OPTIMAL LOT-SIZING POLICY FOR A FAILURE PRONE PRODUCTION SYSTEM WITH INVESTMENT IN PROCESS QUALITY IMPROVEMENT AND LEAD TIME VARIANCE REDUCTION

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ABSTRACT. To survive in today's competitive market, it is not enough to produce low-cost products but also quality-related issues and lead time needs to be considered in the decision-making process. This paper extends the previous research by developing a stochastic economic manufacturing quantity (EMQ) model for a production system which is subject to process shifts from an in-control state to an out-of-control state at any random time. Moreover, we consider the option of investment to increase the process quality and decrease the lead-time variability. Closed-form solutions of the proposed models are obtained by applying the classical optimization technique. Some lemmas and theorems are developed to determine the optimal solution of the decision variables. Numerical results are obtained for each of these models and compared with those of the basic EMQ model without any investment. From the numerical analysis, it has been observed that our proposed model can significantly reduce the cost of the system compared to the basic model.

1. Introduction. People use thousands of products from the time they wake up to the time they fall asleep which can cause serious harm to customers if they are faulty. Therefore, it has become more important to keep an eye on the quality of the products so that customers can receive good quality products for their money (George and Rajagopalan [7]). A poor quality product affects customer confidence as well as the sales of a company. It can even affect the survival of the organization. There is a lot of evidence that clearly shows that every manufacturing company needs constant maintenance of their product making equipment. There are very few companies that constantly improve the quality of their products so that customers can always get quality products at reasonable prices. An organization can reduce the number of refundable products by providing good quality products to its customers which reduces the warranty and rework related costs. In the traditional EMQ model, it is assumed that the system produces defect-less products from the beginning to the end of the production (Choudri et al. [4], Taleizadeh [40]) which may not true assumption for some manufacturing industries (such as decorative
stuff, electronics, primary metal, electrical equipment) where the production system generally deteriorates due to overage. Typically, a production system stays in an in-control state at the beginning; however, due to continuous deterioration over time, the system may shift to an out-of-control state and start producing imperfect items (Lai et al. [19]). These defective products must be reworked or rejected so that they do not reach customers.

Thus, from theoretical as well as practical points of view, it is important and meaningful to study the joint optimization of production quantity and quality principles. The quality control literature focuses on many aspects of quality, such as process control charts, inspection, machine maintenance, quality improvement, repair and replacement; and designing for quality. However, literature that addresses the possible relationship between lot size and quality is very rare. Since quality considerations have a considerable impact on the performance of many manufacturing systems (Cheng [3], Yano [45]) the quality-adjusted EMQ model is a realistic extension of the line of work in this area.

Lead time has a significant influence on production management. Variations in lead time may occur on both the seller's and manufacturer's end. There are various reasons behind this lead-time variation; e.g., transportation delays, bad weather, product quality issues, larger lot-size, stock-out at the supplier level, etc. In order to minimize the impact of time limit variability, manufacturing companies usually keep their safety stock level very high or have a safety lead time. Heard and Plosell [13] blamed the lead-time variation as one of the reasons behind the obstacles to achieving the plant's goals. For all these reasons, companies nowadays have adopted the JIT (Just-In-Time) production system through which they have been able to please their customers and gain a foothold in the competitive market. The goal of this JIT system is to make small lot-size, to provide high-quality products, shorten the lead-time, frequent delivery, etc. For this reason, investments in reducing lead time variability and increasing product quality are worthy of analytical investigation.

2. Literature review. Over the past several decades, frequent research has been conducted to increase the applicability of the EMQ model in real-world manufacturing systems (Hax and Candea [14], Silver and Peterson [37]). The classical EMQ model developed based on the assumption that the production facility is failure-free, i.e., all the items produced are of perfect quality, which is rarely satisfied in reality. In practice, failure-free production is not possible due to long-run process, human mistake, or incomplete process controls, the machine may move to an out-of-control state from an in-control state thereby resulting in the production of defective/imperfect quality items.

In the direction of production models with unreliable machines, numerous studies have been conducted. Rosenblatt and Lee [30] were among the pioneers who looked at a deteriorating production system that is subject to random shipment from an in-control to an out-of-control state. Groenevelt et al. [9] addressed an economic lot-size model to study the effect of machine breakdowns and corrective maintenance. Groenevelt et al. [10] reconsidered this work assuming machine failure rate as exponentially distributed and repair time with general probability distribution. Kim and Hong [18] further considered Rosenblatt and Lees [30] model where the time to shift from in-control to the out-of-control state follows a general distribution and they obtained exact optimal production run length. Sana [31] developed an imperfect production system where the percentage of defective items varies with
the production rate. Liao [21] developed an EMQ model to obtain the optimal production run length when the system deteriorates over time and the system involves perfect and imperfect maintenance. Wen et al. [44] incorporated predictive maintenance in the traditional EMQ model adopting autoregressive integrated moving average (ARIMA) to predict the system's overall health. Afterwards, Nobil et al. [27] developed an EPQ model with multiple machines allowing shortages and scrap of defective items. Giri and Dohi [8] were the first to integrate random machine breakdown within a production system with exponentially distributed failure rate under random corrective and preventive maintenance time. Sett et al. [34] and Sarkar et al. [35] considered imperfection in the production system with prime emphasis on inspection policy. They pointed out the significance of human errors in the production system during human-based inspection processes. Al-Salamah [1] developed an EMQ model with the appearance of random machine breakdown and imperfect quality items. In a recent study, Manna et al. [25] considered random defectiveness in a production-inventory supply chain model where defective items vary with production rate. Cunha et al. [20] proposed an EMQ model with partial backordering and discount for imperfect quality batches. Taheri-Tolgari et al. [41] studied an inventory model with defective quality process and preventive maintenance to establish optimum inventory level for produced items and optimum policy. In the above-mentioned literature, the machine shifting time from an in-control to an out-of-control state was assumed to be either a known random variable or an arbitrary random variable. De et al. [5] considered fuzziness in machine shifting time to investigate a production-inventory model.

Most EMQ models ignore quality issues by setting the system level to an optimal level. But when a machine is operated for a long time, it is normal to produce a number of defective products. To discuss in detail how these defective products affect the lot-size, several researchers have considered the imperfect EMQ model. Porteus [29] one of them, was the first to work on an imperfect EMQ model to determine the relationship between the quality and the lot-size. He made it clear in his paper that while the production system is imperfect, the lot-size should be smaller than that of when the system is perfect. Lee et al. [22] developed a multi-stage inventory model aimed at improving product quality and reducing the percentage of defective items. Hou and Lin [16] considered an imperfect production model where investment was used to improve the process quality. In another model, Hou [17] investigated the impact of simultaneous investment for setup cost reduction and process quality improvement on optimal production run length. Sarkar and Giri [32] considered an integrated production-delivery supply chain model to consider the quality-related issue by assuming an investment option in process quality improvement. Sofiana et al. [33] considered a manufacturing process with inspection and rework and developed two policies of process quality improvement. Initially, they used the same manufacturing facility for rework but in another policy, an additional process facility was considered for the rework.

In each of the papers mentioned above, it is assumed that there will be no deviation of demand and lead time. But in reality, variability in demand and lead time can often be noticed. Gross and Soriano [11] and Vinson [43] developed inventory models dealing with lead-time variability and demonstrated the impact of lead-time reduction on inventory related costs. Furthermore, it is noted that the effect of lead-time deviation on total inventory cost is more visible than demand
deviation. Liberatore [24] and Sphicas [38] investigated the economic order quantity (EOQ) model with constant demand under infinite range variable lead-time. Further, Sphicas and Nasri [39] highlighted that finite-range lead time gives much more noticeable results compared to infinite-range lead time. Subsequently, Nasri et al. [26] considered an EOQ model to study the effects of setup cost reduction with variable lead time. Paknejad et al. [28] reconsidered Sphicas and Nasri’s [39] model to investigate the effects of lead time variability reduction and setup cost reduction. The synergistic impact on inventory costs has been reported by them when lead-time and setup cost reduction policies are investigated together. Chandra and Grabis [2] examined a single-stage inventory model with lead time-dependent procurement costs to optimize inventory and procurement cost assuming. They established a relationship between lead time and procurement costs through linear and nonlinear function where they mentioned that shortening lead-time could improve the level of customer service. Lee and Schwarz [23] formulated an inventory model for a single type of item with periodic review inventory policy under stochastic lead times. Hayya et al. [15] studied an inventory model to demonstrate the impact of the lead-time reduction under order crossover. They used exponential lead-time to show that there is a secondary reduction in the average lead time reduction variant due to order crossover.

2.1. **Research gap and our contribution.** If we take a closer look at the papers mentioned above, we will see that most traditional models that work for production systems were based on the assumption that the raw materials arrive as soon as they ordered, i.e., raw material replenishment is instantaneous, which is quite unrealistic in practice—although there are some EOQ models that consider the replenishment lead time as stochastic or deterministic. However, Sarker and Coates [36] were the first authors who extended the EOQ model with variable lead time to EMQ model with variable lead time. They used a finite range investment option to reduce the system setup cost and addressed some of the benefits associated with reducing the setup cost. However, they do not deal with the situation when the production system goes to out-of-control state and produces some defective units. With the continuous improvement of the production system, the quality of the product can be improved somewhat, but it is not possible to produce 100% non-defective items. This paper considers the realistic assumption that the production process may shift from in-control state to an out-of-control state after a random time interval and it then produce some percentage of defective items. It considers investment options to improve process quality and reduce lead-time variability. Such investment in improving process quality gives better quality output (less error) and smaller lot-size. Reducing lead time variability can also improve output quality as it reduces the size of the production lot.

The rest of the paper is developed as follows: In the next section, notation and assumptions are given. In Section 4, the proposed model is formulated. In Section 5, numerical analysis is done to illustrate the developed model. Finally concluding remarks are given in Section 6.

3. **Notation and assumptions.** We adopt the following notation to develop the proposed model.

3.1. **Notation.** We use the following notation to develop our model.
• **Input parameters**

  | Description                        |
  |-----------------------------------|
  | $D$ average demand rate (quantity unit/time unit) |
  | $P$ production rate (quantity unit/time unit)     |
  | $K$ setup cost per setup ($/setup)                |
  | $H$ holding cost ($/quantity unit/time unit)      |
  | $R$ rework cost ($/unit)                       |
  | $i$ annual fractional cost of capital investment ($) |
  | $B$ backorder cost ($/quantity unit)             |
  | $r$ reorder level (quantity units)               |
  | $c$ lower limit of lead time distribution        |
  | $d$ upper limit of lead time distribution        |
  | $m$ mean lead time                             |
  | $V_0$ original lead time variance               |
  | $θ_0$ original defective percentage in out-of-control state |
  | $N$ number of defective items                   |
  | $δ$ reciprocal of $Γ$                           |
  | $β$ reciprocal of $Δ$                           |

• **Decision variables**

  | Description                        |
  |-----------------------------------|
  | $Q$ production lot-size (quantity unit/cycle) |
  | $θ$ reduced defective percentage in out-of-control state |
  | $V$ lead time variance (time unit)     |
  | $t$ reorder time, $t = reorder level/D$ (time unit) |
  | $T$ cycle time (time unit)            |

• **Optimum variables**

  | Description                        |
  |-----------------------------------|
  | $T^*_{imp}$ cycle time after investment |
  | $Q^*_{imp}$ production lot size after investment |
  | $t^*_{imp}$ reorder time after investment |
  | $V^*_{imp}$ lead time variance after investment |
  | $θ^*_{imp}$ defective percentage after investment |

• **Stochastic variables**

  | Description                        |
  |-----------------------------------|
  | $l$ lead time (time unit)          |
  | $X$ elapsed time until production process shifts from the in-control state to the out-of-control state (time unit) |

• **Functions**

  | Description                        |
  |-----------------------------------|
  | $g(l)$ probability density function of lead time, $l$ |
  | $f(X)$ probability density function of shifting time, $X$ |
  | $φ_θ(θ)$ capital investment allocated to process quality improvement |
  | $φ_V(V)$ capital investment allocated to reduce lead time variance |
  | $E(N)$ expected number of defective items |
  | $E[TC(·)]$ expected total cost function |

3.2. **Assumptions.**

1. A production facility is considered that starts always in an in-control state and produces perfect items.
During the production run time, the system may shift to an out-of-control state at any random time but never breaks down (ref. Rosenblatt and Lee [30] and Sett et al. [34]).

The time after which the system moves is assumed to be exponentially distributed with a mean of $1/\zeta$.

Defective products are identified after the end of the cycle and the system incurs additional costs for related activities (e.g., rework, scrap).

The relationship between process quality $\theta$ and the capital investment in process quality improvement $\phi_V(V)$, can be stated as

$$\phi_\theta(\theta) = \frac{1}{\Gamma} \ln \left( \frac{\theta_0}{\theta} \right) \text{ for } 0 < \theta \leq \theta_0.$$ 

Similarly, the relationship between lead time variance $V$ and capital investment in lead time variability reduction $\phi_V(V)$ is described by

$$\phi_V(V) = \frac{1}{\Delta} \ln \left( \frac{V_0}{V} \right) \text{ for } 0 < V \leq V_0.$$ 

where $\Gamma$ and $\Delta$ are the percentage reduction in $\theta$ and $V$ per dollar increase in investment, respectively. This investment function has been used by previous researchers (Porteus [29], Hou [17], Tiwari et al. [42], and others).

Replenishment lead-time is uniformly distributed over a finite-range (Paknejad et al. [28], Sarker and Coates [36]).

Shortages are fully backlogged.

4. Formulation of the model. In this study, a single stage production system is considered where the system moves from an in-control to an out-of-control state at any random time which is exponentially distributed. Let $X$ be the random time after which the production process shifts to an out-of-control state. Since it is a random variable, it may occur within the production run period $DT/P$ or may occur after period $DT/P$. If $X \geq DT/P$, i.e., the time after which the machine shifts is longer than the production completion time, then there will be no machine shipment resulting in no defective product. If $X \leq DT/P$, then the machine shifts to the out-of-control state and defective items are produced (see Figure 1).

Therefore, the number of defective items per production cycle is

$$N = \begin{cases} 
0 & \text{if } X \geq DT/P, \\
\theta P(DT/P - x) & \text{if } X < DT/P.
\end{cases}$$

Hence, the expected number of defective items can be obtained as follows:

$$E(N) = \int_0^{DT/P} \theta P(DT/P - x)f(X)dx$$

$$= \int_0^{DT/P} \theta P(DT/P - x)\zeta e^{-\zeta x}dx \text{ (assuming } f(X) = \zeta e^{-\zeta x})$$

$$= \theta P \left[ \frac{DT}{P} + \frac{1}{\zeta} \left( e^{-\frac{1}{\zeta DT}} - 1 \right) \right]$$
In most cases, $\zeta$ is very small. So we use McClaurin series to get following approximate result (see, e.g. Rosenblatt and Lee [30]):

$$e^{-\frac{\zeta DT}{P}} \cong 1 - \zeta \frac{DT}{P} + \frac{1}{2} \zeta^2 \left(\frac{DT}{P}\right)^2$$

Using the above approximation we have

$$E(N) = \frac{\theta \zeta T^2 D^2}{2P}$$

Therefore, the expected rework cost per unit time is

$$ERC = \frac{RE(N)}{T} = \frac{R \zeta T D^2}{2P}$$

Now, the expected total cost which consists of setup cost, holding cost, backorder cost, and rework cost, can be obtained in a similar way to Sarker and Coates [36],
as follows:

\[ ETC = E[TC(T, t)] \]

\[
\begin{align*}
\text{Setup cost} & = \frac{K}{T} \\
\text{Holding cost} & = \int_{-\infty}^{\infty} \left[ \frac{HD(1 - D/P)}{2T} \left( T - \frac{l - t}{1 - D/P} \right)^2 \right] g(l)dl \\
\text{Backorder cost} & = \int_{-\infty}^{\infty} \left[ \frac{BD(l - t)^2}{2T(1 - D/P)} \right] g(l)dl + \frac{R}{T} E(N) \\
\text{Rework cost} & = \frac{K}{T} + \frac{HD}{2} \left[ (1 - D/P)T - 2(m - t) \right] + \frac{(H + B)D}{2(1 - D/P)T} \left[ V + (m - t)^2 \right]
\end{align*}
\]

where \( m = \int_{-\infty}^{\infty} g(l)dl \) and \( V = l^2 g(l)dl - m^2 \) are mean and variance of lead time \( l \), respectively.

**Lemma 4.1.** \( E[TC(T, t)] \) is strictly convex in \( T \) and \( t \).

**Proof.** See Appendix A.

Hence the optimal value of \((T, t)\) that minimize \( E[TC(T, t)] \) can be obtained from the necessary conditions \( \partial E[TC(T, t)]/\partial t = 0 \) and \( \partial E[TC(T, t)]/\partial T = 0 \) as

\[
T = \sqrt{\frac{2K(P - D) + V(H + B)PD}{D(P - D)(Z + R\zeta T D/P)}} = T_0 \text{ (say)}
\]

\[
t = m - \frac{H(P - D)}{P(H + B)} T_0 = t_0 \text{ (say)}
\]

where \( Z = \frac{HB}{H + B}(1 - D/P) \).

The optimal production lot-size can be obtained from the relation \( Q = DT \) i.e.,

\[
Q = \sqrt{\frac{2KD(P - D) + V(H + B)PD^2}{(P - D)(Z + R\zeta T D/P)}} = Q_0 \text{ (say)}
\]

Using (2) and (3), one can obtain closed-form solution of the expected total cost as

\[
ETC^* = \sqrt{D \left( \frac{D\zeta R}{P} + Z \right) \left( 2K + \frac{HBDV}{Z} \right)}
\]

From above, it is important to note that the expected total cost and the production lot-size are respectively directly and inversely proportional to the average percentage of defective items produced. Therefore, by reducing the defective items, it is possible to obtain lower expected total cost and larger production lot-size. In the next section, we will develop a model considering defective percentage \( \theta \) as a decision variable.
4.1. The optimal process quality. This section presents the cost model with investments in process quality improvement i.e., in reducing the average percentage of defective items produced ($\theta$). We consider $\phi_\theta(\theta)$ as investment amount to reduce defective percentage from $\theta_0$ to $\theta$. Therefore, we seek to minimize

$$E[TC(T, t, \theta)] = i\phi_\theta(\theta) + E[TC(T, t)]$$

subject to $0 < \theta \leq \theta_0$. 

where the first term is associated with investment cost to improve process quality (by reducing defective percentage) and the second term represents system cost without investment. The restriction in (7) ensures that the reduced defective rate (percentage) should always less than or equal to the initial defective rate (percentage). We consider the investment cost $\phi_\theta(\theta)$ as a convex and strictly decreasing function of $\theta$.

To minimize (6), we first ignore the constraint $0 < \theta \leq \theta_0$ and use classical optimization technique. If the optimal defective rate (percentage) does not satisfy the restriction (7), there is no need to make any investment for process quality improvement, and the results of 4.1 hold.

We have the following investment function (ref Porteus [29]):

$$\theta = \theta_0 e^{-\Gamma \phi_\theta(\theta)} 0 \leq \phi_\theta < \infty$$

Taking logarithm on both sides of (8) we get

$$\phi_\theta = \gamma - \delta \ln(\theta)$$

where $\gamma = \ln(\theta_0)/\Gamma$ and $\delta = 1/\Gamma$.

Using (9), (6) becomes

$$E[TC(T, t, \theta)] = i[\gamma - \delta \ln(\theta)] + \frac{K}{T} + \frac{HD}{2} [(1 - D/P)T - 2(m - t)]$$

$$+ \frac{(H + B)D}{2(1 - D/P)T} [V + (m - t)^2] + \frac{R\theta TD^2}{2P}$$

Lemma 4.2. For $0 < \theta_0 < 1$ and $0 < \Gamma < 1$, $E[TC(T, t, \theta)]$ is strictly convex in $T, t,$ and $\theta$ if the following condition is satisfied:

$$T < \left[ \frac{4i\delta P^2}{D^2R^2\zeta^2\theta^2} \left( \frac{2K + (B + H)V D}{(1 - D/P)} \right) \right]^{1/3}$$

Proof. See Appendix B. 

Lemma 4.2 represents the condition for the expected cost function to be optimum. Under the convexity condition of Lemma 4.2, it is easy to prove that the stationary point $(T, t, \theta)$ is a relative minimum. If Lemma 4.2 holds for a stated imperfect production system, then the following theorem provides the closed form optimal solution to the problem.

Theorem 4.3. (a) The optimal defective percentage, $\theta^*_\text{imp}$ is

$$\theta^*_\text{imp} = \begin{cases} \theta_0 & \text{if } \theta^*_\text{imp} \geq \theta_0, \\ \theta^*_\text{imp} & \text{if } \theta^*_\text{imp} < \theta_0, \end{cases}$$

where $\theta_0$ is the original defective percentage, and

$$\theta^*_\text{imp} = \frac{2i\delta PZ}{R\zeta D} \left( -i\delta + \sqrt{2DZ \left( K + \frac{D(B + H)V}{2(1 - D/P)} \right) + \delta^2 i^2} \right)^{-1}$$
(b) The optimal cycle time $T^\ast_{\text{imp}}$, optimal reorder time $t^\ast_{\text{imp}}$, and optimal production lot-size $Q^\ast_{\text{imp}}$ are given by

$$T^\ast_{\text{imp}} = \begin{cases} T_0 & \text{if } \theta_{\text{imp}} \geq \theta_0, \\ T_{\text{imp}} & \text{if } \theta_{\text{imp}} < \theta_0, \end{cases}$$

$$t^\ast_{\text{imp}} = \begin{cases} t_0 & \text{if } \theta_{\text{imp}} \geq \theta_0, \\ t_{\text{imp}} & \text{if } \theta_{\text{imp}} < \theta_0, \end{cases}$$

$$Q^\ast_{\text{imp}} = \begin{cases} Q_0 & \text{if } \theta_{\text{imp}} \geq \theta_0, \\ Q_{\text{imp}} & \text{if } \theta_{\text{imp}} < \theta_0, \end{cases}$$

where $T_0$, $t_0$, and $Q_0$ are given by (2)-(4), and

$$T_{\text{imp}} = \frac{1}{DZ} \left(-i\delta + \sqrt{2DZ \left(K + \frac{D(B + H)V}{2(1 - D/P)}\right) + \delta^2 i^2}\right) \quad (11)$$

$$t_{\text{imp}} = m - \frac{H(1 - D/P)}{(H + B)DZ} \left(-i\delta + \sqrt{2DZ \left(K + \frac{D(B + H)V}{2(1 - D/P)}\right) + \delta^2 i^2}\right) \quad (12)$$

$$Q_{\text{imp}} = \frac{1}{Z} \left(-i\delta + \sqrt{2DZ \left(K + \frac{D(B + H)V}{2(1 - D/P)}\right) + \delta^2 i^2}\right) \quad (13)$$

(c) The resulting expected annual cost $E[TC^\ast(\theta_{\text{imp}})]$ is

$$E[TC^\ast(\theta_{\text{imp}})] = \begin{cases} E[TC^\ast(\theta_0)] & \text{if } \theta_{\text{imp}} \geq \theta_0, \\ E[TC^\ast(\theta_{\text{imp}})] & \text{if } \theta_{\text{imp}} < \theta_0, \end{cases}$$

where $E[TC^\ast(\theta_0)]$ is $ETC^\ast$ in (5) with $\theta = \theta_0$, and

$$E[TC^\ast(\theta_{\text{imp}})] = i\delta \ln \left(\frac{\theta_0}{\theta_{\text{imp}}}\right) + \sqrt{D \left(2K + \frac{BHDV}{Z}\right) \left(\frac{R\zeta D^2 T_{\text{imp}}}{P} + Z\right)}$$

Proof. See Appendix B.

One can obtain the optimal capital investment in process quality improvement from the following equation:

$$\phi^\ast_0(\theta) = \delta \ln \left(\frac{\theta_0 r \zeta D^2 T_{\text{imp}}^\ast}{2i\delta P}\right) \quad (14)$$

In particular, there is a critical defective percentage level such that if the defective percentage after investment is below that level, improvement in process quality is successful and should be carried out, else there is no need to invest and results of Lemma 4.1 could be considered as optimal solution.
4.2. The optimal lead time variance. We now consider the problem with investment in lead-time variance reduction. We consider $\phi_V(V)$ as the investment cost in reducing the lead-time variance $V_0$ to $V$. The control of lead time variance level is achieved by varying the capital investment amount assigning to reduce lead time variance. That is, reduction of lead time variance level from its original level is possible by introducing capital investment. We have the cost function as follows:

$$\text{Minimize } E[TC(T, t, V)] = i\phi_V(V) + E[TC(T, t)]$$  \hspace{1cm} (15)$$

subject to $0 < V \leq V_0$, \hspace{1cm} (16)

where the first term is associated with investment cost to reduce lead time variability (by reducing lead time variance) and the second term represents the expected total cost of the system without any investment in process quality improvement. The restriction in (16) ensures that the reduced defective percentage rate is always less than or equal to the initial defective percentage. We consider that the investment cost $\phi_V(V)$ is a convex and strictly decreasing function of $V$.

To minimize (15) we first ignore the constraint $0 < V \leq V_0$ and then use classical optimization technique. If the optimal lead time variance does not satisfy the restriction (15), there is no need to make any investment, and the results of Lemma 4.1 holds.

We use following negative exponential function for investment:

$$V = V_0 e^{-\Delta \phi_V(V)} \quad 0 \leq \phi_V < \infty$$ \hspace{1cm} (17)$$

where $\Delta$ is the percentage decrease in $V$ per dollar increase in $\phi_V(V)$.

Taking logarithm on both sides of (17) yields

$$\phi_V = \alpha - \beta \ln(V)$$ \hspace{1cm} (18)$$

where $\alpha = \ln(V_0)/\Delta$ and $\beta = 1/\Delta$.

Using (18), (15) becomes

$$E[TC(T, t, V)] = i[\alpha - \beta \ln(V)] + \frac{K}{T} + \frac{HD}{2} [(1 - D/P)T - 2(m - t)]$$

$$+ \frac{(H + B)D}{2(1 - D/P)T} \ln(V + (m - t))^2 + \frac{R\zeta TD^2}{2P}$$ \hspace{1cm} (19)$$

Lemma 4.4. For $0 < V_0 < 1$ and $0 < \Delta < 1$, the expected total cost function $E[TC(T, t, V)]$ is strictly convex in $T, t,$ and $V$ if the following condition is satisfied:

$$T > \frac{D^2(B + H)^2V^2}{4i\beta \left[2K(1 - D/P) + D(B + H)V\right] (1 - D/P)}$$

Proof. See Appendix C. \hfill \Box

Lemma 4.4 represents the condition for the expected cost function to be optimum. Under the condition of Lemma 4.4, it is easy to show that $(T, t, V)$, is a minimum point. If Lemma 4.4 holds for the stated imperfect production system, then the following theorem provides the closed form optimal solution to the problem.

Theorem 4.5. (a) The optimal lead time variance, $V_{imp}^*$, is

$$V_{imp}^* = \begin{cases} 
V_0 & \text{if } V_{imp} \geq V_0, \\
V_{imp} & \text{if } V_{imp} < V_0.
\end{cases}$$
where $V_0$ is the original lead time variance, and

$$V_{imp} = \frac{2i\beta(1-D/P)}{D^2(H+B)} \left( i\beta + \sqrt{2KD(Z + R\zeta D\theta/P) + \beta^2 i^2} \right)$$

(b) The optimal cycle time $T^\ast_{imp}$, optimal reorder time $t^\ast_{imp}$, and optimal lot-size $Q^\ast_{imp}$ are given by

$$T^\ast_{imp} = \begin{cases} T_0 & \text{if } V_{imp} \geq V_0, \\ T_{imp} & \text{if } V_{imp} < V_0, \end{cases}$$

$$t^\ast_{imp} = \begin{cases} t_0 & \text{if } V_{imp} \geq V_0, \\ t_{imp} & \text{if } V_{imp} < V_0, \end{cases}$$

$$Q^\ast_{imp} = \begin{cases} Q_0 & \text{if } V_{imp} \geq V_0, \\ Q_{imp} & \text{if } V_{imp} < V_0, \end{cases}$$

where $T_0$, $t_0$, and $Q_0$ are given by (2) – (4), and

$$T_{imp} = \frac{i\beta + \sqrt{2KD(Z + R\zeta D\theta/P) + \beta^2 i^2}}{D(Z + R\zeta D\theta/P)}$$

(20)

$$t_{imp} = m - \frac{H(1-D/P) \left[ i\beta + \sqrt{2KD(Z + R\zeta D\theta/P) + \beta^2 i^2} \right]}{(H+B)D^2(Z + R\zeta D\theta/P)}$$

(21)

$$Q_{imp} = \frac{i\beta + \sqrt{2KD(Z + R\zeta D\theta/P) + \beta^2 i^2}}{Z + R\zeta D\theta/P}$$

(22)

(c) The resulting expected annual cost $E[TC^\ast(V_{imp})]$ is

$$E[TC^\ast(V_{imp})] = \begin{cases} E[TC^\ast(V_0)] & \text{if } V_{imp} \geq V_0, \\ E[TC^\ast(V_{imp})] & \text{if } V_{imp} < V_0, \end{cases}$$

where $E[TC^\ast(V_0)]$ is ETC in (5) with $V = V_0$, and

$$E[TC^\ast(V_{imp})] = i\beta \ln \left( \frac{V_0}{V_{imp}} \right) + D \left( 2K + \frac{BHDV_{imp}}{Z} \right) \left( \frac{R\zeta D\theta}{P} + Z \right)$$

Proof. See Appendix C.

One can obtain the optimal capital investment in lead time variability reduction by the following equation:

$$\phi^\ast_V(V) = \beta \ln \left( \frac{V_0(h+B)D}{2i\beta(1-D/P)T^\ast_{imp}} \right)$$

(23)

In particular, there is a critical lead time variance level such that if the variance after investment is below that level, investment in variance reduction is successful and should be carried out; else, there is no need to invest and the solution given in Lemma 4.1 could be considered as the optimal solution.
4.3. Determination of optimal lead time variance and process quality simultaneously. This section considers the case where simultaneous investment in process quality improvement and lead time variance reduction has been carried out. After the addition of invest cost our cost function, which we seek to minimize becomes

\[
\text{Minimize } E[TC(T,t,\theta,V)] = i\phi_\theta(\theta) + i\phi_V(V) + E[TC(T,t)]
\]

subject to

\[
0 < \theta \leq \theta_0 \quad (25)
\]

\[
0 < V \leq V_0. \quad (26)
\]

where \(\phi_\theta(\theta)\) and \(\phi_V(V)\) are given by (9) and (18), respectively.

Initially, we solve the model ignoring both the constraints. If the solution satisfies constraint (25) and (26), then both restrictions are inactive and it is done. Otherwise, any one of those constraints will remain active. In this situation, we minimize (24) subject to the constraint \(V = V_0\) (ignoring the constraint \(0 < \theta \leq \theta_0\)). If the results satisfy the restriction (25), it is optimal (this is the case of no investment in lead time variability reduction, and the results of Theorem 1 can be used to find the optimal solution). If not, then we minimize \(E[TC(T,t,\theta,V)]\) subject to \(\theta = \theta_0\) (ignoring the constraint \(0 < V \leq V_0\)). If the results satisfy the restriction (26), it is optimal (this is the case of no investment in process quality improvement, and the results of Theorem 4.5 hold). If it is not, then both restrictions are active. In this case, we minimize \(E[TC(T,t,\theta,V)]\) subject to \(\theta = \theta_0\) and \(V = V_0\) (this is the case where no investment is made to either process quality improvement or lead time variability reduction, and the results of the EMQ model with finite-range stochastic lead time given by (2)-(5) can be used).

**Lemma 4.6.** For the positive values of \(V_0, \theta_0, \Delta,\) and \(\Gamma, E[TC(T,t,\theta,V)]\) is strictly convex in \(T, t, \theta,\) and \(V\) if the following condition is satisfied:

\[
T > \frac{D^2(B + H)^2V^2}{4i\beta(2K + D(B + H)/(1 - D/P)V)(1 - D/P)^2} \quad (27)
\]

*Proof.* See Appendix D. \(\square\)

Lemma 4.6 represents the condition for the expected cost function to be optimum. The stationary point \((T, t, \theta, V)\) will be minimum if it satisfy the convexity condition given in Lemma 4.6. If Lemma 4.6 holds for the stated imperfect production system, then the following theorem provides the closed form optimal solution to the problem.

**Theorem 4.7.** (a) The optimal defective percentage \(\theta_{imp}\), and optimal lead time variance \(V_{imp}\), are

\[
\theta_{imp}^* = \frac{2i\delta PZ}{R\xi D[i(\beta - \delta) + \sqrt{2KDZ + (\beta - \delta)^2i^2}]}
\]

\[
V_{imp}^* = \frac{2i\beta(1 - D/P)[i(\beta - \delta) + \sqrt{2KDZ + (\beta - \delta)^2i^2}]}{D^2(H + B)Z}
\]
(b) If $0 < \theta \leq 1$ and $0 < V \leq 1$, the resulting optimal cycle time $T^*_\text{imp}$, optimal reorder time $t^*_\text{imp}$, and optimal production lot-size $Q^*_\text{imp}$ are as follows:

\[
T^*_\text{imp} = \frac{i(\beta - \delta) + \sqrt{2KDZ + (\beta - \delta)^2i^2}}{DZ}
\]
\[
t^*_\text{imp} = m - \frac{H(1 - D/P)(i(\beta - \delta) + \sqrt{2KDZ + (\beta - \delta)^2i^2})}{(H + B)DZ}
\]
\[
Q^*_\text{imp} = \frac{i(\beta - \delta) + \sqrt{2KDZ + (\beta - \delta)^2i^2}}{Z}
\]

(c) The resulting expected annual cost $E[TC^*(\theta^*_\text{imp}, V^*_\text{imp})]$ is

\[
E[TC^*(\theta^*_\text{imp}, V^*_\text{imp})] = i\beta \ln \left( \frac{V_0}{V^*_\text{imp}} \right) + i\delta \ln \left( \frac{\theta_0}{\theta^*_\text{imp}} \right) + \sqrt{D \left( 2K + \frac{BHDV^*_\text{imp}}{Z} \right) \left( \frac{R\zeta D\theta^*_\text{imp}}{P} + Z \right)}
\]

Proof. See Appendix D. \qed

One can obtain the optimal capital investment amount for lead time variability reduction and process quality improvement by the following equations:

\[
\phi^*_V(V) = \beta \ln \left( \frac{V_0(h + B)D}{2i\beta(1 - D/P)T^*_\text{imp}} \right)
\]

and

\[
\phi^*_\theta(\theta) = \delta \ln \left( \frac{\theta_0r\zeta T^*_\text{imp}D^2}{2i\delta P} \right)
\]

4.4. Special cases. Here we will discuss some special cases depending on the asymptotic direction of the parameters chosen. We have the expected total cost under simultaneous investment as

\[
E[TC^*(\theta^*_\text{imp}, V^*_\text{imp})] = i\beta \ln \left( \frac{V_0}{V^*_\text{imp}} \right) + i\delta \ln \left( \frac{\theta_0}{\theta^*_\text{imp}} \right) + \sqrt{D \left( 2K + \frac{BHDV^*_\text{imp}}{Z} \right) \left( \frac{R\zeta D\theta^*_\text{imp}}{P} + Z \right)}
\]

with

\[
Q^*_\text{imp} = \frac{i(\beta - \delta) + \sqrt{2KDZ + (\beta - \delta)^2i^2}}{Z}
\]

or alternatively

\[
Q^*_\text{imp} = \left[ D \left( 2K + \frac{DHBV^*_\text{imp}}{Z} \right) / \left( Z + \frac{DR\zeta \theta^*_\text{imp}}{P} \right) \right]^{1/2}
\]
Case I: When $\zeta \to 0$ (i.e., production system is perfect) and $V_{imp}^* = V$ (i.e., lead-time variance is fixed) (30) becomes

$$Q^* = \sqrt[2]{\frac{D}{\left(2K + \frac{DHBV}{Z}\right)/Z}}$$

(31)

which is the economic batch obtained by Sarker and Coates [36].

Case II: When $\zeta \to 0$ (i.e., production system is perfect) and $V_{imp}^* = 0$ (i.e., lead-time is zero) (30) becomes

$$Q^* = \sqrt{\frac{2KD}{H(1 - D/P)}} \sqrt{\frac{H + B}{B}}$$

(32)

which is the economic batch quantity of standard EMQ model with backorder obtained by Elsayed and Boucher [6].

Case III: When $\zeta \to 0$ (i.e., production system is perfect), $V = 0$ (i.e., lead time is zero), $B \to \infty$ (i.e., shortages are not considered), (30) reduces to

$$Q^* = \sqrt{\frac{2KD}{H(1 - D/P)}}$$

(33)

which is the economic batch quantity for classical EMQ model.

5. Numerical analysis. To illustrate the proposed model, a numerical example is considered with the following parameter-values: $D = 5200$ units/year, $P = 20800$ units/year, $K_0 = $100/setup, $h = $10/unit/year, $B = $20/year, $R = $30/unit, $\theta_0 = 0.15$, $i = 0.10$/$/year, $\zeta = 0.95$, $c = 0.01923$ years (1 week), $d = 0.11538$ years (6 weeks), $m = 0.06731$ years, $V_0 = 0.00077$ years, $\Gamma = 0.002$, $\Delta = 0.0005$.

Table 1 provides the results of the model under four different scenarios: (i) EMQ model with stochastic lead-time (EMQ-SLT), (ii) model with stochastic lead-time and process quality improvement (SLT-QI), (iii) model with lead time variability (SLT-VR), and (iv) model with simultaneous process quality improvement and lead time variability reduction (SLT-SI). The results are obtained for uniformly distributed lead-time over five weeks interval. An upper bound of the total cost is obtained in EMQ-SLT model and lower bound of total cost in SLT-SI. Table 1 reflects that the lead-time interval has reduced by 2.68 weeks and mean lead-time by 1.34 weeks due to the investment which results 10.21 reduction in the expected annual total cost and 19.33 reduction in optimal production lot-size. Similarly, due to investment, the process quality is improved by reducing the defective percentage by 84.46 which in turn reduces the expected annual total cost by 4.98. Further, simultaneous investment reduce the lead time interval by 2.58 weeks and mean lead-time by 1.29 weeks, and improve process quality by reducing defective items by 80.79. Besides this, 12.32 reduction in production lot-size and 13.68 reduction in the expected total annual cost are realized.

Table 2 shows the effects of lead-time variability on the optimal solution. Specifically, we compare the results of $EMQ-SLT$, $SLT-QI$, $SLT-VR$, and $SLT-SI$ models for uniformly distributed lead-time interval. For one and two weeks lead time
A mean lead time of 4 weeks. Similarly, simultaneous investment (increases and investment gives up to 89% reduction in lead-time variance is observed for individual variance reduction investment model (SLT-QI) and SLT-SI models are worthwhile. For a mean lead time of 2.5 weeks, 40.07% reduction in lead time variance is observed for individual variance reduction investment model (SLT-VR). Lead time variability reduction percentage increases as the mean lead time increases and investment gives up to 89.01% reduction for a mean lead time of 4.5 weeks. Similarly, simultaneous investment (SLT-SI) model gives 35.02% reduction in lead time variance for a mean lead time of 2.5 weeks, and 88.08% reduction for a mean lead time of 4.5 weeks. This results show that when lead time is long, a

| Model type | Q | V | θ | E[TC(θ)] | reduction in θ (%) | reduction in V (%) | reduction in E[TC(θ)] |
|------------|---|---|----|----------|-------------------|--------------------|---------------------|
| EMQ-SLT    | 556.4 | – | – | 3369.26 | – | – | – |
| Approx.    | 555.6 | – | – | 3371.76 | – | – | – |
| SLT-QI     | 602.3 | 0.02352 | 3205.55 | 84.83 | – | – | 4.92 |
| Approx.    | 602.2 | 0.02331 | 3204.00 | 84.46 | – | – | 4.98 |
| SLT-VR     | Exact 448.8 0.0001660 | – | 3025.87 | – | 78.44 | 10.19 |
| Approx.    | 448.2 0.0001658 | – | 3027.5 | – | 78.47 | 10.21 |
| SLT-SI     | Exact 487.1 0.0001802 0.02903 | 2908.06 | 80.64 | 76.60 | 13.69 |
| Approx.    | 487.1 0.0001801 0.02882 | 2908.43 | 80.79 | 76.61 | 13.68 |

Table 1. Numerical results for different models

| Lead time interval | Model type | Q | V | θ | E[TC(θ)] | reduction in θ (%) | reduction in V (%) | reduction in E[TC(θ)] |
|--------------------|------------|---|---|----|----------|-------------------|--------------------|---------------------|
| 1 week             | EMQ-SLT    | 420 | – | – | 2527 | – | – | – |
| SLT-QI             | 475 | 0.0296 | 2554 | 80.27 | – | – | 4.27 |
| SLT-VR             | 440 | – | – | 2668 | – | – | – |
| SLT-SI             | 475 | 0.0296 | 2554 | 80.27 | – | – | 4.27 |
| 2 weeks            | EMQ-SLT    | 470 | – | – | 2802 | – | – | – |
| SLT-QI             | 475 | 0.0296 | 2554 | 80.27 | – | – | 4.27 |
| SLT-VR             | 440 | – | – | 2668 | – | – | – |
| SLT-SI             | 475 | 0.0296 | 2554 | 80.27 | – | – | 4.27 |
| 3 weeks            | EMQ-SLT    | 470 | – | – | 2802 | – | – | – |
| SLT-QI             | 475 | 0.0296 | 2554 | 80.27 | – | – | 4.27 |
| SLT-VR             | 440 | 0.000166 | 2923 | 81.60 | – | 49.67 | 1.02 |
| SLT-SI             | 475 | 0.0296 | 2554 | 80.27 | – | – | 4.27 |
| 4 weeks            | EMQ-SLT    | 490 | – | – | 3090 | – | – | – |
| SLT-QI             | 475 | 0.0296 | 2554 | 80.27 | – | – | 4.27 |
| SLT-VR             | 440 | 0.000166 | 2923 | 81.60 | – | 49.67 | 1.02 |
| SLT-SI             | 475 | 0.0296 | 2554 | 80.27 | – | – | 4.27 |
| 5 weeks            | EMQ-SLT    | 556 | – | – | 3369.26 | – | – | – |
| SLT-QI             | 602 | 0.0233 | 3294 | 84.46 | – | – | 4.98 |
| SLT-VR             | 448 | 0.000166 | 3028 | – | 78.47 | 10.21 |
| SLT-SI             | 487 | 0.000180 | 2908 | 80.79 | 76.61 | 13.68 |
| 6 weeks            | EMQ-SLT    | 556 | – | – | 3369.26 | – | – | – |
| SLT-QI             | 659 | 0.0213 | 3495 | 85.8 | – | – | 5.21 |
| SLT-VR             | 448 | 0.000166 | 3100 | – | 85.03 | 15.92 |
| SLT-SI             | 487 | 0.000180 | 2981 | 80.8 | 83.77 | 19.15 |
| 7 weeks            | EMQ-SLT    | 664 | – | – | 4028 | – | – | – |
| SLT-QI             | 721 | 0.0194 | 3808 | 87.06 | – | – | 5.46 |
| SLT-VR             | 448 | 0.000166 | 3102 | – | 89.01 | 21.50 |
| SLT-SI             | 487 | 0.000180 | 3043 | 80.8 | 88.08 | 24.45 |

Table 2. Computational results for different lead-time interval intervals, the optimal lead-time variance exceeds the original lead-time variance for both SLT-VR and SLT-SI models, and the investment in lead time variability reduction is not warranted, and hence, the quality improved model (SLT-VR) gives the optimal solution with fixed lead time variance. Beyond two weeks interval, the investment in lead-time variability reduction for both SLT-QI and SLT-SI models are worthwhile. For a mean lead time of 2.5 weeks, 40.07% reduction in lead time variance is observed for individual variance reduction investment model (SLT-VR). Lead time variability reduction percentage increases as the mean lead time increases and investment gives up to 89.01% reduction for a mean lead time of 4.5 weeks. Similarly, simultaneous investment (SLT-SI) model gives 35.02% reduction in lead time variance for a mean lead time of 2.5 weeks, and 88.08% reduction for a mean lead time of 4.5 weeks. This results show that when lead time is long, a
### Table 3. Critical points

| Parameter | SLT-QI | SLT-VR | SLT-SI |
|-----------|--------|--------|--------|
| $\Gamma$  | $\frac{1}{2K_d} \sqrt{\frac{4PZ + D}{PZ + D \theta_0 \zeta (P - D)}}$ | $-\frac{2}{D \theta_0 \zeta D}$ | $\frac{2}{D \theta_0 \zeta D}$ |
| $\Delta$  | $\frac{1}{2K_d} \sqrt{\frac{4PZ + D}{PZ + D \theta_0 \zeta (P - D)}}$ | $-\frac{2}{D \theta_0 \zeta D}$ | $\frac{2}{D \theta_0 \zeta D}$ |
| $i$       | $\frac{1}{2K_d} \sqrt{\frac{4PZ + D}{PZ + D \theta_0 \zeta (P - D)}}$ | $-\frac{2}{D \theta_0 \zeta D}$ | $\frac{2}{D \theta_0 \zeta D}$ |
significant incentive can be attained by the investment. Percentage cost savings in quality improvement model (SLT-QI) and variability reduction (SLT-VR) models compared to fixed model (EMQ-SLT) are observed. However, simultaneous investment model (SLT-SI) gives the maximum percentage of cost savings. In all models, percentage cost saving increases with increasing lead-time variability. It is also seen that, individual investment in lead time variability reduction and process quality improvement results in the lowest lead-time variance and the highest process quality level compared to simultaneous investment model.

5.1. Sensitivity analysis. In this section, we pay our attention to the investigation of the conditions for which investment in process quality improvement and lead-time variability reduction is worthwhile. Specifically, for the process quality improvement model, we assume that the probabilistic condition (7) and the convexity condition in Lemma 4.1 are satisfied. Under this framework, investment in process quality improvement is warranted if and only if \( \theta_{imp} < \theta_0 \), which is equivalent to requiring optimal defective percentage to be strictly less than the original defective percentage. Substituting \( \theta_{imp} \) in (7), we find the critical points of different parameters. In a similar way, we find the critical points of different parameters for lead-time variability reduction model and simultaneous model. The critical points of technology parameters \( \Gamma \) and \( \Delta \), and interest rate \( \delta \) for SLT-QI, SLT-VR, and SLT-SI are provided in Table 3.

Sensitivity of the parameters given in Table 3 is analyzed with respect to lead-time interval by keeping other parameters fixed. The values of the critical points are given in Tables 4 and 5. Note that, as lead-time interval increases from 1 week to 7 weeks, the interest rate that one would be willing to pay to invest in process quality improvement increases from 0.450 to 0.709, and the percentage of defective items one would be willing to accept decreases from 0.0309 to 0.0194. Further, the lower bound of the technology coefficient \( \Gamma \) for which investment in process quality improvement is desirable decreases from 0.000445 to 0.000282. This makes natural sense since high lead-time variability means large amount of shortages and thus, to reduce shortages, one will be willing to pay more investment to cut the lead-time even for the case of higher interest rate.

For lead-time variability reduction case (SLT-VR), when lead-time interval is low (one and two weeks), there is no need to invest money as in that case risk of stock-out situation is less. However, when lead-time interval increases from 3 weeks to 7 weeks, the interest rate that one would be willing to pay to invest in lead-time variability reduction increases from 0.160 to 0.615 and the reduced lead-time variance that one would be willing to accept becomes 0.000166. Similar results are obtained for simultaneous investment case.
Table 5. Critical values for different lead-time intervals for SLT-VR model

| Variables | 1    | 2    | 3    | 4    | 5    | 6    | 7    |
|-----------|------|------|------|------|------|------|------|
| $V_{imp}$ | $-$  | $-$  | $-$  | $-$  | $-$  | $-$  | $-$  |
| $i$       | 0.160| 0.262| 0.375| 0.494| 0.615|      |      |
| $\Delta$  | 0.000313| 0.000191| 0.000133| 0.000101| 0.0000813|      |      |

6. Managerial implications. This research has several practical and managerial implications. We modeled an EMQ problem and presented three models with three different investment scenarios, which can be used as a policy in evaluating investment preferences. The proposed model can be applied for a single item or processing systems for similar items. Our model can be used in places where special attention is paid to product quality (e.g., electronic equipment, semiconductors, and military equipment industries). From the models presented in this paper, several managerial details can be given. Industry managers can make investment decisions to reduce the percentage of defective products through the cost optimization model and to choose the best option to achieve the minimum production cost target. Moreover, if the managers want to maintain the product quality level and lead time together, the simultaneous investment model could be applied there. Managers can use critical points to decide in what condition they should invest in process quality improvement and lead-time variability reduction. In the situation where the lead time variance is large, the manufacturer should increase the production lot size as it can help him/her to reduce the stock out chance. On the other hand, producing more items means running the machine for longer time which in turn will accelerate the production of defective items. Therefore, it is advisable to invest more money to reduce the production of defective items.

7. Conclusions. This paper determines the optimal production policy for a failure prone production system with stochastic lead time, process quality improvement, and lead time variability reduction. Three main contributions of the paper are as follows: First, this is the first study to incorporate finite range stochastic lead time into imperfect EMQ model. Second, investment options in process quality improvement and lead-time variability reduction are examined. Third, we have derived closed form solution for the proposed models.

Numerical experiments for all the models with uniformly distributed lead-time are conducted. From the optimal results, it is seen that significant cost savings over the finite-range stochastic lead time model are realized for the process quality improvement, lead-time variability reduction model, and the simultaneous model. From the practical point of view, this strategy is valid and useful to the business organizations to gain competitive advantages.

Several opportunities for further research are possible. One can extend the model by considering preventive maintenance policy in the production-inventory system. Another interesting research may be conducted by considering lead time in fuzzy sense.

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Appendix A

**Proof of Lemma 4.1:** $E[TC(T, t, \theta)]$ is strictly convex if the principal minors of its Hessian matrix $H_2$ are strictly positive. We have

$$H_2 = \begin{bmatrix}
\frac{\partial^2 E[TC(T, t, \theta)]}{\partial t^2} & \frac{\partial^2 E[TC(T, t, \theta)]}{\partial t \partial \theta} & \frac{\partial^2 E[TC(T, t, \theta)]}{\partial \theta^2} \\
\frac{\partial^2 E[TC(T, t, \theta)]}{\partial t \partial \theta} & \frac{\partial^2 E[TC(T, t, \theta)]}{\partial \theta^2} & 0 \\
\frac{\partial^2 E[TC(T, t, \theta)]}{\partial t^2} & \frac{\partial^2 E[TC(T, t, \theta)]}{\partial t \partial \theta} & \frac{\partial^2 E[TC(T, t, \theta)]}{\partial \theta^2}
\end{bmatrix}$$

We can easily show that the first and second principal minors $|H_{21}|$ and $|H_{22}|$ of the Hessian matrix $H_2$ corresponding to the expected profit function $E[TC(T, t, \theta)]$ in (10) are strictly positive. However, the third principal minor $|H_{23}|$ will give positive value if and only if the convexity condition of Lemma 4.2 holds.

**Proof of Theorem 4.3:** If the convexity condition is satisfied, then the optimal reorder time, defective percentage, and cycle time that minimize $E[TC(T, t, \theta)]$ can be obtained from the necessary conditions $\partial E[TC(T, t, \theta)]/\partial t = 0$ and $\partial E[TC(T, t, \theta)]/\partial \theta = 0$, and $\partial E[TC(T, t, \theta)]/\partial T = 0$. The first two conditions give

$$t = m - \frac{H(1 - D/P)}{(H + B)T}$$

$$\theta = \frac{2i\delta D^2}{R\zeta D^2 T}$$

Using (28) and (29) in the condition $\partial E[TC(T, t, \theta)]/\partial T = 0$, and solving it for $T$, we get

$$T_{imp} = \{-i\delta + \sqrt{2DZ[K + D(B + h)V/(2(1 - D/P))] + \delta^2T^2}\}/DZ$$

Substitution of (30) in (28) and (29) yields $t_{imp}$ and $\theta_{imp}$ of Theorem 4.3.

The optimal production lot-size can be obtained by the relation $Q_{imp} = DT_{imp}$.

Appendix C

**Proof of Lemma 4.4:** Similar to Lemma 4.2 we can easily show that the first and second principal minors of the Hessian matrix corresponding to the expected cost function $E[TC(T, t, V)]$ in (16) is positive. The third principal minor will give positive value if and only if the convexity condition of Lemma 4.4 holds.
Proof of Theorem 4.5: If the convexity condition is satisfied, then the optimal reorder time, defective items, and cycle time that minimize $E[TC(T,t,V)]$ can be obtained from the necessary conditions $\partial E[TC(T,t,V)]/\partial t = 0$ and $\partial E[TC(T,t,V)]/\partial V = 0$, and $\partial E[TC(T,t,V)]/\partial T = 0$. The first two conditions give

$$t = m - \frac{h(1 - D/P)}{(h + B)} T$$

(37)

$$V = \frac{2i\beta(1 - D/P)}{(h + B)D} T$$

(38)

Substituting (31) and (32) in $\partial E[TC(T,t,V)]/\partial T = 0$ and solving it for $T$, we have

$$T_{imp} = \frac{i\beta + \sqrt{2KD(Z + r\zeta D\theta_0/P) + \beta^2T^2}}{D(Z + r\zeta D\theta_0/P)}$$

(39)

Substitution of (33) in (31) and (32) yields $t_{imp}$ and $V_{imp}$ of Theorem 4.5.

Optimal production lot-size can be obtained by the relation $Q_{imp} = DT_{imp}$.

Appendix D

Proof of Lemma 4.6: The proof is similar to that of Lemma 4.4.

Proof of Theorem 4.7: The proof is similar to that of Theorem 4.5.