MEAN-FIELD BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS AND APPLICATIONS

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Abstract. In this paper we study the mean-field backward stochastic differential equations (mean-field bsde) of the form

\[ dY(t) = -f(t, Y(t), Z(t), K(t, \cdot), \mathbb{E}[\varphi(Y(t), Z(t), K(t, \cdot))])dt + Z(t)dB(t) + \int_{\mathbb{R}_0} K(t, \zeta) \tilde{N}(dt, d\zeta), \]

where \( B \) is a Brownian motion, \( \tilde{N} \) is the compensated Poisson random measure. Under some mild conditions, we prove the existence and uniqueness of the solution triplet \( (Y, Z, K) \). It is commonly believed that there is no comparison theorem for general mean-field bsde. However, we prove a comparison theorem for a subclass of these equations. When the mean-field bsde is linear, we give an explicit formula for the first component \( Y(t) \) of the solution triplet. Our results are applied to solve a mean-field recursive utility optimization problem in finance.

1. Introduction

Optimal control of mean-field stochastic differential equation has been studied by a number of researchers lately. To make things more precise let us explain the situation on optimal control of stochastic systems of the following type:

\[
\begin{aligned}
    dX(t) &= b(t, X(t), \mathcal{L}(X(t)), u(t))dt + \sigma(t, X(t), \mathcal{L}(X(t)), u(t))dB(t), \\
    X(0) &= x_0
\end{aligned}
\]

with the performance

\[
J(u) = \mathbb{E}\left[ \int_0^T f(x(t), u(t), \mathcal{L}(X(t)))dt + h(X(T), \mathcal{L}(X(T))) \right],
\]

where \( B \) is the standard \( d \)-dimensional Brownian motion, \( \mathcal{L}(X(t)) \) denotes the probability law of the state \( X(t) \) at time \( t \) and \( b, \sigma, f, h \) are some properly defined function. We refer to Anderson and Djeighiche \cite{Anderson}, \cite{Djeighiche}, Lasry & Lions \cite{Lasry}, Carmona & Delarue \cite{Carmona}, \cite{Delarue} and Agram & Øksendal \cite{Agram}, \cite{Oksendal} for some discussion. In particular, Pham & Wei \cite{Pham} have introduced a dynamic programming approach by using a randomised stopping method.

Due to the presence of the law \( \mathcal{L}(X(t)) \) in the equation and in the performance functional, the process \( X(t) \) is no longer Markovian and it is more effective to use the Pontryagin maximum principle to solve the above mean field stochastic control problem, which will give a mean field backward stochastic differential equation (mean field bsde).

To limit ourselves, we shall only deal with the case that the law \( \mathcal{L}(X(t)) \) appears in its simplest form of expectation (see equation (1.2) below). This simplest mean-field bsde also represents interesting models in finance, for example models of risk measures and recursive utilities.

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Let $c(t) \geq 0$ be a consumption rate process from a given cash flow and let $g(t, Y(t), \mathbb{E}[Y(t)], c(t))$ be a given driver process, assumed to be concave with respect to $Y(t), \mathbb{E}[Y(t)], c(t)$. Then the corresponding recursive utility $U_g(c)$ of the consumption $c$ is the value $Y_g(0)$ at $t = 0$ of the first component $Y_g(t)$ of the solution $Y_g, Z_g, K_g$ of the mean-field bsde
\begin{equation}
\begin{cases}
dY(t) = -g(t, Y(t), \mathbb{E}[Y(t)], c(t))dt + Z(t)dB(t) + \int_{\mathbb{R}_0} K(t, \zeta)\tilde{N}(dt, d\zeta), t \in [0, T], \\
Y(T) = 0.
\end{cases}
\end{equation}

The objective is to find the consumption rate $\hat{c}$ which maximizes the mean-field recursive utility $U_g(c) = Y_g(0)$.

This can be seen as a generalization to mean-field (and jumps) of the classical recursive utility concept of Duffie and Epstein [12]. See also Duffie and Zin [13], Kreps and Parteus [16], El Karoui et al [14], Øksendal and Sulem [20] and Agram and Røse [2].

Backward sde's (bsde's) were first introduced in their linear form by Bismut [7] in connection with a stochastic version of the Pontryagin maximum principle. Subsequently, this theory was extended by Pardoux and Peng [23] to the nonlinear case. The first work applying bsde to finance was the paper by El Karoui et al [14] where they studied several applications to option pricing and recursive utilities. All the above mentioned works are in the Brownian motion framework (continuous case). The discontinuous case is more involved. Tang and Li [27] proved an existence and uniqueness result in the case of a natural filtration associated with a Brownian motion and a Poisson random measure. Barles et al [6] proved a comparison theorem for such equations and later Royer [25] extended comparison theorem under weaker assumptions. Buckdahn et al [8], has studied a MF-BSDE and they obtained a comparison theorem under some conditions.

In this paper, we shall study the following mean field bsde:
\begin{equation}
\begin{cases}
dY(t) = -f(t, y(t), z(t), k(t, \cdot), \mathbb{E}[\varphi(y(t), z(t), k(t, \cdot))])dt + Z(t)dB(t) + \int_{\mathbb{R}_0} K(t, \zeta)\tilde{N}(dt, d\zeta), t \in [0, T], \\
Y(T) = \xi.
\end{cases}
\end{equation}

The notation and conditions will be explained in details in Section 3. The purpose of this paper is the following.

1. To prove new existence and uniqueness results for the above mean-field bsde.
2. To give an explicit formula for the solution when the equation is linear.
3. To prove a comparison principle for a type of mean-field bsde different from those in [8] and under weaker assumptions on the driver.
4. To apply the obtained results to study a mean-field recursive utility optimization problem in finance.

2. Malliavin calculus

In this section we give a brief summary of Malliavin calculus for processes driven by Brownian motion and compensated Poisson random measures. For more details we refer to [11], [15], [19] and [26].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$. The expectation on this probability space is denoted by $\mathbb{E}$ and the conditional expectation $\mathbb{E}(\cdot | \mathcal{F}_t)$ is denoted by $\mathbb{E}^{\mathcal{F}_t}(\cdot) = \mathbb{E}(\cdot | \mathcal{F}_t)$.

Let $B(t) \sim N([0, t], \mathbb{R})$ be a Brownian motion. Let $(\mathcal{N}, \mathbb{E}[\mathcal{N}(\cdot)]) = (\mathcal{N}(\mathbb{R}), \mathbb{E}[\mathcal{N}(\cdot)])$ be a Poisson random measure. Denote by $\nu(\mathcal{B})$ its associated Lévy measure so that $\mathbb{E}[\mathcal{N}(\cdot, \mathcal{B})] = \nu(\mathcal{B})$.

Let $\tilde{N}(\cdot)$ denote the compensated Poisson measure of $\mathcal{N}$ defined by $\tilde{N}(dt, d\zeta) = N(dt, d\zeta) - \nu(d\zeta)dt$.

We assume that $\mathcal{F}_t = \sigma(B(s), \mathcal{N}, 0 \leq s \leq t, B \in \mathcal{B}(\mathbb{R})$ and any square integrable functional $F \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ can be written as
\begin{equation}
F = \sum_{m,n=0}^{\infty} f_{m,n}(f_{m,n}),
\end{equation}
where $f_{m,n}(s, t, \zeta) = f_{m,n}(s_1, \cdots, s_m, t_1, \zeta_1, \cdots, t_n, \zeta_n)$ is a function of $m + n$ variables which is symmetric in the first $m$ variables $s = (s_1, \cdots, s_m)$ and the last $n$-variables $t, \zeta = ((t_1, \zeta_1), \cdots, (t_n, \zeta_n))$ satisfying
\begin{equation}
\int_{[0, T]^{m+n} \times \mathbb{R}^n} |f(s, t, \zeta)|^2 ds_1 \cdots ds_m dt_1 \cdots dt_n \nu(d\zeta_1) \cdots \nu(d\zeta_n) < \infty
\end{equation}
and
\[ I_{m,n}(f_{m,n}) = \int_{[0,T]}^{m+n} f_{m,n}(s,t,\zeta) dB(s_1) \cdots dB(s_m) \tilde{N}(dt_1, d\zeta_1) \cdots \tilde{N}(dt_n, d\zeta_n) \] (2.3)
is the mixed multiple integral. It is easy to see that
\[ \mathbb{E}(F^2) = \sum_{m,n=1}^{\infty} m! n! \int_{[0,T]}^{m+n} |f(s,t,\zeta)|^2 ds_1 \cdots ds_m dt_1 \cdots dt_n \nu(d\zeta_1) \cdots \nu(d\zeta_n). \] (2.4)
We define the Malliavin derivative as $D = (D^1_r, D^2_{r,\zeta})$ (where $D^1$ denotes the partial Malliavin derivative with respect to the Brownian motion and $D^2$ denotes the partial Malliavin derivative with respect to the compensated Poisson process) as follows

**Definition 2.1.** We say that $F$ is in $\mathbb{D}_{1,2}$ if
\[ \sum_{m,n=1}^{\infty} (m + n) m! n! \int_{[0,T]}^{m+n} |f(s,t,\zeta)|^2 ds_1 \cdots ds_m dt_1 \cdots dt_n \nu(d\zeta_1) \cdots \nu(d\zeta_n). \] (2.5)
We define
\[
D^1_{t} I_{m,n}(f_{m,n}) = mI_{m-1,n}(f_{m,n}(r, \cdot, \cdot, \cdot)) = \int_{[0,T]}^{m-n-1} f_{m,n}(s_1, \cdots, s_{m-1}, r; t, \zeta) dB(s_1) \cdots dB(s_{m-1}) \tilde{N}(dt_1, d\zeta_1) \cdots \tilde{N}(dt_n, d\zeta_n);
\] (2.6)
and
\[
D^2_{t,\zeta} I_{m,n}(f_{m,n}) = nI_{m-1,n}(f_{m,n}(\cdot, \cdot, t, \zeta)) = n\int_{[0,T]}^{m-n-1} f_{m,n}(s_1, \cdots, s_m; t_1, \zeta_1, \cdots, t_{n-1}, \zeta_{n-1}, t, \zeta) dB(s_1) \cdots dB(s_m) \tilde{N}(dt_1, d\zeta_1) \cdots \tilde{N}(dt_n, d\zeta_n).
\] (2.7)

When there is confusion, we shall also omit the superscript and write $D_r = D^1_r$ and $D_{t,\zeta} = D^2_{t,\zeta}$.

We give some examples of Malliavin derivatives.

**Example 2.2.**
(i) Suppose that $f, p, q$ and $r$ are given càdlàg $\mathbb{F}$-adapted processes and satisfy the bsde of the form
\[
\begin{cases}
    dp(t) = f(t)dt + q(t)dB(t) + \int_{\mathbb{R}_0} r(t, \zeta) \tilde{N}(dt, d\zeta), & 0 \leq t \leq T, \\
    p(T) = F \in L^2(\Omega, \mathcal{F}_T).
\end{cases}
\] (2.8)
Then for a.a. $t$,
\[ D_t p(t) = q(t), \] (2.9)
and for a.a. $t$ and $\zeta$, we have
\[ D_{t,\zeta} p(t) = r(t, \zeta). \] (2.10)
(ii) If $F = \exp(\int_0^T f(s)dB(s))$, then
\[ D_r \mathbb{E}^{\mathbb{F}}[F] = Ff(r). \]
(iii) If $G = \int_0^T \int_{\mathbb{R}_0} \gamma(s, \zeta) \tilde{N}(ds, d\zeta)$ and $\varphi(G) = \exp(\int_0^T \int_{\mathbb{R}_0} \gamma(s, \zeta) \tilde{N}(ds, d\zeta))$, then
\[ D_{t,\zeta} \varphi(G) = \varphi(G)[\exp(\gamma(t, \zeta)) - 1]. \]
(iv) If $\varphi(G) = \exp(\int_0^T \int_{\mathbb{R}_0} \ln(1 + \gamma(s, \zeta)) \tilde{N}(ds, d\zeta))$, then
\[ D_{t,\zeta} \varphi(G) = \varphi(G) \gamma(t, z). \] (2.11)

**Proof.** First we prove (i). The bsde (2.8) can be written as a forward one.
\[
p(t) = p(0) + \int_0^t f(s)ds + \int_0^t q(s)dB(s) + \int_0^t \int_{\mathbb{R}_0} r(s, \zeta) \tilde{N}(ds, d\zeta), \quad t \in [0, T],
\]
for some given initial value $p(0)$ (determined by the bsde (2.8). The Malliavin derivatives of $p(t)$ are given, for all $r < t$ (if $r > t$ then $D_r p(t) = 0$), by
\[ D_r p(t) = \int_r^t D_r f(s)ds - \int_r^t q(s)dB(s) + q(r), \]
and
\[ D_{r,\zeta} p(t) = \int_r^t D_{r,\zeta} f(s)ds + \int_r^t \int_{\mathbb{R}_0} D_{r,\zeta} r(s, \zeta) \tilde{N}(ds, d\zeta) + r(r, \zeta). \]
Now let $r \to t^-$, then for a.a. $t$,

$$q(t) = D_t p(t),$$

and for a.a. $t$ and $\zeta$,

$$r(t, \zeta) = D_{t, \zeta} p(t).$$

(ii) If $F = \exp(\int_0^T f(s)dB(s))$, taking the conditional expectation, we get

$$\mathbb{E}^{F_t}[F] = \mathbb{E}^{F_t}[\exp(\int_0^T f(s)dB(s) + \int_0^T \nu ds)]$$

$$= \exp(\int_0^T f(s)dB(s)) \mathbb{E}^{F_t}[\exp(\int_0^T f(s)dB(s))]$$

$$:= K_{t,T} \exp(\int_0^T f(s)dB(s)),$$

where $K_{t,T} := \mathbb{E}^{F_t}[\exp(\int_0^T f(s)dB(s))]$. Then its Malliavin derivative

$$D_r \mathbb{E}^{F_t}[F] = K_{t,T} D_r \exp(\int_0^T f(s)dB(s))$$

$$= K_{t,T} \exp(\int_0^T f(s)dB(s)) f(r)$$

$$= \mathbb{E}^{F_t}[D_r F]$$

$$= F(r).$$

(iii) If $\varphi(G) = \exp(\int_{\mathbb{R}_0} \gamma(s, \zeta) \tilde{N}(ds, d\zeta))$, then by the chain rule for Malliavin derivatives of processes driven by compensated Poisson random measures, we get

$$D_{t, \zeta} \varphi(G) = \varphi(G + \gamma(t, \zeta)) - \exp(\gamma(t, \zeta))$$

$$= \exp(G) \exp(\gamma(t, \zeta)) - 1.$$  

(iv) If $\varphi(G) = \exp(\int_{\mathbb{R}_0} \ln(1+\gamma(s, \zeta)) \tilde{N}(ds, d\zeta))$, then by the chain rule for Malliavin derivatives of processes driven by compensated Poisson random measures, we get

$$D_{t, z} \varphi(G) = \varphi(G + D_{t, z} G) - \varphi(G)$$

$$= \exp(G) \exp(D_{t,z} G - 1)$$

$$= \exp(G) [\exp(\ln(1+\gamma(t, z)) - 1]$$

$$= \exp(G) [1 + \gamma(t, z) - 1]$$

$$= \varphi(G) \gamma(t, z).$$

$\square$

3. Mean-field BSDE’S

3.1. Existence and uniqueness of the solution. We define the following spaces for the solution triplet:

- $S^2$ consists of the F-adapted càdlàg processes $Y : \Omega \times [0, T] \to \mathbb{R}$, equipped with the norm

$$\| Y \|_{S^2}^2 := \mathbb{E} \left[ \sup_{t \in [0, T]} |Y(t)|^2 \right] < \infty.$$

- $L^2$ consists of the F-predictable processes $Z : \Omega \times [0, T] \to \mathbb{R}$, with

$$\| Z \|_{L^2}^2 := \mathbb{E} \left[ \int_0^T |Z(t)|^2 dt \right] < \infty.$$

- $L^2_0$ consists of Borel functions $K : \mathbb{R}_0 \to \mathbb{R}$, such that

$$\| K \|_{L^2_0}^2 := \int_{\mathbb{R}_0} |K(\zeta)|^2 d\nu(d\zeta) < \infty.$$

- $H^2_0$ consists of F-predictable processes $K : \Omega \times [0, T] \times \mathbb{R}_0 \to \mathbb{R}$, such that for any fixed $t \in [0, T]$, $K(t, \zeta)$ is any element in $L^2_0$ and

$$\| K \|_{H^2_0}^2 := \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}_0} |K(t, \zeta)|^2 d\nu(d\zeta) dt \right] < \infty.$$

- $L^2(\Omega, \mathcal{F}_T)$ is the set of square integrable random variables which are $\mathcal{F}_T$-measurable.
Let \( f : \Omega \times [0, T] \times \mathbb{R}^2 \times L^2_{\nu} \times \mathbb{R}^d \to \mathbb{R} \), be \( F_t \)-progressively measurable. We consider the following mean-field bsde:

\[
    dY(t) = f(t, Y(t), Z(t), K(t, \cdot), \mathbb{E}[\varphi(Y(t), Z(t), K(t, \cdot))])dt + Z(t)dB(t) + \int_{\mathbb{R}^d} K(t, \zeta)\tilde{N}(dt, d\zeta).
\]

(3.1)

**Definition 3.1.** A process \((Y, Z, K) \in S^2 \times L^2 \times H^2_{\nu}\) is said to be a solution triplet to the mean-field bsde (3.1) with terminal condition \(Y(T) = \xi\) if

\[
    \int_0^T |f(s, Y(s), Z(s), K(s, \cdot), \mathbb{E}[\varphi(Y(s), Z(s), K(s, \cdot))])| \, ds < +\infty \quad \mathbb{P}\text{-a.s.,}
\]

and

\[
    Y(t) = \xi + \int_0^t f(s, Y(s), Z(s), K(s, \cdot), \mathbb{E}[\varphi(Y(s), Z(s), K(s, \cdot))]) \, ds - \int_0^t Z(s) \, dB(s) - \int_0^t \int_{\mathbb{R}^d} K(s, \zeta)\tilde{N}(ds, d\zeta), \quad t \in [0, T].
\]

(3.2)

where \(\xi \in L^2(\Omega, F_T)\) is called the terminal condition and \(f\) is the generator.

Strictly speaking it is \(Y(s-i)\) in the above equation but for simplicity we will drop the minus from now on. To obtain the existence and uniqueness of a solution we make the following set of assumptions.

**Assumption 3.2.** For driver \(f\) we assume

(a) \(f\) is square integrable with respect to \(t\):

\[
    \mathbb{E}[\int_0^T |f(t, 0, 0, 0, 0)|^2 \, dt] < \infty.
\]

(b) There exists a constant \(C > 0\), such that for all \(t \in [0, T]\) and for all \(y_1, y_2, z_1, z_2 \in \mathbb{R}, k_1, k_2 \in L^2_{\nu}\) and \(\mu_1, \mu_2 \in \mathbb{R}^d\),

\[
    |f(t, y_1, z_1, k_1, \mu_1) - f(t, y_2, z_2, k_2, \mu_2)| \\
    \leq C(|y_1 - y_2| + |z_1 - z_2| + ||k_1 - k_2||_{L^2(\nu)} + |\mu_1 - \mu_2|), \quad \mathbb{P}\text{-a.s.}
\]

For the mean functional, we assume

(c) For each \(t \in [0, T]\), the (vector valued) function \(\varphi : \Omega \times [0, T] \times \mathbb{R}^2 \times L^2_{\nu} \to \mathbb{R}^d\) is assumed to be continuously differentiable with bounded partial derivatives, such that

\[
    ||\frac{\partial \varphi}{\partial y}(y, z, k) + \frac{\partial \varphi}{\partial z}(y, z, k) + ||\nabla_k \varphi(y, z, k)||_{L^2} \leq C,
\]

for a given constant \(C > 0\) and \(\nabla_k \varphi(y, z, k)\) is the Fréchet derivative of \(\varphi\) with respect to \(k\).

**Theorem 3.3.** Under the assumption 3.2, the mean-field bsde (3.2) has a unique solution.

**Proof.** For \(t \in [0, T]\) and all \(\beta > 0\), we introduce the norm

\[
    ||(Y, Z, K)||^2_{\beta, L^2_{\nu}} := \mathbb{E}[\int_0^T e^{\beta t} \{||Y(t)||^2 + ||Z(t)||^2 + \int_{\mathbb{R}^d} ||K(t, \zeta)||^2 \nu(d\zeta)\} \, dt].
\]

The space \(\mathbb{H}_\beta\) equipped with this norm is an Hilbert space. Define the mapping \(\Phi : \mathbb{H}_\beta \to \mathbb{H}_\beta\) by \(\Phi(y, z, k) = (Y, Z, K)\) where \((Y, Z, K) \in S^2 \times L^2 \times H^2_{\nu}\) is defined by

\[
    \begin{cases}
        dY(t) = -f(t, y(t), z(t), k(t, \cdot), \mathbb{E}[\varphi(y(t), z(t), k(t, \cdot))]) \, dt \\
        +Z(t) \, dB(t) + \int_{\mathbb{R}^d} K(t, \zeta)\tilde{N}(dt, d\zeta), \quad t \in [0, T], \\
        Y(T) = \xi.
    \end{cases}
\]

To prove the theorem it suffices to prove that \(\Phi\) is contraction mapping in \(\mathbb{H}_\beta\) under the norm \(||\cdot||_{\beta, L^2_{\nu}}\) for sufficiently small \(\beta\). For two arbitrary triplet \((y^1, z^1, k^1), (y^2, z^2, k^2)\) and \((Y^1, Z^1, K^1), (Y^2, Z^2, K^2)\), we denote their difference by \(\bar{y} = y^1 - y^2\) and \(\bar{Z} = Y^1 - Y^2\) and similarly for \(z, k, Z\) and \(K\). Applying Itô’s formula to \(e^{\beta t}||\bar{Y}(t)||^2\)

\[
    \mathbb{E}[\int_0^T e^{\beta t} \{||\bar{Y}(t)||^2 + ||\bar{Z}(t)||^2 + \int_{\mathbb{R}^d} ||\bar{K}(t, \zeta)||^2 \nu(d\zeta)\} \, dt]
\]

\[
    = 2\mathbb{E}[\int_0^T e^{\beta t} \bar{Y}(t) \{f(t, y^1(t), z^1(t), k^1(t, \cdot), \mathbb{E}[\varphi(y^1(t), z^1(t), k^1(t, \cdot))])
\]

\[
    - f(t, y^2(t), z^2(t), k^2(t, \cdot), \mathbb{E}[\varphi(y^2(t), z^2(t), k^2(t, \cdot))])\} \, dt]
\]

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By the Lipschitz property of the map \( f \), the mean value theorem, standard majorization and by choosing \( \beta = 1 + 12C^2 \) (\( C \) depends only on \( C' \) and \( C'' \)), it follows that
\[
\mathbb{E}[\int_0^T e^{\beta t}|\tilde{Y}(t)|^2 + |\tilde{Z}(t)|^2 + \int_{R_0} |\tilde{K}(t, \zeta)|^2 \nu(d\zeta)] dt \\
\leq \frac{1}{2} \mathbb{E}[\int_0^T e^{\beta t}|\tilde{y}(t)|^2 + |\tilde{z}(t)|^2 + \int_{R_0} |\tilde{k}(t, \zeta)|^2 \nu(d\zeta)] dt,
\]
Consequently, we get
\[
\|(|\tilde{Y}, \tilde{Z}, \tilde{K})||^2_2 \leq \frac{1}{2} \|(|\tilde{y}, \tilde{z}, \tilde{k})||^2_2,
\]
and \( \Phi \) is then a contraction mapping. The theorem can now deduced by standard theorem.

**Remark 3.4.** In the above theorem if we take \( d = 3, \varphi_i(x_1, x_2, x_3) = x_i \) for \( i = 1, 2, 3 \), we see that the following mean-field bsde has a unique solution
\[
dY(t) = -f(t, Y(t), Z(t), K(t, \cdot), \mathbb{E}[Y(t)], \mathbb{E}[Z(t)], \mathbb{E}[K(t, \cdot)]) dt \\
+ Z(t) dB(t) + \int_{R_0} K(t, \zeta) \tilde{N}(dt, d\zeta), t \in [0, T],
\]
where \( f : \Omega \times [0, T] \times \mathbb{R}^2 \times L^2_{\mathcal{F}_T} \times \mathbb{R}^3 \rightarrow \mathbb{R} \) satisfies the Assumption 3.2.

3.2. Linear mean-field bsde

In this section, we shall find the closed formula corresponding to the linear mean-field bsde of the form
\[
dY(t) = -[\alpha(t)Y(t) + \beta(t)Z(t) + \int_{R_0} \eta(t, \zeta) K(t, \zeta) \nu(d\zeta) + \gamma(t)] dt \\
+ Z(t) dB(t) + \int_{R_0} K(t, \zeta) \tilde{N}(ds, d\zeta), t \in [0, T],
\]
where the coefficients \( \alpha(t), \beta(t), \beta(t), \eta(t, \cdot), \eta(t, \cdot) \) are given deterministic functions; \( \gamma(t) \) is a given \( \mathcal{F}_{-} \)-adapted process and \( \xi \in L^2(\Omega, \mathcal{F}_T) \) is a given \( \mathcal{F}_T \)-measurable random variable. Applying a result from Øksendal and Sulem [20] or Quenez and Sulem [22], the above linear mean-field bsde (3.3) can be written as follows.
\[
Y(t) = \mathbb{E}[\xi(T, t) + \int_t^T \Gamma(s, t) \{\alpha^2(s) \mathbb{E}[Y(s)] + \beta^2(s) \mathbb{E}[Z(s)] \\
+ \int_{R_0} \eta^2(s, t) \mathbb{E}[K(s, t, \zeta)] \nu(d\zeta) + \gamma(s)\} ds], t \in [0, T],
\]
where \( \Gamma(t, s) \) is the solution of the following linear sde
\[
d\Gamma(t, s) = [\Gamma(t, s^--) + \beta(t) \gamma(t)] dt + \int_{R_0} \eta(t, \zeta) K(t, \zeta) \tilde{N}(ds, d\zeta), s \in [t, T],
\]
\[
\Gamma(t, t) = 1.
\]
Since we are in one dimension, Equation (3.5) can be solved explicitly and the solution is given by
\[
\Gamma(t, s) = \exp\{\int_s^t \beta(r) dr\} \nu(\Gamma(t, s^--)) \\
+ \int_s^t \int_{R_0} \exp(\int_s^r \eta(r) \nu(\zeta) dr) K(s, \zeta) \tilde{N}(dr, d\zeta).
\]
Notice that
\[
\nu(\Gamma(t, s^--)) = \exp(\int_s^t \alpha(r) dr).
\]
To solve (3.4) we take the expectation on both sides of (3.4). Denoting \( \overline{Y}(t) := \mathbb{E}[Y(t)], \overline{Z}(t) := \mathbb{E}[Z(t)], \) and \( \overline{K}(t, \zeta) := \mathbb{E}[K(t, \zeta)], \) we obtain
\[
\overline{Y}(t) = \mathbb{E}[\xi(T, t) + \int_t^T \Gamma(s, t) \{\alpha^2(s) \overline{Y}(s) + \beta^2(s) \overline{Z}(s) \\
+ \int_{R_0} \eta^2(s, t) \overline{K}(s, t, \zeta) \nu(d\zeta) + \gamma(s)\} ds], t \in [0, T].
\]
To find equation for \( \overline{Z}(t) \) and \( \overline{K}(t, \zeta) \) we write the original equation (3.3) as a forward one:
\[
Y(t) = Y(0) + \int_0^t [\alpha^2(s) Y(s) + \beta^2(s) Z(s) + \beta^2(s) Z(s) \\
+ \beta^2(t, \zeta) K(t, \zeta) \nu(d\zeta) + \gamma(s)\} ds], t \in [0, T],
\]
\[
\overline{Y}(t) = \mathbb{E}[Y(t)] = Y(0) + \int_0^t \alpha^2(s) \overline{Y}(s) + \beta^2(s) \overline{Z}(s) \\
+ \beta^2(t, \zeta) K(t, \zeta) \nu(d\zeta) + \gamma(s)\} ds], t \in [0, T].
\]
\[
\overline{Z}(t) = \mathbb{E}[Z(t)] = Z(0) + \int_0^t \beta^2(s) \overline{Z}(s) \\
+ \beta^2(t, \zeta) K(t, \zeta) \nu(d\zeta) + \gamma(s)\} ds], t \in [0, T].
\]
\[
\overline{K}(t, \zeta) = \mathbb{E}[K(t, \zeta)] = K(0, \zeta) + \int_0^t \beta^2(t, \zeta) K(t, \zeta) \nu(d\zeta) + \gamma(s)\} ds], t \in [0, T].
\]
Then Equations (2.9), we compute the Malliavin derivative of $Y(t)$ for all $r < t$, we get

$$D_r Y(t) = \int_r^t D_r [\alpha^1(s)Y(s) + \alpha^2(s)\bar{Y}(s) + \beta^1(s)Z(s) + \beta^2(s)\bar{Z}(s)]$$

$$+ \int_{\mathbb{R}_0} [\eta^1(s, \zeta)K(s, \zeta) + \eta^2(s, \zeta)\bar{K}(s, \zeta)](s, \zeta)\nu(d\zeta) + \gamma(s)ds$$

$$- \int_r^t Z(s)dB(s) + Z(r).$$

Letting $r \to t-$, we get that $Z(t) = D_t Y(t)$. Thus, to find $Z(t)$ we only need to compute $D_t Y(t)$. We shall use the expression (3.4) for $Y(t)$ and the identity

$$D_t \mathbb{E}^{\mathcal{F}_t}[F] = \mathbb{E}^{\mathcal{F}_t}[D_t F].$$

We also notice that $D_t \Gamma(t, T) = \Gamma(t, T)\beta^1(t)$. Then

$$Z(t) = \mathbb{E}^{\mathcal{F}_t}[D_t \xi \Gamma(t, T) + \xi \Gamma(t, T)\beta^1(t) + \int_t^T \Gamma(t, s)\beta^1(t)\alpha^2(s)\bar{Y}(s)$$

$$+ \beta^2(s)Z(s) + \int_{\mathbb{R}_0} \eta^2(s, \zeta)\bar{K}(s, \zeta)\nu(d\zeta) + \gamma(s)]\nu(d\zeta)\nu(s)ds].$$

Taking the expectation, we have

$$\mathbb{Z}(t) = \mathbb{E}[D_t \xi \Gamma(t, T) + \xi \Gamma(t, T)\beta^1(t) + \int_t^T \mathbb{E}(\xi \Gamma(t, s))\beta^1(t)\alpha^2(s)\bar{Y}(s)$$

$$+ \beta^2(s)\mathbb{Z}(s) + \int_{\mathbb{R}_0} \eta^2(s, \zeta)\mathbb{K}(s, \zeta)\nu(d\zeta) + \gamma(s)]ds].$$

Similarly, we have $K(t, \zeta) = D_t \eta Y(t)$ which yields

$$K(t, \zeta) = \mathbb{E}^{\mathcal{F}_t}[D_t \xi \Gamma(t, T) + \xi \Gamma(t, T)\eta^1(t, \zeta) + \int_t^T \mathbb{E}(\xi \Gamma(t, s))\eta^1(t, \zeta)\alpha^2(s)\bar{Y}(s)$$

$$+ \beta^2(s)\mathbb{Z}(s) + \int_{\mathbb{R}_0} \eta^2(s, \zeta)\mathbb{K}(s, \zeta)\nu(d\zeta) + \gamma(s)]ds].$$

Taking the expectation yields

$$\mathbb{K}(t, \zeta) = \mathbb{E}[D_t \xi \Gamma(t, T) + \xi \Gamma(t, T)\eta^1(t, \zeta) + \int_t^T \mathbb{E}(\xi \Gamma(t, s))\eta^1(t, \zeta)\alpha^2(s)\bar{Y}(s)$$

$$+ \beta^2(s)\mathbb{Z}(s) + \int_{\mathbb{R}_0} \eta^2(s, \zeta)\mathbb{K}(s, \zeta)\nu(d\zeta) + \gamma(s)]ds].$$

Equations (3.8), (3.9) and (3.10) can be used to obtain $\bar{Y}, \mathbb{Z}, K$. In fact, we let

$$V(t) = \begin{pmatrix} V_1(t) \\ V_2(t) \\ V_3(t, \zeta) \end{pmatrix} = \begin{pmatrix} \bar{Y}(t) \\ \mathbb{Z}(t) \\ K(t, \zeta) \end{pmatrix} \in L^2 \times L^2 \times H^2,$$

and

$$A(t, s, \zeta) = (A_{ij}(t, s, \zeta))_{1 \leq i, j \leq 3}$$

$$= \begin{pmatrix} \exp\left[\int_s^t \alpha^1(r)dr\right]\alpha^2(s) \\ \exp\left[\int_s^t \alpha^1(r)dr\right]\beta^2(s) \\ \exp\left[\int_s^t \alpha^1(r)dr\right]\eta^2(s, \zeta) \\ \exp\left[\int_s^t \alpha^1(r)dr\right]\beta^1(t)\alpha^2(s) \\ \exp\left[\int_s^t \alpha^1(r)dr\right]\beta^2(t)\beta^2(s) \\ \exp\left[\int_s^t \alpha^1(r)dr\right]\beta^1(t)\eta^2(s, \zeta) \\ \exp\left[\int_s^t \alpha^1(r)dr\right]\eta^1(t, \zeta)\alpha^2(s) \\ \exp\left[\int_s^t \alpha^1(r)dr\right]\eta^1(t, \zeta)\beta^2(s) \\ \exp\left[\int_s^t \alpha^1(r)dr\right]\eta^1(t, \zeta)\eta^2(s, \zeta) \end{pmatrix}. $$

Define a mapping from $V = (V_1, V_2, V_3)^T \in L^2 \times L^2 \times H^2$ to itself as

$$(AV)_i(t, \zeta) = \sum_{j=1}^2 \int_t^T A_{ij}(t, s)V_j(s)ds + \int_t^T \int_{\mathbb{R}_0} A_{3i}(t, s, \zeta)V_3(s, \zeta)\nu(d\zeta)ds.$$

Then Equations (3.8), (3.9) and (3.10) can be written as

$$V = F + AV,$$

where

$$F(t, \zeta) = \begin{pmatrix} \mathbb{E}(\xi \Gamma(t, T)) + \int_t^T \gamma(s)ds \\ \mathbb{E}[D_t \xi \Gamma(t, T) + \beta^1(t)\xi \Gamma(t, T)] + \int_t^T \gamma(s)ds \\ \mathbb{E}[D_t \xi \Gamma(t, T) + \xi \Gamma(t, T)\eta^1(t, \zeta) + \int_t^T \gamma(s)ds] \end{pmatrix}. $$

Equation (3.13) is a linear equation, it can now be solved easily.

$$(I - A)V = F.$$

Or

$$V = (I - A)^{-1}F = \sum_{n=0}^\infty A^n F.$$
Summarizing, we have the following theorem.

**Theorem 3.5 (Closed formula).** Let \( \alpha^1(t), \alpha^2(t), \beta^1(t), \beta^2(t), \eta^1(t, \cdot), \eta^2(t, \cdot) \) are given deterministic functions and \( \gamma(t) \) is \( \mathbb{F} \)-adapted and \( \xi \in L^2(\Omega, \mathcal{F}_T) \). The component of the solution \( Y(t) \) of the linear mean-field bsde (3.3) can be written on its closed formula as follows

\[
Y(t) = \mathbb{E}^F[\xi \Gamma(t, T) + \int_t^T \Gamma(t, s)\{\alpha^2(s), \beta^2(s), \eta^2(s, \xi)\}V(s) + \gamma(s)ds], t \in [0, T], \quad \mathbb{P}\text{-a.s.,}
\]

where

\[
\Gamma(t, s) = \exp\{\int_t^s \beta^1(r)dB(r) + \int_t^s (\alpha^1(r) - \frac{1}{2}(\beta^1(r))^2)dr + \int_t^s \int_{\mathbb{R}_0} (\ln(1 + \eta^1(r, \xi)) - \eta^1(r, \xi))\nu(d\zeta)dr + \int_t^s \int_{\mathbb{R}_0} (\ln(1 + \eta^1(r, \xi))\bar{N}(dr, d\zeta))\}
\]

and

\[
V = \sum_{n=0}^{\infty} A^n F,
\]

where \( A \) and \( F \) are given by (3.11) and (3.14) respectively.

4. A COMPARISON THEOREM FOR MEAN-FIELD BSDE’S

In this section we are interested in a subclass of mean-field bsde. Our idea is to use Picard iteration. So first, we shall prove a convergence result for the Picard iteration.

4.1. Picard iteration. To be able to prove the comparison theorem for mean-field bsde, we consider a mean field bsde and with driver allowed only to depend on the expectation of \( Y(t) \) and independent of the expectations of \( Z(t) \) and \( K(t, \xi) \), as follows

\[
\begin{align*}
dY(t) &= -g(t, Y(t), Z(t), K(t, \cdot), \mathbb{E}[Y(t)]) dt + Z(t)dB(t) + \int_{\mathbb{R}_0} K(t, \xi) \bar{N}(dt, d\xi), t \in [0, T], \\
Y(T) &= \xi.
\end{align*}
\]

We impose the following set of assumptions.

**Assumption 4.1.**

(i) Here \( g: \Omega \times [0, T] \times \mathbb{R}^2 \times L^2_v \times \mathbb{R} \to \mathbb{R} \) is \( \mathbb{F} \)-adapted and satisfies the Lipschitz assumption in the sense that

\[
|g(t, y, z, k, \bar{y}) - g(t, y', z', k', \bar{y}')| \leq C(|y - y'| + |z - z'| + \|k - k'\|_{L^2(\nu)} + |\bar{y} - \bar{y}'|),
\]

for all \( y, z, \bar{y}, y', z', \bar{y}' \in \mathbb{R}, k, k' \in L^2_v \).

(ii) \( \mathbb{E}\left[ \int_0^T |g(t, 0, 0, 0)|^2 dt \right] < \infty \).

(iii) The terminal value \( \xi \in L^2(\Omega, \mathcal{F}_T) \).

The following result is a consequence of Theorem 3.3 with \( d = 1 \) and \( \varphi(x) = x \):

**Theorem 4.2.** Under the above Assumption 4.1, the mean-field bsde (4.1) admits a unique solution \((Y, Z, K) \in S^2 \times L^2 \times H^2_v \).

To prove a comparison theorem, we need the following convergence to hold:

**Lemma 4.3 (Convergence).** Let \((Y, Z, K) \in S^2 \times L^2 \times H^2_v \) satisfies the mean-field bsde

\[
Y(t) = \xi + \int_t^T g(s, Y(s), Z(s), K(s, \cdot), \mathbb{E}[Y(s)]) ds - \int_t^T Z(s)dB(s) - \int_t^T \int_{\mathbb{R}_0} K(s, \zeta) \bar{N}(ds, d\zeta), t \in [0, T],
\]

where \( \xi \) and \( g \) are supposed to satisfy Assumption 4.1. We assume that for all \( n \geq 1 \), the triplet \((Y^n, Z^n, K^n) \) satisfies

\[
Y^n(t) = \xi + \int_t^T g(s, Y^n(s), Z^n(s), K^n(s, \cdot), \mathbb{E}[Y^{n-1}(s)]) ds - \int_t^T Z^n(s)dB(s) - \int_t^T \int_{\mathbb{R}_0} K^n(s, \zeta) \bar{N}(ds, d\zeta), t \in [0, T],
\]

where \( Y^{n-1}(s) \) is known. Thus, the following convergence holds

\[
Y^n(t) \to Y(t), \text{ for each } t \in [0, T].
\]
The proof relies on the classical Picard iteration method. Define \( Y^0(t) = E[\xi] \) and \( Y^n(t) \) given by (4.3) inductively as follows:

\[
Y^{n+1}(t) = \xi + \int_t^T g(s, Y^{n+1}(s), Z^{n+1}(s), K^{n+1}(s, \cdot), E[Y^n(s)]) ds - \int_t^T Z^{n+1}(s) dB(s) - \int_t^T \int_{\mathbb{R}_0} K^{n+1}(s, \xi) N(ds, d\xi), t \in [0, T].
\]

We want to show that the sequence \( (Y^n(t))_{t \geq 0} \) forms a Cauchy sequence. By Itô’s formula, we have

\[
E[|Y^{n+1}(t) - Y^n(t)|^2] \leq \frac{1}{2} E[\int_t^T |Y^{n+1}(t) - Y^n(t)|^2 ds] + \frac{1}{2} E[\int_t^T |Z^{n+1}(t) - Z^n(t)|^2 ds] + E[\int_t^T K^{n+1}(s, \xi) - K^n(s, \xi)|^2 \nu(d\xi) ds]
\]

\[
\leq C E[\int_t^T |Y^{n+1}(t) - Y^n(t)|^2 ds] + E[\int_t^T |Y^n(t) - Y^{n-1}(t)|^2 ds].
\]

This implies

\[
-\frac{d}{dt}(e^{Ct} E[|Y^{n+1}(t) - Y^n(t)|^2]) \leq \frac{1}{2} E[\int_t^T |Y^n(t) - Y^{n-1}(t)|^2 ds].
\]

Integrating both sides from \( u \) to \( T \), yields

\[
E[\int_u^T |Y^{n+1}(t) - Y^n(t)|^2 dt] \leq \frac{1}{2} E[\int_u^T \int_v^T \int_0^t \int_{\mathbb{R}_0} K^{n+1}(s, \xi) - K^n(s, \xi)|^2 \nu(d\xi) ds dt] \leq C^{nT} T^n + \frac{e^{CT} T^n}{n!}.
\]

We conclude that there exists a unique \( \mathbb{F} \)-adapted process \( Y(t) \) such that \( Y^n(t) \) converges to \( Y(t) \) which satisfies equation (4.2). \( \Box \)

We are now ready to state and prove a comparison theorem for mean-field BSDE.

**Theorem 4.4** (Comparison Theorem). Let \( g_1, g_2 : \Omega \times [0, T] \times \mathbb{R}^2 \times L^2 \times \mathbb{R} \) and \( \xi_1, \xi_2 \in L^2(\Omega, \mathcal{F}_T) \) and let \( (Y_i, Z_i, K_i)_{i=1,2} \) be the solutions of the following mean-field BSDE’s

\[
Y_i(t) = \xi_i + \int_t^T g_i(s, Y_i(s), Z_i(s), K_i(s, \cdot), E[Y_i(s)]) ds - \int_t^T Z_i(s) dB(s) - \int_t^T \int_{\mathbb{R}_0} K_i(s, \xi) N(ds, d\xi), t \in [0, T],
\]

\[
\xi_1 \geq \xi_2 \text{ P-a.s.} \quad (4.4)
\]

Assume that the drivers \( (g_i)_{i=1,2} \) are given \( \mathbb{F} \)-predictable processes satisfying Assumption 4.1 and

\[
g_1(t, y_1, z_1, k_1, \nu_1) \succeq g_2(t, y_1, z_1, k_1, \nu_2), \forall t, \nu_1 \succeq \nu_2, \text{ P-a.s.}, \quad (4.5)
\]

and moreover, the following inequality holds

\[
g_2(t, y, z, k_1, E[Y(t)]) - g_2(t, y, z, k_2, E[Y(t)]) \geq \int_{\mathbb{R}_0} \eta^1(t, \xi)(k_1(\xi) - k_2(\xi)) \nu(d\xi), \quad (4.6)
\]

\( \mathbb{P} \)-a.s. for all \( t \).

Then \( Y_1(t) \geq Y_2(t) \text{ P-a.s. for each } t \).

**Proof.** We use Picard iteration and we shall prove that \( Y_1^n(t) \geq Y_2^n(t) \) for all \( n \) and \( t \) by using induction on \( n \). Let \( Y_1^0(t) = E[\xi_1] \) and \( Y_2^0(t) = E[\xi_2] \). Then

\[
Y_1^0(t) \geq Y_2^0(t), \text{ for each } t \geq 0.
\]

Assume

\[
Y_1^n(t) \geq Y_2^n(t), \text{ for each } t \geq 0.
\]

Define the triple \( (Y_i^{n+1}(t), Z_i^{n+1}(t), K_i^{n+1}(t, \cdot))_{i=1,2} \), as follows

\[
Y_i^{n+1}(t) = \xi + \int_t^T g_i(s, Y_i^{n+1}(s), Z_i^{n+1}(s), K_i^{n+1}(s, \cdot), E[Y_i^n(s)]) ds - \int_t^T Z_i^{n+1}(s) dB(s) - \int_t^T \int_{\mathbb{R}_0} K_i^{n+1}(s, \xi) N(ds, d\xi), t \in [0, T],
\]

\[
\int_t^T \int_{\mathbb{R}_0} K_i^{n+1}(s, \xi) N(ds, d\xi), t \in [0, T].
\]
where \( Y^n(t) \) is knowing. Define \( \bar{g}_i(t, y, z, k) := g_i(t, y, z, k, \mathbb{E}[Y^n(t)]) \), then
\[
Y^{n+1}_i(t) = \xi + \int_t^T \bar{g}_i(s, Y^{n+1}_i(s), Z^{n+1}_i(s), K^{n+1}_i(s), \nu(ds, d\zeta)), t \in [0, T].
\]
We have by our assumptions that
\[
\bar{g}_1(t, y, z, k) \geq \bar{g}_2(t, y, z, k), \quad \text{for each } t \geq 0.
\]
By the comparison theorem for BSDE with jumps e.g. Theorem 2.3 in Royer [25], it follows that \( Y^{n+1}_1(t) \geq Y^{n+1}_2(t) \) for all \( t \geq 0 \).
By our convergence result 4.3, we conclude that
\[
Y_1(t) \geq Y_2(t), \quad \text{for each } t \geq 0.
\]

5. Mean-field recursive utility

We consider in this section a mean-field recursive utility process \( Y(t) \), defined to be the first component of the solution triplet \( (Y, Z, K) \) of the following mean-field bsde:
\[
\begin{align*}
\begin{cases}
\begin{aligned}
dY(t) &= -g(t, Y(t), Z(t), K(t, \cdot), \mathbb{E}[Y(t)], \mathbb{E}[Z(t)], \mathbb{E}[K(t, \cdot)], \pi(t))dt \\
&\quad + Z(t)dB(t) + \int_{\mathbb{R}_0} K(t, \zeta)N(dt, d\zeta), t \in [0, T],
\end{aligned}
\end{cases}
Y(T) &= \xi,
\end{align*}
\]
We denote by \( \mathcal{U}, \) the set of all consumption processes. For each \( \pi(t) \in \mathcal{U}, \) the driver \( g : \Omega \times [0, T] \times \mathbb{R}^2 \times L^2_0 \times \mathbb{R}^2 \times \mathcal{L}^2 \times \mathcal{U} \to \mathbb{R} \) and the terminal value \( \xi \) satisfies assumptions (1). Suppose that \( (y, z, k, \bar{y}, \bar{z}, \bar{k}, \bar{\pi}) \mapsto g(t, y, z, k, \bar{y}, \bar{z}, \bar{k}, \bar{\pi}) \) is concave for each \( t \in [0, T] \). The driver
\[
g(t, Y(t), Z(t), K(t, \cdot), \mathbb{E}[Y(t)], \mathbb{E}[Z(t)], \mathbb{E}[K(t, \cdot)], \pi(t))
\]
represents the instantaneous utility at time \( t \) of the consumption rate \( \pi(t) \) \( \geq 0 \), such that
\[
\mathbb{E}[\int_0^T g(t, 0, 0, 0, \pi(t))^2 dt] < \infty, \quad \text{for all } t \in [0, T].
\]
We call a process \( \pi(t) \) a consumption rate process if \( \pi(t) \) is predictable and \( \pi(t) \geq 0 \) for each \( t \) \( \mathbb{P} \)-a.s. Then \( Y(t) = Y_0(0) \) is called a mean-field recursive utility process of the consumption \( \pi(\cdot) \), and the number \( U(\pi) = Y_0(0) \) is called the total mean-field recursive utility of \( \pi(\cdot) \). This is an extension to mean-field (and jumps) of the classical recursive utility concept of Duffie and Epstein [12]. See also Duffie and Zin [13], Kreps and Parteus [16], El Karoui et al [14], Øksendal and Sulem [20] and Agram and Röse [2] and the reference therein. Finding the consumption rate \( \hat{\pi} \) which maximizes its total mean-field recursive utility is an interesting problem in mean-field stochastic control.

5.1. Optimization problem. We discuss now the optimization problem related to the recursive utility. The wealth process \( X(t) = X^\pi(t) \) is given by the following linear sde
\[
\begin{align*}
\begin{cases}
\begin{aligned}
dX(t) &= [b_0(t) - \pi(t)]X(t)dt + \sigma_0(t)X(t)dB(t) \\
&\quad + \int_{\mathbb{R}_0} \gamma_0(t, \zeta)X(t)N(dt, d\zeta), t \in [0, T],
\end{aligned}
\end{cases}
X(0) = x_0,
\end{align*}
\]
where the initial value \( x_0 > 0 \), and the functions \( b_0, \sigma_0, \gamma_0 \) are assumed to be deterministic functions, \( \pi(t) \) is our relative consumption rate at time \( t \), assumed to be a càdlàg \( \mathbb{P} \)-adapted process. We assume that \( \int_0^T \pi(t)dt < \infty \) \( \mathbb{P} \)-a.s. This implies that our wealth process \( X(t) > 0 \) for all \( t \) \( \mathbb{P} \)-a.s. Define the recursive utility process \( Y(t) = Y^\pi(t) \) by the linear mean-field bsde in the unknown triplet \( (Y, Z, K) = (Y^\pi, Z^\pi, K^\pi) \in S^2 \times L^2 \times H^2_0 \), by
\[
\begin{align*}
\begin{cases}
\begin{aligned}
dY(t) &= -[\alpha_0(t)Y(t) + \alpha_1(t)\mathbb{E}[Y(t)] + \beta_0(t)Z(t) + \beta_1(t)\mathbb{E}[Z(t)] \\
&\quad + \int_{\mathbb{R}_0} \{b_0(t, \zeta)K(t, \zeta) + \gamma_0(t, \zeta)\mathbb{E}[K(t, \zeta)]\} \nu(d\zeta) + \ln(\pi(t)X(t))]dt \\
&\quad + Z(t)dB(t) + \int_{\mathbb{R}_0} K(t, \zeta)N(dt, d\zeta), t \in [0, T],
\end{aligned}
\end{cases}
Y(T) = \theta X(T),
\end{align*}
\]
where $\theta = \theta(\omega) > 0$ is a given bounded random variable and $\alpha_0, \alpha_1, \beta_0, \beta_1, \eta_0, \eta_1$ are given deterministic functions with $\eta_0(t, \zeta), \eta_1(t, \zeta) \geq -1$.

From the closed formula (3.16), the first component $Y(t)$ of the solution triplet of the equation (5.3) can be written as

$$Y(t) = \mathbb{E}^{\mathcal{F}_t}[\theta X(T) \Gamma(t, T)] + \int_t^T \Gamma(t, s)\{\alpha_1(s), \beta_1(s), \eta_1(s, \zeta)\}V(s) + \ln(\pi(s)X(s))ds, t \in [0, T]$$

where

$$\Gamma(t, s) = \exp\left\{\int_t^s \beta_0(r)dB(r) + \int_t^s (\alpha_0(r) - \frac{1}{2}(\beta_0(r))^2)dr + \int_t^s \int_{\mathbb{R}^2} (\ln(1 + \eta_0(r, \zeta)) - \eta_0(r, \zeta)) \nu(d\zeta)dr + \int_t^s \int_{\mathbb{R}^2} (\ln(1 + \eta_0(r, \zeta))N(dr, d\zeta)) \right\}.$$

and

$$V = \sum_{n=0}^{\infty} A^n F.$$

We want to maximize the performance functional

$$J(\pi) := \mathbb{E}[Y(0)] = \mathbb{E}[Y(0)].$$

The corresponding Hamiltonian to this optimization problem $H : [0, T] \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathcal{U} \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$, is defined by

$$H(t, x, y, z, k, \pi, \bar{\pi}, \nu, \kappa, r, \lambda) = (b_0 - \pi)xp + \sigma_0xq + \int_{\mathbb{R}^2} g_0(\zeta) \nu(d\zeta) + \lambda[y_0 + \lambda_0 \gamma] + \beta_0z + \beta_1\bar{\pi} + \int_{\mathbb{R}^2} \eta_0(\lambda_0)k(\zeta)\eta_1(\lambda_0)\eta_0(\kappa(\zeta))\nu(d\zeta) + \ln \pi + \ln x$$

where the adjoint processes, for the linear mean field bsde (5.3) and for the linear differential equation $\lambda = \lambda^*$ corresponding to $\pi$, are defined by

$$\begin{cases}
    dp(t) = -[(b_0 - \pi(t))p(t) + \sigma_0(t)q(t) + \int_{\mathbb{R}^2} g_0(t, \zeta)r(t, \zeta)\nu(d\zeta)]dt + q(t)dB(t) + \int_{\mathbb{R}^2} r(t, \zeta)\tilde{N}(dt, d\zeta), t \in [0, T], \\
p(T) = \theta,
\end{cases}$$

and

$$\begin{cases}
    d\lambda(t) = (\alpha_0(t)\lambda(t) + \alpha_1(t)\mathbb{E}[\lambda(t)])dt + (\beta_0(t)\lambda(t) + \beta_1(t)\mathbb{E}[\lambda(t)])dB(t) + \int_{\mathbb{R}^2} \eta_0(t, \zeta)\lambda(t) + \eta_1(t, \zeta)\mathbb{E}[\lambda(t)]\tilde{N}(dt, d\zeta), t \in [0, T], \\
\lambda(0) = 1.
\end{cases}$$

Consequently

$$\lambda(t) = \mathbb{E}^{-1}(t)[1 + \int_0^t \{\mathbb{E}(r)(\alpha_1(r)\mathbb{E}[\lambda(r)]) + \int_{\mathbb{R}^2} \mathbb{E}[\lambda(r)]\nu(d\zeta)dr + \int_{\mathbb{R}^2} \mathbb{E}[\lambda(r)]\nu(d\zeta)dr + \int_{\mathbb{R}^2} \mathbb{E}[\lambda(r)]N(dr, d\zeta)]], t \in [0, T],$$

where

$$\mathbb{E}[\lambda(r)] = \exp\left\{\int_0^t \{(-\alpha_0(r) + \frac{1}{2}(\beta_0(r))^2 - \int_{\mathbb{R}^2} (\ln(1 + \eta_0(r, \zeta)) - \eta_0(r, \zeta)) \nu(d\zeta)dr - \int_{\mathbb{R}^2} \beta_0(r)dB(r) + \int_{\mathbb{R}^2} \ln(1 + \eta_1(r, \zeta))\tilde{N}(dr, d\zeta)\right\},$$

and

$$p(t) = \mathbb{E}^{\mathcal{F}_t}[\theta], t \in [0, T],$$

Now differentiate $H$ with respect to $\pi$, we obtain

$$\frac{\partial}{\partial \pi} H(t) = -p(t) + \frac{\lambda(t)}{\pi(t)}.$$

The first order necessary condition of optimality, yields:
Theorem 5.1. The optimal control $\hat{\pi}(t)$ is given by

$$\hat{\pi}(t) = \frac{\hat{\lambda}(t)}{\hat{p}(t)},$$

where $\hat{\lambda}(t)$ and $\hat{p}(t)$ are the solutions of the equations (5.4) and (5.5) respectively, corresponding to the optimal control $\tilde{\pi}(t)$.

6. Appendix

6.1. Special case of linear mean-field bsde. We first define the measure $Q$ by

$$dQ = M(T)d\mathbb{P}$$
on $F_T$,

where

$$M(t) := \exp(\int_0^t \beta^1(s)dB(s) - \frac{1}{2}\int_0^t \beta_1^2(s)ds + \int_0^t \int_{\mathcal{R}_d} \ln(1 + \eta^1(s, \zeta))\hat{N}(ds, d\zeta) + \int_0^t \int_{\mathcal{R}_d} \{\ln(1 + \eta^1(s, \zeta)) - \eta^1(s, \zeta)\}\nu(d\zeta)ds; \quad 0 \leq t \leq T.$$ 

Then, under the measure $Q$ the process

$$B_Q(t) := B(t) - \int_0^t \beta^1(s)ds, \quad 0 \leq t \leq T,$$

is a Brownian motion, and the random measure

$$\hat{N}_Q(dt, d\zeta) := \hat{N}(dt, d\zeta) - \eta^1(t, \zeta)\nu(d\zeta)dt$$

is the $Q$-compensated Poisson random measure of $N(\cdot, \cdot)$, in the sense that the process

$$\hat{N}_\gamma(t) := \int_0^t \int_{\mathcal{R}_d} \gamma(s, \zeta)\hat{N}_Q(ds, d\zeta)$$

is a local $Q$-martingale, for all predictable processes $\gamma(t, \zeta)$ such that

$$\int_0^T \int_{\mathcal{R}_d} (\gamma(t, \zeta)\eta^1(t, \zeta))^2\nu(d\zeta)dt < \infty.$$

Consider the following linear mean field bsde

$$\begin{cases}
    dY(t) = -[\alpha^1(t)Y(t) + \beta^1(t)Z(t) + \int_{\mathcal{R}_d} \eta^1(t, \zeta)K(t, \zeta)\nu(d\zeta) + \alpha^2(t)\mathbb{E}_Q[Y(t)]]dt + \gamma(t)dB(t) + \int_{\mathcal{R}_d} K(t, \zeta)\hat{N}(ds, d\zeta), t \in [0, T], \\
    Y(T) = \xi,
\end{cases}$$

where $\alpha^1(t), \alpha^2(t), \beta^1(t), \eta^1(t, \cdot)$ are given deterministic functions and $\gamma(t)$ is $\mathbb{F}$-adapted and $\xi \in L^2(\Omega, \mathcal{F}_T)$. Then, by change of measure, the linear mean field bsde (6.3) is equivalent to

$$\begin{cases}
    dY(t) = -[\alpha^1(t)Y(t) + \alpha^2(t)\mathbb{E}_Q[Y(t)] + \gamma(s)]dt + Z(t)dB(t) + \int_{\mathcal{R}_d} K(t, \zeta)\hat{N}_Q(ds, d\zeta), t \in [0, T], \\
    Y(T) = \xi.
\end{cases}$$

Then, $Y(t)$ is given by

$$Y(t) = \mathbb{E}_Q^F[\xi\Gamma'(t, T) + \int_0^T \Gamma'(t, s)\{\alpha^2(s)\mathbb{E}_Q[Y(s)] + \gamma(s)d\zeta\}, t \in [0, T], \text{ a.s.},$$

for $\Gamma'(t, s) = \exp(\int_t^s \alpha^2(r)dr)$.

It remains to find $\mathbb{E}_Q[Y(t)]$. Taking the expectation of both sides of (6.4), we end up with

$$\mathbb{E}_Q[Y(t)] = \exp(-\int_0^t \{\alpha^1(s) + \alpha^2(s)\}ds)Y(0) + \int_0^t \mathbb{E}_v[\gamma(s)]ds \exp(\int_0^t \{\alpha^1(r) + \alpha^2(r)\}dr)ds,$$

for some deterministic value $Y(0)$. 

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