Positivity of higher exterior powers of the tangent bundle

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Abstract

We prove that a smooth projective variety $X$ of dimension $n$ with strictly nef third, fourth or $(n-1)$-th exterior power of the tangent bundle is a Fano variety. Moreover, in the first two cases, we provide a classification for $X$ under the assumption that $\rho(X) \neq 1$.

1 Introduction

Positivity notions are numerous in algebraic geometry: a line bundle can be considered positive, e.g., if it is very ample, ample, strictly nef, nef, big, semiample, effective, pseudoeffective... Some of these notions relate: a very ample line bundle is ample, an ample line bundle is strictly nef and big, a strictly nef line bundle (i.e., a line bundle that has positive intersection with any curve) is nef, a nef line bundle and an effective line bundle are pseudoeffective. These positivity notions, as they tremendously matter in algebraic geometry, have been the subject of a lot of work, to which the books by Lazarsfeld [Laz04a, Laz04b] are a great introduction. Proving new relationships between these various positivity notions is however a rather naive ambition, if not under strong additional assumptions.

From this perspective, the conjecture by Campana and Peternell [CP91] is surprising: they predict that, if $X$ is a smooth projective variety, and the anticanonical divisor $-K_X$ is strictly nef, then $X$ is a Fano manifold. Their conjecture was in fact proven in dimension 2 and 3, by Maeda and Serrano [Mae93, Ser95]. As all Fano manifolds are rationally connected [Cam92, KMM92], an interesting update on the conjecture is the recent proof by Li, Ou and Yang [LOY19, Theorem 1.2] that if $X$ is a smooth projective variety, and the anticanonical divisor $-K_X$ is strictly nef, then $X$ is rationally connected. Their proof uses important results on the Albanese map of varieties with nef anticanonical bundle. Such varieties have been extensively studied too [DPS94, Zha96, PS98, Dem15, CH17, Cao19, CH19].

Positivity notions extend to vector bundles [Laz04b, Definition 6.1.1] in the following fashion: a vector bundle $E$ is strictly nef if the associated line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is strictly nef on $\mathbb{P}(E)$. Instead of asking about the positivity of the top exterior power of the tangent bundle, $-K_X = \bigwedge^{\dim(X)} T_X$, it makes sense to ask about the positivity of intermediate exterior powers $\bigwedge^r T_X$, for $1 \leq r \leq \dim(X)-1$.

For $r = 1$, it is known since Mori [Mor79] that projective spaces are the only smooth projective varieties with ample tangent bundle. They are also the only smooth projective varieties with strictly nef tangent bundle, by [LOY19, Theorem 1.4]. Varieties with nef tangent bundle are, on the other hand, governed by another conjecture of Campana and Peternell [CP91] which has received a lot of attention: see the survey [MnOSC+15], and inter alia [CP91, DPS94, Wat14, Kan17, Kan16, MnOSCW15, Yan, Li17, Dem18, Wat21a, KW].

For $r = 2$, it has been proven that varieties with ample second exterior power of the tangent bundle are projective spaces and quadric hypersurfaces [CS95], varieties with strictly nef second exterior power of the tangent bundle alike.

\textbf{Theorem 1.1.} [LOY19, Theorem 1.5] Let $X$ be a smooth projective variety of dimension $n \geq 2$, such that $\bigwedge^2 T_X$ is strictly nef. Then $X$ is isomorphic to the projective space $\mathbb{P}^n$, or to a smooth quadric hypersurface $Q^n$.

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Partial results were obtained under the nef assumption [Wat21b, Sch]. These results lead us to the following questions.

**Question 1.** Let X be a smooth projective variety of dimension n. Suppose that $\Lambda^r T_X$ is strictly nef for some integer $1 \leq r \leq n$. Is X a Fano variety?

**Question 2.** Let X be a smooth projective variety of dimension n. Suppose that $\Lambda^r T_X$ is nef for some integer $1 \leq r < n$, and that X is rationally connected. Is X a Fano variety?

Note that an affirmative answer to the second question would imply an affirmative answer to the first question, by [LOY19, Theorem 1.2]. Also note that the second question is answered negatively for $r = n$, as there are smooth rationally connected threefolds with $-K_X$ nef but not semiample [Xie]. The first question is answered affirmatively for smooth toric varieties by [Sch]. In this paper, we answer the second question for $r = n - 1$.

**Theorem 1.2.** Let X be a smooth projective variety of dimension $n \geq 2$ such that the vector bundle $\Lambda^{n-1}T_X$ is nef and X is rationally connected. Then X is a Fano variety.

This theorem is reminiscent of [DPS94, Proposition 3.10], which states a dichotomy for varieties X with nef tangent bundle: either X is a Fano manifold, or $\chi(X, O_X) = 0$. The proof similarly involves Chern classes inequalities and the Hirzebruch-Riemann-Roch formula. Note that, building on this theorem, [Wat, Proposition 1.4] very recently gave an affirmative answer to Question 2 in general.

Theorem 1.1 is based on the results of [CMSB02] and [DH17], which instead of the assumption on $\Lambda^3 T_X$, feature a much weaker assumption on the length of rational curves. In a similar spirit, we provide the following partial characterizations and their corollaries.

**Theorem 1.3.** Let X be a smooth projective rationally connected variety of dimension $n \geq 4$ such that for each rational curve C in X, we have $-K_X \cdot C \geq n - 1$. Then either $X \cong \mathbb{P}^2 \times \mathbb{P}^2$, or X is a Fano variety of Picard rank $\rho(X) = 1$.

**Corollary 1.4.** Let X be a smooth projective variety of dimension at least 4 such that the vector bundle $\Lambda^3 T_X$ is strictly nef. Then either $X \cong \mathbb{P}^2 \times \mathbb{P}^2$, or X is a Fano variety of Picard rank $\rho(X) = 1$.

Let us briefly discuss the case when $\rho(X) = 1$. We know that, if X is a cubic or a complete intersection of two quadrics in $\mathbb{P}^n$, the vector bundle $\Lambda^3 T_X$ is ample. These are two examples of del Pezzo manifolds, i.e. Fano n-folds of Picard rank 1 and of index $n - 1$. However, we do not know whether other del Pezzo manifolds have strictly nef $\Lambda^3 T_X$, or whether varieties with strictly nef $\Lambda^3 T_X$ are in general del Pezzo manifolds. We can hardly hope for a characterization of Fano manifolds of Picard rank one on which $-K_X \cdot C \geq n - 1$ for every rational curve C, and it is moreover not clear how to use the positivity of $\Lambda^3 T_X$ beyond that length inequality, cf. Lemma 2.1.

**Theorem 1.5.** Let X be a smooth projective rationally connected variety of dimension $n \geq 6$ such that for each rational curve C in X, we have $-K_X \cdot C \geq n - 2$. Then either X is isomorphic to $\mathbb{P}^3 \times \mathbb{P}^3$ or X is a Fano variety of Picard rank $\rho(X) = 1$.

Studying the possibilities in dimension 5 by hand yields the following result.

**Corollary 1.6.** Let X be a smooth projective variety of dimension at least 5 such that the vector bundle $\Lambda^4 T_X$ is strictly nef. Then either X is isomorphic to one of the following Fano varieties

$$\mathbb{P}^2 \times Q^3; \; \mathbb{P}^2 \times \mathbb{P}^3; \; \mathbb{P}(T_{\mathbb{P}^3}); \; \mathbb{B}l_r(\mathbb{P}^5) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1)); \; \mathbb{P}^3 \times \mathbb{P}^3$$

or X is a Fano variety of Picard rank $\rho(X) = 1$.

These two corollaries were to our knowledge unknown even under the stronger, more classical assumption that $\Lambda^3 T_X$ or $\Lambda^4 T_X$ is ample. The proof of both theorems goes by classifying possible Mori contractions for X. A delicate point is that, while we know that our varieties X with $\rho(X) \geq 2$ admit one Mori contraction by the Cone Theorem, we need to construct by hand a second Mori contraction, e.g., to control higher-dimensional fibres in case of a first fibred Mori contraction. Depending on circumstances, we use unsplit covering families of deformations of rational curves, and a result by Bonavero, Casagrande and Druel [BCD07], or, if X has the right dimension, Theorem 1.2, to produce this second Mori contraction.
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Conventions. We work over the field of complex numbers \( \mathbb{C} \). Varieties (and in particular curves) are always assumed irreducible and reduced. We use the expressions “smooth projective variety” and “projective manifold” interchangeably. We refer to [Deb01] for birational geometry, in particular Mori theory, [Laz04a, Laz04b] for positivity notions, [Ko96] for rational curves and their deformations. We write \( c_i(X) = c_i(T_X) \) for the Chern classes of the tangent bundle of \( X \).

2 A first lemma

We start with a simple lemma.

**Lemma 2.1.** Let \( X \) be a smooth projective variety of dimension \( n \) such that \( \bigwedge^r T_X \) is strictly nef, for some \( 1 \leq r \leq n-1 \). Then any rational curve \( C \) in \( X \) satisfies

\[
-K_X \cdot C \geq n + 2 - r.
\]

**Proof.** The proof goes as [LOY19, Proof of Theorem 1.5]. Let \( f : \mathbb{P}^1 \to C \) be the normalization of the curve. Write

\[
f^* T_X \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n),
\]

with \( (a_i)_{1 \leq i \leq n} \) ordered increasingly. It holds \( a_n \geq 2 \), as \( T_{\mathbb{P}^1} \) maps non-trivially to \( f^* T_X \), and we have \( a_1 + \ldots + a_r > 0 \) because \( \mathcal{O}_{\mathbb{P}^1}(a_1 + \ldots + a_r) \) is a direct summand of the strictly nef vector bundle \( \bigwedge^r f^* T_X \). In particular, \( a_{r+1} \geq a_r \geq 1 \). Hence,

\[
-K_X \cdot C = \deg f^*(-K_X) = a_1 + \ldots + a_n \geq 1 + n - r - 1 + 2 = n + 2 - r.
\]

This result is all the more valuable as, by [LOY19, Theorem 1.2], if \( X \) is a smooth projective variety of dimension \( n \) such that \( \bigwedge^r T_X \) is strictly nef, then it is rationally connected, in particular, it contains numerous rational curves.

We will also need the following result.

**Lemma 2.2.** Let \( X \) be a smooth projective variety of dimension \( n \) such that \( \bigwedge^r T_X \) is nef, for some \( 1 \leq r \leq n-1 \). Then any rational curve \( C \) in \( X \) satisfies \( -K_X \cdot C \geq 2 \).

**Proof.** Let \( f : \mathbb{P}^1 \to C \) be the normalization of the curve. Write

\[
f^* T_X \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n),
\]

with \( (a_i)_{1 \leq i \leq n} \) ordered increasingly. It holds \( a_n \geq 2 \), as \( T_{\mathbb{P}^1} \) maps non-trivially to \( f^* T_X \), and we have \( a_1 + \ldots + a_r \geq 0 \) because \( \mathcal{O}_{\mathbb{P}^1}(a_1 + \ldots + a_r) \) is a direct summand of the nef vector bundle \( \bigwedge^r f^* T_X \). Hence, \( a_{r+1} \geq a_r \geq 0 \), and summing up those inequalities, we obtain the estimate

\[
-K_X \cdot C = a_1 + \ldots + a_n \geq 2.
\]

3 Results on \( \bigwedge^{n-1} T_X \)

The following lemma is the main step in the proof of Theorem 1.2.

**Lemma 3.1.** Let \( X \) be a projective \( n \)-dimensional manifold such that \( \bigwedge^{n-1} T_X \) is nef and \( X \) is rationally connected. Then \( -K_X \) is nef and big.
Proof. By [Laz94b, Theorem 6.2.12(iv)], the anticanonical divisor $-K_X$ is nef. By the Hirzebruch-Riemann-Roch formula, there is a homogeneous polynomial $P$ of degree $n$ in $\mathbb{Q}[X_1, \ldots, X_n]$ with grading $\deg X_i = i$ such that

$$\chi(X, \mathcal{O}_X) = P(c_1(X), \ldots, c_n(X)).$$

Note that, as $\bigwedge^{n-1} T_X = \Omega^n_X \otimes \mathcal{O}_X(-K_X)$, and by [Ful98, Remark 3.2.3(b)], we have

$$c_i \left( \bigwedge^{n-1} T_X \right) = \sum_{j=0}^{i} (-1)^j \binom{n-j}{i-j} c_j(X) c_i(-K_X)^{i-j}.$$  \hfill (⋆)

Let us show by induction that $c_i(X)$ is a rational polynomial in the $c_j(\bigwedge^{n-1} T_X)$, for $0 \leq j \leq i$. Indeed, $c_1(X) = \frac{1}{n} c_1(\bigwedge^{n-1} T_X)$. Assume now that for some $i$, for all $0 \leq j \leq i$, there is a polynomial $P_j \in \mathbb{Q}[X_1, \ldots, X_j]$ such that $c_j(X) = P_j(c_1(\bigwedge^{n-1} T_X), \ldots, c_i(\bigwedge^{n-1} T_X))$. Then, setting

$$P_{i+1}(X_1, \ldots, X_{i+1}) = (-1)^{i+1} X_{i+1} - \sum_{j=0}^{i} (-1)^{i+j+1} \binom{n-j}{i+1-j} P_j(X_1, \ldots, X_j)(P_1(X_1))^{i+1-j},$$

we have $c_{i+1}(X) = P_{i+1}(c_1(\bigwedge^{n-1} T_X), \ldots, c_{i+1}(\bigwedge^{n-1} T_X))$ by (⋆). This perpetuates the induction.

In particular, we have

$$\chi(X, \mathcal{O}_X) = P \left( P_1 \left( c_1 \left( \bigwedge^{n-1} T_X \right) \right), \ldots, P_n \left( c_1 \left( \bigwedge^{n-1} T_X \right) \right) \right),$$

which is a homogeneous polynomial of degree $n$ in $c_1(\bigwedge^{n-1} T_X), \ldots, c_n(\bigwedge^{n-1} T_X)$.

Now, if we suppose that $-K_X$ is not big, then $c_1(\bigwedge^{n-1} T_X)$ is not big. Thus, [DPS94, Corollary 2.7] implies $\chi(X, \mathcal{O}_X) = 0$. But on the other hand, $X$ is rationally connected, so $\chi(X, \mathcal{O}_X) = 1$, a contradiction.

\textbf{Remark 3.2.} If $n = 4$, we cannot write $c_4(X)$ as a polynomial in

$$c_1 \left( \bigwedge^{n-2} T_X \right) = 3c_1(X),
$$

$$c_2 \left( \bigwedge^{n-2} T_X \right) = 3c_1(X)^2 + 2c_2(X),
$$

$$c_3 \left( \bigwedge^{n-2} T_X \right) = c_1(X)^3 + 4c_1(X)c_2(X),
$$

these formulas coming from [Ien, 4.5.2].

\textbf{Lemma 3.3.} Let $X$ be a projective $n$-dimensional manifold such that $\bigwedge^{n-1} T_X$ is nef and $X$ is rationally connected. Then $-K_X$ is ample.

\textbf{Proof of Theorem 1.2.} By Lemma 3.1, $-K_X$ is nef and big. By the base-point-free theorem [Deb01, Theorem 7.32], we dispose of an integer $m$ such that $-mK_X$ is globally generated. Let $\varepsilon : X \to Z$ be the $[-mK_X]$-morphism.

Suppose that it is not finite. By [Kaw91, Theorem 2], any irreducible component $E$ of the exceptional locus is covered by rational curves that are contracted by $\varepsilon$. Let $C$ be one of them: we have $0 = -K_X \cdot C \geq 2$ by Lemma 2.2, a contradiction. So $-K_X$ is ample.

\section{4 Studying Mori contractions}

The strategy for proving Theorems 1.3 and 1.5 is to show that there are only few possible Mori contractions for $X$. In the following, if $R$ is an extremal ray of the Mori cone $NE(X)$, its length denoted by $\ell(R)$ is defined to be the minimal value of $-K_X \cdot C$, for a rational curve $C$ with class in $R$. A Mori contraction is said to be of length $\ell$ if it is a contraction of an extremal ray $R$ with $\ell(R) = \ell$.

Note that here and throughout this paper, we exclusively work with elementary Mori contractions, and simply refer to them as Mori contractions.
4.1 Small contractions

Lemma 4.1. Let $1 \leq r \leq 4$. Let $X$ be a smooth projective variety of dimension at least $r + 1$ such that $\Lambda^r T_X$ is strictly nef. Then $X$ has no small contration.

Proof. Let $n$ be the dimension of $X$. Let $\varphi : X \to Y$ be a birational contraction, $E$ be an irreducible component of the exceptional locus, $F$ an irreducible component of the general fibre of $\varphi|_E$, and $R$ the corresponding extremal ray. Applying Ionescu-Wiśnewski inequality [Ion86, Theorem 0.4], [Wi91a, Theorem 1.1] together with Lemma 2.1 yields

$$\dim E + \dim F \geq n + l(R) - 1 \geq 2n + 1 - r.$$ 

Since $r \leq 4$, we have $\dim E \geq n - 1$, and thus $\varphi$ is a divisorial contraction. \hfill \Box

4.2 Fibred Mori contractions

We move on to studying fibred Mori contractions.

4.2.1 Generalities about fibred Mori contractions

If $X$ is a normal projective variety, and $C$ is a rational curve in $X$, we may denote by $V$ its family of deformations, that is an irreducible component of Chow($X$) containing the point corresponding to $C$. Denoting by $\phi : \text{Univ}(X) \to \text{Chow}(X)$ the universal family and by $ev : \text{Univ}(X) \to X$ the evaluation map, we define

$$\text{Locus}(V) := ev(\phi^{-1}(V)) \subset X.$$ 

We say that $V$ is covering if $\text{Locus}(V) = X$.

We say that $V$ is unsplit if it only parametrizes irreducible cycles. For $x \in \text{Locus}(V)$, we define $V_x := \phi(ev^{-1}(x))$ the family of deformations of $C$ through $x$. We finally define

$$\text{Locus}(V_x) := ev(\phi^{-1}(V_x)) \subset X.$$ 

We use families of deformations of rational curves to prove the following proposition.

Proposition 4.2. Let $X$ be a smooth projective rationally connected variety of dimension $n$. Let $r$ be an integer with $1 \leq r \leq n - 1$. Suppose that $-K_X \cdot C \geq n + 2 - r$ for any rational curve $C$ in $X$. Suppose that there is a fibred Mori contraction $\pi : X \to Y$ with $\dim Y > 0$. Then the general fibre has dimension at most $r - 1$.

If equality holds, then there is a rational curve $C$ in $X$, not contracted by $\pi$, whose family of deformations $V$ is unsplit covering and satisfies $\dim \text{Locus}(V_x) = n + 1 - r$ for $x \in \text{Locus}(V)$ general.

The proof relies on the following lemmas.

Lemma 4.3. Let $X$ be a smooth projective variety. Suppose that $X$ has a fibred Mori contraction $\pi : X \to Y$ with $\dim Y > 0$, and let $C$ be a rational curve such that $\pi(C) \neq \{\text{pt}\}$ and such that its family of deformations $V$ is unsplit. Then, for any $x \in \text{Locus}(V)$,

$$\dim \text{Locus}(V_x) \leq \dim Y.$$ 

Proof of Lemma 4.3. We claim that $\pi|_{\text{Locus}(V_x)}$ is finite onto its image. If it is not, it contracts a curve $B$ to a point: for some ample divisor $H$ on $Y$, we have $B \cdot \pi^*H = 0$. By [AC09, Lemma 4.1], the numerical class of $B \subset \text{Locus}(V_x)$ is a multiple of $[C] \in N_1(X)_\mathbb{Q}$, whence $C \cdot \pi^*H = 0$, which is a contradiction. So $\pi|_{\text{Locus}(V_x)}$ is finite onto its image: this implies $\dim \text{Locus}(V_x) \leq \dim Y$. \hfill \Box

Lemma 4.4. Let $X$ be a smooth projective variety. Suppose that $-K_X \cdot C > 0$ for every rational curve $C \subset X$. Suppose that $X$ has a fibred Mori contraction $\pi : X \to Y$ with $\dim Y > 0$, and let $C$ be a rational curve such that $\pi(C) \neq \{\text{pt}\}$ and such that

$$-K_X \cdot C = \min\{-K_X \cdot B \mid B \text{ rational curve in } X, \pi(B) \neq \{\text{pt}\}\}.$$ 

Then the family of deformations of $C$ is unsplit.
Proof of Lemma 4.4. Let \( \mathcal{V} \) be the family of deformations of \( C \). Suppose that it is splitting. By [Kol96, Explanation IV.2.2], we have
\[
C \equiv \sum_i a_i C_i,
\]
with rational curves \( C_i \) and coefficients \( a_i \geq 1 \) such that \( \sum_i a_i \geq 2 \). Since \(-K_X\) is positive on rational curves, we have \(-K_X \cdot C_i < -K_X \cdot C \) for all \( i \). So, by minimality of \(-K_X \cdot C\), the fibration \( \pi \) contracts all curves \( C_i \). Let \( H \) be an ample divisor on \( Y \). We obtain \( \sum_i a_i C_i \cdot \pi^* H = 0 \), a contradiction. □

Proof of Proposition 4.2. Since \( X \) is rationally connected and \(-K_X\) is Cartier, we dispose of a rational curve \( C \) such that \( \pi(C) \neq \{pt\} \) and \(-K_X \cdot C \geq n+2-r \geq 3 \) is minimal with this condition. Let \( \mathcal{V} \) be the corresponding family of deformations. By Lemma 4.4, it is unsplit.

Fix \( x \in \text{Locus}(\mathcal{V}) \) general. By [Kol96, Proposition IV.2.6] and our assumption, we derive
\[
\dim \text{Locus}(\mathcal{V}) + \dim \text{Locus}(\mathcal{V}_x) \geq -K_X \cdot C + n - 1 \geq 2n + 1 - r.
\]
So \( \dim \text{Locus}(\mathcal{V}_x) \geq n+1-r \).

Let \( d \) denote the dimension of the general fibre of \( \pi \). Then, by Lemma 4.3,
\[
d \leq n - \dim \text{Locus}(\mathcal{V}_x) \leq r - 1.
\]
As for the equality case, if \( d = r - 1 \), then \( \dim \text{Locus}(\mathcal{V}_x) = n - r + 1 \), and so \( C \) is such a rational curve as we claimed existed in the equality case of the proposition. □

Proposition 4.2 has an important consequence.

Corollary 4.5. Let \( X \) be a smooth projective rationally connected variety of dimension \( n \) such that, for some integer \( r \) with \( 1 \leq r \leq n-1 \), one has \(-K_X \cdot C \geq n+2-r \) for any rational curve \( C \subset X \). Suppose that there is a fibred Mori contraction \( \pi : X \to Y \) with \( \dim Y > 0 \). Then \( n \leq 2r - 2 \).

If equality holds, then a general fibre of \( \pi \) has dimension \( r - 1 \), and there is a rational curve \( C \) in \( X \), not contracted by \( \pi \), whose family of deformations \( \mathcal{V} \) is unsplit covering and satisfies \( \dim \text{Locus}(\mathcal{V}_x) = n+1-r \) for \( x \in \text{Locus}(\mathcal{V}) \) general.

Proof. Let \( F \) be a general fiber of \( \pi \). By Proposition 4.2, we have \( r - 1 \geq \dim F \). Adding \( n \) to both sides and applying Ionescu-Wiśniewski inequality (with the exceptional locus \( E = X \) of dimension \( n \)), it holds
\[
n + r - 1 \geq n + \dim F \geq n + \ell(R) - 1 \geq 2n + 1 - r.
\]
If there is an equality, then \( \dim F = r - 1 \), and so we are in the equality case of Proposition 4.2.

In particular, we can find a rational curve \( C \) in \( X \) that is not contracted by \( \pi \), whose family of deformations \( \mathcal{V} \) is unsplit and satisfies \( \dim \text{Locus}(\mathcal{V}_x) = n+1-r \) for \( x \in \text{Locus}(\mathcal{V}) \) general. By [Kol96, Proposition IV.2.6] and Lemma 2.1 again, we have
\[
\dim \text{Locus}(\mathcal{V}) \geq -K_X \cdot C + n - 1 \geq \dim \text{Locus}(\mathcal{V}_x) \geq 2n + 1 - r - n - 1 + r = n,
\]
so \( \mathcal{V} \) is indeed a covering family. □

4.2.2 Fibred Mori contractions for certain varieties of even dimension

The set-up for this paragraph is the following. Let \( r \geq 3 \) be an integer. Let \( X \) be a smooth projective rationally connected variety of dimension \( 2r - 2 \) such that \(-K_X \cdot C \geq r \) for any rational curve \( C \subset X \). Suppose that there is a fibred Mori contraction \( \pi : X \to Y \) with \( \dim Y > 0 \). Let us classify what happens.

Lemma 4.6. Let \( r \geq 3 \) be an integer. Let \( X \) be a smooth projective rationally connected variety of dimension \( 2r - 2 \) such that \(-K_X \cdot C \geq r \) for any rational curve \( C \subset X \). Suppose that there is a fibred Mori contraction \( \pi : X \to Y \) with \( \dim Y > 0 \). Then there is another equidimensional fibred Mori contraction \( \varphi : X \to Z \) with \( \dim Z = r - 1 \).

Proof. We are in the case of equality of Corollary 4.5. In particular, the general fibre \( F \) of \( \pi \) has dimension \( r - 1 \), and there is a rational curve \( C \) in \( X \) that is not contracted by \( \pi \) whose family of deformations \( \mathcal{V} \) is unsplit covering and satisfies \( \dim \text{Locus}(\mathcal{V}_x) = r - 1 \geq (2r - 2) - 3 = \dim X - 3 \).
By [BCD07, Theorem 2, Proposition 1(i)], there is a fibred Mori contraction $\varphi : X \to Z$ whose fibres exactly are the $V$-equivalence classes. By the equality case in Corollary 4.5, the general fibre of $\varphi$ has dimension $r - 1$.

Let $G$ be a fibre of $\varphi$. We claim that $\pi|_G$ is finite. Indeed, if it is not, then there is a curve $B \subset G$ that is contracted by $\pi$. The curve $B$ lies in a $V$-equivalence class, so by [BCD07, Remark 1], as $V$ is unsplit, $B$ is numerically equivalent to a multiple of $C$, so it cannot be contracted by $\pi$, a contradiction! So $\pi|_G$ is finite onto its image, which is contained in $Y$, so $\dim G \leq \dim Y = r - 1$.

So $\varphi$ is indeed equidimensional.

**Proposition 4.7.** Let $r \geq 3$ be an integer. Let $X$ be a smooth projective rationally connected variety of dimension $2r - 2$ such that $-K_X \cdot C \geq r$ for any rational curve $C \subset X$. Suppose that there is an equidimensional fibred Mori contraction $\pi : X \to Y$ with $\dim Y = r - 1$. Then $X \simeq \mathbb{P}^{r-1} \times \mathbb{P}^{r-1}$.

This proposition relies on Lemma 4.10 below.

**Definition 4.8.** Let $X$ and $Y$ be normal projective varieties. We say that a map $\pi : X \to Y$ is a fibration if it is surjective, has connected fibers, and if we have $0 < \dim Y < \dim X$.

Let $\pi : X \to Y$ be a fibration whose general fibre is a projective space. Let $f : \mathbb{P}^1 \to C \subset Y$ be a rational curve whose image lies in the smooth locus of $\pi$. The fibre product $\pi_C$ of $\pi$ by $f$ is the projectivization of a bundle $O_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus O_{\mathbb{P}^1}(a_k)$, with the $(a_i)$ ordered increasingly. A minimal section over $C$ is the section $s : \mathbb{P}^1 \to X$ of $\pi_C$ corresponding to a quotient $O_{\mathbb{P}^1}(a_1)$.

**Remark 4.9.** There may be several minimal sections as soon as $a_1 = a_2$.

**Lemma 4.10.** Let $X$ be a smooth projective variety with a fibration $\pi : X \to Y$ whose general fiber is a projective space. Then for any rational curve $f : \mathbb{P}^1 \to C \subset Y$ whose image lies in the smooth locus of $\pi$, and for any minimal section $s$ of it, it holds $-K_Y \cdot C \geq -K_X \cdot s(\mathbb{P}^1)$. In particular,

$$-K_Y \cdot C \geq \min\{-K_X \cdot C' \mid C' \text{ is a rational curve in } X\}. \quad (**)$$

If there is an equality in $(**)$, then the base change of $\pi$ by $f$ is isomorphic to $\mathbb{P}(O_{\mathbb{P}^1}(a \oplus k)) \to \mathbb{P}^1$. If there is almost an equality, i.e.,

$$-K_Y \cdot C = \min\{-K_X \cdot C' \mid C' \text{ is a rational curve in } X\} + 1,$$

then the base change of $\pi$ by $f$ is isomorphic to to $\mathbb{P}(O_{\mathbb{P}^1}(a \oplus k)) \to \mathbb{P}^1$ or to $\mathbb{P}(O_{\mathbb{P}^1}(a \oplus k-1) \oplus O_{\mathbb{P}^1}(1)) \to \mathbb{P}^1$.

**Proof.** By Tsun’s theorem, the base change $\pi_C$ of $\pi$ by $f$ is the natural projection morphism of the projectivization of a vector bundle $V$ on $\mathbb{P}^1$. We write $V \simeq O_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus O_{\mathbb{P}^1}(a_k)$, with $(a_i)$ ordered increasingly, and consider $s$ the section of $\pi_C$ satisfying $s^*O_{\pi(V)}(1) = O_{\mathbb{P}^1}(a_1)$. The degree of $\text{det}(s^*O_{\pi(V)}(1) \otimes V^*)$ is non-positive, equals zero if and only if $V \simeq O_{\mathbb{P}^1}(a_1)^{\oplus k-1}$, and equals $-1$ if and only if $V \simeq O_{\mathbb{P}^1}(a_1)^{\oplus k-1} \oplus O_{\mathbb{P}^1}(a_1+1)$.

Pulling-back the Euler exact sequence of $\pi_C$ by $s$, we get

$$0 \to O_{\mathbb{P}^1} \to s^*O_{\pi(V)}(1) \otimes V^* \to s^*T_{X/Y} \to 0.$$

Thus, $s^*T_{X/Y}$ has non-positive degree. We also have the tangent bundle exact sequence:

$$0 \to s^*T_{X/Y} \to s^*T_X \to f^*T_Y \to 0,$$

Since $s^*T_{X/Y}$ has non-positive degree, we obtain

$$-K_Y \cdot C \geq -K_X \cdot s(C) \geq \min\{-K_X \cdot C' \mid C' \text{ is a rational curve in } X\}.$$

Moreover, if there is an equality, then we have $-K_Y \cdot C = -K_X \cdot s(C)$, and so $V \simeq O_{\mathbb{P}^1}(a_1)^{\oplus k}$.

If there is almost an equality, then $-K_Y \cdot C = -K_X \cdot s(C)$ or $-K_Y \cdot C = -K_X \cdot s(C) + 1$, so $V \simeq O_{\mathbb{P}^1}(a_1)^{\oplus k}$ or $V \simeq O_{\mathbb{P}^1}(a_1)^{\oplus k-1} \oplus O_{\mathbb{P}^1}(a_1+1)$. \hfill \qed

**Proof of Proposition 4.7.** By [HN13, Theorem 1.3], as $\pi : X \to Y$ is an equidimensional fibration with fibres of dimension $r - 1$, and as it is a Mori contraction of length at least $r$ as well, it is a $\mathbb{P}^{r-1}$-bundle. Let us show that $Y$ is isomorphic to $\mathbb{P}^{r-1}$. Since $X$ is smooth and a projective bundle over $Y$, the variety $Y$ is smooth. By Lemma 4.10, any rational curve $C$ in $Y$ satisfies $-K_Y \cdot C \geq r$. Moreover, $X$ is rationally connected, so $Y$ is too. By [CMSB02, Cor.0.4, 1 $\implies$ 10], we get $Y \simeq \mathbb{P}^{r-1}$.
As $\mathbb{P}^{r-1}$ has trivial Brauer group, there is a vector bundle $V$ of rank $r$ on $Y$ such that $\pi$ identifies with the natural projection $\mathbb{P}(V) \to \mathbb{P}^{r-1}$. Without loss of generality, we can twist $V$ by a line bundle so that $0 \leq \deg_{\Delta} V|_{\Delta} \leq r-1$, for any line $\Delta$ in $\mathbb{P}^{r-1}$. Let $\Delta$ be a line in $\mathbb{P}^{r-1}$. Then $-K_{\mathbb{P}^{r-1}} \cdot \Delta = r$.

By the equality case in Lemma 4.10, the restriction $V|_{\Delta}$ is isomorphic to $L^{\otimes r}$ for some line bundle $L$ on $\Delta$. Hence $\deg L = 0$, so $L = \mathcal{O}_X$. By [OSS80, Theorem 3.2.1], the vector bundle $V$ is globally trivial. Hence, $X \simeq \mathbb{P}^{r-1} \times \mathbb{P}^{r-1}$. □

4.2.3 Fibred Mori contractions for certain fivefolds

The goal in this section is prove the following result.

**Proposition 4.11.** Let $X$ be a smooth projective fivefold such that $\bigwedge^1 T_X$ is strictly nef. Suppose that $\rho(X) > 1$, and that $X$ admits a fibred Mori contraction. Then $X$ is isomorphic to one of the following projective manifolds

\[
\mathbb{P}^2 \times \mathbb{P}^3; \mathbb{P}^2 \times \mathbb{P}^3; \mathbb{P}(T_{\mathbb{P}^3}); \mathbb{P}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1)).
\]

We first establish this classification under the simplifying assumption that $X$ has a $\mathbb{P}^2$-bundle structure, instead of a fibred Mori contraction.

**Lemma 4.12.** Let $X$ be a smooth projective rationally connected fivefold and such that, for any rational curve $C \subset X$, one has $-K_X \cdot C \geq 3$. Suppose that $p : X \to Y$ is a $\mathbb{P}^2$-bundle. Then $Y$ is a smooth projective variety, and $X$ is isomorphic to one of the following projective manifolds

\[
\mathbb{P}^2 \times \mathbb{P}^3; \mathbb{P}^2 \times \mathbb{P}^3; \mathbb{P}(T_{\mathbb{P}^3}); \mathbb{P}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1)).
\]

Among other things, the proof uses the following lemma.

**Lemma 4.13.** Let $V$ be a vector bundle on a smooth quadric hypersurface $Q^n$. If $V$ is trivial on all lines in $Q^n$, then $V$ is trivial.

**Proof.** Note that by [Erm15, Theorem 7], it is enough to show that for any $x, z \in Q^n$, there exists a point $y \in Q^n$ such that the lines $(xy)$ and $(yz)$ belong to $Q^n$. Intersecting with $n - 2$ hyperplanes, we can reduce to $n = 2$, in which case $Q^2 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ is covered by two family of lines corresponding to the two rulings. Hence, the point $y = (pr_1(x), pr_2(z))$ satisfies our requirement. □

**Proof of Lemma 4.12.** Since $X$ is smooth and $X \to Y$ is a projective bundle, $Y$ is smooth as well. Since $X$ is rationally connected, $Y$ is rationally connected and by Lemma 4.10, one has $-K_Y \cdot C \geq 3$ for any rational curve $C$ in $Y$. By [DH17, Cor.1.4], $Y$ is a quadric hypersurface $Q^3$ or the projective space $\mathbb{P}^3$. In either case, $Y$ is rational and so it has trivial Brauer group. Hence, $X = \mathbb{P}(V)$ for some vector bundle $V$ on $Y$.

Let us first assume that $Y$ is a quadric hypersurface $Q^3$. Every line $\Delta$ in $Y$ satisfies $-K_Y \cdot \Delta = 3$. Since $-K_Y$ has degree at least three on any rational curve, by Lemma 4.10 and by its equality case, we have $\mathbb{P}(V|_{\Delta}) \simeq \mathbb{P}^2 \times \Delta$. Hence, for every line $\Delta$ in $Y$, there is an integer $\delta$ such that $V|_{\Delta}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^2}(\delta)^{\oplus 3}$ as a vector bundle on $\Delta \simeq \mathbb{P}^1$. Fixing a single line $\Delta_0$ in $Y$, and noting that $\rho(\Delta_0) = 1$, we have

\[
3\delta - 3\delta_0 = c_1(V) \cdot \Delta - c_1(V) \cdot \Delta_0 = 0,
\]

so the twist $V_0 = V \otimes \mathcal{O}_Y(-\delta_0)$ satisfies $V_0|_{\Delta} = \mathcal{O}_{\Delta}^{\oplus 3}$ for any line $\Delta$ in $Y$. By Lemma 4.13, this vector bundle $V_0$ is trivial on $Y$, and thus $X \simeq \mathbb{P}(V) \simeq \mathbb{P}(V_0) \simeq \mathbb{P}^2 \times \mathbb{P}^3$.

Suppose now that $Y$ is a projective space. By the almost-equality case in Lemma 4.10, for every line $\Delta$ in $Y$,

\[
V|_{\Delta} \simeq \bigoplus_{i=1}^{3} \mathcal{O}_{\mathbb{P}^1}(a_{i,\Delta}),
\]

with either $a_{1,\Delta} = a_{2,\Delta} = a_{3,\Delta}$ or $a_{1,\Delta} = a_{2,\Delta} = a_{3,\Delta} - 1$. Note that the sum $a_{1,\Delta} + a_{2,\Delta} + a_{3,\Delta} = c_1(V) \cdot \Delta$ is independent of the chosen line $\Delta$. If it is divisible by 3, then we are in the first case, else it is congruent to 1 modulo 3 and we are in the second case. In both cases, the $a_{i,\Delta}$ are thus independent of the line $\Delta$. We fix a line $\Delta_0$ in $\mathbb{P}^3$. The twisted bundle $V_0 = V \otimes \mathcal{O}_{\mathbb{P}^3}(-a_{1,\Delta_0})$ now is a uniform bundle of type $(0,0,0)$ or $(0,0,1)$. In the first case, the bundle $V_0$ is globally trivial by [OSS80], and so $X \simeq \mathbb{P}^2 \times \mathbb{P}^3$. In the second case, by [Sat76], the vector bundle $V_0$ is either $\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1)$ or $T_{\mathbb{P}^3}(-1)$, which concludes the classification. □
Let us now study a more general fibred Mori contraction of $X$.

**Lemma 4.14.** Let $X$ be a smooth projective rationally connected fivefold and such that, for any rational curve $C \subset X$, one has $-K_X \cdot C \geq 3$. Suppose that $X$ has a fibred Mori contraction $\pi : X \to Y$. Then $\dim Y \leq 3$.

**Proof.** If $\dim(Y) = 4$, the general fibre of $\pi$ is a smooth curve $C$ with trivial normal bundle. By assumption, 
\[
2 = -K_X \cdot C = \deg_C(-K_C) \geq 3,
\]
absurd. \[\square\]

Let us cover the case when $X$ has a fibred Mori contraction $\pi : X \to Y$ with $1 \leq \dim(Y) \leq 2$.

**Lemma 4.15.** Let $X$ be a smooth projective rationally connected fivefold and such that, for any rational curve $C \subset X$, one has $-K_X \cdot C \geq 3$. Suppose that $X$ has a fibred Mori contraction $\pi : X \to Y$ with $1 \leq \dim Y \leq 2$. Then there is a fibred Mori contraction $p : X \to Z$ that is a $\mathbb{P}^2$-bundle.

**Proof.** We dispose of a rational curve $C$ such that $\pi(C) \neq \{\text{pt}\}$ and $-K_X \cdot C \geq 3$ is minimal with this condition. Let $V$ be the corresponding family of deformations. By Lemma 4.4, $V$ is unsplit. Fix $x \in \text{Locus}(V)$ general. By [Kol96, Proposition IV.2.6] and by assumption, we derive
\[
\dim \text{Locus}(V) + \dim \text{Locus}(V_x) \geq -K_X \cdot C + 5 - 1 \geq 7.
\]
So $\dim \text{Locus}(V_x) \geq 2$. By Lemma 4.3, $\dim \text{Locus}(V_x) \leq \dim Y \leq 2$.

As equality holds, $V$ is a covering family of rational 1-cycles with $\dim \text{Locus}(V_x) = 2 \geq 5 - 3$, so by [BCD07, Theorem 2, Proposition 1(i)], it admits a geometric quotient $p : X \to Z$, that is a fibred Mori contraction, with a general fibre of dimension 2. If a fibre $F$ of $p$ has dimension 3 or more, then since $\dim Y \leq 2$, $\pi|_F$ cannot be finite. So $\pi$ contracts at least a curve $B$ contained in $F$, which is numerically equivalent to a multiple of $C$ as it lies in a $V$-equivalence class [BCD07, Remark 1], a contradiction.

So $p$ is an equidimensional fibred Mori contraction with fibres of dimension 2, of length $-K_X \cdot C \geq 3$. By [HN13, Theorem 1.3], the morphism $p$ is a $\mathbb{P}^2$-bundle. \[\square\]

We are left supposing that $X$ has a fibred Mori contraction $\pi : X \to Y$ with $\dim(Y) = 3$ that is not a $\mathbb{P}^2$-bundle. Let us first prove a few generalities about its fibres.

**Lemma 4.16.** Let $X$ be a smooth projective $n$-dimensional variety with a fibred Mori contraction $\pi$ of length $n - k + 1$ onto a variety $Y$ of dimension $k$. Then the general fibre is isomorphic to $\mathbb{P}^{n-k}$.

**Proof.** The general fibre is a smooth variety $F$ of dimension $n - k$ such that $-K_F \cdot C \geq n - k + 1$ for any rational curve $C$ in $F$, and $-K_F$ is ample. By [CMSB02, Keb02], [HN13, Theorem 2.1], we obtain $F \cong \mathbb{P}^{n-k}$. \[\square\]

We recall and prove a fact mentioned in [HN13, 1.C].

**Lemma 4.17.** Let $X$ be a smooth projective variety of dimension $n \geq 4$ with a fibred Mori contraction $\pi$ of length $n - 2$ onto a threefold $Y$. Suppose that $\pi$ is not equidimensional. Then for any irreducible component $F$ of a fibre of $\pi$ of dimension $n - 2$, the normalization $\tilde{F}$ of $F$ is isomorphic to $\mathbb{P}^{n-2}$.

**Proof.** By the proof of [HN13, Theorem 1.3], and as $\text{Univ}_{n-3}(X/Y) \to \text{Chow}_{n-3}(X/Y)$ is a universal family for the $(n-3)$-cycles of $X$ over $Y$, there is a commutative diagram:

\[
\begin{array}{ccc}
X' & \xrightarrow{\mu'} & X \\
\downarrow{\pi'} & & \downarrow{\pi} \\
Y' & \xrightarrow{\eta} & Y \\
\end{array}
\]

where $\overline{Y}$ is the normalization of the closure of the $\pi$-equidimensional locus of $Y$ in $\text{Chow}_{n-3}(X/Y)$, $\overline{X}$ is the normalization of the universal family over it, $\epsilon'$ is the evaluation map, $Y'$ is a resolution
of $\overline{Y}$, $X'$ is the corresponding normalized fibred product, $\pi'$ is a $\mathbb{P}^{n-3}$ bundle. Note that since $Y$ is $\mathbb{Q}$-factorial, the exceptional loci of $\mu$ and of $\varepsilon$ are unions of surfaces, hence the exceptional locus of $\mu'$ is a union of $\mathbb{P}^{n-3}$-bundles on surfaces. Also note that $\pi$, as a fibred Mori contraction, does not contract any divisor; hence, the indeterminacy locus of $\varepsilon^{-1}$ in $Y$ has dimension zero.

Let $F$ be an irreducible component of dimension $n-2$ of a fibre of $\pi$, let $\nu : \tilde{F} \to F$ be its normalization. Let $\Sigma \subset \overline{Y}$ be one of the surfaces that $\varepsilon$ contracts onto $\pi(F)$, chosen such that $\Gamma := \pi^{-1}(2)$ dominates $F$. Let $S$ be the strict transform of $\Sigma$ by $\eta$, and let $P := \pi'^{-1}(S)$: it is a $\mathbb{P}^{n-3}$-bundle over $S$ and it dominates $\Gamma$. By the universal property of the normalization, we have a map $f : P \to \tilde{F}$, that fits into the following commutative diagram.

$$
\begin{array}{c}
\begin{array}{ccc}
P & \xrightarrow{\eta'} & \Gamma \\
\downarrow & & \downarrow \\
S & \xrightarrow{\pi} & \Sigma \\
\downarrow & \simeq & \downarrow \\
\{\text{pt}\} & \xrightarrow{\pi'} & \pi
\end{array}
\end{array}
\begin{array}{c}
\xrightarrow{\nu} \\

f
\end{array}
$$

Let $\ell$ be a line contained in a fibre of $\pi'|_P$. Let $\mathcal{V}$ be the family of deformation of $f_*\ell$ in $\tilde{F}$.

Let us show that this family satisfies the hypotheses of [HN13, Theorem 2.1]. First, note that $\nu^*(-K_X|_F)$ is ample. Since there is a line in $X'$ numerically equivalent to $\ell$ that is disjoint from all exceptional divisors of $\mu'$, and since $\ell$ is contracted by $\pi'$,

$$\nu^*(-K_X|_F) \cdot f_*\ell = -K_X \cdot \mu'_*\ell = -K_Y \cdot \ell = -K_{X'|\pi'} \cdot \ell = -K_{\mathbb{P}^{n-3}} \cdot \ell = n - 2,$$

Since for any rational curve $C$ in $\tilde{F}$, it holds $\nu^*(-K_X|_F) \cdot C \geq n - 2$ by assumption, the family $\mathcal{V}$ is unsplit. Moreover, it is a covering family, as $\nu$ is birational, $\mu'$ is surjective and the family of deformations of $\ell$ is covering. Hence, by [Kol96, Proposition IV.2.5], for a general point $x \in \tilde{F}$,

$$\dim \mathcal{V} = n - 2 + \dim \text{Locus}(\mathcal{V}_x) + 1 - 3,$$

so we are left to show that $\dim \text{Locus}(\mathcal{V}_x) = n - 2$ to conclude.

Let us take $x$ and $y$ general in $\tilde{F}$. It suffices to show that the image by $\mu'|_P$ of a certain fibre $\mathbb{P}^{n-3}$ of $\pi'|_P$ contains both $x$ and $y$, since then there is a line through any two points in $\mathbb{P}^{n-3}$.

Since $x$ is general and $\Gamma$ dominates $F$, it holds $\dim \varepsilon'^{-1}(x) = \dim \Gamma - \dim F = n - 3 + 2 = (n - 2) = 1$, so there is a one-dimensional family of cycles passing through $x$, parametrized by a curve in $\Sigma$. As there is a finite map $\Sigma \to \text{Chow}_{n-3}(F)$ (a composition of inclusions and a normalization), this is a non-trivial family of divisors. Hence, it must cover $F$, in particular there is one divisor passing through $y$ and $x$. This divisor is dominated by a fibre of $\pi'|_P$, which concludes.

We now use the fact that $\pi$ is not a $\mathbb{P}^2$-bundle (in fact, that $\pi$ is not equidimensional) to construct covering families of rational curves on $X$. Before that, we prove a simple lemma.

**Definition 4.18.** Let $f : X \dasharrow Y$ be a rational map. We say that $f$ is almost holomorphic if there are Zariski open subsets $U \subset X$ and $V \subset Y$ such that $f|_U : U \to V$ is a proper holomorphic map.

**Lemma 4.19.** Let $f : X \dasharrow Y$ be almost holomorphic map. If $Y$ is a curve, then $f$ is holomorphic.

**Proof.** Let $\varepsilon : X' \to X$ be a resolution of indeterminacies for $f$, let $f' : X' \to Y$ be the induced holomorphic map. As $f$ is almost holomorphic, no component of the exceptional locus of $\varepsilon$ is dominant onto $Y$. As $Y$ is curve, this means that the exceptional locus of $\varepsilon$ is sent onto finitely many points in $Y$. So $f'$ factors through $\varepsilon$, i.e., $f$ is holomorphic.

**Lemma 4.20.** Let $X$ be a smooth projective rationally connected fivefold, such that $-K_X \cdot C \geq 3$ for any rational curve $C \subset X$. Suppose that $X$ has a fibred Mori contraction $\pi : X \to Y$ with $\dim Y = 3$. If $\pi$ is not a $\mathbb{P}^2$-bundle, then any rational curve $C \subset X$ such that $\pi(C) \neq \{\text{pt}\}$, and which deforms in an unsplit family, deforms in a family covering $X$. 

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**Proof.** Note that if \( \pi \) is equidimensional, by [HN13, Theorem 1.3] it is a \( \mathbb{P}^2 \)-bundle. Hence, we assume that a variety \( F \) of dimension 3 is contained in a fibre of \( \pi \). By contradiction, we consider a rational curve \( C \subset X \) such that \( \pi(C) \neq \{ \text{pt} \} \), and the family \( V \) of deformations of \( C \) is unsplit and not covering \( X \).

Fix \( x \in \text{Locus}(V) \) general. By Lemma 4.3, \( \dim \text{Locus}(V_x) \leq \dim Y \leq 3 \). Since the family \( V \) is unsplit, \[ \dim \text{Locus}(V) + \dim \text{Locus}(V_x) \geq -K_X \cdot C + 5 - 1 \geq 7, \] in particular as \( V \) is not covering, \( \dim \text{Locus}(V) = 4 \) and \( \dim \text{Locus}(V_x) = 3 \).

Let \( n : \tilde{D} \to D \) denote the normalization of \( D = \text{Locus}(V) \), and let \( \tilde{V} \) be the covering family on \( \tilde{D} \). Note that \( \pi \) induces a fibration of \( \tilde{D} \) onto a variety of smaller dimension that is not a point, in particular \( \rho(\tilde{D}) \geq 2 \). Thus, by [AC09, Corollary 4.4], \( \tilde{D} \) cannot be \( V \)-chain-connected.

Considering the dominant almost holomorphic map \( r : \tilde{D} \to Z \) whose general fibre is a \( V \)-equivalence class [BCD07, Section 2], the variety \( Z \) is thus not a point. Since \( \dim \text{Locus}(\tilde{V}_x) = 3 \) for a general \( x \in \text{Locus}(V) \), the variety \( Z \) must be a curve, in particular, by Lemma 4.19, the map \( r \) is holomorphic.

Note that, as \( D \) is a relatively ample Cartier divisor with respect to \( \pi \), it intersects the three-dimensional variety \( F \) along a surface \( S \). Since \( \dim n^{-1}(S) = 2 > \dim Z = 1 \), the restriction \( r|_{n^{-1}(S)} : n^{-1}(S) \to Z \) cannot be finite. So it contracts a curve \( B \). Its image \( n(B) \) is in a \( V \)-equivalence class, so as \( V \) is unsplit, it is numerically equivalent to a multiple of \( C \). But \( n(B) \subset F \), so this curve is contracted by \( \pi \), a contradiction.

**Definition 4.21.** Let \( f : X \to Y \) be a morphism of normal varieties. We say that \( f \) is *quasiétale* if \( \dim X = \dim Y \), and there is a Zariski closed subset \( Z \) in \( X \) of codimension at least 2 such that \( f : X \setminus Z \to Y \setminus f(Z) \) is étale.

**Remark 4.22.** Note that if \( f : X \to Y \) is a finite quasiétale cover and \( Y \) is smooth, then by Zariski purity of the branch locus [Zar58, Proposition 2], \( f \) is étale.

**Lemma 4.23.** Let \( X \) be a smooth projective rationally connected fivefold, such that \( -K_X \cdot C \geq 3 \) for any rational curve \( C \subset X \). Suppose that \( X \) has a fibred Mori contraction \( \pi : X \to Y \) with \( \dim Y > 0 \). If \( X \) is not a \( \mathbb{P}^2 \)-bundle over any smooth projective base, then \( Y \simeq \mathbb{P}^3 \). Moreover, we have \( \rho(X) = 2 \), and if \( C \) is a line in the smooth locus of \( \pi \) in \( Y \) and \( s \) is a minimal section over \( C \) in \( X \), the class of \( s(\mathbb{P}^1) \) generates the other extremal ray in \( NE(X) \), induces a fibred Mori contraction to a positive dimensional variety too, and satisfies \( -K_X \cdot s(\mathbb{P}^1) = 3 \).

**Proof.** Note that \( \dim(Y) = 3 \), by Lemmas 4.14, 4.15. By the last lemma of [DP], let \( C \) be a minimal free rational curve in the smooth locus \( Y^0 \subset Y \) of \( \pi \). Let \( s \) be a minimal section over \( C \). Lemma 4.10 yields
\[ 4 \geq -K_Y \cdot C \geq -K_X \cdot s(\mathbb{P}^1). \]
The family \( V \) of deformations of \( s(\mathbb{P}^1) \) is unsplit. Indeed, suppose by contradiction that it is splitting. By [Kol96, Explanation IV.2.2] there is a cycle
\[ \sum_i a_i C_i \equiv_{\text{num}} s(\mathbb{P}^1), \]
with \( C_i \) rational curves, \( a_i \geq 1 \) integers, and \( \sum_i a_i \geq 2 \). Then, intersecting with \( -K_X \) yields
\[ 4 \geq -K_X \cdot s(\mathbb{P}^1) \geq 6, \]
a contradiction.

By Lemma 4.20, \( V \) therefore is a covering family. By [Kol96, Proposition IV.2.6], it moreover holds
\[ \dim \text{Locus}(V_x) \geq -K_X \cdot s(\mathbb{P}^1) - 1 \geq 2 = 5 - 3, \]
so by [BCD07, Theorem 2, Proposition 1(i)], there is a geometric quotient \( p : X \to Z \), that is a fibred Mori contraction, with general fibre of dimension at least \( -K_X \cdot s(\mathbb{P}^1) - 1 \). By Lemma 4.14, we have \( \dim Z \leq 3 \) and by Lemma 4.15, we have \( \dim(Z) = 3 \), or \( X \) is a \( \mathbb{P}^2 \)-bundle over some three-dimensional base. So \( \dim Z = 3 \), hence \( -K_X \cdot s(\mathbb{P}^1) = 3 \). It also follows that \( s(\mathbb{P}^1) \) is an extremal class in the Mori cone, as wished.

Again, \( X \) not being a \( \mathbb{P}^2 \)-bundle over any smooth base, \( p \) is not equidimensional by [HN13, Theorem 1.3], so a variety \( F \) of dimension 3 is contained in a fibre of \( p \). By Lemma 4.17, the normalization \( n : \tilde{F} \to F \) satisfies \( \tilde{F} \simeq \mathbb{P}^3 \).
Since $\pi$ and $p$ are distinct Mori contractions, they contract no common numerical class of curve, in particular $\pi|_F : F \to Y$ is finite onto its image, hence finite surjective for dimensional reasons. There is an effective ramification divisor $R \in \text{Pic}(\mathbb{P}^3)$ such that $-K_{\mathbb{P}^3} = n\pi|_F^*(-KY) - R$. As $F$ is an irreducible component of a $\sim$-equivalence class, and as $V$ is unsplit, $F$ contains a deformation of $s(\mathbb{P}^4)$. Let $\tilde{C}$ be the lift to $\tilde{F}$ of a deformation of $s(\mathbb{P}^4)$ that is contained in $F$. Then $-K_{\mathbb{P}^3} \cdot \tilde{C} \geq 4$, and $\pi|_\tilde{F}^*(-KY) \cdot \tilde{C} = -KY \cdot \tilde{C} \leq 4$. So $R \cdot \tilde{C} \leq 0$, but $R \in \text{Pic}(\mathbb{P}^3)$ is effective, thus ample or trivial, so $R$ is trivial. The finite map $\pi|_F \circ n : \mathbb{P}^3 \to Y$ is thus quasiétale. So, its base change $\mathbb{P}^3 \times Y \to Y$ is also quasiétale, as $\pi : X \to Y$ contracts no divisor. But $X$ is rationally connected, hence simply connected, and smooth, so $\mathbb{P}^3 \times Y \to Y$ is an isomorphism. Hence $\pi|_F \circ n : \mathbb{P}^3 \to Y$ is an isomorphism too.

Since $\rho(Y) = 1$, we have $\rho(X) = 2$. Since $Y \cong \mathbb{P}^3$ and $4 \geq -KY \cdot C$, the curve $C$ is a line. \qed

**Lemma 4.24.** Let $X$ be a smooth projective rationally connected fivefold, such that $-K_X \cdot C \geq 3$ for any rational curve $C \subset X$. Suppose that $X$ has a fibred Mori contraction $\pi : X \to Y$ with $\dim(Y) > 0$. If $X$ is not a $\mathbb{P}^2$-bundle over any smooth projective base, then $\rho(X) = 2$ and $X$ has two distinct fibred Mori contractions onto $\mathbb{P}^3$, with corresponding extremal rays generated by the minimal sections $s(\mathbb{P}^4)$, $\sigma(\mathbb{P}^4)$ above lines that lie in each $\mathbb{P}^3$ in the smooth loci of the fibration. Moreover,

$$-K_X \cdot s(\mathbb{P}^4) = -K_X \cdot \sigma(\mathbb{P}^4) = 3.$$

**Proof.** Apply Lemma 4.23 twice. \qed

**Proof of Proposition 4.11.** If $X$ has a $\mathbb{P}^2$-bundle structure, then Lemma 4.12 concludes. Suppose that $X$ is not a $\mathbb{P}^2$-bundle. By Lemma 4.24, $X$ admits exactly two fibred Mori contractions $\pi$ and $p$, both onto $\mathbb{P}^3$. Given the intersection number of $-K_X$ with both extremal rays, and as $\pi_* s(\mathbb{P}^4)$ is a line in $\mathbb{P}^3$ and as $p_* s(\mathbb{P}^4) = 0$, we have

$$-K_X \cdot s(\mathbb{P}^4) = 3 = \pi_* \mathcal{O}_{\mathbb{P}^3}(3) \cdot s(\mathbb{P}^4) = (\pi_* \mathcal{O}_{\mathbb{P}^3}(3) \otimes p_* \mathcal{O}_{\mathbb{P}^3}(3)) \cdot s(\mathbb{P}^4),$$

and similarly

$$-K_X \cdot \sigma(\mathbb{P}^4) = (\pi_* \mathcal{O}_{\mathbb{P}^3}(3) \otimes p_* \mathcal{O}_{\mathbb{P}^3}(3)) \cdot \sigma(\mathbb{P}^4).$$

Hence, as $\rho(X) = 2$, and $s(\mathbb{P}^4)$ and $\sigma(\mathbb{P}^4)$ are independent,

$$\omega_X^* = \pi^* \mathcal{O}_{\mathbb{P}^3}(3) \otimes p^* \mathcal{O}_{\mathbb{P}^3}(3).$$

By Theorem 1.2, $-K_X$ is ample. So $X$ is a Fano fivefold, and we just showed that it has index 3. By the classification in [Wi91b], $X$ must then be a $\mathbb{P}^2$-bundle, which is a contradiction. \qed

### 4.3 Divisorial contractions

Let us classify divisorial Mori contraction of large length.

**Proposition 4.25.** Let $X$ be a smooth projective rationally connected variety of dimension $n$ such that $-K_X \cdot C \geq 3$ for every rational curve $C$. Then $X$ admits no divisorial Mori contraction of length greater or equal to $n - 1$.

**Remark 4.26.** In particular, the assumptions are fulfilled if there is $1 \leq r \leq n - 1$ such that $\bigwedge^r T_X$ is strictly nef, by [LOY19, Theorem 1.2] and Lemma 2.1.

The proof uses the following lemma, that excludes some special contractions of length $n - 1$.

**Lemma 4.27.** Let $X$ be a smooth projective rationally connected variety of dimension $n$ such that $-K_X \cdot C \geq 3$ for every rational curve $C$. Then there is no morphism $X \to Y$ that is a blow-up of a smooth point in a smooth variety.

**Proof of Lemma 4.27.** By contradiction, consider such a smooth blow-up:

$$f : E \subset X \to p \in Y.$$

Note that since $X$ is rationally connected, so $Y$ is too. Let $C$ be a rational curve through $p$.

Since $-f^*KY = -K_X + (n - 1)E$ and since no curve is contained in the blown-up locus $p$, the anticanonical divisor $-KY$ is strictly nef. By bend-and-break [Deb01, Proposition 3.2] on the smooth
Let $−K_Y \cdot C \leq n + 1$. The strict transform $C' \subset X$ of $C$ satisfies $E \cdot C' > 0$. Since $K_X = f^*K_Y + (n-1)E$, we have
\[ 3 \leq -K_X \cdot C' \leq -K_Y \cdot C - (n-1) \leq 2, \]
a contradiction!

**Proof of Proposition 4.25.** By Ionescu-Wiśnewski inequality, if $X$ admits a divisorial Mori contraction of length $\ell \geq n - 1$, the exceptional divisor $E$ and the general fibre $F \subset E$ satisfy:
\[ \dim E + \dim F \geq n + \ell - 1 \geq 2n - 2, \]
i.e., $\ell = n - 1$ and $E = F$ is contracted onto a point. So [AO02, Theorem 5.2] applies and shows that this divisorial Mori contraction of $X$ corresponds to a blow-up of a smooth point in a smooth variety, which contradicts Lemma 4.27.

We now consider divisorial Mori contractions of length $n - 2$.

**Proposition 4.28.** Let $X$ be a smooth projective variety of dimension $n \geq 5$, that is rationally connected and such that $-K_X \cdot C \geq n - 2$ for any rational curve $C \subset X$. Then $X$ has no divisorial Mori contraction contracting the exceptional divisor to a point.

**Remark 4.29.** These assumptions are fulfilled if $\mathbb{A}^4T_X$ is strictly nef, by [LOY19, Theorem 1.2] and Lemma 2.1.

**Proof.** Assume that $\varepsilon : X \to Y$ is a divisorial Mori contraction contracting the exceptional divisor $E$ to a point. Note that as $X$ is rationally connected, there exists a rational curve $C$ that intersects $E$ without being contained in $E$. In particular, $E \cdot C > 0$. Among all such curves, let actually $C$ be one such that $-K_X \cdot C$ is minimal. Then we claim that the family $\nu$ of deformations of $C$ is unsplit. Indeed, suppose by contradiction that it is splitting. By [Kol96, Explanation IV.2.2], we have
\[ C \equiv \sum a_i C_i, \]
with rational curves $C_i$ and coefficients $a_i \geq 1$ such that $\sum a_i \geq 2$. Then $E \cdot C > 0$, so without loss of generality, $E \cdot C_1 > 0$. In particular, $C_1$ intersects $E$ and is not contracted by $\varepsilon$, hence not contained in $E$. Since $-K_X$ has positive degree on all rational curves in $X$, we have $-K_X \cdot C_1 < -K_X \cdot C$, which contradicts the minimality of $-K_X \cdot C$.

By [Kol96, Proposition IV.2.6.1], for a general $x \in \text{Locus}(\nu)$,
\[ \dim \text{Locus}(\nu) + \dim \text{Locus}(\nu_x) \geq n + n - 2 - 1. \]
In particular, $\dim \text{Locus}(\nu_x) \geq n - 3$, and as $X$ is smooth, $E$ is Cartier, hence intersects $\text{Locus}(\nu_x)$ along a subscheme of dimension at least $n - 4 \geq 1$. Let $B$ be a curve in this intersection. It is contained in $E$, hence contracted by $\varepsilon$, hence satisfies $E \cdot B < 0$. On the other hand, it is contained in $\text{Locus}(\nu_x)$, hence is numerically equivalent to a multiple of $C$ by [ACO09, Lemma 4.1]. It has to be a positive multiple, as one sees when intersecting with any ample divisor. But $E \cdot C > 0$, a contradiction.

**Corollary 4.30.** Let $X$ be a smooth projective variety of dimension $n \geq 5$, that is rationally connected and such that $-K_X \cdot C \geq n - 2$ for any rational curve $C \subset X$. Suppose that $\varepsilon : X \to Y$ is a divisorial Mori contraction. Then $Y$ is smooth and $\varepsilon$ is the blow-up of a smooth curve in $Y$.

**Proof.** Recall [Deb01, Proposition 6.10(b)] that the divisorial Mori contraction $\varepsilon$ has a unique exceptional divisor $E$ as its exceptional locus. By [KM98, Lemma 2.62], a ray $\mathbb{R}_+[C]$ associated to $\varepsilon$ satisfies $E \cdot C < 0$, so such $C$ has negative intersection with at least one effective divisor. Moreover, $\varepsilon$ is a Mori contraction of length $n - 2$. So [AO02, Theorem 5.3] applies, showing that $\varepsilon$ either contracts a divisor to a point, or is a blow-up of a smooth curve in a smooth variety $Y$. By Proposition 4.28, only the latter can occur.

Let us finally describe more precisely what happens in the occurrence of Corollary 4.30.

**Lemma 4.31.** Let $X$ be a smooth projective variety of dimension $n \geq 3$, that is rationally connected and such that for some $1 \leq r \leq n - 1$, for any rational curve $C \subset X$, it holds $-K_X \cdot C \geq n + 2 - r$. If there is a morphism $\varepsilon : X \to Y$ that is a blow-up of a smooth curve in the smooth variety $Y$, then $r = n - 1$. 

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Proof. Consider such a smooth blow-up:
\[ f : E \subset X \to \ell \subset Y \]

As \( X \) is rationally connected, so is \( Y \). Fix \( H \) an ample divisor on \( Y \). Let \( C \subset Y \) be a rational curve other than \( \ell \) passing through a point \( p \in \ell \), with \( H \cdot C \) minimal among the degrees of all rational curves intersecting \( \ell \) other than \( \ell \). Fix another point \( q \in C \setminus C \cap \ell \). By bend-and-break [Deb01, Proposition 7.3], as \( Y \) is smooth, if \( -K_Y \cdot C \geq n + 2 \), then there is a connected non-integral 1-cycle that is a deformation of \( C \) passing through \( p \) and \( q \). In particular,
\[
\sum_{i=1}^{k} a_i C_i \equiv \min C,
\]
with rational curves \( C_i \) such that \( p \in C_1, q \in C_{i_0} \) for some \( i_0 \), coefficients \( a_i \geq 1 \), and \( \sum_{i=1}^{k} a_i \geq 2 \). As \( q \notin \ell \), we have that \( C_{i_0} \neq \ell \), so either \( C_1 \neq \ell \), or \( C_1 = \ell \) and \( k \geq 2 \). Intersecting with \( H \), we see that \( H \cdot C_i < H \cdot C \) for all \( i \), in particular for \( C_1 \). If \( C_1 \neq \ell \), then \( H \cdot C_1 \) contradicts the minimality of \( H \cdot C \). If \( C_1 = \ell \), then \( k \geq 2 \) and by connectedness of the rational cycle, there is a curve \( C_i \neq \ell \) that intersects \( C_1 = \ell \). So \( C_{i_1} \neq \ell \) intersects \( \ell \) and contradicts the minimality, as \( H \cdot C_{i_1} < H \cdot C \) again. So \( -K_Y \cdot C \leq n + 1 \).

The strict transform \( C' \subset X \) of \( C \) satisfies \( E \cdot C' > 0 \). Since \( K_X = f^*K_Y + (n - 2)E \), and by assumption,
\[
n + 2 - r \leq -K_X \cdot C' \leq -K_Y \cdot C - (n - 2) \leq 3,
\]
so \( r = n - 1 \).

**Proposition 4.32.** Let \( X \) be a smooth projective variety of dimension \( n \geq 5 \), that is rationally connected and such that \( \bigwedge^3 T_X \) is strictly nef. If there is a morphism \( \epsilon : X \to Y \) that is a blow-up of a smooth curve in the smooth variety \( Y \), then \( X \) is a fivefold and there is a fibred Mori contraction \( \pi : X \to Z \) with \( \dim(Z) > 0 \).

**Proof.** By Lemma 4.31, we have \( n = 5 \). So by Theorem 1.2, \( -K_X \) is ample. The Mori cone \( NE(X) \) is closed, generated by finitely many classes of rational curves. Let \( E \) be the exceptional divisor of \( \epsilon \). Note that there exists an extremal ray \( R = \mathbb{R}_+[C] \) of \( NE(X) \) on which \( E \cdot C < 0 \). Indeed, if there were not such a ray, then \( E \) would be non-positive on all curves in \( X \), which is absurd for an effective divisor. So, let \( R = \mathbb{R}_+[C] \) be an extremal ray on which \( E \cdot C > 0 \).

Denote the associated Mori contraction by \( \pi : X \to Z \). Since \( X \) already had a non-trivial Mori contraction \( \epsilon \), we have \( \dim(Z) > 0 \). Let us prove that \( \pi \) is a fibred Mori contraction.

By Lemma 4.1, \( \pi \) cannot be a small contraction. Assume by contradiction that it is a divisorial contraction. By Corollary 4.30, the variety \( Z \) is smooth and \( \pi \) is a blow-up along a smooth curve of \( Z \). Let \( E' \) be the \( \pi \)-exceptional divisor. Let \( \ell \), respectively \( \ell' \), be the image of \( E \), respectively \( E' \), in \( Y \), respectively \( Z \). Let \( F' \) be a general fibre of \( \pi'|_{E'} \). It has dimension \( n - 2 \). Note that \( F' \) and \( E \) intersect, since \( E \cdot C > 0 \). Hence, \( E \cap F' \) is a subscheme of \( X \) of dimension at least \( n - 3 \). Since \( \epsilon \) and \( \pi \) are distinct Mori contractions, the restriction \( \epsilon|_{E \cap F'} \) must be finite onto its image, which is contained in \( \ell \). So \( n - 3 \leq 1 \), a contradiction!

So \( \pi \) is a fibred Mori contraction.

**Proposition 4.33.** Let \( X \) be a smooth projective variety of dimension \( n \geq 5 \), that is rationally connected and such that \( \bigwedge^3 T_X \) is strictly nef. If there is a morphism \( \epsilon : X \to Y \) that is a blow-up of a smooth curve, then \( Y \cong \mathbb{P}^5 \) and \( \epsilon \) is the blow-up of a line.

**Proof.** By Proposition 4.32, \( X \) is a fivefold and admits a fibred Mori contraction onto a positive dimensional base. So Proposition 4.11 applies, showing that \( X \) belongs to a list of certain varieties of Picard number two. Only one of them has a divisorial Mori contraction, namely \( \text{Bl}_q(\mathbb{P}^5) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^5} \oplus \mathcal{O}_{\mathbb{P}^5} \oplus \mathcal{O}_{\mathbb{P}^5}(1)) \).

5 Results on \( \bigwedge^3 T_X \)

**Proof of Theorem 1.3.** Note that \( -K_X \) is nef, and non-trivial (as it is positive on rational curves, and \( X \) is rationally connected). If \( \rho(X) = 1 \), \( -K_X \) is ample and \( X \) is thus a Fano variety. If \( \rho(X) \geq 2 \), by the Cone Theorem, \( X \) admits a Mori contraction, which by Lemma 4.1 and Proposition 4.25 is a fibred Mori contraction. Corollary 4.5 implies that \( X \) is a fourfold. By Lemma 4.6, \( X \) has an equidimensional fibred Mori contraction to a surface, so by Proposition 4.7, we have \( X \cong \mathbb{P}^2 \times \mathbb{P}^2 \).
Proof of Corollary 1.4. It is straightforward from Lemma 2.1, [LOY19, Theorem 1.2], and Theorem 1.3.

Remark 5.1. It is easy to check that $\Lambda^3 T_{P^n,p}$ is in fact ample.

Example 5.2. Let $X$ be a cubic in $\mathbb{P}^n$ with $n \geq 5$. From the tangent exact sequence

$$0 \to T_X \to T_{P^n}|_X \to \mathcal{O}_X(3) \to 0,$$

we can use [Har77, II.Ex.5.16(d)] to derive the existence of a surjection

$$0 \to F_4 \to \bigwedge^4 T_{P^n}|_X \to \bigwedge^3 T_X \otimes \mathcal{O}_X(3) \to 0.$$

As $T_{P^n}|_X \otimes \mathcal{O}_X(-1)$ is nef, the quotient of its fourth exterior power $\Lambda^3 T_X \otimes \mathcal{O}_X(-1)$ is also nef, and thus $\Lambda^4 T_X$ is ample.

Example 5.3. Let $X$ be the complete intersection of two quadrics in $\mathbb{P}^n$ with $n \geq 6$. From the tangent exact sequence

$$0 \to T_X \to T_{P^n}|_X \to \mathcal{O}_X(2) \oplus \mathcal{O}_X(2) \to 0,$$

we can use [Har77, II.Ex.5.16(d)] to derive the existence of a surjection

$$0 \to F_4 \to \bigwedge^5 T_{P^n}|_X \to \bigwedge^3 T_X \otimes \mathcal{O}_X(4) \to 0.$$

As $T_{P^n}|_X \otimes \mathcal{O}_X(-1)$ is nef, the quotient of its fifth exterior power $\Lambda^3 T_X \otimes \mathcal{O}_X(-1)$ is also nef, and thus $\Lambda^4 T_X$ is ample.

6 Results on $\Lambda^4 T_X$

6.1 Examples

Lemma 6.1. Let $X$ be the fivefold $\mathbb{P}(T_{P^3})$. Then $\Lambda^4 T_X$ is ample.

Proof. Denote the natural projection by $p : X \to \mathbb{P}^3$, the tautological line bundle on $X$ by $\mathcal{O}_X(1)$. By [Har77, II.Ex.5.16(d)], there is an exact sequence

$$0 \to \bigwedge^2 T_X \otimes p^* \bigwedge^2 T_{P^3} \to \bigwedge^4 T_X \to T_{X/\mathbb{P}^3} \otimes p^* \mathcal{O}_{P^3}(-K_{P^3}) \to 0.$$

Let us prove that $E_1 = T_{X/\mathbb{P}^3} \otimes p^* \mathcal{O}_{P^3}(-K_{P^3})$ is ample. We have the relative Euler sequence

$$0 \to \mathcal{O}_X \to p^* \Omega^1_{\mathbb{P}^3} \otimes \mathcal{O}_X(1) \to T_{X/\mathbb{P}^3} \to 0.$$

The bundle $E_1$ is a quotient of $p^* \Omega^1_{\mathbb{P}^3}(4) \otimes \mathcal{O}_X(1)$. But as $T_{P^3}$ is ample, $\mathcal{O}_X(1)$ is ample. Moreover, $\Omega^1_{\mathbb{P}^3}(4) \simeq \bigwedge^2 T_{P^3}$ is ample too, which concludes by [Laz04b, 6.1.16].

Let us prove that $E_2 = \bigwedge^2 T_{X/\mathbb{P}^3} \otimes p^* \bigwedge^2 T_{P^3}$ is ample. This would settle the ampleness of $\Lambda^4 T_X$ by [Laz04b, 6.1.13(ii)]. From [Har77, II.Ex.5.16(d)] and the relative Euler sequence, we derive

$$0 \to T_{X/\mathbb{P}^3} \to p^* T_{P^3}(-4) \otimes \mathcal{O}_X(2) \to \bigwedge^2 T_{X/\mathbb{P}^3} \to 0.$$

Since $E_2$ is a quotient of $p^* (T_{P^3}(-4) \otimes \bigwedge^2 T_{P^3}) \otimes \mathcal{O}_X(2)$, we are left proving that the latter is ample. Notice that $T_{P^3}(-1)$ is globally generated and thus nef. So the bundle $T_{P^3}(-3) \otimes \bigwedge^2 T_{P^3} = T_{P^3}(-1) \otimes \bigwedge^2 T_{P^3}(-1)$ is nef as well. Finally, $\mathcal{O}_X(1)$ is ample, and we see that $\mathcal{O}_X(1) \otimes p^* \mathcal{O}_{P^3}(-1)$ is a quotient of $p^* T_{P^3}(-1)$ (dualizing the relative Euler exact sequence and twisting by $\mathcal{O}_X(1)$), hence it is nef. We conclude by [Laz04b, 6.2.12(iv)].

Lemma 6.2. Let $X$ be the fivefold $\mathbb{P}(\mathcal{O}_{P^3} \oplus \mathcal{O}_{P^3} \oplus \mathcal{O}_{P^3}(1))$. Then $\Lambda^4 T_X$ is ample.

Remark 6.3. Note that $\mathbb{P}(\mathcal{O}_{P^3} \oplus \mathcal{O}_{P^3} \oplus \mathcal{O}_{P^3}(1))$ is isomorphic to the blow-up of line in $\mathbb{P}^5$ [EH16, Section 9.3.2].
Proof. Denote the natural projection by $p : X \to \mathbb{P}^3$, the tautological line bundle on $X$ by $\mathcal{O}_X(1)$. By [Har77, II.Ex.5.16(d)], there is an exact sequence

$$0 \to \bigwedge^2 T_{X/\mathbb{P}^3} \otimes p^* T_{\mathbb{P}^3} \to \bigwedge^4 T_X \to T_{X/\mathbb{P}^3} \otimes p^* \mathcal{O}_{\mathbb{P}^3}(-K_{\mathbb{P}^3}) \to 0.$$  

Let us prove that $E_1 = T_{X/\mathbb{P}^3} \otimes p^* \mathcal{O}_{\mathbb{P}^3}(-K_{\mathbb{P}^3})$ is ample. We have the relative Euler sequence

$$0 \to \mathcal{O}_X \to p^*(\mathcal{O}_{\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3}(-1)) \to \mathcal{O}_X(1) \to T_{X/\mathbb{P}^3} \to 0.$$  

The bundle $E_1$ is a quotient of $p^*(\mathcal{O}_{\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3}(4) \otimes \mathcal{O}_{\mathbb{P}^3}(4)) \otimes \mathcal{O}_X(1)$. Since $\mathcal{O}_{\mathbb{P}^3}(3) \otimes \mathcal{O}_{\mathbb{P}^3}(4) \otimes \mathcal{O}_{\mathbb{P}^3}(4)$ is ample and $\mathcal{O}_X(1)$ is nef and $p$-ample, the bundle $E_1$ is thus ample.

Let us prove that $E_2 = \bigwedge^2 T_{X/\mathbb{P}^3} \otimes p^* \bigwedge^2 T_{\mathbb{P}^3}$ is ample. From [Har77, II.Ex.5.16(d)] and the relative Euler sequence, we derive

$$0 \to T_{X/\mathbb{P}^3} \to p^*(\mathcal{O}_{\mathbb{P}^3}(-1) \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \otimes \mathcal{O}_{\mathbb{P}^3}) \otimes \mathcal{O}_X(2) \to \bigwedge^2 T_{X/\mathbb{P}^3} \to 0.$$  

It is thus enough to prove that $p^* \bigwedge^2 T_{\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \otimes \mathcal{O}_X(2)$ is ample, which is clear since $\bigwedge^2 T_{\mathbb{P}^3}(-1) = (\bigwedge^2 T_{\mathbb{P}^3})(-2)$ is globally generated and thus nef, and since $p^* \mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathcal{O}_X(2)$ is ample.

\[\blacksquare\]

Remark 6.4. It is easy check to that $\bigwedge^4 T_{\mathbb{P}^2 \times \mathbb{P}^3}$, $\bigwedge^4 T_{\mathbb{P}^2 \times \mathbb{Q}^3}$, $\bigwedge^4 T_{\mathbb{P}^2 \times \mathbb{P}^3}$ are ample.

6.2 Proof of Theorem 1.5 and Corollary 1.6

Proof of Theorem 1.5. Note that $-K_X$ is nef, and non-trivial (as it is positive on rational curves, and $X$ is rationally connected). If $\rho(X) = 1$, $-K_X$ is ample and $X$ is thus a Fano variety. If $\rho(X) \geq 2$, by the Cone Theorem, $X$ admits a Mori contraction. By Lemma 4.1, it cannot be a small contraction.

Suppose that it is a divisorial contraction. By Corollary 4.30 and Lemma 4.31, it is a smooth blow-up of a smooth curve in a fivefold, but we are assuming that $X$ has dimension at least six, a contradiction!

So $X$ has no divisorial contraction. Thus, it has a fibred Mori contraction onto a positive dimensional variety. Corollary 4.5 implies that $X$ is a fivefold or a sixfold. By assumption, $X$ is thus a sixfold. By Lemma 4.6, $X$ has an equidimensional fibred Mori contraction to a threefold, so by Proposition 4.7, we have $X \simeq \mathbb{P}^3 \times \mathbb{P}^3$, which concludes.

\[\blacksquare\]

Proof of Corollary 1.6. By Theorem 1.5, is is enough to consider the case when $X$ is a fivefold. In particular, by Theorem 1.2, $X$ is a Fano variety. Again, if $\rho(X) = 1$, there is nothing to prove.

If $\rho(X) \geq 2$, by the Cone Theorem, $X$ admits a Mori contraction. By Lemma 4.1, it cannot be a small contraction.

Suppose that it is a divisorial contraction. By Corollary 4.30, it is a smooth blow-up of a smooth curve, and by Proposition 4.33, $X \simeq \text{Bl}_Y \mathbb{P}^3$.

Otherwise, it is a fibred Mori contraction onto a positive dimensional variety. Since $X$ is a fivefold such that $\bigwedge^4 T_X$ is strictly nef, Proposition 4.11 applies and concludes.

\[\blacksquare\]

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