Non-Abelian topological phases in three spatial dimensions from coupled wires

Thomas Iadecola, Titus Neupert, Claudio Chamon, and Christopher Mudry

1 Physics Department, Boston University, Boston, Massachusetts 02215, USA
2 Kavli Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106, USA
3 Joint Quantum Institute and Condensed Matter Theory Center, Department of Physics, University of Maryland, College Park, Maryland 20742, USA
4 Department of Physics, University of Zurich, Winterthurerstrasse 190, 8057 Zurich, Switzerland
5 Condensed Matter Theory Group, Paul Scherrer Institute, CH-5232 Villigen PSI, Switzerland

(Dated: December 19, 2017)

Starting from an array of interacting fermionic quantum wires, we construct a family of non-Abelian topologically ordered states of matter in three spatial dimensions (3D). These states of matter inherit their non-Abelian topological properties from the $su(2)_k$ conformal field theories that characterize the constituent interacting quantum wires in the decoupled limit. Thus, the resulting topological phases can be viewed as 3D generalizations of the (bosonic) $su(2)_k$ Read-Rezayi sequence of fractional quantum Hall states. Focusing in detail on the $su(2)_2$ case, we first review how to determine the nature of the non-Abelian topological order (in particular, the topological degeneracy on the torus) in the two-dimensional (2D) case, before generalizing this approach to the 3D case. We also investigate the 2D boundary of the 3D phases, and show for the $su(2)_2$ case that there are anomalous gapless surface states protected by an analog of time-reversal symmetry, similar to the massless Dirac surface states of the noninteracting 3D topological insulator.

CONTENTS

I. Introduction 2
   A. Motivation 2
   B. Outline and summary of results 3

II. Non-Abelian bosonization of a single wire 5
   A. Free-fermion wire 5
   B. Intra-wire interactions 6

III. Non-Abelian topological order in two dimensions 7
   A. Definition of the class of models 7
   B. Parafermion representation of the interwire interactions 7
   C. Case study: $su(2)_2$
      1. Quasilocal chirality-resolved $Z_2$ gauge symmetry 9
      2. $su(2)_2$ primary fields 10
      3. String operators and topological degeneracy on the two-torus 12

IV. Weak non-Abelian topological order in three dimensions 19
   A. Physical setup for 3D wire arrays 19
   B. Interwire couplings for weak non-Abelian topological order 19

V. Strong non-Abelian topological order in three dimensions 21
   A. Analyzing the opening of a gap when $k = 2$ 22
   B. String and membrane operators when $k = 2$
      1. Primary operators in a single wire 24
      2. “Spin-1” string and membrane operators 24
      3. “Spin-$1 \frac{1}{2}$” string and membrane operators 27
      4. An aside on energetics 28
   5. Topological degeneracy on the three-torus 29

VI. Surface theory of the 3D non-Abelian phase 34
   A. One-loop RG analysis 36
   B. Mean-field theory for $k = 2$ 37

VII. Conclusions 40

Acknowledgments 41

A. The parafermion current algebra 41
   1. Gaussian algebra 41
   2. Parafermion algebra 42
   3. Parafermion representation of the $su(2)_k$ current algebra 43

B. The $Z_k$ conformal field theory 44
   1. Example: $Z_2$ (Ising CFT) 44

C. Commutation between string operators and the Hamiltonian; “Analytic” proof of state exclusion for the 2D case 45
   1. Introduction 45
   2. Calculation 46

D. Diagrammatics for operator algebra in the Ising CFT 48

E. Independence of string-operator algebra on arbitrary phase factors 50

References 51
I. INTRODUCTION

A. Motivation

In recent decades, topological order has emerged as a novel paradigm for describing states of matter. Motivated by the study of the fractional quantum Hall effect and chiral spin liquids, theoretical investigations uncovered a rich landscape of topologically ordered phases in two spatial dimensions. The unifying features common to all phases in this landscape are 1) the degeneracy of the ground state when the system is defined on a manifold with nonzero genus [1], and 2) the (intimately related) existence of fractionalized excitations in the gapped bulk [2]. The theoretical understanding of these topologically ordered phases has been placed on a firm mathematical footing rooted in the apparatus of modular tensor categories [3–7]. While numerous problems remain open to investigation, such as the inclusion of symmetries [8–11] and the description of topological phases starting from interacting electrons [12–18], this mathematical framework provides an indispensable point of reference in the ongoing effort to understand strongly interacting topological states of matter in two spatial dimensions.

The theoretical proposal [19, 20] and experimental discovery [21–24] of three-dimensional topological insulators protected by time-reversal symmetry (TRS) underscores the natural question of whether a similar understanding of topological order in three spatial dimensions could be achieved. Numerous examples of topologically ordered phases in three spatial dimensions are known, of which discrete gauge theories and their twisted counterparts are perhaps the most elementary [25–28]. There also exists a procedure, the Crane-Yetter/Walker-Wang construction [29–32], that can be used to build certain topological phases in three spatial dimensions. Despite this progress, the question of what kinds of strongly interacting topological phases can exist in (3+1)-dimensional spacetime (3D) is far from settled. This is especially true of non-Abelian topological orders. Furthermore, it is (for the most part) unclear how such topological phases emerge as low-energy descriptions of condensed matter systems, which are conventionally made of electrons and spins that interact in decidedly non-exotic ways.

In this paper, we propose a family of non-Abelian topological phases in (3+1)-dimensional spacetime. This family of phases can be viewed as completing the following series of analogies between topological phases in two- and three-dimensional space. We begin with the integer quantum Hall effect (IQHE) [33]. This is a topological phase in (2+1)-dimensional spacetime (2D) whose electromagnetic response is encoded by a $U(1)$ Chern-Simons effective action at level $1$ [34–36]. It features a unique ground state on the torus, and, on a manifold with boundary, has gapless chiral Dirac fermion edge states [37] described by the affine Lie algebra $u(1)_1$ (see Fig. 1). The noninteracting $\mathbb{Z}_2$ topological insulator (TI) in 3D can be viewed as inheriting many of its defining properties from the IQHE. For example, although the noninteracting TI respects time-reversal symmetry (TRS) while the IQHE does not, the gapped surface states that emerge when TRS is broken on the surface of the TI feature a Hall response that is exactly half of what is expected in the IQHE case [38]. Thus, a “magnetic domain wall” that separates regions with opposite TRS-breaking fields on the TI surface binds the same gapless chiral Dirac fermion mode that constitutes the edge state of the IQHE (see Fig. 1). This is a direct consequence of the axiom electromagnetic response (signaled by a $\theta$ term in the effective action) in 3D that characterizes the bulk of the TI [39–41]. When TRS is preserved, the noninteracting $\mathbb{Z}_2$ TI features a single massless Dirac fermion on its surface. In pure 2D, the existence of a TRS theory of a single massless Dirac fermion is forbidden by the fermion doubling theorem [42]. However, on the surface of a $\mathbb{Z}_2$ TI, its presence is necessary to ensure that the TRS-breaking surface contribution of the $\theta$ term does not spoil TRS on the surface. If TRS is broken on the surface, then a mass term for the Dirac fermion is symmetry-allowed, and the aforementioned surface quantum Hall effect develops. In this sense, the Dirac fermion surface states of the $\mathbb{Z}_2$ TI are anomalous, and their gaplessness is protected by TRS.

Recent work on so-called fractional TIs (FTIs) in 3D has borne out this analogy to the interacting setting. Indeed, these FTIs can be defined as systems in (3+1)-dimensional spacetime with TRS whose bulk axion electromagnetic response is characterized by axion angles $\theta$ that are rational multiples of $\pi$. Consistency with TRS then demands the presence of topological order in the bulk [43, 44]. In the case $\theta = \pi/k$ with $k \in \mathbb{Z}$,
one finds that breaking TRS on the surface of an FTI yields a gapped surface state with Hall conductivity $\sigma_{xy} = (1/2k) e^2/h$. Consequently, a magnetic domain wall on the surface binds a chiral Luttinger-liquid mode described by the affine Lie algebra $u(1)\kappa$, which is precisely the edge state of the $\nu = 1/k$ Laughlin state in the fractional quantum Hall effect (FQHE) \cite{45}. Moreover, preserving TRS on the surface necessitates the presence of fractionalized gapless excitations on the surface \cite{44}. Hence, FTIs feature fractionalized analogues of the anomalous gapless surface states of the noninteracting $\mathbb{Z}_2$ TI.

Given the two preceding analogies between the (F)QHE in 2D and the (F)TI in 3D, the following natural question arises. Are there TI-like analogues in 3D of the 2D non-Abelian quantum Hall states? One can focus, for example, on asking this question for the (bosonic) Read-Rezayi quantum Hall sequence in (2+1)-dimensional spacetime \cite{46, 47}. These non-Abelian topological phases are described by an SU$(2)$ Chern-Simons term at level $k$, and feature chiral edge states described by the affine Lie algebra $su(2)\kappa$ \cite{45, 48}. Hence, the analogous topological phase in (3+1)-dimensional spacetime would need to satisfy the following three properties. First, it should be time-reversal invariant in the bulk. Second, it should be topologically ordered, in the sense that the ground state manifold on the three-torus must have dimension greater than one. Third, a domain wall between regions on the surface in which TRS is broken in opposite ways should bind a chiral $su(2)\kappa$ mode. Fourth, there should be gapless surface states protected by TRS. Is it possible to construct such a topological phase? If so, what is the nature of the bulk topological order? These are the questions we address in this paper.

We address the question of the existence of 3D analogues of the $su(2)\kappa$ fractional quantum Hall states in 2D by attempting to build them from scratch. In particular, we employ a coupled-wire construction based on non-Abelian current algebras to construct a topological phase with the desired properties. In this approach, the topological phase in (3+1)-dimensional spacetime is constructed by coupling together many sub-systems, each of which lives in (1+1)-dimensional spacetime (1D), with appropriate many-body interactions. Coupled-wire constructions have been used to construct a variety of strongly-correlated phases in 2D, including non-Fermi liquids \cite{49–51} as well as Abelian and non-Abelian quantum Hall states \cite{52–59}. Moreover, this approach has recently been generalized to 3D, yielding a variety of phases including Weyl semimetals \cite{60, 61}, fractional topological insulators \cite{62}, and strongly-correlated phases described by emergent Abelian gauge theories \cite{63}. The utility of this approach lies in the fact that numerous analytic techniques exist for quantum field theories in (1+1)-dimensional spacetime, enabling the description of a wide variety of strongly interacting states of matter in a controlled manner. We also argue by example in this work that the coupled-wire approach can be used as a means to search for and characterize candidates for new topological phases of matter, like the family of 3D $su(2)\kappa$ phases constructed here.

### B. Outline and summary of results

We now provide an overview of the organization of the paper and summarize the results.

In Sec. II, we review how to bosonize a multi-flavor fermionic wire in terms of the currents associated with the non-Abelian internal symmetry group of the wire \cite{64}. This bosonization scheme has been used to address a wide variety of physical problems in 1D, including the multi-channel Kondo effect \cite{65–67} and marginally-perturbed conformal field theories (CFTs) \cite{68}. In Ref. \cite{69} it was also used as a starting point for the construction of a series of non-Abelian topological phases in 2D. In Sec. II B, we show how to add intra-wire interactions to drive the fermionic wire to a strong-coupling fixed point described by an $su(2)\kappa$ CFT. This treatment is crucial for what follows, as these CFTs are used as building blocks for the coupled-wire constructions of the subsequent sections: the non-Abelian topologically ordered phases in 2D and 3D that we construct later in the paper inherit their non-Abelian character from the $su(2)\kappa$ CFTs.

Next, in Sec. III, we describe how to construct non-Abelian topological phases of matter in 2D starting from a one-dimensional array of decoupled $su(2)\kappa$ CFTs. This section serves as a prelude to Sec. IV, where the 3D topological phase is constructed. While the $su(2)\kappa$ topological phases constructed in Sec. III are not new, this section serves two important purposes, on which we now elaborate.

First, Sec. III establishes the approach we later take to construct the 3D topological phases of the following section. This approach can be described as follows. We use $su(2)\kappa$ current-current interactions to couple channels in neighboring wires that have opposite chirality. These couplings can be viewed as arising from continuum limits of microscopic interactions between the spin sectors of neighboring wires (see, e.g., Refs. \cite{70} and \cite{71}). Focusing on the specific example of $su(2)_2$, we argue that, in the strong-coupling limit, these interactions gap the bulk of the array of coupled wires, leaving chiral $su(2)_2$ modes on the boundaries when the model is defined on a cylinder. Once we have shown how to gap the bulk of the array, we move on to characterize the bulk topological order within the coupled-wire construction. (In the quantum Hall parlance, this topological phase is related to the Moore-Read state for bosons at filling factor $\nu = 1$.) The procedure for doing so hinges on using the primary operators of the unperturbed CFTs in each wire to construct nonlocal “string operators” that commute with the interaction term and satisfy a nontrivial algebra among themselves. These string operators can then be used to determine the topological ground-state degeneracy of the coupled-wire theory on the torus. More
specifically, these string operators can be used to construct a representation of the ground-state manifold of the coupled-wire theory at strong coupling.

Second, the calculation of the ground-state degeneracy of the $su(2)_2$ topological phase in 2D constructed in Sec. III serves to clarify precisely what is meant by a "non-Abelian" topological phase in the context of this paper. (Moreover, such a calculation was not presented in previous coupled-wire constructions of similar topological phases, see e.g., Refs. [57] and [69].) In particular, we will show that the algebra of the nonlocal string operators mentioned previously suggests the algebra of Wilson loops in a $Z_2$ gauge theory. Namely, there are four nonlocal string operators that break into two sets of anticommuting operators. Naive intuition derived from Abelian gauge theory then suggests that the ground-state degeneracy on the torus should be fourfold. However, one finds that one of these four putative ground states cannot reside in the ground-state manifold. The reason for this has deep connections to the non-Abelian algebra of primary operators in the CFT [5], and has come up before in less microscopic studies of related topological phases [72]. In this way, we conclude that the topological degeneracy of the $su(2)_2$ topological phase in 2D is three, rather than four. This exclusion of states from the ground-state manifold based on non-Abelian operator algebras is at the heart of what distinguishes non-Abelian topological phases from Abelian ones, and appears again with a vengeance in the (3+1)-dimensional case.

In Sec. IV, we attempt a naive generalization of the results of Sec. III to (3+1)-dimensional spacetime. The starting point for the 3D wire construction is a two-dimensional array of parallel interacting quantum wires that realize $su(2)_k$ CFTs at low energies (see Fig. 2). We then couple the $su(2)_k$ CFTs in neighboring wires with current-current interactions, exactly as in Sec. III. When these current-current interactions open a gap in the array of wires, the result is a quantum phase of matter that supports weak topological order, being simply a stack of many copies of the 2D phases considered in Sec. III. The $su(2)_2$ case features an extensive ground-state degeneracy that is inherited directly from its 2D precursor.

In Sec. V, we propose a different interwire interaction that precludes the possibility of viewing the system as a stack of decoupled planes. This interaction is still based on current-current interactions, but is defined on plaquettes of the square lattice, rather than bonds as in Secs. III and IV. We move on to consider the $su(2)_2$ case in detail, building a set of nonlocal operators that can be used to define a manifold of topologically degenerate ground states. Most of the important aspects of the analysis of Sec. III carry through to Sec. IV, albeit with a few crucial modifications that we discuss in detail as they arise.

The most important modification involved in generalizing the 2D results of Sec. III to the 3D setting of Sec. V is that one must construct nonlocal "membrane" operators (see Fig. 3), in addition to the string operators of the two-dimensional case, in order to characterize the resulting topological order. These membrane operators can be viewed as the two-dimensional worldsheets of deconfined stringlike excitations, while the string operators can be viewed as the one-dimensional worldlines of deconfined pointlike excitations. The existence of stringlike excitations is crucial for topological order in 3D, since deconfined point particles in three-dimensional space must have trivial braiding [77]. (Indeed, this is the case for the present topological phase: all of the string operators will be shown to commute with one another.)
non-Abelian algebra of string operators in the 2D example generalizes to a non-Abelian algebra of string and membrane operators in 3D. Moreover, there is also a non-Abelian algebra among the membranes. Carrying through the naive Abelian gauge theory counting for the (3+1)-dimensional case and removing states that are excluded based on the non-Abelian operator algebra, we find 20 degenerate ground states of the $su(2)_2$ topological phase in (3+1)-dimensional spacetime when space is a three-torus.

Finally, we investigate in Sec. VI the surface states of the 3D topological phase constructed in Sec. V. It is readily seen that, when open boundary conditions are imposed in one of the two directions of the array of wires, there are gapless $su(2)_k$ modes left on the exposed two-dimensional surfaces. The goal of Sec. VI is to better understand the fate of these “dangling” modes when they are coupled by marginally relevant local interactions. When these interactions break the TRS analogue, it is straightforward to see that a “magnetic” domain wall (i.e., a domain wall between two regions of the surface in which the TRS analogue is broken in different ways) binds a chiral $su(2)_k$ current (see Fig. 3).

To probe the phase diagram of the surface when the TRS analogue is preserved, we perform a one-loop renormalization group (RG) analysis of the surface. We find that all couplings on the surface can flow to strong coupling simultaneously, even though neighboring couplings do not commute in general. Furthermore, we allow for surface couplings that break the $SU(2)$ spin-rotation symmetry, and find that the surface couplings nevertheless flow towards an $SU(2)$-symmetric strong-coupling fixed point in a large region of parameter space. Next, we investigate the nature of the $SU(2)$-symmetric strong-coupling fixed point by an explicit self-consistent mean-field calculation for the $su(2)_2$ case. We find that the surface states are indeed gapless when the TRS analogue is imposed, and that they do not break the symmetry spontaneously (at least at the level of mean-field theory). While the question of whether or not there exist gapped, symmetry-preserving surface states of these topological phases, which would then likely exhibit surface topological order [73, 78–84], is interesting, we do not pursue it in this work. However, some investigation along these lines has been carried out in Ref. [74].

In summary, we are able to demonstrate in this paper that it is possible to construct (3+1)-dimensional analogues of the $su(2)_k$ non-Abelian (bosonic) quantum Hall states. For the special case of $su(2)_2$, we partially characterize the non-Abelian topological order by explicit calculation, in addition to characterizing the associated surface states. Thus, we arrive at the surprising result that non-Abelian topological states of matter based on conformal field theory can be constructed in (3+1)-dimensional spacetime.

II. NON-ABELIAN BOSONIZATION OF A SINGLE WIRE

A. Free-fermion wire

Consider a one-dimensional wire containing $N_c$ “colors” of spinful fermions. Its action $S_{0,\text{wire}}$ is the integral over time $t$ and the coordinate $z$ along the wire of the Lagrangian density

$$\mathcal{L}_{0,\text{wire}} := 2 \sum_{\sigma = \uparrow, \downarrow} \sum_{\alpha = 1}^{N_c} \bigg( \chi_{L,\sigma,\alpha}^{*} i \partial_{t} \chi_{L,\sigma,\alpha} + \chi_{R,\sigma,\alpha}^{*} i \partial_{t} \chi_{R,\sigma,\alpha} \bigg).$$

(2.1)

The derivatives $\partial_{M} \equiv \partial_{z_{M}}$ ($M = L, R$) are taken with respect to the chiral (light-cone) coordinates

$$z_{L} \equiv t + z, \quad z_{R} \equiv t - z.$$  

(2.2)

We assume periodic boundary conditions along the wire, i.e., in the $z$-direction. The four Grassmann-valued fields $\chi_{L,\sigma,\alpha}^{*}$, $\chi_{L,\sigma,\alpha}$, $\chi_{R,\sigma,\alpha}^{*}$, $\chi_{R,\sigma,\alpha}$ are independent of each
Such a wire has the internal symmetry $U(2N_c)_L \times U(2N_c)_R$. The central idea of the series of coupled-wire constructions presented in this paper is to decompose the Lie algebra associated with this symmetry using the following identity (or “conformal embedding”) [85],

$$u(2N_c)_1 = u(1) \oplus su(2)_N \oplus su(N_c)_2,$$  \hspace{1cm} (2.3)

where we have employed the notation $q_k$ for the affine Lie algebra at level $k$ associated with the connected, compact, and simple Lie group $G$. (For a review of affine Lie algebras, see, e.g., Ref. [85].) Equation (2.3) tells us that the theory (2.1) has three conserved currents $j^a_L$, $J^a_R$, and $J^a_R$ corresponding to the affine Lie algebra $u(1)$, $su(2)_N$, and $su(N_c)_2$, respectively. (Note that, of course, there are analogous conserved currents $j^a_L$, $J^a_R$, and $J^a_R$ for the left-handed sector.) We use indices $a = 1, 2, 3$ to label the generators of $SU(2)$ and $a = 1, \cdots, N_c^2 - 1$ to label the generators of $SU(N_c)$. In terms of the complex fermions, these currents are given by

$$j_M := \sum_{\sigma = \uparrow, \downarrow} \sum_{a = 1}^{N_c} \chi^a_{M, \sigma} \chi_{M, \sigma, a},$$  \hspace{1cm} (2.4a)

$$j^a_M := \frac{1}{2} \sum_{\sigma, \sigma' = \uparrow, \downarrow} \sum_{a = 1}^{N_c} \chi^a_{M, \sigma} \sigma^a_{\sigma \sigma'} \chi_{M, \sigma', a},$$  \hspace{1cm} (2.4b)

$$J^a_M := \sum_{\sigma = \uparrow, \downarrow} \sum_{a, a' = 1}^{N_c} \chi^a_{M, \sigma} T^a_{\sigma \sigma'} \chi_{M, \sigma', a},$$  \hspace{1cm} (2.4c)

with $M = L, R$. The $U(1)$ currents $j_M$ with $M = L, R$ are associated with charge conservation. The $SU(2)$ currents $j^a_M$ with $M = L, R$ and $a = 1, 2, 3$ are associated with the spin-rotation symmetry. The $SU(N_c)$ currents $J^a_M$ with $M = L, R$ and $a = 1, \cdots, N_c^2 - 1$ are associated with the color isospin-rotation symmetry. The generators $\sigma^a/2$ of $SU(2)$ and $T^a$ of $SU(N_c)$ obey the normalizations and the independent algebras

$$\text{tr} \left( \sigma^a \sigma^b \right) = 2 \delta^{ab}, \quad \left[ \sigma^a, \sigma^b \right] = 2i \epsilon^{abc} \sigma^c,$$  \hspace{1cm} (2.5a)

$$\text{tr} \left( T^a T^b \right) = \frac{1}{2} \delta^{ab}, \quad \left[ T^a, T^b \right] = i f^{abc} T^c,$$  \hspace{1cm} (2.5b)

where $\epsilon^{abc}$ is the Levi-Civita symbol and $f^{abc}$ are the structure constants of $SU(N_c)$. With these definitions, one can build the energy-momentum tensor for the free theory defined by the Sugawara construction [65–67, 86] for the energy-momentum tensor in the M-moving sector,

$$T_M[u(2N_c)_1] = T_M[u(1)] + T_M[su(2)_N] + T_M[su(N_c)_2].$$  \hspace{1cm} (2.6a)

With these definitions, it follows that the Hamiltonian density associated with the free Lagrangian density (2.1) is given by

$$\mathcal{H}_{0, \text{wire}} = 2\pi \sum_{M=L,R} \left( T_M[u(1)] + T_M[su(2)_N] + T_M[su(N_c)_2] \right).$$  \hspace{1cm} (2.7)

Rewriting the free theory (2.1) in terms of the currents (2.4) amounts to performing a non-Abelian bosonization of the free theory. This rewriting highlights the fact that a theory of multiple flavors of free fermions can be broken up into independent charge $u(1)$, spin $su(2)_N$, and color $su(N_c)_2$ sectors.

**B. Intra-wire interactions**

Having rewritten the free theory (2.1) in terms of the non-Abelian currents (2.4), we now wish to isolate the $su(2)_N$ spin degrees of freedom by removing the $u(1)$ charge and $su(N_c)_2$ color degrees of freedom from the low-energy sector of the theory. We accomplish this by adding interactions that gap out the latter pair of degrees of freedom.

To gap out the charge sector, we add to the free Lagrangian density (2.1) the interaction term

$$\mathcal{L}_{\text{int}}[u(1)] := -\lambda_{u(1)} \cos \left( \sqrt{2} \left( \phi_R + \phi_L \right) \right).$$  \hspace{1cm} (2.8a)

The chiral bosonic fields $\phi_M$ are defined by the Abelian bosonization identity

$$j_M = -\frac{1}{\sqrt{2} \pi} \partial_M \phi_M. \hspace{1cm} (2.8b)$$

In the fermionic language, the interaction (2.8a) is interpreted as an Umklapp process. It is marginally relevant in the renormalization group (RG) sense, i.e., it flows to strong coupling under RG and gaps the charge sector when $\lambda_{u(1)} > 0$.

To gap out the color sector, we add to the free La-
sights that are interesting and useful in their own right. The analysis yields in-
su

theories as building blocks for non-Abelian topological

Thus, by adding the interactions (2.8) and (2.9) to the
current interaction is also marginally relevant, flowing to strong coupling for \( \lambda_{su(N_c)_2} > 0 \).

At the strong-coupling fixed point dominated by the in-
the interactions (2.8) and (2.9), the effective Hamiltonian density
for the low-energy sector becomes

\[
\mathcal{H}_{0, \text{eff}} := 2\pi \left( T_L[su(2)_{N_c}] + T_R[su(2)_{N_c}] \right).
\]

This is nothing but the Hamiltonian description of the
su

Wess-Zumino-Witten (WZW) CFT [64, 87] with the central charge

\[
\sigma = \frac{3 N_c}{2 + N_c}.
\]

Thus, by adding the interactions (2.8) and (2.9) to the
free theory (2.1), we can convert a quantum wire contain-
ing \( N_c \) colors of spinful fermions into a highly nontrivial
conformal field theory. The coupled-wire constructions
presented in this paper use arrays of these
su

WZW theories as building blocks for non-Abelian topological phases.

III. NON-ABELIAN TOPOLOGICAL ORDER IN TWO DIMENSIONS

As preparation for Sec. IV, we construct a class of
su

quantum liquids in two spatial dimensions and show, for the case of \( k = 2 \), how to compute their topo-
logical degeneracy on the torus. This analysis yields in-
sights that are interesting and useful in their own right.

A. Definition of the class of models

We begin with a one-dimensional array \( \Lambda \) of parallel nonchiral spinful fermionic quantum wires aligned along the \( z \)-direction, each of which is described by the La-
grangian density (2.1) (see Fig. 4). The cardinality of
the one-dimensional lattice \( \Lambda \) is

\[
|\Lambda| \equiv L_y + 1.
\]

We set \( N_c = k \), where \( N_c \) is the number of “colors” of
fermions in each wire. Each wire has an internal sym-
metry \( U(2k)_L \times U(2k)_R \), with respect to which we carry
out the bosonization procedure of Sec. II. We then gap
the \( u(1) \) and \( su(k)_2 \) sectors with the intra-wire interactions
discussed in Sec. II A, leaving behind an \( su(2)_k \) cur-
rent algebra for each of the left- and right-moving chiral
sectors in every wire. In the Heisenberg picture and in
two-dimensional Minkowski space, we denote the chiral
su

currents by \( \lambda_{M,y}((z)_M) \) where \( M = L, R \) labels the
chirality, \( a = 1, 2, 3 \) labels the \( SU(2) \) generators, \( y \) labels
the wire, and \( z_M \) is defined in Eq. (2.2).

We couple nearest-neighbor wires with the \( su(2)_k \) in-
teraction (see Fig. 4)

\[
\hat{\mathcal{L}}_{bs} \equiv -\mathcal{H}_{bs} := -\lambda \sum_{y=0}^{L_y} \hat{J}_{L,y+1}^- \hat{J}_{R,y}^- + \text{H.c.,} \tag{3.2a}
\]

where \( \sigma_{BC} = 0, 1 \) for periodic and open boundary con-
ditions, respectively. In Eq. (3.2a), we have introduced the linear combinations

\[
\hat{J}_{M,y}^\pm \equiv \hat{J}_{M,y}^0 \pm i \hat{J}_{M,y}^2.
\]

When periodic boundary conditions are imposed in the
\( y \)-direction, i.e., when \( \sigma_{BC} = 0, 1 \), each chiral current is
paired with exactly one current of the opposite chirality in a neighboring wire, and, hence, the full array of quantum wires may become gapped in the strong-coupling limit \( |\lambda| \gg 0 \). When open boundary conditions are imposed in the \( y \)-direction, i.e., when \( \sigma_{BC} = 1 \), there is a left-moving \( su(2)_k \) current at \( y = 0 \) and a right-moving \( su(2)_k \) current at \( y = L_y \) that are fully decoupled from the bulk. This edge structure is reminiscent of that of the \( su(2)_k \) non-Abelian Chern-Simons theories [88, 89] and that of the \( \mathbb{Z}_k \) Read-Rezayi quantum Hall states [47].

B. Parafermion representation of the interwire
interactions

The interaction (3.2a) can be better understood by
rewriting the \( su(2)_k \) currents in terms of auxiliary degrees of
freedom. This rewriting must preserve the \( su(2)_k \) cur-
rent algebra, which is encoded in the operator product

FIG. 4. (Color online) Schematic of the coupled-wire con-
struction for \( su(2)_k \) non-Abelian topological orders in two
spatial dimensions. Grey ovals represent quantum wires,
while red and blue circles represent chiral \( su(2)_k \) currents.
expansion (OPE) [85]

\[ \mathcal{J}_{L,y}(v) \mathcal{J}_{L,y}(w) \sim \delta_{y,\bar{y}} \left( \frac{k/2}{v^2 - w^2} + i e^{i\theta} \frac{\mathcal{J}_{L,y}(w)}{v - w} \right) , \]

(3.3)

for the holomorphic sector \( M = L \), and similarly for the antiholomorphic sector \( M = R \). (Here, we employ complex coordinates \( v \equiv t + i z \), obtained from the chiral coordinate \( z_L \) defined in Eq. (2.2) by the analytic continuation \( z \rightarrow i z \), and \( \bar{v} \equiv t - i z \), obtained from the chiral coordinate \( z_R \) also defined in Eq. (2.2) by the same analytic continuation.) The group indices \( a, \bar{a} = 1, 2, 3 \), and summation over the repeated index \( b = 1, 2, 3 \) is implied. The symbol \( \sim \) denotes equality up to nonsingular terms in the limit \( v \rightarrow w \).

As shown by Zamolodchikov and Fateev [90] (see Appendix A), the current algebra (3.3) can be represented in terms of \( Z_k \) parafermion and chiral boson operators as follows [85]:

\[ \hat{J}^\pm_{M,y} = \sqrt{k} \hat{\Psi}^\dagger_{M,y} \; ; \quad e^{\pm i\sqrt{k/\epsilon} \hat{\phi}_{M,y}} \],

(3.4a)

\[ \hat{J}^\mp_{M,y} = \sqrt{k} e^{-i\sqrt{k/\epsilon} \hat{\phi}_{M,y}} \; ; \quad \hat{\Psi}_{M,y} \],

(3.4b)

\[ \hat{J}^\pm_{M,y} = \frac{i}{\sqrt{2}} \partial_M \hat{\phi}_{M,y} \],

(3.4c)

where \( : \cdot : \) denotes normal ordering with respect to the many-body ground state of \( \hat{H}_{0,\text{eff}} \) within each wire. Here, the \( Z_k \) parafermions \( \hat{\Psi}_{M,y} \) satisfy the equal-time algebra

\[ \hat{\Psi}_{M,y}(t, z) \hat{\Psi}^\dagger_{M',y'}(t', z') = \hat{\Psi}^\dagger_{M',y'}(t', z') \hat{\Psi}_{M,y}(t, z) e^{-i \frac{2\pi}{k} \delta_{y,y'} \left( -1 \right)^M \left[ \text{sgn}(z - z') + \epsilon_{M,M'} \right]} + i \frac{2\pi}{k} \text{sgn}(y - y') , \]

(3.4d)

\[ \hat{\Psi}^\dagger_{M,y}(t, z) \hat{\Psi}^\dagger_{M',y'}(t', z') = \hat{\Psi}^\dagger_{M',y'}(t', z') \hat{\Psi}^\dagger_{M,y}(t, z) e^{-i \frac{2\pi}{k} \delta_{y,y'} \left( -1 \right)^M \left[ \text{sgn}(z - z') + \epsilon_{M,M'} \right]} + i \frac{2\pi}{k} \text{sgn}(y - y') , \]

(3.4e)

\[ \hat{\Psi}_{M,y}(t, z) \hat{\Psi}^\dagger_{M',y'}(t', z') = \hat{\Psi}^\dagger_{M',y'}(t', z') \hat{\Psi}_{M,y}(t, z) e^{i \frac{2\pi}{k} \delta_{y,y'} \left( -1 \right)^M \left[ \text{sgn}(z - z') + \epsilon_{M,M'} \right]} - i \frac{2\pi}{k} \text{sgn}(y - y') . \]

(3.4f)

The sign function above is defined such that \( \text{sgn}(0) = 0 \). The left- and right-moving labels \( M = L, R \) are defined with the convention that \( \epsilon_{M,M'} \) is the antisymmetric Levi-Civita symbol obeying \( \epsilon_{L,R} = -\epsilon_{L,L} = -1 \). Moreover, \( (-1)^R = -(-1)^L \equiv 1 \). The algebra of the \( su(2)_k \) currents holds so long as the equal-time algebra

\[ \left[ \hat{\phi}_{M,y}(t, z), \hat{\phi}_{M',y'}(t', z') \right] = -2\pi \left[ (-1)^M \delta_{y,y'} \delta_{M,M'} \text{sgn}(z - z') + \delta_{y,y'} \epsilon_{M,M'} - \text{sgn}(y - y') \right] , \]

(3.4g)

is imposed in the chiral bosonic sector. In particular, one verifies that currents defined in different wires commute with one another at equal times when the definitions (3.4) are imposed. Furthermore, one can show that all equal-time commutators between \( su(2)_k \) currents differing by their \( L \) and \( R \) labels also vanish. Finally, the chiral parafermions commute with the chiral bosons at equal times.

The representation (3.4) of the \( su(2)_k \) current algebra provides a convenient interpretation of the interactions (3.2a) in terms of fractionalized degrees of freedom, as we discuss below. However, there are several caveats to keep in mind. Chief among these is the fact that the factorization (3.4a)–(3.4c) of the \( su(2)_k \) currents re-expresses a set of local operators (the currents) in terms of products of auxiliary degrees of freedom (the parafermions and the chiral bosons). While the \( su(2)_k \) currents admit a local expression [Eq. (2.4b)] in terms of the original degrees of freedom used to define the theory (the electrons) these auxiliary degrees of freedom do not. This fact will be important when we construct the nonlocal string operators that allow us to calculate the topological degeneracy in Sec. III C. Furthermore, we note that this parafermion representation is not unique in two ways. First, as it factorizes a local (observable) operator into the product of two operators, there is an ambiguity with the choice of the phase assigned to each operator-valued factor. (This is an explicit manifestation of the nonlocality of the auxiliary degrees of freedom.) The choice for this phase cannot have observable consequences. Second, the dependence on the labels \( y \neq y' \) of the equal-time algebra is not unique since many distinct choices accomodate the fact that any two currents belonging to two distinct wires \( y \) and \( y' \) must always commute. Hence, the dependence on the labels \( y \neq y' \) of the parafermion equal-time algebra cannot have observable consequences. We demonstrate that this is true for the case of \( su(2)_2 \) in Appendix E.

We work with the normalization convention for which the operator \( \exp(i \alpha \hat{\phi}_{M,y}) \), for any real-valued number, has the anomalous scaling dimension \( \alpha^2 \). With this convention, the chiral vertex operator \( \exp(i \sqrt{1/k} \hat{\phi}_{M,y}) \), which annihilates a chiral Abelian quasiparticle, has anomalous scaling dimension \( 1/k \). In turn, the chiral parafermion operator \( \hat{\Psi}_{M,y} \) must have the anomalous scal-
given by Eq. (3.4d) with $k_{\text{parafermions}}$). Their equal-time exchange algebra is

$$su(2)_k \simeq u(1)_k \oplus \mathbb{Z}_k,$$

(3.5a)

which states that an $SU(2)$ WZW theory at level $k$ can be interpreted as a direct product of a chiral boson and a $\mathbb{Z}_k$ parafermion conformal field theory.

With these definitions, the interactions (3.2a) take the form

$$\hat{L}_{bs} = -\hat{H}_{bs} = -\lambda \sum_{y=0}^{L_y} \left( e^{+i\sqrt{k} \phi_{L,y}} \hat{\psi}_{L,y} \hat{\psi}_{R,y+1}^{+} + \text{H.c.} \right),$$

(3.6)

possible in the limit $\lambda \to \infty$. We will see an explicit example of this gapping mechanism in the next section, where we study the case $k = 2$ in detail.

C. Case study: $su(2)_2$

In this section, we work through the example of $k = 2$ in detail. First, we will show how the interaction (3.6) leads to a gapped state of matter. Next, we will characterize the topological order in this gapped state of matter by imposing periodic boundary conditions in the $y$ and $z$-directions and constructing nonlocal string operators that commute with the interaction $\hat{H}_{bs}$ defined by Eq. (3.2a). These string operators will label the topologically degenerate ground states in the limit $\lambda \to \infty$.

The Lagrangian density in this case is (omitting the normal ordering of the vertex operators)

$$\hat{L}_{bs} = -\hat{H}_{bs} := -\lambda \sum_{y=0}^{L_y} \left( e^{+i\sqrt{2} \phi_{L,y}} \hat{\psi}_{L,y} \hat{\psi}_{R,y+1}^{+} + \text{H.c.} \right)$$

(3.7a)

which should be compared with Eq. (3.6). The chiral operators

$$\hat{\psi}_{M,y}(t,z) \equiv \hat{\psi}_{M,y}(t,z) \equiv \hat{\psi}_{M,y}^{\dagger}(t,z)$$

(3.8a)

with $M = L, R$ are Majorana operators (i.e., $\mathbb{Z}_2$ parafermions). Their equal-time exchange algebra is given by Eq. (3.4d) with $k = 2$. We also impose the normalization

$$\lim_{z' \to z} \hat{\psi}_{M,y}(t,z) \hat{\psi}_{M,y}^{\dagger}(t,z') \equiv \lim_{z' \to z} \delta(z - z') = \mathcal{N}_{\delta},$$

(3.8b)

where $\mathcal{N}_{\delta}$ is a constant with dimension [1/length]. The chiral bosons $\hat{\phi}_{M,y}$ obey the equal-time algebra (3.4g), as before. Furthermore, the chiral Majorana operators and
the chiral bosons commute at equal times:

\[ [\hat{\psi}_{M,y}(t, z), \hat{\psi}_{M',y'}(t, z')] = 0. \]  (3.9)

The rewriting of the interaction (3.7a) presented in Eq. (3.7b) provides an intuitive illustration of the discussion in Sec. III B of how the interaction (3.6) leads to a gap when periodic boundary conditions are imposed. In this case, when the bosonic field \( \hat{\phi}_{t,y} - \hat{\phi}_{t,y+1} \) becomes locked to an extremum of the sine potential, a Majorana mass term is induced for the fermionic degrees of freedom. The simultaneous gapping of the Majorana modes and locking of the bosonic fields is consistent due to the independence of the \( u(1)_2 \) and \( \mathbb{Z}_2 \) sectors of the \( su(2)_2 \) theory.

1. Quasilocal chirality-resolved \( \mathbb{Z}_2 \) gauge symmetry

Observe that the interaction (3.7) is invariant under the \( M \)- and \( y \)-resolved \( \mathbb{Z}_2 \) gauge transformation

\[
\hat{\psi}_{M,y}(t, z) \mapsto e^{i\alpha_{M,y}} \hat{\psi}_{M,y}(t, z), \]  (3.10a)

\[
\hat{\psi}_{M,y}(t, z) \mapsto e^{i\alpha_{M,y}} \hat{\psi}_{M,y}(t, z), \]  (3.10b)

acts only on the chiral boson sector of the theory and implements the transformation (3.10b), and where the operator

\[
\hat{\mathcal{Z}}_\alpha(t) = \prod_{M=L,R} \prod_{y=0}^{L_y} \hat{\mathcal{Z}}_{\alpha_{M,y}}(t) \]  (3.13)

acts only on the Ising (i.e., \( \mathbb{Z}_2 \)) sector and implements the transformation (3.10a). The action of the operator \( \hat{U}_\alpha(t) \) on the chiral bosons follows from the fact that

\[
\hat{U}_{\alpha_{M,y}}(t) \hat{\phi}_{M',y'}(t, z) \hat{U}_{\alpha_{M,y}}(t) = \hat{\phi}_{M',y'}(t, z) + \sqrt{2} \alpha_{M,y} \hat{\psi}_{M,y}(t, z), \]  (3.14)

holds for any pair of chiralities \( M, M' = L, R \), for any pair of wires \( y, y' \), and for any \( t \) and \( z \) [see Eq. (3.4g)]. The action of the operator \( \hat{\mathcal{Z}}_\alpha(t) \) follows from the definition of \( \hat{\mathcal{Z}}_{\alpha_{M,y}}(t) \) in terms of the fermion parity operator in the wire \( y \), which is somewhat involved and will not be presented here.

2. \( su(2)_2 \) primary fields

To construct the excitations of the coupled-wire theory, we will use the primary operators of the underlying \( su(2)_2 \) theory defined on each quantum wire in Fig. 4. In accordance with the identity (3.5a), the primary operators of the underlying theories for any \( k \) [90]. For \( k = 2 \), there are three primary operators labeled by the “angular momenta” 0, \( \frac{1}{2} \), and 1 with scaling dimensions 0, \( \frac{1}{4} \), and \( \frac{1}{2} \), respectively. The “spin-0” primary, with scaling dimension 0, is simply the identity operator in the \( M \)-moving channel of wire \( y \). The “spin-\( \frac{1}{2} \)” primary, with scaling dimension \( \frac{3}{2} \), is defined to be the product [90]

\[
\hat{\Phi}^{(1)}_{\alpha_{M,y}}(t, z) := \hat{\sigma}_{M,y}(t, z) \hat{\phi}_{M,y}(t, z) \]  \hat{\sigma}_{M,y}(t, z) \hat{\psi}_{M,y}(t, z), \]  (3.15a)

where the operator \( \hat{\sigma}_{M,y}(t, z) \) is the chiral “twist field” from the \( \mathbb{Z}_2 \) sector (c.f. Appendix B), which has scaling dimension 1/16. Adding this scaling dimension to that of the vertex operator \( e^{i\pi/2} \hat{\phi}_{M,y}(t, z) \) gives the appropriate scaling dimension \( \frac{3}{16} \) (c.f. Appendix A1). Finally, the
“spin-1” primary, with scaling dimension $\frac{1}{2}$, can be written as [90]

$$\hat{\Phi}_{M,y}^{(1)}(t,z) := e^{+i\frac{1}{\sqrt{2}} \hat{\Phi}_{M,y}(t,z)}.$$  \hfill (3.15b)

Note that the scaling dimension of the vertex operator $e^{+i\frac{1}{\sqrt{2}} \hat{\Phi}_{M,y}}$ is already $\frac{1}{2}$, so the only operator from the $\mathbb{Z}_2$ sector that can appear in this expression is the identity.

It is important to note that the above expressions for the primary operators carry chirality labels $M = L, R$ and wire labels $y = 0, \ldots, L_y$. In other words, these operators are defined within either the holomorphic (L) or antiholomorphic (R) sector of a single wire $y$. Consequently, the primary operators (3.15a) and (3.15b) have nonvanishing conformal spin, equal to their scaling dimensions, and are hence nonlocal (i.e., they cannot be regularized on a 1D lattice by an operator with finite support). However, the operators $\hat{\Phi}_{L,y}^{(1)} \hat{\Phi}_{R,y}^{(1)}$ and $\hat{\Phi}_{L,y}^{(1)} \hat{\Phi}_{R,y'}^{(1)}$, being products of holomorphic and antiholomorphic operators with the same scaling dimensions, have vanishing conformal spin and are local. We will use these local building blocks to construct the nonlocal string operators that encode the ground-state degeneracy of the coupled-wire theory.

In order to compute commutators of these string operators with the Hamiltonian (3.7) and with each other, we need to establish the algebra of the primary operators with the Hamiltonian (3.7) and with each other, ground-state degeneracy of the coupled-wire theory.

\[\psi(\bar{v},\bar{y})(v',y') = \delta_{y,y'} \psi(v',\bar{y}) + \cdots, \quad (3.16a)\]

\[\psi(v,y)(\bar{v}',\bar{y}) = \delta_{y,y'} \psi(v,\bar{y}) + \cdots, \quad (3.16b)\]

\[\psi(v,y)(v',\bar{y}) = \psi(v',\bar{y}) \psi(v,\bar{y}) = 0 + \cdots, \quad (3.16c)\]

where the structure constants obey the symmetry condition

$$C_{\sigma\psi}^\sigma = C_{\sigma\psi}^\sigma,$$  \hfill (3.16d)

and $\cdots$ stands for nonsingular terms. Determining the equal-time algebra of the twist fields and the Majorana fields requires one to restrict the above OPE to the real line in the complex plane. Because of the symmetry condition (3.16d) on the structure constants, exchanging the order of the fields $\hat{\psi}_{L,y}(v)$ and $\hat{\psi}_{L,y'}(v')$ on the left-hand side of Eqs. (3.16a), say, is equivalent to exchanging $v$ and $v'$. However, when we restrict the OPE to equal times, in

\[\hat{\psi}_{L,y}(t,z) \hat{\psi}_{R,y'}(t',z') = \hat{\psi}_{R,y}(t,z) \hat{\psi}_{L,y'}(t',z') + \cdots, \quad (3.17a)\]

\[\hat{\psi}_{R,y}(t,z) \hat{\psi}_{L,y'}(t',z') = \hat{\psi}_{L,y}(t,z) \hat{\psi}_{R,y'}(t',z') + \cdots, \quad (3.17b)\]

\[\hat{\psi}_{L,y}(t,z) \hat{\psi}_{R,y'}(t',z') = \hat{\psi}_{R,y}(t,z) \hat{\psi}_{L,y'}(t',z') + \cdots, \quad (3.17c)\]

for any pair of wires $y$ and $y'$ and for any $z \neq z'$. The appearance of the phase $\pi/2$ is fixed by the OPE (3.16a) and (3.16b) and the sign $\text{sgn}(z-z')$ is used to keep track of the handedness of the exchange. This choice of sign convention for the phase $\pi/2$ is equivalent to a choice of analytic continuation into the complex plane in order to regularize the equal-time exchange of the two operators.

The equal-time algebra of two twist operators is more subtle. For any pair of wires $y$ and $y'$, the OPE of two twist fields in the complex plane is given by [c.f. Eq. (B5b)]

$$\hat{\sigma}_{L,y}(v) \hat{\sigma}_{L,y'}(v') = \delta_{y,y'} \frac{C_{\sigma\sigma}^1}{(v-v')^{1/8}} + \delta_{y,y'} \frac{C_{\sigma\psi}^\psi}{(v-v')^{3/8}} \psi_{L,y}(v), \quad (3.18a)$$

$$\hat{\sigma}_{R,y}(\bar{v}) \hat{\sigma}_{R,y'}(\bar{v}') = \delta_{y,y'} \frac{C_{\sigma\sigma}^1}{(\bar{v}-\bar{v}')^{1/8}}$$

FIG. 5. Counterclockwise monodromy of two operators $\hat{O}_1(t_0, z_1)$ and $\hat{O}_2(t_0, z_2)$ in the complex plane. When the operators $\hat{O}_1$ and $\hat{O}_2$ are evaluated at equal times, their exchange can be viewed as monodromy in the complex plane, provided that the handedness of the monodromy is specified. We adopt the convention that the (holomorphic) operator with the larger value of $z$ is passed counterclockwise around the operator with the smaller value of $z$, resulting in the factors of $\text{sgn}(z-z')$ that appear in the exchange algebras for the primary operators in this section.

formation about the handedness of this exchange is lost, see Fig. 5. We therefore adopt the following convention for their equal-time operator algebra in two-dimensional Minkowski space. We make the choice

$$\hat{\psi}_{L,y}(t,z) \hat{\sigma}_{L,y'}(t',z') = \hat{\sigma}_{L,y'}(t',z') \hat{\psi}_{L,y}(t,z) \times e^{-\frac{i}{2} \frac{\pi}{2} \delta_{y,y'} \text{sgn}(z-z')}, \quad (3.17a)$$

$$\hat{\psi}_{R,y}(t,z) \hat{\sigma}_{R,y'}(t',z') = \hat{\sigma}_{R,y'}(t',z') \hat{\psi}_{R,y}(t,z) \times e^{+\frac{i}{2} \frac{\pi}{2} \delta_{y,y'} \text{sgn}(z-z')}, \quad (3.17b)$$

$$\hat{\psi}_{L,y}(t,z) \hat{\sigma}_{R,y'}(t',z') = \hat{\sigma}_{R,y}(t,z) \hat{\psi}_{L,y'}(t',z'), \quad (3.17c)$$

for any pair of wires $y$ and $y'$ and for any $z \neq z'$. The appearance of the phase $\pi/2$ is fixed by the OPE (3.16a) and (3.16b) and the sign $\text{sgn}(z-z')$ is used to keep track of the handedness of the exchange. This choice of sign convention for the phase $\pi/2$ is equivalent to a choice of analytic continuation into the complex plane in order to regularize the equal-time exchange of the two operators.
\[ \hat{\sigma}_{L,y}(v) \hat{\sigma}_{R,y'}(v') = \hat{\sigma}_{R,y}(v) \hat{\sigma}_{L,y'}(v') = 0 + \cdots. \]  
(3.18c)

Since there are two singular terms appearing on the righthand side of Eqs. (3.18a) and (3.18b), the product of two chiral twist fields must be defined with care. In particular, correlation functions involving multiple chiral twist fields are not well-defined unless the fusion channel \(1\) or \(\psi\) is specified [91]. We choose an equal-time operator algebra that reflects this ambiguity in the definition of chiral correlation functions involving the twist field. Thus, we define the equal-time algebra

\[
\begin{align*}
\hat{\sigma}_{L,y}(t,z) \hat{\sigma}_{L,y'}(t,z') &= \hat{\sigma}_{L,y}(t,z') \hat{\sigma}_{L,y}(t,z) \\
&\times \left\{ e^{-i \frac{\pi}{2} \delta_{y,y'} \text{sgn}(z-z')} \quad \text{if } \sigma \times \sigma = \mathbb{1}, \\
&\quad \text{or } e^{+i \frac{\pi}{2} \delta_{y,y'} \text{sgn}(z-z')} \quad \text{if } \sigma \times \sigma = \psi, \right. \\
\hat{\sigma}_{R,y}(t,z) \hat{\sigma}_{R,y'}(t,z') &= \hat{\sigma}_{R,y}(t,z') \hat{\sigma}_{R,y}(t,z) \\
&\times \left\{ e^{+i \frac{\pi}{2} \delta_{y,y'} \text{sgn}(z-z')} \quad \text{if } \sigma \times \sigma = \mathbb{1}, \\
&\quad \text{or } e^{-i \frac{\pi}{2} \delta_{y,y'} \text{sgn}(z-z')} \quad \text{if } \sigma \times \sigma = \psi, \right. \\
\hat{\sigma}_{L,y}(t,z) \hat{\sigma}_{R,y'}(t,z') &= \hat{\sigma}_{R,y'}(t,z') \hat{\sigma}_{L,y}(t,z), \quad (3.19c)
\end{align*}
\]

in two-dimensional Minkowski space for any pair of wires \(y\) and \(y'\) and for any \(z \neq z'\). We have used the shorthand notation \(\sigma \times \sigma = \mathbb{1}\) and \(\sigma \times \sigma = \psi\) to distinguish the two possible fusion outcomes. It is important to stress here that this equal-time algebra is not well-defined unless one specifies a fusion channel. This ambiguity is essential. Its origin is physical, and it reflects the non-Abelian nature of the twist field. We will see in the next section that this ambiguity has important consequences for the topological degeneracy.

3. String operators and topological degeneracy on the two-torus

We shall consider two distinct wires \(y\) and \(y'\) and a coordinate \(z\) along any one of these wires. Periodic boundary conditions are imposed both along the \(y\)-direction and along the \(z\)-direction. Hence, the one-dimensional array of wires has the topology of a torus.

We are going to construct the equal-time algebra

\[
\left\{ \hat{\Gamma}^{(1)}_1, \hat{\Gamma}^{(1)}_2 \right\} = 0
\]  
(3.20)

for a first pair of nonlocal operators \(\hat{\Gamma}^{(1)}_1\) and \(\hat{\Gamma}^{(1)}_2\). This pair will be shown to commute with the interaction (3.7).

The nonlocal, nonunitary operator \(\hat{\Gamma}^{(1)}_2\) can be thought of as creating a pair of pointlike “spin-\(\frac{1}{2}\)” excitations, transporting them in opposite directions around a noncontractible cycle of the torus along the \(y\)-direction, and then annihilating them. Likewise, the nonlocal operator \(\hat{\Gamma}^{(1)}_1\) can be thought of as implementing a similar process for a pair of pointlike “spin-1” excitations around a noncontractible cycle of the torus along the \(z\)-direction.

Similarly, we are going to construct the equal-time algebra

\[
\left\{ \hat{\Gamma}^{(\frac{1}{2})}_1, \hat{\Gamma}^{(\frac{1}{2})}_2 \right\} = 0
\]  
(3.21)

for a second pair of nonlocal operators \(\hat{\Gamma}^{(\frac{1}{2})}_1\) and \(\hat{\Gamma}^{(\frac{1}{2})}_2\). This pair will also be shown to commute with the interaction (3.7), modulo appropriate regularization of the operator \(\hat{\Gamma}^{(\frac{1}{2})}_2\), as we will discuss. The nonlocal, unitary operator \(\hat{\Gamma}^{(1)}_1\) can be thought of as creating a pair of “spin-\(\frac{1}{2}\)” excitations, transporting them in opposite directions around a noncontractible cycle of the torus along the \(y\)-direction, and then annihilating them. The nonlocal, nonunitary operator \(\hat{\Gamma}^{(\frac{1}{2})}_2\) can be thought of as implementing the same process for a pair of “spin-\(\frac{1}{2}\)” excitations around a noncontractible cycle of the torus along the \(z\)-direction.

If we denote a ground state of the interaction (3.7) by \(|\Omega\rangle\), we shall demonstrate that the three states

\[
|\Omega\rangle, \quad |\hat{\Gamma}^{(\frac{1}{2})}_1\rangle := \hat{\Gamma}^{(\frac{1}{2})}_1|\Omega\rangle, \quad |\hat{\Gamma}^{(\frac{1}{2})}_2\rangle := \hat{\Gamma}^{(\frac{1}{2})}_2|\Omega\rangle
\]  
(3.22)

are linearly independent ground states of the interaction (3.7). The proof of this claim relies on the vanishing equal-time commutators

\[
\left[ \hat{\Gamma}^{(1)}_1, \hat{\Gamma}^{(1)}_1 \right] = 0,
\]  
(3.23)

\[
\left[ \hat{\Gamma}^{(\frac{1}{2})}_1, \hat{\Gamma}^{(\frac{1}{2})}_1 \right] = 0,
\]  
(3.24)

and

\[
\left[ \hat{\Gamma}^{(1)}_2, \hat{\Gamma}^{(\frac{1}{2})}_2 \right] = 0.
\]  
(3.25)

Crucially, however, the exchange algebra of the nonlocal operators \(\hat{\Gamma}^{(\frac{1}{2})}_1\) and \(\hat{\Gamma}^{(\frac{1}{2})}_2\) suffers from the same ambiguity as that found on the right-hand side of Eq. (3.19). This is why one cannot infer from Eqs. (3.20)–(3.25) that the state

\[
|\hat{\Gamma}^{(\frac{1}{2})}_1\Gamma^{(\frac{1}{2})}_2\rangle|\Omega\rangle
\]  
(3.26)

is linearly independent from the states (3.22). (See also Appendix C.)

Proof. The proof consists of three steps.

Step 1: “Spin-1” string operators. The first string operators that we will construct are the “spin-1” string operators. We begin with strings running along the \(y\)-direction, perpendicular to the wires. These strings are
built from the local bilinears
\[
\hat{O}_y^{(1)}(t, z) := \hat{\Phi}^{(1)}_{L, y}(t, z) \hat{\Phi}^{(1)}_{R, y}(t, z)
\]
\[= -i \frac{1}{\sqrt{2}} \hat{\phi}_{L, y}(t, z) e^{+i \frac{1}{\sqrt{2}} \phi_{R, y}(t, z)},
\]
for any \(0 < z < L_z\) (hereafter, we suppress the normal order).

Now define the nonlocal operator \(k\) as a result of Eq. (3.4g) for \(M = R\). This result reflects the fact that the spin-1 primary operator in the \(su(2)_2\) has trivial self-

monodromy. We have established Eq. (3.23) provided we

dering of the vertex operators). Being a product of local unitary operators [the \(u(1)_2\) vertex operators], \(\hat{O}_y^{(1)}(t, z)\) is also a local unitary operator. Using Eq. (3.4g) for \(k = 2\), we see that a product of “spin-1” bilinears in neighboring wires commutes with the part of the interaction (3.7) that connects the two wires, since

\[
\hat{O}_y^{(1)}(t, z) \hat{O}_y^{(1)}(t, z) e^{+i \sqrt{2} \left[ \hat{\phi}_{L, y}(t, z) - \hat{\phi}_{R, y+1}(t, z) \right]} = e^{+i \sqrt{2} \left[ \hat{\phi}_{L, y}(t, z) - \hat{\phi}_{R, y+1}(t, z) \right]} \hat{O}_y^{(1)}(t, z) \hat{O}_y^{(1)}(t, z), \tag{3.28}
\]

and because \(\hat{O}_y^{(1)}(t, z)\) commutes with any operator from the \(\mathbb{Z}_2\) sector of the theory. Thus, the nonlocal string operator

\[
\hat{\Gamma}^{(1)}_{1}(t, z) := \prod_{y=0}^{L_y} \hat{O}_y^{(1)}(t, z) \tag{3.29}
\]

commutes with the interaction (3.7) for any value of \(0 \leq z < L_z\) when periodic boundary conditions are imposed in the \(y\)-direction. The nonlocal operator (3.29) is a member of the family

\[
\hat{\Gamma}^{(1)}_{1}(t, z_1, \cdots, z_{L_y}) = \hat{\Phi}^{(1)}_{L, 1}(t, z_1) \hat{\Phi}^{(1)}_{R, 1}(t, z_2) \hat{\Phi}^{(1)}_{L, 2}(t, z_3) \cdots \hat{\Phi}^{(1)}_{L, L_y}(t, z_{L_y}) \hat{\Phi}^{(1)}_{R, L_y}(t, z_1) \tag{3.30}
\]

of operators, which all commute with the Hamiltonian defined by Eq. (3.2a) for any values of \(0 \leq z_1, \cdots, z_{L_y} < L_z\) when periodic boundary conditions are imposed in the \(y\)-direction. Any “spin-1” string operator from the family (3.30) can be viewed as creating a pair of “spin-

1” excitations and transporting one of them around a noncontractible loop that encircles the torus in the \(y\)-direction (a noncontractible cycle along the \(y\)-direction), before annihilating it with its partner.

To construct a “spin-1” string running along the \(z\)-direction, parallel to the wires, we consider the unitary operator

\[
\hat{O}_{M, y}^{(1)}(t, z_1, z_2) := \hat{\Phi}^{(1)}_{M, y}(t, z_2) \hat{\Phi}^{(1)}_{M, y}(t, z_1)
\]

\[= \exp \left( -i \frac{1}{\sqrt{2}} \int_{z_1}^{z_2} dz \partial_z \tilde{\phi}_{M, y}(t, z) \right), \tag{3.31a}
\]

for any \(0 \leq z_1, z_2 < L_z\) and \(M = L, R\). Hence, \(\hat{O}_{M, y}^{(1)}(t, z_1, z_2)\) is a bilocal unitary operator that also obeys

\[
\hat{O}_{M, y}^{(1)}(t, z_1, z_2) e^{+i \sqrt{2} \left[ \hat{\phi}_{L, y}(t, z) - \hat{\phi}_{R, y+1}(t, z) \right]} = e^{+i \sqrt{2} \left[ \hat{\phi}_{L, y}(t, z) - \hat{\phi}_{R, y+1}(t, z) \right]} \hat{O}_{M, y}^{(1)}(t, z_1, z_2) \times e^{+i \frac{2 \pi}{\sqrt{2}} \int_{z_1}^{z_2} dz' \tilde{\phi}_{M, y}(t, z')}, \tag{3.32}
\]

as a result of Eq. (3.4g) for \(k = 2\). (A similar expression holds for \(M = R\).) Now define the nonlocal operator

\[
\hat{\Gamma}_{2, M, y}(t) := \hat{\Gamma}_{M, y}^{(1)}(t, 0, L_z), \tag{3.33}
\]

which commutes with the interaction (3.7) by Eq. (3.32). This “spin-1” string operator can be viewed as transporting a “spin-1” excitation around a noncontractible loop that encircles the torus in the \(z\)-direction (a noncontractible cycle along the \(z\)-direction).

The equal-time commutation relation between the string operators (3.29) with \(0 < z < L_z\) and (3.33) is computed using Eq. (3.4g) for \(k = 2\). It is simply the commutative rule

\[
\hat{\Gamma}_{1}^{(1)}(t, z) \hat{\Gamma}_{2, M, y}^{(1)}(t) = \hat{\Gamma}_{2, M, y}^{(1)}(t) \hat{\Gamma}_{1}^{(1)}(t, z), \tag{3.34}
\]

for any \(M = L, R\). This result reflects the fact that the spin-1 primary operator in the \(su(2)_2\) has trivial self-

monodromy. We have established Eq. (3.23) provided we
make the identifications
\[ \hat{\Gamma}_1^{(1)}(t, z) \rightarrow \hat{\Gamma}_1^{(1)}, \quad \text{and} \quad \hat{\Gamma}_2^{(1)}(t, y) \rightarrow \hat{\Gamma}_1^{(1)}, \]
for some choice of chirality M and wire y.

**Step 2:** “Spin-1/2” string operators. We next construct string operators associated with the spin-1/2 primary of the \(su(2)_k\) theory. We proceed according to a strategy similar to the one used for the “spin-1” strings. To construct a “spin-1/2” string along the \(y\)-direction, let \(0 < z, z' < L_z\) and consider the local “spin-1/2” bilinears
\[ \tilde{O}^{(1)}_y(t, z) := \Phi_{L,y}^{(1)}(t, z) \Phi_{R,y}^{(1)}(t, z) \]
\[ = e^{-i\frac{z}{\sqrt{\phi}} \Phi_{L,y}(t,z)} e^{i\frac{z'}{\sqrt{\phi}} \Phi_{R,y}(t,z)} \times \sigma_{L,y}(t, z) \sigma_{R,y}(t, z), \] (3.36a)

where we have defined the operator
\[ \Phi^{(1)*}_y(t, z) := e^{-i \frac{z}{\sqrt{\phi}} \Phi_{L,y}(t,z)} \sigma_{L,y}(t, z) \] (3.36b)
in which the adjoint operation pertains only to the \(u(1)_k\) vertex operator. Using Eqs. (3.4g) and (3.17), we find that the equal-time product of such bilinears over all wires, namely
\[ \hat{\Gamma}_1^{(1)}(t, z) \rightarrow \hat{\Gamma}_1^{(1)}, \quad \hat{\Gamma}_1^{(1)}(t, z) \rightarrow \hat{\Gamma}_1^{(1)}(t, z) \]
commutes with the interaction (3.7) for any value \(0 < z < L_z\) when periodic boundary conditions are imposed in the \(y\)-direction. This nonlocal, nonunitary operator is a member of the family

\[ \Phi^{(1)*}_y(t, z) := e^{-i \frac{z}{\sqrt{\phi}} \Phi_{L,y}(t,z)} \sigma_{L,y}(t, z), \] (3.37)

of operators that commute with the Hamiltonian defined by Eq. (3.2a) for any values of \(0 < z_1, \ldots, z_{L_y} < L_z\) when periodic boundary conditions are imposed in the \(y\)-direction. Any “spin-1/2” string operator from the family (3.38) can be interpreted as creating a pair of “spin-1/2” excitations and transporting one of them around a noncontractible cycle along the \(y\)-direction, before annihilating it with its partner.

We first observe that the operators \(\hat{\Gamma}_1^{(1)}(t, z)\) and \(\hat{\Gamma}_1^{(1)}(t, z')\) commute with one another for any \(z\) and \(z'\), as one can show using the equal-time algebra (3.4g),
\[ \hat{\Gamma}_1^{(1)}(t, z) \hat{\Gamma}_1^{(1)}(t, z') = \hat{\Gamma}_1^{(1)}(t, z') \hat{\Gamma}_1^{(1)}(t, z). \] (3.39)

We have established Eq. (3.24) provided we make the identifications
\[ \hat{\Gamma}_1^{(1)}(t, z) \rightarrow \hat{\Gamma}_1^{(1)}, \quad \hat{\Gamma}_1^{(2)}(t, z') \rightarrow \hat{\Gamma}_1^{(2)}. \] (3.40)

We claim that the “spin-1/2” string \(\hat{\Gamma}_1^{(2)}\) can be interpreted as an operator that “twists,” from antiperiodic to periodic, the boundary conditions of a “spin-1” excitation that encircles the torus in the \(z\)-direction. To see that this is the case, we use the chiral boson algebra of Eq. (3.4g) to show that the equal-time operator algebra
\[ \hat{\Gamma}_1^{(1)}(t, y) \hat{\Gamma}_1^{(2)}(t, z) \rightarrow -\hat{\Gamma}_1^{(2)}(t, z) \hat{\Gamma}_1^{(1)}(t, y) \] (3.41)
holds for any choice of chirality \(M = L, R\) and wire \(y\). We further recall that the operator \(\hat{\Gamma}_1^{(2)}(t, y)\) transports a “spin-1” excitation around the torus along the \(z\)-direction. Thus, Eq. (3.41) shows that the amplitude for transporting a “spin-1” excitation around the torus and then applying the operator \(\hat{\Gamma}_1^{(2)}(t, z)\) differs by a minus sign from the amplitude for applying the operator \(\hat{\Gamma}_1^{(2)}(t, z)\) and then transporting a “spin-1” excitation around the torus. This is precisely the action of an operator that twists the boundary conditions of a “spin-1” excitation.

In deriving Eq. (3.41), we have established Eq. (3.20) provided that we make the identifications
\[ \hat{\Gamma}_1^{(1)}(t, y) \rightarrow \hat{\Gamma}_1^{(1)}, \quad \hat{\Gamma}_1^{(2)}(t, z) \rightarrow \hat{\Gamma}_1^{(2)}. \] (3.42)

for some choice of chirality M and wire y.

Next, we seek an operator that twists the boundary conditions of a “spin-1” excitation encircling the torus along the \(y\)-direction. We proceed in direct analogy with Eq. (3.31a) by defining the (nonlocal, nonunitary) operator
\[ \tilde{O}^{(1)}_{M,y}(t, z_1, z_2) := \Phi_{M,y}^{(1)}(t, z_2) \Phi_{M,y}^{(1)}(t, z_1) \]
\[ = \exp \left( -i \frac{1}{2 \sqrt{2}} \int_{z_1}^{z_2} dz \partial_z \Phi_{M,y}(t, z) \right) \times \sigma_{M,y}(t, z_2) \sigma_{M,y}(t, z_1). \] (3.43)

We seek to define a string operator by taking \(z_1 \rightarrow 0\) and \(z_2 \rightarrow L_z\). However, one must be careful in taking these limits since Eq. (3.43) contains two chiral \(\mathbb{Z}_2\) twist fields in the same wire. Due to the ambiguity of the OPE (3.19), such a product is ill-defined unless a fusion channel is specified. Meanwhile, the product of \(u(1)_k\) vertex operators is unambiguous. By analogy with the
construction of \( \hat{\Gamma}_2^{(1)} \) in Eq. (3.31a), we would like to define the string operator \( \hat{\Gamma}_2^{(\frac{1}{2})} \) in such a way as to leave the system in the vacuum sector. Hence, the natural choice is to specify that the two \( \hat{\sigma}_{M,y'} \) operators in Eq. (3.43) fuse to the identity operator \( I \). In addition to providing a sensible parallel with the construction of \( \hat{\Gamma}_1^{(1)} \), this choice agrees with the choice made in the construction of the operator that tunnels an \( e/4 \) quasiparticle across a quantum point contact in the Moore-Read state [91].

This motivates the definition of the “spin-\( \frac{1}{2} \)” string

\[
\lim_{\epsilon \to 0} \hat{\psi}_{L,y}(t,\epsilon) \hat{\psi}_{R,y+1}(t,\epsilon) \hat{\sigma}_{L,y'}(t,\epsilon) = \lim_{\epsilon \to 0} \hat{\sigma}_{L,y'}(t,\epsilon) \hat{\psi}_{L,y}(t,\epsilon) \hat{\psi}_{R,y+1}(t,\epsilon) \hat{\sigma}_{L,y'}(t,\epsilon) = \lim_{\epsilon \to 0} \hat{\sigma}_{L,y'}(t,\epsilon) \hat{\psi}_{L,y}(t,\epsilon) \hat{\psi}_{R,y+1}(t,\epsilon) \times \begin{cases} +1, & y \neq y', \\ -1, & y = y', \end{cases}
\]

follows from the algebra (3.17), while

\[
e^{i\sqrt{\frac{1}{2}} \left[ \hat{\sigma}_{L,y}(t,z) - \hat{\sigma}_{R,y+1}(t,z) \right]} \exp \left( -i \frac{1}{2\sqrt{2}} \int_0^{L_z} dz \partial_z \hat{\sigma}_{L,y'}(t,z) \right) = \exp \left( -i \frac{1}{2\sqrt{2}} \int_0^{L_z} dz \partial_z \hat{\sigma}_{L,y'}(t,z) \right) e^{i\sqrt{\frac{1}{2}} \left[ \hat{\sigma}_{L,y}(t,z) - \hat{\sigma}_{R,y+1}(t,z) \right]} \times \begin{cases} +1, & y \neq y', \\ -1, & y = y', \end{cases}
\]

follows from the algebra (3.4g). (Similar expressions hold for \( M = R \).) Consequently, \( \hat{\Gamma}_2^{(\frac{1}{2})}(t,\epsilon) \) commutes with the interaction (3.7) in the limit \( \epsilon \to 0 \) [92].

Moreover, we can also show that \( \hat{\Gamma}_2^{(\frac{1}{2})}(t,\epsilon) \) twists the boundary conditions of a “spin-1” excitation encircling the torus along the \( y \)-direction. To do this, we use the algebra (3.4g) to compute the exchange relation (in the limit \( \epsilon \to 0 \))

\[
\hat{\Gamma}_2^{(\frac{1}{2})}(t,\epsilon) \hat{\Gamma}_2^{(1)}(t,z) = -\hat{\Gamma}_2^{(1)}(t,z) \hat{\Gamma}_2^{(\frac{1}{2})}(t,\epsilon), \quad (3.47)
\]

which holds for any chirality \( M \) and wire \( y' \). This exchange relation has an interpretation similar to Eq. (3.41). Thus, we have established Eq. (3.21) provided we make the identifications

\[
\hat{\Gamma}_1^{(1)}(t,z) \to \hat{\Gamma}_1^{(1)}, \quad \hat{\Gamma}_2^{(\frac{1}{2})}(t,\epsilon) \to \hat{\Gamma}_2^{(\frac{1}{2})}, \quad (3.48)
\]

for infinitesimal \( \epsilon > 0 \). By assumption \( y \neq y' \). Hence, the operators \( \hat{\Gamma}_{2,y} \) \( \to \hat{\Gamma}_2^{(1)} \) and \( \hat{\Gamma}_{2,y'} \) \( \to \hat{\Gamma}_2^{(\frac{1}{2})} \) commute with one another in a trivial way. This establishes Eq. (3.25).
eigenvalue of $\tilde{\mathcal{H}}_{bs}$ as $|\Omega\rangle$. Second, we will prove that the many-body states \(3.49\) are linearly independent. In doing so, we will have established that the ground state degeneracy on the torus of the interaction $\tilde{\mathcal{H}}_{bs}$ is threefold.

First, we recall that $\hat{\Gamma}^{(1)}_1(z)$ commutes with the interaction $\tilde{\mathcal{H}}_{bs}$ defined in Eqs. (3.7). Hence, the many-body state $|\hat{\Gamma}^{(1)}_1\rangle$ defined in Eq. (3.49b) is a ground state of the interaction $\tilde{\mathcal{H}}_{bs}$. Making sure to treat the limit $\epsilon \rightarrow 0$ with care, we show in Appendix C that the many-body state $|\hat{\Gamma}^{(2)}_2\rangle$ defined in Eq. (3.49b) is also a ground state of the interaction $\tilde{\mathcal{H}}_{bs}$. Now, we are going to show that the three many-body states \(3.49\) are linearly independent.

The operators $\hat{\Gamma}^{(1)}_1$ and $\hat{\Gamma}^{(1)}_2$ commute with the interaction $\tilde{\mathcal{H}}_{bs}$ and with each other [recall Eq. (3.23)]. They are thus simultaneously diagonalizable. Consequently, we can choose $|\Omega\rangle$ to be a simultaneous eigenstate of the pair of operators $\hat{\Gamma}^{(1)}_1$ and $\hat{\Gamma}^{(1)}_2$. Both $\hat{\Gamma}^{(1)}_1$ and $\hat{\Gamma}^{(1)}_2$ are unitary, i.e., there should exist the pair of unimodular complex numbers $\omega^{(1)}_1 \neq 0$ and $\omega^{(1)}_2 \neq 0$ such that

$$\hat{\Gamma}^{(1)}_1 |\Omega\rangle = \omega^{(1)}_1 |\Omega\rangle,$$

and

$$\hat{\Gamma}^{(1)}_2 |\Omega\rangle = \omega^{(1)}_2 |\Omega\rangle,$$  \hspace{1cm} (3.50a)

respectively.

Because of the anticommutator \(3.20\), we find the eigenvalue

$$\hat{\Gamma}^{(1)}_2 |\hat{\Gamma}^{(1)}_1\rangle = -\omega^{(1)}_1 |\hat{\Gamma}^{(1)}_1\rangle.$$  \hspace{1cm} (3.51)

Hence, $|\Omega\rangle$ and $|\hat{\Gamma}^{(1)}_1\rangle$ are simultaneous eigenstates of the unitary operator $\hat{\Gamma}^{(1)}_2$ with distinct eigenvalues. As such, $|\Omega\rangle$ and $|\hat{\Gamma}^{(1)}_1\rangle$ are orthogonal. Similarly, because of the anticommutator \(3.21\), we find the eigenvalue

$$\hat{\Gamma}^{(1)}_1 |\hat{\Gamma}^{(1)}_2\rangle = -\omega^{(1)}_2 |\hat{\Gamma}^{(1)}_2\rangle.$$  \hspace{1cm} (3.52)

Hence, $|\Omega\rangle$ and $|\hat{\Gamma}^{(1)}_2\rangle$ are simultaneous eigenstates of the unitary operator $\hat{\Gamma}^{(1)}_1$ with distinct eigenvalues. As such, $|\Omega\rangle$ and $|\hat{\Gamma}^{(1)}_2\rangle$ are orthogonal.

To complete the proof that $|\Omega\rangle$, $|\hat{\Gamma}^{(1)}_1\rangle$, and $|\hat{\Gamma}^{(1)}_2\rangle$ are linearly independent, it suffices to show that $|\hat{\Gamma}^{(2)}_1\rangle$ and $|\hat{\Gamma}^{(2)}_2\rangle$ are orthogonal. Because of the commutator \(3.24\), we find the eigenvalue

$$\hat{\Gamma}^{(1)}_1 |\hat{\Gamma}^{(2)}_2\rangle = +\omega^{(1)}_1 |\hat{\Gamma}^{(2)}_2\rangle.$$  \hspace{1cm} (3.53)

Hence, $|\hat{\Gamma}^{(1)}_2\rangle$ and $|\hat{\Gamma}^{(2)}_2\rangle$ are simultaneous eigenstates of the unitary operator $\hat{\Gamma}^{(1)}_1$ with the pair of distinct eigenvalues $+\omega^{(1)}_1$ and $-\omega^{(1)}_1$. As such, $|\hat{\Gamma}^{(1)}_1\rangle$ and $|\hat{\Gamma}^{(2)}_2\rangle$ are orthogonal.

We note that the commutator \(3.25\) could equally well have been used to show that $|\hat{\Gamma}^{(2)}_1\rangle$ and $|\hat{\Gamma}^{(2)}_2\rangle$ are simultaneous eigenstates of the unitary operator $\hat{\Gamma}^{(2)}_1$ with the pair of distinct eigenvalues $+\omega^{(2)}_2$ and $-\omega^{(2)}_2$.

As promised, we have shown that the ground-state manifold of the interaction $\tilde{\mathcal{H}}_{bs}$ on the torus is threefold degenerate.

It is useful to pause at this stage to interpret this lower bound on the ground state degeneracy and how it comes about. Naively, given two pairs of anticommuting non-local operators, all of which commute with the Hamiltonian, i.e., given Eqs. \(3.20\) and \(3.21\) there are at most four degenerate ground states. In the case of Kitaev's toric code \(93\), the dimensionality of the ground state manifold saturates this upper bound. However, in the case of the two-dimensional state of matter that we have constructed here, we argue that this is not the case. The reason for this is intimately related to the nonunitarity of the string operators $\hat{\Gamma}^{(2)}_1(z)$ and $\hat{\Gamma}^{(2)}_2(z)$.

In particular, we assert that neither of the naively-expected fourth states, namely

$$|\hat{\Gamma}^{(1)}_2 \hat{\Gamma}^{(2)}_2\rangle := \lim_{\epsilon \rightarrow 0} \hat{\Gamma}^{(1)}_2 (z) \hat{\Gamma}^{(2)}_2 (z, y, \epsilon) |\Omega\rangle,$$  \hspace{1cm} (3.54a)

and

$$|\hat{\Gamma}^{(2)}_2 \hat{\Gamma}^{(1)}_1\rangle := \lim_{\epsilon \rightarrow 0} \hat{\Gamma}^{(1)}_2 (z, y, \epsilon) \hat{\Gamma}^{(2)}_1 (z) |\Omega\rangle,$$  \hspace{1cm} (3.54b)

belongs to the ground-state manifold of the interaction $\tilde{\mathcal{H}}_{bs}$. Note that the limit $\epsilon \rightarrow 0$ above is to be taken after forming the products $\hat{\Gamma}^{(2)}_1(z) \hat{\Gamma}^{(2)}_2 (z, y, \epsilon)$ and $\hat{\Gamma}^{(2)}_2(z, y, \epsilon) \hat{\Gamma}^{(1)}_1 (z)$, as discussed in Footnote \[92\] and Appendix C. If the operator products $\hat{\Gamma}^{(2)}_1(z) \hat{\Gamma}^{(2)}_2 (z, y, \epsilon)$ and $\hat{\Gamma}^{(2)}_2(z, y, \epsilon) \hat{\Gamma}^{(1)}_1 (z)$ were to commute with the interaction $\tilde{\mathcal{H}}_{bs}$ in the limit $\epsilon \rightarrow 0$, as they would in an Abelian topological phase, then there would be no obstruction to the states $|\hat{\Gamma}^{(2)}_1 \hat{\Gamma}^{(2)}_2\rangle$ and $|\hat{\Gamma}^{(2)}_1 \hat{\Gamma}^{(2)}_2\rangle$ belonging to the ground-state manifold. The proof that such an obstruction exists in the present (non-Abelian) case is undertaken in two complementary ways in the present work. The first, which we call the “algebraic” approach, relies on diagrammatic techniques developed in Appendix D, and is presented below. The second, which we call the “analytic” approach, is carried out in Appendix C. Both the “algebraic” and “analytic” proofs rely on the fact, discussed in Appendix C, that the operator products $\hat{\Gamma}^{(2)}_1(z) \hat{\Gamma}^{(2)}_2 (z, y, \epsilon)$ and $\hat{\Gamma}^{(2)}_2(z, y, \epsilon) \hat{\Gamma}^{(1)}_1 (z)$ are not bound to commute with the interaction $\tilde{\mathcal{H}}_{bs}$ in the limit $\epsilon \rightarrow 0$. We now proceed with the “algebraic” version of the proof, and refer the reader to Appendices D and C for more details.
Proof ("algebraic"). We introduce the projection operator
\[ \hat{\mathcal{P}}_{\text{GSM}} = N^{-1}_1 \hat{1} \langle \hat{1} | 
\]
\[ + N^{-1}_{\frac{1}{2}} \hat{1} \langle \hat{1} | \hat{\Gamma}^{(\frac{1}{2})}_1 | \hat{\Gamma}^{(\frac{1}{2})}_1 | 
\]
\[ + N^{-1}_{\frac{1}{2}} \hat{1} \langle \hat{1} | \hat{\Gamma}^{(\frac{1}{2})}_2 | \hat{\Gamma}^{(\frac{1}{2})}_2 | + \cdots \] (3.55)
onto the ground state manifold. Here, \( N_1 \) is the squared norm of the state \( \langle \hat{1} | \hat{\Omega} \rangle \), \( N_{\frac{1}{2}} \) is the squared norm of the state \( \langle \hat{\Gamma}^{(\frac{1}{2})}_1 | \hat{\Omega} \rangle \), \( N_{\frac{1}{2}} \) is the squared norm of the state \( \langle \hat{\Gamma}^{(\frac{1}{2})}_2 | \hat{\Omega} \rangle \), and \( \cdots \) is a sum over any remaining elements from the orthonormal basis of the ground state manifold. By definition, any one of the three states \( \langle \hat{1} | \hat{\Gamma}^{(\frac{1}{2})}_1 \rangle \), \( \langle \hat{\Gamma}^{(\frac{1}{2})}_1 | \hat{\Omega} \rangle \), and \( \langle \hat{\Gamma}^{(\frac{1}{2})}_2 | \hat{\Omega} \rangle \) defined in Eq. (3.49) is invariant under the action of
\[ \hat{\mathcal{P}}_{\text{GSM}} = \hat{\mathcal{P}}_{\text{GSM}}^2. \] (3.56)
Hence, we may write
\[ | \hat{\Gamma}^{(\frac{1}{2})}_1 \rangle = \hat{\mathcal{P}}_{\text{GSM}} | \hat{\Gamma}^{(\frac{1}{2})}_1 \rangle = \hat{\mathcal{P}}_{\text{GSM}} \hat{\Gamma}^{(\frac{1}{2})}_1 (z) \hat{\mathcal{P}}_{\text{GSM}} | \Omega \rangle , \] (3.57a)
\[ | \hat{\Gamma}^{(\frac{1}{2})}_2 \rangle = \hat{\mathcal{P}}_{\text{GSM}} | \hat{\Gamma}^{(\frac{1}{2})}_2 \rangle = \hat{\mathcal{P}}_{\text{GSM}} \lim_{\epsilon \to 0} \hat{\Gamma}^{(\frac{1}{2})}_{2,R,y'}(\epsilon) \hat{\mathcal{P}}_{\text{GSM}} | \Omega \rangle . \] (3.57b)
On the other hand,
\[ \hat{\mathcal{P}}_{\text{GSM}} \hat{O} \hat{\mathcal{P}}_{\text{GSM}} = 0 \] (3.58)
must hold for any operator \( \hat{O} \) such that \( \hat{O} \) returns an excited state when applied to any state from the ground-state manifold.

We are first going to show that the operators \( \hat{\Gamma}^{(\frac{1}{2})}_1 (z) \) and \( \hat{\Gamma}^{(\frac{1}{2})}_{2,R,y'}(\epsilon) \) do not commute in the limit \( \epsilon \to 0 \). After that, we will elaborate on why the state \( | \hat{\Gamma}^{(\frac{1}{2})}_1 \rangle \) \( | \hat{\Gamma}^{(\frac{1}{2})}_2 \rangle \) does not belong to the ground-state manifold of the interaction \( \hat{H}_{\text{gs}} \).

We begin by considering the exchange algebra of the string operators \( \hat{\Gamma}^{(\frac{1}{2})}_1 (z) \) and \( \hat{\Gamma}^{(\frac{1}{2})}_{2,R,y'}(\epsilon) \) defined in Eqs. (3.37) and (3.44), respectively. Specifically, we consider the product
\[ \hat{\Gamma}^{(\frac{1}{2})}_1 (z) \hat{\Gamma}^{(\frac{1}{2})}_{2,R,y'}(\epsilon) \propto \left( \prod_{y=0}^{L_y} \hat{\sigma}_{L,y}(z) \hat{\sigma}_{R,y}(z) \right) \times \hat{\mathcal{P}}_1 \hat{\sigma}_{R,y'}(0) \hat{\sigma}_{R,y'}(\epsilon) \hat{\mathcal{P}}_1, \] (3.59)
where \( \epsilon > 0 \) is infinitesimal and we have also omitted the operators in the \( u(1)_2 \) sector appearing in the definition (3.44), as these operators commute with all operators in the \( \mathbb{Z}_2 \) sector. Using the fact that twist operators in different wires (and in different chiral sectors of the same wire) commute, we deduce that
\[ \hat{\Gamma}^{(\frac{1}{2})}_1 (z) \hat{\Gamma}^{(\frac{1}{2})}_{2,R,y'}(\epsilon) \propto \left( \prod_{y \neq y'} \hat{\sigma}_{L,y}(z) \hat{\sigma}_{R,y}(z) \right) \hat{\sigma}_{L,y'}(z) \times \hat{\mathcal{P}}_1 \hat{\sigma}_{R,y'}(0) \hat{\sigma}_{R,y'}(\epsilon) \hat{\mathcal{P}}_1. \] (3.60)
Since all operators in the first line of the right-hand side above commute with all operators in the second line, computing the exchange algebra of the operators \( \hat{\Gamma}^{(\frac{1}{2})}_1 \) and \( \hat{\Gamma}^{(\frac{1}{2})}_{2,R} \) boils down to considering the following product of operators,
\[ \lim_{z_2 \to z_1 + \epsilon} \hat{\mathcal{P}}_1 \hat{\sigma}_{R,y'}(z_1) \hat{\sigma}_{R,y'}(z_2) \hat{\mathcal{P}}_1. \] (3.61)
Using the prescriptions of Appendix D, we find that the process of commuting the leftmost operator, \( \hat{\sigma}_{R,y'}(z_1) \), past the remaining two operators is represented by the diagram
\[ z_2 \quad z \quad z_1 \quad \sigma \quad \sigma \]
(3.62)
Untwisting the legs of this fusion diagram, we find
where the $F$- and $R$-symbols are given in Appendix D. The diagrammatic relation expressed in Eq. (3.63) can be rewritten as the algebraic statement

$$
\hat{\sigma}_{R,y'}(z) \hat{P}_1 \hat{\sigma}_{R,y}(z_1) \hat{P}_1 = e^{+i \frac{\pi}{4}} \hat{P}_\psi \hat{\sigma}_{R,y'}(z_1) \hat{P}_\psi \hat{\sigma}_{R,y'}(z),
$$

(3.64)

where $\hat{P}_\psi$ is a projection operator that projects the product $\hat{\sigma}_{R,y'}(z_1) \hat{\sigma}_{R,y'}(z_2)$ into the fusion channel $\sigma = \psi$. Taking the limits $z_2 \to z_1 + \epsilon$ and $z_1 \to 0$ and restoring the operators $\hat{\sigma}_{M,y}(z)$ present in Eq. (3.60) (as well as the operators from the $u(1)_2$ sector that were omitted there), we arrive at the relation

$$
\hat{\Gamma}_1^{(\frac{1}{2})}(z) \hat{\Gamma}_2^{(\frac{1}{2})}_{2,R,y'}(\epsilon) = e^{+i \frac{3\pi}{4}} \hat{\Gamma}_2^{(\frac{1}{2})}_{2,R,y'}(\epsilon) \hat{\Gamma}_1^{(\frac{1}{2})}(z),
$$

(3.65a)

in the limit $\epsilon \to 0$, where we have defined the operator

$$
\hat{\Gamma}_2^{(\frac{1}{2})}_{2,R,y'}(\epsilon) := \exp \left( -i \frac{1}{2\sqrt{2}} \int_0^{L_z} dz \partial_z \hat{\sigma}_{R,y'}(t,z) \right) \times \hat{P}_\psi \hat{\sigma}_{R,y'}(0) \hat{\sigma}_{R,y'}(\epsilon) \hat{P}_\psi,
$$

(3.65b)

which is identical to the operator $\hat{\Gamma}_2^{(\frac{1}{2})}$ defined in Eq. (3.44), except that the product $\hat{\sigma}_{R,y'}(0) \hat{\sigma}_{R,y'}(\epsilon)$ is evaluated in the fusion channel $\psi$ rather than the fusion channel $1$. This difference is fundamental. Since the two twist operators entering the operator $\hat{\Gamma}_2^{(\frac{1}{2})}$ fuse to $\psi$, this operator can be interpreted as adding an extra Majorana fermion to the state on which it acts. Acting with $\hat{\Gamma}_2^{(\frac{1}{2})}$ on any of the states $|1\rangle, |\hat{\Gamma}_1^{(\frac{1}{2})}\rangle, |\hat{\Gamma}_2^{(\frac{1}{2})}\rangle, \ldots$ in the ground-state manifold of the interaction $\hat{H}_{bs}$ can then be viewed as creating an excited state of the interaction $\hat{H}_{bs}$ with one extra fermion. In other words, we have

$$
\hat{P}_{GSM} \lim_{\epsilon \to 0} \hat{\Gamma}_2^{(\frac{1}{2})}_{2,R,y'}(\epsilon) \hat{P}_{GSM} = 0.
$$

(3.66)

This relation is crucial in what follows. Note also the difference between the phase on the RHS of Eq. (3.65a) and that on the RHS of Eq. (3.64), which comes from commutators in the $u(1)_2$ sector.

We are now prepared to exclude the state $|\hat{\Gamma}_1^{(\frac{1}{2})}\hat{\Gamma}_2^{(\frac{1}{2})}\rangle$ from the ground-state manifold of the interaction $\hat{H}_{bs}$. Applying Eq. (3.65a) to the definition (3.54a) of the state $|\hat{\Gamma}_1^{(\frac{1}{2})}\hat{\Gamma}_2^{(\frac{1}{2})}\rangle$, we obtain

$$
|\hat{\Gamma}_1^{(\frac{1}{2})}\hat{\Gamma}_2^{(\frac{1}{2})}\rangle = e^{+i \frac{3\pi}{4}} \lim_{\epsilon \to 0} \hat{\Gamma}_2^{(\frac{1}{2})}_{2,R,y'}(\epsilon) |\hat{\Gamma}_1^{(\frac{1}{2})}\rangle.
$$

(3.67)

If the state $|\hat{\Gamma}_1^{(\frac{1}{2})}\hat{\Gamma}_2^{(\frac{1}{2})}\rangle$ is in the ground-state manifold of the interaction $\hat{H}_{bs}$, then it cannot be a null vector of $\hat{P}_{GSM}$. However, using Eqs. (3.57) and (3.66), we find that

$$
\hat{P}_{GSM} |\hat{\Gamma}_1^{(\frac{1}{2})}\hat{\Gamma}_2^{(\frac{1}{2})}\rangle = e^{+i \frac{3\pi}{4}} \hat{P}_{GSM} \lim_{\epsilon \to 0} \hat{\Gamma}_2^{(\frac{1}{2})}_{2,R,y'}(\epsilon) \hat{P}_{GSM} |\hat{\Gamma}_1^{(\frac{1}{2})}\rangle
$$

(3.68)

$$
= 0.
$$

Thus, the state $|\hat{\Gamma}_1^{(\frac{1}{2})}\hat{\Gamma}_2^{(\frac{1}{2})}\rangle$ does not lie in the ground-state manifold of the interaction $\hat{H}_{bs}$. Similarly, the state $|\hat{\Gamma}_2^{(\frac{1}{2})}\hat{\Gamma}_1^{(\frac{1}{2})}\rangle$ defined in Eq. (3.54b) is excluded from the ground-state manifold. We note in passing that a related line of reasoning was used in Ref. [72] to exclude certain states from the ground-state manifold of the gauged $p+i p$ superconductor (see also Ref. [94]).
In summary, we have shown that the $su(2)_2$ coupled-wire construction in (2+1)-dimensional spacetime has a threefold topological degeneracy on the two-torus. The proof that this topological degeneracy is threefold and not fourfold relied on the observation that the “spin-$1/2$” string operators obey the non-Abelian exchange algebra (3.65a). This algebra, whereby exchanging the two operators does not simply produce a phase factor, but instead enacts a nontrivial transformation on the operators themselves, is the essence of what it means to be a non-Abelian topological phase. We will see that a similar, algebraic structure can be employed in (3+1) dimensions to encode the topological degeneracy on the two-torus. The wire construction in (2+1)-dimensional spacetime has a common physical setup, which we describe below.

IV. WEAK NON-ABELIAN TOPOLOGICAL ORDER IN THREE DIMENSIONS

In this section and the next, we consider 3D generalizations of the class of 2D models defined in Sec. III. In Sec. IV we construct a family of weak topological phases that can be viewed as stacks of the topological phases discussed in Sec. III In Sec. V, we consider a more complicated interaction that may yield a truly 3D non-Abelian topological phase. These two families of phases share a common physical setup, which we describe below.

A. Physical setup for 3D wire arrays

Consider a square lattice $\Lambda$ of wires, each described by the Lagrangian density (2.1). We want to break the degrees of freedom in any one of the identical wires up into four groups, two of which contain only right-moving degrees of freedom and two of which contain only left-moving degrees of freedom (see Fig. 2). Consequently, let each wire (2.1) contain $N_c = 2k$ colors of fermions, so that the full symmetry group of each wire is $U(4k)_L \times U(4k)_R$. However, we wish to employ the conformal embedding (2.3) to write down the couplings in our theory in terms of currents. Thus, let us consider only couplings that are symmetric under the subgroup $U(2k)_M \times U(2k)_M \subset U(4k)_M$ with $M = L, R$. Note that the central charges associated with the groups $U(2k)_M \times U(2k)_M$ and $U(4k)_M$ are identical [85]. Thus, we can use couplings with either symmetry to fully gap the theory.] Then, we can use the identity

$$u(2k)_1 = u(1) \oplus su(2)_k \oplus su(k)_2$$

(4.1)

to define the M-moving chiral currents $\hat{J}^\gamma_M$, $\hat{J}^\gamma_M$, and $\hat{J}^\gamma_M$, which are given by Eqs. (2.4) with the substitution $N_c \rightarrow k$. Because we are considering couplings that are symmetric under rotations in $[U(2k) \times U(2k)]_L \times [U(2k) \times U(2k)]_R$, there are actually two copies of each of the chiral currents $\hat{J}^\gamma_M$, $\hat{J}^\gamma_M$, and $\hat{J}^\gamma_M$ with $M = L, R$ in each wire. Therefore, we adopt an additional label $\gamma = 1, 2$ to distinguish the chiral currents $\hat{J}^\gamma_M$, $\hat{J}^\gamma_M$, and $\hat{J}^\gamma_M$ from one another. The label $\gamma$ is somewhat redundant in that it will always transform trivially under all the symmetries that we shall impose. We only use it to keep track of the two independent copies of each set of currents.

B. Interwire couplings for weak non-Abelian topological order

Next, as in Sec. IIIA, we gap out the $u(1)$ and $su(k)_2$ degrees of freedom by turning on intra-wire interactions of the form (2.8) and (2.9), respectively, for each $\gamma = 1, 2$. The remaining $su(2)_k$ degrees of freedom are coupled in the following way. First, we define a square lattice $\tilde{\Lambda}$, whose unit cell is enlarged with respect to that of the square lattice $\Lambda$. Let the unit cell of $\tilde{\Lambda}$ contain four quantum wires, which we label by an index $J = A, B, C, D$. This enlarged unit cell is depicted in Fig. 6. Each $su(2)_k$ current operator $\hat{J}^\gamma_{\gamma', M, l, \tilde{r}}$ thus carries the labels $\gamma = 1, 2$ and $M = L, R$, as well as a label $\tilde{r} \in \tilde{\Lambda}$ to specify the unit cell and a label $l = A, B, C, D$ to specify a wire within a unit cell. We then write down the many-body “backscattering” current-current interactions encoded by the Lagrangian density
\begin{align}
\hat{\mathcal{L}}_{bs}[su(2)_k] & \equiv -\hat{\mathcal{H}}_{bs}[su(2)_k] := -\lambda_{su(2)_k} \sum_{\bar{\gamma} \in \Lambda} \left( \hat{\mathcal{L}}_{2,R,A\bar{\gamma},1,L,D} + \hat{\mathcal{L}}_{2,L,D,\bar{\gamma},1,R,A} + \hat{\mathcal{L}}_{2,L,R,\bar{\gamma},1,R,C} + \hat{\mathcal{L}}_{2,R,C,\bar{\gamma},1,L,B} 
+ \hat{\mathcal{L}}_{2,L,A,\bar{\gamma},1,R,B} + \hat{\mathcal{L}}_{2,R,B,\bar{\gamma},1,L,A} + \hat{\mathcal{L}}_{2,R,D,\bar{\gamma},1,L,C} + \hat{\mathcal{L}}''_{2,L,C,\bar{\gamma},1,R,D} \right), \quad (4.2a)
\end{align}

where we have assigned to each of the eight nearest-neighbor bonds shown in Fig. 6 the bond operators

\begin{align}
\hat{\mathcal{L}}_{\gamma,M,J,\bar{\gamma},M',J'} & := \sum_{a=1}^{2} \hat{J}_{\gamma,M,J,\bar{\gamma},M',J'}^a \hat{J}_{\gamma',M',J',\bar{\gamma}}^a, \quad (4.2b) \\
\hat{\mathcal{L}}'_{\gamma,M,J,\bar{\gamma},M',J'} & := \sum_{a=1}^{2} \hat{J}_{\gamma,M,J,\bar{\gamma},M',J'}^a \hat{J}_{\gamma',M',J',\bar{\gamma}}^a, \quad (4.2c) \\
\hat{\mathcal{L}}''_{\gamma,M,J,\bar{\gamma},M',J'} & := \sum_{a=1}^{2} \hat{J}_{\gamma,M,J,\bar{\gamma},M',J'}^a \hat{J}_{\gamma',M',J',\bar{\gamma}}^a, \quad (4.2d)
\end{align}

and where the lattice vectors $2\hat{x}$ and $2\hat{y}$ connect neighboring unit cells along the $x$- and $y$-directions, respectively. The interactions described by these bond operators have the same form as those appearing in Eq. (3.2a), which were used in the 2D case. The answer to the question of whether or not the interactions (4.2a) yield a fully gapped spectrum is thus the same as in 2D. In particular, the argument for the existence of a gap in the case $k = 2$ is identical.

The current-current interactions (4.2a) possess an antunitary involutive symmetry that effectively plays the same role as that of time-reversal symmetry for TIs. We call this symmetry an antiferromagnetic time-reversal symmetry. (This symmetry will also appear in Sec. V.)

To see this, note that Fig. 6 is invariant under interchanging the colors red and blue, and then translating by either of the half-lattice vectors $\hat{x}$ or $\hat{y}$. Formally, we define the symmetry operations

\begin{align}
T_{\text{eff},\hat{x}} & := T \times T_{\hat{x}}, \quad (4.3a) \\
T_{\text{eff},\hat{y}} & := T \times T_{\hat{y}}, \quad (4.3b)
\end{align}

where the time-reversal operation $T$ acts in the usual way on the spinful fermions (i.e., $T^2 = -1$), and acts on the $su(2)_k$ currents [see Eqs. (2.4)] as

\begin{align}
T \hat{J}_{\gamma,M,J,\bar{\gamma},\hat{\gamma}}^a T^{-1} = -\hat{J}_{\gamma,M,J,\bar{\gamma}}^a, \quad (4.3c)
\end{align}

for any $a = 1, 2, 3, \gamma = 1, 2, M = L, R, J = A, B, C, D,$ and $\bar{\gamma} \in \Lambda$, with $\hat{J}_{\gamma,M,J,\bar{\gamma}}^a \equiv R$ and $\hat{J}_{\gamma,M,J,\bar{\gamma}}^a \equiv L$, and where the half-lattice translation operators $T_{\hat{x}}$ and $T_{\hat{y}}$ act as

\begin{align}
T_{\hat{x}} \hat{T}_{\gamma,M,J,\bar{\gamma}}^a T_{\hat{x}}^{-1} = \begin{cases} 
\hat{J}_{\gamma,M,B,\bar{\gamma}}, & J = A, \\
\hat{J}_{\gamma,M,A,\bar{\gamma} + 2\hat{x}}, & J = B, \\
\hat{J}_{\gamma,M,D,\bar{\gamma} + 2\hat{x}}, & J = C, \\
\hat{J}_{\gamma,M,C,\bar{\gamma}}, & J = D,
\end{cases} \quad (4.3d)
\end{align}

and

\begin{align}
T_{\hat{y}} \hat{T}_{\gamma,M,J,\bar{\gamma}}^a T_{\hat{y}}^{-1} = \begin{cases} 
\hat{J}_{\gamma,M,D,\bar{\gamma}}, & J = A, \\
\hat{J}_{\gamma,M,C,\bar{\gamma}}, & J = B, \\
\hat{J}_{\gamma,M,B,\bar{\gamma} + 2\hat{y}}, & J = C, \\
\hat{J}_{\gamma,M,A,\bar{\gamma} + 2\hat{y}}, & J = D,
\end{cases} \quad (4.3e)
\end{align}

respectively. The full Lagrangian in the presence of the current-current interactions (4.2a) is invariant under the symmetry operations $T_{\text{eff},\hat{x}}$ and $T_{\text{eff},\hat{y}}$ when periodic boundary conditions are imposed in the $x$- and $y$-directions. If we use mixed boundary conditions, such as ones that are open along the $x$-direction and periodic along the $y$-direction (or vice versa), then only the symmetry $T_{\text{eff},\hat{x}}$ (or $T_{\text{eff},\hat{y}}$) remains. In Sec. VI, we will see that the remaining symmetry $T_{\text{eff},\hat{y}}$ protects gapless surface states on the boundaries at $x = 0$ and $x = L_x$. Analogous non-onsite implementations of time reversal symmetry have arisen in studies of anti-ferromagnetic TIs [75, 76] and in coupled-wire models of topological-insulator and topological-superconductor surfaces [73, 74].

One can interpret the coupled-wire array with the in-
interaction (4.2a) as a stack of 2D topological phases like the ones defined and studied in Sec. (III). To see this, one need only group the terms in the sum in Eq. (4.2a) in an appropriate way, as indicated in Fig. 7. Then, one sees that Eq. (4.2a) breaks up into a sum over decoupled planes, each consisting of a set of wires coupled by interactions of the form (3.2a). Each plane thus forms its own 2D non-Abelian topological phase. More specifically, any ground-state wavefunction of the coupled-wire array can be written in the form

$$|\Psi_{\text{gs}}\rangle := \bigotimes_i |\Psi_{\text{gs}_i}\rangle,$$  

(4.4)

where the index \(i\) runs over the layers in the stack. Here, \(|\Psi_{\text{gs}_i}\rangle\) is a ground-state wavefunction of the 2D topological phase supported by layer \(i\) of the stack depicted in Fig. 7. Since each layer carries with it a threefold topological degeneracy when periodic boundary conditions are imposed in all directions, the full 3D system then carries a weak topological degeneracy of

$$3 \times \text{# of layers.}$$  

(4.5)

This subextensive degeneracy reflects the direct-product structure of the ground-state wavefunction (4.4). If one uses this wavefunction to calculate the entanglement entropy between adjacent layers in the stack by partitioning the stack along a plane parallel to both layers, one inevitably obtains zero. For any pair \(i\) and \(j\) of shaded layers from the stack depicted in Fig. 7, the ground state wavefunctions \(|\Psi_{\text{gs}_i}\rangle\) and \(|\Psi_{\text{gs}_j}\rangle\) are not correlated. The result is a topological ground-state degeneracy that scales with the number of layers.

When periodic boundary conditions are imposed, each plane of coupled wires defined in this way is related to its neighbors by the symmetries \(\mathcal{T}_{\text{eff}, \bar{x}, \bar{y}}\) defined in Eq. (4.3). Thus, one can view this 3D stack of 2D non-Abelian topological phases as a weak topological phase respecting the symmetries \(\mathcal{T}_{\text{eff}, \bar{x}, \bar{y}}\).

V. STRONG NON-ABELIAN TOPOLOGICAL ORDER IN THREE DIMENSIONS

The interaction (4.2a) depicted in Fig. 6 can be decomposed into a sum of interactions between wires living in decoupled planes. We now propose an alternative interaction that avoids this problem, thereby realizing a truly 3D non-Abelian topological phase. The proposed interaction preserves the structure of the individual wires in the decoupled limit, as described in Sec. IV A and depicted in Fig. 2. It is given by

$$\hat{\mathcal{H}}_{\square} := -\lambda \left( \sum_{p_A \in \Lambda} \hat{\mathcal{H}}_{p_A} + \sum_{p_B \in \Lambda} \hat{\mathcal{H}}_{p_B} + \sum_{p_C \in \Lambda} \hat{\mathcal{H}}_{p_C} + \sum_{p_D \in \Lambda} \hat{\mathcal{H}}_{p_D} \right),$$  

(5.1a)

where

$$\hat{\mathcal{H}}_{p_A} := \sum_{a,b,c,d=1}^{2} \left( \hat{J}^a_{2,1,1,R,A,p_A} \hat{J}^a_{1,1,R,B,p_A} \right) \left( \hat{J}^b_{2,1,1,R,C,p_A} \hat{J}^b_{1,1,R,D,p_A} \right) \left( \hat{J}^c_{2,2,R,D,p_A} \hat{J}^c_{1,2,L,C,p_A} \right) \left( \hat{J}^d_{2,2,R,A,p_A} \hat{J}^d_{1,2,L,D,p_A} \right),$$  

(5.1b)

$$\hat{\mathcal{H}}_{p_B} := \sum_{a,b,c,d=1}^{2} \left( \hat{J}^a_{2,1,1,L,B,p_B} \hat{J}^a_{1,1,L,A,p_B} \right) \left( \hat{J}^b_{2,1,1,L,C,p_B} \hat{J}^b_{1,1,L,D,p_B} \right) \left( \hat{J}^c_{2,2,L,D,p_B} \hat{J}^c_{1,2,L,C,p_B} \right) \left( \hat{J}^d_{2,2,L,A,p_B} \hat{J}^d_{1,2,L,D,p_B} \right),$$  

(5.1c)

$$\hat{\mathcal{H}}_{p_C} := \sum_{a,b,c,d=1}^{2} \left( \hat{J}^a_{2,1,1,R,D,p_C} \hat{J}^a_{1,1,R,L,p_C} \right) \left( \hat{J}^b_{2,1,1,R,A,p_C} \hat{J}^b_{1,1,R,C,p_C} \right) \left( \hat{J}^c_{2,2,R,C,p_C} \hat{J}^c_{1,2,L,A,p_C} \right) \left( \hat{J}^d_{2,2,R,L,p_C} \hat{J}^d_{1,2,L,A,p_C} \right),$$  

(5.1d)

$$\hat{\mathcal{H}}_{p_D} := \sum_{a,b,c,d=1}^{2} \left( \hat{J}^a_{2,1,1,L,D,p_D} \hat{J}^a_{1,1,L,C,p_D} \right) \left( \hat{J}^b_{2,1,1,L,C,p_D} \hat{J}^b_{1,1,L,B,p_D} \right) \left( \hat{J}^c_{2,2,L,C,p_D} \hat{J}^c_{1,2,L,B,p_D} \right) \left( \hat{J}^d_{2,2,L,B,p_D} \hat{J}^d_{1,2,L,A,p_D} \right).$$  

(5.1e)

Here, the labels 1, 2, L, R, and A, \(\cdots\), D are as in Fig. 6. Moreover, we have defined a new label \(p_J, J = A, \cdots, D\) as follows. The label \(p_A\) denotes a four-wire plaquette in the lattice \(\Lambda\) in which the four labels \(J = A, B, C, D\) cycle through \(A, B, C, D\) clockwise from the top left (this is the boxed plaquette in Fig. 2). The remaining plaquette labels are defined analogously, with \(p_B\) labeling a plaquette whose top-left wire carries the label \(B\) and so on.

The plaquette interaction (5.1) avoids the problem of the current-current interaction (4.2a) in that it is not possible to rewrite the former as a sum over decoupled planes. The crucial difference between the two interactions that allows this to happen is the fact that the in-
teration (5.1) is a sum of products of current-current bilinears, whereas the interaction (4.2a) is merely a sum of such bilinears.

In the remainder of this section, we will analyze the consequences of the plaquette interaction (5.1) in the strong-coupling limit $\lambda \to \infty$ when periodic boundary conditions are imposed in all directions. We will focus our attention on the case $k = 2$, although generalizations to $k > 2$ along the lines of Sec. (III B) are possible. First, we will show that the interaction (5.1) opens a gap at strong coupling. Second, we will define a set of string and membrane operators associated with excitations of the gapped theory, and argue on the basis of their algebra that the ground state of the gapped system is topologically degenerate. The algebra obeyed by these operators will reveal the non-Abelian character of the phase.

A. Analyzing the opening of a gap when $k = 2$

To see how the plaquette interaction (5.1) opens a gap at strong coupling when $k = 2$, it is useful to rewrite it in

$$\hat{J}_{\gamma,M,J,p_K}^\pm = \sqrt{2} \hat{\psi}_{\gamma,M,J,p_K} : e^{\mp i \sqrt{2} \phi_{\gamma,M,J,p_K}} : \ .$$ (5.2a)

where $\hat{\psi}_{\gamma,M,J,p_K}$ is a chiral Majorana fermion and $\phi_{\gamma,M,J,p_K}$ is a chiral bosonic field, for any $\gamma = 1, 2, M = L, R$, and $J, K = A, B, C, D$. On the Majorana operators, we impose the equal-time algebra

$$\hat{\phi}_{\gamma,M,J,p_K}(t,z) \hat{\phi}_{\gamma',M',J',p_K'}(t,z') = \hat{\phi}_{\gamma',M',J',p_K'}(t,z') \hat{\phi}_{\gamma,M,J,p_K}(t,z)$$

This equal-time algebra is a generalization of Eqs. (3.4d)–(3.4f). As before, we use the conventions $\text{sgn}(0) = 0$ and $\epsilon_{L,R} = -\epsilon_{R,L} = -1$. Note that, on the one hand, the Majorana operator $\hat{\psi}_{\gamma,M,J,p_K}$ is labeled by the tuple $(J,p_K) \in \hat{\Lambda}$, which is the square lattice with a four-wire unit cell depicted in Fig. 6. On the other hand, the coordinates $\{r := (x,y) \in \Lambda\}$, which label the square lattice with a single-wire unit cell, enter the phase factors in Eqs. (5.3a). This notation is consistent because the tuple $(J,p_K)$, $J, K = A, B, C, D$, uniquely specifies an $r \in \Lambda$. Finally, we impose the equal-time algebra

$$\left[\hat{\phi}_{\gamma,M,J,p_K}(t,z), \hat{\phi}_{\gamma',M',J',p_K'}(t,z')\right] = -i 2 \pi \left(\frac{1}{2} \delta_{M,M'} \delta_{\gamma,\gamma'} \delta_{r,r'} \text{sgn}(z - z') - (1 - \delta_{M,M'} \delta_{\gamma,\gamma'} \delta_{r,r'}) (-1)^M \delta_{M,M'} + \epsilon_{M,M'}\right),$$ (5.3b)

on the chiral bosons. This algebra is a generalization of Eq. (3.4g).

The interaction (5.1) is amenable to analysis upon substituting in the decompositions (5.2). For instance, the first term in Eq. (5.1a) becomes

$$\hat{H}_p = 16 \hat{\psi}_{2,L,A,p_A} \hat{\psi}_{1,R,B,p_A} \hat{\psi}_{2,L,B,p_A} \hat{\psi}_{1,R,C,p_A} \hat{\psi}_{1,L,C,p_A} \hat{\psi}_{2,R,D,p_A} \hat{\psi}_{1,L,D,p_A} \hat{\psi}_{2,R,A,p_A}$$

$$\times \sin \left(\frac{\phi_{2,L,A,p_A} - \phi_{1,R,B,p_A}}{\sqrt{2}}\right) \sin \left(\frac{\phi_{2,L,B,p_A} - \phi_{1,R,C,p_A}}{\sqrt{2}}\right) \sin \left(\frac{\phi_{1,L,C,p_A} - \phi_{2,R,D,p_A}}{\sqrt{2}}\right) \sin \left(\frac{\phi_{1,L,D,p_A} - \phi_{2,R,A,p_A}}{\sqrt{2}}\right).$$ (5.4)

Similar expressions hold for the remaining three plaquette terms in the summand in Eq. (5.1a). Each such plaquette term factorizes into a $u(1)_2$ piece (the product of four sine potentials) and a $\mathbb{Z}_2$ piece (the eight-Majorana interaction). By analogy with Eq. (3.7) in the 2D case, we would like to argue that the $u(1)_2$ and $\mathbb{Z}_2$ sectors can simultaneously acquire expectation values and become gapped. To see that this is indeed possible, it is useful to
consider the \( u(1)_2 \) and \( Z_2 \) sectors separately.

First, one can show by direct calculation that the four
eight-Majorana plaquette terms appearing in Eq. (5.1a)
commute with one another. Hence, it is possible for each
eight-Majorana plaquette term to acquire an expectation
value such that the total energy in the Majorana sector is
minimized. When this occurs, we can replace the product
of eight Majorana operators that enters each plaquette
term with a constant. What remains is an effective in-


teraction for the \( u(1)_2 \) sector consisting of four types of
plaquette terms, each a product of four sines.

\[
\sin \left( \frac{\hat{\phi}_{2L,A,p_A} - \hat{\phi}_{1L,B,p_A}}{\sqrt{2}} \right) \sin \left( \frac{\hat{\phi}_{2L,B,p_A} - \hat{\phi}_{1L,C,p_A}}{\sqrt{2}} \right) \sin \left( \frac{\hat{\phi}_{1L,C,p_A} - \hat{\phi}_{2R,D,p_A}}{\sqrt{2}} \right) \sin \left( \frac{\hat{\phi}_{1L,D,p_A} - \hat{\phi}_{2R,A,p_A}}{\sqrt{2}} \right)
\]

where \( s_i = \pm 1 \) is an \( i \)-dependent sign whose value is unimportant for the purposes of this argument. Here, we have
defined the eight-component vector

\[
\hat{\varphi}_{p_A} := \left( \hat{\phi}_{2L,A,p_A} \hat{\phi}_{2R,A,p_A} \hat{\phi}_{2L,B,p_A} \hat{\phi}_{1L,B,p_A} \hat{\phi}_{1L,C,p_A} \hat{\phi}_{1R,C,p_A} \hat{\phi}_{1L,D,p_A} \hat{\phi}_{2R,D,p_A} \right)^T
\]

and the \( 8 \times 8 \) diagonal matrix

\[
\mathcal{K} := \text{diag}(+1, -1, +1, -1, +1, -1, +1, -1).
\]

The eight-component vectors \( T_i \), \( i = 1, \cdots, 8 \) with entries \( \pm 1 \) specify the linear combinations of bosonic fields that
enter each cosine term.

One can verify using the algebra (5.3b) that the cos-

ine terms appearing in Eq. (5.6) do not commute with
one another. One might worry that this implies that
these eight cosines cannot be minimized simultaneously,
so that the \( u(1)_2 \) sector is not gapped, even when the \( Z_2 \)
sector is. However, Haldane showed [95] that a weaker
condition than strict commutation is sufficient for a set of
cosine potentials to be simultaneously minimizable. For
the cosine terms in Eq. (5.6), this condition reads

\[
T_i^T \mathcal{K} T_j = 0, \quad i, j = 1, \cdots, 8.
\]

Once the eight vectors \( T_i \) have been determined, it is
straightforward to check that the above condition indeed
holds for the interaction (5.6). Thus, the eight linear
combinations \( T_i^T \mathcal{K} \hat{\varphi}_{p_A} \) can simultaneously assume classical
configurations that minimize the expectation value of
the effective interaction (5.6). One can repeat this ex-
cise for the three plaquettes \( p_B, p_C, \) and \( p_D \) and verify that they, too, can each assume classical configurations
that minimize the energy in the \( u(1)_2 \) sector. Finally,
one checks that a version of the criterion (5.7) holds for
bosonic fields that are shared between adjacent plaqu-
ettes, ensuring that the adjacent plaquette terms do not
interfere with one another in this minimization process.

Next, we address the question of whether this effective
interaction is capable of opening a gap in the \( u(1)_2 \) sec-
tor. To do this, we successively apply the trigonometric
identities

\[
2 \sin a \sin b = \cos(a - b) - \cos(a + b),
\]

\[
2 \cos a \cos b = \cos(a - b) + \cos(a + b),
\]

until each product of four sines becomes a sum of \( eight \).cosines. For instance, applying this procedure to the \( u(1)_2 \) part of the plaquette term (5.4) yields

\[
\frac{1}{8} \sum_{i=1}^{8} s_i \cos \left( \frac{1}{\sqrt{2}} T_i^T \mathcal{K} \hat{\varphi}_{p_A} \right),
\]

\[
\frac{1}{8} \sum_{i=1}^{8} s_i \cos \left( \frac{1}{\sqrt{2}} T_i^T \mathcal{K} \hat{\varphi}_{p_A} \right)
\]

Until each product of four sines becomes a sum of \( eight \) cosines. For instance, applying this procedure to the \( u(1)_2 \) part of the plaquette term (5.4) yields

\[
\frac{1}{8} \sum_{i=1}^{8} s_i \cos \left( \frac{1}{\sqrt{2}} T_i^T \mathcal{K} \hat{\varphi}_{p_A} \right)
\]

Until each product of four sines becomes a sum of \( eight \) cosines. For instance, applying this procedure to the \( u(1)_2 \) part of the plaquette term (5.4) yields

\[
\frac{1}{8} \sum_{i=1}^{8} s_i \cos \left( \frac{1}{\sqrt{2}} T_i^T \mathcal{K} \hat{\varphi}_{p_A} \right)
\]

Until each product of four sines becomes a sum of \( eight \) cosines. For instance, applying this procedure to the \( u(1)_2 \) part of the plaquette term (5.4) yields

\[
\frac{1}{8} \sum_{i=1}^{8} s_i \cos \left( \frac{1}{\sqrt{2}} T_i^T \mathcal{K} \hat{\varphi}_{p_A} \right)
\]

Until each product of four sines becomes a sum of \( eight \) cosines. For instance, applying this procedure to the \( u(1)_2 \) part of the plaquette term (5.4) yields

\[
\frac{1}{8} \sum_{i=1}^{8} s_i \cos \left( \frac{1}{\sqrt{2}} T_i^T \mathcal{K} \hat{\varphi}_{p_A} \right)
\]

Until each product of four sines becomes a sum of \( eight \) cosines. For instance, applying this procedure to the \( u(1)_2 \) part of the plaquette term (5.4) yields

\[
\frac{1}{8} \sum_{i=1}^{8} s_i \cos \left( \frac{1}{\sqrt{2}} T_i^T \mathcal{K} \hat{\varphi}_{p_A} \right)
\]

Until each product of four sines becomes a sum of \( eight \) cosines. For instance, applying this procedure to the \( u(1)_2 \) part of the plaquette term (5.4) yields

\[
\frac{1}{8} \sum_{i=1}^{8} s_i \cos \left( \frac{1}{\sqrt{2}} T_i^T \mathcal{K} \hat{\varphi}_{p_A} \right)
\]

Until each product of four sines becomes a sum of \( eight \) cosines. For instance, applying this procedure to the \( u(1)_2 \) part of the plaquette term (5.4) yields

\[
\frac{1}{8} \sum_{i=1}^{8} s_i \cos \left( \frac{1}{\sqrt{2}} T_i^T \mathcal{K} \hat{\varphi}_{p_A} \right)
\]

Until each product of four sines becomes a sum of \( eight \) cosines. For instance, applying this procedure to the \( u(1)_2 \) part of the plaquette term (5.4) yields

\[
\frac{1}{8} \sum_{i=1}^{8} s_i \cos \left( \frac{1}{\sqrt{2}} T_i^T \mathcal{K} \hat{\varphi}_{p_A} \right)
\]

Until each product of four sines becomes a sum of \( eight \) cosines. For instance, applying this procedure to the \( u(1)_2 \) part of the plaquette term (5.4) yields

\[
\frac{1}{8} \sum_{i=1}^{8} s_i \cos \left( \frac{1}{\sqrt{2}} T_i^T \mathcal{K} \hat{\varphi}_{p_A} \right)
\]
the interaction (5.1) and the Wen-Plaquette model suggests that one should be able to construct string operators in the \(x-y\) plane that are in one-to-one correspondence with deconfined pointlike excitations. (This reasoning is similar to the reasoning that guided our work in Ref. [63], which constructed Abelian 3D topological phases from coupled wires with interactions meant to resemble those of the toric code.) Sec. V B is devoted to the construction of these operators, and others associated with the \(z\)-direction, as well as membrane operators corresponding to stringlike excitations.

Of course, the fact that the Majorana operators in the interaction (5.1) arise from a conformal field theory, and not from a representation of the Pauli algebra as in the Wen-Plaquette model, implies that the topological order (if any) obtained from the interaction (5.1) must be richer than “simple” \(Z_2\) topological order. In particular, the conformal field theories that furnish the Majorana operators also furnish more complicated non-Abelian operators like the Ising twist field, from which the coupled-wire model inherits its non-Abelian character.

Finally, we note that applying the above reasoning to the interaction (4.2) in Sec. IV B does not allow one to draw comparisons with a truly 2D lattice model. Instead, the \(Z_2\) sector of the interaction (4.2) resembles an array of decoupled 1D Kitaev chains [97]. This is yet another way to see that the interaction (4.2) cannot yield strong topological order.

B. String and membrane operators when \(k = 2\)

In this section we build string and membrane operators to characterize the different topological sectors of the coupled-wire theory. This discussion parallels the analysis of the 2D case presented in Sec. III C, albeit with complications owing to the increase in dimensionality.

1. Primary operators in a single wire

Since each wire consists of four independent chiral \(su(2)_2\) CFTs in the decoupled limit (see, e.g., Fig. 6), there are a number of chiral primary operators in each wire. In particular, there is a “spin-0”, a “spin-\(\frac{1}{2}\)”, and a “spin-1” primary for each set of labels \((\gamma, M, J, p_K)\), with \(\gamma = 1, 2\), \(M = L, R\), and \(J, K = A, B, C, D\). Fixing some \(J, K\), and plaquette \(p_K\) specifies a single wire, and the remaining labels \(\gamma\) and \(M\) enumerate the four chiral CFTs defined within that wire. Each chiral sector of the wire is equipped with a “spin-0” primary operator, which is simply the identity operator in that sector. The nontrivial primary operators in each CFT are the “spin-\(\frac{1}{2}\)” primary,

\[
\hat{\Phi}_{\gamma, M, J, p_K}^{(1)}(t, z) := \hat{\sigma}_{\gamma, M, J, p_K}(t, z) e^{\frac{i}{\sqrt{2}} \sum_{\gamma, J, p_K} \hat{\sigma}_{\gamma, M, J, p_K}(t, z)}.
\]

and the “spin-1” primary

\[
\hat{\Phi}_{\gamma, M, J, p_K}^{(1)}(t, z) := e^{\frac{i}{\sqrt{2}} \sum_{\gamma, J, p_K} \hat{\sigma}_{\gamma, M, J, p_K}(t, z)}.
\]

[compare with Eqs. (3.15a) and (3.15b) in Sec. III C 2]. The algebraic properties of these primary fields, including their OPEs and exchange algebras, are summarized in Sec. III C 2. The key information to retain from that discussion is that all non-Abelian properties of these primary operators stem from the presence of the Ising twist operator \(\delta_{\gamma, M, J, p_K}\) in the definition of the “spin-\(\frac{1}{2}\)” primary. The vertex operators from the \(u(1)_2\) sector have Abelian fusion rules.

As in Sec. III C, our strategy for building string operators will be to find appropriate nonchiral products of primary fields from which to build nonlocal string and membrane operators that characterize the topological order. We proceed with this program below.

2. “Spin-1” string and membrane operators

In direct analogy with Sec. III C, we build “spin-1” string operators acting parallel to the \(x-y\) plane of the square lattice \(\Lambda\) by taking products of the unitary operators

\[
\hat{\sigma}_{\gamma', M, J, p_K}^{(1)}(t, z) := \hat{\Phi}_{\gamma', M, J, p_K}^{(1)}(t, z) \hat{\Phi}_{\gamma, M, J, p_K}^{(1)}(t, z) \quad \text{ (5.10)}
\]

for \(\gamma, \gamma' = 1, 2\), \(J, K = A, B, C, D\), and for any \(0 < z < L_z\) (we continue to suppress the normal ordering of vertex operators). For any choice of \(\gamma, \gamma', J, K\), the operator \(\hat{\sigma}_{\gamma', M, J, p_K}^{(1)}\) fails to commute with exactly three plaquette terms in the interaction (5.1) [see Fig. 9(a)].

We refer to these plaquettes as “defective.” A calculation analogous to Eq. (3.28) shows that acting with an additional operator \(\hat{\sigma}_{\gamma', M, J, p_K}^{(1)}\) in a neighboring wire heals some of these defective plaquettes, while creating others. By repeating this calculation, one can verify that it is possible to separate pairs of defective plaquettes arbitrarily far from one another without creating any additional defective plaquettes [see Fig. 9(b)]. Thus, we can interpret each pair of defective plaquettes as a pointlike “spin-1” excitation. The fact that one can separate these excitations arbitrarily far from one another without creating additional defective plaquettes indicates that they are deconfined. (For a more detailed discussion of deconfinement and the energetics of these excitations, see Sec. V B 4.)

For the purposes of building string operators, we need to identify a set of three orthogonal cycles of the three-torus. The cycle winding around the torus in the \(z\)-direction consists of traversing the torus along a single wire, as was done in the 2D case. For cycles parallel to the \(x-y\) plane, we need to choose a canonical set of paths
The "spin-1" string operators acting along the paths $\mathcal{P}_x$ and $\mathcal{P}_y$ are given by

$$\hat{\Gamma}_a^{(1)}(t, z) := \prod_{(\gamma, \gamma', J, J')} \hat{O}_{\gamma \gamma', J, J'}^{(1)}(t, z), \quad (5.11)$$

where $\hat{a} = \hat{x}, \hat{y}$. A calculation analogous to Eqs. (3.28) in Sec. III C shows that both families of string operators commute with the interaction (5.1) when periodic boundary conditions are imposed in all directions. Like their 2D counterpart defined in Eq. (3.29), these string operators can be interpreted as creating a pair of pointlike "spin-1" excitations, transporting them around either the $\mathcal{P}_x$- or $\mathcal{P}_y$-cycle of the three-torus, and then annihilating them. Also like their 2D counterpart, these string operators can be rewritten as products of primary operators at different $z$ points, similar to Eq. (3.30) in the 2D case. We
can thus view these “spin-1” excitations as being free to move in all three spatial dimensions.

To construct a “spin-1” string operator running along the $z$-direction, we define the unitary operator

$$
\hat{O}_{\gamma,M,J,p,K}^{(1)}(t,z_1,z_2) := \hat{\Phi}_{\gamma,M,J,p,K}^{(1)}(t,z_2) \hat{\Phi}_{\gamma,M,J,p,K}^{(1)}(t,z_1)
$$

$$
= \exp \left( -i \frac{1}{\sqrt{2}} \int_{z_1}^{z_2} dz \hat{\phi}_{\gamma,M,J,p,K}(t,z) \right). \quad (5.12)
$$

This definition is in direct parallel with Eq. (3.31a) in the 2D case. Similar to the 2D case, one can verify by direct calculation that the nonlocal operator

$$
\hat{\Gamma}_{z,\gamma,M,J,p,K}^{(1)}(t) := \hat{O}_{\gamma,M,J,p,K}^{(1)}(t,0,L_z) \quad (5.13)
$$

commutes with the interaction (5.1). This completes the definitions of the “spin-1” string operators we will consider.

In 3D, we can also build nontrivial membrane operators whose algebra with the string operators can indicate the presence of topological order. Our general strategy for defining membrane operators parallel to the $x$-$z$ and $y$-$z$ planes is to apply $z$-string operators of the form (5.13) along paths in the families $[\mathcal{P}_{\hat{x}}]$ and $[\mathcal{P}_{\hat{y}}]$, respectively. This yields the “spin-1” membrane operators

$$
\hat{\Sigma}_{\hat{a}}^{(1)}(t)(z) := \prod_{(\gamma,\gamma',J,p,K) \in \mathcal{P}_{\hat{a}}} \hat{O}_{\gamma',R,M,J,p,K}^{(1)}(t,0,L_z), \quad (5.14)
$$

where $\hat{a} = \hat{x}, \hat{y}$ and $\hat{a}_\perp$ is defined such that $\hat{x}_\perp = \hat{y}$ and $\hat{y}_\perp = \hat{x}$. The choice of chirality $M = R$ above is arbitrary. Here, we have adopted a convention whereby any membrane carries the label of a path normal to the membrane. A depiction of one such membrane is shown in Fig. 12. Constructing a membrane parallel to the $x$-$y$ plane is achieved by simply acting with the bilinears (5.10) in all wires according to the prescription

$$
\hat{\Sigma}_{\hat{z}}^{(1)}(t,z) := \prod_{p,A \subset A} \prod_{J = A}^{D} \hat{O}_{11,J,p_A}^{(1)}(t,z) \hat{O}_{22,J,p_A}^{(1)}(t,z). \quad (5.15)
$$

A partial implementation of any of these membrane operators, obtained by restricting its support to an open surface, leaves a line of defective plaquettes along the boundary of the surface. Thus, we can interpret the
membranes on which these operators act as worldsheets of stringlike excitations of the coupled-wire theory.

Using the definitions (5.11), (5.13), (5.14), and (5.15), and the equal-time algebra (5.3b), one can show that the “spin-1” string operators \( \Gamma_{\hat{a}} \) and membrane operators \( \Sigma^{(1)}_{\hat{b}} \) commute with one another for all \( \hat{a}, \hat{b} = \hat{x}, \hat{y}, \hat{z} \). Moreover, one verifies that this equal-time algebra is independent of the details of how one defines the paths and surfaces on which the string and membranes act, i.e., deforming the path along which a string operator acts, or the surface on which a membrane operator acts, has no effect on the equal-time algebra as long as these deformations leave the intersection of the path and surface intact.

3. “Spin-\( \frac{1}{2} \)” string and membrane operators

String operators corresponding to the “spin-\( \frac{1}{2} \)” primary operator can be constructed as follows. Similarly to the case of “spin-1” strings, “spin-\( \frac{1}{2} \)” strings acting along paths in the \( x \)-\( y \) plane are built out of the bilinear operators

\[
\hat{O}^{(\frac{1}{2})}_{\gamma', \gamma, J, p_K}(t, z) := \hat{\Phi}^{(\frac{1}{2})}_{\gamma', \gamma, J, p_K}(t, z) \hat{\Phi}^{(\frac{1}{2})}_{\gamma', \gamma, J, p_K}(t, z),
\]

(5.16)

where the operator \( \hat{\Phi}^{(\frac{1}{2})}_{\gamma', \gamma, J, p_K}(t, z) \) is defined in Eq. (5.8).

The “spin-\( \frac{1}{2} \)” string operators along the paths \( \mathcal{P}_{\hat{x}, \hat{y}} \) are then defined by

\[
\hat{\Sigma}_{\hat{a}}^{(\frac{1}{2})}(t, z) := \prod_{(\gamma', \gamma, J, p_K) \in \mathcal{P}_{\hat{a}}} \hat{O}^{(\frac{1}{2})}_{\gamma', \gamma, J, p_K}(t, z),
\]

(5.17)

where \( \hat{a} = \hat{x}, \hat{y} \). “Spin-\( \frac{1}{2} \)” string operators along the \( z \)-direction are again defined by analogy with the 2D case,

\[
\hat{\Gamma}_{\hat{z}}^{(\frac{1}{2})}(t, \epsilon) := \exp \left( -\frac{i}{2\sqrt{2}} \int_0^{L_z} \mathrm{d}z \partial_z \hat{\Sigma}_{\gamma, M, J, p_K}(t, z) \right) \times \hat{P}_1 \partial_{\gamma, M, J, p_K}(t, 0) \hat{\Sigma}_{\gamma, M, J, p_K}(t, \epsilon) \hat{P}_1,
\]

(5.18)

where the choice of \( \gamma, M, J, \) and \( p_K \) is arbitrary. The operator \( \hat{P}_1 \) above is the counterpart to the projector onto the fusion channel \( \sigma \times \sigma = 1 \) that appears in Eq. (3.34).

This definition of the operator \( \Gamma_{\hat{a}}^{(\frac{1}{2}) M, J, p_K}(t, \epsilon) \) is subject to the same caveats as its 2D analogue, which was defined in Eq. (3.44). In particular, the limit \( \epsilon \rightarrow 0 \) must be taken carefully, as discussed in footnote [92] and Appendix C. As in the 2D case, we will only take the limit \( \epsilon \rightarrow 0 \) at the end of calculations.

With these definitions, one verifies using Eq. (5.3b), that these “spin-\( \frac{1}{2} \)” string operators have the following equal-time algebra with the “spin-1” membrane operators. Any “spin-\( \frac{1}{2} \)” string operator defined along a noncontractible cycle parallel to the \( x \)-\( y \) plane anticommutes with a “spin-1” membrane operator orthogonal to the noncontractible cycle,

\[
\hat{\Gamma}_{\hat{a}}^{(\frac{1}{2})}(t, z) \hat{\Sigma}_{\hat{a}}^{(1)}(t) = -\hat{\Sigma}_{\hat{a}}^{(1)}(t) \hat{\Gamma}_{\hat{a}}^{(\frac{1}{2})}(t, z),
\]

(5.19a)

for any \( \hat{a} = \hat{x}, \hat{y} \). In contrast, any “spin-1” string operator defined along a noncontractible cycle in the \( x \)-\( y \) plane commutes with any “spin-1” membrane operator such that the noncontractible cycle and membrane are not pairwise orthogonal, i.e.,

\[
\hat{\Gamma}_{\hat{a}}^{(\frac{1}{2})}(t, z) \hat{\Sigma}_{\hat{b}}^{(1)}(t) = \hat{\Sigma}_{\hat{a}}^{(1)}(t) \hat{\Gamma}_{\hat{b}}^{(\frac{1}{2})}(t, z),
\]

(5.19b)

for \( (\hat{a}, \hat{b}) = (\hat{x}, \hat{y}) \) or \( (\hat{a}, \hat{b}) = (\hat{y}, \hat{x}) \). This equal-time algebra holds independently of local deformations of the paths and surfaces on which the string and membrane operators are defined, so long as these deformations leave the intersections of these paths and surfaces unchanged. The “spin-\( \frac{1}{2} \)” string operator acting along the \( \hat{z} \)-direction anticommutes with any “spin-1” membrane operator acting on a surface that is orthogonal to the \( \hat{z} \)-direction,

\[
\hat{\Gamma}_{\hat{z}}^{(\frac{1}{2})}(t, \epsilon) \hat{\Sigma}_{\hat{z}}^{(1)}(t, z) = -\hat{\Sigma}_{\hat{z}}^{(1)}(t, z) \hat{\Gamma}_{\hat{z}}^{(\frac{1}{2})}(t, \epsilon),
\]

(5.20a)

for any infinitesimal \( \epsilon > 0 \). The “spin-\( \frac{1}{2} \)” string operator acting along the \( \hat{z} \)-direction commutes with any \( \psi \)-membrane operator acting on a surface orthogonal to the \( x \)- or \( y \)-directions (for simplicity, we assume that the \( \hat{z} \)-string does not intersect with the \( x \)- and \( y \)-membranes),

\[
\hat{\Gamma}_{\hat{z}}^{(\frac{1}{2})}(t, \epsilon) \hat{\Sigma}_{\hat{a}}^{(1)}(t) = \hat{\Sigma}_{\hat{a}}^{(1)}(t) \hat{\Gamma}_{\hat{z}}^{(\frac{1}{2})}(t, \epsilon),
\]

(5.20b)

for any \( \hat{a} = \hat{x}, \hat{y} \) and for any infinitesimal \( \epsilon > 0 \).

Once we have constructed the “spin-\( \frac{1}{2} \)” string operators, we can also investigate the braiding statistics of pointlike particles in the coupled-wire theory. For example, the mutual statistics of “spin-\( \frac{1}{2} \)” and “spin-1” excitations can be deduced from exchange relations like

\[
\hat{\Gamma}_{\hat{y}}^{(1)}(t, z) \hat{\Gamma}_{\hat{x}}^{(\frac{1}{2})}(t, z') = \hat{\Gamma}_{\hat{x}}^{(\frac{1}{2})}(t, z') \hat{\Gamma}_{\hat{y}}^{(1)}(t, z) e^{-i \frac{\pi}{2} \text{sgn}(z-z') \epsilon + i \frac{\pi}{2} \text{sgn}(z-z')} = \hat{\Gamma}_{\hat{x}}^{(\frac{1}{2})}(t, z') \hat{\Gamma}_{\hat{y}}^{(1)}(t, z),
\]

(5.21)

where we used the equal-time algebra (5.3b), which demonstrates that “spin-1” and “spin-\( \frac{1}{2} \)” particles braid trivially in the three-dimensional model. Likewise, the self-statistics of “spin-\( \frac{1}{2} \)” excitations can be deduced from exchange
relations like

\[ \tilde{\Gamma}_{g}^{(\frac{1}{2})}(t, z) \tilde{\Gamma}_{g}^{(\frac{1}{2})}(t, z') = \tilde{\Gamma}_{g}^{(\frac{1}{2})}(t, z') \tilde{\Gamma}_{g}^{(\frac{1}{2})}(t, z) \times \begin{cases} e^{-i \frac{2\pi}{3\sigma} \text{sgn}(z-z')} e^{+i \frac{2\pi}{3\sigma} \text{sgn}(z-z')}, & \text{if } \sigma \times \sigma = \mathbb{1}, \\ e^{+i \frac{2\pi}{3\sigma} \text{sgn}(z-z')} e^{-i \frac{2\pi}{3\sigma} \text{sgn}(z-z')}, & \text{if } \sigma \times \sigma = \psi \end{cases} (5.22) \]

where we used (5.3b) and the counterparts to the equal-time algebra (3.19). The meaning of the two cases distinguished above, namely the cases \( \sigma \times \sigma = \mathbb{1} \) and \( \sigma \times \sigma = \psi \), is as follows. When two “spin-\( \frac{1}{2} \)” strings act along the noncontractible cycles \( \mathcal{P}_{x} \) and \( \mathcal{P}_{y} \), they necessarily coincide in exactly two chiral channels located astride a bond of the square lattice [see Fig. 8(b)]. Each of these chiral channels is acted upon by two \( \sigma \) operators from the \( \mathbb{Z}_2 \) sector, one from the \( \mathcal{P}_{x} \) string and one from the \( \mathcal{P}_{y} \) string. The outcome of fusing the two \( \sigma \) fields in each of the two channels is correlated. If one pair of \( \sigma \) fuses to \( \mathbb{1} \) or \( \psi \), then the other pair must fuse in this channel as well. Otherwise, extra excitations are created. The upshot of this discussion is that all pointlike particles in the three-dimensional theory have trivial braiding with one another. This fact is consistent with the fact that any deconfined point particle in three spatial dimensions must be either a fermion or a boson. As we will see below, however, there is no such restriction for the braiding of a pointlike excitation with a line-like excitation.

The logic for the construction of “spin-\( \frac{1}{2} \)” membranes parallels the logic for “spin-1” membranes. A “spin-\( \frac{1}{2} \)” membrane parallel to the \( x-y \) plane is defined by

\[ \tilde{\Sigma}_{\hat{z}}^{(\frac{1}{2})}(t, z) := \prod_{J,P_{A}} \tilde{\Theta}_{11,J,P_{A}}^{(\frac{1}{2})}(t, z) \tilde{\Theta}_{22,J,P_{A}}^{(\frac{1}{2})}(t, z), \tag{5.23} \]

while the membrane operators orthogonal to the \( x-y \) plane can be chosen to be

\[ \tilde{\Sigma}_{\hat{a}}^{(\frac{1}{2})}(t, \epsilon) := \prod_{(\gamma,J,P_{K}) \in \mathcal{P}_{a \perp}} \exp \left( -i \frac{1}{2\sqrt{2}} \int_{0}^{L_{z}} dz \partial_{z} \tilde{\Theta}_{\gamma,R,J,P_{K}}(t, z) \right) \tilde{\prod}_{1} \tilde{\sigma}_{\gamma,R,J,P_{K}}(t, 0) \tilde{\sigma}_{\gamma,R,J,P_{K}}(t, \epsilon) \tilde{\prod}_{1}, \tag{5.24} \]

for any \( \hat{a} = \hat{x}, \hat{y} \). The choice of chirality \( M = R \) is arbitrary. The definition (5.24) of the operator \( \tilde{\Sigma}_{\hat{a}}^{(\frac{1}{2})}(t, \epsilon) \) is subject to the same caveats as the definition (5.18) of the operator \( \tilde{\Gamma}_{\hat{a}}^{(\frac{1}{2})}(t, \epsilon) \). As before, we refer the reader to Footnote [92] and to Appendix C for details.

Using the algebra (5.3b) and the definitions (5.23) and (5.24), one can show that the equal-time algebra between any pair of “spin-\( \frac{1}{2} \)” membrane “spin-1-string operators is mere commutations except for the three anticommuting exceptions

\[ \tilde{\Sigma}_{\hat{a}}^{(\frac{1}{2})}(t, \epsilon) \tilde{\Gamma}_{\hat{a}}^{(\frac{1}{2})}(t, z) = -\tilde{\Gamma}_{\hat{a}}^{(\frac{1}{2})}(t, z) \tilde{\Sigma}_{\hat{a}}^{(\frac{1}{2})}(t, \epsilon), \tag{5.25a} \]

\[ \tilde{\Sigma}_{\hat{z}}^{(\frac{1}{2})}(t, z) \tilde{\Gamma}_{\hat{z}}^{(\frac{1}{2})}(t) = -\tilde{\Gamma}_{\hat{z}}^{(\frac{1}{2})}(t) \tilde{\Sigma}_{\hat{z}}^{(\frac{1}{2})}(t, z), \tag{5.25b} \]

for \( \hat{a} = \hat{x}, \hat{y} \), and for infinitesimal \( \epsilon > 0 \). Thus, any of the “spin-\( \frac{1}{2} \)” membrane operators can be interpreted as twisting the boundary conditions of a pointlike “spin-1” excitation encircling the three-torus along any noncontractible cycle orthogonal to the membrane.

Finally, we also have the algebra between “spin-\( \frac{1}{2} \)” membranes and “spin-1” membranes given by

\[ \tilde{\Sigma}_{\hat{a}}^{(1)}(t) \tilde{\Sigma}_{\hat{a}}^{(\frac{1}{2})}(t, \epsilon) = \tilde{\Sigma}_{\hat{a}}^{(\frac{1}{2})}(t, \epsilon) \tilde{\Sigma}_{\hat{a}}^{(1)}(t), \tag{5.26a} \]

for any \( \hat{a} = \hat{x}, \hat{y} \) (recall that \( \hat{x} \perp \hat{y} \)) for infinitesimal \( \epsilon > 0 \). The system-size-dependent integers \( N_{\hat{z}} \) and \( N_{\hat{y}} \) are the number of wires contained in the path \( \mathcal{P}_{\hat{x}} \) and \( \mathcal{P}_{\hat{y}} \), respectively. One can show that \( N_{\hat{x}} \) and \( N_{\hat{y}} \) are even for paths \( \mathcal{P}_{\hat{x}} \) and \( \mathcal{P}_{\hat{y}} \) that encompass the entire system, so long as the system contains an integer number of unit cells.

4. An aside on energetics

Before moving on to discuss the derivation of the topological degeneracy from the algebra of the string and membrane operators constructed in Secs. VB 2 and VB 3, it is necessary to address some aspects of the energetics of the excitations associated with these operators. We will focus first on the case of the pointlike “spin-1” excitations, and then on the case of the pointlike “spin-\( \frac{1}{2} \)” excitations.
The plaquette excitations generated by the “spin-1” operator \( \hat{O}_{\gamma\gamma',J,p_K}^{(1)} \) are solitons in a particular linear combination of scalar fields in the \( u(1)_2 \) sector. For example, the operator \( \hat{O}_{11,C,p_A}^{(1)} \) acts nontrivially on only two scalar fields,

\[
\begin{align}
\frac{1}{\sqrt{2}} \hat{O}_{11,C,p_A}^{(1)} (z') \phi_{1,R,C,p_A} (z) \hat{O}_{11,C,p_A}^{(1)} (z') &= \frac{1}{\sqrt{2}} \phi_{1,R,C,p_A} (z) - 2 \pi \Theta (z - z') + \text{const.}, \\
\frac{1}{\sqrt{2}} \hat{O}_{11,C,p_A}^{(1)} (z') \phi_{1,L,C,p_A} (z) \hat{O}_{11,C,p_A}^{(1)} (z') &= \frac{1}{\sqrt{2}} \phi_{1,L,C,p_A} (z) - 2 \pi \Theta (z - z') + \text{const.},
\end{align}
\]

where we have used the algebra (5.3b). Here, \( 2 \pi \Theta (z - z') = \pi \left[ \text{sgn} (z - z') + 1 \right] \) is the Heaviside step function that represents a sharp soliton of height \( 2 \pi \). The additive constant terms arise from the algebra in Eq. (5.3b) and are multiples of \( 2 \pi \).

The introduction of these soliton profiles costs infinite energy after the limit \( \lambda \to \infty \) of the interaction (5.1), holding the kinetic energy fixed, has been taken. This is consistent with the expectation that the theory has an infinite energy gap in this limit. Acting in neighboring wires with additional operators \( \hat{O}_{\gamma\gamma',J,p_K}^{(1)} \) in the manner of Fig. 9 leads to cancellations of soliton profiles in any linear combination of scalar fields that is affected by two successive such operators. For this reason, a string of operators \( \hat{O}_{\gamma\gamma',J,p_K}^{(1)} \) costs energy only at its end points (end wires). The property that this energy cost (possibly infinite in the limit of infinitely strong interactions relative to the kinetic energy) is localized around the two end points (end wires) of the string that justifies interpreting the associated “spin-1” excitations as being deconfined.

The case of the “spin-1/2” excitations is similar, but complicated by the appearance of operators from the \( \mathbb{Z}_2 \) sector in the operators \( \hat{O}_{\gamma\gamma',J,p_K}^{(2)} \). In this case, the \( u(1)_2 \) part of the operator \( \hat{O}_{\gamma\gamma',J,p_K}^{(2)} \) also creates a soliton, but one with height \( \pi \) rather than \( 2 \pi \). Such an object cannot, by itself, be created by a local operator. However, the combination of \( u(1)_2 \) and \( \mathbb{Z}_2 \) operators entering \( \hat{O}_{\gamma\gamma',J,p_K}^{(2)} \) is local. While an explicit calculation of the energy cost that results from acting with the operator \( \hat{O}_{\gamma\gamma',J,p_K}^{(2)} \) in a single wire for any value of the coupling \( \lambda \) is beyond the scope of this work, we expect that it will be infinite in the limit \( \lambda \to \infty \), judging from the action of this operator on the \( u(1)_2 \) sector. If this energy cost is localized around the two end points (end wires) of the string, we may then interpret the associated “spin-1/2” excitations as being deconfined. Although proving deconfinement of “spin-1/2” excitations is beyond the scope of this work because of the \( \mathbb{Z}_2 \) sector of the theory, we expect it judging from the action of \( \hat{O}_{\gamma\gamma',J,p_K}^{(2)} \) on the \( u(1)_2 \) sector.

The interaction (5.1) possesses an extensive number of symmetries. String operators acting along closed loops do not leave any solitons in the linear combinations of scalar fields that enter the interaction (5.1) and hence create no excitations. Indeed, such operators commute with the interaction (5.1). As such, they play a role similar to the one played by the local gauge symmetry of the toric code. In fact, in the coupled-wire construction of 3D Abelian topological phases carried out in Ref. [63], it was shown explicitly that such symmetries were present for a class of interactions studied there. However, a complicating difference between the interaction (5.1) and the ones studied in Ref. [63] is that the latter consisted of sums over local terms that were all pairwise commuting, whereas this is not the case for the plaquette interactions entering Eq. (5.1). Consequently, it is not possible to find a closed form for the ground states of Eq. (5.1) as was done in Ref. [63].

When the interaction (5.1) is weakly perturbed by the kinetic energy of the chiral modes in the quantum wires, i.e., when \( 0 < \lambda < \infty \), the string operators acting along closed loops like the ones depicted in Fig. 11 fail to commute with the Hamiltonian. However, because the kinetic energy is a local perturbation, it cannot lift the topological degeneracy, which we derive in Sec. V B 4 when the interaction is infinitely stronger than the kinetic energy, to any finite order in perturbation theory. A quantitative effect of weakly perturbing interaction (5.1) by the kinetic energy of the chiral modes is to render the excitation energy of open strings finite. This finite excitation energy is shared between the end points (end wires) and a string tension. However, the string tension is not strong enough to confine the excitations localized around the end points (end wires) of a sufficiently weak kinetic energy, as there is a one-to-one correspondence between the topological degeneracy and the deconfinement of excitations. Using the results of the previous sections, we now derive the topological ground-state degeneracy of the array of quantum wires coupled by the interwire interactions (5.4), for the case of \( su(2)_2 \) current-current interactions.

5. Topological degeneracy on the three-torus

Using the results of the previous sections, we now derive the topological ground-state degeneracy of the array of quantum wires coupled by the interwire interactions (5.4), for the case of \( su(2)_2 \) current-current interactions.
We assume periodic boundary conditions in $x$, $y$, and $z$, so that the array of coupled wires has the topology of a three-torus ($T^3$). The logic of our derivation of this lower bound follows closely the logic of the corresponding derivation in the two-dimensional case discussed in Sec. III C. It hinges on the exchange algebra of the following set of nonlocal operators, which is summarized in Table II. There are three nonlocal and unitary “spin-1” string operators (5.11) and (5.13), for which we use the short-hand notation

$$
\hat{\Gamma}_{\hat{x}}^{(1)}, \hat{\Gamma}_{\hat{y}}^{(1)}, \hat{\Gamma}_{\hat{z}}^{(1)},
$$

respectively. There are three nonlocal and nonunitary “spin-$\frac{1}{2}$” string operators (5.17) and (5.18) for which we use the short-hand notation

$$
\hat{\Gamma}_{\hat{x}}^{(2)}, \hat{\Gamma}_{\hat{y}}^{(2)}, \hat{\Gamma}_{\hat{z}}^{(2)},
$$

respectively. There are three nonlocal and unitary “spin-1” membrane operators (5.14) and (5.15) for which we use the short-hand notation

$$
\hat{\Sigma}_{\hat{x}}^{(1)}, \hat{\Sigma}_{\hat{y}}^{(1)}, \hat{\Sigma}_{\hat{z}}^{(1)},
$$

respectively. There are three nonlocal and nonunitary “spin-$\frac{1}{2}$” membrane operators, defined in Eqs. (5.23) and (5.24), for which we use the short-hand notation

$$
\hat{\Sigma}_{\hat{x}}^{(2)}, \hat{\Sigma}_{\hat{y}}^{(2)}, \hat{\Sigma}_{\hat{z}}^{(2)},
$$

respectively. Each of these twelve nonlocal operators commutes with the plaquette interaction (5.1), except for the three operators $\hat{\Gamma}_{\hat{x}}^{(2)}$, $\hat{\Sigma}_{\hat{x}}^{(2)}$, and $\hat{\Sigma}_{\hat{y}}^{(2)}$, defined in Eqs. (5.18) and (5.24), respectively. These three operators are regularized by the parameter $\epsilon$, and therefore must be treated in a manner similar to the operator $\hat{\Gamma}_{\hat{2},M,y}^{(2)}(\epsilon)$ in the 2D case. Nevertheless, an analysis along the lines of the one presented in Appendix C for the 2D case reveals that these three $\epsilon$-regularized operators can be used to define states in the ground-state manifold of the interaction (5.1). We will elaborate on this statement below.

The derivation of the topological degeneracy begins by observing that the “spin-1” string and “spin-1” membrane operators appearing in Eqs. (5.28a) and (5.28c) all commute with one another. Thus, we can choose a many-body ground state

$$
|\Omega\rangle \equiv |\mathbb{I}\rangle
$$

that is a simultaneous eigenstate of all “spin-1” string and “spin-1” membrane operators, namely

$$
\hat{\Gamma}_{\hat{x}}^{(1)} |\mathbb{I}\rangle = \omega_{\hat{x}}^{\mathbb{G}} |\mathbb{I}\rangle, \hat{\Gamma}_{\hat{y}}^{(1)} |\mathbb{I}\rangle = \omega_{\hat{y}}^{\mathbb{G}} |\mathbb{I}\rangle, \hat{\Gamma}_{\hat{z}}^{(1)} |\mathbb{I}\rangle = \omega_{\hat{z}}^{\mathbb{G}} |\mathbb{I}\rangle,
$$

on the one hand, and

$$
\hat{\Sigma}_{\hat{x}}^{(1)} |\mathbb{I}\rangle = \omega_{\hat{x}}^{\mathbb{S}} |\mathbb{I}\rangle, \hat{\Sigma}_{\hat{y}}^{(1)} |\mathbb{I}\rangle = \omega_{\hat{y}}^{\mathbb{S}} |\mathbb{I}\rangle, \hat{\Sigma}_{\hat{z}}^{(1)} |\mathbb{I}\rangle = \omega_{\hat{z}}^{\mathbb{S}} |\mathbb{I}\rangle,
$$

on the other hand, must hold for the nonvanishing eigenvalues

$$
\omega_{\hat{x}}^{\mathbb{G}}, \omega_{\hat{y}}^{\mathbb{G}}, \omega_{\hat{z}}^{\mathbb{G}}, \omega_{\hat{x}}^{\mathbb{S}}, \omega_{\hat{y}}^{\mathbb{S}}, \omega_{\hat{z}}^{\mathbb{S}} \in U(1).
$$

Not all choices of $|\Omega\rangle$ are equivalent. Similarly to the argument presented in Appendix C for the 2D case, depending on the topological sector in which the state $|\Omega\rangle$ resides, it is possible for the state created by acting upon $|\Omega\rangle$ with certain combinations of the nonlocal, nonunitary operators $\hat{\Gamma}_{\hat{x},M,y}^{(2)}$ and $\hat{\Sigma}_{\hat{x},y}^{(2)}$ to have norm zero or infinity. In other words, not all combinations of the eigenvalues (5.30c) label states in the ground-state manifold.

Next, we define a set of many-body states obtained by acting on the state $|\mathbb{I}\rangle$ with the “spin-$\frac{1}{2}$” string and “spin-$\frac{1}{2}$” membrane operators from Eqs. (5.28b) and (5.28d), respectively. There are

$$
4^3 = 64
$$

![Table II](https://example.com/table.png)

Summary of the algebra of the string and membrane operators (5.28). Entries corresponding to a pair of operators that commute are labeled with a +. Entries corresponding to a pair of operators that anticommute are labeled with a −. Entries corresponding to a pair of operators that neither commute nor anticommute are labeled with a X. (Compare with Table I.) The operator algebra contained in the left 6 × 6 subblock of the table is derived in Secs. V B 2 and V B 3. The operator algebra contained in the right 6 × 6 subblock of the table is derived in Sec. V B 5.
TABLE III. The 20 orthogonal states of the form (5.32) that span the ground-state manifold of the \( su(2) \) coupled-wire theory in \((3 + 1)\)-dimensional spacetime, as well as the eigenvalues of these states under the “spin-1" string and membrane operators. The states are labeled according to the notation \( |O\rangle = \tilde{O} |\Omega\rangle \). The 6-tuple of signs \( \pm \) indicating the eigenvalues of a state \( |\tilde{O}\rangle \) is obtained by evaluating the list of matrix elements \( \langle \tilde{O} | (\tilde{\Gamma}_x^{(2)}) (\tilde{\Gamma}_y^{(2)}) (\tilde{\Gamma}_z^{(2)}) |\Omega\rangle \) and dividing each element in the list by its magnitude.

states, since for any choice of a noncontractible cycle of the three-torus \((\tilde{x}, \tilde{y}, \text{or} \tilde{z})\), there are four nonlocal operators we can apply to the state \(|\Omega\rangle \). For example, fixing the noncontractible cycle \(\tilde{x}\), we can insert the identity operator \( I \), the “spin-\( \frac{1}{2} \)” string operator \( \hat{\Gamma}^{(\frac{1}{2})}_x \), the “spin-\( \frac{1}{2} \)” membrane operator \( \hat{\Sigma}^{(\frac{1}{2})}_x \), or the product \( \hat{\Gamma}^{(\frac{1}{2})}_x \hat{\Sigma}^{(\frac{1}{2})}_x \).

Alternatively, we can label all 2\(^6\) = 64 states according to the rule

\[
\langle \tilde{O} | (\hat{\Sigma}^{(\frac{1}{2})}_x) (\hat{\Sigma}^{(\frac{1}{2})}_y) (\hat{\Sigma}^{(\frac{1}{2})}_z) (\hat{\Gamma}^{(\frac{1}{2})}_x) (\hat{\Gamma}^{(\frac{1}{2})}_y) (\hat{\Gamma}^{(\frac{1}{2})}_z) |\Omega\rangle = 0.
\]

where \( \sigma^\Gamma_x, \sigma^\Sigma_x, \sigma^\Sigma_y, \sigma^\Sigma_z = 0, 1 \). Any of the above states involving one or more of the \( \epsilon \)-regularized operators \( \hat{\Gamma}^{(\frac{1}{2})}_x, \hat{\Sigma}^{(\frac{1}{2})}_x, \hat{\Sigma}^{(\frac{1}{2})}_y \), and \( \hat{\Sigma}^{(\frac{1}{2})}_y \) carries an implicit limit \( \epsilon \to 0 \). As in the 2D case (see Footnote [92] and Appendix C), this limit should be taken after forming the product of the relevant string and/or membrane operators.

Not all states of the form (5.32) belong to the ground state manifold, as we are going to show explicitly. The counting based on Eq. (5.31) is “naive” because it is based purely on the number of noncontractible cycles of the manifold on which the theory is defined, and on the number of string or membrane operators that can act along each noncontractible cycle. In the following, we are going to show that a majority of the states (5.32) must be excluded from the ground-state manifold, on grounds similar to the reason for which we had to exclude the “extra” state \( |\hat{\Gamma}^{(\frac{1}{2})}_1 \hat{\Gamma}^{(\frac{1}{2})}_2 \rangle \) that appeared in the two-dimensional example discussed in Sec. III C 3. In the end, there will be a total of

\[
D_{\text{g3}} = 20
\]

states that survive projection into the ground-state manifold. These states are listed, along with their eigenvalues under the “spin-1" string and membrane operators, in Table III. We emphasize that all of these 20 ground

states are mutually orthogonal, as they are simultaneous eigenstates of the unitary “spin-1" string and membrane operators with different eigenvalues.

The 20 states in Table III have in common the fact that they are created by acting on the state \( |\Omega\rangle \) with a product of commuting operators (these correspond to entries marked with a + in the right 6 \times 6 block of Table II). [98] Conversely, the 64 – 20 = 44 excluded states not appearing in Table III are created by acting on the state \( |\Omega\rangle \) with a product of noncommuting operators (these correspond to entries marked with a X in the right 6 \times 6 block of Table II). In the two-dimensional case studied in Sec. III C 3, it was precisely the noncommutativity of the string operators \( \hat{\Gamma}^{(\frac{1}{2})}_1 \) and \( \hat{\Gamma}^{(\frac{1}{2})}_2 \) that led to the exclusion of the state \( |\hat{\Gamma}^{(\frac{1}{2})}_1 \hat{\Gamma}^{(\frac{1}{2})}_2 \rangle \) from the ground-state manifold. In the proof below, we follow this “algebraic" approach, identifying all noncommuting pairs of “spin-\( \frac{1}{2} \)” string and “spin-\( \frac{1}{2} \)” membrane operators from Eqs. (5.28b) and (5.28d), respectively. The noncommuting “spin-\( \frac{1}{2} \)” string-membrane pairs consist of strings and membranes that are perpendicular to one another, intersecting in a point. The noncommuting “spin-\( \frac{1}{2} \)” membrane-membrane pairs consist of perpendicular membranes, whose intersection is a line. Whenever a noncommuting pair of operators acts on one of the states
in Table III, we will show that the resulting state must be excluded from the ground-state manifold. (A complementary “analytic” proof that these states must be excluded, along the lines of Appendix C, could also be undertaken, but we do not do this here.)

We now proceed with the proof. Of key importance is the projection operator

$$\hat{P}_{GSM} := \mathcal{N}^{-1}_1 \hat{1} + \mathcal{N}^{-1}_{\frac{1}{2}} \hat{1}_{\frac{1}{2}} \hat{g}_{\frac{1}{2}} + \mathcal{N}^{-1}_{\frac{1}{2}} \hat{g}_{\frac{1}{2}} \hat{g}_{\frac{1}{2}} + \mathcal{N}^{-1}_{\frac{1}{2}} \hat{g}_{\frac{1}{2}} \hat{g}_{\frac{1}{2}} + \mathcal{N}^{-1}_{\frac{1}{2}} \hat{g}_{\frac{1}{2}} \hat{g}_{\frac{1}{2}} + \ldots$$

(3.54)

onto the ground state manifold, c.f. Eq. (3.55). Here, \(\mathcal{N}_0\) is the squared norm of the state \(\hat{O}\), and \(\ldots\) is a sum over the remaining elements of the orthonormal basis of the ground state manifold, including the states listed in Table III. By construction, \(\hat{P}_{GSM}\) leaves any state in Table III invariant, and, being a projector, satisfies

$$\hat{P}_{GSM}^2 = \hat{P}_{GSM}.$$

(3.55)

Furthermore, the projector \(\hat{P}_{GSM}\) satisfies

$$\hat{P}_{GSM} \hat{O} \hat{P}_{GSM} = 0$$

(3.56)

for any operator \(\hat{O}\) whose action on any of the states in Table III produces an excited state. To prove that the 44 states in question must be excluded from the ground state manifold, we will show for two particular classes of operators \(\hat{O}\) that Eq. (3.56) holds. This will turn out to be sufficient to exclude the offending states.

The first class of operators arises when we consider products of perpendicular “spin-\(\frac{1}{2}\)” strings and “spin-\(\frac{1}{2}\)” membranes. This includes the three operators

$$\hat{\Sigma}_{\frac{1}{2}} \hat{g}_{\frac{1}{2}} \hat{g}_{\frac{1}{2}}, \quad \hat{\Sigma}_{\frac{1}{2}} \hat{g}_{\frac{1}{2}} \hat{g}_{\frac{1}{2}}, \quad \hat{\Sigma}_{\frac{1}{2}} \hat{g}_{\frac{1}{2}} \hat{g}_{\frac{1}{2}},$$

(5.37a)

as well as products of the form

$$\hat{\Sigma}_{\frac{1}{2}} \hat{g}_{\frac{1}{2}} \hat{g}_{\frac{1}{2}} \hat{O},$$

(5.37b)

for \(\hat{\Sigma}_{\frac{1}{2}} \hat{g}_{\frac{1}{2}} \hat{g}_{\frac{1}{2}} \hat{O}\) and any string or membrane operator \(\hat{O}\) that commutes with \(\hat{\Sigma}_{\frac{1}{2}} \hat{g}_{\frac{1}{2}} \hat{g}_{\frac{1}{2}} \hat{O}\). The operators in Eq. (5.37a) share the common trait that the domain of intersection of the string and membrane operators contains three \(\hat{\sigma}\) operators per chiral channel. The exchange algebra relevant to this case was computed in Sec. III C 3. By small variations on the calculation presented in Eqs. (3.63) and (3.64), one verifies the relations

$$\hat{\Sigma}_{\frac{1}{2}} \hat{g}_{\frac{1}{2}} \hat{g}_{\frac{1}{2}} = e^{+i \hat{\pi}} \hat{\Sigma}_{\frac{1}{2}} \hat{g}_{\frac{1}{2}} \hat{g}_{\frac{1}{2}},$$

(5.38a)

$$\hat{g}_{\frac{1}{2}} \hat{g}_{\frac{1}{2}} \hat{g}_{\frac{1}{2}} = e^{+i \hat{\pi}} \hat{g}_{\frac{1}{2}} \hat{g}_{\frac{1}{2}} \hat{g}_{\frac{1}{2}},$$

(5.38b)

$$\hat{g}_{\frac{1}{2}} \hat{g}_{\frac{1}{2}} \hat{g}_{\frac{1}{2}} = e^{+i \hat{\pi}} \hat{g}_{\frac{1}{2}} \hat{g}_{\frac{1}{2}} \hat{g}_{\frac{1}{2}},$$

(5.38c)

where the operators \(\hat{\Sigma}_{\frac{1}{2}} \hat{g}_{\frac{1}{2}} \hat{g}_{\frac{1}{2}} \hat{O}\) and \(\hat{g}_{\frac{1}{2}} \hat{g}_{\frac{1}{2}} \hat{g}_{\frac{1}{2}} \hat{O}\) are defined in the same way as the operator \(\hat{\Sigma}_{\frac{1}{2}} \hat{g}_{\frac{1}{2}} \hat{g}_{\frac{1}{2}} \hat{O}\).
in the limit $\epsilon \to 0$, for $\hat{a} = \hat{x}, \hat{y}$, and for $\hat{x}_\perp = \hat{y}$. The system-size-dependent integers $N_{\hat{x}}$ and $N_{\hat{y}}$ are the number of wires contained in the path $\mathcal{P}_{\hat{x}}$ and $\mathcal{P}_{\hat{y}}$, respectively. The operators $\tilde{\Sigma}_{\hat{x}}^{(4)}$ and $\tilde{\Sigma}_{\hat{y}}^{(4)}$ are again defined by direct analogy with the operator $\tilde{\Gamma}_{\hat{z}}^{(4)}$ appearing in Eq. (3.64), i.e., by replacing any appearance of $\tilde{P}_{\hat{1}} \tilde{\sigma}_{\gamma,\beta,J,PK}(0) \tilde{\sigma}_{\gamma,\beta,J,PK}(\epsilon) \tilde{P}_{\hat{1}}$ with $\tilde{P}_{\hat{y}} \tilde{\sigma}_{\gamma,\beta,J,PK}(0) \tilde{\sigma}_{\gamma,\beta,J,PK}(\epsilon) \tilde{P}_{\hat{y}}$. The reason for using the primes here is to distinguish these operators from $\tilde{\Sigma}_{\hat{x}}^{(4)}$ and $\tilde{\Sigma}_{\hat{y}}^{(4)}$, where only some of the appearances of $\tilde{P}_{\hat{1}} \tilde{\sigma}_{\gamma,\beta,J,PK}(0) \tilde{\sigma}_{\gamma,\beta,J,PK}(\epsilon) \tilde{P}_{\hat{1}}$ are replaced by $\tilde{P}_{\hat{y}} \tilde{\sigma}_{\gamma,\beta,J,PK}(0) \tilde{\sigma}_{\gamma,\beta,J,PK}(\epsilon) \tilde{P}_{\hat{y}}$. Regardless of these slight differences in definition, the operators $\tilde{\Sigma}_{\hat{x}}^{(4)}$ and $\tilde{\Sigma}_{\hat{y}}^{(4)}$ create excited states when acting on the vacuum $|\Omega\rangle$. Consequently, we have

$$\tilde{P}_{\text{GSM}} \tilde{\Sigma}_{\hat{a}}^{(4)} \tilde{P}_{\text{GSM}} = 0,$$

in the limit $\epsilon \to 0$ for $\hat{a} = \hat{x}, \hat{y}$. The operator $\tilde{\Sigma}_{\hat{x}}^{(4)} \tilde{\Sigma}_{\hat{y}}^{(4)}$, which involves four $\tilde{\sigma}$ operators in the same chiral channel contained in the intersection of the two membranes, can be treated similarly. The exchange algebra of the membrane operators $\tilde{\Sigma}_{\hat{x}}^{(4)}$ and $\tilde{\Sigma}_{\hat{y}}^{(4)}$ can be determined by considering the diagram

$$e^{i\frac{\pi}{4}}.$$

The algebraic interpretation of this diagrammatic statement is

$$\tilde{\Sigma}_{\hat{x}}^{(4)} \tilde{\Sigma}_{\hat{y}}^{(4)} = e^{i\frac{\pi}{4}} \tilde{\Sigma}_{\hat{y}}^{(4)} \tilde{\Sigma}_{\hat{x}}^{(4)}.$$
in the limit $\epsilon \to 0$, where the operators $\hat{\Sigma}_{\hat{a}}^{(2)}$ and $\hat{\Sigma}_{\hat{a}}^{(2)}$ also appear in Eqs. (5.38). Explicitly, we have

$$
\hat{\Sigma}_{\hat{a}}^{(2)} := \prod_{(\gamma, J, P_K) \in \mathcal{P}_a \setminus (\mathcal{P}_a \cap \mathcal{P}_a)} \exp\left(-i \frac{1}{2\sqrt{2}} \int_0^{L_z} dz \partial_z \hat{\phi}_{\gamma, R, J, P_K}(t, z) \hat{P}_1 \hat{\sigma}_{\gamma, R, J, P_K}(0) \hat{\sigma}_{\gamma, R, J, P_K}(\epsilon) \hat{P}_1 \right)
\times \prod_{(\gamma, J, P_K) \in \mathcal{P}_a \cap \mathcal{P}_a} \exp\left(-i \frac{1}{2\sqrt{2}} \int_0^{L_z} dz \partial_z \hat{\phi}_{\gamma, R, J, P_K}(t, z) \hat{P}_1 \hat{\sigma}_{\gamma, R, J, P_K}(0) \hat{\sigma}_{\gamma, R, J, P_K}(\epsilon) \hat{P}_1 \right)
\sim \left( \prod_{(\gamma, J, P_K) \in \mathcal{P}_a \setminus (\mathcal{P}_a \cap \mathcal{P}_a)} \exp\left(-i \frac{1}{2\sqrt{2}} \int_0^{L_z} dz \partial_z \hat{\phi}_{\gamma, R, J, P_K}(t, z) \hat{P}_1 \hat{\sigma}_{\gamma, R, J, P_K}(0) \hat{\sigma}_{\gamma, R, J, P_K}(\epsilon) \hat{P}_1 \right)
\times \prod_{(\gamma, J, P_K) \in \mathcal{P}_a \cap \mathcal{P}_a} \exp\left(-i \frac{1}{2\sqrt{2}} \int_0^{L_z} dz \partial_z \hat{\phi}_{\gamma, R, J, P_K}(t, z) \hat{P}_1 \hat{\sigma}_{\gamma, R, J, P_K}(0) \hat{\sigma}_{\gamma, R, J, P_K}(\epsilon) \hat{P}_1 \right) \right) \psi_{\gamma, R, J, P_K}(0) + \cdots,
$$

for $\hat{a} = \hat{x}, \hat{y}$ and for infinitesimal $\epsilon > 0$ (recall that $\hat{x}_\perp = \hat{y}$ and $\hat{y}_\perp = \hat{x}$), where in the second line we have performed the OPE in the wires where the two membranes intersect. Thus, the chiral channel in which the two membranes intersect [see Fig. 8(b)] contains a pair of fermion excitations due to the product $\hat{\psi}_{\gamma, R, J, P_K}(0) \hat{\psi}_{\gamma, R, J, P_K}(0)$. Note that both fermion operators are evaluated at $z = 0$ purely due to an (arbitrary) choice made in the definitions (5.24) of the “spin-$\frac{1}{2}$” membrane operators. All of our algebraic results would still hold if we had replaced $z = 0 \to z = z_\perp + \epsilon$ and $\epsilon \to z = z_\perp + \epsilon$, say. Hence, the two fermion excitations in the product $\hat{\psi}_{\gamma, R, J, P_K}(0) \hat{\psi}_{\gamma, R, J, P_K}(0)$ can actually be separated by arbitrarily large distances along the $z$-axis. Consequently, we have

$$\hat{P}_{\text{GSM}} \hat{\Sigma}_{\hat{a}}^{(2)} \hat{P}_{\text{GSM}} = e^{+i \frac{\pi}{2}} \hat{P}_{\text{GSM}} \hat{\Sigma}_{\hat{a}}^{(2)} \hat{P}_{\text{GSM}} = 0,
$$

in the limit $\epsilon \to 0$. Using Eqs. (5.41), (5.45), (5.42), and (5.47), one can show that the seven states listed in Table V are eliminated from the ground-state manifold.

Finally, the 44–24–7 = 13 remaining “extra” states of the form (5.32) can also be eliminated using appropriate combinations of Eqs. (5.38), (5.39), (5.41), (5.45), (5.42), and (5.47). These states are listed in Table VI. In all cases, the reason for elimination is the same. Each state is created by acting on one of the states in Table III with an operator that creates an excess of fermion excitations.

To summarize, we have shown that of the 36 = 64 states labeled by the eigenvalues of the “spin-1” string or “spin-1” membrane operators, only the 20 listed in Table III truly reside in the ground-state manifold once the exchange algebra of the “spin-$\frac{1}{2}$” string and “spin-$\frac{1}{2}$” membrane operators is taken into account. This exchange algebra is highly nontrivial, because reordering a product of “spin-$\frac{1}{2}$” string and/or “spin-$\frac{1}{2}$” membrane operators not only produces simple multiplicative phase factors, but enacts nontrivial unitary operations within the space spanned by the operator products. As in the two-dimensional case discussed in Sec. III C 3, this reduction of the number of states in the ground-state manifold from the naive value lies at the heart of the distinction between Abelian and non-Abelian topological states of matter.

VI. SURFACE THEORY OF THE 3D NON-ABELIAN PHASE

Let us now remove the periodic boundary conditions imposed in the previous section and replace them with boundary conditions that are open along the $x$-direction and periodic along the $y$-direction. The bulk of the coupled-wire theory in $(3+1)$-dimensional spacetime then possesses the “time-reversal” symmetry $T_{\text{eff}, \hat{x}}$ defined in Eq. (4.3b). [The operation $T_{\text{eff}, \hat{x}}$ defined in Eq. (4.3a) is not a symmetry anymore under open boundary conditions along the $x$-direction.] In this case, the plaquette interaction (5.1) is then not sufficient to gap all gapless $su(2)_k$ modes; even if this interaction succeeds in gapping the bulk, there must remain gapless modes that are confined to the surfaces at $x = 0$ and $x = L_x$.

In this section, we investigate the fate of these gapless surface modes when they are coupled by marginally relevant current-current interactions. From now on, we shall only consider the surface at $x = 0$. 
Eigenvalues | Modes on the surface represent the interactions $(5.38)$ in the $x$-direction, while periodic boundary conditions are imposed along the $y$- and $z$-directions, the plaquette interaction $(5.1)$ leaves gapless modes on the surfaces at $x = 0$ and $x = L_x$. The purple bonds connecting chiral modes on the surface represent the interactions $(6.2a)$ in the $SU(2)$-symmetric limit $(6.2b)$. When $\lambda = \lambda'$, the symmetry $T_{eff, g}$ is present.

The surface at $x = 0$ supports gapless modes that can be represented by a quadratic form for the currents that generate the copy $\gamma = 1$ of the $su(2)_k$ spin-1 affine Lie algebra with the M-moving currents $J^a_{\gamma=1,M,y}$ (the label $\gamma = 2$ applies to the surface $x = L_x$, see Fig. 13). From now on, we will drop the explicit reference to the label $\gamma = 1$. Hence, the Hamiltonian density for the gapless modes on the surface at $x = 0$ is the linear combination

$$H_{x=0} := 2\pi \left(T_{L,A,x=0}[su(2)_k] + T_{R,B,x=0}[su(2)_k]\right),$$

where

$$T_{M,J,x=0}[su(2)_k] = \frac{1}{2 + k} \sum_{y=0}^{L_y} \sum_{a=1}^{3} J^a_{M,J,y} J^a_{M,J,y}$$

is the energy-momentum tensor for the M-moving mode on sublattice $J = A, B$. Here, the priming of the sum over $y$ indicates that only even wires are to be summed over. Summing only over even $y$ and over $J = A, B$ accounts for the “dangling” gapless modes on the $x = 0$ surface that do not couple to any neighbors via the couplings depicted in Fig. 6 when open boundary conditions are imposed in the $x$-direction. We assume that $L_y$ is odd, so that the total number of wires (i.e., $L_y + 1$) is even.

The surface theory at $x = 0$, whose energy-momentum tensor has the chiral components $(6.1)$, can be viewed as a conformal field theory in $(1 + 1)$-dimensional spacetime with an extensive central charge. We would like to decrease this central charge to a finite number in the thermodynamic limit ($L_y \to \infty$). To this end, we perturb the gapless theory $(6.1)$ with the interactions

$$\mathcal{L}_{bs,x=0} := - \sum_{y=0}^{L_y} \sum_{a=1}^{3} (\lambda^a J^a_{R,y} J^a_{L,y+1} + \lambda'^a J^a_{L,y} J^a_{R,y+1} + J^a_{R,y} J^a_{L,y+1}).$$

(6.2a)

To investigate the nature of the surface more closely, we allow the possibility that this surface interaction breaks explicitly the $SU(2)$ symmetry. The choices

$$\lambda \equiv \lambda^a, \quad \lambda' \equiv \lambda'^a, \quad a = 1, 2, 3,$$

(6.2b)

restore the explicit $SU(2)$ symmetry. These couplings are depicted in Fig. 13.

For the isotropic point $(6.2b)$, it is readily shown that there are two gapped phases, one for $\lambda > \lambda' \geq 0$ and one for $0 \leq \lambda < \lambda'$, that are related to one another by the “time-reversal” symmetry $T_{eff, g}$ defined in Eq. (4.3b). Indeed, when $\lambda' = 0$ and $\lambda > 0$ (or vice versa) the interactions $(6.2a)$ are marginally relevant, flowing to strong coupling and opening a gap, as they do in the bulk. Furthermore, if we define a “magnetic” domain wall at $y = 0$

| State | Eigenvalues |
|-------|-------------|
| $|\Sigma^x_z|\Sigma^y_z|\Sigma^z_x\Gamma^x_z\rangle$ | $(-, -, -, +, +)$ |
| $|\Sigma^x_y|\Sigma^y_y|\Sigma^z_x\Gamma^y_z\rangle$ | $(-, -, -, +, +)$ |
| $|\Sigma^x_z|\Sigma^y_y|\Sigma^z_y\Gamma^z_x\rangle$ | $(-, -, -, +, +)$ |
| $|\Sigma^x_y|\Sigma^y_y|\Sigma^z_y\Gamma^z_x\rangle$ | $(-, -, -, +, +)$ |

TABLE VI. The 13 orthogonal states of the form $(5.32)$ that are excluded based on appropriate combinations of Eqs. $(5.38)$, $(5.39)$, $(5.41)$, $(5.45)$, $(5.42)$, and $(5.47)$, as well as the eigenvalues of these states under the “spin-1” string and membrane operators. The notation for states and eigenvalues is as in Table III.
by allowing \( \lambda \) and \( \lambda' \) to acquire the \( y \)-dependent profiles

\[
\lambda_y = \begin{cases} 
\lambda_{\infty} > 0, & \text{if } y < 0, \\
0, & \text{otherwise}, 
\end{cases} \quad (6.3a)
\]

and

\[
\lambda'_y = \begin{cases} 
0, & \text{if } y < 0, \\
\lambda'_{\infty} > 0, & \text{otherwise}, 
\end{cases} \quad (6.3b)
\]

respectively, one finds that a single chiral \( su(2)_k \) current is localized at the domain wall. This is reminiscent of the surface physics of the usual 3D TI, where a domain wall between different TRS-broken regions on the surface binds a chiral \( u(1)_1 \) current (i.e., a chiral Dirac fermion mode). In the present setting, the role of TRS is played by the non-onsite symmetry operation \( T_{\text{eff},g} \). This non-onsite implementation of an effective TRS is common to other coupled-wire construction. For example, counterparts to such an effective nonlocal TRS can be found in the coupled-wire constructions presented in Refs. [73] and [74].

The remainder of this section is devoted to elucidating the nature of the surface theory for the \( T_{\text{eff},g} \)-symmetric (but not necessarily \( SU(2) \)-symmetric) case

\[
\lambda^a = \lambda'^a, \quad a = 1, 2, 3. \quad (6.4)
\]

First, we present a one-loop renormalization group (RG) analysis, valid for small magnitudes of \( \lambda^a \) and \( \lambda'^a \) with \( a = 1, 2, 3 \). This analysis sheds light on the phase diagram of the surface, particularly on the response of the surface theory to \( SU(2) \)-breaking perturbations. Second, we present a mean-field analysis of the surface theory for the case \( k = 2 \). This analysis demonstrates that the point \( \lambda = \lambda' > 0 \), a strongly interacting quantum field theory when expressed in terms of the original fermionic modes, is a continuous quantum critical point that can be described by two noninteracting modes. The first mode is a gapless complex-valued fermion realizing a single Dirac cone in the low-energy limit. The second mode is a gapless real-valued fermion realizing a single Majorana cone in the low-energy limit. These low-energy modes are surface states of the \( su(2)_2 \) coupled-wire theory that are protected by the symmetry \( T_{\text{eff},g} \).

## A. One-loop RG analysis

We now perform a one-loop RG analysis of the surface interaction \( (6.2a) \) in the presence of both \( \lambda^a \) and \( \lambda'^a \) with \( a = 1, 2, 3 \) under the assumption that these couplings are small. Hence, the bare surface interaction \( (6.2a) \) is a small perturbation to the critical surface theory with the energy-momentum tensor \( (6.1) \). The RG calculation itself is standard, and makes use of the current-current OPEs \( (3.3) \) (see, e.g., [99]). The resulting RG equations describing the flow of the couplings \( \lambda^a \) and \( \lambda'^a \) as functions of the cutoff length scale \( \ell \) are

\[
\frac{d\lambda^a}{d\ell} = +2\pi \lambda^b \lambda^c, \quad (6.5a)
\]

\[
\frac{d\lambda'^a}{d\ell} = +2\pi \lambda'^b \lambda'^c, \quad (6.5b)
\]

for \( 1 \leq a < b < c \leq 3 \) and cyclic permutations thereof. Note that at the \( SU(2) \)-symmetric point \( (6.2b) \), the RG flows \( (6.5) \) indicate that the couplings \( \lambda \) and \( \lambda' \) are marginally relevant, as is the case in the bulk.

The one-loop renormalization-group flows for the triplet \( \lambda^1, \lambda^2, \) and \( \lambda^3 \) have decoupled from those of the triplet \( \lambda'^1, \lambda'^2, \) and \( \lambda'^3 \). In fact, they are identical at the one-loop level. We expect this to be true to all orders in perturbation theory, since the interaction \( (6.2a) \) is form-invariant under composing the transformation \( T_{\text{eff},g} \) with the interchange of \( \lambda^a \) and \( \lambda'^a \). Thus, if the flow starts from an initial condition such that \( \lambda^a = \lambda'^a \) for all \( a = 1, 2, 3 \) (as must be the case for a surface that does not explicitly break the symmetry \( T_{\text{eff},g} \)), the asymmetry \( \lambda - \lambda' = 0 \) for all \( \ell > 0 \). If initial conditions are chosen such that \( \lambda^a, \lambda'^a > 0 \) for all \( a = 1, 2, 3 \) is in fact \( SU(2) \)-symmetric. Thus, even if the initial conditions do not satisfy the conditions \( (6.2b) \), the strong-coupling fixed point does. We will illustrate this below for the \( U(1) \)-symmetric case, which is easier to visualize as the phase diagram is then two- rather than three-dimensional.

We now focus on the \( T_{\text{eff},g} \)-symmetric case \( (\lambda^a = \lambda'^a \) for all \( a = 1, 2, 3 \) and investigate the fate of \( SU(2) \) symmetry under the RG flows \( (6.5) \). By analyzing vector-field plots for the differential equations \( (6.5) \), one can convince oneself that the strong-coupling fixed point reached from initial conditions \( \lambda^a > 0 \) for \( a = 1, 2, 3 \) is in fact \( SU(2) \)-symmetric. Thus, even if the initial conditions do not satisfy the conditions \( (6.2b) \), the strong-coupling fixed point does. We will illustrate this below for the \( U(1) \)-symmetric case, which is easier to visualize as the phase diagram is then two-dimensional.

Let us analyze in greater detail the \( T_{\text{eff},g} \) and \( U(1) \)-symmetric case

\[
\lambda^a = \lambda'^a, \quad \lambda^1 = \lambda^2 \equiv \lambda_\perp, \quad \lambda^3 \equiv \lambda_\parallel, \quad (6.6a)
\]

for which the one-loop renormalization-group flows \( (6.5) \) simplify to

\[
\frac{dX}{d\ell} = +2\pi Y^2, \quad \frac{dY}{d\ell} = +2\pi XY, \quad (6.6b)
\]

where either \( (X, Y) = (\lambda_\perp, \lambda_\parallel) \) or \( (X, Y) = (\lambda_\parallel, \lambda_\perp) \). (Recall that the RG flows of \( \lambda^a \) and \( \lambda'^a \) are decoupled.) These one-loop renormalization-group flows are shown in Fig. 14. The separatrix \( X^2 - Y^2 = 0 \) is typical of the Kosterlitz-Thouless renormalization group flows. The RG flow diagram in Fig. 14 indicates that the fixed points for the interacting surface modes when \( \lambda^a = \lambda'^a \) are \( (i)\)
the $SU(2)$-symmetric strong-coupling fixed point
\[ \lambda^a = \lambda'^a \equiv \lambda \to \infty, \tag{6.7} \]

(ii) the strong-coupling fixed point
\[ -\lambda^1 = -\lambda'^1 = -\lambda^2 = -\lambda'^2 = \lambda^3 = \lambda'^3 \equiv \lambda \to \infty, \tag{6.8} \]

that follows from performing a global $SU(2)$ rotation by $\pi/2$ about the quantization axis, and (iii) the line of fixed points
\[ \lambda^1 = \lambda'^1 = \lambda^3 = \lambda'^3 \equiv \lambda \parallel < 0. \tag{6.9} \]

Case (i) [and, upon making a global $SU(2)$ rotation, case (ii)] is the $SU(2)$-isotropic strong-coupling fixed point discussed earlier. Cases (i) and (ii) dominate the phase diagram in the sense that any initial conditions in three of the four quadrants of the $X$-$Y$ plane will lead to one of those strong-coupling fixed points.

The line of stable fixed points in case (iii) constitutes what we call a “sliding parafermion liquid” (SPL). This set of fixed points is gapless—even if the wires are initially coupled with some finite values of $\lambda_\perp$ and $\lambda_\parallel$, the wires decouple as the theory flows to the infrared. (The use of the term “parafermion” refers to the fact that the decoupled chiral $su(2)_k$ CFTs contain parafermion degrees of freedom.) Thus, the SPL resembles the so-called “sliding Luttinger liquids,” which are another class of non-Fermi liquid in $(2+1)$ dimensions [49–51, 55, 100]. These sliding phases have in common the fact that certain classes of perturbations are either irrelevant or marginally irrelevant and hence flow to zero in the infrared. It would be interesting to investigate the SPL phase in more detail, but at present such a study is beyond the scope of this work.

In summary, we have shown that, for a variety of initial conditions on the couplings $\lambda^a = \lambda'^a$, the interactions (6.2a) lead to an $SU(2)$-symmetric strong-coupling RG fixed point, even if the interaction itself breaks $SU(2)$ symmetry explicitly. The $SU(2)$-broken fixed points (like the SPL) may constitute interesting strongly-correlated gapless phases (i.e., non-Fermi liquids).

### B. Mean-field theory for $k = 2$

Having established the stability of the $SU(2)$-symmetric strong-coupling fixed point, we now wish to investigate the nature of this fixed point. Specifically, we would like to know whether this fixed point is gapped or gapless when the symmetry $T_{\text{eff}}$ is not explicitly broken. This is equivalent to asking whether the phase transition between the two $T_{\text{eff}}$-conjugate gapped phases $\lambda > \lambda' \geq 0$ and $0 \leq \lambda < \lambda'$ is discontinuous or continuous. Moreover, we would like to determine whether the symmetry $T_{\text{eff}}$ might be spontaneously broken by the interacting surface theory.

For general $k = 3, 4, \ldots$, the answers to these questions are difficult to determine. Rewriting the interaction (6.2a) at the $SU(2)$-symmetric point (6.2b) in terms of the parafermion representation (3.4) recasts it as a correlated hopping process like (3.6). However, unlike in the $T_{\text{eff}}$-breaking case studied in Sec. III, the current-current interactions on neighboring bonds do not commute for general $k$, owing to the presence of the nonzero couplings $\lambda' = \lambda$. Furthermore, since not all $su(2)_k$ CFTs admit a free-field description, performing detailed calculations is intractable in general. (Although it may be possible to make progress using certain methods from the theory of integrable systems, like the thermodynamic Bethe ansatz, which has been used to study perturbed $su(2)_k$ and $Z_k$ CFTs [68, 101].)

However, the case $k = 2$ is special, for it is the simplest nontrivial example in which the $su(2)_k$ current algebra has a free-fermion representation. In the $k = 2$ case, we can rewrite the $su(2)_2$ current operators as [see also Eqs. (3.4) for $k = 2$]

\begin{align*}
\hat{J}^+_M,y &= \sqrt{2} \hat{\psi}_{M,y} \hat{\tilde{\xi}}_{M,y}, \\
\hat{\tilde{J}}_M,y &= \sqrt{2} \xi^\dagger_{M,y} \hat{\psi}_{M,y}, \\
\hat{\tilde{J}}_M^\dagger,y &= \frac{1}{\sqrt{2}} \partial_M \hat{\phi}_{M,y}. 
\end{align*}

On the one hand, the operator $\hat{\psi}_{M,y}$ either creates or annihilates an M-moving Majorana mode with $M = L, R$ standing for left and right, respectively. On the other hand, the creation operators $\hat{\xi}_{M,y}^\dagger$ and annihilation operators $\hat{\xi}_{M,y}$ create and annihilate M-moving complex Dirac modes, respectively. Moreover, they are related to the chiral boson operators $\hat{\phi}_{M,y}$ through the vertex operator
\[ \hat{\xi}_{M,y} = : e^{+i \sqrt{1/2} \hat{\phi}_{M,y}} :, \tag{6.10d} \]
where \( M = L, R \). Finally, the chiral currents \( \hat{J}_{M,y}^{i} \) can be reexpressed in terms of the pair of Dirac fermion operators \( \hat{\xi}_{M,y}^{i} \) and \( \hat{\xi}_{M,y}^{i} \), using the bosonization identity

\[
\hat{J}_{M,y}^{i} = \frac{1}{2\pi\sqrt{2}} \partial_{M}\hat{\phi}_{M,y}^{i},
\]

(6.10e)

where \( M = L, R \) and \((-1)^{R} = (-1)^{L} = 1\), that defines the two chiral fermionic densities. Note that the wire-dependence of the chiral bosonic commutation relations (3.4g) was chosen so as to ensure that fermionic vertex operators in different wires anticommute at equal times. This is to say that one can use Grassmann coherent states to represent the partition function associated with the Hamiltonian (6.1) as a path integral over Grassmann variables with the free Lagrangian density [102]

\[
\mathcal{L}_{0,x=0} = \sum_{y=0}^{L_{y}} 2 \left( \psi_{L,y+1}^{i} i \partial_{L} \psi_{L,y+1}^{i} + \psi_{R,y}^{i} i \partial_{R} \psi_{R,y}^{i} \right)
\]

\[
+ \sum_{y=0}^{L_{y}} 2 \left( \xi_{L,y+1}^{i} i \partial_{L} \xi_{L,y+1}^{i} + \xi_{R,y}^{i} i \partial_{R} \xi_{R,y}^{i} \right),
\]

(6.11)

\[
\mathcal{L}_{bs,x=0} = -\lambda \sum_{y=0}^{L_{y}} \left[ \left( \xi_{L,y+1}^{i} \xi_{R,y}^{i} + \xi_{L,y+1}^{i} \psi_{R,y}^{i} + \xi_{L,y+1}^{i} \psi_{L,y+1}^{i} \right) - (2\pi)^{2} \xi_{L,y+1}^{i} \xi_{R,y}^{i} \right],
\]

(6.12)

where we have set \( \lambda_{\perp} = \lambda_{||} \equiv \lambda \) and \( \lambda'_{\perp} = \lambda'_{||} \equiv \lambda' \) as we are considering the \( SU(2) \)-symmetric limit. Next, we decouple this interaction with a Hubbard-Stratonovich transformation. That is to say, for each directed bond \((y, y+1)\), we introduce the complex-valued auxiliary field \( \Delta_{\xi,y}(t,z) \), together with the real-valued auxiliary field \( \Delta_{\psi,y}(t,z) \). Similarly, for each directed bond \((y + 1, y + 2)\), we introduce the complex-valued auxiliary field \( \Delta_{\xi,y}(t,z) \), together with the real-valued auxiliary field \( \Delta_{\psi,y}(t,z) \). We then introduce the auxiliary Lagrangian density

\[
\mathcal{L}_{bs,x=0}^{aux} = \frac{1}{2\pi} \sum_{y=0}^{L_{y}} \left[ \left( \Delta_{\xi,y} + \Delta_{\xi,y}^{*} \right) \Delta_{\psi,y} - (2\pi)^{2} \left( \Delta_{\xi,y} + \Delta_{\xi,y}^{*} i \lambda \xi_{L,y+1}^{i} \xi_{R,y}^{i} - \Delta_{\xi,y} i \lambda \xi_{R,y}^{i} \xi_{L,y+1}^{i} \right) \right]
\]

\[
- \sum_{y=0}^{L_{y}} \left[ \left( i \xi_{R,y}^{i} \xi_{L,y+1}^{i} - i \xi_{L,y+1}^{i} \xi_{R,y}^{i} \right) \Delta_{\psi,y} + (\Delta_{\xi,y}^{*} + \Delta_{\xi,y}) i \psi_{L,y+1}^{i} \psi_{R,y}^{i} \right]
\]

\[
+ (\lambda \rightarrow \lambda', \Delta_{\xi,y} \rightarrow \Delta'_{\xi,y}, i \rightarrow -i, R \leftrightarrow L, y \rightarrow y + 1),
\]

(6.13)
The auxiliary Lagrangian density (6.13) reduces to the original Lagrangian density (6.12) when the equations of motion for the auxiliary fields, namely,
\[ \Delta_{\xi,y} = -i\lambda \xi_{L,y+1}^* \xi_{R,y}, \] (6.14a)
\[ \Delta_{\psi,y} = +i\lambda \psi_{L,y+1} \psi_{R,y}, \] (6.14b)
and similarly for the primed fields, are imposed. Note that the phases of the complex auxiliary fields \( \Delta_{\xi,y} \) and \( \Delta_{\psi,y} \) can be removed by a gauge transformation, e.g. \( \xi_{R,y} \rightarrow e^{i\theta} \xi_{R,y} \).

Imposing the symmetry \( T_{\text{eff},y} \) forces the constraints \( \Delta_{\xi} = \Delta'_{\xi} \) and \( \Delta_{\psi} = \Delta'_{\psi} \) among the Hubbard-Stratonovich fields. However, it is important to remember that, within a mean-field treatment of the theory with the interaction (6.13), \( T_{\text{eff},y} \)-symmetry can be broken spontaneously if Eqs. (6.14) develop vacuum expectation values that are not symmetric under \( \Delta_{\xi,\psi} \leftrightarrow \Delta'_{\xi,\psi} \). Checking the self-consistency of such spontaneous-symmetry-breaking solutions is one of the primary goals of the present mean-field calculation.

At this point, the standard way to proceed is to integrate out both the Dirac fields \( \xi_M \) and the Majorana fields \( \psi_M \) and then to solve for the saddle point of the effective action involving only the Hubbard-Stratonovich fields. We first focus on the Majorana contribution to Eq. (6.13). Taking the continuum limit in the \( y \)-direction and linearizing around \( k_y = \pi/2 \) yields the full Euclidean action

\[ S^\text{aux mf} = \int \frac{d\omega}{2\pi} \int \frac{d^2 k}{(2\pi)^2} \left( c^*_k \ c_{-k} \right) \left( \begin{array}{cc} i\omega + v_\xi k_y & k_z - im_\xi \\
 -k_z + im_\xi & i\omega - v_\xi k_y \end{array} \right) \left( \begin{array}{c} c_k \\
 c_{-k}^* \end{array} \right), \] (6.15a)

for any directed bond \( \langle y, y + 1 \rangle \) with \( y \) even, and
\[ \psi_{L,y+1} = c_y^* + c_y, \quad \psi_{R,y+2} = i \left( c_{y+1}^* - c_{y+1} \right), \] (6.15c)
for any directed bond \( \langle y + 1, y + 2 \rangle \) with \( y \) even, followed by taking the Fourier transform
\[ c_y(k_z) = \frac{1}{\sqrt{N}} \sum_{k_y} e^{i k_y k_z} c_{k_y}(k_z). \] (6.15f)

A similar treatment of the Dirac contribution to Eq. (6.13) yields the full Euclidean action

\[ S^\text{aux mf} = \int \frac{d\omega}{2\pi} \int \frac{d^2 k}{(2\pi)^2} \left( \xi_{R,k}^* \xi_{L,k} \right) \left( \begin{array}{cc} i\omega + k_z & +i v_\psi k_y + m_\psi \\
 -i v_\psi k_y + m_\psi & i\omega - k_z \end{array} \right) \left( \begin{array}{c} \xi_{R,k} \\
 \xi_{L,k}^* \end{array} \right), \] (6.16a)

for the case \( k = 2 \) will be gapless so long as \( T_{\text{eff},y} \) is not broken spontaneously. This is because the masses \( m_\xi \) and \( m_\psi \) vanish when \( \Delta_{\xi} = \Delta'_{\xi} \) and \( \Delta_{\psi} = \Delta'_{\psi} \) [see Eqs. (6.15b) and (6.16c)]. Thus, what remains to be checked is that, upon integration over the fermions, the saddle point of the resulting effective action has self-consistent solutions such that \( \Delta_{\xi} = \Delta'_{\xi} \) and \( \Delta_{\psi} = \Delta'_{\psi} \). Integrating out the real and complex fermions, we find the following set of four self-consistency equations for the masses and velocities:

\[ S^\text{aux mf} = S^\text{aux mf} + S^\text{aux mf}_\xi \] (6.17)
\[ m_\psi = \frac{4 \lambda \lambda'}{\lambda + \lambda'} \int \frac{d^2 k}{(2\pi)^2} \frac{m_\xi}{\sqrt{v_\psi^2 k_y^2 + k_z^2 + m_\psi^2}} - 2\pi^2 m_\xi + \frac{\lambda - \lambda'}{\lambda + \lambda'} (v_\psi + 2\pi^2 v_\xi), \]
\[ m_\xi = \frac{4 \lambda \lambda'}{\lambda + \lambda'} \int \frac{d^2 k}{(2\pi)^2} \frac{m_\psi}{\sqrt{v_\psi^2 k_y^2 + k_z^2 + m_\psi^2}} + \frac{\lambda - \lambda'}{\lambda + \lambda'} v_\xi, \]
\[ v_\psi = \frac{4 \lambda \lambda'}{\lambda + \lambda'} \int \frac{d^2 k}{(2\pi)^2} \frac{v_\lambda k_y^2}{\sqrt{v_\psi^2 k_y^2 + k_z^2 + m_\psi^2}} - 2\pi^2 v_\xi + \frac{\lambda - \lambda'}{\lambda + \lambda'} (m_\psi + 2\pi^2 m_\xi), \]
\[ v_\xi = \frac{4 \lambda \lambda'}{\lambda + \lambda'} \int \frac{d^2 k}{(2\pi)^2} \frac{v_\lambda k_y^2}{\sqrt{v_\psi^2 k_y^2 + k_z^2 + m_\psi^2}} + \frac{\lambda - \lambda'}{\lambda + \lambda'} m_\xi. \]

Equations (6.18) constitute a set of four coupled self-consistency equations that must be solved simultaneously for the four unknowns \( m_\xi, m_\psi, v_\xi, \) and \( v_\psi. \) For general values of \( \lambda \) and \( \lambda', \) this must be done numerically. We find that nontrivial solutions of Eqs. (6.18) exist, and that they exhibit the following general features. When \( \lambda = \lambda', \) we find that \( m_\xi = m_\psi = 0 \) for all \( \lambda > 0. \) Thus, at the mean-field level, the surface of the \( su(2)_2 \) non-Abelian coupled-wire construction is a gapless liquid with both Dirac and Majorana degrees of freedom, so long as the symmetry \( T_{\text{eff}}, \bar{g} \) is not broken explicitly. When \( \lambda \neq \lambda', \) we find solutions where the masses \( m_\xi \) and \( m_\psi \neq 0. \) This agrees with our earlier hypothesis that the surface develops a gap when the symmetry \( T_{\text{eff}}, \bar{g} \) is broken explicitly.

VII. CONCLUSIONS

In this paper, we have proposed a means of constructing a family of non-Abelian topological phases in \((3+1)\)-dimensional spacetime. These phases inherit their non-Abelian character from the underlying \( su(2)_k \) CFTs that describe the constituent interacting fermionic quantum wires in the decoupled limit. For the special case of \( su(2)_2, \) we showed explicitly how to construct a set of nonlocal operators that can be used to label a set of degenerate ground states and to cycle between states in this set, thus shedding light on some aspects of the topological order. This calculation relies on the operator algebra of the underlying CFTs that furnish the low-energy degrees of freedom for the coupled-wire construction, thus making explicit the connection between these CFTs and the emergent topological phase. We also examined the phase diagram of the surface for this family of topological phases, and showed explicitly for the case of \( su(2)_2 \) that they are gapless and protected by a nonlocal analogue of TRS.

There are many open questions to be pursued in light of this work. First and foremost, a deeper study of the precise nature of the topological order in the \( su(2)_2 \) example is necessary in order to fully specify the phase. (Indeed, the question of what are the minimal data necessary in order to uniquely determine an arbitrary topological order in 3D is itself not settled.) For example, one can determine the topological spins of the pointlike excitations, and study the braiding of multiple looplike excitations in this model. One can also ask whether the presence of the time-reversal analogues \( T_{\text{eff}}, \bar{x}, \bar{g} \) enriches the topological order in this phase, i.e., whether the action of the symmetry on excitations provides additional topological information [10]. Moreover, one could consider cases where \( k > 2 \) in more detail, and ask whether these also yield candidates for non-Abelian topological order in 3D. More broadly, it would be interesting to determine whether and how such phases could be represented within the Crane-Yetter/Walker-Wang construction, or the formalism of discrete non-Abelian gauge theory. Another avenue to pursue would be to try to construct these phases via parton constructions like the ones that have been carried out for some non-Abelian quantum Hall states [103, 104], and for the Abelian FTIs in \((3+1)\)-dimensional spacetime [43, 44]. Coupled-layer constructions of non-Abelian topological phases, extending the work of Ref. [105], could also be considered.

It would also be interesting to investigate how to construct microscopic lattice models that yield the \( su(2)_k \) topological phases in \((3+1)\)-dimensional spacetime proposed in this work. One clue for how to proceed relies on the fact that \( su(2)_k \) CFTs can be obtained as continuum limits of certain spin-\( \frac{k}{2} \) chains [106]. Microscopic spin-spin interactions can then be derived whose continuum limits give rise to the current-current interactions used to gap out an array of such spin chains; for example. Such interactions were constructed for the \((2+1)\)-dimensional case in Ref. [70].

Another line of inquiry is to investigate more deeply the nature of the \( su(2)_k \) surface states for \( k > 2. \) While spontaneous breaking of the nonlocal TRS analogue (and the concomitant opening of a gap on the surface) is always a possibility, it could be that these surface states constitute novel stable fractionalized non-Fermi liquid phases. The investigation of this class of models would
likely need to rely on nonperturbative techniques, and could provide insights into conformal field theories in (2+1)-dimensional spacetime.

**ACKNOWLEDGMENTS**

We thank D. Aasen, M. Barkeshli, P. Bonderson, F. Burnell, M. Cheng, M. Metlitski, M. Oshikawa, Z. Wang, X.-G. Wen, and D. Williamson for helpful discussions. T.I. gratefully acknowledges the hospitality of the KITP, where a significant portion of this work was completed, and thanks the organizers of the “Symmetry, Topology, and Quantum Phases of Matter: From Tensor Networks to Physical Realizations” and “Synthetic Quantum Matter” programs, both of which were supported in part by the National Science Foundation under Grant No. NSF PHY11-25915. T.I. was supported by a KITP Graduate Fellowship, the National Science Foundation Graduate Research Fellowship Program under Grant No. DGE-1247312, the Laboratory for Physical Sciences, Microsoft, and a JQI Postdoctoral Fellowship. C.C. was supported by DOE Grant DE-FG02-06ER46316.

**Appendix A: The parafermion current algebra**

We are going to review how the affine Lie algebra of level $k = 1, 2, 3, \cdots$ for the compact connected Lie group $SU(2)$ can be represented in terms of parafermions as was done by Zamolodchikov and Fateev in Ref. [90].

### 1. Gaussian algebra

For any $\kappa > 0$, define the Euclidean action

$$S := \frac{\kappa}{2} \int \! d^2x \, (\partial \phi)^2$$

(A1)

for the real-valued scalar field $\phi$ and the positive number $0 < \kappa \in \mathbb{R}$. Its two-point function is

$$\langle \phi(x) \phi(y) \rangle = -\frac{1}{4\pi \kappa} \ln |x - y|^2$$

(A2)

up to an additive dimensionful constant that depends on the boundary condition imposed on the Laplacian. If we trade the complex coordinates $v \in \mathbb{C}$ and $w \in \mathbb{C}$ in two-dimensional Euclidean space for the Cartesian coordinates $x \in \mathbb{R}^2$ and $y \in \mathbb{R}^2$, respectively, then

$$|x - y|^2 = (v - w) (\overline{v} - \overline{w})$$

(A3)

and

$$\langle \phi(x) \phi(y) \rangle = -\frac{1}{4\pi \kappa} [\ln(v - w) + \ln(\overline{v} - \overline{w})]$$

(A4a)

Another set of chiral Abelian OPEs follows from making the Ansatz

$$\varphi(v, \overline{v}) = \phi_L(v) + \phi_R(\overline{v})$$

(A7a)

$$\langle \partial_v \phi_L(v) \phi_L(w) \rangle = -\frac{1}{4\pi \kappa} \frac{1}{v - w}$$

(A7b)

$$\langle \partial_v \phi_R(\overline{v}) \phi_R(\overline{w}) \rangle = -\frac{1}{4\pi \kappa} \frac{1}{\overline{v} - \overline{w}}$$

(A7c)

$$\langle \phi_R(v) \phi_L(\overline{w}) \rangle = 0$$

(A7d)

The holomorphic, $\phi_L$, and antiholomorphic, $\phi_R$, fields are uniquely defined up to the addition of holomorphic and antiholomorphic functions, respectively. One then deduces from

$$\langle e^{+ia \phi_L(v)} e^{-ia \phi_L(w)} \rangle = \frac{1}{(v - w)^{\frac{2}{1 + \kappa}}}$$

(A8a)

$$\langle e^{+ia \phi_L(v)} e^{-ia \phi_L(w)} \rangle = 0$$

(A8b)

$$\langle e^{+ia \phi_R(\overline{v})} e^{-ia \phi_R(\overline{w})} \rangle = \frac{1}{(\overline{v} - \overline{w})^{\frac{2}{1 + \kappa}}}$$

(A8c)

$$\langle e^{+ia \phi_R(\overline{v})} e^{-ia \phi_R(\overline{w})} \rangle = 0$$

(A8d)

$$\langle e^{\pm ia \phi_L(v)} e^{\pm ia \phi_L(w)} \rangle = 0$$

(A8e)

that

$$e^{+ia \phi_L(v)} e^{-ia \phi_L(w)} = \frac{1}{(v - w)^{\frac{2}{1 + \kappa}}}$$
\[ e^{-i \phi_R(\overline{v})} e^{i \phi_R(\overline{w})} = \frac{1}{(\overline{v} - \overline{w})^{\frac{\Delta}{2\pi i}}} (\partial_{\overline{w}} \phi_R)(\overline{w}) + \cdots \]  
\[ e^{+i \phi_R(\overline{v})} e^{-i \phi_R(\overline{w})} = \frac{1}{(\overline{v} - \overline{w})^{\frac{\Delta}{2\pi i}}} (\partial_{\overline{w}} \phi_R)(\overline{w}) + \cdots \]  
\[ (A9a) \]

\[ (A9b) \]

are the only chiral Abelian OPEs between the vertex fields \( e^{\pm i \phi_R(v)} \) and \( e^{\pm i \phi_R(\overline{v})} \) that are proportional to the identity operator to leading order.

At last, we shall need the OPEs

\[ \partial_v \phi_L(w) e^{+i \phi_L(v)} = -\frac{ia}{4\pi \kappa (v - w)} e^{+i \phi_L(w)} + \cdots, \]

\[ \partial_{\overline{v}} \phi_R(\overline{w}) e^{+i \phi_R(\overline{v})} = -\frac{ia}{4\pi \kappa (\overline{v} - \overline{w})} e^{+i \phi_R(\overline{w})} + \cdots. \]

\[ (A10a) \]

\[ (A10b) \]

In the following, we make the choice

\[ \kappa = \frac{1}{8\pi}. \]

With this choice, the conformal weights of the vertex fields \( \exp(ia \phi_L) \) and \( \exp(ia \phi_R) \) are

\[ (h_\kappa, \overline{h}_\kappa) \equiv (a^2, 0), \quad (h_\kappa, \overline{h}_\kappa) \equiv (0, a^2), \]

\[ (A12) \]

respectively. Moreover, the proportionality constant on the right-hand side of Eq. (A10) is \(-2ai\).

### 2. Parafermion algebra

Let \( k = 0, 1, 2, \ldots \) be a positive integer. Define the holomorphic conformal weights

\[ \Delta_l = \frac{l(k - l)}{k}, \quad l = 0, \ldots, k - 1. \]

\[ (A13a) \]

We posit the family of \( k \) local parafermion fields

\[ I, \Psi_1(v), \ldots, \Psi_{k-1}(v), \]

\[ (A13b) \]

where \( I \) is the identity operator with the scaling dimension \( \Delta_0 \equiv 0 \). For any \( m, n = 0, \ldots, k - 1 \), we impose the OPEs [90]

\[ \Psi_m(v) \Psi_n(v') = \mathcal{C}^{\Psi_m \Psi_n^{m+n}}_{\Psi_o \Psi_{m+n}^{m+n}} (v') \frac{(v - v')^{\Delta_m - \Delta_{m+n}}}{(v - v')^{\Delta_m - \Delta_{m+n}}} + \cdots \]

\[ (A13c) \]

with the understanding that \( m + n \) is defined modulo \( k \), i.e.,

\[ \Psi_0 \equiv \Psi_k \equiv I. \]

\[ (A13d) \]

The complex-valued number \( \mathcal{C}^{\Psi_m \Psi_n^{m+n}}_{\Psi_o \Psi_{m+n}^{m+n}} \) is called a structure constant. Demanding that the OPEs for the parafermions are associative fixes this structure constant to be the positive roots of [90]

\[ \frac{\Gamma(m + n + 1) \Gamma(k - m + 1) \Gamma(k - n + 1)}{\Gamma(m + 1) \Gamma(n + 1) \Gamma(k - m - n + 1) \Gamma(k + 1)} \]

\[ (A13e) \]

provided the normalization conditions

\[ \mathcal{C}^{\Psi_k \Psi_{k-m}}_{\Psi_o \Psi_{k-m}} = 1, \quad m = 0, \ldots, k - 1 \]

\[ (A13f) \]

are imposed.

An important consequence of (A13e) is the symmetry

\[ \mathcal{C}^{\Psi_m \Psi_n}_{\Psi_o \Psi_{m+n}} = \mathcal{C}^{\Psi_n \Psi_m}_{\Psi_o \Psi_{m+n}}, \quad m, n = 0, \ldots, k - 1, \]

\[ (A14) \]

under interchanging \( m \) and \( n \). This is why

\[ \Psi_n(v') \Psi_m(v) = (-1)^{\Delta_{m+n} - \Delta_m - \Delta_n} \Psi_{m+n}(v) \Psi_n(v'), \]

\[ (A15a) \]

where

\[ \Delta_{m+n} - \Delta_m - \Delta_n = -\frac{2mn}{k} \equiv S_{m,n}^{(k)}. \]

\[ (A15b) \]

We shall call \( \pi S_{m,n}^{(k)} \) the mutual (self) statistical angle between the parafermion \( m \) and the parafermion \( n \neq m \) (when \( n = m \)).

Because the OPE between \( \Psi_m \) and \( \Psi_{k-m} \) gives the identity operator, we shall use the notation

\[ \Psi_m^I \equiv \Psi_{k-m} \]

\[ (A16a) \]

for \( m = 1, \ldots, k - 1 \). The self statistical angle of the parafermion \( m \) is

\[ S_{m,m}^{(k)} = -\frac{2m^2}{k}. \]

\[ (A16b) \]

The self statistical angle of the parafermion \( k - m \) is

\[ S_{k-m,k-m}^{(k)} = -\frac{2(k-m)^2}{k} = S_{m,m}^{(k)} \mod \mathbb{Z}. \]

\[ (A16c) \]

The mutual statistics between parafermion \( m \) and \( k - m \) is

\[ S_{m,k-m}^{(k)} = -\frac{2m(k-m)}{k} = -S_{m,m}^{(k)} \mod \mathbb{Z}. \]

\[ (A16d) \]
3. Parafermion representation of the $su(2)_k$ current algebra

The $su(2)_k$ current algebra is defined by the holomorphic current algebra \[85\]
\[ J^a(v) J^b(w) = \frac{(k/2)^{g_{ab}}}{(v-w)^2} + \frac{ie^{abc}}{(v-w)} J^c(w) + \cdots \] (A17)
for any $a, b = 1, 2, 3$ together with its antiholomorphic copy. Without loss of generality, we consider only this holomorphic current algebra. In the basis
\[ J^\pm = J^1 \pm iJ^2, \quad J^3, \] (A18a)
the holomorphic current algebra (A17) reads
\[ J^\pm(v) J^\pm(w) = 0 + \cdots, \] (A18b)
\[ J^+(v) J^-(w) = \frac{k}{(v-w)^2} + \frac{2}{(v-w)} J^3(w) + \cdots, \] (A18c)
\[ J^3(v) J^\pm(w) = \pm \frac{1}{(v-w)} J^\pm(w) + \cdots, \] (A18d)
\[ J^3(v) J^3(w) = \frac{(k/2)}{(v-w)^2} + \cdots. \] (A18e)

We are going to verify that this current algebra can be represented in terms of the Gaussian boson $\phi$ from Sec. A1 and the pair of parafermions $\Psi_1 \equiv \Psi$ and $\Psi_{k-1} \equiv \Psi^\dagger$ from Sec. A2.

We make the Ansatz
\[ J^+(v) = \mathcal{N} \Psi_1(v) e^{+i\sqrt{k} \phi(v)} = \mathcal{N} \Psi(v) e^{+i\sqrt{k} \phi(v)}, \] (A19a)
where we impose on $\partial_v \phi$ the Gaussian algebra
\[ \partial_v \phi(v) \partial_w \phi(w) = -\frac{2}{(v-w)^2} + \cdots, \] (A20a)
while we impose on $\Psi$ and $\Psi^\dagger$ the parafermion algebra
\[ \Psi(v) \Psi(w) = \frac{C_\Psi^I}{(v-w)^2(k-1)/k} + \cdots, \] (A20b)
\[ \Psi^\dagger(v) \Psi^\dagger(w) = \frac{C_\Psi^I \Psi^\dagger}{(v-w)^2(k-1)/k} + \cdots, \] (A20c)
\[ \Psi(v) \Psi^\dagger(w) = \frac{1}{(v-w)^2(k-1)/k} + \cdots. \] (A20d)

The OPE (A18e) follows from the Ansatz (A19c) with the OPE (A20a). Because of the OPE (10), we have the OPE
\[ \partial_v \phi(v) e^{\pm i\sqrt{k} \phi(v)} = \mp i \sqrt{k} \frac{2}{(v-w)^k} e^{\pm i\sqrt{k} \phi(w)}. \] (A21)

The OPE (A18d) follows from the Ansatz (A19) with the OPE (A21). We thus see that the multiplicative factor $\sqrt{k}/k$ entering the argument of the vertex fields $\exp(\pm i\sqrt{k} \phi)$ is fixed by the condition that the two currents have the holomorphic conformal weight one. In turn, the normalization factor $\mathcal{N}$ is fixed by the following considerations. Because of the OPEs (A13) and (A9), we have the OPE
\[ J^+(v) J^-(w) = \mathcal{N}^2 \Psi_1(v) \Psi_{k-1}(w) e^{\pm i\sqrt{k} \phi(v)} e^{-i\sqrt{k} \phi(w)} = \left( \frac{\mathcal{N}^2}{(v-w)^{k+1/k} + \cdots} \right) \frac{1}{(v-w)^k} \left( 1 + i \sqrt{k} (v-w) (\partial_w \phi(w) + \cdots) \right), \] (A22)

The leading singularity on the right-hand side of this OPE agrees with the one on the right-hand side of Eq. (A18c) if
\[ \mathcal{N}^2 = k. \] (A23)

Finally, the vanishing OPE (A18b) follows from the fact that the OPE between any two vertex fields such that the $\mathbb{C}$-valued prefactors to the fields $\phi(v)$ and $\phi(w)$ in the arguments of the vertex fields are not of opposite sign, vanishes to leading order.

We close Sec. A3 by observing that the Ansatz (A19) is not unique. Indeed, the transformation
\[ \Psi(v) \mapsto \Psi(v) e^{i\alpha}, \] (A24a)
\[ \Psi^\dagger(v) \mapsto \Psi^\dagger(v) e^{-i\alpha}, \] (A24b)
\[ \phi(v) \mapsto \phi(v) - \sqrt{k} \alpha, \] (A24c)
leaves the $su(2)_k$ currents \( (A18a) \) invariant for any choice of the number $\alpha$. The number $\alpha$ is defined modulo $2\pi$ and takes $k$ inequivalent values $2\pi n/k$, $n = 0, \cdots, k - 1$.

### Appendix B: The $\mathbb{Z}_k$ conformal field theory

The parafermions defined in Appendix A represent currents in the $\mathbb{Z}_k$ conformal field theory (CFT). The $\mathbb{Z}_k$ CFT describes the long-wavelength properties of the critical point of a two-dimensional lattice model of classical $\mathbb{Z}_k$ spins. It is characterized by the primary fields \[85\]

$$\Phi_n(v), \quad m = 0, \cdots, k, \quad m + n = 0 \mod 2, \quad (B1)$$

with $n \in \mathbb{Z}$. The integer $n$ must be restricted to the range $(-m, m]$ using the relations

$$\Phi_n(v) \equiv \Phi_{n+2k}(v) \equiv \Phi_{n-k}(v). \quad (B2)$$

Hence, the number of unique primary fields for a given $k$ is restricted. The holomorphic conformal weight of the primary field $\Phi_n(v)$ is given by \[85\]

$$\Delta_n := m(m + 2) - \frac{n^2}{4k}, \quad (B3)$$

which is nonnegative for $-m < n \leq m$. Among the primary fields $\Phi_n(v)$, there are the so-called “identity field”

$$1 := \Phi_0^0, \quad (B4a)$$

the “twist fields”

$$\sigma_m := \Phi_m^0, \quad m = 1, \cdots, k - 1, \quad (B4b)$$

and the parafermions

$$\Psi_m := \Phi_0^{2m} \equiv \Phi_{2m-k}^k, \quad m = 1, \cdots, k - 1, \quad (B4c)$$

which were introduced in Appendix A. The twist fields are of particular importance, as they are the continuum analogues of the lattice $\mathbb{Z}_k$ spins.

The $\mathbb{Z}_k$ primary fields obey the “fusion algebra” \[85\]

$$\Phi_n \times \Phi_{n'} = \sum_{l=|m-m'|}^{\min(m+m',2k-m-m')} \Phi_{n+n'}^{l}, \quad (B5a)$$

This fusion algebra is a shorthand notation for the OPEs

$$\Phi_n^m(v) \Phi_{n'}^{m'}(w) = \sum_{\substack{l=|m-m'| \\ l+m+m'=0 \mod 2}} C_{\Phi_n^m \Phi_{n'}^{m'}}^{\Phi_l^0} (v-w)^{S_{m,n'}^{m',l}(n+n')} \Phi_{n+n'}^l(v), \quad (B5b)$$

where

$$S_{m,n'}^{m',l}(n+n') := \Delta_{n+n'} - \Delta_n - \Delta_{m'}, \quad (B5c)$$

and the structure constants $C_{\Phi_n^m \Phi_{n'}^{m'}}^{\Phi_l^0}$ are fixed by associativity of the algebra. The quantity $2\pi S_{m,n'}^{m',l}(n+n')$ is the phase acquired, in the channel where $m$ and $m'$ fuse to $l$, when the complex coordinate $v$ is rotated around the complex coordinate $w$.

1. **Example: $\mathbb{Z}_2$ (Ising CFT)**

When the conditions \( (B2) \) are imposed, the $\mathbb{Z}_2$ CFT (also known as the Ising CFT, as it describes the critical point of the classical Ising model in two dimensions) has the three primary fields

$$1, \quad \sigma_1 \equiv \sigma, \quad \Psi_1 \equiv \psi. \quad (B6)$$

According to Eq. \( (B3) \), their holomorphic conformal weights are

$$\Delta_1 = 0, \quad \Delta_\sigma := \frac{1}{16}, \quad \Delta_\psi = \frac{1}{2}, \quad (B7)$$

respectively. According to Eq. \( (B5a) \), the primaries obey the fusion algebra

$$\sigma \times \sigma = 1 + \psi, \quad (B8a)$$
$$\psi \times \psi = 1, \quad (B8b)$$
$$\sigma \times \psi = \sigma, \quad (B8c)$$

in addition to the trivial fusion rules

$$1 \times a = a \quad (B8d)$$
for \( a = 1, \sigma, \psi \).

Appendix C: Commutation between string operators and the Hamiltonian: “Analytic” proof of state exclusion for the 2D case

1. Introduction

We are given the Hamiltonian

\[
\hat{H}_{\text{bs}} := \int_0^{L_y} \, dz \, \hat{H}_{\text{bs}}
\]

and we are told that it commutes with two nonlocal operators \( \hat{\Gamma}_1^{(1)} \) and \( \hat{\Gamma}_2^{(1)} \). Moreover, we are told that \( \hat{\Gamma}_1^{(1)} \) and \( \hat{\Gamma}_2^{(1)} \) commute pairwise. Hence, we can label any eigenstate of the Hamiltonian \( \hat{H}_{\text{bs}} \) by the simultaneous eigenvalues \( \omega_1^{(1)} \) and \( \omega_2^{(1)} \) of the operators \( \hat{\Gamma}_1^{(1)} \) and \( \hat{\Gamma}_2^{(1)} \). In particular, we can label the basis for the ground-state manifold by

\[
\{ |\omega_1^{(1)}, \omega_2^{(1)}, \cdots \rangle \}
\]

where the \( \cdots \) allow for additional sources of degeneracies. We shall demand that this basis is orthonormal.

In order to establish the set to which the eigenvalues \( \omega_1^{(1)} \) and \( \omega_2^{(1)} \) belong, we note that we are given two nonlocal operators

\[
\begin{align*}
\hat{\Gamma}_1^{(1)}(z) &= \prod_{y=0}^{L_y} \hat{\sigma}_{L,y}(z) \hat{\sigma}_{R,y}(z), \\
\hat{\Gamma}_2^{(1)}(z) &= \prod_{y=0}^{L_y} \hat{\sigma}_{L,y}(z) \hat{\sigma}_{R,y}(z).
\end{align*}
\]

The operator \( \hat{\Gamma}_1^{(1)}(z) \) is a discrete product of a countable number of operators acting along a closed \( y \)-cycle of the two-torus. It requires no regularization for its definition, and it is nonunitary. It anticommutes with \( \hat{\Gamma}_2^{(1)}(z) \), and commutes with \( \hat{\Gamma}_1^{(1)}(z) \) and with the Hamiltonian \( \hat{H}_{\text{bs}} \). In contrast, the operator \( \hat{\Gamma}_2^{(1)}(z) \) is a nonlocal operator defined within one chiral channel of the wire \( y \). It acts along an open string (along the \( z \)-cycle coinciding with wire \( y \)) that fails to close by the infinitesimal amount \( \epsilon > 0 \). It is nonunitary and it anticommutes with \( \hat{\Gamma}_1^{(1)}(z) \) in the limit \( \epsilon \to 0 \).

If both \( \hat{\Gamma}_1^{(1)}(z) \) and \( \lim_{\epsilon \to 0} \hat{\Gamma}_2^{(1)}(z) \) were to commute with the Hamiltonian, then so would their product. The ground-state manifold would then be four-dimensional, with the orthogonal basis

\[
|\Omega, \cdots \rangle := |\omega_1^{(1)}, \omega_2^{(1)}, \cdots \rangle,
\]

\[
\hat{\Gamma}_1^{(1)}(z) |\Omega, \cdots \rangle = \mathcal{N}_1 |\omega_1^{(1)}, -\omega_2^{(1)}, \cdots \rangle,
\]

\[
\lim_{\epsilon \to 0} \hat{\Gamma}_2^{(1)}(z) |\omega, \cdots \rangle = \mathcal{N}_2 |\omega_1^{(1)}, \omega_2^{(1)}, \cdots \rangle,
\]

\[
\hat{\Gamma}_1^{(1)}(z) |\omega_1^{(1)}, \omega_2^{(1)}, \cdots \rangle = \mathcal{N}_1 |\omega_1^{(1)}, -\omega_2^{(1)}, \cdots \rangle \equiv \mathcal{N}_2 |\omega_1^{(1)}, \omega_2^{(1)}, \cdots \rangle.
\]

We demand that the states on the left-hand side can be normalized. This can only be achieved if the normalizations \( \mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3 \) are neither zero nor infinity, for the basis \( \{ |\omega, \cdots \rangle \} \) is orthonormal. However, the logical possibility that one or more of these normalizations are zero or infinity cannot be excluded. In this appendix, we will assume \( \mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3 \) to be nonvanishing and finite. This assumption amounts to choosing the “highest-weight state” \( |\omega, \cdots \rangle \) is known. Since we do not have this knowledge, we leave its value unspecified for the moment.

Given that we do not know the value of \( \mathcal{N}_1 \), we proceed by an alternate route. This line of reasoning makes use of the fact that it is not correct to think of the operator \( \lim_{\epsilon \to 0} \hat{\Gamma}_2^{(1)}(z) \) as commuting with the Hamiltonian \( \hat{H}_{\text{bs}} \). It is a nonlocal operator that changes the topological sector of the state on which it acts, and can potentially exhibit different limiting behavior as a function of \( \epsilon \) when acting on states belonging to different topological sectors. Thus, the limit \( \epsilon \to 0 \) must be treated carefully when multiplying the operators \( \hat{\Gamma}_1^{(1)}(z) \) and \( \hat{\Gamma}_2^{(1)}(z) \). Indeed, instead of the set of states \( \{ |\omega, \cdots \rangle \} \), we can also consider the following set of states,

\[
|\Omega, \cdots \rangle := |\omega_1^{(1)}, \omega_2^{(1)}, \cdots \rangle,
\]

\[
\hat{\Gamma}_1^{(1)}(z) |\omega, \cdots \rangle = \lim_{\epsilon \to 0} \hat{\Gamma}_2^{(1)}(z) |\omega_1^{(1)}, \omega_2^{(1)}, \cdots \rangle,
\]

\[
\lim_{\epsilon \to 0} \hat{\Gamma}_2^{(1)}(z) |\omega, \cdots \rangle = \lim_{\epsilon \to 0} \hat{\Gamma}_1^{(1)}(z) |\omega_1^{(1)}, \omega_2^{(1)}, \cdots \rangle.
\]

The only difference between the states \( \{ |\omega, \cdots \rangle \} \) and the states \( \{ |\omega, \cdots \rangle \} \) in Eq. \( \{ |\omega, \cdots \rangle \} \) is that the limit \( \epsilon \to 0 \) is taken after forming the product \( \hat{\Gamma}_1^{(1)}(z) \hat{\Gamma}_2^{(1)}(z) \) in Eq. \( \{ |\omega, \cdots \rangle \} \). We adopt the point of view that the dimension of the ground-state
manifold of the Hamiltonian (C1) cannot depend on the choice of when [i.e., before or after forming the product $\hat{\Gamma}_{1}^{(1/2)} \hat{\Gamma}_{2,R,y}^{(1/2)}(\epsilon)$] the limit $\epsilon \to 0$ is taken. Hence, the number of ground states present in Eqs. (C4) and (C5) must agree with one another. For this reason, we ask how many of the states (C5) are indeed ground states of the interaction (C1). This allows us to scrutinize the limiting behavior of operator products without losing important information related to the nonlocality of its constituent operators. We will show that the state (C5d) cannot be in the ground-state manifold of the interaction (C1). Logical consistency then demands that $N_{12} = 0$ or $\infty$ in Eqs. (C4), as these are the only two possibilities that would exclude the state (C4d) from the ground-state manifold.

The nonunitary operator $\hat{\Gamma}_{2,R,y}^{(1/2)}(\epsilon)$ does not commute with the interaction $\hat{H}_{bs}$ defined by Eq. (C1). The purpose of this appendix is to determine whether the states (C5c) and (C5d), which involve taking the limit $\epsilon \to 0$, indeed belong to the ground-state manifold of the interaction (C1) once this limit is taken. More precisely, we define

$$\left[\hat{H}_{bs}, \hat{\Gamma}_{2,R,y}^{(1/2)}(\epsilon)\right] := \hat{D}_{2,R,y}(\epsilon), \tag{C6a}$$

where the operator $\hat{D}_{2,R,y}(\epsilon)$ is nonlocal, as we shall see below, and nonvanishing in general. We further define

$$\left[\hat{H}_{bs}, \hat{\Gamma}_{1}^{(1/2)}(\epsilon) \hat{\Gamma}_{2,R,y}^{(1/2)}(\epsilon)\right] = \hat{\Gamma}_{1}^{(1/2)}(\epsilon) \hat{D}_{2,R,y}(\epsilon) =: \hat{D}_{12,R,y}(z, \epsilon). \tag{C6b}$$

We are going to show that

$$\lim_{\epsilon \to 0} \hat{D}_{2,R,y}(\epsilon) \rvert_{\Omega, \ldots} = 0. \tag{C7a}$$

Equation (C7a) is equivalent to the statement

$$\lim_{\epsilon \to 0} \left[\hat{H}_{bs}, \hat{\Gamma}_{2,R,y}^{(1/2)}(\epsilon)\right] \rvert_{\Omega, \ldots} = (\hat{H}_{bs} - E_{\Omega}) \hbar_{bs}(\epsilon) \rvert \hat{\Gamma}_{2,R,y}^{(1/2)}, \ldots \rangle = 0, \tag{C7b}$$

where $E_{\Omega}$ is the energy eigenvalue of the state $\rvert \Omega, \ldots \rangle$. From this it immediately follows that the state $\rvert \hat{\Gamma}_{2,R,y}^{(1/2)}, \ldots \rangle$ indeed belongs to the ground-state manifold of the interaction (C1).

We are also going to show that the state

$$\lim_{\epsilon \to 0} \hat{D}_{12}(z, \epsilon) \rvert_{\Omega, \ldots} \tag{C8a}$$

has infinite norm as $z \to 0$. Equation (C8a) is equivalent to the statement that

$$\lim_{z \to 0} \lim_{\epsilon \to 0} \left[\hat{H}_{bs}, \hat{\Gamma}_{1}^{(1/2)}(z) \hat{\Gamma}_{2,R,y}^{(1/2)}(\epsilon)\right] \rvert_{\Omega, \ldots} = \lim_{z \to 0} \left(\hat{H}_{bs} - E_{\Omega}\right) \rvert \hat{\Gamma}_{1}^{(1/2)} \hat{\Gamma}_{2,R,y}^{(1/2)}, \ldots \rangle \tag{C8b}$$

is a state with infinite norm. That this divergence occurs as $z \to 0$ is especially problematic. In order for the product $\hat{\Gamma}_{1}^{(1/2)}(z) \hat{\Gamma}_{2,R,y}^{(1/2)}(\epsilon)$ of string operators to yield a topologically-degenerate ground state when acting on the state $\rvert \Omega, \ldots \rangle$, the resulting state cannot depend on the quantities $z$ and $\epsilon$ in an observable way as $z \to 0$ and $\epsilon \to 0$. If this were the case, then the states $\rvert \Omega, \ldots \rangle$ and $\rvert \hat{\Gamma}_{1}^{(1/2)} \hat{\Gamma}_{2,R,y}^{(1/2)}, \ldots \rangle$ could be distinguished by simply evaluating the string operator $\hat{\Gamma}_{1}^{(1/2)}(z)$ near the point $z = 0$.

Hence, proving that the state defined in Eq. (C8a) is not normalizable will allow us to conclude that the state $\rvert \hat{\Gamma}_{1}^{(1/2)} \hat{\Gamma}_{2,R,y}^{(1/2)}, \ldots \rangle$ does not belong to the ground-state manifold of the interaction (C1).

We are left with the conclusion of the paper, namely that the ground-state manifold of the interaction (C1) includes the states (C5a)–(C5c), and excludes the state (C5d). From now on, we ignore the $\cdots$ representing additional degeneracies for the ground-state manifold.

2. Calculation

We first prove Eq. (C7a). We begin by calculating $\hat{D}_{2,R,y}(\epsilon)$. For finite $\epsilon > 0$, we have

$$\hat{H}_{bs} \hat{\Gamma}_{2,R,y}^{(1/2)}(\epsilon) = \hat{\Gamma}_{2,R,y}^{(1/2)}(\epsilon) \hat{H}_{bs} \times \begin{cases} +1, & z > \epsilon, \\ +i, & z = \epsilon, \\ -1, & z < \epsilon. \end{cases} \tag{C9}$$

We now use the definition (C6a), along with the identity

$$\hat{A} \hat{B} = \hat{B} \hat{A} f(z, \epsilon) \iff [\hat{A}, \hat{B}] = \hat{B} \hat{A} [f(z, \epsilon) - 1], \tag{C10}$$

which gives

$${\hat{D}_{2,R,y}(\epsilon) = -4i \int_{0}^{\epsilon} dz \sin\left(\frac{1}{\sqrt{2}} \left(\hat{\phi}_{R,y}(z) - \hat{\phi}_{L,y}(z)\right)\right) \hat{U} \hat{V}_{L,y}(z) \hat{U}_{R,y}(z) \hat{P}_{1} \hat{\sigma}_{R,y}(0) \hat{\sigma}_{R,y}(\epsilon) \hat{P}_{1}, \tag{C11}$$

up to a contribution from the set of measure zero where $z = \epsilon$, which we will ignore.
To prove Eq. (C7a), we compute the leading contribution to $\hat{D}_2(\epsilon)$ as $\epsilon \to 0$. For $\epsilon$ infinitesimal, we may replace the integral in Eq. (C11) by the value of the integrand at the midpoint of the integration domain,

$$\hat{D}_{2,R,y}(\epsilon) \approx -4i \epsilon \sin \left( \frac{1}{\sqrt{2}} \left[ \hat{\phi}_{R,y} \left( \frac{\epsilon}{2} \right) - \hat{\phi}_{L,y+1} \left( \frac{\epsilon}{2} \right) \right] \right) \hat{U} \hat{\psi}_{L,y+1} \left( \frac{\epsilon}{2} \right) \hat{\psi}_{R,y} \left( \frac{\epsilon}{2} \right) \hat{P}_1 \hat{\sigma}_{R,y}(0) \hat{\sigma}_{R,y}(\epsilon) \hat{P}_1.$$  \hspace{1cm} (C12)

We now perform the (equal-time) OPE

$$\sin \left( \frac{1}{\sqrt{2}} \left[ \hat{\phi}_{R,y} \left( \frac{\epsilon}{2} \right) - \hat{\phi}_{L,y+1} \left( \frac{\epsilon}{2} \right) \right] \right) \hat{U} = \sin \left( \frac{1}{\sqrt{2}} \left[ \hat{\phi}_{R,y} \left( \frac{\epsilon}{2} \right) - \hat{\phi}_{L,y+1} \left( \frac{\epsilon}{2} \right) \right] \right) \exp \left( -\frac{i}{2\sqrt{2}} \int_0^{L_z} dz \partial_z \hat{\phi}_{R,y}(z) \right).$$  \hspace{1cm} (C13)

Inserting the OPEs

$$\lim_{\epsilon \to 0} e^{\frac{i}{\sqrt{2}} \hat{\phi}_{R,y}(\frac{\epsilon}{2})} e^{-\frac{i}{\sqrt{2}} \hat{\phi}_{R,y}(L_z)} \sim \frac{1}{1} e^{\frac{i}{\sqrt{2}} \hat{\phi}_{R,y}(\frac{\epsilon}{2})},$$  \hspace{1cm} (C14a)

$$\lim_{\epsilon \to 0} e^{-\frac{i}{\sqrt{2}} \hat{\phi}_{R,y}(\frac{\epsilon}{2})} e^{\frac{i}{\sqrt{2}} \hat{\phi}_{R,y}(0)} \sim \frac{1}{1} e^{-\frac{i}{\sqrt{2}} \hat{\phi}_{R,y}(\frac{\epsilon}{2})},$$  \hspace{1cm} (C14b)

$$\lim_{\epsilon \to 0} e^{\frac{i}{\sqrt{2}} \hat{\phi}_{R,y}(\frac{\epsilon}{2})} e^{\frac{i}{\sqrt{2}} \hat{\phi}_{R,y}(0)} \sim \frac{1}{1} e^{\frac{i}{\sqrt{2}} \hat{\phi}_{R,y}(\frac{\epsilon}{2})},$$  \hspace{1cm} (C14c)

$$\lim_{\epsilon \to 0} e^{-\frac{i}{\sqrt{2}} \hat{\phi}_{R,y}(\frac{\epsilon}{2})} e^{-\frac{i}{\sqrt{2}} \hat{\phi}_{R,y}(\frac{\epsilon}{2})} \sim \frac{1}{1} e^{-\frac{i}{\sqrt{2}} \hat{\phi}_{R,y}(\frac{\epsilon}{2})},$$  \hspace{1cm} (C14d)

where “$\sim$” denotes equality up to constant factors and nonsingular terms, and using the fact that $L_z \sim 0$ by periodic boundary conditions, we find

$$\sin \left( \frac{1}{\sqrt{2}} \left[ \hat{\phi}_{R,y} \left( \frac{\epsilon}{2} \right) - \hat{\phi}_{L,y+1} \left( \frac{\epsilon}{2} \right) \right] \right) \hat{U} \sim \frac{1}{1} \sin \left( \frac{1}{\sqrt{2}} \left[ \hat{\phi}_{R,y} \left( \frac{\epsilon}{2} \right) - \hat{\phi}_{L,y+1} \left( \frac{\epsilon}{2} \right) \right] \right).$$  \hspace{1cm} (C15)

Next, we perform the OPE

$$\hat{P}_1 \hat{\sigma}_{R,y}(0) \hat{\sigma}_{R,y}(\epsilon) \hat{P}_1 \sim \frac{1}{1}.$$  \hspace{1cm} (C16)

Inserting this pair of OPEs into Eq. (C12), we find

$$\lim_{\epsilon \to 0} \hat{D}_{2,R,y}(\epsilon) |\Omega\rangle \sim \lim_{\epsilon \to 0} e^{5/8} \sin \left( \frac{1}{\sqrt{2}} \left[ \hat{\phi}_{R,y} \left( \frac{\epsilon}{2} \right) - \hat{\phi}_{L,y+1} \left( \frac{\epsilon}{2} \right) \right] \right) \hat{\psi}_{L,y+1} \left( \frac{\epsilon}{2} \right) \hat{\psi}_{R,y} \left( \frac{\epsilon}{2} \right) |\Omega\rangle = 0.$$  \hspace{1cm} (C17)

The form of the operator appearing on the RHS above is not important. All that matters is that its expectation value in the state $|\Omega\rangle$ is not singular in the limit $\epsilon \to 0$. Also of crucial importance is the factor $e^{5/8}$ that sends $\lim_{\epsilon \to 0} \hat{D}_{2,R,y}(\epsilon) |\Omega\rangle \to 0$ as $\epsilon \to 0$. Hence, we may conclude that the state $|\Gamma_2^{(z)}\rangle$, defined in Eq. (C5c), is a ground state.

We now turn to the state $|\Gamma_1^{(z)}\rangle$, defined in Eq. (C5d), and ask if it, too, is a ground state. We will see that it cannot be a ground state by proving that the state defined in Eq. (C8a) has infinite norm as $z \to 0$ and $\epsilon \to 0$. We proceed by setting $z = z_0 = 0$ from the outset. Using Eq. (C12),

$$\hat{P}_1 \hat{\sigma}_{R,y}(0) \hat{\sigma}_{R,y}(\epsilon) \hat{P}_1 \sim \frac{1}{1}.$$  \hspace{1cm} (C18)

$$\hat{P}_1 \hat{\sigma}_{R,y}(0) \hat{\sigma}_{R,y}(\epsilon) \hat{P}_1 \sim \frac{1}{1}.$$  \hspace{1cm} (C19)
Using the OPEs (C13) in conjunction with the OPEs
\[ \hat{\sigma}_{R,y}(0) \hat{\psi}_{R,y} \left( \frac{\epsilon}{2} \right) \sim \frac{1}{\epsilon^{1/2}} \hat{\sigma}_{R,y}(0), \]
\[ \hat{\sigma}_{L,y+1}(0) \hat{\psi}_{L,y+1} \left( \frac{\epsilon}{2} \right) \sim \frac{1}{\epsilon^{1/2}} \hat{\sigma}_{L,y+1}(0), \]
we find
\[ \hat{\Gamma}^{(2)}_{1}(0) \hat{D}_{2,R,y}(\epsilon) \sim \frac{1}{\epsilon^{3/8}} \sin \left( \frac{1}{\sqrt{2}} \left[ \hat{\varphi}_{R,y} \left( \frac{\epsilon}{2} \right) - \hat{\varphi}_{L,y+1} \left( \frac{\epsilon}{2} \right) \right] \right) \hat{\Gamma}^{(2)}_{1}(0). \]

In contrast to the RHS of Eq. (C17), we now have the product between a local operator and a nonlocal operator on the RHS. Furthermore, the real-valued prefactor is a function of \( \epsilon \) that diverges as \( \epsilon \to 0 \). We conclude that
\[ \lim_{\epsilon \to 0} \hat{D}_{12}(0, \epsilon) |\Omega\rangle = \hat{\Gamma}^{(2)}_{1}(0) \hat{D}_{2,R,y}(\epsilon) |\Omega\rangle \sim \frac{1}{\epsilon^{1/8}} \sin \left( \frac{1}{\sqrt{2}} \left[ \hat{\varphi}_{R,y} \left( \frac{\epsilon}{2} \right) - \hat{\varphi}_{L,y+1} \left( \frac{\epsilon}{2} \right) \right] \right) \hat{\Gamma}^{(2)}_{1}(0) \]

is a state with infinite norm, as advertised, provided that the operator \( \sin \left( \frac{1}{\sqrt{2}} \left[ \hat{\varphi}_{R,y}(\epsilon/2) - \hat{\varphi}_{L,y+1}(\epsilon/2) \right] \right) \) does not annihilate the state \( \hat{\Gamma}^{(2)}_{1}(\epsilon) \). (Determining whether or not this is the case again requires an explicit expression for the state \( |\Omega\rangle \), which we do not have at our disposal.) In that case, we conclude that the state \( \hat{\Gamma}^{(2)}_{1}(\epsilon) \hat{\Gamma}^{(2)}_{1}(\epsilon) \) cannot be a ground state of the interaction \( \hat{H}_{bs} \) defined by Eq. (C1).

Appendix D: Diagrammatics for operator algebra in the Ising CFT

The discussion surrounding Eqs. (3.17) in the main text concerns how to infer the exchange algebra of two chiral primary operators in the Ising CFT from their operator product expansion. This exchange algebra is simple to determine in cases where the two primary operators have a unique fusion product, as in the case of the \( \sigma \) and \( \psi \) operators in Eqs. (3.17). However, when the two primary operators do not have a unique fusion product, as occurs in the case of two \( \sigma \) operators [see the OPEs in Eqs. (3.18)], the exchange algebra depends on the fusion channel in which the product of the pair of operators is evaluated [see the exchange algebra in Eqs. (3.19)]. This poses a challenge for calculations. It is necessary to keep track of both fusion and braiding in a way that respects consistency conditions between the two. This challenge is the essence of the difference between Abelian and non-Abelian excitations in quantum field theory.

To this end, it is expedient to make use of the diagrammatic calculus developed in, e.g., Refs. [3, 5, 107, 108] to represent chiral algebras associated with rational conformal field theories (RCFTs). In this Appendix, we review aspects of this calculus, as they relate to the wire constructions of two- and three-dimensional non-Abelian topological phases discussed in this work. For simplicity, we focus on the example of the Ising CFT, although generalizations to other RCFTs are straightforward.

We first define the data necessary to compute the exchange algebra of chiral primary fields in a general RCFT. These are the fusion rules, the \( R \)-symbols, and the \( F \)-symbols. The fusion rules of the \( \mathbb{Z}_k \) RCFTs were given in Eq. (B5a), and for the special case of the Ising (\( \mathbb{Z}_2 \)) CFT in Eqs. (B8).

In general, for chiral primary fields \( a, b, \) and \( c \), the fusion rules take the form
\[ a \times b = \sum_c N^{c}_{ab} c, \]
with \( N^{c}_{ab} \) nonnegative integers. The diagrammatic representation of a product of two chiral primary fields \( a \) and \( b \) that fuse to \( c \) is

\[ \text{Diagram (D1b)} \]

The requirement that the fusion algebra (D1a) be associative imposes the constraints
\[ \sum_d N^{d}_{ab} N^{c}_{de} = \sum_f N^{e}_{af} N^{c}_{bc}. \]

For many interesting RCFTs, including all of the \( \mathbb{Z}_k \) CFTs [c.f. Eq. (B5a)], the fusion coefficients \( N^{c}_{ab} = 0 \) or 1. For simplicity, we will restrict ourselves to this class of RCFTs, which is known as the class of RCFTs without fusion multiplicity since the nonnegative integers \( N^{c}_{ab} = 0 \) are never larger than one.

Read from bottom to top, diagram (D1b) is an element of the vector space \( V_{c}^{ab} \), which is known as a “splitting space.” Read from top to bottom, it is an element of the vector space \( V_{ab}^{c} \), which is known as a “fusion space.”
These vector spaces are dual to one another, and we will use the terms “fusion” and “splitting” interchangeably unless otherwise noted. The \( R \)-symbols are defined to be unitary maps
\[
R^{ab} \colon V^{ba}_e \to V^{ab}_e
\]
that implement the diagrammatic braiding operation
\[
\begin{array}{c}
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array}
\end{array} = R^{ab} \begin{array}{c}
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array}
\end{array} .
\]

Note that we have defined the diagrammatic action of the \( R \)-symbols in such a way that the left leg of the fusion tree passes over the right leg. If instead the right leg passes over the left leg, then the inverse \( R \)-symbol \((R^{ab})^{-1}\) appears. The \( R \)-symbols are essential for determining how primary operators in an RCFT behave under exchange.

The final data necessary to determine the exchange algebra of primary operators in an RCFT are the \( F \)-symbols. These are required if exchange of chiral primary fields is to be associative. Associativity of the fusions rules \((D1a)\) is encoded by Eq. \((D1c)\). Equation \((D1c)\) suggests that one defines the splitting space \(V^{abc}_d\) that encodes the fusion of three chiral fields \(a, b, c\) into one chiral field \(d\) by demanding that
\[
\sum_a V^{ab}_e \otimes V^{ec}_d = \sum_f V^{af}_d \otimes V^{bc}_f \equiv V^{abc}_d
\]
holds. The \( F \)-symbols are then defined to be unitary maps
\[
[F^{abc}]_{ef} \colon V^{ab}_e \otimes V^{ec}_d \rightarrow V^{af}_d \otimes V^{bc}_f
\]
that implement the diagrammatic operation
\[
\begin{array}{c}
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array}
\end{array} = [F^{abc}]_{ef} \begin{array}{c}
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array}
\end{array} .
\]
The \( F \)-symbols \( F^{abc}_d \) are thus automorphisms (i.e., changes of basis) of the splitting space \(V^{abc}_d\). The fusion rules, \( F \)-symbols, and \( R \)-symbols define a mathematical structure known as a braided fusion category (BFC). This structure can be used as a starting point for an axiomatic formulation of RCFT [5].

For the Ising RCFT, whose fusion rules are given in Eqs. \((B8)\), the \( R \)-symbols are given by
\[
R^{\sigma\psi}_1 = -1, \quad \text{(D4a)}
\]
\[
R^{\sigma\psi}_2 = R^{\sigma\psi}_3 = +i, \quad \text{(D4b)}
\]
with all other \( R \)-symbols trivial (i.e., equal to +1). Note that, up to complex conjugation, these \( R \)-symbols coincide with the phases acquired in Eqs. \((3.17)\) and \((3.19)\) when the corresponding chiral primary fields are exchanged. This is by design. The \( R \)-symbols reflect the monodromy of products of chiral primary fields in the corresponding RCFT. The \( F \)-symbols for the Ising RCFT are given by
\[
F^{\psi\sigma\psi}_\sigma = F^{\psi\sigma\psi}_\psi = F^{\sigma\psi\sigma}_\sigma = -1, \quad \text{(D5a)}
\]
\[
F^{\psi\sigma\psi}_\sigma = F^{\psi\sigma\psi}_\psi = F^{\sigma\psi\sigma}_\sigma = -1, \quad \text{(D5b)}
\]
\[
F^{\sigma\sigma\sigma}_\sigma = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \text{(D5c)}
\]
with all other \( F \)-symbols trivial (i.e., equal to +1).

We will now demonstrate, using the example of the Ising RCFT, how to translate diagrams like those appearing in Eqs. \((D2b)\) and \((D3c)\) into algebraic statements. Performing this translation requires one to fix a chiral sector of the CFT. We choose to work with the chiral sector \( M = R \). Once this choice is made, the starting point for this “dictionary” is to compare the diagram corresponding to the action of a particular \( R \)-symbol, say
\[
\psi
\]
with its algebraic analogue, namely Eq. \((3.17b)\),
\[
\hat{\psi}_R(z) \hat{\sigma}_R(z') = \hat{\sigma}_R(z') \hat{\psi}_R(z) e^{+i \frac{\pi}{2} \text{sgn}(z-z')} \quad \text{sgn}(z-z') = , \quad \text{(D6a)}
\]
where we have suppressed the coordinate \( t \) as we assume all operators to be evaluated at equal times, and where we have suppressed the wire labels \( y, y' \) as we are working within a single chiral sector of a single CFT. Comparing Eqs. \((D6a)\) and \((D6b)\), we see that the phases only coincide if the diagram \((D6a)\) is interpreted such that the coordinate \( z \) attached to the \( \psi \) branch is larger than the coordinate \( z' \) attached to the \( \sigma \) branch [i.e., if \( \text{sgn}(z-z') = +1 \)]. We thus establish

Rule 1: In the operator product corresponding to a fusion tree, the spatial coordinates \( z, z' \), at which the operators are evaluated, are ordered according to the positions of the corresponding branches of the fusion tree on the axis pointing into the page.

As a sanity check of this rule, we note that if the \( \psi \) branch instead passed over the \( \sigma \) branch in the dia-
gram in Eq. (D6a), we would use \((R_{a}^{\sigma})^{-1} = -i\) instead, in accordance with Eq. (D2b), but the ordering of the legs would now dictate that \(\text{sgn}(z - z') = -1\) in Eq. (D6b). Thus, Rule 1 ensures a meaningful correspondence between the \(R\)-symbols in the diagrammatics and the phases acquired under exchanging two operators in the CFT.

Next, we need to establish a convention for ordering the operators in an algebraic expression based on a fusion tree, and vice versa. There are various ways of doing this, but we choose to use

**Rule 2:** In the operator product corresponding to a fusion tree, the operators are ordered from left to right according to the order from right to left of the corresponding branches of the fusion tree, before any braiding is performed.

In Rule 2, the word “before” is interpreted under the assumption that the diagram is read from bottom to top. In this way, the ordering of operators in Eq. (D6b) agrees with the ordering of the branches of the fusion tree in Eq. (D6a).

With Rules 1 and 2 in place, we can now reliably translate fusion diagrams into equations and vice versa. For example, the correspondence

\[
\begin{array}{c}
\psi_{M,y}(t,z)\sigma_{M',y'}(t,z') = \sigma_{M',y'}(t,z')\psi_{M,y}(t,z) e^{i\frac{\pi}{2}(-1)^{M'}\delta_{M,M'}\delta_{y,y'}\text{sgn}(z-z')} e^{i\epsilon_{M,M'}\delta_{y,y'}\varphi} e^{i\epsilon_{y-y'}\theta_{M,M'}},
\end{array}
\]

is used in Eqs. (3.63) and (3.64) of the main text, while the correspondence

\[
\begin{array}{c}
\sigma_{R}(z)\sigma_{R}(z_{1})\sigma_{R}(z_{2})
\end{array}
\]

is used in Eqs. (5.44) and (5.45).

**Appendix E: Independence of string-operator algebra on arbitrary phase factors**

We have made extensive use of the fact that the OPE of two operators in the same wire determines the algebra of these two operators under exchange. However, in certain situations [e.g. Eqs. (5.3) and (3.4)], we found it important (on physical grounds) to modify the exchange algebra between operators in different wires. We will now show that, despite their importance in calculating local quantities (such as the masses and velocities of the Majorana and Dirac modes on the gapless surface of the 3D phase in Sec. VIB), these modifications have no effect on topological features like the ground state degeneracy.

We proceed with an explicit example that illustrates how this comes about for the \(su(2)_{2}\) case in 2D studied in Sec. III C. We begin by rewriting the exchange algebra (3.17), but this time allowing for operators in different wires to have nontrivial commutation with one another. Hence, we posit that

\[
\begin{array}{c}
\psi_{M,y}(t,z)\sigma_{M',y'}(t,z') = \sigma_{M',y'}(t,z')\psi_{M,y}(t,z) e^{i\frac{\pi}{2}(-1)^{M'}\delta_{M,M'}\delta_{y,y'}\text{sgn}(z-z')} e^{i\epsilon_{M,M'}\delta_{y,y'}\varphi} e^{i\epsilon_{y-y'}\theta_{M,M'}},
\end{array}
\]

where \((-1)^{R} \equiv (-1)^{L} \equiv 1, \epsilon_{R,L} = -\epsilon_{L,R} = 1, \epsilon_{R,R} = \epsilon_{L,L} = 0\). The reason why the choice (E1) has no effect on the topological features of the phase is that all of these features depend on the algebra of string operators, which are constructed from bilinears in the operators \(\hat{\psi}_{M,y}\) and \(\hat{\sigma}_{M,y}\). In particular, for Majorana and twist-field operators in the same wire \(y\), we have

\[
\begin{array}{c}
\hat{\psi}_{R,y}(t,z)\psi_{L,y}(t,z)\sigma_{R,y}(t,z')\sigma_{L,y}(t,z') = \sigma_{R,y}(t,z')\hat{\sigma}_{L,y}(t,z)\psi_{R,y}(t,z)\psi_{L,y}(t,z) \\
\times e^{i\epsilon_{L,R}\varphi} e^{i\epsilon_{y-y'}\theta_{L,L}+\epsilon_{R,L}+\epsilon_{R,R}}
\end{array}
\]

\((E2)\)

For Majorana and twist-field operators in different wires \(y \neq y'\), we find that

\[
\begin{array}{c}
\hat{\psi}_{R,y}(t,z)\psi_{L,y'}(t,z)\sigma_{R,y'}(t,z')\sigma_{L,y'}(t,z') = \sigma_{R,y'}(t,z')\hat{\sigma}_{L,y'}(t,z)\psi_{R,y}(t,z)\psi_{L,y}(t,z) \\
\times e^{i\epsilon_{y-y'}\theta_{L,L}+\epsilon_{R,L}+\epsilon_{R,R}}
\end{array}
\]

\((E3)\)
holds so long as the angles $\theta_{M,M'}$ satisfy

$$\theta_{L,R} + \theta_{L,L} + \theta_{R,R} + \theta_{R,L} \in 2\pi \mathbb{Z}. \quad (E4a)$$

For general choices of the angles $\theta_{M,M'}$, Eq. (E4a) is automatically satisfied if

$$\theta_{R,R} = -\theta_{L,L}, \quad \theta_{L,R} = -\theta_{R,L}. \quad (E4b)$$

Thus, when string operators are built from bilinears like $\hat{\psi}_{R,y} \hat{\psi}_{L,y}$ and $\hat{\sigma}_{R,y} \hat{\sigma}_{L,y}$, the additional phases in the exchange algebra (E1) drop out of all calculations.

The calculations of the previous paragraph generalize readily to other combinations of primary operators, and to the three-dimensional case. The key observation in all cases is that string (and membrane) operators are built either from nonchiral bilinears of primary operators, like the ones studied in the previous paragraph, or from operators like $\hat{U}_s(t)$ [defined in Eq. (3.12)] that act only within one channel of one wire.

[1] X.-G. Wen, Phys. Rev. B 40, 7387 (1989).
[2] M. Oshikawa and T. Senthil, Phys. Rev. Lett. 96, 066001 (2006).
[3] D. Friedan and S. Shenker, Nucl. Phys. B 281, 509 (1987).
[4] J. Fröhlich, in Nonperturbative Quantum Field Theory, edited by G. ’t Hooft, A. Jaffe, G. Mack, P. Mitter, and R. Stora (Springer US, 1988), vol. 185 of NATO ASI Series, pp. 71–100.
[5] G. Moore and N. Seiberg, Comm. Math. Phys. 123, 177 (1989).
[6] J. Fröhlich and F. Gabbiani, Rev. Math. Phys. 2, 251 (1990).
[7] D. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik, Tensor categories (Mathematical Surveys and Monographs, American Mathematical Society, 2015).
[8] M. Barkeshli, P. Bonderson, M. Cheng, and Z. Wang, arXiv:1410.4540 (unpublished).
[9] J. C. Teo, T. L. Hughes, and E. Fradkin, Ann. Phys. 360, 349 (2015).
[10] M. Barkeshli, P. Bonderson, C.-M. Jian, M. Cheng, and K. Walker, arXiv:1612.07792 (unpublished).
[11] R. Thorngren and D. V. Else, arXiv:1612.00846 (unpublished).
[12] Z.-C. Gu and X.-G. Wen, Phys. Rev. B 90, 115141 (2014).
[13] M. Cheng, Z. Bi, Y.-Z. You, and Z.-C. Gu, arXiv:1501.01313 (unpublished).
[14] A. Kapustin, R. Thorngren, A. Turzillo, and Z. Wang, J. High Energy Phys. 2015, 52 (2015).
[15] D. Gaiotto and A. Kapustin, Int. J. Mod. Phys. A 31, 1645044 (2016).
[16] B. Ware, J. H. Son, M. Cheng, R. V. Mishmash, J. Alicea, and B. Bauer, Phys. Rev. B 94, 115127 (2016).
[17] M. Tarantino and L. Fidkowski, Phys. Rev. B 94, 115115 (2016).
[18] D. J. Williamson, N. Bultinck, J. Haegeman, and F. Verstraete, arXiv:1609.02897 (unpublished).
[19] L. Fu, C. L. Kane, and E. J. Mele, Phys. Rev. Lett. 98, 106803 (2007).
[20] J. E. Moore and L. Balents, Phys. Rev. B 75, 121306 (2007).
[21] D. Hsieh, D. Qian, L. Wray, Y. Xia, Y. S. Hor, R. J. Cava, and M. Z. Hasan, Nature (London) 452, 970 (2008).
[22] Y. L. Chen, J. G. Analytis, J. H. Chu, Z. K. Liu, S. K. Mo, X. L. Qi, H. J. Zhang, D. H. Lu, X. Dai, Z. Fang, S. C. Zhang, I. R. Fisher, Z. Hussain, Z. X. Shen, Science 325, 178 (2009).
[23] D. Hsieh, Y. Xia, D. Qian, L. Wray, F. Meier, J. H. Dil, J. Osterwalder, L. Patthey, A. V. Fedorov, H. Lin, et al., Phys. Rev. Lett. 103, 146401 (2009).
[24] D. Hsieh, Y. Xia, D. Qian, L. Wray, J. Dil, F. Meier, J. Osterwalder, L. Patthey, J. Checkelsky, N. Ong, et al., Nature (London) 460, 1101 (2009).
[25] R. Dijkgraaf and E. Witten, Comm. Math. Phys. 129, 393 (1990).
[26] J. C. Wang and X.-G. Wen, Phys. Rev. B 91, 035134 (2015).
[27] Y. Wan, J. C. Wang, and H. He, Phys. Rev. B 92, 045101 (2015).
[28] D. V. Else and C. Nayak, arXiv:1702.02148 (unpublished).
[29] L. Crane and D. Yetter, in Quantum Topology, edited by L. H. Kauffman and R. A. Baadhio (1993), p. 120.
[30] K. Walker and Z. Wang, Frontiers of Physics 7, 150 (2012).
[31] Z. Wang and X. Chen, Phys. Rev. B 95, 115142 (2017).
[32] D. J. Williamson and Z. Wang, Ann. Phys. 377, 311 (2017).
[33] The Quantum Hall Effect, edited by R. E. Prange and S. M. Girvin (Springer, New York, 1987).
[34] X.-G. Wen, F. Wilczek, and A. Zee, Phys. Rev. B 39, 11413 (1989).
[35] X.-G. Wen, Phys. Rev. B 354, 369 (1991).
[36] J. Fröhlich and T. Kerler, Nucl. Phys. B 364, 517 (1991).
[37] J. Fröhlich and A. Zee, Nucl. Phys. B 364, 517 (1991).
[38] J. Fröhlich and A. Zee, Nucl. Phys. B 364, 517 (1991).
[39] B. I. Halperin, Phys. Rev. B 25, 1799 (1982).
[40] F. Wilczek, Phys. Rev. Lett. 58, 1799 (1987).
[41] X.-L. Qi, T. L. Hughes, and S.-C. Zhang, Phys. Rev. B 78, 195124 (2008).
[42] A. M. Essin, J. E. Moore, and D. Vanderbilt, Phys. Rev. Lett. 102, 146805 (2009).
[43] H. Nielsen and M. Ninomiya, Phys. Lett. B 150, 219 (1981).
[44] J. Maciejko, X.-L. Qi, A. Karch, and S.-C. Zhang, Phys. Rev. Lett. 105, 246809 (2010).
[45] B. Swingle, M. Barkeshli, J. McGreevy, and T. Senthil, Phys. Rev. B 83, 195139 (2011).
[46] X. G. Wen, Phys. Rev. B 43, 11025 (1991).
[47] G. Moore and N. Read, Nucl. Phys. B 360, 362 (1991).
[48] N. Read and E. Rezayi, Phys. Rev. B 59, 8084 (1999).
[49] X.-G. Wen, Phys. Rev. Lett. 66, 802 (1991).
[50] V. J. Emery, E. Fradkin, S. A. Kivelson, and T. C. Lubensky, Phys. Rev. Lett. 85, 2160 (2000).
[50] R. Mukhopadhyay, C. L. Kane, and T. C. Lubensky, Phys. Rev. B 63, 081103 (2001).
[51] A. Vishwanath and D. Carpentier, Phys. Rev. Lett. 86, 676 (2001).
[52] D. Poilblanc, G. Montambaux, M. Héritter, and P. Lederer, Phys. Rev. Lett. 58, 270 (1987).
[53] V. M. Yakovenko, Phys. Rev. B 43, 11353 (1991).
[54] D.-H. Lee, Phys. Rev. B 50, 10788 (1994).
[55] S. L. Sondhi and K. Yang, Phys. Rev. B 63, 054430 (2001).
[56] C. L. Kane, R. Mukhopadhyay, and T. C. Lubensky, Phys. Rev. Lett. 88, 036401 (2002).
[57] J. C. Y. Teo and C. L. Kane, Phys. Rev. B 89, 085101 (2014).
[58] R. S. K. Mong, D. J. Clarke, J. Alicea, N. H. Lindner, P. Fendley, C. Nayak, Y. Oreg, A. Stern, E. Berg, K. Shtengel, et al., Phys. Rev. X 4, 011036 (2014).
[59] T. Neupert, C. Chamon, C. Mudry, and R. Thomale, Phys. Rev. B 93, 195136 (2016).
[60] M. M. Vazifeh, Europhys. Lett. 102, 67011 (2013).
[61] T. Meng, Phys. Rev. B 92, 115152 (2015).
[62] E. Sagi and Y. Oreg, Phys. Rev. B 92, 195137 (2015).
[63] T. Iadecola, T. Neupert, C. Chamon, and C. Mudry, Phys. Rev. B 93, 195136 (2016).
[64] E. Witten, Comm. Math. Phys. 92, 455 (1984).
[65] I. Affleck, Nucl. Phys. B 336, 517 (1990).
[66] I. Affleck and A. W. W. Ludwig, Nucl. Phys. B 352, 849 (1991).
[67] I. Affleck and A. W. W. Ludwig, Nucl. Phys. B 360, 641 (1991).
[68] A. M. Tsvelik, Phys. Rev. Lett. 113, 066401 (2014).
[69] P.-H. Huang, J.-H. Chen, P. R. S. Gomes, T. Neupert, C. Chamon, and C. Mudry, Phys. Rev. B 93, 205123 (2016).
[70] P.-H. Huang, J.-H. Chen, A. E. Feiguin, C. Chamon, and C. Mudry, Phys. Rev. B 95, 144413 (2017).
[71] J.-H. Chen, C. Mudry, C. Chamon, and A. M. Tsvelik, ArXiv e-prints (2017), 1708.06446.
[72] M. Oshikawa, Y. B. Kim, K. Shtengel, C. Nayak, and S. Tewari, Ann. Phys. 322, 1477 (2007).
[73] D. F. Mross, A. Essin, and J. Alicea, Phys. Rev. X 5, 011011 (2015).
[74] S. Sahoo, Z. Zhang, and J. C. Y. Teo, Phys. Rev. B 94, 165142 (2016).
[75] R. S. K. Mong, A. M. Essin, and J. E. Moore, Phys. Rev. B 81, 245209 (2010).
[76] C. Fang, M. J. Gilbert, and B. A. Bernevig, Phys. Rev. B 88, 085406 (2013).
[77] J. Leinaas and J. Myrheim, Nuovo Cim. B 37, 1 (1977).
[78] C. W. von Keyserlingk, F. J. Burnell, and S. H. Simon, Phys. Rev. B 87, 045107 (2013).
[79] A. Vishwanath and T. Senthil, Phys. Rev. X 3, 011016 (2013).
[80] C. Wang and T. Senthil, Phys. Rev. B 87, 235122 (2013).
[81] C. Wang, A. C. Potter, and T. Senthil, Phys. Rev. B 88, 115137 (2013).
[82] F. J. Burnell, X. Chen, L. Fidkowski, and A. Vishwanath, Phys. Rev. B 90, 245122 (2014).
[83] X. Chen, L. Fidkowski, and A. Vishwanath, Phys. Rev. B 89, 165132 (2014).
[84] M. A. Metlitski, C. L. Kane, and M. P. A. Fisher, Phys. Rev. B 92, 125111 (2015).
[85] P. Di Francesco, P. Mathieu, and D. Sénéchal, Conformal field theory, Graduate texts in contemporary physics (Springer, New York, 1997).
[86] H. Sugawara, Phys. Rev. 170, 1659 (1968).
[87] J. Wess and B. Zumino, Phys. Lett. B 37, 95 (1971).
[88] E. Witten, Comm. Math. Phys. 121, 351 (1989).
[89] D. C. Cabra, E. Fradkin, G. L. Rossini, and F. A. Schaipsnik, Int. J. Mod. Phys. A 15, 8457 (2000).
[90] A. B. Zamolodchikov and V. A. Fateev, Zh. Eksp. Teor. Fiz. 89, 380 (1985).
[91] P. Fendley, M. P. A. Fisher, and C. Nayak, Phys. Rev. B 75, 045317 (2007).
[92] There is a caveat here that has extremely important implications for the derivation of the topological degeneracy, and that we discuss in detail in Appendix C. The caveat is that, although the operator \( \hat{p}^{(1)}_{\pm,\mu}(t, \epsilon) \) commutes with the interaction (3.7) in the limit \( \epsilon \to 0 \), this need not be (and, in fact, is not) true of the operator products \( \hat{p}^{(1/2)}_{1}(t, z) \hat{p}^{(1)}_{2,\mu}(t, \epsilon) \) and \( \hat{p}^{(1/2)}_{2,\mu}(t, \epsilon) \hat{p}^{(1/2)}_{1}(t, z) \) in the same limit. As discussed in Appendix C, the reason for this unusual limiting behavior as a function of the regulator \( \epsilon \) has to do with the fact that \( \hat{p}^{(1/2)}_{1}(t, z) \) and \( \hat{p}^{(1/2)}_{2,\mu}(t, \epsilon) \) are nonlocal, and nonunitary, operators that change the topological sectors of the states on which they act, and is intrinsically related to the non-Abelian nature of the topological phase. To keep clear track of this limiting behavior, we will always defer the evaluation of the limit \( \epsilon \to 0 \) to the end of all calculations.
[93] A. Kitaev, Ann. Phys. 303, 2 (2003).
[94] N. Read and D. Green, Phys. Rev. B 61, 10267 (2000).
[95] F. D. M. Haldane, Phys. Rev. Lett. 74, 2090 (1995).
[96] X.-G. Wen, Phys. Rev. Lett. 90, 016803 (2003).
[97] A. Y. Kitaev, Phys.-Usp. 44, 131 (2001).
[98] Any of the states in Table III involving one or more of the \( \epsilon \)-regularized operators \( \hat{p}^{(1/2)}_{1}, \hat{p}^{(1/2)}_{2}, \) and \( \hat{p}^{(1/2)}_{\sigma} \) can be shown to be a ground state of the interaction \( \hat{H}_{\text{int}}[su(2)_{1}] \) by an argument along the lines of Appendix C.
[99] J. Cardy, Scaling and renormalization in statistical physics, vol. 5 (Cambridge University Press, 1996).
[100] C. S. O’Hern, T. C. Lubensky, and J. Toner, Phys. Rev. Lett. 83, 2745 (1999).
[101] V. Fateev and A. Zamolodchikov, Phys. Lett. B 271, 91 (1991).
[102] E. Fradkin, Field Theories of Condensed Matter Physics (Cambridge University Press, 2013).
[103] X.-G. Wen, Phys. Rev. B 60, 8827 (1999).
[104] M. Barkeshli and X.-G. Wen, Phys. Rev. B 81, 155302 (2010).
[105] C.-M. Jian and X.-L. Qi, Phys. Rev. X 4, 041043 (2014).
[106] I. Affleck, Nucl. Phys. B 265, 409 (1986).
[107] C. Vafa, Phys. Lett. B 206, 421 (1988).
[108] E. Verlinde, Nucl. Phys. B 300, 360 (1988).