Abstract. In an earlier paper, we have introduced the Tosha-degree of an edge in a graph without multiple edges and studied some properties. In this paper, we extend the definition of Tosha-degree of an edge in a graph in which multiple edges are allowed. Also, we introduce the concepts - zero edges in a graph, T-line graph of a multigraph, Tosha-adjacency matrix, Tosha-energy, edge-adjacency matrix and edge energy of a graph $G$ and obtain some results.

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1. Introduction

For standard terminology and notion in graphs and matrices, we refer the reader to the text-books of Harary [2] and Bapat [1]. The non-standard will be given in this paper as and when required.

Throughout this paper, $G = (V, E)$ denotes a graph (finite and undirected) and $V = V(G)$ and $E = E(G)$ denote vertex set and edge set of $G$, respectively. The degree of a vertex $v \in V(G)$, denoted by $d(v)$ or $d_G(v)$, is the number of edges incident on $v$, with self-loops counted twice. A vertex of degree one is a pendant vertex and an edge incident onto a pendant vertex is a pendant edge. A graph $G$ is $r$-regular if every vertex
of $G$ has degree $r$. The minimum degree $\delta(G)$ of a graph $G$ is the minimum degree among all the vertices of $G$ and the maximum degree $\Delta(G)$ of $G$ is the maximum degree among all the vertices of $G$.

Two non-distinct edges in a graph are adjacent if they are incident on a common vertex. We consider that an edge in a graph is not adjacent to itself. The letters $k, l, m, n$, and $r$ denote positive integers or zero.

The line graph $L(G)$ of a simple graph with at least one edge is the graph $(W, F)$, where there is a one-to-one correspondence $\phi$ from $E$ to $W$ such that there is an edge between $\phi(\alpha)$ and $\phi(\beta)$ if and only if the edges $\alpha$ and $\beta$ are adjacent. We identify the set $W$ by $E$.

The adjacency matrix of a graph $G$ with $n$ vertices is denoted by $A(G)$. If $A(G)$ is an $n \times n$ matrix and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A(G)$, the energy of $G$ is defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$  

In our earlier paper [4], we have introduced the Tosha-degree of an edge in a graph without multiple edges, Rajendra-Reddy index of a graph and Tosha-degree equivalence graph of a graph, and studied some properties. In this paper, we define Tosha-degree of an edge in a graph in which multiple edges are allowed. The aim of this paper is to introduce the concepts: zero edges in a graph, $T$-line graph of a multigraph, Tosha-adjacency matrix, Tosha-energy, edge-adjacency matrix and edge energy of a graph $G$ and obtain some results.

A signed graph is an ordered pair $\Sigma = (G, \sigma)$, where $G = (V, E)$ is a graph called the underlying graph of $\Sigma$ and $\sigma : E \to \{+, -\}$ is a function. A marking of $\Sigma$ is a function $\mu : V(G) \to \{+, -\}$.

In [4], we have also defined the Tosha-degree equivalence graph of a graph which is motivated us to extend this notion to signed graphs as follows: The Tosha-degree equivalence signed graph (See [3]) of a signed graph $\Sigma = (G, \sigma)$ as a signed graph $T(\Sigma) = (T(G), \sigma')$, where $T(G)$ is the underlying graph of $T(\Sigma)$ is the Tosha-degree equivalence graph of $G$, where for any edge $e_1e_2$ in $T(\Sigma)$, $\sigma'(e_1e_2) = \sigma(e_1)\sigma(e_2)$. Hence, we shall call a given signed graph $\Sigma$ as Tosha-degree equivalence signed graph if it is isomorphic to the Tosha-degree equivalence signed graph $T(\Sigma')$ of some sigraph $\Sigma'$ (See [3]). In [3], we offered a switching equivalence characterization of signed graphs that are switching equivalent to Tosha-degree equivalence signed graphs and $k^{th}$ iterated Tosha-degree equivalence signed graphs. Further, we have presented the structural characterization of Tosha-degree equivalence signed graphs.
2. Tosha-degree of an edge in a graph

In [4], R. Rajendra and P.S.K. Reddy have defined the Tosha-degree of an edge in a graph without multiple edges as follows: The Tosha-degree of an edge $\alpha$ in a graph $G$ without multiple edges, denoted by $T(\alpha)$, is the number of edges adjacent to $\alpha$ in $G$, with self-loops counted twice. Here we allow graphs with multiple edges (multi-graphs) and the new definition of the Tosha-degree of an edge in a graph (with or without multiple edges) is given below:

**Definition 1.** Let $\alpha$ be an edge in a graph $G$. The Tosha-degree of $\alpha$, denoted by $T(\alpha)$ or $T_G(\alpha)$, is the number of edges adjacent to $\alpha$ in $G$, where self-loops and edges parallel to $\alpha$ are counted twice.

By the Definition 1, for any edge $\alpha$ in a graph $G$, $T(\alpha) \geq 0$.

**Definition 2.** A graph $G$ is said to be a Tosha-regular graph if all edges are of equal Tosha-degree. We say that $G$ is $l$-Tosha-regular, if $T(\alpha) = l$, for all $\alpha \in E(G)$.

The following proposition is proved for graphs without parallel edges in [4]. This result is true for graphs having parallel edges also with respect to the Definition 1.

**Proposition 1.** [4] Let $\alpha$ be an edge in a graph $G$ with end vertices $u$ and $v$.

(i) If $\alpha$ is not a self-loop, then

$$T(\alpha) = d(u) + d(v) - 2$$  \hspace{1cm} (1)

(ii) If $\alpha$ is a self-loop, then $u = v$ and

$$T(\alpha) = d(u) - 2$$  \hspace{1cm} (2)

**Proof.** The proof follows by the definition 1, and the definition of degree of a vertex.

**Observation:** By the Proposition 1, for an edge $\alpha$ in a graph $G$, it follows that,

(a) if $\alpha$ is not a self-loop, then

$$2(\delta(G) - 1) \leq T(\alpha) \leq 2(\Delta(G) - 1);$$

(b) if $\alpha$ is a self-loop, then

$$\delta(G) - 2 \leq T(\alpha) \leq \Delta(G) - 2.$$

**Corollary 1.** [4] If $G$ is a simple graph and $\alpha$ is an edge in $G$, then

$$T(\alpha) = d_{L(G)}(\alpha)$$  \hspace{1cm} (3)

where $d_{L(G)}(\alpha)$ is the degree of $\alpha$ as a vertex in the line graph $L(G)$ of $G$.

**Proof.** Follows from the definition of $L(G)$ and Eq.(1).
Corollary 2. In a simple graph $G$, the number of odd Tosha-degree edges is even.

Proof. In any graph the number of odd degree vertices is even. So, the number of odd degree vertices in the line graph $L(G)$ of $G$ is even. Since the vertices in $L(G)$ are corresponding to the edges in $G$, by Eq.(3) it follows that, the number of odd Tosha-degree edges in $G$ is even.

Remark 1. The Corollary 2 may not be true for the graphs having self-loops. There are graphs with odd number of edges and all edges are of odd Tosha-degree. For eg., consider the graph $G$ given in Figure 1. The graph $G$ has three edges namely, $\alpha$, $\beta$ and $\gamma$. We observe that $T(\alpha) = 1$, $T(\beta) = 3$, $T(\gamma) = 1$ and hence all the edges in $G$ are of odd Tosha-degree.

$$\begin{array}{c}
\alpha \\
\beta \\
\gamma
\end{array}$$

$G$

Figure 1: Graph containing odd number of odd Tosha-degree edges.

Observation: Let $\alpha$ be an edge in a simple graph $G$. The addition of a parallel edge $\beta$ to $\alpha$ gives a count plus two to the Tosha-degree of $\alpha$ and to the edges parallel to $\alpha$, and a count plus one to non-parallel edges adjacent to $\alpha$ in the new graph $G+\beta$ and Tosha-degrees of all other edges are unaltered $G+\beta$. Hence an odd (even) Thosa-degree edge $\gamma$ remains odd (even) Tosha-degree in $G+\beta$, if it is not adjacent to $\alpha$ or $\gamma = \alpha$ in $G$.

Corollary 3. If $\alpha$ and $\beta$ are parallel edges in a graph $G$, then $T(\alpha) = T(\beta)$ in $G$.

Proof. The proof follows by Proposition 1.

2.1. $T$-line graph of a multigraph

Definition 3. A multigraph is a graph in which multiple edges (parallel edges) are permitted between any pair of vertices. All multigraphs in this paper are loopless.

We say that two distinct edges $\alpha$ and $\beta$ in a multigraph $G$ are $k$-adjacent if they are adjacent and share $k$ end vertices.

We say that two distinct vertices $u$ and $v$ in a multigraph $G$ are $r$-adjacent if they are adjacent and the number of edges between them is $r$ (i.e., $r$ edges have common end vertices $u$ and $v$).

From the Definition 3, it follows that, when two distinct edges $\alpha$ and $\beta$ are $k$-adjacent in a multigraph $G$, we have,

$$k = \begin{cases} 
1, & \text{if } \alpha \text{ and } \beta \text{ are not parallel;} \\
2, & \text{if } \alpha \text{ and } \beta \text{ are parallel.}
\end{cases}$$
Definition 4. Given a multigraph $G = (V,E)$, the T-line graph of $G$ denoted by $TL(G)$, is a graph with vertex set $E$; two distinct vertices $\alpha$ and $\beta$ are $k$-adjacent in $TL(G)$ if and only if their corresponding edges in $G$ are $k$-adjacent.

From the Definition 4, it is clear that,

(a) $TL(G)$ is also a multigraph,

(b) if $G$ is a simple graph, then $TL(G)$ is nothing but $L(G)$.

Proposition 2. Let $G$ be a multigraph and $\alpha$ be a vertex in $TL(G)$ (so $\alpha$ is an edge in $G$). Then

$$d_{TL(G)}(\alpha) = d_G(u) + d_G(v) - 2 = T_G(\alpha)$$

where $u$ and $v$ are end vertices of $\alpha$ in $G$.

Proof. Proof follows by the definitions 1, 3 and 4, and propositions 1 and 2.

Corollary 4. In a multigraph $G$, the number of odd Tosha-degree edges is even.

Proof. In any graph(multigraph) the number of odd degree vertices is even. So, the number of odd degree vertices in the line graph $TL(G)$ of $G$ is even. Since the vertices in $TL(G)$ are corresponding to the edges in $G$, by Eq.(4), the number of odd Tosha-degree edges in $G$ is even.

3. Zero edges in a graph

Definition 5. In a graph $G$, an edge $\alpha$ is said to be a zero edge if its Tosha degree is zero i.e., $T(\alpha) = 0$.

Observations: The edge in the complete graph $K_2$ is a zero edge. The self-loop in the graph containing only one vertex and a self-loop attached to that vertex, is a zero edge.

Proposition 3. A simple connected graph $G$ has a zero edge if and only if $G \cong K_2$.

Proof. Suppose that $G$ is a simple connected graph having a zero edge, say $\alpha = uv$, where $u$ and $v$ are end vertices of $\alpha$. Then

$$d(u) + d(v) - 2 = 0$$

Since $G$ is connected, $d(u) \geq 1$ and $d(v) \geq 1$; from Eq.(5), $d(u) = 1$ and $d(v) = 1$. Therefore, there is no other edge in $G$ incident to $u$ and $v$. So $G$ has only one edge $\alpha$. Since $G$ is connected, $G \cong K_2$.

Conversely, if $G \cong K_2$, then clearly $G$ is a simple connected graph having only one edge whose Tosha-degree is zero.
Corollary 5. A simple connected graph $G$ with two or more edges, has no zero edge. Hence $T(\alpha) \geq 1$, $\forall \alpha \in E(G)$.

**Proof.** Follows from Proposition 3.

Corollary 6. A simple graph $G$ has no zero edge if and only if either $G \not\cong K_2$ or no component of $G$ is isomorphic to $K_2$ or no component of $G$ is of only one vertex with a self-loop.

**Proof.** Follows from Proposition 3.

4. Degree colorable graphs

In this section we consider self-loop free graphs (multigraphs).

**Definition 6.** A graph $G$ is degree colorable if no two adjacent vertices have the same degree.

**Theorem 1.** If all the edges of a graph $G$ are of odd Tosha-degree, then $G$ is a degree colorable graph with even number of vertices.

**Proof.** Suppose that $G$ is a graph in which all the edges are of odd Tosha-degree. By the corollaries 2 and 4, it follows that $G$ has an even number of vertices. Let $\alpha$ be an edge in $G$ with end vertices $u$ and $v$. Then by Eq.(1) and Eq.(4),

$$T(\alpha) = d(u) + d(v) - 2.$$ 

Since $T(\alpha)$ is odd, $d(u) \neq d(v)$. Thus, no two adjacent vertices in $G$ have the same degree. Therefore $G$ is a degree colorable graph.

By Theorem 1, the following corollary is immediate.

**Corollary 7.** An $l$-Tosha-regular graph, where $l$ is an odd positive integer, is degree colourable.

**Remark 2.** There are degree colorable non-Tosha-regular graphs with odd number of vertices. The following graph is an example for such graphs, in which the edges are indicated by respective Tosha-degrees.

5. Tosha-even graphs

**Definition 7.** A graph $G$ is said to be Tosha-even if all its edges are of even Tosha-degree.

We recall the following proposition from [4].

**Proposition 4.** [4, Proposition 2.15] If $G$ is an Euler graph, then all edges in $G$ are of even Tosha-degree.
Corollary 8. Euler graphs are Tosha-even.

Proof. Follows from the Proposition 4.

Remark 3. The converse of the Corollary 8 is not true in general. There are connected graphs with even number of vertices and all vertices are of odd degree, for instance, $K_4$. Such graphs are not Euler graphs, but are Tosha-even.

Proposition 5. There exist degree colorable Tosha-even graphs that are not Euler graphs.

Proof. The following graph $G$ (see Figure 3) is an example of a degree colorable Tosha-even graph which is not an Euler graph. In $G$, the vertices and edges are indicated by their degrees and Tosha-degrees, respectively. We see that all vertices of $G$ are of odd degree and hence $G$ is not an Euler graph. But all edges are of Tosha-even, so $G$ is a Tosha-even graph.

6. Tosha-adjacency matrix of a graph

Definition 8. If $G$ is a graph with $n$ vertices $v_1, \ldots, v_n$ and no parallel edges. The Tosha-adjacency matrix of the graph $G$ is an $n \times n$ matrix $A_T(G) = (t_{ij})$ defined over the ring of integers such that

$$t_{ij} = \begin{cases} T(v_iv_j), & \text{if } v_iv_j \in E \\ 0, & \text{otherwise}. \end{cases}$$

Observations:

Figure 2: A degree colorable non-Tosha-regular graph with 3 vertices.

Figure 3: A degree colorable Tosha-even graph which is not an Euler graph.
(i) By the definition of the Tosha-degree of an edge, we have

\[ T(v_i v_j) = \begin{cases} 
  d(v_i) + d(v_j) - 2, & \text{if } v_i v_j \in E \text{ and } i \neq j; \\
  d(v_i) - 2, & \text{if } v_i v_j \in E \text{ and } i = j; \\
  0, & \text{if } v_i v_j \notin E.
\end{cases} \]

Therefore, \( t_{ij} = t_{ji} \). Therefore \( A_T(G) \) is a real symmetric matrix.

(ii) The entries along the principal diagonal of \( A_T(G) \) are all 0s if and only if either \( G \) has no self-loops or \( G \) has only self-loops that are zero edges. Hence if either \( G \) has no self-loops or \( G \) has only self-loops that are zero edges, then \( tr(A_T(G)) = 0 \). In this case, if \( \mu_1, \mu_2, \ldots, \mu_n \) are the eigenvalues of \( A_T(G) \), then

\[ \sum_{i=1}^{n} \mu_i = 0. \]

(iii) If \( G \) has no zero edges, then the degree of a vertex equals the number of non-zero entries in the corresponding row or column; and the non-zero entry in the \( ij \)-th place gives the Tosha-degree of the corresponding edge incident to \( i \)-th and \( j \)-th vertices.

(iv) For a zero edge free graph \( G \), the adjacency matrix \( A(G) \) can be obtained from the Tosha-adjacency matrix \( A_T(G) \) by replacing all the non-zero entries by 1s. This is possible because, in a zero edge free graph Tosha-degrees of edges are non-zero. Thus, reconstrction of the graph from the Tosha-adjacency matrix is possible if the given graph has no zero edges.

Throughout this section \( G \) denotes a graph with no parallel edges.

**Theorem 2.** If a graph \( G \) with \( n \) vertices is \( l \)-Tosha-regular, then

\[ A_T(G) = l \cdot A(G). \]

**Proof.** Suppose that \( G \) is \( l \)-Tosha-regular. Then \( T(\alpha) = l \), for all \( \alpha \in E(G) \). Let \( A(G) = (a_{ij}) \) and \( A_T(G) = (t_{ij}) \) be the adjacency matrix and the Tosha-adjacency matrix of \( G \), respectively. Then by the definition of the Tosha-adjacency matrix \( A_T(G) \), we have

\[ t_{ij} = \begin{cases} 
  l, & \text{if } v_i v_j \in E \\
  0, & \text{otherwise}
\end{cases} = l \cdot a_{ij}. \]

Therefore, \( A_T(G) = l \cdot A(G) \).
Corollary 9. If a graph $G$ with $n$ vertices is $r$-regular, then

$$A_T(G) = 2(r - 1)A(G).$$

Proof. If a graph $G$ with $n$ vertices is $r$-regular, then $G$ is $2(r - 1)$-Tosha-regular (by [4, Corollary 2.6]) and hence by Theorem 2, $A_T(G) = 2(r - 1)A(G)$.

Corollary 10. A graph $G$ is 1-Tosha-regular if and only if $A_T(G) = A(G)$.

Proof. ($\Leftarrow$:) Suppose that for a graph $G$, $A_T(G) = A(G)$. Then by the definitions of $A_T(G)$ and $A(G)$, it follows that, $T(\alpha) = 1$, $\forall \alpha \in E(G)$. Hence, $G$ is 1-Tosha-regular. ($\Rightarrow$:) Follows by Theorem 2.

7. Tosha-energy of a graph

Definition 9. Let $G$ be a graph with $n$ vertices $v_1, \ldots, v_n$ and no parallel edges. Let $\mu_1, \mu_2, \ldots, \mu_n$ be the eigenvalues of the Tosha-adjacency matrix $A_T(G)$ of $G$. The Tosha-energy of $G$, denoted by $E_T(G)$, is defined as

$$E_T(G) = \sum_{i=1}^{n} |\mu_i|. \quad (6)$$

Throughout this section $G$ denotes a graph with no parallel edges.

Proposition 6. The Tosha-energy of an $l$-Tosha-regular graph $G$ with $n$ vertices is given by

$$E_T(G) = l \cdot E(G) \quad (7)$$

where $E(G)$ is the energy of $G$.

Proof. Let $G$ be an $l$-Tosha-regular graph with $n$ vertices. Then by the Theorem 2, the Tosha-adjacency matrix of $G$ is

$$A_T(G) = l \cdot A(G) \quad (8)$$

where $A(G)$ is the adjacency matrix of $G$. For brevity we write $A$ for $A(G)$ and $A_T$ for $A_T(G)$. We consider two cases: (i) When $l > 0$ and (i) When $l = 0$.

Case (i): When $l > 0$. Let $\mu$ be an eigenvalue of $A_T$. From Eq.(8) we have,

$$\det(A_T - \mu I) = 0 \iff \det(lA - \mu I) = 0$$

$$\iff l^n \det(A - \frac{\mu}{l} I) = 0$$

$$\iff \det(A - \frac{\mu}{l} I) = 0.$$
Therefore, $\mu$ is an eigenvalue of $A_T$ if and only if $\frac{\mu}{l}$ is an eigenvalue of $A$. Let $\mu_1, \mu_2, \ldots, \mu_n$ be the eigenvalues of the $A_T$. Then $\frac{\mu_1}{l}, \frac{\mu_2}{l}, \ldots, \frac{\mu_n}{l}$ are the eigenvalues of $A$ and the Tosha-energy of $G$ is

$$
E_T(G) = \sum_{i=1}^{n} |\mu_i| = l \cdot \sum_{i=1}^{n} \left| \frac{\mu_i}{l} \right| = l \cdot E(G).
$$

Case (ii): When $l = 0$. From Eq.(7), $A_T = 0$ and so zero is the only eigenvalue of $A_T$ of multiplicity $n$. In this case, $E_T(G) = 0 = 0 \cdot E(G)$.

**Corollary 11.** The Tosha-energy of an $r$-regular graph $G$ with $n$ vertices is given by

$$
E_T(G) = 2(r - 1)E(G)
$$

where $E(G)$ is the energy of $G$.

**Proof.** Let $G$ be an $r$-regular graph with $n$ vertices. By [4, Corollary 2.6] $G$ is a $2(r - 1)$-Tosha-regular graph. Then by Proposition 6, the proof follows.

**Corollary 12.**

(i) For the complete graph $K_n$ on $n > 1$ vertices,

$$
E_T(K_n) = 2(n - 2)E(K_n) = 4(n - 1)(n - 2).
$$

(ii) For the cycle graph $C_n$ on $n > 1$ vertices,

$$
E_T(C_n) = 2E(C_n) = 4 \sum_{i=0}^{n-1} \left| \cos \left( \frac{2\pi i}{n} \right) \right|.
$$

(iii) For the complete bipartite graph $K_{m,n}$,

$$
E_T(K_{m,n}) = (m + n - 2)E(K_{m,n}) = 2(m + n - 2)\sqrt{mn}.
$$

**Proof.** (i) The eigen values of $A(K_n)$ are given below:

| eigen value | multiplicity |
|-------------|--------------|
| $n - 1$     | $n - 1$      |
| $-1$        | $1$          |

Therefore

$$
E(K_n) = |n - 1| + (n - 1)| - 1| = 2(n - 1).
$$
Since $K_n$ is an $(n - 1)$-regular graph, from Eq.(9) we have,

$$E_T(K_n) = 2(n - 2)E(K_n) = 2(n - 2) \cdot 2(n - 1) = 4(n - 1)(n - 2).$$

(ii) The eigen values of $A(C_n)$ are

$$2 \cos \left( \frac{2\pi i}{n} \right), \quad i = 0, 1, \ldots, n - 1.$$

Therefore

$$E(C_n) = 2 \sum_{i=0}^{n-1} \left| \cos \left( \frac{2\pi i}{n} \right) \right|.$$

Since $C_n$ is an 2-regular graph, from Eq.(9) we have,

$$E_T(C_n) = 2 \cdot E(C_n) = 4 \sum_{i=0}^{n-1} \left| \cos \left( \frac{2\pi i}{n} \right) \right|.$$

(iii) The eigen values of $A(K_n)$ are given below:

| eigen value | multiplicity |
|-------------|--------------|
| $-\sqrt{mn}$ | 0 | $\sqrt{mn}$ |
| 1 | $n + m - 2$ | 1 |

Therefore

$$E(K_{m,n}) = 2\sqrt{mn}.$$

Since $K_{m,n}$ is an $(m + n - 2)$-Tosha-regular graph, from Eq.(8) we have,

$$E_T(K_{m,n}) = (m + n - 2)E(K_{m,n}) = 2(m + n - 2)\sqrt{mn}.$$

**Corollary 13.** (i) For the path $P_2$ of 2 vertices, $E_T(P_2) = 0$.

(ii) For the path $P_3$ of 3 vertices, $E_T(P_3) = E(P_3) = 2\sqrt{2}$.

Proof. Since $P_2 = K_{1,1}$ and $P_3 = K_{2,1}$, (i) and (ii) follow immediately from Corolla-

**Theorem 3.** Let $G$ be a simple connected graph with at least one edge. Then

$$A_T(G) = A(G) \iff G = P_3.$$

Proof. ($\Leftarrow$) If $G = P_3$, then it has two edges and each of these are of Tosha-degree 1. Therefore, it is 1-Tosha-regular and hence by Theorem 2, $A_T(G) = A(G)$.

($\Rightarrow$) Suppose that $A_T(G) = A(G)$. Then $G$ is 1-Tosha-regular and hence

$$T(v_iv_j) = 1, \quad \forall v_i, v_j \in E(G).$$
\[ \Rightarrow d(v_i) + d(v_j) - 2 = 1, \ \forall \ v_iv_j \in E(G) \]

Therefore, for any edge \( \alpha \) in \( G \) with end vertices \( u \) and \( v \),

\[ d(u) = 3 - d(v) \]  

(10)

Since \( G \) is connected, \( d(v) > 0 \) and \( d(u) > 0 \), and from Eq.(10) we have, \( d(u) < 3 \); which implies

\[ d(u) = 1 \text{ or } 2. \]  

(11)

Let \( u \) be an arbitrary vertex in \( G \). Since \( G \) is a simple connected graph with at least one edge, \( u \) is an end vertex of at least one edge say \( \alpha \). Let \( v \) be the other end vertex of \( \alpha \) in \( G \). Then by Eq.(10) and Eq.(11), either \( d(u) = 1 \) and \( d(v) = 2 \) or \( d(u) = 1 \) and \( d(v) = 2 \).

If \( d(u) = 1 \) and \( d(v) = 2 \), there is another vertex \( w \) adjacent to \( v \) and \( d(w) = 1 \) (by above argument). There are no other vertices adjacent to the vertices \( u, v \) and \( w \). So, \( G \) is a path with 3 vertices. A similar argument can be used for the case \( d(u) = 1 \) and \( d(v) = 2 \), to show that \( G \) is \( P_3 \).

8. Edge-adjacency matrix and edge-energy of a graph

**Definition 10.** We say that two distinct edges \( \alpha \) and \( \beta \) in a graph \( G \) (where self-loops and parallel edges are allowed) are \( k \)-adjacent if they are adjacent and share \( k \) end vertices. We consider that an edge in a graph is not adjacent to itself.

**Definition 11.** If \( G \) is a graph with \( m \) edges \( e_1, \ldots, e_m \). The edge-adjacency matrix of the graph \( G \) is an \( m \times m \) matrix \( A_{E}(G) = (x_{ij}) \) defined over the ring of integers such that

\[ x_{ij} = \begin{cases} k, & \text{if } e_i \text{ and } e_j \text{ are } k \text{-adjacent;} \\ 0, & \text{otherwise.} \end{cases} \]

**Observations:**

(i) \( A_{E}(G) \) is a \( \{0, 1, 2\} \)-matrix and it is real symmetric. If \( G \) is a simple graph, then \( A_{E}(G) \) is a \( \{0, 1\} \)-matrix.

(ii) The entries along the principal diagonal of \( A_{E}(G) \) are all 0s. Therefore, \( tr(A_{E}(G)) = 0 \). Hence if \( \nu_1, \nu_2, \ldots, \nu_m \) are the eigenvalues of \( A_{E}(G) \), then

\[ \sum_{i=1}^{m} \nu_i = 0. \]

(iii) If \( G \) has no self-loops, then the Tosha-degree of an edge equals the sum of entries in the corresponding row or column of \( A_{E}(G) \).
**Proposition 7.** For a multigraph $G$, the edge-adjacency matrix of $G$ is the adjacency matrix of the $T$-line graph of $G$. That is,

$$A_E(G) = A(TL(G)).$$

**Proof.** Follows by the definitions 4 and 11.

**Corollary 14.** For a simple graph $G$, the edge-adjacency matrix of $G$ is the adjacency matrix of the line graph of $G$. That is,

$$A_E(G) = A(L(G)).$$

**Proof.** For simple graph $G$, $TL(G) = L(G)$ and so by Proposition 7 the result follows.

**Definition 12.** Let $G$ be graph with $m$ edges $e_1, \ldots, e_m$. Let $\nu_1, \nu_2, \ldots, \nu_m$ be the eigenvalues of the edge-adjacency matrix $A_E(G)$ of $G$. The edge-energy of $G$, denoted by $E_E(G)$, is defined as

$$E_E(G) = \sum_{i=1}^{m} |\nu_i|.$$ (12)

**Corollary 15.** For a multigraph $G$, the edge-energy of $G$ is the energy of the $T$-line graph of $G$. That is,

$$E_E(G) = E(TL(G)).$$

**Proof.** Follows by Proposition 7.

**Corollary 16.** For a simple graph $G$, the edge-energy of $G$ is the energy of the line graph of $G$. That is,

$$E_E(G) = E(L(G)).$$

**Proof.** Follows by Corollary 14.

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