Classical and quantum massive string

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Abstract

The classical and the quantum massive string model based on a modified BDHP action is analyzed in the range of dimensions $1 < d < 25$. The discussion concerning classical theory includes a formulation of the geometrical variational principle, a phase-space description of the two-dimensional dynamics, and a detailed analysis of the target space geometry of classical solutions. The model is quantized using ”old” covariant method. In particular an appropriate construction of DDF operators is given and the no-ghost theorem is proved. For a critical value of one of free parameters of the model the quantum theory acquires an extra symmetry not present on the classical level. In this case the quantum model is equivalent to the noncritical Polyakov string and to the old Fairlie-Chodos-Thorn massive string.

1 Introduction

The first quantum model of massive string was proposed twenty years ago by Chodos and Thorn [1]. It was originally formulated as the tensor product of $d$-copies of the standard free field representation of the Virasoro algebra and one copy of the free field Fairlie representation with the central charge $c = 26 - d$. Although satisfying all consistency conditions of a free string model it had some important drawbacks. First of all it contained a non physical continuous internal quantum number. Secondly because of the lack of the
path integral representation it was not clear how to implement the idea of "joining-splitting" interaction. The attempts to construct string amplitudes by means of the operator formalism known from the critical dual models lead to tree amplitudes with a continuous range of intercepts. Finally rather ad hoc phase space formulation made the physical interpretation of the classical system obscure. For these reasons the FCT string did not received much attention at that time.

Soon after the Polyakov paper \[2\] on conformal anomaly in the BDH string \[3\], the Fairlie realization of the Virasoro algebra reappeared in the quantization of Liouville theory \[4\]. One of the early attempts to clarify the relation between noncritical string and the Liouville theory suggested by Polyakov’s work, was made by Marnelius \[5\]. He considered a modified string model determined by the world sheet action

\[
S[M, g, \varphi, x] = -\frac{\alpha}{2\pi} \int_M \sqrt{-g} \, d^2 z \, g^{ab} \partial_a x^\mu \partial_b x^\nu \eta_{\mu\nu}
- \frac{\beta}{2\pi} \int_M \sqrt{-g} \, d^2 z \, \left( g^{ab} \partial_a \varphi \partial_b \varphi + 2 R_g \varphi \right) - \mu \int_M \sqrt{-g} \, d^2 z \, \exp(\varphi).
\]

The analysis of \[5\] was devoted to the case of non-vanishing cosmological constant. As a side remark it was pointed out that for \(\mu = 0\) and \(\beta = \frac{25-d}{48}\) the canonically quantized model should be equivalent with the FCT string \[1\].

The action \(\text{(1)}\) regarded as a two-dimensional conformal field theory action describes a special case of the induced (Liouville) 2-dim gravity coupled to the conformal matter. This system has been extensively studied over last few years, both as a noncritical dual model \[6\] (this application being restricted by the famous \(c = 1\) barrier) and as a dilaton gravity toy model for analyzing formation and evaporation of black holes \[7\]. (For a review with references to the literature see \[8\].)

The basic idea of the present paper is to consider the action \(\text{(1)}\) with \(\mu = 0\) as a world sheet action of relativistic one-dimensional extended object, rather than a conformal field theory input of the general dual model construction. The main result is that such action leads in the range of dimensions \(1 < d < 25\) to a consistent classical and quantum relativistic string theory. Since for all admissible values of \(\beta\) parameter physical states on first excited level are massive we call this model the massive string. Our motivation to consider it stems from the recent analysis of noncritical Polyakov string \[10\], where it was shown that the Polyakov sum over random surfaces leads in the range \(1 < d < 25\) to a consistent quantum mechanics of one-dimensional relativistic objects, equivalent to the old FCT string model. One can expect that the same quantum theory can be obtained from the modified BDHHP action \(\text{(1)}\), by standard quantization techniques.

One of the results of the present paper is that for a critical value of the parameter \(\beta\) these two procedures are indeed equivalent. This yields a new insight into the symmetry structure of the noncritical Polyakov string and gives some new hints for constructing a consistent joining-splitting interaction in this model. Although the problem of interacting noncritical string was our main motivation for analyzing the free massive string, the model is very interesting by its own. Both the classical and the quantum theory exhibit nontrivial structures and rise many interesting physical and geometrical questions.

The organization of the paper is as follows. In Section 2 we formulate a geometrical variational principle for the massive string. In order to emphasize the geometrical character of the theory we use reparameterization invariant boundary conditions. This
clarifies the geometrical origin of the constraint removing the continuous internal degree of freedom from the model. It also considerably simplifies the phase-space analysis of the two-dimensional dynamics of the system which is given in Section 3.

The target space interpretation of the classical model as well as its relation to the notion of classical causality is analyzed in Section 4. It is shown in particular that if one assumes the strong classical causality condition then the massive string coincides with the Nambu-Goto and the BDHP classical models. It turns out however that the model admits "stringy" interpretation with a weaker notion of causality, which allows for an essentially wider space of classical solutions. In this respect the massive string can be seen as an extension of the classical string models.

One of important features of the classical massive string is that it does not admit the target space light-cone gauge. Due to the nonzero central term in the classical (Poisson bracket) algebra

\[ \{L_m, L_n\} = i(m - n)L_{m+n} - 4\beta im^3\delta_{m,-n} . \]

the constraints are of the second kind. Thus, in contrast to the BDHP and the Nambu-Goto string models, solving constraints before quantization is a prohibitively difficult task.

In Section 5 we use the covariant operator method to quantize the model. Following Brower’s ideas we construct the DDF operators and show that they generate the full space of physical states. The metric structure on this space is then completely determined by general results concerning unitary irreducible highest weight representations of the Virasoro algebra. The corresponding no-ghost theorem is formulated for the whole range of free parameters of the model.

Section 6 contains conclusions and a brief discussion of some open problems.

2 Variation principle

A classical open CTP string trajectory will be described by a set \((M, g, \varphi, x)\) where \(M\) is a rectangle-like 2-dim oriented manifold with distinguished "initial" \(\partial_i M\) and "final" \(\partial_f M\) opposite boundary components, \(g\) is a pseudo-Riemannian metric on \(M\), \(\varphi : M \to \mathbb{R}\) is a scalar function on \(M\) and \(x : M \to \mathbb{R}^d\) is a map from \(M\) into \(d\)-dim Minkowski space. It is assumed that the metric \(g\) on \(M\) has a trivial causal structure such that the "initial" and "final" boundary components are space-like while the other components \(\partial M \setminus (\partial_i M \cup \partial_f M)\) are time-like.

On the space of classical trajectories we consider the action functional

\[
S[M, g, \varphi, x] = -\frac{\alpha}{2\pi} \int_M \sqrt{-g} \, d^2 z \, g^{ab} \partial_a x^\mu \partial_b x^\nu \eta_{\mu\nu} - \frac{\beta}{2\pi} \int_M \sqrt{-g} \, d^2 z \left( g^{ab} \partial_a \varphi \partial_b \varphi + 2R_g \varphi \right),
\]

where \(R_g\) is the scalar curvature of \(g\), and \(\eta^{\mu\nu} = \text{diag}(-1, +1, ..., +1)\). The constant \(\beta\) is dimension-less in units of \(\hbar\), and the constant \(\alpha\) is conventionally expressed in terms of the slope parameter \(\alpha'\) with dimension of length-squared \(\alpha = \frac{1}{2\alpha'}\).
The group of global symmetries of the action functional (2) consists of Poincare transformations in the target space and constant rescalings of the internal metric $g$.

The action is invariant with respect to general diffeomorphisms $f : M \to M'$ preserving the initial and final boundary components and their orientations. The transformation rule for the scalar curvature

$$R_{e^eg} = e^{-e}(R_g + \Box_g e) ,$$

implies that it is also invariant with respect to the rescalings of metrics $g \to e^g$ with a conformal factor $e$ satisfying the equation

$$\Box_g e \equiv -\frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} g^{ab} \partial_b e) = 0 .$$

Before formulating variational principle it is convenient to restrict this huge gauge invariance by introducing some partial gauge fixings. This will be done in two steps, first partially restricting reparameterization invariance and then introducing gauge fixing for the rescaling symmetry.

Let us fix a model manifold $M$ and a normal direction $n$ along the boundary $\partial M$. For any string trajectory $(N, g, \varphi, x)$ there exists a diffeomorphism $f : M \to N$ such that the normal direction $n_{f^* g}$ of the metric $f^* g$ is proportional to $n$, and the boundary components meet orthogonally at corners with respect to $f^* g$. This means that fixing $M$ and imposing these conditions on metrics provides a good partial gauge fixing for the reparameterization invariance, which we shall call the $(M,n)$-gauge. In this gauge the space of classical trajectories is the Cartesian product $\mathcal{M}_M^n \times \mathcal{W}_M \times \mathcal{E}_M^d$ where $\mathcal{M}_M^n$ is the space of metrics with the normal direction $n_g \propto n$ and with right angles at the corners, $\mathcal{W}_M$ is the space of real-valued scalar functions on $M$, and $\mathcal{E}_M^d$ is the space of maps from $M$ into d-dim Minkowski target space. The reparameterizations form the group $\mathcal{D}_M^n$ of diffeomorphisms of $M$ preserving corners and the normal direction $n$. The action of $\mathcal{D}_M^n$ on $\mathcal{M}_M^n \times \mathcal{W}_M \times \mathcal{E}_M^d$ is given by

$$(g, \varphi, x) \xrightarrow{f \in \mathcal{D}_M^n} (f^* g, \varphi \circ f, x \circ f) .$$

In the $(M,n)$-gauge the rescalings also form a group which can be identified with the abelian group $\mathcal{K}_\partial M$ of 1-forms on $\partial M$ satisfying $\int \tilde{\kappa} = 0$. For any $g \in \mathcal{M}_M^n$ and $\tilde{\kappa} \in \mathcal{K}_\partial M$ there exists a unique solution $\varrho[g, \tilde{\kappa}]$ to the boundary problem

$$\Box_g \varrho = 0 , \quad (n^a_g \partial_a \varrho) e^g = |\partial M \tilde{\kappa} ,$$

satisfying the normalization condition

$$\int_M \sqrt{-g} d^2 z \varrho = 0 .$$

e^g in the boundary condition above denotes the einbein (or the volume form) of the 1-dim metric induced on $\partial M$. The action of $\mathcal{K}_\partial M$ on the space of trajectories $\mathcal{M}_M^n \times \mathcal{W}_M \times \mathcal{E}_M^d$ is given by

$$(g, \varphi, x) \xrightarrow{\tilde{\kappa} \in \mathcal{K}_\partial M} (e^{\varrho[g, \tilde{\kappa}]} g, \varphi, x) .$$
It follows that in the \((M, n)\)-gauge the group of gauge transformations is a semi-simple product of \(D^n_M\) and \(K_{\partial M}\). Let us observe that the transformation rule for the geodesic curvature \(\kappa_g\) of the boundary \(\partial M\)

\[
\kappa_{\epsilon \kappa g} = e^{-\frac{\epsilon}{2}}(\kappa_g + \frac{1}{2}n^a_g \partial g),
\]

implies that for a fixed \(\bar{\kappa} \in K_{\partial M}\) the condition \(\kappa_{\epsilon \kappa g} = \bar{\kappa}\) is a good gauge fixing for the gauge symmetry (4). Choosing \(\bar{\kappa} = 0\) one gets the \(D^n_M\)-invariant gauge fixing condition

\[
\kappa_g = 0,
\]

which we shall call the geodesic gauge. It can be thought of as a counterpart of the constant curvature gauge in the BDHP model. In this partial gauge the space of trajectories is given by the product \(M^{n0}_M \times W_M \times E^d_M\) where \(M^{n0}_M\) is the space of metrics from \(M^n_M\) with zero geodesic curvature. Let us note that the same space of metrics has been obtained in the Polyakov sum over bordered surfaces from the requirement of a proper structure of boundary conditions for the Faddeev-Popov operator [12]. In the geodesic gauge the group of gauge transformations reduces to \(D^n_M\) and its action on the space of trajectories

\[
\delta g_{ab} = \frac{1}{2} \int_{M} \sqrt{-g} d^2z \left[ \alpha \left( \partial_a x^\mu \partial_b x_\mu - \frac{1}{2} g_{ab} g^{cd} \partial_c x^\nu \partial_d x_\nu \right) \right. \\
+ \beta \left( \partial_a \varphi \partial_b \varphi - \frac{1}{2} g_{ab} g^{cd} \partial_c \varphi \partial_d \varphi - 2 \Box_g \varphi g_{ab} - 2 \nabla_a \nabla_b \varphi \right] \delta g_{ab} \\
+ \frac{1}{2} \int_{D_M} ds t^a t^b \delta g_{ab} n^c \partial_c \varphi - \frac{1}{2} \int_{D_M} ds t^a t^b \delta g_{ab} n^c \partial_c \varphi ,
\]

where \(\partial M_t, \partial M_s\) denote the time-like and the space-like boundary components, respectively. Let us note that in the geodesic gauge variations in the metric sector satisfy the boundary conditions

\[
n^a t^b \delta g_{ab} = 0, \quad n^a \nabla_a (g^{bc} \delta_{bc}) - n^a \nabla^b \delta g_{ab} = 0,
\]

along all boundary components.

The requirement of the vanishing bulk variation leads to the following equations of motion:

\[
\Box_g x^\mu = 0, \quad \Box_g \varphi + R = 0, \quad T_{ab} = 0,
\]
where $T_{ab}$ is the energy-momentum tensor:

$$T_{ab} = \frac{\alpha}{\pi} \left( \partial_a x^\mu \partial_b x_\mu - \frac{1}{2} g_{ab} g^{cd} \partial_c x^\mu \partial_d x_\mu \right) + \frac{\beta}{\pi} \left( \partial_a \varphi \partial_b \varphi - \frac{1}{2} g_{ab} g^{cd} \partial_c \varphi \partial_d \varphi - 2 \Box g \varphi g_{ab} - 2 \nabla_a \nabla_b \varphi \right).$$

The trace part of (10) yields

$$\Box g \varphi = 0,$$

which, together with the equation (9) implies

$$R_g = 0.$$  (12)

In order to obtain a well posed variational problem one has to impose boundary conditions in the $x$- and $\varphi$-sector for which the boundary terms in the expression (9) vanish. Since the classical system under consideration is supposed to describe a free open string with some additional internal structure the variations $\delta x^\mu$ and $\delta \varphi$ should be arbitrary at the "ends". This leads to the following boundary conditions along $\partial M_i$:

$$n^a \partial_a x^\mu = 0, \quad n^a \partial_a \varphi = 0.$$  (13)

On the space-like boundary components one needs Dirichlet type boundary conditions describing initial and final configurations of the system. $D_M^\partial$-invariant boundary conditions of this type were first introduced in [11]. More recently these boundary conditions have been analyzed in the context of the covariant path integral quantization of noncritical Polyakov string [10]. The derivation of the gauge invariant boundary condition in the $x$- and $\varphi$-sector of the present model is essentially the same. The basic idea is to impose the Dirichlet boundary conditions in the parameterization in which the einbeins induced on the space-like boundary components are constant.

For every classical trajectory $(g, \varphi, x) \in M_M^0 \times W_M \times E_M^d$ we define the initial configuration as a set $(e^0_i, \varphi_i, x_i)$ where

$$(e^0_i)^2 = g_{ab} t^a t^b (dt)^2_{\partial M_i}, \quad \varphi_i = \varphi_{\partial M_i}, \quad x^\mu_i = x^\mu_{\partial M_i}. \quad (14)$$

All possible initial configurations form the space

$$\mathcal{P}_i \equiv M_i \times \mathcal{W}_i \times \mathcal{E}_i^d,$$

where $M_i$ consists of all einbeins on $\partial M_i$, $W_i$ is the space of real-valued functions on $\partial M_i$, and $E_i^d$ is the space of maps $x_i : \partial M_i \to \mathbb{R}^d$. Due to the boundary conditions (13) the functions from $W_i$ and $E_i^n$ satisfy Neumann boundary conditions at the ends of $\partial M_i$.

The initial configuration defined by (14) is $D_M^\partial$-covariant, i.e. for $f \in D_M^\partial$ one has

$$(e^0_i f, (\varphi \circ f)_i, (x \circ f)_i) = (\gamma^*_i e^0_i, \varphi_i \circ \gamma_i, x_i \circ \gamma_i), \quad \gamma_i = f_{\partial M_i}.$$

Due to this property one can define the space $\mathcal{C}_i$ of gauge invariant initial configurations as the quotient

$$\mathcal{C}_i \equiv \frac{M_i \times W_i \times E_i^d}{\mathbb{R}_+ \times D_i},$$
where $\mathcal{D}_i$ is the group of orientation preserving diffeomorphisms of the initial boundary component $\partial M_i$, and the action of $\mathbf{R}_+ \times \mathcal{D}_i$ on $\mathcal{M}_i \times \mathcal{W}_i \times \mathcal{E}_i^d$ is given by

$$(e_i, \varphi_i, x_i) \xrightarrow{(\lambda, \gamma) \in \mathbf{R}_+ \times \mathcal{D}_i} (\tilde{\lambda} \gamma^* e_i, \varphi_i \circ \gamma, x_i \circ \gamma) \ .$$  

The group $\mathbf{R}_+$ of constant rescalings of einbeins has been introduced in order to avoid over complete boundary conditions. In fact using the Gauss-Bonnet theorem one can easily show that for the zero-curvature metrics from $\mathcal{M}_M^n$ with fixed geodesic curvature of $\partial M$ the internal length of the boundary cannot be arbitrary.

The space of gauge invariant final configurations $\mathcal{C}_f$ is defined in a similar way. It is convenient to use a common description for both spaces introducing a model interval $L$ and the quotient space

$$\mathcal{C}_L \equiv \frac{\mathbf{M}_L \times \mathcal{W}_L \times \mathcal{E}_L^d}{\mathbf{R}_+ \times \mathcal{D}_L}$$

canonically isomorphic with $\mathcal{C}_i$ and $\mathcal{C}_f$. If we assume the induced orientations on $\partial M_i$ and $\partial M_f$, the isomorphisms are given by

$$\mathcal{C}_L \ni [(e_i, \varphi_i, x_i, \kappa_i)] \rightarrow [(\gamma_i^* e_i, \varphi_i \circ \gamma_i, x_i \circ \gamma_i)] \in \mathcal{C}_i \ ,$$

$$\mathcal{C}_L \ni [(\tilde{e}_i, \tilde{\varphi}_i, \tilde{x}_i)] \rightarrow [(\gamma_i^* \tilde{e}_i, \tilde{\varphi}_i \circ \gamma_i, \tilde{x}_i \circ \gamma_i)] \in \mathcal{C}_f \ ,$$

where $\gamma_i : \partial M_i \rightarrow L$ is an arbitrary orientation preserving diffeomorphism, $\gamma_f : \partial M_f \rightarrow L$ is an arbitrary orientation reversing diffeomorphism, and the square brackets denote the gauge orbits of corresponding elements of $\mathcal{P}_i, \mathcal{P}_f$, and $\mathcal{P}_L$.

The gauge invariant boundary conditions for classical trajectory $(g, \varphi, x) \in \mathcal{M}_M^n \times \mathcal{W}_M \times \mathcal{E}_M^d$ are then defined by

$$[((e_i, \varphi_i, x_i, \kappa_i)] = c_i \ , \quad [((e_f, \varphi_f, x_f, \kappa_f)] = c_f \ ,$$  

where $c_i, c_f \in \mathcal{C}_L$.

We shall check whether the variational problem (8) is well posed with the boundary conditions (16). For this purpose it is convenient to introduce a more tractable description of the space of gauge invariant configurations. Let $\tilde{e}$ be an einbein on the model interval $L$. One can easily check that for every $(\mathbf{R}_+ \times \mathcal{D}_L)$-orbit $c \in \mathcal{C}_L$ there exists a unique $(\tilde{\varphi}, \tilde{x}) \in \mathcal{W}_L \times \mathcal{E}_L^d$ such that $c = [(\tilde{e}, \tilde{\varphi}, \tilde{x})]$. In fact the condition $\tilde{e} = \tilde{e}$ defines a smooth gauge slice for the action (15), and provides 1-1 parametrization of $\mathcal{C}_L$

$$\mathcal{W}_L \times \mathcal{E}_L^d \ni (\tilde{\varphi}, \tilde{x}) \rightarrow [(\tilde{e}, \tilde{\varphi}, \tilde{x})] \in \mathcal{C}_L \ .$$

Let $c_i = [(\tilde{e}, \tilde{\varphi}_i, \tilde{x}_i, \tilde{\kappa}_i)]$ and $c_f = [(\tilde{e}, \tilde{\varphi}_f, \tilde{x}_f, \tilde{\kappa}_f)]$. Then the boundary conditions (16) take the form

$$x_i = \tilde{x}_i \circ \tilde{\gamma}_i[g] \ , \quad \varphi_i = \tilde{\varphi}_i \circ \tilde{\gamma}_i[g] \ ,$$

$$x_f = \tilde{x}_f \circ \tilde{\gamma}_f[g] \ , \quad \varphi_f = \tilde{\varphi}_f \circ \tilde{\gamma}_f[g] \ ,$$

where the diffeomorphisms

$$\tilde{\gamma}_i[g] : \partial M_i \rightarrow L \ , \quad \tilde{\gamma}_f[g] : \partial M_f \rightarrow L$$
are uniquely determined by the equations
\[ \hat{\gamma}_i[g] \propto e_i \], \quad \hat{\gamma}_f[g] \propto e_f \]  
and the convention concerning orientation.

The analysis of the boundary conditions (17) is the same for both space-like boundary components and we restrict our considerations to the initial boundary. Let \( s \in [0, 1] \) be an arbitrary parameterization of \( \partial M_i \). The variation of (17) yields
\[ \delta x_i(s) = (\partial_s x_i)(s)\delta\hat{\gamma}_i[g](s) \], \quad \delta\phi_i(s) = (\partial_s \phi_i)(s)\delta\hat{\gamma}_i[g](s) \]  
where
\[ \delta\hat{\gamma}_i[g](s) = \frac{\delta\hat{\gamma}_i[g](s)}{\partial_s \hat{\gamma}_i[g](s)} \].

In the parameterization \( \hat{s} \in [0, 1] \) of \( L \) such that \( \hat{e} = \text{const} \cdot d\hat{s} \) the solution to the equation (18) can be easily calculated:
\[ \hat{\gamma}_i[g](s) = \frac{1}{l_i} \int_0^s e_i(s') ds' \],

where \( e_i(s) = \sqrt{g(\partial_s, \partial_s)} \) and \( l_i = \frac{1}{\int_0^1 e_i(s) ds} \). Using the equation above one gets
\[ \delta\hat{\gamma}_i[g](s) = -\frac{\delta\hat{\gamma}_i[g](s)}{e_i(s)} + \frac{1}{e_i(s)} \int_0^s \delta e_i(s') ds' \]

and
\[ \frac{\delta g(\partial_s, \partial_s)}{\sqrt{g(\partial_s, \partial_s)}} = 2\partial_s(e_i \delta\hat{\gamma}_i[g]) + 2\frac{\delta l_i}{l_i} e_i \].

Inserting (19), and (20) into the \( \partial M_i \)-boundary terms in (3) one obtains
\[ \frac{1}{\pi} \int_{\partial M_i} e_i \, ds \left[ \alpha n^a \partial_a x^\mu \delta x_\mu + \beta n^a \partial_a \phi \delta \phi + \beta t^a t^b \delta g_{ab} n^c \partial_c \phi \right] = \]
\[ = \frac{1}{\pi} \int_0^1 e_i \, ds \left[ \alpha n^a \partial_a x^\mu \partial_\mu x_\mu + \beta n^a \partial_a \phi \partial_\phi \phi - 2\beta \partial_a(n^a \partial_a \phi) \right] \delta\hat{\gamma}_i[g] \]
\[ + \frac{\delta l_i}{l_i} 2\beta \frac{1}{\pi} \int_0^1 e_i(s) \, ds n^a \partial_a \phi \].

Note that for the metric variations \( \delta g \) satisfying the conditions (4) the variations \( \delta\hat{\gamma}_i[g](s) \), \( \delta l_i \) are arbitrary and independent. It follows that the \( \partial M_i \)-boundary terms vanish if and only if the following conditions are satisfied
\[ \alpha n^a t^b \partial_a x^\mu \partial_\mu x_\mu + \beta n^a t^b \partial_a \phi \partial_\phi \phi + 2\beta n^a t^b \nabla_a \nabla_b \phi = 0 \]  
\[ \int_{\partial M_i} e_i \, ds n^a \partial_a \phi = 0 \].
where in the derivation of (21) the identity \( n^a t^b \nabla_a \nabla_b \varphi = \kappa g^a \partial_a \varphi - t^a \partial_a (n^b \partial_b \varphi) \) and the condition \( \kappa_g = 0 \) have been used.

The conditions (21,22) should be regarded as (off-shell) constraints on possible boundary values of dynamical variables. The first condition implies that a part of the Euler-Lagrange equations (10) – the \( T_{nt} \) component of the energy-momentum tensor along the space-like boundary – should be regarded as an off-shell condition. The second condition means that the zero mode of the momenta of \( \varphi \) should be zero off-shell. This removes the unwanted continuous internal degree of freedom and is very important for the "stringy" interpretation of the model. Since the both conditions are consistent with the equations of motions the variational problem is well posed.

### 3 Two-dimensional dynamics

In string models one has two conceptually different notions of evolution: the inner time evolution, and the evolution in the target space. Although in some cases (e.g. Nambu-Goto string in the light-cone gauge) the inner and the target times can be identified, the interpretation of classical trajectories of two-dimensional system as histories of a one-dimensional extended object in the target space, is a difficult problem and has to be analyzed in each model separately. In this section we shall concentrate on the two-dimensional dynamics of the massive string model, leaving the discussion of the target space evolution to the next section.

In general the space of states and the time evolution of a classical system are given in terms of Cauchy data for the Euler-Lagrange equations of the corresponding variational principle. This formulation of dynamics assumes the existence of an evolution parameter for which the Cauchy problem is well posed. In the case of systems with reparameterization invariance there is in general no gauge independent notion of (inner) time. The standard method of dealing with this problem is to formulate the variational principle with some fixed choice of time parameter. In this formulation the initial and final boundary conditions are gauge dependent. The structure of the Cauchy data for the resulting Euler-Lagrange equations can be analyzed by the phase space Dirac method. If the Hamiltonian of the classical system obtained in this way weakly vanishes one gets a consistent formulation independent of the choice of inner time. The classical system determined by the action functional (1) has been analyzed within this framework in a number of papers [13].

In contrast to the scheme above the formulation of the variational problem given in the previous section is gauge invariant. In consequence the problem is well defined on the quotient space

\[
\frac{\mathcal{M}^{a0} \times \mathcal{W}_M \times \mathcal{E}_d^d}{\mathcal{D}_M^d}.
\]

For a given choice of boundary conditions the solution \((g, \varphi, x)\) of the Euler-Lagrange equations is determined up to the \( \mathcal{D}_M^d \)-action and can be seen as a point in the space (23). The interpretation of the Euler-Lagrange equations as dynamical equations requires an introduction of an evolution parameter. The uniformization of metrics from \( \mathcal{M}^{a0}_M \) allows for a \( \mathcal{D}_M^d \)-invariant definition of the inner time as the corresponding Teichmüller parameter. As we shall see this leads to a consistent phase space formulation. The advantage of the gauge invariant description of the 2-dim dynamics is that it minimalizes the number
of dynamical variables and therefore the number of constraints in the corresponding phase space formulation. Moreover the dynamical variables, the constraints, and the equations involved have a clear geometrical interpretation.

In order to analyze the dynamical content of the model one needs some parametrization of the quotient space (23). With our choice of the inner time the conformal gauge is especially convenient:

\begin{equation}
\gamma = e^{\theta \hat{g}_t} , \quad (\hat{M}_t, \hat{g}_t) = \begin{pmatrix} 0, t \end{pmatrix} \times \begin{pmatrix} 0, \pi \end{pmatrix} , \quad t \in \mathbb{R}_+ .
\end{equation}

In this gauge each point in the quotient (23) is uniquely represented by a set \((t, \varrho, \varphi, x) \in \mathbb{R}_+ \times \mathcal{W}_M \times \mathcal{W}_M \times \mathcal{E}_M^2\).

In the conformal gauge the action functional (2) takes the form

\begin{equation}
S[t, \varrho, \varphi, x] = \int_0^t dt \int_0^\pi d\sigma \left[ \frac{\alpha}{2} (\dot{\varphi}^2 + \varphi'^2) + \frac{\beta}{2} (\dot{\varphi}^2 + \varphi'^2 + 2\dot{\varphi} \dot{\varphi}' - 2\varphi \varphi'') \right] ,
\end{equation}

where dot and prime stand for the partial derivatives with respect to the parameters \(\tau\) and \(\sigma\) respectively. By simple calculations one gets the equations of motions (8,11,12)

\begin{equation}
-\ddot{x}^\mu + x^{\mu''} = 0 , \quad -\ddot{\varphi} + \varphi'' = 0 , \quad -\ddot{\varrho} + \varrho'' = 0 ,
\end{equation}

and the energy momentum tensor

\begin{align}
T_{\tau\tau} &= \frac{\alpha}{2\pi} (\dot{x}^2 + x'^2) + \frac{\beta}{2\pi} (\dot{\varphi}^2 + \varphi'^2) + \frac{\beta}{\pi} \dot{\varphi} \dot{\varphi}' + \frac{\beta}{\pi} \varphi \varphi' - 2\varphi \varphi'' , \\
T_{\sigma\sigma} &= \frac{\alpha}{2\pi} (\dot{x}^2 + x'^2) + \frac{\beta}{2\pi} (\dot{\varphi}^2 + \varphi'^2) + \frac{\beta}{\pi} \dot{\varphi} \dot{\varphi}' + \frac{\beta}{\pi} \varphi \varphi' - 2\varphi \varphi'' , \\
T_{\sigma\tau} &= T_{\tau\sigma} = \frac{\alpha}{\pi} \dot{x} x' + \frac{\beta}{\pi} (\dot{\varphi} \varphi' + \dot{\varphi}' \varphi + \varphi \varphi'') .
\end{align}

The boundary conditions along the time-like boundary components are given by

\begin{equation}
x^{\mu''} = 0 , \quad \varphi' = 0 , \quad \varrho' = 0 .
\end{equation}

Using natural identification of \(\partial M_i\) and \(\partial M_f\) with the model interval \([0, \pi]\) the boundary conditions (3,10) along the space-like boundary components can be written as follows:

\begin{equation}
\begin{align*}
\dot{\varrho}_i &= \varrho \partial M_i = 0 , \quad \dot{\varrho}_f = \varrho \partial M_f = 0 , \\
x_i &= \dot{x}_i \circ \tilde{\gamma}_i[\varrho_i] , \quad x_f = \dot{x}_f \circ \tilde{\gamma}_f[\varrho_f] , \\
\varphi_i &= \varphi_i \circ \tilde{\gamma}_i[\varrho_i] , \quad \varphi_f = \varphi_f \circ \tilde{\gamma}_f[\varrho_f] ,
\end{align*}
\end{equation}

where \(\varrho_i = \varrho \partial M_i, \varrho_f = \varrho \partial M_f\), and the diffeomorphisms

\begin{equation}
\tilde{\gamma}_i[\varrho_i] : [0, \pi] \rightarrow [0, \pi] , \quad \tilde{\gamma}_f[\varrho_f] : [0, \pi] \rightarrow [0, \pi]
\end{equation}

are uniquely determined by the equations

\begin{equation}
\partial_{\sigma} \tilde{\gamma}_i[\varrho_i] \propto \exp \frac{\varrho_i}{2} , \quad \partial_{\sigma} \tilde{\gamma}_f[\varrho_f] \propto \exp \frac{\varrho_f}{2} .
\end{equation}
Let us note that with the boundary conditions (29,30) there is no dynamical degree of freedom in the metric sector. Indeed the only classical solutions to the equation (26) satisfying (29,30) are $\varrho_{\text{cl}} = \text{const}$. Inserting these solutions into (27) and (31) one can easily check that $\varrho_{\text{cl}}$ completely decouples. The resulting system is determined by the equations of motions

\begin{align}
- \ddot{x} + x'' & = 0 , \\
- \ddot{\varphi} + \varphi'' & = 0 , \\
\frac{\alpha}{2\pi} (\dot{x}^2 + x'^2) + \frac{\beta}{2\pi} (\dot{\varphi}^2 + \varphi'^2) - 2\frac{\beta}{\pi} \varphi'' & = 0 ,
\end{align}

and the constraints

\begin{align}
\frac{\alpha}{\pi} \dot{x}' + \frac{\beta}{\pi} \dot{\varphi}' - 2\frac{\beta}{\pi} \dot{\varphi} & = 0 , \\
\int_0^\pi d\sigma \dot{\varphi} & = 0 .
\end{align}

The phase-space analysis of the system above is straightforward. The non vanishing Poisson bracket relations are

\begin{align}
\{ p^\mu(\sigma), x'(\sigma') \} & = \eta^{\mu\nu} \delta(\sigma - \sigma') , \\
\{ \omega(\sigma), \varphi(\sigma') \} & = \delta(\sigma - \sigma') .
\end{align}

The Hamiltonian generating the inner time evolution of the system

\[ \tilde{H} \equiv \int_0^\pi d\sigma \left( \frac{\pi}{2\alpha} p^2 + \frac{\alpha}{\pi} x'^2 + \frac{\pi}{2\beta} \omega^2 + \frac{\beta}{\pi} \varphi'^2 \right) , \]

leads to Hamilton’s equations

\begin{align}
\dot{f} & = \{ \tilde{H}, f \} , \\
\dot{x}^\mu(\sigma) & = \frac{\pi}{\alpha} p^\mu(\sigma) , \quad \dot{\varphi}(\sigma) = \frac{\pi}{\beta} \omega(\sigma) , \\
\dot{p}^\mu(\sigma) & = \frac{\alpha}{\pi} x''^\mu(\sigma) , \quad \dot{\omega}(\sigma) = \frac{\beta}{\pi} \varphi''(\sigma) .
\end{align}

The phase-space constraints are given by

\begin{align}
H(\sigma) & \equiv \frac{\pi}{2\alpha} p^2 + \frac{\alpha}{2\pi} x'^2 + \frac{\pi}{2\beta} \omega^2 + \frac{\beta}{2\pi} \varphi'^2 - 2\frac{\beta}{\pi} \varphi'' = 0 , \\
V(\sigma) & \equiv p \cdot x + \omega \varphi' - 2\omega' = 0 , \\
\omega_0 & \equiv \frac{1}{\pi} \int_0^\pi d\sigma \omega = 0 .
\end{align}

For any canonical variable $Z = x^\mu, p^\mu, \varphi, \omega$ and for $H(\sigma), V(\sigma)$ we define the mode expansions

\[ Z(\sigma) = \sum_{n=0}^{\infty} Z_n \cos n\sigma \quad , \quad H(\sigma) = \sum_{n=0}^{\infty} H_n \cos n\sigma \quad , \quad V(\sigma) = \sum_{n=1}^{\infty} V_n \sin n\sigma .\]
In order to simplify the analysis of the algebra of constraints $H_k, V_k, \omega_0$ we introduce

\[ L_0 \equiv \pi H_0 = \tilde{H}, \quad L_{\pm k} \equiv \frac{\pi}{2}(H_k \pm iV_k), \quad k > 0, \]

\[ \alpha^\mu_0 \equiv \frac{\pi}{\sqrt{\alpha}} \rho^\mu_0, \quad \alpha^\mu_{\pm k} \equiv \frac{1}{2} \left( \frac{\pi}{\sqrt{\alpha}} p^\mu_k \mp ik\sqrt{\alpha} x^\mu_k \right), \quad k > 0, \]

\[ \beta_0 \equiv \frac{\pi}{\sqrt{\beta}} \omega_0, \quad \beta_{\pm k} \equiv \frac{1}{2} \left( \frac{\pi}{\sqrt{\beta}} \omega_k \mp ik\sqrt{\beta} \varphi_k \right), \quad k > 0. \]

By straightforward calculations one gets

\[ L_k = L_k^x + L_k^\varphi, \tag{38} \]

where

\[ L_k^x \equiv \frac{1}{2} \sum_{-\infty}^{+\infty} \alpha_{-n} \cdot \alpha_{k+n}, \quad L_k^\varphi \equiv \frac{1}{2} \sum_{-\infty}^{+\infty} \beta_{-n} \cdot \beta_{k+n} - 2\sqrt{\beta} i k \beta_k. \]

The Poisson brackets imply the relations

\[ \{\alpha^\mu_0, \alpha^\nu_0\} = 2m^2 \delta_{\mu,\nu}, \]

\[ \{L^\mu_x, \beta_0\} = 0. \]

Calculating the algebra of constraints one gets

\[ \{L_m, L_n\} = i(m - n)L_{m+n} - 4\beta im^3 \delta_{m,-n}, \tag{39} \]

\[ \{L_m, \beta_0\} = 0. \]

An important property of the algebra above is that \( \{L_m\}_{m>0} \) is a family of second kind constraints.

We close this section by the formulae for the conserved charges related to the global Poincare symmetry. Using the Noether method one gets the conserved currents:

\[ j^\mu = \alpha \pi \sqrt{-g} g^{ab} \partial_b x_\mu, \]

\[ j^{a\mu} = \alpha \pi \sqrt{-g} g^{ab} \partial_b x_{[\mu} x_{\nu]} \]

In the phase space the total energy momentum of the string is given by

\[ P^\mu = \int_{\Gamma} n^a j^\mu_a \, ds = \int_0^\pi p^\mu(\sigma) \, d\sigma = \pi p^\mu_0, \]

where \( \Gamma \) is an arbitrary curve connecting opposite sides of the strip of parameters \( \sigma, \tau \).

The total angular momentum of the string reads

\[ M_{\mu\nu} = \int_{\Gamma} n_a j^a_{\mu\nu} = \int_0^\pi (p^\mu(\sigma) x^\nu(\sigma) - p^\nu(\sigma) x^\mu(\sigma)) \, d\sigma, \]

\[ = P^\mu x^\nu_0 - P^\nu x^\mu_0 + i \sum_{n=1}^{\infty} (\alpha^\mu_{-n} \alpha^\nu_n - \alpha^\nu_{-n} \alpha^\mu_n). \]
4 Target space dynamics

In this subsection we shall discuss the space-time interpretation of classical solutions of the massive string model. For that purpose let us briefly recall the assumptions concerning the target space behavior of classical solutions in the Nambu-Goto and the BDHP string models. The Nambu-Goto string in the orthogonal gauge and the BDHP string in the conformal gauge are determined by the same set of equations

\[ -\dddot{x}^\mu + x^{\mu''} = 0, \]  
\[ \dot{x}^2 + x'^2 = 0, \]  
\[ \dot{x} \cdot x' = 0, \]

with the boundary conditions

\[ x'^\mu(0,\tau) = x'^\mu(\pi,\tau) = 0. \]

A simple consequence of (41) and (43) is that the ends of string move with the speed of light. Another one is that the string world sheet is time-like i.e.

\[ \det \partial_a x^\mu \partial_b x_\mu = -(\dot{x}^2)^2 \leq 0. \]

Note that this is the property one has to assume for all trajectories in the Nambu-Goto string model. Indeed only for time-like trajectories the Nambu-Goto action is a real-valued functional and the orthogonal gauge is well defined [14]. In the BDHP model the action is real and well defined on trajectories \((g,x)\) with \(g\) Lorentzian and non degenerate on the whole strip of parameters (including boundaries). Only for such trajectories one can prove the validity of the conformal gauge. In this model the property (44) is a consequence of the constraint equations (41,42).

In the relativistic theory of classical point-like particles the causality principle is formulated as the condition for the energy-momentum of the particle to be time-like. In the case of relativistic one-dimensional extended objects the notion of causal motion is less obvious and depends on the way such objects may interact with themselves and other classical systems. In the commonly accepted formulation of causality in the classical Nambu-Goto and BDHP models the string is regarded as a collection of points which may individually interact. One says that a string trajectory is causal if there exists a parameterization such that all points of the string move with the speed less or equal to the speed of light [13, 14]. Such property of string trajectory we shall call micro-causality. If we assume that classical strings may interact only as a whole the causality principle is much less restrictive - it requires the spectral condition

\[ P^2 \leq 0, \]

where \(P^\mu\) is the total energy-momentum of the string. Trajectories satisfying the spectral condition above will be called macro-causal.

The notion of micro-causality plays an important role in the classical Nambu-Goto and BDHP string models. First of all for micro-causal trajectories one can show the validity of the light-cone gauge which in order allows to find all micro-causal solutions of the system [10, 13, 14, 16]. Secondly, the micro-causality implies the spectral condition for classical solutions [13]. The inverse implication does not seem to be true but we do not know any
macro-causal solution of the Nambu-Goto model which is not micro-causal. What can be easily shown is that the system (40–43) admits tachyonic motions. A simple example in the three-dimensional target space is given by

\[
\begin{align*}
t(\sigma, \tau) &= \cos(\tau + \sigma) + \cos(\tau - \sigma), \\
x(\sigma, \tau) &= \frac{1}{2} \left( \sin^2(\tau + \sigma) + \sin^2(\tau - \sigma) \right), \\
y(\sigma, \tau) &= \tau - \frac{1}{4} \left( \sin 2(\tau + \sigma) + \sin 2(\tau - \sigma) \right).
\end{align*}
\]

To conclude this brief discussion of causality in the classical Nambu-Goto model let us mention that it is related to a rather subtle structure of the phase space which has no impact on quantum string models obtained by the covariant quantization techniques. As far as the system (40–43) is regarded as a two-dimensional \(\sigma\)-model its reduced phase space can be identified with the space of all classical solutions. If we however consider the same system as a classical string model the corresponding phase space is ”smaller” and consists of solutions satisfying some causality assumptions. In the so called old covariant approach the ”big” phase space is quantized. It turns out however that the quantum spectral condition automatically appears due to the properties of the Fock space holomorphic representation (the tachyonic ground state appearing there is of a different origin and has nothing to do with classical tachyonic motions). As we shall see the same phenomenon takes place in the covariant quantization of the massive string model.

Let us now turn to the classical solutions of the model derived in the previous subsections. The general solution to the equations of motion (32,33) satisfying the boundary conditions (29) can be written in the following standard form

\[
\begin{align*}
x^\mu(\sigma, \tau) &= \frac{1}{2} \left( f^\mu(\tau + \sigma) + f^\mu(\tau - \sigma) \right), \\
\phi(\sigma, \tau) &= \frac{1}{2} \left( h(\tau + \sigma) + h(\tau - \sigma) \right),
\end{align*}
\]

where \(f^\mu(z), h(z)\) are arbitrary functions such that

\[
f^\mu(z + 2\pi) = f^\mu(z) + \frac{2\pi}{\alpha} P^\mu, \quad h(z + 2\pi) = h(z) + \frac{2\pi^2}{\beta} \omega_0,
\]

and \(P^\mu, \omega_0\) are the total energy-momentum of the string and the zero mode of the Liouville momentum, respectively.

Inserting the general solution to the constraint equations (34–36) one gets the equation for the functions \(f^\mu, h\)

\[
\frac{\alpha}{2} f'^2 + \frac{\beta}{2} h'^2 - 2\beta h'' = 0 \tag{45}
\]

and the periodicity condition \(h(z + 2\pi) = h(z)\). The equation above can be seen as the Hill’s equation \[16\]

\[
8\beta H'' = -\frac{\alpha}{2} f'^2 H
\]

with respect to the function \(H = \exp(-\frac{1}{4} h)\). The problem is to find conditions for the function \(f'^2\) under which the equation above admits strictly positive periodic solutions. Since we are not aware of any simple method to solve this problem we shall restrict
ourselves in the present paper to a simple ansatz giving a subclass of solutions, large enough to exhibit peculiar features of the system at hand.

We start with the discussion of micro-causal solutions to the equation (45). For such solutions the ends of string cannot move with the speed greater than the speed of light and one has \( f^2(z) \leq 0 \). Then by the constraint equation (46) \( H'' \geq 0 \), and the function \( H' \) is monotonic. Since \( H' \) is periodic this is possible only for \( h = \text{const} \). One obtains a rather surprising conclusion that the micro-causality does not allow for the Liouville excitations in the classical solutions of the massive string model. Moreover the micro-causal solutions of this model precisely coincide with the micro-causal solutions of the Nambu-Goto and BDHP models. It means that with the micro-causality principle imposed all three classical string models are identical.

Let us now consider macro-causal solutions. A large subclass of such solution can be obtained by the following light-cone ansatz. In the Minkowski target space we introduce the light-cone coordinates \( x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^{d-1}) \) in a two-dimensional time-like subspace and the transverse coordinates \( \{x^{\text{tr}}_{i} \}_{i=1}^{d-2} \) in its orthogonal complement. In these coordinates the Lorentzian scalar product reads \( x \cdot y = -x^+ y^- - x^- y^+ + x^{\text{tr}}_i x^{\text{tr}}_i \).

For \( f^+ \) of the form
\[
f^+(z) = \frac{P^+_i}{\alpha} z + c^+ ,
\]
the constraint equation (45) can be solved with respect to \( f^- \)
\[
f^-(z) = \frac{1}{P^+_i} \int_0^z \left( \frac{\alpha}{2} f'^2_{\text{tr}} + \frac{\beta}{2} h'^2 - 2\beta h'' \right) dz' + c^- .
\]

It follows that the collection \( (f^+, f^-, f^\text{tr}, h) \) where \( f^\text{tr}, h \) are arbitrary functions satisfying the periodicity conditions
\[
f_{\text{tr}}^i(z + 2\pi) = f_{\text{tr}}^i(z) + \frac{2\pi}{\alpha} P_{\text{tr}}^i , \quad h(z + 2\pi) = h(z) , \]
and \( f^+, f^- \) are given by the formulae (47,48), is a classical solution of the massive string. Solutions obtained in this way are macro-causal. Indeed, calculating \( 2P^+ P^- \) one gets
\[
2P^+ P^- = \frac{\alpha}{\pi} \int_{-\pi}^{\pi} \left( \frac{\alpha}{2} f'^2_{\text{tr}} + \frac{\beta}{2} h'^2 - 2\beta h'' \right) dz = P_{\text{tr}}^2 + \frac{\alpha}{\pi} \int_{-\pi}^{\pi} \left( \frac{\alpha}{2} \left( f'_{\text{tr}} - \frac{P_{\text{tr}}}{\alpha} \right)^2 + \frac{\beta}{2} h'^2 \right) dz ,
\]
which implies the spectral condition \( P^2 = -2P^+ P^- + P_{\text{tr}}^2 \leq 0 \).

Let us note that what we really need to linearize the constraint equation (45) in the light-cone coordinates is the condition for the function \( (f^+)' \) to be strictly positive. With this condition satisfied one can calculate \( f^- \) in terms of \( f^+, f^\text{tr}, \) and \( h \):
\[
f^-(z) = \int_0^z \frac{\alpha}{2} f'^2_{\text{tr}} + \frac{\beta}{2} h'^2 - 2\beta h'' \, dz' + c^- .
\]
It is convenient to formulate the ansatz above in a slightly different way. Let us first observe that any function $f^+$ with strictly positive and periodic derivative can be expressed in the form

$$f^+(z) = \frac{P^+}{\alpha} \gamma(z) + c^+ ,$$

where $\gamma$ is an orientation preserving diffeomorphism of the real line such that

$$\gamma(2k\pi) = 2k\pi \text{ for } k \in \mathbb{Z} ,
$$

and

$$\gamma'(z + 2\pi) = \gamma'(z) .
$$

By explicit calculation one can show that for any diffeomorphism $\gamma$ satisfying (49) and (50), and for any light-cone solution $(f^+ = \frac{P^+}{\alpha}z + c^+, f^-, f_{tr}, h)$ the collection

$$(f^+ \circ \gamma, f^- \circ \gamma + \hat{\gamma}, f_{tr} \circ \gamma, h \circ \gamma + 2 \log \gamma') ,
$$

where

$$\hat{\gamma}(z) = \frac{2\beta}{P^+} \int_0^z \left( \frac{3}{2} \frac{\gamma''}{\gamma^3} - 2 \frac{\gamma'''}{\gamma^2} \right) dz'$$

is a new solution of (43) with required periodicity properties.

Note that except of the non-homogeneous part $\hat{\gamma}$ in the expression for the new $f^-$, the formula (51) is the transformation rule for classical variables with respect to the conformal transformations. The appearance of the non-homogeneous term $\hat{\gamma}$ results from the fact that the constraints of classical massive string are not conformally invariant.

Calculating $2P^+P^-$ for the deformed light-cone solution (51) one gets

$$2P^+P^- = p_+^2 + \frac{\alpha}{\pi} \int_{-\pi}^{\pi} \left( \frac{\alpha}{2} \left( f_{tr}' - \frac{P_{tr}}{\alpha} \right)^2 + \frac{\beta}{2} h'^2 \right) dz - \frac{2\beta}{P^+} \int_{-\pi}^{\pi} \frac{\gamma''}{\gamma^3} dz .$$

It follows that the transformation (51) may lead to tachyonic solutions. Indeed a simple example of a tachyonic string motion is given by $f_{tr} = h = 0$ and $\gamma \neq id$.

In order to illustrate some features of the macro-causal solutions let us consider a particular case of the light-cone solution in 3-dimensional Minkowski space time given by the functions

$$f^+(z) = \frac{P^+}{\alpha} z ,
$$

$$f^z_{tr}(z) = a \cos mz ,
$$

$$h(z) = b \cos nz .
$$

The corresponding string world sheet is described by

$$t(\sigma, \tau) = \left( \frac{P^+}{\sqrt{2\alpha}} + \frac{\alpha a^2 m^2 + \beta b^2 n^2}{4\sqrt{2}P^+} \right) \tau
- \frac{\alpha a^2 m}{16\sqrt{2}P^+} \left( \sin 2m(\tau + \sigma) + \sin 2m(\tau - \sigma) \right)
- \frac{\beta b^2 n}{16\sqrt{2}P^+} \left( \sin 2n(\tau + \sigma) + \sin 2n(\tau - \sigma) \right)$$
\[ x(\sigma, \tau) = \frac{a}{2} (\cos m(\tau + \sigma) + \cos m(\tau - \sigma)) \]
\[ y(\sigma, \tau) = \left( \frac{P^+}{\sqrt{2\alpha}} - \frac{\alpha a^2 m^2 + \beta b^2 n^2}{4\sqrt{2}P^+} \right) \tau \]
\[ + \frac{\alpha a^2 m}{16\sqrt{2}P^+} (\sin 2m(\tau + \sigma) + \sin 2m(\tau - \sigma)) \]
\[ + \frac{\beta b^2 n}{16\sqrt{2}P^+} (\sin 2n(\tau + \sigma) + \sin 2n(\tau - \sigma)) \]
\[ - \frac{\beta b n}{\sqrt{2}P^+} (\sin n(\tau + \sigma) + \sin n(\tau - \sigma)) \]

For the parameters \(\alpha = 1, \beta = \frac{11}{24}, a = 0.6, b = 1, P^+ = 0.8, n = 3, m = 1\) the parametric plot of the string history is presented on Fig.1. We shall analyze more closely this particular string motion.

First of all, as one could expect from the discussion above, some parts of the string oscillate with the speed greater than the speed of light. In consequence for the amplitudes of this oscillations high enough one has several disjoint pieces of the string on a fixed target time hyperplane. The configuration of the string at the time \(t = T\) corresponding to the upper bound of the plot on Fig.1 is shown on Fig.2.a. This strange behavior of string trajectory is in apparent contradiction even with the weaker principle of macro-causality. What saves the day is that all classical macro-observables of the string are concentrated on its one "material" piece. For all other "ghost" pieces the total energy-momentum and the angular momentum are zero. This can be easily seen from the structure of the equal time lines in the space of parameters i.e. curves \(\gamma(s) = (\sigma(s), \tau(s))\) determined by the implicit equation

\[ t(\sigma(s), \tau(s)) = T \]

For the configuration of Fig.2.a. these lines are drown on Fig.2.b.

It is clear that calculating appropriate line integrals of conserved currents one gets nonzero result only for the curve \(m\) connecting opposite sides of the strip of parameters. The only "material" piece of the string is that corresponding to the curve \(m\) (the line \(M\) on Fig.2.a.). Due to this property the strange time evolution of "ghost" pieces of the string (they appear, disappear, split and join with each other and with the "material" piece) is in a perfect agreement with the macro-causality principle and the global conservation laws.
Another important feature of the solution (52), closely related to the structure of equal
time lines, is that the ”ghosts” pieces are inessential from the point of view of the Cauchy
problem for the target space dynamic. Indeed in order to determine the evolution of the
system it is enough to know ”positions” and ”velocities” of all classical variables at each
point of the ”material” piece of the string at a fixed target time. Solving the equations
of motions in both time directions one obtains all ”future ghosts” (lines $F$ on Fig.2.a.)
and all ”past ghosts” (lines $P$ on Fig.2.a.). It means that the macro-causality principle
allows for a consistent formulation of target space dynamics, and therefore, for a ”stringy”
interpretation of the solution (52).

The question arises whether the discussion of one particular macro-causal solution
given above is valid in general. The main property of the equal time lines required for the
”stringy” interpretation is that for each moment of the target space time one has only one
”material” piece of the string. Analyzing the geometry of levels of the time component
$x^0(\sigma, \tau)$ of a general macro-causal solution one can easily show that this property holds
except instants corresponding to saddle points of $x^0(\sigma, \tau)$. At these moments ”ghost”
pieces join or separate from the ”material” one so the identification of the ”material”
piece is ambiguous. This ambiguity however does not lead to any ambiguity in the Cauchy
problem – all possible choices of ”material” piece at an ”interaction time” lead to the same
string trajectory and the same values of macro-observables.

It follows from the considerations above that one can provide a consistent ”stringy” in-
terpretation for all macro-causal solutions of the massive string model. The space of these
solutions can be regarded as the reduced phase space of the massive string. It is essentially
bigger that the reduced phase space of the Nambu-Goto and the BDHP string models.
It seems that the generalized light cone ansatz provides a good local parameterization of
the reduced phase space but the analysis of singularities of this parameterizations is a
difficult open problem. Since solving constraints before quantization seems to be a very
difficult task the only available method is to represent the algebra of constraints on the
quantum level. In the next section we shall discuss the covariant quantization of the ”big”
phase space. As in the case of the Nambu-Goto string the quantum spectral condition
automatically appears.

5 First quantized massive string

In these section we discuss the covariant operator quantization of the massive string.
Following standard prescriptions of string theory [17, 18] we start with the CCR algebra

$$
[a^\mu_m, a^\nu_n] = m \eta^\mu\nu \delta_{m,-n},
$$

$$
[\beta_m, \beta_n] = m \delta_{m,-n}, \quad m, n \neq 0,
$$

$$
[P_\mu, x^\nu_0] = -i \delta^\nu_0 .
$$

supplemented by the conjugation properties

$$(P_\mu)^+ = P_\mu, \quad (x^\nu_0)^+ = x^\nu_0, \quad (\alpha^\mu_m)^+ = \alpha^-_{m}, \quad (\beta_m)^+ = \beta_{-m} .
$$

The space of states is a direct sum of the Fock spaces $F_p$ along $d$-dimensional spectrum
of momentum operators:

$$
\mathcal{H} = \int d^dp F_p .
$$
In each $F_p$ there is a unique vacuum state $\Omega_p$ satisfying
\[
\alpha_m^\mu \Omega_p = \beta_m \Omega_p = 0 \quad , \quad m > 0 \quad ; \quad P_\mu \Omega_p = p_\mu \Omega_p .
\]
Despite of the conclusions from the classical analysis we do not impose any condition on the square of momentum.

Because of the presence of the Lorentzian metric in (53), the scalar product generated on $\mathcal{H}$ is not positive. For this reason we shall consider the Fock part of $\mathcal{H}$ in a purely algebraic way i.e. we assume that all vectors in $\mathcal{H}$ are given by polynomials in creation operators.

In order to define the constraints operators (38) we introduce the standard normal ordering of quadratic expressions:
\[
: \alpha_m^\mu \alpha_n^\nu : = \begin{cases} 
\alpha_m^\mu \alpha_n^\nu & m < 0 \\
\alpha_n^\nu \alpha_m^\mu & m \geq 0
\end{cases} ,
\]
with similar rules for $\beta_m \beta_n$ and mixed products. With this definition the action of the normally ordered constraints operators
\[
L_n = \frac{1}{2} \sum_{m=-\infty}^{+\infty} : \alpha_{-m} \cdot \alpha_{n+m} : + \frac{1}{2} \sum_{m=-\infty}^{+\infty} : \beta_{-m} \beta_{n+m} : - 2 \sqrt{\beta} i k \beta_k + 2 \beta \delta_{n,0} ,
\]
on the vacuum state is well defined. The action on states from $\mathcal{H}$ is then uniquely determined by the commutation relations with excitation operators:
\[
[L_n, \alpha_m^\mu] = - m \alpha_{m+n}^\mu ,

[L_n, \beta_m] = - m \beta_{m+n} + 2 i n^2 \sqrt{\beta} \delta_{n,-m} .
\]
Due to the normal ordering the central term of the algebra of constraints gets shifted by $\frac{1}{12} (d + 1) \left( n - m \right) L_{n+m} + \frac{1}{12} (d + 1 + 48 \beta) (n^3 - n) \delta_{n,-m} .
\]
The physical subspace $\mathcal{H}_{phy} \subset \mathcal{H}$ is defined as the set of all vectors $\psi$ satisfying
\[
(L_n - \delta_{n,0} a_0) \psi = 0 .
\]
The parameter $a_0$ in the conditions above is left arbitrary at the moment. It’s value will be restricted by the no-ghost theorem.

In this representation of the quantum theory the algebra of Poincare charges is realized by the translation operators $P^\mu$ and the Lorentz generators:
\[
M^{\mu \nu} = (P^\mu x_0^\nu - P^\nu x_0^\mu) + i \sum_{n>0} \left( \alpha_n^{\mu} \alpha_n^{\nu} - \alpha_n^{\nu} \alpha_n^{\mu} \right) .
\]

Following the DDF approach we introduce the formal operator series
\[
X^\mu(\theta) = q_0^\mu + \alpha_0^\mu \theta + \sum_{m \neq 0} \frac{i}{m} \alpha_m^\mu e^{-im\theta} ,

\Phi(\theta) = \sum_{m \neq 0} \frac{i}{m} \beta_m e^{-im\theta} ,

P^\mu(\theta) = (X^\mu)'(\theta) = \sum_{m=-\infty}^{\infty} \alpha_m^\mu e^{-im\theta} ,

\Pi(\theta) = \Phi'(\theta) = \sum_{m \neq 0} \beta_m e^{-im\theta} ,
\]
where $q_0 = \sqrt{\alpha_0}$ is the operator canonically conjugate to $\alpha_0$ and $\theta$ is a real parameter.

From (53) one gets

$$[X^\mu(\theta), X'^\nu(\theta')] = -2\pi i \eta^\mu e(\theta - \theta') ,$$

$$[\Phi(\theta), \Phi(\theta')] = -2\pi i (e(\theta - \theta') + (\theta' - \theta)) ,$$

$$[P_\mu(\theta), X'^\nu(\theta')] = -2\pi i \delta^\nu_\mu \delta(\theta - \theta') ,$$

$$[\Pi(\theta), \Phi(\theta')] = -2\pi i (\delta(\theta - \theta') - 1) ,$$

$$[P_\mu(\theta), P_\nu(\theta')] = -2\pi i \eta_{\mu\nu} \delta(\theta - \theta') ,$$

$$[\Pi(\theta), \Pi(\theta')] = -2\pi i \delta'(\theta - \theta') ,$$

Commutation relations with constraint operators follow from (56)

$$[L_m, X^\mu(\theta)] = -i P^\mu(\theta) e^{im\theta} ,$$

$$[L_m, \Phi(\theta)] = -i \Pi(\theta) e^{im\theta} + 2m \sqrt{\beta} e^{im\theta} ,$$

$$[L_m, P_\mu(\theta)] = -i \frac{d}{d\theta} (P_\mu(\theta) e^{im\theta}) ,$$

$$[L_m, \Pi(\theta)] = -i \frac{d}{d\theta} (\Pi(\theta) e^{im\theta}) + 2im^2 \sqrt{\beta} e^{im\theta} .$$

For a fixed light-like vector $k$ ($k^2 = 0$) we define a basis of DDF operators as follows.

Let $k'$ be a light-like vector satisfying the condition $k \cdot k' = -1$ and $\{e_i\}_{i=1}^{i=d-2}$ a basis in the Euclidean subspace orthogonal to both $k$ and $k'$. We define $d - 2$ families of the transverse operators [19]

$$A^i_m(\theta) = \frac{1}{2\pi} \int_0^{2\pi} d\theta : e_i \cdot P(\theta) e^{imk \cdot X(\theta)} : ,$$

and one family of the longitudinal (Brower) vortices [20]

$$\tilde{B}_m(k) = \frac{1}{2\pi} \int_0^{2\pi} d\theta : (k' \cdot P(\theta) - \frac{im}{2} \log'(k \cdot P(\theta))) e^{imk \cdot X(\theta)} : .$$

In addition we introduce the operator corresponding to the Liouville degree of freedom [14]

$$C_m(k) = \frac{1}{2\pi} \int_0^{2\pi} d\theta : (\Phi(\theta) - 2\sqrt{\beta} \log'(k \cdot P(\theta))) e^{imk \cdot X(\theta)} : .$$

The primes over logarithmic terms denote derivatives with respect to $\theta$.

In order to have well defined operators the definition of the normal ordering [14] has to be supplemented by the following rules

$$e^{imk \cdot X(\theta)} : = e^{imk \cdot X_-(\theta)} e^{imk \cdot X_0(\theta)} e^{imk \cdot X_+(\theta)} ,$$

$$\xi \cdot P(\theta) e^{imk \cdot X(\theta)} : = \xi \cdot P_-(\theta) e^{imk \cdot X(\theta)} + e^{imk \cdot X(\theta)} \xi \cdot P_+(\theta) ,$$

where

$$X_\pm(\theta) = \pm \sum_{m>0} \frac{i}{m} \alpha_{\pm m} e^{\mp im\theta} , \quad X_0(\theta) = q_0 + \alpha_0 \theta ,$$

$$P_\pm(\theta) = \frac{1}{2} \alpha_0 + \sum_{m>0} \alpha_{\pm m} e^{\mp im\theta} , \quad \Pi_\pm(\theta) = \sum_{m>0} \beta_{\pm m} e^{\mp im\theta} .$$
The logarithmic terms should be understood as power series expansions around the eigenvalue of the zero-mode $k \cdot \alpha_0$. With these prescriptions, the power series present in (58–60) are reduced to polynomials in excitations on the subspaces $F_p \subset \mathcal{H}$ with $p$ satisfying $k \cdot p = \sqrt{\alpha}$.

Calculating the algebra of DDF operators (58,59,60) one gets

$$[A^i_m(k), A^j_n(k)] = m \delta^{ij} \delta_{m,-n} \ ,$$
$$[C_m(k), C_n(k)] = m \delta_{m,-n} \ ,$$
$$[\bar{B}_m(k), B_n(k)] = (n - m) \bar{B}_{n+m}(k) + 2n^3 \delta_{m,-n} \ ,$$
$$[\bar{B}_n(k), A^i_m(k)] = -mA^{i}_{m+n}(k) \ ,$$
$$[\bar{B}_n(k), C_m(k)] = -mC_{n+m}(k) + 2in^2 \sqrt{\beta} \delta_{n,-m} \ .$$

This algebra can be diagonalized by the following shift of the Brower vertex (20)

$$B_n(k) = \bar{B}_n(k) - \mathcal{L}_n(k) + \delta_{n,0} \ ,$$

where

$$\mathcal{L}_n(k) = \frac{1}{2} \sum_{m=-\infty}^{+\infty} \frac{d-2}{d} A^i_m(k) A^i_{n+m}(k) + \frac{1}{2} \sum_{m=-\infty}^{+\infty} C_{n+m}(k) C_{m+n}(k) + 2in \sqrt{\beta} C_n(k) + 2\beta \delta_{n,0} \ .$$

In the new basis $A^i_m(k), B_m(k), C_m(k)$ the only nonzero commutators are

$$[A^i_m(k), A^j_n(k)] = m \delta^{ij} \delta_{m,-n} \ ,$$
$$[B_n(k), B_m(k)] = (n - m) B_{n+m}(k) + \frac{1}{12}(n^3 - n)(25 - d - 48\beta) \delta_{n,-m} \ ,$$
$$[C_m(k), C_n(k)] = m \delta_{m,-n} \ .$$

Using (57) one can find out the commutation relations of the DDF operators with the constraints (58)

$$[L_n, A^i_m(k)] = [L_n, B_m(k)] = [L_n, C_m(k)] = 0 \ ,$$

for all $m, n \in \mathbb{Z}$. It follows that acting on vacuum states $\Omega_p$ such that $k \cdot p = \sqrt{\alpha}$ the DDF operators $A^i_m(k), B_m(k), C_m(k)$ generate off-shell physical states i.e. states satisfying the physical off-shell condition

$$L_n \Psi = 0 \ , \ n > 0 \ .$$

All states obtained in this way (for different $k$) we shall call the DDF states.

As a preparation to the no-ghost theorem we introduce (for a fixed light-like vector $k$) the family of operators (21)

$$F_m(k) = \frac{1}{2\pi} \int_0^{2\pi} d\theta : e^{imk \cdot \chi(\theta)} : \ .$$

The commutation relations read

$$[A^i_m(k), F_n(k)] = [C_m(k), F_n(k)] = [F_m(k), F_n(k)] = 0 \ ,$$
$$[B_m(k), F_n(k)] = -n F_{n+m}(k) \ ,$$

$$[A^i_m(k), C_n(k)] = [C_m(k), A^i_n(k)] = [A^i_m(k), B_n(k)] = [C_m(k), B_n(k)] = 0 \ .$$
In contrast to the DDF operators $F_m(k)$ do not commute with the constraints

$$[L_m, F_n(k)] = -m F^n_m(k),$$

$$F^n_m(k) = \frac{1}{2\pi} \int_0^{2\pi} d\theta \, e^{in\theta} :e^{imkX(\theta)}: .$$

For a vacuum state $\Omega_p$ with $p \cdot k = \sqrt{\alpha}$ one has

$$F^n_m(k)\Omega_p = 0 \ , \ m > 0, n > m \ ,$$

$$F^n_m(k)\Omega_p = \Omega_p .$$

Let us consider the states of the form

$$\Psi^{(N)}_{(a, b, c, f)} = \prod_{i=1}^{d-2} \left( A^i_{-m_i} \right)^{a^i_{m_i}} \cdot \ldots \cdot \left( A^i_{-1} \right)^{a^i_1} \cdot \tilde{B}^{b_i}_{-1} \cdot \ldots \cdot \tilde{B}^{b_i}_{-1} \cdot \tilde{C}^{c_i}_{-1} \cdot \ldots \cdot \tilde{C}^{c_i}_{-1} \cdot F^{f_1}_{-k} \cdot \ldots \cdot F^{f_1}_{-k} \tilde{\Omega}_p + \cdots ,$$

where $\{a^i\}, \{b\}, \{c\}, \{f\}$ are arbitrary finite sequences of nonnegative integers such that the eigenvalue

$$N = \sum_{i=1}^{d-2} \sum_{r_i=1}^{m_i} r_i a^i_{r_i} + \sum_{s=1}^{n} s b_s + \sum_{t=1}^{l} t f_t + \sum_{u=1}^{k} u f_u$$

of the level operator $R$ is fixed. The virtue of the operators $F_m(k)$ is that for $N = 0, 1, 2, \ldots$ the states (65) constitute a basis in the space $\mathcal{H}_p$. In order to check this property one can look at the structure of the operators $A^i_{-m}(k), B^i_{-m}(k), C^i_{-m}(k), F^i_{-m}(k)$ in terms of fundamental creation operators:

$$A^i_{-m}(k) = \alpha^i_{-m} - \alpha^i_0 k \cdot \alpha_{-m} + "more" ,$$

$$B^i_{-m}(k) = k' \cdot \alpha_{-m} - (k' \alpha_0 - \frac{1}{2} m (m + 1)) k \cdot \alpha_{-m} + "more" ,$$

$$C^i_{-m}(k) = \beta_{-m} - 2i \sqrt{\beta} mk \cdot \alpha_{-m} + "more" ,$$

$$F^i_{-m}(k) = k \cdot \alpha_{-m} + "more" .$$

The terms denoted by "more" are of higher order in the creation operators $\alpha_{-n}, \beta_{-n}$ with $n < m$. The structure of the "leading terms" in (66) implies that the states (65) are linearly independent. Counting their number level by level one can show that they form a basis of $\mathcal{H}_p$. This statement holds if we replace the Brower vortices $\tilde{B}_n(k)$ in the formula (65) by the shifted ones $B_n(k)$.

Now we proceed to the no-ghost theorem for the massive string. As in Brower’s proof of the no-ghost theorem for the Nambu-Goto string [20] the problem can be considered in two steps. We first show that the on-shell DDF states exhaust all physical states. Then using the algebra of the DDF operators we shall analyze the metric on the subspace $\mathcal{H}_{DDF}$.

We start with the following lemma

**Lemma** Let $p \neq 0$. A state $\Psi \in \mathcal{H}_p$ is an off-shell physical state if and only if it is a DDF state.

Since $p \neq 0$ there exists a light-like vector $k$ such that $k \cdot p = \sqrt{\alpha}$ and one can use the states (65) as a basis in $\mathcal{H}_p$. It follows that any state $\Psi \in \mathcal{H}$ can be written as a sum.
\[ \Psi = \Psi_{\text{DDF}} + \Psi_F, \] where \( \Psi_{\text{DDF}} \in \mathcal{H}_{\text{DDF}} \) and \( \Psi_F \) contains nonzero excitations \( F_{-m} \). We shall show that the conditions

\[ L_m \Psi = L_m \Psi_F = 0 \quad n > 0, \]

imply \( \Psi_F = 0 \). In the expression for \( \Psi_F \) in terms of the basis \([63]\) there is a term containing the operator \( F_{-m}(k) \) with maximal \( m \) and raised to the maximal power \( f_m \):

\[ \Psi_F = A F_{-m}^{f_m}(k) \cdots F_1^{f_1}(k) \Omega_{p+N_A k} + BF_{-m}^{f_m}(k) \cdots F_1^{f_1}(k) \Omega_{p+N_B k} + \ldots, \]

where \( f_m > f_m' \) and \( A, B \) contains only DDF creation operators. Due to \([63]\) and \([64]\) one has

\[ (L_1)^{f_1} \cdots (L_m)^{f_m} \Psi_F = \prod_{j=1}^{m} f_j ! (-j)^{f_j} A \Omega_{p+N_A k}, \]

and consequently \( A \Omega_{p+N_A k} = 0 \). Hence the first term in the expansion \([67]\) must vanish.

Repeating this procedure for next terms we prove \( \Psi_F = 0 \).

It follows from the Lemma that the space of physical states \( \mathcal{H}_{\text{ph}} \) coincides with the space of on-shell DDF states or, which is the same, with the space of DDF states created from physical vacua.

Let \( \Omega_p \) with \( p \neq 0 \) be a physical vacuum i.e. \( p^2 = -2\alpha(2\beta - a_0) \). Consider the subspace \( \mathcal{H}_{\text{DDF}}(p, k) \subset \mathcal{H}_{\text{ph}} \) of states generated from \( \Omega_p \) by the DDF operators with some \( k, k \cdot p = \sqrt{\alpha} \). The metric on \( \mathcal{H}_{\text{DDF}}(p, k) \) is completely determined by the algebra \([62]\) and the hermicity properties of the operators \( A_n(k), B_n(k), C_n(k) \). Due to the diagonal form of the algebra \([62]\) the space \( \mathcal{H}_{\text{DDF}}(p, k) \) is isomorphic to the symmetric tensor product of the spaces \( \mathcal{H}_{\text{AC}}(p, k) \) and \( \mathcal{H}_{\text{B}}(p, k) \) generated from \( \Omega_p \) by the algebra of \( A_n(k), C_n(k) \) and the algebra of \( B_n(k) \) operators, respectively. Since the metric on \( \mathcal{H}_{\text{AC}}(p, k) \) is positive, the metric structure on \( \mathcal{H}_{\text{DDF}}(p, k) \) depends on the metric structure on \( \mathcal{H}_{\text{B}}(p, k) \).

Calculating the action of the operator \( B_0(k) \) \([61]\) on the physical vacuum one gets

\[ B_0 \Omega_p = (1 - a_0) \Omega_p. \]

It follows that \( \mathcal{H}_{\text{B}}(p, k) \) is the Verma module \( \mathcal{V}_{c,h} \) of the Virasoro algebra with the central charge

\[ c = 25 - d - 48\beta, \]

and the weight

\[ h = 1 - a_0. \]

In general the metric on \( \mathcal{V}_{c,h} \) may be degenerate. Taking the quotient by the subspace of null vectors one gets the irreducible highest weight representation \( \mathcal{H}_{c,h} \) \([24]\). It is known \([23, 20]\) that the representation \( \mathcal{H}_{c,h} \) is unitary i.e. the metric on \( \mathcal{H}_{c,h} \) is positively defined, if and only if one of the two following conditions is satisfied

\[ c \geq 1 \quad h \geq 0, \]

or

\[ c = c_m \quad h = h_{rs}(m) \quad \text{for} \quad m = 2, 3, \ldots; \quad 1 \leq r \leq m - 1; \quad 1 \leq s \leq r, \]
where
\[ c_m \equiv 1 - \frac{6}{m(m+1)}, \quad h_{rs}(m) \equiv \frac{(m+1)r - ms)^2 - 1}{4m(m+1)}. \]

One gets the following

**Theorem** The space of physical states in the massive string model is ghost free if and only if one of the following two conditions is satisfied:

\[ a_0 \leq 1, \quad 0 < \beta \leq \frac{24 - d}{48}; \quad (68) \]

or

\[ \beta = \beta_m, \quad a_0 = a_{rs}(m) \quad \text{for} \quad m = 2, 3, \ldots; \quad 1 \leq r \leq m - 1; \quad 1 \leq s \leq r; \quad (69) \]

where
\[ \beta_m \equiv \frac{24 - d}{48} + \frac{1}{8m(m+1)}, \quad a_{rs}(m) \equiv 1 - \frac{(m+1)r - ms)^2 - 1}{4m(m+1)}. \]

Due to the structure of the operator \( L_0 \) the physical mass spectrum of the massive string is bounded from below. The theorem above implies that there are no excited tachyonic states in the physical spectrum. Indeed, for all admissible values of \( \beta, a_0 \) only the vacuum states with \( m^2 = 2\alpha(2\beta - a_0) \) may be tachyonic.

### 6 Conclusions

The main result of the present paper is that in the range of dimensions \( 1 < d < 25 \) the action functional (2) leads to a new consistent classical and quantum theory of one-dimensional relativistic extended objects.

Our derivation of the classical model is based on a new reparameterization invariant formulation of the variational principle. The virtue of this approach is a clear geometrical interpretation of the classical system and a simple phase space formulation of the 2-dim dynamics. The most interesting result of this part of the work is the derivation of the constraint \( \omega_0 = 0 \). In the standard formulation [3, 13] the origin of this constraint is not clear and one has to introduce it by hand in order to remove the unphysical internal degree of freedom. In the present approach it appears as a necessary consistency condition for the variational principle to be well posed and diffeomorphism invariant.

Let us note that the Euclidean counterpart of the variational problem formulated in Section 2 is interesting by its own. One can easily check that minimal surfaces form a special subclass of solutions to such problem with \( \varphi = 0 \). It would be interesting to find some local and global geometric characterizations of solutions with \( \varphi \neq 0 \).

The analysis of the classical causality given in Section 4 leads to the following conclusions. First of all if one assumes the micro-causality principle the classical massive string model coincides with the Nambu-Goto and the BDHP models. Secondly if one admits the spectral condition for the total energy-momentum of string as a weaker notion of causality then the space of macro-causal solutions is essentially bigger than the space of micro-causal ones. The important point is that macro-causal solutions still can be given a
consistent ”stringy” interpretation. Both results mean that the classical massive string is in a way a minimal generalization of the Nambu-Goto and the BDHP models.

Our discussion of the target space dynamics leaves some open questions. First of all it is desirable to have a detailed description of the reduced phase space of the massive string defined as a space of all solutions of the constraint equation (45) satisfying the spectral condition. The light cone ansatz which parameterizes almost all micro-causal solutions is not a good parametrization for macro-causal ones. It seems that the generalized light-cone ansatz may provide at least a local parametrization of the massive string phase space. A justification of this conjecture and a clarification of the global geometric structure are still open problems. Similar questions are also interesting in the case of old string models, where the difference between the micro- and macro-causal solutions is unknown.

As was shown in Section 5 the quantum massive string model can be obtained by covariant quantization techniques. The main results of this section are the explicit construction of physical states by suitably modified DDF method and the no-ghost theorem yielding necessary and sufficient conditions for a consistent quantum theory.

One of the problems not analyzed in this paper is the structure of null states. Although all technical ingredients required are known from the conformal field theory with the central charge $0 < c \leq 1$ a comprehensive analysis of this point is rather involved and we restrict ourselves only to few remarks. First of all for $0 < \beta < \frac{25-d}{48}$ there are no null states. In this range the symmetry structure and the number of dynamical degrees of freedom of the quantum theory is the same as in the classical one.

The largest subspace of null states appears for the critical values $\beta_c = \beta_2 = \frac{25-d}{48}$, $a_c = a_{11}(2) = 1$. In this case the model is equivalent to the old FCT string [1] and to the noncritical Polyakov string [10] with extra constraint $\omega_0 = 0$. All states generated by the shifted longitudinal operators $B_n$ are null and decouple from the physical Hilbert space. The resulting quantum system has effectively one ”functional” degree of freedom less than the classical one. This phenomenon can be seen as an extra gauge symmetry (anti-anomaly) of the quantum model. Due to this special structure of null states one can use the ”quantum” light cone gauge to describe the physical states of the Polyakov noncritical string [10]. Let us stress however that since this symmetry is not present in the classical model the light cone gauge cannot be used to solve classical constraints.

Another open problem of the quantum massive string is its spin spectrum. It is given in terms of the decomposition of the physical Hilbert space into irreducible unitary representations of the Poincare algebra. Such decomposition depends on the structure of null states and is particularly interesting for the discrete series (69). As a simple illustration of the problem let us consider the structure of $SO(d-1)$ multiplet at first excited level in the case of the critical massive string (i.e. corresponding to $\beta_c = \frac{25-d}{48}, a_c = 1$).

For a given on-shell vacuum state $\Omega_p$ there exists a Lorentz frame such that $p = \lambda k - k'(k \cdot p = 1)$ with $\lambda = 1 - 2\beta$ (for the sake of simplicity we put $\alpha = 1$). Acting with DDF operators one gets the set of states

$$\langle a' \rangle \equiv A_1^i \Omega_p = \alpha_{-1}^i \Omega_{p-k}$$,
$$\langle b \rangle \equiv B_1 \Omega_p = (k' \cdot \alpha_{-1} + 2i\sqrt{\beta} \beta_{-1} + 2\beta k \cdot \alpha_{-1}) \Omega_{p-k}$$,
$$\langle c \rangle \equiv C_1 \Omega_p = (\beta_{-1} - 2i\sqrt{\beta} k \cdot \alpha_{-1}) \Omega_{p-k}$$.
The little group of the massive vector \( p - k = -2\beta k - k' \) is generated by

\[
M^{ie} \equiv -i \sum_{n=1}^{\infty} \frac{1}{n} \left( (e \cdot \alpha_n)\alpha_{-n} - (e \cdot \alpha_{-n})\alpha_n^i \right),
\]

where \( e = -2\beta k + k' \) is orthogonal to \( p - k \). By simple calculations one checks that

\[
M^{ie}|a^i\rangle = \delta^{ij}2\sqrt{\beta}|c\rangle - i\delta^{ij}|b\rangle,
M^{ie}|b\rangle = 0,
M^{ie}|c\rangle = 2\sqrt{\beta}|a^i\rangle.
\]

Consequently up to the null longitudinal state \( |b\rangle \) the states \( |a^i\rangle, |c\rangle \) form a linear multiplet with respect to the little group \( SO(d - 1) \) of the massive vector \( p - k \).

In many respects the critical massive string is especially interesting. First of all it provides solution to the problem which was our original motivation – the application of standard quantization techniques to this classical system yields the noncritical Polyakov string. This relates the critical massive string with the Polyakov sum over random surfaces in the range \( 1 < d < 25 \). Secondly the critical massive string has the largest subspace of null states and the structure of the quantum theory is the same as in the critical Nambu-Goto and the BDHP string models. Finally the rescaling gauge symmetry of the model, for which the assumption of vanishing cosmological constant is crucial, implies that the two-dimensional gravity completely decouples. Note that the last feature holds for all admissible (not necessary critical) values of \( \beta \) and \( a_0 \) and yields a chance to overcome the \( c = 1 \) barrier. All these properties makes the critical massive string a promising candidate for a consistent interacting string theory in physical dimensions. Of course the most interesting open question is whether such theory exists.

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