Proximity Factors of Lattice Reduction-Aided Precoding for Multiantenna Broadcast

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Abstract—Lattice precoding is an effective strategy for multiantenna broadcast. In this paper, we show that approximate lattice precoding in multiantenna broadcast is a variant of the closest vector problem (CVP) known as $\eta$-CVP. The proximity factors of lattice reduction-aided precoding are defined, and their bounds are derived, which measure the worst-case loss in power efficiency compared to sphere precoding. Unlike decoding applications, this analysis does not suffer from the boundary effect of a finite constellation, since the underlying lattice in multiantenna broadcast is indeed infinite.

I. INTRODUCTION

Broadcast is referred to as the application where a single transmitter sends different messages to many users simultaneously. It may arise, for example, in the downlink of a multiuser communication system where the base station wants to communicate with the users in the area of coverage. The multi-input multi-output (MIMO) technology offers a new opportunity for developing efficient broadcast strategies.

The capacity of a MIMO broadcast channel has been determined in [1], where it was shown that the so-called dirty-paper coding is instrumental to achieving the capacity. Multiple antennas allow to pre-cancel the interuser interference, which is known at the transmitter in the broadcast application. The lattice method represents a major approach to cancelling known interference [2], and specifically, to precoding for MIMO broadcast. However, dirty-paper coding suffers from high complexity.

Hochwald et al. [3] formulated precoding as a decoding problem at the transmitter. Their technique, termed “vector perturbation”, corresponds to solving the closest lattice vector problem (CVP). It requires the use of the sphere precoder [4], whose average complexity grows quickly with the system size. Earlier, the idea of precoding via an algorithmic search over modulo equivalent points was proposed by Fischer et al. for the intersymbyol interference channel [5]. More recently, reference [6] considered some practical issues in the implementation of vector perturbation.

To reduce the complexity, lattice reduction (LR) can be used, i.e., an approximate solution is found by zero-forcing (ZF) or successive interference cancelation (SIC) on a reduced lattice $\mathbb{Z}^n$. Another scheme of approximate lattice precoding was proposed in [9] (we will show that it is actually equivalent to LR-aided ZF in [8]), which was shown to achieve the full diversity order.

In contrast to the complexity analysis of sphere decoding [10], the complexity of sphere precoding is not available in literature. Moreover, the signal-to-noise (SNR) gap between sphere precoding and LR-aided precoding has not been analyzed, although it has been done for decoding [11]. In this paper, we investigate these aspects of lattice precoding algorithms. We view the precoding problem as a variant of the CVP known as $\eta$-CVP. This view enables us to derive the proximity factors for lattice precoding, which measure the worst-case loss in power efficiency of LR-aided precoding schemes.

The paper is organized as follows: Section II presents the model of MIMO broadcast using lattice precoding, and investigates its complexity. In Section III the analysis of the proximity factors is given. Section V is a discussion.

Notation: The transpose, inverse, pseudoinverse of a matrix $\mathbf{B}$ by $\mathbf{B}^T$, $\mathbf{B}^{-1}$, and $\mathbf{B}^+$, respectively, and the Euclidean length $\|\mathbf{u}\| = \sqrt{\mathbf{u}^T \mathbf{u}}$. $[x]$ rounds to a closest integer.

II. LATTICE PRECODING FOR MIMO BROADCAST

Consider a MIMO broadcast system including one transmitter, equipped with $n$ antennas, and $n$ receivers, each equipped with a single antenna [3]. For convenience, we use the real-valued signal model

$$\mathbf{y} = \mathbf{H}\mathbf{s} + \mathbf{n},$$  

(1)

where $\mathbf{y}$ is the received signal vector at the users, $\mathbf{H} \in \mathbb{R}^{n \times n}$ is a full-rank channel matrix, $\mathbf{s}$ is the transmitted signal, and $\mathbf{n}$ is the noise vector. The entry $h_{i,j}$ of $\mathbf{H}$ indicates the channel coefficient between transmit antenna $i$ and user $j$. $\mathbf{s}$ is derived from the data vector $\mathbf{x} = [x_1, ..., x_n]^T$. We assume that $\mathbf{x} \in [-A/2, A/2]^n$ is taken from the intersection of a finite hypercube and an integer lattice. The entries of $\mathbf{n}$ are i.i.d. Gaussian with variance $\sigma^2$ each.
A. Lattice Preliminaries

An n-dimensional lattice in the m-dimensional Euclidean space \( \mathbb{R}^m \) is the set of integer linear combinations of \( n \) independent vectors \( \mathbf{b}_1, \ldots, \mathbf{b}_n \in \mathbb{R}^m \):

\[
\mathcal{L}(\mathbf{B}) = \left\{ \sum_{i=1}^{n} x_i \mathbf{b}_i \mid x_i \in \mathbb{Z}, i = 1, \ldots, n \right\}.
\]

The matrix \( \mathbf{B} = [\mathbf{b}_1 \cdots \mathbf{b}_n] \) is a basis of the lattice \( \mathcal{L}(\mathbf{B}) \). In matrix form, \( \mathcal{L}(\mathbf{B}) = \{ \mathbf{Bx} : x \in \mathbb{Z}^n \} \). For any point \( \mathbf{y} \in \mathbb{R}^m \) and any lattice \( \mathcal{L}(\mathbf{B}) \), the distance of \( \mathbf{y} \) to the lattice is \( \text{dist}(\mathbf{y}, \mathbf{B}) = \min_{x \in \mathbb{Z}^n} \| \mathbf{y} - \mathbf{Bx} \| \). A shortest vector of a lattice \( \mathcal{L}(\mathbf{B}) \) is a non-zero vector in \( \mathcal{L}(\mathbf{B}) \) with the smallest Euclidean norm. The length of the shortest vector, often referred to as the minimum distance, of \( \mathcal{L}(\mathbf{B}) \) is denoted by \( \lambda_1 \).

A lattice has infinitely many bases. In general, every matrix \( \mathbf{B} = \mathbf{BU} \) is also a basis, where \( \mathbf{U} \) is an unimodular matrix, i.e., \( \det(\mathbf{U}) = \pm 1 \) and all elements of \( \mathbf{U} \) are integers. The aim of lattice reduction is to find a good basis for a given lattice. In many applications, it is advantageous to have the basis vectors as short as possible. The celebrated LLL algorithm is the first polynomial (average) time algorithm which finds a vector not much longer than the shortest nonzero vector.

Let \( \hat{\mathbf{b}}_1, \ldots, \hat{\mathbf{b}}_n \) be the Gram-Schmidt vectors corresponding to a basis \( \mathbf{b}_1, \ldots, \mathbf{b}_n \), where \( \hat{\mathbf{b}}_i \) is the projection of \( \mathbf{b}_i \) orthogonal to the vector space generated by \( \mathbf{b}_1, \ldots, \mathbf{b}_{i-1} \). These are the vectors found by the Gram-Schmidt algorithm for orthogonalization. Gram-Schmidt orthogonalization (GSO) is closely related to QR decomposition \( \mathbf{B} = \mathbf{QR} \). More precisely, one has the relations \( \mu_{j,i} = r_{i,j}/r_{i,i} \) and \( \hat{\mathbf{b}}_i = r_{i,i} \mathbf{q}_i \), where \( \mathbf{q}_i \) is the \( i \)th column of \( \mathbf{Q} \).

A basis \( \mathbf{B} \) is LLL reduced if

\[
|\mu_{i,j}| \leq 1/2
\]

for \( 1 \leq j < i \leq n \), and

\[
\| \hat{\mathbf{b}}_i \|^2 \geq (\delta - \mu_{i,i}^2) \| \hat{\mathbf{b}}_{i-1} \|^2
\]

for \( 1 < i \leq n \), where \( 1/4 < \delta \leq 1 \) is a factor selected to achieve a good quality-complexity tradeoff.

We now define a variant of the CVP.

**Definition 1 (\( \eta \)-CVP):** Given a lattice \( \mathcal{L}(\mathbf{B}) \) and a vector \( \mathbf{y} \in \mathbb{R}^m \), find a vector \( \mathbf{B\hat{x}} \in \mathcal{L}(\mathbf{B}) \) such that \( \| \mathbf{y} - \mathbf{B\hat{x}} \| \leq \eta \text{dist}(\mathbf{y}, \mathbf{B}) \).

B. Sphere Precoding

In this method, the transmitted signal is given by [3]

\[
\mathbf{s} = \mathbf{B}(\mathbf{x} - \mathbf{A}) = \mathbf{Bx} \mod \mathcal{L}(\mathbf{AB}), \tag{4}
\]

where \( \mathbf{B} \triangleq \mathbf{H}^{-1} \), and \( \mathbf{A} \) is an integer vector, chosen to minimize the transmission power:

\[
\hat{\mathbf{A}} = \arg \min_{\mathbf{A}} \| \mathbf{B}(\mathbf{x} - \mathbf{A}) \|^2. \tag{5}
\]

Note that \( \mathbf{s} \in \mathcal{V}(\mathcal{L}(\mathbf{AB})) \) (the Voronoi region). The receivers apply the modulo function each, obtaining

\[
\mathbf{y} \mod \mathbf{A} = \mathbf{HB}(\mathbf{x} - \hat{\mathbf{A}}) + \mathbf{n} \mod \mathbf{A} = (\mathbf{x} - \hat{\mathbf{A}}) + \mathbf{n} \mod \mathbf{A}. \tag{6}
\]

Namely, the data arrive at individual users free of interuser interference; the only effect is noise. To solve the CVP (5), the sphere precoding algorithm originally proposed for decoding purposes was used.

It is worth pointing out several distinctions between the CVP’s in decoding and precoding:

- Decoding gets easier for weaker noise, while noise has no impacts on the hardness of lattice precoding.
- The constellation in decoding is often finite, while the lattice in precoding is infinite. Thus, the boundary errors in decoding will not an issue in precoding.
- The received signal in decoding has a Gaussian distribution centered at a lattice point, while the input to precoding is roughly uniformly distributed on a fundamental parallelepiped.

For these reasons, sphere precoding incurs more computational complexity than sphere decoding at the same dimension \( n \). Fincke and Pohst [12] proposed an algorithm to enumerate the lattice points in a sphere, running on an LLL-reduced lattice, but their complexity estimate was loose. Kannan’s algorithm [13] for HKZ reduction can be used to preprocess the lattice, giving a CVP algorithm with \( n^{n/2+o(n)} \) complexity. Hanrot and Stehlé’ improved the CVP complexity analysis to \( n^{n/2+o(n)} \) [14].

On the other hand, Jaldén and Ottersten [10] showed that the average complexity of sphere decoding is exponential with the dimension for any fixed SNR; the constant within the exponent, though, does decrease with SNR, meaning lower complexity at higher SNR. However, the encouraging results for lattice decoding do not extend to precoding. Noise, which is crucial to the decreasing complexity of sphere decoding, does not even arise in lattice precoding. Since the input is largely uniformly distributed in the fundamental parallelepiped, the worst-case bound is a sensible measure of complexity. Moreover, the paper [10] assumed a finite constellation, rendering the analysis inapplicable to an infinite lattice, which is nonetheless the case for precoding problems.

To conclude, the worst-case complexity of sphere precoding is super-exponential.

C. Approximate Lattice Precoding

1) SIC Precoding: To obtain a fast precoder, Windpassinger et al. [8] approximated the CVP by using lattice reduction, i.e., the closest vector is replaced with Babai’s approximations [7]. Let \( \hat{\mathbf{B}} \) designate the reduced basis, i.e., \( \hat{\mathbf{B}} = \mathbf{BU} \), where \( \mathbf{U} \) is a unimodular matrix. Performing the QR decomposition \( \hat{\mathbf{B}} = \mathbf{QR} \), where \( \mathbf{Q} \) has orthogonal columns and \( \mathbf{R} \) is an upper triangular matrix with nonnegative diagonal elements.
Let \( \mathbf{u} = \mathbf{Q}^{\dagger} \hat{\mathbf{B}} \mathbf{x} / A \). An estimate of \( \hat{l} \) is then found by the SIC procedure:

\[
\hat{l}_n = \left[ \frac{u_n/r_{n,n}}{r_{n,n}} \right], \\
\hat{l}_i = \left[ \frac{u_i - \sum_{j=i+1}^{n} r_{i,j} \hat{j}_j}{r_{i,i}} \right], \quad i = n-1, ..., 1. \tag{7}
\]

The transmitted signal is given by

\[
\mathbf{s} = \mathbf{B} \mathbf{x} - A \hat{\mathbf{B}} \hat{l}.
\tag{8}
\]

At the receivers, the modulo operation is applied, yielding

\[
y \mod A = \mathbf{H}(\mathbf{B} \mathbf{x} - A \hat{\mathbf{B}} \hat{l}) + \mathbf{n} \mod A \\
= \mathbf{x} - A \mathbf{U} \hat{l} + \mathbf{n} \mod A \tag{9}
\]

\[= \mathbf{x} + \mathbf{n} \mod A. \]

2) ZF Precoding: Let \( \mathbf{u} = \mathbf{B} \mathbf{x} / A \). An estimate of \( \hat{l} \) is found by ZF as follows

\[
\hat{l} = \left[ \hat{\mathbf{B}}^{-1} \mathbf{u} \right] = \left[ \frac{\mathbf{U}^{-1} \mathbf{x}}{A} \right].
\]

The transmitted signal is given by

\[
\mathbf{s} = \mathbf{B} \mathbf{x} - A \hat{\mathbf{B}} \hat{l} \\
= \hat{\mathbf{B}} \left( \mathbf{U}^{-1} \mathbf{x} - A \left[ \frac{\mathbf{U}^{-1} \mathbf{x}}{A} \right] \right) \\
= \hat{\mathbf{B}} \left( \mathbf{U}^{-1} \mathbf{x} \mod A \right). \tag{10}
\]

The second line of (10) represents the transmission scheme in [8], while the third line corresponds to the transmission scheme in [9]. Therefore, the schemes proposed in [8,9] are equivalent. To the best of our knowledge, this equivalence is not known in literature. At the receivers, the modulo operation is applied, yielding the same as (9).

D. Reduction Criteria

To summarize, the purpose of approximate lattice precoding is to find a sub-optimal solution \( \hat{l} \) that can reduce the norm \( \|\mathbf{s}\| \). With lattice reduction, the transmitted vector \( \mathbf{s} \) falls into the fundamental parallelepiped (for ZF) or the rectangle spanned by the Gram-Schmidt vectors of the reduced basis \( \mathbf{AB} \) (for SIC). In both cases, the transmission power is proportional to the second moment of the V oronoi region of \( \mathcal{L}(\mathbf{B}) \), and \( \sigma^2(\mathcal{V}) \) is the second moment of \( \mathcal{V} \). Then, the SNR gap is asymptotically given by

\[
\rho = \left\{ \begin{array}{ll} \\
\frac{\|\mathbf{B}\|^2}{12 \sigma^2(\mathcal{V})}, & \text{for ZF;} \\
\frac{\|\hat{\mathbf{B}}\|^2}{12 \sigma^2(\mathcal{V})}, & \text{for SIC.} \\
\end{array} \right. \tag{14}
\]

Unfortunately, it is difficult to compute \( \rho \), and we resort to the proximity factors of LLL reduction-aided precoding, which measure the worst-case loss in power efficiency relative to sphere precoding. More formally, we define the proximity factor as

\[
F_p \triangleq \left\{ \begin{array}{ll} \\
\sup_{\|\mathbf{x}\| = 1} \frac{\|\mathbf{x}\|_2}{\|\mathbf{x}_{\mathcal{V}}\|_2}, & \text{for ZF;} \\
\sup_{\|\mathbf{x}\| = 1} \frac{\|\mathbf{x}\|_2}{\|\mathbf{x}_{\mathcal{V}}\|_2}, & \text{for SIC.} \\
\end{array} \right. \tag{15}
\]

Obviously, \( \rho \leq F_p \). This viewpoint implies that the precoding problem is \( \eta \)-CVP:

\[
\|\mathbf{B}(\mathbf{x} - \mathbf{A} \hat{l})\| \leq \eta \min_{\mathbf{y} \in \mathcal{V}} \|\mathbf{B}(\mathbf{x} - \mathbf{A} \mathbf{y})\|, \tag{16}
\]

and consequently, \( F_p \leq \eta^2 \). Babai derived the value of \( \eta \) in the case of \( \delta = 3/4 \). In what follows, we will derive the bounds in the general case. Let \( \alpha = 1/(\delta - 1/4) \).

**Lemma 1:** If the lattice basis is LLL-reduced, then SIC solves \( \eta \)-CVP for \( \eta = \eta_n = \alpha^{n/2}/\sqrt{\alpha - 1} \).

**Proof:** Let \( \mathbf{B} \) be a LLL reduced basis and \( \hat{\mathbf{B}} = \hat{\mathbf{B}} \mu^T \) be the GSO of the basis \( \mathbf{B} \). Given a vector \( \mathbf{y} \in \mathbb{R}^m \), we write \( \mathbf{y} \) as a linear combination of the GS vectors \( \mathbf{y} = \sum_{i=1}^{n} \beta_i \hat{\mathbf{b}}_i \). Let \( \mathbf{u} = \sum_{i=1}^{n} \beta_i \hat{\mathbf{b}}_i \) be the nearest neighbor of \( \mathbf{y} \) in \( \mathcal{L}(\mathbf{B}) \). Let \( \theta \) be the integer nearest to \( \beta_n \) and \( \mathbf{y}' = \sum_{i=1}^{n-1} \beta_i \hat{\mathbf{b}}_i + \theta \hat{\mathbf{b}}_n \), and \( \mathbf{v} = \theta \hat{\mathbf{b}}_n \). For \( n = 1 \), SIC can find the closest vector \( \mathbf{u} \). For \( n \geq 2 \), we have

\[
\|\mathbf{y} - \mathbf{y}'\| = \|\theta - \beta_n\| \|\hat{\mathbf{b}}_n\| \leq \frac{\|\hat{\mathbf{b}}_n\|}{2}, \tag{17}
\]

\[1\text{In practice, power normalization is applied at the transmitter in vector perturbation [3], yet such a scaling factor has no impact on the "coding gain".}
We have spanned by the other Since If solves By (18) and (21),

Let \( w \) be the estimate of \( u \) found by SIC. From (17), we obtain

\[
\|y - w\|^2 \leq \frac{1}{4} \sum_{i=1}^{n} \|\hat{b}_i\|^2 .
\]

According to (3), we have

\[
\|y - w\| \leq \frac{1}{2} \sqrt{\frac{\alpha^{n-1}}{\alpha - 1}} \|\hat{b}_n\| .
\]

If \( p_n = \theta \), then

\[
\|y' - w\| \leq \eta_{n-1} \|y' - u\| \leq \eta_{n-1} \|y - u\| .
\]

By (18) and (21),

\[
\|y - w\| = \left( \|y - y'\|^2 + \|y' - w\|^2 \right)^{1/2} \leq (1 + \eta_n^{-2}) \|y - u\| < \eta_n \|y - u\| .
\]

If \( p_n \neq \theta \), then

\[
\|y - u\| \geq \frac{\|\hat{b}_n\|}{2} .
\]

Combining this inequality with (20), we obtain

\[
\|y - w\| \leq \frac{\alpha^{n-1}}{\alpha - 1} \|y - u\| .
\]

For \( \alpha = 2 \), \( \eta_n = \alpha^{n/2}/\sqrt{\alpha - 1} \) reduces to Babai’s upper bound 2\(n/2\).

**Lemma 2:** If the lattice basis is LLL-reduced, then ZF solves \( n \)-CVP with \( \eta = \eta_n = 1 + 2n (3\sqrt{\alpha}/2)^{n-1} \).

**Proof:** Let \( B \) be a LLL-reduced basis. Let \( \theta_i \) be the angle between \( b_i \) and the linear space \( S(\{b_1,...,b_{i-1},b_{i+1},...,b_n\}) \) spanned by the other \( n-1 \) basis vectors. Recall the following bound (11)

\[
\sin \theta_i \geq \left( \frac{2}{3\sqrt{\alpha}} \right)^{n-1} .
\]

Since

\[
\sin \theta_i = \min_{\mathbf{m} \in S} \frac{\|\mathbf{m} - \mathbf{b}_i\|}{\|\mathbf{b}_i\|},
\]

we have

\[
\|\mathbf{m} - \mathbf{b}_i\| \geq \left( \frac{2}{3\sqrt{\alpha}} \right)^{n-1} \|\mathbf{b}_i\| , \ \forall \mathbf{m} \in S.
\]

Let \( w \) be the lattice point found by ZF. Then

\[
w - y = \sum_{i=1}^{n} \beta_i b_i ,
\]

where \( |\beta_i| \leq 1/2 \), for \( 1 \leq i \leq n \). Let \( u \) be the nearest neighbor of \( y \) in \( L(B) \). We may write

\[
u - w = \sum_{i=1}^{n} \phi_i b_i ,
\]

where \( \phi_i \in \mathbb{Z} \). We assume \( u \neq w \). Let \( \|\phi_k b_k\| = \max_i \|\phi_i b_i\| \). Then

\[
\|u - w\| \leq n \|\phi_k b_k\| .
\]

Meanwhile,

\[
u - y = (u - w) + (w - y) = (\phi_k + \beta_k) (b_k - m) ,
\]

where

\[
m = -\frac{1}{\phi_k + \beta_k} \sum_{j \neq k} (\phi_j + \beta_j) b_j .
\]

By (26) and \( |\beta_i| \leq 1/2 \), we have

\[
\|u - y\| \geq \frac{\|\phi_k\|}{2 (3\sqrt{\alpha}/2)^{n-1}} \|b_k\| .
\]

Combining (29) and (20), we have

\[
\|u - w\| \leq n \|\phi_k b_k\| \leq 2n (3\sqrt{\alpha}/2)^{n-1} \|u - y\| .
\]

It is easy to see that

\[
\|y - w\| \leq \|y - u\| + \|u - w\| \leq \left( 1 + 2n (3\sqrt{\alpha}/2)^{n-1} \right) \|y - u\| .
\]

For \( \alpha = 2 \), we have \( \eta_n = 1 + 2n (9/2)^{n-1} \). From the two lemmas, we have the following theorem for the proximity factors:

**Theorem 1:** According to Lemmas 1 and 2 we have

\[
F_{P,SIC} \leq \frac{\alpha^n}{\alpha - 1} ,
\]

and

\[
F_{P,ZF} \leq \left( 1 + 2n (3\sqrt{\alpha}/2)^{n-1} \right)^2 .
\]

These results show that the worst-case loss in power efficiency of approximate lattice precoders is bounded above by a function of the dimension of the lattice alone.
IV. DISCUSSION

Our main contribution in this paper was to view the LR-aided precoding problem as $\eta$-CVP, compared to the viewpoint of bounded distance decoding for LR-aided decoding \[11\]. This viewpoint allowed us to derive the proximity factors, which measure the worst-cased bound for approximate lattice precoding. Since the underlying lattice is infinite, this analysis is rigorous, and it follows that LR-aided precoding also achieves full diversity. The derived bounds may not be tight, but nonetheless give more insights. Improving the bounds is the future work.

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