Abstract: In this paper, we present a new, network flow LP model of the standard Assignment Problem (AP) polytope. The model is not meant to be competitive with existing standard procedures for solving the AP, as its complexity order of size is $O(m^9)$, where $m$ is the number of assignments. However, it allows for hard combinatorial optimization problems (COPs) to be solved as Assignment Problems (APs), including, in particular, the Quadratic, Cubic, Quartic, Quintic, and Sextic Assignment Problems, as well as the Traveling Salesman Problem and many of its variations. Hence, in particular, the model re-affirms “$P = NP$.” Illustrations are provided for the Linear Assignment (LAP), Quadratic Assignment (QAP), and Traveling Salesman (TSP) problems. Issues pertaining to the extended formulations “barriers” for the LP modeling of hard COPs are not discussed in this paper because the developments are focused on the Assignment Problem polytope only, and also the applicability/non-applicability of those “barriers” are thoroughly addressed in a separate paper in which it is shown that, in an optimization context, these “barriers” have no pertinence for a model which projects to the AP polytope, provided appropriate costs can be attached to the non-superfluous variables of the model. Hence, the issues of the “barriers” are left out of this paper essentially for the sake of space.

†: Diaby, M., M. Karwan, and L. Sun [2024]. On modeling NP-Complete problems as polynomial-sized linear programs: Escaping/Side-stepping the “barriers.” Available at: https://arxiv.org/abs/2304.07716 [cc.CC].

Keywords: Assignment Problem; Linear Programming; Quadratic Assignment Problem; Traveling Salesman Problem.

1 Introduction

The Assignment Problem is one of the most basic problems in Operations Research and Mathematical Programming. The problem is concerned with the assignments of objects of one class (say, workers) to objects of another class (say, tasks) in such a way that every object of either class is matched exactly once. The assignment of a pair of objects to each
other incurs a cost or a profit. The optimization problem is to find a “full assignment” which has minimum cost or maximum profit. The problem has a very broad range of applicability in industry in general. It is also one of the most “well-solved” problems of Operations Research, as very efficient (low-degree polynomial-time) solution algorithms have been known for it for some time, starting with the classical Hungarian/Kuhn-Munkres algorithm (Kuhn [1955]; Munkres [1957]). The problem is a fundamental problem for Mathematics in general also because of its connection to the Birkhoff polytope and permutation matrices (Birkhoff [1946]). Moreover, many of the well-known hard combinatorial optimization problems are essentially Assignment Problems (APs) with alternate objective cost functions. Good, extensive treatments of the problem can be found in (Burkhard [2009]; Bazaraa et al. [2010; pp. 535-550]), among others.

The model presented in this paper is not meant to be competitive with existing standard procedures for solving the Assignment Problem. Instead, it is a reformulation of the Assignment Problem polytope in a higher-dimensional space which allows for other COPs (including many of the hard ones, in particular) to be also directly solved as Assignment Problems (in that higher-dimensional space). The basic ideas of the modeling date back to our seminal models of the late 2000’s (Diaby [2006; 2007]). We are not aware of any counter-examples to these models or of any direct claims (i.e., based on the models themselves) against them. Hence, our efforts since those initial developments had been almost exclusively focused on attempts to develop lower-dimensional equivalents for them (the only exception being Diaby and Karwan [2016]) and at clarifying the fact that the extended formulations “barriers” are not applicable to them. Unfortunately, none of our “size reductions” efforts resulted in a correct model. However, the non-applicability of extended formulations “barriers” to the modeling approach has been thoroughly and clearly established in our most recent paper on this topic (see Diaby et al. [2024]).

The completeness of insight we gained through all of our unsuccessful attempts at “size reduction” is what has lead to the broadened perspective on the familiar network flow modeling paradigm developed in this paper, and the clear positioning of the approach within a well-established, conventional Operations Research paradigm. The computational complexity bound on the size of the model is $O(m^9)$ (where $m$ is the number of assignments). However the model has smaller numbers of variables and constraints, respectively, than the basic ones it draws from (specifically, Diaby [2006; 2007], and Diaby and Karwan [2016]). In addition to the classic Assignment Problem (i.e., the Linear Assignment Problem), in particular, the proposed model allows for the solution of the Quadratic, Cubic, Quartic, Quintic, and Sextic Assignment Problems, as well as many of the other hard COPs (including the Traveling Salesman Problem and many of its variations) as polynomial-sized LPs. Hence, an important consequence of the modeling approach is the clear, unequivocal re-affirmation of “$P = NP$.” Illustrations of the cost “perturbations”/associations which need to be undertaken in relation to our higher-dimensional modeling variables are provided for the Linear Assignment (LAP), Quadratic Assignment (QAP), and Traveling Salesman (TSP) problems. Issues pertaining to the extended formulations “barriers” to modeling hard COPs as LPs (such as in Fiorini et al. [2015], for example) are not discussed in this paper for two reasons. One is that the developments in the paper are focused on the Assignment Problem polytope only. The other is that the applicability/non-applicability of the “barriers” are
fully addressed in a separate paper (Diaby et al. [2024]) in which it is shown that in an optimization context, these “barriers” have no pertinence for a model which projects to the AP polytope, provided appropriate costs can be attached to the non-superfluous variables of the model. In essence, in an optimization context, these “barriers” are valid for “component projections” only, after all the redundant/superfluous variables and constraints have been removed from a model (see Diaby et al. [2024]). Hence, the issues of the “barriers” are left out of this paper for the sake of space.

The plan of this paper is as follows. In section 2, we discuss our higher-dimensional network flow abstraction of Assignment solutions. The proposed LP reformulation is described in section 3. The structure and integrality of the model are developed and shown in section 4. Some illustrative examples of how costs can be “perturbed” and attached to the modeling variables in order to solve hard COPs (in particular) are provided in section 5. Finally, some concluding remarks are discussed in section 6.

2 Flow representation of Assignment solutions

The graph which underlies the modeling in this paper is illustrated in Figure 1. It is essentially a graphical matrix/“tableau” form representation of the Assignment Problem (AP; Burkhard [2009]; Bazarraa et al. [2010; pp. 535-550]). However, it does not have isolated nodes. In the Assignment Problem, objects of one class must be assigned to objects of another class. For example, one of the classes of “objects” may be workers (W), and the other, tasks (T). A node of the graph pairs two objects, one from each class, (for example, (w, t) ∈ (W, T)), to represent their assignment to each other. Hence, each row of nodes of the graph represents all the possible pairings for an object from one of the classes (for example, a worker), while each column of nodes represents all the possible pairings for an object from the other class (for example, a task). We generically refer to a row of nodes as a “level” of the graph, and to a column of nodes, as a “stage” of the graph.

Each arc of the graph in our modeling links nodes involving different levels at consecutive stages of the graph, with the tail node having the lower stage index. Hence, our modeling graph is directed, and multipartite. For convenience, given their centrality in our exposition, we use the special notation “⟨·⟩” in representing the arcs of the graph. Specifically, a given arc ((u,p), (v,p+1)) of the graph will be represented by “⟨u,p,v⟩” throughout the remainder of this paper.

We assume that the Assignment Problem is balanced (i.e., that the two sets of objects being matched have the same cardinality). This causes no loss of generality, since fictitious stages (or levels) can be added to the graph as needed in order to compensate for a deficit of stages (or levels), with costs of zero attached to them. Also, no node or arc is excluded from our graph. This also does not cause any loss of generality, since assignments which are prohibited in a given context can be handled in a linear optimization model by associating large (“Big-M”) costs to them.

We refer to our modeling graph as the “Multipartite Assignment Problem Graph (MAPG).” A formal statement of the graph and the path structures of it which underlie our modeling will be discussed in the remainder of this section.
Notation 1 (General Notation)

1. \( m \) : Number of assignments to be made.

2. \( \mathcal{M}_m \) : Set of “full assignments”/bipartite matchings/Assignment solutions of an \( m \)-Assignment Problem (\( m \)-AP).

3. \( \mathcal{AP}_m \) : \( m \)-Assignment polytope (Polytope which has the member of \( \mathcal{M}_m \) as its extreme points.)

4. \( L := \{1, \ldots, m\} \) (Index set for the levels of the MAPG).

5. \( S := \{1, \ldots, m\} \) (Index set for the stages of the MAPG).

6. \( N := \{(l, s) \in (L, S)\} \) (Set of nodes of the MAPG).

7. \( A := \{(u, p, v) : \nexists (u, p), (v, p + 1) \in N^2, (\forall p \in S \setminus \{m\}), (\forall (u, v) \in L^2 (u \neq v))\} \) (Set of arcs of the MAPG).

8. \( B^t \) : Transpose of matrix \( B \).

9. \( \text{Ext}(B) \) : Set of extreme points of polyhedron \( B \).

10. \( \mathbb{R}_{\geq} \) : Set of nonnegative real numbers.

11. \( \mathbb{N} \) : Set of natural numbers (excluding “0”).

Definition 2 (“Arc Separation”) (\( \forall p, q \in S (p \neq q) \) (\( \forall (i_p, p, i_{p+1}) \), \( (i_q, q, i_{q+1}) \) \( \in A \) :
Definition 3 ("Graph-Path") $(\forall g, q \in S \ (g + 1 < q < m))$:

1. A set of arcs $B := \{(i_p, p, i_{p+1}) \in A, (\forall p = g, \ldots, q\}$ is called a "graph-path (of the MAPG) (between $\langle i_g, g, i_{g+1} \rangle$ and $\langle i_q, q, i_{q+1} \rangle$)" if it has no repeating head or tail of the arcs.

Specifically, $B$ is a graph-path (of the MAPG) between $\langle i_g, g, i_{g+1} \rangle$ and $\langle i_q, q, i_{q+1} \rangle$ if

$$\forall r, s \in \{g, \ldots, q\} \ (r < s)$$

$$[(s = r + 1 \implies (i_s = i_{r+1} \land i_r \neq i_{r+1} \neq i_{s+1})) \land$$

$$(s > r + 1 \implies i_r \neq i_{r+1} \neq i_s \neq i_{s+1})].$$

2. The notation $gPath(B, \langle i_g, g, i_{g+1} \rangle, \langle i_q, q, i_{q+1} \rangle)$ is used to say “$B$ is a graph-path (of the MAPG) between $\langle i_g, g, i_{g+1} \rangle$ and $\langle i_q, q, i_{q+1} \rangle$.”

3. A graph-path which starts at an arc at stage $g = 1$ and ends at an arc at stage $q = m - 1$ is referred to as a "spanning-graph-path."

4. The notation “$sgPath(B, (i_1, i_2, i_{m-1}, i_m))$" stands for “$B$ is a spanning-graph-path between $\langle i_1, 1, i_2 \rangle$ and $\langle i_{m-1}, m - 1, i_m \rangle$.”

5. The set of all the spanning-graph-paths of the MAPG is denoted as $\Gamma_m$ and explicitly expressed as

$$\Gamma_m := \{G_{i_1,i_2,i_{m-1},i_m}^\gamma \in A^{m-1} : (sgPath(G_{i_1,i_2,i_{m-1},i_m}^\gamma, (i_1, i_2, i_{m-1}, i_m)),$$

$$(\forall i_1, i_2, i_{m-1}, i_m \in L \ (i_1 \neq i_2 \neq i_{m-1} \neq i_m)), (\forall \gamma_{i_1,i_2,i_{m-1},i_m} \in \{1, \ldots, (m - 4)\})\}.$$ 

The significance of the graph-paths is shown in the following theorem.

**Theorem 4 (Graph Paths $\iff$ AP Solutions)** The mapping $f_1 : \Gamma_m \to \mathcal{M}_m$ is bijective.

**Proof.** First, the number of quadruplets $(i_1, i_2, i_{m-1}, i_m) \in L^4$ with pairwise-distinct members is $(m \cdot (m - 1) \cdot (m - 2) \cdot (m - 3))$. The number of permutations of the members of the remaining $L \setminus \{i_1, i_2, i_{m-1}, i_m\}$ set of levels is $(m - 4)!$. Hence, $|\Gamma_m| = m \cdot (m - 1) \cdot (m - 2) \cdot (m - 3) \cdot (m - 4)! = m!$.

Second, it is a well-known result that the number of full assignments of an $m$-AP is $|\mathcal{M}_m| = m!$.

Hence, thirdly, we have that $|\Gamma_m| = |\mathcal{M}_m| = m!$, and the theorem follows from this directly. $\blacksquare$
Our modeling consists of developing a set of constraints which induces spanning-graph-paths over the MAPG.

3  \( O(m^9) \) network flow model of the AP polytope

We use a minimum-cost network flow modeling framework (see Bazaraa et al. [2010; pp. 453-512], among others) to formulate a higher-dimensional linear program (LP) model of the Assignment Problem (AP; Burkhard [2009]; Bazaraa et al. [2010; pp. 535-550]) polytope. Our modeling variables respectively involve multiple arcs of its underlying graph (i.e., the MAPG). Our Kirchhoff Equations (KEs)/“flow-balance”/“mass-conservation” constraints are “complex” in that they are parametrized in terms of the arcs which index the specific variables involved in them respectively. We refer to these as “Generalized Kirchhoff Equations (GKEs).” They induce a structure of differently-labeled, super-imposed, but non-separable/non-independent layers of flows through the flow graph (MAPG). By being layered, these flows are akin to “commodity flows” in a multicommodity flow context. On the other hand, however, their not being separable makes these flows unlike “commodity flows” also.

Other constraints of our model which also enforce flow-balance/mass-conservation requirements across the stages of the MAPG pertain to what are essentially “boundary conditions,” in as much as every pair of arcs in the modeling is a potential source. We refer to these as the “Flow Consistency Constraints.” A third class of constraints serve to enforce flow-balance/mass-conservation conditions across the levels of the flow graph, MAPG. We refer to these as the “Visit Requirements Constraints.” Finally, there is a constraint which initiates a unit flow at the first stage of the flow graph, and there is a class of constraints which serve to preclude implicitly-zero (logically-zero) variables from being considered in the Assignment decision-making, as required for flow connectedness or for flow not to “re-visit” a level.

3.1 Model variables

Notation 5 (Modeling variables) \((\forall \langle i_g, g, j_g \rangle, \langle i_p, p, j_p \rangle, \langle i_q, q, j_q \rangle \in A) : \)

1. \( x_{(i_g,g,j_g)(i_p,p,j_p)(i_q,q,j_q)} \) : Variable indicating the simultaneous assignments of levels \( i_g, j_g, i_p, j_p, i_q, \) and \( j_q \) to stages \( g, g + 1, p, p + 1, q, \) and \( q + 1, \) respectively.

2. \( x((\langle i_g, g, j_g \rangle, \langle i_p, p, j_p \rangle, \langle i_q, q, j_q \rangle) : \) Function that returns the \( x \)-variable with the arc indices arranged in increasing order of the stage indices. Specifically,

\[
\begin{align*}
x((\langle i_g, g, j_g \rangle, \langle i_p, p, j_p \rangle, \langle i_q, q, j_q \rangle)) :=
\begin{cases}
x_{(i_g,g,j_g)(i_p,p,j_p)(i_q,q,j_q)} & \text{if } g < p < q; \\
x_{(i_g,g,j_g)(i_q,q,j_q)(i_p,p,j_p)} & \text{if } g < q < p; \\
x_{(i_p,p,j_p)(i_g,g,j_g)(i_q,q,j_q)} & \text{if } p < g < q; \\
x_{(i_p,p,j_p)(i_q,q,j_q)(i_g,g,j_g)} & \text{if } p < q < g; \\
x_{(i_q,q,j_q)(i_g,g,j_g)(i_p,p,j_p)} & \text{if } q < g < p; \\
x_{(i_q,q,j_q)(i_p,p,j_p)(i_g,g,j_g)} & \text{if } q < p < g; \\
0 & \text{otherwise.}
\end{cases}
\end{align*}
\]
\((\pi(\{\cdot\})\) is used for the purpose of simplifying the exposition only.\)

We interpret variable \(x_{\langle i_g, g, j_g \rangle \langle i_p, p, j_p \rangle \langle i_q, q, j_q \rangle}\) as the amount of flow that traverses all three of the arcs \(\langle i_g, g, j_g \rangle\), \(\langle i_p, p, j_p \rangle\), and \(\langle i_q, q, j_q \rangle\) jointly, and we refer to it as the “joint-flow” of the three arcs. This notion is further elaborated on in the definition below.

**Definition 6 (“Joint-Flow”)**

1. The joint-flow of three arcs, \(\langle i_g, g, j_g \rangle\), \(\langle i_p, p, j_p \rangle\) and \(\langle i_q, q, j_q \rangle\) of the MAPG, is the value of \(x_{\langle i_g, g, j_g \rangle \langle i_p, p, j_p \rangle \langle i_q, q, j_q \rangle}\). (The domain of \(x\) is specified in section 3.2).
2. Two arcs \(\langle i_g, g, j_g \rangle\) and \(\langle i_p, p, j_p \rangle\) are said to “have joint-flow” in a feasible solution \(x\) to our model (described in section 3.2) if there exists a third arc \(\langle i_q, q, j_q \rangle\) such that \(\pi(\{\langle i_g, g, j_g \rangle, \langle i_p, p, j_p \rangle, \langle i_q, q, j_q \rangle\})\) is positive.
3. The joint-flow of two arcs, \(\langle i_g, g, j_g \rangle\) and \(\langle i_p, p, j_p \rangle\), is the sum of the values of all the variables which are indexed by the two arcs and one additional arc (respectively) at a given stage of the MAPG. (The enforcement of the consistency of this joint-flow is discussed in section 3.2 below.)
4. The notation “\(\text{jointFlow}(x, \langle i_g, g, j_g \rangle, \langle i_p, p, j_p \rangle)\)” stands for “Arcs \(\langle i_g, g, j_g \rangle\) and \(\langle i_p, p, j_p \rangle\) have (positive) joint-flow in \(x\).”

**3.2 Model constraints**

As discussed above, our proposed LP is essentially a min-cost network flow model. However, it is based on a more complex notion of “flow.” One unit of this “complex flow” is initiated into the first stage of our flow graph with different “labels” effected to it over the first four stages of the MAPG by an “Initial Flow” constraint. These “labeled flows” are respectively propagated throughout the MAPG in a connected manner and with additional labels progressively attached to them through our parametrized Kirchhoff Equations (i.e., the GKEs). Hence, our proposed formulation captures the essence of a shortest path model. There is also a stipulation in the model (the “Visit Requirements” constraints) that flow “visit”/propagate to each level of the graph equally. With this “cross-level” balancing/conservation stipulation, the model also captures the essence of an Assignment Problem formulation (since the underlying graph is basically a graphical representation of an Assignment tableau and the GKEs also constrain flow to “visit” each stage equally).

**3.2.1 Statement of the constraints**

The constraints of our model are as follows.

- **Initial Flow (IF) constraints.**

\[
\sum_{i_1=1}^{m} \sum_{j_1=1}^{m} \sum_{j_2=1}^{m} \sum_{j_3=1}^{m} x_{\langle i_1, 1, j_1 \rangle \langle j_1, 2, j_2 \rangle \langle j_2, 3, j_3 \rangle} = 1.
\] (1)
• Generalized Kirchhoff Equations (GKEs)

\[
\sum_{k=1}^{m} \gamma \left( \{ \langle i_g, g, j_g \rangle, \langle i_q, q, j_q \rangle, \langle i_p, p, k \rangle \} \right) - \\
\sum_{k=1}^{m} \gamma \left( \{ \langle i_g, g, j_g \rangle, \langle i_q, q, j_q \rangle, \langle k, p-1, i_p \rangle \} \right) = 0; \quad (\forall g, p, q \in S \\
((g < q < m) \land (p \neq g, g + 1, q, q + 1, m))) \quad (\forall i_g, j_g, i_q, j_q, i_p \in L).
\] (2)

• Flow Consistency (FC) constraints.

\[
\sum_{k=1}^{m} \sum_{t=1}^{m} \gamma \left( \{ \langle i_g, g, j_g \rangle, \langle i_q, q, j_q \rangle, \langle k, p, t \rangle \} \right) - \\
\sum_{k=1}^{m} \sum_{t=1}^{m} \gamma \left( \{ \langle i_g, g, j_g \rangle, \langle i_q, q, j_q \rangle, \langle k, p + \Delta^S_{gq}(p), t \rangle \} \right) = 0; \\
(\forall g, p, q \in S ((g < q < m) \land (p < m - \Delta^S_{gq}(p)) \land (p \neq g, g + 1, q, q + 1))), \\
(\forall i_g, j_g, i_q, j_q \in L). \quad (\text{Where } \Delta^S_{gq}(p) := \min_{\delta \in \{1, \ldots, m-p\}} \{ \delta : (p + \delta) \notin \{g, q\} \}).
\] (3)

• Visit Requirements (VR) constraints.

\[
g^{g-1} \sum_{r=1}^{m} \sum_{k=1}^{m} \gamma \left( \{ \langle i_g, g, j_g \rangle, \langle i_q, q, j_q \rangle, \langle u, r, k \rangle \} \right) + \\
\sum_{r=g+2}^{m} \sum_{k=1}^{m} \gamma \left( \{ \langle i_g, g, j_g \rangle, \langle i_q, q, j_q \rangle, \langle k, r-1, u \rangle \} \right) - \\
g^{g-1} \sum_{r=1}^{m} \sum_{k=1}^{m} \gamma \left( \{ \langle i_g, g, j_g \rangle, \langle i_q, q, j_q \rangle, \langle (u + \Delta^L_{i_g,j_g,i_q,j_q}(u)), r, k \rangle \} \right) - \\
\sum_{r=g+2}^{m} \sum_{k=1}^{m} \gamma \left( \{ \langle i_g, g, j_g \rangle, \langle i_q, q, j_q \rangle, \langle k, r-1, (u + \Delta^L_{i_g,j_g,i_q,j_q}(u)) \rangle \} \right) = 0; \\
(\forall g, q \in S (g < q < m)), \quad (\forall i_g, j_g, i_q, j_q, u \in L \\
((u \leq m - \Delta^L_{i_g,j_g,i_q,j_q}(u)) \land (u \neq i_g, j_g, i_q, j_q))). \\
(\text{Where } \Delta^L_{i_g,j_g,i_q,j_q}(u) := \min_{\delta \in \{1, \ldots, m-u+1\}} \{ \delta : (\delta + u) \notin \{i_g, j_g, i_q, j_q\} \}).
\] (4)
• “Implicit-Zeros (IZ)” constraints.

\[
x_{(i_g,g,j_g)(i_p,p,j_p)(i_q,q,j_q)} = 0 \text{ if } \neg(g < p < q) \lor \neg(i_g \neq j_g, i_p, j_p, i_q, j_q) \lor \\
\neg(j_g \neq j_p, i_q, j_q) \lor \neg(i_p \neq j_p, i_q, j_q) \lor \neg(j_p \neq j_q) \lor \\
\neg(i_q \neq j_q) \lor \neg((p > g + 1) \land (i_p \neq j_g)) \lor \neg((q > p + 1) \land (i_q \neq j_p)) \lor \\
\neg((p = g + 1) \land (i_p = j_g)) \lor \neg((q = p + 1) \land (i_q = j_p))\].
\] (5)

• Nonnegativity (NN) constraints.

\[
x_{(i_g,g,j_g)(i_p,p,j_p)(i_q,q,j_q)} \geq 0; \quad (\forall i_g, g, j_g, i_p, p, j_p, i_q, q, j_q \in \{1, \ldots, m\}).
\] (6)

One unit of flow is initiated and “labeled” with arcs at the first four stages of our flow graph (MAPG) by constraint (1). This flow is propagated through the stages of the graph by constraints (2), which are a parametrized form of the standard “mass/joint-flow balance” equations known as the Kirchhoff Equations. We refer to these constraints as the Generalized Kirchhoff Equations (GKEs). They stipulate that the joint-flow of two given arcs which enters a node must be equal to the joint-flow of the two arcs which leaves the node. In constraints (3), \((p + \Delta^{S}_{gq}(p))\) is the the index of the first stage after \(p\) which is distinct from stages \(g\) and \(q\) respectively. Hence, these constraints stipulate that the joint-flow that traverses any stage of the MAPG must be the same for all the stages of the MAPG. These constraints are non-redundant only for boundary joint-flows conditions only. They essentially ensure that the joint-flow of two given nodes is balanced across either of those nodes. In Visit Requirements constraints (4), \((u + \Delta^{L}_{i_g,j_g,i_q,j_q}(u))\) is the the index of the first level after \(u\) which is distinct from levels \(i_g, j_g, i_q, j_q\), respectively. Hence, these constraints stipulate that the joint-flow of two given arcs which propagates to a given level of the MAPG is the same for all of the levels of the MAPG. Hence, these constraints ensure the “mass/joint-flow balance” conditions across the levels of the MAPG. The IZ Constraints (5) ensure that joint-flow is not broken and does not “re-vist” at the level of the individual \(x\)-variables. Finally, Nonnegativity constraints (6) are the usual nonnegativity constraints on the modeling variables.

Notation 7 (Polytope \(Q\); Set \(Q_I\))

1. \(Q := \{ x \in \mathbb{R}^m : x \text{ satisfies (1) - (6)} \} \) (“the LP polytope”).

2. \(Q_I := \{ x \in Q : x \text{ is integral} \} \) (Set of the integral points of the LP polytope).

Lemma 8 (Valid Constraints for \(Q\)) The following constraints are valid for the LP poly-
tope, \( Q:\)

\[
\begin{align*}
&\sum_{k=1}^{m} \sum_{t=1}^{m} x_{i_g,g,j_g}^{(k,p,t)}(i_q,q,j_q) - \sum_{k=1}^{m} \sum_{t=1}^{m} x_{i_g,g,j_g}^{(k,r,t)}(i_q,q,j_q) = 0; \\
&(\forall g, p, q, r \in S ((q < m) \land (g < p < q) \land (g < r < q))), (\forall i_g, j_g, i_q, j_q \in L). \quad (7)
\end{align*}
\]

\[
\begin{align*}
&\sum_{k=1}^{m} x_{i_g,g,j_g}^{(i_q,q,j_q)} - \sum_{k=1}^{m} x_{i_g,g,j_g}^{(i_q,q,j_q)} = 0; \\
&(\forall g, q \in S (g + 1 < q < m)), (\forall i_g, j_g, i_q, j_q \in L). \quad (8)
\end{align*}
\]

\[
\begin{align*}
&\sum_{k=1}^{m} \sum_{i_p=1}^{m} \sum_{j_p=1}^{m} \sum_{i_q=1}^{m} \sum_{j_q=1}^{m} x_{i_g,g,j_g}^{(i_p,p,j_p)}(i_q,q,j_q) - \sum_{k=1}^{m} \sum_{i_h=1}^{m} \sum_{j_h=1}^{m} \sum_{i_q=1}^{m} \sum_{j_q=1}^{m} x_{i_h,h,j_h}^{(i_r,r,j_r)}(i_s,s,j_s) = 0; \\
&(\forall g, p, q, h, r, s \in S ((g < p < q < m) \land (h < r < s < m))). \quad (9)
\end{align*}
\]

\[
\begin{align*}
&\sum_{k=1}^{m} \sum_{i_p=1}^{m} \sum_{j_p=1}^{m} \sum_{i_q=1}^{m} \sum_{j_q=1}^{m} x_{i_g,g,j_g}^{(i_p,p,j_p)}(i_q,q,j_q) = 1; \\
&(\forall g, p, q \in S (g < p < q < m)). \quad (10)
\end{align*}
\]

**Proof.**

1. *Flow-Consistency* constraints \([3]\) stipulate that the total of the joint-flow of two given arcs \(\langle i_g, g, j_g \rangle\) and \(\langle i_q, q, j_q \rangle\) which propagates through/traverses a stage of the MAPG is the same for all the stages of the MAPG. Hence, these constraints (constraints \([3]\)) can be equivalently expressed (although less parsimoniously) as

\[
\begin{align*}
&\sum_{k=1}^{m} \sum_{t=1}^{m} x(\{i_g, g, j_g\}, \{i_q, q, j_q\}, \{k, p, t\}) - \\
&\sum_{k=1}^{m} \sum_{t=1}^{m} x(\{i_g, g, j_g\}, \{i_q, q, j_q\}, \{k, r, t\}) = 0; \\
&(\forall g, q, p, r \in S ((g < q < m) \land (p, r \notin \{g, q\}) \land (p \neq r))), (\forall i_g, j_g, i_q, j_q \in L). \quad (11)
\end{align*}
\]

Constraints \([7]\) are a special case of constraints \([11]\) for the case in which “\( p \)” and “\( r \)”
are respectively strictly between “g” and “q.”

2. Constraints (8) are obtained by using Implicit-Zeros constraints (5) to exclude implicitly-zero variables from constraints (4) for the special case of \( p = g + 1 \) and \( r = q - 1 \).

3. Summing over all the levels involved in (11) and then using the associativity of addition to recursively re-group terms according to pairs of stages gives:

\[
\sum_{i_g=1}^{m} \sum_{j_g=1}^{m} \sum_{i_q=1}^{m} \sum_{j_q=1}^{m} \left( \sum_{i_h=1}^{m} \sum_{j_h=1}^{m} \mathcal{X}(\{i_g, g, j_g\}, \{i_q, q, j_q\}, \{i_h, h, j_h\}) \right)
\]

(Using (11))

\[
= \sum_{i_q=1}^{m} \sum_{j_q=1}^{m} \sum_{i_h=1}^{m} \sum_{j_h=1}^{m} \left( \sum_{i_g=1}^{m} \sum_{j_g=1}^{m} \mathcal{X}(\{i_q, q, j_q\}, \{i_h, h, j_h\}, \{i_g, g, j_g\}) \right)
\]

(Re-grouping)

4. From Implicit-Zeros constraints (5), we have

\[
(\forall g \in S \ (g < m - 2)) \ (\forall i_g, j_g, i_{g+1}, j_{g+1}, i_{g+2}, j_{g+2} \in L)
\]

\[
((i_{g+1} \neq j_{g+1}) \lor (i_{g+2} \neq j_{g+1}) \lor \neg(i_{g+2} \neq i_g, j_g, i_{g+1}, j_{g+1}, i_{g+2})) \implies \\
\mathcal{X}(i_g, g, j_g)(i_{g+1}, g, j_{g+1})(i_{g+2}, g, j_{g+2}) = 0.
\]

(Using (12))

Constrains (9) are a special case of the last in the sequence of the equalities above in which \( g < p < q < m \) and \( h < r < s < m \).

4. From Implicit-Zeros constraints (5), we have

\[
(\forall g \in S \ (g < m - 2)) \ (\forall i_g, j_g, i_{g+1}, j_{g+1}, i_{g+2}, j_{g+2} \in L)
\]

\[
\mathcal{X}(i_g, g, j_g)(i_{g+1}, g, j_{g+1})(i_{g+2}, g, j_{g+2}) = 0.
\]

Constraints (10) follows directly from the combination of (9) and (13).
Corollary 9 ($0 \leq x \leq 1$) It follows directly from constraints (10) (of Lemma 8) and Non-negative constraints (6) that:

1. $x \in Q \implies x \in [0, 1]^m$.
2. $x \in Q_I \implies x \in \{0, 1\}^m$.

4 Structure of the LP polytope

In this section, we will develop the structure of $Q$ and establish its integrality. Roughly, a pair of arcs that have joint-flow propagates flow that spans the stages (by GKE constraints (2)) and the levels (by Visit Requirements constraints (4)) of our underlying flow graph. This induces chains of arcs of the graph which can be grouped into sets, each of which is such that every triplet of its members indexes a positive flow ($x$-) variable. Sets in the collection thus created may be “overlapping”/non-disjoint. The existence and characterizations of this “overlapping-chains” structure of $Q$ are discussed in section 4.1 and used in section 4.2 in order to prove the integrality of $Q$.

4.1 “Overlapping-chains” structure

Theorem 10 (Structure of Integral Points) Every integral point $x$ of the LP polytope corresponds to a unique spanning-graph-path of the MAPG and is such that its positive components are indexed by triplets of arcs in this spanning-graph-path of the MAPG only.

Specifically,

$$(\forall x \in Q) [x \in Q_I \iff (\exists! (i^x_1, i^x_2, i^x_m) \in L^4) (\exists! G_{i^x_1, i^x_2, i^x_m} \in \Gamma_m)]$$

$$((\forall g, p, q \in S) (\forall (u_g, g, v_g), (u_p, p, v_p), (u_q, q, v_q) \in A)$$

$$[ (B_1 \implies x_{(u_g, g, v_g), (u_p, p, v_p), (u_q, q, v_q)} = 1) \wedge (B_2 \implies x_{(u_g, g, v_g), (u_p, p, v_p), (u_q, q, v_q)} = 0)])];$$

Where $B_1$: “$(g < p < q < m)$ \wedge ((u_g, g, v_g), (u_p, p, v_p), (u_q, q, v_q) \in G_{i^x_1, i^x_2, i^x_m})$$,$

and $B_2$: “$-(g < p < q < m) \vee -((u_g, g, v_g), (u_p, p, v_p), (u_q, q, v_q) \in G_{i^x_1, i^x_2, i^x_m})$$.$

Proof. ($\implies$) From Corollary 9

$$x \in Q_I \implies ((\forall g, p, q \in S (g < p < q < m))$$

$$(\forall i_g, i_{g+1}, i_p, i_{p+1}, i_q, i_{q+1} \in L) [x_{(i_g, g, i_{g+1}), (i_p, p, i_{p+1}), (i_q, q, i_{q+1})} \in \{0, 1\}]).$$
and Valid Constraints \[^{10}\] imply
\[
x \in Q_I \iff ((\forall g, p, q \in S \ (g < p < q < m)) \ (\exists i_g, i_{g+1}, i_p, i_{p+1}, i_q, i_{q+1} \in L^6) \ (\forall (j_g, j_{g+1}, j_p, j_{p+1}, j_q, j_{q+1}) \in L^6) \ [((j_g, j_{g+1}, j_p, j_{p+1}, j_q, j_{q+1}) = (i_g, i_{g+1}, i_p, i_{p+1}, i_q, i_{q+1}) \implies x_{(j_g, j_{g+1}, j_p, j_{p+1}, j_q, j_{q+1})} = 1) \land ((j_g, j_{g+1}, j_p, j_{p+1}, j_q, j_{q+1}) \neq (i_g, i_{g+1}, i_p, i_{p+1}, i_q, i_{q+1})) \implies x_{(j_g, j_{g+1}, j_p, j_{p+1}, j_q, j_{q+1})} = 0)]).
\]

By \[^{15}\] and Valid Constraints \[^{10}\], there is exactly one positive component of \(x \in Q_I\) for each ordered triplet from among \((m - 1)\) stages of the MAPG. In other words, denoting the number of positive components of \(x \in Q\) by “\(npc(x)\),” \[^{15}\] implies
\[
(\forall x \in Q_I) \ [npc(x) = \frac{(m - 1)!}{3!(m - 4)!}].
\]

By Implicit-Zeros constraints \[^{5}\], the \((i_g, i_{g+1}, i_p, i_{p+1}, i_q, i_{q+1})\)'s of \[^{15}\] are also subject to the following stipulations:

1. Case 1: Positive components of \(x\) which are such that the three arcs indexing them are at consecutive stages of the MAPG. We must have:
   \[
   (\forall g, p, q \in S \ ((g + 1 = p) \land (p + 1 = q < m))) \ [(i_p = i_{g+1}) \land (i_q = i_{p+1}) \land (i_g \neq i_{g+1}, i_p, i_{p+1}, i_q, i_{q+1}) \land (i_{g+1} \neq i_{p+1}, i_q, i_{q+1}) \land (i_p \neq i_{p+1}, i_q, i_{q+1}) \land (i_{p+1} \neq i_q, i_{q+1}) \land (i_q \neq i_{q+1})].
   \] (17)

2. Case 2: Positive components of \(x\) which are such that only the first two of the three arcs indexing them are at consecutive stages of the MAPG. We must have:
   \[
   (\forall g, p, q \in S \ ((g + 1 = p) \land (p + 1 < q < m))) \ [(i_p = i_{g+1}) \land (i_g \neq i_{g+1}, i_p, i_{p+1}, i_q, i_{q+1}) \land (i_{g+1} \neq i_{p+1}, i_q, i_{q+1}) \land (i_p \neq i_{p+1}, i_q, i_{q+1}) \land (i_q \neq i_{q+1})].
   \] (18)

3. Case 3: Positive components of \(x\) which are such that only the last two of the three arcs indexing them are at consecutive stages of the MAPG. We must have:
   \[
   (\forall g, p, q \in S \ ((g + 1 < p) \land (p + 1 = q < m))) \ [(i_q = i_{p+1}) \land (i_g \neq i_{g+1}, i_p, i_{p+1}, i_q, i_{q+1}) \land (i_{g+1} \neq i_p, i_{p+1}, i_q, i_{q+1}) \land (i_p \neq i_{p+1}, i_q, i_{q+1}) \land (i_{p+1} \neq i_q, i_{q+1}) \land (i_q \neq i_{q+1})].
   \] (19)

4. Case 4: Positive components of \(x\) which are such that the none of the three arcs
indexing them are at consecutive stages of the MAPG. We must have:

\[
(\forall g, p, q \in S \ ((g + 1 < p) \land (p + 1 < q < m))) \\
[(i_g \neq i_{g+1} \neq i_p \neq i_{p+1} \neq i_q \neq i_{q+1})].
\] (20)

and the connectivities stipulated in (17)-(19) imply that, in \( x \in Q_I \), there can be (only) exactly one arc at a given stage of the MAPG which can have joint-flow with arcs at other stages of the MAPG. Hence, denoting the cardinality of the set of arcs involved in the positive components of \( x \in Q \) by “npa(x),” the following is true:

\[
(\forall x \in Q_I) \ [npa(x) = m - 1].
\] (21)

One easily verifies that the unique set of arcs which satisfies (15)-(21) can be written as

\[
P_x := \{(i_r, r, i_{r+1}) \in A, \ (\forall r \in \{1, \ldots, m - 1\})\}.
\] (22)

By Graph-Path Definition 3, (16)-(21) imply that \( P_x \) is a spanning-graph-path of the MAPG. Moreover, one easily verifies that \( P_x \) is unique for satisfying (15)-(21) for a given \( x \in Q_I \), as we have discussed earlier in this proof.

\[\text{(} \iff \text{)}\] Assume \( x \in \mathbb{R}^m \) satisfies the existentially quantified formula of the theorem. Then, each component of \( x \) belongs to \{0, 1\}. Also, one easily verifies that \( x \) satisfies the constraints set (1)-(6) of \( Q \). Hence, such \( x \) is an integral point of \( Q \). In other words, we must have \( x \in Q_I \).

**Corollary 11 (Integral Points \( \leftrightarrow \) AP Solutions)**

1. \( f_3 : Q_I \rightarrow \Gamma_m \) is bijective. (Follows directly from Theorem 10)
2. \( f_4 : Q_I \rightarrow M_m \) is bijective. (Follows by transitivity from the combination of Part (1) above and Theorem 4)

**Theorem 12** \( Q_I \subseteq Ext(Q) \).

**Proof.** First, from Corollary 11.2, a given member of \( Q_I \) cannot be a convex combination of other members of \( Q_I \).

(13) and (15) imply that if \( x \in Q \setminus Q_I \), then there must exist at least one ordered triplet of stages of the MAPG which involves arcs indexing two or more positive components of \( x \). In other words, (13) and (15) imply

\[
(\forall x \in Q \setminus Q_I) \ (\exists g, p, q \in S \ (g < p < q < m)) \ (\exists \kappa \in \mathbb{N} \ (\kappa > 1)) \ (\forall k \in \{1, \ldots, \kappa\}) \\
(\exists (i^{k}_{g}, i^{k}_{g+1}, i^{k}_{p}, i^{k}_{p+1}, i^{k}_{q}, i^{k}_{q+1}) \in L^{k}) \ [x(i^{k}_{g}, i^{k}_{g+1}, i^{k}_{p}, i^{k}_{p+1}, i^{k}_{q}, i^{k}_{q+1}) > 0].
\] (23)

Hence, letting “npc(x)” denote the number of positive components of \( x \in Q \), (13) and (23) imply

\[
(\forall x \in Q \setminus Q_I) \ [npc(x) > \frac{(m - 1)!}{3!(m - 4)!}].
\] (24)
\[ \text{(24) and (16)} \] imply that a non-integral point of \( Q \) cannot have a positive weight in a convex combination of \( x \in Q_I \), since there must exist at least one zero entry of \( x \in Q_I \) which corresponds to a positive entry of such a convex combination representation.

Hence, in conclusion, there cannot exist a convex combination representation of \( x \in Q_I \) in terms of other points of \( Q \). Hence, every \( x \in Q_I \) must be an extreme point of \( Q \). \[ \square \]

We focus next on extending the result of Theorem 10 to other points of \( Q \). For this purpose, the following two key notions are needed.

**Definition 13 ("Joint-Flow-Cover")** Let \( x \in Q \). The "joint-flow-cover" of two given arcs having separation (Definition 2) greater than 0 and joint-flow in \( x \) (Definition 6) is defined as the set comprised of the two arcs and all the arcs between them that have joint-flow with them.

Letting \( JFC_x((i, g, i_{g+1}), (i, q, i_{q+1})) \) denote the joint-flow-cover of \( ((i, g, i_{g+1}), (i, q, i_{q+1})) \) \( \in \mathbb{A}^2 \) in \( x \in Q \), the definition is as follows:

\[
(\forall x \in Q) (\forall \delta \in \mathbb{N} (\delta < m - 2)) (\forall g, q \in S (g < q < m)) (\forall (i, g, i_{g+1}), (i, q, i_{q+1}) \in A
((SEP(\langle i, g, i_{g+1} \rangle, \langle i, q, i_{q+1} \rangle) = \delta) \land (jointFlow(x, \langle i, g, i_{g+1} \rangle, \langle i, q, i_{q+1} \rangle) \text{ TRUE}))),
\]

\[
JFC_x((i, g, i_{g+1}), (i, q, i_{q+1})) := \{((i, g, i_{g+1}), (i, q, i_{q+1})) \cup
\{u, v \in A : x(i, g, i_{g+1})(u, v) \langle i, q, i_{q+1} \rangle > 0, (\forall p \in [g + 1, q - 1], (\forall u, v \in L)\}
\]

**Definition 14 ("Induced-Path")** Let \( x \in Q \). Let \( g, q \in S (g < q < m) : \)

1. \( \forall (i, g, i_{g+1}), (i, q, i_{q+1}) \in A, \) a graph-path of the MAPG between \( \langle i, g, i_{g+1} \rangle \) and \( \langle i, q, i_{q+1} \rangle \), \( B = \{h \in \{g, \ldots, q\} \}, \) is called an “induced-path in \( x \) between \( \langle i, g, i_{g+1} \rangle \) and \( \langle i, q, i_{q+1} \rangle \)” (or alternatively, “an \( x \)-induced-path between \( \langle i, g, i_{g+1} \rangle \) and \( \langle i, q, i_{q+1} \rangle \)” if

\[
(\forall p, r, s \in S (g \leq p < r < s \leq q)) [x(i, p, i_{p+1})(i, r, i_{r+1})(i, s, i_{s+1}) > 0].
\]

2. The number of \( x \)-induced-paths between \( \langle i, g, i_{g+1} \rangle \) and \( \langle i, q, i_{q+1} \rangle \) \( \langle \langle i, g, i_{g+1} \rangle, (i, q, i_{q+1}) \rangle \) \( (i, g, i_{g+1}), \) \( (i, q, i_{q+1}) \) \( \in A (g + 1 < q < m) \), is denoted by “\( \nu_{x,\langle i, g, i_{g+1} \rangle, \langle i, q, i_{q+1} \rangle} \)”.

3. The \( k \)-th \( \nu_{x,\langle i, g, i_{g+1} \rangle, \langle i, q, i_{q+1} \rangle} \) \( x \)-induced-path between \( \langle i, g, i_{g+1} \rangle \) and \( \langle i, q, i_{q+1} \rangle \) \( \langle \langle i, g, i_{g+1} \rangle, \) \( (i, q, i_{q+1}) \rangle \) \( \in A (g + 1 < q < m) \land (\nu_{x,\langle i, g, i_{g+1} \rangle, \langle i, q, i_{q+1} \rangle} > 0) \) is denoted by “\( \nu_{x,\langle i, g, i_{g+1} \rangle, \langle i, q, i_{q+1} \rangle} \)”.

4. The notation \( iPath(B, x, \langle i, g, i_{g+1} \rangle, \langle i, q, i_{q+1} \rangle) \) stands for “\( B \) is an indcued-path in \( x \) between \( \langle i, g, i_{g+1} \rangle \) and \( \langle i, q, i_{q+1} \rangle \)”.

Theorem 15 below shows the relation between the joint-flow-cover of two given arcs in a given \( x \in Q \) and the set of the induced-paths between the two arcs in the given \( x \).
**Theorem 15 (Induced-Paths Make-up of Q)** The joint-flow-cover (Definition 13) of two given arcs having a positive separation (Definition 2) and joint-flow in \( x \in Q \) (Definition 6) resolves into a unique collection of \( x \)-induced paths (Definition 14) between the two arcs.

In other words,

\[
\forall x \in Q \quad (\forall g, q \in S \ (g < q < m)) \quad (\exists \alpha \in \mathbb{N} (\delta < m - 2)) \quad (\forall \langle i_g, g, i_{g+1} \rangle, \langle i_q, q, i_{q+1} \rangle \in A) \quad \text{SEP}((\langle i_g, g, i_{g+1} \rangle, \langle i_q, q, i_{q+1} \rangle) = \delta \land \text{jointFlow}(x, \langle i_g, g, i_{g+1} \rangle, \langle i_q, q, i_{q+1} \rangle))
\]

\[
(\exists! \nu_x, \nu_{x, (i_g, g), (i_q, q)}, (i_q, q, i_{q+1}) \in \mathbb{N} (\nu_x, (i_g, g), (i_q, q, i_{q+1}) \leq (m - 4)!))
\]

\[
(\exists! \{P^\alpha_{x, (i_g, g), (i_q, q, i_{q+1})}, (i_q, q, i_{q+1}) \} : (\forall \alpha \in \{1, \ldots, \nu_x, (i_g, g), (i_q, q, i_{q+1})\}))
\]

\[
[JFC_x((i_g, g, i_{g+1}), (i_q, q, i_{q+1})) = \bigcup_{\alpha = 1} P^\alpha_{x, (i_g, g), (i_q, q, i_{q+1})} \neq \emptyset].
\]

**Proof.** Let \( x \in Q \). Let \( g, q \in S \ (g < q < m) \). Let \( (i_g, g, i_{g+1}), (i_q, q, i_{q+1}) \in A \). Assume \( \text{SEP}((i_g, g, i_{g+1}), (i_q, q, i_{q+1})) = \delta > 0 \) and that \( \text{jointFlow}(x, (i_g, g, i_{g+1}), (i_q, q, i_{q+1})) \) is TRUE.

1. (jointFlow\((x, (i_g, g, i_{g+1}), (i_q, q, i_{q+1})) \) TRUE) \( \implies \)

\[
JFC_x((i_g, g, i_{g+1}), (i_q, q, i_{q+1})) \neq \emptyset.
\]

2. \( \nu_x, (i_g, g), (i_q, q, i_{q+1}) \leq (m - 4)! \). (See proof of Theorem 4).

The proof of the equality predicated in the theorem is by induction on the arc separation \( \delta \). We will show that if there is a \( \delta_0 \geq 2 \) such that the theorem is true for every \( \delta \in [1, \delta_0] \), then the theorem must also be true for \( \delta = \delta_0 + 1 \). Our base case will consist of direct proofs for the cases of \( \delta = 1 \) and \( \delta = 2 \).

**B. Base Case.** We will show that the theorem is true for \( \delta = 1 \) and \( \delta = 2 \), respectfully.

**B.1 Case of \( \delta = 1 \).**

By Arc Separation Definition 2 we have:

\[
(g < q) \land (\delta = 1) \implies q = g + \delta + 1 = g + 1 + 1 = g + 2.
\]  \( \text{(25)} \)

**B.1.1 We will develop an explicit expression for JFC\(_x((i_g, g, i_{g+1}), (i_q, q, i_{q+1})) \) when \( q = g + 2 \).**

\( \text{[25]} \) implies (since \( q = g + 2 \))

\[
\{g + 1, \ldots, q - 1\} = \{g + 1\}.
\]  \( \text{(26)} \)
Implicit-Zeros constraints \[5\] stipulate that

\[(\forall r, s \in S \ (r < m - 2)) \ (\forall \langle i_r, r, j_r \rangle, \langle i_s, s, j_s \rangle, \langle i_{r+2}, r + 2, j_{r+2} \rangle \in A)\]

\[\neg((s = r + 1) \land i_s = j_r) \lor \neg((s = r + 1) \land j_s = i_{r+2}) \Rightarrow x_{\langle i_r, r, j_r \rangle \langle i_s, s, j_s \rangle \langle i_{r+2}, r + 2, j_{r+2} \rangle} = 0.\] \hspace{1cm} (27)

(26) and (27) imply

\[(x_{\langle i_g, g, i_{g+1} \rangle \langle i_p, p, i_{p+1} \rangle \langle i_{g+2}, g+2, i_{g+3} \rangle}) > 0 \implies (p = g + 1) \land (i_p = i_{g+1}) \land (i_{p+1} = i_{g+2}).\] \hspace{1cm} (28)

(26), (27), (28), and Joint-Flow-Cover Definition \[13\] imply

\[JFC_x(\langle i_g, g, i_{g+1} \rangle, \langle i_q, q, i_{q+1} \rangle) = JFC_x(\langle i_g, g, i_{g+1} \rangle, \langle i_{g+2}, g+2, i_{g+3} \rangle) = \{\langle i_g, g, i_{g+1} \rangle, \langle i_{g+2}, g+2, i_{g+3} \rangle, \langle i_q, q, i_{q+1} \rangle\}.\] \hspace{1cm} (29)

(B.1.2) We will develop an explicit expression for the \textit{x-induced-paths between} \[\langle i_g, g, i_{g+1} \rangle\]\ and \[\langle i_q, q, i_{q+1} \rangle\] when \[q = g + 2\].

By Implicit-Zeros constraints \[5\],

\[x_{\langle i_g, g, i_{g+1} \rangle \langle i_{g+1}, g+1, i_{g+2} \rangle \langle i_{g+2}, g+2, i_{g+3} \rangle} > 0 \implies i_g \neq i_{g+1} \neq i_{g+2} \neq i_{g+3}.\] \hspace{1cm} (30)

(29), (30), and Induced-Path Definition \[14\] imply that

\[P_x(\langle i_g, g, i_{g+1} \rangle, \langle i_q, q, i_{q+1} \rangle) = P_x(\langle i_g, g, i_{g+1} \rangle, \langle i_{g+2}, g+2, i_{g+3} \rangle) = \{\langle i_g, g, i_{g+1} \rangle, \langle i_{g+2}, g+2, i_{g+3} \rangle, \langle i_q, q, i_{q+1} \rangle\}\]

is an \textit{x-induced-path between} \[\langle i_g, g, i_{g+1} \rangle\] and \[\langle i_q, q, i_{q+1} \rangle = \langle i_{g+2}, g+2, i_{g+3} \rangle.\] \hspace{1cm} (31)

The uniqueness of \[\{P_x(\langle i_g, g, i_{g+1} \rangle, \langle i_{g+2}, g+2, i_{g+3} \rangle)\}\] follows from (26) directly.

(B.1.3) In conclusion, it follows from (26), (31), and the uniqueness of the set of \textit{x-induced-paths between} \[\langle i_g, g, i_{g+1} \rangle\] and \[\langle i_{g+2}, g+2, i_{g+3} \rangle\] (i.e., \[\{P_x(\langle i_g, g, i_{g+1} \rangle, \langle i_{g+2}, g+2, i_{g+3} \rangle)\}\]) that the theorem is true for the case of \[\delta = 1\].

(B.2) Case of \[\delta = 2\].

By Arc Separation Definition \[2\] we have:

\[(g < q) \land (\delta = 2) \implies q = g + \delta + 2 = g + 2 + 1 = g + 3.\] \hspace{1cm} (32)

Hence, by premise,

\[\text{jointFlow}(x, \langle i_g, g, i_{g+1} \rangle, \langle i_{g+3}, g + 3, i_{g+4} \rangle)\] is TRUE. \hspace{1cm} (33)

(B.2.1) We will develop an explicit expression for \[JFC_x(\langle i_g, g, i_{g+1} \rangle, \langle i_q, q, i_{q+1} \rangle)\] when \[q = g + 3\].
(32) implies (since $q = g + 3$)
\[
\{g + 1, \ldots, q - 1\} = \{g + 1, g + 2\}. \tag{34}
\]

$GKE$ constraints (2) and Implicit-Zeros constraints (5) stipulate that
\[
(\forall s \in S \ (s < m - 3)) \ (\forall \langle i_s, s, i_{s+1} \rangle, \langle i_{s+3}, s + 3, i_{s+4} \rangle \in A) \ (\forall k \in L)
[x_{\langle i_s, s, i_{s+1} \rangle, i_{s+3}, s + 3, i_{s+4} + 1, k} - x_{\langle i_s, s, i_{s+1} \rangle, i_{s+3}, s + 3, i_{s+4} + 3, i_{s+4} + 4} = 0]. \tag{35}
\]

By Joint-Flow Definition (6), (33) and (35) imply
\[
(\exists L \subset L \ (L \neq \emptyset)) \ (((u \in L) \implies (x_{\langle i_g, g, i_{g+1} \rangle, i_{g+1}, g+1, u \rangle, g+3, i_{g+4} > 0) \land
x_{\langle i_g, g, i_{g+1} \rangle, i_{g+1}, g+1, u \rangle, g+3, i_{g+4} + 3, i_{g+4} + 4} > 0)) \land ((u \in L \setminus L) \implies
(x_{\langle i_g, g, i_{g+1} \rangle, i_{g+1}, g+1, u \rangle, g+3, i_{g+4} + 3, i_{g+4} + 4} = 0 \land x_{\langle i_g, g, i_{g+1} \rangle, i_{g+1}, g+1, u \rangle, g+3, i_{g+4} + 3, i_{g+4} + 4} = 0)). \tag{36}
\]

Using $GKE$ constraints (2) and Implicit-Zeros constraints (3) on arc pair $((i_g, g, i_{g+1}),$
$\langle i_{g+1}, g + 1, u \rangle)$ and node $(i_{g+3}, g + 3)$ gives:
\[
\sum_{k \in L} x_{\langle i_g, g, i_{g+1} \rangle, i_{g+1}, g+1, u \rangle, g+3, i_{g+4} + 3, i_{g+4} + 1} - x_{\langle i_g, g, i_{g+1} \rangle, i_{g+1}, g+1, u \rangle, g+3, i_{g+4} + 3, i_{g+4} + 3} = 0. \tag{37}
\]

(36) and (37) imply
\[
(\forall u \in L) \ [x_{\langle i_g, g, i_{g+1} \rangle, i_{g+1}, g+1, u \rangle, g+3, i_{g+4} + 3, i_{g+4} + 1} > 0]. \tag{38}
\]

Similarly, using $GKE$ constraints (2) and Implicit-Zeros constraints (5) on arc pair
$((u, g + 2, i_{g+3}), (i_{g+3}, g + 3, i_{g+4}))$ and node $(i_{g+1}, g + 1)$ gives:
\[
x_{\langle i_{g+1}, g+1, u \rangle, g+2, i_{g+3}, i_{g+3}, g+3, i_{g+4} + 4} = 0. \tag{39}
\]

(36) and (39) imply
\[
(\forall u \in L) \ [x_{\langle i_{g+1}, g+1, u \rangle, g+2, i_{g+3}, i_{g+3}, g+3, i_{g+4} + 4} > 0]. \tag{40}
\]

By Joint-Flow-Cover Definition (13), (34), (36), (38), and (40) imply
\[
JFC_{x}(\langle i_g, g, i_{g+1} \rangle, i_{q}, q, i_{q+1}) = JFC_{x}(\langle i_g, g, i_{g+1} \rangle, i_{g+3}, g + 3, i_{g+4})
= \{\langle i_g, g, i_{g+1} \rangle, \langle i_{g+3}, g + 3, i_{g+4} \rangle\} \cup \bigcup_{u \in L} \{\langle i_{g+1}, g + 1, u \rangle, \langle u, g + 2, i_{g+3} \rangle\}
= \bigcup_{u \in L} \{\langle i_g, g, i_{g+1} \rangle, \langle i_{g+1}, g + 1, u \rangle, \langle u, g + 2, i_{g+3} \rangle, \langle i_{g+3}, g + 3, i_{g+4} \rangle\}. \tag{41}
\]
(B.2.2) We will show that each \( \{(i_g, g, i_{g+1}), (i_{g+1}, g + 1, u), (u, g + 2, i_{g+3}), (i_{g+3}, g + 3, i_{g+4})\} \) \( (u \in L) \) is an \( x \)-induced-path between \( (i_g, g, i_{g+1}) \) and \( (i_{g+3}, g + 3, i_{g+4}) \).

Define
\[
\forall u \in L, \ P^u_{x,(i_g,g,i_{g+1}),(i_{g+3},g+3,i_{g+4})} := \\
\{(i_g, g, i_{g+1}), (i_{g+1}, g + 1, u), (u, g + 2, i_{g+3}), (i_{g+3}, g + 3, i_{g+4})\}. \tag{42}
\]

Then, (41) can be re-written as
\[
JFC_x((i_g, g, i_{g+1}), (i_{g+3}, g + 3, i_{g+4})) = \bigcup_{u \in L} P^u_{x,(i_g,g,i_{g+1}),(i_{g+3},g+3,i_{g+4})}. \tag{43}
\]

(B.2.3) In conclusion, it follows directly from the uniqueness of \( \{ P^\alpha_{x,(i_g,g,i_{g+1}),(i_{g+3},g+3,i_{g+4})} : \forall \alpha \in \{1, \ldots, |L|\} \} \) and statement (47) that the theorem is true for the case of \( \delta = 2 \).

(C) Inductive Step. We will show that if there is a \( \delta \geq 2 \) such that the theorem holds for every \( \delta \in [1, \overline{\delta}] \), then the theorem must also hold for \( \delta = \overline{\delta} + 1 \). For this purpose, assume the theorem holds for all \( \delta \in [1, \overline{\delta}] \) for a given \( \overline{\delta} \geq 2 \). Let \( \delta = \overline{\delta} + 1 \), and assume
\[
(q = g + \overline{\delta} + 2 < m) \land (jointFlow(x, (i_q, g, i_{g+1}), (i_q, q, i_{q+1}))) \text{ is TRUE}). \tag{48}
\]
Let:
\[ W := \{(u,v) \in L^2 : (x_{(i_g,g+1)}(i_q+1,g+1,u)(v,q-1,i_q) > 0) \land (x_{(i_g,g+1,u)}(v,q-1,i_q)(i_q+1,i_q) > 0) \land (x_{(i_g,g+1,u)}(v,q+1,i_q)(i_q+1,i_q+1) > 0) \} \].

\[ U := \{u \in L : (\exists v \in L) [(u,v) \in W] \}. \]

\[ V := \{v \in L : (\exists u \in L) [(u,v) \in W] \}. \]

Then, it follows immediately from Joint-Flow-Cover Definition the connectivity stipulated by the GKE constraints and the Implicit-Zeros constraints, and Valid Constraints, that:

1. \( W, U, \) and \( V \) are (respectively) non-empty, i.e,
\[ (W \neq \emptyset) \land (U \neq \emptyset) \land (V \neq \emptyset). \]

2. All of the joint-flow between \( \langle i_g, g, i_{g+1} \rangle \) and \( \langle i_q, q, i_{q+1} \rangle \) must propagate through the \( (\langle i_{g+1}, g + 1, u \rangle, \langle v, q - 1, i_q \rangle) \) pairs \( (u,v) \in W \) only.

3. Since it may be the case that some of the joint-flow of a \( (\langle i_{g+1}, g + 1, u \rangle, \langle v, q - 1, i_q \rangle) \) pair \( (u,v) \in W \) does not propagate onto \( \langle i_g, g, i_{g+1} \rangle \) and/or \( \langle i_q, q, i_{q+1} \rangle \),
\[ ((JFC_x(\langle i_g, g, i_{g+1} \rangle, \langle i_q, q, i_{q+1} \rangle)) \setminus \{\langle i_g, g, i_{g+1} \rangle, \langle i_q, q, i_{q+1} \rangle \}) \subseteq \bigcup_{(u,v) \in W} JFC_x(\langle i_{g+1}, g + 1, u \rangle, \langle v, q - 1, i_q \rangle). \]

4. Since it may be the case that some of the joint-flow of a \( (\langle i_g, g, i_{g+1} \rangle, \langle v, q - 1, i_q \rangle) \) pair \( v \in V \) does not propagate onto \( \langle i_q, q, i_{q+1} \rangle \),
\[ ((JFC_x(\langle i_g, g, i_{g+1} \rangle, \langle i_q, q, i_{q+1} \rangle)) \setminus \{\langle i_g, g, i_{g+1} \rangle, \langle i_q, q, i_{q+1} \rangle \}) \subseteq \bigcup_{v \in V} JFC_x(\langle i_g, g, i_{g+1} \rangle, \langle v, q - 1, i_q \rangle) \setminus \{\langle i_g, g, i_{g+1} \rangle \}. \]

5. Since it may be the case that some of the joint-flow of a \( (\langle i_{g+1}, g + 1, u \rangle, \langle i_q, q, i_{q+1} \rangle) \) pair \( u \in U \) does not propagate onto \( \langle i_g, g, i_{g+1} \rangle \),
\[ ((JFC_x(\langle i_g, g, i_{g+1} \rangle, \langle i_q, q, i_{q+1} \rangle)) \setminus \{\langle i_g, g, i_{g+1} \rangle, \langle i_q, q, i_{q+1} \rangle \}) \subseteq \bigcup_{u \in U} JFC_x(\langle i_{g+1}, g + 1, u \rangle, \langle i_q, q, i_{q+1} \rangle) \setminus \{\langle i_q, q, i_{q+1} \rangle \}. \]
(C.1) **Focusing on** $W$. We will show the relationship between $JFC_x(⟨i_g, g, i_{g+1}⟩, ⟨i_q, q, i_{q+1}⟩)$ and the $x$-induced-paths involving the pairs $(u, v) ∈ W$.

From *Arc Separation* Definition 2 and statement (48) (i.e., the fact that $q = g + δ + 2$), we have

$$(∀(u, v) ∈ W) [SEP(⟨i_{g+1}, g + 1, u⟩, ⟨v, q - 1, i_q⟩)]$$

$$= SEP(⟨i_g, g, i_{g+1}⟩, ⟨i_q, q, i_{q+1}⟩) - 2$$

$$= (δ + 1) - 2$$

$$= δ - 1$$

$$≥ 2 - 1$$

$$≥ 1]. \tag{56}$$

Hence $SEP(⟨i_{g+1}, g + 1, u⟩, ⟨v, q - 1, i_q⟩) ∈ [1, δ)$. Hence, by premise, (49), (52), and (56) imply

$$([JFC_x(⟨i_{g+1}, g + 1, u⟩, ⟨v, q - 1, i_q⟩)]$$

$$= \cup_{νx,⟨i_{g+1}, g + 1, u⟩, ⟨v, q - 1, i_q⟩} \cup_{α_{uw}=1} P_{α_{uw}}^{α_{wv}}_{x,⟨i_{g+1}, g + 1, u⟩, ⟨v, q - 1, i_q⟩} ≠ ∅]. \tag{57}$$

By the connectivity stipulated by the $GKE$ constraints (2) and *Implicit-Zeros* constraints (5), (53) and (57) imply:

$$(∃!(H_{uw} ⊆ \{1, ..., νx,⟨i_{g+1}, g + 1, u⟩, ⟨v, q - 1, i_q⟩\}, (∀u, v) ∈ W)$$

$$[JFC_x(⟨i_{g+1}, g + 1, u⟩, ⟨i_q, q, i_{q+1})\{⟨i_g, g, i_{g+1}⟩, ⟨i_q, q, i_{q+1}⟩\}) =$$

$$= \cup_{(u,v)∈W} P_{x, ⟨i_{g+1}, g + 1, u⟩, ⟨v, q - 1, i_q⟩}^k ≠ ∅]. \tag{58}$$

For each $(u, v) ∈ W$, let $ν_{x,⟨i_{g+1}, g + 1, u⟩, ⟨v, q - 1, i_q⟩} := |H_{uw}|$, and assume (without loss of generality) that the $P_{x, ⟨i_{g+1}, g + 1, u⟩, ⟨v, q - 1, i_q⟩}^k ((u, v) ∈ W)$ have been re-labeled in such a way that the members of $H_{uw}$ are indexed as 1, 2, ..., $ν_{x,⟨i_{g+1}, g + 1, u⟩, ⟨v, q - 1, i_q⟩}$. Then, (58) can be re-written as:

$$JFC_x(⟨i_{g+1}, g + 1, u⟩, ⟨i_q, q, i_{q+1})\{⟨i_g, g, i_{g+1}⟩, ⟨i_q, q, i_{q+1}⟩\} =$$

$$= \cup_{(u,v)∈W} P_{x, ⟨i_{g+1}, g + 1, u⟩, ⟨v, q - 1, i_q⟩}^β ≠ ∅]. \tag{59}$$
(C.2) **Focusing on V.** We will establish the joint-flow relationships between \( \langle i_g, g, i_{g+1} \rangle \) and the members of the \( P_{x,(i_{g+1},g+1,u),(v,q-1,i_q)} \) induced-paths of expression [50].

From Arc Separation Definition 2 and statement [48], we have:

\[
(\forall v \in V) \ [SEP((i_g, g, i_{g+1}), (v, q-1, i_q)) = SEP((i_q, q, i_{q+1})) - 1 = (q - g - 1 - 1) = q - g - 2 = (g + \delta + 2) - g - 2 = \delta]. (60)
\]

By premise, (51), (52), and (60) imply

\[
(\forall v \in V) \ (\exists ! \{ \gamma_v \subseteq \{1, \ldots, \nu_{x,(i_{g+1},g+1,u),(v,q-1,i_q)} \} \})
\]

\[
[JFC_{x,(i_g, g, i_{g+1}), (v, q-1, i_q)} = \bigcup_{\gamma_v=1}^{\nu_{x,(i_g, g, i_{g+1}), (v, q-1, i_q)}} P_{x,(i_g, g, i_{g+1}), (v, q-1, i_q)} \neq \emptyset]. \quad (61)
\]

By Induced-Path Definition 14, (61) implies

\[
(\forall v \in V) \ (\forall \gamma_v \subseteq \{1, \ldots, \nu_{x,(i_g, g, i_{g+1}), (v, q-1, i_q)} \})
\]

\[
(x_{(i_{g+1},g+1),(i_r,i_{r+1}),(i_s,i_{s+1})} > 0). \quad (62)
\]

(In other words, every triplet of arcs comprised of \( \langle i_g, g, i_{g+1} \rangle \) and two arcs belonging to \( P_{x,(i_g, g, i_{g+1}), (v, q-1, i_q)} \) (\( \forall v \in V \), \( \forall \gamma_v \subseteq \{1, \ldots, \nu_{x,(i_g, g, i_{g+1}), (v, q-1, i_q)} \} \)) indexes a positive component of \( x \).

Also from Induced-Path Definition 14 we have

\[
(\forall v \in V) \ (\forall \gamma_v \subseteq \{1, \ldots, \nu_{x,(i_g, g, i_{g+1}), (v, q-1, i_q)} \})
\]

\[
(\exists \{ u_v^\gamma \subseteq U \} \ [\langle i_{g+1}, g + 1, u_v^\gamma \rangle \in P_{x,(i_{g+1},g+1),(v,q-1,i_q)}]). \quad (63)
\]

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Using (63) and Induced-Path Definition 14, (61) can be re-written as

\[
(\forall v \in V) \ (\exists! \{P_{x,(i_g+1, g+1, u_v^v),(v,q-1,i_q)}^v, (\forall \gamma_v \in \{1, \ldots, \nu_{x,(i_g,g,i_g+1)},(v,q-1,i_q)\}\})
\]

\[
[JFC_x((i_g, g, i_g+1), (v, q - 1, i_q)) = \{\langle i_g, g, i_g+1 \rangle\} \cup \
\bigcup_{\gamma_v=1} P_{x,(i_g+1, g+1, u_v^v),(v,q-1,i_q)}^v \neq \emptyset];
\]

Where: (\forall v \in V) (\forall \gamma_v \in \{1, \ldots, \nu_{x,(i_g,g,i_g+1)},(v,q-1,i_q)\}) \]

\[
[iPath(P_{x,(i_g+1, g+1, u_v^v),(v,q-1,i_q)}, x, \langle i_g+1, g + 1, u_v^v \rangle, \langle v, q - 1, i_q \rangle)] (64)
\]

(54), (59), and (64) imply

\[
\{P_{x,(i_g+1, g+1, u),(v,q-1,i_q)}^v, (\forall \beta_{uv} \in \{1, \ldots, \nu_{x,(i_g+1, g+1, u),(v,q-1,i_q)}\}, (\forall (u, v) \in W)\} \subseteq \
\{P_{x,(i_g+1, g+1, u_v^v),(v,q-1,i_q)}^v, (\forall \gamma_v \in \{1, \ldots, \nu_{x,(i_g,g,i_g+1)},(v,q-1,i_q)\}), (\forall v \in V)\}. (65)
\]

(62), (64), and (65) imply

\[
(\forall (u, v) \in W) (\forall \beta_{uv} \in \{1, \ldots, \nu_{x,(i_g+1, g+1, u),(v,q-1,i_q)}\}) \]

\[
(\forall (i_r, r, i_r+1), (i_s, s, i_s+1) \in P_{x,(i_g+1, g+1, u),(v,q-1,i_q)}^v \ (g + 1 \leq r < s \leq q - 1)) \]

\[
[x(i_g,g,i_g+1)(i_r,r,i_r+1)(i_s,s,i_s+1) > 0]. (66)
\]

(In other words, every triplet of arcs comprised of \((i_g, g, i_g+1)\) and any two members of \(P_{x,(i_g+1, g+1, u),(v,q-1,i_q)}^v \ (\forall (u, v) \in W), (\forall \beta_{uv} \in \{1, \ldots, \nu_{x,(i_g+1, g+1, u),(v,q-1,i_q)}\}\)) indexes a positive component of \(x\).)

(C.3) Focusing on \(U\). We will establish the joint-flow relationships between \((i_q, q, i_q+1)\) and the members of the \(P_{x,(i_g+1, g+1,u),(v,q-1,i_q)}^v\) induced-paths of expression 59.

From Arc Separation Definition 2 and statement 48, we have:

\[
(\forall u \in L) \ [SEP((i_g+1, g + 1, u), (i_q, q, i_q+1))
\]

\[
= SEP((i_g, g, i_g+1), (i_q, q, i_q+1)) - 1
\]

\[
= q - g - 1 - 1 = q - g - 2
\]

\[
= (g + \delta + 2) - g - 2
\]

\[
= \delta]. \quad (67)
\]
By premise, (50), (52), and (67) imply
\[
(\forall u \in U) \left( \exists! \{ P^r_{x,(i_{g+1},g+1,u)}, (i_q,q,i_{q+1}) \} \implies (\forall \tau_u \in \{1, \ldots, \nu_{x,(i_{g+1},g+1,u)}, (i_q,q,i_{q+1})\}) \right)
\]
\[
[JFC_x((i_{g+1},g+1,u), (i_q,q,i_{q+1})) = \bigcup_{\tau_u=1} P^r_{x,(i_{g+1},g+1,u)}, (i_q,q,i_{q+1}) \neq \emptyset].
\]

By Induced-Path Definition 14, (68) implies
\[
(\forall u \in U) \ (\forall \tau_u \in \{1, \ldots, \nu_{x,(i_{g+1},g+1,u)}, (i_q,q,i_{q+1})\})
\]
\[
(\exists! v^r_u \in V) \ [v^r_u \cdot q - 1, i_q \in P^r_{x,(i_{g+1},g+1,u)}, (i_q,q,i_{q+1})].
\]

Using (70) and Induced-Path Definition 14, (68) can be re-written as
\[
(\forall u \in U) \ (\exists! \{ P^r_{x,(i_{g+1},g+1,u)}, (v^r_u,q-1,i_q) \} \implies (\forall \tau_u \in \{1, \ldots, \nu_{x,(i_{g+1},g+1,u)}, (v^r_u,q-1,i_q)\}) \right)
\]
\[
[JFC_x((i_{g+1},g+1,u), (i_q,q,i_{q+1})) = \bigcup_{\nu_{x,(i_{g+1},g+1,u)}, (v^r_u,q-1,i_q)} P^r_{x,(i_{g+1},g+1,u)}, (v^r_u,q-1,i_q) \neq \emptyset];
\]

Where: (\forall u \in U) \ (\forall \tau_u \in \{1, \ldots, \nu_{x,(i_{g+1},g+1,u)}, (v^r_u,q-1,i_q)\})
\[
[iPath(P^r_{x,(i_{g+1},g+1,u)}, (v^r_u,q-1,i_q)), x, (i_{g+1},g+1,u), (v^r_u,q-1,i_q)].
\]

(55), (59), and (71) imply
\[
\{ P^r_{x,(i_{g+1},g+1,u)}, (v,q-1,i_q) \} \ (\forall \beta_{uv} \in \{1, \ldots, v_{x,(i_{g+1},g+1,u)}, (v,q-1,i_q)\}) \subseteq \{ P^r_{x,(i_{g+1},g+1,u)}, (v^r_u,q-1,i_q) \} \ (\forall \tau_u = 1 \in \{1, \ldots, v_{x,(i_{g+1},g+1,u)}, (q,q,i_{q+1})\}) \ (\forall u \in U).}
\[\text{(69), (71), and (72) imply}
\]
\[
(\forall (u, v) \in W) \ (\forall \beta_{uv} \in \{1, \ldots, \nu_x, (i_{q+1}, g+1, u), (v, q-1, i_q)\})
\]
\[
(\forall i_r, r, i_{r+1}), (i_s, s, i_{s+1}) \in P^\beta_{x,(i_{q+1}, g+1, u),(v, q-1, i_q)} \ (g+1 \leq r < s \leq q-1)
\]
\[
[x(i_r, r, i_{r+1}) (i_s, s, i_{s+1}) (i_q, q, i_{q+1}) > 0].
\]  

(73)

(In other words, every triplet of arcs comprised of \((i_q, q, i_{q+1})\) and any two members of \(P^\beta_{x,(i_{q+1}, g+1, u),(v, q-1, i_q)}\) \((u, v) \in W; \beta_{uv} \in \{1, \ldots, \nu_x, (i_{q+1}, g+1, u), (v, q-1, i_q)\}\) indexes a positive component of \(x\).)

(C.4) **Synthesizing.** Define

\[
(\forall (u, v) \in W) \ (\forall \beta_{uv} \in \{1, \ldots, \nu_x, (i_{q+1}, g+1, u), (v, q-1, i_q)\}),
\]
\[
P^\beta_{uv}_{x,(i_g, g, i_{g+1}),(i_q, q, i_{q+1})} := \{\langle i_g, g, i_{g+1} \rangle, \langle i_q, q, i_{q+1} \rangle\} \cup P^\beta_{uv}_{x,(i_{q+1}, g+1, u),(v, q-1, i_q)}. \tag{74}
\]

By Induced-Path Definition [14], statements [59], (66), and (73) imply

\[
(\forall (u, v) \in W) \ (\forall \beta_{uv} \in \{1, \ldots, \nu_x, (i_{q+1}, g+1, u), (v, q-1, i_q)\})
\]
\[
[iPath(P^\beta_{uv}_{x,(i_g, g, i_{g+1}),(i_q, q, i_{q+1})}, x, \langle i_g, g, i_{g+1} \rangle, \langle i_q, q, i_{q+1} \rangle)]. \tag{75}
\]

Hence, using [59] and (75), we have that

\[
JFC_{x,(i_g, g, i_{g+1}),(i_q, q, i_{q+1})} = \bigcup_{(u, v) \in W} \nu_x, (i_{q+1}, g+1, u), (v, q-1, i_q) \bigcup_{\beta_{uv}=1} P^\beta_{uv}_{x,(i_g, g, i_{g+1}),(i_q, q, i_{q+1})};
\]

Where:

\[
(\forall (u, v) \in W) \ (\forall \beta_{uv} \in \{1, \ldots, \nu_x, (i_{q+1}, g+1, u), (v, q-1, i_q)\})
\]
\[
[iPath(P^\beta_{uv}_{x,(i_g, g, i_{g+1}),(i_q, q, i_{q+1})}, x, \langle i_g, g, i_{g+1} \rangle, \langle i_q, q, i_{q+1} \rangle)]. \tag{76}
\]

The uniqueness of \(\{P^\beta_{x,(i_g, g, i_{g+1}),(i_q, q, i_{q+1})}, (\forall (u, v) \in W), (\forall \beta_{uv} \in \{1, \ldots, \nu_x, (i_{q+1}, g+1, u), (v, q-1, i_q)\}\) follows from the uniqueness stipulation in [59]. It follows from this and (76) that the theorem holds true for an arc separation of \(\delta = \delta + 1\).

Hence, the inductive step (and therefore, the theorem) is proven. ■

### 4.2 Integrality of the LP polytope, Q

In the remainder of this section, we will focus on the induced-paths of points of the LP polytope (Q) which span the stages of the MAPG. We refer to these as “spanning-induced-paths.”

**Definition 16 (Spanning-Induced-Path)** Let \(x \in Q\) :
1. An $x$-induced-path over the MAPG is referred to as a “spanning-$x$-induced-path” (or equivalently, a “spanning-induced-path-in-$x$”) if it begins at an arc at stage $g = 1$ and ends at an arc at stage $q = m - 1$.

2. The number of spanning-$x$-induced-paths between $(i_1, 1, i_2)$ and $(i_{m-1}, m - 1, i_m)$ $(\forall i_1, i_2, i_{m-1}, i_m \in L)$ is denoted “$\pi_{x,(i_1,i_2,i_{m-1},i_m)}$”.

3. The $k^{th}$ $(k \leq \pi_{x,(i_1,i_2,i_{m-1},i_m)})$ spanning-$x$-induced-paths between $(i_1, 1, i_2)$ and $(i_{m-1}, m - 1, i_m)$ $(\forall i_1, i_2, i_{m-1}, i_m \in L \; (\pi_{x,(i_1,i_2,i_{m-1},i_m)} > 0))$ is denoted “$\mathcal{P}^k_{x,(i_1,i_2,i_{m-1},i_m)}$”.

4. $\forall(i_1, 1, i_2), (i_{m-1}, m - 1, i_m) \in A$, the notation $siPath(B, x, (i_1, i_2, i_{m-1}, i_m))$ stands for “$B$ is a spanning-$x$-induced path between $(i_1, 1, i_2)$ and $(i_{m-1}, m - 1, i_m)$.”

5. The set of all the spanning-$x$-induced-paths is denoted $\Pi_x$, and expressed as $\Pi_x := \{ \mathcal{P}_{x,(i_1,i_2,i_{m-1},i_m)}^{\alpha_{i_1,i_2,i_{m-1},i_m}} \in A^{m-1} : siPath(\mathcal{P}_{x,(i_1,i_2,i_{m-1},i_m)}^{\alpha_{i_1,i_2,i_{m-1},i_m}}, x, (i_1, i_2, i_{m-1}, i_m)), \forall \alpha_{i_1,i_2,i_{m-1},i_m} \in 1, \ldots, \pi_{x,(i_1,i_2,i_{m-1},i_m)} \}; \forall (i_1, 1, i_2), (i_{m-1}, m - 1, i_m) \in A \}$

Where: $\pi_{x,(i_1,i_2,i_{m-1},i_m)} \leq (m - 4)!$.

**Definition 17** (“Characteristic Vectors”): $\widehat{x}; \widehat{\Pi}_x$ Let $x \in Q$. We denote the characteristic vector of $\mathcal{P}_{x,(i_1,i_2,i_{m-1},i_m)}^{\alpha_{i_1,i_2,i_{m-1},i_m}} \in \Pi_x$ $(\alpha_{i_1,i_2,i_{m-1},i_m} \in 1, \ldots, \pi_{x,(i_1,i_2,i_{m-1},i_m)}) \; (\pi_{x,(i_1,i_2,i_{m-1},i_m)} > 0)$ by $\widehat{x}_{x,(i_1,i_2,i_{m-1},i_m)}^{\alpha_{i_1,i_2,i_{m-1},i_m}}$.

In other words,

$\forall (\alpha_{i_1,i_2,i_{m-1},i_m} \in 1, \ldots, \pi_{x,(i_1,i_2,i_{m-1},i_m)}) \Rightarrow \pi_{x,(i_1,i_2,i_{m-1},i_m)} > 0)$

$\Rightarrow \mathcal{P}_{x,(i_1,i_2,i_{m-1},i_m)}^{\alpha_{i_1,i_2,i_{m-1},i_m}} \in L^{m-1} \Rightarrow \forall \alpha_{i_1,i_2,i_{m-1},i_m} \in 1, \ldots, \pi_{x,(i_1,i_2,i_{m-1},i_m)} \); \forall (i_1, 1, i_2), (i_{m-1}, m - 1, i_m) \in A \}

$[(p < r < s < m) \land (\langle i_p, p, j_p \rangle, \langle i_r, r, j_r \rangle, \langle i_s, s, j_s \rangle) \in \mathcal{P}_{x,(i_1,i_2,i_{m-1},i_m)}^{\alpha_{i_1,i_2,i_{m-1},i_m}}] \Rightarrow (\widehat{x}_{x,(i_1,i_2,i_{m-1},i_m)}^{\alpha_{i_1,i_2,i_{m-1},i_m}} - 1)^{\langle i_g,j_g,i_p,j_p,i_q,j_q \rangle} = 1 \land (\widehat{x}_{x,(i_1,i_2,i_{m-1},i_m)}^{\alpha_{i_1,i_2,i_{m-1},i_m}})_{\langle i_g,j_g,i_p,j_p,i_q,j_q \rangle} = 0] \Rightarrow (\widehat{x}_{x,(i_1,i_2,i_{m-1},i_m)}^{\alpha_{i_1,i_2,i_{m-1},i_m}})_{\langle i_g,j_g,i_p,j_p,i_q,j_q \rangle} = 0] \Rightarrow (\widehat{x}_{x,(i_1,i_2,i_{m-1},i_m)}^{\alpha_{i_1,i_2,i_{m-1},i_m}})_{\langle i_g,j_g,i_p,j_p,i_q,j_q \rangle} = 0]

(77)

We will denote the set of all the characteristic vectors of the spanning-induced-paths of $x \in Q$ by $\widehat{\Pi}_x$.

**Theorem 18** ($\widehat{\Pi}_x \leftrightarrow Q^t$) The following following relationships are true:

1. $\forall x \in Q \left\lfloor L \subseteq Q^t \right.$.
2. \((\forall x \in Q) \left[ |\hat{\Pi}_x| \leq m! \right].\)
3. \((\exists x \in Q) \left[ f_5 : \hat{\Pi}_x \to \mathcal{H}_n \right. \text{ is injective (and not surjective)} \left. \right].\)
4. \((\exists x \in Q) \left[ f_6 : \hat{\Pi}_x \to \mathcal{H}_n \right. \text{ is bijective} \left. \right].\)

**Proof.**

1. Let \(x \in Q\). Let \((i_1, i_2, i_{m-1}, i_m) \in L^4 (\pi_{x,(i_1,i_2,i_{m-1},i_m)} > 0)\). Let \(\alpha_{i_1,i_2,i_{m-1},i_m} \in \{1, \ldots, \pi_{x,(i_1,i_2,i_{m-1},i_m)}\}\). By definition, \(\hat{x}_{x,(i_1,i_2,i_{m-1},i_m)}^{\alpha_{i_1,i_2,i_{m-1},i_m}} \in \{0,1\}^{m^n}\). Also, one easily verifies that \(\hat{x}_{x,(i_1,i_2,i_{m-1},i_m)}^{\alpha_{i_1,i_2,i_{m-1},i_m}}\) satisfies \((4)-(6)\). Hence, \(\hat{x}_{x,(i_1,i_2,i_{m-1},i_m)}^{\alpha_{i_1,i_2,i_{m-1},i_m}}\) is integral and a member of \(Q\). Hence, in other words,

\[
(\forall (i_1, i_2, i_{m-1}, i_m) \in L^4 (\pi_{x,(i_1,i_2,i_{m-1},i_m)} > 0)) \Rightarrow (\forall \alpha_{i_1,i_2,i_{m-1},i_m} \in \{1, \ldots, \pi_{x,(i_1,i_2,i_{m-1},i_m)}\}) [\hat{x}_{x,(i_1,i_2,i_{m-1},i_m)}^{\alpha_{i_1,i_2,i_{m-1},i_m}} \in Q].
\]

2. In particular, from “Structure-of-Integral-Points” Theorem \((\forall x \in Q) \left[ |\hat{\Pi}_x| = 1 < m! \right].\) On the other hand, it is trivial to construct \(x \in Q\) such that \(|\hat{\Pi}_x| = m!\). Part (2) of the theorem follows from these directly.

3. Parts (3) follows directly from the combination of Corollary \((\forall x \in Q) \left[ |\hat{\Pi}_x| = 1 < m! \right].\) and Part (2) of the theorem when it is the case that \(|\hat{\Pi}_x| < m!\).

4. Parts (4) follows directly from the combination of Corollary \((\forall x \in Q) \left[ |\hat{\Pi}_x| = 1 < m! \right].\) and Part (2) of the theorem when it is the case that \(|\hat{\Pi}_x| = m!\).

**Lemma 19** Let \(x \in Q\). Three arcs, one at each of the stages 1, 2, and 3 of the MAPG, have joint-flow in \(x\) iff there exists at least one spanning-x-induced-path over the MAPG which includes all three of the arcs.

In other words, excluding the trivial cases which violate constraints \((27)\) for convenience, the following is true:

\[
(\forall x \in Q) \left( (\forall \langle i_1, i_2 \rangle, \langle i_2, i_3, i_4 \rangle, \langle i_3, 3, i_4 \rangle \in A) \Rightarrow \right.
\]

\[
[x_{\langle i_1, i_2 \rangle}^{\langle i_2, i_3, i_4 \rangle} > 0 \iff ((\exists i_{m-1}, i_m \in L) (\exists P \in A^{m-1}) \left[ siPath(P, x, \langle i_1, i_2, i_{m-1}, i_m \rangle) \land ((\langle i_1, 1, i_2 \rangle, \langle i_2, 2, i_3 \rangle, \langle i_3, 3, i_4 \rangle \in P)] \right].
\]

**Proof.** Let \(x \in Q\). Let \(\langle i_1, i_2 \rangle, \langle i_2, i_3 \rangle, \langle i_3, 3, i_4 \rangle \in A\).
Letting \( q = 1, p = 2, r = m - 1 \) and using Implicit-Zeros constraints \((5)\) in Valid Constraints \((11)\) gives:

\[
x_{\langle i_1, i_2 \rangle \langle i_2, i_3 \rangle \langle i_3, i_4 \rangle} - \sum_{k=1}^{m} \sum_{t=1}^{m} x_{\langle i_1, i_2 \rangle \langle i_3, i_4 \rangle} (k, m-1, t) = 0. \tag{78}
\]

\((78)\) implies

\[
x_{\langle i_1, i_2 \rangle \langle i_2, i_3 \rangle \langle i_3, i_4 \rangle} > 0 \iff (\exists (i_{m-1}, i_m) \in L^2) \ [x_{\langle i_1, i_2 \rangle \langle i_3, i_4 \rangle} (i_{m-1}, m-1, i_m) > 0]. \tag{79}
\]

"Induced-Path-Make-up-of-Q" Theorem \([15]\) Implicit-Zeros constraints \((5)\), and \((79)\) imply

\[
(\forall (i_1, i_2, i_3, i_4, i_{m-1}, i_m) \in L^2) \ [x_{\langle i_1, i_2 \rangle \langle i_3, i_4 \rangle} (i_{m-1}, m-1, i_m) > 0 \iff ((\exists P \in A^{m-1}) [siPath(P, x, (i_1, i_2, i_{m-1}, i_m)) \wedge ((i_1, 1, i_2), (i_3, 3, i_4) \in P)])]. \tag{80}
\]

By Induced-Path Definition \((14)\) we have

\[
(\forall i_{m-1}, i_m \in L) (\forall P \in A^{m-1}) \ [\ (siPath(P, x, (i_1, i_2, i_{m-1}, i_m) \wedge ((i_1, 1, i_2), (i_3, 3, i_4) \in P)) \implies ((\exists u, v \in L) \ [x_{\langle i_1, i_2 \rangle \langle u, 2, v \rangle \langle i_3, i_4 \rangle} > 0 \wedge (u, 2, v) \in P)]). \tag{81}
\]

By Implicit-Zeros constraints \((5)\),

\[
(\forall i_1, i_2, i_3, i_4, u, v \in L) \ [x_{\langle i_1, i_2 \rangle \langle u, 2, v \rangle \langle i_3, i_4 \rangle} > 0 \implies (u = i_2 \wedge v = i_3)]. \tag{82}
\]

\((80), (81), \) and \((82)\) imply

\[
x_{\langle i_1, i_2 \rangle \langle i_2, i_3 \rangle \langle i_3, i_4 \rangle} > 0 \iff (\exists i_{m-1}, i_m \in L) (\exists P \in A^{m-1}) \ [siPath(P, x, (i_1, i_2, i_{m-1}, i_m)) \wedge ((i_1, 1, i_2), (i_2, 2, i_3), (i_3, 3, i_4) \in P)]. \tag{83}
\]

We will now establish the integrality of \( Q \).

**Theorem 20** Every \( x \in Q \) is a convex combination of points in \( Q_I \). In other words,

\[
(\forall x \in Q) (\exists \kappa \in \mathbb{N}) (\exists y^1, \ldots, y^\kappa \in Q_I) (\exists \lambda_1, \ldots, \lambda_\kappa \in (0, 1]) \ [x = \sum_{k=1}^{\kappa} \lambda_k y^k \wedge \sum_{k=1}^{\kappa} \lambda_k = 1].
\]

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Proof. Let \( x \in Q \). Constraints \((1)\) and Lemma \([19]\) imply

\[
\Pi_x \neq \emptyset. \tag{84}
\]

We will assume for convenience (without loss of generality) that the members of \( \hat{\Pi}_x \) are indexed as \( y^1, \ldots, y^{\pi_x} \), where

\[
\pi_x := \sum_{i_1, i_2, i_{m-1}, i_m \in L} \pi_x(i_1, i_2, i_{m-1}, i_m) = |\hat{\Pi}_x|.
\]

Theorems \([12]\) and \([18.1]\) imply

\[
\hat{\Pi}_x \subseteq Q_I \subseteq Ext(Q). \tag{85}
\]

From *Induced-Path* Definition \([14]\) every triplet of arcs of an *induced-path* indexes a positive variable, and also, the *Visit Requirements* constraints \((4)\) stipulate that the amount of *joint-flow* from a given pair of arcs which “reaches”/propagates to a given level of the MAPG must be the same for all the levels. Hence, let

\[
\Phi_x := \{ \varphi^\ell, \ell \in \mathbb{N} \}
\]

be the set of all the possible representations of \( x \) in terms of extreme-points of \( Q \), and

\[
\Lambda_x := \{ (\lambda_1, \ldots, \lambda_\pi) \in (0,1]^\pi_x : (\exists \ell \in \mathbb{N}) \\
(\exists \varphi^\ell \in \Phi_x) (\forall k \in \{1, \ldots, \pi_x\}) \ [y^k \text{ has weight } \lambda_k \text{ in } \varphi^\ell] \}
\]

be the set of all \( \pi_x \)-tuple of scalars on \((0,1]\) which can be weights for the \( y^k \)'s \((\forall k \in \{1, \ldots, \pi_x\})\) in a convex combination representation of \( x \) in terms of the extreme points of \( Q \). Then, by the convexity of \( Q \) (see Bazaraa et al. [2010; pp. 45-82]), the stipulation that the amount of *joint-flow* from a given pair of arcs which propagates to a given level of the MAPG must be the same for all the levels of the MAPG (i.e., the *Visit Requirements* constraints \((4)\)) and \((85)\) imply

\[
(\forall x \in Q) \ [\Lambda_x \neq \emptyset]. \tag{86}
\]

If is is the case that

\[
(\exists (\lambda_1, \ldots, \lambda_{\pi_x}) \in \Lambda_x) \ [x = \sum_{k=1}^{\pi_x} \lambda_k y^k \wedge \sum_{k=1}^{\pi_x} \lambda_k = 1], \tag{87}
\]

then, the theorem is proved.

Hence, for the purpose of contradiction, assume

\[
(\nexists (\lambda_1, \ldots, \lambda_{\pi_x}) \in \Lambda_x) \ [x = \sum_{k=1}^{\pi_x} \lambda_k y^k \wedge \sum_{k=1}^{\pi_x} \lambda_k = 1]. \tag{88}
\]
Then, the convexity of $Q$, (85) and (88) imply
\[(\forall \lambda_1, \ldots, \lambda_{\pi} x) \in \Lambda x) \; (\exists \eta \in \mathbb{N}) \; (\exists \mu_1, \ldots, \mu_\eta \in (0, 1]) \; (\exists z^1, \ldots, z^\eta \in Q \setminus \text{Conv}(\tilde{\Pi}_x))
\]
\[[(x = \sum_{k=1}^{\pi} \lambda_k y^k + \sum_{i=1}^{\eta} \mu_i z^i) \land (\sum_{k=1}^{\pi} \lambda_k + \sum_{i=1}^{\eta} \mu_i = 1)]. \tag{89}
\]
Consider the feasibility of a set of tuples $(\lambda_1, \ldots, \lambda_{\pi} x) \in \Lambda x), (\mu_1, \ldots, \mu_\eta) \in (0, 1]^\eta, (z^1, \ldots, z^\eta) \in (Q \setminus \text{Conv}(\tilde{\Pi}_x))^\eta$ satisfying (89). In light of (86), we have the following possibilities.

**Case 1:** Assume $\sum_{k=1}^{\pi} \lambda_k = 1.$

Then, from (89), we would have
\[\sum_{k=1}^{\pi} \lambda_k = 1 \implies \sum_{i=1}^{\eta} \mu_i = 0 \implies \mu_1 = \ldots = \mu_\eta = 0. \tag{90}
\]
(90) implies (87) which contradicts (88).

Hence, it cannot be the case that $\sum_{k=1}^{\pi} \lambda_k = 1.$

**Case 2:** Assume $0 < \sum_{k=1}^{\pi} \lambda_k < 1.$

Then, from (89), we have
\[0 < \sum_{i=1}^{\eta} \mu_i < 1. \tag{91}
\]

*Initial-Flow* constraint (1) and (89) imply
\[\sum_{i_1, i_2, i_3, i_4 \in L} \sum_{k=1}^{\pi} \lambda_k y^k_{(i_1,i_2,i_3,i_4)} + \sum_{i_1, i_2, i_3, i_4 \in L} \sum_{\eta} \mu_i z^i_{(i_1,i_2,i_3,i_4)} = 1. \tag{92}
\]

Corollary 9.1, the premise (i.e., $0 < \sum_{k=1}^{\pi} \lambda_k < 1$), (91), and (92) imply

\[(0 < \sum_{i_1, i_2, i_3, i_4 \in L} \sum_{k=1}^{\pi} \lambda_k y^k_{(i_1,i_2,i_3,i_4)} < 1) \land \tag{93}
\]
\[(0 < \sum_{i_1, i_2, i_3, i_4 \in L} \sum_{\eta} \mu_i z^i_{(i_1,i_2,i_3,i_4)} < 1). \tag{94}
\]
and \( (89) \) (i.e., the premised fact that \( z^1, \ldots, z^\eta \notin \text{Conv}(\widehat{\Pi}_x) \)) imply

\[
(\exists i \in \{1, \ldots, \eta\}) \ (\exists i_1, i_2, i_3, i_4 \in L) \ [(z^z_{i_1,1,i_2}, z^z_{i_2,2,i_3}, z^z_{i_3,3,i_4}) > 0) \land

((\forall i_{m-1}, i_m \in L) \ [(\exists P \in \Lambda^{m-1}) \ [\text{siPath}(P, z, (i_1, i_2, i_{m-1}, i_m) \land

\langle i_1, 1, i_2 \rangle, \langle i_2, 2, i_3 \rangle, \langle i_3, 3, i_4 \rangle \in P])].
\]

\( (95) \) contradicts Lemma 19.

Hence, it cannot be the case that \( 0 < \sum_{k=1}^{\pi_x} \lambda_k < 1. \)

It follows from the infeasibilities of both Cases 1 and 2 above that the premise \( (88) \) must be false. Hence, it must be the case that \( (87) \) is true. The theorem follows from this and the arbitrariness of \( x. \)

Corollary 21 \( Q = \text{Conv}(Q_I). \)

5 Some illustrative applications

As discussed earlier in this paper, many of the well-known combinatorial optimization problems (COPs) are essentially Assignment Problems (APs) with alternate objective cost functions. In general, the correct accounting of these costs cannot be done in the space of the natural variables traditionally used in formulating the AP, the reason being that those natural variables do not (respectively) contain enough information for that purpose. Hence, in order to model COP’s in general in the space of the natural variables of the AP polytope, additional constraints must be added to the standard AP constraints set, thereby destroying the “nice” structure of the polytope. We believe that that is the most fundamental “root” of the notorious difficulties of the hard COPs in particular, at least from an Operations Research/Optimization perspective. Hence, in a sense, one must “think-outside-the-box” of the “natural spaces” of hard COPs in general for any hope of being able to overcome their difficulties. The more complex variables used in this paper allow for the costs for many of the COPs (other than the AP) to be correctly captured without the need for additional constraints beyond those required for a “full assignment” solution. The key to this is to be able to attach costs to these higher-dimensional modeling variables using the information that is contained in them in such a way that the total cost associated to a given extreme point of the polytope induced by our proposed model \( (Q) \) is equal to that of the combinatorial configuration corresponding to the assignment solution represented by the given extreme point. In this way therefore, in particular, the quadratic, cubic, quartic, quintic, and sextic assignment problems, as well as the TSP and many of its variations can be modeled as LPs over our proposed polytope, \( Q. \) In this section, we will provide illustrations for the Linear Assignment (LAP), Quadratic Assignment (QAP), and Traveling Salesman (TSP) problems.
5.1 Linear Assignment Problem

The LAP is one of the most well-studied problems in Operations Research and Mathematics in general. Generically, the problem is to match objects of one class (say, objects of “type” $I$) to objects of another class (say, objects of “type” $J$). The assignment of object $i \in I$ to object $j \in J$ incurs a cost of $w_{ij}$. The problem is to find an assignment that matches each object of either class exactly once, and in such a way that the total cost of all the assignments is minimized.

The following theorem shows that costs based on the $w_{ij}$’s ($i \in I; j \in J$) can be attached to our modeling variables in such a way that the total cost of a “full assignment” is correctly accounted at the extreme points of our proposed model. With these “adjusted” costs, the LAP can be solved as a linear program (LP) over our proposed polytope, $Q$, as shown in the following theorem.

**Theorem 22** Let $c^{LAP} \in \mathbb{R}^{n^p}$ be a vector of costs defined in terms of the assignment costs, $w$, as follows:

\[
(\forall \langle i_g, j_g \rangle, \langle i_p, j_p \rangle, \langle i_q, j_q \rangle) \in A \ (x_{(i_g,j_g)}(i_p,j_p)(i_q,j_q)) \text{ not implicitly-zero by } (5),
\]

\[
c_{(i_g,j_g)}(i_p,j_p)(i_q,j_q) := \begin{cases} 
 w_{i_g,g} + w_{i_p,p} + w_{i_q,q} + w_{j_q,q+1} & \text{if } [(g = 1) \land (p = 2) \land (q = 3)]; \\
 w_{j_q,q+1} & \text{if } [(g = 1) \land (p = 2) \land (q \in [4, m - 1])]; \\
 0 & \text{otherwise.}
\end{cases}
\]

Then, the following is true:

\[
\forall x \in \text{Ext}(Q) \ [\mathcal{V}_{LAP}(x) := c^{LAP}x = \sum_{(i_g,j_g)(i_p,j_p)(i_q,j_q) \in A^3} c_{(i_g,j_g)}(i_p,j_p)(i_q,j_q)x_{(i_g,j_g)}(i_p,j_p)(i_q,j_q)}
\]

correctly accounts the total cost of the Assignment solution corresponding to $x$.

**Proof.** Let $x \in \text{Ext}(Q)$. Then, by Corollary 21, $x \in Q_I$. Hence, by Theorem 10, we have:

\[
(\exists i_1^x, i_2^x, i_{m-1}^x, i_m^x) \ (\exists G_{i_1^x, i_2^x, i_{m-1}^x, i_m^x} \in G_m)
\]

\[
[(\forall g, p, q \in S) \ (\forall \langle u_g, g, v_g \rangle, \langle u_p, p, v_p \rangle, \langle u_q, q, v_q \rangle \in A) \ [(x_{(u_g,g,v_g)}(u_p,p,v_p)(u_q,q,v_q) = 1) \iff ((g < p < q < m) \land ((u_g, g, v_g), (u_p, p, v_p), (u_q, q, v_q) \in G_{i_1^x, i_2^x, i_{m-1}^x, i_m^x})].
\]

$G_{i_1^x, i_2^x, i_{m-1}^x, i_m^x}$ can be expressed as $G_{i_1^x, i_2^x, i_{m-1}^x, i_m^x} = \{(i_{r}^x, r, i_{r+1}^x), r = 1, \ldots, m - 1 \}$ (Definition 3). The unique “full assignment” solution corresponding to $x$ is represented by the set of
nodes of the MAPG, \( a_x : = \{(i_r^x, r) \in N, r = 1, \ldots, m\} \). The total cost incurred by \( a_x \) is:

\[
T \text{CLAP}(a_x) = \sum_{r=1}^{m} w_{i_r^x, r}.
\]  \hspace{1cm} (96)

We will now account the total cost incurred by \( x \) using \( c^{\text{LAP}} \). We have:

| Component, \( x(1, 2, 3)(1, 2, 3)(1, 2, 3) \) | Cost, \( c^{\text{LAP}}(1, 2, 3)(1, 2, 3)(1, 2, 3) \) |
|-----------------------------------------------|-----------------------------------------------|
| \( g = 1 \land p = 2 \land q = 3 \)          | \( w_{i_1^x, 1} + w_{i_2^x, 2} + w_{i_3^x, 3} + w_{i_4^x, 4} \) |
| \( g = 1 \land p = 2 \land q = 4 \)          | \( w_{i_5^x, 5} \) |
| \( \cdots \)                                  | \( \cdots \)                                  |
| \( p = 1 \land r = m - 2 \land s = m - 1 \)  | \( w_{i_m^x, m} \) |
| Total cost attached to \( x \), \( V_{\text{LAP}}(x) = \) | \( \sum_{r=1}^{m} w_{i_r^x, r} \) |

It is easy to observe that \( V_{\text{LAP}}(x) = T \text{CLAP}(a_x) \). The theorem follows from this and the arbitrariness of \( x \). \( \blacksquare \)

### 5.2 Quadratic assignment problem

The QAP is perhaps one of the three top-most-studied problems in Operations Research. The two best-recognized seminal papers for the problem are those by Koopmans and Beckmann (1957) and Lawler (1963), respectively. \( NP\)-hardness was established in the 1970’s (Sahni and Gonzales (1976)). One of the earliest reviews can be found in Pardalos et al. (1994).

The constraints of the QAP constraints are the same as those of the LAP. The difference between the two problems however, is in the objective function which is linear in the LAP, whereas it is nonlinear for the QAP. We will use the facilities location/allocation context in the seminal paper of Koopmans and Bechmanns (1957) for the purpose of our illustration here. The two sets of objects to be matched are (generically) “departments” and “sites/locations.” There is a nonlinear assignment interaction cost component generically referred as the “material handling” cost. In addition to this there is a fixed cost associated with each “department”/“site” pairing decision. To apply our model to this context, let \( L \) and \( S \) (Notations 1.4–1.5) stand for the sets of “departments” and “sites,” respectively. Let the inter-departmental volumes of flows be denoted as \( f_{ij} \) (\( \forall (i, j) \in L^2 : i \neq j \)), and the inter-site distances be denoted by \( d_{rs} \) (\( \forall (r, s) \in S^2 : r \neq s \)). Letting \( h_{(i, r)(j, s)} \) denote the material handling cost of assigning departments \( i \) and \( j \) to sites \( r \) and \( s \), respectively, the expression for the \( h_{(i, r)(j, s)} \)’s is as follows:

\[
\forall (i, j) \in L^2 : i \neq j, \forall (r, s) \in S^2 : r \neq s,
\]

\[
h_{(i, r)(j, s)} = f_{ij}d_{rs} + f_{ji}d_{sr}.
\]  \hspace{1cm} (97)
There is also a fixed cost, \( a_{ir} \), which is incurred when \( i \in L \) is assigned to \( r \in S \). The optimization problem is to find an assignment which minimizes the total cost of the material handling and fixed costs. This problem can be solved as an LP over our proposed polytope \( Q \), by attaching proper costs to our modeling variables, as shown in the following theorem.

**Theorem 23** Let \( c^{QAP} \in \mathbb{R}^{m^2} \) be a vector of costs defined in terms of the QAP material handling and fixed costs, as follows:

\[
\forall \langle i_g, g, j_g \rangle, \langle i_p, p, j_p \rangle, \langle i_q, q, j_q \rangle \in A \ (x_{\langle i_g, g, j_g \rangle \langle i_p, p, j_p \rangle \langle i_q, q, j_q \rangle} \) not implicitly-zero by (5)),
\[
\begin{align*}
\forall (i_g, g, j_g) & \in A (x_{\langle i_g, g, j_g \rangle \langle i_p, p, j_p \rangle \langle i_q, q, j_q \rangle}) \\
& = \sum_{\langle i_g, g, j_g \rangle \langle i_p, p, j_p \rangle \langle i_q, q, j_q \rangle \in A^3} c^{QAP}_{\langle i_g, g, j_g \rangle \langle i_p, p, j_p \rangle \langle i_q, q, j_q \rangle} x_{\langle i_g, g, j_g \rangle \langle i_p, p, j_p \rangle \langle i_q, q, j_q \rangle} \\
& \quad \text{correctly accounts the total material handling and fixed costs} \\
& \quad \text{of the Assignment solution corresponding to } x. \\
\end{align*}
\]

**Proof.** Let \( x \in Ext(Q) \). Then, by Corollary \[21 \], \( x \in Q_I \). Hence, by Theorem \[10 \] we have:

\[
(\exists i^*_1, i^*_2, i^*_{m-1}, i^*_m \in G_m) \ \forall g, p, q \in S \ (\forall u_{g, p, v, q} \in A) \ (x_{\langle u_{g, p, v, q} \rangle = 1}) \iff (g < p < q < m) \wedge ((u_{g, p, v, q} \in G_{i^*_1, i^*_2, i^*_{m-1}, i^*_m}).
\]

\( G_{i^*_1, i^*_2, i^*_{m-1}, i^*_m} \) can be expressed as \( G_{i^*_1, i^*_2, i^*_{m-1}, i^*_m} = \{ \langle i^r, r \rangle \in N, r = 1, \ldots, m-1 \} \) (Definition \[3 \]). The unique “full assignment” solution corresponding to \( x \) is represented by the set of nodes of the MAPG, \( a_x := \{ \langle i^r, r \rangle \in N, r = 1, \ldots, m \} \). The total material handling and
fixed cost incurred by $a_x$ is:

$$TCQAP(a_x) = \sum_{r=1}^{m} o_{x,r} + \sum_{r=1}^{m-1} \sum_{s=r+1}^{m} h_{(i,x),r}(i,s).$$

(99)

We will now account the total cost incurred by $x$ using $c^{QAP}$. We have:

| Component, $x_{(i_p,j_p)}(i_p,j_p)(i_q,j_q)$ | Cost, $c^{QAP}_{(i_p,j_q),(i_q,j_q)}$ |
|---------------------------------------------|--------------------------------------|
| $g; p$                                      | $q$                                  |
| $1; 2$                                      | $3$                                  |
| $h_{(i,x),1}(i,x,2)$                       | $h_{(i,x),1}(i,x,3)$                 |
| $h_{(i,x),1}(i,x,4)$                       | $\vdots$                            |
| $m-1$                                      | $h_{(i,x),1}(i,x,m)$                |
| $r; r+1$                                    | $r+2$                                |
| $h_{(i,x),r}(i,x,r+1)$                     | $h_{(i,x),r}(i,x,r+2)$               |
| $h_{(i,x),r}(i,x,r+4)$                     | $\vdots$                            |
| $m-1$                                      | $h_{(i,x),r}(i,x,m)$                |
| $m-4; m-3$                                  | $m-2$                                |
| $h_{(i,x),m-4}(i,x,m-4)$                   | $h_{(i,x),m-4}(i,x,m-3)$             |
| $h_{(i,x),m-4}(i,x,m-2)$                   | $h_{(i,x),m-4}(i,x,m-1)$             |
| $m-1$                                      | $h_{(i,x),m-4}(i,x,m)$              |
| $m-3; m-2$                                  | $m-1$                                |
| $h_{(i,x),m-3}(i,x,m-3)$                   | $h_{(i,x),m-3}(i,x,m-2)$             |
| $h_{(i,x),m-3}(i,x,m-1)$                   | $h_{(i,x),m-3}(i,x,m)$              |
| $h_{(i,x),m-2}(i,x,m-2)$                   | $h_{(i,x),m-2}(i,x,m-1)$             |
| $h_{(i,x),m-2}(i,x,m)$                     | $h_{(i,x),m-2}(i,x,m)$              |
| $h_{(i,x),m-1}(i,x,m-1)$                   | $h_{(i,x),m-1}(i,x,m)$              |
| $h_{(i,x),m-1}(i,x,m)$                     | $h_{(i,x),m-1}(i,x,m)$              |
| $m$                                        | $m$                                  |
| $\sum_{r=1}^{m} o_{x,r} + \sum_{r=1}^{m-1} \sum_{s=r+1}^{m} h_{(i,x),r}(i,s)$ | $V^{QAP}(x)$ |

Hence, $V^{QAP}(x) = TCQAP(a_x)$, and the theorem follows from this and the arbitrariness of $x$. □

### 5.3 Traveling Salesman Problem

The traveling salesman problem (TSP) has been one of the most-studied problems over the past several decades. Many books that have been written on the problem and its variants include Lawler et al. (1985) and Diaby and Karwan (2016), among many others. An early, classical review paper is Balas and Toth (1985). The problem is simple to state: Starting
Let the Assignment Problem which is the subject of this paper. By attaching proper costs to our orders-of-visits/"times-of-travel" be $S = \{1, \ldots, m\}$, and the set of orders-of-visits/“times-of-travel” be $S = \{1, \ldots, m\}$, the problem of finding TSP tours then reduces to that of assigning each city in $L$ to an order-of-visit in $S$, which is the generic Assignment Problem which is the subject of this paper. By attaching proper costs to our modeling variables, the TSP can be solved in our higher-dimensional space as an LP, as we show in the following theorem.

**Theorem 24** Let $c_TSP \in \mathbb{R}^{n^3}$ be a vector of costs defined in terms of the TSP travel costs, $d$, as follows:

$$
\forall (i_g, g, j_g), (i_p, p, j_p), (i_q, q, j_q) \in N \{ x(i_g, g, j_g) \langle (i_p, p, j_p) \langle (i_q, q, j_q) \}
$$

The objective of the problem is to minimize the total cost of the travels. By setting one of the cities as the starting point and ending point of all the travels, the tour finding problem over the remaining cities reduces to that of finding an Assignment solution. In this illustration, we will fix city “0” as the starting and ending point of all the travels. Letting $m = n - 1$, the set of the remaining cities to visit be $L = C \setminus \{0\} = \{1, \ldots, m\}$, and the set of orders-of-visits/“times-of-travel” be $S = \{1, \ldots, m\}$, the problem of finding TSP tours then reduces to that of assigning each city in $L$ to an order-of-visit in $S$, which is the generic Assignment Problem which is the subject of this paper. By attaching proper costs to our modeling variables, the TSP can be solved in our higher-dimensional space as an LP, as we show in the following theorem.

Then, the following is true:

$$\forall x \in Ext(Q)$$

$$[V_{TSP} (x) := c_TSP x = \sum_{(i_g, g, j_g), (i_p, p, j_p), (i_q, q, j_q) \in A^3} c_{TSP}^{(i_g, g, j_g), (i_p, p, j_p), (i_q, q, j_q)} x_{G_{i_g, g, j_g}, (i_p, p, j_p), (i_q, q, j_q)}]$$

correctly accounts the total travel cost of the TSP tour corresponding to $x$.

**Proof.** Let $x \in Ext(Q)$. Then, by Corollary [21] $x \in Q_I$. Hence, by Theorem [10] we have:

$$\exists! (i_1^x, i_2^x, i_{m-1}^x, i_m^x) \exists! G_{i_1^x, i_2^x, i_{m-1}^x, i_m^x} \in G_m$$

$$[(\forall g, p, q \in S) (\forall (u_g, g, v_g), (u_p, p, v_p), (u_q, q, v_q) \in A) [(x_{(u_g, g, v_g), (u_p, p, v_p), (u_q, q, v_q)} = 1) \iff ((g < p < q < m) \land ((u_g, g, v_g), (u_p, p, v_p), (u_q, q, v_q) \in G_{i_1^x, i_2^x, i_{m-1}^x, i_m^x})].$$

$G_{i_1^x, i_2^x, i_{m-1}^x, i_m^x}$ can be expressed as $G_{i_1^x, i_2^x, i_{m-1}^x, i_m^x} = \{(i_r^x, r, i_{r+1}^x), r = 1, \ldots, m-1\}$ (Definition [3]). The unique “full assignment” solution corresponding to $x$ is represented by the set of nodes of the MAPG, $a_x := \{(i_r^x, r) \in N, r = 1, \ldots, m\}$.

The (unique) Hamiltonian cycle/TSP tour

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corresponding to \( a_x \) is the sequence \( \tau_x := \text{“}0\rightarrow i_1^x \rightarrow \ldots \rightarrow i_m^x \rightarrow \text{“}0\). The total cost of the travels involved in this sequence is:

\[
TCTSP(\tau_x) = d_{0,i_1^x} + d_{i_m^x,0} + \sum_{r=1}^{m-1} d_{i_r^x,i_{r+1}^x}.
\] (100)

We will now account the total cost incurred by \( x \) using \( c^{TSP} \). We have:

| Component, \( x(i_g,g,j_g)(i_p,p,j_p)(i_q,q,j_q) \) | Cost, \( c^{TSP} (i_g,g,j_g)(i_p,p,j_p)(i_q,q,j_q) \) |
|-------------------------------------------------|--------------------------------------------------|
| \( g = 1 \land p = 2 \land q = 3 \)           | \( d_{0,i_1^x} + d_{i_2^x,i_3^x} + d_{i_3^x,i_4^x} \) |
| \( g = 1 \land p = 2 \land q = 4 \)           | \( d_{i_3^x,i_4^x} \)                             |
| :                                              | :                                                |
| \( p = 1 \land r = m - 2 \land s = m - 2 \)   | \( d_{i_{m-2}^x,i_{m-1}^x} \)                   |
| \( g = 1 \land p = 2 \land q = m - 1 \)       | \( d_{i_{m-1}^x,i_m^x} + d_{i_m^x,0} \)         |
| Total cost attached to \( x \), \( V_{TSP}(x) = \) | \( d_{0,i_1^x} + d_{i_m^x,0} + \sum_{r=1}^{m-1} d_{i_r^x,i_{r+1}^x} \). |

Hence, \( V_{TSP}(x) = TCTSP(\tau_x) \), and the theorem follows from this and the arbitrariness of \( x \). ■

6 Conclusions

We have presented a new, network flow modeling-based linear programming (LP) reformulation of the well-known Assignment Problem (AP) polytope. The model is very-large-scale, having a \( O(m^9) \) computational complexity of size. We have illustrated a simple cost transformation procedure which allows for the quadratic assignment (QAP) and traveling salesman (TSP) problems to be solved as LPs in the space of the model can be extended straightforwardly to the cubic, quartic, quintic, and sextic assignment problems, as well as many of the other hard combinatorial optimization problems (COPs). Hence, the model represents a new affirmation of “\( P = NP \).” From a more Operations Research perspective, one important issue that may be fruitful for further research is the question of the significance of “side constraints” (the “Achilles heel,” so to speak, for traditional network flow modeling) within the context of the “complex flow” modeling we have introduced. We believe an examination of this in particular could potentially lead to a very useful broadening of the area of network flow modeling in general.
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