Gauge Equivalence in Two–Dimensional Gravity

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ABSTRACT

Two-dimensional quantum gravity is identified as a second-class system which we convert into a first-class system via the Batalin-Fradkin (BF) procedure. Using the extended phase space method, we then formulate the theory in most general class of gauges. The conformal gauge action suggested by David, Distler and Kawai is derived from a first principle. We find a local, light-cone gauge action whose Becchi-Rouet-Stora-Tyutin invariance implies Polyakov’s curvature equation $\partial_- R = \partial^3 g_{++} = 0$, revealing the origin of the $SL(2, R)$ Kac-Moody symmetry. The BF degree of freedom turns out be dynamically active as the Liouville mode in the conformal gauge, while in the light-cone gauge the conformal degree of freedom plays that rôle. The inclusion of the cosmological constant term in both gauges and the harmonic gauge-fixing are also considered.

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1 INTRODUCTION

In spite of many efforts in studying two-dimensional (2D) quantum gravity [1–4], there are still some fundamental problems to be solved. First, the conformal gauge formulation developed by David, Distler and Kawai (DDK) [3, 4] crucially relies upon the validity of their conjecture on the functional measure that a Liouville action may be substituted for the path-integral Jacobian associated with the transition to a translation invariant one. There have been indeed several attempts [5, 6] to justify the conjecture, but a simple, rigorous proof is lacking. Second, the origin of the $SL(2,\mathbb{R})$ Kac-Moody algebra in the light-cone gauge, discovered by Knizhnik, Polyakov and Zamolodchikov (KPZ) [1, 2], has remained obscure in the BRST quantization. Furthermore, there is no direct proof for the equivalence between the conformal gauge and light-cone gauge formulations, although there are various theoretical indications for the equivalence [7].

To overcome these problems, we formulate in this paper Polyakov’s string theory [9] at sub-critical dimensions [10] or 2D gravity in most general class of gauges, according to our previous proposal [11] to quantize the theory as an anomalous gauge theory. The strategy to achieve our aim is as follows: We rely on the recent result [14] on the most general form of the BRST anomaly in 2D gravity in the extended phase space (EPS) of Batalin, Fradkin and Vilkovisky (BFV) [15]. The anomaly found there expresses the fact that due to anomalous Schwinger terms the first-class (generalized) Virasoro constraints have converted into second-class constraints. This observation is our starting point to consider 2D gravity as an anomalous gauge theory [16]. We shall exactly follow the approach of Refs. [17, 18] to theories suffering from chiral gauge anomalies. This method is based on the Hamiltonian formalism developed by Batalin and Fradkin (BF) [19] to quantize systems under second-class constraints.

The heart of the BF method [19] is to rewrite a system under second-class constraints into a gauge symmetric one by adding to the EPS the compensating fields, the BF fields. This idea can be simply extended [17, 18] to quantization of anomalous gauge theories.

1A similar program based on the anti-bracket formalism [12] has been considered in Ref. [13].
because they can be regarded as second-class-constrained systems. Remarkably, the re-conversion from the anomalous second-class constraints back into the effectively first-class ones can be at least formally performed without considering gauge fixing. One expects therefore that, once the corresponding program has been successfully applied to Polyakov’s string, one arrives at a quantum theory for 2D gravity formulated in most general class of gauges. This is indeed possible as we will see.

A new and subtle feature in quantizing 2D gravity as an anomalous gauge theory is that the compensating fields themselves contribute to the anomaly in the gauge algebra. As a result, some of the coefficients involved in the anomaly-compensating mechanism can not be explicitly determined unless one fixes a gauge. Therefore, the quantum nilpotency of the BRST charge should be carefully examined for each gauge choice. The desired gauge equivalence, formally ensured by the BFV theorem \[15, 21\], can be achieved only in this manner.

We shall consider here three gauge choices: the conformal \[3, 4\], light-cone \[1, 2\] and harmonic gauge \[20\] fixings. The master action of BFV contains two Liouville-type actions: One is for the Liouville mode arising from the compensating field, and the other for the conformal factor of the metric, and they have a definite Weyl-transformation property. In the conformal gauge, the master action becomes the effective action suggested by DDK \[3, 4\], on one hand. In the light-cone gauge, on the other hand, the master action becomes a local action, a part of which looks like a Liouville action, and the conformal mode acts as the gravitational analog of the Liouville mode. The BRST invariance in this gauge-fixed theory naturally leads to Polyakov’s curvature equation \[1, 2\] as well as to the \(SL(2, R)\) Kac-Moody symmetry. The KPZ condition can be obtained by investigating the quantum nilpotency condition on the BRST charge. So our local, light-cone gauge action contains the same information as that of Polyakov’s non-local action. We also present a careful treatment of the cosmological constant term in both the gauges.

In contrast to those gauge choices, the metric and ghost variables become dynamically active in the harmonic gauge \[20\]. In the BFV formalism this is equivalent to keep certain Legendre terms in the master action.
In section 2, we would like to briefly outline the derivation \[14\] of the most general form of anomaly on the EPS in Polyakov’s theory. We apply in section 3 the canonical method of Ref. \[18\] to the present case: By an appropriately chosen canonical pair of BF variables, we re-convert the system into a first-class one. The EPS variables are usually non-covariant. For covariant gauges, e.g. for the conformal gauge, it is obviously convenient to use covariant quantities. In section 4 we partially fix the gauge to define covariant quantities such as the metric variables and the covariant ghost fields. Section 5, 6 and 7 are respectively devoted to explicitly consider the gauge-fixed theory in the conformal \[3, 4\], light-cone \[1, 2\], and harmonic \[20\] gauges. Discussions and summary are given in section 8. In Appendix we derive the algebraic properties in the light-cone gauge that are used in section 6.

### 2 ANOMALY IN THE EPS

Here we would like to briefly outline the derivation \[14\] of the most general form of anomaly on the EPS in Polyakov’s theory \[9\]. We begin by defining the EPS of the theory described by the classical action

\[
S_X = -\frac{1}{2} \int d^2\sigma \sqrt{-g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu ,
\]

with \(\alpha, \beta = 0, 1\) and \(\mu = 0, \cdots, D - 1\),

where we mostly follow the notation of Ref. \[22\]. We shall make an ADM decomposition for the 2D metric variables \(g_{\alpha\beta}\), parametrizing

\[
\lambda^\pm = \frac{\sqrt{-g} \pm g_{01}}{g_{11}} , \quad \xi = \ln g_{11} .
\]

In this notation, the Weyl transformation corresponds to a translation in the \(\xi\)-variable.

The conjugate momenta of \(\lambda^\pm\) and \(\xi\), which we denote respectively by \(\pi^\lambda_{\pm}\) and \(\pi_\xi\), vanish identically, yielding the primary constraints

\[
\pi^\lambda_{\pm} \approx 0 , \quad \pi_\xi \approx 0 .
\]

\(^2\)We ignore the cosmological constant term here. It will be included later.
The Dirac algorithm further leads to the secondary constraints, the Virasoro constraints (we use the abbreviations \( \dot{f} = \partial_\tau f \), \( f' = \partial_\sigma f \), \( \sigma^\pm = \tau \pm \sigma \), and \( \partial_\pm = \partial_\tau \pm \partial_\sigma \) with \( \partial_\pm \sigma^\pm = 2 \)),

\[ \varphi_\pm \equiv \frac{1}{4}(P \pm X')^2 \approx 0, \tag{4} \]

where \( P_\mu \) denotes the conjugate momentum of \( X^\mu \) and given by

\[ P_\mu = -\sqrt{-g}g^{\alpha \mu} \partial_\alpha X^\mu, \tag{5} \]

and the constraints \( \varphi_\pm \) satisfy under Poisson bracket the Virasoro algebra

\[ \{\varphi_\pm(\sigma), \varphi_\pm(\sigma')\}_{PB} = \pm (\varphi_\pm(\sigma) + \varphi_\pm(\sigma')) \delta'(\sigma - \sigma'), \tag{6} \]

\[ \{\varphi_+(\sigma), \varphi_-(\sigma')\}_{PB} = 0. \]

So, at the classical level, the theory is a system under the five first-class constraints defined in Eqs. (3) and (4).

According to these five first-class constraints, we define the EPS by adding to the classical phase space the ghost-auxiliary field sector which consists of the canonical pairs

\[ \left( C^A, \overline{P}_A \right), \left( P^A, \overline{C}_A \right), \left( N^A, B_A \right), \tag{7} \]

where \( A (= \lambda^\pm, \xi, \pm) \) labels the first-class constraints. \( C^A \) and \( P^A \) are the BFV ghost fields carrying one unite of the ghost number, \( \text{gh}(C^A) = \text{gh}(P^A) = 1 \), while \( \text{gh}(\overline{C}_A) = \text{gh}(\overline{P}_A) = 1 \) for their canonical momenta, \( \overline{P}_A \) and \( \overline{C}_A \). The last canonical pairs in (7) are auxiliary fields and carry no ghost number. We assign 0 to the canonical dimension of \( X^\mu, \lambda^\pm \) and \( \xi \), and correspondingly +1 to \( P_\mu, \pi_\lambda^\pm \), and \( \pi_\xi \). The canonical dimensions of \( C^\lambda, C^\xi, \overline{P}_\pm, \overline{P}_\lambda, \overline{C}_\xi \) are fixed only relative to that of \( C^\pm \). Let \( c \equiv \text{dim}(C^\pm) \), we then have

\[ \text{dim}(C^\pm) = \text{dim}(C^\xi) = 1 + c, \quad \text{dim}(\overline{P}_\pm) = 1 - c, \quad \text{dim}(\overline{P}_\lambda) = \text{dim}(\overline{P}_\xi) = -c. \tag{8} \]

The BRST charge \( Q \) can be easily constructed from the constraints given in Eqs. (3) and (4) as

\[ Q = \int d\sigma \left[ C^+_\lambda \pi^\lambda_+ + C^-_\lambda \pi^-_+ + C^\xi \pi_\xi 
+ C^+ (\varphi_+ + \overline{P}_+ C'^+) + C^- (\varphi_- - \overline{P}_- C'^-) + P^A B_A \right], \tag{9} \]
which satisfies the super-Poisson bracket (PB) relation

$$\{Q, Q\}_\text{PB} = 0.$$  \hspace{1cm} (10)

In quantum theory, the operator $Q$ should be suitably regularized, and $Q^2$ expressed by the super-commutator $[Q, Q]/2$ may fail to vanish due to an anomaly [23]. In Ref. [24, 14], we imposed super-Jacobi identities among super-commutation relations while assuming that super-commutators can be expanded in $\bar{\hbar}$. Thus we assumed that the anomalous commutator relation

$$[Q, Q] = i\hbar^2\Omega + O(\hbar^3)$$  \hspace{1cm} (11)

satisfies the super-Jacobi identity

$$[Q, [Q, Q]] = 0.$$  \hspace{1cm} (12)

In the lowest order in $\hbar$, Eq. (12) reads

$$\delta\Omega = 0,$$  \hspace{1cm} (13)

where $\delta$ is the BRST transformation given by the Poisson bracket:

$$\delta A \equiv -\{Q, A\}_\text{PB}.$$  \hspace{1cm} (14)

Eq. (13) exhibits the consistency condition on $\Omega$ and has to be solved to find the most general form of $Q^2$ in the EPS. The BRST non-trivial solution to (13), which does not depend on the choice of regularizations and gauges, is found to be [14]

$$\Omega = K \int d\sigma \left[ C^+ \partial^3_\sigma C^+ - C^- \partial^3_\sigma C^- \right]$$  \hspace{1cm} (15)

for Polyakov’s theory (1). It is known from the explicit calculation in the conformal gauge [23] that

$$K = \frac{(26 - D)}{24\pi},$$  \hspace{1cm} (16)

and that the $O(\hbar^3)$-term in Eq. (11) is absent.
Due to the very nature of our method to find anomalous Schwinger terms, the above result can be obtained without any reference to a two-dimensional metric, and the result is valid for any space-time dimension, as emphasized in Ref. [14]. The $Q^2$-anomaly (11) expresses the anomalous conversion of the first-class nature of the generalized Virasoro constraints

$$\Phi_\pm = \{P_\pm, Q\}_\text{PB} = \varphi_\pm \pm 2\overline{P}_\pm \partial_\sigma C_\pm \pm \partial_\sigma \overline{P}_\pm C_\pm$$

into the second-class ones:

$$[\Phi_\pm(\sigma), \Phi_\pm(\sigma')] = \pm i\hbar(\Phi_\pm(\sigma) + \Phi_\pm(\sigma'))\delta'(\sigma - \sigma') \mp i K \hbar^2 \delta'''(\sigma - \sigma') .$$

(18)

The last terms in Eq. (18) are the anomalous Schwinger terms in the Virasoro algebra.

The Hamiltonian plays a secondary role in the present case because the canonical Hamiltonian $H_C$ identically vanishes. Indeed, if $H_C = 0$, the total Hamiltonian in the BFV formalism is a BRST-commutator

$$H_T = \{Q, \Psi\}_\text{PB} ,$$

(19)

where $\Psi$ is a gauge-fermion [13].

We fix the anomalous Schwinger term in $[Q, H_T]$ by imposing the super-Jacobi identity [24, 14]

$$2 [Q, [Q, H_T] ] + [H_T, [Q, Q] ] = 0 .$$

(20)

Assuming that the Schwinger term again can be written as

$$[Q, H_T] = \frac{i}{2} \hbar^2 \Gamma + O(\hbar^3) ,$$

(21)

one easily finds that the super-Jacobi identities among Poisson brackets and the consistency condition on $\Omega$ given in (13) yield

$$\Gamma = \{\Omega, \Psi\}_\text{PB} .$$

(22)

For the standard form of the gauge fermion

$$\Psi = \int d\sigma[\overline{c}^\Lambda \chi^\Lambda + \overline{p}^\Lambda N^\Lambda ] , (A = \lambda^\pm, \xi, \pm) ,$$

(23)
with the gauge-fixing functions $\chi^A$, we can compute the Poisson bracket on the right-hand side of Eq. (22) unambiguously, because $\Omega$ contains only $C^\pm$ (see Eq. (15)). The only assumption we need is that $\chi$’s do not depend on $\mathcal{F}_\pm$. It leads to the unique solution
\[
\Gamma = -2K \int d\sigma [ (\partial_\sigma N^+ \partial^2_\sigma C^+) - ( + \rightarrow - ) ] .
\] (24)
This result is independent of the gauge-fixing functions $\chi$’s as long as the above assumption (which is about the weakest one imposed on $\Psi$) is satisfied.

3 SYMMETRIZATION

Given the most general form of anomaly in the EPS of the theory, expressed in Eqs. (15) and (24), we next apply the BF algorithm to re-convert the anomalous system back into a gauge symmetric one, i.e., first-class under commutator \(^\text{3}\). To this end, we introduce a canonical pair of BF fields $(\theta, \pi_\theta)$ to cancel the anomaly, and construct new effective Virasoro constraints $\tilde{\Phi}_\pm$ by adding an appropriate polynomial of BF fields to $\Phi_\pm$. This new contribution is fixed by requiring that $\tilde{\Phi}_\pm$ satisfy the anomaly-free Virasoro algebra under commutator and reduce to $\Phi_\pm$ (mod coboundary term) when the new fields are set equal to zero. The new constraints take the form \([10, 18]\)
\[
\tilde{\Phi}_\pm \equiv \Phi_\pm + \frac{\kappa}{2} ( \frac{\Theta^2_\pm}{4} - \Theta'_\pm ) + \frac{\mu^2}{2} V ,
\] (25)
where $\Theta_\pm$ is given in Eq. (17), $\kappa$ and $\alpha$ are coupling constants which will be determined later. The last term \(^\text{4}\) in Eq. (25) would correspond to the inclusion of the cosmological constant term, $-\mu^2 \sqrt{-g}$, in the string action (1). The new BRST charge is given by
\[
\tilde{\mathcal{Q}} = \int d\sigma [ \mathcal{C}_\chi^+ \pi_\chi^\lambda + \mathcal{C}_\chi^- \pi_\chi^- + \mathcal{C}_\xi^\xi \pi_\xi
\]
\[ + C^+ (\tilde{\varphi}^+ + \overline{\mathcal{P}}^+ C^+) + C^- (\tilde{\varphi}^- - \overline{\mathcal{P}}^- C^-) + \mathcal{P}^A \mathcal{B}_A \] , \( (A = \lambda^\pm, \xi, \pm) \), \hspace{1cm} (26)

where
\[ \tilde{\varphi}_\pm = \varphi_\pm + \frac{\kappa}{2} \left( \frac{1}{4} \Theta^2_{\pm} - \Theta'_{\pm} \right) + \frac{\mu^2}{2} \mathcal{V} . \]

This \( \tilde{Q} \) generates the BRST transformations
\[ \begin{aligned}
\delta X^\mu &= \frac{1}{2} \left[ C^+ (P + X')^\mu + C^- (P - X')^\mu \right] , \\
\delta P^\mu &= \frac{1}{2} \partial_\sigma \left[ C^+ (P + X')^\mu - C^- (P - X')^\mu \right] , \\
\delta \theta &= \frac{1}{2} \left[ C^+ \Theta^+ - C^- \Theta^- + 2C^+ - 2C^- \right] , \\
\delta \pi_\theta &= \frac{\kappa}{4} \partial_\sigma \left[ C^+ \Theta^+ + C^- \Theta^- + 2C^+ + 2C^- \right] + \frac{\mu^2 \alpha}{2} (C^+ + C^-) \mathcal{V} , \\
\delta C^\pm &= \pm C^\pm \partial_\sigma C^\pm , \\
\delta \overline{\mathcal{P}}^\pm &= -\tilde{\Phi}^\pm , \delta \lambda^\pm = \lambda^\pm , \delta \xi = \xi , \\
\delta \pi_\lambda^\pm &= \delta \pi_\xi = 0 , \delta \mathcal{C}^\pm = \delta \mathcal{C}^\xi = 0 , \\
\delta \overline{\mathcal{P}}^\lambda^\pm &= -\pi^\lambda , \delta \overline{\mathcal{P}}^\xi = -\pi^\xi , \delta N^A = \mathcal{P}^A , \delta \overline{\mathcal{C}}_A = -\mathcal{B}_A , \\
\delta \mathcal{P}^A &= \delta \mathcal{B}_A = 0 .
\end{aligned} \] (27)

The new Hamiltonian is defined by
\[ \tilde{H}_T = \frac{1}{i} \left[ \tilde{Q} , \Psi \right] , \] (28)

and the new BRST charge \( \tilde{Q} \) and total Hamiltonian \( \tilde{H}_T \) are required to satisfy the anomaly-free algebra
\[ \left[ \tilde{Q} , \tilde{Q} \right] = \left[ \tilde{Q} , \tilde{H}_T \right] = 0 , \] (29)

for a suitable choice of \( \kappa \) and \( \alpha \), which will be determined after having fixed a gauge.
4 GAUGE-FIXED ACTION AND COVARIANTIZATION

It should be emphasized that the BRST charge (26) is obtained prior to gauge-fixing. The gauge-fixing appears in defining the total Hamiltonian $\tilde{H}_T$ as in Eq. (28). The BRST invariant master action of the theory (1) can then be written as

$$S = \int d^2\sigma \left( \pi_+ \dot{\lambda}^+ + \pi_- \dot{\lambda}^- + \pi_\xi \dot{\xi} + \pi_\theta \dot{\theta} + P_\mu \dot{X}^\mu + \mathcal{P}_A \dot{C}^A \right) - \int d\tau \tilde{H}_T. \quad (30)$$

(Except for the harmonic gauge discussed later, we cancel the Legendre term $\mathcal{C}_A \dot{P}^A + \mathcal{B}_A \dot{N}^A$ in the action (30) by shifting the gauge fermion as $\Psi \rightarrow \Psi + \int d\sigma \mathcal{C}_A \dot{N}^A$.) The BFV theorem [13, 21] formally ensures that physical quantities in the quantum theory, based on the action (30) along with $\tilde{Q}$ (26), are $\Psi$-independent.

It is not straightforward to recognize 2D gravity in the action (30) because the geometrical meaning of the 2D metric variables is lost. To recover it, we must go to configuration space. This requires elimination of various phase space variables by means of the equations of motion, and for that we have to specify a gauge. In the standard form of the gauge fermion defined in Eq. (23), we have five gauge conditions $\chi^A$ ($A = \lambda^\pm, \xi, \pm$). To identify the 2D metric variables as well as the reparametrization ghosts and the Weyl ghost, we use two of them to impose the geometrization conditions [14]

$$\chi^\pm = \lambda^\pm - N^\pm, \quad (31)$$

while making an (inessential) assumption that $\chi^\pm$ and $\chi^\xi$ do not contain

$$\pi_\pm, \pi_\xi, \mathcal{C}_\pm, \mathcal{P}_\pm, N_\lambda^\pm, N^\xi, \mathcal{P}_\pm, \text{ and } \mathcal{B}_\pm.$$

One finds that

$$\lambda^\pm = N^\pm, N_\lambda^\pm = \dot{\lambda}^\pm, N^\xi = \dot{\xi},$$

$$\mathcal{P}^\pm = C_\lambda^\pm = \dot{C}^\pm \pm C^\pm \partial_\alpha N^\pm \mp \partial_\alpha C^\pm N^\pm, \quad (32)$$
can be still unambiguously derived. Then one can verify that the covariant variables defined by
\[ C^0 \equiv C^0/N_0, \quad C^1 = C^1 - N^1C_0/N_0, \tag{33} \]
\[ C_W \equiv C^\xi - C^0N^\xi - C^1\partial_\xi \xi - 2 \partial_\sigma C^0N^1 - 2 \partial_\sigma C, \tag{34} \]
\[ g_{\alpha\beta} \equiv \begin{pmatrix} -N^+N^- & (N^+ - N^-)/2 \\ (N^+ - N^-)/2 & 1 \end{pmatrix} \exp \xi, \tag{35} \]
obey the covariant BRST transformation rules
\[ \delta g_{\alpha\beta} = C^\gamma \partial_\gamma g_{\alpha\beta} + \partial_\alpha C^\gamma g_{\gamma\beta} + \partial_\beta C^\gamma g_{\alpha\gamma} + C_W g_{\alpha\beta}, \tag{36} \]
\[ \delta C^\alpha = C^\gamma \partial_\gamma C^\alpha, \quad \delta C_W = C^\gamma \partial_\gamma C_W, \tag{37} \]
where \( C^\pm = C^0 \pm C^1 \) and similarly for other quantities. Not that the \( \tau \)-derivatives in Eqs. (36) and (37) are exactly those which appear in Eq. (32).

At this stage, we are left with three unspecified gauge conditions, which correspond to two reparametrization and one Weyl symmetries. We shall consider in the following sections three gauges to illustrate our formulation of 2D gravity, which would clarify the relations to other approaches.

5 CONFORMAL GAUGE

The conformal gauge is defined by (31) and
\[ \chi^\pm = N^\pm - \hat{N}^\pm, \tag{38} \]
where \( \hat{N}^\pm \) and \( \hat{\xi} \) are background fields which define a background metric \( \hat{g}_{\alpha\beta} \) (see Eq. (35)). We substitute the gauge fermion \( \Psi \) (23) with the gauge-fixing functions (31) and (38) into the master action (30) to obtain the gauge-fixed action. The momentum variables can be eliminated by means of the equations of motions, e.g.,
\[ \pi_\theta = \frac{\kappa}{N^+ + N^-} [\dot{\theta} - \frac{1}{2}(N^+ - N^-)\theta' - (N^+ - N^-)'], \]
\[ P_\mu = \frac{2}{N^+ + N^-} [\dot{X}_\mu - \frac{1}{2}(N^+ - N^-)X'_\mu], \]
\[ \overline{P}_\pm = -\overline{C}_\pm. \tag{39} \]
Defining in terms of the BFV anti-ghosts, $\overline{C}_\pm$ and $\overline{C}_\xi$, the covariant anti-ghosts $\overline{b}_{\alpha\beta}$ (symmetric and traceless) by
\begin{equation}
\overline{b}_{00} = -(N^+)^2 \overline{C}_+ - (N^-)^2 \overline{C}_- , \quad \overline{b}_{01} = -N^+ \overline{C}_+ + N^- \overline{C}_- , \quad \overline{b}_{11} = -\overline{C}_+ - \overline{C}_- ,
\end{equation}
and the Weyl anti-ghost $\overline{C}_W$ by
\begin{equation}
\overline{C}_W = \frac{\overline{C}_\xi}{\sqrt{-g}} ,
\end{equation}
we obtain the conformal gauge action in configuration space:
\begin{align}
S_{CG} & = S_X + S_\phi + S_g + S_{gf+gh} , \\
S_\phi & = \int d^2\sigma \sqrt{-g} \left[ -\frac{\kappa}{2} \left( \frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + R \phi \right) - \mu^2 V \right] , \\
S_g & = \frac{\kappa}{2} \int d^2\sigma \sqrt{-g} \left[ \frac{1}{2} g^{\alpha\beta} \partial_\alpha \xi \partial_\beta \xi - R \xi - 2 g_{11} \frac{g_{01}}{g_{11}} \{ (g_{01})' \}^2 \right] , \\
S_{gf+gh} & = - \int d^2\sigma \left\{ B_\xi (N^0 e^\xi - \hat{N}^0 e^\hat{\xi}) + B_+ (N^+ - \hat{N}^+) + B_- (N^- - \hat{N}^-) \right. \\
& \quad + B_\xi (N^0 e^\xi - \hat{N}^0 e^\hat{\xi}) + B_+ (N^+ - \hat{N}^+) \\
& \quad \left. + \overline{C}_W (C_W + \nabla_\alpha C^\alpha) \right\} ,
\end{align}
with $\phi \equiv \theta - \xi$, $V \equiv \exp[\alpha \phi + (\alpha - 1)\xi]$, where $\nabla_\alpha$ is the covariant derivative, and $R$ the curvature scalar.

We next examine the quantum nilpotency of the BRST charge. To this end, we calculate the commutators of the generalized Virasoro operators (25), where all the operators are supposed to be normal ordered, and find that they satisfy the algebra

\begin{align}
[\tilde{\Phi}_\pm (\sigma) , \tilde{\Phi}_\pm (\sigma') ] & = \pm i (\tilde{\Phi}_\pm (\sigma) + \tilde{\Phi}_\pm (\sigma')) \delta'(\sigma - \sigma') \\
& \quad \pm i \frac{\mu^2}{2} \left( \alpha - \frac{\alpha^2}{4\pi\kappa} - 1 \right) (V(\sigma) + V(\sigma')) \delta'(\sigma - \sigma') \\
& \quad \mp i \left( \frac{D + 1 - 26}{24\pi} + \kappa \right) \delta''(\sigma - \sigma') ,
\end{align}
\begin{align}
[\tilde{\Phi}_+ (\sigma) , \tilde{\Phi}_- (\sigma') ] & = i \frac{\mu^2}{2} \left( \alpha - \frac{\alpha^2}{4\pi\kappa} - 1 \right) \partial_\sigma V(\sigma) \delta(\sigma - \sigma') .
\end{align}

Note that the BF fields non-trivially contribute to the commutator anomaly by one unit of the central charge in addition to the $\kappa$-dependent classical contribution. The coupling
relations needed for the closure of the algebra and hence to ensure the nilpotency of the BRST charge are

\[ 1 = \alpha - \frac{\alpha^2}{4\pi\kappa}, \quad (44) \]

\[ \kappa = \frac{(25 - D)}{24\pi}, \quad (45) \]

which lead to

\[ \alpha = \alpha_\pm = \frac{25 - D \pm \sqrt{(25 - D)(1 - D)}}{12}, \quad (46) \]

in accord with the result of Ref. [4].

The conformal-gauge action given in Eq. (42) contains two Liouville-type modes, \( \phi \) and \( \xi \). The BRST transformation of \( \phi \) is covariant and given by

\[ \delta\phi = C^\alpha \partial_\alpha \phi - C_W, \quad (47) \]

and so it plays the rôle of the conformal degree of freedom. Note also that \( V \) appearing in the cosmological constant term transforms as a world scalar

\[ \delta V = C^\alpha \partial_\alpha V - C_W V. \quad (48) \]

On the contrary to \( \phi \) and \( V \), the \( \xi (= \ln g_{11}) \) is a non-covariant object, and \( S_g \) is a non-covariant expression. The origin of \( S_g \) is related to the fact that the manifest 2D covariance is violated in the class of regularization schemes we approve. In order to restore the 2D covariance, one has to add an appropriately chosen non-covariant counterterm to the action. \( S_g \) is nothing but this counteraction, and therefore, the present approach has a built-in mechanism to keep the 2D covariance.

This counterterm can be removed, if one wishes, as follows. By using the equations of motion in the conformal gauge, we first express the counteraction in terms of the phase space variable:

\[ S_g = -\frac{\kappa}{4} \int d^2\sigma \left( V_N^+ G_- + V_N^- G_+ + N^{+'} G_+ + N^{-'} G_- \right), \quad (49) \]

\[ ^5 \text{The normal-ordering prescription, for example, is such a scheme.} \]

\[ ^6 \text{In the operator language, } S_X + S_g \text{ is thus reparametrization invariant, but not Weyl invariant. It is the Liouville action } S_\phi \text{ that acts as a Wess-Zumino-Witten term to recover the Weyl symmetry.} \]
where we have used the relations

\[ \sqrt{-\hat{g}}R = -\partial_{\sigma}(V_N^+ + V_N^-) + \frac{1}{2} \partial_{\sigma}(G_+ - G_-), \]

\[ V_N^\pm = \frac{1}{2} G_{\pm N}^\pm + \partial_{\sigma}N^\pm, \]

\[ G_{\pm} = \frac{1}{N_0} \left[ \pm N^\xi + (N^0 \mp N^1) \xi' \mp 2 N'^1 \right]. \] (50)

Further, the right-hand side of Eq. (49) can be written as

\[ \frac{i\kappa}{2} \int d^2\sigma [\eta, \Psi_{CG}], \] (51)

where \( \Psi_{CG} \) is the gauge fermion in the conformal gauge (\( \Psi \) (23) with \( \chi \)'s given in (31) and (38)), and

\[ \eta = \frac{1}{4} G_+^2 C^+ + G_+(C^+)'+ \frac{1}{4} G_-^2 C^- + G_-(C^-)' - \frac{1}{2}(G_+ - G_-)C^\xi. \] (52)

Therefore, the counterterm \( S_g \) can be canceled by replacing the BRST charge \( \tilde{Q} \) by \( \tilde{Q} - \kappa \int d\sigma (\eta/2) \). In configuration space, the redefined BRST charge is found to be

\[ \tilde{Q}_{CG} = \tilde{Q} - \frac{\kappa}{2} \int d\sigma \eta \]

\[ = \int d\sigma \sqrt{-\hat{g}} \left\{ \left[ \frac{C^0}{2} \hat{g}^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu - \hat{g}^{0\alpha} \partial_\alpha X^\mu C^\gamma \partial_\gamma X_\mu \right] \right. 

+ \frac{\kappa}{2} \left[ \frac{K^0}{\sqrt{-\hat{g}}} C^\gamma \partial_\gamma \phi - \hat{g}^{0\alpha} \partial_\alpha \phi \left( C^0 \frac{\hat{g}_{01}}{\hat{g}_{11}} + C^1 \right) \frac{\hat{g}'_{11}}{\hat{g}_{11}} 

+ 2 \hat{g}^{0\alpha} \partial_\alpha \phi \left( C^0 \frac{\hat{g}_{01}}{\hat{g}_{11}} + C^1 \right)' - \frac{\hat{g}^0}{\hat{g}_{11}} C^0 \right\} \right. 

\left. \left. + (\tilde{b}_{00} \hat{g}^{00} + \tilde{b}_{11} \hat{g}^{11})(C^0 \hat{C}^1 + C^1 \hat{C}^0) - \frac{\tilde{b}_{11}}{\hat{g}_{11}} C^0 \hat{C}^0 

+ (\tilde{b}_{01} \hat{g}^{01} - \tilde{b}_{11} \hat{g}^{11}) C^1 \hat{C}^0 \right\}, \] (53)

where

\[ \sqrt{-\hat{g}} R(\hat{g}) = \partial_{\sigma}K^\alpha, \]

\[ K^0(\hat{g}) = \frac{1}{\sqrt{-\hat{g}}} \left( \hat{g}_{11} - 2\hat{g}_{01} + \frac{\hat{g}_{01} \hat{g}'_{11}}{\hat{g}_{11}} \right), \]

\[ K^1(\hat{g}) = \frac{1}{\sqrt{-\hat{g}}} \left( \hat{g}'_0 - \frac{\hat{g}_0 \hat{g}'_{11}}{\hat{g}_{11}} \right). \]
To obtain the second equation in Eq. (53), we have frequently used the equations of motion in the conformal gauge. (The covariant quantities are defined in section 4.) On the flat Minkowski space, the BRST current given in Eq. (53) reduces to the well known form

\[
\tilde{J} = \{ C^+ (\partial_+ X)^2 + \frac{\kappa}{2} (\partial_+ \phi)^2 - \kappa \partial_+^2 \phi + \frac{\mu^2}{2} e^{\alpha \phi} + \tilde{b}_+ \partial_+ C^+ \} + \{ [+] \to [-] \} .
\] (54)

The re-definition of the BRST charge discussed above has the following interpretation. If we remove all the BF fields, the re-defined BRST charge develops a BRST anomaly, which is not of the form given in Eq. (15), but it is

\[
\Omega - K \int d\sigma \, \delta \eta ,
\] (55)

where \( \eta \) is defined in Eq. (52), and the absence of the BF fields reduces the anomaly coefficient from \( \kappa \) to \( K \). As shown in Ref. [14], this shift by a coboundary term exactly gives rise to the covariant expressions for the BRST anomalies

\[
\Omega_{\text{cov}} = K \int d\sigma \left[ \sqrt{-g} R C^0 C_W + \sqrt{-g} g^{\alpha \gamma} C_W \partial_\alpha C_W \right] ,
\]
\[
\Gamma_{\text{cov}} = -K \int d\sigma \sqrt{-g} R C_W ,
\] (56)

instead of the non-covariant expressions (15) and (24). This means that, if the underlying regularization schemes respect the reparametrization invariance, one may begin with the BRST anomalies given by Eq. (56), and end up with the covariant effective action \( S_X + S_\phi + S_{gf+gh} \).

In summary, we have obtained the effective action in the conformal gauge, \( S_X + S_\phi + S_{gf+gh} \), which is equivalent to the the DDK action [3, 4], though these two actions have a slightly different cosmological constant term. It should be remarked, however, that in Ref. [4] a conjecture was needed to use the translation invariant measure for the Liouville mode which is embedded as a component of the 2D metric variables. In contrast to this, the Liouville mode \( \phi \) in our approach originates from the anomaly-compensating degree of freedom, and the functional measure in the path-integral quantization on the EPS is fixed as the canonical measure which is translation invariant by construction.
6 LIGHT-CONE GAUGE

The light-cone gauge, \( g_{++} = -1/2, \ g_{--} = 0 \), is realized by the gauge-fixing functions

\[
\begin{align*}
\chi^+ &= N^+ - 1, \quad \chi^- = [(N^- + 1)e^\xi] - 2, \\
\chi^\xi &= \xi - \theta, \quad (57)
\end{align*}
\]

along with those given in Eq. (31). The last condition in Eq. (57) eliminates the \( \phi \) (= \( \xi - \theta \)) field, and instead, \( g_{11} = \exp \xi = g_{++} + 1 \) behaves as the light-cone gauge analog of the Liouville mode. As a result, the canonical structure becomes rather complicated, and we have to make a suitable change of the BFV variables to make our analysis simple. We shall begin by considering the classical theory, and establish the \( SL(2, R) \) Kac-Moody symmetry from the original BRST invariance already formulated on the BFV basis. We then quantize the system in the operator formalism to see whether we can consistently maintain the current algebra.

To obtain the gauge-fixed action, we substitute the gauge fermion (23) with \( \chi \)'s given in Eqs. (31) and (57) into the master action (30), and then eliminate again all the non-dynamical fields by using the equations of motion such as

\[
\begin{align*}
\pi_\theta &= \frac{\kappa}{2} \left( \partial_- g_{11} - ( \ln g_{11} )' \right) - g_{11} \sigma^- \left( C^+ + C^- \right), \\
\mathcal{F}_+ &= -\mathcal{C}_+ - \mathcal{F}_- = -g_{11} \mathcal{C}_-, \\
\mathcal{C}_\xi &= \frac{1}{2} \left( C^+ + C^- \right) \partial_- g_{11} - C^- ( \ln g_{11} )' + ( C^+ - C^- )'.
\end{align*}
\]

The gauge-fixed action is then given by

\[
\begin{align*}
S_{LG} &= S_X + S_g + S_{gh}, \\
S_X &= \int d^2 \sigma \frac{1}{2} \left[ ( g_{11} - 1 ) \left( \partial_- X \right)^2 + \partial_- X \cdot \partial_+ X \right], \\
S_g &= \int d^2 \sigma \frac{\kappa}{4g_{11}} \left[ ( \partial_- g_{11} )^2 - 2 \left( \partial_- g_{11} \right)( \ln g_{11} )' + 4 \left( \ln g_{11} \right)'' \right], \\
S_{gh} &= -\int d^2 \sigma \left[ b_{++} \partial_- C^+ + b \partial_- c_+ \right], \quad (59)
\end{align*}
\]

where we have introduced the new ghost variables

\[
c^+ = C^+, \ c_+ = C^- + g_{++} ( C^+ + C^- ) + \frac{\sigma^-}{2} \partial_+ C^+,
\]
in order to simplify the ghost sector. \((g_{++} = g_{11} - 1 = \exp \xi - 1)\) The cosmological constant term becomes a constant in the light-cone gauge (as shown in Appendix A), and so we have suppressed it in the action. The light-cone gauge action (59) is local, and should be compared with the non-local action of Polyakov \([\]\). The form of \(S_X\) and \(S_{\phi}\) may be expected from a general consideration on the light-cone gauge, whatever the \(Q^2\)-anomaly in Eq. (15) looks like. It is \(S_g\) which contains the information of the anomaly, and has not been derived before.

The equations of motion which follow from the action read:

\[
\begin{align*}
\partial_- \partial_+ X^u &= -\partial_- (g_{11} - 1) \partial_- X^u, \\
\partial_- c^+ &= \partial_- c_+ = 0, \quad \partial_- b_{++} = \partial_- b = 0, \\
\frac{\kappa}{4} \partial_-^2 g_{11} &= \frac{g_{11}}{4} (\partial_- X)^2 + \frac{\kappa}{2g_{11}} \left[ \left( \frac{\Theta^0}{4} \right)^2 - \left( \Theta^0 \right)' \right],
\end{align*}
\]

(61)

where \(\Theta^0_0 \equiv 2 \left( \ln g_{11} \right)' - \partial_- g_{11} \). Note that the new ghosts satisfy the free equations of motion. Rewriting (27) in terms of the new variables and using the equations of motion (61), we obtain the BRST transformations in the light-cone gauge

\[
\begin{align*}
\delta X &= \frac{1}{2} c^+ \partial_+ X + \frac{1}{2} (c_+ - \frac{\sigma^-}{2} \partial_+ c^+) \partial_- X, \\
\delta g_{++} &= \frac{1}{2} c^+ \partial_+ g_{++} + \partial_+ c^+ g_{++} + \frac{1}{2} (c_+ - \frac{\sigma^-}{2} \partial_+ c^+) \partial_- g_{++} - \frac{1}{2} \partial_+ (c_+ - \frac{\sigma^-}{2} \partial_+ c^+), \\
\delta c^+ &= \frac{1}{2} c^+ \partial_+ c^+, \\
\delta c_+ &= \frac{1}{2} c^+ \partial_+ c_+ + \frac{1}{2} \partial_+ c^+ c_+, \\
\delta b_{++} &= T_{++}^{\text{total}} \equiv T_X + T_g + T_{b_{++}} + T_b, \\
\delta b &= \frac{\kappa}{4} \partial_-^2 g_{++} + \frac{\mu^2}{2} - \frac{1}{2} \partial_+ b c^+, 
\end{align*}
\]

(62)

where we have defined:

\[
\begin{align*}
T_X &\equiv \frac{1}{4} (\partial_+ X + g_{++} \partial_- X)^2, \\
T_g &\equiv \frac{\kappa}{8} \left[ (\partial_- g_{++})^2 - 2 g_{++} \partial_-^2 g_{++} - (2 - \sigma^- \partial_-) \partial_+ \partial_- g_{++} \right],
\end{align*}
\]

(63)
\[ T_{b++} \equiv -\frac{1}{2} \partial_+ b_{++} c^+ - b_{++} \partial_+ c^+ , \]
\[ T_b \equiv \frac{1}{2} \partial_+ b c_+ . \]

In terms of these stress tensors, the BRST charge can be written as
\[ \tilde{Q}_{\text{LG}} = \int d\sigma^+ \left[ c^+ \left( T_X + T_g + \frac{1}{2} T_{b++} + T_b \right) + c_+ \left( \frac{\kappa}{4} \partial^2 g_{++} + \frac{\mu^2}{2} \right) \right] . \] (64)

This coincides with the BRST charge of Ref. [26] except for the contribution from the cosmological constant term.

The \( SL(2, R) \) Kac-Moody symmetry in the light-cone gauge arises in the present formalism as follows: First, one derives Polyakov’s fundamental identity, \[ 1, 2 \]
\[ \partial_3 - g_{++} = 0 , \] (65)
by considering the BRST variation of the equation of motion, \( \delta (\partial_- b) = 0 \). Then one expands the gravitational field \( g_{++} \) according to
\[ g_{++} = -\frac{1}{2\kappa} \left[ J^+(\sigma^+) - 2\sigma^- J^0(\sigma^+) + (\sigma^-)^2 J^-(\sigma^+) \right] . \] (66)

Substituting Eq. (66) into \( T_g \) given in Eq. (63), one finds that the stress tensor of the gravity sector takes the Sugawara form:
\[ T_g = \frac{1}{2\kappa} \left[ (J^0)^2 - J^+ J^- \right] - \frac{1}{2} \partial_+ J^0 . \] (67)

At the same time, one also expands \( \delta g_{++} \) in Eq. (62) in powers of \( \sigma^- \) to find the BRST transformation of the currents:
\[ \delta J^+ = \frac{1}{2} c^+ \partial_+ J^+ + \partial_+ c^+ J^+ - 2 c_+ J^0 + \kappa \partial_+ c_+ , \]
\[ \delta J^0 = \frac{1}{2} c^+ \partial_+ J^0 + \partial_+ c^+ J^0 - c_+ J^- + \frac{\kappa}{4} \partial^2 c^+ , \]
\[ \delta J^- = \frac{1}{2} c^+ \partial_+ J^- . \] (68)

In Appendix B it is shown how these \( \delta J^a \) imply the \( SL(2, R) \) current algebra
\[ \{ J^a(\sigma^+) , J^b(\sigma'^+) \}_{\text{PB}} = f^{abc} J^c(\sigma^+) \delta(\sigma^+ - \sigma'^+) - \kappa \eta^{ab} \delta'(\sigma^+ - \sigma'^+) , \] (69)
where $f^{ab}_{c} = - f^{ba}_{c}$ and $\eta^{ab} = \eta^{ba}$ have non-vanishing components, $f^{+0}_{+} = f^{0-}_{-} = 1$, $f^{+0}_{-} = 2$, $\eta^{00} = -1$, $\eta^{+-} = 2$.

Now to achieve the full quantum treatment of the system, we first assume that the current algebra (69) is realized under commutator too:

$$[J^a(\sigma^+), J^b(\sigma'^+)] = i f^{ab}_{c} J^c(\sigma^+) \delta(\sigma^+ - \sigma'^+) - i \kappa \eta^{ab} \delta'(\sigma^+ - \sigma'^+).$$

(70)

And to find the quantum operator corresponding to $T_g$, we consider

$$T \equiv \eta_{ab} : J^a J^b := \eta_{ab} (J^a - J^b - J^a + J^b + J^a J^b - J^a - J^b) ,$$

(71)

where the normal ordering prescription is used to regularize the products of the currents (see Appendix C). The algebraic relations among $J^a$ and $T$ given in Eq. (A.10) in Appendix C imply that the quantum operator for $T_g$ has to be defined as

$$T_g(\sigma^+) = -\frac{\pi}{2\pi \kappa - 1} T(\sigma^+) - \frac{1}{2} \partial_+ J^0(\sigma^+) .$$

(72)

Then from the Sugawara construction, one obtains the commutation relations

$$[J^a(\sigma^+), T_g(\sigma'^+)] = \frac{i}{2} \partial_+ \{ J^a(\sigma^+) \delta(\sigma^+ - \sigma'^+) \} + i f^{a0}_{b} J^b(\sigma^+) \delta'(\sigma^+ - \sigma'^+)$$

$$- i \kappa \eta^{a0} \delta''(\sigma^+ - \sigma'^+) ,$$

(73)

$$[ T_g(\sigma^+), T_g(\sigma'^+) ] = i \left( T_g(\sigma^+) + T_g(\sigma'^+) \right) \delta'(\sigma^+ - \sigma'^+)$$

$$- i \frac{c_g}{24\pi} \delta''''(\sigma^+ - \sigma'^+) .$$

(74)

The central charge $c_g$ is given by

$$c_g = \frac{6\pi \kappa}{2\pi \kappa - 1} + 24\pi \kappa = \frac{3k}{k - 2} + 6k ,$$

(75)

where $k \equiv 4\pi \kappa$ is the central charge of the current algebra. The stress tensors for the matter and ghost sectors contained in the BRST charge operator

$$\tilde{Q}_{LG} = \int d\sigma^+ \left[ c^+ (T_X + T_g + \frac{1}{2} T_{b++} + T_b) - c_+ (J^- - \frac{\mu^2}{2}) \right]$$

(76)
have the central charges, \( c_X = D \) and \( c_{gh} = -28 \), respectively. Therefore, the quantum nilpotency condition, \([\bar{Q}_{\text{LG}}, \bar{Q}_{\text{LG}}]\) = 0 implies the KPZ condition [2]

\[
\frac{3k}{k-2} + 6k + D - 28 = 0.
\]

Before we close this section, we would like to summarize the content of this section. We have derived the local gauge-fixed action (59) from the master action (30). Then using the conservation of the BRST charge (26) and the equation of motion for the \( b \) ghost (see Eq. (61)), we have obtained the constant curvature equation (65), which makes the decomposition of \( g_{++} \) into the chiral currents \( J^a \) possible. Since \( g_{++} = g_{11} - 1 = \exp \xi - 1 \), the BRST transform of \( g_{++} \) is already known. Expressing \( \delta g_{++} \) in terms of \( J^a \), we have deduced \( \delta J^a \), which is given in Eq. (68). At the same time, we have re-written the gauge-fixed form of the BRST charge (64) in terms of \( J^a \), and compared \( \{ J^a, \bar{Q}_{\text{LG}} \} \) with the \( \delta J^a \) which has been deduced from \( \delta g_{++} \). In doing so, we have derived the \( SL(2, R) \) current algebra (69) at the level of Poisson bracket. We may fairly say that the \( SL(2, R) \) Kac-Moody symmetry is to be traced back to the original BRST symmetry, which is defined on the extended phase space before the gauge-fixing.

Having arrived at this stage, one can follow Refs. [1, 2, 26] for the rest of the analyses. As a by-product, we have found that a cosmological constant can be included in the light-cone gauge and it does not affect the KPZ condition (77) (if the relation between \( \alpha \) and \( \kappa \) given in Eq. (44) is satisfied in the light-cone gauge too).

7 **HARMONIC GAUGE**

In the BFV formalism it is also possible to take the harmonic gauge [20] where the metric variables become dynamically active. To realize such gauge, we must keep the Legendre terms for the multiplier fields in the master action. Let us first consider the gauge-fixing corresponding to reparametrization symmetry. The relevant part in the master action is given by

\[
S^{R}_{g+\phi} = \int d^2 \sigma \ ( \mathcal{B}_\pm \dot{\chi}^\pm + \mathcal{C}_\pm \dot{\chi}^\pm - \mathcal{B}_\pm \chi_\pm - \mathcal{C}_\pm \delta \chi_\pm ),
\]

(78)
where the summation is taken over the constraint-label $\pm$. We would like this action to be written in terms of the GL(2)-covariant variables, and to identify it with the gauge-fixing action \[20\]

$$
\int d^2\sigma \left[ B_\alpha \partial_\beta (\sqrt{-g} g^{\alpha\beta}) + \overline{C}_\alpha \partial_\beta (\sqrt{-g} g^{\alpha\beta}) \right],
$$

(79)

where the BRST transformations for the covariant variables are given in Eqs. (36) and (37). The action (78) indeed becomes identical to (79), if we require the relations

$$
B_0 = N^0 (N^0 B_0 + N^1 B_1) + 2 (N^0 \overline{C}_0 + N^1 \overline{C}_1) \mathcal{P}^0 + \mathcal{C}_1 (N^0 \mathcal{P}_1 - N^1 \mathcal{P}_0),
$$

$$
B_1 = N^0 B_1 + \mathcal{C}_1 \mathcal{P}_0,
$$

$$
\overline{C}_0 = N^0 (N^0 \overline{C}_0 + N^1 \overline{C}_1),
$$

$$
\overline{C}_1 = N^0 \overline{C}_1,
$$

(80)

and the gauge-fixing functions are given by

$$
\chi^0 = N^1 N^0 \mathcal{V} - N^0 N^1 \mathcal{V}, \quad \chi^1 = N^1 N^1 \mathcal{V} - N^0 N^0 \mathcal{V}. \quad (81)
$$

As for the gauge-fixing corresponding to the Weyl symmetry, we may impose the flat-curvature condition given by

$$
S_{gf+gh}^W = \int d^2\sigma \left[ B_W \sqrt{-g} R + \overline{C}_W \delta (\sqrt{-g} R) \right].
$$

(82)

To see that the covariant action (82) can be obtained from the one defined in terms of the non-covariant BFV variables, we use the non-covariant decomposition of the curvature:

$$
\sqrt{-g} R = \partial_\alpha K^\alpha,
$$

$$
K^0 = \frac{1}{N^0} \dot{N}^\xi + L,
$$

$$
K^1 = -\frac{1}{N^0} [N^1 N^\xi + N^+ N^- \xi' + (N^+ N^-)'],
$$

(83)

where $N^0_{\lambda} = \dot{\lambda}^0 = \dot{N}_{\lambda}^0$, and

$$
L = \frac{1}{N^0^2} \left[ (N^0 - N^0_{\lambda}) N^x + (N^1 N^0_{\lambda} - N^0 N^1_{\lambda}) \xi' - N^0 N^1 N^\xi' 
+ 2 (N^1 N^0_{\lambda} - N^0 N^1_{\lambda}) \xi' \right].
$$

(84)
Imposing the relations

\[ B_W = B_\xi N^0 + \overline{C}_\xi \mathcal{P}^0, \quad \overline{C}_W = N^0 \overline{C}_\xi, \]

\[ \chi^\xi = -N^0(L + K^{1'}) , \]

one finds that the BFV action

\[ S_{gf+gh}^W = \int d^2\sigma \left( B_\xi \dot{N}^\xi + \overline{C}_\xi \dot{P}^\xi - B_\xi \chi^\xi - \overline{C}_\xi \delta \chi^\xi \right) \]

is identical to the covariant one (82).

The total action is then given by

\[ S_{HG} = S_X + S_\phi + S_g + S_{gf+gh}^R + S_{gf+gh}^W. \]

We will not discuss here the full quantum treatment of the theory in the harmonic gauge, though it is certainly an interesting problem.

8 DISCUSSIONS AND SUMMARY

We have treated 2D gravity as an anomalous gauge theory and applied the canonical formalism of Refs. [17, 18] which is based on the extended phase space method of BF [19]. In doing so, we are able to formulate 2D gravity in most general class of gauges. The conformal, light-cone, and harmonic gauges have been explicitly considered to illustrate how to get the gauge-fixed action in those gauges from the master action (30). This derivation of the gauge-fixed theories in 2D gravity might be related to the observation of Ref. [8] that the Wess-Zumino-Witten model based on $SL(2,R)$ plays a crucial rôle to understand the relation between those theories.

It is well known that in the path-integral quantization on configuration space, one encounters the ambiguity in defining a local measure. A possible way to avoid it is to impose some invariance on the measure. For 2D gravity, the measure is usually fixed so as to be reparametrization invariant. Without the anomaly-compensating fields, it would definitely lead to a complicated, translation non-invariant measure for the conformal factor of the metric. This is the origin of the DDK conjecture. The path-integral quantization based on the EPS, however, does not suffer from this problem, because that measure in
this approach is uniquely fixed as the Liouville measure which is translation invariant by construction.

The presence of the non-covariant counterterm $S_g$ in the conformal gauge indicates that the path-integral measure to be used for this case is not reparametrization invariant. The path-integral Jacobian associated with the transition to the reparametrization invariant measure can be calculated by using Fujikawa’s method [25] in principle. At least in the $\hat{N}^\pm = 1$ gauge, we can verify that the Jacobian in question exactly cancels $S_g$. This is another derivation of the equivalence between the DDK approach and ours. The derivation based on the operator language has been presented in section 5.

We have delivered a BRST formulation of 2D gravity in the light-cone gauge, thereby clarifying the origin of the $SL(2, R)$ current algebra. It is nothing but the original BRST invariance from which the $SL(2, R)$ Kac-Moody symmetry (69) originates, as one can convince oneself from our derivation of the $SL(2, R)$ current algebra in section 6. It should be reminded once again that, in the previous derivations of the BRST charge [26], one either starts with Polyakov’s non-local, light-cone effective action which has an $SL(2, R)$ symmetry, or bases on an $SL(2, R)$ current algebra from the beginning.

We have shown that the $SL(2, R)$ symmetry can be maintained in the full quantum treatment in accord with the result of Refs. [1, 2, 26]. We also have shown, both in the conformal and light-cone gauges, that the quantum nilpotency condition on the BRST charge is not affected in the presence of a cosmological constant term.

In conclusion, our approach to 2D gravity gives a common basis to formulate the theory in different gauges, and is therefore suitable to study different dynamical aspects of the theory; it is certainly useful to analyze 2D cosmology [27] in different gauges for instance. An extension of our approach to 2D chiral gravity [28] is straightforward.

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APPENDIX

Here we derive the algebraic identities in the light-cone gauge, that have been used in section 6.

A. The cosmological constant term

Let us first show that the cosmological constant term becomes trivial in the light-cone gauge. Consider the BRST variation of $\mathcal{V} = e^{\alpha \theta}$ and $g_{11} = \exp(\xi)$. We regularize $\mathcal{V}$ by taking the normal ordering to obtain

$$-i [ \tilde{Q}, : \mathcal{V} : ] = C^\alpha \partial_\alpha : \mathcal{V} : + \frac{2}{g_{11}} C^{\alpha'} g_{\alpha 1} : \mathcal{V} :, \quad (A.1)$$

$$\delta e^\xi = e^\xi C^\xi = C^\alpha \partial_\alpha g_{11} + 2 C^{\alpha'} g_{\alpha 1}, \quad (A.2)$$

where we have used the relation (44), the equations of motion in the light-cone gauge (58), and the relations for the ghosts in that gauge

$$C^+ = C^0 + C^1, \quad C^- = \left( \frac{2}{g_{11}} - 1 \right) C^0 - C^1. \quad (A.3)$$

(See Eqs. (33) and (35)) From Eqs. (A.1) and (A.2) one sees that $\mathcal{V}$ and $g_{11} = \exp(\xi)$ have the same transformation property, and therefore, one should identify $\mathcal{V}$ with $g_{11} = \exp(\xi)$ though the gauge condition in the light-cone gauge $\theta = \xi$. It follows then that the cosmological constant term becomes really constant because

$$S_{\text{cosm}} = -\frac{\mu^2}{2} \int d^2 \sigma \left( N^+ + 1 \right) e^{\alpha \theta} = -\mu^2 \int d^2 \sigma \frac{e^{\alpha \theta}}{g_{11}}, \quad (A.4)$$

and does not contribute to the the equation of motion for $g_{11}$. It does, however, contribute to the BRST charge as given in Eq. (64):

$$\tilde{Q}_{\text{cosm}} = \frac{\mu^2}{2} \int d\sigma \left( C^+ + C^- \right) e^{\alpha \theta}$$

$$= \frac{\mu^2}{2} \int d\sigma \left( c_+ + c^- - \frac{\sigma^-}{2} \theta_+ c^+ \right) = \frac{\mu^2}{2} \int d\sigma^+ c_+, \quad (A.5)$$

where we have used Eqs. (A.3), (60), and the equations of motion (61).
B. The derivation of the \( SL(2, R) \) current algebra

The relevant part of the BRST charge \( \tilde{Q}_{LG} \) can be written in terms of the currents \( J^a \):

\[
\tilde{Q}_g = - \int d\sigma^+ \left[ c^+( \frac{1}{2\kappa} \eta_{ab} J^b + \frac{1}{2} \partial_+ J^0 ) + c_+ J^- \right].
\]  

(A.6)

The BRST transform of \( J^a \), which has been read off from \( \delta g_{++} \) in Eq. (62), is given in Eq. (68), on one hand. On the other hand, \( \delta J^a \) is defined by

\[
\delta J^a (\sigma^+) = \{ J^a (\sigma^+), \tilde{Q}_g \}_PB
\]

\[
= - \int d\sigma'^+ \left[ \frac{1}{\kappa} c^+ (\sigma'^+) \eta_{bc} \{ J^a (\sigma^+), J^b (\sigma'^+) \}_PB J^c (\sigma'^+) 
\right.
\]

\[
+ c^+ (\sigma'^+) \{ J^a (\sigma^+), \frac{1}{2} \partial_+ J^0 (\sigma'^+) + J^- (\sigma'^+) \}_PB \].
\]  

(A.7)

Comparing this with \( \delta J^a \) given in Eq. (68) and assuming that the brackets of the currents become at most linear in \( J^a \), we obtain the Poisson bracket relation (70).

C. The Sugawara construction for the quantum \( SL(2, R) \) current algebra

We define the positive and negative frequency parts of the currents by

\[
J^{a,\pm}(\sigma^+) = \int d\sigma'^+ \delta^{(\pm)}(\sigma^+ - \sigma'^+) J^a (\sigma'^+) ,
\]  

(A.8)

with \( \delta^{(\pm)}(\sigma^+) = \frac{\pm i}{2\pi(\sigma^+ \pm i0^+)} \),

and employ the normal ordering prescription (with respect to \( J^{a,\pm} \)) to define the products of the currents. The assumption that the current algebra (69) persists at the quantum level yields

\[
\left[ J^a (\sigma^+), J^{b,\pm}(\sigma'^+) \right] = i f^{abc} J^c (\sigma^+) \delta^{(\pm)}(\sigma^+ - \sigma'^+) - \frac{2\pi \kappa - 1}{2\pi} \partial_+ \left\{ J^a (\sigma^+) \delta(\sigma^+ - \sigma'^+) \right\},
\]  

(A.9)

which can be used to obtain the commutator with \( T = \eta_{ab} : J^a J^b : \)

\[
\left[ J^a (\sigma^+), T(\sigma'^+) \right] = -i \frac{2\pi \kappa - 1}{2\pi} \partial_+ \left\{ J^a (\sigma^+) \delta(\sigma^+ - \sigma'^+) \right\}.
\]  

(A.10)

If we define

\[
\hat{T} = - \frac{\pi}{2\pi \kappa - 1} T,
\]  

(A.11)
we find the commutation relation

\[
[\hat{T}(\sigma^+), \hat{T}(\sigma'^+)] = i (\hat{T}(\sigma^+) + \hat{T}(\sigma'^+)) \delta' (\sigma^+ - \sigma'^+) - \frac{i\kappa}{4(2\pi\kappa - 1)} \delta''' (\sigma^+ - \sigma'^+). \quad (A.12)
\]

These relations give rise to the fundamental commutators (73) and (74) that are needed to show the nilpotency condition of the BRST charge \( \tilde{Q}_{LG} \) (76).
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