Conformal cosmological black holes: restoring determinism to Einstein theory

Fayçal Hammad, Dilek Çiftci, and Valerio Faraoni

1Department of Physics and Astronomy and STAR Research Cluster, Bishop’s University, 2600 College Street, Sherbrooke, Quebec, Canada J1M 1Z7
2Physics Department, Champlain College-Lennoxville, 2580 College Street, Sherbrooke, Quebec, Canada J1M 0C8
3Department of Physics, Namık Kemal University, Tekirdağ, Turkey

A widespread solution-generating technique of general relativity consists of conformally transforming known “seed” solutions. It is shown that these new solutions always solve the field equations of a pathological Brans-Dicke theory. Furthermore, when interpreted as effective Einstein equations, those field equations exhibit, in the case of a cosmological “background”, an induced imperfect fluid as an additional effective source besides the original sources of the “seed” solutions. As an application, the charged non-rotating Thakurta black hole conformal to Reissner-Nordström is used to demonstrate the fragility of the inner Cauchy horizon when this black hole is embedded in the universe (even accounting for the separation of black hole and Hubble scales). Similarly, the charged McVittie spacetime representing a charged black hole embedded in a cosmological “background” with varying Hubble parameter does not exhibit a real Cauchy horizon. These arguments speak in favor of restoring determinism to Einstein theory, which was questioned in recent research.

I. INTRODUCTION

Conformal transformations of the spacetime metric play an important role in general relativity (GR) in the study of conformal infinity and in the construction of Penrose-Carter diagrams, in alternative theories of gravity where different conformal frames (the Jordan and the Einstein frames) provide alternative representations of these theories, and in highlighting the true nature of some of the quasi-local definitions of mass in GR and in scalar-tensor gravity. Conformal transformations are also used in GR as a technique to generate new analytical solutions of the Einstein equations starting from known ones, particularly in the case of electrovacuum, and other solutions such as spherical perfect fluid solutions. In these cases, the metric \( g_{ab} \) usually describes a black hole (Schwarzschild, Kerr, or their charged generalizations) and the conformal factor \( \Omega \) is chosen as the scale factor of a spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) universe, which comes to constitute the cosmological “background”. This procedure has generated the Thakurta, Sultana-Dyer, McClure-Dyer, and other solutions such as spherical perfect fluid solutions.

The new metric \( \tilde{g}_{ab} \) is not a solution of the Einstein equations with the same form of matter source for which the original metric \( g_{ab} \) is a solution. In fact, under the conformal transformation (1.2), the Ricci tensor changes according to

\[
\tilde{R}_{ab} = R_{ab} - 2\nabla_a \nabla_b \ln \Omega - g_{ab} \nabla^c \nabla^f \ln \Omega
\]

where \( \Omega \) is a nowhere-vanishing, regular, conformal factor that is still a solution of the Einstein equations. In order for this new solution to be of any physical interest, however, the conformal factor \( \Omega \) must be chosen judiciously. While in scalar-tensor gravity a Brans-Dicke-like field \( \phi \) is already present in the theory and determines completely the conformal factor \( \Omega \) as a function of \( \phi \), in GR the form of \( \Omega \) is left completely free, as long as it produces interesting new solutions. Over the years, there has been increasing interest in generating solutions of the Einstein equations which describe black holes embedded in cosmological “backgrounds”.

We use quotation marks because, due to the non-linearity of the field equations, one cannot split a metric into a “background” and a “deviation” from it in a covariant way (apart from algebraically special geometries, such as the Kerr-Schild ones).

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while the trace of this equation gives
\[ \mathcal{R} = \Omega^{-2} \left( \mathcal{R} - \frac{6 \Box \Omega}{\Omega} \right), \tag{1.4} \]
so that Eq. 1.1 becomes
\[ \mathcal{G}_{ab} = \kappa T_{ab} - \frac{2}{\Omega} \left( \nabla_a \nabla_b \Omega - g_{ab} \Box \Omega \right) \]
\[ + \frac{1}{\Omega^2} \left( 4 \nabla_a \Omega \nabla_b \Omega - g_{ab} \nabla_c \Omega \nabla^c \Omega \right) \]
\[ \equiv \kappa \left( T_{ab} + T^{(\Omega)}_{ab} \right). \tag{1.5} \]
A vacuum solution \( g_{ab} \) (with \( \mathcal{R}_{ab} = 0 \)) is transformed into a non-vacuum one with \( \mathcal{R}_{ab} \neq 0 \). The derivatives of the scale factor \( \Omega \) act as an effective form of matter in the right-hand side of the Einstein equations. Since generating new solutions in this way amounts to the “Synge procedure” of imposing the form of the metric and then running the Einstein equations to determine the matter that makes the chosen metric a solution, there is a priori little hope that this artificially created effective matter \( T^{(\Omega)}_{ab} \) will be physically meaningful. The right-hand side of the tilded Einstein equations \( \mathcal{G}_{ab} \) contains, in addition to “standard” terms quadratic in the gradient \( \nabla_a \Omega \), terms linear in the second derivatives \( \nabla_a \nabla_b \Omega \) and \( \Box \Omega \). These terms make the sign of the effective energy density undefined and \( T^{(\Omega)}_{ab} \) will not, in general, satisfy any energy condition. Indeed, the “cosmological black hole” geometries generated this way are often reported to exhibit negative energy densities in certain spacetime regions \( [11, 12] \). Furthermore, the new solutions thus obtained might actually exhibit singularities that may or may not be present in the original solution. Indeed, as can be seen from 1.1, the new Ricci scalar \( \mathcal{R} \) might possess a singularity whenever the inverse metric component in \( \Box \Omega = g^{ab} \nabla_a \nabla_b \Omega \) becomes singular. Nevertheless, these solutions of GR are still seen as interesting at least as toy models of black holes, and they are put to use, for example, in recent investigations of the Hawking radiation and thermodynamics of dynamical black holes \( [24, 26] \), see also the related references \( [27, 28] \). The situation does not always have to be so dire, however: for example, the Husain-Martinez-Nuñez solution of GR \( [9] \) is conformal to the Fisher solution (with the scale factor of the FLRW “background” universe as the conformal factor), but has as the matter source a canonical, minimally coupled, free scalar field which satisfies the weak and null energy conditions. The same can be said about its generalization in which the scalar acquires an exponential potential, known as the Fonarev solution \( [10] \). It may even happen that an unphysical solution of the Einstein equations is conformally transformed into a physically interesting one, as is the case for the fluid spheres of Ref. \( [15] \).

The GR geometries realized by effective \( \Omega \)-matter which violates the energy conditions can somehow be “rehabilitated” if they can be regarded as solutions of the field equations of a different theory of gravity. While this may be possible on a case-by-case basis, here we point out a general occurrence. It is known that the Brans-Dicke-like scalar field of scalar-tensor gravity non-minimally coupled to the Ricci curvature acts as a form of effective matter that can violate all the energy conditions in the way described by Eq. 1.15, therefore it is natural to look at scalar-tensor gravity as the ambient theory for these “bad” solutions. Indeed, provided that \( g_{ab} \) is an (electro)vacuum solution of the Einstein equations, the geometry \( g_{ab} = \Omega^2 g_{ab} \) can always be seen as a solution of a Brans-Dicke theory with Brans-Dicke coupling \( \omega = -3/2 \). This theory is known to be pathological in the sense that the Brans-Dicke field \( \phi \) is non-dynamical and the Cauchy problem is ill-posed. This pathology is then just a reflection of the fact that the conformal factor is forced arbitrarily into the geometry by imposing that \( g_{ab} \) describe a central object in a cosmological “background” (or some similar condition), and this fact is obviously bound to have some consequences.

The outline of the rest of this paper is as follows. Sec. II discusses these general aspects; Sec. III discusses an ambiguity present in the literature about cosmological black hole geometries of the kind described above. The following section focuses on a particular GR solution of this type which is of special interest in both GR and scalar-tensor gravity, the non-rotating Thakurta solution. The main use of this solution is as a counterexample to study the (absence of the) inner Cauchy horizon in cosmological black holes, a subject of great interest in recent research threatening determinism in GR. Section IV proposes another non-conformally static cosmological black hole as a useful example in the debate about inner Cauchy horizons in GR black holes. Finally, Sec. VI contains the conclusions.

II. GR SEED GEOMETRIES AND \( \omega = -3/2 \) BRANS-DICKE GRAVITY

Assume that \( g_{ab} \) is an (electro)vacuum solution of the Einstein equations \( [11] \) obtained from the Einstein-Hilbert action
\[ S = \int d^4 x \sqrt{-g} \left[ \frac{1}{2 \kappa} (\mathcal{R} - \Lambda) + \mathcal{L}_{(m)} [g_{ab}, \psi] \right], \tag{2.1} \]
where \( \Lambda \) is the cosmological constant and \( \mathcal{L}_{(m)} [g_{ab}, \psi] \) is the matter Lagrangian, with \( \psi \) denoting collectively the matter fields. Consider the conformal metric \( \tilde{g}_{ab} = \Omega^2 g_{ab} \); by using Eqs. 1.3, 1.3, and
\[ \sqrt{-\tilde{g}} = \Omega^4 \sqrt{-g}, \tag{2.2} \]
3 See, however, Refs. 20 exploring it.
\[
\frac{\Box \Omega}{\Omega^3} = \frac{\Box \Omega}{\Omega} - \frac{2\hat{g}^{ab}\hat{\nabla}_a \Omega \hat{\nabla}_b \Omega}{\Omega^2},
\]

and introducing the scalar field
\[
\phi = \Omega^{-2}
\]
(which will become a Brans-Dicke scalar), one easily obtains
\[
\sqrt{-g} R = \sqrt{-g} \left( \phi \hat{\nabla}^a \nabla_a \phi \nabla_b \phi \right) - 3 \partial_c \left( \sqrt{-g} \phi \nabla^c \partial_a \phi \right). \tag{2.5}
\]

In terms of \( \hat{g}_{ab} \) and \( \phi \), the Einstein-Hilbert action \( S \) becomes
\[
S = \int d^4 x \sqrt{-\hat{g}} \left\{ \frac{1}{2k} \left( \phi \hat{\nabla}^a \nabla_a \phi \nabla_b \phi - V(\phi) \right) + \hat{\mathcal{L}}_{(m)} [\hat{g}_{ab}, \psi] \right\}, \tag{2.6}
\]
where the total divergence in the right-hand side of Eq. \(2.4\) (which only contributes a boundary term to the action), has been dropped, the cosmological constant has become the mass potential
\[
V(\phi) = \frac{\Lambda}{2k} \phi^2 \equiv \frac{\mu^2 \phi^2}{2}, \tag{2.7}
\]
and the matter Lagrangian density is now \( \sqrt{-g} \hat{\mathcal{L}}_{(m)} = \phi^{-2} \sqrt{-\hat{g}} \hat{\mathcal{L}}_{(m)} [\hat{g}_{ab}, \psi] \). This is a (Jordan frame) Brans-Dicke action \(31\) with coupling parameter \( \omega = -3/2 \). Its variation with respect to \( \hat{g}_{ab} \) and \( \phi \) generates the field equations
\[
\hat{\nabla}^a \nabla_a \hat{\nabla}^b \hat{\nabla}_b \phi = \frac{\kappa}{\phi} \hat{\nabla}_{ab} \hat{\nabla}^a \nabla_b \phi - \frac{3}{2\phi} \hat{g}^{ab} \hat{\nabla}_a \phi \hat{\nabla}_b \phi \left( \hat{\nabla}_c \phi \nabla^c \phi \right) + \frac{1}{\phi} \left( \hat{\nabla}_a \hat{\nabla}_b \phi - \hat{g}_{ab} \Box \phi \right) - \frac{V}{2\phi} \hat{g}_{ab}, \tag{2.8}
\]
\[
\Box \phi = \frac{\phi}{3} \left( \hat{\nabla}^a \frac{dV}{d\phi} \right) + \frac{1}{2\phi} \hat{g}^{cd} \hat{\nabla}_c \phi \hat{\nabla}_d \phi. \tag{2.9}
\]

The tilded energy-momentum tensor \( \hat{T}_{ab} \) in \(2.8\) is related to the original energy-momentum tensor \( T_{ab} \) by \( \hat{T}_{ab} = \phi T_{ab} \). Notice the important fact, over which we shall come back below, that the absence of a radial matter flow in the original spacetime, \( T_{01} = 0 \), does not prevent such a radial flow from emerging in the new frame even with conformal factor which depends only on time. This general pattern stems from the fact that the \((0,1)\) component of the second derivative \( \hat{\nabla}_a \hat{\nabla}_b \phi \) is not zero. Physically, this could be understood as the result of the original radial dependence of the metric transformed into an effective radial flow due to the stretching of spacetime in a time-dependent way. This is illustrated by the expression \(4.11\) of the energy flux density \( q_0 \), which would identically vanish only for a time-independent conformal factor.

In general, just as there is an induced energy flow in the form of a non-vanishing \( T_{01}^{(\phi)} \), the field equations for the matter fields \( \psi \) also acquire extra terms due to the new form of the matter Lagrangian, which picks up an explicit dependence on the scalar field \( \phi \). This fact does not, however, arise for conformally invariant matter fields, as is the case for the Maxwell field whose Lagrangian density \( -\sqrt{-g} F_{ab} F^{ab}/4 \) is invariant under conformal transformations. As such, the Maxwell equations in vacuo are also conformally invariant. We shall come back to this observation in Sec. \( IV \) where we examine the charged Thakurta black hole.

Now, Brans-Dicke theory with the particular value \( -3/2 \) of the \( \omega \)-parameter is known to be pathological: the Brans-Dicke scalar \( \phi \) (corresponding approximately to the inverse of the gravitational coupling) is not dynamical. In fact, by taking the trace of Eq. \(2.8\),
\[
\hat{R} = \frac{\kappa}{\phi} \hat{\nabla}_{ab} \nabla^a \nabla_b \phi - \frac{3}{2\phi} \hat{g}^{ab} \hat{\nabla}_a \phi \hat{\nabla}_b \phi + \frac{3\Box \phi}{\phi} + \frac{2V}{\phi}, \tag{2.10}
\]
and substituting it into Eq. \(2.7\) reduces the latter to \( \hat{T} = 0 \), and therefore \( T = 0 \), which is identically satisfied with a conformally invariant form of matter in the original frame and, in particular, (electro)vacuum. The usual Brans-Dicke dynamical equation for \( \phi \) is thus completely lost and this field is not even subject to a first order constraint, becoming completely arbitrary. Correspondingly, the Cauchy problem for \( \omega = -3/2 \) is ill-posed \(32\), see also \(33\ 34\). This feature resurfaced recently in the literature with the revival of Palatini \( f(R) \) gravity as an alternative to dark energy to explain the current acceleration of the universe, because this theory is equivalent to \( \omega = -3/2 \) Brans-Dicke gravity with a special potential \(35\). These properties are not surprising because, while the geometry \( g_{ab} \) solves the Einstein equations (with no matter or with just the Maxwell field), the conformal factor \( \Omega \) is completely arbitrary and is introduced \textit{ad hoc} without being required to satisfy any rule or physical equation. Requiring \( \Omega \) to coincide with the scale factor of a “background” FLRW universe does introduce some physics into this picture, but this is still an artificial way to force a geometry to do what we want. While the goal of the transformation \(1.2\) is to generate new solutions \( g_{ab} \) of GR with some desired properties, these can always be seen also as solutions of the (pathological) \( \omega = -3/2 \) Brans-Dicke gravity, possibly with a mass potential.

The Einstein frame representation of the theory using variables \( (\hat{g}_{ab}, \phi) \) gives back the Einstein-Hilbert action with no dynamics for the field \( \Omega \) (cf. the first of Refs. \(36\)).
III. RELATIONS BETWEEN CONFORMAL GR SOLUTIONS

Let the metric $g_{ab}$ be an (electro)vacuum solution of the Einstein equations which can be expressed in various coordinate systems. Let $g_{\mu\nu}$ and $g_{\mu'\nu'}$ denote, respectively, the metric components in two coordinate systems (for example, consider the Schwarzschild metric in Schwarzschild, isotropic, Kerr-Schild, or Eddington-Finkelstein coordinates [21]). By conformally transforming $g_{ab}$ with a conformal factor $\Omega$, one obtains the two expressions $g_{\mu\nu} = \Omega^2 g_{\mu\nu}$ and $g_{\mu'\nu'} = \Omega^2 g_{\mu'\nu'}$ of the same metric. This stems from the one-to-one character of the conformal transformations (1.2) in the case of (electro)vacuum, as shown in Ref. [22]. There are, however, incorrect claims to the contrary in the literature. For example, in Ref. [12], the Schwarzschild metric in Schwarzschild coordinates $(t, r, \theta, \phi)$ and in isotropic coordinates $(t, \rho, \theta, \phi)$, respectively,

$$ds^2_{(S)} = -(1 - \frac{2m}{r}) dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\Omega^2_{(2)}$$

(3.1)

$$= -\left(\frac{1 - \frac{2m}{r}}{1 + \frac{2m}{r}}\right)^2 dt^2 + \left(1 + \frac{m}{2\rho}\right)^4 (d\rho^2 + \rho^2 d\Omega^2_{(2)})$$

(3.2)

(where $d\Omega^2_{(2)} = d\theta^2 + \sin^2 \theta d\phi^2$ is the line element on the unit 2-sphere) is conformally transformed. If the line element in the form (3.1) is used, one obtains the non-rotating Thakurta metric

$$ds^2_{(T)} = a^2(t) \left[-\left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\Omega^2_{(2)}\right]$$

(3.3)

where $a(t)$ is the scale factor of the FLRW “background” universe into which the Schwarzschild black hole gets embedded. In Ref. [12], the line element

$$ds^2 = a^2(t) \left[-\left(\frac{1 - \frac{2m}{r}}{1 + \frac{2m}{r}}\right)^2 dt^2 + \left(1 + \frac{m}{2\rho}\right)^4 (d\rho^2 + \rho^2 d\Omega^2_{(2)})\right]$$

(3.4)

obtained by conformally transforming the Schwarzschild line element in its form (3.1) with the same conformal factor $a$, is presented as a new GR solution alternative to the Thakurta one. However, the usual coordinate transformation

$$\rho \to r = \rho \left(1 + \frac{m}{2\rho}\right)^2$$

(3.5)

turns the line element (3.4) into (3.3).

In contrast with the two forms (3.3) and (3.4), the non-rotating Thakurta and the Sultana-Dyer [11] solutions are genuinely different from each other, in spite of being both conformal to Schwarzschild because they are generated using two different conformal factors in Eq. (1.2). This fact (remarked in Ref. [14]) is not obvious in the coordinate systems normally used in the literature. The Sultana-Dyer line element is [11]

$$ds^2_{(SD)} = a^2(\tau) \left[-d\tau^2 + \frac{2m}{r} (d\tau + dr)^2 + dr^2 + r^2 d\Omega^2_{(2)}\right]$$

$$= a^2(\tau) \left[-\left(1 - \frac{2m}{r}\right) dr^2 + \frac{4m}{r} d\tau dr\right.$$ 

$$+ \left(1 + \frac{2m}{r}\right) dr^2 + r^2 d\Omega^2_{(2)}\right],$$

(3.6)

where $a(\tau) = \tau^2$ and $m > 0$ is the mass of the original Schwarzschild black hole [11]. It is already clear from this last expression of the Sultana-Dyer metric that the latter is fundamentally different from the conformal Schwarzschild metric (3.3) that has a conformal factor depending only on time. To make this difference more apparent, let us introduce a new time coordinate $t$ defined by

$$\tau(t, r) = t + 2m \ln \left|\frac{r}{2m} - 1\right|,$$

(3.7)

which will be interpreted as the conformal time of the FLRW “background” universe. Differentiation gives

$$d\tau = dt + \frac{2mdr}{r (1 - 2m/r)},$$

(3.8)

and substitution into Eq. (3.6) turns this line element into the diagonal form

$$ds^2 = a^2(t, r) \left[-\left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\Omega^2_{(2)}\right].$$

(3.9)

In these coordinates, the Sultana-Dyer line element is explicitly conformal to Schwarzschild, with conformal factor

$$\Omega = a(t, r) = \tau^2(t, r) = \left(t + 2m \ln \left|\frac{r}{2m} - 1\right|\right)^2,$$

(3.10)

which is clearly different from the conformal factor of the non-rotating Thakurta metric (3.3), which depends only on time.

IV. THAKURTA GEOMETRY AND STRONG COSMIC CENSORSHIP

A. Uncharged non-rotating Thakurta metric

As we shall see below, the other expression of the Thakurta line element (3.4) looks superficially like that of the McVittie metric [38] but, while the mass coefficient
in the McVittie metric is \( M = m/a(t) \) (with constant \( m \)), that of the line element \( ds^2 \) is strictly constant. The difference is crucial because, allowing \( M \) to be different from its McVittie form \( m/a(t) \), implies that there is a (purely spatial) radial energy flux with density \( q^a \) described by an imperfect fluid term in the matter stress-energy tensor.

\[
T^{(\text{fluid})}_{ab} = (P + \rho) u_a u_b + P g_{ab} + q_a u_b + q_b u_a .
\] (4.1)

Therefore, instead of a McVittie metric, the non-rotating Thakurta spacetime is a generalized McVittie geometry of the class presented in Ref. [39] and studied in Ref. [40].

Conversely, the non-rotating Thakurta solution becomes valid at the black hole horizon \( r = 2m \), with \( m \) a constant, and constant charge \( Q \) embedded in a cosmological spacetime.

As follows from the discussion of Sec. II, being conformal to the Schwarzschild black hole, the Thakurta metric represents a spacetime filled with an artificially created effective matter that might exhibit negative energy densities in certain spacetime regions. In addition, given that the inverse metric \( g^{ab} \) of the original Schwarzschild line element used to create such a spacetime is singular at the black hole horizon \( r = 2m \), we expect that the non-rotating Thakurta spacetime will also possess a singularity at that same coordinate location. In fact, the coordinate singularity becomes, as we shall see, a true spacetime singularity in the conformally mapped geometry.

Using the action \( \mathcal{L} \) and defining an energy-momentum tensor for the matter part by Eq. (4.1), we find that to satisfy the Einstein equations, the components of the four-velocity vector and the energy flux density should be, respectively,

\[
u_a = \left( -a \sqrt{1 - \frac{2m}{r}}, 0, 0, 0 \right) ,
\] (4.2)

\[
q_a = \left( 0, \frac{-m \dot{a}}{4 \pi a^2 r^2 \left( 1 - \frac{2m}{r} \right)^{3/2}}, 0, 0 \right) .
\] (4.3)

On the other hand, the required energy density and pressure of the artificial fluid are found to be, respectively,

\[
\rho = -\frac{3 \dot{a}^2}{8 \pi a^4 \left( 1 - \frac{2m}{r} \right)} ,
\] (4.4)

\[
P = \frac{\dot{a}^2 - 2a \ddot{a}}{8 \pi a^4 \left( 1 - \frac{2m}{r} \right)} .
\] (4.5)

The Ricci scalar of the Thakurta metric, computed directly from Eq. (4.3), is

\[
\mathcal{R} = \frac{6 \left( \dot{f} + H^2 \right)}{a^2 \left( 1 - \frac{2m}{r} \right)} .
\] (4.6)

As expected, the Ricci scalar is singular at \( r = 2m \). This singularity translates into a singular fluid as both the energy density and pressure and diverge there as well.

All the previous results concerning the possibility of a negative energy density and a singular character of the fluid necessary for the creation of the non-rotating Thakurta spacetime remain valid in the case of a charged and/or rotating Thakurta spacetime. In what follows, we examine this case in detail.

### B. Charged non-rotating Thakurta metric

The charged Thakurta metric of a black hole of constant mass \( m \) and constant charge \( Q \) embedded in a cosmological background of scale factor \( a(t) \) reads

\[
ds^2_{(\text{CNSRT})} = a^2(t) \left[ -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \right] ,
\] (4.7)

where

\[
f(r) = 1 - \frac{2m}{r} + \frac{Q^2}{r^2} .
\] (4.8)

This metric being conformal to the RN metric, and given the conformal invariance of the electromagnetic field \( F_{ab} \), the corresponding expression of the latter for this geometry is the same as the one of the RN spacetime \( F_{ab} = \partial_a A_b - \partial_b A_a \), with four-potential

\[
A_a = \left( -\frac{Q}{r}, 0, 0, 0 \right) .
\] (4.9)

The energy-momentum tensor \( T^{(\text{em})}_{ab} \) of the electromagnetic field that appears on the right-hand side of (4.9) is
then the same as the one sourcing the RN metric. However, as for the uncharged Thakurta metric (3.3), an imperfect fluid source of the form (4.11) is now needed in addition to the electromagnetic energy-momentum tensor, with a four-velocity $u_a$ and an energy flux density $q_a$ given by

$$u_a = \left(-a\sqrt{f(r)}, 0, 0, 0\right), \quad (4.10)$$

$$q_a = \left(0, -\frac{\dot{a}(mr - Q^2)}{4\pi a^3 f(r)} \sqrt{f(r)}, 0, 0\right). \quad (4.11)$$

The energy density and pressure of such a fluid are

$$\rho(t, r) = \frac{3\dot{a}^2}{8\pi a^2 f(r)}; \quad (4.12)$$

$$P(t, r) = \frac{\dot{a}^2 - 2a\dot{a}}{8\pi a f(r)}. \quad (4.13)$$

The energy-momentum tensors of the electromagnetic field and of the imperfect fluid appearing on the right-hand side of the Einstein equations and responsible for sourcing the metric thus read

$$T_{00}^{(fluid)} = \frac{3\dot{a}^2}{8\pi a^2}, \quad T_{00}^{(em)} = \frac{Q^2 f(r)}{8\pi^2},$$

$$T_{11}^{(fluid)} = \frac{\dot{a}^2 - 2a\dot{a}}{8\pi a^2 f(r)}, \quad T_{11}^{(em)} = -\frac{Q^2}{8\pi r^2},$$

$$T_{22}^{(fluid)} = \frac{r^2(\dot{a}^2 - 2a\dot{a})}{8\pi a^2 f(r)}, \quad T_{22}^{(em)} = \frac{Q^2}{8\pi r^2},$$

$$T_{33}^{(fluid)} = T_{22}^{(fluid)} \sin^2 \theta, \quad T_{33}^{(em)} = T_{22}^{(em)} \sin^2 \theta,$$

$$T_{01}^{(fluid)} = \frac{\dot{a}(mr - Q^2)}{4\pi a^3 f(r)}. \quad (4.14)$$

The origin of this imperfect fluid is, again, the nonvanishing second derivative $\nabla_a \nabla^a V(t)$ of the time-dependent conformal factor $a(t)$. Also, as for the noncharged Thakurta metric (3.3), a spacetime singularity arises at $r = m \pm \sqrt{m^2 - Q^2}$, given that the Ricci scalar is

$$\mathcal{R} = \frac{6(\dot{H} + H^2)}{a^2 f(r)}. \quad (4.15)$$

Since the charged Thakurta metric (4.7) is conformal to the Reissner-Nordström metric, which solves the vacuum Maxwell equations $\nabla_a F^{ab} = 0$ and $\nabla_a F^{bc} = 0$, one is also guaranteed to satisfy the Maxwell equations. This can be understood from our discussion of Sec. II about the conformal invariance of such equations. Physically, although there is an induced effective matter flow, the latter is uncharged, as shown by the expressions (4.10), (4.11). Therefore, the Maxwell equations are preserved due to the absence of induced currents.

### C. Cauchy horizon of charged black holes and determinism in GR

The most general spherically symmetric and asymptotically flat solution of the coupled Einstein-Maxwell equations is the Reissner-Nordström (RN) geometry describing a charged black hole. This static solution has a null event horizon (the outermost black hole horizon) which encloses a null Cauchy horizon. Cauchy horizons are surfaces through which the geometry can be continued but cannot be predicted by prescribing regular initial data. In other words, they are null hypersurfaces that constitute the boundary of the domain of validity of the Cauchy problem for spacetime. The existence of such a boundary, avoided only by the strong cosmic censorship conjecture [14], implies the loss of determinism in such a spacetime. Therefore, the initial value problem of vacuum GR fails in the interior of a charged black hole and the theory ceases to be predictive and deterministic, a completely unacceptable shortcoming for any fundamental physical theory or for its solutions. Realistic astrophysical black holes are not charged nor static: they are electrically neutral and they rotate, but static charged black holes have been used in the recent Ref. [43] as toy models for realistic black holes.

Fortunately, there is a long history of indications that the Cauchy horizon inside the RN black hole is an artifact of the perfect symmetries of the latter: it is fragile and it disappears when these symmetries are broken or the RN solution is perturbed. Specifically, photons arriving to the Cauchy horizon from larger radii are infinitely blueshifted (a phenomenon known as mass inflation) and a mass inflation singularity develops when this phenomenon is taken into account [16]. The Cauchy horizon is then unstable with respect to perturbations of the RN solution, which decay outside the event horizon but grow in the region inside of it because of the infinite blueshift, transforming the Cauchy horizon into a singularity through which the spacetime cannot be continued. In [43] (see also [47]) it was pointed out, through a clever study of the quasinormal modes, that adding a positive cosmological constant $\Lambda$ to the picture, the resulting Reissner-Nordström-de Sitter (RNdS) solution of the Einstein equations exhibits a decay rate of the perturbations outside the black hole horizon which is quite different from that of the RN black hole (exponential instead of power-law [15]). Since the decay rate outside the black hole horizon is tied to mass inflation near the Cauchy horizon, the latter is stabilized by the cosmological constant and determinism is again in jeopardy. This fact is worrisome since, ultimately, no black hole is isolated but it is embedded in the universe and the asymptotics are not Minkowskian, but cosmological. Therefore, the RNdS model is a more realistic model of a black hole than a RN one and the introduction of $\Lambda$ in [43] is well justified. Even though the asymptotics are usually neglected for astrophysical black holes evolving on temporal and spatial scales much smaller than the Hubble...
radius, this cannot always be done in problems of principle, as Ref. [43] shows, because even a tiny cosmological constant can make a profound difference.

The conclusions of [43] have been challenged in Refs. [48, 49]. In [48] it is pointed out that scalar field perturbations around a charged black hole necessarily involve a charged scalar field, and that its decay rate outside the black hole horizon is not altered with respect to the RN case. Ref. [49] studies instead electrically neutral but rotating black holes in a de Sitter background and shows that the Cauchy horizon is again unstable for this more realistic situation. Here we point out a different, non-perturbative way in which the Cauchy horizon is destroyed by modifying the black hole model to make it more realistic. While the perturbations of the RNdS geometry described by quasinormal modes break the symmetries, the real universe is not described by an exact de Sitter model. While de Sitter is the late-time attractor of many dark energy models attempting to explain the current acceleration of the universe (including that caused by a cosmological constant) [50], and de Sitter space may ultimately turn out to be its final asymptotic state, the real universe contains dark and ordinary matter, radiation, neutrinos and other forms of mass-energy and is not completely empty. The matter stress-energy tensor $T_{ab}$ in the right-hand side of the Einstein equations with cosmological constant $\Lambda$

$$\mathcal{R}_{ab} - \frac{1}{2} g_{ab} \mathcal{R} = 8\pi T_{ab} - \Lambda g_{ab} \quad (4.16)$$

cannot be neglected entirely. As a consequence, the cosmological model describing our universe is not a pure de Sitter space, but rather a FLRW one. Then, a better model of a black hole with non-Minkowskian asymptotics is one in which this object is embedded in a non-static FLRW universe.

There are immediately two problems arising with such models. First, while the RNdS geometry is the unique spherical, static, and asymptotically de Sitter solution of the coupled Einstein-Maxwell equations, there is no unique solution with dynamical FLRW asymptotics. While a few exact solutions of the Einstein equations (and their charged versions) are known, they are special and they usually suffer from some physical pathologies (see [4] for a review). We argue that they are still better models of charged black holes than the RNdS space in the sense that they model the cosmological asymptotics in a general (instead of locally static) way.

The second problem is that, while in stationary situations (such as for the RNdS model) black hole horizons are static and null surfaces, for dynamical black holes one must consider instead apparent horizons (AHs), which are dynamical and are spacelike or timelike. However, AHs are foliation-dependent, as exemplified dramatically by the fact that in the Schwarzschild spacetime there exist foliations without AHs [51]. This problem is somehow alleviated by the recent realization that, in spherical symmetry, all spherically symmetric foliations (to which we restrict here) possess the same AHs [52]. In any case, the recent detections of gravitational waves from black hole mergers by the LIGO interferometers [53] are based in an essential way on the use of marginally trapped surfaces and AHs. In fact, due to the low signal to noise ratio, gravitational wave signals are matched to banks of templates for the gravitational waveforms, which are produced by numerical simulations identifying black holes with their apparent, not event, horizons. Event horizons are essentially useless for this practical task. The new and promising gravitational wave science is based on AHs when these waves are generated by mergers of black holes with other objects.

Keeping in mind the two caveats above, one can nevertheless study particular solutions of GR describing charged black holes embedded in FLRW universes as more general toy models than RNdS. We discuss two examples in which, changing the background from static Minkowski or de Sitter to FLRW, the inner Cauchy horizon disappears. This is a further indication that the Cauchy horizon is very fragile and is not expected to occur in nature, restoring determinism to GR.

### D. Charged Thakurta geometry: determinism restored

As is well known [37], the RNdS geometry has a Cauchy horizon nested inside an event horizon, with radii

$$r_{\pm} = m \pm \sqrt{m^2 - Q^2}; \quad (4.17)$$

these horizons are null surfaces [37]. Since the null structure is left unchanged by conformal transformations, one would expect these two null horizons to be mapped into null horizons of the CNRT geometry [47], but they are mapped into null spacetime singularities instead. In fact, the Ricci scalar of the metric (4.17) is given by (4.15) and it diverges at the would-be Cauchy and event horizons $r = r_{\pm}$. The main point here is that the Cauchy horizon of the RN black hole disappears by embedding it into a non-static FLRW universe. The conformal transformation from the RN to the CNRT black hole brings an improvement if $|Q| < m$. It is well known [37] that, at small radii, the RN geometry exhibits a negative energy, as measured by the Misner-Sharp-Hernandez mass. In spherical symmetry, the Misner-Sharp-Hernandez mass $M_{MSH}$ contained in a sphere of areal radius $R$ is

$$1 - \frac{2M_{MSH}}{R} = \nabla^e R \nabla_e R. \quad (4.18)$$

For the RN black hole, this quantity is

$$M_{MSH} = m - \frac{Q^2}{2r} \quad (4.19)$$

5 This fact was noted in [12].
and it is negative for small radii $r < Q^2/(2m)$. The Misner-Sharp-Hernandez mass of the CNRT geometry is computed either directly or by using the transformation property under conformal transformations

$$\tilde{M}_{\text{MSH}} = \Omega M_{\text{MSH}} - \frac{R^3}{2\Omega} \nabla^c \Omega \nabla_c \Omega - R^2 \nabla^c \Omega \nabla_c R. \quad (4.20)$$

In either way, one obtains for CNRT

$$\tilde{M}_{\text{MSH}} = a \left( M_{\text{MSH}} + \frac{3aH^2r^3}{2f} \right). \quad (4.21)$$

This quantity is non-negative in the entire physical range $r > r_+$ if $|Q| \leq m$. In fact, since for the RN black hole $M_{\text{MSH}} \geq 0$ when $r \geq m/2$, it follows that $\tilde{M}_{\text{MSH}} > 0$ for any $r > r_+$. In the supercritical case $|Q| > m$ in which the RN geometry contains a naked singularity, instead, $\tilde{M}_{\text{MSH}}$ becomes arbitrarily negative at small radii (the physical range of values of the radial coordinate is now $r > 0$).

E. Apparent horizons

The areal radius of the CNRT geometry \[1.7\] is

$$R(t, r) = a(t)r, \quad (4.22)$$

and, as usual in spherical symmetry, the AH radii are located by the roots of the equation \[54, 56\]

$$\nabla^c R \nabla_c R = 0. \quad (4.23)$$

Since $\nabla_c R = \dot{a} r \delta_{c0} + a \delta_{c1}$, this equation corresponds to

$$\nabla^c R \nabla_c R = \frac{1}{f} \left( f^2 - H^2 r^2 \right) = 0, \quad (4.24)$$

where $H \equiv \dot{a}/a$ is the Hubble parameter. Since $r > r_+$, we have $f > 0$ and, taking the positive sign in the square root of Eq. \[4.24\], the AHs correspond to the roots of

$$f(r) \equiv 1 - \frac{2m}{r} + \frac{Q^2}{r^2} = Hr > 0, \quad (4.25)$$

therefore it is clear that the AHs (when they exist) do not coincide with the null spacetime singularities at $r_+$ (which correspond to $f = 0$ instead). Equation \[4.25\] corresponds to the cubic

$$Hr^3 - r^2 + 2mr - Q^2 = 0, \quad (4.26)$$

but it is more interesting to discuss Eq. \[4.25\] graphically. The AHs correspond to the intersections between the graph of the function $y = f(r)$ and the straight line $y = Hr$. A qualitative graphical analysis determines when roots exist and the number of these roots lying in the physical region $r > r_+$. Since

$$f'(r) = \frac{2}{r^2} \left( m - \frac{Q^2}{r} \right), \quad (4.27)$$

the function $f(r)$, which tends to $+\infty$ as $r \to 0^+$, decreases for $0 < r < r_{\text{min}}$, has a minimum $f_{\text{min}} = 1 - m^2/Q^2$ at $r_{\text{min}} = Q^2/m$, and it increases for $r > r_{\text{min}}$, asymptoting to 1 as $r \to +\infty$. For reference, we consider in all cases a FLRW universe which begins with a Big Bang at which the Hubble parameter $H$ diverges and expands for an infinite time. For definiteness, we use a dust-dominated FLRW universe with scale factor $(t/t_0)^{2/3}$. Then, the slope $H(t)$ of the straight line through the origin $y = Hr$ decreases as time evolves, from positive infinity near the Big Bang to zero as $t \to +\infty$. We discuss separately the subcritical, critical, and supercritical situations $|Q| < m$, $|Q| = m$, and $|Q| > m$, respectively.

1. $|Q| < m$

When the CNRT metric is conformal to a subcritical RN black hole, the function $f(r)$ vanishes at $r_* = m \pm \sqrt{m^2 - Q^2}$ and its minimum $f_{\text{min}} = 1 - m^2/Q^2$ is negative. There are three possibilities, reported in Fig. II (which is drawn for the parameter choice $|Q| = m/2$, $t_0 = 5m$).

The straight line $y = Hr$ intersects the curve $y = f(r)$ at only one point if the slope $H(t)$ is sufficiently large, that is, at early times near the Big Bang. This intersection corresponds to the unphysical region $r < r_-$ and there are no AHs.

As time goes by, the slope of the straight line $y = Hr$ decreases and the latter becomes tangent to the curve $y = f(r)$ at a critical time $t_*$, at which a pair of AHs is created. These AHs necessarily have a radius $r_* > r_+$, as is clear from Fig. II. This critical situation occurs when the slopes of the straight line and of the curve $f(r)$
As time progresses ($t > t_*$) the two roots separate, becoming two distinct intersections between the two curves, which correspond to two distinct AHs of radii $r_{1,2}$ (labelled so that $r_2 > r_1$). As time grows and $t \to +\infty$, the line $y = Hr$ becomes closer and closer to the horizontal and the smallest root $r_1 \to r_+,$ while $r_2 \to +\infty$. The AH corresponding to the largest root $r_2$ is cosmological and $r_2$ reduces to the radius of the cosmological AH of the spatially flat FLRW universe $r_2 \approx 1/H$ (or $R_2 \approx a/H$) as $r_2 \to +\infty,$ which happens as $t \to +\infty$. The smaller root $r_1$ corresponds to a black hole AH which always covers the null spacetime singularity ($r > r_+$) but approaches it as $t \to +\infty$. The behaviour of the areal radii of these AHs versus the comoving time of the “background” universe is given in Fig. 2 for the parameters choice $|Q| = m/2, \ t_0 = 5m.$ This phenomenology of AHs is well known and is dubbed “C-curve” in the literature [56].

2. $|Q| = m$

In this case the CNRT metric is conformal to an extremal RN black hole in which Cauchy and event horizons coincide. The null spacetime singularities of the CNRT geometry at $r_\pm = m$ coincide and there are only two spacetimes disconnected by it. Now the function

$$f(r) = \left(1 - \frac{m}{r}\right)^2$$  \hspace{1cm} (4.29)

is non-negative and vanishes only at its minimum, achieved at $r_\pm = m$. Repeating the graphical analysis (see Fig. 3), at early times and high values of $H$, there is only one root $r_1$ with $0 < r_1 < m,$ which lies in the unphysical region, and there are no AHs. As time reaches a critical value $t_*$, two AHs appear, corresponding to a double root $r_* > m$ and to the straight line $y = Hr$ being tangent to the curve $y = f(r).$ At later times $t > t_*$, there are two distinct roots $r_{1,2}$ with $m < r_1 < r_2.$ As the universe evolves and $t \to +\infty, r_1 \to m$ and $r_2 \to +\infty.$ The AH at $r_1$ is interpreted as a black hole AH, while the one at radius $r_2$ is interpreted as a cosmological AH, which approaches the usual FLRW AH of areal radius $R_2 = a/H$ at late times. Qualitatively, the situation is similar to that of the previous case $|Q| < m$.

3. $|Q| > m$

In this case the CNRT geometry is conformal to a RN supercritical solution which does not have horizons and exhibits a naked singularity at $r = 0.$ The physical range of the coordinate $r$ is now the entire interval $r > 0.$ The function $f(r)$ can be written as

$$f(r) = \frac{1}{r^2} \left[(r - m)^2 + Q^2 - m^2\right]$$  \hspace{1cm} (4.30)

and is always positive, with positive minimum

$$f_{\text{min}} = f\left(\frac{Q^2}{m}\right) = 1 - \frac{m^2}{Q^2}.$$  \hspace{1cm} (4.31)

The equation locating the AHs, $f(r) = Hr > 0$ can still be satisfied. Now the situation is qualitatively different from the previous cases.
FIG. 4. The intersections between $y = f(r)$ and $y = Hr$ for $|Q| > m$. At early times (solid line) there is only one AH. As time progresses (dashed line), a pair of AHs appears and there are three of them. At later times a pair of AHs merge and disappear, leaving only a cosmological AH (dash-dotted line).

Referring to Fig. 4 for illustration, one sees that at early times, when the slope of the line $y = Hr$ is large, there is only one root (a cosmological AH) in the region $r < Q^2/m$: this spacetime region hosts a naked singularity. Later on, at a critical time $t_1$, we have a single root $r_1 < Q^2/m$ and a double root $r_2 > Q^2/m$. As time progresses, this double root splits in two and there are three distinct AHs with radii $r_1, r_2, r_3$ satisfying

$$0 < r_1 < \frac{Q^2}{m} < r_2 < r_3.$$  

As time progresses and the slope of the straight line decreases, $r_1$ increases and approaches $Q^2/m$, while $r_2$ decreases approaching $Q^2/m$, and $r_3$ increases without limit. At a critical time $t_2 > t_1$, $r_1$ and $r_2$ merge, corresponding to the annihilation of these two AHs, while $r_3$ (corresponding to a cosmological AH) still exists. At times $t > t_2$, there is only one intersection between $y = f(r)$ and $y = Hr$, with radius $r_3 \to +\infty$ as $t \to +\infty$. This surviving AH is a nearly-FLRW cosmological AH. The behaviour of the areal radii of the AHs versus the comoving time of the “background” universe is given in Fig. 4 for the parameter choice $|Q| = 3m/2$ and $t_0 = 5m$.

This behaviour of the AHs is the alternative to the “C-curve” phenomenology most often seen in the literature on AHs, and is called “S-curve” behaviour \[56\]. It was found for the first time in the Husain-Martinez-Ñuñez solution of GR sourced by a free, canonical and minimally coupled scalar field [56]. The appearance of a naked singularity in the supercritical CNRT geometry is not too surprising, since the latter is conformal to a naked singularity spacetime. (This is also the case for the Husain-Martinez-Ñuñez spacetime, which is conformal to the Fisher scalar field solution hosting a naked singularity [56].)

V. CAUCHY HORIZON AND MCVITTIE METRIC

As discussed above, the central assumption made in Refs. \[45, 47\] is a charged black hole embedded in a static de Sitter background, as well as a constant charge assigned to the black hole. While the weakness of the latter assumption will be dealt with elsewhere, our goal in this section is to deal with the former assumption using yet another spacetime representing a black hole embedded in a cosmological background. Indeed, we know from the Friedmann equation corresponding to an FLRW universe that, whenever there is matter, the universe cannot describe a de Sitter background as the Hubble parameter governed by such an equation can never be constant. Then, this fact allows one to argue that by embedding the RN black hole in a more “realistic” background, the Cauchy horizon would always be hidden behind a singularity. It turns out that this is what happens whenever the background is allowed to be non-static as is the case for the McVittie spacetime.

The McVittie spacetime was introduced long ago in order to study the competition between cosmic expansion and local dynamics \[38\]. It is regarded as describing a black hole embedded in a FLRW universe and, recently, it has been the subject of considerable attention \[12, 13, 57\]. A charged version of the McVittie cosmological black hole was introduced in \[58\], generalized in \[59\], and further studied in \[12, 13, 60, 61\]. The line element and Maxwell
field assume the form
\[ ds^2 = -\left[ 1 - \frac{(m^2 - Q^2)}{4a^2 r^2} \right]^2 \left[ 1 + \frac{m^2}{2ar} - \frac{Q^2}{4a^2 r^2} \right]^2 dt^2 + a^2(t) \left[ 1 + \frac{m}{2ar} - \frac{Q^2}{4a^2 r^2} \right]^2 \left( dr^2 + r^2 d\Omega^2_2 \right) , \] (5.1)

where the parameters \( m > 0 \) and \( Q \) describe the mass and the electric charge, respectively, while \( a(t) \) is the scale factor of the "background" FLRW universe. The line element (5.1) interpolates between the RN spacetime (obtained for \( a \equiv 1 \)) and the spatially flat FLRW metric (obtained for large values of \( r \)). The geometry reduces to the spatially flat FLRW one if \( m = Q = 0 \).

The AHs of the charged McVittie metric have been studied in [13, 61]. The areal radius is
\[ R(t, r) = m + a(t) r + \frac{m^2 - Q^2}{4a(t) r} , \] (5.3)
with \( R \geq m \) if \( |Q| \leq m \). The Ricci scalar
\[ R = 6 \left[ 2H^2 + H \left( 1 + \frac{m}{2ar} + \frac{(m^2 - Q^2)}{4a^2 r^2} \right) \right] , \] (5.4)
(where \( H(t) \equiv \dot{a}/a \) is singular at \( R_+ = m + \sqrt{m^2 - Q^2} \) if \( |Q| \leq m \)). This is the location of the outer apparent horizon of the RN geometry. This spacelike singularity splits the spacetime into two completely disconnected portions.

The line element of McVittie spacetime in \((t, R, \theta, \phi)\) coordinates is
\[ ds^2 = -\left( 1 - \frac{2m}{R} + \frac{Q^2}{R^2} - H^2 R^2 \right) dt^2 - \frac{2HRdRdt}{\sqrt{1 - \frac{2m}{R} + \frac{Q^2}{R^2}}} + \frac{2HRdRd\Omega^2}{\sqrt{1 - \frac{2m}{R} + \frac{Q^2}{R^2}}} + R^2 d\Omega^2 . \] (5.5)

Equation (5.5) locating the AHs becomes the quartic
\[ \mathcal{G}(t, R) = H^2 R^4 - R^2 + 2mR - Q^2 = 0 . \] (5.6)

For large radii one obtains the asymptotic root \( R \simeq H^{-1} \), which corresponds to the cosmological AH of the FLRW background. If \( H \rightarrow 0 \), there are only the two roots \( R_+ = m \pm \sqrt{m^2 - Q^2} \), where the smaller one, \( R_- \), is always located inside the spherical singularity and \( R_+ \) lies outside of it. In order to locate the AHs numerically, one needs to fix the FLRW background. However, in order to have a general result, which would be independent of the particular conformal factor \( a(t) \), we shall first repeat the procedure applied on the charged Thukurta metric in subsection 5.4 and investigate the occurrence of a null internal horizon that we would identify with the Cauchy horizon. After locating these various AHs we shall investigate the possibility of identifying one of them – the internal one – as a Cauchy horizon.

The solutions to Eq. (5.6) can be found graphically as shown in Fig. 6 by detecting the intersections of the parabola \( H^2 R^4 \) (shown in red, blue, then green for consecutive times corresponding to smaller and smaller values of \( H(t) \)) with the parabola \( R^2 - 2mR + Q^2 \) (solid black curve). For different moments in the evolution of the universe described by the scale factor \( a(t) \) we obtain the pattern shown in Fig. 6.

In order for an AH detected by (5.6) to be null, its gradient needs to satisfy \( \nabla_a \mathcal{G} \nabla^a \mathcal{G} = 0 \). In terms of the metric, this reads
\[ g^{RR} (\partial_R \mathcal{G})^2 + 2g^{Rt} \partial_R \mathcal{G} \partial_t \mathcal{G} + g^{tt} (\partial_t \mathcal{G})^2 = 0 . \] (5.7)

Because \( \mathcal{G}(t, R) = 0 \) at an AH, using (5.6) we can compute the needed components of the inverse metric:
\[ g^{RR} = 0 , \quad g^{Rt} = -1 , \quad g^{tt} = -\frac{1}{H^2 R^2} . \] (5.8)

On the other hand, computing the partial derivatives in (5.5), we find,
\[ \partial_R \mathcal{G} = 4H^2 R^3 - 2R + 2m , \quad \partial_t \mathcal{G} = 2RH^4 . \] (5.9)

Substituting the results (5.8) and (5.9) in (5.7), we find the following condition for one of the AHs of McVittie spacetime to be null,
\[ H (4H^2 R^3 - 2R + 2m) + H R^2 = 0 . \] (5.10)
Thus, we conclude that a given apparent horizon of the McVittie spacetime is not necessarily null. Instead, an algebraic equation in \( R \) and \( H \) has to be satisfied. One should keep in mind, though, the important fact that, because a Cauchy horizon is a null hypersurface, one needs two equations to be satisfied in order to be able to identify an AH with a Cauchy horizon. On one hand, for a given Hubble expansion \( H \), Eq. (5.6) gives for all times \( t \) the corresponding location \( R \) of the AH. On the other hand, Eq. (5.10) is required for such a horizon to be null at whatever location it happens to be and at any corresponding time. However, satisfying both equations (5.6) and (5.10) can only happen at a finite number of instants of time \( t \) because extracting \( t \) in terms of \( R \) from the first and then substituting in the second leads to an algebraic equation in \( R \) alone and hence can only yield discrete pairs \((t_0, R_0)\).

These results show that, whenever the background is not static and not artificially created by a conformal transformation like in a McVittie spacetime, the would-be Cauchy horizon would only exist at certain instants of time. In the next subsection, we illustrate the occurrence of the various apparent horizons using a concrete model of an expanding universe.

**A. AHs in a charged McVittie spacetime with scale factor \( a(t) = a_0 t^p \)**

In keeping with the spirit of Ref. [43], we choose a dark-energy dominated and accelerating FLRW “background” with scale factor \( a(t) = a_0 t^p \) with \( p > 0 \) and, as a specific example, \( p = 3 \). The behaviour of the AH radii (in units of \( m \)) are plotted in Figs. 7 and 8 as functions of the comoving time \( t \) (also measured in units of \( m \)) for the particular parameter choice \( Q = \pm m/2 \). (For ease of illustration, the scale is different in the two figures.)

Figure 7 reports the AH radii as given by (5.6) in the spacetime region \( R > m + \sqrt{m^2 - Q^2} \) above the spacetime singularity. This is again a “C-curve” phenomenology. There are no AHs in this spacetime region at early times. Then, a black hole AH and a cosmological AH form as a pair at a critical time. The cosmological horizon expands forever, while the black hole horizon asymptotes to the spacetime singularity at \( R = m + \sqrt{m^2 - Q^2} \) (the horizontal dashed line). The (blue) dashed, oblique line of equation \( R = H^{-1} - m \) is an asymptote for the cosmological AH at late times and large radii.

![Figure 7](image1.png)

**FIG. 7.** The AH radii in the charged McVittie spacetime with FLRW scale factor \( a(t) = a_0 t^p \). A black hole AH and a cosmological AH are born at a critical time. The cosmological horizon expands forever, while the black hole AH asymptotes to the spacetime singularity at \( R = m + \sqrt{m^2 - Q^2} \) (the horizontal black dashed line). The (blue) dashed, oblique line of equation \( R = H^{-1} - m \) is an asymptote for the cosmological AH at late times and large radii.

The extremal case \( |Q| = m \) can be discussed analytically. In this case the singularity is located at \( R = m \) and Eq. (5.6) for the AHs can be solved exactly, giving

\[
R_{AH}^{(\pm)} = \frac{1 \pm \sqrt{1 - 4mH}}{2H}.
\]  

(5.11)

In a universe with scale factor \( a(t) = a_0 t^p \), the cosmological AH at late times and large radii, as remarked in Ref. [61], embedding the RN black hole in a time-dependent cosmological “background” (not a locally static de Sitter one) has the effect of making the Cauchy horizon disappear. This fact is consistent with the known instability of this horizon in the RN spacetime [40].
logical and black hole AHs $R^{(±)}_{AH}$ and $R^{(−)}_{AH}$ are created at the critical time $t_0 = 4pm$ and exist for all times $t > t_0$. Since $R > m_0$, for $t > t_0$ we have

$$m_0 < R^{(−)}_{AH} < R^{(±)}_{AH} < \frac{1}{H}.$$  \hspace{0.5cm} (5.12)

Again, no inner black hole Cauchy horizon exists, restoring determinism to GR.

VI. CONCLUSIONS

We have investigated cosmological black holes obtained by conformally transforming either the neutral Schwarzschild black hole or the charged Reissner-Nordström black hole. The first case consists of the uncharged non-rotating Thakurta spacetime, while the second one consists of the charged version of this geometry. The general pattern emerging from obtaining conformal solutions by using “seed” solutions of the Einstein equations has been studied. The analysis shows that, while the resulting action after such a transformation is no longer an Einstein-Hilbert type action but a Brans-Dicke action with the pathological Brans-Dicke parameter $\omega = −3/2$, the resulting Brans-Dicke field equations may nevertheless be interpreted as effective Einstein field equations with an effective imperfect fluid playing the role of an additional source besides the electrovacuum associated with the seed. The imperfect character of the induced effective fluid is manifested by the emergence of an energy flow and is unavoidable as long as the chosen embedding background is evolving, i.e., the scale factor is time-dependent. This pattern is general and arises for both charged and uncharged black holes embedded in cosmological “backgrounds” obtained by this conformal technique. However, while an induced effective energy flow is automatically obtained even if it is absent in the original seed solution, no charged flow emerges even for charged black hole seeds, as a consequence of the conformal invariance of the vacuum Maxwell equations.

This technique has then been used to tackle the problem of determinism in GR by building the charged cosmological black hole in the form of the charged non-rotating Thakurta spacetime. Such a spacetime requires the presence of a neutral, but imperfect, fluid as a source. Nevertheless, the corresponding black hole is more realistic than the RNds black hole as the former might be chosen to be embedded in a FLRW universe, in contrast to the latter which lives in a de Sitter “background”. We found that the Cauchy horizon of such a spacetime always hides, in the non-extremal case, behind a singularity.

The same analysis has been performed on another type of charged black hole embedded in a cosmological “background”, the McVittie geometry. We found that whenever the background is not static nor artificially created by a conformal transformation as in the case of the charged non-rotating Thakurta spacetime, the would-be Cauchy horizon appears only at certain instants of time. Hence, those locations could not really qualify as the loci of a real Cauchy horizon that would put determinism within GR in jeopardy. Indeed, Cauchy horizons are necessarily null hypersurfaces [1], whereas the would-be inner Cauchy horizon of McVittie spacetime is mainly a non-null AH, except at discrete instants of time.

The use of the charged McVittie spacetime (excluding the case in which it reduces to the RNds space for $H = \text{const.}$) has conceptual weaknesses. First, before the critical time at which the black hole/cosmological AH pair is created, there is a naked singularity and this solution of the Einstein equations cannot be obtained as the development of regular Cauchy data. Second, AHs ultimately depend on the foliation [51], although all spherically symmetric foliations (the only ones of practical importance here) determine the same AHs [52]. Third, while the RN solution is the most general spherical and locally static electrovacuum solution of the Einstein equations with positive $\Lambda$, in the presence of a fluid there is no general solution and the charged Thakurta and McVittie geometries cannot claim such a degree of generality. In spite of these shortcomings, a time-dependent FLRW “background” is a more general setup for a charged black hole than the de Sitter one. Once (local) staticity is removed, there is no trace of inner Cauchy horizons in charged black holes, according to our models. This result agrees with those of [48, 49] which restore determinism to Einstein theory.

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