We say that a nonselfadjoint operator algebra is *partly free* if it contains a free semigroup algebra. Motivation for such algebras occurs in the setting of what we call *free semigroupoid algebras*. These are the weak operator topology closed algebras generated by the left regular representations of semigroupoids associated with finite or countable directed graphs. We expand our analysis of partly free algebras from previous work and obtain a graph-theoretic characterization of when a free semigroupoid algebra with countable graph is partly free. This analysis carries over to norm closed *quiver algebras*. We also discuss new examples for the countable graph case.

Every finite or countable directed graph $G$ recursively generates a Fock space Hilbert space and a family of partial isometries. These operators are of Cuntz-Krieger-Toeplitz type and also arise through the left regular representations of semigroupoids determined by directed graphs. This was initially discovered by Muhly [23], and, in the case of finite graphs, more recent work with Solel [24] considered the norm closed algebras generated by these representations, which they called *quiver algebras*. In [19], we developed a structure theory for the weak operator topology closed algebras $\mathcal{L}_G$ generated by the left regular representations coming from both finite and countable directed graphs; we called these algebras *free semigroupoid algebras*. In doing so, we found a unifying framework for a number of classes of algebras which appear in the literature, including; noncommutative analytic Toeplitz algebras $\mathcal{L}_n$ [2, 10, 11, 20, 25, 26] (the prototypical free semigroup algebras), the classical analytic Toeplitz algebra $H^\infty$ [14, 16], and certain finite dimensional digraph algebras [19]. But this approach gives rise to a diverse collection of new examples which include finite dimensional algebras, algebras with free behaviour, algebras which can be represented as matrix function algebras, and examples which mix these possibilities. The general theme of our work in [19] was a marriage of simple graph-theoretic properties with properties of the operator algebra. Furthermore, our technical analyses were chiefly spatial in nature; for instance, we proved the graph is a complete unitary invariant of both the free semigroupoid algebra and the quiver algebra.

In the next section we give a short introduction to free semigroupoid algebras and discuss a number of examples. The second section contains an expanded analysis of a
subclass called \textit{partly free algebras}, which are characterized by containment of a copy of a free semigroup algebra. Specifically, we extend analysis from \cite{19} to the case of countable graphs, obtaining a new graph-theoretic description of when a free semigroupoid algebra is partly free. In fact, we find that it is unusual for an algebra coming from a countable graph to not be partly free. We present a number of new illustrative examples for the countable graph case. Our analysis works equally well for quiver algebras, and we obtain a graph-theoretic condition for when these norm closed algebras are partly free.

1. Free Semigroupoid Algebras

Let $G$ be a finite or countable directed graph, with edge set $E(G)$ and vertex set $V(G)$. Let $\mathbb{F}^+(G)$ be the \textit{free semigroupoid} determined by $G$; that is, $\mathbb{F}^+(G)$ consists of the vertices, which act as units, and all allowable finite paths in $G$, with the natural operations of concatenation of allowable paths. Given a path $w$ in $\mathbb{F}^+(G)$ we write $w = ywx$ when the initial and final vertices of $w$ are, respectively, $x$ and $y$. Let $\mathcal{H}_G = l^2(\mathbb{F}^+(G))$ be the Hilbert space with orthonormal basis indexed by elements of $\mathbb{F}^+(G)$. For each edge $e \in E(G)$ and vertex $x \in V(G)$, we may define partial isometries and projections on $\mathcal{H}_G$ by the following actions on basis vectors:

$$L_e \xi_w = \begin{cases} \xi_{ew} & \text{if } ew \in \mathbb{F}^+(G) \\ 0 & \text{otherwise} \end{cases}$$

and

$$L_x \xi_w = \begin{cases} \xi_{xw} = \xi_w & \text{if } w = xw \in \mathbb{F}^+(G) \\ 0 & \text{otherwise} \end{cases}$$

These operators may thus be regarded as ‘partial creation operators’ acting on a generalized Fock space Hilbert space. There is an equivalent tree perspective \cite{19} which gives an appealing visual interpretation of the actions of these operators. The family $\{L_e, L_x\}$ also arises through the left regular representation $\lambda_G : \mathbb{F}^+(G) \to \mathcal{B}(\mathcal{H}_G)$, with $\lambda_G(e) = L_e$, and $\lambda_G(x) = L_x$. The associated \textit{free semigroupoid algebra} is the weak operator topology closed algebra generated by this family,

$$\mathcal{L}_G = \text{wot-Alg} \{L_e, L_x : e \in E(G), x \in V(G)\}$$

$$= \text{wot-Alg} \{\lambda_G(w) : w \in \mathbb{F}^+(G)\}.$$ 

Remark 1.1. In the case of finite graphs, Muhly and Solel \cite{23, 24} considered the norm closed algebras $A_G$ generated by such a family, calling them \textit{quiver algebras}. For both finite and countable graphs, we considered the classification problem for $A_G$ in \cite{19}, and in Section 2.1 we derive partly free conditions for $A_G$. Recently the C*-algebras generated by families of partial isometries associated with directed graphs have been studied heavily. The set of generators for these algebras are sometimes referred to as Cuntz-Krieger $E$-families (for instance see \cite{4, 15, 21, 22}). On the other hand, the generators of free semigroupoid algebras are of Cuntz-Krieger-Toeplitz type in the sense that the C*-algebra generated by a family $\{L_e\}$ is generally the extension of a Cuntz-Krieger C*-algebra by the compact operators.

There is also a right regular representation $\rho_G : \mathbb{F}^+(G) \to \mathcal{B}(\mathcal{H}_G)$ determined by $G$, which yields partial isometries $\rho_G(w) \equiv R_w'$ for $w \in \mathbb{F}^+(G)$ acting on $\mathcal{H}_G$ by the equations $R_w' \xi_v = \xi_{vw}$, where $w'$ is the word $w$ in reverse order, with similar conditions. The
corresponding algebra is
\[ \mathfrak{A}_G = \text{wot-Alg} \{ R_e, R_x : e \in E(G), x \in V(G) \} \]
\[ = \text{wot-Alg} \{ \rho_G(w) : w \in \mathbb{F}^+(G) \}. \]

Given edges \( e, f \in E(G) \), observe that \( L_e R_f \xi_w = \xi_{ewf} = R_f L_e \xi_w \), for all \( w \in \mathbb{F}^+(G) \), so that \( L_e R_f = R_f L_e \), and similarly for the vertex projections. In fact, the commutant of \( \mathfrak{A}_G \) coincides with \( \mathfrak{L}_G = \mathfrak{A}'_G \) [19]. Further, the commutant of \( \mathfrak{L}_G \) coincides with \( \mathfrak{A}_G \), and thus \( \mathfrak{L}_G \) is its own second commutant.

A useful ingredient in the proof is the observation that the algebras \( \mathfrak{A}_G \) and \( \mathfrak{L}_G \) are naturally unitarily equivalent, where \( G^t \) is the transpose directed graph obtained from \( G \) by reversing the directions of all edges. An important technical device obtained here is the existence of Fourier expansions for elements of \( \mathfrak{L}_G \). Specifically, if \( A \in \mathfrak{L}_G \) and \( x \in V(G) \), then \( A \xi_x = \sum_{w \in \mathbb{F}^+(G)} a_w \xi_w \), with \( a_w \in \mathbb{C} \), and Cesaro type sums associated with the formal sum \( \sum_{w \in \mathbb{F}^+(G)} a_w L_w \), converge in the strong operator topology to \( A \). We write, \( A \sim \sum_{w \in \mathbb{F}^+(G)} a_w L_w \). As a notational convenience, for projections determined by vertices \( x \in V(G) \), we put \( P_x \equiv L_x \) and \( Q_x \equiv R_x \).

In [19] we proved that \( G \) is a complete unitary invariant of both \( \mathfrak{L}_G \) and \( \mathfrak{A}_G \). Thus, different directed graphs really do yield different algebras. We finish this section by discussing a number of examples from simple graphs.

**Examples 1.2.** (i) The algebra generated by the graph with a single vertex \( x \) and single loop edge \( e = xex \) is unitarily equivalent to the classical analytic Toeplitz algebra \( H^\infty \) [14, 16]. Indeed, the Hilbert space in this case may be naturally identified with the Hardy space \( H^2 \) of the unit disc, and under this identification \( L_e \) is easily seen to be unitarily equivalent to the unilateral shift \( U_+ \).

(ii) The noncommutative analytic Toeplitz algebras \( \mathfrak{L}_n \), \( n \geq 2 \) [2, 9, 10, 11, 20, 25, 26], the fundamental examples of free semigroup algebras, arise from the graphs with a single vertex and \( n \) distinct loop edges. For instance, in the case \( n = 2 \) with loop edges \( e = xex \neq f = xfx \), the Hilbert space is identified with unrestricted 2-variable Fock space \( \mathcal{H}_2 \). The operators \( L_e, L_f \) are equivalent to the natural creation operators on \( \mathcal{H}_2 \), also known as the Cuntz-Toeplitz isometries. Further, \( P_x = I \), and thus \( \mathfrak{L}_G \simeq \mathfrak{L}_2 \).

(iii) As an example of a simple matrix function algebra, we may consider the graph \( G \) with vertices \( x, y \) and edges \( e = xex, f = yfx \). Then \( \mathfrak{L}_G \) is generated by \( \{ L_e, L_f, P_x, P_y \} \). If we make the natural identifications \( \mathcal{H}_G = P_x \mathcal{H}_G \oplus P_y \mathcal{H}_G \simeq H^2 \oplus H^2 \) (respecting word length), then
\[ L_e \simeq \begin{bmatrix} U_+ & 0 \\ 0 & 0 \end{bmatrix}, \quad L_f \simeq \begin{bmatrix} 0 & 0 \\ U_+ & 0 \end{bmatrix}, \quad P_x \simeq \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad P_y \simeq \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}. \]

Thus, \( \mathfrak{L}_G \) is seen to be unitarily equivalent to
\[ \mathfrak{L}_G \simeq \begin{bmatrix} H^\infty & 0 \\ H_0^\infty & CI \end{bmatrix} \]
where \( H_0^\infty \) is the subalgebra of \( H^\infty \) functions \( h \) with \( h(0) = 0 \).

(iv) By simply adding a directed edge \( g = xgy \) to the previous graph, we obtain a very different algebra \( \mathfrak{L}_{G'} \). In fact, \( \mathfrak{L}_{G'} \) is a unitally partly free algebra in the sense of the
next section because it contains isometries with mutually orthogonal ranges; for instance, $U = L_e^2 + L_f L_g$ and $V = L_e L_g + L_g L_e$ are isometries which satisfy $U^*V = 0$.

(v) If $G$ is a finite graph with no directed cycles, then the Fock space $H_G$ is finite-dimensional and so too is $L_G$. As an example, consider the graph with three vertices and two edges, labelled $x_1, x_2, x_3, e, f$ where $e = x_2 e x_1, f = x_3 f x_1$. Then the Fock space is spanned by the vectors $\{\xi_{x_1}, \xi_{x_2}, \xi_{x_3}, \xi_e, \xi_f\}$ and with this basis the general operator $X = \alpha L_{x_1} + \beta L_{x_2} + \gamma L_{x_3} + \lambda L_e + \mu L_f$ in $L_G$ is represented by the matrix

\[
X \simeq \begin{bmatrix}
\alpha & \beta & \gamma \\
\lambda & \beta & \gamma \\
\mu & 0 & \gamma
\end{bmatrix}.
\]

Algebraically, $L_G$ is isometrically isomorphic to the so-called digraph algebra in $M_3(\mathbb{C})$ consisting of the matrices

\[
\begin{bmatrix}
\alpha & 0 & 0 \\
\lambda & \beta & 0 \\
\mu & 0 & \gamma
\end{bmatrix}.
\]

Recall that a digraph algebra $A(H)$ is a unital subalgebra of $M_n(\mathbb{C})$ which is spanned by some of the standard matrix units of $M_n(\mathbb{C})$. The graph $H$ is transitive and reflexive and is such that the edges of $H$ naturally label the relevant matrix units.

(vi) Let $n \geq 1$ and consider the cycle graph $C_n$ which has $n$ vertices $x_1, \ldots, x_n$ and $n$ edges $e_n = x_1 e_n x_n$ and $e_k = x_{k+1} e_k x_k$ for $k = 0, \ldots, n-1$. The cycle algebra $L_{C_n}$ may be identified with the wot-closed semicrossed product $\mathbb{C}^n \times^\beta \mathbb{Z}_+$ associated with the cyclic shift automorphism $\beta$ of $\mathbb{C}^n$ \[13\]. To see this identify $L_{x_i} H_G$ with $H^2$ for each $i$ in the natural way (respecting word length). Then $H_G = L_{x_1} H_G \oplus \cdots \oplus L_{x_n} H_G \simeq \mathbb{C}^n \otimes H^2$ and the operator $\alpha_1 L_{e_1} + \cdots + \alpha_n L_{e_n}$ is identified with the operator matrix

\[
\begin{bmatrix}
0 & & & & \alpha_n T_z \\
\alpha_1 T_z & 0 & & & \\
\alpha_2 T_z & 0 & \ddots & & \\
& \ddots & \ddots & \ddots & \\
& & & \alpha_{n-1} T_z & 0
\end{bmatrix}.
\]

Writing $H^\infty(\mathbb{C}^n)$ for the subalgebra of $H^\infty$ arising from functions of the form $h(\mathbb{C}^n)$ with $h$ in $H^\infty$, the algebra $L_{C_n}$ is readily identified with the matrix function algebra

\[
\begin{bmatrix}
H^\infty(\mathbb{C}^n) & z^{n-1} H^\infty(\mathbb{C}^n) & \cdots & z H^\infty(\mathbb{C}^n) \\
z H^\infty(\mathbb{C}^n) & H^\infty(\mathbb{C}^n) & \cdots & \\
& \ddots & \ddots & \\
z^{n-1} H^\infty(\mathbb{C}^n) & \cdots & H^\infty(\mathbb{C}^n)
\end{bmatrix}.
\]

This in turn is identifiable with the crossed product above. In fact, this matrix function algebra is the wot-closed variant of the matrix function algebra $B_n$ of De Alba and Peters \[13\] for the norm closed semicrossed product $\mathbb{C}^n \times^\beta \mathbb{Z}_+$. Such identifications are the Toeplitz versions of the identification of the graph $C^*$-algebra of $C_n$ with $M_n(C(T))$.\]
2. Partly Free Algebras

We say that a wot-closed operator algebra \( \mathfrak{A} \) is partly free if it contains the free semigroup algebra \( \mathfrak{L}_2 \) as a subalgebra in the sense of the following definition.

**Definition 2.1.** A wot-closed algebra \( \mathfrak{A} \) is partly free if there is an inclusion map \( \mathfrak{L}_2 \hookrightarrow \mathfrak{A} \) which is the restriction of an injection between the generated von Neumann algebras. If the map can be chosen to be unital, then \( \mathfrak{A} \) is said to be unitally partly free.

These notions parallel the requirement that a C*-algebra contain the Cuntz algebra \( \mathcal{O}_2 \), or that a discrete group contain a free group. Theorems 2.3 and 2.4 determine when the algebras \( \mathfrak{L}_G \) are partly free and unitally partly free for both finite and countable directed graphs. We require the following structural result on partial isometries in \( \mathfrak{L}_G \).

**Lemma 2.2.** [19] The initial projections of partial isometries \( V \) in \( \mathfrak{L}_G \) are sums of projections from \( \{P_x : x \in V(G)\} \). Specifically,

\[
V^*V = \sum_{x \in I} P_x \quad \text{where} \quad I = \{x \in V(G) : V\xi \neq 0\}.
\]

In fact, we proved much more in [19]. All partial isometries \( V \) in \( \mathfrak{L}_G \) satisfy a standard form, written as \( V = \sum_{x \in I} \oplus L_{\eta_x} \), where \( \{\eta_x\}_{x \in I} \) are unit wandering vectors for \( \mathfrak{R}_G \) supported on distinct \( Q_x \mathcal{H}_G \), and hence the initial projections satisfy \( L_{\eta_x}L_{\eta_x} = P_x \).

The cycle algebras \( \mathfrak{L}_{C_n}, 1 \leq n < \infty \), arise in the general theory, as does the algebra \( \mathfrak{L}_{C_\infty} \) generated by an infinite graph analogue of \( C_n \). Let \( C_\infty \) be the directed graph with vertex set \( \{x_k : k \geq 1\} \) indexed by the natural numbers, and edge set \( \{e_k = x_{k+1}e_kx_k : k \geq 1\} \). In the finite graph case, the cycle algebras \( \mathfrak{L}_{C_n} \) are the key examples of free semigroupoid algebras which are not partly free. However, \( \mathfrak{L}_{C_\infty} \) is unitally partly free, and this gives an indication of how pervasive these algebras are in the countable graph case.

**Lemma 2.3.** The algebras \( \mathfrak{L}_{C_n}, 1 \leq n < \infty \), are not partly free. However, \( \mathfrak{L}_{C_\infty} \) is unitally partly free.

**Proof.** We proved this result in [19] for \( 1 \leq n < \infty \) by showing \( \mathfrak{L}_{C_n} \) does not contain pairs of partial isometries \( U, V \) which satisfy condition \((iii)\) of Theorem 2.5. For \( n = \infty \), we shall construct a pair of isometries \( U, V \) in \( \mathfrak{L}_{C_\infty} \) with orthogonal ranges. For \( k \geq 1 \), let \( u_k \) and \( v_k \) be the unique finite paths in \( \mathcal{F}^+(C_\infty) \) with \( u_k = x_{2k}u_kx_k \) and \( v_k = x_{2k+1}v_kx_k \). Then we may define \( U \) and \( V \) by

\[
U = \sum_{k \geq 1} \oplus L_{u_k} \quad \text{and} \quad V = \sum_{k \geq 1} \oplus L_{v_k},
\]

where the sums converge in the strong operator topology. A unital injection of \( \mathfrak{L}_2 \) into \( \mathfrak{L}_{C_\infty} \) is then defined by mapping the two generators of \( \mathfrak{L}_2 \) to \( U \) and \( V \). 

In fact it turns out there are two ways in which an algebra \( \mathfrak{L}_G \) can be partly free; namely \( G \) must either contain a double-cycle or a proper infinite (directed) path, in the sense of the following definition.

**Definition 2.4.** A cycle at vertex \( x \) is a path of edges \( w = xwx \) for which only the initial edge has source vertex \( x \). The graph \( G \) contains a double-cycle if there are distinct cycles.
\[ w_i = xw_ix, \; i = 1, 2, \text{ at some vertex } x \text{ in } G. \] 
By a proper infinite (directed) path in \( G \), we mean an infinite word \( \omega = \cdots e_{i_3}e_{i_2}e_{i_1} \) in the edges of \( G \) such that no edges are repeated and every finite segment corresponds to an allowable finite directed path in \( G \).

A little thought shows that the only way a graph \( G \) fails to have either a double-cycle or such a proper infinite path is if either there are no infinite paths at all, or, if every infinite path in the graph becomes periodic; in this case say that \( G \) has the periodic path property. If, on the other hand, \( G \) does not have the periodic path property, we shall say that \( G \) has the aperiodic path property meaning that there exists an aperiodic infinite path. This in turn means that there exists a proper infinite path or a double-cycle.

We define the saturation at a vertex \( x \) in \( G \) to be the set, \( \text{sat}(x) \), that consists of \( x \), together with all finite and infinite paths which start at \( x \), and all vertices that are final vertices for paths starting at \( x \). Consider a graph which is a downward directed tree in which every branch has finite length and different branches may intersect. Also perform the following surgery: add arbitrary cycles to some vertices without creating double-cycles.

Note there can be infinite branchings. This ‘aperiodic looped tree graph’ does not have the aperiodic path property and furthermore such a graph is a ‘fairly typical’ graph without that property. In fact, \( G \) does not have the aperiodic path property if and only if the saturation graph for each vertex has this kind of structure.

For the sake of brevity in the next proof we shall assume the finite vertex case \([19]\).

**Theorem 2.5.** The following assertions are equivalent for a finite or countable directed graph \( G \):

(i) \( G \) has the aperiodic path property.

(ii) \( \mathcal{L}_G \) is partly free.

(iii) There are nonzero partial isometries \( U, V \in \mathcal{L}_G \) with

\[ U^*U = V^*V, \; UU^* \leq U^*U, \; VV^* \leq V^*V, \; U^*V = 0. \]

**Proof.** Since von Neumann algebra isomorphisms are spatial, condition (iii) is a reformulation of (ii). Thus it suffices to prove the equivalence of (i) and (iii).

To see \( (iii) \implies (i) \), first suppose \( U \) and \( V \) are partial isometries in \( \mathcal{L}_G \) satisfying (iii) with \( U^*U = P_x = V^*V \) for some \( x \in V(G) \). Then there is at least one cycle \( w = xwx \) over \( x \), for otherwise \( U = P_xUP_x = 0 \) or \( U = P_x \) and \( 0 = P_xU = V \). Suppose \( w \) is the only cycle over \( x \). Then the compression algebra \( P_x\mathcal{L}_GP_x \) forms the wot-closed subalgebra of \( \mathcal{L}_G \) consisting of elements \( X \in \mathcal{L}_G \) with Fourier expansions of the form \( X \sim \sum_{k \geq 0} a_kL_{w^k} \). Let \( \mathcal{H}_x = P_x\mathcal{H}_G \). Then \( P_x\mathcal{L}_G|_{\mathcal{H}_x} \) is evidently unitarily equivalent to \( \mathcal{L}_{\mathcal{C}_1} \simeq H^\infty \). But \( P_x\mathcal{L}_G|_{\mathcal{H}_x} \) contains a pair of non-zero partial isometries \( U|_{\mathcal{H}_x} = P_xU|_{\mathcal{H}_x}, V|_{\mathcal{H}_x} = P_xV|_{\mathcal{H}_x} \) satisfying condition (iii), and this contradicts Lemma \([2.3]\). Thus, we deduce the existence of at least two cycles over \( x \), and hence \( G \) contains a double-cycle.

For the general case, let \( U \) and \( V \) be partial isometries in \( \mathcal{L}_G \) satisfying (iii) with \( U^*U = V^*V = \sum_{x \in \mathcal{I}} P_x \) as in Lemma \([2.2]\). Suppose, by way of contradiction, that (i) fails so that \( G \) has the structure discussed above. The vertices \( x \) in the index set \( \mathcal{I} \) are scattered over this graph. Plainly, either there exists such an \( x \) with no other \( y \) in \( \mathcal{I} \) in the saturation of \( x \) or there is a cycle \( C \) containing several such edges such that the saturation of \( C \) contains no further vertices \( x \) in \( \mathcal{I} \). In the first case we can consider the compression of the given \( U \).
and $V$ to $\mathcal{H}_x$, and in the second case the compression to $\mathcal{H}_C = \sum_{x \in \mathcal{V}(C)} P_x \mathcal{H}_G$. Observe that the saturations of $x$ and $C$ may be infinite. However, the conditions in $(iii)$ allow us to see that these compressions are indeed partial isometries because the hypothesis implies that the compression is the same (modulo a zero summand) as the restriction to the subspace $\mathcal{L}_G \mathcal{H}_x$ or $\mathcal{L}_G \mathcal{H}_C$. Now we get our contradiction by arguing as in the previous paragraph. Indeed, the first case is trivial and in the second case the compression of the algebra is unitarily equivalent to a cycle graph algebra $\mathcal{L}_c$.

Towards the implication $(i) \Rightarrow (iii)$, suppose $\omega$ is a proper infinite path in $G$. Let $\mathcal{J}$ be the set of all vertices in $G$ for which the saturation at each $x \in \mathcal{J}$ contains part, and hence an entire tail, of $\omega$. Let $\{x_k, y_k\} \rightarrow \{z_k\}$ be a two-to-one map from the countable set $\mathcal{J}$ onto itself, such that the vertices $x_k, y_k \in \mathcal{J}$ belong to the saturation of $z_k \in \mathcal{J}$. Thus, there are distinct finite paths $u_k = x_k u_k z_k, v_k = y_k v_k z_k$. By design, $U = \sum_k \oplus L_{u_k}, V = \sum_k \oplus L_{v_k}$ are partial isometries in $\mathcal{L}_G$ with orthogonal ranges, initial projection $U^* U = P_{\mathcal{J}} = V^* V$, and $U = P_{\mathcal{J}} U P_{\mathcal{J}}, V = P_{\mathcal{J}} V P_{\mathcal{J}}$.

Next, suppose distinct cycles $w_1, w_2$ form a double-cycle over a vertex $x$ in $G$. Let $\mathcal{J}$ be the set of all vertices for which the saturation at each vertex in $\mathcal{J}$ contains this double-cycle. Let us enumerate the (possibly finite) vertices of $\mathcal{J}$ as $\{x_k\}_k$, and let $r_k = x r_k x$ be a path from $x_k$ to $x$. For each vertex $x_k$ in $\mathcal{J}$ choose, without repeating any choices, two paths $u_k, v_k$ amongst the set $\{w_1^m w_2 : m \geq 1\}$. Then again by design, $U = \sum_k \oplus L_{u_k r_k}, V = \sum_k \oplus L_{v_k r_k}$ are partial isometries in $\mathcal{L}_G$ with orthogonal ranges, initial projection $U^* U = P_{\mathcal{J}} = V^* V$, and $U = P_{\mathcal{J}} U P_{\mathcal{J}}, V = P_{\mathcal{J}} V P_{\mathcal{J}}$. Therefore, in both cases we have shown that $(i) \Rightarrow (iii)$, and this completes the proof.

It follows from this result that the free semigroupoid algebras coming from countable graphs are typically partly free. Indeed, the condition on a countable graph $G$ which forces $\mathcal{L}_G$ to not be partly free is quite restrictive as outlined in the previous discussion. Below we discuss a number of examples.

The following is the unital version of the previous theorem. We shall say $G$ has the uniform aperiodic path property if the saturation at every vertex includes an aperiodic infinite path.

**Theorem 2.6.** The following assertions are equivalent for a finite or countable directed graph $G$:

(i) $G$ has the uniform aperiodic path property.

(ii) $\mathcal{L}_G$ is unitally partly free.

(iii) There are isometries $U, V$ in $\mathcal{L}_G$ with

$$U^* V = 0.$$ 

**Proof.** Once again, condition $(iii)$ is a restatement of $(ii)$, so it suffices to prove the equivalence of $(i)$ and $(iii)$.

For $(iii) \Rightarrow (i)$, the proof of Theorem 2.5 can be adapted to show that the saturation at every vertex in the index set of vertices $\mathcal{I}$, determining the initial projection for the partial isometries $U, V$, includes a double-cycle or a proper infinite path. Hence we may apply this argument in the current case with $\mathcal{I} = \mathcal{V}(G)$ as we are dealing with isometries...
here, \( U^*U = V^*V = I = \sum_{x \in V(G)} P_x \), and it follows that \( G \) has the uniform aperiodic path property.

To see \((i) \Rightarrow (iii)\), consider the last two paragraphs in the proof of Theorem 2.3. As \( G \) satisfies the uniform aperiodic path property, it follows that we may decompose the vertex set for \( G \) into disjoint subsets \( V(G) = \bigcup_i J_i \), where each \( J_i \) is obtained as in one of these two cases; double-cycles, or proper infinite paths. In either case we can define partial isometries \( U_i, V_i \) in \( \mathcal{L}_G \) with orthogonal ranges, \( U_i = P_{J_i}U_iP_{J_i} \), \( V_i = P_{J_i}V_iP_{J_i} \), and \( U_i^*U_i = P_{J_i} = V_i^*V_i \). Thus, the operators \( U = \sum_i \oplus U_i \), \( V = \sum_i \oplus V_i \) are isometries in \( \mathcal{L}_G \) with mutually orthogonal ranges.

We next add to the short list of known hyper-reflexive algebras \([3, 5, 7, 8, 10]\) by extending our result from \([19]\) to the case of countable graphs. Given an operator algebra \( \mathfrak{A} \), a measure of the distance to \( \mathfrak{A} \) is given by

\[
\beta_{\mathfrak{A}}(X) = \sup_{L \in \text{Lat } \mathfrak{A}} ||P_L^*X P_L||,
\]

where \( P_L \) is the projection onto the subspace \( L \) and \( \text{Lat } \mathfrak{A} \) is the lattice of invariant subspaces for \( \mathfrak{A} \). Evidently, \( \beta_{\mathfrak{A}}(X) \leq \text{dist}(X, \mathfrak{A}) \), and the algebra \( \mathfrak{A} \) is said to be hyper-reflexive if there is a constant \( C \) such that \( \text{dist}(X, \mathfrak{A}) \leq C \beta_{\mathfrak{A}}(X) \) for all \( X \).

The free semigroup algebras \( \mathcal{L}_n \) were proved to be hyper-reflexive by Davidson \((n = 1 \ [8]\) and Davidson and Pitts \((n \geq 2 \ [10]\)). Furthermore, motivated by the \( \mathcal{L}_n \) case Bercovici \([5]\) proved an algebra is hyper-reflexive with distant constant no greater than 3 whenever its commutant contains a pair of isometries with orthogonal ranges.

**Corollary 2.7.** Let \( G \) be a finite or countable directed graph such that the transpose graph \( G^t \) satisfies the uniform aperiodic path property; equivalently, \( \mathcal{L}_{G^t} \) is unitally partly free. Then \( \mathcal{L}_G \) is hyper-reflexive with distant constant at most 3.

**Proof.** This is a direct consequence of Bercovici’s result \([5]\) since \( \mathcal{L}_G' = \mathcal{R}_G \simeq \mathcal{L}_{G^t} \).

**Problem 2.8.** Is \( \mathcal{L}_G \) hyper-reflexive for every directed graph \( G \)?

We next present some simple examples for the countable graph case. The focus will be on the new aspect discovered here; the relevance of proper infinite paths.

**Examples 2.9.** (i) Let \( G \) be the directed graph with vertices \( \{x_k : k \in \mathbb{Z}\} \) indexed by the integers and directed edges \( \{e_k = x_{k+1}e_kx_k : k \in \mathbb{Z}\} \). We could also add (possibly infinite) directed paths \( w_k = x_kw_k \). Then every vertex saturation in \( G \) contains a proper infinite tail \( \omega_k = \cdots e_{k+1}e_k \) for some \( k \in \mathbb{Z} \). Thus \( G \) satisfies the uniform aperiodic path property, and \( \mathcal{L}_G \) is unitally partly free.

Consider the interesting special case that occurs when all the paths \( w_k = x_kw_kx_k \) are loop edges. Evidently \( G^t \) and \( G \) are isomorphic, hence the commutant \( \mathcal{L}_G' = \mathcal{R}_G \simeq \mathcal{L}_{G^t} \simeq \mathcal{L}_G \) is unitarily equivalent to \( \mathcal{L}_G \) and is also unitally partly free. As a variation of this case, instead let \( H \) be the subgraph \( H = \{x_k, e_k, w_k : k \geq 1\} \). Then \( \mathcal{L}_H \) is unitally partly free, but \( \mathcal{L}_H' \simeq \mathcal{L}_{H^t} \) is not even partly free.

(ii) A non-discrete example is given by the graph \( Q \) consisting of vertices \( \{x_q : q \in \mathbb{Q}\} \) indexed by the rational numbers, and directed edges \( e_{qp} = x_qe_{qp}x_p \) whenever \( p \leq q \). This example satisfies the uniform aperiodic path property, in fact there is an abundance of
infinite non-overlapping directed paths emanating from each vertex, so \( \mathcal{L}_Q \) is unitally partly free. However, notice that the quiver algebra \( \mathcal{A}_G \) is not even partly free (see Section 2.1). Further note that \( Q^t \) is graph isomorphic to \( Q \), thus \( \mathcal{L}_Q \simeq \mathcal{L}_Q^t \simeq \mathcal{L}_Q' \) is unitarily equivalent to its commutant.

(iii) The following example was suggested to us by Ken Davidson. For \( n \geq 2 \), let \( \mathbb{F}_n^+ \) be the unital free semigroup on \( n \) noncommuting letters, written as \( \{1, 2, \ldots, n\} \), with unit \( \phi \). Let \( G_n \) be the doubly-bifurcating (sideways) infinite tree with vertices \( \{x_w : w \in \mathbb{F}_n^+\} \) indexed by words in \( \mathbb{F}_n^+ \), and directed edges \( \{e_{iw} = x_{iw}e_{iw}x_w : w \in \mathbb{F}_n^+, 1 \leq i \leq n\} \) determined by the directions \( w \mapsto iw \). Then \( G_n \) satisfies the uniform aperiodic path property (observe that \( G_1 = C_\infty \)), and hence \( \mathcal{L}_{G_n} \) is unitally partly free. An interesting point here is that the graph \( G_n \) itself has the structure of the full Fock space Hilbert space \( \ell^2(\mathbb{F}_n^+) \) traced out by its left creation operators. Whereas, the Fock space \( \mathcal{H}_{G_n} \) consists of infinitely many disjoint infinite-dimensional components, indexed by elements of \( \mathbb{F}_n^+ \). The transpose graph \( G_n^t \) is quite different from \( G_n \). In fact the commutant algebra \( \mathcal{L}_{G_n^t} \simeq \mathcal{H}_{G_n}^t = \mathcal{L}'_{G_n} \) is not partly free.

(iv) Let \( G \) be the directed graph with vertices \( \{x_k : k \geq 1\} \) and edges \( \{e_k = x_ke_kx_1 : k \geq 1\} \). This graph has no infinite paths or double-cycles, hence \( \mathcal{L}_G \) is not partly free. A variation of this example, turning the \( e_k \) into non-overlapping paths of length \( k \) that only intersect at vertex \( x_1 \), produces a non-partly free \( \mathcal{L}_G \) with graph containing non-overlapping finite paths of arbitrarily large length.

(v) Let \( G = \{x_1, x_2, e_k = x_2e_kx_1 : k \geq 1\} \) with \( e_k \) distinct edges. Then \( \mathcal{L}_G \) is not partly free and is unitarily equivalent to its commutant \( \mathcal{L}_G' \simeq \mathcal{L}_G^t \).

(vi) Let \( G \) be the directed graph with vertices \( \{x_k : k \in \mathbb{Z}\} \) and directed edges \( \{e_k : k \in \mathbb{Z}\} \) where

\[
e_k = \begin{cases} x_{2m+1}e_kx_{2m} & \text{if } k = 2m \\ x_{2m+1}e_kx_{2m+2} & \text{if } k = 2m+1. \end{cases}
\]

Then \( \mathcal{L}_G \) is not partly free and is unitarily equivalent to its commutant \( \mathcal{L}_G' \simeq \mathcal{L}_G^t \).

2.1. Partly Free Quiver Algebras. Using Theorems 2.5 and 2.6 we may readily deduce graph-theoretic conditions for quiver algebras \( \mathcal{A}_G \) to be partly free. We require the following structural result for partial isometries in \( \mathcal{A}_G \) for the countable graph case. An immediate consequence is that \( \mathcal{A}_G \) only contains isometries when \( G \) has finitely many vertices.

**Lemma 2.10.** Let \( G \) be a countable directed graph. If \( V \) is a partial isometry in \( \mathcal{A}_G \), then its initial projection \( V^*V = \sum_{x \in \mathcal{I}} P_x \) is the sum of only finitely many \( P_x \).

**Proof.** In fact, if \( V \) is a partial isometry in \( \mathcal{L}_G \) for which \( \mathcal{I} \) is an infinite set, then dist\((V, \mathcal{A}_G) \geq 1. Indeed, let \( q(L) \) belong to the set of polynomials \( A = \text{Alg}\{L_w : w \in \mathbb{F}^+(G)\} \) in the \( L_e \) and \( L_x = P_x \). As \( \mathcal{I} \) is infinite, there is a \( y \in \mathcal{I} \) such that \( q(L)P_y = 0 \). Hence \( 1 = \|VP_y\| = \|VP_y - q(L)P_y\| \leq \|V - q(L)\|, \) and this proves the claim because \( A \) is (norm) dense in \( \mathcal{A}_G \). The lemma follows since \( \mathcal{L}_G \) contains \( \mathcal{A}_G \).

Let \( \mathcal{A}_2 \) be the quiver algebra generated by the graph with a single vertex and two distinct loop edges. This is the noncommutative disc algebra of Popescu [26, 27], also considered by Arias [1], and Muhly-Solel [23, 24]. Say that \( \mathcal{A}_G \) is partly free, or unitally partly free, if the maps in Definition 2.1 are injections of \( \mathcal{A}_2 \) into \( \mathcal{A}_G \), and are restrictions of injections of the generated \( \mathcal{C}^* \)-algebras.
We say that a graph $G$ has the double-cycle property if $G$ contains a double-cycle, and $G$ satisfies the uniform double-cycle property when every vertex saturation $\text{sat}(x)$ includes a double-cycle. Compare the following results with Theorems 2.5 and 2.6 and notice how the proper infinite path phenomena only arises in the wot-closed case. In particular, there are many examples in the countable graph case for which $L_G$ is partly free, but $A_G$ is not.

**Theorem 2.11.** The following assertions are equivalent for a finite or countable directed graph $G$:

(i) $G$ has the double-cycle property.
(ii) $A_G$ is partly free.
(iii) There are nonzero partial isometries $U, V$ in $A_G$ with

$$U^*U = V^*V, \quad UU^* \leq U^*U, \quad VV^* \leq V^*V, \quad U^*V = 0.$$ 

**Proof.** As in Theorem 2.5 it suffices to establish the equivalence of (i) and (iii). But (i) $\Rightarrow$ (iii) is clear since $A_G$ will include $U = L_{w_1}, V = L_{w_2}$, where $w_1, w_2$ are distinct cycles over a common vertex, when (i) holds. On the other hand, as $L_G$ contains $A_G$, condition (iii) implies $G$ satisfies the aperiodic path property by Theorem 2.5. But recall from the proof of (iii) $\Rightarrow$ (i) in Theorem 2.5 that the proper infinite path part of this property can only occur when the initial vertex set $I$ is infinite. Hence by Lemma 2.10, $G$ contains a double-cycle and (i) holds. \[\blacksquare\]

In the unital case, the graph can only have finitely many vertices.

**Theorem 2.12.** The following assertions are equivalent for a finite or countable directed graph $G$:

(i) $G$ has finitely many vertices and satisfies the uniform double-cycle property.
(ii) $A_G$ is unitally partly free.
(iii) There are isometries $U, V$ in $A_G$ with

$$U^*V = 0.$$ 

**Proof.** This result follows from Theorem 2.6 as the previous result follows from Theorem 2.5, other than the extra vertex condition on $G$. In particular, (iii) implies $G$ satisfies the uniform infinite path property, but since $U, V$ are isometries the initial vertex set in this case is the entire vertex set of $G$, that is $I = V(G)$. Therefore, $G$ can only have finitely many vertices by Lemma 2.11 and must satisfy the uniform double-cycle property. \[\blacksquare\]

**Acknowledgements.** We would like to thank Ken Davidson for organizing a workshop on nonselfadjoint operator algebras at the Fields Institute in Toronto (July 2002), where the authors had a number of productive conversations. The first named author would also like to thank members of the Department of Mathematics at Purdue University for kind hospitality during preparation of this article.

**Note Added in Proof.** Problem 2.8 has been answered in the affirmative for all finite graphs \[17\]. The problem remains open for general $L_G$. We also mention that the ideal structure of $L_G$ has been analyzed in \[18\]. Further, Ephrem \[15\] has recently identified graph conditions
which characterize when a Cuntz-Krieger graph $C^*$-algebra has type I representation theory. Interestingly, the condition he obtains for finite graphs is equivalent to the graph not having any double-cycles.

**References**

[1] A. Arias, *Multipliers and representations of noncommutative disc algebras*, Houston J. Math., **25** (1999), 99-120.

[2] A. Arias, G. Popescu, *Factorization and reflexivity on Fock spaces*, Int. Equat. Oper. Th. **23** (1995), 268–286.

[3] W. Arveson, *Interpolation problems in nest algebras*, J. Func. Anal. **20** (1975), 208–233.

[4] T. Bates, J. Hong, I. Raeburn, W. Szymenski, *The ideal structure of the $C^*$-algebras of infinite graphs*, e-print arxiv.math.OA/0109142 preprint, 2001.

[5] H. Bercovici, *Hyper-reflexivity and the factorization of linear functionals*, J. Func. Anal. **158** (1998), 242–252.

[6] A. Beurling, *On two problems concerning linear transformations in Hilbert space*, Acta Math. **81** (1949), 239–255.

[7] E. Christensen, *Perturbations of operator algebras II*, Indiana U. Math. J. **26** (1977), 891–904.

[8] K.R. Davidson, *The distance to the analytic Toeplitz operators*, Illinois J. Math. **31** (1987), 265–273.

[9] K.R. Davidson, E. Katsoulis, D.R. Pitts, *The structure of free semigroup algebras*, J. reine angew. Math. **533** (2001), 99-125.

[10] K.R. Davidson, D.R. Pitts, *Invariant subspaces and hyper-reflexivity for free semi-group algebras*, Proc. London Math. Soc. **78** (1999), 401–430.

[11] K.R. Davidson, D.R. Pitts, *The algebraic structure of non-commutative analytic Toeplitz algebras*, Math. Ann. **311** (1998), 275–303.

[12] K.R. Davidson, *Nest Algebras*, Longman Scientific & Technical, London, 1988.

[13] L.M. De Alba, J. Peters, *Classification of semicrossed products of finite-dimensional $C^*$-algebras*, Proc. Amer. Math. Soc. **95** (1985), 557-564.

[14] R. Douglas, *Banach algebra techniques in operator theory*, Springer-Verlag, New York, 1998.

[15] M. Ephrem, *Characterizing liminal and type I graph $C^*$-algebras*, arXiv:math.OA/0211241 preprint, 2003.

[16] K. Hoffmann, *Banach spaces of analytic functions*, Dover Publications Inc., New York, 1988.

[17] F. Jaeck, S.C. Power, *The semigroupoid algebras of finite graphs are hyper-reflexive*, in preparation, 2003.

[18] M.T. Jury, D.W. Kribs, *Ideal structure in free semigroupoid algebras from directed graphs*, preprint, 2003.

[19] D.W. Kribs, S.C. Power, *Free semigroupoid algebras*, preprint, 2002.

[20] D.W. Kribs, *Factoring in non-commutative analytic Toeplitz algebras*, J. Operator Theory **45** (2001), 175-193.

[21] A. Kumjian, D. Pask, I. Raeburn, *Cuntz-Krieger algebras of directed graphs*, Pacific J. Math **184** (1998), 161-174.

[22] A. Kumjian, D. Pask, I. Raeburn, J. Renault, *Graphs, Groupoids, and Cuntz-Krieger algebras*, J. Funct. Anal. **144** (1997), 505-541.

[23] P.S. Muhly, *A finite dimensional introduction to operator algebra*, A. Katavolos (ed.), Operator Algebras and Applications, 313-354, Kluwer Academic Publishers, 1997.

[24] P.S. Muhly, B. Solel, *Tensor algebras, induced representations, and the Wold decomposition*, Can. J. Math. **51** (4), 1999, 850-880.

[25] G. Popescu, *Multi-analytic operators and some factorization theorems*, Indiana Univ. Math. J. **38** (1989), 693-710.

[26] G. Popescu, *Multi-analytic operators on Fock spaces*, Math. Ann. **303** (1995), 31–46.

[27] G. Popescu, *Noncommuting disc algebras and their representations*, Proc. Amer. Math. Soc. **124** (1996), 2137–2148.
[28] S.C. Power, *Approximately finitely acting operator algebras*, J. Func. Anal. **189** (2002), 409-469.

Addresses:

**DEPARTMENT OF MATHEMATICS AND STATISTICS**

**UNIVERSITY OF GUELPH**

**GUELPH, ONTARIO**

**CANADA N1G 2W1**

**DEPARTMENT OF MATHEMATICS AND STATISTICS**

**LANCASTER UNIVERSITY**

**LANCASTER, ENGLAND**

**UK LA1 4YW**

E-mail addresses:

kribs@math.purdue.edu

s.power@lancaster.ac.uk

2000 Mathematics Subject Classification. 47L55, 47L75.