Neron-Severi group preserving lifting of K3 surfaces and applications

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1 Introduction

The Torelli theorem for complex K3 surfaces states that the isomorphism classes of algebraic complex K3 surfaces can be classified by the location of a non-zero holomorphic 2 form in the second complex singular cohomology. Also it states that the automorphism group of a given algebraic complex K3 surface is a subgroup of the orthogonal group of the second integral singular cohomology preserving the line of holomorphic 2 forms and the ample cone. (22) By the Torelli theorem, we can express many geometric properties of complex K3 surfaces in the language of lattices. For example, there is a criterion for a complex K3 surface to be a Kummer surface or an Enriques K3 surface in terms of the Neron-Severi lattice. (10, 17) Precisely a complex K3 surface is an Enriques surface if and only if there is a primitive embedding of the twice of the Enriques lattice into the Neron-Severi group such that the orthogonal complement has no vector of self intersection -2. And a K3 surface is a Kummer surface if and only if there is a primitive embedding of the Kummer lattice into the Neron-Severi group.

The Torelli theorem provides with more precise results on the classification especially for K3 surfaces of high Picard number. In particular, isomorphism classes of a complex K3 surfaces of Picard number 20 is completely classified by the isomorphism class of positive definite even lattice of rank 2. (8)

The Torelli theorem is valid only on complex K3 surfaces or K3 surfaces over a field of characteristic 0. Over a field of positive characteristic, all K3 surfaces are divided into two classes, K3 surfaces of finite height and K3 surfaces of infinite height. A K3 surface of infinite height is a supersingular K3 surface if the Picard number is 22. It is known that a K3 surface of infinite height is supersingular if $p \geq 5$ or it has an elliptic fibration. (3, 4) In particular, a K3 surface of infinite height of the Picard number greater than 5 is supersingular. For supersingular K3 surfaces over a field of odd characteristic, the crystalline Torelli theorem can replace the Torelli theorem. Assume $k$ is an algebraically closed field of odd characteristic $p$ and $X$ is a supersingular K3 surface over $k$. $W$ is the ring of Witt vectors of $k$. The second crystalline cohomology $H^2_{\text{cris}}(X/W)$ is equipped with a canonical Frobenius linear morphism and a $W$-bilinear pairing induced by the cup product. The crystalline Torelli theorem states that the isomorphism class of $X$
is determined by the isomorphism class of the F-crystal lattice $H^2_{\text{cris}}(X/W)$. It also states that the automorphism group of $X$ is the subgroup of the orthogonal group of the Neron-Severi lattice preserving the ample cone and the period space. In a previous work, using the crystalline Torelli theorem, we prove that a supersingular K3 surface is an Enriques K3 surface if and only if the artin invariant is less than 6 when $p \geq 23$. ([9]) However, when $X$ is a K3 surface of finite height, the F-crystal lattice $H^2_{\text{cris}}(X/W)$ is completely determined by the height of $X$ and we do not have nice replacement of the Torelli theorem. Instead, for a K3 surface of finite height, there is a smooth lifting to $W$ on which every line bundle on $X$ can extend. ([19], [12]) For such a lifting, the reduction map from the Neron-Severi group of the generic fiber is to the Neron-Severi group of the special fiber, $X$ is an isomorphism. Hence the Neron-Severi group of $X$ is isomorphic to the Neron-Severi group of a K3 surface over a field of characteristic 0. Moreover using the reduction isomorphism of the Neron-Severi groups, we may import many results on complex K3 surfaces expressed in terms of Neron-Severi groups.

In this paper, by this means, we prove that a criterion to be an Enriques K3 surface or a Kummer surface for a complex K3 surface or a supersingular K3 surface also holds for a K3 surface of finite height.

**Theorem 3.5.** Assume $k$ is an algebraically closed field of characteristic $p \neq 2$. If there exists a primitive embedding $\Gamma(2) \hookrightarrow NS(X)$ such that the orthogonal complement does not have a vector of self intersection -2, then $X$ has an Enriques involution.

**Theorem 3.6.** Assume $k$ is an algebraically closed field of characteristic $p \neq 2$. A K3 surface over $k$, $X$ is a Kummer surface if and only if there exists a primitive embedding of the Kummer lattice into $(X)$.

Then we prove that every Kummer surface is an Enriques K3 surface. ([10], [9])

**Theorem 3.7.** Assume $k$ is an algebraically closed field of characteristic $p \neq 2$. A Kummer surface $X$ over $k$ has an Enriques involution.

For another application, we give a classification of K3 surfaces of Picard number 20 over a field of odd characteristic. Let $S_p$ be the set of isomorphic classes of even positive definite lattices of rank 2 such that the discriminant is a non-zero square modulo $p$. For each $M \in S_p$, there is a unique singular K3 surface $X_M$ of which the transcendental lattice is isomorphic to $M$. $X_M$ is defined over $\mathbb{Q}$ and good reduction of Picard number 20 over $k$. We denote this K3 surface over $k$ by $X_{k,M}$.

**Theorem 4.9.** Let $k$ be an algebraically closed field of characteristic $p > 2$. The correspondence $M \mapsto X_{k,M}$ is a bijection from $S_p$ to the set of isomorphic classes of K3 surfaces of Picard number 20 over $k$. Every K3 surface of Picard number 20 over $k$ has a model over a finite field.
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2 Neron-Severi group preserving lifting of K3 surfaces of finite height

In this section we review the result of Nygaard and Ogus on the Neron-Severi group preserving lifting. ([18], [19]) Assume $k$ is an algebraically closed field of characteristic $p > 2$. $W = W(k)$ is the ring of Witt-Vectors of $k$ and $W_n = W/p^n$. $K$ is the fraction field of $W$. Let $X$ be a K3 surface of finite height $h$ over $k$. Assume $S$ is the formal deformation space of $X$ over $W$. $S$ is formally smooth 20 dimensional. Assume $X_A \in S(A)$, is a deformation of $X$ to a local artin algebra $A$ whose residue field is $k$. For a local artin $A$-algebra $R$, we define

$$\psi_{X_A}(R) = H^2_{fl}(X_A \otimes R, \mu_{p^\infty}) = \lim_{\to} H^2_{fl}(X_A \otimes R, \mu_{p^r}).$$

The covariant group functor $\psi_{X_A}$ is the enlarged formal Brauer group of $X_A$. ([2], [19]) $\psi_{X_A}$ is a $p$-divisible group on $A$ and the connected part of $\psi_{X_A}$ is the formal Brauer group of $X_A$, $\hat{Br}_{X_A}$ and the étale part of $\psi_{X_A}$ is $H^2_{fl}(X, \mu_{p^\infty})$. Here $H^2_{fl}(X, \mu_{p^\infty})$ is an étale $p$-divisible group of corank $22 - 2h$ and is isomorphic to $(\mathbb{Q}_p/\mathbb{Z}_p)^{22 - 2h}$. Note that $22 - 2h$ is the length of the slope 1 part of $H^2_{cris}(X/W)$. We have an exact sequence of $p$-divisible groups

$$0 \to \hat{Br}_{X_A} \to \psi_{X_A} \to H^2_{fl}(X, \mu_{p^\infty}) \to 0. \tag{2.1}$$

The restriction of $\psi_{X_A}$ to $k$ via the reduction $\text{Spec} \, k \to \text{Spec} \, A$ is $\psi_{X} = \hat{Br}_{X} \oplus H^2_{fl}(X, \mu_{p^\infty})$. Let $\mathcal{U}$ be the formal deformation space of $\psi_{X}$ over $W$. $\mathcal{U}$ is formally smooth of $21$-$h$ dimension over $W$. There is a canonical morphism $\lambda : S \to \mathcal{U}$ given by $X_A \to \psi_{X_A}$.

Theorem 2.1 ([19], p.493). $\lambda$ is smooth.

In particular if $X$ is ordinary, $\gamma$ is an isomorphism. $\gamma_A : S(A) \to \mathcal{U}(A)$ is surjective for any local artin $W$-algebra $A$. Assume $X_A \in S(A)$ is a deformation of $X$ over a local artin algebra $A$. We consider an exact sequence on the étale topology of $X$,

$$0 \to 1 + m_A \mathcal{O}_{X_A} \to \mathbb{G}_{m, X_A} \to \mathbb{G}_{m, X} \to 0. \tag{2.2}$$

Since $A$ is artinian and $\mathbb{G}_{m, X}$ is $p$-torsion free, for a sufficiently large $r$, the $p^r$-times map of the above sequence gives an exact sequence

$$0 \to 1 + m_A \mathcal{O}_{X_A} \to \mathbb{G}_{m, X_A}/\mathbb{G}_{m, X_A}^{p^r} \to \mathbb{G}_{m, X}/\mathbb{G}_{m, X}^{p^r} \to 0 \tag{2.3}$$

Let $\beta_r : H^1(X, \mathbb{G}_X/\mathbb{G}_X^{p^r}) \to H^2(X, 1 + m_A \mathcal{O}_{X_A})$ be the connecting homomorphism. Then the connecting homomorphism of ([22], $H^1(X, \mathbb{G}_X) \to H^2(X, 1 + m_A \mathcal{O}_A)$ factors
All the line bundles of $X \lambda X \to \text{Spf} X \to \text{Spf} \lambda X$ define a map \(W\) projective system and there exists a preimage of $\chi$. By this proposition, if the sequence \(2.1\) splits, $\psi$ extends to $\alpha$. We obtain an exact sequence

$$0 \to \Br(X_A) \to H^2_{fl}(X_A, \mu_p) \to H^2_{fl}(X, \mu_p) \to 0.$$  

The image of $\gamma : H^2_{fl}(X, \mu_p) \to H^2_{fl}(X, \mu_p)$ is annihilated by $p^n$. For $\alpha \in \text{Im} \gamma$, let us choose an arbitrary preimage of $\alpha$, $\vec{\alpha} \in H^2_{fl}(X_A, \mu_p)$. Since $A$ is artinian, when $r$ is sufficiently large, $p^n \vec{\alpha}$ is well defined and $p^n \vec{\alpha} \in H^2(X, 1 + m_A \mathcal{O}_{X_A})$. In this way, we define a map

$$\gamma_r : H^2_{fl}(X, \mu_p) \to H^2(X, 1 + m_A \mathcal{O}_{X_A}), \alpha \mapsto p^n \vec{\alpha}$$

and we set

$$\gamma = \lim_{\to} \gamma_r : H^2(X, \mathbb{Z}_p(1)) \to \Br(X_A).$$

It is known that the map

$$\text{Ext}_A^1(H^2_{fl}(X, \mu_p), \Br(X_A)) \to \text{Hom}_{\mathbb{Z}_p}(H^2(X, \mathbb{Z}_p(1)), \Br(X_A))$$

sending the extension \(2.1\) to $\gamma$ is isomorphic. (\[18\], p.217) If the extension \(2.1\) splits, $\gamma$ is the zero map.

**Proposition 2.2** (\[18\], p.218). When $H^2_{fl}(X, \mu_p)$ is identified to $H^1(X, \mathbb{G}_m, \mathbb{G}_m^p)$, $\beta_r$ is equal to $\gamma_r$.

By this proposition, if the sequence \(2.1\) splits, $\beta_r = 0$, so the connecting homomorphism $\text{Pic}(X) \to H^2(X, 1 + m_A \mathcal{O}_{X_A})$ vanishes and $\text{Pic}(X_A) \to \text{Pic}(X)$ is surjective.

**Theorem 2.3** (\[19\], p.505, \[12\]). Assume $X$ is a K3 surface of finite type over $k$. There exists a smooth lifting of $X$ to $W$, $X' \to \text{Spec} W$ such that the reduction map $\text{Pic}(X \otimes K) \to \text{Pic}(X)$ is an isomorphism.

**Proof.** There exists a smooth lifting of $X$ to $W$, $X' \to \text{Spec} W$. (\[5\], p.63) We set $X'_n = X' \otimes W_n$. A $p$-divisible group over $W_n$, $\Br_{X'} = H^2_{fl}(X, \mu_p)$ is a lifting of $\psi_X$ and gives a map $g_n : \text{Spec} W_{n+1} \to U$. Since $\Br_{X', W} | W_n = \Br_{X', W_n}$, $g_n$ forms a projective system and there exists a $W$-map $g = \lim g_n : \text{Spf} W \to U$. Also since $\lambda : S \to U$ is smooth, $g$ has a lifting $h : \text{Spf} W \to S$. Hence we obtain a formal scheme $\mathcal{X} \to \text{Spf} W$ such that $\psi_{X} = \Br_X = H^2_{fl}(X, \mu_p)$ and all the classes of $\text{Pic}(X)$ extends to $\mathcal{X}$ by proposition \(2.2\). In particular, an ample line bundle of $X$ extends to $\mathcal{X}$ and $\mathcal{X} \to \text{Spf} W$ is algebraizable. Let $X \to \text{Spec} W$ be the algebraic model of $X \to \text{Spf} W$. All the line bundles of $X$ can be extended to $X$ (\[7\], p.204) and the reduction map $\text{Pic}(X \otimes K) \to \text{Pic}(X)$ is isomorphic. \(\square\)
We call a lifting of $X$ to $W$ satisfying the above theorem a Neron-Severi group preserving lifting of $X$. Let us fix a Neron-Severi group preserving lifting $X$ over $W$ of $X$ and set $X_K = X \otimes K$. We identify the Neron-Severi lattice of $X_K$ and the Neron-Severi lattice of $X$ via the reduction map $NS(X_K) \to NS(X)$. Let us fix a class $h \in NS(X)$ which is ample on both of $X$ and $X_K$.

**Lemma 2.4.** A class $v \in NS(X)$ is effective on $X$ if and only if it is effective on $X_K$.

*Proof.* Assume $v \in NS(X)$ is an effective class on $X$ and $D$ is an effective curve associated to $v$. We put $D = \sum n_i C_i$, here $C_i$ is an integral curve. Let $w_i$ be the class in $NS(X)$ which represents $C_i$. Then $(h, w_i) > 0$ and $(w_i, w_i) \geq -2$ by the adjunction formula. Then by the Riemann-Roch theorem, $w_i$ is an effective class in $NS(X_K)$, so $v = \sum n_i w_i$ is also effective in $NS(K_X)$. The converse follows from the same argument.

An effective class $v \in NS(X)$ is indecomposable if it can not be expressed as a sum of two non-zero effective classes. The above lemma asserts that $v \in NS(X)$ is an indecomposable effective class if and only if $v \in NS(X_K)$ is indecomposable effective. A nodal class of $X$ is a vector $v \in NS(X)$ which represents a smooth rational curve.

**Lemma 2.5.** $v \in NS(X)$ is a nodal class if and only if $v \in NS(X_K)$ is a nodal class.

*Proof.* $v \in NS(X)$ is a nodal class if and only if $(v, v) = -2$ and $v$ is indecomposable.

**Corollary 2.6.** Assume $X$ is a K3 surface of finite height over $k$ and of Picard number at least 12. $X$ contains a smooth rational curve.

*Proof.* It follows from the above lemma and [11], p.687

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3 Kummer surfaces and Enriques surfaces

Assume $k$ is an algebraically closed field of characteristic $p > 2$. $Y$ is an Enriques surface over $k$ and $f : X \to Y$ is the K3 cover of $Y$. Since $f$ is étale, $f^* T_{Y/k} = T_{X/k} = \Omega^1_{X/k}$ and $H^0(Y, \Omega^1_{Y/k}) = H^2(T, T_{Y/k}) = 0$. Since $f_* \Omega^1_{X/k} = \Omega^1_{Y/k} \otimes T_{Y/k}$, $\dim H^1(Y, T_{Y/k}) = 10$. It follows that the deformation space of $Y$ to $k$-artin local algebras with residue field $k$ is formally smooth of 10 dimension. Let $S = \text{Spf} k[[t_1, \ldots, t_{10}]]$ be the deformation space of $Y$. Suppose $A$ is an artin local $k$-algebra with a residue map $A \to k$ and $Y_A \in S(A)$ is a deformation of $Y$ to $A$. $i : X \to X_A$ is a canonical embedding.

**Lemma 3.1.** The canonical map $i^* : \text{Pic}(Y_A) \to \text{Pic}(Y)$ is an isomorphism.

*Proof.* We set $A = k \oplus m$, here $m$ is the maximal ideal of $A$. Consider an exact sequence

$$0 \to 1 + m\mathcal{O}_{Y_A} \to \mathbb{G}_{m, Y_A} \to \mathbb{G}_{m, Y} \to 0.$$  

Since $A$ is a finite algebra over $k$, $H^2(Y, 1 + m\mathcal{O}_{Y_A})$ is an extension of finite number of $H^2(Y, \mathcal{O}_Y)$, so $H^3(Y, 1 + m\mathcal{O}_Y) = 0$. By the same reason, $H^1(Y, 1 + m\mathcal{O}_Y) = 0$. Hence $H^1(Y, \mathbb{G}_{m, Y_A}) \to H^1(Y, \mathbb{G}_{m, Y})$ is an isomorphism.

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Let $\mathcal{Y} \to S$ be the universal family over the deformation space $S$. By the lemma \[3.1\] all the line bundles of $Y$ extend to $\mathcal{Y}$. In particular, an ample divisor extends to $\mathcal{Y}$ and $\mathcal{Y} \to S$ is algebraizable. Let $S = \text{Spec} \ k[[t_1, \cdots, t_{10}]]$ and $Y_S \to S$ be the algebraic model of the formal scheme $\mathcal{Y} \to S$. If $f_S : X_S \to Y_S$ is the K3 cover of $Y_S$, $\pi : X_S \to S$ is a family of K3 surfaces equipped with an Enriques involution over $S$.

**Lemma 3.2** (c.f. \[16\], p.383). If $X$ is a supersingular K3 surface of Artin invariant 1 over $k$, $X$ has an Enriques involution.

**Proof.** Let $E$ be a supersingular elliptic curve over $k$. $X$ is isomorphic to the Kummer surface of an abelian surface $E \times E$. Assume $a$ is a non-zero 2-torsion point of $E$. An involution $E \times E \to E \times E, (x,y) \mapsto (-x + a, y + a)$ commutes with $-id_{E \times E}$ and induces a fixed point free involution on $X$. Hence $X$ is an Enriques K3 surface. 

Assume $Y$ is an Enriques surface whose K3 cover $X$ is a supersingular of Artin invariant 1. For the family of K3 surfaces $\pi : X_S \to S$, we consider a stratification on $S$,

$$S = M_1 \supset M_2 \supset \cdots \supset M_{10} \supset M_{11} = \Sigma_{10} \supset \Sigma_9 \supset \cdots \supset \Sigma_1. \quad \text{(3.1)}$$

Here $M_i$ is the locus of fibers of height at least $i$ and $\Sigma_i$ is the locus of supersingular fibers of Artin invariant at most $i$. All $M_i$ and $\Sigma_i$ are closed subset of $S$ and we assume they are all reduced. Each step on the stratification is defined by one equation, so the dimension drops at most 1 at each step. Since the central fiber of $\pi$ is $X$, the unique closed point of $S$ is contained in $\Sigma_1$. On the other hands, Enriques surfaces with supersingular K3 covers of Artin invariant 1 are classified by Enriques involutions of the supersingular K3 lattice of Artin invariant 1 which are discrete. Therefore the family of Enriques surfaces with supersingular K3 covers of Artin invariant 1 is discrete and $\Sigma_1$ consists of 1 point. Hence there are exactly 10 steps of dimension down on (3.1). If an Enriques K3 surface is of finite height, the height is at most 6. Also, if an Enriques K3 surface is supersingular, the Artin invariant is at most 5. (\[9\], p.8) It follows that $M_6 = M_{10}$ and $\Sigma_{10} = \Sigma_5$, and

$$S = M_1 \supset M_2 \supset \cdots \supset M_6 \supset \Sigma_5 \supset \Sigma_4 \supset \cdots \supset \Sigma_1$$

is the effective stratification.

**Theorem 3.3.** For each $h$ $(1 \leq h \leq 6)$, there exists an Enriques K3 surface of height $h$ over $k$. For each $\sigma$ $(1 \leq \sigma \leq 5)$, there exists a supersingular Enriques K3 surface of Artin invariant $\sigma$ over $k$.

**Proof.** When $h$ is as above, we choose a point $x \in M_h - M_{h+1}$ or $x \in M_6 - \Sigma_5$ when $h = 6$. The fiber over $x$, $X_x = X_S \times_S k(x)$ is an Enriques K3 surface of height $h$ over $k(x)$. Let $X_B \to \text{Spec} \ B$ be an integral model of $X_x \to \text{Spec} \ k(x)$, where $B$ is an integral domain of finite type over $k$ and there is an imbedding $B \hookrightarrow k(x)$ such that $X_B \otimes k(x)$ is isomorphic to $X_x$. We may assume every fiber of $X_B \to \text{Spec} \ B$ is an Enriques K3
surface of height $h$ by the Grothendieck specialization theorem. Hence a closed fiber of $X_B \to \text{Spec} \, B$ is an Enriques K3 surface of height $h$ over $k$. In a similar way, we can construct a supersingular Enriques K3 surface of Artin invariant $\sigma$ for $1 \leq \sigma \leq 5$.

In a previous work, ([9]) we show that if the characteristic of the base field is $p \geq 23$, a supersingular K3 surface has an Enriques involution if and only if the Artin invariant is less than 5. We also show that if one supersingular K3 surface of artin invariant $\sigma$ over $k$ is an Enriques K3 surface, then every supersingular K3 surface of artin invariant $\sigma$ over $k$ is supersingular. By the above theorem, we obtain the following result for any odd base characteristic.

**Corollary 3.4.** A supersingular K3 surface over $k$ is an Enriques K3 surface if and only if the Artin invariant is less than 5.

The Numerical lattice of an Enriques surface $Y$ is even unimodular $\mathbb{Z}$-lattice of signature $(1,9)$. Such a lattice is unique up to isomorphism and we denote this lattice by $\Gamma$. Assume $f : X \to Y$ is the K3 cover of $Y$ and $g : X \to X$ is the involution associated to the double cover $f$. There is a primitive embedding $\Gamma(2) \simeq f^*(NS(Y)) \hookrightarrow NS(X)$

and $f^*(NS(Y)) = NS(X)^g$. Let $\Gamma'$ be the orthogonal complement of $\Gamma$ in $NS(X)$. Then $g^*$ acts on $\Gamma'$ by $-1$ and $\Gamma'$ is negative definite. $\Gamma'$ does not contain a vector of self intersection -2 by the Riemann-Roch theorem.

**Theorem 3.5.** Assume $k$ is an algebraically closed field of characteristic $p \neq 2$. If there exists a primitive embedding $\Gamma(2) \hookrightarrow NS(X)$ such that the orthogonal complement does not have a vector of self intersection -2, then $X$ has an Enriques involution.

**Proof.** If $p = 0$ or $X$ is supersingular, the claim holds. ([10], [9]) Now assume $p$ is an odd prime and $X$ is a K3 surface of finite height. We fix a Neron-Severi group preserving lifting of $X$ over $W$. $\pi : X \to \text{Spec} \, W$. Since $NS(X) = NS(X_K)$ and $X_K$ is defined over a field of characteristic 0, $X$ has an Enriques involution $g$ over $K$. We assume $g$ is defined over a finite extension $L$ of $K$ and $V$ is the ring of integers $L$. $g$ extends to $X \otimes V$ and induces an involution $\tilde{g}$ on $X$. ([13], p.672) Since $g^*$ is -1 on $H^0(X_K, \Omega^2_{X_K/K})$, $g^*$ is -1 on $\pi_* \Omega^2_{X_K/K} \otimes V$ and $\tilde{g}^*$ is -1 on $H^0(X, \Omega^2_{X/K})$. Also $\tilde{g}^*$ fixes a sublattice of $NS(X)$ of rank 10. Hence $\tilde{g}$ is an Enriques involution of $X$.

Suppose let $F_2$ is a field consisting of 2 elements. Put $I = F_2^4 = \{a_1, \ldots, a_{16}\}$. We assume $x_1, \ldots, x_4$ is a coordinate of $I$. Let $L$ be the set of linear equations in $x_i$. $L$ consists of 32 equations. Let $Q$ be the set of zero sets of all equations in $L$. $Q$ consists of the empty set, $I$ and other 30 subsets of 8 elements in $I$. We consider a $\mathbb{Z}$-lattice $M$ of rank 16 generated by $a_i$ with $(a_i, a_j) = -2\delta_{ij}$. Put $v_\alpha = \frac{1}{2} \sum_{x \in \alpha} x \in M \otimes \mathbb{Q}$ for $\alpha \in Q$. We set $J = M + \sum_{\alpha \in Q} \mathbb{Z} \cdot v_\alpha \subset M \otimes \mathbb{Q}$. $J$ is an overlattice of $M$ and $J/M = (\mathbb{Z}/2)^5$. $J$ is called the Kummer lattice. A complex K3 surface is a Kummer surface if and only if there exists a primitive embedding of $J$ into $NS(X)$. ([17])
Theorem 3.6. Assume $k$ is an algebraically closed field of characteristic $p > 2$. A K3 surface over $k$, $X$ is a Kummer surface if and only if there exists a primitive embedding of the Kummer lattice into $NS(X)$.

Proof. For the only if part, we refer to [17]. Suppose $X$ is supersingular. $X$ is a Kummer surface if and only if the Artin invariant of $X$ is 1 or 2. ([20], p.77) Assume the artin invariant of $X$ is greater than 2. $J \otimes \mathbb{Z}_p$ is a unimodular $\mathbb{Z}_p$-lattice of rank 16 with a square discriminant. By [24], p.396, there is no embedding of $J \otimes \mathbb{Z}_p$ into $NS(X) \otimes \mathbb{Z}_p$. Hence if there is an embedding of $J$ into $NS(X)$, $X$ is a supersingular Kummer surface.

Suppose $X$ is of finite height. Let $X$ be a Neron-Severi group preserving lifting of $X$ to $W$ and $X_K$ be the generic fiber of $X \rightarrow Spec W$. Since $J \hookrightarrow NS(X) = NS(X_K)$, $X_K$ is a Kummer surface. Therefore $X_K$ contains 16 mutually disjoint smooth rational curves $C_1, \ldots, C_{16}$ and $\frac{1}{2} \sum C_i \in NS(X_K)$. The reduction of each $C_i$ is a smooth rational curve in $X$ (Lemma 2.5). We denote the reduction of $C_i$ by $\bar{C}_i$. Let $X' \rightarrow X$ be the double cover ramified along the 16 rational curves $\bar{C}_i$. The preimage of each $\bar{C}_i$ in $X'$ is a (-1)-curve. Let $A$ be the surface taken by blowing down the 16 (-1)-curves of $X'$. It is easy to see that $A$ is an abelian surface and $X$ is the Kummer surface of $A$.

Theorem 3.7. Assume $k$ is an algebraically closed field of characteristic $p \neq 2$. Every Kummer surface $X$ over $k$ has an Enriques involution.

Proof. If $p = 0$ or $X$ is supersingular, we refer to [10] and [9]. Assume $p$ is an odd prime and $X$ is of finite height. Let $\pi : X \rightarrow Spec W$ is a Neron-Severi group preserving lifting of $X$ and $X_K$ be the generic fiber of $\pi$. If $X$ is a Kummer surface, there exists a primitive embedding of the Kummer lattice $J \hookrightarrow NS(X) = NS(X_K)$ and $X_K$ is a Kummer surface. Since $X_K$ is defined over a field of characteristic 0, $X_K$ has an Enriques involution and there exists a primitive embedding $\Gamma \hookrightarrow NS(X_K) = NS(X)$ such that the orthogonal complement has no vector of self intersection -2. By theorem 3.6, $X$ has an Enriques involution. 

4 Classification of K3 surfaces of Picard number 20

A singular K3 surface is a complex K3 surface of Picard number 20. For a complex K3 surface $X$, we denote the transcendental lattice of $X$ by $T(X)$. The transcendental lattice of a singular K3 surface is an even positive definite lattice of rank 2. Conversely when $M$ is an even positive definite lattice of rank 2, there exists a unique singular K3 surface $X_M$ up to isomorphism such that $T(X_M)$ is isomorphic to $M$. ([5]) It is also known that any singular K3 surface has a model over a number field. For any lattice $M$, we denote the discriminant of $M$ and the discriminant group of $M$ by $d(M)$ and $l(M)$ respectively. The order of $l(M)$ is $|d(M)|$. Let $S$ be the set of isomorphic classes of even positive definite lattices of rank 2. We set $\mathfrak{s} = \{l(M)|M \in S\}$. For a given $l \in \mathfrak{s}$, we denote $S_l = \{M \in S|l(M) = l\}$. $S_l$ is a finite set for any $l \in \mathfrak{s}$. The Neron-Severi group of $X_M$ only depends on $l(M)$. ([15], p.112) If $l = l(M)$, we denote the Neron-Severi group of $X_M$ by $N_l$. For $l \in \mathfrak{s}$ there are only finitely many singular K3 surfaces whose Neron-Severi groups are isomorphic to $N_l$. We denote the set of singular K3 surfaces with the Neron-Severi group $N_l$ by $X_l$. 


Assume \( k \) is an algebraically closed field of characteristic \( p > 2 \) and the cardinality of \( k \) is equal to or less than the cardinality of \( \mathbb{C} \). \( W \) is the ring of Witt vectors of \( k \) and \( K \) is the fraction field of \( W \). We fix an isomorphism \( \bar{K} \cong \mathbb{C} \).

**Lemma 4.1.** Assume \( X \) is a K3 surface of Picard number 20 over \( k \). Then \( \text{NS}(X) = N_l \) for some \( l \in \mathfrak{s} \). There are only finitely many K3 surfaces over \( k \) up to isomorphism of which the Neron-Severi group is isomorphic to \( \text{NS}(X) \).

*Proof.* Since the Picard number of \( X \) is 20, \( X \) is ordinary. In particular, \( X \) is of finite height. Assume \( X/W \) is a Neron-Severi group preserving lifting of \( X \). Then \( X_{\bar{K}} = X \otimes \bar{K} \) is a singular K3 surface and \( \text{NS}(X_{\bar{K}}) = \text{NS}(X) \). Therefore \( \text{NS}(X) = N_l \) for some \( l \in \mathfrak{s} \). Now assume \( X' \) is another K3 surface over \( k \) such that \( \text{NS}(X) = \text{NS}(X') \). Assume \( X' \) is a Neron-Severi group preserving lifting of \( X' \) and \( X'_k = X' \otimes \bar{K} \). Then \( X'_k \in X_k \). Suppose \( X'_k \) is isomorphic to \( X_{\bar{K}} \). Then there exists a finite extension \( L \) of \( K \) such that \( X'_L \) is isomorphic to \( X_{\bar{K}} \). Assume \( V \) is the ring of integers of \( L \). The isomorphism \( X_L \cong X'_L \) extends to an isomorphism of the integral models \( X' \otimes V \cong X \otimes V \) ([13]). Hence their special fibers \( X \) and \( X' \) are isomorphic. \( \square \)

**Lemma 4.2.** Assume \( X \) is a K3 surface of Picard number 20. \( \text{NS}(X) \otimes \mathbb{Z}_p \) is unimodular \( \mathbb{Z}_p \)-lattice of square discriminant.

*Proof.* By the flat duality, \( H^3_{fl}(X, \mu_p) = 0 \). Considering an exact sequence on the flat topology on \( X \)

\[
0 \to \mu_p \to \mathbb{G}_{m,X} \xrightarrow{\times p} \mathbb{G}_{m,X} \to 0,
\]

the \( p \)-times map on the Brauer group of \( X \), \( Br_X = H^2(X, \mathbb{G}_{m,X}) \) is surjective. Hence \( Br_X \) does not have a non-divisible \( p \)-torsion and the \( p \)-Tate module of \( Br_X \), \( T_p(\text{Br}_X) \) is a free \( \mathbb{Z}_p \)-module. Since the Picard number of \( X \) is equal to the \( \mathbb{Z}_p \)-rank of \( H^2(X, \mathbb{Z}_p(1)) \), by the exact sequence

\[
0 \to \text{NS}(X) \otimes \mathbb{Z}_p \to H^2(X, \mathbb{Z}_p(1)) \to T_p(\text{Br}_X) \to 0, \quad (\text{[6]}, \text{p.629})
\]

\( T_p(\text{Br}_X) = 0 \) and \( \text{NS}(X) \otimes \mathbb{Z}_p \cong H^2(X, \mathbb{Z}_p(1)) \). Because the cup product pairing of \( H^2(X, \mathbb{Z}_p(1)) \) is unimodular of square discriminant, ([21], p.363) so is \( \text{NS}(X) \otimes \mathbb{Z}_p \). \( \square \)

**Lemma 4.3.** Let \( F \) be a local field of mixed characteristic \((0,p)\) and \( k \) be an algebraic closure of the residue field of \( F \). \( X_F \) is a K3 surface defined over \( F \). Assume \( X_F \) has good reduction and \( X \) is the reduction of \( X_F \). Then the embedding

\[
\text{NS}(X_F \otimes \bar{F}) \otimes \mathbb{Z}_l \hookrightarrow \text{NS}(X \otimes k) \otimes \mathbb{Z}_l
\]

is primitive for any prime number \( l \neq p \).

*Proof.* We may assume

\[
\text{NS}(X_F) = \text{NS}(X_F \otimes \bar{F}) \quad \text{and} \quad \text{NS}(X) = \text{NS}(X \otimes k).
\]
For a prime \( l \neq p \), the canonical embedding

\[
NS(X_F) \otimes \mathbb{Z}_l \hookrightarrow H^2(X_F(\mathbb{C}), \mathbb{Z}_l) \cong H^2_{\text{ét}}(X_F \otimes \bar{F}, \mathbb{Z}_l) = H^2_{\text{ét}}(X \otimes k, \mathbb{Z}_l)
\]
factors through

\[
NS(X_F) \otimes \mathbb{Z}_l \hookrightarrow NS(X) \otimes \mathbb{Z}_l \hookrightarrow H^2_{\text{ét}}(X \otimes k, \mathbb{Z}_l).
\]

Since the embedding \( NS(X) \hookrightarrow H^2(X_F(\mathbb{C}), \mathbb{Z}) \) is primitive, \( (NS(X)/NS(X_F)) \otimes \mathbb{Z}_l \) has no torsion.

**Remark 4.4.** In the above lemma, the quotient group \( NS(X)/NS(X_F) \) may have a non-trivial \( p \)-torsion. It is known that if the ramification index of \( F \) is less than \( p-1 \), \( NS(X)/NS(X_F) \) is torsion-free. \((\text{[23]}, \text{p.132})\)

**Lemma 4.5.** Assume \( X_F \) is a singular K3 surface defined over a number field \( F \). For \( v \) is a finite place of \( F \), \( X_F \) has potential good reduction at \( v \). We suppose the residue characteristic of \( v \) is \( p > 2 \) and \( d(NS(X_F \otimes \bar{F})) \) is a unit modulo \( p \). The reduction of \( X \) at \( v \) is of finite height if and only if \( d(NS(X_F \otimes \bar{F})) \) is a square modulo \( p \). If this condition is satisfied, when \( X \) is the reduction of \( X_F \) at \( v \) and \( k \) is an algebraic closure of the residue field of \( v \), \( NS(X_v \otimes k) = NS(X \otimes \bar{F}) \).

**Proof.** \( X \) has potential good reduction at \( v \) by \([14]\), p.2. Since the Picard number of \( X \) is at least 20 and \( d(NS(X_F \otimes \bar{F})) \) is non-zero modulo \( p \), \( X \) is supersingular of artin invariant 1 or ordinary. If \( d(NS(X_F \otimes \bar{F})) \) is zero modulo \( p \), there is not embedding of \( NS(X_F \otimes \bar{F}) \otimes \mathbb{Z}_p \) into the Neron-Severi group of a supersingular K3 surface. \((\text{[24]}\) Hence \( X \) is ordinary. On the other hand, if \( d(NS(X_F \otimes \bar{F})) \) is non-zero non-square modulo \( p \), the discriminant of \( \mathbb{Z}_p \)-lattice \( H^2(X, \mathbb{Z}_p(1)) \) can not be square unit. Hence \( X \) is supersingular. If \( X \) is ordinary, by the lemma \([13]\) \( NS(X \otimes k)/NS(X \otimes \bar{F}) = 0 \).

Assume \( k \) is an algebraically closed field of characteristic \( p > 2 \) and the cardinality of \( k \) is less than or equal to the cardinality of \( \mathbb{C} \).

**Corollary 4.6.** Every K3 surface \( X \) over \( k \) of Picard number 20 has a model over a finite field. The model over a finite field is a reduction of a singular K3 surface defined over a number field.

**Proof.** We fix an embedding \( \mathbb{Q} \hookrightarrow \mathbb{Q}_p \) and a valuation preserving embedding \( \mathbb{Q}_p \hookrightarrow \overline{K} \simeq \mathbb{C} \). Assume \( X \) is a Neron-Severi group preserving lifting of \( X \). \( X_{\overline{K}} = X \otimes \overline{K} \) is a singular K3 surface and has a model over a number field. We assume \( X'_F \) is a singular K3 surface defined over a number field \( F \) and \( X'_F \otimes \overline{K} \simeq X_{\overline{K}} \). Let \( v \) be the place of \( F \) corresponding to the chosen embedding \( \overline{Q} \hookrightarrow \mathbb{Q}_p \). By the lemma \([4.5]\) we may assume \( X'_F \) has good reduction at \( v \). Let \( X' \) be a smooth integral model over the ring of integers of \( F_v \). Set \( X'_L \) is the special fiber of \( X' \). There is a finite extension \( L \) of \( K \) such that \( X'_L \otimes L \simeq X \otimes L \). Assume \( V \) is the ring of integers of \( L \). Then we have \( X \otimes V \simeq X' \otimes V \) and \( X \simeq X'_L \otimes k \).
Theorem 4.7. Let $X_F$ and $X'_F$ be two singular K3 surfaces defined over a number field $F$. Assume $X_F$ and $X'_F$ have good reduction at a place $v$ whose residue characteristic is $p > 2$. Let $X$ and $X'$ be the reduction of $X_F$ and $X'_F$ over $k$. Assume $X$ is ordinary and isomorphic to $X'$. Then $X_F \otimes \bar{F}$ and $X'_F \otimes \bar{F}$ are isomorphic.

Proof. It is enough to show that $X_F \otimes \bar{F}_v$ is isomorphic to $X'_F \otimes \bar{F}_v$. We set $L = KF_v \subset \bar{K}$ and $V$ is the ring of integers of $L$. $L$ is a finite extension of $K$. Assume $X$ and $X'$ are smooth integral models of $X_F$ and $X'_F$ over $V$ respectively. $X$ and $X'$ are the special fibers of $X$ and $X'$ respectively. We may assume $NS(X) = NS(X_L) = NS(X_K)$ and $NS(X') = NS(X'_L) = NS(X'_K)$.

Suppose $X$ is isomorphic to $X'$ and $X''$ is a Neron-Severi group preserving lifting of $X$ to $W$. Let $S$ be the formal deformation space of $X$ over $W$ and $U$ be the formal deformation space of the enlarged Brauer group $\psi_X$ over $W$. Since $X$ is ordinary, the canonical function $\lambda : S \to U$ is an isomorphism and

$$\lambda(A) = \Ext^1_{\hat{Z}}(H^2_1(X, \mu_{p^\infty}), \hat{\mathbb{G}}_m(A))$$

for any artin local $W$-algebra $A$. ([18], p.217) Moreover by the isomorphism (2.4), there exists a bijection

$$\lambda(A) : S(A) \to \Hom_{\hat{Z}}(H^2_1(X, \mathbb{Z}_p(1)), \hat{\mathbb{G}}_m(A)).$$

For a deformation $X_A \in S(A)$ over $A$, the composition of the cycle map and $\lambda_A(X_A)$,

$NS(X) \to H^2(X, \mathbb{Z}_p(1)) \to \hat{\mathbb{G}}_m(A)$

(4.1)

is the boundary map of the exact sequence (2.2).

Lemma 4.8 (c.f. [12], p.5). For an artin local algebra $A$, there is a unique deformation $X_A$ of $X$ to $A$ such that $\Pic(X_A) \to \Pic(X)$ is isomorphic.

Proof. If the reduction map $\Pic(X_A) \to \Pic(X)$ is surjective, the boundary map [4.11] is zero. But since the Picard number of $X$ is 20, $NS(X) \otimes \mathbb{Z}_p \to H^2(X, \mathbb{Z}_p(1))$ is an isomorphism, (lemma [1.2]) so $\lambda_A(X_A)$ is the zero map. Therefore $X_A$ satisfying the given condition is unique.

Since $X$ and $X'' \otimes V$ are lifting of $X$ to $V$, they define two morphisms $W$-morphisms $f, f'' : \text{Spf} V \to S$. For both of $X$ and $X'' \otimes V$, the reduction maps of the Picard groups are isomorphic. Hence when $m$ is the maximal ideal of $V$, $f \otimes V/m^n$ is equal to $f \otimes V/m^n$ as a map $\text{Spec} V/m^n \to S$ by the above lemma. Therefore $f = f''$ and $X$ and $X'' \otimes V$ induce the same formal scheme over $\text{Spf} V$. Then $X$ is isomorphic to $X'' \otimes V$ by [7], p.207. In particular, $X \otimes \bar{F}_v$ is isomorphic to $X'' \otimes \bar{F}_v$. By the same reason, $X' \otimes \bar{F}_v$ is isomorphic to $X'' \otimes \bar{F}_v$ and the proof is completed.
We summarize all the above results to give a classification of K3 surfaces of Picard number 20 over \( k \). Let \( S_p \) be the set of isomorphic classes of positive definite even \( \mathbb{Z} \)-lattices of rank 2 such that the discriminant is a non-zero square modulo \( p \). We fix embeddings \( \overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}_p \) and \( \overline{\mathbb{Q}}_p \rightarrow \overline{K} \cong \mathbb{C} \). For each \( M \in S_p \), there is a unique singular K3 surface \( X_M \) over \( \overline{\mathbb{Q}} \) such that \( T(X_M) \) is isomorphic to \( M \) via the given embedding \( \overline{\mathbb{Q}} \rightarrow \mathbb{C} \). Let \( X_{k,M} \) be the reduction of \( X_M \) over \( k \) as in the proof of corollary 4.6.

**Theorem 4.9.** Let \( k \) be an algebraically closed field of characteristic \( p > 2 \). The correspondence \( M \mapsto X_{k,M} \) is a bijection from \( S_p \) to the set of isomorphic classes of K3 surfaces of Picard number 20 over \( k \). Every K3 surface of Picard number 20 over \( k \) has a model over a finite field.

**References**

[1] Artin, M. Supersingular K3 surfaces, Ann. Sci. École Norm. Sup. (4) 7, 1974, 545–567.

[2] Artin, M. and Mazur, B. Formal groups arising from algebraic varieties, Ann. Sci. École Norm. Sup(4) 10, 1977, 87–131.

[3] Artin, M. and Swinnerton-Dyer, H. P. F. The Shafarevich-Tate conjecture for pencils of elliptic curves on K3 surfaces, Invent. Math. 20, 1973, 249–266.

[4] Charles, F. The Tate conjecture for K3 surfaces over finite fields, math-arxiv:1206.4002.

[5] Deligne, P. Relèvement des surfaces K3 en caractéristique nulle, Surfaces Algébriques, Lecture notes in Mathematics 868, 1981, 58–79.

[6] Illusie, L. Complexe de de Rham-Witt et cohomologie cristalline, Ann. ENS 4serie 12, 1979, 501–661.

[7] Illusie, L. Grothendieck’s existence theorem in formal geometry, Funeamental Algebraic Geometry, Mathematical surveys and monographs 123, 2005, 179–234.

[8] Inose, H. and Shioda, T. On singular K3 surfaces, Complex analysis and algebraic geometry, 1977, Iwanami Shoten, 119–136.

[9] Jang, J. An Enriques involution of a supersingular K3 surface over odd characteristic, matharxiv:1301.1118.

[10] Keum, J. Every algebraic Kummer surface is the K3-cover of an Enriques surface. Nagoya Math. J. 118, 1990, 99–110.

[11] Kovács, J. The cone of curves of a K3 surface. Math. Ann. 300, 1994, 681–691.

[12] Lieblich, M. and Maulik, D. A note on the cone conjecture for K3 surfaces in positive characteristic. matharxiv:1102.3377
[13] Matsusaka, T. and Mumford, D. Two fundamental theorems on deformations of Polarized varieties. American Journal of Mathematics 86, 1964, 668–684.

[14] Matsumoto, Y. On good reduction of some K3 surfaces related to abelian surfaces, matharxiv:1202.2421

[15] Morrison, D.R. On K3 surfaces with large Picard number, Invent. Math 75, 1984, 105–121.

[16] Mukai, S. and Namikawa, Y. Automorphisms of Enriques surfaces which acts trivially on the cohomology groups, Invent. Math. 77, 1984, 383–397.

[17] Nikulin, V.V. Kummer surface, Izv. Akad. Nauk SSSR 39, 1975, 278–293.

[18] Nygaard, N.O. The Tate conjecture for ordinary K3 surfaces over finite fields, Invent. Math 74, 1983, 213–237.

[19] Nygaard, N.O. and Ogus, A. Tate conjecture for K3 surfaces of finite height, Ann. of Math.(2) 122, 1985, 461–507.

[20] Ogus, A. Supersingular K3 crystal, Astérisque 64, 1979, 3–86

[21] Ogus, A. A crystalline Torelli theorem for supersingular K3 surfaces, Prog. Math. 36, 1983, 361-394.

[22] Piatetski-Shapiro, I and Shafarevich, I.R. A Torelli theorem for algebraic surfaces of type K3, Math. USSR Izv. 5, 1971, 547–588.

[23] Raynaud, M. p-torsion du schéma de Picard, astérisque 64, 1979, 87–148.

[24] Shimada, I. On Normal K3 surfaces, Michigan Math. J. 55, 2007, 395–416

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