A mechanical analog of Bohr’s atom based on de Broglie’s double-solution approach

P. Jamet and A. Drezet

Univ. Grenoble Alpes, CNRS, Grenoble INP, Institut Neel, F-38000 Grenoble, France

(*Electronic mail: aurelien.drezet@neel.cnrs.fr)

(*Electronic mail: pierre.jamet@neel.cnrs.fr)

(Dated: October 28, 2021)

Motivated by recent developments of hydrodynamical quantum mechanical analogs [J. W. M. Bush, Annu. Rev. Fluid Mech. 47, 269–292 (2015)] we provide a relativistic model for a classical particle coupled to a scalar wave-field through a holonomic constraint. In presence of an external Coulomb field we define a regime where the particle is guided by the wave in a way similar to the old de Broglie phase-wave proposal. Moreover, this dualistic mechanical analog of the quantum theory is reminiscent of the double-solution approach suggested by de Broglie in 1927 and is able to reproduce the Bohr-Sommerfeld semiclassical quantization formula for an electron moving in a atom.

The old atomic model proposed by Bohr for a single electron orbiting in a Coulombian potential constitutes a paradigmatic example of paradoxical physics conflicting with classical intuitions. Here, modifying a old proposal made by de Broglie to explain orbital quantization using a pilot-wave guiding the particle, we develop a realistic model to make sense of Bohr’s theory. Our approach considers a classical particle nonlinearly coupled to a scalar wave-field through a holonomic constraint. In presence of an external Coulomb field we define a regime where the particle is guided by the wave and reproduces the well known Bohr-Sommerfeld quantization rule for circular orbits.

I. INTRODUCTION

One of the most remarkable feature of quantum mechanics is the prediction of stable electronic motions in atoms. Nowadays, and as every physics students knows, atomic orbitals are easily computed by solving the stationary Schrödinger’s equation in a central Coulomb potential. However, in the early times of the quantum era Bohr (following results by Planck, Einstein, Nicholson and Sommerfeld) already obtained a semiclassical model of an electron circular orbit in the Hydrogen atom. The method of Bohr, which was subsequently generalized by Sommerfeld to elliptical and relativistic motions starts with the quantization of the action variable \( J = \oint P dq = 2\pi n\hbar \) (with \( P \) and \( q \) two conjugate canonical variables, \( n \in \mathbb{N} \) and \( \hbar \) the reduced Planck constant) calculated along a periodic motion. However, despite tremendous success and theoretical agreements in reproducing spectrometry experiments with various atoms the Bohr-Sommerfeld model seated on an unclear basis mixing elements of classical mechanics with unjustified quantization rules. As we know, progress in the physical understanding started when de Broglie added a circulatory wave propagation going along with the particle orbital motion (the idea to introduce oscillations of an electric or mechanical medium to explain atomic spectra was originally proposed by Nicholson and Brillouin). In this approach, the momentum \( P \) of the particle is associated with the wave vector \( k \) of the wave through the formula \( P = \hbar k \) and the quantization \( J = 2\pi n \) becomes a stationary condition for the phase \( \phi \) of the wave around the closed orbit: \( \Delta \phi = \frac{1}{\hbar} kdq = 2\pi n \). De Broglie was strongly motivated by the analogy existing between Fermat’s principle in optics and Maupertuis’s least-action theorem in classical mechanics (an analogy already exploited by Hamilton and Jacobi). Generally, the textbook historical explanations concerning de Broglie’s method stop here and the mathematical development goes then with the more precise Schrödinger equation propagating in the configuration space and thereby abandoning the dualistic association of a wave with a particle trajectory advocated by de Broglie. Following the work of Born, the wave ultimately becomes probabilistic and the motion of the particle in space-time completely disappears from the quantum formalism. Nevertheless, de Broglie didn’t agree with these developments and tried to obtain a more physical and deterministic interpretation of the wave mechanics in which particles and waves move together in space-time. In particular, he and later Bohm developed a ‘pilot-wave interpretation’ of quantum mechanics which is empirically equivalent to the standard quantum approach.

In the last decades interest for the pioneering mechanical modeling of de Broglie resurrected with the development of new fluid mechanical analogs of quantum mechanics by Couder and coworkers (for reviews see) in which a particle droplet bouncing on a vibrating oil bath reproduces several paradigmatic features of quantum mechanics such as the wave-particle duality in double slit experiments, the quantum tunneling effect and most importantly for us: Stationary quantized states in external potentials. These works are remarkable by their analogy with an early proposal by de Broglie named ‘double solution’ in which a wave field \( u(t, \mathbf{x}) \) propagating in the usual 4D space-time guides a field singularity acting as a particle and synchronized with the \( u \)-wave during its motion (see also for recent quantum hydrodynamical models similar to de Broglie’s proposal).

In a previous article, based on early proposals by Boudaoud et al. we developed a mechanical analog of the double solution proposal by using the motion of a sliding
bead, i.e., a ‘particle’, on a vibrating string to model wave-particle duality. In this approach the transverse wave $u(t,x)$ propagating along the $x$ direction carries the particle motion $x_p(t)$ along the same direction as the wave. A phase matching condition between the sliding particle and the wave is leading to a quantum-like guidance of the particle by the wave in a way reminiscent of the so-called de Broglie-Bohm quantum interpretation. This first model ideally reproduces some features of the phase-wave introduced by de Broglie in 1923 at least for the simple case corresponding to the linear and uniform motion as it was first considered by de Broglie.

In the present work we extend the previous 1D model and propose a more sophisticated approach with a complex wave field $u(t,x) \in \mathbb{C}$ propagating in the 3 dimensional space. The model takes into account the presence of an external Coulomb field acting on the charged particle with trajectory $x_p(t)$. This proposal is built in a fully covariant and relativistic framework which agrees with the original methodology proposed by de Broglie. In turn, this allows us to study the quantized circular motion of a relativistic particle in the Coulomb field and allows us to reproduce the well-known Bohr-Sommerfeld quantization formula for this Hydrogen-like atom. We stress that, like in de Broglie’s double solution, our model reveals the fundamental role played by a group and phase contributions in the $u$-wave. In turn, this decomposition imposes strong constraints on the physical properties of the particle, i.e., its orbital motion but also its mass and electric charge.

The layout of this work is as follow: In Sec. II after a short reminder concerning de Broglie original idea about phase-waves we describe our relativistic model starting from a covariant Lagrangian formulation. In Sec. II we discuss the solution of our systems of equations leading to an entangled dynamics between particle and wave (the particle being guided by the wave). We show how to recover the Bohr-Sommerfeld quantization approach, i.e., at least for circular orbits, and subsequently discuss the constraints and limitations of our model in Sec. IV.

II. THE RELATIVISTIC ATOM

A. The historical de Broglie’s derivation of the Bohr-Sommerfeld quantization condition

In 1923-1924, Louis de Broglie proposed a model in order to reproduce the circular uniform motion of an electron in an atom, which in turn explained the famous Bohr quantization hypothesis.

$$L_c = nh$$

with $n \in \mathbb{N}$ and $L_c = m_e v_e r_e$ the orbital angular momentum of the electron for an orbit of radius $r_e$ and velocity $v_e$.

To understand de Broglie’s insight let us consider an electron of mass $m_e$ orbiting with constant velocity $v_e$ around a nucleus. de Broglie’s idea, centered around the notion of wave-particle duality, was first to match the relativistic rest energy of the electron $m_e c^2$ with an oscillatory energy $\hbar \omega_c$, i.e., a local ‘clock’, so that the electron undergoes an internal motion of the form

$$e^{-i\omega_c \tau} = e^{-i \omega'_c t}$$

with $\omega'_c = \omega_c \sqrt{1 - v^2 / c^2}$ the frequency of the electron as seen by an external observer in the laboratory reference frame (here $\tau$ is the proper time associated with the internal clock and $t = \sqrt{\frac{v}{\sqrt{1 - v^2 / c^2}}}$ is the relativistically dilated time interval in the laboratory frame where the clock is moving at a constant speed $v$). Then, he introduced a phase wave on the electron’s path

$$e^{-i\omega(t-x_0/v)}.$$ (3)

with $x = r_e \varphi$ a coordinate along the circular orbit ($\varphi$ is the azimuthal angle in the plane of the orbit). In vacuum this phase wave has a velocity $v_\varphi = c^2 / v_c > v_c$, so that it will catch up with the electron after a time $\delta t$:

$$v_\varphi \delta t = L + v_c \delta t$$ (4)

with $L$ the length of the orbit. We find that this time $\delta t$ is

$$\delta t = \frac{L}{v_\varphi - v_c} = \frac{v_c L}{c^2} \left( 1 - \frac{v^2}{c^2} \right).$$ (5)

De Broglie introduced his famous ‘phase-harmony’ condition telling that the phase wave and the internal oscillation of the electron must be locked in phase in order to have a stable motion. On top of that, these two phases must be multiples of $2\pi$, since we are on a circular path and we have to impose a periodicity condition:

$$\omega'_c \delta t = \frac{m_e c^2}{\hbar} \left( 1 - \frac{v^2}{c^2} \right) = 2\pi n$$ (6)

Now given that $L = 2\pi r_e$, and $P_\varphi = m_e v_r / \sqrt{1 - v^2 / c^2}$ the relativistic linear momentum, we finally get:

$$2\pi P_\varphi r_e = \oint P_\varphi dx = 2\pi L_c = nh$$ (7)

Thus we get back Bohr’s quantization condition.

This derivation is sketchy since de Broglie originally assumed no external potential acting on the particle. The model actually corresponds to the case of a particle constrained to move along a closed loop. In a more realistic case the particle is moving along a circular orbit in a central potential $U(r)$. The phase/action velocity is given by

$$v_\varphi = \frac{E_e}{P_\varphi} = \frac{c^2}{v_e} \frac{E_e}{E_e - U}$$ (8)

where

$$P_\varphi = \frac{m_e v_e}{\sqrt{1 - v^2 / c^2}}, \quad E_e = \frac{m_e c^2}{\sqrt{1 - v^2 / c^2}} + U(r)$$ (9)
are the particle linear momentum and total energy respectively. The particle velocity is given by Hamilton’s equation

\[ v_c = \frac{\partial E_c}{\partial P_c} = \frac{c}{E_c} - \frac{P_c}{E_c - U} \]  

(10)

which is subsequently identified with Rayleigh’s group velocity \( v_g = \frac{\partial \omega}{\partial k} \) using the iconic quantum relations \( P_c = \hbar k \), \( E_c = \hbar \omega \). Moreover, the phase matching condition Eq. [4] still holds and instead of Eqs. [6,7] we get:

\[-L_c \delta t = P_c dx = 2\pi L_c = nh \]

(11)

which again recovers Bohr’s quantization condition. In this formula \( L_c = P_c v_c - E_c = -m_e \sqrt{1 - v_c^2/c^2} - U \) is the Lagrangian of the particle and \(-L_c/\hbar\) plays the role of the clock frequency \( \omega_c = \sqrt{1 - v_c^2/c^2} \) used in the original de Broglie deduction.

This idea of a moving clock synchronized with a guiding wave is the hallmark of de Broglie’s conception of quantum mechanics. If we generally write \( S_c(t) = \int_0^t dt' L_c(t') \) the action integral along the trajectory \( C \) followed by the particle the phase harmony condition of de Broglie reads

\[ \frac{d}{dt} S_c = L_c = \hbar \frac{d}{dt} \varphi \]

(12)

where \( \hbar \frac{d}{dt} \varphi := -\hbar \omega_c(t) = -\hbar \omega_c(t) \sqrt{1 - v_c^2/c^2} \) generally defines a time dependent frequency \( \omega_c(t) \) along the path.

In the following years after his PhD thesis de Broglie tried to make sense of his phase-wave hypothesis. As the name suggests, this generally superluminal wave is not strictly a physical object and does not carry any energy, but is rather a clever way to make sense of quantum features such as wave-particle duality and quantization conditions. His goal was to build a more realistic mechanical model using a ‘physical’ wave \( u(t,\mathbf{x}) \) that one could interpret as being the particle. He called ‘double solution’ his new proposal [26] in which a wave-field \( u \) carrying the particle energy coexisted with the more conventional \( \psi \)-field used in quantum (wave) mechanics. In the later sections of this paper, we propose a physical model for this de Broglie wave using a complex scalar field \( u(t,\mathbf{x}) \) which we can decompose as a group-wave carrying the energy (i.e., the particle), and a phase-wave which is actually predating the \( \psi \)-wave solution of Schrödinger’s equation.

B. A relativistic atomic model coupling a wave-field and a particle

In this section, we will derive the equations of motion of our model using natural units \((c = \hbar = 1)\) for simplicity, and reintroduce the physical constants when necessary. Furthermore, we consider the Minkowski metric \( \eta_{\mu,\nu} \) with signature \((1, -1, -1, -1)\) in the following. We start with the relativistic action

\[ I = -\int \left[ m_p \left( |z(t)|^2 - \Omega_p^2 |z(t)|^2 \right) \right] d\tau \]

\[ + \left\{ -\mathcal{N}(\tau) [z(t) - u(x_p(t))] \right\}^* \]

\[ + \mathcal{N}^*(\tau) [z(t) - u(x_p(t))] \right\} d\tau \]

\[ - e \int A(x_p(t)) \xi_p(t) d\tau + T \int (Du)^2 + \left( i e A(x_p(t)) \right)^2 d^4x \]

(13)

Let us specify what each of these terms represent. The first term

\[ -\int \left[ m_p \left( |z(t)|^2 - \Omega_p^2 |z(t)|^2 \right) \right] d\tau \]

(14)

which defines a time dependent frequency \( \omega_c(t) \) along the path.

In the following years after his PhD thesis de Broglie tried to make sense of his phase-wave hypothesis. As the name suggests, this generally superluminal wave is not strictly a physical object and does not carry any energy, but is rather a clever way to make sense of quantum features such as wave-particle duality and quantization conditions. His goal was to build a more realistic mechanical model using a ‘physical’ wave \( u(t,\mathbf{x}) \) that one could interpret as being the particle. He called ‘double solution’ his new proposal [26] in which a wave-field \( u \) carrying the particle energy coexisted with the more conventional \( \psi \)-field used in quantum (wave) mechanics. In the later sections of this paper, we propose a physical model for this de Broglie wave using a complex scalar field \( u(t,\mathbf{x}) \) which we can decompose as a group-wave carrying the energy (i.e., the particle), and a phase-wave which is actually predating the \( \psi \)-wave solution of Schrödinger’s equation.

In this section, we will derive the equations of motion of our model using natural units \((c = \hbar = 1)\) for simplicity, and reintroduce the physical constants when necessary. Furthermore, we consider the Minkowski metric \( \eta_{\mu,\nu} \) with signature \((1, -1, -1, -1)\) in the following. We start with the relativistic action

\[ I = -\int \left[ m_p \left( |z(t)|^2 - \Omega_p^2 |z(t)|^2 \right) \right] d\tau \]

\[ + \left\{ -\mathcal{N}(\tau) [z(t) - u(x_p(t))] \right\}^* \]

\[ + \mathcal{N}^*(\tau) [z(t) - u(x_p(t))] \right\} d\tau \]

\[ - e \int A(x_p(t)) \xi_p(t) d\tau + T \int (Du)^2 + \left( i e A(x_p(t)) \right)^2 d^4x \]

(13)

Let us specify what each of these terms represent. The first term

\[ -\int \left[ m_p \left( |z(t)|^2 - \Omega_p^2 |z(t)|^2 \right) \right] d\tau \]

(14)

which defines a time dependent frequency \( \omega_c(t) \) along the path.
The $u-$ field could thus be seen as the transverse motion along the $z$ (vertical) direction of the membrane and the point on the surface is labeled by 2D coordinates $x,y$. This mechanical analogy makes sense for small transverse vibrations (i.e. like in the 1D mechanical analog of 28). Here, our covariant model works in a 4D space-time and the $u-$ field is complex rather than real but this generalization is not mandatory and only makes the framework more elegant and symmetrical. The internal vibration $z(t)$ is just, if we follow this membrane analogy, the height at which the particle is located on top of the surface, i.e., $z(t) := z_p(t)$. This transverse motion should not be confused with the $x_p(t)$ particle motion which in the 2D analogy is just the set $t,x_p(t),y_p(t)$ describing the in-plane motion of the particle. In 4D it is more judicious to call $z(t)$ an internal motion acting in a different space (also the proper time label $\tau$ helps to make the theory fully covariant). The other variables have also a clear physical meaning in this analogy: For instance, $\Omega_p$ is a mechanical pulsation associated with a vertical restoring force acting upon the particle. Taking the non-relativistic limit of Eq. 14 and using $z(t) := z_p(t)$ we obtain

$$\int [-m_p + \frac{1}{2} m_p x_p(t)^2 + \frac{1}{2} m_p \sigma (\dot{z}_p(t)^2 - \Omega_p^2 z_p(t)^2)] d\tau $$

which is indeed describing the motion of an harmonic oscillator 28. Moreover, the presence of the electric field acting on the particle with charge $e$ can easily be included in this mechanical analogy. Finally, the charge $\dot{e}'$ associated with the oscillating medium is more difficult to physically interpret in the membrane analogy. Still, the $\dot{e}'$ charge together with the complex nature of the $u-$ field allow us to introduce gauge invariance and covariant derivative $D_\mu = \partial_\mu + ie A_\mu (x)$ in the formulation which is fine in the context of a fundamental quantum theory. Moreover, the model developed here is so robust that even the case $\dot{e}' = 0$ can be used to model a Bohr atom as we will see in Sec. IIIIC. In such a case the membrane mechanical analogy is perfectly valid and could be used for developing a possible experimental demonstrator.

Now going back to our model, we can see that obtaining the equations of motion is straightforward using the Euler-Lagrange equations (the full derivation of which can be found in the appendix). We choose to consider cases where there is no longer any interaction between the particle and the field, i.e.

$$\mathcal{N}(\tau) = 0, \mathcal{N}^*(\tau) = 0,$$

as was motivated in 28. This regime, hereafter referred to as transparency, strongly simplifies the dynamics. The goal here is to find stable solutions for the motion of the particle, and construct a wave-particle duality model. We can ultimately turn back to investigating the chaotic regimes where the field and the particle start interacting with each other, and as such consider dynamical cases like, for example, atomic transitions and photon emissions: This will be the subject of future works.

In this transparency regime we get the following equations: First we obtain the condition

$$z(\tau) - u(t,x_p) = 0,$$

which is very general in our theory (i.e., independent of the transparency regime) and models the holonomic constraint that we impose between the field and the particle. This constraint is central in our theory since, as show below, it allows us to recover the phase harmony condition introduced by de Broglie.

Moreover, for $z(\tau)$ we have also the simple equation

$$\ddot{z}(\tau) + \Omega_p^2 z(\tau) = 0$$

which gives us a relativistic harmonic motion of the form

$$z(\tau) = z_0 e^{-i\Omega_p \tau}.$$  

Assuming Eq. 18 and Eq. 20 we deduce the equation of motion for the position of the particle

$$m_p (1 + \sigma \Omega_p^2 |z_0|^2 ) \ddot{x}_p (t) = e F_{\mu \nu} (x_p (t)) x_p^\nu (t)$$

where $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic tensor. This equation is completely analogous to Newton’s second principle, where the Lorentz force accelerating the particle comes directly from the applied external electromagnetic field (the standard Lorentz force involved in Eq. 21 reads $F = e (E + v_p \times B)$), with the introduction of an effective and constant mass term $m_{eff} := m_p (1 + \sigma \Omega_p^2 |z_0|^2 )$ to account for the oscillatory motion $z(\tau)$. We mention that if we leave the transparency regime this effective mass is generally varying with time along the trajectory. Furthermore, in this general regime we have an additional contributing force in the right hand side of Eq. 21 reading $\mathcal{N}^* \partial_\mu u^\mu (x_p) + \mathcal{N} \partial_\mu u (x_p)$. This force depends on the $u-$field gradient leading to corrections on the classical-like equation of motion Eq. 21 (the dynamics is derived in the Appendix).

We emphasize that in the transparency regime (where Eq. 21 holds true) the dynamics is also derived from the effective action:

$$S_{eff} = \int [-m_{eff} - e A (x_p (\tau)) x_p (\tau)] d\tau.$$  

Finally, for the $u-$field we have a d’Alembert-like equation with covariant derivatives

$$D^2 u(x) = 0.$$
It is interesting to note that if we leave the transparency regime
\( \mathcal{N} = 0 \) Eq. 23 becomes
\[
D^2 u(x) + \omega_0^2 u(x) = 0,
\]
where \( \omega_0 \) is the 'Compton' frequency usually acting as a mass.

For the present work exploiting the continuous spectrum of the wave equation the condition \( \omega_0 = 0 \) is the simplest choice (as shown in Sec. III C). This makes sense since in our model the frequency \( \omega_0 \) is not identified with the particle mass but rather characterizes the \( u \)-field. Moreover, for generality we mention that if we add in the action \( I \) a term \( -T \int \omega_0^2 u(t) \delta^4(x-x_0) \) we obtain the field equation:
\[
D^2 u(x) + \omega_0^2 u(x) = - \frac{1}{T} \int d\tau \mathcal{N}(\tau) \delta^4(x-x_0),
\]
which transforms the term inside the cosine function of Eq. 29 into
\[
(\omega + \epsilon)[r \varphi - k - \eta \omega / \omega + \epsilon], \quad \epsilon = \epsilon_+ + \epsilon_-, \quad \eta = \epsilon_+ - \epsilon_-.
\]

The term in front of \( t \) is the group velocity \( v_g \) of our total wave, and following de Broglie’s atomic model, we identify it with the particle velocity
\[
v_p := k - \eta \omega / \omega + \epsilon := v_p := \frac{P_p - eA_\varphi}{E_p - eV}.
\]
where \( P_p, E_p \) are the particle linear momentum and energy respectively. Here, in the definition of the particle velocity \( v_p \) we include contributions of the external field \( A^\mu := [V,A] \). In this work we only consider the scalar Coulomb field \( A^0 := V(r) \) and remove the magnetic vector potential \( A \). We only stress that we can introduce an azimuthal magnetic vector potential \( A_\varphi = A_\varphi(r) \hat{\varphi} \) to describe the Zeeman effect for our 'electron' motion in the atom (this will be developed in a subsequent work). With this hypothesis we see that if we compute the wave-field at the position \( x = x_p(t) \) of the particle at time \( t \) we obtain
\[
u(t, x_p(t)) = u_0 e^{i(kr_p - \omega t)}.
\]
showing that the wave amplitude remains constant at the position of the particle and that its phase is a linear function of \( t \). Moreover, according to the original de Broglie hypothesis we have \( P_p = k \) and \( E_p = \omega \). Yet, as Eq. 34 suggests it appears judicious to assume more general relations:
\[
P_p = bk, \quad E_p = b\omega
\]
where \( b \) is a dimensionless constant.

Furthermore, we impose \( \eta = 0 \) and \( A_\varphi = 0 \) in Eq. 33 and therefore
\[
be = b \varepsilon_\pm = -eV(r_p)
\]
where \( V(r_p) = \frac{e}{4\pi r_p} = \frac{|e|}{4\pi r_p} \) is the external electric Coulomb potential acting upon the moving particle of charge \( e = -|e| \).
We will justify the consistency of Eqs. [33] [36] more rigorously once we derive the dynamics of the particle in Sec. [III B] but it is important to note that we still work with a wave-particle duality in mind, which is why Eqs. [33] [36] are so crucial to our model.

Moreover, being on a circular path, we have from Eq. [28] quantization conditions which impose

\[ P_p = bk = b \frac{m_+ - m_-}{2r_p}, \]

\[ E_p = b \omega = b \frac{m_+ + m_-}{2r_p} + eV. \]  

(37)

For convenience we introduce in the following the notations

\[ n = b \frac{m_+ - m_-}{2}, \quad N = b \frac{m_+ + m_-}{2}, \]  

(38)

i.e.,

\[ bm_\pm = N \pm n. \]

(39)

Importantly, the first equality of Eq. [37] reads

\[ 2\pi r_p P_p = \oint P_p dx = 2\pi n \]  

(40)

(with \( dx = r_p d\varphi \)) which is clearly reminiscent of the Bohr-Sommerfeld quantization formula Eq. [7] as described by de Broglie.

Moreover, we now see the importance of the \( b \) constant in Eq. [38]. Indeed, if we select \( b = 1 \) as suggested by de Broglie we obtain \( n = 0, \pm 1/2, \pm 1, \pm 3/2, \ldots \in \mathbb{Z}/2 \) which introduces half-integer numbers in addition to integers usually considered in the Bohr-Sommerfeld semi-classical quantum theory. This might give rise to richer dynamics than those considered historically and one could for example speculate that this is related to the existence of an half-integer spin in quantum mechanics (this issue was already discussed by Sommerfeld and Heisenberg in the old quantum theory) [33]. Moreover, observe that if we select the condition \( b = 2 \) we recover exactly the Bohr-Sommerfeld theory. Therefore, we obtain \( n = 0, \pm 1, \pm 2, \ldots \in \mathbb{Z} \) as it should be. As we will show in Sec. [III B] the choice of the \( b \) parameter is further constrained by the value given to the electric charge \( e = \sqrt{(4\pi\alpha)} \), i.e., as deduced from the particle dynamics.

### B. The quantized particle dynamics

We are now interested in solutions of Eq. [21] for circular motions of the particle. First, the term \( m_\gamma (1 + \sigma \Omega_\gamma^2 |z|^2) \) is constant for a given field and behaves as an effective mass \( m_\gamma \). We also remove most of the components of the electromagnetic tensor \( F_{\mu\nu} \) since we only have a static scalar potential \( A_\gamma(x) = V(x) = \frac{|z|}{4\pi r} \), which gives us in the end

\[ m_\text{eff} \dot{x}_p = -e\nabla V(x_p) = -\frac{\alpha}{r_p^2} \dot{r}_p, \]

(41)

where we introduced the Sommerfeld fine-structure constant \( \alpha = \frac{e^2}{4\pi} \). This equation also reads

\[ \frac{d}{dt} (m_\text{eff} \gamma p_p) = -m_\text{eff} \gamma \frac{v_p^2}{r_p} = -\frac{\alpha}{r_p^2} \dot{r}_p, \]  

(42)

or equivalently

\[ \gamma m_\text{eff} v_p^2 r_p = \alpha, \]  

(43)

with \( \gamma = 1/\sqrt{1 - v_e^2} \) the Lorentz boost factor. Furthermore, in a central potential the orbital angular momentum is a constant of motion and we have

\[ J = \oint P_p dx = 2\pi n, \]  

(44)

and given that \( P_p = \nabla S_\text{eff} = \gamma m_\text{eff} v_p \) (with \( S_\text{eff} \) given by Eq. [22]), this leads to

\[ \gamma m_\text{eff} v_p r = n. \]  

(45)

Here the angular orbital momentum \( n := L_z \) is a parameter that characterizes the orbits, our hypothesis being that the orbits are indeed quantized to recover the Bohr-Sommerfeld quantization rule and therefore \( n \) is expected to be an integer. Without lack of generality we will from now consider \( n \geq 0 \) assuming an anticlockwise motion of the particle along the orbit. With this constraint we have naturally \( m_+ \geq m_- \geq 0 \) in Eq. [38].

In our model this quantization condition must actually match Eq. [40]. Having this in mind, we will add an index \( n \) to all our orbital quantities, while also removing most of the \( p \) indices so as not to clutter the equations.

In the end, we can write the velocity

\[ v_n = \frac{\alpha}{n}, \quad n = 0, 1, 2, \ldots \]  

(46)

leading to the momentum

\[ P_n = m_\text{eff} \left( \frac{\alpha}{n} \right) \sqrt{1 - \frac{\alpha^2}{n^2}}. \]  

(47)

Similarly, we deduce the radius of the orbit

\[ r_n = n^2 a_0 \sqrt{1 - \frac{\alpha^2}{n^2}}. \]  

(48)

with \( a_0 = 1/(m_\gamma \alpha) \) a typical distance, which is identified to an ‘effective’ Bohr radius. Finally, we obtain the energy of our atomic system

\[ E_n = -\partial_j S_\text{eff} = \frac{m_\text{eff}}{\sqrt{1 - \frac{\alpha^2}{n^2}}} - \frac{\alpha}{r_n} = m_\text{eff} \sqrt{1 - \frac{\alpha^2}{n^2}}. \]  

(49)

All the previous formulas require \( \alpha < 1 \) for consistency. Of course, in the non relativistic limit Eq. [49] reduces to \( E_n \approx m_\text{eff} - \frac{m_\text{eff} \alpha^2}{2n^2} \) which is the famous Bohr quantization energy.
spectra.
By combining Eqs. 47 and 49 we obtain
\[ v_n = \frac{P_n}{E_n + \frac{\sigma}{r_n}}, \]
which is identical to Eq. 33 and justifies our identification \( v_n = v_n \) in Sec. II.A. Alternatively, from Eq. 37 we get
\[ P_n = b \sqrt{\frac{n - \alpha}{r_n}}, \quad E_n = b \omega = \frac{N - \alpha}{r_n}. \] (51)

Inversely, we can express \( \omega_\pm \) as functions of physical parameters associated with the particle:
\[ b \omega_\pm = \frac{m_{\text{eff}}}{\sqrt{1 - \frac{\alpha^2}{n^2}}}(1 \pm \frac{\alpha}{n} - \frac{\alpha^2}{n^2}) \] (52)

Note, that from Eq. 52 and assuming \( \omega_\pm \geq 0 \) we deduce the constraint \( \alpha < (\sqrt{5} - 1)/2 \) that is obviously satisfied with \( \alpha \approx 1/137 \).

All these results are particularly interesting, since we recover the known semi-classical formulas, with an added relativistic correction in the term \( \sqrt{1 - \alpha^2/n^2} \), the same correction that appears in Sommerfeld’s extension of Bohr’s atomic model for circular orbits.

Before moving to the rigorous solutions for the \( u \)-field, we have two more important equations of motion for the particle to derive. Let us write the Lagrangian for the particle
\[ \mathcal{L}_n = -m_{\text{eff}} \sqrt{1 - \frac{\alpha^2}{n^2}} + \frac{\alpha}{r_n}. \] (53)

Using a Legendre transform this also reads
\[ \mathcal{L}_n t = (P_n v_n - E_n) t = b(kv_n - \omega) t. \] (54)

In the end, if we inject the Lagrangian \( \mathcal{L}_n \) into the \( u \)-field given by Eq. 34 and use the constraint given by Eq. 18 we get de Broglie’s phase harmony condition
\[ e^{-i \Omega_p \sqrt{1 - \frac{\alpha^2}{n^2}} = u_0 e^{i \frac{2\pi}{N}}}. \] (55)

This condition implies \( u_0 = u_0 \) which is linking the amplitude of the internal vibration with one of the wave. From Eq. 55 we also deduce
\[ b \Omega_p = m_{\text{eff}} \left(1 - \frac{\alpha^2}{n^2} - \frac{\alpha}{n^2} \right) \] (56)

which shows that in the present model the effective mass is quantized and depends on the quantum number \( n \). More precisely, if we consider the fundamental constants of the model \( \Omega_p \) and \( \sigma \) Eq. 56 equivalently reads
\[ |z_0|^2 = \frac{1}{m_{\text{eff}}} \frac{b \Omega_p}{1 - \frac{\alpha^2}{n^2} - m_p}. \] (57)
can be easily checked that with this value of $\alpha$ we still fulfill Eq. (60) by slightly modifying the value of $b \approx 1$. More precisely, writing Eq. (60) as

$$m_{\pm} = \frac{b\hbar^2}{\alpha} \pm \tilde{n},$$

with $\tilde{n} = \frac{m - m_{\pm}}{2} \in \mathbb{N}$ we see that the condition for $b/\alpha$ to be an integer $q$ reads $b = \frac{q}{137 + \xi} \approx \frac{q}{137}$ (1 - $\frac{\xi}{137}$). If we take $q = 137$ we get $b \approx 1 - \frac{\xi}{137} \approx 1 - 3.10^{-4}$. This value is indeed very close from $b = 1$ and therefore in this model $n = b\hbar$ must be an integer up to a fluctuation $\delta n/\tilde{n} = \frac{\xi}{137}$ in order to satisfy Eq. (61). This defines a small deviation with respect to the Bohr and Sommerfeld quantization postulate and therefore shows Eq. (61). This defines a small deviation with respect to the Bohr and Sommerfeld quantization postulate and therefore shows the limitation of our model. Finally, we point out that the case $b \approx 1$ is only the simpler choice. However, similar conclusions concerning Eq. (61) if we consider the integers $q = b/\alpha = 2, 3,...$. In particular, if $1/\alpha \in \mathbb{N}$ and $b \in \mathbb{N}$ we again obtain a selection rule forcing the use of integer values for $n$.

C. Solutions for the $u$-field

In order to solve the wave equation for $u$ in a central potential, we first write it as $u = u_+ + u_-$ with the eigenmodes

$$u_{\pm}(t, x) = \phi_{\pm}(x)e^{-i\omega_{\pm}t}.$$  

(62)

solutions of Eq. (23). The wave equation for a eigenstate reads:

$$\left( \omega_{\pm} + \frac{\beta}{r} \right)^2 \phi_{\pm}(x) = 0.$$  

(63)

where we defined the modified fine structure constant $\beta = \frac{\xi}{2} e^{i\phi} = \frac{\xi}{2}$ taking into account the fact that the charge $e' = \xi e$ associated with the $u$-field is not necessarily identical to $e$ (i.e., $\xi \neq 1$). In spherical coordinates $r, \theta, \phi$, we seek separable eigenmodes having the general form:

$$\phi_{\pm}(x) = A_{\pm} R_{\pm}(r) Y_{l_{\pm}, m_{\pm}}(\theta, \phi).$$  

(64)

where $Y_{l_{\pm}, m_{\pm}}(\theta, \phi)$ is a spherical harmonic function with integer quantum number $l, m$ (i.e., $|m| \leq l$):

$$Y_{l,m}(\theta, \phi) := \ell^{m}_{l}(\cos \theta)\exp{i m \phi}.$$  

(65)

with $\ell^{m}_{l}(\cos \theta)$ an associated Legendre polynomial (the irrelevant normalization constant has been here absorbed in the $A_{\pm}$ constant). For the present problem the radial function $R_{\pm}(r)$ follows the equation

$$\left[ \frac{1}{r^2} \partial_r (r^2 \partial_r) - \ell_{\pm} (\ell_{\pm} + 1) - \frac{\beta^2}{r^2} + \frac{2\beta \omega_{\pm}}{r} + \omega_{\pm}^2 \right] R_{\pm}(r) = 0.$$  

(66)

Using the substitution

$$\ell_{\pm} (\ell_{\pm} + 1) - \beta^2 = \ell'_{\pm} (\ell'_{\pm} + 1)$$  

(67)

we get

$$l'_{\pm} = -\frac{1}{2} + \sqrt{\left( l_{\pm} + \frac{1}{2} \right)^2 - \beta^2}$$  

(68)

(which reduces to $l_{\pm}$ if $\beta = 0$) and equation (66) becomes

$$\left[ \frac{1}{r^2} \partial_r (r^2 \partial_r) - \ell'_{\pm} (\ell'_{\pm} + 1) + \frac{2\beta \omega_{\pm}}{r} + \omega_{\pm}^2 \right] R_{\pm}(r) = 0.$$  

(69)

The solution for the radial function reads

$$R_{\pm}(r) = e^{\omega_{\pm} r} \tilde{M}(\ell'_{\pm} + 1 - i\beta, 2\ell'_{\pm} + 2, -2i\omega_{\pm} r)$$  

(70)

where $M(a, b, z)$ is the Kummer confluent hypergeometric function which is a regular solution of $z^2 \frac{d^2 M}{dz^2} + (b - z) \frac{dM}{dz} - a = 0$.

The asymptotic solution for large values of $r$ is

$$R_{\pm}(r) \approx C_{\pm} \sin \left( \omega_{\pm} r - \frac{\xi}{2} l'_{\pm} + \delta_{\pm} \right) \omega_{\pm} r$$  

(71)

with $\delta_{\pm} = \beta \ln (2\omega_{\pm} r) + \eta_{\pm}, \eta_{\pm} = \arg(\Gamma(l'_{\pm} + 1 - i\beta))$ and $C_{\pm}$ a normalization constant reading:

$$C_{\pm} = \frac{e^\frac{\eta_{\pm}}{2} e^{-\beta \pi /2} \Gamma(2\ell'_{\pm} + 2)}{(2\omega_{\pm})^{\ell'_{\pm}}}. \Gamma(l'_{\pm} + 1 - i\beta).$$  

(72)

Having obtained the eigensolutions for our wave equation, we return to the case of the transparency regime. As we explained, we need a superposition of two counter-propagating modes $u_{\pm}$ in order to reproduce the total field of Eq. (29). Since our two modes can be written as

$$u_{\pm}(t, r) = A_{\pm} R_{\pm}(r) \ell^{m}_{l_{\pm}}(\cos \theta)e^{i(\pm m_{\pm} \phi - \omega_{\pm} t)}$$  

(73)

with $A_{\pm}$ two normalization constants and $m_{\pm} \geq 0$. We have

$$\ell^{m}_{l_{\pm}}(\cos \theta) = (-1)^{m_{\pm}} \frac{l_{\pm}(l_{\pm} - m_{\pm} + 1)!}{(l_{\pm} + m_{\pm})!} \ell^{m}_{l_{\pm}}(\cos \theta).$$  

(74)

with

$$\ell^{m}_{l_{\pm}}(0) = (-1)^{m_{\pm}} \frac{l_{\pm}(l_{\pm} + 1)!}{(l_{\pm} + m_{\pm})!} \ell^{m}_{l_{\pm}}(0)$$  

(75)

if $l_{\pm} + m_{\pm}$ is odd. It follows for a specific orbit in the equatorial plane (with radius $r_n$ and a polar angle $\theta = \pi/2$) that in order to recover Eq. (29) we must impose:

$$A_{\pm} R_{\pm}(r_n) \ell^{m}_{l_{\pm}}(0) = \frac{u_0}{2}.$$  

(76)

These two constants $A_{\pm}$, which are defined for the specific radius $r_n$, will also give us the link between the amplitudes of the field $u$ and the oscillatory motion $z$ of the particle through the constraint:

$$z_0 = \frac{u_0}{2}.$$  

(77)

Having done that and using Eq. (51), the final field on the particle’s orbit is

$$u(t, r_n, \pi/2, \phi) = z_0 e^{i\frac{\beta}{N}(\phi - \frac{\omega_{\pm} t}{N})} \times \cos \left[ \frac{N}{b} \left( r_n \phi - \frac{n}{N} \right) \right]$$  

(78)
which recovers Eq. [29]

Some comments must be done concerning the \( u \)-field solutions obtained so far.

First, observe that the form of the equation for the \( u \)-modes was not very constraining. Indeed, the value of the charge \( e' \) is not specified. In particular, one can consider the chargeless fluid \( e' = 0 \) and the solution \( R_l(r) \) reads now:

\[
R_l(r) = e^{i\omega_0 r/l_\pm} M(l_\pm + 1, 2l_\pm + 2, -2i\omega_0 r)
\]

where

\[
j_l(x) = (-1)^l \left( \frac{1}{\pi} \right)^{l+1/2} \sin^l \left( \frac{\pi x}{2} \right)
\]

are well known spherical Bessel functions. For large value of \( r \) we have

\[
R_l(r) \simeq \frac{1}{(2\omega_\pm)^{1/2}} \frac{2l_\pm + 1}{l_\pm!} \sin \left( \frac{\omega_\pm r - \frac{\pi}{2} l_\pm}{\omega_\pm r} \right),
\]

which agrees with Eq. [71] with \( l'_\pm = l_\pm, \eta_\pm = \delta_\pm = 0 \).

Alternatively, and as briefly alluded to in Sec. [11], in-

\[\text{Figures 2-5).}\]

\[\text{Similarly shown (i.e., red curves) as parametric curves with radius}
\]

\[\text{Figures 1. Parametric representation of the } u \text{-wave along the particle}
\]

\[\text{trajectory in the } x - y \text{ equatorial plane. The particle trajectory for}
\]

\[\text{Figures } n = 1, 2, 3 \text{ are the dashed (gray color) circles with constant radius}
\]

\[\text{Figures given by Eq. [45]. The } u \text{-wave for } n = 1, 2, 3 \text{ are represented (i.e., blue curves) as}
\]

\[\text{parametric curves } x_n(\phi) = R_n(\phi) \cos \phi, y_n(\phi) =
\]

\[\text{with } R_n(\phi) = r_n + \Delta \Re \left[ i(t = r_n + \theta = \frac{\pi}{2}, \phi) \right] \text{ and where } \Delta \text{ is a}
\]

\[\text{constant used for graphical convenience. For comparison, the de Broglie phase guiding wave at}
\]

\[\text{the given time } t = 0 \text{ are similarly shown (i.e., red curves) as parametric curves with}
\]

\[\text{radius } R_n(\phi) = r_n + \Delta_0 \cos (n\phi) \text{ where } \Delta_0 \text{ is here chosen real (see Eq. [76]).}
\]

\[\text{In this figure we imposed } b = 1, \alpha = \beta = 1/3, \omega_0 = 0 \text{ and we used the maximal}
\]

\[\text{values of the quantum numbers } l_\pm = m_\pm \text{ (see also Figs. [25]).}
\]

\[\text{Figure 2. Intensity map of the } u \text{-field in the } x - y \text{ equatorial plane for}
\]

\[\text{the quantum number } n = 1 \text{ (ground state). The arbitrary normalized}
\]

\[\text{field intensity is defined as } I_n(x, y) = |u(t = 0, r, \theta = \frac{\pi}{2}, \phi)|^2 \text{ with } |x = r \cos \phi, y = r \sin \phi|.
\]

\[\text{The parameters and conditions for calculating the}
\]

\[\text{field are the same as for Fig. [1]. The black circle represents the}
\]

\[\text{particle trajectory and the spatial dimensions are normalized to the}
\]

\[\text{Bohr radius value } \alpha_0.
\]

\[\text{The Legendre associated polynomials } P_{\pm m_\pm}^\pm (\cos \theta) \text{ that are involved in Eq. [72]. Indeed, the mathematical structure of}
\]

\[\text{the wave doesn’t constrain very much the choice of the } l_\pm \text{ values}
\]

\[\text{despite the fact that we must have } m_\pm \leq l_\pm. \text{This once again shows that many solutions for the}
\]

\[\text{field are the same as for Fig. [1]. The black circle represents the}
\]

\[\text{particle trajectory and the spatial dimensions are normalized to the}
\]

\[\text{Bohr radius value } \alpha_0.
\]

\[\text{An other related comment about the } u \text{-modes concerns}
\]

\[\text{Figure 2. Intensity map of the } u \text{-field in the } x - y \text{ equatorial plane for}
\]

\[\text{the quantum number } n = 1 \text{ (ground state). The arbitrary normalized}
\]

\[\text{field intensity is defined as } I_n(x, y) = |u(t = 0, r, \theta = \frac{\pi}{2}, \phi)|^2 \text{ with } |x = r \cos \phi, y = r \sin \phi|.
\]

\[\text{The parameters and conditions for calculating the}
\]

\[\text{field are the same as for Fig. [1]. The black circle represents the}
\]

\[\text{particle trajectory and the spatial dimensions are normalized to the}
\]

\[\text{Bohr radius value } \alpha_0.
\]

\[\text{An other related comment about the } u \text{-modes concerns}
\]
A mechanical analog of Bohr’s atom based on de Broglie’s double-solution approach

$P_{m_+}^m (\cos \theta) = (-1)^{m_+} (2m_+ - 1)!! \sin^{m_+} \theta$ and $P_{m_-}^m (\cos \theta) = (2m_- - 1)!! \sin^{m_-} \theta$. Indeed, with this choice the $u-$wave is strongly confined in the equatorial plane $\theta = \pi/2$ containing the orbit for $l \gg 1$. We actually believe or hope that in the high quantum number limit the equatorial plane acts as a dynamical attractor for the particle trajectory in the non-transparent regime $N \neq 0$. In other words, due to the force $-N^* \nabla u + cc.$ acting upon the particle, the strong field gradient in the spatial region $\theta \simeq \pi/2$ is expected to attract the particle in the equatorial plane. Further work on stability is necessary to confirm or not this hypothesis.

To illustrate the complete dynamics we first show in Fig. 1 at the given time $t = 0$ a parametric representation of the propagative $u-$field along the particle trajectory for the three first energy levels $n = 1, 2$ and 3. We compare the total field (blue curves) with the phase field (red curves) used in the paradigmatic de Broglie atomic model. In particular, we see that the total field involves fast oscillations with shorter wavelengths that are associated with the group-wave, i.e., envelope wave, propagation (see Eq. 76). This situation is reminiscent of the analysis already obtained in our previous article for a 1D string mechanical analog. Furthermore, the longer wavelength modulations in the $u-$field (red curves) are associated with the faster-than-light phase-wave of de Broglie in full agreement with the 1D string model. For the illustrations we used the particular conditions $b = 1$, $\alpha = \beta = 1/3$, $\omega_0 = 0$ leading to easy observation of the field modulations. Furthermore, we used the maximal values of the quantum numbers $l \equiv m_\pm$. As explained, this choice is motivated by the semiclassical approximation which in quantum mechanics is described by the Brillouin-Wentzel-Kramers (BWK) theory working for high quantum numbers. Graphically, the $u-$field intensity is actually strongly confined near the particle trajectory as shown in Figs. 2, 3, 4, and 5 in the $x-y$ equatorial plane for $n = 1, 2, 3$ respectively. The same effect occurs in the $x-z$ plane as shown in Fig. 5 for the case $n = 1$ with a doughnut shape for the intensity profile. This analysis shows that already for small quantum numbers the $u-$field is strongly confined near the Bohr-Sommerfeld trajectory. This feature is of course an interesting specificity of our model. More studies are needed to further understand the implications of these results for discussing more complicated atomic motions.
IV. PERSPECTIVES AND CONCLUSIONS

The model proposed in this article is directly motivated by the first ‘phase-harmony’ wave-mechanics proposed by de Broglie between 1923-1925[10]. Both models are dualistic in nature coupling a point-like particle to an extended guiding-wave. The key idea of the phase harmony hypothesis is the synchronization between the local clock associated with the particle and the wave-field computed at the position of the particle. In our model, this condition is summarized by the holonomic constraint Eq. 18 [\( \tau = u(\tau_p) \)] locking the phase and amplitude of the \( u \)-wave with those of the internal clock oscillation \( z(\tau) \).

We emphasize that for de Broglie in his early work the physical meaning of the phase-wave was not very clear. The specificity of our model is the introduction of a \( u \)-field having a physical content like the classical electromagnetic field or the gravitational metric in general relativity. This is clearly reminiscent of the double-solution program developed by de Broglie[8,9]. More precisely, in the double solution of de Broglie one postulates the existence of a physical \( u \)-field guiding the particle considered as a localized ‘accident’ in the wave (i.e., a singularity[11]). Our model is more specific than the one proposed by de Broglie. First, there is indeed a phase-wave coming from the sum of two counterpropagating modes \( u_\pm \) in Eq. 29. However, we have also a group wave (the cosine term in Eq. 29) and both are essential for guiding the particle. Indeed, the subluminal group wave guides the particle since the constancy of its amplitude during the particle motion allows us to fulfill one part of the holonomic constraint: \( |z(\tau)| = |u(\tau_p(\tau))| \). The superluminal phase-wave also guides the particle since it fixes the phase-harmony condition: \( \arg [z(\tau)] = \arg [u(\tau_p(\tau))] \), and it allows us to fix the dynamics of the particle in order to recover the Bohr-Sommerfeld quantization formula \( \int p_x \, dx = 2\pi n \). This separation of the \( u \)-wave into a phase and group contributions is thus fundamental in our approach since it explains why we must consider two quantum numbers \( n \) and \( N \gg n \) (see Eq. [38] instead of only one quantum number \( n \) in the old de Broglie theory[12,13]).

Remarkably, our ‘entangled’ wave/particle dynamics, with these two quantum numbers, leads to strong constraints on the particle properties like the effective ‘dressed’ mass \( m_{eff} \) (i.e., Eq. 50), the internal oscillator amplitude \( |z_0| \) (i.e., Eq. 57) which depend on the number \( n \), and the electric charge \( e = \sqrt{4\pi\alpha} \) which depends on both \( n \) and \( N \) through the fine structure constant \( \alpha = \frac{e^2}{\hbar c} \) (see Eq. 58). As we showed in Sec III-B the constraint imposed on the particle charge \( e \) allows us to define a selection rule for the quantum number \( n \) in the regime \( b = 1 \). With this choice we get an interesting relation (i.e., Eq. 61) prohibiting half integer quantum numbers \( n = 1/2, 3/2, \ldots \) in the Bohr-Sommerfeld formula \( \int p_x \, dx = 2\pi n \). Moreover, we stress once more that the value of the \( b \) constant used in our model is not imposed by the theory itself but must be better seen as an initial or boundary condition for the whole coupled system particle-wave.

We also stress that the present theory requires us to find two eingensolutions \( u_\pm (t, \mathbf{x}) = \phi_\pm (x)e^{-i\omega_\pm \mathbf{p} \mathbf{x}} \) obeying a Klein-Gordon wave equation \( D^2 u(x) + \omega_\pm^2 u(x) = 0 \), i.e., Eq. 79 where \( D_\mu = \partial_\mu + ieA_\mu \) depends on a field charge \( e' = \xi e \) that is in general different from the particle charge \( e \). The theory is however not constraining very much the choice of the parameters \( \omega_\pm \) and \( e' \) in Eq. 79. Our main requirement is to be able to find a combination \( \omega = \omega_+ + \omega_0 \) such that we can recover the Bohr-Sommerfeld quantization spectrum \( E_n \) through the equality \( E_n = b \omega_0 \). Moreover, since we are considering the continuous spectrum \( \omega_\pm \geq \omega_0 \) we get from the definition \( E_n \geq b \omega_0 \) and thus with Eq. 49:

\[
E_n = m_{eff} \sqrt{1 - \frac{\omega^2}{c^2}} \geq m_{eff} \sqrt{1 - \frac{\alpha^2}{c^2}} \geq b \omega_0.
\]  

(81)

It is thus always possible to find a particle mass \( m_p \) in the effective mass \( m_{eff} \) of the form \( m_p (1 + \sigma \Omega^2 (|z_0|^2) \) in order to fulfill this condition. This is the case in particular if \( \omega_0 = 0 \) which is also the simplest choice.

It is interesting to watch the problem of the form of the \( u \)-wave equation from a different perspective. Indeed, let us write the field \( u_\pm \) in polar coordinates:

\[
u_\pm (t, \mathbf{r}) = f_\pm (t, \mathbf{r})e^{i\Phi_\pm (t, \mathbf{r})}
\]  

(82)

(with \( f_\pm \) and \( \Phi_\pm \) real) which lets us separate the Klein-Gordon equation (25) into two parts:

\[\partial^\mu [\partial_\mu \Phi_\pm + e'A_\mu] = 0 \]  

(83)

and

\[\partial^\mu [\partial_\mu \Phi_\pm + e'A_\mu] = 0 \]  

(84)

The second equation is reminiscent of the quantum version of the Hamilton-Jacobi equation introduced by de Broglie in his double solution and pilot-wave mechanics[12,13]. The term \( Q_\pm = \frac{\partial^\mu}{\partial f_\pm} \) is called quantum potential and characterizes the difference between the quantum Eq. [84] and the classical equation \( \partial^2 \Phi_\pm + e'A_\mu = 0 \) or in other words the difference between wave mechanics and the Eikonal equation of geometrical optics.

In the present case, with \( \omega_\pm = -\partial \Phi_\pm \) and \( k_\pm = \pm \frac{1}{\alpha} \partial_\Phi_\pm \), we obtain:

\[\omega_\pm + \frac{\beta}{r} - k_\pm^2 = \alpha_0^2 + Q_\pm. \]  

(85)

Moreover, from Eq. [31] we have along the particle orbit \( r = r_n, \theta = \pi/2 \) the condition \( k_\pm = \omega_\pm + \frac{\alpha}{r_n} = \frac{m_\pm}{r_n} \), and by comparing with Eq. [55] we obtain:

\[Q_\pm (t, r_n, \phi, \theta = \pi/2) = \frac{m_\pm}{r_n} - \left( \frac{m_\pm + \beta}{r_n} \right)^2 - \omega_0^2 \]  

(86)

Remarkably, if we impose \( \omega_0 = 0 \) and \( b\beta = \alpha \) (i.e., \( be' = e \)) we obtain rigorously \( Q_\pm (t, r_n, \phi, \theta = \pi/2) = \frac{\partial f_\pm}{\partial f_\pm} = -\frac{\nabla^2 f_\pm}{f_\pm} = 0 \) along the particle trajectory. In other words the quantum Hamilton-Jacobi Eq. [84] reduces to the classical one along the orbit. We believe this is another motivation for the case \( be' = ...
A mechanical analog of Bohr’s atom based on de Broglie’s double-solution approach 12

\[ (\partial + ieA)^2 \Psi(x) = 0 \]  
\( (\partial + ieA)^2 u(x) = 0 \)  

(87)

where \( \epsilon' = \epsilon \).

A last point that we want to briefly comment concerning our model is about causality. Indeed, in order to work our model involves two waves \( u_+ \) specially tuned in order to reproduce the phase matching condition of de Broglie and thereby the Bohr-Sommerfeld quantization formula. Moreover, this specific field \( u = u_+ + u_- \) is in some sense conspiratorial or better ‘superdeterministic’. This issue about superdeterminism has recently been the subject of many interesting discussions in the context of Bell inequality and quantum nonlocality. It is therefore not unreasonable to further study this possibility in order to develop a more sophisticated quantum model using a \( u- \) field. This idea will be developed in a subsequent work in preparation.

To summarize our work, we developed a model for a \( u- \) wave (i.e., solution of Eq. 87) guiding a particle. The guiding dynamics is reminiscent of the old phase-wave introduced by de Broglie in order to justify the Bohr-Sommerfeld quantization formula. This pilot-wave theory, our model for the \( u- \) wave is valid for any integer quantum numbers \( n = 1,2,3, \ldots \) and not only for large integers \( n \gg 1 \) required in the BWK semiclassical approximation of Eq. 89. This shows the limitation of our mechanical analogy of quantum mechanics.

A different interesting feature of our model is the quantization of the constant \( \alpha^{-1} \) which is required in order to satisfy the set of coupled equations. This property is remarkable since it shows that coupling a guiding wave to a particle in order to reproduce quantum mechanics can lead to strong constraints on the physical parameters. This is to be expected since our \( u- \) wave needs interferences and a resonance condition in order to reproduce Bohr’s quantization formula. This feature was ignored in the original phase-wave model of de Broglie where a mechanical description like the one proposed in Sec. II B was missing. Here, we have two quantum numbers \( m_\alpha \) rather than one as it was in the first proposal of de Broglie. This is due to the fact that we need two waves to reproduce the guidance formula of de Broglie. In the end, this constraint on \( \alpha \) has strong physical consequences. If a model like ours has to be taken seriously it apparently implies a strong fine-tuning on the parameters used, which might be related to the freedom we have regarding the parameter \( b \) in our model. We don’t have here an explanation for this fact but it suggests a cosmological explanation perhaps related to some (weak) anthropic principle.

At the same time, the model for the \( u- \) wave demonstrates that it is in principle possible to reproduce some important features of quantum mechanics with a classical and deterministic analogy. Furthermore, unlike in the conventional Copenhagen interpretation where the very notion of an orbit is ill defined, here the particle path is deterministic and continuous in the four-dimensional space-time. We believe this result to be in the direct continuation of early works by de Broglie and more recent ones on hydrodynamic mechanical analogs by Coudre and Bush.

In this context, our theory offers some interesting potentialities in order to develop realistic mechanical demonstrator for the Bohr atomic model. As we explained in Sec. II B, our model is a direct generalization of our previous article where a 1D model is used for a transverse wave propagating along an elastic string and guiding a particle. In Sec. II B, we pointed out that a 2D mechanical analog using a vibrating membrane coupled to a particle could constitute a realistic demonstrator of our model. We believe that other physical analogies could be developed along this direction. For example, we could imagine an hydrodynamic analog with a particle coupled to acoustic waves propagating in a spherical or toroidal tank. Optical traps and tweezers using laser beams are also good candidates. Optical vortices with well defined angular orbital momenta can be nowadays easily generated (e.g., 53,55) at least in 2D. Since our model can easily be developed in 2D space this suggests interesting experimental
developments. In particular, optical traps in liquids have the potentiality, i.e., coupled to small Brownian particles in water, to create stabilized circular orbital motions of particles fulfilling the Bohr-Sommerfeld quantum condition. We believe that all these interesting issues deserve further analysis.

To conclude it is interesting to go back to de Broglie’s double solution research program: In the 1950’s de Broglie returned to his double solution after 25 years. In his new version of the theory he wrote the u-field as

$$u(x) = u_0(x) + v(x)$$  \(\text{(91)}\)

where \(u_0(x)\) was a strongly singular wave associated with the particle and \(v(x)\) a base wave guiding the point-like singularity and solution of a linear wave equation such as the Klein-Gordon or Schrödinger equation. For de Broglie this base wave is proportional to the usual quantum variable \(\Psi\) of quantum mechanics: \(v(x) = C\Psi(x)\) (with \(C\) a constant).

Our model shows strong similarities with this idea since our \(u\)-field indeed guides the particle (acting as a kind of singularity). Moreover, our dynamics is satisfying the action-reaction principle since there is a coupling between the wave and the particle such that if the transparency regime \(\mathcal{N} = 0\) is not satisfied a new \(u\)-field solution of Eq. \(26\), i.e., \(D^2u(x) = -\frac{1}{\hbar} \int d\tau.\mathcal{N}(\tau)\delta^4(x-x_p)\) will be emitted by the particle and this in turn will modify the motion of the point-like singularity. In the transparency regime considered in this article the wave and the particle peacefully ignore each other in order to satisfy a guidance condition which is reminiscent of the pilot-wave interpretation (at least in the semi-classical regime). We believe that the knowledge of the source field emitted if \(\mathcal{N} \neq 0\) could play a role in order to describe optical transitions between the different energy levels of the atom. This requires to include radiation damping due to electromagnetic self-interaction of the moving electron and clearly opens interesting possibilities for future extensions of the present model.

**Appendix A: Derivation of the equations of motion**

We first introduce an affine parameter \(\lambda\) along the particle trajectory such that \(\sqrt{(x')^2} d\lambda = d\tau\) (in the following \(f'\) denotes a derivative \(\frac{df}{d\lambda}\)) and write the action \(\mathcal{L}\) of the whole system:

\[
\mathcal{L}(y,y') = -\left[\frac{m_p \sigma}{2} \left( \frac{|\mathcal{z}'(\lambda)|^2}{(x')^2} - \Omega_p^2 |\mathcal{z}(\lambda)|^2 \right) \right] \sqrt{(x')^2} d\lambda
\]

\[
+ \left\{ \mathcal{N}(\lambda) \right\} [z(\lambda) - u(x_p(\lambda))]^* \\
+ \mathcal{N}^*(\lambda) \left[ \mathcal{z}(\lambda) - u(x_p(\lambda)) \right] \sqrt{(x')^2} d\lambda
\]

\[
- e \int A(x_p(\lambda)) x_p(\lambda) d\lambda + T \int (\mathit{Du})(\mathit{Du})^* d^4x.
\]

(A1)

In order to write the Euler-Lagrange equations for the ‘particle’ variables \(y(\lambda) := [z(\lambda), \mathcal{z}^*(\lambda), \mathcal{N}(\lambda), \mathcal{N}^*(\lambda)\) and \(x(\lambda)\) we write the previous action integral as

$$I = \int d\lambda.\mathcal{L}(y,y') + T \int (\mathit{Du})(\mathit{Du})^* d^4x$$ where the four volume integral \(T \int (\mathit{Du})(\mathit{Du})^* d^4x\) is actually irrelevant since it is independent of \(y\) and \(y'\). The Lagrangian function \(\mathcal{L}(y,y')\) reads:

$$\mathcal{L}(y,y') = -\left[ \frac{m_p \sigma}{2} \left( \frac{|\mathcal{z}'(\lambda)|^2}{(x')^2} - \Omega_p^2 |\mathcal{z}(\lambda)|^2 \right) \right] \sqrt{(x')^2} d\lambda$$

\[
+ \left\{ \mathcal{N}(\lambda) \right\} [z(\lambda) - u(x_p(\lambda))]^* \\
+ \mathcal{N}^*(\lambda) \left[ \mathcal{z}(\lambda) - u(x_p(\lambda)) \right] \sqrt{(x')^2} d\lambda
\]

\[
- eA(x_p(\lambda)) x_p(\lambda) d\lambda + T \int (\mathit{Du})(\mathit{Du})^* d^4x.
\]

(A2)

This allows us to write Euler-Lagrange equations \(\frac{d}{dx} \frac{\partial \mathcal{L}}{\partial y^*} = \frac{\partial \mathcal{L}}{\partial y}\):

- \(i\) For \(\mathcal{N}\) we obtain:

$$\frac{\partial \mathcal{L}}{\partial \mathcal{N}^*} = \sqrt{(x')^2} \left( z(\lambda) - u(x_p(\lambda)) \right) = 0,$$ \(\text{(A3)}\)

which leads to the holonic condition \(z(\lambda) - u(x_p(\lambda)) = 0\) (a similar equation is obtained for the complex conjugate variables).

- \(ii\) For \(z\) we have:

$$\frac{\partial \mathcal{L}}{\partial z^*} = -\frac{m_p \sigma}{2} \left( \Omega_p^2 z(\lambda) + \mathcal{N}(\lambda) \right) \sqrt{(x')^2},$$ \(\text{(A4)}\)

$$\frac{\partial \mathcal{L}}{\partial z} = \frac{m_p \sigma}{2} \left( \frac{z'(\lambda)}{\sqrt{(x')^2}} + \Omega_p^2 z(\lambda) \sqrt{(x')^2} \right) = \mathcal{N}(\lambda) \sqrt{(x')^2}$$ \(\text{(A5)}\)

$$\frac{m_p \sigma}{2} \left( \frac{d}{d\lambda} \left( \frac{z'(\lambda)}{\sqrt{(x')^2}} \right) + \Omega_p^2 z(\lambda) \sqrt{(x')^2} \right) = \mathcal{N}(\lambda) \sqrt{(x')^2}$$ \(\text{(A6)}\)

In particular if we choose \(\lambda = \tau\) where \(\tau\) is the proper time we have \(\sqrt{(x')^2} = 1\) and we obtain

$$m_p \frac{\sigma}{2} \left( \frac{d}{d\tau} z(\tau) + \Omega_p^2 z(\tau) \right) = \mathcal{N}(\tau).$$ \(\text{(A7)}\)

Again, we stress that similar equations are easily obtained for the complex conjugate variables \(z^*, \mathcal{N}^*\).

- \(iii\) Similarly, for \(x_p\) (which is the only real valued variable in the dynamics) we get:

$$\frac{\partial \mathcal{L}}{\partial x_p^*} = - \left\{ \mathcal{N} \frac{\partial \mathcal{N}^*}{\partial x_p} + \mathcal{N}^* \frac{\partial \mathcal{N}}{\partial x_p} \right\} \sqrt{(x')^2} - e \varepsilon \mathcal{A}_x v'$$ \(\text{(A8)}\)

$$\frac{\partial \mathcal{L}}{\partial x_p} = -m_p \left( 1 + \frac{\sigma}{2} \left( \frac{|\mathcal{z}'(\lambda)|^2}{(x')^2} + \Omega_p^2 |\mathcal{z}(\lambda)|^2 \right) \right) \frac{x_p'}{(x')^2}$$

$$- \left( e \varepsilon \mathcal{A}_\mu + \mathcal{N}^*(z - u) + \mathcal{N}(z^* - u^*) \right) \frac{x_p'}{(x')^2}.$$ \(\text{(A9)}\)
Moreover after introducing the holonomic conditions \( z(\lambda) = u(x(\lambda)) \), \( z'(\lambda) = u'(x(\lambda)) \) we deduce:

\[
\frac{m_p}{\sqrt{(\dot{x})^2}} \frac{d}{d\tau} \left( 1 + \frac{\sigma}{2} \left( \frac{\dot{z}(\lambda)^2}{(\dot{x})^2} + \Omega_2^2 |z(\lambda)|^2 \right) \right) \frac{\dot{x}_\mu}{\sqrt{(\dot{x})^2}} = \mathcal{N}^\mu \partial_\mu u + \mathcal{N}^\mu \partial_\mu u^* + eF_{\mu\nu} \dot{x}^\nu
\]

Euler-Lagrange equation.

In particular if \( \lambda = \tau \) we obtain:

\[
\frac{m_p}{\sqrt{\dot{x}}^2} \frac{d}{d\tau} \left( 1 + \frac{\sigma}{2} \left( \frac{\dot{z}(\tau)^2}{\dot{x}(\tau)^2} + \Omega_2^2 |z(\tau)|^2 \right) \right) \frac{d}{d\tau} \dot{x}_\mu(\tau) = \mathcal{N}^\mu \partial_\mu u + \mathcal{N}^\mu \partial_\mu u^* + eF_{\mu\nu} \dot{x}^\nu
\]

\( \text{A11} \)

\(-iV\) For the field variables \( u(x) \) and \( u(x)^* \) we rewrite the action integral as: \( I = \int d^4x \mathcal{L}(u, \partial u, u^*, \partial u^*, x) + \ldots \) where the dots are irrelevant terms independent of \( u, \partial u, u^*, \partial u^* \) variables, and \( \mathcal{L}(u, \partial u, u^*, \partial u^*) \) is the Lagrangian density:

\[
\mathcal{L}(u, \partial u, u^*, \partial u^*, x) = T(Du)(Du)^* + \int (\mathcal{N}(\lambda) [z(\lambda) - u(x_p(\lambda))] ) + \mathcal{N}^*(\lambda) [z(\lambda) - u(x_p(\lambda))] ) \sqrt{(\dot{x})^2} \delta^4(x - x_p(\lambda)) d\lambda
\]

\( \text{A12} \)

where an integral along the trajectory \( C \) of the particle has been included for convenience (it represents an explicit \( x \) dependence in \( \mathcal{L} \)). We deduce

\[
\frac{1}{T} \frac{\partial \mathcal{L}}{\partial \dot{u}^*} = - \int (\mathcal{N}(\lambda) \sqrt{(\dot{x})^2} \delta^4(x - x_p(\lambda)) - ieA_\mu(\partial^\mu + iA^\mu) - u
\]

\( \text{A13} \)

and

\[
\frac{1}{T} \frac{\partial \mathcal{L}}{\partial \partial_\mu u^*} = (\partial^\mu + iA^\mu) u
\]

\( \text{A14} \)

\[
\frac{1}{T} \frac{\partial \mathcal{L}}{\partial \partial_\mu u} = \partial_\mu \partial^\mu u + ieA_\mu(A^\mu u)
\]

\( \text{A15} \)

which leads to the Euler-Lagrange equation:

\[
0 = \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu u^*} - \frac{\partial \mathcal{L}}{\partial \partial_\mu u} = T \left[ \partial_\mu \partial^\mu u + ieA_\mu(A^\mu u) + ieA_\mu(\partial^\mu + iA^\mu) u \right] + \int (\mathcal{N}(\lambda) \sqrt{(\dot{x})^2} \delta^4(x - x_p(\lambda))
\]

\( \text{A16} \)

and in the end we get

\[
D_\mu D^\mu u = - \int (\mathcal{N}(\lambda) \sqrt{(\dot{x})^2} \delta^4(x - x_p(\lambda))
\]

\( \text{A17} \)

The line integral along \( C \) can be written in a simpler form if \( \lambda = \tau \) and we obtain

\[
D_\mu D^\mu u = - \int (\mathcal{N}(\tau) \sqrt{(\dot{x})^2} \delta^4(x - x_p(\tau)).
\]

\( \text{A18} \)

Alternatively, we can use \( \lambda = t' \) where \( t' \) is a laboratory time for the particle and we then obtain

\[
D_\mu D^\mu u(t, x) = - \int (\mathcal{N}(t') \sqrt{1 - v^2(t')} \delta(t - t') \times \delta^3(x - x_p(t')
\]

\( \text{A19} \)

We stress that we can directly introduce a covariant form of the Euler-Lagrange equations and get the same wave equation, by taking \( D_\mu u \) and \( D^\mu u^* \) as our independent variables instead of \( \partial u \) and \( \partial u^* \).

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

REFERENCES

1. E. Schrödinger, "Quantisierung als Eigenwertproblem", Ann. der Phys. (Berlin) 79, 361–376 (1926).
2. N. Bohr, "On the constitution of atoms and molecules", Philosophical Magazine 26, 1–24 (1913). For an historical discussion of the Bohr-Sommerfeld theory see [3].
3. J. Mehra, H. Rechenberg, The historical development of quantum theory, Vol. 1, pt. 1 (Springer, Berlin, 1982).
4. A. Sommerfeld, Atomic structure and spectral lines (Methuen, London, 1923).
5. J. W. Nicholson, "The Spectrum of Nebulium", Monthly Notices of the Royal Astronomical Society, 72, 49-64 (1911).
6. M. Brillouin, "Actions mécaniques à hétéridité discontinue par propagation; essai de théorie dynamique de l'atome à quanta", C. R. Acad. Sci. 168, 1318–1320 (1919).
7. L. de Broglie, "Ondes et quanta" C. R. Acad. Sci. 177, 507–10 (1923).
8. L. de Broglie, Recherches sur la théorie des quanta, Ann. Phys. (Paris) 10, 22-128 (1925).
9. M. Born, "Quantenmechanik der Stossvorgänge", Z. Phys. 38, 803–27 (1926).
10. G. Bacciaguluppi, A. Valentini, Quantum theory at the crossroads: Reconsidering the 1927 Solvay Conference (Cambridge Univ. Press, Cambridge, 2009).
11. D. Bohm, "A suggested interpretation of the quantum theory in terms of "hidden" variables. I.", Phys. Rev. 85, 166–179 (1952).
12. D. Bohm and B. J. Hiley, The unevented Universe (Routledge, London, 1993).
13. J. W. Bush, "The new wave of pilot-wave theory", Phys. Today 68, 47-53 (2015).
14. J. W. Bush, "Pilot-wave hydrodynamics", Annu. Rev. Fluid Mech. 47, 269 (2015).
15. J. W. Bush, A. U. Oza, "Hydrodynamic quantum analogs", Rep. Prog. Phys. 84, 017001 (2020).
16. J. W. Bush, Y. Couder, T. Gilet, P. A. Milewski and A. Nachbin, "Introduction to focus issue on hydrodynamic quantum analogs", Chaos 28, 096601 (2018).
A mechanical analog of Bohr’s atom based on de Broglie’s double-solution approach

17 Y. Couder and E. Fort, “Single-particle diffraction and interference at a macroscopic scale” Phys. Rev. Lett. 97, 154101 (2006).
18 A. Eddi, E. Fort, F. Moisy, and Y. Couder, “Unpredictable tunneling of a classical wave-particle association”, Phys. Rev. Lett. 102, 240401 (2009).
19 A. Nachbin, P. A. Milewski, and J. W. M. Bush, “Tunneling with a hydrodynamic pilot-wave model”, Phys. Rev. Fluids 2, 034801 (2017).
20 E. Fort, A. Eddi, A. Boudaoud, J. Moukhtar, and Y. Couder, “Path-memory induced quantization of classical orbits”, Proc. Natl. Acad. Sci. USA 107, 17515 (2010).
21 D. M. Harris, J. Moukhtar, E. Fort, Y. Couder, and J. W. M. Bush, “Wavelike statistics from pilot-wave dynamics in a circular corral”, Phys. Rev. E 88, 011001(R) (2013).
22 T. Gilet, “Quantumlike statistics of deterministic wave-particle interactions in a circular cavity”, Phys. Rev. E 93, 042202 (2016).
23 T. Shinbrot, “Dynamic pilot wave bound states”, Chaos 29, 113124 (2019).
24 L. de Broglie, “La mécanique ondulatoire et la structure atomique de la matière et du rayonnement”, J. Phys. Radium 8, 225-241 (1927); translated in: L. de Broglie, and L. Brillouin, Selected papers on wave mechanics (Blackie and Son, Glasgow, 1928).
25 L. de Broglie, Une tentative d’interprétation causale et non linéaire de la mécanique ondulatoire: la théorie de la double solution (Gauthier-Villars, Paris 1956); translated in: L. de Broglie, Nonlinear wave mechanics: A causal interpretation (Elsevier, Amsterdam, 1960).
26 V. Dagan, J. W. M. Bush, “Hydrodynamic quantum field theory: the free particle”, Comptes Rendus Mécanique 348, 555-571 (2020).
27 M. Durey, J. W. M. Bush, “Classical pilot-wave dynamics: The free particle”, Chaos 31, 033136 (2021).
28 A. Drezet, P. Jamet, D. Bertschy, A. Ralko, and C. Poulain, “Mechanical analog of quantum bradyons and tachyons”, Phys. Rev. E 102, 052206 (2020).
29 A. Boudaoud, Y. Couder, and M. Ben Amar, “A self-adaptive oscillator”, Eur. J. Phys. B 9, 159-165 (1999).
30 C. Borghesi, “Dualité onde-corps formée par une masselotte oscillante dans un milieu élastique : étude théorique et similitudes quantiques”, Ann. Fond. de Broglie 42, 161-196 (2017).
31 A. Drezet, “The guidance theorem of de Broglie”, Ann. Fond. de Broglie 46, 65-85 (2021).
32 S. Hossenfelder and T. Palmer, “Rethinking Superdeterminism”, Frontiers in Physics 8, Article 139 (2020).
33 L. Vervoort, “Supercorrelation, an interpretation of quantum non-locality”, AIP Conf. Proc. 1508, 507-513 (2012).
34 C. L. Lewis, “Explicit gauge covariant Euler-Lagrange equation”, Am. J. Phys. 77, 839-843 (2009).
35 A. Ashkin, “Acceleration and trapping of particles by radiation pressure”, Phys. Rev. Lett. 24, 156–159 (1970).
36 Y. Gorodetski, A. Drezet, C. Genet, and T. W. Ebbesen, “Generating far-field orbital angular momenta from near-field optical chirality”, Phys. Rev. Lett. 110, 203906 (2013).