Is there the color-flavor locking in the instanton induced quark-antiquark pairing in QCD vacuum?

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By means of the functional integral method we show that in the case of the quark-antiquark pairing at zero temperature and zero chemical potential (in the vacuum) the singlet pairing is more preferable than that with the color-flavor locking (CFL).

1 Introduction

Recently, the physical properties of the existence of the four-quark interactions (color superconductivity, the quark-antiquark pairing and the formation of unusual bound states of quark matter, etc...) were widely discussed[1−10]. In an earlier work[1], Mark Alford, Krishna Rajagopal and Frank Wilczek have proposed that due to the condensation of quarks the chiral flavor SU(3) symmetries are broken and the LL condensate “locks” SU(3)_L rotations to color rotations while the RR condensate “locks” SU(3)_R rotations to color rotations:

$$SU(3) \otimes SU(3)_L \otimes SU(3)_R \otimes U(1)_B \rightarrow SU(3)_{C+L+R}$$

In this work the problem of the color-flavor locking in the quark-antiquark pairing is considered. We calculate the free energy densities of the QCD vacuum due to the quark-antiquark pairing in two cases: the singlet pairing and the pairing with the color-flavor locking. By comparing these values of the free energy densities we conclude that the singlet pairing is more preferable.

Let us start from the four-fermion interaction Lagrangian of the four quark fields

$$L_{\text{int}} (x) = \frac{1}{2} \bar{\psi}^A (x) \psi_B (x) U_{AC}^{BD} \bar{\psi}^C (x) \psi_D (x),$$

where $\psi_A (x)$ and $\bar{\psi}^A (x)$ are the quark and antiquark fields. $A, B, C, D$ are the sets of indices

$$A = (\alpha, a, i), \quad B = (\beta, b, j), \quad C = (\gamma, c, k), \quad D = (\delta, d, l),$$

$\alpha, \beta, \gamma, \delta = 1, 2, 3, 4$ are the Dirac spinor indices, $a, b, c, d = 1, 2, 3...N_C$ are the color symmetry indices and $i, j, k, l = 1, 2, 3...N_f$ are the flavor ones. The coupling constants $U_{AC}^{BD}$ are antisymmetric under the permutations of the indices

$$U_{AC}^{BD} = -U_{AC}^{DB} = -U_{CA}^{BD} = U_{CA}^{DB}.$$

The functional integral of the free and interacting massless quark fields are

$$Z_0 = \int [D\psi] [D\bar{\psi}] \exp \left\{ -i \int d^4x \bar{\psi}^A (x) \left( \partial_x \right)_A^B \psi_B (x) \right\}$$

and

$$Z = \int [D\psi] [D\bar{\psi}] \exp \left\{ -i \int d^4x \bar{\psi}^A (x) \left( \partial_x \right)_A^B \psi_B (x) \right\}
\exp \left\{ \frac{i}{2} \int d^4x \bar{\psi}^A (x) \psi_B (x) U_{AC}^{BD} \bar{\psi}^C (x) \psi_D (x) \right\}.$$ (3)

Introduce the composite bosonic field $\Phi_A^B (x)$ and the functional integral

$$Z_0^B = \int [D\Phi] \exp \left\{ -i \int d^4x \Phi_A^B (x) U_{AC}^{BD} \Phi_D^C (x) \right\}$$ (4)
By means of the Hubbard - Stratonovich transformation, we rewrite $Z$ in the form of a functional integral over the composite bosonic fields:

$$Z = \frac{Z_0}{Z_0} \int [D\Phi] \exp \{iS_{\text{eff}}[\Phi]\}$$

with the effective action $S_{\text{eff}}[\Phi]$:

$$S_{\text{eff}}[\Phi] = -\frac{1}{2} \int d^4x \Phi^B_A(x) U^{BD}_{AC} \Phi^C_D(x) + W[\Delta],$$

$$W[\Delta] = \sum_{n=1}^{\infty} W^{(n)}[\Delta]$$

$$W^{(1)} = \int dx \Delta^B_A(x) S^A_B(x),$$

$$W^{(2)} = -\frac{1}{2} \int dx_1 \int dx_2 \Delta^B_{A_1}(x_1) S^A_{B_1}(x_1 - x_2) \Delta^B_{A_2}(x_2) S^A_{B_2}(x_2 - x_1),$$

$$W^{(3)} = \frac{1}{3} \int dx_1 \int dx_2 \int dx_3 \Delta^B_{A_1}(x_1) S^A_{B_1}(x_1 - x_2) \Delta^B_{A_2}(x_2) S^A_{B_2}(x_2 - x_3) \Delta^B_{A_3}(x_3) S^A_{B_3}(x_3 - x_1),$$

$$W^{(4)} = \ldots$$

and $\Delta^B_A$ play the role of the order parameters of the system of quarks

$$\Delta^A_B(x) = U^{BD}_{AC} \Phi^C_D(x).$$

From the effective action (6) we can derive the field equation by means of the variational principle

$$\Delta^D_C(x) = U^{BD}_{AC} \frac{\delta W[\Delta]}{\delta \Delta^A_B(x)} = \Delta^D_C(x) G^A_B(x, x)$$

with the interacting Green’s function $G^A_B(x, y)$ satisfying the Schwinger-Dyson equations

$$G^A_B(y, x) = G^A_B(y - x) - \int dz G^C_B(y - z) \Delta^D_C(z) G^A_B(z, x).$$

$S^A_B(y - x)$ is the Green’s function of free quarks.

### 2 Equations for the order parameters

We consider the special class of solution of the equation (9)

$$\Delta^A_B(x) = \Delta^A_B = \text{constant},$$

$$G^A_B(x, y) = G^A_B(x - y).$$

In this case (10) becomes

$$G^A_B(x - y) = S^A_B(x - y) - \int dz S^C_B(x - z) \Delta^D_C(z) G^A_B(z, x).$$

Then equation (9) can be written in the form

$$\Delta^C_D = U^{BD}_{CA} \frac{1}{(2\pi)^4} \int d^4p \tilde{G}^A_B(p),$$

Denote $\tilde{G}^A_B(p)$ the Fourier component of $G^A_B(x - x)$. 
In the case of the singlet pairing the order parameters $\Delta_B^A$ equal

$$\Delta_B^A = \delta_\alpha^\beta \Delta_{ij}^{\alpha\beta} = \delta_\alpha^\beta \delta_i^j \Delta$$

with some constant $\Delta$. It follows that in the limiting case with zero temperature ($T = 0$) and zero chemical potential ($\mu = 0$), the Green’s function has the form

$$\left[ \frac{1}{G(p)} \right]^A_B = \left[ \frac{1}{S(p)} \right]^A_B + \Delta_B^A.$$  

(14)

In the case of the color-flavor locking the order parameters $\Delta_B^A$ equal

$$\Delta_B^A = \delta_\alpha^\beta \left\{ \delta_i^\alpha \delta_j^\beta \Delta + \delta_i^\beta \delta_j^\alpha \Omega \right\}.$$  

(15)

Then the Green’s function has the expression of the form

$$\tilde{G}_B^A(p) = (\hat{A})^{\alpha\beta} \delta_i^\alpha \delta_j^\beta + (\hat{B})^{\alpha\beta} \delta_i^\beta \delta_j^\alpha.$$  

(16)

(17)

It is clearly that the chiral symmetry is broken due to the existence of $\Delta$. In the case of the color-flavor locking the order parameters $\Delta_B^A$ equal

$$\Delta_B^A = \delta_\alpha^\beta \left\{ \delta_i^\alpha \delta_j^\beta \Delta + \delta_i^\beta \delta_j^\alpha \Omega \right\}.$$  

(18)

Then the Green’s function has the expression of the form

$$\tilde{G}_B^A(p) = \left( \hat{A} \right)^{\alpha\beta} \delta_i^\alpha \delta_j^\beta + \left( \hat{B} \right)^{\alpha\beta} \delta_i^\beta \delta_j^\alpha.$$  

(19)

$X, Y, W$ and $Z$ are scalar functions need to be determined.

In the systems with the instanton induced interaction between quarks and anti-quarks the coupling constants equal $[4]

$$U_{AC}^{BD} = \left( \delta_i^\alpha \delta_j^\alpha \delta_l^\gamma \delta_k^\gamma \right) \left\{ U_1 \left[ \left( \sigma_\mu \right)_\alpha^\gamma \left( \sigma_\mu \right)_\gamma^\delta \right] \delta_i^\delta \delta_j^\gamma \delta_k^\delta + \left( \sigma_\mu \right)_\gamma^\delta \delta_i^\gamma \delta_j^\delta \delta_k^\gamma \right\}$$

+ $U_2 \left[ \left( \sigma_\mu \right)_\gamma^\delta \delta_i^\gamma \delta_j^\delta \delta_k^\gamma \right] \left\{ \left( \sigma_\mu \right)_\gamma^\delta \delta_i^\gamma \delta_j^\delta \delta_k^\gamma \right\}.$  

(20)

In order to determine $X, Y, W$ and $Z$ we insert (17), (18) and (20) into (13). Because the integrals of the terms containing the factor $\hat{\rho}$ vanish $X$ and $Z$ give no contribution. For $Y$ and $W$ we have the expressions

$$Y = \frac{1}{2} \left\{ \frac{M + \Delta + \Omega}{p^2 + (M + \Delta + \Omega)^2} + \frac{M + \Delta - \Omega}{p^2 + (M + \Delta - \Omega)^2} \right\},$$

$$W = \frac{1}{2} \left\{ \frac{M + \Delta + \Omega}{p^2 + (M + \Delta + \Omega)^2} - \frac{M + \Delta - \Omega}{p^2 + (M + \Delta - \Omega)^2} \right\}.$$  

(21)

It follows that $\Delta$ and $\Omega$ are determined by

$$\Delta = (N_C - 1) (N_C U_1 + U_2) \frac{1}{(2\pi)^4} \int d^4p Y$$

+ $[(N_C - 1) U_1 + U_2] \frac{1}{(2\pi)^4} \int d^4p W,$  

(22)

$$\Omega = -U_2 \frac{1}{(2\pi)^4} \int d^4p W.$$  

(23)

$N_C$ is number of colors. In reference [4] it was shown that the constants $U_1$ and $U_2$ equal

$$U_1 = \frac{4g (2N_C - 1)}{2N_C (N_C^2 - 1)}, \quad U_2 = \frac{g (N_C - 4)}{2N_C (N_C^2 - 1)}.$$  

(24)

with some coupling constant $g$. Using this result with $N_C = 3$ we rewrite (22) and (23) in the more compact form.
\[ \Delta + \Omega = \frac{g}{24} \frac{1}{16 \pi^2} \int p^2 dp^2 \left[ 42 \frac{M + \Delta + \Omega}{p^2 + (M + \Delta + \Omega)^2} + 22 \frac{M + \Delta - \Omega}{p^2 + (M + \Delta - \Omega)^2} \right], \quad (25) \]

\[ \Delta - \Omega = \frac{g}{24} \frac{1}{16 \pi^2} \int p^2 dp^2 \left[ 44 \frac{M + \Delta + \Omega}{p^2 + (M + \Delta + \Omega)^2} + 20 \frac{M + \Delta - \Omega}{p^2 + (M + \Delta - \Omega)^2} \right]. \quad (26) \]

If there is no CFL, \( \Omega = 0 \), then the two equations for \( \Delta \) and \( \Omega \) reduce to one and the same equation for \( \Delta \)

\[ \Delta = \frac{g}{6 \pi^2} \int p^2 dp^2 \frac{M + \Delta}{p^2 + (M + \Delta)^2}. \quad (27) \]

For the convenient we set

\[ \Delta_1 = \Delta + \Omega, \quad \Delta_2 = \Delta - \Omega \]

then after taking the integrations over \( p^2 \) we rewrite the equations (25), (26) and (27) in the form

\[ \Delta_1 = \frac{1}{32} \frac{g}{6 \pi^2} \left\{ 21 (M + \Delta_1) \left[ 1 - (M + \Delta_1)^2 \ln \frac{1 + (M + \Delta_1)^2}{(M + \Delta_1)^2} \right] \right. \]

\[ \left. + 11 (M + \Delta_2) \left[ 1 - (M + \Delta_2)^2 \ln \frac{1 + (M + \Delta_2)^2}{(M + \Delta_2)^2} \right] \right\}, \quad (29) \]

\[ \Delta_1 = \frac{1}{32} \frac{g}{6 \pi^2} \left\{ 22 (M + \Delta_1) \left[ 1 - (M + \Delta_1)^2 \ln \frac{1 + (M + \Delta_1)^2}{(M + \Delta_1)^2} \right] \right. \]

\[ \left. + 10 (M + \Delta_2) \left[ 1 - (M + \Delta_2)^2 \ln \frac{1 + (M + \Delta_2)^2}{(M + \Delta_2)^2} \right] \right\}, \quad (30) \]

\[ \Delta = \frac{g}{6 \pi^2} (M + \Delta) \left[ 1 - (M + \Delta)^2 \ln \frac{1 + (M + \Delta)^2}{(M + \Delta)^2} \right]. \quad (31) \]

The values of \( \Delta_1, \Delta_2 \) and \( \Delta \) are plotted in Figures 1, 2 and 3 with certain values of \( g \) and \( M \). Note that in the case of massless quarks, \( M = 0 \), from (29), (30) and (31) it follows that

\[ \Delta_1 = \Delta_2 = \Delta \]

This means that \( \Omega \) must equal to zero. Thus we conclude that in the case of the massless quarks there is no CFL.

### 3 The free energy densities

In order to choose the more preferable one among two pairing mechanisms in the case \( M \neq 0 \) we must calculate the free energy density\(^{[7,8]}\)

\[ F = \frac{1}{(2\pi)^4} \int d^4p \text{Tr} \left\{ \frac{1}{2} \left( \tilde{\Delta S}(p) - [\tilde{\Delta S}(p)]^2 + [\tilde{\Delta S}(p)]^3 - [\tilde{\Delta S}(p)]^4 + \ldots \right) \right. \]

\[ \left. - \left( \tilde{\Delta S}(p) - \frac{1}{2} [\tilde{\Delta S}(p)]^2 + \frac{5}{3} [\tilde{\Delta S}(p)]^3 - \frac{1}{4} [\tilde{\Delta S}(p)]^4 + \ldots \right) \left\} \right\}, \quad (33) \]

If we define a new Green’s function \( \tilde{G}(p, \alpha) \)
\[
\left[ \frac{1}{G(p, \alpha)} \right]_{B}^{A} = \left[ \frac{1}{S(p)} \right]_{B}^{A} + \alpha \Delta_{B}^{A} \tag{34}
\]

then we can write the free energy \( F \) as

\[
F = \frac{1}{(2\pi)^4} \int d^4 p \left[ \hat{\Delta} \left\{ \frac{1}{2} \tilde{G}(p) - \int d\alpha \tilde{G}(p, \alpha) \right\} \right]. \tag{35}
\]

We see here that the free energy is a function of the order parameters and then with the support of the calculation of \( \Delta_{B}^{A} \) in Section II we can write down the free energy densities of the two cases \( F(\Delta) \) and \( F(\Delta_{1}, \Delta_{2}) \) as follow

\[
F(\Delta) = \Delta (M + \Delta) \left\{ 1 - (M + \Delta)^2 \ln \frac{1 + (M + \Delta)^2}{(M + \Delta)^2} \right\}
\]

\[
- \left\{ \frac{1}{2} (M + \Delta)^2 + \frac{1}{2} \left[ 1 - (M + \Delta)^4 \right] \ln \left[ 1 + (M + \Delta)^2 \right] \right\}
\]

\[
+ \left\{ \frac{1}{2} M^2 + \frac{1}{2} \left[ 1 - M^4 \right] \ln \left[ 1 + M^2 \right] + \frac{1}{2} M^4 \ln M^4 \right\}, \tag{36}
\]

\[
F(\Delta_{1}, \Delta_{2}) = \frac{2}{3} F(\Delta_{1}) + \frac{1}{3} F(\Delta_{2}) \tag{37}
\]

The values of \( F(\Delta_{1}, \Delta_{2}) \) and \( F(\Delta) \) with \( \Delta_{1}, \Delta_{2} \) and \( \Delta \) given by (29), (30) and (31) are plotted in Figures 4 and 5. We can conclude that the value of \( F(\Delta) \) is always lower than that of \( F(\Delta_{1}, \Delta_{2}) \). This means that the singlet pairing is more preferable than that with the color-flavor locking.

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**Figure captions**

Figure 1: \( \Delta \) as a function of coupling constant \( g \) (in GeV\(^{-2}\)) at \( M = 0.1 \text{MeV} \)

Figure 2: \( \Delta_{1} \) as a function of coupling constant \( g \) (in GeV\(^{-2}\)) at \( M = 0.1 \text{MeV} \)

Figure 3: \( \Delta_{2} \) as a function of coupling constant \( g \) (in GeV\(^{-2}\)) at \( M = 0.1 \text{MeV} \)

Figure 4: The values of \( F(\Delta_{1}, \Delta_{2}) \) depend on \( \Delta_{1} \) and \( \Delta_{2} \)

Figure 5: The values of \( F(\Delta) \) as a function of \( \Delta \)
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