Loop Equations for + and − Loops
in $c = \frac{1}{2}$ Non-Critical String Theory

Ryuichi NAKAYAMA * and Toshiya SUZUKI †

Department of Physics, Faculty of Science,
Hokkaido University, Sapporo 060, Japan

Abstract

New loop equations for all genera in $c = \frac{1}{2}$ non-critical string theory are constructed. Our loop equations include two types of loops, loops with all Ising spins up (+ loops) and those with all spins down (− loops). The loop equations generate an algebra which is a certain extension of $W_3$ algebra and are equivalent to the $W_3$ constraints derived before in the matrix-model formulation of 2d gravity. Application of these loop equations to construction of Hamiltonian for $c = \frac{1}{2}$ string field theory is considered.

* nakayama@phys.hokudai.ac.jp
† tsuzuki@particle.phys.hokudai.ac.jp
1 Introduction

Loop equations frequently appear in 2d quantum gravity. For example, they were used to show that one-matrix model has various critical points that correspond to non-unitary conformal fields coupled to 2d gravity. In the case of $(p,q)$ conformal fields coupled to 2d gravity it was shown that loop equations can be written in the form of $W_p$ constraints on generating function of correlation functions for scaling operators. Loop equations in other models such as ADE models, O(n) models on a random lattice have also been studied.

String field theory (SFT) is important for nonperturbative study of string theories. Several years ago a new type of SFT was considered for $c = 0$ noncritical string (2d pure gravity). Much effort has been devoted to extending this formalism to other $c \leq 1$ strings. Satisfactory formalism, however, seems to be still elusive.

In the study of SFT it was realized that hamiltonian in SFT is in close relationship with loop equations. Therefore investigation of loop equations is an important step toward construction of SFT. In this paper we will concentrate on $c = 1/2$ string, i.e. continuum limit of Ising model coupled to 2d gravity. In this theory many types of boundary conditions for Ising spins can be considered. Apparently it seems that by restricting the spin configurations on the loops differently we obtain different versions of SFT. Here we will consider only two types of spin configurations on loops; all Ising spins on loops are either up or down. Throughout this paper we will call a loop with all spins up a $+$ loop, and a loop with all spins down a $-$ loop. The main purpose of this paper is to construct loop equations in $c = 1/2$ string for $+$ and $-$ loops. Such loop equations should be constructed in such a way that they are equivalent to $W_3$ constraints derived in the matrix model formulation of 2d gravity.

c = 1/2 SFT for the above boundary conditions has already been presented in [1]. In this reference they assumed a certain Hamiltonian for $c = 1/2$ SFT from the outset and derived loop equations as Schwinger-Dyson equations (SDE) in SFT. Their loop equations, however, turned out to be a set of two decoupled Virasoro constraints and the connection of their loop equations with $W_3$ constraints is not clear. The purpose of the present work is to derive loop equations which are directly related to $W_3$ constraints.

The loop operators $w_+(l), w_-(l)$ which create $+$ and $-$ loops of length $l$ can be formally expanded in terms of the scaling operators. Therefore we can expect that if we consider only one type of loops (e.g. $+$ loops), the
structure of loop equations will be very similar to that of $W_3$ constraints in matrix models. Indeed loop equations for + loops can be derived in this way. It turns out that two independent source functions $J_+^{(1)}(l)$, $J_+^{(2)}(l)$ have to be introduced for a + loop of length $l$. However, it is rather intricate to include − loops into the loop equations. We will do this by using the relationship between $w_+(l)$ and $w_-(l)$ through analytic continuation in $l$. We will show that in this procedure we have to carefully treat the singular terms in two-loop amplitudes. As we will see later the structure of the resultant loop equations is very intriguing.

In sec 2 we will write down the loop equations for + loops and show these equations satisfy consistency conditions. The differential operators in the loop equations generate the continuum version of $W_3$ algebra. In sec 3 we derive a relationship between + and − loops in terms of analytic continuation in the length variable $l$ of loops. We point out the problem which arises in Laplace transformation of this relation and derive correct formulae. By using these formulae the loop equations for + and − loops are obtained in sec 4. In sec 5 we will show that these loop equations satisfy the consistency conditions. These loop equations generate a generalization of $W_3$ algebra. An interesting structure of the loop equations (e.g. $T_+ - V_-$ followed by Bogoliubov transformation) is noticed. In sec 6 we will consider SFT as an application of our loop equations. The problem of consistency condition on string field Hamiltonian is discussed. In sec 7 we will give a brief summary and discussions. A reduced version of the loop equations is also derived by eliminating extra degrees of freedom $J_\pm = (J_+^{(1)} - J_+^{(2)})/2$. In appendix A one of the loop equations $lU_+(l)Z_{+-} = 0$ obtained in sec 4 is shown to be equivalent to the Virasoro constraint in matrix models. In appendix B an explicit form of the $W_3$ current $X_+(l)$ defined in sec 5 is presented and the algebra that $U_\pm$ and $X_\pm$ generate is displayed. Some of the results in this paper were presented in [19].

2 Loop Equations for + Loops

A generating function for Green functions of scaling operators $O_n (n \neq 0 \mod 3)$ in $c = 1/2$ string theory,

$$
\tau(\mu) = \tau(\mu_1, \mu_2, \mu_4, \mu_5, \mu_7, \mu_8, \cdots)
= \exp\{\sum_{n=0}^{\infty}(\mu_{3n+1}O_{3n+1} + \mu_{3n+2}O_{3n+2})\}
$$

(2.1)
satisfies the $W_3$ constraints

$$L_n \tau(\mu) = 0 \quad n = -1, 0, 1, \cdots \quad (2.2)$$
$$W_n \tau(\mu) = 0 \quad n = -2, -1, 0, 1, \cdots \quad (2.3)$$

where $L_n$ and $W_n$ are differential operators with respect to $\mu$'s, which generate the $W_3$ algebra \[20\]

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{1}{12}n(n^2 - 1)\delta_{n,-m}, \quad (2.4)$$
$$[L_n, W_m] = (2n - m)W_{n+m}, \quad (2.5)$$

$$[W_n, W_m] = -9(n - m)U_{n+m} - \frac{1}{10}n(n^2 - 1)(n^2 - 4)\delta_{n,-m}$$
$$+ (n - m)\left\{ \frac{3}{2}(n^2 + 4nm + m^2) + \frac{27}{2}(n + m) + 21 \right\}L_{n+m}.$$

$$\left( U_n = \sum_{k \leq -2} L_k L_{n-k} + \sum_{k \geq -1} L_{n-k} L_k \right) \quad (2.6)$$

These results for $c = 1/2$ string were first conjectured in the study of SDE for large-N matrix models \[2\] \[3\] and confirmed in \[14\]. These $L_n$'s and $W_n$'s can be succinctly expressed in terms of a complex $Z_3$-twisted scalar field $\phi(z)$

$$T(z) = -: \partial_z \phi^*(z) \partial_z \phi(z) : + \frac{1}{9z^2} \equiv \sum_n z^{-n-2}L_n,$$
$$W(z) = (\partial_z \phi(z))^3 + (\partial_z \phi^*(z))^3 \equiv \sum_n z^{-n-3}W_n, \quad (2.7)$$

where the fields $\phi(z)$, $\phi^*(z)$ have the following mode expansions \[8\]

$$\phi(z) = \sum_{n=0}^{\infty} \left( z^{n+\frac{1}{2}} \mu_{3n+1} \sqrt{g} - z^{-n-\frac{1}{2}} \frac{3\sqrt{g}}{3n + 2} \frac{\partial}{\partial \mu_{3n+2}} \right),$$
$$\phi^*(z) = \sum_{n=0}^{\infty} \left( -z^{n+\frac{1}{2}} \mu_{3n+2} \sqrt{g} + z^{-n-\frac{1}{2}} \frac{3\sqrt{g}}{3n + 1} \frac{\partial}{\partial \mu_{3n+1}} \right). \quad (2.8)$$

Here $g$ is a string-coupling constant.

The relationship between the SDE for matrix models and the loop equations is well known \[3\] \[8\] \[10\] \[11\]. The functional differential operators appearing in loop equations generate a 'continuum' Virasoro algebra.\[7\] From the above results we are naturally led to introduce two independent source functions

\[1\] See eqs(2.7),(2.8) below
\[2\] Normal ordering : · · · is defined by regarding $\mu$ as an annihilation operator and $\partial/\partial \mu$ a creation operator.
$J^{(1)}_{+}(l), J^{(2)}_{+}(l)$ to construct loop equations for $+$ loops. Actually it is not difficult to figure out that loop equations take the following forms

$$IT_{+}(l) Z_{+}[J^{(1)}_{+}, J^{(2)}_{+}] = 0,$$
$$l^2 W_{+}(l) Z_{+}[J^{(1)}_{+}, g^{(2)}_{+}] = 32\{6\delta''(l) - t\delta(l)\} Z_{+}[J^{(1)}_{+}, J^{(2)}_{+}]$$

Here $Z_{+}$ is a generating function for amplitudes of $+$ loops and $T_{+}(l), W_{+}(l)$ are given by

$$T_{+}(l) = \{D^{(1)}_{+} * D^{(2)}_{+} + g \sum_{r=1}^{2}(lJ^{(r)}_{+}) \triangleq D^{(r)}_{+}\}(l),$$

$$W_{+}(l) = \sum_{r=1}^{2}\{D^{(r)}_{+} * D^{(r)}_{+} * D^{(r)}_{+} + 3g(lJ^{(3-r)}_{+}) \triangleq (D^{(r)}_{+} * D^{(r)}_{+})$$
$$+3g^2(lJ^{(3-r)}_{+}) \triangleq ((lJ^{(3-r)}_{+}) \triangleq D^{(r)}_{+})\}(l)$$

Here $D^{(r)}_{+}(l)$ stands for $\delta/\delta J^{(r)}_{+}(l)$. The symbols $*$ and $\triangleq$ represent the following integrals.

$$(f * g)(l) \equiv \int_{0}^{l} dl' f(l') g(l - l'), \quad (f \triangleq g)(l) \equiv \int_{0}^{\infty} dl' f(l') g(l' + l).$$

Later we will also need the integral

$$(f \triangleright g)(l) \equiv \int_{0}^{\infty} dl f(l' + l) g(l')$$

It has to be stressed that $J^{(1)}_{+}$ and $J^{(2)}_{+}$ in the above formulae are not the source functions for the loop of length $l$ with a fixed boundary condition (+ spins). Only the linear combination $J_{+}(l) = (J^{(1)}_{+}(l) + J^{(2)}_{+}(l))/2$ gives the source function for such a loop. $J_{+}$ is decomposed into $J^{(1)}_{+}$ and $J^{(2)}_{+}$ according to the powers of $l$ contained.

$$J^{(1)}_{+}(l) \sim l^{-1/3-n}, \quad J^{(2)}_{+}(l) \sim l^{-2/3-n} \quad (n \in \mathbb{Z})$$

For example, the disk amplitudes

$$f^{(r)}_{+}(l) = D^{(r)}_{+}(l) \ln Z|_{J=0,g=0}$$

are given \footnote{Henceforth $\sim$ will denote laplace transform} in laplace transformed forms \footnote{by}

$$\tilde{f}^{(1)}_{+}(\zeta) = \int_{0}^{\infty} dt e^{-\zeta t} f^{(1)}_{+}(l) = (\zeta - \sqrt{\zeta^2 - 4} t)^{4/3},$$

$$\tilde{f}^{(2)}_{+}(\zeta) = \int_{0}^{\infty} dt e^{-\zeta t} f^{(2)}_{+}(l) = (\zeta + \sqrt{\zeta^2 - 4} t)^{4/3}$$

\footnote{Henceforth $\sim$ will denote laplace transform}
Here $t$ is a cosmological constant.

It is not difficult to show that the disk amplitudes (2.17) satisfy the loop equations (2.9), (2.10). Conversely the loop equations determine the disk amplitudes completely. To show that these are actually equivalent to the $W_3$ constraints (2.2), (2.3) we have to factor out $Z_+$ into a singular piece $Z_{+}^{\text{sing}}$ and a regular piece $Z_{+}^{\text{reg}}$ as in the $c = 0$ case. \[6\] We can show after a certain amount of calculation that

$$\tau(\mu) = Z_{+}^{\text{reg}}[J_{+}^{(1)}(l) + c_1 t l^{-4/3} + c_2 l^{-7/3}, J_{+}^{(2)}(l)]$$  \hspace{1cm} (2.18)

with $c_1, c_2$ some constants satisfies $W_3$ constraints (2.2), (2.3).

Before we consider $-\text{ loops}$, we will comment on the algebra of $T_+ (l)$ and $W_+ (l)$. By explicit calculation we can show that these generate the ‘continuum’ $W_3$ algebra

$$[T_+ (l), T_+ (l')] = g(l - l')T_+(l + l'), \hspace{1cm} (2.19)$$

$$[T_+ (l), W_+ (l')] = g(2l - l')W(l + l'), \hspace{1cm} (2.20)$$

$$[W_+ (l), W_+ (l')] = 9g(l - l')(T_+ * T_+)(l + l') + 18g(l - l')(V_+ < T_+)(l + l')$$
$$- \frac{3}{2}g^2(l - l')(l'^2 + 4ll' + l'^2)T_+(l + l'). \hspace{1cm} (2.21)$$

Here $V_+ (l)$ is defined by

$$V_+ (l) = \{ g \sum_{r=1}^{2}(lJ_+^{(r)})*D_+^{(r)} + g^2(lJ_+^{(1)})*(lJ_+^{(2)})\}(l) \hspace{1cm} (2.22)$$

and corresponds to Virasoro generators $L_n$ with $n \leq -2$, while $T_+ (l)$ to $L_n$ with $n \geq -1$: we have

$$[T_+ (l), V_+ (l')] = g(l + l')T_+(l - l')\theta(l - l') + g(l + l')V_+(l' - l)\theta(l' - l), \hspace{1cm} (2.23)$$

$$[V_+ (l), V_+ (l')] = -g(l - l')V_+(l + l'), \hspace{1cm} (2.24)$$

where $\theta$ is a step function.

The algebra (2.19) ∼ (2.21) is not strictly closed because $V_+$ appears on the right-hand side of (2.21). Nonetheless this is sufficient for consistency of the loop equations (2.9), (2.10). \[6\] We can also show as in \[7\] that the right-hand side of (2.10) does not make (2.9), (2.10) inconsistent.

\footnote{Strictly speaking, for the proof of consistency we need more information than (2.3). Actually by explicitly solving (2.9) and (2.11) for $Z_+$ as perturbation series in $g$ we can show that $T_+ (l)Z_+ = -g\delta (l) \frac{\partial}{\partial t} Z_+$. A similar equation was also noticed in \[13\].}
3 Relation between Loop Amplitudes for $+$ Loops and $-$ Loops

In this section we will derive formulae which relate $+$ and $-$ loops by analytic continuation in variables $l$, $\zeta$. In the next section by using these formulae we will derive loop equations for both types of loops, $+$ and $-$ loops.

Let us denote the loop operators which create a loop of length $l$ with $+$ spins and that with $-$ spins by $w_+(l)$ and $w_-(l)$, respectively. These are related by

$$w_-(l) = -w_+(e^{3\pi i l})$$  \hspace{1cm} (3.1)

This relation is implicit in the representation for loop amplitudes in terms of heat kernels.\cite{[17],[22]} For example one- and two-loop amplitudes can be written as

$$< w_{\pm}(l) > \propto \int_{t}^{\infty} dx < x|e^{\pm l L}|x >,$$

$$< w_+(l_1)w_-(l_2) > \propto \int_{t}^{\infty} dx \int_{-\infty}^{t} dy < x|e^{l_1 L}|y > < y|e^{\pm l_2 L}|x >$$  \hspace{1cm} (3.3)

Here $L$ is a third-order Lax operator given by

$$L = \frac{1}{2} \left\{ -\left( \frac{d}{dx} \right)^3 + 3x^{1/3} \frac{d}{dx} \right\}$$  \hspace{1cm} (3.4)

These representations can be obtained by computing correlation functions in two-matrix model \cite{[16]} by using the method of orthogonal polynomials and taking continuum limits of them.\footnote{Loop amplitudes in $(p,q)$ gravity were computed in \cite{[23]}} A $-$ loop of length $l$ appears as a $+$ loop of 'length $-l$' in these representations.

The reason for an extra minus sign on the right-hand side of (3.1) can be understood if we expand $w_{\pm}(l)$ in powers of $l$.

$$w_+(l) = \sum_{n=0}^{\infty} O_n^{(1)} l^{n+1/3} + \sum_{n=0}^{\infty} O_n^{(2)} l^{n+2/3},$$

$$w_-(l) = \sum_{n=0}^{\infty} O_n^{(1)} (-1)^n l^{n+1/3} - \sum_{n=0}^{\infty} (-1)^n O_n^{(2)} l^{n+2/3}$$  \hspace{1cm} (3.5)

Here $O_{3n+1} \equiv O_n^{(1)}$ and $O_{3n+2} \equiv O_n^{(2)}$ are scaling operators. $O_{2n}^{(1)}$ and $O_{2n+1}^{(2)}$ are $Z_2$-even operators, while $O_{2n+1}^{(1)}$ and $O_{2n}^{(2)}$ are odd. We have $\exp\{3\pi i\}$ instead of $\exp\{\pi i\}$ in (3.1), because loop amplitudes have to be real functions.
To derive loop equations we need to laplace transform the relation (3.1). As far as loop amplitudes can be expanded in fractional powers of $l$ as in (3.3), we can easily show that

$$\tilde{w}_-(\zeta) = \tilde{w}_+(e^{3\pi i} \zeta)$$  \hspace{1cm} (3.6)

This relation (3.6) is valid for $n$-loop amplitudes with $n \geq 3$. In the case of one- and two-loop amplitudes we have to be careful in the treatment of singular terms. One-loop amplitudes can be expanded in powers of $l$, although first few terms are singular. Laplace transformation of these singular terms can be performed as usual in the sense of analytic continuation and the relation (3.6) is still valid. On the other hand two-loop amplitudes contain the singular terms which are not expandable.

$$< w_+(l_1)w_\pm(l_2) >_{c,sing} = \frac{\sqrt{3}}{2\pi} g \int_0^{l_1/3} (l_1^{1/3} \pm l_2^{1/3})$$  \hspace{1cm} (3.7)

The laplace transforms of these are given by

$$< \tilde{w}_+(\zeta_1)\tilde{w}_+(\zeta_2) >_{c,sing} = \frac{g(\zeta_1/\zeta_2)^{2/3} + 2(\zeta_1/\zeta_2)^{1/3} + 2(\zeta_2/\zeta_1)^{1/3} + (\zeta_2/\zeta_1)^{2/3} - 6}{3(\zeta_1 - \zeta_2)^2},$$  \hspace{1cm} (3.8)

$$< \tilde{w}_+(\zeta_1)\tilde{w}_-(\zeta_2) >_{c,sing} = \frac{g(\zeta_1/\zeta_2)^{2/3} - 2(\zeta_1/\zeta_2)^{1/3} - 2(\zeta_2/\zeta_1)^{1/3} + (\zeta_2/\zeta_1)^{2/3} + 3}{3(\zeta_1 + \zeta_2)^2}. $$  \hspace{1cm} (3.9)

These yield the following relation

$$< \tilde{w}_+(\zeta_1)\tilde{w}_-(\zeta_2) >_{c} = < \tilde{w}_+(\zeta_1)\tilde{w}_+(e^{3\pi i} \zeta_2) >_{c} + \frac{3g}{(\zeta_1 + \zeta_2)^2}$$  \hspace{1cm} (3.10)

We should also decompose the loop operators $w_+(l)$, $w_-(l)$ according to the powers of $l$.

$$w_\pm(l) = w_\pm^{(1)}(l) + w_\pm^{(2)}(l),$$  \hspace{1cm} (3.11)

$$w_\pm^{(1)}(l) \sim l^{1/3+n}, \quad w_\pm^{(2)}(l) \sim l^{2/3+n}, \quad (n \in \mathbb{Z})$$  \hspace{1cm} (3.12)

It is natural to put the second term on the right-hand side of (3.10) into the relation for $< \tilde{w}^{(1)}(l)\tilde{w}^{(2)}(l) >$ and we obtain the following relations.

$$< \tilde{w}_+^{(r_1)}(\zeta_1) \cdots \tilde{w}_+^{(r_n)}(\zeta_n)\tilde{w}_-^{(s_1)}(\zeta_1) \cdots \tilde{w}_-^{(s_m)}(\zeta_m) >_{c}$$

$$= < \tilde{w}_+^{(r_1)}(\zeta_1) \cdots \tilde{w}_+^{(r_n)}(\zeta_n)\tilde{w}_+^{(s_1)}(e^{3\pi i} \zeta_1) \cdots \tilde{w}_+^{(s_m)}(e^{3\pi i} \zeta_m) >_{c},$$

$$< \tilde{w}_+^{(r)}(\zeta)\tilde{w}_-^{(s)}(\zeta) >_{c} = < \tilde{w}_+^{(r)}(\zeta)\tilde{w}_+^{(s)}(e^{3\pi i} \zeta) >_{c} + \frac{3}{2} g \frac{\delta_{r+s,3}}{(\zeta + \xi)^2}$$  \hspace{1cm} (3.13)

Here $n + m \neq 2$ and $r, s, r_i, s_i = 1, 2$. These formulae will play an important role in the next section.
4 Inclusion of $-$ Loops into Loop Equations

We will derive the loop equations for loops with both types of spin configurations. For that purpose we have to introduce two more source functions $J_{-}^{(1)}(l), J_{-}^{(2)}(l)$ for $-$ loops. We start with

$$D_{+}^{(1)}(l_2)\frac{1}{Z_{+}} l_1 T_{+}(l_1) Z_{+} = 0$$

and set $J_{+}^{(1)} = J_{+}^{(2)} = 0$. By laplace transforming this ($\int_{0}^{\infty} dl_{1} \int_{0}^{\infty} dl_{2} \exp\{-\zeta_{1} l_{1} - \zeta_{2} l_{2}\}$) and rewriting this in terms of loop operators, we obtain

$$-\frac{\partial}{\partial \zeta_{1}}[< \tilde{w}_{+}^{(1)}(\zeta_{1})\tilde{w}_{+}^{(2)}(\zeta_{1})\tilde{w}_{-}^{(1)}(\zeta_{2})>_c + < \tilde{w}_{+}^{(1)}(\zeta_{1})\tilde{w}_{+}^{(1)}(\zeta_{2})>_c < \tilde{w}_{+}^{(2)}(\zeta_{1})>_c + < \tilde{w}_{+}^{(1)}(\zeta_{1})>_c < \tilde{w}_{+}^{(2)}(\zeta_{1})>_c + g \frac{\partial}{\partial \zeta_{2}}\left\{ \frac{1}{\zeta_{1} + \zeta_{2}}( < \tilde{w}_{+}^{(1)}(\zeta_{1})>_c - < \tilde{w}_{+}^{(1)}(\zeta_{2})>_c)\right\}] = 0.$$  

(4.2)

Here we replace $\zeta_{2}$ by $e^{3\pi i} \zeta_{2}$ and use the relation (3.13) to rewrite the above equation. This yields

$$-\frac{\partial}{\partial \zeta_{1}}[< \tilde{w}_{+}^{(1)}(\zeta_{1})\tilde{w}_{+}^{(2)}(\zeta_{1})\tilde{w}_{-}^{(1)}(\zeta_{2})>_c + < \tilde{w}_{+}^{(1)}(\zeta_{1})\tilde{w}_{+}^{(1)}(\zeta_{2})>_c < \tilde{w}_{+}^{(2)}(\zeta_{1})>_c + < \tilde{w}_{+}^{(1)}(\zeta_{1})>_c < \tilde{w}_{+}^{(2)}(\zeta_{1})>_c + g \frac{\partial}{\partial \zeta_{2}}\left\{ \frac{1}{\zeta_{1} + \zeta_{2}}( < \tilde{w}_{+}^{(1)}(\zeta_{1})>_c + < \tilde{w}_{-}^{(1)}(\zeta_{2})>_c)\right\}] = 0.$$  

(4.3)

This equation can be furthermore rewritten as follows

$$D_{-}^{(1)}(l_2)\frac{1}{Z_{-}}\int_{0}^{l_1} dl_{1} D_{+}^{(1)}(l_{1}) D_{+}^{(2)}(l_{1} - l_{1}') - \frac{1}{2} g \int_{0}^{l_1} dl_{1}' J_{-}^{(1)}(l_{1}') D_{+}^{(1)}(l_{1} - l_{1}')$$

$$- g \int_{0}^{\infty} dl_{1}' J_{-}^{(1)}(l_{1}') D_{+}^{(1)}(l_{1}') Z_{-} |_{J_{-}^{(r)} = 0} = 0.$$  

(4.4)

Here $Z_{-} = [J_{+}^{(1)}, J_{+}^{(2)}, J_{-}^{(1)}, J_{-}^{(2)}]$ is a generating function for both types of loops.

By operating $D_{+}^{(r)}$ on eq (4.1) arbitrary times and repeating this procedure we finally arrive at the new loop equation

$$l U_{+}(l) Z_{+-} = 0,$$

(4.5)

$$l U_{-}(l) Z_{+-} = 0,$$

(4.6)

where $U_{\pm}(l)$ is the following operator.

$$U_{\pm}(l) = \{ D_{\pm}^{(1)} * D_{\pm}^{(2)} + g \sum_{r=1}^{2} (l J_{\pm}^{(r)}) \circ D_{\pm}^{(r)} - g \sum_{r=1}^{2} (l J_{\pm}^{(r)}) \triangleright D_{\pm}^{(r)}$$

9
\[ +\alpha g \sum_{r=1}^{2} (lJ_{\pm}^{(r)}) \ast D_{\pm}^{(r)} + (\alpha^2 - 1)g^2(lJ_{\pm}^{(1)}) \ast (lJ_{\pm}^{(2)}) \} (l), \]
\[ (\alpha = \frac{1}{2}). \] (4.7)

Eq (4.6) is obtained from eq (4.5) by $Z_2$ symmetry for Ising spins. Let us note that if we set $J^{(1)}_\pm = J^{(2)}_\pm = 0$, the loop equations (4.5), (4.6) reduce to those for only + loops (2.9).

Although we have derived loop equation (4.3) by using the method of analytical continuation (3.13), more direct proof will be desirable. In Appendix A we will prove that loop equation (4.5) is equivalent to the Virasoro constraint (2.2). We also checked that disk and cylinder amplitudes satisfy (4.5).

Let us make a remark on the value of $\alpha$. We might formally repeat the procedure after (4.1) without Laplace trasforming the loop equations. Because the singular parts of two-loop amplitudes (3.8), (3.9) respect the relation (3.1), this relation might be used for any amplitudes. In this case we would again end up with the loop equations (4.5), (4.6) but the value of $\alpha$ in (4.7) would then be replaced by 1. This discrepancy can be resolved if we take into account the fact that the $\int dl$ integrals appearing in the loop equations are divergent, when the lengths $l$ of loops in one-loop amplitudes vanish, and we should not make the analytic continuation (3.1) in such divergent integrals. To the contrary Laplace-transformed loop equations like (4.2) are algebraic and cause no problems. Hence we should put $\alpha = -\frac{1}{2}$ as above.

5 Consistency Conditions

In this section we will show that $U_+(l)$ and $U_-(l)$ defined in the previous section satisfy consistency conditions. Let us define the operators

\[ \hat{U}_+(l) \equiv T_+(l) - V_-(l), \quad \hat{U}_-(l) \equiv T_-(l) - V_+(l), \] (5.1)

where $T_-(l)$ and $V_-(l)$ are obtained from $T_+(l)$ and $V_+(l)$, respectively, by interchanging $J^{(r)}_+$ and $D^{(r)}_+$ with $J^{(r)}_-$ and $D^{(r)}_-$. Because $J^{(r)}_+$ and $J^{(r)}_-$ are independent functions, we can show by using the commutation relations (2.19), (2.28), and (2.24) that $U_\pm(l)$ generate the algebra

\[ [\hat{U}_\pm(l_1), \hat{U}_\pm(l_2)] = g(l_1 - l_2)\hat{U}_\pm(l_1 + l_2), \] (5.2)

\[ [\hat{U}_+(l_1), \hat{U}_-(l_2)] = -g(l_1 + l_2)\hat{U}_+(l_1 - l_2)\theta(l_1 - l_2) + g(l_1 + l_2)\hat{U}_-(l_2 - l_1)\theta(l_2 - l_1). \] (5.3)
These two generators can be combined into a single one: the operator

\[ \hat{U}(l) \equiv \hat{U}_+(l)\theta(l) - \hat{U}_-(l)\theta(-l) \quad (-\infty < l < \infty) \]  

(5.4)

generates a whole ‘continuum’ Virasoro algebra.

The functional differential operators \( U_+(l), U_-(l) \) which appear in loop equations (4.5), (4.6) turn out to be related to \( \hat{U}_+(l) \) and \( \hat{U}_-(l) \) by a Bogoliubov transformation

\[
D_{\pm}^{(r)}(l) \rightarrow D_{\pm}^{(r)}(l) - \frac{1}{2}glJ_{\pm}^{(3-r)}(l),
\]

\[
J_{\pm}^{(r)}(l) \rightarrow J_{\pm}^{(r)}(l) \quad (r = 1, 2)
\]  

(5.5)

Because this transformation does not change the algebra, \( U_+(l) \) and \( U_-(l) \) generate two coupled Virasoro algebras (5.2),(5.3), and consistency conditions are satisfied. At present the meaning of this transformation is not clear to us.

The analysis in the preceding and present sections can be generalized to \( W_+(l) \). We define

\[
Y_-(l) = \sum_{r=1}^{2}[3g((IJ_+^{(r)}) \triangleright D_{-}^{(3-r)} \triangleright D_{-}^{(3-r)} + 3g^2((IJ_+^{(r)}) * (IJ_+^{(r)})) \triangleright D_{-}^{(3-r)} \\
+ g^3(IJ_+^{(r)}) * (IJ_+^{(r)}) * (IJ_+^{(r)}))(l),
\]

(5.6)

and construct

\[ \hat{X}_+(l) \equiv W_+(l) - Y_-(l). \]  

(5.7)

By applying transformation (5.5) on \( \hat{X}_+(l) \), we obtain \( X_+(l) \). We will present it in Appendix B explicitly. Similarly we can construct \( X_-(l) \). The operator \( X_\pm(l) \) together with \( U_\pm \) generate a closed algebra. This is also presented in Appendix B.

To summarize, loop equations for \( c = 1/2 \) string are given by

\[
lU_+(l)Z_{+-} = lU_-(l)Z_{+-} = 0,
\]

\[
l^2X_+(l)Z_{+-} = l^2X_-(l)Z_{+-} = 32(6\delta''(l) - t\delta(l))Z_{+-}
\]  

(5.8)

This is the main result of this paper.

---

\[ ^6 \] We can also show that \( U_\pm(l)Z_{+-} = -g\delta(l) \frac{\partial}{\partial l}Z_{+-} \) by starting from the equation in footnote 4 and applying the same method as in sec 4
6 String Field Theory for $c = 1/2$ String

The loop equations obtained in the previous section determine the loop amplitudes completely. In this section we will discuss applications of these loop equations to string field theory.

Let us first consider a theory which contains only loops with + spins. In the string field theory we consider here[6], the generating function of loop amplitudes will be expressed in the form

$$Z_+ [J_+^{(1)}, J_+^{(2)}] = \lim_{D \to \infty} e^{-DH} \exp \left\{ \int_0^\infty dl \sum_{r=1,2} J_+^{(r)}(l) \Psi_+^{(r)}(l) \right\} |0>$$  \hspace{1cm} (6.1)

The parameter $D$ may be interpreted as geodesic distance on the world sheet. Here $\Psi_+^{(r)}(l)$ is a creation operator for a loop of length $l$ with + spins and satisfies with the corresponding annihilation operators $\Psi_+^{(r)}(l)$ the commutation relations

$$[\Psi_+^{(r)}(l), \Psi_+^{(r)^\dagger}(l')] = \delta_{r,r'} \delta(l - l')$$  \hspace{1cm} (6.2)

Hamiltonian $H$ is a functional of $\Psi_+^{(r)}$, $\Psi_+^{(r)^\dagger}$ and related to the loop equations. To determine its form we note that the existence of the limit (6.1) implies string field SDE

$$\lim_{D \to \infty} \frac{\partial}{\partial D} < 0| e^{-DH} \exp \left\{ \int_0^\infty dl \sum_{r=1,2} J_+^{(r)}(l) \Psi_+^{(r)}(l) \right\} |0>= 0.$$  \hspace{1cm} (6.3)

This can be rewritten as a differential equation for $Z_+$

$$\hat{H} [J_+^{(r)}, D_+^{(r)}] Z_+ [J_+^{(1)}, J_+^{(2)}] = 0,$$  \hspace{1cm} (6.4)

where $\hat{H}$ is obtained from $H$ by replacing $\Psi_+^{(r)}(l)$ and $\Psi_+^{(r)^\dagger}(l)$ by $J_+^{(r)}(l)$ and $D_+^{(r)}(l)$, respectively, and interchanging the ordering of $J$’s and $D$’s.

In the case of pure gravity ($c = 0$), $\hat{H}$ is given by[3]

$$\hat{H} = \int_0^\infty dl J(l) \{ lT(l) - \rho(l) \},$$  \hspace{1cm} (6.5)

where

$$T(l) = \int_0^l D(l')D(l-l') + g \int_0^\infty dl' J(l')D(l+l')$$  \hspace{1cm} (6.6)

and $\rho(l)$ is a tadpole term. If the generating function $Z[J]$ satisfies the loop equation

$$lT(l)Z[J] = \rho(l)Z[J]$$  \hspace{1cm} (6.7)

$Z[J]$ also satisfies SDE (6.4). The converse can also be proved[3].

12
Therefore a simple generalization to $c = 1/2$ string will be to take as $\hat{H}$

$$\hat{H} = \int_0^\infty dl J_+(l)\{l^2W_+(l) - \rho(l)\} \tag{6.8}$$

$$+ g \int_0^\infty dl_1 \int_0^\infty dl_2 l_1 l_2 (l_1 + l_2)\{aJ_+(l_1)J_+(l_2) + b\bar{J}_+(l_1)\bar{J}_+(l_2)\} T_+(l_1 + l_2),$$

where $a, b$ are some constants not yet determined and the tadpole term is given by

$$\rho(l) = 32(6\delta''(l) - t\delta(l)). \tag{6.9}$$

$J_+(l)$ and $\bar{J}_+(l)$ are defined by

$$J_+ \equiv \frac{1}{2}(J_+^{(1)} + J_+^{(2)}), \quad \bar{J}_+ \equiv \frac{1}{2}(J_+^{(1)} - J_+^{(2)}). \tag{6.10}$$

The dimensions of $\Psi_+^{(r)}, \Psi_+^{(r)\dagger}$ are given by $L^{4/3}, L^{-7/3}$, respectively and that of $g$ is $L^{-14/3}$, where $L$ denotes the dimension of the length of a loop. Hence the dimension of $\hat{H}$ is $L^{-2/3}$ and we get the result $[D] \sim L^{2/3}$. However, for the reason which will be stated in the next few paragraphs we cannot draw a definite conclusion about the dimension of geodesic distance $D$ in this paper. This construction of Hamiltonian can also be extended in an obvious way to the theory where loops are included. In (6.8) $W_+$ and $T_+$ have to be replaced by $X_+$ and $U_+$, respectively and terms with + and − interchanged should be added.

By construction our $Z_+[J_+^{(1)}, J_+^{(2)}]$ satisfies loop equations (2.9), (2.10) and we may stop at this point. But if our string field theory is to have geometrical meaning, each interaction in the Hamiltonian should give proper decomposition of world sheets into propagators, vertices and tadpoles. In other words, the Hamiltonian should provide us with Feynman rules for calculation of loop amplitudes with geodesic distances among loops fixed.

In [7] a consistency condition for string field Hamiltonian was proposed. Let us consider a world sheet with the topology of a cylinder. Suppose that the minimum geodesic distance of the two boundaries is $D$. We are able to compute an amplitude for such a geometry by sewing two transfer matrices by a disk. Starting from the two boundaries two loops propagate by splitting and disappearing and eventually two loops from the two boundaries meet at a point. Let the geodesic distance between this point and the two boundaries be $D_1$ and $D_2$ ($D_1 + D_2 = D$). After the two loops meet at a point, they

---

7 Similar Hamiltonian is constructed by a different method in [24]. Main difference between our Hamiltonian and theirs is that their Hamiltonian does not contain terms corresponding to the singular terms of one- and two-loop amplitudes.

8 For discussion of the dimension of geodesic distance $D$ see [4], [8], [24], [26].
merge and the loop will keep splitting and disappearing to form a disk. In [4] it was proposed to require that the resulting amplitude should not depend on how the geodesic distance $D$ is decomposed into two. This is a reasonable requirement if we are to compute such amplitudes in string field theory.

We performed such a consistency check of our Hamiltonian (6.8) and found the consistency condition in the above sense does not seem to be satisfied. The obstruction seems to lie in the fact that our loop equations contain extra degrees of freedom $\bar{J}_\pm = (J_{\pm}^{(1)} - J_{\pm}^{(2)})/2$ in addition to $J_\pm = (J_{\pm}^{(1)} + J_{\pm}^{(2)})/2$. In the next section we will construct loop equations for $J_\pm$ only.

7 Discussions

In this paper we derived the loop equations (5.8) in $c = \frac{1}{2}$ string, where Ising spins on the loops are all either up or down. These loop equations are equivalent to the $W_3$ constraints [2][3][14] which were derived in large-N matrix models and the consistency conditions of the loop equations yield an algebra which is a generalization of the $W_3$ algebra. Although our loop equations contain only the same information as the $W_3$ constraints, they are expected to provide us with a geometrical setting for constructing $c = 1/2$ SFT. Unfortunately the purpose of constructing SFT in sec 6 was not completely fulfilled. In this section we will construct loop equations for only $J_\pm = (J_{\pm}^{(1)} + J_{\pm}^{(2)})/2$ by eliminating $\bar{J}_\pm = (J_{\pm}^{(1)} - J_{\pm}^{(2)})/2$ from (5.8) and speculate on possible application of the result to SFT.

Let us first consider loop equations (2.9), (2.10) for $+$ loops. Because we need loop equations only for loops corresponding to the source $J_+$, we rewrite $T_+(l)$ and $W_+(l)$ in terms of $J_+$, $\bar{J}_+$ and $D_+ \equiv D_+^{(1)} + D_+^{(2)}$, $\bar{D}_+ \equiv D_+^{(1)} - D_+^{(2)}$.

\[
T_+ = \frac{1}{4} D_+ \ast D_+ - \frac{1}{4} \bar{D}_+ \ast \bar{D}_+ + g(lJ_+) \ast D_+ + g(l\bar{J}_+) \ast \bar{D}_+, \quad (7.1)
\]

\[
W_+ = -3(D_+ \ast + 2g(lJ_+)\ast T_+ + (D_+ \ast + 3g(lJ_+)\ast)^2 D_+ \\
-3g(lJ_+) \ast (D_+ \ast \bar{D}_+ - g(l\bar{J}_+) \ast D_+ + g(lJ_+) \ast \bar{D}_+) \\
+3gD_+ \ast ((lJ_+) \ast \bar{D}_+) + 3g^2(lJ_+) \ast ((l\bar{J}_+) \ast \bar{D}_+). \quad (7.2)
\]

This result shows that we can eliminate $\bar{D}_+$ from the loop equations by setting $\bar{J}_+ = 0$. By using (7.2) and the equation in footnote 4, we obtain

\[
l^2\{(D_+ \ast + 3g(lJ_+)\ast)^2 D_+\}(l)Z_+ \\
= -3gl^2 \frac{\partial}{\partial l} D_+(l)Z_+ + 32(6\delta''(l) - t\delta(l))Z_+. \quad (7.3)
\]
This agrees with the result in [8], where the right-hand side in (7.3) was set to zero (denoted \( \approx 0 \)) as being terms either with a support at \( l = 0 \) or proportional to backgrounds.

Similarly we can derive loop equations for \( J_+ \) and \( J_- \) from (5.8). We obtain

\[
I^2 \{ D_+ * D_+ * D_+ + 3gD_+ * ((lJ_+)<< D_+) + 3g(lJ_+) \langle (D_+ * D_+) \\
-3g(lJ_-) * D_+ * D_- - 3gD_+ * ((lJ_-)\triangleright D_-) - 3g(lJ_-) \triangleright (D_- * D_-) \\
-9g^2(lJ_-) \triangleright ((lJ_-) \langle D_-) + 9g^2(lJ_+) \langle ((lJ_+)\langle D_+) \\
-9g^2(lJ_-) \triangleright ((lJ_-) \lhd D_-) - 9g \delta \delta K_+ \}
\]

\[
\approx 0 \quad (7.4)
\]

and the counterpart obtained by interchanging + and − in (7.4). These are generalizations of (7.3) to + and − loops. Let us note that while \( X_+(l) \) in (5.3) contain terms which are cubic in \( J_+ \)'s, these terms are canceled altogether in the above equation.

This loop equation may serve as a starting point for construction of SFT for \( c = \frac{1}{2} \) string. We can show that (7.4) can be decomposed into a set of loop equations.

\[
\{ D_+ * D_+ + 3g(lJ_+) \langle D_+ - 3g(lJ_-) * D_+ - 3g(lJ_-) \triangleright D_- + \frac{\delta}{\delta K_+} \} Z_{+-} \approx 0,
\]

\[
\{ D_+ * \frac{\delta}{\delta K_+} + 3g(lJ_+) \langle \frac{\delta}{\delta K_+} - 3g(lJ_-) \triangleright \frac{\delta}{\delta K_-} \} Z_{+-} \approx 0
\]

(7.5)

Here \( K_+(l), K_-(l) \) are source functions for some auxiliary loop fields. It is understood that \( K_\pm(l) \) are set to zero after differentiation in (7.5). These are generalization of the loop equations considered in [8].

\[
\left\{ \frac{\delta}{\delta J_0} * \frac{\delta}{\delta J_n} + (lJ_0) \langle \frac{\delta}{\delta J_n} + \frac{\delta}{\delta J_{n+1}} \right\} Z|_{J_i=0} \approx 0 \quad (n = 0, 1, 2)
\]

(7.6)

Here \( J_0 \) is identical to \( J_+ \), while \( J_1 \) is a source for a + loop with a single operator \( \mathcal{H} \) insertion, and \( J_2 \) that with double \( \mathcal{H} \) insertion. Construction of SFT Hamiltonian based on loop equation (7.5) is under study.

Whether loop equation (7.4) is compatible with that of Ishibashi and Kawai [7]

\[
l\{ D_+ * D_+ + D_+ \triangleright D_- + g(lJ_+) \langle D_+ \} (l) Z = 0
\]

(7.7)

is an important problem. This is under investigation and we hope to report on this in the future [27]. If these loop equations are equivalent, the present
work will provide us with a connection of Ishibashi-Kawai’s SFT for $c = 1/2$ gravity to $W_3$ constraints. If not, $c = 1/2$ SFT has to be constructed from loop equations (7.4).

The structures of the loop equations (5.8) are quite interesting. For example the Virasoro constraint $U_+(l)$ is obtained by performing Bogoliubov transformation (5.5) on $T_+(l) - V_-(l)$. These mathematical structures may worth further investigation. It should also be pointed out that the Virasoro generators $U_\pm$ involve terms quadratic in $J$s and the $W_3$ currents $X_\pm$ contain cubic $J$ terms.

We can extend the present analysis to $c = 1 - \frac{6}{m(m+1)}$ string. It is known that this theory can be constructively defined in terms of $(m - 1)$-matrix-chain model and that the generating function of correlation functions in this theory satisfies $W_m$ constraints. So corresponding to the $n$-th matrix $M_n$ ($n = 1, \ldots, m-1$) we need to introduce $m-1$ source functions $J^{(r)}_n(l) \quad (r = 1, \ldots, m-1)$. It will be straightforward to construct loop equations at least for two matter configurations on the loops corresponding to the ends $M_1, M_{m-1}$ of the matrix chain.

Acknowledgements

One of the authors (R. N.) thanks J. Ambjørn, N. Ishibashi, H. Kawai, N. Kawamoto and C. Kristjansen for discussions.

This work is supported in part by Grant-in-Aid for Scientific Research (No.07640364) from the Ministry of Education, Science and Culture, Japan.

A Equivalence of Loop Eq (4.5) to the Virasoro Constraint (2.2)

In this appendix we will show that the loop equation (4.3) is equivalent to the Virasoro constraint (2.2).

The generating function $Z_{+-}$ can be factorized into two parts.

$$Z_{+-} = Z_{+-}^{sing} Z_{+-}^{reg} \quad (A.1)$$

The singular part $Z_{+-}^{sing}$ is an exponential of quadratic forms of $J$’s and given
by
\[
\ln Z_{+}^{\text{sing}} = \frac{\sqrt{3}}{2\pi} g \int_0^{\infty} dl \int_0^{\infty} dl' \frac{l^{1/3}l'^{2/3}}{l + l'} \{J_+^{(1)}(l)J_+^{(2)}(l') + J_-^{(1)}(l)J_-^{(2)}(l')\}
- \frac{\sqrt{3}}{2\pi} g \int_0^{\infty} dl \int_0^{\infty} dl' \frac{l^{1/3}l'^{2/3}}{l - l'} \{J_+^{(1)}(l)J_-^{(2)}(l') + J_-^{(1)}(l)J_+^{(2)}(l')\}
+ \frac{2^{4/3}}{3\Gamma(2/3)} \int_0^{\infty} dl \left(4l^{-7/3} - tl^{-1/3}\right) \{J_+^{(2)}(l) + J_-^{(2)}(l)\}
\]
(A.2)

Here the symbol \(\tilde{f}\) means a principal value of an integral.

When \(\tilde{D}_\pm^{(r)}(\zeta)\) operates on the regular part \(Z_{+}^{\text{reg}}\), it is assumed to have the following expansions.

\[
\tilde{D}_+^{(r)}(\zeta)Z_{+}^{\text{reg}} = g \sum_{n=0}^{\infty} \zeta^{-n-1-r/3} \partial \frac{\partial}{\partial \mu_{3n+r}} Z_{+}^{\text{reg}},
\]
(A.3)

\[
\tilde{D}_-^{(r)}(\zeta)Z_{+}^{\text{reg}} = g \sum_{n=0}^{\infty} (-1)^{n+r+1} \zeta^{-n-1-r/3} \partial \frac{\partial}{\partial \lambda_{3n+r}} Z_{+}^{\text{reg}}
\]
(A.4)

Here \(\mu_n\) and \(\lambda_n\) are source variables for scaling operators and defined by

\[
\int_0^{\infty} dl J_+^{(r)}(l)l^{n+r/3} = \frac{1}{g} \Gamma(n + 1 + \frac{r}{3})\mu_{3n+r},
\]
(A.5)

\[
\int_0^{\infty} dl J_-^{(r)}(l)l^{n+r/3} = \frac{1}{g} \Gamma(n + 1 + \frac{r}{3})\lambda_{3n+r}(-1)^{n+r}
\]
(A.6)

By using these definitions we will rewrite the loop equation (4.5) into a differential equation for \(Z_{+}^{\text{reg}}\). After shifting the source variables according to

\[
\mu_1 \rightarrow \mu_1 + (2)^{1/3}t, \quad \mu_7 \rightarrow \mu_7 - \frac{3}{7}2^{1/3},
\]
\[
\lambda_1 \rightarrow \lambda_1 + (2)^{1/3}t, \quad \lambda_7 \rightarrow \lambda_7 - \frac{3}{7}2^{1/3},
\]
(A.7)

we obtain

\[
\partial \zeta \left\{ g^2 \sum_{n,m=0}^{\infty} \zeta^{-n-m-3} \frac{\partial}{\partial \mu_{3n+1}} \frac{\partial}{\partial \mu_{3m+2}}
+ g \sum_{r=1}^{\infty} \sum_{n=0}^{\infty} \zeta^{m-n-2} \frac{\partial}{\partial \mu_{3n+r}} \lambda_{3m+r}
+ g \sum_{r=1}^{\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \zeta^{m-n-2} \lambda_{3m+r} \frac{\partial}{\partial \mu_{3n+r}} - \frac{\partial}{\partial \lambda_{3n+r}}
+ \frac{1}{9} \zeta \left(\mu_1 + \lambda_1\right) \left(\mu_2 + \lambda_2\right) Z_{+}^{\text{reg}} = 0
\}
\]
(A.8)
Let us note that the third term is composed of positive powers of \( \zeta \). This term, however, shows that \( Z_{+}^{reg} \) depends on \( \mu \) and \( \lambda \) only through the combination \( \mu_n + \lambda_n \). Then (A.8) reduces to the Virasoro constraint (2.2) on \( \tau(\mu + \lambda) = Z_{+}^{reg} \).

**B ‘Extended’ \( W_3 \) Algebra**

The operator \( X_+(l) \) defined in sec 5 is given by

\[
X_+(l) = \sum_{r=1}^{2}\{D^{(r)}_+ * D^{(r)}_+ * D^{(r)}_+ + 3g(lJ^{(3-r)}_+ \triangleright (D^{(r)}_+ * D^{(r)}_+) \\
- \frac{3}{2}g(lJ^{(3-r)}_+ * D^{(r)}_+ * D^{(r)}_+ - 3g((lJ^{(r)}_-) \triangleright D^{(3-r)}_-) \triangleright D^{(3-r)}_-) \\
+ 3g^2(lJ^{(3-r)}_+ \triangleright ((lJ^{(3-r)}_+ \triangleright D^{(r)}_+ + \frac{3}{4}g^2(lJ^{(3-r)}_-) * (lJ^{(3-r)}_-) \triangleright D^{(r)}_+ \\
- 3g^2((lJ^{(r)}_-) * (lJ^{(r)}_-) \triangleright D^{(3-r)}_- - \frac{9}{8}g^3(lJ^{(r)}_-) * (lJ^{(r)}_-) * (lJ^{(r)}_-) \\
- \frac{9}{4}g^3(lJ^{(r)}_+) \triangleright ((lJ^{(r)}_+) \triangleright (lJ^{(r)}_-) + \frac{9}{4}g^3(lJ^{(r)}_+) \triangleright ((lJ^{(r)}_+) \triangleright (lJ^{(r)}_-))\}(l) \\
\]

(B.1)

In the remaining part of this Appendix we will show the algebra that \( U_\pm \) and \( X_\pm \) generate. First of all \( T_\pm, V_\pm, W_\pm \) and \( Y_\pm \) generate the following algebra. Some formulae which are not presented here can be found in (2.19)~(2.21), (2.23), (2.24). The index \( \pm \) will be suppressed here.

\[
[T(l), Y(l')] = g(2l + l')W(l - l')\theta(l - l') + g(2l + l')Y(l' - l)\theta(l' - l), \quad (B.2)
\]
\[
[V(l), W(l')] = -g(2l + l')Y(l - l')\theta(l - l') - 2(2l' + l)W(l' - l)\theta(l' - l), \quad (B.3)
\]
\[
[V(l), Y(l')] = -g(2l - l')Y(l + l'), \quad (B.4)
\]
\[
[Y(l), Y(l')] = -9g(l - l')(V * V)(l + l') - 18g(l - l')(V \triangleright T)(l + l') \\
+ \frac{3}{2}g^2(l - l')(l^2 + 4ll' + l'^2)V(l + l'), \quad (B.5)
\]
\[
[Y(l), W(l')] = \theta(l' - l)\{-9g(l + l')(T * T)(l' - l) \\
- 18g(l + l')(V \triangleleft T)(l' - l) + \frac{3}{2}g^2(l + l')(l^2 - 4ll' + l'^2)T(l' - l)\} \\
+ \theta(l - l')\{-9g(l + l')(V * V)(l - l') - 18g(l + l')(V \triangleright T)(l - l') \\
+ \frac{3}{2}g^2(l + l')(l^2 - 4ll' + l'^2)V(l - l')\} \quad (B.6)
\]
By using these relations we can derive the algebra of $\hat{U}_\pm = T_\pm - V_\mp$ and $\hat{X}_\pm = W_\pm - Y_\mp$.

$$[\hat{U}_\pm(l), \hat{X}_\pm(l')] = g(2l - l')\hat{X}_\pm(l + l'), \quad (B.7)$$

$$[\hat{U}_\pm(l), \hat{X}_\pm(l')] = -g(2l + l')\hat{X}_\pm(l - l')\theta(l - l') + g(2l + l')\hat{X}_\pm(l' - l)\theta(l' - l), \quad (B.8)$$

$$[\hat{X}_\pm(l), \hat{X}_\pm(l')] = -\frac{3}{2}g^2(l - l')(l^2 + 4ll' + l'^2)\hat{U}_\pm(l + l')$$

$$+9g(l - l')\{T_\pm \ast \hat{U}_\pm + V_\mp \ast \hat{U}_\mp + 2V_\pm \ast \hat{U}_\mp + 2V_\mp \ast \hat{U}_\pm + 2V_\pm \ast \hat{U}_\mp \ast\} \theta(l - l')$$

$$-2V_\mp \ast \hat{U}_\pm \hat{U}_\mp \ast\} (l + l'), \quad (B.9)$$

$$[\hat{X}_+(l), \hat{X}_-(l')] = \theta(l - l')[\frac{3}{2}g^2(l + l')(l^2 - 4ll' + l'^2)\hat{U}_+(l - l')$$

$$-9g(l + l')\{T_+ \ast \hat{U}_+ + V_- \ast \hat{U}_- + 2V_+ \ast \hat{U}_- + 2V_- \ast \hat{U}_+ \ast\} \theta(l - l')\]$$

$$-(l \leftrightarrow l', + \leftrightarrow -) \quad (B.10)$$

Other commutation relations are given in (5.2) and (5.3). The algebra of $U_\pm$ and $X_\pm$ can be derived from the above by transformation (5.3).
References

[1] V. Kazakov, Mod. Phys. Lett. A4 (1989) 2125.

[2] M. Fukuma, H. Kawai and R. Nakayama, Int. J. Mod. Phys. A6 (1991) 1385.

[3] R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. B348 (1991) 435.

[4] I. K. Kostov, Nucl. Phys. B326 (1989) 583; ibid. B376 (1992) 539; Phys. Lett. B266 (1991) 42; I. Kostov and M. Staudacher, Nucl. Phys. B384 (1992) 459; B. Eynard and C. Kristjansen, Nucl. Phys. B455 (1995) 577.

[5] M. Kaku and K. Kikkawa, Phys. Rev. D10 (1974) 1110, 1823; W. Siegel, Phys. Lett. B151 (1985) 391, 396; E. Witten, Nucl. Phys. B268 (1986) 253; Hata, K. Itoh, T. Kugo, H. Kunitomo and K. Ogawa, Phys. Lett. B172 (1986) 186, 195; A. Neveu and P. West, Phys. Lett. B168 (1986) 192; M. Kaku, Int. J. Mod. Phys. A9 (1994) 139.

[6] N. Ishibashi and H. Kawai, Phys. Lett. B314 (1993) 190.

[7] N. Ishibashi and H. Kawai, Phys. Lett. B322 (1994) 67.

[8] M. Ikehara, N. Ishibashi, H. Kawai, T. Mogami, R. Nakayama and N. Sasakura, Phys. Rev. D50 (1994) 7467; A Note on String Field Theory in the Temporal Gauge, proceedings of the Workshop on Quantum Field Theory, Integrable Models and Beyond, Yukawa Institute for Theoretical Physics, Kyoto University, 14-18 Feb 1994, Prog. Theor. Phys. Supp. 118 (1995) 241.

[9] R. Nakayama and T. Suzuki, Phys. Lett. B354 (1995) 69.

[10] Y. Watabiki, Nucl. Phys. B441 (1995) 119; Phys. Lett. 346 (1995) 46.

[11] M. Ikehara, Phys. Lett B348 (1995) 365; Prog. Theor. Phys. 93 (1995) 1141.

[12] A. Jevicki and J. Rodrigues, Nucl. Phys. B421 (1994) 278.

[13] F. Sugino and T. Yoneya, Stochastic Hamiltonian for Noncritical String Field Theories, UT-Komaba-95-8, Oct 1995.

[14] E. Gava and K. S. Narain, Phys. Lett. B263 (1991) 213.
[15] V.A. Kazakov, I.K. Kostov and A.A. Migdal, Phys. Lett. **B157** (1985) 295; J. Ambjørn, B. Durhuus and J. Fröhlich, Nucl. Phys. **B257** [FS14] (1985) 433; F. David, Nucl. Phys. **B257** [FS14] (1985) 543.

[16] V.A.Kazakov, Phys. Lett. **A119** (1986) 140; D.V.Boulatov and V.A.Kazakov, Phys. Lett. **B186** (1987) 379.

[17] E. Brézin and V. Kazakov, Phys. Lett. **B236** (1990) 144; M. Douglas and S. Shenker, Nucl. Phys. **B335** (1990) 635; D. J. Gross and A. Migdal, Phys. Rev. Lett. **64** (1990) 127; Nucl. Phys. **B340** (1990) 333.

[18] T. Banks, M. Douglas, N. Seiberg and S. Shenker, Phys. Lett. **B238** (1990) 43.

[19] R. Nakayama and T. Suzuki, *New Loop Equations in Ising Model Coupled to 2d Gravity and String Field Theory*, in the Proceedings of the Workshop held in honor of the 60th birthday of Professor Keiji Kikkawa “Frontiers in Quantum Field Theory”, Osaka, Japan, Dec 1995.

[20] A.B. Zamolodchikov, Theor. Math. Phys. **65** (1985) 1205.

[21] I. Kostov, Phys. Lett. **B266** (1991) 317.

[22] G. Moore, N. Seiberg and M. Staudacher, Nucl. Phys. **B362** (1991) 665.

[23] J.-M. Daul, V.A. Kazakov and I.K. Kostov, Nucl. Phys. **B409** (1993) 311; M. Anazawa, A. Ishikawa and H. Itoyama, Phys. Lett. **B362** (1995) 59; M. Anazawa and H. Itoyama, *Macroscopic N Loop Amplitude for Minimal Models Coupled to Two-Dimensional Gravity*, preprint OUHET-222 (Nov 1995).

[24] J. Ambjørn and Y. Watabiki, *Non-critical string field theory for 2d quantum gravity coupled to (p,q)-conformal fields*, NBI and TIT preprint (1996).

[25] H. Kawai, N. Kawamoto, T. Mogami and Y. Watabiki, Phys. Lett. **306B** (1993) 19; S.S. Gubser and I.R. Klebanov, Nucl. Phys. **B416** (1994) 827; J. Ambjørn and Y. Watabiki, Nucl. Phys. **B445** (1995) 129.

[26] J. Ambjørn, J. Jurkiewicz and Y. Watabiki, Nucl. Phys. **B454** 313; S. Catteral, G. Thorleifsson, M. Bowick and V. John, Phys. Lett. **B354** (1995) 58.

[27] R. Nakayama and T. Suzuki, in preparation.