On finding an obstacle embedded in the rough background medium via the enclosure method in the time domain

Masaru Ikehata

Laboratory of Mathematics, Institute of Engineering, Hiroshima University, Higashi 739–8527, Japan

E-mail: ikehata@amath.hiroshima-u.ac.jp

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Abstract

A mathematical method for through-wall imaging via wave phenomena in the time domain is introduced. The method makes use of a single reflected wave over a finite time interval and gives us a criterion whether a penetrable obstacle exists or not in a general rough background medium. Moreover, if the obstacle exists, the lower and upper estimates of the distance between the obstacle and the center point of the support of the initial data are given. As an evidence of the potential of the method two applications are also given.

Keywords: enclosure method, inverse obstacle scattering, through-the-wall, rough background

1. Introduction

Assume that there is a large wall between an observer and an unknown object. The wall is opaque and thus the observer can not see the object directly. How can the observer find the object? Consider the case when the wall is electromagnetically penetrable. Then there is no doubt that everyone thinks about the use of electromagnetic wave. Generate the electromagnetic wave at the place where the observer is. And observe the reflected wave from the object at the same place over a finite time interval. The observed wave should include information about the object. How can one extract the information from the observed wave? The purpose of this paper is to develop a mathematical method for through-wall imaging by using the governing equation of the wave from the beginning to end. Originally the governing equation should be the Maxwell system. In this paper, as a first step we consider two single equations for scalar waves.
1.1. Finding discontinuity in a medium with a rough refractive index

Let us formulate the first problem. Let \( n = 1, 2, 3 \). Let \( \alpha \in L^\infty(\mathbb{R}^n) \) and satisfy \( \inf_{x \in \mathbb{R}^n} \alpha(x) > 0 \). Let \( 0 < T < \infty \). Given \( f \in L^2(\mathbb{R}^n) \), let \( u = u_f(x, t) \) be the weak solution of

\[
\begin{align*}
\alpha(x) \partial_t^2 u - \Delta u &= 0 &\text{in} &\mathbb{R}^n \times ]0, T[, \\
 u(x, 0) &= 0 &\text{in} &\mathbb{R}^n, \\
 \partial_t u(x, 0) &= f(x) &\text{in} &\mathbb{R}^n.
\end{align*}
\] (1.1)

The notion of the weak solution is taken from [6]. See section 2.1 in this paper.

We assume that \( \alpha \) takes the form

\[
\alpha(x) = \begin{cases} 
\alpha_0(x), & \text{if} \ x \in \mathbb{R}^n \setminus D, \\
\alpha_0(x) + h(x), & \text{if} \ x \in D,
\end{cases}
\] (1.2)

where \( D \subset \mathbb{R}^n \) is a bounded open subset with Lipschitz boundary. The function \( \alpha_0 \) belongs to \( L^\infty(\mathbb{R}^n) \) and satisfies \( m_0^2 \leq \alpha_0(x) \leq M_0^2 \) a.e. \( x \in \mathbb{R}^n \) for positive constants \( m_0 \) and \( M_0 \); \( h \) belongs to \( L^\infty(D) \) and satisfies one of (A.I) and (A.II) listed below:

(A.I) \( \exists C > 0 \ h(x) \geq C \ a.e. \ x \in D; \)
(A.II) \( \exists C > 0 \ - h(x) \geq C \ a.e. \ x \in D. \)

\( D \) is a model of an unknown penetrable obstacle with material parameter \( \alpha_0 + h \) embedded in the background medium with material parameter \( \alpha_0 \). The distribution of \( \alpha_0 \) models various penetrable walls in the space and \( D \) is something hidden in the walls or a space between the walls and various penetrable obstacles.

Let \( B \) be an open ball satisfying \( \overline{B} \cap \overline{D} = \emptyset \). Let \( f \in L^2(\mathbb{R}^n) \) satisfy \( f(x) = 0 \) a.e. \( x \in \mathbb{R}^n \setminus B \) and \( \inf_{x \in \partial B} f(x) > 0 \). Generate \( u_f \) by the initial data \( f \). In this paper, we consider the following inverse problems under the assumption that \( \alpha_0 \) is known and that both \( D \) and \( h \) are unknown.

**Problem 1.** Find a criterion whether \( D = \emptyset \) or not in terms of only \( u_f \) on \( B \) over time interval \( ]0, T[ \).

**Problem 2.** Assume that \( D \neq \emptyset \). Extract information about \( D \) from \( u_f \) on \( B \) over time interval \( ]0, T[ \) for a fixed large \( T \).

Note that the correspondence \( (D, h) \rightarrow u |_{B \times ]0, T[} \) is nonlinear and thus both problems become nonlinear problems. The existence of variation of \( \alpha_0 \) outside \( D \) plays a role of the wall in which a wave can propagate. This paper aims at developing a through-the-wall imaging method via the governing equation on the wave used.

There are a lot of studies from the engineering side for through-the-wall imaging using electromagnetic waves. See [1, 3] for a survey on through-wall imaging and various engineering approaches. Roughly speaking, one can say that those approaches introduce various processing of the reflected signal from the wall and targets behind the wall. For example, in [1] the compressive sensing incorporating invariance of the antenna location due to the geometry of the assumed wall has been applied to a wall clutter mitigation technique for the signal. In [2] under the assumption that the wall is a single uniform one, an approach which employs an imaging function incorporating geometrical optics (Snell’s law) for the wave...
propagation through the wall has been considered and tested numerically. In [23] an algorithm to find a moving human in a simple situation using the time-of-flight approach is introduced. In [4] the idea of the linear sampling method in the frequency domain has been applied to through-wall imaging and tested numerically in two dimensions under the assumption that the wall is infinitely long in one direction. They employ the concrete form of the Green’s function for the wave propagation through the background medium and thus, in this sense, their approach should belong to a class of PDE approaches.

In this paper, we employ the enclosure method introduced by the author himself in [9, 10] as a guiding principle for attacking the problem mentioned above. It is a direct method which connects the unknown discontinuity and the observation data through the so-called indicator function. In [11] the author initiated to apply the method to inverse obstacle problems whose governing equations are given by the heat and wave equations in one-space dimension. Now we have several applications of the method to inverse obstacle scattering problems in three-space dimensions whose governing equations are given by the wave equations [12–15]. See [16] for a review of the results in those papers. However, imaging an obstacle through-the-wall is still out of the range of the results in those papers.

Now let us describe the result.

Let \( \tau > 0 \) and define

\[
 w(x, \tau) = \int_0^T e^{-\tau t} u(x, t) \, dt \quad \text{in} \quad \mathbb{R}^n. \tag{1.3}
\]

Let \( v = v(x, \tau) \in H^1(\mathbb{R}^n) \) be the weak solution of

\[
 \Delta v - \alpha_0 \tau^2 v + \alpha_0 f = 0 \quad \text{in} \quad \mathbb{R}^n. \tag{1.4}
\]

Define

\[
 I_f(\tau, T) = \int_B \alpha_0 f (w - v) \, dx.
\]

The following result gives us solutions to Problems 1 and 2.

**Theorem 1.1.** We have:

(i) if \( D = \emptyset \), then for all \( T > 0 \) it holds that

\[
 \lim_{\tau \to \infty} e^{\tau T} I_f(\tau, T) = 0;
\]

(ii) if \( D \neq \emptyset \) and \( h \) satisfies (A.I), then for all \( T > 2M_0 \text{dist}(D, B) \) it holds that

\[
 \lim_{\tau \to \infty} e^{\tau T} I_f(\tau, T) = -\infty;
\]

(iii) if \( D \neq \emptyset \) and \( h \) satisfies (A.II), then for all \( T > 2M_0 \text{dist}(D, B) \) it holds that

\[
 \lim_{\tau \to \infty} e^{\tau T} I_f(\tau, T) = \infty.
\]

Moreover, in case of both (ii) and (iii) we have, for all \( T > 2M_0 \text{dist}(D, B) \)

\[
 \lim_{\tau \to \infty} \inf_{\tau} \frac{1}{2\tau} \log |I_f(\tau, T)| \geq -M_0 \text{dist}(D, B) \tag{1.5}
\]

and
Note that \(a_0\) and \(h\) are just essentially bounded on \(\mathbb{R}^n\) and \(D\), respectively. In particular, \(a_0\) can be a model for various background media such as multilayered media with complicated interfaces or unions of various domains with different refractive indexes. It will be impossible to apply any approach based on the idea of geometrical optics to this case. See [8, 18, 20, 21] for such approach in the case when the scattering kernel which is the observation data in the Lax-Phillips scattering theory is given under the assumptions that \(a_0(x) = 1\) a.e. \(x \in \mathbb{R}^n\), \(\partial D\) is smooth and \(h \in C^\infty(\overline{D})\).

Let \(p\) be the center of \(B\) and \(\eta\) the radius. Define \(d_{\partial D}(p) = \inf_{x \in \partial D} |x - p|\). Since \(\text{dist}(D, B) = d_{\partial D}(p) - \eta\), estimates (1.5) and (1.6) give us an upper and lower estimate of \(d_{\partial D}(p)\) provided \(m_0\) and \(M_0\) are known and \(T\) is sufficiently large. Thus we obtain the upper and lower estimation of the sphere \(|x - p| = d_{\partial D}(p)\) whose exterior encloses the object.

Estimates (1.5) and (1.6) suggest a new direction of the enclosure method in the case when the background medium is inhomogeneous and quite complicated: give up to find a precise quantity in the observation data which is related to the exact location of unknown obstacles; instead give lower and upper estimates rigorously like (1.5) and (1.6) for \(\text{dist}(D, B)\).

Some further remarks are in order.

- In Theorem 1.1 it suffices to know \(v\) on \(B\) not whole \(v\). However, without knowing \(a_0\) outside \(B\) it is impossible to compute \(v\) on \(B\) in advance. In the last section we suggest an experimental computation procedure of \(v\) on \(B\) without detailed knowledge of \(a_0\) outside \(B\). It seems that this will be useful for the daily security of the interior of the room, house and building which have complicated environment. For the purpose Theorem 1.1 will be suitable since \(a_0\) is just essentially bounded and we never assume any other regularity.

- If \(a_0(x) = 1\) a.e. \(x \in \mathbb{R}^n\), then one can choose \(M_0 = m_0 = 1\) and thus (1.5) and (1.6) imply that

\[
\lim_{\tau \to \infty} \frac{1}{2\tau} \log |I_f(\tau, T)| = \text{dist}(D, B).
\]

This coincides with a result (1.20) in [13].

In the next section we apply the idea developed here to more special case and show (1.5) and (1.6) can be replaced with a single formula.

### 1.2. Finding discontinuity in a dissipative medium

Let \(n = 1, 2, 3\). Let \(q \in L^{\infty}(\mathbb{R}^n)\) satisfy \(q(x) \geq 0\) a.e. \(x \in \mathbb{R}^n\). Let \(0 < T < \infty\). Given \(f \in L^2(\mathbb{R}^n)\), let \(u = u_f(x, t)\) be the weak solution of

\[
\begin{cases}
\partial_t^2 u - \Delta u + q(x) \partial_t u = 0 & \text{in } \mathbb{R}^n \times [0, T[, \\
u(x, 0) = 0, & \text{in } \mathbb{R}^n, \\
\partial_t u(x, 0) = f(x), & \text{in } \mathbb{R}^n.
\end{cases}
\]

\[\text{(1.7)}\]
We assume that \( q \) takes the form
\[
q(x) = \begin{cases} 
q_0(x), & \text{if } x \in \mathbb{R}^n \setminus D, \\
q_0(x) + h(x), & \text{if } x \in D,
\end{cases}
\]
where \( D \) is a bounded open set of \( \mathbb{R}^n \) with Lipschitz boundary. The function \( q_0 \) belongs to \( L^\infty(\mathbb{R}^n) \) and satisfies \( q_0(x) \geq 0 \) a.e. \( x \in \mathbb{R}^n \).

The function \( h \) on \( D \) in (1.8) has to belong to \( L^2(D) \). We assume that \( h \) satisfies one of (A.I) and (A.II).

Let \( \tau > 0 \). Define \( w \) by the formula (1.3) in which \( u \) is replaced with the solution of (1.7).

Let \( v \in H^1(\mathbb{R}^n) \) be the weak solution of
\[
\left( \triangle - \tau^2 - \tau q_0 \right) v + f = 0 \text{ in } \mathbb{R}^n.
\]

Let \( B \) and \( f \) be the same as those of Theorem 1.1. Define
\[
J_f(\tau, T) = \int_B f(w - v)dx.
\]

The following result is new and suggests that, even in a rough dissipative medium the solution of (1.7) carries information about \( D \) clearly like the wave equation outside \( D \).

**Theorem 1.2.** We have:

(i) if \( D = \emptyset \), then for all \( T > 0 \) it holds that
\[
\lim_{\tau \to \infty} e^{TJ_f(\tau, T)} = 0;
\]

(ii) if \( D \neq \emptyset \) and \( h \) satisfies (A.I), then for all \( T > 2\text{dist}(D, B) \) it holds that
\[
\lim_{\tau \to \infty} e^{TJ_f(\tau, T)} = -\infty;
\]

(iii) if \( D \neq \emptyset \) and \( h \) satisfies (A.II), then for all \( T > 2\text{dist}(D, B) \) it holds that
\[
\lim_{\tau \to \infty} e^{TJ_f(\tau, T)} = \infty.
\]

Moreover, in case of both (ii) and (iii) we have, for all \( T > 2\text{dist}(D, B) \)
\[
\lim_{\tau \to \infty} \frac{1}{2\tau} \log \left| J_f(\tau, T) \right| = -\text{dist}(D, B).
\]

Roughly speaking, we see that (1.9) as \( \tau \to \infty \) corresponds to (1.4) with \( \alpha_0 = 1 \) and thus formula (1.10) is reasonable.

Comparing Theorem 1.2 with the previous results in [13, 14] for the wave equation outside \( D \), we see (ii) and (iii) suggest us that that assumptions (A.I) (stronger dissipation) and (A.II) (weaker dissipation) correspond to the Dirichlet and Neumann boundary conditions imposed at \( \partial D \), respectively.

A brief outline of this paper is as follows. In section 2 first we specify the meaning of the weak solution used in the formulation of the problems. Second we establish two basic integral identities for \( w \) given by (1.3). Theorem 1.1 is proved in section 3. The proof starts with deriving the lower and upper estimates for the indicator function: \( \tau \to J_f(\tau, T) \) as \( \tau \to \infty \).
from the basic identities in section 2. Next, by virtue of the governing equation (1.4) we see that \( v \) has point-wise explicit lower and upper estimates. Combining those, we obtain all the conclusions stated in Theorem 1.1. The proof of Theorem 1.2 is given in section 4 which is a combination of a reduction and similar argument done in the proof of Theorem 1.1. In section 5 we give some remarks concerned with a ‘practical use’ of Theorem 1.1 and present a sharp form of Theorem 1.1 in the case when the space dimension is 1 and \( \alpha \) has a special but important form.

2. Preliminaries

2.1. On the solution class

In this section we specify the meaning of the weak solutions of (1.1) and (1.7) at the same time. It is based on Theorem 1 given on p 558 in [6] which we have already used for the formulation of the weak solution of (1.1) in [13].

Set \( V = H^{1}(\mathbb{R}^n) \) and \( H = L^2(\mathbb{R}^n) \). Applying Theorem 1, we know that given \( u^0 \in V \) and \( u^1 \in H \), there exists a unique \( u \) satisfying

\[
\begin{align*}
&\int_{0}^{T} \int_{\mathbb{R}^n} C(u'(t)), \phi > + b_0(u'(t), v) + a(u(t), \phi) = 0, \quad \text{a.e. } t \in ]0, T[, \\
&\int_{0}^{T} \int_{\mathbb{R}^n} \frac{d}{dt} C(u'(t)), \phi > + b_0(u'(t), v) + a(u(t), \phi) = 0, \quad \text{a.e. } t \in ]0, T[, \\
&\int_{0}^{T} \int_{\mathbb{R}^n} C(u'(t)), \phi > + b_0(u'(t), v) + a(u(t), \phi) = 0, \quad \text{a.e. } t \in ]0, T[,.
\end{align*}
\]

(2.1)

and \( u(0) = u^0 \) and \( u'(0) = u^1 \), where

\[
\int_{\mathbb{R}^n} \nabla u \cdot \nabla v dx, \quad b_0(u, v) = \int_{\mathbb{R}^n} quv dx, \quad u, v \in V,
\]

and \( C: H \rightarrow H \) is the bounded linear operator defined by

\[
C(u) = au, \quad u \in H.
\]

Note that this \( C \) satisfies (5.11) on p 553 in [6] under the condition \( \inf_{x \in \mathbb{R}^n} \alpha(x) > 0 \).

Since \( q \geq 0 \), \( b_0 \) satisfies (5.8) on p 553 with \( \beta_0 = 0 \) in their notation. However, equation (2.1) is homogeneous, i.e., the source term is 0, and by virtue of this, their proof also covers this case.

In this paper, we say that this \( u \) for \( u^0 = 0 \) and \( u^1 = f \) with \( q = 0 \) and \( \alpha = 1 \) is the weak solutions of (1.1) and (1.7), respectively.

We see that \( w \) given by

\[
w = \int_{0}^{T} e^{-\tau \alpha} u dt
\]

belongs to \( V \) and applying integration by parts to (2.1) multiplied by \( e^{-\tau t} \), we obtain, for all \( \phi \in V \)

\[
\int_{\mathbb{R}^n} \{ \nabla w \cdot \nabla \phi + (\tau^2 \alpha + \tau q) w \phi \} dx - \int_{\mathbb{R}^n} a \phi dx + e^{-\tau T} \int_{\mathbb{R}^n} F \phi dx = 0,
\]

where \( \tau > 0 \).
where

\[ F(x) = \alpha(x)u'(x, T) + (\alpha(x)\tau + q(x))u(x, T). \]

This means that \( w \) is the weak solution of the following equation:

\[ \left( \triangle - \alpha \tau^2 - qt \right) w + af = e^{-iT} F \text{ in } \mathbb{R}^n. \] (2.2)

### 2.2. Two basic identities

In this section we consider only the case when \( q(x) = 0 \) a.e. \( x \in \mathbb{R}^n \). Then, it follows from (2.2) that \( w \) satisfies

\[ \triangle w - \alpha \tau^2 w + af = \alpha e^{-iT} F \text{ in } \mathbb{R}^n, \] (2.3)

where \( F(x, \tau) = u'(x, T) + \tau u(x, T). \)

In what follows \( f \) just belongs to \( L^2(\mathbb{R}^n) \); \( v \in H^1(\mathbb{R}^n) \) satisfies (1.4).

**Proposition 2.1.** We have

\[
\int_{\mathbb{R}^n} f \left( \int_{\mathbb{R}^n} \left( a_0 - a \right) v + aR \right) \, dx = \tau^2 \int_{\mathbb{R}^n} \left( a_0 - a \right) v^2 \, dx \\
+ \int_{\mathbb{R}^n} \left( |VR|^2 + \alpha \tau^2 R^2 \right) \, dx + e^{-iT} \left( \int_{\mathbb{R}^n} aFR \, dx - \int_{\mathbb{R}^n} aFv \, dx \right). \]

(2.4)

where \( R = w - v \).

**Proof.** From (1.4) we have

\[
\int_{\mathbb{R}^n} a_0 v \, dx = \tau^2 \int_{\mathbb{R}^n} \nabla v \cdot \nabla w \, dx.
\]

On the other hand, from (2.3) we have

\[
\int_{\mathbb{R}^n} aR \, dx = \tau^2 \int_{\mathbb{R}^n} \nabla w \cdot \nabla v \, dx + e^{-iT} \int_{\mathbb{R}^n} aFv \, dx.
\]

Therefore we obtain

\[
\int_{\mathbb{R}^n} f \left( a_0 w - av \right) \, dx = \tau^2 \int_{\mathbb{R}^n} \left( a_0 - a \right) v \, dx + e^{-iT} \int_{\mathbb{R}^n} aF \, dx.
\]

(2.5)

Write

\[
\tau^2 \int_{\mathbb{R}^n} \left( a_0 - a \right) v \, dx = \tau^2 \int_{\mathbb{R}^n} \left( a_0 - a \right) v^2 \, dx + \tau^2 \int_{\mathbb{R}^n} \left( a_0 - a \right) vR \, dx.
\]

(2.6)

It follows from (2.3) and (1.4) that \( R \) satisfies

\[ \triangle R - \alpha \tau^2 R + (a_0 - a) \tau^2 v - (a_0 - a)f = \alpha e^{-iT} F \text{ in } \mathbb{R}^n. \] (2.7)

Thus we have

\[
\tau^2 \int_{\mathbb{R}^n} \left( a_0 - a \right) v R \, dx = \int_{\mathbb{R}^n} \left( |VR|^2 + \alpha \tau^2 R^2 \right) \, dx + \int_{\mathbb{R}^n} \left( a_0 - a \right) f R \, dx + e^{-iT} \int_{\mathbb{R}^n} aFR \, dx.
\]
A combination of this and (2.6) in (2.5) yields
\[
\int_{\mathbb{R}^n} f (a_0 w - av) dx = \tau^2 \int_{\mathbb{R}^n} (a_0 - a) v^2 dx + \int_{\mathbb{R}^n} \left( |VR|^2 + \alpha^2 R^2 \right) dx \\
+ \int_{\mathbb{R}^n} (a_0 - a) \mathcal{F} dx + e^{-\tau T} \left( \int_{\mathbb{R}^n} aF R dx - \int_{\mathbb{R}^n} aF v dx \right).
\]
Since \((a_0 w - av) - (a_0 - a) R = (a_0 - a) v + aR\), we obtain (2.4).

**Proposition 2.2.** We have
\[
\int_{\mathbb{R}^n} f \left\{ (a - a_0) w - a_0 R \right\} dx = \tau^2 \int_{\mathbb{R}^n} \frac{a_0}{a} (a - a_0) v^2 dx \\
+ \int_{\mathbb{R}^n} |VR|^2 + \alpha^2 \left( R + \left( 1 - \frac{a_0}{a} \right) v \right)^2 dx + e^{-\tau T} \left( \int_{\mathbb{R}^n} aF R dx + \int_{\mathbb{R}^n} aF v dx \right). \tag{2.8}
\]
where \(R = w - v\).

**Proof.** We recall that we have equation (2.5). Instead of (2.6) we write
\[
\tau^2 \int_{\mathbb{R}^n} (a - a_0) v w dx = \tau^2 \int_{\mathbb{R}^n} (a - a_0) w^2 dx + \tau^2 \int_{\mathbb{R}^n} (a_0 - a) w R dx. \tag{2.9}
\]
From (2.7) we see that \(R\) satisfies
\[
\Delta R = -a_0 \tau^2 R + (a_0 - a) \tau^2 w - (a_0 - a) f = \alpha e^{-\tau T} F \text{ in } \mathbb{R}^n.
\]
Thus we have
\[
\tau^2 \int_{\mathbb{R}^n} (a_0 - a) w R dx = \int_{\mathbb{R}^n} \left( |VR|^2 + a_0 \alpha^2 R^2 \right) dx \\
+ \int_{\mathbb{R}^n} (a_0 - a) \mathcal{F} dx + e^{-\tau T} \int_{\mathbb{R}^n} aF R dx.
\]
Now a combination of this and (2.9) in (2.5) yields
\[
\int_{\mathbb{R}^n} f (av - a_0 w) dx = \tau^2 \int_{\mathbb{R}^n} (a - a_0) w^2 dx + \int_{\mathbb{R}^n} \left( |VR|^2 + a_0 \tau^2 R^2 \right) dx \\
+ \int_{\mathbb{R}^n} (a_0 - a) \mathcal{F} dx + e^{-\tau T} \left( \int_{\mathbb{R}^n} aF R dx + \int_{\mathbb{R}^n} aF v dx \right). \tag{2.10}
\]
Finally write \((av - a_0 w) - (a_0 - a) R = (a - a_0) w - a_0 R\) and
\[
a_0 R^2 + (a - a_0) w^2 = a \left| w - \frac{a_0}{a} v \right|^2 + \frac{a_0}{a} (a - a_0) v^2.
\]
Combining these with (2.10) we obtain (2.8).

**3. Proof of Theorem 1.1**

First we derive two asymptotic estimates.
Proposition 3.1. We have, as $\tau \to \infty$

$$I_f(\tau, T) \leq \tau^2 \int_{\mathbb{R}^n} \frac{a_0}{\alpha} (a_0 - a)v^2 \, dx + O\left(\tau^{-1}e^{-\tau T}\right) \quad (3.1)$$

and

$$I_f(\tau, T) \geq \tau^2 \int_{\mathbb{R}^n} (a_0 - a)v^2 \, dx + O\left(\tau^{-1}e^{-\tau T}\right). \quad (3.2)$$

Proof. We start with describing two simple estimates for $v$ and $R = w - v$. It follows from (1.4) that

$$\int_{\mathbb{R}^n} \left(|Vv|^2 + a_0 \tau^2 v^2 - a_0 v \right) \, dx = 0,$$

that is,

$$\int_{\mathbb{R}^n} \left(|Vv|^2 + a_0 \left(\tau v - \frac{f}{2\tau}\right) \right)^2 \, dx = \frac{1}{4\tau^2} \int_{\mathbb{R}^n} a_0 f^2 \, dx.$$

It is easy to see that from this we obtain, as $\tau \to \infty$

$$\int_{\mathbb{R}^n} \left(|Vv|^2 + a_0 \tau^2 v^2\right) \, dx = O\left(\tau^{-2}\right)$$

and in particular,

$$\|v\|_{L^2(\mathbb{R}^n)} = O\left(\tau^{-2}\right). \quad (3.3)$$

Next rewrite (2.4) as

$$\tau^2 \int_{\mathbb{R}^n} (a_0 - a)v^2 \, dx + \int_{\mathbb{R}^n} \left(|VR|^2 + a \left(\tau R - \frac{f - e^{-\tau F}}{2\tau}\right) \right)^2 \, dx$$

$$= \int_{\mathbb{R}^n} (a_0 - a)v \, dx + \frac{1}{4\tau^2} \int_{\mathbb{R}^n} a \left(f - e^{-\tau F}\right)^2 \, dx.$$

Since $F = u'(x, T) + \tau u(x, T)$ and $\|u'(\cdot, T)\|_{L^2(\mathbb{R}^n)} + \|u(\cdot, T)\|_{L^2(\mathbb{R}^n)} < \infty$, it follows from this and (3.3) that

$$\int_{\mathbb{R}^n} \left(|VR|^2 + a \left(\tau R - \frac{f - e^{-\tau F}}{2\tau}\right) \right)^2 \, dx = O\left(\tau^{-2}\right)$$

and hence

$$\int_{\mathbb{R}^n} \left(|VR|^2 + a \tau^2 R^2\right) \, dx = O\left(\tau^{-2}\right).$$

In particular, we have

$$\|R\|_{L^2(\mathbb{R}^n)} = O\left(\tau^{-2}\right). \quad (3.4)$$

Applying (3.3) and (3.4) to (2.4) and (2.8), we obtain, as $\tau \to \infty$

$$\int_{\mathbb{R}^n} f \left\{ (a_0 - a)v + aR \right\} \, dx = \tau^2 \int_{\mathbb{R}^n} (a_0 - a)v^2 \, dx$$

$$+ \int_{\mathbb{R}^n} \left(|VR|^2 + a \tau^2 R^2\right) \, dx + O\left(\tau^{-1}e^{-\tau T}\right) \quad (3.5)$$
and

\[
\int_{\mathbb{R}^n} f \left\{ (\alpha - \alpha_0)w - \alpha_0 R \right\} dx = \tau^2 \int_{\mathbb{R}^n} \frac{\alpha_0}{\alpha} (\alpha - \alpha_0) v^2 dx \\
+ \int_{\mathbb{R}^n} \left| VR \right|^2 + \alpha \tau^2 \left| R + \left( 1 - \frac{\alpha_0}{\alpha} \right) v \right|^2 dx + \mathcal{O} \left( \tau^{-1} e^{-\tau \tau} \right). \tag{3.6}
\]

Here note that

\[
(\alpha_0 - \alpha) + \frac{\alpha_0}{\alpha} (\alpha - \alpha_0) = -\frac{1}{\alpha} (\alpha - \alpha_0)^2 
\tag{3.7}
\]

and \((\alpha_0 - \alpha)v + \alpha R + (\alpha - \alpha_0)w - \alpha_0 R = 2(\alpha - \alpha_0)R\). Thus, summing (3.5) and (3.6) up, we obtain

\[
2 \int_{\mathbb{R}^n} f (\alpha - \alpha_0) R dx + \tau^2 \int_{\mathbb{R}^n} \frac{\alpha - \alpha_0}{\alpha} v^2 dx = \int_{\mathbb{R}^n} \left| VR \right|^2 + \alpha \tau^2 R^2 \right\} dx \\
+ \int_{\mathbb{R}^n} \left| VR \right|^2 + \alpha \tau^2 \left| R + \left( 1 - \frac{\alpha_0}{\alpha} \right) v \right|^2 dx + \mathcal{O} \left( \tau^{-1} e^{-\tau \tau} \right). \tag{3.8}
\]

From the assumption on \(f\) we have \(\text{supp } f = \overline{B}\). Since \(\alpha(x) = \alpha_0(x)\) a.e. \(x \in \mathbb{R}^n \setminus D\) and \(B \cap \overline{D} = \emptyset\), we have \(\alpha(x) = \alpha_0(x)\) a.e. \(x \in B\). Therefore the first integral in the left-hand side on equation (3.8) vanishes. Then (3.8) gives

\[
\int_{\mathbb{R}^n} \left| VR \right|^2 + \alpha \tau^2 R^2 \right\} dx + \int_{\mathbb{R}^n} \left| VR \right|^2 + \alpha \tau^2 \left| R + \left( 1 - \frac{\alpha_0}{\alpha} \right) v \right|^2 dx \\
= \tau^2 \int_{\mathbb{R}^n} \frac{\alpha - \alpha_0}{\alpha} v^2 dx + \mathcal{O} \left( \tau^{-1} e^{-\tau \tau} \right)
\]

and hence

\[
\int_{\mathbb{R}^n} \left| VR \right|^2 + \alpha \tau^2 R^2 \right\} dx \leq \tau^2 \int_{\mathbb{R}^n} \frac{\alpha - \alpha_0}{\alpha} v^2 dx + \mathcal{O} \left( \tau^{-1} e^{-\tau \tau} \right). \tag{3.9}
\]

Now it follows from (3.5), (3.7) and (3.9) that (3.1) and (3.2) are valid. \(\square\)

**Remark 3.1.** We note that (3.7) gives

\[
\alpha_0 - \alpha \leq \frac{\alpha_0}{\alpha} (\alpha_0 - \alpha).
\]

Thus (3.1) and (3.2) are reasonable.

Now assume that \(D = \emptyset\). Then \(\alpha_0 = \alpha\) and thus (3.1) and (3.2) yield

\[
I_f (\tau, T) = \mathcal{O} \left( \tau^{-1} e^{-\tau \tau} \right).
\]

This gives (i) of Theorem 1.1.

The proof of (ii) is as follows. Since \(h\) satisfies (A.I), it follows from (3.1) and (3.2) that

\[
I_f (\tau, T) \leq -A \tau^2 \|v\|_{L^2(D)}^2 + \mathcal{O} \left( \tau^{-1} e^{-\tau \tau} \right), \tag{3.10}
\]
where $A = -(m_0^2/M^2)C$, $M = \sqrt{\text{ess. sup}_{x \in \mathbb{R}^n} \alpha(x)}$ and $C$ comes from (A.1); 
\[ I_{\tau}(\tau, T) \geq -A' \tau^2 \|v\|_{L^2(D)}^2 + O(\tau^{-1}e^{-\tau}). \tag{3.11} \]
where $A' = \|h\|_{L^2(D)} > 0$.

Thus it suffices to give a lower and upper estimate of $v$ over $D$. Given $\lambda > 0$ define

\[ G_{\lambda}(\xi) = \begin{cases} 
\frac{1}{2\lambda}e^{-\lambda|\xi|}, & \text{if } n = 1, \\
\frac{1}{2\pi}K_0(\lambda|\xi|), & \text{if } n = 2, \\
\frac{e^{-\lambda|\xi|}}{4\pi|\xi|}, & \text{if } n = 3, 
\end{cases} \tag{3.12} \]

where $K_0$ is the modified Bessel function of the second kind of order 0 (see [19]). It seems that the following lemma is closely related to the maximum principle or comparison principle [7]. However, our final purpose is to consider the electromagnetic wave which satisfies a system. So in appendix we give a proof without making use of such principles.

**Lemma 3.1.** Let $f \in L^2(\mathbb{R}^n)$ and satisfy $f(x) \geq 0$ a.e. $x \in \mathbb{R}^n$. Let $v \in H^1(\mathbb{R}^n)$ be the weak solution of (1.4). We have

\[ v(x) \geq \int_{\mathbb{R}^n} \alpha_0(y)f(y)G_{M_\tau}(x - y)dy \text{ a.e. } x \in \mathbb{R}^n, \tag{3.13} \]

and

\[ v(x) \leq \int_{\mathbb{R}^n} \alpha_0(y)f(y)G_{m_\tau}(x - y)dy \text{ a.e. } x \in \mathbb{R}^n. \tag{3.14} \]

Let us continue the proof of (ii). Since the case when $n = 1, 2$ can be treated easily, hereafter we only consider the case when $n = 3$. By the mean value theorem [5] we have

\[ \frac{1}{4\pi} \int_B e^{-\tau|x-y|} \frac{\varphi(tM_0\eta)}{|x-y|} dy = \frac{\varphi(tM_0\eta)}{(M_0\tau)^3} e^{-\tau|x-p|} \frac{|x-p|}{|x-p|}, \quad x \in \mathbb{R}^3 \setminus \overline{B}, \tag{3.15} \]

where $p$ and $\eta$ are the center and radius of $B$, respectively and $\varphi(\xi) = \xi \cosh \xi - \sinh \xi$.

By [17], we know that there exists a positive constant $C'$ and number $\mu \in \mathbb{R}$ such that, for all $\tau \gg 1$

\[ \tau^\mu e^{2\tau|p|} \int_0^{\tau^\mu} \frac{e^{-2\tau|x-p|}}{|x-p|} dx \geq C'. \]

Applying this and (3.15) to a lower bound derived from the lower bounds for $\alpha_0$ and $f$ on $B$ for the right-hand side on (3.13), we obtain, for all $\tau \gg 1$

\[ \|v\|_{L^2(D)}^2 \geq C^2\tau^{2\mu} \left( \frac{\varphi(tM_0\eta)}{(M_0\tau)^3} e^{-\tau|x-p|} \right)^2 \int_D \frac{e^{-2\tau|x-p|}}{|x-p|^2} dx \]

\[ \geq C^2\tau^{2\mu} \left( \frac{\varphi(tM_0\eta)}{(M_0\tau)^3} e^{-\tau|x-p|} \right)^2 \int_D \frac{e^{-2\tau|x-p|}}{|x-p|^2} dx, \tag{3.16} \]

where $C = \text{ess. inf}_{\xi \in \mathbb{R}^n} \varphi(\xi)$. Since as $\xi \to \infty \varphi(\xi) \sim e^\xi/2$, from (3.16) we obtain, for all $\tau \gg 1$ \[ \|v\|_{L^2(D)}^2 \geq C^* \tau^{-(4+\mu)} e^{-2\tau t(d_B(p)+\eta)} \], where $C^*$ is a positive constant. Since
\(d_{\text{dist}}(p, q) = \text{dist}(D, B)\), we finally obtain, for all \(\tau \gg 1\)
\[
\|v\|_{L^2(I)}^2 \geq C^* e^{-\mu_1 \tau} e^{-2M_0 \text{dist}(D, B)}.
\] (3.17)

Now a combination of this and (3.10) yields \(\lim_{\tau \to \infty} e^{\tau^2} I_\tau(\tau, T) = -\infty\) if \(T > 2M_0 \text{dist}(D, B)\).

Moreover, it is easy to obtain from (3.14) that, for a positive constant \(C^*\) and all \(\tau > 0\) we have
\[
\|v\|_{L^2(I)}^2 \leq C^* e^{-2\mu_0 \tau \text{dist}(D, B)}.
\] (3.18)

Now it is easy to verify that a combination of (3.10) and (3.17) yields (1.5); a combination of (3.11) and (3.18) yields (1.6).

A similar argument based on (3.1) and (3.2) works also for the case when \(h\) satisfies (A.II).

### 4. Proof of Theorem 1.2

From (2.2) we have
\[
\left(\Delta - \tau^2 - \tau q\right)w + f = e^{-\tau T} F \text{ in } \mathbb{R}^n
\] (4.1)
and \(F(x, T) = u(x) + (\tau + q(x))u(x, T)\).

Define
\[
\tilde{a} = \tilde{a}(x, \tau) = 1 + \frac{q}{\tau}, \quad \tilde{f} = \tilde{f}(x, \tau) = \left(1 + \frac{q}{\tau}\right)^{-1} f, \quad \tilde{F} = \tilde{F}(x, \tau) = \left(1 + \frac{q}{\tau}\right)^{-1} F.
\]

Then (4.1) becomes
\[
\left(\Delta - \tau^2 \tilde{a}\right)w + \tilde{a} \tilde{f} = \tilde{a} e^{-\tau T} \tilde{F} \text{ in } \mathbb{R}^n.
\]

Define
\[
\tilde{a}_0 = \tilde{a}_0(x, \tau) = 1 + \frac{q_0}{\tau}, \quad \tilde{f}_0 = \tilde{f}_0(x, \tau) = \left(1 + \frac{q_0}{\tau}\right)^{-1} f.
\]

Let \(v \in H^1(\mathbb{R}^n)\) be the solution of (1.9). Then (1.9) becomes
\[
\left(\Delta - \tilde{a}_0 \tau^2\right)v + \tilde{a}_0 \tilde{f}_0 = 0 \text{ in } \mathbb{R}^n.
\] (4.2)

Note that \(\tilde{a}_0 \tilde{f}_0 = f = \tilde{a} \tilde{f} = F\).

Define \(R = w - v\). We start with the following two integral identities which can be derived along the same line as that of Propositions 2.2 and 2.3 and thus we omit the description of their proofs:
\[
J_f(\tau, T) = \tau^2 \int_{\mathbb{R}^n} (\tilde{a}_0 - \tilde{a}) v^2 dx + \int_{\mathbb{R}^n} \left(\nabla R^2 + \tilde{a}_0 \tau^2 R^2\right) dx
\]
\[
+ e^{-\tau T} \left(\int_{\mathbb{R}^n} FR dx - \int_{\mathbb{R}^n} F v dx\right);
\] (4.3)
\[-J_f(\tau, T) = \tau^2 \int_{\mathbb{R}^n} \frac{\tilde{a}_0}{\tilde{a}} (\tilde{a} - \tilde{a}_0) v^2 dx + \int_{\mathbb{R}^n} \left( |VR|^2 + \tilde{a} \tau^2 \left| R + \left( 1 - \frac{\tilde{a}_0}{\tilde{a}} \right) v \right|^2 \right) dx + e^{-\tau T} \left( \int_{\mathbb{R}^n} FR dx + \int_{\mathbb{R}^n} Fv dx \right). \tag{4.4} \]

Note also that
\[\tau^2 \int_{\mathbb{R}^n} (\tilde{a}_0 - \tilde{a}) v^2 dx = \tau \int_{\mathbb{R}^n} (q_0 - q) v^2 dx \]
and
\[-\tau^2 \int_{\mathbb{R}^n} \frac{\tilde{a}_0}{\tilde{a}} (\tilde{a} - \tilde{a}_0) v^2 dx = \tau \int_{\mathbb{R}^n} \frac{\tau + q_0}{\tau + q} (q_0 - q) v^2 dx. \]

Then, applying a similar argument as done in the proof of Proposition 3.1 to (4.3) and (4.4) we obtain

**Proposition 4.1.** We have, as \(\tau \to \infty\)
\[J_f(\tau, T) \leq \tau \int_{\mathbb{R}^n} \frac{\tau + q_0}{\tau + q} (q_0 - q) v^2 dx + O\left( \tau^{-1} e^{-\tau T} \right) \tag{4.5} \]
and
\[J_f(\tau, T) \geq \tau \int_{\mathbb{R}^n} (q_0 - q) v^2 dx + O\left( \tau^{-1} e^{-\tau T} \right). \tag{4.6} \]

Thus it suffices to prepare the following lower and upper estimates for \(v\).

**Lemma 4.1.** Let \(f \in L^2(\mathbb{R}^n)\) and satisfy \(f(x) \geq 0\ a.e. \ x \in \mathbb{R}^n\). Let \(v \in H^1(\mathbb{R}^n)\) be the weak solution of (1.9). We have
\[v(x) \geq \int_{\mathbb{R}^n} f(y) G_{L_0(\tau)}(x - y) dy \text{ a.e. } x \in \mathbb{R}^n \tag{4.7} \]
and
\[v(x) \leq \int_{\mathbb{R}^n} f(y) G_{L_0(\tau)}(x - y) dy \text{ a.e. } x \in \mathbb{R}^n, \tag{4.8} \]
where \(G_{L_0(\tau)} = G_\lambda \big|_{\lambda = L_0(\tau)}, G_\tau = G_\lambda \big|_{\lambda = \tau}\) with \(G_\lambda\) given by (3.12),
\[L_0(\tau) = \sqrt{1 + \frac{L_0}{\tau}} \tag{4.9} \]
and \(L_0 = \text{ess. sup}_{x \in \mathbb{R}^n} q_0(x)\).

**Proof.** Since \(v\) satisfies (4.2) and \(L_0(\tau)\) given by (4.9) satisfies \(\tilde{a}_0(x) \leq L_0(\tau)^2\) a.e. \(x \in \mathbb{R}^n\), from (3.13) in Lemma 3.1 one obtains (4.7). The proof of (4.8) is as follows. \(v\) has the expression \(v = v_0 - \epsilon_0\), where \(v_0 \in H^1(\mathbb{R}^n)\) solves \((\Delta - \tau^2)v_0 + f = 0\) in \(\mathbb{R}^n\) and \(\epsilon_0 \in H^1(\mathbb{R}^n)\) solves
\[(\Delta - \tau^2 - \tau q_0)\epsilon_0 + \tau q_0 v_0 = 0 \text{ in } \mathbb{R}^n. \tag{4.10} \]
\( v_0 \) has the explicit form
\[
v_0(x) = \int_{\mathbb{R}^n} f(y)G_{x-y}dy \geq 0
\]
and hence \( \tau_0 v_0 \geq 0 \). Applying (4.7) to (4.10) we obtain
\[
e_0(x) \geq \int_{\mathbb{R}^n} \tau \delta_0(y)v_0(y)G_{L_\delta(\tau)}(x-y)dy
\]
and hence \( e_0(x) \geq 0 \) a.e. \( x \in \mathbb{R}^n \). Therefore we obtain \( v \leq v_0 \) and thus (4.8).

Now noting \( L_\delta(\tau) \rightarrow 1 \) as \( \tau \rightarrow \infty \), it is not difficult to deduce all the conclusions in theorem 1.2 from (4.5), (4.6) and Lemma 4.1 as done in the proof of Theorem 1.1.

5. Remarks and further problems

5.1. How to compute the indicator function without knowledge about \( \alpha_0 \) outside \( B \)

In this work we think that \( D \) is an invader into a space with the refractive index \( \alpha_0 \). However, it will be difficult to know the detail of \( \alpha_0 \) and find \( v \) on \( B \) which is the solution of equation (1.4). In this section we describe an experimental computation procedure of the indicator function from the observed data in the two spaces one of which has an invader and another does not have an invader yet.

Let \( V = V(x,t) \) be the weak solution of
\[
\begin{cases}
\alpha_0(x) \partial_t^2 V - \Delta V = 0 & \text{in } \mathbb{R}^n \times [0, T], \\
V(x,0) = 0 & \text{in } \mathbb{R}^n, \\
\partial_t V(x,0) = \chi_B(x) & \text{in } \mathbb{R}^n.
\end{cases}
\]
(5.1)

Generate wave \( V_e \) by the initial data \( f = \chi_B \) in the space which has no invader and observe \( V_e \) on \( B \) over time interval \( [0, T] \). We assume that \( V_e \) on \( B \) is given by \( V \) on \( B \).

From Proposition 3.1 in the case when \( \alpha_0 = \alpha \) we obtain
\[
\int_B \alpha_0(v_e - v)dx = O\left( \tau^{-1}e^{-\tau T} \right),
\]
where
\[
v_e(x, \tau) = \int_0^\tau e^{-\tau \tau}V_e(x, t)dt.
\]
Thus, we obtain
\[
I_f(\tau, T) \equiv \int_B \alpha_0(w - v_e)dx = I_f(\tau, T) + O\left( \tau^{-1}e^{-\tau T} \right).
\]
Then, it follows from Proposition 3.1 that, as \( \tau \rightarrow \infty \)
\[
I_f(\tau, T) \leq \tau^2 \int_{\mathbb{R}^n} \frac{\alpha_0}{\alpha}(\alpha_0 - \alpha)v^2dx + O\left( \tau^{-1}e^{-\tau T} \right)
\]
and
\[
I_f(\tau, T) \geq \tau^2 \int_{\mathbb{R}^n} (\alpha_0 - \alpha)v^2dx + O\left( \tau^{-1}e^{-\tau T} \right).
\]
Therefore, one can transplant all the results in Theorem 1.1 into the present case and we obtain
Theorem 5.1. We have:

(i) if \( D = \emptyset \), then for all \( T > 0 \) it holds that
\[
\lim_{\tau \to \infty} e^{\tau I_f(\tau, T)} = 0;
\]

(ii) if \( D \neq \emptyset \) and \( h \) satisfies (A.I), then for all \( T > 2M_0 \text{dist}(D, B) \) it holds that
\[
\lim_{\tau \to \infty} e^{\tau I_f(\tau, T)} = -\infty;
\]

(iii) if \( D \neq \emptyset \) and \( h \) satisfies (A.II), then for all \( T > 2M_0 \text{dist}(D, B) \) it holds that
\[
\lim_{\tau \to \infty} e^{\tau I_f(\tau, T)} = \infty.
\]

Moreover, in case of both (ii) and (iii) we have, for all \( T > 2M_0 \text{dist}(D, B) \)
\[
\begin{cases}
\liminf_{\tau \to \infty} \frac{1}{2\tau} \log |I_f(\tau, T)| \geq -M_0 \text{dist}(D, B), \\
\limsup_{\tau \to \infty} \frac{1}{2\tau} \log |I_f(\tau, T)| \leq -m_0 \text{dist}(D, B).
\end{cases}
\]

Note that, if we know \( \alpha_0 \) on \( B \), then one can compute the indicator function \( \tau \mapsto I_f(\tau, T) \) from the experimental data and the values of \( \alpha_0 \) on \( B \).

We just need the following qualitative knowledge:

(i) the governing equation of the observed wave in the space which has a penetrable obstacle takes the form (1.1) and its refractive index is given by (1.2);

(ii) the governing equation of the observed wave in the space which has no obstacle yet takes the form (5.1) with \( \alpha_0 \) in (1.2).

Summing up, we can say that: one can know the existence of something added to a reference space by comparing the ‘snapshot’ \( u \) on \( B \) with a ‘reference snapshot’ \( V_e \) on \( B \) even in the case when: the reference space has a complicated rough refractive index \( \alpha_0 \); \( \alpha_0 \) is unknown outside \( B \).

5.2. One-space dimensional case

Finally let us describe one non trivial application of the method presented in the proof of Theorem 1.1 in the case when the space dimension is one and the background medium is not homogeneous.

We assume that \( \alpha_0 \) is piecewise constant and takes the form
\[
\alpha_0(x) = \begin{cases} 1, & \text{if } x < a \text{ or } b < x \\ k_0, & \text{if } a < x < b, \end{cases}
\]  
(5.2)

where \(-\infty < a < b < \infty\) and \( k_0 \) is a positive constant.

We choose \( f = x_0 \), where \( x_0 \) denotes the characteristic function of open interval \( B = [p - \epsilon, p + \epsilon] \) with a fixed \( p \) satisfying \( p + \epsilon < a \). \( f \) is a simple model of the disturbance given at \( t = 0 \) from the left side of the wall \([a, b]\).
We assume that $D = ]c, d[$ with, for simplicity $b < c < d < \infty$. This means that obstacle $D$ is located behind the wall $]a, b[$ from the observer.

Define

$$
\varphi = a - (p + \epsilon) + \sqrt{k_0}(b - a) + (c - b).
$$

The quantity $2\varphi$ coincides with the time of flight of the signal which propagates as

$$
x_0 = p + \epsilon \rightarrow x_1 = a \rightarrow x_2 = b \rightarrow x_3 = c \rightarrow x_4 = b \rightarrow x_5 = a \rightarrow x_6 = p + \epsilon,
$$

where the propagation speed of the signal in $]p + \epsilon, a[$ and $]b, c[$ is 1, in $]a, b[$ is $1/\sqrt{k_0}$.

Note that knowing $c$ is equivalent to knowing $\varphi$ provided the wall thickness $b - a$ and the propagation speed of the wave inside the wall $1/\sqrt{k_0}$ are known.

Let $a$ be the same as (1.2) with $n=1$, $h \in L^\infty(D)$ and $a_0$ given by (5.2). Let $\nu$ be the weak solution of (1.4) with $f = \chi_B$. In this case we have the expression

$$
I_f(\tau, T) = \int_B (w - \nu)dx.
$$

**Theorem 5.2.** Let $T > 2\varphi$. We have:

(i) if $h$ satisfies (A.I), then

$$
\lim_{\tau \rightarrow \infty} e^{\tau T} I_f(\tau, T) = -\infty;
$$

(ii) if $h$ satisfies (A.II), then

$$
\lim_{\tau \rightarrow \infty} e^{\tau T} I_f(\tau, T) = \infty.
$$

Moreover, in case of both (i) and (ii) we have

$$
\lim_{\tau \rightarrow \infty} \frac{1}{2\tau} \log \left| I_f(\tau, T) \right| = -\varphi. \tag{5.3}
$$

The proof is based on the following asymptotic formula of the solution of (1.4) which corresponds to Lemma 3.1 and the argument of the proof done in that of Theorem 1.1 to the present case.

**Lemma 5.1.** We have, as $\tau \rightarrow \infty$

$$
2\pi e^{\pi \varphi} \int_D v^2 dx = 1 + O(\tau^{-2}). \tag{5.4}
$$

**Proof.** A direct computation shows that $\nu$ has the expression
\[
\nu(x, \tau) = \begin{cases}
A(\tau)e^{\tau x}, & \text{if } x < p - \epsilon, \\
B(\tau)e^{\tau x} + C(\tau)e^{-\tau x} + \frac{1}{\tau^2}, & \text{if } p - \epsilon < x < p + \epsilon, \\
D(\tau)e^{\tau x} + G(\tau)e^{-\tau x}, & \text{if } p + \epsilon < x < a, \\
H(\tau)e^{\sqrt{k_0} \tau} + K(\tau)e^{-\sqrt{k_0} \tau}, & \text{if } a < x < b, \\
L(\tau)e^{-\tau x}, & \text{if } b < x,
\end{cases}
\]

where

\[
A(\tau) = D(\tau) + \frac{e^{-\tau(p-\epsilon)} - e^{-\tau(p+\epsilon)}}{2\tau^2}, \quad
B(\tau) = A(\tau) - \frac{e^{-\tau(p-\epsilon)}}{2\tau^2},
\]

\[
C(\tau) = -\frac{e^{\tau(p+\epsilon)}}{2\tau^2}, \quad
G(\tau) = \frac{e^{\tau(p+\epsilon)} - e^{\tau(p-\epsilon)}}{2\tau^2};
\]

\[
H(\tau) = \frac{\sqrt{k_0} + 1}{2\sqrt{k_0}}D(\tau)e^{\tau(1+\sqrt{k_0})a} + \frac{\sqrt{k_0} - 1}{2\sqrt{k_0}}G(\tau)e^{-\tau(1+\sqrt{k_0})a};
\]

\[
K(\tau) = \frac{\sqrt{k_0} - 1}{2\sqrt{k_0}}D(\tau)e^{\tau(1-\sqrt{k_0})a} + \frac{\sqrt{k_0} + 1}{2\sqrt{k_0}}G(\tau)e^{-\tau(1-\sqrt{k_0})a};
\]

\[
D(\tau) = -\frac{(k_0 - 1)e^{-\tau(a-\epsilon)e^{-\tau}}}{2\tau^2} \left\{ 1 - \left( \frac{\sqrt{k_0} - 1}{\sqrt{k_0} + 1} \right)^2 e^{-2\sqrt{k_0}(b-a)} \right\};
\]

\[
L(\tau) = e^{tb}e^{-\tau\sqrt{k_0}(b-a)}e^{-\tau(a-\epsilon)} \times \left\{ 1 - \frac{1 - e^{-2\tau \epsilon}}{2\tau^2} \left( \frac{\sqrt{k_0} - 1}{\sqrt{k_0} + 1} \right)^2 \left( 1 + e^{-2\tau \sqrt{k_0}(b-a)} \right) \left( 1 - e^{-2\tau \epsilon} \right) \right\}.
\]

Thus, as \( \tau \to \infty \)

\[
L(\tau) = e^{tb}e^{-\tau\sqrt{k_0}(b-a)}e^{-\tau(a-\epsilon)} \left( 1 + O(\tau^{-2}) \right).
\]

Since \( \nu(x) = L(\tau)e^{-\tau x} \) for \( x > b \), we have

\[
\int_c^d \nu^2 \, dx = \frac{(L(\tau)e^{-\tau x})^2}{2\tau} \left( 1 - e^{-2\tau(d-c)} \right).
\]

Then (5.5) yields (5.4). \( \square \)

Theorem 5.2 suggests that, if \( \nu \in H^1(\mathbb{R}^n) \) is the solution of (1.4) and one knows the leading term of \( \| \nu \|_{L_2(D)} \) as \( \tau \to \infty \) like (5.4) in Lemma 5.1, then one can obtain a formula in three-dimensions like (5.3) in Theorem 5.2 instead of estimates (1.5) and (1.6). For the determination of the leading term, usually, one has to solve the eikonal equation \( |\nabla \Psi|^2 = a_0(x) \). In some restricted cases it is possible to solve the equation, see [22]. The case treated in Theorem 5.2 is just simplest one of such cases. However, we do not go into such
research direction further since the solvability of the eikonal equation requires some regularity for \( \alpha_0 \) and we are seeking a method that works for finding an obstacle embedded in a rough background medium. We think that the method presented in this paper is the first one for the purpose. In a forthcoming paper we will consider the original problem which is formulated by the Maxwell system.

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Appendix A. Proof of Lemma 3.1.

We make use of the following elementary fact and an iteration process.

**Lemma A.** Let \( \lambda > 0 \). Given \( f \in L^2(\mathbb{R}^n) \) let \( v \in H^1(\mathbb{R}^n) \) be the weak solution of

\[
\left( \triangle - \lambda^2 \right)v + f = 0 \quad \text{in} \quad \mathbb{R}^n.
\]

Then we have

\[
\|v\|_{L^2(\mathbb{R}^n)} \leq \lambda^{-2} \|f\|_{L^2(\mathbb{R}^n)}\quad(A.1)
\]

and

\[
\|\nabla v\|_{L^2(\mathbb{R}^n)} \leq (2\lambda)^{-1}\|f\|_{L^2(\mathbb{R}^n)}.
\]

**Proof of (3.13).** In what follows, \( G_{M_0,\tau} = G_\tau \mid_{\tau=M_0,\tau} \) with \( G_\tau \) given by (3.12).

Rewrite (1.4) as

\[
\left\{ \triangle - (M_0\tau)^2 \right\} v_1 + \left\{ \alpha_0 f + \tau^2 \left( M_0^2 - \alpha_0 \right) v_1 \right\} = 0 \quad \text{in} \quad \mathbb{R}^n.
\]

Let \( v_1 \in H^1(\mathbb{R}^n) \) be the weak solution of \( \left\{ \triangle - (M_0\tau)^2 \right\} v_1 + \alpha_0 f = 0 \) in \( \mathbb{R}^n \). Since \( v_1 \) has the expression

\[
v_1(x) = \int_{\mathbb{R}^n} a_0(y)f(y)G_{M_0,\tau}(x-y)dy,
\]

we have \( v_1(x) \geq 0 \).

Let \( j = 1, \ldots \). Given \( v_j \in H^1(\mathbb{R}^n) \) let \( v_{j+1} \in H^1(\mathbb{R}^n) \) be the weak solution of

\[
\left\{ \triangle - (M_0\tau)^2 \right\} v_{j+1} + \left\{ \alpha_0 f + \tau^2 \left( M_0^2 - \alpha_0 \right) v_j \right\} = 0 \quad \text{in} \quad \mathbb{R}^n.
\]

Then, \( v_{j+1} - v_j \) for \( j \geq 2 \) satisfies \( \left\{ \triangle - (M_0\tau)^2 \right\} (v_{j+1} - v_j) = -\tau^2 (M_0^2 - \alpha_0) (v_j - v_{j-1}) \) in \( \mathbb{R}^n \) and from (A.1) we obtain

\[
\|v_{j+1} - v_j\|_{L^2(\mathbb{R}^n)} \leq \left( 1 - \frac{m_0^2}{M_0^2} \right) \|v_j - v_{j-1}\|_{L^2(\mathbb{R}^n)}
\]
and hence, for \( j = 1, \ldots \)
\[
\| v_{j+1} - v_{j} \|_{L^2(\mathbb{R}^n)} \leq \left( 1 - \frac{m_0^2}{M_0^2} \right)^{j-1} \| v_2 - v_1 \|_{L^2(\mathbb{R}^n)}.
\]

Similarly we have
\[
\| v_2 - v_1 \|_{L^2(\mathbb{R}^n)} \leq \left( 1 - \frac{m_0^2}{M_0^2} \right) \| v_1 \|_{L^2(\mathbb{R}^n)}
\]
and applying (A.1) to \( v_1 \) on this right-hand side, we obtain
\[
\| v_2 - v_1 \|_{L^2(\mathbb{R}^n)} \leq \left( 1 - \frac{m_0^2}{M_0^2} \right) \frac{1}{\tau^2} \| f \|_{L^2(\mathbb{R}^n)}.
\]

Thus we obtain, for \( j = 1, \ldots \)
\[
\| v_{j+1} - v_{j} \|_{L^2(\mathbb{R}^n)} \leq \left( 1 - \frac{m_0^2}{M_0^2} \right)^{j} \frac{1}{\tau^2} \| f \|_{L^2(\mathbb{R}^n)}.
\]

Moreover, from (A.2) we have
\[
\| \nabla (v_{j+1} - v_{j}) \|_{L^2(\mathbb{R}^n)} \leq \frac{\tau (M_0^2 - m_0^2)}{2M_0} \| v_j - v_{j-1} \|_{L^2(\mathbb{R}^n)}
\]
and similarly
\[
\| \nabla (v_{j+1} - v_{j}) \|_{L^2(\mathbb{R}^n)} \leq \frac{M_0}{2\tau} \left( 1 - \frac{m_0^2}{M_0^2} \right)^{j} \| f \|_{L^2(\mathbb{R}^n)}.
\]

Therefore, the sequence \( \{v_j\} \) in \( H^1(\mathbb{R}^n) \) converges to
\[
v \equiv \sum_{j=1}^{\infty} (v_{j+1} - v_{j}) + v_1 \text{ in } H^1(\mathbb{R}^n) \tag{A.5}
\]
and we have:
\[
\| v - v_1 \|_{L^2(\mathbb{R}^n)} \leq \frac{M_0^2}{\tau^2m_0^2} \left( 1 - \frac{m_0^2}{M_0^2} \right) \| f \|_{L^2(\mathbb{R}^n)};
\]
\[
\| \nabla v - \nabla v_1 \|_{L^2(\mathbb{R}^n)} \leq \frac{M_0^3}{2\tau m_0^3} \left( 1 - \frac{m_0^2}{M_0^2} \right) \| f \|_{L^2(\mathbb{R}^n)}.
\]

Thus taking the limit of (A.4) as \( j \to \infty \), we see that \( v \) satisfies (A.3) and thus is the weak solution of (1.4).

Since \( v_2 - v_1 \) has the expression
\[
v_2(x) - v_1(x) = \tau^2 \int_{\mathbb{R}^n} \left( M_0^2 - \alpha_0(y) \right) v_1(y) G_{M_0,\tau}(x - y) dy,
\]
we have \( v_2(x) - v_1(x) \geq 0 \). For \( j \geq 2 \) we have also
\[
v_{j+1}(x) - v_j(x) = \tau^2 \int_{\mathbb{R}^n} \left( M_0^2 - \alpha_0(y) \right) \left( v_j(y) - v_{j-1}(y) \right) G_{M_0,\tau}(x - y) dy
\]
and thus by induction we obtain, for all \( j \geq 2 \) \( v_{j+1}(x) - v_j(x) \geq 0 \). Therefore from (A.5) and the almost convergence property of a subsequence of \( \{ \sum_{j=1}^{\infty} (v_{j+1} - v_j) \} \) we conclude that \( v(x) \geq v_1(x) \) a.e. \( x \in \mathbb{R}^n \). This completes the proof of (3.14). \( \square \)

**Proof of (3.14).** Let \( v_0 \in H^1(\mathbb{R}^n) \) solve

\[
\left\{ \Delta - (m_0 \tau^2) \right\} v_0 + \alpha_0 f = 0 \text{ in } \mathbb{R}^n.
\]

\( v_0 \) has the expression

\[
v_0(x) = \int_{\mathbb{R}^n} a_0(y)f(y) G_{m_0 \tau}(x-y) dy,
\]

where \( G_{m_0 \tau} = G_{\lambda \tau} |_{\lambda = m_0 \tau} \). Thus \( v_0 \geq 0 \). Then, \( v \) has the expression \( v = v_0 - \epsilon_0 \), where \( \epsilon_0 \in H^1(\mathbb{R}^n) \) solves \( \Delta \epsilon_0 - \alpha_0 \tau^2 \epsilon_0 + \tau^2 (\alpha_0 - m_0^2) \epsilon_0 = 0 \) in \( \mathbb{R}^n \). Since \( \tau^2 (\alpha_0 - m_0^2) \geq 0 \), applying (3.13) to the equation above, we obtain \( \epsilon_0 \geq 0 \). Therefore we obtain \( v \leq v_0 \). \( \square \)

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