GEOMETRY OF QUANTUM DYNAMICS AND OPTIMAL CONTROL FOR MIXED STATES

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Abstract. Geometric effects make evolution time vary for different evolution curves that connect the same two quantum states. Thus, it is important to be able to control along which path a quantum state evolve to achieve maximal speed in quantum calculations. In this paper we establish fundamental relations between Hamiltonian dynamics and Riemannian structures on the phase spaces of unitarily evolving finite-level quantum systems. In particular, we show that the Riemannian distance between two density operators equals the infimum of the energy dispersions of all possible evolution curves connecting the two density operators. This means, essentially, that the evolution time is a controllable quantity. The paper also contains two applied sections. First, we give a geometric derivation of the Mandelstam-Tamm estimate for the evolution time between two distinguishable mixed states. Secondly, we show how to equip the Hamiltonians acting on systems whose states are represented by invertible density operators with control parameters, and we formulate conditions for these that, when met, makes the Hamiltonians transport density operators along geodesics.

1. Introduction

Geometric quantum mechanics is a branch of physics that has received much attention lately. This is due in large part to the crucial role geometry plays in quantum information and quantum computing [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. The performance of a quantum computer relies greatly on the efficiency of its algorithms, and the ability to control which route an evolving state should take when joining two given ones. This paper concerns fundamental aspects of the latter issue.

A quantum system prepared in a pure state is usually modeled on a projective Hilbert space, and if the system is closed its state will evolve unitarily. Aharonov and Anandan [11] showed that for unitary evolutions there is a geometric quantity which, like Berry’s phase [12, 13], is independent of the particular Hamiltonian used to transport a pure state along a given route. More precisely, they showed that the energy dispersion (i.e. $1/\hbar$ times the path integral of the energy uncertainty) of an evolving state equals the Fubini-Study length of the curve traced out by the state. Using this, Aharonov and Anandan gave a new geometric interpretation of the time-energy uncertainty relation.

Quantum computing and quantum information are theories developed for manipulating mixed quantum states, i.e., statistical ensembles of pure states. Such states are usually represented by density operators. Many metrics on spaces of density operators have been invented to capture various physical, mathematical,
or information theoretical aspects of quantum mechanics. In this paper we make use of a construction inspired by Montgomery [14] to provide the spaces of isospectral density operators with Riemannian metrics, and we show that these metrics admit a generalization of the energy dispersion result of Aharonov and Anandan to evolutions of quantum systems in mixed states. Specifically, we show that the energy dispersion of an evolving mixed state is bounded from below by the length of the curve traced out by the density operator of the state. Furthermore, we show that every curve of isospectral density operators is generated by a Hamiltonian for which the energy dispersion equals the curve’s length. The latter result allows us to express the distance between two mixed states in terms of a measurable quantity, and we use it to derive a time-energy uncertainty principle for mixed states.

Generically, mixed states of finite-level quantum systems are represented by invertible density operators. In a concluding section we define control parameters for Hamiltonians of systems whose states are represented by invertible density operators. In a concluding section we define control parameters for Hamiltonians of systems whose states are represented by invertible density operators. In a concluding section we define control parameters for Hamiltonians of systems whose states are represented by invertible density operators. In a concluding section we define control parameters for Hamiltonians of systems whose states are represented by invertible density operators. In a concluding section we define control parameters for Hamiltonians of systems whose states are represented by invertible density operators.

2. Geometry of unitary quantum dynamics

In this paper we consider finite dimensional quantum systems in mixed states that evolve unitarily. They will be modeled on a Hilbert space $\mathcal{H}$ of dimension $n$, and their states will be represented by density operators. Evolving mixed states will be represented by curves of density operators, all of which, for convenience, are assumed to be defined on an unspecified domain $0 \leq t \leq \tau$. We write $D(\mathcal{H})$ for the space of density operators on $\mathcal{H}$.

2.1. Riemannian structures on orbits of density operators. A density operator that evolves unitarily remains in a single orbit of the left conjugation action of the unitary group $U(\mathcal{H})$ of $\mathcal{H}$ on $D(\mathcal{H})$. The orbits of this action are in bijection with the set of possible spectra for density operators on $\mathcal{H}$, where by the spectrum of a density operator with $k$-dimensional support we mean the nonincreasing sequence $\sigma = (p_1, p_2, \ldots, p_k)$ of its, not necessarily distinct, positive eigenvalues. Throughout this paper we fix $\sigma$, and write $D(\sigma)$ for the corresponding orbit.

To furnish $D(\sigma)$ with a geometry, let $\mathcal{L}(\mathbb{C}^k, \mathcal{H})$ be the space of linear maps from $\mathbb{C}^k$ to $\mathcal{H}$, equipped with the Hilbert-Schmidt inner product, and $P(\sigma)$ be the diagonal $k \times k$ matrix with diagonal $\sigma$. Set $S(\sigma) = \{ \psi \in \mathcal{L}(\mathbb{C}^k, \mathcal{H}) : \psi^\dagger \psi = P(\sigma) \}$, and define $\pi : S(\sigma) \to D(\sigma)$ by $\pi(\psi) = \psi \psi^\dagger$. The fibration $\pi$ is a principal bundle with left acting gauge group $U(\sigma)$ consisting of all unitaries in $U(k)$ that commute with $P(\sigma)$. (The action is $U \cdot \psi = \psi U^\dagger$.) Moreover, the real part of the Hilbert-Schmidt product restricts to a gauge invariant Riemannian metric on $S(\sigma)$:

$$G(X, Y) = \frac{1}{2} \text{Tr}(X^\dagger Y + Y^\dagger X).$$

We equip $D(\sigma)$ with the unique metric $g$ that makes $\pi$ a Riemannian submersion.

The Lie algebra if the gauge group is $u(\sigma)$. It consists of all antiHermitian $k \times k$ matrices that commute with $P(\sigma)$. A connection for $\pi$ is given by the bundle of kernels of the mechanical connection form $\mathcal{A}$ on $S(\sigma)$ defined by $\mathcal{A}_\psi = \mathbb{I}_\psi^{-1} J_\psi$, where $\mathbb{I}_\psi : u(\sigma) \to u(\sigma)^*$ and $J_\psi : T_\psi S(\sigma) \to u(\sigma)^*$ are the locked inertia tensor

$$\text{tr}_{\mathbb{C}^k} \mathcal{A}_\psi X \psi = -\frac{1}{2} \mathbb{I}_\psi^{-1} \mathbb{I}_\psi X \psi.$$
and moment map, respectively:

$$\Pi_\psi \xi \cdot \eta = G(\psi \xi^\dagger, \psi \eta^\dagger), \quad J_\psi(X) \cdot \xi = G(X, \psi \xi^\dagger).$$

Vectors tangent to $S(\sigma)$ are called horizontal if they are annihilated by $A$, and a curve in $S(\sigma)$ is called horizontal if all of its velocity vectors are horizontal. Recall that for every curve $\rho$ in $D(\sigma)$ and every $\psi_0$ in the fiber over $\rho(0)$ there is a unique horizontal curve in $S(\sigma)$ that starts at $\psi_0$ and projects onto $\rho$. This curve is the horizontal lift of $\rho$ extending from $\psi_0$.

2.2. A geometric uncertainty estimate. Suppose $\hat{A}$ is an observable on $\mathcal{H}$. Let $X_A$ be the projection to $D(\sigma)$ of the gauge invariant vector field $X_\hat{A}$ on $S(\sigma)$ defined by

$$X_\hat{A}(\psi) = \frac{d}{d\varepsilon} \left[ \exp\left(\frac{\varepsilon}{\hbar} \hat{A}\right) \psi \right]_{\varepsilon=0}.$$

We say that $\hat{A}$ is parallel at a density operator $\rho$ if $X_\hat{A}$ is horizontal along the fiber over $\rho$, and we say that $\hat{A}$ is parallel along a curve $\rho$ if $\hat{A}(t)$ is parallel at $\rho(t)$ for every instant $t$. The precision to which the value of $\hat{A}$ can be known is quantified by its uncertainty function

$$\Delta A(\rho) = \sqrt{\text{Tr}(\hat{A}^2(\rho)) - \text{Tr}(\hat{A}\rho)^2}.$$  

In [15] the current authors have proven that $\Delta A$ is bounded from below by $\hbar$ times the norm of the vector field $X_A$:

1. $$\Delta A(\rho) \geq \hbar \sqrt{g(X_A(\rho), X_A(\rho))},$$

2. $$\Delta A(\rho) = \hbar \sqrt{g(X_A(\rho), X_A(\rho))} \text{ if } \hat{A} \text{ is parallel at } \rho.$$

The main argument is the following. For each $\psi$ in $S(\sigma)$ there is a canonical identification between $U(\sigma)$ and the fiber of $\pi$ containing $\psi$, namely $U \mapsto \psi U^\dagger$. The metric on $U(\sigma)$ obtained by pulling back $G$ via this identification is independent of $\psi$. Restricted to $u(\sigma)$ it is given by

$$\xi \cdot \eta = \frac{1}{2} \text{Tr}((\xi^\dagger \eta + \eta^\dagger \xi) P(\sigma)).$$

Define the $u(\sigma)$-valued field $\xi_A$ on $D(\sigma)$ by $\pi^* \xi_A = A \circ X_\hat{A}$, and write $\xi_A^\perp$ for $\xi_A$ followed by projection onto the orthogonal complement of $i1_k$ in $u(\sigma)$. Then

$$\Delta A^2 = \hbar^2 (g(X_A, X_A) + \xi_A^\perp \cdot \xi_A^\perp).$$

Now [1] follows from the observation that $\xi_A^\perp \cdot \xi_A^\perp \geq 0$, and [2] from the fact that $\xi_A(\rho) = 0$ if $X_\hat{A}$ is horizontal along the fiber over $\rho$. Note that for pure states, $\xi_A^\perp = 0$ regardless of $\hat{A}$ since $u(1)$ is spanned by $i1_1$.

2.3. Distance, geodesics, and energy dispersion. The distance between two density operators with common spectrum $\sigma$ is defined as the infimum of the lengths of all curves in $D(\sigma)$ that connects them. There is at least one such curve whose length equals the distance, since $D(\sigma)$ is compact, and all such curves are geodesics. Moreover, horizontal lifting of curves is length preserving, $\pi$ being a Riemannian submersion, and a curve in $D(\sigma)$ is a geodesic if and only if its horizontal lift is
a geodesic in $S(\sigma)$, see [16]. Here we show that the distance between two density operators $\rho_0$ and $\rho_1$ with common spectrum $\sigma$ satisfies

$$\text{dist}(\rho_0, \rho_1) = \frac{1}{\hbar} \inf_{\hat{H}} \int_0^\tau \Delta H(\rho) \, dt,$$

where the infimum is taken over all Hamiltonians $\hat{H}$ for which the following von Neumann equation is solvable:

$$\dot{\rho} = \frac{1}{i\hbar} [\hat{H}, \rho] = X_H(\rho), \quad \rho(0) = \rho_0, \quad \rho(\tau) = \rho_1.$$

The length of a curve $\rho$ in $D(\sigma)$ is

$$\text{Length}[\rho] = \int_0^\tau \sqrt{g(\dot{\rho}, \dot{\rho})} \, dt.$$

If $\dot{\rho} = X_H(\rho)$ for some Hamiltonian $\hat{H}$, then, by (1), the length of $\rho$ is a lower bound for the energy dispersion:

$$\text{Length}[\rho] \leq \frac{1}{\hbar} \int_0^\tau \Delta H(\rho) \, dt.$$

There is a Hamiltonian $\hat{H}$ that generates a horizontal lift of $\rho$ because the unitary group of $\mathcal{H}$ acts transitively on $\mathcal{L}(\mathbb{C}^k, \mathcal{H})$. For such a Hamiltonian we have equality in (6) by (2). Moreover, we can take $\rho$ to be a shortest geodesic. Then,

$$\text{dist}(\rho_0, \rho_1) = \frac{1}{\hbar} \int_0^\tau \Delta H(\rho) \, dt.$$

Assertion (4) follows. We refer to [17] for a prescription how to produce a parallel transporting Hamiltonian from a given one, without affecting the evolution curve.

2.4. A time-energy uncertainty relation. The Mandelstam-Tamm time-energy uncertainty relation [18] provide a limit on the speed of dynamical evolution. For systems prepared in pure states it implies that the minimum time it takes for a state to evolve to an orthogonal state is bounded from below by $\pi\hbar/2$ times the inverse of the average energy uncertainty of the system. Recently, Jones and Kok [19, 20] showed that the same inequality holds for mixed states, when orthogonality is replaced by distinguishability [21, 22]. Their proof involves an estimate of the rate of change of the statistical distance between density operators. Here we give a short geometric proof of this inequality.

Consider a quantum system with Hamiltonian $\hat{H}$, and suppose $\rho$ is a solution to (5). If $\rho_0$ and $\rho_1$ are distinguishable, then

$$\langle \Delta H \rangle \tau \geq \frac{\pi\hbar}{2}, \quad \langle \Delta H \rangle = \frac{1}{\tau} \int_0^\tau \Delta H(\rho) \, dt.$$

To see this, let $\psi_0$ in $\pi^{-1}(\rho_0)$ and $\psi_1$ in $\pi^{-1}(\rho_1)$ be such that $\text{dist}(\rho_0, \rho_1) = \text{dist}(\psi_0, \psi_1)$. The operators $\rho_0$ and $\rho_1$ have orthogonal supports, being distinguishable, and the same is true for $\psi_0$ and $\psi_1$ since the the support of $\psi_0$ equals the support of $\rho_0$, and likewise for $\psi_1$ and $\rho_1$. A compact way to express this is

$$\psi_0^\dagger \psi_1 = 0, \quad \psi_1^\dagger \psi_0 = 0.$$
If we consider \( \psi_0 \) and \( \psi_1 \) elements in the unit sphere in \( \mathcal{L}(\mathbb{C}^k, \mathcal{H}) \), they are a distance of \( \pi/2 \) apart. In fact, \( \psi(t) = \cos(t)\psi_0 + \sin(t)\psi_1 \), with domain \( 0 \leq t \leq \pi/2 \), is a length minimizing unit speed curve from \( \psi_0 \) to \( \psi_1 \). Consequently,

\[
\text{dist}(\rho_0, \rho_1) \geq \pi/2.
\]

The relation \( (7) \) now follows from \( (6) \) and \( (8) \). Also note that the estimate \( (8) \) cannot be improved. Direct computations yield \( \psi^\dagger \psi = P(\sigma) \) and \( \psi^\dagger \dot{\psi} = 0 \). Therefore, \( \psi \) is a horizontal curve in \( S(\sigma) \), and hence \( (8) \) is, in fact, an equality. From this it also follows that the estimate \( (7) \) is saturated by any parallel Hamiltonian that generate \( \rho = \psi\psi^\dagger \).

3. Optimal Hamiltonians for mixed states of full rank

Generically, the number of independent kets in a mixed state equals the dimension of the Hilbert space. Such mixed states are represented by invertible density operators. From now on, we assume that the density operators in \( D(\sigma) \) are invertible.

3.1. Lie algebra controlled Hamiltonians. To achieve optimal computational speed in quantum computers it is desirable that the Hamiltonians transport states along shortest possible paths. Here we classify the Hamiltonians that transport an invertible density operator \( \rho_0 \) along geodesics, and we provide conditions for control parameters of these Hamiltonians that, when satisfied, makes the Hamiltonians transport a \( \psi_0 \) in the fiber over \( \rho_0 \) along horizontal geodesics.

As control space we choose the matrix Lie algebra \( u(n) \), equipped with the metric given by \( (3) \). For each curve \( \xi \) in \( u(n) \) we define \( \dot{H}_\xi \) by

\[
\dot{H}_\xi = i\hbar\psi_0 P(\sigma)^{-1/2} \exp - \left( \int_0^t \xi dt \right) \xi \exp \left( \int_0^t \xi dt \right)^\dagger P(\sigma)^{-1/2}\psi_0^\dagger.
\]

where \( P(\sigma)^{-1/2} \) is the diagonal \( n \times n \) matrix whose \( j^{\text{th}} \) diagonal entry is \( 1/\sqrt{P_j} \), and \( \exp \) is the negative time-ordered exponential. Also, let \( \psi_\xi \) be the solution to the Schrödinger equation of \( \dot{H}_\xi \) extending from \( \psi_0 \). In the next section we show that every curve \( \rho \) extending from \( \rho_0 \) equals \( \psi_\xi \psi_\xi^\dagger \), for some curve \( \xi \) in \( u(n) \), and that \( \psi_\xi \) is a geodesic if and only if \( \dot{\xi} = \text{ad}_\xi^\ast \xi \), where \( \text{ad}_\xi^\ast \eta = \xi \cdot [\xi, \eta] \). Furthermore, we show that \( \psi_\xi \) is horizontal if and only if \( \xi \) is contained in the orthogonal complement \( u(\sigma)^\perp \) of \( u(\sigma) \) in \( u(n) \).

3.2. Evolution operators that generate horizontal geodesics. The group \( \mathcal{U}(\mathcal{H}) \) acts freely and transitively on \( S(\sigma) \) from the left, and the metric on \( \mathcal{U}(\mathcal{H}) \) obtained by declaring the diffeomorphism \( U \mapsto U \psi_0 \) an isometry is the left invariant metric \( X : Y = \text{Tr}((X^\dagger Y + Y^\dagger X)P(\sigma)) \). An evolution curve \( U \psi_0 \) is a geodesic in \( S(\sigma) \) if and only if \( U \) is a geodesic in \( \mathcal{U}(\mathcal{H}) \) extending from the identity operator. The second order geodesic equation in \( \mathcal{U}(\mathcal{H}) \) can be reduced to a first order equation in \( u(n) \) as follows. Define an isomorphism \( \phi : u(n) \rightarrow \mathcal{U}(\mathcal{H}) \) by \( \phi(U) = \psi_0 P(\sigma)^{-1/2} U P(\sigma)^{-1/2} \psi_0^\dagger \). Equip \( u(n) \) with the metric that makes \( \phi \) an isometry, and for each \( \xi \) in \( u(n) \) write \( \chi_\xi \) for the vector field on \( \mathcal{U}(\mathcal{H}) \) made up of left translates of \( d\phi(\xi) \). For a given curve \( U \) in \( \mathcal{U}(\mathcal{H}) \) define a curve \( \xi \) in \( u(n) \) by \( \dot{U} = \chi_\xi(U) \). The Hamiltonian associated with \( \xi \), that transport \( \psi_0 \) along \( U \psi_0 \) in \( S(\sigma) \), is \( \dot{H}_\xi \) given by \( (9) \). Now, \( U \) is a geodesic if and only if \( \xi \) satisfies the
Arnold-Euler equation $\dot{\xi} = ad^*_\xi \xi$, see [23] and Appendix A. Moreover, the inclusion of $U(\sigma)$ in $U(n)$ is an isometric embedding, when the former is equipped with the left invariant metric determined by (3). A straightforward verification shows that $U\psi_0$ is horizontal if and only if $\xi$ is contained in $u(\sigma)^\perp$, see Appendix A.

3.3. Geodesic orbit spaces and almost pure states. If $\sigma$ contains precisely two different, possibly degenerate, eigenvalues, every geodesic in $D(\sigma)$ is generated by a time independent Hamiltonian. This since $ad^*_\xi \xi = 0$ holds for every $\xi$ in $u(\sigma)^\perp$.

To see this let $\eta$ be any element in $u(n)$, and write
$$
\xi = \begin{bmatrix}
0 & \xi_{12} \\
-\xi_{12}^\dagger & 0
\end{bmatrix},
\eta = \begin{bmatrix}
\eta_{11} & \eta_{12} \\
-\eta_{12}^\dagger & \eta_{22}
\end{bmatrix}.
$$

Then
$$
\xi \cdot [\xi, \eta] = \frac{1}{2} \left( p_1 \text{Tr}[\xi_{12}\xi_{12}^\dagger, \eta_{11}] + p_2 \text{Tr}[\xi_{12}\xi_{12}^\dagger, \eta_{22}] \right) = 0
$$
since commutators have vanishing trace. (The corresponding result does not hold if $\sigma$ contains at least three distinct eigenvalues.) Another way to put this is to say that $D(\sigma)$ is a geodesic orbit space, i.e., a Riemannian homogeneous space in which each geodesic is an orbit of a one-parameter subgroup of its isometry group.

By an almost pure state we mean a mixture of two pure quantum states in which one state is present in greater proportion than the other. Here we apply the above results to almost pure qubit systems.

Two independent qubits are modeled by the standard basis elements $e_1$ and $e_2$ in $\mathbb{C}^2$. Consider an ensemble of qubits prepared so that the proportion of qubits in state $e_j$ is $p_j$, where $p_1 > p_2$. The initial state of the ensemble is represented by the density operator $\rho_0 = \text{diag}(p_1, p_2)$. Chose $\psi_0 = \text{diag}(\sqrt{p_1}, \sqrt{p_2})$ in the fiber over $\rho_0$, and let $\xi$ be an arbitrary constant curve in $u(\sigma)^\perp$:
$$
\xi(t) = \begin{bmatrix}
0 & \varepsilon e^{i\theta} \\
-\varepsilon e^{-i\theta} & 0
\end{bmatrix}, \quad \varepsilon > 0, \quad 0 \leq t \leq 1.
$$

The solution $\psi_\xi$ to the Schrödinger equation of $\hat{H}_\xi = i\hbar \xi$ that extends from $\psi_0$ is a horizontal geodesic, and the projection $\rho = \psi_\xi \psi_\xi^\dagger$ is a geodesic extending from $\rho_0$. Explicitly,
$$
\rho(t) = \begin{bmatrix}
p_1 \cos^2 \varepsilon t + p_2 \sin^2 \varepsilon t & e^{i\theta}(p_2 - p_1) \cos \varepsilon t \sin \varepsilon t \\
e^{-i\theta}(p_2 - p_1) \cos \varepsilon t \sin \varepsilon t & p_1 \sin^2 \varepsilon t + p_2 \cos^2 \varepsilon t
\end{bmatrix}.
$$

The curve $\rho$ is a shortest geodesic between its end points provided that $\varepsilon$ is small enough, and $\text{dist}(\rho(0), \rho(1)) = \varepsilon$.

4. Conclusion

The classic time-energy uncertainty relation by Mandelstam and Tamm implies that the evolution time between two distinguishable mixed states is bounded from below by a factor which is inversely proportional to the average energy uncertainty. In this paper we have shown that there is a fundamental relation between the length of evolution curves of mixed states, as measured by a specific Riemannian metric, and the energy dispersions of the Hamiltonians that generate the evolution curves. Our work thus indicates that the evolution time estimate derived from the Mandelstam-Tamm relation has a purely geometric origin. In fact, we have provided a geometric derivation of the same estimate.
In quantum computing it is desirable to have greatest possible control over the evolution of states. This to achieve maximum computational speed. Generically, mixed states of finite-level quantum systems are represented by invertible density operators. In the paper’s concluding section, we have focused on quantum systems whose states are represented by invertible density operators. There we have described how Hamiltonians acting on such systems can be equipped with control parameters, and we have provided conditions for these that, if met, guarantees that the Hamiltonians transport density operators along geodesics.

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**Appendix A. Constrained Arnold-Euler equation**

Suppose the density operators with spectrum $\sigma$ are invertible. Fix $\psi_0$ in $S(\sigma)$, and identify $U(n)$ and $U(H)$ according to the isomorphism $\phi : U(n) \rightarrow U(H)$ given by

$$\phi(U) = \psi_0 P(\sigma)^{-1/2} U P(\sigma)^{-1/2} \psi_0^\dagger.$$ 

Then $U(n)$ acts freely and transitively from the left on $S(\sigma)$. Thus $U \mapsto \phi(U)$ is a diffeomorphism from $U(n)$ to $S(\sigma)$. Let $Y_\xi$ be the push-forward of the one-parameter family of left invariant vector fields on $U(n)$ generated by a curve $\xi$ in $\mathfrak{u}(n)$:

$$Y_\xi(\phi(U)\psi_0) = \psi_0 P(\sigma)^{-1/2} U \xi P(\sigma)^{1/2}.$$ 

We assert that the integral curves of $Y_\xi$ are horizontal geodesics if and only if $\xi$ is contained in $\mathfrak{u}(\sigma)^\perp$ and $\dot{\xi} = \text{ad}_\xi^* \xi$. The latter equation is called the Arnold-Euler equation [23].

The mechanical connection is such that a tangent vector is horizontal if and only if it is orthogonal to the fibers of $\pi$. The tangent space at $\phi(U)\psi_0$ of the fiber of $\pi$ is spanned by the vectors $\phi(U)\psi_0 \eta^\dagger$, where $\eta$ run through the matrices in $\mathfrak{u}(\sigma)$. A straightforward computation yields

$$G(\nabla_\psi \dot{\psi} = X_\xi(\psi) + \nabla_{X_\xi} X_\xi(\psi).$$

Moreover, by the Kozul formula [21 Prop 2.3],

$$2G(\nabla_{X_\xi} X_\xi, X_\eta) = X_\xi G(X_\xi, X_\eta) + X_\xi G(X_\eta, X_\xi)$$
$$- X_\eta G(X_\xi, X_\xi) - G(X_\xi, [X_\xi, X_\eta])$$
$$+ G(X_\xi, [X_\eta, X_\xi]) + G(X_\eta, [X_\xi, X_\xi]).$$

$$= -\xi \cdot [\xi, \eta] + \xi \cdot [\eta, \xi]$$
$$= -2 \text{ad}_\xi^* \xi \cdot \eta$$
$$= -2G(X_{\text{ad}_\xi^* \xi}, X_\eta)$$
for every $\eta$ in $u(n)$. Accordingly, $\nabla_{\psi} \dot{\psi} = X_{\dot{\xi} - \text{ad}^*_\xi \xi} (\psi)$. Thus, $\psi$ is a geodesic if and only if $\dot{\xi} = \text{ad}^*_\xi \xi$.

References
[1] J. Pachos, P. Zanardi, and M. Rasetti. Non-abelian berry connections for quantum computation. Phys. Rev. A, 61:010305, Dec 1999.
[2] P. Zanardi and M. Rasetti. Holonomic quantum computation. Phys. Lett. A, 264(2–3):94–99, 1999.
[3] A. Ekert, M. Ericsson, P. Hayden, H. Inamori, J. A. Jones, D. K. L. Oi, and V. Vedral. Geometric quantum computation. J. Mod. Opt., 47:2501–2513, 2000.
[4] P. Zanardi, P. Giorda, and M. Cozzini. Information-theoretic differential geometry of quantum phase transitions. Phys. Rev. Lett., 99:100603, Sep 2007.
[5] A. T. Rezakhani, D. F. Abasto, D. A. Lidar, and P. Zanardi. Intrinsic geometry of quantum adiabatic evolution and quantum phase transitions. Phys. Rev. A, 82:012321, Jul 2010.
[6] Erik Sjöqvist, D M Tong, L Mauritz Andersson, Björn Hessmo, Markus Johansson, and Kuldirp Singh. Non-adiabatic holonomic quantum computation. New J. Phys., 14(10):103035, 2012.
[7] J. A. Jones, V. Vedral, A. Ekert, and G. Castagnoli. Geometric quantum computation using nuclear magnetic resonance. Nature, 403(5516):869–871, 2000.
[8] G. Falci, R. Fazio, G. M. Palma, J. Siewert, and V. Vedral. Detection of geometric phases in superconducting nanocircuits. Nature, 407(5516):355–358, 2000.
[9] L.-M. Duan, J. I. Cirac, and P. Zoller. Geometric manipulation of trapped ions for quantum computation. Science, 292(5522):1695–1697, 2001.
[10] A. Recati, T. Calarco, P. Zanardi, J. I. Cirac, and P. Zoller. Holonomic quantum computation with neutral atoms. Phys. Rev. A, 66:032309, Sep 2002.
[11] J. Anandan and Y. Aharonov. Geometry of quantum evolution. Phys. Rev. Lett., 65:1697–1700, 1990.
[12] M. V. Berry. Quantal phase factors accompanying adiabatic changes. Proc. R. Soc. Lond. A, 392(1802):45–57, 1984.
[13] B. Simon. Holonomy, the quantum adiabatic theorem, and berry's phase. Phys. Rev. Lett., 51:2167–2170, 1983.
[14] R. Montgomery. Heisenberg and isoholonomic inequalities. In Symplectic geometry and mathematical physics (Aix-en-Provence, 1990), volume 99 of Progr. Math., pages 303–325. Birkhäuser Boston, Boston, MA, 1991.
[15] O. Andersson and H. Heydari. Geometric uncertainty relation for mixed quantum states. arXiv:1302.2074, 2013.
[16] R. Hermann. A sufficient condition that a mapping of riemannian manifolds be a fibre bundle. Proc. Am. Math. Soc., 11(2):236–242, 1960.
[17] O. Andersson and H. Heydari. Operational geometric phase for mixed quantum states. arXiv:1302.1838 (accepted for publication in New J. Phys.), 2013.
[18] L. I. Mandelstam and I. E. Tamm. The uncertainty relation between energy and time in nonrelativistic quantum mechanics. J. Phys. (USSR), 9, 1945.
[19] Philip J. Jones and Pieter Kok. Geometric derivation of the quantum speed limit. Phys. Rev. A, 82:022107, Aug 2010.
[20] Marcin Zwierz. Comment on “geometric derivation of the quantum speed limit”. Phys. Rev. A, 86:016101, Jul 2012.
[21] B.-G. Englert. Fringe visibility and which-way information: An inequality. Phys. Rev. Lett., 77:2154–2157, Sep 1996.
[22] D. Markham, J. A. Miszczak, Z. Puchała, and K. Życzkowski. Quantum state discrimination: A geometric approach. Phys. Rev. A, 77:042111, Apr 2008.
[23] V. Arnold. Sur la géométrie différentielle des groupes de lie de dimension infinie et ses applications à l’hydrodynamique des fluides parfaits. Ann. Inst. Fourier (Grenoble), 16:319–361, 1966.
[24] S. Kobayashi and K. Nomizu. Foundations of differential geometry. Vol. I. Wiley Classics Library. John Wiley & Sons Inc., New York, 1996.
