Foliated and leaf preserving diffeomorphisms of simplest Morse-Bott foliations on lens spaces

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Abstract. Let \( \mathcal{F} \) be a Morse-Bott foliation on the solid torus \( T = S^1 \times D^2 \) into 2-tori parallel to the boundary and one singular circle \( S^1 \times 0 \). A diffeomorphism \( h : T \to T \) is called foliated (resp. leaf preserving) if for each leaf \( \omega \in \mathcal{F} \) its image \( h(\omega) \) is also leaf of \( \mathcal{F} \) (resp. \( h(\omega) = \omega \)). Gluing two copies of \( T \) by some diffeomorphism between their boundaries, one gets a lens space \( L_{p,q} \) with a Morse-Bott foliation \( \mathcal{F}_{p,q} \) obtained from \( \mathcal{F} \) on each copy of \( T \). Denote by \( D_{\text{fol}}(T, \partial T) \) and \( D_{\text{lp}}(T, \partial T) \) respectively the groups of foliated and leaf preserving diffeomorphisms of \( T \) fixed on \( \partial T \). Similarly, let \( D_{\text{fol}}(L_{p,q}) \) and \( D_{\text{lp}}(L_{p,q}) \) be respectively the groups of foliated and leaf preserving diffeomorphisms of \( \mathcal{F}_{p,q} \). In a recent joint paper of the author it is shown that \( D_{\text{lp}}(T, \partial T) \) is weakly contractible (all homotopy groups vanish), which allowed also to compute the weak homotopy type of \( D_{\text{lp}}(L_{p,q}) \). In the present paper it is proved that \( D_{\text{lp}}(T, \partial T) \) is a strong deformation retract of \( D_{\text{fol}}(T, \partial T) \). As a consequence the weak homotopy type of \( D_{\text{fol}}(L_{p,q}) \) for all possible pairs \( (p, q) \) is computed.

1. Introduction

The paper is devoted to computations of homotopy types of diffeomorphisms groups of a solid torus and lens spaces preserving foliations by level sets of simplest Morse-Bott functions whose sets of critical points consist of extreme circles only.

Note that the homotopy types of diffeomorphisms groups of compact manifolds in dimensions 1, 2 are described completely, [37, 7, 8, 13]; in dimension 3 there is a lot of information, e.g. [16, 11, 18]; while in dimensions \( \geq 4 \) only very specific cases are computed, e.g. [32, 36, 14, 6, 25, 2]. Computations for the groups of leaf preserving diffeomorphisms of foliations are usually related with perfection properties of such groups proved independently by T. Rybicki [34] and T. Tsuboi [39], which extended the results by M.-R. Herman [17], W. Thurston [38], J. Mather [29, 30], D. B. A. Epstein [9] on simplicity of groups of compactly supported isotopic to the identity diffeomorphisms. For foliations with singularities there is much less information, e.g. [10, 35, 28, 26].

In a recent series of joint papers by the author with O. Khokliuk [23, 24, 22, 21] there were developed several techniques for computations of homotopy types of diffeomorphisms groups of Morse-Bott and more general classes of «singular» foliations in higher dimensions. In particular, [21] computes the homotopy types of diffeomorphisms groups of leaf preserving diffeomorphisms for mentioned above foliations of the solid torus and lens spaces. In this

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paper we find the relationship between those groups and the groups of diffeomorphisms sending leaves to leaves, see Theorems 2.1, 2.2, 3.1, 3.4 below.

Some definitions and notations. Let $\mathcal{F}$ be a partition of a set $M$. Then a map $h: M \to M$ is $\mathcal{F}$-foliated if for each leaf $\omega \in \mathcal{F}$ its image $h(\omega)$ is a (possibly distinct from $\omega$) leaf of $\mathcal{F}$ as well. Also $h$ is $\mathcal{F}$-leaf preserving, if $h(\omega) = \omega$ for each leaf $\omega \in \mathcal{F}$. More generally, if $\mathcal{G}$ is a partition of a set $N$, then a map $h: M \to N$ is $(\mathcal{F}, \mathcal{G})$-foliated, if for each leaf $\omega \in \mathcal{F}$ its image $h(\omega)$ is a leaf of $\mathcal{G}$.

All manifolds and their diffeomorphisms are assumed to be of class $C^\infty$. If $M$ is a manifold, and $X \subset M$ is a subset, then we will denote by $\mathcal{D}^{\text{fol}}(\mathcal{F}, X)$, (resp. $\mathcal{D}^{\text{lp}}(\mathcal{F}, X)$), the groups of $\mathcal{F}$-foliated, (resp., $\mathcal{F}$-leaf preserving), $C^\infty$ diffeomorphisms of $M$ endowed with the corresponding strong $C^\infty$ Whitney topologies. If $X = \emptyset$, then we will omit it from notation and denote the above groups by $\mathcal{D}^{\text{fol}}(\mathcal{F})$ and $\mathcal{D}^{\text{lp}}(\mathcal{F})$ respectively.

Throughout the paper $D^2 = \{|z| \leq 1\}$ is the unit disk in the complex plane and $S^1 = \partial D^2$ the unit circle. For an abelian group $A$ we will denote by $\text{Dih}(A)$ the dihedral extension of $A$, i.e. the semidirect product $A \rtimes \mathbb{Z}_2$ corresponding to the natural action of $\mathbb{Z}_2$ on $A$ generated by the isomorphism $o: A \to A, o(a) = -a$. For example, $\text{Dih}(\mathbb{Z}_n)$ is the dihedral group $\mathbb{D}_n$, and $\text{Dih}(\text{SO}(2)) = \text{O}(2)$.

2. Solid torus

Let $T = S^1 \times D^2$ be the solid torus. Consider the following Morse-Bott function $f: T \to \mathbb{R}, f(w, z) = |z|^2$, and let $\mathcal{F} = \{f^{-1}(t) \mid t \in [0; 1]\}$ be the partition of $T$ into the level sets of $f$. Then $f^{-1}(0) = S^1 \times 0$ is the «central circle» of $T$ consisting of all critical points of $f$, and thus being a non-degenerate critical submanifold. For all other values $t \in (0; 1)$, $f^{-1}(t) = S^1 \times \{|z|^2 = t\}$ is a 2-torus (product of two circles) «parallel» to $\partial T$. In particular, $f^{-1}(1) = \partial T$.

N. Ivanov [19] proved that the group $\mathcal{D}(T, \partial T)$ of all diffeomorphism of $T$ fixed on the boundary is contractible. Also in a recent joint paper [21] of the author with O. Khokhliuk it is computed the weak homotopy type of $\mathcal{D}^{\text{lp}}(\mathcal{F})$ and shown that all homotopy groups of $\mathcal{D}^{\text{lp}}(\mathcal{F}, \partial T)$ vanish. One of the main results of this paper accomplishes the results of [21] with the following theorem:

**Theorem 2.1.** The pair of groups $(\mathcal{D}^{\text{lp}}(\mathcal{F}), \mathcal{D}^{\text{lp}}(\mathcal{F}, \partial T))$ is a strong deformation retract of the pair $(\mathcal{D}^{\text{fol}}(\mathcal{F}), \mathcal{D}^{\text{fol}}(\mathcal{F}, \partial T))$.

The proof identifies the groups $\mathcal{D}^{\text{fol}}(\mathcal{F})$ and $\mathcal{D}^{\text{lp}}(\mathcal{F})$ with the stabilizers of $f$ with respect to the natural actions «left-right» and «right» actions of diffeomorphisms groups of $T$ and $[0; 1]$ on $C^\infty (T, [0; 1])$, see §5, and then uses the author’s result from [27] claiming that those stabilizers are homotopy equivalent. The above identification exploits the well-known Whitney theorem on even smooth functions, see Lemma 4.3. In fact, it will be established a more general statement concerning smooth fiber-wise definite homogeneous functions of degree 2 on total spaces of vector bundles, see Theorem 6.7.

As a consequence of [19, 21] we will get a description of homotopy types of the mentioned above groups, see Theorem 2.2 below. To formulate the corresponding statement we need to define certain subgroups of $\mathcal{D}^{\text{lp}}(\mathcal{F})$ which will also be used later in the paper.

Consider the following subgroup of $\text{GL}(2, \mathbb{Z})$ consisting of matrices for which the vector $(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix})$ is eigen with eigen value $\pm 1$:

$$
\mathcal{G} := \left\{ \begin{pmatrix} \varepsilon & 0 \\ m & \delta \end{pmatrix} \mid m \in \mathbb{Z}, \varepsilon, \delta \in \{\pm 1\} \right\}.
$$

(2.1)
Then $G$ is generated by the matrices:
\[
D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -\lambda, \quad T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]
satisfying the following identities:
\[
\Lambda^2 = M^2 = E, \quad \Lambda D \Lambda = MDM = D^{-1}, \quad T = \Lambda M = \Lambda, \quad TD = DT. \tag{2.3}
\]
Hence, $G = \langle D, \Lambda \rangle \times \langle T \rangle \cong \text{Dih}(\mathbb{Z}) \times \mathbb{Z}_2$. It is easy to check that every $A = \begin{pmatrix} \varepsilon & 0 \\ m & \delta \end{pmatrix} \in G$ yields the following diffeomorphism $g_A \in D^{lp}(F)$,
\[
g_A(w, z) = \begin{cases}
(w^\varepsilon, w^m z), & \text{if } \delta = 1, \\
(w^\varepsilon, w^m \bar{z}), & \text{if } \delta = -1,
\end{cases}
\tag{2.4}
\]
so that the correspondence $A \mapsto g_A$ is a monomorphism $G \subset D^{lp}(F)$, and we will identify $G$ with its image in $D^{lp}(F)$. It will be convenient to denote $d := g_D, \lambda := g_\Lambda, \mu := g_M, \tau := g_T$, so
\[
d(w, z) = (w, wz), \quad \lambda(w, z) = (\bar{w}, z), \\
\mu(w, z) = (w, \bar{z}), \quad \tau(w, z) = (\bar{w}, \bar{z}).
\tag{2.5}
\]
Further note that $D^{lp}(F)$ contains another subgroup $R$, called the rotation subgroup, isomorphic to the 2-torus $S^1 \times S^1$ and consisting of diffeomorphisms $\rho_{\alpha, \beta}, (\alpha, \beta) \in S^1 \times S^1$, given by
\[
\rho_{\alpha, \beta}(w, z) = (\alpha w, \beta z), \quad (w, z) \in T = S^1 \times D^2. \tag{2.6}
\]
Let $T = \langle R, G \rangle$ be the subgroup of $D^{lp}(F)$ generated by $R$ and $G$. Evidently, $R$ is the identity path component of $T$, whence $\pi_0 T \cong G$.

**Theorem 2.2.** In the following diagrams of inclusions the arrows denoted by $(w.h.e.$ are (weak) homotopy equivalences (the new statements are written in bold):
\[
\begin{array}{cccc}
\{ \text{id}_T \} & \xrightarrow{(w.h.e)} & D^{lp}(F, \partial T) & \xrightarrow{(w.h.e)} & D^{lp}(F) \\
\downarrow h.e & & \downarrow h.e & & \downarrow h.e \\
D(T, \partial T) & \xrightarrow{(w.h.e)} & D^{fol}(F, \partial T) & \xrightarrow{(w.h.e)} & D^{fol}(F)
\end{array}
\]

**Proof.** The proof that the inclusion $\{ \text{id}_T \} \subset D(T, \partial T)$ is a homotopy equivalence, i.e. contractibility of $D(T, \partial T)$, is given in [19, Theorem 2]. The fact, that the inclusion $T \subset D(T)$ is a homotopy equivalence, is classical and follows from contractibility of $D(T, \partial T)$, see [21] for exposition of those results. Also for the proof that the corresponding induced map $\pi_0 T = G \to \pi_0 D(T)$ is an isomorphism see e.g. [40, Theorem 14]. The upper horizontal weak homotopy equivalences are established in [21], and the right vertical homotopy equivalence are contained in Theorem 2.1. This implies that the lower arrows are weak homotopy equivalences as well. \qed

**3. Lens spaces**

Recall that any 3-manifold $L_\xi$ obtained by gluing two copies of the solid torus $T$ by some diffeomorphism of their boundaries $\xi : \partial T \to \partial T$ is called a lens space. We assume that the reader is familiar with lens spaces and will briefly recall their basic properties, see e.g. [33, 4, 3, 12, 18].
More precisely, let $L := T \times \{0, 1\}$ be two copies of $T$. Identify each point $(x, 0)$ with $(\xi(x), 1)$ and denote the obtained quotient space by $L_\xi$. Let also $p : L \to L_\xi$ be the corresponding quotient map, $T_i := p(T \times \{i\})$, $i = 0, 1$, and $C_i := S^1 \times 0 \times i$ be the «central circle» of the torus $T_i$. It is well known that $L_\xi$ admits a smooth structure, and if $\xi' : \partial T \to \partial T$ is another diffeomorphism satisfying either of the following conditions:

- $\xi$ is isotopic to $\xi'$;
- $\xi' = g \circ \partial T \circ \xi \circ h \circ \partial T$, where $g, h$ are certain diffeomorphisms of $T$;
- $\xi' = \xi^{-1}$

then $L_\xi$ and $L_{\xi'}$ are diffeomorphic. This finally implies that one can always assume that $\xi$ is defined by the formula:

$$\xi(w, z) = (w^p z^q, w^q z^p), \quad (w, z) \in \partial T,$$

for some matrix $X = \begin{pmatrix} r & p \\ s & q \end{pmatrix} \in \text{GL}(2, \mathbb{Z})$ with $|X| = rq - ps = -1$, and in this case $L_\xi$ is denoted by $L_{p,q}$. Moreover, if $\xi' : \partial T \to \partial T$ is given by the matrix $X' = \begin{pmatrix} r' & p' \\ s' & q' \end{pmatrix}$ with $r = r'(mod\ p)$ and $q = q'(mod\ p)$, then $L_\xi$ and $L_{\xi'}$ are still diffeomorphic. This implies that for $p = 0, 1, 2$ there exists a unique (up to a diffeomorphism) lens space so that

- $L_{0,1} \cong S^1 \times S^2$ with $X = \Lambda = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\xi(w, z) = \lambda(w, z) = (w, z)$;
- $L_{1,0} \cong S^3$ with $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\xi(w, z) = (z, w)$;
- $L_{2,1} \cong \mathbb{R}P^3$ with $X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\xi(w, z) = (wz^2, w^2 z)$ or with $X' = XD^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\xi'(w, z) = (\bar{w}z^2, \bar{z})$.

For $p > 2$ one can assume that $1 \leq q < p$, $\text{gcd}(p, q) = 1$, and $q$ can be replaced with $q'$ such that $qq' = 1(mod\ p)$.

**Subgroups $L_{p,q}$ and $\hat{L}_{p,q}$ of $D(L_{p,q})$.** We will now recall the definition of a certain subgroup $L_{p,q} \subset D(L_{p,q})$ considered in [21], and also define some other group $\hat{L}_{p,q}$ containing $L_{p,q}$. They will play the principal role in our main results.

1) First note that every diffeomorphism $\phi \in D(L_\xi)$ which preserves the common boundary $\partial T_0 = \partial T_1$ induces a diffeomorphism $\phi : L \to L$ such that $p \circ \phi = \phi \circ p : L \to L_\xi$.

a) Moreover, if $\phi$ preserves each torus $T_i$, $i = 0, 1$, then $\phi|_{\partial T_i} \circ (x, i) = (\phi_i(x), i)$ for unique diffeomorphisms $\phi_0, \phi_1 : T \to T$ satisfying the identity $\xi \circ \phi_0|_{\partial T} = \phi_1 \circ \xi : \partial T \to \partial T$, i.e. making commutative the diagram (a) in (3.2):

$$\begin{array}{ccc}
\partial T & \xrightarrow{\phi_0} & \partial T \\
\downarrow{\xi} & & \downarrow{\xi} \\
\partial T & \xrightarrow{\phi_1} & \partial T \\
\end{array}$$

(b) Similarly, if $\phi$ exchanges $T_0$ and $T_1$, then $\phi|_{\partial T \setminus \{\xi\}} \circ (x, i) = (\phi_i(x), 1 - i)$ for unique diffeomorphisms $\phi_0, \phi_1 : T \to T$ satisfying the identity $\phi_0|_{\partial T} = \xi \circ \phi_1 \circ \xi : \partial T \to \partial T$, i.e. making commutative the diagram (b) in (3.2). In that case we will write down $\phi$ as follows:

$$\phi : \begin{pmatrix} 0 & \phi_0 \\ 1 & \phi_1 \end{pmatrix}^T 0 \quad \phi_1 0,$$
2) For \((\alpha, \beta) \in S^1 \times S^1\) let \(\rho_{\alpha,\beta} : T \to T\), \(\rho_{\alpha,\beta}(w, z) = (\alpha w, \beta z)\), be the diffeomorphism given by (2.6). Evidently,
\[
\xi \circ \rho_{\alpha,\beta}(w, z) = (\alpha^r \beta^p w^r z^p, \alpha^s \beta^q w^s z^q) = \rho_{\xi(\alpha,\beta)} \circ \xi(w, z),
\]
and we have the following diffeomorphism \(\hat{\rho}_{\alpha,\beta} : \frac{1}{\xi(\alpha,\beta)} \frac{\rho_{\alpha,\beta}}{\rho_{\xi(\alpha,\beta)}} \frac{1}{\xi(w, z)}\) of \(L_{p,q}\), called a rotation. Then the group \(\mathcal{R} = \{\rho_{\alpha,\beta} \mid \alpha, \beta \in S^1\} \subset \mathcal{D}(L_{p,q})\) of all rotations is isomorphic to \(S^1 \times S^1 = \partial T\).

3) We will recall now the definition of two diffeomorphisms \(\sigma_+\) and \(\sigma_-\) exchanging \(T_0\) and \(T_1\). F. Bonahon [3, Theorem 1] proved that every diffeomorphism \(h\) of \(L_{p,q}\) is isotopic to a diffeomorphism which leaves the common boundary \(T_0\) and \(T_1\) invariant, and thus preserving or exchanging those tori. Moreover, for \(p > 2\) the group \(\pi_0 \mathcal{D}(L_{p,q})\) is generated by the isotopy classes of the diffeomorphism \(\hat{\tau} : \frac{1}{\xi} \xrightarrow{\tau} \frac{1}{\xi}\), and \(\sigma_+, \sigma_-\) (more precisely by that \(\sigma_-\) or \(\sigma_+\) which is defined for the given \((p, q)\)).

a) Suppose \(-q^2 - ps = -1\) for some \(s \in \mathbb{Z}\), so \(\xi\) is given by the matrix \(X = \begin{pmatrix} \alpha & \beta \\ s & q \end{pmatrix}\). This property holds when \(L_{p,q}\) is either \(L_{0,1} = S^1 \times S^2\), or \(L_{1,0} = S^3\), or \(L_{2,1} = \mathbb{R}P^3\), or \(q^2 = 1(mod p)\) for \(p > 2\). Notice that \(X^2 = E\), so \(\xi^2 = \text{id}_{\partial T}\), i.e. \(\text{id}_{\partial T} = \xi \circ \text{id}_{\partial T} \circ \xi\), and we have a well defined diffeomorphism \(\sigma_+ : \frac{1}{\xi} \xrightarrow{\tau} \frac{1}{\xi}\) of \(L_{p,q}\) exchanging \(T_0\) and \(T_1\). One easily checks that \(\sigma_+\) preserves orientation and \(\sigma_- = \text{id}_{L_{p,q}}\).

b) Suppose \(q^2 - ps = -1\) for some \(s \in \mathbb{Z}\), so \(\xi\) is given by the matrix \(X = \begin{pmatrix} \alpha & \beta \\ s & q \end{pmatrix}\). This holds if \(L_{p,q}\) is either \(L_{1,0} = S^3\), or \(L_{2,1} = \mathbb{R}P^3\) or if \(q^2 = -1(mod p)\) for \(p > 2\). One easily checks that \(\Lambda = XMX\), so \(\Lambda = \xi \circ \mu \circ \xi\), and we have another diffeomorphism \(\sigma_- : \frac{1}{\xi} \xrightarrow{\tau} \frac{1}{\xi}\) of \(L_{p,q}\) exchanging \(T_0\) and \(T_1\). One easily checks that \(\sigma_-\) reverses orientation, and is periodic of period 4.

4) We will now define the group \(\mathcal{L}_{p,q}\) and put \(\hat{\mathcal{L}}_{p,q} := \langle \mathcal{L}_{p,q}, \sigma_-, \sigma_+ \rangle\) to be the group generated by \(\mathcal{L}_{p,q}\) and those of \(\sigma_-\) and \(\sigma_+\) which are defined for the given values \((p, q)\).

Let \(d, \lambda, \mu, \tau = \lambda \circ \mu \in \mathcal{D}^2(\mathcal{F})\) be diffeomorphisms of \(T\) given by (2.5) and corresponding to matrices (2.2).

a) Let \(L_{p,q} = L_{0,1} = S^1 \times S^2\), so \(X = \Lambda = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\). Then it follows from (2.3) that \(\mathcal{D}(L_{0,1})\) contains the following diffeomorphisms:
\[
\tilde{d} : \frac{1}{\xi} \xrightarrow{d} \frac{1}{\xi}, \quad \tilde{\lambda} : \frac{1}{\xi} \xrightarrow{\lambda} \frac{1}{\xi}, \quad \tilde{\mu} : \frac{1}{\xi} \xrightarrow{\mu} \frac{1}{\xi}, \quad \tilde{\tau} = \tilde{\lambda} \circ \tilde{\mu} : \frac{1}{\xi} \xrightarrow{\tau} \frac{1}{\xi}.
\]
Notice that in this case only \(\sigma_- : \frac{1}{\xi} \xrightarrow{\lambda} \frac{1}{\xi}\) is defined. Define the following groups
\[
\mathcal{L}_{0,1} := \langle \mathcal{R}, \tilde{d}, \tilde{\lambda}, \tilde{\mu} \rangle \cong \langle \mathcal{R}, \mathcal{G} \rangle = \mathcal{T}, \quad \hat{\mathcal{L}}_{0,1} := \langle \mathcal{L}_{p,q}, \sigma_- \rangle.
\]
Then \(\mathcal{R}\) is the identity path component of each of them. Moreover, one also easily checks that \(\sigma_- \circ \tilde{d} = \tilde{d}^{-1} \circ \sigma_- : \frac{1}{\xi} \xrightarrow{\lambda} \frac{1}{\xi}\) and \(\sigma_-\) commutes with \(\tilde{\lambda}\) and \(\tilde{\mu}\). Then together with the first and second identities in (2.3) we get that
\[
\pi_0 \mathcal{L}_{0,1} = \mathcal{G}, \quad \pi_0 \hat{\mathcal{L}}_{0,1} = \langle \mathcal{G}, \sigma_- \rangle \cong \langle \tilde{d} \rangle \times \langle \tilde{\lambda}, \tilde{\mu}, \sigma_- \rangle \cong \mathbb{Z} \times (\mathbb{Z}_2)^3,
\]
where the semidirect product corresponds to the homomorphism \((\mathbb{Z}_2)^3 \to \text{Aut}(\mathbb{Z}) = \mathbb{Z}_2, (a, b, c) \cdot n = (-1)^{a+b+c}n\), for \(a, b, c \in \mathbb{Z}_2\) and \(n \in \mathbb{Z}\).
b) Let \( L_{p,q} = L_{1,0} = S^3 \), so \( X = \left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right) \). Then \( X^2 = E, X\Lambda = MX, XM = \Lambda X \), which implies that \( D^p(F_{1,0}) \) contains the following diffeomorphisms of order two: \( \hat{\lambda} : \frac{0 \lambda}{0 \mu} 1 \), \( \hat{\mu} : \frac{0 \mu}{0 \lambda} 1 \).

In this case both \( \sigma_- : \frac{0 \lambda}{1 \mu} 1 \) and \( \sigma_+ : \frac{0 \mu}{1 \lambda} 1 \) are defined, and one easily checks that

\[
\begin{align*}
\sigma_+^2 &= \text{id}_{L_{1,0}}, & \sigma_-^2 &= \hat{\lambda} \hat{\mu}, & \sigma_+ \sigma_- \sigma_+ &= \sigma_-^1, \\
\sigma_+ \sigma_- &= \hat{\lambda}, & \sigma_- \sigma_+ &= \hat{\mu}, & \hat{\lambda} \rho_{\alpha,\beta} &= \rho_{\alpha,\beta} \hat{\lambda}, & \hat{\mu} \rho_{\alpha,\beta} &= \rho_{\alpha,\beta} \hat{\mu}.
\end{align*}
\]

Define the following groups

\[
\mathcal{L}_{0,1} := \langle \mathcal{R}, \hat{\lambda}, \hat{\mu} \rangle, \quad \hat{\mathcal{L}}_{0,1} := \langle \mathcal{L}_{0,1}, \sigma_-, \sigma_+ \rangle.
\]

Then \( \mathcal{R} \) is the identity path component of each of them, and it follows from the above identities that

\[
\begin{align*}
\mathcal{L}_{0,1} &= \langle \rho_{\alpha,1}, \hat{\lambda}, \alpha \in S^1 \rangle \times \langle \rho_{1,\beta}, \hat{\mu}, \beta \in S^1 \rangle = O(2) \times O(2), \\
\pi_0 \mathcal{L}_{1,0} &= \langle \hat{\lambda}, \hat{\mu} \rangle = \mathbb{Z}_2 \times \mathbb{Z}_2, \\
\hat{\mathcal{L}}_{0,1} &= \langle \mathcal{R}, \hat{\lambda}, \hat{\mu}, \sigma_-, \sigma_+ \rangle = \langle \mathcal{R}, \sigma_- \sigma_+ \rangle, \\
\pi_0 \hat{\mathcal{L}}_{1,0} &= \langle \sigma_-, \sigma_+ \rangle \cong \text{Dih}(\mathbb{Z}_4) = D_4.
\end{align*}
\]

c) Let \( L_{p,q} = L_{2,1} = \mathbb{RP}^3 \), so \( X = \left( \begin{smallmatrix} 1 & 0 \\ 0 & 3 \end{smallmatrix} \right) \). Then \( X^2 = -X, XT = TX \), and \( D^p(F_{2,1}) \) contains the following diffeomorphisms of order two: \( \hat{\theta} : \frac{0 \lambda}{1 \mu} 1 \) and \( \hat{\tau} : \frac{0 \tau}{1 \tau} 1 \).

Since \( (p,q) = (2,1) \), we have that \( q^2 = -1 \mod 2 \) and thus the diffeomorphism \( \sigma_- : \frac{0 \lambda}{1 \mu} 1 \)

is defined.

On the other hand, \( L_{2,1} \) is also diffeomorphic to the lens space \( L_{2,-1} \) given by the matrix \( X' = XD^{-1} = \left( \begin{smallmatrix} 1 & 0 \\ -1 & 3 \end{smallmatrix} \right) \) for which \( \sigma'_+ : \frac{0 \text{id}}{1 \text{id}} 1 \) is also defined. To transfer \( \sigma'_+ \) to a diffeomorphism of \( L_{2,1} \) we need to «conjugate» it by \( d \).

More precisely, note that \( E = X'X' = XD^{-1}XD^{-1}, \) whence \( D = XD^{-1}X \), and we get the following diffeomorphism \( \sigma'_+ : \frac{0 \text{id}}{1 \text{id}} 1 \) of \( L_{2,1} \). Define the following groups

\[
\mathcal{L}_{2,1} := \langle \mathcal{R}, \hat{\theta}, \hat{\tau} \rangle, \quad \hat{\mathcal{L}}_{2,1} := \langle \mathcal{L}_{2,1}, \sigma_-, \sigma_+ \rangle.
\]

Then again \( \mathcal{R} \) is the identity path component of each of them. Moreover,

\[
\begin{align*}
\sigma_+^2 &= \text{id}, & \sigma_+ \sigma_- \sigma_+ &= \sigma_-^1, & \sigma_-^2 &= \hat{\tau}, & \sigma_+ \sigma_- &= \hat{\theta}, & \hat{\theta} \hat{\tau} &= \hat{\tau} \hat{\theta},
\end{align*}
\]

which imply that \( \hat{\mathcal{L}}_{2,1} := \langle \mathcal{R}, \hat{\theta}, \hat{\tau}, \sigma_-, \sigma_+ \rangle = \langle \mathcal{R}, \sigma_- \sigma_+ \rangle, \) and

\[
\pi_0 \mathcal{L}_{2,1} = \langle \hat{\theta}, \hat{\tau} \rangle = \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \pi_0 \hat{\mathcal{L}}_{2,1} = \langle \sigma_- \sigma_+ \rangle \cong \text{Dih}(\mathbb{Z}_4) = D_4.
\]

d) In all other cases \( p > 2 \) and \( D(L_{p,q}) \) still contains the diffeomorphism \( \hat{\tau} : \frac{0 \tau}{1 \tau} 1 \).

Then we put

\[
\mathcal{L}_{p,q} := \langle \mathcal{R}, \hat{\tau} \rangle \cong \text{Dih}(\mathcal{R}), \quad \hat{\mathcal{L}}_{p,q} := \langle \mathcal{L}_{p,q}, \sigma_-, \sigma_+ \rangle.
\]

Note that \( \mathcal{R} \) is the identity path component of each of these groups. Let us specify them more precisely.
Suppose $q^2 = 1 \pmod{p}$, so $\sigma_+: 0 \xrightarrow{\id} 1$ is defined. Then $\sigma_+ \circ \hat{\tau} = \hat{\tau} \circ \sigma_+$, whence

$$\hat{\mathcal{L}}_{p,q} = \langle \mathcal{R}, \hat{\tau}, \sigma_+ \rangle, \quad \pi_0\hat{\mathcal{L}}_{p,q} = \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Suppose $q^2 = -1 \pmod{p}$, so $\sigma_-: 0 \rightarrow \frac{\lambda}{\mu} 1 \rightarrow 0$ is defined. Then $\sigma_-^{2} = \hat{\tau}$, whence

$$\hat{\mathcal{L}}_{p,q} = \langle \mathcal{R}, \sigma_- \rangle, \quad \pi_0\hat{\mathcal{L}}_{p,q} = \mathbb{Z}_4.$$

Otherwise, $q^2 \neq \pm 1 \pmod{p}$, so neither $\sigma_-$ nor $\sigma_+$ are defined, and

$$\hat{\mathcal{L}}_{p,q} = \mathcal{L}_{p,q} = \langle \mathcal{R}, \hat{\tau} \rangle \cong \text{Dih}(\mathcal{R}), \quad \pi_0\hat{\mathcal{L}}_{p,q} = \pi_0\mathcal{L}_{p,q} = \mathbb{Z}_2.$$

**Smale conjecture.** Recall that Smale conjecture for a manifold $M$ with a «good» Riemannian metric is a statement that the inclusion $\text{Isom}(M) \subset \mathcal{D}(M)$ of the group of isometries of $M$ into the group of all its diffeomorphisms is a homotopy equivalence.

It is also well known that each lens space $L_{p,q}$ (except for $L_{0,1} = S^1 \times S^2$) is also the quotient of the unit 3-sphere $S^3$ in $\mathbb{C}^2$ by a free action of $\mathbb{Z}_p$ generated by the diffeomorphism $\delta_{p,q} : S^3 \rightarrow S^3$, $\delta_{p,q}(z_1, z_2) = (e^{2\pi i/p} \cdot z_1, e^{2\pi i q/p} \cdot z_2)$. In particular, $L_{p,q}$ has a natural metric, called elliptic, induced from the standard metric on $S^3$ via the corresponding covering map $S^3 \rightarrow L_{p,q}$. The group of isometries $\text{Isom}(L_{p,q})$ of this metric is described in [20, Theorem 2.3], which, in particular, implies that $\mathcal{L}_{p,q} \subset \hat{\mathcal{L}}_{p,q} \subset \text{Isom}(L_{p,q})$, so the above groups consist of isometries, and these three groups coincide exactly when neither $\sigma_-$ nor $\sigma_+$ is defined. Thus, (3.3)

$$\mathcal{L}_{p,q} = \hat{\mathcal{L}}_{p,q} = \text{Isom}(L_{p,q}) \iff p > 2 \text{ and } q^2 \neq \pm 1 \pmod{p}.$$  

Note further that the Smale conjecture is proved

(a) for $L_{1,0} = S^3$, so the inclusion $O(4) \subset \mathcal{D}(S^3)$ is a homotopy equivalence, A. Hatcher [16];

(b) for all lens spaces $L_{p,q}$ with $p > 2$ in the book [18];

(c) for all lens spaces except for $S^1 \times S^2$ by another methods used Ricci flow in the preprint by R. Bamler and B. Kleiner [1].

Also in [15] A. Hatcher also proved that $\mathcal{D}(S^1 \times S^2) = \mathcal{D}(L_{0,1})$ has the homotopy type of $\Omega(O(3)) \times O(3) \times O(2)$.

**Simplest Morse-Bott foliation on $L_{p,q}$.** Recall that in §2 we defined a Morse-Bott foliation $\mathcal{F}$ on the solid torus $T$ by the central circle and 2-tori «parallel» to the boundary, see §2. In particular, $\partial T$ is a leaf of $\mathcal{F}$. Since $\xi$ identifies $\partial T \times \{0\}$ with $\partial T \times \{1\}$, it induces the foliation $\mathcal{F}_{p,q}$ on $L_{p,q}$ whose leaves are the images of the corresponding leaves of foliations on those tori. In particular, $\mathcal{F}_{p,q}$ has two singular leaves (the central circles) $C_0$ and $C_1$ and all other leaves are 2-tori parallel each other.

One can also define a Morse-Bott function $g : L_{p,q} \rightarrow \mathbb{R}$ such that $\mathcal{F}_{p,q}$ coincides with the partition of $L_{p,q}$ into the level sets of $g$. For example, define the function $\hat{g} : L \rightarrow [0; 2]$ by

$$\hat{g}(w, z) = \begin{cases} |z|^2, & (w, z) \in T \times \{0\}, \\ 2 - |z|^2, & (w, z) \in T \times \{1\}. \end{cases}$$

Then $\hat{g}$ induces a Morse-Bott $C^\infty$ function $g : L_{p,q} \rightarrow [0; 2]$ such that $\hat{g} = g \circ p$ and the set of critical points of $g$ is a union of two circles $g^{-1}(0) = C_0$ and $g^{-1}(1) = C_1$. Evidently, $T_0 = g^{-1}([0; 1])$, $T_1 = g^{-1}([1; 2])$, so $\partial T_0 = \partial T_1 = g^{-1}(1)$. 

Let $\mathcal{D}^{\text{fol}}(\mathcal{F}_{p,q})$ be the group of $\mathcal{F}_{p,q}$-foliated diffeomorphisms of $L_{p,q}$, $\mathcal{D}^{\text{bp}}(\mathcal{F}_{p,q})$ be its (normal) subgroup consisting of $\mathcal{F}_{p,q}$-leaf preserving ones, and

$$\mathcal{D}^{\text{fol}}_+(\mathcal{F}_{p,q}) = \{ h \in \mathcal{D}^{\text{bp}}(\mathcal{F}_{p,q}) \mid h(C_i) = C_i, i = 0, 1 \}$$

be the another normal subgroup of $\mathcal{D}^{\text{fol}}(\mathcal{F}_{p,q})$ leaving invariant each central circle $C_i$. If $\mathcal{D}^{\text{fol}}_+(\mathcal{F}_{p,q})$ does not coincide with all the group $\mathcal{D}^{\text{fol}}(\mathcal{F}_{p,q})$, then $\mathcal{D}^{\text{fol}}_+(\mathcal{F}_{p,q})$ obviously has index 2 in $\mathcal{D}^{\text{fol}}(\mathcal{F}_{p,q})$.

Since the above diffeomorphisms $\rho_{\alpha, \beta}$, $d$, $\lambda$, $\tau$ of $\mathcal{T}$ belong to $\mathcal{D}^{\text{bp}}(\mathcal{F})$, it follows that $\mathcal{L}_{p,q} \subset \mathcal{D}^{\text{bp}}(\mathcal{F}_{p,q})$, while $\sigma_+, \sigma_- \in \mathcal{D}^{\text{fol}}(\mathcal{F}_{p,q}) \setminus \mathcal{D}^{\text{fol}}_+(\mathcal{F}_{p,q})$, and thus we get the following inclusions of subgroups:

$$\mathcal{D}^{\text{bp}}(\mathcal{F}_{p,q}) \supset \mathcal{D}^{\text{fol}}_+(\mathcal{F}_{p,q}) \supset \mathcal{D}^{\text{fol}}(\mathcal{F}_{p,q})$$

(3.4)

Our second results of this paper is the following:

**Theorem 3.1.** The group $\mathcal{D}^{\text{bp}}(\mathcal{F}_{p,q})$ is a strong deformation retract of $\mathcal{D}^{\text{fol}}_+(\mathcal{F}_{p,q})$.

In fact, we will prove it for a more general class of foliations by level sets of Morse-Bott functions on a closed manifold having only extreme critical submanifolds, see Theorem 7.7. Now, using previous results we can deduce homotopy types of all the groups in (3.4), see Theorem 3.4 below.

**Theorem 3.2** ([21]). The inclusion $\mathcal{L}_{p,q} \subset \mathcal{D}^{\text{bp}}(\mathcal{F}_{p,q})$ is a weak homotopy equivalence.

**Lemma 3.3.** The following conditions are equivalent:

(a) there are no diffeomorphisms $h \in \mathcal{D}(L_{p,q})$ exchanging $\mathbf{T}_0$ and $\mathbf{T}_1$;

(b) $p > 2$ and $q^2 \neq \pm 1 \text{(mod } p)$, i.e. exactly when neither $\sigma_-$ nor $\sigma_+$ are defined;

(c) $\mathcal{L}_{p,q} = \mathcal{D}_{p,q}$;

(d) $\mathcal{D}^{\text{fol}}_+(\mathcal{F}_{p,q}) = \mathcal{D}^{\text{fol}}(\mathcal{F}_{p,q})$, i.e. every $h \in \mathcal{D}^{\text{fol}}(\mathcal{F}_{p,q})$ leaves invariant each $C_0$ and $C_1$.

**Proof.** In fact, the equivalence (a)$\Leftrightarrow$(b) is well known, (b)$\Leftrightarrow$(c) follows from the definition of $\mathcal{L}_{p,q}$, and the implication (d)$\Rightarrow$(b) is evident.

(a)$\Rightarrow$(d). Let $M = L_{p,q} \setminus (C_0 \cup C_1) = g^{-1}(\{0; 2\})$. Then $g$ has no critical points in $M$ and each level set $g^{-1}(t)$, $t \in (0; 2)$, is diffeomorphic to a 2-torus. Moreover, using standard technique with some gradient flow of $g$, one can construct a diffeomorphism $\psi : M \rightarrow T^2 \times (0; 2)$ such that $f \circ \psi^{-1}(y, t) = t$, $t \in (0; 2)$.

Now suppose that (d) fails, so there exists $h \in \mathcal{D}^{\text{fol}}_+(\mathcal{F}_{p,q}) \setminus \mathcal{D}^{\text{fol}}(\mathcal{F}_{p,q})$. If $h(g^{-1}(1)) = 1$, then $h$ exchanges $\mathbf{T}_0$ and $\mathbf{T}_1$, so (a) also fails. Suppose $h(g^{-1}(1)) = g^{-1}(\tau)$ for some $\tau \in (0; 2)$ and let $\mu : (0; 2) \rightarrow (0; 2)$ be a diffeomorphism fixed near 0 and 2 and such that $\mu(\tau) = 1$, so it gives the diffeomorphism $k' : T^2 \times (0; 2) \rightarrow T^2 \times (0; 2)$, $k'(y, t) = k(y, \mu(t))$. Notice also that $h(M) = M$, whence we have a diffeomorphism $k = \psi \circ h \circ \psi^{-1} : T^2 \times (0; 2) \rightarrow T^2 \times (0; 2)$ such that $k(T^2 \times 1) = T^2 \times \tau$. Define the following diffeomorphism $h' \in \mathcal{D}(L_{p,q})$ by

$$h'(x) = \begin{cases} \psi^{-1} \circ k' \circ k \circ \psi(x), & x \in M, \\ h(x), & x \in C_0 \cup C_1. \end{cases}$$

Then $h' \in \mathcal{D}^{\text{fol}}(\mathcal{F}_{p,q}) \setminus \mathcal{D}^{\text{fol}}_+(\mathcal{F}_{p,q})$ coincides with $h$ near $C_0 \cup C_1$ and leaves invariant $g^{-1}(1)$. □
**Theorem 3.4.** 1) Suppose \( p > 2 \) and \( q^2 \neq \pm1 \) (mod \( p \)). Then we have the following commutative diagram of inclusions in which the arrows denoted by \((\text{w.h.e.})\) are (weak) homotopy equivalences (and new statements of this paper are written in bold):

\[
\begin{array}{cccc}
D^p(F_{p,q}) & \xrightarrow{\text{h.e. \ (Lemma 3.1)}} & D^\text{fol}(F_{p,q}) & \xrightarrow{\text{h.e. \ (w.h.e.) \ \text{Theorem 3.2)}} & D^\text{fol}(F_{p,q}) \\
\text{w.h.e. \ \text{Theorem 3.2}} & & & \xrightarrow{\text{w.h.e. \ \text{Theorem 3.2}}}
\end{array}
\]

\[
L_{p,q} = \hat{L}_{p,q} = \langle R, \hat{T} \rangle \xrightarrow{\text{Isom}} \text{Isom}(L_{p,q}) \xrightarrow{\text{h.e. \ \text{Smale conjecture}}} D(L_{p,q})
\]

In particular, each of those groups has the homotopy type of the disjoint union of two 2-tori \( R \sqcup \hat{R} \).

2) For all other values of \((p,q)\), i.e. when either \( \sigma_- \) or \( \sigma_+ \) is defined, the inclusion of pairs

\[
(\hat{L}_{p,q}, L_{p,q}) \subset (D^\text{fol}(F_{p,q}), D^\text{fol}(F_{p,q}))
\]

is a weak homotopy equivalence. In particular, each path components of those groups is also homotopy equivalent to a 2-torus \( R \).

**Proof.** 1) The left vertical and all horizontal arrows explained at the diagram. They imply that the right vertical arrow is also a weak homotopy equivalence.

2) By Lemma 3.3, \( \hat{L}_{p,q} \neq L_{p,q} \) and \( D^\text{fol}(F_{p,q}) \neq D^\text{fol}(F_{p,q}) \). Hence \( \hat{L}_{p,q} \), resp. \( D^\text{fol}(F_{p,q}) \), is a union of two adjacent classes by the group \( L_{p,q} \), resp. \( D^\text{fol}(F_{p,q}) \). By 1) the inclusion \( j : L_{p,q} \subset D^\text{fol}(F_{p,q}) \) is a weak homotopy equivalence, i.e. \( j : \pi_0L_{p,q} \to \pi_0D^\text{fol}(F_{p,q}) \) is a bijection, and \( j \) yields a weak homotopy equivalence between the corresponding identity path components. Hence so must be the inclusion of pairs (3.5). \( \square \)

**Structure of the paper.** §4 collects preliminary results about left-right actions of groups of diffeomorphisms. §6 is devoted to properties of fiberwise homogeneous functions on the total spaces of vector bundles. In particular, in §6 we prove Theorem 6.7 including Theorem 2.1 as a particular case. The proof uses a famous Whitney lemma on \( C^\infty \) even functions, see Lemma 4.3.

Further in §7 we consider high-dimensional analogues of lens spaces: obtained by gluing unit disk bundles over some manifolds by some diffeomorphism of their boundaries (assuming that such a diffeomorphism exists). We prove there Theorem 7.7 including Theorem 3.1 as a particular case.

### 4. Preliminaries

**Morphisms of topological groups and monoids.** Let \( j : A \to X \) be a topological inclusion, i.e. a homeomorphism of \( A \) onto some subset of \( X \). It will be convenient to say that \( A \) is a (strong) deformation retract of \( X \) with respect to \( j \), if so is the image \( j(A) \). We will need the following simple lemmas.

**Lemma 4.1.** Let \( 1 \to A \xleftarrow{\alpha} B \xrightarrow{p} C \to 1 \) be a short exact sequence of continuous homomorphisms of topological groups in which \( \alpha \) is a topological embedding. Suppose there exists a continuous homomorphism \( \theta : C \to B \) being a section of \( p \), i.e. \( p \circ \theta = \text{id}_C \).

1) Then the map \( \zeta : B \to A \times C, \zeta(b) = (\theta(p(b^{-1})), b, p(b)) \), is a **homeomorphism**.

2) Suppose that there exists a strong deformation retract of \( C \) into the unit \( e_C \) of \( C \). Then \( A \) is a strong deformation retract of \( B \) with respect to the inclusion \( \alpha \). If, in addition,
\[ \theta(C) \text{ is contained in some subgroup } B' \subset B, \text{ and denote } A' = \alpha^{-1}(A \cap B'). \text{ Then the pair } (A, A \cap B') \text{ is a strong deformation retract of } (B, B') \text{ via the inclusion } \alpha. \]

**Proof.** The first statement is trivial. Suppose \( H : C \times [0;1] \to C \) is a strong deformation retraction of \( C \) into \( e_C \), i.e. \( H_0 = \text{id}_C, \ H_t(e_C) = e_C \) for \( t \in [0;1] \), and \( H_1(C) = \{e_C\} \). Then the map

\[ G : B \times [0;1] \to B, \quad G(b, t) = \theta(H(p(b^{-1}), 1-t))) \cdot b, \quad (4.1) \]

is a strong deformation retraction of \( B \) onto \( A \), so \( G_0 = \text{id}_B, \ G_t \) is fixed on \( A \) for \( t \in [0;1] \), and \( H_1(B) \subset A \).

Moreover, if \( \theta(C) \subset B' \), then \( (4.1) \) shows that if \( b \in B' \), then \( G(b,t) \in B' \) as well. In other words, \( B' \) is invariant under \( G \), and thus \( G \) induces a strong deformation retraction of \( B' \) onto \( j(A \cap B') \).

**Lemma 4.2.** Let \( \sigma : A \to B \) be a homomorphism of monoids, so \( \sigma(e_A) = e_B \) and \( \sigma(aa') = \sigma(a)\sigma(a') \) for all \( a, a' \in A \). Then for every invertible \( a \in A \), its image \( \sigma(a) \) is invertible in \( B \).

**Proof.** We have that \( e_B = \sigma(e_A) = \sigma(aa^{-1}) = \sigma(a)\sigma(a^{-1}) \). Therefore, \( (\sigma(a))^{-1} = \sigma(a^{-1}) \).

**Smooth even functions.** We will need the following statement.

**Lemma 4.3** (H. Whitney, [41]). Let \( a > 0 \), \( I_a = [-a; a] \) and \( \gamma \in C^\infty(I_a, \mathbb{R}) \) be an even function, that is \( \gamma(-t) = \gamma(t) \) for all \( t \in I_a \). Then there exists a unique \( \phi \in C^\infty([0; a], \mathbb{R}) \) such that \( \gamma(t) = \phi(t^2) \) for all \( t \in \mathbb{R} \).

Moreover, let \( C^\infty_e(I_a, \mathbb{R}) \) be the space of even \( C^\infty \) functions on \( I_a \). Then the correspondence \( \gamma \to \phi \) is an \( \mathbb{R} \)-linear map \( \delta : C^\infty_e(I_a, \mathbb{R}) \to C^\infty([0; a], \mathbb{R}) \) being continuous between the \( C^\infty \) topologies.

**Sketch of proof.** Notice that \( \phi : [0; a] \to \mathbb{R} \) is uniquely defined by \( \phi(t) = \gamma(\sqrt{t^2}) \), \( t \in [0; a] \). Such formula implies that \( \phi \) is \( C^\infty \) only for \( t \neq 0 \) and the main difficulty was to show that \( \phi \) is in fact \( C^\infty \) near 0 as well. Smoothness of \( \phi \) is proved by Whitney, and we need to derive the second statement about continuity of \( \delta \).

Uniqueness of \( \phi \) easily implies that \( \delta \) is an \( \mathbb{R} \)-linear map. Hence it suffices to verify continuity of \( \delta \) at the zero function 0 only. One easily checks by induction that the identity \( \gamma(t) = \phi(t^2) \) implies that for every for \( r \geq 1 \) there exists some constant \( A_r > 0 \) depending only on \( r \) such that:

\[ \sup_{t \in [0; \alpha]} \left| \frac{d^r \gamma}{dt^r}(t) \right| \leq A_r \sum_{i=0}^{r+1} \sup_{t \in I_a} \left| \frac{d^i \phi}{dt^i}(t) \right|. \]

This implies, that for each \( r \geq 0 \) the map \( \delta \) is continuous from \( C^{r+1} \) topology of \( C^\infty_e(I_a, \mathbb{R}) \) into \( C^r \) topology of \( C^\infty([0; a], \mathbb{R}) \). Hence it is continuous between \( C^\infty \) topologies.

For example, notice that \( \gamma'(t) = 2t \phi'(t^2) \). Hence \( \gamma'(0) = 0 \), and thus, by Hadamard lemma, \( \gamma'(t) = t \delta(t) \), where \( \delta(t) = \frac{1}{2} \gamma''(st)ds \). Therefore \( \phi'(t^2) = \frac{1}{2} \delta(t) = \frac{1}{2} \int_0^t \gamma''(st)ds \), and

\[ \sup_{t \in [0; a]} |\phi'(t)| \leq \frac{1}{2} \sup_{t \in I_a} |\gamma''(t)|. \]

We leave the other cases \( r \geq 2 \) for the reader.
5. Stabilizers of functions under actions of diffeomorphisms groups

**Left-right actions of diffeomorphisms groups.** Let $M$ be a smooth compact manifold. Then the product $\mathcal{D}(\mathbb{R}) \times \mathcal{D}(M)$ of groups of diffeomorphisms naturally acts from the left on the space of smooth functions $C^\infty(M, \mathbb{R})$ by the following action map, see e.g. [31, §3] for detailed discussions and references:

$$
\mu : \mathcal{D}(\mathbb{R}) \times \mathcal{D}(M) \times C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R}), \quad \mu(\phi, h, f) = \phi \circ f \circ h^{-1}.
$$

It is usually referred as *left-right*. Notice also that $\mathcal{D}(M) = \text{id}_{\mathbb{R}} \times \mathcal{D}(M)$ is a subgroup of $\mathcal{D}(\mathbb{R}) \times \mathcal{D}(M)$, and thus we have an induced (still *left*) action

$$
\mu : \mathcal{D}(M) \times C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R}), \quad \mu(h, f) = f \circ h^{-1},
$$

which will be referred below as *right*\(^1\). In terms of «arrows» these actions «move down» the horizontal arrow $f$:

\[
\begin{array}{ccc}
M & \xrightarrow{f} & \mathbb{R} \\
\downarrow h & & \downarrow \phi \\
M & \xrightarrow{\phi \circ f \circ h^{-1}} & \mathbb{R}
\end{array}
\]

Let $f \in C^\infty(M, \mathbb{R})$ and $X \subset M$ be a subset. Denote by $\mathcal{D}(M, X)$ the subgroup of $\mathcal{D}(M)$ consisting of diffeomorphisms fixed on $X$. Then one can define its stabilizers with respect to the above left-right and right actions:

$$
\mathcal{S}_{LR}(f, X) := \{ (\phi, h) \in \mathcal{D}(\mathbb{R}) \times \mathcal{D}(M, X) \mid \phi \circ f \circ h^{-1} = f \},
$$

$$
\mathcal{S}_{R}(f, X) := \{ h \in \mathcal{D}(M, X) \mid f \circ h^{-1} = f \}.
$$

If $X$ is empty, then we will omit it from the notation.

Notice also that we have a canonical inclusion $j : \mathcal{S}_{R}(f, X) \subset \mathcal{S}_{LR}(f, X)$, $j(h) = (\text{id}_{\mathbb{R}}, h)$, and therefore sometimes we will identify $\mathcal{S}_{R}(f, X)$ with its image $\{\text{id}_{\mathbb{R}}\} \times \mathcal{S}_{R}(f, X)$.

**Example 5.1.** Define $\phi, f, h, q : \mathbb{R} \to \mathbb{R}$ by $\phi(t) = 4t$, $f(x) = x^2$, $h(x) = 2x$, $q(x) = -x$. Then the identities $4x^2 = (2x)^2$ and $(-x)^2 = x^2$, mean respectively that $\phi \circ f = f \circ h$ and $f \circ q = f$, so $(\phi, h) \in \mathcal{S}_{LR}(f)$ and $q \in \mathcal{S}_{R}(f)$.

We will describe now the geometrical meaning of the above right- and left-right stabilizers in terms of level sets of $f$. Let $\mathcal{F} = \{ f^{-1}(t) \}_{t \in \mathbb{R}}$ be the partition of $M$ into the level sets of $f$, $\phi \in \mathcal{D}(\mathbb{R})$, and $h \in \mathcal{D}(M)$. Then it is easy to see that the following conditions are equivalent:

(R1) $f \circ h^{-1} = f$, so $h \in \mathcal{S}_{R}(f)$;

(R2) $f = f \circ h$;

(R3) $h(f^{-1}(t)) = f^{-1}(t)$ for every $t \in \mathbb{R}$, i.e. $h$ is an $\mathcal{F}$-leaf preserving diffeomorphism.

Similarly, the following conditions are also equivalent:

(LR1) $\phi \circ f \circ h^{-1} = f$, so $h \in \mathcal{S}_{LR}(f)$;

(LR2) $\phi \circ f = f \circ h$;

(LR3) $h(f^{-1}(t)) = f^{-1}(\phi(t))$ for all $t \in \mathbb{R}$.

\(^1\)In fact, it will become a right action if we define it by $\mu(h, f) = f \circ h$. However, for certain reason it is convenient to mean the terms «left-right» and «right» to refer the sides at which we append the corresponding diffeomorphisms to $f$. 
In particular, condition (LR3) implies that \( h \) is an \( \mathcal{F} \)-foliated diffeomorphism. However, if \( h' \) is an \( \mathcal{F} \)-foliated diffeomorphism, then a priori we can not claim that there exists \( \phi \in \mathcal{D}(\mathbb{R}) \) such that \( (\phi, h') \in \mathcal{S}_{LR}(f) \).

Let us clarify relations between stabilizers and foliated diffeomorphisms.

**Lemma 5.2.** Let \( f \in C^\infty(M, \mathbb{R}) \) and \( A := f(M) \subset \mathbb{R} \) be its image. Then for every \( (\phi, h) \in \mathcal{S}_{LR}(f) \), we have that \( \phi(A) = A \). Moreover, if \( \psi \in \mathcal{D}(\mathbb{R}) \) is another diffeomorphism such that \( \psi = \phi \) on \( A \), then \( (\psi', h) \in \mathcal{S}_{LR}(f) \).

**Proof.** Let \( a \in A \), so \( a = f(x) \) for some \( x \in M \). Then \( \phi(a) = \phi \circ f(x) = f \circ h(x) \in A \), so \( \phi(A) \subset A \). Applying the same arguments for the inverse \( (\phi^{-1}, h^{-1}) = (\phi, h)^{-1} \in \mathcal{S}_{LR}(f) \), we obtain that \( \phi^{-1}(A) \subset A \), whence \( \phi(A) = A \). Moreover, if \( \psi = \phi \) on \( A \), then for each \( x \in M \) we have that \( \psi \circ f(x) = \phi \circ f(x) = f \circ h(x) \). Thus \( (\psi, h) \in \mathcal{S}_{LR}(f) \) as well. \( \square \)

Let \( f \in C^\infty(M, \mathbb{R}) \) and \( X \subset M \) a subset. Then its image \( A := f(M) \subset \mathbb{R} \) is a finite union of closed intervals, and therefore a submanifold of \( \mathbb{R} \). Let \( \mathcal{D}(A) \) be the group of diffeomorphisms of \( A \). In view of Lemma 5.2, it is more natural to regard \( f \) as a surjective function \( f \in C^\infty(M, \mathbb{R}) \), and study the stabilizer of \( f \) with respect to the corresponding «left-right» action of \( \mathcal{D}(A) \times \mathcal{D}(M) \). Therefore it is more convenient to consider instead of \( \mathcal{S}_{LR}(f) \) the following groups:

\[
\mathcal{S}_{LR}^{img+}(f, X) := \{(\phi, h) \in \mathcal{D}^+(A) \times \mathcal{D}(M, X) \mid \phi \circ f = f \circ h \},
\]

\[
\mathcal{S}_{LR}^{img}(f, X) := \{(\phi, h) \in \mathcal{D}(A) \times \mathcal{D}(M, X) \mid \phi \circ f = f \circ h \}.
\]

Then \( \mathcal{S}(f) \equiv \text{id}_A \times \mathcal{S}(f) \) is a subgroup of \( \mathcal{S}_{LR}^{img+}(f) \subset \mathcal{S}_{LR}^{img}(f) \), and the properties (LR1)-(LR3) still hold for \( \mathcal{S}_{LR}^{img}(f) \) instead of \( \mathcal{S}_{LR}(f) \). Moreover, we have the following lemma.

**Lemma 5.3.** Let \( p_1 : \mathcal{D}(A) \times \mathcal{D}(M) \to \mathcal{D}(A) \) and \( p_2 : \mathcal{D}(A) \times \mathcal{D}(M) \to \mathcal{D}(M) \) be natural projections, so \( p_1(\phi, h) = \phi \) and \( p_2(\phi, h) = h \). Then the following statements hold.

1. \( p_1 \) and \( p_2 \) are homomorphisms;
2. \( \mathcal{S}(f) = \mathcal{D}^{lp}(\mathcal{F}) \);
3. \( \ker(p_1|_{\mathcal{S}_{LR}^{img}(f)}) = \text{id}_A \times \mathcal{S}(f) \);
4. \( p_2(\mathcal{S}_{LR}^{img}(f)) \subset \mathcal{D}^{fol}(\mathcal{F}) \);
5. if \( (\phi, h), (\phi', h') \in \mathcal{S}_{LR}(f) \), then \( \phi = \phi' \), i.e. the map \( p_2|_{\mathcal{S}_{LR}^{img}(f)} : \mathcal{S}_{LR}^{img}(f) \to \mathcal{D}(M) \) is injective.

**Proof.** Statements (1) and (3) are trivial, (2) coincides with (R3), while (4) coincides with (LR3).

(5) Notice that the kernel of \( p_2|_{\mathcal{S}_{LR}^{img}(f)} \) consists of pairs of the form \( (\phi, \text{id}_M) \in \mathcal{S}_{LR}^{img}(f) \) satisfying \( \phi \circ f = f \), whence \( \phi \in \mathcal{D}(A) \) is fixed on the image \( A \) of \( f \), so \( \phi = \text{id}_A \). \( \square \)

In particular, we get the following commutative diagram in which the upper row is exact at the two middle items (though \( p_2 \) is not necessarily surjective):

\[
\begin{array}{cccc}
1 & \mathcal{S}(f) & \mathcal{S}_{LR}^{img}(f) & \mathcal{D}(A) \\
\downarrow j: h \mapsto (\text{id}_A, h) & \downarrow & \downarrow p_1 & \downarrow p_2 \\
\mathcal{D}^{lp}(\mathcal{F}) & \mathcal{D}^{fol}(\mathcal{F}) \\
\end{array}
\]
Condition (J). In [27] the author gave wide conditions on $f$ under which the inclusion $j : S_R(f) \subset S_{LR}^{img+}(f)$ is a homotopy equivalence, see Theorem 5.6 below. We will briefly recall that result. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $C^\infty$ function. Say that $f$ has property (J) at $u \in \mathbb{R}^n$ if there exists a neighborhood $U$ of $u$ and $C^\infty$ functions $\alpha_1, \ldots, \alpha_n : U \to \mathbb{R}$ such that $f(x) - f(u) = \sum_{i=1}^n f'_{x_i}(x) \alpha_i(x)$, $x \in U$.

Example 5.4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $C^\infty$ homogeneous function of order $k$, that is $f(tx) = t^k f(x)$ for all $t \geq 0$ and $x \in \mathbb{R}^n$. Then, by the well known Euler identity

$$f = f'_{x_1} \frac{x_1}{n} + \cdots + f'_{x_n} \frac{x_n}{n},$$

so $f$ has property (J) at the origin $0 \in \mathbb{R}^n$.

Equivalently, denote by $C^\infty_u(\mathbb{R}^n)$ the algebra of germs at $u$ of $C^\infty$ functions $\mathbb{R}^n \to \mathbb{R}$, and for each $f \in C^\infty_u(\mathbb{R}^n)$ let $J_u(f)$ be the ideal in $C^\infty_u(\mathbb{R}^n)$ generated by partial derivatives of $f$. Then property (J) means that the germ at $u$ of the function $g(x) = f(x) - f(u)$ belongs to $J_u(f)$. The following simple lemma shows that the property (J) does not depend on local coordinates at $u$, and so it is well defined for functions on manifolds.

Lemma 5.5. Let $h = (h_1, \ldots, h_n) : (\mathbb{R}^m, v) \to (\mathbb{R}^n, u)$ be a germ of a $C^\infty$ map, and $h^* : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^m)$, $h^*(f) = f \circ h$, be the induced algebra homomorphism. Then

$$J_u(h^*(f)) \subset h^*(J_u(f)).$$

In particular, if $h$ is a diffeomorphism, then $f \in J_u(f)$ iff $h^*(f) \in J_u(h^*(f))$.

Proof. Let $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$. Then for each $k = 1, \ldots, m$ we have that

$$\frac{\partial (h^*(f))}{\partial y_k}(y) = \frac{\partial (f \circ h)}{\partial y_k}(y) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(h(y)) \frac{\partial h}{\partial y_k}(y) = \sum_{i=1}^n h^*(f'_{x_i})(y) \frac{\partial h}{\partial y_k}(y),$$

i.e. partial derivatives of $h^*(f)$ are linear combinations with smooth coefficients of the images of partial derivatives $h^*(f'_{x_i})$ of $f$. Hence $J_v(h^*(f)) \subset h^*(J_u(f))$. \qed

Theorem 5.6 ([27, Theorem 1.3]). Let $M$ be a smooth connected compact manifold, $f \in C^\infty(M, \mathbb{R})$, and $A = f(A)$ be its image. Suppose that

(a) $f$ takes a constant value at each connected component of $\partial M$ and has only finitely many critical values;

(b) $f$ has property (J) at every critical point $x \in M$.

Then id$_A \times S_R(f)$ is a strong deformation retract of $S_{LR}^{img+}(f)$.

Remarks to the proof. Let $Q = \{a_0, a_1, \ldots, a_n\} \subset A$ be all the values of $f$ at critical points and boundary components of $M$. Condition (a) implies that $Q$ is finite. Let also $\mathcal{D}(A, Q)$ be the subgroup of $\mathcal{D}(A)$ fixed on $Q$. One easily checks that if $(\phi, h) \in S_{LR}^{img+}(f)$, then $\phi \in \mathcal{D}(A, Q)$, i.e. $p_1(S_{LR}^{img+}(f)) \subset \mathcal{D}(A, Q)$. Moreover, each $\phi \in \mathcal{D}(A, Q)$ preserves orientation of $A$, and $\mathcal{D}(A, Q)$ is convex in $C^\infty(A, A)$, and therefore contractible.

The main technical result of [27, Theorem 1.3] is based on condition (b) and claims that there exists a continuous homomorphism $\theta : \mathcal{D}(A, Q) \to S_{LR}^{img+}(f)$ such that $\phi \circ f = f \circ \theta(\phi)$. In other words, $\theta$ is a section of $p_1 : S_{LR}^{img+}(f) \to \mathcal{D}(A, Q)$, so the statement of this theorem follows from Lemma 4.1. \qed
6. Fiberwise homogeneous functions on vector bundles

In this section we prove Theorem 6.7 including Theorem 2.1 as a particular case.

**Smooth homogeneous functions.** Recall that a continuous function \( f : \mathbb{R}^n \to \mathbb{R} \) is **homogeneous** of some degree \( k \geq 0 \), whenever \( f(tv) = t^k f(v) \) for all \( t > 0 \) and \( v \in \mathbb{R}^n \).

The following simple lemma seems to be a classical result. However, the author did not find precise references, though there are discussions on this question in internet resources, e.g. [5].

**Lemma 6.1.** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a homogeneous \( C^k \) function of integer degree \( k \geq 0 \). Then \( f \) is a homogeneous polynomial of degree \( k \).

**Proof.** If \( k = 0 \), then the assumption that \( f \) is continuous and homogeneous of degree 0, means that \( f(tv) = t^0 f(v) = f(v) \) for all \( t \geq 0 \) and \( v \in \mathbb{R}^n \). In particular, for \( t = 0 \) we get that \( f(v) = f(0) \) for all \( v \in \mathbb{R}^n \), so \( f \) is constant.

For \( k > 0 \), applying \( \frac{\partial}{\partial v_i} \), \( i = 1, \ldots, n \), to both sides of the identity \( f(tv) = t^k f(v) \), we get \( tf_{v_i}(tv) = t^k f'_{v_i}(tv) \), whence \( f'_{v_i}(tv) = t^{k-1} f''_{v_i}(tv) \), i.e. every partial derivative \( f'_{v_i} \) is a homogeneous \( C^{k-1} \) function. Then, by induction on \( k \), \( f'_{v_i} \) must be a homogeneous polynomial of degree \( k - 1 \). Hence, by Euler’s identity, \( f(v) = \frac{1}{k} \sum_{i=1}^n v_i f'_{v_i}(v) \) is a homogeneous polynomial of degree \( k \).

More generally, let \( p : E \to B \) be a smooth vector bundle of rank \( n \) over a manifold \( B \). We will identify \( B \) with the image of the zero section of \( p \) in \( E \). A continuous function \( f : E \to \mathbb{R} \) is called **homogeneous** of degree \( k \geq 0 \) (or \( k \)-**homogeneous**) whenever \( f(tx) = t^k f(x) \) for all \( t \geq 0 \) and \( x \in E \).

Assume that \( f : E \to \mathbb{R} \) is homogeneous. Then \( B \subset f^{-1}(0) \), however, in general, this inclusion in non-strict. We will say that \( f \) is **definite**, whenever \( f(x) > 0 \) for all \( x \in E \setminus B \). In particular, \( B = f^{-1}(0) \).

**Corollary 6.2.** Let \( p : E = B \times \mathbb{R}^n \to B \) be a trivial vector bundle, and \( f : E \to \mathbb{R} \) be a \( k \)-homogeneous \( C^r \) function with \( k \leq r \). Then

\[
 f(y, v_1, \ldots, v_n) = \sum_{i_1, \ldots, i_n \in \{0, \ldots, k\}} \alpha_{i_1, \ldots, i_n}(y) v_1^{i_1} \cdots v_n^{i_n},
\]

where each \( \alpha_{i_1, \ldots, i_n} : B \to \mathbb{R} \) is some \( C^{r-k} \) function.

**Proof.** Notice that \( k \)-homogenuity of \( f \) means that for every \((y, u) \in B \times \mathbb{R}^n \) and \( t \geq 0 \) we have that \( f(y, tu) = t^k f(y, u) \). Now the proof follows from the Lemma 6.1.

**Corollary 6.3.** Let \( f : E \to \mathbb{R} \) be a \( C^\infty \) homogeneous function of degree \( k \geq 2 \). Then \( f \) satisfies condition (J) at each \( x \in B \).

**Proof.** Due to Lemma 5.5 one can pass to a local trivialization of \( p \) at \( x \), and thus assume that \( p : E \to B \) is a trivial vector bundle. Since \( k \geq 2 \), every point of \( B \) is critical. Then by (6.1) and «Euler’s identity with respect to the coordinates \( v_1, \ldots, v_n \)» we have that \( f(y, v) = \frac{1}{k} \sum_{i=1}^n v_i f'_{v_i}(y, v) \). Hence \( f \) satisfies property (J) at \( x \).

**Stabilizers of homogeneous functions.** Let \( p : E \to B \) be a smooth vector bundle of rank \( n \) over a manifold \( B \). The following statement is a particular case of [27, Theorem 1.3]. However, the proof is essentially simpler and it additionally takes to account the behavior of diffeomorphisms on \( \partial T \).
Let $f : E \to [0; +\infty)$ be a $C^\infty$ function such that 1 is a regular value of $f$, so $T = f^{-1}([0; 1])$ is a submanifold of $E$, and $F = \{ f^{-1}(t) \mid t \in [0; 1] \}$ be the partition of $T$ into level sets of $f$.

**Lemma 6.4** (c.f. [27, Theorem 1.3]). Let $k \geq 1$, $f : E \to \mathbb{R}$ be a $k$-homogeneous $C^\infty$ function, $T = f^{-1}([0; 1])$, and $F = \{ f^{-1}(t) \mid t \in [0; 1] \}$ be the partition of $T$ into level sets of $f$. Then there exists a homomorphism $\theta : D^+([0; 1]) \to D^{\text{fol}}(F, \partial T)$ such that $\phi \circ f = f \circ \theta(h)$ for all $\phi \in D^+([0; 1])$. In particular, $(\phi, \theta(\phi)) \in S_{LR}^\text{img}(f)$.

If $T$ is compact, then $\theta$ is continuous, and the pair $(S_{R}(f), S_{R}(f, \partial T))$ is a strong deformation retract of the pair $(S_{LR}^\text{img}(f), S_{LR}^\text{img}(f, \partial T))$ with respect to the natural inclusion $j : S_{R}(f) \equiv \{ \text{id}_{[0; 1]} \} \times S_{R}(f) \subset S_{LR}^\text{img}(f)$.

**Proof.** Let $\phi \in D^+([0; 1])$, so $\phi : [0; 1] \to [0; 1]$ is a $C^\infty$ function such that $\phi(0) = 0$, $\phi(1) = 1$, and $\phi' > 0$. Then by the arguments of Hadamard lemma:

$$\phi(t) = \int_0^t \phi'(u)du = \begin{cases} \text{replace } u = st, \text{then } du = sdt \end{cases} = t \int_0^1 \phi'(st)dt = tg_\phi(t),$$

where $g_\phi : [0; 1] \to \mathbb{R}$ is $C^\infty$. Moreover, $g_\phi(t) > 0$ for all $t \in [0; 1]$ and $g_\phi(0) = \phi'(0)$.

Define the map

$$\theta : D^+([0; 1]) \to C^\infty(T, T), \quad \theta(\phi)(x) = [g_\phi(f(x))]^{1/k} x.$$  

We claim that the image of $\theta$ is contained in $D^{\text{fol}}(F)$, and the induced map $\theta : D^+([0; 1]) \to D^{\text{fol}}(F)$ is the desired homomorphism.

1) First we show that $\theta$ is a homomorphism of monoids with respect to the natural composition of maps. Then by Lemma 4.2 it sends invertible elements to invertible, and therefore $\theta(h)$ will be a diffeomorphism of $T$.

a) Indeed, if $\phi = \text{id}_{[0; 1]}$, then $g_\phi(t) \equiv 1$, whence $\theta(\text{id}_{[0; 1]}) = \text{id}_T$.

b) Furthermore, let $\phi_0, \phi_1 \in D^+([0; 1])$ be two diffeomorphisms. Then $\phi_0(t) = tg_0(t)$ and $\phi_1(t) = tg_1(t)$ for unique $C^\infty$ functions $g_0, g_1 : [0; 1] \to \mathbb{R}$ such that

$$\theta(\phi_0)(x) = [g_0(f(x))]^{1/k} x, \quad \theta(\phi_1)(x) = [g_1(f(x))]^{1/k} x.$$  

Hence

$$\theta(\phi_1) \circ \theta(\phi_0)(x) = \theta(\phi_1)([g_0(f(x))]^{1/k} x) = [g_1(f([g_0(f(x))]^{1/k} x))]^{1/k} \cdot [g_0(f(x))]^{1/k} x = [g_1(g_0(f(x)) \cdot f(x))]^{1/k} \cdot [g_0(f(x))]^{1/k} x.$$  

On the other hand, $\phi_1 \circ \phi_0(t) = \phi_0(t) \cdot g_1(\phi_0(t)) = t \cdot g_0(t) \cdot g_1(tg_0(t))$, whence

$$\theta(\phi_1 \circ \phi_0)(x) = [g_1(f(x))]^{1/k} x = [g_0(f(x)) \cdot g_1(f(x) \cdot g_0(f(x)))]^{1/k} x = \theta(\phi_1) \circ \theta(\phi_0)(x).$$

2) Let us verify that $(\phi, \theta(\phi)) \in S_{LR}(f)$. Indeed,

$$f \circ \theta(\phi)(x) = f([g_\phi \circ f(x])^{1/k} x) = (g_\phi \circ f(x)) \cdot f(x) = \phi \circ f(x).$$

This implies that $(\phi, \theta(\phi)) \in S_{LR}(f)$, and in particular, $\theta(\phi) \in D^{\text{fol}}(F)$. 

3) Finally, note that \( g_0(1) = \phi(1)/1 = 1 \), whence if \( x \in \partial \mathbf{T} = f^{-1}(1) \), then \( \theta(\phi)(x) = x \), so \( \theta(\phi) \) is fixed on \( \partial \mathbf{T} \). Thus, \( \theta(\phi) \in D^{\text{fol}}(\mathcal{F}, \partial \mathbf{T}) \).

4) Suppose \( \mathbf{T} \) is compact. Then continuity of \( \theta \) directly follows from the formulas for \( \theta(h) \).

Consider the following short exact sequence, see (5.1):

\[
1 \to S_{\mathcal{R}}(f) \overset{j}{\to} S_{\mathcal{L}}^{\text{img}}(f) \overset{p_2}{\to} D^+([0; 1]) \to 1.
\]

Then the map \( \tilde{\theta} : D^+([0; 1]) \to S_{\mathcal{L}}^{\text{img}}(f) \), \( \tilde{\theta}(\phi) = (\phi, \theta(\phi)) \), is a continuous section of \( p_1 \) and its image is contained in \( S_{\mathcal{L}}^{\text{img}}(f, \partial \mathbf{T}) \). Moreover, \( D^+([0; 1]) \) is contractible into \( \text{id}_{[0; 1]} \) via the homotopy

\[
H : D^+([0; 1]) \times [0; 1] \to D^+([0; 1]), \quad H(\phi, t) = (1 - t)\phi + t\text{id}_{[0; 1]}.
\]

Then, by Lemma 4.1, the pair \( (S_{\mathcal{R}}(f), S_{\mathcal{R}}(f, \partial \mathbf{T})) \) is a strong deformation retract of

\[
(S_{\mathcal{L}}^{\text{img}}(f), S_{\mathcal{L}}^{\text{img}}(f, \partial \mathbf{T}))
\]

with respect to the inclusion map \( j \). \( \square \)

**Foliated maps for homogeneous functions.** Let \( p_i : E_i \to B_i \) for \( i = 0, 1 \) be a smooth vector bundle of some rank \( n_i \) over a compact manifold \( B_i \) and \( f_i : E_i \to [0; +\infty) \) be a \( C^\infty \) function such that \( 1 \in \mathbb{R} \) is a regular value for \( f \). Then \( \mathbf{T}_i := f_i^{-1}([0; 1]) \) is a submanifold of \( E_i \). Let also

\[
\mathcal{F}_i = \{ f_i^{-1}(t) \mid t \in [0; 1] \}
\]

be the partition of \( \mathbf{T}_i \) into the level sets of \( f \).

A map \( h : \mathbf{T}_0 \to \mathbf{T}_1 \) will be called \( (\mathcal{F}_0, \mathcal{F}_1) \)-foliated if for each leaf \( \omega \) of \( \mathcal{F}_0 \) its image \( h(\omega) \) is contained in some leaf of \( \mathcal{F}_1 \). Denote by \( C^\infty(\mathcal{F}_0, \mathcal{F}_1) \) the subset of \( C^\infty(\mathbf{T}_0, \mathbf{T}_1) \) consisting of \( (\mathcal{F}_0, \mathcal{F}_1) \)-foliated maps. If \( \mathbf{T}_0 \) and \( \mathbf{T}_1 \) are diffeomorphic, then we denote by \( D(\mathcal{F}_0, \mathcal{F}_1) \) the set of all \( (\mathcal{F}_0, \mathcal{F}_1) \)-foliated diffeomorphisms.

**Lemma 6.5.** Suppose the functions \( f_0 \) and \( f_1 \) are \( 2 \)-homogeneous and definite.

1) Then for each \( (\mathcal{F}_0, \mathcal{F}_1) \)-foliated map \( h : \mathbf{T}_0 \to \mathbf{T}_1 \) there exists a unique function \( \sigma(h) : [0; 1] \to [0; 1] \) such that \( \sigma(h) \circ f_0 = f_1 \circ h \). Moreover, if \( h \) is \( C^\infty \), then \( \sigma(h) \) is also \( C^\infty \) and the correspondence \( h \mapsto \sigma(h) \) is a continuous map \( \sigma : C^\infty(\mathbf{T}_0, \mathbf{T}_1) \to C^\infty([0; 1], [0; 1]) \).

2) If \( \mathbf{T}_0 \) and \( \mathbf{T}_1 \) are diffeomorphic, then \( \sigma(h) \in D^+([0; 1]) \) is a preserving orientation diffeomorphism of \( [0; 1] \) for each \( h \in D(\mathcal{F}_0, \mathcal{F}_1) \).

3) Suppose \( p_0 = p_1 \) is the same vector bundle and \( f_0 = f_1 : E_0 = E_1 \to \mathbb{R} \), so \( \mathbf{T}_0 = \mathbf{T}_1 \), \( \mathcal{F}_0 = \mathcal{F}_1 \), and thus \( D(\mathcal{F}_0, \mathcal{F}_1) = D^{\text{fol}}(\mathcal{F}_0) \). Then \( \sigma : C^\infty(\mathcal{F}_0, \mathcal{F}_0) \to C^\infty([0; 1], [0; 1]) \) is a (continuous) homomorphism of monoids. In particular, \( \sigma(D^{\text{fol}}(\mathcal{F}_0)) \subset D^+([0; 1]) \).

**Proof.** (1) The assumption that \( h : \mathbf{T}_0 \to \mathbf{T}_1 \) is \( (\mathcal{F}_0, \mathcal{F}_1) \)-foliated means that for each leaf \( A_t = f_0^{-1}(t), \) \( t \in [0; 1] \) of \( \mathcal{F}_0 \) there exist a unique leaf \( B_{t'} = f_1^{-1}(t') \) such that \( h(A_t) \subset B_{t'} \). Define the function \( \phi : [0; 1] \to [0; 1] \) by \( \phi(t) = t' \). Then \( \phi \circ f_0 = f_1 \circ h \), and then we put \( \sigma(h) = \phi \). Hence such \( \sigma(h) \) is unique.

Suppose \( h \) is \( C^\infty \). To prove that \( \sigma(h) \) is \( C^\infty \) as well we will use Whitney Lemma 4.3. Fix any point \( w \in \partial \mathbf{T}_0 \), so \( f_0(w) = 1 \), and consider the path \( \eta : [-1; 1] \to \mathbf{T}_0 \) given by \( \eta(t) = tw \). Then

\[
f_0 \circ \eta(t) = f_0(tw) = t^2f_0(w) = t^2.
\]

In particular, for each \( t \in [-1; 1] \) the points \( \eta(t) \) and \( \eta(-t) \) belong to the same leaf \( f_0^{-1}(t^2) \) of \( \mathcal{F}_0 \).
Define the following $C^\infty$ function $\gamma = f_1 \circ h \circ \eta : [-1, 1] \to \mathbb{R}$. Since $h$ sends level sets of $f_0$ to level sets of $f_1$, the points $h(\eta(t))$ and $h(\eta(-t))$ also belong to the same level set of $f_1$, that is

$$\gamma(-t) = f_1 \circ h \circ \eta(-t) = f_1 \circ h \circ \eta(t) = \gamma(t).$$

Hence $\gamma$ is an even function. Then, due to Lemma 4.3, there exists a unique $C^\infty$ function $\phi : [0; 1] \to [0; 1]$ such that $\gamma(t) = \phi(t^2)$. Thus

$$f_1 \circ h \circ \eta(t) = \gamma(t) = \phi(t^2) = \phi \circ f_0 \circ \eta(t). \quad (6.3)$$

We claim that $f_1 \circ h \equiv \phi \circ f_0$ on all of $T_0$. Indeed, let $y \in T_0$ and $f_0(y) = s$ for some $s \in [0; 1]$. Then $f_0(\eta(\sqrt{s})) = s$ as well, so the points $y$ and $\eta(\sqrt{s})$ belong to the same level set $f_0^{-1}(s)$ of $f_0$. Hence $h(y)$ and $h(\eta(\sqrt{s}))$ also belong to the same level set of $f_1$. Therefore

$$f_1 \circ h(y) = f_1 \circ h \circ \eta(\sqrt{s}) = \phi \circ f_0 \circ \eta(\sqrt{s}) = \phi \circ f_0(y). \quad (6.4)$$

Continuity of the correspondence $h \mapsto \phi$ also follows from Lemma 4.3.

(2) Suppose $h$ is a diffeomorphism. We should show that then $\phi$ is a preserving orientation diffeomorphism of $[0; 1]$. It suffices to check the following three properties of $\gamma$.

(a) $\gamma'(t) > 0$ for $t > 0$;

(b) $\gamma(t) = t^2 \delta(t)$ for some $C^\infty$ function $\delta : [0; 1] \to \mathbb{R}$ such that $\delta(0) > 0$.

Assuming they are proved, let us show that $\phi \in D^+(0; 1]$. Indeed, it is evident that $\phi(0) = 0$ and $\phi(1) = 1$, so we need to verify that $\phi' > 0$. Since $\phi(s) = \gamma(\sqrt{s})$ for $s \in [0; 1]$, we get from (a) that $\phi'(s) > 0$ for $s \in (0; 1]$. Furthermore, as $\phi(0) = 0$, we have by the Hadamard lemma that $\phi(s) = s \psi(s)$ for some $C^\infty$ function $\psi : [0; 1] \to \mathbb{R}$ such that $\psi(0) = \phi'(0)$. Then, due to (b), $t^2 \delta(t) = \gamma(t) = \phi(t^2) = t^2 \psi(t^2)$, whence $\delta(t) = \psi(t^2)$, and therefore $\phi'(0) = \psi(0) = \delta(0) > 0$.

Proof of (a). Note that if $\zeta : [a, b] \to T_0$ is a smooth path not passing through $B$, then for each $t \in [a; b]$ the following conditions are equivalent:

- $t$ is a regular point of the function $f_0 \circ \zeta : [a; b] \to \mathbb{R}$;

- $\zeta$ is transversal at $t$ to the level set $f_0^{-1}(f_0(\zeta(t)))$.

Since $f_0 \circ \eta(t) = t^2$, it follows that the function $f_0 \circ \eta : [-1, 1] \to \mathbb{R}$ has a unique critical point $t = 0$, and thus $\eta$ is transversal to all level sets $f_0^{-1}(s)$ for all $s \in (0; 1]$.

On the other hand, as $h$ sends level sets of $f_0$ to level sets of $f_1$, the path $h \circ \eta$ must also be transversal to the level sets of $f_1$ for all $t \neq 0$. More precisely, if $\eta$ is transversal at some $t \neq 0$ to the leaf $L = f_0^{-1}(t^2)$, then $h \circ \eta$ is transversal at $t$ to the leaf

$$h(L) = f_1^{-1}(f_1 \circ h \circ \eta(t)) = f_1^{-1}(\gamma(t)).$$

This implies that the function $\gamma = f_1 \circ h \circ \eta$ has a unique critical point $t = 0$.

Moreover, since $\gamma(-1) = \gamma(1) = 1$ and $\gamma(0) = 0$, we see that $\gamma$ decreases on $[-1, 0)$ and increases on $(0; 1]$. In particular, $\gamma'(t) < 0$ for $t < 0$ and $\gamma'(t) > 0$ for $t > 0$.

Proof of (b). Let $q_0 : U \times \mathbb{R}^n \to U$ and $q_1 : V \times \mathbb{R}^n \to V$ be vector bundle trivializations of $p$ over open neighborhood $U$ of $\eta(0)$ and $V$ of $h(\eta(0))$ respectively. Then $h$ has local representation as an embedding $h = (h_0, h_1, \ldots, h_n) : U \times \mathbb{R}^n \supset W \to V \times \mathbb{R}^n$ of some open neighborhood $W$ of $\eta(0)$ in $U \times \mathbb{R}^n$, where $h_0 : W \to V$ is a $C^\infty$ map, and each $h_i : W \to \mathbb{R}$ is a $C^\infty$ function.

One can assume that the image of $\eta$ is contained in $W$, and $w = (x, \bar{u}) \in U \times \mathbb{R}^n$ are the coordinates of $w$, so $\eta(t) = (x, t \bar{u})$. 
Then by Corollary 6.2 the restriction of $f_1$ to each fiber of $p$ is a $2$-homogeneous polynomial, i.e. $f_1(y, v) = \sum_{1 \leq i,j \leq n} a_{ij}(y)v_iv_j$ for all $(y, v) \in V \times \mathbb{R}^n$. One can assume that $a_{ij} = a_{ji}$, so we get a symmetric matrix $A(y) = (a_{ij})$, such that $f_1(y, v) = vA(y)v^t$, where $v = (v_1, \ldots, v_n)$, and $v^t$ is the transposed vector column. Hence

$$\gamma(t) = f_1 \circ h \circ \eta(t) = f_1(h(x, t\bar{u})) = \bar{h}(x, t\bar{u}) \cdot A(h_0(x, t\bar{u})) \cdot \hat{h}(x, t\bar{u}),$$

where $\bar{h} = (h_1, \ldots, h_n)$. By the Hadamard lemma $h_i(x, u) = \sum_{j=1}^n h_{ij}(x, u)u_j$ for some $C^\infty$ functions $h_{ij} : W \to \mathbb{R}$ such that $h_{ij}(x, 0) = \frac{\partial}{\partial u_j}h_i$. Let $J(x, u) = (h_{ij}(x, u))$ the matrix whose $i$th row consists of the functions $h_{i1}, \ldots, h_{in}$. Notice that $J(x, 0)$ is the Jacobi matrix of the composition

$$(x \times \mathbb{R}^n) \cap W \xrightarrow{h} V \times \mathbb{R}^n \xrightarrow{p^2} \mathbb{R}^n.$$ 

Since $h$ is a diffeomorphism leaving invariant zero section $B$, it follows that $J(x, 0)$ is non-degenerate.

One can also write $\hat{h}(x, u) = J(x, u)u^t$, whence

$$\gamma(t) = t\bar{u} \cdot J(x, t\bar{u})^t \cdot A(h_0(x, t\bar{u})) \cdot J(x, t\bar{u}) \cdot t\bar{u},$$

which implies that

$$\delta(t) = \gamma(t)/t^2 = uJ(x, tu)^t \cdot A(h_0(x, t\bar{u})) \cdot J(x, tu)u^t.$$ 

Therefore

$$\delta(0) = u \underbrace{J(x, 0)^t \cdot A(h_0(x, 0)) \cdot J(x, 0)}_{B(x)}u^t = uB(x)u^t.$$ 

The assumption that $f_1$ is definite means that $A(h_0(x, 0))$ is non-degenerate, whence $B$ is symmetric and non-degenerate as well, and therefore $\delta(0) = uB(x)u^t > 0$.

(3) Suppose $p_0 = p_1$ is the same vector bundle and $f_0 = f_1 : E_0 = E_1 \to \mathbb{R}$. Since $\text{id}_{[0,1]} \circ f_0 = f_0 \circ \text{id}_\mathbb{T}$, it follows from uniqueness of $\sigma$, that $\sigma(\text{id}_\mathbb{T}) = \text{id}_{[0,1]}$. Moreover, if $h, h' \in \mathcal{D}^{\text{fol}}(\mathcal{F}_0)$, then

$$\sigma(h') \circ \sigma(h) \circ f_0 = \sigma(h') \circ f_0 \circ h = f_0 \circ h' \circ h = \sigma(h' \circ h) \circ f_0.$$ 

(6.5)

Now uniqueness of $\sigma$ implies that $\sigma(h') \circ \sigma(h) = \sigma(h' \circ h)$. Thus $\sigma$ is a homomorphism of monoids.

**Example 6.6.** Notice that the assumption that $f_0$ and $f_1$ are of the same homogeneity order is essential for $\phi$ to be a diffeomorphism. Indeed, let $f_0, f_1 : B \times \mathbb{R}^n \to [0; 1]$ be given by $f_0(w, x) = ||x||^2$ and $f_1(x) = ||x||^4$. Then $f_0$ is 2-homogeneous, while $f_1$ is 4-homogeneous, $\mathcal{F}_0 = \mathcal{F}_1$, and $\mathcal{T}_0 = f_0^{-1}([0; 1]) = f_1^{-1}([0; 1]) = \mathcal{T}_1$. Let also $h = \text{id}_{B \times \mathbb{R}^n}$. Then $h$ is a $(\mathcal{F}_0, \mathcal{F}_1)$-foliated diffeomorphism, while $\phi : [0; 1] \to [0; 1], \phi(t) = t^2$, is a unique $C^\infty$ map satisfying $\phi \circ f_0 = f_1 \circ h$. However $\phi$ is not a diffeomorphism.

It seems plausible that Lemma 6.5 holds for definite homogeneous functions of the same degree $> 2$, but one needs an analogue of Whitney Lemma 4.3 for «evenness of higher order». 

Fiberwise definite 2-homogeneous functions. Let \( p: E \to B \) be a smooth vector bundle of rank \( n \) over a compact manifold \( B \), \( f: E \to \mathbb{R} \) be a definite 2-homogeneous \( C^\infty \) function, \( T = f^{-1}([0; 1]) \), and \( \mathcal{F} = \{ f^{-1}(t) \mid t \in [0; 1] \} \) the partition of \( T \) into level sets of \( f \). Then we have two homomorphisms \( \theta: \mathcal{D}^+([0; 1]) \to \mathcal{D}^{fol}(\mathcal{F}) \) and \( \sigma: \mathcal{D}^{fol}(\mathcal{F}) \to \mathcal{D}^+([0; 1]) \), defined in Lemmas 6.4 and 6.5 respectively, such that \( \sigma(h) = f \circ h \) and \( \phi \circ f = f \circ \theta(\phi) \) for all \( h \in \mathcal{D}^{fol}(\mathcal{F}) \) and \( \phi \in \mathcal{D}^+([0; 1]) \), so we have well-defined homomorphisms

\[
\hat{\theta}: \mathcal{D}^{fol}(\mathcal{F}) \to S^{img}_{LR}(f), \quad \hat{\sigma}(h) = (\sigma(h), h), \\
\hat{\theta}: \mathcal{D}^+([0; 1]) \to S^{img}_{LR}(f), \quad \hat{\theta}(\phi) = (\phi, \theta(\phi)).
\] (6.6)

Notice that the foliation described in Theorem 2.1 corresponds to the definite homogeneous function \( f : S^1 \times \mathbb{R}^2 \to \mathbb{R}, f(w, x, y) = x^2 + y^2 \), on the trivial vector bundle \( p : S^1 \times \mathbb{R}^2 \to S^1 \) of rank 2 over the circle. Therefore Theorem 2.1 is a particular case of the following:

**Theorem 6.7.** The homomorphism

\[
p_2 : S_{LR}(f) \to \mathcal{D}^{fol}(\mathcal{F}), \quad p_2(\phi, h) = h,
\]

induces an isomorphism of the following short exact sequences:

\[
\begin{array}{c}
1 \to S_R(f) \xrightarrow{j} S^{img}_{LR}(f) \xrightarrow{p_1} \mathcal{D}^+([0; 1]) \to 1 \\
\quad \| \quad \| \\
1 \to \mathcal{D}^{lp}(\mathcal{F}) \xhookleftarrow{\mathcal{D}^{fol}(\mathcal{F})} \xrightarrow{\sigma} \mathcal{D}^+([0; 1]) \to 1
\end{array}
\] (6.7)

and its inverse is \( \hat{\sigma} \). Moreover, the pair \( (\mathcal{D}^{lp}(\mathcal{F}), \mathcal{D}^{lp}(\mathcal{F}, \partial \mathcal{T})) \) is a strong deformation retract of \( (\mathcal{D}^{fol}(\mathcal{F}), \mathcal{D}^{fol}(\mathcal{F}, \partial \mathcal{T})) \).

**Proof.** Evidently, \( p_2 \circ \hat{\sigma}(h) = h \) for all \( h \in \mathcal{D}^{fol}(\mathcal{F}) \), in particular, \( p_2 \) is surjective. Since it is also injective, it follows that \( p_2 \) and \( \hat{\sigma} \) are mutually inverse isomorphisms of topological groups. Moreover, by Lemma 5.3, \( S_R(f) = \mathcal{D}^{lp}(\mathcal{F}) \), and \( p_2 \circ j = \text{id}_{S_R(f)} \). It is also evident that

\[
p_2(S^{img}_{LR}(f, \partial \mathcal{T})) = \mathcal{D}(\mathcal{F}, \partial \mathcal{T}).
\]

Now, by Lemma 6.4, the pair \( (S_R(f), S_R(f, \partial \mathcal{T})) \) is a strong deformation retract of the pair \( (S^{img}_{LR}(f), S^{img}_{LR}(f, \partial \mathcal{T})) \) with respect to the inclusion \( j : S_R(f) \subset S^{img}_{LR}(f) \). Hence the pair \( (\mathcal{D}^{lp}(\mathcal{F}), \mathcal{D}^{lp}(\mathcal{F}, \partial \mathcal{T})) \) is a strong deformation retract of \( (\mathcal{D}^{fol}(\mathcal{F}), \mathcal{D}^{fol}(\mathcal{F}, \partial \mathcal{T})) \). \( \square \)

7. High-dimensional analogues of lens spaces

In this section we prove Theorem 7.7 including Theorem 3.1 as a particular case.

For \( i = 0, 1 \) let \( \pi_i: E_i \to B_i \) be a smooth vector bundle of some rank \( n_i \) over a compact manifold \( B_i \) and \( f_i: E_i \to \mathbb{R} \) be a definite \( k_i \)-homogeneous \( C^\infty \) function for some \( k_i > 0 \). Put

\[
T_i := f_i^{-1}([0; 1]), \quad U_i := f_i^{-1}([0; 1]) = T_i \setminus \partial T_i, \quad L := U_0 \sqcup U_1.
\]

Denote by \( \mathcal{F}_i = \{ f_i^{-1}(t) \mid t \in [0; 1] \} \) the partition of \( T_i \) into the level sets of \( f \).

**Lemma 7.1.** Suppose there is a diffeomorphism \( \psi: \partial T_0 \to \partial T_1 \). Then the map

\[
\xi: U_0 \setminus B_0 \to U_1 \setminus B_1, \quad \xi(x) = \sqrt[2k]{1 - f_0(x) \cdot \psi(x/\sqrt[k]{f_0(x)})},
\] (7.1)
is a diffeomorphism such that
\[ f_0(x) + f_1(\xi(x)) = 1, \quad x \in U_0 \setminus B_0, \tag{7.2} \]
which is the same as \( \xi(f_0^{-1}(t)) = f_1^{-1}(1 - t) \) for all \( t \in (0; 1) \), so \( \xi \) is a \((\mathcal{F}_0, \mathcal{F}_1)\)-foliated diffeomorphism.

**Proof.** Let \( x \in U_0 \setminus B_0 \), and \( y = x/\sqrt[3]{f_0(x)} \). Then
\[ f_0(y) = f_0(x/\sqrt[3]{f_0(x)}) = f_0(x)/\sqrt[3]{f_0(x)} = 1, \]
i.e. \( y \in \partial T_0 \), so \( \xi \) is a well-defined map. Moreover, since \( \psi(y) \in \partial T_1 \), \( f_1(\psi(y)) = 1 \), and therefore
\[ f_1(\xi(x)) = f_1(1 - f_0(x) \cdot \psi(y)) = (1 - f_0(x)) \cdot f_1(\psi(y)) = 1 - f_0(x). \]
\[ \square \]

**Example 7.2.** Notice that even if \( \partial T_0 \) and \( \partial T_1 \) are diffeomorphic, the bases \( B_0 \) and \( B_1 \) may have distinct topological type. The following example is inspired by surgery theory of manifolds. Fix any \( a, b \geq 0 \) and let \( p_0 : S^a \times \mathbb{R}^{b+1} \to S^a \) and \( p_1 : S^b \times \mathbb{R}^{a+1} \to S^b \) be trivial vector bundles over spheres \( S^a \) and \( S^b \). Consider the following 2-homogeneous functions \( f_0 : S^a \times \mathbb{R}^{b+1} \to \mathbb{R} \) and \( f_1 : S^b \times \mathbb{R}^{a+1} \to \mathbb{R} \) given by
\[ f_0(x, u_1, \ldots, u_{b+1}) = \sum_{i=1}^{b+1} u_i^2, \quad f_1(v, v_1, \ldots, v_{a+1}) = \sum_{i=1}^{a+1} v_i^2. \]
Then both \( f_0^{-1}(1) \) and \( f_1^{-1}(1) \) are diffeomorphic with \( S^a \times S^b \), though the bases \( S^a \) and \( S^b \) are not homeomorphic for \( a \neq b \).

**Remark 7.3.** Recall that each lens space \( L_\xi \) is glued from two solid tori by some diffeomorphism \( \xi \) between their boundaries. Though \( L_\xi \) admits a smooth structure, such a definition has the following disadvantage: suppose we have a \( h : L_\xi \to M \) into some other manifold. Therefore, even if we know that the restriction of \( h \) on each tori \( T_i \) is \( C^\infty \), the full map \( h \) is not necessary even differentiable, and checking of its smoothness might be rather complicated. For that reason, in what follow we will glue our analogues of lens spaces by diffeomorphism between open sets, as in Lemma 7.1.

Suppose that there exists a diffeomorphism \( \psi : \partial T_0 \to \partial T_1 \). Let \( L_\xi = U_0 \cup_\xi U_1 \) be the space obtained by gluing \( U_0 \) with \( U_1 \) via the diffeomorphism \( \xi : U_0 \setminus B_0 \to U_1 \setminus B_1 \) from Lemma 7.1. Let also \( p : L \to L_\xi \) be the corresponding quotient map and \( p_i := p|U_i : U_i \to L_\xi \) be the restriction maps. Then it is evident that \( L_\xi \) is a manifold, and each \( p_i \) is an open embedding. One can regard the pair \( \{p_0, p_1\} \) as a \( C^\infty \) atlas for \( L_\xi \) with \( \xi \) being a transition map. Define the following function
\[ \hat{f} : L \to [0; 1], \quad \hat{f}(x) = \begin{cases} f_0(x), & x \in U_0, \\ 1 - f_1(x), & x \in U_1. \end{cases} \]
It follows from (7.2) that \( \hat{f}(\xi(x)) = \hat{f}(x) \) for all \( x \in U_0 \setminus B_0 \), whence \( \hat{f} \) yields a well-defined \( C^\infty \) function \( f : L_\xi \to [0; 1] \) such that \( \hat{f} = f \circ p \). It will be convenient to define the following

\[ \text{Footnote 2: Usually an atlas of a manifold } M \text{ is a collection of open embeddings } \mathbb{R}^n \supset U_i \xrightarrow{\psi_i} M, i \in \Lambda, \text{ from open subsets of } \mathbb{R}^n \text{ such that } M = \bigcup_{i \in \Lambda} \psi_i(U_i). \text{ However, all the theory of manifolds will not be changed if one extend a notion of an atlas allowing each } U_i \text{ to be an open subset of some } n\text{-manifold such that the corresponding transition functions are smooth maps.} \]
diffeomorphism \( q : [0; 1] \to [0; 1], q(t) = 1 - t \). Then we get the following commutative diagram:

\[
\begin{array}{c}
\text{U}_0 \backslash B_0 & \xrightarrow{\xi} & \text{U}_0 & \xrightarrow{f_0} & [0; 1] \\
\text{U}_1 \backslash B_1 & \xrightarrow{\xi} & \text{U}_1 & \xrightarrow{f_1} & [0; 1]
\end{array}
\]

**Example 7.4.** If \( E_i = S^1 \times \mathbb{R}^2 \to S^1, i = 0, 1, \) are trivial vector bundles of rank 2 over the circle, and the functions \( f_0 = f_1 : S^1 \times \mathbb{R}^2 \to \mathbb{R} \) coincide and are given by the formula \( f_0(w, x, y) = x^2 + y^2, \) then the map \( \text{L}_\xi \) satisfies the property \((J)\) at each \( x \) such that \( \phi \mid_{\text{L}_\xi} \). Further, we will also have that \( \text{L}_\xi \) is a particular case of Theorem 7.7 below.

**Lemma 7.5.** There exists a continuous homomorphism

\[
\theta : \mathcal{D}^+([0; 1]) \to \mathcal{D}^+_\text{fol}(\mathcal{F})
\]

such that \( \phi \circ f = f \circ \theta(h) \) for all \( \phi \in \mathcal{D}^+([0; 1]) \). Therefore \( \mathcal{S}_R(f) \) is a strong deformation retract of \( \mathcal{S}^\text{img}(f) \) with respect to the natural inclusion \( j : \mathcal{S}_R(f) \subset \mathcal{S}^\text{img}(f) \).

**Proof.** By definition, \( f_i : E_i \to \mathbb{R}, i = 0, 1, \) is a \( k_i \)-homogeneous \( C^\infty \) function, so it satisfies the property \((J)\) at each \( x \in B_i \). Moreover, \( f_0 \) and \( f_1 \) are local representations of \( f \) in the «local chart» \( U_0 \) and \( U_1 \), it follows that \( f \) has property \((J)\) at each \( x \in L_0 \cup L_1 \). Now the result follows from Theorem 5.6. However, we will give an explicit proof similar to the proof of Lemma 6.4.

1) Let \( \phi \in \mathcal{D}^+([0; 1]) \). It will be convenient to put \( \phi_0 = \phi \). As \( \phi_0([0; 1]) = [0; 1] \), it follows from Lemma 6.4 that the map

\[
h_0 : U_0 \to U_0, \quad h_0(x) = \sqrt[k]{g_0(f_0(x))} x,
\]

is a diffeomorphism satisfying \( \phi_0 \circ f_0 = f_0 \circ h_0 \), where \( g_0 : [0; 1] \to (0; +\infty) \) is a unique \( C^\infty \) function such that \( \phi_0(t) = g_0(t)t \).

Similarly, we have that \( \phi([0; 1]) = [0; 1] \). Consider another diffeomorphism \( \phi_1 \in \mathcal{D}^+([0; 1]) \), \( \phi_1(t) = q \circ \phi \circ q(t) = 1 - \phi(1 - t) \). Then \( \phi_1([0; 1]) = [0; 1] \), whence, again by Lemma 6.4, the map

\[
h_1 : U_1 \to U_1, \quad h_1(x) = \sqrt[k]{g_1(f_1(x))} x,
\]

is a diffeomorphism satisfying \( \phi_1 \circ f_1 = f_1 \circ h_1 \), where \( g_1 : [0; 1] \to (0; +\infty) \) is a unique \( C^\infty \) function such that \( \phi_1(t) = g_1(t)t \).

We claim that

\[
\xi \circ h_0(x) = h_1 \circ \xi(x), \quad x \in U_0 \backslash B_0.
\]

This will imply that \( h_0 \) and \( h_1 \) yield a well-defined diffeomorphism

\[
\theta(\phi) : \text{L}_\xi \to \text{L}_\xi, \quad \theta(\phi)(x) = \begin{cases} h_0(p_0^{-1}(x)), & x \in p_0(U_0), \\ h_1(p_1^{-1}(x)), & x \in p_1(U_1). \end{cases}
\]

Moreover, we will also have that \( \phi \circ f = f \circ \theta(\phi) \).
Before proving (7.3) let us establish several simple identities:

\[
\frac{h_0(x)}{\sqrt[k]{f_0 \circ h_0(x)}} = \frac{\sqrt[k]{g_0(f_0(x)) \cdot x}}{\sqrt[k]{g_0(f_0(x)) \cdot x}} = \frac{g_0(f_0(x))}{\sqrt[k]{g_0(f_0(x)) \cdot x}} = \frac{x}{\sqrt[k]{f_0(x)}},
\]

(7.4)

\[
(g_1 \circ q \circ f_0(x)) \cdot (q \circ f_0(x)) = \phi_1 \circ q \circ f_0(x) = q \circ \phi_0 \circ f_0(x) = q \circ f_0 \circ h_0(x).
\]

(7.5)

Then

\[
h_1 \circ \xi(x) = \sqrt[k]{g_1 \circ f_1 \circ \xi(x) \cdot \xi(x)} = \sqrt[k]{g_1 \circ f_1 \circ \xi(x) \cdot \psi(x/\sqrt[k]{f_0(x)})} = \\
\sqrt[k]{g_1 \circ q \circ f_0(x) \cdot \sqrt[k]{q \circ f_0(x) \cdot \psi(x/\sqrt[k]{f_0(x)})}} = \\
\sqrt[k]{q \circ f_0 \circ h_0(x) \cdot \psi(\frac{\sqrt[k]{h_0(x)}}{\sqrt[k]{f_0(h_0(x))}})} = \xi \circ h_0(x).
\]

(7.6)

The correspondence \( \phi \mapsto \theta(\phi) \) is a homomorphism, since so is the correspondence \( \phi_0 \mapsto h_0 \). Continuity of \( \theta \) follows from the formulas for \( h_0 \) and \( h_1 \).

2) The proof that \( \mathcal{S}_\text{LR}(f) \) is a strong deformation retract of \( \mathcal{S}_\text{LR}^{\text{img+}}(f) \) is literally the same as in step 4) of the proof of Lemma 6.4.

For \( t \in [0; 1]\) let \( L_t := f^{-1}(t) \), and \( \mathcal{F} = \{L_t\}_{t \in [0; 1]} \) be the foliation on \( L \) by level sets of \( f \). Then \( L_t = p(f_0^{-1}(t)) \) for \( t \in [0; 1] \) and \( L_t = p(f^{-1}(1 - t)) \) for \( t \in (0; 1) \). In particular, \( L_0 = p(B_0) \) and \( L_1 = p(B_1) \) are singular leaves, and each \( L_t \), \( t \in (0; 1) \), is diffeomorphic with \( \partial \mathcal{T}_0 \). Let \( \mathcal{D}_+^{\text{fol}}(\mathcal{F}) \) be the group of \( \mathcal{F} \)-foliated diffeomorphisms leaving invariant each \( L_0 \) and \( L_1 \).

**Lemma 7.6.** Suppose \( f_0 \) and \( f_1 \) are definite and 2-homogeneous. Then there is a unique continuous homomorphism \( \sigma : \mathcal{D}_+^{\text{fol}}(\mathcal{F}) \rightarrow \mathcal{D}([0; 1]) \) such that \( \sigma(h) \circ f = f \circ h \), i.e. \( (\sigma(h), h) \in \mathcal{S}_\text{LR}^{\text{img}}(f) \).

Also, \( h \in \mathcal{D}_+^{\text{fol}}(\mathcal{F}) \) if and only if \( \sigma(h) \in \mathcal{D}_+^+([0; 1]) \), and moreover, \( \sigma(\mathcal{D}_+^{\text{fol}}(\mathcal{F})) = \mathcal{D}_+^+([0; 1]) \). Hence, if \( \mathcal{D}_+^{\text{fol}}(\mathcal{F}) \neq \mathcal{D}_+^{\text{fol}}(\mathcal{F}) \), then \( \sigma \) is surjective.

**Proof.** 1) Let \( h \in \mathcal{D}_+^{\text{fol}}(\mathcal{F}) \). We should construct a diffeomorphism \( \phi : [0; 1] \rightarrow [0; 1] \) satisfying \( \phi \circ f = f \circ h \) and then put \( \sigma(h) \). As \( h \) is \( \mathcal{F} \)-foliated, for each \( t \in [0; 1] \) there exists a unique \( t' \in [0; 1] \) such that \( h(L_t) = L_{t'} \). Then, by condition (LR3), \( \phi \) must be defined by \( \phi(t) = t' \). In particular, \( \phi \) is uniquely determined by \( h \), and we need to check that \( \phi \) is a diffeomorphism. Consider two cases.

a) First assume that \( h \in \mathcal{D}_+^{\text{fol}}(\mathcal{F}) \), i.e. \( h(L_i) = L_i \) for \( i = 0, 1 \). Then \( p(U_i) = L_0 \setminus L_{1-i} \) is also invariant under \( h \). Hence \( h_i \) yields a \( \mathcal{F}_i \)-foliated diffeomorphism

\[
h_i = p_i \circ h \circ p_i : U_i \xrightarrow{p_{i-1}} p(U_i) \xrightarrow{h} p(U_i) \xrightarrow{p_{i-1}} U_i, \quad i = 0, 1,
\]
such that $h_1 \circ \xi = \xi \circ h_0$. Then by Lemma 6.5, there exists a unique diffeomorphism $\phi_i : [0; 1) \to [0; 1)$ such that $\phi_i \circ f_i = f_i \circ h_i$. Thus we get the following commutative diagram:

It implies that $\phi_0 = q \circ \phi_1 \circ q^{-1}$, i.e. $\phi_0(t) = 1 - \phi_1(1 - t)$ for $t \in (0; 1)$. Therefore, we get a well defined diffeomorphism $\phi \in D^+([0; 1])$ given by either of the formulas:

$$\phi(t) = \begin{cases} \phi_0(t), & t \in [0; 1), \\ 1 - \phi_1(1 - t), & t \in (0; 1] \end{cases} \tag{7.6}$$

and satisfying $\phi \circ f = f \circ h$.

b) Suppose that $h(L_i) = L_{1-i}$ for $i = 0, 1$. Then $h$ yields a $(\mathcal{F}_1, \mathcal{F}_{1-i})$-foliated diffeomorphism

$$h_i = p_{1-i} \circ h \circ p_i : U_i \xrightarrow{p_i} p(U_i) \xrightarrow{h} p(U_{1-i}) \xrightarrow{p_{1-i}} U_{1-i}, \quad i = 0, 1,$$

such that $\xi^{-1} \circ h_0 = h_1 \circ \xi$. Therefore, by Lemma 6.5, there exists a diffeomorphism $\phi_i : [0; 1) \to [0; 1)$ such that $\phi_i \circ f_i = f_{1-i} \circ h_i$. Thus we get the following commutative diagram:

In particular, $q \circ \phi_0 = \phi_1 \circ q$, i.e. $1 - \phi_0(t) = \phi_1(1 - t)$ for $t \in (0; 1)$. Therefore, we get a well defined reversing orientation diffeomorphism $\phi \in D([0; 1])$ given by

$$\phi(t) = \begin{cases} q \circ \phi_0(t) = 1 - \phi_0(t), & t \in [0; 1), \\ \phi_1 \circ q(t) = \phi_1(1 - t), & t \in (0; 1] \end{cases} \tag{7.7}$$

and satisfying $\phi \circ f = f \circ h$.

2) Due to Lemma 6.5 and formulas for $\phi$, the correspondence $h \mapsto \phi$ is a well-defined continuous map $\sigma : D_{\text{fol}}^+(\mathcal{F}) \to D([0; 1])$, $\sigma(h) = \phi$. Moreover, as in (6.5), uniqueness of $\sigma$ implies that $\sigma$ is a homomorphism. Also, by the construction $\sigma(h) \in D^+(\mathcal{F})$ if and only if $h \in D_{\text{fol}}^+(\mathcal{F})$.

Further notice, that due to Lemma 7.5, $\sigma(\theta(\phi)) = \phi$ for all $\phi \in D^+([0; 1])$. Indeed, we have that $\phi \circ f = f \circ \theta(\phi)$, and $\sigma(\theta(\phi)) \circ f = f \circ \theta(\phi)$. Then by uniqueness of $\sigma$ we should have that $\sigma(\theta(\phi)) = \phi$. This implies that $\sigma(D_{\text{fol}}^+(\mathcal{F})) = D^+([0; 1])$. 
Finally, suppose that $D^+_{\text{fol}}(F) \neq D^+_{\text{fol}}(F)$. Then $\sigma$ must map the unique adjacent class $D^+_{\text{adj}}(F) \setminus D^+_{\text{adj}}(F)$ of $D^+_{\text{adj}}(F)$ by $D^+_{\text{adj}}(F)$, onto the unique adjacent class $D(\{0;1\}) \setminus D^+(\{0;1\})$ of $D([0;1])$ by $D^+(\{0;1\})$. Hence $\sigma$ is surjective.

**Theorem 7.7.** Suppose $f_0$ and $f_1$ are definite and 2-homogeneous. Then the projection $p_2 : S^\text{img}_{LR}(f) \to D^+_{\text{fol}}(F)$ induces an isomorphism of the following short exact sequences:

\[
\begin{align*}
1 & \longrightarrow S_R(f) \overset{j}{\longrightarrow} S^\text{img}_{LR}(f) \overset{p_1}{\longrightarrow} D^+(\{0;1\}) \longrightarrow 1 \\
1 & \longrightarrow D^b(F) \overset{\sigma}{\longrightarrow} D^+_{\text{fol}}(F) \overset{\theta}{\longrightarrow} D^+(\{0;1\}) \longrightarrow 1
\end{align*}
\]

where $\hat{\sigma}(h) = (\sigma(h), h)$ is the inverse to $p_2$, and $\hat{\theta}(\phi) = (\phi, \theta(\phi))$ is the inverse to $p_1$. In particular, $D^b(F)$ is a strong deformation retract of $D^+_{\text{fol}}(F)$.

Moreover, if $D^+_{\text{adj}}(F) \neq D^+_{\text{adj}}(F)$, then we have isomorphism of another pair of short exact sequences:

\[
\begin{align*}
1 & \longrightarrow S_R(f) \overset{j}{\longrightarrow} S^\text{img}_{LR}(f) \overset{p_1}{\longrightarrow} D([0;1]) \longrightarrow 1 \\
1 & \longrightarrow D^b(F) \overset{\sigma}{\longrightarrow} D^+_{\text{adj}}(F) \overset{\theta}{\longrightarrow} D([0;1]) \longrightarrow 1
\end{align*}
\]

**Proof.** Evidently, $p_2 \circ \hat{\sigma}(h) = h$ for all $h \in D^+_{\text{adj}}(F)$, in particular, $p_2$ is surjective and maps $S^\text{img}_{LR}(f)$ onto $D^+_{\text{adj}}(F)$. Since it is also injective (see Lemma 5.3), it follows that $p_2$ and $\hat{\sigma}$ are mutually inverse isomorphisms of topological groups. Moreover, again by Lemma 5.3, $S_R(f) = D^b(F)$ and $p_2 \circ j = \text{id}_{S_R(f)}$. Hence $p_2$ yields isomorphisms of the corresponding short exact sequences in (7.8) and (7.9).

By Lemma 7.5, $S_R(f)$ is a strong deformation retract of $S^\text{img}_{LR}(f)$ with respect to the inclusion $j : S_R(f) \subset S^\text{img}_{LR}(f)$, whence $D^b(F)$ is a strong deformation retract of $D^+_{\text{adj}}(F)$.

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