Curvatures of Smooth and Discrete Surfaces

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Abstract. We discuss notions of Gauss curvature and mean curvature for polyhedral surfaces. The discretizations are guided by the principle of preserving integral relations for curvatures, like the Gauss/Bonnet theorem and the mean-curvature force balance equation.

Keywords. Discrete Gauss curvature, discrete mean curvature, integral curvature relations.

The curvatures of a smooth surface are local measures of its shape. Here we consider analogous quantities for discrete surfaces, meaning triangulated polyhedral surfaces. Often the most useful analogs are those which preserve integral relations for curvature, like the Gauss/Bonnet theorem or the force balance equation for mean curvature. For simplicity, we usually restrict our attention to surfaces in euclidean three-space $E^3$, although some of the results generalize to other ambient manifolds of arbitrary dimension.

This article is intended as background for some of the related contributions to this volume. Much of the material here is not new; some is even quite old. Although some references are given, no attempt has been made to give a comprehensive bibliography or a full picture of the historical development of the ideas.

1. Smooth curves, framings and integral curvature relations

A companion article [Sul08] in this volume investigates curves of finite total curvature. This class includes both smooth and polygonal curves, and allows a unified treatment of curvature. Here we briefly review the theory of smooth curves from the point of view we will later adopt for surfaces.

The curvatures of a smooth curve $\gamma$ (which we usually assume is parametrized by its arclength $s$) are the local properties of its shape, invariant under euclidean motions. The only first-order information is the tangent line; since all lines in space are equivalent, there are no first-order invariants. Second-order information (again, independent of parametrization) is given by the osculating circle; the one corresponding invariant is its curvature $\kappa = 1/r$. 
(For a plane curve given as a graph \( y = f(x) \) let us contrast the notions of curvature \( \kappa \) and second derivative \( f'' \). At a point \( p \) on the curve, we can find either one by translating \( p \) to the origin, transforming so the curve is horizontal there, and then looking at the second-order behavior. The difference is that for curvature, the transformation is an euclidean rotation, while for second derivative, it is a shear \((x,y) \mapsto (x, y - ax)\). A parabola has constant second derivative \( f'' \) because it looks the same at any two points after a shear. A circle, on the other hand, has constant curvature because it looks the same at any two points after a rotation.)

A plane curve is completely determined (up to rigid motion) by its (signed) curvature \( \kappa(s) \) as a function of arclength \( s \). For a space curve, however, we need to look at the third-order invariants; these are the torsion \( \tau \) and the derivative \( \kappa' \), but the latter of course gives no new information. Curvature and torsion now form a complete set of invariants: a space curve is determined by \( \kappa(s) \) and \( \tau(s) \).

Generically speaking, while second-order curvatures usually suffice to determine a hypersurface (of codimension 1), higher-order invariants are needed for higher codimension. For curves in \( \mathbb{R}^d \), for instance, we need \( d - 1 \) generalized curvatures, of order up to \( d \), to characterize the shape.

Let us examine the case of space curves \( \gamma \subset \mathbb{E}^3 \) in more detail. At every point \( p \in \gamma \) we have a splitting of the tangent space \( T_p \mathbb{E}^3 \) into the tangent line \( T_p \gamma \) and the normal plane. A framing along \( \gamma \) is a smooth choice of a unit normal vector \( N_1 \), which is then completed to the oriented orthonormal frame \((T, N_1, N_2)\) for \( \mathbb{R}^3 \), where \( N_2 = T \times N_1 \). Taking the derivative with respect to arclength, we get a skew-symmetric matrix (an infinitesimal rotation) that describes how the frame changes:

\[
\begin{pmatrix}
T \\ N_1 \\ N_2
\end{pmatrix}' =
\begin{pmatrix}
0 & \kappa_1 & \kappa_2 \\
-\kappa_1 & 0 & \tau \\
-\kappa_2 & -\tau & 0
\end{pmatrix}
\begin{pmatrix}
T \\ N_1 \\ N_2
\end{pmatrix}.
\]

Here, \( T'(s) = \sum \kappa_i N_i \) is the curvature vector of \( \gamma \), while \( \tau \) measures the twisting of the chosen orthonormal frame.

If we fix \( N_1 \) at some basepoint along \( \gamma \), then one natural framing is the parallel frame or Bishop frame [Bis75] defined by the condition \( \tau = 0 \). Equivalently, the vectors \( N_i \) are parallel-transported along \( \gamma \) from the basepoint, using the Riemannian connection on the normal bundle induced by the immersion in \( \mathbb{E}^3 \). One should note that this is not necessarily a closed framing along a closed loop \( \gamma \); when we return to the basepoint, the vector \( N_1 \) has been rotated through an angle called the writhe of \( \gamma \).

Other framings are also often useful. For instance, if \( \gamma \) lies on a surface \( M \) with unit normal \( \nu \), it is natural to choose \( N_1 = \nu \). Then \( N_2 = \eta := T \times \nu \) is called the cornormal vector, and \((T, \nu, \eta)\) is the Darboux frame (adapted to \( \gamma \subset M \subset \mathbb{E}^3 \)). The curvature vector of \( \gamma \) decomposes into parts tangent and normal to \( M \) as \( T' = \kappa_n \nu + \kappa_g \eta \). Here, \( \kappa_n = -\nu' \cdot T \) measures the normal curvature of \( M \) in the direction \( T \), and is independent of \( \gamma \), while \( \kappa_g \), the geodesic curvature of \( \gamma \) in \( M \), is an intrinsic notion, showing how \( \gamma \) sits in \( M \), and is unchanged if we isometrically deform the immersion of \( M \) into space.
When the curvature vector of \( \gamma \) never vanishes, we can write it as \( T' = \kappa N \), where \( N \) is a unit vector, the principal normal, and \( \kappa > 0 \). This yields the orthonormal Frenet frame \((T, N, B)\), whose twisting \( \tau \) is the torsion of \( \gamma \).

The total curvature of a smooth curve is \( \int \kappa \, ds \). In \cite{Sul08} we review a number of standard results: For closed curves, the total curvature is at least \( 2\pi \) (Fenchel) and for knotted space curves the total curvature is at least \( 4\pi \) (Fáry/Milnor). For plane curves, we can consider instead the signed curvature, and find that \( \int \kappa \, ds \) is always an integral multiple of \( 2\pi \). Suppose (following Milnor) we define the total curvature of a polygonal curve simply to be the sum of the turning angles at the vertices. Then, as we explain in \cite{Sul08}, all these theorems on total curvature remain true. Our goal, when defining curvatures for polyhedral surfaces, will be to ensure that similar integral relations remain true.

2. Curvatures of smooth surfaces

Given a (two-dimensional, oriented) surface \( M \) smoothly immersed in \( \mathbb{E}^3 \), we understand its local shape by looking at the Gauss map \( \nu : M \to \mathbb{S}^2 \) given by the unit normal vector \( \nu = \nu_p \) at each point \( p \in M \). The derivative of the Gauss map at \( p \) is a linear map from \( T_p M \) to \( T_{\nu_p} \mathbb{S}^2 \). Since these spaces are naturally identified, being parallel planes in \( \mathbb{E}^3 \), we can view the derivative as an endomorphism \( -S_p : T_p M \to T_p M \). The map \( S_p \) is called the shape operator (or Weingarten map).

The shape operator is the complete second-order invariant (or curvature) which determines the original surface \( M \). (This statement has been left intentionally a bit vague, since without a standard parametrization like arclength, it is not quite clear how one should specify such an operator along an unknown surface.) Usually, however, it is more convenient to work not with the operator \( S_p \) but instead with scalar quantities. Its eigenvalues \( \kappa_1 \) and \( \kappa_2 \) are called the principal curvatures, and (since they cannot be globally distinguished) it is their symmetric functions which have the most geometric meaning.

We define the Gauss curvature \( K := \kappa_1 \kappa_2 \) as the determinant of \( S_p \) and the mean curvature \( H := \kappa_1 + \kappa_2 \) as its trace. Note that the sign of \( H \) depends on the choice of unit normal \( \nu \), so often it is more natural to work with the vector mean curvature (or mean curvature vector) \( \mathbf{H} := H \nu \). Furthermore, some authors use the opposite sign on \( S_p \) and thus \( \mathbf{H} \), and many use \( H = (\kappa_1 + \kappa_2)/2 \), justifying the name “mean” curvature. Our conventions mean that the mean curvature vector for a convex surface points inwards (like the curvature vector for a circle). For a unit sphere oriented with inward normal, the Gauss map \( \nu \) is the antipodal map, \( S_p = I \), and \( H = 2 \).

The Gauss curvature is an intrinsic notion, depending only on the pullback metric on the surface \( M \), and not on the immersion into space. That is, \( K \) is unchanged by bending the surface without stretching it. For instance, a developable surface like a cylinder or cone has \( K \equiv 0 \) because it is obtained by bending a flat plane. One intrinsic characterization of \( K(p) \) is obtained by comparing the circumferences \( C_\varepsilon \) of (intrinsic) \( \varepsilon \)-balls around \( p \)
to the value $2\pi \varepsilon$ in $E^2$. We get
\[
\frac{C_\varepsilon}{2\pi \varepsilon} = 1 - \frac{\varepsilon^2}{6} K(p) + O(\varepsilon^3).
\]

Mean curvature, on the other hand, is certainly not intrinsic, but it has a nice variational interpretation. Consider a variation vectorfield $\xi$ on $M$; for simplicity assume $\xi$ is compactly supported away from any boundary. Then $H = -\delta \text{Area}/\delta \text{Vol}$ in the sense that
\[
\delta_{\xi} \text{Vol} = \int \xi \cdot \nu \, dA, \quad \delta_{\xi} \text{Area} = -\int \xi \cdot H \nu \, dA.
\]
With respect to the $L^2$ inner product $\langle \xi, \eta \rangle := \int \xi_p \cdot \eta_p \, dA$ on vectorfields, the vector mean curvature is thus the negative gradient of the area functional, often called the first variation of area: $H = -\nabla \text{Area}$. (Similarly, the negative gradient of length for a curve is its curvature vector $\kappa N$.)

Just as $\kappa$ is the geometric version of second derivative for curves, mean curvature is the geometric version of the Laplacian $\Delta$. Indeed, if a surface $M$ is written locally near a point $p$ as the graph of a height function $f$ over its tangent plane $T_p M$, then $H(p) = \Delta f$. Alternatively, we can write $H = \nabla_M \cdot \nu = \Delta_M x$, where $x$ is the position vector in $E^3$ and $\Delta_M$ is the Laplace/Beltrami operator, the intrinsic surface Laplacian.

We can flow a curve or surface to reduce its length or area, by following the gradient vectorfield $\kappa N$ or $H\nu$; the resulting parabolic (heat) flow is slightly nonlinear in a natural geometric way. This so-called mean-curvature flow has been extensively studied as a geometric smoothing flow. (See, among many others, [GH86, Gra87] for the curve-shortening flow and [Bra78, Hui84, Ilm94, Eck04, Whi05] for higher dimensions.)

3. Integral curvature relations for surfaces

For surfaces, we will consider various integral curvature relations that relate area integrals over a region $D \subset M$ to arclength integrals over the boundary $\gamma := \partial D$. First, the Gauss/Bonnet theorem says that, when $D$ is a disk,
\[
2\pi - \iint_D K \, dA = \oint_{\gamma} \kappa_p \, ds = \oint_{\gamma} T' \cdot \eta \, ds = -\oint_{\gamma} \eta' \cdot dx.
\]
Here, $dx = T \, ds$ is the vector line element along $\gamma$, and $\eta = T \times \nu$ is again the cornormal. In particular, this theorem implies that the total Gauss curvature of $D$ depends only on a collar neighborhood of $\gamma$: if we make any modification to $D$ supported away from the boundary, the total curvature is unchanged (as long as $D$ remains topologically a disk). We will extend the notion of (total) Gauss curvature from smooth surfaces to more general surfaces (in particular polyhedral surfaces) by requiring that this property remain true.

Our other integral relations are all proved by Stokes’s Theorem, and thus require only that $\gamma$ be the boundary of $D$ in a homological sense; for these $D$ need not be a disk. First consider the vector area
\[
A_\gamma := \frac{1}{2} \oint_{\gamma} \mathbf{x} \times dx = \frac{1}{2} \oint_{\gamma} \mathbf{x} \times T \, ds = \iint_D \nu \, dA.
\]
The right-hand side represents the total vector area of any surface spanning $\gamma$, and the relation shows this to depend only on $\gamma$ (and this time not even on a collar neighborhood). The integrand on the left-hand side depends on a choice of origin for the coordinates, but because we integrate over a closed loop, the integral is independent of this choice. Both sides of this vector area formula can be interpreted directly for a polyhedral surface, and the equation remains true in that case. We note also that this vector area $A_\gamma$ is one of the quantities preserved when $\gamma$ evolves under the Hasimoto or smoke-ring flow $\dot{\gamma} = \kappa B$. (Compare [LP94, PSW07, Hof08].)

A simple application of the fundamental theorem of calculus to the tangent vector of a curve $\gamma$ from $p$ to $q$ shows that

$$T(q) - T(p) = \int_p^q T'(s) \, ds = \int_p^q \kappa N \, ds.$$  

This can be viewed as a balance between elastic tension forces trying to shrink the curve, and sideways forces holding it in place. It is the key step in verifying that the vector curvature $\kappa N$ is the first variation of length.

The analog for a surface patch $D$ is the mean-curvature force balance equation

$$\oint_D \kappa \nu \times dx = \oint_D H \nu \, dA = \int D \kappa N \, dA.$$  

Again this represents a balance between surface tension forces acting in the conormal direction along the boundary of $D$ and what can be considered as pressure forces (especially in the case of constant $H$) acting normally across $D$. We will use this equation to develop our analog of mean curvature for discrete surfaces.

The force balance equation can be seen to arise from the translational invariance of curvatures. It has been important for studying surfaces of constant mean curvature; see for instance [KKS89, Kus91, GKS03]. The rotational analog is the following torque balance:

$$\oint_D \mathbf{x} \times \kappa \eta \, ds = \oint_D \mathbf{x} \times (\nu \times dx) = \oint_D H (\mathbf{x} \times \nu) \, dA = \int_D \mathbf{x} \times H \, dA.$$  

Somewhat related is the following equation:

$$\oint_D \mathbf{x} \cdot \eta \, ds = \oint_D \mathbf{x} \cdot (\nu \times dx) = \int_D (H \cdot \mathbf{x} - 2) \, dA.$$  

It gives, for example, an interesting expression for the area of a minimal ($H \equiv 0$) surface.

4. Discrete surfaces

For us, a discrete or polyhedral surface $M \subset \mathbb{E}^3$ will mean a triangulated surface with a continuous map into space, linear on each triangle. In more detail, we start with an abstract combinatorial triangulation—a simplicial complex—representing a 2-manifold with boundary. We then pick positions $p \in \mathbb{E}^3$ for all the vertices, which uniquely determine a linear map on each triangle; these maps fit together to form the polyhedral surface.

The union of all triangles containing a vertex $p$ is called $\text{Star}(p)$, the star of $p$. Similarly, the union of the two triangles containing an edge $e$ is $\text{Star}(e)$.  

4.1. Gauss curvature

It is well known how the notion of Gauss curvature extends to such discrete surfaces $M$. (Banchoff [Ban67, Ban70] was probably the first to discuss this in detail, though he notes that Hilbert and Cohn-Vossen [HCV32, §29] had already used a polyhedral analog to motivate the intrinsic nature of Gauss curvature.) Any two adjacent triangles (or, more generally, any simply connected region in $M$ not including any vertices) can be flattened—developed isometrically into the plane. Thus the Gauss curvature is supported on the vertices $p \in M$. In fact, to keep the Gauss/Bonnet theorem true, we must take

$$\int \int_D K dA := \sum_{p \in D} K_p, \quad \text{with} \quad K_p := 2\pi - \sum \theta_i.$$ 

Here, the angles $\theta_i$ are the interior angles at $p$ of the triangles meeting there, and $K_p$ is often known as the angle defect at $p$. If $D$ is any neighborhood of $p$ contained in $\text{Star}(p)$, then $\oint_{\partial D} \kappa_g ds = \sum \theta_i$; when the triangles are acute, this is most easily seen by letting $\partial D$ be the path connecting their circumcenters and crossing each edge perpendicularly. Analogous to our intrinsic characterization of Gauss curvature in the smooth case, note that the circumference of a small $\varepsilon$ ball around $p$ here is exactly $2\pi\varepsilon - \varepsilon K_p$.

This version of discrete Gauss curvature is quite natural, and seems to be the correct analog when Gauss curvature is used intrinsically. But the Gauss curvature of a smooth surface in $\mathbb{E}^3$ also has extrinsic meaning; for instance the total absolute Gauss curvature is proportional to the average number of critical points of different height functions. For such considerations, Brehm and Kühl [BK82] suggest the following: when a vertex $p$ is extreme on the convex hull of its star, but the star itself is not a convex cone, then we should think of $p$ as having both positive and negative curvature. We let $K_p^+$ be the curvature at $p$ of the convex hull, and set $K_p^- := K_p^+ - K_p \geq 0$. Then the absolute curvature at $p$ is $K_p^+ + K_p^-$, which is greater than $|K_p|$. This discretization, is of course, also based on preserving an integral curvature relation—a different one. (See also [BK97, vDA95].)

We can use the same principle as before—preserving the Gauss/Bonnet theorem—to define Gauss curvature, as a measure, for much more general surfaces. For instance, on a piecewise smooth surface, we have ordinary $K$ within each face, a point mass (again the angle defect) at each vertex, and a linear density along each edge, equal to the difference in the geodesic curvatures of that edge within its two incident faces. Indeed, clothes are often designed from pieces of (intrinsically flat) cloth, joined so that each vertex is intrinsically flat and thus all the curvature is along the edges; corners would be unsightly in clothes.

Returning to polyhedral surfaces, we note that $K_p$ is clearly an intrinsic notion (as it should be) depending only on the angles of each triangle and not on the precise embedding into $\mathbb{E}^3$. Sometimes it is useful to have a notion of combinatorial curvature, independent of all geometric information. Given just a combinatorial triangulation, we can pretend that each triangle is equilateral with angles $\theta = 60^\circ$. (Such a euclidean metric with cone points at certain vertices exists on the abstract surface, independent of whether or not it could be embedded in space. See the survey [Tro07].)
The curvature of this metric, $K_p = \frac{\pi}{3} (6 - \deg p)$, can be called the combinatorial (Gauss) curvature of the triangulation. (See [Thu98, IK + 08] for combinatorial applications of this notion.) In this context, the global form $\sum K_p = 2\pi\chi(M)$ of Gauss/Bonnet amounts to nothing more than Euler’s formula $\chi = V - E + F$. (We note that Forman has proposed a combinatorial Ricci curvature [For03]; although for smooth surfaces, Ricci curvature is Gauss curvature, for discrete surfaces Forman’s combinatorial curvature does not agree with ours, so he fails to recover the Gauss/Bonnet theorem.)

Our discrete Gauss curvature $K_p$ is of course an integrated quantity. Sometimes it is desirable to have instead a curvature density, dividing $K_p$ by the surface area associated to the vertex $p$. One natural choice is $A_p := \frac{1}{3} \text{Area(Star}(p))$, but this does not always behave nicely for irregular triangulations. One problem is that, while $K_p$ is intrinsic, depending only on the cone metric of the surface, $A_p$ depends also on the choice of which pairs of cone points are connected by triangle edges. One fully intrinsic notion of the area associated to $p$ would be the area of its intrinsic Voronoi cell in the sense of Bobenko and Springborn [BS05]; perhaps this would be the best choice for computing Gauss curvature density.

4.2. Vector area

The vector area formula

$$A_{\gamma} := \frac{1}{2} \oint_{\gamma} \mathbf{x} \times d\mathbf{x} = \iint_D \nu dA$$

needs no special interpretation for discrete surfaces; both sides of the equation make sense directly, since the surface normal $\nu$ is well-defined almost everywhere. However, it is worth interpreting this formula for the case when $D$ is the star of a vertex $p$. More generally, suppose $\gamma$ is any closed curve (smooth or polygonal), and $D$ is the cone from $p$ to $\gamma$ (the union of all line segments $pq$ for $q \in \gamma$). Fixing $\gamma$ and letting $p$ vary, we find that the volume enclosed by this cone is an affine linear function of $p \in \mathbb{E}^3$, and thus

$$A_p := \nabla_p \text{Vol} D = \frac{A_{\gamma}}{3} = \frac{1}{6} \oint_{\gamma} \mathbf{x} \times d\mathbf{x}$$

is independent of the position of $p$. We also note that any such cone $D$ is intrinsically flat except at the cone point $p$, and that $2\pi - K_p$ is the cone angle at $p$.

4.3. Mean curvature

The mean curvature of a discrete surface $M$ is supported along the edges. If $e$ is an edge, and $e \subset D \subset \text{Star}(e) = T_1 \cup T_2$, then we set

$$H_e := \iint_D H dA = \oint_{\partial D} \eta ds = e \times \nu_1 - e \times \nu_2 = J_1 e - J_2 e.$$ 

Here $\mathbf{\nu}_i$ is the normal vector to the triangle $T_i$, and the operator $J_i$ rotates by $90^\circ$ in the plane of that triangle. Note that $|H_e| = 2 \sin(\theta_e/2) |e|$, where $\theta_e$ is the exterior dihedral angle along the edge, defined by $\cos \theta_e = \nu_1 \cdot \nu_2$. 
No nonplanar discrete surface has $H_e = 0$ along every edge. But this discrete mean curvature can cancel out around vertices. We set

$$2H_p := \sum_{e \ni p} H_e = \iint_{\text{Star}(p)} H \, dA = \oint_{\partial \text{Star}(p)} \eta \, ds.$$  

The area of the discrete surface is a function of the vertex positions; if we vary only one vertex $p$, we find that $\nabla_p \text{Area}(M) = -H_p$. This mirrors the variational characterization of mean curvature for smooth surfaces, and we see that a natural notion of discrete minimal surfaces is to require $H_p \equiv 0$ for all vertices [Bra92, PP93].

Suppose that the vertices adjacent to $p$, in cyclic order, are $p_1, \ldots, p_n$. Then we can express $A_p$ and $H_p$ explicitly in terms of these neighbors. We get

$$3A_p = 3\nabla_p \text{Vol} = \iint_{\text{Star}(p)} \nu \, dA = \frac{1}{2} \oint_{\partial \text{Star}(p)} x \times dx = \frac{1}{2} \sum_i p_i \times p_{i+1}$$

and similarly

$$2H_p = \sum_{i} H_{pp_i} = -2\nabla_p \text{Area} = \sum_{i} J_i (p_{i+1} - p_i) = \sum_{i} (\cot \alpha_i + \cot \beta_i) (p - p_i),$$

where $\alpha_i$ and $\beta_i$ are the angles opposite edge $pp_i$ in the two incident triangles. This latter equation is the famous “cotangent formula” [PP93, War08] which also arises naturally in a finite-element discretization of the Laplacian.

Suppose we change the combinatorics of a discrete surface $M$ by introducing a new vertex $p$ along an existing edge $e$ and subdividing the two incident triangles. Then $H_p$ in the new surface equals the original $H_e$, independent of where along $e$ we place $p$. This allows a variational interpretation of $H_e$.

### 4.4. Minkowski mixed volumes

A somewhat different interpretation of mean curvature for convex polyhedra is found in the context of Minkowski’s theory of mixed volumes. (In this simple form, it dates back well before Minkowski, to Steiner [Ste40].) If $X$ is a smooth convex body in $\mathbb{E}^3$ and $B_t(X)$ denotes its $t$-neighborhood, then its Steiner polynomial is:

$$\text{Vol}(B_t(X)) = \text{Vol} X + t \text{Area} \partial X + \frac{t^2}{2} \int_{\partial X} H \, dA + \frac{t^3}{3} \int_{\partial X} K \, dA.$$  

Here, by Gauss/Bonnet, the last integral is always $4\pi$.

When $X$ is instead a convex polyhedron, we already understand how to interpret each term except $\int_{\partial X} H \, dA$. The correct replacement for this term, as Steiner discovered, is $\sum_{e} \theta_e |e|$. This suggests $H_e := \theta_e |e|$ as a notion of total mean curvature for the edge $e$.

Note the difference between this formula and our earlier $|H_e| = 2 \sin(\theta_e/2) |e|$. Either one can be derived by replacing the edge $e$ with a sector of a cylinder of length $|e|$ and arbitrary (small) radius $r$. We have $\iint H \, dA = H_e$, so that

$$\left| \iint H \, dA \right| = |H_e| = 2 \sin(\theta_e/2) |e| < \theta_e |e| = H_e = \iint H \, dA.$$
The difference is explained by the fact that one formula integrates the scalar mean curvature while the other integrates the vector mean curvature. Again, these two discretizations both arise through preservation of (different) integral relations for mean curvature.

See [Sul08] for a more extensive discussion of the analogous situation for curves: although as we have mentioned, the sum of the turning angles $\psi_i$ is often the best notion of total curvature for a polygon, in certain situations the “right” discretization is instead the sum of $2\sin \psi_i/2$ or $2\tan \psi_i/2$.

The interpretation of curvatures in terms of the mixed volumes or Steiner polynomial actually works for arbitrary convex surfaces. (Compare [Sch08] in this volume.) Using geometric measure theory—and a generalized normal bundle called the normal cycle—one can extend both Gauss and mean curvature in a similar way to quite general surfaces. See [Fed59, Fu94, CSM03, CSM06].

4.5. Constant mean curvature and Willmore surfaces

A smooth surface which minimizes area under a volume constraint has constant mean curvature; the constant $H$ can be understood as the Lagrange multiplier for the constrained minimization problem. A discrete surface which minimizes area among surfaces of fixed combinatorial type and fixed volume will have constant discrete mean curvature $H$ in the sense that at every vertex, $H_p = H A_p$, or equivalently $\nabla_p \text{Area} = -H \nabla_p \text{Vol}$.

In general, of course, the vectors $H_p$ and $A_p$ are not even parallel: they give two competing notions of a normal vector to the discrete surface at the vertex $p$. Still,

$$h_p := \frac{|\nabla_p \text{Area}|}{|\nabla_p \text{Vol}|} = \frac{|H_p|}{|A_p|} = \frac{\int_{\text{Star}(p)} H \, dA}{\int_{\text{Star}(p)} \nu \, dA}$$

gives a better notion of mean curvature density near $p$ than, say, the smaller quantity $\frac{|H_p|^2}{A_p}$ (used fifteen years ago in [HKS92]). Several related discretizations are by now built into Brakke’s Evolver; see the discussion in [Bra07]. Recently, Bobenko [Bob05, Bob08] has described a completely different approach to discretizing the Willmore energy, which respects the Möbius invariance of the smooth energy.

4.6. Relation to discrete harmonic maps

As mentioned above, we can define a discrete minimal surface to be a polyhedral surface with $H_p \equiv 0$. An early impetus to the field of discrete differential geometry was the realization (starting with [PP93]) that discrete minimal surfaces are not only critical points for area (fixing the combinatorics), but also have other properties similar to those of smooth minimal surfaces.
For instance, in a conformal parameterization of a smooth minimal surface, the coordinate functions are harmonic. To interpret this for discrete surfaces, we are led to the question of when a discrete map should be considered conformal. In general this is still open. (Interesting suggestions come from the theory of circle packings, and this is an area of active research. See for instance [Ste05] [BH03] [Bob08] [KSS06] [Spr06].)

However, we should certainly agree that the identity map is conformal. A polyhedral surface $M$ comes with an embedding $\text{Id}_M : M \to \mathbb{E}^3$ which we consider as the identity map. Indeed, we then find (following [PP93]) that $M$ is discrete minimal if and only if $\text{Id}_M$ is discrete harmonic. Here a polyhedral map $f : M \to \mathbb{E}^3$ is called \textit{discrete harmonic} if it is a critical point for the Dirichlet energy, written as the following sum over the triangles $T$ of $M$:

$$E(f) := \sum_{T} |\nabla f_T|^2 \text{Area}_M(T).$$

We can view $E(f) - \text{Area}_f(M)$ as a measure of nonconformality. For the identity map, $E(\text{Id}_M) = \text{Area}(M)$ and $\nabla_p E(\text{Id}_M) = \nabla_p \text{Area}(M)$, confirming that $M$ is minimal if and only if $\text{Id}_M$ is harmonic.

5. Vector bundles on polyhedral manifolds

We now give a general definition of vector bundles and connections on polyhedral manifolds; this leads to another interpretation of the Gauss curvature for a polyhedral surface.

A polyhedral $n$-manifold $P^n$ means a CW-complex which is homeomorphic to an $n$-dimensional manifold, and which is regular and satisfies the intersection condition. (Compare [Zie08] in this volume.) That is, each $n$-cell (called a \textit{facet}) is embedded in $P^n$ with no identifications on its boundary, and the intersection of any two cells (of any dimension) is a single cell (if nonempty). Because $P^n$ is topologically a manifold, each $(n-1)$-cell (called a \textit{ridge}) is contained in exactly two facets.

\textbf{Definition.} A \textit{discrete vector bundle} $V^k$ of rank $k$ over $P^n$ consists of a vectorspace $V_f \cong \mathbb{E}^k$ for each facet $f$ of $P$. A \textit{connection} on $V^k$ is a choice of isomorphism $\phi_r$ between $V_f$ and $V_{f'}$ for each ridge $r = f \cap f'$ of $P$. We are most interested in the case where the vectorspaces $V_f$ have inner products, and the isomorphisms $\phi_r$ are orthogonal.

Consider first the case $n = 1$, where $P$ is a polygonal curve. On an arc (an open curve) any vector bundle is trivial. On a loop (a closed curve), a vector bundle of rank $k$ is determined (up to isomorphism) simply by its \textit{holonomy} around the loop, an automorphism $\phi : \mathbb{E}^k \to \mathbb{E}^k$.

Now suppose $P^n$ is linearly immersed in $\mathbb{E}^d$ for some $d$. That is, each $k$-face of $P$ is mapped homeomorphically to a convex polytope in an affine $k$-plane in $\mathbb{E}^d$, and the star of each vertex is embedded. Then it is clear how to define the discrete tangent bundle $T_f$ (of rank $n$) and normal bundle $N_f$ (of rank $d-n$). Namely, each $T_f$ is the $n$-plane parallel to the affine hull of the facet $f$, and $N_f$ is the orthogonal $(d-n)$-plane. These inherit inner products from the euclidean structure of $\mathbb{E}^d$. 
There are also natural analogs of the Levi-Civita connections on these bundles. Namely, for each ridge $r$, let $\alpha_r \in [0, \pi)$ be the exterior dihedral angle between the facets $f_i$ meeting along $r$. Then let $\phi_r : \mathbb{E}^d \to \mathbb{E}^d$ be the simple rotation by this angle, fixing the affine hull of $r$ (and the space orthogonal to the affine hull of the $f_i$). We see that $\phi_r$ restricts to give maps $T_f \to T_{f'}$ and $N_f \to N_{f'}$; these form the connections we want. (Note that $T \oplus N = \mathbb{E}^d$ is a trivial vector bundle over $P^n$, but the maps $\phi_r$ give a nontrivial connection on it.)

Consider again the example of a closed polygonal curve $P^1 \subset \mathbb{E}^d$. The tangent bundle has rank 1 and trivial holonomy. The holonomy of the normal bundle is some rotation of $\mathbb{E}^{d-1}$. For $d = 3$ this rotation of the plane $\mathbb{E}^2$ is specified by an angle equal (modulo $2\pi$) to the writhe of $P^1$. (To define the writhe of a curve as a real number, rather than just modulo $2\pi$, requires a bit more care, and requires the curve $P$ to be embedded.)

Next consider a two-dimensional polyhedral surface $P^2$ and its tangent bundle. Around a vertex $p$ we can compose the cycle of isomorphisms $\phi_e$ across the edges incident to $p$. This gives a self-map $\phi_p : T_f \to T_f$. This is a rotation of the tangent plane by an angle which—it is easy to check—equals the discrete Gauss curvature $K_p$.

Now consider the general case of the tangent bundle to a polyhedral manifold $P^n \subset \mathbb{E}^d$. Suppose $p$ is a codimension-two face of $P$. Then composing the ring of isomorphisms across the ridges incident to $p$ gives an automorphism of $\mathbb{E}^n$ which is the local holonomy, or curvature of the Levi-Civita connection around $p$. We see that this is a rotation fixing the affine hull of $p$. To define this curvature (which can be interpreted as a sectional curvature in the two-plane normal to $p$) as a real number and not just modulo $2\pi$, we should look again at the angle defect around $p$, which is $2\pi - \sum \beta_i$ where the $\beta_i$ are the interior dihedral angles along $p$ of the facets $f$ incident to $p$.

In the case of a hypersurface $P^{d-1}$ in $\mathbb{E}^d$, the one-dimensional normal bundle is locally trivial: there is no curvature or local holonomy around any $p$. Globally, the normal bundle is of course trivial exactly when $P$ is orientable.

References

[Ban67] Thomas F. Banchoff, Critical points and curvature for embedded polyhedra, J. Differential Geometry 1 (1967), 245–256.

[Ban70] ———, Critical points and curvature for embedded polyhedral surfaces, Amer. Math. Monthly 77 (1970), 475–485.

[BH03] Philip L. Bowers and Monica K. Hurdal, Planar conformal mappings of piecewise flat surfaces, Visualization and Mathematics III (H.-C. Hege and K. Polthier, eds.), Springer, 2003, pp. 3–34.

[Bis75] Richard L. Bishop, There is more than one way to frame a curve, Amer. Math. Monthly 82 (1975), 246–251.

[BK82] Ulrich Brehm and Wolfgang Kühnel, Smooth approximation of polyhedral surfaces regarding curvatures, Geom. Dedicata 12:4 (1982), 435–461.

[BK97] Thomas F. Banchoff and Wolfgang Kühnel, Tight submanifolds, smooth and polyhedral, Tight and taut submanifolds (Berkeley, CA, 1994), Math. Sci. Res. Inst. Publ., vol. 32, Cambridge Univ. Press, Cambridge, 1997, pp. 51–118.
[Bob05] Alexander I. Bobenko, *A conformal energy for simplicial surfaces*, Combinatorial and computational geometry, Math. Sci. Res. Inst. Publ., vol. 52, Cambridge Univ. Press, Cambridge, 2005, pp. 135–145.

[Bob08] ———, *Surfaces from circles*, Discrete Differential Geometry (A. I. Bobenko, P. Schröder, J. M. Sullivan, G. M. Ziegler, eds.), Oberwolfach Seminars, vol. 38, Birkhäuser, 2008, this volume, to appear; [arXiv.org/0707.1318](https://arxiv.org/0707.1318).

[Bra78] Kenneth A. Brakke, *The motion of a surface by its mean curvature*, Mathematical Notes, vol. 20, Princeton University Press, Princeton, N.J., 1978.

[Bra92] ———, *The Surface Evolver*, Experiment. Math. 1:2 (1992), 141–165.

[Bra07] ———, *The Surface Evolver*, www.susqu.edu/brakke/evolver, online documentation, accessed September 2007.

[BS05] Alexander I. Bobenko and Boris A. Springborn, *A discrete Laplace–Beltrami operator for simplicial surfaces*, preprint, 2005; [arXiv:math.DG/0503219](https://arxiv.org/0503219).

[CSM03] David Cohen-Steiner and Jean-Marie Morvan, *Restricted Delaunay triangulations and normal cycle*, SoCG ’03: Proc. 19th Sympos. Comput. Geom., ACM Press, 2003, pp. 312–321.

[CSM06] ———, *Second fundamental measure of geometric sets and local approximation of curvatures*, J. Differential Geom. 74:3 (2006), 363–394.

[Eck04] Klaus Ecker, *Regularity theory for mean curvature flow*, Progress in Nonlinear Differential Equations and their Applications, 57, Birkhäuser Boston Inc., Boston, MA, 2004.

[Fed59] Herbert Federer, *Curvature measures*, Trans. Amer. Math. Soc. 93 (1959), 418–491.

[For03] Robin Forman, *Bochner’s method for cell complexes and combinatorial Ricci curvature*, Discrete Comput. Geom. 29:3 (2003), 323–374.

[FSK97] George Francis, John M. Sullivan, Robert B. Kusner, Kenneth A. Brakke, Chris Hartman, and Glenn Chappell, *The minimax sphere eversion*, Visualization and Mathematics (H.-C. Hege and K. Polthier, eds.), Springer, Heidelberg, 1997, pp. 3–20.

[Fu94] Joseph H. G. Fu, *Curvature measures of subanalytic sets*, Amer. J. Math. 116:4 (1994), 819–880.

[GH86] Michael E. Gage and Richard S. Hamilton, *The heat equation shrinking embedded plane curves*, J. Differential Geom. 23:1 (1986), 69–96.

[GKS03] Karsten Groveß-Brauckmann, Robert B. Kusner, and John M. Sullivan, *Triunduloids: Embedded constant mean curvature surfaces with three ends and genus zero*, J. reine angew. Math. 564 (2003), 35–61; [arXiv:math.DG/0102183](https://arxiv.org/0102183).

[Gra87] Matthew A. Grayson, *The heat equation shrinks embedded plane curves to round points*, J. Differential Geom. 26:2 (1987), 285–314.

[HCV32] David Hilbert and Stephan Cohn-Vossen, *Anschauliche Geometrie*, Springer, Berlin, 1932, second printing 1996.

[HKS92] Lucas Hsu, Robert B. Kusner, and John M. Sullivan, *Minimizing the squared mean curvature integral for surfaces in space forms*, Experimental Mathematics 1:3 (1992), 191–207.

[Hof08] Tim Hoffmann, *Discrete Hashimoto surfaces and a doubly discrete smoke-ring flow*, Discrete Differential Geometry (A. I. Bobenko, P. Schröder, J. M. Sullivan, G. M. Ziegler, eds.), Oberwolfach Seminars, vol. 38, Birkhäuser, 2008, this volume, to appear; [arXiv.org/math.DG/0007150](https://arxiv.org/0007150).
Curvatures of Smooth and Discrete Surfaces

[Hui84] Gerhard Huisken, *Flow by mean curvature of convex surfaces into spheres*, J. Differential Geom. 20:1 (1984), 237–266.

[IK⁺⁰⁸] Ivan Izmestiev, Robert B. Kusner, Günter Rote, Boris A. Springborn, and John M. Sullivan, *Torus triangulations . . .*, in preparation.

[Ilm94] Tom Ilmanen, *Elliptic regularization and partial regularity for motion by mean curvature*, Mem. Amer. Math. Soc. 108:520 (1994), x+90 pp.

[KKS89] Nicholas J. Korevaar, Robert B. Kusner, and Bruce Solomon, *The structure of complete embedded surfaces with constant mean curvature*, J. Differential Geom. 30:2 (1989), 465–503.

[KSS06] Liliya Kharevych, Boris A. Springborn, and Peter Schröder, *Discrete conformal maps via circle patterns*, ACM Trans. Graphics 25:2 (2006), 412–438.

[Kus91] Robert B. Kusner, *Bubbles, conservation laws, and balanced diagrams*, Geometric analysis and computer graphics, Math. Sci. Res. Inst. Publ., vol. 17, Springer, New York, 1991, pp. 103–108.

[LP94] Joel Langer and Ron Perline, *Local geometric invariants of integrable evolution equations*, J. Math. Phys. 35:4 (1994), 1732–1737.

[MD⁺⁰³] Mark Meyer, Mathieu Desbrun, Peter Schröder, and Alan H. Barr, *Discrete differential-geometry operators for triangulated 2-manifolds*, Visualization and Mathematics III (H.-C. Hege and K. Polthier, eds.), Springer, Berlin, 2003, pp. 35–57.

[PP93] Ulrich Pinkall and Konrad Polthier, *Computing discrete minimal surfaces and their conjugates*, Experiment. Math. 2:1 (1993), 15–36.

[PSW07] Ulrich Pinkall and Boris A. Springborn and Steffen Weissmann, *A new doubly discrete analogue of smoke ring flow and the real time simulation of fluid flow*, J. Phys. A, 2007, to appear; arXiv.org/0708.0979.

[Sch08] Peter Schröder, *What can we measure?*, Discrete Differential Geometry (A. I. Bobenko, P. Schröder, J. M. Sullivan, G. M. Ziegler, eds.), Oberwolfach Seminars, vol. 38, Birkhäuser, 2008, this volume, to appear.

[Spr06] Boris A. Springborn, *A variational principle for weighted Delaunay triangulations and hyperideal polyhedra*, preprint, 2006, arXiv.org/math.GT/0603097.

[Ste⁴⁰] Jakob Steiner, *Über parallele Flächen*, (Monats)Bericht Akad. Wiss. Berlin (1840), 114–118; Ges. Werke II, 2nd ed. (AMS/Chelsea, 1971), 171–176; Bibliothek, bbaw.de/Bbaw/Bibliothek-digital/digitalequellen/schriften/anzeige/index.html?band=08-verh/1840&seite=int=114.

[Ste⁰⁵] Kenneth Stephenson, *Introduction to circle packing*, Cambridge Univ. Press, Cambridge, 2005.

[Sul08] John M. Sullivan, *Curves of finite total curvature*, Discrete Differential Geometry (A. I. Bobenko, P. Schröder, J. M. Sullivan, G. M. Ziegler, eds.), Oberwolfach Seminars, vol. 38, Birkhäuser, 2008, this volume, to appear; arXiv.org/math/0606007.

[Thu⁹⁸] William P. Thurston, *Shapes of polyhedra and triangulations of the sphere*, The Epstein birthday schrift, Geom. Topol. Monogr., vol. 1, Geom. Topol. Publ., Coventry, 1998, pp. 511–549.

[Tro⁰⁷] Marc Troyanov, *On the moduli space of singular euclidean surfaces*, preprint, 2007, arXiv:math/0702666.
[vDA95] Ruud van Damme and Lyuba Alboul, *Tight triangulations*, Mathematical methods for curves and surfaces (Ulvik, 1994), Vanderbilt Univ. Press, Nashville, TN, 1995, pp. 517–526.

[War08] Max Wardetzky, *Convergence of the cotangent formula: An overview*, Discrete Differential Geometry (A. I. Bobenko, P. Schröder, J. M. Sullivan, G. M. Ziegler, eds.), Oberwolfach Seminars, vol. 38, Birkhäuser, 2008, this volume, to appear.

[Whi05] Brian White, *A local regularity theorem for mean curvature flow*, Ann. of Math. (2) 161:3 (2005), 1487–1519.

[Zie08] Günter M. Ziegler, *Polyhedral surfaces of high genus*, Discrete Differential Geometry (A. I. Bobenko, P. Schröder, J. M. Sullivan, G. M. Ziegler, eds.), Oberwolfach Seminars, vol. 38, Birkhäuser, 2008, this volume, to appear; arXiv.org/math/0412093.