Schrödinger-Cat-Likeness in Adiabatic Approximation for Generalized Quantum Rabi Model without and with $A^2$-Term

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Abstract

We give a mathematical procedure to obtain the adiabatic approximation for the generalized quantum Rabi Hamiltonian both without and with a quadratic interaction. We consider the Hamiltonian as the energy of a model describing the interaction system of a two-level artificial atom and a one-mode microwave photon in circuit QED. In the case without the quadratic interaction, we show in the adiabatic approximation that whether each bare eigenstate forms a Schrödinger-cat-like entangled state or not depends on whether the energy bias of the atom is zero or non-zero, and then, the effect of the tunnel splitting of the atom is ignored. On the other hand, in the case with the quadratic interaction, we show in the adiabatic approximation that all the physical eigenstates obtained by the (meson) pair theory form individual Schrödinger-cat-like entangled states for every energy bias. We conclude that this fact comes from the effect of the tunnel splitting.

1 Introduction

Quantum electrodynamics (QED) describes the interaction between light and matter. QED is a great success as a quantum field theory (QFT) for electrodynamics, which certifies that QFT is useful and excellent to explain the electromagnetic force caused by the exchanging of photons. The exchanged photon is called a virtual photon \[ \text{[1]} \]. QED enables us predict some quantities such as the Lamb shift, the difference in energy between the two energy levels of the two orbitals $2S_{1/2}$ and $2P_{1/2}$, of hydrogen atom with extreme accuracy \[ 2, 3, 4, 5, 6, 7 \]. The Lamb shift is caused by the fact that the different orbitals interact with the vacuum fluctuations of the radiation field. The vacuum fluctuation is originated from the annihilation and creation of virtual photons; therefore, the Lamb shift results from the fact that the atom is dressed with the cloud of virtual photons. Even the ground state is a non-zero photon state; however, the photons with which it is dressed are virtual and not directly observed (cf. Complement B\textsubscript{III}.2 of Ref.\[5\]). It is well known that the vacuum fluctuation due to Heisenberg’s uncertainty principle brings the generation of virtual particles from the quantum vacuum \[ 8 \]. The state dressed with the virtual photons is called the bare state. Some method have been considered to derive physical states, which are experimentally observable states, from the bare states \[ 5, 9 \]. After the success of QED, some physicists developed the analogy for QED, and applied it to nuclear models. They then had to meet and straggle troubles of the strong interaction. Following Yukawa’s theory \[ 10 \], nucleons are connected by a strong force, called nuclear force, and it is made by the fact that nucleons exchange $\pi$-mesons (i.e., pion). Namely, nucleon and $\pi$-meson respectively play individual roles of electron and photon in QED. In the early 1940s, (meson) pair theory were studied by Wentzel \[ 11, 12 \] to consider the nuclear forces under the strong coupling regime \[ 13 \]. On another note, according to quantum chromodynamics (QCD), quark and gluon in QCD respectively play roles of electron and photon in QED. Hadrons are classified into mesons and baryons consisting of...
quarks. Thus, the well-known problem that whether we can derive Yukawa’s theory for the nuclear force from QCD arises.

The recent technology of circuit QED can make a quantum simulation of cavity QED. Quantum simulation is to simulate a target quantum system by a controllable quantum system [14]. In particular, it enables us experimentally to demonstrate the amazingly strong interaction between a two-level artificial atom and a one-mode light on a superconducting circuit: Cavity QED has supplied us with stronger interaction than the standard QED does [15, 16]. Experimental physicists demonstrate the interaction using a two-level atom coupled with a one-mode light in a mirror cavity. The solid-state analogue of the strong interaction in a superconducting system was theoretically proposed [17, 18], and it has been experimentally demonstrated [19, 20, 21]. The atom, the light, and the mirror resonator in cavity QED are respectively replaced by an artificial atom, a microwave, and a microwave resonator on a superconducting circuit. The artificial atom is a superconducting LC circuit based on some Josephson junctions. This replaced cavity QED is the so-called circuit QED [22, 23]. The circuit QED has been intensifying the coupling strength so that its region is beyond the strong coupling regime [24, 25, 26, 27, 28, 29].

Yoshihara et al. succeeded in demonstrating the deep-strong coupling regime, and experimentally showed how the theory using the quantum Rabi model can well describe a physical set-up of circuit QED [29, 30, 31]. The set-up consists of a two-level artificial atom interacting to a one-mode photon of a microwave cavity. The notion of the deep-strong coupling regime is proposed in Ref. [32], and the strength of that regime is so large that it exceeds the strength of the ultra-strong coupling regime for the atom-photon interaction in circuit QED. Braak gives an analytics solution of the eigenvalue problem for the quantum Rabi model [33]. In Ref. [34], meanwhile, Ashhab and Nori give a physical establishment of the adiabatic approximation for the bare eigenstates of the quantum Rabi model. The adiabatically approximated eigenstates make the Schrödinger-cat-like entangled states. The adiabatic approximation is very handy to analyze the quantum Rabi model, and thus, the Schrödinger-cat-likeness is beginning to investigate [36] using it. In Ref. [37], we show a mathematical theory so that the adiabatic approximation is actually obtained under the strong-coupling limit in the norm resolvent sense.

We are interested in a quantum simulation of some phenomena predicted in nuclear physics on superconducting circuit. In particular, this paper deals with the (meson) pair theory for the generalized quantum Rabi Hamiltonian, which is also called asymmetric quantum Rabi Hamiltonian. It consists of the two-level atom Hamiltonian, the one-mode photon Hamiltonian, and the interaction between the atom and the photon. We give our attention to the non-zero energy bias in the atom Hamiltonian. In the case where the energy bias is equal to zero, the generalized quantum Rabi model is the quantum Rabi model. The energy-bias parameter is easily tunable in experiments of circuit QED with the cutting-edge technology. Thus, we treat it as a tunable parameter. We consider the bare (physical) eigenstates of the generalized quantum Rabi Hamiltonian without (with) the quadratic interaction. The quadratic interaction is often called the $A^2$-term. We then show how we can mathematically obtain the adiabatic approximation for the generalized quantum Rabi Hamiltonian both without and with the $A^2$-term. Based on this mathematical theory, in the case without the $A^2$-term, we show that whether the adiabatically approximated bare eigenstates are formed as the Schrödinger-cat-like entangled states or not depends on whether the energy bias is zero or non-zero. As its result, we point out that the effect of the tunnel splitting of the two-level atom disappears. On the other hand, in the case with the $A^2$-term, we renormalize it using (meson) pair theory [11, 12, 13], and show that all the adiabatically approximated physical eigenstates are formed as the Schrödinger-cat-like entangled states for every energy bias. We realize that this fact results from the effect of the tunnel splitting of the two-level atom.
2 Mathematical Set-Ups

In this section, we prepare and recall some mathematical notations and notions to explain and consider our problem. For their details, see Refs. \[38\], \[39\] for instance.

For a separable Hilbert space \( \mathcal{H} \) we denote its inner product by \( \langle \ , \rangle_{\mathcal{H}} \). The norm \( \| \cdot \|_{\mathcal{H}} \) is naturally introduced by \( \| \psi \|_{\mathcal{H}} = \sqrt{\langle \psi , \psi \rangle_{\mathcal{H}}} \) for every vector \( \psi \) in the Hilbert space \( \mathcal{H} \). An operator \( A \) acting in the Hilbert space is the linear map from a linear subspace \( D(A) \subset \mathcal{H} \) to the Hilbert space \( \mathcal{H} \). The subspace \( D(A) \) is called the domain of the operator \( A \). In particular, when the operator \( A \) satisfies that there is a positive constant \( M \) so that the inequality, \( \| A \psi \|_{\mathcal{H}} \leq M \| \psi \|_{\mathcal{H}} \), can hold for any vector \( \psi \in \mathcal{H} \), we say the operator \( A \) is bounded. Then, for every bounded operator \( A \), the operator norm is given by \( \| A \|_{\text{op}} := \sup_{\psi \neq 0} \| A \psi \|_{\mathcal{H}} / \| \psi \|_{\mathcal{H}} \). The inequality, \( \| A \psi \|_{\mathcal{H}} \leq \| A \|_{\text{op}} \| \psi \|_{\mathcal{H}} \), holds then. On the other hand, in the case \( \| A \|_{\text{op}} = \infty \), we say the operator \( A \) is unbounded. In quantum theory, an observable \( A \) corresponds to a self-adjoint operator, that is, it satisfies the domain identity, \( D(A) = D(A^*) \), and the action identity, \( A \psi = A^* \psi \) for every vector \( \psi \in D(A) \), where \( A^* \) is the adjoint operator of the operator \( A \). It is convenient to consider the resolvent \( (A - iz)^{-1} \) for an unbounded self-adjoint operator \( A \) for every complex number \( z \) with \( \Im z \neq 0 \). Let \( A_n \) be a sequence of self-adjoint operators. When there is a self-adjoint operator \( A \) so that the limit, \( \lim_{n \to \infty} \| (A_n - iz)^{-1} - (A - iz)^{-1} \|_{\text{op}} = 0 \), holds for every complex number \( z \) with \( \Im z \neq 0 \), the operators \( A_n \) are said to converge to the operator \( A \) in the norm resolvent sense [38], and we often denote the convergence by \( A_n \xrightarrow{\text{n.r.s.}} A \) as \( n \to \infty \) in this paper.

We sometimes represent by \( |E \rangle \) a vector in the state space \( \mathcal{H} \). We often use Dirac’s bra-ket notation \( \langle E_1 | E_2 \rangle \) for the inner product \( |\langle E_2 , E_1 \rangle_{\mathcal{H}} | \) of vectors \( E_1 \) and \( E_2 \), i.e., \( \langle E_1 | E_2 \rangle := \langle E_2 | E_1 \rangle_{\mathcal{H}} \). So, the notation \( \langle E_1 | A | E_2 \rangle \) stands for the inner product \( \langle E_2 , A | E_1 \rangle_{\mathcal{H}} \) for vectors \( E_1 \) and \( E_2 \) and an operator \( A \), i.e., \( \langle E_1 | A | E_2 \rangle := \langle E_2 | A | E_1 \rangle_{\mathcal{H}} \).

Let \( A_j \) be an operator acting in Hilbert spaces \( \mathcal{H}_j, j = 1,2 \). We often omit the tensor sign \( \otimes \) from the tensor product \( A_1 \otimes A_2 \) and denote the tensor product \( A_1 \otimes A_2 \) by \( A_1 A_2 \), i.e., \( A_1 A_2 = A_1 \otimes A_2 \). We simply write \( A_1 \otimes I_{\mathcal{H}_2} \) as \( A_1 \) for the identity operator \( I_{\mathcal{H}_2} \), and \( I_{\mathcal{H}_1} \otimes A_2 \) as \( A_2 \) for the identity matrix \( I_{\mathcal{H}_1} \). Correspondingly, we also omit the tensor symbol \( \otimes \) form the tensor product of vectors in \( \mathcal{H}_1 \otimes \mathcal{H}_2 \).

The state space of the two-level atom system coupled with one-mode light is given by \( \mathbb{C}^2 \otimes L^2(\mathbb{R}) \), where \( \mathbb{C}^2 \) is the 2-dimensional unitary space, and \( L^2(\mathbb{R}) \) the Hilbert space consisting of the square-integrable functions. We sometimes omit the tensor sign \( \otimes \) from vectors in the state space \( \mathbb{C}^2 \otimes L^2(\mathbb{R}) \). We use the notation \( |\uparrow\rangle \) for the up-spin state and the notation \( |\downarrow\rangle \) for the down-spin state, which are defined by \( |\uparrow\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( |\downarrow\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) in \( \mathbb{C}^2 \). We use standard notations for the Pauli matrices, \( \sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \text{and} \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \). We denote by \( |n\rangle \) the Fock state in \( L^2(\mathbb{R}) \) with the photon number \( n = 0,1,2, \cdots \). That is, \( |0\rangle := (w^2/\pi)^{1/4} \exp[-(wx)^2/2] \) and \( |n\rangle := \sqrt{w \gamma_n} H_n(wx) \exp[-(wx)^2/2] \in L^2(\mathbb{R}) \), where \( H_n(x) \) is the Hermite polynomial of variable \( x \), \( \gamma_n = \pi^{1/4}(2^n n!)^{-1/2} \), and \( w = \sqrt{m \omega/\hbar} \) for the frequency \( \omega \) of a one-mode photon. We omit the tensor sign, \( \otimes \), from the tensor product, \( |s\rangle \otimes |n\rangle \) with \( s = \uparrow, \downarrow, n = 0,1,2, \cdots, \) and use a compact notation, \( |s|n\rangle \), for the tensor product. We respectively denote by \( a \) and \( a^\dagger \) the annihilation and creation operators of one-mode photon defined by \( a|0\rangle := 0, a|n\rangle := \sqrt{n}|n-1\rangle \), and \( a^\dagger |n\rangle := \sqrt{n+1}|n+1\rangle \). The spin-annihilation operator \( \sigma_- \) and the spin-creation operator \( \sigma_+ \) are defined by \( \sigma_{\pm} := \frac{1}{2}(\sigma_x \pm i\sigma_y) \). The identity \( 2 \times 2 \) matrix \( \sigma_0 \) is given by \( \sigma_0 = \sigma_+ \sigma_- + \sigma_- \sigma_+ \).
3 Some Reviews and Our Problem

We introduce the parameters $\omega$ and $g$, respectively, playing roles of a frequency of a one-mode photon in a cavity and a coupling strength between an artificial two-level atom and the photon in the cavity. For every frequency $\omega$ and coupling strength $g$, the Hamiltonian of the generalized quantum Rabi model reads

$$H_{\text{GQR}}(\omega, g) := H_{\text{atm}}(\varepsilon) + H_{\text{ptn}}(\omega) + \hbar g \sigma_x (a + a^\dagger)$$

with the two-level atom Hamiltonian $H_{\text{atm}}(\varepsilon)$ and the one-mode photon Hamiltonian $H_{\text{ptn}}(\omega)$ defined by

$$H_{\text{atm}}(\varepsilon) := \frac{\hbar}{2} (\omega a \sigma_x - \varepsilon \sigma_z) \quad \text{and} \quad H_{\text{ptn}}(\omega) := \hbar \omega \left( a^\dagger a + \frac{1}{2} \right),$$

where $\hbar \omega_a$ and $\hbar \varepsilon$ are respectively the tunnel splitting and energy bias between the states $|\uparrow\rangle$ and $|\downarrow\rangle$, of the two-level atom.

We recall the expression of the photon annihilation operator $a$ and creation operator $a^\dagger$ using the position operator $x$ and momentum operator $p$:

$$a = \sqrt{\frac{\omega}{2\hbar}} x + i \sqrt{\frac{1}{2\hbar \omega}} p \quad \text{and} \quad a^\dagger = \sqrt{\frac{\omega}{2\hbar}} x - i \sqrt{\frac{1}{2\hbar \omega}} p.$$

Then, we have another expression of the photon Hamiltonian $H_{\text{ptn}}(\omega)$ as

$$H_{\text{ptn}}(\omega) = \frac{1}{2} p^2 + \frac{\omega^2}{2} x^2$$

using the canonical commutation relation $[x, p] = i\hbar$.

In the case where the energy bias is zero (i.e., $\varepsilon = 0$), the generalized quantum Rabi Hamiltonian $H_{\text{GQR}}(\omega, g)$ becomes the quantum Rabi Hamiltonian. We denote it by $H_{\text{QR}}(\omega, g)$. The quantum Rabi Hamiltonian $H_{\text{QR}}(\omega, g)$ has the parity symmetry,

$$[H_{\text{QR}}(\omega, g), \Pi] = 0,$$

for the parity operator $\Pi = (-1)^{a^\dagger a} \sigma_z$.

To introduce the form of the generalized quantum Rabi Hamiltonian $H_{\text{GQR}}(\omega_c, g)$ that we consider in this paper, we define a unitary matrix $U_{xx}$ by

$$U_{xx} := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

The Hamiltonian $H_{\text{GQR}}(\omega, g)$ is given by

$$H_{\text{GQR}}(\omega, g) := U_{xx} H_{\text{GQR}}(\omega_c, g) U_{xx}^*$$

$$= H_{\text{atm}}(\varepsilon) + H_{\text{ptn}}(\omega) + \hbar g \sigma_z (a + a^\dagger)$$

with the atom Hamiltonian $H_{\text{atm}}(\varepsilon)$ given by

$$H_{\text{atm}}(\varepsilon) := -\frac{\hbar}{2} (\omega a \sigma_x - \varepsilon \sigma_z).$$

In this paper, we employ the one-mode photon frequency $\omega_c$ and the coupling strength $g$ as parameters $\omega$ and $g$, respectively, and we also call $H_{\text{GQR}}(\omega_c, g) := U_{xx} H_{\text{GQR}}(\omega_c, g) U_{xx}^*$ the generalized quantum Rabi Hamiltonian, and $H_{\text{QR}}(\omega_c, g) := U_{xx} H_{\text{QR}}(\omega_c, g) U_{xx}^*$ the quantum Rabi Hamiltonian. Using Eq. (3.4), we have the parity symmetry in the case $\varepsilon = 0$,

$$0 = U_{xx} [H_{\text{QR}}(\omega_c, g), \Pi] U_{xx} = [H_{\text{QR}}(\omega_c, g), \mathcal{P}],$$

where $\mathcal{P}$ is a parity operator.
where $P = -(-1)^a \sigma_x$. This determines the form of the eigenstates of the quantum Rabi Hamiltonian $\mathcal{H}_{QR}(\omega_c, g)$ as

\[
|\uparrow\rangle \left( |\text{even}\rangle + |\text{odd}\rangle \right) \pm |\downarrow\rangle \left( |\text{even}\rangle - |\text{odd}\rangle \right)
\]

(3.8)

for proper states $|\text{even}\rangle$ and $|\text{odd}\rangle$ with the individual forms,

\[
|\text{even}\rangle = \sum_{n: \text{even}} c_n^{\text{even}} |n\rangle \quad \text{and} \quad |\text{odd}\rangle = \sum_{n: \text{odd}} c_n^{\text{odd}} |n\rangle.
\]

Physically based on the argument in Ref.\[34\], the Schrödinger-cat-likeness appears in the adiabatic approximation for the quantum Rabi Hamiltonian $\mathcal{H}_{QR}(\omega_c, g)$ (i.e., $\epsilon = 0$): all the eigenstates of the quantum Rabi Hamiltonian can be approximated by the Schrödinger-cat-like states,

\[
\frac{1}{\sqrt{2}} \left( |\uparrow\rangle D(-g/\omega_c) |n\rangle + |\downarrow\rangle D(g/\omega_c) |n\rangle \right),
\]

\[
\frac{1}{\sqrt{2}} \left( |\uparrow\rangle D(-g/\omega_c) |n\rangle - |\downarrow\rangle D(g/\omega_c) |n\rangle \right),
\]

(3.9)

in the deep-strong coupling regime. Here, $D(g/\omega_c)$ is the displacement operator defined by $D(g/\omega_c) := \exp \left( g (a^\dagger - a)/\omega_c \right)$. Eqs.\[3.9\] are well known as the adiabatic approximation (e.g., see Eq.(5) of Ref.\[33\]). The eigenenergies of the both eigenstates in Eq.\[3.9\] are almost $h\omega_c(n + 1/2) - hg^2/\omega_c$; but, every true eigenstates are non-degenerate besides some cases. For instance, the adiabatically approximated eigenstates with the lowest energy $h\omega_c/2 - hg^2/\omega_c$ apparently seem to be degenerate; however, the ground state of the quantum Rabi Hamiltonian actually is unique for every coupling strength $g$ \[40\].

As shown in Ref.\[37\] \[41\], the adiabatic approximation given by Eq.\[3.9\] is mathematically justified in the following: We define the unitary operator $U(g/\omega_c)$ by $U(g/\omega_c) := \sigma_+ \sigma_- D(g/\omega_c) + \sigma_- \sigma_+ D(-g/\omega_c)$. We note the equation, $U(g/\omega_c)^* = U(-g/\omega_c)$. Then, we obtain the unitary transformation,

\[
U(g/\omega_c) \left( \mathcal{H}_{QR}(\omega_c, g) + hg^2/\omega_c \right) U(g/\omega_c)^* = H_{\text{ptn}}(\omega_c) - \frac{h}{2} \varepsilon \sigma_z - \frac{h}{2} \omega c \left\{ \sigma_+ D(g/\omega_c)^2 + \sigma_- D(-g/\omega_c)^2 \right\},
\]

(3.10)

where $-hg^2/\omega_c$ is the self-energy. Taking the strong coupling limit $g \to \infty$, we have the limits, $\lim_{g \to \infty} hg^2/\omega_c = \infty$ and $\lim_{g \to \infty} g/\omega_c = \infty$. Thus, the energy $hg^2/\omega_c$ plays a role of a counter term for mass renormalization (i.e., for the bare-photon divergence) in the strong coupling limit. The displacement operators $D(\pm g/\omega_c)$ decay to the zero operator in a mathematically proper sense \[37\] \[41\] in the strong-coupling limit. Developing this fact and using Theorem VIII.19(a) of Ref.\[33\], we can prove that the unitarily transformed Hamiltonian $U(g/\omega_c) \left( \mathcal{H}_{QR}(\omega_c, g) + hg^2/\omega_c \right) U(g/\omega_c)^*$ converges to the Hamiltonian $H_{\text{ptn}}(\omega_c) - h\varepsilon \sigma_z/2$ in the norm resolvent sense:

\[
U(g/\omega_c) \mathcal{H}_{QR}(\omega_c, g) + hg^2/\omega_c \to^* \text{ n.r.s. } H_{\text{ptn}}(\omega_c) - \frac{h}{2} \varepsilon \sigma_z \quad \text{as} \quad g \to \infty.
\]

Thanks to Theorem VIII.23(b) of Ref.\[33\], each eigenstate of the Hamiltonian $\mathcal{H}_{QR}(\omega_c, g)$ is well approximated by that of the Hamiltonian $U(g/\omega_c)^* \left( H_{\text{ptn}}(\omega_c) - h\varepsilon \sigma_z /2 \right) U(g/\omega_c) - hg^2/\omega_c$. This mathematical procedure with $\varepsilon = 0$ secures the adiabatic-approximation formulas given
by Eq. (3.9). On the other hand, in the case where $\varepsilon \neq 0$, all the eigenstates of the generalized quantum Rabi Hamiltonian $H_{GQR}(\omega_c, g)$ is well approximated by the states,

$$\begin{align*}
\forall \ket{\pm} & \in \{\ket{f}, \ket{g}\}, \ket{\pm} D \left( -g/\omega_c \right) \ket{n}, \\
\ket{\mp} & \in \{\ket{f}, \ket{g}\}, \ket{\mp} D \left( g/\omega_c \right) \ket{n}.
\end{align*}$$

(3.11)

The first adiabatically approximated eigenstates $\ket{\pm} D \left( -g/\omega_c \right) \ket{n}$ gives the eigenenergy $h\omega_c (n + 1/2) - \hbar \varepsilon /2 - \hbar g^2/\omega_c$, and the second one gives the eigenenergy $h\omega_c (n + 1/2) + \hbar \varepsilon /2 - \hbar g^2/\omega_c$. Following the adiabatic-approximation formulas (3.11), whether the energy bias is positive or negative causes an energy level crossing. At last, we realized that i) the limit Hamiltonian, $U(\omega_c)(H_{\text{ptn}}(\omega_c) - \hbar \varepsilon \sigma_z/2)U(\omega_c) - \hbar g^2/\omega_c$, as well as its eigenstates and eigenenergies does not include the tunnel splitting $\hbar \varepsilon_a$ of two-level atom, but the energy bias $\hbar \varepsilon$; ii) the eigenstates in Eq.(3.9)

are the Schrödinger-cat-like, but the eigenstates in Eq.(3.11) are not.

Here, we make a remark on a physical role of the displacement operator $D(\pm g/\omega_c)$ to introduce our problem. The appearance of the displacement operator in Eqs.(3.9) and (3.11) makes coherent states. However, they are for bare photons; and in fact, the photon-field fluctuation $\Delta \Phi$ increases the ground-state expectation $N^G_{\omega_c}(\omega_c, g)$ of the number of photons. More precisely, Eqs.(3.9) and (3.11) say that the ground-state expectation $N^G_{\omega_c}(\omega_c, g) = \langle E_0^{GQR} \vert a^+ a \vert E_0^{GQR} \rangle$ increases as the coupling strength $g$ grows larger, i.e., $N^G_{\omega_c}(\omega_c, g) \sim g^2/\omega_c^2$ as $g \to \infty$, where $\vert E_0^{GQR} \rangle$ is the ground state of the generalized quantum Rabi Hamiltonian $H_{GQR}(\omega_c, g)$. This increase results from the mathematical establishment of the adiabatic approximation. Actually, it is pushed up by the fluctuation $\Delta \Phi$ of the photon field $\Phi := (a + a^+)/\sqrt{2\omega_c}$ in the ground state since the inequality,

$$\langle \Delta \Phi \rangle^2 \leq \frac{2N^G_{\omega_c}(\omega_c, g) + 1}{\omega_c},$$

is obtained in the same way as in Appendix B of Ref. [37]. The mathematical establishment of the adiabatic approximation also says that $(\Delta \Phi)^2 \sim (1 + 4g^2/\omega_c^2)/2\omega_c$ as $g \to \infty$ for $\varepsilon = 0$; $(\Delta \Phi)^2 \sim 1/2\omega_c$ as $g \to \infty$ for $\varepsilon \neq 0$. Therefore, the Schrödinger-cat-likeness is caused by bare photons, and it is not observable directly.

We now try to derive physical states from the adiabatically approximated bare states given in Eqs.(3.9) and (3.11). The Hamiltonians $H_{\pm \text{vH}}$ of the van Hove model for the neutral scalar field theory with a fixed sources [42] are given by $H_{\pm \text{vH}} := H_{\text{ptn}}(\omega_c) \pm \hbar g(a + a^+)$. We denote by $\ket{n_{\pm \text{vH}}}$ the eigenstate of the van Hove Hamiltonians $H_{\pm \text{vH}}$. Since each eigenstate is given by $\ket{n_{\pm \text{vH}}} = D(\mp g/\omega_c)\ket{n}$, the ground-state expectation is calculated as $\langle 0_{\pm \text{vH}} \vert a^+ a \vert 0_{\pm \text{vH}} \rangle = g^2/\omega_c^2$. It increases in association with the growth of the coupling strength $g$ as it looks as it appears to be. However, we find unitary operators $U_{\pm \text{vH}}$ to derive physical states from bare states for the van Hove Hamiltonians $H_{\pm \text{vH}}$, and then, we have the renormalized van Hove Hamiltonian given by $U^R_{\pm \text{vH}}(H_{\pm \text{vH}} + \hbar g^2/\omega_c)U^R_{\pm \text{vH}}$. Here, the energy $-\hbar g^2/\omega_c$ is the self-energy of the van Hove Hamiltonian, and we have to make the so-called mass renormalization [9]. In addition to this, Ref.[9] tells us that the unitary operators are given by $U_{\pm \text{vH}} = D(\mp g/\omega_c)$, and each physical state $\ket{n_{\pm \text{vH}}}$ of the bare state $\ket{n_{\pm \text{vH}}}$ is given by $\ket{n^R_{\pm \text{vH}}} = U^R_{\pm \text{vH}}\ket{n_{\pm \text{vH}}}$. Eventually, the physical state $\ket{n^R_{\pm \text{vH}}}$ gets itself satisfying $\ket{n^R_{\pm \text{vH}}} = \ket{n}$. The photon in the physical state $\ket{n^R_{\pm \text{vH}}}$ is the so-called dressed photon, which sometimes called real photon. Thus, we can expect no dressed photon in the physical ground state $\ket{0^R_{\pm \text{vH}}}$, i.e., $\langle 0^R_{\pm \text{vH}} \vert a^+ a \vert 0^R_{\pm \text{vH}} \rangle = 0$. Therefore, $\langle 0_{\pm \text{vH}} \vert a^+ a \vert 0_{\pm \text{vH}} \rangle = g^2/\omega_c^2$ is the expectation value of the number of the bare photons including virtual photons in the bare ground state.

Following the argument in Complement BIII.2 of Ref.[5], we can think that the photons in the ground state are virtual photons since the ground state expectation $N^G_{\omega_c}(\omega_c, g)$ increases as the coupling strength $g$ grows larger: $N^G_{\omega_c}(\omega_c, g) \sim g^2/\omega_c^2$ as $g \to \infty$. Using the representation
in Ref. 32, we define the annihilation operator \( \alpha \) and creation operator \( \alpha^\dagger \) by \( \alpha^\dagger := \sigma_z \alpha \). Then, we have the matrix-valued CCR, \([\alpha, \alpha^\dagger] = 1\), and the expression,

\[
\mathcal{H}_{\text{GQR}}(\omega_c, g) = \hbar \omega_c \left( \alpha^\dagger \alpha + \frac{1}{2} \right) + h g (\alpha + \alpha^\dagger) + \mathcal{H}_{\text{atm}}(\varepsilon).
\]

In the sufficiently strong coupling regime, the atom Hamiltonian \( \mathcal{H}_{\text{atm}}(\varepsilon) \) can be regarded as the perturbation of the van Hove Hamiltonian \( \hbar \omega_c (\alpha^\dagger \alpha + 1/2) + h g (\alpha + \alpha^\dagger) \). This is the very idea of the adiabatic approximation. Based on this fact, in a similar way to the van Hove model’s case, we define the unitary operator \( U_{\text{GQR}} \) for deriving physical states from bare states by \( U_{\text{GQR}} := \sigma_+ \sigma_- U_{\text{vH}} + \sigma_- \sigma_+ U_{\text{vH}} \). Then, for each bare state \( \psi \) of the generalized quantum Rabi Hamiltonian \( \mathcal{H}_{\text{GQR}}(\omega_c, g) \), we have the physical eigenstates \( \psi_{\text{ren}} \) by \( \psi_{\text{ren}} = U_{\text{GQR}}^* \psi \).

Since the renormalized Hamiltonian \( \mathcal{H}_{\text{GQR}}^\text{ren}(\omega_c, g) \) of the generalized quantum Rabi Hamiltonian \( \mathcal{H}_{\text{GQR}}(\omega_c, g) \) is given by \( \mathcal{H}_{\text{GQR}}^\text{ren}(\omega_c, g) = U_{\text{GQR}}^* (\mathcal{H}_{\text{GQR}}(\omega_c, g) + h g^2 / \omega_c) U_{\text{GQR}} \) and \( U_{\text{GQR}} = U(g/\omega_c) \), we have

\[
\mathcal{H}_{\text{GQR}}^\text{ren}(\omega_c, g) = \mathcal{H}_{\text{ptn}}(\omega_c) - \frac{\hbar}{2} \varepsilon \sigma_z - \frac{\hbar}{2} \omega_a \{ \sigma_+ U_{\text{vH}}^* U_{\text{vH}} + \sigma_- U_{\text{vH}}^* U_{\text{vH}} \}.
\]  

We immediately realize that RHS of Eq. (3.12) is RHS of Eq. (3.10). In the case \( \varepsilon = 0 \), using Eq. (3.9), all the normalized eigenstates \( \psi \) of the quantum Rabi Hamiltonian \( \mathcal{H}_{\text{QR}}(\omega_c, g) \) are approximated by \( |\uparrow\rangle |n_{\text{vH}}\rangle + |\downarrow\rangle |n_{\text{vH}}\rangle / \sqrt{2} \) or \( (|\uparrow\rangle |n_{\text{vH}}\rangle - |\downarrow\rangle |n_{\text{vH}}\rangle) / \sqrt{2}, n = 0, 1, 2, \ldots \). Thus, the physical eigenstates \( \psi_{\text{ren}} \) (i.e., eigenstates of the renormalized Hamiltonian \( \mathcal{H}_{\text{GQR}}^\text{ren}(\omega_c, g) \) are approximated by

\[
\frac{1}{\sqrt{2}} \left( |\uparrow\rangle |n_{\text{ren}}^\text{vH}\rangle + |\downarrow\rangle |n_{\text{ren}}^\text{vH}\rangle \right) = \frac{1}{\sqrt{2}} \left( |\uparrow\rangle + |\downarrow\rangle \right) |n\rangle,
\]

\[
\frac{1}{\sqrt{2}} \left( |\uparrow\rangle |n_{\text{ren}}^\text{vH}\rangle - |\downarrow\rangle |n_{\text{ren}}^\text{vH}\rangle \right) = \frac{1}{\sqrt{2}} \left( |\uparrow\rangle - |\downarrow\rangle \right) |n\rangle.
\]

In the same way, in the case \( \varepsilon \neq 0 \), using Eq. (3.11), we have the adiabatically approximated physical eigenstates

\[
|\uparrow\rangle |n_{\text{ren}}^\text{vH}\rangle = |\uparrow\rangle |n\rangle,
\]

\[
|\downarrow\rangle |n_{\text{ren}}^\text{vH}\rangle = |\downarrow\rangle |n\rangle,
\]

for sufficiently large coupling strength. Thus, the adiabatically approximated physical states in Eqs. (3.13) and (3.14) can no longer make any coherent state of dressed photons, and are no longer macroscopic. That is, Schrödinger-cat-likeness does not appear in those physical eigenstates.

As observed above, if we employ the unitary operator \( U_{\text{GQR}} \) to derive physical states from bare states for the (generalized) quantum Rabi model, physical eigenstates are approximated by eigenstates of the free part of the (generalized) quantum Rabi Hamiltonian (i.e., \( g = 0 \)). Moreover, the adiabatic approximations given by Eqs. (3.13) and (3.14) tell us that the (generalized) quantum Rabi Hamiltonian cannot make us expect any dressed photon in the physical ground state even for sufficiently large coupling strength. Here, we point out the following properties:

P1) The derivations of Eqs. (3.9) and (3.11) tell us that whether the adiabatically approximated eigenstates of the generalized quantum Rabi Hamiltonian are formed as the Schrödinger-cat-like entangled states or not depends on whether the energy-bias parameter is zero or non-zero, in other words, whether the parity symmetry given by Eq. (3.7) is conserved or not.
P2) The limit Hamiltonian in the norm resolvent sense says that the effect of the tunnel splitting of the two-level atom can be ignored in the adiabatic approximations.

P3) Following the theory of van Hove model, the Schrödinger-cat-likeness disappears from the adiabatically approximated physical eigenstates.

We are interested in a vestige of the Schrödinger-cat-likeness of bare photons in the physical eigenstates.

Meanwhile, we have to consider the quadratic interaction (i.e., $A^2$-term) of the photon field in the case where our physical system of the two-level atom interacting with the one-mode photon field is in the very strong coupling regime such as the deep-strong coupling regime. For such a situation, we should consider the Hamiltonian

$$ H_{A^2}(\varepsilon) := H_{GQR}(\omega_c, g) + \hbar g C_g \left( a + a^\dagger \right)^2, \quad (3.15) $$

where $C_g$ is a positive function of the coupling strength $g$ satisfying $\lim_{g \to \infty} g C_g = \infty$. We assume the following conditions:

$$ \lim_{g \to \infty} g^{-1/3} C_g = \infty, \quad (3.16) $$

and there is a non-negative constant $C_\infty$ so that

$$ \lim_{g \to \infty} g^{-1} C_g = C_\infty. \quad (3.17) $$

For instance, if we estimate $C_g$ at $(\text{const}) \times g$, then Eqs. (3.16) and (3.17) hold.

In this paper, we investigate an effect caused by the $A^2$-term in the physical eigenstates of our total Hamiltonian $H_{A^2}(\varepsilon)$. To obtain the physical states, we employ the (meson) pair theory in nuclear physics \[9, 11, 12, 13, 43\]. Then, we have a unitary operator $U_{A^2}$ such that the unitarily transformed Hamiltonian $U_{A^2}^* H_{A^2}(\varepsilon) U_{A^2}$ becomes the renormalized Hamiltonian for the physical eigenstates. Following (meson) pair theory, we obtain the unitary operator $U_{A^2}$ given by the Hopfield-Bogoliubov transformation $U_{HB}$, i.e., $U_{A^2} = U_{HB}$, as shown in Ref. [37], so that we obtain the renormalized Hamiltonian $H_{A^2}^{\text{ren}}(\varepsilon)$ as

$$ H_{A^2}^{\text{ren}}(\varepsilon) := U_{A^2}^* H_{A^2}(\varepsilon) U_{A^2} = H_{\text{atm}}(\varepsilon) + H_{\text{ptn}}(\omega_g) + \hbar \tilde{\omega} \sigma_z \left( a + a^\dagger \right) = H_{GQR}(\omega_{\tilde{g}}, \tilde{g}), \quad (3.18) $$

where $\omega_g$ and $\tilde{g}$ are respectively renormalized photon frequency and the renormalized coupling strength given by

$$ \omega_g = \frac{\sqrt{\omega_c^2 + 4 \omega_c g C_g}}{\omega_c g} \quad \text{and} \quad \tilde{g} = g \sqrt{\frac{\omega_g}{\omega_c}}. $$

We briefly review how to obtain Eq. (3.18) in the next section.

Similarly to the argument for the generalized quantum Rabi model, we introduce the unitary operator $U(\tilde{g}/\omega_g)$ by

$$ U(\tilde{g}/\omega_g) := \sigma_+ \sigma_- D(\tilde{g}/\omega_g) + \sigma_- \sigma_+ D(-\tilde{g}/\omega_g), \quad (3.19) $$

where $D(\tilde{g}/\omega_g)$ is the displacement operator defined by $D(\tilde{g}/\omega_g) := \exp \left[ \tilde{g} \left( a^\dagger - a \right) / \omega_g \right]$. Then, we have the unitary transformation,

$$ U \left( \tilde{g}/\omega_g \right) \left( H_{A^2}^{\text{ren}}(\varepsilon) + \hbar \tilde{\omega}^2 / \omega_g \right) U \left( \tilde{g}/\omega_g \right)^* = H_{\text{ptn}}(\omega_g) - \frac{\hbar}{2} \varepsilon \sigma_z - \frac{\hbar}{2} \omega_a \left\{ \sigma_+ D(\tilde{g}/\omega_g)^2 + \sigma_- D(-\tilde{g}/\omega_g)^2 \right\}. \quad (3.20) $$
For Eq. (3.20), we realize the followings: Since the limit
\[
\lim_{g \to \infty} \frac{\tilde{g}^2}{\omega_g} = \frac{\hbar}{4C_\infty}
\]  
(3.21)
follows from Eq. (3.17), the self-energy \( \hbar \tilde{g}^2 / \omega_g \) does not work as a counter term when we take the strong coupling limit \( g \to \infty \). Instead, we meet a trouble of divergence for the Hamiltonian \( H_{\text{ptn}}(\omega_g) \) due to the divergence \( \lim_{g \to \infty} \omega_g = \infty \). In addition to this trouble, the third term of RHS of Eq. (3.20) does not vanish as the coupling strength \( g \) tends to the infinity because the displacement operators \( D(\tilde{g}/\omega_g) \) and \( D(\tilde{g}/\omega_g)^* \) do not decay to zero because of the limit,
\[
\lim_{g \to \infty} \frac{\tilde{g}}{\omega_g} = \lim_{g \to \infty} \omega_g^{-1/4} \left( \omega_c g^{-4/3} + 4g^{-1/3} C_g \right)^{-3/4} = 0,
\]
by Eq. (3.16).

In this paper, coping with the trouble and difference, we will consider the properties corresponding to \( P_1 \), \( P_2 \), and \( P_3 \) for our total Hamiltonian \( \mathcal{H}_{\text{ren}}(\varepsilon) \).

### 4 From Bare Eigenstates to Physical Eigenstates

Following (meson) pair theory \cite{9 11 12 13 43}, we obtain physical eigenstates from bare ones. We review it in brief. For more details on how to apply (meson) pair theory to our model, see the argument in Ref. \cite{37}. Our Hamiltonian \( \mathcal{H}_{A^2}(\varepsilon) \) has the matrix representation as
\[
\mathcal{H}_{A^2}(\varepsilon) = \begin{pmatrix}
H_{A^2}^+ - \hbar \varepsilon / 2 & -\hbar \omega_1 / 2 \\
-\hbar \omega_1 / 2 & H_{A^2}^- + \hbar \varepsilon / 2
\end{pmatrix},
\]
where
\[
H_{A^2}^\pm = H_{\text{ptn}}(\omega_c) \pm \hbar g \left( a + a^\dagger \right) + \hbar C_g(\varepsilon) \left( a + a^\dagger \right)^2.
\]
Using Eqs. (3.2) and (3.3), we can rewrite the Hamiltonians \( H_{A^2}^\pm \) as
\[
H_{A^2}^\pm = \frac{1}{2} p^2 + \frac{\omega_c^2}{2} x^2 \pm \hbar g \sqrt{\frac{2\omega_c}{\hbar}} x + 2C_g g \omega_c x^2 = \frac{1}{2} p^2 + \frac{\omega_c^2}{2} x^2 \pm \hbar g \sqrt{\frac{2\omega_c}{\hbar}} x.
\]
We define new photon’s annihilation operator \( b \) and creation operator \( b^\dagger \) by
\[
b := \sqrt{\frac{\omega_g}{2\hbar}} x + i \sqrt{\frac{1}{2\hbar \omega_g}} p \quad \text{and} \quad b^\dagger := \sqrt{\frac{\omega_g}{2\hbar}} x - i \sqrt{\frac{1}{2\hbar \omega_g}} p.
\]
(4.1)

Then, we have expression of the Hamiltonians \( H_{A^2}^\pm \) using the new annihilation and creation operators, \( b \) and \( b^\dagger \), as
\[
H_{A^2}^\pm = \hbar \omega_g \left( b^\dagger b + \frac{1}{2} \right) \pm \hbar \tilde{g} \left( b + b^\dagger \right).
\]
(4.2)

Making the correspondence between the normalized eigenstates of the Hamiltonian \( (1/2)p^2 + (\omega_c^2/2)x^2 \) to those of the Hamiltonian \( (1/2)p^2 + (\omega_c^2/2)x^2 \), we eventually obtain the so-called Hopfield-Bogoliubov transformation \( U_{\text{HB}} \), and then, reach the unitary transformation,
\[
\begin{align*}
U_{\text{HB}} a U_{\text{HB}}^* &= b = \frac{1}{2} (c_1 + c_2) a + \frac{1}{2} (c_1 - c_2) a^\dagger, \\
U_{\text{HB}} a^\dagger U_{\text{HB}}^* &= b^\dagger = \frac{1}{2} (c_1 - c_2) a + \frac{1}{2} (c_1 + c_2) a^\dagger.
\end{align*}
\]
where \( c_1 = \sqrt{\frac{\omega_g}{\omega_c}} \) and \( c_2 = \sqrt{\frac{\omega_c}{\omega_g}} \). We note that the Hopfield-Bogoliubov transformation \( U_{\text{HB}} \) is concretely defined by Eqs.(50) and (57) of Ref.[37] or Eqs.(12.17)-(12.19) of Ref.[9] in (meson) pair theory. Use the Hopfield-Bogoliubov transformation \( U_{\text{HB}} \), and we obtain the unitary transformation

\[
U_{\text{HB}}^* \mathcal{H}_{A^2}(\varepsilon) U_{\text{HB}} = \mathcal{H}_{\text{GQR}}(\omega_g, \tilde{g}).
\] (4.3)

This is our renormalized Hamiltonian \( \mathcal{H}_{A^2}^{\text{ren}}(\varepsilon) \) in Eq.(3.18).

We denote by \( |E_{\nu}(\varepsilon)\rangle \) eigenstates of our total Hamiltonian \( \mathcal{H}_{A^2}(\varepsilon) \) with eigenenergy \( E_{\nu} \), i.e., \( \mathcal{H}_{A^2}(\varepsilon)|E_{\nu}(\varepsilon)\rangle = E_{\nu}|E_{\nu}(\varepsilon)\rangle \). We make the order of the eigenenergies as \( E_1 \leq E_2 \leq \cdots \leq E_{\nu} \leq E_{\nu+1} \leq \cdots \). These eigenstates, \( |E_{\nu}(\varepsilon)\rangle \), \( \nu = 0, 1, 2, \cdots \), are bare states. According to (meson) pair theory, we should derive the physical eigenstates \( |E_{\nu}^{\text{ren}}(\varepsilon)\rangle \) from the bare ones by

\[
|E_{\nu}^{\text{ren}}(\varepsilon)\rangle := U_{\text{HB}}^*|E_{\nu}(\varepsilon)\rangle.
\] (4.4)

Then, we have \( \mathcal{H}_{A^2}^{\text{ren}}(\varepsilon)|E_{\nu}^{\text{ren}}(\varepsilon)\rangle = E_{\nu}|E_{\nu}^{\text{ren}}(\varepsilon)\rangle \). In next section, we will give the adiabatic approximation for these physical eigenstates \( |E_{\nu}^{\text{ren}}(\varepsilon)\rangle \) by taking the strong coupling limit.

5 Schrödinger-Cat-Likeness in Adiabatic Approximation

In this section, we show the Schrödinger-cat-likeness in the adiabatic approximation. We will give a mathematical proof of the adiabatic approximation in the next section.

To make an energy renormalization for \( H_{\text{ptn}}(\omega_g) \) as \( g \to \infty \), we introduce a function \( \Delta_g \) of the coupling strength \( g \) such that there is a positive function \( \delta_g \) satisfying the conditions,

\[
|\omega_c - (\omega_g - \Delta_g)| \leq \delta_g \omega_c,
\] (5.1)

\[
\lim_{g \to \infty} \delta_g = 0.
\] (5.2)

These conditions yield the limit

\[
\lim_{g \to \infty} (\omega_g - \Delta_g) = \omega_c.
\] (5.3)

For example, take \( \Delta_g \) as \( \Delta_g = \sqrt{\omega_g^2 - 4\omega_c\sqrt{gC_g}C_c} \) for every coupling strength \( g \) with \( \sqrt{\omega_c/gC_g} \neq 2 \). Then, we have the equations,

\[
\omega_c \left\{ 1 - \frac{1}{|1 - \sqrt{\omega_c/4gC_g}|} \right\} = \omega_c \left\{ 1 - \frac{1}{\sqrt{(\sqrt{\omega_c/4gC_g})^2 + 1^2 - \sqrt{\omega_c/gC_g}}} \right\}
\]

\[
= \omega_c \left\{ 1 - \frac{1}{\sqrt{1 + (\omega_c/4gC_g) - \sqrt{\omega_c/gC_g}}} \right\},
\] (5.4)

and

\[
\omega_c - (\omega_g - \Delta_g)
\]

\[
= \omega_c \left\{ 1 - \frac{2}{\sqrt{1 + (\omega_c/4gC_g) + \sqrt{1 + (\omega_c/4gC_g) - \sqrt{\omega_c/gC_g}}}} \right\}.
\] (5.5)
Meanwhile, since we have the inequalities,
\[
2\sqrt{1 + (\omega_c/4gC_g)} - \sqrt{\omega_c/gC_g} \\
\leq \sqrt{1 + (\omega_c/4gC_g)} + \sqrt{1 + (\omega_c/4gC_g)} - \sqrt{\omega_c/gC_g} \\
\leq 2\sqrt{1 + (\omega_c/4gC_g)},
\]
we reach the inequalities,
\[
1 - \frac{1}{\sqrt{1 + (\omega_c/4gC_g)}} - \frac{2}{\sqrt{1 + (\omega_c/4gC_g)} + \sqrt{1 + (\omega_c/4gC_g)} - \sqrt{\omega_c/gC_g}} \\
\leq 1 - \frac{1}{\sqrt{1 + (\omega_c/4gC_g)}},
\]
(5.6)

By Eqs. (5.4), (5.5), and (5.6), we have the following two inequalities,
\[
\omega_c \left\{ 1 - \frac{1}{1 - \sqrt{\omega_c/gC_g}} \right\} \leq \omega_c - (\omega_g - \Delta_g) \leq \omega_c \left\{ 1 - \frac{1}{\sqrt{1 + (\omega_c/4gC_g)}} \right\}.
\]

Here, we note the inequality, \(1 - \sqrt{\omega_c/gC_g} \leq 1\). These inequalities suggest us that we chose the function \(\delta_g\) as
\[
\delta_g = \max \left\{ \left| 1 - \frac{1}{1 - \sqrt{\omega_c/gC_g}} \right|, \left| 1 - \frac{1}{\sqrt{1 + (\omega_c/4gC_g)}} \right| \right\}.
\]

As proved in the next section, we have the adiabatic approximation of the renormalized Hamiltonian \(\mathcal{H}_{A2}^{\text{ren}}(\varepsilon)\):
\[
\mathcal{H}_{A2}^{\text{ren}}(\varepsilon) \approx U(\tilde{g}/\omega_g)\left\{ H_{\text{ptn}}(\omega_g) + \mathcal{H}_{\text{atm}}(\varepsilon) \right\} U(\tilde{g}/\omega_g) - \hbar \frac{\tilde{g}^2}{\omega_g}.
\]
(5.7)

We note that the whole atom energy \(\mathcal{H}_{\text{atm}}(\varepsilon)\) remains in the adiabatic approximation. Thanks to Theorem VIII.23(b) of Ref. [38], we can obtain the adiabatic approximation of the eigenstates and their corresponding eigenenergies in the following.

In the case \(\varepsilon = 0\), the adiabatic approximation of all the physical eigenstates \(|\mathcal{E}_\nu^{\text{ren}}(0)\rangle\) of the renormalized Hamiltonian \(\mathcal{H}_{A2}^{\text{ren}}(0)\) are given by the same formulas of the Schrödinger-cat-like entangled states as in Eq. (3.9):
\[
\begin{cases}
\frac{1}{\sqrt{2}} \left( |\uparrow\rangle D\left(-\tilde{g}/\omega_g\right) |n\rangle + |\downarrow\rangle D\left(\tilde{g}/\omega_g\right) |n\rangle \right), \\
\frac{1}{\sqrt{2}} \left( |\uparrow\rangle D\left(-\tilde{g}/\omega_g\right) |n\rangle - |\downarrow\rangle D\left(\tilde{g}/\omega_g\right) |n\rangle \right).
\end{cases}
\]
(5.8)
We denote by $|\mathcal{E}_n^{\text{app},+}(0)\rangle$ the first expression in Eq. (5.8), and by $|\mathcal{E}_n^{\text{app},-}(0)\rangle$ the second one. The eigenenergy of the approximated eigenstate $|\mathcal{E}_n^{\text{app},+}(0)\rangle$ is

$$h\omega_g \left( n + \frac{1}{2} \right) - \frac{h\omega_a}{2} - \frac{\tilde{g}^2}{\omega_g} \approx h(\omega_c + \Delta_g)n + \frac{h(\omega_g - \omega_a)}{2} - \frac{h}{4C_{\infty}},$$

and that of the approximated eigenstate $|\mathcal{E}_n^{\text{app},-}(0)\rangle$ is

$$h\omega_g \left( n + \frac{1}{2} \right) + \frac{h\omega_a}{2} - \frac{\tilde{g}^2}{\omega_g} \approx h(\omega_c + \Delta_g)n + \frac{h(\omega_g + \omega_a)}{2} - \frac{h}{4C_{\infty}},$$

which says that the tunnel splitting of the two-level atom remains in the adiabatic approximation.

In the case $\epsilon \neq 0$, all the physical eigenstates $|\mathcal{E}_n^{\text{ren}}(\epsilon)\rangle$ of the renormalized Hamiltonian $\mathcal{H}^{\text{ren}}_{A^2}(\epsilon)$ have the following adiabatic approximation:

$$c_{\epsilon,\omega_a} \left( -\omega_a|\uparrow\rangle D(-\tilde{g}/\omega_g)|n\rangle + (\epsilon - \sqrt{\omega_d^2 + \omega_a^2})|\downarrow\rangle D(\tilde{g}/\omega_g)|n\rangle \right),$$

$$c_{-\epsilon,\omega_a} \left( -\omega_a|\uparrow\rangle D(-\tilde{g}/\omega_g)|n\rangle + (\epsilon + \sqrt{\omega_d^2 + \omega_a^2})|\downarrow\rangle D(\tilde{g}/\omega_g)|n\rangle \right),$$

where the positive constant $c_{\pm \epsilon,\omega_a}$ is given by $1/c_{\pm \epsilon,\omega_a}^2 = 2(\omega_a^2 + \epsilon^2 \mp \sqrt{\omega_d^2 + \epsilon^2})$. Eqs. (5.9) show up the Schrödinger-cat-like entangled states. We denote by $|\mathcal{E}_n^{\text{app},+}(\epsilon)\rangle$ the first expression in Eq. (5.9), and by $|\mathcal{E}_n^{\text{app},-}(\epsilon)\rangle$ the second one. The eigenenergy of the adiabatically approximated eigenstates $|\mathcal{E}_n^{\text{app},\pm}(\epsilon)\rangle$ is

$$\mp \frac{h}{2} \sqrt{\omega_a^2 + \epsilon^2} + h\omega_g \left( n + \frac{1}{2} \right) - \frac{\tilde{g}^2}{\omega_g}$$

$$\approx \mp \frac{h}{2} \sqrt{\omega_a^2 + \epsilon^2} + h(\omega_c + \Delta_g)n + \frac{h}{2}\omega_g - \frac{h}{4C_{\infty}}.$$  

The energy difference between the two adiabatically approximated eigenstates, $|\mathcal{E}_n^{\text{app},-}(\epsilon)\rangle$ and $|\mathcal{E}_n^{\text{app},+}(\epsilon)\rangle$, is $\mathcal{E}_n^{\text{app},-}(\epsilon) - \mathcal{E}_n^{\text{app},+}(\epsilon) = h\sqrt{\omega_a^2 + \epsilon^2}$. Since the energy difference between $\mathcal{E}_n^{\text{app},\pm}(\epsilon)$ and $\mathcal{E}_{n+1}^{\text{app},\pm}(\epsilon)$ is $\mathcal{E}_{n+1}^{\text{app},\pm}(\epsilon) - \mathcal{E}_n^{\text{app},\pm}(\epsilon) = h(\omega_g + \Delta_g)$, we can obtain

$$0 < \mathcal{E}_n^{\text{app},-}(\epsilon) - \mathcal{E}_n^{\text{app},+}(\epsilon) \leq \mathcal{E}_{n+1}^{\text{app},\pm}(\epsilon) - \mathcal{E}_n^{\text{app},\pm}(\epsilon)$$

by controlling the parameters, $\omega_a$, $\epsilon$, $\omega_c$, $g$, and $C_g$.

We give an application example of the adiabatic approximation given by Eqs. (5.8) and (5.9).

Take the energy-bias parameter $\epsilon$ as the transverse axis now. The adiabatic approximation in Eq. (5.8) says that whether the energy bias is positive or negative causes an energy level crossing with respect to the $\epsilon$-axis because the model does not have the $A^2$-term. On the other hand, the adiabatic-approximation formula, Eq. (5.9), with Eq. (5.10), says that the $A^2$-term makes an avoided crossing with respect to the $\epsilon$-axis.

In the same way as the proof of Eq. (21) of Ref. [37], we can obtain the expression of the ground-state expectation $N_0^{\text{ren}} := \langle \mathcal{E}_0^{\text{ren}} |a^\dagger a|\mathcal{E}_0^{\text{ren}}\rangle$ of the number of dressed photons and estimate it as

$$N_0^{\text{ren}} = \frac{h^2 g^2}{\omega_g^2} \sum_{\nu=0}^{\infty} \left| \langle \mathcal{E}_\nu^{\text{ren}} | \sigma_+ | \mathcal{E}_0^{\text{ren}} \rangle \right|^2 \left( \frac{1}{E_\nu - E_0 + h\omega_g} \right)^2 \leq \frac{g^2}{\omega_g^2} \sum_{\nu=0}^{\infty} \left( \langle \mathcal{E}_\nu^{\text{ren}} | \sigma_+ | \mathcal{E}_0^{\text{ren}} \rangle \right)^2 = \frac{g^2}{\omega_g^2} \| \sigma_+ |\mathcal{E}_0^{\text{ren}}\rangle \|^2_{L^2(\mathbb{R})} \leq \frac{g^2}{\omega_a^2}.$$
We define the modified ground-state expectation $N_0^{\text{app}}$ using the adiabatic approximation, i.e., $N_0^{\text{app}} := \langle c_0^{\text{app},+}(\varepsilon) | a^\dagger a | c_0^{\text{app},+}(\varepsilon) \rangle$. Then, the immediate calculation gives us the expression,
\[
N_0^{\text{app}} = \frac{\omega_c^2}{\omega_g^2} = \omega_c^{-1/2} \left( \omega_c g^{-1/3} + 4g^{-1/3}C_g \right)^{-3/2},
\]
and thus, we have the limit, $\lim_{g \to \infty} N_0^{\text{ren}} = \lim_{g \to \infty} N_0^{\text{app}} = 0$, with $N_0^{\text{ren}} \leq N_0^{\text{app}}$. Meanwhile, for the one-mode photon field $\Phi^{\text{ren}} := \left( a + a^\dagger \right)/\sqrt{2\omega_g}$, our adiabatic approximation immediately shows that the fluctuation $\Delta \Phi^{\text{ren}}$ decays to zero as $g \to \infty$.

We have information on the dressed photon in the ground state in the following. The symbol $\geq$ stands for one of the (in)equality symbols, $\geq$, $\leq$, or $\approx$. Eq. (5.12) says that the (in)equality, $N_0^{\text{app}} \geq 1$, is equivalent to the (in)equality,
\[
1/4 \left( \frac{g}{\omega_c} \right)^{-1} \left\{ \left( \frac{g}{\omega_c} \right)^{4/3} - 1 \right\} \geq C_g.
\]
Particularly, in the case where the function $C_g$ is given by $C_g = C g$ with a constant $C$, Eq. (5.13) can be written as
\[
1/4 \omega_c \left( \frac{g}{\omega_c} \right)^{-2} \left\{ \left( \frac{g}{\omega_c} \right)^{4/3} - 1 \right\} \geq C.
\]
Namely, following the (meson) pair theory, if we can make the constant $C$ so small that it satisfies the condition, $\left( \omega_c / 4g^2 \right)^{4/3} - 1 \geq C$, then there is a possibility that the ground state of the generalized quantum Rabi model has some dressed photons. We will explain the reason why we are interested in Eqs. (5.13) and (5.14) in Section 7.

6 A Proof of Adiabatic Approximation for $\mathcal{H}_{A^2}^{\text{ren}}(\varepsilon)$

We define the modified photon Hamiltonian $\tilde{H}_\text{ptn}(\omega)$ for every frequency $\omega$ by removing the zero-point energy, that is,
\[
\tilde{H}_\text{ptn}(\omega) := H_\text{ptn}(\omega) - \hbar \omega / 2 = \hbar \omega a^\dagger a.
\]
In this section, we will take frequencies, $\omega_c$, $\omega_g$, $\omega_c - (\omega_g - \Delta g)$, or $\omega_c - \omega_g$ as $\omega$. With this modification, we slightly modify the renormalized Hamiltonian $\mathcal{H}_{A^2}^{\text{ren}}(\varepsilon)$ as $\tilde{H}_{A^2}^{\text{ren}}(\varepsilon) := \mathcal{H}_{A^2}^{\text{ren}}(\varepsilon) - \hbar \omega_g / 2$. By Eq. (5.18), we have the expression,
\[
\tilde{H}_{A^2}^{\text{ren}}(\varepsilon) = \mathcal{H}_{\text{GQR}}(\omega_g, \bar{g}) - \hbar \omega_g / 2 = \mathcal{H}_\text{atm}(\varepsilon) + \tilde{H}_\text{ptn}(\omega_g) + h\bar{g} \sigma_z (a + a^\dagger).
\]
All the eigenstates of the slightly modified Hamiltonian $\tilde{H}_{A^2}^{\text{ren}}(\varepsilon)$ are completely same as those of the original Hamiltonian $\mathcal{H}_{A^2}^{\text{ren}}(\varepsilon)$. Thus, we prove our desired results for the Hamiltonian $\mathcal{H}_{A^2}^{\text{ren}}(\varepsilon)$.

Correspondingly, we denote the modified free Hamiltonian $\tilde{\mathcal{H}}_0$ of the atom-photon system by
\[
\tilde{\mathcal{H}}_0 := \mathcal{H}_\text{atm}(\varepsilon) + \tilde{H}_\text{ptn}(\omega_c).
\]
All the eigenenergies of the atom Hamiltonian $\mathcal{H}_\text{atm}(\varepsilon)$ are $\pm (\hbar / 2) \sqrt{\omega_a^2 + \varepsilon^2}$, and thus, we have its operator norm, $\| H_\text{atm}(\varepsilon) \|_{\text{op}} = (\hbar / 2) \sqrt{\omega_a^2 + \varepsilon^2}$. Since we can rewrite the Hamiltonian $\tilde{H}_\text{ptn}(\omega_c)$ as $\tilde{H}_\text{ptn}(\omega_c) = \mathcal{H}_0 - \mathcal{H}_\text{atm}(\varepsilon)$, we have
\[
\| \tilde{H}_\text{ptn}(\omega_c) \Psi \|_{C^2 \otimes L^2(\mathbb{R})} \leq \| \mathcal{H}_0 \Psi \|_{C^2 \otimes L^2(\mathbb{R})} + \| \mathcal{H}_\text{atm}(\varepsilon) \|_{\text{op}} \| \Psi \|_{C^2 \otimes L^2(\mathbb{R})} \leq \| \mathcal{H}_0 \Psi \|_{C^2 \otimes L^2(\mathbb{R})} + \frac{\hbar}{2} \sqrt{\omega_a^2 + \varepsilon^2} \| \Psi \|_{C^2 \otimes L^2(\mathbb{R})}
\]
for every vector $\Psi \in C^2 \otimes L^2(\mathbb{R})$. In particular, we set $\Psi = (\tilde{H}_0 - ih)^{-1}\Phi$ for every vector $\Phi \in C^2 \otimes L^2(\mathbb{R})$, and insert it into the above inequality. Then, we have the inequalities,

$$
\|\tilde{H}_{ptn}(\omega_c)(\tilde{H}_0 - ih)^{-1}\Phi\|_{C^2 \otimes L^2(\mathbb{R})}
\leq \|\tilde{H}_0(\tilde{H}_0 - ih)^{-1}\Phi\|_{C^2 \otimes L^2(\mathbb{R})} + \frac{\hbar}{2} \sqrt{\omega_n^2 + \varepsilon^2}\|\tilde{H}_0 - ih\|^2\|\Phi\|_{C^2 \otimes L^2(\mathbb{R})}
\leq \|\tilde{H}_0(\tilde{H}_0 - ih)^{-1}\Phi\|_{C^2 \otimes L^2(\mathbb{R})} + \frac{\hbar}{2} \sqrt{\omega_n^2 + \varepsilon^2}\|\tilde{H}_0 - ih\|^2\|\Phi\|_{C^2 \otimes L^2(\mathbb{R})}.
$$

(6.1)

To estimate the operator norms, $\|\tilde{H}_0(\tilde{H}_0 - ih)^{-1}\|_{op}$ and $\|\tilde{H}_0 - ih\|_{op}$, we make a general argument. Let $H$ be an arbitrary self-adjoint energy operator (i.e., Hamiltonian). We recall Theorem VIII.6 of Ref. [38] or Theorems 7.14 and 7.17 of Ref. [39]: There is a spectral family (i.e., the set of projection-valued measures) $P^H_\xi$ for the Hamiltonian $H$ so that

$$
H = \int_{-\infty}^{\infty} \xi dP^H_\xi.
$$

Using the properties of the projection-valued measures $P^H_\xi$, we have the following estimates,

$$
\|H(H - ih)^{-1}\Phi\|_{C^2 \otimes L^2(\mathbb{R})}^2 = \int_{-\infty}^{\infty} \frac{\xi^2}{\xi - ih} d\|P^H_\xi\|_{C^2 \otimes L^2(\mathbb{R})}^2 
\leq \int_{-\infty}^{\infty} d\|P^H_\xi\|_{C^2 \otimes L^2(\mathbb{R})} = \Phi^2_{C^2 \otimes L^2(\mathbb{R})}
$$

and

$$
\|(H - ih)^{-1}\Phi\|_{C^2 \otimes L^2(\mathbb{R})}^2 = \int_{-\infty}^{\infty} \frac{1}{\xi - ih} d\|P^H_\xi\|_{C^2 \otimes L^2(\mathbb{R})}^2 
\leq \frac{1}{\hbar^2} \int_{-\infty}^{\infty} d\|P^H_\xi\|_{C^2 \otimes L^2(\mathbb{R})} = \frac{1}{\hbar^2} \Phi^2_{C^2 \otimes L^2(\mathbb{R})},
$$

for every vector $\Phi \in C^2 \otimes L^2(\mathbb{R})$. These estimates bring us the two operator-norm inequalities,

$$
\|H(H - ih)^{-1}\|_{op} \leq 1 \quad \text{and} \quad \|(H - ih)^{-1}\|_{op} \leq 1/\hbar.
$$

Inserting the inequalities, $\|\tilde{H}_0(\tilde{H}_0 - ih)^{-1}\|_{op} \leq 1$ and $\|(\tilde{H}_0 - ih)^{-1}\|_{op} \leq 1/\hbar$, into Eq. (6.1), we reach the inequality

$$
\|\tilde{H}_{ptn}(\omega_c)(\tilde{H}_0 - ih)^{-1}\|_{op} \leq 1 + \frac{1}{2} \sqrt{\omega_n^2 + \varepsilon^2}.
$$

(6.2)

Meanwhile, since we assume Eq. [5.1], we have

$$
\|\tilde{H}_{ptn}(\omega_c - (\omega_g - \Delta_g))(\tilde{H}_0 - ih)^{-1}\Phi\|_{C^2 \otimes L^2(\mathbb{R})}
= \hbar|\omega_c - (\omega_g - \Delta_g)| \|a^\dagger a(\tilde{H}_0 - ih)^{-1}\Phi\|_{C^2 \otimes L^2(\mathbb{R})}
\leq \hbar\delta_g|\omega_c| \|a^\dagger a(\tilde{H}_0 - ih)^{-1}\Phi\|_{C^2 \otimes L^2(\mathbb{R})}
= \delta_g\|\tilde{H}_{ptn}(\omega_c)(\tilde{H}_0 - ih)^{-1}\Phi\|_{C^2 \otimes L^2(\mathbb{R})},
$$

which implies the operator-norm inequality,

$$
\|\tilde{H}_{ptn}(\omega_c - (\omega_g - \Delta_g))(\tilde{H}_0 - ih)^{-1}\|_{op} \leq \delta_g\|\tilde{H}_{ptn}(\omega_c)(\tilde{H}_0 - ih)^{-1}\|_{op}.
$$

(6.3)
Using the 2nd resolvent identity in Theorem 5.13(b) of Ref. [39] and the equation \( \tilde{D} G \), we denote the displacement operators in the domain of the photon number operator \( a \). We define an operator \( \sigma \parallel \tilde{D} G \). Therefore, in the weak operator topology. For details on these topologies, see §VI.1 of Ref. [38].

Combining Eqs. (6.2) and (6.3), we reach the inequality,

\[
\| \tilde{H}_{\text{ptn}}(\omega_c - (\omega_g - \Delta_g)) (\tilde{H}_0 - i\hbar)^{-1} \|_{\text{op}} \leq \delta_g \left( 1 + \frac{1}{2} \sqrt{\omega_g^2 + \varepsilon^2} \right). \tag{6.4}
\]

Let \( G = \tilde{g}/\omega_g \), i.e., \( G := \tilde{g}/\omega_g \). Then, we immediately know that this quantity \( G \) decays to the zero as the coupling strength \( g \) tends to the infinity, i.e., \( \lim_{g \to \infty} G = 0 \). For simplicity, we denote the displacement operators \( D(\pm \tilde{g}/\omega_g) \) by \( D_\pm(G) \), i.e., \( D_\pm(G) := D(\pm \tilde{g}/\omega_g) \). We know the expression, \( D_\pm(G) = e^{\pm iG (\pm i(a - a^\dagger))} \). Since the operator \( i(a - a^\dagger) \) is self-adjoint on the domain of the photon number operator \( a a^\dagger \), the operator \( D_\pm(G) = e^{\pm iG (\pm i(a - a^\dagger))} \) is a strongly continuous one-parameter unitary group by Theorem VIII.7 of Ref. [38]. Namely, we have the limit, \( \lim_{G \to 0} (1 - D_\pm(G)^2) \Psi = 0 \), for every vector \( \Psi \in \mathbb{C}^2 \otimes L^2(\mathbb{R}) \). Thus, the operator \( \sigma_+ (1 - D_+(G)^2) + \sigma_- (1 - D_-(G)^2) \) goes to the zero operator as \( g \to \infty \) in the strong operator topology, and therefore, in the weak operator topology. For details on these topologies, see §VI.1 of Ref. [38].

We note that the inequality, \( \| \sigma_+ (1 - D_+(G)^2) + \sigma_- (1 - D_-(G)^2) \|_{\text{op}} \leq 2 \), holds, and that the resolvent \( (\tilde{H}_0 - i\hbar)^{-1} \) is compact. Therefore, by applying Theorem in Appendix A of Ref. [37], we obtain the limit,

\[
\lim_{g \to \infty} \left\| (\tilde{H}_0 - i\hbar)^{-1} \left\{ \sigma_+ (1 - D_+(G)^2) + \sigma_- (1 - D_-(G)^2) \right\} (\tilde{H}_0 - i\hbar)^{-1} \right\|_{\text{op}} = 0. \tag{6.5}
\]

From now on, we denote the operator \( \sigma_+ D_+(G)^2 + \sigma_- D_-(G)^2 \) by \( \Xi_0(g) \), and moreover, the operator \( \sigma_x - \Xi_0(g) \) by \( \Xi_1(g) \):

\[
\Xi_0(g) := \sigma_+ D_+(G)^2 + \sigma_- D_-(G)^2, \quad \Xi_1(g) := \sigma_x - \Xi_0(g) = \sigma_+ (1 - D_+(G)^2) + \sigma_- (1 - D_-(G)^2). \]

Moreover, we define a Hamiltonian \( \tilde{H}(g) \) by

\[
\tilde{H}(g) := \tilde{H}_{\text{ptn}}(\omega_g - \Delta_g) - \frac{\hbar}{2} \varepsilon \sigma_z - \frac{\hbar}{2} \omega_a \Xi_0(g).
\]

We define an operator \( R \) by the difference between the resolvents of \( \tilde{H}(g) \) and \( \tilde{H}_0 \), that is,

\[
R := (\tilde{H}(g) - i\hbar)^{-1} - (\tilde{H}_0 - i\hbar)^{-1}.
\]

Using the 2nd resolvent identity in Theorem 5.13(b) of Ref. [39] and the equation \( \tilde{H}_{\text{ptn}}(\omega_c) - \tilde{H}_{\text{ptn}}(\omega_g - \Delta_g) = \tilde{H}_{\text{ptn}}(\omega_c - (\omega_g - \Delta_g)) \), we can calculate the expression of the difference operator \( R \) as

\[
R = (\tilde{H}(g) - i\hbar)^{-1} \left\{ \tilde{H}_{\text{ptn}}(\omega_c - (\omega_g - \Delta_g)) - \frac{\hbar}{2} \omega_a \Xi_1(g) \right\} (\tilde{H}_0 - i\hbar)^{-1}. \tag{6.6}
\]

Insert this into the equation \( (\tilde{H}(g) - i\hbar)^{-1} = (\tilde{H}_0 - i\hbar)^{-1} + R \), then we have the equation,

\[
(\tilde{H}(g) - i\hbar)^{-1} = (\tilde{H}_0 - i\hbar)^{-1} + (\tilde{H}(g) - i\hbar)^{-1} \tilde{H}_{\text{ptn}}(\omega_c - (\omega_g - \Delta_g))(\tilde{H}_0 - i\hbar)^{-1}
- \frac{\hbar}{2} \omega_a (\tilde{H}(g) - i\hbar)^{-1} \Xi_1(g) (\tilde{H}_0 - i\hbar)^{-1}. \tag{6.7}
\]
Inserting Eq. (6.7) into Eq. (6.6), we have the decomposition,

\[
R = (\tilde{H}_0 - ih)^{-1} \left\{ \tilde{H}_{\text{ptn}}(\omega_c - (\omega_g - \Delta_g)) - \frac{\hbar \omega_a}{2} \tilde{1}(g) \right\} (\tilde{H}_0 - ih)^{-1} \\
+ (\tilde{H}(g) - ih)^{-1} \tilde{H}_{\text{ptn}}(\omega_c - (\omega_g - \Delta_g)) (\tilde{H}_0 - ih)^{-1} \\
\left\{ \tilde{H}_{\text{ptn}}(\omega_c - (\omega_g - \Delta_g)) - \frac{\hbar \omega_a}{2} \tilde{1}(g) \right\} (\tilde{H}_0 - ih)^{-1} \\
- \frac{\hbar \omega_a}{2} (\tilde{H}(g) - ih)^{-1} \tilde{1}(g) (\tilde{H}_0 - ih)^{-1} \\
\left\{ \tilde{H}_{\text{ptn}}(\omega_c - (\omega_g - \Delta_g)) - \frac{\hbar \omega_a}{2} \tilde{1}(g) \right\} (\tilde{H}_0 - ih)^{-1}.
\]

Eventually, we can decompose the difference operator \( R \) as

\[
R = \sum_{j=1}^{6} I_j, \quad \text{and thus, } ||R||_{op} = \sum_{j=1}^{6} ||I_j||_{op}, \quad (6.8)
\]

where

\[
I_1 = (\tilde{H}_0 - ih)^{-1} \tilde{H}_{\text{ptn}}(\omega_c - (\omega_g - \Delta_g))(\tilde{H}_0 - ih)^{-1},
\]

\[
I_2 = -\frac{\hbar \omega_a}{2} (\tilde{H}_0 - ih)^{-1} \tilde{1}(g)(\tilde{H}_0 - ih)^{-1},
\]

\[
I_3 = (\tilde{H}(g) - ih)^{-1} \left\{ \tilde{H}_{\text{ptn}}(\omega_c - (\omega_g - \Delta_g))(\tilde{H}_0 - ih)^{-1} \right\}^2,
\]

\[
I_4 = -\frac{\hbar \omega_a}{2} (\tilde{H}(g) - ih)^{-1} \tilde{H}_{\text{ptn}}(\omega_c - (\omega_g - \Delta_g))(\tilde{H}_0 - ih)^{-1} \tilde{1}(g)(\tilde{H}_0 - ih)^{-1},
\]

\[
I_5 = -\frac{\hbar \omega_a}{2} (\tilde{H}(g) - ih)^{-1} \tilde{1}(g)(\tilde{H}_0 - ih)^{-1} \tilde{H}_{\text{ptn}}(\omega_c - (\omega_g - \Delta_g))(\tilde{H}_0 - ih)^{-1},
\]

\[
I_6 = \left( -\frac{\hbar \omega_a}{2} \right)^2 (\tilde{H}(g) - ih)^{-1} \tilde{1}(g)(\tilde{H}_0 - ih)^{-1} \tilde{1}(g)(\tilde{H}_0 - ih)^{-1}.
\]

Using Eq. (6.4) and the inequalities, \( ||\tilde{H}_0 - ih||_{op} \leq 1/h, ||\tilde{H}(g) - ih||_{op} \leq 1/h, ||\tilde{1}(g)||_{op} \leq 2 \), individual operators \( I_j \) are bounded from above in the following:

\[
||I_1||_{op} \leq ||(\tilde{H}_0 - ih)^{-1}||_{op} ||\tilde{H}_{\text{ptn}}(\omega_c - (\omega_g - \Delta_g))(\tilde{H}_0 - ih)^{-1}||_{op} \leq \frac{\delta_g}{h} \left( 1 + \frac{1}{2} \sqrt{\omega_a^2 + \epsilon^2} \right),
\]

\[
||I_2||_{op} \leq \frac{\hbar \omega_a}{2} ||(\tilde{H}_0 - ih)^{-1} \tilde{1}(g)(\tilde{H}_0 - ih)^{-1}||_{op},
\]

\[
||I_3||_{op} \leq ||(\tilde{H}(g) - ih)^{-1}||_{op} ||\tilde{H}_{\text{ptn}}(\omega_c - (\omega_g - \Delta_g))(\tilde{H}_0 - ih)^{-1}||^2_{op} \leq \frac{\delta_g^2}{h} \left( 1 + \frac{1}{2} \sqrt{\omega_a^2 + \epsilon^2} \right)^2.
\]
\[ \| I_4 \|_{\text{op}} \leq \frac{h}{2} \omega_a \| (\widetilde{H}(g) - i\hbar)^{-1} \|_{\text{op}} \| \widetilde{H}_{\text{ptn}}(\omega_c - (\omega_g - \Delta_g))(\widetilde{H}_0 - i\hbar)^{-1} \|_{\text{op}} \]
\[ \times \| \Xi_1(g) \|_{\text{op}} \| (\widetilde{H}_0 - i\hbar)^{-1} \|_{\text{op}} \leq \frac{\omega_a \delta_g}{\hbar} \left( 1 + \frac{1}{2} \sqrt{\omega_a^2 + \varepsilon^2} \right), \]
\[ \| I_5 \|_{\text{op}} \leq \frac{h}{2} \omega_a \| (\widetilde{H}(g) - i\hbar)^{-1} \|_{\text{op}} \| \Xi_1(g) \|_{\text{op}} \| (\widetilde{H}_0 - i\hbar)^{-1} \|_{\text{op}} \]
\[ \times \| \widetilde{H}_{\text{ptn}}(\omega_c - (\omega_g - \Delta_g))(\widetilde{H}_0 - i\hbar)^{-1} \|_{\text{op}} \leq \frac{\omega_a \delta_g}{\hbar} \left( 1 + \frac{1}{2} \sqrt{\omega_a^2 + \varepsilon^2} \right), \]
\[ \| I_6 \|_{\text{op}} \leq \frac{h^2 \omega_a^2}{4} \| (\widetilde{H}(g) - i\hbar)^{-1} \|_{\text{op}} \| \Xi_1(g) \|_{\text{op}} \| (\widetilde{H}_0 - i\hbar)^{-1} \|_{\text{op}} \]
\[ \| \Xi_1(g) \|_{\text{op}} \| (\widetilde{H}_0 - i\hbar)^{-1} \|_{\text{op}} \leq \frac{h^2 \omega_a^2}{2} \| (\widetilde{H}_0 - i\hbar)^{-1} \|_{\text{op}} \| \Xi_1(g)(\widetilde{H}_0 - i\hbar)^{-1} \|_{\text{op}}. \]

Eqs. (6.2) and (6.5) tell us that all the operator norms of operators \( I_j \) converges to zero as taking strong coupling limit, i.e., \( \lim_{g \to \infty} \| I_j \|_{\text{op}} = 0 \). Therefore, by Eq. (6.8), we obtain our desired limit, \( \lim_{g \to \infty} \| R \|_{\text{op}} = 0 \). Namely, we succeed in proving the convergence,
\[ \widetilde{H}(g) \xrightarrow{\text{n.r.s.}} \widetilde{H}_0 \text{ as } g \to \infty. \]  
(6.9)

We recall the unitary transformation given in Eq. (3.10), and replace the parameters \( \omega_c \) and \( g \) with the parameters \( \omega_g \) and \( \bar{g} \) in it, respectively. This comes up with the identity,
\[ U(\bar{g}/\omega_g) H_{\text{GQR}}(\omega_g, \bar{g}) U(\bar{g}/\omega_g)^* \]
\[ = H_{\text{ptn}}(\omega_g) - \frac{\hbar}{2} \varepsilon \sigma_z - \frac{\hbar}{2} \omega_a \Xi_0(g) - \frac{\hbar \omega_g}{2} \]
\[ = \widetilde{H}_{\text{ptn}}(\omega_g) - \frac{\hbar}{2} \varepsilon \sigma_z - \frac{\hbar}{2} \omega_a \Xi_0(g) - \widetilde{H}_{\text{ptn}}(\Delta_g) \]
\[ = \widetilde{H}(g) - \frac{\hbar \bar{g}^2}{\omega_g}. \]  
(6.10)

Applying Eq. (6.9) to the Hamiltonian \( \widetilde{H}(g) \) in RHS of Eq. (6.10) yields the limit
\[ U(\bar{g}/\omega_g) H_{\text{GQR}}(\omega_g, \bar{g}) U(\bar{g}/\omega_g)^* - \widetilde{H}_{\text{ptn}}(\Delta_g) - \frac{\hbar \omega_g}{2} \]
\[ \xrightarrow{\text{n.r.s.}} \widetilde{H}_0 - \frac{\hbar}{4C_{\infty}} \text{ as } g \to \infty. \]  
(6.11)

Since \( H_{\text{A}_2}^{\text{ren}}(\varepsilon) = H_{\text{GQR}}(\omega_g, \bar{g}) \) by Eq. (3.18), the renormalized Hamiltonian \( H_{\text{A}_2}^{\text{ren}}(\varepsilon) \) can be well approximated by
\[ U(\bar{g}/\omega_g)^* \left\{ \widetilde{H}_0 + \widetilde{H}_{\text{ptn}}(\Delta_g) + \frac{\hbar \omega_g}{2} \right\} U(\bar{g}/\omega_g) - \frac{\hbar}{4C_{\infty}} \]
\[ = U(\bar{g}/\omega_g)^* \left\{ H_{\text{atm}}(\varepsilon) + \widetilde{H}_{\text{ptn}}(\omega_c + \Delta_g) + \frac{\hbar \omega_g}{2} \right\} U(\bar{g}/\omega_g) - \frac{\hbar}{4C_{\infty}} \]
\[ \approx U(\bar{g}/\omega_g)^* \left\{ H_{\text{atm}}(\varepsilon) + \widetilde{H}_{\text{ptn}}(\omega_g) + \frac{\hbar \omega_g}{2} \right\} U(\bar{g}/\omega_g) - \frac{\hbar \bar{g}^2}{\omega_g} \]
\[ = U(\bar{g}/\omega_g)^* \left\{ H_{\text{atm}}(\varepsilon) + \widetilde{H}_{\text{ptn}}(\omega_g) \right\} U(\bar{g}/\omega_g) - \frac{\hbar \bar{g}^2}{\omega_g}, \]
that is, by the limit Hamiltonian as in Eq. (5.7). Here, we used \( \widetilde{H}_{\text{ptn}}(\omega_c) + \widetilde{H}_{\text{ptn}}(\Delta_g) = \widetilde{H}_{\text{ptn}}(\omega_c + \Delta_g) \), and approximations, \( \omega_g \approx \omega_c + \Delta_g \) and \( \hbar/4C_{\infty} \approx \hbar \bar{g}^2/\omega_g \), respectively secured by Eqs. (5.3) and (3.11).
7 Conclusion and Discussion

We have considered a mathematical establishment of the adiabatic approximation for the generalized quantum Rabi Hamiltonian both without and with the $A^2$-term. In the case without the $A^2$-term, we have shown in the adiabatic approximation that whether each bare eigenstate forms a Schrödinger-cat-like entangled state or not depends on whether the energy bias in the atom Hamiltonian is zero or non-zero. On the other hand, in the case with the $A^2$-term, we have renormalized the $A^2$-term by employing (meson) pair theory, and then, we mathematically established the adiabatic approximation for the renormalized Hamiltonian. Moreover, we have shown in the adiabatic approximation that the Schrödinger-cat-likeness appears in the both cases where the energy bias is zero and where it is non-zero.

At the end of this section, we explain the reason why we take the interest in Eqs. (5.13) and (5.14). We showed in Ref.[37] that if the $A^2$-term effect is sufficiently small, then the renormalized Hamiltonian of the generalized quantum Rabi model has the chance to have some dressed photons (real photons) in the ground state. Based on the adiabatic approximation given by Eqs. (5.8) and (5.9), the approximated ground-state expectation $N_{0}^{\text{app}}$ can be calculated as $N_{0}^{\text{app}} = \frac{g^2}{\omega^2_g}$ as in Eq.(5.12). Therefore, each of the adiabatically approximated eigenstates has the expression as

$$|E_{n}^{\text{app}, \pm}(0)\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle D\left(-\sqrt{N_{0}^{\text{app}}}\right) |n\rangle \pm |\downarrow\rangle D\left(+\sqrt{N_{0}^{\text{app}}}\right) |n\rangle)$$

for $\varepsilon = 0$ and

$$|E_{n}^{\text{app}, \pm}(\varepsilon)\rangle = c_{\pm \varepsilon, \omega_a} \left(|\uparrow\rangle D\left(-\omega_a\right) (-\sqrt{N_{0}^{\text{app}}} |n\rangle + (\varepsilon + \sqrt{\varepsilon^2 + \omega_a^2}) |\downarrow\rangle D\left(+\sqrt{N_{0}^{\text{app}}}\right) |n\rangle)$$

for $\varepsilon \neq 0$. We note that the approximated ground-state expectation $N_{0}^{\text{app}}$ can be expressed as $N_{0}^{\text{app}} = \langle E_{0}^{\text{app}, -}(\varepsilon) | a^\dagger a | E_{0}^{\text{app}, -}(\varepsilon) \rangle$, and that the state $|E_{0}^{\text{app}, -}(\varepsilon)\rangle$ is the 1st excited state for sufficiently small $|\varepsilon|$ by Eq.(5.11). Therefore, whether the Schrödinger-cat-likeness for each eigenstate can be observed depends on the number of dressed photons in the ground state or the 1st excited state. In a sense, namely, the ‘size’ of the Schrödinger-cat-likeness is determined by the number of dressed photons in the ground state. This is the reason why we are interested in Eqs. (5.13) and (5.14).

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