SPECTRA AND VARIANCE OF QUANTUM RANDOM VARIABLES

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Abstract. We study essentially bounded quantum random variables and show that the Gelfand spectrum of such a quantum random variable $\psi$ coincides with the hypoconvex hull of the essential range of $\psi$. Moreover, a notion of operator-valued variance is introduced, leading to a formulation of the moment problem in the context of quantum probability spaces in terms of operator-theoretic properties involving semi-invariant subspaces and spectral theory.

1. Introduction

Some of the most basic and useful properties of classical random variables are altered when passing from real- or complex-valued measurable functions to operator-valued measurable functions (that is, from classical to quantum random variables). In earlier works [5, 6, 7], a certain operator-valued formulation of the notion of expectation of a quantum random variable was considered. In the present paper, we consider a similar formulation for the variance of a quantum random variable. As in these earlier investigations, the noncommutativity of operator algebra will lead to some structure that simply does not appear in the classical setting.

It is a basic fact of functional analysis that the essential range of an essentially bounded random variable coincides with the spectrum of a certain element in an abelian von Neumann algebra. Specifically, if $\psi : X \to \mathbb{C}$ is an essentially bounded function on a probability space $(X, \mathcal{F}(X), \mu)$, then the essential range of $\psi$ is precisely the spectrum of $\psi$, where one considers $\psi$ as an element of the von Neumann algebra $L^\infty(X, \mathcal{F}(X), \mu)$. We will arrive at a similar result for essentially bounded quantum random variables on quantum probability spaces using higher dimensional spectra. However, it will turn out that our investigation of quantum variance will also involve notions from spectral theory. In particular, the quantum moment problem admits a characterisation entirely within spectral terms.

In this paper, $(X, \mathcal{F}(X))$ denotes an arbitrary measurable space, $\mathcal{H}$ denotes a $d$-dimensional Hilbert space, $\mathcal{B}(\mathcal{H})$ denotes the $C^*$-algebra of linear operators acting on $\mathcal{H}$, and $\mathcal{B}(\mathcal{H})_+$ denotes the cone of positive operators. The predual of $\mathcal{B}(\mathcal{H})$ is denoted by $\mathcal{T}(\mathcal{H})$, the space of trace-class operators. Since $\mathcal{H}$ is finite dimensional, $\mathcal{B}(\mathcal{H})$ and $\mathcal{T}(\mathcal{H})$ coincide as sets. A set function $\nu : \mathcal{F}(X) \to \mathcal{B}(\mathcal{H})$ is a positive operator-valued measure (POVM) if

1. $\nu(E) \in \mathcal{B}(\mathcal{H})_+$ for every $E \in \mathcal{F}(X)$,
2. $\nu(X) \neq 0$, and
3. for every countable collection $\{E_k\}_{k \in \mathbb{N}} \subseteq \mathcal{F}(X)$ with $E_j \cap E_k = \emptyset$ for $j \neq k$ we have

\[ \nu\left( \bigcup_{k \in \mathbb{N}} E_k \right) = \sum_{k \in \mathbb{N}} \nu(E_k). \]

If, in addition, $\nu(X) = 1 \in \mathcal{B}(\mathcal{H})$, then $\nu$ is called a quantum probability measure. The convergence in (1) above is normally assumed to be with respect to the ultraweak topology; however, because $\mathcal{B}(\mathcal{H})$ has finite dimension, the convergence in (1) may be taken with respect to any of the usual
operator topologies on \( \mathcal{B}(\mathcal{H}) \). The POVM \( \nu : \mathcal{F}(X) \to \mathcal{B}(\mathcal{H}) \) induces the classical (i.e., scalar-valued) measure \( \mu \) via \( \mu = \frac{1}{d} \text{Tr} \circ \nu \), where \( \text{Tr} \) is the canonical trace on \( \mathcal{B}(\mathcal{H}) \). Note that if \( \nu \) is a quantum probability measure, then \( \mu \) is a classical probability measure. A function \( \psi : X \to \mathcal{B}(\mathcal{H}) \) is said to be measurable (i.e., a quantum random variable) if, for every pair \( \xi, \eta \in \mathcal{H} \), the complex-valued function \( x \mapsto \langle \psi(x) \xi, \eta \rangle \) is measurable (i.e., a random variable) in the classical sense.

The predual of the von Neumann algebra \( L^\infty(X, \mu) \odot \mathcal{B}(\mathcal{H}) \) is given by \( L^1_{\mathcal{H}}(X, \mu) \) [12, Theorem IV.7.17]. In particular, if \( \Psi \in L^\infty(X, \mu) \odot \mathcal{B}(\mathcal{H}) \), then there is a bounded quantum random variable \( \psi : X \to \mathcal{B}(\mathcal{H}) \) such that, for each \( f \in L^1_{\mathcal{H}}(X, \mu) \), the complex number \( \Psi(f) \) is given by

\[
\Psi(f) = \int_X \tau(f(x)\psi(x))
\]

where \( \tau = \frac{1}{d} \text{Tr} \) is the normalised trace on \( \mathcal{B}(\mathcal{H}) \). Although \( \psi \) is not unique, it is unique up to a set of \( \mu \)-measure zero. We therefore identify \( \Psi \) and \( \psi \) and consider the elements of \( L^\infty(X, \mu) \odot \mathcal{B}(\mathcal{H}) \) to be bounded quantum random variables \( \psi : X \to \mathcal{B}(\mathcal{H}) \). We furthermore adopt the following notation:

\[
L^\infty_{\mathcal{H}}(X, \nu) = L^\infty(X, \mu) \odot \mathcal{B}(\mathcal{H}) \cong L^\infty(X, \mu) \otimes \mathcal{M}_d(\mathbb{C})
\]

where \( \mathcal{M}_d(\mathbb{C}) \) is the space of \( d \times d \) matrices over \( \mathbb{C} \). Lastly, all homomorphisms and isomorphisms of \( C^* \)-algebras are assumed, without saying so each time, to be unital and \( * \)-preserving. If \( Z \) is a compact Hausdorff space, then \( C(Z) \) is the unital abelian \( C^* \)-algebra of all continuous functions \( f : Z \to \mathbb{C} \).

## 2. Basic Properties of Measurability and Quantum Expectation

Some elementary but useful facts concerning measurable functions are noted in this section.

**Theorem 2.1.** The following two statements are equivalent for a function \( \psi : X \to \mathcal{B}(\mathcal{H}) \).

1. \( \psi \) is measurable.
2. \( \psi^{-1}(U) \) is a measurable set, for every open set \( U \subseteq \mathcal{B}(\mathcal{H}) \).

**Proof.** Fix an orthonormal basis \( \{e_1, \ldots, e_d\} \) of \( \mathcal{H} \). Because \( \psi \) is measurable if and only if each coordinate function \( \psi_{ij}(x) = \langle \psi(x)e_j, e_i \rangle \) is measurable [8, Section III], and because \( \mathcal{B}(\mathcal{H}) \) is topologically equivalent to \( \mathbb{C}^{d^2} \) in the product topology, we may assume without loss of generality that \( \psi : X \to \mathbb{C}^{d^2} \) and \( \psi(x) = (\psi_{11}(x), \psi_{12}(x), \ldots, \psi_{dd}(x)) \) for \( x \in X \). Furthermore, every open set \( U \subseteq \mathbb{C}^{d^2} \) will be viewed as a product \( U = \prod_{i,j=1}^d U_{ij} \) of open sets \( U_{ij} \subseteq \mathbb{C} \). Suppose now that \( \psi \) is measurable. As each \( \psi_{ij} \) is therefore measurable, we have that \( \psi_{ij}(U_{ij}) \in \mathcal{F}(X) \) for every open set \( U_{ij} \subseteq \mathbb{C}^{d^2} \). Thus, if \( U = \bigcup_{i,j=1}^d U_{ij} \) is open in \( \mathbb{C}^{d^2} \), then \( \psi^{-1}(U) = \bigcap_{i,j=1}^d \psi_{ij}^{-1}(U_{ij}) \) is a measurable set. Conversely, if \( \psi^{-1}(U) \) is a measurable set, then for a fixed ordered pair \( (k, \ell) \) and any open set \( U_{k\ell} \subseteq \mathbb{C} \), we have \( \psi^{-1}(U_{k\ell}) = \psi^{-1}(U) \), where \( U = \bigcup_{i,j=1}^d U_{ij} \) is the open set for which \( U_{ij} = \mathbb{C} \) for all \( (i, j) \neq (k, \ell) \). Thus, \( \psi_{k\ell} \) is a measurable function.

Mimicking the classical definition of a regular probability measure, we have the following.

**Definition 2.2.** A quantum probability measure \( \nu : \mathcal{F}(X) \to \mathcal{B}(\mathcal{H}) \) is regular if for every \( E \in \mathcal{F}(X) \),

\[
\nu(E) = \inf \{ \nu(U) \mid U \subseteq X \text{ is open, and } E \subseteq U \} = \sup \{ \nu(K) \mid K \text{ is compact, and } K \subseteq E \}.
\]
We note that because $B(H)$ is a von Neumann algebra, the infimum and supremum in the definition of regular measure above exist. Furthermore, because the normalised trace $\tau$ on $B(H)$ is a normal linear functional, the induced classical probability measure $\mu = \tau \circ \nu$ on $(X, F(X))$ is regular if the quantum probability measure $\nu$ is. This leads to the next result which is the quantum analogue of the classical Lusin theorem.

**Theorem 2.3.** If $\psi : X \to B(H)$ is a quantum random variable and if $\nu$ is a regular quantum probability measure on $(X, F(X))$, where $F(X)$ is the $\sigma$-algebra of Borel sets of a locally compact Hausdorff space $X$, then for every $\epsilon > 0$ there is a continuous function $\vartheta : X \to B(H)$ with compact support such that $\mu(\{x \in X \mid \psi(x) \neq \vartheta(x)\}) < \epsilon$.

**Proof.** Let $\{\psi_{ij}\}_{i,j=1}^d$ be the set of coordinate functions defined in the proof of Theorem 2.1. Because $\nu$ is a regular measure and $\tau$ is a normal state, the induced measure $\mu = \tau \circ \nu$ is also regular. Hence, the classical Lusin theorem may be invoked to obtain, for each $i$ and $j$, a continuous function $\vartheta_{ij} : X \to \mathbb{C}$ with compact support and such that $\mu(D_{ij}) < \epsilon/d^2$, where $D_{ij} = \{x \in X \mid \psi_{ij}(x) \neq \vartheta_{ij}(x)\}$. Let $\vartheta : X \to B(H)$ be the continuous map induced by the coordinate functions $\vartheta_{ij}$, and define $D$ to be the set $D = \{x \in X \mid \psi(x) \neq \vartheta(x)\}$ which is measurable by Theorem 2.1. Because $D \subseteq \bigcup_{i,j=1}^d D_{ij}$, we deduce that $\mu(D) \leq \sum_{i,j=1}^d \mu(D_{ij}) < \epsilon$. \hfill $\Box$

The following theorem and two definitions summarise the results of [6, Section III] relevant for our purposes; see also [3, 7] for additional details.

**Theorem 2.4.** If $\nu$ is a quantum probability measure, then $\nu$ is absolutely continuous with respect to the induced classical measure $\mu$, and there exists a quantum random variable denoted by $\frac{d\nu}{d\mu}$ such that

\[(2) \quad \int_E \left\langle \frac{d\nu}{d\mu}(x)\xi, \xi \right\rangle d\mu(x) = \langle \nu(E)\xi, \xi \rangle,\]

for all $E \in F(X)$ and all $\xi \in H$.

The Borel function $\frac{d\nu}{d\mu}$ is called the principal Radon-Nikodým derivative of $\nu$ and is a positive operator for $\mu$-almost all $x \in X$.

**Definition 2.5.**

1. A quantum random variable $\psi$ is $\nu$-integrable if for every density operator $\rho$ the complex-valued function

$$\psi_\rho(x) = \text{Tr} \left( \rho \left( \frac{d\nu}{d\mu}(x) \right)^{1/2} \psi(x) \left( \frac{d\nu}{d\mu}(x) \right)^{1/2} \right), \quad x \in X,$$

is $\mu$-integrable.

2. The integral of a $\nu$-integrable function $\psi : X \to B(H)$ is defined to be the unique operator acting on $H$ having the property that

$$\text{Tr} \left( \rho \int_X \psi d\nu \right) = \int_X \psi_\rho d\mu,$$

for every density operator $\rho$.

**Definition 2.6.** If $\nu : F(X) \to B(H)$ is a quantum probability measure, then the map $E_\nu : L^\infty(\nu) \to B(H)$ defined by

$$E_\nu[\psi] = \int_X \psi d\nu$$
is called the quantum expectation of $\psi$ with respect to $\nu$.

A version of the following example first appeared in [6] Example 3.4; see also [5] Theorem 2.3(4).

**Example 2.7.** Let $X = \{x_1, \ldots, x_n\}$ and let $\mathcal{F}(X)$ be the power set of $X$. If $h_1, \ldots, h_n \in \mathcal{B}(\mathcal{H})_+$ are such that $h_1 + \cdots + h_n = 1 \in \mathcal{B}(\mathcal{H})$, and $\nu$ satisfies $\nu(\{x_j\}) = h_j$ for $j = 1, \ldots, n$, then for every $\psi : X \to \mathcal{B}(\mathcal{H})$,

$$E_{\nu}[\psi] = \int_X \psi \, d\nu = \sum_{j=1}^n h_j^{1/2} \psi(x_j) h_j^{1/2}.$$  

Thus one can view $E_{\nu}[\psi]$ as a quantum averaging of $\psi$.

Recall [9] Chapter 3 that a linear map $\varphi : A \to B$ of unital C$^*$-algebras is a unital completely positive (ucp) map if $\varphi$ is positive (ucp) map if

$$E = \int_X \psi \, d\nu = \sum_{j=1}^n h_j^{1/2} \psi(x_j) h_j^{1/2}.$$  

Thus one can view $E_{\nu}[\psi]$ as a quantum averaging of $\psi$.

Recall [9] Chapter 3 that a linear map $\varphi : A \to B$ of unital C$^*$-algebras is a unital completely positive (ucp) map if $\varphi(1_A) = 1_B$ and the induced linear maps

$$\varphi \otimes \text{id}_n : A \otimes M_n(\mathbb{C}) \to B \otimes M_n(\mathbb{C})$$

are positive for every $n \in \mathbb{N}$.

**Theorem 2.8.** Quantum expectation is a completely positive operation. That is, the linear map $E_{\nu} : L^\infty_{\mathcal{H}}(X, \nu) \to \mathcal{B}(\mathcal{H})$ is a unital completely positive map, for every quantum probability measure $\nu$.

**Proof.** The linearity of the map $E_{\nu}$ follows readily by definition. Because the algebra $L^\infty(X, \mu)$ is a unital abelian C$^*$-algebra, where $\mu$ is induced by a quantum probability measure $\nu$, the Gelfand transform $\Gamma : L^\infty(X, \mu) \to C(Z_\nu)$ is a unital C$^*$-algebra isomorphism, where $Z_\nu$ is the maximal ideal space of $L^\infty(X, \mu)$. The topological space $Z_\nu$ is necessarily compact, Hausdorff, and totally disconnected. Hence, if $\mathcal{H}$ has finite dimension $d$, then $L^\infty_{\mathcal{H}}(X, \nu)$ and $C(Z_\nu) \otimes \mathcal{B}(\mathcal{H})$ are isomorphic C$^*$-algebras via the unital $*$-isomorphism $\Gamma_{\mathcal{H}} = \Gamma \otimes \text{id}_{\mathcal{B}(\mathcal{H})} : L^\infty_{\mathcal{H}}(X, \nu) \to C(Z_\nu) \otimes \mathcal{B}(\mathcal{H})$. Because the map $E_{\nu} \circ \Gamma^{-1}_{\mathcal{H}} : C(Z_\nu) \otimes \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is unital and completely positive [6] Theorem 3.5 and because the homomorphism $\Gamma^{-1}_{\mathcal{H}}$ is completely positive, the linear map $E_{\nu} : L^\infty_{\mathcal{H}}(X, \nu) \to \mathcal{B}(\mathcal{H})$ is necessarily completely positive.

Theorem 2.8 gives rise to the following operator inequality.

**Corollary 2.9** (Schwarz Inequality). If the operators $h_1, \ldots, h_n \in \mathcal{B}(\mathcal{H})_+$ satisfy $h_1^2 + \cdots + h_n^2 = 1$, then for all $z_1, \ldots, z_n \in \mathcal{B}(\mathcal{H})$,

$$\left( \sum_{j=1}^n h_j z_j h_j \right)^* \left( \sum_{j=1}^n h_j z_j h_j \right) \leq \sum_{j=1}^n h_j z_j^* z_j h_j.$$  

**Proof.** Let $X = \{x_1, \ldots, x_n\}$, and let $\mathcal{F}(X)$ be the power set of $X$. If $\nu$ is the quantum probability measure for which $\nu(\{x_j\}) = h_j^2$, and if the quantum random variable $\psi : X \to \mathcal{B}(\mathcal{H})$ is defined by $\psi(x_j) = z_j$, for each $j = 1, \ldots, n$, then $\psi \in L^\infty_{\mathcal{H}}(X, \nu)$ and $E_{\nu}[\psi] = h_1 z_1 h_1 + \cdots + h_n z_n h_n$ as in Example 2.7. By the Schwarz inequality for completely positive linear maps [9] Proposition 3.3, we have $E_{\nu}[\psi]^* E_{\nu}[\psi] \leq E_{\nu}[\psi^* \psi]$, which is precisely inequality [8].

3. The Essential Range of Quantum Random Variables

**Definition 3.1.** Let $\psi : X \to \mathcal{B}(\mathcal{H})$ be a quantum random variable. The essential range of $\psi$ is the set ess-ran $\psi$ of all operators $\lambda \in \mathcal{B}(\mathcal{H})$ for which $\mu(\psi^{-1}(U)) > 0$, for every neighbourhood $U$ of $\lambda$.

The essential range of $\psi : X \to \mathcal{B}(\mathcal{H})$ is closed and the $\mu$-measure of the set $\{x \in X \mid \psi(x) \notin \text{ess-ran } \psi\}$ is zero. Thus, if $\psi_1$ and $\psi_2$ determine the same element $\psi \in L^\infty_{\mathcal{H}}(X, \nu)$, then $\psi_1$ and $\psi_2$ have the same essential range. Our aim is to identify the essential range with certain spectral elements of $\psi$. 


**Definition 3.2.** If $\mathcal{A}$ is a unital $C^*$-algebra and $a \in \mathcal{A}$, let $C^*(a)$ be the unital $C^*$-subalgebra generated by $a$. For $d \in \mathbb{N}$, the set

$$\text{Spec}^d(a) = \{ g(a) \mid g : C^*(a) \to M_d(\mathbb{C}) \text{ is a homomorphism} \}$$

is called the Gelfand spectrum of $a$.

Of course, for many elements $a$, it will be the case that $\text{Spec}^d(a)$ is empty. A notable exception occurs with (essentially) bounded measurable functions $\psi : X \to \mathbb{C}$, in which case $\text{Spec}^1(\psi)$, where $\psi$ is considered to be an element of the abelian von Neumann algebra $L^\infty(X, \mu)$, coincides with the essential range of $\psi$ [8]. However, the case of quantum random variables (see Theorem 3.4 below) requires the notion of hypoconvexity [10, Definition 1.6].

**Definition 3.3.** A nonempty compact subset $Q \subset B(H)$ is hypoconvex if

1. $u^*\lambda u \in Q$ for every unitary $u \in B(H)$ and $\lambda \in Q$, and
2. $\sum_{j=1}^m p_j \lambda_j \in Q$ for all $\lambda_1, \ldots, \lambda_m \in Q$ and projections $p_1, \ldots, p_m \in B(H)$ satisfying $p_1 + \cdots + p_m = 1$ and $p_j \lambda_j = \lambda_j p_j$ for each $j$.

If $Q \subset B(H)$ is an arbitrary nonempty compact set, then $Q^\sim$ denotes the hypoconvex hull of $Q$, namely, the smallest hypoconvex set that contains $Q$.

If $\mathcal{H} = \mathbb{C}$, then the two conditions above for the hypoconvexity of a compact set $Q$ are trivially satisfied. Hence, the notion of hypoconvex set is distinguished from compactness only at dimension $d = 2$ and higher.

**Theorem 3.4.** The following two statements are equivalent for a Hilbert space $\mathcal{H}$ of dimension $d$ and a quantum random variable $\psi \in L^\infty_\mathcal{H}(X, \nu)$.

1. $\lambda \in \text{Spec}^d(\psi)$.
2. There exists a unitary $v : \mathbb{C}^d \to \mathcal{H}$ such that $v\lambda v^{-1} \in (\text{ess-ran} \psi)^\sim$.

**Proof.** Consider the isomorphism $\Gamma_{\mathcal{H}} : L^\infty_\mathcal{H}(X, \nu) \to C(Z_\nu) \otimes B(H)$ defined in the proof of Theorem 2.8 and fix a computational basis $\mathcal{C} = \{e_1, \ldots, e_d\}$ of $\mathcal{H}$. Let $\pi_\mathcal{C} : B(H) \to M_d(\mathbb{C})$ be the isomorphism that sends each rank-1 operator $e_i \otimes e_j \in B(H)$ to the canonical matrix unit $E_{ij}$ of $M_d(\mathbb{C})$. Thus, $\rho_\mathcal{C} = \text{id}_{C(Z_\nu)} \otimes \pi_\mathcal{C}$ is an isomorphism of $C(Z_\nu) \otimes B(H)$ and $C(Z_\nu) \otimes M_d(\mathbb{C})$ that maps each $f \in C(Z_\nu) \otimes B(H)$ to a $d \times d$ matrix $\rho_\mathcal{C}(f) = [f_{ij}]_{i,j}$ of continuous maps $f_{ij} : Z_\nu \to \mathbb{C}$.

Suppose now that $\psi \in L^\infty_\mathcal{H}(X, \nu)$ and let $f = \Gamma_{\mathcal{H}}(\psi)$ and $[f_{ij}]_{i,j} = \rho_\mathcal{C}(f) = \rho_\mathcal{C} \circ \Gamma_{\mathcal{H}}(\psi)$. Define a subset $\Omega_{\psi, \mathcal{C}} \subset M_d(\mathbb{C})$ by setting

$$\Omega_{\psi, \mathcal{C}} = \{ \omega([f_{ij}])_{i,j=1}^d \in M_d(\mathbb{C}) \mid \omega : C^* \{ \{ f_{ij} \}_{i,j} \} \to \mathbb{C} \text{ is a homomorphism} \}.$$ 

Note that by restricting the domain of a homomorphism $\omega : C(Z_\nu) \to \mathbb{C}$ to the unital $C^*$-subalgebra $C^* \{ \{ f_{ij} \}_{i,j} \}$ of $C(Z_\nu)$ we obtain an inclusion $\Delta_{\psi, \mathcal{C}} \subseteq \Omega_{\psi, \mathcal{C}}$, where

$$\Delta_{\psi, \mathcal{C}} = \{ \omega([f_{ij}])_{i,j=1}^d \in M_d(\mathbb{C}) \mid \omega : C(Z_\nu) \to \mathbb{C} \text{ is a homomorphism} \}.$$ 

On the other hand, because every homomorphism $\omega_0 : C^* \{ \{ f_{ij} \}_{i,j} \} \to \mathbb{C}$ is, by the fact that $\mathbb{C}$ is 1-dimensional, irreducible, there is a homomorphism $\omega : C(Z_\nu) \to \mathbb{C}$ such that $\omega|_{C^* \{ \{ f_{ij} \}_{i,j} \}} = \omega_0$. Hence, $\Delta_{\psi, \mathcal{C}} = \Omega_{\psi, \mathcal{C}}$. Because homomorphisms $C(Z_\nu) \to \mathbb{C}$ are point evaluations $g \mapsto g(z_0)$ for $z_0 \in Z_\nu$, the set $\Omega_{\psi, \mathcal{C}}$ is the range of the matrix-valued function $z \mapsto [f_{ij}(z)]_{i,j=1}^d$, which we have denoted by $\rho_\mathcal{C} \circ \Gamma_{\mathcal{H}}(\psi)$. Thus, $\Omega_{\psi, \mathcal{C}} = \text{Spec}^d(\rho_\mathcal{C} \circ \Gamma_{\mathcal{H}}(\psi))$ [10, Lemma 2.3]. The Gelfand spectrum is an isomorphism invariant; hence, $\text{Spec}^d(\psi) = \Omega_{\psi, \mathcal{C}}^\sim$.

Now let $v_\mathcal{C} : \mathbb{C}^d \to \mathcal{H}$ be the unitary operator that sends the $j$-th coordinate vector of $\mathbb{C}^d$ to the unit vector $e_j \in \mathcal{H}$ in the computational basis $\mathcal{C} = \{e_1, \ldots, e_d\}$ of $\mathcal{H}$. If $\lambda = [\lambda_{ij}]_{i,j} \in \Omega_{\psi, \mathcal{C}}$, then

$$\sum_{j=1}^m p_j \lambda_j \in Q$$

for all $\lambda_1, \ldots, \lambda_m \in Q$ and projections $p_1, \ldots, p_m \in B(H)$ satisfying $p_1 + \cdots + p_m = 1$ and $p_j \lambda_j = \lambda_j p_j$ for each $j$. Therefore, $\text{Spec}^d(\psi)$ is hypoconvex.
then there is a $z_0 \in Z_\nu$ such that $\lambda_{ij} = f_{ij}(z_0) = \langle f(z_0)e_j,e_i \rangle$. Hence, $\nu_{\nu} \lambda \nu_{\nu}^{-1}$ is an element of the range of $f = \Gamma_{\mathcal{H}_b}(\nu)$, which coincides with the essential range of $\psi$. As the map $s \to \nu_{\nu} sv_{\nu}^{-1}$ is an automorphism of $\mathcal{B}(\mathcal{H})$, it is also true that if $\lambda \in \Omega_{\nu_{\nu}}$, then $\nu_{\nu} \lambda \nu_{\nu}^{-1}$ is an element of the hypoconvex hull of the range of $f$. This completes the proof that $\lambda \in \text{Spec}^d(\psi)$ only if there exists a unitary operator $v : \mathbb{C}^d \to \mathcal{H}$ such that $v \nu_{\nu}^{-1} \in (\text{ess-ran} \psi)^\dagger$.

Conversely, for each choice of computational basis $\mathcal{C} = \{e_1,\ldots,e_d\}$ of $\mathcal{H}$ there is an isometry $v_{\nu} : \mathbb{C}^d \to \mathcal{H}$ that sends the $j$-th coordinate vector of $\mathbb{C}^d$ to the unit vector $e_j \in \mathcal{H}$. Hence, if $\nu_{\nu} \lambda \nu_{\nu}^{-1} \in (\text{ess-ran} \psi)^\dagger$, then $\lambda \in \text{Spec}^d(\psi)$. $\square$

The expectation $E_x[\psi]$ of $\psi$ is just one of many operators $\varphi(\psi) \in \mathcal{B}(\mathcal{H})$ obtained by evaluating $\psi$ at a ucp map $\varphi : L^\infty_{\mathcal{H}_b}(X,\nu) \to \mathcal{B}(\mathcal{H})$; that is, $E_x[\psi] \in \{ \Phi(\psi) \mid \Phi : L^\infty_{\mathcal{H}_b}(X,\nu) \to \mathcal{B}(\mathcal{H}) \}$.

Theorem 3.5 below clarifies the relationship between operators of this type and the essential range of $\psi$.

**Theorem 3.5.** If $\varphi : L^\infty_{\mathcal{H}_b}(X,\nu) \to \mathcal{B}(\mathcal{H})$ is a ucp map, then for every $\psi \in L^\infty_{\mathcal{H}_b}(X,\nu)$ there exist $x_1,\ldots,x_m \in X$ (not necessarily distinct) and $t_1,\ldots,t_m \in \mathcal{B}(\mathcal{H})$ such that

$$\varphi(\psi) = \sum_{j=1}^m t_j^* \psi(x_j) t_j \quad \text{and} \quad \sum_{j=1}^m t_j^* t_j = 1 \in \mathcal{B}(\mathcal{H}).$$

**Proof.** Because $L^\infty_{\mathcal{H}_b}(X,\nu)$ is isomorphic to $L^\infty(X,\mu) \otimes \mathcal{M}_d(\mathbb{C})$, every $\psi \in L^\infty_{\mathcal{H}_b}(X,\nu)$ may be represented as a $d \times d$ matrix whose entries are taken from an abelian C*-algebra; that is, $\psi$ is $d$-normal. Select an orthonormal basis $\{e_1,\ldots,e_d\}$ of $\mathcal{H}$ and consider the $d \times d$ matrix $\omega = [\langle \varphi(\psi)e_j,e_i \rangle]_{i,j=1}^d$. Thus, $\omega$ is an element of the matricial range $W^d(\psi)$, namely the set of all $d \times d$ matrices of the form $\Phi(\psi)$ for some ucp map $\Phi : L^\infty_{\mathcal{H}_b}(X,\nu) \to \mathcal{M}_d(\mathbb{C})$. By [3, Theorem 3.9], there are $\lambda_1,\ldots,\lambda_q \in \text{Spec}^d(\psi)$ (not necessarily distinct) and $s_1,\ldots,s_q \in \mathcal{M}_d(\mathbb{C})$ such that $\omega = \sum_j s_j^* \lambda_j s_j$ and $\sum_j s_j^* s_j = 1$. By Theorem 3.4 there exist unitaries $v_j : \mathbb{C}^d \to \mathcal{H}$ such that $v_j \lambda_j v_j^* \in (\text{ess-ran} \psi)^\dagger$. For each $j = 1,\ldots,q$ there are $x_1^{(j)},\ldots,x_{n_j}^{(j)} \in X$ and pairwise-orthogonal projections $p_1^{(j)},\ldots,p_{n_j}^{(j)} \in \mathcal{B}(\mathcal{H})$ such that each $p_i^{(j)}$ commutes with $\psi(x_i^{(j)})$ and $v_j \lambda_j v_j^* = \sum_{i=1}^{n_j} \psi(x_i^{(j)}) p_i^{(j)}$.

Let $u : \mathbb{C}^d \to \mathcal{H}$ be the unitary for which $u^* z u = [\langle z e_j,e_i \rangle]_{i,j}$ for all $z \in \mathcal{B}(\mathcal{H})$. In particular, $\omega = u^* \varphi(\psi) u$ and each $s_j = u^* r_j u$ for a unique $r_j \in \mathcal{B}(\mathcal{H})$. Thus,

$$\varphi(\psi) = \sum_{j=1}^q \sum_{i=1}^{n_j} u s_j^* \lambda_j s_j u^* = \sum_{j=1}^q \sum_{i=1}^{n_j} (p_i^{(j)} u^* r_j)^* \psi(x_i^{(j)}) (p_i^{(j)} u^* r_j)$$

$$= \sum_{\ell=1}^m t_\ell^* \psi(x_\ell) t_\ell,$$

where $\{t_\ell\} = \{p_i^{(j)} u^* r_j\}_{i,j}$ and $\{x_\ell\} = \{x_i^{(j)}\}_{i,j}$ are relabelings and renumberings of the operators and points in the decomposition of $\varphi(\psi)$ above. $\square$

Recall that a subset $K \subseteq \mathcal{B}(\mathcal{H})$ is C*-convex if $\sum_{j=1}^m t_j^* z_j t_j \in K$, for every $z_1,\ldots,z_m \in K$ and $t_1,\ldots,t_m \in \mathcal{B}(\mathcal{H})$ with $\sum_{j=1}^m t_j^* t_j = 1 \in \mathcal{B}(\mathcal{H})$. If $S \subset \mathcal{B}(\mathcal{H})$ is a nonempty set, then $C^*\text{conv}(S)$ is
the smallest C*-convex set that contains $S$. Thus, Theorem 3.5 leads immediately to the following corollary which is a generalization of Theorem 2.3(8).

**Corollary 3.6.** If $\psi \in L_{H}^{\infty}(X, \nu)$, then

$$\{ \Phi(\psi) \mid \Phi : L_{H}^{\infty}(X, \nu) \to B(H) \text{ is a ucp map} \} = C^{*}\mathrm{conv}(\mathrm{ess-ran} \ \psi).$$

In particular, $E_{\nu}[\psi] \in C^{*}\mathrm{conv}(\mathrm{ess-ran} \ \psi)$.

## 4. Quantum Variance

**Definition 4.1.** If $\psi \in L_{H}^{\infty}(X, \nu)$ is a quantum random variable, then

1. the **left variance** of $\psi$ with respect to $\nu$ is the operator
   $$\text{Var}_{\ell_{\nu}}[\psi] = E_{\nu}[\psi^* \psi] - E_{\nu}[\psi^*] E_{\nu}[\psi],$$

2. the **right variance** of $\psi$ with respect to $\nu$ is the operator
   $$\text{Var}_{r_{\nu}}[\psi] = E_{\nu}[\psi \psi^*] - E_{\nu}[\psi] E_{\nu}[\psi^*],$$

and

3. the **variance** of $\psi$ with respect to $\nu$ is the operator
   $$\text{Var}_{\nu}[\psi] = \frac{1}{2} (\text{Var}_{\ell_{\nu}}[\psi] + \text{Var}_{r_{\nu}}[\psi]).$$

The Schwarz inequality ensures that all three of the variances defined above are positive operators. However, this occurrence of positivity is a consequence of the fact that $\psi$ is essentially bounded and that $L_{H}^{\infty}(X, \nu)$ is a von Neumann algebra. In contrast, variance is defined classically for square-integrable random variables rather than essentially bounded random variables, as it is a result of Chebyshev’s inequality that square-integrable random variables are necessarily integrable (i.e., $L^{2} \subset L^{1}$). To similarly define the variance of an arbitrary quantum random variable $\psi$, it is necessary to fulfil the **second moment condition** that $\psi^* \psi$ be $\nu$-integrable. The obvious question is whether or not the second moment condition implies that $\psi$ is itself $\nu$-integrable; the theorem below answers this question affirmatively.

**Theorem 4.2.** Suppose that $\psi : X \to B(H)$ is a quantum random variable. If $\psi^* \psi$ is $\nu$-integrable, then $\psi$ is $\nu$-integrable.

**Proof.** Let $\rho$ be a density operator and consider the functions $(\psi^* \psi)_{\rho}$ and $\psi_{\rho}$ on $X$ defined by

$$ (\psi^* \psi)_{\rho}(x) = \text{Tr} \left( \rho^{1/2} \left( \frac{d\nu}{d\mu}(x) \right)^{1/2} [\psi^* \psi(x)] \left( \frac{d\nu}{d\mu}(x) \right)^{1/2} \rho^{1/2} \right),$$

and

$$ \psi_{\rho}(x) = \text{Tr} \left( \rho^{1/2} \left( \frac{d\nu}{d\mu}(x) \right)^{1/2} \psi(x) \left( \frac{d\nu}{d\mu}(x) \right)^{1/2} \rho^{1/2} \right),$$

which coincide with Definition 2.5 using elementary properties of the trace. Let us also define, using the constant function $\iota(x) = 1 \in B(H)$, the scalar-valued function

$$\iota_{\rho}(x) = \text{Tr} \left( \rho \frac{d\nu}{d\mu}(x) \right).$$

Note that $\iota_{\rho}$ is $\mu$-integrable. To complete the proof we shall require the following two tracial inequalities for arbitrary $y, z \in B(H)$:

1. ([13] Theorem 1]) $| \text{Tr}(y) | \leq \text{Tr}(|y|)$, and
2. ([11] Theorem 1]) $\text{Tr}(|yz^*|) \leq \frac{1}{2} \text{Tr}(|y|^2) + \frac{1}{2} \text{Tr}(|z|^2)$. 

Consider the function \( w : X \to \mathcal{B}(\mathcal{H}) \) defined by \( w(x) = \left( \frac{d\nu}{d\mu}(x) \right)^{1/2} \). The two tracial inequalities above imply that
\[
|\psi_\rho(x)| = |\text{Tr}(w(x)^*\psi(x)w(x))| = |\text{Tr}(\psi(x)w(x)w^*(x))| \\
\leq \text{Tr}(|\psi(x)w(x)|w^*(x)|) \leq \frac{1}{2} \text{Tr}(|\psi(x)w(x)|^2) + \frac{1}{2} \text{Tr}(|w(x)|^2) \\
= \frac{1}{2} \text{Tr}(w(x)^*\psi(x)^*\psi(x)w(x)) + \frac{1}{2} \text{Tr}(w^*(x)w(x)) = \frac{1}{2} \psi^*\psi_\rho(x) + \frac{1}{2} \nu_\rho(x).
\]
Thus, \( |\psi_\rho| \) is bounded above by the average of the two nonnegative \( \mu \)-integrable functions \( (\psi^*\psi)_\rho \) and \( \nu_\rho \). Hence, \( \psi_\rho \in L^1(X,\mu) \). As this is true for every density operator \( \rho \), we deduce that \( \psi \) is \( \nu \)-integrable.

**Corollary 4.3.** The three variances in Definition 4.1 can be defined for quantum random variables \( \psi \) for which \( \psi^*\psi \) is \( \nu \)-integrable.

Notwithstanding the extension of the variance domains as indicated in Corollary 4.3, it is not necessarily true that the left or right variance is positive. In other words, there is no natural analogue of the Schwarz inequality from essentially bounded quantum random variables to square-integrable quantum random variables.

We now turn our attention to essentially bounded quantum random variables having variance zero. Although random variables having variance zero are trivially constant in the classical case, we will show that a much richer structure exists for quantum random variables having variance zero.

One family of quantum random variables that have variance zero is the following. For \( z \in \mathcal{B}(\mathcal{H}) \), let \( \psi_z : X \to \mathcal{B}(\mathcal{H}) \) denote the constant function defined by \( \psi_z(x) = z \) for every \( x \in X \). Because of the noncommutativity of operator algebra, quantum averaging of \( z \), by way of \( z \mapsto \mathbb{E}_\nu[\psi_z] \), may in fact alter \( z \). This phenomenon was observed in [5, Theorem 2.3(8)], where it was shown that, in general, one only has \( \mathbb{E}_\nu[\psi_z] \in C^*\text{conv}\{z\} \). However, if \( \mathbb{E}_\nu[\psi_z] = z \), namely if quantum averaging does not disturb \( z \), then the variance of \( \psi_z \) is zero; this is the immediate analogue of the fact that scalars (i.e., constant random variables) have variance zero.

**Proposition 4.4.** If \( \mathbb{E}_\nu[\psi_z] = z \), then \( \text{Var}_\nu[\psi_z] = 0 \).

**Proof.** By [5, Theorem 2.3(8)], the set of all \( y \in \mathcal{B}(\mathcal{H}) \) for which \( \mathbb{E}_\nu[\psi_y] = y \) is a unital \( C^* \)-subalgebra of \( \mathcal{B}(\mathcal{H}) \). Hence, \( \mathbb{E}_\nu[\psi_z] = z \) implies that \( \text{Var}_{\nu^*}[\psi] = \text{Var}_\nu[\psi] = 0 \).

The following result is a concise spectral characterisation of variance zero in the case of essentially bounded quantum random variables.

**Theorem 4.5.** The following two statements are equivalent for the quantum random variable \( \psi \in L^\infty_\nu(X,\nu) \).

1. \( \text{Var}_\nu[\psi] = 0 \).
2. There exist a unitary \( u : \mathcal{H} \to \mathbb{C}^d \) and a \( \lambda \in \text{Spec}^d(\psi) \) such that \( u^*\lambda u = \mathbb{E}_\nu[\psi] \).

**Proof.** The condition that \( \text{Var}_\nu[\psi] = 0 \) is equivalent to the two equations \( \mathbb{E}_\nu[\psi^*\psi] = \mathbb{E}_\nu[\psi]^*\mathbb{E}_\nu[\psi] \) and \( \mathbb{E}_\nu[\psi\psi^*] = \mathbb{E}_\nu[\psi]\mathbb{E}_\nu[\psi]^* \) holding simultaneously, which in turn is equivalent to \( \psi \) belonging to the multiplicative domain of the ucp map \( \mathbb{E}_\nu \) [3, Theorem 3.18]. Because the multiplicative domain of \( \mathbb{E}_\nu \) is a unital \( C^* \)-subalgebra of \( L^\infty_\nu(X,\nu) \) and contains \( \psi \), the restriction of \( \mathbb{E}_\nu \) to \( C^*(\psi) \) is a homomorphism. Thus, by selecting an orthonormal basis \( \{\phi_1, \ldots, \phi_d\} \) of \( \mathcal{H} \) and in letting \( u : \mathcal{H} \to \mathbb{C}^d \) be the unitary operator that sends each \( \phi_j \) to \( e_j \), we have that \( \lambda = u^*\mathbb{E}_\nu[\psi]u \) is an element of \( \text{Spec}^d(\psi) \). Conversely, if there exist \( \lambda \in \text{Spec}^d(\psi) \) and a unitary \( u : \mathcal{H} \to \mathbb{C}^d \) such that \( u^*\lambda u = \mathbb{E}_\nu[\psi] \), then the restriction of \( \mathbb{E}_\nu \) to \( C^*(\psi) \) is a homomorphism and so \( \text{Var}_\nu[\psi] = 0 \). 

The classical Hamburger moment problem, named after the German mathematician Hans Ludwig Hamburger, is as follows. Suppose that \( \{g_k\}_{k \in \mathbb{N}} \) is a sequence of real numbers. Does there exist a positive Borel measure \( \mu \) on the real line such that
\[
g_k = \int_{-\infty}^{\infty} x^k \, d\mu.
\]
This problem, as well as many variants of it, has been extensively studied for almost a century. If \( \{g_k\}_{k \in \mathbb{N}} \) is given, then we say that \( \mu \) is a solution to the moment problem for \( \{g_k\}_{k \in \mathbb{N}} \) if (1) holds.

A condition for a unique solution to one variant of the moment problem that is very well known in classical probability is found in [2, Theorem 30.1], namely if \( g = \{g_k\}_{k \in \mathbb{N}} \) is given and satisfies
\[
P_g(t) = \sum_{k=0}^{\infty} \frac{g_k t^k}{k!} < \infty
\]
for all \( t \in \mathbb{R} \), then there is a unique probability measure \( P \) on \( \mathbb{R} \) that is the solution to the moment problem for \( \{g_k\}_{k \in \mathbb{N}} \). In the case that \( \{g_k\}_{k \in \mathbb{N}} \) is a multiplicative moment sequence, meaning that \( g_k = (g_1)^k \) for all \( k \) for some \( g_1 \in \mathbb{R} \), the unique solution to the moment problem is trivial. That is, (2) implies the solution to the moment problem is unique since \( P_g(t) = e^{g_1 t} < \infty \) for all \( t \in \mathbb{R} \).

If \( \mu \) denotes the probability measure on \( \mathbb{R} \) supported on \( g_1 \) and \( Y \) is a random variable with \( P \{ Y = g_1 \} = 1 \), then \( \int_{\mathbb{R}} Y^k \, dP = (g_1)^k \). Thus, only constant random variables have multiplicative moment sequences.

Suppose now that \( \{g_k\}_{k \in \mathbb{N}} \) is a sequence in \( \mathcal{B}(\mathcal{H}) \). We say that a quantum probability measure \( \nu \) on the Borel sets of \( \mathcal{B}(\mathcal{H}) \) is a solution to the quantum moment problem for \( \{g_k\}_{k \in \mathbb{N}} \) if there exists a Borel subset \( X \subseteq \mathcal{B}(\mathcal{H}) \) and a quantum random variable \( \psi : X \to \mathcal{B}(\mathcal{H}) \) such that
\[
E_\nu [\psi^k] = \int_X \psi^k \, d\nu = g_k
\]
for all \( k = 0, 1, 2, \ldots \). A natural question to ask is if we can develop an operator-theoretic criterion to determine when the quantum moment problem for a multiplicative moment sequence has only a trivial solution.

By Stinespring’s dilation theorem for unital completely positive linear maps [9, Theorem 4.1], we deduce that for every quantum probability measure \( \nu \), there exist a Hilbert space \( \mathcal{K}_\nu \), an isometry \( \psi : \mathcal{K}_\nu \to \mathcal{H} \), and a homomorphism \( \Delta_\nu : L_1(\mathcal{K}_\nu) \to \mathcal{B}(\mathcal{K}_\nu) \) such that
\[
\begin{align*}
(1) & \quad E_\nu [\psi] = \nu^* \Delta_\nu (\psi) \nu \quad \text{for every } \psi \in L_1(\mathcal{K}_\nu), \\
(2) & \quad \text{Span} \{ \Delta_\nu(\psi) \nu \xi | \psi \in L_1(\mathcal{K}_\nu), \xi \in \mathcal{H} \} \quad \text{is dense in } \mathcal{K}_\nu.
\end{align*}
\]
The two conditions above determine the triple \( (\mathcal{K}_\nu, \Delta_\nu, \nu) \) up to unitary equivalence [9, Proposition 4.2], and so we may refer unambiguously to the triple \( (\mathcal{K}_\nu, \Delta_\nu, \nu) \) as the minimal Stinespring dilation of the quantum expectation \( E_\nu \). (Here, “minimal” is in reference to the second condition, which is to say that the Hilbert space \( \mathcal{K}_\nu \) is no larger than it needs to be.)

A second operator-theoretic concept that we will employ is that of a semi-invariant subspace. A subspace \( \mathcal{M} \) of a Hilbert space \( \mathcal{K} \) is said to be semi-invariant for an operator \( z \in \mathcal{B}(\mathcal{K}) \) if \( \mathcal{M} = \mathcal{L}_0^z \cap \mathcal{L}_1 \) for some \( z \)-invariant subspaces \( \mathcal{L}_0 \) and \( \mathcal{L}_1 \).

**Theorem 5.1.** Let \( \nu \) be a quantum probability measure, and let \( \psi \in L_1^Z(\mathcal{K}, \nu) \). Assume that \( \{g_k\}_{k \in \mathbb{N}} \subset \mathcal{B}(\mathcal{H}) \) is a sequence of operators with \( g_k = E_\nu [\psi^k] \), and let \( (\mathcal{K}_\nu, \Delta_\nu, \nu) \) be a minimal Stinespring dilation of the quantum expectation \( E_\nu \). Then the following two statements are equivalent.

1. \( g_k = (g_1)^k \) for every \( k \in \mathbb{N} \).
2. \( \psi(\mathcal{H}) \) is a semi-invariant subspace for \( \Delta_\nu(\psi) \).
Proof. By a result of Sarason [11, Lemma 0], a subspace $\mathcal{M}$ of $\mathcal{K}_\nu$ is semi-invariant for $\Delta_\nu(\psi)$ if and only if $p\Delta_\nu(\psi)^k|_{\mathcal{M}} = (p\Delta_\nu(\psi))_\mathcal{M}^k$ for every $k \in \mathbb{N}$, where $p \in \mathcal{B}(\mathcal{K}_\nu)$ is the projection with range $\mathcal{M}$. In the case at hand, the dimension of $\mathcal{M}$ is necessarily $d$. Further, any projection $p \in \mathcal{B}(\mathcal{K}_\nu)$ of rank $d$ can be factored as $p = vv^*$ for some isometry $v : \mathcal{H} \to \mathcal{H}_\nu$ and, conversely, for every isometry $v : \mathcal{H} \to \mathcal{H}_\nu$ the operator $vv^*$ is a projection of rank $d$. Because $\Delta_\nu$ is a homomorphism, Sarason’s criterion is, for an isometry $v$ of rank $d$ of $\mathcal{K}_\nu$, equivalent to: $v(\mathcal{H})$ is semi-invariant for $\Delta_\nu(\psi)$ if and only if $v^*\Delta_\nu(v^*(\psi)k)v = (v^*\Delta_\nu(\psi)k)v$ for every $k \in \mathbb{N}$. Thus, because $g_k = v^*\Delta_\nu(v^*(\psi)k)v$ for all $k \in \mathbb{N}$, the proof of the theorem is complete. \qed

In order to state our final result, we recall the following definition [3, Definition 5.1].

**Definition 5.2.** Assume that $\mathcal{A}$ is a unital $C^*$-algebra and that $k \in \mathbb{N}$. The $k \times k$ matricial spectrum of $a \in \mathcal{A}$ is the subset $\sigma^k(a) \subset M_k(\mathbb{C})$ defined by

$$
\sigma^k(a) = \{ \varphi(a) | \varphi : \mathcal{A} \to M_k(\mathbb{C}) \text{ is ucp and } \varphi|_{\mathcal{R}(a)} \text{ is a homomorphism} \},
$$

where $\mathcal{R}(a)$ is the rational Banach subalgebra of $\mathcal{A}$ generated by $a$.

To be more precise, the algebra $\mathcal{R}(a)$ is the norm-closure of the abelian algebra of all elements of the form $p(a)q(a)^{-1}$, where $p$ and $q$ are complex polynomials such that $q$ has no roots in the spectrum $\sigma(a)$ of $a$.

If one considers the classical $d = 1$ case, then every point $\lambda$ in the spectrum of $\psi$ gives rise to a measure $\mu$ for which $\lambda^k = \int_X \psi^k d\mu$ for every $k \in \mathbb{N}$. Indeed, this measure $\mu$ is a point-mass measure corresponding to a point evaluation of $\psi$ that yields the complex number $\lambda$ (which is basically the situation described at the end of the second paragraph of this section). The matter is simplified somewhat by the fact that $\sigma^1(\psi) = \text{Spec}^1(\psi)$ if $\psi$ is a classical random variable. However, in higher dimensions, the matricial spectrum $\sigma^d(\psi)$ is generally much larger than the Gelfand spectrum $\text{Spec}^d(\psi)$ and, consequently, the quantum moment problem for multiplicative moment sequences entails certain obstructions not seen at the classical level. Our final result, Theorem 5.3, illustrates the obstruction in that it demonstrates that a correction by a unitary quantum random variable is necessary prior to integration. The underlying complicating factor is that an analogue of the Riesz Representation Theorem holds only for a certain subset of ucp maps on $L^\infty_{\mathcal{F}}(X, \nu)$ [6, Corollary 4.5].

If $(X, \mathcal{F}(X))$ is the Borel space of a compact metric space $X$, and if $\psi : X \to M_d(\mathbb{C})$ is continuous, then below we consider $\psi$ as an element of the unital $C^*$-algebra $C(X) \otimes M_d(\mathbb{C})$, and the matricial spectra of $\psi$ are defined relative to this choice of $C^*$-algebra.

**Theorem 5.3.** If $\psi : X \to M_d(\mathbb{C})$ is a continuous quantum random variable on a compact metric space $X$, and if $\lambda \in \sigma^d(\psi)$, then there exist sequences $\{v_n\}_{n \in \mathbb{N}}$ and $\{w_n\}_{n \in \mathbb{N}}$ of quantum probability measures and quantum random variables, respectively, such that

1. $w_n(x)$ is unitary for all $x \in X$ and every $n \in \mathbb{N}$, and
2. $\lim_{n \to \infty} \|\lambda^k - \mathbb{E}_{w_n}[w_n^*\psi^k w_n]\| = 0$ for every $k \in \mathbb{N}$.

Proof. By hypothesis there is a ucp map $\vartheta : C(X) \otimes M_d(\mathbb{C}) \to M_d(\mathbb{C})$ such that the restriction of $\vartheta$ to the rational algebra $\mathcal{R}(\psi)$ is a homomorphism. Let $\nu^*\Delta_\nu$ be a minimal Stinespring representation of $\vartheta$. Because $X$ is a metric space, the $C^*$-algebra $C(X) \otimes M_d(\mathbb{C})$ is separable; hence, the minimal Stinespring dilation $\Delta$ of $\vartheta$ takes place on a representing Hilbert space $\mathcal{H}_\Delta$ that is separable. Hence, by Voiculescu’s Theorem [11, Corollary II.5.9], $\Delta$ is approximately unitarily equivalent to a direct sum $\tilde{\Delta}$ of a countable family of irreducible representations of $C(X) \otimes M_d(\mathbb{C})$. Because $C(X) \otimes M_d(\mathbb{C})$ is homogenous, every irreducible representation of it is a point evaluation. Thus, there are a countable subset $X_1 \subseteq X$, a separable Hilbert space $\mathcal{H}_\Delta = \bigoplus_{x \in X_1} \mathbb{C}^d_x$, where $\mathbb{C}^d_x = \mathbb{C}^d$. 


for each \( x \in X_1 \), and a sequence \( \{u_n\}_{n \in \mathbb{N}} \) of unitary operators \( u_n : \mathcal{H}_\Delta \to \mathcal{H}_{\tilde{\Delta}} \) such that, for every \( f \in C(X) \otimes M_d(\mathbb{C}) \),

\[
\tilde{\Delta}(f) = \bigoplus_{x \in X_1} f(x) \quad \text{and} \quad \lim_{n \to \infty} \|\Delta(f) - u_n^* \tilde{\Delta}(f) u_n\| = 0.
\]

Hence,

\[
\lim_{n \to \infty} \|\vartheta(f) - (u_n v)^* \tilde{\Delta}(f)(u_n v)\| = 0
\]

for every \( f \in C(X) \otimes M_d(\mathbb{C}) \).

The isometry \( u_n v : \mathbb{C}^d \to \bigoplus_{x \in X_1} \mathbb{C}^d \) acts as \( u_n v \xi = \bigoplus_{x \in X_1} v_{n,x} \xi \) for \( \xi \in \mathbb{C}^d \), for some \( v_{n,x} \in M_d(\mathbb{C}) \) and has the property that \( \sum_{x \in X_1} v_{n,x}^* v_{n,x} = 1 \) for each \( n \). By the Polar Decomposition, there is a unitary operator \( w_{n,x} \in \mathcal{B}(\mathbb{C}^d) = M_d(\mathbb{C}) \) such that \( v_{n,x} = w_{n,x}|v_{n,x}| \). Now define \( w_n : X \to M_d(\mathbb{C}) \) by \( w_n(x) = w_{n,x} \), if \( x \in X_1 \), and \( w_n(x) = 1 \) if \( x \notin X_1 \). Because point sets are closed in a Hausdorff space, each \( w_n \) is a measurable function. By defining \( \nu_n : \mathcal{F}(X) \to M_d(\mathbb{C}) \) by

\[
\nu_n = \sum_{x \in X_1} \delta_{\{x\}} v_{n,x}^* v_{n,x},
\]

where \( \delta_{\{x\}} \) is a classical point-mass measure concentrated at \( \{x\} \), we see that \( \nu_n \) is a quantum probability measure such that \( (u_n v)^* \tilde{\Delta}(f)(u_n v) = E_{\nu_n} [w_n^* f w_n] \) for every \( f \in C(X) \otimes M_d(\mathbb{C}) \). Thus, using the fact that \( \vartheta \) is a homomorphism on \( \mathcal{R}(\psi) \), we have that

\[
\lambda^k = \vartheta(\psi)^k = \vartheta(\psi^*) = \lim_{n \to \infty} E_{\nu_n} \left[ w_n^* \psi^k w_n \right]
\]

for every \( k \in \mathbb{N} \).

In the following special case, one can dispense with the sequences \( \{\nu_n\}_n \) and \( \{w_n\}_n \), and the quantum moment problem for multiplicative moment sequences is solved exactly rather than asymptotically.

**Corollary 5.4.** If \( \psi : X \to M_d(\mathbb{C}) \) is a quantum random variable on \( X = \{x_1, \ldots, x_n\} \) and if \( \lambda \in \sigma^d(\psi) \), then there exist a quantum probability measure \( \nu \) on \( (X, \mathcal{F}(X)) \), where \( \mathcal{F}(X) \) is the power set of \( X \), and a unitary-valued quantum random variable \( v : X \to M_d(\mathbb{C}) \) such that \( \lambda^k = E_{\nu} [w^* \psi^k w] \) for every \( k \in \mathbb{N} \).

**Proof.** The \( C^* \)-algebra \( C(X) \otimes M_d(\mathbb{C}) \) has, in this case, finite dimension. Therefore, the minimal Stinespring dilation of every ucp map on \( C(X) \otimes M_d(\mathbb{C}) \) has a representing Hilbert space of finite dimension. Hence, in the proof of Theorem 5.3, the representations \( \Delta \) and \( \tilde{\Delta} \) may be assumed to be equal.

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