly diffeomorphic to $G \times N$ for some $N$. Note, however, that the result in [2] yields only a topological covering and does not further describes the covering group.
Theorem 1.2 is obtained in Section 4 by applying the main results and arguments from [10]. As mentioned above, this yields in Section 5 our characterization of Lorentzian manifolds acted upon with a dense orbit by a simple noncompact Lie group. In Section 6 we establish the characterization of G-spaces of the form \((G \times H)/\Gamma\) provided by Theorem 1.3 for this, one of the main ingredients is given by Theorem 1.1. We observe that the arguments used in Section 6 are based on those found in [11].

To obtain Theorem 1.4 in Section 7 an application of Theorem 1.1 is used again. But now, the notion of the algebraic hull and the deep result about it found in [17] are also fundamental. Ultimately, this is somewhat to be expected, since a computation of the algebraic hull for frame bundles over \(M\) is in fact an important step to build Gromov's machinery for geometric \(G\)-actions on \(M\). This is evident in our proof of Proposition 2.3.

We want to observe that the conclusion of Theorem 1.4 can be paraphrased by saying that the manifold \(M\) has infinitesimally the structure of either \(\text{SO}(p,q+1)\) or \(\text{SO}(p+1,q)\). It remains the question as to whether or not \(M\) can be related more precisely to either of these groups. This problem will be pursued elsewhere (see [7]) by requiring some additional conditions.

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2. Killing fields on manifolds with a simple group of isometries

By Theorem 4.17 from [13], if the \(G\)-action on \(M\) has a dense orbit and preserves a finite volume pseudo-Riemannian metric, then the action is locally free and so the orbits define a foliation that we have agreed to denote with \(\mathcal{O}\). In this case, it is well known that the bundle \(T\mathcal{O}\) tangent to the foliation \(\mathcal{O}\) is a trivial vector bundle isomorphic to \(M \times \mathfrak{g}\), under the isomorphism given by:

\[
M \times \mathfrak{g} \to T\mathcal{O} \\
(x, X) \mapsto X^*_x.
\]

For every \(x \in M\), this induces an isomorphism between the fiber \(T_x\mathcal{O}\) and \(\mathfrak{g}\), which we will refer to as the natural isomorphism. Furthermore, if we consider on \(M \times \mathfrak{g}\) the product \(G\)-action, where the \(G\)-action on \(\mathfrak{g}\) is the adjoint one, then such isomorphism is \(G\)-equivariant. More precisely, we have:

\[
dg(X^*) = \text{Ad}(g)(X)^*
\]

for every \(g \in G\) and \(X \in \mathfrak{g}\). Note that for \(X\) in the Lie algebra of a group acting on a manifold, we denote with \(X^*\) the vector field on the manifold whose one-parameter group of diffeomorphisms is given by \((\exp(tX))_t\) through the action on the manifold.

For any given pseudo-Riemannian manifold \(N\), we will denote with \(\text{Kill}(N,x)\) the Lie algebra of germs at \(x\) of local Killing vector fields defined in a neighborhood of \(x\), and with \(\text{Kill}_0(N,x)\) we will denote the Lie subalgebra consisting of those germs that vanish at \(x\). The following result is a consequence of the Jacobi identity and the fact that the Lie derivative of a metric with respect to its Killing fields vanishes. In the rest of this work, for a vector space \(W\) with a nondegenerate symmetric bilinear form, we will denote with \(\mathfrak{so}(W)\) the Lie algebra of linear maps on \(W\) that are skew-symmetric with respect to the bilinear form.
Lemma 2.1. Let $N$ be a pseudo-Riemannian manifold and $x \in N$. Then, the map:

$$\lambda_x : \text{Kill}_k(N, x) \to \mathfrak{so}(T_x N)$$

$$\lambda_x(Z)(v) = [Z, V]_x,$$

where $V$ is any vector field such that $V_x = v$, is a well defined homomorphism of Lie algebras.

From now on, for a given point $x$ of a pseudo-Riemannian manifold, the map $\lambda_x$ will denote the homomorphism from the previous lemma.

For the proof of our next result we will present some facts about infinitesimal automorphisms and Killing fields, and we refer to [1] for further details. The tangent bundle of order $k$ of a manifold $N$ is the smooth bundle $T^{(k)}N$ whose fiber at $x$ is the space $T_x^{(k)}N$ of $(k - 1)$-jets at $x$ of vector fields of $N$. For every (local) diffeomorphism $\varphi : N_1 \to N_2$ mapping $x_1$ to $x_2$ we have a linear isomorphism:

$$T^{(k)}_{x_1}N_1 \to T^{(k)}_{x_2}N_2$$

$$j_{x_1}^{k-1}(X) \mapsto j_{x_2}^{k-1}(d\varphi(X))$$

that depends only on the jet $j^k_{x_1}(\varphi)$. For $N_1 = N_2 = N$, $x_1 = x_2 = x$ this yields the group $D^{(k)}_x(N)$ of $k$-jets at $x$ of local diffeomorphisms fixing $x$, whose group structure is induced by the composition of maps. In the case when $N = \mathbb{R}^n$ and $x = 0$, we will denote this group with $\text{GL}^{(k)}(n)$. We also recall that the $k$-th order frame bundle over $N$ is the smooth bundle $L^{(k)}(N)$ that consists of the $k$-jets at 0 of local diffeomorphisms $\mathbb{R}^n \to N$; any such a jet $j^k_{x_1}(\varphi)$ thus defines a linear isomorphism $T^{(k)}_0\mathbb{R}^n \to T^{(k)}_{\varphi(0)}N$. The structure group of $L^{(k)}(N)$ is $\text{GL}^{(k)}(n)$.

With respect to vector fields, we denote with $D^{(k)}_x(N)$ the space of $k$-jets at $x$ of vector fields vanishing at $x$, and we use the special notation $\mathfrak{g}^{(k)}(n)$ when $N = \mathbb{R}^n$ and $x = 0$. If $N$ carries a pseudo-Riemannian metric, for every $x \in N$, we will denote with $\text{Aut}^{k}(N, x)$ the subgroup of $D^{(k)}_x(N)$ consisting of those $k$-jets that preserve the metric up to order $k$ at $x$. Correspondingly, for vector fields we denote with $\text{Kill}^{k}(N, x)$ the space of $k$-jets at $x$ of vector fields on $N$ that preserve the metric up to order $k$ at $x$; the subspace of those $k$-jets whose 0-jet vanishes is denoted with $\text{Kill}^{k}_0(N, x)$. The next result provides a natural representation of $D^{(k)}_x(N)$ from which the Lie algebras of this group and of $\text{Aut}^{k}(N, x)$ are described in terms of $D^{(k)}_x(N)$ and $\text{Kill}^{k}_0(N, x)$, respectively; the proof is elementary, but it is detailed in Section 2 and 4 of [1].

Lemma 2.2. For a smooth manifold $N$ and any given point $x \in N$ the following properties hold for every $k \geq 1$:

1. The map

$$\Theta_x : D^{(k)}_x(N) \to \text{GL}(T^{(k)}_x N)$$

$$\Theta_x(j^k_x(\varphi))(j^{k-1}_x(X)) = j^{k-1}_x(d\varphi(X))$$

is a Lie group monomorphism.

2. The assignment $[j^k_x(X), j^k_x(Y)] = -j^k_x([X,Y])$ yields a well defined Lie algebra structure on $D^{(k)}_x(N)$. 
The map:

\[ \theta_x : \mathcal{D}_x^k(N) \to \mathfrak{gl}(T_x^{(k)}N) \]

\[ \theta_x(j_x^k(X))(j_x^{k-1}(Y)) = -j_x^{k-1}([X,Y]) \]

is a Lie algebra monomorphism for the Lie algebra structure on \( \mathcal{D}_x^k(N) \) given by \([\cdot,\cdot]^k\). Furthermore, \( \theta_x(\mathcal{D}_x^k(N)) = \text{Lie}(\Theta_x(\mathcal{D}_x^k(N))) \).

(4) If \( N \) has a pseudo-Riemannian metric, then we have \( \theta_x(\text{Kill}_0^k(N,x)) = \text{Lie}(\Theta_x(\text{Aut}^k(N,x))) \).

In particular, with respect to the homomorphisms \( \Theta_x \) and \( \theta_x \), the Lie algebra of \( \text{Aut}^k(N,x) \) is realized by \( \text{Kill}_0^k(N,x) \) with the Lie algebra structure given by \([\cdot,\cdot]^k\).

The following result is due to Gromov in the analytic case (see [2]) and it was extended to the smooth case in [10]. We present here a fairly detailed proof based on the results from [1]. In congruence with our notation for \( M \), in the rest of this work we will use \( \mathcal{O} \) to denote the foliation by \( \hat{G} \)-orbits in \( \hat{M} \) for the lifted \( \hat{G} \)-action on every covering space \( \hat{M} \) of \( M \); this will be so for any covering whether finite or not.

**Proposition 2.3.** Suppose that the \( G \)-action on \( M \) has a dense orbit and preserves a finite volume pseudo-Riemannian metric. Then, there exists a dense subset \( S \subset \hat{M} \) such that for every \( x \in S \) the following properties are satisfied.

1. There is a homomorphism of Lie algebras \( \rho_x : \mathfrak{g} \to \text{Kill}(\hat{M},x) \) which is an isomorphism onto its image \( \rho_x(\mathfrak{g}) = \mathfrak{g}(x) \).
2. \( \mathfrak{g}(x) \subset \text{Kill}_0(\hat{M},x) \), i.e. every element of \( \mathfrak{g}(x) \) vanishes at \( x \).
3. For every \( X, Y \in \mathfrak{g} \) we have:

\[ [\rho_x(X),Y^*] = [X,Y]^* = -[X^*,Y^*], \]

in a neighborhood of \( x \). In particular, the elements in \( \mathfrak{g}(x) \) and their corresponding local flows preserve both \( \mathcal{O} \) and \( T\mathcal{O}^\perp \) in a neighborhood of \( x \).

4. The homomorphism of Lie algebras \( \lambda_x \circ \rho_x : \mathfrak{g} \to \mathfrak{so}(T_x\hat{M}) \) induces a \( \mathfrak{g} \)-module structure on \( T_x\hat{M} \) for which the subspaces \( T_x\mathcal{O} \) and \( T_x\mathcal{O}^\perp \) are \( \mathfrak{g} \)-submodules.

**Proof.** Since the proof builds on the notions and results found in [1], we will mostly follow its notation. We will be careful to define the objects considered but we refer to [1] for further details.

For every \( k \geq 1 \), let us denote with \( \sigma^k : L^{(k)}(M) \to Q_k \) the GL\((k)\)-equivariant map that defines the \( k \)-th order extension of the geometric structure defined by the pseudo-Riemannian metric on \( M \).

Consider the set:

\[ \mathcal{G}^k = \{ j_x^{k-1}(X^*) : X \in \mathfrak{g}, x \in M \}, \]

which, by the local freeness of the \( G \)-action, is a smooth subbundle of \( T^{(k)}M \). In fact, we have \( \mathcal{G} = T\mathcal{O} \), and as with this bundle there is a trivialization given by:

\[ M \times \mathfrak{g} \to \mathcal{G}^k \]

\[ (x,X) \mapsto j_x^{k-1}(X^*). \]
The corresponding trivialization of the frame bundle of $G^k$ is given by:

$$M \times \text{GL}(g) \to L(G^k)$$

$$(x, A) \mapsto A_x.$$  

where $A_x(X) = j_x^k((AX)^*)$. Note that we have taken $g$ as the standard fiber of the bundle $G^k$.

Choose a subspace $G_0$ of $T_0^{(k)} \mathbb{R}^n$ isomorphic to $g$. We will fix such subspace as well as an isomorphism with $g$ through which we will identify these two spaces. Let us now consider:

$$L^{(k)}(M, G^k) = \{ \alpha \in L^{(k)}(M) : \alpha(G_0) = G^k_x \text{ if } \alpha \in L^{(k)}(M)_x \}$$

which is a smooth reduction of $L^{(k)}(M)$ to the subgroup of $\text{GL}^{(k)}(n)$ that preserves the subspace $G_0$; we will denote such subgroup with $\text{GL}^{(k)}(n, G_0)$. Recall from the remarks preceding Lemma 2.22 that for every $\tilde{j}_0^k(\varphi) \in L^{(k)}(M)$ we obtain a linear isomorphism:

$$T_0^{(k)} \mathbb{R}^n \to T_{\varphi(0)}^0 M$$

$$j_0^{k-1}(X) \mapsto j_{\varphi(0)}^{-1}(d\varphi(X)).$$

In particular, if we let:

$$f_k : L^{(k)}(M, G^k) \to L(G^k)$$

$$\tilde{j}_0^k(\varphi) \mapsto \tilde{j}_0^k(\varphi)|_{G_0},$$

then, by the identification between $G_0$ and $g$, we can consider $f_k$ as a well-defined smooth principal bundle morphism that covers the identity. The associated homomorphism of structure groups for $f_k$ is given by:

$$\pi_k : \text{GL}^{(k)}(n, G_0) \to \text{GL}(g)$$

$$\tilde{j}_0^k(\varphi) \mapsto \tilde{j}_0^k(\varphi)|_{G_0},$$

which is clearly surjective. Note that we have used again our identification between $g$ and $G_0$.

Fix $\mu$ an arbitrary ergodic component for the $G$-action on $M$ for the pseudo-Riemannian volume. Then, there is a measurable reduction $P$ of $L^{(k)}(M, G^k)$ so that $\sigma^k(P)$ is ($\mu$-a.e. over $M$) a single point $q_0 \in Q_k$. Furthermore, the structure group of $P$ is the subgroup of $\text{GL}^{(k)}(n, G_0)$ that stabilizes $q_0$, and in particular it is algebraic. This claim is a consequence of the fact that the $\text{GL}^{(k)}(n, G_0)$-action on $Q_k$ is algebraic, which in turn follows from the fact that a pseudo-Riemannian metric is a geometric structure of algebraic type; we refer to Section 4 and the proof of Proposition 8.4 of [1] for further details.

On the other hand, since $\pi_k$ is a surjection and $f_k$ is $G$-equivariant, by Proposition 8.2 of [1], there exist reductions $Q_1$ and $Q_2$ of $L^{(k)}(M, G^k)$ and $L(G^k)$, respectively, to subgroups $L_1 \subset \text{GL}^{(k)}(n, G_0)$ and $\widetilde{\text{Ad}(G)^k} \subset \text{GL}(g)$, such that $f_k(Q_1) \subset Q_2$ ($\mu$-a.e. over $M$) and such that $\pi_k(L_1)$ is a finite index subgroup of $\widetilde{\text{Ad}(G)^k}$. Here $L_1$ can be chosen to be the algebraic hull of $L^{(k)}(M, G^k)$ for the $G$-action on $M$ with respect to the ergodic measure $\mu$. This claim uses the well-known fact that $\widetilde{\text{Ad}(G)^k}$ is the algebraic hull of $M \times \text{GL}(g)$ for the product action.
It is not difficult to see that this can be chosen so that $Q_2 = M \times \text{Ad}(G)^{\mathbb{Z}}$ (μ-a.e. over $M$) with respect to the above identification $M \times \text{GL}(g) \cong L(G^k)$. We can also assume that $Q_1 \subset P$, μ-a.e. over $M$, because the reduction $Q_1$ is the smallest one to an algebraic subgroup.

The above discussion ensures that for μ-a.e. $x \in M$, we have the following relations:

$$f_k((Q_1)_x) \subset (Q_2)_x = \{ x \} \times \text{Ad}(G)^{\mathbb{Z}}$$

$$(Q_1)_x \subset (P)_x \subset L^{(k)}(M, G^k)_x$$

$$\sigma^k((P)_x) = \{ q_0 \}$$

Let us now fix a point $x$ such that these conditions hold. Choose $\alpha_x \in (Q_1)_x$ and let $f_k(\alpha_x) = (x, k_x)$ where $k_x \in \text{Ad}(G)^{\mathbb{Z}}$. Since $\pi_k$ is surjective, there exists $k_x \in \text{GL}^{(k)}(n, G_0)$ such that $\pi_k(k_x) = k_x$. In particular, by the $\pi_k$-equivariance of $f_k$ we have $f_k(\alpha_x k_x^{-1}) = (x, e)$. We also have by the same reason:

$$f_k(\alpha_x g k_x^{-1}) = f_k(\alpha_x g k_x^{-1} k_x g k_x^{-1}) = (x, k_x \pi_k(g) k_x^{-1}),$$

for every $g \in L_1$. Also, the inclusion $(Q_1)_x \subset L^{(k)}(M, G^k)_x$ implies that, for every $g \in L_1$, the $k$-jets of diffeomorphisms $\alpha_x k_x^{-1}, \alpha_x g k_x^{-1}$ considered as linear isomorphisms $T_{0}^{(k)} \mathbb{R}^n \rightarrow T_{x}^{(k)} M$ map $G_0$ onto $G^k_x$. Hence, from the definition of $f_k$ it follows that $\alpha_x g \alpha_x^{-1} = (\alpha_x g k_x^{-1})(\alpha_x k_x^{-1})^{-1}$ is a $k$-jet of local diffeomorphism of $M$ at $x$ whose associated isomorphism $T_{x}^{(k)} M \rightarrow T_{x}^{(k)} M$ maps $G^k_x$ onto itself by the assignment:

$$j_{x}^{(k-1)}(X^*) \mapsto j_{x}^{(k-1)}((k_x \pi_k(g) k_x^{-1} X^*).$$

for which we have used the above trivialization $M \times \text{GL}(g) \cong L(G^k)$. Since $\pi_k(L_1)$ has finite index in $\text{Ad}(G)^{\mathbb{Z}}$ it contains the identity component $\text{Ad}(G)$, and because $k_x \in \text{Ad}(G)^{\mathbb{Z}}$ the group $k_x \pi_k(L_1) k_x^{-1}$ also contains $\text{Ad}(G)$. It follows that $\alpha_x L_1 \alpha_x^{-1}$ is a subgroup of $D_{x}^{(k)}_1(M)$ for which the homomorphism from Lemma 2.2(1) induces a homomorphism:

$$H_x : \alpha_x L_1 \alpha_x^{-1} \rightarrow \text{GL}(G^k_x)$$

$$\alpha_x g \alpha_x^{-1} \mapsto \Theta_x(\alpha_x g \alpha_x^{-1})|g^k_x$$

whose image contains $\text{Ad}(G) \subset \text{GL}(g)$ with respect to the identification between $g$ and $G^k_x$ given by the above isomorphism $M \times g \cong G^k$. This implies that the induced Lie algebra homomorphism:

$$h_x : \text{Lie}(\alpha_x L_1 \alpha_x^{-1}) \rightarrow g(\mathbb{G}^k_x)$$

has image $\text{ad}(g)$, again with respect to the referred identification between $g$ and $G^k_x$.

On the other hand, we have for every $g \in L_1$:

$$\sigma^k((\alpha_x g \alpha_x^{-1}) \alpha_x) = \sigma^k(\alpha_x g) = \sigma^k(\alpha_x)$$

because $\sigma^k((Q_1)_x) \subset \sigma^k((P)_x) = \{ q_0 \}$ is a single point. But this identity proves that every such $k$-jet $\alpha_x g \alpha_x^{-1}$ preserves the pseudo-Riemannian metric up order $k$ (see [1]). In other words, $\alpha_x L_1 \alpha_x^{-1}$ is a subgroup of $\text{Aut}^k(M, x)$ and by Lemma 2.2 we also have that $\text{Lie}(\alpha_x L_1 \alpha_x^{-1})$ is a Lie subalgebra of $\text{Kill}^k_0(M, x)$. 


From the above remarks, it follows that there is a Lie algebra homomorphism \( \hat{\rho}^k_x : \mathfrak{g} \to \text{Kill}^k_0(M, x) \) such that:

\[
(*) \quad \theta_x(\hat{\rho}^k_x(X))(j_x^{-1}(Y^*)) = j_x^{-1}([X, Y]^*) \quad \text{for every } X, Y \in \mathfrak{g}.
\]

For \( k \) fixed, the existence of the homomorphism \( \hat{\rho}^k_x \) has been established for \( \mu \)-a.e. \( x \in M \), where \( \mu \) is an arbitrary ergodic component of the pseudo-Riemannian volume of \( M \). Thus, for \( k \) fixed, it follows that the homomorphism \( \hat{\rho}^k_x \) exists for every \( x \in S_k \), where \( S_k \) is some subset of \( M \) which is conull with respect to the pseudo-Riemannian volume of \( M \). Finally, if we let \( S_0 = \cap_{k=1}^\infty S_k \), then \( S_0 \) is conull with respect to the pseudo-Riemannian volume and for every \( x \in S_0 \) and every \( k \geq 1 \) there exist a homomorphism \( \hat{\rho}^k_x : \mathfrak{g} \to \text{Kill}^k_0(M, x) \) satisfying (*)

In [6], the notion of \( \mathfrak{e} \)-regular point for a metric in a manifold is introduced. Such regular points satisfy two key properties relevant to our discussion. First, the set of regular points \( U \) of \( M \) is an open dense subset. Second, for \( x \in U \) there is some integer \( k(x) \geq 1 \) so that, for \( k \geq k(x) \), every element of \( \text{Kill}^k_0(M, x) \) extends uniquely to an element of \( \text{Kill}^k_0(M, x) \). The first property is found in [6] and the second one is proved in [1], both just using smoothness. Note that the results in [6] are stated for Riemannian manifolds but, as remarked in [1], the ones we consider here apply without change to general pseudo-Riemannian manifolds. The upshot of these remarks is that for every \( x \in U \), there is some \( k(x) \geq 1 \) so that the map:

\[
J^k_x : \text{Kill}^k_0(M, x) \to \text{Kill}^k_0(M, x)
\]

\[
X \mapsto j^k_x(X),
\]

is a linear isomorphism for every \( k \geq k(x) \). Note that in this case, for the usual brackets in \( \text{Kill}^k_0(M, x) \) and the brackets \([\cdot, \cdot]^k\) in \( \text{Kill}^k_0(M, x) \) considered above, the map \( J^k_x \) is a Lie algebra anti-isomorphism.

For \( S_0 \) and \( U \) as above, consider the dense subset \( S = S_0 \cap U \subset M \). Next choose \( x \in S \) and \( k \geq \max(k(x), 2) \). Then, the map \( J^k_x \) is a Lie algebra anti-isomorphism, and there exists a Lie algebra homomorphism \( \hat{\rho}^k_x : \mathfrak{g} \to \text{Kill}^k_0(M, x) \) satisfying (*)

If we let \( \rho_x = -(J^k_x)^{-1} \circ \hat{\rho}^k_x : \mathfrak{g} \to \text{Kill}^k_0(M, x) \), then \( \rho_x \) defines a Lie algebra homomorphism such that:

\[
j^k_x^{-1}((\rho_x(X), Y^*)) = j^k_x^{-1}([X, Y]^*),
\]

for every \( X, Y \in \mathfrak{g} \). For this, we have used (*) and the definition of \( \theta_x \) from Lemma 2.2. Since \( k - 1 \geq 1 \) and because germs of Killing fields are determined by any jet of order at least 1, we conclude that, at our chosen point \( x \), \( \rho_x \) satisfies (from our statement) (1), (2) and the identity in (3) in a neighborhood of \( x \) with \( \tilde{M} \) replaced with \( M \). The identity in (3) now proves that every element of \( \mathfrak{g}(x) \) preserves the tangent bundle to \( \mathcal{O} \) in a neighborhood of \( x \): i.e. the corresponding Lie derivatives map sections of \( T\mathcal{O} \) into sections of \( T\mathcal{O} \). By Proposition 2.2 of [6], we conclude that the local flows of the elements of \( \mathfrak{g}(x) \) preserve \( \mathcal{O} \) as well in a neighborhood of \( x \). Since the elements of \( \mathfrak{g}(x) \) are Killing fields, we conclude that they (and their local flows) also preserve the normal bundle \( T\mathcal{O}^{\perp} \) in a neighborhood of \( x \). This completes the proof of our statement for the dense subset \( S \subset M \) and for \( \tilde{M} \) replaced with \( M \) in (1)-(4). Finally, this yields the statement for \( \tilde{M} \) for the dense subset which is the inverse image of \( S \) under the covering map since such map is a local isometry. □
Remark 2.4. The conclusions of Proposition 2.3 hold without change for some dense subset $S \subset M$ by replacing $\tilde{M}$ with $M$ in (1)-(4). In fact, our proof first obtains the required Killing fields on $M$ which are then translated into corresponding ones on $\tilde{M}$.

In this work, we will be dealing with and interested in the case where the $G$-orbits in $M$ are nondegenerate submanifolds with respect to the pseudo-Riemannian metric. In this case, the $\tilde{G}$-orbits on $\tilde{M}$ are nondegenerate as well and we have a direct sum decomposition $T\tilde{M} = TO \oplus TO^\perp$. When this holds, we can consider the $\mathfrak{g}$-valued 1-form $\omega$ on $\tilde{M}$ that is given, at every $x \in \tilde{M}$, by the composition $T_x\tilde{M} \to T_xO \cong \mathfrak{g}$, where the first map is the natural projection and the second map is the natural isomorphism. From this, we then define the $\mathfrak{g}$-valued 2-form given by $\Omega = d\omega|_{\tilde{M} \subseteq O}$. The following result will provide us with a criterion, in terms of $\Omega$, for the normal bundle $TO^\perp$ to be integrable. At the same time, we relate $\Omega$ with the $\mathfrak{g}$-module structures from Proposition 2.3. This result is very well known and it is essentially contained in [2], [3] and [10], but we include its proof here for the sake of completeness.

Lemma 2.5. Let $G$, $M$ and $S$ be as in Proposition 2.3. If we assume that the $G$-orbits are nondegenerate, then:

1. For every $x \in S$, the maps $\omega_x : T_x\tilde{M} \to \mathfrak{g}$ and $\Omega_x : \wedge^2 T_xO^\perp \to \mathfrak{g}$ are both homomorphisms of $\mathfrak{g}$-modules, for the $\mathfrak{g}$-module structures from Proposition 2.3.

2. The normal bundle $TO^\perp$ is integrable if and only if $\Omega = 0$.

Proof. In the rest of the proof, let $X \in \mathfrak{g}$ and $x \in S$ be fixed but arbitrarily given.

For $Z$ a vector field over $\tilde{M}$, let $Z^\perp, Z^\perp$ be its $TO$ and $TO^\perp$ components, respectively. Since $\rho_x(X)$ is a Killing field preserving $O$ and $TO^\perp$ it follows that:

$$[\rho_x(X), Z]^\perp = [\rho_x(X), Z^\perp],$$

$$[\rho_x(X), Z]^\perp = [\rho_x(X), Z^\perp].$$

Denote with $\alpha : T_xO \to \mathfrak{g}$ the inverse map of $X \mapsto X^*_x$. Then we have:

$$\omega_x(X \cdot Z_x) = \omega_x([\rho_x(X), Z]_x)$$

$$= \alpha([\rho_x(X), Z]^\perp)_x$$

$$= \alpha([\rho_x(X), \omega(Z)^*_x])$$

$$= \alpha([X, \omega(Z)]^*_x)$$

$$= [X, \omega_x(Z)]_x$$

$$= X \cdot \omega_x(Z_x),$$

thus showing that $\omega_x$ is a homomorphism of $\mathfrak{g}$-modules. Here we used in the second and third identities the definition of $\omega$, and in the fourth identity the formula from Proposition 2.3(3); the rest follows from the definition of the $\mathfrak{g}$-module structures involved.

Next, observe that for every pair of sections $Z_1, Z_2$ of $TO^\perp$ we have:

$$\Omega(Z_1 \wedge Z_2) = Z_1(\omega(Z_2)) - Z_2(\omega(Z_1)) - \omega([Z_1, Z_2])$$

$$= -\omega([Z_1, Z_2]),$$

which clearly implies (2).
Now let \( u, v \in T_xO^\perp \) be given and choose \( U, V \) sections of \( TO^\perp \) extending them, respectively. Hence, using that \( \omega \) is a homomorphism of \( \mathfrak{g} \)-modules, the Jacobi identity and the above expression relating \( \Omega \) and \( \omega \), we obtain:

\[
\Omega_x(X \cdot (u \wedge v)) = \Omega_x((X \cdot u) \wedge v) + \Omega_x(u \wedge (X \cdot v)) \\
= \Omega_x([\rho_x(X), U] \wedge V) + \Omega_x(U \wedge [\rho_x(X), V]) \\
= -\omega_x([\rho_x(X), U], V) - \omega_x([U, [\rho_x(X), V]]) \\
= -\omega_x([\rho_x(X), [U, V]]) \\
= -\omega_x(X \cdot [U, V]_x) \\
= -[X, \omega_x([U, V])] \\
= [X, \Omega_x(U \wedge V)] \\
= X \cdot \Omega_x(u \wedge v),
\]

thus showing that \( \Omega_x \) is a homomorphism of \( \mathfrak{g} \)-modules. Note that we have used that both \([\rho_x(X), U], [\rho_x(X), V] \) are sections of \( TO^\perp \).

The following result allows us to relate the metric \( TO \) coming from \( M \) to suitable metrics on \( G \). The proof presented here is due to Gromov (see [2]) and provides our first application of Proposition 2.3.

**Lemma 2.6.** Suppose that the \( G \)-action on \( M \) has a dense orbit and preserves a finite volume pseudo-Riemannian metric. Then, for every \( x \in M \) and with respect to the natural isomorphism \( \mathfrak{g} \cong T_xO \), the metric of \( M \) restricted to \( T_xO \) defines an \( \text{Ad}(G) \)-invariant symmetric bilinear form on \( \mathfrak{g} \) independent of the point \( x \).

**Proof.** With the above-mentioned trivialization of \( TO \), the metric \( h \) on \( M \) restricted to the orbits and pulled back to \( M \times \mathfrak{g} \) yields a map:

\[
\psi : M \to \mathfrak{g}^* \otimes \mathfrak{g}^* \\
x \mapsto B_x
\]

where \( B_x(X, Y) = h_x(X^*_x, Y^*_x) \).

By Remark 2.4, there is a dense subset \( S \subset M \) so that the conclusions of Proposition 2.3 are satisfied for every \( x \in S \) with the tangent spaces and Killing fields of \( \tilde{M} \) replaced by those of \( M \). Hence, for every \( x \in S \) the inner product \( h_x \) is preserved, in the sense of Proposition 2.3(4), by the Killing vector fields that belong to \( \mathfrak{g}(x) \). In particular, for every \( x \in S \) and \( X, Y, Z \in \mathfrak{g} \) we have:

\[
h_x([\rho_x(X), Y^*_x], Z^*_x) = -h_x(Y^*_x, [\rho_x(X), Z^*_x]),
\]

which, by Proposition 2.3(3) yields:

\[
h_x([X, Y]^*_x, Z^*_x) = -h_x(Y^*_x, [X, Z]^*_x).
\]

In other words, for every \( x \in S \) we have:

\[
B_x([X, Y], Z) = -B_x(Y, [X, Z]),
\]

for all \( X, Y, Z \in \mathfrak{g} \). This implies that \( \psi(x) = B_x \) is an \( \text{Ad}(G) \)-invariant form on \( \mathfrak{g} \) for every \( x \in S \). By the density of \( S \) in \( M \), we conclude that the image of \( \psi \) lies in the set of \( \text{Ad}(G) \)-invariant forms.
On the other hand, at every \( x \in M \) and for \( g \in G \), \( X, Y \in g \) we have:

\[
\psi(gx)(X, Y) = h_{gx}(X_{gx}^*, Y_{gx}^*)
\]

\[
= h_x(dg^{-1}_{gx}(X^*)_{gx}, dg^{-1}_{gx}(Y^*))
\]

\[
= h_x(\text{Ad}(g^{-1})(X)_{gx}^*, \text{Ad}(g^{-1})(Y)_{gx}^*)
\]

\[
= \psi(x)(\text{Ad}(g^{-1})(X), \text{Ad}(g^{-1})(Y)),
\]

which shows that \( \psi \) is \( G \)-equivariant. Note that we used in the second identity that \( G \) preserves the metric, and in the third identity we used the remarks at the beginning of this section.

The \( G \)-equivariance of \( \psi \) and the fact that its image lies in \( G \)-fixed points implies that \( \psi \) is \( G \)-invariant. Then, the result follows from the existence of a dense \( G \)-orbit. \( \square \)

As a simple application of the previous result, we prove the nondegeneracy of the orbits when the manifold acted upon has a suitably bounded dimension.

**Lemma 2.7.** Suppose that the \( G \)-action on \( M \) has a dense orbit and preserves a finite volume pseudo-Riemannian metric. If \( \dim(M) < 2 \dim(G) \), then the \( G \)-orbits are nondegenerate with respect to the metric on \( M \).

**Proof.** By Lemma 2.6, for every \( x \in M \) the metric restricted to \( T_xO \) corresponds to an \( \text{Ad}(G) \)-invariant form in \( g \). The kernel of such a form is an ideal and so the metric \( h_x \) restricted to \( T_xO \) is either nondegenerate or zero.

Suppose that \( h_x \) is zero when restricted to \( T_xO \) for some \( x \in M \). Then, \( T_xO \) lies in the null cone of \( T_xM \) for the metric \( h_x \). Hence, for \((m_1, m_2)\) the signature of \( M \), we have \( \dim(G) = \dim(T_xO) \leq \min(m_1, m_2) \). And this implies \( 2 \dim(G) \leq m_1 + m_2 = \dim(M) \), which is impossible. \( \square \)

### 3. Proof of Theorem 1.1 and some consequences

We start this section by proving Theorem 1.1.

**Proof of Theorem 1.1.** Assuming that \( TO^\perp \) is integrable, let \( F \) be the induced foliation. We will first prove that \( F \) is totally geodesic, i.e. its leaves are totally geodesic submanifolds of \( M \). We will denote with \( h \) the metric on \( M \) preserved by \( G \).

First note that, if \( Y, Z \) are local sections of \( TO^\perp \) that preserve the foliation, then we have for every \( X \in g \):

\[
X^*(h(Y, Z)) = h([X^*, Y], Z) + h(Y, [X^*, Z]) = 0,
\]

because our choices imply that \( [X^*, Y] \) and \( [X^*, Z] \) are section of \( TO \). In particular, for every \( Y, Z \) as above the function \( h(Y, Z) \) is constant along the \( G \)-orbits. In the notation of [5], we conclude that \( h \) is a bundle-like metric for the foliation \( O \). Hence, by the remarks in page 79 of [5] it follows that \( h \) induces a transverse metric to the foliation \( O \). By the construction of such transverse metric from \( h \) and the arguments in the proof of Proposition 3.2 of [5], it is easy to conclude that the foliation \( O \) is given by pseudo-Riemannian submersions that define the transverse metric. More precisely, at every point in \( M \) there is an open subset \( U \) of \( M \) and a pseudo-Riemannian submersion \( \pi : U \to B \) such that the fibers of \( \pi \) define the foliation \( O \) restricted to \( U \). We observe that the results of [5] are stated for...
Riemannian metrics, but those that we use here extend without change to arbitrary pseudo-Riemannian metrics.

We will now use the properties of the structural equations from [8] for a pseudo-Riemannian submersion \( \pi : U \to B \) as above. Again, the results in [8] are stated for Riemannian submersions, but the ones that we will use are easily seen to hold for pseudo-Riemannian submersions as well. For \( \pi \) as above, let \( A \) be the associated fundamental tensor defined in [8]. In particular, by the definition of \( A \), the second fundamental form for the leaves of \( \mathcal{F} \) is given by \( A_X Y \), for \( X, Y \) tangent to \( \mathcal{F} \). But by Lemma 2 of [8] we have for \( X, Y \) tangent to \( \mathcal{F} \) the identity \( A_X Y = \frac{1}{2} [X, Y] \top \), where \( Z \top \) denotes the projection of \( Z \) onto \( TO \). Hence, the integrability of \( TO \) to \( \mathcal{F} \) shows that \( A \) vanishes on vector fields tangent to \( \mathcal{F} \), thus showing that the leaves of \( \mathcal{F} \) are totally geodesic.

Choose a leaf \( N \) of \( \mathcal{F} \). Then, one can prove fairly easy that every geodesic in \( M \) which is tangent at some point to \( N \) remains in \( N \) for every value of the parameter of the geodesic; this uses the fact that \( N \) is a maximal integral submanifold of \( TO \) and that the leaves of \( \mathcal{F} \) are totally geodesic. Hence, the completenes of \( M \) implies that of \( N \).

For our chosen leaf \( N \) of \( \mathcal{F} \), consider the \( G \)-action map restricted to \( G \times N \). This defines a smooth map \( \varphi : G \times N \to M \) which is \( G \)-equivariant. By Lemma 2.6 it follows easily that \( \varphi \) is a local isometry for \( G \times N \) endowed with the product metric where \( G \) carries a suitable bi-invariant metric. In particular, \( G \times N \) is complete and so we conclude from Corollary 29 in page 202 from [9] that \( \varphi \) is an isometric covering map. Hence, the universal covering map of \( M \) is given by \( \tilde{\varphi} : \tilde{G} \times \tilde{N} \to M \) and the \( \tilde{G} \)-action on \( M \) lifted to \( \tilde{G} \times \tilde{N} \) is the left action on the first factor.

We now claim that \( \pi_1(M) \subset \text{Iso}(\tilde{G}) \times \text{Iso}(\tilde{N}) \), i.e. that every element in \( \pi_1(M) \) preserves the factors in the product \( \tilde{G} \times \tilde{N} \). To see this, let \( \gamma \in \pi_1(M) \) be given with \( \gamma = (\gamma_1, \gamma_2) \) its component decomposition. Observe that in \( \tilde{G} \times \tilde{N} \) we have:

\[
T_{(g,x)}O = T_g \tilde{G}, \quad T_{(g,x)}O^\perp = T_x \tilde{N},
\]

for every \((g, x) \in \tilde{G} \times \tilde{N}\). Since \( \gamma \) commutes with the \( \tilde{G} \)-action, it preserves both \( TO \) and \( TO^\perp \) and so:

\[
d\gamma(u) = d\gamma_1(u) + d\gamma_2(u) \in TO
\]

\[
d\gamma(v) = d\gamma_1(v) + d\gamma_2(v) \in TO^\perp,
\]

for every \( u \in T \tilde{G} \) and \( v \in T \tilde{N} \). We conclude that \( d\gamma_2(TO) = 0 \) and \( d\gamma_1(TO^\perp) = 0 \), which implies that \( \gamma_1 \) is independent of \( \tilde{N} \) and \( \gamma_2 \) is independent of \( \tilde{G} \). This yields our claim about \( \pi_1(M) \).

On the other hand, since \( \tilde{G} \) carries a bi-invariant metric, by the results from Section 4 of [10] we know that the connected component of the identity of \( \text{Iso}(\tilde{G}) \) is given by \( \text{Iso}_0(\tilde{G}) = L(\tilde{G})R(\tilde{G}) \) (the left and right translations) and that it is a finite index subgroup of \( \text{Iso}(\tilde{G}) \). Hence, the group \( \Lambda = \pi_1(M) \cap (\text{Iso}_0(\tilde{G}) \times \text{Iso}(\tilde{N})) \) has finite index in \( \pi_1(M) \), and so the induced map \( (\tilde{G} \times \tilde{N})/\Lambda \to M \) is a finite covering. Moreover, every \( \gamma \in \Lambda \) can be written as \( \gamma = (L_{g_1} R_{g_2}, \gamma_2) \), where \( g_1, g_2 \in \tilde{G} \) and \( \gamma_2 \in \text{Iso}(\tilde{N}) \), and since such \( \gamma \) commutes with the \( \tilde{G} \)-action we conclude that:

\[
(g_1 g_2, \gamma_2(x)) = (L_{g_1} R_{g_2}, \gamma_2)(g(e, x)) = g((L_{g_1} R_{g_2}, \gamma_2)(e, x)) = (gg_1 g_2, \gamma_2(x))
\]
for every $g \in \tilde{G}$ and $x \in \tilde{N}$, which implies $g_1 \in Z(\tilde{G})$. Hence, $L_{g_1} = R_{g_1}$ and then
\[\gamma \in R(\tilde{G}) \times \text{Iso}(\tilde{N}),\]
thus showing that $\Lambda \subset R(\tilde{G}) \times \text{Iso}(\tilde{N})$.

Also note that the covering map $G \times \tilde{N} \rightarrow M$ realizes $\pi_1(G) \subset \Lambda$, which induces a covering map $G \times \tilde{N} \rightarrow (\tilde{G} \times \tilde{N})/\Lambda$. We claim that $G \times \tilde{N} \rightarrow (\tilde{G} \times \tilde{N})/\Lambda$ is a normal covering map. For this we need to check that $\pi_1(G)$ is a normal subgroup of $\Lambda$ under the inclusion $z \mapsto (R_z, e)$. But by the above remarks, every $\gamma \in \Lambda$ can be written as $\gamma = (R_{g_1}, \gamma_2)$, where $g_1 \in G$ and $\gamma_2 \in \text{Iso}(\tilde{N})$, from which we obtain:
\[\gamma(R_z, e)\gamma^{-1} = (R_{g_1}R_zR_{g_1}^{-1}, \gamma_2\gamma_2^{-1}) = (R_z, e)\]
since $\pi_1(G)$ is central in $\tilde{G}$. It follows that the group of deck transformations for $G \times \tilde{N} \rightarrow (\tilde{G} \times \tilde{N})/\Lambda$ is given by the group $\Gamma = \Lambda/\pi_1(G)$ and that we also have $(G \times \tilde{N})/\Gamma = (\tilde{G} \times \tilde{N})/\Lambda$.

From the above, we conclude that $\Gamma \subset R(G) \times \text{Iso}(\tilde{N}) = G \times \text{Iso}(\tilde{N})$ is a group of deck transformations of $G \times \tilde{N} \rightarrow M$ that induces a $G$-equivariant finite covering map $(G \times \tilde{N})/\Gamma \rightarrow M$ that satisfies the properties required to obtain Theorem 1.1. □

The integrability of the normal bundle can be ensured for suitable relations between the Lie group $G$ and the geometry of the manifold on which it acts, thus providing the following results. In what follows, the signature of $G$, as a pseudo-Riemannian manifold, is always considered with respect to a bi-invariant metric. Note that if $(n_1, n_2)$ is the signature of some bi-invariant metric on $G$, then the signature of any other bi-invariant metric is either $(n_1, n_2)$ or $(n_2, n_1)$.

**Corollary 3.1.** Suppose that the $G$-action on $M$ has a dense orbit and preserves a finite volume complete pseudo-Riemannian metric. If the $G$-orbits are nondegenerate and either one of the following holds:

1. there is no Lie algebra embedding of $\mathfrak{g}$ into $\text{so}(T_xO^\perp)$ for every $x \in M$, or
2. for $n_0 = \min(n_1, n_2)$ and $m_0 = \min(m_1, m_2)$, where $(n_1, n_2)$ and $(m_1, m_2)$ are the signatures of $G$ and $M$, respectively, we have $\text{rank}_\mathbb{R}(\mathfrak{g}) > m_0 - n_0$,

then the conclusion of Theorem 1.1 holds.

**Proof.** Let us consider a subset $S \subset \tilde{M}$ given as in Proposition 2.3.

First suppose that condition (1) is satisfied. Then, for every $x \in S$, the $\mathfrak{g}$-module structure on $T_xO^\perp$ given by Proposition 2.3(4) is trivial. By Lemma 2.3(1), being a homomorphism of $\mathfrak{g}$-modules, the map $\Omega_x$ is trivial for every $x \in S$. Since $S$ is dense, we conclude that $\Omega = 0$ and so $TO^\perp$ is integrable by Lemma 2.3(2). Hence, Theorem 1.1 can be applied.

Let us now assume that (2) holds. Note that by Lemma 2.3, the signature of $TO$ is either $(n_1, n_2)$ or $(n_2, n_1)$. If we let $(k_1, k_2)$ be the signature of $TO^\perp$, then it is easily seen that:
\[\min(k_1, k_2) \leq m_0 - n_0.\]
Since the real rank of $\text{so}(T_xO^\perp)$ is precisely $\min(k_1, k_2)$ the result in this case follows from the first part. □

**Corollary 3.2.** Suppose that the $G$-action on $M$ has a dense orbit and preserves a finite volume complete pseudo-Riemannian metric. Let $n$ be the dimension of the smallest $\mathfrak{g}$-module $V$ such that $\wedge^2 V$ contains a $\mathfrak{g}$-submodule isomorphic to $\mathfrak{g}$. If $\dim(M) < \dim(G) + n$, then the conclusion of Theorem 1.1 holds.
Proof. First observe that the Lie brackets in $\mathfrak{g}$ define a surjective homomorphism of $\mathfrak{g}$-modules from $\Lambda^2 \mathfrak{g}$ onto $\mathfrak{g}$. Hence, $\Lambda^2 \mathfrak{g}$ contains a submodule isomorphic to $\mathfrak{g}$, thus implying that $n \leq \dim(\mathfrak{g})$. Then, Lemma 2.7 shows that the $G$-orbits in $M$ are nondegenerate with respect to the metric of $M$.

Let $S \subset \tilde{M}$ be a subset given as in Proposition 2.3. From our hypotheses and since the $G$-orbits are nondegenerate, we have $\dim(T_xO^\perp) < n$. In particular, for every $x \in S$ and for the $\mathfrak{g}$-module structure from Proposition 2.3(4), $\Lambda^2 T_xO^\perp$ does not contain a $\mathfrak{g}$-submodule isomorphic to $\mathfrak{g}$. Hence, Lemma 2.5(1) implies that $\Omega_x = 0$ for every $x \in S$ and so that $\Omega = 0$. By Lemma 2.5(2) the bundle $TO^\perp$ is integrable and so Theorem 1.1 can be applied.

On a compact manifold with Riemannian normal bundle, we can obtain the conclusions of Theorem 1.1 without having to a priori assume that the manifold is complete and that the normal bundle is integrable.

Corollary 3.3. Suppose that the $G$-action on $M$ has a dense orbit and preserves a pseudo-Riemannian metric. If $M$ is compact, the $G$-orbits are nondegenerate and the normal bundle $TO^\perp$ is Riemannian, then the conclusions of Theorem 1.1 hold. Moreover, we can assume that $\tilde{N}$ is Riemannian homogeneous and that $\Gamma \subset G \times \text{Iso}_0(\tilde{N})$.

Proof. The proof is a refinement of that of Theorem 1.1 so we will follow the notation of the latter.

First observe that the integrability of $TO^\perp$ follows from the proof of Corollary 3.1(1) since $\mathfrak{so}(T_xO^\perp)$ is compact for every $x \in \tilde{M}$. In particular, we have the hypotheses of Theorem 1.1 except for the completeness of $M$. With this respect, it is easy to check that the compactness of $M$ and the fact that $TO^\perp$ is Riemannian imply that the geodesics in $M$ perpendicular to the $G$-orbits are complete. This completeness is enough for the rest of the arguments in the proof of Theorem 1.1 to apply.

Finally, we observe that the existence of a dense $G$-orbit implies that $\tilde{N}$ has a dense orbit by its local isometries and, being Riemannian, we conclude that it is homogeneous. The latter follows from the infinitesimal characterization of homogeneous Riemannian manifolds obtained in [12], and the fact that the orthogonal group is compact for definite metrics; we refer to [10] for further details. But for a homogeneous Riemannian manifold the group of isometries has finitely many connected components, and so we can intersect $\Gamma$ with $G \times \text{Iso}_0(\tilde{N})$ to obtain the last claim after passing to a finite covering.

4. Manifolds with a transverse Riemannian structure: Proof of Theorem 1.2

In this section we will characterize those actions that preserve a Riemannian structure transverse to the $G$-orbits. We start with the following basic result relating transverse geometric structures for the foliation by $G$-orbits with geometric structures on the normal bundle to such orbits.

Lemma 4.1. Suppose that the $G$-action on $M$ has a dense orbit and preserves a finite volume pseudo-Riemannian metric. Also assume that the $G$-orbits are nondegenerate. If $H$ is a subgroup of $\text{GL}(k, \mathbb{R})$, where $k = \dim(M) - \dim(G)$, then there is a one-to-one correspondence between the $G$-invariant $H$-reductions...
of $L(TM/TO)$ and the $G$-invariant $H$-reductions of $L(TO^\perp)$. In particular, every transverse $H$-structure for the foliation $O$ by $G$-orbits induces a $G$-invariant $H$-reduction of $L(TO^\perp)$.

Proof. From the decomposition $TM = TO \oplus TO^\perp$ we obtain a natural $G$-equivariant isomorphism $TM/TO \to TO^\perp$ which clearly yields the first claim.

Next, we recall that a transverse $H$-structure to the foliation $O$ is given by a reduction $P$ of $L(TM/TO)$ which is invariant under the local flows of vectors fields tangent to the foliation $O$. In particular, $P$ is invariant under the $G$-action on $L(TM/TO)$ thus showing the last claim. \qed

The following result is obtained by applying the main theorems from [10].

Proposition 4.2. Suppose that the $G$-action on $M$ has a dense orbit and preserves a pseudo-Riemannian metric. Also assume that $M$ is compact. If the normal bundle to the orbits $TO^\perp$ is Riemannian, then there exist:

1. a finite covering map $\hat{M} \to M$,
2. a connected Lie group $H$ with a compact subgroup $K$, and
3. a discrete cocompact subgroup $\Gamma \subset G \times H$ such that $\Gamma T$ is dense in $G \times H$.

for which the $G$-action on $M$ lifts to $\hat{M}$ so that $\hat{M}$ is $G$-equivariantly diffeomorphic to $(G \times K\backslash H)/\Gamma$. Furthermore, if $G$ has finite center and real rank at least $2$, then we can assume that $G \times H$ is a finite center isotypic semisimple Lie group and $\Gamma$ is an irreducible lattice.

Proof. Let us denote with $(m_1, m_2)$ and $(n_1, n_2)$ the signatures of $M$ and $G$, respectively. Also, let us denote $m_0 = \min(m_1, m_2)$ and $n_0 = \min(n_1, n_2)$. Observe that since $TO^\perp$ is Riemannian, the bundle $TO$ is nondegenerate. Hence, by Lemma 2.6 the signature of $TO$ is either $(n_1, n_2)$ or $(n_2, n_1)$; this is because the signature of every $\text{Ad}(G)$-invariant metric on $g$ is the signature of either the Killing form or its negative. Without loss of generality, we can assume that the signature of $TO$ is precisely $(n_1, n_2)$.

Theorems A and B from [10] provide precisely our required conclusions when $n_0 = m_0$ holds, except for explicitly stating the property that $\Gamma T$ is dense in $G \times H$. The latter is obtained as follows. The conclusion of Theorem A from [10] ensures the existence of a $G$-invariant ergodic smooth measure, which in turn implies the existence of a dense $G$-orbit in $(G \times H)/\Gamma$. From this, it is easily seen that we necessarily have the density of the $G$-orbit of $(e, e)\Gamma$ in $(G \times H)/\Gamma$. Since the map $G \times H \to (G \times H)/\Gamma$ is a covering, we conclude that the inverse image under this covering of such orbit, which is $\Gamma T$, is dense in $G \times H$.

We now consider the possibilities for $n_0$. We will now assume that the signatures are given in the form $(+,-)$. If $n_0 = n_2$, then the fact that $TO^\perp$ is Riemannian implies that $(m_1, m_2) = (n_1 + k, n_2)$, where $k$ is the rank of $TO^\perp$. In particular, $n_0 = m_0$ in this case and the result follows from [10].

In case $n_0 = n_1$, we can replace the metric on $TO^\perp$ by its negative to obtain a new $G$-invariant metric for which the signature of $M$ is $(m_1, m_2) = (n_1, n_2 + k)$, where $k$ is again the rank of $TO^\perp$. Hence, for this new metric $n_0 = m_0$ and so the result follows also from [10]. \qed

We now proceed to prove Theorem 1.2.
Proof of Theorem 1.2. Assume that (1) holds. Let us take $\widehat{M} = (G \times K \backslash H)/\Gamma$ and verify that it satisfies (2).

Note that, with respect to the quotient map $G \times H \to (G \times H)/\Gamma$, the set $G\Gamma$ projects onto the $G$-orbit of the class of the identity. Hence, there is a dense $G$-orbit in $\widehat{M}$.

Next, endow $G \times H$ with the product metric given by a bi-invariant metric on $G$ and a Riemannian metric on $H$ which is left $K$-invariant and right $H$-invariant. The latter exists because $K$ is compact. This induces a pseudo-Riemannian metric on $G \times K \backslash H$ which is left $G$-invariant and right $\Gamma$-invariant. Note that the $G$-orbits are nondegenerate with normal bundle given by the tangent bundle to the factor $K \backslash H$. Furthermore, by construction the projection $G \times K \backslash H \to K \backslash H$ is a pseudo-Riemannian submersion and so it defines a transverse Riemannian structure for the foliation by $G$-orbits. Since the left $G$-action and the $\Gamma$-action commute with each other, the transverse Riemannian structure on $G \times K \backslash H$ induces a corresponding one on the double coset $(G \times K \backslash H)/\Gamma$ which is $G$-invariant. This provides the geometric structures required by (2).

Let us now assume that (2) holds, so that in particular $\widehat{M}$ has a dense $G$-orbit and carries the indicated geometric structures. Since the $G$-orbits in $\widehat{M}$ are nondegenerate then we have an orthogonal decomposition $T\widehat{M} = TO \oplus TO^\perp$ which is invariant under $G$. The existence of a transverse Riemannian structure for the foliation by $G$-orbits in $\widehat{M}$ and Lemma 4.1 imply the existence of a $G$-invariant Riemannian metric on $TO^\perp$. Hence, if we replace with such Riemannian metric the metric on $TO^\perp$ induced from $\widehat{M}$, we obtain a new $G$-invariant pseudo-Riemannian metric on $\widehat{M}$ for which $TO^\perp$ is Riemannian. Hence, the $G$-action on $\widehat{M}$ satisfies the hypotheses of Proposition 4.2, and the latter provides a finite covering of $\widehat{M}$, and thus of $\widehat{M}$, with the properties required by (1).

Finally, let us assume that $G$ has finite center and real rank at least 2. First observe that (1) replaced by the conditions in the last part of the statement still implies (2). The only nontrivial property to check is the existence of a dense $G$-orbit in $(G \times K \backslash H)/\Gamma$ for $\Gamma$ an irreducible lattice. But in this case, $G\Gamma$ is dense in $G \times H$ (see for example Lemma 6.3 from [10]) which yields the required dense orbit. On the other hand, that (2) implies (1) with the properties from the last part of the statement is a consequence of Proposition 4.2. \hfill \Box

5. Actions on Lorentzian manifolds

In this section, we present a characterization of the $G$-actions on $M$ preserving a Lorentzian metric.

**Theorem 5.1.** Let $G$ be a connected noncompact simple Lie group acting faithfully on a compact manifold $M$. Then the following conditions are equivalent:

1. The group $G$ is locally isomorphic to $SL(2,\mathbb{R})$ and there is a $G$-equivariant finite covering map $(G \times K \backslash H)/\Gamma \to M$ where $H$ is a connected Lie group with a compact subgroup $K$ and $\Gamma \subset G \times H$ is a discrete cocompact subgroup such that $G\Gamma$ is dense in $G \times H$.

2. There is a finite covering map $\widehat{M} \to M$ for which the $G$-action on $\widehat{M}$ lifts to a $G$-action on $\widehat{M}$ with a dense orbit and preserving a Lorentzian metric.
Proof. Let us assume (1) and consider the finite covering \( \hat{M} = (G \times K \setminus H)/\Gamma \to M \). Endow \( G \times H \) with the product metric given by the bi-invariant metric coming from the Killing form in \( g \) and a left \( K \)-invariant and right \( H \)-invariant Riemannian metric on \( H \). Then consider the induced \( G \)-invariant metric on \( \hat{M} \). Since \( G \) is locally isomorphic to \( \text{SL}(2, \mathbb{R}) \), such metric on \( \hat{M} \) is Lorentzian. As in the proof of Theorem 1.2, the density of \( G\Gamma \) implies the existence of a dense \( G \)-orbit in \( \hat{M} \). This proves (2).

Let us now assume that (2) holds. By Lemma 2.6, for every \( x \in M \), the metric in \( T_x O \) corresponds to an \( \text{Ad}(G) \)-invariant symmetric bilinear form on \( g \). Such form is either 0 or nondegenerate. The former case implies the existence of a null tangent subspace of dimension at least 3, which is impossible; in particular, the \( G \)-orbits are nondegenerate. Since \( G \) is noncompact and simple, we conclude the existence of an \( \text{Ad}(G) \)-invariant form on \( g \) which is Lorentzian. But \( \mathfrak{s}(2, \mathbb{R}) \) is the only simple Lie algebra admitting such a form, which implies that \( G \) is locally isomorphic to \( \text{SL}(2, \mathbb{R}) \). We also conclude that \( TO^\perp \) is Riemannian and so the rest of the claims in (1) follow from Proposition 4.2. \( \square \)

6. MANIFOLDS WITH A TRANSVERSE PARALLELISM: PROOF OF THEOREM 1.3

The following result is an immediate consequence of the properties of Lie foliations. A proof can be found in [10].

Lemma 6.1. Let \( X \) be a compact manifold carrying a foliation with a transverse Lie structure. If the foliation has a dense leaf, then the lifted foliation to any finite covering space of \( X \) has a dense leaf as well.

We now obtain the next result which describes actions preserving a metric and a transverse parallelism. Its proof is based on some of the arguments found in [11].

Proposition 6.2. Suppose that the \( G \)-action on \( M \) has a dense orbit and preserves a pseudo-Riemannian metric. Also assume that \( M \) is compact. If the foliation by \( G \)-orbits is nondegenerate (with respect to the pseudo-Riemannian metric) and carries a transverse parallelism, then there exist:

1. a finite covering map \( \hat{M} \to M \),
2. a connected Lie group \( H \), and
3. a discrete cocompact subgroup \( \Gamma \subset G \times H \) such that \( G\Gamma \) is dense in \( G \times H \), for which the \( G \)-action on \( M \) lifts to \( \hat{M} \) so that \( \hat{M} \) is \( G \)-equivariantly diffeomorphic to \( (G \times H)/\Gamma \). Furthermore, if \( G \) has finite center and real rank at least 2, then we can assume that \( G \times H \) is a finite center isotypic semisimple Lie group and \( \Gamma \) is an irreducible lattice.

Proof. By Lemma 4.1 the transverse parallelism to the \( G \)-orbits yields a \( G \)-invariant trivialization of \( L(TO^\perp) \). Hence, there is a family of \( G \)-invariant sections of \( TO^\perp \), say \( X_1, \ldots, X_k \), that defines a basis of \( TO^\perp \) on every fiber. Let us consider the \( G \)-invariant Riemannian metric on \( TO^\perp \) for which these vector fields are orthonormal at every point. Because of the orthogonal decomposition \( TM = TO \oplus TO^\perp \), if we replace the metric on \( TO^\perp \) induced from \( M \) with the \( G \)-invariant Riemannian metric thus defined from the parallelism, then we obtain a pseudo-Riemannian metric \( h \) on \( M \) which is \( G \)-invariant, defines the same orthogonal complement \( TO^\perp \) to the orbits and such that this orthogonal complement is Riemannian. In the rest of this proof we will consider \( M \) endowed with this new metric \( h \). Hence, by Corollary 3.3
there is a simply connected homogeneous Riemannian manifold $\tilde{N}$ and a discrete cocompact subgroup $\Gamma \subset G \times \text{Iso}_0(\tilde{N})$ for which there is a $G$-equivariant finite covering map:

$$\tilde{M} = (G \times \tilde{N})/\Gamma \rightarrow M.$$  

Note that by the proof of Corollary 3.3 the normal bundle $TO^\perp$ is integrable. In particular, the vector fields $X_i$ given above satisfy:

$$[X_i, X_j] = \sum_{r=1}^k f_{ij}^r X_r$$

for some smooth functions $f_{ij}^r$ defined on $M$. The $G$-invariance of the parallelism then implies that the functions $f_{ij}^r$ are $G$-invariant and so constant because of the existence of a dense $G$-orbit. We conclude that the linear span over $\mathbb{R}$ of the vector fields $X_1, \ldots, X_k$ is a Lie algebra.

On the other hand, the fact that the vector fields $X_i$ are preserved by the $G$-action is easily seen to imply that such fields are foliate in the notation of [5]. Hence, the parallelism $X_1, \ldots, X_k$ defines a transverse Lie structure for the foliation by $G$-orbits in $M$. Clearly, this transverse Lie structure induces a corresponding one on $\tilde{M}$. Let $H$ be a simply connected Lie group that models the transverse Lie structure on $\tilde{M}$ and consider a corresponding development $D : \tilde{G} \times \tilde{N} \rightarrow H$. In particular, $D$ is a submersion whose fibers have connected components given precisely by subsets of the form $\tilde{G} \times \{x\}$. Hence, we conclude that $D_\{e\} \times \tilde{N} : \tilde{N} \rightarrow H$ defines a local diffeomorphism.

By the proofs of Corollary 3.3 and Theorem 1.1 the manifold $\tilde{N}$ is the (isometric) universal covering space of a leaf $N$ for the foliation in $M$ defined by $TO^\perp$. Hence, if we let $\tilde{X}_1, \ldots, \tilde{X}_k$ be the pull backs to $\tilde{N}$ of the restrictions of the fields $X_1, \ldots, X_k$ to $N$, then the induced fields on $\tilde{G} \times \tilde{N}$ (which we will denote with the same symbols) define the transverse Lie structure on $\tilde{G} \times \tilde{N}$. Furthermore, from the previous construction of the metric $h$, the metric on $\tilde{N}$ is given by the condition of the fields $\tilde{X}_1, \ldots, \tilde{X}_k$ being orthonormal at every point. In particular, if we let $\mathfrak{h}$ be the Lie algebra of $H$, then there is a basis $v_1, \ldots, v_k$ of $\mathfrak{h}$ such that the development $D$ maps the vector field $\tilde{X}_i$ into $v_i$, for every $i = 1, \ldots, k$. Without loss of generality we will assume that the transverse $H$-structures are modeled by taking $H$ with its right translations, and so transverse Lie parallelisms are modeled on right invariant vector fields. With this convention, each $v_i$ is considered as a right invariant vector field on $H$.

From the above remarks, if we endow $H$ with the right invariant Riemannian metric for which the vector fields $v_1, \ldots, v_k$ are orthonormal at every point, then $D_\{e\} \times \tilde{N} : \tilde{N} \rightarrow H$ is a local isometry. By Corollary 29 in page 202 of [9] and since $\tilde{N}$ is complete, we conclude that $D_\{e\} \times \tilde{N} : \tilde{N} \rightarrow H$ is an isometry and so it induces in $\tilde{N}$ a Lie group structure with respect to which it is an isomorphism. Hence, we can replace the Riemannian manifold $\tilde{N}$ with $H$ carrying the above right invariant Riemannian metric and assume that the natural projection $\tilde{G} \times H \rightarrow H$ is a development for the transverse Lie structure on $\tilde{M}$. Moreover, by the definition of $\tilde{M}$, the space $G \times \tilde{N}$ covers $\tilde{M}$, and so we can assume that the natural projection $\pi : G \times H \rightarrow H$ is also a development for the transverse Lie structure on $\tilde{M}$ with a corresponding holonomy representation $\hat{\rho} : \Gamma \rightarrow H$. 
Since $\Gamma \subset G \times \text{Iso}_0(\widetilde{N}) = G \times \text{Iso}_0(H)$, if we choose $\gamma \in \Gamma$, then we can write
\[
\gamma = (R_{\gamma_1}, \gamma_2), \text{ where } \gamma_1 \in G \text{ and } \gamma_2 \in \text{Iso}_0(H).
\]
And so, the $\hat{\rho}$-equivariance of $\pi$ yields for every $(g, x) \in G \times H$:
\[
\gamma_2(x) = \pi(g \gamma_1, \gamma_2(x)) = \pi((g, x)\gamma) = \pi(g, x)\hat{\rho}(\gamma) = xy
\]
where $y \in H$. It follows that $\gamma_2$ is given by a right translation by an element in $H$, and so we have $\Gamma \subset G \times R(H) = G \times H$. Since $\widetilde{M} = (G \times H)/\Gamma \rightarrow M$ is a finite covering, by Lemma 6.1 we conclude that $\widetilde{M}$ has a dense $G$-orbit and so $G\Gamma$ is dense in $G \times H$ as in the proof of Proposition 4.2. This concludes the proof of the first part of the statement.

For the last claim, if $G$ has finite center and real rank at least 2, then $H$ is semisimple by the main results from [10]. Also, by modding out by a suitable central group we can assume that $H$ has finite center (see [10]). The arguments at the end of Section 6 from [10] also prove that $\Gamma$ is an irreducible lattice. Finally, recall that an irreducible lattice in a semisimple Lie group can only exist if the group is isotypic.

\[\square\]

We can now prove our characterization of actions with a transverse parallelism.

**Proof of Theorem 1.3** First, let us assume that (1) holds and consider the finite covering $\widetilde{M} = (G \times H)/\Gamma \rightarrow M$. As in the proof of Theorem 1.2 we conclude that $\widetilde{M}$ has a dense $G$-orbit. Note that the right invariant vector fields on $H$ define both a transverse Lie structure on $G \times H$ for the foliation given by the factor $G$, and a right $H$-invariant Riemannian metric on $H$. If we consider the product pseudo-Riemannian metric on $G \times H$ using a bi-invariant metric on $G$, then we also obtain a pseudo-Riemannian metric for which the foliation given by the factor $G$ is nondegenerate. Both of these geometric structures on $G \times H$ are left $G$-invariant and right $\Gamma$-invariant and so descend to corresponding $G$-invariant geometric structures on $\widetilde{M}$ thus establishing (2).

If we now assume (2), then Proposition 6.2 applied to $\widetilde{M}$ yields (1). The last claim is also a consequence of Proposition 6.2.

\[\square\]

7. **Proof of Theorem 1.4**

Let us consider $G$ and $M$ as in Proposition 2.3 with $S \subset M$ a dense subset provided by this result and Remark 2.4. With the notation of Proposition 2.3 the representation $\lambda_x \circ \rho$ leaves invariant the subspace $T_xO^\perp$ thus defining its $g$-module structure. This induces a homomorphism $g \rightarrow so(T_xO^\perp)$ obtained by restricting $\lambda_x \circ \rho$ to the submodule $T_xO^\perp$. By composition with such homomorphism, we can consider $\Omega_x$ obtained from Lemma 2.5 as a map $\wedge^2T_xO^\perp \rightarrow so(T_xO^\perp)$.

For a vector space $W$ with inner product $(\cdot, \cdot)$, we will say that a bilinear map $T : W \times W \rightarrow \mathfrak{gl}(W)$ is of curvature type if it satisfies the following conditions for every $x, y, z, v, w \in W$:

1. $T(x, y) = -T(x, y)$,
2. $(T(x, y)v, w) = -\langle v, T(x, y)w \rangle$,
3. $(T(x, y)z + T(y, z)x + T(z, x)y) = 0$,
4. $(T(x, y)v, w) = (T(v, w)x, y)$.
Note that (1) and (2) together are equivalent to \( T \) inducing a map \( \wedge^2 W \to \mathfrak{so}(W) \). Also, by the proof of Proposition 36 in page 75 of [3] we know that (1), (2) and (3) together imply (4).

With the above notation, we have the following result.

**Lemma 7.1.** Let \( G, M \) and \( S \subset M \) be as in Proposition 2.3 and Remark 2.4. For every \( x \in S \), define the bilinear operation \([\cdot, \cdot]\) on \( T_x M \) by the assignments:

- \([X^*_x, Y^*_x]_0 = [X, Y]_x^*\), for every \( X^*_x, Y^*_x \in T_x \mathcal{O}, (X, Y \in \mathfrak{g})\),
- \([X^*_x, v]_0 = -v \circ X^*_x = X(v)\) for every \( X^*_x \in T_x \mathcal{O}, v \in T_x \mathcal{O}^\perp, (X \in \mathfrak{g})\),
- \([v_1, v_2]_0 = \Omega_x(v_1, v_2)^*_x\) for every \( v_1, v_2 \in T_x \mathcal{O}^\perp\).

Then, \([\cdot, \cdot]\) yields a Lie algebra structure on \( T_x M \) if and only if \( \Omega_x \) is of curvature type when considered as a map \( \wedge^2 T_x \mathcal{O}^\perp \to \mathfrak{so}(T_x \mathcal{O}^\perp) \). In this case, the representation of \( \mathfrak{g} \) in \( T_x M \), given by Proposition 2.3(4), preserves the Lie algebra structure of \( T_x M \).

**Proof.** Note that by the above remarks, \( \Omega_x \) is of curvature type if and only if it satisfies the above condition (3). Also observe that \([\cdot, \cdot]_0\) always defines a skew-symmetric bilinear form in \( T_x M \). Hence, for the first claim, we need to show that \([\cdot, \cdot]_0\) satisfies the Jacobi identity if and only if \( \Omega_x \) satisfies (3).

For the Jacobi identity to hold we only need to verify the following cases.

- \( u_1, u_2, u_3 \in T_x \mathcal{O} \). Note that \([\cdot, \cdot]_0\) maps \( T_x \mathcal{O} \times T_x \mathcal{O} \) into \( T_x \mathcal{O} \) in such a way that it defines a skew-symmetric operation that corresponds to the Lie brackets of \( \mathfrak{g} \) under the natural isomorphism \( \mathfrak{g} \to T_x \mathcal{O} \) given by \( X \mapsto X^*_x \). Hence, the Jacobi identity always holds in this case.

- \( u_1, u_2 \in T_x \mathcal{O} \) and \( u_3 \in T_x \mathcal{O}^\perp \). In this case we can write \( u_1 = X^*_x \) and \( u_2 = Y^*_x \) for some \( X, Y \in \mathfrak{g} \). Then, by the definition of \([\cdot, \cdot]_0\):

\[
[[X^*_x, Y^*_x]_0, u_3]_0 = [[X, Y]_x^*, u_3]_0 = [X, Y](u_3) = X(Y(u_3)) - Y(X(u_3)) = X([Y^*_x, u_3]_0) - Y([X^*_x, u_3]_0) = [X^*_x, [Y^*_x, u_3]_0]_0 - [Y^*_x, [X^*_x, u_3]_0]_0
\]

which proves that the Jacobi identity holds in this case in general.

- \( u_1 \in T_x \mathcal{O} \) and \( u_2, u_3 \in T_x \mathcal{O}^\perp \). We can now choose \( X \in \mathfrak{g} \) such that \( u_1 = X^*_x \). Hence, using from Lemma 2.3 the fact that \( \Omega_x \) is a homomorphism of \( \mathfrak{g} \)-modules, we obtain:

\[
[X^*_x, [u_2, u_3]_0]_0 = [X^*_x, \Omega_x(u_2, u_3)]_0 = [X, \Omega_x(u_2, u_3)]_x^*
= \Omega_x(X(u_2), u_3)]_x^* + \Omega_x(u_2, X(u_3)]_x^*
= \Omega_x([X^*_x, u_2]_0, u_3]_0 + \Omega_x(u_2, [X^*_x, u_3]_0)]_x^*
= [[X^*_x, u_2]_0, u_3]_0 + [u_2, [X^*_x, u_3]_0]_0,
\]

which yields again the Jacobi identity without extra conditions.

- \( u_1, u_2, u_3 \in T_x \mathcal{O}^\perp \). The definition of \([\cdot, \cdot]_0\) yields now:

\[
[[u_1, u_2]_0, u_3]_0 = [\Omega_x(u_1, u_2)]^*_x, u_3]_0 = \Omega_x(u_1, u_2)(u_3),
\]

and so the Jacobi identity is satisfied in this case exactly when \( \Omega_x \) satisfies condition (3).
The above proves the equivalence in the statement, and so it remains to obtain the last claim. For this we need to show that:

\[ X([u_1, u_2]) = X(u_1, u_2) + [u_1, X(u_2)]_0 \]

for every \( X \in \mathfrak{g} \) and \( u_1, u_2 \in T_x M \). This is now dealt with through the following cases which just basically apply the definitions involved and properties already considered.

- \( u_1, u_2 \in T_x O \). Then, we can write \( u_1 = Y^*_x, u_2 = Z^*_x \) for some \( Y, Z \in \mathfrak{g} \) and:

\[
\begin{align*}
X([Y^*_x, Z^*_x]) &= X([Y, Z])_x = [\rho_x(X), [Y, Z]]_x = [X, [Y, Z]]_x^* \\
&= [[X, Y], Z]^*_x + [Y, [X, Z]]_x^* \\
&= [X, Y]^*_x, Z^*_x + [X, Z]^*_x]_0 \\
&= [[\rho_x(X), Y^*_x], Z^*_x]_0 + [Y^*_x, [\rho_x(X), Z^*_x]]_0 \\
&= [X(Y^*_x), Z^*_x]_0 + [X^*_y, X(Z^*_x)]_0 \\
&\end{align*}
\]

- \( u_1 \in T_x O \) and \( u_2 \in T_x O^\perp \). Now we can write \( u_1 = Y^*_x \) for \( Y \in \mathfrak{g} \) and:

\[
\begin{align*}
X([Y^*_x, u_2]) &= X([Y, u_2]) = [X, Y](u_2) + Y(X(u_2)) \\
&= [[X, Y]^*_x, u_2]_0 + [Y^*_x, X(u_2)]_0 \\
&= [[\rho_x(X), Y^*_x], u_2]_0 + [Y^*_x, X(u_2)]_0 \\
&= [X(Y^*_x), u_2]_0 + [Y^*_x, X(u_2)]_0 \\
&\end{align*}
\]

- \( u_1, u_2 \in T_x O^\perp \). We now have:

\[
\begin{align*}
X([u_1, u_2]) &= X(\Omega_x(u_1, u_2)) = [\rho_x(X), \Omega_x(u_1, u_2)]_x \\
&= [X, \Omega_x(u_1, u_2)]^*_x \\
&= \Omega_x(X(u_1, u_2)) + \Omega_x(u_1, X(u_2))_x^* \\
&= [X(u_1), u_2]_0 + [u_1, X(u_2)]_0 \\
&\end{align*}
\]

\( \square \)

The following result will allow us to prove Theorem 1.4. It also provides an explicit description of the Lie algebra structure considered in Theorem 1.4.

**Theorem 7.2.** Suppose that \( G \) has finite center and real rank at least 2, and that the \( G \)-action on \( M \) preserves a finite volume complete pseudo-Riemannian metric. Also assume that \( G \) acts ergodically on \( M \) and that the foliation by \( G \)-orbits is nondegenerate. Denote with \( L \) the algebraic hull for the \( G \)-action on the bundle \( L(TO^\perp) \) and with \( \mathfrak{l} \) its Lie algebra. In particular, there is an embedding of Lie algebras \( \mathfrak{m} \rightarrow \mathfrak{so}(p, q) \), where \( (p, q) \) is the signature of the metric of \( M \) restricted to \( TO^\perp \). If this embedding is surjective, then one of the following occurs:

1. the conclusion of Theorem 1.4 holds, or
2. \( G \) is locally isomorphic to \( \text{SO}_0(p, q) \), \( \dim(M) = (p + q)(p + q + 1)/2 \), for some \( x \in M \) the bilinear map \( \Omega_x \) is nonzero and of curvature type and the Lie algebra structure on \( T_x M \) obtained from Lemma 7.1 is isomorphic to either \( \mathfrak{so}(p, q + 1) \) or \( \mathfrak{so}(p + 1, q) \).

**Proof.** Since the \( G \)-orbits are nondegenerate, \( G \) preserves a pseudo-Riemannian metric on \( TO^\perp \). Hence, the algebraic hull of \( L(TO^\perp) \) for the \( G \)-action can be embedded into the structure group \( O(p, q) \) for such metric, where \( (p, q) \) is the signature of \( TO^\perp \). This yields the first claim.
Let us now assume that the induced embedding \( I \hookrightarrow \mathfrak{so}(p, q) \) is surjective. This implies that \( L \) is a finite index subgroup of \( O(p, q) \). By Section 3 of \cite{17}, we can write \( L = ZS \) where \( S \) is semisimple without compact factors, \( Z \) is compact and centralizes \( S \) and the product is almost direct. In particular, we have a direct product \( I = Z \times S \). Note that for \( p + q \leq 2 \) the conclusion of Theorem 1.1 holds by Corollary 3.1(1). Hence, we can assume from now on that \( p + q \geq 3 \). Then, the only cases in which \( \mathfrak{so}(p, q) \) is not simple is for \( \mathfrak{so}(4) \cong \mathfrak{so}(3) \times \mathfrak{so}(3) \) and \( \mathfrak{so}(2, 2) \cong \mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{sl}(2, \mathbb{R}) \). And so, one of the following holds:

- \( s = 0 \) and \( \hat{s} \cong \mathfrak{so}(p, q) \) is compact, or
- \( \hat{s} = 0 \) and \( s \cong \mathfrak{so}(p, q) \) is noncompact.

In the first case, we conclude that \( \mathfrak{so}(T_xO^\perp) \) is compact for every \( x \in M \), thus implying that the conclusion of Theorem 1.1 holds by Corollary 3.1(1). In particular, we can assume that \( I = s \cong \mathfrak{so}(p, q) \). Also, by the arguments in Section 3 of \cite{17}, there is a surjection \( g \to s \) of Lie algebras which implies that \( g \cong \mathfrak{so}(p, q) \). Hence, \( G \) is locally isomorphic to \( SO_0(p, q) \), and because of the decomposition \( TM = TO \oplus TO^\perp \) we also have \( \dim(M) = \dim(SO_0(p, q)) + \dim(T_xO^\perp) = \dim(SO_0(p, q)) + p + q = (p + q)(p + q + 1)/2 \).

On the other hand, for \( S \subset M \) given as in Proposition 2.3 if \( \Omega|_S = 0 \), then the conclusion of Theorem 1.1 holds by Lemma 2.5(2). Then, we can choose \( x \in S \) such that \( \Omega_x \neq 0 \). In particular, by Lemma 2.5(1) the \( g \)-module \( T_xO^\perp \) is a nontrivial one. If we consider the representation \( g \to \mathfrak{so}(T_xO^\perp) \) that defines the \( g \)-module structure of \( T_xO^\perp \) from Proposition 2.3 then the above remarks show that this representation is an isomorphism. We conclude that \( T_xO^\perp \) is a \( g \)-module isomorphic to \( \mathbb{R}^{p, q} \) with respect to the isomorphism \( g \to \mathfrak{so}(T_xO^\perp) \) thus obtained.

For \( \langle \cdot, \cdot \rangle_{p, q} \) the metric in \( \mathbb{R}^{p, q} \) preserved by \( \mathfrak{so}(p, q) \) we have the following.  

Claim: For every \( c \in \mathbb{R} \), the map \( T_c : \wedge^2 \mathbb{R}^{p, q} \to \mathfrak{so}(p, q) \) given by:

\[
T_c(u \wedge v) = c \langle v, \cdot \rangle_{p, q} u - c \langle u, \cdot \rangle_{p, q} v,
\]

where \( u, v \in \mathbb{R}^{p, q} \), is a well defined homomorphism of \( \mathfrak{so}(p, q) \)-modules. Moreover, these maps exhaust all the \( \mathfrak{so}(p, q) \)-module homomorphisms \( \wedge^2 \mathbb{R}^{p, q} \to \mathfrak{so}(p, q) \).

The only nontrivial part of the claim is the last statement, which we will now prove. Let \( T : \wedge^2 \mathbb{R}^{p, q} \to \mathfrak{so}(p, q) \) be a homomorphism of \( \mathfrak{so}(p, q) \)-modules. Consider the map \( T \circ T_1^{-1} \) which is a homomorphism of \( \mathfrak{so}(p, q) \)-modules \( \mathfrak{so}(p, q) \to \mathfrak{so}(p, q) \). Hence, for \( n = p + q \) and by complexifying, the map \( T \circ T_1^{-1} \) yields a homomorphism \( \hat{T} : \mathfrak{so}(n, \mathbb{C}) \to \mathfrak{so}(n, \mathbb{C}) \) of \( \mathfrak{so}(n, \mathbb{C}) \)-modules such that \( \hat{T}|_{\mathfrak{so}(p, q)} = T \circ T_1^{-1} \). In particular, \( \hat{T}(\mathfrak{so}(p, q)) \subset \mathfrak{so}(p, q) \) and so \( \hat{T} \) commutes with the conjugation \( \sigma \) of \( \mathfrak{so}(n, \mathbb{C}) \) whose fixed point set is \( \mathfrak{so}(p, q) \), i.e. \( \hat{T} \circ \sigma = \sigma \circ \hat{T} \). Note that since \( \mathfrak{g} \cong \mathfrak{so}(p, q) \) is simple with real rank at least 2, the Lie algebra \( \mathfrak{so}(n, \mathbb{C}) \) is simple as well. Hence, the irreducibility of \( \mathfrak{so}(n, \mathbb{C}) \) as \( \mathfrak{so}(n, \mathbb{C}) \)-module implies, by Schur’s Lemma, that there is a complex number \( c \) such that \( \hat{T} = c \text{Id}_{\mathfrak{so}(n, \mathbb{C})} \). But then, the condition \( \hat{T} \circ \sigma = \sigma \circ \hat{T} \) implies that \( c \) is real and so \( T = T_c \).

By the Claim and Lemma 2.5(1), with respect to the above established isomorphisms \( g \to \mathfrak{so}(T_xO^\perp) \) and \( T_xO^\perp \cong \mathbb{R}^{p, q} \), we conclude that the map \( \Omega_x \) corresponds to a homomorphism \( T_c \) for some real \( c \neq 0 \). It is straightforward to check that \( T_c \) is of curvature type; in fact, it yields the curvature of the pseudo-Riemannian manifolds of constant sectional curvature \( c \) and signature \( (p, q) \) (see Corollary 43 in page 80 of \cite{9}). Hence, Lemma \cite{7} applies and it provides a Lie algebra structure...
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on $T_xM$. It remains to show that such structure is isomorphic to either $\mathfrak{so}(p, q + 1)$ or $\mathfrak{so}(p + 1, q)$.

By our identifications, the Lie algebra structure on $T_xM$ is isomorphic to the one obtained on $\mathfrak{so}(p, q) \oplus \mathbb{R}^{p-q}$ from the map $T_x$ with the Lie brackets $\{\cdot, \cdot\}$ defined with formulas similar to those in Lemma 7.1 replacing $\Omega_x$ with $T_x$. Also, it is straightforward to show that the map given by:

$$(\mathfrak{so}(p, q) \oplus \mathbb{R}^{p-q}, \{\cdot, \cdot\}) \rightarrow \mathfrak{so}((\mathbb{R}^{p+q+1}, I_{p,q}(c)$$

$$X + u \rightarrow \begin{pmatrix} X \cr u^t I_{p,q} \cr 0 \end{pmatrix},$$

is an isomorphism of Lie algebras, where:

$I_{p,q}(c) = \begin{pmatrix} I_{p,q} & 0 \\ 0 & c \end{pmatrix}.$

Since $\mathfrak{so}((\mathbb{R}^{p+q+1}, I_{p,q}(c))$ is clearly isomorphic to either $\mathfrak{so}(p, q + 1)$ or $\mathfrak{so}(p + 1, q)$ this yields (2) from our statement. □

We can now complete the following proof.

**Proof of Theorem 1.4.** Since the hypotheses of Theorem 7.2 are a subset of those of Theorem 1.4, the conclusions of the former hold. Since $M$ is weakly irreducible its universal covering space cannot split isometrically and so part (2) of Theorem 7.2 necessarily holds at some $x \in M$. By the definition of the Lie algebra structure in $T_xM$ from Lemma 7.1 we conclude that (1) and (2) of Theorem 1.4 are satisfied.

Finally we observe that, by the proof of Theorem 7.2 we can assume that $x \in S$ where $S \subset M$ is given by Proposition 2.3. Consider the Lie algebra $\mathfrak{g}(x)$ of local Killing fields provided by Proposition 2.3 Then, the last claim of Lemma 7.1 and the definition of the representations involved implies the last claim of Theorem 1.4 for the Lie algebra $\mathfrak{g}(x)$. □

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