On Thompson Sampling with Langevin Algorithms

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Abstract

Thompson sampling is a methodology for multi-armed bandit problems that is known to enjoy favorable performance in both theory and practice. It does, however, have a significant limitation computationally, arising from the need for samples from posterior distributions at every iteration. We propose two Markov Chain Monte Carlo (MCMC) methods tailored to Thompson sampling to address this issue. We construct quickly converging Langevin algorithms to generate approximate samples that have accuracy guarantees, and we leverage novel posterior concentration rates to analyze the regret of the resulting approximate Thompson sampling algorithm. Further, we specify the necessary hyperparameters for the MCMC procedure to guarantee optimal instance-dependent frequentist regret while having low computational complexity. In particular, our algorithms take advantage of both posterior concentration and a sample reuse mechanism to ensure that only a constant number of iterations and a constant amount of data is needed in each round. The resulting approximate Thompson sampling algorithm has logarithmic regret and its computational complexity does not scale with the time horizon of the algorithm.

1 Introduction

Sequential decision making under uncertainty has become one of the fastest developing fields of machine learning. A central theme in such problems is addressing exploration-exploitation tradeoffs [Auer et al., 2002, Lattimore and Szepesvári, 2020], wherein an algorithm must balance between exploiting its current knowledge and exploring previously unexplored options.

The classic stochastic multi-armed bandit problem has provided a theoretical laboratory for the study of exploration/exploitation tradeoffs [Lai and Robbins, 1985]. A vast literature has emerged that provides algorithms, insights, and matching upper and lower bounds in many cases. The dominant paradigm in this literature has been that of frequentist analysis; cf. in particular the analyses devoted to the celebrated upper confidence bound (UCB) algorithm [Auer et al., 2002]. Interestingly, however, Thompson sampling, a Bayesian approach first introduced almost a century ago [Thompson, 1933] has been shown to be competitive and sometimes outperform UCB algorithms in practice [Scott, 2010, Chapelle and Li, 2011]. Further, the fact that Thompson sampling, being a Bayesian method, explicitly makes use of prior information, has made it particularly popular in industrial applications [see, e.g., Russo et al., 2017, and the references therein].

Although most theory in the bandit literature is focused on non-Bayesian methods, there is a smaller, but nontrivial, theory associated with Thompson sampling. In particular, Thompson sampling has been shown to achieve optimal risk bounds in multi-armed bandit settings with Bernoulli rewards and beta
priors \cite{Kaufmann2012, Agrawal2013}, Gaussian rewards with Gaussian priors \cite{Agrawal2013}, one-dimensional exponential family models with uninformative priors \cite{Korda2013}, and finitely-supported priors and observations \cite{Gopalan2014}. Thompson sampling has further been shown to asymptotically achieve optimal instance-independent performance \cite{Russo2016}.

Despite these appealing foundational results, the deployment of Thompson sampling in complex problems is often constrained by its use of samples from posterior distributions, which are often difficult to generate in regimes where the posteriors do not have closed forms. A common solution to this has been to use approximate sampling techniques to generate samples from approximations of the posteriors \cite{Russo2017, Chapelle2011, Gomez-Uribet2016, Lu2017}. Such approaches have been demonstrated to work effectively in practice \cite{Riquelme2018, Urteaga2018}, but it is unclear how to maintain performance over arbitrary time horizons while using approximate sampling. Indeed, to the best of our knowledge the strongest regret guarantees for Thompson sampling with approximate samples are given by \cite{Lu2017} who require a model whose complexity grows with the time horizon to guarantee optimal performance. Further, it was recently shown theoretically by \cite{Phan2019} that a naive usage of approximate sampling algorithms with Thompson sampling can yield a drastic drop in performance.

Contributions In this work we analyze Thompson sampling with approximate sampling methods in a class of multi-armed bandit algorithms where the rewards are unbounded, but their distributions are log-concave. In Section 3 we derive posterior contraction rates for posteriors when the rewards are generated from such distributions and under general assumptions on the priors. Using these rates, we show that Thompson sampling with samples from the true posterior achieves finite-time optimal frequentist regret. Further, the regret guarantee we derive has explicit constants and explicit dependencies on the dimension of the parameter spaces, variance of the reward distributions, and the quality of the prior distributions.

In Section 4 we present a simple counterexample demonstrating the relationship between the approximation error to the posterior and the resulting regret of the algorithm. Building on the insight provided by this example, we propose two approximate sampling schemes based on Langevin dynamics to generate samples from approximate posteriors and analyze their impact on the regret of Thompson sampling. We first analyze samples generated from the unadjusted Langevin algorithm (ULA) and specify the runtime, hyperparameters, and initialization required to achieve an approximation error which provably maintains the optimal regret guarantee of exact Thompson sampling over finite-time horizons. Crucially, we initialize the ULA algorithm from the approximate sample generated in the previous round to make use of the posterior concentration property and ensure that only a constant number of iterations are required to achieve the optimal regret guarantee. Under slightly stronger assumptions, we then demonstrate that a stochastic gradient variant called stochastic gradient Langevin dynamics (SGLD) requires only a constant batch size in addition to the constant number of iterations to achieve logarithmic regret. Since the computational complexity of this sampling algorithm does not scale with the time horizon, the proposed method is a true “anytime” algorithm. Finally, we conclude in Section 5 by validating these theoretical results in numerical simulations where we find that Thompson sampling with our approximate sampling schemes maintain the desirable performance of exact Thompson sampling.

Our results suggest that the tailoring of approximate sampling algorithms to work with Thompson sampling can overcome the phenomenon studied in \cite{Phan2019}, where approximation error in the samples can yield linear regret. Indeed, our results suggest that it is possible for Thompson sampling to achieve order-optimal regret guarantees with an efficiently implementable approximate sampling algorithm.

2 Preliminaries

In this work we analyze Thompson sampling strategies for the $K$-armed stochastic multi-armed bandit (MAB) problem. In such problems, there is a set of $K$ options or “arms,” $A = \{1, ..., K\}$, from which a player must choose at each round $t = 1, 2, ...$. After choosing an arm $A_t \in A$ in round $t$, the player receives a real-valued reward $X_{A_t}$ drawn from a fixed yet unknown distribution associated with the arm, $p_a$. The random rewards...
A player’s objective in MAB problems is to maximize her cumulative reward over any fixed time horizon. The measure of performance most commonly used in the MAB literature is known as the expected regret, which is simply the expectation of the accrued reward minus the reward that would have been accrued had the learner selected the action with the highest mean reward during all steps. We remark that the analysis of Thompson sampling has often been focused on a different quantity known as the Bayes regret, which is simply the expectation of the accrued reward minus the reward that would have been accrued had the learner selected the action with the highest mean reward during all steps. However, in an effort to demonstrate that Thompson sampling is an effective alternative to frequentist methods like UCB, we analyze the frequentist regret.

Assumption 1 (Strong log-concavity of the family \( p_a(X | \theta_a) \)). Assume that \( \log p_a(X; \theta_a) \) is strongly concave around \( \theta_a^* \) for all \( X \in \mathbb{R} \) with parameter \( m_a > 0 \). We also assume that \( \nabla_{\theta} \log p_a(X; \theta_a) \) is continuously differentiable in \( \theta_a \) for all \( X \):

\[
(\nabla_{\theta} \log p_a(x; \theta_a) - \nabla_{\theta} \log p_a(x; \theta_a^*))^T (\theta_a - \theta_a^*) \geq m_a \| \theta_a - \theta_a^* \|^2_2, \quad \forall \theta_a \in \mathbb{R}^{d_a}, x \in \mathbb{R}.
\]

Additionally we make assumptions on the true distribution of the rewards:

Assumption 2 (Assumption on true reward distribution \( p_a(X | \theta_a^*) \)). For every \( a \in A \) assume that \( p_a(X; \theta_a^*) \) is strongly log-concave in \( X \) with some parameter \( \nu_a \), and that \( \nabla_{\theta} \log p_a(x; \theta_a^*) \) is \( L_a \)-Lipschitz in \( X \):

\[
(\nabla_x \log p_a(x; \theta_a^*) - \nabla_x \log p_a(x'; \theta_a^*))^T (x - x') \geq \nu_a \| x - x' \|_2^2, \quad \forall x, x' \in \mathbb{R}.
\]

Finally, we assume that for each arm \( a \in A \) there is an \( L_a \)-Lipschitz function \( f_a : \mathbb{R}^{d_a} \rightarrow \mathbb{R} \) such that for all \( \theta_a \in \mathbb{R}^{d_a} \), \( f_a(\theta_a) = E_{x \sim p_a(x | \theta_a)} [X] \). To further simplify notation, we define \( \bar{r}_a := f_a(\theta_a^*) \).

We now review Thompson sampling, the pseudo-code for which is presented in Algorithm 1. A key advantage of Thompson sampling over frequentist algorithms for multi-armed bandit problems is its use of prior information in the form of priors. In this paper, we assume that the prior distributions \( \pi_a(\theta_a) \) over the parameters of the arms have smooth log-concave densities:

Assumption 3 (Assumptions on the prior distribution). For every \( a \in A \) assume that \( \pi_a(\theta_a) \) is log-concave and assume that \( \nabla_{\theta} \log \pi_a \) is continuously differentiable for all \( \theta_a \in \mathbb{R}^{d_a} \).

Given the priors, Thompson sampling proceeds by maintaining a posterior distribution over the parameters of each arm \( a \) at each round \( t \). Letting \( T_a(t) \) be the number of samples received from arm \( a \) after \( t \) rounds, we let \( \mu_a^{T_a(t)} \) denote the probability measure associated with the posterior at round \( t \). In each round \( t = 1, 2, \ldots \), the algorithm samples from the posterior of each arm: \( \theta_{a,t} \sim \mu_a^{T_a(t)} \). It then chooses the arm for which the sample has the highest value:

\[
A_t = \arg\max_{a \in A} f(\theta_{a,t}).
\]

A player’s objective in MAB problems is to maximize her cumulative reward over any fixed time horizon \( T \). The measure of performance most commonly used in the MAB literature is known as the expected regret \( R(T) \), which corresponds to the expected difference between the accrued reward and the reward that would have been accrued had the learner selected the action with the highest mean reward during all steps \( t = 1, \ldots, T \).

Recalling that \( \bar{r}_a \) is the mean reward for arm \( a \in A \), the regret is given by:

\[
R(T) := \mathbb{E} \left[ \sum_{t=1}^{T} \bar{r}_{a^*} - \bar{r}_{A_t} \right],
\]

where \( \bar{r}_{a^*} = \max_{a \in A} \bar{r}_a \). Without loss of generality, we assume throughout this paper that the optimal arm, \( a^* = \arg\max_{a \in A} \bar{r}_a \), is arm 1. Further, we assume that the optimal arm is unique: \( \bar{r}_1 > \bar{r}_a \) for \( a > 1 \).

\[1\] We remark that the analysis of Thompson sampling has often been focused on a different quantity known as the Bayes regret, which is simply the expectation of \( R(T) \) over the priors: \( \mathbb{E}_\pi [R(T)] \). However, in an effort to demonstrate that Thompson sampling is an effective alternative to frequentist methods like UCB, we analyze the frequentist regret \( R(T) \).
Algorithm 1 Thompson sampling

Input: Priors $\pi_a$ for $a \in \mathcal{A}$.

1. Set $\mu_a,t = \pi_a$ for $a \in \mathcal{A}$ for $t = 0, 1, \cdots$ do
   2. Sample $\theta_{a,t} \sim \mu_a^{(T_a(t))}$ according to Algorithm 2.
   3. Choose action $A_t = \text{argmax}_{a \in \mathcal{A}} f_a(\theta_{a,t})$.
   4. Receive reward $X_{A_t}$.
   5. Update posterior distribution for arm $A_t$: $\mu_a^{(T_a(t+1))}$.

Traditional treatment of Thompson sampling algorithms often overlooks one of the most critical aspects: ensuring compatibility between the mechanism that produces samples from the posterior distributions and the algorithm’s regret guarantees. This issue is usually addressed by assuming that the prior distributions and the reward distributions are conjugate pairs. Although this approach is simple and prevalent in the literature [see, e.g., Russo et al., 2017], it fails to capture more complex distributional families for which this assumption may not hold. Indeed, it was recently shown in Phan et al. [2019] that if the samples come from distributions that approximate the posteriors with a constant error, the regret may grow at a linear rate. A more nuanced understanding of the relationship between the quality of the samples and the regret of the algorithms is, however, still lacking.

In the following sections we analyze Thompson sampling in two settings. In the first, the algorithm uses samples from the true posterior distribution, $\mu_a^{(T_a(t))}$, at each round. In the second, Thompson sampling makes use of samples coming from two approximate sampling schemes that we propose, such that the samples can be seen as coming from approximations of the posteriors, $\hat{\mu}_a^{(T_a(t))}$. We refer to the former as exact Thompson sampling, and the latter as approximate Thompson sampling.

For the analysis of exact Thompson sampling in Section 3 we derive posterior concentration theorems which characterize the rate at which the posterior distributions for the arms $\mu_a^n$ converge to delta functions centered at $\theta_a^*$ as a function of the number of samples received from the arm. We then use these rates to show that Thompson sampling in this family of multi-armed bandit problems achieves the optimal finite-time regret. Further, our results demonstrate an explicit dependence on the quality of the priors and other problem-dependent constants, which improve upon prior works.

In Section 4, we propose two efficiently implementable Langevin-MCMC-based sampling schemes for which the regret of approximate Thompson sampling still achieves the optimal logarithmic regret. To do so, we derive new results for the convergence of Langevin-MCMC-based sampling schemes in the Wasserstein-$p$ distance which we then use to prove optimal regret bounds.

3 Exact Thompson Sampling

In this section we first derive posterior concentration rates on the parameters of the reward distributions when the data, priors, and likelihoods satisfy our assumptions. We then make use of these concentration results to give finite-time regret guarantees for exact Thompson sampling in log-concave bandits.

3.1 Posterior Concentration Results

Core to the analysis of Thompson sampling is understanding the behavior of the posterior distributions over the parameters of the arms’ distributions as the algorithm progresses and samples from the arms are collected.

The literature on understanding how posteriors evolve as data is collected goes back to Doob [1949] and his proof of the asymptotic normality of posteriors. More recently, there has been a line of work [see, e.g., van der Vaart and van Zanten, 2008; Ghosal and van der Vaart, 2007] that analyzes the asymptotic rates of convergence of posteriors in various regimes, mostly following the framework first developed in Ghosal et al. [2000] for finite- and infinite-dimensional models. Finite-time rates remain an active area of research but have been developed using information theoretic arguments [Shen and Wasserman, 2001, Mou et al.] and more recently through the analysis of stochastic differential equations [Mon et al., 2019], though in both cases the
In this section we derive posterior concentration rates for parameters in $d$-dimensions and for a large class of priors and likelihoods by analyzing the moments of a stochastic differential equation for which the posterior is the limiting distribution. Our results expand upon the recent derivation of novel contraction rates for posterior distributions presented in Mou et al. [2019] to hold for a finite number of samples and may be of independent interest. We make use of these concentration results to show that Thompson sampling with such priors and likelihoods results in order-optimal regret guarantees.

Let $F_{n,a} : \mathbb{R}^d \rightarrow \mathbb{R}$ be $F_{n,a}(\theta_a) = \frac{1}{n} \sum_{i=1}^{n} \log p_a(X_i, \theta_a)$ be the sample average likelihood function, and $F_a(\theta_a) = E[\log p_a(X, \theta_a)]$ be the expected likelihood function. Classic results [Øksendal 2003] guarantee that, as $t \rightarrow \infty$ the distribution $P_t$ of $\theta_t$ which evolves according to:

$$d\theta_t = \frac{1}{2} \nabla_{\theta} F_{n,a}(\theta_t)dt + \frac{1}{2n} \nabla_{\theta} \log \pi(\theta_t)dt + \frac{1}{\sqrt{n}} dB_t,$$

is given by:

$$\lim_{t \rightarrow \infty} P_t(\theta|X_1, ..., X_n) \propto \exp(-nF_{n,a}(\theta) - \pi_a(\theta)),$$

almost surely. By construction, this limiting distribution is also the posterior distribution over $\theta$ given our prior and likelihood functions. Thus, by analyzing the limiting properties of $\theta_t$ as it evolves according to the stochastic differential equation, we can derive properties of the posterior distribution.

We first show that with high probability the gradient of $F_{n,a}(\theta^*)$ concentrates around zero (given the data $X_1, ..., X_n$). More exactly:

**Proposition 1.** The event $G_{a,n}(\delta)$, defined below, holds with probability at least $1 - \delta$:

$$G_{a,n}(\delta) = \left\{ \| \nabla_{\theta} F_{a,n}(\theta_a^*) \| \leq L_a \left[ \frac{2d_a \log(\frac{2}{\delta})}{m_a \nu_a} \right] \right\}.$$

The proof follows from the concentration of Lipschitz functions under strongly log-concave densities and relies on Assumption 2. It can be found in Appendix B. Conditioning on this high-probability event, we then analyze how the potential function,

$$V(\theta_t) = \frac{1}{2} e^{\alpha t} \| \theta_t - \theta^* \|_2^2,$$

evolves along trajectories of the stochastic differential equation, where $\alpha > 0$. By bounding the supremum of $V(\theta_t)$, we construct bounds on the higher moments of the random variable $\| \theta - \theta^* \|$ where $\theta \sim \mu^{(n)}$. These moment bounds translate directly into the posterior concentration bound of $\theta \sim \mu^{(n)}_a$ around $\theta^*$ presented in the following theorem the proof of which is deferred to Appendix B.

**Theorem 1.** Suppose that Assumptions 1-3 hold, then for $\delta_1, \delta_2 \in (0, 1)$, with probability at least $1 - \delta_1$ (with respect to the samples $X^{(n)}_a = X_{a,1}, ..., X_{a,n}$):

$$\mathbb{P}_{\theta_a \sim \mu_a^{(n)}}(\| \theta_a - \theta_a^* \|_2 < \sqrt{\frac{2e}{m_a n} \left( d_a + B_a + 32 \log(1/\delta_2) + \frac{2d_a L_a^2 \log(2/\delta_1)}{m_a \nu_a} \right)}) < \delta_2,$$

where $B_a = \max_{\theta \in \mathbb{R}^d} \log \pi_a(\theta) - \log \pi_a(\theta_a^*)$.

Thus, conditioned on the event $G_{a,n}(\delta_1)$, the posterior distribution over the parameters of the arms will concentrate at rate $\frac{1}{\sqrt{n}}$, where $n$ is the number of times the arm has been pulled.
We remark that the posterior concentration result we present in Theorem 1 has a number of desirable properties. Through the presence of \( B_a \), it reflects an explicit dependence on the quality of the prior. In particular, \( B_a = 0 \) if the prior is properly centered such that its mode is at \( \theta^* \) or if the prior is uninformative or nearly flat everywhere. We further remark that the concentration result also scales with the variance of \( \theta_a \) which is on the order of \( d_a/m_a n \). The bound also has an explicit dependence on the quality of the data received from the arm through its dependence on \( 1/\delta_1 \). Lastly, we remark that this concentration result holds for any \( n > 0 \) and the constants are explicitly defined in terms of the smoothness and structural assumptions on the priors, likelihoods, and reward distributions. This makes it more amenable for use in constructing regret guarantees, since we do not have to wait for a burn-in period for the result to hold, as in Shen and Wasserman [2001] and Mou et al. [2019]. Moreover, the dependence on the dimension of the parameter space and constants are explicit.

### 3.2 Exact Regret for Thompson Sampling

We now show that, under our assumptions, Thompson sampling with samples from the true posterior enjoys optimal finite-time regret guarantees. To provide these results we proceed as is common in regret proofs for multi-armed bandits by upper bounding the number of times a sub-optimal arm \( a \in A \) is pulled up to time \( T \), denoted \( T_a(T) \). Without loss of generality we assume throughout this section that arm 1 is the optimal arm, and define the filtration associated with a run of the algorithm as \( F_t = \{ A_1, X_1, A_2, X_2, ..., A_t, X_t \} \).

To upper bound the expected number of times a sub-optimal arm is pulled up to time \( T \), we first define the low-probability event that the mean calculated from the value of \( \theta_{a,t} \) sampled from the posterior at time \( t \leq T \), \( r_{a,t}(T_a(t)) \), is greater than \( \bar{r}_1 - \epsilon \) (recall that \( \bar{r}_1 \) is the optimal arm’s mean): \( E_a(t) = \{ r_{a,t}(T_a(t)) \geq \bar{r}_1 - \epsilon \} \) for some \( \epsilon > 0 \). Given these events, we proceed to decompose the expected number of pulls of a sub-optimal arm \( a \in A \) as:

\[
E[T_a(T)] = E \left[ \sum_{t=1}^{T} \mathbb{1}(A_t = a) \right] = E \left[ \sum_{t=1}^{T} \mathbb{1}(A_t = a, E_a^c(t)) \right] + E \left[ \sum_{t=1}^{T} \mathbb{1}(A_t = a, E_a(t)) \right].
\]

These two terms satisfy the following bounds:

**Lemma 1** (Bounding I and II). For a sub-optimal arm \( a \in A \), we have that:

\[
I \leq \ell + E \left[ \sum_{s=\ell}^{T-1} \frac{1}{p_{a,s}} - 1 \right];
\]

\[
II \leq 1 + E \left[ \sum_{s=1}^{T} \mathbb{1}(p_{a,s} > \frac{1}{T}) \right],
\]

where \( \ell > 0 \) is a positive integer and \( p_{a,s} = \mathbb{P}(r_{a,s}(s) > \bar{r}_1 - \epsilon | F_{t-1}) \), for some \( \epsilon > 0 \).

The proof of these results follows from a modification of the standard proofs for the regret of Thompson sampling and can be found in Appendix 1. Lemmas 11 and 12. To be able to use the posterior concentration result from Theorem 1 to further upper bound these expressions, we first define for each sub-optimal arm \( a \in A \) the event \( G_a(T) \) as the union of the high probability events defined in Proposition 4:

\[
G_a(T) = \left( \bigcap_{s=1}^{T-1} G_{a,s}(1/T^2) \right) \bigcap \left( \bigcap_{s=1}^{T-1} G_{1,s}(1/T^2) \right),
\]

This event is designed to ensure the posterior concentration in Theorem 1 holds (\( \delta_1 = 1/T^2 \)) for all time steps and for all arms with probability at least \( 1 - 1/T^2 \). Given this definition, we then show (in Lemma 13) that
if the likelihood and reward distributions satisfy Assumptions 1-3, the regret of exact Thompson sampling can be decomposed as:

$$\mathbb{E}[R(T)] \leq \sum_{a>1} \Delta_a \mathbb{E} \left[ T_a(T) \mid G_a(T) \right] + 2\Delta_a.$$

Invoking Lemma 1 on this conditional expectation allows us to invoke our posterior concentration results. We note that in contrast with simple subgaussian concentration bounds, our posterior concentration rates have a bias term decreasing at a rate of $1/\sqrt{\text{number of samples}}$. In our analysis we carefully track and control the effects of this bias term ensuring it does not compromise our log-regret guarantees. Indeed, using the posterior concentration in the bounds from Lemma 1 it can be shown that if we choose $\ell = \left\lceil \frac{8eL^2_{1,r}}{m\Delta_1^2} (D_1 + 32 \log 2) \right\rceil$, we have:

$$I \leq \left\lceil \frac{8eL^2_{1,r}}{m\Delta_1^2} (D_1 + 64 \log 2) \right\rceil + 1;$$

$$II \leq \frac{8eL^2_{a,r}}{m\Delta_1^2} (D_a + 32 \log(T)),$$

where $D_a = d_a + B_a + \frac{4d_aL^2_a \log 2T}{m_a \nu_a}$. Finally, combining all these observations we obtain the following regret guarantees:

**Theorem 2** (Regret of Exact Thompson Sampling). *When the likelihood and true reward distributions satisfy Assumptions 1-3, we have that the expected regret after $T > 0$ rounds of Thompson sampling with exact sampling satisfies:*

$$\mathbb{E}[R(T)] \leq \sum_{a>1} 4\Delta_a + \frac{C \ell^2}{m_a \Delta_a} \left( d_a + B_a + \log(T) + \frac{d_aL^2_a \log(2T)}{m_a \nu_a} \right) + \frac{C \ell^2}{m_1 \Delta_1} \left( d_1 + B_1 + 2 \log(2) + \frac{d_1L^2_1 \log(2T)}{m_1 \nu_1} \right),$$

*where $C$ is a universal constant which is independent of problem-dependent parameters.*

The proof of the theorem is deferred to Appendix D where we also provide the exact values of the universal constant $C$. We remark that this regret bound gives an $O\left(\frac{\log(T)}{\Delta} \right)$ asymptotic regret guarantee, but holds for any $T > 0$. This further highlights that Thompson sampling is a competitive alternative to UCB algorithms since it achieves the optimal problem-dependent rate for multi-armed bandit algorithms first presented in Lai and Robbins [1985].

Our bound also has explicit dependencies on the dimension of the parameter space of the likelihood distributions for each arm, as well as on the quality of the priors through the presence of $B_a$ and $B_1$. We note that the dependence on the priors does not distinguish between “good” and “bad” priors. Indeed, the parameter $B_a > 0$ is worst case, and does not capture the potential advantages of good priors in Thompson sampling, as we observe in our numerical experiments in Section 5. Finally, we note that our regret bound scales with the variances of the reward and likelihood families since $\frac{1}{m_a}$ and $\frac{1}{\nu_a}$ reflect the variance of the likelihoods in $\theta_a$ and the rewards $X_a$ respectively.

Thus, through the use of the posterior contraction rates we are able to get finite-time regret bounds for Thompson sampling with multi-dimensional log-concave families and arbitrary log-concave priors. This generalizes the result of Korda et al. [2013] to a more general class of priors and higher dimensional parametric families.
4 Approximate Thompson Sampling

In this section we present two approximate sampling schemes for generating samples from approximations of the posteriors at each round. For both, we give the values of the hyperparameters and computation time needed to guarantee an approximation error which does not result in a drastic change in the regret of the Thompson sampling algorithm.

Before doing so, however, we first present a simple counterexample to illustrate that in the worst case, Thompson sampling with approximate samples incurs an irreducible regret dependent on the error between the posterior and the approximation to the posterior. In particular, by allowing the approximation error to decrease over time, we extract a relationship between the order of the regret and the level of approximation.

Example 1. Consider a Gaussian bandit instance of two arms $A = \{1, 2\}$ having mean rewards $\bar{r}_1$ and $\bar{r}_2$ and known unit variances. Further assume that the unknown parameters are the means of the distributions such that $\theta^*_a = \bar{r}_a$, and consider the case where the learner makes use of a zero-mean, unit-variance Gaussian prior over $\theta_a$ for $a = 1, 2$. Under these assumptions, after obtaining samples $X_{a,1}, \ldots, X_{a,n}$, the posterior updates satisfy the following well-known formula:

$$P_{a,n}(\theta_a) \propto N\left(\frac{n}{n+1}, \frac{1}{n+1}\right).$$

Let $\bar{r}_1 = 1$ and $\bar{r}_2 = 0$ such that arm 1 is optimal. We now show there exists an approximate posterior $\tilde{P}_{a,t}$ of arm 2, satisfying $TV(\tilde{P}_{2,t}, P_{2,t}) \leq n^{-\alpha}$ and such that if samples from $P_{1,t}$ and $P_{2,t}$ were to be used by a Thompson sampling algorithm, its regret would satisfy $R(T) = \Omega(T^{1-\alpha}).$

We substantiate this claim by a simple construction. Let $\tilde{P}_{a,t}$ be $(1-n^{-\alpha}) P_{a,t} + n^{-\alpha} \delta_2$, where $\delta_2$ denotes a delta mass centered at 2. $\tilde{P}_{a,t}$ is a mixture distribution between the true posterior and a point mass.

Clearly, for all $t \geq C$ for some universal constant $C$, with probability at least $n^{-\alpha}$ the posterior sample from arm 2 will be larger than the sample from arm 1. Since $t > n$, $t^{-\alpha} < n^{-\alpha}$ for $\alpha > 0$ and since the suboptimality gap equals 1, we conclude $R(T) = \Omega(\sum_{t=1}^T t^{-\alpha})$. Thus, to incur logarithmic regret, one needs $TV(\tilde{P}_{2,t}, P_{2,t}) = \Omega\left(\frac{1}{n}\right)$.

Algorithm 2 (Stochastic Gradient) Langevin Algorithm for Arm $a$

Input : Data $\{x_{a,1}, \ldots, x_{a,n}\}$;
MCMC sample $\theta_{a,t-1}$ from last round

3 Set $\theta_0 = \theta_{a,t-1}$ for $a \in A$

for $i = 0, 1, \ldots, N$ do

4 Subsample $S \subseteq \{x_{a,1}, \ldots, x_{a,n}\}$

Compute $\nabla \hat{U}(\theta_{h(n)}) = -\frac{n}{|S|} \sum_{x_i \in S} \nabla \log p_a(x_i; \theta_i) - \nabla \log \pi_a(\theta_i)$.

Sample $\theta_{(i+1)h(n)} \sim N\left(\theta_{h(n)} - \hat{h}(n) \nabla \hat{U}(\theta_{h(n)}), 2\hat{h}(n)I\right)$.

Output : $\theta_{a,t} = \theta_{N\hat{h}(n)}$

Example 1 builds on the insights in Phan et al. [2019], who showed that constant approximation error can incur linear regret, which highlights the fact that to achieve logarithmic regret the total variation distance between the approximation of the posterior $\tilde{\mu}_a(n)$ and the true posterior $\mu_a(n)$ must decrease as samples are collected. In particular it illustrates that the rate at which the approximation error decreases is directly linked to the resulting regret bound.

Given this result, we first propose an unadjusted Langevin algorithm (ULA) [Durmus and Moulines, 2017], which generates samples from an approximate posterior which monotonically approaches the true posterior as data is collected and provably maintains the regret guarantee of exact Thompson sampling. Important to this effort, we demonstrate that the number of steps inside the ULA procedure does not scale with the time horizon, though the number of gradient evaluations scale with the number of times an arm has been pulled.
To obtain this result, we propose a stochastic gradient Langevin dynamics (SGLD) \cite{Welling2011} variant of ULA which has appealing computational benefits: under slightly stronger assumptions, SGLD takes a constant number of iterations as well as a constant number of data samples in the stochastic gradient estimate while maintaining the order-optimal regret of the exact Thompson sampling algorithm.

### 4.1 Convergence of (Stochastic Gradient) Langevin Algorithms

As described in Algorithm 2, in each round \( t \) we run the (stochastic gradient) Langevin algorithm for \( N \) steps to generate a sample of desirable quality for each arm. We initialize the algorithm for each arm in the \( t \)-th round using the sample obtained for the arm in the previous \((t-1)\)-th round. Within each iteration, the (non-stochastic) ULA algorithm takes the entire dataset \( \{X_{a,1}, \cdots, X_{a,n}\} \) to evaluate the gradient of the posterior and propose the next iterate. The SGLD algorithm, on the other hand, first subsamples the dataset, \( S \subset \{X_{a,1}, \cdots, X_{a,n}\} \), and forms a stochastic estimate of the true gradient. Both algorithms are described in Algorithm 2 (with ULA corresponding to taking \( S = \{X_{a,1}, \cdots, X_{a,n}\} \)).

**Remark 1.** We remark that we are able to keep the number of iterations, \( K \), for both algorithms constant by initializing the current round of the approximate sampling algorithm using the sample from the last round of the Thompson sampling algorithm. If we initialized the algorithm independently from the prior, we would need \( O(\log T_a(t)) \) iterations to achieve this result, which would in turn yield a Thompson sampling algorithm for which the computational complexity grows with the time horizon. We note that this warm-starting complicates the regret proof for the approximate Thompson sampling algorithms since the samples used by Thompson sampling are no longer independent.

To perform and analyze the gradient-based MCMC method, we first make a Lipschitz smoothness assumption on the log-likelihood:

**Assumption 4** (Lipschitz smoothness of the likelihood \( \log p_a(X|\theta) \): for ULA). Assume the Lipschitz smoothness of \( \log p_a(x; \theta) \) and \( \log \pi_a(\theta) \) in variable \( \theta \) so that for any \( x \), \( \theta_a \), and \( \theta'_a \):

\[
\| \nabla_\theta \log p_a(x; \theta_a) - \nabla_\theta \log p_a(x; \theta'_a) \| \leq L_a^U \| \theta_a - \theta'_a \|.
\]

If the likelihood satisfies Assumption 4, we prove (in Theorem 3 in the Appendix) that running ULA with exact gradients provides appealing convergence properties. In particular, for a number of iterations independent of the number of rounds \( t \) or the number of samples from an arm, \( n = T_a(t) \), ULA converges to an accuracy in Wasserstein-\( p \) distance which maintains the logarithmic regret of the exact algorithm (for more information on such metrics see \cite{Villani2009}). We note parenthetically that working with the Wasserstein-\( p \) distance provides us with a tighter MCMC convergence analysis (than with the total variation distance used in Example 1) that helps in conjunction with the regret bounds.

However, in each iteration, the number of gradient computations for ULA scales with the number of times an arm has been pulled, \( T_a(t) \). To tackle this issue, we sub-sample the data at each iteration and use a stochastic gradient MCMC method \cite{Ma2015}. To be able to get convergence guarantees despite the larger variance this method incurs, we make a slightly stronger Lipschitz smoothness assumption on the parametric family of likelihoods.

**Assumption 5** (Joint Lipschitz smoothness of the family \( \log p_a(X|\theta) \): strengthened for SGLD). Assume a joint Lipschitz smoothness condition, which strengthens Assumptions 2 and 4 to impose the Lipschitz smoothness on the entire bilinear function \( \log p_a(x; \theta) \). For any \( x \), \( x' \), \( \theta_a \), and \( \theta'_a \):

\[
\| \nabla_\theta \log p_a(x; \theta_a) - \nabla_\theta \log p_a(x'; \theta'_a) \| \leq L_a^U \| \theta_a - \theta'_a \| + L^\pi_a \| x - x' \|.
\]

Under this stronger assumption, we prove the fast convergence of the SGLD method in the following Theorem 4. Specifically, we demonstrate that for a suitable choice of stepsize \( h^{(n)} \), number of iterations \( K \), and subset \( S \), samples generated by Algorithm 2 are distributed sufficiently close to the true posterior to ensure the optimal regret guarantee. By examining the number of iterations \( K \) and size of the subset \( |S| \), we confirm that the algorithmic and sample complexity of our method do not grow with the number of rounds \( t \), as advertised.
Theorem 3 (SGLD Convergence). Assume that the family \( \log p_a(x; \theta) \), prior distributions, and true reward distributions satisfy assumptions 1-5. If we take the batch size \( k = \mathcal{O}\left(\frac{(L^2)^2}{m_a
u_a} d_a\right) \), step size \( h^{(n)} = \mathcal{O}\left(\frac{1}{n} (L^2)^2 d_a\right) \) and number of steps \( N = \tilde{\mathcal{O}}\left(\frac{(L^2)^2}{m_a^2} d_a\right) \) in the SGLD algorithm, then for \( \delta_1 \in (0, 1) \), with probability at least \( 1 - \delta_1 \) with respect to \( X_{a,1}, \ldots, X_{a,n} \), we have convergence of the SGLD algorithm in the Wasserstein-\( p \) distance. In particular, between the \( n \)-th and the \((n + 1)\)-th pull to arm \( a \), samples \( \theta_{a,t} \) approximately follow the posterior

\[
W_p\left(\hat{\mu}_a^{(n)}, \mu_a^{(n)}\right) \leq \sqrt{\frac{8}{nm_a}} \left(\frac{2dL_a^2 \log (2/\delta_1)}{m_a\nu_a} + d_a + B_a + 16p\right)^{\frac{1}{2}},
\]

where \( \hat{\mu}_a^{(n)} \) is the probability measure associated with any of the sample(s) \( \theta_{a,t} \) between the \( n \)-th and the \((n + 1)\)-th pull of arm \( a \).

We restate this theorem and make the constants explicit in Theorem 6 in the appendix, but remark that this bound in Wasserstein-\( p \) distance guarantees us the same posterior concentration result as in Theorem 1, except with a larger leading constant:

Lemma 2. Suppose that Assumptions 1-5 hold, then for \( \delta_1, \delta_2 \in (0, 1) \), with probability at least \( 1 - \delta_1 \) (with respect to the samples \( X_{a}^{(n)} = X_{a,1}, \ldots, X_{a,n} \)), the sample \( \theta_{a,t} \) resulting from running SGLD with \( N \) steps, a step size of \( h^{(n)} \), and a batch-size \( k \) as defined in Theorem 3 satisfies:

\[
\mathbb{P}_{\theta_{a,t} \sim \hat{\mu}_a^{(n)}} \left(\|\theta_{a,t} - \theta_a^*\|_2 > \sqrt{\frac{18e}{m_a n} \left(d_a + B_a + 32 \log (1/\delta_2) + \frac{2d_aL_a^2 \log (2/\delta_1)}{m_a\nu_a}\right)}\right) < \delta_2,
\]

where \( B_a = \max_{\theta \in \mathbb{R}^d} \log \pi_a(\theta) - \log \pi_a(\theta_a^*) \).

By scrutinizing the stepsizes \( h^{(n)} \) and the accuracy level of the sample distribution \( W_p\left(\hat{\mu}_a^{(n)}, \mu_a^{(n)}\right) \), we note that we are taking smaller steps to get increasingly accurate MCMC samples as more data are being collected. This is due to the need of decreasing the error incurred by discretizing the continuous Langevin dynamics and stochastically estimating the gradient of the log posterior. However, the number of iterations and subsampled gradients are not increasing since the concentration of the posterior provides us with stronger contraction of the continuous Langevin dynamics and requires less work because \( \mu_a^{(n)} \) and \( \mu_a^{(n+1)} \) are closer.

The proof of Theorem 3 hinges upon the contraction of the continuous Langevin dynamics as well as controlling the \( p \)-th order moments of the stochastic gradient error and the discretization error. Our proof (Lemma 7 in particular) leverages and strengthens recent works in the MCMC literature [see, e.g., Dalalyan and Karagulyan 2019, Durmus and Moulines 2016, Cheng and Bartlett 2018, Ma et al. 2019, Vempala and Wibisono 2019].

4.2 Thompson Sampling Regret with (Stochastic Gradient) Langevin Algorithms

Given that the concentration results of the samples from ULA and SGLD have the same form as that of exact Thompson sampling, we now show that approximate Thompson sampling achieves the same finite-time optimal regret guarantees (up to a constant factor) as the exact Thompson sampling algorithm.

We remark that the proof of the regret for approximate Thompson sampling is essentially the same—given our concentration results—as that of the regret for exact Thompson sampling. However, more care has to be taken to deal with the fact that the samples from the approximate posteriors are no longer independent due to the fact that we warm-start in our proposed sampling algorithms using a previous sample. We cope with this issue by constructing concentration rates (of a similar form as in Lemma 2) on the distributions of the samples given the initial sample is sufficiently well behaved (see Lemmas 9 and 10). We then show that this happens with sufficiently high probability to maintain our regret guarantees in Lemma 15, which in turn allows us to prove the following Theorem in Appendix D.2.
Theorem 4 (Regret of Thompson sampling with a (stochastic gradient) Langevin algorithm). When the likelihood and true reward distributions satisfy Assumptions 1-5: we have that the expected regret after $T > 0$ rounds of Thompson sampling with the (stochastic gradient) ULA method with the hyperparameters and runtime as described in Theorem 3 satisfies:

$$
\mathbb{E}[R(T)] \leq \sum_{a>1} 6\Delta_a + \frac{CL_{a,r}^2}{m_a \Delta_a} \left( d_a + B_a + \log(T) + \frac{d_a L_a^2 \log(2T)}{m_a \nu_a} \right) \\
+ \frac{CL_{1,r}^2}{m_1 \Delta_a} \left( d_1 + B_1 + 2 \log(2) + \frac{d_1 L_1^2 \log(2T)}{m_1 \nu_1} \right),
$$

where $C$ is a universal constant that is independent of problem dependent parameters.

We note that Theorem 3 allows for SGLD to be implemented with a constant number of steps per iteration and a constant batch-size with only the step-size decreasing linearly with the number of samples. Combining this with our regret guarantee shows that an anytime algorithm for Thompson sampling with approximate samples can indeed achieve logarithmic regret.

5 Numerical Experiments

We empirically corroborate our theoretical results with numerical experiments of approximate Thompson sampling in log-concave multi-armed bandit instances. We benchmark against both UCB and exact Thompson sampling across three different multi-armed bandit instances, where in the first instance, the priors reflect correct ordering of the mean rewards for all arms; in the second instance, the priors are agnostic of the ordering; in the third instance, the priors reflects the complete opposite ordering. See Appendix E for details of the experimental settings.

As suggested in our theoretical analysis in Section 4, we use a constant number of steps for both ULA and SGLD (with constant number of data points in the stochastic gradient evaluation) to generate samples from the approximate posteriors. The regret of the three algorithms averaged across 100 runs is displayed in Figure 1, where we see approximate Thompson sampling with samples generated by ULA and SGLD perform competitively against both exact Thompson sampling and UCB across all three instances.

![Figure 1: Performance of exact and approximate Thompson sampling vs UCB on Gaussian bandits with (a) “good priors” (priors reflecting the correct ordering of the arms’ means), (b) the same priors on all the arms’ means, and (c) “bad priors” (priors reflecting the exact opposite ordering of the arms’ means). The shaded regions represent the 95% confidence interval around the mean regret across 100 runs of the algorithm.](image)

We observe significant performance gains from the (approximate) Thompson sampling approach over the deterministic UCB algorithm when the priors are suggestive or even non-informative of the appealing arms.
When the priors are adversarial to the algorithm, the UCB algorithm outperforms the Thompson sampling approach as expected. (This case corresponds to the constant $B_a$ in the Theorems 2 and 4 being large). Also as the theory predicts, we observe little difference between the exact and the approximate Thompson sampling methods in terms of the regret. If we zoom in and scrutinize further, we can see that SGLD slightly outperforms the exact Thompson sampling method in the adversarial prior case. This might be due to the added stochasticity from the approximate sampling techniques, which improves the robustness against bad priors.

6 Conclusions

Although Thompson sampling has been used successfully in real-world problems for decades and has been shown to have appealing theoretical properties there remains a lack of understanding of how approximate sampling affects its regret guarantees.

In this work we derived new posterior contraction rates for log-concave likelihood families with arbitrary log-concave priors which capture key dependencies between the posterior distributions and various problem-dependent parameters such as the prior quality and the parameter dimension. We then used these rates to show that exact Thompson sampling in MAB problems where the reward distributions are log-concave achieves the optimal finite-time regret guarantee for MAB bandit problems from Lai and Robbins [1985]. As a direction for future work, we note that although our regret bound demonstrates a dependence on the quality of the prior, it still is unable to capture the potential advantages of good priors.

We then demonstrated that Thompson sampling using samples generated from ULA, and under slightly stronger assumptions, SGLD, could still achieve the optimal regret guarantee with constant algorithmic as well as sample complexity in the stochastic gradient estimate. Thus, by designing approximate sampling algorithms specifically for use with Thompson sampling, we were able to construct a computationally tractable anytime Thompson sampling algorithm from approximate samples with end-to-end guarantees of logarithmic regret.

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A Notation

Before presenting our proofs, we first include a table summarizing our notation.

| Symbol | Meaning |
|--------|---------|
| $\mathcal{A}$ | set of arms in bandit environment |
| $K$ | number of arms in the bandit environment $|\mathcal{A}|$ |
| $T$ | Time horizon |
| $A_t$ | arm pulled at time $t$ by the algorithm $A_t \in \mathcal{A}$ |
| $T_a(t)$ | number of times arm $a$ has been pulled by time $t$ |
| $X_{A_t}$ | reward from choosing arm $A_t$ at time $t$ |
| $\theta_a$ | parameters of likelihood functions such that, $\theta_a \in \mathbb{R}^{d_a}$ |
| $d_a$ | dimension of parameter space for arm $a$ |
| $p_a(x; \theta_a)$ | parametric family of reward distributions for arm $a$ |
| $\pi_a(\theta_a)$ | prior distribution over the parameters for arm $a$ |
| $\mu_a^{(n)}$ | probability measure associated with the posterior over the parameters of arm $a$ after $n$ samples from arm $a$ |
| $\mu_a^{(n)}$ | probability measure resulting from an approximate sampling method which approximates $\mu_a^{(n)}$ |
| $\theta_a^*$ | true parameter value for arm $a$ |
| $\theta_{a,t}$ | sampled parameter for arm $a$ at time $t$ of the Thompson Sampling algorithm: $\theta_{a,t} \sim \mu_a^{(n)}$ |
| $\bar{r}_a$ | mean of the reward distribution for arm $a$: $\bar{r}_a = \mathbb{E}[X_a | \theta_a^*]$ |
| $f_a(\theta_a)$ | function such that $\bar{r}_a = f_a(\theta_a^*)$ |
| $r_{a,t}(T_a(t))$ | estimate of mean of arm $a$ at round $t$: $r_{a,t}(T_a(t)) = f(\theta_{a,t})$ |
| $L_{a,r}$ | Lipschitz constant for $f_a$ |
| $m_a$ | Strong log-concavity parameter of the family $p_a(x; \theta)$ in $\theta$ for all $x$. |
| $\nu_a$ | Strong log-concavity parameter of the true reward distribution $p_a(x; \theta^*)$ in $x$. |
| $F_{n,a}(\theta_a)$ | Averaged log likelihood over the data points: $F_{n,a}(\theta_a) = \frac{1}{n} \sum_{i=1}^{n} \log p_a(X_i, \theta_a)$ |
| $L_a$ | Lipschitz constant for the true reward distribution $p_a(x; \theta^*)$ in $x$. |
| $g_a^*$ | maximum value of the log-potential function of arm $a$: $g_a^* = \max_{\theta \in \mathbb{R}^{d}} \log \pi_a(\theta)$ |
| $B_a$ | reflects the quality of the prior: $B_a = g_a^* - \log \pi_a(\theta^*)$ |

We also define a few notations used within the approximate sampling Algorithm 2.

| Symbol | Meaning |
|--------|---------|
| $N$ | number of steps of the approximate sampling algorithm |
| $h^{(n)}$ | step size of the approximate sampling algorithm after $n$ samples from the arm |
| $\theta_{i_k}^{(n)}$ | MCMC sample generated within $i$-th iteration of Algorithm 2 |
| $\mu_{i_k}^{(n)}$ | measure of $\theta_{i_k}^{(n)}$ |
| $k$ | batch-size of the stochastic gradient Langevin algorithm |

B Posterior Concentration Proof

To begin the proof of Theorem 1, we first prove that under our assumptions, the gradients of the population likelihood function concentrates.

**Proposition 2.** If the prior distribution over $\theta_a$ satisfies Assumption 3, then we have:

$$\sup_{\theta \in \mathbb{R}^{d_a}} \nabla \log \pi_a(\theta_a)^T(\theta_a - \theta_a^*) \leq g_a^* - \log \pi_a(\theta_a^*),$$

where $g_a^* = \max_{\theta \in \mathbb{R}^{d}} \log \pi_a(\theta_a)$. 

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Proof. Let \( \log \pi_a(\theta_a) = g(\theta_a) \). From the concavity of \( g \), we know that

\[
\nabla g(\theta_a)^T (\theta_a - \theta_a^*) \leq g(\theta_a) - g(\theta_a^*)
\]

Since this holds for all \( \theta \in \mathbb{R}^{d_a} \), we take the supremum of both sides and get that:

\[
\sup_{\mathbb{R}^{d_a}} \nabla g(\theta_a)^T (\theta_a - \theta_a^*) \leq g^* - g(\theta_a^*)
\]

Let \( B_a := g_a^* - \log \pi_a(\theta_a^*) \). If the prior is centered on the correct value of \( \theta_a^* \), then \( B_a = 0 \). Our posterior concentration rates will depend on \( B_a \).

Before proving the posterior concentration result we first present a lemma on the concentration of the empirical likelihood function at \( \theta_a^* \):

**Proposition 3.** The event \( G_{a,n}(\delta) \), defined below, holds with probability at least \( 1 - \delta \):

\[
G_{a,n} = \left\{ \| \nabla \theta F_{a,n}(\theta_a^*) \| \leq L_a \frac{\sqrt{2d_a \log \left( \frac{d_a}{n} \right)}}{\sqrt{n
a}} \right\}
\]

Proof. Recall that the true density \( p_a(x; \theta_a^*) \) is \( \nu_a \)-strongly log-concave in \( x \) and that \( \nabla \theta \log p_a(x; \theta_a^*) \) is \( L_a \)-Lipschitz in \( x \). Notice that \( \nabla \theta F_a(\theta_a^*) = 0 \) since \( \theta_a^* \) is the point maximizing the population likelihood.

Let’s consider the random variable \( Z = \nabla \theta \log p_a(X; \theta_a^*) \). Since \( \mathbb{E}[Z] = \nabla \theta F_a(\theta_a^*) \), the random variable \( Z \) is centered.

We start by showing \( Z \) is a subgaussian random vector. Let \( v \in S_{d_a} \) be an arbitrary point on the \( d_a \)-dimensional sphere and define the function \( V : \mathbb{R}^{d_a} \to \mathbb{R} \) as \( V(x) = \langle \nabla \theta \log p_a(x; \theta_a^*), v \rangle \). This function is \( L_a \)-Lipschitz. Indeed let \( x_1, x_2 \in \mathbb{R}^{d_a} \) be two arbitrary points in \( \mathbb{R}^{d_a} \):

\[
\begin{align*}
|V(x_1) - V(x_2)| &= |\langle \nabla \theta \log p_a(x_1; \theta_a^*) - \nabla \theta \log p_a(x_2; \theta_a^*), v \rangle | \\
&\leq ||\nabla \theta \log p_a(x_1; \theta_a^*) - \nabla \theta \log p_a(x_2; \theta_a^*)||_2 ||v||_2 \\
&= ||\nabla \theta \log p_a(x_1; \theta_a^*) - \nabla \theta \log p_a(x_2; \theta_a^*)||_2 \\
&\leq L_a ||x_1 - x_2||
\end{align*}
\]

The first inequality follows by Cauchy-Schwarz, the second inequality by the Lipschitz assumption on the gradients. After a simple application of Proposition 2.18 in [Ledoux, 2001], we conclude that \( V(x) \) is subgaussian with parameter \( \frac{L_a}{\sqrt{n
a}} \).

Since the projection of \( Z \) onto an arbitrary direction \( v \) of the unit sphere is subgaussian, with a parameter independent of \( v \), we conclude the random vector \( Z \) is subgaussian with the same parameter \( \frac{L_a}{\sqrt{n
a}} \). Consequently, the vector \( \nabla \theta F_{a,n}(\theta_a^*) \), being an average of \( n \) i.i.d. subgaussian vectors with parameter \( \frac{L_a}{\sqrt{n
a}} \) is also subgaussian with parameter \( \frac{L_a}{\sqrt{n
a}} \).

Since \( \nabla \theta F_{a,n}(\theta_a^*) \) is a subgaussian vector with parameter \( \frac{L_a}{\sqrt{n
a}} \), Lemma 1 of [Jin et al., 2019] implies it is norm subgaussian with parameter \( \frac{L_a}{\sqrt{n
a}} \). Definition 3 of the same source implies:

\[
\mathbb{P}(\|\nabla \theta F_{a,n}(\theta_a^*)\| \geq t) \leq 2 \exp \left( -\frac{t^2n
a}{2L_a^2} \right), \quad \forall t \in \mathbb{R}
\]

Equating \( 2 \exp \left( -\frac{t^2n
a}{2L_a^2} \right) = \delta \), yields \( t = L_a \frac{\sqrt{2\log(\frac{d}{n})}d_a}{n
a} \). The result follows.

Given these results we now prove Theorem 1. For clarity, we restate the theorem below:
Theorem B.1. Suppose that Assumptions 1-3 hold, then for \( \delta_1, \delta_2 \in (0, 1) \), with probability at least \( 1 - \delta_1 \) (with respect to the samples \( X_{a}^{(n)} = X_{a,1}, ..., X_{a,n} \)):

\[
P_{\theta_a \sim \mu_{a}^{(n)}} \left( \| \theta_a - \theta_a^* \|_2 > \sqrt{\frac{2e}{m a_n} \left( d_a + B_a + 32 \log 1/d_2 + \frac{2d_a L_2^a \log 2/\delta_1}{m a_n} \right)} \right) < \delta_2.
\]

Proof. The proof makes use of the techniques used to prove Theorem 1 in Mou et al. [2019]: analyzing how a carefully designed potential function evolves along trajectories of the s.d.e. By a careful accounting of the moments will imply that \( V_\theta \) has subgaussian tails with a rate that we make explicit.

Consider the s.d.e.:

\[
d\theta_t = \frac{1}{2} \nabla_\theta F_a(\theta_t) dt + \frac{1}{2n} \nabla_\theta \log \pi(\theta_t) dt + \frac{1}{\sqrt{n}} dB_t,
\]

and the potential function given by:

\[
V(\theta) = \frac{1}{2} e^{\alpha t \| \theta - \theta^* \|^2} - \frac{1}{2n} e^{\alpha t \| \theta - \theta^* \|^2} ds.
\]

for a choice of \( \alpha > 0 \). The idea is that bounds on the \( p \)-th moments of \( V(\theta_t) \) can be translated into bounds on the \( p \)-th moments of \( V(\theta) \) where \( \theta \sim \mu_{a}^{(n)} \), due to the fact that \( \lim_{n \to \infty} \theta_t = \theta \sim \mu_{a}^{(n)} \). The square-root growth in \( p \) of these moments will imply that \( \| \theta - \theta^* \|_2 \) has subgaussian tails with a rate that we make explicit.

We begin by using Ito’s Lemma on \( V(\theta_t) \):

\[
V(\theta_t) = T_1 + T_2 + T_3 + T_4,
\]

where:

\[
T_1 = \frac{1}{2} \int_0^t e^{\alpha s \langle \theta^* - \theta_s, \nabla_\theta F_a(\theta_s) \rangle} ds + \frac{\alpha}{2} \int_0^t e^{\alpha s \| \theta_s - \theta^* \|^2} ds
\]

\[
T_2 = \frac{1}{2n} \int_0^t e^{\alpha s \langle \theta^* - \theta_s, \nabla_\theta \log \pi(\theta_s) \rangle} ds
\]

\[
T_3 = \frac{d}{2n} \int_0^t e^{\alpha s} ds
\]

\[
T_4 = \frac{1}{\sqrt{n}} \int_0^t e^{\alpha s} (\theta_s - \theta^*, dB_s)
\]

Let us first upper bound \( T_1 \):

\[
T_1 = -\frac{1}{2} \int_0^t e^{\alpha s \langle \theta^* - \theta_s, \nabla_\theta F_a(\theta_s) \rangle} ds + \frac{\alpha}{2} \int_0^t e^{\alpha s \| \theta_s - \theta^* \|^2} ds
\]

\[
= -\frac{1}{2} \int_0^t e^{\alpha s \langle \theta^* - \theta_s, \nabla_\theta F_a(\theta_s) - \nabla_\theta F_a(\theta^*) \rangle} ds + \frac{\alpha}{2} \int_0^t e^{\alpha s \| \theta_s - \theta^* \|^2} ds
\]

\[
\leq \frac{\alpha - m}{2} \int_0^t e^{\alpha s \| \theta_s - \theta^* \|^2} ds - \frac{1}{2} \int_0^t e^{\alpha s \langle \theta^* - \theta_s, \nabla_\theta F_a(\theta^*) \rangle} ds
\]

\[
\leq \frac{\alpha - m}{2} \int_0^t e^{\alpha s \| \theta_s - \theta^* \|^2} ds + \frac{1}{2} \int_0^t e^{\alpha s \| \theta^* - \theta_s \| \| \nabla_\theta F_a(\theta^*) \|} ds
\]

\[
\leq \frac{\alpha - m}{2} \int_0^t e^{\alpha s \| \theta_s - \theta^* \|^2} ds + \frac{1}{2} \int_0^t e^{\alpha s \| \theta^* - \theta_s \| \epsilon(n, \delta_1)} ds,
\]

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where in (i) we use the strong-concavity property from Assumption 1 and in (ii) and (iii) we use Cauchy-Shwartz and the high-probability upper bound on $\|\nabla_\theta F_n(\theta^*)\|$ we have from our conditioning on $G_{a,n}(\delta_1)$.

Using Young’s inequality for products, where the constant is $m$, gives:

$$T1 \leq \frac{2\alpha - m}{4} \int_0^t e^{\alpha s}\|\theta_s - \theta^*\|^2 ds + \frac{\epsilon(n, \delta_1)^2}{4m} \int_0^t e^{\alpha s} ds$$

Finally, choosing $\alpha = m/2$ the first term on the RHS vanishes. Evaluating the integral in the second term on the RHS gives:

$$T1 \leq \frac{\epsilon(n, \delta_1)^2}{2m^2} (e^{\alpha t} - 1) \leq \frac{\epsilon(n, \delta_1)^2}{m^2} e^{\alpha t}.$$ 

Given our assumption on the prior, our choice of $\alpha = m/2$ and simple algebra, we can upper bound $T2$ and $T3$ as:

$$T2 = \frac{1}{2n} \int_0^t e^{\alpha s}\langle \theta_s - \theta^*, \nabla_\theta \log \pi(\theta_s) \rangle ds \leq \frac{B}{2\alpha n} (e^{\alpha t} - 1) \leq \frac{B}{nm} e^{\alpha t}$$

$$T3 = \frac{d}{2n} \int_0^t e^{\alpha s} ds \leq \frac{d}{nm} e^{\alpha t}.$$

We proceed to bound $T4$. Let’s start by defining:

$$M_t = \int_0^t e^{\alpha s}\langle \theta_s - \theta^*, dB_s \rangle,$$

so that:

$$T4 = \frac{M_t}{\sqrt{n}}.$$ 

Combining all the upper bounds of $T1, T2, T3$, and $T4$ we have that:

$$V(\theta_t) \leq \left( \frac{\epsilon(n, \delta_1)^2}{m^2} + \frac{d + B}{nm} \right) e^{\alpha t} + \frac{M_t}{\sqrt{n}}.$$ 

To find a bound for the $p$-th moments of $V$, we upper bound the $p$-th moments of the supremum of $M_t$ where $p \geq 1$:

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |M_t|^p \right] \leq (8p)^{\frac{p}{2}} \mathbb{E} \left[ (M_t, M_t)^{\frac{p}{2}}_T \right]$$

$$= (8p)^{\frac{p}{2}} \mathbb{E} \left[ \left( \int_0^T e^{2\alpha s}\|\theta_s - \theta^*\|^2 ds \right)^{\frac{p}{2}} \right]$$

$$\leq (8p)^{\frac{p}{2}} \mathbb{E} \left[ \left( \sup_{0 \leq t \leq T} e^{\alpha t}\|\theta_t - \theta^*\|^2 \int_0^T e^{\alpha s} ds \right)^{\frac{p}{2}} \right]$$

$$= (8p)^{\frac{p}{2}} \mathbb{E} \left[ \left( \sup_{0 \leq t \leq T} e^{\alpha t}\|\theta_t - \theta^*\|^2 \left( e^{\alpha T} - 1 \right) \right)^{\frac{p}{2}} \right]$$

$$\leq (8pe^{\alpha T})^{\frac{p}{2}} \mathbb{E} \left[ \left( \sup_{0 \leq t \leq T} e^{\alpha t}\|\theta_t - \theta^*\|^2 \right)^{\frac{p}{2}} \right]$$

Inequality (i) is a direct consequence of the Burkholder-Gundy-Davis inequality [Ren, 2008]. (ii) follows by pulling out the supremum out of the integral, (iii) holds because $e^{\alpha T} - 1 \leq e^{\alpha T}$. 

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Now, let us look at the moments of $V(\theta_t)$. To simplify notation, let

\[ U_t = \left( \frac{e(n, \delta_1)^2}{m^2} + \frac{d + B}{nm} \right) e^{\alpha t} \]

Note that $U_t$ is an increasing function of $t$.

\[
\mathbb{E} \left[ \left( \sup_{0 \leq t \leq T} V(\theta_t) \right)^p \right]^{\frac{1}{p}} \leq \mathbb{E} \left[ \left( \sup_{0 \leq t \leq T} U_t + \sup_{0 \leq t \leq T} \frac{|M_t|}{\sqrt{n}} \right)^p \right]^{\frac{1}{p}}
\]

\[
\leq \mathbb{E} \left[ \left( \sup_{0 \leq t \leq T} U_t + \sup_{0 \leq t \leq T} \frac{|M_t|}{\sqrt{n}} \right)^p \right]^{\frac{1}{p}}
\]

Via the Minkowski Inequality, we can expand the above as:

\[
\mathbb{E} \left[ \left( \sup_{0 \leq t \leq T} V(\theta_t) \right)^p \right]^{\frac{1}{p}} \leq U_T + \mathbb{E} \left[ \left( \sup_{0 \leq t \leq T} \frac{|M_t|}{\sqrt{n}} \right)^p \right]^{\frac{1}{p}}
\]

Using our upper bound on the supremum of $M_t$ gives:

\[
\mathbb{E} \left[ \left( \sup_{0 \leq t \leq T} V(\theta_t) \right)^p \right]^{\frac{1}{p}} \leq U_T + \mathbb{E} \left[ \left( \frac{8pe^{\alpha T}}{an} \right)^{\frac{p}{2}} \left( \sup_{0 \leq t \leq T} e^{\alpha t} ||\theta_t - \theta^*||_2^2 \right)^{\frac{p}{2}} \right]^{\frac{1}{p}}
\]

(1)

Let’s bound the second term on the RHS of the expression above:

\[
\mathbb{E} \left[ \left( \frac{8pe^{\alpha T}}{an} \right)^{\frac{p}{2}} \left( \sup_{0 \leq t \leq T} e^{\alpha t} ||\theta_t - \theta^*||_2^2 \right)^{\frac{p}{2}} \right]^{\frac{1}{p}} \leq \mathbb{E} \left[ \left( \frac{2^{p-1}}{2} \left( \frac{8pe^{\alpha T}}{an} \right)^{\frac{p}{2}} \left( \sup_{0 \leq t \leq T} e^{\alpha t} ||\theta_t - \theta^*||_2^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \right]
\]

\[
\leq 2^{\frac{p+2}{p}} \mathbb{E} \left[ \left( \frac{8pe^{\alpha T}}{an} \right)^p \right]^{\frac{1}{p}} + \frac{1}{2} \mathbb{E} \left[ \left( \sup_{0 \leq t \leq T} e^{\alpha t} ||\theta_t - \theta^*||_2^2 \right)^p \right]^{\frac{1}{p}}
\]

\[
\leq 16 \mathbb{E} \left[ \left( \frac{pe^{\alpha T}}{an} \right)^p \right]^{\frac{1}{p}} + \frac{1}{2} \mathbb{E} \left[ \left( \sup_{0 \leq t \leq T} e^{\alpha t} ||\theta_t - \theta^*||_2^2 \right)^p \right]^{\frac{1}{p}}
\]

Inequality (i) follows from using Young’s inequality for products on the term inside the expectation with constant $2^{p-1}$, inequality (ii) is a consequence of Minkowski Inequality and (iii) because $2^{\frac{p+2}{p}} \leq 2$. We note now that the second term $I$ on the right hand side above is exactly:

\[
\frac{1}{2} \mathbb{E} \left[ \left( \sup_{0 \leq t \leq T} V(\theta_t) \right)^p \right]^{\frac{1}{p}}
\]

Plugging this into Equation (1) and rearranging gives:

\[
\frac{1}{2} \mathbb{E} \left[ \left( \sup_{0 \leq t \leq T} V(\theta_t) \right)^p \right]^{\frac{1}{p}} \leq U_T + \frac{16pe^{\alpha T}}{an},
\]

which finally results in:

\[
\mathbb{E} \left[ \left( \sup_{0 \leq t \leq T} V(\theta_t) \right)^p \right]^{\frac{1}{p}} \leq 2 \left( \frac{e(n, \delta_1)^2}{m^2} + \frac{d + B}{mn} + \frac{16p}{mn} \right) e^{\alpha T}.
\]
Given this control on the moments of the supremum of $V(\theta_t)$ (recall $V(\theta) = \frac{1}{2}e^{\alpha t||\theta - \theta^*||^2_2}$), we finally construct the bound on the moments of $||\theta_T - \theta^*||$:

$$E[||\theta_T - \theta^*||^p]^{\frac{1}{p}} = E\left[e^{-\frac{\alpha t}{2} V(\theta_t)}\right]^{\frac{1}{p}}$$

$$(i) \leq E\left[e^{-\frac{\alpha t}{2}} \left(\sup_{0 \leq t \leq T} V(\theta_t)\right)\right]^{\frac{1}{p}}$$

$$= e^{-\frac{\alpha T}{2}} \left(E\left[ \left(\sup_{0 \leq t \leq T} V(\theta_t)\right)\right]\right)^{\frac{1}{p}}$$

$$(ii) \leq e^{-\frac{\alpha T}{2}} \left(2 \left(\frac{\epsilon(n, \delta_1)^2}{m^2} + \frac{d + B}{mn} + \frac{16p}{mn}\right) e^{\alpha T}\right)^{\frac{1}{2}}$$

$$= \sqrt{2} \left(\frac{\epsilon(n, \delta_1)^2}{m^2} + \frac{d + B}{mn} + \frac{16p}{mn}\right)^{\frac{1}{2}}$$

$$(iii) \leq \sqrt{2} \left(\frac{2dL_n^2 \log 2/\delta_1}{mn^2} + \frac{d + B}{mn} + \frac{16p}{mn}\right)^{\frac{1}{2}}$$

$$= \sqrt{\frac{2}{mn}} \left(\frac{2dL_n^2 \log 2/\delta_1}{mn} + d + B + 16p\right)^{\frac{1}{2}}.$$

Inequality $(i)$ follows from taking the supremum of $V(\theta_t)$, inequality $(ii)$ from plugging in the upper bound from Equation 2, and $(iv)$ is a consequence of Proposition 3.

Taking the limit as $T \to \infty$ and using Fatou’s Lemma, we therefore have that the moments of $E[||\theta - \theta^*||^p]^{\frac{1}{p}}$, with probability at least $1 - \delta_1$, grow at a rate of $\sqrt{p}$:

$$E[||\theta - \theta^*||^p]^{\frac{1}{p}} \leq \lim inf_{T \to \infty} E[||\theta_T - \theta^*||^p]^{\frac{1}{p}}$$

$$= \sqrt{2} \left(\frac{2dL_n^2 \log 2/\delta_1}{mn} + d + B + 16p\right)^{\frac{1}{2}}. \quad (3)$$

To simplify notation, let $D = \left(\frac{2dL_n^2 \log 2/\delta_1}{mn} + d + B\right)$. Therefore we have:

$$E[||\theta - \theta^*||^p]^{\frac{1}{p}} \leq \sqrt{\frac{2}{mn}} (D + 16p) \quad (4)$$

The result $(5)$ guarantees us that the norm of the uncentered random variable $\theta - \theta^*$ has subgaussian tails. We make the parameters explicit via Markov’s inequality:

$$P_{\theta \sim \mu_n^{(n)}} (||\theta - \theta^*|| > \epsilon) \leq \frac{E[||\theta - \theta^*||^p]}{\epsilon^p} \leq \left(\frac{\sqrt{2(D + 16p)}}{\sqrt{mn} \epsilon}\right)^p.$$

Choosing $p = 2 \log 1/\delta_2$ and letting

$$\epsilon = e^{\frac{1}{2} \sqrt{\frac{2}{mn}} (D + 16p)}$$

gives us our desired solution. With probability at least $1 - \delta_1$:

$$20$$
\[ \mathbb{P}_{\theta \sim \mu_a^{(n)}} \left( \| \theta - \theta^* \|_2 > \sqrt{\frac{2e}{mn} \left( d + B + 32 \log 1/\delta_2 + \frac{2dL^2\log 2/\delta_1}{mn} \right)} \right) < \delta_2. \]

\[ \square \]

C Proofs for Approximate MCMC Sampling

We restate the assumptions required of the likelihood for the convergence of the MCMC sampling methods.

**Assumption 4** (Lipschitz smoothness of the likelihood \( \log p_a(x|\theta) \): for ULA). Assume the Lipschitz smoothness of \( \log p_a(x; \theta) \) in variable \( \theta \) that for any \( \theta, \theta' \):

\[ \| \nabla_{\theta} \log p_a(x; \theta) - \nabla_{\theta} \log p_a(x; \theta') \| \leq L^U_a \| \theta - \theta' \|. \]

**Assumption 5** (Joint Lipschitz smoothness of the family \( \log p_a(X|\theta_a) \): strengthened for SGLD). Assume a joint Lipschitz smoothness condition, which strengthens upon Assumptions 2 and 4 to impose the Lipschitz smoothness on the entire bilinear function \( \log p_a(x; \theta) \). For any \( x, x', \theta, \theta' \):

\[ \| \nabla_{\theta} \log p_a(x; \theta) - \nabla_{\theta} \log p_a(x'; \theta') \| \leq L^U_a \| \theta - \theta' \| + L^* a \| x - x' \|. \]

We also include the Lipschitz smoothness assumption on the prior, which has been omitted for succinctness in the main text. 

**Assumption 6** (Lipschitz smoothness of the prior \( \pi_a(\theta_a) \): for ULA and SGLD). For each arm \( a \in \mathcal{A} \), for all \( \theta, \theta' \in \mathbb{R}^{d_a} \) the prior has \( L^\pi_a \) Lipschitz gradients:

\[ \| \nabla_{\theta} \log \pi_a(\theta_a) - \nabla_{\theta} \log \pi_a(\theta'_a) \| \leq L^\pi_a \| \theta - \theta' \|. \]

C.1 Convergence of the unadjusted Langevin algorithm (ULA)

If function \( \log p_a(x; \theta) \) satisfies the Lipschitz smoothness condition in Assumption 4, then we can leverage gradient based MCMC algorithms to generate samples with convergence guarantees in the \( \rho \)-Wasserstein distance. As stated in Algorithm 2, we initialize ULA in the \( n \)-th round from the last iterate in the \((n-1)\)-th round.

**Theorem 5** (ULA Convergence). Assume that the likelihood \( \log p_a(x; \theta) \) satisfies Assumptions 2, 3, 4. We take step size \( h^{(n)} = \frac{1}{\sqrt{n}} \frac{m_a}{(L^U_a + \frac{1}{2} L^* a)} \frac{1}{d_a} = O \left( \frac{m_a}{n \sqrt{d_a}} \right) \) and number of steps \( N = \tilde{O} \left( \frac{(L^U_a)^2}{m_a^2} d_a \right) \left( N = 160 \left( \frac{L^U_a + L^* a}{m_a^2} \right)^2 d_a \max \{ \log(d_a), 1 \} \right) \) in the first round and \( N = 640 \left( \frac{L^U_a + \frac{1}{2} L^* a}{m_a^2} \right)^2 d_a \) for \( n \geq 2 \) in Algorithm 2. If the posterior distribution satisfy the concentration inequality that \( \mathbb{P}_{\theta \sim \mu_a^{(n)}} \left( \| \theta - \theta^* \| \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{n}} \tilde{D} \), then for any positive even integer \( p \), we have convergence of the ULA algorithm in \( W_p \) distance to the posterior \( \mu_a^{(n)} \):

\[ W_p \left( \mu_a^{(n)} \right) \leq \frac{2}{\sqrt{n p}} \tilde{D}, \quad \forall \tilde{D} \geq \sqrt{\frac{32p}{m_a^2}}. \]

Proof of Theorem 5

We use induction to prove this theorem.

- For \( n = 1 \), we initialize at \( \theta_0 \) which is within a \( \sqrt{\frac{d_a}{m_a}} \)-ball from the maximum of the posterior, \( \theta^*_a = \arg \max p_a(\theta|x_1) \), where \( p_a(\theta|x_1) \propto p_a(x_1; \theta)p_a(\theta) \) and negative \( \log p_a(\theta|x_1) \) is \( m_a \)-strongly convex and \( (L^U_a + L^* a) \)-Lipschitz smooth. Invoking Lemma 8, we obtain that for \( d\mu_a^{(1)} = \mu_a(\theta|x_1)d\theta \), Wasserstein-\( p \) distance between the posterior and the point mass at its mode: \( W_p \left( \mu_a^{(1)}, \delta(\theta^*_a) \right) \leq 5 \sqrt{\frac{d_a p}{m_a}} \). Therefore,
Focusing on the large

Therefore, we invoke Lemma \ref{lem:convergence} with initial condition \( \mu_0 = \delta (\theta_p) \), to obtain the convergence in the \( N \)-th iteration of Algorithm \ref{alg:ULA} after the first pull to arm \( a \):

\[
W_p \left( \mu_{h(1)}, \delta (\theta_0) \right) \leq W_p \left( \mu_{h(1)}^1, \delta (\theta_p^1) \right) + \| \theta_0 - \theta_p^1 \| \leq 6 \sqrt{\frac{d_p \hat{m}}{m_a}}.
\]

We then substitute in the strong convexity \( m_a \) for \( \hat{m} \) and the Lipschitz smoothness \( (L^U_a + L^\pi_a) \) for \( \tilde{L} \). Plugging in the step size \( h(1) = \frac{1}{160 \sqrt{d_p \hat{m}}} \), we have substituted in the strong convexity \( m_a \) for \( \hat{m} \) and the Lipschitz smoothness \( (L^U_a + L^\pi_a) \) for \( \tilde{L} \). Plugging in the step size \( h(1) = \frac{1}{32 (L^U_a + L^\pi_a)^2} \hat{m}^2 \leq \min \left\{ \frac{1}{32 (L^U_a + L^\pi_a)^2} \frac{1}{1024 (L^U_a + L^\pi_a)^2} \frac{\tilde{D}^2}{d_p} \right\} \), and number of steps \( N = \frac{1}{m_a} \max \{ \log(d_a), 1 \} \), we have substituted in the strong convexity \( m_a \) for \( \hat{m} \) and the Lipschitz smoothness \( (L^U_a + L^\pi_a) \) for \( \tilde{L} \). We now prove that after the \( n \)-th pull and before the \( (n+1) \)-th pull, it is guaranteed that \( W_p \left( \mu_{h(n)}, \delta (\theta_p) \right) \leq \frac{2}{\sqrt{n}} \tilde{D} \).

We first obtain from the assumed posterior concentration inequality:

\[
W_p \left( \mu_{h(n)}, \delta (\theta^*) \right) \leq \mathbb{E}_{\theta \sim \mu_{h(n)}} \| \theta - \theta_p^* \| \leq \frac{1}{\sqrt{n}} \tilde{D}.
\]

Therefore, for \( n \geq 2 \),

\[
W_p \left( \mu_{h(n)}, \delta (\theta_p) \right) \leq W_p \left( \mu_{h(n)}, \delta (\theta^*) \right) + W_p \left( \mu_{h(n)}, \delta (\theta^*) \right) \leq \frac{3}{\sqrt{n}} \tilde{D}.
\]

We combine this bound with the induction hypothesis and obtain that

\[
W_p \left( \mu_{h(n)}, \delta (\theta_p) \right) \leq W_p \left( \mu_{h(n)}, \delta (\theta_p) \right) + W_p \left( \mu_{h(n)}, \delta (\theta_p) \right) \leq \frac{8}{\sqrt{n}} \tilde{D}.
\]

From Lemma \ref{lem:convergence}, we know that for \( \hat{m} = n \cdot m_a \) and \( \tilde{L} = n \cdot L^U_a + L^\pi_a \), with initial condition \( \mu_0 = \mu_{h(n)} \), with accurate gradient,

\[
W_p \left( \mu_{h(n)}, \mu_{h(n)} \right) \leq \left( 1 - \frac{\hat{m}}{8} h(n) \right)^{p-1} W_p \left( \mu_{h(n)}, \mu_{h(n)} \right) + 2^{p-1} \tilde{D}^p \left( d_a \right)^{p/2} \left( n \right)^{p/2}.
\]

If we take step size \( h(n) = \frac{1}{32 (L^U_a + L^\pi_a)^2} \hat{m}^2 \leq \min \left\{ \frac{1}{32 (L^U_a + L^\pi_a)^2} \frac{1}{1024 (L^U_a + L^\pi_a)^2} \frac{\tilde{D}^2}{d_a} \right\} \) and number of steps taken in the ULA algorithm from \( (n-1) \)-th pull till \( n \)-th pull to be: \( \hat{N} \geq \frac{20}{m_a} \frac{1}{h(n)} \),

\[
W_p \left( \mu_{h(n)}, \mu_{h(n)} \right) \leq \left( 1 - \frac{\hat{m}}{8} h(n) \right)^{p-1} \frac{8}{\tilde{D}} \left( d_a \right)^{p/2} \left( n \right)^{p/2} \leq \frac{2 \tilde{D}^p}{n^{p/2}}.
\]

leading to the result that \( W_p \left( \mu_{h(n)}, \mu_{h(n)} \right) \leq \frac{2}{\sqrt{n}} \tilde{D} \).

Since at least one round would have past from the \( (n-1) \)-th pull to the \( n \)-th pull to arm \( a \), taking number of steps in each round \( t \) to be \( N = \frac{20}{m_a} \frac{1}{h(t)} \) suffices.

Therefore,

\[
N = \begin{cases} 
160 \left( \frac{L^U_a + L^\pi_a}{m_a^2} \right)^2 d_a \max \{ \log(d_a), 1 \}, & n = 1; \\
640 \left( \frac{L^U_a + L^\pi_a}{m_a^2} \right)^2 d_a, & n \geq 2.
\end{cases}
\]

Focusing on the large \( n \) scenarios and assuming that \( \frac{1}{n} \tilde{D}^p \leq L^U_a \), we obtain that \( N = \tilde{O} \left( \frac{\left( L^U_a \right)^2}{m_a^2} d_a \right) \).
C.2 Convergence of the stochastic gradient Langevin algorithm (SGLD)

If \( \log p_a(x; \theta) \) satisfies a stronger joint Lipschitz smoothness condition in Assumption 5, similar guarantees can be obtained for stochastic gradient MCMC algorithms.

**Theorem 6 (SGLD Convergence).** Assume that the family \( \log p_a(x; \theta) \) satisfies Assumptions 7, 8. We take number of data samples in the stochastic gradient estimate \( k = 32 \left( \frac{L_s^2}{m \sigma_a^2} \right)^2 d_a \), step size \( h^{(n)} = \frac{1}{16} \left( \frac{L_s^2}{m \sigma_a^2} \right)^2 \frac{1}{d_a} = O \left( \frac{m_a}{n \left( \frac{L_s^2}{m_a} \right)^2} \right) \) and number of steps \( N = \tilde{O} \left( \frac{L_s^2}{m_a^2} d_a \right) \) in the first round and \( N = 1280 \left( \frac{L_s^2 + h L_s^2}{m_a^2} \right)^2 d_a \) for \( n \geq 2 \) in Algorithm 2. If the posterior distribution satisfy the concentration inequality that \( \mathbb{E}_{\theta \sim p_a^{(1)}} \left[ \| \theta - \theta^* \|^p \right]^{\frac{1}{p}} \leq \frac{1}{\sqrt{n}} \tilde{D} \), then for any positive even integer \( p \), we have convergence of the ULA algorithm in \( W_p \) distance to the posterior \( \mu_a^{(n)} : W_p \left( \hat{\mu}_a^{(n)}, \mu_a \right) \leq \frac{\sqrt{2}}{\sqrt{n}} \tilde{D} \), \( \forall \tilde{D} \geq \sqrt{32p} m_a \).

**Proof of Theorem 6.** Similar to Theorem 5, we use induction to prove this theorem. After the first pull to arm \( a \), we take the same 160 \( \left( \frac{L_s^2 + h L_s^2}{m_a^2} \right)^2 d_a \) number of steps to converge to \( W_p \left( \mu_a^{(1)}, \mu_a^{(1)} \right) \leq 2D_p \).

Assume that after the \((n-1)\)th pull and before the \(n\)th pull to the arm \( a \), the SGLD algorithm guarantees that \( W_p \left( \hat{\mu}_a^{(n-1)}, \mu_a^{(n-1)} \right) \leq \frac{\sqrt{2}}{\sqrt{n}} \tilde{D} \). We prove that after the \(n\)th pull and before the \((n+1)\)th pull, it is guaranteed that \( W_p \left( \hat{\mu}_a^{(n)}, \mu_a^{(n)} \right) \leq \frac{\sqrt{2}}{\sqrt{n}} \tilde{D} \). Following the proof of Theorem 5, we combine the assumed posterior concentration inequality and the induction hypothesis to obtain:

\[
W_p \left( \hat{\mu}_a^{(n)}, \mu_a^{(n)} \right) \leq W_p \left( \hat{\mu}_a^{(n)}, \mu_a^{(n-1)} \right) + W_p \left( \mu_a^{(n-1)}, \hat{\mu}_a^{(n-1)} \right) + 2p \sqrt{\frac{L \hat{d}}{m^p} (d_a p)^{p/2}} \left( h^{(n)} \right)^{p/2} + 2p^3 \frac{\Delta_p}{m^p}.
\]

Denote function \( U \) as the negative log-posterior density over parameter \( \theta \). From Lemma 3, we know that for \( \hat{m} = n \cdot m_a \) and \( \hat{L} = n \cdot L_s^2 + L_s^2 \), with initial condition that \( \mu_0 = \hat{\mu}_a^{(n-1)} \), if the difference between the stochastic gradient \( \nabla \hat{U} \) and the exact one \( \nabla U \) is bounded as \( \mathbb{E} \left[ \left\| \nabla U(\hat{\theta}) - \nabla \hat{U}(\theta) \right\|^p \right] \leq \Delta_p \), then

\[
W_p \left( \mu_{ih^{(n)}}, \mu_a^{(n)} \right) \leq \left( 1 - \frac{\hat{m}}{8} h^{(n)} \right)^{p-i} W_p \left( \hat{\mu}_a^{(n-1)}, \mu_a^{(n)} \right) + 2p^2 \frac{\hat{L} \hat{d}}{m^p} (d_a p)^{p/2} \left( h^{(n)} \right)^{p/2} + 2p^3 \frac{\Delta_p}{m^p}.
\]

We demonstrate in the following Lemma 3 that

\[
\Delta_p \leq 2 \frac{p^{p/2}}{k_{p/2}} \left( \frac{\sqrt{d_a \hat{L}^*}}{\sqrt{\hat{v}_a}} \right)^p.
\]

**Lemma 3.** Denote \( \hat{U} \) as the stochastic estimator of \( U \). Then for stochastic gradient estimate with \( k \) data points,

\[
\mathbb{E} \left[ \left\| \hat{U}(\theta) - \nabla U(\theta) \right\|^p \right] \leq 2 \frac{n^{p/2}}{k_{p/2}} \left( \frac{\sqrt{d_a \hat{L}^*}}{\sqrt{\hat{v}_a}} \right)^p.
\]

If we take the number of samples in the stochastic gradient estimator \( k = 32 \left( \frac{L_s^2}{m \sigma_a^2} \right)^2 d_a \), then \( \Delta_p \leq 2^{2p-5} \frac{\hat{m}^p \hat{D}_p}{n^{p/2}} \). Consequently, \( 2^{2p+3} \frac{\Delta_p}{\hat{m}^{p/2}} \leq \frac{1}{4} \frac{\hat{D}_p}{n^{p/2}} \).

If we take step size \( h^{(n)} = \frac{1}{32} \frac{\hat{m}^2}{n \sigma_a^2} \frac{1}{d_a} \leq \min \left\{ \frac{\hat{m}}{32 \hat{L}^2}, \frac{1}{1024} \frac{\hat{m}^2}{\hat{L}^2} \frac{\hat{d}_a p}{d_a} \right\} \) and number of steps taken in the SGLD algorithm from \((n-1)\)th pull till \(n\)th pull to be: \( \hat{N} \geq \frac{40}{\hat{m}} \frac{1}{n^{p/2}} \),

\[
W_p \left( \hat{\mu}_a^{(n)}, \mu_a^{(n)} \right) = W_p \left( \mu_{\hat{N} h^{(n)}}, \mu_a^{(n)} \right) \leq \left( 1 - \frac{\hat{m}}{8} h^{(n)} \right)^{p-\hat{N}} \frac{8 \hat{D}_p}{n^{p/2}} + 2p^2 \frac{\hat{L} \hat{d}}{m^p} (d_a p)^{p/2} \left( h^{(n)} \right)^{p/2} + 2p^3 \frac{\Delta_p}{m^p}.
\]

\leq \frac{2 \hat{D}_p}{n^{p/2}}.
leading to the result that $W_p \left( \hat{\mu}_a^{(n)}, \mu_a^{(n)} \right) \leq \frac{2}{\sqrt{n}} \tilde{D}$. Since at least one round would have past from the $(n - 1)$-th pull to the $n$-th pull to arm $a$, taking number of steps in each round $t$ to be $N = \frac{40}{m \alpha} \frac{1}{\sqrt{n}}$ suffices. Therefore,

$$N = \begin{cases} 
160 \left( \frac{L^U_a + L^s_a}{m_a} \right)^2 d_a \max \{ \log(d_a), 1 \}, & n = 1; \\
1280 \left( \frac{L^U_a + L^s_a}{m_a} \right)^2 d_a, & n \geq 2.
\end{cases}$$

Focusing on the large $n$ scenarios and assuming that $\frac{1}{n} L^u_a \leq L^u_a$, we obtain that $N = \tilde{O} \left( \frac{(L^U_a)^2}{m_a^2} d_a \right)$.

**Proof of Lemma 3.** We first develop the expression:

$$\mathbb{E} \left[ \left\| \nabla U(\theta) - \nabla \hat{U}(\theta) \right\|^p \right] = n^p \mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^n \nabla \log p(x_i, \theta_a) - \frac{1}{k} \sum_{j=1}^k \nabla \log p(x_j, \theta_a) \right\|^p \right]$$

$$= n^p \frac{k^p}{k^p} \mathbb{E} \left[ \left\| \sum_{j=1}^k \left( \frac{1}{n} \sum_{i=1}^n \nabla \log p(x_i, \theta_a) - \nabla \log p(x_j, \theta_a) \right) \right\|^p \right].$$

We note that

$$\nabla \log p(x_j, \theta_a) - \frac{1}{n} \sum_{i=1}^n \nabla \log p(x_i, \theta_a) = \frac{1}{n} \sum_{i \neq j} \left( \nabla \log p(x_j, \theta_a) - \nabla \log p(x_i, \theta_a) \right).$$

By the joint Lipschitz smoothness Assumption 5, we know that $\nabla \log p(x, \theta_a)$ is a Lipschitz function of $x$:

$$\| \nabla \log p(x_j, \theta_a) - \nabla \log p(x_i, \theta_a) \| \leq L^*_a \| x_j - x_i \|.$$

On the other hand, the data $x$ follows the true distribution $p(x; \theta^*)$, which by Assumption 2 is $\nu_a$-strongly log-concave. Applying Theorem 3.16 in [Wainwright 2019], we obtain that $(\nabla \log p(x_j, \theta_a) - \nabla \log p(x_i, \theta_a))$ is $\frac{4L^*_a}{\nu_a^2}$-sub-Gaussian. Leveraging the Azuma-Hoeffding inequality for martingale difference sequences [Wainwright 2019], we obtain that sum of the $(n - 1)$ sub-Gaussian random variables:

$$\left( \nabla \log p(x_j, \theta_a) - \frac{1}{n} \sum_{i=1}^n \nabla \log p(x_i, \theta_a) \right),$$

is $\frac{2\sqrt{n - 1}L^*_a}{n\nu_a^2}$-sub-Gaussian. In the same vein, $(\sum_{j=1}^k \left( \frac{1}{n} \sum_{i=1}^n \nabla \log p(x_i, \theta_a) - \nabla \log p(x_j, \theta_a) \right))$ is $\frac{2\sqrt{k(n - 1)}L^*_a}{n\nu_a^2}$-sub-Gaussian. We then invoke the $\frac{2\sqrt{k(n - 1)}L^*_a}{n\nu_a^2}$-sub-Gaussianity of

$$\left\| \sum_{j=1}^k \left( \frac{1}{n} \sum_{i=1}^n \nabla \log p(x_i, \theta_a) - \nabla \log p(x_j, \theta_a) \right) \right\|$$

and have

$$\mathbb{E} \left[ \left\| \sum_{j=1}^k \left( \frac{1}{n} \sum_{i=1}^n \nabla \log p(x_i, \theta_a) - \nabla \log p(x_j, \theta_a) \right) \right\|^p \right] \leq 2 \left( \frac{2\sqrt{d_a k(n - 1)}L^*_a}{en\nu_a} \right)^p.$$
Therefore,

$$E \left[ \left\| \nabla U(\theta) - \nabla \hat{U}(\theta) \right\|^p \right] = \frac{n^p}{k^p} E \left[ \left\| \sum_{j=1}^{k} \left( \frac{1}{n} \sum_{i=1}^{n} \nabla \log p(x_i, \theta_a) - \nabla \log p(x_j, \theta_a) \right) \right\|^p \right]$$

$$\leq 2 \frac{n^{p/2}}{k^{p/2}} \left( \frac{2 \sqrt{d_a \rho L_a^2}}{\epsilon \sqrt{\rho_a}} \right)^p \leq 2 \frac{n^{p/2}}{k^{p/2}} \left( \frac{\sqrt{d_a \rho L_a^2}}{\sqrt{\rho_a}} \right)^p.$$

\[\square\]

### C.3 Convergence of (Stochastic Gradient) Langevin Algorithm within Rounds

In this section, we examine convergence of the (stochastic gradient) Langevin algorithm to the posterior distribution over $a$-th arm at the $n$-th round. Since only the $a$-th arm and $n$-th round are considered, we drop these two indices in the notation whenever suitable. We also define some notation that will only be used within this subsection. For example, we focus on the $\theta$ parameter and denote the posterior measure $d\mu_a^{(n)}(x; \theta) = d\mu^*(\theta) = \exp (-U(\theta)) d\theta$ as the target distribution.

| Symbol | Meaning |
|--------|---------|
| $\mu^*$ | posterior distribution, $\mu_a^{(n)}$ |
| $U$ | potential (i.e., negative log posterior density) |
| $\theta_t^*$ | minimum of the potential $U$ (or mode of the posterior $\mu^*$) |
| $\theta_t$ | interpolation between $\theta_{ih(n)}$ and $\theta_{(i+1)h(n)}$, for $t \in [ih(n), (i+1)h(n)]$ |
| $\mu_t$ | measure associated with $\theta_t$ |
| $\theta_t^a$ | an auxiliary stochastic process with initial distribution $\mu^c$ and follows dynamics [12] |
| $\hat{m}$ | strong convexity of the potential $U, nm_a$ |
| $\hat{L}$ | Lipschitz smoothness of the potential $U, nL_a^U + L_a^*$ |

We also formally define the Wasserstein-$p$ distance used in the main text. Given a pair of distributions $\mu$ and $\nu$ on $\mathbb{R}^d$, a coupling $\gamma$ is a joint distribution over the product space $\mathbb{R}^d \times \mathbb{R}^d$ that has $\mu$ and $\nu$ as its marginal distributions. We let $\Gamma(\mu, \nu)$ denote the space of all possible couplings of $\mu$ and $\nu$. With this notation, the Wasserstein-$p$ distance is given by

$$W^p(\mu, \nu) = \inf_{\gamma \in \Gamma(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \|x - y\|^p d\gamma(x, y). \quad (8)$$

We use the following (stochastic gradient) Langevin algorithm to generate approximate samples from the posterior distribution $\mu_{a}^{(n)}(\theta)$ at $n$-th round. For $i = 0, \cdots, T$,

$$\theta_{(i+1)h(n)} \sim \mathcal{N} \left( \theta_{ih(n)} - h(n) \nabla \hat{U}(\theta_{ih(n)}), 2h(n)I \right), \quad (9)$$

where $\nabla \hat{U}(\theta_{ih(n)})$ is a stochastic estimate of $\nabla U(\theta_{ih(n)})$. We prove in the following Lemma 4 the convergence of this algorithm within $n$-th round.

**Lemma 4.** Assume that the potential $U$ is $\hat{m}$-strongly convex and $\hat{L}$-Lipschitz smooth. Further assume that the $p$-th moment between the true gradient and the stochastic one satisfies:

$$E \left[ \left\| \nabla U(\theta_{ih(n)}) - \nabla \hat{U}(\theta_{ih(n)}) \right\|^p \left| \theta_{ih(n)} \right. \right] \leq \Delta_p.$$ 

Then for $\mu_{ih(n)}$ following the (stochastic gradient) Langevin algorithm with $h \leq \frac{\hat{m}}{32L^2}$,

$$W^p_p \left( \mu_{ih(n)}, \mu^* \right) \leq \left( 1 - \frac{\hat{m}}{8} h(n) \right)^{-i} W^p_p \left( \mu_0, \mu^* \right) + 2^{i-1} \frac{\hat{m}^p \hat{L}_p}{\hat{m}^p} (d\rho)^{p/2} \left( \frac{h(n)}{n} \right)^{p/2} + 2^{2p+3} \frac{\Delta_p}{\hat{m}^p}. \quad (10)$$

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Remark 2. When $\Delta_p = 0$, Lemma 4 provides convergence rate of the unadjusted Langevin algorithm (ULA) with the exact gradient.

Proof of Lemma 4 We first interpolate a continuous time stochastic process, $\theta_t$, between $\theta_{ih(n)}$ and $\theta_{(i+1)h(n)}$. For $t \in [ih(n), (i+1)h(n)]$,

$$d\theta_t = \nabla U(\theta_{ih(n)})dt + \sqrt{2}dB_t,$$

where $B_t$ is standard Brownian motion. This process connects $\theta_{ih(n)}$ and $\theta_{(i+1)h(n)}$ and approximates the following stochastic differential equation which maintains the exact posterior distribution:

$$d\theta_t^* = \nabla U(\theta_t^*)dt + \sqrt{2}dB_t.$$  \hspace{1cm} (12)

For a $\theta_t^*$ initialized from $\mu^*$ and following equation (12), $\theta_t^*$ will always have distribution $\mu^*$.

We therefore design a coupling between the two processes: $\theta_t$ and $\theta_t^*$, where $\theta_t$ follows equation (11) (and thereby interpolates Algorithm 2) and $\theta_t^*$ initializes from $\mu^*$ and follows equation (12) (and thereby preserves $\mu^*$). By studying the difference between the two processes, we will obtain the convergence rate in terms of the Wasserstein-$p$ distance.

For $t = ih(n)$, we let $\theta_{ih(n)}$ to couple optimally with $\theta_{ih(n)}^*$, so that for

$$\theta_{ih(n)}, \theta_{ih(n)}^* \sim \gamma^* \in \Gamma_{opt}(\mu_{ih(n)}, \mu_{ih(n)}^*),$$

$$\mathbb{E} \left[ \left\| \theta_{ih(n)} - \theta_{ih(n)}^* \right\|^p \right] = W_p^p(\mu_{ih(n)}, \mu^*).$$

For $t \in [ih(n), (i+1)h(n)]$, we choose a synchronous coupling $\tilde{\gamma}(\theta_t, \theta_t^*|\theta_{ih(n)}^*, \theta_{ih(n)}^*) \in \Gamma(\mu_t(\theta_{ih(n)}), \mu_t^*(\theta_{ih(n)}))$ for the laws of $\theta_t$ and $\theta_t^*$. (A synchronous coupling simply means that we use the same Brownian motion $B_t$ in defining $\theta_t$ and $\theta_t^*$.) We then obtain that for any pair $(\theta_t, \theta_t^*) \sim \tilde{\gamma}$,

$$\frac{d\|\theta_t - \theta_t^*\|^p}{dt} = \|\theta_t - \theta_t^*\|^{p-2}\langle \theta_t - \theta_t^*, \frac{d\theta_t}{dt} - \frac{d\theta_t^*}{dt} \rangle$$

$$= p\|\theta_t - \theta_t^*\|^{p-2}\langle \theta_t - \theta_t^*, -\nabla U(\theta_t) + \nabla U(\theta_t^*) \rangle$$

$$+ p\|\theta_t - \theta_t^*\|^{p-2}\langle \theta_t - \theta_t^*, \nabla U(\theta_t) - \nabla \tilde{U}(\theta_{ih(n)}) \rangle$$

$$\leq -p\bar{m}\|\theta_t - \theta_t^*\|^{p} + p\|\theta_t - \theta_t^*\|^{p-1}\|\nabla U(\theta_t) - \nabla \tilde{U}(\theta_{ih(n)})\|^p$$

$$\leq -p\bar{m}\|\theta_t - \theta_t^*\|^{p} + \frac{p-1}{2}\|\theta_t - \theta_t^*\|^p + \frac{1}{p}\left(\frac{\bar{m}}{2(p-1)}\right)^{p-1}\|\nabla U(\theta_t) - \nabla \tilde{U}(\theta_{ih(n)})\|^p$$

$$\leq -p\bar{m}\|\theta_t - \theta_t^*\|^{p} + \frac{p-1}{m^p-1}\|\nabla U(\theta_t) - \nabla \tilde{U}(\theta_{ih(n)})\|^p,$$

where equation (15) follows from Young’s inequality.

Equivalently, we can obtain

$$\frac{d\|e^{\frac{1}{m}t}\|\theta_t - \theta_t^*\|^p}{dt} \leq e^{\frac{1}{m}t} \frac{2p-1}{m^p-1}\|\nabla U(\theta_t) - \nabla \tilde{U}(\theta_{ih(n)})\|^p.$$  \hspace{1cm} (16)

By the fundamental theorem of calculus,

$$\|\theta_t - \theta_t^*\|^p \leq e^{-\frac{p}{m}(t-ih(n))}\|\theta_{ih(n)} - \theta_{ih(n)}^*\|^p + \frac{2p-1}{m^p-1}\int_{ih(n)}^t e^{-\frac{p}{m}(t-s)}\|\nabla U(\theta_s) - \nabla \tilde{U}(\theta_{ih(n)})\|^p ds. \hspace{1cm} (17)$$
Taking expectation on both sides, we obtain that
\[
\mathbb{E}[\|\theta_t - \theta_t^*\|^p] = \mathbb{E}\left[\mathbb{E}\left[\|\theta_t - \theta_t^*\|^p \mid \theta_{ih^{(n)}}, \theta_{ih^{(n)}}^*\right]\right]
\leq e^{-\frac{\mu}{m^p}(t - ih^{(n)})}\mathbb{E}\left[\|\theta_{ih^{(n)}} - \theta_{ih^{(n)}}^*\|^p\right]
\leq 2^{p-1} \int_{ih^{(n)}}^t e^{-\frac{\mu^2}{2m^p}(t-s)}\mathbb{E}\left[\left\|\nabla U(\theta_s) - \nabla \hat{U}(\theta_{ih^{(n)}})\right\|\right]^p ds.
\] (18)

In the above expression, the integral and expectation are exchanged using Tonelli’s theorem, since
\[
\left\|\nabla U(\theta_s) - \nabla \hat{U}(\theta_{ih^{(n)}})\right\|^p
\] is positive measurable.

We further expand the expected error \(\mathbb{E}\left[\left\|\nabla U(\theta_s) - \nabla \hat{U}(\theta_{ih^{(n)}})\right\|^p\right]\):
\[
\mathbb{E}\left[\left\|\nabla U(\theta_s) - \nabla \hat{U}(\theta_{ih^{(n)}})\right\|^p\right]
= \mathbb{E}\left[\left\|\nabla U(\theta_s) - \nabla U(\theta_{ih^{(n)}}) + \nabla U(\theta_{ih^{(n)}}) - \nabla \hat{U}(\theta_{ih^{(n)}})\right\|^p\right]
\leq \frac{1}{2}\mathbb{E}\left[\left\|2\left(\nabla U(\theta_s) - \nabla U(\theta_{ih^{(n)}})\right)\right\|^p\right] + \frac{1}{2}\mathbb{E}\left[\left\|2\left(\nabla U(\theta_{ih^{(n)}}) - \nabla \hat{U}(\theta_{ih^{(n)}})\right)\right\|^p\right]
= 2^{p-1}\mathbb{E}\left[\left\|\nabla U(\theta_s) - \nabla U(\theta_{ih^{(n)}})\right\|^p\right] + 2^{p-1}\mathbb{E}\left[\left\|\nabla U(\theta_{ih^{(n)}}) - \nabla \hat{U}(\theta_{ih^{(n)}})\right\|^p\left|\theta_{ih^{(n)}}\right\|^p\right]
\leq 2^{p-1}\hat{L}^p \cdot \mathbb{E}\left[\left\|\theta_s - \theta_{ih^{(n)}}\right\|^p\right] + 2^{p-1} \Delta_p.
\] (19)

Plugging into equation (17), we have that
\[
\mathbb{E}[\|\theta_t - \theta_t^*\|^p]
\leq e^{-\frac{\mu}{m^p}(t - ih^{(n)})}\mathbb{E}\left[\|\theta_{ih^{(n)}} - \theta_{ih^{(n)}}^*\|^p\right]
+ 2^{2p-2} \frac{\hat{L}^p}{m^p-1} \int_{ih^{(n)}}^t e^{-\frac{\mu^2}{2m^p}(t-s)}\mathbb{E}\left[\|\theta_s - \theta_{ih^{(n)}}\|^p\right] ds + 2^{2p-2}(t - ih^{(n)}) \frac{\Delta_p}{m^p-1}.
\] (20)

We provide an upper bound for \(\int_{ih^{(n)}}^t e^{-\frac{\mu^2}{2m^p}(t-s)}\mathbb{E}\left[\|\theta_s - \theta_{ih^{(n)}}\|^p\right] ds\) in the following lemma.

**Lemma 5.** For \(h^{(n)} \leq \frac{\hat{m}}{32L^2}\), and for \(t \in [ih^{(n)}, (i+1)h^{(n)}]\),
\[
\int_{ih^{(n)}}^t e^{-\frac{\mu^2}{2m^p}(t-s)}\mathbb{E}\left[\|\theta_s - \theta_{ih^{(n)}}\|^p\right] ds
\leq 2^{3p-3}\hat{L}^p \left(t - ih^{(n)}\right)^{p+1} W_p (\mu_{ih^{(n)}}, \mu^*) + \frac{8p}{2} \left(t - ih^{(n)}\right)^{p+1} (dp)^{p/2} + 2^{2p-2}(t - ih^{(n)})^{p+1} \cdot \Delta_p.
\] (21)

Applying this upper bound to equation (20), we obtain that for \(h^{(n)} \leq \frac{\hat{m}}{32L^2}\), and for \(t \in [ih^{(n)}, (i+1)h^{(n)}]\),
\[
\mathbb{E}[\|\theta_t - \theta_t^*\|^p] \leq e^{-\frac{\mu^2}{2m^p}(t - ih^{(n)})}\mathbb{E}\left[\|\theta_{ih^{(n)}} - \theta_{ih^{(n)}}^*\|^p\right] + 2^{5p-5} \frac{\hat{L}^p}{m^p-1} \left(t - ih^{(n)}\right)^{p+1} W_p (\mu_{ih^{(n)}}, \mu^*)
\leq 2^{5p-3} \frac{\hat{L}^p}{m^p-1} \left(t - ih^{(n)}\right)^{p/2+1} (dp)^{p/2} + 2^{4p-4} \frac{\hat{L}^p}{m^p-1} (t - ih^{(n)})^{p+1} \cdot \Delta_p
\leq \left(1 - \frac{\hat{m}}{4} \left(t - ih^{(n)}\right)^p\right) \mathbb{E}\left[\|\theta_{ih^{(n)}} - \theta_{ih^{(n)}}^*\|^p\right] + 2^{5p-5} \frac{\hat{L}^p}{m^p-1} \left(t - ih^{(n)}\right)^{p+1} W_p (\mu_{ih^{(n)}}, \mu^*)
\leq 2^{5p-3} \frac{\hat{L}^p}{m^p-1} \left(t - ih^{(n)}\right)^{p/2+1} (dp)^{p/2} + 2^{2p}(t - ih^{(n)}) \frac{\Delta_p}{m^p-1}.
\]
Recognizing that \( \hat{\gamma}(\theta_t, \theta_t^*) = \mathbb{E}(\theta_{ih(n)}, \theta_{ih(n)}^*) \sim \gamma \cdot [\gamma(\theta_t, \theta_t^*|\theta_{ih(n)}, \theta_{ih(n)}^*)] \) is a coupling, we achieve the upper bound for \( W_p^p(\mu_t, \mu^*) \):

\[
W_p^p(\mu_t, \mu^*) \leq \mathbb{E}_{(\theta_t, \theta_t^*)} \sim \hat{\gamma}[[\theta_t - \theta_t^*]^p]
\]

\[
\leq \left(1 - \frac{\hat{m}}{4} \left(t - ih(n)\right)\right)^p \mathbb{E}_{(\theta_{ih(n)}, \theta_{ih(n)}^*)} \sim \gamma \cdot [\gamma(\theta_{ih(n)}, \theta_{ih(n)}^*)]^p
\]

\[
+ 2^{5p-5} \frac{L^p_{m-1}}{\hat{m}^{p-1}} \left(t - ih(n)\right)^{p+1} W_p^p(\mu_{ih(n)}, \mu^*) + 2^{5p-3} \frac{\hat{L}^p}{\hat{m}^{p-1}} \left(t - ih(n)\right)^{p/2+1} (dp)^{p/2}
\]

\[
+ 2^{2p}(t - ih(n)) \frac{\Delta_p}{\hat{m}^{p-1}}.
\]

Taking \( t = (i + 1)h(n) \), the recurring bound reads

\[
W_p^p(\mu_{(i+1)h(n)}, \mu^*) \leq \left(1 - \frac{\hat{m}}{8} h(n)\right)^p W_p^p(\mu_{ih(n)}, \mu^*) + 2^{5p-3} \frac{\hat{L}^p}{\hat{m}^{p-1}} (dp)^{p/2} \left(h(n)\right)^{p/2+1} + \frac{4p}{\hat{m}^{p-1}} h(n) \Delta_p.
\]

We finish the proof by invoking the recursion \( i \) times:

\[
W_p^p(\mu_{ih(n)}, \mu^*) \leq \left(1 - \frac{\hat{m}}{8} h(n)\right)^p W_p^p(\mu_{(i-1)h(n)}, \mu^*) + 2^{5p-3} \frac{\hat{L}^p}{\hat{m}^{p-1}} (dp)^{p/2} \left(h(n)\right)^{p/2+1} + \frac{4p}{\hat{m}^{p-1}} h(n) \Delta_p
\]

\[
\leq \left(1 - \frac{\hat{m}}{8} h(n)\right)^i W_p^p(\mu_0, \mu^*)
\]

\[
+ \sum_{k=0}^{i-1} \left(1 - \frac{\hat{m}}{8} h(n)\right)^{p-k} \left(2^{5p-3} \frac{\hat{L}^p}{\hat{m}^{p-1}} (dp)^{p/2} \left(h(n)\right)^{p/2+1} + \frac{4p}{\hat{m}^{p-1}} h(n) \Delta_p\right)
\]

\[
\leq \left(1 - \frac{\hat{m}}{8} h(n)\right)^p W_p^p(\mu_0, \mu^*) + 2^{5p} \frac{\hat{L}^p}{\hat{m}^{p}} (dp)^{p/2} \left(h(n)\right)^{p/2} + 2^{2p+3} \frac{\Delta_p}{\hat{m}^p}.
\]

\(\square\)
C.3.1 Supporting proofs for Lemma \[\text{[4]}\]

**Proof of Lemma \[\text{[4]}\]** We use the update rule of ULA to develop $\int_{ih(n)}^t e^{-\frac{pm}{2}(t-s)} \mathbb{E} \left[ \| \theta_s - \theta_{ih(n)} \|^p \right] ds$:

$$
\int_{ih(n)}^t e^{-\frac{pm}{2}(t-s)} \mathbb{E} \left[ \| \theta_s - \theta_{ih(n)} \|^p \right] ds
$$

$$
= \int_{ih(n)}^t e^{-\frac{pm}{2}(t-s)} \mathbb{E} \left[ \left( s - \frac{m}{2} \right) \left( \nabla U(\theta_{ih(n)}) - \nabla U(\theta_{ih(n)}) - \nabla \hat{U}(\theta_{ih(n)}) \right) + \sqrt{2}(B_s - B_{ih(n)}) \right]^p ds
$$

$$
\leq 2^{p-2} (t - ih(n))^p \int_{ih(n)}^t e^{-\frac{pm}{2}(t-s)} \mathbb{E} \left[ \| \nabla U(\theta_{ih(n)}) - \nabla \hat{U}(\theta_{ih(n)}) \|^p \right] ds
$$

$$
+ 2^{3p/2-1} \int_{ih(n)}^t e^{-\frac{pm}{2}(t-s)} \mathbb{E} \left[ \| B_s - B_{ih(n)} \|^p \right] ds
$$

$$
+ 2^{p-2} (t - ih(n))^p \int_{ih(n)}^t e^{-\frac{pm}{2}(t-s)} \mathbb{E} \left[ \| \nabla U(\theta_{ih(n)}) - \nabla \hat{U}(\theta_{ih(n)}) \|^p \right] ds
$$

$$
\leq 2^{p-2} \hat{L} \left( t - ih(n) \right)^{p+1} \mathbb{E} \left[ \| \theta_{ih(n)} - \theta_{\mu}^p \|^p \right] + 2^{3p/2-1} \int_{ih(n)}^t \mathbb{E} \left[ \| B_s - B_{ih(n)} \|^p \right] ds
$$

$$
+ 2^{p-2} (t - ih(n))^{p+1} \Delta_p,
$$

where $\theta_{\mu}^p$ is the fixed point of $U$. We then use the following lemma to simplify the above expression.

**Lemma 6.** The integrated $p$-th moment of the Brownian motion can be bounded as:

$$
\int_{ih(n)}^t \mathbb{E} \left[ \| B_s - B_{ih(n)} \|^p \right] ds \leq 2 \left( \frac{dp}{e} \right)^{p/2} \left( t - ih(n) \right)^{p/2+1}.
$$

We also provide bound for the $p$-th moment of $\| \theta_{ih(n)} - \theta_{\mu}^p \|$.

**Lemma 7.** For $\theta_{ih(n)} \sim \mu_{ih(n)}$,

$$
\mathbb{E} \left[ \| \theta_{ih(n)} - \theta_{\mu}^p \|^p \right] \leq 2^{p-1} W_p^p \left( \mu_{ih(n)} \right) + 10^p \left( \frac{dp}{m} \right)^{p/2}.
$$

Plugging the results into equation (25), we obtain that for $h^{(n)} \leq \frac{m}{\sqrt{2} L^2}$, and for $t \in [ih(n), (i+1)h(n)]$,

$$
\int_{ih(n)}^t e^{-\frac{pm}{2}(t-s)} \mathbb{E} \left[ \| \theta_s - \theta_{ih(n)} \|^p \right] ds
$$

$$
\leq 2^{3p-3} \hat{L} \left( t - ih(n) \right)^{p+1} W_p^p \left( \mu_{ih(n)} \right) + 40^p \left( \frac{dp}{m} \right)^{p/2}
$$

$$
+ \left( \frac{8}{e} \right)^{p/2} \left( t - ih(n) \right)^{p/2+1} + 2^{2p-2} (t - ih(n))^{p+1} \cdot \Delta_p
$$

$$
\leq 2^{3p-3} \hat{L} \left( t - ih(n) \right)^{p+1} W_p^p \left( \mu_{ih(n)} \right) + \frac{8p}{2} (t - ih(n))^{p/2+1} (dp)^{p/2} + 2^{2p-2} (t - ih(n))^{p+1} \Delta_p.
$$

**Proof of Lemma \[\text{[4]}\]** The Brownian motion term can be upper bounded by higher moments of a normal random variable:

$$
\int_{ih(n)}^t \mathbb{E} \left[ \| B_s - B_{ih(n)} \|^p \right] ds \leq \left( t - ih(n) \right) \mathbb{E} \left[ B_t - B_{ih(n)} \right] (t - ih(n))^{p/2+1} \mathbb{E} \| v \|^p,
$$

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where \( v \) is a standard \( d \)-dimensional normal random variable. We then invoke the \( \sqrt{d} \) sub-Gaussianity of \( \|v\| \) and have (assuming \( p \) to be an even integer):

\[
\mathbb{E} \|v\|^p \leq \frac{p^p}{2^{p/2} (p/2)!} \|v\|^p \leq \frac{e^{1/2p} \sqrt{2\pi p (p/e)^p}}{2^{p/2} \sqrt{p (p/2)!}} \leq 2 \left( \frac{dp}{e} \right)^{p/2}.
\]

Proof of Lemma 8. For the \( \mathbb{E} \|\theta_{ih(n)} - \theta_{U}^*\|^p \) term, we note that any coupling of a distribution with a delta measure is their product measure. Therefore, \( \mathbb{E} \|\theta_{ih(n)} - \theta_{U}^*\|^p \) relates to the \( p \)-Wasserstein distance between \( \mu_{ih(n)} \) and the delta measure at the fixed point \( \theta^*_U \), \( \delta(\theta^*_U) \):

\[
\mathbb{E} \|\theta_{ih(n)} - \theta_{U}^*\|^p = W^p_p(\mu_{ih(n)}, \delta(\theta^*_U)) \leq (W^p_p(\mu_{ih(n)}, \mu^*) + W^p_p(\mu^*, \delta(\theta^*_U)))^p 
\leq 2p^{-1} W^p_p(\mu_{ih(n)}, \mu^*) + 2p^{-1} W^p_p(\mu^*, \delta(\theta^*_U)).
\]

We then bound \( W^p_p(\mu^*, \delta(\theta^*_U)) \) in the following lemma.

**Lemma 8.** Assume the posterior \( \mu^* \) is \( \bar{m} \)-strongly log-concave. Then for \( \theta_{U}^* = \arg\max \mu^* \),

\[
W^p_p(\mu^*, \delta(\theta^*_U)) \leq 5p \left( \frac{dp}{\bar{m}} \right)^{p/2}.
\]

Therefore,

\[
\mathbb{E} \left\| \theta_{ih(n)}^{(n)} - \theta^*_U \right\|^p \leq 2^{p-1} W^p_p(\mu_{ih(n)}, \mu^*) + \frac{10p}{2} \left( \frac{dp}{\bar{m}} \right)^{p/2}.
\]

Proof of Lemma 8. We first decompose \( W^p_p(\mu^*, \delta(\theta^*_U)) \) into two terms:

\[
W^p_p(\mu^*, \delta(\theta^*_U)) \leq W^p_p(\mu^*, \delta([\theta_{\sim \mu^*}]) + \|\theta_{U}^* - \mathbb{E}_{\theta_{\sim \mu^*}} [\theta]\|.
\]

By the celebrated relation between mean and mode for 1-unimodal distributions [see, e.g., Basu and DasGupta, 1996 Theorem 7], we can first bound the difference between mean and mode:

\[
(\theta_{U}^* - \mathbb{E}_{\theta_{\sim \mu^*}} [\theta])^T \Sigma^{-1} (\theta_{U}^* - \mathbb{E}_{\theta_{\sim \mu^*}} [\theta]) \leq 3.
\]

where \( \Sigma \) is the covariance matrix of \( \mu^* \). Therefore,

\[
\|\theta_{U}^* - \mathbb{E}_{\theta_{\sim \mu^*}} [\theta]\|^2 \leq \frac{3}{\bar{m}}.
\]

We then bound \( W^p_p(\mu^*, \delta([\theta_{\sim \mu^*}])) \). Since the coupling between \( \mu^* \) and the delta measure \( \delta([\theta_{\sim \mu^*}]) \) is their product measure, we can directly obtain that the \( p \)-Wasserstein distance is the \( p \)-th moments of \( \mu^* \):

\[
W^p_p(\mu^*, \delta([\theta_{\sim \mu^*}])) = \int \|\theta - \mathbb{E}_{\theta_{\sim \mu^*}} [\theta]\|^p d\mu^*(\theta).
\]

We invoke the Herbst argument [see, e.g., Ledoux, 1999] to obtain the \( p \)-th moment bound. We first note that for an \( \bar{m} \)-strongly log-concave distribution, it has a log Sobolev constant of \( \bar{m} \). Then using the Herbst argument, we know that \( x \sim \mu^* \) is a sub-Gaussian random vector with parameter \( \sigma^2 = \frac{1}{2\bar{m}} \):

\[
\int e^{\lambda^T (\theta - \mathbb{E}_{\theta_{\sim \mu^*}} [\theta])} d\mu^*(\theta) \leq e^{\frac{\lambda^2}{4\bar{m}}}, \quad \forall \|u\| = 1.
\]
Hence θ is $2\sqrt{\frac{d}{m}}$ norm-sub-Gaussian, which implies that

$$\left(\mathbb{E}_{\theta \sim \mu^*} \left[ \|\theta - \mathbb{E}_{\theta \sim \mu^*}[\theta]\|^p \right]\right)^{1/p} \leq 2^{1/e} \sqrt{\frac{dp}{m}}$$

Combining equations (30) and (31), we obtain the final result that

$$W_p^p (\mu^*, \delta(\theta_U^n)) \leq \left(2^{1/e} \sqrt{\frac{dp}{m}} + \sqrt{\frac{3}{m}}\right)^p$$

$$\leq 5^p \left(\frac{dp}{m}\right)^{p/2}.$$

**Lemma 9.** Assume that the likelihood $\log p_a(x; \theta)$ satisfies the Lipschitz smoothness Assumption 4 and the strong convexity Assumption 1 and that arm $a$ has been chosen $n = T_a(t)$ times up to iteration $t$ of the Thompson sampling algorithm. Further, assume that we choose the stepsize $h^{(n)} = \frac{1}{32} \frac{m_a}{n(L_a^0 + L_a^2)} \frac{1}{d_a} = \mathcal{O} \left(\frac{m_a}{n(L_a^0 + L_a^2)} \frac{1}{d_a}\right)$ and the number of steps $N = \tilde{O} \left(\frac{(L_a^0 + L_a^2)^2}{m_a^2} d_a\right)$ $(N = 160 \frac{(L_a^0 + L_a^2)^2}{m_a^2} d_a \max\{\log(d_a), 1\})$ in the first round and $N = 640 \frac{(L_a^0 + L_a^2)^2}{m_a^2} d_a$ for $n \geq 2$ in Algorithm 2, then:

$$P_{\theta \sim \theta_a^{(n)}} \left(\|\theta_a - \theta_a^*\|_2 > \sqrt{\frac{8e}{m_a n} \left(2d_a + 2B_a + 32 \log 1/\delta_3 + 32 \log 1/\delta_2 + 4d_1L_a^2 \log 2/\delta_1\right)} \right) Z_{t-1} < \delta_2$$

where $Z_{t-1} = \|\theta_a, t-1 - \theta_a^*\| \leq C(n)$ where:

$$C(n) = \sqrt{\frac{18e}{m_a n} \left(2d_aL_a^2 \log 2/\delta_1\right) + d_a + B_a + 32 \log 1/\delta_3} \frac{1}{2},$$

and $\theta_a, t-1$ is the sample from the previous round of the Thompson sampling algorithm for arm $a$.

**Proof.** We begin as in the proof of Theorem 3 except that we now take $\mu_0 = \delta_{\theta_a, t-1}$, where $\theta_a, t-1$ is the sample from the previous step of the algorithm:

$$W_p^p (\mu_{\tilde{h}^{(n)}}, \mu_a^{(n)}) \leq \left(1 - \frac{m}{8} \tilde{h}^{(n)}\right)^{p/2} W_p^p (\delta(\theta_a, t-1), \mu_a^{(n)}) + \frac{80p}{2} \frac{L_p}{m^p} (dp)^{p/2} \left(\tilde{h}^{(n)}\right)^{p/2}.$$

We first use the triangle inequality on the first term on the RHS:

$$W_p^p (\delta(\theta_a, t-1), \mu_a^{(n)}) \leq W_p^p (\delta(\theta_a, t-1), \delta_{\theta_a^*}) + W_p^p (\delta(\theta_a^*), \mu_a^{(n)})$$

$$= \|\theta_a^* - \theta_a, t-1\| + \frac{1}{2} W_p^p (\delta(\theta_a^*), \mu_a^{(n)})$$

$$\leq C(n) + \frac{D}{\sqrt{n}}$$

where we have used the fact that $\|\theta_a^* - \theta_a, t-1\| \leq C(n)$ by assumption, and the definition of $\tilde{D}$ from the proof of Theorem 5.

$$\tilde{D} = \sqrt{\frac{2}{m_a} \left(2d_aL_a^2 \log 2/\delta_1\right) + d_a + B_a + 16p}.$$

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Since:
\[ C(n) = \sqrt{\frac{18e}{m_a n} \left( \frac{2d_a L_a^2 \log 2/\delta_1}{m_a \nu_a} + d_a + B_a + 32 \log 1/\delta_3 \right)^{\frac{1}{2}}}, \]

We can further develop this upper bound:
\[ W_p \left( \delta_{\theta_{a,t-1}}, \mu_a^{(n)} \right) \leq \frac{\bar{D}}{\sqrt{n}} + C(n) \]
\[ \leq 8 \sqrt{\frac{2}{m_a n} \left( \frac{4d_a L_a^2 \log 2/\delta_1}{m_a \nu_a} + 2d_a + 2B_a + 32 \log 1/\delta_3 + 16p \right)^{\frac{1}{2}}}, \]

where to derive this result we have used the fact that \( \sqrt{2(x+y)} \geq \sqrt{x} + \sqrt{y} \).

Letting \( \bar{D} = \sqrt{\frac{2}{m} \left( \frac{4d_a L_a^2 \log 2/\delta_1}{m_a \nu_a} + 2d_a + 2B_a + 32 \log 1/\delta_3 + 16p \right)^{\frac{1}{2}}} \), we see that our final result is:
\[ W_p \left( \delta_{\theta_{a,t-1}}, \mu_a^{(n)} \right) \leq \frac{8}{\sqrt{n}} \bar{D}, \]

where \( \bar{D} < \bar{D} \). Using the same choice of \( h^{(n)} \) and number of steps \( N \) as in the proof or Theorem 5 guarantees us that:
\[ W_p^{p} \left( \mu_{ih^{(n)}}, \mu_a^{(n)} \right) \leq \left( \frac{\bar{D}}{\sqrt{n}} \right)^p \]

Further combining this with the triangle inequality, and the fact that \( \bar{D} < \bar{D} \) gives us that:
\[ W_p (\mu_{ih^{(n)}}, \delta_{\theta^*}) \leq \frac{\bar{D}}{\sqrt{n}} + \frac{\bar{D}}{\sqrt{n}} \leq 2 \frac{\bar{D}}{\sqrt{n}}, \]

which, by the same derivation as in the proof of Theorem 1, gives us that:
\[ \mathbb{P}_{\theta_{a,t} \sim \tilde{\mu}_a^{(n)}} \left( \| \theta_{a,t} - \theta_a^* \|_2 > \frac{8e}{m_a n} \left( 2d_a + 2B_a + 32 \log 1/\delta_3 + 32 \log 1/\delta_2 + \frac{4d_a L_a^2 \log 2/\delta_1}{m_a \nu_a} \right) \right| Z_{t-1} \right) < \delta_2 \]

We remark that via an identical argument, the following Lemma holds as well:

**Lemma 10.** Assume that the family \( \log p_a(x; \theta) \) satisfies the joint Lipschitz smoothness Assumption 2 and the strong convexity Assumption 3 and that arm \( a \) has been chosen \( n = T_a(t) \) times up to iteration \( t \) of the Thompson sampling algorithm. If we take the number of data samples in the stochastic gradient estimate \( k = 32 \left( \frac{L_a^2}{m_a \nu_a} \right) d_a \), step size \( h^{(n)} = \frac{1}{32} \left( \frac{m_a}{n \left( L_a^2 + \frac{1}{4} L_a^2 \right)} \right)^{\frac{1}{2}} \), and number of steps \( N = \tilde{O} \left( \left( \frac{L_a^2}{m_a^2} \right)^2 d_a \right) \left( N = 160 \left( \frac{L_a^2 + L_a^2}{m_a^2} \right)^2 d_a \max(1, \log(d_a)) \right) \) in the first round and \( N = 1280 \left( \frac{L_a^2 + L_a^2}{m_a^2} \right)^2 d_a \) for \( n \geq 2 \) in Algorithm 2.

\[ \mathbb{P}_{\theta \sim \tilde{\mu}_a^{(n)}} \left( \| \theta - \theta_a^* \|_2 > \sqrt{\frac{8e}{m_a n} \left( 2d_a + 2B_a + 32 \log 1/\delta_3 + 32 \log 1/\delta_2 + \frac{4d_a L_a^2 \log 2/\delta_1}{m_a \nu_a} \right)} \right| Z_{t-1} \right) < \delta_2 \]

where \( Z_{t-1} = \{ \| \theta_{a,t-1} - \theta_a^* \| \leq C(n) \} \) where:
\[ C(n) = \sqrt{\frac{18e}{m_a n} \left( \frac{2d_a L_a^2 \log 2/\delta_1}{m_a \nu_a} + d_a + B_a + 32 \log 1/\delta_3 \right)^{\frac{1}{2}}}, \]

and \( \theta_{a,t-1} \) is the sample from the previous round of the Thompson sampling algorithm for arm \( a \).
D Regret Proofs

We now present the proof of logarithmic regret of Thompson sampling under our assumptions with samples from the true posterior and from the approximate sampling schemes discussed in Section 4. To provide the regret guarantees for Thompson sampling with samples from the true posterior and from approximations to the posterior, we proceed as is common in regret proofs for multi-armed bandits by upper-bounding the number of times a sub-optimal arm \( a \in A \) is pulled up to time \( T \), denoted \( T_a(T) \). Without loss of generality we assume throughout this section that arm 1 is the optimal arm, and define the filtration associated with a run of the algorithm as \( \mathcal{F}_t = \{ A_1, X_1, A_2, X_2, ..., A_t, X_t \} \).

To upper bound the expected number of times a sub-optimal arm is pulled up to time \( T \), we first define the event \( E_a(t) = \{ r_{a,t}(T_a(t)) \geq \bar{r}_1 - \epsilon \} \) for some \( \epsilon > 0 \). This captures the event that the mean calculated from the value of \( \theta_a \) sampled from the posterior at time \( t \leq T, r_{a,t}(T_a(t)) \), is greater than \( \bar{r}_1 - \epsilon \) (recall \( \bar{r}_1 \) is the optimal arm’s mean). Given these events, we proceed to decompose the expected number of pulls of a sub-optimal arm \( a \in A \) as:

\[
\mathbb{E}[T_a(T)] = \mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{I}(A_t = a) \right] = \mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{I}(A_t = a, E_a(t)) \right] + \mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{I}(A_t = a, E_a(t)) \right].
\]

In Lemma 11 we upper bound (I), and then bound term (II) in Lemmas 12.

We note that this proof follows a similar structure to that of the regret bound for Thompson sampling for Bernoulli bandits and bounded rewards in [Agrawal and Goyal, 2012]. However, to give the regret guarantees that incorporate the quality of the priors as well as the potential errors and lack of independence resulting from the approximate sampling methods we discuss in Section 4, the proof is more complex.

Lemma 11 (Bounding I). For a sub-optimal arm \( a \in A \), we have that:

\[
I = \mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{I}(A_t = a, E_a(t)) \right] \leq \ell + \mathbb{E} \left[ \sum_{s=\ell}^{T-1} \frac{1}{p_{1,s}} - 1 \right].
\]

where \( \ell > 0 \) is a positive integer and \( p_{a,s} = \mathbb{P}(r_{a,t}(s) > \bar{r}_1 - \epsilon | \mathcal{F}_{t-1}) \), for some \( \epsilon > 0 \).

Proof. To bound term I of (32), we first recall \( A_t \) is the arm achieving the largest sample reward mean at round \( t \). Further, we define \( A'_t \) to be the arm achieving the maximum sample mean value among all the suboptimal arms:

\[
A'_t = \arg\max_{a \in A, a \neq 1} r_a(t, T_a(t)).
\]

Since \( \mathbb{E}[\mathbb{I}(A_t = a, E_a(t))] = \mathbb{P}(A_t = a, E_a(t)) \), we aim to bound \( \mathbb{P}(A_t = a, E_a(t) | \mathcal{F}_{t-1}) \). We note that the following inequality holds:

\[
\mathbb{P}(A_t = a, E_a(t) | \mathcal{F}_{t-1}) \leq \mathbb{P}(A'_t = a, E_a(t) | \mathcal{F}_{t-1})(\mathbb{P}(r_1(t, T_1(t)) \leq \bar{r}_1 - \epsilon | \mathcal{F}_{t-1}))
\]

\[
= \mathbb{P}(A'_t = a, E_a(t) | \mathcal{F}_{t-1})(1 - \mathbb{P}(E_1(t) | \mathcal{F}_{t-1})).
\]

We also note that the term \( \mathbb{P}(A'_t = a, E_a(t) | \mathcal{F}_{t-1}) \) can be bounded as follows:

\[
\mathbb{P}(A_t = 1, E_a(t) | \mathcal{F}_{t-1}) \overset{(i)}{\geq} \mathbb{P}(A'_t = a, E_a(t), E_1(t) | \mathcal{F}_{t-1})
\]

\[
= \mathbb{P}(A'_t = a, E_a(t) | \mathcal{F}_{t-1}) \mathbb{P}(E_1(t) | \mathcal{F}_{t-1})
\]

Inequality (i) holds because \( \{ A'_t = a, E_a(t), E_1(t) \} \subseteq \{ A_t = 1, E_a(t), E_1(t) \} \). The equality is a consequence of the conditional independence of \( E_1(t) \) and \( \{ A'_t = a, E_a(t) \} \) (conditioned on \( \mathcal{F}_{t-1} \)).

2The conditional independence property holds for all of our sampling mechanisms because the sample distributions for the two distinct arms (a, 1) are always conditionally independent on \( \mathcal{F}_{t-1} \)
Assuming \( \mathbb{P}(E_1(t)|\mathcal{F}_{t-1}) > 0 \) and\(^3\) putting inequalities [33] and [34] together gives the following upper bound for \( \mathbb{P}(A_t = a, E^c_a(t)|\mathcal{F}_{t-1}) \):

\[
\mathbb{P}(A_t = a, E^c_a(t)|\mathcal{F}_{t-1}) \leq \mathbb{P}(A_t = 1, E^c_a(t)|\mathcal{F}_{t-1}) \left( \frac{1 - \mathbb{P}(E_1(t)|\mathcal{F}_{t-1})}{\mathbb{P}(E_1(t)|\mathcal{F}_{t-1})} \right).
\]

Letting \( P(E_1(t)|\mathcal{F}_{t-1}) := p_{1,T_1(t)} \) and noting that \( \{A_t = 1, E^c_a(t)\} \subseteq \{A_t = 1\} \):

\[
\mathbb{P}(A_t = a, E^c_a(t)|\mathcal{F}_{t-1}) \leq \mathbb{P}(A_t = 1|\mathcal{F}_{t-1}) \left( \frac{1}{p_{1,T_1(t)}} - 1 \right). \tag{35}
\]

Now, letting \( \ell \in \mathbb{N} \) we use this to give an upper bound on the term of interest:

\[
\mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{I}(A_t = a, E^c_a(t)) \right] \overset{(i)}{=} \mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{I}(A_t = a, E^c_a(t)) \mathbb{I}(T_1(t) < \ell) \right] + \mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{I}(A_t = a, E^c_a(t)) \mathbb{I}(T_1(t) \geq \ell) \right]
\]

\[
\overset{(ii)}{=} \ell + \mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{I}(A_t = a, E^c_a(t)) \mathbb{I}(T_1(t) \geq \ell) \right]
\]

\[
\overset{(iii)}{=} \ell + \mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{E} \left[ \mathbb{I}(A_t = a, E^c_a(t)) \mathbb{I}(T_1(t) \geq \ell)|\mathcal{F}_{t-1} \right] \right]
\]

\[
\overset{(iv)}{=} \ell + \mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{P}(A_t = 1|\mathcal{F}_{t-1}) \left( \frac{1}{p_{1,T_1(t)}} - 1 \right) \mathbb{I}(T_1(t) \geq \ell) \right]
\]

\[
\overset{(v)}{=} \ell + \mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{E} \left[ \mathbb{I}(A_t = 1)|\mathcal{F}_{t-1} \right] \left( \frac{1}{p_{1,T_1(t)}} - 1 \right) \mathbb{I}(T_1(t) \geq \ell) \right]
\]

\[
\overset{(vi)}{=} \ell + \mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{I}(A_t = 1) \mathbb{I}(T_1(t) \geq \ell) \left( \frac{1}{p_{1,T_1(t)}} - 1 \right) \right]
\]

\[
\overset{(vii)}{=} \ell + \mathbb{E} \left[ \sum_{s=1}^{T-1} \frac{1}{p_{1,s}} - 1 \right].
\]

Here the equality (i) follows by linearity of expectation, inequality (ii) follows by noting \( I_1 \leq \ell \). Equality (iii) is a consequence of the tower property, equality (iv) by noting \( \mathbb{I}(T_1(t) \geq \ell) \) is \( \mathcal{F}_{t-1} \)-measurable and that \( \mathbb{E} \mathbb{I}(A_t = a, E^c_a(t)) |\mathcal{F}_{t-1} = \mathbb{P}(A_t = a, E^c_a(t)|\mathcal{F}_{t-1}) \). Inequality (v) follows by noting that since by Equation [35] \( \mathbb{P}(A_t = a, E^c_a(t)|\mathcal{F}_{t-1}) \leq \mathbb{P}(A_t = 1|\mathcal{F}_{t-1}) \left( \frac{1}{p_{1,T_1(t)}} - 1 \right) \), multiplying both sides by the \( \mathcal{F}_{t-1} \)-measurable indicator \( \mathbb{I}(T_1(t) \geq \ell) \) the inequality is preserved. Equality (vi) follows by definition. Equality (vii) by the tower property. The last inequality (viii) is a consequence of upper bounding the indicators \( \mathbb{I}(A_t = 1) \) by 1 and using the conditioning on \( \mathbb{I}(T_1(t) \geq \ell) \). \( \square \)

Given the bound on (I) from [32], we now present the tighter of two bounds on (II) which is used to provide regret guarantees for Thompson sampling with exact samples from the posteriors.

\(^3\)In all the cases we consider, including approximate sampling schemes, this property holds. In that case, since the Gaussian noise in the Langevin diffusion ensures all sets of the form \((a, b)\) have nonzero probability mass.
Lemma 12 (Bounding II - exact posterior). For a sub-optimal arm \( a \in A \), we have that:

\[
II = \mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{I}(A_t = a, E_a(t)) \right] \leq 1 + \mathbb{E} \left[ \sum_{s=1}^{T} \mathbb{I}(p_{a,s} > \frac{1}{T}) \right].
\]

where \( p_{a,s} = P(r_{a,t}(s) > \bar{r}_1 - \epsilon | \mathcal{F}_{t-1}) \), for some \( \epsilon > 0 \).

Proof. The upper bound for term \( II \) in (32) follows the exact same proof as in Agrawal and Goyal [2012], and we recreate it for completeness below. Let \( T = \{ t : p_{a,T_a(t)} > \frac{1}{T} \} \), then:

\[
\mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{I}(A_t = a, E_a(t)) \right] \leq \mathbb{E} \left[ \sum_{t \in T} \mathbb{I}(A_t = a) \right] + \mathbb{E} \left[ \sum_{t \notin T} \mathbb{I}(E_a(t)) \right] \tag{36}
\]

By definition, term \( I \) in (36) satisfies:

\[
\sum_{t \in T} \mathbb{I}(A_t = a) = \sum_{t \in T} \mathbb{I}(A_t = a, p_{a,T_a(t)} > \frac{1}{T}) \leq \sum_{s=1}^{T} \mathbb{I}(p_{a,s} > \frac{1}{T})
\]

To address term \( II \) in (36), we note that, by definition: \( \mathbb{E}[\mathbb{I}(E_a(t)) | \mathcal{F}_{t-1}] = p_{a,T_a(t)} \). Therefore, using the definition of the set of times \( T \), we can construct this simple upper bound:

\[
\mathbb{E} \left[ \sum_{t \notin T} \mathbb{I}(E_a(t)) \right] = \mathbb{E} \left[ \sum_{t \notin T} \mathbb{I}(E_a(t)) | \mathcal{F}_{t-1} \right] = \mathbb{E} \left[ \sum_{t \notin T} p_{a,t} \right] \leq \sum_{t \notin T} \frac{1}{T} \leq 1
\]

Using the two upper bounds for terms \( I \) and \( II \) in (36) gives out desired result:

\[
\mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{I}(A_t = a, E_a(t)) \right] \leq 1 + \mathbb{E} \left[ \sum_{s=1}^{T} \mathbb{I}(p_{a,s} > \frac{1}{T}) \right]
\]

\( \square \)

D.1 Regret of Exact Thompson Sampling

We now present two last technical lemmas for use in the proof of the regret of exact Thompson sampling. We first define for each sub-optimal arm \( a \in A \) the event \( G_a(T_a(T), T_1(T)) \) as the union of the high probability events defined Proposition 3:

\[
G_{a,1}(T) = \left( \bigcap_{s=1}^{T-1} G_{a,s}(\delta_1) \right) \bigcap \left( \bigcap_{s=1}^{T-1} G_{1,s}(\delta_1) \right),
\]

where \( \delta_1 \in (0, 1) \).
Lemma 13. Suppose the likelihood and reward distributions satisfy Assumptions 1-3. Then the regret of a Thompson sampling algorithm can be decomposed as:

$$ E[R(T)] \leq \sum_{a>1} \Delta_a \mathbb{E} \left[ T_a(T) \bigg| G_{a,1}(T) \right] + 2\Delta_a $$

(37)

Proof. To prove this lemma, we begin with the canonical regret decomposition:

$$ E[R(T)] = \sum_{a \in A} \Delta_a \mathbb{E} [T_a(T)] $$

By construction $p_G = \mathbb{P}(G_{a,1}(T)^c) \leq 2T\delta_1$, and we can trivially upper bound $(1 - p_G) = \mathbb{P}(G_{a,1}(T))$ by 1. This gives:

$$ E[T_a(T)] = \mathbb{E} \left[ T_a(T) \bigg| G_{a,1}(T) \right] (1 - p_G) + \mathbb{E} \left[ T_a(T) \bigg| G_{a,1}(T)^c \right] p_G $$

$$ \leq \mathbb{E} \left[ T_a(T) \bigg| G_{a,1}(T) \right] + 2\delta_1 T \mathbb{E} \left[ T_a(T) \bigg| G_{a,1}(T)^c \right] $$

$$ \leq \mathbb{E} \left[ T_a(T) \bigg| G_{a,1}(T) \right] + 2\delta_1 T^2, $$

where, in the last line we used the fact that $T_a(T)$ is trivially less than $T$. Choosing $\delta_1 = 1/T^2$ completes the proof.

The second technical lemma upper bounds the first term on the right hand side in (37).

Lemma 14. Suppose the likelihood, true reward distributions, and priors satisfy Assumptions 1-3, then:

$$ \ell + \sum_{s=\ell}^{T-1} \mathbb{E} \left[ \frac{1}{p_{1,s}} - 1 \bigg| G_{a,s}(\delta_1) \right] \leq \left[ \frac{8\epsilon L^2_{1,r}}{m \Delta_a^2} (D_1 + 64 \log 2) \right] + 1 $$

(38)

$$ \sum_{s=1}^{T} \mathbb{E} \left[ \mathbb{I} \left( p_{a,s} > \frac{1}{T} \right) \bigg| G_{a,s}(\delta_1) \right] \leq \frac{8\epsilon L^2_{a,r}}{m \Delta_a^2} (D_a + 32 \log(T)) $$

(39)

Where for $a \in A$, $D_a$ is given by:

$$ D_a = B_a + d_a + \frac{2d_a L^2_a \log 2 / \delta_1}{m a \nu_a} $$

Proof. We begin by showing that (38) holds. To do so, we first note that, by definition $p_{1,s}$ satisfies:

$$ p_{1,s} = \mathbb{P}(r_{1,t}(s) > \bar{r}_1 - \epsilon | F_{t-1}) $$

(40)

$$ = 1 - \mathbb{P}(r_{1,t}(s) < -\epsilon | F_{t-1}) $$

(41)

$$ \geq 1 - \mathbb{P}(|r_{1,t}(s) - \bar{r}_1| > \epsilon | F_{t-1}) $$

(42)

$$ \geq 1 - \mathbb{P}_{\theta \sim \mu_1(s)} \left( \frac{\epsilon}{L_{1,r}} \right) $$

(43)

where the last inequality follows from the fact that $r_{1,t}(s)$ and $\bar{r}_1$ are $L_{a,r}$-Lipschitz functions of $\theta \sim \mu_{1}(s)$ and $\theta^*$ respectively.
We then use the fact that conditioned on $G_{1,s}(\delta_1)$, the posterior distribution $P_{\theta \sim \mu(s)}$ satisfies the concentration bound from Theorem 1. Therefore, we have that:

$$P_{\theta \sim \mu(s)}\left(\|\theta - \theta_s\| > \frac{\epsilon}{L_{1,r}}\right) \leq \exp\left(-\frac{1}{32} \left(\frac{mne^2}{2eL_{1,r}^2} - D_1\right)\right),$$

(44)

where we use the constant $D_1$ defined in the proof of Theorem 1 to simplify notation. We remark that this bound is not useful unless:

$$n > \frac{2eL_{1,r}^2}{\epsilon^2 m} D_1.$$

Thus, choosing $\epsilon = (\bar{r}_1 - \bar{r}_a)/2 = \Delta_a/2$, we can choose $\ell$ as:

$$\ell = \left\lfloor \frac{8eL_{1,r}^2}{m\Delta_a^2} (D_1 + 32 \log 2) \right\rfloor.$$

With this choice of $\ell$, we proceed as follows:

$$\sum_{s=0}^{T-1} \mathbb{E}\left[\frac{1}{p_{1,s}} - 1 \mid G_{1,s}(\delta_1)\right] \leq \sum_{s=0}^{T-1} \frac{1}{1 - \frac{1}{2}\delta(s)} - 1$$

$$\leq \int_{s=1}^{\infty} \frac{1}{1 - \frac{1}{2}\delta(s)} - 1 ds$$

where:

$$\delta(s) = \exp\left(-\frac{1}{32} \left(\frac{mne^2}{2eL_{1,r}^2} s\right)\right),$$

and the first inequality follows from our choice of $\ell$ and the second by upper bounding the sum by an integral.

To finish, we write $\delta(s) = \exp(-c*s)$, and solve the integral to find that:

$$\int_{s=1}^{\infty} \frac{1}{1 - \frac{1}{2}\delta(s)} - 1 ds = \frac{\log 2 - \log (2e^c - 1)}{c} + 1 \leq \frac{\log 2}{c} + 1.$$

plugging in our values for $\ell$, and $c$ gives the desired solution:

$$\ell + \sum_{s=0}^{T-1} \mathbb{E}\left[\frac{1}{p_{1,s}} - 1 \mid G_{1,s}(\delta_1)\right] \leq \left\lfloor \frac{8eL_{1,r}^2}{m\Delta_a^2} (D_1 + 64 \log 2) \right\rfloor + 1$$

To show that (39) holds, we do a similar derivation as in (43):

$$\sum_{s=1}^{T} \mathbb{E}\left[\mathbb{I}\left(\|\theta - \theta_s\| > \frac{1}{T}\right) \mid G_{a,s}(\delta_1)\right]$$

$$= \sum_{s=1}^{T} \mathbb{E}\left[\mathbb{I}\left(\|\theta - \theta_s\| > \Delta_a - \epsilon/|F_{t-1}| > \frac{1}{T}\right) \mid G_{a,s}(\delta_1)\right]$$

$$= \sum_{s=1}^{T} \mathbb{E}\left[\mathbb{I}\left(\|\theta - \theta_s\| > \frac{\Delta_a}{2} \mid |F_{t-1}| > \frac{1}{T}\right) \mid G_{a,s}(\delta_1)\right]$$

$$\leq \sum_{s=1}^{T} \mathbb{E}\left[\mathbb{I}\left(\|\theta - \theta_s\| > \frac{\Delta_a}{2} \mid |F_{t-1}| > \frac{1}{T}\right) \mid G_{a,s}(\delta_1)\right]$$

$$\leq \sum_{s=1}^{T} \mathbb{E}\left[\mathbb{I}\left(\|\theta - \theta_s\| > \frac{\Delta_a}{2L_{a,r}} \mid \|G_{a,s}(\delta_1)\| > \frac{1}{T}\right) \mid G_{a,s}(\delta_1)\right].$$

37
Since on the event $G_{a,s}(\delta_1)$, the posterior concentration result from Theorem 1 holds, it remains to upper bound the number of pulls $\bar{n}$ of arm $a$ such that for all $n \geq \bar{n}$:

$$\mathbb{P}_{\theta \sim \mu_{\alpha}}\left(\|\theta - \theta^*\| > \frac{\Delta_a}{2L_{a,r}}\right) \leq \frac{1}{T}.$$

Since the posterior for arm $a$ after $n$ pulls of arm $a$ has the same form as in (44), we can choose $\bar{n}$ as:

$$\bar{n} = \frac{8eL_{a,r}^2}{m\Delta_a^2} (D_a + 32 \log(T)).$$

This completes the proof.

For clarity, we restate the theorem below:

**Theorem D.1.** When the likelihood and true reward distributions satisfy Assumptions 1-3: we have that the expected regret after $T > 0$ rounds of Thompson sampling with exact sampling satisfies:

$$\mathbb{E}[R(T)] \leq \sum_{a>1} \frac{CL_{a,r}^2}{m_a\Delta_a} \left( d_a + B_a + \log(T) + \frac{d_aL_a^2 \log 2T}{m_a\nu_a} \right)$$

$$+ \frac{CL_{1,r}^2}{m_1\Delta_a} \left( d_1 + B_1 + 2 \log 2 + \frac{d_1L_1^2 \log 2T}{m_1\nu_1} \right) + 4\Delta_a$$

Where $C$ is a universal constant.

**Proof.** To begin, we invoke Lemma 13, which shows that we only need to bound the number of times a suboptimal arm $a \in \mathcal{A}$ is chosen on the nice event $G(T_a(T), T_1(T))$. We then invoke Lemmas 11 and 12 to find that:

$$\mathbb{E}\left[ T_a(T) \bigg| G(T_a(T), T_1(T)) \right] \leq 1 + \ell$$

$$+ \sum_{s=\ell}^{T-1} \mathbb{E}\left[ \frac{1}{p_{1,s}} - 1 \bigg| G_{1,s}(\delta_1) \right] + \sum_{s=1}^{T} \mathbb{E}\left[ I\left(1 - p_{a,s} > \frac{1}{T}\right) \bigg| G_{a,s}(\delta_1) \right]$$

Now, invoking Lemma 14, we use the upper bounds for terms (I) and (II) in the regret decomposition and expanding $D_a$ and $D_1$ to give that:

$$\mathbb{E}[R(T)] \leq \sum_{a>1} \frac{8eL_{a,r}^2}{m_a\Delta_a} \left( d_a + B_a + 32 \log(T) + \frac{4d_aL_a^2 \log 2T}{m_a\nu_a} \right)$$

$$+ \frac{8eL_{1,r}^2}{m_1\Delta_a} \left( d_1 + B_1 + 64 \log 2 + \frac{4d_1L_1^2 \log 2T}{m_1\nu_1} \right) + 4\Delta_a$$

\qed
D.2 Regret of Approximate Sampling

For the proof of Theorem 4, we proceed similarly as for the proof of Theorem 2, but require another intermediate lemma to deal with the fact that the samples from the arms are no longer conditionally independent given the filtration (due to the fact that we use the last sample as the initialization of the filtration). To do so, we first define the event:

\[ Z_a(T) = \bigcap_{t=1}^{T-1} Z_{a,t}, \]

where:

\[ Z_{a,t} = \left\{ \| \theta_{a,t} - \theta_a^* \| < \sqrt{\frac{18e}{m_a T_a(t)}} \left( \frac{2d_a L_a^2 \log 2/\delta_1}{m_a \nu_a} + d_a + B_a + 32 \log 1/\delta_3 \right)^{1/2} \right\}, \]

**Lemma 15.** Suppose the likelihood and reward distributions satisfy Assumptions 1-5. Then the regret of a Thompson sampling algorithm with approximate sampling can be decomposed as:

\[ \mathbb{E}[R(T)] \leq \sum_{a > 1} \Delta_a \mathbb{E} \left[ T_a(T) \left| G_a(T), T_1(T) \cap Z_a(T) \cap Z_1(T) \right. \right] + 4\Delta_a \quad (47) \]

**Proof.** We begin by invoking Lemma 13 to have that:

\[ \mathbb{E}[R(T)] \leq \sum_{a > 1} \Delta_a \mathbb{E} \left[ T_a(T) \left| G_a,1(T) \right. \right] + 2\Delta_a \quad (48) \]

We can now further condition on the event \( Z_a(T) \cap Z_1(T) \) for each \( a \in \mathcal{A} \), where we note that by construction \( p_Z = \mathbb{P}(Z_a(T)^c \cup Z_1(T)^c | G_a,1(T)) \leq \mathbb{P}(Z_a(T)^c | G_a,1(T)) + \mathbb{P}(Z_1(T)^c | G_a,1(T)) = 2T\delta_3 \) (since via Lemma 2 the probability of each event in \( Z_a(T)^c \) and \( Z_1(T)^c \) given \( G_a,1(T) \) is less than \( \delta_3 \)).

Therefore, we must have that:

\[
\mathbb{E}[T_a(T)|G_a,1(T)] \leq \mathbb{E} \left[ T_a(T) \left| G_a,1(T) \cap Z_a(T) \cap Z_1(T) \right. \right] + \mathbb{E} \left[ T_a(T) \left| G_a,1(T) \cap (Z_a(T)^c \cup Z_1(T)^c) \right. \right] p_Z \\
\leq \mathbb{E} \left[ T_a(T) \left| G_a,1(T) \cap Z_a(T) \cap Z_1(T) \right. \right] + 2T\delta_3 \mathbb{E} \left[ T_a(T) \left| G_a,1(T) \cap (Z_a(T)^c \cup Z_1(T)^c) \right. \right] \\
\leq \mathbb{E} \left[ T_a(T) \left| G_a,1(T) \cap Z_a(T) \cap Z_1(T) \right. \right] + 2\delta_3 T^2,
\]

where in the first line we use the fact that \( \mathbb{P}(G_a,1(T)) \leq 1 \) and in the last line we used the fact that \( T_a(T) \) is trivially less than \( T \). Choosing \( \delta_1 = 1/T^2 \) and \( \delta_3 = 1/T^2 \), and combining with the result of Lemma 13 completes the proof.

With this lemma in hand, we can now proceed as in Lemma 14 to finalize the proof of Theorem 4.

**Lemma 16.** Suppose the likelihood, true reward distributions, and priors satisfy Assumptions 1-5, and the samples are generated from the sampling schemes described in Theorem 3 and Theorem 4, then:

\[
\ell + \sum_{s=1}^{T-1} \mathbb{E} \left[ \frac{1}{\hat{p}_{1,s}} - 1 \left| G_{1,a}(T) \cap Z_1(T) \right. \right] \leq \frac{32eL^2_r}{m \Delta_a^2} (2D_1 + 64 \log 2 + 32 \log (1/\delta_3)) + 1 \quad (49)
\]

\[
\sum_{s=1}^{T} \mathbb{E} \left[ \mathbb{I} \left( \hat{p}_{a,s} > \frac{1}{T} \right) \left| G_{1,a}(T) \cap Z_a(T) \right. \right] \leq \frac{32eL^2_r a}{m \Delta_a^2} (2D_a + 32 \log (T) + 32 \log (1/\delta_3)), \quad (50)
\]
where \( \hat{\mu}_{a,s} \) is the distribution of a sample from the approximate posterior \( \hat{\mu}_a \) after \( s \) samples have been collected, and for \( a \in A \), \( D_a \) is given by:

\[
D_a = B_a + d_a + \frac{2d_a L_a^2 \log 2/\delta_1}{m_a \nu_a}.
\]

**Proof.** We begin by showing that (19) holds. To do so, we first note that, by definition \( \hat{\mu}_{1,s} \) satisfies:

\[
\hat{\mu}_{1,s} = \mathbb{P}(r_{1,t}(s) > \tilde{r}_1 - \epsilon | F_{t-1}) = 1 - \mathbb{P}(r_{1,t}(s) - \tilde{r}_1 < -\epsilon | F_{t-1}) \leq 1 - \mathbb{P}(|r_{1,t}(s) - \tilde{r}_1| > \epsilon | F_{t-1}) \geq 1 - \mathbb{P}_{\hat{\mu}_{1,s}}(\|\theta - \theta^*\| > \frac{\epsilon}{L_{1,r}}),
\]

where the last inequality follows from the fact that \( r_{1,t}(s) \) and \( \tilde{r}_1 \) are \( L_{a,r} \)-Lipschitz functions of \( \theta \sim \mu_1^{(s)} \) and \( \theta^* \) respectively.

We then use the fact that conditioned on \( G_{1,a}(T) \cap Z_1(T) \), the approximate posterior distribution \( \mathbb{P}_{\hat{\mu}_{1,s}}(\cdot) \) satisfies the concentration bound from Lemmas 10 and Lemma 9. Therefore, we have that:

\[
\mathbb{P}_{\hat{\mu}_{1,s}}(\|\theta - \theta^*\| > \frac{\epsilon}{L_{1,r}}) \leq \exp \left( -\frac{1}{32} \left( \frac{m \epsilon^2}{8 eL_{1,r}^2} - \bar{D}_1 \right) \right),
\]

where \( \bar{D}_1 = 2D_1 + 32 \log 1/\delta_3 \), and \( D_1 \) is as in the proof of Theorem 4 to simplify notation. We remark that this bound is not useful unless:

\[
\bar{n} > \frac{8eL_{1,r}^2}{\epsilon^2 m \bar{D}_1}.
\]

Thus, choosing \( \epsilon = (\tilde{r}_1 - \tilde{r}_a)/2 = \Delta_a/2 \), we can choose \( \ell \) as:

\[
\ell = \left\lfloor \frac{32eL_{1,r}^2}{m \Delta_a^2} (\bar{D}_1 + 32 \log 2) \right\rfloor.
\]

With this choice of \( \ell \), we proceed exactly as in the proof of Lemma 14 to find that:

\[
\ell + \sum_{s=\ell}^{T-1} \mathbb{E} \left[ \left( \frac{1}{p_{1,s}} - 1 \right) | G_{1,a}(T) \cap Z_1(T) \right] \leq \left\lfloor \frac{32eL_{1,r}^2}{m \Delta_a^2} (\bar{D}_1 + 64 \log 2) \right\rfloor + 1.
\]

To show that (50) holds, we conduct a similar derivation as in (54):

\[
\sum_{s=1}^{T} \mathbb{E} \left[ \left( p_{a,s} > \frac{1}{T} \right) | G_{1,a}(T) \cap Z_a(T) \right] \leq \sum_{s=1}^{T} \mathbb{E} \left[ \left( \mathbb{P}_{\hat{\mu}_{a,s}}(\|\theta - \theta^*\| > \frac{\Delta_a}{2L_{a,r}}) > \frac{1}{T} \right) | G_{1,a}(T) \cap Z_a(T) \right].
\]

Since on the event \( G_{1,a}(T) \cap Z_a(T) \), the posterior concentration result from Lemmas 10 and Lemma 9 holds, it remains to upper bound the number of pulls \( \bar{n} \) of arm \( a \) such that for all \( n \geq \bar{n} \):

\[
\mathbb{P}_{\hat{\mu}_{a,n}}(\|\theta - \theta^*\| > \frac{\Delta_a}{2L_{a,r}}) \leq \frac{1}{T}.
\]

Since the posterior for arm \( a \) after \( n \) pulls of arm \( a \) has the same form as in (44), we can choose \( \bar{n} \) as:

\[
\bar{n} = \frac{32eL_{a,r}^2}{m \Delta_a^2} (\bar{D}_a + 32 \log(T)).
\]

This completes the proof. \( \square \)
Putting the results of Lemmas 15 and 16 together gives us our final theorem:

**Theorem D.2** (Regret of Thompson sampling with (stochastic gradient) Langevin algorithm). When the likelihood and true reward distributions satisfy Assumptions 1-5: we have that the expected regret after \( T > 0 \) rounds of Thompson sampling with the (stochastic gradient) ULA method with the hyper-parameters and runtime as described in Theorem 3 satisfies:

\[
\mathbb{E}[R(T)] \leq \sum_{a > 1} CL_a r \left( d_a + B_a + \log(T) + \frac{d_a L_a^2 \log 2T}{m_a \nu_a} \right) + \sum_{a > 1} 32 \left( d_1 + B_1 + 1 + \frac{L_1^2 \log 2T}{m_1 \nu_1} \right) + 6\Delta_a,
\]

where \( C \) is a universal constant that is independent of problem dependent parameters.

**Proof.** To begin, we invoke Lemma 15, which shows that we only need to bound the number of times a suboptimal arm \( a \in \mathcal{A} \) is chosen on the ‘nice’ event \( G(T_a(T), T_1(T)) \cap Z_1(T) \cap Z_a(T) \) where the gradient of the log likelihood has concentrated and the approximate samples have been in high probability regions of the posteriors. We then invoke Lemmas 11 and 12, to find that:

\[
\mathbb{E} \left[ T_a(T) \mid G_a,1(T) \cap Z_1(T) \cap Z_a(T) \right] \leq 1 + \ell + \sum_{s=1}^{T-1} \mathbb{E} \left[ \frac{1}{p_{1,s}} - 1 \mid G_{1,a}(T) \cap Z_1(T) \right] \color{red}{(I)} + \sum_{s=1}^{T} \mathbb{E} \left[ \mathbb{I} \left( 1 - p_{a,s} > \frac{1}{T} \right) \mid G_{1,a}(T) \cap Z_1(T) \right] \color{red}{(II)}
\]

Now, invoking Lemma 14, we use the upper bounds for terms (I) and (II) in the regret decomposition, use our choice of both \( \delta_1 \) and \( \delta_3 = 1/T^2 \), expanding \( D_a \) and \( D_1 \), and use the fact that \( \lceil x \rceil \leq x + 1 \) to give that:

\[
\mathbb{E}[R(T)] \leq \sum_{a > 1} \frac{32e L_a^2}{m_a \Delta_a} \left( 2d_a + 2B_a + 96 \log(T) + \frac{8d_a L_a^2 \log 2T}{m_a \nu_a} \right) + \frac{32e L_1^2}{m_1 \Delta_a} \left( 2d_1 + 2B_1 + 64 \log 2 + 64 \log T + \frac{8d_1 L_1^2 \log 2T}{m_1 \nu_1} \right) + 6\Delta_a.
\]

Using the fact that \( \log(T) < \log(2T) \) and that \( \frac{d_a L_a^2}{m_a \nu_a} > 1 \), allows us to simplify into our desired result which makes the bound similar up to leading constants to that in Theorem 2.

\[\Box\]

### E Details in the Numerical Experiments

We benchmark the effectiveness of approximate Thompson sampling against both UCB and exact Thompson sampling across three different Gaussian multi-armed bandit instances with 10 arms. We remark that the use of Gaussian bandit instances is due to the fact that the closed form for the posteriors allows for us to properly benchmark against exact Thompson sampling and UCB, though our theory applies to a broader family of prior/likelihood pairs.

In all three instances we keep the reward distributions for each arm fixed such that their means are evenly spaced from 0 to 10 (\( \bar{r}_1 = 1, \bar{r}_2 = 2, \) and so on), and their variances are all 1. In each instance we use different priors over the means of the arms to analyze whether the approximate Thompson sampling
algorithms preserve the performance of exact Thompson sampling. In the first instance, the priors reflect the correct orderings of the means. We use Gaussian priors with variance 4, and means evenly spaced between 5 and 10 such that $E_{\pi_1}[X] = 5$, and $E_{\pi_{10}}[X] = 10$. In the second instance, the prior for each arm is a Gaussian with mean 7.5 and variance 4. Finally, the third instance is ‘adversarial’ in the sense that the priors reflects the complete opposite ordering of the means. In particular, the priors are still Gaussians such that their means are evenly spaced between 5 and 10 with variance 4, but this time $E_{\pi_1}[X] = 10$, and $E_{\pi_{10}}[X] = 5$.

As suggested in our theoretical analysis in Section 4, we use a constant number of steps for both ULA and SGLD to generate samples from the approximate posteriors. In particular, for ULA, we take $N = 100$ and double that number for SGLD $N = 200$. We also choose the stepsize for both algorithms to be $\frac{1}{\sqrt{2T_{a}(t)}}$. For SGLD, we use a batch size of $\min(T_{a}(t), 32)$. The regret is calculated as $\sum_{t=1}^{T} \bar{r}_{10} - \bar{r}_{A_t}$ for the three algorithms and is averaged across 100 runs. Finally, for the implementation of UCB, we used the time-horizon tuned UCB [Lattimore and Szepesvári, 2020] and the known variance, $\sigma^2$ of the arms in the upper confidence bounds (to maintain a level playing field between algorithms):

$$UCB_a(t) = \frac{1}{T_a(t)} \sum_{i=1}^{t-1} X_{A_i}\mathbb{1}(A_i = a) + \sqrt{\frac{4\sigma^2 \log 2T}{T_a(t)}}$$