GAUGE–ININVARIANT CHARGED, MONOPOLE AND DYON FIELDS IN GAUGE THEORIES

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Abstract

We propose explicit recipes to construct the euclidean Green functions of gauge–invariant charged, monopole and dyon fields in four–dimensional gauge theories whose phase diagram contains phases with deconfined electric and/or magnetic charges. In theories with only either abelian electric or magnetic charges, our construction is an euclidean version of Dirac’s original proposal, the magnetic dual of his proposal, respectively. Rigorous mathematical control is achieved for a class of abelian lattice theories. In theories where electric and magnetic charges coexist, our construction of Green functions of electrically or magnetically charged fields involves taking an average over Mandelstam strings or the dual magnetic flux tubes, in accordance with Dirac’s flux quantization condition. We apply our construction to ’t Hooft–Polyakov monopoles and Julia–Zee dyons. Connections between our construction and the semiclassical approach are discussed.
1. Introduction

In this paper, we study a variety of gauge theories in four dimensions with the following features: their phase diagrams contain phases with deconfined electric or magnetic charges; their particle spectra thus contain electrically or magnetically charged particles or dyons. They may exhibit phase transitions characterized by condensation of charged particles or magnetic monopoles. Some of them exhibit duality symmetries.

One is interested in studying various aspects of the phase diagram and of the dynamics in particular phases of such theories. Obviously, one would like to study these theories analytically. However, explicit analytical results are often only available for physically rather unrealistic theories with supersymmetries. In order to study more realistic theories, one therefore often resorts to numerical investigations of lattice approximations of such theories. Usually, such investigations are based on a euclidean (imaginary–time) formulation of quantum field theory obtained from a real–time formulation by Wick rotation. In the lattice approximation, one replaces Euclidean space–time by a (finite, but arbitrarily large) lattice.

In order to study the mass spectrum of a quantum field theory, one considers euclidean Green functions of gauge–invariant (physical) fields of the theory that couple a vacuum (ground) state to some one–particle state. Masses can be calculated from exponential decay rates of certain two–point euclidean Green functions. Signals for a phase transition, e.g., one between a Coulomb– and a confining phase, can be detected in an analysis of asymptotic behaviour of suitable euclidean Green functions. For example, the transition from a Coulomb– to a confining phase in an abelian gauge theory is reflected in the appearance of long–range order in the two–point euclidean Green functions of magnetic monopoles.

On a more foundational level, we are longing for a complete description of quantum field theories in terms of the (euclidean) Green functions of gauge–invariant interpolating fields. For example, to describe the deconfined phase of an abelian gauge theory, we would like to construct Green functions of charged fields.

For QED–like theories without dynamical magnetic monopoles, a proposal for a gauge–invariant charged field has been made, many years ago, by Dirac [1]. For lattice theories of this type, Dirac’s proposal has been studied carefully e.g. in [2,3]. For the convenience of the reader, the main results of this analysis are summarized in sect. 3.

There are, however, plenty of physically interesting theories with deconfined, electrically charged particles and dynamical magnetic monopoles. The problem of con-
structing gauge–invariant electrically or magnetically charged interpolating fields for such theories has not been solved in adequate generality and is quite non–trivial. The difficulties encountered in studying this problem are a consequence of the Dirac quantization condition. We report partial results towards a solution of this problem in sect. 4.

The main purpose of this paper is to describe constructions of gauge–invariant electrically or magnetically charged interpolating fields and of dyon fields in (lattice) gauge theories with dynamical electric charges and dynamical magnetic monopoles and to exhibit duality transformations converting electrically into magnetically charged fields (and vice versa). Our work is primarily kinematical: we propose fairly explicit recipes for how to construct the euclidean Green functions of such fields. But we do not engage in any mathematically careful analysis of the properties of these Green functions. Instead, we gather evidence supporting various conjectures on their behaviour. Part of this evidence comes from previous, mathematically precise work on lattice gauge theory, another part is based on more heuristic arguments and conventional wisdom.

All in all, we believe we arrive at a fairly consistent picture of how electrically or magnetically charged and dyon fields should be constructed.

We also review duality properties of some gauge theories and analyze how duality transformations act on charged and dyon fields. This part of our analysis makes contact with issues that have been quite topical, during the past few years; (see e.g.[4]).

Finally, we outline a formal construction of Green functions of monopole– and dyon fields in the continuum $SU(2)$ Georgi–Glashow model and indicate how our construction is related to the (semi–)classical analyses of ’t Hooft and Polyakov [5] and of Julia and Zee [6].

Next, we present brief summaries of the different sections of this paper.

In section 2, we introduce the gauge theories studied in this paper. We consider three classes of models (A, B, and C). The models in class A are non–compact abelian lattice gauge theories with electrically charged matter fields, but without dynamical magnetic monopoles, i.e., models of lattice QED. The models in class B are compact abelian lattice gauge theories with electrically charged matter fields. As originally pointed out by Polyakov [7], they describe dynamical magnetic monopoles coexisting with electrically charged particles. The models in class C are related to (lattice approximations or formal continuum limits of) the Georgi–Glashow model. We define our models in terms of euclidean action functionals.
We introduce some key notation and review some facts on the phase diagrams of these models.

In section 3, we construct gauge–invariant charged fields for models of class A, in accordance with Dirac’s proposal and with the result of [2,3]. We also construct monopole fields in compact, pure abelian lattice gauge theories. These theories are dual, in the sense of Kramers–Wannier duality [8], to some models of class A, and this suggests a dual variant of Dirac’s proposal as the right definition of monopole fields.

We start section 3 with a short recapitulation of Osterwalder–Schrader reconstruction and of an analysis of superselection sectors based on euclidean Green functions of charged fields and monopole fields. We then recall (an euclidean version of) Dirac’s proposal [1] for gauge–invariant charged fields,

$$\Phi(x, E) = \Phi_x e^{i(A, E(x))}$$

where $\Phi_x$ is a charged matter field, and $A$ is a (non–compact, i.e., real–valued) abelian gauge potential; furthermore, $E(x)$ is a c–number one–form related to the electrostatic Coulomb field of a point charge. (Thus, $E(x)$ has a source at the point $x$ whose charge is the same as the charge of $\Phi$). We study the “infra–particle nature” of charged particles in such theories.

Our construction of monopole fields and our analysis of their Green functions in compact abelian gauge theories without matter fields follows from the results on electrically charged field by using Kramers–Wannier duality; (this is made explicit in sect. 3.3).

In sect. 4, we study electrically charged particles and dynamical magnetic monopoles in compact abelian lattice gauge theories with matter fields. We start by explaining the origin of Dirac’s quantization condition,

$$q_e \cdot q_m = 2\pi n$$

$n = 0, \pm 1, \pm 2, \ldots$. We then construct gauge–invariant electrically charged fields as averages of charged fields multiplied by Mandelstam string operators (exponential of the gauge field integrated along a path, called Mandelstam string [9]), the average being taken over a suitable space of Mandelstam strings. Among the more subtle points appearing in this paper is the one to come up with a good definition of “averages over Mandelstam strings”. In an analysis of Green functions of these fields based on successively integrating out the large–frequency (short–distance) modes of the fields, one observes that the large–distance effective theory
becomes increasingly similar to the one describing a model in class A. In particular, the functional integral expressions for these Green functions resemble the ones for the Green functions of charged fields constructed according to Dirac’s proposal, in models of class A. Suitable averaging over Mandelstam strings yields operators which, under renormalization, approach ones related to (1.1). The magnetic monopoles get suppressed, and Dirac’s flux quantization condition becomes irrelevant.

We then proceed to define monopole fields, or, more precisely, euclidean Green functions of such fields, in terms of certain averages over ’t Hooft disorder operators [10]. We also present a heuristic analysis of properties of euclidean Green functions of electrically charged– and monopole fields and compare our results with these in section 3.

We conclude section 4 with an analysis of dyons and of an $SL(2,\mathbb{Z})$ duality group in a compact abelian lattice gauge theory with a topological term, related to the instanton number, in the action. Our analysis is based on previous work of Cardy and Rabinovici [11].

In section 5, we outline a construction of physical interpolating fields for ’t Hooft–Polyakov monopoles and Julia–Zee dyons in the Georgi–Glashow model on the lattice and in the formal continuum limit of this model. The role of ’t Hooft’s $\mathbb{Z}_2$ monopoles and the corresponding disorder operators in our construction is explained in detail.

Some technical points are relegated to two short appendices.

2. The models

The models we consider in this paper are lattice gauge theories. (Some comments on a continuum gauge theory are added in the last section, for the Georgi–Glashow model.)

Our (euclidean space–time) lattice is $\mathbb{Z}_{1/2}^4$, where the subscript $1/2$ indicates that the coordinates of the sites are half–integers. We will also have to consider sublattices (more precisely cell subcomplexes) of $\mathbb{Z}_{1/2}^4$.

Lattice fields can be defined as follows:

A scalar field $\Phi$ is a map from the sites, $i$, of the lattice to a normed vector space $V_H$.

A fermion field $\Psi$ is an anticommuting map from the sites to the orthonormal frames of a vector space $V_s \otimes V_F$, the fermion space, where $V_s$, the spin space, carries a representation of the Dirac–Clifford algebra.
A gauge field \( g \) is a map from the links \( <ij> \) of the lattice to a group \( G \), the gauge group.

If \( W \) is an additive abelian group and \( k \) is a positive integer, one can define \( k \)-forms with values in \( W \) as maps, \( F_k \), from oriented \( k \)-dimensional cells, \( c_k \) of the lattice to \( W \) satisfying \( F(-c_k) = -F(c_k) \), where \( c_k \) denotes the cell obtained from \( c_k \) by reversing the orientation.

We denote by \( d \) the lattice exterior differential

\[
dF(c_{k+1}) = \sum_{c_k \in \partial c_{k+1}} F(c_k)
\]

and by * the Hodge–star:

Let \( c_{4-k}^* \) denote the cell in the dual lattice, \( \mathbb{Z}^4 \), dual to \( c_k \). Then

\[
*F(c_{4-k}^*) = F(c_k)
\]

We also introduce the codifferential \( \delta = (-)^{d(k+1)}d^* \) and the Laplacian \( \Delta = d\delta + \delta d \). If \( \Lambda \) is a sublattice we denote the restriction of lattice operators to forms over \( \Lambda \) by a subindex \( \Lambda \), e.g. \( d_\Lambda \) instead of \( d \).

If \( W \) is a Hilbert space with scalar product \( (\cdot, \cdot) \) one can define a scalar product among \( k \)-forms \( F \) and \( F' \) by

\[
(F, F') = \sum_{c_k} (F(c_k), F'(c_k))
\]

The restriction of the scalar product (2.1) to forms defined on a sublattice \( \Lambda \) is denoted by \( (\cdot, \cdot)_{\Lambda} \). We define the \( \ell_2 \)-norm of \( F \) by

\[
||F|| = \sqrt{(F, F)}
\]

The vacuum functional of a gauge theory with gauge field \( g \) and with matter fields \( \Phi, \Psi \) is given in terms of formal integration “measures”

\[
d\mu(g, \Phi, \bar{\Psi}, \Psi) = \frac{1}{Z} \prod_{<ij>} dg_{<ij>} \prod_i d\Phi_i d\bar{\Psi}_i d\Psi_i e^{-S(g, \Phi, \bar{\Psi}, \Psi)}
\]

where \( dg_{<ij>} \) is the Haar measure on \( G \), \( d\Phi_i \) is the Lebesgue measure on \( V_H \), \( d\bar{\Psi}_i d\Psi_i \) denotes Berezin integration, \( S \) is the total action, and \( Z \) is a normalization factor, the partition function. [Mathematically, the measure (2.2) is first defined for a finite lattice \( \Lambda \subset \mathbb{Z}^4_{\frac{1}{2}} \), with suitable boundary conditions at \( \partial \Lambda \); subsequently, one takes the limit \( \Lambda \rightarrow \mathbb{Z}^4_{\frac{1}{2}} \).]
Expectation values w.r.t. the measure (2.2) are denoted by $\langle \cdot \rangle$.

### 2.1 Actions

We consider the following classes of models.

**Class A: Non-compact abelian gauge theories with matter fields.**

As an example we discuss the abelian Higgs model.

The fields of this model are a real gauge field, $A$, and a complex scalar field, $\Phi$. The action is given by

$$S(A, \Phi) = S_0(A) + S_1(A, \Phi) + S_{gf}(A)$$

where

$$S_0(A) = \frac{1}{2\beta} ||dA||^2,$$

$$S_1(A, \Phi) = \frac{\kappa}{2} \sum_{<ij>} |\Phi_i - e^{iA_{<ij>} \Phi_j}|^2 + \lambda \sum_i (|\Phi_i|^2 - 1)^2$$

(2.3)

$S_{gf}(A)$ is a gauge fixing term, e.g.

$$S_{gf}(A) = \frac{1}{2} ||\delta A||^2$$

From now on, this gauge fixing term will usually not be written explicitly in our formulas, anymore. In the limit $\lambda \rightarrow \infty$ this model reduces to the Stückelberg model. For simplicity, we only discuss the model at $\lambda = \infty$, in this paper. It will be referred to as model A.

**Remark 2.1** A second interesting model in class A is spinor QED on the lattice; (see [2] for a rather detailed discussion). The fields of QED are the real gauge field $A$ and a four-component fermion field $\Psi = \{\Psi_\alpha, \alpha = 1, \ldots, 4\}$. The action is given by

$$S(A, \bar{\Psi}, \Psi) = S_0(A) + S_1(A, \bar{\Psi}, \Psi) + S_{gf}(A)$$

$$S_1(A, \bar{\Psi}, \Psi) = \kappa \sum_{<ij>} \bar{\Psi}_{i} \Gamma_{<ij>} e^{iA_{<ij>}} \Psi_j + \sum_i \bar{\Psi}_i \Psi_i$$

(2.4)

where $\Gamma_{<ij>}$ are matrices on $V_F$ given by

$$\Gamma_{<ij>} = 1 \pm \gamma_\mu$$

if $<ij>$ is directed in the $\pm \mu$ direction, $\mu = 1, \ldots, d$, where $\gamma_\mu$ are Euclidean Dirac matrices.

**Remark 2.2** In these models, we can omit gauge fixing by working directly with a measure defined on gauge equivalence classes.
\[ [A] = \{ A' : A' - A = d\xi \}, \] (2.5)

where \( \xi \) is a real scalar field [12].

Class B: Compact abelian gauge theories with matter fields.

As an example we analyze the compact abelian Higgs model (model B).

In the “Villain formulation”, the basic fields are a \( U(1) \)-valued 0–form \( \varphi \), a \( U(1) \)-valued 1–form \( \theta \), a \( 2\pi \mathbb{Z} \)-valued 1–form \( \ell \) and a \( 2\pi \mathbb{Z} \)-valued 2–form \( n \). The action is given by

\[
S(\theta, n, \varphi, \ell) = S_0(\theta, n) + S_1(\theta, \varphi, \ell)
\]

\[
S_0(\theta, n) = \frac{\beta}{2} ||d\theta + n||^2 , \quad S_1(\theta, \varphi, \ell) = \frac{\kappa}{2} ||d\varphi + q\theta + \ell||^2 ,
\] (2.6)

where \( q \) is the charge of the matter field; it is an integer.

Alternatively, we can use the “Wilson formulation”:

\[
S_0(\theta) = \frac{\beta}{2} \sum_p (1 - \cos d\theta)_p , \quad S_1(\theta, \varphi) = \frac{\kappa}{2} \sum_{<ij>} (1 - \cos (d\varphi + q\theta)_{<ij>}).
\] (2.7)

Class C: Non abelian gauge theories coupled to matter fields breaking the gauge group to a Cartan subgroup containing \( U(1) \).

As an example we analyze the Georgi–Glashow model (model C).

Its basic fields are an \( SU(2) \)-valued gauge field \( g \) and a Higgs field \( \Phi \), of unit length, in the adjoint representation of \( SU(2) \). The action is given by

\[
S(g, \Phi) = S_0(g) + S_1(g, \Phi)
\]

\[
S_0(g) = \beta \sum_p \left( 1 - \chi(g_{\theta_p}) \right) , \quad S_1(g, \Phi) = \beta_H \sum_{<ij>} (\Phi, U_H(g_{<ij>})\Phi)
\] (2.8)

where \( \chi \) denotes the character of the fundamental representation, and \( U_H(\cdot) \) denotes the adjoint representation.

2.2 Phase diagrams

For small \( \kappa \) and \( \beta^{-1} \), model A has a Coulomb phase with a massless photon [13,12,14]. Furthermore, it is known that, for small \( \beta \) and sufficiently large \( \kappa \), it has a superconducting Higgs phase [15].
Model B, with \( q = 1 \), has a confining / Higgs phase, for \( \beta \) small or \( \kappa \) large. Furthermore one expects a massless phase for \( \beta \) and \( \kappa \) small [16].

For \( q \neq 1 \), a phase transition line is expected to separate the confining and the Higgs phases, furthermore, for \( q \) sufficiently large, a Coulomb phase is known to appear [13,12] for intermediate values of \( \beta \), for arbitrary large \( \kappa \). If we add a topological \( \Theta \)-term,

\[
\frac{i\Theta}{4\pi^2q} \sum_{c_4} [(d\theta + n) \wedge (d\theta + n)](c_4),
\]

where the wedge product on the lattice can be defined as in [17], to the action of model B, in the limit \( \kappa \rightarrow \infty \), we recover the \( \mathbb{Z}_q \) model discussed by Cardy and Rabinovici, which is expected [11] to exhibit Higgs, Coulomb, confinement and “oblique confinement” phases, related to an (approximate) symmetry under an \( SL(2,\mathbb{Z}) \) duality group.

Model C has been rigorously shown to have a Coulomb phase for \( \beta_H = \infty \) and \( \beta \) large. The Coulomb phase is expected [18] to extend into a region of large values of \( \beta_H \) and \( \beta \). A confining phase is also known to exist [15], for small values of \( \beta \). Charged particles and monopoles are expected to exist in the Coulomb phase of these models as (infra–)particle excitations. This is the main issue discussed in this paper.

For models in class C, magnetic monopoles are expected to exist as massive particles also in the continuum limit (provided it exists). These are the celebrated at’t Hooft–Polyakov monopoles. In the models of class B, monopoles are a lattice version of the (singular) Dirac monopoles.

3. Charges or monopoles

To motivate our constructions of charged and monopole fields for the models introduced in the previous section, we outline a general strategy exploited in [19,20,21,2] (see also [22]) to construct charged (and soliton) fields and superselction sectors directly from correlation functions defined through euclidean functional integrals. We discuss this construction on the lattice; but similar ideas have been applied to continuum models, although only formally.

3.1 Reconstruction of charged and soliton quantum fields from euclidean Green functions

Let \( \mathcal{O}(C) \) denote a euclidean “observable”, i.e., a neutral, gauge–invariant, local function of the basic fields with support on a compact connected set of cells \( C \),
such as a Wilson loop, a "string field", etc.
We consider expectation values of such euclidean observables

\[ G_m(C_1, ..., C_m) = \langle \mathcal{O}(C_1)...\mathcal{O}(C_m) \rangle. \]

Let \( r \) denote reflection in the time–zero plane, \( f \) a complex–valued function and \( \bar{f} \) its complex conjugate. We define the Osterwalder-Schrader (O.S.) involution \( \Theta_{OS} \) as an antilinear map satisfying

\[ \Theta_{OS} f(\Phi_i) = \bar{f}(\Phi_{ri}), \Theta_{OS} f(g_{<ij>}) = \bar{f}(g_{<rirj>}), \]

\[ \Theta_{OS} \Psi_i = \bar{\Psi}_{ri} \gamma_0, \Theta_{OS} \bar{\Psi}_i = \gamma_0 \Psi_{ri}. \] (3.1)

Let \( \mathcal{F}_+ \) denote the set of linear combinations of euclidean observables \( \mathcal{O}(C) \) with \( C \) supported in the positive–time lattice. If, for every \( F \in \mathcal{F}_+ \),

\[ \langle (\Theta_{OS} F) F \rangle \geq 0 \]

then the correlation functions \( \{G_m\} \) are said to be O.S. positive.

**Theorem 3.1 (O.S. reconstruction).** If the correlation functions \( \{G_m(C_1, ..., C_m)\} \)

i) lattice–translation invariance, and

ii) O.S. positivity

then one can reconstruct from \( \{G_m\} \)

a) a separable Hilbert space, \( \mathcal{H}_0 \) of physical states,

b) a vector of unit norm, \( \Omega \in \mathcal{H}_0 \), the vacuum,

c) a self–adjoint transfer matrix \( T \), with unit norm, and unitary space translation operators \( U_{\mu}, \mu = 1, ..., d - 1 \), acting on \( \mathcal{H}_0 \) and leaving \( \Omega \) invariant.

We define \( T(t) = T^t, t \in \mathbb{Z}_+, U(\vec{a}) = \prod_{\mu=1}^{d-1} U_{\mu}^{a\mu}, \vec{a} = (a_1, ..., a_{d-1}) \in \mathbb{Z}^{d-1} \).

If, moreover, the correlation functions satisfy

iii) cluster properties

then

d) \( \Omega \) is the unique vector in \( \mathcal{H}_0 \) invariant under \( T \) and \( U_{\mu} \).

From the explicit proof of the theorem it follows that there is a set of vectors \( \{|C_1, ..., C_m >\} \) in \( \mathcal{H}_0 \), with \( C_j \) contained in the positive time lattice, \( j = 1, ..., m \), such that the set of linear combinations, denoted \( \hat{\mathcal{F}}_+ \), is dense in \( \mathcal{H}_0 \). On these vectors, the scalar product is defined by
\[ \langle C_1, \ldots, C_m | C_1', \ldots, C_n' \rangle = \Theta_{OS}[O(C_1) \ldots O(C_m)]O(C_1') \ldots O(C_n') \]  

(3.2)

Field operators \( \hat{O}(C_1, \ldots, C_\ell) \), with \( C_j \) contained in the strip \( \{ x : x^0 \in (0, t), t \in \mathbb{Z}_+ \} \), can be defined on \( T(t) \hat{F}_+ \) by setting

\[
\hat{O}(C_1, \ldots, C_\ell)T(t)|C_1', \ldots, C_m' > = |C_1, \ldots, C_\ell, (C_1')_t, \ldots, (C_m')_t >
\]  

(3.3)

where \((\cdot)_t\) denotes translation by \( t \) in the time direction. The operators \( \hat{O}(C) \) are the quantum mechanical operators reconstructed from the euclidean observables \( O(C) \). The algebra, \( \mathcal{A} \), generated by the operators \( \hat{O}(C_1, \ldots, C_\ell)T(t) \) defined above is called "lattice observable algebra".

If the space of the physical states of the model contains charged or soliton states, such states are not contained in the Hilbert space constructed above. The construction underlying Theorem 3.1 reproduces only the vacuum sector of such models. Suppose, however, that the Hilbert space, \( \mathcal{H} \), of physical states of the model can be decomposed into orthogonal sectors \( \mathcal{H}_q \) invariant under \( T, U_\mu \) and the lattice observable algebra \( \mathcal{A} \), i.e.,

\[ \mathcal{H} = \bigoplus_q \mathcal{H}_q. \]

Here \( \mathcal{H}_0 \) is defined to be the subspace given by \( \overline{\mathcal{A}\Omega} \). It is called vacuum sector. A sector \( \mathcal{H}_q \perp \mathcal{H}_0 \) is called a "charged sector" if there are no lattice translation–invariant vectors in \( \mathcal{H}_q \).

To construct the charged sectors, suppose that we can find an enlarged set of correlation functions \( \{ G_{n,m} \} \) obtained by taking expectation values involving, besides euclidean observables, "charged" order– and disorder–fields, denoted by \( \mathcal{C}(\Gamma) \) and by \( \mathcal{D}(\Gamma) \), respectively, with support in a connected (but, in general, non–compact) set of cells \( \Gamma \). Correlation functions with non vanishing total "charge" \( q \) are obtained from those of vanishing total charge,by removing the support of the charge of a "compensating" field of charge \(-q\) to infinity.

Assume that the correlation functions \( \{ G_{n,m}(C_1, \ldots, C_n, \Gamma_1, \ldots, \Gamma_m) \} \) still satisfy the hypotheses of the O.S. reconstruction theorem.

Denote by \( \mathcal{H}, T, U_\mu, \Omega, \hat{O}(C), \hat{C}(\Gamma), \hat{D}(\Gamma) \) the Hilbert space, the transfer matrix, the translation operators, the vacuum and the quantum fields obtained via O.S. reconstruction from the correlation functions \( \{ G_{n,m} \} \). If all the correlation functions of non-vanishing total "charge" vanish, and clustering holds, then the Hilbert space \( \mathcal{H} \) contains charged sectors, because the scalar product between two charged
states of unequal charge, defined in analogy with (3.2), vanishes; (for more precise
definitions and proofs see [ ]).
The quantum fields \( \hat{C}(\Gamma), \hat{D}(\Gamma) \) map the vacuum sector to a charged sector.

**Remark 3.1** Some of the quantum fields might couple the vacuum to one–particle
states, or infra–particle states. This can be seen as follows: Provided that \( T \geq 0 \),
the mass operator can be defined by

\[
M = -\ln(T[\{H^{(0)} \ominus C\Omega\}]) \tag{3.4}
\]

where \( H^{(0)} \) denotes the fibre of \( H \) of zero total momentum, (i.e. a generalized
vector \( |\Psi> \) belongs to \( H^{(0)} \) iff \( U_\mu|\Psi> = |\Psi> \)). Then we have the following
result.

**Theorem 3.2** A field operator \( \hat{A} \) acting on \( H \) couples the vacuum \( \Omega \) to a stable
massive one–particle state iff

\[
<\hat{A}\Omega, T(t)U(a)\hat{A}\Omega> - <\hat{A}\Omega, \Omega> <\Omega, \hat{A}\Omega> \sim \frac{e^{-m(\hat{A}) t}}{t^{d-1}}, \quad \text{as } t \to \infty \tag{3.5}
\]

with \( m(\hat{A}) > 0 \), for any \( a \in \mathbb{Z}^{d-1} \). This decay law is called Ornstein–Zernike
decay.(For infra–particles the exponent in the denominator on the r.h.s. of (3.5)
has a positive small correction.)

For many lattice models, the large \( t \) behaviour involved in (3.5) can be analysed
in terms of expansions methods [23,19,21].

3.2 **Charged fields in non–compact abelian models**

In this section we recapitulate the basic steps of the construction of charged fields
in the models of class A [2,3], following the scheme outlined above. We do this
for the sake of completeness and in order to elucidate the difficulties arising in
attempts to extend our construction to models in class B.

Let \( E \) be a real–valued 1–form with support on an infinite, connected sublattice
of the time–zero hyperplane,\( \Lambda_0 \), of the lattice \( \mathbb{Z}^4 \).

Furthermore we assume that

\[
\delta E = \delta_0 \tag{3.6}
\]

where

\[
(\delta_0)_i = \begin{cases} 
1 & \text{if } i = 0 \\
0 & \text{otherwise}
\end{cases}
\]
\[ E_{<ij>} \sim d(<ij>,0)^{-2}, \text{as } d(<ij>,0) \to \infty \]  

(3.7)

Here 0 is the origin in \( \mathbb{Z}^4 \) and \( d(<ij>,0) \) denotes the euclidean distance between \( <ij> \) and 0. As an example, one may consider

\[ E = d_{\Lambda_0} \Delta_{\Lambda_0}^{-1} \delta_0 \]  

(3.8)

We denote by \( E(x) \) the 1-form \( E \) translated by \( x \); \( E(x) \) describes the electrostatic Coulomb field surrounding a source of charge 1 located at \( x \). We define the charged fields of the Higgs model by

\[ \Phi(x,q,E) = (\Phi_x)^q e^{iq(A,E(x))}, \quad \Phi(x,-q,E) = (\Phi_x)^q e^{-iq(A,E(x))}, \quad q \in \mathbb{N} \]  

(3.9)

(See fig.1) Typical observables, \( \mathcal{O} \), are Wilson loops

\[ \mathcal{O}(\alpha C) = \prod_{<ij> \in C} e^{iA_{<ij>} \alpha} \]  

(3.10)

where \( C \) is a loop, \( \alpha \in \mathbb{R}/\{0\} \), and “string fields”

\[ \mathcal{O}(C_{xy}) = \Phi_x \prod_{<ij> \in C_{xy}} e^{iA_{<ij>} \Phi_y} \]  

(3.11)

where \( C_{xy} \) is a path from \( x \) to \( y \).

Correlation functions \( G_{n,m}^E \) are defined by

\[ G_{n,m}^E(x_1 q_1, ..., x_n q_n, C_1, ..., C_m) = \langle \prod_{i=1}^n \Phi(x_i, q_i, E) \prod_{j=1}^m \mathcal{O}(C_j) \rangle \]  

(3.12)

For later purposes it is convenient to define a generalization of (3.12): For two different 1-form \( E, E' \) satisfying (3.6), we set

\[ G_{n+m,r}^E(x_1 q_1, E, ..., x_n q_n E, x'_1 q'_1, E', ..., x'_m q'_m, E', C_1, ..., C_r) = \]  

\[ \equiv \langle \prod_{i=1}^n \Phi(x_i, q_i, E) \prod_{j=1}^m \Phi(x'_i, q'_i, E') \prod_{\ell=1}^r \mathcal{O}(C_{\ell}) \rangle \]  

(3.13)

The correlation functions \( G_{n,m} \) can be expressed as sums over configurations of “electric currents”, by first integrating over the matter fields and then integrating over \( A \). For example
\[ G_{n,0}^{E}(x_1, q_1, ..., x_n, q_n) = \frac{\sum_{\rho: \delta(\rho + \tilde{E})=0} z(\rho) e^{-\frac{\beta}{2}(\rho + \tilde{E}, \Delta^{-1}(\rho + \tilde{E}))}}{\sum_{\rho: \delta\rho=0} z(\rho) e^{-\frac{\beta}{2}(\rho, \Delta^{-1}\rho)}} \tag{3.14} \]

where \( \rho \) is an integer valued 1-current, \( z(\rho) \) is a certain statistical weight determined by the action of the model, and

\[ \tilde{E} = \sum_{i=1}^{n} q_i E(x_i) \]

Since, in the denominator of (3.18), \( \delta \rho = 0 \), the support of the currents \( \rho \) is given by a set of loops. Hence the denominator can be interpreted as the partition function of a gas of current loops interacting via the four-dimensional (lattice) Coulomb potential \( \Delta^{-1} \). In the numerator of (3.18), open currents appear, with sources at \( \{x_i\} \), besides current loops. At the sources, the electric currents spread out in fixed time planes, as described by \( E(x_i) \). The currents \( \rho \) can be interpreted as the Euclidean worldlines of charged particles; currents supported on loops correspond to worldlines of virtual particle-antiparticle pairs, open currents with connected support correspond to the worldlines of particles created at one end of the line and annihilated at the other one.

**Theorem 3.3** The correlation functions \( G_{n,m}^{E} \) defined in (3.12) are lattice translation invariant and O.S. positive. Furthermore, for \( \beta \) large enough or for strictly positive \( \beta \) and \( \kappa \) small enough, clustering holds and all correlation functions with non-zero total charge vanish.

**Idea of proof** Invariance under lattice translations follows from the existence of the thermodynamic limit of the measure corresponding to (2.3) derived by correlation inequalities [15,16]. O.S. positivity of those measures can be proved as in [15], and O.S. positivity of charged correlations follows from the fact that \( E(x) \) is localized in a fixed-time plane, so that e.g.

\[ G_{2,0}^{E}(rx, -q, y, q) = \langle \Theta_{OS}((\Phi_x)^q e^{iq(A,E(x)))}(\Phi_y)^q e^{iq(A,E(y))} \rangle \]

for \( x^0, y^0 > 0 \). (There is a slight subtlety concerning the choice of boundary conditions for correlation functions of charged fields in a bounded space-time volume. It can be dealt with in a way similar to that explained in [21]; see also section 4.2) Cluster properties are a consequence of (generalizations of) the bounds stated in the next theorem, for details see [2,3].

**Theorem 3.4** In model A, for \( \beta \) large enough or \( \kappa \) sufficiently small, and \( |x - y| \) large enough,
\[ G_{2,0}(x, -1, E, y, 1, E') \leq \exp\{-c_1(\beta, \kappa)(\tilde{E}, \tilde{E})\} \]
\[ \leq \exp\{-\left(\frac{\beta}{2} - c_2(\beta, \kappa)\right)(\tilde{E} - E_{dip}, \Delta^{-1}(\tilde{E} - E_{dip}))\}c_3(\beta, \kappa)|x-y| \]  
(3.15)

and for \( x = ry \)

\[ \exp\{-\frac{\beta}{2}((\tilde{E} + \rho_{min}), \Delta^{-1}(\tilde{E} + \rho_{min}))\}c(\kappa)|x-y| \leq G_{2,0}(x, -1, E, y, 1, E') \]  
(3.16)

where \( c_i(\beta, \kappa) \) tends to 0 exponentially, as \( \beta \to \infty \), and, for fixed \( \beta, c_i(\beta, \kappa) \sim O(\kappa) \), as \( \kappa \downarrow 0, i = 1, 2, 3, \) and \( c(\kappa) \leq 1, c(\kappa) \sim O(\kappa) \), as \( \kappa \downarrow 0 \); \( \rho_{min} \) is a current of minimal length and flux 1 connecting \( x \) to \( y \), \( \tilde{E} = E(x) - E'(y) \) and

\[ E_{dip} = d\Delta^{-1}(\delta_x - \delta_y) \]  
(3.17)

The upper bound in Theorem 3.2 shows that charged (infra–)particles in the Higgs model have strictly positive mass in the range of coupling constants indicated in the theorem. The lower bound proves that, in the same range of coupling constants, their mass is finite.

The method of proof is based on a combination of a Peierls– and renormalization group argument, following [12,24]. For \( \kappa = \infty \) the lower bound in (3.16) can be obtained more easily from Jensen’s inequality, using representation (3.14).

**Remark 3.2** It can be shown, following [25], that, for \( \beta \) sufficiently small and \( \kappa \) sufficiently large, clustering holds, and

\[ G_{2,0}(x, 1, E, y, -1, E') \geq e^{-O(\beta)(\tilde{E}, (\Delta + O(\beta))^{-1}\tilde{E})} > \text{const.} \]  
(3.18)

uniformly in \( |x-y| \). Hence correlation functions of non zero total charge do not vanish. This is a manifestation of charged–particle condensation typical for the Higgs phase.

From the O.S. reconstruction theorem we obtain a Hilbert space, denoted by \( \mathcal{H}(E) \), a transfer matrix \( T_E \) and a dense set of vectors:

\[ |x_1, q_1, \ldots, x_n, q_n, C_1, \ldots, C_m \rangle_E, \quad q_i \in \mathbb{Z}\backslash\{0\} \]  
(3.19)

corresponding to charged fields of charges \( \{q_i\} \) inserted at points \( \{x_i\} \) and local observables located at \( \{C_j\} \), with \( x_i \) and \( C_j \) in the positive time lattice.
From (generalizations of) the lower bounds (3.16) it follows that \( \| T_E \| \neq 0 \), i.e. the states (3.19) have finite energy.

One can define non-local charged fields by

\[
\hat{\Phi}(\vec{x}, q, E) | x_1, q_1, \ldots x_n, q_n, C_1, \ldots, C_m >_E = | x, q, x_1, \ldots, x_n, q_n, C_1, \ldots, C_m >_E
\]

(3.20)

with \( x^0 < x^0_1 < \ldots < x^0_n \).

We now consider the region of coupling constant space where all correlation functions of non-zero total charge vanish. Then \( \mathcal{H}(E) \) decomposes into orthogonal sectors labelled by the total electric charge \( q \):

\[
\mathcal{H}(E) = \bigoplus_q \mathcal{H}_q(E).
\]

Cluster properties show that the sectors \( \mathcal{H}_q(E) \), \( q \neq 0 \), are charged sectors, in the sense described in sect.3.1.

From the above construction it is clear that, a priori, the charged sectors \( \mathcal{H}_q(E) \) depend on the choice of the distribution \( E \). It is natural to ask if \( \mathcal{H}_q(E) \) is orthogonal to \( \mathcal{H}_q(E') \), for \( E \neq E' \). In order to answer this question, one considers the generalized correlation functions (3.16).

One can define a scalar product between states in \( \mathcal{H}_q(E) \) and \( \mathcal{H}_q(E') \) by

\[
E < x, q | x', q >_{E'} = G_{2,0}(rx, -q, E, x', q, E')
\]

From generalizations of the upper bounds (3.19), (3.20) it follows that \( \mathcal{H}_q(E) \perp \mathcal{H}_q(E') \) if

\[
\left( (E - E'), \Delta^{-1}(E - E') \right)
\]

(3.21)

diverges and one easily realizes that this divergence occurs if \( E \) and \( E' \) do not have the same “behaviour at infinity”.

There is an interesting choice of \( E \) and \( E' \) which naturally leads to a divergence in (3.21). For example, if one chooses \( E \) to be supported in a spatial cone \( S \) with apex in 0 and opening solid angle less than \( 4\pi \), the states obtained via O.S. reconstruction are the lattice approximation of the states discussed by Buchholz [26] in the algebraic approach to Q.E.D.. If the field \( E' \) is chosen to be localized in a disjoint spatial cone \( S' \), then (3.21) diverges. In particular, if we choose \( S' \) to be the cone obtained by a rotation of \( S \) then this divergence shows that in the
continuum limit (if it exists) the rotations cannot be unitarily implemented on “Buchholz states”. For more details see [2,3,19].

We conclude with some remarks about particle structure analysis on charge $\pm 1$ sectors. By inspection of the proof of Theorem 3.2 (see [2,3]), one can argue that for $|x - y|$ large:

$$
G_{2,0}^{E_0}(x, -1, y, 1) \sim \sum_{\rho_{xy}} z(\rho_{xy}) \int d\mu_{\text{ren}}(A) e^{i(A, \rho_{xy} + \bar{E})}
$$

where $\rho_{xy}$ is a current of flux 1 and connected support $\bar{E} = E(x) - E(y)$, $z(\rho_{xy})$ is a statistical weight and $d\mu_{\text{ren}}(A)$ is a positive measure of the form

$$
d\mu_{\text{ren}}(A) = \frac{1}{Z_{\text{ren}}} d\mu_{\beta_{\text{ren}}}(A) e^{\mathcal{I}(dA)}
$$

In (3.23), $d\mu_{\beta_{\text{ren}}}(A)$ is a gaussian measure with mean 0 and covariance $\beta_{\text{ren}}(\delta d)^{-1}$ (+ gauge fixing), where $\beta_{\text{ren}}$ is a renormalized coupling constant $[\beta_{\text{ren}}(\kappa, \beta) = \beta + O(c(\kappa))]$ and $\mathcal{I}(dA)$ is a sum of gauge-invariant “irrelevant” terms, in the jargon of the renormalization group. Hence the large distance behaviour of $G_{2,0}^{E_0}$, which is independent of the irrelevant terms, $\mathcal{I}$, should essentially be given by a product of two factors: one is due to the fluctuating current line $\rho_{xy}$, and, from the analysis in terms of excitation expansions of [23], it is expected to produce an Ornstein–Zernike decay

$$
|x - y|^{-3/2} e^{-m(\kappa, \beta)|x - y|}
$$

corresponding to a particle of mass $m(\kappa, \beta) \sim -\ln c(\kappa) + O(\beta)$. The second factor can be argued to contribute another power correction to the exponential law

$$
\exp[-\frac{\beta_{\text{ren}}}{2} (\bar{E}, \Delta^{-1}\bar{E})] \sim |x - y|^{-c\beta_{\text{ren}}}, c > 0
$$

It is due to the soft photons accompanying an infra–particle. Therefore, as $x^0 \nearrow \infty$, one expects that

$$
< \phi(0, \pm 1, E)\Omega, \phi(x, \pm 1, E)\Omega >
\sim \frac{e^{-mx^0}}{(x^0)^{3/2+c\beta_{\text{ren}}}}, \quad x^0 \nearrow \infty
$$

Equation (3.24) exhibits the infraparticle nature of the charged particles in the Higgs model. In fact, the vacuum expectation value of a charged field vanishes

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in the region of coupling constant space considered, and a comparison with the
general formula (3.5) shows that the mass operator $M$, as defined in (3.4), does
not have a discrete eigenvalue corresponding to a sharp one–particle state.

Remark 3.3 (Q.E.D.) In lattice Q.E.D., the electron field is defined by

$$\Psi(x, 1, E) = \Psi_x e^{i(A, E(x))}$$

and the conjugate euclidean field is

$$\Psi(x, -1, E) \equiv \bar{\Psi}_x \gamma_0 e^{-i(A, E(x))}$$

Typical local observables are Wilson loops and string fields. Correlation functions
of charged fields and local observables satisfy the hypotheses of the Reconstruction
Theorem 3.1 and, for $\kappa$ sufficiently small, bounds analogous to those in Theorem
$\alpha[5\ 3.3$ (as discussed in $[2])$.

This permits one to reconstruct non–local electron–positron field operators
$\Psi_\alpha(\vec{x}, q, E), q = \pm 1$, which are defined by

$$\hat{\Psi}_\alpha(\vec{x}, q, E)|_{x_1, q_1, \alpha_1, \ldots, x_n q_n \alpha_n, C_1, \ldots C_m > E} =$$

$$= |x, q, \alpha, x_1 \alpha_1, \ldots, x_n q_n \alpha_n, C_1, \ldots, C_m > E$$

for $x = (\frac{1}{2}, \vec{x}), x \neq x_j$.

The field $\Psi_\alpha(\vec{x}, q, E)$ can be viewed as the lattice approximation of the formal
operator (1.1) introduced by Dirac. An analogue of equation (3.24) is expected to
hold for $\kappa$ small enough, on the basis of the proof of bounds on electron-positron
correlations. It would exhibit the infraparticle nature of electrons and positrons.

3.3 Monopoles and duality

Model B, in Villain form, at $\kappa = 0$ is dual to model A in the limit $\kappa \nearrow \infty$
and it describes an interacting theory of “photons” and Dirac monopoles. Since
monopoles can be viewed as solitons in such a model, we may appeal to the strategy
outlined in sect. 3.1 to construct a monopole field operator by introducing disorder
fields, whose expectation values are the Euclidean Green functions of the Dirac
monopoles $[2,3,19]$. For this purpose we introduce a real 3–form $B$ on the lattice
$Z^4$, given by $B = 2\pi^* E$ and define the disorder field by

$$D_\omega(x_1, q_1, \ldots, x_n, q_n, B) = e^{-\frac{\beta}{2} \{||d\theta+n+\delta \Delta^{-1} (\tilde{B}-\omega)||^2-||d\theta+n||^2\}} \quad (3.25)$$
where \( x_i \in \mathbb{Z}^4, q_i \in \mathbb{Z}\setminus\{0\} \),

\[
\tilde{B} = \sum_i q_i B(x_i)
\]

and \( \omega \) is a \( 2\pi \mathbb{Z} \)-valued 3–form satisfying

\[
d(\tilde{B} - \omega) = 0. \tag{3.26}
\]

We now explain why expectation values of \( D_\omega \) are correlation functions of monopoles. First, we use the Hodge decomposition to rewrite

\[
n = d\Delta^{-1}\delta n + \delta\Delta^{-1}m \tag{3.27}
\]

where

\[
m = dn \tag{3.28}
\]

Let \( n[m] \) be an integer–valued solution of the cohomological equation (3.28). Then every other solution is of the form \( n[m] + d\ell \), where \( \ell \) is a to \( 2\pi \mathbb{Z} \)-valued 1-form. We define a real–valued gauge field \( A \) by setting

\[
A = \theta + \ell + \delta\Delta^{-1}n[m] \tag{3.29}
\]

and one can easily verify that

\[
\langle D_\omega(x_1, q_1, \ldots, x_n, q_n, B) \rangle = \sum_{m:dm=0} \int \prod_{<ij>} dA_{<ij>} e^{-\beta\frac{1}{2}\|dA + \delta\Delta^{-1}(m + \tilde{B} - \omega)\|^2} \sum_{m:dm=0} \int \prod_{<ij>} dA_{<ij>} e^{-\beta\frac{1}{2}\|dA + \delta\Delta^{-1}m\|^2} \tag{3.30}
\]

\( A \) is the euclidean “photon field”, and the Hodge–dual of \( (m - \omega) \) is supported on a set of lines in the dual lattice \( \mathbb{Z}_4 \), which can be interpreted as Euclidean worldlines of Dirac monopoles; \( \omega \) itself can be viewed as a Dirac string if we take

\[
\omega = \sum_i \omega_{x_i} + \omega_{\infty}
\]

where \( ^*\omega_{x_i} \) has support in an open line at constant time with one end at \( x_i \) and the other end joining a compensating current \( \omega_{\infty} \) at infinity (monopole b.c.). Then \( B(x_i) - \omega_{x_i} \) is exactly the lattice approximations of the magnetic field of a monopole located at \( x_i \), together with its Dirac string \( \omega_{x_i} \).
In the presence of the disorder field $D_\omega$ the Euclidean observables of the model must be modified so that expectation values do not depend on the choice of the “Dirac string” $\omega$. In particular the Wilson loop $O(C)$ is now replaced by

$$O_\omega(S) = O(C) \prod_{p \in S: \partial S = C} e^{-i(\delta \Delta^{-1})_\omega}$$  \hspace{1cm} (3.31)

where $S$ is a surface; see [19,2].

Correlation functions in the $U(1)$ gauge theory are then defined by

$$G_{n,m}^B(x_1, q_1, \ldots, x_n, q_n; S_1, \ldots, S_m) = \langle D_\omega(x_1, q_1, \ldots, x_n, q_n, B) \prod_{i=1}^m O_\omega(S_i) \rangle$$  \hspace{1cm} (3.32)

By duality e.g.

$$\langle D_\omega(x_1, q_1, \ldots, x_n, q_n, B) \rangle = \lim_{\kappa \rightarrow \infty} \langle \prod_{i=1}^n \Phi(x_i, q_i, E) \rangle_\kappa$$  \hspace{1cm} (3.33)

where $\langle \cdot \rangle_\kappa$ denotes the expectation value in the Higgs model $A$ and $E = *^* B$. O.S. reconstruction theorem applied to $G_{n,m}^B$ provide us with non–local Dirac monopole fields $M(\vec{x}, q, B)$ and, for $\beta$ large enough, Dirac monopole sectors $H_q(B)$. Furthermore a particle–structure analysis along the line of sect. 3.2 exhibits the infraparticle nature of the Dirac monopole of charge $\pm 1$.

Remark 3.3 The monopole construction outlined above can be applied to every $U(1)$–gauge theory without matter fields. This constructions and variants thereof have been applied in numerical simulations of lattice theories in [27], with the aim of detecting phase transitions between phases corresponding to different behaviour at large distances of the monopole Green functions, which has been described in the dual picture in Theorem 3.3 and Remark 3.2.

4. Charges and monopoles

We start this section with an outline of the new problems appearing in an attempt to extend the construction sketched in the previous sections to models where charges and monopoles coexist (class B,C).

4.1 Dirac quantization condition

We have seen that, for $\kappa = 0$, model B (in Villain form) can be written explicitly in terms of a “photon field” $A$ and a “monopole field” $*m$. The Dirac strings of
the virtual monopole loops in the partition function sweep out surfaces described by the support of $*n$. The term $S_1$ in the action introduced in eq.(2.6) couples the gauge field $\theta$ to a charged matter field. In our expression for the partition function, the charged matter gives rise to charged loops describing the worldlines of virtual particle–antiparticle pairs. We must ask whether the statistical weights of these charged loops depend on the location of the Dirac strings of virtual monopoles.
To answer this question, let us rewrite the action (2.6) in terms of $A, m, \varphi, \ell$. We obtain, for $q = 1$,

$$S = \frac{\beta}{2}||dA||^2 + \frac{\beta}{2}(m, \Delta^{-1}m) + \frac{\kappa}{2}||d\varphi + A - \delta\Delta^{-1}n[m] + \ell||^2$$  \hspace{1cm} (4.1)

Independence of the Dirac strings corresponds, as recalled above, to independence of the choice of the two–form $n[m]$ satisfying $dn[m] = m$, as in (3.28).
Let $\zeta$ be a $\mathbb{Z}$-valued 1-form. Then by Poisson summation formula,

$$\sum_\ell e^{-\frac{\kappa}{2}||d\varphi + A - \delta\Delta^{-1}n[m] + \ell||^2} = \sum_\zeta e^{-\frac{\kappa}{2}||\zeta||^2} e^{i(\zeta, d\varphi + A - \delta\Delta^{-1}n[m])}$$

Integrating over $\varphi$ we obtain

$$\delta \zeta = 0.$$

We conclude, using the Poincaré lemma, that there exists a $\mathbb{Z}$-valued 2-form, $\chi$, such that

$$\zeta = \delta \chi,$$

so that

$$(\zeta, \delta\Delta^{-1}n[m]) = (\delta \chi, \delta\Delta^{-1}n[m]) = (\chi, d\delta\Delta^{-1}n[m]) = (\chi, n[m]) - (\chi, \delta\Delta^{-1}m)$$

where the last equality follows from Hodge decomposition.
The second term depends only on $m$ and

$$e^{i(\chi, n[m])} = 1.$$

This is nothing but the Dirac quantization condition, since the electric charges appearing in the partition functions are $q_e \in \mathbb{Z}$ and the magnetic charges are $q_m \in 2\pi\mathbb{Z}$, so that
From the above proof it is clear that if we consider the expectation value of $e^{i(\theta,E)}$, where the “electric current” 1-form $E$ is not integer valued, as introduced in sect 3.2, we would encounter an inconsistent dependence on the choice of Dirac strings. This problem is often neglected in the physics literature, where virtual monopole loops are sometimes ignored, assuming that monopoles are “very heavy”, see e.g. [28]. But, in order to arrive at a fully consistent definition of gauge–invariant charged field Green functions, one must cope with it.

The natural suggestion is to replace the electric field $E(x)$ of sect. 3.2 by an integer–valued electric current, a “Mandelstam string”, starting at $x$ and ending at the location of some compensating charge which will eventually be sent to infinity.

A naive idea would be to simply replace the variable $\phi_x e^{i(A,E(x))}$, used in the construction of charged states in model A, by

$$c_x^R e^{i\phi_x e^{i(\theta,\gamma_x^R)}}$$

in model B, and then take the limit $R \to \infty$. In (4.3), $\gamma_x^R$ is a unit 1-form with support on a straight line in a fixed–time plane from $x$ to some point at a distance $R$ and then joining that point to a fixed point in the time–zero plane, and $c_x^R$ is some normalization factor.

Consider the 2-point function: Integrating out $\varphi$ we obtain

$$\langle e^{i\varphi_x e^{i(\theta,\gamma_x^R)}} e^{-i\varphi_y e^{-i(\theta,\gamma_y^R)}} \rangle = \frac{1}{Z} \int \prod_{<ij>} d\theta_{<ij>} e^{-S_0(\theta)} \sum_{\rho: \delta \rho = \delta_x - \delta_y} z(\rho) e^{i(\theta,\gamma_x^R - \gamma_y^R + \rho)}$$

where $\rho$ are $\mathbb{Z}$-valued currents whose statistical weight is denoted by $z(\rho)$, and $Z$ is the partition function. The phase factors appearing in (4.4) define Wilson loops, and, in the region of coupling constant space where monopoles are particle excitations, the expectation value of the Wilson loop is known to exhibit perimeter decay. Hence we expect (4.4) to vanish exponentially fast as $R \to \infty$.

This decay is dominantly due to the self–energy of the strings $\gamma$. This effect could eliminated by adjusting $c$. But we would then be left with an interaction term between the strings which, being attractive and extending over the full string, tends to infinity in the limit $R \to \infty$, and this appears to render the renormalized
Green function divergent. Furthermore, since the interaction term depends on the time distance between strings, it cannot be renormalized away without violating O.S. positivity. This problem could be solved if we replace a “straight Mandelstam string” by a sum over “fluctuating Mandelstam strings”, weighted by a measure which is concentrated on strings fluctuating so strongly that, with probability one, their “interaction energy” remains finite in the limit $R \nearrow \infty$.

A proposal for a “natural” measure can be inferred from a representation of correlation functions of a $U(1)$ scalar field, $\chi$, with coupling constant $\beta_\chi$, coupled to an external $U(1)$ gauge field $\theta$, in terms of random walks. E.g., in $d$ dimensions, with $< \cdot > (\theta)$ denoting the corresponding expectation value,

$$
\langle e^{i\chi_x} e^{-i\chi_y} \rangle (\theta) = \sum_{\omega_{xy}} \frac{(2d)^{-|\omega_{xy}|} - 1}{\beta} \frac{Z(\theta|\omega_{xy})}{Z(\theta)} e^{i(\omega_{xy}, \theta)}
$$

(4.5)

where $\omega_{xy}$ is a path (“string”) from $x$ to $y$, $|\omega_{xy}|$ its length, $Z(\theta)$ is the partition function of the system, and $Z(\theta|\omega_{xy})$ is a partition function modified by $\omega_{xy}$, see [29].

In $d \geq 3$ dimensions, for $\beta_\chi$ sufficiently large and $\theta = 0$, the string $\omega_{xy}$ is known to be rough. In fact

$$
\langle e^{i\chi_x} e^{-i\chi_y} \rangle (0) \longrightarrow \text{const}
$$

as $|x - y| \rightarrow \infty$, and it is believed that it remains in the rough phase for a class of positive measure of $U(1)$-external gauge fields, (roughly speaking, if $d\theta$ is sufficiently small in average).

This suggests that two–point functions of a $U(1)$ scalar field might yield an appropriate measure on “Mandelstam strings”, if $\beta_\chi$ is large enough.

For a better understanding of our construction in the compact models, we show how to reproduce the results already obtained in the non–compact models following the above ideas.

4.2 The non–compact model revisited

Let $\Gamma(R, t), R \in \mathbb{Z}_+, t \in \mathbb{Z}_{1/2}$, be a cube of height $2|t|$ in the time $(x^0–)$ direction and with sides of length $R$ in coordinate planes $x^0 = \text{const}$ and centered at the origin; see fig.2. We define $\Lambda(R, t)$ to be given by $\partial \Gamma(R, t) \cap \{x : x^0 \gtrsim 0\}$, for $t \gtrsim 0$. (To simplify our notation, the restriction of a lattice operator to $\Lambda(R, t)$ will be denoted with a subscript $\Lambda$, the specific sublattice which we are referring to
being identified from context.) We introduce a real-valued 0-form $\lambda$ on $\Lambda(R,t)$, with action given by

$$S_\Lambda(\lambda) = \frac{\beta\lambda}{2} ||d\Lambda \lambda + A||^2_\Lambda$$

(4.6)

where $(,)_{\Lambda}$ is the inner product in $\Lambda(R,t)$, and $||f||^2_\Lambda = (f,f)_{\Lambda}$.

Denote by $\langle \cdot \rangle_{\Lambda(R,x^0)}(A)$ the (normalized) $\lambda$–expectation corresponding to the action $S_\Lambda(\lambda)$. Set $R_\pm = (\pm \frac{1}{2}, R, 0, 0)$ and, for $x^0 > 0, y^0 < 0$, define

$$G(x, y) = \lim_{R \to \infty} \langle \langle e^{i\lambda x} e^{-i\lambda R_+^+} \rangle_{\Lambda(R,x^0)}(A)\langle e^{-i\lambda y} e^{i\lambda R_-^-} \rangle_{\Lambda(R,y^0)}(A) e^{iA < R^+_+ R^-_->} \Phi_x \Phi_y \rangle$$

(4.7)

where $\langle \cdot \rangle$ is the expectation value of model A.

By explicit computation

$$\langle e^{i\lambda x} e^{-i\lambda R_+} \rangle_{\Lambda(R,x^0)}(A) = e^{-\frac{1}{2\beta}((\delta_x - \delta_{R^+_+}), \Delta_{\Lambda}^{-1}(\delta_x - \delta_{R^+_+}))} e^{i(E_\Lambda(x) - E_\Lambda(R^+_+), A)}$$

(4.8)

where

$$E_\Lambda(x) = d\Delta^{-1}_{\Lambda} \delta_x$$

(4.9)

Hence

$$G(x, y) = \lim_{R \to \infty} \langle \langle e^{i(E_\Lambda(x) - E_\Lambda(y),A) \Phi_x \Phi_y} \rangle [e^{i(E_\Lambda(R^+_+ - E_\Lambda(R^+_+ + \delta_{<R^+_+ R^-->},A))}]angle$$

$$= c \langle e^{i(E(x) - E(y),A) \Phi_x \Phi_y} \rangle$$

(4.10)

where $\delta_{<R^+_+ R^-->}$ denotes the 1–form with support on the link $< R^+_+ R^-->$, whose value is 1 on that link, and

$$c = \lim_{R \to \infty} \langle [e^{i(E_\Lambda(R^+_+ - E_\Lambda(R^+_+ + \delta_{<R^+_+ R^-->},A))}angle.$$ 

To prove (4.10), we note that the form $E_\Lambda(x) - E_\Lambda(y)$ decays like $d^{-3}$ in the limit $R \not\to \infty$, and $E_\Lambda(x) \to E(x)$, where $E$ is given in (3.8).

We observe that the two–point function of the auxiliary matter field $e^{i\lambda}$ exactly reproduces the exponential $e^{i(A,E)}$ needed in the construction of charged states. Furthermore our construction respects O.S. positivity and, when the limit $R \not\to \infty$ is taken, lattice translation invariance is restored.

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Sectors corresponding to different choices of the one–form $E$ can be reproduced by modifying $S_{\Lambda}(\lambda)$.

### 4.3 Charged fields in compact models

It is natural to generalize the construction of the previous section to the compact model $B$.

We introduce a $U(1)$-valued scalar field $\chi$ and a $2\pi\mathbb{Z}$-valued 1–form $r$ on the lattices $\Lambda(R, t)$ defined in the previous section. An action functional for $\chi$ and $r$ is defined by

$$S_{\Lambda}(\chi, r) = \beta \frac{e^{iY} + e^{-i\chi} + e^{i\chi} + e^{-iY}}{2}$$

We denote by $\langle \cdot \rangle_{\Lambda(R, t)}(\theta)$ the expectation value w.r.t. the action (4.12).

With notations analogous to those of the previous section, we propose the following definition of the two–point function of the charge–1 field in model B:

$$G(x, y) = \lim_{R \to \infty} c_{x, q}^R c_{y, 1}^R \langle e^{i\chi_x} e^{-i\chi_y} \rangle_{\Lambda(R, x_0)}(\theta) \langle e^{i\chi_x} e^{-i\chi_y} \rangle_{\Lambda(R, y_0)}(\theta) e^{i\theta < R_+ R_- + e^{i\varphi_x} - e^{i\varphi_y}}$$

(See fig.2) In (4.13) $c_{x, q}^R$ is a normalization factor given by

$$c_{x, q}^R = \left\{ \langle e^{i\chi_x} e^{-i\chi_y} \rangle_{\Lambda(R, x_0)}(\theta) \langle e^{i\chi_x} e^{-i\chi_y} \rangle_{\Lambda(R, y_0)}(\theta) e^{i\theta < R_+ R_- + e^{i\varphi_x} - e^{i\varphi_y}} \right\}^{-\frac{1}{2}}$$

(4.14)

where $q$ is an integer and $x_\pm = (\pm \frac{1}{2}, \vec{x})$. An approximate evaluation of (4.14) suggests that $c_{x, q}^R$ might be bounded in $R$ (see Appendix A).

The charge–1 two–point function $\langle e^{i\chi_x} e^{-i\chi_y} \rangle_{\Lambda(R, x_0)}(\theta)$ is periodic in $\theta$, with period 1. Hence, it has a Fourier representation

$$\langle e^{i\chi_x} e^{-i\chi_y} \rangle_{\Lambda(R, x_0)}(\theta) = \int d\mu_{R_+}(j_x) e^{i(j_x, \theta)}$$

(4.15)

where $j_x$ are $\mathbb{Z}$–valued one–forms satisfying

$$\delta j_x = \delta_x - \delta_{R_+}$$

and $d\mu_{R_+}(j_x)$ is a complex measure on the space of 1–forms $j_x$. One can integrate out the matter field $\varphi$ in (4.13) and express the contribution as a weighted sum of

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$\mathbb{Z}$-valued 1-currents. In the numerator of (4.13), a contribution due to a 1-current $\omega$ satisfying

$$\delta \omega = \delta x - \delta y$$

will appear. As a result of this representation of expectation values in terms of sums over integer-valued currents, equation (4.13) yields a representation of the Green function $G(x, y)$ as an expectation value of a weighted sum of Wilson loops

$$W_\omega(j_x, j_y) = e^{i(j_x + j_y + \delta_{<R_+R_->} + \omega, \theta)}$$  \hspace{1cm} (4.16)

(See fig.2)

To analyze $\langle e^{-i\chi} e^{i\chi_{R_+}} \rangle_{\Lambda(R, x^0)}(\theta)$, we perform a change of variables analogous to (3.36). Define

$$v = d_\Lambda r$$  \hspace{1cm} (4.17)

and let $r(v)$ be a fixed $2\pi\mathbb{Z}$-valued solution of equation (4.17). We then define a real-valued scalar field $\lambda$ by

$$\lambda = \chi + \delta_\Lambda \Delta_\Lambda^{-1} r[v]$$

Changing variables from $\chi, r$ to $\lambda, v$ we obtain

$$\langle e^{-i\chi} e^{i\chi_{R_+}} \rangle_{\Lambda(R, x^0)}(\theta) =$$

$$= \sum_{v, dv=0} \int \prod_i d\lambda_i e^{-\frac{\beta\chi}{2} ||d\Lambda \lambda + \delta_\Lambda \Delta_\Lambda^{-1} v + \theta||^2_\Lambda} e^{i(\lambda - \delta_\Lambda \Delta_\Lambda^{-1} r[v], \delta_x - \delta_{R_+})} \Lambda$$

$$= e^{-\frac{\beta\chi}{2} (\delta_x, \Delta_\Lambda^{-1}(\delta_x, \Delta_\Lambda^{-1}(\delta_x, \Delta_\Lambda^{-1}(\delta_x, \Delta_\Lambda^{-1})))} e^{-i(E_\Lambda(x, R_+), \theta)} F(E_\Lambda(x, R_+)|\theta)$$  \hspace{1cm} (4.18)

where

$$F(E|\theta) = \sum_{v, dv=0} e^{-\frac{\beta\chi}{2} (v + d\theta, \Delta_\Lambda^{-1}(v + d\theta))} e^{-i(E_\Lambda, r[v])} \Lambda$$  \hspace{1cm} (4.19)

and

$$E_\Lambda(x, R_+) = E_\Lambda(x) - E_\Lambda(R_+).$$

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The action (4.12) describes a $U(1)$ spin system in the presence of the external
gauge field $\theta$; the first two factors on the r.h.s. of (4.18) correspond to the spin–
wave (gaussian) approximation of the correlation function, the last one to the
contribution of the vortices, described by $v$, to the spin system.

For $\beta\chi$ sufficiently large and $d\theta = 0$, the spin system is known to be in a phase
with long range order, where the spin–wave approximation is known to capture
the essential physics at large distances. It is argued in [29], as remarked above,
that, in the presence of the gauge field, the system remains in a phase with LRO,
provided $d\theta$ is sufficiently “small” in average. Hence, for $\beta\chi, \beta$ sufficiently large, we
expect the vortices to form a dilute gas, so that the factor $F$ in (4.18) is expected
to represent a “small” correction to the spin–wave approximation. However, the
presence of $F$ is crucial for the periodicity in $\theta$ which ensures independence of the
Dirac strings of virtual monopole loops.

Following the arguments of sect. 3.2 (but see also [2,14]), one can argue that, for $\kappa$
small enough, the contribution of the matter field at large distances essentially
yields a renormalized $U(1)$ gauge theory with a coupling constant

$$\beta'_{\text{ren}} = \beta + O(\kappa^4)$$

which, according to [12,24,3], and sect. 3.4, is expected to renormalize to a non–
compact gauge theory with coupling constant

$$\beta_{\text{ren}} = \beta'_{\text{ren}} + e^{O(\beta'_{\text{ren}})}$$

in the scaling limit, provided $\beta'_{\text{ren}}$ is large enough. Accordingly, the 2–point function is expected to behave like

$$G(x, y) \sim \frac{1}{Z} \int \mathcal{D}A e^{-\beta_{\text{ren}}||dA||^2} \sum_{\rho_{xy}:\delta \rho_{xy} = \delta_x - \delta_y} z(\rho_{xy})e^{i(A, \rho_{xy} + E(x) - E(y))}$$

$$F^G(E(x)||dA) F^G(-E(y)||dA), \quad (4.20)$$

at large distances, where $F^G$ is a gaussian approximation of $F$ estimated in
Appendix A by

$$F^G(E||dA) \sim e^{-[e^{-O(\beta\chi)}||E||^2 + e^{-O(\beta\chi)}||E\cdot \delta \Delta^{-1}dA||^2]} \quad (4.21)$$

Appealing to universality in the scaling limit, we believe that the large distance
properties of $F$ are indeed correctly captured by the gaussian approximation, $F^G$. 

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Green functions with vanishing total charge $G_{m+n}(x_1, q_1, \ldots, x_m, q_m, y_1, q'_1, \ldots, y_n, q'_n)$, $x_i^0 > 0, y_i^0 < 0$, are defined by

$$G_{m+n}(x_1, q_1, \ldots, x_m, q_m, y_1, q'_1, \ldots, y_n, q'_n) =$$

$$\lim_{R \to \infty} \left( \prod_{j=1}^{m} c^R_{x_j, q_j} e^{-i q_j x_j} e^{i q_j x_j R_+} \right) \Lambda(R, x_0^j) \langle \theta \rangle i^{j} x_j$$

$$\prod_{k=1}^{n} c^R_{y_k, q'_k} \left( e^{-i q'_k y_k} e^{i q'_k y_k R_-} \right) \Lambda(R, y_0^k) \langle \theta \rangle e^{i q'_k y_k} e^{i q_k \theta} \langle R_+ R_- \rangle,$$

(4.22)

where $q = \sum_{j=1}^{n} q_j$ and $c^R_{c,q}$ is as in (4.14).

The large distance behaviour described by (4.20) implies cluster properties enabling us to reconstruct charged sectors and non-local charged field operators $\hat{\Phi}(\vec{x}, q, E)$ for model B. Equation (4.20) also suggests that the particles of charge $\pm 1$ are infraparticles (see sect.3.2), since the contribution due to $F^G$ does not change qualitatively the large distance behaviour [30]. The same ideas also apply straightforwardly to model B with $q \neq 1$.

These observations suggest that the Green functions of charged fields in the present model have large-scale properties similar to those in gauge theories without dynamical monopoles, constructed according to Dirac’s original proposal, provided $\beta_\chi$ is chosen large enough. On short distance scales, the Dirac quantization condition implies integral quantization of electric flux accompanying a charged field $\hat{\Phi}(x, q, E)$ in all models with dynamical monopoles, while electric flux is not quantized in Dirac’s construction. But, on large distance scales, the constraint of integer electric flux becomes ”irrelevant”, (i.e. this constraint renormalize to zero). The phenomenon described here is analogous to the phenomenon of symmetry enhancement studied in [12,31].

**Remark 4.1** The construction described above can be generalized to Green functions of charged fields in deconfined (non-abelian Coulomb) phases of non-abelian gauge theories with some gauge group $G$, by simply replacing the auxiliary two-point correlation functions of the $U(1)$ scalar field $\chi$ coupled to the $U(1)$ gauge field by two-point correlation functions of a non linear sigma model with target $G$ coupled to the gauge field. These correlation functions can be expressed [29,32] in terms of sums over path-ordered exponentials of the gauge field. It is useful to recall that an inequality analogous to the one quoted after (4.5) holds. If $G$ contains a $U(1)$ subgroup and the field, $g$, of the non-linear sigma model, transforms under a representation, $U$, corresponding to a character non trivial on this subgroup,
then, at weak coupling, the two–point function in an external gauge field of zero curvature, \( \langle U(g_x)U(g^{-1}_y) \rangle (0) \) is bounded uniformly from below in \( |x - y| \). One expects that this long range order persists in the model coupled to the \( G \)-gauge field, provided the curvature of the gauge field is small enough in average [29].

4.4 Monopoles in compact models with matter

In this section, we propose a construction of monopole Green functions in compact models with matter (class B) obtained through a duality transformation. First we identify the model dual to B. It can be described in terms of a \( \mathbb{Z} \)-valued 2-form \( \gamma \) and a \( \mathbb{Z} \)-valued 1-form \( \alpha \) (defined on the dual lattice) with an action given by

\[
S(\alpha, \gamma) = \frac{1}{2\beta} ||d\alpha - q\gamma||^2 + \frac{1}{2\kappa} ||d\gamma||^2
\]

We can replace \( \alpha \) by a real-valued 1-form \( A \) by inserting the constraint

\[
\sum m \delta (A - m) = \sum \rho e^{i(A,\rho)}
\]

where \( m \) is a \( \mathbb{Z} \)- and \( \rho \) is a \( 2\pi \mathbb{Z} \)-valued 1-form, and we have applied the Poisson summation formula.

The partition function of the dual model can then be written as

\[
Z = \sum_{[\gamma]} \int \prod_{<ij>} dA_{<ij>} e^{-\frac{1}{2\beta} ||dA - q\gamma||^2 - \frac{1}{2\kappa} ||d\gamma||^2} \sum_{\rho, \delta \rho = 0} e^{i(A,\rho)},
\]

where we exploit the gauge–invariance of the \( A \) measure to impose the constraint \( \delta \rho = 0 \).

The interpretation of formula (4.24) is clear: it describes an abelian gauge theory with charged matter fields and dynamical monopoles satisfying Dirac’s quantization condition. The 1-form \( \rho \) describes the worldlines of electric point charges (with values in \( 2\pi \mathbb{Z} \)), the 1-form \( *d\gamma \) describes the worldlines of magnetic point charges (which are integer–valued). This interpretation is consistent with the feature of duality transformations that they exchange electric and magnetic charge.

Accordingly, a Green function of charged fields in the dual model corresponds to a monopole Green function in the original model. If we try to follow the construction of the previous section we encounter the following problem: we can reabsorb the “Mandelstam strings” appearing in an expansion of the \( \chi \)-correlation functions, as in (4.5) and (4.13), in a shift of the current \( \rho \). One may say that these “Mandelstam
strings” are “screened” by the charge loops appearing in the dual model. Starting from the action (4.23), we meet the same phenomenon in the following way: Since the gauge field $\alpha$ is integer-valued and the electric charge of the auxiliary field $\chi$ is $2\pi$, the dynamics of $\chi$ is independent of $\alpha$, (i.e., coupling $\chi$ to $\alpha$ does not have any physical effects).

A way out of this difficulty appears when $q \neq 1$, because the magnetic charges appearing in the model are multiples of $q$, so that one can introduce electric loops of charge $\frac{2\pi}{q}$, still satisfying Dirac’s quantization condition.

Hence, instead of inserting correlation functions described as sums over a single fluctuating string of charge $2\pi$, one can introduce correlation functions described as sums over $q$ distinct fluctuating strings of charge $\frac{2\pi}{q}$ which are not screened.

For this purpose we introduce a scalar field $\chi$ with value in $[-\pi q, \pi q]$ and a $2\pi \mathbb{Z}$-valued 1–form $r$ with action

$$S_{\Lambda}(\chi, r) = \frac{\beta \chi}{2} \left| | \frac{d\Lambda \chi - 2\pi A}{q} + r | \right|_{\Lambda}^2$$

and denote the expectation value w.r.t. the action (4.25) by $\langle \cdot \rangle_{\Lambda(R,t)}(A)$, (see sect. 4.2). The charge–1 two–point correlation function $\langle e^{i\chi_x} e^{-i\chi_{R^+}} \rangle_{\Lambda(R,x^0)}(A)$ has a Fourier representation

$$\langle e^{i\chi_x} e^{-i\chi_{R^+}} \rangle_{\Lambda(R,x^0)}(A) = \int d\mu_{R^+}(j_x) e^{i(j_x,A)}$$

where $j_x$ are $\frac{2\pi \mathbb{Z}}{q}$-valued 1–forms satisfying

$$\delta j_x = 2\pi \delta x - 2\pi \delta R^+$$

and $d\mu_{R^+}(j_x)$ is a complex measure on $j_x$. As a 2–point function of charged fields in the model dual to B, for $q \neq 1$, we propose to consider

$$G^*(x, y) = \lim_{R \to \infty} c_x^R c_y^R \langle \langle e^{i\chi_x} e^{-i\chi_{R^+}} \rangle_{\Lambda(R,x^0)}(\alpha) \langle e^{i\chi_y} e^{-i\chi_{R^-}} \rangle_{\Lambda(R,y^0)}(\alpha) \rangle^* = \lim_{R \to \infty} c_x^R c_y^R \langle \langle e^{i\chi_x} e^{-i\chi_{R^+}} \rangle_{\Lambda(R,x^0)}(A) \langle e^{i\chi_y} e^{-i\chi_{R^-}} \rangle_{\Lambda(R,y^0)}(A) e^{i(\omega, A)} e^{i2\pi A < R^+ R^- >} \rangle^*$$

where $\langle \cdot \rangle^*$ is the expectation value of the dual model, and $\omega$ is a $2\pi \mathbb{Z}$-valued 1–form satisfying

$$\delta \omega = 2\pi (\delta x - \delta y)$$
(See fig.3) Note that $G^*$ is independent of the choice of $\omega$ satisfying (4.28).

The corresponding two–point function in the original compact model is obtained by applying an inverse duality transformation, by replacing the Wilson loops

$$W_\omega(j_x, j_y) = e^{i(j_x + j_y + 2\pi \delta_{<R_+R_->}} + \omega, A)$$

appearing in an expansion of (4.27) (as discussed in the previous section, see (4.16)) by ’t Hooft disorder loops.

To construct the ’t Hooft loop, we have to identify a surface whose boundary is given by the loop $L = j_x + j_y + 2\pi \delta_{<R_+R_->} + \omega$. Let us denote its $\frac{2\pi\mathbb{Z}}{q}$–valued Poincaré dual by $\Sigma(j_x, j_y; \omega)$. The ’t Hooft disorder field, $D_\omega(j_x, j_y)$, is obtained by shifting $d\theta$ by $\Sigma$ in the original model, i.e.,

$$D_\omega(j_x, j_y) = e^{-\frac{\beta}{2}(|d\theta + \Sigma(j_x, j_y; \omega) + n||^2 - ||d\theta + n||^2)}$$

Combining (4.26) and (4.27), the two–point monopole function in model B is given by

$$G^*(x, y) = \lim_{R \to \infty} \int d\mu(j_x, j_y) \langle \int d\mu_{R_+}(j_x) d\mu_{R_-}(j_y) D_\omega(j_x, j_y) \rangle$$

Equation (4.29) expresses $G^*(x, y)$ as an expectation value of a weighted sum of ’t Hooft disorder fields $D_\omega(j_x, j_y)$. In order to better understand the meaning of (4.29), notice that, as in sect.4.2, one can easily show that

$$\langle e^{i\chi x} e^{-i\chi R_+} \rangle_{\Lambda(R, x^0)}(A) = e^{-\frac{\beta}{2\pi}((\delta_x - \delta_{R_+}) - \Delta \chi^{-1}((\delta_x - \delta_{R_+})))} e^{i(E_\Lambda(x, R_+), A)} F(E_\Lambda(x, R_+) | \frac{2\pi}{q} dA).$$

From (4.26) and (4.30) it follows that $d\mu(j_x)$ behaves, for large $\beta_x$, as an approximate Dirac $\delta$ measure around $E_\Lambda(x, R_+)$. This, in turn, implies that $d\Sigma(j_x, j_y; \omega)$ is peaked around

$$B_\Lambda(x, R_+) + B_\Lambda(R_-, y) + \omega + 2\pi \delta_{<R_+R_->}, \quad \text{with} \quad B_\Lambda = E_\Lambda$$

Here we see the dual version of the phenomenon occurring for charged fields: the magnetic current appearing in the definition of monopole Green functions is forced to be $2\pi\mathbb{Z}$ valued so as to respect Dirac quantization condition. But its average, at sufficiently large scales, approximates the ordinary magnetic field $B$, of the classical monopole.
The deviation from a $\delta$-measure of $d\mu_{R_+}(j_x)$ is “small”, for $\beta_\chi$ large. This may explain why, in numerical simulations [33], a naive application of the definition of monopole Green functions in theories without matter fields appears to give reasonable results.

Redefining $\beta_\chi = q^2 \tilde{\beta}_\chi$, keeping $\tilde{\beta}_\chi$ fixed and taking the limits $q \nearrow \infty, \kappa \searrow 0$ one recovers the monopole Green functions discussed in sect. 3.3.

Using a Peierls type argument and renormalization group techniques discussed in [2,3,12,24], we may argue that the monopole Green functions in a theory with matter fields have the same qualitative large distance behaviour as the monopole Green functions in a pure $U(1)$ gauge theory, if $\beta$ and $\kappa^{-1}$ or $q^2/\beta$ are sufficiently large. (In the dual model the renormalization of the electric currents $\rho$ requires $\beta$ to be large, and the renormalization of the monopole currents $d\gamma$ requires $\kappa^{-1} + O(q^2/\beta)$ to be large.)

4.5 Dyons

Model B in Villain form is self–dual in the limit $\kappa \nearrow \infty$, where it becomes a $\mathbf{Z}_q$ gauge theory. If a topological term (2.9) is added one can show, see [11], that a modified duality transformation is an approximate symmetry. When combined with the $2\pi$–shift of $\Theta$ it generates an $SL(2,\mathbf{Z})$ duality group of approximate (large–scale) symmetries of the model.

Here we briefly describe how to derive the duality symmetry and then discuss its action on charged and monopole Green functions. Green functions of dyon fields are obtained in a natural way.

A wedge product on the lattice compatible with O.S. positivity can be defined [17] as follows: Label a (positively oriented) $k$-cell in the lattice by a site, $x$, and a set of $k$ directions $\underline{\mu} = \{\mu_1, \ldots, \mu_k\}$; i.e.,

$$c_k = (x; \mu_1, \ldots, \mu_k) = x + \sum_{i=1}^{k} \xi^{\mu_i} e_{\mu_i}, \quad (4.31)$$

where $\xi^{\mu_i} \in [0, 1]$ and $e_{\mu_i}$ is the unit vector in the $\mu_i$ direction.

Given a $p$–form $A$ and a $q$–form $B$ we define

$$A \wedge B(x; \underline{\mu}) = \sum_{\underline{\mu}_j \in \underline{\mu}} \epsilon_{\mu_1 \ldots \mu_{p+q}} \frac{1}{2} \{A(x; \mu_1, \ldots, \mu_p)B(x + e_{\mu_1} + \ldots + e_{\mu_p}; \mu_{p+1}, \ldots, \mu_{p+q})$$

$$+ \frac{1}{2} A(x; r\mu_1, \ldots, r\mu_p)B(x + e_{r\mu_1} + \ldots + e_{r\mu_p}; r\mu_{p+1}, \ldots, r\mu_{p+q}) \} \quad (4.32)$$
where $r$ inverts the time direction, leaving the other ones unchanged. It is easy to verify that
\[ d(A \wedge B) = dA \wedge B + (-1)^p A \wedge dB \] (4.33)
but in general
\[ A \wedge B \neq -(-)^{p+q} B \wedge A \]
Furthermore, on the lattice, we do not have an analogue of the continuum identity
\[ \int (A \wedge B) = (A,^*B) = (^*A, B) \] (4.34)
for $p + q = 4$, where $^*$ is the Hodge dual and $(,)$ denotes the inner product on forms in $\mathbf{R}^4$.
From the definition (4.32) it follows, however, that we can define two modified star operations, $A \mapsto ^*A, A \mapsto A^*$, such that for $p + q = 4$,
\[ \sum_{c_4} (A \wedge B)(c_4) = (A,^*B) = (^*A, B) \] (4.35)
and with the property that if $\delta^*A = 0$, then $\delta^*A = 0 = \delta A^*$, but $A^* \neq ^*A$ and $A^{**} \neq A, ^*A \neq A$. Furthermore, if $A$ is $\mathbf{Z}$-valued, $A^*$ and $^*A$ are $\mathbf{Z}/2$-valued.
Using the rule (4.33) one can rewrite the topological term (2.9) in terms of the magnetic currents $m = dn$ : up to boundary terms vanishing if 0–Dirichlet b.c. are imposed on $d\theta$ and $n$, it can be rewritten as
\[ i \frac{\Theta}{4\pi^2} q \sum_{c_4} (m \wedge \theta + \theta \wedge m + n \wedge n)(c_4) \] (4.36)
Its meaning is clear: the first two terms associate to every magnetic current $m$ a line of total electric flux $\Theta/2\pi q$ along the support of $m^*$ and $^*m$ (Witten effect [34]); the last term is the self–intersection form of the surface $^*n$.
To discuss the duality group it is convenient to define
\[ A_\pm = \frac{A \pm A^*}{2} \]
for each $k$–form $A$, and to introduce, following [11], the complex coupling constant parameter
\[ \zeta = -i \frac{2\pi \beta}{q} + \frac{\Theta}{2\pi} \] (4.37)
and, integrating over the matter field, the partition function is given by

$$Z = \sum_n \int \prod_{<ij>} d\theta_{<ij>} e^{-S_0(\theta,n)} \sum_{\rho: \delta \rho = 0} e^{iq(\theta,\rho)}$$

(4.39)

The SL(2, Z) group acts on \(\zeta\) by

$$\zeta \rightarrow \frac{A\zeta + B}{C\zeta + D}, \quad AD - BC \neq 0$$

Its generators are called \(S: \zeta \rightarrow \zeta^{-1}\), and \(T: \zeta \rightarrow \zeta + 1\).

We now show why \(S\) is an approximate symmetry of the partition function.

Introducing real 2–forms \(\lambda, \bar{\lambda}\), denoting by \([\sigma]\) the equivalence class of \(\mathbb{Z}\)–valued two–forms, i.e., \([\sigma] = \{\sigma' - \sigma = \delta V, V(c_3) \in \mathbb{Z}\}\), and performing the change of

variables \(\{\theta, n\} \rightarrow \{A, m\}\), as in previous section, one obtains that

$$Z = \sum_{m:dm=0} \sum_{\sigma} \int \prod_{<ij>} dA_{<ij>} \prod_p d\lambda_p \prod_p d\bar{\lambda}_p$$

$$e^{-\frac{i}{2\pi q\zeta}||\lambda||^2} e^{\frac{i}{2\pi q\bar{\zeta}}||\bar{\lambda}||^2} e^{i(dA+n[m]:\frac{\lambda}{2\pi} + \frac{\bar{\lambda}}{2\pi} + q\sigma)}$$

Integrating out \(A\) we obtain the constraint

$$\delta\left(\frac{\lambda}{2\pi} + \frac{\bar{\lambda}}{2\pi} + q\sigma\right) = 0$$

(4.40)

If the identity \(A^{**} = A\) were true then it would follow from (4.40) that there exists a real–valued 3–form, \(\xi\), such that

$$\lambda = (\delta \xi - 2\pi q\sigma)_+ , \quad \bar{\lambda} = (\delta \xi - 2\pi q\sigma)_-$$

Formally, \(A^*\) and \(A^*\) have the same continuum limit. On large scales, we may therefore replace \(*\) by \(\ast\). Then \(A_+\) and \(A_-\) are just the selfdual and anti–selfdual components of \(A\) and, indeed, \(A^{**} = A\).

So, within this approximation,

$$Z \sim \sum_{m:dm=0} \sum_{[\sigma]} \int \prod_c d\xi_c e^{-\frac{i}{2\pi q\zeta}||\xi - 2\pi q\sigma||^2} e^{\frac{i}{2\pi q\zeta}||\xi - 2\pi q\sigma||^2} e^{i(n[m]:\frac{\xi}{2\pi}, \delta \xi)}.$$

(4.41)
Passing to the dual lattice and introducing a $U(1)$–valued 1–form $\tilde{\theta}$ and a $\mathbb{Z}$–valued 1–form $\ell$ and setting

$$\xi = q^* \tilde{\theta} + 2\pi q^* \ell,$$

we can rewrite (4.41) as

$$Z \sim \sum_{m:dm=0} \sum_\sigma \int \prod_{<ij>} d\tilde{\theta}_{<ij>} e^{-\frac{iq}{2\pi\zeta} \|(d\tilde{\theta} - 2\pi^* \sigma)_+\|} e^{\frac{iq}{2\pi\zeta} \|(d\tilde{\theta} - 2\pi^* \sigma)_-\|} e^{i(q(\frac{m}{2\pi}, \tilde{\theta}))}$$

(4.42)

Equation (4.42) proves that the $S$ generator $SL(2, \mathbb{Z})$ induces an approximate symmetry exchanging

$$\zeta \to \frac{1}{\zeta} \quad m \to -2\pi^* \rho \quad \rho \to \frac{m^*}{2\pi}.$$

Analogously, the $T$–generator induces an approximate symmetry exchanging

$$\zeta \to \zeta + 1 \quad n \to n \quad \rho \to \rho - \frac{m^*}{2\pi} - \frac{m^*}{2\pi}$$

as one can easily see from (4.15) (4.36); the symmetry becomes exact for $T^2$ since $\frac{m^*}{\pi}$ and $\frac{m^*}{\pi}$ are actually $\mathbb{Z}$–valued, as the original $\rho$ current.

Our construction of the charged 2–point function, $G(x, y)$, is based on the introduction of a weighted sum of Wilson loops

$$W_\omega(j_x, j_y) = e^{i q(\theta_{j_x} + j_y + \omega + \delta < R_+ R_- >)}$$

(4.43)

where $\omega$ is a current line from $x$ to $y$, and our construction of the monopole 2–point function $G^*(x, y)$ is based on the introduction of a weighted sum of ’t Hooft loops

$$D_\omega(j_x, j_y) =$$

$$e^{-i \frac{q}{2\pi} \{(||d\theta + n + \Sigma(j_x, j_y, \omega)||^2 - ||(d\theta - n)||^2) + i \frac{q}{2\pi} \{(||d\theta + n + \Sigma(j_x, j_y, \omega)||^2 - ||(d\theta - n)||^2)\}}$$

(4.44)

with the notations of previous sections.

The above discussion makes it clear that the $S$ generator of $SL(2, \mathbb{Z})$ approximately exchanges the two correlation functions in the dual models, while the $T$ generator acts (approximately) by multiplying the ’t Hooft loop (4.44) by the Wilson loop (4.43), raised to the power $-2$, hence producing the 2–point function of a dyon, whose electric charge is $-2q$ at $\Theta = 0$. 

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5. 't Hooft–Polyakov monopoles

5.1 Preliminaries

In order to explain our proposal for the construction of Green functions for the quantum 't Hooft–Polyakov monopole, it is useful to recall the definition of topological invariants characterizing the classical 't Hooft–Polyakov monopole in the Georgi–Glashow model and then to exhibit their lattice counterparts.

The field content of the classical Georgi–Glashow model consists of a connection \( A \) on a principal \( SU(2) \)-bundle, \( P_{SU(2)} \), on \( \mathbb{R}^4 \) and a scalar field \( \Phi \) with values in \( su(2) = \text{Lie}(SU(2)) \), which is a section of a vector bundle associated to \( P_{SU(2)} \) with fiber \( su(2) \) equipped with the adjoint action of \( SU(2) \).

The Lagrangian density of the model is given by

\[
\mathcal{L} = \frac{1}{2} |F_A|^2 + \frac{1}{2} |D_A \Phi|^2 - \frac{\lambda}{8} (|\Phi|^2 - 1)^2 \quad (5.1)
\]

where \( F_A \) is the curvature of \( A \), and \( D_A \) is the covariant derivative

\[
D_A \Phi = d\Phi + [A, \Phi]
\]

Moreover \( |\cdot| \) denotes the Killing norm on \( su(2) \). Let us assume that \( A \) and \( \Phi \) verify appropriate regularity conditions, and \( |1 - |\Phi(\vec{x})||, |D_A \Phi(\vec{x}), F_A(\vec{x}) \) have decay properties at \( \infty \) discussed in detail in [35]. Then, at fixed time, a finite–energy static configuration \((A, \Phi)\), with \( A_0 = 0 \), defines a homotopy class, \([A, \Phi] \in \pi_2(SU(2)/U(1)) \sim \mathbb{Z}\), labelled by an integer topological charge

\[
N = \frac{1}{4\pi} \int_{\mathbb{R}^3} \text{Tr}(F_A \wedge D_A \Phi) = \frac{1}{4\pi} \int_{S^2_{\infty}} \text{Tr}(\tilde{\Phi} F_A) \quad (5.2)
\]

where \( S^2_{\infty} \) is the 2–sphere at infinity. Furthermore, the field \( \Phi \) defines a homotopy class \([\Phi] \in \pi_2(S^2)\) which can be characterized by the Kronecker index

\[
N = \frac{1}{4\pi} \int_{S^2_R} \text{Tr}(\tilde{\Phi} \wedge d\tilde{\Phi} \wedge d\tilde{\Phi}); \quad (5.3)
\]

with \( \tilde{\Phi} = \Phi/|\Phi| \), and \( S_R = \{|x| = R\} \) provided \( R \) is sufficiently large. Then (5.3) is independent of \( R \), and \([\Phi] = [(A, \Phi)]\). A solution of the equations of motion with \( N \neq 0 \) is called a classical 't Hooft–Polyakov monopole.

The magnetic charge of a Dirac monopole in \( \mathbb{R}^3 \) is given by the first Chern number of a \( U(1) \)-bundle over a topological 2–sphere containing the monopole.[This geometrical definition differs from the one used previously by a factor \( 1/(2\pi) \).] We
show how one can associate to a configuration \((A, \Phi)\) and a cube \(c\) in \(\mathbb{R}^3\) a \(U(1)\) bundle in such a way that the first Chern number of the bundle can be identified with the magnetic charge contained in \(c\). We then relate the magnetic charge to the topological charge previously defined.

We choose a cube \(c\) in \(\mathbb{R}^3\); its boundary, \(\partial c\), is homeomorphic to a 2–sphere. The restriction of an \(SU(2)\) connection \(A\) to \(\partial c\) can be viewed as a connection \(A|_{\partial c}\) on an \(SO(3)\) bundle over \(\partial c\). There is then an \(SO(3)\) gauge transformation mapping \((A, \Phi)|_{\partial c}\) to \((\bar{A}, |\Phi|_3)|_{\partial c}\). Projecting \(\bar{A}\) to a Cartan subalgebra of \(su(2)\) one obtains a \(U(1)\) connection, \(a\). This projection is called “abelian projection”. It has been introduced by ’t Hooft [36] to express Yang–Mills theories in terms of abelian gauge fields, charges and monopoles. We identify the magnetic charge contained inside \(c\) with the first Chern number of the curvature, \(F_a\), of \(a\).

The relation between \(\Phi, \bar{A}\) and \(a\) can be described in terms of the homotopy exact sequence

\[
0 \sim \pi_2(SO(3)) \rightarrow \pi_2(S^2) \xrightarrow{\partial} \pi_1(U(1)) \xrightarrow{i_*} \pi_1(SO(3)) \rightarrow \pi_1(S^2) \sim 0
\]

where

\[
\partial : n \in \mathbb{Z} \sim \pi_2(S^2) \rightarrow 2n \in \mathbb{Z} \sim \pi_1(U(1))
\]

and

\[
i_* : n \in \mathbb{Z} \sim \pi_1(U(1)) \rightarrow n \text{mod } 2 \in \mathbb{Z}_2 \sim \pi_1(SO(3))
\]

The group \(\pi_1(U(1))\) classifies \(U(1)\)–bundles over the (topological) 2–sphere \(S^2\), here identified with \(\partial c\). The integer \(n \in \pi_1(U(1))\) is given in terms of a connection \(a\) on a principal \(U(1)\)–bundle over \(\partial c\) by

\[
n = \frac{1}{2\pi} \sum_{p \in \partial c} \int_F a
\]

(5.4)

where \(p\) denotes faces of the cube \(c\).

The group \(\pi_1(SO(3))\) classifies \(SO(3)\)–bundles over \(S^2 \simeq \partial c\). We have the following relation between \(n \text{ mod } 2 \in \pi_1(SO(3))\) and a connection \(\bar{A}\) on a principal \(SO(3)\) bundle over \(\partial c\):

\[
e^{i\pi n} = \exp[i\{\arg \sum_{p \in \partial c} Tr P(e^{i\oint_{\partial p} \bar{A}(p)})\}]
\]

(5.5)

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where $\bar{A}^{(p)}$ is the connection 1–form of an $SU(2)$ bundle over $p$ obtained by lifting the $SO(3)$–bundle over $p$. If $e^{i\pi n} \neq 1$ one cannot extend the $SO(3)$–bundle to the interior of the cube $c$. This signals the presence of an odd number of singular $Z_2$–monopoles of $SO(3)$ inside $c$.

If $\Phi$ is a map from $S^2_R$ to $S^2$ of Kronecker index $N$ then the first Chern number corresponding to $\partial[\Phi]$ is $2N$. Hence ’t Hooft–Polyakov monopoles have even magnetic charge and, with an abuse of language, following [37], the term ’t Hooft–Polyakov monopole stands for an arbitrary monopole configuration with even magnetic charge. The most general configuration inside a cube $c$ in $R^3$ is a combination of ’t Hooft–Polyakov monopoles and $Z_2$ monopoles characterized by odd magnetic charge.

Next we explain how to define the magnetic charge on the lattice by means of the abelian projection, following [38].

In model C, we start by choosing a unitary gauge by means of an $SO(3)$ gauge transformation $W$. In this gauge the fields are given by

$$\bar{\Phi}_i = W_i \Phi_i W_i^{-1} = \sigma_3$$

$$\bar{g}_{<ij>} = W_i g_{<ij>} W_j^{-1}$$

(5.6)

We write the $SU(2)$–valued gauge field $\bar{g}_{<ij>}$ as a product

$$\bar{g}_{<ij>} = C_{<ij>} u_{<ij>}(\theta)$$

with

$$C_{<ij>} = \begin{pmatrix}
(1 - |c_{<ij>}|^2)\frac{1}{2} & -c_{<ij>}^* \\
-\frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

$$u_{<ij>}(\theta) = e^{i\frac{\theta_{<ij>} - \sigma_3}{2}},$$

(5.7)

where $c_{<ij>} \in \mathbb{C}$, $c_{<ij>}^*$ denotes its complex conjugate and $\frac{\theta_{<ij>}}{2} = \arg(\bar{g}_{<ij>})_{11}$. Under a $U(1)$–gauge transformation $\left\{ e^{-iA_j, \frac{\sigma_3}{2}} \right\}$ of the original fields, $\theta$ and $c$ transform as

$$\theta \to \theta + d\Lambda, \quad c_{<ij>} \to c_{<ij>} e^{i\frac{\Lambda_j}{2}} e^{i\frac{\Lambda_i}{2}}$$

Hence $\theta$ is a $U(1)$–gauge field, and $c$ is a charged field of charge 1.

We define the magnetic charge in a lattice cube $c$ by
\( m_c(\theta) = \frac{1}{2\pi} \sum_{p \in \partial c} (d\theta)_p, \)  

where \( d\theta \) is restricted to the range \([-2\pi, 2\pi]\). Equation (5.8) is the lattice analogue of eq. (5.4). A \( \mathbb{Z}_2 \)– charge in a cube \( c \) is defined by

\[ e^{i\pi z_c(g)} = e^{i\sum_{p \in \partial c} \text{arg} \chi(g_{\partial p}).} \]  

(5.9)

A plaquette \( p \) where

\[ e^{i\text{arg} \chi(g_{\partial p})} = -1 \]

can be identified as the location of a Dirac string of a \( \mathbb{Z}_2 \)–monopole intersecting the plane containing \( p \).

The relation established between (5.4) and (5.5) can be translated to the lattice as

\[ e^{i\pi z_c(g)} = e^{i\pi m_c(\theta)}. \]  

(5.10)

With the help of the abelian projection, the phase transitions in model C have been analyzed numerically [39] in terms of condensations of magnetic currents.

5.2 Monopole Green functions

In model C, the magnetic charge of 't Hooft–Polyakov monopoles is even and the electric charge of the matter field is integer. Hence we are facing a situation analogous to the one discussed in sect 4.3, for \( q = 2 \), in models of class B. Following the ideas in that section, we propose to construct the two–point function of a monopole of magnetic charge 2 by summing over pairs of fluctuating strings of magnetic charge 1 with end points at the location of the monopole.

As discussed in the previous section, these strings can be identified as Dirac strings of \( \mathbb{Z}_2 \)–monopoles. They can be introduced by means of 't Hooft disorder fields. For a 2–surface \( \Sigma \) bounded by a lattice loop \( L \), the disorder field is defined by

\[ D(\Sigma) = e^{-[S_0(g e^{i\Sigma 3}) - S_0(g)]} \]  

(5.11)

where \( S_0 \) is given in (2.8), \( \Sigma \) is a 2–form with values in \{0, \pi\}, whose support is dual to the surface \( \Sigma \).

The expectation value of \( D(\Sigma) \) depends only on \( L \). Clearly \( e^{i\Sigma 3} \) takes values in \{0, 1\} \( \sim \mathbb{Z}_2 \), which is the center of the gauge group \( SU(2) \).
In order to construct the monopole two–point function we start, by introducing a $U(1)$ scalar field $\chi$ on the sublattice $\Lambda(R,t)$. The field $\chi$ is minimally coupled to a real gauge field $A$ with coupling constant $\pi$. The action for $\chi$ is given by $S_{\Lambda}(\chi,r)$, a functional defined in (4.25), with $q = 2$.

Since $S_{\Lambda}(\chi,r)$ is periodic in $A$ with period 2, $\langle e^{i\chi x} e^{-i\chi R_+} \rangle_{\Lambda(R,x)}(A)$ is the Fourier transform of a complex measure $d\mu_{R_+}(j_x)$ on the space of currents $j_x$ with values in $\pi\mathbb{Z}$ which are constrained by $\delta j_x = 2\pi(\delta_x - \delta_{R_+})$; see formula (4.26). We propose to define the two–point function for a 't Hooft–Polyakov monopole of magnetic charge 2 by

$$G^*(x,y) = \lim_{R \to \infty} c^R_x c^R_y \left\langle \int d\mu_{R_+}(j_x) d\mu_{R_-}(j_y) D_\omega(j_x,j_y) \right\rangle$$  \hspace{1cm} (5.12)

with $\Sigma$ constrained by

$$^*d\Sigma(j_x,j_y;\omega) = j_x + j_y + \omega + 2\pi\delta_{<R_+,R_->},$$  \hspace{1cm} (5.13)

where $\omega$ is a 3–form taking values in $\{0,2\pi\}$ whose support is dual to a path connecting $x$ to $y$.

As the definition (5.12) involves a 't Hooft disorder field, the expectation value on the r.h.s. is independent of the choice of $\Sigma$, given $\omega$, and, since $\omega$ takes value in $\{0,2\pi\}$, also of $\omega$.

In Appendix B it is shown, using the abelian projection, that the disorder field $D_\omega(j_x,j_y)$ introduces a source for 't Hooft–Polyakov monopole currents. For this purpose, a representation of $\langle D_\omega(j_x,j_y) \rangle$ as a sum over configurations of magnetic currents of even charge, with boundary given by $\{x,y\}$, is derived. This representation makes precise the idea that, in the abelian projection, the effect of the disorder field is to introduce a shift of $d\theta$ by $2\Sigma(j_x,j_y;\omega)$. Hence, according to definition (5.8), the magnetic charge in a cube $c$ is changed by $2(d\Sigma)_c$.

**Remark 5.1** In (5.10) one may replace the matrix $\sigma_3$ by $\Phi_x$ at the plaquette $p = (x;\mu_1,\mu_2)$, since, in the abelian projection, $\Phi_x$ is replaced by $\sigma_3$, and the computation of the magnetic charge is then unchanged.

**Remark 5.2** The above construction cannot be adapted to obtain Green functions for the $\mathbb{Z}_2$–monopoles. For such Green functions, the current $\omega$ would take values in $\{0,\pi\}$, and the expectation value of the disorder field $D_\omega(j_x,j_y)$ would depend on $\omega$, hence on the Dirac strings of the $\mathbb{Z}_2$–monopoles, which is unphysical (see Appendix B).
One can argue that, in the Coulomb phase of model C correlation functions of non-zero total magnetic charge vanish (as proved at $\beta_H = \infty$, where the model reduces to a $U(1)$ theory), so that we expect the appearence of superselection sectors labelled by an even magnetic charge. According to the previous section, one may identify these sectors as 't Hooft–Polyakov monopole sectors. In the confinement phase of the model, Green functions of non-zero total charge are expected not to vanish due to “monopole condensation” (this is proved at $\beta_H = \infty$). The large distance behaviour of the two–point monopole Green function could then be used to detect phase transitions in the Georgi–Glashow model, as in the $U(1)$ gauge theory.

5.3 Continuum

In the final section we sketch how to define Green functions for quantum ‘t Hooft–Polyakov monopole fields in the formal continuum limit of the Georgi–Glashow model.

It turns out that it is more convenient to work in a first–order formalism for the Yang–Mills field. (A formal relation of this formalism to ordinary Yang Mills and BF theories is discussed in [40]).

We introduce a 2–form $B$ with values in $su(2)$, which is a section of the vector bundle of 2–forms associated to $P_{SU(2)}$. We replace the Yang–Mills euclidean action

$$S_0(A) = \frac{1}{2} \int |F_A|^2 d^4x$$

by

$$S_0(A, B) = i \int \text{Tr}(F_A \wedge B) + \frac{1}{2} \int |B|^2 d^4x$$  \hspace{1cm} (5.14)$$

In the functional integral approach, integrating over $B$ with the (white–noise) gaussian measure corresponding to the second term in (5.14), formally yields the standard weighting factor, $e^{-S_0(A)}$, for Yang–Mills fields.

In the first order formalism, we use

$$D_\omega(j_x, j_y) = [e^{i \int \text{Tr}B \wedge \tilde{\Phi}(j_x j_y \omega)}]_{\text{ren}}$$  \hspace{1cm} (5.15)$$
as our ‘t Hooft disorder field, with the notations of the previous section adapted to the continuum. The notation $[\ ]_{\text{ren}}$ indicates that an ultraviolet multiplicative
renormalization is necessary. For the lattice theory, (5.15) corresponds to the choice mentioned in Remark 5.1.

We wish to comment on the relation between our construction of monopole Green functions, based on expression (5.15) for the disorder operator, and the conventional semi–classical analysis of ’t Hooft–Polyakov monopoles. For this purpose, we consider a monopole two–point function with \( y = rx \) (where, as usual, \( r \) denotes time reflection), and \( x = (x^0, \vec{0}) \). We are interested in the asymptotic behaviour of this Green function, as \( x^0 \) becomes large. One strategy to analyze this behaviour is to attempt to evaluate the functional integral defining the Green function with the help of semi–classical techniques. In a semi–classical approximation, an evaluation of the expectation value

\[
\lim_{R \to \infty} \langle \int d\mu_{R+}(j_x)d\mu_{R-}(j_y)D\omega(j_x, j_y) \rangle
\]

of the disorder operator in the formal functional measure of the Georgi–Glashow model is accomplished through an expansion of the functional measure around a back–ground field configuration. In the unitary gauge of the abelian projection, the disorder operator creates a “mean background gauge field” with curvature \( F_{\tilde{A}}(z) \) approximately given by

\[
\sigma_3 \tilde{B}(\vec{z})
\]

for \( z = (z^0, \vec{z}) \), with \( |z^0| << x^0 \), where \( \tilde{B}(\vec{z}) \) is the rotation–covariant, static magnetic field generated by a magnetic monopole of magnetic charge 2, located at the origin.

If, instead of the unitary gauge, we choose a gauge with the property that

\[
\tilde{\Phi}(z) = \frac{\vec{z} \cdot \vec{\sigma}}{|\vec{z}|}
\]

the mean background gauge field has a field strength \( F_{\tilde{A}}(z) \) approximately given by

\[
\tilde{B}^{H-P}(\vec{z}) = \frac{\vec{z} \cdot \vec{\sigma}}{|\vec{z}|}
\]

for \( z = (z^0, \vec{z}) \), with \( |z^0| << x^0, |\vec{z}| \) large.

The field in (5.17) describes the large–distance behaviour of the field strength of the classical ’t Hooft–Polyakov monopole solution [5].
We should comment on the notion of “mean background (gauge) field” used in the arguments above: Every field configuration contributing to the expectation value of a disorder field $D_\omega(j_x, j_y)$ actually satisfies the non-abelian version of Dirac’s quantization condition [28,37]. Such a field configuration is therefore singular near the support of the magnetic flux currents, $j_x$ and $j_y$. However, after integrating over $j_x$ and $j_y$ with the complex measure $d\mu_{R_+}(j_x)d\mu_{R_-}(j_y)$ and taking the limit $R \to \infty$, and after integrating out the high-frequency (short-distance) modes of the fields, i.e., after ultraviolet renormalization (“coarse graining”), the resulting background field configuration approaches the one described above. This is because the constraint of flux quantization of the Dirac strings is softened under renormalization and is actually expected to scale to zero in the limit of very large distances scales. This phenomenon is analogous to the one discussed in the abelian models of class B.

Taking it for granted that, on large distance scales, the background field configuration described in (5.16), (5.17) dominates the functional integral appearing in the (numerator of the) monopole Green function $G(x, rx)$, as $x^0$ becomes large, the relation of our approach to the conventional semi-classical analysis of (e.g., the mass of) the ’t Hooft–Polyakov monopole becomes clear.

We conclude this section with a comment on dyons. If a topological term

$$i\frac{\Theta}{16\pi^2} \int Tr(F_A \wedge F_A)$$

is added to the action of the Georgi–Glashow model then the excitation spectrum of the model contains dyons.

Dyon Green functions can be constructed with the help of disorder fields. The construction is analogous to the one discussed in the context of the Cardy–Rabinovici model, in sect. 4.5. If $\Theta$ is not an integer multiple of $2\pi$ dyons carry a fractional charge. As $\Theta$ approaches an integer multiple of $2\pi$, the dyons correspond to those first described by Julia and Zee [41]. The spectrum of the theory is periodic in $\Theta$, with period $2\pi$.

**Appendix A** (Gaussian evaluation of $F(E|dA)$)

[In this appendix all symbols referring explicitly to the sublattice $\Lambda(R, t)$ are omitted, e.g., we write $d$ instead of $d_\Lambda$ etc..].

To analyse $F(E|dA)$ it is convenient to work on the dual lattice and introduce an auxiliary gauge field, $C$: 

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\[
F(E|dA) = \frac{\int \prod_{<ij>} dC_{<ij>} e^{-\frac{1}{\beta}||dC||^2} \sum_{\rho; \delta \rho = 0} e^{-i(C,^*dA)} e^{i(dC - E, \sigma(\rho))} \delta \rho}{\int \prod_{<ij>} dC_{<ij>} e^{-\frac{1}{2\beta}||dC||^2} \sum_{\rho; \delta \rho = 0} e^{-i(C,^*dA)} e^{i(dC, \sigma(\rho))}}
\]  

(A.1)

where, comparing with (4.18), \( \rho = v^* \) and \( \sigma(\rho) \) is a fixed, integer–valued solution of

\[
\delta \sigma(\rho) = \rho.
\]

Following the techniques developed in [12] (see also [2,3]), we divide a configuration of currents, \( \rho \), into (not necessarily connected) networks, \( \{\rho_\alpha\} \), such that the distance \( d(\rho_\alpha, \rho_\beta) \geq 2, \alpha \neq \beta \).

We now renormalize the activities of the current networks, \( \{\rho_\alpha\} \), using the method of complex translations.

Let \( B(\rho_\alpha) \) be a set of links in the support of \( \rho_\alpha \) such that two links in \( B(\rho_\alpha) \) do not belong to a common plaquette and such that

\[
\sum_{<ij> \in B(\rho_\alpha)} |\rho_{<ij>|}^2 \geq c(\rho_\alpha, \rho_\alpha),
\]

where

\[
c^{-1} = \text{card} \{ \ell' : \ell' \neq \ell, \{\ell, \ell'\} \in \partial p \text{ for some } p \} = 18.
\]

We set

\[
\rho'_\alpha = \rho_\alpha |_{B(\rho_\alpha)} \quad \rho' = \sum_\alpha \rho'_\alpha
\]

and perform the complex translation

\[
C \rightarrow C + i \frac{\beta \chi}{n} \rho' + i \beta \chi \Delta^{-1} * dA
\]

where

\[
n = \text{card}\{p : \partial p \in <ij>\} = 6
\]

As a result every current network \( \rho_\alpha \) obtains a complex activity

\[
z_A(\rho_\alpha, C, E) = e^{-\frac{\beta}{2}[||\rho'_\alpha||^2 + 2(\rho_\alpha, \Delta^{-1} * dA)]} e^{i(d\bar{C} - E, \sigma(\rho))}
\]

in the numerator of (A.1), and an activity \( z_A(\rho_\alpha; C) \equiv z_A(\rho_\alpha, C, 0) \) in the denominator of (A.1), where \( \bar{C} = C - \frac{1}{n}(\delta dC)|_{\rho'_\alpha} \).
If \( dA \) is sufficiently small so that
\[
|z_A(\rho_\alpha, C)|, |z_A(\rho_\alpha, C, E)| \leq e^{-c\beta\chi ||\rho_\alpha||^2}
\]
the current networks form a dilute gas, and we can exponentiate their contributions in the form of a Mayer series:
\[
F(E|dA) = \int \prod_{<ij>} dC_{<ij>} e^{-\frac{1}{2\beta\chi} ||dC||^2 + \sum_{\mathcal{L}} a_T(\mathcal{L}) z_A(\mathcal{L}, C, E)}
\]
where \( \mathcal{L} \) is a collection of current networks \( \rho \) in which a single \( \rho \) can occur an arbitrary number of times, \( a_T(\mathcal{L}) \) is a standard combinatorial factor which enforces all \( \mathcal{L} \) to have connected supports and
\[
z_A(\mathcal{L}, C, E) = \prod_{\rho \in \mathcal{L}} z_A(\rho, C, E).
\]
We give an approximate evaluation of (A.3) by keeping only the leading contribution in \( \mathcal{L} \), which correspond to plaquette terms, and expanding the activities \( z_A \) to second order in \( C, A, E \).  

\( F(E|dA) \) is then approximated by
\[
\int \prod_{<ij>} dC_{<ij>} e^{\left\{-\frac{1}{2\beta\chi} ||dC||^2 - \sum_p e^{-O(\beta\chi)} (1 + \frac{\beta^2}{2} \langle A^T \rangle_p^2)(d\bar{C} - \ast E)_p^2 + i\beta\chi \langle \ast A^T \rangle_p (d\bar{C})_p \right\}}
\]
\[
\int \prod_{<ij>} dC_{<ij>} e^{\left\{-\frac{1}{2\beta\chi} ||dC||^2 - \sum_p e^{-O(\beta\chi)} (1 + \frac{\beta^2}{2} \langle A^T \rangle_p^2)(d\bar{C})_p^2 + i\beta\chi \langle \ast A^T \rangle_p (d\bar{C})_p \right\}}
\]
where \( A^T = \delta \Delta^{-1} dA \), and we have used that
\[
(d\bar{C}, \ast E) = 0.
\]
Explicit evaluation of (A.4) to second order in \( A \) gives
\[
F^G(E|dA) = \exp[-e^{-O(\beta\chi)} (\frac{1}{2} ||E||^2 + \frac{\beta^2}{2} ||A^T \cdot E||^2)]
\]

Appendix B

In this appendix we exhibit a representation of \( \langle D_\omega(j_x, j_y) \rangle \) in model C in terms of magnetic currents. We start by replacing the \( SU(2) \) gauge field \( g \) with a couple of new variables \( \{U, \sigma\} \).  \( U \) is the gauge coset variable given on a link \( <ij> \) by
\( g_{<ij> \Gamma} \), where \( \Gamma \sim \mathbb{Z}_2 \) is the center of the gauge group \( SU(2) \); \( \sigma \) is a 2–form with values in \( \{0, \pi\} \simeq \mathbb{Z}_2 \).

The variables \( U \) and \( \sigma \) are not completely independent: it can be proved [42] that \( e^{i\pi z_c(g)} \), defined in (5.9), is a function of the coset field \( U \), which we denote by \( e^{i\pi z_c(U)} \), and the following constraint holds:

\[
e^{i\pi z_c(U)} = e^{i\pi (d\sigma)_c}
\]  

for every cube \( c \).

The partition function of model C can be rewritten as

\[
Z = \sum_{\sigma} \int dU_{<ij>} \prod_i d\Phi_i \quad e^{\sum_p |\chi|(U_{\partial p}) e^{i\sigma_p} - S_1(U, \Phi) \prod_c \delta(e^{i[\pi z_c(U)-(d\sigma)_c]})}
\]

where

\[
|\chi|(U_{\partial p}) = |\chi(g_{\partial p})|.
\]

The introduction of the disorder field \( D(\Sigma) \) induces a shift of \( \sigma_p \) by \( \Sigma_p \). We perform a duality transformation in the \( \mathbb{Z}_2 \) variable. Let \( \tau \) denote the 1–form with values in \( \{0,1\} \) dual to \( \sigma \). Defining

\[
S_0(U; d\tau) = -\sum_p \text{lnch}[|\chi|(U_{\partial p})] + * (d\tau)_p \text{ltanh}[|\chi|(U_{\partial p})],
\]

the expectation value of ’t Hooft disorder field can be written

\[
\langle D(\Sigma) \rangle = \frac{1}{Z} \sum_{\tau} \int dU_{<ij>} \prod_i d\Phi_i e^{-[S_0(U; d\tau)+S_1(U, \Phi)]} \prod_c e^{i[\pi z_c(U)+(d\Sigma)_c](*\tau)_c}.
\]

To exhibit the wordlines of magnetic currents we perform the abelian projection. We can decompose \( U_{<ij>} \) as in the l.h.s. of (5.7), but with \( \theta_{<ij>} \) restricted to the range \((-\pi, \pi)\).

As a consequence of \( U(1) \)–gauge invariance, we can expand

\[
\int \prod_{<ij>} dC_{<ij>} e^{-[S_1(C, \sigma_3) + S_0(C e^{i\sigma_3 \tilde{\tau}^2} d\tau)]}
\]  

(B.5)
as a Fourier series in $d\theta$. The Fourier coefficients are denoted by $F(n; d\tau)$, where $n$ is an integer valued 2–form.

Defined the 1–form $\ell$ by

$$\delta n = \ell$$

(B.6)

we decompose the 2–form $n$ as

$$n = n[\ell] + \star d\xi$$

where $n[\ell]$ is an integer–valued solution of (B.6) and $\xi$ a $\mathbb{Z}$–valued 1–form in the dual lattice. Furthermore, we define a $\mathbb{Z}/2$–valued 1–form $\alpha$ in the dual lattice by

$$\alpha = \xi + \frac{1}{2}\tau$$

(B.7)

and we adopt the notation:

$$F(n[\ell] + \star d\xi; d\tau) \equiv F(n[\ell]|\star d\alpha)$$

(B.8)

Making use of eq.(5.10), we obtain

$$\langle D(\Sigma) \rangle = \frac{1}{Z} \sum_{[\alpha]} \sum_{\ell,\delta\ell = 0} F(n[\ell]|\star d\alpha)$$

\[
\int \prod_{<ij>} d\theta_{<ij>} e^{i(\star \alpha, 2d\Sigma + 2\pi m(\theta))} e^{i(\theta, \ell)}
\]

(B.9)

where $[\alpha]$ denotes a gauge equivalence class of $\alpha$.

In eq.(B.9) we can replace $\alpha$ by a real–valued 1–form $A$ by inserting the term $\sum_{\rho; \delta\rho = 0} e^{i(A,\rho)}$, where $\rho$ is a $4\pi\mathbb{Z}$–valued 1–form.

We split $m(\theta)$ into a component of even magnetic charge $m_e(\theta)$ and a component of odd magnetic charge $m_o(\theta)$. Shifting $\rho$ by $2\pi m_e(\theta)$, we obtain the following identity:

$$\langle D_\omega(j_x, j_y) \rangle = \frac{1}{Z} \sum_{\ell,\delta\ell = 0} \int d[A] \sum_{\rho; \delta\rho = 0} F(n[\ell]|\star dA) \int \prod_{<ij>} d\theta_{<ij>} e^{i(\theta, \ell)} e^{i(A, 2j_x + 2j_y + 2\omega + \rho + 2\pi m_o(\theta) + 4\pi \delta_{R_+, R_-})}$$

(B.10)

where $d[A]$ denotes formal integration over gauge equivalence classes of $A$. In eq.(B.10) worldlines of ’t Hooft–Polyakov monopoles are described by $2\omega + \rho$ and
they exhibit sources at \( \{x\} \) and \( \{y\} \). Analogously \( 2\pi m_o \) describe the virtual trajectories of \( \mathbb{Z}_2 \) monopoles. The representation (B.10) shows also explicitely the independence of the choice of \( \omega \) in the construction of Green functions of ’t Hooft–Polyakov monopoles.

Taking \( \omega \) with values in \( \{0, \pi\} \), a similar representation shows that our construction of Green functions cannot be adapted to \( \mathbb{Z}_2 \)–monopoles, because a change in \( \omega \) cannot be reabsorbed by field redefinition.

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References

[1] P.A.M. Dirac, Can. J. Phys. 33, 650 (1955).
[2] J. Fröhlich, P.A. Marchetti in “The Algebraic Theory of Superselection Sectors. Introduction and Recent Results”. D. Kastler ed., World Scientific 1990.
[3] J. Fröhlich, P.A. Marchetti, Europhys. Lett. 2, 933 (1986) and unpublished notes.
[4] N. Seiberg, E. Witten, Nucl. Phys. B426, 19; B431, 484 (1994).
[5] G. ’t Hooft, Nucl. Phys. B79, 276 (1984); A.M. Polyakov JETP Lett. 20, 194 (1974).
[6] B. Julia, A. Zee, Phys. Rev. D11, 2227 (1975).
[7] A.M. Polyakov, Nucl. Phys.B120; 429 (1977).
[8] H.A. Kramers, G.H. Wannier, Phys. Rev. 60, 252 (1941); F. Wegner, J. Math. Phys. 12, 2259 (1971); G. ’t Hooft, Nucl. Phys. B138, 1 (1978).
[9] S. Mandelstam, Ann. Phys. 19, 1 (1962).
[10] G. ’t Hooft, Nucl. Phys. B138, 1 (1978); Nucl. Phys. B153, 141 (1979).
[11] J. L. Cardy, E. Rabinovici, Nucl. Phys. B205, 1 (1982); J.L. Cardy, Nucl. Phys. B205, 17 (1982).
[12] J. Fröhlich, T. Spencer, Commun Math. Phys. 83, 411 (1982).
[13] A. Guth, Phys. Rev. D21, 2291 (1980).
[14] D. Brydges, E. Seiler, Journal Stat. Phys. 42, 405 (1986).
[15] E. Seiler, “Gauge Theories as a Problem in Constructive Quantum Field Theory and Statistical Mechanics”, Lecture Notes in Physics 159, (1982).
[16] see e.g. C. Borgs, F. Nill, “The Phase Diagram of Abelian Lattice Higgs Model; A Review of Rigorous Results”. Preprint ETH 87–0069; J. Bricmont, J. Fröhlich, Nucl. Phys. B230, 407 (1984).
[17] J. Fröhlich, P.A. Marchetti, Commun. Math. Phys. 121, 177 (1989); V. Müller Z. Phys. C51, 665 (1991).
[18] R. Brower et al., Phys. Rev. D25, 3319 (1982).
[19] J. Fröhlich, P.A. Marchetti, Commun. Math. Phys. 112, 343 (1987).
[20] J. Fröhlich, P.A. Marchetti, Commun. Math. Phys. 116, 127 (1988).
[21] J. Fröhlich, P.A. Marchetti, Commun. Math. Phys. 121, 177 (1989).
[22] E.C. Marino, B. Schroer, J.A. Swieca, Nucl. Phys. B200, [FS 14], 472, (1982); K. Fredenhaegen, M. Marcu, Commun. Math. Phys. 92, 81 (1983); K. Szlachanyi, Commun. Math. Phys. 108, 319 (1987); J. Fröhlich, P.A. Marchetti, Lett. Math. Phys. 16, 347 (1988); Nucl. Phys. B355, 1 (1990); P.A. Marchetti, Europhys. Lett. 4, 633 (1987).
[23] J. Bricmont, J. Fröhlich, Nucl. Phys. B251, [FS 13], 517 (1985); Commun. Math. Phys. 98, 553 (1985); P.A. Marchetti, Commun. Math. Phys. 117, 501 (1988).
[24] J. Fröhlich in “Scaling and Self–similarity in Physics”, J. Fröhlich ed., “Progress in Physics”, Birkauser, 1983.
[25] T. Kennedy, C.King, Phys. Rev. Lett. 55, 776 (1985); Commun. Math. Phys. 104, 345 (1986).
[26] D. Buchholz, Commun. Math. Phys. 85, 49 (1982).
[27] L. Polley, U.J. Wiese, Nucl. Phys.B356, 629 (1991); M.I. Polikarpov, L. Polley, U.J. Wiese, Phys. Lett. B253, 212 (1991); L. Del Debbio, A. di Giacomo, G. Paffuti, Phys. Lett. B349, 513 (1995).
[28] S. Coleman in “The Unity of the Fundamental Interactions” (Erice 1981), A. Zichichi ed., Plenum Press, 1983.
[29] D. Durhuus, J. Fröhlich, Commun. Math. Phys. 75, 103 (1988).
[30] D. Marenduzzo, Tesi di laurea, University of Padova.
[31] J. Fröhlich in “Unified Theories of Elementary Particles. Critical Assessments and Prospects”. P. Breitenhohner, H.P. Durr eds., Springer Lecture Notes in Physics vol. 160 (1982).
[32] D. Brydges, J. Fröhlich, T. Spencer, Commun. Math. Phys. 83, 123 (1982).
[33] L. Del Debbio, A. Di Giacomo, G. Paffuti, L. Pieri, Phys. Lett. B355, 255 (1995); A. Di Giacomo, G. Paffuti, Phys. Rev. D 56, 6816 (1997); A. I. Veselov, M.I. Polikarpov, M.N. Chernodub, JETP Letters 63, 411 (1996).
[34] E. Witten, Phys. Lett. B86, 283 (1979).
[35] A. Jaffe, C. Taubes “Vortices and Monopoles. Structure of a static gauge theories”. Progress in Phys. 2; A. Jaffe, D. Ruelle eds., Birkhauser 1980.

[36] G. ’t Hooft, Nucl. Phys. B120, 455 (1981).

[37] F. Englert in “Hadron Structure and Lepton–Hadron Interactions” (Cargese 1977) M. Levy et al. eds., Plenum Press, 1979.

[38] A.S. Kronfeld, G. Schierholz, U.J. Wiese, Nucl. Phys. B293, 461 (1987).

[39] A.S. Kronfeld, M.L. Laursen, G. Schierholz, U.J. Wiese, Phys. Lett. B198, 516 (1987).

[40] A.S. Cattaneo et al., “Four–Dimensional Yang–Mills theory as a Deformation of Topological BF theory”; hep–th/9705123, to appear in Commun. Math. Phys.

[41] B. Julia, A. Zee, Phys. Rev. D11, 2227 (1975).

[42] G. Mack, V.B. Petkova, Z. Phys. C12, 177 (1982).
Figures
(One space dimension is suppressed)
Fig. 2.

Fig. 3.