SOLVABILITY OF A CLASS OF COMPLEX GINZBURG-LANDAU EQUATIONS IN PERIODIC SOBOLEV SPACES

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Abstract. This paper is concerned with the Cauchy problem for the complex Ginzburg-Landau type equation $u_t = (\delta_1 + i\delta_2)\Delta u - i\mu|u|^{2\sigma}u$, $(t, x) \in (0, \infty) \times \mathbb{R}^d$, where $\delta_1 > 0$, $\delta_2, \mu \in \mathbb{R}$ and $d \in \mathbb{N}$. Existence and uniqueness of spatially periodic solutions to the problem are established in a space which corresponds to the Sobolev space on the $d$-dimensional torus when $0 < \sigma < \infty (d = 1, 2)$ and $0 < \sigma < 1/(d - 2)$ $(d \geq 3)$. The result improves the case $p = 2$ of the result in the space $W^{1,p}$ given by Gao-Wang [2, Theorem 1] in which it is assumed that $d < p$ and $\sigma < p/d$.

1. Introduction. In this paper we consider the Cauchy problem for a class of complex Ginzburg-Landau equations

$$
\begin{aligned}
&\frac{\partial u}{\partial t} = (\delta_1 + i\delta_2)\Delta u - i\mu|u|^{2\sigma}u, \quad (t, x) \in (0, \infty) \times \mathbb{R}^d, \\
&u(0, x) = u_0(x), \\
&x \in \mathbb{R}^d,
\end{aligned}
$$

(CGL)

where $i = \sqrt{-1}$, $\sigma > 0$, $\delta_1 > 0$, $\delta_2, \mu \in \mathbb{R}$ and $d \in \mathbb{N}$. We discuss existence and uniqueness of global (in time) solutions to (CGL)$_0$ with initial value $u_0 \in W^{1,2}_{\text{per}}(\mathbb{R}^d)$, where $W^{m,p}_{\text{per}}(\mathbb{R}^d)$ $(m \in \mathbb{N} \cup \{0\}, 1 \leq p \leq \infty)$ is the Sobolev space of spatially periodic functions defined as follows (cf. Kozlov-Maz’ya-Rossmann [6, Section 2.1.1]):

$$
W^{m,p}_{\text{per}}(\mathbb{R}^d) := \{u \in W^{m,p}_{\text{loc}}(\mathbb{R}^d); u(\cdot) = u(\cdot + n) \text{ for all } n \in \mathbb{Z}^d\},
$$

$$
||u||_{m,p} := \left(\sum_{|\alpha| \leq m} \int_{(0,1)^d} |D^\alpha u(x)|^p dx\right)^{1/p} (1 \leq p < \infty),
$$

$$
||u||_{m,\infty} := \max_{|\alpha| \leq m} \left(\text{ess. sup}_{x \in (0,1)^d} |D^\alpha u(x)|\right).
$$

Since spatial periodic functions can be regarded as those on the $d$-dimensional torus $\mathbb{T}^d$, the problem (CGL)$_0$ can be also translated to the problem on $\mathbb{T}^d$. In this paper we shall use $W^{m,p}_{\text{per}}(\mathbb{T}^d)$ instead of $W^{m,p}(\mathbb{T}^d)$ (Sobolev space on $\mathbb{T}^d$) because our interest is solvability of (CGL)$_0$ on $\mathbb{T}^d$ and our treatment is based on functions on $\mathbb{T}^d$.

The problem (CGL)$_0$ is the limiting case of the following usual complex Ginzburg-Landau equation as $\kappa \downarrow 0$ (and $\gamma = 0$):

$$
\begin{aligned}
&\frac{\partial u}{\partial t} = (\delta_1 + i\delta_2)\Delta u - (\kappa + i\mu)|u|^{2\sigma}u + \gamma u, \quad (t, x) \in (0, \infty) \times \mathbb{R}^d, \\
&u(0, x) = u_0(x), \\
&x \in \mathbb{R}^d,
\end{aligned}
$$

(CGL)$_\kappa$
where $\delta_1 > 0$, $\kappa > 0$, $\delta_2, \mu, \gamma \in \mathbb{R}$ and $\sigma > 0$ are constants (see e.g., Ginibre-Velo [3]; note that the problem on $\mathbb{T}^d$ is studied by Levermore-Oliver [7] and the problem on a general domain is considered by Okazawa-Yokota [10]).

Gao-Wang [2, Theorem 1] succeeded in establishing local and global existence of solutions to (CGL)$_0$ in $W^{1,p}(\mathbb{T}^d)$ with $p > d$ and $p > \sigma d$, respectively. If we regard functions on $\mathbb{T}^d$ as periodic functions on $\mathbb{R}^d$ as stated above, then their result is translated as follows; note that their result can deal with the general case $1 < p < \infty$.

Gao-Wang’s result. Let $u_0 \in W^{1,p}(\mathbb{R}^d)$, $\delta_1 > 0$, $\delta_2 \in \mathbb{R}$ and $\sigma \in \mathbb{N}$.

(I) Assume that $W^{1,p}(\mathbb{R}^d) \hookrightarrow W^{0,\infty}(\mathbb{R}^d)$, i.e.,

$$p > d.$$ (1)

Then there exists a unique local solution of (CGL)$_0$.

(II) Assume further that

$$\frac{-2}{\sqrt{1 + (\delta_2/\delta_1)^2} + 1} < p - 2 < \frac{2}{\sqrt{1 + (\delta_2/\delta_1)^2} - 1} \quad \text{and} \quad p > \sigma d.$$ (2)

Then there exists a unique global solution of (CGL)$_0$.

We focus our eyes on the case $p = 2$. If $p = 2$, then $d$ satisfying (1) is restricted to $d = 1$ and the combination of $\sigma$ and $d$ satisfying (2) exists only when $\sigma = 1$ and $d = 1$. In other words, Gao and Wang have not dealt with the case $d \geq 2$ or $\sigma \neq 1$.

The purpose of this paper is to relax the conditions (1) and (2) when $p = 2$ and to extend the restriction from $\sigma \in \mathbb{N}$ to $\sigma \in \mathbb{R}^+: = (0, \infty)$. Here we define local and global solutions of (CGL)$_0$ as follows.

**Definition 1.1.** A function $u$ is said to be a local solution on $[0, T)$ of (CGL)$_0$ if

(i) $u \in C([0, T); W^{1,2}_{\text{per}}(\mathbb{R}^d)) \cap C((0, T); W^{2,2}_{\text{per}}(\mathbb{R}^d)) \cap C^1((0, T); W^{0,2}_{\text{per}}(\mathbb{R}^d));$

(ii) $u$ satisfies (CGL)$_0$ on $[0, T)$ in $W^{0,2}_{\text{per}}(\mathbb{R}^d)$.

In particular, if $T = \infty$, then $u$ is said to be a global solution of (CGL)$_0$.

Now we state local and global existence of solutions to (CGL)$_0$ in the following two theorems, respectively.

**Theorem 1.2** (Local existence). Let $u_0 \in W^{1,2}_{\text{per}}(\mathbb{R}^d)$ and $\delta_1 > 0$. Assume that $\sigma$ satisfies

$$0 < \sigma < \infty \quad (d = 1, 2), \quad 0 < \sigma < \frac{1}{d-2} \quad (d \geq 3).$$ (3)

Then there exists a unique local solution on $[0, T)$ of (CGL)$_0$ for some $T > 0$. Also, let $u$ and $v$ be local solutions on $[0, T)$ of (CGL)$_0$ with $u(0) = u_0$, $v(0) = v_0$ and $\|u(t)\|_{1,2}, \|v(t)\|_{1,2} \leq M$ ($t \in [0, T)$) for some $M > 0$. Then

$$\|u(t) - v(t)\|_{1,2} \leq L e^{\omega t} \|u_0 - v_0\|_{1,2}, \quad t \in (0, T),$$ (4)

where $L$ and $\omega$ are positive constants depending only on $\delta_1, \delta_2, \mu, \sigma, d, T$ and $M$.

**Remark 1.** The case $\sigma = 1/(d-2)$ ($d \geq 3$) is excluded from Theorem 1.2. But local existence and uniqueness of mild solutions to (CGL)$_0$ hold (see Lemma 3.2) in the case. It is open whether the mild solution is the solution in the sense of Definition 1.1 (see also Remark 3).

**Theorem 1.3** (Global existence). Let $u_0 \in W^{1,2}_{\text{per}}(\mathbb{R}^d)$ and $\delta_1 > 0$. Assume that $\sigma$ satisfies the condition (3). Assume that $\sigma$ satisfies the condition (3). Assume further that either (a) or (b) is fulfilled:

(a) $\delta_2 \mu > 0$;

(b) $\delta_2 \mu \leq 0$ and $\frac{|\delta_2|}{\delta_1} < \frac{\sqrt{2\sigma + 1}}{\sigma}$. 
Then there exists a unique global solution of \((\text{CGL}_0)\).

This theorem can be regarded as a limiting case of the results for \((\text{CGL})\) as \(\kappa \downarrow 0\).

Indeed, in [3, 7, 9, 10] global existence of solutions to \((\text{CGL})\) was established under the condition \((a)\) or the following \((b)\):

\((b)\kappa\); \(\kappa > 0\) and \(\frac{2\delta_2}{\delta_1} + \left| \frac{\mu}{\kappa} \right| - 1 \left( \frac{\delta_2\mu}{\delta_1\kappa} - 1 \right) < \frac{\sqrt{2\sigma + 1}}{\sigma}\).

To prove Theorems 1.2 and 1.3 we prepare fundamental estimates for \((\text{CGL})\) which are given in Section 2. In Section 3 we first construct a mild solution of \((\text{CGL}_0)\) and we next prove local existence of solutions (Theorem 1.2). Section 4 is devoted to the proof of global existence (Theorem 1.3). In Section 5 we give some remarks on the inviscid limit (as \(\delta_1 \downarrow 0\)) of solutions to \((\text{CGL}_0)\).

2. Preliminaries. For \(\delta_1 > 0\) and \(\delta_2 \in \mathbb{R}\) we define \(G_t\) as follows:

\[
G_t = G_t(x) := \frac{1}{[4\pi(\delta_1^2 + i\delta_2)]^{\frac{d}{2}}} e^{-\frac{|x|^2}{4\delta_1^2 + 4\delta_2}} t > 0, x \in \mathbb{R}^d, \tag{5}
\]

First we show that \(G_t\) plays a fundamental role in solving

\[
\frac{\partial u}{\partial t} = (\delta_1 + i\delta_2)\Delta u, \quad (t, x) \in (0, \infty) \times \mathbb{R}^d. \tag{6}
\]

The following three lemmas can be proved by the direct calculations.

**Lemma 2.1.** Let \(G_t\) be as in \((5)\) with \(\delta_1 > 0\). Then

(i) \(\int_{\mathbb{R}^d} G_t \, dx = 1\) for every \(t > 0\);

(ii) \(G_t \in C^\infty((0, \infty) \times \mathbb{R}^d)\) is a solution of \((6)\).

**Lemma 2.2.** Let \(G_t\) be as in \((5)\) with \(\delta_1 > 0\). Then for every \(t > 0\),

\[
\|G_t\|_{L^1(\mathbb{R}^d)} = \left[ 1 + \left( \frac{\delta_2}{\delta_1} \right)^2 \right]^\frac{d}{4} =: K, \tag{7}
\]

\[
\left\| \frac{\partial G_t}{\partial x_j} \right\|_{L^1(\mathbb{R}^d)} = \frac{K}{\sqrt{\delta_1}} \frac{1}{\sqrt{t}}, \quad j = 1, \ldots, d, \tag{8}
\]

\[
\|\Delta G_t\|_{L^1(\mathbb{R}^d)} \leq \frac{Kd}{2} \left( \frac{1}{\sqrt{\delta_1^2 + \delta_2^2}} + \frac{1}{\delta_1} \right) \frac{1}{t}. \tag{9}
\]

Now we define \(*\) by the convolution with respect to spatial variables:

\[
(v * w)(x) := \int_{\mathbb{R}^d} v(x - y)w(y) \, dy \quad \text{for } v \in L^1(\mathbb{R}^d), \; w \in W^{0,p}_{\text{per}}(\mathbb{R}^d).
\]

**Lemma 2.3.** Let \(p \in [1, \infty]\). Then for all \(v \in L^1(\mathbb{R}^d)\) and \(w \in W^{0,p}_{\text{per}}(\mathbb{R}^d)\),

\[
v * w \in W^{0,p}_{\text{per}}(\mathbb{R}^d) \quad \text{and} \quad \|v * w\|_{L^1(\mathbb{R}^d)} \leq \|v\|_{L^1(\mathbb{R}^d)} \|w\|_{L^1(\mathbb{R}^d)} \tag{10}
\]

**Remark 2.** If \(v \notin L^1(\mathbb{R}^N)\), then the convolution operator \(w \mapsto v * w\) cannot be defined as the bounded operator from \(W^{0,p}_{\text{per}}(\mathbb{R}^d)\) to \(W^{0,q}_{\text{per}}(\mathbb{R}^d)\) for any \(1 \leq p, q < \infty\). This fact requires \(v \in L^1(\mathbb{R}^d)\).

In the next lemma we describe that \((\delta_1 + i\delta_2)\Delta\) generates an analytic semigroup on \(W^{0,2}_{\text{per}}(\mathbb{R}^d)\) which can be represented by the convolution operator.

**Lemma 2.4.** Let \(G_t\) be as in \((5)\) with \(\delta_1 > 0\). Define \(T(t) := I\) and

\[
T(t)v := G_t * v = \int_{\mathbb{R}^d} G_t(x - y)v(y) \, dy, \quad v \in W^{0,2}_{\text{per}}(\mathbb{R}^d), \; t > 0.
\]
Then $\mathcal{T}(t)$ is a uniformly bounded $C_0$ semigroup on $W^{0,2}_{\text{per}}(\mathbb{R}^d)$ and its infinitesimal generator is given by $(\delta_1 + i\delta_2)\Delta$ with domain $W^{2,2}_{\text{per}}(\mathbb{R}^d)$. Moreover, $\mathcal{T}(t)$ can be extended to an analytic semigroup, and so

$$u(t) := \mathcal{T}(t)u_0, \ t \geq 0$$

is a unique solution of (6) with initial condition $u(0) = u_0 \in W^{0,2}_{\text{per}}(\mathbb{R}^d)$.

**Proof.** First we show that $\mathcal{T}(t)$ is a uniformly bounded $C_0$ semigroup and $\mathcal{T}(t)$ is differentiable for $t > 0$. From (7) and (10) it follows that

$$\|\mathcal{T}(t)v\|_{0,2} \leq \|G_t\|_{L^1(\mathbb{R}^d)}\|v\|_{0,2} = K\|v\|_{0,2}, \ v \in W^{0,2}_{\text{per}}(\mathbb{R}^d), \ t > 0$$

and hence $\mathcal{T}(t)$ is uniformly bounded; note that $\mathcal{T}(0) = I$. Using the Fourier transform, we can verify that $\mathcal{T}(t+s) = \mathcal{T}(t)\mathcal{T}(s)$ for $t, s \geq 0$. Moreover, we see by Lemma 2.1 that $\lim_{t\to 0} \mathcal{T}(t)v = v$ and $\mathcal{T}(t)$ is differentiable for $t > 0$.

Next, in the same way as in the proof of [1, Theorems 7.1], we can show that for $\delta_1 > 0$, $\delta_2 \in \mathbb{R}$ and $f \in W^{0,2}_{\text{per}}(\mathbb{R}^d)$ there exists a solution $u \in W^{2,2}_{\text{per}}(\mathbb{R}^d)$ of the equation $u - (\delta_1 + i\delta_2)\Delta u = f$. Therefore we see that the infinitesimal generator of $\mathcal{T}(t)$ is given by $A := (\delta_1 + i\delta_2)\Delta$ with domain $D(A) = W^{2,2}_{\text{per}}(\mathbb{R}^d)$ and $1 \in \rho(A)$ (cf. Hempel-Voigt [4, Remark 2.1 (c)]). Moreover, (9) and (10) give

$$\| (\delta_1 + i\delta_2)\Delta (G_t * v) \|_{0,2} \leq \frac{Kd}{2} \left( 1 + \frac{\sqrt{\delta_1^2 + \delta_2^2}}{\delta_1} \right) \frac{1}{t} \|v\|_{0,2}, \ t > 0.$$ 

Thus by Pazy [11, Theorem 2.5.2] we obtain the conclusion. 

We can also obtain the following two lemmas for $W^{1,2}_{\text{per}}(\mathbb{R}^d)$.

**Lemma 2.5** (Sobolev’s embeddings). Let $d \in \mathbb{N}$.

(i) If $d \geq 3$, then $W^{1,2}_{\text{per}}(\mathbb{R}^d) \subset W^{2}\ast_{\text{per}}(\mathbb{R}^d)$, where $2\ast := (d - 2)/(2d)$.

(ii) If $d = 2$, then $W^{1,2}_{\text{per}}(\mathbb{R}^d) \subset W^{q}_{\text{per}}(\mathbb{R}^d)$ for every $q \in [2, \infty)$.

(iii) If $d = 1$, then $W^{1,2}_{\text{per}}(\mathbb{R}^d) \subset W^{0}\ast_{\text{per}}(\mathbb{R}^d)$.

Moreover all the above injections are continuous.

**Lemma 2.6** (The Gagliardo-Nirenberg inequality). Assume that $\sigma$ satisfies (3). Then there exists a constant $C = C(\sigma, d) > 0$ such that for every $w \in W^{1,2}_{\text{per}}(\mathbb{R}^d)$,

$$\|w\|_{0,\sigma + 2} \leq C\|w\|_{1.2}^{\frac{d\sigma}{d\sigma + 2}}\|w\|_{0,2}^{1 - \frac{d\sigma}{d\sigma + 2}}.$$ 

Finally we can give useful inequalities (cf. [10, Lemma 4.3] and [9, Lemma 6.2]).

**Lemma 2.7.** Let $\sigma > 0$. Then for every $u, v \in W^{0,4\sigma + 2}_{\text{per}}(\mathbb{R}^d)$,

$$\|u^{2\sigma}v - |u|^{2\sigma}v\|_{0,2} \leq (2\sigma + 1)(\|u\|_{0,4\sigma + 2} + \|v\|_{0,4\sigma + 2})^{2\sigma} \|u - v\|_{0,4\sigma + 2}.$$ 

**Lemma 2.8.** Let $\sigma > 0$. Then for every $u \in W^{2,2}_{\text{per}}(\mathbb{R}^d) \cap W^{1,4\sigma + 2}_{\text{per}}(\mathbb{R}^d)$,

$$\left| \text{Im} \int_{(0,1)^d} (-\Delta u) |u|^{2\sigma} \bar{u} \, dx \right| \leq \frac{\sigma}{\sqrt{2\sigma + 1}} \text{Re} \int_{(0,1)^d} (-\Delta u) |u|^{2\sigma} \bar{u} \, dx.$$ 

3. **Local existence.** The purpose of this section is to show Theorem 1.2, i.e., existence and uniqueness of local solutions to (CGL). For the purpose we first prove existence of mild solutions to (CGL)$_0$ and we next prove Theorem 1.2.
3.1. Mild solutions.

**Definition 3.1.** Let $T > 0$. Then a function $u \in C([0, T]; W^{1, 2}_{\text{per}}(\mathbb{R}^d))$ is said to be a *mild solution* on $[0, T]$ of $(\text{CGL})_0$ if $u$ satisfies

$$u(t) = \begin{cases} u_0, & t = 0, \\ G_t * u_0 + \int_0^t G_{t-s} * f(u(s)) \, ds, & t \in (0, T], \end{cases}$$

where $*$ denotes the convolution w.r.t. spatial variables, $G_t$ is defined by (5) and

$$f(u(t)) := -i\mu |u(t)|^{2\sigma} u(t).$$

**Lemma 3.2.** Let $u_0 \in W^{1, 2}_{\text{per}}(\mathbb{R}^d)$ and assume that

$$0 < \sigma < \infty \ (d = 1, 2), \quad 0 < \sigma \leq \frac{1}{d-2} \ (d \geq 3).$$

Then there exists $T > 0$ and a unique mild solution on $[0, T]$ of $(\text{CGL})_0$.

**Proof.** We prove this lemma by the contraction mapping principle. Let $T > 0$ and

$$B_R := \left\{ v \in C([0, T]; W^{1, 2}_{\text{per}}(\mathbb{R}^d)); \sup_{t \in [0, T]} \|v(t)\|_{1, 2} \leq K\|u_0\|_{1, 2} + 1 =: R \right\}. $$

For every $v \in B_R$, we set $(Sv)(0) := u_0$ and

$$(Sv)(t) := G_t * u_0 + J(t), \quad \text{where} \quad J(t) := \int_0^t G_{t-s} * f(v(s)) \, ds,$$

Then, by virtue of (7), (8) and (10), we can show that for $u \in B_R$,

$$\|Sv(t)\|_{1, 2} \leq K\|u_0\|_{1, 2} + C(t)R^{2\sigma+1},$$

where

$$C(t) := K\sqrt{t} \left( t + \frac{4d}{\delta_1 \pi} \right)^{\frac{1}{2}} C_1^{2\sigma+1} |\mu|,$$

and $C_1$ is a positive constant satisfying $\|w\|_{0, 4\sigma+2} \leq C_1\|w\|_{1, 2}$ for $w \in W^{1, 2}_{\text{per}}(\mathbb{R}^d)$. Indeed, let $v \in B_R$ and $t \in (0, T]$. By (7) and (10) we obtain

$$\|G_t * u_0\|_{0, 2} \leq K\|u_0\|_{0, 2},$$

$$\left\| \frac{\partial (G_t * u_0)}{\partial x_j} \right\|_{0, 2} = \left\| G_t * \frac{\partial u_0}{\partial x_j} \right\|_{0, 2} \leq K\left\| \frac{\partial u_0}{\partial x_j} \right\|_{0, 2}.$$

Therefore we have

$$\|G_t * u_0\|_{1, 2} \leq K\|u_0\|_{1, 2}. \quad (13)$$

On the other hand, we see from (7) and (10) that

$$\|J(t)\|_{0, 2} \leq Kt \sup_{s \in [0, t]} \|f(v(s))\|_{0, 2},$$

$$\left\| \frac{\partial J(t)}{\partial x_j} \right\|_{0, 2} \leq \frac{2K}{\sqrt{\delta_1 \pi}} \sqrt{t} \sup_{s \in [0, t]} \|f(v(s))\|_{0, 2}.$$

Combining these inequalities and using Lemma 2.5, we obtain

$$\|J(t)\|_{1, 2} \leq C(t) \sup_{s \in [0, t]} \|v(s)\|_{1, 2}^{2\sigma+1}. \quad (14)$$

In view of (13) and (14) we have (12). Next we show that for $u, v \in B_R$,

$$\sup_{t \in [0, T]} \|Su(t) - Sv(t)\|_{1, 2} \leq C(T)(2\sigma + 1)(2R)^{2\sigma} \sup_{t \in [0, T]} \|u(t) - v(t)\|_{1, 2}. \quad (15)$$
As in the proof of (14), we see from Lemma 2.7 that

\[ \|Su(t) - Sv(t)\|_{1,2} \leq K \sqrt{t} \left( t + \frac{4d}{\delta_1 \pi} \right) \frac{1}{2} \frac{\|\mu\|}{\sup_{s \in [0,t]} \|u(s)\|} \|u(s) - v(s)\|_{1,2} \]

\[ \leq C(t)(2\sigma + 1)(2R)^{2\sigma} \sup_{s \in [0,t]} \|u(s) - v(s)\|_{1,2}. \]

Taking the supremum on \([0, T]\), we have (15). Therefore if we take \(T\) sufficiently small, then the mapping \(S\) is a contraction on \(B_R\). Consequently, the contraction mapping principle yields that there exists a unique solution of (11) in \(B_R\).

Finally we show uniqueness of mild solutions to (CGL). Let \(u\) and \(v\) be two mild solutions on \([0, T]\) of (CGL). Then we can see from Lemma 2.7, (7) and (8) that

\[ \|u(t) - v(t)\|_{1,2} \leq \bar{C}(T)(2\sigma + 1)(2M)^{2\sigma} \int_0^t \frac{\|u(s) - v(s)\|_{1,2}}{\sqrt{t - s}} ds, \]

(16)

where

\[ \bar{C}(T) := K \left( T + \frac{d}{\delta_1 \pi} \right)^{\frac{1}{2}} \frac{C_1^{2\sigma + 1}}{2\sigma} \|\mu\|, \]

\[ M := \max \left\{ \sup_{t \in [0,T]} \|u(t)\|_{1,2}, \sup_{t \in [0,T]} \|v(t)\|_{1,2} \right\}. \]

Multiplying (16) to \(1/\sqrt{r-t}\) for \(r \in (0, T]\) and integrating in \(t\) over \([0, r]\):

\[ \int_0^r \frac{\|u(t) - v(t)\|_{1,2}}{\sqrt{r - t}} dt \leq \bar{C}(T)(2\sigma + 1)(2M)^{2\sigma} \int_0^r \int_0^t \frac{\|u(s) - v(s)\|_{1,2}}{\sqrt{(r - t)(t - s)}} ds dt \]

\[ = \bar{C}(T)(2\sigma + 1)(2M)^{2\sigma} \pi \int_0^r \|u(s) - v(s)\|_{1,2} ds. \]

(17)

Combining (17) with (16), we have

\[ \|u(t) - v(t)\|_{1,2} \leq [\bar{C}(T)(2\sigma + 1)(2M)^{2\sigma}]^{\frac{1}{2}} \pi \int_0^t \|u(s) - v(s)\|_{1,2} ds. \]

Hence by Gronwall’s inequality we obtain that \(\|u(t) - v(t)\|_{1,2} \leq 0\). Therefore we conclude that \(u = v\) on \([0, T]\). \(\square\)

3.2. Local solutions. We show that the mild solution of (CGL) is the local one of (CGL) in the sense of Definition 1.1.

Proof of Theorem 1.2. Let \(T(t)\) be as in Lemma 2.4, \(T > 0\) and \(\tilde{u}\) the mild solution on \([0, T]\) of (CGL) obtained in Lemma 3.2:

\[ \tilde{u}(t) = T(t)u_0 + \int_0^t T(t-s)f(\tilde{u}(s))ds. \]

Then it suffices by Puzy [11, Corollary 4.3.3] and Lemma 2.4 to show that \(f(\tilde{u}) \in L^1(0, T; W_{\text{per}}^{0,2}(\mathbb{R}^d))\) is locally Hölder continuous on \((0, T]\). Setting

\[ \tilde{M} := \sup_{t \in [0,T]} \|\tilde{u}(t)\|_{1,2}, \]

we see from Lemma 2.5 that \(f(\tilde{u}) \in L^1(0, T; W_{\text{per}}^{0,2}(\mathbb{R}^d))\) and for \(s \in [0, T]\),

\[ \|f(\tilde{u}(s))\|_{0,2} = |\mu|\|\tilde{u}(s)\|^2_{0,4\sigma + 1} \leq |\mu|\tilde{M}^{2\sigma + 1}C_1^{2\sigma + 1}. \]

(18)

Let \(t \in [0, T]\) and \(h \in (0, T - t]\). Then we have

\[ \tilde{u}(t + h) - \tilde{u}(t) = \int_0^h (\delta_1 + i\delta_2) \Delta [G_s * \tilde{u}(t)] ds + \int_t^{t + h} [G_{t+h-s} * f(\tilde{u}(s))] ds. \]

(19)
Here Lemmas 2.2 and 2.3 give
\[
\left\| \int_0^h (\delta_1 + i\delta_2) \Delta [G_s \ast \tilde{u}(t)] \, ds \right\|_{0,2} \leq \sqrt{\delta_1^2 + \delta_2^2} \sum_{j=1}^d \int_0^h \left\| \frac{\partial G_s}{\partial x_j} \ast \frac{\partial \tilde{u}}{\partial x_j}(t) \right\| \, ds \\
\leq 2K M d \sqrt{\frac{\delta_1^2 + \delta_2^2}{\delta_1 \pi}} \sqrt{h} \tag{20}
\]
and (18) yields
\[
\left\| \int_t^{t+h} [G_{t+h-s} \ast f(\tilde{u}(s))] \, ds \right\|_{0,2} \leq K |\mu| M^{2\sigma+1} C_1^{2\sigma+1} h. \tag{21}
\]
Combining (20) and (21) with (19), we have
\[
\|\tilde{u}(t+h) - \tilde{u}(t)\|_{0,2} \leq K M \left( \frac{2d\sqrt{\delta_1^2 + \delta_2^2}}{\delta_1 \pi} + |\mu| M^{2\sigma} C_1^{2\sigma+1} \sqrt{T} \right) \sqrt{h} =: M_1 \sqrt{h}.
\]
Therefore it follows from Lemmas 2.6 and 2.7 that
\[
\|f(\tilde{u}(t+h)) - f(\tilde{u}(t))\|_{0,2} \leq (2\sigma + 1) |\mu| C_1^{2\sigma} C(2M)^{2\sigma + \frac{d\sigma}{2\pi \pi}} \|\tilde{u}(t+h) - \tilde{u}(t)\|_{0,2}^{1 - \frac{d\sigma}{2\pi \pi}} \\
\leq (2\sigma + 1) |\mu| C_1^{2\sigma} C(2M)^{2\sigma + \frac{d\sigma}{2\pi \pi}} M_1^{1 - \frac{d\sigma}{2\pi \pi}} \sqrt{h}^{1 - \frac{d\sigma}{2\pi \pi}}. \tag{22}
\]
Noting that $1 - \frac{d\sigma}{2\sigma + 1} > 0$ by virtue of the condition (3), we see that $f(\tilde{u})$ is locally Hölder continuous on $[0, T]$. Consequently, we conclude that $\tilde{u}$ is a local solution on $[0, T]$ of (CGL)$_0$. Uniqueness follows from that of mild solutions.

Finally we prove (4). Let $M$ be as Proof of Lemma 3.2 and let $u, v$ be local solutions on $[0, T]$ of (CGL)$_0$ with initial data $u_0, v_0$, respectively. As in the proof of (16), we have
\[
\|u(t) - v(t)\|_{1,2} \leq K \|u_0 - v_0\|_{1,2} + \bar{C}(T)(2\sigma + 1)(2M)^{2\sigma} \int_0^t \frac{\|u(s) - v(s)\|_{1,2}}{\sqrt{T-s}} \, ds.
\]
In the same way as in Proof of Lemma 3.2 we have (4):
\[
\|u(t) - v(t)\|_{1,2} \leq Le^{\omega t} \|u_0 - v_0\|_{1,2},
\]
where $L := K \left[ 1 + 2\sqrt{T\bar{C}(T)(2\sigma + 1)(2M)^{2\sigma}} \right]$ and $\omega := \bar{C}(T)(2\sigma + 1)(2M)^{2\sigma}$. This completes the proof of Theorem 1.2. \hfill \Box

Remark 3. In Lemma 3.2 we have proved local existence of mild solutions to (CGL)$_0$ even if $\sigma = 1/(d - 2)$ ($d \geq 3$). However, the estimate (22) does not imply the Hölder continuity of $f(\tilde{u})$ when $\sigma = 1/(d - 2)$.

4. Global existence. To extend the local solution $u$ of (CGL)$_0$ to the global one we give uniform estimates in $t$ for $\|u(t)\|_{0,2}$ and $\|\nabla u(t)\|_{0,2}$.

Lemma 4.1. Let $\delta_1 > 0$. Let $u$ be a local solution on $[0, T]$ of (CGL)$_0$ with initial value $u_0 \in W^{1,2}_{\text{per}}(\mathbb{R}^d)$. Then
\[
\|u(t)\|_{0,2} \leq \|u_0\|_{0,2}, \quad t \in [0, T). \tag{23}
\]
Assume further that the assumption in Theorem 1.3 is satisfied. Then there exists a constant $L_0 > 0$ depending only on $\delta_1, \delta_2, \mu$ and $\sigma$ such that
\[
\|\nabla u(t)\|_{0,2}^2 \leq \|\nabla u_0\|_{0,2}^2 + L_0 \|u_0\|_{1,2}^{2(\sigma+1)}, \quad t \in [0, T). \tag{24}
\]
Proof. Multiplying the equation in \((CGL)_0\) by \(\bar{u(t)}\), integrating it over \((0, 1)^d\), taking its real part and using integration by parts, we obtain
\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|_{0, 2}^2 = \Re \int_{(0, 1)^d} [\langle \delta_1 + i\delta_2 \rangle \Delta u(t) - i\mu |u(t)|^{2\sigma} u(t) \bar{u(t)}] dx
\]
\[
= -\delta_1 \|\nabla u(t)\|_{0, 2}^2 \leq 0.
\]
Thus we obtain (23). To prove (24) we set
\[
Z(t) := \int_{(0, 1)^d} [\Delta u(t)] |u(t)|^{2\sigma} \bar{u(t)}) dx.
\]
Multiplying the equation in \((CGL)_0\) by \(-\Delta u(t)\) and \(|u(t)|^{2\sigma} \bar{u(t)}\), in the same way as in the proof of (23), we have
\[
\frac{1}{2} \cdot \frac{d}{dt} \|\nabla u(t)\|_{0, 2}^2 = -\delta_1 \|\Delta u(t)\|_{0, 2}^2 - \Re \left[ i\mu \int_{(0, 1)^d} [\Delta u(t)] |u(t)|^{2\sigma} \bar{u(t)} dx \right]
\]
\[
= -\delta_1 \|\Delta u(t)\|_{0, 2}^2 - \mu \Im Z(t)
\]  
(25)
and
\[
\frac{1}{2(\sigma + 1)} \cdot \frac{d}{dt} \|u(t)\|_{0, 2\sigma + 2}^{2(\sigma + 1)} = -\Re \left[ (\delta_1 + i\delta_2) \int_{(0, 1)^d} [\Delta u(t)] |u(t)|^{2\sigma} \bar{u(t)} dx \right]
\]
\[
= -\frac{1}{\delta_1} \Re Z(t) + \delta_2 \Im Z(t),
\]
(26)
respectively. Noting that (3) implies \(W_{per}^{2, 2}(\mathbb{R}^d) \subset W_{per}^{1, 4\sigma + 2}(\mathbb{R}^d)\), we see from (25), (26) and Lemma 2.8 that \(\Re Z(t) \geq 0\) and for any \(k \geq 0\),
\[
\frac{d}{dt} \left[ \frac{1}{2} \|\nabla u(t)\|_{0, 2}^2 + \frac{k}{2(\sigma + 1)} \|u(t)\|_{0, 2\sigma + 2}^{2(\sigma + 1)} \right]
\]
\[
= -\delta_1 \|\Delta u(t)\|_{0, 2}^2 - k\delta_1 \Re Z(t) - (\mu - k\delta_2) \Im Z(t)
\]
\[
\leq (C_{(\sigma)} |\mu - k\delta_2| - k\delta_1) \Re Z(t),
\]
(27)
where \(C_{(\sigma)} := \sigma / \sqrt{2\sigma + 1} > 0\). By virtue of the conditions on \(\delta_1, \delta_2, \mu\) and \(\sigma\) in Theorem 1.3, we can choose \(k \geq 0\) satisfying
\[
C_{(\sigma)} |\mu - k\delta_2| - k\delta_1 \leq 0.
\]
(28)
In fact, we have (28) by taking
\[
k := \begin{cases}
\frac{\mu}{\delta_2} & \text{if } \delta_2 \mu > 0, \\
\frac{C_{(\sigma)} |\mu|}{\delta_1 - C_{(\sigma)} |\delta_2|} & \text{if } \delta_2 \mu \leq 0 \text{ and } \left| \frac{\delta_2}{\delta_1} \right| < \frac{\sqrt{2\sigma + 1}}{\sigma} = \frac{1}{C_{(\sigma)}}.
\end{cases}
\]
Combining (28) with (27) yields
\[
\frac{d}{dt} \left[ \frac{1}{2} \|\nabla u(t)\|_{0, 2}^2 + \frac{k}{2(\sigma + 1)} \|u(t)\|_{0, 2\sigma + 2}^{2(\sigma + 1)} \right] \leq 0.
\]
Thus from Lemma 2.5 we obtain
\[
\frac{1}{2} \|\nabla u(t)\|_{0, 2}^2 \leq \frac{1}{2} \|\nabla u_0\|_{0, 2}^2 + \frac{k}{2(\sigma + 1)} \|u_0\|_{0, 2\sigma + 2}^{2(\sigma + 1)}
\]
\[
\leq \frac{1}{2} \|\nabla u_0\|_{0, 2}^2 + \frac{kC^2_{2(\sigma + 1)}}{2(\sigma + 1)} \|u_0\|_{0, 1, 2}^{2(\sigma + 1)},
\]
where \(C_2\) is a positive constant satisfying \(\|w\|_{0, 2\sigma + 2} \leq C_2 \|w\|_{1, 2}\) for \(w \in W_{per}^{1, 2}(\mathbb{R}^d)\). This is nothing but the desired inequality (24).

We are in a position to complete the proof of Theorem 1.3.
Proof of Theorem 1.3. Using Lemma 4.1, we have
\[ \|u(t)\|_{1,2}^2 \leq \|u_0\|_{1,2}^2 + L_0\|u_0\|_{1,2}^{2(\sigma+1)}, \quad t \in [0, T). \]
Therefore we see from the standard argument that \( u \) can be extended to the global solution of \((CGL)\).

We finish the proof of Theorem 1.3. \( \square \)

5. Remarks on the inviscid limits. In this section we consider the inviscid limit of solutions to \((CGL)\) (cf. Huang-Wang [5], Ogawa-Yokota [8] and their references therein).

Letting \( \delta_1 \downarrow 0 \) in \((CGL)\), we obtain the Cauchy problem for nonlinear Schrödinger equations
\[
\begin{aligned}
\frac{\partial u}{\partial t} + i\delta_2 \Delta v - i\mu \|u\|^{2\sigma} v, \quad (t, x) \in (0, \infty) \times \mathbb{R}^d, \\
v(0, x) = u_0(x), \quad x \in \mathbb{R}^d.
\end{aligned}
\]
(NLS)

Here we point out some remarks in order.

Remark 4. For \( \varepsilon > 0 \) let \( u_\varepsilon \) be a unique global solution of \((CGL)\) with \( \delta_1 = \varepsilon \) and \( u_\varepsilon(0) = u_0 \in W^{1,2}_{\text{per}}(\mathbb{R}^d) \). For example, we consider the case that \( \sigma \) satisfies the condition (3) and \( \delta_2 \mu > 0 \). It follows from (24) that for every \( t \in [0, \infty) \),
\[ \|\nabla u_\varepsilon(t)\|_{0,2}^2 \leq \|\nabla u_0\|_{0,2}^2 + L_0\|u_0\|_{1,2}^{2(\sigma+1)}; \]

note that \( L_0 \) is the constant independent of \( \varepsilon \). This implies that the family \( \{u_\varepsilon\}_{\varepsilon>0} \) is bounded in \( L^\infty(0, \infty; W^{1,2}_{\text{per}}(\mathbb{R}^d)) \). Hence there exist a sequence \( \{\varepsilon_n\} \) (\( \varepsilon_n \downarrow 0 \) as \( n \to \infty \)) and a function \( u \in L^\infty(0, \infty; W^{1,2}_{\text{per}}(\mathbb{R}^d)) \) such that
\[ u_{\varepsilon_n} \to u \quad (n \to \infty) \quad \text{weakly* in } \mathcal{L}^\infty(0, T; W^{0,2}_{\text{per}}(\mathbb{R}^d)) \quad \forall T > 0. \]

Since the embedding \( W^{1,2}_{\text{per}}(\mathbb{R}^d) \hookrightarrow W^{0,2}_{\text{per}}(\mathbb{R}^d) \) is compact, we can show that
\[ u_{\varepsilon_n}(t) \to u(t) \quad (n \to \infty) \quad \text{strongly in } W^{0,2}_{\text{per}}(\mathbb{R}^d) \quad \text{a.a. } t \in (0, \infty). \] (29)

Consequently, we conclude that \( u \) is a global weak solution of (NLS).

Remark 5. In the one-dimensional case we can derive the convergence rate of (29). In the same way as in [2], we can show from Lemma 2.5 that for every \( t \in (0, \infty) \),
\[ \|u_{\varepsilon_n}(t) - u(t)\|_{0,2} \leq \left( \frac{\varepsilon_n^{-\frac{1}{2}}}{2} \right) [\|\nabla u_0\|_{0,2}^2 + L_0\|u_0\|_{1,2}^{2(\sigma+1)}] e^{-\omega_1 t}, \]

where \( \omega_1 := (2\sigma + 1)\mu(2C_3)^{2\sigma}[\|u_0\|_{1,2}^2 + L_0\|u_0\|_{1,2}^{2(\sigma+1)}]^{\sigma} \geq 0 \) and \( C_3 \) is a positive constant satisfying \( \|w\|_{0,\infty} \leq C_3\|w\|_{1,2} \) for \( w \in W^{1,2}_{\text{per}}(\mathbb{R}) \).

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