ENLARGEMENT OF SUBGRAPHS OF INFINITE GRAPHS
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Abstract. We consider changes in properties of a subgraph of an infinite graph resulting from the addition of open edges of Bernoulli percolation on the infinite graph to the subgraph. We give the triplet of an infinite graph, one of its subgraphs, and a property of the subgraphs. Then, in a manner similar to the way Hammersley’s critical probability is defined, we can define two values associated with the triplet. We regard the two values as certain critical probabilities, and compare them with Hammersley’s critical probability. In this paper, we focus on the following cases of a graph property: being a transient subgraph, having finitely many cut points or no cut points, being a recurrent subset, or being connected. Our results depend heavily on the choice of the triplet.

1. Introduction

A connected graph is called transient (resp. recurrent) if the simple random walk on it is transient (resp. recurrent). Benjamini, Gurel-Gurevich and Lyons [3] showed the cerebrating result claiming that the trace of the simple random walk on a transient graph is recurrent almost surely. If a connected subgraph of an infinite connected graph is transient, then the infinite connected graph is transient. Therefore, the trace is somewhat “smaller” than the graph on which the simple random walk runs. Now we consider the following questions: How far are a transient graph \( G \) and the trace of the simple random walk on \( G \)? More generally, how far are \( G \) and a recurrent subgraph \( H \) of \( G \)? How many edges of \( G \) do we need to add to \( H \) so that the enlargement of \( H \) becomes transient?

There are numerous choices of edges of \( G \) to be added to \( H \). If we add finitely many edges to \( H \), then the enlarged graph is also recurrent. Therefore, we add infinitely many edges to \( H \) and consider whether the enlarged graph is transient. In this paper, we add infinitely many edges of \( G \) to \( H \) randomly. Specifically, we add open edges of Bernoulli bond percolation on \( G \) to \( H \), and consider the probability that the enlargement of \( H \) is transient.

We more precisely state our purpose as follows. Let \( \mathbb{P}_p \) be the Bernoulli measure on the space of configurations of Bernoulli bond percolation on \( G \) such that each edge of \( G \) is open with probability \( p \in (0, 1) \). Consider the probability that the number of vertices of \( G \) connected by open edges from a fixed vertex is infinite under \( \mathbb{P}_p \). Then Hammersley’s critical probability \( p_c \) is the infimum of \( p \) such that the probability is positive. Similarly, we consider the probability that the enlarged graph is transient under \( \mathbb{P}_p \) and either of the following two values: the infimum of \( p \) such that the probability
finite simple graph. A simple graph is an unoriented graph in which neither
satisfies a certain condition. See Theorem 1.7 for details.

Framework and Main results.

1.1.

In this paper, a graph is a locally-finite simple graph. A simple graph is an unoriented graph in which neither
is positive, or the infimum of \( p \) such that the probability is one. We regard these two values as certain critical probabilities, and compare them with Hammersley’s critical probability.

We also consider questions of this kind, not only for transience, but also for other graph properties. Let \( G \) be an infinite connected graph and \( H \) be a subgraph of \( G \). Let \( \mathcal{P} \) be a property of the subgraphs of \( G \). Assume that \( G \) satisfies \( \mathcal{P} \) and \( H \) does not. Let \( \mathcal{U}(H) \) be the graph obtained by adding open edges of Bernoulli bond percolation on \( G \) to \( H \). (See Definition 1.1 for a precise definition.) Let \( \mathbb{P}_p \) be the Bernoulli measure on the space of configurations of Bernoulli bond percolation on \( G \) such that each edge of \( G \) is open with probability \( p \in (0, 1) \). Then we consider the probability that \( \mathcal{U}(H) \) satisfies \( \mathcal{P} \) under \( \mathbb{P}_p \). Let \( p_{c,1}(G,H,\mathcal{P}) \) (resp. \( p_{c,2}(G,H,\mathcal{P}) \)) be the infimum of \( p \) such that the probability is positive (resp. one). For example, if \( \mathcal{P} \) is infinite and \( H \) is a subgraph consisting of a vertex of \( G \) and no edges, then \( p_{c,1}(G,H,\mathcal{P}) = p_c(G) \) and \( p_{c,2}(G,H,\mathcal{P}) = 1 \).

The main purpose of this paper is to compare \( p_{c,1}(G,H,\mathcal{P}) \) and \( p_{c,2}(G,H,\mathcal{P}) \) with Hammersley’s critical probability \( p_c(G) \). We focus on the following cases that \( \mathcal{P} \) is: being a transient subgraph, having finitely many cut points or no cut points, being a recurrent subset, or being connected. \( p_{c,1}(G,H,\mathcal{P}) \) and \( p_{c,2}(G,H,\mathcal{P}) \) depend heavily on the choice of \((G,H,\mathcal{P})\). Assume \( \mathcal{P} \) is a transient subgraph. Then there is a triplet \((G,H,\mathcal{P})\) such that \( p_{c,1}(G,H,\mathcal{P}) = 1 \). On the other hand, there is a triplet \((G,H,\mathcal{P})\) such that \( p_{c,2}(G,H,\mathcal{P}) = 0 \). There is also a triplet \((G,H,\mathcal{P})\) such that \( p_{c,1}(G,H,\mathcal{P}) = p_{c,2}(G,H,\mathcal{P}) = p_c(G) \). See Theorem 1.4 for details. We also consider the case that \( H \) is chosen randomly, specifically, \( H \) is the trace of the simple random walk on \( G \).

Finally, we refer to related results. Benjamini, Häggström and Schramm [4] considered questions of this kind with a different motivation to ours. Their original motivation was considering the conjecture that for all \( d \geq 2 \), there is no infinite cluster in Bernoulli percolation on \( \mathbb{Z}^d \) with probability one at the critical point. If an infinite cluster of Bernoulli percolation \( C_\infty \) satisfies \( p_c(C_\infty) < 1 \) \( \mathbb{P}_p \) a.s. for any \( p \), then the conjecture holds. A question related to this is considering what kinds of conditions on a subgraph \( H' \) of \( \mathbb{Z}^d \) assure \( p_c(H') < 1 \). They introduced the concept of percolating everywhere (See Definition 6.1 for a precise definition,) and considered whether the following claim holds: if we add Bernoulli percolation to a percolating everywhere graph, then the enlarged graph is connected, and moreover, \( p_c(\text{the enlarged graph}) < 1 \), \( \mathbb{P}_p \) a.s. for any \( p \). This case can be described using our terminology as follows. \( G \) is \( \mathbb{Z}^d \), \( H \) is a percolating everywhere subgraph, and \( \mathcal{P} \) is connected and \( p_c(\mathcal{U}(H)) < 1 \). They showed that if \( d = 2 \), then \( p_{c,2}(G,H,\mathcal{P}) = 0 \), and conjectured that it also holds for all \( d \geq 2 \). Recently, Benjamini and Tassion [5] showed the conjecture for all \( d \geq 2 \) by a method different from [4]. In this paper, we will discuss the values \( p_{c,i}(G,H,\mathcal{P}) \), \( i = 1, 2 \), for percolating everywhere subgraphs \( H \) of \( G \). \( G \) is not necessarily assumed to be \( \mathbb{Z}^d \), and the result depends on whether \( G \) satisfies a certain condition. See Theorem 1.7 for details.

1.1. Framework and Main results. In this paper, a graph is a locally-finite simple graph. A simple graph is an unoriented graph in which neither
multiple edges or self-loops are allowed. \( V(X) \) and \( E(X) \) denote the sets of vertices and edges of a graph \( X \), respectively. If we consider the \( d \)-dimensional integer lattice \( \mathbb{Z}^d \), then it is the nearest-neighbor model.

Let \( G \) be an infinite connected graph. In this paper, we consider Bernoulli bond percolation and do not consider site percolation. Denote a configuration of percolation by \( \omega = (\omega_e)_{e \in E(G)} \in \{0,1\}^{E(G)} \). We say that an edge \( e \) is open if \( \omega_e = 1 \) and closed otherwise. We say that an event \( A \subset \{0,1\}^{E(G)} \) is increasing (resp. decreasing) if the following holds: if \( \omega = (\omega_e) \in A \) and \( \omega'_e \geq \omega_e \) (resp. \( \omega'_e \leq \omega_e \)) for any \( e \in E(G) \), then \( \omega' \in A \). Let \( C_x \) be the open cluster containing \( x \in V(G) \). We remark that \( \{x\} \subset V(C_x) \) holds. By convention, we often denote the set of vertices \( V(C_x) \) by \( C_x \). Let \( p_c(G) \) be Hammersley’s critical probability of \( G \). That is, for some \( x \in V(G) \),

\[
p_c(G) = \inf \{ p \in (0,1) : \mathbb{P}_p(|C_x| = +\infty) > 0 \}.
\]

This value does not depend on the choice of \( x \).

**Definition 1.1** (Enlargement of subgraph). Let \( H \) be a subgraph of \( G \). Let \( \mathcal{U}(H) = \mathcal{U}_\omega(H) \) be a random subgraph of \( G \) such that

\[
V(\mathcal{U}(H)) := \bigcup_{x \in V(H)} V(C_x) \quad \text{and} \quad E(\mathcal{U}(H)) := E(H) \cup \left( \bigcup_{x \in V(H)} E(C_x) \right).
\]

If \( H \) is connected, then \( \mathcal{U}(H) \) is also connected. If \( H \) consists of a single vertex \( x \) with no edges, then \( \mathcal{U}(H) \) is identical to \( C_x \).

In this paper, a property \( \mathcal{P} \) is a subset of the class of subgraphs of \( G \) which is invariant under any graph automorphism of \( G \). We consider a property which is well-defined only on a class of subgraphs of \( G \), and call the class the scope of the property. For example, being a transient subgraph is defined only for connected subgraphs of \( G \), and the scope of being transient is the class of connected subgraphs of \( G \). We denote \( X \in \mathcal{P} \) (resp. \( X \notin \mathcal{P} \)) if a subgraph \( X \) of \( G \) is in the scope of \( \mathcal{P} \) and satisfies (resp. does not satisfy) \( \mathcal{P} \). Let \( \mathcal{F} \) be the cylindrical \( \sigma \)-algebra on the configuration space \( \{0,1\}^{E(G)} \).

**Assumption 1.2.** We assume that an infinite connected graph \( G \), a subgraph \( H \) of \( G \), and a property \( \mathcal{P} \) satisfy the following:

(i) \( G, H \) and \( \mathcal{U}(H) \) are in the scope of \( \mathcal{P} \).

(ii) \( G \in \mathcal{P} \) and \( H \notin \mathcal{P} \).

(iii) The event that \( \mathcal{U}(H) \in \mathcal{P} \) is \( \mathcal{F} \)-measurable and increasing.

If \( H \) is chosen according to a probability law \( (\Omega', \mathcal{F}', \mathbb{P}') \), then we assume that (i) and (ii) above hold \( \mathbb{P}' \)-a.s., and the event \( \mathcal{U}(H) \in \mathcal{P} \) is \( \mathcal{F}' \otimes \mathcal{F} \)-measurable and increasing for \( \mathbb{P}' \)-a.s. \( \mathcal{F}' \otimes \mathcal{F} \) denotes the product \( \sigma \)-algebra of \( \mathcal{F}' \) and \( \mathcal{F} \).

In Section 2, we will check that the event \( \{\mathcal{U}(H) \in \mathcal{P}\} \) is \( \mathcal{F} \)-measurable for those properties, and give an example of \( (G, H, \mathcal{P}) \) such that \( \mathcal{U}(H) \in \mathcal{P} \) is not \( \mathcal{F} \)-measurable.

**Definition 1.3** (A certain kind of critical probability).

\[
p_{c,1}(G, H, \mathcal{P}) := \inf \{ p \in [0,1] : \mathbb{P}_p(\mathcal{U}(H) \in \mathcal{P}) > 0 \}.
\]

\[
p_{c,2}(G, H, \mathcal{P}) := \inf \{ p \in [0,1] : \mathbb{P}_p(\mathcal{U}(H) \in \mathcal{P}) = 1 \}.
\]
If $H$ obeys a law $\mathbb{P}'$, then we define $p_{c,i}(G,H,\mathcal{P})$, $i = 1,2$, by replacing $\mathbb{P}_p$ above with the product measure $\mathbb{P}' \otimes \mathbb{P}'_p$ of $\mathbb{P}'$ and $\mathbb{P}'_p$.

The main purpose of this paper is to compare the values $p_{c,i}(G,H,\mathcal{P})$, $i = 1,2$, with $p_c(G)$. If $H$ is a single vertex and $\mathcal{P}$ is being an infinite graph, then the definitions of $p_{c,1}(G,H,\mathcal{P})$ and $p_c(G)$ are identical and, hence, $p_{c,1}(G,H,\mathcal{P}) = p_c(G)$. It is easy to see that $p_{c,2}(G,H,\mathcal{P}) = 1$. In this paper, we focus on each of the following properties: (i) being a transient subgraph, (ii) having finitely many cut points or having no cut points, (iii) being a recurrent subset, and (iv) being a connected subgraph. The scopes for properties (i) and (ii) are connected subgraphs of $G$, and the scopes for (iii) and (iv) are all subgraphs.

We now state our main results informally. The following four assertions deal with the four properties (i) - (iv) above respectively. Some of the assertions are special cases of full and precise versions of them appearing in Sections 3 to 6.

**Theorem 1.4.** Let $\mathcal{P}$ be being a transient graph. Then,

(i-a) There is a pair $(G,H)$ such that $p_{c,1}(G,H,\mathcal{P}) = 1$.

(i-b) There is a pair $(G,H)$ such that $p_{c,2}(G,H,\mathcal{P}) = 0$.

(ii) Let $G = \mathbb{Z}^d$, $d \geq 3$. Then

(ii-a) For any $\epsilon > 0$, there exists a subgraph $H_\epsilon$ such that $p_{c,2}(\mathbb{Z}^d,H_\epsilon,\mathcal{P}) \leq \epsilon$.

(ii-b) If $H$ is the trace of the simple random walk on $\mathbb{Z}^d$, then $p_{c,1}(\mathbb{Z}^d,H,\mathcal{P}) = p_{c,2}(\mathbb{Z}^d,H,\mathcal{P}) = p_c(\mathbb{Z}^d)$.

(iii) Let $G$ be an infinite tree. Then

(iii-a) $p_{c,1}(G,H,\mathcal{P}) = p_c(G)$ for any $H$.

(iii-b) There is a subgraph $H$ such that $p_{c,2}(G,H,\mathcal{P}) = p_c(G)$.

(iii-c) There is a subgraph $H$ such that $p_{c,2}(G,H,\mathcal{P}) = 1$.

We now consider the number of cut points. Let $P_{x,y}$ be the law of two independent simple random walks on $G$ which start at $x$ and $y$, respectively.

**Theorem 1.5.** Let $G = \mathbb{Z}^d$, $d \geq 5$. Let $H$ be the trace of the two-sided simple random walk on $\mathbb{Z}^d$. Let $P_{0,0}^0 \otimes \mathbb{P}_p$ be the product measure of $P_{0,0}^0$ and $\mathbb{P}_p$. Then,

(i) If $p < p_c(G)$, then $U(H)$ has infinitely many cut points $P_{0,0}^0 \otimes \mathbb{P}_p$-a.s.

(ii) If $p > p_c(G)$, then $U(H)$ has no cut points $P_{0,0}^0 \otimes \mathbb{P}_p$-a.s.

In particular, if $\mathcal{P}$ is having finitely many cut points, or having no cut points, then

$$p_{c,1}(\mathbb{Z}^d,H,\mathcal{P}) = p_{c,2}(\mathbb{Z}^d,H,\mathcal{P}) = p_c(\mathbb{Z}^d).$$

The term cut points above is similar to the notion of a cut point of a random walk. See Definition 4.1 for a precise definition. It is known that the trace of the two-sided simple random walk on $\mathbb{Z}^d$ has infinitely many cut points $P_{0,0}^0$-a.s. (Cf. Lawler [14, Theorem 3.5.1]) The result above means that in the subcritical regime, there remain infinitely many cut points that are not bridged by open bonds of percolation.

Now we consider the case that $\mathcal{P}$ is being a recurrent subset. In this paper, we regard this as a subgraph and consider the induced subgraph of the subset.
Theorem 1.6. Let $\mathcal{P}$ be being a recurrent subset. Then,

(i-a) There is a pair $(G, H)$ such that $p_{c,1}(G, H, \mathcal{P}) = 1$.

(i-b) There is a pair $(G, H)$ such that $p_{c,2}(G, H, \mathcal{P}) = 0$.

(ii) Let $G = \mathbb{Z}^d$ and $H$ be the trace of the simple random walk on $\mathbb{Z}^d$. Then

(ii-a) $p_{c,1}(\mathbb{Z}^d, H, \mathcal{P}) = p_{c,2}(\mathbb{Z}^d, H, \mathcal{P}) = p_c(\mathbb{Z}^d), d \geq 5$.

(ii-b) $p_{c,1}(\mathbb{Z}^d, H, \mathcal{P}) = p_{c,2}(\mathbb{Z}^d, H, \mathcal{P}) = 0$, $d = 3, 4$.

(iii) Let $G$ be an infinite tree. Then $p_{c,1}(G, H, \mathcal{P}) = 1$ for any $H$.

The following concerns the connectedness of the enlargement of a percolating everywhere subgraph.

Theorem 1.7. Let $\mathcal{P}$ be being connected and $H$ be a percolating everywhere subgraph of an infinite connected graph $G$.

(i) Assume $G$ satisfies the following: for any infinite subsets $A, B \subset V(G)$ satisfying $V(G) = A \cup B$ and $A \cap B = \emptyset$, the number of edges connecting a vertex of $A$ and a vertex of $B$ is infinite. Then

$$p_{c,1}(G, H, \mathcal{P}) = p_{c,2}(G, H, \mathcal{P}).$$

(ii) Otherwise, there is a percolating everywhere subgraph $H$ such that

$$p_{c,1}(G, H, \mathcal{P}) = 0 \text{ and } p_{c,2}(G, H, \mathcal{P}) = 1.$$

The remainder of this paper is organized as follows. Section 2 states some preliminary results including the measurability of $\{U(H) \in \mathcal{P}\}$. We consider the case that $\mathcal{P}$ is being a transient graph, the case that $\mathcal{P}$ is a property concerning the number of cut points of graphs, the case that $\mathcal{P}$ is being a recurrent subset, and the case that $\mathcal{P}$ is being connected and $H$ is percolating everywhere, in Sections 3 to 6 respectively.

2. Preliminaries

This section consists of three subsections. First we give a lemma estimating $p_{c,i}(G, H, \mathcal{P})$. Then we state some results concerning random walk and percolation. Finally we discuss the measurability of the event $\{U(H) \in \mathcal{P}\}$.

2.1. A lemma. Roughly speaking, in the following, we will show that under a certain condition, $p_{c,i}(G, H, \mathcal{P})$ can be arbitrarily small, if there is a “suitable” subgraph $H$. Let $\mathcal{N}(v)$ be the set of neighborhoods of a vertex $v$.

Lemma 2.1. Fix an infinite connected graph $G$ and a property $\mathcal{P}$ for subgraphs of $G$. Let $i = 1, 2$. Assume that there is a subgraph $H$ of $G$ such that

$$p_{c,i}(G, H, \mathcal{P}) < 1, \text{ and }$$

$$v \in V(H) \text{ or } \mathcal{N}(v) \subset V(H), \text{ for any } v \in V(G). \tag{2.1}$$

Then for any $\epsilon > 0$ there is a subgraph $H_\epsilon$ such that $p_{c,i}(G, H_\epsilon, \mathcal{P}) \leq \epsilon$. 

Proof. We show this assertion for \( i = 1 \). Let \( \Phi : \{0, 1\}^{E(G)} \times \{0, 1\}^{E(G)} \to \{0, 1\}^{E(G)} \) be the map defined by
\[
\Phi(\omega_1, \omega_2) = \omega_1 \lor \omega_2.
\]
(Here and henceforth \( \omega_1 \lor \omega_2 \) means the maximum of \( \omega_1 \) and \( \omega_2 \).) Then the push-forward measure of the product measure \( \mathbb{P}_{q_1} \otimes \mathbb{P}_{q_2} \) on \( \{0, 1\}^{E(G)} \times \{0, 1\}^{E(G)} \) by \( \Phi \) is
\[
\mathbb{P}_{q_1} + q_2 - q_1 q_2.
\]

Since \( p_{c,1}(G, H, \mathcal{P}) < 1 \), we have that for any \( q_2 > 0 \), there is \( q_1 < p_{c,1}(G, H, \mathcal{P}) \) such that
\[
q_1 + q_2 - q_1 q_2 > p_{c,1}(G, H, \mathcal{P}).
\]

It is easy to see that
\[
\mathcal{U}_{\omega_2}(\mathcal{U}_{\omega_1}(H)) \subset \mathcal{U}_{\omega_1 \lor \omega_2}(H).
\]
By (2.1),
\[
\mathcal{U}_{\omega_2}(\mathcal{U}_{\omega_1}(H)) = \mathcal{U}_{\omega_1 \lor \omega_2}(H).
\]
Therefore,
\[
\mathbb{P}_{q_1} \otimes \mathbb{P}_{q_2}(\mathcal{U}_{\omega_2}(\mathcal{U}_{\omega_1}(H)) \in \mathcal{P}) = \mathbb{P}_{q_1 + q_2 - q_1 q_2}(\mathcal{U}(H) \in \mathcal{P}) > 0.
\]
Since \( q_1 < p_{c,1}(G, H, \mathcal{P}) \), there is a configuration \( \omega_1 \) such that \( \mathcal{U}_{\omega_1}(H) \notin \mathcal{P} \) and

\[
\mathbb{P}_{q_2}(\mathcal{U}_{\omega_2}(\mathcal{U}_{\omega_1}(H)) \in \mathcal{P}) > 0.
\]
Hence
\[
p_{c,1}(G, \mathcal{U}_{\omega_1}(H), \mathcal{P}) \leq q_2.
\]
We can show this for \( i = 2 \) in the same manner. \( \square \)

2.2. Random walk and percolation. We now define recurrent and transient subsets of \( G \) by following Lawler and Limic [15, Section 6.5]. Here and henceforth \( ((S_n)_{n \geq 0}, (P^x)_{x \in V(G)}) \) denotes the simple random walk on \( G \). We regard a recurrent subset as a subgraph and consider the induced subgraph of the recurrent subset.

**Definition 2.2** (recurrent subset). We say that a subset \( A \) of \( V(G) \) is a **recurrent subset** if for some \( x \in V(G) \)
\[
P^x(S_n \in A \text{ i.o. } n) > 0.
\]
Otherwise, \( A \) is called a **transient subset**. This definition does not depend on choices of a vertex \( x \in V(G) \).

For a graph \( X \), we let \( d_X(x, y) \) be the graph distance between \( x \) and \( y \) in \( X \) and
\[
B_X(x, n) := \{ y \in V(X) : d_X(x, y) \leq n \}.
\]

We now briefly state the notion of Cayley graphs. Let \( \Gamma \) be a finitely generated countable group and \( S \) be a symmetric finite generating subset of \( \Gamma \) which does not contain the unit element. Then the **Cayley graph of \( \Gamma \) with respect to \( S \)** is the graph such that the set of vertices is \( \Gamma \) and the set of edges is \( \{ (x, y) \subset \Gamma : x^{-1} y \in S \} \). This graph depends on choices of \( S \). In this paper, all results concerning Cayley graphs of groups do not depend on...
choices of $S$. We say that a graph $G$ has the degree of growth $d \in (0, +\infty)$ if for any vertex $x$ of $G$,
\[ 0 < \liminf_{n \to \infty} \frac{|B_G(x, n)|}{n^d} \leq \limsup_{n \to \infty} \frac{|B_G(x, n)|}{n^d} < +\infty. \]

Lemma 2.3. Let $G$ be a Cayley graph of a finitely generated group with the degree of growth $d$. Let $o$ be the unit element of the finitely generated group. Assume $p_c(G) < 1$ and $p \in (p_c(G), 1)$. Then,
(i) There is a unique infinite cluster $C_\infty$, $\mathbb{P}_p$-a.s.
(ii) $C_\infty$ is a recurrent subset of $G$, that is,
\[ P^o(S_n \in C_\infty \ i.o. \ n) > 0, \quad \mathbb{P}_p \text{-a.s.} \quad (2.2) \]
(iii)
\[ P^o(S_n \in C_\infty \ i.o. \ n) = 1, \quad \mathbb{P}_p \text{-a.s.} \quad (2.3) \]

Let $T_A$ be the first hitting time of $(S_n)_n$ to a subset $A \subseteq V(G)$.

Proof. By Woess [19, Theorem 12.2 and Proposition 12.4], Cayley graphs of a finitely generated group with polynomial growth is amenable graphs. Therefore, by Bollobás and Riordan [6, Theorem 4 in Chapter 5], the number of infinite clusters is $0$ $\mathbb{P}_p$-a.s. or it is $1$ $\mathbb{P}_p$-a.s. By $p > p_c(G)$, the latter holds. Thus we have (i).

We will show (ii). Let $P^o \otimes \mathbb{P}_p$ be the product measure of $P^o$ and $\mathbb{P}_p$.

\[ P^o \otimes \mathbb{P}_p (S_n \in C_\infty \ i.o. \ n) = \lim_{N \to \infty} P^o \otimes \mathbb{P}_p \left( \bigcup_{n \geq N} \{ S_n \in C_\infty \} \right). \]

Using the shift invariance of Bernoulli percolation and the Markov property for simple random walk,
\[ P^o \otimes \mathbb{P}_p \left( \bigcup_{n \geq N} \{ S_n \in C_\infty \} \right) = P^o \otimes \mathbb{P}_p \left( \bigcup_{n \geq 0} \{ S_N^{-1} \cdot S_{N+n} \in C_\infty \} \right) \]
\[ = P^o \otimes \mathbb{P}_p \left( \bigcup_{n \geq 0} \{ S_n \in C_\infty \} \right). \]

Here $S_N^{-1}$ is the inverse element of $S_N$ as group. Hence,
\[ P^o \otimes \mathbb{P}_p (S_n \in C_\infty \ i.o. \ n) = P^o \otimes \mathbb{P}_p \left( \bigcup_{n \geq 0} \{ S_n \in C_\infty \} \right). \]

Since
\[ \{ S_n \in C_\infty \ i.o. \ n \} \subseteq \bigcup_{n \geq 0} \{ S_n \in C_\infty \}, \]
we have
\[ P^o (S_n \in C_\infty \ i.o. \ n) = P^o \left( \bigcup_{n \geq 0} \{ S_n \in C_\infty \} \right) = P^o (T_{C_\infty} < +\infty) > 0, \mathbb{P}_p \text{-a.s.} \]

Thus we have (2.2).
By [19, Corollary 25.10] all bounded harmonic functions on $G$ are constant. By following the proof of [15, Lemma 6.5.7], we have (2.3). □

2.3. Measurability of $U(H) \in \mathcal{P}$. Recall that $\mathcal{F}$ is the cylindrical $\sigma$-algebra of $\{0,1\}^{E(G)}$. First, we consider the case that $H$ is a non-random subgraph.

**Lemma 2.4.** (i) Let $H$ be a recurrent subgraph of a transient graph $G$. Then the event that $U(H)$ is a transient subgraph of $G$ is $\mathcal{F}$-measurable.

(ii) Let $H$ be a recurrent subgraph of a transient graph $G$. Then the number of cut points of $U(H)$ is an $\mathcal{F}$-measurable function. See Definition 4.1 in Section 4 for the definition of cut points.

(iii) Let $H$ be a transient subset of a transient graph $G$. Then the event that $U(H)$ is a recurrent subset is $\mathcal{F}$-measurable.

(iv) Let $H$ be a non-connected subgraph of an infinite connected graph $G$. Then the event that $U(H)$ is connected is $\mathcal{F}$-measurable.

**Proof.** (i) Let $R_{\text{eff}}(x, U(H) \setminus B_{U(H)}(x, n))$ be the effective resistance from $x$ to the outside of $B_{U(H)}(x, n)$. It suffices to show that $R_{\text{eff}}(x, U(H) \setminus B_{U(H)}(x, n))$ is an $\mathcal{F}$-measurable function for each $n$. Since $U(H)$ is a connected subgraph of $G$, $B_{U(H)}(x, n)$ is contained in $B_G(x, n)$. Therefore, $R_{\text{eff}}(x, U(H) \setminus B_{U(H)}(x, n))$ is determined by configurations in $B_G(x, n)$ and hence is $\mathcal{F}$-measurable.

(ii) It suffices to show that for any $z \in V(G)$, the event that $z \in U(H)$ and $z$ is a cut point of $U(H)$ is $\mathcal{F}$-measurable. $z$ is a cut point of $U(H)$ if and only if $z$ is a cut point of $U(H \cap B_G(z, n))$ for any $n$.

(iii) By Fubini’s theorem, it suffices to see that $\{S_n \in U(H)\}$ is $\mathcal{F}_{\text{SRW}} \otimes \mathcal{F}$-measurable. This follows from

$$\{S_n \in U(H)\} = \bigcup_{y \in V(G)} \{S_n = y\} \times \{y \text{ is connected to } H \text{ by an open path}\}.$$ 

(iv) If $x, y \in V(U(H))$ are connected in $U(H)$, then there is $n$ such that $x$ and $y$ are in a connected component of $U(H) \cap B_G(x, n)$. This event is determined by configurations of edges in $B_G(x, n + 1)$. □

We now consider the case that $H$ is a random subgraph of $G$. Let $\mathcal{F}_{\text{SRW}}$ be the $\sigma$-algebra on the path space defined by the simple random walk on $G$ and $\mathcal{F}_{\text{SRW}} \otimes \mathcal{F}$ be the product $\sigma$-algebra of $\mathcal{F}_{\text{SRW}}$ and $\mathcal{F}$. The following easily follows from that the event that the trace of the simple random walk is identical with a given connected subgraph $H$ is $\mathcal{F}_{\text{SRW}}$-measurable.

**Lemma 2.5.** Assume that the event $U(H)$ satisfies $\mathcal{P}$ is $\mathcal{F}$-measurable for any infinite connected subgraph $H$. Let $H$ be the trace of the simple random walk. Then the event $U(H)$ satisfies $\mathcal{P}$ is $\mathcal{F}_{\text{SRW}} \otimes \mathcal{F}$-measurable.

**Example 2.6** (A triplet $(G, H, \mathcal{P})$ such that the event $\{U(H) \in \mathcal{P}\}$ is not measurable). We first show that there is a non-measurable subset of $\{0,1\}^N$ with respect to the cylindrical $\sigma$-algebra of $\{0,1\}^N$. Here and henceforth, $\mathbb{N}$ denotes the set of natural numbers. Let $\phi : \{0,1\}^N \to \{0,1\}^N$ be the
one-sided shift and $A$ be an uncountable subset of $\{0, 1\}^\mathbb{N}$ such that (i)

$$\bigcup_{n \geq 0} \phi^{-n}(A) = \{0, 1\}^\mathbb{N} \setminus \left( \bigcup_{n \geq 1} \{x \in \{0, 1\}^\mathbb{N} : \phi^n(x) = x \} \right),$$

and (ii) for any $x, y \in A$ and any $n \geq 1$ $y \neq \phi^n(x)$.

Assume that $A$ is measurable. Let $\ell$ be the product measure of the probability measure $\mu$ on $\{0, 1\}$ with $\mu(\{0\}) = \mu(\{1\}) = 1/2$. Since $\phi^{-1}(A) \cap \phi^{-j}(A) = \emptyset$ for $i \neq j$,

$$\ell \left( \bigcup_{i \geq 0} A \right) = \sum_{i \geq 0} \ell(\phi^{-i}(A)).$$

Since $\cup_{n \geq 1}\{x \in \{0, 1\}^\mathbb{N} : \phi^n(x) = x\}$ is countable, $\ell(\cup_{i \geq 0} A) = 1$. Since $\phi$ preserves $\ell$, we see that

$$\ell(\phi^{-i}(A)) = \ell(A)$$

for any $i$, and

$$\sum_{i \geq 0} \ell(\phi^{-i}(A)) = 0 \text{ or } +\infty.$$

But this is a contradiction. Hence $A$ is not measurable.

Let $G$ be the connected subgraph of $\mathbb{Z}^2$ whose vertices are

$$\{(x, 0) : x \geq -2\} \cup \{(y, 1) : y \geq -1\} \cup \{(-1, -1)\}.$$

Then any graph automorphism of $G$ is the identity map between vertices of $G$. Let $H$ be the connected subgraph of $G$ whose vertices are

$$\{(x, 0) : x \geq -2\} \cup \{(-1, 1)\} \cup \{(-1, -1)\}.$$

Then

$$E(G) \setminus E(H) = \{(n, 0), (n, 1) : n \in \mathbb{N}\}.$$

Let $\tilde{\omega}$ be the projection of $\omega \in \{0, 1\}^{E(G)}$ to $\{0, 1\}^{E(G) \setminus E(H)}$. Regard $E(G) \setminus E(H)$ as $\mathbb{N}$. Let $\mathcal{P}$ be the property that a graph is isomorphic to a graph in the class $\{U_\omega(H) : \tilde{\omega} \in A\}$. Then

$$\{U(H) \in \mathcal{P}\} = A \times \{0, 1\}^{E(H)}.$$

This event is not measurable with respect to the cylindrical $\sigma$-algebra of $\{0, 1\}^{E(G)}$.

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**Figure 1.** Graph of $H$. The dotted lines are $E(G) \setminus E(H)$.

### 3. $\mathcal{P}$ is being a transient graph

In this section, we consider the case that $\mathcal{P}$ is being a transient graph and assume that $H$ is connected.
3.1. The case that $H$ is a fixed subgraph.

**Theorem 3.1** (Extreme cases). (i) There is a graph $G$ such that

$$0 < p_c(G) < p_{c,1}(G, H, \mathcal{P}) = 1.$$  

(ii) There is a graph $G$ such that for any infinite recurrent subgraph $H$ of $G$,

$$p_{c,2}(G, H, \mathcal{P}) = 0.$$  

We remark that if $H$ is finite, then $p_{c,2}(G, H, \mathcal{P}) = 1$.

**Proof.** (i) Let $G$ be the graph which is constructed as follows: Take $\mathbb{Z}^2$ and attach a transient tree $T$ such that $p_c(T) = 1$ to the origin of $\mathbb{Z}^2$. This appears in Häggström and Mossel [10, Section 6]. Then for any recurrent subgraph $H$,

$$p_c(G) < p_{c,1}(G, H, \mathcal{P}) = 1.$$  

(ii) Let $G$ be an infinite connected line-graph in Benjamini and Gurel-Gurevich [2, Section 2]. In their paper, it is given as a graph having multilines, but we can construct a simple graph by adding a new vertex on each edge.

Let $H$ be an infinite connected recurrent subgraph of $G$. Then $\mathbb{N} \subset V(H)$. Let $p > 0$. If the number of edges between $k$ and $k+1$ is $2k^3$, then

$$\mathbb{P}_p(\text{two open consecutive edges connecting } k \text{ and } k+1 > k^2) \rightarrow 1, \ k \rightarrow \infty, \text{ exponentially fast.}$$  

Hence

$$\mathbb{P}_p\left(\bigcap_{k \geq 1} \{ \text{two open consecutive edges connecting } k \text{ and } k+1 > k^2 \} \right) > 0.$$  

By this and the recurrence/transience criterion by effective resistance (See [19, Theorem 2.12] for example.),

$$\mathbb{P}_p(\mathcal{U}(H) \text{ is transient}) > 0.$$  

Since $H$ is infinite, we can use the 0-1 law and have

$$\mathbb{P}_p(\mathcal{U}(H) \text{ is transient}) = 1.$$  

We give rough figures of the two graphs in the proof above.

The proof of (ii) above heavily depends on the fact that $G$ has unbounded degrees. Now we consider a case that $G$ has bounded degrees.

**Theorem 3.2.** Let $G = \mathbb{Z}^d, d \geq 3$. Then for any $\epsilon > 0$ there is a recurrent subgraph $H_\epsilon$ such that $p_{c,2}(G, H_\epsilon, \mathcal{P}) \leq \epsilon$. 
Proof. Let $H$ be a recurrent subgraph of $\mathbb{Z}^d$ such that $V(H) = V(G)$. By [19, (2.21)] such $H$ exists. If $p > p_c(\mathbb{Z}^d)$, then $\mathcal{U}(H)$ contain the unique infinite open cluster a.s. By Grimmett, Kesten and Zhang [9], $\mathcal{U}(H)$ is transient. Hence $\quad p_{c,2}(G,H,\mathcal{P}) \leq p_c(\mathbb{Z}^d) < 1.$

Now the assertion follows from this and Lemma 2.1. \qed

In the proof of Theorem 3.2, we choose a subgraph $H$ such that $V(H) = V(G)$ and apply Lemma 2.1. However, if $H$ is a connected proper subgraph of an infinite tree $T$ with $\text{deg}(x) \geq 2, \forall x \in V(T)$, then (2.1) in Lemma 2.1 fails.

Theorem 3.3. Let $T$ be an infinite transient tree. Then

(i) If $T'$ is a recurrent subtree of $T$, then

$$p_{c,1}(T,T',\mathcal{P}) = p_c(T).$$

(ii) If $T'$ is an infinite recurrent subtree of $T$, then

$$p_{c,2}(T,T',\mathcal{P}) = \sup \{ p_c(H) : H \text{ is a transient subtree in } T \text{ and } E(H) \cap E(T') = \emptyset \}.$$ 

Proof. (i) By Peres [17, Exercise 14.7], if $p > p_c(T)$, then

$$\mathbb{P}_p(C_v \text{ is transient}) > 0$$

for any $v \in T$. Since $C_v \subset \mathcal{U}(T')$ for any $v \in T$,

$$\mathbb{P}_p(\mathcal{U}(T') \text{ is transient}) > 0.$$

Therefore, $p_{c,1}(T,T',\mathcal{P}) \leq p_c(T)$.

If $p < p_c(T)$, then $\mathbb{P}_p$-a.s., $\mathcal{U}(T')$ is an infinite tree obtained by attaching at most countably many finite trees to $T'$. Hence, $\mathcal{U}(T')$ is also a recurrent graph $\mathbb{P}_p$-a.s. Therefore, $p_{c,1}(T,T',\mathcal{P}) \geq p_c(T)$.

(ii) Assume that there is a transient subtree $H$ of $T$ such that $E(H) \cap E(T') = \emptyset$ and $p < p_c(H)$. There is a finite path from $o$ to a vertex of $H$. Since $H$ is transient, the probability that random walk starts at $o$ and, then, goes to a vertex of $H$ and remains in $H$ after the hitting to $H$ is positive. Hence the probability that $\mathcal{U}(T')$ is still recurrent is positive. Hence $p \leq p_{c,2}(T,T',\mathcal{P}).$

Assume that

$$p > \sup \{ p_c(H) : H \text{ is a transient subtree in } T \text{ and } E(H) \cap E(T') = \emptyset \}.$$
Since there are infinitely many transient connected subtrees $H$ of $T$ such that $E(H) \cap E(T') = \emptyset$, $U(T')$ contains at least one infinite transient cluster in $H$. □

Hereafter $T_d$ denotes the $d$-regular tree, $d \geq 2$. By Theorem 3.3,

**Corollary 3.4.** Let $T = T_d, d \geq 3$ and $T'$ be a recurrent subgraph. Then

$$p_{c,1}(T, T', \mathcal{P}) = p_{c,2}(T, T', \mathcal{P}) = p_c(T).$$

The value $p_{c,2}$ depends on choices of a subgraph $T'$ as the following example shows.

**Example 3.5.** Let $T$ be the graph obtained by attaching a vertex of $T_3$ to a vertex of $T_4$.

(i) If $T'$ is a subgraph of $T_3$ which is isomorphic to $L = (\mathbb{N}, \{\{n, n+1\} : n \in \mathbb{N}\}$), then

$$p_{c,2}(T, T', \mathcal{P}) = p_c(T_3) = \frac{1}{2}.$$

(ii) If $T'$ is a subgraph of $T_4$ which is isomorphic to $L$, then

$$p_{c,2}(T, T', \mathcal{P}) = p_c(T_4) = \frac{1}{3}.$$

We give a short remark about stability with respect to rough isometry. Let $G$ be the graph obtained by attaching one vertex of the triangular lattice to a vertex of the $d$-regular tree $T_d$. If $d = 3$, then

$$p_c(G) = p_c(\text{triangular lattice}) = 2 \sin \left(\frac{\pi}{18}\right) < \frac{1}{2},$$

and

$$p_{c,1}(G, H, \mathcal{P}) = p_c(T_3) = \frac{1}{2}.$$

If $d$ is large, then

$$p_c(G) = p_{c,1}(G, H, \mathcal{P}) = \frac{1}{d-1}.$$

As this remark shows, there is a pair $(G, H)$ such that

$$p_c(G) < p_{c,1}(G, H, \mathcal{P}) < 1.$$

We are not sure that there is a pair $(G, H)$ such that

$$0 < p_{c,1}(G, H, \mathcal{P}) < p_c(G).$$

**3.2. The case that $H$ is the trace of the simple random walk.**

**Theorem 3.6.** Let $G$ be a Cayley graph of a finitely generated countable group with the degree of growth $d \geq 3$. Let $H$ be the trace of the simple random walk on $G$. Then

$$p_{c,1}(G, H, \mathcal{P}) \geq p_c(G).$$

Hereafter $E^\mu$ denotes the expectation with respect to a probability measure $\mu$. 
Proof. Let $p < p_c(G)$. We show that the volume growth of $\mathcal{U}(H)$ is (at most) second order. We assume that simple random walks start at a vertex $o$. Since $B_{\mathcal{U}(H)}(o,n)$ is contained in $B_G(o,n)$,

$B_{\mathcal{U}(H)}(o,n) \subset \bigcup_{x \in V(H) \cap B_G(o,n)} C_x$.

By Mensikov, Molchanov and Sidrenko [16],

$E^p[C_o] < +\infty$, $p < p_c(G)$.

Therefore,

$E^{p_o \otimes P} [B_{\mathcal{U}(H)}(o,n)] \leq E^{p_o} [E^{p_o} [B_H(o,n)]]$.

Using Hebisch and Saloff-Coste [11, Theorem 5.1] and summation by parts,

$E^{p_o} [V(H) \cap B_G(o,n)] \leq \sum_{x \in B_G(o,n)} \sum_{m \geq 0} P^o(S_m = x)
= \sum_{x \in B_G(o,n)} d_G(o,x)^{2-d} = O(n^2)$.

Using this and Fatou’s lemma,

$E^{p_o \otimes P} \left[ \liminf_{n \to \infty} \frac{B_{\mathcal{U}(H)}(o,n)}{n^2} \right] < +\infty$.

Hence

$\liminf_{n \to \infty} \frac{B_{\mathcal{U}(H)}(o,n)}{n^2} < +\infty$, $P^o \otimes P$-a.s.

The assertion follows from this and [19, Lemma 3.12]. □

Let $G = \mathbb{Z}^d$, $d \geq 3$ and $H$ be the trace of the simple random walk on $G$. Then by Lemma 2.3 and the transience of infinite cluster by [9],

$p_{c,2}(G, H, \mathcal{P}) \leq p_c(G)$.

By this and Theorem 3.6,

Corollary 3.7. Let $G = \mathbb{Z}^d$, $d \geq 3$, and $H$ be the trace of the simple random walk on $G$. Then

$p_{c,1}(G, H, \mathcal{P}) = p_{c,2}(G, H, \mathcal{P}) = p_c(G)$.

4. $\mathcal{P}$ IS A PROPERTY CONCERNING CUT POINTS

In this section, we assume that $G$ is a transient graph and $H$ is a recurrent subgraph of $G$.

Definition 4.1 (cut point). We say that a vertex $x \in V(G)$ is a cut point if we remove an edge $e$ containing $x$, then the graph splits into two infinite connected components.

The graph appearing in the proof of Theorem 3.1 (ii) (see Figure 3) has a vertex such that if we remove it, then the graph splits into two connected components. However, it is not a cut point in the sense of the above definition.
Theorem 4.2. Let $G$ be a Cayley graph of a finitely generated countable group with the degree of growth $d \geq 5$. Let $H$ be the trace of the two-sided simple random walk on $G$. Then if $p < p_c(G)$, then $\mathcal{U}(H)$ has infinitely many cut points, $P^{o,o} \otimes P_p$-a.s.

Let

$$S^i(A) := \{S^i_n : n \in A\} \text{ for } A \subset \mathbb{N}.$$ 

Proof. Fix a vertex $o$. First we will show that

$$P^{o,o} \otimes P_p (o \text{ is a cut point of } \mathcal{U}(H)) > 0. \tag{4.1}$$

We give a rough sketch of proof of (4.1). First we show there exists a vertex $z$ such that two simple random walks starting at $o$ and $z$ respectively do not intersect with positive probability. Then we “make” vertices in a large box closed and show the two random walks do not return to the large box with positive probability. Finally we choose a path connecting the two traces in a suitable way.

Using [16], $d \geq 5$, and [11, Theorem 5.1],

$$\sum_{i,j \geq 0} P^{o,o} \otimes P_p (\mathcal{U}((S^1_n) \cap \mathcal{U}((S^2_n)) \neq \emptyset) = \sum_{x \in V(G)} \sum_{i,j \geq 0} P^{o}(S_{i+j} = x)P_p(x \in C_o) < +\infty.$$ 

Hence for large $N$

$$P^{o,o} \otimes P_p (\mathcal{U}(S^1([N, +\infty))) \cap \mathcal{U}(S^2([N, +\infty))) \neq \emptyset) \leq \sum_{i,j \geq N} P^{o,o} \otimes P_p (\mathcal{U}((S^1_n) \cap \mathcal{U}((S^2_n)) \neq \emptyset) < 1.$$ 

Let

$$A := \{\mathcal{U}(S^1([1, +\infty))) \cap \mathcal{U}(S^2([0, +\infty))) = \emptyset\}.$$ 

Then there is a vertex $z \in V(G)$ such that $P^{z,o}(A) > 0$. If $z = o$, then (4.1) holds. Assume $z \neq o$. Let $B := B_G(o, 3d_G(o, z))$ and $C$ be the event that all edges in $B$ are closed.

Since $p < 1$ and $A$ is decreasing, $P^{z,o} \otimes P_p (A \cap C) > 0$.

Since $S^1$ and $S^2$ are transient, there is $N$ such that

$$P^{z,o} \otimes P_p \left( A \cap C \cap \bigcap_{i=1,2} \{S^i((N, \infty)) \cap B = \emptyset\} \right) > 0.$$ 

Now we can specify two finite paths of $S^i_j$. There are vertices $x^i_j, i = 1, 2, j = 0, \cdots, N$ such that

$$P^{z,o} \otimes P_p \left( A \cap C \cap \bigcap_{i=1,2} \{S^i_j = x^i_j, \forall j, S^i((N, \infty)) \cap B = \emptyset\} \right) > 0.$$ 

Now we can pick up a path in $B$ connecting $\{x^1_j\}_j \cap B$ and $\{x^2_j\}_j \cap B$. We can let $S^1_j = x^1_j$, for $-m_i < j < 0$, $i = 1, 2$, and $x^1_{-m_1} = x^2_{-m_2} = y_0$.

$$P^{y_0,y_0} \otimes P_p \left( A \cap C \cap \bigcap_{i=1,2} \{S^i_j = x^i_j, \forall j, S^i((N, \infty)) \cap B = \emptyset\} \right) > 0.$$ 


This event is contained in the event \( \{ \mathcal{U}(S^1([-m_1, +\infty))) \cap \mathcal{U}(S^2([-m_2, +\infty))) = \emptyset \} \)
and hence we have (4.1).

Let \( \mathcal{S} \) be the generating set of the Cayley graph \( G \). Consider the following
transformation \( \Theta \) on \( \mathcal{S} \times \{0, 1\}^{E(G)} \) defined by
\[
\Theta ((a_j)_j, (\omega_e)_e) := ((a_{j+1})_j, (\omega_{ae})_e).
\]
Here we let \( ae := \{ ax, ay \} \) for an edge \( e = \{ x, y \} \) and a point \( a \in \mathcal{S} \). We
have that \( \Theta \) preserves \( P^{o,o} \otimes \mathbb{P}_p \).

Define a transformation \( \varphi_a \) on \( \{0, 1\}^{E(G)} \) by
\[
\varphi_a ((\omega_e)_e) := (\omega_{ae})_e.
\]
By following the proof of [6, Lemma 1 in Chapter 5], the family of maps
\( \{ \varphi_a : a \in \mathcal{S} \} \) is ergodic. By Kakutani [12, Theorem 3], \( \Theta \) is ergodic with
respect to \( P^{o,o} \otimes \mathbb{P}_p \).

By applying the Poincaré recurrence theorem (See Pollicott and Yuri [18, Theorem 9.2] for example.), to the dynamical system \( (\mathcal{S} \times \{0, 1\}^{E(G)}, P^{o,o} \otimes \mathbb{P}_p, \Theta) \), we have
\[
P^{o,o} \otimes \mathbb{P}_p \left( \mathcal{U}(\tilde{S}((−\infty, n])) \cap \mathcal{U}(\tilde{S}([n + 1, +\infty)))) = \emptyset \text{ infinitely many } n \in \mathbb{Z} \right) = 1,
\]
where we let
\[
\tilde{S}_n := \begin{cases} S^1_n, & n \geq 0 \\ S_{−n}^2, & n < 0 \end{cases}
\]
□

The following considers this problem at the critical point in high dimensions. It is pointed out by Itai Benjamini. (personal communication)

**Theorem 4.3.** Let \( G = \mathbb{Z}^d, d \geq 11 \). Let \( H \) be the trace of the two-sided simple random walk on \( \mathbb{Z}^d \). Let \( p = p_c \). Then \( \mathcal{U}(H) \) has infinitely many cut points \( P^{o,o} \otimes \mathbb{P}_{p_c(\mathbb{Z}^d)} \)-a.s.

**Proof.** We will show that
\[
\sum_{x \in \mathbb{Z}^d} \sum_{i,j \geq 0} P^0(S_{i+j} = x) \mathbb{P}_p(x \in C_0) < +\infty.
\]
In below \( c, c' \) and \( c'' \) are constants depending only on \( d \) and \( p \). Fitzner and
van der Hofstad [7, Theorem 1.4] claims that the decay rate for the two-point function \( \mathbb{P}_{p_c(\mathbb{Z}^d)}(0 \leftrightarrow x) \) is \( |x|^{2-d} \) as \( |x| \to +\infty \), if \( d \geq 11 \). Therefore,
\[
\mathbb{P}_p(x \in C_0) \leq c \cdot d_{2d}(0, x)^{2-d}, \text{ for any } x.
\]
Since \( P^0(S_k = x) = 0 \) if \( k < d_{2d}(0, x) \),
\[
\sum_{x \in \mathbb{Z}^d} \sum_{i,j \geq 0} P^0(S_{i+j} = x) \mathbb{P}_p(x \in C_0) \leq \sum_k c' k^{1-d/2} \left( \sum_{l=1}^k c l^{2-d} \{x : d_{2d}(0, x) = l\} \right)
\]
\[
= c'' \sum_{k \geq 1} k^{3-d/2} < +\infty.
\]
The rest of the proof goes in the same way as in the proof of Theorem 4.2. □

The following deals with supercritical phases.
Proposition 4.4. Let $G = \mathbb{Z}^d, d \geq 3$. Let $H$ be the trace of the two-sided simple random walk on $\mathbb{Z}^d$. If $p > p_c(G)$, then $U(H)$ has no cut points $P^{0,0} \otimes P_p$-a.s.

Proof. Using the two-arms estimate by Aizenman, Kesten and Newman [1],

$$P_p(0 \in C_{\infty} \text{ and } 0 \text{ is a cut point of } C_{\infty}) = 0.$$ 

Using the shift invariance of $P_p$, the unique infinite cluster $C_{\infty}$ has no cut points $P_p$-a.s. \hfill \Box

5. $\mathcal{P}$ IS BEING A RECURRENT SUBSET

In this section, we assume that $G$ is a transient graph. Recall Definition 2.2. We regard a recurrent subset as a subgraph and consider the induced subgraph of the recurrent subset. In other words, if $A$ is a recurrent subset of $V(G)$, then we consider the graph such that the set of vertices is $A$ and the set of edges $\{\{x, y\} \in E(G) : x, y \in A\}$.

5.1. The case that $H$ is a fixed subgraph. We proceed with this subsection as in Subsection 3.1. The following correspond to Theorem 3.1.

Theorem 5.1 (Extreme cases). (i) There is a graph $G$ such that for any transient subset $H$ of $G$, 

$$0 < p_c(G) < p_{c,1}(G, H, \mathcal{P}) = 1.$$ 

(ii) There is a graph $G$ such that for any infinite transient subset $H$ of $G$, 

$$p_{c,2}(G, H, \mathcal{P}) = 0.$$ 

We show this in the same manner as in the proof of Theorem 3.1.

Proof. Even if we add one edge to a transient subset, then the enlarged graph is also a transient subset. If not, the random walks hit an added vertex infinitely often, a.s., which contradicts that $G$ is a transient graph. Therefore, we can show (i) in the same manner as in the proof of Theorem 3.1 (i).

Let $G$ be the graph defined in the proof of Theorem 3.1 (ii). Then 

$$|N \cap V(U(H))| = +\infty.$$ 

Hence $U(H)$ is a recurrent subset, $P_p$-a.s. for any $p > 0$. \hfill \Box

Second we consider the case $G$ is $\mathbb{Z}^d, d \geq 3$. Lemma 2.3 implies that

Proposition 5.2. Let $G = \mathbb{Z}^d, d \geq 3$. Then for any transient subset $H$ of $G$ 

$$p_{c,1}(G, H, \mathcal{P}) \leq p_c(G).$$ 

Third we consider the case that $G$ is a tree $T$.

Theorem 5.3. Let $T$ be an infinite tree and $H$ be a transient subset of $T$. Then, $p_{c,1}(T, H, \mathcal{P}) = 1.$
Proof. For \( e \in E(T) \) and \( x \in e \), we let \( T_{e,x} \) be the connected subtree of \( T \) such that \( x \in V(T_{e,x}) \) and \( e \notin E(T_{e,x}) \). Since \( H \) is a transient subset, there are an edge \( e \) and a vertex \( x \in e \) such that \( T_{e,x} \) is a transient subgraph of \( T \) and \( V(H) \cap V(T_{e,x}) = \{x\} \). Then we can take an infinite path \((x_0, x_1, x_2, \ldots)\) in \( T_{e,x} \) such that \( x_0 = x \), and for each \( i \geq 0 \), \( \{x_i, x_{i+1}\} \in E(T_{x_{i-1},x_i}) \), and \( T_{x_{i-1},x_i} \) is a transient subgraph. If \( p < 1 \), then there is a number \( i \) such that \( \mathcal{U}(H) \) does not intersect with \( T_{\{x_{i-1},x_i\},x_i} \), \( \mathbb{P}_p \)-a.s. Hence \( \mathcal{U}(H) \) is a transient subset of \( T \), \( \mathbb{P}_p \)-a.s. \( \square \)

We do not give an assertion corresponding to Theorem 3.2. We are not sure that there is a recurrent subset such that the induced subgraph of it satisfies (2.1) in Lemma 2.1.

5.2. The case that \( H \) is the trace of the simple random walk.

Theorem 5.4. Let \( G \) be a Cayley graph of a finitely generated countable group with the degree of growth \( d \geq 3 \). Let \( H \) be the trace of the simple random walk on \( G \). Then

(i) If \( d \geq 5 \),

\[ p_{c,1}(G,H,\mathcal{P}) = p_{c,2}(G,H,\mathcal{P}) = p_c(G). \]

(ii) If \( d = 3, 4 \),

\[ p_{c,1}(G,H,\mathcal{P}) = p_{c,2}(G,H,\mathcal{P}) = 0. \]

Proof. Let \( o \) be the unit element of the group. Let

\[ \theta(x) := \sum_{n \geq 0} P^o(S_n = x) \quad \text{and} \quad \theta_p(x) := \sum_{n \geq 0} P^o \otimes \mathbb{P}_p(S_n \in C_x). \]

We remark that \( \theta(x) = \theta_0(x) \). First we show \( p_c(G) \leq p_{c,1}(G,H,\mathcal{P}) \). Let \( p < p_c(G) \). It follows from [11, Theorem 5.1] and [16] that

\[ \theta_p(x) = O \left(d_G(o,x)^{2-d}\right). \]

By following the proof of [15, Theorem 6.5.10],

\[ E^{P^o \otimes \mathbb{P}_p} \left[ \sum_{x \in \mathcal{U}(H)} \theta(x) \right] = \sum_{x \in V(G)} \theta_p(x) \theta(x) = O \left( \sum_{x \in V(G)} d_G(o,x)^{4-2d} \right). \]

Using \( d \geq 5 \) and summation by parts,

\[ \sum_{x \in V(G)} d_G(o,x)^{4-2d} < +\infty. \]

Hence

\[ P^o \otimes \mathbb{P}_p (S_n \in \mathcal{U}(H), \text{ i.o. } n) = 0 \]

and \( p \leq p_{c,1}(G,H,\mathcal{P}) \). Thus we have

\[ p_c(G) \leq p_{c,1}(G,H,\mathcal{P}). \]

If \( p_c(G) < 1 \), then by Lemma 2.3,

\[ p_c(G) \geq p_{c,2}(G,H,\mathcal{P}). \]

If \( p_c(G) = 1 \), this clearly holds. Thus we see (i).
We show (ii) by following the proof of [15, Theorem 6.5.10]. Let $S^1$ and $S^2$ be two independent simple random walks on $G$. Let

$$Z_k := \left| \left( B_G(o, 2^{k}) \setminus B_G(o, 2^{k-1}) \right) \cap S^1 \left( [0, T^1_{V(G) \setminus B_G(o, 2^{k})}] \right) \right| \cap S^2 \left( [0, T^2_{V(G) \setminus B_G(o, 2^{k})}] \right).$$

Let $E_k$ be the event that $Z_k$ is strictly positive.

In below, $c_i$, $1 \leq i \leq 8$, are positive constants depending only on $G$. It follows from a generalized Borel-Cantelli lemma that if

$$(1) \sum_{k \geq 1} P^{o,o}(E_{3k}) = +\infty \quad \text{and}$$

(2) For some constant $c_1$

$$P^{o,o}(E_{3k} \cap E_{3m}) \leq c_1 P^{o,o}(E_{3k}) P^{o,o}(E_{3m}), \quad k \neq m$$

hold, then $E_{3k}$ holds i.o. $k$, $P^{o,o}$-a.s. and assertion (ii) follows.

[11, Theorem 5.1] states that

$$c_2 \cdot d_G(o, x)^{2-d} \leq \theta(x) \leq c_3 \cdot d_G(o, x)^{2-d}. $$

By Grigoryan and Telcs [8, Proposition 10.1] the elliptic Harnack inequality holds. Therefore, we have (2).

Now we show (1).

$$E^{p-o}[Z_k] = \sum_{x \in B_G(o, 2^{k}) \setminus B_G(o, 2^{k-1})} P^o \left( T_x \leq T_{V(G) \setminus B_G(o, 2^{k})} \right)^2$$

$$= c_4 \sum_{x \in B_G(o, 2^{k}) \setminus B_G(o, 2^{k-1})} \theta_{B_G(o, 2^{k})}(x)^2.$$

Since $\theta_{B_G(o, 2^{k})}(x) \geq c_5 2^{(2-d)}$ for any $x \in B_G(o, 3 \cdot 2^{k-2}) \setminus B_G(o, 2^{k-1})$,

$$E^{p-o}[Z_k] \geq c_6 2^{(4-2d)} \left| B_G(o, 3 \cdot 2^{k-2}) \setminus B_G(o, 2^{k-1}) \right|.$$ 

Using this and an isoperimetric inequality (Cf. [11, Theorem 7.4]),

$$E^{p-o}[Z_k] \geq c_7 2^{(4-d)}. \quad (5.1)$$

We have

$$E^{p-o}[Z_k^2] = \sum_{x \in B_G(o, 2^{k}) \setminus B_G(o, 2^{k-1})} P^o \left( T_x \vee T_y \leq T_{V(G) \setminus B_G(o, 2^{k})} \right)^2.$$

Since

$$P^o \left( T_x \vee T_y \leq T_{V(G) \setminus B_G(o, 2^{k})} \right) \leq P^o \left( T_x \leq T_y \leq T_{V(G) \setminus B_G(o, 2^{k})} \right) + P^o \left( T_y \leq T_x \leq T_{V(G) \setminus B_G(o, 2^{k})} \right)$$

$$\leq \theta(x) \theta(x^{-1} y) + \theta(y) \theta(y^{-1} x)$$

$$= O \left( 2^{(2-d)} (1 + d_G(x, y))^{2-d} \right),$$

we have that by using summation by parts

$$\sum_{y \in B_G(o, 2^{k}) \setminus B_G(o, 2^{k-1})} (1 + d_G(x, y))^{4-2d} = \begin{cases} O(2^k) & d = 3, \\ O(k) & d = 4. \end{cases}$$
Therefore,
\[ E^{P^{o,o}}(Z^2_k) = \begin{cases} \frac{O(4^k)}{k} & d = 3, \\ \frac{O(k)}{k} & d = 4. \end{cases} \]  
(5.2)

Using (5.1), (5.2) and the second moment method, for \( d = 3, 4 \),
\[ P^{o,o}(E_k) = P^{o,o}(Z_k) \geq \frac{E^{P^{o,o}}(Z^2_k)}{E^{P^{o,o}}(Z^2_k)} \geq \frac{c_k}{k}. \]
Thus we have (1). \( \square \)

6. \( P \) is being connected

We say that a subgraph \( H \) of \( G \) is connected if for any two vertices \( x \) and \( y \) of \( H \) there are vertices \( x_0, \ldots, x_n \) of \( H \) such that \( x_0 = x, x_n = y \), and \( \{x_{i-1}, x_i\} \) is an edge of \( H \) for each \( i \).

By Definition 1.1, if \( H \) is connected, then \( \mathcal{U}(H) \) is also connected. On the other hand, if \( H \) is not connected, then \( \mathcal{U}(H) \) can be non-connected. For example, if \( (V(G), E(G)) = (\mathbb{Z}, \{n, n + 1 : n \in \mathbb{Z}\}) \) and \( (V(H), E(H)) = (\mathbb{Z}, \emptyset) \), then
\[ \mathbb{P}_p(\mathcal{U}(H) \text{ is connected}) = 0, \quad p < 1. \]

The following is introduced by [4].

**Definition 6.1** (percolating everywhere). We say that a subgraph \( H \) of \( G \) is percolating everywhere if \( V(H) = V(G) \) and every connected component of \( H \) is infinite.

We introduce a notion concerning connectivity. For \( A, B \subset V(G) \), we let
\[ E(A, B) := \{y, z \in E(G) : y \in A, z \in B\}. \]

**Definition 6.2.** We say that \( G \) satisfies (TI) if for every \( A, B \subset V(G) \) satisfying
\[ V(G) = A \cup B, \quad A \cap B = \emptyset \text{ and } |A| = |B| = +\infty, \]
\( E(A, B) \) is an infinite set.

**Example 6.3.** (i) \( \mathbb{Z}^d, d \geq 2 \), satisfy (TI).
(ii) \( \mathbb{T}_d, d \geq 2 \), does not satisfy (TI).
(iii) The trace of the two-sided simple random walk on \( \mathbb{Z}^d, d \geq 5 \), does not satisfy (TI) a.s.

**Proof.** Let \( a_1 \in A \) and \( b_1 \in B \). Since \( \mathbb{Z}^d \) is connected, there is a path \( \gamma_1 \) connecting \( a_1 \) and \( b_1 \). Then \( \gamma_1 \) contains at least one edge in \( E(A, B) \) and is contained in a box \( B_{\mathbb{Z}^d}(o, N_1) \). Since \( A \) and \( B \) are infinite, there are points \( a_2 \in A \cap (\mathbb{Z}^d \setminus B_{\mathbb{Z}^d}(o, N_1)) \) and \( b_2 \in B \cap (\mathbb{Z}^d \setminus B_{\mathbb{Z}^d}(o, N_1)) \). There is a path \( \gamma_2 \) connecting \( a_2 \) and \( b_2 \) in \( \mathbb{Z}^d \setminus B_{\mathbb{Z}^d}(o, N_1) \). Then \( \gamma_2 \) contains at least one edge in \( E(A, B) \) and is contained in a box \( B_{\mathbb{Z}^d}(o, N_2) \). Since \( A \) and \( B \) are infinite, we can repeat this procedure and have infinitely many disjoint paths \( (\gamma_n)_{n \geq 1} \). Thus we have (i). Since \( \mathbb{T}_d \) and trace have infinitely many cut points, (ii) and (iii) hold. \( \square \)
Theorem 6.4. (i) If $G$ is (TI), then for any percolating everywhere subgraph $H$
$$p_{c,1}(G, H, \mathcal{P}) = p_{c,2}(G, H, \mathcal{P}).$$
If the number of connected components of $H$ is finite, then
$$p_{c,1}(G, H, \mathcal{P}) = p_{c,2}(G, H, \mathcal{P}) = 0.$$
(ii) If $G$ does not satisfy (TI), then there is a percolating everywhere subgraph $H$ such that
$$p_{c,1}(G, H, \mathcal{P}) = 0 \text{ and } p_{c,2}(G, H, \mathcal{P}) = 1.$$

Proof. Let $\mathbb{P}_p^H$ be the product measure on $\{0, 1\}^{E(G)}$ such that $\mathbb{P}_p^H(\omega_e = 1) = 1$ if $e \in E(H)$ and $\mathbb{P}_p^H(\omega_e = 1) = p$ if $e \notin E(H)$. Denote $x \leftrightarrow y$ if $x$ and $y$ are connected by an open path in this percolation model.

Define $x \sim y$ if and only if $\mathbb{P}_p^H(x \leftrightarrow y) = 1$. This is an equivalent definition. Let $[x]$ be the equivalent class containing $x$. Let $G'$ be the quotient graph of $G$ by $\sim$. This is a connected graph which may have multilines. The number of edges between two vertices are finite.

If $|V(G')| = 1$, then
$$\mathbb{P}_p(\mathcal{U}(H) \text{ is connected}) = 1.$$ Assume that $|V(G')| \geq 2$, $V(G') = A \cup B$ and $A \cap B = \emptyset$. Let $A'$ (resp. $B'$) be a subset of $V(G)$ such that the equivalent class of each element is in $A$ (resp. $B$). Then
$$V(G) = A' \cup B', \quad A' \cap B' = \emptyset \quad \text{and} \quad |A'| = |B'| = +\infty.$$ By (TI), $E(A', B') = +\infty$. Since the number of edges between two vertices of $G'$ are finite, $E(A, B) = +\infty$.

Define $p([x], [y])$ be the probability $[x]$ and $[y]$ are connected by an open edge with respect to the induced measure of $\mathbb{P}_p$ by the quotient map. Then
$$p([x], [y]) = 1 - (1 - p)^{|E([x],[y])|}.$$ Hence
$$\sum_{[x] \in A, [y] \in B} p([x], [y]) \geq p \sum_{[x] \in A, [y] \in B} 1 \{[x] \text{ and } [y] \text{ are connected by an edge of } G\} = +\infty.$$ By Kalikow and Weiss [13, Theorem 1],
$$\mathbb{P}_p(\text{the random graph on } G' \text{ is connected}) \in \{0, 1\}.$$ Each connected component of $H$ is contained in an equivalent class, and conversely, each equivalent class contains each connected component of $H$, due to the percolating everywhere assumption. Therefore, $\mathcal{U}(H)$ is connected if and only if the random graph on $G'$ is connected. Hence, for any $p > 0$
$$\mathbb{P}_p(\mathcal{U}(H) \text{ is connected}) \in \{0, 1\}$$ and hence
$$p_{c,1}(G, H, \mathcal{P}) = p_{c,2}(G, H, \mathcal{P}).$$ If the number of connected components of $H$ is finite, then $E(A', B') < +\infty$ for any decomposition $V(G') = A' \cup B'$. Therefore, $|V(G')| = 1$, and hence,
$$p_{c,1}(G, H, \mathcal{P}) = p_{c,2}(G, H, \mathcal{P}) = 0.$$
Thus we have (i).

Assume that $G$ does not satisfy (TI). Then there is two infinite disjoint sets $A$ and $B$ such that $V(G) = A \cup B$ and $E(A, B) < +\infty$. Since $(A, E(A, A))$ and $(B, E(B, B))$ may have finite connected components, we modify $A$ and $B$. Let $\partial A$ and $\partial B$ be the inner boundaries of $A$ and $B$, respectively. For any vertex in $(A, E(A, A))$, there is a vertex in $\partial A$ such that they are connected in $(A, E(A, A))$. Since $A$ and $B$ are infinite, there are $a_0 \in \partial A$ and $b_0 \in \partial B$ such that infinitely many vertices of $A$ are connected to $a_0$ in $A$ and infinitely many vertices of $B$ are connected to $b_0$ in $B$.

Let $E_{A,B} := \{(a, b) \in E_A : b \leftrightarrow b_0 \text{ in } E(G) \setminus E_A\}$, where $E_A := \{(a, b) \in E(A, B) : a \leftrightarrow a_0 \text{ in } A\}$.

Let $H$ be a subgraph of $G$ such that $E(H) = E(G) \setminus E_{A,B}$. Then $a_0$ and $b_0$ are not connected in $H$. Assume that there is a vertex $x$ such that it is not connected to $a_0$ in $H$. Then consider a path $\gamma$ from $x$ to $b_0$ in $G$. Let $\{a, b\} \in E_{A,B}$ be the first edge which $\gamma$ intersects with $E_{A,B}$. Since $a \leftrightarrow a_0$ in $A$, $\gamma$ pass $b$ before it pass $a$. There is a path from $b$ to $b_0$ which does not pass any edges of $E_{A,B}$. Hence, there is a path from $x$ to $b_0$ in $H$. Therefore, there are just two connected components of $H$, and due to the choices of $a_0$ and $b_0$, they are both infinite. Thus $H$ is percolating everywhere.

Since $E_{A,B}$ is finite, $p_{c,1}(G, H, \mathcal{P}) = 0$. Since $E_{A,B}$ is non-empty, $p_{c,2}(G, H, \mathcal{P}) = 1$. Thus we have (ii). 

We are not sure whether if $G$ satisfies (TI) and $H$ is a percolating everywhere subgraph with infinitely many connected components then $p_{c,1}(G, H, \mathcal{P}) = p_{c,2}(G, H, \mathcal{P}) = 0$.

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