TRIMMING OF FINITE METRIC SPACES

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Abstract. We define a class of trim metric spaces and show that each finite metric space is isometric to the leaf space of a metric forest with trim base.

1. Introduction

Finite metric spaces naturally arise in mathematics, informatics and phylogenetics, see [DHM], [Li], [SS]. We introduce here a class of trim metric spaces and show that every finite metric space is the leaf space of a metric forest with trim base. We use the language of pseudometrics and begin by recalling the relevant terminology. A pseudometric space is a pair consisting of a set $X$ and a mapping $d : X \times X \to \mathbb{R}$ (the pseudometric) such that for all $x, y, z \in X$, $d(x, x) = 0$, $d(x, y) = d(y, x) \geq 0$, $d(x, y) + d(y, z) \geq d(x, z)$.

Pseudometric spaces $(X, d)$ and $(X', d')$ are isometric if there is a bijection $X \to X'$ carrying $d$ to $d'$. A map $q : X \to X'$ is non-expansive if $d(x, y) \geq d'(q(x), q(y))$ for all $x, y \in X$. A metric space is a pseudometric space $(X, d)$ such that $d(x, y) > 0$ for all distinct $x, y \in X$ (and then $d$ is a metric).

Given points $x, y, z$ of a pseudometric space $(X, d)$, we say following K. Menger [Mc] that $x$ lies between $y$ and $z$ if $x, y, z$ are pairwise distinct and

\[(1.0.1) \quad d(x, y) + d(x, z) = d(y, z) .\]

We say that a finite pseudometric space $(X, d)$ is trim if either $\text{card}(X) \leq 1$ or every point of $X$ lies between two other points of $X$ (that is for each $x \in X$, there are distinct $y, z \in X \setminus \{x\}$ satisfying (1.0.1)). It is easy to give examples of trim pseudometric spaces. For instance, any finite set with $\geq 3$ elements and zero pseudometric $d = 0$ is trim. On the other hand, the class of trim metric spaces is quite narrow. In particular, there are no trim metric spaces having just two points or three points. A finite subset of a Euclidean space with $\geq 2$ points and with the induced metric cannot be trim. Indeed, such a subset must contain a pair of points lying at the maximal distance; these points cannot lie between other points of the subset. Consequently, trim finite metric spaces having more than one point do not isometrically embed into Euclidean spaces.

We give here three examples of trim metric spaces:

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(1) the set of words of a fixed finite length \( \geq 2 \) in a given finite alphabet with the Hamming distance defined as the number of positions at which the letters of two words differ;
(2) a subset of a Euclidean circle \( C \subset \mathbb{R}^2 \) meeting each half-circle in \( C \) in at least three points; here the distance between points is the length of the shorter arc in \( C \) connecting these points;
(3) the 4-point metric space \( \{a, b, c, e\} \) with metric \( d \) defined by
\[
d(a, b) = d(c, e) = r, \quad d(a, c) = d(b, e) = s, \quad d(a, e) = d(b, c) = r + s,
\]
where \( r, s > 0 \) are real numbers. This is a special case of Example 2 arising from 4-point subsets of \( C \) formed by two pairs of diametrically opposite points.

Our main construction derives from any finite pseudometric space \( X \) a trim finite metric space \( c(X) \), called the trim core of \( X \), and a surjective non-expansive map \( q_X : X \to c(X) \). The construction of \( c(X) \) and \( q_X \) is functorial: each isometry of finite pseudometric spaces \( \varphi : X \to X' \) induces an isometry \( c(\varphi) : c(X) \to c(X') \) such that \( q_X \circ \varphi = c(\varphi) \circ q_X \) : \( X \to c(X') \). If \( X \) is a trim metric space, then \( c(X) = X \) and \( q_X \) is \( \text{id}_X \).

Pseudometric spaces whose trim core is a point can be described in terms of trees. By a tree, we mean a metric tree, i.e., a connected graph without cycles whose edges are endowed with non-negative real numbers called the lengths. The set of vertices of a tree \( \tau \) is the disjoint union \( \partial \tau = \{a, \ldots, \} \) if \( \tau \) is a vertex of \( \tau \) which is distinct from * and has degree 1. The set of all leaves of \( \tau \) is denoted by \( \partial \tau \); the pseudometric space \( (\partial \tau, d_\tau|_{\partial \tau}) \) is the leaf space of \( \tau \). A metric forest \( \zeta \) consists of a metric space \( B = (B, d_B) \) called the base and a family of rooted trees \( (\zeta_b, \ast_a)_{a \in B} \) called the components. By vertices, edges and leaves of \( \zeta \) we mean the vertices, edges and leaves of the components of \( \zeta \). The leaf space of \( \zeta \) is the disjoint union \( \partial \zeta = \bigcup_{a \in B} \partial \zeta_a \) with the following pseudometric \( d_\zeta \): for any \( x \in \partial \zeta_a, y \in \partial \zeta_b \) with \( a, b \in B \),
\[
(1.0.2) \quad d_\zeta(x, y) = \begin{cases} 
    d_\zeta_a(x, y) & \text{if } a = b, \\
    d_\zeta_a(x, \ast_a) + d_B(a, b) + d_\zeta_b(y, \ast_b) & \text{if } a \neq b.
\end{cases}
\]

For example, any metric space \( B \) determines a metric forest \( \{[0, 1]_a\}_{a \in B} \) where \([0, 1]_a\) is a copy of the segment \([0, 1]\) with vertices 0, 1, root 0, and zero length of the only edge. The leaf space of this metric forest is isometric to \( B \).

A metric forest is finite if its base is a finite metric space and all its components are finite rooted trees.

**Theorem 1.1.** The trim core of a finite pseudometric space \( X \) is a point if and only if \( \text{card}(X) = 1 \) or \( X \) is isometric to the leaf space of a finite tree.
Theorem 1.2. Any finite pseudometric space $X$ determines (in a canonical way) a finite metric forest with leaf space $X$ and base $c(X)$.

For finite pseudometric spaces $X, Y$, we write $X \geq Y$ if there is a finite metric forest with leaf space $X$ and base $Y$. Theorem 1.2 implies that $X \geq c(X)$ for any finite pseudometric space $X$. We now explain that this property may be used to characterize $c(X)$ at least up to isometry.

Theorem 1.3. For any finite metric forest $\zeta$ with trim base $B$, we have $c(\partial \zeta) = B$ (up to isometry).

Corollary 1.4. If a finite pseudometric space $X$ and a trim finite metric space $B$ satisfy $X \geq B$, then $c(X) = B$ (up to isometry).

We say that two finite pseudometric spaces are trim equivalent if their trim cores are isometric. The next theorem generalizes Theorem 1.3 to arbitrary finite metric forests.

Theorem 1.5. The leaf space and the base of any finite metric forest are trim equivalent to each other.

We define the trim core in Section 2 and prove Theorems 1.1–1.5 in Sections 3–5.

2. Trimming and the trim core

2.1. Trim spaces. Given a set $X$ and a map $d : X \times X \to \mathbb{R}$, we will use the same symbol $d$ for the map $X \times X \times X \to \mathbb{R}$ defined by

$$d(x, y, z) = \frac{d(x, y) + d(x, z) - d(y, z)}{2}$$

for all $x, y, z \in X$. We derive from any pseudometric $d : X \times X \to \mathbb{R}$ a function $d : X \to \mathbb{R}$: if $X$ has only one point, then $d = 0$; if $X$ has two points $x, y$, then $d(x) = d(y) = d(x, y)/2$; if $X$ has $\geq 3$ points, then for all $x \in X$,

$$d(x) = \inf_{y, z \in X \setminus \{x\}, y \neq z} d(x, y, z) \geq 0.$$

We say that a pseudometric space $(X, d)$ is trim if $d = 0$. It is clear that for finite $X$ this definition is equivalent to the one in the introduction.

2.2. Metric quotient. Any pseudometric space $(X, d)$ determines a metric space $(\tilde{X}, \tilde{d})$ called the metric quotient of $(X, d)$. Here $\tilde{X} = X/\sim_d$ where $\sim_d$ is the equivalence relation on $X$ defined by $x \sim_d y$ if $d(x, y) = 0$ for $x, y \in X$. The metric $\tilde{d}$ in $\tilde{X}$ is uniquely defined by $\tilde{d}(\tilde{x}, \tilde{y}) = d(x, y)$ where $x, y$ are any points of $X$ and $\tilde{x}, \tilde{y} \in \tilde{X}$ are their equivalence classes.

The metric quotient of a trim pseudometric space may be non-trim. For example, let $X = \{x, y, a, b\}$ and let the distance between any point of the set $\{x, y\}$ with any point of $\{a, b\}$ be 1 while all the other distances between points of $X$ be 0. The resulting pseudometric space $X$ is trim (for instance, $x$ lies between $y$ and $a$). However, the metric space $\tilde{X}$ has two points and is not trim.
2.3. Drift. Given a set \( X \), a map \( d : X \times X \to \mathbb{R} \) and a function \( f : X \to \mathbb{R} \), we define a map \( d^f : X \times X \to \mathbb{R} \) by the following rule: for any \( x, y \in X \),

\[
d^f(x, y) = \begin{cases} 
0 & \text{if } x = y, \\
(d(x, y) + f(x) + f(y)) & \text{if } x \neq y.
\end{cases}
\]

We say that \( d^f \) is obtained from \( d \) by a drift. The idea here is that each point of \( X \) is carried away from all the other points by the distance \( f(x) \).

**Lemma 2.1.** If \((X, d)\) is a pseudometric space and \( f : X \to \mathbb{R} \) is a function such that \( f(x) \geq -\tilde{d}(x) \) for all \( x \in X \), then \((X, d^f)\) is a pseudometric space. Moreover, if \( \text{card}(X) \geq 3 \) or \( \text{card}(X) = 2 \) and \( f \) is a constant function, then \( d^f = \tilde{d} + f \).

**Proof.** We first prove that for any distinct \( x, y \in X \),

\[
(2.3.1) \quad \tilde{d}(x) + \tilde{d}(y) \leq \tilde{d}(x, y).
\]

If \( X = \{x, y\} \), then (2.3.1) follows from the definition of \( \tilde{d} \). If \( \text{card}(X) \geq 3 \), we pick any \( z \in X \setminus \{x, y\} \). Then \( \tilde{d}(x) \leq \tilde{d}(x, y, z) \) and \( \tilde{d}(y) \leq \tilde{d}(y, x, z) \). So,

\[
\tilde{d}(x) + \tilde{d}(y) \leq \tilde{d}(x, y, z) + \tilde{d}(y, x, z) = (\tilde{d}(x, y) + \tilde{d}(y, x))/2 = \tilde{d}(x, y).
\]

We now check that \( d^f : X \times X \to \mathbb{R} \) is a pseudometric. Clearly, \( d^f \) is symmetric and, by definition, \( d^f(x, x) = 0 \) for all \( x \in X \). Formula (2.3.1) and the assumptions on \( f \) imply that for any distinct \( x, y \in X \),

\[
\tilde{d}(x) + \tilde{d}(y) \leq \tilde{d}(x, y) + f(x) + f(y) \geq \tilde{d}(x) + f(x) + \tilde{d}(y) + f(y) \geq 0.
\]

The triangle inequality for \( d^f \) may be rewritten as \( d^f(x, y, z) \geq 0 \) for any \( x, y, z \in X \). If \( x, y, z \) are pairwise distinct, then

\[
(2.3.2) \quad d^f(x, y, z) = \tilde{d}(x, y, z) + f(x).
\]

Hence \( d^f(x, y, z) \geq \tilde{d}(x) + f(x) \geq 0 \). If \( x = y \) or \( x = z \), then \( d^f(x, y, z) = 0 \); if \( y = z \), then \( d^f(x, y, z) = d^f(x, y) \geq 0 \).

The equality \( d^f = \tilde{d} + f \) follows from (2.3.2) if \( \text{card}(X) \geq 3 \) and from the definitions if \( \text{card}(X) = 2 \) and \( f = \text{const} \).

2.4. Trimming. Given a pseudometric space \((X, d)\), we can apply Lemma 2.1 to \( f = -\tilde{d} : X \to \mathbb{R} \). This gives a pseudometric in \( X \) denoted \( d^* \). By definition, for any \( x, y \in X \),

\[
(2.4.1) \quad d^*(x, y) = \begin{cases} 
0 & \text{if } x = y, \\
(d(x, y) - \tilde{d}(x) - \tilde{d}(y)) & \text{if } x \neq y.
\end{cases}
\]

Lemma 2.1 implies that \( d^* = 0 \) so that \((X, d^*)\) is a trim pseudometric space. Let \( t(X) = (X, d^*) \) be its metric quotient. We say that \( t(X) \) is obtained from \((X, d)\) by **trimming**. Note that the metric space \( t(X) \) may be non-trim.

By the definition of the metric quotient, \( \bar{X} = X/\sim_d \) and the projection \( p_X : X \to \bar{X} \) is a non-expansive surjection. It is bijective if and only if \( d^* \) is a metric. In this case, \( \bar{X} = X, \tilde{d}^* = d^* \) and \( t(X) \) is a trim metric space.

Though it is not used in the sequel, we compute the equivalence relation \( \sim_{d^*} \) in \( X \) directly from \( d \). Note first that if \( 1 \leq \text{card}(X) \leq 3 \), then all elements of \( X \) are related by \( \sim_{d^*} \) and \( t(X) = \{pt\} \) (exercise).
Lemma 2.2. Suppose that \( \text{card}(X) \geq 3 \). Two distinct elements \( x, y \in X \) are related by \( \sim_{t^n} \) if and only if for any distinct \( x', x'' \in X \setminus \{x\} \) and any distinct \( y', y'' \in X \setminus \{y\} \), we have
\[
2d(x, y) + d(x', x'') + d(y', y'') \leq d(x, x') + d(x, x'') + d(y, y') + d(y, y'').
\]

Proof. It follows from the definitions that
\[
x \sim_{t^n} y \iff d^*(x, y) = 0 \iff d(x, y) = d(x) + d(y).
\]
By (2.3.1), the latter equality holds iff \( d(x, y) \leq d(x) + d(y) \). Recall that
\[
d(x) = \inf_{x', x'' \in X \setminus \{x\}, x' \neq x''} d(x, x', x'') \quad \text{and} \quad d(y) = \inf_{y', y'' \in X \setminus \{y\}, y' \neq y''} d(y, y', y'').
\]
Thus,
\[
x \sim_{t^n} y \iff d(x, y) \leq d(x, x'') + d(y, y'')
\]
for any distinct \( x', x'' \in X \setminus \{x\} \) and distinct \( y', y'' \in X \setminus \{y\} \). Substituting here the defining expression of the function \( d : X \times X \times X \to \mathbb{R} \), we obtain the claim of the lemma. \( \square \)

2.5. The trim core. Starting from a pseudometric space \( X \) and iterating the trimming, we obtain metric spaces \( \{t^n(X)\}_{n \geq 1} \) and non-expansive surjections
\[(2.5.1) \quad X = t^0(X) \xrightarrow{p_n} t^1(X) \xrightarrow{p_1} t^2(X) \xrightarrow{p_2} \cdots \]
where \( p_n = p_{tn}(X) \) for all \( n \geq 0 \). We say that \( X \) has finite height if \( t^n(X) \) is a trim metric space for some \( n \geq 0 \). The minimal such \( n \) is the height of \( X \) and the corresponding metric space \( c(X) = t^n(X) \) is the trim core of \( X \). For a pseudometric space \( X \) of finite height \( n \geq 0 \), the map
\[
q_X = p_{n-1} \circ \cdots \circ p_1 \circ p_0 : X \to t^n(X) = c(X)
\]
is a non-expansive surjection while the maps \( \{p_m\}_{m \geq n} \) are isometries. Note that \( c(t(X)) = c(X) \) and, if \( n \geq 1 \), then \( \text{height}(t^n(X)) = n - 1 \). Clearly, \( \text{height}(X) = 0 \) if and only if \( X \) is a trim metric space and then \( c(X) = t(X) = X \).

These definitions and results apply, in particular, to any finite pseudometric space \( X \). Indeed, for \( N = \text{card}(X) \), at least one of the surjections \( p_0, p_1, \ldots, p_{N-1} \) in (2.5.1) must be bijective. Hence, \( \text{height}(X) \leq N \) and \( c(X) = t^N(X) \).

2.6. Remarks. 1. If a pseudometric space \( (X, d) \) has \( \geq 3 \) points, then the function \( d^* : X \to \mathbb{R} \) can be computed by a slightly shorter formula
\[
d^*(x) = \inf_{y, z \in X \setminus \{x\}} d(x, y, z)
\]
for any \( x \in X \). To deduce this from the definition of \( d^* \), it suffices to note that by (2.3.1), we have \( d^*(x) \leq d^*(x, y) = d(x, y) \) for all \( y \in X \setminus \{x\} \).

2. The maps (2.5.1) induce a sequence of equivalence relations \( \{\sim_n\}_{n \geq 1} \) in any pseudometric space \( (X, d) \): points of \( X \) are related by \( \sim_n \) if their images in \( t^n(X) \) are equal. Clearly, \( \sim_1 = \sim_{t^0} \) and \( x \sim_n y \implies x \sim_{n+1} y \) for all \( x, y \in X \).

3. For any \( X, d, f \) as in Lemma 2.1 the pseudometric space \( X^f = (X, df) \) satisfies \( (df)^* = d^* : X \times X \to \mathbb{R} \). Therefore \( t(X^f) = t(X) \), i.e., the trimming
turns $X$ and $X^f$ into the same metric space. Consequently, if $X$ has a finite height, then so does $X^f$ and $c(X^f) = c(X)$.

4. Consider a finite set $X$ and a map $h : X \times X \to \mathbb{R}$ such that $h(x, y) = h(y, x)$ for all $x, y \in X$. For $r \in \mathbb{R}$, we define a map $h_r : X \times X \to \mathbb{R}$ by

\[
h_r(x, y) = \begin{cases} 
0 & \text{if } x = y, \\
h(x, y) + r & \text{if } x \neq y.
\end{cases}
\]

It is clear that $h_r$ is a pseudometric in $X$ for all sufficiently big $r$. The metric space $(X, h) = (X, h_r)$ is easily seen to be independent of the choice of $r$. This defines the trimming of the pair $(X, h)$ and allows us to define the trim core of this pair to be the trim metric space $c((t(X, h)))$.

3. Proof of Theorem

3.1. Lemmas. We begin with two lemmas.

**Lemma 3.1.** Any pseudometric space $X = (X, d)$ determines for all $n \geq 1$ a metric forest with base $t^n(X)$ and leaf space $X$.

**Proof.** Set $X_0 = X$ and $d_0 = d$. For $i \geq 1$, let $X_i$ and $d_i : X_i \times X_i \to \mathbb{R}$ be the underlying set and the metric of the metric space $t^i(X)$. For all $i \geq 0$, consider the function $d_i : X_i \to \mathbb{R}$ and the map $p_i = p_{i+1} : X_i \to X_{i+1}$. We fix $n \geq 1$ and form a graph whose set of vertices is the disjoint union $\bigcup_{i=0}^n X_i$. Each $v \in X_i$ with $i \leq n - 1$ is connected to $p_i(v) \in X_{i+1}$ by an edge $e_v$ of length $d_i(v) \geq 0$. It is clear that the resulting graph, $\zeta$, is a disjoint union of trees, and each of these trees has precisely one vertex belonging to $X_n$. We take this vertex as the root. Then every point of $X_n$ is the root of a unique tree component of $\zeta$. Thus, $\zeta$ is a metric forest with base $X_n$. Note that the vertices of $\zeta$ lying in $X_0 = X$ have degree 1. The surjectivity of $p_0, p_1, \ldots, p_{n-2}$ implies that the vertices of $\zeta$ lying in $X_1 \sqcup X_2 \sqcup \cdots \sqcup X_{n-1}$ have degree $\geq 2$. Thus, $\partial \zeta = X$. We need only to show that the path pseudometric $d_\zeta$ in $\partial \zeta$ coincides with the original metric $d$ in $X$.

Observe that any $x \in X$ determines a sequence of points $\{x_i \in X_i\}_{i=0}^n$ by

\[
x_0 = x \text{ and } x_{i+1} = p_i(x_i) \text{ for } i \leq n - 1.
\]

Clearly, $x$ is a degree 1 vertex of the tree $\zeta_{x_n}$ with root $x_n$. We pick any $x, y \in X$ and prove that $d_\zeta(x, y) = d(x, y)$. It suffices to handle the case $x \neq y$. Suppose first that $x_n = y_n$ so that $x$ and $y$ are vertices of the same tree $\mu = \zeta_{x_n}$. Let $1 \leq m \leq n$ be the smallest index such that $x_m = y_m$. The shortest path from $x$ to $y$ in $\mu$ is formed by the edges

\[
e_{x_0}, e_{x_1}, \ldots, e_{x_{m-1}}, e_{y_{m-1}}, \ldots, e_{y_1}, e_{y_0}.
\]

By (I.0.2) and the definition of the path pseudometric $d_\mu$ in $\mu$,

\[
d_\zeta(x, y) = d_\mu(x, y) = \sum_{i=0}^{m-1} (d_i(x_i) + d_i(y_i)).
\]

By the definition of the metric $d_{i+1}$ in $X_{i+1}$, for all $i = 0, 1, \ldots, m - 1$, we have

\[
d_{i+1}(x_i) + d_{i+1}(y_i) = d_i(x_i, y_i) - d_{i+1}(x_{i+1}, y_{i+1}).
\]
Substituting these expressions in (3.1.1), we obtain
\[ d_\zeta(x, y) = d_0(x_0, y_0) - d_n(x_n, y_n) = d_0(x_0, y_0) = d(x, y). \]

Suppose now that \( x_n \neq y_n \) so that \( x, y \) lie in different trees \( \mu = \zeta_{x_n}, \eta = \zeta_{y_n} \) with roots respectively \( x_n \) and \( y_n \). The shortest path from \( x \) to \( x_n \) in \( \mu \) is formed by the edges \( \{e_i\}_{i=0}^{n-1} \) and similarly for \( y \). Hence,
\[ d_\mu(x, x_n) = \sum_{i=0}^{n-1} d_i(x_i) \quad \text{and} \quad d_\eta(y, y_n) = \sum_{i=0}^{n-1} d_i(y_i) \]
where \( d_\mu, d_\eta \) are the path pseudometrics in \( \mu, \eta \), respectively. By (1.0.2),
\[ d_\zeta(x, y) = d_\mu(x, x_n) + d_n(x_n, y_n) + d_\eta(y, y_n) \]
\[ = d_n(x_n, y_n) + \sum_{i=0}^{n-1} (d_i(x_i) + d_i(y_i)). \]
Substituting (3.1.2) for all \( i \), we obtain
\[ d_\zeta(x, y) = d_0(x_0, y_0) = d(x, y). \]

**Lemma 3.2.** Any pseudometric space \( X \) of finite height determines (in a canonical way) a metric forest with base \( c(X) \) and leaf space \( X \).

**Proof.** Set \( n = \text{height}(X) \geq 0 \). If \( n = 0 \), then \( c(X) = X \) and the metric forest \( \{[0,1]_a \}_{a \in X} \) described in the introduction satisfies the conditions of the lemma. If \( n \geq 1 \), then this lemma is a special case of Lemma 3.1. \( \blacksquare \)

3.2. **Proof of Theorem 1.2** Theorem 1.2 is a direct consequence of Lemma 3.2 since all finite metric spaces have finite height.

4. **Proof of Theorem 1.1**

4.1. **Lemmas.** Given a (non-rooted) tree \( \tau \), we let \( \tau^* \) be the tree obtained from \( \tau \) by changing to zero the lengths of all edges adjacent to degree 1 vertices while keeping the lengths of all other edges.

**Lemma 4.1.** Let \( \tau \) be a finite non-rooted tree with leaf space \( (X, d = d_\tau|_X) \). If \( \tau \) has no vertices of degree 2 and \( \text{card}(X) \geq 3 \), then the pseudometric \( d^* \) in \( X \) defined by (2.4.1) coincides with the path pseudometric of the tree \( \tau^* \).

**Proof.** We first compute the function \( d : X \to \mathbb{R} \) determined by the pseudometric \( d = d_\tau \) in \( X \). Pick any \( x \in X \) and let \( e_x \) be the unique edge of \( \tau \) adjacent to \( x \). Let \( L \geq 0 \) be the length of \( e_x \). We claim that \( d(x) = L \). To see it, we compute \( d(x, y, z) \in \mathbb{R} \) for any \( y, z \in X \setminus \{x\} \). Since \( \tau \) is a tree, there are injective paths \( p_x, p_y, p_z \) in \( \tau \) leading respectively from \( x, y, z \) to a vertex \( v \) of \( \tau \) and meeting solely in \( v \). Denoting by \( \ell(p) \) the length of a path \( p \), we obtain
\[ d(x, y) = d_\tau(x, y) = \ell(p_x) + \ell(p_y), \]
\[ d(x, z) = d_\tau(x, z) = \ell(p_x) + \ell(p_z), \]
\[ d(y, z) = d_\tau(y, z) = \ell(p_y) + \ell(p_z). \]
Consequently,
\[ d(x, y, z) = \frac{d(x, y) + d(x, z) - d(y, z)}{2} = \ell(p_x). \]

Since the path \( p_x \) has to traverse the edge \( e_x \), we have \( \ell(p_x) \geq L \). Thus, 
\[ d(x, y, z) \geq L. \]
To prove that \( d^\bullet(x, y) = L \), it remains to find distinct \( y, z \in X \setminus \{x\} \) such that the path \( p_x \) above consists only of the edge \( e_x \) so that \( d(x, y, z) = \ell(p_x) = L \). Such \( y, z \) do exist because the second endpoint of \( e_x \) has degree \( \geq 3 \).

The computation of \( d \) above and the definition of \( d^\bullet \) imply that \( d^\bullet \) is the path pseudometric of \( \tau^\bullet \). □

Lemma 4.2. For any finite pseudometric space \( X \), the following three conditions are equivalent:

(i) \( c(X) = \{pt\} \);
(ii) \( X \) is the leaf space of a finite rooted tree;
(iii) \( \text{card}(X) = 1 \) or \( X \) is the leaf space of a finite non-rooted tree.

Proof. The implication (i) \( \Rightarrow \) (ii) follows from Theorem 1.2. To prove the implication (ii) \( \Rightarrow \) (iii), suppose that \( X \) is the leaf space of a finite rooted tree \( \tau \) with root \( * \). If the degree of \( * \) is \( \neq 1 \), then the leaf spaces of \( \tau \) viewed as a rooted tree and a non-rooted tree coincide and we get (iii). If \( * \) has degree 1, then \( * \) is adjacent to a unique edge \( e \) connecting \( * \) to another vertex, \( v \), of \( \tau \). If \( v \) has degree 1, then \( \tau \) has no edges other than \( e \) and \( X = \{v\} \), so that \( \text{card}(X) = 1 \). If \( v \) has degree \( \geq 2 \), then \( X \) is the leaf space of the rooted tree obtained from \( \tau \) by deleting \( * \) and \( e \) and taking \( v \) for the root. This new rooted tree has less edges than \( \tau \) and the implication (ii) \( \Rightarrow \) (iii) follows by induction.

We now prove the implication (iii) \( \Rightarrow \) (i). If \( \text{card}(X) = 1 \), then \( c(X) = X = \{pt\} \). Suppose now that \( \text{card}(X) \geq 2 \) and \( X \) is the leaf space of a finite non-rooted tree \( \tau \). If \( \tau \) has a vertex of degree 2, then eliminating this vertex and uniting its adjacent edges into a single edge (the lengths add up) we obtain a new tree with the same leaf space. Similarly, if \( \tau \) has an edge of length zero with both endpoints of degree \( \neq 1 \), then contracting this edge, we obtain a new tree with the same leaf space. Thus, without loss of generality we can assume that \( \tau \) has no vertices of degree 2 and the length of each edge of \( \tau \) with both endpoints of degree \( \neq 1 \) is positive. We prove that \( c(X) = \{pt\} \) by induction on \( \text{card}(X) \). If \( \text{card}(X) = 2 \), then \( c(X) = t(X) = \{pt\} \). If \( \text{card}(X) \geq 3 \), then by Lemma 4.1 the pseudometric \( d^\bullet \) in \( X \) coincides with the path pseudometric of the tree \( \tau^\bullet \). The assumptions on \( \tau \) imply that the equivalence relation \( \sim_{d^\bullet} \) in \( X \) relates precisely those degree 1 vertices of \( \tau \) whose adjacent edges share the second endpoint. This equivalence relation is non-trivial: an easy induction on the number of edges shows that a finite tree having at least three vertices and no vertices of degree 2 must have distinct degree 1 vertices whose adjacent edges share an endpoint (cf. Lemma 5.1 below). Hence the set \( \tilde{X} = X/\sim_{d^\bullet} \) has less elements than \( X \). The pseudometric space \( t(X) = (\tilde{X}, d^\bullet) \) is the leaf space of the tree obtained from \( \tau^\bullet \) by deleting all but one vertices (and the adjacent edges) in
each equivalence class in $X$. By the definition of the trim core and the induction assumption, $c(X) = c(t(X)) = \{pt\}$. 

4.2. **Proof of Theorem 1.1** Theorem 1.1 follows from the equivalence of the conditions (i) and (iii) in Lemma 4.2.

5. **Proof of Theorems 1.3 and 1.5**

5.1. **Conventions and lemmas.** Throughout Section 5 all trees and forests are assumed to be finite. Every leaf $x$ of a rooted tree $\tau$ is adjacent to a single edge of $\tau$ whose other vertex is called the neighbor of $x$ (the neighbor may happen to be the root of $\tau$ but cannot be a leaf of $\tau$). Two leaves of a rooted tree are contiguous is they have the same neighbor. We say that a rooted tree is reduced if it has no vertices of degree 2 except possibly the root. The number of edges of a tree $\tau$ is denoted by $|\tau|$.

**Lemma 5.1.** Any reduced rooted tree $\tau$ with $|\tau| \geq 2$ has at least one pair of contiguous leaves.

**Proof.** We proceed by induction on $|\tau|$. If $|\tau| = 2$, then $\tau$ has a vertex of degree 2 which has to be the root because $\tau$ is reduced; the other two vertices of $\tau$ are contiguous leaves. The induction step goes as follows. Suppose that $|\tau| \geq 3$ and let $*$ be the root of $\tau$. If $*$ has degree 1, then eliminating $*$ and the adjacent edge, and taking the second vertex of this edge as the new root, we obtain a reduced rooted tree $\tau'$ with $|\tau'| = |\tau| - 1 \geq 2$. By the induction assumption, $\tau'$ has contiguous leaves. The same leaves are contiguous in $\tau$. If $*$ has degree $m \geq 2$, then $\tau$ is a union of $m$ reduced rooted trees $\{\tau_i\}_{i=1}^m$ meeting in the common root $*$. Clearly, $|\tau_i| < |\tau|$ for all $i$ and either $|\tau_i| = 1$ for all $i$ or there is an $i$ such that $|\tau_i| \geq 2$. In the former case, the vertices of $\{\tau_i\}_{i=1}^m$ distinct from $*$ are contiguous leaves of $\tau$. If $|\tau_i| \geq 2$, then by the induction assumption, $\tau_i$ has contiguous leaves. The same leaves are contiguous in $\tau$. 

The following claim is a version of Lemma 4.1 for metric forests.

**Lemma 5.2.** Let $\zeta$ be a metric forest whose components are reduced rooted trees and whose base $B = (B, d_B)$ is a trim metric space with $\geq 2$ points. Consider the leaf space $(\partial\zeta, d = d_\zeta)$ of $\zeta$. Then the pseudometric space $(\partial\zeta, d^*)$ is the leaf space of the metric forest $\zeta^*$ obtained from $\zeta$ by changing to zero the lengths of all edges adjacent to leaves (and keeping the lengths of all the other edges).

**Proof.** We compute the function $d : \partial\zeta \to \mathbb{R}$. Pick any $a \in \partial\zeta_a \subset \partial\zeta$ where $a \in B$. Let $e_x$ be the unique edge of $\zeta_a$ adjacent to $x$. Let $L$ be the length of $e_x$. We claim that $d(x) = L$. We first show that $d(x, y, z) \geq L$ for all $y, z \in \partial\zeta \setminus \{x\}$.

Case 1: $y, z \in \partial\zeta_a$. In this case the computations in the proof of Lemma 4.1 (for $\tau = \zeta_a$) apply and give $d(x, y, z) \geq L$.

Case 2: $y \in \partial\zeta_a$ and $z \in \partial\zeta_b$ with $b \in B \setminus \{a\}$. Then there are injective paths $p_x, p_y, p_z$ in $\zeta_a$ leading respectively from $x, y, z_a$ to a vertex $v$ of $\zeta_a$ and meeting solely in $v$. Then

$$d(x, y) = d_{\zeta_a}(x, y) = \ell(p_x) + \ell(p_y),$$
$$d(x, z) = d_c(x, z) = \ell(p_x) + \ell(p_z) + d_B(a, b) + d_{\zeta_a}(z, *_b),$$
$$d(y, z) = d_c(y, z) = \ell(p_y) + \ell(p_z) + d_B(a, b) + d_{\zeta_a}(z, *_b).$$

Consequently,

$$d(x, y, z) = \frac{d(x, y) + d(x, z) - d(y, z)}{2} = \ell(p_x) \geq L.$$ 

Case 3: $y \in \partial \zeta_a$ with $b \in B \setminus \{a\}$ and $z \in \partial \zeta_a$. This case is similar to Case 2.

Case 4: $y, z \in \partial \zeta_a$ with $b \in B \setminus \{a\}$. Then $d(y, z) = d_{\zeta_a}(y, z)$ and

$$d(x, y) = d_c(x, y) = d_{\zeta_a}(x, *_a) + d_B(a, b) + d_{\zeta_a}(y, *_b),$$
$$d(x, z) = d_c(x, z) = d_{\zeta_a}(x, *_a) + d_B(a, b) + d_{\zeta_a}(z, *_b).$$

Using the triangle inequality for $d_{\zeta_a}$, we deduce that $d(x, y, z) \geq d_{\zeta_a}(x, *_a) \geq L$.

Case 5: $y \in \partial \zeta_a$ and $z \in \partial \zeta_c$ with distinct $b, c \in B \setminus \{a\}$. Then

$$d(x, y) = d_c(x, y) = d_{\zeta_a}(x, *_a) + d_B(a, b) + d_{\zeta_a}(y, *_b),$$
$$d(x, z) = d_c(x, z) = d_{\zeta_a}(x, *_a) + d_B(a, c) + d_{\zeta_c}(z, *_c),$$
$$d(y, z) = d_c(y, z) = d_{\zeta_b}(y, *_b) + d_B(b, c) + d_{\zeta_c}(z, *_c).$$

Therefore

$$d(x, y, z) = d_{\zeta_a}(x, *_a) + \frac{d_B(a, b) + d_B(a, c) - d_B(b, c)}{2} \geq d_{\zeta_a}(x, *_a) \geq L.$$ 

To prove that $d(x) = L$ it remains to exhibit distinct $y, z \in \partial \zeta \setminus \{x\}$ such that $d(x, y, z) = L$. Suppose first that the edge $e_x$ connects $x$ to $*_a$. Since $B$ is trim, $a$ lies between certain points $b, c \in B$. Pick any $y \in \partial \zeta_b$ and $z \in \partial \zeta_c$. The computation in Case 5 yields $d(x, y, z) = d_{\zeta_a}(x, *_a) = L$. Suppose now that $e_x$ connects $x$ to a vertex $v \neq *_a$ of $\zeta_a$. Since $\zeta_a$ is reduced, degree$(v) \geq 3$. This implies the existence of a leaf $y \in \partial \zeta_a \setminus \{x\}$ such that certain injective paths $p_y, p_x$ in $\zeta_a$ lead respectively from $y, *_a$ to $v$ and meet solely in $v$. Pick any $b \in B \setminus \{a\}$ and $z \in \partial \zeta_b$. The computation in Case 2 (with $p_x = e_x$) shows that $d(x, y, z) = \ell(p_x) = L$.

The computation of $d$ above and the definition of $d^*$ imply that $d^*$ is the pseudometric in $\partial \zeta = \partial \zeta^*$ determined by the metric forest $\zeta^*$.

Lemma 5.3. Let $\zeta$ be a metric forest whose base $B$ is a trim metric space with $\geq 2$ points. Then at least one of the following two claims hold:

(i) there is a metric forest with less edges than $\zeta$, same base $B$ as $\zeta$ and the leaf space $\partial \zeta$ or $t(\partial \zeta)$.

Proof. Denote by $|\zeta|$ the number of edges of $\zeta$. The definition of a metric forest implies that $|\zeta| \geq \text{card}(B)$. If $|\zeta| = \text{card}(B)$, then each tree $\zeta_a$ with $a \in B$ is just an edge with two vertices (one of them being the root) and a certain length $f(a) \geq 0$. This defines a function $f : B \rightarrow \mathbb{R}$ such that $\partial \zeta = B^f$. So, (i) holds.

We say that a metric forest is reduced if all its components are reduced rooted trees having no edges of length zero with both endpoints of degree $\neq 1$. If $\zeta$ is not reduced, then eliminating vertices of degree 2 and contracting edges as in the proof of Lemma [7.2] we obtain a metric forest satisfying (ii).
It remains to prove (ii) in the case where $|\zeta| > \text{card}(B)$ and $\zeta$ is reduced. Since all components of $\zeta$ are reduced, Lemma 5.2 implies that the pseudometric space $(\partial\zeta, d^\bullet)$ (where $d = d_\zeta$) is the leaf space of the metric forest $\zeta^\bullet$ obtained from $\zeta$ by changing to zero the lengths of all edges adjacent to leaves. Since $\zeta$ is reduced, the lengths of all other edges of $\zeta^\bullet$ are non-zero. This implies that two distinct leaves $u, v$ of $\zeta$ are related by the equivalence relation $\sim_{d^\bullet}$ (i.e., satisfy $d^\bullet(u, v) = 0$) if and only if they are contiguous leaves of a tree component of $\zeta$. Then the metric space $t(\partial\zeta, d_\zeta) = (\tilde{\partial}\zeta, \tilde{d}^\bullet)$ is the leaf space of the metric forest $\zeta'$ obtained from $\zeta^\bullet$ by deleting all but one vertices (and the adjacent edges) in each equivalence class of leaves. Since $\zeta$ is reduced, the lengths of all other edges of $\zeta^\bullet$ are non-zero. This implies that two distinct leaves $u, v$ of $\zeta$ are related by the equivalence relation $\sim_{d^\bullet}$ (i.e., satisfy $d^\bullet(u, v) = 0$) if and only if they are contiguous leaves of a tree component of $\zeta$. Then the metric space $t(\partial\zeta, d_\zeta) = (\tilde{\partial}\zeta, \tilde{d}^\bullet)$ is the leaf space of the metric forest $\zeta'$ obtained from $\zeta^\bullet$ by deleting all but one vertices (and the adjacent edges) in each equivalence class of leaves. To check that $|\zeta'| < |\zeta|$, it suffices to note that the equivalence relation $\sim_{d^\bullet}$ in $\partial\zeta$ is non-trivial. Indeed, the condition $|\zeta| > \text{card}(B)$ implies that $\zeta$ has a tree component with $\geq 2$ edges and by Lemma 5.1 such a component has contiguous leaves.

5.2. Proof of Theorem 1.3. If $B = \{pt\}$, then $\zeta$ is a rooted tree and by Lemma 4.2, $c(\partial\zeta) = \{pt\} = B$. If $\text{card}(B) \geq 2$, then applying Lemma 5.3 recursively, we obtain that for a sufficiently big integer $N \geq 1$, the metric space $t^N(\partial\zeta)$ is obtained from $B$ by a drift. By Remark 2.6.3, $t^{N+1}(\partial\zeta) = t(t^N(\partial\zeta)) = t(B)$. Since $B$ is trim, $t(B) = B$. Thus, $t^{N+1}(\partial\zeta) = B$ is trim and $c(\partial\zeta) = B$.

5.3. Proof of Theorem 1.5. We first prove the transitivity of the relation $\geq$ between finite pseudometric spaces (cf. the introduction). Consider metric forests $\zeta = \{\zeta_a\}_{a \in A}$ and $\eta = \{\eta_b\}_{b \in B}$ with bases $A$ and $B$ respectively. We claim that if $A = \partial\eta$, then $\partial\zeta \geq B$. To see it, if for each $b \in B$ glue the trees $\eta_b$ and $\{\zeta_a\}_{a \in A}$ where $a$ runs over the leaves of $\eta_b$. The gluing goes by identifying each such $a$ with the root of $\zeta_a$ and produces a tree $\mu_b$. We take the root of $\eta_b$ as the root of $\mu_b$. Clearly, the metric forest $\{\mu_b\}_{b \in B}$ has the same leaf space as $\zeta$. Hence, $\partial\zeta \geq B$.

To prove Theorem 1.5 consider a metric forest $\zeta$ with base $A$. Then $\partial\zeta \geq A$ and, by Theorem 1.2, $A \geq c(A)$. Therefore $\partial\zeta \geq c(A)$, so that there is a metric forest with leaf space $\partial\zeta$ and base $c(A)$. Since $c(A)$ is trim, Theorem 1.3 implies that $c(\partial\zeta) = c(A)$. Thus, $\partial\zeta$ is trim equivalent to $A$.

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