TETRAHEDAL GEOMETRY FROM AREAS

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Abstract. We solve a very classical problem: providing a description of the geometry of a Euclidean tetrahedron from the initial data of the areas of the faces and the areas of the medial parallelograms of \cite{9} or equivalently of the pseudofaces of \cite{5}. In particular, we derive expressions for the dihedral angles, face angles and (an) edge length, the remaining parts being derivable by symmetry or by identities in the classic compendium of results on tetrahedral geometry of Richardson \cite{6}.

We also provide an alternative proof using (bi)vectors of the result of \cite{10} that four times the sum of the squared areas of the medial parallelograms is equal to the sum of the squared areas of the faces.

For a number of years the authors of this paper have been interested in creating models for topological quantum field theory \cite{3,2} and quantum gravity \cite{1} in three and four dimensions using extensions of the quantum theory of spin. Spin is the quantum version of angular momentum. Classically angular momentum is represented as the cross product of two vectors, $\vec{r} \times \vec{p}$.

The cross product of two vectors is strictly speaking a bivector, or antisymmetric rank two tensor, which can be thought of as an oriented area element, regardless of how many dimensions the ambient space has. This gave rise to the suggestion of considering models for physical theories on triangulated manifolds in which the geometric data live on the 2-dimensional faces of the triangulation. This suggestion has proven very influential, particularly in quantum gravity (cf. \cite{1,4}).

The cross product of two vectors is not so commonly thought of as a bivector because in three dimensions it can be identified with its dual vector. This amounts to describing an oriented area element by a normal vector of length equal to the area, with the direction of the normal as one is taught in physics classes, using the right-hand rule (a choice of orientation on the ambient 3-dimensional space). Most models for TQFTs and gravity in four dimensions \cite{3,2,1,4} involve not only assigning a spin to each 2-dimensional face, but another spin to each 3-simplex to give so-called (quantum) 15j-symbols (10 spins for the 2-dimensional faces, and 5 for the 3-dimensional faces of a 4-simplex). Motivated by the suggestion that the extra spin corresponded to (the area of) a parallelogram with vertices on the midpoints of
four edges, in 1999, the second author gave a formula for the areas of these “medial parallelogram” in terms of the six edge lengths [9]. His observation, made soon after, that for any tetrahedron, the sum of the squares of the four faces of the parallelogram was always four times the sum of the areas of the three medial parallelograms, was eventually published [10] in the context of its generalization to a relation between squared hypervolumes of faces and squared hypervolumes of medial sections of simplexes, in the sense of Talata [7], for simplexes of any dimension.

Independently in 2005, motivated by purely classical geometric considerations, McConnell [5], in work published only on a private website in 2012, considered a notion equivalent to the medial parallelogram, which he termed “pseudofaces”, the quadrilateral obtained by projecting the tetrahedron onto a plane, which (albeit described differently in his paper) may be taken without loss of generality to be the plane containing the medial parallelogram. The medial parallelogram is then the Varignon parallelogram [8] of McConnell’s pseudoface, its area being exactly half that of the pseudoface. Thus the relation, discovered by McConnell, corresponding to that cited above is simply that the sum of the squares of the face areas is the sum of the squares of the pseudoface areas.

Considerations of what sort of quantum geometry we were constructing led to the classical problem of recovering the shape of a tetrahedron from its oriented areas. Along the way, we will see that the (bi)vector area elements of the medial parallelograms contain the same information as sums of the (bi)vector elements of the faces, and thus contain information about the dihedral angles. Recovering the shape of the tetrahedron reduces to finding its six edge lengths.

Aspects of this paper, in particular our new proof of the relation involving sums of squared areas, can be thought of as “using physics to do math,” although on a much more modest scale than, say, Gromov-Witten theory.

It is the purpose of this note to show explicitly how any six of the seven inequivalent areas thus associated with a tetrahedron determine the dihedral angles, face angles, and finally the edge lengths, thus completely determining the geometry of the tetrahedron.

We fix notation following the classic compendium of trigonometric results on tetrahedra by Richardson [6], omitting those for which the present work has no use:

\( OABC \) is the tetrahedron;
\( \Delta_0, (\text{resp. } \Delta_1, \Delta_2, \Delta_3) \) the areas of the faces opposite to \( O \), (resp. \( A, B, C \));
\( a, b, c, x, y, z \) the lengths of \( OA, OB, OC, BC, CA, \) and \( AB \), respectively;
\( A, (\text{resp. } B, C, X, Y, Z) \) the dihedral angles whose edge is (of length) \( a \) (resp. \( b, c, x, y, z \));
\( \alpha_1, \beta_2 \) and \( \gamma_3 \) the angles \( BOC, COA, AOB \);
\( \alpha_0, \beta_3 \) and \( \gamma_2 \) the angles \( BAC, OAB, CAO \);
\( \alpha_3, \beta_0 \) and \( \gamma_1 \) the angles \( OBA, ABC, CBO \);
\( \alpha_2, \beta_1 \) and \( \gamma_0 \) the angles \( OCA, BCO, ACB \), respectively.
As Richardson [6] observes of his notational choices, $\alpha_i$ is not adjacent to either $a$ or $x$, $\beta_i$ is not adjacent to either $b$ or $y$, and $\gamma_i$ is not adjacent to either $c$ or $z$, for $i = 0, 1, 2, 3$, and that the subscripts are the same as those of the faces $\Delta_i$ in which the angles lie.

Finally we need notations for the areas not considered by Richardson [6] as they only arose in the work of Yetter [9, 10] or McConnell [5].

Following McConnell [5], $P, Q,$ and $R$ are the areas of the projections of the tetrahedron onto planes parallel to $a$ and $x$, $b$ and $y$, and $c$ and $z$, respectively, while the areas of the medial parallelograms lying in the plane parallel to those pairs of edges midway between them are $L, M$ and $N$, respectively.

Before turning to the derivation of dihedral angles, face angles and (an) edge length from the areas we provide a new proof the following:

**Theorem 1.** For any tetrahedron

$$4(L^2 + M^2 + N^2) = \Delta_0^2 + \Delta_1^2 + \Delta_2^2 + \Delta_3^2$$

*Proof.* Regard the tetrahedron as a region in a fluid at rest, and for simplicity exerting a constant pressure of 1. The net force acting on the region is zero, but this can be expressed as the sum of forces acting by inward normal vectors to the faces, with magnitude equal to the area of the face. Thus letting the addition of the vector mark to our notation for the face areas indicate these inward normal we have

$$\vec{\Delta}_0 + \vec{\Delta}_1 + \vec{\Delta}_2 + \vec{\Delta}_3 = \vec{0},$$

from which it follows that the sum of any two inward normals is equal to the negative of the sum of the other two inward normals (for example $\vec{\Delta}_0 + \vec{\Delta}_1 = -\vec{\Delta}_2 - \vec{\Delta}_3$).

The same argument applies to the regions bounded by a medial parallelogram, two triangles (each a quarter of the area of the face containing it) and two trapezoids (each three-quarters the area of the face containing it). If we let the addition of a vector mark to the name of the area of a medial parallelogram indicate the normal vector to the parallelogram in the direction of the non-incident edge containing $O$, we then have

$$\vec{L} + \frac{3}{4} \vec{\Delta}_3 + \frac{3}{4} \vec{\Delta}_2 + \frac{1}{4} \vec{\Delta}_1 + \frac{1}{4} \vec{\Delta}_0 = \vec{0},$$

$$\vec{M} + \frac{3}{4} \vec{\Delta}_3 + \frac{3}{4} \vec{\Delta}_1 + \frac{1}{4} \vec{\Delta}_2 + \frac{1}{4} \vec{\Delta}_0 = \vec{0},$$

and

$$\vec{N} + \frac{3}{4} \vec{\Delta}_2 + \frac{3}{4} \vec{\Delta}_1 + \frac{1}{4} \vec{\Delta}_3 + \frac{1}{4} \vec{\Delta}_0 = \vec{0}.$$ 

Using the previous observation to rewrite the summands with a coefficient of $\frac{1}{4}$ or those with a coefficient of $\frac{3}{4}$ then gives

$$2\vec{L} + \vec{\Delta}_3 + \vec{\Delta}_2 = \vec{0} = 2\vec{L} - \vec{\Delta}_1 - \vec{\Delta}_0,$$

$$2\vec{M} + \vec{\Delta}_3 + \vec{\Delta}_1 = \vec{0} = 2\vec{M} - \vec{\Delta}_2 - \vec{\Delta}_0,$$
\[2\vec{N} + \Delta_2 + \Delta_1 = 0 = 2\vec{N} - \Delta_3 - \Delta_0.\]

Armed with these relations, we can now consider dot products:

From the tetrahedron we have that

\[-\Delta_i^2 = -|\vec{\Delta}_i|^2 = \sum_{j \neq i} \vec{\Delta}_i \cdot \vec{\Delta}_j\]

while from the last set of equations we obtain

\[4L^2 = |\vec{\Delta}_3 + \vec{\Delta}_2|^2 = |\vec{\Delta}_1 + \vec{\Delta}_0|^2,\]
\[4M^2 = |\vec{\Delta}_3 + \vec{\Delta}_1|^2 = |\vec{\Delta}_2 + \vec{\Delta}_0|^2,\]
\[4N^2 = |\vec{\Delta}_2 + \vec{\Delta}_1|^2 = |\vec{\Delta}_3 + \vec{\Delta}_0|^2.\]

Adding all six equations expressing four times a squared parallelogram area as a magnitude of a sum of face vectors gives

\[8(L^2 + M^2 + N^2) = \sum_{i,j,i < j} |\vec{\Delta}_i + \vec{\Delta}_j|^2 = \sum_{i,j,i < j} |\vec{\Delta}_i|^2 + 2\vec{\Delta}_i \cdot \vec{\Delta}_j + |\vec{\Delta}_j|^2\]

Now, in the last expression, the square of each face area occurs three times. But the middle terms involving a dot product can be rearranged to give the negatives of each squared face area once by the relations from the tetrahedron, giving us exactly twice the desired equation.

\[\square\]

In our notation, the basic results of McConnell [5] which motivated his definition of pseudofaces, together with the relationship between pseudofaces and medial parallelograms give

\[\Delta_0^2 + \Delta_2^2 - \Delta_0 \Delta_2 \cos X = P^2 = \Delta_2^2 + \Delta_3^2 - \Delta_2 \Delta_3 \cos A = 4L^2\]
\[\Delta_0^2 + \Delta_2^2 - \Delta_0 \Delta_2 \cos Y = Q^2 = \Delta_1^2 + \Delta_3^2 - \Delta_1 \Delta_3 \cos B = 4M^2\]
\[\Delta_0^2 + \Delta_3^2 - \Delta_0 \Delta_3 \cos Z = R^2 = \Delta_1^2 + \Delta_2^2 - \Delta_1 \Delta_2 \cos C = 4N^2\]

McConnell’s proof [5] of this identity is to observe that the pseudoface is the union of two triangles sharing a common edge, and that the union of the altitudes of these, together with the altitudes of two faces of which they are projections, when translated into some plane perpendicular to the common edge, form a euclidean triangle, from which the result follows by the ordinary law of cosines and the usual formula for areas of triangles in terms of altitude and base. However, observe that when the dot product is expressed in terms of the magnitudes of vectors and the cosine of the angle between them, and it is recalled that dihedral angles are the angle between normal vectors to the faces, these are exactly the six equations we just added to give the result of [10] relating the squares of face and parallelogram areas.

We can now solve these equations to express the cosines of all of the dihedral angles in terms of the seven inequivalent areas:
\[
\cos A = \frac{P^2 - \Delta_2^2 - \Delta_3^2}{\Delta_2 \Delta_3} = \frac{4L^2 - \Delta_2^2 - \Delta_3^2}{\Delta_2 \Delta_3}
\]
\[
\cos B = \frac{Q^2 - \Delta_1^2 - \Delta_3^2}{\Delta_1 \Delta_3} = \frac{4M^2 - \Delta_1^2 - \Delta_3^2}{\Delta_1 \Delta_3}
\]
\[
\cos C = \frac{R^2 - \Delta_1^2 - \Delta_2^2}{\Delta_1 \Delta_2} = \frac{4N^2 - \Delta_1^2 - \Delta_2^2}{\Delta_1 \Delta_2}
\]
\[
\cos X = \frac{P^2 - \Delta_0^2 - \Delta_1^2}{\Delta_0 \Delta_1} = \frac{4L^2 - \Delta_0^2 - \Delta_1^2}{\Delta_0 \Delta_1}
\]
\[
\cos Y = \frac{Q^2 - \Delta_0^2 - \Delta_2^2}{\Delta_0 \Delta_2} = \frac{4M^2 - \Delta_0^2 - \Delta_2^2}{\Delta_0 \Delta_2}
\]
\[
\cos Z = \frac{R^2 - \Delta_0^2 - \Delta_3^2}{\Delta_0 \Delta_3} = \frac{4N^2 - \Delta_0^2 - \Delta_3^2}{\Delta_0 \Delta_3}
\]

From this and the classical result that the dihedral angles on edges meeting at any vertex are the angles of a spherical triangle with the face angles at the vertex as the central angles, we can use the spherical law of cosines to express the angles between the edges in terms of the areas. We give the face angles for the face opposite \( O \), leaving the others to the interested reader.

\[
\cos \alpha_0 = \frac{\cos A + \cos Y \cos Z}{\sin Y \sin Z}
\]
\[
\cos \beta_0 = \frac{\cos B + \cos X \cos Z}{\sin X \sin Z}
\]
\[
\cos \gamma_0 = \frac{\cos C + \cos X \cos Y}{\sin X \sin Y}
\]

One can substitute the previous expressions for the dihedral angles in terms of areas to obtain formulas directly expressing the face angles in terms of the areas.

\[
\cos \alpha_0 = \sqrt{\frac{[\Delta_0^2 P^2 - 2\Delta_0^2 \Delta_2^2 - \Delta_0^2 \Delta_3^2 + (Q^2 - \Delta_0^2 - \Delta_2^2)(R^2 - \Delta_0^2 - \Delta_3^2)]^2}{[\Delta_0^2 \Delta_2^2 - (Q^2 - \Delta_0^2 - \Delta_2^2)^2][\Delta_0^2 \Delta_3^2 - (R^2 - \Delta_0^2 - \Delta_3^2)^2]}}
\]
\[
\cos \beta_0 = \sqrt{\frac{[\Delta_1^2 Q^2 - 2\Delta_1^2 \Delta_3^2 - \Delta_1^2 \Delta_0^2 + (P^2 - \Delta_1^2 - \Delta_3^2)(R^2 - \Delta_1^2 - \Delta_0^2)]^2}{[\Delta_1^2 \Delta_3^2 - (P^2 - \Delta_1^2 - \Delta_3^2)^2][\Delta_1^2 \Delta_0^2 - (R^2 - \Delta_1^2 - \Delta_0^2)^2]}}
\]
\[
\cos \gamma_0 = \sqrt{\frac{[\Delta_2^2 R^2 - 2\Delta_2^2 \Delta_0^2 - \Delta_2^2 \Delta_3^2 + (P^2 - \Delta_2^2 - \Delta_0^2)(Q^2 - \Delta_2^2 - \Delta_3^2)]^2}{[\Delta_2^2 \Delta_0^2 - (P^2 - \Delta_2^2 - \Delta_0^2)^2][\Delta_2^2 \Delta_3^2 - (Q^2 - \Delta_2^2 - \Delta_3^2)^2]}}
\]

And similarly for the sines:
\[
\sin \alpha_0 = \sqrt{1 - \frac{\Delta_0^2 P^2 - \Delta_0^2 \Delta_1^2 - \Delta_0^2 \Delta_2^2 + (Q^2 - \Delta_0^2 - \Delta_1^2)(R^2 - \Delta_0^2 - \Delta_2^2))^2}{\Delta_0^2 \Delta_1^2 - (Q^2 - \Delta_0^2 - \Delta_1^2)^2}[\Delta_0^2 \Delta_2^2 - (R^2 - \Delta_0^2 - \Delta_2^2)^2]}
\]

\[
\sin \beta_0 = \sqrt{1 - \frac{\Delta_0^2 Q^2 - \Delta_0^2 \Delta_1^2 - \Delta_0^2 \Delta_2^2 + (P^2 - \Delta_0^2 - \Delta_1^2)(R^2 - \Delta_0^2 - \Delta_2^2))^2}{\Delta_0^2 \Delta_1^2 - (P^2 - \Delta_0^2 - \Delta_1^2)^2}[\Delta_0^2 \Delta_2^2 - (R^2 - \Delta_0^2 - \Delta_2^2)^2]}
\]

\[
\sin \gamma_0 = \sqrt{1 - \frac{\Delta_0^2 R^2 - \Delta_0^2 \Delta_1^2 - \Delta_0^2 \Delta_2^2 + (P^2 - \Delta_0^2 - \Delta_1^2)(Q^2 - \Delta_0^2 - \Delta_2^2))^2}{\Delta_0^2 \Delta_1^2 - (P^2 - \Delta_0^2 - \Delta_1^2)^2}[\Delta_0^2 \Delta_2^2 - (Q^2 - \Delta_0^2 - \Delta_2^2)^2]}
\]

Having thus determined the angles between the edges, we can now use the ordinary law of sines and the area formula for a triangle in side-angle-side form to express the edge lengths in terms of the areas:

Consider the triangle opposite O, with area \(\Delta_0\). The area is given by

\[
\Delta_0 = \frac{1}{2} \sin \alpha_0 yz.
\]

Now by the law of sines, we have

\[
y = x \frac{\sin \beta_0}{\sin \alpha_0} \quad \text{and} \quad z = x \frac{\sin \gamma_0}{\sin \alpha_0}.
\]

So we may rewrite the area as

\[
\Delta_0 = \frac{1}{2} \frac{\sin \beta_0 \sin \gamma_0}{\sin \alpha_0} x^2.
\]

From which we obtain

\[
x = \sqrt{\frac{2 \Delta_0 \sin \alpha_0}{\sin \beta_0 \sin \gamma_0}}.
\]

Substituting the expressions above for the sines, then expresses the length \(x\) directly in terms of the areas. Unfortunately the resulting expression, even after all readily apparent simplifications have been performed, is too large to conveniently typeset. As a practical matter computing the sines and substituting the results into the last expression would be the convenient way to find lengths from areas.

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