NONCOMMUTATIVITY IN A SIMPLE TOY MODEL

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Abstract: We discuss various symmetry properties of the reparametrization invariant toy model of a free non-relativistic particle and show that its commutativity and noncommutativity (NC) properties are the artifact of the underlying symmetry transformations. For the case of the symmetry transformations corresponding to the noncommutative geometry, the mass parameter of the toy model turns out to be noncommutative in nature. By exploiting the Becchi-Rouet-Stora-Tyutin (BRST) symmetry transformations, we demonstrate the existence of this NC and show its cohomological equivalence with its commutative counterpart. A connection between the usual gauge symmetry transformations corresponding to the commutative geometry and the quantum groups, defined on the phase space, is also established for the present model at the level of Poisson bracket structure. We show that, for the NC geometry, such a kind of quantum group connection does not exist.

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1 Introduction

During the past few years, there has been an upsurge of interest in the study of physics, formulated on the general $D$-dimensional noncommutative spaces (i.e. $[x_i, x_j] \neq 0; i, j = 1, 2, ..., D$). Such an interest has been thriving because of some very exciting modern advances in the realm of string theory, matrix models, quantum gravity, black hole physics, etc., which have brought out a few intriguing aspects of the noncommutativity (NC) in the spacetime structure (see, e.g., [1-4] and references therein). This kind of NC has also been discussed in the simpler settings where the redefinitions (and/or the change of variables) play a very decisive role in the construction of the NC in the spacetime. To be more precise, these redefinitions allow a mapping between the commutative spacetime to the noncommutative spacetime. Both versions of the spacetime, at times, describe the same physical system because of the fact that the dynamical equations of motion remain unchanged. Many examples of quantum mechanical models, particle mechanics, Landau problem, etc., belong to this category where the choices of the gauge, redefinitions, restricted symmetries, etc., lead to the NC in the spacetime structure (see, e.g., [5-13] and references therein).

Since time immemorial, the basic concepts of symmetries have played some notable roles in the development of theoretical physics. In a recent set of papers [10,11,13], the continuous symmetries have been exploited for some models to demonstrate the existence of the NC. The purpose of our present paper is to exploit a set of non-standard symmetry transformations for the toy model of a massive non-relativistic (NR) free particle and show the existence of a NC in the spacetime structure. The insights gained from this toy model might turn out to be quite useful in the discussion of the more subtle reparametrization invariant models of the relativistic particles and strings. The above NR system is rendered reparametrization invariant by hand because the “time” parameter $t$ of this model is treated on a par with the space variable $x$ (see, e.g., [14] for details). As a consequence, both these variables constitute the configuration space in which a trajectory of the above toy model is parametrized by a new evolution parameter $\tau$. This model is shown to be endowed with the first-class constraints in the language of Dirac’s prescription for the classification of constraints [15]. These constraints generate a gauge symmetry transformation which turns out to be equivalent to the reparametrization symmetry transformation for the model in a certain specific limit. This equivalence occurs because of the fact that both the above symmetry transformations owe their origin to the same constraint(s) (cf. Sect. 3 below).

A variant of the above gauge symmetry transformation, that leads to the existence of the NC, has been christened by us as the non-standard symmetry transformation because it is not generated by the first-class constraints of the theory and the transformations for the space and time variables are put by hand * to generate the NC in the transformed spacetime.

*The logical reason behind this kind of choice (that leads to NC in spacetime) is given in Sec. 5 in terms of the BRST transformations and their cohomological properties. In fact, a whole set of transformations are allowed by the cohomological considerations of the BRST transformations but only a small subset (cf. (4.5) below) corresponds to the symmetry transformation for the Lagrangian of the present toy model.
frames. The rest of the transformations for the dynamical variables are derived by demanding the consistency among (i) the equations of motion, (ii) the expressions for the canonical momenta, and (iii) the basic transformations for the space and time variables. The above consistency requirements enforce this non-standard symmetry transformation to reduce to a specific form (cf. (4.5) below) of the usual gauge transformation. However, in the process, the mass parameter of the toy model becomes noncommutative to retain the NC associated with the original non-standard symmetry transformations †. Quite independently, the above specific form of the gauge transformations (cf. (4.5)) can also be obtained directly from the gauge symmetry transformations that (i) are generated by the first-class constraints, and (ii) correspond to the commutative geometry. Thus, the NC and commutativity, present in this model, are different facets of the continuous symmetry transformations. This observation is consistent with the result of [9] where the equivalence between the NC and commutativity for the present toy model has been demonstrated in the language of the Dirac bracket formalism for two different choices of the gauge conditions (which, in turn, are connected with each-other by a specific kind of gauge transformation).

To the best of our knowledge, the reason behind the existence of such kind of an equivalence between the NC and commutativity has not yet been discussed in the modern literature by exploiting the continuous symmetry transformations alone.

One of the central results of our present paper is to demonstrate, the above kind of equivalence, by exploiting the Becchi-Rouet-Stora-Tyutin (BRST) symmetry transformations and their cohomological considerations. We show, in particular, that the NC and commutativity of this model are cohomologically equivalent (cf. (5.7), (5.8) and (4.5) below). We also consider the connection of the commutativity and NC of this toy model within the framework of quantum groups, defined in the phase space of this model. We demonstrate that the commutativity associated with the gauge symmetry transformations (cf. (3.3) below) can be captured in the language of the quantum groups in the phase space. As it turns out, the above commutativity is associated with the quantum group $SL_{q,q^{-1}}(2)$ when the phase space variables are transformed under it. This connection is demonstrated at the level of the Poisson bracket structures alone. We also show that the special type of NC associated with the non-standard gauge type transformations (cf. (4.1) below) cannot be captured by the quantum groups (that have been considered on the phase space). This is proven, once again, only at the level of the Poisson bracket structure. The equivalence between the NC and commutativity, shown by BRST approach, cannot arise in the realm of the quantum groups because we do not discuss the symmetry property of the Lagrangian in the framework of the latter. Rather, we focus only on the Poisson bracket structure.

Our present investigation is interesting and essential on the following grounds. First and foremost, our present model provides a very simple setting to discuss the NC in spacetime

†It turns out that the mass parameter becomes noncommutative only with the space variable $x(\tau)$ but it remains commutative with the other configuration variable $t(\tau)$. In fact, to retain the NC of the original non-standard symmetry transformations, the mass parameter remains commutative with all the dynamical as well as the auxiliary variables of the theory but it becomes noncommutative with the space variable.
structure. This topic, as is obvious from our earlier discussions, is an active area of research in theoretical high energy physics. Second, our model is one of the simplest models to possess the celebrated reparametrization invariance. Thus, its study would provide some useful insights that would propel us to our central endeavour of studying the reparametrization invariant models of (super)gravity and (super)strings. Third, our present study provides a meeting-ground for the discussion for the two types of NCs that are associated with (i) the quantum groups, and (ii) the Snyder-type spacetime considerations. Fourth, the existence of the NC in our model owes its origin to the continuous symmetry transformations which are somewhat new in their contents and textures. Finally, the key idea of our present investigation has already been generalized to more interesting systems (see, e.g., [22,23]). Thus, our present investigation is the first step towards our main goal of developing a general theoretical scheme to produce NCs in the spacetime structure in a consistent fashion.

The contents of our present paper are organized as follows. In Sec. 2, we discuss the continuous reparametrization symmetry transformations for the toy model of a free massive non-relativistic particle. Section 3 is devoted to the discussion of the gauge symmetry transformations for this system which are equivalent to the reparametrization transformations in a specific limit. We discuss a non-standard set of continuous symmetry transformations for the variables of the first-order Lagrangian of this model in Sec. 4. The (anti-)BRST transformations for this toy model and comments on the cohomological equivalence of the NC and commutativity, are considered in Sec. 5. Section 6 deals with the connection between the above commutative and noncommutative symmetry transformations and the quantum group symmetry transformations in the phase space of the above toy model where emphasis is laid on the Poisson bracket structures. Finally, we make some concluding remarks in Sec. 7 and point out a few future directions for further investigations.

2 Reparametrization Symmetry Transformations

We begin with the action integral \( S_t \) for the \((1 + 1)\)-dimensional system of a free non-relativistic (NR) particle with mass \( m \), as given below

\[
S_t = \int dt \ L_0^{(t)}(x, \dot{x}) \equiv \int dt \ \frac{m\dot{x}^2}{2}, \tag{2.1}
\]

where the Lagrangian \( L_0^{(t)}(x, \dot{x}) \), for the free NR particle, depends only on the square of the velocity \( \dot{x} = \frac{dx}{dt} \) variable \( \dot{x} \) that is constructed from the displacement vector \( x \) and the evolution parameter “time” \( t \). It is evident that the above action is not endowed with the reparametrization symmetry. However, this symmetry can be brought in by treating the “time” parameter \( t \) as a dynamical variable \( t(\tau) \) on a par with the displacement variable \( x(\tau) \) (see, e.g., [14]) where \( \tau \) is the new evolution parameter. The action integral \( S_t \) (integrated over the element \( dt \) with Lagrangian \( L_0^{(t)}(x, \dot{x}) \)) of (2.1) can be transformed to the action
integral \( S_\tau \) (integrated over the element \( d\tau \) with Lagrangian \( L_0^{(\tau)}(x, \dot{x}, t, \dot{\tau}) \)) as (see, e.g., [9])

\[
S_\tau = \int d\tau \, L_0^{(\tau)}(x, \dot{x}, t, \dot{\tau}) \equiv \int d\tau \, \frac{m\dot{x}^2}{2t},
\]

(2.2)

where, now, we have \( \dot{x} = dx/d\tau, \dot{\tau} = dt/d\tau \) and the Jacobian \( J \) (in \( dt = Jd\tau \)) is given by \( J = dt/d\tau \). It can be checked that this system is endowed with the first-class constraint

\[
p_x^2 + 2mp_t \approx 0,
\]

(2.3)

in the language of Dirac’s prescription [15] for the classification of constraints where the conjugate momenta \( p_x(\tau) \) and \( p_t(\tau) \), corresponding to the configuration variables \( x(\tau) \) and \( t(\tau) \), are defined in terms of the Lagrangian function \( L_0^{(\tau)}(x, \dot{x}, t, \dot{\tau}) \), as

\[
p_x = \frac{\partial L_0^{(\tau)}}{\partial \dot{x}} = \frac{m\dot{x}}{t}, \quad p_t = \frac{\partial L_0^{(\tau)}}{\partial \dot{\tau}} \equiv -\frac{m\dot{x}^2}{2t^2}.
\]

(2.4)

The symbol \( \approx \) in (2.3) stands for the weak equality (see, e.g., [15] for details). The first-order \( (L_f^{(\tau)}) \)- and the second-order \( (L_s^{(\tau)}) \) Lagrangians can be derived from \( L_0^{(\tau)} \) by exploiting (i) the Legendre transformation, and (ii) the explicit expressions for the momenta \( p_x \) and \( p_t \) as given in (2.4). The ensuing Lagrangians, thus obtained, are as follows

\[
L_f^{(\tau)} = p_x\dot{x} + p_t\dot{\tau} - \frac{1}{2} E (p_x^2 + 2mp_t),
\]

\[
L_s^{(\tau)} = \frac{\dot{x}^2}{2E} + \frac{m\dot{x}^2}{2t} \left[ \frac{Em}{t} - 1 \right],
\]

(2.5)

where \( E(\tau) \) is a Lagrange multiplier that takes care of the constraint (2.3). At this stage, a few comments are in order. First, the massless limit (i.e. \( m \to 0 \)) for the Lagrangian in (2.1) and/or (2.2) is not well defined. However, the Lagrangians in (2.5) do allow the existence of such a limit. Second, all the Lagrangians \( L_0^{(\tau)}, L_f^{(\tau)} \) and \( L_s^{(\tau)} \) are equivalent as can be seen by exploiting the equations of motion \( \dot{x} = E p_x, \dot{t} = E m, \dot{p}_x = 0, \dot{p}_t = 0 \) that emerge from the first-order Lagrangian \( L_f^{(\tau)} \) and the expressions for the momenta \( p_x \) and \( p_t \) given in (2.4). It will be noted that the free motion (i.e. \( \dot{p}_x = 0, \dot{p}_t = 0 \)) of the NR particle leads to the second-order equations of motion: \( \ddot{x}E - \dot{x}\dot{E} = 0, \ddot{t} - \dot{x}\dot{t} = 0 \) which emerge from \( L_f^{(\tau)} \) and \( L_s^{(\tau)} \). Third, all the three Lagrangians in (2.2) and (2.5) are reparametrization invariant. For the sake of simplicity \(^\dagger\), however, let us check the invariance of the first-order Lagrangian \( L_f^{(\tau)} \) under the infinitesimal reparametrization transformation \( \tau \to \tau' = \tau - \epsilon(\tau) \) where \( \epsilon(\tau) \) is an infinitesimal local parameter. All the variables of \( L_f^{(\tau)} \) undergo the following change under the above infinitesimal reparametrization transformation

\[
\delta_\tau x = \epsilon \dot{x}, \quad \delta_\tau t = \epsilon \dot{\tau}, \quad \delta_\tau p_x = \epsilon \dot{p}_x, \quad \delta_\tau p_t = \epsilon \dot{p}_t, \quad \delta_\tau E = \frac{d}{d\tau} [\epsilon E],
\]

(2.6)

\(^\dagger\)In contrast to the Lagrangians \( L_0^{(\tau)} \) and \( L_s^{(\tau)} \), the first-order Lagrangian \( L_f^{(\tau)} \) is simpler in the sense that it has no variable (and its derivative(s) w.r.t. \( \tau \)) in the denominator. Furthermore, it is endowed with the largest number of variables (i.e. \( x, p_x, t, p_t, E \)) that allow more freedom for theoretical discussions.
where the transformation $\delta_\tau$ for a generic variable $\phi(\tau) \equiv x(\tau), t(\tau), p_x(\tau), p_t(\tau), E(\tau)$ is defined as $\delta_\tau \phi(\tau) = \phi'(\tau) - \phi(\tau)$. It is elementary to verify that the first-order Lagrangian $L_f^{(\tau)}$ remains quasi-invariant under the transformation $\delta_\tau$ as it changes to a total derivative w.r.t. $\tau$ (i.e. $\delta_\tau L_f^{(\tau)} = (d/d\tau) [\epsilon L_f^{(\tau)}]$). Similarly, one can check that, under (2.6), the other Lagrangians transform as: $\delta_\tau L_0^{(\tau)} = (d/d\tau) [\epsilon L_0^{(\tau)}]$ and $\delta_\tau L_s^{(\tau)} = (d/d\tau) [\epsilon L_s^{(\tau)}]$.

3 Gauge Symmetry Transformations

It is well-known that the existence of the first-class constraints on a physical system entails upon the Lagrangian of the system to be endowed with some gauge type of continuous symmetry transformations. These symmetry transformations are generated by the first-class constraints themselves. For instance, for the present toy model described by the first-order Lagrangian $L_f^{(\tau)}$, there are only two first-class constraints. These are explicitly written as $\Pi_E \approx 0, p_x^2 + 2m p_t \approx 0$ where $\Pi_E$ is the conjugate momentum corresponding to the Lagrange multiplier variable $E(\tau)$. The generator $G$ for the gauge transformations can be written, in terms of the above first-class constraints, as

$$G = \Pi_E \dot{\xi} + \frac{\xi}{2} (p_x^2 + 2mp_t),$$

(3.1)

where $\xi(\tau)$ is the local infinitesimal gauge parameter for the above transformations. The explicit form of the gauge transformation $\delta_g$ for the generic local variable $\phi(\tau) = x(\tau), t(\tau), p_x(\tau), p_t(\tau), E(\tau)$ can be written, in terms of the above generator $G$, as

$$\delta_g \phi(\tau) = -i [\phi(\tau), G],$$

(3.2)

where, for the sake of explicit computation, the canonical commutation relations: $[x, p_x] = i, [t, p_t] = i, [E, \Pi_E] = i$ have to be exploited with the understanding that all the other canonical commutators are zero. Ultimately, one obtains the following gauge transformations for all the $\tau$ dependent dynamical variables

$$\delta_g x = \xi \ p_x, \quad \delta_g t = \xi \ m, \quad \delta_g p_x = 0, \quad \delta_g p_t = 0, \quad \delta_g E = \dot{\xi}.$$  (3.3)

Under the above infinitesimal continuous gauge transformations, the first-order Lagrangian $L_f^{(\tau)}$ remains quasi-invariant because it changes to a total derivative. This statement can be written explicitly in the mathematical form as

$$\delta_g L_f^{(\tau)} = \frac{d}{d\tau} \left[ \frac{\xi}{2} p_x^2 \right].$$

(3.4)

The above gauge transformation is the symmetry transformation for the other Lagrangians too. In the explicit form, it can be checked that the following infinitesimal gauge transformations ($\delta_g^{(0)}$) for the case of the Lagrangian $L_0^{(\tau)}$, lead to: $\delta_g^{(0)} x = \xi \ (\frac{mp_t}{\Pi_E}), \delta_g^{(0)} t = \xi \ m, \delta_g^{(0)} L_0^{(\tau)} = \frac{d}{d\tau} \left[ \frac{\xi}{2} \left( \frac{mp_t}{\Pi_E} \right)^2 \right]$. Similarly, for the second-order Lagrangian $L_s^{(\tau)}$, the infinitesimal gauge transformations ($\delta_g^{(s)}$) on the dynamical variables and the Lagrangian yield: $\delta_g^{(s)} x = \xi \ (\frac{mp_t}{\Pi_E}), \delta_g^{(s)} t = \xi \ m, \delta_g^{(s)} E = \dot{\xi}, \delta_g^{(s)} L_s^{(\tau)} = \frac{d}{d\tau} \left[ \frac{\xi}{2} \left( \frac{mp_t}{\Pi_E} \right)^2 \right]$.
It will be noted that the continuous gauge transformations (3.3) can also be obtained from the reparametrization symmetry transformations (2.6) if we exploit the Euler-Lagrange equations of motion $\dot{x} = EP_x, \dot{t} = Em, \dot{p}_x = 0, \dot{p}_t = 0$ (emerging from the first-order Lagrangian $L_{f}^{(r)}$) and identify $\epsilon(\tau) = \xi(\tau)/E(\tau)$. The root cause behind the existence of the above connection between the continuous gauge- and reparametrization symmetries is the fact that both these symmetries are generated by the first-class constraints of the theory.

The gauge-transformations (3.3) have been derived by exploiting the generator $G$, the basic canonical commutators and the generic transformation in (3.2). However, it can be proven non-trivially that these gauge transformations $\delta_g E = \xi, \delta_g p_x = 0, \delta_g p_t = 0$ are correct. For this purpose, we take, as inputs, only the transformations for the basic variables $x$ and $t$ and demand their consistency with the equations of motion and the expressions for momenta in (2.4). For instance, the equations of motion $\dot{t} = Em, \dot{p}_x = \dot{x}/E$ and the expression $p_t = -(mx^2)/(2t^2)$ imply that $\delta_g \dot{t} = m\delta_g E, \delta_g p_x = \delta_g (\dot{x}/E), \delta_g p_t = -\frac{m}{2} \delta_g (\dot{x}^2/t^2)$. Using $\delta_g t = \xi m, \delta_g x = \xi p_x$, we obtain $\delta_g E = \dot{\xi}, \delta_g p_x = (\dot{\xi}/E)(p_x - \dot{x}/E) + (\dot{\xi}/E) \dot{p}_x$. However, the equations of motion $\dot{p}_x = 0$ and $p_x = (\dot{x}/E)$, imply that $\delta_g p_x = 0$. Similarly, the requirement of the consistency among (i) the basic gauge transformations for $x$ and $t$, (ii) the expressions (2.4), and (iii) the equations of motion, leads to $\delta_g p_t = 0$. The explicit computation

$$\delta_g p_t = -\frac{m\dot{x}\dot{\xi}}{t^2} - \frac{m\dot{x}}{t} - \frac{m\dot{x}\xi}{t^2} \dot{p}_x,$$

(3.5)
demonstrates that $\delta_g p_t = 0$ because of the fact that $\dot{p}_x = 0$ and $p_x = (m\dot{x})/t$. It should be noted that the transformations $\delta_g p_x = 0, \delta_g p_t = 0$ are also consistent with the other equations of motion: $\dot{p}_x = 0, \dot{t} = 0, p_x^2 + 2mp_t = 0$. This is due to the fact that $\delta_g \dot{p}_x = 0, \delta_g \dot{t} = 0$ and $2p_x \delta_g p_x + 2m \delta_g p_t = 0$ are automatically satisfied. Thus, we note that the definitions in (2.4), the gauge transformations in (3.3) and the equations of motion (deduced from $L_{f}^{(r)}$) are consistent with one-another. This non-trivial trick for the derivation of the symmetry transformations will be exploited in the next section too.

4 Non-Standard Symmetry Transformations and Noncommutativity

The usual gauge transformations $\delta_g$ in (3.3) imply that the transformed frame is now characterized by $x^{(g)} = x + \xi p_x, t^{(g)} = t + \xi m$. With the help of (i) the canonical Poisson brackets $\{x, p_x\}_{(PB)} = 1, \{t, p_t\}_{(PB)} = 1, \{x, x\}_{(PB)} = 0, \{x, t\}_{(PB)} = 0, \{p_x, p_x\}_{(PB)} = 0, \{t, t\}_{(PB)} = 0, \{p_t, p_t\}_{(PB)} = 0$, etc., and, (ii) treating the mass parameter to be commutative with everything, it can be seen that, in the gauge transformed frame too, we have the commutative geometry (i.e. $[x^{(g)}, t^{(g)}] = 0$) because of the fact that $\{x^{(g)}, t^{(g)}\}_{(PB)} = 0$. To bring in the noncommutative geometry, let us focus on some non-standard transformations $\tilde{\delta}_g$

$$x(\tau) \rightarrow x^{(ns)}(\tau) = x(\tau) + \zeta(\tau)m \Rightarrow \tilde{\delta}_g x(\tau) = \zeta(\tau)m, \tag{4.1}$$
$$t(\tau) \rightarrow t^{(ns)}(\tau) = t(\tau) + \zeta(\tau)p_x(\tau) \Rightarrow \tilde{\delta}_g t(\tau) = \zeta(\tau)p_x(\tau),$$

where $\zeta(\tau)$ is an infinitesimal transformation parameter. These transformations have been
chosen because (i) they lead to the NC in the transformed frame due to \( \{x^{(ns)}, t^{(ns)}\}_{(PB)} = \zeta(\tau) \), and (ii) they have some relevance in the context of the nilpotent BRST symmetries and BRST cohomology (cf. Sec. 5 below). Let us have a close look at (4.1). Here the infinitesimal increments in the space and time variables have connections with the corresponding increments in the gauge transformations (3.3). In fact, the increments of the latter are exchanged in the former (i.e. \( \tilde{\delta}_g x = \delta_g t, \tilde{\delta}_g t = \delta_g x \)). This is, moreover, consistent with the gauge choice(s) made in [9]. As we have argued earlier for the non-trivial way of deriving the usual gauge transformations (3.3), we derive here, exactly in a similar fashion, the non-standard transformations \( \tilde{\delta}_g \) for all the variables of the first-order Lagrangian \( L_f^{(\tau)} \).

The basic inputs for such a derivation are the requirements of the consistency among (i) the equations of motion, (ii) the expressions for the momenta (2.4), and (iii) the basic transformations (4.1) for the space and time variables that lead to the NC. Taking all these considerations into account, we obtain the following transformations \( \tilde{\delta}_g \):

\[
\begin{align*}
\tilde{\delta}_g x &= \zeta m, \\
\tilde{\delta}_g t &= \zeta p_x, \\
\tilde{\delta}_g E &= \frac{1}{m}(\zeta p_x), \\
\tilde{\delta}_g p_x &= -\frac{\zeta}{Em} \left( \frac{\dot{x}}{E} p_x - m^2 \right) \equiv -\frac{\zeta}{Em} \left( p_x^2 - m^2 \right), \\
\tilde{\delta}_g p_t &= \frac{m\dot{x}}{E} \dot{\zeta} \left( \frac{\dot{x}}{E} p_x - m \right) \equiv \frac{p_x}{Em^2} \dot{\zeta} \left( p_x^2 - m^2 \right).
\end{align*}
\] (4.2)

A few comments are in order now. First, the above transformations have been deduced by exploiting the basic transformations: \( \hat{\delta}_g x = \zeta m, \hat{\delta}_g p_x = \zeta p_x \) and the equations of motion \( \dot{x} = Ep_x, \dot{t} = Em, \dot{p}_x = 0 \) as well as the definition \( p_t = -(m\dot{x}^2)/(2\dot{t}^2) \). Second, the expressions (2.4) for the momenta \( p_x \) and \( p_t \) have been utilized to express the r.h.s. of the above transformations, ultimately, in terms of the physical variables \( m, E \) and \( p_x \). Third, it can be readily seen that the above transformations are consistent with the equation of motion \( p_x^2 + 2mp_t = 0 \) because \( \tilde{\delta}_g p_t = -(p_x/m) \tilde{\delta}_g p_x \). Fourth, the consistency between the above transformations and the equations of motion \( \dot{p}_x = 0, \dot{p}_t = 0 \) (i.e. \( \tilde{\delta}_g \dot{p}_x \equiv (d/d\tau) \tilde{\delta}_g p_x = 0, \tilde{\delta}_g \dot{p}_t \equiv (d/d\tau) \tilde{\delta}_g p_t = 0 \)) leads to the following restrictions

\[
\begin{align*}
\frac{d}{d\tau}(\tilde{\delta}_g p_x) &= 0 \Rightarrow \frac{1}{Em^2}(p_x^2 - m^2)(\dot{\zeta} \dot{E} - \ddot{\zeta} E) = 0, \\
\frac{d}{d\tau}(\tilde{\delta}_g p_t) &= 0 \Rightarrow -\frac{p_x}{Em^2}(p_x^2 - m^2)(\dot{\zeta} \dot{E} - \ddot{\zeta} E) = 0.
\end{align*}
\] (4.3)

It is obvious, from the above, that for \( E \neq 0, m \neq 0 \), we have the following solutions

\[
(i) \quad p_x = m, \quad \text{and/or} \quad (ii) \quad \dot{\zeta} E - \ddot{\zeta} E = 0.
\] (4.4)

Let us focus first on the solution (i) of (4.4). In this case, the above non-standard transformations \( \hat{\delta}_g \) reduce to a new specific transformation \( \delta_g^{(sp)} \) (i.e. \( \hat{\delta}_g \rightarrow \delta_g^{(sp)} \)), under which, we obtain the following symmetry (i.e. \( \delta_g^{(sp)} L_f^{(\tau)} = (d/d\tau)((\zeta m^2)/(2)) \)) transformations

\[
\delta_g^{(sp)} x = \zeta m, \quad \delta_g^{(sp)} t = \zeta m, \quad \delta_g^{(sp)} E = \dot{\zeta}, \quad \delta_g^{(sp)} p_x = 0, \quad \delta_g^{(sp)} p_t = 0.
\] (4.5)
It is evident that the original NC (i.e. \{x^{(ns)}, t^{(ns)}\}_{(PB)} = \zeta(\tau); \text{ with } \zeta(\tau) \neq 0), discussed earlier under the transformations (4.1), can be retained (for \( p_x = m \)) if and only if \( \{x, px\}_{(PB)} = 1 \rightarrow \{x, m\}_{(PB)} = 1 \). This demonstrates that the above non-standard symmetry enforces the mass parameter \( m \) of our toy model to become noncommutative only with the configuration variable \( x(\tau) \). However, it retains its commutative nature with all the other variables of the theory. Such kind of NC for the mass parameter has also appeared for the free motion of a NR particle on a quantum-line in the framework of “quantum groups”[16,17-19]. The solution \( p_x = -m \) in (i) of (4.4) does not correspond to an interesting choice for a free NR “particle” because it would mean a negative momentum.

There is a drastically different way to look at the solution \( p_x = m \). This is in the language of the gauge transformations listed in (3.3) which yields the transformations (4.5) in the limit \( p_x = m \) and \( \xi(\tau) = \zeta(\tau) \). In fact, in this limit, one finds the same transformations for \( x \) as well as \( t \) (i.e. \( x \rightarrow x^{(g)} = x + \zeta m, t \rightarrow t^{(g)} = t + \zeta m \)) and, as expected, one obtains the commutative geometry (i.e. \( \{x^{(g)}, t^{(g)}\}_{(PB)} = 0 \)). It will be noted that, for the standard gauge symmetry transformations (3.3), the mass parameter \( m \) is assumed to be commutative with everything, right from the beginning. This establishes the fact that the NC, present in this toy model, is only the artifact of the underlying symmetry transformations. In particular, the symmetry transformations in (4.5) can be interpreted in two different ways which lead to NC and commutativity in the theory. This observation is consistent with the result obtained in [9] for this model where the Dirac bracket considerations have been taken into account for the different choices of the gauge conditions. In fact, it has been shown explicitly in [9] that the NC and commutativity for this toy model owe their origin to the different choices of the gauge conditions which, in turn, are connected to each other by a special kind of gauge transformation.

Finally, let us concentrate on the solution (ii) of (4.4). It is clear that the solution (ii) implies \( \tilde{\zeta}/\dot{\zeta} = (\dot{E}/E) \equiv R \). This ratio can be chosen in many different ways. For the choice \( R = \pm K \), we obtain the solutions \( \zeta(\tau) = \zeta(0)e^{\pm K \tau}, E(\tau) = E(0)e^{\pm K \tau} \) where \( K \) is a constant. However, these solutions do not lead to any interesting symmetry property of the first-order Lagrangian \( L_f^{(r)} \). One can choose \( R = \tau \), which leads to \( E(\tau) = E(0)e^{\pm \tau^2/2} \) and \( \zeta(\tau) = \tau \pm \sum_{n=1}^{\infty}[(2n - 1)!!]/[2n + 1]!\tau^{2n+1} \). However, these solutions also do not lead to any interesting symmetry properties of \( L_f^{(r)} \). At the moment, we do not know any solution for \( E(\tau) \) and \( \zeta(\tau) \) that corresponds to any symmetry properties of \( L_f^{(r)} \). Thus, we are led to conclude that \( p_x = m \) is the only interesting solution corresponding to a symmetry transformation of the Lagrangian for our present reparametrization invariant toy model and it leads to the existence of the NC in the spacetime structure.

5 (Anti-)BRST Symmetry Transformations and Noncommutativity

The “classical” local and continuous gauge symmetry transformations \( \delta_\alpha \) in (3.3) can be traded with the “quantum” local and continuous gauge transformations which are nothing
but the off-shell nilpotent \((s^2_{(a)b} = 0)\) (anti-)BRST symmetry transformations \(s_{(a)b}\). To obtain the above nilpotent transformations, the infinitesimal gauge parameter \(\xi\) is replaced by an anticommuting number \((\eta)\) and the (anti-)ghost fields \((c)c\). These off-shell nilpotent \((s^2_{(a)b} = 0)\) and anticommuting \((s_{b}s_{ab} + s_{ab}s_{b} = 0)\) (anti-)BRST transformations \(s_{(a)b}\) are

\[
\begin{align*}
  s_{b}x &= c_{p_{x}}, \\
  s_{b}t &= c_{m}, \\
  s_{b}E &= \dot{c}, \\
  s_{b}c &= iB, \\
  s_{b}B &= 0,
\end{align*}
\]

\[
\begin{align*}
  s_{ab}x &= \ddot{c}_{p_{x}}, \\
  s_{ab}t &= \ddot{c}_{m}, \\
  s_{ab}E &= \ddot{c}_{\dot{c}}, \\
  s_{ab}c &= -iB, \\
  s_{ab}B &= 0,
\end{align*}
\]

(5.1)

(5.2)

where \(B(\tau)\) is the Nakanishi-Lautrup auxiliary variable. The above local, continuous and off-shell nilpotent \((s^2_{(a)b} = 0)\) symmetry transformations leave the (anti-)BRST invariant Lagrangian \(\eta\) (see, e.g., [18] for the details on such kind of Lagrangians)

\[
L_b^{(r)} = p_{x} \dot{x} + p_{t} \dot{t} - \frac{1}{2} E (p_{x}^{2} + 2mp_{t}) + B\dot{E} + \frac{1}{2} B^{2} - i\dot{c}\dot{c},
\]

(5.3)

quasi-invariant because this Lagrangian \(L_b^{(r)}\) transforms as follows

\[
s_{b}L_b^{(r)} = \frac{d}{d\tau} \left[ \frac{c}{2} \left( p_{x}^{2} + 2mp_{t} \right) + B\dot{c} \right], \quad s_{ab}L_b^{(r)} = \frac{d}{d\tau} \left[ \frac{\ddot{c}}{2} \left( p_{x}^{2} + 2mp_{t} \right) + B\ddot{c} \right],
\]

(5.4)

under the (anti-)BRST transformations \(s_{(a)b}\). The generators for the above off-shell nilpotent (anti-)BRST transformations are the off-shell nilpotent \((Q_{(a)b}^{2} = 0)\), anticommuting \((Q_{b}Q_{ab} + Q_{ab}Q_{b} = 0)\) and conserved \((\dot{Q}_{(a)b} = 0)\) (anti-)BRST charges \(Q_{(a)b}\) as given below

\[
Q_{b} = B\dot{c} + \frac{c}{2}(p_{x}^{2} + 2mp_{t}) \equiv B\dot{c} - \dot{B}c, \quad Q_{ab} = B\ddot{c} + \frac{\ddot{c}}{2}(p_{x}^{2} + 2mp_{t}) \equiv B\ddot{c} - \ddot{B}c,
\]

(5.5)

where the Euler-Lagrange equation \(\dot{B} = -\frac{1}{2}(p_{x}^{2} + 2mp_{t})\), emerging from the (anti-)BRST invariant Lagrangian (5.3), has been exploited. A few comments are in order at this juncture. First, the above expressions in (5.5) are the generalizations of the expression in (3.1). Second, the nilpotency of the charges in (5.5) can be proven by exploiting the canonical (anti-)commutators \([x, p_{x}] = i, [t, p_{t}] = i, [E, B] = i, \{c, \dot{c}\} = +1, \{\dot{c}, \dot{c}\} = -1\) in the computations \(Q_{b}^{2} = \frac{1}{2}\{Q_{b}, Q_{b}\} = 0, Q_{ab}^{2} = \frac{1}{2}\{Q_{ab}, Q_{ab}\} = 0\). Third, the generic transformations in (3.2) can be generalized to \(s_{(a)b}\phi = i[\phi, Q_{(a)b}]\) where the subscripts \((+)-\) on the square brackets stand for the (anti)commutators for a given generic variable \(\phi\) being

*We follow here the notations and conventions adopted in [20]. In fact, in its full blaze of glory, the nilpotent \((\delta_{(a)b}^{2} = 0)\) (anti-)BRST transformations \(\delta_{(a)b}\) are the product of an anticommuting spacetime independent parameter \(\eta\) (with \(\eta c = -\eta c, \eta \dot{c} = -\dot{\eta}c, \) etc.) and the transformations \(s_{(a)b}\) (with \(s^2_{(a)b} = 0)\).

||The on-shell \((\dot{c} = \ddot{c} = 0)\) nilpotent \((\tilde{s}^{2}_{(a)b} = 0)\) (anti-)BRST transformations \(\tilde{s}_{(a)b}\), namely: \(\tilde{s}_{b}x = c_{p_{x}}, \tilde{s}_{b}t = c_{m}, \tilde{s}_{b}p_{x} = 0, \tilde{s}_{b}p_{t} = 0, \tilde{s}_{b}E = \ddot{c}, \tilde{s}_{b}c = 0, \tilde{s}_{b}B = -i\dot{E} \) and \(\tilde{s}_{ab}x = \ddot{c}_{p_{x}}, \tilde{s}_{ab}t = \ddot{c}_{m}, \tilde{s}_{ab}p_{x} = 0, \tilde{s}_{ab}p_{t} = 0, \tilde{s}_{ab}E = \ddot{c}, \tilde{s}_{ab}c = 0, \tilde{s}_{ab}B = +i\dot{E}\) leave the Lagrangian \(L_{b}^{(r)} = p_{x}\dot{x} + p_{t}\dot{t} - \frac{1}{2} E (p_{x}^{2} + 2mp_{t}) - \frac{1}{2} \dot{E}^{2} - i\dot{c}\dot{c}\) quasi-invariant. The symmetries \(\tilde{s}_{(a)b}\) and the Lagrangian \(\tilde{L}_{b}^{(r)}\) are obtained from (5.1), (5.2) and (5.3) by the substitution \(B = -\dot{E}\) that emerges as the equation of motion from the Lagrangian (5.3).
(fermionic) bosonic in nature. Fourth, the physicality criteria \( Q_{(a)b} |\text{phys} > = 0 \) imply that the physical states (i.e. \( |\text{phys} > \)) are the subset of the total Hilbert space of states which are annihilated by the operator form of the first-class constraints of the theory. This statement can be succinctly expressed in the mathematical form due to the requirement that the condition \( Q_{(a)b} |\text{phys} > = 0 \) implies the following

\[
(i) \quad \Pi_E (= B) |\text{phys} > = 0, \quad \text{and} \quad (ii) \quad (p_x^2 + 2mp_t)(\sim \hat{B}) |\text{phys} > = 0. \tag{5.6}
\]

The above restrictions on the physical states are in agreement with the Dirac’s prescription for the consistent quantization of a physical system endowed with the first-class constraints. In more precise words, the operator form of the primary constraint \( \Pi_E = B \) annihilates the physical states of the theory (i.e. \( B |\text{phys} > = 0 \Rightarrow \Pi_E |\text{phys} > = 0 \)). The requirement that this constraint condition should remain intact w.r.t. “time” (i.e. \( \tau \)) evolution of the system leads to the annihilation of the physical states by the secondary-constraint (i.e. \( (p_x^2 + 2mp_t) |\text{phys} > = 0 \Rightarrow \sim \hat{B}(= \Pi_E) |\text{phys} > = 0 \)) of the theory. Thus, it is clear that the physicality criteria \( Q_{(a)b} |\text{phys} > = 0 \) imply, in one stroke, the annihilation of the physical states by both the primary and the secondary constraints of the theory.

We dwell a bit on the derivation of the NC by exploiting (i) the BRST transformations for the variables \( x \) and \( t \), and (ii) the BRST cohomology connected with these transformations. To elaborate it, let us re-express the off-shell nilpotent BRST transformations for the space and time variables in (5.1) as

\[
x(\tau) \rightarrow x^{(b)}(\tau) = x(\tau) + c(\tau) p_x(\tau) \equiv x(\tau) + s_b[x(\tau)], \tag{5.7}
\]
\[
t(\tau) \rightarrow t^{(b)}(\tau) = t(\tau) + c(\tau) m \equiv t(\tau) + s_b[t(\tau)].
\]

It is clear that (i) the transformed variables \( (x^{(b)}, t^{(b)}) \) and the original untransformed variables \( (x, t) \) belong to the same cohomology class w.r.t. \( s_b \) (because of the fact that the BRST transformation \( s_b \) is a nilpotent \( (s_b^2 = 0) \) operator), and (ii) the space-time geometry is \textit{commutative} even in the BRST transformed frames (characterized by \( x^{(b)} \) and \( t^{(b)} \)) because the Poisson brackets \( \{x^{(b)}, t^{(b)}\}_{(PB)} = \{x, t\}_{(PB)} = 0 \). Let us now focus on the non-standard BRST-type transformations corresponding to (4.1), namely;

\[
x(\tau) \rightarrow \bar{x}^{(b)}(\tau) = x(\tau) + c(\tau) m \equiv x(\tau) + s_b[t(\tau)], \tag{5.8}
\]
\[
t(\tau) \rightarrow \bar{t}^{(b)}(\tau) = t(\tau) + c(\tau) p_x(\tau) \equiv t(\tau) + s_b[x(\tau)],
\]

which can be thought of as the generalizations of the non-standard symmetry transformations given in (4.1). It can be readily seen that the above transformations (i.e. (5.8)) are also cohomologically equivalent w.r.t. the nilpotent BRST transformations \( s_b \) but they lead to the existence of a \textit{noncommutative} geometry because of the NC in \( \bar{x}^{(b)} \) and \( \bar{t}^{(b)} \) (i.e. \( \{\bar{x}^{(b)}, \bar{t}^{(b)}\}_{(PB)} = c(\tau) \)). The above statement is trivially true (cf. (4.5)) for the case \( p_x = m \) too (where the mass parameter \( m \) becomes noncommutative with \( x(\tau) \)). This establishes \( (\text{vis-à-vis} \) transformations (5.7)) the cohomological equivalence of the commutativity and NC in the language of the nilpotent \( (s_b^2 = 0) \) BRST transformations (5.1). The above
arguments could be repeated with the nilpotent anti-BRST transformations $s_{ab}$ as well. It will be noted that, out of all the set of transformations in (5.8), only a small subset of transformations corresponding to $p_x = m$, turns out to be the symmetry transformation of the Lagrangian of the present toy model as can be seen explicitly from (4.5).

### 6 Quantum Groups on Phase Space and Noncommutativity

In this section, we very briefly discuss the connection of our earlier work [19] on the dynamics of a $q$-deformed particle, based on a consistently developed differential calculus on the $q$-deformed phase space, and our present reparametrization invariant NR toy model. To begin with, it can be noted that the following relations on the phase space [19]

$$x_i x_j = x_j x_i, \quad p_i p_j = p_j p_i, \quad x_i p_j = q p_j x_i,$$  \hspace{1cm} (6.1)

remain form invariant under the following transformations on 2N-dimensional phase space

$$x_i \rightarrow x_i^{(q)} = A x_i + B p_i, \quad p_i \rightarrow p_i^{(q)} = C x_i + D p_i,$$  \hspace{1cm} (6.2)

where (i) the original 2N-dimensional phase space (corresponding to the N-dimensional configuration space characterized by $x_i$ (with $i = 1, 2, 3, ..., N$), is parametrized by $x_i$ and $p_i$, (ii) the transformed phase space variables ($x_i^{(q)}$ and $p_i^{(q)}$) characterize the transformed co-tangent manifold, (iii) the elements $A, B, C$ and $D$ belong to the $2 \times 2$ quantum group $GL_{q,p}(2)$ matrix that obey the following braiding relations (see, e.g., [21])

$$AB = pq AB, \quad CD = p DC, \quad AC = q CA, \quad BD = q DB,$$  
$$BC = (q/p) CB, \quad AD - DA = (p - q^{-1}) BC = (q - p^{-1}) CB,$$ \hspace{1cm} (6.3)

where $q$ and $p$ are the non-zero complex numbers (i.e. $q, p \in \mathbb{C}/\{0\}$) corresponding to the most general type of quantum group deformation of the general linear group $GL(2)$ of $2 \times 2$ non-singular matrices **, (iv) the exact form invariance of (6.1) is assured only for the condition $pq = 1$ in relations (6.3). In other words, the form invariance of $q$-relations (6.1) is guaranteed under the quantum group $GL_{q,q^{-1}}(2)$ which is a special case of the general quantum group $GL_{q,p}(2)$ but is quite different from the single parameter deformed quantum group $GL_q(2)$ (see, e.g., [21]), (v) the relations in (6.3) reduce to the following relationships for the quantum group $GL_{q,q^{-1}}(2)$, namely;

$$AB = q^{-1} BA, \quad CD = q^{-1} DC, \quad BC = q^2 CB,$$  
$$AC = q CA, \quad BD = q DB, \quad AD = DA,$$ \hspace{1cm} (6.4)

which are found to be useful in the proof of the form invariance of (6.1), (vi) in the above proof, the elements $A, B, C$ and $D$ are assumed to commute with the phase variables ($x_i, p_i$).

**It can be easily seen that for $q = p = 1$, we obtain the commutative ordinary numbers as elements $A, B, C$ and $D$ of an ordinary group $GL(2)$. This is evident from the relations (6.3). Furthermore, for the condition $p = q$, we retrieve the braiding relations for the quantum group $GL_q(2)$ from (6.3).
The brackets have been consistently generalized, with a specific choice of the  

To be precise, the transformations (6.2) correspond to two sets of transformations for  

These brackets have been consistently generalized, with a specific choice of the  

where, as is obvious, the elements $A, B, C$ and $D$ obey the $q$-algebraic relations (6.4). The  

These brackets have been consistently generalized, with a specific choice of the $q$-deformed  

Using these $q$-canonical Poisson brackets and taking into account the commutativity of the  

This shows that the NCs exist in the spacetime as well as in the momentum coordinates.  

These NCs lie in the transformed frames due to the quantum group transformations (6.5)  

This shows that the NCs exist in the spacetime as well as in the momentum coordinates.  

These transformations, however, become canonical transformations if we choose $q^2 = 1$ (i.e. $q = \pm 1$) and the $q$-determinant of the $GL_{q,q^{-1}}(2)$ matrix to be one. That is to say, the following choice  

entails upon the quantum group to reduce from $GL_{q,q^{-1}}(2)$ to $SL_{q,q^{-1}}(2)$. It is easy to check that if we choose $C = 0$ in (6.8), the NC in the momentum coordinates disappears and only the NC persists in the spacetime coordinates. Similarly, the other interesting possibilities can be explored by specific choices of the elements of the quantum group.

Now let us focus on the gauge transformations in (3.3) which imply that $x \rightarrow x^{(g)} = x + \xi_x, t \rightarrow t^{(g)} = t + \xi_m, p_x \rightarrow p_x^{(g)} = p_x, p_t \rightarrow p_t^{(g)} = p_t$. It can be seen, using the canonical
brackets in the untransformed space (i.e. \( \{x_i, x_j\}_{(PB)} = \{p_i, p_j\}_{(PB)} = 0, \{x_i, p_j\}_{PB} = \delta_{ij} \)) that the following Poisson brackets are valid in the transformed phase space

\[
\{x_i^{(g)}, x_j^{(g)}\}_{(PB)} = 0, \quad \{p_i^{(g)}, p_j^{(g)}\}_{(PB)} = 0, \quad \{x_i^{(g)}, p_j^{(g)}\}_{(PB)} = \delta_{ij}, \quad (6.10)
\]

due to the gauge transformations written in equation (3.3). This demonstrates that the gauge transformations in (3.3) are the canonical transformations and they correspond to the commutative geometry of the spacetime structure. On the contrary, if we concentrate on the non-standard symmetry transformations in (4.1), even for \( p_x = m \), we have the following brackets in the transformed frames

\[
\{x_i^{(ns)}, x_j^{(ns)}\}_{(PB)} = \zeta(\tau) \varepsilon_{ij}, \quad \{p_i^{(ns)}, p_j^{(ns)}\}_{(PB)} = 0, \quad \{x_i^{(ns)}, p_j^{(ns)}\}_{(PB)} = \delta_{ij}, \quad (6.11)
\]

where we have used the basic canonical Poisson brackets of the original untransformed phase space and we have exploited the NC of the mass parameter with the space variable (i.e. \( \{x(\tau), m\}_{(PB)} = 1 \)). Here totally antisymmetric Levi-Civita tensor in two dimensions has been chosen such that \( \varepsilon_{12} = +1 = \varepsilon_{12} \). A close look at (6.8) and (6.11) shows that, under no possible restrictions, the NC of the transformations (4.1) can be captured by the quantum group NC illustrated by the Poisson bracket \( \{x_i^{(q)}, x_j^{(q)}\}_{(PB)} = (1 - q^2)AB \delta_{ij} \) in (6.8). This is due to the fact that \( \delta_{ij} \) and \( \varepsilon_{ij} \), present in (6.8) and (6.11), respectively, have diametrically opposite mathematical properties which cannot be reconciled in any manner.

7 Conclusions

In our present investigation, we have concentrated only on a set of continuous symmetry transformations as a tool for the discussion of the NC and commutativity in the context of the toy model of a free massive NR particle. The reparametrization symmetry, in some sense, is enforced on this model by treating the “time” parameter as a configuration space variable that depends on a monotonically increasing evolution parameter \( \tau \). One of the key new features of our discussion is the result that the NC for this toy model appears because of a set of non-standard symmetry transformations (cf. (4.1),(4.2),(4.5)). In particular, the symmetry transformations (4.5) can be understood in two different ways which correspond to NC and commutativity. That is to say (i) when (4.5) is derived as the special case of the gauge transformations (3.3), it corresponds to a commutative geometry, and (ii) when (4.5) is derived from the non-standard symmetry transformations (4.1) and (4.2), it corresponds to the NC of spacetime where (a) the mass parameter becomes noncommutative with the space variable \( x(\tau) \) alone, and (b) there exists a restriction (i.e. \( p_x = m \)) in (4.2).

The reason behind the existence of the above kind of “restricted” symmetry transformations can be explained in the language of the BRST cohomology. In fact, as it turns out, the BRST transformations (cf. (5.8), (5.7)) that lead to the NC and commutativity for this toy model, belong to the same cohomology class w.r.t. the BRST charge \( Q_b \) (or equivalently w.r.t. the nilpotent BRST transformations \( s_b \)) ††. However, there is a whole set of

††The BRST cohomology, corresponding to the transformations (4.5), can be readily discussed.
transformations that is allowed by the BRST cohomology alone. The transformations that correspond to the symmetry property of the Lagrangian are restricted to be a subset of the above set of cohomologically equivalent transformations in the phase space where \( p_x = m \). This explains, in a new way, the earlier claims of [8,9] that the NC and commutativity owe their origin to the gauge transformations for the general reparametrization invariant theories that, of course, include our present toy model, too (see, e.g.,[9]).

We have discussed the NC and commutativity of the spacetime associated with the present toy model within the framework of the quantum groups defined on the phase space of our present model. At the level of the Poisson bracket structure, we have shown that the commutativity (i.e. \( \{x_i^{(g)}, x_j^{(g)}\}_{(PB)} = 0 \)) of the model, corresponding to the gauge symmetry transformations (3.3), is equivalent to the transformations of the phase space variables of the model under \( SL_{q,q-1}(2) \) where the deformation parameter is restricted to be \( q = \pm 1 \). This kind of restriction has also been found in [12,17,18]. We have been able to establish theoretically, however, that the NC (i.e. \( \{x_i^{(ns)}, x_j^{(ns)}\}_{(PB)} = \zeta(\tau)\varepsilon_{ij} \)) associated with the non-standard symmetry transformations in (4.1) cannot be captured by the NC of spacetime structure associated with the quantum group (cf. (6.8)). This is primarily due to the fact that: whereas the NC due to the quantum group is associated with \( \delta_{ij} \), the NC due to the non-standard symmetry transformations in (4.1) is connected with \( \varepsilon_{ij} \). Both these mathematical quantities, having their own specific properties, cannot be reconciled to each-other simultaneously in any (mathematically consistent and correct) manner.

Our present discussion has been generalized to the examples of the reparametrization invariant (i) free massive relativistic particle [22], and (ii) its interaction with the electromagnetic field in the background [23]. It will be worthwhile to point out that the models of the free (non-)relativistic particles and spinning relativistic particles have also been considered in the framework of quantum groups [16-19]. The latter framework is based on a different kind of NC in spacetime structure. It will be an interesting endeavour to find out some connections between the above two approaches. The presence of the Snyder’s NC has also been shown in the context of the mechanical model of the two-time physics [24]. It will be a nice venture to explore the possible connection(s) of our approach with that of [24] for this very interesting model in the language of continuous symmetry properties. These are some of the future directions that are under investigation at the moment and our results would be reported in our forthcoming future publications [25].

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