Ring of physical states in the M(2,3) Minimal Liouville gravity

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Abstract

We consider the M(2,3) Minimal Liouville gravity, whose states in the gravity sector are represented by irreducible modules of the Virasoro algebra. We present a recursive construction for BRST cohomology classes. This construction is based on using an explicit form of singular vectors in irreducible modules of the Virasoro algebra. We construct an algebra of operators acting on the BRST cohomology space. The operator algebra of physical states is established by use of these operators.

1. Introduction

The Liouville gravity is a dynamic theory of the metric on certain two-dimensional manifold whose action is induced by a critical matter, i.e., matter described by a conformal field theory (CFT). Simple reaction of conformal theories to the scaling of the metric leads to the universal form of the effective action of the generated gravity called Liouville gravity \cite{1}. In the David and Distler-Kawai (DDK) approach \cite{2} the Liouville gravity can be represented as a tensor product of a conformal matter theory, the Liouville theory, and a ghost system. Schematically, the action for the Liouville gravity can be written in the form

\[ S = S^M + S^L + S^{gh}. \]  \hspace{1cm} (1.1)

The consistency condition by David and Distler-Kawai impose a restriction on the central charges of these theories. The restriction reads that the total central charge of the critical matter, the Liouville theory and the ghosts system vanishes

\[ c_L + c_M + c_{gh} = 0. \]  \hspace{1cm} (1.2)

Thus in the critical gravity these three field theories are formally decoupled and only interact due to the conformal anomaly cancellation condition \ref{1.2}.

In this paper we consider the particular case of the Liouville gravity, namely, a Minimal Liouville gravity where the conformal matter is a Minimal CFT \cite{3}. In this case it is possible to investigate a space of physical states in detail.

The simplest states the ghost number\textsuperscript{1} 1 are the matter highest weight vectors "dressed" by an appropriate Liouville highest weight vectors such that the total conformal dimension of the state (including ghosts) is equal to 0. The simple structure of these states make it possible to study the corresponding operators in detail. In particular, their three- and four-point functions and an operator algebra have been found explicitly \cite{4,5}.

Lian and Zuckerman \cite{6} have realized that for a given matter highest weight state there exists infinitely many additional states with arbitrary ghost numbers. It is an interesting problem to consider additional states in more detail and investigate the operator algebra of the corresponding operators.

Note, that there are two versions of the Liouville gravity. In the first version the gravity sector is realized by a theory of free scalar field. From the mathematical point of view, it may be said that the space of states in gravity sector is direct sum of the Feigin-Fuchs modules \cite{7}. In this case the operator algebra has been investigated by Kanno and Sarmadi \cite{8}.

\textsuperscript{1}See section\textsuperscript{2} for the definition of the ghost number.
In the second version of the Liouville gravity the space of states in the gravity sector is represented by irreducible modules [9]. It has been shown [6] that these two versions of the Liouville gravity possess different spaces of physical states.

In this paper we consider the second formulation of the Liouville gravity. It is an interesting problem to consider the additional states. The natural way to construct physical states is the BRST procedure. In the BRST quantization procedure physical states are identified with BRST cohomology classes. Generally, relative cohomology classes are called physical. In this paper we present a recursive procedure to construct relative cohomology classes. This generalizes construction of [10]

However we show that the definition of physical states as relative cohomology classes is not completely suitable. The problem is that the operator algebra of relative cohomology classes is not associative. In order to avoid this problem we extend the space of states and consider absolute cohomology classes. It is convenient to start from relative cohomology classes and construct the absolute cohomology classes using certain procedure.

We define certain operators that acts on the cohomology space. These operators allow us to calculate absolute cohomology and investigate the operator algebra of absolute cohomology classes.

2. Notation

The natural way to quantize the theory (1.1) with the constraint (1.2) is the BRST procedure. Let \((b, c)\) be the conformal ghost system of weights \((2, -1)\). The ghost fields \(b\) and \(c\) admit the following Laurent expansions:

\[
b(z) = \sum_{n=-\infty}^{\infty} b_n z^{-n+2}, \quad c(z) = \sum_{n=-\infty}^{\infty} c_n z^{-n-1},
\]

where the coefficients \(b_n\) and \(c_n\) form an algebra with the only nonzero anticommutation relation

\[
\{b_m, c_n\} = \delta_{m+n,0}.
\]

We denote the Fock representation of the ghost system by \(\Lambda^{bc}\). Define a vacuum state \(|0\rangle_g\) in \(\Lambda^{bc}\) by the conditions

\[
b_m |0\rangle_g = 0, \quad m \geq -1, \quad c_n |0\rangle_g = 0, \quad n \geq 2.
\]

This vacuum is an \(SL(2, \mathbb{C})\) invariant with the conformal dimension 0. We assign a ghost number 0 to this vacuum. The ghost field \(b\) decreases the ghost number by 1, while \(c\) increases the ghost number by 1. It is convenient to define a vacuum \(|v\rangle_g = c_1 |0\rangle_g\) of the conformal dimension \(-1\) and the ghost number 1.

Let us consider a conformal field theory (CFT) with the central charge \(c = 26\). In this paper we consider only chiral part of the CFT. Let \(T(z)\) denote stress tensor. The modes \(L_n\) of the stress tensor are given by the Laurent expansion

\[
T(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}}.
\]

It was shown in [3] that operators \(L_n\) satisfy commutation relations

\[
[L_n, L_m] = (n - m)L_{n+m} + \delta_{n+m,0} \frac{n^3 - n}{12} c,
\]

and generate the Virasoro algebra (Vir). Then the space of states in a conformal field theory is representation of Vir.

Let \(\mathcal{M}\) be any representation of the CFT. Introduce a Hilbert space

\[
C^{abs}_k (\mathcal{M}) = \mathcal{M} \otimes \Lambda^{bc}.
\]

Denote by \(C^{abs}_k (\mathcal{M})\) the subspace of states of the definite ghost number \(k\). The superscript ‘abs’ stresses the fact that the space \(C^{abs}_k (\mathcal{M})\) forms an absolute BRST complex with respect to the BRST operator:

\[
Q = \oint : T(z) + \frac{1}{2} T^{gh}(z) : c(z) : \frac{c_0}{2} = \sum_{n=-\infty}^{\infty} L_{-n} c_n - \frac{1}{2} \sum_{m,n=-\infty}^{\infty} (m - n) : c_m c_n b_{n+m} : \frac{c_0}{2},
\]

2
where $T^\text{gh}(z)$ is the stress tensor for the ghost system. It is well known that $Q^2 = 0$ if and only if $c_M = 26$. We denote the cohomology space of $C^\text{abs}_*(\mathcal{M})$ by $H^\text{abs}_*(\mathcal{M})$.

Let us introduce the subcomplex

$$C^\text{rel}_*(\mathcal{M}) = \{ w \in C^\text{abs}_*(\mathcal{M}) \mid b_0 w = (L_0 + L^\text{gh}_0) w = 0 \}. $$

This subcomplex is called the relative BRST complex. We denote its cohomology by $H^\text{rel}_*(\mathcal{M})$.

In this paper we restrict our consideration to the $M(2,3)$ Minimal gravity, where the conformal matter is a Minimal CFT $M_{2,3}$ with the central charge $c_M = 0$. The only matter primary field possess the conformal dimension $\Delta_M = 0$ and the only representation in the Hilbert space is the identity representation $\mathbb{1}$. One may say that the matter sector of the model is trivial. The gravitational sector is represented by the direct sum of the Virasoro irreducible modules $L_\Delta$ with the central charge $c_L = 26$ and the highest weights $\Delta \in \mathbb{C}$.

Thus, the case

$$\mathcal{M} = \bigoplus_{\Delta \in \mathbb{C}} (\mathbb{1} \otimes L_\Delta) = \bigoplus_{\Delta \in \mathbb{C}} L_\Delta$$

corresponds to the Minimal Liouville Gravity $M(2, 3)$.

Let $|L_\Delta\rangle$ be the highest weight vector in the Vir irreducible module $L_\Delta$. It is convenient to define a vacuum vector in $C^\text{rel}_*(L_\Delta)$ by

$$\Psi_\Delta = |L_\Delta\rangle \otimes |v^g\rangle,$$  \hspace{1cm} (2.1)

In the BRST quantization procedure the physical states $w$ are defined to be BRST cohomology classes, that is $Qw = 0$ where the states $w$ are not BRST exact.

3. Relative BRST complex

3.1. Lian–Zuckerman theorems

First, we describe Lian–Zuckerman theorem for irreducible Vir representations. Let us consider the relative BRST complex $C^\text{rel}_*(L_\Delta)$, where $L_\Delta$ is the irreducible Vir module with the highest weight $\Delta$. The cohomology $H^\text{rel}_*(L_\Delta)$ depend on the value of the highest weight $\Delta$. More precisely, $H^\text{rel}_*(L_\Delta)$ is non-trivial for $\Delta$ belongs to some countable set $E = \{a_n, b_n\}$ of complex numbers. This numbers are given in terms of the Kac conformal dimensions

$$\Delta_{r,s} = \frac{25 - (3r + 2s)^2}{24}$$

as

$$a_n = \Delta_{1,1+3(n-1)}, \quad n \geq 0 \quad \text{and} \quad b_n = \Delta_{1,2+3(n-1)}, \quad n \geq 1.$$  \hspace{1cm} (3.1)

This numbers appear in the study of the structure of Verma modules with $c = 26$ [11]. By $V_\Delta$ denote the Verma module with highest weight $\Delta$ and central charge $c = 26$. The Verma module $V_{a_0}$ is irreducible: $V_{a_0} = L_{a_0}$. But the Verma modules $V_{a_1}$ and $V_{b_1}$ possess a null vector of the weight $a_0$. It means that both $V_{a_1}$ and $V_{b_1}$ contain the module $V_{a_0}$ as a submodule. This can be continued: there is an infinite ladder of Verma modules $V_{a_k}$, $V_{b_k}$, $k = 1, 2, \ldots$ that contain the modules $V_{a_{k-1}}$ and $V_{b_{k-1}}$ as submodules. This can be represented by the following embedding diagram [11]:

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By the (embedding) level of the modules \(V_{\alpha_k}, V_{\beta_k}\) we shall call the value \(k\). An arrow connecting two nodes \(\Delta \rightarrow \Delta'\) represents the fact that the module \(V_{\Delta'}\) is a submodule of the module \(V_{\Delta}\). In this case, the highest vector \(|V_{\Delta'}\rangle\) as a vector in the module \(V_{\Delta}\) is a null vector of the form \(D_{\Delta',\Delta}|V_{\Delta}\rangle\) with the operator \(D_{\Delta',\Delta}\) being a linear combination of products of the Virasoro generators \(L_{-k}, k > 0\).

Lian and Zuckerman proved in [6] that the relative cohomology classes \(H^*_{\text{rel}}(L_{\Delta})\) are non-trivial if and only if \(\Delta \in E\). For each \(\Delta \in E\) the dimension of the cohomology space \(H^*_{\text{rel}}(L_{\Delta})\) is given by

\[
\dim H^*_{\text{rel}}(L_{\alpha_n}) = \dim H^*_{\text{rel}}(L_{\beta_n}) = \begin{cases} 1, & k = -n + 1, n + 1, \\ 2, & k = -n + 3, -n + 5, \ldots, n - 1, \\ 0, & \text{otherwise}, \end{cases} \quad (3.2)
\]

where we assume that \(n > 0\). In the case \(n = 0\) the dimension of the cohomology space is

\[
\dim H^*_{\text{rel}}(L_{\alpha_0}) = \delta_{k,1}.
\]

Second, we will need Lian–Zuckerman theorem for Verma modules. Consider the BRST complex \(C^*_{\text{rel}}(V_{\Delta})\). In this case Lian and Zuckerman also proved in [6] that the relative cohomology space \(H^*_{\text{rel}}(V_{\Delta})\) is non-trivial if and only if \(\Delta \in E\). The dimension of \(H^*_{\text{rel}}(V_{\Delta})\) is the following

\[
\dim H^*_{\text{rel}}(V_{\alpha_n}) = \dim H^*_{\text{rel}}(V_{\beta_n}) = \begin{cases} 1, & k = n + 1, \\ 0, & \text{otherwise}. \end{cases} \quad (3.3)
\]

For any \(\Delta\) there is a map \(V_{\Delta} \rightarrow L_{\Delta}\) from Verma module with highest weight \(\Delta\) to irreducible Vir module with the same highest weight. It induces maps \(C^*_{\text{rel}}(V_{\Delta}) \rightarrow C^*_{\text{rel}}(L_{\Delta})\) and \(H^*_{\text{rel}}(V_{\Delta}) \rightarrow H^*_{\text{rel}}(L_{\Delta})\). Comparing the ghost numbers we see that the image of unique class in \(H^*_{\text{rel}}(V_{\Delta})\) is a cohomology class in \(H^*_{\text{rel}}(L_{\Delta})\) of the highest ghost number. The cohomology class of the highest ghost number in space \(H^*_{\text{rel}}(L_{\Delta})\) will be called the highest cohomology \(\Delta^2\).

### 3.2. Recursive Construction of the Basis States

We suggest that there exists a relation between expressions for physical states and a form of the corresponding singular vectors. This relation leads to an explicit recursive construction of cohomology classes. More precisely, we show that all cohomology classes can be determined in terms of the highest ones.

#### 3.2.1. The Highest Cohomology Classes

The construction of the highest cohomology classes simplifies due to

**Proposition 1** All the highest cohomology classes can be obtained by applying operators \(c_{-1}, c_{-2}, \ldots\) to the vacuum vectors \(\Psi_{\Delta}\) introduced in [12].

This proposition easily follows from Proposition 1.11 in [15]. An essential part of the proof of this proposition is the following construction of the highest cohomology classes. Let \(K^n\) be the vector space of all (possibly infinite) linear combinations of antisymmetric monomials

\[
c_{-i_1}, \ldots, -i_n = c_{-i_1}c_{-i_2} \cdots c_{-i_n}, \quad (3.4)
\]

Let \(d(c_{-i_1}, \ldots, -i_n) = i_1 + i_2 + \cdots + i_n\) be the degree of the monomial \(\Delta\). Let us define a differential \(\delta : K^n \rightarrow K^{n+1}\) as follows

\[
\delta(c_{-i}) = \sum_{\alpha + \beta = i} (\alpha - \beta)c_{-\alpha, -\beta}, \quad \delta(c_{-i_1}, \ldots, -i_n) = \sum_{j=1}^n (-1)^{j-1} d(c_{-i_j})c_{-i_1, \ldots, -i_{j-1}, \ldots, -i_n}.
\]

\(\Delta^2\) As was already mentioned in the introduction in another approach the space of states in gravity sector is represented by free field (Feigin-Fuchs) modules. The space of physical states in this theory is \(H^*_{\text{rel}}(F)\), where \(F\) is a free field module with central charge \(c_F = 26\). In the case \(c > 25\), the free field module is isomorphic to either a Verma module or a contragradient Verma module (see [12]). In the first case cohomology \(H^*_{\text{rel}}(F) = H^*_{\text{rel}}(V_\Delta)\) and corresponds to highest cohomology of \(H^*_{\text{rel}}(L_\Delta)\). One can show that in second case cohomology \(H^*_{\text{rel}}(F)\) corresponds to lowest cohomology of \(H^*_{\text{rel}}(L_\Delta)\). This remark allows us to compare our results with those of [8], [13], [14]
It is easy to prove that $\delta^2 = 0$. We denote the cohomology space of this complex by $H^\ast(K)$. This complex is isomorphic to the standard cohomology complex of the Lie algebra Vir$_{>0} = \{L_1, L_2, \ldots\}$ and was studied well. The Goncharova theorem [16] states that the dimensions of the homology spaces $H^n(K)$ is

$$\dim H^n(K) = \begin{cases} 1, & n = 0, \\ 2, & n > 0. \end{cases}$$

Each cohomology space $H^n(K)$ is generated by two vectors $u_n, v_n$ with the degrees

$$d(u_n) = \frac{3n^2 - n}{2}, \quad d(v_n) = \frac{3n^2 + n}{2}. \quad (3.5)$$

For example,

$$H^1(K) = \langle c_{-1}, c_{-2} \rangle \quad \text{and} \quad H^2(K) = \langle c_{-1}c_{-4}, c_{-2}c_{-5} - 3c_{-3}c_{-4} \rangle.$$

The vectors $u_n$ and $v_n$ are defined modulo $\delta$ exact terms.

Let us return to the construction of the highest cohomology classes for the Vir irreducible modules $\mathcal{L}_a$. Consider the vector $u_n\Psi_a = |\mathcal{L}_a\rangle \otimes u_n|v^g\rangle$

of the ghost number $n + 1$ and the conformal dimension 0 (since $a_n + d(u_n) - 1 = 0$ where $a_n$ in given in (3.1) and $d(u_n)$ is given in (3.5)). It is clear that the vector $u_n\Psi_a$ is BRST closed, i.e., $Q(u_n\Psi_a) = 0$. Moreover, one can show that this vector is not BRST exact. Thus, the state

$$O_{a_n} = u_n\Psi_a,$$

is a representative of the relative cohomology $H^*_{rel}(\mathcal{L}_a)$.

The highest cohomology classes of any Vir irreducible module $\mathcal{L}_b$ can be constructed in a similar manner. We give an explicit form of several highest cohomology classes

$$O_{a_1}^{a_1} = H_{a_1}^{a_1}\Psi_{a_1} = c_{-1}\Psi_{a_1}, \quad O_{a_2}^{a_2} = H_{a_2}^{a_2}\Psi_{a_2} = c_{-1}c_{-4}\Psi_{a_2},$$

$$O_{b_1}^{b_1} = H_{b_1}^{b_1}\Psi_{b_1} = c_{-2}\Psi_{b_1}, \quad O_{b_2}^{b_2} = H_{b_2}^{b_2}\Psi_{b_2} = (c_{-2}c_{-5} - 3c_{-3}c_{-4})\Psi_{b_2}. \quad (3.6)$$

### 3.2.2. Recursion equations

In this subsection we use the following notation. Let us introduce the BRST complex $C_\ast(V_\Delta)$, where $V_\Delta$ is the Verma module with highest weight $\Delta$. Let $|V_\Delta\rangle$ be the highest weight vector in this Verma module. We define a vacuum vector in this complex by

$$\Psi^V_\Delta = |V_\Delta\rangle \otimes |v^g\rangle.$$

To emphasize the difference between this vacuum vector and the vacuum vector defined in (2.1) we denote the latter by $\Psi^V_\Delta$.

The cohomology classes, except the highest ones, possess recursive construction[3]. This construction makes it possible to find the expressions for cohomology classes related to the highest weights of the Verma modules in the embedding diagram one by one downwards starting from the top. Let us perform several first steps explicitly. Then we shall describe the $n$-th step.

**Embedding level 0.**

The top node of the embedding diagram corresponds to the highest weight $a_0 = 1$. The dimension of the cohomology space is $\dim H^0_{rel}(\mathcal{L}_{a_0}) = \delta_{\delta, 1}$. It is straightforward to find a representative of cohomology space. But for further references we give an additional construction. Let us consider the relative BRST complex $C_\ast^{rel}(V_{a_0})$. It is easy to check that the state

$$H^{a_0}_{a_0}\Psi^V_{a_0} = \Psi^V_{a_0} \quad (3.7)$$

is a representative of the cohomology classes. Taking into account our discussion in the last part of Subsection 3.1 one can check that the state
is a representative of the cohomology classes $H^{\text{rel}}(L_{a_{0}})$. This state is the simplest and was the subject of the most studies of correlation functions. The corresponding operator contains a primary field in the matter sector (in our case it is an identity operator) dressed by an appropriate Liouville primary field.

**Embedding level 1.**

Nodes at the first level of the embedding diagram corresponds to the highest weights $a_{1} = 0$ and $b_{1} = -1$. Let us consider the cohomology space $H^{\text{rel}}(L_{a_{1}})$ associated with the former. By Lian–Zuckerman results (3.2), one can realize that non-trivial cohomology classes belong to $H^{\text{rel}}_{0}(\mathcal{L}_{a_{1}})$ and $H^{\text{rel}}_{2}(\mathcal{L}_{a_{1}})$. The highest cohomology class $O_{a_{1}}^{0} \in H^{\text{rel}}_{2}(\mathcal{L}_{a_{1}})$ is given in (3.6). The cohomology class from the space $H^{\text{rel}}_{3}(\mathcal{L}_{a_{1}})$ can be constructed as follows:

**Step 1.** Let us consider the relative BRST complex $C_{s}^{\text{rel}}(V_{a_{1}})$. The Verma module $V_{a_{1}}$ contains a singular vector at the first level. This singular vector has the form $\Psi_{0}$, which we use to

\[ O_{a_{0}}^{0} = H_{a_{0}}^{a_{0}} \Psi_{a_{0}}^{L} = \Psi_{a_{0}}^{L} \]  

(3.8)

de
denotes a representative of the cohomology classes $H^{\text{rel}}(L_{a_{0}})$. This state is the simplest and was the subject of the most studies of correlation functions. The corresponding operator contains a primary field in the matter sector (in our case it is an identity operator) dressed by an appropriate Liouville primary field.

**Step 2.** We will show that the state (3.9) is also BRST exact. Indeed, due to Lian-Zuckerman results (3.3) the only non-trivial cohomology class in the space $H^{\text{rel}}_{0}(V_{a_{1}})$ has the ghost number 2, while the state (3.9) has the ghost number 1. Therefore, this state is BRST exact. Hence, there exists a certain operator $H_{a_{1}}^{a_{0}}$ such that

\[ Q(H_{a_{1}}^{a_{0}} \Psi_{a_{1}}^{V}) = O_{a_{0}}^{a_{0}} \]  

(3.10)

The right side of this equation is completely specified by the unique cohomology class at the previous level and the embedding structure of the Verma modules. Therefore, in the sequel this equation will be called the recursive equation. A solution of the equation (3.10), namely, the operator $H_{a_{1}}^{a_{0}}$ is used to obtain a representative of the cohomology space $H^{\text{rel}}_{0}(\mathcal{L}_{a_{1}})$ in the following:

**Step 3.** We assert that the state $O_{a_{1}}^{a_{0}} = H_{a_{0}}^{a_{0}} \Psi_{a_{1}}^{V}$ represents a cohomology class in $H^{\text{rel}}_{0}(\mathcal{L}_{a_{0}})$. Indeed by (3.10) and (3.9) we get

\[ Q(O_{a_{1}}^{a_{0}}) = H_{a_{0}}^{a_{0}} D_{a_{0},a_{1}}(\mathcal{L}_{a_{1}}) \otimes |v^{\theta}) = 0, \]

since $D_{a_{0},a_{1}}(\mathcal{L}_{a_{1}}) = 0$. Moreover, one can show that the state $O_{a_{1}}^{a_{0}}$ is not BRST exact. Thus it represents a cohomology class.

We have obtained all cohomology classes in the space $H^{\text{rel}}(\mathcal{L}_{a_{0}})$ and show the connection between these classes and singular vectors in the irreducible Vir module $\mathcal{L}_{a_{1}}$.

The cohomology classes in the space $H^{\text{rel}}_{0}(\mathcal{L}_{b_{0}})$ can be obtained in a similar manner. The highest cohomology class is given in (3.6). The other part of the construction is the same as we used in the space $H^{\text{rel}}_{0}(\mathcal{L}_{a_{1}})$. Let us list the results. There is a singular vector at the second level in the Verma module $V_{b_{1}}$. This vector has the form $D_{a_{0},b_{1}}(\mathcal{L}_{b_{1}})$ for some operator $D_{a_{0},b_{1}}$. Here we have $D_{a_{0},b_{1}} = (L_{-1}^{2} + (2/3)L_{-2})$. The recursive equation reads

\[ Q(H_{b_{1}}^{a_{0}} \Psi_{b_{1}}^{V}) = O_{a_{0}}^{a_{0}}_{b_{1}}, \quad O_{a_{0}}^{a_{0}}_{b_{1}} = H_{a_{0}}^{a_{0}} D_{a_{0},b_{1}}(\mathcal{L}_{b_{1}}) \Psi_{b_{1}}^{V}, \]  

(3.11)

The solution of this equation allows us to specify the operator $H_{b_{1}}^{a_{0}}$ and, hence, the representative $O_{a_{0}}^{a_{0}} = H_{b_{1}}^{a_{0}} \Psi_{a_{0}}^{L}$ of the cohomology space $H^{\text{rel}}_{0}(\mathcal{L}_{b_{1}})$.  

**Embedding level 2**

Nodes at this level of the embedding diagram corresponds to the highest weights $a_{2} = -4$ and $b_{2} = -6$. We consider the cohomology space $H^{\text{rel}}(\mathcal{L}_{a_{2}})$. The dimensions and ghost numbers of the cohomology classes are given by Lian–Zuckerman results (3.2). The highest cohomology class $O_{a_{2}}^{a_{2}}$ is given in (3.6).
Let us consider the cohomology space $H_{1}^{rel}(\mathcal{L}_{a_2})$. Its two basic cohomology classes can be constructed in a similar manner as on the previous level. There are two singular vectors in the Verma module $V_{a_2}$. The first one $D_{a_1,a_2}|V_{a_2}$ is at the level 4 and the second one $D_{b_1,a_2}|V_{a_2}$ is on the level 3. In this case the recursive equations are

\begin{align*}
Q(H_{a_1}^{a_2} \Psi_{a_2}^{V}) &= O_{a_1|a_2}^{a_2}, \quad O_{a_1|a_2}^{a_2} = H_{a_1}^{a_2}D_{a_1,a_2} \Psi_{a_2}^{V}, \\
Q(H_{b_2}^{b_1} \Psi_{a_2}^{V}) &= O_{b_1|a_2}^{b_1}, \quad O_{b_1|a_2}^{b_1} = H_{b_1}^{b_1}D_{b_1,a_2} \Psi_{a_2}^{V}.
\end{align*}

(3.12)

The operators $H_{a_1}^{a_2}$ and $H_{b_1}^{b_1}$ specify the representatives $O_{a_2}^{a_1} = H_{a_2}^{a_1} \Psi_{a_2}^{L}$ and $O_{a_2}^{b_1} = H_{a_2}^{b_1} \Psi_{a_2}^{V}$ of the cohomology classes $H_{1}^{rel}(\mathcal{L}_{a_2})$.

The construction of the representative of the cohomology space $H_{1}^{rel}(\mathcal{L}_{a_2})$ is a little more tricky.

**Step 1.** Let us consider the relative BRST complex $C_{1}^{rel}(V_{a_2})$. As we discussed, there are two singular vectors in the Verma module $V_{a_2}$. Let us introduce the states

\[ O_{a_1|a_2}^{a_0} = H_{a_1}^{a_0}D_{a_1,a_2} \Psi_{a_2}^{V}, \quad O_{b_1|a_2}^{a_0} = H_{b_1}^{a_0}D_{b_1,a_2} \Psi_{a_2}^{V}. \]

One can consider these states as $H_{a_1}^{a_0} \Psi_{a_1}^{V}$ and $H_{b_1}^{a_0} \Psi_{b_1}^{V}$, which are specified at the previous level. Thus, by relations (3.9), (3.10) and (3.11) we obtain

\begin{align*}
Q(O_{a_1|a_2}^{a_0}) &= H_{a_1}^{a_0}D_{a_0,a_1}D_{a_1,a_2} \Psi_{a_2}^{V}, \quad Q(O_{b_1|a_2}^{a_0}) = H_{b_1}^{a_0}D_{a_0,b_1}D_{b_1,a_2} \Psi_{a_2}^{V}.
\end{align*}

(3.13)

As we discussed above, there are a singular vectors of conformal dimension $a_0 = 1$ in the Verma modules $V_{a_1}$ and $V_{b_1}$. This modules and, therefore, the singular vectors are contained in the Verma module $V_{a_2}$. Since the Verma module $V_{a_2}$ contains the only singular vector of the highest weight $a_0$, one get an operator identity $D_{a_0,a_1}D_{a_1,a_2} = D_{a_0,b_1}D_{b_1,a_2}$. Thus, from (3.13) we obtain

\[ Q(O_{a_1|a_2}^{a_0} - O_{b_1|a_2}^{a_0}) = 0. \]

The remaining part of the construction is the same as we used on the previous level. Thus, let us list the results. The recursive equation is

\[ Q(H_{a_2}^{a_0} \Psi_{a_2}^{V}) = O_{a_1|a_2}^{a_0} - O_{b_1|a_2}^{a_0}, \]

(3.14)

and $O_{a_2}^{a_0} = H_{a_2}^{a_0} \Psi_{a_2}^{L}$ is a cohomology class in the space $H_{1}^{rel}(\mathcal{L}_{a_2})$. As we can see the equation (3.14) determine the cohomology class at the second level by cohomology classes at the first level.

**Embedding level $n$**

We suggest that one can obtain the recursive equations for cohomology classes of the space $H_{*}(\mathcal{L}_{a_n})$. The basic cohomology classes and their ghost numbers $N^{g}$ can be represented in the form of the following diagram:

\[ O_{a_n}^{a_1} \quad O_{a_n}^{a_2} \quad \ldots \quad O_{a_n}^{a_n-1} \]

\[ O_{a_n}^{a_0} \]

\[ O_{a_n}^{b_1} \quad O_{a_n}^{b_2} \quad \ldots \quad O_{a_n}^{b_n-1} \]

\[ N^{g} = \quad -n + 1, \quad -n + 3, \quad -n + 5, \quad \ldots \quad n - 1, \quad n + 1. \]

Let $\gamma_k$ and $\delta_k$ be either $a_k$ or $b_k$. Define the set of operators $H_{a_n}^{\gamma_k}$ by

\[ O_{a_n}^{\gamma_k} = H_{a_n}^{\gamma_k} \Psi_{a_n}^{L}. \]

These operators are polynomials in Vir algebra generators $L_n$ and ghosts $c_n, b_n$ with $n < 0$. Let us introduce the set of states

\[ O_{\delta_{n-1}|a_n}^{\gamma_j} = H_{\delta_{n-1}}^{\gamma_j}D_{\delta_{n-1}, a_n} \Psi_{a_n}^{V}. \]
One can consider these states as the states $H^{\gamma_j}_{a_{n-1}} \Psi^V_{a_{n-1}}$, i.e., as a cohomology classes on the previous level $n-1$. Thus, by the assumption of the recursion procedure the set of the operators $H^{\gamma_j}_{a_{n-1}}$ is supposed to be specified. The operators $D_{\Delta}\gamma_{a_{n-1}}$ are specified by the embedding structure of singular vectors in the Verma module $V_{a_n}$. Now we can formulate

**Proposition 2** For the cohomology classes of the space $H^*_{rel}(L_{a_n})$ the set of recursive equations reads

$$Q(H^{\gamma_j}_{a_{n-1}} \Psi^V_{a_{n-1}}) = \begin{cases} O^{\gamma_j}_{a_{n-1}|a_{n}} - O^{\gamma_j}_{b_j|a_{n}}, & j = 0, \ldots, n - 2 \\ O^{\gamma_{n-1}}_{\gamma_{n-1}|a_{n}}, & j = n - 1 \\ 0, & j = n, \end{cases}$$

and states $O^{\gamma_j}_{a_{n}} = H^{\gamma_j}_{a_{n}} \Psi^L_{a_{n}}$ form a basis in the cohomology space $H^*_{rel}(L_{a_n})$.

One can formulate a similar proposition for the cohomology space $H^*_{rel}(L_{b_n})$. As a consequence of the Proposition 2 we show that forms for all cohomology classes can be determined from the highest ones and the operators $D_{\Delta\gamma_{a_{n}}}$.

### 3.3. Operators Acting on the Relative Cohomology Space

In the previous subsection we constructed the basis in the cohomology space $H^*_{rel}(L_{a_n})$. To investigate the operator algebra it is convenient to choose another set of representatives of the cohomology classes. We construct a new basis by introducing certain operators acting on the cohomology space.

Let us consider two operators

$$X = \frac{1}{2} \sum_{n=-\infty}^{\infty} n c_{-n} c_n, \quad X_+ = \frac{1}{2} \sum_{n=-\infty}^{\infty} n^3 c_{-n} c_n.$$  

These operators commute with the BRST charge $Q$. Indeed, it is easy to check that the operator $X$ is equal to $[Q, c_0]$ and commutes with the BRST charge due to the property $Q^2 = 0$. The commutation of the operator $X_+$ with the BRST charge can be verified straightforwardly. Thus these operators act on the relative cohomology space, i.e., if $w$ represents a BRST cohomology class of the ghost number $k$, then $Xw$ and $X_+w$ represent cohomology classes of the ghost numbers $k + 2$ in the same space.

The operators $X$ and $X_+$ satisfy the relation

$$X \cdot X_+ = 0$$

on the cohomology space $H^*_{rel}(L_\Delta)$. This relation can be verified as follows. Let us introduce the operator

$$\hat{Y} = \frac{1}{12} \sum_{i+j+k=0, i,j,k\neq 0} (i-j)(j-k)(k-i)c_i c_j c_k.$$  

(3.15)

It is straightforward to check that the product $X \cdot X_+$ is equal to $[\hat{Y}, Q]$. Therefore, for any cohomology class $w$, we have $XX_+w = [\hat{Y}, Q]w = -Q\hat{Y}(w)$, i.e. $XX_+w$ is equal to 0 in the cohomology space.

The operators $X$ and $X_+$ with relation $XX_+ = 0$ generate an algebra acting on the cohomology space $H^*_{rel}(L_\Delta)$. Operators $X$ and $X_+$ can be used to construct representatives of the cohomology classes. We construct a representatives of the cohomology classes by use of this operators. Consider the space $H^*_{rel}(L_{a_n})$. To simplify notation denote by $O_{a_n}$ the cohomology class $O^{a_0}_{a_0}$ of the lowest ghost number. A basis in the cohomology space can be obtained by means of

**Theorem 1** Cohomology classes

$$O_{a_n}, \quad XO_{a_n}, \quad X^2O_{a_n}, \ldots, X^nO_{a_n}, \quad X_+O_{a_n}, \quad X^2O_{a_n}, \ldots, X_{n-1}^nO_{a_n}$$

form a basis in the cohomology space $H^*_{rel}(L_{a_n})$.  

5From more abstract point of view, we study an action of the cohomology of Virasoro algebra on semi-infinite cohomology. The operators $X$ and $X_+$ form a basis in the two-dimensional space $H^2(Vir, Vir_0, \mathbb{C})$, where $Vir_0 = \{L_0, c\}$. The whole algebra $H^2(Vir, Vir_0, \mathbb{C})$ is generated by $X$ and $X_+$ with one relation $XX_+ = 0$. 

8
The Theorem \[\text{I}\] can be illustrated as follows

\[\begin{array}{c}
X & \xrightarrow{X \cdot O_{a_n}} & XO_{a_n} \\
X_+ & \xrightarrow{X_+ \cdot XO_{a_n}} & X_+O_{a_n} \\
& \vdots & \vdots \\
& \xrightarrow{X_\ast \cdot X^{n-1}O_{a_n}} & X_\ast X^{n-1}O_{a_n} \\
& \xrightarrow{X \cdot X^nO_{a_n}} & X^nO_{a_n}
\end{array}\]

\[N^a = -n + 1, \quad -n + 3, \quad -n + 5, \quad \ldots \quad n - 1, \quad n + 1.\]

The similar theorem holds for the cohomology space \(H^*_s(\mathcal{L}_{n})\). Proof of this theorem will be published elsewhere \[\text{II}\]. Let us check the Theorem \[\text{I}\] in the first nontrivial example \(H^*_s(\mathcal{L}_{a_2})\). By explicit form of the cohomology classes (see Appendix) we obtain

\[X(O_{a_2}^0) = -\frac{5}{3} \cdot O_{a_2}^1 - O_{a_2}^a, \quad X_+(O_{a_2}^0) = \frac{7}{3} \cdot O_{a_2}^1 - O_{a_2}^a,\]

\[X^2(O_{a_2}^0) = 240 O_{a_2}^2, \quad X_+^2(O_{a_2}^0) = -3696 O_{a_2}^2\] \hspace{1cm} (3.16)

This implies that the cohomology classes \(O_{a_2}, XO_{a_2}, X^2O_{a_2}, X_+O_{a_2}\) form a basis in the cohomology space \(H^*_s(\mathcal{L}_{a_2})\). Also it follows from (3.16) that the basis introduced in Theorem \[\text{I}\] differs from that of Proposition \[\text{II}\].

From Theorem \[\text{I}\] follows that \(O_{a_2}^n = \lambda X^nO_{a_2}^n\), where \(\lambda \neq 0\), i.e. the highest cohomology class can be obtained from lowest one by use of operator \(X\). This fact concerns only highest and lowest cohomology classes therefore it should have analogue in free field approach due to remark \[\text{II}\]. Indeed this formula is equivalent to the equation (2.12) in [14].

Let us recall that all cohomology classes except highest ones can be obtained from the previous level cohomology classes by the recursive construction. On the other hand, it follows from the Theorem \[\text{I}\] that all the cohomology classes (including the highest one) in the space \(H^*_s(\mathcal{L}_{a_n})\) can be obtained from the cohomology class with the lowest ghost number \(O_{a_n}\) by applying the operators \(X, X_+\). Therefore all cohomology classes can be found from the simplest cohomology \(O_{a_0}^n \in H^*_s(\mathcal{L}_{a_0})\) (see \[\text{III}\]).

### 3.4. Operator Algebra

One can construct physical operators from the states in the Hilbert space using state-operator correspondence. Every state in the Hilbert space has an image in the space of local operators. For example, any Liouville highest weight vector corresponds to a Liouville primary field of the same conformal dimension.

For any cohomology class \(O\) we can consider a unique local operator \(O(z)\), commuting with the BRST charge \(Q\). This operator doesn’t depend on point \(z\) modulo BRST exact terms. Indeed,

\[\partial O(z) = L_{-1}O(z) = L_{-1}O = [Q, b_{-1}]O = Qb_{-1}O.\]

It is well known that any nontrivial BRST cohomology class \(O\) has zero conformal dimension. The prove is simple. Suppose that \(L_0 O = \Delta O\) and \(\Delta \neq 0\). Hence, we have

\[O = \Delta^{-1}L_0 O = \Delta^{-1}[Q, b_0]O = Q (\Delta^{-1}b_0 O).\]

The general form of the operator product expansion (OPE) of any two local operators \(O_1(z)\) and \(O_2(0)\) reads

\[O_1(z)O_2(0) = \sum_{n=-\infty}^{\infty} A_n(0) z^n,\]

---

6 The idea of the proof is the following (this idea is due to B. Feigin) The algebra generated by \(X, X_+\) with relation \(XX_+ = 0\) is the cohomology algebra \(H^*(\mathfrak{Vir}, \mathfrak{Vir}_0, \mathbb{C})\). The homology space \(H_*(\mathfrak{Vir}, \mathfrak{Vir}_0, \mathbb{C})\) is a cofree module over this algebra. The irreducible Vir module \(\mathbb{C}\) is dual to some infinite complex \(\mathcal{K}\) by means of duality between \(c = 0\) and \(c = 26\) Virasoro modules. Hence the \(H^*/2^s(\mathfrak{Vir}, \mathfrak{Vir}_0, \mathbb{C})\) is free module over the algebra \(H^*(\mathfrak{Vir}, \mathfrak{Vir}_0, \mathbb{C})\). The irreducible Liouville module \(\mathcal{L}_{a_0}\) is quasi-isomorphic to the truncation of the complex \(\mathcal{K}\). Therefore the space \(H^*_s(\mathcal{L}_{a_0}) = H^*/2^s(\mathfrak{Vir}, \mathfrak{Vir}_0, \mathcal{L}_{a_0})\) is cyclic module over algebra \(H^*(\mathfrak{Vir}, \mathfrak{Vir}_0, \mathbb{C})\).
where the expansion coefficients $A_n(0)$ are some operators of the conformal dimension $n$. Since the operators the $O_1(z)$ and $O_2(0)$ commute with BRST charge $Q$, the expansion coefficients $A_n(z)$ commute with it as well. Since there is no BRST cohomology classes of non-zero conformal dimension, only $A_0(z)$ may give a BRST nontrivial cohomology class. Let us denote $A_0(z)$ by $O_3(z)$. We obtain a ring structure, defined by the operator product expansion modulo BRST exact terms

$$O_1(z)O_2(0) = O_3(0) + [Q, \ldots],$$

which will be denoted by

$$O_1 \cdot O_2 = O_3.$$  \hspace{1cm} (3.17)

We assert that the operator algebra on the relative cohomology space is not associative. Let us consider the simplest non-trivial example and show that

$$(O_{a_1}^{a_1} \cdot O_{a_2}^{a_2}) \cdot O_{a_3}^{a_3} \neq O_{a_1}^{a_1} \cdot (O_{a_2}^{a_2} \cdot O_{a_3}^{a_3}).$$  \hspace{1cm} (3.18)

The l.h.s of (3.18) is equal to 0 in the cohomology space. Indeed, from Liouville fusion rules and ghost number conservation follows that $O_{a_1}^{a_1} \cdot O_{a_2}^{a_2} \cdot O_{a_3}^{a_3}$ belongs to $H_{rel}^4(L_{a_1})$ and has ghost number 4. Due to Lian-Zuckerman results $H_{a_1}^4(L_{a_1}) = 0$. Therefore $O_{a_1}^{a_1} \cdot O_{a_1}^{a_1}$ is equal to 0, then l.h.s of (3.18) is equal to 0.

The r.h.s of (3.18) can be evaluated using the explicit form of the operators given in Appendix. The r.h.s can be shown to be equal to $240O_{a_2}^{a_2}$. This calculation proves that the operator algebra of relative cohomology classes is not associative.

The absence of the associativity of the operator algebra is quite undesirable. Let us investigate this problem in more detail. Up to now we discussed the relative BRST cohomology classes $w$, modulo $Qw$ where both elements $w$ and $w'$ are annihilated by $b_0$. Note that there exist states of the form $Q\tilde{w}$ such that $b_0\tilde{w} \neq 0$. For example, the state $O_{a_1}^{a_1}$ is of this kind

$$O_{a_1}^{a_1} = Q \left(c_0 b_{-1} \Psi_{a_1}\right).$$

Any correlation function that contains the such states vanishes. One can say that these states are not physical.

Thus we need to exclude such states from the physical spectrum. In order to exclude undesirable states we consider the absolute BRST complex in the next section.

4. Absolute cohomology

4.1. Basic cohomology classes

In this section we consider the absolute BRST complex $C^{abs}(L_{\Delta})$. Lian and Zuckerman proved that the cohomology classes $H^*_a(L_{\Delta})$ are nontrivial if and only if $\Delta \in E$. Recall that $E$ is the set of the highest weights appearing in the embedding diagram of Verma modules (3.1). It is possible to extend Lian and Zuckerman consideration and formulate

**Theorem 2** If $\Delta \in E$, the dimension of the cohomology space $H^*_a(L_{\Delta})$ are given by

$$\dim H^a_k(L_{a_n}) = \dim H^a_k(L_{b_n}) = \begin{cases} 1, & k = -n + 1, \ldots, n - 1, \\ 1, & k = -n + 4, \ldots, n + 2, \\ 0, & \text{otherwise}. \end{cases}$$

where we assume that $n > 0$. In the case $n = 0$ the dimension of the cohomology space is

$$\dim H^a_k(L_{a_0}) = \delta_{k,1} + \delta_{k,2}.$$  \hspace{1cm} (4.1)

**Proof.** We prove the theorem for $H^a_k(L_{a_n})$ (the other case can be done in similar manner). Following [15], [6] one has a long exact sequence containing relative and absolute cohomology spaces

$$\ldots \to H^a_{k-1} \to H^a_{k-2} \to \cdots \to H^a_k \to H^a_{k+1} \to \cdots$$

(4.1)
It follows from the exactness of this sequence that

$$\dim H^k_{\text{abs}}(L_a) = \dim \text{im}(\gamma_k) + \dim \text{im}(\beta_k) = \dim \ker(\alpha_k) + \dim H^k_{\text{rel}}(L_a) - \dim \text{im}(\alpha_{k-1}).$$

Since the dimension of the cohomology space $\dim H^k_{\text{rel}}(L_a)$ is known \cite{32}, it is sufficient to study the maps $\alpha_k$.

The map $\alpha_*: H^1_{\text{rel}}(L_\Delta) \to H^1_{\text{rel}}(L_\Delta)$ is defined by the action of the ghost operator $c_0$ and then the action of the BRST charge $Q$. Note that $[Q, c_0] = X$. Therefore the map $\alpha_*$ is equivalent to the action of the operator $X$. Thus, from Theorem 1 we conclude, that the kernel of the map $\alpha_*$ is spanned on the cohomology classes of the form $X^jO_a, (1 \leq j < n)$ and $X^nO_a$. The image of this map is spanned on the cohomology classes $X^jO_a, (1 \leq j \leq n)$. From this consideration the dimension of cohomology spaces can be easily obtained.

The meaning of the long exact sequence \cite{41} is the following. Any absolute cohomology class can be either a class from $H^1_{\text{rel}}(L_\Delta)$, or represented by a state of the form $c_0w + w'$, where $w \in H^1_{\text{rel}}(L_\Delta)$ and $w' \in C^*_a(L_\Delta)$. Some relative cohomology classes are not absolute cohomology classes. Indeed, one can consider the action of the BRST operator on state of the form $c_0w + w'$ and obtain

$$Q(c_0w + w') = Xw + Qw'.$$

Thus, every relative cohomology class of the form $Xw$ is BRST exact in the absolute cohomology space. From the Theorem 1 it follows that the relative cohomology spaces

$$O_a, \ X + O_a, \ X^2O_a, \ldots \ X^{n-1}O_a$$

aren’t of the form $Xw$ and therefore form a basis in the space $H^1_{\text{rel}}(L_a) \cap H^a_{\text{abs}}(L_a)$. In order to extend the set \cite{43} to basis we need to add some states in form $c_0w + w'$, where $w \in H^1_{\text{rel}}(L_a)$ and $w' \in C^*_a(L_a)$. It follows from \cite{12} that if $Q(c_0w + w') = 0$ then $Xw = 0$ in $H^1_{\text{rel}}(L_a)$. By Theorem 1 such $w$ is a linear of combination of cohomology classes $X^jO_a, X^2O_a, \ldots, X^{n-1}O_a, X^nO_a$. Then all additional basic states have form $c_0X^iO_a, c_0X^2O_a, \ldots, c_0X^{n-1}O_a, c_0X^nO_a$ modulo $C^*_a(L_a)$.

Let us introduce a new operator

$$Y = \frac{1}{12} \sum_{i+j+k=0} (i-j)(j-k)(k-i)c_ic_jc_k$$

with the ghost number 3. It is easy to check, that this operator commutes with the BRST charge $Q$ and thus acts on the cohomology space. Moreover, if $w \in H^1_{\text{rel}}(L_a)$, then $Yw = c_0X^iO_a + w'$ with $w' \in C^*_a(L_a)$.

It follows from previous consideration that the set \cite{43} can be expanded to form a basis in the absolute cohomology space $H^a_{\text{abs}}(L_a)$ by adding the following cohomology classes

$$YO_a, \ YX^1O_a, \ YX^2O_a, \ldots, \ YX^{n-1}O_a.$$

Indeed, $YO_a = c_0X^1O_a, \ YX^1O_a = c_0X^2O_a, \ldots, \ YX^{n-1}O_a = c_0X^nO_a$ modulo $C^*_a(L_a)$. It remains to prove that $YX^{n-1}O_a \neq 0$ in cohomology space $H^a_{\text{abs}}(L_a)$. This can be proven by duality arguments similar to remark 6.

Our discussion of the cohomology classes of the absolute BRST complex $C^a_{\text{abs}}(L_a)$ can be summarized by the following diagram

$$N^g = -n + 1, \ -n + 3, \ -n + 5, \ldots, \ -n + 2.$$
Let $\gamma_n$ be either $a_n$ or $b_n$. It is convenient to denote the basic cohomology classes in the following form

$$O_{\gamma_n}^i = (n - i - 1)!X_+^i O_{\gamma_n} \quad N_{\gamma_n}^i = (n - i - 1)!Y X_+^i O_{\gamma_n}, \quad 0 \leq i < n. \quad (4.4)$$

We will see that this representation simplify the structure constants of the operator algebra. We clearly have

$$X_+ O_{\gamma_n}^i = (n - i - 1)O_{\gamma_n}^{i+1}, \quad X_+ N_{\gamma_n}^i = (n - i - 1)O_{\gamma_n}^{i+1}. \quad (4.5)$$

Ghost numbers for these cohomology classes are given by

$$N^g(O_{\gamma_n}^i) = 2i - n + 1, \quad N^g(N_{\gamma_n}^i) = 2i - n + 4.$$

### 4.2. Operator Algebra

The operator product expansion provides the ring structure \((3.17)\) on the absolute cohomology space. By the basic assumptions this ring is associative and commutative. There is an unit element in the ring namely the identity operator $O_{a_1}(z) = \mathbb{I}(z)$.

As we will show an operator algebra in the absolute cohomology space is almost determined by the operators $X_+$ and $Y$. Thus it is useful to consider these operators first. Due to operator-state correspondence any operator that acts on the Hilbert space of states has an image acting in the space of local operators. One can show that the image of the operator $X_+$ is given by the contour integral

$$X_+ = -\oint dz c(z) \partial^3 c(z), \quad (4.6)$$

where we omit BRST exact terms. As a consequence of this representation we conclude that the operator $X_+$ differentiates the product of any two local operators, that is

$$X_+(O_1 \cdot O_2) = (X_+ O_1) \cdot O_2 + O_1 \cdot (X_+ O_2). \quad (4.7)$$

Now consider the operator $Y$. It is straightforward to check that on the space of local operators the application of the $Y$ is equivalent to the zero mode of the operator product with $\frac{1}{2} c \partial c \partial^2 c$, i.e.

$$YO(0) = \frac{1}{2} \text{Res}_{z = 0} \left( \frac{c(z) \partial c(z) \partial^2 c(z):O(0)}{z} \right)$$

modulo BRST exact terms. Note that $\frac{1}{2} c \partial c \partial^2 c$: is local operator corresponding to the cohomology class $N_{a_1}^0$:

$$N_{a_1}^0 = \frac{1}{2} c \partial c \partial^2 c.$$ 

Therefore on the cohomology space we have

$$YO = N_{a_1}^0 \cdot O. \quad (4.8)$$

The study of the operator algebra is simplified due to fusion rules. First we consider the case when operators are of the form $O_{a_1}^i$. The corresponding states form a basis in $H_{a_1}^{rel}(L_{a_n}) \cap H_{a_1}^{abs}(L_{a_n})$. Taking into account the fusion rules for degenerate Virasoro representations [3] and the ghost number conservation, we have

$$O_{a_{k+1}}^i \cdot O_{a_{l+1}}^j = \sum_{n=0}^{i+j} \lambda_n O_{a_{k+i+j-n}}^{i+j-n}, \quad (4.9)$$

with some structure constants $\lambda_n$ depending on all indexes $i, j, k, l$.

The operators $X_+$ and $Y$ almost determine structure constants of the operator algebra due to the next two propositions.

**Proposition 3** Suppose that

$$O_{a_{k+1}}^0 \cdot O_{a_{l+1}}^0 = O_{a_{k+l+1}}^0. \quad (4.10)$$

Then the subring $H_{a_1}^{rel}(L_{a_n}) \cap H_{a_2}^{abs}(L_{a_n})$ is isomorphic to the polynomial ring in two generators $O_{a_2}^0$ and $O_{a_2}^1$. Moreover, we have

$$O_{a_{n+1}}^i = (O_{a_2}^0)^{n-i}(O_{a_2}^1)^i. \quad (4.11)$$
In the assumption of this proposition we suppose that the operator product is non-degenerate, while the corresponding coefficient (see (4.9)) can be removed by certain normalization of the lowest cohomology classes $O_{a_n}$.

**Proof of Proposition 3**

It is enough to prove (4.11). For $n = 0$ the equation (4.11) is equivalent to the fact that $O_{a_1}^0$ is a unit element. For $n = 1$ the equation (4.11) is obvious. From the assumption (4.10) follows that

$$O_{a_{n+1}}^0 = O_{a_n}^0 \cdot O_{a_2}^0 = O_{a_{n-1}}^0 \cdot O_{a_2}^0 = \ldots = (O_{a_2}^0)^n$$

(4.12)

Applying $X_+$ to both sides and using (4.7), (4.5) we get

$$nO_{a_{n+1}}^1 = n(O_{a_2}^0)^{n-1} \cdot O_{a_2}^1, \quad O_{a_{n+1}}^1 = (O_{a_2}^0)^{n-1} \cdot O_{a_2}^1.$$

Applying $X_+$ to both sides again and using (4.7), (4.5) we get

$$(n - 1)O_{a_{n+1}}^2 = (n - 1)(O_{a_2}^0)^{n-2} \cdot O_{a_2}^2 \cdot O_{a_2}^1 = (O_{a_2}^0)^{n-2} \cdot (O_{a_2}^1)^2.$$

Similarly, applying $X_+ \ n$ times to both sides of (4.12) we get (4.11).

It is interesting to compare our results with those of Kanno and Sarmadi [8], where irreducible modules in the Liouville sector are replaced by free fields (or Feigin-Fuchs) modules. As was mentioned in remark 2, the cohomology classes from [8] correspond to our relative cohomology classes with the highest and the lowest ghost numbers. For example $w^n$ in their notation corresponds to $O_{a_{n+1}}^0$ with $n \geq 0$. The results of [8] confirm that the products of the cohomology classes of the form $O_{a_{n+1}}^0$ are non-degenerate and the assumption of Proposition 3 is satisfied.

The structure constants of the operator algebra on the space $\oplus_{n \geq 0} H^*_{\text{abs}}(\mathcal{L}_{a_n})$ are given in Proposition 4

**Proposition 4** Under the assumption of Proposition 3 we have

$$O_{a_{k+1}}^{k-i} \cdot O_{a_{l+1}}^{l-j} = O_{a_{k+l+i+1}}^{k+l-i-j}, \quad N_{a_{k+1}}^{k-i} \cdot O_{a_{l+1}}^{l-j} = N_{a_{k+l+1}}^{k+l-i-j},$$

$$N_{a_{k+1}}^{k-i} \cdot N_{a_{l+1}}^{l-j} = 0.$$  \hspace{1cm} (4.13)

**Proof of Proposition 4**

The first equality evidently follows from (4.11). Let us multiply the first equality by $N_{a_1}^0$. By (4.8) and (4.1) we obtain

$$N_{a_{k+1}}^{k-i} \cdot O_{a_{l+1}}^{l-j} = N_{a_1}^0 \cdot O_{a_{k+1}}^{k-i} \cdot O_{a_{l+1}}^{l-j} = N_{a_1}^0 \cdot O_{a_{k+l+1}}^{k+l-i-j} = N_{a_{k+l+1}}^{k+l-i-j}.$$

The last statement in (4.13) easily follows from the equalities $N_{a_{k+1}}^{k-i} = YO_{a_{k+1}}^{k-i} = N_{a_1}^0 \cdot O_{a_{k+1}}^{k-i}$ and $N_{a_1}^0 \cdot N_{a_1}^0 = YN_{a_1}^0(z) = 0$. \hfill \Box

Note, that we consider an operator algebra for the states from the subspace $\oplus_{n \geq 0} H^*_{\text{abs}}(\mathcal{L}_{a_n})$ only. It is sufficient, because there is an isomorphism between the cohomology spaces $H^*_{\text{abs}}(\mathcal{L}_{a_n})$ and $H^*_{\text{abs}}(\mathcal{L}_{b_n})$. This isomorphism is realized by the operator product with

$$O_{b_1}^0(z) = (\partial + \frac{2}{3} : bc:) \Phi_{b_1}(z),$$

where $\Phi_{b_1}(z) = \Phi_{1.2}(z)$ is a Liouville primary field corresponding to the state $|\mathcal{L}_{b_1}\rangle$ from the Hilbert space. Indeed, by the fusion rules for degenerate Virasoro representations [3] and the ghost number conservation, we have

$$O_{b_1}^0 \cdot O_{a_{n+1}}^i = \lambda_{b_1, a_{n+1}}^i O_{b_{n+1}}^i, \quad O_{b_1}^i \cdot O_{a_{n+1}}^0 = \lambda_{b_1, b_{n+1}}^i O_{a_{n+1}}^i,$$

(4.14)

where $\lambda_{b_1, a_{n+1}}^i$ and $\lambda_{b_1, b_{n+1}}^i$ are the structure constants. To show that these constants are not equal to 0, we multiply both sides of the first equation in (4.14) by the operator $O_{b_1}^0$. Taking into account the associativity of the operator algebra, we have

$$(O_{b_1}^0)^2 \cdot O_{a_{n+1}}^i = \lambda_{b_1, a_{n+1}}^i O_{a_1}^0 \cdot O_{a_{n+1}}^i = \lambda_{b_1, a_{n+1}}^i \lambda_{b_1, b_{n+1}}^i O_{a_{n+1}}^i.$$
\[(O_{b_1}^0)^2 = -14/9 C^{(1,1)}_{(1,2),(1,2)} \mathcal{I}(z), \quad (4.15)\]

where \(C^{(1,1)}_{(1,2),(1,2)}\) is the Liouville structure constant ( [18], [9]), we conclude that

\[\lambda_{a_1+n+1}^i \lambda_{b_1+n+1}^i = -14/9 C^{(1,1)}_{(1,2),(1,2)} \neq 0.\]

It is possible to renormalise local operators \(O_{b_n}^0\) such that

\[O_{b_1}^0 \cdot O_{b_{n+1}}^0 = O_{b_{n+1}}^0.\]

Applying operators \(X_+\) and \(Y\) to both sides we get

\[O_{b_1}^0 \cdot O_{b_{n+1}}^0 = O_{b_{n+1}}^0, \quad O_{b_1}^0 \cdot N_{a_{n+1}}^i = N_{b_{n+1}}^i \quad (4.16)\]

Using Proposition 4 and formulae (4.15), (4.16) one can easily calculate operator product of any two local fields.

### 5. Discussion

One of the main aims of this paper is to clarify the difference between absolute and relative cohomology classes. For mathematical reasons, it is convenient to compute relative cohomology first and then pass to absolute cohomology. In some cases (for example in the case of free field modules in the Liouville sector [8]) there is an isomorphism \(H^\text{abs}_{k} \cong H^\text{rel}_{k} + c_0 H^\text{rel}_{k-1}\). In particular, any relative cohomology class is also an absolute cohomology class. In our case the relation between absolute and relative cohomology classes is more complicated.

It was proved in Proposition 3 that the structure of the operator algebra in the space \(H^\text{rel}_{k} \cap H^\text{abs}_{n} \cong \mathbb{C}[a, b]\) and the isomorphism is realized by

\[O_{a_{n+1}}^i \mapsto a^{-i} b^i.\]

It is well known that \(sl_2\) acts on \(\mathbb{C}[a, b]\). Thus one can expect that \(sl_2\) acts in the space \(H^\text{rel}_{k} \cap H^\text{abs}_{n}\). It is easy to check, that the operator \(X_+\) corresponds to the \(sl_2\) increasing generator

\[X_+ \mapsto b \partial / \partial a.\]

We may expect that there exists an operator \(X_-\) corresponding to the \(sl_2\) decreasing generator

\[X_- \mapsto a \partial / \partial b.\]

A construction of this operator is an open problem. It seems that there is no decreasing operator \(X_-\) such that \([Q, X_-] = 0\) and \(X_-\) acts nonzero on the cohomology space. It is expected that there exists operator \(X_-\) such that \([Q, X_-] \neq 0\), but \(X_-\) acts on certain representatives of \(H^\text{rel}_{k} \cap H^\text{abs}_{n}\). This is similar to \(sl_2\) action on the space of harmonic forms on a Kahler manifold [19].

In ref. [20] Lian and Zuckerman have recognized that absolute cohomology has a structure of Gerstenhaber algebra. In other words they define a bracket \(\{u, v\}\). This bracket differentiates an operator product and provides the structure of a Lie algebra on the absolute cohomology space. It would be interesting to calculate this bracket in our case. The simplest example is \(\{N_{a_1}^0, O\} = X_+ O\). This equality is equivalent to (4.10) and explains the fact that \(X_+\) differentiates operator product.

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7. Appendix

In this Appendix we give an explicit form of some states in the relative cohomology space. Explicit form of several the highest cohomology classes are given in (3.6). Here we consider the other states. These examples are obtained by recursion procedure described above. Let us start from the

**Embedding level 1.**

Let us consider the cohomology space \( H^\text{rel}_a(L_{a_1}) \). It is easy to check that the cohomology class of the ghost number 0 has the following form

\[
O^a_{a_1} = H^a_{a_1} \Psi^F = b_{-1} \Psi^F
\]

One can show that \( Q(O^a_{a_1}) = L_{-1} \Psi^F = 0 \), since \( L_{-1}|V_{a_1}\rangle \) is a singular vector in the Verma module \( V_{a_1} \) and, therefore, in the irreducible module \( L_{-1}|L_{a_1}\rangle = 0 \).

In the space \( H^\text{rel}_b(L_{b_1}) \) the cohomology class of the ghost number 0 is

\[
O^b_{b_1} = H^b_{b_1} \Psi^F = \left( b_{-1} L_{-1} + \frac{2}{3} b_{-2} \right) \Psi^F
\]

It is easy to check that \( Q(O^b_{b_1}) = (L_{-1}^2 + (2/3)L_{-2}) \Psi^F = 0 \), since \( (L_{-1}^2 + (2/3)L_{-2})|V_{b_1}\rangle = 0 \) is a singular vector in the Verma module \( V_{b_1} \) and, therefore, in the irreducible module \( (L_{-1}^2 + (2/3)L_{-2})|L_{b_1}\rangle = 0 \).

**Embedding level 2.**

Here we only consider the cohomology space \( H^\text{rel}_a(L_{a_2}) \). The highest cohomology class is given in (3.6). Let us consider the rest of them.

It is straightforward to check that cohomology classes of the ghost number 1 can be written as follows

\[
O^a_{a_2} = H^a_{a_2} \Psi^F = \left( \frac{5}{3} c_{-1} b_{-1}^2 L_{-1}^3 - \frac{20}{3} c_{-1} b_{-2}^3 L_{-1}^2 - 4 c_{-1} b_{-2} L_{-2} - \frac{52}{3} c_{-1} b_{-3} L_{-1} + 3 c_{-2} b_{-1} L_{-1}^2 + \frac{64}{3} c_{-3} b_{-1} L_{-1} - \frac{76}{3} c_{-1} b_{-3} - 20 c_{-3} b_{-2} + \frac{144}{3} c_{-4} b_{-1} \right) \Psi^F
\]

\[
O^b_{a_2} = H^b_{a_2} \Psi^F = \left( -6 c_{-2} b_{-1} L_{-1} + 14 c_{-3} b_{-1} L_{-1} - 12 c_{-2} b_{-3} + 16 c_{-4} b_{-1} \right) \Psi^F.
\]

Indeed, one can check that \( Q(O^a_{a_2}) = D_{b_1,a_2} \Psi^F = 0 \) and \( Q(O^b_{a_2}) = D_{b_1,a_2} \Psi^F = 0 \). The explicit forms of the operators \( D_{a_1,a_2} \) and \( D_{b_1,a_2} \) can be found in [11].

The cohomology class with the ghost number -1 is

\[
O^a_{a_2} = \left( -\frac{2}{3} b_{-2} b_{-1} (L_{-1}^2 + 6 L_{-2}) + \frac{2}{3} b_{-2} b_{-1} L_{-1} - \frac{4}{3} b_{-4} b_{-1} + 4 b_{-3} b_{-2} \right) \Psi^F,
\]

and one can check that

\[
Q(O^a_{a_2}) = H^a_{a_1} D_{a_1,a_2} \Psi^F - H^b_{b_1} D_{b_1,a_2} \Psi^F = 0
\]

which is in agreement with the result of our recursive construction procedure.

Due to remark 2 the formula (7.1) for the lowest cohomology (7.1) have an analogue is the free field approach. Indeed this formula is equivalent to the formula (2.9) in [14] obtained by the different method.

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