OPTIMAL CONVERGENCE BEHAVIOR OF ADAPTIVE FEM DRIVEN BY SIMPLE \((h - h/2)\)-TYPE ERROR ESTIMATORS

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Abstract. For some Poisson-type model problem, we prove that adaptive FEM driven by the \((h - h/2)\)-type error estimators from [Ferraz-Leite, Ortner, Praetorius, Numer. Math. 116 (2010)] leads to convergence with optimal algebraic convergence rates. Besides the implementational simplicity, another striking feature of these estimators is that they can provide guaranteed lower bounds for the energy error with known efficiency constant 1.

1. Introduction

Let \(\Omega \subset \mathbb{R}^d\) with \(d \geq 2\) be a bounded Lipschitz domain with polyhedral boundary \(\Gamma := \partial \Omega\). Given \(f \in L^2(\Omega)\), let \(u \in H^1_0(\Omega)\) be the unique weak solution of the model problem
\[
-\text{div}(A \nabla u) = f \text{ in } \Omega \quad \text{subject to Dirichlet boundary conditions } \ u = 0 \text{ on } \Gamma,
\]
where \(A : \Omega \rightarrow \mathbb{R}^{d \times d}\) is piecewise constant on some initial conforming triangulation \(T_0\) and maps into the space of symmetric positive definite matrices.

Based on a conforming simplicial triangulation \(T_\ell\), we consider the \(H^1\)-conforming FE space of \(T_\ell\)-piecewise polynomials of degree \(p \geq 1\). Let \(u_\ell\) be the corresponding FEM solution. Throughout, the index \(\ell \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}\) denotes the step of the adaptive algorithm. Due to singularities of the (unknown) exact solution, uniform mesh-refinement usually leads a suboptimal convergence behavior of the energy norm error \(\|A^{1/2} \nabla (u - U_\ell)\|_\Omega\), where \(\| \cdot \|_\Omega := \| \cdot \|_{L^2(\Omega)}\). However, the appropriate grading of the triangulation \(T_\ell\) has the potential to lead to the optimal convergence rate \(O((\#T_\ell)^{-p/d})\) with respect to the number of elements \(\#T_\ell\). Such a mesh-grading can automatically be generated by adaptive mesh-refining algorithms of the type
\[
\text{SOLVE} \rightarrow \text{ESTIMATE} \rightarrow \text{MARK} \rightarrow \text{REFINE}
\]

In the last two decades, the mathematical understanding of adaptive algorithms has matured. Starting with the first convergence results in [Dör96, MNS00], it is meanwhile known that the adaptive algorithm, driven by the canonical residual error estimator, leads to linear convergence with optimal algebraic rates; see, e.g., [Ste07, CKNS08, FFP14]. The same result holds for any estimator, which is locally equivalent to the residual error estimator [KSL11, CFPP14], where the analysis strongly exploits this local equivalence.
Examples for locally equivalent estimators include hierarchical error estimators, averaging estimators, and equilibrated fluxes.

The current work considers \((h-h/2)\)-type error estimators which are only globally, but not locally equivalent to residual error estimators. This error estimation strategy is a well-known technique; see [HNW87] for ordinary differential equations and the works of Bank [BW85, BS93, Ban96] or the monograph [AO00, Chapter 5] in the context of the finite element method. Let \(T_\ell\) be the uniform refinement of \(T_\ell\).

Let \(\hat{u}_\ell\) be the corresponding FE solution. The natural \((h-h/2)\)-error estimator

\[
\tilde{\mu}_\ell := \|A^{1/2}\nabla (\hat{u}_\ell - u_\ell)\|_\Omega
\]

is a computable quantity which can be used to estimate the error \(\|A^{1/2}\nabla (u - u_\ell)\|_\Omega\). According to the Galerkin orthogonality, it holds that

\[
\|A^{1/2}\nabla (u - \hat{u}_\ell)\|_\Omega^2 + \tilde{\mu}_\ell^2 = \|A^{1/2}\nabla (u - u_\ell)\|_\Omega^2
\]

From this, it is easy to see that

\[
\tilde{\mu}_\ell \leq \|A^{1/2}\nabla (u - u_\ell)\|_\Omega \leq (1 - q_{\text{sat}}^2)^{-1/2}\tilde{\mu}_\ell
\]

where the upper bound requires and is even equivalent to the so-called saturation assumption

\[
\|A^{1/2}\nabla (u - \hat{u}_\ell)\|_\Omega \leq q_{\text{sat}} \|A^{1/2}\nabla (u - u_\ell)\|_\Omega \quad \text{with some uniform} \quad 0 < q_{\text{sat}} < 1.
\]

We remark that the saturation assumption dates back to the early work [BW85], but may fail to hold in general [BEK96, DN02] and is essentially equivalent to asymptotic behavior of the FEM; see the discussion in [FP08, Section 5.2]. However, under certain assumptions on the polynomial degree \(p\) and/or the mesh-refinement (e.g., \(d = 2\) with bisec5-refinement or \(d = 2\) with \(p \geq 2\) and bisec3-refinement), one can rigorously prove that

\[
(\tilde{\mu}_\ell^2 + \text{osc}_\ell(f)^2)^{1/2} \leq (\|A^{1/2}\nabla (u - u_\ell)\|_\Omega^2 + \text{osc}_\ell(f)^2)^{1/2} \leq C_{\text{rel}} (\tilde{\mu}_\ell^2 + \text{osc}_\ell(f)^2)^{1/2},
\]

where \(\text{osc}_\ell(f)\) denote the data oscillations; see Theorem 4 below, where we extend an idea from [Dör96, MNS00]. However, having to compute \(\hat{u}_\ell\), it is not attractive to compute the less accurate \(u_\ell\); cf. [1]. In this work, we thus consider variants of the \((h-h/2)\) error estimator from [FOP10], which avoid this computation, e.g.,

\[
\eta_\ell = \|(1 - \pi_\ell)A^{1/2}\nabla \hat{u}_\ell\|_\Omega^2 + \text{osc}_\ell(f)^2)^{1/2},
\]

where \(\pi_\ell\) is the \(T_\ell\)-elementwise \(L^2\)-projection onto polynomials of degree \(p - 1\) (see 28 below for further variants). We prove that

\[
\eta_\ell \leq (\|A^{1/2}\nabla (u - u_\ell)\|_\Omega^2 + \text{osc}_\ell(f)^2)^{1/2} \leq C_{\text{rel}}C_{\text{LH2}} \eta_\ell,
\]

i.e., \(\eta_\ell\) is a computable guaranteed lower bound for the total error even with known constant 1. Using this estimator (or one of its variants 28 in the adaptive algorithm (see Algorithm 3 for the precise statement), we prove that the error estimator (or equivalently: the total error) is linearly convergent with optimal algebraic rates, i.e.,

\[
\eta_{\ell+n} \leq C_{\text{lin}} q_{\text{lin}}^n \eta_\ell \quad \text{for all} \ \ell, n \in \mathbb{N}_0
\]

May 7, 2018

2
For NVB in 2D, each triangle $T \in \mathcal{T}$ has one reference edge, indicated by the double line (left). Bisection of $T$ is achieved by halving the reference edge. The reference edges of the sons are always opposite to the new node. Recursive application of this refinement rule leads to conforming triangulations. It needs three bisections per element to halve all edges of a triangle. Five bisection create an interior node and hence a discrete element bubble function within $T$.

and, for all possible algebraic rates $s > 0$,

$$\eta_\ell \leq C_{\text{opt}} \left( \# \mathcal{T}_\ell \right)^{-s}$$

with certain constants $C_{\text{lin}}, C_{\text{opt}} > 0$ and $0 < q_{\text{lin}} < 1$. Possible algebraic rates are, as usually, characterized in terms of certain approximation classes which are the same as those for residual error estimators. In explicit terms, the simple $(h - h/2)$-type error estimators thus yield the same optimal convergence behavior as the residual error estimators, even though these two types of estimators are not locally equivalent.

Outline. In Section 2, we collect the mathematical framework to formally state our main results. To this end, we formulate the precise assumptions on the conforming triangulations and the mesh-refinement (Section 2.1), define the employed FEM spaces (Section 2.2), introduce the considered $(h - h/2)$-type error estimators (Section 2.3) and the corresponding adaptive algorithm (Algorithm 3 as a precise specification of (2)), and formulate the main result (Theorem 4 which gives the formal statement of (9) as well as (10)–(11)). For the proof of Theorem 4, we rely on certain properties of the residual error estimator. These are collected and proved in Section 3 where we slightly improve the discrete reliability estimate from [Ste07, CKNS08] as well as the discrete efficiency estimate from [Dör96, MNS00]. The proof of Theorem 4 is given in Section 4. Finally, we underline the theoretical findings by some numerical experiments in Section 5.

General notation. Throughout, we write $a \lesssim b$ to abbreviate $a \leq Cb$ with some generic constant $C > 0$ which is clear from the context. Moreover, $a \simeq b$ abbreviates $a \lesssim b \lesssim a$. Mesh-related quantities have the same index, e.g., $u_\star$ is the FEM solution corresponding to the triangulation $\mathcal{T}_\star$, and $\mathcal{E}_\star$ is the set of facets of the triangulation $\mathcal{T}_\star$. Throughout, we make the following convention: If $\mathcal{T}_\star$ is a triangulation and $\alpha_\star(T, \cdot) \in \mathbb{R}$ is defined for all $T \in \mathcal{T}_\star$, then

$$\alpha_\star(\cdot) := \alpha_\star(\mathcal{T}_\star, \cdot), \quad \text{where} \quad \alpha_\star(\mathcal{U}_\star, \cdot)^2 := \sum_{T \in \mathcal{U}_\star} \alpha_\star(T, \cdot)^2 \quad \text{for all} \quad \mathcal{U}_\star \subseteq \mathcal{T}_\star.$$  

Finally, $\| \cdot \|_\omega^2 := \int_\omega (\cdot)^2 \, dx$ abbreviates the $L^2$-norm over a measurable set $\omega$ (with respect to either the $d$-dimensional Lebesgue measure or the $(d-1)$-dimensional surface measure).

2. Main result

2.1. Conforming triangulations and mesh-refinement. Throughout, $\mathcal{T}_\star$ denotes a conforming triangulation of $\Omega$ into non-degenerate compact simplices. In particular, we
Figure 2. For NVB in 3D, each tetrahedron $T \in \mathcal{T}_\bullet$ is assigned with a permutation $(z_{\pi(1)}, z_{\pi(2)}, z_{\pi(3)}, z_{\pi(4)})$ of its vertices \{$z_1, z_2, z_3, z_4$\} and a type $\tau \in \{0, 1, 2\}$. The numbers in the figure are the positions of the nodes in the corresponding tuple. Bisection of $T$ is achieved by halving the reference edge between $z_{\pi(1)}$ and $z_{\pi(4)}$ indicated by the bold line. The permutations as well as the types of its sons depend on the permutation and the type of $T$. Recursive application of this refinement rule leads to conforming triangulations.

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avoid hanging nodes. The triangulation is called $\gamma$-shape regular, if

$$\max_{T \in \mathcal{T}_\bullet} \frac{\text{diam}(T)}{h_T} \leq \gamma.$$  

(13)

Here, $\text{diam}(T)$ denotes the Euclidean diameter of $T$ and $h_T := |T|^{1/d}$ with $|T|$ being its $d$-dimensional volume. Note that $\gamma$-shape regularity implies that $h_T \leq \text{diam}(T) \leq \gamma h_T$.

For given $\mathcal{T}_\bullet$, let $\mathcal{N}_\bullet$ be the set of nodes and $\mathcal{E}_\bullet$ be the set of facets. For $E \in \mathcal{E}_\bullet$, we define $h_E := |E|^{1/(d-1)}$ with $|\cdot|$ being the $d$-dimensional surface measure. Note that $h_E \simeq \text{diam}(T)$ if $E \subset T \in \mathcal{T}_\bullet$, where the hidden constants depend only on $\gamma$. Finally, $\mathcal{E}_\Omega$ denotes the set of all interior facets, i.e., $E \in \mathcal{E}_\Omega \subset \mathcal{E}_\bullet$ satisfies that $E = T \cap T'$ for certain simplices $T, T' \in \mathcal{T}_\bullet$.

Throughout, we employ newest vertex bisection (NVB) to refine triangulations locally; see [Ste08, KPP13] for details on the refinement algorithm. Figure 1 and Figure 2 give an illustration for $d = 2$ and $d = 3$, respectively. For a conforming triangulation $\mathcal{T}_\bullet$ and $\mathcal{M}_\bullet \subseteq \mathcal{T}_\bullet$, let $\mathcal{T}_0 := \text{nvb}(\mathcal{T}_\bullet, \mathcal{M}_\bullet)$ be the coarsest conforming triangulation such that all marked elements $T \in \mathcal{M}_\bullet$ have been refined, i.e., $\mathcal{M}_\bullet \subseteq \mathcal{T}_0 \setminus \mathcal{T}_\bullet$. We write $\mathcal{T}_0 \in \text{nvb}(\mathcal{T}_\bullet)$, if there exists $n \in \mathbb{N}_0$, conforming triangulations $\mathcal{T}(0), \ldots, \mathcal{T}(n)$, and corresponding sets of marked elements $\mathcal{M}_j \subseteq \mathcal{T}_j$ such that

- $\mathcal{T}_0 = \mathcal{T}(0)$,
- $\mathcal{T}(j+1) = \text{nvb}(\mathcal{T}(j), \mathcal{M}(j))$ for all $j = 0, \ldots, n - 1$,
- $\mathcal{T}_0 = \mathcal{T}(n)$.

May 7, 2018
i.e., $\mathcal{T}_s$ is obtained from $\mathcal{T}_0$ by finitely many refinement steps.

The analysis of the $(h-h/2)$-type error estimators requires a stronger mesh-refinement. We suppose that we are given some initial conforming triangulation $\mathcal{T}_0$. For $\mathcal{T}_s \in \text{nvb}(\mathcal{T}_0)$, let $\mathcal{T}_s := \text{refine}(\mathcal{T}_s, \mathcal{M}_s)$ be an NVB refinement which satisfies:

(M1) There exists a uniform constant $C_{\text{son}} > 0$ such that $\#\{T' \in \mathcal{T}_0 : T' \subseteq T\} \leq C_{\text{son}}$ for all $T \in \mathcal{T}_s$, i.e., the number of sons per element is uniformly bounded.

(M2) If $\mathcal{T}_s \in \text{nvb}(\mathcal{T}_0)$, $\mathcal{M}_s \subseteq \mathcal{T}_s$, and $\mathcal{T}_b := \text{refine}(\mathcal{T}_b, \mathcal{M}_b)$, it holds that

$$\{T' \in \mathcal{T}_0 : T' \subseteq T\} = \{T' \in \mathcal{T}_b : T' \subseteq T\}$$

for all $T \in \mathcal{M}_s \cap \mathcal{M}_b$, i.e., refinement of a marked element is independent of its neighbors.

Further, we suppose that it satisfies one of the following constrains:

(M3) All facets of $T \in \mathcal{M}_s$ contain an interior node $z \in \mathcal{N}_0$.

(M3') All facets of $T \in \mathcal{M}_s$ as well as $T$ contain an interior node $z \in \mathcal{N}_0$.

As above, we let $\mathcal{T}_0 \in \text{refine}(\mathcal{T}_0)$ be the set of all possible refinements.

For $d = 2$, (M3) corresponds to refinement of marked elements $T \in \mathcal{M}_s$ by at least 3 bisections, while (M3') follows from at least 5 bisections; cf. Figure 1. Obviously, these refinements also satisfy (M1)–(M2); cf. Figure 1. For $d = 3$, each $T = \text{conv}\{z_1, \ldots, z_4\} \in \mathcal{T}_s$ is assigned with a permutation $(z_{\pi(1)}, z_{\pi(2)}, z_{\pi(3)}, z_{\pi(4)})$ of its nodes $\{z_1, z_2, z_3, z_4\}$ and a type $\tau \in \{0, 1, 2\}$; see Figure 2. To achieve (M3), one can bisect each marked element $T \in \mathcal{M}_s$ depending on its type as follows:

**$\tau = 0$:** First, bisect $T$ uniformly into 8 sons, then bisect all resulting sons which do not contain $z_{\pi(1)}$ nor $z_{\pi(4)}$, finally, bisect all resulting sons which either contain the two nodes $z_{\pi(2)}$ and $\frac{1}{2}(z_{\pi(1)} + z_{\pi(3)})$ or the two nodes $z_{\pi(3)}$ and $\frac{1}{2}(z_{\pi(2)} + z_{\pi(4)})$.

Altogether, $T$ is split into 18 sons with 14 nodes.

**$\tau = 1$:** First, bisect $T$ uniformly into 8 sons, then bisect all resulting sons which do not contain $z_{\pi(1)}$ nor $z_{\pi(4)}$, finally, bisect all resulting sons which contain $z_{\pi(2)}$.

Altogether, $T$ is split into 18 sons with 14 nodes.

**$\tau = 2$:** First, bisect $T$ uniformly into 8 sons, then bisect all resulting sons which do not contain $z_{\pi(1)}$ nor $z_{\pi(4)}$, finally, bisect all resulting sons which contain $z_{\pi(2)}$.

Altogether, $T$ is split into 20 sons with 14 nodes.

The resulting sons of $T$ are visualized in Figure 3. Note that the proposed strategy satisfies (M1)–(M2) with $C_{\text{son}} = 20$.

**Remark 1.** (i) We came up with this refinement by considering all possible configurations of the element $T$ in our MATLAB implementation of 3D NVB. Indeed, it is sufficient to consider only 4 node permutations instead of $4!$, since the others can be obtained by rotating the element. Hence, the number of all possible configurations is $4 \cdot 3 = 12$. This refinement leads to 5 two-dimensional NVBs of each facet of $T$ as in Figure 4. In particular, uniform refinement with $\mathcal{M}_s = \mathcal{T}_s$ leads to a conforming triangulation. For $\mathcal{M}_s \neq \mathcal{T}_s$, further bisections are required to obtain conformity. However, since uniform refinement automatically guarantees conformity, only non-marked elements have to be additionally bisected.

(ii) Using our MATLAB implementation of 3D NVB, we saw that it is not possible to satisfy (M3') strictly in the sense that refine($\cdot$) generates exactly one interior node per facet and exactly one interior node in each marked element $T \in \mathcal{M}_s$. Indeed, this is only
Figure 3. Starting with the same configuration for \( T \) as in Figure 2, the resulting sons for the 3D refinement guaranteeing (M3) are depicted. The outcome depends on the type \( \tau \) of \( T \). The type of the sons is indicated by their color, green for \( \tau = 0 \), blue for \( \tau = 1 \), and red for \( \tau = 2 \). Finally, two nodes are highlighted in magenta. The product of their hat functions is a discrete element bubble function within \( T \).

possible for \( T \) being of type \( \tau \in \{0, 2\} \), while type \( \tau = 1 \) enforces even three interior nodes on one facet, if interior nodes on each facet and inside of \( T \) are generated.

2.2. Finite element method. The Lax–Milgram theorem proves existence and uniqueness of \( u \in H^1_0(\Omega) \) with

\[
\int_{\Omega} A \nabla u \cdot \nabla v \,dx = \int_{\Omega} fv \,dx \quad \text{for all } v \in H^1_0(\Omega),
\]

which is the variational formulation of (1). Given a triangulation \( \mathcal{T}_\bullet \) and \( p \in \mathbb{N} \), define the space of \( \mathcal{T}_\bullet \)-piecewise polynomials

\[
P^p(\mathcal{T}_\bullet) := \{ v_\bullet \in L^2(\Omega) : \forall T \in \mathcal{T}_\bullet \quad v_\bullet|_T \text{ is a polynomial of degree } \leq p \}. \tag{15}
\]

Define \( S^p(\mathcal{T}_\bullet) := P^p(\mathcal{T}_\bullet) \cap H^1(\Omega) = P^p(\mathcal{T}_\bullet) \cap C(\Omega) \) as well as the \( H^1 \)-conforming FE space

\[
S^p_0(\mathcal{T}_\bullet) := P^p(\mathcal{T}_\bullet) \cap H^1_0(\Omega) = \{ v_\bullet \in S^p(\mathcal{T}_\bullet) : v_\bullet|_{\Gamma} = 0 \}. \tag{16}
\]

The Lax–Milgram theorem proves existence and uniqueness of \( u_\bullet \in S^p_0(\mathcal{T}_\bullet) \) such that

\[
\int_{\Omega} A \nabla u_\bullet \cdot \nabla v_\bullet \,dx = \int_{\Omega} fv_\bullet \,dx \quad \text{for all } v_\bullet \in S^p_0(\mathcal{T}_\bullet). \tag{17}
\]

Recall the Galerkin orthogonality

\[
\int_{\Omega} A \nabla (u - u_\bullet) \cdot \nabla v_\bullet \,dx = 0 \quad \text{for all } v_\bullet \in S^p_0(\mathcal{T}_\bullet), \tag{18}
\]

which results in the Pythagoras theorem

\[
\| A^{1/2} \nabla (u - u_\bullet) \|^2_{\Omega} + \| A^{1/2} \nabla (u_\bullet - v_\bullet) \|^2_{\Omega} = \| A^{1/2} \nabla (u - v_\bullet) \|^2_{\Omega} \quad \text{for all } v_\bullet \in S^p(\mathcal{T}_\bullet). \tag{19}
\]

2.3. Simple \((h - h/2)\)-type error estimators. Given a triangulation \( \mathcal{T}_\bullet \), let \( \widehat{\mathcal{T}}_\bullet := \text{refine}(\mathcal{T}_\bullet, \mathcal{T}_\bullet) \) be the uniform refinement. Recall the natural \( h - h/2 \) error estimator

\[
\widetilde{\mu}_\bullet = \| A^{1/2} \nabla (\widehat{u}_\bullet - u_\bullet) \|_{\Omega}. \tag{20}
\]

May 7, 2018 6
One drawback of \( \tilde{u}_* \) is that it requires to compute two FE solutions \( \hat{u}_* \in S_0^p(\hat{T}_*) \) and \( u_* \in S_0^p(T_*) \), even though the Pythagoras theorem (19) predicts that
\[
\| A \nabla (u - \hat{u}_*) \|^2_\Omega + \| \tilde{\mu}_* \|^2_\Omega = \| A^{1/2} \nabla (u - u_*) \|^2_\Omega,
\]
i.e., \( \hat{u}_* \) is more accurate than \( u_* \). One remedy is to replace \( u_* \) by some (cheap) postprocessing of \( \hat{u}_* \) as proposed in [FOP10]. Let \( I_* : C(\Omega) \to S^p(T_*) \) denote the nodal interpolation operator. Let \( \pi_* : L^2(\Omega) \to \mathcal{P}^{p-1}(T_*) \) be the \( L^2(\Omega) \)-orthogonal projection onto \( \mathcal{P}^{p-1}(T_*) \).

Recalling the convention (12), we define, for all \( T \in T_* \) and all \( \hat{u}_* \in S^p(\hat{T}_*) \),
\[
\mu_*(T, \hat{u}_*) := \| A^{1/2} \nabla (1 - I_*) \hat{u}_* \|_T \quad \text{and} \quad \lambda_*(T, \hat{u}_*) := \| (1 - \pi_*) A^{1/2} \nabla \hat{u}_* \|_T.
\]
Since \( A^{1/2} \) is \( T_* \)-piecewise constant and \( \pi_* \) acts elementwise and componentwise, we immediately see the alternative representation
\[
\lambda_*(T, \hat{u}_*) = \| A^{1/2} (1 - \pi_*) \nabla \hat{u}_* \|_T.
\]

The following lemma is proved in [FOP10] Prop. 3 for \( p = 1 \) and the Poisson model problem by use of scaling arguments, but also holds for general \( p \geq 1 \) and our model problem (1).

**Lemma 2 (simple \((h - h/2)\)-type error estimators).** There exists \( C_{hh2} \geq 1 \) such that there holds local equivalence
\[
\lambda_*(T, \hat{u}_*) \leq \mu_*(T, \hat{u}_*) \leq C_{hh2} \lambda_*(T, \hat{u}_*) \quad \text{for all } T \in T_* \text{ and all } \hat{u}_* \in S_0^p(T_*).
\]
Moreover, for \( \hat{u}_* = \hat{u}_* \), there holds global equivalence
\[
\lambda_*(\hat{u}_*) \leq \tilde{\mu}_* = \| A^{1/2} \nabla (\hat{u}_* - u_*) \|_\Omega \leq \| A^{1/2} \nabla (1 - I_*) \hat{u}_* \|_\Omega = \mu_*(\hat{u}_*) \leq C_{hh2} \lambda_*(\hat{u}_*)
\]
as well as efficiency
\[
C_{hh2}^{-1} \mu_*(\hat{u}_*) \leq \lambda_*(\hat{u}_*) \leq \tilde{\mu}_* \leq \| A^{1/2} \nabla (u - u_*) \|_\Omega.
\]
The constant \( C_{hh2} \) depends only on \( d, A, p, \) and \( \gamma \)-shape regularity of \( T_* \).

**Sketch of proof.** Note that \( A^{1/2} I_* \hat{u}_*, A^{1/2} \nabla u_* \in \mathcal{P}^{p-1}(T_*) \) and that \( \pi_* \) is also the \( T_* \)-elementwise best approximation onto \( \mathcal{P}^{p-1}(T_*) \). This proves the first estimate in (24) as well as the first estimate in (25). The second estimate in (25) follows from \( \mathcal{P}^p(T_*) \subseteq \mathcal{P}^{p-1}(\hat{T}_*) \) and the best approximation property of the Galerkin solution in the energy norm, since \( u_* \) is also a Galerkin approximation to \( \hat{u}_* \). Since (26) is a direct consequence of (24) and (25), it only remains to prove the second estimate in (24), which also implies the third estimate in (25).

Let \( T \in T_* \). Note that \( \mu_*(T, \cdot) \) and \( \lambda_*(T, \cdot) \) are seminorms on \( \mathcal{P}^{p-1}(\hat{T}_*) \). Recall that seminorms on finite-dimensional spaces are equivalent if the kernels coincide. For \( \hat{u}_* \in \mathcal{P}^{p-1}(\hat{T}_*) \), it holds that \( \mu_*(T, \hat{u}_*) = 0 \) and \( \lambda_*(T, \hat{u}_*) = 0 \), if and only if \( \hat{u}_* |_T \in \mathcal{P}^{p-1}(\{ T \}) \).

Hence, we derive the equivalence (24). A scaling argument proves that the constant \( C_{hh2} \) depends only on \( d, A, p, \) and \( \gamma \)-shape regularity of \( T_* \), while \( C_{hh2} \geq 1 \) is obvious. \( \Box \)
Let $\Pi_\bullet : L^2(\Omega) \to P_\bullet^{\max(0,p-2)}(\mathcal{T}_\bullet)$ be the $L^2(\Omega)$-orthogonal projection onto $P_\bullet^{\max(0,p-2)}(\mathcal{T}_\bullet)$. With the convention \[(M3')\]
and the convention (12), let
\[
\eta_{\bullet}(T,\hat{v}_\bullet)^{2} := h_T^2 \sum_{T' \in \mathcal{T}_\bullet} \|f + \text{div}(A\nabla_{\bullet} \hat{v}_\bullet)\|^2_{T'}, \quad \text{and} \quad \text{osc}_{\bullet}(T)^2 := h_T^2 \| (1 - \Pi_\bullet) f \|_{T}. \tag{28}
\]

Further, we abbreviate $\text{osc}_{\bullet}^{2} := \sum_{T \in \mathcal{T}_\bullet} \text{osc}_{\bullet}(T)^2$. Note that $\text{res}_{\bullet}(T, v_\bullet)^{2} = h_T^2 \|f + \text{div}(A\nabla_{\bullet} v_\bullet)\|^2_{T}$ for all $v_\bullet \in S_0(\mathcal{T}_\bullet)$. Then, we consider the following *a posteriori* error estimators

| $\eta_{\bullet}(T,\hat{v}_\bullet)^{2}$ | requirements | refinement |
|--------------------------------|--------------|------------|
| $\lambda_{\bullet}(T,\hat{v}_\bullet)^{2} + \text{res}_{\bullet}(T,\hat{v}_\bullet)^{2}$ | $p \geq 1$ | $d \geq 2$ | (M3) |
| $\lambda_{\bullet}(T,\hat{v}_\bullet)^{2} + \text{osc}_{\bullet}(T)^{2}$ | $p \geq 1$ | $d \geq 2$ | (M3') |
| $\lambda_{\bullet}(T,\hat{v}_\bullet)^{2} + \text{osc}_{\bullet}(T)^{2}$ | $p \geq 2$ | $d \in \{2,3\}$ | (M3) |
| $\mu_{\bullet}(T,\hat{v}_\bullet)^{2} + \text{res}_{\bullet}(T,\hat{v}_\bullet)^{2}$ | $p \geq 1$ | $d \geq 2$ | (M3) |
| $\mu_{\bullet}(T,\hat{v}_\bullet)^{2} + \text{osc}_{\bullet}(T)^{2}$ | $p \geq 1$ | $d \geq 2$ | (M3') |
| $\mu_{\bullet}(T,\hat{v}_\bullet)^{2} + \text{osc}_{\bullet}(T)^{2}$ | $p \geq 2$ | $d \in \{2,3\}$ | (M3) |

### 2.4. Adaptive algorithm.
We analyze the following adaptive strategy which is driven by one of the error estimators $\eta_{\bullet}$ from (28).

**Algorithm 3. Input:** Conforming triangulation $\mathcal{T}_0$ of $\Omega$, adaptivity parameter $0 < \theta \leq 1$.

**Loop:** For all $\ell = 0, 1, 2, \ldots$, iterate the following steps (i)–(iv):

(i) Compute the discrete solution $\hat{u}_{\ell} \in S_0(\mathcal{T}_{\ell})$, where $\mathcal{T}_{\ell} := \text{refine} (\mathcal{T}_{\ell-1})$.

(ii) Compute the indicators $\eta_{\ell}(T,\hat{u}_{\ell})$ for all $T \in \mathcal{T}_{\ell}$.

(iii) Determine some $\mathcal{M}_{\ell} \subseteq \mathcal{T}_{\ell}$ with minimal cardinality such that $\theta \eta_{\ell}(\hat{u}_{\ell})^2 \leq \eta_{\ell}(\mathcal{M}_{\ell},\hat{u}_{\ell})^2$.

(iv) Generate $\mathcal{T}_{\ell+1} := \text{refine} (\mathcal{T}_{\ell},\mathcal{M}_{\ell})$.

**Output:** Sequences of successively refined triangulations $\mathcal{T}_{\ell}$, discrete solutions $\hat{u}_{\ell}$, and corresponding error estimators $\eta_{\ell}(\hat{u}_{\ell})$, for all $\ell \geq 0$.

### 2.5. Main result.
Given the initial triangulation $\mathcal{T}_0$, we define the following two approximation classes for $s > 0$: With the error estimator $\eta_{\bullet}$ from (28) used for Algorithm 3 and the convention (12), let
\[
\|u\|_{A_{\bullet}^{s}} := \sup_{N \in \mathbb{N}_0} \min_{\mathcal{T}_N \in \text{wh}(\mathcal{T}_0)} (N + 1)^s \eta_{\bullet}(\hat{u}_s) \in [0, \infty]. \tag{29}
\]
Moreover, let
\[
\|u\|_{A_{\bullet}^{s}} := \sup_{N \in \mathbb{N}_0} \min_{\mathcal{T}_N \in \text{wh}(\mathcal{T}_0)} (N + 1)^s \left( \min_{v_\bullet \in S_0(\mathcal{T}_s)} \| A^{1/2} \nabla (u - v_\bullet) \|_{\Omega} + \text{osc}_{\bullet} \right) \in [0, \infty]. \tag{30}
\]

Note that the definition of $\|u\|_{A_{\bullet}^{s}}$ is independent of the error estimator $\eta_{\bullet}$.

By definition, $\|u\|_{A_{\bullet}^{s}} < \infty$ and $\|u\|_{A_{\bullet}^{s}} < \infty$ imply that the quantity $\eta_{\bullet}(\hat{u}_s)$ and the *total error* on the optimal meshes $\mathcal{T}_\star$ decay at least with rate $O((\#\mathcal{T}_\star)^{-s})$. The following main theorem states that each possible rate $s > 0$ is in fact realized by Algorithm 3. The proof requires some technical preparations and is thus postponed to Section 4.
Theorem 4. Let $\eta_\bullet$ be one of the error estimators from \([28]\). Then, the error estimator $\eta_\bullet(\hat{u}_\bullet)$ is reliable and efficient, i.e., there exist constants $C_{\text{eff}}, C_{\text{rel}} > 0$ such that

\[
C_{\text{eff}}^{-1} \eta_\bullet(\hat{u}_\bullet) \leq \left( \min_{v \in S(T_\bullet)} \| A^{1/2} \nabla (u - v) \|_2^2 + \text{osc}_2^2 \right)^{1/2} \leq C_{\text{rel}} \eta_\bullet(\hat{u}_\bullet) \quad \text{for } T_\bullet \in \text{nvb}(T_0).
\]  

In particular, this implies that

\[
C_{\text{rel}}^{-1} \| u \|_{A^{\lambda_{\text{opt}}}} \leq \| u \|_{A^\gamma} \leq C_{\text{eff}} \| u \|_{A^{\text{opt}}}.
\]

For arbitrary $0 < \theta \leq 1$, the error estimator sequence generated by Algorithm 3 converges linearly, i.e., there exist constants $0 < q_{\text{lin}} < 1$ and $C_{\text{lin}} \geq 1$ such that

\[
\eta_{\ell+n}(\hat{u}_{\ell+j}) \leq C_{\text{lin}} q_{\text{lin}}^j \eta_\ell(\hat{u}_\ell) \quad \text{for all } \ell, n \in \mathbb{N}_0.
\]

Moreover, there exists a constant $0 < \theta_{\text{opt}} < 1$ such that for all $0 < \theta < \theta_{\text{opt}}$, the estimator $\eta_\bullet(\hat{u}_\bullet)$ converges at optimal algebraic rate, i.e., for all $s > 0$ there exist constants $c_{\text{opt}}, C_{\text{opt}} > 0$ such that

\[
c_{\text{opt}} \| u \|_{A^\gamma} \leq \sup_{\ell \in \mathbb{N}_0} (\#T_\ell - \#T_{\ell} + 1)^s \eta_\ell(\hat{u}_\ell) \leq C_{\text{opt}} \| u \|_{A^{\lambda_{\text{opt}}}}.
\]

All involved constants $C_{\text{rel}}, C_{\text{eff}}, C_{\text{lin}}, q_{\text{lin}}$, and $\theta_{\text{opt}}$ depend only on $d, A, p, C_{\text{son}},$ and $\gamma$-shape regularity of $T_0$, whereas $C_{\text{lin}}$ and $q_{\text{lin}}$ depend additionally on $\theta$, and $C_{\text{opt}}$ depends furthermore on $s$. The constant $c_{\text{opt}}$ depends only on $C_{\text{son}}, \#T_0$, and $s$.

Remark 5. (i) Recall that $\lambda_\bullet(\hat{u}_\bullet) \leq \| A^{1/2} \nabla (u - u_\bullet) \|_\Omega$ according to \([26]\). For $\eta_\bullet^2 = \lambda_\bullet(\hat{u}_\bullet)^2 + \text{osc}_2^2$, this yields that $C_{\text{eff}} = 1$ in \([31]\), i.e., the estimator $\eta_\bullet$ is a guaranteed lower bound for the unknown total error with constant $1$.

(ii) In general, one expects an optimal convergence rate of $s = p/d$ for the error. Asymptotically, this leads to $\| A^{1/2} \nabla (u - u_\bullet) \|_\Omega = C(\#T_\bullet)^{-p/d}$ for some constant $C > 0$. If uniform refinement $\text{refine}(T_\ell, T_\ell)$ bisects all elements into exactly $C_{\text{son}}$ elements, this suggests that $\| A^{1/2} \nabla (u - \hat{u}_\ell) \|_\Omega = C_{\text{son}}(\#T_\ell)^{-p/d}$. In particular, one obtains that $q_{\text{sat}} = C_{\text{son}}^{-p/d}$ in \([6]\). Together with \([31]\) and \([28]\), this yields the asymptotical upper bound

\[
\| A^{1/2} \nabla (u - u_\ell) \|_\Omega \leq (1 - C_{\text{son}}^{-2p/d})^{-1/2} \eta_\ell.
\]

For $\eta_\bullet^2 = \mu_\bullet(\hat{u}_\bullet)^2 + \text{osc}_\bullet^2$, the estimator $\eta_\bullet$ is hence an upper bound for the unknown total error in \([31]\) with known asymptotical reliability constant $C_{\text{rel}} = (1 - C_{\text{son}}^{-2p/d})^{-1/2}$.

(iii) Note that the approximation norm $\| u \|_{A^{\lambda_{\text{opt}}}}$ is the same as for residual error estimators; cf. \([\text{CKNS08}, \text{KS11}, \text{FFP14}, \text{CFPP14}]\). In explicit terms, the $(h - h/2)$-type estimators from \([28]\) thus lead to the same algebraic convergence rates as the residual error estimators.

(iv) Alternatively, one could define the approximation classes $\| u \|_{A^\gamma}$ and $\| u \|_{A^{\text{opt}}}$ with $\text{refine}(\cdot)$ instead of $\text{nvb}(\cdot)$, i.e.,

\[
\| u \|_{A^\gamma} := \sup_{N \in \mathbb{N}_0} \min_{T_\bullet \in \text{refine}(T_0)} \left( N + 1 \right)^s \eta_\bullet(\hat{u}_\bullet),
\]

\[
\| u \|_{A^{\text{opt}}} := \sup_{N \in \mathbb{N}_0} \min_{T_\bullet \in \text{refine}(T_0)} \left( N + 1 \right)^s \left( \min_{v \in S(T_\bullet)} \| A^{1/2} \nabla (u - v) \|_\Omega + \text{osc}_\bullet \right).
\]
Clearly, (31) gives that \( \|u\|_{\tilde{\mathcal{M}}_u} \approx \|u\|_{\tilde{\mathcal{M}}_{\text{opt}}}. \) Moreover, \( \|u\|_{\tilde{\mathcal{M}}_u} \leq \|u\|_{\tilde{\mathcal{M}}_u} \) follows from \( \text{refine}(\mathcal{T}_0) \subseteq \text{nvb}(\mathcal{T}_0). \) Arguing as in [CFP14] Prop. 4.15, we prove that

\[
\tilde{c}_{\text{opt}} \|u\|_{\tilde{\mathcal{M}}_u} \leq \sup_{\ell \in \mathcal{T}_0} (\#\mathcal{T}_\ell - \#\mathcal{T}_0 + 1)^s \eta_\ell(\hat{u}_\ell),
\]

where \( \tilde{c}_{\text{opt}} \) depends only on \( C_{\text{son}}, \#\mathcal{T}_0, \) and \( s. \) Together with (31), we conclude

\[
\frac{\tilde{c}_{\text{opt}}}{C_{\text{opt}}} \|u\|_{\tilde{\mathcal{M}}_u} \leq \|u\|_{\tilde{\mathcal{M}}_u} \leq \|u\|_{\tilde{\mathcal{M}}_u}
\]

### 3. Residual error estimator

As an auxiliary tool, we consider the residual error estimator. Because of its later application, we use the notation \( \mathcal{T}_\Delta \) and \( \mathcal{T}_0 \in \text{nvb}(\mathcal{T}_\Delta) \) for a given triangulation \( \mathcal{T}_\Delta \in \text{nvb}(\mathcal{T}_0) \) and a corresponding refinement. Recall the definition of \( \text{res}_\Delta(\cdot) \) and \( \text{osc}_\Delta(\cdot) \) from (27). We define, for all \( v_\Delta \in \mathcal{S}_0^0(\mathcal{T}_\Delta), \)

\[
\varrho_\Delta(\tau, v_\Delta)^2 := \begin{cases} \text{res}_\Delta(T, v_\Delta)^2 & \text{for } \tau = T \in \mathcal{T}_\Delta, \\ h_E \|[A \nabla v_\Delta \cdot n]|_E^2 & \text{for } \tau = E \in \mathcal{E}_\Omega. \end{cases}
\]

Generalizing the convention (12), we define, for all \( v_\Delta \in \mathcal{S}_0^0(\mathcal{T}_\Delta), \)

\[
\varrho_\Delta(v_\Delta) := \varrho_\Delta(\mathcal{T}_\Delta \cup \mathcal{E}_\Omega, v_\Delta), \quad \text{where } \varrho_\Delta(U_\Delta, v_\Delta)^2 := \sum_{\tau \in U_\Delta} \varrho_\Delta(\tau, v_\Delta)^2 \text{ for all } U_\Delta \subseteq \mathcal{T}_\Delta \cup \mathcal{E}_\Omega.
\]

It is well-known [AO00, Ver13] that \( \varrho_\Delta(\cdot) \) is reliable and efficient in the sense that

\[
C_{\text{rel}}^{-1} \|A^{1/2}\nabla(u - u_\Delta)\|_\Omega + \text{osc}_\Delta \leq \varrho_\Delta(u_\Delta) \leq C_{\text{eff}} \left( \|A^{1/2}\nabla(u - u_\Delta)\|_\Omega + \text{osc}_\Delta \right),
\]

where \( C_{\text{rel}}, C_{\text{eff}} > 0 \) depend only on \( d, A, p, \) and \( \gamma \)-shape regularity of \( \mathcal{T}_\Delta. \)

The next lemma recalls the discrete reliability estimate which originally goes back to [Ste07]. While the proof of [Ste07] relied on the refined elements \( \mathcal{T}_\Delta \setminus \mathcal{T}_\Delta \) plus one additional layer of elements for the localized upper bound, the proof of [CKNS08] involves only the refined elements \( \mathcal{T}_\Delta \setminus \mathcal{T}_\Delta. \) Even though [Ste07, CKNS08] consider an element-based formulation of the residual error estimator, their ideas of the proof also yield the following slightly stronger estimate for our variant of \( \varrho_\Delta(\cdot) \) (which is indexed by elements and interior facets). While [CKNS08] Lemma 3.6] would also involve non-refined facets of refined elements on the right-hand side of (12), we only require refined facets.

**Lemma 6 (discrete reliability of residual error estimator).** It holds that

\[
\|A^{1/2}\nabla(u_\Delta - u_\Delta)\|_\Omega \leq C_{\text{del}} \varrho_\Delta((\mathcal{T}_\Delta \setminus \mathcal{T}_\Delta) \cup (\mathcal{E}_\Omega \setminus \mathcal{E}_\Delta), u_\Delta).
\]

The constant \( C_{\text{del}} > 0 \) depends only on \( d, A, \) and \( \gamma \)-shape regularity of \( \mathcal{T}_\Delta. \)

**Sketch of proof.** Recall the (discrete) Galerkin orthogonality

\[
\int_\Omega A \nabla(u_\Delta - u_\Delta) \cdot \nabla v_\Delta \, dx = 0 \quad \text{for all } v_\Delta \in \mathcal{S}_0^0(\mathcal{T}_\Delta).
\]

May 7, 2018
For arbitrary \( v_{\Delta} \in S_{0}^{p}(T_{\Delta}) \), define \( w_{\Delta} := (u_{\Delta} - u_{\Delta}) - v_{\Delta} \). The discrete formulation (17) for \( u_{\Delta} \) proves that

\[
\|A^{1/2}\nabla (u_{\Delta} - u_{\Delta})\|_{\Omega}^{2} = \int_{\Omega} A \nabla (u_{\Delta} - u_{\Delta}) \cdot \nabla w_{\Delta} \, dx \tag{17} \equiv \int_{\Omega} f w_{\Delta} \, dx - \sum_{T \in T_{\Delta}} \int_{T} A \nabla u_{\Delta} \cdot \nabla w_{\Delta} \, dx.
\]

For \( T \in T_{\Delta} \), integration by parts and \( w_{\Delta} \in S_{0}^{p}(T_{\Delta}) \) yield that

\[- \int_{T} A \nabla u_{\Delta} \cdot \nabla w_{\Delta} \, dx = \int_{T} \text{div}(A \nabla u_{\Delta}) w_{\Delta} \, dx - \int_{\partial T \setminus \Gamma} A \nabla u_{\Delta} \cdot n \, w_{\Delta} \, ds.\]

Combining these identities, we see that

\[
(43) \quad \|A^{1/2}\nabla (u_{\Delta} - u_{\Delta})\|_{\Omega}^{2} = \sum_{T \in T_{\Delta}} \int_{T} (f + \text{div}(A \nabla u_{\Delta})) w_{\Delta} \, dx - \sum_{E \in \{E \in T_{\Delta} \}} \int_{E} [A \nabla u_{\Delta} \cdot n] w_{\Delta} \, ds.
\]

To proceed, we will choose \( v_{\Delta} = J_{\Delta}(u_{\Delta} - u_{\Delta}) \), where \( J_{\Delta} : H^{1}(\Omega) \to S^{p}(T_{\Delta}) \) is a Scott-Zhang projector \([SZ90]\). For the convenience of the reader, we recall the construction of \( J_{\Delta} \): Let \( L_{\Delta} \subseteq \Omega \) be the set of Lagrange nodes of \( S^{p}(T_{\Delta}) \). Let \( \{ \phi_{z} : \Delta E \setminus \Omega \to \mathbb{R} \} \) be the corresponding nodal basis of \( S^{p}(T_{\Delta}) \), i.e., with Kronecker’s delta, it holds that \( \phi_{z}(\Delta) = \delta_{z\Delta} \) for all \( z \in L_{\Delta} \). If \( z \in L_{\Delta} \) is on the skeleton \( \bigcup_{E \in \Delta E} E \), choose a facet \( \tau_{z} := E \in \Delta E \) with \( z \in \tau_{z} \) subject to the following constraints (which further specify the constraints from \([SZ90]\)):

- If \( z \in \Gamma \), then choose \( \tau_{z} = E \subset \Gamma \).
- If \( z \in E \in \Delta E \cap \Delta \Omega \), then choose \( \tau_{z} = E \) (which is not necessarily unique).
- Otherwise, choose an arbitrary \( \tau_{z} = E \in \Delta E \setminus \Delta \Omega \) with \( z \in E \).

If \( z \) is not on the skeleton, then there exists a unique element \( T \in T_{\Delta} \) such that \( z \) lies in the interior of \( \tau_{z} := T \). Consider the nodal basis \( \{ \phi_{z} \} \) restricted to \( \Delta E \) and let \( \{ \psi_{z} \} \subset \Delta \Omega \) be the corresponding dual basis, i.e., \( \int_{\tau_{z}} \psi_{z} \phi_{z} \, dx = \delta_{z} \) for all \( z, z' \in E \cap \Delta E \). Then, the Scott-Zhang projector is defined by

\[
J_{\Delta} v := \sum_{z \in L_{\Delta}} \left( \int_{\tau_{z}} v \psi_{z} \, dx \right) \phi_{z}.
\]

According to \([SZ90]\), \( J_{\Delta} \) has the following properties for all \( w \in H^{1}(\Omega) \), all \( v_{\Delta} \in \Delta S^{p}(T_{\Delta}) \), and all \( T \in T_{\Delta} \), where \( \omega_{\Delta}(T) := \bigcup \{ T' \in T_{\Delta} : T \cap T' \neq \emptyset \} \) denotes the element patch:

- **projection property:** \( w = w_{\Delta} \) on \( \omega_{\Delta}(T) \) implies that \( J_{\Delta} w = w_{\Delta} \) on \( T \);

- **preservation of discrete traces:** \( w = w_{\Delta} \) on \( \Gamma \) implies that \( J_{\Delta} w = w_{\Delta} \) on \( \Gamma \);

- **L2 approximation property:** \( \|w - J_{\Delta} w\|_{T} \lesssim h_{T} \|\nabla w\|_{\omega_{\Delta}(T)} \);

- **H1 stability:** \( \|\nabla (w - J_{\Delta} w)\|_{T} \lesssim \|\nabla w\|_{\omega_{\Delta}(T)} \).

In addition, our choice of \( \tau_{z} \) yields further structure: Let \( v_{\Delta} \in \Delta S_{0}^{p}(T_{\Delta}) \) and \( z \in L_{\Delta} \).

- If \( z \in \Gamma \), it holds that \( v_{\Delta}|_{\tau_{z}} = 0 \) and hence \((J_{\Delta}v_{\Delta})(z) = 0 = v_{\Delta}(z)) \).

- If \( z \) is in the interior of \( T \), then \( v_{\Delta}|_{\tau_{z}} = J_{\Delta}v_{\Delta}(z) \) and hence \( (J_{\Delta}v_{\Delta})(z) = v_{\Delta}(z) \) follows from the previous step.

- If \( z \in \tau_{z} = T \in T_{\Delta} \), then \( v_{\Delta}|_{\tau_{z}} = J_{\Delta}v_{\Delta}(z) \) and hence \((J_{\Delta}v_{\Delta})(z) = v_{\Delta}(z) \) by choice of the dual basis.
Overall, this proves that \( v_\Delta - J_\Delta v_\Delta = 0 \) on all \( T \in \mathcal{T}_\Delta \cap \mathcal{T}_\Delta \) as well as on all \( E \in \mathcal{E}_\Delta \cap \mathcal{E}_\Delta \). For \( v_\Delta := u_\Delta - u_\Delta \) and \( w_\Delta := v_\Delta - J_\Delta v_\Delta \), we plug this into (43) and observe that
\[
\| A^{1/2} \nabla (u_\Delta - u_\Delta) \|_T^2 = \sum_{T \in \mathcal{T}_\Delta \setminus \mathcal{T}_\Delta} (f + \text{div}(A \nabla u_\Delta))w_\Delta \, dx - \sum_{E \in \mathcal{E}_\Delta \setminus \mathcal{E}_\Delta} \int_E [A \nabla u_\Delta \cdot n] w_\Delta \, ds.
\]
With the usual arguments (see, e.g., [AO00, Ver13]), this leads to (42). \( \square \)

Next, we recall that the error estimator \( \varphi_\Delta(\cdot) \) depends (locally) Lipschitz continuously on the discrete functions. The following result is obtained analogously to [CKNS08, Prop. 3.3], where the proof relies only on the trace inequality plus inverse estimates.

**Lemma 7 (local stability of residual error estimator).** Let \( v_\Delta, w_\Delta \in S_0^p(\mathcal{T}_\Delta) \). Let \( T, T' \in \mathcal{T}_\Delta \) and \( E := T \cap T' \in \mathcal{E}_\Delta \). Then, it holds that
\[
\begin{align*}
\varphi_\Delta(T, v_\Delta) &\leq \varphi_\Delta(T, w_\Delta) + C_{\text{stab}} \| A^{1/2} \nabla (v_\Delta - w_\Delta) \|_T, \\
\varphi_\Delta(E, v_\Delta) &\leq \varphi_\Delta(E, w_\Delta) + C_{\text{stab}} \| A^{1/2} \nabla (v_\Delta - w_\Delta) \|_{T \cup T'}.
\end{align*}
\]

The constant \( C_{\text{stab}} > 0 \) depends only on \( d, A, p, \) and \( \gamma \)-shape regularity of \( \mathcal{T}_\Delta \). \( \square \)

**Remark 8.** We note that (44a) is also satisfied if \( v_\Delta, w_\Delta \in S_0^p(\hat{\mathcal{T}}_\Delta) \). In this case the constant \( C_{\text{stab}} > 0 \) depends additionally on \( C_{\text{son}} \).

The following lemma is proved along the lines of [FOP10, Prop. 2] and adapts the classical efficiency proof by using cleverly chosen bubble functions. We note that the idea goes back to the seminal works [Dör96, MNS00].

**Lemma 9 (local discrete efficiency of residual error estimator).** Let \( T, T', T'' \in \mathcal{T}_\Delta \setminus \mathcal{T}_\Delta \) and \( E = T' \cap T'' \in \mathcal{E}_\Delta \). Let \( p \geq 1 \). If \( E \) contains an interior node \( z \in \mathcal{N}_\Delta \), then it holds that
\[
\begin{align*}
C_{\text{eff}}^{-1} \varphi_\Delta(E, v_\Delta) &\leq \| A^{1/2} \nabla (u_\Delta - v_\Delta) \|_{T \cup T''} + \varphi_\Delta(\{T', T''\}, v_\Delta).
\end{align*}
\]

If one of the following cases is satisfied
\begin{itemize}
  \item \( d = 2 \) and \( p \geq 2 \),
  \item \( d = 3, p \geq 2, \) and each facet of \( T \) contains an interior node \( z \in \mathcal{N}_\Delta \),
  \item \( T \) contains an interior node \( z \in \mathcal{N}_\Delta \),
\end{itemize}
then it holds that
\[
\begin{align*}
C_{\text{eff}}^{-1} \varphi_\Delta(T, v_\Delta) &\leq \| A^{1/2} \nabla (u_\Delta - v_\Delta) \|_T + \text{osc}_\Delta(T).
\end{align*}
\]

The constant \( C_{\text{eff}} > 0 \) depends only on \( d, A, p, \) and \( \gamma \)-shape regularity of \( \mathcal{T}_\Delta \) and \( \mathcal{T}_\Delta \).

**Proof of (45b).** Since NVB is a binary refinement rule, there exists a coarsest refinement \( T_\Delta \in \text{nvb}(\mathcal{T}_\Delta) \) such that \( E \) contains an interior node \( z \in \mathcal{N}_\Delta \). Choose the corresponding hat function \( \phi_{*,z} \in S_0^1(\mathcal{T}_\Delta) \) as discrete facet bubble function
\[
\beta_E := \phi_{*,z} \in S_0^1(\mathcal{T}_\Delta) \subseteq S_0^1(\mathcal{T}_\Delta).
\]
In particular, $\beta_E \in H^1_0(T' \cup T'')$ and $|\text{supp}(\beta_E)| \simeq |T' \cup T''|$. Since $u_{\bullet} \in \mathcal{P}^p(T_{\bullet})$, a scaling argument shows the existence of some $r_{\bullet} \in \mathcal{P}^{p-1}(T' \cup T'')$ such that

$$r_{\bullet}|_E = [A\nabla u_{\bullet} \cdot n]|_E \quad \text{and} \quad ||r_{\bullet}|_{T' \cup T''} \lesssim h_E^{1/2} \|[A\nabla u_{\bullet} \cdot n]|_E. $$

Choose $v := r_{\bullet} \beta_E \in \mathcal{P}^p(T_{\bullet})$ and note that $v \in H^1_0(T' \cup T'')$. Let $\text{div}_{\bullet}$ denote the $T_{\bullet}$-piecewise divergence operator. A scaling argument and integration by parts prove that

$$||[A\nabla u_{\bullet} \cdot n]|^2_E \lesssim ||[A\nabla u_{\bullet} \cdot n]|_{T' \cup T''}^2 \lesssim \int_{T' \cup T''} [A\nabla u_{\bullet} \cdot n] v \, dx$$

Choose $v := r_{\bullet} \beta_E \in \mathcal{P}^p(T_{\bullet})$ and note that $v \in H^1_0(T' \cup T'')$. Let $\text{div}_{\bullet}$ denote the $T_{\bullet}$-piecewise divergence operator. A scaling argument and integration by parts prove that

$$||[A\nabla u_{\bullet} \cdot n]|_E \lesssim ||[A\nabla u_{\bullet} \cdot n]|_{T' \cup T''} \lesssim \int_{T' \cup T''} [A\nabla u_{\bullet} \cdot n] v \, dx$$

(47) \[ \|A\nabla u_{\bullet} \cdot n\|_E^2 \lesssim \|A\nabla u_{\bullet} \cdot n\|_{T' \cup T''} \lesssim \int_{T' \cup T''} [A\nabla u_{\bullet} \cdot n] v \, dx \]

This leads to

$$q_{\bullet}(E, u_{\bullet}) = h_E^{1/2} \|[A\nabla u_{\bullet} \cdot n]|_E \lesssim \|A^{1/2} \nabla (u_{\bullet} - u_{\star})\|_{T' \cup T''} + h_E \|f + \text{div}_{\bullet}(A\nabla u_{\bullet})\|_{T' \cup T''}$$

and concludes the proof. $\square$

**Proof of (45).** The proof is split into three steps.

**Step 1.** First, we consider $d = 2$ and $p \geq 2$. Since NVB is a binary refinement rule, there exists a coarsest refinement $T_{\star} \in \text{nvb}(T_{\bullet})$, where $T$ is only bisected once into triangles $T_1, T_2 \in T_{\star}$, i.e., there exists $E = \text{conv}\{z_1, z_2\} \in E_{\star}$ which bisects the interior of $T$, such that $z_1 \in N_{\star} \subset N_{\bullet}$ and $z_2 \in N_{\star}\backslash N_{\bullet} \subset N_{\star}\backslash N_{\bullet}$. With the corresponding hat functions $\phi_{\star,j} \in \mathcal{P}^1(T_1 \cup T_2)$, define the discrete bubble function

$$\beta_{T} = \sum_{j=1}^{2} \phi_{\star,j} \in \mathcal{P}^2(T_1 \cup T_2) \cup H^1_0(T) \subset S^2_0(T_{\bullet}) \quad \text{with \text{supp}(\beta_{T}) = T};$$

we note that $\beta_{T}$ is, in fact, the “classical” edge bubble for the new edge $E$. Recall that $\Pi_{\bullet}$ is the $L^2(\Omega)$-orthogonal projection onto $\mathcal{P}^{\max(0,p-2)}(T_{\bullet})$. Let $q_{\bullet} := \Pi_{\bullet}(f + \text{div}_{\bullet}(A\nabla u_{\bullet})) = \Pi_{\bullet} f + \text{div}_{\bullet}(A\nabla u_{\bullet}) \in \mathcal{P}^{\max(0,p-2)}(T_{\bullet})$, where $\text{div}_{\bullet}$ is the $T_{\bullet}$-piecewise divergence and hence $\text{div}_{\bullet}(A\nabla u_{\bullet}) \in \mathcal{P}^{\max(0,p-2)}(T_{\bullet})$. Define $v := q_{\bullet} \beta_{T} \in H^1_0(T) \cap S^2_0(T_{\bullet})$. A scaling argument proves that

$$\|q_{\bullet}\|_{T}^2 \simeq \|q_{\bullet}^{1/2}\|_{T}^2 = \int_{T} (q_{\bullet} - (f + \text{div}(A\nabla u_{\bullet}))) v \, dx + \int_{T} (f + \text{div}(A\nabla u_{\bullet})) v \, dx.$$
The first integral is estimated by
\[
\int_T (q_\bullet - (f + \text{div}(A\nabla u_\bullet))) \, v \, dx \leq \|q_\bullet - (f + \text{div}(A\nabla u_\bullet))\|_T \|v\|_T = \|(1 - \Pi_\bullet)f\|_T \|v\|_T.
\]
For the second integral, integration by parts and \(\frac{1}{2}(z_{\tau(1)} + z_{\tau(4)})\) and \(\frac{1}{2}(z_{\tau(2)} + z_{\tau(3)})\) of Figure 3 provide a discrete element bubble function.

Since \(\Pi = \text{nvb}(T)\), there exists a coarsest refinement such that \(T\) is refined as depicted in Figure 3. Consider the product of hat functions for the highlighted nodes \(\frac{1}{2}(z_{\tau(1)} + z_{\tau(4)})\) and \(\frac{1}{2}(z_{\tau(2)} + z_{\tau(3)})\) of Figure 3. This provides a discrete element bubble function \(\beta_T \in \mathcal{P}^2(T)\) and \(\text{nvb}(T)\).

Arguing as in Step 1, we conclude (45b).

Since \(\text{nvb}(T)\) is a binary refinement rule, there exists a coarsest refinement such that \(\Phi^\bullet\) is refined as depicted in Figure 3. Then, we suppose that each facet of \(T\) contains an interior node and may thus serve as an element bubble function. Arguing along the lines of Step 1, we conclude (45b).

**Step 2.** For \(d = 3\) and \(p \geq 2\), we suppose that each facet of \(T\) contains an interior node. Since \(\text{nvb}(T)\) is a binary refinement rule, there exists a coarsest refinement \(\Phi^\bullet \in \text{nvb}(T)\) with this property. Then, \(T\) is refined as depicted in Figure 3. Consider the product of hat functions for the highlighted nodes \(\frac{1}{2}(z_{T(1)} + z_{T(4)})\) and \(\frac{1}{2}(z_{T(2)} + z_{T(3)})\) of Figure 3. This provides a discrete element bubble function \(\beta_T \in \mathcal{P}^2(T)\) and \(\text{nvb}(T)\).

Arguing as in Step 1, we conclude (45b).

**Step 3.** Finally, suppose that \(d \geq 2\), \(p \geq 1\), and \(T\) contains an interior node \(z \in \mathcal{N}_\bullet\).

Since \(\text{nvb}(T)\) is a binary refinement rule, there exists a coarsest refinement \(\Phi^\bullet \in \text{nvb}(T)\) such that \(T\) contains an interior node \(z \in \mathcal{N}_\bullet\). In particular, \(\text{nvb}(T)\) is refined as depicted in Figure 3. Then, we suppose that each facet of \(T\) contains an interior node and may thus serve as an element bubble function. Arguing along the lines of Step 1, we conclude (45b).

**4. Proof of Theorem 4**

**4.1. Proof of efficiency and reliability (31).** Recall the different estimators from (28). The proof is split into several steps.

**Step 1.** We recall that the residual error estimator (10) satisfies that
\[
\|A^{1/2}\nabla(u - u_\bullet)\|_\Omega + \text{osc}_\bullet \leq g_\bullet(u_\bullet) \simeq g_\bullet(\mathcal{E}_\bullet, u_\bullet) + g_\bullet(\Phi^\bullet, u_\bullet). 
\]
Moreover, the stability from Remark 8 implies that
\[
\|A^{1/2}\nabla(\tilde{u}_\bullet - u_\bullet)\|_\Omega + \varphi_\bullet(\Phi^\bullet) \simeq \|A^{1/2}\nabla(\tilde{u}_\bullet - u_\bullet)\|_\Omega + \text{res}_\bullet(\Phi^\bullet, \tilde{u}_\bullet). 
\]

**Step 2.** According to (26), it holds that
\[
\lambda_\bullet(\tilde{u}_\bullet) \leq \|A^{1/2}\nabla(\tilde{u}_\bullet - u_\bullet)\|_\Omega \leq \|A^{1/2}\nabla(u - u_\bullet)\|_\Omega.
\]

May 7, 2018 14
Moreover, it holds that
\[
\text{res}_\bullet(T_\bullet, \hat{\nu}_\bullet)^2 \lesssim \|A^{1/2} \nabla (\hat{u}_\bullet - u_\bullet)\|_\Omega^2 + \varrho_\bullet(T_\bullet)^2 \lesssim \|A^{1/2} \nabla (u - u_\bullet)\|_\Omega^2 + \text{osc}_\bullet^2.
\]
In any case (cf. (28)), the estimator equivalence (25) proves efficiency
\[
\eta(\hat{\nu}_\bullet)^2 \lesssim \|A^{1/2} \nabla (u - u_\bullet)\|_\Omega^2 + \text{osc}_\bullet^2.
\]

**Step 3.** Recall that the refinement employed in Algorithm 3 satisfies (M3). Therefore, Lemma 9 implies that
\[
\varrho_\bullet(\mathcal{E}_\bullet) \lesssim \|A^{1/2} \nabla (\hat{u}_\bullet - u_\bullet)\|_\Omega + \varrho_\bullet(T_\bullet).
\]
Hence, we are led to
\[
\|A^{1/2} \nabla (u - u_\bullet)\|_\Omega + \text{osc}_\bullet \lesssim \varrho_\bullet(\mathcal{E}_\bullet) + \varrho_\bullet(T_\bullet) \lesssim \|A^{1/2} \nabla (\hat{u}_\bullet - u_\bullet)\|_\Omega + \varrho_\bullet(T_\bullet).
\]
In the first and fourth case of (28), the equivalence (25) of the \((h - h/2)\)-type error estimators shows that \(\eta(\hat{\nu}_\bullet) \simeq \|A^{1/2} \nabla (\hat{u}_\bullet - u_\bullet)\|_\Omega + \text{res}_\bullet(T_\bullet, \hat{\nu}_\bullet)\). This yields that
\[
\|A^{1/2} \nabla (\hat{u}_\bullet - u_\bullet)\|_\Omega + \varrho_\bullet(T_\bullet) \lesssim \|A^{1/2} \nabla (\hat{u}_\bullet - u_\bullet)\|_\Omega + \text{res}_\bullet(T_\bullet, \hat{\nu}_\bullet) \simeq \eta(\hat{\nu}_\bullet).
\]
In the other cases of (28), the equivalence (25) shows that \(\eta(\hat{\nu}_\bullet) \simeq \|A^{1/2} \nabla (\hat{u}_\bullet - u_\bullet)\|_\Omega + \text{osc}_\bullet\). We recall that according to (28), it holds that either \(p \geq 2\) or that the refinement ensures (M3’). Therefore, Lemma 9 implies again that
\[
\varrho_\bullet(T_\bullet) \lesssim \|A^{1/2} \nabla (\hat{u}_\bullet - u_\bullet)\|_\Omega + \text{osc}_\bullet.
\]
Then, we are led to
\[
\|A^{1/2} \nabla (\hat{u}_\bullet - u_\bullet)\|_\Omega + \varrho_\bullet(T_\bullet) \lesssim \|A^{1/2} \nabla (\hat{u}_\bullet - u_\bullet)\|_\Omega + \text{osc}_\bullet \simeq \eta(\hat{\nu}_\bullet).
\]
In any case, this proves that
\[
\|A^{1/2} \nabla (u - u_\bullet)\|_\Omega^2 + \text{osc}_\bullet^2 \lesssim \eta(\hat{\nu}_\bullet)^2.
\]
This concludes the proof. \(\square\)

4.2. Proof of (33)–(34). In the following, we verify that the \(\lambda_\bullet\)-based error estimators \(\eta_\bullet\) from (28) satisfy the axioms of adaptivity from [CFPP14]. To prove linear convergence with optimal rates for the \(\mu_\bullet\)-based error estimators, we then exploit the local equivalence (24). We stress that unlike the various a posteriori error estimators in [KS11, CFPP14], the \((h - h/2)\)-type estimators \(\eta_\bullet\) are not locally equivalent to the residual error estimator. Throughout, let \(T_\bullet \in \text{nvb}(T_0)\).

**Lemma 10 (local stability of \(\lambda_\bullet(\cdot)\)).** Let \(T_\bullet \in \text{nvb}(T_\bullet)\). For all \(\hat{\nu}_\bullet \in S_0^0(\hat{T}_\bullet)\) and all \(\hat{v}_\bullet \in S_0^0(\hat{T}_\circ)\), it holds that
\[
|\lambda_\bullet(T, \hat{v}_\circ) - \lambda_\bullet(T, \hat{\nu}_\bullet)| \leq \|A^{1/2} \nabla (\hat{v}_\circ - \hat{\nu}_\bullet)\|_T \quad \text{for all } T \in T_\bullet \cap T_\circ.
\]
In particular, this implies that
\[
|\lambda_\bullet(T_\bullet \cap T_\circ, \hat{\nu}_\circ) - \lambda_\bullet(T_\bullet \cap T_\circ, \hat{\nu}_\bullet)| \leq \|A^{1/2} \nabla (\hat{\nu}_\circ - \hat{\nu}_\bullet)\|_{\cup(T_\bullet \cap T_\circ)}.
\]
Further, there exists \(C_{\text{stab}} > 0\) such that the \(\lambda_\bullet\)-based estimators \(\eta_\bullet\) from (28) satisfy that
\[
|\eta_\bullet(S, \hat{\nu}_\circ) - \eta_\bullet(S, \hat{\nu}_\bullet)| \leq C_{\text{stab}} \|A^{1/2} \nabla (\hat{\nu}_\circ - \hat{\nu}_\bullet)\|_S \quad \text{for all } S \subseteq T_\bullet \cap T_\circ.
\]

May 7, 2018 15
The constant $C_{\text{sth}}$ depends only on $d$, $A$, $p$, $C_{\text{son}}$, and shape-regularity of $T_0$.

Proof. We prove the lemma in two steps.

Step 1. Note that $\pi_0$ and $\pi_*$ coincide on $T$. The triangle inequality thus proves that

$$\lambda_o(T, \widehat{\nu}_o) = \|(1 - \pi_*) A^{1/2} \nabla \widehat{\nu}_o\|_T \leq \lambda_*(T, \widehat{\nu}_*) + \|(1 - \pi_*) A^{1/2} \nabla (\widehat{\nu}_o - \widehat{\nu}_*)\|_T.$$ 

The same argument shows that

$$\lambda_*(T, \widehat{\nu}_*) = \|(1 - \pi_*) A^{1/2} \nabla \widehat{\nu}_*\|_T \leq \lambda_o(T, \widehat{\nu}_o) + \|(1 - \pi_*) A^{1/2} \nabla (\widehat{\nu}_o - \widehat{\nu}_*)\|_T.$$ 

Together with $\|(1 - \pi_*) A^{1/2} \nabla (\widehat{\nu}_o - \widehat{\nu}_*)\|_T \leq \|A^{1/2} \nabla (\widehat{\nu}_o - \widehat{\nu}_*)\|_T$, we conclude the proof of (53), which immediately yields (54).

Step 2. If $\eta_*(\widehat{\nu}_o)^2 = \lambda_*(\widehat{\nu}_o)^2 + \text{osc}_o^2$, (55) follows from Step 1 and the fact that $\text{osc}_o(T) = \text{osc}_*(T)$ for all $T \in T_o \cap T_0$. If $\eta_*(\widehat{\nu}_o)^2 = \lambda_*(\widehat{\nu}_o)^2 + \text{res}_o(\widehat{\nu}_o)^2$, we note that $\text{res}_o(T, \widehat{\nu}_*) = \text{res}_*(T, \widehat{\nu}_*)$ for all $T \in T_o \cap T_0$. Therefore, (55) follows from Step 1 and Remark 8 with $T_k = T_0$. □

Lemma 11 (local reduction of $\lambda_*(\cdot)$). Let $M_* \subseteq T_*$ and $T_o \in \text{nvb}(T_0)$ with $T_0 = \text{refine}(T_*, M_*)$. For all $\widehat{\nu}_o \in \mathcal{S}_0(T_o)$, it holds that

$$\lambda_o\{\{T' \in T_o : T' \subset T\}, \widehat{\nu}_o\} = 0 \quad \text{for all } T \in M_*.$$ 

In particular, this implies that

$$\lambda_o\{\{T' \in T_o : T' \subset \bigcup M_*\}, \widehat{\nu}_o\} \leq \|A^{1/2} \nabla (\widehat{\nu}_o - \widehat{\nu}_*)\|_{\bigcup M_*}.$$ 

Further, there exist constants $0 < q_{\text{red}} < 1$ and $C_{\text{red}} > 0$ such that the $\lambda_*$-based estimators $\eta_* \text{ from (28)}$ satisfy that

$$\eta_o\{\{T' \in T_o : T' \subset \bigcup M_*\}, \widehat{\nu}_o\} \leq q_{\text{red}} \eta_*(M_*, \widehat{\nu}_*) + C_{\text{red}} \|A^{1/2} \nabla (\widehat{\nu}_o - \widehat{\nu}_*)\|_{\bigcup M_*}.$$ 

The constant $q_{\text{red}}$ depends only on $d$, while $C_{\text{red}}$ depends additionally on $A$, $p$, $C_{\text{son}}$, and shape-regularity of $T_0$.

Proof. We prove the lemma in two steps.

Step 1. Recall that NVB is a binary refinement rule. Therefore, $T \in M_*$ and (M2) imply that $T_o|_T := \{T' \in T_o : T' \subset T\}$ is finer than $T_k|_T = T_o|_T$. This proves that $\|(1 - \pi_*) A^{1/2} \nabla \widehat{\nu}_o\|_T = 0$ and hence (56). The triangle inequality, the fact that orthogonal projections have operator norm one, and the Young inequality prove for all $\delta > 0$ that

$$\lambda_o\{\{T' \in T_o : T' \subset \bigcup M_*\}, \widehat{\nu}_o\}^2 \leq (1 + \delta^{-1}) \lambda_o\{\{T' \in T_o : T' \subset \bigcup M_*\}, \widehat{\nu}_o\}^2 + (1 + \delta) \|\|A^{1/2} \nabla (\widehat{\nu}_o - \widehat{\nu}_*)\|_{\bigcup M_*}\|^2 \leq (1 + \delta) \|\|A^{1/2} \nabla (\widehat{\nu}_o - \widehat{\nu}_*)\|_{\bigcup M_*}\|^2.$$ 

With $\delta \to 0$, this concludes the proof of (57).

Step 2. Since $\|(1 - \pi_*)(\cdot)\|_T \leq \|(1 - \pi_*)(\cdot)\|_T$ for all $T \in T_o$ and each marked element is bisected at least once, we have that

$$\text{osc}_o\{\{T' \in T_o : T' \subset \bigcup M_*\}\} \leq 2^{-1/d} \text{osc}_o(M_*).$$

May 7, 2018

16
Moreover, Remark 8 with the fact that each marked element is bisected at least once yields that

\[
\text{res}_o(\{T' \in T_o : T' \subset \bigcup M_o\}, \hat{\nu}_o) \leq 2^{-1/d} \text{res}_o(M_o, \hat{\nu}_o) + C\|A^{1/2}\nabla (\hat{\nu}_o - \hat{\nu}_*)\|_{\bigcup M_o}.
\]

(60)

Together with Step 1 and the Young inequality, (59) and (60) conclude the proof. \(\square\)

**Lemma 12 (discrete reliability of \(\eta_\bullet(\cdot)\)).** There exists \(C_{\text{drl}} > 0\) such that

\[
\|A^{1/2}\nabla (\hat{\nu}_o - \hat{\nu}_*)\|_\Omega \leq C_{\text{drl}} \eta_\bullet(T_o \setminus T_o, \hat{\nu}_*) \quad \text{for all } T_o \in \text{nvb}(T_\bullet).
\]

(61)

The constant \(C_{\text{drl}}\) depends only on \(A\), \(p \geq 1\), and \(\gamma\)-shape regularity of \(T_0\). \(\square\)

**Proof.** Due to the local equivalence (24), it suffices to consider the \(\mu_\bullet\)-based estimators from (28). The proof is split into three steps.

**Step 1.** Let \(\nu_\bullet := L_\bullet(\hat{\nu}_*) \in S_\bullet^p(T_\bullet)\). We apply the discrete reliability (42) of the residual error estimator for \(T_\bullet = \hat{T}_o\) and \(T_\bullet = \hat{T}_\bullet\). Together with (local) stability (44) of the residual error estimator, this proves that

\[
\|A^{1/2}\nabla (\hat{\nu}_o - \hat{\nu}_*)\|_\Omega \overset{(42)}{\lesssim} \hat{\varrho}_\bullet((\hat{T}_\bullet \setminus T_o) \cup (\hat{E}_0^\Omega \setminus E_0^\Omega), \hat{\nu}_*)
\]

\[
\overset{(43)}{\lesssim} \hat{\varrho}_\bullet((\hat{T}_\bullet \setminus T_o) \cup (\hat{E}_0^\Omega \setminus E_0^\Omega), \nu_\bullet) + \|A^{1/2}\nabla (\hat{\nu}_o - \nu_\bullet)\|_{(\hat{T}_\bullet \setminus T_o)},
\]

since the patch of a refined facet \(\hat{\nu}_o \in E_0^\Omega \setminus E_0^\Omega\) belongs to \(\hat{T}_\bullet \setminus \hat{T}_o\). Next, we show that \(\bigcup(\hat{T}_\bullet \setminus T_o) \subset \bigcup(T_\bullet \setminus T_o)\). Let \(\hat{T} \in \hat{T}_\bullet \setminus \hat{T}_o\) and \(T \in T_\bullet\) be the unique father element, i.e., \(\hat{T} \subset T\). If \(T \in T_o\), then (M2) implies that \(\hat{T} \in \hat{T}_o\), which contradicts the assumption \(\hat{T} \notin \hat{T}_o\). This concludes the desired inclusion. Since the local weights of the residual error estimator are decreasing for (uniform) mesh-refinement, this yields that

\[
\hat{\varrho}_\bullet(\hat{T}_\bullet \setminus T_o, \nu_\bullet) \leq \varrho_\bullet(T_o \setminus T_o, \nu_\bullet).
\]

According to the discrete efficiency (45a) of the residual error estimator for \(T_o = \hat{T}_o\) and \(T_\bullet = \hat{T}_\bullet\), it holds that

\[
\hat{\varrho}_\bullet(E_0^\Omega \setminus E_0^\Omega, \nu_\bullet) \lesssim \|A^{1/2}\nabla (\hat{\nu}_o - \nu_\bullet)\|_{(\hat{T}_\bullet \setminus T_o)} + \hat{\varrho}_\bullet(\hat{T}_\bullet \setminus T_o, \nu_\bullet).
\]

Combining the last three estimates and using that \(\bigcup(\hat{T}_\bullet \setminus T_o) \subset \bigcup(T_\bullet \setminus T_o)\), we are led to

\[
\|A^{1/2}\nabla (\hat{\nu}_o - \hat{\nu}_*)\|_\Omega \overset{(62)}{\lesssim} \varrho_\bullet(T_o \setminus T_o, \nu_\bullet) + \|A^{1/2}\nabla (\hat{\nu}_o - \nu_\bullet)\|_{(T_o \setminus T_\bullet)}.
\]

**Step 2.** For arbitrary \(p \geq 1\), we may use stability (44) of the residual error estimator to see that

\[
\varrho_\bullet(T_o \setminus T_o, \nu_\bullet) \overset{(44)}{\lesssim} \hat{\varrho}_\bullet(\{\hat{T}' \in \hat{T}_o : \hat{T}' \setminus \bigcup(T_o \setminus T_o)\}, \nu_\bullet)
\]

\[
\lesssim \hat{\varrho}_\bullet(\{\hat{T}' \in \hat{T}_o : \hat{T}' \setminus \bigcup(T_o \setminus T_o)\}, \hat{\nu}_o) + \|A^{1/2}\nabla (\hat{\nu}_o - \nu_\bullet)\|_{(T_o \setminus T_\bullet)}
\]

\[
\overset{(62)}{\lesssim} \text{res}_o(T_o \setminus T_o, \hat{\nu}_o) + \|A^{1/2}\nabla (\hat{\nu}_o - \nu_\bullet)\|_{(T_o \setminus T_\bullet)}.
\]

Combining this with (62) and the definition of \(\mu_\bullet(T_o \setminus T_o, \hat{\nu}_*)\), we prove (61) for \(\eta^2_\bullet = \mu^2_\bullet + \text{res}^2_\bullet\).
Step 3. If \( p \geq 2 \) or if the refinement ensures (M3'), we use the discrete efficiency \((\ref{eq:efficientdiscrete})\) of the residual error estimator for \( \mathcal{T}_0 = \hat{\mathcal{T}}_0 \) and \( \mathcal{T}_\bullet = \mathcal{T}_\star \) to see that
\[
g_\bullet(\mathcal{T}_\star \setminus \mathcal{T}_0, v_\bullet) \lesssim \| A \|^{1/2} \nabla (\hat{u}_\bullet - u_\bullet) \|_{\Omega(\mathcal{T}_\star \setminus \mathcal{T}_0)} + \text{osc}_\bullet(\mathcal{T}_\star \setminus \mathcal{T}_0).
\]
Combining this with \((\ref{eq:errorest})\), we prove \((\ref{eq:generalquasiorthogonality})\) for \( \eta_\bullet^2 = \mu_\bullet^2 + \text{osc}_\bullet^2 \).

Lemma 13 (general quasi-orthogonality for \( \eta_\bullet(\cdot) \)). Consider Algorithm 3 with \( \eta_\bullet \) from \((\ref{eq:Ademdist})\). Then, it holds that
\[
\sum_{j = \ell}^\infty \| A \|^{1/2} \nabla (\hat{u}_{\ell+1} - \hat{u}_{\ell}) \|_{\Omega}^2 \leq C_{\text{rel}}^2 \eta_\ell(\hat{u}_\ell)^2 \quad \text{for all } \ell \in \mathbb{N}_0,
\]
where \( C_{\text{rel}} > 0 \) is the reliability constant from \((\ref{eq:reliability})\).

Proof. For \( \mathcal{T}_0 \in \text{refine}(\mathcal{T}_0) \), there holds the Pythagoras identity
\[
\| A \|^{1/2} \nabla (u_\ell - \hat{u}_\ell) \|_{\Omega}^2 = \| A \|^{1/2} \nabla (\hat{u}_\ell - \hat{u}_\ell) \|_{\Omega}^2.
\]
Applying this for \( \mathcal{T}_0 = \mathcal{T}_{j+1} \) and \( \mathcal{T}_\bullet = \mathcal{T}_j \), we are led to
\[
\sum_{j = \ell}^N \| A \|^{1/2} \nabla (\hat{u}_{j+1} - \hat{u}_j) \|_{\Omega}^2 = \sum_{j = \ell}^N \left( \| A \|^{1/2} \nabla (u_\ell - \hat{u}_j) \|_{\Omega}^2 - \| A \|^{1/2} \nabla (u_{j+1} - \hat{u}_{j+1}) \|_{\Omega}^2 \right)
\]
\[
\leq \| A \|^{1/2} \nabla (u - \hat{u}_\ell) \|_{\Omega}^2 - \| A \|^{1/2} \nabla (u_{N+1} - \hat{u}_{N+1}) \|_{\Omega}^2
\]
\[
\leq \| A \|^{1/2} \nabla (u - \hat{u}_\ell) \|_{\Omega}^2.
\]
According to the Pythagoras theorem \((\ref{eq:pythagoras})\) and reliability \((\ref{eq:reliability})\), last term satisfies that
\[
\| A \|^{1/2} \nabla (u - \hat{u}_\ell) \|_{\Omega}^2 \leq \| A \|^{1/2} \nabla (u - u_\ell) \|_{\Omega}^2 \leq C_{\text{rel}}^2 \eta_\ell(\hat{u}_\ell)^2.
\]
As \( N \to \infty \), we conclude the proof. \( \square \)

Proof of \((\ref{eq:consistency})-(\ref{eq:control})\). We prove the assertion in three steps.

Step 1: First, we consider only the \( \lambda_\bullet \)-based estimators from \((\ref{eq:Ademdist})\). With \( \mathcal{S}_{\ell+1,\ell} := \{ T' \in \mathcal{T}_{\ell+1} : T' \not\subset \bigcup \mathcal{M}_\ell \} \) being the sons of the non-marked elements, it holds that
\[
\eta_{\ell+1}(\hat{u}_{\ell+1})^2 = \eta_{\ell+1}(\mathcal{S}_{\ell+1,\ell}, \hat{u}_{\ell+1})^2 + \eta_{\ell+1}(\mathcal{T}_{\ell+1} \setminus \mathcal{S}_{\ell+1,\ell}, \hat{u}_{\ell+1})^2.
\]
Stability \((\ref{eq:stability})\) with \( \mathcal{T}_\bullet = \mathcal{T}_\circ = \mathcal{T}_{\ell+1} \), reduction \((\ref{eq:reduction})\) with \( \mathcal{T}_\circ = \mathcal{T}_\ell \) and \( \mathcal{T}_\bullet = \mathcal{T}_\ell \), and the Young inequality show for arbitrary \( \delta > 0 \) that
\[
\eta_{\ell+1}(\hat{u}_{\ell+1})^2 \leq (1 + \delta) \eta_{\ell+1}(\mathcal{S}_{\ell+1,\ell}, \hat{u}_{\ell+1})^2 + q_{\text{red}} \eta_{\ell}(\mathcal{M}_\ell, \hat{u}_\ell)^2
\]
\[
+ (1 + \delta^{-1})(C_{\text{stb}}^2 + C_{\text{red}}^2) \| A \|^{1/2} \nabla (\hat{u}_{\ell+1} - \hat{u}_\ell) \|_{\Omega}^2.
\]
Due to the facts that \( h_{\ell+1} \leq h_\ell \) and \( \| (1 - \pi_{\ell+1})(\cdot) \|_T \leq \| (1 - \pi_\ell)(\cdot) \|_T \) for all \( T \in \mathcal{T}_{\ell+1} \), we have that
\[
\eta_{\ell+1}(\mathcal{S}_{\ell+1,\ell}, \hat{u}_{\ell+1})^2 \leq \eta_{\ell}(\mathcal{T}_\ell \setminus \mathcal{M}_\ell, \hat{u}_\ell)^2 = \eta_{\ell}(\hat{u}_\ell)^2 - \eta_{\ell}(\mathcal{M}_\ell, \hat{u}_\ell)^2.
\]
Together with the Dörfler marking in Algorithm 3 (iii), we derive the estimator reduction
\[
\eta_{\ell+1}(\hat{u}_{\ell+1})^2 \leq (1 + \delta)(1 - (1 - q_{\text{red}}^2)\theta) \eta_{\ell}(\hat{u}_\ell)^2
\]
\[
+ (1 + \delta^{-1})(C_{\text{stb}}^2 + C_{\text{red}}^2) \| A \|^{1/2} \nabla (\hat{u}_{\ell+1} - \hat{u}_\ell) \|_{\Omega}^2.
\]

May 7, 2018 18
According to [CFPP14, Prop. 4.10], general quasi-orthogonality (63), reliability (64), and estimator reduction (66) yield linear convergence (33) for the $\lambda_\ast$-based estimators from (28).

**Step 2:** Again, we only consider the $\lambda_\ast$-based estimators from (28). The first inequality in (34) follows immediately from (38) and (39). We prove the second inequality. Similarly as in (65), one shows that
\[
\eta_\circ(T_0, \hat{u}_0) \lesssim \eta_\circ(T_\ast, \hat{u}_\ast) + \|A^{1/2} \nabla (\hat{u}_o - \hat{u}_\ast)\|_\Omega
\]
for arbitrary $T_\ast \in \text{nvb}(T_0)$ and $T_0 \in \text{nvb}(T_\ast)$. Then, discrete reliability (61) immediately implies quasi-monotonicity (67)
\[
\eta_\circ(u_0) \lesssim \eta_\circ(\hat{u}_\ast).
\]
Altogether, stability (55), discrete reliability (61), quasi-monotonicity (67), and the overlay estimate [CKNS08, Eq. (2.2)] for $\text{nvb}()$ allow to apply optimality of Dörfler marking [CFPP14, Prop. 4.12] and the comparison lemma [CFPP14, Lem. 4.14], which show the following: There exists a constant $0 < \theta_0^\lambda < 1$ such that for $0 < \theta < \theta_0^\lambda$, $s > 0$, and all meshes $T_\ast$, there exists a refinement $\tilde{T}_\ast \in \text{nvb}(T_\ast)$ such that
\[
\eta_\circ(\tilde{u}_\ast) \lesssim \frac{1}{s} \eta_\circ(\tilde{u}_\ast)^{-1/s}.
\]
For $0 < \theta < \theta_0^\lambda$, and arbitrary $\ell \in \mathbb{N}$ with $\ell_\ast \neq T_0$, the closure estimate [CKNS08, Section 2.6] for refine(), and minimality of the set of marked elements $M_\ell$ yield that
\[
\#T_\ell - \#T_0 + 1 \leq \sum_{j=0}^{\ell-1} \#M_j \leq \sum_{j=0}^{\ell-1} \#(T_j \setminus \tilde{T}_j) \leq \sum_{j=0}^{\ell-1} (\#\tilde{T}_j - \#T_j) \lesssim \|u\|_{H^s} \sum_{j=0}^{\ell-1} \eta_\circ(\hat{u}_j)^{-1/s}.
\]
With the linear convergence (33), one can elementarily show that
\[
\sum_{j=0}^{\ell-1} \eta_\circ(\hat{u}_j)^{-1/s} \lesssim \eta_\circ(\hat{u}_\ell)^{-1/s};
\]
see, e.g., [CFPP14, Lemma 4.9]. Since $\eta_\circ(\hat{u}_0) \leq \|u\|_{H^s}$, this concludes (34) for the $\lambda_\ast$-based estimators from (28).

**Step 3:** Finally, we consider the $\mu_\ast$-based estimators from (28). Recall the local equivalence of $\lambda_\ast$ and $\mu_\ast$ which immediately transfers to the corresponding estimators from (28). Hence, $\mu_\ast$-based Dörfler marking $\eta_\circ(\hat{u}_\ast) \leq \eta_\circ(M_\ast, \hat{u}_\ast)$ with parameter $\theta$ and marked elements $M_\ast$ implies $\lambda_\ast$-based Dörfler marking with parameter $C_{\text{bd2}}^{-1} \theta$ and the same marked elements $M_\ast$ for the corresponding $\lambda_\ast$-based estimator and vice versa.

Therefore, $\mu_\ast$-based Dörfler marking implies linear convergence of the corresponding $\lambda_\ast$-based estimator and by equivalence also linear convergence of the $\mu_\ast$-based estimator. Moreover, for sufficiently small $\theta$, the $\lambda_\ast$-based estimator converges with optimal algebraic rates and hence does the $\mu_\ast$-based estimator. Details are left to the reader. □
5. Numerical experiments

In this section, we present three examples in two dimensions to empirically verify our theoretical results. For all examples we choose the L-shaped domain

\[(69) \Omega = (-1, 1)^2 \setminus ([0, 1] \times [-1, 0]).\]

The uniform initial mesh \(\mathcal{T}_0\) consists of 12 triangles. We run Algorithm 3 either with \(\theta = 1\) for uniform refinement or with \(\theta = 0.5\) for adaptive refinement based on the indicator

\[(70) \eta(T, \hat{u}_*)^2 := \lambda(T, \hat{u}_*)^2 + \text{osc}_s(T)^2\]

from (28). We consider the model problem (1) with \(A = I\), where we now allow inhomogeneous Dirichlet conditions. In our examples, we replace the Dirichlet data by its nodal interpolant for the numerical calculations. In all figures, we plot the error \((\|\nabla (u - u_*)\|^2 + \text{osc}_s)^{1/2}\) (if available) as well as the overall error estimators

\[\lambda_* := \left( \sum_{T \in \mathcal{T}_*} [\lambda(T, \hat{u}_*)^2 + \text{res}_s(T, \hat{u}_*)^2] \right)^{1/2}, \quad \lambda'_* := \left( \sum_{T \in \mathcal{T}_*} [\lambda_s(T, \hat{u}_*)^2 + \text{osc}_s(T)^2] \right)^{1/2},\]

\[\mu_* := \left( \sum_{T \in \mathcal{T}_*} [\mu_s(T, \hat{u}_*)^2 + \text{res}_s(T, \hat{u}_*)^2] \right)^{1/2}, \quad \mu'_* := \left( \sum_{T \in \mathcal{T}_*} [\mu_s(T, \hat{u}_*)^2 + \text{osc}_s(T)^2] \right)^{1/2}\]

for uniform (unif.) and adaptive (adap.) refinement with respect to the number of elements \(N\) of \(\mathcal{T}_*\). We use either three or five bisections for refinement of a marked element; cf. Figure 1. This guarantees (M3) or (M3'). Although the configuration \(p = 1\) (and (M3) with the chosen indicator (70)) is not covered by our theory (cf. the table from (28)), we also present the corresponding convergence plots in Figure 4(a), 8(a), and 10(a). Note that for uniform refinement with (M3), the convergence order \(O(N^{-s})\) with \(s > 0\) corresponds to \(O(h^{2s})\), where \(h := \max_{T \in \mathcal{T}_*} h_T\). This does not hold for uniform refinement with (M3'), since one refinement step leads to element sons of different levels. In particular, the uniform convergence rate seem to be slightly worse than expected. However, plotted over the maximal mesh-size \(h\), one obtains the expected rates (not displayed).

5.1. Experiment with known smooth solution. We prescribe the exact solution

\[(71) u(x_1, x_2) = (1 - 10x_1^2 - 10x_2^2)e^{-5(x_1^2 + x_2^2)} \quad \text{with } x = (x_1, x_2) \in \mathbb{R}^2.\]

This also defines inhomogeneous Dirichlet conditions and the right-hand side \(f\) is calculated appropriately. Since \(u\) is smooth, uniform as well as adaptive mesh refinement with (M3) or (M3') lead to optimal convergence behavior of \(O(N^{-1/2})\) and \(O(N^{-1})\) for \(S^1\)-FEM and \(S^2\)-FEM, respectively; see Figure 4 for (M3) and Figure 5 for (M3'). In Figure 6 and Figure 7, we consider corresponding reliability and efficiency indices for adaptive refinement, which empirically confirm Remark 5(i) and (ii) (for inhomogeneous Dirichlet conditions). Note that \(C_{\text{son}} = 4\) for (M3) and \(C_{\text{son}} = 6\) for (M3').

5.2. Experiment with known solution with generic singularity. We prescribe the exact solution in polar coordinates by

\[(72) u(x_1, x_2) = r^{2/3} \sin(2\varphi/3) \quad \text{with } r \in [0, \infty), \varphi \in [0, 2\pi).\]
May 7, 2018

Figure 4. Experiment from Section 5.1 with known smooth solution. We use the (minimal) refinement with (M3) for both, uniform and adaptive refinement.

Figure 5. Experiment from Section 5.1 with known smooth solution. We use the (minimal) refinement with (M3') for both, uniform and adaptive refinement.

Hence, \( f = 0 \) and the solution defines inhomogeneous Dirichlet conditions. Furthermore, \( \text{osc}_* = 0 \) and for \( S^1\text{-FEM} \) even \( \text{res}_* = 0 \). This implies that \( \lambda_* = \lambda''_* \) and \( \mu_* = \mu''_* \) for \( S^1\text{-FEM} \). It is well known that \( u \) has a generic singularity at the reentrant corner \((0, 0)\), which leads to reduced regularity \( u \in H^{1+2/3-\varepsilon}(\Omega) \) for all \( \varepsilon > 0 \). According to approximation theory we therefore get a reduced convergence order \( O(N^{-1/3}) \) for uniform refinement (M3) which is indeed observed in Figure 5. Our adaptive Algorithm 3 recovers the optimal convergence rates for \( S^1\text{-FEM} \) and \( S^2\text{-FEM} \), which are plotted in Figure 8 for (M3) and Figure 9 for (M3'). Hence, these figures also verify Theorem 4. In Figure 6

(a) \( S^1\text{-FEM with (M3).} \)

(b) \( S^2\text{-FEM with (M3).} \)

(a) \( S^1\text{-FEM with (M3').} \)

(b) \( S^2\text{-FEM with (M3').} \)
Figure 6. Reliability index \( \|\nabla (u - u_\bullet)\|_\Omega^2 + \text{osc}_\bullet^2 \|/\mu'_\bullet \) (upper half) and efficiency index \( \lambda''_\bullet / (\|\nabla (u - u_\bullet)\|_\Omega^2 + \text{osc}_\bullet^2) \) (lower half) for the example (Ex. 1) in Section 5.1 and for the example (Ex. 2) in Section 5.2, and for adaptive refinement with (M3) and (M3'). The (asymptotic) upper bounds for the reliability indices predicted in Remark 5 are highlighted with dashed black lines.

Figure 7. Reliability index \( \|\nabla (u - u_\bullet)\|_\Omega / \mu'_\bullet \hat{u}_\bullet \) (upper half) and efficiency index \( \lambda'_\bullet/\|\nabla (u - u_\bullet)\|_\Omega \) (lower half) for the example (Ex. 1) in Section 5.1 and for the example (Ex. 2) in Section 5.2, and for adaptive refinement with (M3) and (M3'). The (asymptotic) upper bounds for the reliability indices predicted in Remark 5 are highlighted with dashed black lines.
and Figure 7 we consider corresponding reliability and efficiency indices for adaptive refinement, which empirically confirm Remark 5 (i) and (ii) (for inhomogeneous Dirichlet conditions). Note that $C_{\text{son}} = 4$ for (M3) and $C_{\text{son}} = 6$ for (M3').

5.3. Experiment with unknown solution with generic singularity. For this example we define $f = 1$ in $\Omega$ and $u = 0$ on $\Gamma$. The solution is unknown. Therefore, we only plot the estimators in Figure 10 for (M3) and in Figure 11 for (M3'). All estimators
Figure 10. Experiment from Section 5.3 with unknown solution with generic singularity. We use the (minimal) refinement with (M3) for both, uniform and adaptive refinement.

Figure 11. Experiment from Section 5.3 with unknown solution with generic singularity. We use the (minimal) refinement with (M3') for both, uniform and adaptive refinement.

are efficient and reliable. Hence, the convergence rate of our numerical solution is observed by the asymptotics of the estimators. As in Example 5.2 uniform mesh refinement leads to a suboptimal convergence rate, whereas Algorithm 3 reproduces the optimal rates by adaptive mesh refinement.
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