AGENDA MANIPULATION-PROOFNESS, STALEMATES, AND REDUNDANT ELICITATION IN PREFERENCE AGGREGATION.
EXPOSING THE BRIGHT SIDE OF ARROW’S THEOREM.

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Abstract. This paper provides a general framework to explore the possibility of agenda manipulation-proof and proper consensus-based preference aggregation rules, so powerfully called in doubt by a disputable if widely shared understanding of Arrow’s ‘general possibility theorem’. We consider two alternative versions of agenda manipulation-proofness for social welfare functions, that are distinguished by ‘parallel’ vs. ‘sequential’ execution of agenda formation and preference elicitation, respectively. Under the ‘parallel’ version, it is shown that a large class of anonymous and idempotent social welfare functions that satisfy both agenda manipulation-proofness and strategy-proofness on a natural domain of single-peaked ‘meta-preferences’ induced by arbitrary total preference preorders are indeed available. It is only under the second, ‘sequential’ version that agenda manipulation-proofness on the same natural domain of single-peaked ‘meta-preferences’ is in fact shown to be tightly related to the classic Arrowian ‘independence of irrelevant alternatives’ (IIA) for social welfare functions. In particular, it is shown that using IIA to secure such ‘sequential’ version of agenda manipulation-proofness and combining it with a very minimal requirement of distributed responsiveness results in a characterization of the ‘global stalemate’ social welfare function, the constant function which invariably selects universal social indifference. It is also argued that, altogether, the foregoing results provide new significant insights concerning the actual content and the constructive implications of Arrow’s ‘general possibility theorem’ from a mechanism-design perspective.

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1. Introduction

The agenda is a key part of any decision problem, and is specified by its two components. The first component of the agenda, its content, is defined by the alternative options it contains. The second component, its structure, is the rule governing the process of option-scrutiny that is required in order to solve the given decision problem. Thus, such a structure amounts to a total preorder of the available options, resulting in an ordered partition of options that channels their sequential scrutiny. It follows that agenda control is perforce a crucial issue, and has at least two dimensions, namely agenda-content...
and agenda-structure control. Therefore agenda manipulation, or the exercise of agenda control to influence the final decision, is also a ‘two-dimensional’ activity which must be taken into account, to be possibly prevented or restrained.

As it happens, concerns for agenda manipulation along both of its dimensions have always played a distinguished role in the literature on collective decision-making, but have scarcely if ever been the target of explicit treatment within formal models of preference aggregation. To be sure, it is well-known that under transitivity of the relevant preference relation, preference-maximizing choices are path-independent, namely do not depend on the sequence of intermediate choices and rejections out of the sequence of subsets which is dictated by the agenda-structure. Hence, whenever the aggregation rule is a social welfare function, meaning that both the individual preference relations to be aggregated and the ‘social’ or aggregate preference are transitive (and total), agenda-structure control is virtually inconsequential, and agenda manipulation along the agenda-structure dimension is automatically prevented. On the contrary, agenda-content manipulation is of course still possible even if all the relevant preference relations are transitive. Since we are going to focus precisely on social welfare functions, in the sequel we shall only discuss agenda-content manipulation, disregarding entirely agenda-structure manipulation. Therefore, in the rest of the present paper we shall simply identify agendas with their content. Accordingly, ‘agenda manipulation’ is henceforth used, for the sake of simplicity, as a synonym for ‘agenda-content manipulation’.

Indeed, in his classic Social Choice and Individual Values (1963) Arrow used precisely the need to prevent agenda manipulation as the main argument in favor of his own ‘Independence of Irrelevant Alternatives’ (IIA), as a key condition for proper (or non-trivial) consensus-based social welfare functions, which are the main focus of that work. Since then, the notion that IIA should be regarded as a basic ‘nonmanipulability property’ has been further reinforced by the rise and enormous proliferation of models focussing on strategic manipulation issues in preference aggregation, to become eventually almost common place. Yet, within standard models of preference aggregation including social welfare functions it is just preference relations that agents provide as inputs while the relevant agenda is a parameter of the aggregation rule. But then, such an aggregation rule is nothing else than the relevant strategic game-form. It follows that agenda-manipulation amounts to a structural manipulation of the very ‘aggregation game’, literally a game-changer. Hence, the exact connection between agenda manipulation-proofness and IIA is not amenable to a proper game-theoretic scrutiny unless the preference aggregation model is expanded to involve the agenda formation process itself. In particular, such an expanded model is needed to establish whether the full force of IIA is actually necessary to prevent agenda manipulation for social welfare functions that are at least minimally outcome-unbiased and agent-inclusive.

1By contrast, most contributions to agenda control in the political science literature are focussed on agenda-structure manipulation: see e.g. Schwartz (1986), part IV, and Austen-Smith, Banks (2005) for an extensive treatment of agenda-structure control models in political science. Miller (1995) explicitly distinguishes the two dimensions of agenda control but then provides a review of some models offering a joint treatment of them.

2A dictatorial social welfare function is of course unanimity-respecting and thus in a sense also consensus-based, but only trivially so.
Thus, some explicit formulation of the agenda formation process has to be introduced in the relevant preference aggregation model. In the present work two main types of agenda formation protocols are considered. Both of them rely on a prespecified admissible set of outcomes out of which the actual agenda has to be defined. Moreover, in order to avoid any sort of infinite regress, we can safely assume that outcome-admissibility is established by another (possibly ‘democratic’, but distinct) procedure in the first agenda formation protocol, however, agents provide at once both their preferences on admissible outcomes and their proposals concerning the agenda. In the second one, on the contrary, a first stage is devoted to specifying the actual agenda, and is followed by a second, preference-elicitation stage where the agents express their preferences on the previously chosen agenda. Accordingly, two distinct formulations of Agenda Manipulation-Proofness (AMP) are introduced, and their distinctive impact on the design of preference aggregation rules that guarantee at least a minimal amount of outcome-unbiased distributed responsiveness to individual preferences is explored and discussed at length. In particular, it is shown that under the first formulation, AMP and Strategy-Proofness on a comprehensive single-peaked domain of ‘meta-preferences’ most naturally induced by basic preferences on outcomes are shared by a large class of preference aggregation rules on the full domain of arbitrary profiles of total preference preorders. Such a class of strategy-proof aggregation rules includes social welfare functions which indeed satisfy a remarkable combination of valuable properties. Namely, anonymity, monotonicity and a basic version of Pareto-optimality, possibly even (weak) neutrality, though occasionally producing stalemate as an output to some preference profiles exhibiting certain specific patterns of strong conflict (e.g. Condorcet cycles). By contrast, the second version of AMP turns out to be strictly related to IIA, and the combination of IIA with a couple of much weaker, indeed minimal, requirements of unbiased and distributed responsiveness provides a characterization of the Global Stalemate constant social welfare function, namely the social welfare functions which has universal indifference as its unique possible output. Moreover, if the Weak Pareto property or even just idempotence (namely, ‘respect for unanimity’) is adjoined to IIA, an Arrowian impossibility result is obtained. Thus, in order to secure agenda manipulation-proofness of a social welfare function one may consider two basic alternative approaches having strikingly different consequences. One of those approaches relies on the

3It is worth recalling here that Dahl (1956) famously suggested to label as ‘populist democracy’ the doctrine that advocates reliance on the simple majority rule to settle every issue including the identification of the admissible issues for a possible public agenda. Arguably, one might invoke a generalized notion of ‘populist democracy’ as the advocacy of a unique ‘democratic’ decision rule to settle every issue, including every aspect pertaining to agenda control. In that connection, assuming that admissibility of outcome sets is subject to a distinct protocol (if possibly also ‘democratic’ in some appropriate sense) also amounts to preventing any ‘populist’ interpretation of the overall decision mechanism.

4As mentioned below, such a result relies heavily on the main theorem of Savaglio, Vannucci (2021) concerning strategy-proof aggregation rules in median join-semilattices.

5A stalemate is defined as ‘social indifference’ among a set of alternative social states including a pair \(x, y\) such that \(x\) is unanimously strictly preferred to \(y\). Thus, by definition, a stalemate admits of violations of the Weak Pareto principle (which enforces strict social preference for \(x\) versus \(y\) under the aforementioned situation).

6The above mentioned social welfare functions are the quota rules, including the Condorcet-Kemeny median rule which is indeed neutral when the number of agents is odd.

7Such a result, which amounts to a considerable strengthening of a previous characterization of the same constant social welfare function due to Hansson (1969), relies heavily on Wilson (1972) and Savaglio, Vannucci (2021).
introduction of IIA: it was correctly identified by Arrow’s seminal contribution, and paves the way to his classic characterization of dictatorial social welfare functions. That result signals an important obstruction to the design of social welfare functions as democratic preference aggregation protocols. The other approach, however, has no connection whatsoever to IIA, and is consistent with a large class of anonymous, unanimity-respecting social welfare functions that also retain a basic version of Pareto optimality involving nonstrict preferences. We argue that the very contrast between those two approaches and their respective results makes it possible to single out and appreciate the constructive implications of Arrowian ‘impossibility theorems’ concerning the design of preference aggregation rules, as a significant part of their actual meaning and content.

The rest of this paper is organized as follows: section 2 collects the formal description of the model and the results; section 3 consists in a brief discussion of a few most strictly related contributions (the interested reader is addressed to the supplementary Appendix for a more extensive and detailed discussion of a carefully selected sample of the massive amount of related literature); section 4 provides some concluding remarks, and prospects for future research.

2. Model and results

1. Preliminaries.

Let $A$ be a nonempty finite set of alternative social states with $|A| \geq 3$, $\mathcal{R}_A$ the set of all total preorders (i.e. reflexive, transitive and connected binary relations) on $A$, $\mathcal{L}_A \subseteq \mathcal{R}_A$ the set of all linear orders (or antisymmetric total preorders on $A$), and $\mathcal{P}(A)$ the set of parts of $A$, or possible agendas from $A$. Let $N = \{1, \ldots, n\}$ denote a finite population of agents/voters. We assume that $n \geq 3$ in order to avoid tedious qualifications. The subsets of $N$ are also referred to as coalitions, and $(\mathcal{P}(N), \subseteq)$ denotes the partially ordered set of coalitions induced by set-inclusion. An order filter of $(\mathcal{P}(N), \subseteq)$ is a set $F \subseteq \mathcal{P}(N)$ of coalitions such that for any $S \in F$ and any $T \subseteq N$, if $S \subseteq T$ then $T \in F$. The basis of order filter $F$ is the set of inclusion-minimal elements/coalitions of $F$, and is denoted by $F^{\text{min}}$.

Each agent $i \in N$ is endowed with a total preference preorder $R_i \in \mathcal{R}_A$ (whose asymmetric component or strict preference is denoted by $P(R_i)$), and proposes an agenda $A_i \subseteq A$. A social welfare function for $(N, A)$ if a function $f : \mathcal{R}_A^N \to \mathcal{R}_A$. We shall also consider two types of social welfare functions enriched with an endogenous agenda formation process. According to the class of parallel rules agents release concurrently their entire inputs consisting of a total preorder on the set of all admissible alternatives and of a proposed agenda: preference aggregation and agenda formation are also mutually concurrent processes. Notice that this also entails that typically, namely whenever the selected agenda is a proper subset of $A$, the elicited individual preferences turn out to be redundant. Thus, a parallel agenda-formation-enriched (PAFE) social welfare function for $(N, A)$ is an aggregation rule $f : (\mathcal{P}(A) \times \mathcal{R}_A)^N \to \mathcal{P}(A) \times \mathcal{R}_A$ (with projections $f_1$ and $f_2$ on $\mathcal{P}(A)$ and $\mathcal{R}_A$, respectively).

In particular, such a PAFE $f$ is said to be decomposable if and only if it can be decomposed into two component aggregation rules: an agenda formation rule $f^{(1)} : \mathcal{P}(A)^N \to \mathcal{P}(A)$ and a social welfare function $f^{(2)} : \mathcal{R}_A^N \to \mathcal{R}_A$. Conversely, for any $f'$ be an agenda formation rule $f'$ for $(N, A)$ and any $\mathcal{R}_A$ the partially ordered set of coalitions induced by set-inclusion. An order filter of $(\mathcal{P}(N), \subseteq)$ is a set $F \subseteq \mathcal{P}(N)$ of coalitions such that for any $S \in F$ and any $T \subseteq N$, if $S \subseteq T$ then $T \in F$. The basis of order filter $F$ is the set of inclusion-minimal elements/coalitions of $F$, and is denoted by $F^{\text{min}}$.

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In other terms, the two projections of $f$, namely $f_1 : (\mathcal{P}(A) \times \mathcal{R}_A)^N \to \mathcal{P}(A)$ and
social welfare function $f$ for $(N, A)$ a decomposable PAFE social welfare function $f \simeq f' \times f$ can be defined in an obvious way. Hence, any social welfare function can be regarded as a component of a decomposable PAFE social welfare function by combining it with an agenda formation rule.

Observe that a decomposable PAFE $f$ also induces a family of functions $\mathcal{F}_f := \left\{ f_B^{(2)} : \mathcal{R}_A^N \to \mathcal{R}_B \mid B \in f^{(1)}[\mathcal{P}(A)] \right\}$ where $f_B^{(2)}(R_N) := (f^{(2)}(R_N))|_B$ for any $R_N \in \mathcal{R}_A^N$. Accordingly, the possible values of functions in $\mathcal{F}_f$ are given by a family of total preorders, namely $\left\{ f_B^{(2)}(R_N) \right\}_{R_N \in \mathcal{R}_A^N, B \in \mathcal{P}(A)}$: thus, for every $R_N \in \mathcal{R}_A^N$ and $B \in \mathcal{P}(A)$, $f_B^{(2)}(R_N) \in \mathcal{R}_B \subseteq \bigcup_{B \in \mathcal{P}(A)} \mathcal{R}_B$. In particular, we shall focus on sovereign agenda-formation rules $f^{(1)} : \mathcal{P}(A)^N \to \mathcal{P}(A)$ (namely, such that for every $C \subseteq A$ there exists $B_N \in \mathcal{P}(A)^N$ with $f^{(1)}(B_N) = C$).

Under the class of sequential rules, on the contrary, agents release their inputs in two steps: first they provide concurrently their proposed agendas to be aggregated into a shared agenda, then they submit concurrently their preferences on the previously determined actual agenda as their input to preference aggregation itself. Notice that in this case, no redundancy in preference elicitation is to be expected. Thus, a sequential agenda-formation-enriched (SAFE) social welfare function for $(N, A)$ is in fact an agenda-contingent social welfare function, namely a pair $\hat{f} = (\hat{f}^1, \mathcal{F}(\hat{f}^1))$ consisting of an agenda formation rule $\hat{f}^1 : \mathcal{P}(A)^N \to \mathcal{P}(A)$, and a family $\mathcal{F}(\hat{f}^1) = \left\{ \hat{f}_B : \mathcal{R}_B^N \to \mathcal{R}_B \right\}_{B \in \hat{f}^1[\mathcal{P}(A)]}$ of possible social welfare functions, one for each possible agenda selected by $\hat{f}^1$. A particular case of special interest obtains when the family of agenda-contingent social welfare functions is the uniform family induced by its $\hat{f}_A$, namely $\mathcal{F}(\hat{f}^1) := \left\{ \hat{f}_B : \mathcal{R}_B^N \to \mathcal{R}_B \mid \hat{f}_B := (\hat{f}_A)|_B \right\}_{B \in \hat{f}^1[\mathcal{P}(A)]}$.

In any case, again, the values of possible social welfare functions according to $\hat{f}$ are in $\bigcup_{B \in \mathcal{P}(A)} \mathcal{R}_B$.

As a consequence, SAFE and (decomposable) PAFE social welfare functions essentially share $\bigcup_{B \in \mathcal{P}(A)} \mathcal{R}_B$ as their common outcome space.

Therefore, it transpires that SAFE and PAFE social welfare functions for a pair $(N, A)$ consist of functions whose domains and codomains result from various combinations of ‘building blocks’ chosen from the collection given by $\mathcal{P}(A)$ and the sets $\mathcal{R}_B$ of all total preorders on $B$, for any $B \subseteq A$. As it

\[ f_2 : (\mathcal{P}(A) \times \mathcal{R}_A^N)^N \to \mathcal{R}_A^N \text{ are such that for every } R_N, R'_N \in (\mathcal{R}_A^N)^N \text{ and } B_N, B'_N \in (\mathcal{P}(A))^N : \]

\[ f_1(B_N, R_N) = f_1(B_N, R'_N) := f^{(1)}(B_N) \]

and

\[ f_2(B_N, R_N) = f_2(B'_N, R_N) := f^{(2)}(R_N). \]

Strictly speaking $f$ and $f^{(1)} \times f^{(2)}$ are isomorphic functions, hence we also use the notation $f \simeq f^{(1)} \times f^{(2)}$ to denote that fact.

This is in fact a most important feature that distinguishes PAFE and SAFE social welfare functions. In principle, one could also consider a preference-first version of SAFE social welfare functions. However, that version would require a lot of redundancy in preference elicitation, either by eliciting preferences on the entire set $A$ (or even, most impractically, eliciting specific preferences on every relevant subset of $A$). Now, the first and more practical option gives rise to an aggregation procedure that is essentially equivalent to a PAFE social welfare function (since preferences on $A$ would be adapted to subsets by restriction). That is why in the present work SAFE social welfare functions are actually identified with their agenda-first variety.
turns out, such ‘building blocks’ share a common structure: all of them are median join-semilattices, and that fact will play a key role in the subsequent analysis of the behaviour of PAFE and SAFE social welfare functions. Accordingly, we turn now to providing a precise definition of median join-semilattices, and establishing the previous claim on \(\mathcal{P}(A)\) and the sets of the family \(\{R_B\}_{B \subseteq A}\).

**Definition 1.** A (finite) **join-semilattice** is a pair \(\mathcal{X} = (X, \leq)\) where \(X\) is a (finite) set and \(\leq\) is a partial order (i.e. a reflexive, transitive and antisymmetric binary relation) such that the least upper bound or join \(x \lor y\) (with respect to \(\leq\)) is well-defined in \(X\) for all \(x, y \in X\) and thus \(\lor : X \times X \to X\) is a well-defined associative and commutative function that also satisfies idempotency, namely \(x \lor x = x\) for every \(x \in X\).

**Remark 1.** Thus a join-semilattice \(\mathcal{X} = (X, \leq)\) can also be regarded as a pair \((X, \lor)\) where \(\lor : X \times X \to X\) is an associative, commutative and idempotent operation such that, for any \(x, y \in X\), \(x \lor y = x\) iff \(y \leq x\). Note that a partial meet-operation \(\land : X \times X \to X\) is also definable in \(\mathcal{X}\) by means of the following rule: for any \(x, y \in X\), \(x \land y\) is the (necessarily unique, whenever it exists) \(z \in X\) such that: (i) \(x \lor z = x\), \(y \lor z = y\), and (ii) \(v \lor z = z\) for every \(v \in X\) which satisfies (i).

Observe that a finite join-semilattice \(\mathcal{X} = (X, \leq)\) has a (unique) universal upper bound or top element \(1 = \lor X = \land \varnothing\), and its co-atoms are those elements \(x \in X\) such that \(x \ll 1\) (i.e. \(x < 1\) and there is no \(z \in X\) such that \(x < z < 1\)): the set of co-atoms of \(\mathcal{X} = (X, \leq)\) is denoted by \(C_X\). An element \(x \in X\) is **meet-irreducible** if for any \(Y \subseteq X\), \(x = \land Y\) entails \(x \in Y\). Moreover, for any \(Y \subseteq X\), \(\lor Y\), respectively is well-defined if and only if there exists \(z \in X\) such that \(y \leq z\) for all \(y \in Y\), namely the elements of \(Y\) have a common upper bound. The set of all meet-irreducible elements of \(\mathcal{X} = (X, \leq)\) will be denoted by \(M_X\). Notice that, by construction, for every \(x \in X\), \(x = \land M(x)\) where \(M(x) := \{m \in M_X : x \leq m\}\). By construction, a co-atom is also a meet-irreducible element, but the converse need not be true. When co-atoms and meet-irreducibles do in fact *coincide* the join-semilattice is said to be *coatomistic*.10

**Definition 2.** (Median join-semilattice) A (finite) **join-semilattice** \(\mathcal{X} = (X, \leq)\) is a **median join-semilattice** if it also satisfies the following pair of conditions:

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10Dually, a (finite) **meet-semilattice** is a pair \(\mathcal{X} = (X, \leq)\) where \(X\) is a (finite) set, and partial order \(\leq\) is such that the greatest lower bound or meet \(x \land y\) (with respect to \(\leq\)) is well-defined in \(X\) for all \(x, y \in X\) and thus \(\land : X \times X \to X\) is a well-defined associative and commutative function that also satisfies idempotency, namely \(x \land x = x\) for every \(x \in X\).

An element \(x \in X\) of a meet-semilattice is **join-irreducible** if for any \(x = \lor Y\) entails \(x \in Y\) for any (finite) \(Y \subseteq X\) such that \(\lor Y\), is well-defined. The set of all join-irreducible elements of \(\mathcal{X} = (X, \leq)\) is denoted by \(J_X\). The **atoms** of \(\mathcal{X}\) are those elements \(x \in X\) such that \(0 \ll x\) (i.e. \(0 < x\) and there is no \(z \in X\) such that \(0 < z < x\) where \(0 = \land X = \land \varnothing\)): the set of atoms of \(\mathcal{X}\) is denoted by \(A_X\). Clearly, \(A_X \subseteq J_X\). The semilattice \(\mathcal{X}\) is **atomistic** if \(A_X = J_X\).

Notice that, by construction, for every \(x \in X\), \(x = \lor J(x)\) where \(J(x) := \{j \in J_X : j \leq x\}\). If a join-semilattice \(\mathcal{X} = (X, \leq)\) is also a meet-semilattice then \(\mathcal{X}\) is a lattice and absorption laws hold, namely for any \(x, y \in X\), \(x \lor (y \land x) = x \land (y \lor x)\).
i) **upper distributivity**: for all $u \in X$, and for all $x, y, z \in X$ such that $u$ is a lower bound of \{\(x, y, z\), \(x \lor (y \land z) = (x \lor y) \land (x \lor z)\) (or, equivalently, \(x \land (y \lor z) = (x \land y) \lor (x \land z)\)) holds i.e. \((\uparrow u, \leq_{\uparrow u})\) -where \(\leq_{\uparrow u}\) denotes the restriction of \(\leq\) to \(\uparrow u\)- is a distributive lattice.\(^{11}\)

(ii) **co-coronation** (or meet-Helly property): for all $x, y, z \in X$ if \(x \land y, y \land z\) and \(x \land z\) exist, then \((x \land y \land z)\) also exists.

In fact, it is easily checked that if $X = (X, \leq)$ is a median-semilattice then the partial function \(\mu_X : X^3 \to X\) defined as follows: for all $x, y, z \in X$, \(\mu_X(x, y, z) = (x \lor y) \land (y \lor z) \land (x \lor z)\)
is in fact a well-defined ternary operation on $X$, the **median** of $X$ which satisfies the following two characteristic properties (see Sholander (1952, 1954)):

\[
(\mu_1) \quad \mu_X(x, y, x) = x \quad \text{for all } x, y \in X
\]

\[
(\mu_2) \quad \mu_X(\mu_X(x, y, v), \mu_X(x, y, w), z) = \mu_X(\mu_X(v, w, z), x, y)
\]

for all $x, y, v, w, z \in X$.

A pair \((X, \mu)\) where $\mu$ is a ternary operation on $X$ that satisfies \((\mu_1)\) and \((\mu_2)\) is also said to be a **median algebra**.

Relying on $\mu_X$, a ternary (median-induced) betweenness relation
\[B_{\mu_X} := \{(x, z, y) \in X^3 : z = \mu_X(x, y, z)\}\]can also be defined on $X$.\(^{12}\) The pair \((X, B_{\mu_X})\) is also said to be a **median (ternary) space**.

**Remark 2.** It is worth emphasizing here that any finite median join-semilattice is naturally endowed with two equivalent metrics, and that a further betweenness relation can be defined on it relying on such metrics. However, it turns out that such a metric-based betweenness is in fact equivalent to the median-based betweenness $B_{\mu_X}$ introduced above in the text (see e.g. Sholander (1954), Avann (1961)).

Thus, $B_{\mu_X}$ is indeed a most natural ‘intrinsic’ betweenness relation and can also be regarded as ‘the’ natural metric betweenness attached to $X$. Relying on such a betweenness $B_{\mu_X}$, a ‘natural’ notion of **single-peakedness** for preference preorders on $X = (X, \leq)$ can be defined as follows.

\(^{11}\)A partially ordered set \((Y, \leq)\) is a **distributive lattice** iff, for any $x, y, z \in X$, $x \land y$ and $x \lor y$ exist, and $x \land (y \lor z) = (x \land y) \lor (x \land z)$ (or, equivalently, $x \lor (y \land z) = (x \lor y) \land (x \lor z)$). Moreover, a (distributive) lattice $X$ is said to be **lower (upper) bounded** if there exists $\bot \in X$ (\(\top \in X\)) such that $\bot \leq x$ (\(x \leq \top\)) for all $x \in X$, and bounded if it is both lower bounded and upper bounded. A bounded distributive lattice \((X, \leq)\) is **Boolean** if for each $x \in X$ there exists a **complement** namely an $x' \in X$ such that $x \lor x' = \top$ and $x \land x' = \bot$.

\(^{12}\)It should be recalled that such a median operation $\mu$ is also well-defined in any **distributive lattice** $X = (X, \leq)$. Thus, every (finite) distributive lattice is in particular a (finite) median join-semilattice (and a finite) median meet-semilattice as well.

\(^{13}\)In a (finite) median join-semilattice $X = (X, \leq)$ a metric $d_r : X \times X \to \mathbb{Z}_+$ can be defined in a natural way by the following rule: for any $x, y \in X$, $d_r(x, y) = 2r(x \lor y) - r(x) - r(y)$, where $r$ is a **rank function** of $X$, namely a function $r : X \to \mathbb{Z}_+$ such that, for any $x, y \in X$, $r(y) = r(x) + 1$ whenever $x$ is an immediate $\leq$-predecessor of $y$.

This metric turns out to be equivalent to the metric $\delta_{C(X)}$ induced on $X$ by the length of shortest path between any two elements on the graph defined by the Hasse diagram of $X$ (the simple undirected graph having $X$ as its set of vertices, with edges connecting each pair consisting of a vertex and one of its immediate $\leq$-predecessors).
Definition 3. (Single-peaked preference preorders on a median join-semilattice). Let $X = (X, \preceq)$ be a finite median join-semilattice and $\succ$ a preorder i.e. a reflexive and transitive binary relation on $X$ (we shall denote by $\succ$ and $\sim$ its asymmetric and symmetric components, respectively). Then, $\succ$ is said to be **single-peaked** with respect to betweenness relation $B_{\mu, X}$ (or $B_{\mu, X}$-single-peaked) if and only if

- $U$-(i) there exists a unique maximum of $\succ$ in $X$, its top outcome -denoted $\text{top}(\succ)$ - and
- $U$-(ii) for all $x, y, z \in X$, if $x = \text{top}(\succ)$ and $z = \mu_X(x, y, z)$ then not $y \succ z$.

We denote by $U_{B, \mu}$ the set of all $B_{\mu, X}$-single-peaked preorders on $X$. An $N$-profile of $B_{\mu}$-single-peaked preorders is a mapping from $N$ into $U_{B, \mu}$. We denote by $U^{N}_{B_{\mu, X}}$ the set of all $N$-profiles of $B_{\mu, X}$-lu preorders.

Moreover, a set $D \subseteq U^{N}_{B_{\mu, X}}$ of preorders which are single-peaked w.r.t. $B_{\mu, X}$ is a rich single-peaked domain for $\mathcal{X}$ if for all $x, y \in X$ there exists $\succ \in D$ such that $\text{top}(\succ) = x$ and $UC(\succ, y) = \{ z \in X : z = \mu(x, y, z) \}$ (where $UC(\succ, y) := \{ y \in X : x \succ y \}$ is the upper contour of $\succ$ at $y$).

An aggregation rule $f$ for $(N, X)$ is **strategy-proof** on $U^{N}_{B_{\mu, X}}$ iff for all $B_{\mu, X}$-single-peaked $N$-profiles $(\succ_i)_{i \in N} \in U^{N}_{B_{\mu, X}}$, and for all $i \in N$, $y_i \in X$, and $(x_j)_{j \in N} \in X^N$ such that $x_j = \text{top}(\succ_j)$ for each $j \in N$, not $f((y_i, \{x_j\}_{j \in N}) \succ_i f((x_j)_{j \in N})$. Finally, an aggregation rule $f : X^N \rightarrow X$ is $B_{\mu, X}$-monotonic iff for all $i \in N$, $y_i \in X$, and $(x_j)_{j \in N} \in X^N$,

$$f((x_j)_{j \in N}) = \mu_X(x_i, f((x_j)_{j \in N}), f(y_i), (x_j)_{j \in N})$$

In particular, let $\mathcal{X} = (X, \preceq)$ be a finite join-semilattice and $M_{\mathcal{X}}$ the set of its meet-irreducible elements, and for any $x^N \in X^N$, and any $m \in M_{\mathcal{X}}$, poset $N_m(x^N) := \{ i \in N : x_i \leq m \}$. Then, the following properties of an aggregation rule can also be introduced:

**$M_{\mathcal{X}}$-Independence:** an aggregation rule $f : X^N \rightarrow X$ is $M_{\mathcal{X}}$-independent if and only if for all $x^N, y^N \in X^N$ and all $m \in M_{\mathcal{X}}$: if $N_m(x^N) = N_m(y^N)$ then $f(x^N) \leq m$ if and only if $f(y^N) \leq m$.

**Isotony:** an aggregation rule $f : X^N \rightarrow X$ is isotonic if $f(x^N) \leq f(x'^N)$ for all $x^N, x'^N \in X^N$ such that $x_i \leq x'_i$ for each $i \in N$.

It can be easily shown (see Monjardet (1990)) that the conjunction of $M_{\mathcal{X}}$-Independence and Isotony is equivalent to the following condition:

**Monotonic $M_{\mathcal{X}}$-Independence:** An aggregation rule $f : X^N \rightarrow X$ is monotonically $M_{\mathcal{X}}$-independent if and only if for all $x^N, y^N \in X^N$ and all $m \in M_{\mathcal{X}}$: if $N_m(x^N) \subseteq N_m(y^N)$ then $f(x^N) \leq m$ implies $f(y^N) \leq m$.\(^{14}\)

\(^{14}\) $B_{\mu, X}$-monotonicity of $f$ amounts to requiring all of its projections $f_j$ to be gate maps to the image of $f$ (see van de Vel (1993), p.98 for a definition of gate maps). The introduction of $B_{\mu, X}$-monotonic functions in a strategic social choice setting is essentially due to Danilov (1994).

\(^{15}\) The notions of $J_{\mathcal{X}}$-Independence and Monotonic $J_{\mathcal{X}}$-Independence are defined similarly by dualization for a finite median meet-semilattice $\mathcal{X} = (X, \preceq)$ as follows: for all $x^N, y^N \in X^N$ and all $j \in J_{\mathcal{X}}$, if

$$N_j(x^N) := \{ i \in N : j \leq x_i \} \subseteq N_j(y^N) := \{ i \in N : j \leq y_i \}$$

then $j \leq f(x^N)$ implies $j \leq f(y^N)$.
We are now ready to establish the following claim.

Claim 1. \((\mathcal{P}(A), \subseteq), (\mathcal{R}_A, \subseteq), (\mathcal{R}_B, \subseteq)\) for any \(B \subseteq A\), and

\[
\bigcup_{B \in \mathcal{P}(A)} (\mathcal{R}_B, \subseteq) \text{ are median join-semilattices.}
\]

Proof. Let us define the join of two total preorders \(R, R' \in \mathcal{R}_A\) as the transitive closure \(\sqcup\) of their set-theoretic union. Then, by construction, \(\mathcal{X} := (\mathcal{R}_A, \sqcup)\) is a join-semilattice, and satisfies both upper distributivity (by Claim (P.1) of Janowitz (1984)), and co-coronation (by Claims (P.3) and (P.5) of Janowitz (1984)). It follows that \((\mathcal{R}_A, \sqcup)\) thus defined is indeed a median join-semilattice (whose median ternary operation is denoted here \(\mu'\)), and its meet-irreducibles are the total preorders \(R_{A_1, A_2} \in \mathcal{R}_A\) having just two (non-empty) indifference classes \(A_1, A_2\) such that (i) \((A_1, A_2)\) is a two-block ordered partition of \(A\), written \((A_1, A_2) \in \Pi_2^A\), namely \(A_1 \cup A_2 = A\), \(A_1 \cap A_2 = \emptyset\) and (ii) \([xR_{A_1, A_2}y \text{ and not } yR_{A_1, A_2}x]\) if and only if \(x \in A_1\) and \(y \in A_2\). It can be easily checked that such total preorders \(R_{A_1, A_2}\) with \((A_1, A_2) \in \Pi_2^A\) are also the co-atoms of \((\mathcal{R}_A, \sqcup)\). Of course, the very same argument applies to \((\mathcal{R}_B, \sqcup)\), for every \(B \subseteq A\). Moreover, the partially ordered set \(\mathcal{X}' := (\mathcal{P}(A), \subseteq)\) of agendas is of course a bounded distributive lattice with respect to set-theoretic union \(\cup\) and intersection \(\cap\). Hence \((\mathcal{P}(A), \sqcup)\) is in particular a median join-semilattice. As a consequence, the product join-semilattice \(\mathcal{X} \times \mathcal{X}' := (\mathcal{R}_A \times \mathcal{P}(A), \sqcup \times \sqcup)\) is also a median join-semilattice: indeed, the ternary product-operation \(\mu_{\mathcal{X}} \times \mu_{\mathcal{X}'} : (\mathcal{R}_A^T \times \mathcal{P}(A))^3 \to \mathcal{R}_A \times \mathcal{P}(A)\) inherits the characteristic median properties \(\mu(i), \mu(ii)\) (as previously defined above) from its components. Finally, \(\bigcup_{B \in \mathcal{P}(A)} (\mathcal{R}_B, \subseteq)\) is a median join-semilattice with join \(\sqcup\), because it is a (disjoint) sum (or co-product) of the family \(\{(\mathcal{R}_B, \subseteq)\}_{B \subseteq A}\) of median join-semilattices, and its median operation \(\mu_{\mathcal{X}}\) is defined as follows: for any \(B, C, D \in \mathcal{P}(A)\), and \(R^B \in \mathcal{R}_B, R^C \in \mathcal{R}_C, R^D \in \mathcal{R}_D\),

\[
\mu_{\mathcal{X}}(R^B, R^C, R^D) = (R^B \sqcup R^C) \cap (R^C \sqcup R^D) \cap (R^B \sqcup R^D).
\]

Therefore, in particular, \(\mathcal{X}' := (\bigcup_{B \in \mathcal{P}(A)} \mathcal{R}_B, \sqcup)\) is also endowed with a ‘natural’ metric \(d\) (namely \(d = d_r = \delta_{C(\mathcal{X}')}\)). But then, any preference relation \(R \in \bigcup_{B \in \mathcal{P}(A)} \mathcal{R}_B\) induces in a ‘natural’ way a reflexive preference relation \(R_R\) on \(\bigcup_{B \in \mathcal{P}(A)} \mathcal{R}_B\) which has \(R\) itself as its unique maximum and is single-peaked with respect to \(d\), being defined as follows: for any \(R', R'' \in (\bigcup_{B \in \mathcal{P}(A)} \mathcal{R}_B) \setminus \{R\}\), \(R'R_RR''\) holds if and only if \(R'\) lies on a geodesic from \(R\) to \(R''\) on the Hasse diagram \(C(\mathcal{X}')\). Moreover, it can be shown that any such \(R_R\) is also transitive.\[16\]

\[16\]See e.g. Sholander (1954), Section 3, property 3.6 for a proof.
Thus, it turns out that \( \bigcup_{B \in \mathcal{P}(A)} \mathcal{R}_B \) can also be ‘naturally’ endowed with a set \( \mathcal{D}^{\text{sp}(d)} := \{ \mathcal{R}_R : R \in \bigcup_{B \in \mathcal{P}(A)} \mathcal{R}_B \} \) of preorders on \( \bigcup_{B \in \mathcal{P}(A)} \mathcal{R}_B \) which are single-peaked with respect to the ‘intrinsic’ metric \( d \) of itself.\footnote{Notice that \( \mathcal{D}^{\text{sp}(d)} \) includes the set \( \mathcal{R}^{\text{sp}(d)} \) of all total preorders (hence in particular all the linear orders) on \( \bigcup_{B \in \mathcal{P}(A)} \mathcal{R}_B \) which are single-peaked with respect to \( d \). Moreover, \( \mathcal{R}^{\text{sp}(d)} \) includes in turn the subclass \( \mathcal{R}^{\text{msp}(d)} \) of all metric single-peaked total preorders on \( \bigcup_{B \in \mathcal{P}(A)} \mathcal{R}_B \) (namely those total preorders which are entirely determined by the \( d \)-distance from the peak).}

### 2. Main results.

We are now eventually ready to provide two distinct definitions of agenda-manipulation proofness to be matched, respectively, to PAFE and SAFE social welfare functions as defined above.

**Definition** Agenda Manipulation-Proofness of a PAFE social welfare function (AMP\(_P\)). A PAFE social welfare function \( f : (\mathcal{P}(A)^N \times \mathcal{R}_A)^N \to \mathcal{P}(A) \times \mathcal{R}_A \) with projections \( f_1, f_2 \) is AMP\(_P\) if for all \( i \in N \), \( R_N \in \mathcal{R}_A^N \), and \( B_N, B'_N \in \mathcal{P}(A)^N \) such that \( C = f_1(B_N, R_N) \subseteq f_1(B'_N, R_N) = D \), \( f_2(B_N, R_N) \in \mathcal{R}_R f(R_N) \in \mathcal{R}_D \) if and only if \( f_2(B'_N, R_N) \in \mathcal{R}_R f(R_N) \in \mathcal{R}_D \).

Accordingly, a social welfare function \( f : \mathcal{R}_A^N \to \mathcal{R}_A \) is said to be AMP\(_P\) if the PAFE \( f \simeq f^0 \times f \) is AMP\(_P\) for every sovereign agenda formation rule \( f^0 : \mathcal{P}(A)^N \to \mathcal{P}(A) \).

In other words, AMP\(_P\) requires that at every preference profile \( R_N = (R_i)_{i \in N} \) on the entire set \( A \) of admissible alternatives each agent \( i \) be indifferent (according to her preference \( R_{R_i} \), on preference preorders on \( A \) as induced by her actual preference \( R_i \) on \( A \)) between the restriction of the social preference \( f(R_N) \) to an arbitrary agenda \( C \), no matter if that agenda is the actually selected agenda \( D \) or just a subagenda of \( D \).

**Definition** Agenda Manipulation-Proofness of a SAFE social welfare function (AMP\(_S\)).

A SAFE social welfare function \( \hat{f} = (f^0, \mathcal{F}(f^0)) \) (with \( \mathcal{F}(f^0) = \{ f_B^0 : \mathcal{R}_B^N \to \mathcal{R}_B \} \) as defined above) is AMP\(_S\) if for all \( i \in N \), \( R_N \in \mathcal{R}_A^N \), and \( B_N, B'_N \in \mathcal{P}(A)^N \), \( C, D \in \mathcal{P}(A) \) such that \( C = f^0(B_N) \subseteq f^0(B'_N) = D \), \( f_C^0((R_N)_C) \in \mathcal{R}_B f(R_N) \in \mathcal{R}_D \) if and only if \( f_D^0((R_N)_D) \in \mathcal{R}_B f(R_N) \in \mathcal{R}_D \).

Accordingly, a social welfare function \( f : \mathcal{R}_A^N \to \mathcal{R}_A \) is AMP\(_S\) if for every sovereign agenda formation rule \( f^0 : \mathcal{P}(A)^N \to \mathcal{P}(A) \) the SAFE social welfare function \( \hat{f} = (f^0, \mathcal{F}) \) with uniform family \( \mathcal{F} = \{ (f_A)_B : \mathcal{R}_B^N \to \mathcal{R}_B \} \) is induced by \( f \) is AMP\(_S\).

Thus, AMP\(_S\) requires that at every preference profile \( R_N = (R_i)_{i \in N} \) on the entire set \( A \) of admissible alternatives each agent \( i \) be indifferent (according to her preference \( R_{R_i} \), on preference preorders on \( A \) as induced by her actual preference \( R_i \) on \( A \)) between the social preference \( f^0_C((R_N)_C) \) at the restriction
of $R_N$ to any selected agenda $C$, and the restriction to $C$ of the social preference $f_D^R((R_N)|_D)$ at the restriction of $R_N$ to any other selected agenda $D \supseteq C$.

As mentioned above, in his classic work (Arrow (1963)) Arrow refers to the need to prevent agenda manipulation as the main argument to support the requirement of Independence of Irrelevant Alternatives for social welfare functions, that is defined as follows.

**Definition Independence of Irrelevant Alternatives (IIA).**

A social welfare function $f_A : \mathcal{R}_A^N \rightarrow \mathcal{R}_A$ satisfies IIA iff for all $R_N, R'_N \in \mathcal{R}_A^N$, and $B \in \mathcal{P}(A)$,

$$(f(R_N))|_B = (f(R'_N))|_B$$

whenever $(R_N)|_B = (R'_N)|_B$.

Therefore, we have just introduced three distinct conditions that are meant to address the same problem, namely preventing agenda manipulation. A first fact about such conditions is worth mentioning at the outset: when regarded as conditions on social welfare functions both AMP$_P$ and AMP$_S$ only make reference to an arbitrary single preference profile on $A$, while IIA concerns an arbitrary pair of preference profiles on $A$. That contrast is quite remarkable, because reference to a single preference profile is a feature that seems to make full sense, in view of Arrow’s overt intention to put aside all the issues related to possible strategic misrepresentation of preferences. Notice, however, that in Arrow’s work the notion of agenda manipulation-proofness is only introduced in a quite informal way. Accordingly, our next task is to explore the precise relationship of IIA to each one of the agenda manipulation-proofness properties introduced above.

Let us start from AMP$_P$. Indeed, our first finding is that IIA is not at all related to AMP$_P$.

**Proposition 1.** Let $f : \mathcal{R}_A^N \rightarrow \mathcal{R}_A$ be a social welfare function and $f : (\mathcal{R}_A \times \mathcal{P}(A))^N \rightarrow \mathcal{R}_A \times \mathcal{P}(A)$ a decomposable PAFE social welfare function for $(N,A)$ such that $f \simeq f^0 \times f$ where $f^0$ is an arbitrary sovereign agenda-formation rule. Then, $f$ is AMP$_P$ (and consequently $f$ is also AMP$_P$, by definition).

**Proof.** Straightforward: let $R_N \in \mathcal{R}_A^N$, and $B_N, B'_N \in \mathcal{P}(A)^N$ such that $C = f_1(B_N, R_N) \subseteq f_1(B'_N, R_N) = D$. By decomposability of $f$, $f_2(B_N, R_N) = f_2(B'_N, R_N) = f(R_N)$. Hence, for every $i \in N$, both $f_2(B_N, R_N)|_C \circ R_i \circ f_2(B'_N, R_N)|_C$ and $f_2(B'_N, R_N)|_C \circ R_i \circ f_2(B'_N, R_N)|_C$ hold by reflexivity of $R_i$, and the thesis follows. 

Observe that, when formally considered as a condition for a PAFE social welfare function $f$, AMP$_P$ is in fact an interprofile condition because it involves two profiles $(B_N, R_N), (B'_N, R_N)$ in $(\mathcal{R}_A \times \mathcal{P}(A))^N$. However, the projection of AMP$_P$ to the social welfare component $f$ of $f$ collapses in fact to an intraprofile condition since it involves a single profile $R_N \in \mathcal{R}_A^N$. Now, IIA is of course an interprofile condition for social welfare functions involving arbitrary pairs of profiles in $\mathcal{R}_A^N$. Therefore, ostensibly, AMP$_P$ and IIA are mutually unrelated as conditions for social welfare functions.

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18 See Fishburn (1973) for a careful classification of structural, interprofile and intraprofile conditions for social welfare functions and related constructs.
Let us now turn to the relationship between AMP of a SAFE social welfare function and IIA. In order to accomplish that task, we shall take advantage of the notion of projection of a preference profile for a social welfare function. Indeed, let \( B \subseteq A \), \( f : \mathcal{R}_B^N \rightarrow \mathcal{R}_B \) and \( R_N \in \mathcal{R}_B^N \): then, \( R_N \) is a \textit{projective profile} for \( f \) if there exists \( i \in N \) such that \( f(R_N) = R_i \). Next, we introduce a considerably weakened version of IIA, namely its restriction to projective profiles in the following sense:

\[ \text{Independence of Irrelevant Alternatives at Projective Profiles (IIAP).} \]

A social welfare function \( f : \mathcal{R}_A^N \rightarrow \mathcal{R}_A \) satisfies IIAP iff for all \( B \in \mathcal{P}(A) \) and \( R_N, R'_N \in \mathcal{R}_A^N \), if \( R_N \) and \( R'_N \) are projective profiles for \( f \) and \( (R_N)|_B = (R'_N)|_B \) then \( f(R_N)|_B = (f(R'_N))|_B \).

Observe that IIAP is indeed \textit{strictly weaker} than IIA. To check the validity of that statement just consider the social welfare function \( f_{BC}^B \) for \( (N, A) \) defined as follows: for \( R_N \in \mathcal{R}_A^N \), \( f_{BC}^B(R_N) := f_{BC}(R_N) \) (where \( f_{BC} \) denotes the Borda-Count scoring aggregation rule) if \( R_N \) is not a projective profile for \( f_{BC} \) and \( f_{BC}^B(R_N) := R_i \) otherwise. By construction, \( R_N \in \mathcal{R}_B^N \) is projective for \( f_{BC}^B \) if and only if \( f_{BC}^B(R_N) = R_i \), hence \( f_{BC}^B \) does satisfy IIAP. However, \( f_{BC}^B \) clearly violates IIA: it is easily checked that there exist profiles \( R_N, R'_N \) that are not projective for \( f_{BC} \) and such that \( (f_{BC}^B(R_N))|_B \neq (f_{BC}^B(R'_N))|_B \) for some \( B \subseteq A \).

We are now ready to show that AMP is in fact tightly connected to IIA, as established by the following proposition.

\[ \textbf{Proposition 2.} \text{ Let } f : \mathcal{R}_A^N \rightarrow \mathcal{R}_A \text{ be a social welfare function for } (N, A). \text{ Then, (i) if } f \text{ satisfies IIA then } f \text{ is also AMP; (ii) if } f \text{ is AMP then } f \text{ satisfies IIAP.} \]

\textit{Proof.} (i) Let \( f : \mathcal{R}_A^N \rightarrow \mathcal{R}_A \) be a social welfare function for \( (N, A) \) that satisfies IIA. Then, take any sovereign agenda-formation rule \( f^0 : \mathcal{P}(A)^N \rightarrow \mathcal{P}(A) \) and consider the uniform SAFE social welfare function \( \hat{f} = (f^0, \mathcal{F}(f^0)) \) induced by \( f \), namely with \( \mathcal{F}(f^0) := \{ f_B : \mathcal{R}_B^N \rightarrow \mathcal{R}_B \}_{B \subseteq A} \) defined as follows: for every \( B \subseteq A \), and \( Q_N \in \mathcal{R}_B^N \), \( f_B(Q_N) := (f(R_N))|_B \) where \( R_N \in \mathcal{R}_A \) is such that \( (R_N)|_B = Q_N \). Clearly, such an \( f_B \) is well-defined precisely because \( f \) satisfies IIA. But then, any \( R_N \in \mathcal{R}_A^N \), and \( B_N, B'_N \in \mathcal{P}(A)^N \), \( C, D \in \mathcal{P}(A) \) such that \( C = f^0(B_N) \subseteq f^0(B'_N) = D \), and consider \( f_C((R_N)|_C) \) and \( (f_{D((R_N)|_D))}|_C \). By definition \( f_C((R_N)|_C) = (f(R_N))|_C = (f(R_N))|_C = (f_{D((R_N)|_D))}|_C \) whence \( f_C((R_N)|_C)R_{R_i}(f_{D((R_N)|_D))}|_C \text{ if } (f_{D((R_N)|_D))}|_C \) for all \( i \in N \) i.e. \( \hat{f} \) satisfies AMP hence by definition \( f \) is also AMP.

(ii) Suppose that \( f \) is AMP yet it violates IIAP. Thus, there exist \( R_N, R'_N \in \mathcal{R}_A^N \) and \( B \subseteq A \) such that \( R_N, R'_N \) are projective profiles for \( f \) and \( (R_N)|_B = (R'_N)|_B \) yet \( (f(R_N))|_B \neq (f(R'_N))|_B \), hence \( f(R_N) \neq f(R'_N) \). Moreover, by projectivity of \( R_N \) and \( R'_N \), there exist \( i, j \in N \) such that \( R_i = f(R_N) \), \( R'_j = f(R'_N) \) whence, in particular, \( (R_i)|_B = (f(R_N))|_B \neq (f(R'_N))|_B = (R'_j)|_B \). However, \( (R_N)|_B = (R'_N)|_B \) implies that \( R_i|_B = R'_i|_B \) and \( R_j|_B = R'_j|_B \). Now, let \( f^0 \) be a sovereign agenda formation rule and \( \hat{f} = (f^0, \mathcal{F}) \) a uniform SAFE social welfare function with \( \mathcal{F} = \{ f_B : \mathcal{R}_B^N \rightarrow \mathcal{R}_B | f_B = (f_A)|_B \}_{B \in \mathcal{P}(A)} \) and \( f = f_A \in \mathcal{F} \). Clearly \( f_B((R_N)|_B) = f_B((R'_N)|_B) \).
But then, either \( (f(R_N))_{|B} \neq f_B((R_N)_{|B}) \) or \( (f(R'_N))_{|B} \neq f_B((R'_N)_{|B}) \).

Suppose w.l.o.g. that \( (f(R_N))_{|B} \neq f_B((R_N)_{|B}) \). Then, by definition of \( R_{R_i}, R_{i|B} = (f(R_N))_{|B} \)
implies
\[
(f(R_N))_{|BR_{R_i}}(f_B((R_N)_{|B})) \quad \text{and not} \quad (f_B((R_N)_{|B}))R_{R_i}(f(R_N))_{|B}
\]
whence AMP\(_S\) fails, a contradiction. \(\square\)

**Remark 3.** One might also wonder whether IIA itself is also a necessary condition for any social welfare function \( f \) to be AMP\(_S\). But that is clearly not the case. To see that, just consider for any \( R^* \in \mathcal{R}_A, \emptyset \neq B^* \subset A \) and \( i \in N \), the social welfare function \( f^{IR^*B^*} \) defined as follows: for every \( R_N \in \mathcal{R}^N_A \),
\[
f^{IR^*B^*}(R_N) := \left\{ \begin{array}{ll}
R^* & \text{if } R_{j\mid A \setminus B^*} = R^*_{\mid A \setminus B^*} \text{ for some } j \in N \setminus \{i\} \\
\text{and } R_i & \text{otherwise}
\end{array} \right.
\]

It is easy to check that \( f^{IR^*B^*} \) is AMP\(_S\) because for any \( C \subseteq D \subseteq A \), and \( R_N \in \mathcal{R}^N_A \), \( (f^{IR^*B^*}(R_N))_{|C} = ((f^{IR^*B^*}(R_N))_{|D}) \).

However, \( f^{IR^*B^*} \) violates IAAP (hence, in particular, IIA as well). Indeed, consider profiles \( R_N, R'_N \in \mathcal{R}^N_A \) such that \( R_{j\mid B^*} = R'_{j\mid B^*} \) for all \( j \in N \), \( R_{i\mid B^*} \neq R'_{i\mid B^*} \), \( R_{j\mid A \setminus B^*} \neq R'_{j\mid A \setminus B^*} \) for each \( j \in N \setminus \{i\} \), and \( R'_k = R^* \) for some \( k \in N \setminus \{i\} \). Now, both \( R_N \) and \( R'_N \) are projective profiles for \( f^{IR^*B^*} \) since \( f^{IR^*B^*}(R_N) = R_i \neq R^* \) while \( f^{IR^*B^*}(R'_N) = R^* = R'_k \). In particular, \( f^{IR^*B^*}(R_N)_{|B^*} = R_{i|B^*} \neq R'_i \) since \( f^{IR^*B^*}(R'_N)_{|B^*} \) though \( (R_N)_{|B^*} = (R'_N)_{|B^*} \), hence IAAP is violated.

Finally, we can proceed to the next main task of the present analysis, which is to explore the class of social welfare functions which are agenda manipulation-proof and do satisfy at least some minimal combination of *outcome-unbiasedness* and *distributed responsiveness* to agents’ preferences, as specified below.

A basic unbiasedness requirement is embodied in the standard sovereignty property, as defined below.

**Sovereignty (S)** A social welfare function \( f : \mathcal{R}^N_A \to \mathcal{R}_A \) for \( (N, A) \) is **sovereign** if for each \( R \in \mathcal{R}_A \) there exists \( R_N \in \mathcal{R}^N_A \) such that \( f(R_N) = R \).

A further, and weaker, unbiasedness condition is implicit in the following property.

**Weak Sovereignty (WS)** A social welfare function \( f : \mathcal{R}^N_A \to \mathcal{R}_A \) for \( (N, A) \) is **weakly sovereign** if for any \( x, y \in A \) there exists \( R_N \in \mathcal{R}^N_A \) such that \( x f(R_N) y \).

Clearly, WS ensures a minimal degree of *outcome-unbiasedness* and *responsiveness* of the relevant social welfare function, but it is consistent both with fairly distributed responsiveness-patterns involving a large number of agents, and with extremely concentrated responsiveness-patterns involving very few agents, or even just a single agent.

In order to make precise such distributed responsiveness requirement, we introduce the *responsiveness correspondence* of a social welfare function as defined below.
Responsiveness Correspondence of a social welfare function Let \( f : \mathcal{R}_A^N \to \mathcal{R}_A \) be a social welfare function for \((N, A)\). Then, its responsiveness correspondence \( F_f : A \times A \to \mathcal{P}(N) \) is defined as follows: for every \( x, y \in A \),
\[
F_f(x, y) := \left\{ S \subseteq N : \text{there exists } R^x_y \in \mathcal{R}_A^S \text{ such that for all } R_N \in \mathcal{R}_A^N, \right. \\
\left. \text{if } [xR_i y] \text{ iff } xR^x_i y \text{ for every } i \in S \text{ then } f(R_N)y \right\}.
\]

Minimally Distributed Responsiveness (MDR) A social welfare function \( f : \mathcal{R}_A^N \to \mathcal{R}_A \) for \((N, A)\) satisfies minimally distributed responsiveness if whenever \( \{i\} \in F_f(x, y) \) for some \( i \in N \) and some pair of distinct \( x, y \in A \) it must be the case that there exist \( S \subseteq N \setminus \{i\} \) and \( v, z \in A, v \neq z \) such that \( S \in F_f(v, z) \).

In plain words, if the nonstrict preference of a single agent \( i \) between two distinct alternatives \( x, y \) has to be accepted as part of the social preference, then the nonstrict preference between two distinct alternatives \( v \) and \( z \) of some other coalition not including \( i \) is also entitled to acceptance as part of the social preference. Thus, arguably, the combination of WS and MDR amounts in fact to an appropriate minimal requirement of unbiased distributed responsiveness.

Let us now recall a few (mostly classic) requirements for social welfare functions.

A social welfare function \( f : \mathcal{R}_A^N \to \mathcal{R}_A \) satisfies
- **Anonymity (AN)** if for every \( R_N \in \mathcal{R}_A^N \) and every permutation \( \sigma \) of \( N \), \( f(x^N) = f(R_{\sigma(N)}) \) (where \( R_{\sigma(N)} = (R_{\sigma(1)}, ..., R_{\sigma(n)}) \));
- **Idempotence (ID)** if \( f(R_N) = R \) whenever \( R_N \) is such that \( R_i = R \) for each \( i \in N \);
- **Neutrality (NT)** if \( [xf(R_N)y] \iff yf(R'_N)x \) for any \( x, y \in A \) and \( R_N, R'_N \in \mathcal{R}_A^N \) such that \( xR_i y \iff yR'_i x \) for each \( i \in N \);
- **Weak Neutrality (WNT)** if \( [f(R_N) \subseteq R] \iff [f(R_N) \subseteq R'] \) for any two-indifference-class \( R, R' \in \mathcal{R}_A \) and \( R_N, R_N' \in \mathcal{R}_A^N \) such that \( R_i \subseteq R \) iff \( R_i \subseteq R' \) for every \( i \in N \);
- **Weak Pareto Principle (WP)** if for every \( x, y \in A \) and \( R_N \in \mathcal{R}_A^N \), \( xP(R_i)y \) for every \( i \in N \) then \( xP(f(R_N))y \);
- **Basic Pareto Principle (BP)** if for every \( x, y \in A \) and \( R_N \in \mathcal{R}_A^N \), \( xR_i y \) for every \( i \in N \) then \( xf(R_N)y \);
- **Local Separation (LS)** if for every \( x, y \in A \) there exist \( R_N, R'_N \in \mathcal{R}_A^N \) such that \( f(R_N)_{\{x, y\}} \neq f(R'_N)_{\{x, y\}} \).

It should be emphasized, for future reference, that LS implies WS, while WP and LS are mutually independent.

Moreover, for any domain \( D \) of (preference) preorders on \( \mathcal{R}_A \), a social welfare function \( f : \mathcal{R}_A^N \to \mathcal{R}_A \) is strategy-proof on \( D \) iff for every \( i \in N, R_N \in D^N, R_N \in \mathcal{R}_A^N \) and \( R' \in \mathcal{R}_A \), \( f(R_N)R_i f((R'_i, R_N \setminus \{i\})) \).

Moreover, a social welfare function \( f : \mathcal{R}_A^N \to \mathcal{R}_A \) is said to be
dictatorial (respectively, inversely dictatorial) if there exists \( i \in N \) such that for all \( R_N \in \mathcal{R}_A^N \) and \( x, y \in A \), \( xf(R_N)y \) only if \( xR_i y \) (respectively, \( yR_i x \)), weakly paretian if it satisfies WP, and weakly anti-paretian if \( yP(f(R_N))x \) for every \( x, y \in A \) and \( R_N \in \mathcal{R}_A^N \) such that \( xP(R_i)y \) for every \( i \in N \), consensual if it satisfies ID, and properly consensual if it satisfies ID and MDR.
The **global stalemate** social welfare function for \((N, A)\) is the constant function \(f_U^A : \mathcal{R}_A^N \to \mathcal{R}_A\) such that \(f_U^A(R_N) = U_A\) for every \(R_N \in (\mathcal{R}_A^T)^N\), where \(U_A := A \times A\), the universal indifference relation.

We are now ready to establish which social welfare functions among those that ensure at least a modicum of unbiased distributed responsiveness do also satisfy the agenda manipulation-proofness requirements \(\text{AMP}_P\) and \(\text{AMP}_S\), respectively.

Concerning \(\text{AMP}_P\) social welfare functions, we can rely on the following recent result (see Savaglio, Vannucci (2019, 2021)).

**Theorem 1.** (Savaglio, Vannucci (2021)) Let \(\mathcal{X} = (X, \leq)\) be a finite median join-semilattice, \(M_X\) the set of its meet-irreducible elements, \(B_\mu\) its median-induced betweenness, and \(f : X^N \to X\) an aggregation rule. Then, the following statements are equivalent:

(i) \(f\) is strategy-proof on \(D^N\) for every rich domain \(D \subseteq U_{B_\mu}\) of locally unimodal preorders on \(w.r.t. B_\mu\) on \(X\);

(ii) \(f\) is monotonically \(M_X\)-independent;

(iii) there exists a family \(F_{M_X} = \{F_m : m \in M_X\}\) of order filters of \((\mathcal{P}(N), \subseteq)\) such that \(f(x_N) = f_{F_{M_X}}(x_N) := \bigwedge \{m \in M_X : N_m(x_N) \in F_m\}\) for all \(x_N \in X^N\).

**Remark 4.** Thus, in particular, let \(\mathcal{X} = (\mathcal{R}_A^T, \sqcup)\) be the join-semilattice of total preorders on finite set \(A\), \(\mu\) its median ternary operation and \(B_\mu\) the corresponding betweenness as previously defined, \(I^{(2)}_A\) the set of all total preorders on \(A\) with two indiference classes and \(f : \mathcal{R}_A^N \to \mathcal{R}_A\) an aggregation rule (namely, a social welfare function) for \((N, \mathcal{R}_A)\). Then, \(M_X = I^{(2)}_A\) and \(f\) is strategy-proof on \(D^N\) for every rich domain \(D \subseteq U_{B_\mu}\) of locally unimodal preorders \(w.r.t.\) \(B_\mu\) on \(\mathcal{R}_A\) iff there exists a family \(F_{M_X} = \{F_{A_1A_2} : (A_1, A_2) \in I^{(2)}_A\}\) of order filters of \((\mathcal{P}(N), \subseteq)\) such that 

\[
f(R_N) = f_{F_{M_X}}(R_N) := \bigwedge \{R_{A_1A_2} \in M_X : \{i \in N : R_{i} \subseteq R_{A_1A_2}\} \in F_{A_1A_2}\}\)

for all \(R_N \in \mathcal{R}_A^N\).

**Remark 5.** Hence, the collection of such strategy-proof social welfare functions includes the following subclasses:

- Inclusive quorum systems, namely functions \(f_{F_{M_X}}\) such that every order filter \(F_{R_{A_1A_2}}\) is transversal i.e. \(S \cap T \neq \emptyset\) for all \(S, T \in F_{R_{A_1A_2}}\) and \(\bigcup_{R_{A_1A_2} \in M_X} F_{R_{A_1A_2}} = N\) (observe that such a class includes any rule such that for every \(R_{A_1A_2} \in M_X\), \(F_{R_{A_1A_2}}\) is simple-majority collegial i.e. there exists a minimal simple majority coalition \(S_{A_1A_2} \subseteq N\), \(|S_{A_1A_2}| = \left\lfloor \frac{|N|+2}{2} \right\rfloor\) with \(F_{R_{A_1A_2}} = \{T \subseteq N : S_{A_1A_2} \subseteq T\}\). Generally speaking, inclusive quorum systems need not be anonymous or neutral.
• Outcome-biased aggregation rules, namely functions \( f_{FM_X} \) where \( F_{RA_1A_2} = \emptyset \) for some \( RA_1A_2 \in M_X \) (observe that they include the subclass of those aggregation rules such that for some total preorder \( \mathcal{R} \in \mathcal{R}_A \), including possibly a linear order, \( F_{RA_1A_2} = \emptyset \) for every \( RA_1A_2 \in M_X \) such that \( \mathcal{R} \subseteq RA_1A_2 \).

• Weakly-neutral aggregation rules, namely functions \( f_{FM_X} \) where \( F_{RA_1A_2} = F_{RA_1A_2}' \) whenever \( RA_1A_2 \wedge RA_1A_2' \) exists.

• Quota aggregation rules, i.e. functions \( f_{FM_X} \) such that for each \( RA_1A_2 \in M_X \) there exists an integer \( q_{[RA_1A_2]} \leq |N| \) with \( F_m = \{ T \subseteq N : q_{[RA_1A_2]} \leq |T| \} \) (such rules are clearly anonymous, but not necessarily weakly-neutral: they are of course weakly-neutral as well if, furthermore, \( F_{RA_1A_2} = F_{RA_1A_2}' \) whenever \( RA_1A_2 \wedge RA_1A_2' \) exists). Quota aggregation rules are said to be positive if \( q_{[RA_1A_2]} > 0 \) for every \( RA_1A_2 \in M_X \). The subclass of positive and weakly-neutral quota aggregation rules includes as a prominent example the co-majority social welfare function \( f_{\text{maj}} \) defined as follows: for every \( RA_1A_2 \in \mathcal{R}_A \)

\[
f_{\text{maj}}(RA_1A_2) := \land_{S \in W^{maj}}(\lor_{i \in S} A_i)
\]

where \( W^{maj} := \{ S \subseteq N : |S| \geq \frac{n+1}{2} \} \).

• The global stalemate social welfare function \( f^{UA} \) for \( (N, A) \) which obtains when \( F_m = \emptyset \) for all \( m \in M_X \).

It is worth noticing that a large subclass of such social welfare functions \( f_{FM_X} \) (including positive quota aggregation rules and inclusive quorum systems) satisfy the Basic Pareto Principle (BP), as made precise by the following claim.

**Claim 2.** Let \( f_{FM_X} \) be a social welfare function as defined above such that \( F_{RA_1A_2} \) is a nontrivial proper order filter (i.e. \( \emptyset \notin F_{RA_1A_2} \neq \emptyset \)) for every \( RA_1A_2 \in M_X \). Then \( f_{FM_X} \) satisfies BP.

**Proof.** Suppose that \( x, y \in A \) and \( RA_1A_2 \in \mathcal{R}_A \) are such that \( xR_iy \) for every \( i \in N \), yet not \( xf_{FM_X}y \).

Namely, by construction,

\[
(x, y) \notin \bigwedge \{ RA_1A_2 \in M_X : \{ i \in N : R_i \subseteq RA_1A_2 \} \in FA_1A_2 \}.
\]

Hence, there exists \( RA_1A_2 \in M_X \) such that \( \{ i \in N : R_i \subseteq RA_1A_2 \} \in FA_1A_2 \) and \( (x, y) \notin RA_1A_2 \).

However, by assumption, \( FA_1A_2 \) is nonempty and every \( T \in FA_1A_2 \) is itself nonempty: thus, \( N \in FA_1A_2 \). But then \( (x, y) \in R_i \subseteq RA_1A_2 \) for any \( i \in T \), a contradiction. \( \square \)

**Remark 6.** It should be emphasized that BP and WP are independent alternative ways of weakening the (strong) Pareto principle\(^{19}\). To see that, just consider social welfare functions \( f^{UN} \) and \( f^{LX} \) for \( (N, A) \) defined informally as follows: if \( RA_1A_2 \) is such that \( Ri = R \) for each \( i \in N \) then \( f^{UN}(RA_1A_2) = R \), otherwise

\(^{19}\) A social welfare function \( f : \mathcal{R}_A^N \rightarrow \mathcal{R}_A \) satisfies the (strong) Pareto principle iff \( xP(f(R_i))y \) for any \( x, y \in A \) and \( RA_1A_2 \in \mathcal{R}_A \) such that \( xR_iy \) for every \( i \in N \), and \( xP(R_i)jy \) for some \( j \in N \).
Proposition 3. There exist social welfare functions \( f : \mathcal{R}_A \rightarrow \mathcal{A} \) which satisfy AMP, AN, ID, WNT, BP and are strategy-proof on the domain \( \mathcal{D}^{sp(d)} \) of single-peaked preorders on \( \bigcup_{B \in \mathcal{P}(A)} \mathcal{R}_B \).

Proof. Since \( (\bigcup_{B \in \mathcal{P}(A)} \mathcal{R}_B, \subseteq) \) is a median join-semilattices by Claim 2 above, it follows that Theorem 1 above applies hence positive weakly-neutral quota social welfare functions as defined above satisfy AN, ID, WNT and are strategy-proof on the domain \( \mathcal{D}^{sp(d)} \) of single-peaked preorders on \( \bigcup_{B \in \mathcal{P}(A)} \mathcal{R}_B \). Moreover, they also satisfy AMP by Claim 1, and BP by Claim 3, and the thesis is established. \( \square \)

Remark 7. Observe that several domains of preference profiles are being considered here. The first one consists of profiles \( \mathcal{R}_N = (R_i)_{i \in N} \) of arbitrary total preorders on the set \( A \) of basic alternatives. The second domain consist of profiles \( \mathcal{R}_N = (R_i)_{i \in N} \) of single-peaked (partial) preorders on the ground set \( \mathcal{R}_A \) of the median join-semilattice of total preorders on \( A \) (with single-peakedness induced by the median betweenness of \( \mathcal{R}_A \), and \( \mathcal{R}_N \) induced by \( \mathcal{R}_N \)). The third domain amounts to the subdomain of the second one which only includes the metric single-peaked profiles \( \tilde{\mathcal{R}}_N = (\tilde{R}_i)_{i \in N} \) of preorders on \( \mathcal{R}_A \) that are entirely determined by profiles \( \mathcal{R}_N \) through their graphic distance from the top. Accordingly, we can consider different notions of WP and BP: namely,

(1) WP (BP) of each ‘social preference’ \( f(\mathcal{R}_N) \) with respect to \( \mathcal{R}_N \) : the requirement that, at any \( \mathcal{R}_N \), \( f(\mathcal{R}_N) \) should faithfully reflect any unanimous preference for an alternative in \( A \) over another one;

(2) WP (BP) of each ‘social preference’ \( f(\mathcal{R}_N) \) with respect to \( \mathcal{R}_N \) (or \( \tilde{\mathcal{R}}_N \)) : the requirement that, at any \( \mathcal{R}_N \), \( f(\mathcal{R}_N) \) should be consistent with unanimous preferences according to \( \mathcal{R}_N \) (or \( \tilde{\mathcal{R}}_N \)), which means that there should be no alternative ‘social preference’ \( \mathcal{R}' \in \mathcal{R}_A \) that is unanimously preferred over \( f(\mathcal{R}_N) \) according to \( \mathcal{R}_N \) (or \( \tilde{\mathcal{R}}_N \)).

Two most remarkable points are to be made here concerning the social welfare functions mentioned in the previous Proposition. First, such social welfare functions fail to satisfy WP with respect to the first and second domains, consisting respectively of arbitrary profiles \( \mathcal{R}_N \) of total preorders on \( A \), and of single-peaked profiles \( \mathcal{R}_N \) of preorders on \( \mathcal{R}_A \). Second, the very same social welfare functions do satisfy WP with respect to the domain consisting of metric single-peaked profiles \( \tilde{\mathcal{R}}_N \) of total preorders on \( \mathcal{R}_A \).
Remark 8. It is worth noticing that all of the anonymous, idempotent and strategy-proof social welfare functions mentioned in the previous proposition (including those which satisfy the Basic Pareto Principle BP e.g. positive quota social welfare functions) admit a stalemate as one of the possible outcomes, arising from certain specific patterns of strong conflict among individual preferences. By definition, such (contingent) stalemates give rise to violations of the Weak Pareto principle (WP) by the chosen ‘social preference’ both with respect to the ‘basic’ preference profiles $R_N$ of total preorders on the set $A$ of alternatives, and with respect to general single-peaked domains $R_N$ on $R_A$ induced by the former $R_N$ profiles. Thus, the foregoing social welfare functions may also be regarded as valuable sources of information and advice concerning the ‘general interest’ (or ‘common good’). In many cases, they provide an explicit description of the alternatives that best represent the ‘common good’, or define anyway clear improvements on the status quo. But occasionally they may also help to pursue the ‘general interest’ by pointing to situations of pathologically strong social conflict: they do that precisely by returning outcomes that allow for ‘inefficient’ choices when fed with inputs encoding such a sort of social conflict.\textsuperscript{20} To put it in other terms, any violation of WP by such social welfare functions might be regarded as a sort of ‘error message’ calling for public intervention (e.g. promoting an improved access to key relevant information for the general public, implementing some appropriate redistribution policies, or just relying on some contingent agenda manipulation activities of the sort thoroughly analyzed and discussed in Schwartz (1986)\textsuperscript{21} in order to ensure outcome-efficiency).

Concerning the study of agenda manipulation-proofness for SAFE social welfare functions and their agenda, Proposition 2 implies that IIA does in fact enforce AMPs. Unfortunately, the side effects of IIA on proper consensus-based social welfare functions are simply devastating. That is well-known thanks to a cluster of theorems originating with Arrow’s famous ‘general possibility theorem’. We recall here just a selected sample of four key results from that cluster. The first pair consists of two characterizations of \textit{dictatorial} social welfare functions: both of them rely on the combination of IIA with some further condition which at first sight would seem to be rather uncontroversial or at least undemanding.

\textbf{Theorem 2.} (i) (Arrow’s Theorem (Arrow (1963))) A social welfare function $f : R_A^N \rightarrow R_A$ satisfies IIA and WP if and only if $f$ is dictatorial;

\textsuperscript{20}See also Saari (2008) on the connection between conflict, cycling and inefficiency.
\textsuperscript{21}Notice, however, that in IIA-based models of collective choice as advocated by Schwartz (1986) such agenda-structure manipulation activities are treated as \textit{normal} and endemic to every democratic aggregation protocol. Of course, that is pretty much the same conclusion as that typically suggested by authors that regard Arrow’s theorem as an indictment of democratic preference aggregation protocols, à la Riker (1982) (see footnote 26 below). By contrast, within the IIA-free models considered in the present work, such agenda-structure manipulation processes can (and should) be considered as \textit{local}, \textit{contingent} subroutines appended to general democratic aggregation protocols in order to increase their effectiveness to cope with certain \textit{specific} sorts of conflicts related to Condorcet cycles.
ii) (Hansson’ s Non-Constancy Theorem (Hansson (1973))) A social welfare function \( f : \mathcal{R}_A^N \rightarrow \mathcal{R}_A \) satisfies IIA, LS and is not inversely dictatorial if and only if \( f \) is dictatorial.

The third theorem focusses instead on the combination of IIA with a definitely compelling and uncontroversial condition, to point out the exceedingly strong and unpalatable restrictions that combination engenders on the admissible social welfare functions.

**Theorem 3.** (Wilson (1972)) Let \( f : \mathcal{R}_A^N \rightarrow \mathcal{R}_A \) be a social welfare function that satisfies IIA and WS. Then, \( f \) is either dictatorial, or inversely dictatorial, or else \( f = f^{UA} \) i.e. \( f \) is the global stalemate social welfare function for \((N, A)\).

**Remark 9.** Notice that the original proof of Hansson’s Non-Constancy Theorem in Hansson (1973) relies in fact on Arrow’s Theorem. Moreover, a careful inspection of that proof makes it clear that the only social welfare function that is neither dictatorial nor inversely dictatorial and satisfies IIA is the Global Stalemate function. In other terms, Hansson’s proof shows that Arrow’s Theorem implies Wilson’s Theorem (it should be recalled here that Hansson’s Non-Constancy Theorem was first published in a 1972 working paper, independently of Wilson’s Theorem). But then, since Wilson’s Theorem obviously implies Arrow’s Theorem, the foregoing observation confirms that the two of them are in fact equivalent.

The fourth theorem shows that combining IIA with two widely accepted conditions for ‘democratic’ aggregation rules such as anonymity (AN) and neutrality (NT) results in a characterization of the global stalemate social welfare function.

**Theorem 4.** (Hansson (1969a)) Let \( f : \mathcal{R}_A^N \rightarrow \mathcal{R}_A \) be a social welfare function that satisfies IIA, AN and NT. Then, \( f = f^{UA} \) i.e. \( f \) is the global stalemate social welfare function for \((N, A)\).

The following alternative characterization of the global stalemate social welfare function combines IIA with two weak conditions following from anonymity and neutrality such as WS and MDR to the effect of emphasizing the inordinate strength of IIA.

**Proposition 4.** A social welfare function \( f : \mathcal{R}_A^N \rightarrow \mathcal{R}_A \) satisfies IIA, WS and MDR if and only if \( f \) is the global stalemate social welfare function i.e. \( f = f^{UA} \).

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22It should be emphasized that Hansson’s Non-Constancy Theorem (which was established independently of Wilson’s Theorem) amounts to replacing the ‘global stalemate’ clause of Wilson’s Theorem with a weaker clause (violation of the LS condition i.e. of ‘Strong Non-Constancy’ in Hansson’s own original terminology). Incidentally, a close inspection of Hansson’s proof shows that it can also be deployed to imply the stronger Wilson’s ‘global stalemate’ clause. See also Malawski, Zhou (1994) and Cato (2012) for related work on preference aggregation without WP.
Proof. \(\implies\) Suppose that social welfare function \(f\) satisfies IIA, WS and MDR. But then, it follows from Wilson’s Theorem as mentioned above that \(f\) is dictatorial, inversely dictatorial, or the global stalemate constant function \(f^{UA}\). However, a dictatorial social welfare function clearly violates MDR: indeed, suppose \(i \in N\) is such that, for every \(x, y \in A\) and \(R_N \in \mathcal{R}_N^N\), \(xf(R_N)y\) entails \(x_{R_i}y\). Moreover, by WS, for every \(x, y \in A\) there exists \(R_N \in \mathcal{R}_N^N\) such that \(xf(R_N)y\), whence \(x_{R_i}y\) holds. Then, by MDR there exists \(S \subseteq N \setminus \{i\}\) and a pair of distinct \(v, z \in A\) such that \(S \in F_f(v, z)\) i.e. there exists \(R_S^vz \in \mathcal{R}_i^N\) such that \(vf(R_N)z\) for every \(R_N \in \mathcal{R}_i^N\) with \(R_S(v, z) = R_S^vz\). Now, consider a profile \(R_N \in \mathcal{R}_A^N\) such that \(z_{R_N}v\), not \(v_{R_N}z\) and \(R_S(v, z) = R_S^vz\). By definition of \(F_f\), \(vf(R_N)z\). However, since \(f\) is dictatorial, not \(v_{R_N}z\) implies not \(vf(R_N)z\), a contradiction. Thus, \(f\) is not dictatorial, as required.

Similarly, suppose there exists \(i \in N\) such that for every \(x, y \in A\) and \(R_N \in \mathcal{R}_A^N\), \(xf(R_N)y\) entails \(y_{R_i}x\). Again, it follows from WS that for every \(x, y \in A\) there exists \(R_N \in \mathcal{R}_A^N\) such that \(xf(R_N)y\), whence \(y_{R_i}x\) holds, by our assumption. Then, by MDR there exists \(S \subseteq N \setminus \{i\}\) and a pair of distinct \(v, z \in A\) such that \(S \in F_f(v, z)\). Now, consider a profile \(R_N \in \mathcal{R}_A^N\) such that \(z_{R_N}v\), not \(v_{R_N}z\) and \(R_S(v, z) = R_S^vz\). By definition of \(F_f\), \(vf(R_N)z\). However, by assumption, not \(z_{R_N}v\) implies not \(vf(R_N)z\), a contradiction. Thus, \(f\) is not inversely dictatorial either. Therefore, it follows from Wilson’s Theorem that \(f = f^{UA}\), the global stalemate function.

\(\iff\) It can be easily shown that the global stalemate social welfare function \(f^{UA}\) satisfies IIA, WS and MDR. Indeed, take any sovereign agenda formation rule \(f, \text{ posit } \mathcal{F}(f) := \{f_B := f^{UB}\}_{B \subseteq A}\), and consider the corresponding PAFE social welfare function \(f = (f, \mathcal{F}(f))\). By definition, \(f_A := f^{UA}\) which obviously satisfies IIA, being a constant function defined on \(\mathcal{R}_A^N\). Thus, by Proposition 2, \(f\) satisfies AMP\(S\) and consequently, by definition, \(f^{UA}\) also satisfies AMP\(S\). Moreover, for any \(x, y \in A\) and \(R_N \in \mathcal{R}_A^N\), \(xf^{UA}(R_N)y\) hence WS is trivially satisfied by \(f^{UA}\). Finally, observe that the responsiveness correspondence \(F_f^{UA}\) is such that \(F_f^{UA}(x, y) = \mathcal{P}(N)\) for all \(x, y \in A\). But then, for any \(x, y \in A\) and any \(i, j \in N, \{N, \{i\}, \{j\}\} \subseteq F_f^{UA}(x, y)\). It follows that \(f^{UA}\) also satisfies MDR.

\(\square\)

Remark 10. Notice that AN and NT do indeed imply WS and MDR, while the converse does not hold: to see this, consider the social welfare function \(f^*\) such that for some \(1, 2, 3 \in N\), and for every \(x, y \in A\), \(R_N \in \mathcal{R}_A^N\), \(xf^*(R_N)y\) iff either \(x_{R_1}y\) or \(\neg x_{R_2}y\). It follows that Proposition 4 amounts to an extension of Hansson’s characterization of the Global Stalemate social welfare function \(f^{UA}\) via AN, NT and IIA (Hansson(1969a)).

Corollary 1. There is no social welfare function \(f: \mathcal{R}_A^N \to \mathcal{R}_A^T\) that satisfies IIA, S and MDR. Thus, in particular, there is no idempotent social welfare function that satisfies IIA and MDR.

Proof. Suppose that on the contrary there exists a social welfare function \(f\) which satisfies IIA, S, and MDR. Since S clearly implies WS, it follows from Proposition 4 above that \(f = f^{UA}\), a contradiction because by definition \(f^{UA}\) does not satisfy S. The second statements follows trivially since any idempotent social welfare function does satisfy S.

\(\square\)
So, we have a further impossibility result that follows just from the combination of IIA and MDR with Sovereignty, with no role at all for the Weak Pareto Principle.

It should also be mentioned that, relying on other well-known results from the extant literature, further elaborations on the role of IIA established by the foregoing theorems can be easily produced. For instance, a further result in Wilson (1972) shows that any social welfare function which satisfies IIA must produce ‘social preferences’ invariably composed by some combination of at most five different types of patches corresponding respectively to ‘locally imposed strict preferences’, ‘minimal (local) stalemates’, ‘non-minimal (local) stalemates’, ‘locally dictatorial preferences’ and ‘locally inversely-dictatorial preferences’. Furthermore, it is also well-known that there exists a quite general model-theoretic rationale underlying such results (see e.g. Lauwers, Van Liedekerke (1995) for details). Namely, it is sufficient to join either IIA and the Weak Pareto Principle (WP) or IIA and Idempotence (ID) to force the set of decisive coalitions of a social welfare function \( f \) for \( (N, A) \) to be an ultrafilter on \( N \), a fact which in turn implies that \( f \) is dictatorial since \( N \) is by assumption finite.

Thus, our results confirm Arrow’s core intuition concerning the relationship of IIA to agenda manipulation-proofness. At the same time, they also circumscribe that relationship to the case of an ‘agenda-first’ sequential coupling of agenda formation and preference aggregation (the AMP\(_S\) version of agenda manipulation-proofness fr SAFE social welfare functions). Specifically, Proposition 4 shows that IIA both ensures AMP\(_S\) for any SAFE social welfare function, and implies a condition (IIAP) that is required to secure AMP\(_S\). It follows that Arrow’s ‘general possibility theorem’ can be regarded as the seminal result of a cluster of theorems showing that reliance on IIA to achieve agenda manipulation-proofness in the AMP\(_S\) version for SAFE social welfare function has an exhorbitant, unacceptable cost. To be sure, it remains to be seen whether relaxing IIA to IIAP makes it possible to achieve AMP\(_S\) without ruling out at all proper consensus-based social welfare functions, and if so to what extent. But in any case, such a cluster of theorems (as combined with Propositions 1 and 3 above) point to PAFE social welfare functions as a feasible working alternative to secure the possibility of reliable and proper consensus-based preference aggregation rules.

3. Discussion.

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23See Wilson (1972), Theorem 5. Binmore (1976) is an interesting further extension of that theorem, showing that in order to avoid the grim consequences of the latter the domain of a social welfare function which satisfies IIA should be dramatically restricted. Specifically, its domain should not include all the preference profiles of total preorders consistent with at least one arbitrarily fixed ‘party structure’ (namely, a partition of agents into ‘parties’ defined as sets of agents whose preferences are at least partially concordant on every pair of alternatives).

24A decisive coalition of a social welfare function \( f \) is any coalition \( C \subseteq N \) that can enforce the unanimous preference of its members between each pair of alternatives as the actual social preference.

25An ultrafilter (or maximal lattice-filter) on \( N \) is a nonempty set \( \mathcal{F} \subseteq \mathcal{P}(N) \setminus \{\emptyset\} \) such that for every \( C, D \in \mathcal{F} \):

(i) \( C \cap D \in \mathcal{F} \)

and (ii) either \( C \in \mathcal{F} \) or \( N \setminus C \in \mathcal{F} \). Since \( N \) is finite, every ultrafilter \( \mathcal{F} \) on \( N \) is principal, namely \( \mathcal{F} = \{C \subseteq N : i \in C\} \) for some \( i \in N \). It follows that (i) is a decisive coalition for \( f \), and consequently \( f \) is a dictatorial social welfare function.
Thus, it is abundantly clear that IIA is a powerful obstruction to each one of the following two basic requirements for any proper consensus-based social welfare function, namely (i) \textit{weak Pareto-optimality} and (ii) \textit{minimally distributed responsiveness}. But the main motivation for introducing IIA is \textit{precisely} the attempt to prevent agenda manipulation of a social welfare function when the elicited preferences only concern alternatives of the agenda. Hence, it is insisting on IIA to ensure agenda manipulation-proofness (as expressed by AMP$_S$) that gives rise \textit{by itself} to a bleak scenario concerning the construction of properly consensual social welfare functions.

Let us then briefly summarize the widely shared views on the import of Arrow’s theorem which typically follow from a firm endorsement of IIA as grounded on the assumption that IIA amounts to a key requirement for \textit{any} reasonable voting rule in order to prevent agenda manipulation. Since Arrow’s IIA is a property shared by several commonly used voting rules including the simple majority rule (arguably, a paragon of ‘democratic’ voting rules), it seems to follow that Arrow’s theorem does validate the following challenging, momentous statement. Namely, the assertion that \textit{any attempt} to use \textit{democratic} voting rules to articulate a \textit{consistent formulation of the collective interest} with a view to identify and select policies which best promote it is \textit{doomed to failure}. That is so precisely because Arrow’s theorem shows that under IIA (\textit{Weak}) \textit{Pareto optimality can only be achieved through dictatorship}. Therefore, since dictatorship is obviously to be rejected as a means to define proper consensus-based ‘social preferences’, insisting to prevent agenda manipulation (specifically, agenda-content manipulation) entails reliance on some aggregation rule which might license ‘social preferences’ that occasionally reverse unanimously held individual strict preferences between some pairs of alternatives. And that is also deemed to be not acceptable. But then, the only alternative left is \textit{to allow for agenda manipulation} (specifically, agenda-content manipulation) to the effect of undermining \textit{reliability} of the aggregation rule, since the representation of the ‘general interest’ provided by such a rule will typically reflect just successful manipulable activities, and possibly nothing else. Either way, the aim of producing a consistent, faithful and credible formulation of the ‘general interest’ cannot be apparently fulfilled. To put it bluntly, under that view majority voting cannot be relied upon to discover the public interest or ‘general will’ because of its possible cycles, and \textit{nothing else can work} because of the same vulnerability to agenda manipulation. As a consequence, in actual practice \textit{there is apparently not such a thing as ‘general interest’ to discover, express and implement as a guide or benchmark for public policy} (see e.g. Schwartz (1970)). It also follows, accordingly, that there is apparently no way to feed work in mechanism design and/or institutional design with well-grounded and reliable criteria summarizing the ‘general interest’, aimed at improving the effectiveness of democratic institutions.

\footnote{Riker (1982) is probably the most outspoken and consistent presentation of such a view available in print. While Dahl (1956) tentatively labeled ‘populist democracy’ the doctrine that identifies the exercise of sovereignty with exclusive and unrestrained reliance on majority voting, Riker assumes that Arrow’s theorem licences the imputation of the same, allegedly hopeless, limitations of the majority rule to every possible ‘democratic’ preference aggregation rule. Consequently, virtually all the social-choice-theoretic work in mechanism design is unceremoniously identified with ‘populism’ and dismissed as a hopeless endeavour. Riker’s suggested and clearly preferred alternative is acceptance of the ‘Schumpeterian’ view that modern ‘democratic politics’ is -and has to be- nothing else than (i) competitive electoral selection of
That scenario has been variously described as the impossibility of ‘rational’ collective decisions, or the impossibility of a reliable and significant consensus-based expression of the ‘collective good’ (or ‘general interest’ or ‘public will’). This is in fact the most familiar understanding of Arrow’s ‘impossibility theorem’. Such an interpretation projects a ‘dark’ view of the content and significance of Arrow’s theorem concerning the viability and effectiveness of democratic protocols since it suggests, in short, that there is no way to use voting methods, decision systems or preference aggregation rules of any sort to help improving the effectiveness and deliberative quality of current democratic protocols.

Notice however that, again, all of the above rests crucially on the working assumption that IIA is both reasonable and virtually inescapable.

But then, Propositions 1 and 3 show that there exists in fact an alternative way to achieve agenda manipulation-proofness via $AMP_P$: such an alternative makes it possible to devise anonymous and idempotent social welfare functions that satisfy a basic version of the Pareto principle, and are—in a compelling sense—both agenda manipulation-proof and strategy-proof. Thus, there is indeed an effective way out of the strictures identified by Arrow’s theorem. From that perspective, Arrow’s result actually provides constructive information about the design of social welfare functions and preference aggregation rules: in that sense, there is also a bright side of Arrow’s theorem. Access to the latter requires three basic steps:

(i) reliance on (possibly redundant) preference elicitation concerning an entire set of prefixed admissible alternatives in order to ensure agenda manipulation-proofness without any recourse to IIA;

(ii) a mild relaxation of the Pareto Principle to BP allowing for occasional stalemates (namely, social indifference over a set of alternatives including Pareto-dominated outcomes), and a concurrent reinterpretation of possible violations of WP and other, stronger, versions of the Pareto principle as ‘warning signals’ pointing to the need for remedial actions including policies to correct blatant disparities of access to information and/or other key resources;

The ruling elite, and (ii) pervasive and relentless activities of agenda manipulation on the part of elected officials and representatives, in view of more or less special interests, with no effective role left for a public representation of the common interest as a shared, consensus-based benchmark. Accordingly, analyzing ‘democratic politics’ from such a perspective is regarded as the main task of ‘democratic theory’ proper, and the plain endorsement of that view is the defining feature of ‘liberalism’ (see Riker (1982), and Schofield (1985) for a thorough technical treatment of social choice theoretic models in a multidimensional spatial setting that shares at least some of Riker’s views, while refraining from the latter’s most ideological overtones).

27 See e.g. Buchanan (1954) for an early example of that view, insisting on the alleged impossibility of collective decisions replicating the ‘rationality’ of individual decisions. A much more elaborated argument to a similar effect is offered by Bordes, Tideman (1991), showing that IIA is a consequence of a strong restriction on general voting rules called ‘regularity’ that is however scarcely compelling unless systematic irredundancy of preference elicitation is again tacitly assumed. A somewhat more balanced view, advocating a combination of IIA with less demanding criteria of ‘rationality’ for both collective and individual decisions is advanced by Schwartz (1986).

28 See e.g. the highly influential Riker (1982) as discussed above (footnote 25).

29 An influential tentative list of basic, substantive requirements for democratic decision protocols including ‘political equality’, ‘deliberation’, ‘participation’, and ‘agenda control’ is due to Dahl (1979).

30 See for instance Schwartz (1986), p.33 or Bordes, Tideman (1991)) for a remarkably clear, adamant endorsement of that view.
(iii) refocussing on a further condition (‘monotonic $M_X$-independence’) in order to address strategic manipulation issues: such a condition can be regarded as a combination of a mild monotonicity property and a considerably weakened version of IIA.

Broadly speaking, some form of each one of the foregoing steps was previously considered or at least evoked in the extant literature (a discussion of that matter is provided in the next section). What is new here is, arguably, their joint consideration as made possible by a model that combines agenda formation and preference aggregation. The resulting analysis shows that a sound parallel-coupling of (‘full domain’) social welfare functions to their own agenda-formation processes makes it possible to jointly achieve anonymity, respect for unanimity, a basic form of Pareto optimality, and agenda manipulation-proofness together with strategy-proofness on a suitably large and very natural domain of single-peaked ‘meta-preferences’ (or ‘preferences on preferences’). As observed above, it is also remarkable that the latter strategy-proofness property turns out to be essentially equivalent to an ‘independence’ condition which amounts to a considerable weakening of IIA.

3. Related work

As mentioned in the introduction, agenda manipulation-proofness was used by Arrow as the main motivation for introducing IIA as a basic requirement for social welfare functions in his seminal work (Arrow (1963)). Since then, it has become quite common to use ‘Arrowian’ as a qualifier for social welfare functions or aggregation rules which satisfy some version of ‘independence of irrelevant alternatives’ (and possibly some further basic requirement such as idempotence i.e. ‘respect for unanimity’). Hence, the amount of literature which is broadly related to the topics covered by the present paper is simply enormous. Therefore, we shall confine ourselves to a very brief comment focussing on the contributions that raise points most strictly related to the content of the present work (the interested reader is addressed to the supplementary Appendix for a more extensive and detailed discussion).

Monjardet (1990) first introduces the notion of (monotonic) $M_X$-independence presenting it as a generalized counterpart of IIA, but is not concerned with nonmanipulability properties of any sort. Dietrich, List (2007b) shows the equivalence between a similar notion of monotonic independence as

31 Remarkably, Przeworski (2011) suggests both relaxations of IIA and some judicious use of agenda-structure manipulation as possible remedies to cope with possible violations of WP, and secure a reliable representation of the collective interest based on ‘proximity’ to individual preferences. However, his somewhat informal treatment leaves it unexplained whether or not he also proposes to combine such remedial moves (and if so, how).

32 See, among many others, Aleskerov (1999), Sethuraman, Teo, Vohra (2003), Nehring, Puppe (2010). An alternative (and perhaps more appropriate) usage is the one that rather contrasts Arrowian (or multi-profile) and Bergson-Samuelson (or single-profile) social welfare functions, and goes as follows. Let $N, A$ be two (finite) sets and $\mathcal{R}_A$ the set of all total preorders on $A$. An Arrowian social welfare function for $(N, A)$ is a function $f : \mathcal{R}_N^N \to \mathcal{R}_A$, while a Bergson-Samuelson social welfare function for $(N, A)$ is a function $f : \{r^N\} \to \mathcal{R}_A$, with $r^N \in \mathcal{R}_N^N$. Moreover, their strict counterparts are obtained by replacing $\mathcal{R}_A$ with $\mathcal{L}_A$ i.e. the set of all linear orders (namely, antisymmetric total preorders) on $A$. 


specialized to propositions of a certain class of formal languages and strategy-proofness of judgment aggregation rules. Savaglio, Vannucci (2019) shows the equivalence of monotonic $M_X$-independence and strategy-proofness for arbitrary aggregation rules in bounded distributive lattices, and Vannucci (2019) applies that equivalence to rating aggregation rules. Nehring, Puppe (2007, 2010) rely on (finite) property spaces to define a class of generalized IIA properties by reducing each one of them to a specific family of binary issues. Thus, IIA and $M_X$-independence are easily obtained as special cases of that construct (but the foregoing papers do not consider the latter property, and are not concerned with either agenda manipulation-proofness or strategy-proofness of social welfare functions). Neither agenda manipulation-proofness nor strategy-proofness properties are considered in Huang (2014). However, that work provides an escape route from Arrow’s ‘general possibility theorem’ relying on the combination of a newly defined weakened version of IIA and the same basic Pareto optimality condition employed in the present work, to the effect of allowing stalemates (or ‘singularities’ in Huang’s own terminology).

Sato (2015) is partly concerned with the relationship between agenda manipulation-proofness and IIA, but only considers a special class of strict social welfare functions enjoying a certain continuity property (‘bounded response’). Curiously enough, the preference-approval aggregators studied in Kruger, Sanver (2021), that are not meant to address agenda manipulation issues but rather an entirely different problem (the joint aggregation of rankings and binary ratings), are in fact isomorphic to a restricted-domain version of our PAFE social welfare functions: such a domain-restriction is indeed necessary to capture the required consistency between individual rankings and ratings. Finally, it should also be noticed that the replacement of IIA with (monotonic) $M_X$-independence that is suggested in the present work amounts in general to a remarkable change and enlargement of the information-base that is made available for the definition of the relevant social welfare function. That is so because when expressed in terms of properties of $R_A$ with $|A| = m \geq 3$, the size of the set of actually relevant binary issues (for each individual preference relation of any profile) is $m(m - 1)$ for IIA, and $2(2^{m-1} - 1)$ for $M_X$-Independence. That move is of course consonant with a long standing argument in the social choice-theoretic literature, that counts Sen (2017) and Saari (2008) among its most committed and distinguished advocates. Yet, the present enlargement rests on a peculiar combination of two characteristic features that single it out from many other proposed enrichments of the information-base. Namely, it relies on a standard input-format consisting of profiles of total preorders, and at the same time exploits the structure of the set of total preorders themselves i.e. the objects to be aggregated.

4. Concluding remarks

The IIA condition for preference aggregation rules was first introduced by Arrow in order to ensure their agenda manipulation-proofness, but when combined with a few minimal reasonable conditions it results in a characterization of dictatorial social welfare functions. That is the basic content of Arrow’s general possibility theorem. Under the assumption that IIA is indeed the only way to block agenda manipulation, that theorem does also imply that reliable and proper consensus-based social welfare functions do not exist. Now, that interpretation projects a negative, disagreeable shadow on the perceived consequences of Arrow’s theorem since it suggests that no meaningful consensus-based
formulation of ‘general interest’ is available as a guide and benchmark to promote and assess public decisions and policies, and improve the design and/or implementation of democratic protocols. Thus, it is not unfair to describe all that as the ‘dark’ side of Arrow’s theorem. The present work shows however that, as a matter of fact, agenda manipulation-proofness of a social welfare function is indeed available without any appeal to IIA provided that agenda formation and preference elicitation are simultaneous. In the latter case some anonymous, idempotent, agenda manipulation-proof, minimally efficient and weakly-neutral social welfare functions do exist. Moreover, a much relaxed independence condition that they do satisfy ensures their strategy-proofness as well. But then, from the perspective provided by such positive results, Arrow’s theorem may also be regarded as a most constructive contribution to the design of preference aggregation rules, in that it suggests that agenda formation and preference elicitation are better not coupled sequentially. That is precisely the ‘bright’ side of Arrow’s theorem that the present work is meant to highlight and emphasize.

To be sure, the consensus-based, agenda manipulation-proof, and strategy-proof preference aggregation rules we have shown to be available require a significant increase of the amount of information to be extracted from preference profiles, and processed. Thus, reliance on such aggregation rules also involves a careful consideration of computational complexity issues, and their implications. Moreover, while individual strategy-proofness issues concerning social welfare functions have been also considered in the present work, the further problems arising from coalitional strategy-proofness requirements for preference aggregation rules have been deliberately put aside. Finally, since we have shown that IIAP only (a weaker version of IIA) is actually necessary to ensure ‘sequential’ agenda manipulation-proofness in the required sense, it remains to be seen whether further interesting aggregation rules are consistent with IIAP. Such most significant and intriguing issues are however best left as challenging topics for future research.

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33Indeed, it is well-known that the computation of median total preorders for arbitrary profiles of total preorders is NP-complete (i.e. it belongs to the class of the hardest problems whose solutions are polynomial-time verifiable or ‘easy’ to verify, but apparently worst-case ‘hard’ to compute). Specifically, if the size of $N$ is suitably larger than the size of $A$, computing a median total preorder is NP-complete for arbitrary profiles of total preorders or linear orders (Hudry (2012)) and NP-hard (i.e. ‘easy’ to reduce to a NP-complete problem) for arbitrary profiles of binary relations (Wakabayashi (1998)).

34Concerning the relationships between individual and coalitional strategy-proofness for general aggregation rules see e.g. Vannucci (2016).
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5. Appendix

The present supplementary Appendix provides a rather detailed discussion of some related previous contributions collecting them under two distinct subsections that correspond to two focal points of the present analysis both related to IIA, namely ‘Agenda manipulation-proofness and IIA’ and ‘Strategy-proofness and weakenings of IIA’.

(I) Agenda manipulation-proofness and IIA.

The role of preference elicitation on the entire set of admissible alternatives in order to ensure transitivity properties of the ‘social preferences’ and the resulting violation of IIA has been repeatedly pointed out, and contrasted with peacemeal elicitation of preferences on specific agendas of admissible alternatives, which is conducive to IIA-consistency and violation of transitivity properties of ‘social preferences’ (see e.g. Sen (1977) where that contrast is discussed with reference to several versions of the simple majority rule and the Borda Count scoring rule). Unfortunately, preference elicitation and agenda formation are typically not modelled together in the extant literature: specifically, agenda manipulation is usually left unmodelled and thus given a quite informal treatment. As a consequence, even the agenda-content and agenda-structure dimensions of agenda manipulation are typically not neatly and consistently distinguished. Thus, following the lead of a much discussed, misleading example proposed in Arrow (1963), use of the label ‘IIA’ has also been occasionally stretched to also refer (improperly) to requirements on social preference rankings on a certain subset of alternatives across several distinct admissible agendas for a fixed profile of individual preferences on the largest admissible agenda (see e.g. Ray (1973), Fishburn (1973), Sen (1977), Schwartz (1986), Bordes, Tideman (1991), Young (1995) for discussions related to such topic). Furthermore, without an explicit joint modelling of agenda formation and preference elicitation it is virtually impossible to distinguish not only between parallel coupling and sequential coupling of those two processes, but also between preference-first and agenda-first sequential coupling. In the previous section of the present work it has been shown that parallel-coupling allows for agenda manipulation-proofness of social welfare functions without any recourse to IIA, while under agenda-first sequential coupling agenda manipulation-proofness is in fact strictly related to IIA. But then, what about preference-first sequential coupling of agenda formation?

A partial exception due to Dietrich (2016) is available in the related framework of judgment aggregation (to be discussed below) where agenda manipulation is modeled as sensitivity of aggregate judgments on issues to agenda-content alterations (including expansions), with no explicit role for preferences. Notably, that notion of agenda manipulation-proofness is shown to be tightly connected to a version of IIA, and results in a characterization of dictatorial judgment aggregation rules when combined with a unanimity-respecting condition for general finite agendas.

Actually, Young (1995) also discusses at length a condition he calls ‘local stability’ or local independence of irrelevant alternatives (LIIA) which is also satisfied by median-based aggregation rules (but not by positionalist rules such as the Borda Count). When applied to a social welfare function \( f : \mathcal{R}_A^N \rightarrow \mathcal{R}_A \) LIIA may be formulated as follows:

\[
I_R := \left\{ I \subseteq A : I = \{ z : xRzRy \} \text{ for some } x, y \in A \right\},
\]

Thus LIIA is in fact an intraprofile property, rather than an interprofile property like IIA and its relaxed versions (again, see Fishburn (1973) for a classic, exhaustive classification of standard social choice-theoretic properties for preference aggregation rules).
and preference elicitation if outputs are just ‘social choice sets’ out of the entire admissible set that might be not representable as optima of an underlying total preorder of social preferences? Under such circumstances, the possibility of agenda-structure manipulation re-enters the picture. In that connection, the main theorem of Hansson (1969b) concerning generalized social choice correspondences (GSCCs) \(^{37}\) and its reformulation and extension due to Denicolò (2000) are indeed relevant and most helpful. In fact, the foregoing Hansson’s result relies on an extended version of IIA for GSCCs that are not necessarily generated through maximization of the total preorders that express social preferences. Specifically, it implies that any social choice correspondence \(F\) on a set \(A\) (with \(|A| \geq 3\)) that satisfies WP and such an extended IIA property can be represented as the choice of maxima of the social preferences in the range of a social welfare function \(f\) which satisfies IIA and WP if and only if both \(F\) and \(f\) are dictatorial (see Hansson (1969b), Theorem 3 \(^{38}\)). In a similar vein, Ferejohn, McKelvey (1983) shows that even substituting transitivity and totality of social preferences with social choice sets that are Von Neumann-Morgenstern solutions of an asymmetric social dominance/preference relation, insistence on IIA (in a slightly strenghtened monotonic version that implies WP, actually) results in the alternative between allowance for at least one agent with unlimited veto power and allowance for agenda-structure manipulation as defined in the Introduction.

(II) ‘Strategy-proofness and weakenings of IIA’. The other major theme in the present work is that, once agenda manipulation-proofness of properly consensus-based social welfare function is secured through parallel-coupling of agenda formation and preference elicitation (with no role at all for IIA), the strategy-proofness issue for such social welfare functions does also admit a sensible formulation and a positive solution. Specifically, the latter requires just (a) focussing on the ‘right’ individual preferences (which must be preferences on the outcomes of a social welfare function, hence preferences on social preferences over outcomes i.e. ultimately ‘meta-preferences on basic preferences’) and (b) observing that basic preferences on alternatives induce in a natural way single-peaked ‘meta-preferences’ on the ‘preference space’ which in turn ensure strategy-proofness of the proper consensus-based social welfare functions mentioned above. Moreover, it turns out that in such a setting strategy-proofness is in fact equivalent to the combination of a very mild monotonicity condition on the influence of coalitions (namely the requirement that adding support to a previously positive decision on a certain binary

\(^{37}\)Or ‘group decision functions’ in the original terminology of Hansson (1969b).

A generalized social choice correspondence for \((N, A)\) is a function \(f : \mathcal{R}_N^N \rightarrow C_A\) where \(A \subseteq \mathcal{P}(A) \setminus \{\emptyset\}\) with \(A \in A\) and \(C_A\) is the set of all functions \(C : A \rightarrow \mathcal{P}(A) \setminus \{\emptyset\}\) such that \(C(B) \subseteq B\) for every \(B \in A\).

A social choice correspondence for \((N, A)\) is a function \(f : \mathcal{R}_A^N \rightarrow \mathcal{P}(A) \setminus \{\emptyset\}\) i.e. a generalized social choice correspondence such that \(A = \{A\}\).

A social choice correspondence for \((N, A)\) whose range consists of singleton-sets is also said to be a social choice function, and usually written \(f : \mathcal{R}_A^N \rightarrow A\).

\(^{38}\)See also Denicolò (2000) for a simplified presentation of Hansson’s theorem, and a detailed formulation of its consequences for social choice correspondences and social welfare functions as just mentioned in the text.
issue should never result in a decision reversal) and an *independence* condition that amounts to a *much weakened version of IIA*.

Thus, in a sense, a certain version of IIA ultimately reenters the picture but (i) in a *much weakened* and *very specific form* and (ii) with reference to *strategy-proofness*, an issue that (as opposed to agenda manipulation-proofness) was explicitly *put aside* in the original Arrowian analysis of social welfare functions (see Arrow (1963), p.7).

Now, both of those tenets run counter to some views that are apparently still widely held in the literature, and to which we now turn. To begin with, the exceptional strength of IIA is sometimes downplayed or in any case not fully appreciated. One reason for that may be the (correct) perception of the relationship of IIA to agenda-content manipulation-proofness as combined with the (incorrect) view that sequential-coupling of agenda formation to preference elicitation is the only available possibility. It is also possibly the case that IIA is occasionally confused with its earlier counterpart named ‘Postulate of Relevancy’ which is due to Huntington (1938), and is explicitly quoted by Arrow himself as a source of inspiration and ‘a condition analogous to’ IIA (Arrow (1963), p. 27). Notice, however, that while Huntington’s ‘Postulate of Relevancy’ may well be quite similar in spirit to IIA, it is in fact *much weaker* than the latter because it relies on a *common language of linearly ordered grades* (indeed, numbers) to express *absolute judgments* (as opposed to merely comparative ones). A third line of reasoning in support of IIA originates from a misleading interpretation of a well-known theorem due

39To be sure, it is also well-known that IIA is *so strong* that there are also weakened versions of IIA which imply *dictatorship* for the relevant preference aggregation rule. An obvious example concerning social welfare functions is the restriction of IIA to subsets of alternatives of a fixed cardinality (see Blau (1971)). More significantly, there are certain consequences of IIA (weaker than IIA itself) that imply dictatorship of preference aggregation rules when coupled with the Weak Pareto condition or indeed any *non-constancy constraint*. Notice that this fact holds not only for social welfare functions, but also for preference aggregation rules admitting any *total binary relation* as their output (with no transitivity or ‘consistency’ requirement at all!). That is the case of so-called *Independent Decisiveness* of aggregation rule *f* requiring that any coalition which is able to enforce its strict preference over a certain ordered pair of alternatives (*x*, *y*) for some preference profile (no matter what the preferences of others over *x*, *y* happen to be) must also be *decisive* for (*x*, *y*): see Sen (1993), Denicolò (1998), Quesada (2002). For another example of an Arrow-like theorem for aggregation rules in the same vein (albeit in a much more general setting) see Daniëls, Pacuit (2008).

40Indeed, there is arguably no other way to make full sense of the following statement from a well-known and highly respected scholar: ‘Independence of Irrelevant Alternatives and therewith Binary Independence are eminently reasonable assumptions to make in a realistic study of collective choice. I know of no real-world collective-choice process that violates either condition. Both formalize the idea that collective choices depend only on such preferential data as could be revealed by voting.’ (Schwartz (1986), 33). In that connection, similar comments apply to Bordes, Tideman (1991).

41The examples considered by Huntington (1938) concern in fact competing teams of equal size, and the relevant numbers/scores are uniquely determined by the measurement of individual performances of each team’s members. Thus, the alternatives to be ranked are teams, while the agents are the *shared classifiers* for distinct members of each team (e.g. *first* members of some team, *second* members of some team, and so on). Huntington essentially contrasts team ranking by aggregation of members’ ratings and members’ rankings, respectively, observing that the former method typically *does satisfy* the ‘Postulate of Relevancy’ while the latter *does not*.

42*Majority judgment* as recently introduced by Balinski and Laraki (Balinski, Laraki (2011)) denotes a family of aggregation and voting mechanisms which typically satisfy the ‘Postulate of Relevancy’ while violating IIA (see also Vannucci (2019) for a detailed discussion of strategy-proofness properties of majority judgment).
to Satterthwaite (1975) that establishes a tight connection between strategy-proof *strict* social choice functions and *strict* social welfare functions that satisfy IIA. Indeed, Satterthwaite himself claims that such a theorem ‘creates a strong new justification for [WP and] IIA as conditions that an ideal social welfare function should satisfy’ (Satterthwaite (1975), p. 207, with some minor editing of mine). Notice, however, that the ‘IIA-nonmanipulability’ connection identified and discussed by Satterthwaite concerns IIA as a property of a strict social welfare function and nonmanipulability of the strict social choice function attached to the former, and such nonmanipulability amounts to strategy-proofness (and obviously not agenda manipulation-proofness) of the latter. To be sure, further interesting elaborations on such connections between IIA and nonmanipulable aggregation rules are provided in Sato (2015). Specifically, Sato considers four notions of nonmanipulability for strict social welfare functions in order to formulate both agenda manipulation-proofness and strategy-proofness requirements, respectively. Then, relying on the Kendall metric for linear orders, he introduces a weak continuity condition for strict social welfare functions called Bounded Response. The main result of Sato (2015) implies the equivalence of the following statements concerning a *strict* social welfare function $f$ for $(N,A)$: (1) $f$ satisfies Bounded Response and at least one of the four distinct agenda manipulation-proofness or strategy-proofness conditions mentioned above; (2) $f$ satisfies Bounded Response and each one of the foregoing nonmanipulability conditions; (3) $f$ satisfies Adjacency-restricted Monotonicity (AM) and the Arrowian IIA condition. Thus, even factoring in AM (a very mild requirement that is virtually undisputable) it turns out that IIA is in particular a necessary condition of strategy-proofness only for a specific class of ‘weakly continuous’ and *strict* social welfare functions. In short, a closer inspection of both Satterthwaite (1975) and Sato (2015) confirms that the most interesting results they contribute
are in fact silent on necessary and/or sufficient conditions for strategy-proofness of general, unrestricted social welfare functions.

All of the above suggests that both agenda manipulation-proofness and strategy-proofness of a proper consensus-based social welfare function do indeed require that IIA be either just dropped or at the very least considerably relaxed.

The independence condition used in the present paper, namely $M_X$-Independence, can be indeed regarded as a drastic relaxation of IIA when applied to social welfare functions. It was first introduced by Monjardet (1990) and explicitly related to IIA and Arrowian aggregation models, but not at all to strategy-proofness issues (or, for that matter, to agenda manipulation-proofness issues).

Unsurprisingly, several alternative weakenings of IIA have been proposed in the earlier literature. An entire set of substantially relaxed versions of IIA was first introduced and discussed by Hansson (1973) with no reference whatsoever to nonmanipulability issues of any sort. The strongest of them (i.e. the least dramatic relaxation of IIA, denoted by Hansson as Strong Positionalist Independence (SPI)) requires invariance of aggregate preference between any two alternatives $x, y$ for any pair of preference profiles such that their restrictions to $\{x, y\}$ are identical, and for every agent/voter the supports of the respective closed preference intervals having $x$ and $y$ as their extrema are also identical. Incidentally, SPI has been recently rediscovered, relabeled as Modified IIA, and provided with a new motivation by Maskin (2020). Indeed, Maskin points out that SPI enforces resistance of the relevant aggregation rule to certain sorts of ‘vote splitting’ effects, thereby connecting SPI to manipulation issues, including strategic manipulation. Notice, however, that Maskin’s proposal is aimed at strategy-proofness of the ‘maximizing’ social choice function induced by a certain social welfare function (as opposed to strategy-proofness of the social welfare function itself). In a similar vein, another weakening of IIA that is even stronger than SPI has been proposed by Saari under the label ‘Intensity form of IIA’ (IIIA). IIIA requires invariance of aggregate preference between any two alternatives $x, y$ for any pair of preference profiles such that for every agent/voter the rank (or score) difference between $x$ and $y$ is left unchanged from one profile to the other (see Saari (1995) and (1998)).

A further weakening of IIA in a quite different vein is due to Huang (2014), under the label Weak Arrow’s Independence (WIIA). In plain words, a social welfare function $f$ satisfies WIIA if, for any pair $R_N, R_N'$ of profiles of total preorders and any pair $x, y$ of alternatives such that the preferences

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48 It should be noted, however, that conditions strictly related to $M_X$-Independence are deployed in Dietrich, List (2007b) to study strategy-proofness properties in judgment aggregation as discussed below in the present section.

49 Afriat (1987) -first appeared as a 1973 conference paper- is also to be credited for an early criticism of IIA. That criticism is mainly motivated by the understanding of an Arrowian social welfare function as a way of modeling the process of ‘voting for an order’ and, again, with no explicit reference whatsoever to agenda manipulation issues.

50 The label comes from the fact that SPI is of course satisfied by ‘positionalist’ or score-based aggregation rules including the Borda Count rule (which assigns to every alternative $x$ a score given by the sum of its individual ranks, defined as the sizes of the sets of alternatives which are classified as strictly worse than $x$ itself).

51 Arguably, Saari’s IIIA can also be regarded as a formalization of the criticism of IIA originally advanced by Dahl (1956) with his advocacy of aggregation rules based on intensity of individual preferences. Notice that IIIA is indeed satisfied by some positional aggregation rules such as the Borda Count but also by majority judgment as discussed above.
between $x$ and $y$ of every agent $i$ in $N$ are the same in $R_N$ and $R'_N$, the following condition holds: if $x$ is strictly preferred to $y$ according to social preference $f(R_N)$ then $x$ is preferred (i.e. either strictly preferred or indifferent) to $y$ according to social preference $f(R'_N)$. Notice the main difference between WIIA and virtually all of the other weakenings of IIA considered in the present work: while the other weakenings strengthen the hypothetical clause of IIA and leave its consequent unaltered, WIIA keeps the hypothetical clause of IIA unaltered and weakens its consequent. It should also be emphasized that the overt motivation of WIIA is just finding an escape route from the strictures of Arrow’s theorem without any explicit consideration of agenda manipulation-proofness or strategy-proofness properties.

On the other hand, it is worth mentioning that Huang’s positive proposal for social welfare functions (what he denotes as ‘Weak Arrow’s Framework’) consists in replacing IIA and WP with WIIA and BP (the Basic Pareto principle -or Weak Pareto Condition in Huang’s own terminology- which is also used in our Proposition 3), and allowing for social indifference classes that contain both a strictly Pareto dominated alternative and some of its strict Pareto improvements (such classes are precisely the stalemates previously considered in the present work, that are denoted as singularities by Huang).

Remarkably, even at a first glance one conspicuous difference between $M_X$-Independence and SPI (or IIIA and WIIA) stands out immediately: the former relies heavily on the structure of the outcome set, while SPI, IIIA and WIIA only impinge upon the relevant preference profiles, completely disregarding any specific feature/structure of the relevant outcome set (namely the set of all total preorders of the set $A$ of basic alternatives).

This crucial difference and its significant import can be further clarified and fully appreciated by reconsidering all the relaxations of IIA mentioned above from the common perspective of ‘aggregation by binary issues’ that encompasses them all.

The binary aggregation model originates with Wilson (1975) and has been further extended by Rubinstein, Fishburn (1986) a finite number of $k$ issues are considered for a collective yes/no judgment (the output) to be based on some profile of individual yes/no judgments on each issue (the input), under some feasibility constraints (usually the same, but possibly different) imposed, respectively, on inputs and outputs. Thus, the basic aggregation rules for a set $N$ of agents and a set $K = \{1, \ldots, k\}$ of binary issues are given by functions $f : X^N \to X$ with $X \subseteq \{0, 1\}^K$. This model has also been shown to be equivalent to the basic model of judgment aggregation where the judgments to be aggregated amount to acceptance/rejection of every element of an agenda of interconnected formulas of a suitable formal language representing propositions (see Dokow, Holzman (2009)). Indeed, several versions of Arrow’s general (im)possibility theorem for social welfare functions have been explicitly shown to follow as a special interesting case under both the feasible binary aggregation and the judgment aggregation

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52 To be sure, the original work by Wilson only considers the finite case, but Wilson’s framework can also be extended to an infinite number of issues, and to non-binary issues (see Dokow, Holzman (2010c)). Indeed, one such extension is covered in Rubinstein, Fishburn (1986). Since the present paper is only concerned with finite social welfare functions, however, we shall only consider the basic binary aggregation model with a finite number of issues.

53 In more recent contributions coming from the computational social choice and artificial intelligence research communities ‘integrity constraints’ is the most commonly used label to denote such constraints (see e.g. Grandi, Endriss (2013)).
frameworks (see e.g. Dokow, Holzman (2010a, 2010b) for the former and Dietrich, List (2007a), Van Hees (2007), Mongin (2008), Daniëls, Pacuit (2008), Porello (2010) for the latter).

An additional and most convenient perspective for the finite version of the binary aggregation model of our concern here is provided by some joint work of Nehring and Puppe (see in particular Nehring, Puppe (2007),(2010)). To be sure, Nehring, Puppe (2007) is mainly concerned with *strategy-proof social choice functions* as defined on profiles of total preorders on finite sets. Conversely, Nehring, Puppe (2010) is focussed on an ‘abstract’ class of Arrowian aggregation problems including preference aggregation and, more specifically, social welfare functions, but it does not address issues concerning their strategy-proofness properties. However, social choice functions with the top-only property may be regarded as aggregation rules endowed with a specific domain of total preorders, and the class of Arrowian aggregation rules considered in Nehring, Puppe (2010) does include the case of preference aggregation rules in finite median semilattices. Specifically, Nehring and Puppe attach to any *finite* outcome space X a certain finite hypergraph \( \mathbb{H} = (X, \mathcal{H}) \) denoted as *property space*, where the set \( \mathcal{H} \subseteq \mathcal{P}(X) \setminus \{ \emptyset \} \) of (nonempty) hyperedges or *properties* of outcomes/states in X is complementation-closed and separating (namely \( H^c := X \setminus H \in \mathcal{H} \) whenever \( H \in \mathcal{H} \), and for every two distinct \( x, y \in X \) there exists \( H_{x^+y^-} \in \mathcal{H} \) such that \( x \in H_{x^+y^-} \) and \( y \notin H_{x^+y^-} \)). Such a property space \( \mathbb{H} \) models the set of all binary properties of outcomes that are regarded as relevant for the decision problem at hand. Thus, *binary issues are modeled here as pairs* \( (H, H^c) \) of complementary properties and, as it is easily checked, both the feasible binary aggregation and the judgment aggregation models can be immediately reformulated as aggregation models in property spaces. Then, a betweenness relation \( B_{\mathbb{H}} \subseteq X^3 \) is introduced by stipulating that \( B_{\mathbb{H}}(x, y, z) \) holds precisely when y satisfies all the properties shared by x and z. Moreover, *single-peaked* preference domains on X can be defined relying on \( B_{\mathbb{H}} \). In particular, \( B_{\mathbb{H}} \) is said to be *median* if for every \( x, y, z \in X \) there exists a unique \( m_{xyz} \in X \) such that \( B_{\mathbb{H}}(x, m_{xyz}, y), B_{\mathbb{H}}(x, m_{xyz}, z), \) and \( B_{\mathbb{H}}(y, m_{xyz}, z) \) hold. Of course, a main advantage of that second-order representation of binary issues is the possibility to focus on several different property spaces which are defined on the very same ground set of alternative states.

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54 In particular, Mongin (2008) introduces a specific weakening of IIA for the standard judgment aggregation model, by restricting the scope of IIA to atomic propositional formulas, and still obtains a version of Arrow’s (im)possibility theorem under WP. As previously mentioned, Daniëls, Pacuit (2008) offers another characterization of dictatorial rules in a quite general judgment aggregation framework using just some consequences of IIA as combined with non-constancy and neutrality conditions. Furthermore, it has been shown that Arrowian characterizations of dictatorial aggregation rules by IIA and idempotence hold for other disparate domains including arbitrary single-valued choice functions on finite sets (Shelah (2005)) and task assignments (Dokow, Holzman (2010c)).

55 A social choice function \( f : (R_A)^N \rightarrow A \) satisfies the *top-only property* if \( f(R_N) = f(R'_N) \) whenever \( t(R_i) = t(R'_i) \) for each \( i \in N \), and \( |t(R_i)| = |t(R'_i)| = 1 \) for all \( i \in N \) (with \( t(R) := \{ x \in A : x R y \text{ for all } y \in A \} \)).

56 In particular, a nonempty subset \( Y \subseteq X \) is said to be *convex* for \( \mathbb{H} = (X, \mathcal{H}) \) if for every \( x, y \in Y \) and \( z \in X \), if \( B_{\mathbb{H}}(x, z, y) \) then \( z \in Y \), and *prime* (or a halfspace) for \( \mathbb{H} \) if both \( Y \) and \( X \setminus Y \) are convex for \( \mathbb{H} \) and \( \{ Y, X \setminus Y \} \subseteq \mathcal{H} \).

57 In that case, \( \mathbb{H} \) is said to be a *median property space*, \( (X, m_{\mathbb{H}}) \) (where \( m_{\mathbb{H}} : X^3 \rightarrow X \) is defined by the rule \( m_{\mathbb{H}}(x, y, z) = m_{xyz} \) for every \( x, y, z \in X \) is a *median algebra*, and for each \( u \in X \) the pair \( (X, \vee_u) \) (where \( x \vee_u y = y \) iff \( m_{\mathbb{H}}(x, y, u) = y \) for some \( u \in X \)) is a *median join-semilattice* having \( u \) as its maximum.)
The following key results are obtained by Nehring and Puppe: (i) the class of all idempotent social choice functions which are strategy-proof on the domain of single-peaked preferences thus defined are characterized in terms of voting by binary issues through a certain combinatorial property of the families of winning coalitions for the relevant issues and (ii) if the property space is median then such combinatorial property is definitely met, and consequently non-dictatorial neutral and/or anonymous strategy-proofs aggregation rules including the simple majority rule are available (Nehring, Puppe (2007), Theorems 3 and 4). Furthermore, in Nehring, Puppe (2010) the very same theoretical framework is deployed to analyze preference aggregation and social welfare functions. In particular, several ‘classical’ properties for social welfare conditions including the Arrowian Independence of Irrelevant Alternatives (IIA) property can be reformulated in more general terms which depend on the specification of the relevant property space. It follows that several versions of such a generalized IIA condition can be considered. But then, as it turns out, (iii) the versions of generalized IIA attached to median property spaces are consistent with anonymous and (weakly) neutral social welfare functions including those induced by majority-based aggregation rules (Nehring, Puppe (2010), Theorem 4). Interestingly, a specific example of a median property space for the set of all total preorders is also provided by Nehring and Puppe, namely the one whose issues consist in asking for each non-empty \( Y \subseteq X \) and any total preorder \( R \) whether or not \( Y \) is a lower contour of \( R \) with respect to some outcome \( x \in X \). By contrast, it can be easily checked that when translated into the property-space framework SPI and IIIA correspond to non-median property spaces.

The overlappings between such results and those presented here are remarkable, along with some sharp differences which make them mutually independent. Since any finite median semilattice is indeed

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58 The combinatorial property mentioned in the text is the so-called ‘Intersection Property’ which requires that for every minimal inconsistent set of properties, it must be the case that any selection of winning coalitions for the corresponding binary issues has a non-empty intersection.

59 Specifically, given a property space \( H = (\mathcal{R}_A, H) \), such a generalized IIA for a social welfare function \( f : (\mathcal{R}_A)^N \rightarrow \mathcal{R}_A \) can be defined as follows: for every \( H \in H \) and \( R_N, R'_N \in (\mathcal{R}_A)^N \) such that \( \{ i \in N : R_i \in H \} = \{ i \in N : R'_i \in H \} \), if \( f(R_N) \in H \) then \( f(R'_N) \in H \) as well. Of course the original Arrowian version of such a generalized IIA may be obtained by taking \( H := \{ H(x,y) : x,y \in A \} \) with \( H(x,y) := \{ R \in \mathcal{R}_A : xRy \} \). That is so because it is well-known that in the Arrowian aggregation framework IIA is equivalent to its binary version (namely, its restriction to arbitrary pairs of alternatives).

60 Thus, the property space suggested here is \( H^0 := (\mathcal{R}_A, H^0) \), where

\[
H^0 := \{ H_L : \emptyset \neq L \subseteq A \} and
H_L := \begin{cases} R \in \mathcal{R}_A : \text{for some } x \in A \\ L = \{ y \in A : xRy \} \end{cases}.
\]

61 Indeed, the most natural property-space attached to SPI is

\[
H_{SPI} := \{ H_{(x,y,B)} : x,y \in A \} \quad \text{with}
H_{(x,y,B)} := \begin{cases} R \in \mathcal{R}_A : \{ a \} \times B \subseteq R and \\ B \times \{ b \} \subseteq R \\ \text{if } \{ a,b \} = \{ x,y \} \end{cases}.
\]

Similarly, the most natural property-space attached to IIIA is

\[
H_{IIIA} := \{ H_{(x,y,k)} : x,y \in A, k \leq |A| - 2 \} \quad \text{with}
H_{(x,y,k)} := \begin{cases} R \in \mathcal{R}_A : \{ a \} \times B \subseteq R and \\ B \times \{ b \} \subseteq R \\ \text{if } \{ a,b \} = \{ x,y \} \end{cases}.
\]
an example of a finite median algebra and is consequently representable as a median property space. all of the Nehring and Puppe’s results mentioned above do apply to finite median semilattices as a special case. Notice however that our results provide a characterization of (finite) strategy-proof social welfare functions which is also both more explicit (it includes a polynomial description of some such rules) and more comprehensive (it is a complete characterization in that it is not limited to sovereign and idempotent ones). Moreover, our treatment of social welfare functions can also be translated in terms of a median property space, but a different one from that considered by Nehring and Puppe. In fact, in our case the set of relevant properties correspond to the meet-irreducibles of the semilattice of total preorders, namely the total preorders having just two indifference classes, or equivalently the binary ordered classifications of basic alternatives as good or bad, respectively. Accordingly, the collection of relevant issues consist in asking, for each binary good/bad classification of basic alternatives and any total preorder \( R \), whether the latter is consistent with the given binary classification or not. Thus, proper consensus-based social welfare functions that are agenda manipulation-proof and even strategy-proof can be defined by binary aggregation, provided that the set of relevant binary issues is carefully selected, and in fact expanded if the basic alternatives are more than three: notice that, when expressed in terms of properties of \( R \) with \( |A| = m \), the size of the set of actually relevant binary issues (for each individual preference relation of any profile) is \( m(m-1) \) for IIA, and \( 2(2^{m-1} - 1) \) for \( M_X \)-Independence. This point is strongly consonant with one of the main arguments in Saari (2008), lamenting the enormous loss of information enforced by the Arrowian II.A. It also amounts to a special

\[
H(x,y,k) := \begin{cases} 
R \in \mathcal{R}_A : \text{either } I_{x,y} = \{z \in A : xP(R)zP(R)y\} \\
\text{and } k = |I_{x,y}| \\
or I_{x,y} = \{z \in A : yP(R)zP(R)x\} \\
\text{and } |I_{y,z}| = k
\end{cases}
\]

It can be shown that \( H_{SP1} \) and \( H_{IIA} \) are not median property spaces since both of them contain minimal inconsistent subsets of properties of size three. To check validity of that statement, just consider any triplet \( \{H_{x,y,0}, H_{y,z,0}, H_{z,x,0}\} \subseteq H_{SP1} \) with \( x \neq y \neq z \neq x \).

\footnote{Specifically, a finite median join-semilattice can be regarded as a generic instance of a finite ‘pointed’ median algebra, having one of its elements singled out (that point corresponds to the top element of the semilattice).

\footnote{For instance, it is always possible to represent a (finite) median algebra as a (finite) property space by taking as properties its prime sets as defined through its median betweenness (see e.g. Bandelt, Hedliková (1983), Theorem 1.5, and footnote 52 above for a definition of prime sets). It is important to observe that in general a finite median algebra or ternary space admits of several representations by distinct median property spaces. By contrast, a ternary (finite) algebra or space which is not median can only be represented by (finite) property spaces which are not median.

\footnote{Thus, the appropriate version of generalized IIA in our own model is \( H^* := (\mathcal{R}_A, H^*) \) with \( H^* := \{H_{A_1,A_2} : A_1 \neq \emptyset \neq A_2 \} \)
\begin{align*}
A_1 \cap A_2 = \emptyset, A_1 \cup A_2 = A \\
&\quad \text{or } \quad H_{A_1,A_2} := \{R \in \mathcal{R}_A : R \subseteq R_{A_1,A_2}\}
\end{align*}
\text{and } R_{A_1,A_2} \text{ is of course the two-indifference-class total preorder having } A_1 \text{ and } A_2 \text{ as top and bottom indifference classes, respectively. Notice that both } H^* \text{ and Nehring-Puppe’s } H^2 \text{as previously defined (see footnote 56 above) are median property spaces, while the original Arrowian } H \text{ is not.}
instance of a recurrent theme in the social choice-theoretic literature, namely emphasizing the link between Arrow’s theorem and the strictures of the preference-information base enforced by IIA and the other Arrowian axioms (see e.g. the classic Sen (2017) for extensive elaborations on that topic). Notice, however, that while changes and/or enrichments of the Arrowian input-format figure prominently among the invoked remedies for the aforementioned strictures, the relaxations of IIA we have been considering stick to a fixed, standard input-format consisting of profiles of total preorders.

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