A variant of the Dressing Method applied to nonintegrable multidimensional nonlinear Partial Differential Equations

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Abstract

We describe a variant of the dressing method giving alternative representation of multidimensional nonlinear PDE as a system of Integro-Differential Equations (IDEs) for spectral and dressing functions. In particular, it becomes single linear Partial Differential Equation (PDE) with potentials expressed through the field of the nonlinear PDE. The absence of linear overdetermined system associated with nonlinear PDE creates an obstacle to obtain evolution of the spectral data (or dressing functions): evolution is defined by nonlinear IDE (or PDE in particular case). As an example, we consider generalization of the dressing method applicable to integrable (2+1)-dimensional $N$-wave and Davey-Stewartson equations. Although represented algorithm does not supply an analytic particular solutions, this approach may have a perspective development.

1 Introduction

Completely integrable multidimensional Partial Differential Equations (PDEs) represent attractive subject of intensive study during last decades after the paper [1]. This popularity is due to their remarkable mathematical properties and variety of physical applications, which may be found in literature. Investigation approach considered in this paper is, in some sense, associated with so-called $S$-integrable PDEs [2], i.e. nonlinear PDEs which may be "linearized" using special technique, such as Inverse Spectral Transform (IST) [3, 4, 5]. It is well known that IST is not the only method to study $S$-integrable PDEs. One may refer to Sato Theory [6, 7, 8, 9], Symmetry Approach [10, 11], Dressing Method [12, 13, 14, 15]. The later, in turn, has several formulations: Zakharov-Shabat method [12], local Riemann problem [13], nonlocal Riemann and $\bar{\partial}$-problem [14, 15, 16].

Classical $S$-integrable systems are basically (1+1)- and (2+1)-dimensional. Only special types of multidimensional $S$-integrable examples are known, such as self-dual Yang-Mills equations [17, 18, 19, 20, 21] and the Plebanski heavenly equation [22, 23, 24]. Recently a new type of multidimensional partially integrable systems have been found [25], for which integration algorithm is based on the integral operator with nontrivial kernel, which is a variant of
the dressing method. This recent result encourage us to search for other improvements of the
dressing method.

It is well known, that dressing method has been originally developed to construct nonlinear
PDEs together with their solutions. Variant of the dressing method suggested here does not
allow one to find analytic solutions for nonlinear PDEs. However

1. it gives an alternative representation of largely arbitrary nonlinear PDE as nonlinear
system of Integro-Differential Equations (IDEs). In particular case, this system becomes
single linear PDE where potentials are expressed through the spectral function from one
hand and through the field of original nonlinear PDE from another hand;

2. it relates a single linear spectral evolution equation (written for some spectral function)
with largely arbitrary nonlinear PDE.

This is an interesting result of the paper. However, the fact that one has single linear equation
associated with given nonlinear PDE (instead of overdetermined linear system, like in
$S$-integrable case) results in system of nonlinear IDEs (or PDEs) defining evolution of the
dressing function, which is disadvantage of our representation. Remember that dressing functions
of $S$-integrable PDE satisfy linear PDE. As a consequence, our (largely arbitrary) PDE
may not be derived as compatibility conditions of linear overdetermined system.

In some sence, similar purpose (but different approach) was sought in series of papers
generalizing known (2+1)- and (1+1)-dimensional completely integrable equations. These are
generalization of Kadomtsev-Petviashvili equation (KP) using deformation of the classical
Inverse Spectral Transform (IST) [28], generalizations of Korteweg-de Vries equation (KdV) and
Nonlinear Shrödinger equation (NLS) [29], generalization of Benjamin-Omo equation (BO) [30].
In these papers evolution of spectral data is defined by nonlinear nonlocal equations (spectral
data are replaced by dressing functions in our case).

Here we start with dressing method based on the integral equation in the form [26, 27],
where we introduce an integral operator with different type of kernel allowing us to increase
dimensionality of PDE. As a consequence, an arbitrary function of $x_i$ (independent variables of
nonlinear PDE) appears in the dressing algorithm (see function $\hat{\Phi}(\lambda_1; x)$ in Sec.2.1) enforcing us
to introduce an extra constrain in the form of largely arbitrary nonlinear IDE for $\hat{\Phi}(\lambda_1; x)$, see
eq(40). Fixing function $\hat{\Phi}(\lambda_1; x)$, this constrain provides possibility to write single nonlinear
PDE for single field $u$ expressible in terms of the dressing and spectral functions. Note that
similar extra constrain has been introduced in [25], but arbitrary function there has quite
different origin.

Below we concentrate on multidimensional generalizations of dressing algorithm for (2+1)-
dimensional $N$-wave equation and Davey-Stewartson equation (DS). However, generalized version
is applicable to largely arbitrary nonlinear PDE.

In the next section (Sec.2) we give general algorithm deriving nonlinear $N$-wave type PDE.
We introduce an extra constrain allowing to write single nonlinear PDE for single field. Character-
ization of solution space for derived nonlinear PDE is given in Sec.2.1. Sec.3 considers similar
generalization of dressing method for DS. Finally we represent some conclusions in Sec.4.
2 Derivation of multidimensional nonlinear $N$-wave equation

We start with usual integral equation

\[
\Phi(\lambda) = \int \Psi(\lambda, \nu; x) U(\nu; x) d\nu = \Psi(\lambda, \nu; x) * U(\nu; x) = \Psi * U,
\]

where $*$ means integration over spectral parameter appearing in both functions. There are two types of parameters in this equation. First, already mentioned spectral parameters denoted by Greek letters $\lambda$, $\mu$, $\nu$ (for instance $\lambda = (\lambda_1, \ldots, \lambda_{\text{dim}\lambda})$), and, second, additional parameters denoted by $x$, $x = (x_1, \ldots, x_{\text{dim}\ x})$. These additional parameters are independent variables of resulting nonlinear PDE. Besides, we reserve $k$ for scalar Fourier type parameter appearing in integral representations of some functions. All functions are $Q \times Q$ matrices. We always assume $\text{dim}\ x = M$ and $\text{dim}^\lambda = \text{dim} \mu = \text{dim} \nu = M + 1$, where $M$ is dimensionality of resulting nonlinear PDE.

Eq.(1) is a linear equation for the spectral function $U(\lambda; x)$, where operator $\Psi(\lambda, \mu; x) *$ is required to be uniquely invertible, $\Phi$ is a diagonal matrix function specified below. Integration is over whole space of vector spectral parameter $\nu$. Function $\Psi(\lambda, \mu; x)$ is defined by the following formulae introducing $x$-dependence:

\[
\partial_{x_n} \Psi_{\alpha\beta}(\lambda, \mu; x) + \left(h^n_{\alpha}(\lambda) + g^n_{\alpha\beta}(\mu)\right) \Psi_{\alpha\beta}(\lambda, \mu; x) = \Phi_{\alpha}(\lambda; x) B^n_{\alpha\beta} C_{\alpha\beta}(\mu; x),
\]

\[
1 \leq n \leq M.
\]

Here $C(\mu; x)$ is a new function, which will be characterized below; $h^n(\lambda)$ are diagonal and $g^n(\mu)$ are arbitrary matrix functions of argument; $B^n$ are diagonal constant matrices. Short form of eq.(2) reads:

\[
L^n_{\alpha\beta}(\lambda, \mu) \Psi_{\alpha\beta}(\lambda, \mu; x) = \Phi_{\alpha}(\lambda; x) B^n_{\alpha\beta} C_{\alpha\beta}(\mu; x),
\]

\[
L^n_{\alpha\beta}(\lambda, \mu)(*) = \left(\partial_{x_n} + h^n_{\alpha}(\lambda) + g^n_{\alpha\beta}(\mu)\right)(*), \quad 1 \leq n \leq M.
\]

Remark that derivatives $\partial_{x_i} \Psi(\lambda, \mu; x)$ are not separated functions of spectral parameters, unlike the $S$-integrable case [25, 27]. Been overdetermined system of PDEs for function $\Psi(\lambda, \mu; x)$, eqs. (3) imply compatibility conditions, which are following:

\[
L^n_{\alpha\beta}(\lambda, \mu) \left(\Phi_{\alpha}(\lambda; x) B^n_{\alpha\beta} C_{\alpha\beta}(\mu; x)\right) = L^j_{\alpha\beta}(\lambda, \mu) \left(\Phi_{\alpha}(\lambda; x) B^n_{\alpha\beta} C_{\alpha\beta}(\mu; x)\right), \quad n \neq j.
\]

Without loss of generality, we put $j = 1$, and

\[
B^1 = I, \quad h^1(\lambda) = g^1(\lambda) = 0,
\]

where $I$ is the unit matrix. Since each term in expanded form of eqs.(4) is separated function of parameters $\lambda$ and $\mu$, these equations with $j = 1$ are equivalent to two sets of equations:

\[
\partial_{x_n} \Phi(\lambda; x) + h^n(\lambda) \Phi(\lambda; x) - \partial_{x_1} \Phi(\lambda; x) B^n = 0, \quad 1 \leq n \leq M,
\]

\[
\partial_{x_n} C(\mu; x) + C^{1n}(\mu; x) - B^n \partial_{x_1} C(\mu; x) = 0, \quad 1 \leq n \leq M,
\]

where

\[
C^{1n}_{\alpha\beta}(\mu; x) = C_{\alpha\beta}(\mu; x) g^n_{\alpha\beta}(\mu).
\]
Eqs. (6-8) define \( \Phi \) and \( C \). We refer to functions \( \Phi \), \( C \) and \( \Psi \) as dressing functions, where \( \Psi \) is expressed in terms of \( \Phi \) and \( C \) due to eqs. (2).

Thus we have specified all functions appearing in eqs. (1) and (2). Now we demonstrate how linear integral equation (1) is related with appropriate multidimensional nonlinear PDE written for fields expressible in terms of spectral function \( U(\lambda; x) \) and dressing functions.

System of nonlinear equations is generated by eq. (6). Derivation is very similar to derivation of classical integrable equations [25, 27]. First of all, we use representation for \( \Phi \) as \( \Psi \) eq. (1). Then using equations (2) for derivatives \( \Psi_x \), we end up with homogeneous equations in the form

\[
\Psi(\lambda, \mu; x) * E_n(\mu; x) = 0, \\
E_n(\lambda, x) = U_{x_n}(\lambda, x) - U_{x_1}(\lambda, x)B^n + U(\lambda, x)[B^n, u(x)] - \mathcal{G}^n(\lambda, x), \quad 1 < n \leq M,
\]

where function \( u \) is related with spectral function by the formula

\[
u(x) = C(\lambda, x) * U(\lambda, x)
\]

and functions \( \mathcal{G}^n \) satisfy the following equations:

\[
\Psi(\lambda, \mu; x) * \mathcal{G}^n(\mu; x) = \Psi^n(\lambda, \mu; x) * U(\mu; x), \quad 1 < n \leq M;
\]

\[
\Psi^n_{\alpha\beta}(\lambda, \mu; x) = \Psi^n_{\alpha\beta}(\lambda, \mu; x) g^n_{\alpha\beta}(\mu).
\]

Later, function \( u \) will be field of nonlinear PDE.

Eqs. (11,12) along with eq. (1) will be used in Sec. 2.1 to analyze solution space of nonlinear system. Inverting operator \( \Psi \) in eqs. (9) one gets

\[
E_n(\lambda; x) := U_{x_n}(\lambda; x) - U_{x_1}(\lambda; x)B^n + U(\lambda; x)[B^n, u(x)] - \mathcal{G}^n(\lambda; x) = 0, \quad 1 < n \leq M. \tag{13}
\]

In the case of classical dressing method, nonlinear integrable PDE can be received for function \( u \) applying \( C(\lambda; x)* \) to (13) and using eq. (7) for \( C_{x_n}, n > 1 \). Doing the same one gets in our case:

\[
E^n_1(x) := u_{x_n}(x) - u_{x_1}(x)B^n + u(x)[B^n, u(x)] = \tilde{\mathcal{H}}^n(x), \tag{14}
\]

\[
\tilde{\mathcal{H}}^n(x) = [B^n, u^1(x)] - C_{1n}(\mu; x) * U(\mu; x) + C(\mu; x) * \mathcal{G}^n(\mu; x), \quad 1 < n \leq M,
\]

where function \( u^1 \) is related with spectral function by the formula similar to eq. (10):

\[
u^1(x) = C_{x_1}(\lambda; x) * U(\lambda; x).
\]

Functions \( u^1 \) and \( \tilde{\mathcal{H}}^n \) are ”intermediate” functions which will be eliminated from the final system of nonlinear PDEs.

System (14) has an obvious limit to classical (2+1)-dimensional S-integrable N-wave equation. In fact, if \( g^n = 0 \) for all \( n \), (i.e. \( \mathcal{G}^n(\lambda; x) = 0 \), \( \tilde{\mathcal{H}}^n(x) = [B^n, u^1(x)] \)), then we may eliminate \( u_1 \) using two equations (14): \( E^n_m \) and \( E^m_n, n \neq m \):

\[
[u_{x_n}, B^m] - [u_{x_m}, B^n] + B^m u_{x_1} B^n - B^n u_{x_1} B^m - [[u, B^m], [u, B^n]] = 0. \tag{16}
\]

This is the classical (2+1)-dimensional completely integrable N-wave equation, which has acceptable reduction \( u_{\alpha\beta} = \bar{u}_{\alpha\beta} \), where bar means complex conjugation, see, for instance, [5]. System (13) with \( \mathcal{G}^n = 0 \) becomes linear overdetermined system for eq. (16), where \( U(\lambda; x) \) is
a spectral function, i.e. eq.(16) is compatibility condition for \( E_n \) and \( E_m \). This is well-known common feature of \( S \)-integrable models: they may be derived both algebraically through compatibility condition of overdetermined linear system and using dressing method.

However, if \( g^n \neq 0 \) for all \( n \), then \( G^n \neq 0 \) and system (13) may not be considered as a linear overdetermined system, since it has set of spectral functions, such as \( U(\lambda; x) \) and \( G^n(\lambda; x) \). As a consequence, nonlinear eqs. (14) have extra functions \( \tilde{H}^n(x) \) and may not be received as compatibility condition of the system (13) through commutation of linear operators appearing in (13). So, similar to [25], the only way to derive system (14) from eq.(13) is the dressing method.

The derived system (14) consists of \((M - 1)\) equations and \( M \) fields, which are \( u \) and \( \tilde{H}^n, \ 1 < n \leq M \). In other words, it is not complete. In order to write a single nonlinear PDE for field \( u \) we involve another important deviation from the classical approach.

Let us split \( C(\lambda; x) \) into two factors:
\[
C_{\alpha\beta}(\mu; x) = G^1_{\alpha}(\mu_1; x)G^2_{\alpha\beta}(\mu; x),
\]
\[
\partial_{x_n} G^1(\mu_1; x) - B^n \partial_{x_1} G^1(\mu_1; x) = 0, \ 1 < n \leq M,
\]
\[
\partial_{x_n} G^2(\mu; x) + G^{1n}(\mu; x) = 0, \ 1 < n \leq M, \ G^{1n}_{\alpha\beta}(\mu; x) = G_{\alpha\beta}(\mu; x)g^n_{\alpha\beta}(\mu),
\]
\[
\partial_{x_1} G^2(\mu; x) = 0
\]
where eqs.(17b-d) appear due to the eq.(7). Multiply eq.(13) by \( G^2(\lambda; x) \) from the left and integrate over \( \lambda = (\lambda_2, \ldots, \lambda_{M+1}) \). One gets
\[
\tilde{E}_n(\lambda_1; x) := \tilde{U}_{x_n}(\lambda_1; x) - \tilde{U}_{x_1}(\lambda_1; x)B^n + \tilde{U}(\lambda_1; x)[B^n, u(x)] - \tilde{F}^n(\lambda_1; x) = 0, \ 1 < n \leq M,
\]
where
\[
\tilde{U}(\lambda_1; x) = \int G^2(\lambda; x)U(\lambda; x)d\lambda,
\]
\[
\tilde{U}^{1n}(\lambda_1; x) = -\int G^2_{x_n}(\lambda; x)U(\lambda; x)d\lambda = \int G^{1n}(\lambda; x)U(\lambda; x)d\lambda,
\]
\[
\tilde{G}^n(\lambda_1; x) = \int G^2(\lambda; x)G^n(\lambda; x)d\lambda, \ \tilde{F}^n(\lambda_1; x) = \tilde{G}^n(\lambda_1; x) - \tilde{U}^{1n}(\lambda_1; x),
\]
\[
1 < n \leq M.
\]
We will see in the next section that off-diagonal parts of \( \tilde{U}(\lambda_1; x) \) and \( \tilde{G}^n(\lambda_1; x) \) have arbitrary dependence on \( x \). Thus we are able to introduce one more relation among them. For instance, let
\[
\sum_{i=2}^{M} S^i_{\alpha\beta} \left( \tilde{F}^i_{\alpha\beta}(\lambda_1; x) - \lambda_1 \tilde{U}_{\alpha\beta}(\lambda_1; x)(B^i_{\alpha} - B^i_{\beta}) \right) = 0, \ \alpha \neq \beta,
\]
where \( S^i_{\alpha\beta} \) are constants. Then eq.(18) gives \( \alpha \neq \beta \):
\[
\sum_{i=2}^{M} S^i_{\alpha\beta} \left( \partial_{x_i} \tilde{U}_{\alpha\beta}(\lambda_1; x) - \partial_{x_1} \tilde{U}_{\alpha\beta}(\lambda_1; x)B^i_{\beta} + \sum_{\gamma=1}^{Q} \tilde{U}_{\alpha\gamma}(\lambda_1; x)u_{\gamma\beta}(x)(B^i_{\gamma} - B^i_{\beta}) - \lambda_1 \tilde{U}_{\alpha\beta}(\lambda_1; x)(B^i_{\alpha} - B^i_{\beta}) \right) = 0,
\]
\[
\sum_{i=2}^{M} S^i_{\alpha\beta}(B^i_{\alpha} - B^i_{\beta}) = 0.
\]
This equation is a linear equation for the spectral function $\tilde{U}^{of}$; additional relation (22) is introduced to eliminate diagonal part of $\tilde{U}$ from the nonlinear term of (21).

Multiply this equation by $G\alpha_1^1(\lambda_1; x)$ from the left, integrate over $\lambda_1$ and assume that $G\lambda_2^1(\lambda_1; x) = \lambda_1 G\lambda_1^1(\lambda_1; x)$:

$$\sum_{i=2}^M S^i_{\alpha\beta}(\partial_{x_i} u_{\alpha\beta}(x) - \partial_{x_1} u_{\alpha\beta}(x) B^i_\beta + \sum_{\gamma\neq\alpha\neq\beta}^{Q} u_{\alpha\gamma}(x) u_{\gamma\beta}(x) (B^i_\gamma - B^i_\beta)) = 0, \quad \alpha \neq \beta$$  \hspace{1cm} (23)

which becomes $N$-wave equation if, along with (22), one requires

$$S^i_{\alpha\beta} = S^i_{\beta\alpha}, \quad u_{\beta\alpha} = \bar{u}_{\alpha\beta}.$$  \hspace{1cm} (24)

Thus, nonlinear eq.(23) is equivalent to linear eq.(21) where spectral function $\tilde{U}^{of}(\lambda_1; x)$ is related with dressing functions by the eqs.(1-8,17,19). Detailed discussion of this relation is represented in the next subsection.

### 2.1 Analysis of the system (1-8,17,19,21)

In this section we characterize solution space of nonlinear equation (23) in terms of dressing functions $\Psi$, $\Phi$ and $C$. First step is solving equations (2,6,7) for $\Psi(\lambda, \mu; x)$, $\Phi(\lambda; x)$ and $C(\mu; x)$. Eq.(2) is nonhomogeneous equation for $\Psi(\lambda, \mu; x)$, so we take the following solution:

$$\Psi_{\alpha\beta}(\lambda, \mu; x) = \partial_{x_1}^{-1} \left( \Phi_{\alpha}(\lambda; x) C_{\alpha\beta}(\mu; x) \right) + \delta_{\alpha\beta} \delta(\lambda - \mu) e^{-\sum_{j=2}^{M+1} \left( h_{\alpha}^j(\lambda) + g_{\alpha\beta}^j(\mu) \right)}, \quad \delta(\lambda - \mu) = \prod_{i=1}^{M+1} \delta(\lambda_i - \mu_i)$$  \hspace{1cm} (25)

(remember that dimension of spectral parameters is $M + 1$), where $\delta_{\alpha\beta}$ is Kronecker delta symbol, first term is a particular solution of nonhomogeneous equation, while the second term is particular solution of homogeneous equation associated with eq.(2). Function (25) is not general solution of (2), but this is enough for our algorithm.

Solutions of eqs. (6,7) in view of (17) read

$$\Phi_{\alpha}(\lambda; x) = \int \Phi^0_{\alpha}(\lambda, k) e^{K^\alpha_{\alpha}(\lambda, k; x)} dk, \quad K^\alpha_{\alpha}(\lambda, k; x) = kx_1 + \sum_{j=2}^M (kB^j_\alpha - h^j_{\alpha}(\lambda)) x_j; \quad \text{ (26)}$$

$$C_{\alpha\beta}(\mu; x) = G^1_{\alpha}(\mu_1; x) G^2_{\alpha\beta}(\mu; x), \quad \text{ (27)}$$

$$G^1_{\alpha}(\mu_1; x) = e^{K^\alpha_{\alpha}(\mu_1; x)}, \quad G^2_{\alpha\beta}(\mu; x) = e^{K^\alpha_{\alpha}(\mu; x) C^0_{\alpha\beta}(\mu)}, \quad K^\alpha_{\alpha}(\mu_1; x) = \mu_1 \left( x_1 + \sum_{i=2}^M B^i_{\alpha} x_i \right), \quad K^\alpha_{\alpha}(\mu; x) = -\sum_{j=2}^M g^j_{\alpha\beta}(\mu) x_j,$$

where parameter $k$ is scalar.

Hereafter we take

$$\Phi^0(\lambda, k) = \delta(\lambda_2 - k) I.$$  \hspace{1cm} (28)
Thus expression (25) may be written in explicit form:

\[
\Psi_{\alpha\beta}(\lambda, \mu; x) = \frac{e^{K_\phi^f(\lambda, \lambda_2; \xi) + K_{G^1}(\mu_1; x) + K_{G^2}(\mu_3; x)C_{\alpha\beta}^0(\mu)}}{\lambda_2 + \mu_1} + \delta_{\alpha\beta}\delta(\lambda - \mu)e^{-\sum_{j=2}^{M} (h_\phi^j(\lambda) + g_{\alpha\beta}^j(\mu))},
\]

(29)

Due to the last term in eq.(29), eq.(1) has term \( e^{-\sum_{j=2}^{M} (h_\phi^j(\lambda) + g_{\alpha\beta}^j(\mu))}U_{\alpha\beta}(\lambda; x) \). However, we would like to eliminate factor ahead of \( U \) in this term for convenience of subsequent constructions. To do this, we multiply eqs.(1,11) by \( e_{j=2}^{M} (h_\phi^j(\lambda) + g_{\alpha\beta}^j(\lambda))x_j \):

\[
E^U(\lambda; x) := U(\lambda; x) = -\partial_{x_1}^{-1} \left( \Phi^1(\lambda; x)C(\mu; x) \right) * U(\mu; x) + \Phi^1(\lambda; x),
\]

(30)

\[
E^{G^n}(\lambda; x) := G^n(\lambda; x) = -\partial_{x_1}^{-1} \left( \Phi^1(\lambda; x)C(\mu; x) \right) * G^n(\mu; x) + \partial_{x_1}^{-1} \left( \Phi^1(\lambda; x)C^{1n}(\mu; x) \right) * U(\mu; x) + U^n(\lambda; x), \quad n > 1,
\]

(31)

where

\[
\Phi^1(\lambda; x) = \sum_{j=2}^{M} (h_\phi^j(\lambda) + g_{\alpha\beta}^j(\lambda)) \Phi(\lambda; x) = e^{K_{\Phi^1}(\lambda; x)},
\]

(32)

\[
K_{\Phi^1}(\lambda; x) = \lambda_2 x_1 + \sum_{j=2}^{M} (\lambda_2 B^j_{\alpha} + g_{\alpha\beta}^j(\lambda)) x_j,
\]

\[
U_{\alpha\beta}^n(\lambda; x) = g_{\alpha\beta}^n(\lambda)U_{\alpha\beta}(\lambda; x).
\]

(33)

Below we need function

\[
G_{\alpha\beta}^{1n}(\lambda; x) = G_{\alpha\beta}^2(\lambda; x)g_{\alpha\beta}^n(\lambda).
\]

(34)

Applying \( \int d\bar{\lambda}G^2(\lambda; x) \) to eqs (30,31) and \( \int d\bar{\lambda}G^{1n}(\lambda; x) \) to (30) one gets equations for \( \hat{U} \), \( \hat{G}^n \) and \( \hat{U}^n \):

\[
\hat{U}(\lambda_1; x) = -\int \partial_{x_1}^{-1} \left( \hat{\Phi}(\lambda_1; x)G^1(\mu_1; x) \right) \hat{U}(\mu_1; x)d\mu_1 + \hat{\Phi}(\lambda_1; x),
\]

(35)

\[
\hat{G}^n(\lambda_1; x) = -\int \partial_{x_1}^{-1} \left( \hat{\Phi}(\lambda_1; x)G^1(\mu_1; x) \right) \left( \hat{G}^n(\mu_1; x) - \hat{U}^{1n}(\mu_1; x) \right) d\mu_1 + \hat{U}^{2n}(\lambda_1; x), \quad 1 < n \leq M,
\]

(36)

\[
\hat{U}^{1n}(\lambda_1; x) = -\int \partial_{x_1}^{-1} \left( \hat{\Phi}^{1n}(\lambda_1; x)G^1(\mu_1; x) \right) \hat{U}(\mu_1; x)d\mu_1 + \hat{\Phi}^{1n}(\lambda_1; x), \quad 1 < n \leq M,
\]

(37)

where

\[
\hat{\Phi}(\lambda_1; x) = \int G^2(\lambda; x)\Phi^1(\lambda; x)d\bar{\lambda}, \quad \hat{\Phi}^{1n}(\lambda_1; x) = \int G^{1n}(\lambda; x)\Phi^1(\lambda; x)d\bar{\lambda},
\]

(38)

\[
\hat{U}^{2n}(\lambda_1; x) = \int G^2(\lambda; x)U^n(\lambda; x)d\bar{\lambda} = \int G^{2n}(\lambda; x)U(\lambda; x)d\bar{\lambda},
\]

\[
G_{\alpha\beta}^{2n}(\lambda; x) = G_{\alpha\beta}^2(\lambda; x)g_{\alpha\beta}^n(\lambda).
\]
Equation for $\hat{U}^{2n}$ follows from eq.(30) after applying $\int d\tilde{\lambda}G^{2n}(\lambda; x)$:

$$\hat{U}^{2n}(\lambda_{1}; x) = -\int \partial^{-1}_{x_{1}} \left( \Phi^{2n}(\lambda_{1}; x)G^{1}(\mu_{1}; x) \right) \hat{U}(\mu_{1}; x)d\mu_{1} + \Phi^{2n}(\lambda_{1}; x), \quad n > 1,$$

(39)

$$\Phi^{2n}(\lambda_{1}; x) = \int G^{2n}(\lambda; x)\Phi^{1}(\lambda; x)d\tilde{\lambda}.$$

By construction, function $\Phi(\lambda_{1}; x)$ has arbitrary dependence on variables $x$, if, for instance, $g^{i}_{\alpha\beta}(\lambda) = \lambda_{i+1}g^{i}_{\alpha\beta}$, where $g^{i}_{\alpha\beta}$ are constants, $i = 2, \ldots, M$. Due to this fact $\Phi(\lambda_{1}; x)$ may solve equation (20). Let us transform eq.(20) substituting eqs.(35-39):

$$\sum_{i=2}^{M} S_{\alpha\beta}^{i} \left\{ \partial_{x_{1}} \Phi_{\alpha\beta}(\lambda_{1}; x) - \partial_{x_{1}} \Phi_{\alpha\beta}(\lambda_{1}; x)B_{\beta}^{i} - \lambda_{1} \Phi_{\alpha\beta}(\lambda_{1}; x)(B_{\alpha}^{i} - B_{\beta}^{i}) - \int \sum_{\gamma=1}^{Q} \left[ \frac{\partial^{-1}_{x_{1}}}{U} \left( \Phi_{\alpha\gamma}(\lambda_{1}; x)G^{1}_{\gamma}(\mu_{1}; x) \right) \hat{F}_{\gamma\beta}(\mu_{1}; x) + \partial^{-1}_{x_{1}} \left( (\partial_{x_{1}} \Phi_{\alpha\gamma}(\lambda_{1}; x) - \partial_{x_{1}} \Phi_{\alpha\gamma}(\lambda_{1}; x)B_{\beta}^{i})G^{1}_{\gamma}(\mu_{1}; x) \right) \hat{U}_{\gamma\beta}(\mu_{1}; x) - \lambda_{1} \partial^{-1}_{x_{1}} \left( \Phi_{\alpha\gamma}(\lambda_{1}; x)G^{1}_{\gamma}(\mu_{1}; x) \right) \hat{U}_{\gamma\beta}(\mu_{1}; x)(B_{\alpha}^{i} - B_{\beta}^{i}) \right] d\mu_{1} \right\} = 0$$

where $\alpha \neq \beta$ and eqs.(35-39) give us

$$\hat{F}^{n}(\lambda_{1}; x) = -\int \partial^{-1}_{x_{1}} \left( \Phi(\lambda_{1}; x)G^{1}(\mu_{1}; x) \right) \hat{F}^{n}(\mu_{1}; x)d\mu_{1} - \int \partial^{-1}_{x_{1}} \left( \Phi_{x_{1}}(\lambda_{1}; x) - \Phi_{x_{1}}(\lambda_{1}; x)B^{n}_{x_{1}} \right) G^{1}(\mu_{1}; x) \hat{U}(\mu_{1}; x)d\mu_{1} + \Phi_{x_{n}}(\lambda_{1}; x) - \Phi_{x_{1}}(\lambda_{1}; x)B^{n}_{x_{1}}, \quad 1 < n \leq M.$$ Deriving eqs.(40,41), we took into account an obvious relation

$$\hat{\Phi}_{\alpha\beta}^{2n} - \hat{\Phi}_{\alpha\beta}^{1n} = \partial_{x_{n}} \Phi_{\alpha\beta} - \partial_{x_{1}} \Phi_{\alpha\beta} B_{\beta}^{n}, \quad \alpha \neq \beta, \quad 1 < n \leq M.$$ (42)

Note, that diagonal elements of $\Phi$,

$$\hat{\Phi}_{\alpha\alpha}(\lambda_{1}; x) = \int C^{0}_{\alpha\alpha}(\lambda)e^{\lambda_{1}\sum_{i=2}^{M} B_{\alpha}^{i}x_{i}} \lambda_{1} d\tilde{\lambda},$$

(43)

may be arbitrary functions of single independent variable and $\hat{\Phi}_{\alpha\alpha}^{2n} - \hat{\Phi}_{\alpha\alpha}^{1n} = 0$.

System (35,40,41) represent a complete nonlinear system of equations allowing to find $\hat{U}(\lambda_{1}; x)$ and $\Phi(\lambda_{1}; x)$. Since $u(x) = \int G^{1}(\lambda_{1}; x)\hat{U}(\lambda_{1}; x)d\lambda_{1}$, this system is alternatives form of the nonlinear equation (23). In particular case $C^{0}(\mu) = \delta(\mu)C^{0}(\tilde{\mu})$, one has $G^{1}(0; x) = I$, and this system reduces to PDE for $\varphi(\lambda_{1}; x) = \partial^{-1}_{x_{1}}\Phi(\lambda_{1}; x)$ (below $\alpha \neq \beta$):
\[
\hat{F}^n(\lambda_1; x) = -\varphi(\lambda_1; x)\hat{F}^n(0; x) - \\
\left(\varphi_{x_1}(\lambda_1; x) - \varphi_{x_1}(\lambda_1; x)B^n\right)u(x) + \varphi_{x_1x_1}(\lambda_1; x) - \varphi_{x_1x_1}(\lambda_1; x)B^n,
\]

\(1 < n \leq M,\)

Eqs. (44,46) with \(\lambda_1 = 0\) give us

\[
u(x) = \left(1 + \varphi(0; x)\right)^{-1}\varphi_{x_1}(0; x),
\]

\[
\hat{F}^n(0; x) = \left(1 + \varphi(0; x)\right)^{-1}\left[\varphi_{x_1x_1}(0; x) - \varphi_{x_1x_1}(0; x)B^n - \\
\left(\varphi_{x_1}(0; x) - \varphi_{x_1}(0; x)B^n\right)u(x)\right]
\]

We see that eq.(45) is linear PDE for \(\varphi^{of}(\lambda_1; x)\), \(\lambda_1 \neq 0\), with ”boundary” function \(\varphi(0; x)\) satisfying (47a). By construction, if \(\lambda_1 = 0\), then eq.(45) is projected into (23), i.e. calculation of evolution of \(\varphi^{of}(0; x)\) is equivalent to solving original nonlinear PDE (23). However, from another point of view, this evolution may be found as \(\lim_{\lambda_1 \to 0} \varphi^{of}(\lambda_1; x)\).

The simplest algorithm for numerical construction of particular solutions to (23) is following. For given arbitrary \(\varphi(\lambda_1; x)|_{x_M=0}\) we find \(u(x)|_{x_M=0}\) and \(\tilde{G}^n(0; x)|_{x_M=0} - \tilde{U}^{1n}(0; x)|_{x_M=0}\) using (47). Then solve (45) for \(\varphi^{of}_{x_M}(\lambda_1; x)|_{x_M=0}\). Using Tailor formulae we approximate

\[
\varphi^{of}(\lambda_1; x)|_{x_M=\Delta t} \approx \varphi^{of}(\lambda_1; x)|_{x_M=0} + \Delta t \varphi^{of}_{x_M}(\lambda_1; x)|_{x_M=0}.
\]

Evolution of diagonal elements \(\varphi_{\alpha\alpha}(\lambda_1; x)\) is fixed by \(\varphi_{\alpha\alpha}(\lambda_1; x)|_{x_M=0}\) due to (43). Substitute this result into (47) we find \(u(x)|_{x_M=\Delta t}\) and \(\tilde{G}^n(0; x)|_{x_M=\Delta t} - \tilde{U}^{1n}(0; x)|_{x_M=\Delta t}\). Then eq.(45) gives \(\varphi^{of}_{x_M}(\lambda_1; x)|_{x_M=\Delta t}\), and so on. Solving the Initial Value Problem (IVP) (i.e. construction of \(u^{of}(x)\) for given initial data \(u^{of}(x)|_{x_M=0}\) is more complicated and will not be considered here, since it seems to be not simpler then direct numerical solving of IVP for (23).

Let us remark in the end of this section, that eq.(20) is not the only admissible constrain. Instead of zero in the rhs of this equation one might use expression \(L(\tilde{U}^{of}(\lambda_1; x); u^{of}(x))\) which is linear differential operator applied to \(\tilde{U}^{of}(\lambda_1; x)\). Coefficients of this operator depend on field \(u^{of}(x)\) and its derivatives. Then expression \(L(\tilde{U}^{of}(\lambda_1; x); u^{of}(x))\) appears in the rhs of (21). The only requirement to \(L\) is that after multiplying eq.(21) by \(G^0(\lambda_1; x)\) from the left and integrating over \(\lambda_1\) one gets nonlinear PDE for \(u^{of}\). This new PDE (which replaces eq.(23)) may be largely arbitrary nonlinear PDE for \(u^{of}\). So, as for now, represented multidimensional version of the dressing method is not the method for solving of nonlinear PDE, but it gives a new representation of nonlinear PDE. This situation is equivalent to the situation appearing when Fourier method is applied to PDE other then linear PDE with constant coefficients.

### 3 Derivation of multidimensional Nonlinear Shrödinger Equation

In the previous section we demonstrated that (largely) arbitrary nonlinear PDE can be transformed using a variant of multidimensional generalization of the dressing method for \((2+1)\)-dimensional \(N\)-wave equation. In this section we show that similar construction may be performed starting with the dressing method for \((2+1)\)-dimensional DS. We use notations of the
Sec.2. For simplicity, we take \( Q = 2 \), i.e. consider \( 2 \times 2 \) matrix equations. Variables \( x_i \) are introduced by the following system:

\[
\begin{align*}
\partial_{x_n} \Psi_{\alpha\beta}(\lambda; \mu; x) + \left( h^n_\alpha(\lambda) + g^n_{\alpha\beta}(\mu) \right) \Psi_{\alpha\beta}(\lambda; \mu; x) &= \Phi_\alpha(\lambda; x) B^n_\alpha C_{\alpha\beta}(\mu; x), \quad 1 \leq n < M \\
\partial_{x_M} \Psi_{\alpha\beta}(\lambda; \mu; x) + \left( h^M_\alpha(\lambda) + g^M_{\alpha\beta}(\mu) \right) \Psi_{\alpha\beta}(\lambda; \mu; x) &= \partial_{x_1} \Phi_\alpha(\lambda; x) B^M_\alpha C_{\alpha\beta}(\mu; x) - \Phi_\alpha(\lambda; x) B^M_\alpha \partial_{x_1} C_{\alpha\beta}(\mu; x),
\end{align*}
\]

where the first equation is identical to (2). Since \( Q = 2 \), only two \( B^i \) are linearly independent, so we may put \( B^i = 0, \, i > 2 \) without loss of generality. Let, in addition, \( B^1 = I, \, B^M = B^2 = \text{diag}(1, -1), \, h^1 = g^1 = 0 \). Compatibility of (49) results in (compare with Sec.(2)):

\[
\begin{align*}
\partial_{x_2} \Phi(\lambda; x) + h^2(\lambda) \Phi(\lambda; x) - \partial_{x_1} \Phi(\lambda; x) B^2 &= 0, \\
\partial_{x_n} \Phi(\lambda; x) + h^n(\lambda) \Phi(\lambda; x) &= 0, \quad 2 < n < M \\
\partial_{x_M} \Phi(\lambda; x) + h^M(\lambda) \Phi(\lambda; x) - \partial^2_{x_1} \Phi(\lambda; x) B^2 &= 0,
\end{align*}
\]

\[
\begin{align*}
\partial_{x_2} C(\mu; x) + C^{12}(\mu; x) - B^2 \partial_{x_1} C(\mu; x) &= 0, \\
\partial_{x_n} C(\mu; x) + C^{1n}(\mu; x) &= 0, \quad 2 < n < M, \\
\partial_{x_M} C(\mu; x) + C^{1M}(\mu; x) + B^2 \partial^2_{x_1} C(\mu; x) &= 0,
\end{align*}
\]

where

\[
C^{1n}(\mu; x) = C_{\alpha\beta}(\mu; x) g^n_{\alpha\beta}(\mu), \quad 1 < n \leq M. \tag{52}
\]

Eqs.(50-52) define \( \Phi \) and \( C \).

System of nonlinear equations is generated by eq.(50). Derivation is very similar to derivation carried out in Sec.2. First of all, we use representation for \( \Phi \) as \( \Psi^* \), see eq.(1). Then using equations (49) for derivatives \( \Psi_{x_n} \) and inverting \( \Psi^* \) we end up with system of linear equations in the form

\[
\begin{align*}
E_2(\lambda; x) &:= U_{x_2}(\lambda; x) - U_{x_1}(\lambda; x) B^2 + U(\lambda; x)[B^2, u(x)] - \mathcal{G}^2(\lambda; x) = 0, \\
E_n(\lambda; x) &:= U_{x_n}(\lambda; x) - \mathcal{G}^n(\lambda; x) = 0, \\
&\quad 2 < n < M \\
E_M(\lambda, x) &:= U_{x_M}(\lambda, x) - U_{x_1 x_1}(\lambda, x) B^2 + U(\lambda, x)(u(x)[B^2, u(x)] - 2u x_1(x) B^2 + [u^1, B^2]) + U_{x_1}(\lambda, x)[B^2, u] - \mathcal{G}^M(\lambda; x) = 0
\end{align*}
\]

where functions \( u \) and \( u^1 \) are related with spectral functions by the formula

\[
u(x) = C(\lambda, x) * U(\lambda, x), \quad u^1(x) = C_{x_1}(\lambda, x) * U(\lambda, x)
\]

and functions \( \mathcal{G}^n \) satisfy the following equations:

\[
\begin{align*}
\Psi(\lambda, \mu; x) * \mathcal{G}^n(\mu; x) &= \Psi^n(\lambda, \mu; x) * U(\mu; x), \\
\Psi_{\alpha\beta}(\lambda, \mu; x) &= \Psi_{\alpha\beta}(\lambda, \mu; x) g^n_{\alpha\beta}(\mu), \quad 1 < n \leq M.
\end{align*}
\]

Later, function \( u \) will be field in the nonlinear PDE.
In the case of classical dressing method, nonlinear integrable PDE can be received for function $u$ applying $C(\lambda; x)\ast$ and $C_{x\lambda}(\lambda; x)\ast$ to (53), applying $C(\lambda; x)\ast$ to (55) and using eqs.(51) for $C_{xn}, n > 1$. Doing the same one gets in our case:

$$E_{02}^n(x) := u_{x2}(x) - u_{x1}(x)B^2 + u(x)[B^2, u(x)] = \tilde{H}_{02}^n(x),$$

$$E_{12}^n(x) := u_{x1}^2(x) - u_{x1}(x)B^2 + u_1(x)[B^2, u(x)] = \tilde{H}_{12}^n(x),$$

$$E_M^n(x) := u_{xM}(x) - u_{x_{1}x}(x)B^2 + u(x)\left(u(x)[B^2, u(x)] - 2u_{x1}B^2\right) + u_{x1}(x)[B^2, u(x)] = \tilde{H}_{0M}^n(x),$$

where function $u^2$ is related with spectral function by the formula similar to eq.(10):

$$u^2(x) = C_{x1x}(\lambda; x)\ast U(\lambda; x).$$

(61)

Remark that eqs.(58) coincide with (14) where $n = 2$ and $\tilde{H}_{02} = \tilde{H}_{02}^n$. Functions $u_1$, $u^2$ and $\tilde{H}_{in}$ are "intermediate" functions which will be eliminated from the final nonlinear PDE.

System (58-60) has an obvious limit to classical (2 + 1)-dimensional S-integrable DS. In fact, if $g^2 = g^M = 0$, i.e. $G^2(\lambda; x) = G^M(\lambda; x) = 0$, then $\tilde{H}(x) = [B^2, u^1(x)], \tilde{H}_{12}(x) = [B^2, u^2(x)], \tilde{H}_{0M}(x) = -2u_{x1}^2(x)B^2 - u(x)[u^1(x), B^2] + u^1(x)[B^2, u(x)] - [B^2, u^2(x)] - C^1M(\mu; x)\ast U(\mu; x) + C(\mu; x)\ast G^M(\mu; x),$ then we may eliminate $u_1$ and $u_2$ from eq.(60) using equations (58) and (59):

$$\mathcal{E} := [u_{x_{M+1}}^f, \sigma] - u_{x_{1x1}}^f - u_{x_{2x2}}^f - 8u_{12}u_{21}u^f - 4\varphi u^f = 0$$

$$\varphi_{x_{2x2}} - \varphi_{x_{1x1}} = 4(u_{12}u_{21})_{x_{1x1}}, \varphi = (u_{11} + u_{22})_{x_{1}},$$

(62)

(63)

where

$$u = \begin{pmatrix}
    u_{11} & u_{12} \\
    u_{21} & u_{22}
\end{pmatrix},$$

(64)

which is DS if $x_M = it$, $i^2 = -1$, $u_{21} = \bar{u}_{12}$. Eqs. (53) and (55) with $G^2 = G^M$ = 0 become linear overdetermined system for this equation where spectral function is $U(\lambda; x)$, i.e. eq.(62) is compatibility condition for $E_2$ and $E_M$.

However, if $g^2 \neq 0$ and $g^M \neq 0$, then system (53,55) may not be considered as a linear overdetermined system, since it has set of spectral functions, such as $U(\lambda; x)$ and $G^n(\lambda; x)$. As a consequence, nonlinear eqs. (58-60) have extra fields $\tilde{H}_{in}(x), i = 0, 1,$ and may not be received as compatibility condition of the system (53,55) through commutation of linear operators. So, similar to Sec.2, the only way to derive nonlinear system (58-60) from eqs.(53,55) is the dressing method.

Similar to Sec.2.1, we can take largely arbitrary equation for $\tilde{U}^f(\lambda; x)$ resulting to largely arbitrary nonlinear PDE for field $u^f$. For example, we want to construct such linear equation for $\tilde{U}^f(\lambda; x)$ that after multiplying it by $G^1(\lambda_1; x)$ and integrating over $\lambda_1$ one gets

$$u_{x_{2M}}^f - \Delta u^f B^2 + u^f u_{12}u_{21}B^2 = 0, \Delta = \sum_{i=1}^{M-1} \partial_{x_{1i}}^2,$$

(65)
which becomes multidimensional NLS if $x_M = it$, $i^2 = -1$, $u_{21} = \bar{u}_{12}$. Let $G^1_{2x}(\lambda_1; x) = \lambda_1 G^1(\lambda_1; x)$. Appropriate linear equation is following

$$
\begin{align*}
\hat{U}_{xM}^0(\lambda_1; x) - \Delta \hat{U}_{xM}^0(\lambda_1; x)B^2 + \hat{U}_{xM}^0(\lambda_1; x)u_{12}u_{21} - B^2\lambda^2_{1} \hat{U}_{xM}^0(\lambda_1; x) - \\
2\left( \lambda_1 \hat{U}_{x^2}^0(\lambda_1; x) + \lambda_1 B^2 \hat{U}_{x^2}^0(\lambda_1; x) + \lambda^2_{1} \hat{U}_{xM}^0(\lambda_1; x) \right)B^2 = 0
\end{align*}$$

(66)

Thus nonlinear eq.(65) is equivalent to linear eq.(66) where $\hat{U}$ is expressed in terms of the dressing functions by the system (1,49-52). Detailed discussion of this relation is given in the next subsection.

### 3.1 Analysis of the system (1,49-52,66)

In this section we characterize solution space of nonlinear equation (65) in terms of the dressing functions. First step is solving equations (49-51) for $\Psi(\lambda, \mu; x)$. Solutions of eqs. (50,51) in view of (17) read

$$
\Phi^\alpha(\lambda; x) = e^{\Phi^\alpha(\lambda; x)} = K^\alpha(\lambda, x) = \lambda_2(x_1 + x_2B^2) + x_M \lambda_2B^2 - \sum_{j=2}^{M} h^j_\alpha(\lambda)x_j,
$$

(68)

$$
C_{\alpha\beta}(\mu; x) = G^1_{\alpha}(\mu_1; x)G^2_{\alpha\beta}(\mu; x),
$$

(69)

$$
G^1_{\alpha}(\mu_1; x) = e^{G^1_{\alpha}(\mu_1; x)} = e^{K^G_{\alpha}(\mu_1; x)}C^0_{\alpha\beta}(\mu),
$$

(70)

Thus expression (67) can be written in explicit form:

$$
\Psi_{\alpha\beta}(\lambda, \mu; x) = \frac{e^{\Phi^\alpha(\mu_1; x) + K^G_{\alpha}(\mu_1; x)}\Phi^\beta(\mu; x)}{\lambda_2 + \mu_1} + \delta_{\alpha\beta}(\lambda - \mu)e^{-\sum_{j=2}^{M} (h^j_\alpha(\lambda) + g^j_{\alpha\beta}(\mu))},
$$

(70)

Equations (30-39) have the same form with

$$
\Phi^1_{\alpha\beta}(\lambda; x) = e^{K^G_{\alpha}(\lambda; x)} = K^\alpha(\lambda; x) = \lambda_2(x_1 + x_2B^2) + x_M \lambda_2B^2 + \sum_{j=2}^{M} g^j_{\alpha\beta}(\lambda)x_j.
$$

(71)

Function $\Phi^of$ satisfies equation (66) where $\hat{U}$ is related with $\Phi$ by eq.(35). Note, that diagonal elements of $\Phi$ may be arbitrary functions of single independent variable, similar to Sec.2. In particular case $C^0(\mu) = \delta(\mu_1)C^0(\mu)$, eq.(35) reduces to PDE (44) so that $u$ is defined by the formula (47a). We see that eq.(66) in view of (44) is linear PDE for $\varphi^{of}(\lambda_1; x)$ with "boundary" function $\varphi(0; x)$ satisfying (47a). Remark made in the end of Sec.2.1 regarding numerical construction of particular solutions is relevant for this section as well.
4 Conclusions

We applied a variant of the dressing method to derive a special representation for a largely arbitrary multidimensional nonlinear PDEs nonintegrable in classical sense. Although we have considered only $N$-wave equation and NLS, reducible from the linear eqs. (21) and (66) respectively, different linear equation for the spectral function $\hat{U}(\lambda_1; x)$ may be used. The only requirement is that after multiplying this equation by $G^1(\lambda_1; x)$ and integrating over $\lambda_1$ one gets nonlinear PDE for $\omega^{ef}$.

We introduced several modifications in the classical dressing method:

1. Eqs. (2) (or (49)) with functions $h^n(\lambda)$ and $g^n(\mu)$ showing that derivatives $\Psi_{x_j}(\lambda, \mu; x)$ are not separated functions of spectral parameters.

2. Eq. (17) splitting $C(\lambda; x)$.

3. Extra constrain (20) (or (66) together with (35)) defining structure of PDE (23) (or (65)). This constrain is equation for function $\Phi^{ef}(\lambda_1, x)$ (see for instance eqs. (40, 45) of the Sec.2.1) and has no spectral origin.

At the present form, multidimensional dressing method doesn’t give explicit solutions for nonlinear PDEs, but represents them in different form. We expect perspective development of the ideas outlined in this paper.

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References

[1] C.S. Gardner, J.M. Green, M.D. Kruscal, R.M. Miura, Phys. Rev. Lett, 19, 1095 (1967)

[2] F. Calogero in What is integrability by V.E. Zakharov, Springer, 1, (1990),

[3] V.E. Zakharov, S.V. Manakov, S.P. Novikov and L.P. Pitaevsky, Theory of Solitons. The Inverse Problem Method, (Plenum Press, 1984)

[4] M.J. Ablowitz and H. Segur, Solitons and Inverse Scattering Transform, (SIAM, Philadelphia, 1981)

[5] M.J. Ablowitz and P.A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, (Cambridge University Press, Cambridge, 1991)

[6] E. Date, M. Kashi, M. Jimbo and T. Miwa, Nonlinear integrable Systems - Classical Theory and Quantum Theory, ed. M. Jimbo and T. Miwa (World Scientific, Singapore, 1983), p. 39

[7] E. Date, M. Kashi M. Jimbo and T. Miwa, Publ. RIMS, Kyoto Univ. 19 (1983) 943

[8] M. Sato and Y. Sato, Nonlinear Partial Differential Equations in Applied Science, ed. H. Fujita, P. D. Lax and G. Strang (Kinokuniya/North-Holland, Tokyo, 1983) p. 259
[9] Y.Ohta, J.Satsuma, D.Takahashi and T. Tokihiro, Progr. Theor.Phys. Suppl., No.94, p.210 (1988).

[10] A.V.Mikhailov, A.B.Shabat, V.V.Sokolov, in "What is integrability?" by V.Zakharov, Springer-Verlag., 115 (1991)

[11] Y.Kodama, A.V.Mikhailov, in "Algebraic Aspects of integrability" by I.M.Gelfand and Fokas, Birkhauser, 173 (1996)

[12] V.E.Zakharov and A.B.Shabat, Funct.Anal.Appl. 8, 43 (1974)

[13] V.E.Zakharov and A.B.Shabat, Funct.Anal.Appl. 13, 13 (1979)

[14] V.E.Zakharov and S.V.Manakov, Funct.Anal.Appl. 19, 11 (1985)

[15] L.V.Bogdanov and S.V.Manakov, J.Phys.A:Math.Gen. 21, L537 (1988)

[16] B.Konopelchenko, Solitons in Multidimensions (World Scientific, Singapore, 1993)

[17] C. N. Yang and R. L. Mills, Phys. Rev 96, 191-195 (1954).

[18] A.A.Belavin, A.M.Polyakov, A.S.Schwartz and Yu.S.Tyupkin, Phys.Lett, 59B, 85 (1975)

[19] A.A.Belavin and V.E.Zakharov, Phys. Lett., 73B, 53 (1978)

[20] M.F. Atiyah, V.G. Drinfeld, N.J. Hitchin, Yu. I. Manin, Phys. Lett.A 65 185 (1978)

[21] V.G. Drinfeld, Yu.I. Manin, Journal of Nuclear Phys. 29:1646-1654,1979

[22] J. F. Plebanski, J. Math. Phys. 16, 2395-2402 (1975).

[23] L.V.Bogdanov and B.G.Konopelchenko, Phys. Lett. A 345 (2005) 137-143.

[24] S.V.Manakov and P.M.Santini, Inverse scattering transform for vector fields and for the heavenly equation; arXiv:nlin.SI/0512043.

[25] A.I.Zenchuk, P.M.Santini, nlin.SI/0512062

[26] P.M.Santini, M.J.Ablowitz and A.S.Fokas, J.Math.Phys. 25, 2614 (1984).

[27] A.Zenchuk J.Physics A: Math.Gen. 37, (2004) 6557

[28] E.S.Benilov and S.P.Burtsev, J.Phys.A:Math.Gen., 19 (1986) L177

[29] A.S.Fokas and M.J.Ablowitz, Stud.Appl.Math., 80 (1989) 253

[30] D.J.Kaup, T.I.Lakoba and Y.Matsuno, Inverse Problems, 15 (1999) 215