The exponent of the longitudinal structure function $F_L$ at low $x$

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Abstract We present a set of formula to extract exponents of the longitudinal structure function and reduced cross section from the Regge-like behavior at small $x$. The exponents are found to be independent of $Q^2$ at NNLO analysis. As a result, we show that the reduced cross-sectional exponents do not have the same behavior at some values of $x$. This difference predicts the non-linear effects and some evidence for shadowing and antishadowing at LHeC. Also the ratio $\frac{F_2}{\sigma}$ is calculated and compared with the corresponding HERA data. Our calculations show a very good agreement with the DIS experimental data throughout the small values of $x$.

1 Introduction

The appropriate framework for the theoretical description of the small-$x$ behavior of the structure functions is the Regge approach. The Regge theory gives a good description of the structure functions, where the high-energy scattering can be described by power-like behavior at small-$x$. The following parameterization of the deep inelastic scattering structure function $F_2(x, Q^2)$ defined by

$$F_2(x, Q^2) \simeq \sum_i A_i(Q^2)x^{-\lambda_i} \quad (1)$$

that the singlet part of the structure function is controlled by pomeron exchange. Here the $i = 0$ term is hard-pomeron and $i = 1$ is soft-pomeron exchange [1, 2]. The effective intercept behavior, at small values of $x$, exhibited for the fast growth of the singlet structure function. The exponent $\lambda_s$ is found to be $\simeq 0.33$ in Refs. [3, 4]. It can be recast into the symbolic form as

$$F_2(x, Q^2) = A_s(Q^2)x^{-\lambda_s}. \quad (2)$$

According to the Regge theory the charm component $F_2^c(x, Q^2)$ of $F_2(x, Q^2)$ is governed entirely by hard pomeron exchange. In perturbative quantum chromodynamics (pQCD) the charmed quark originates from a gluon in the proton. Therefore, the small-$x$ behavior of the gluon distribution function is dominated with hard-pomeron intercept as

$$G(x, Q^2) = A_g(Q^2)x^{-\lambda_g}, \quad (3)$$

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where \( \lambda_g \simeq 0.42 \) [1–9]. This implies that the gluon distribution is dominated by the hard pomeron behavior. Indeed this steep behavior of the gluon distribution generates a similar steep behavior of \( F_2 \) at small \( x \) where \( \lambda_s \neq \lambda_g \) in high-order corrections.

We now consider the proton’s longitudinal structure function \( F_L(x, Q^2) \). It is known that the dominant source for the longitudinal structure function, at small-\( x \), is the gluon density. It is become traditional to believe that the longitudinal intercept has the same behavior of the gluon density. It is tempting, however, to explore the possibility of obtaining analytical solutions of the longitudinal intercept in the restricted domain of small \( x \) at least. In this paper, we suggest the power-like behavior of the longitudinal structure function as

\[
F_L(x, Q^2) = A_L(Q^2)x^{-\lambda_L} \tag{4}
\]

at high-order corrections [10–13].

In pQCD, the Altarelli–Martinelli equation for longitudinal structure function in terms of coefficient functions is given by [14–16]

\[
gluonic \text{ term:} \quad x^{-1}F_L = \langle e^2 \rangle C_{L,g} \otimes g,
\]

\[
singlet + \text{gluon terms } n_f = 4 \quad x^{-1}F_L = \langle e^2 \rangle (C_{L,q} \otimes q_s + C_{L,g} \otimes g),
\]

\[
light quarks + \text{heavy terms } n_f = 3 \quad x^{-1}F_L = \langle e^2 \rangle (C_{L,q} \otimes q_s + C_{L,g} \otimes g) + x^{-1}F_L^c. \tag{5}
\]

At small \( x \), the nonsinglet contribution \( F_L^{ns} \) is negligible and can be ignored. Here \( q_s \) and \( g \) are the flavour singlet and gluon distribution function, where \( \langle e^2 \rangle \) stand for the average of the charge \( e^2 \) for the active quark flavours \( \langle e^2 \rangle = n_f^{-1} \sum_{i=1}^{n_f} e_i^2 \) and \( n_f \) denotes the number of active light flavours. The \( \otimes \) symbol denotes the convolution integral which turns into a simple multiplication in Mellin \( N \)-space and the notation is given by \( a(x) \otimes b(x) = \int_x^1 \frac{dz}{z} a(z) b(\frac{z}{x}) \).

The coefficient functions can be expressed as \( C_{L,q,g} (\alpha_s, x) = \sum_{n=1}^{\infty} (\frac{\alpha_s}{4\pi})^n C_{L,q,g}^{(n)} (x) \) [15,16], where \( C_{L,q,g}^{(1)} \), \( C_{L,q,g}^{(2)} \) and \( C_{L,q,g}^{(3)} \) are LO, NLO and NNLO contributions, respectively. The explicit forms of the first-order until third-order coefficient functions are given in Appendix A. Here \( n \) denotes the order in \( \alpha_s \) as at NNLO analysis the running coupling constant has the following form:

\[
\alpha_s(t) = \frac{4\pi}{\beta_0 t} \left[ 1 - \frac{\beta_1 \ln t}{\beta_0^2} \frac{t}{\beta_0} + \frac{1}{\beta_0^3 t^2} \left( \frac{\beta_1^2}{\beta_0} (\ln^2 t - \ln t - 1) + \beta_2 \right) \right]. \tag{6}
\]

Here \( \beta_i \)‘s are the high-order corrections to the QCD \( \beta \)-function and \( t = \ln \frac{Q^2}{\Lambda^2} \) where \( \Lambda \) is the QCD cut-off parameter. In Eq. (5), we use the NLO expression for the longitudinal charm structure function \( F_L^c \) [17–23] where the charm cross section is generated by photon–gluon fusion. This is called the fixed flavour number scheme (FFNS) and incorporates the correct threshold behavior for \( Q^2 \sim m_c^2 \) and extended to the zero mass variable flavour number scheme (ZM-VFNS) above this threshold [24–26]. In the framework of this scheme, we consider the heavy flavor physics in the DGLAP [27–29] dynamics. Further simplification...
is obtained by neglecting the contributions caused by incoming light quark and antiquarks at small values of $x$.

The contribution of the longitudinal structure function $F_L$ to the cross section can be sizeable only at large values of the inelasticity $y$ [30,31]. The reduced cross section is defined as

$$\sigma_r \equiv F_2(x, Q^2) - \frac{y^2}{Y_+} F_L(x, Q^2),$$

where $y = Q^2/sx$ is the inelasticity with $s$ is the $ep$ center of mass energy squared and $Y_+ = 1 + (1 - y)^2$. At small-$y$, the relation $\sigma_r \approx x^{-\lambda_L}$ holds to a very good approximation as the cross section rises with decreasing $x$. However, at very high-$y$ a characteristic bending of the cross section is attributed to the longitudinal structure function contribution [32–37].

In this paper, we suggest analytical solutions of the high-order corrections for the longitudinal structure function exponent at small $x$. The results have been included in the reduced cross section exponent. The behavior of these exponents are compared with the gluon and singlet exponents where hard pomeron is dominant.

2 Behavior of $F_L$

2.1 Gluonic term

The perturbative predictions for the gluonic longitudinal structure function can be written as

$$F_{Lg}^g(x, Q^2) = \langle e^2 \rangle C_{Lg}(\alpha_s, x) \otimes G(x, Q^2),$$

where $G(x, Q^2) = xg(x, Q^2)$. The evolution of $\partial F_{Lg}^g(x, Q^2)/\partial \ln x$ at fixed $Q^2$ is obtained by the following form:

$$\frac{\partial F_{Lg}^g(x, Q^2)}{\partial \ln x} = \langle e^2 \rangle \left\{ \frac{\partial G(x, Q^2)}{\partial \ln x} (C_{Lg}(x, \alpha_s) \otimes x^{\lambda_g}) + G(x, Q^2) \frac{\partial}{\partial \ln x} (C_{Lg}(x, \alpha_s) \otimes x^{\lambda_g}) \right\},$$

where $C_{Lg}(x, \alpha_s) = \frac{\alpha_s}{4\pi} C_{Lg}^{LO}(x) + (\frac{\alpha_s}{4\pi})^2 C_{Lg}^{NLO}(x) + (\frac{\alpha_s}{4\pi})^3 C_{Lg}^{NNLO}(x)$. Here we used the Regge-like behavior of the gluon distribution function in Eq. (8). Using Eqs. (8) and (9) and simplifying derivative of the longitudinal structure function, we get

$$\frac{\partial \ln F_{Lg}^g(x, Q^2)}{\partial \ln x} = \frac{\partial \ln G(x, Q^2)}{\partial \ln x} + \frac{\partial \ln I_g(x, Q^2)}{\partial \ln x},$$

where $I_g(x, Q^2) = \langle e^2 \rangle C_{Lg}(x, \alpha_s) \otimes x^{\lambda_g}$. We note that exponents $\lambda_g$ and $\lambda_L$ are given as the derivatives

$$\lambda_g = \frac{\partial \ln G(x, Q^2)}{\partial \ln (1/x)}$$

and

$$\lambda_L = \frac{\partial \ln F_L(x, Q^2)}{\partial \ln (1/x)}.$$
Therefore, the longitudinal exponent with respect to the gluonic term is defined as follows:

$$\lambda_L = \lambda_g + \frac{\partial \ln I_g(x, Q^2)}{\partial \ln (1/x)}, \quad (12)$$

where

$$I_g(x, Q^2) = \langle e^2 \rangle \int_x^1 \frac{dz}{z} \left[ \frac{\alpha_s}{4\pi} C^{LO}_{L,g}(x) + \left(\frac{\alpha_s}{4\pi}\right)^2 C^{NLO}_{L,g}(x) \right] + \left(\frac{\alpha_s}{4\pi}\right)^3 C^{NNLO}_{L,g}(x) \right] x^{\lambda_g}. \quad (13)$$

2.2 Singlet + gluon terms

The standard collinear factorization formula for the longitudinal structure function in terms of singlet and gluon structure function at small-$x$ is given by

$$F_L(x, Q^2) = C_{L,q}(\alpha_s, x) \otimes F^q_L(x, Q^2) + \langle e^2 \rangle C_{L,g}(\alpha_s, x) \otimes G(x, Q^2). \quad (14)$$

Taking the derivative of Eq. (14) with respect to $\ln x$ for each value of constant $Q^2$, we get

$$\frac{\partial F_L(x, Q^2)}{\partial \ln x} = \frac{\partial F_2(x, Q^2)}{\partial \ln x} I_s(x, Q^2) + F^q_L(x, Q^2) \frac{\partial I_s(x, Q^2)}{\partial \ln x} + \frac{\partial G(x, Q^2)}{\partial \ln x} I_g(x, Q^2) G(x, Q^2) \frac{\partial I_g(x, Q^2)}{\partial \ln x}, \quad (15)$$

where $I_s(x, Q^2) = C_{L,q}(\alpha_s, x) \otimes x^{\lambda_s}$. Exploiting the small-$x$ behavior of the distribution functions according to the hard-pomeron. Then Eq. (14) can be rewritten as

$$F_L(x, Q^2) = F^q_L(x, Q^2) I_s(x, Q^2) + G(x, Q^2) I_g(x, Q^2). \quad (16)$$

Now, using Eqs. (15) and (16), the longitudinal exponent $\lambda_L$ is found directly from the singlet and gluon exponents, namely

$$\lambda_L = \frac{I_s(x, Q^2) + \partial I_s(x, Q^2)/\partial \ln (1/x) + K(x, Q^2)[I_g(x, Q^2) + \partial I_g(x, Q^2)/\partial \ln (1/x)]}{I_s(x, Q^2) + K(x, Q^2)I_g(x, Q^2)}, \quad (17)$$

where $K(x, Q^2) = G(x, Q^2)/F^q_L(x, Q^2)$. We observe that Eq. (17) implies a relationship between the longitudinal exponent and singlet-gluon exponents for even $n_f$. Thus, an analytical expression for the longitudinal exponent $\lambda_L$ is suggested at LO, NLO and NNLO.

2.3 Light + charm terms

In a similar manner, the charm contribution to the longitudinal structure function is considered and the longitudinal exponent can be determined at small $x$ with the help of the light and gluon exponents as

$$F^\text{Total}_L = F^\text{Light}_L (= F^q_L + F^S_L) + F^\text{Heavy}_L. \quad (18)$$

where $F^\text{Light}_L = \text{Eq. (14)}$ with $\langle e^2 \rangle = \frac{2}{5}$ for $n_f = 3$ (number of active light flavours).
With respect to the recent measurements of HERA [40], the charm contribution to the structure function at small $x$ is a large fraction of the total. This behavior is directly related to the growth of the gluon distribution at small $x$ [20–23] as

$$F^c_L(x, Q^2, m^2_c) = 2e_c^2 \alpha_s(\mu^2) \frac{2\pi}{2\pi} \int_{1-\frac{1}{n}}^{1-x} dz C_{g,L}^c(1-z, \zeta)$$

$$\times G \left( \frac{x}{1-z}, \mu^2 \right), \quad (19)$$

where $a = 1 + 4\xi(\xi = \frac{m^2_c}{\mu^2})$ and $\mu$ is the mass factorization scale. The factorization scale is equal to the renormalization scales $\mu^2 = 4m^2_c$ or $\mu^2 = 4m^2_c + Q^2$. Here $C_{g,L}^c$ is the charm coefficient functions in LO and NLO analyses [41–47] as

$$C_{g,L}(z, \zeta) \rightarrow C_{g,L}^0(z, \zeta) + a_s(\mu^2) \left[ C_{g,L}^1(z, \zeta) + C_{g,L}^2(z, \zeta) \ln \frac{\mu^2}{m^2_c} \right]. \quad (20)$$

The explicit form of the charm longitudinal coefficient function is given in Appendix B.

Here $a_s(\mu^2) = \frac{a_s(\mu^2)}{4\pi}$ and in the NLO analysis

$$\alpha_s(\mu^2) = \frac{4\pi}{\beta_0 \ln(\mu^2/\Lambda^2)} - \frac{4\pi \beta_1 \ln \ln(\mu^2/\Lambda^2)}{\beta_0^2} \frac{\ln(\mu^2/\Lambda^2)}{\ln(\mu^2/\Lambda^2)} \quad (21)$$

with $\beta_0 = 11 - \frac{2}{3} n_f$ and $\beta_1 = 102 - \frac{38}{3} n_f$.

After doing the integration over $z$, Eq. (19) can be rewritten as

$$F^c_L(x, Q^2, m^2_c) = G(x, Q^2)[C_{g,L}^c(x, Q^2) \otimes x^\lambda_c]$$

$$= G(x, \mu^2)I_c(x, Q^2), \quad (22)$$

where

$$I_c(x, Q^2) = 2e_c^2 \alpha_s(\mu^2) \frac{2\pi}{2\pi} \int_{1-\frac{1}{n}}^{1-x} dz C_{g,k}^c(1-z, \zeta)(1-z)^\lambda_c \ln(1-z)$$

$$\times G \left( \frac{x}{1-z}, \mu^2 \right). \quad (23)$$

The $x$-derivative of the longitudinal structure function is defined by

$$\frac{\partial F^c_L(x, Q^2)}{\partial \ln x} = \text{Eq.}(15) + \frac{\partial F^c_L(x, Q^2)}{\partial \ln x} = \text{Eq.}(15)$$

$$+ \frac{\partial G(x, Q^2)}{\partial \ln x} I_c(x, Q^2) + G(x, Q^2) \frac{\partial I_c(x, Q^2)}{\partial \ln x}. \quad (24)$$

Following the suggestion of the power-like behavior of the logarithmic $x$-derivative of the distribution functions, we have the longitudinal exponent $\lambda_c$ for $n_f = 3 + \text{charm}$, as

$$\lambda_c = \frac{I_c(x, Q^2)\lambda_c + \partial I_c(x, Q^2)\partial \ln(1/x) + K(x, Q^2)[I_c(x, Q^2) + I_c(x, Q^2)\lambda_c + \partial I_c(x, Q^2)\partial \ln(1/x)][I_c(x, Q^2) + I_c(x, Q^2)]}{I_c(x, Q^2) + K(x, Q^2)[I_c(x, Q^2) + I_c(x, Q^2)]. \quad (25)$$

Therefore, Eqs. (12), (17) and (25) are a set of formulas to extract the longitudinal exponent from the singlet and gluon exponents at gluonic, singlet+gluon and light+charm terms, respectively, in LO, NLO and NNLO.
Fig. 1 The longitudinal exponent $\lambda_L$ given by Eq. (12) versus $x$ at four fixed $Q^2$ values at LO, NLO and NNLO analysis, compared with the gluon exponent $\lambda_g = 0.42$ (dot line). The predictions for $\lambda_L$ at NLO and NNLO are almost equal ($\lambda_{L,NLO} \simeq \lambda_{L,NNLO}$).

We now discuss how the presented results give the exponents for the longitudinal structure functions at small $x$. In Ref. [4], the authors have suggested that singlet and gluon effective exponents can be reasonably defined by color dipole model and hard-pomeron exponents. The exponents of $\lambda_s$ and $\lambda_g$ are found to be $\simeq 0.33$ and $\simeq 0.42$, respectively [4]. Based on the coefficient functions [15,16] and effective exponents, we present result for the longitudinal exponents at LO, NLO and NNLO using Eqs. (12), (17) and (25), respectively.

In Eq. (12), the longitudinal exponent behavior for the gluonic contribution is determined. After doing the integration and using the required coefficient functions, the longitudinal exponents, in the range $10^{-5} \leq x \leq 10^{-2}$ and $2 \leq Q^2 \leq 45$ GeV$^2$, are determined in Fig. 1. In this figure, the obtained results are compared with $\lambda_g = 0.42$. We observe that $\lambda_L \leq \lambda_g$ at NLO and NNLO analyses. For comparison, the predictions for $\lambda_L$ at NLO and NNLO are almost equal. In all the graphs, $\lambda_L$ is equal to $\lambda_g$ at very low $x$ values. For all values of $x$, we observe that $\lambda_L = \lambda_g$ only at LO analysis. In this case, the longitudinal exponent is hard-pomeron dominated. Therefore, the averaged value to all exponents has the effective constraint where the effective longitudinal exponent has the following value as

$$\lambda_{L,\text{Gluonic}} \simeq 0.41.$$ (26)

In the following, the longitudinal exponent is obtained using the singlet and gluon terms from Eq. (17) with respect to the exponents in Fig. 2. In this figure, the longitudinal exponent $\lambda_L$ is plotted against $x$ for different values of $Q^2$ in comparison with singlet ($\lambda_s = 0.33$) and gluon ($\lambda_g = 0.42$) exponents at LO, NLO and NNLO analysis. Since $\lambda_L$ is an analytical function of $x$, it cannot be exactly constant at small $x$. This is due to the coefficient function and dispersion of data. Nevertheless, we observe that $\lambda_L$ does not strongly depend on $x$ at
Fig. 2  The longitudinal exponent $\lambda_L$ given by Eq. (17) versus $x$ at four fixed $Q^2$ values at LO, NLO and NNLO analysis, compared with the gluon exponent $\lambda_g = 0.42$ (dot line) and singlet exponent $\lambda_s = 0.33$ (dash line).

$x < 0.01$ $[48–50]$. In fact, it is more likely that exponent depends weakly on $x$. However, the averaged value to all exponents in this case has the following value at NNLO as

$$\lambda_{\text{Singlet+Gluon}} \simeq 0.40.$$ (27)

In Fig. 3, the values of longitudinal exponent are shown as a function of $x$ at four different fixed $Q^2$ values with respect to the light ($n_f = 3$)-charm coefficient functions. After doing some derivation of the heavy quarks, we observe that the longitudinal exponents have the same behavior as discussed in Figs. 1 and 2. The merit of this plot, in comparison with another one, is mainly due to its relation with the charm distribution. The data have the property that the charm structure function requires a hard-pomeron component $[1,2,5–7,40]$. The averaged value to all exponents for the light and charm distribution at NNLO has the following value:

$$\lambda_{\text{Light+charm}} \simeq 0.38.$$ (28)

It can be clearly seen that the longitudinal exponents decrease as active flavours increases, but with a somewhat smaller rate. It can be well described by

$$\lambda_s < \lambda_{\text{Light+charm}} < \lambda_{\text{Singlet+Gluon}} < \lambda_{\text{Gluonic}} \leq \lambda_g.$$ (29)

Furthermore, these solutions predict that $\lambda_L \neq \lambda_g$ in a wide range of $x - Q^2$ values at high-order corrections.

In Fig. 4, we compare these predictions for longitudinal exponents as a function of $Q^2$. The exponent $\lambda_L$ of the longitudinal structure function is observed that depends weakly on $Q^2$. It can be represented by a constant $\lambda_L$ which is almost independent of $x$ and $Q^2$. This is consistent with the hard-pomeron defined by Donnachie and Landshoff $[1,2,5–7]$. So the
The longitudinal exponent \( \lambda_L \) given by Eq. (25) versus \( x \) at four fixed \( Q^2 \) values at NLO and NNLO analysis, compared with the gluon exponent \( \lambda_g = 0.42 \) (dot line) and singlet exponent \( \lambda_s = 0.33 \) (dash line). The renormalization scale is \( \mu = \sqrt{4m_c^2 + Q^2} \).

The longitudinal exponent \( \lambda_L \) plotted against \( Q^2 \) with respect to the gluonic terms, singlet+gluon terms and light+charm terms, compared with the gluon exponent \( \lambda_g = 0.42 \) (dot line) and singlet exponent \( \lambda_s = 0.33 \) (dash line).

simplest form to the small \( x \) behavior of the longitudinal structure function corresponds to \( F_L \sim x^{-\lambda_L} \). Therefore we use the hard-pomeron component of the longitudinal structure function which is defined in Eq. (29).
3 Reduced cross section

The extraction of the reduced double differential cross section is based on two proton structure functions \( F_2(x, Q^2) \) and \( F_L(x, Q^2) \). When \( y \to 1 \), the reduced cross section \( \sigma_r \) tends to \( F_2 - F_L \). An important advantage of HERA is used to perform an extraction of the longitudinal structure function with respect to the extrapolation and derivative methods [32–34, 51–55].

As discussed in Sect. 2, the behavior of the proton structure functions \( F_2 \) and \( F_L \) are \( x^{-\lambda_s} \) and \( x^{-\lambda_L} \) at fixed \( Q^2 \), respectively. On this basis, the reduced cross-sectional distribution can be parametrized as

\[
\sigma_r = A_2(Q^2)x^{-\lambda_s} - \frac{y^2}{Y_+}A_L(Q^2)x^{-\lambda_L}. \tag{30}
\]

We analyze the reduced cross-sectional behavior with a power-like behavior at small \( x \) at fixed \( Q^2 \) as

\[
\sigma_r(x, Q^2) \equiv A_\sigma(Q^2)x^{-\lambda_\sigma}, \tag{31}
\]

where

\[
\lambda_\sigma = \frac{\partial \ln \sigma_r(x, Q^2)}{\partial \ln(1/x)}, \tag{32}
\]

To do this, the derivative of \( \ln \sigma_r \), taken at fixed \( Q^2 \), is given by

\[
\left. \frac{\partial \ln \sigma_r(x, Q^2)}{\partial \ln x} \right|_{Q^2} = \left[ \frac{\partial \ln F_2(x, Q^2)}{\partial \ln x} \frac{F_L}{F_2 Y_+} + \frac{\partial \ln F_L(x, Q^2)}{\partial \ln x} \frac{F_L}{F_L Y_+} \right] / \left[ 1 - \frac{F_L}{F_2 Y_+} \right]. \tag{33}
\]

Hence, the reduced cross-sectional exponent is defined by an analytical expression as

\[
\lambda_\sigma = \left[ \lambda_s - \lambda_L \frac{F_L}{F_2} \frac{y^2}{Y_+} + \frac{F_L}{F_2} \frac{\partial}{\partial \ln x} \frac{y^2}{Y_+} \right] / \left[ 1 - \frac{F_L}{F_2 Y_+} \right], \tag{34}
\]

when \( 0 < y < 1 \). For \( y \to 0 \), the reduced cross-sectional exponent tends to the limit

\[
\lambda_\sigma \to \lambda_s, \tag{35}
\]

and tends to the limit

\[
\lambda_\sigma \approx \frac{\lambda_s - \lambda_L F_L}{1 - \frac{F_L}{F_2}}, \tag{36}
\]

when \( y \to 1 \). We note that the behavior of \( \frac{\partial}{\partial \ln x} \frac{y^2}{Y_+} \) in Eq. (34) is controlled at two limited regions (Eqs. (35) and (36)). In Fig. 5, the behavior of \( \frac{\partial}{\partial \ln x} \frac{y^2}{Y_+} \) at fixed \( s \) and \( Q^2 \) values is shown that leads to rapid depletion and enhancement in the small-\( x \) region \((10^{-6} < x < 10^{-3})\). To better illustration this behavior at small \( x \), the reduced cross-sectional exponent \( \lambda_\sigma \) is plotted versus the \( x \) variable (see Fig. 6). It can be clearly seen that this result is dependent on the ratio of the structure functions behavior. In color dipole model [56–58], a strict bound for the ratio \( F_L/F_2 \) is defined as \( \frac{F_L}{F_2} \leq 0.27 \). For realistic dipole-proton cross section [59], the bound is reduced to 0.22. From the new measurement of \( F_L \) at HERA, a phenomenological model derives the ratio of structure functions which lead to the bound 0.12 in a wide range of \( Q^2 \).
The calculated values $\Delta = \frac{\partial}{\partial \ln x} \frac{\vec{y}^2}{F_1}$ plotted against $x$ at four fixed $Q^2$ values at $\sqrt{s} = 300.9$ GeV (H1 2001 [32–34]).

The behavior of the reduced cross-sectional exponent $\lambda_\sigma$ plotted against $x$ at four fixed $Q^2$ values with respect to the ratio of the structure functions $F_L/F_2$ in color dipole model (dash line) [56,57] and phenomenological model (solid line) [38,39]. In Fig. 6, the effects of these bounds for the reduced cross-sectional exponent have been presented. For a constant $Q^2$, the reduced cross-sectional exponent has the same behavior of the singlet exponent at $x > 10^{-3}$ and $x < 10^{-5}$. There is some violation at $10^{-5} < x < 10^{-3}$. In this range, a depletion and then an enhancement is observable in all figures as $x$ decreases.
In Fig. 7, the form $x^{\lambda_\sigma} \equiv \frac{\sigma_r(x, Q^2)}{A_{\sigma}(Q^2)}$ for the reduced cross-sectional parametrization at small $x$ is plotted. For fixed $Q^2$, the reduced cross section at HERA data [32–34] rises with decreasing $x$ as $x \to 10^{-3}$. The increase of $F_L$ towards small-$x$ is consistent with the high-order QCD corrections. This behavior is reflecting the decrease of the reduced cross section towards small-$x$. In Fig. 7, this characteristic of the reduced cross section is observed with respect to the depletion behavior at this region. This behavior is consistence with the available HERA data [32–34]. Thus, we observe a continuous increase then decrease towards small $x$.

Therefore, we observe that for low $x$ values the differences between the results are dependence on the two used models (color dipole model and phenomenological model). The oscillating behaviors in Figs. 6 and 7 can be explained by the shadowing and antishadowing effects. The depletion is called shadowing correction to this behavior and can be explain by the gluon recombination. Also the positive antishadowing effect coexists with the shadowing effect in the QCD recombination processes. As enhancement observed in these figures is called antishadowing. So the significant nonlinear effects can be apparent in the behavior of the reduced cross-sectional exponent.

In H1 analysis, the measured reduced cross section is represented as

$$\sigma_r(x, Q^2) \equiv F_2(x, Q^2) \left[ 1 - \frac{y^2}{Y_+} \frac{R}{1 + R} \right],$$

where the value $R$ is generally assumed that is constant for all $Q^2$ bins. In HERA analysis, the observations are obtained with the general methods such as derivative method, offset method and fitted method [32–34]. We now discuss how the presented results give an analytical
Our results for the ratio $F_2^2/\sigma$, using Eq. (40) and its comparison with the H1 2001 data [32–34] as accompanied with total errors analysis for the ratio $F_2/\sigma$ with respect to the effective exponents at small $x$. To obtain the ratio $F_2/\sigma$, the derivative of the reduced cross section, taken at fixed $Q^2$, is used as

$$\frac{\partial \sigma_r}{\partial \ln y} \bigg|_{Q^2} = \frac{\partial F_2}{\partial \ln y} \bigg|_{Q^2} - \frac{\partial F_L}{\partial \ln y} \frac{y^2}{Y_+}.$$ (38)

Using the fact that cross section and distribution functions have a power-law behavior with an effective exponent. Considering the relationship between the functions and effective exponents gives a similar relation as we have

$$\lambda_\sigma \sigma_r = \lambda_s F_2 - F_L \cdot 2y^2 \cdot \frac{2-y}{Y_+} - \lambda_L F_L \cdot \frac{y^2}{Y_+}.$$ (39)

Now, using Eqs. (7) and (39), the ratio $F_2/\sigma$ is found directly from the exponents, namely

$$\frac{F_2}{\sigma} = \frac{\lambda_\sigma - \left[\lambda_L + \frac{2}{T_+}(2-y)\right]}{\lambda_s - \left[\lambda_L + \frac{2}{T_+}(2-y)\right]}.$$ (40)

Here we used the pomeron value of the exponents assumed for small $x$ region by the available H1 data. In Fig. 8, a comparison is made between our obtained values and the available data. The results of analytical solutions with respect to the exponents for the ratio $F_2/\sigma$ clearly show significant agreement over a wide range of $x$ and $Q^2$ values. The ratio $F_2/\sigma$ has the same behavior for $x \geq 10^{-4}$ when we used the color dipole and phenomenological models at moderate $Q^2$ values. One can see that these predictions increase as $x$ decreases. The prediction of new data in this region is between the upper and lower bounds with respect to the the color dipole model and phenomenological model, respectively. At very small $x$ a depletion for the ratio $F_2/\sigma$ can be explain when the nonlinear corrections have to be take
into account. Extension of current result to the nonlinear effect is also a valuable task to follow it in future.

4 Conclusion

In this section, a set of new formulas connecting the longitudinal exponent with the gluon and singlet exponents at small $x$ have been presented. Based upon the hard pomeron behavior of the gluon and singlet exponents, the behavior of the longitudinal exponent at high-order corrections is considered. We found that longitudinal exponent behavior is dependent on the active flavors. The value of the longitudinal exponent is similar to the one predicted for the singlet and gluons. This exponent is almost independent of $x$ for $x < 10^{-2}$. We see that $\lambda_s < \lambda_L \leq \lambda_g$. This exponent as a function of $Q^2$ is consistent with the hard pomeron behavior. Thus, the behavior of $F_L$ at small $x$ is consistent with a dependence $F_L(x, Q^2) = A_L(Q^2)x^{-\lambda_L}$ throughout that region.

Also we analyse the behavior of the exponent for the reduced cross section. The behavior of $\Delta(\equiv \frac{\partial}{\partial \ln x} \frac{1}{Y^2})$ at high and very low-$x$ values is considered as this behavior is linear and equal to zero. But in the region $10^{-6} < x < 10^{-3}$ (at four $Q^2$ value determined), the behavior of this function ($\Delta$) can no longer be neglected. The deviation of this expression from zero shows the importance of non-linear effects. A depletion in the low $x$ (high $y$) is called shadowing whereas an enhancement is called anti-shadowing [60–62].

The oscillating behavior for $\lambda_\sigma$ can be explained by new effects at low-$x$, such as the nonlinear recombination. The behavior of the function $x^{-\lambda_\sigma}$ increase as $x$ decreases. The negative shadowing and the positive anti-shadowing corrections to this behavior can be explain by the non-linear effects to the structure functions. In view of these results for the exponents, we may infer some evidence for non-linear effects at LHeC [63].

Considering these determined exponents and using the derivatives methods to find the ratio $\frac{F_2}{\sigma}$ and finally comparing with the H1 data, one concludes that this new method is capable of determining the ratio $\frac{F_2}{\sigma}$ with considerable precision.

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Appendix A

The coefficient functions involved in the Altarelli–Martinelli equation for gluon distribution at LO up to NNLO are

$$C_{L,g}^{LO} = 8n_f x(1 - x),$$
$$C_{L,g}^{NLO} = n_f \{(94.74 - 49.20x)x_1L_1^2 + 864.8x_1L_1 + 1161xL_1L_0 + 60.06xL_0^2 + 39.66x_1L_0 - 5.333(1/x - 1)\},$$
$$C_{L,g}^{NNLO} = n_f \{(144L_1^2 - 47024/27L_1^3 + 6319L_1^2 + 53160L_1)x_1 + 72549L_0L_1 + 88238L_0^2L_1 + (3709 - 33514x) - 9533x^2)x_1 + 66773xL_0^2 - 1117L_0 + 45.37L_0^2 - 5360/27L_0^3 - (2044.70x_1 + 409.506L_0)1/x\}.$$
\[ x_n^2 ((32/3L_1^3 - 1216/9L_1^3 - 592.3L_1 + 1511xL_1)x_1 + 311.3L_0L_1 + 14.24L_0^2L_1 + (577.3 - 729x)x_1 + 30.78xL_0^3 + 366L_0 + 1000/9L_0^2 + 160/9L_0^3 + 88.50371/x_1) + 1.550L_1^2 + 19.72xL_1 - 66.745x + 0.615x^2 x_1 + xf^{\mu\nu}_{\alpha\beta} n_f^2 \left( (-0.0105L_1^3 + 20/27xL_0^4 \right) + (280/81 + 2.260)xL_3^3 - (15.40 - 2.201x)xL_0^2 - (71.66 - 0.121x)xL_0 \right), \tag{41} \]

where \( x_1 = 1 - x \), \( L_0 = \ln x \) and \( L_1 = \ln x_1 \). The charge factor \( fl^{\mu\nu}_{\alpha\beta} \) is given by \( \frac{(e^2)^2}{(e^2)} \) which \( \langle e^k \rangle = n_f^{-1} \sum_i^\infty e_i^k \).

The singlet contribution to the longitudinal coefficient functions can be written as

\[
C_{L,q}^{LO} = 4C_F x, \\
C_{L,q}^{NLO} = \left( 128/9xL_1^2 - 46.50xL_1 - 84.094L_0L_1 - 37.338 \right) + 89.53x + 33.82x^2 + xL_0(32.90 + 18.41L_0) - 128/9L_0 - 0.012\delta(x_1) + 16/27n_f(6xL_1 - 12xL_0 - 25x + 6) + n_f((15.94 - 5.212x)x_1^2L_1 + (0.421 + 1.520x)L_0^2 + 28.09xL_0 - (2.371/x - 19.27)x_1^3). \\
C_{L,q}^{NNLO} = 512/27L_1^3 - 177.40L_1^3 + 650.6L_1^2 - 2729L_1 - 2220.5 - 7884x + 4168x^2 - (844.7L_0 + 517.3L_1)L_0L_1 + (195.6L_1 - 125.3)x_1L_1^3 + 208.3xL_0^3 - 1355.7L_0 - 7456/27L_0^2 - 1280/81L_3^3 + 0.113\delta(x_1) + n_f((1024/81L_3^3 - 112.35L_1^2 + 344.1L_1 + 408.4 - 9.345x - 919.3x^2 + 239.7 + 20.63L_1)x_1L_1^3 + (887.3 + 294.5L_0 - 59.14L_1)L_0L_1 - 1792/81xL_0^3 + 200.73L_0 + 64/3L_0^2 + 0.006\delta(x_1) + n_f^2(3xL_1^2 + (6 - 25x)L_1 - 19 + (317/6 - 12\zeta_2)x - 6xL_0L_1 + 6xL_i_2 + 9xL_0^2 - (6 - 50x)L_0)/64 + fL^{\mu\nu}_{\alpha\beta} n_f((107 + 321.05x - 54.62x^2)x_1 - 26.717 + 9.773L_0 + (363.8 + 68.32L_0)xL_0 - 320/81L_0^2(2 + L_0)x + n_f((1568/27L_1^3 - 3968/9L_1^2 + 5124L_1)x_1^2 + (2184L_0 + 6059x_1)L_0L_1 - (795.6 + 1036x)x_1^2 - 143.6x_1L_0 + 2848/9L_0^2 - 1600/27L_0^3 - (885.53x_1 + 182L_0)x/x_1) + n_f^2((-32/9L_1^2 + 340/27xL_0^2 + 249/27L_0^2 + 45/27L_0^3) + 41/27L_0^3}}\right].
\]
\[ +29.52L_1 x_1^2 + (35.18L_0 + 73.06x_1)L_0 L_1 -35.24x_1^2 - (14.16 - 69.84x_1)x_1^2 - 69.41x_1L_0 - 128/9 L_0^2 + 40.2391/x x_1^2 + f l_{11}^{ps} n_f ((107 \\
+ 321.05x - 54.62x^2)x_1 - 26.717 + 9.773L_0 + (363.8 + 68.32L_0)x L_0 - 320/81 L_0^2(2 + L_0))x. \]

(42)

Here, \(N_C = 3\) and \(C_F = \frac{N_C^2 - 1}{2N_C} = \frac{4}{3}\). Also the charge factor \(f l_{11}^{ps} = f l_{11}^g - f l_{11}^{ns}\), where \(f l_{11}^{ns} = 3\langle e\rangle\) [15,16].

**Appendix B**

In the LO analysis, the longitudinal coefficient function can be found in the following form:

\[ C_{0, L}^g (z, \zeta) = -4z^2\zeta \ln \frac{1 + \beta}{1 - \beta} + 2\beta z(1 - z), \]

(43)

where \(\beta^2 = 1 - \frac{4z \zeta}{1 - z}\).

At NLO, \(O(\alpha_{em}\alpha_s^2)\), the contribution of the photon–gluon component is usually presented in terms of the coefficient functions \(C_{k,g}^4, \bar{C}_{k,g}^4\). The NLO longitudinal coefficient function is only available as computer code which at the high-energy regime \((\zeta << 1)\) we used the compact form of these coefficients [41–47].

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