REFINEMENTS OF THE ONE DIMENSIONAL FREE POINCARÉ INEQUALITY

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ABSTRACT. We present two extensions of the one dimensional free Poincaré inequality similar in spirit to two classical refinements.

1. INTRODUCTION

The classical Poincaré inequality for a probability measure \( \mu \) on \( \mathbb{R}^d \) states that there is a constant \( \lambda > 0 \), such that for any real-valued compactly supported smooth function \( \phi \) defined on \( \mathbb{R}^d \),

\[
\lambda \text{Var}_\mu(\phi) \leq \int |\nabla \phi|^2 d\mu,
\]

where \( \text{Var}_\mu(\phi) = \int \phi^2 d\mu - (\int \phi d\mu)^2 \) is the variance of \( \phi \) with respect to \( \mu \). Another, well known, interpretation of this inequality is to view \( \lambda \) as the spectral gap of the generator \( L \) of the Dirichlet form \( \Gamma(\phi, \phi) = \int |\nabla \phi|^2 d\mu \) for which \( \mu \) is an invariant measure.

For the standard Gaussian measure \( \mu \) in \( \mathbb{R}^d \), an extension of the classical Poincaré inequality due to Houdré-Kagan [12] states that, for any smooth compactly supported function \( \phi \) on \( \mathbb{R}^d \) and \( n \geq 1 \),

\[
\sum_{k=1}^{2n} (-1)^{k+1} \frac{1}{k!} \int |\nabla^k \phi|^2 d\mu \leq \text{Var}_\mu(\phi) \leq \sum_{k=1}^{2n-1} (-1)^{k+1} \frac{1}{k!} \int |\nabla^k \phi|^2 d\mu.
\]

This last inequality which can be viewed as a Taylor type expansion for \( \text{Var}_\mu(f) \) was extended to a general Markov operators framework by Ledoux in [16].

In a different direction (1.1) was also extended by Brascamp-Lieb [4, Theorem 4.1] to measures on \( \mathbb{R}^d \) of the form \( \mu(dx) = e^{-V(x)} dx \) with \( V''(x) \) positive definite at each point \( x \in \mathbb{R}^d \). The extension asserts that

\[
\text{Var}_\mu(\phi) \leq \int \langle (V'')^{-1} \nabla \phi, \nabla \phi \rangle d\mu,
\]

for any compactly supported function \( \phi \). For one dimensional measures, a further extension of (1.1) in the spirit of (1.2) is also possible (see [22]).

With the recent interest in high dimensional phenomena, it is quite natural to ask what happens with these functional inequalities in the limit. One such setup is to apply the classical inequalities to some standard random matrices models and then analyze the limiting object. Since large random matrices have deep connections with free probability, it is also natural to interpret the limiting inequalities as the free counterparts of the classical inequalities. This is particularly true in view of the random matrix approach, as developed in [11], to the Biane-Voiculescu [3] transportation inequality and the free Log-Sobolev inequality, which first appeared in [2], and was subsequently analyzed with random matrices in [1].

In [21] and [17] some of these inequalities are studied as stand alone inequalities and analyzed using tools from mass transportation. In [17], a version of the Poincaré inequality is introduced using random
matrix heuristics but proved without references to random matrix models. For the standard semicircular law \( \alpha(dx) = 1_{[-2,2]}(x)\sqrt{4-x^2}dx \), this Poincaré inequality states that for any smooth function \( \phi \) on \([-2,2] \),

\[
\int_{-2}^{2} \int_{-2}^{2} \left( \frac{\phi(x) - \phi(y)}{x - y} \right)^2 \frac{(4 - xy)}{4\pi^2 \sqrt{(4 - x^2)(4 - y^2)}} \, dx \, dy \leq \int (\phi')^2 \, dx.
\]

Note that left-hand side of (1.4), which replaces the classical variance term, is essentially the fluctuation quantity of random matrices. Further, note that (1.4) has a different flavor from its classical counterpart. For example, in case of the standard Gaussian measure, (1.1) is the expression of the spectral gap of the Ornstein-Uhlenbeck operator. In the free case, and as shown in [17], (1.4) is equivalent to

\[
\mathcal{N} \leq \mathcal{L}
\]

where \( (\mathcal{L}\phi)(x) = -(4 - x^2)\phi''(x) + x\phi'(x) \) and \( \mathcal{N} \) are respectively the Jacobi operator and the counting number operator for the orthonormal basis of Chebyshev polynomials \( T_n(x/2) \) of \( L^2(\beta) \), where \( \beta \) is the arcsine measure \( \beta(dx) = 1_{[-2,2]}(x)\frac{dx}{\pi\sqrt{4-x^2}} \). Here we interpret the operators as unbounded operators on \( L^2(\beta) \) which is to be contrasted with the classical case where the left-hand side is simply a projection operator.

The inequality (1.4) can be realized as the limiting case of the classical Poincaré inequality applied to the distribution of the GUE-ensemble. On the other hand, the inequality (1.1) is valid for measures \( \mu(dx) = e^{-V(x)}dx \) on \( \mathbb{R}^d \) with \( V''(x) \geq \lambda > 0 \), as easily seen from (1.3). Now, let \( V : \mathbb{R} \to \mathbb{R} \) be such that \( V'' \geq \lambda > 0 \), and let \( \mu_n(dX) = \frac{1}{Z_n(V)} e^{-n\text{Tr}(V)dX} \) be the corresponding probability measure on \( n \times n \) Hermitian matrices. Let further \( \mu_V \) be the equilibrium measure, i.e., the unique probability measure minimizing the functional

\[
E_V(\mu) = \int V \, d\mu - \int \int \log |x - y| \mu(dx) \mu(dy).
\]

Since \( V \) is convex, the support of \( \mu_V \) is an interval and, up to rescaling, we may assume for simplicity that this support is \([-2,2] \). In this setup, applying the Poincaré inequality to the measures \( \mu_n \) and functions of the form \( \Phi(X) = \text{Tr}(\phi(X)) \), with \( \phi : \mathbb{R} \to \mathbb{R} \) smooth and compactly supported lead (see [17] for details) to the limiting inequality:

\[
\lambda \int_{-2}^{2} \int_{-2}^{2} \left( \frac{\phi(x) - \phi(y)}{x - y} \right)^2 \frac{(4 - xy)}{4\pi^2 \sqrt{(4 - x^2)(4 - y^2)}} \, dx \, dy \leq \int (\phi')^2 \, d\mu_V.
\]

This inequality was further investigated in [18] in relation to other free functional inequalities such as the transportation, the Log-Sobolev, and the HWI ones. The main tool involved there is the counting number operator \( \mathcal{N} \) alluded to above and given by

\[
(\mathcal{N}\phi)(x) = \int y\phi'(y) \beta(dy) + x \int \phi'(y) \beta(dy) - (4 - x^2) \int \frac{\phi'(x) - \phi'(y)}{x - y} \beta(dy).
\]

A first primary purpose of the present paper is to refine the inequality (1.4), which is the free Poincaré inequality for the semicircular law in the spirit of the classical refinement (1.2). The corresponding statement is that for any smooth function \( \phi \) on \([-2,2] \), and any positive integer \( k \),

\[
\sum_{l=1}^{2k} \frac{(-1)^{l-1}}{l} \Vert \partial^{(l-1)} \phi' \Vert_{\alpha \otimes l}^2 \leq \int_{-2}^{2} \int_{-2}^{2} \left( \frac{\phi(x) - \phi(y)}{x - y} \right)^2 \frac{(4 - xy)}{4\pi^2 \sqrt{(4 - x^2)(4 - y^2)}} \, dx \, dy \leq \sum_{l=1}^{2k-1} \frac{(-1)^{l-1}}{l} \Vert \partial^{(l-1)} \phi' \Vert_{\alpha \otimes l}^2,
\]

where \( \partial \) is the non-commutative derivative introduced in [26], and where the \( \partial^{(l)} \) are its higher versions. The above inequality is, in fact, a consequence of an exact representation with remainder (depending on \( \mathcal{M} \), the counting number operator for the rescaled Chebyshev polynomials of the second kind) for the sandwiched term. This is contained in Theorem 8. The proof of this result is based on two main ingredients, a first one is the basic relation between the operator \( \mathcal{N} \) and \( \mathcal{M} \) which appears in Theorem 3 and states that

\[
(\mathcal{N}\phi,\phi) = 2\langle (\mathcal{M} + I)^{-1} \phi',\phi' \rangle_{\alpha},
\]
where the inner product on the left-hand side is the one in $L^2(\beta)$, while on the right-hand side is the one in $L^2(\alpha)$. This statement, by itself, is enough to get the free Poincaré inequality (1.4) which follows from that $\mathcal{M}$ is a non-negative operator.

The second ingredient is based on an idea exposed in [22] which gives a refinement of the Brascamp-Lieb inequality (1.3) in the spirit of the expansion from (1.2). At its roots there are some commutation relations. To wit a bit on this idea, the starting point is the fact that

$$ \langle (\mathcal{M} + I)^{-1}\phi', \phi' \rangle_\alpha = \langle \phi', \phi' \rangle - \langle \mathcal{M}(\mathcal{M} + I)^{-1}\phi', \phi' \rangle_\alpha $$

and that $\mathcal{M} = \partial^*\partial$, where $\partial^*$ is the adjoint of $\partial$. This can then be continued with

$$ \langle \mathcal{M}(\mathcal{M} + I)^{-1}\phi', \phi' \rangle_\alpha = \langle \partial(\mathcal{M} + I)^{-1}\phi', \partial\phi' \rangle_{\alpha \otimes \alpha} = (\langle \mathcal{M}^{(2)} + 2I \rangle^{-1}\partial\phi', \partial\phi' \rangle_{\alpha \otimes \alpha}, $$

where $\mathcal{M}^{(2)}$ is an extension of the operator $\mathcal{M}$ to tensors, in a natural way, as $\mathcal{M}^{(2)}(P \otimes Q) = (\mathcal{M}P) \otimes Q + P \otimes (\mathcal{M}Q)$, for any polynomials $P$ and $Q$. Along the way, we also used an important commutation relation between $\mathcal{M}$ and the derivative operator $\partial$. Now, an iteration leads to the the basic expansion

$$ \langle (\mathcal{M} + I)^{-1}\phi', \phi' \rangle_\alpha = \langle \phi', \phi' \rangle - (\langle \mathcal{M}^{(2)} + 2I \rangle^{-1}\partial\phi', \partial\phi' \rangle_{\alpha \otimes \alpha}. $$

This procedure can then be pursued to get more terms as detailed in Section 5.

As a second purpose, we wish to extend the free Poincaré inequality (1.6) to a free Brascamp–Lieb inequality similar to (1.3) in the form (presented in Theorem 11)

$$ \int_{-2}^{2} \int_{-2}^{2} \left( \frac{\phi(x) - \phi(y)}{x - y} \right)^2 \frac{(4 - xy)dx dy}{4\pi^2 \sqrt{(1 - x^2)(4 - y^2)}} \leq \int \phi'^2 \frac{d\mu_V}{V^n}, $$

which holds for any smooth function $\phi$ on $[-2, 2]$. The main idea in proving (1.8) is similar to one outlined by Helffer in [9] and consists in writing the left-hand side of (1.8) as $\langle (\mathcal{M} + V'' + 1\phi, \phi)_{\nu_V} \rangle$, for some operator $\mathcal{M}_V$, presumably unbounded and non-negative definite. To see what the candidate for $\mathcal{M}_V$ should be, we use heuristics from the classical result applied to random matrices. Once this operator is settled, then the proof follows once it is shown that $\langle (\mathcal{M} + V'' + 1\phi, \phi)_{\nu_V} \rangle$ does not depend on the potential $V$. If this is indeed the case, then choosing our favorite potential, namely, $V(x) = x^2/2$, then the left-hand side of (1.8) is nothing but (1.7). To some extent, at the bottom of this argument is the fact that the variance term on the left-hand side of (1.8) is universal, by which we mean universality of the fluctuations of random matrices.

Both extensions provided here are sharp, i.e., we can find non-trivial functions for which equality is attained.

The paper is organized as follows. Section 2 introduces the main notations and some preliminary facts. Section 3 contains the main operators and their interrelations which are partially import from [18]. Section 4 is an intermezzo containing an interpolation equality which parallels the classical one. It is also a motivation for a brief description for the free Ornstein-Uhlenbeck semigroup seen from a different perspective. This, in turn, provides yet another (and simple) proof of the free one dimensional Poincaré inequality for the semicircular law. Section 5 gives the main refinement associated with the semicircular law, extending the operator $\mathcal{M}$ to tensors and unearthing the main commutation relations. These are then used in the expansion of the variance like term, finally leading to the free version of Houdré-Kagan (1.2). Next, Section 6 is a purely heuristic section which motivates the introduction of the main operator associated to the equilibrium measure for a potential $V$. It also serves as a quick recapitulation of Helffer’s arguments (from [9]) on obtaining the Brascamp-Lieb result. These are, finally, used in Section 7 to prove the free version of Brascamp-Lieb (1.3).

2. Preliminaries

2.1. Random matrices and logarithmic potentials. The random matrix ensembles we deal with are prescribed by the probability measures on the set $\mathcal{H}_n$ of $n \times n$ Hermitian matrices determined by a potential $V : \mathbb{R} \to \mathbb{R}$ via

$$ \mathbb{P}_V^0(dX) = \frac{1}{Z_n(V)} e^{-n\text{Tr}V(X)}dX. $$

(2.1)
Here,
\[ Z_n(V) = \int e^{-n\operatorname{Tr}(X)}dX \]
is simply the normalizing constant which makes \( P^n \) a probability measure.

It is known, see [5] or [14], that for any such \( V \)
\[ -\lim_{n \to \infty} \frac{1}{n^2} \log Z_n(V) = E_V = \inf \{ E_V(\mu) : \mu \in \mathcal{P}(\mathbb{R}) \} , \]
where
\[ E_V(\mu) := \int Vd\mu - \iint \log |x - y|\mu(dx)\mu(dy), \]
and \( \mathcal{P}(\mathbb{R}) \) is the set of probability measures on \( \mathbb{R} \). For \( V \) having enough growth at infinity, for instance, if
\[ \lim_{|x| \to \infty} (V(x) - 2\log |x|) = \infty, \]
this minimization problem is known to have a unique solution \( \mu_V \) and standard references on this are [5, 14, 23]. The variational characterization of the measure \( \mu_V \) is
\[ V(x) \geq 2 \int \log |x - y|\mu(dy) + C \quad \text{quasi-everywhere on } \mathbb{R} \]
and
\[ V(x) = 2 \int \log |x - y|\mu(dy) + C \quad \text{quasi-everywhere on } \text{supp}\mu. \]
Therefore, taking the derivative in the second line of (2.3), it follows that on the support of \( \mu_V \) (assuming the support is a finite union of intervals),
\[ V'(x) = \text{p.v.} \int \frac{2}{x - y}\mu_V(dy), \]
where, as usual, \( \text{p.v.} \) stands for the Cauchy principal value. In this paper we limit ourselves to a smooth \( V \) which is also convex, in which case the support of the measure \( \mu_V \) is a single interval (see [23]). We can, in fact, weaken the smoothing condition on \( V \) and for the main result, it suffices for \( V \) to be \( C^4 \)–regular. To shorten the notations, we also denote the principal value of a measure \( \nu \) by \( H\nu \), i.e.,
\[ (H\nu)(x) := \text{p.v.} \int \frac{2}{x - y}\nu(dy). \]

In addition to (2.2), another important convergence property is that for any bounded continuous function \( g \) on the real line,
\[ \int \frac{1}{n}\operatorname{Tr}(g(X)) P^n_V(dX) \overset{n \to \infty}{\longrightarrow} \int g d\mu_V. \]
In fact, something even stronger takes place here, namely, \( \frac{1}{n}\operatorname{Tr}(g(X)) \) converges almost surely to \( \int g d\mu_V \), as it can be seen, for instance, from [10]. Above, the \( \text{GUE}\left(\frac{1}{\sqrt{n}}\right) \) ensemble corresponds to \( V(x) = x^2/2 \).

Let us now turn to some of the basic operators which play an important rôle in the treatment of the free Poincaré inequality. There are two important measures on \([-2, 2] \), the semicircular one and the arcsine one, respectively defined by
\[ \alpha(dx) = \frac{\sqrt{4 - x^2}}{2\pi}dx, \quad \text{and} \quad \beta(dx) = \frac{dx}{\pi\sqrt{4 - x^2}}. \]
Most of the action takes place around the arcsine measure \( \beta \) and so we use \( \langle \cdot, \cdot \rangle \) to denote the inner product in \( L^2(\beta) \), while for any other measure \( \mu \), \( \langle \cdot, \cdot \rangle_\mu \) is the inner product in \( L^2(\mu) \). The reason for dealing with the interval \([-2, 2] \) is that the semicircular law \( \alpha \) has mean zero and variance one. Another important reason is unfolded in [7] and [18] and stems from the prominent rôles played by the Chebyshev polynomials in analyzing the logarithmic potentials. Thus, for convex potentials \( V \) the support of the equilibrium measure is a single interval. Thus, by rescaling, namely replacing \( V(x) \) by \( V(cx + b) \) for appropriate \( c > 0 \) and \( b \) real, the support of \( \mu_V \) can always be arranged to be \([-2, 2] \).
Another measure which plays a rôle in the sequel is

$$\omega(dx, dy) = 1_{[-2, 2]}(x)1_{[-2, 2]}(y) \frac{(4 - xy)}{4\pi^2 \sqrt{(4 - x^2)(4 - y^2)}} dx dy.$$  

We introduce next the appropriate orthogonal basis associated to the measures $\alpha$ and $\beta$. These are

$$\phi_n(x) = T_n \left(\frac{x}{2}\right) \quad \text{and} \quad \psi_n(x) = U_n \left(\frac{x}{2}\right), \quad \text{for } n \geq 0.$$  

Here $T_n(x)$ is the $n$th Chebyshev polynomials of the first kind defined via $T_n(\cos \theta) = \cos(n\theta)$, while $U_n$ is the $n$th Chebyshev polynomials of the second kind defined via $U_n(\cos \theta) = \frac{\sin((n+1)\theta)}{\sin \theta}$. Adjusting a little the polynomials $T_n$ as $\tilde{T}_0 = T_0$ and $\tilde{T}_n(x) = \sqrt{2}T_n(x)$, it is easily seen that $\{\tilde{T}_n(x/2)\}_{n \geq 0}$ is an orthonormal basis for $L^2(\beta).$ Similarly, $\{U_n(x/2)\}_{n \geq 0}$ forms an orthonormal basis for $L^2(\alpha).$ Other relations between these functions, of later use and, which can be checked effortlessly include

$$\phi'_n = \frac{n}{2} \psi_{n-1},$$

and

$$-2\psi'_{n-1}(x) + \frac{x}{2} \psi'_n(x) = n\psi_n(x).$$

A further fact, used several times below, is the following relationship:

$$\frac{\psi_n(x) - \psi_n(y)}{x - y} = \sum_{k=0}^{n-1} \psi_k(x)\psi_{n-1-k}(y),$$

which, for instance, can be deduced from the expression for the generating function of the Chebyshev polynomials of the second kind given by:

$$\sum_{n=0}^{\infty} r^n U_n(x) = \frac{1}{1 - 2rx + r^2}, \quad r \in (-1, 1).$$

2.2. Random matrices and fluctuations. By the study of the fluctuations associated to the random matrix models introduced above, we mean the study of the limiting behavior of $\text{Tr}(\phi(M)) - \mathbb{E}[\text{Tr}(\phi(M))]$, as $n$ tends to $\infty$, e.g., see [14] and [15]. Assuming that $\mu_V$ is supported on $[-2, 2]$, the variance of this random variable, with respect to $\mathbb{P}^n_V$ is given in the limit by

$$\int_{-2}^{2} \int_{-2}^{2} \left(\frac{\phi(x) - \phi(y)}{x - y}\right)^2 \frac{(4 - xy)}{4\pi^2 \sqrt{(4 - x^2)(4 - y^2)}} dx dy = \int \int \left(\frac{\phi(x) - \phi(y)}{x - y}\right)^2 \omega(dx, dy),$$

a quantity which plays in our context a rôle analogous to the one of the variance in the classical setting.

2.3. Semicircular systems. Here we summarize a few facts about the $R$-transform and introduce the notion of a semicircular system.

A non-commutative probability space is a pair $(\mathcal{A}, \phi)$, where $\mathcal{A}$ is a unital $\ast$-algebra and $\phi$ is a trace on it such that $\phi(1) = 1.$ For basic notions of freeness we refer the reader to [25]. Nevertheless, we mention here the version of $R$-transform in the spirit of [20]. All non-commutative variables $a, b$ considered in this section are assumed to be self-adjoint, i.e. $a^* = a$ and $b^* = b$.

Now, given non-commutative variables $a_1, a_2, \ldots, a_n$ in $\mathcal{A}$, the moment generating function of $(a_1, a_2, \ldots, a_n)$ is the formal power series in non-commuting variables $z_1, z_2, \ldots, z_n$ described by

$$M_{a_1, a_2, \ldots, a_n}(z_1, z_2, \ldots, z_n) = \sum_{s=1}^{\infty} \sum_{i_1, i_2, \ldots, i_s=1}^{n} \phi(a_{i_1} a_{i_2} \ldots a_{i_s}) z_{i_1} z_{i_2} \ldots z_{i_s}.$$  

The $R$-transform is also a formal power series in non-commuting variables $z_1, z_2, \ldots, z_n$ described by

$$R_{a_1, a_2, \ldots, a_n}(z_1, z_2, \ldots, z_n) = \sum_{s=1}^{\infty} \sum_{i_1, i_2, \ldots, i_s=1}^{n} k_s(a_{i_1} a_{i_2} \ldots a_{i_s}) z_{i_1} z_{i_2} \ldots z_{i_s},$$
where the \( k_s \) are the free cumulants. The moment generating function and the \( R \) transforms are related by
\[
M = R[\text{Zeta}], \quad R = M[\text{Zeta}],
\]
where \( \text{Zeta} \) is described in [24] and also in [20] in terms of the lattice of the non-crossing partitions. Here
\[
\text{Zeta}(z_1, z_2, \ldots, z_n) = \sum_{s=1}^{\infty} \sum_{i_1, i_2, \ldots, i_s=1}^{n} z_{i_1} z_{i_2} \cdots z_{i_s}
\]
and
\[
\text{Moeb}(z_1, z_2, \ldots, z_n) = \sum_{s=1}^{\infty} \sum_{i_1, i_2, \ldots, i_s=1}^{n} (-1)^{s+1} \frac{(2s-2)!}{(s-1)! s!} z_{i_1} z_{i_2} \cdots z_{i_s}
\]
are the Zeta and Moebius functions in \( n \) variables associated to the lattice of non-crossing partitions (see [20, Eqs. 3.10 and 3.11]). The only point we need to make here is that \( R \) determines \( M \), and that vice versa \( M \) determines \( R \). In particular, if \( R_{a_1, a_2, \ldots, a_n}(z_1, z_2, \ldots, z_n) = R_{b_1, b_2, \ldots, b_n}(z_1, z_2, \ldots, z_n) \), then \( M_{a_1, a_2, \ldots, a_n}(z_1, z_2, \ldots, z_n) = M_{b_1, b_2, \ldots, b_n}(z_1, z_2, \ldots, z_n) \) which means that \( \phi(a_1 a_2 \ldots a_n) = \phi(b_1 b_2 \ldots b_n) \), or otherwise stated, the mixed moments are the same.

A main property of the \( R \)-transform is that \( (a_1, a_2, \ldots, a_n) \) and \( (b_1, b_2, \ldots, b_m) \) are free if and only if
\[
R_{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m}(z_1, z_2, \ldots, z_n, z'_1, z'_2, \ldots, z'_m) = R_{a_1, a_2, \ldots, a_n}(z_1, z_2, \ldots, z_n) + R_{b_1, b_2, \ldots, b_m}(z'_1, z'_2, \ldots, z'_m).
\]
The second property is that if \( a \) is a standard semicircular element (i.e. \( \phi(a^k) = \int a^k \alpha(dx) \)), then
\[
R_a(z) = z^2.
\]

Next, we say that a tuple \( (b_1, b_2, \ldots, b_n) \) is a standard semicircular system if the variables are free and each of them is a standard semicircular element. In particular, in light of (2.14) and (2.15), this means that
\[
R_{b_1, b_2, \ldots, b_n}(z_1, z_2, \ldots, z_n) = z_1^2 + z_2^2 + \cdots + z_n^2.
\]

Further, we say that a tuple \( (a_1, a_2, \ldots, a_n) \) of centered variables (i.e. \( \phi(a_i) = 0, 1 \leq i \leq n \)) is a semicircular system if
\[
R_{a_1, a_2, \ldots, a_n}(z_1, z_2, \ldots, z_n) = \sum_{i,j=1}^{n} c_{ij} z_i z_j.
\]
As it turns out, the coefficients \( c_{ij} \) are determined by \( c_{ij} = \phi(a_i a_j) \), hence the matrix \( C = \{c_{ij}\}_{i,j=1}^{n} \) is simply the covariance matrix of the tuple. Moreover, since \( \phi \) is a trace, \( C \) is a real valued symmetric non-negative definite matrix. Note that this notion of semicircular system mimics the classical notion of a (multidimensional) Gaussian random variable, in that the logarithm of the characteristic function is a quadratic function. Also, as in the classical case, a semicircular system is completely determined by the covariance matrix, by which we mean that the mixed moments of \( (a_1, a_2, \ldots, a_n) \) are determined by the covariance matrix and the inversion formula (2.13).

The main results to be used, are contained in the following statement.

**Proposition 1.**

(1) Let \( (b_1, b_2, \ldots, b_m) \) be a semicircular system with covariance \( C \). Let \( D \) be an \( n \times m \) matrix and let \( a_i = \sum_{j=1}^{m} d_{ij} b_j \). Then \( (a_1, a_2, \ldots, a_n) \) is a semicircular system with covariance matrix \( \tilde{C} = DCD^t \).

(2) Let \( C \) be an \( n \times n \) real symmetric and non-negative definite matrix, and let \( D = C^{1/2} \). Then for any standard semicircular system \( (b_1, b_2, \ldots, b_n) \), the tuple \( (a_1, a_2, \ldots, a_n) \) with \( a_i = \sum_{j=1}^{n} d_{ij} b_j \) is a semicircular system with covariance matrix \( \tilde{C} \). In particular, for any symmetric non-negative definite matrix \( C \), there exists a semicircular system with covariance matrix \( \tilde{C} \).

(3) If two semicircular systems have the same covariance matrix, then they have the same moments. More precisely, if \( (a_1, a_2, \ldots, a_n) \) and \( (b_1, b_2, \ldots, b_n) \) are two semicircular systems with the same covariance matrix, then \( \phi(a_{i_1} \ldots a_{i_s}) = \phi(b_{i_1} b_{i_s}) \), for any \( 1 \leq i_1, i_2, \ldots, i_s \leq n \) and any \( s \geq 1 \).
Proof. (1) First, by the very definition of the $R$ transform and the linearity of the cumulants, it follows that

$$R_{a_1,a_2,...,a_n}(z_1,z_2,...,z_n) = \sum_{s=1}^{\infty} \sum_{i_1,i_2,...,i_s=1}^{n} k_s(a_{i_1},a_{i_2},...,a_{i_s})z_{i_1}z_{i_2}...z_{i_s}$$

$$= \sum_{s=1}^{\infty} \sum_{i_1,i_2,...,i_s=1}^{n} \sum_{j_1,j_2,...,j_s=1}^{m} d_{i_1,j_1}d_{i_2,j_2}...d_{i_s,j_s}k_s(b_{j_1},b_{j_2},...,b_{j_s})z_{i_1}z_{i_2}...z_{i_s}.$$

Next, by the definition of $R_{b_1,b_2,...,b_m}(z_1,z_2,...,z_m)$, and the quadratic assumption in $z_1,...,z_m$, we infer that

$$k_s(b_{j_1},b_{j_2},...,b_{j_s}) = 0 \text{ if } s \neq 2 \text{ and } k_2(b_j,b_l) = c_{jl}.$$ 

In turn, this implies that (denoting by $\tilde{C}$ the entries of $C$)

$$R_{a_1,a_2,...,a_n}(z_1,z_2,...,z_n) = \sum_{i_1,i_2=1}^{n} \sum_{j_1,j_2=1}^{m} d_{i_1,j_1}d_{i_2,j_2}c_{j_1,j_2}z_{i_1}z_{i_2} = \sum_{i_1,i_2=1}^{n} \tilde{c}_{i_1,i_2}z_{i_1}z_{i_2},$$

which is precisely what needed to be proved.

(2) This follows from the previous item combined with the fact that the covariance matrix of a standard semicircular system is the identity matrix.

(3) This is the uniqueness of the moment generating function as it follows, for example, from (2.13). □

3. The main operators

We are now ready to introduce the main operators of interest. For a $C^2$ function, $\phi : [-2, 2] \to \mathbb{R}$, set

$$(\mathcal{E}\phi)(x) = -\int \log |x-y|\phi(y)\beta(dy),$$

$$(\mathcal{F}\phi)(x) = -\int \log |x-y|\phi(y)\alpha(dy),$$

$$(\mathcal{N}\phi)(x) = \int y\phi'(y)\beta(dy) + x\int \phi'(y)\beta(dy) - (4-x^2)\int \frac{\phi'(x) - \phi'(y)}{x-y}\beta(dy),$$

$$(\mathcal{M}\phi)(x) = 2\text{p.v.} \int \frac{\phi(x) - \phi(y)}{(x-y)^2}\alpha(dy) = \lim_{\epsilon \to 0} \int_{|x-y| \geq \epsilon} \frac{\phi(x) - \phi(y)}{(x-y)^2}\alpha(dy).$$

Given a measure $\mu$ on $[-2, 2]$, let

$$L^2_0(\mu) = \left\{ f \in L^2(\mu) : \int f d\mu = 0 \right\}.$$

Below, is a list of relationships between the operators just defined which is mainly imported from [18, Proposition 1].

**Proposition 2.**

(1) $\mathcal{E}$ maps $C^2([-2, 2])$ to $C^2([-2, 2])$ and can be extended to a bounded self-adjoint operator from $L^2(\beta)$ into itself.

(2) For any $C^2$ function $\phi \in L^2_0(\beta)$,

$$(\mathcal{E}\mathcal{N}\phi)(x) = \phi,$$

$$(\mathcal{N}\mathcal{E}\phi)(x) = \phi.$$ 

(3) $\mathcal{E}\phi_0 = 0$, and for $n \geq 1$, $\mathcal{E}\phi_n = \phi_n/n$. Moreover, for $n \geq 0$, $\mathcal{N}\phi_n = n\phi_n$. In other words, $\mathcal{N}$ is the counting number operator for the Chebyshev basis $\{\phi_n\}_{n \geq 0}$ of $L^2(\beta)$, and it can be canonically extended to a self-adjoint operator on $L^2(\beta)$, which when restricted to $L^2_0(\beta)$ has inverse $\mathcal{E}$. 
(4) For any $C^1$ functions, $\phi$ and $\psi$, on $[-2, 2]$,

\begin{equation}
\langle N \phi, \psi \rangle = 2 \iint \frac{(\phi(x) - \phi(y))(\psi(x) - \psi(y))}{(x - y)^2} \omega(dx, dy),
\end{equation}

and in particular, $\langle N \phi, \psi \rangle = \langle \phi, N \psi \rangle$.

(5) If $V$ is a $C^3$ potential on $[-2, 2]$ whose equilibrium measure $\mu_V$ has support $[-2, 2]$, then

\begin{equation}
d\mu_V = \left( 1 - \frac{1}{2} N V \right) d\beta.
\end{equation}

(6) The operator $M$ is the counting number operator for the basis $(\psi_n)_{n \geq 0}$ of $L^2(\alpha)$ and it has a natural extension as a self-adjoint operator on $L^2(\alpha)$. In other words, for any $n \geq 0$, $M \psi_n = n \psi_n$. In addition, for any $C^1$ function $\phi$ on $[-2, 2]$,

\begin{equation}
\langle M \phi, \phi \rangle_\alpha = \int \int \left( \frac{\phi(x) - \phi(y)}{x - y} \right)^2 \alpha(dx)\alpha(dy).
\end{equation}

Proof. Only the last part of this theorem is not covered in [18, Proposition 1]. To prove it, we proceed as follows: Take $\phi$ to be a $C^2$ function on $[-2, 2]$ and note that the variational characterization (2.4) gives

\begin{equation}
p.v. \int \frac{2}{x - y} \alpha(dy) = x.
\end{equation}

This can then be used to remove the singularity in the definition of $M$ by observing that

\begin{equation}
(M \phi)(x) = 2 \int \frac{\phi(x) - \phi(y) - \phi'(x)(x - y)}{(x - y)^2} \alpha(dy) + \phi'(x)p.v. \int \frac{2}{x - y} \alpha(dy)
= -2 \frac{d}{dx} \int \frac{\phi(x) - \phi(y)}{x - y} \alpha(dy) + \phi'(x)x.
\end{equation}

Next, to show that $M$ is the counting number operator for $\psi_n$, notice that, from (2.11) and from the orthogonality of $\psi_n$ with respect to the inner product associated with the measure $\alpha$,

\[ \int \frac{\psi_n(x) - \psi_n(y)}{x - y} \alpha(dy) = \psi_{n-1}(x), \]

which, in turn, using (2.10) leads to

\[ M \psi_n(x) = -2 \psi'_{n-1}(x) + x \psi'_n(x) = n \psi_n(x). \]

In other words, $M$ is the counting number operator for the orthonormal basis $\psi_n$ of $L^2(\alpha)$. Finally, to prove (3.5), use the first line of (3.7) combined with (3.6) to justify the following chain of equalities,
satisfied by any $C^2$ function $\phi$ on $[-2, 2]$:

\begin{equation}
\langle M\phi, \phi \rangle_\alpha = 2 \int \int \frac{(\phi(x) - \phi(y))\phi(x) - \phi'(x)\phi(x)(x-y)}{(x-y)^2} \alpha(dy)\alpha(dx) + \int x\phi'(x)\phi(x)\alpha(dx)
\end{equation}

For instance, it turns out that (2.11) can be nicely rewritten in the form

\begin{equation}
\left( \phi(x) - \phi(y) \right)^2 \frac{(\phi(x) - \phi(y))(x-y)}{(x-y)^2} \alpha(dy)\alpha(dx) + \int x\phi'(x)\phi(x)\alpha(dx)
\end{equation}

This equality for $C^2$ functions can be used in combination with standard results of the theory of Dirichlet forms [6] to justify that $M$ has a unique essentially self-adjoint extension. Moreover, standard approximation arguments prove that (3.5) is valid for any $C^1$ function $\phi$.

Let us record separately the following important identity.

**Theorem 3.** For any smooth function $\phi$ on $[-2, 2]$,

\begin{equation}
\langle N\phi, \phi \rangle = 2 \langle (M+I)^{-1}\phi', \phi' \rangle_\alpha.
\end{equation}

**Proof.** By polarization, (3.9) is equivalent to

\begin{equation}
\langle N\phi, \psi \rangle = 2 \langle (M+I)^{-1}\phi', \psi' \rangle_\alpha,
\end{equation}

which, by simple approximations, needs only to be verified for $\phi = \phi_n$ and $\psi = \phi_m$. Now, since $N\phi_n = n\phi_n$, (2.9) combined with $(M+I)^{-1}\psi_{n-1} = n^{-1}\psi_{n-1}$ and orthogonality, lead to the desired conclusion.

Voiculescu in [26] introduced the non-commutative derivative $\partial : \mathbb{C}[X] \to \mathbb{C}[X] \otimes \mathbb{C}[X]$ which is given by

\begin{equation}
\partial 1 = 0, \quad \partial(X) = 1 \otimes 1, \quad \partial(m_1 m_2) = \partial(m_1)(1 \otimes m_2) + (m_1 \otimes 1)\partial(m_2).
\end{equation}

Particularly useful, is the fact that the non-commutative derivative of $P = X^m$ can naturally be identifies as

\begin{equation}
\partial P = \frac{P(x) - P(y)}{x - y}.
\end{equation}

For instance, it turns out that (2.11) can be nicely rewritten in the form

\begin{equation}
\partial \psi_n = \sum_{l=0}^{n-1} \psi_l \otimes \psi_{n-l-1}.
\end{equation}

Let us now introduce the natural trace $\alpha$ on $\mathbb{C}[X]$ by

\begin{equation}
\alpha(P) = \int P d\alpha
\end{equation}
and denote by $\alpha^{\otimes k}$ its natural extension to $\mathbb{C}[X]^{\otimes k}$. Also introduce the trace $\omega$ on $\mathbb{C}[X] \otimes \mathbb{C}[X]$ through

$$\omega(P \otimes Q) = \int P(x)Q(y)\omega(dx, dy).$$

With these notations, (3.3) can be translated into

$$\langle N\phi, \psi \rangle = 2\omega(\partial\phi \times \partial\psi).$$

In the language of Dirichlet forms this simply indicates that $N$ is the generator of the Dirichlet form $D(P, Q) = 2\omega(\partial P \times \partial Q)$.

In a similar vein, for the operator $M$

$$(3.11)$$

$$\langle M\phi, \psi \rangle_\alpha = (\alpha \otimes \alpha)(\partial\phi \times \partial\psi).$$

Next, introducing the dual operator $\partial^*$ (see [26]) via

$$\langle \partial^*(\phi \otimes \eta), \psi \rangle_\alpha = \langle \phi \otimes \eta, \partial\psi \rangle_{\alpha \otimes \alpha},$$

a relation which has to be satisfied for $C^1$ functions $\phi, \psi$, and $\eta$, we see that

$$\langle \partial^*(\psi_a \otimes \psi_b) = \psi_{a+b+1} \text{ for } a, b \geq 0. \rangle$$

4. An Interpolation Formula for the Semicircular Law

Let us start by recalling a classical interpolation result, e.g., see [13] and the references therein.

**Proposition 4.** Let $f, g : \mathbb{R}^d \to \mathbb{R}$ be smooth compactly supported functions, then

$$(4.1)$$

$$\mathbb{E}[f(X)g(X)] - \mathbb{E}[f(X)]\mathbb{E}[g(X)] = \int_0^1 \mathbb{E}[\langle \nabla f(\sqrt{1-s}X + \sqrt{s}Z), \Sigma \nabla g(\sqrt{1-s}Y + \sqrt{s}Z) \rangle] ds$$

where $X, Y, Z$ are $d$-dimensional iid $N(0, \Sigma)$ random vectors.

Replacing $X_n, Y_n, Z_n$ by iid $GUE(\frac{1}{n})$ ensembles and taking $f(A) = g(A) = Tr_n\phi(A)$ the above yields

$$\text{Var}(Tr_n\phi(X_n)) = \int_0^1 \frac{1}{n} Tr_n \left[ \phi'(\sqrt{1-s}X_n + \sqrt{s}Z_n) \phi'(\sqrt{1-s}Y_n + \sqrt{s}Z_n) \right] ds.$$

Upon taking the limit, as $n \to \infty$, and using fluctuation results for random matrices [14, 15] or (2.12) combined with the general result of Voiculescu on freeness [25], lead to the following formal result.

**Proposition 5.** Let $\phi : [-2, 2] \to \mathbb{R}$ be a smooth function. Then,

$$(4.2)$$

$$\int \int \left( \frac{\phi(x) - \phi(y)}{x - y} \right)^2 \omega(dx, dy) = \int_0^1 \tau(\phi'(\sqrt{1-s}x + \sqrt{s}z)\phi'(\sqrt{1-s}y + \sqrt{s}z)) ds,$$

where $x, y, z$ are free semicircular random variables on some non-commutative probability space $(\mathcal{A}, \tau)$.

**Proof.** A proof of (4.2) has already been given through random matrix manipulations. However, here is a different and more direct approach: From (3.3), the left-hand side of (4.2) is:

$$(4.3)$$

$$\int \int \left( \frac{\phi(x) - \phi(y)}{x - y} \right)^2 \omega(dx, dy) = \frac{1}{2} \langle N\phi, \psi \rangle.$$

To deal with the right-hand side of (4.2), start by observing that for a fixed $s \in [0, 1]$, the pair $(\sqrt{1-s}x + \sqrt{s}z, \sqrt{1-s}y + \sqrt{s}z)$ is a semicircular system as introduced in Section 2.3. The covariance matrix of $(\sqrt{1-s}x + \sqrt{s}z, \sqrt{1-s}y + \sqrt{s}z)$ is easy to compute and is equal to:

$$\left[ \begin{array}{cc} 1 & s \\ s & 1 \end{array} \right].$$
Since according to Proposition 1, the mixed moments do not depend on the particular realization of the semicircular system as long as the covariance matrix is the same, a different system, namely \( t(\sqrt{1-s^2}x + sz, z) \) can be chosen. With this choice,
\[
\tau(\Phi(\sqrt{1-s^2}x + \sqrt{s}z, \sqrt{1-s^2}y + \sqrt{s}z)) = \tau(\Phi(\sqrt{1-s^2}x + sz, z)),
\]
for any non-commutative polynomial \( \Phi \) in two variables. In particular, for any smooth functions \( \phi \) and \( \psi \) on \([-2, 2]\), it follows that
\[
\tau(\phi'(\sqrt{1-s^2}x + \sqrt{s}z)\psi'(\sqrt{1-s^2}y + \sqrt{s}z)) = \tau(\phi'(\sqrt{1-s^2}x + sz)\psi'(z)).
\]
From this last fact, combined with the change of variable \( s = e^{-t} \), the right-hand side of (4.2) becomes
\[
\int_0^1 \tau(\phi'(\sqrt{1-s^2}x + sz)\psi'(z))ds = \int_0^\infty e^{-t}\tau(\phi'(\sqrt{1-e^{-2t}x + e^{-t}z})\psi'(z))dt.
\]
Next, define the operator \( P_t \) via
\[
\langle P_t\phi, \psi \rangle_\alpha = \tau(\phi(\sqrt{1-e^{-2t}x + e^{-t}z})\psi(z)).
\]
From the covariance structure of semicircular systems pointed above, it is easy to check that \( (P_t)_{t \geq 0} \) form a semigroup of bounded selfadjoint operators. Denote by \( -A \) its generator, which we now plan to identify. To this end, take \( \phi(x) = x^a \) and \( \psi(x) = x^b \), with \( a, b \geq 0 \) integers, and compute
\[
\langle Ax^a, x^b \rangle = \frac{d}{dt} \bigg|_{t=0} \tau(\langle z + \sqrt{2}tx - tz^a z^b \rangle) = 2 \sum_{l_1+l_2+l_3=a-2} \tau(z^{l_1}xz^{l_2}xz^{l_3}z^b) - a\tau(z^a z^b).
\]
Using the freeness of \( x \) and \( z \), continue with
\[
2 \sum_{l_1+l_2+l_3=a-2} \tau(z^{l_2})\tau(z^{l_1+l_3+b}) - a\tau(z^a z^b) = 2a \sum_{l=0}^{a-2} (l+1)\tau(z^{l+b})\tau(z^{a-2-l}) - a\tau(z^a z^b),
\]
and thus, since \( z \) is semicircular under \( \tau \), arrive at
\[
- A\phi = 2D(I \otimes \alpha)(\partial \phi) - x\phi',
\]
where the operator \( D \) is the derivative operator.

Taking this last identity on functions \( \phi = \psi_n \), combined with (3.10) and the fact that the sequence \( \{\psi_n\}_{n \geq 0} \) is orthogonal with respect to inner product associated to \( \alpha \), as well as (2.10) lead to \( A\psi_n = n\psi_{n+1} \), which shows that \( A = M \). Hence, for smooth functions \( \phi \) on \([-2, 2]\), the right-hand side of (4.2) can now be written as
\[
\int_0^\infty e^{-t}\tau(\phi'(\sqrt{1-e^{-2t}x + e^{-t}z})\phi'(z))dt = \int_0^\infty e^{-t}\langle e^{-tM}\phi', \phi' \rangle_\alpha dt
\]
\[
= \int_0^\infty (e^{-t(M+I)}\phi', \phi' \rangle_\alpha dt = \langle (M+I)^{-1}\phi', \phi' \rangle_\alpha.
\]
To conclude, the left-hand side of (4.2) is \( \frac{1}{2}\langle N\phi, \phi \rangle \) while its right-hand side is \( \langle (M+I)^{-1}\phi', \phi' \rangle_\alpha \), and therefore the remaining of the statement follows from Theorem 3.

Note that \( (P_t)_{t \geq 0} \) is nothing but the free Ornstein-Uhlenbeck semigroup first introduced in [2].

We can now state the following consequence:

**Corollary 6 (The Free Poincaré Inequality).** For any smooth function \( \phi : [-2, 2] \to \mathbb{R} \),
\[
\int_{-2}^2 \int_{-2}^2 \frac{(\phi(x) - \phi(y))^2}{x - y} \frac{(4 - xy)}{4\pi^2(4 - x^2)(4 - y^2)} \, dx \, dy \leq \int_{-2}^2 \phi'(x)^2 \alpha(dx).
\]
Equality is only attained for linear functions \( \phi \).
Proof. From the Cauchy-Schwarz inequality,
\[ \tau(\phi'(\sqrt{1- \bar{s}x + \sqrt{s}z})\phi'(\sqrt{1- \bar{s}y + \sqrt{s}z})) \leq (\tau(\phi'(\sqrt{1- \bar{s}x + \sqrt{s}z}))^2)^{1/2} \tau((\phi'(\sqrt{1- \bar{s}y + \sqrt{s}z}))^2)^{1/2}. \]
Both, \( \sqrt{1- \bar{s}x + \sqrt{s}z} \) and \( \sqrt{1- \bar{s}y + \sqrt{s}z} \) are semicircular elements of variance 1, thus, for any \( s \in [0, 1] \),
\[ \tau(\phi'(\sqrt{1- \bar{s}x + \sqrt{s}z})\phi'(\sqrt{1- \bar{s}y + \sqrt{s}z})) \leq \int \phi'(x)^2 \alpha(dx), \]
which combined with (4.2) finishes the proof.

A different proof of (4.6), follows directly from (3.3), Theorem 3 and the fact that \( \mathcal{M} \) is non-negative. Equality is easily seen to be attained when \( \phi' \) is in the kernel of \( \mathcal{M} \), meaning that \( \phi' \) is a constant function, in other words, \( \phi \) must be a linear. \( \square \)

5. First Refinement

We now wish to extend the operator \( \mathcal{M} \) to tensors. To do so, let \( \mathbb{C}[X]^\otimes(k) := \mathbb{C}[X] \otimes \mathbb{C}[X] \otimes \cdots \otimes \mathbb{C}[X] \) where the tensor product is taken \( k \) times. The non-commutative derivative \( \partial : \mathbb{C}[X] \to \mathbb{C}[X] \otimes \mathbb{C}[X] \), as it appears in [26], is given by
\[ \partial 1 = 0, \quad \partial(X) = 1 \otimes 1, \quad \partial(m_1 m_2) = \partial(m_1)(1 \otimes m_2) + (m_1 \otimes 1)\partial(m_2). \]
On monomials \( X^a, a \geq 1 \), this becomes
\[ \partial(X^a) = \sum_{p+q=a-1} X^p \otimes X^q \]
which is clearly equal to zero for \( a = 0 \). The higher derivatives \( \partial(k) : \mathbb{C}[X] \to \mathbb{C}[X]^\otimes(k) \) are defined inductively by \( \partial(k) = (\partial \otimes I^{k-1})\partial(k-1) \), and it is easy to check that for \( 0 \leq k \leq n \) and \( 0 \leq p \),
\[ \partial(p+n) = (I^\otimes(k) \otimes \partial(p) \otimes I^\otimes(n-k))\partial(n). \]

Now, extend the operator \( \mathcal{M} \) on \( \mathbb{C}[X] \) to an operator \( \mathcal{M}^{(k)} \) on \( \mathbb{C}[X]^\otimes(k) \) via:
\[ \mathcal{M}^{(k)}(P_1 \otimes P_2 \cdots \otimes P_k) = (\mathcal{M}P_1) \otimes P_2 \cdots \otimes P_k + P_1 \otimes (\mathcal{M}P_2) \cdots \otimes P_k + P_1 \otimes P_2 \cdots \otimes (\mathcal{M}P_k). \]
Equivalently, this is characterized by
\[ \mathcal{M}^{(a+b)} = \mathcal{M}^{(a)} \otimes I^\otimes(b) + I^\otimes(a) \otimes \mathcal{M}^{(b)}, \]
for any \( a, b \geq 0 \), with also \( \mathcal{M}^{(0)} = I \) and \( \mathcal{M}^{(1)} = \mathcal{M} \).

The following are important properties verified by the operators defined thus far.

**Proposition 7.** For any \( k \geq 1 \),
\[ \partial \odot I^\otimes(k-1)\mathcal{M}^{(k)} = (\mathcal{M}^{(k+1)} + I)(\partial \odot I^\otimes(k-1)) \]
while for any polynomials \( \phi, \psi \in \mathbb{C}[X] \),
\[ \langle \mathcal{M}^{(k)}\phi^{(k-1)}\phi, \psi^{(k-1)}\psi \rangle_{\mathcal{M}^{(a)}} = k\langle \phi^{(k)}\phi, \psi^{(k)}\psi \rangle_{\mathcal{M}^{(a+k+1)}}. \]
In particular, \( \mathcal{M}^{(k)}\phi^{(k-1)}\phi = 0 \) if and only if \( \phi \) is a polynomial of degree \( k-1 \).

In addition to verifying these properties, the operator \( \mathcal{M}^{(k)} \) is essentially self-adjoint and non-negative on \( L^2(\alpha^{\otimes(k)}) \), and for any \( a > k-1 \),
\[ (\mathcal{M}^{(k)} + aI)^{-1}\phi^{(k-1)} = \phi^{(k-1)}(\mathcal{M} + (1 + a - k)I)^{-1}. \]

**Proof.** The proof of (5.3) is done by induction. For \( k = 1 \), we need to prove that
\[ \partial\mathcal{M} = (\mathcal{M}^{(2)} + I)\partial, \]
and this is going to be verified on polynomials \( \psi_1 \). Since
\[ \partial\psi_1 = \sum_{a+b=l-1} \psi_a \odot \psi_b, \]
and since $\mathcal{M}$ is the counting number operator,
\[
\partial (\mathcal{M} \psi_l) = l \sum_{a+b=l-1} \psi_a \otimes \psi_b, \quad \text{while} \quad \mathcal{M}^{(2)} \partial \psi_l = (l-1) \sum_{a+b=l-1} \psi_a \otimes \psi_b,
\]
which is exactly the case $k = 1$.

Now assume $k \geq 2$ and use (5.2) to write
\[
(5.7) \quad (\partial \otimes I^{(k-1)}) \mathcal{M}^{(k)} = (\partial \otimes I^{(k-1)})(\mathcal{M}^{(k-1)} \otimes I) + (\partial \otimes I^{(k-1)})(I^{(k-1)} \otimes \mathcal{M}).
\]
Now, the induction step justifies that
\[
(\partial \otimes I^{(k-1)})(\mathcal{M}^{(k-1)} \otimes I) = ((\partial \otimes I^{(k-2)})\mathcal{M}^{(k-1)}) \otimes I = ((\mathcal{M}^{(k)} + I)(\partial \otimes I^{(k-2)})) \otimes I
\]
\[
= (\mathcal{M}^{(k)}) \otimes I)(\partial \otimes I^{(k-1)}) + \partial \otimes I^{(k-1)},
\]
while the last term of (5.7) is
\[
(\partial \otimes I^{(k-1)})(I^{(k-1)} \otimes \mathcal{M}) = (I^{(k-1)} \otimes \mathcal{M})(\partial \otimes I^{(k-1)}),
\]
completing the induction step for (5.3).

For $k = 1$, the equality (5.4) readily follows from (5.6) and since $\mathcal{M}$ is the counting number operator. Alternatively, this is just the same as (3.11).

For $k \geq 2$, using (5.3), one easily shows that
\[
(5.8) \quad \mathcal{M}^{(k)} \partial^{(k-1)} = \partial^{(k-1)} (\mathcal{M} - k + 1).
\]
The rest of the proof reduces to showing that
\[
\langle \partial^{(k-1)} (\mathcal{M} - (k - 1)I) \phi, \partial^{(k-1)} \psi \rangle_{\alpha^{(k)}} = k \langle \partial^{(k)} \phi, \partial^{(k)} \psi \rangle_{\alpha^{(k+1)}}.
\]
To do so, it is enough to take $\phi = \psi_l$ and $\psi = \psi_{l'}$ for some $l, l' \geq k - 1$, i.e., to show that
\[
(l - k + 1) \langle \partial^{(k-1)} \psi_l, \partial^{(k-1)} \psi_{l'} \rangle_{\alpha^{(k)}} = k \langle \partial^{(k)} \psi_l, \partial^{(k)} \psi_{l'} \rangle_{\alpha^{(k+1)}}.
\]
Now, an elementary calculation based on (3.10) reveals that
\[
(5.9) \quad \partial^{(k)} \psi_l = \sum_{a_1+a_2+\cdots+a_{k+1}=l-k} \psi_{a_1} \otimes \psi_{a_2} \otimes \cdots \otimes \psi_{a_{k+1}},
\]
where the summation is over all possible writings of $l-k = a_1 + a_2 + \cdots + a_{k+1}$, with all $a_1, a_2, \ldots, a_{k+1} \geq 0$. It remains to show that
\[
(l - k + 1) N_{k,l} N_{k,l+1} = k N_{k,l-k} \delta_{l,l'},
\]
where $N_{k,l}$ is the number of writings of $l = a_1 + a_2 + \cdots + a_{k+1}$, with all $a_1, a_2, \ldots, a_{k+1} \geq 0$. This follows from the fact that
\[
N_{k,l} = \binom{l+k}{k},
\]
which is well known and easy to verify. The last part, namely (5.5), is obtained in a straightforward fashion from (5.8) combined with the fact that $\mathcal{M}$ is a self-adjoint non-negative operator.

Using the above proposition, a refinement of the Poincaré inequality for the semicircular law can now be stated formally.

**Theorem 8.** For $k \geq 1$, and any smooth function $\phi$ on $[-2, 2]$,
\[
(5.11) \quad \frac{1}{2} \langle \mathcal{N} \phi, \phi \rangle = \|\phi'\|_{\alpha}^2 - \frac{1}{2} \|\partial \phi'\|_{\alpha^2}^2 + \frac{1}{3} \|\partial^{(2)} \phi'\|_{\alpha^3}^2 + \cdots + \frac{(-1)^{k-1}}{k!} \|\partial^{(k-1)} \phi'\|_{\alpha^k}^2
\]
\[
+ \frac{(-1)^k}{k!} (\mathcal{M}^{(k)} (\mathcal{M}^{(k)} + kI)^{-1} \partial^{(k-1)} \phi', \partial^{(k-1)} \phi')_{\alpha^{k}}.
\]
Moreover, $\phi$ is a polynomial of degree $k$ if and only if
\[
\frac{1}{2} \langle \mathcal{N} \phi, \phi \rangle = \|\phi'\|_{\alpha}^2 - \frac{1}{2} \|\partial \phi'\|_{\alpha^2}^2 + \frac{1}{3} \|\partial^{(2)} \phi'\|_{\alpha^3}^2 + \cdots + \frac{(-1)^{k-1}}{k!} \|\partial^{(k-1)} \phi'\|_{\alpha^k}^2.
\]
Proof. We prove (5.11) by induction starting with (3.9). For simplicity of notation, denote \( \phi' \) by \( \psi \) and note that
\[
\langle (I + \mathcal{M})^{-1}\psi, \psi \rangle_\alpha = \langle \psi, \psi \rangle_\alpha - \langle \mathcal{M}(I + I)^{-1}\psi, \psi \rangle_\alpha.
\]
Now, using (5.3), \( \partial(M + I) = (M^{(2)} + 2I)\partial \), which then leads to
\[
\langle \mathcal{M}(M + I)^{-1}\psi, \psi \rangle_\alpha = \langle \partial(M + I)^{-1}\psi, \partial\psi \rangle_{\alpha \otimes 2} = \langle (M^{(2)} + 2I)^{-1}\partial\psi, \partial\psi \rangle_{\alpha \otimes 2}.
\]
Moreover, by repeating this argument, the last term above becomes:
\[
\langle (M^{(2)} + 2I)^{-1}\partial\psi, \partial\psi \rangle_{\alpha \otimes 2} = \frac{1}{2} \langle \partial\psi, \partial\psi \rangle_{\alpha \otimes 2} - \frac{1}{2} \langle \mathcal{M}^{(2)}(M^{(2)} + 2I)^{-1}\partial\psi, \partial\psi \rangle_{\alpha \otimes 2}.
\]
Now that we saw the mechanics on how to proceed, we can formally prove the inductive step by showing that the formula (5.11) for \( k \geq 1 \), implies the case \( k + 1 \). To do so, using (5.8), (5.4) and again (5.8), allow to first justify that
\[
\langle \mathcal{M}^{(k)}(\mathcal{M}^{(k)} + kI)^{-1}\partial^{(k-1)}\psi, \partial^{(k-1)}\psi \rangle_{\alpha \otimes k} = \langle \mathcal{M}^{(k)}(M + I)^{-1}\psi, \partial^{(k-1)}\psi \rangle_{\alpha \otimes k}
\]
\[
= k \langle \partial^{(k)}(M + I)^{-1}\psi, \partial^{(k)}\psi \rangle_{\alpha \otimes k} = k \langle \mathcal{M}^{(k+1)}(k + 1I)^{-1}\partial^{(k)}\psi, \partial^{(k)}\psi \rangle_{\alpha \otimes k}
\]
\[
= \frac{k}{k + 1} \langle \partial^{(k)}\psi, \partial^{(k)}\psi \rangle_{\alpha \otimes k} - \frac{k}{k + 1} \langle \mathcal{M}^{(k)}(M^{(k+1)} + (k + 1I)^{-1}\partial^{(k)}\psi, \partial^{(k)}\psi \rangle_{\alpha \otimes k}.
\]
Therefore,
\[
\frac{1}{k} \langle \mathcal{M}^{(k)}(\mathcal{M}^{(k)} + kI)^{-1}\partial^{(k-1)}\psi, \partial^{(k-1)}\psi \rangle_{\alpha \otimes k}
\]
\[
= \frac{1}{k + 1} \langle \partial^{(k)}\psi \rangle_{\alpha \otimes k} - \frac{1}{k + 1} \langle \mathcal{M}^{(k)}(M^{(k+1)} + (k + 1I)^{-1}\partial^{(k)}\psi, \partial^{(k)}\psi \rangle_{\alpha \otimes k},
\]
proving the main induction step.

It is also clear that the last term in (5.11) is zero since for a polynomial \( \phi \) of degree \( k \), \( \partial^{(k-1)}\phi' \) is constant, and \( \mathcal{M}^{(k)} \) vanishes on constants. \( \square \)

As a consequence of Theorem 8, we also have the following result.

Corollary 9. For any \( k \geq 1 \) and any smooth function \( \phi \) on \([-2, 2]\),
\[
\sum_{l=1}^{2k} \frac{(-1)^{l-1}}{l} \langle \partial^{(l-1)}\phi', \partial^{(l-1)}\phi \rangle_{\alpha \otimes l} \leq \frac{1}{2} \langle \mathcal{N} \phi, \phi \rangle \leq \sum_{l=1}^{2k-1} \frac{(-1)^{l-1}}{l} \langle \partial^{(l-1)}\phi', \partial^{(l-1)}\phi \rangle_{\alpha \otimes l}.
\]

Above, equality is attained on the left-hand side for any polynomial \( \phi \) of degree \( 2k \), while on the right-hand side it is attained for any polynomial \( \phi \) of degree \( 2k - 1 \).

Also,
\[
\frac{1}{2} \langle \mathcal{N} \phi, \phi \rangle = \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l} \langle \partial^{(l-1)}\phi', \partial^{(l-1)}\phi \rangle_{\alpha \otimes l},
\]
provided the series converges (for instance, this is always the case if \( \phi \) is a polynomial).

6. Heuristics

The main purpose of the present section is to give heuristic arguments justifying the presence of the operators involved in the proof of the free Brascamp-Lieb inequality.

We start with the classical case. On \( \mathbb{R}^d \), consider a probability measure of the form \( \mu(dx) = e^{-V(x)}dx \), where \( V \) is a smooth function on \( \mathbb{R}^d \). The measure \( \mu_V \) is the invariant measure of the operator \( L = -\Delta + \nabla V \cdot \nabla \) which in turn is a generalization of the Ornstein-Uhlenbeck operator. A simple integration by parts leads to
\[
\mathbb{E}_\mu[(L\phi)g] = \mathbb{E}_\mu[(\nabla \phi, \nabla g)].
\]
One of the classical approaches to the Brascamp-Lieb inequality is due to Helffer [9] and we quickly review it here. First, from (6.1) with $f$ replaced by $Lf$,

$$\langle L\phi, L\phi \rangle_{L^2(\mu)} = \langle \nabla L\phi, \nabla \phi \rangle_{L^2(\mu)}. \tag{6.2}$$

Second, with the natural (component-wise) extension of $L$ to several dimensions,

$$\nabla L = L\nabla + \text{Hess}V \nabla = (L + \text{Hess}V) \nabla. \tag{6.3}$$

In particular, if $K := L + \text{Hess}V$, then

$$\nabla L = K \nabla, \tag{6.4}$$

and so, using inverses whenever these are defined,

$$K^{-1} \nabla = \nabla L^{-1}. \tag{6.5}$$

If $f$ is a smooth function such that \( f d\mu = 0 \), (6.2) with $\phi = L^{-1} f$ and (6.4) lead to

$$\text{Var}_\mu(f) = \langle K^{-1} \nabla f, \nabla f \rangle_{L^2(\mu)}. \tag{6.6}$$

Since, $L$ is a non-negative operator and since $\text{Hess}V$ is positive definite, it follows that $\text{Hess}V \leq K$ and so $K^{-1} \leq (\text{Hess}V)^{-1}$, from which the Brascamp-Lieb inequality

$$\text{Var}_\mu(f) \leq \langle (\text{Hess}V)^{-1} \nabla f, \nabla f \rangle_{L^2(\mu)} \tag{6.7}$$

follows naturally.

We now wish to apply these types of arguments to the case of $\mu = P^V_n$. To do so, let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be non-constant, compactly supported and smooth, and let $f(X) = \text{Tr}\phi(X)$, for any $X \in \mathcal{H}_n$. For a better understanding, we actually back up a step and start with

$$\text{Var}_{P^V_n}(f) = \langle \nabla ((L^0_V)^{-1}) \text{Tr}\phi), \nabla \text{Tr}\phi \rangle_{P^V_n} = \langle ((K^0_V)^{-1}) \nabla \text{Tr}\phi, \nabla \text{Tr}\phi \rangle_{P^V_n}. \tag{6.8}$$

We next wish to understand what happens if we let $n$ tend to infinity in (6.5). The limit of the left-hand side is determined by the fluctuations of random matrices, and (say, provided that $V$ is a polynomial of even degree with equilibrium measure $\mu_V$ having support $[-2, 2]$), it is given by:

$$\lim_{n \rightarrow \infty} \text{Var}_{P^V_n}(f) = \int_{-2}^{2} \int_{-2}^{2} \left( \frac{\phi(x) - \phi(y)}{x - y} \right)^2 \frac{(4 - xy)}{4\pi^2(4 - x^2)(4 - y^2)} dx dy. \tag{6.9}$$

For the right-hand side of (6.7) observe first that $\nabla \text{Tr}\phi(X) = \phi'(X)$. Now,

$$-L^0_V \text{Tr}\phi(X) = \Delta \text{Tr}\phi(X) - n \text{Tr}\text{V}(X) \cdot \nabla \text{Tr}\phi(X) = \Delta \text{Tr}\phi(X) - n \text{Tr}(V'(X)\phi'(X)), \tag{6.10}$$

hence we need to identify the limit of the operators $K^0_V$.

Hence, the Laplacian on matrices needs to be computed, and we do so for monomials of the form $\phi(X) = X^a$ and then extend the result by linearity. By definition,

$$\Delta \Phi(X) = \sum_\gamma \frac{d^2}{dh^2} \Phi(X + hE_\gamma), \tag{6.11}$$

where $E_\gamma$ is an orthonormal basis of $\mathcal{H}_n$. In fact, a basis consists of the matrices $E_{jj}$ which have 1 on the $j$th position on the diagonal and 0 elsewhere, $E_{jk}$ with $j < k$ with $1/\sqrt{2}$ for the $(j, k)$th and $(k, j)$th entries and 0 otherwise, and $\tilde{E}_{jk}$ with $i/\sqrt{2}$ for the $(j, k)$th entry and $-i/\sqrt{2}$ for the $(k, j)$th entry and 0 otherwise. Using this basis,

$$\frac{1}{2}(\Delta \text{Tr}\phi)(X) = \sum_{1 \leq j \leq k \leq n} \sum_{1 \leq l_1 + l_2 + l_3 = a - 2} \text{Tr}(X^{l_1}E_{jk}X^{l_2}E_{jk}X^{l_3}) + \sum_{1 \leq j < k \leq n} \sum_{1 \leq l_1 + l_2 + l_3 = a - 2} \text{Tr}(X^{l_1} \tilde{E}_{jk}X^{l_2} \tilde{E}_{jk}X^{l_3}) \tag{6.12}$$

$$= \sum_{1 \leq j \leq k \leq n} \sum_{1 \leq l_1 + l_2 + l_3 = a - 2} \text{Tr}(X^{l_1}E_{jk}X^{l_2}E_{jk}) + \sum_{1 \leq j < k \leq n} \sum_{1 \leq l_1 + l_2 + l_3 = a - 2} \text{Tr}(X^{l_1} \tilde{E}_{jk}X^{l_2} \tilde{E}_{jk}) \tag{6.13}$$

$$= \sum_{l=0}^{a-2} (l + 1) \sum_{1 \leq j \leq k \leq n} \text{Tr}(X^lE_{jk}X^{a-2-l}E_{jk}) + \sum_{l=0}^{a-2} (l + 1) \sum_{1 \leq j < k \leq n} \text{Tr}(X^l \tilde{E}_{jk}X^{a-2-l} \tilde{E}_{jk}). \tag{6.14}$$
Let $F_{jk}$ be the matrix with 1 on the $(j,k)$th entry and 0 otherwise. Then $E_{jk} = (F_{jk} + F_{kj})/\sqrt{2}$ and $	ilde{E}_{jk} = i(F_{jk} - F_{kj})/\sqrt{2}$, for $j < k$. A small computation reveals that for two matrices $A$ and $B$,

\[(6.8)\]

\[
\sum_{1\leq j \leq k \leq n} \text{Tr}(AE_{jk}BE_{jk}) + \sum_{1 < j < k \leq n} \text{Tr}(AE_{jk}B\tilde{E}_{jk}) = \sum_{j,k=1}^{n} \text{Tr}(AF_{jk}BF_{jk})
\]

\[
= \sum_{j,k=1}^{n} \sum_{u_1,u_2,u_3,u_4=1} A_{u_1u_2}(F_{jk})_{u_2u_3}B_{u_3u_4}(F_{kj})_{u_4u_1}
\]

\[
= \sum_{j,k=1}^{n} A_{jj}B_{kk} = \text{Tr}(A)\text{Tr}(B).
\]

Therefore,

\[(\Delta \text{Tr}\phi)(X) = 2 \sum_{l=0}^{a-2} (l + 1)\text{Tr}(X^l)\text{Tr}(X^{a-2-l}).\]

Summing up our findings, for $\phi(x) = x^a$, $a \geq 2$,

\[-L^a_V\text{Tr}\phi(X) = 2 \sum_{l=0}^{a-2} (l + 1)\text{Tr}(X^l)\text{Tr}(X^{a-2-l}) - n a \text{Tr}(V'(X)X^{a-1})
\]

\[= \sum_{l=0}^{a-2} (l + 1)\text{Tr}(X^l)\text{Tr}(X^{a-2-l}) + \sum_{l=0}^{a-2} (a - 1 - l)\text{Tr}(X^l)\text{Tr}(X^{a-2-l}) - n a \text{Tr}(V'(X)X^{a-1})
\]

\[= \text{Tr} \otimes \text{Tr}(\partial\phi'(X)) - n \text{Tr}(V'(X)\phi'(X)),
\]

which, by linearity, is then true for all polynomials $\phi$. Taking the gradient, then gives

\[-\nabla(L^a_V\text{Tr}\phi)(X) = \nabla \text{Tr} \otimes \text{Tr}(\partial\phi'(X)) - n(V'\phi')' = (D \otimes \text{Tr})\partial\phi'(X) + (\text{Tr} \otimes D)\partial\phi'(X) - n(V'\phi')'(X).
\]

In particular, using the operator $K^a_V$ which satisfies $\nabla(L^a_V\text{Tr}\phi)(X) = K^a_V \nabla \text{Tr}\phi(X)$, we now obtain

\[\frac{1}{n} K^a_V \nabla \text{Tr}\phi(X) = -\frac{1}{n} (D \otimes \text{Tr} + \text{Tr} \otimes D)\partial\phi'(X) + (V'\phi')'(X),
\]

and therefore, since $\partial\phi$ is a symmetric tensor,

\[\frac{1}{n} K^a_V \phi'(X) = -\frac{2}{n} D(I \otimes \text{Tr})\partial\phi'(X) + (V'\phi')'(X).
\]

Finally, since $\text{Tr}\psi(X)/n$ converges to $\mu_V(\psi)$, heuristically $K^a_V \phi'/n$ converges to

\[K \phi' = -2D(I \otimes \mu_V)\partial\phi' + (V'\phi')'
\]

and replacing $\phi'$ by $\phi$, motivates the following definition:

\[(6.9)\]

\[K_V \phi := -D[2(I \otimes \mu_V)\partial\phi - V'\phi] = -D[2(I \otimes \mu_V)\partial\phi] + V'\phi' + V''\phi.
\]

It is interesting to remark that using, for instance, [26, Corollary 4.4 and Proposition 3.5] one can justify the following equality

\[-D[2(I \otimes \mu_V)\partial\phi] + V'\phi' = \partial_V\phi,
\]

where $\partial_V^e$ is the adjoint of the operator $\partial$, i.e., $\langle \partial_V^e(\phi \otimes \psi), \eta \rangle_{\mu_V} = \langle \phi \otimes \psi, \partial\eta \rangle_{\mu_V \otimes \mu_V}$ and the inner product generated by a state $\mu$ on polynomials $C(X)$ is $\langle \phi, \psi \rangle_{\mu} = \mu(\phi \psi)$, for any polynomials $\phi, \psi$ with the convention that $\tilde{\psi} = \sum_i a_i X^i$ if $\psi = \sum_i a_i X^i$. The extension to tensor products is done via the usual procedure:

\[\langle \phi_1 \otimes \phi_2, \psi_1 \otimes \psi_2 \rangle_{\mu \otimes \mu} = \mu(\phi_1 \tilde{\psi}_1)\mu(\phi_2 \tilde{\psi}_2),
\]

and the representation therefore obtained has the flavor of a generalization of the non-commutative Ornstein-Uhlenbeck operator.
Finally, heuristically, taking the limit in (6.7), it follows that
\[(6.10)\]
\[
\frac{1}{2} \langle Nf, f \rangle = \langle K_V^{-1} f', f' \rangle_{L^2(\mu_V)}. \]

7. The Free Brascamp-Lieb Inequality

From the heuristics of the previous section, let
\[K_V \phi := D \left[ -2(I \otimes \mu_V)\partial \phi + V'\phi \right],\]
and set
\[M_V \phi := -2D(I \otimes \mu_V)\partial \phi + V'\phi'.\]
Again, if \(V(x) = x^2/2\), then \(M_V\) is the counting number operator for the Chebyshev polynomials of the second kind. It is clear that,
\[K_V \phi = M_V \phi + V''\phi,\]
and it is trivial that, for any function \(f\), the multiplication operator
\[A_f \phi = f \phi,\]
extends to a self-adjoint operator on \(L^2(\mu_V)\). As shown next, the operator \(M_V\) is a non-negative operator on \(L^2(\mu_V)\).

**Proposition 10.** Assume \(\mu_V\) has support \([-2, 2]\). The operator \(M_V\) is given on \(C^2\) functions by
\[(7.1)\]
\[
(M_V \phi)(x) = 2p.v. \int \frac{\phi(x) - \phi(y)}{(x-y)^2} \mu_V(dy) := 2 \lim_{\epsilon \to 0} \int_{|x-y| \geq \epsilon} \frac{\phi(x) - \phi(y)}{(x-y)^2} \mu_V(dy),
\]
and for \(C^1\) functions, \(\phi, \psi\) on \([-2, 2]\),
\[(7.2)\]
\[
(M_V \phi, \psi)_V = \int \frac{(\phi(x) - \phi(y))(\psi(x) - \psi(y))}{(x-y)^2} \mu_V(dx) \mu_V(dy).
\]
Moreover, \(M_V\) can be extended to an unbounded non-negative essentially self-adjoint operator on \(L^2(\mu_V)\) whose domain includes the set of \(C^1\) functions on \([-2, 2]\).

In addition, if \(V'' \geq 0\), and \(V\) is \(C^2\) on \([-2, 2]\), then the operator \(K_V\) has a self-adjoint extension such that for some \(\delta > 0, K_V \geq \delta I\). In particular, \(K_V\) is an essentially self-adjoint operator on \(L^2(\mu_V)\) which is invertible with a bounded inverse on \(L^2(\mu_V)\).

**Proof.** The statements in the first part of this proposition, namely, (7.1) and (7.2) follow from arguments similar to those involved in the proof of (3.5) from Proposition 2. Start with a \(C^2\) function \(\phi\) on \([-2, 2]\) and notice that
\[
\partial \phi = \frac{\phi(x) - \phi(y)}{x-y},
\]
and thus
\[\[I \otimes \mu_V\](\partial \phi)(x) = \int \frac{\phi(x) - \phi(y)}{x-y} \mu_V(dy).\]
Next, \(\phi\) is a \(C^2\) function which when combined with the variational characterization of the equilibrium measure from (2.4), leads to
\[(7.3)\]
\[
-2 \frac{d}{dx} \int \frac{\phi(x) - \phi(y)}{x-y} \mu_V(dy) = 2 \int \frac{\phi(x) - \phi(y) - \phi'(x)(x-y)}{(x-y)^2} \mu_V(dy)
= 2p.v. \int \frac{\phi(x) - \phi(y)}{(x-y)^2} \mu_V(dy) - \phi'(x)p.v. \int \frac{2}{x-y} \mu_V(dy)
= 2p.v. \int \frac{\phi(x) - \phi(y)}{(x-y)^2} \mu_V(dy) - V'(x)\phi'(x),
\]

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giving (7.1). In turn, (7.2) follows easily for $C^2$ functions exactly as in (3.8), replacing $\alpha$ by $\mu_V$. Next, the extension to a self-adjoint operator is deduced from the fact that the Dirichlet form
\[
D(\phi, \phi) = \int \left( \frac{\phi(x) - \phi(y)}{x - y} \right)^2 \mu_V(dx)\mu_V(dy)
\]
is positive and closable, therefore $\mathcal{M}_V$, its generator, according to [6] must be essentially self-adjoint and non-negative. This last fact and standard approximations of $C^1$ functions with $C^2$ functions proves (7.2) for $C^1$ functions.

Since $V$ is a $C^2$ function, the multiplicative operator $\mathcal{A}_V$ is a bounded operator on $L^2(\mu_V)$ and this implies, for instance, that $K_V$ has a non-negative extension with the same domain as $\mathcal{M}_V$. In addition, we claim that $V'' > 0$ on a set of positive measure (with respect to $\mu_V$). Indeed if otherwise, then $V''$ would be identically 0 on $[-2, 2]$ which means, for example, that $V'$ is constant on $[-2, 2]$. Since the support of $\mu_V$ is $[-2, 2]$, it follows that (e.g., see [23, Theorem 1.11 Chapter IV] or [18, Equation (4.3)])

\[
\int V'(x)\beta(dx) = 0 \text{ and } \int xV'(x)\beta(dx) = 2.
\]

These equalities cannot be satisfied if $V'$ is constant on $[-2, 2]$. Therefore, there must be a subset $A \subset [-2, 2]$ with $\mu_V(A) > 0$ and a positive $\epsilon > 0$ such that $V''(x) > \epsilon$, for all $x \in A$. On the other hand, for $x, y \in [-2, 2], \frac{1}{(x-y)^2} \geq \frac{1}{16}$, and then
\[
\langle K_V \phi, \phi \rangle \geq \frac{1}{16} \int (\phi(x) - \phi(y))^2 \mu_V(dx)\mu_V(dy) + \epsilon \int A \phi^2d\mu_V
\]
\[
= \epsilon \int A \phi^2d\mu_V + \frac{1}{8} \int \phi^2d\mu_V - \frac{1}{8} \left( \int \phi d\mu_V \right)^2.
\]

Next, we wish to show that there exists $\delta > 0$ such that
\[
\epsilon \int A \phi^2d\mu_V + \frac{1}{8} \int \phi^2d\mu_V - \frac{1}{8} \left( \int \phi d\mu_V \right)^2 \geq \delta \int \phi^2d\mu_V,
\]
or equivalently,
\[
(1 + 8\epsilon - 8\delta) \int A \phi^2d\mu_V + (1 - 8\delta) \int A^c \phi^2d\mu_V \geq \left( \int \phi d\mu_V \right)^2.
\]

To show this is possible, notice that if $A$ has full measure, then we can take $\delta = \epsilon$ and we are done. If not, then $\mu_V(A) > 0$ and $\mu_V(A^c) > 0$. By the Cauchy-Schwarz inequality,
\[
(1 + 8\epsilon - 8\delta) \int A \phi^2d\mu_V + (1 - 8\delta) \int A^c \phi^2d\mu_V \geq \frac{1 + 8\epsilon - 8\delta}{\mu_V(A)} \left( \int A \phi d\mu_V \right)^2 + \frac{1 - 8\delta}{\mu_V(A^c)} \left( \int A^c \phi d\mu_V \right)^2,
\]
and then $(a^2 + b^2)(c^2 + d^2) \geq (ac + bd)^2$, with $a = \sqrt{\frac{1 + 8\epsilon - 8\delta}{\mu_V(A)}} \int A \phi d\mu_V, b = \sqrt{\frac{1 - 8\delta}{\mu_V(A^c)}} \int A^c \phi d\mu_V, c = \sqrt{\frac{\mu_V(A)}{1 + 8\epsilon - 8\delta}}, d = \sqrt{\frac{\mu_V(A^c)}{1 - 8\delta}}$ yields
\[
(1 + 8\epsilon - 8\delta) \int A \phi^2d\mu_V + (1 - 8\delta) \int A^c \phi^2d\mu_V \geq \frac{1}{\mu_V(A) + \mu_V(A^c)} \left( \int \phi d\mu_V \right)^2.
\]

Thus, we just need to choose $\delta > 0$ such that
\[
\frac{\mu_V(A)}{1 + 8\epsilon - 8\delta} + \frac{\mu_V(A^c)}{1 - 8\delta} < 1,
\]
which is certainly possible since the above quantity is continuous in $\delta$, and since for $\delta = 0$,
\[
\frac{\mu_V(A)}{1 + 8\epsilon} + \mu_V(A^c) < \mu_V(A) + \mu_V(A^c) = 1.
\]

The rest now follows.
Theorem 11. Let the support of the equilibrium measure $\mu_V$ be $[-2, 2]$ and let $V$ be $C^4$ with $V'' \geq 0$ on $[-2, 2]$. Then, for any $C^1$ function $\phi$ on $[-2, 2]$,

\[ \langle N\phi, \phi \rangle = 2 \langle K^{-1}_V \phi', \phi' \rangle_{L^2(\mu_V)}. \]

Moreover, the following version of Brascamp-Lieb inequality holds true: For any $C^1$ function $\phi$ on $[-2, 2]$,

\[ \int_{-2}^2 \int_{-2}^2 \left( \frac{\phi(x) - \phi(y)}{x - y} \right)^2 \frac{(4 - xy)}{4\pi^2 \sqrt{(4 - x^2)(4 - y^2)}} \, dx \, dy \leq \int \frac{\phi'^2}{\nu'} \, d\mu_V, \]

with equality for $\phi(x) = V'(x) + C, C \in \mathbb{R}$.

We use the $C^4$ regularity of $V$ at a single place in the proof, and so most likely this assumption can be reduced to $C^3$ regularity, but we are not pursuing this here.

Proof. We want to show that the right-hand side of (7.5) is in fact independent of $V$, so it suffices to check it for a potential which is smooth, convex and whose equilibrium measure is also supported on $[-2, 2]$. That candidate is precisely $V'(x) = x^2/2$ and then the rest of the statement is just Theorem 3.

Let the operator $F_V$ be defined as

\[ (F_V \phi)(x) = 2 \int \log |x - y| \phi(y) \mu_V(dy). \]

From (3.4), we know that $d\mu_V = (1 - \frac{1}{2} NV) d\beta$, hence for any $C^2$ function $\phi$ on $[-2, 2]$,

\[ (F_V \phi)(x) = 2 \int \log |x - y| \phi(y) (1 - \frac{1}{2} NV(y)) \beta(dy) = -2 \mathcal{E} \left( \left( 1 - \frac{1}{2} NV \right) \phi \right)(x) = -2 \left( \mathcal{E} A(1 - \frac{1}{2} NV) \phi \right)(x). \]

Now, taking the derivative yields,

\[ (D F_V \phi)(x) = 2p.v. \int \frac{\phi(y)}{x - y} \mu_V(dy) = -2 \int \frac{\phi(x) - \phi(y)}{x - y} \mu_V(dy) + 2p.v. \int \frac{\phi(x)}{x - y} \mu_V(dy) \]

\[ = -2 \int \frac{\phi(x) - \phi(y)}{x - y} \mu_V(dy) + \phi(x) V'(x). \]

Taking another derivative and using (7.3) give,

\[ (D^2 F_V \phi)(x) = (M_V \phi)(x) - V'(x) \phi'(x) + (\phi V'')(x) \]

\[ = (M_V \phi)(x) + V''(x) \phi(x) = (K_V \phi)(x). \]

Thus, $K_V = -2 D^2 \mathcal{E} A(1 - \frac{1}{2} NV)'$ on $C^2([-2, 2])$. To finish the proof we want to take inverses. To do so requires to properly define the inverses and to this end, we look at the following diagram:

\[ \begin{array}{c}
C^2([-2, 2]) \xrightarrow{A(1 - \frac{1}{2} NV)} C^2([-2, 2]) \xrightarrow{\mathcal{E}} C^2([-2, 2]) \cap L^2_0(\beta) \xrightarrow{D} C^1([-2, 2]) \xrightarrow{D} C([-2, 2]).
\end{array} \]

We need to justify that the composition is well defined here. In the first place, and for instance, from (7.4), and the very definition of the operator $N'$, it follows that

\[ 1 - \frac{1}{2} (NV)(x) = \frac{4 - x^2}{2} \int \frac{V'(x) - V'(y)}{x - y} \beta(dy). \]

This proves two things. In the first place, since $V$ is convex, we learn that $u(x) := \int \frac{V'(x) - V'(y)}{x - y} \beta(dy)$, is strictly positive and also $C^2$ on $[-2, 2]$ (this is the only place where the $C^4$ condition on $V$ is used). Thus, the first operator is well defined. The second operator is also well defined by Proposition 2, while the other operators are self-explanatory. The inverses are written as follows:

\[ \begin{array}{c}
C([-2, 2]) \xrightarrow{I} C^1([-2, 2]) \xrightarrow{I_0} C^2([-2, 2]) \cap L^2_0(\beta) \xrightarrow{N} C([-2, 2]) \xrightarrow{A(1 - \frac{1}{2} NV)} L^2(\nu_V),
\end{array} \]

where

\[ (I\psi)(x) = \int_{-2}^x \psi(y) \, dy \quad \text{and} \quad (I_0 \psi)(x) = (I\psi)(x) - \int I\psi \, d\beta, \]
with $\nu_V(dx) = (4 - x^2)\mu_V(dx)$. Clearly, by definition, the operator $N$ maps $C^2([-2, 2])$ into $C([-2, 2])$, while $A_{1/(1 - \frac{1}{2}N)}$ sends $C([-2, 2])$ into $L^2(\nu_V)$, because of (7.8). The natural choice here would be to have the operator $A_{1/(1 - \frac{1}{2}N)}$ map $C([-2, 2])$ into $L^2(\mu_V)$, but this works only for functions which vanish (like a power greater than $1/2$ of $x$) at the endpoints of $[-2, 2]$. Therefore, instead of restricting the domain of definitions of all the other operators to accommodate this fact, we change the range where $A_{1/(1 - \frac{1}{2}N)}$ takes values. On the other hand, the operator $K_V^{-1}$ is a bounded operator from $L^2(\mu_V)$ into itself by Proposition 10, hence it can be taken as a bounded operator from $L^2(\mu_V)$ into $L^2(\nu_V)$.

Thus, we can now argue that on the set of continuous functions on $[-2, 2]$, $K_V^{-1} = -\frac{1}{2}A_{1/(1 - \frac{1}{2}N)} NI_0I$. This, combined with \[ \langle \phi, \psi \rangle_{\mu_V} = \langle A_{1/(1 - \frac{1}{2}N)}\phi, \psi \rangle, \]
now results with
\[ \langle K_V^{-1}\phi', \phi' \rangle_{L^2(\mu_V)} = -\frac{1}{2} \langle A_{1 - \frac{1}{2}N}A_{1/(1 - \frac{1}{2}N)}NI_0I D\phi, D\phi \rangle = -\frac{1}{2} \langle NI_0I D\phi, D\phi \rangle, \]
which is independent of $V$! The rest of (7.5) follows as we pointed out by taking $V(x) = x^2/2$ and using Theorem (3).

To verify (7.6), notice that it suffices to show that for any $\phi \in L^2(\mu_V)$,
\[ \langle (M_V + A_{V''})^{-1}\phi, \phi \rangle_{\mu_V} \leq \left( \frac{1}{\sqrt{\mu_V}} \phi, \phi \right)_{\mu_V}. \]
Since $K_V$ is invertible, any $\phi \in L^2(\mu_V)$ can be written as $\phi = (M_V + V'')\psi$, for some $\psi \in L^2(\mu_V)$, so we need to check that
\[ \langle (M_V + V'')\psi, \psi \rangle_{\mu_V} \leq \langle (V'')^{-1}(M_V + V'')\psi, (M_V + V'')\psi \rangle_{\mu_V}. \]
If the right-hand side is infinite, there is nothing to prove. If it is finite, then write it as
\[ \langle (V'')^{-1}(M_V + V'')\psi, (M_V + V'')\psi \rangle_{\mu_V} = \langle (M_V + V'')\psi, \psi \rangle_{\mu_V} + \langle (V'')^{-1/2}M_V\psi, (V'')^{-1/2}M_V\psi \rangle_{\mu_V} + \langle M_V\psi, \psi \rangle_{\mu_V}, \]
from which (7.10) follows immediately. There is, however, a small detail we need to take care of, namely justifying that if the left-hand side of the above is finite, the equality above is well defined. This essentially boils down to showing that all terms on the right-hand side are finite. This is indeed so because the finiteness of the left-hand side is equivalent to $(V'')^{-1/2}(M_V + V'')\psi \in L^2(\mu_V)$ which, in particular, since $V''$ is continuous and $\psi \in L^2(\mu_V)$, is equivalent to $(V'')^{-1/2}M_V\psi \in L^2(\mu_V)$. This is sufficient to guarantee the validity of the equation above, ensuring in particular that the middle term on the right-hand side is finite.

For the case of equality, according to (3.3) we have to show that
\[ \langle NV', V' \rangle = 2 \int V''d\mu_V. \]
To do so, use [18, Eq (1.32)] which gives
\[ \langle N\phi, \psi \rangle + \langle N\psi, \phi \rangle = \left( \int \phi' d\beta \right) \left( \int x\psi'(x)\beta(dx) \right) + \left( \int x\phi'(dx)\beta(dx) \right) \left( \int \psi' d\beta \right). \]
Taking $\phi = V'$ and $\psi = V$, results with
\[ \langle NV', V' \rangle + \langle NV, V'' \rangle = \left( \int V''d\beta \right) \left( \int xV'(x)\beta(dx) \right) + \left( \int xV''(x)\beta(dx) \right) \left( \int V'd\beta \right). \]
On the other hand, since the equilibrium measure $\mu_V$ is supported on $[-2, 2]$, invoking the equation (7.4) yields the equality
\[ \langle NV', V' \rangle + \langle NV, V'' \rangle = 2 \int V''d\beta, \]
or written differently,
\[ \langle NV', V' \rangle = 2 \int V'' \left( 1 - \frac{1}{2}N \right) d\beta, \]
which combined with (3.4) is precisely the statement of (7.11).

The curious reader may wonder how the free Brascamp-Lieb looks like in the case where the support of the measure \( \mu_V \) is not \([-2, 2]\). Assuming that the equilibrium measure \( \mu_V \) has support \([a, b]\) and that \( V = C^4 \) on \([a, b]\) with \( V''(x) \geq 0 \) for \( x \in [a, b] \), the analog of inequality (7.6) takes the form

\[
\int_a^b \int_a^b \left( \frac{\phi(x) - \phi(y)}{x - y} \right)^2 \frac{-2ab + (a + b)(x + y) - 2xy}{8\pi^2 \sqrt{(x - a)(b - x)(y - a)(b - y)}} \, dx \, dy \leq \int \frac{\phi'^2}{V''} \, d\mu_V,
\]

for any smooth function \( \phi \) on \([a, b]\). The proof of this is simply done by a linear rescaling, namely reducing everything to the case \([a, b] = [-2, 2]\). More precisely, take \( \theta(x) = (b - a)x/4 + (a + b)/2 \) which maps \([-2, 2]\) into \([a, b]\). With this, (7.12) reduces to (7.6) for \( V(x) = V(\theta(x)) \), \( \phi(x) = \phi(\theta(x)) \). Notice here that the equilibrium measure \( \mu_V \) is determined by \( \mu_V(A) = \mu_V(\{x \in [a, b] : \theta^{-1}(x) \in A\}) \), for any \( A \subset [-2, 2] \).

Equality in (7.12) is attained for functions \( \phi \) of the form \( \phi(x) = c_1 + c_2 V'(x) \), for some constants \( c_1, c_2 \).

We close this section with an extension of [17, Eq. 10.16] which seems mysterious there, but is demystified by the free Brascamp-Lieb inequality discussed here. This inequality is related to the Wishart random matrix models and the main potential \( V \) is defined only on the positive axis. The interested reader can take a look at [17] for more details.

**Corollary 12.** Let \( Q : [0, \infty) \to \mathbb{R} \) be a continuous function and let \( V(x) = Q(x) - s \log(x) \), for \( s > 0 \), be such that \( \lim_{x \to \infty} (V(x) - 2 \log(x)) = \infty \). Let the support of \( \mu_V \) be \([a, b]\) and on \([a, b]\), let \( Q''(x) \geq 0 \). Then, for any smooth function \( \phi \) on \([a, b]\),

\[
\int_a^b \int_a^b \left( \frac{\phi(x) - \phi(y)}{x - y} \right)^2 \frac{-2ab + (a + b)(x + y) - 2xy}{8\pi^2 \sqrt{(x - a)(b - x)(y - a)(b - y)}} \, dx \, dy \leq \int \frac{x^2 \phi'^2}{s + x^2 Q''(x)} \, d\mu_V(dx),
\]

with equality for \( \phi(x) = c_1 (Q'(x) - s/x) + c_2 \), for some constants \( c_1 \) and \( c_2 \).

In particular we obtain [17, Eq. 10.16]

\[
s \int_a^b \int_a^b \left( \frac{\phi(x) - \phi(y)}{x - y} \right)^2 \frac{-2ab + (a + b)(x + y) - 2xy}{8\pi^2 \sqrt{(x - a)(b - x)(y - a)(b - y)}} \, dx \, dy \leq \int \frac{x^2 \phi'^2}{s + x^2} \, d\mu_V(dx).
\]

If \( Q(x) = r x + t \), for some constants \( r \) and \( t \), (7.14) is sharp with equality attained for \( \phi(x) = c_1 + c_2/x \).

**Proof.** From (7.12) and since \( V''(x) = Q''(x) + s/x^2 \geq s/x^2 \) one immediately deduces (7.13). Equality in (7.14) is attained if \( Q''(x) = 0 \) and \( \phi(x) = c_1 + c_2/x \).

\[\square\]

8. **Final Remarks**

It is clearly of interest to discuss a multidimensional version of the free Poincaré inequality and extensions, as for instance in the spirit of [22]. This requires more work and it will eventually be done in a separate publication.

There is a version of the free Poincaré inequality, introduced by Biane in [1], and in the one dimensional case it is different from the one presented here. It is interesting to point out that in several dimensions, the fluctuations of jointly independent random matrices, more precisely the limiting variance of the fluctuations, are the main ingredients for the formulation of the free Poincaré inequality. This already appears in the literature in two different forms. One is in [19], which describes it in terms of second order freeness. The other is investigated in [8], and the variance term is given in a form similar to the one presented in (7.5).

There are however some noticeable differences between the one dimensional case and the multidimensional case. If we interpret the variance term in the Poincaré inequality described in (7.5) as \( \langle K^{-1}_V \phi', \phi' \rangle_{\mu_V} \), then the key statement is that, as long as the support of the measure \( \mu_V \) is \([-2, 2]\), this variance does not depend on the other details of the potential \( V \). This is, in some sense, reminiscent of the universality of fluctuations in random matrix theory. As it turns out, this fact does not seem to take place in several dimensions which means that the approach for proving Theorem 11 is not going to work.
To fully understand the multidimensional case it seems desirable to unify the two points of view mentioned above, namely, the second order freeness and the analog of the variance through the inverse of a properly defined operator, at least for some natural cases of potentials.

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