Euclidean Action of Electrically Charged Black Holes in the Eddington-inspired Born-Infeld Gravity

A M Kusuma and H S Ramadhan
Departemen Fisika, FMIPA, Universitas Indonesia
Jl. Margonda Raya, Depok 16424, Indonesia.
Email: aulia.martha81@ui.ac.id, hramad@sci.ui.ac.id

Abstract. We investigate the Euclidean action of electrically charged black hole in the Eddington-inspired Born-Infeld (EiBI) gravity utilizing a semiclassical approach. The Euclidean action is computed in the canonical ensemble. We found that by imposing an appropriate set of boundary terms, the expression of the black hole's entropy could be found. The obtained form of the entropy is also found to be consistent with previous results.

1. Introduction
The proposal that black holes could be considered as thermal objects started as early as 1973, when Bekenstein conjectured that there should be a generalized second law of black hole mechanics [1]. This conjecture was partly motivated by Hawking's black hole horizon area increase theorem ($\delta A \geq 0$). Which resembles that of the never decreasing nature of entropy ($\delta S \geq 0$) [2]. Around the same time this conjecture was being proposed, Bardeen, Carter and Hawking have successfully shown that there is a set of mathematical relations that can be used to explain black hole mechanics, and these relations are analogous to the laws of thermodynamics [3]. Later on, Hawking would give proof that black holes radiate at certain thermal spectrum ($T_H = \pi / 2\pi$), and consequently black hole would have an entropy that is quarter of its area as well ($A_{BH} = A / 4 = \pi r_s^2$) [4]. The fact that these thermodynamics variables could be obtained from the different approach and still yield the same result further cemented the fact that black hole is a thermodynamic object [5].

On the other hand, a modified theory of gravity was proposed in 2010 by Banados and Ferreira, the theory known as the Eddington-inspired Born-Infeld (EiBI) gravity [6]. As the name itself suggest, this particular model of gravity was constructed in an affine and detrimental form, resembling that of Eddington's earlier proposal about affine gravity and Born-Infeld theory of nonlinear electrodynamics [7-8]. EiBI gravity is constructed in the Palatini formalism, thus it is able to avoid some of the problems that are usually encountered in models of gravity that are constructed from pure metric formalism, such as the ghost-like instabilities [9].

In this study, we intend to investigate whether it is feasible to obtain the entropy of an asymptotically flat EiBI black hole with Maxwell electrodynamics from a semiclassical approach, namely by evaluating its Euclidean action. The paper is organized as follows. In section 2 we briefly review the solution of electrically charged, static, black holes in EiBI gravity. In section 3 we calculate the Euclidean action on canonical ensemble. Finally, we conclude our work in section 4.
2. Obi Black Holes with Maxwell Electrodynamics

The solution of EiBI black holes with Maxwell electrodynamics has been studied in [10 - 11]. In this section we are going to briefly review the results and highlight certain part of solutions that would be used in the Euclidean action evaluation later. The action of EiBI gravity is given by

\[ S = \frac{1}{8\pi\kappa} \int d^4x \left( \sqrt{-g_{\mu\nu}} + \kappa R_{\mu\nu}(\Gamma) - \lambda \sqrt{-g} \right) + S_M[g, \phi_M]. \]  

(1)

Employing the Palatini formalism, the gravitational field equations are

\[ \sqrt{-q} q^{\mu\nu} = \lambda \sqrt{-g} g^{\mu\nu} - 8\pi\kappa \sqrt{-g} T^{\mu\nu}, \]  

(2)

and

\[ q^{\mu\nu} = g^{\mu\nu} + \kappa T^{\mu\nu}, \]  

(3)

for \( g_{\mu\nu} \) and \( q_{\mu\nu} \) Respectively.

Assuming static, spherically symmetric configuration, we choose Ansatz

\[ ds_g^2 = -\psi^2(r) f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega^2, \]  

(4)

\[ ds_q^2 = -G^2(r) F(r) dt^2 + \frac{1}{F(r)} dr^2 + H^2(r) d\Omega^2, \]  

(5)

where \( ds_g^2 \) and \( ds_q^2 \) Are the physical and auxiliary metrics, respectively.

We are going to couple the EiBI black hole with Maxwell electrodynamics as its source of matter, and the Lagrangian density is given by

\[ \mathcal{L}_M = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}. \]  

(6)

Based on \( \mathcal{L}_M \), we can obtain the energy-momentum tensor as follows

\[ T_{\mu\nu} = \frac{1}{4\pi} \left( g^{\alpha\beta} F_{\alpha\mu} F_{\nu\beta} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right). \]  

(7)

The electromagnetic field strength tensor is defined as

\[ F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}. \]  

(8)

Considering the Maxwell potential \( A^\mu = \{ \phi(r), 0,0,0 \} \) the non-vanishing component of \( F_{\mu\nu} \) reads

\[ F_{01} = -A_{0,1} = -F_{10}. \]  

(9)

From the Maxwell’s field equation (\( \partial_{\nu} F^{\mu\nu} = 0 \)), the non-vanishing Maxwell electromagnetic field strength tensor is obtained to be

\[ F_{\mu\nu} = \frac{C_0 \psi(r)}{r^2}, \]  

(10)

where \( C_0 \) Is an undetermined constant, which later would be found to be equal to \( Q \), the electric charge. Substituting the non-vanishing component of the energy - momentum tensor into Eq. (3), we will obtain a set of equations that should be eliminated so that it reduces to a relatively straightforward relation between functions in the physical and auxiliary metric. Solving these relations to the equation of motion would yield the following solutions

\[ \psi(r) = \frac{\sqrt{\lambda} r^2}{\sqrt{\lambda r^4 + \kappa Q^2}}, \]  

(11)

\[ F_{\mu\nu} = \frac{Q \sqrt{\lambda}}{\sqrt{\lambda r^4 + \kappa Q^2}}. \]  

(12)
\[ f(r) = r \sqrt{\frac{\lambda r^4 + \kappa Q^2}{\lambda r^4 - \kappa Q^2}} \left( \frac{(3r^2 - Q^2 - \Lambda r^4)\sqrt{\lambda r^4 + \kappa Q^2}}{3r^3} \right) \]
\[
+ \frac{4}{3} \frac{iQ^3}{\sqrt{\lambda \kappa}} \text{EllipticF} \left[ i \text{ArcSinh} \left[ r \left( \frac{i\sqrt{\lambda}}{\sqrt{Q\sqrt{\kappa}}} \right), -1 \right] \right] - 2\sqrt{\lambda}M \\
+ \frac{1}{3} \frac{Q^3}{\pi \sqrt{\lambda \kappa}} (\text{Gamma}[1/4])^2. \\
\]

We can also express the relations between the auxiliary metric and the physical metric functions as follows:

\[ G(r) = \psi(r) \left( \frac{\lambda r^4 - \kappa Q^2}{r^4} \right), \quad (14) \]
\[ F(r) = f(r) \left( \frac{r^4}{\lambda r^4 - \kappa Q^2} \right), \quad (15) \]
\[ H(r) = \frac{1}{r} \sqrt{\lambda r^4 + \kappa Q^2}. \quad (16) \]

In this work we are going to focus on electrically charged EiBI black holes in asymptotically flat space \((\Lambda = 0, \lambda = 1)\).

3. **Euclidean action calculation of asymptotically flat EiBI Black Holes with Maxwell Electrodynamics**

To evaluate the Euclidean action, we start by "Euclideanizing" the initial form of the action [5]. The Euclideanizing is done by employing Wick rotation \((t \to i\tau)\) to the previously defined line elements, so that we have

\[ ds_g^2 = \psi^2(r)f(r)d\tau^2 + \frac{1}{f(r)}dr^2 + r^2d\Omega_2^2, \quad (17) \]
\[ ds_q^2 = G^2(r)F(r)d\tau^2 + \frac{1}{F(r)}dr^2 + H^2(r)d\Omega_2^2. \quad (18) \]

In this particular model, the Hawking temperature gives [12]

\[ T_H = \frac{\kappa}{2\pi} = \frac{1}{4\pi} \left[ \frac{\partial_r(g_{00})}{\sqrt{g_{00}g_{11}}} \right], \]
\[ = - \frac{Q^2 - r^2}{4\pi r \sqrt{Q^2 \kappa + r^4}} \quad (19) \]

The period of the black hole is then obtained to be

\[ \beta = - \frac{4\pi r \sqrt{r^4 + Q^2 \kappa}}{Q^2 - r^2}. \quad (20) \]

Note that to obtain \(T_H\), and \(\beta\) consequently, we have utilized the black hole’s mass \(M\) which can be obtained from \(f(r_+) = 0\) and would give
\[ M = -\frac{(Q^2 - 3r^2)\sqrt{r^4 + Q^4}}{6r^3} + \frac{2}{3} \frac{iQ^3}{\sqrt{Q^4}} \text{EllipticF} \left[ i \text{ArcSinh} \left( r \frac{i}{\sqrt{Q \sqrt{K}}} \right), -1 \right] \]
\[ + \frac{\sqrt{Q^3 \text{Gamma} \left[ \frac{1}{4} \right]^2}}{6\sqrt{\pi}}. \]  

(21)

We wish to study the black hole in the situation in the canonical ensemble. Thus, the Euclidean action of electrically charged black hole holes in asymptotically flat EiBI gravity configuration for this case can be written as

\[ I = I_b - I_s + I_{GHY} - I_{ct}, \]  

(22)

where \( I_b \) is the bulk action, \( I_s \) is the surface term, \( I_{GHY} \) is the Gibbons-Hawking-York boundary term and \( I_{ct} \) is the counter term. The bulk action is the Wick rotated form of Eq. (1), while \( I_s \) is the surface term that needs to be calculated so that the black hole manifests in the canonical ensemble. The \( I_{GHY} \) and \( I_{ct} \) are the boundary terms that need to be evaluated so that we have a well-defined form of Euclidean action [13]. The reason why Gibbons-Hawking-York-like form is used as the boundary term would be explained later. First, we evaluate the bulk action with the line elements given in Eqs. (17)-(18), along with the Wick-rotated form of the Maxwell electric field strength tensor on the matter’s Lagrangian density, which gives

\[ I_b = \frac{1}{8\pi \kappa} \int d^4 x (-\sqrt{q} + \lambda \sqrt{g} - 8\pi \kappa \sqrt{q} \mathcal{L}_M), \]

\[ = \frac{\beta}{2} \left[ \frac{Q^2 \sqrt{r^4 + Q^4}}{3r^3} - \frac{4}{3} \frac{iQ^3}{\sqrt{Q^4}} \text{EllipticF} \left[ i \text{ArcSinh} \left( r \frac{i}{\sqrt{Q \sqrt{K}}} \right), -1 \right] \right] |^r_{r_b}. \]  

(23)

Before we continue with the integral evaluation, it is interesting to note that the result above would be finite valued even when \( r = r_b \), unlike the usual cases in black hole thermodynamics where some form of divergences would arise in the bulk action. For the first term that appears on the equation above it is evident that the result would be null, whereas the elliptic function that appears on the second term of the result above would yield the following form

\[ \frac{4}{3} \frac{iQ^3}{\sqrt{Q^4}} \text{EllipticF} \left[ i \text{ArcSinh} \left( r \frac{i}{\sqrt{Q \sqrt{K}}} \right), -1 \right] = \frac{\sqrt{Q^3 \text{Gamma} \left[ \frac{1}{4} \right]^2}}{6\sqrt{\pi}}. \]  

(24)

Thus, the bulk action is obtained as follows

\[ I_b = \beta \left( \frac{Q^2 \sqrt{r^4 + Q^4}}{6r^3} - \frac{2}{3} \frac{iQ^3}{\sqrt{Q^4}} \text{EllipticF} \left[ i \text{ArcSinh} \left( r \frac{i}{\sqrt{Q \sqrt{K}}} \right), -1 \right] \right. \]
\[ \left. - \frac{\sqrt{Q^3 \text{Gamma} \left[ \frac{1}{4} \right]^2}}{6\sqrt{\pi}} \right). \]  

(25)
In a fixed ensemble variation of the matter part of the action must be done to ensure that the charge is fixed. The surface term for the matter part of the action is given by

$$I_s = -\frac{1}{4\pi} \int d^3x \sqrt{h} F^{\mu\nu} \eta_{\mu\nu} A_\nu. \quad (26)$$

The Wick-rotated form of the Maxwell potential could be written as follows

$$A_\tau = \left[ i Q \frac{\Gamma(\frac{1}{4})}{\sqrt{\kappa}} \right] \left[ i \text{ArcSinh} \left( r \frac{i}{Q \sqrt{\kappa}} \right) \right] - 1 \right]^{r_b}_{r_s},$$

$$= - \frac{\sqrt{Q^3} \Gamma(\frac{1}{4})}{4\sqrt{\pi}} \left[ i Q \frac{\Gamma(\frac{1}{4})}{\sqrt{\kappa}} \right] \left[ i \text{ArcSinh} \left( r \frac{i}{Q \sqrt{\kappa}} \right) \right] - 1 \right]^{r_b}_{r_s}. \quad (27)$$

Employing the result above, along with the induced metric from the physical metric, the surface term reads

$$I_s = \beta \left[ - \frac{\sqrt{Q^3} \Gamma(\frac{1}{4})}{4\sqrt{\pi}} \left[ i \text{ArcSinh} \left( r \frac{i}{Q \sqrt{\kappa}} \right) \right] - 1 \right]^{r_b}_{r_s}. \quad (28)$$

Before we construct the boundary term and counterterm action for this particular configuration of black hole in EiBI gravity, first let us note that the action given in Eq. (1) could also be written in its alternative form [9]

$$I_b = \frac{1}{16\pi\kappa} \int d^4x \sqrt{-q} \left( \kappa R - 2 + \left( q^{\mu\nu} g_{\mu\nu} - 2\lambda \frac{(-g)}{\sqrt{-q}} \right) \right) + I_m. \quad (29)$$

From Eq. (2), we have

$$\sqrt{-q} q^{\mu\nu} g_{\mu\nu} = 4\lambda \sqrt{-g} - 8\pi\kappa \sqrt{-g} T. \quad (30)$$

If we were to define $\sqrt{-g}/\sqrt{-q}$ as $\epsilon$, we may express the equation above as follows

$$q^{\mu\nu} g_{\mu\nu} = 4\lambda \epsilon - 8\pi\kappa \epsilon T. \quad (31)$$

From the electromagnetic Ansatz, we know that $T$ is null. Thus, once we substitute the expression above into Eq. (29) and evaluate for the case of asymptotically flat space configuration, we will obtain

$$I_b = \frac{1}{16\pi\kappa} \int d^4x \sqrt{-q} (\kappa R - 2 + 2\epsilon) + I_m,$n

$$= \frac{1}{16\pi} \int d^4x \sqrt{-q} \left( R - \frac{2}{\kappa} \right) + \frac{1}{8\pi\kappa} \int d^4x \sqrt{-g} + I_m. \quad (32)$$

We can see that the form above has a somewhat similar structure with the usual Einstein-Hilbert action. This leads us to use a boundary term and a counterterm that follows the usual Gibbons-Hawking-York (GHY) boundary form [5], [13 - 14]. Each of these terms would have the following form,

$$I_{\text{GHY}} = \frac{1}{8\pi} \int d^3x \sqrt{h} K, \quad (33)$$

$$I_{\text{GHY}} = \frac{1}{8\pi} \int d^3x \sqrt{h_0} K_0. \quad (34)$$
To proceed with the GHY boundary term evaluation, we construct the trace of the extrinsic curvature from Eq. (18). The result gives

\[
I_{\text{GHY}} = \beta \left( \frac{3M}{2} - \frac{\sqrt{\kappa Q^2 + r_b^4}}{r_b} \right).
\] (35)

The usual procedure to evaluate Eq. (34) to match the period of the black hole under consideration in the space-time in which it resides. In our case, we would do the rescaling between the physical and auxiliary metric of the black holes. Consider Eqs (11) and (13) for \( A = 0, \lambda = 1 \) case. As \( r \to r_b \), each of these functions would be equal to 1. Thus, the Ricci scalar and tensor shall be constructed from the following boundary metric

\[
h_{ij}^0 = \{1, r_b^2, r_b^2 \sin^2 \theta\}.
\] (36)

Using the quantities above, the counter term gives

\[
I_{\text{ct}} = \beta_o (-r_b).
\] (37)

By interpreting that \( \beta_o \) resides in the physical metric, we shall rescale it so that it matches with the auxiliary metric of the black hole,\[
\beta_o \sqrt{\psi^2(r_b)f(r_b)} = \beta \sqrt{G^2(r_b)F(r_b)} ,
\] (38)

\[
\beta_o = \frac{\beta}{\psi(r_b)} \sqrt{G^2(r_b)F(r_b)} \left( \frac{1}{f(r_b)} \right) ,
\] (39)

and with some approximation we obtain the following form

\[
\beta_o \cong \beta \left( -\frac{M}{r_b} + \frac{\sqrt{\kappa Q^2 + r_b^4}}{r_b} \right).
\] (40)

Substituting the results above into Eq. (37) and omitting the null terms (terms that becomes zero as \( r \to r_b \)), leave us with the following results

\[
I_{\text{ct}} = \beta \left( M - \frac{r_b^3 \sqrt{\kappa Q^2 + r_b^4}}{r_b^4 - \kappa Q^2} \right).
\] (41)

Combining the results that we have obtained so far from Eqs. (21), (23), (35), and (41) yields the Euclidean action as follows

\[
I = \beta \left( \frac{\sqrt{Q^2\kappa + r^4(Q^2 + 3r^2)}}{12r^3} + 2 \left[ \frac{iQ^3}{3 \sqrt{\kappa}} \right] \text{EllipticF} \left[ i \text{ArcSinh} \left( \frac{r}{\sqrt{Q^2\kappa}} \right), 1 \right] 
+ \frac{1}{6} \frac{Q^3}{\pi \sqrt{\kappa}} \left( \Gamma \left[ \frac{1}{4} \right] \right)^2 \right) \] (42)

Based on the obtained Euclidean action, we may calculate the black hole's entropy, which would yield the following form

\[
S = \beta \left( \frac{\partial I}{\partial \beta} \right)_A - I ,
\] (43)

\[
= \beta \left( \frac{\partial I}{\partial r_+} \right)_A \left( \frac{\partial \beta}{\partial r_+} \right)_A - I ,
\] (44)
The result above is consistent with the form found previously in [15]. We can see that the entropy for our case is different from that of the ordinary Einstein-Hilbert gravity, due to the existence of \( \kappa \). Nevertheless, as \( \kappa \rightarrow 0 \) we can see that the entropy would reduce to the usual \( \pi r_s^2 \) result. Note that the form given in Eq. (43) can also be obtained from the black hole's entropy and event horizon area relation calculated from the auxiliary metric

\[
S = \frac{A}{4} = \frac{1}{4} \int q_{\theta\theta} q_{\phi\phi} d\theta d\phi.
\]  

(45)

4. Conclusion

In this work, we have calculated the Euclidean action for asymptotically flat EiBI black holes with Maxwell electrodynamics. We found that by employing appropriate ensemble in the Euclidean action evaluation, a finite and well-defined expression of the Euclidean action could be obtained. The Euclidean action then is used to obtain the black hole's entropy, which was found to slightly differ from the usual \( \pi r_s^2 \) expression in general relativity due to the coupling of \( \kappa \) and \( Q \) on this model of modified gravity. The form of the entropy is found to be consistent with the previous work on this matter. Further works on the interlinking between the first law of black hole mechanics and the Smarr formula for this particular type of black hole in EiBI gravity would seem to be an interesting inquiry to pursue.

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