Voigt waves in homogenized particulate composites based on isotropic dielectric components

Tom G Mackay

School of Mathematics and Maxwell Institute for Mathematical Sciences, University of Edinburgh, Edinburgh EH9 3JZ, UK
and
NanoMM—Nanoengineered Metamaterials Group, Department of Engineering Science and Mechanics, Pennsylvania State University, University Park, PA 16802-6812, USA

E-mail: T.Mackay@ed.ac.uk

Received 5 May 2011, accepted for publication 9 August 2011
Published 1 September 2011
Online at stacks.iop.org/JOpt/13/105702

Abstract
Homogenized composite materials (HCMs) can support a singular form of optical propagation, known as Voigt wave propagation, while their component materials do not. In essence, Voigt waves represent coalescent degenerate eigenmodes of the corresponding propagation matrix. Crucially, the existence of Voigt waves stems from the non-Hermitian nature of this propagation matrix, which in turn is a manifestation of the dissipative nature of the HCM. This phenomenon was investigated for biaxial HCMs arising from nondissipative isotropic dielectric component materials. The biaxiality of these HCMs stems from the oriented spheroidal shapes of the particles which make up the component materials. An extended version of the Bruggeman homogenization formalism was used to investigate the influence of component particle orientations, shapes and sizes, as well as volume fractions of the component materials, upon Voigt wave propagation. Our numerical studies revealed that the directions in which Voigt waves propagate are highly sensitive to the orientations of the component particles and to the volume fractions of the component materials, but less sensitive to the shapes of the component particles and less sensitive still to the sizes of the component particles. Furthermore, whether or not an HCM supports Voigt wave propagation at all is critically dependent upon the sizes of the component particles and, in certain cases, upon the volume fractions of the component materials.

Keywords: Voigt waves, Bruggeman homogenization formalism, optical singularities

(Some figures in this article are in colour only in the electronic version)

1. Introduction

A composite material, comprising random distributions of two (or more) particulate component materials, may be regarded as effectively homogeneous provided that the particles which constitute the component materials are much smaller than the wavelengths involved [1, 2]. Depending upon the shapes, sizes and distributions of the component particles, such homogenized composite materials (HCMs) can exhibit optical properties not exhibited at all by their component materials, or at least not exhibited to the same extent by their components. Indeed, through judicious design, HCMs arising from commonplace component materials may be conceptualized which support rather exotic (and potentially useful) optical phenomena, while their component materials do not. A prime example is provided by HCMs which support plane wave propagation with negative phase velocity [3, 4]. Another example—which provides the setting for the present study—is furnished by HCMs that support a singular form of optical propagation known as Voigt wave propagation [5].
In order to describe what constitutes a Voigt wave, it is helpful to first consider the nonsingular case of optical propagation in a linear, homogeneous, anisotropic, dielectric material. Generally two independent plane waves, with orthogonal polarizations and different phase speeds, can propagate in a given direction [6]. However, as first reported by Voigt in 1902 [7] and later described by others [8–12], in certain dissipative biaxial materials there are particular directions along which these two waves coalesce to form a single plane wave. A prominent feature of this coalescent Voigt wave is that its amplitude has a linear dependence upon propagation direction. In essence, Voigt waves represent coalescent degenerate eigenmodes of the corresponding propagation matrix. Crucially, the existence of Voigt waves stems from the non-Hermitian nature of this propagation matrix, which in turn is a manifestation of the dissipative nature of the biaxial material. These waves differ fundamentally from the non-coalescent degenerate eigenmodes of the Hermitian matrices that correspond to nondissipative biaxial materials [13]. A precise mathematical description of a Voigt wave is provided in the appendix. Biaxial materials give rise to more possibilities for Voigt waves to occur [14, 15], but herein our attention is restricted to the simpler case of biaxial dielectric materials.

While Voigt waves represent a fundamental topic in the optics of anisotropic (and biaxialotropic) materials, they have yet to be exploited in applications. However, the ability to design metamaterials that support Voigt wave propagation already exists, courtesy of the homogenization approach outlined in section 2. And the ability to fabricate such HCM-metamaterials for use at optical wavelengths may well already lie within the grasp of present-day nanotechnology. One may therefore speculate that Voigt waves are ripe for exploitation.

Previously, the standard Bruggeman homogenization formalism was used to establish that certain dissipative biaxial HCMs can support Voigt wave propagation [5]. A follow-up study based on the second-order strong-permittivity-fluctuation theory—which represents a higher-order formulation of the standard Bruggeman formalism wherein two-point statistical correlations between particles of the component materials are taken into account [16]—emphasized the importance of correlation length for Voigt wave propagation [17]. Both of these earlier studies concerned HCMs arising from two uniaxial dielectric component materials (which cannot themselves support Voigt wave propagation). However, a biaxial dielectric HCM can also arise from isotropic dielectric components in instances where the shapes of the component material particles are non-spherical. For example, if each of the two component materials comprises an assembly of oriented spheroidal particles then the corresponding HCM will be biaxial, in general [18]. This is the scenario explored here. We implement an extended version of the standard Bruggeman formalism [19], which utilizes a recently-developed extended depolarization dyadic formalism [20] in order to take into account the nonzero sizes of component particles. This approach allows us to investigate the influence of the non-electromagnetic attributes of the component materials—these being the orientations, shapes and sizes of the component materials as well as the volume fractions of the component materials—upon the propagation of Voigt waves.

In the notation adopted, vectors are represented in boldface, with the \( \hat{\cdot} \) symbol denoting a unit vector. Thus, the unit Cartesian vectors are written as \( \hat{x}, \hat{y} \) and \( \hat{z} \). Double underlining with normal typeface signifies a 3 × 3 dyadic; and the identity 3 × 3 dyadic is \( I = \hat{\mathbf{x}} \hat{\mathbf{x}} + \hat{\mathbf{y}} \hat{\mathbf{y}} + \hat{\mathbf{z}} \hat{\mathbf{z}} \). The superscript T denotes the dyadic or vector transpose. Double underlining with blackboard bold typeface signifies a 6 × 6 dyadic. The permittivity and permeability of free space are written as \( \epsilon_0 \) and \( \mu_0 \), respectively. The free-space wavenumber is \( k_0 = \omega \sqrt{\epsilon_0 \mu_0} \), with \( \omega \) being the angular frequency.

### 2. Homogenization formalism

#### 2.1. Component materials

We consider optical propagation in an HCM derived from two component materials, labelled \( a \) and \( b \). The component materials are both taken to be isotropic dielectric materials with permittivity dyadics \( \epsilon_a = \epsilon_0 \epsilon_a I \) and \( \epsilon_b = \epsilon_0 \epsilon_b I \). The volume fraction of material \( a \) is \( f_a \) while that of material \( b \) is \( f_b = 1 - f_a \).

Each component material comprises a randomly-distributed assembly of spheroidal particles. All material \( a \) particles have the same orientation and all material \( b \) particles have the same orientation, but these two orientations are generally different. For simplicity, both material \( a \) and \( b \) particles are assumed to have the same shape. The surfaces of the component particles, relative to their centres, are prescribed by the vector

\[
\mathbf{r}_\ell = \eta_\ell U_\ell \hat{\mathbf{r}} \quad (\ell = a, b). \tag{1}
\]

Here, \( \hat{\mathbf{r}} \) is the radial vector prescribing the surface of the unit sphere, the real-symmetric surface dyadic \( U_\ell \) maps the spherical surface onto a spheroidal one, and \( \eta_\ell > 0 \) is a linear measure of particle size. In conformity with the homogenization regime, \( \eta_\ell \) must be much smaller than the wavelengths involved, but—unlike in conventional approaches taken in homogenization studies [21]—we shall not insist that \( \eta_\ell \) is vanishingly small. Furthermore, let us assume that \( \eta_a = \eta_b \), and henceforth simply write \( \eta \) in lieu of \( \eta_\ell \). A fractal-like distribution of the spheroidal component particles is envisaged, so that spaces between the particles are eliminated. In consonance with this representation, the size parameter \( \eta \) is interpreted as an upper bound on the linear dimensions of the component particles [21].

The symmetry axis of the spheroidal particles comprising material \( a \) is taken to lie in the \( xy \) plane at an angle \( \varphi \) to the \( x \) axis. Thus, the surface dyadic for material \( a \) may be expressed as

\[
U_\ell = \frac{1}{\sqrt{U_x U_z}} R(\varphi) \cdot \left[ U_x \hat{\mathbf{x}} \hat{\mathbf{x}} + U (\hat{\mathbf{y}} \hat{\mathbf{y}} + \hat{\mathbf{z}} \hat{\mathbf{z}}) \right] \cdot R^T(\varphi)
\]

\((U, U_{\ell} > 0), \tag{2}\)

where the orthogonal rotation dyadic

\[
R(\varphi) = \cos \varphi (\hat{\mathbf{x}} \hat{\mathbf{x}} + \hat{\mathbf{y}} \hat{\mathbf{y}}) + \sin \varphi (\hat{\mathbf{y}} \hat{\mathbf{y}} - \hat{\mathbf{x}} \hat{\mathbf{x}}) + \hat{\mathbf{z}} \hat{\mathbf{z}}. \tag{3}
\]
Without loss of generality, the material \( b \) particles are assumed to be aligned with the \( x \) axis. Thus, the surface dyadic for material \( b \) may be expressed as

\[
U_{bb} = \frac{1}{\sqrt{U_b U_b}}[\mathbf{U_b} \hat{\mathbf{x}} \hat{\mathbf{x}} + U_b (\hat{\mathbf{y}} \hat{\mathbf{y}} + \hat{\mathbf{z}} \hat{\mathbf{z}})] \quad (U, U_i > 0). \tag{4}
\]

### 2.2. Homogenized composite material

As the symmetry axes of the spheroidal particles comprising components \( a \) and \( b \) are not generally aligned, the resulting HCM is a biaxial dielectric material with a symmetric permittivity dyadic of the form

\[
\epsilon_{HCM} = \epsilon_0 \left[ \epsilon_1 \hat{\mathbf{x}} \hat{\mathbf{x}} + \epsilon_2 \hat{\mathbf{y}} \hat{\mathbf{y}} + \epsilon_3 (\hat{\mathbf{y}} \hat{\mathbf{y}} + \hat{\mathbf{z}} \hat{\mathbf{z}}) \right]. \tag{5}
\]

We implement an extended version of the Bruggeman homogenization formalism in order to estimate \( \epsilon_{HCM} \). Accordingly, \( \epsilon_{HCM} \) is extracted from the nonlinear dyadic equation [22]

\[
\epsilon_1 \left( \epsilon_{HCM} - \epsilon_{HCM} \right) \cdot \left( U + D_a \right) \cdot (\epsilon_{HCM} - \epsilon_{HCM}) = 0, \quad (a, b), \tag{6}
\]

using standard numerical techniques, such as the Jacobi method [23]. The depolarization dyadics \( D_{ab} \), herein may be regarded as sums of two terms; that is,

\[
D_{ab} = D_{0ab}^0 + D_{1ab}^0 \quad (\ell = a, b). \tag{7}
\]

The term \( D_{0ab}^0 \) represents the depolarization contribution arising from a vanishingly small particle described by the surface dyadic \( U_{ab} \), as given by the double integral [24, 25]

\[
D_{0ab}^0 = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \frac{(U_{ab}^{-1} \cdot \hat{\mathbf{q}})(U_{ab}^{-1} \cdot \hat{\mathbf{q}})}{(U_{ab}^{-1} \cdot \hat{\mathbf{q}}) \cdot \epsilon_{HCM} \cdot (U_{ab}^{-1} \cdot \hat{\mathbf{q}})} \sin \theta \, d\theta \, d\phi \quad (\ell = a, b), \tag{8}
\]

wherein the unit vector \( \hat{\mathbf{q}} = \sin \theta \cos \phi \hat{\mathbf{x}} + \sin \theta \sin \phi \hat{\mathbf{y}} + \cos \theta \hat{\mathbf{z}} \). The depolarization contribution which derives from the nonzero size of the component particles is provided by the term \( D_{1ab}^0 \). It is most conveniently expressed in terms of elements of the \( 6 \times 6 \) dyadic \( \tilde{D}_{ab} \), per

\[
\left[ \tilde{D}_{ab} \right]_{mn} = \delta_{mn} \quad (m, n \in \{1, 2, 3\}), \tag{9}
\]

with [20]

\[
\left[ \tilde{D}_{ab}^+ \right]_{mn} = \frac{\omega^2}{4\pi \mu_0} \times \int_0^{2\pi} \int_0^{\pi} \sin \theta \times \left[ \frac{1}{\kappa_+ - \kappa_-} \left( \exp(\text{i} \eta \phi) - 1 \right) \right] \times \left[ \det[\hat{\mathbf{A}}(U_{ab}^{-1} \cdot \hat{\mathbf{q}})] \left[ \tilde{G}_{ab}^+(U_{ab}^{-1} \cdot \hat{\mathbf{q}}) \right]^{q = \kappa_+} + \det[\hat{\mathbf{A}}(-U_{ab}^{-1} \cdot \hat{\mathbf{q}})] \left[ \tilde{G}_{ab}^+(U_{ab}^{-1} \cdot \hat{\mathbf{q}}) \right]^{q = \kappa_-} \right] d\theta \, d\phi \quad (\ell = a, b). \tag{10}
\]

Herein \( \kappa_{\pm} \) are the \( q^2 \) roots of \( \det[\hat{\mathbf{A}}(U_{ab}^{-1} \cdot \hat{\mathbf{q}})] = 0 \), the vector \( \hat{\mathbf{q}} = q \hat{\mathbf{q}} \), while the \( 6 \times 6 \) dyadics are

\[
\tilde{G}_{ab}^+(p) = \left[ \frac{\varepsilon_{HCM}}{\varepsilon_{HCM}} \cdot (p/\omega) \cdot \varepsilon_{HCM} \right] \tag{11}
\]

and

\[
\tilde{G}_{ab}^{-1}(p) = \lim_{|p| \to \infty} \tilde{G}_{ab}^{-1}(p). \tag{12}
\]

Analytical evaluations of the integrals in equations (8) and (10) are available for relatively simple anisotropic HCMs [26, 27], but for general biaxial HCMs numerical methods are needed to evaluate these integrals.

### 3. Voigt wave propagation

Let us turn now to the possibility of Voigt wave propagation in the HCM. All propagation directions relative to the symmetry axes of the HCM should be considered. It is expedient to do so indirectly, by investigating Voigt wave propagation along the \( z \) axis for all possible orientations of the HCM. Thus, we introduce the HCM permittivity dyadic in the rotated coordinate frame

\[
\tilde{\epsilon}_{HCM}(\alpha, \beta, \gamma) = \tilde{R}(\gamma) \cdot \tilde{R}(\beta) \cdot \tilde{R}(\alpha) \cdot \tilde{\epsilon}_{HCM} \cdot \tilde{R}^{-1}(\beta) \cdot \tilde{R}^{-1}(\gamma) \quad \text{(13)}
\]

where the orthogonal rotation dyadic

\[
\tilde{R}(\beta) = \cos \beta (\hat{\mathbf{x}} \hat{\mathbf{x}} + \hat{\mathbf{z}} \hat{\mathbf{z}}) + \sin \beta (\hat{\mathbf{z}} \hat{\mathbf{z}} - \hat{\mathbf{x}} \hat{\mathbf{x}}) + \hat{\mathbf{y}} \hat{\mathbf{y}}, \tag{15}
\]

and \( \alpha, \beta \) and \( \gamma \) are the three Euler angles [28].

In order for Voigt waves to propagate along the \( z \) axis of the biaxial dielectric material described by the permittivity dyadic (13), the following two necessary and sufficient conditions must be satisfied [29]:

(i) \( Y(\alpha, \beta, \gamma) = 0 \), and

(ii) \( W(\alpha, \beta, \gamma) \neq 0 \),

where the scalars are

\[
Y(\alpha, \beta, \gamma) = -\varepsilon_1^4 + \varepsilon_1^4 + 2\varepsilon_2\varepsilon_3[2\varepsilon_1\varepsilon_2 - (\varepsilon_1 - \varepsilon_2)\varepsilon_3] \tag{16}
\]

and

\[
W(\alpha, \beta, \gamma) = \varepsilon_2^2\varepsilon_3 - (\varepsilon_1 - \varepsilon_2)\varepsilon_3. \tag{17}
\]

Let us note that conditions (i) and (ii) cannot be satisfied by isotropic or uniaxial dielectric materials.
4. Numerical studies

4.1. Preliminaries

We now investigate the scope for Voigt wave propagation in the biaxial HCM, in terms of the shapes, sizes and orientations of the component material particles, as well as the volume fractions of the component materials, by means of representative numerical calculations. Two stages are involved: first, $\epsilon_{\text{HCM}}$ is estimated using the extended Bruggeman formalism; and second, the quantities $Y(\alpha, \beta, \gamma)$ and $W(\alpha, \beta, \gamma)$ are calculated as functions of the Euler angles. More specifically, the angular coordinates ($\alpha, \beta, \gamma$) of the zeros of $|Y|$ and the corresponding values of $|W|$ are computed. Notice that the angular coordinate $\gamma$ may be eliminated from our investigation because propagation parallel to the $z$ axis (of the rotated coordinate system) is independent of rotation about that axis.

In the following, we chose $\epsilon_a = 1.5$ and $\epsilon_b = 12$ as the relative permittivities of the component materials $a$ and $b$. Since $\epsilon_{a,b} \in \mathbb{R}$, the component materials are nondissipative. However, under the extended Bruggeman homogenization formalism with $\eta > 0$, the relative permittivity parameters of the HCM are complex-valued. The imaginary parts of the HCM’s relative permittivity parameters are indicative of losses due to scattering from the macroscopic coherent field [30]. Similar accommodations of scattering loss occur generally in higher-order approaches to homogenization, whenever the component materials are characterized by a (nonzero) length scale. Other examples of formalisms that deliver (nontrivially) complex-valued HCM constitutive parameters even though the component materials are specified by real-valued constitutive parameters include the Maxwell Garnett formalism extended to cater for component particles of nonzero volume [31, 32], and the strong-permittivity-fluctuation theory [33] wherein the statistical distribution of the component particles is characterized by a correlation length. Parenthetically, we note that these higher-order homogenization approaches generally do not scale in a physically-intuitive manner [34], but the physical significance of this aspect is unclear.

We take the spheroid parameters $U_1 = 1 + \rho$ and $U = 1 - (\rho/18)$, and consider the range $0 < \rho < 9$. Thus, the component particles become increasingly elongated as the eccentricity parameter $\rho$ increases from zero, whereas in the limit $\rho \to 0$ the particle shape becomes spherical.

4.2. HCM constitutive parameters

The extended Bruggeman estimates of the HCM’s relative permittivity parameters $\epsilon_{x,y,z,t}$ are plotted as functions of spheroid orientation angle $\varphi$ and volume fraction $f_a$ in figure 1. Here, we fixed the relative size parameter $k_0\eta$ and the eccentricity parameter $\rho$. We now investigate the scope for Voigt wave propagation parallel to the boundaries $\varphi = 0^\circ$ and $90^\circ$, and $f_a = 0$ and 1, but away from these boundary values we have $\text{Re}[\epsilon_i] < 0$ with a local minimum occurring at approximately $\varphi = 45^\circ$ and $f_a = 0.26$. The imaginary parts of $\epsilon_{x,y,z}$ attain their largest values in the vicinity of $f_a = 0.26$, and vanish at $f_a = 0$ and 1. As $\varphi$ increases from zero, $\text{Im}[\epsilon_i]$ generally increases, $\text{Im}[\epsilon_i]$ generally decreases, and $\text{Im}[\epsilon_i]$ is largely unchanged. The imaginary part of $\epsilon_t$ is null-valued along the boundaries $\varphi = 0^\circ$ and $90^\circ$, and $f_a = 0$ and 1, but away from these boundary values $\text{Im}[\epsilon_t]$ exhibits a local maximum which coincides with the local minimum exhibited by $\text{Re}[\epsilon_t]$. The following two limits, which hold for arbitrary $f_a \in (0, 1)$, are especially noteworthy. (a) As $\varphi \to 0^\circ$, we find that $\epsilon_t \neq \epsilon_s = \epsilon_i$; i.e., the HCM is uniaxial. Accordingly the HCM cannot support Voigt wave propagation in this limit [29]. (b) As $\varphi \to 90^\circ$, we find that $\epsilon_s$, $\epsilon_i$, and $\epsilon_t$ are all different from zero, i.e., the HCM’s biaxial structure is orthorhombic.

Next we turn to the dependence upon the component particle shapes and sizes. In figure 2, $\epsilon_{x,y,z,t}$ are plotted against relative size parameter $k_0\eta$ and the eccentricity parameter $\rho$. Here, we fixed the volume fraction $f_a = 0.25$ and the spheroid orientation angle $\varphi = 45^\circ$. The real parts of $\epsilon_{x,y,z}$ generally increase as $\eta$ increases, whereas $\text{Re}[\epsilon_t]$ is largely independent of $\eta$. As $\rho$ increases, $\text{Re}[\epsilon_t]$ generally increases but $\text{Re}[\epsilon_{x,y,z}]$ generally decrease. The most conspicuous feature of the plots of $\text{Re}[\epsilon_{x,y,z,t}]$ is that these quantities increase uniformly from zero as $\eta$ increases from zero. Also, $\text{Im}[\epsilon_{x,y,z}]$ are fairly insensitive to increasing $\rho$ but $\text{Im}[\epsilon_t]$ increases markedly as $\rho$ increases. Let us note that in the limit $\rho \to 0$, we have $\epsilon_s = \epsilon_i = \epsilon_t$ = 0; i.e., the HCM is an isotropic dielectric material, regardless of the size parameter $\eta$ (or the orientation angle $\varphi$ and volume fraction $f_a$).

4.3. Orientations for Voigt waves

Before presenting in detail the results of our numerical study of Voigt wave propagation in the HCM characterized in figures 1 and 2, the following should be pointed out. In general, for a given HCM the Voigt wave conditions $Y = 0$ and $W \neq 0$ are satisfied at two distinct orientations, specified by the angular coordinates $\alpha = \alpha_{1,2}$ and $\beta = \beta_{1,2}$. However, for the numerical investigations reported herein we found that $\alpha_1 \approx \alpha_2$ and $\beta_1 \approx \beta_2$, with the differences between the two orientations being at most approximately 1% and often much less. Consequently, each curve of $\alpha$, corresponding to a zero of $|Y|$, presented in the following figures 3, 6 and 7 would appear to perceive. This matter is elaborated upon in due course, in the discussion of figures 4 and 5.

Let us now begin by considering the effects of volume fraction $f_a$, and spheroid orientation angle $\varphi$, on Voigt wave propagation in the HCM characterized in figures 1 and 2. The angular coordinates $\alpha$ and $\beta$ of the zeros of $|Y|$, along with the corresponding values of $|W|$, are plotted versus $f_a \in (0, 1)$ in figure 3 for $\varphi = 30^\circ$, 60$^\circ$ and 90$^\circ$. The $\alpha$ coordinate for $\varphi = 90^\circ$ changes abruptly at $f_a = 0.48$, from being very close to 90$^\circ$ for $f_a < 0.48$ to being very close to 180$^\circ$ for $f_a > 0.48$. The change in the $\alpha$ coordinate for mid-ranges of $f_a$ becomes progressively less abrupt as $\varphi$ decreases: for $\varphi = 60^\circ$...
Figure 1. The extended Bruggeman estimates of relative permittivity parameters of the HCM plotted versus volume fraction $f_a \in (0, 1)$ and spheroid orientation angle $\varphi \in (0^\circ, 90^\circ)$. The relative size parameter $k_0 \eta = 0.2$ and the eccentricity parameter $\rho = 9$. The corresponding graph of $\alpha$ is approximately sigmoidal while for $\varphi = 30^\circ$ the graph is nearly linear. The $\beta$ coordinate is also highly sensitive to the volume fraction. Indeed, for $\varphi = 90^\circ$, the value of $\beta$ ranges from $0^\circ$ at $f_a = 0.48$ to $90^\circ$ in the limits $f_a \to 0$ and $1$. This sensitivity becomes less pronounced as $\varphi$ decreases. From the corresponding graphs of $|W|$, we can deduce for which values of $f_a$ the HCM supports Voigt wave propagation. In the limits $f_a \to 0$ and $1$ we see that $|W| \to 0$ and therefore the HCM cannot support Voigt waves in these limits. There are three further points for $\varphi = 90^\circ$, namely...
$f_a = 0.2, 0.48$ and $0.83$, at which $|W| = 0$ and the HCM cannot support Voigt wave propagation.

As mentioned earlier, each curve in figure 3 (and indeed in figures 6 and 7) actually represents two closely-spaced but distinct directions for Voigt wave propagation. In order to better appreciate this feature, in figure 4 the curves of $\alpha_1$ and $\alpha_2$ for $\varphi = 90^\circ$ in figure 3 are reproduced at much higher resolution. The corresponding curves of $\beta_{1,2}$ are indistinguishable even at this higher resolution. At values of $f_a < 0.48$, the graphs of $\alpha_1$ and $\alpha_2$ are mirror images with

**Figure 2.** As figure 1 except that the HCM’s relative permittivity parameters are plotted versus the relative size parameter $k_0 \eta \in (0, 0.2)$ and the eccentricity parameter $\rho \in (0, 9)$. The volume fraction $f_a = 0.25$ and spheroid orientation angle $\varphi = 45^\circ$. 
The angular coordinates $\alpha$, $\beta$, and the absolute value of the quantity $W$, plotted versus volume fraction $f_a$ for spheroid orientation angle $\phi = 90^\circ$ (blue, dashed curves), $60^\circ$ (green, solid curves) and $30^\circ$ (red, broken dashed curves). The size parameter $\eta = 0.2/k_0$ and the eccentricity parameter $\rho = 9$.

In order to illustrate the two distinct orientations for Voigt wave propagation for $\phi \neq 90^\circ$, in figure 5 a representative example is provided for $\phi = 30^\circ$, with $f_a = 0.2$ and $0.83$, correspond to the zeros of $W$ observed in figure 3.

Next the influence of the component particles’ shape on the propagation of Voigt waves is considered. The angular coordinates $\alpha$ and $\beta$ of the zeros of $|Y|$, along with the corresponding values of $|W|$, are plotted as functions of the eccentricity parameter $\rho \in (0, 9)$ in figure 6 for the spheroid orientation angle $\phi \in [30^\circ, 60^\circ, 90^\circ]$. The volume fraction was fixed at $f_a = 0.25$ and the size parameter $\eta = 0.2/k_0$. The values of $\alpha$ for each value of $\phi$ are quite different, but
Figure 6. The angular coordinates $\alpha$, $\beta$, and the absolute value of the quantity $W$, plotted versus the eccentricity parameter $\rho$ for spheroid orientation angle $\varphi = 90^\circ$ (blue, dashed curves), $60^\circ$ (green, solid curves) and $30^\circ$ (red, broken dashed curves). The size parameter $\eta = 0.2/k_0$ and the volume fraction $f_a = 0.25$.

Figure 7. The angular coordinates $\alpha$, $\beta$, and the absolute value of the quantity $W$, plotted versus the relative size parameter $k_0 \eta$ for spheroid orientation angle $\varphi = 90^\circ$ (blue, dashed curves), $60^\circ$ (green, solid curves) and $30^\circ$ (red, broken dashed curves). The volume fraction $f_a = 0.25$ and the eccentricity parameter $\rho = 9$.

these values are almost independent of $\rho$. The values of $\beta$ for each value of $\varphi$ are also quite different, but for the values of this angular coordinate decrease gradually as $\rho$ increases. The corresponding values of $|W|$ increase as $\rho$ increases, most rapidly for $\varphi = 90^\circ$ and least rapidly for $\varphi = 30^\circ$. Furthermore, $|W| \to 0$ as $\rho \to 0$, regardless of $\varphi$, as would be expected since the HCM becomes an isotropic dielectric material in this limit.

Lastly, we turn to the effect of the component particle sizes, as accommodated by the extended Bruggeman formalism. In figure 7, graphs of the angular coordinates $\alpha$ and $\beta$ at which $|Y| = 0$, and the corresponding values of $|W|$, against the relative size parameter $k_0 \eta \in (0, 0.2)$ are displayed for the spheroid orientation angle $\varphi \in \{30^\circ, 60^\circ, 90^\circ\}$. Here the volume fraction $f_a = 0.25$ and the eccentricity parameter $\rho = 9$. While the size parameter $\eta$ has very little influence upon the orientations for Voigt waves, as indicated by the nearly horizontal graphs of $\alpha$ and $\beta$, it does have a profound influence upon whether or not Voigt waves can propagate. Since $|W| \to 0$ as $\eta \to 0$, we infer that the nondissipative HCM—arising from vanishingly small component particles—does not support Voigt wave propagation. In contrast, for nonzero values of $\eta$ we see that $|W| > 0$ and therefore Voigt wave propagation is supported.

5. Closing remarks

Engineered materials, in the form of HCMs, may be conceptualized which support Voigt wave propagation while their component materials do not. The case considered here
involved remarkably simple component materials, namely nondissipative isotropic dielectric materials, in contrast to earlier studies on Voigt-wave-supporting HCMs which involved dissipative uniaxial dielectric components [5, 17]. In the present case, the ability of the HCMs to support Voigt waves relied upon the shapes, orientations and nonzero sizes of the component particles, as well as the volume fractions of the component materials. In order to cater for such component materials, an extended version [20] of the well-established Bruggeman homogenization formalism [19, 21] was needed. We note that the homogenization approaches adopted in earlier Voigt wave studies, to wit the Maxwell Garnett formalism [5] and strong-permittivity-fluctuation theory [17], could not be used here because they cannot accommodate component particles with differing orientations or component particles of nonzero size.

Our numerical investigations revealed that the directions in which Voigt waves may propagate are highly sensitive to the orientations of the component particles and to the volume fractions of the component materials, but less sensitive to the sizes of the component particles and less sensitive still to the volumes of the component particles. Furthermore, whether or not a HCM supports Voigt wave propagation is critically dependent upon the sizes of the component particles and, in certain cases, upon the volume fractions. For the scenarios considered here, the two directions which support Voigt wave propagation are very close together, in general. This is a consequence of the component materials being nondissipative. For dissipative component materials, these two directions can be more widely spaced [17].

This study further emphasizes the role of the micro- and/or nano-structure in determining the macroscopic optical properties of engineered materials, and paves the way for a study of Voigt wave propagation in bianisotropic HCMs, which—courtesy of their enormous parameter space—present more possibilities for Voigt waves to occur.

### Appendix

We provide here a mathematical description of Voigt waves, propagating in a dissipative biaxial dielectric material, as specified by a permittivity dyadic of the form given in equation (14). This dyadic is symmetric but non-Hermitian. Furthermore, we emphasize that the material is not chiral nor does it support Faraday rotation.

Without loss of generality, let us consider plane waves propagating along the z axis. The corresponding electromagnetic field phasors may be expressed as

\[
\begin{align*}
\begin{bmatrix} E_x(r) \hat{x} + E_y(r) \hat{y} + E_w(r) \hat{z} \\ H_x(r) \hat{x} + H_y(r) \hat{y} + H_w(r) \hat{z} \end{bmatrix} &= \begin{bmatrix} e_x \hat{x} + e_y \hat{y} + e_w \hat{z} \exp(ik_0kz) \\ h_x \hat{x} + h_y \hat{y} + h_w \hat{z} \exp(ik_0kz) \end{bmatrix},
\end{align*}
\]

(18)

with \( k \) being the relative wavenumber. After combining equations (14) and (18) with the Maxwell curl equations, and then solving for the \( z \)-directed phasor components, we arrive at the matrix equation governing propagation [29]:

\[
\frac{1}{\epsilon_{33}} \begin{bmatrix} \epsilon_{11} \epsilon_{33} - \epsilon_{13}^2 & \epsilon_{12} \epsilon_{33} - \epsilon_{13} \epsilon_{23} \\ \epsilon_{12} \epsilon_{33} - \epsilon_{13} \epsilon_{23} & \epsilon_{22} \epsilon_{33} - \epsilon_{23}^2 \end{bmatrix} \begin{bmatrix} e_x \\ e_y \end{bmatrix} = k^2 \begin{bmatrix} e_x \\ e_y \end{bmatrix}.
\]

(19)

In the usual (i.e., non-Voigt wave) biaxial scenario, the propagation matrix on the left side of equation (19) has two eigenvalues, \( k = k_+ \), say, each being associated with only one eigenvector. Suppose that \( k_+ \) is associated with the eigenvector \([e_{x+}, e_{y+}]^T\), and \( k_- \) is associated with the eigenvector \([e_{x-}, e_{y-}]^T\). Then the \( x \) and \( y \) components of the corresponding plane wave solution may expressed as

\[
\begin{bmatrix} E_x(z) \\ E_y(z) \end{bmatrix} = C_+ \begin{bmatrix} e_{x+} \\ e_{y+} \end{bmatrix} \exp(ik_+z) + C_- \begin{bmatrix} e_{x-} \\ e_{y-} \end{bmatrix} \exp(ik_-z),
\]

(20)

with \( C_\pm \) being amplitude coefficients.

In the Voigt wave scenario, the propagation matrix on the left side of equation (19) has only one eigenvalue, say \( k = k_v \), which is associated with a single eigenvector, say \([e_{vx}, e_{vy}]^T\). The \( x \) and \( y \) components of the Voigt wave solution may expressed as

\[
\begin{bmatrix} E_x(z) \\ E_y(z) \end{bmatrix} = \left( C_0 + z C_\beta \right) \begin{bmatrix} e_{vx} \\ e_{vy} \end{bmatrix} + C_\beta \begin{bmatrix} e_{wx} \\ e_{wy} \end{bmatrix} \exp(ik_vz),
\]

(21)

where \([e_{wx}, e_{wy}]^T\) is a generalized eigenvector and \( C_{0,\beta} \) are amplitude coefficients.

More generally, including cases involving bianisotropic materials, Voigt waves arise whenever the algebraic multiplicity of an eigenvalue of the corresponding propagation matrix exceeds its geometric multiplicity. Further details can be found elsewhere [35].

### References

[1] Lakhtakia A (ed) 1996 Selected Papers on Linear Optical Composite Materials (Bellingham, WA: SPIE Optical Engineering Press)
[2] Milton G W 2002 The Theory of Composites (Cambridge: Cambridge University Press)
[3] Mackay T G and Lakhtakia A 2006 Correlation length and negative phase velocity in isotropic dielectric-magnetic materials J. Appl. Phys. 100 063533
[4] Mackay T G and Lakhtakia A 2004 Plane waves with negative phase velocity in Faraday chiral mediums Phys. Rev. E 69 026602
[5] Mackay T G and Lakhtakia A 2003 Voigt wave propagation in biaxial composite materials J. Opt. A: Pure Appl. Opt. 5 91–5
[6] Born M and Wolf E 1980 Principles of Optics 6th edn (Oxford: Pergamon)
[7] Voigt W 1902 On the behaviour of pleochroitic crystals along directions in the neighbourhood of an optic axis Phil. Mag. 4 90–7
[8] Pancharatnam S 1957 Light propagation in absorbing crystals possessing optical activity Proc. Industr. Acad. Sci. B 46 280–302
[9] Khapalyuk A P 1962 On the theory of circular optical axes Opt. Spectrosc. (USSR) 12 52–4
[10] Fedorov F I and Goncharenko A M 1963 Propagation of light along the circular optical axes of absorbing crystals Opt. Spectrosc. (USSR) 14 51–3
[11] Agranovich V M and Ginzburg V L 1984 Crystal Optics with Spatial Dispersion, and Excitons (Berlin: Springer)
[12] Grechushnikov B N and Konstantinova A F 1988 Crystal optics of absorbing and gyrotropic media Comput. Math. Appl. 16 637–55
[13] Berry M V and Dennis M R 2003 The optical singularities of birefringent dichroic chiral crystals Proc. R. Soc. A 459 1261–92
[14] Lakhtakia A 1998 Anomalous axial propagation in helicoidal bianisotropic media Opt. Commun. 157 193–201
[15] Berry M V 2005 The optical singularities of bianisotropic crystals Proc. R. Soc. A 461 2071–98
[16] Zhuck N P 1994 Strong–fluctuation theory for a mean electromagnetic field in a statistically homogeneous random medium with arbitrary anisotropy of electrical and statistical properties Phys. Rev. B 50 15636–45
[17] Mackay T G and Lakhtakia A 2004 Correlation length facilitates Voigt wave propagation Waves Random Media 14 L1–11
[18] Mackay T G and Weiglhofer W S 2000 Homogenization of biaxial composite materials: dissipative anisotropic properties J. Opt. A: Pure Appl. Opt. 2 426–32
[19] Goncharenko A V 2003 Generalizations of the Bruggeman equation and a concept of shape-distributed particle composites Phys. Rev. E 68 041108
[20] Mackay T G 2004 Depolarization volume and correlation length in the homogenization of anisotropic dielectric composites Waves Random Mediat 14 485–98
Mackay T G 2006 Waves Random Complex Media 16 85 (erratum)
[21] Mackay T G and Lakhtakia A 2008 Electromagnetic fields in linear bianisotropic mediums Prog. Opt. 51 121–209
[22] Mackay T G 2005 Linear and nonlinear homogenized composite mediums as metamaterials Electromagnetics 25 461–81
[23] Buchanan J L and Turner P R 1992 Numerical Methods and Analysis (New York: McGraw-Hill)
[24] Michel B 1997 A Fourier space approach to the pointwise singularity of an anisotropic dielectric medium Int. J. Appl. Electromagn. Mech. 8 219–27
[25] Michel B and Weiglhofer W S 1997 Pointwise singularity of dyadic Green function in a general bianisotropic medium Arch. Elektron. Übertrag. 51 219–23
Michel B and Weiglhofer W S 1998 Arch. Elektron. Übertrag. 52 31 (erratum)
[26] Weiglhofer W S 1998 Electromagnetic depolarization dyadics and elliptic integrals J. Phys. A: Math. Gen. 31 7191–6
[27] Mackay T G 2008 On extended homogenization formalisms for nanocomposites J. Nanophoton. 2 021850
[28] Arfken G B and Weber H J 1995 Mathematical Methods for Physicists 4th edn (London: Academic)
[29] Gerardin J and Lakhtakia A 2001 Conditions for Voigt wave propagation in linear, homogeneous, dielectric mediums Optik 112 493–5
[30] Van Kranendonk J and Sipe J E 1977 Foundations of the macroscopic electromagnetic theory of dielectric media Prog. Opt. 15 245–350
[31] Shanker B and Lakhtakia A 1993 Extended Maxwell Garnett model for chiral-in-chiral composites J. Phys. D: Appl. Phys. 26 1746–58
[32] Prinkey M T, Lakhtakia A and Shanker B 1994 On the extended Maxwell–Garnett and the extended Bruggeman approaches for dielectric-in-dielectric composites Optik 96 25–30
[33] Tsang L, Kong J A and Newton R W 1982 Application of strong fluctuation random medium theory to scattering of electromagnetic waves from a half-space of dielectric mixture IEEE Trans. Antennas Propagat. 30 292–302
[34] Bohren C F 2009 Do extended effective-medium formulas scale properly? J. Nanophoton. 3 039501
[35] Mackay T G and Lakhtakia A 2010 Electromagnetic Anisotropy and Bianisotropy: A Field Guide (Singapore: World Scientific)