Statistical Topology of Real Polynomials.

I: Two Variables

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Abstract

We investigate average gradient degree of normal random polynomials of fixed algebraic degree \( n \). In particular, for polynomials of two variables, asymptotics of the average gradient degree for large values of \( n \) is determined.
1 Introduction

Since the seminal work of M.Kac, random polynomials became an object of permanent study in mathematics and physics. For polynomials of one variable, several important numerical characteristics, including average real roots number, are already understood well enough.

Much less is known about random polynomials of several variables. In fact, only recently an important breakthrough was made by M.Shub and S.Smale, who have found a suitable probabilistic setting in higher dimensions and, in particular, computed the average real roots number having thus generalized the aforementioned classical results of M.Kac. Their results were, in turn, clarified and developed by E.Kostlan. The state of arts field is described in a review paper of A.Edelman and E.Kostlan.

It should be emphasized that despite these important contributions, many natural problems for random polynomials of several variables still remain completely uninvestigated. Up to the authors’ knowledge, there are no investigations of numerical invariants specific for higher dimensions, such as Betti numbers of level surfaces or topological mapping degree. At the same time, random polynomials inevitably appear in various mathematical and physical problems of intrinsically multidimensional nature, where consideration of such characteristics becomes urgent. This direction of research may be (and in our exposition actually will be) called statistical topology of random polynomials, in accordance with the title of this text. As usual, an important aspect of such investigations is concerned with asymptotical behaviour of statistical characteristics.

In the present note we are mainly interested in gradient degree (topological degree of gradient mapping), which is perhaps the simplest and most visual topological characteristic of multidimensional polynomials. As is well known, gradient degree plays an important role in the qualitative study of polynomial topology. In particular, it enables one to compute certain topological invariants of level surfaces. For example, from results of it follows readily that knowledge of the average of gradient degree enables one to compute the average Euler characteristic of level surfaces of random polynomials.

In other words, the average gradient degree definitely belongs to the core of (the just christened discipline of) statistical topology and it seemed appropriate for us to start with its investigation in the simplest case of two variables.

Our concrete aim was to investigate the asymptotics of average of modulus of gradient degree (which we will call simply average absolute degree) for certain natural distributions of coefficients of a random real polynomial in two variables and below we describe some concrete results in this direction.

Actually, we work in a wider context of computing the average absolute topological degree of a random polynomial mapping defined by a pair of random polynomials with independent identically distributed (i.i.d.) coefficients. Since topological degree is nothing else than the algebraic number of real solutions (counted with weights equal to the signs of Jacobian determinant) of the corresponding system of algebraic equations, our results may be considered as complementary to those about the average real roots number obtained by S.Shub, S.Smale and E.Kostlan. In this note we only present first results in this direction, leaving their development and applications for further publications. Also, no attempt is made to provide formally exhaustive proofs because we are going to present full proofs of more general results in the forthcoming paper. As a rule, we only outline methods and indicate crucial points of proofs, to the extent hopefully sufficient for making it clear why those results are true. Although the whole setting was suggested by
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2 General setting

We will now describe a method of computing $E(\text{deg grad } |)$ for most natural distributions of coefficients. Having in view physical applications we mainly deal with the case when coefficients are independent standard normal random variables. For small values of algebraic degree of polynomials we will compute $E(\text{deg grad } |)$ explicitly. We also reformulate related probabilistic problems in terms of random walks and formulate a general hypothesis on asymptotics of the average gradient degree for large values of algebraic degree of random polynomials under consideration.

We start by considering polynomials in two indeterminates. Thus let $F_n$ be the ring of real polynomials in two indeterminates of the algebraic degree $n$. Every $P \in F_n$ may be written in the standard form $P(x, y) = \sum_{k+l=0}^{n} a_{kl} x^k y^l$ with $k, l \in \mathbb{Z}_+$. As usual, the gradient of such polynomial is a couple of polynomials $\text{grad } P = (\partial P/\partial x, \partial P/\partial y)$. Sometimes we will denote it by $P'$ and identify with the corresponding polynomial endomorphism of $\mathbb{R}^2$.

Recall that if $\text{grad } P$ is a proper mapping, i.e. full preimage of every compact set is compact, then the topological degree $\text{deg grad } P$ is well defined as the algebraic number of preimages of any regular value of the gradient mapping \[1\]). As is well known, the property of being proper is generic. In other words, polynomials with non-proper gradients are “rare” and can be ignored in any probabilistic considerations on $F_n$. More precisely, such ”bad” polynomials constitute an algebraic subset, hence a subset of measure zero in $F_n$ \[4\].

Consider now an $F_n$-valued random variable $P(\omega)$ with $a_{kl}(\omega)$ certain real valued random variables. We will always assume that those random variables are (stochastically) independent \[3\] (e.g. i.i.d. coefficients). Often $a_{kl}(\omega)$ will be taken to be central normals (Gaussian random variables with zero mean and given covariance matrix \[5\]) and then $P(\omega)$ is called simply a normal random polynomial.

For almost all values of $\omega$ one obtains a proper gradient mapping with the well defined topological degree and we are interested in statistical properties of the corresponding random variable $\text{gd } P(\omega) = \text{deg grad } P(\omega)$. Of course, its average vanishes for trivial symmetry reasons but it is the average of its modulus which is really important in applications. Thus we denote the latter by $E(\text{agd } P)$ or $\bar{\text{agd }} P$ and try to compute it explicitly, at least for some specially chosen distributions of coefficients. Then it becomes possible to determine its asymptotical behaviour and there is some hope that the latter remains valid for a wider variety of situations, as often happens with asymptotical results.

It turns out that we can perform this programme even in a more general context of average topological degree $E(\text{deg } f(\omega))$ for certain normal random polynomial endomorphisms of $\mathbb{R}^2$ defined by a pair of i.i.d. random polynomials $(P(\omega), Q(\omega))$, and this is what we are going to describe in the sequel. Results about $\text{agd}$ will appear as corollaries.
We want to simplify our task by reducing this problem to consideration of random binary forms, that is random homogeneous polynomials in two indeterminates with independent distributions of coefficients. The possibility of such reduction follows from a simple topological lemma. Recall that the leader $P^*$ of a polynomial $P \in \mathbb{F}_n$ is defined to be the binary form equal to the sum of monomials of the highest degree actually entering into $P$. Of course, its degree may be less than $n$. Recall that a polynomial endomorphism $(p, Q)$ of the plane is named non-degenerate, if the two leaders $P^*, Q^*$ have no non-trivial common zeroes. It is evident that non-degeneracy is a "generic" property so in computations of averages one can deal only with non-degenerate pairs $(P, Q)$.

Lemma 1. If $f = (P, Q)$ is a non-degenerate polynomial endomorphism of the plane then $\deg f = \deg f^*$, where $f^* = (P^*, Q^*)$.

The proof is very simple. First of all, a non-degenerate mapping is proper, which follows from the well-known fact that the sum of moduli of leaders of a non-degenerate endomorphism majorates terms of lower degree on circles of large radii. Thus the degree is well defined. Moreover, the conclusion follows from the same fact by application of Rouché principle.

As was noticed, while computing $E(\deg)$ we can exclude degenerate mappings and then the degree is always equal to that of the leader. Also, the set of those polynomials with leaders having not the maximal possible degree has the measure zero, so we can consider only pairs of polynomials with leaders both having the degree $n$. Thus we see that it is sufficient to solve the problem for random binary forms. In fact, all what was said refers to any reasonable distributions of coefficients. In further analysis we restrict ourselves with three types of independent continuous distributions of coefficients, which already appeared in previous works on random polynomials:

1) standard normals $\mathcal{N}$;
2) central normals with special covariance matrix from $\mathbb{E}$;
3) i.i.d. coefficients uniformly distributed on $(-1,1)$.

All of them have common features with respect to our issue and results on asymptotics of average degree should be similar. Thus we will mainly deal with the case of independent standard normals and briefly mention necessary changes in other cases. Similar results may be also obtained for various discrete distributions of coefficients, where the situation is even simpler. In fact, for certain simplest discrete central distributions, such as symmetric $\{-1,1\}$-valued coin, the problem becomes purely combinatorial and for small values of $n$ may be solved explicitly with the aid of computer but we will not dwell on these elementary versions.

From now on we concentrate on binary forms and perform further simplifications. It turns out helpful to introduce a natural subdivision of the space of events according to the number of real zeroes of binary forms considered.

Again, by "genericity" arguments it is clear that our forms can be supposed to have only simple zeroes. Evidently, the number of these zeroes can be an arbitrary integer $r$ not exceeding $n$ and having the same parity as $n$. We denote by $A_{rs}$ such subset of events when the first leader has precisely $r$ real zeroes and the second one has $s$ real zeroes.

From the very definition of the average it is clear that it is equal to the weighted sum of averages over subsets $A_{rs}$ with the weights equal to probabilities of subsets $A_{rs}$. Thus our task may be performed in two steps:

1. compute probabilities of events $A_{rs}$;
2. compute the average $E(\deg)$ with the respect to subset $A_{rs}$.
The first step is a purely geometrical problem and reduces to estimation of certain integrals over multidimensional domains appearing as components of complement to the discriminant hypersurface \([1]\). For small values of \(n\) these computations may be done explicitly. In general this is a difficult analytical problem \([13]\). Luckily, it turns out that precise values of these probabilities are not important in studying asymptotics of the average degree, as will be explained below.

Thus, it is the second step which we develop here in some detail. Our main tool is combinatorics arising from a natural analogy of the problem with random walks on the real line. The latter issue is well studied and necessary combinatorial results together with asymptotical analysis may be successfully borrowed from the theory of random walks. We benefit from this circumstance in the next section.

Before proceeding with combinatorics we would like to point out that appearance of one-dimensional random walks in our problem is not occasional and may be illustrated by the following heuristical argument. Recent results of \([13]\) suggest that in certain cases the most probable number of common real zeroes of the random pair \((P, Q)\) is \(n\). This suggests that one could restrict himself with only such pairs and try to estimate average degree over this "representative" subset of events. In doing so it is reasonable to suggest that at every root the probability of any of two possible values of the sign of Jacobian \(J(P, Q)\) is one half.

Of course, these probabilistical assumptions should be justified but accepting them for a moment, it becomes evident that we deal with the estimation of average deviation of paths in a symmetric random walk on the line and one can use known results \([5]\). We found it remarkable that asymptotics obtained in this way is in agreement with precise results of the next section, which supports validity of such "oversimplified" approach.

The big advantage of this approach is that the same heuristical considerations are applicable in all dimensions. In particular, inspired by asymptotical consistence of the both methods in two-dimensional case, we were led to certain general hypotheses on asymptotics of averages. One of such hypotheses is presented in the conclusion of the paper.

### 4 Average degree via random walks

First we reduce computation of topological degree for a non-degenerate pair of binary forms to a simple geometric procedure. Recall that due to homogeneity, zero sets of binary forms consist of straight lines containing the origin. Paint all zero-lines of the first binary form \(P\) with one colour, and choose a different colour for zero-lines of the second form \(Q\). Then we can orient the unit circle \(T \subset \mathbb{C}\) and walk around it in the chosen direction counting at the same time the number \(d\) of alternances of colours inductively, as follows.

If after a line of one colour comes a line of the other colour, and only in this case, we increase the value of \(d\) by one. In other words, pairs of consecutive lines of the same colour do not contribute to \(d\). It is trivial to check that the resulting value of \(d\) does not depend on the line from which to start. Thus with any pair of binary forms we may associate a natural number.

**Lemma 2.** If the pair \((P, Q, )\) is non-degenerate then \(|\text{deg}(P, Q)| = d\).

The proof again follows from invariance properties of topological degree. First of all, notice that any two consecutive lines of the same colour may be brought into coincidence by a non-degenerate homotopy creating no common zeroes. The degree is unchanged under this procedure and now we have a square of linear form in the representation of our binary form. As non-negative factors of components do not affect the value of degree, we can omit these two lines and proceed inductively. It is clear that eventually we reach the situation when numbers of lines
of each colour coincide, say are equal to certain $k$, and colours are mutually alternating along unit circle. In order to compute degree in this case it remains to notice that such configuration of lines is non-degenerately homotopic to zero-lines configuration of ”realification” of the map $z^k : \mathbb{C} \to \mathbb{C}, z \mapsto z^k$. The latter map evidently has the degree equal to $k$, which finishes the proof.

This understood, we come very close to the desired reduction to a one-dimensional problem. Namely, we dehomogenize our forms by dividing over one of variables and immediately observe that computation of $d$ is equivalent to computation of average alternance number of real roots of two i.i.d. random polynomials of $n$-th degree. In order to avoid troubles with probabilities of $A_{rs}$ let us consider first hyperbolic polynomials, that is those having all their roots situated on the real axis. An equivalent way of saying this is that we consider $n$-forms represented as products of linear forms with i.i.d coefficients. For distributions listed in previous section coefficients of these linear forms are centered (have vanishing means) and it turns out that their explicit form is not important.

Indeed, it becomes clear that the symmetrical role of both polynomials implies that the number of alternances under consideration may be interpreted in terms of a symmetric random walk on the real line. Dealing with random walks we will freely use standard geometric terminology of \cite{5}.

Thus we consider a symmetric random walk on the real line and want to compute the average number of alternances, in the sense explicated above, over the set of all trajectories of length $2n$ ending on the horizontal axis. This may be done explicitly via simple combinatorics. Namely, the number of such trajectories is evidently equal to

$$C_{2n}^{2n} = \frac{(2n)!}{n!n!}.$$ 

Denoting by $\epsilon_k$ discrete entities equal to 1 or $-1$ we see that all what is needed is to compute the sum

$$\sum (-1)^k \epsilon_k$$

with the condition that sum of all $\epsilon_k$ is equal to zero.

**Lemma 3.** The average value of alternances is equal to

$$\frac{\sum_{k=0}^{n}(C_k^n)^2 \cdot |2k - n|}{C_{2n}^{2n}}.$$ 

In order to obtain this formula, observe first that the value of the above sum is evidently determined by the number $k$ of $\epsilon_j = 1$ with odd $j$. In fact, then the degree is found to be equal to $|2k - n|$. Now, these $k$ places are arbitrary within $n$ odd numbers from 1 to $2n - 1$, and the same holds for the choice of places of the rest $n - k$ epsilons equal to 1 within 2, 4, \ldots , 2n. This gives us exactly the sum in the numerator.

Now one can analyze this explicit formula in the manner of \cite{5}. Asymptotics of every summand in the numerator divided by denominator may be found by Stirling’s formula. In particular, one observes that for ”central” binomial coefficients (corresponding to $k = t - 1, t + 1$ for $n = 2t$ and $k = t, t + 1$ for $n = 2t + 1$) this asymptotics is $n^{-1/2}$. Now, one can expect that the linear factor $|2k - n|$ will change asymptotics to that of $\sqrt{n}$. This phenomenon is well known is random walks theory \cite{5} and it also occurs in our situation.

**Lemma 4.** The average absolute degree of two normal random hyperbolic binary $n$-forms, for large $n$, is asymptotically equivalent to $\sqrt{\frac{2}{\pi}}$.
We omit details of the proof because it involves some tedious but quite standard transformations of this combinatorial sum accompanied by a simple application of saddle-point asymptotic estimation. The most remarkable fact here is that this asymptotics turns out to be universal and enters into formulation of general result for two variables.

The latter is obtained in a similar way. We only have to take into account summation with weights equal to probabilities of $A_{rs}$. Averages $\tilde{d}_{rs}$ over $A_{rs}$ are computed using the same combinatorics as in Lemma 3.

**Lemma 5.** For binary $n$-forms with $r$ and $s$ real zeroes, respectively, the average value of alternances is

$$
\sum_{k=0}^{\lfloor \frac{r+s}{2} \rfloor} \left( C_{\frac{r+s}{2} k} \right) \left( C_{\frac{r+s}{2} (r-s/2)} \right) |2k+1| \frac{r-s}{2} - \frac{r+s}{2}.
$$

It is easy to check that order of growth of these averages is always not bigger than that of the average from Lemma 3. Remembering that the average absolute topological degree is the weighted sum of these averages with positive "probabilistic" (i.e. summing up to 1) coefficients, we conclude that the ultimate order of growth will be that of Lemma 3, which yields the main result of this note.

**Theorem.** The average absolute topological degree of the pair of normal random $n$-polynomials, for large $n$, grows as $\sqrt{n}$.

5 Concluding remarks

We would like to indicate that the results above have some immediate applications and generalizations. For example, exactly in the same manner one can compute the average topological degree of a pair of quasi-homogeneous polynomials [1] and then determine its order of growth for large values of the weighted algebraic degree. Similar results may be obtained for random harmonic polynomials and random entire functions.

Also, considering the aforementioned average absolute gradient degree of a random $n$-polynomial of two variables, we obtain the growth of $\sqrt{n-1}$. Now, using the well known relation between gradient degree and Euler characteristic of level surfaces and, in particular, formulas from [2], one may apply this result for finding asymptotics of Euler characteristics of random algebraic surfaces. Description of underlying topological results and precise statements about average Euler characteristic are left for future publications.

Finally, one could try to generalize the theorem to higher dimensions. Following the lines described at the end of Section 3, it is possible to guess certain higher-dimensional results. We would like to finish this note by the formulation of one plausible general hypothesis. Consider the orthogonally invariant distribution of coefficients of a random $n$-polynomial of $m$-variables introduced by M.Shub and S.Smale [3]. Denote by by $g(n, m)$ its average absolute gradient degree.

**Hypothesis.** For fixed $m$ and large $n$, $g(n, m)$ grows as $(n-1)^{m/4}$.

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