First-passage time of run-and-tumble particles

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Abstract. We solve the problem of first-passage time for run-and-tumble particles in one dimension. Exact expression is derived for the mean first-passage time in the general case, considering external force fields and chemotactic fields, giving rise to space-dependent swim speed and tumble rate. Agreement between theoretical formulae and numerical simulations is obtained in the analyzed case studies —constant and sinusoidal force fields, constant gradient chemotactic field. Reported findings can be useful to get insights into very different phenomena involving active particles, such as bacterial motion in external fields, intracellular transport, cell migration, animal foraging.

1 Introduction

First-passage problems are of great interest in many disciplines, from physics and chemistry to engineering and biology [1,2]. After the seminal paper of Kramers on kinetic reactions [3], a lot of efforts have been devoted to analyze first-passage processes, or the related narrow-escape problems [4], in a variety of cases, both using theoretical modeling and numerical analysis [5–11]. The basic question is to find the mean first-passage time (MFPT), i.e. the mean time needed for a particle (that can be an atom, molecule, cell, animal or also a signal) to reach for the first time a specific site. Usually one considers processes in which the particle’s motion is described by Brownian dynamics and the probability density function (PDF) obeys a Fokker-Planck equation [12]. This encompasses many real processes involving thermal passive objects, such as molecules, atoms or colloidal particles. In recent years a new kind of systems, characterized by an active nature, have been found to describe many interesting situations in real living world, from bacterial baths [13,14] to cells migration [15] or animals’ movements [16]. Such systems are characterized by self-propulsion mechanisms inducing a persistent motion, that can be described by simple random walk models [17,18]. Among them the run-and-tumble model [14] has been found to be a very powerful modeling of real situations, allowing for analytic treatments [19–21] and easy implementation in numerical simulations [22–25]. In the simple version of the model, particles perform a straight line motion at constant speed alternating tumble events (assumed to be random Poissonian processes), in which particle’s orientation randomly changes. The run-and-tumble model captures the behavior of motile bacteria, such as E. coli [13,14], but is also related to many different physical processes, such as the motion of electrons in metals (Lorentz kinetic model) [26,27] the propagation of signals in transmission lines (telegraph equation) [28], the dynamics instability of microtubules [29,30], the Dirac equation for relativistic particles in one spatial dimension [31,32].

In this paper we analyze the first-passage problem for run-and-tumble particles in one dimension. We consider the general case in which external potentials and chemotactic fields are present, inducing a space dependence of particles’ speed and tumble rate. A general expression of the mean first-passage time is given in the general case, and specialized to some interesting case studies —linear and sinusoidal potentials, constant gradient chemotactic field. Well known expression valid for Brownian particles is recovered in the limit of high tumble rate and particle’s velocity.

2 First-passage time

We consider run-and-tumble particles in one dimension. Denoting with \( P_R(x,t) \) and \( P_L(x,t) \) the PDFs of right-oriented and left-oriented particles, with \( v_R \) and \( v_L \) their swim speeds and with \( \alpha_R \) and \( \alpha_L \) the tumble rates, we can write the continuity equations as follows [19,20,33–35]:

\[
\partial_t P_R = -\partial_x (v_R P_R) - \frac{\alpha_R}{2} P_R + \frac{\alpha_L}{2} P_L, \\
\partial_t P_L = \partial_x (v_L P_L) + \frac{\alpha_R}{2} P_R - \frac{\alpha_L}{2} P_L.
\]

We treat here the very general case in which speeds \( v_{R,L}(x) \) and tumble rates \( \alpha_{R,L}(x) \) are space-dependent quantities, for example due to the presence of external...
force fields or chemotactic fields. We can write the equations for the total PDF $P(x,t) = P_+(x,t) + P_-(x,t)$ and the current $J(x,t) = v_+(x)P_+(x,t) - v_-(x)P_-(x,t)$ as

$$\partial_t P = - \partial_x J$$

and

$$\partial_t J = -(v_r + v_L) \frac{v_r v_L}{v_r + v_L} P + \frac{P}{2} (\alpha_r v_r - \alpha_L v_L) - (v_r - v_L) \partial_x J - \frac{J}{2} \left[ \frac{\alpha_r + \alpha_L + (v_r + v_L)}{v_r + v_L} \partial_x \left( \frac{v_r - v_L}{v_r + v_L} \right) \right],$$

where, for the sake of simplicity, the dependence on space and time of the different quantities has not explicitly indicated. We want to determine the mean first-passage time, that is the mean time needed for a particle starting its motion at $x = 0$ to reach for the first time the position $L$. In order to solve the problem we consider initial conditions $P_r(x,0) = \delta(x)$, $P_+(x,0) = 0$ and reflecting/absorbing boundary conditions, respectively, at $x = 0$ and $x = L$: $J(0,t) = 0$ and $P_i(L,t) = 0$, i.e. $J(L,t) = P(L,t)v_r(L)$. By noting that the survival probability that the particle has not yet been absorbed at time $t$ is $P(t) = \int_0^L dx P(x,t)$ and the probability density of the first-passage time is $\varphi(t) = -\partial_t P$, the MFPT $\tau = \int dt \varphi(t)$ can be written as

$$\tau = \int_0^L dx \frac{Q(x)}{v_r},$$

where

$$Q(x) = \int_0^\infty dt P(x,t).$$

From eqs. (3), (4) the quantity $Q(x)$ obeys the following equation:

$$-(v_r + v_L) \partial_x \left( \frac{2v_r v_L}{v_r + v_L} Q \right) + (\alpha_r v_r - \alpha_L v_L) Q = \alpha_r + \alpha_L + (v_r + v_L) \partial_x \left( \frac{v_r - v_L}{v_r + v_L} \right) - 2v_r \delta(x),$$

with boundary conditions $Q(L)v_r(L) = 1$. It is worth noting that the above equation is similar to that obtained for the PDF of a run-and-tumble particle in the stationary case [33, 21, 19]. We have then reduced the problem of MFPT to the search of a solution of a stationary equation with particular boundary conditions. We introduce the following quantities:

$$A(x) = \exp \left\{ \int_0^x dy \frac{\alpha_r v_r - \alpha_v v_L}{2v_r v_L} \right\}$$

and

$$B(x) = \int_x^L \frac{dy}{A(y)} \left[ \frac{\alpha_r + \alpha_L}{v_r + v_L} + \partial_y \left( \frac{v_r - v_L}{v_r + v_L} \right) \right].$$

The solution of eq. (7) can be written as

$$Q(x) = \frac{v_r(x) + v_L(x)}{2v_r(x)v_L(x)} \frac{A(x)}{v_r(L) + v_L(L)} \frac{1}{A(L)} + B(x)$$

The mean first-passage time $\tau$ is finally obtained from eq. (5).

Some interesting limits are:

i) The free particles case (absence of external force fields and chemotactic fields), i.e. $v_{r,L}(x) = v$ and $\alpha_{r,L} = 0$, leads to

$$\tau_{\text{free}} = \frac{L}{v} + \frac{L^2}{2D_0},$$

where $D_0 = \nu^2/\alpha$. It is worth noting that the MFPT can be written as the sum of a pure ballistic term plus a diffusive term.

ii) In the case of no tumbling, $\alpha_{r,L} = 0$, we get

$$\tau_{\text{no-tumble}} = \int_0^L dx \frac{v_r(x)}{v_r},$$

which is the time a particle with space-dependent velocity $v_r(x)$ takes to go form 0 to $L$.

In the following we will consider the two cases in which particles are immersed in an external potential field (sect. 3) or in a chemotactic field (sect. 4). A few case studies will be analyzed, deriving theoretical expressions and comparing them with numerical simulations. The latter are performed considering run-and-tumble particles which obey the 1D version of the equation of motion $\partial_x r = v \hat{e} + \mu f$, with $\hat{e}$ the unit vector indicating the direction of motion of the particle, $\mu$ the particles mobility and $f(x)$ the external force. The direction $\hat{e}$ is updated with rate $\gamma = \gamma \hat{e} \cdot \nabla c(x)$ (the chemotactic field) randomly choosing, in a uniform way, the new one $\hat{e} = \pm \hat{x}$ in 1D (see the following sections for details).

3 External forces

We first analyze the case in which particles feel an external potential field and chemotactic effects are absent, i.e. the tumbling rates of right-oriented and left-oriented particles can be considered equal and uniform

$$\alpha_{r,L}(x) = \alpha$$

and the speeds are given by

$$v_r(x) = v + \mu f(x),$$

$$v_L(x) = v - \mu f(x),$$

with $v$ the free speed of the particles, $\mu$ the particles mobility and $f(x) = -\partial_x V(x)$ the force due to an external field.
potential \( V(x) \). In such a case, eqs. (8), (9), (10) become

\[
A(x) = \exp \left\{ \int_0^x dy \frac{\mu f(y)}{D(y)} \right\},
\]

\[
B(x) = \int_x^L dy \left( \alpha + \mu \frac{\partial_y f}{v A(y)} v A(y) \right)^{-1},
\]

\[
Q(x) = \frac{v A(x)}{\alpha D(x)} \left[ \frac{v - \mu f(L)}{v A(L)} + B(x) \right],
\]

where \( D(x) \) is the local diffusion coefficient \( D(x) = [v^2 - \mu^2 f^2(x)]/\alpha \).

It is worth noting that the Brownian case is obtained in the limit \( \alpha, v \to \infty \) with constant \( D_0 = v^2/\alpha \), leading to the well known expression of the mean escape time for Brownian particles \([12,36]\)

\[
\tau_b = \frac{1}{D_0} \int_0^L dx \ e^{-\beta V(x)} \int_x^L dy \ e^{\beta V(y)},
\]

where \( \beta = \mu/D_0 = \mu \alpha/v^2 \).

We consider here two kinds of external potentials. The first one is a linear potential

\[
V(x) = f_0 \ x
\]

where \( f_0 = \Delta/L \) (\( \Delta \) is the barrier height). In this case the mean escape time \( \tau \) has an explicit expression

\[
\tau = \frac{v(v + \mu f_0)}{\alpha (\mu f_0)^2} \left( e^{\mu f_0 L/D} - 1 \right) - \frac{L}{\mu f_0},
\]

where \( D = (v^2 - \mu^2 f_0^2)/\alpha \). In the free limit, \( f_0 \to 0 \), one has \( \tau = \tau_{\text{free}} \) —see eq. (11)— while for \( f_0 \to v/\mu \) one has \( D \to 0 \) and the MFPT diverges as \( \tau \simeq (2/\alpha) \exp(vL/D) \). In fig. 1 (upper panel) the MFPT \( \tau \) is reported as a function of \( f_0 \) for different values of \( L \). Theoretical predictions are in agreement with numerical simulations.

In the Brownian limit, \( \alpha, v \to \infty \) at constant \( D_0 = v^2/\alpha \), one has

\[
\tau_b = \frac{D_0}{(\mu f_0)^2} \left( e^{\mu f_0 L/D_0} - 1 \right) - \frac{L}{\mu f_0}.
\]

We observe that \( \tau_b < \tau \), indicating that in the Brownian limit particles take less time to reach absorbing boundaries. It is worth noting, however, that the comparison is made considering equal diffusivities, and this does not imply that in real situation, at room temperature, thermally activated particles are faster than self-propelling ones. Indeed, just to make an example, considering parameters’ values suitable for \( E. \ coli \) \( (v \approx 30 \mu \text{m/s}, \mu \approx 60 \mu \text{m} \text{pN}^{-1} \text{s}^{-1}, \alpha \approx 1 \text{s}^{-1}) \) one has that thermal diffusivity at room temperature is about four order of magnitude less than the self-propelling one \( D \approx 10^4 \mu \text{m}^2 \text{s}^{-1} \), or, in other words, in order to have the same diffusivity one has to consider a temperature of \( 10^6 \) K.

As second example we consider a sinusoidal potential

\[
V(x) = \frac{\Delta}{2} \left( 1 - \cos \pi x/L \right),
\]

where \( \Delta \) again is the barrier height. The corresponding force on particles is \( f(x) = -f_0 \sin \pi x/L \), where \( f_0 = \pi \Delta/2L \). We can write an explicit expression for the quantity \( A \) defined by eq. (8)

\[
A(x) = \exp \left\{ \frac{\alpha L [W(x) - W(0)]}{\pi v \sqrt{1 - \eta^2}} \right\},
\]

where \( \eta = \mu f_0/v \) and the quantity \( W \) is given by \( W(x) = \tan^{-1}[\eta \cos(\pi x/L)/\sqrt{1 - \eta^2}] \). The MFPT \( \tau \) can be obtained by numerical integration of eq. (5), by using eqs. (9), (10). In fig. 1 (lower panel) the MFPT \( \tau \) is reported as a function of \( f_0 \) for different values of \( L \) (see footnote 1). It is worth noting that, as already stated above,
one observes a divergence of the escape time $\tau$ when the external force equals the self-propelling one $v/\mu$ ($f_0 \to 1$ in reduced unit). Indeed, due to the finite velocity of run-and-tumble particles, they can be easily confined by strong enough external fields, inducing interesting trapping phenomena, recently investigated in random environments, such as speckle fields \[37\] or random distribution of obstacles \[23,38\].

### 4 Chemotaxis

We now analyze the effects of chemotactic fields. Bacteria, such as *E. coli*, are able to sense the environment and, through a chemical internal circuit, modify their tumble rate in order to achieve a net preferential motion towards high nutrient concentration \[13\]. The adopted strategy consists in evaluating changes in ambient chemical concentration performing a time-integral of concentration $c$ filtered by a suitable kernel function \[13\]. A simplified expression of the tumble rate $\alpha$ as a function of chemotactic field $c(x)$ is obtained in the limit of weak concentration gradient, giving rise to space dependent right and left tumble rates

$$
\alpha_R(x) = \alpha - \gamma v \partial_x c(x),
$$

$$
\alpha_L(x) = \alpha + \gamma v \partial_x c(x),
$$

where $\alpha$ is the tumble rate in the absence of chemotactants and $\gamma$ the strength of the particles reaction to the chemicals \[14,33\]. It is worth noting, however, that modeling chemotaxis through spatially varying tumble rates could be a quite crude approximation of the mechanism adopted by real organisms, which instead involve more complex time integration \[14\]. By considering force-free particles

$$
v_{n.L}(x) = v,
$$

we have the following expression for the quantities $A, B$, and $Q$:

$$
A(x) = \exp \left\{ \int_0^x dy \, \gamma \partial_y c(y) \right\},
$$

$$
B(x) = \int_x^L dy \, \alpha \left| v A(y) \right|^{-1},
$$

$$
Q(x) = \frac{A(x)}{v} \left[ \frac{1}{A(L)} + B(x) \right]
$$

and the MFPT takes the form

$$
\tau = \int_0^L dx \, \frac{v}{c} \left[ e^{-\gamma c(L)} + \frac{\alpha}{v} \int_x^L \, dy \, e^{-\gamma c(y)} \right].
$$

We consider here the simple case of constant gradient concentration

$$
c(x) = c_0 \, x.
$$

Fig. 2. Mean first-passage time $\tau$ of run-and-tumble particles in chemotactic fields as a function of field strength. The two curves refer to the cases of chemoattractant boundary, $c_0 > 0$ (full symbols) and chemorepellent boundary, $c_0 < 0$ (open symbols). Lines are theoretical predictions and symbols are from numerical simulations. Quantities are expressed in reduced units (see footnote\(^1\)).

The MFPT reads

$$
\tau = \frac{L}{\gamma c_0 D_0} - \frac{1}{\gamma^2 c_0^2 D_0} \left( 1 - \frac{v \gamma c_0}{\alpha} \right) \left( 1 - e^{-\gamma c_0 L} \right).
$$

For $c_0 > 0$ the maximum chemoattractant concentration is at the absorbing boundary $L$, resulting in a net preferential swimming towards such a point. For $c_0 < 0$ the maximum is at $x = 0$ and reaching the boundary is made more difficult by the chemotactic field. For weak concentration gradient one has:

$$
\tau \approx \tau_{\text{free}} - c_0 \frac{\gamma L^2}{2v} \left( 1 - \frac{L \alpha}{3v^2} \right),
$$

which, in the absence of chemicals, $c_0 = 0$, reduce to the free case $\tau = \tau_{\text{free}}$. In Fig. 2 the MFPT $\tau$, eq. (31), is reported as a function of $|c_0|$ for the two cases of positive and negative $c_0$, corresponding, respectively, to chemoattractant and chemorepellent boundary.

We conclude this section discussing a different form of chemotaxis, in which the cell sets its tumble rate considering the absolute attractant concentration instead of its temporal changes. This is the case, for example, of mutant bacteria lacking particular enzymes required to perform temporal comparisons of concentration \[33\]. If we assume instantaneous sensing we can write tumble rates as

$$
\alpha_{n.L}(x) = \alpha - \tilde{\gamma} \, c(x),
$$

leading to the following expression for MFPT:

$$
\tau = \tau_{\text{free}} - \frac{\gamma}{v^2} \int_0^L dx \, x \, c(x)
$$

which, in the case of linear concentration $c(x) = c_0 \, x$, reduces to

$$
\tau = \tau_{\text{free}} - \frac{\gamma c_0 L^3}{3v^2}.
$$
However, real cells measure chemical concentrations in a finite time, and then a more appropriate form of tumbling rates in the case of mutant bacteria is [33]

\[
\alpha_p(x) = \alpha - \tilde{\gamma} c(x) + \gamma v \partial_x c(x),
\]

\[
\alpha_L(x) = \alpha - \tilde{\gamma} c(x) - \gamma v \partial_x c(x).
\]

The ratio \( \gamma/\tilde{\gamma} \) is proportional to the average time the cell spends to measure chemical concentration. The MFTP now reads

\[
\tau = \int_0^L \frac{dx}{v} e^{-\gamma c(x)}
\]

\[
\times \left[ e^{\gamma c(L)} + \int_x^L \frac{dy}{v} e^{\gamma c(y)} \left( \alpha - \tilde{\gamma} c(y) \right) \right]
\]

and, for constant gradient field \( c(x) = c_0 x \)

\[
\tau = \frac{L}{\gamma c_0 v^2} \left[ \frac{\tilde{\gamma}}{\gamma} \left( \frac{\gamma c_0 L}{2} - 1 \right) - \alpha \right]
\]

\[
+ \frac{e^{\gamma c_0 L} - 1}{\gamma c_0 v^2} \left[ \alpha + v \gamma c_0 + \tilde{\gamma} \left( 1 - \gamma c_0 L \right) \right].
\]

For weak concentration gradient one has

\[
\tau \simeq \tau_{\text{free}} + c_0 \frac{\gamma L^2}{2v} \left[ 1 + \frac{L}{3v} \left( \alpha - \frac{2 \tilde{\gamma}}{\gamma} \right) \right],
\]

indicating a reverse taxis (\( \tau > \tau_{\text{free}} \)) when \( \tilde{\gamma}/\gamma < 3v/2L + \alpha/2, \) i.e. when the average time required for the cell to measure chemical concentration is long enough [33].

5 Conclusions

First-passage time problems for active particles ruled by run-and-tumble dynamics are solved in one dimension, considering generic space-dependent external forces and chemotactic fields. Exact expressions are given for the mean first-passage time and tested with simulations in simple case studies. A diffusion limit is obtained for high particle’s velocity and tumbling rate at constant diffusivity, where well-known expressions for Brownian particles are recovered. Our results can be of interest to many disciplines, allowing a better understanding of phenomena involving persistent motion, such as barrier escaping of active particles [39,40], intracellular transport [41], foraging and predation phenomena in the animal kingdom [42–45].

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References

1. P. Hänggi, P. Talkner, M. Borkovec, Rev. Mod. Phys. 62, 251 (1990).
2. S. Redner, A Guide to First-Passage Processes (Cambridge University Press, Cambridge, 2001).
3. H.A. Kramers, Physica 7, 284 (1940).
4. Z. Schuss, J. Sci. Comput. 53, 194 (2012).
5. A. Engel, F. Moss, Phys. Rev. A 38, 571 (1988).
6. S. Condamin et al., Nature 450, 77 (2007).
7. M. Gitterman, Phys. Rev. E 62, 6065 (2000).
8. Z. Schuss, A. Singer, D. Holcman, Proc. Natl. Acad. Sci. U.S.A. 104, 16098 (2007).
9. O. Bénichou, R. Voituriez, Phys. Rev. Lett. 100, 168105 (2008).
10. G. Oshanin, M. Tamm, O. Vasilyev, J. Chem. Phys. 132, 235101 (2010).
11. T.G. Mattos, C. Mejia-Monasterio, R. Metzler, G. Oshanin, Phys. Rev. E 86, 031143 (2012).
12. H. Risken, The Fokker-Planck Equation: Methods of Solution and Applications (Springer, Berlin, 1989).
13. H.C. Berg, E. Coh in Motion (Springer-Verlag, New York, 2004).
14. M.E. Cates, Rep. Prog. Phys. 75, 42601 (2012).
15. M. Le Berre et al., Phys. Rev. Lett. 111, 198101 (2013).
16. T. Vicsek, A. Zafeiris, Phys. Rep. 517, 71 (2012).
17. E.A. Codling, M.J. Plank, S.J. Benhamou, J. R. Soc. Interface 5, 813 (2008).
18. P. Romanczuk, M. Bär, W. Ebeling, B. Lindner, L. Schimansky-Geier, Eur. Phys. J. Special Topics 202, 1 (2012).
19. J. Tailleur, M. Cates, Phys. Rev. Lett. 100, 218103 (2008).
20. K. Martens et al., Eur. Phys. J. E 35, 84 (2010).
21. L. Angelani, EPL 102, 20004 (2013).
22. M.B. Wan, C.J. Olson Reichhardt, Z. Nussinov, C. Reichhardt, Phys. Rev. Lett. 101, 018102 (2008).
23. C. Reichhardt, C.J. Olson Reichhardt, Phys. Rev. E 90, 012701 (2014).
24. L. Angelani, R. Di Leonardo, G. Ruocco, Phys. Rev. Lett. 102, 048104 (2009).
25. M. Paoluzzi, R. Di Leonardo, L. Angelani, J. Phys.: Condens. Matter 25, 415102 (2013).
26. H.A. Lorentz, Arch. Néerl. III, 336 (1905).
27. H.A. Lorentz, Collected Papers (Martinus Nijhoff, The Hague, 1936).
28. S. Goldstein, J. Mech. Appl. Math. 4, 129 (1951).
29. M. Dogterom, S. Leibler, Phys. Rev. Lett. 70, 1347 (1993).
30. D.J. Bicout, Phys. Rev. E 56, 6556 (1997).
31. B. Gaveau, T. Jacobson, M. Kac, L.S. Schulman, Phys. Rev. Lett. 53, 419 (1984).
32. V. Balakrishnan, S. Lakshmibala, New J. Phys. 7, 11 (2005).
33. M.J. Schnitzer, Phys. Rev. E 84, 84201 (2011).
34. J. Tailleur, M.E. Cates, Phys. Rev. Lett. 100, 218103 (2008).
35. M. Paoluzzi, R. Di Leonardo, L. Angelani, to be published in Condens. Matter (2014) [arXiv:cond-mat/1401.6944].
36. O. Chepizhko, F. Peruani, Phys. Rev. Lett. 111, 160604 (2013).
37. P.S. Burada, B. Lindner, Phys. Rev. E 85, 032102 (2012).
40. N. Koumakis, A. Lepore, C. Maggi, R. Di Leonardo, Nat. Commun. 4, 2588 (2013).
41. P.C. Bressloff, J.M. Newby, Rev. Mod. Phys. 85, 135 (2013).
42. G.M. Viswanathan, M.G.E. da Luz, E.P. Raposo, H.E. Stanley, The Physics of Foraging: An Introduction to Random Searches and Biological Encounters (Cambridge University Press, Cambridge, 2011).
43. S. Redner, P.L. Krapivsky, Am. J. Phys. 67, 1277 (1999).
44. A. Gabel, S.N. Majumdar, N.K. Panduranga, S. Redner, J. Stat. Mech. P05011 (2012).
45. L. Angelani, Phys. Rev. Lett. 109, 118104 (2012).