GLOBAL WELL-POSEDNESS OF THE RADIAL CONFORMAL NONLINEAR
WAVE EQUATION WITH INITIAL DATA IN A CRITICAL SPACE

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ABSTRACT. In this note we prove global well-posedness and scattering for the conformal, defocusing, nonlinear wave equation with radial initial data in a critical Besov space. We also prove a polynomial bound on the scattering norm.

1. INTRODUCTION

Consider the defocusing wave equation
\begin{equation}
    u_{tt} - \Delta u + |u|^{4} u = 0, \quad u(0, x) = u_0, \quad u_t(0, x) = u_1,
\end{equation}
where \((u_0, u_1) \in B^{\frac{d}{2} + 1, 2}_{1, 1} \times B^{\frac{d}{2} - \frac{1}{2}, 2}_{1, 1}\) is radially symmetric, in any dimension \(d \geq 3\). Previous work on this problem has addressed the \(d = 3\) and \(d = 5\) cases. In this paper we prove

\textbf{Theorem 1.} When \(d \geq 4\), if \((u_0, u_1) \in B^{\frac{d}{2} + 1, 2}_{1, 1} \times B^{\frac{d}{2} - \frac{1}{2}, 2}_{1, 1}\) is radially symmetric, then (1.1) has a global solution that scatters. Moreover, if
\begin{equation}
    \|u_0\|_{B^{\frac{d}{2} + 1, 2}_{1, 1}} + \|u_1\|_{B^{\frac{d}{2} - \frac{1}{2}, 2}_{1, 1}} = A,
\end{equation}
then
\begin{equation}
    \|u\|_{L^2_t(\mathbb{R}^{d+1})} \lesssim A + A^{\frac{4}{d-1}}.
\end{equation}

In general, the nonlinear wave equation,
\begin{equation}
    u_{tt} - \Delta u + |u|^{p-1} u = 0,
\end{equation}
has the scaling symmetry
\begin{equation}
    u(t, x) \mapsto \lambda^{\frac{2}{p-1}} u(\lambda t, \lambda x).
\end{equation}
That is, if \(u(t, x)\) solves (1.4), then the right hand side of (1.5) solves (1.4) for any \(\lambda > 0\). Then set
\begin{equation}
    s_c = \frac{d}{2} - \frac{2}{p-1}.
\end{equation}
Then the \(H^{s_c}(\mathbb{R}^d) \times H^{s_c-1}(\mathbb{R}^d)\) norm of the initial data is invariant under (1.5). So for \(p - 1 = \frac{d}{2-1}\), \(s_c = \frac{d}{2}\). The results of [LS95] completely determine the local behavior of (1.1). Moreover, global well-posedness and scattering hold for small initial data. See Theorem 2 (where we recall the work of [LS95]). Moreover, [LS95] proved that this result is sharp, that is, local well-posedness for (1.1) does not hold for initial data in \(H^s \times H^{s-1}\) for any \(s < \frac{d}{2}\).
Remark 1. The $L^2_{t,x}(\mathbb{R} \times \mathbb{R}^d)$ norm is invariant under the scaling (1.5).

It is conjectured that global well-posedness and scattering hold for (1.1) for initial data in $\dot{H}^{1/2} \times \dot{H}^{-1/2}$. This has been resolved in the affirmative when $d = 3$ for radially symmetric initial data. See [Dod18a] and also [Dod18c].

The Besov spaces $B^s_{1,1}$ are defined by the Littlewood–Paley partition of unity.

$$\|f\|_{B^s_{1,1}(\mathbb{R}^d)} = \sum_{j \in \mathbb{Z}} 2^{js} \|P_j f\|_{L^1(\mathbb{R}^d)}.$$  (1.7)

The Sobolev embedding theorem implies $B^{s+\frac{1}{2}}_{1,1} \subset \dot{H}^{\frac{1}{2}}$ and $B^{s-\frac{1}{2}}_{1,1} \subset \dot{H}^{-\frac{1}{2}}$. The space $B^{s+\frac{1}{2}}_{\frac{1}{2}} \times B^{s-\frac{1}{2}}_{\frac{1}{2}}$ for the initial data is also invariant under (1.5).

Study of dispersive partial differential equations with initial data in a Besov space has proved to be quite fruitful.

Proposition 1. When $d = 3$, for every radial initial data $(u_0, u_1) \in B^{\frac{3}{4}}_{1,1} \times B^{\frac{1}{4}}_{1,1}(\mathbb{R}^3)$, let $u$ be the solution to (1.1). Then there exists a function $A : [0, \infty) \to [0, \infty)$ such that

$$\|u\|_{L^4_{t,x}(\mathbb{R} \times \mathbb{R}^3)} \leq A(\|(u_0, u_1)\|_{B^{\frac{3}{4}}_{1,1} \times B^{\frac{1}{4}}_{1,1}(\mathbb{R}^3)}).$$  (1.8)

When $d = 5$, for every radial initial data $(u_0, u_1) \in B^{\frac{5}{4}}_{1,1} \times B^{\frac{1}{4}}_{1,1}(\mathbb{R}^5)$, let $u$ be the solution to (1.1). Then there exists a function $A : [0, \infty) \to [0, \infty)$ and a parameter $\delta_1 > 0$ that depends on the initial data $(u_0, u_1)$ such that

$$\|u\|_{L^4_{t,x}(\mathbb{R} \times \mathbb{R}^5)} \leq A(\|(u_0, u_1)\|_{B^{\frac{5}{4}}_{1,1} \times B^{\frac{1}{4}}_{1,1}(\mathbb{R}^5)}, \delta_1).$$  (1.9)

Proof. See [Dod18b] for the result in dimension $d = 3$ and [MYZ20] in dimension $d = 5$. □

In this paper we extend Proposition 1 to any dimension and in addition obtain polynomial bounds in dimensions $d \geq 4$. We do not obtain an improved bound in dimension $d = 3$.

Polynomial bounds were also obtained for the nonlinear Schrödinger equation with initial data in a critical Besov space in [Dod21]. As in the nonlinear wave case, polynomial bounds were obtained for solutions to the nonlinear Schrödinger equation

$$iu_t + \Delta u = |u|^{p-1}u,$$  (1.10)

when $1 < p < 3$, but when $p = 3$ only bounds of the form (1.8) were obtained.

There are many initial data that fall into the category of radially symmetric $B^{\frac{3}{4}+\frac{1}{2}}_{\frac{1}{2}} \times B^{\frac{1}{4}-\frac{1}{2}}_{\frac{1}{2}}$. Suppose for example that $u_0$ is a Gaussian function and $u_1 = 0$. Since the Gaussian function is smooth, rapidly decreasing, as are all its derivatives, the initial data satisfies (3.2), and therefore the work of [Str68] implies that the solution to (1.1) with $u_0 = Ce^{-|x|^2}$ and $u_1 = 0$ satisfies

$$\|u\|_{L^{2(\frac{d+1}{d-1})}_{t,x}(\mathbb{R} \times \mathbb{R}^d)} \lesssim C + C^{c(d)},$$  (1.11)

where $c(d) > 1$ depends only on the dimension.

Remark 2. We will recall the work of [Str68] in Lemma 1.
The same bounds also hold for initial data rescaled under (1.15). Thus, (1.11) also holds for $u_1 = 0$ and $u_0 = C\lambda^{\frac{d-1}{2}}e^{-\lambda|x|^2}$, for any $\lambda > 0$. Moreover, the solution to (1.1) with such initial data also satisfies (1.11).

Now take $\lambda_1 \ll 1$ and $\lambda_2 \gg 1$, and assume $C_1, C_2, C_0 > 0$ are constants. Then take $u_1 = 0$ and

$$u_0 = C_1\lambda_1^{\frac{d-1}{2}}e^{-\lambda_1^2|x|^2} + C_0e^{-|x|^2} + C_2\lambda_2^{\frac{d-1}{2}}e^{-\lambda_2^2|x|^2}.$$ \hspace{1cm} (1.12)

In this case, it is reasonable to conjecture that the bounds on a solution to (1.1) will be some polynomial function of $C_1 + C_2 + C_0$, by (1.11) and the heuristic that solutions to (1.1) at very different frequencies are basically decoupled. However, if one merely plugs (1.12) into Lemma [1] in the case of initial data given by (1.1), the scattering size bounds on the right hand side of (1.11) will depend on $\lambda_1^{-1}$ and $\lambda_2$. However taking $\lambda_1 \searrow 0$ and $\lambda_2 \nearrow \infty$ ought to increase the strength of the decoupling.

By proving Theorem [1] we resolve this issue.

**Remark 3.** Throughout this paper, a solution to (1.1) refers to a strong solution. That is, $u$ satisfies Duhamel’s principle,

$$\begin{align*}
(u(t), u_t(t)) &= S(t)(u_0, u_1) - \int_0^t S(t - \tau)(0, |u|_{\tau}^{-1}u) d\tau.
\end{align*}$$ \hspace{1cm} (1.13)

The notation $S(t)(f, g)$ denotes $(u(t), u_t(t))$, where $u$ is a solution to the linear wave equation,

$$u_{tt} - \Delta u = 0, \quad u(0, x) = f(x), \quad u_t(0, x) = g(x).$$ \hspace{1cm} (1.14)

Global well-posedness has the canonical definition, taking a solution to mean a strong solution.

Scattering implies that as $t \to +\infty$ or $t \to -\infty$, the solution looks like a solution to (1.14). That is, there exist $(u_0 ^\pm, u_1 ^\pm) \in H^{1/2} \times H^{-1/2}$ and $(u_{0} ^\pm, u_{1} ^\pm) \in H^{1/2} \times H^{-1/2}$ such that

$$\lim_{t \to \infty} \|S(-t)(u(t), u_t(t)) - (u_0 ^\pm, u_1 ^\pm)\|_{H^{1/2} \times H^{-1/2}} = 0$$ \hspace{1cm} (1.15)

lim_{t \to -\infty} \|S(-t)(u(t), u_t(t)) - (u_0 ^\pm, u_1 ^\pm)\|_{H^{1/2} \times H^{-1/2}} = 0.

2. Strichartz estimates and local well-posedness

Global well-posedness and scattering for (1.1) is equivalent to obtaining the bound

$$\|u\|_{L_{t,x}^{\frac{2(d+1)}{d-1}}(\mathbb{R} \times \mathbb{R}^d)} < \infty.$$ \hspace{1cm} (2.1)

**Proof of equivalence.** This fact follows from the Strichartz estimates.

**Theorem 2** (Strichartz estimates for the wave equation). Let $I$ be a time interval and let $u : I \times \mathbb{R}^d \to \mathbb{C}$ be a Schwartz solution to the wave equation

$$u_{tt} - \Delta u = F, \quad u(t_0) = u_0, \quad u_t(t_0) = u_1,$$ \hspace{1cm} (2.2)

for some $t_0 \in I$. Then

$$\|u\|_{L_t^q L_x^r(I \times \mathbb{R}^d)} + \|\nabla u\|_{C_t^0 H_x^{s+1}(I \times \mathbb{R}^d)} + \|\partial_t u\|_{C_t^0 H_x^{-s-1}(I \times \mathbb{R}^d)} \lesssim \|u_0\|_{H_x^s(I \times \mathbb{R}^d)} + \|u_1\|_{H_x^{-s}(I \times \mathbb{R}^d)} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(I \times \mathbb{R}^d)},$$ \hspace{1cm} (2.3)

whenever $s \geq 0$, $2 \leq q, \tilde{q} \leq \infty$ and $2 \leq r, \tilde{r} < \infty$ obey the scaling condition

$$\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - s = \frac{1}{\tilde{q}} + \frac{d}{\tilde{r}} = 2.$$ \hspace{1cm} (2.4)
and the wave admissibility conditions
\[
\frac{1}{q} + \frac{d-1}{2r}, \frac{1}{q} + \frac{d-1}{2r} \leq \frac{d-1}{4}.
\]

**Proof.** This theorem is copied from [Tao06]. The proof of this theorem is found in [Kap89], [LS95], [Sog95], [SS00], [KT98].

When \( s = \frac{1}{2} \), \((q, r) = (\frac{2(d+1)}{d-1}, \frac{2(d+1)}{d+1})\) satisfies (2.4) and (2.5), as does \((q', r') = (\frac{2(d+1)}{d+3}, \frac{2(d+1)}{d+3})\). Furthermore, by Hölder’s inequality,
\[
\|u\|_{L_{t,x}^{2(d+1)}} \leq \|u\|_{L_{t,x}^{2(d+1)}(\mathbb{R} \times \mathbb{R}^d)}.
\]

Therefore, standard small data arguments and the Lebesgue dominated convergence theorem imply

**Theorem 3.** The nonlinear wave equation (1.1) is globally well-posed and scattering for initial data sufficiently small, that is, \(\|u_0\|_{\dot{H}^{1/2}} + \|u_1\|_{\dot{H}^{-1/2}} < \epsilon_0(d)\), where \(0 < \epsilon_0(d) \ll 1\). For any \((u_0, u_1) \in \dot{H}^{1/2} \times \dot{H}^{-1/2}\), there exists some \(T(u_0, u_1) > 0\) such that (1.1) is locally well-posed on an interval \((-T, T)\).

**Proof.** This theorem was proved in [LS95].

Now suppose \(u\) has a solution to (1.1) on \([0, T)\) that satisfies
\[
\|u\|_{L_{t,x}^{2(d+1)}((0, T) \times \mathbb{R}^d)} < \infty.
\]

By Duhamel’s principle,
\[
(u(t), u_t(t)) = S(t)(u_0, u_1) - \int_0^t S(t - \tau)(0, |u|^{\frac{4}{d-1}} u) d\tau.
\]

For any fixed \((u_0, u_1) \in \dot{H}^{1/2} \times \dot{H}^{-1/2}\),
\[
S(t)(u_0, u_1) \to S(T)(u_0, u_1) \quad \text{in} \quad \dot{H}^{1/2} \times \dot{H}^{-1/2}.
\]

Also, by (2.7), Theorem 2 and the dominated convergence theorem, for any \(\epsilon > 0\) there exists \(\delta(T, \epsilon) > 0\) sufficiently small such that, for any \(t \in [T - \delta, T)\),
\[
\|\int_{T - \delta}^t S(t - \tau)(0, |u|^{\frac{4}{d-1}} u) d\tau\|_{\dot{H}^{1/2} \times \dot{H}^{-1/2}} \lesssim \epsilon^{1 + \frac{1}{d-1}}.
\]

As in (2.9),
\[
\int_0^{T - \delta} S(t - \tau)(0, |u|^{\frac{4}{d-1}} u) d\tau \to \int_0^{T - \delta} S(T - \tau)(0, |u|^{\frac{4}{d-1}} u) d\tau,
\]
in \(\dot{H}^{1/2} \times \dot{H}^{-1/2}\), by (2.7). Therefore, \((u(t), u_t(t))\) converges in \(\dot{H}^{1/2} \times \dot{H}^{-1/2}\) as \(t \rightarrow T\). By Theorem 3 the solution to (1.1) can therefore be extended past \(T\).

Then if (2.11) holds, set
\[
(u_0^+, u_1^+) = (u_0, u_1) - \int_0^\infty S(-\tau)(0, |u|^{\frac{4}{d-1}} u) d\tau.
\]

Theorem 2 and (2.11) imply that \((u_0^+, u_1^+) \in \dot{H}^{1/2} \times \dot{H}^{-1/2}\) and furthermore,
\[
\lim_{t \to \infty} \|S(t) (u_0^+, u_1^+) - (u(t), u_t(t))\|_{\dot{H}^{1/2} \times \dot{H}^{-1/2}} = 0.
\]
An identical argument shows that a solution to (1.1) satisfying (1.1) scatters backward in time. Therefore, (2.1) implies that scattering holds.

The converse will not be needed here, so the proof will merely be sketched. The idea is that if scattering holds, $S(t)(u_0^+, u_1^+)$ will dominate the solution for large $t$ and $S(t)(u_0^-, u_1^-)$ will dominate for $|t|$ large, $t$ negative. By standard perturbative arguments combined with Strichartz estimates, this implies that there exists some $T$ sufficiently large such that
\begin{equation}
\|u\|_{L^{2(d+1)}\cap ((-\infty, -T]\times \mathbb{R}^d)} < \infty.
\end{equation}
Now then, since $u \in C^0_t(\mathbb{R}; \dot{H}^{1/2})$ and $u_t \in C^1_t(\mathbb{R}; \dot{H}^{-1/2})$, for any $t_0 \in [-T, T]$, the local well-posedness result of [LS95] holds on some interval $(t_0 - \delta(t_0), t_0 + \delta(t_0))$, with
\begin{equation}
\|u\|_{L^{2(d+1)}((t_0 - \delta(t_0), t_0 + \delta(t_0)) \times \mathbb{R}^d)} \leq 1.
\end{equation}
Since $[-T, T]$ is compact, every open cover has a finite subcover, so (2.15) implies
\begin{equation}
\|u\|_{L^{2(d+1)}((-T, T)\times \mathbb{R}^d)} < \infty.
\end{equation}

\section{The Conformal Energy}

Recall the scattering result of [Str68].

\textbf{Theorem 4.} Let $v$ be a solution to
\begin{equation}
v_{tt} - \Delta v + |v|^{d-1} v = 0, \quad v(0, x) = v_0, \quad v_1(0, x) \in v_1,
\end{equation}
such that
\begin{equation}
\|\nabla v_0\|_L^2 + \|\nabla v_1\|_L^2 \leq \int |x|^2 |v|^{\frac{2(d+1)}{d-1}} \, dx. < \infty.
\end{equation}
Then the solution to (3.1) is global and satisfies
\begin{equation}
\|v\|_{L^{2(d+1)}(\mathbb{R} \times \mathbb{R}^d)} < \infty.
\end{equation}

\textbf{Proof.} The proof uses the conservation of the conformal energy, see the exposition in [Tao06], and follows directly from the lemma of [Kha01].

\textbf{Lemma 1.} The conformal energy
\begin{equation}
\mathcal{E}(v) = \frac{1}{4} \int_{\mathbb{R}^d} |(t + |x|)L v + (d - 1)v|^2 + |(t - |x|)L v + (d - 1)v|^2
\end{equation}
\begin{equation}
+ 2|t^2 + |x|^2|\nabla v|^2 \, dx + \frac{d - 1}{4(d + 1)} \int (t^2 + |x|^2)|v|^{\frac{2(d+1)}{d-1}} \, dx,
\end{equation}
is conserved for a solution to (3.1), where $L = (\partial_t + \frac{\nabla}{|x|} \cdot \nabla)$ and $L = (\partial_t - \frac{\nabla}{|x|} \cdot \nabla)$.

Indeed, if Lemma 1 is true, $E(v)(0) < \infty$ implies
\begin{equation}
\|v\|_{L^{2(d+1)}(\mathbb{R} \times \mathbb{R}^d)} < \infty,
\end{equation}
which proves the Theorem.
Proof of Lemma. Define the tensors
\[
T^{00}(t, x) = \frac{1}{2} |\partial_t v|^2 + \frac{1}{2} |\nabla v|^2 + \frac{d-1}{2(d+1)}|v|^{2(d+1)},
\]
(3.6)
\[
T^{0j}(t, x) = T^{j0}(t, x) = - (\partial_t v)(\partial_x v),
\]
\[
T^{jk}(t, x) = (\partial_x v \partial_x v) - \frac{\delta_{jk}}{2} (|\nabla v|^2 - |\partial_t v|^2) - \delta_{jk} \frac{d-1}{2(d+1)}|v|^{2(d+1)}.
\]
The tensor functions satisfy the differential equations
\[
\partial_t T^{00}(t, x) + \partial_{x_j} T^{0j}(t, x) = 0, \quad \partial_t T^{0j}(t, x) + \partial_{x_k} T^{jk}(t, x) = 0.
\]
The differential equations (3.7) imply that the quantity
\[
Q(t) = \int (t^2 + |x|^2)T^{00}(t, x) - 2tx_j T^{0j}(t, x) + (d-1)t v(\partial_t v) - \frac{d-1}{2}|v|^2 dx,
\]
is conserved. The Einstein summation convention is observed. Indeed, by (3.7),
\[
\frac{d}{dt} Q(t) = 2t \int T^{00}(t, x)dx - \int (t^2 + |x|^2)\partial_x T^{0j}(t, x)dx - 2t \int x_j T^{0j}(t, x)dx + 2t \int x_j \partial_{x_k} T^{jk}(t, x)dx
\]
\[
+ (d-1) \int v(\partial_t v)dx + (d-1)t \int (\partial_t v)^2 dx + (d-1)t \int v(\Delta v - |v|^{\frac{d-4}{d-1}}v)dx - (d-1) \int v(\partial_t v)dx.
\]
Integrating the second term in (3.9) by parts,
\[
= 2t \int T^{00}(t, x)dx - 2t \int \delta_{jk} T^{jk}(t, x)dx + (d-1)t \int (\partial_t v)^2 dx + (d-1)t \int v(\Delta v - |v|^{\frac{d-4}{d-1}}v)dx.
\]
Since \(\delta_{jk} \delta_{jk} = d\),
\[
= 2t \int T^{00}(t, x)dx - 2t \int |\nabla v|^2 dx + dt \int (|\nabla v|^2 - |\partial_t v|^2)dx + \frac{d(d-1)t}{d+1} \int |v|^{2(d+1)} dx
\]
\[
+ (d-1)t \int |\partial_t v|^2 dx - (d-1)t \int |\nabla v|^2 dx - (d-1)t \int |v|^{2(d+1)} dx.
\]
Doing some algebra,
\[
= 2t \int T^{00}(t, x)dx - t \int |\nabla v|^2 dx - t \int |\partial_t v|^2 dx - \frac{d-1}{d+1} t \int |v|^{2(d+1)} dx = 0.
\]
Therefore, \(Q(t)\) is conserved.

Now then,
\[
\int (t^2 + |x|^2)T^{00}(t, x)dx - \int 2tx_j T^{0j}(t, x)dx
\]
\[
= \int (t^2 + |x|^2)(\frac{1}{2} |\partial_t v|^2 + \frac{1}{2} |\nabla v|^2 + \frac{d-1}{2(d+1)}|v|^{2(d+1)})dx
\]
\[
= \frac{1}{4} \int (t + |x|)^2|L v|^2 dx + \frac{1}{4} \int (t - |x|)^2|L v|^2 dx + \frac{1}{2} t^2 + |x|^2 |\nabla v|^2 dx + \frac{d-1}{2(d+1)} t(t^2 + |x|^2)|v|^{2(d+1)} dx.
\]
Next, integrating by parts,
\begin{equation}
\frac{1}{2}((t + |x|)Lv, (d - 1)v)_{L^2} + \frac{1}{2}((t - |x|)Lv, (d - 1)v)_{L^2}
\end{equation}
\begin{equation}
= (d - 1)t \int (\partial_t v) v dx + (d - 1) \int v(x \cdot \nabla v) dx = (d - 1)t \int (\partial_t v) v dx - \frac{d(d - 1)}{2} \int |v|^2 dx.
\end{equation}
Since
\begin{equation}
- \frac{d(d - 1)}{2} \int |v|^2 dx + (d - 1)^2 \int |v|^2 dx = \frac{d}{2} \int |v|^2 dx,
\end{equation}
(3.13)–(3.15) imply that $Q(t)$ is equal to the right hand side of (3.14).

\section{4. Proof of Theorem 1}

\textbf{Proof of Theorem 1} Since $B^{\frac{d}{2} + \frac{1}{2}}_{1,1} \subset \dot{H}^{1/2}$ and $B^{\frac{d}{2} - \frac{1}{2}}_{1,1} \subset \dot{H}^{-1/2}$, (1.1) is locally well-posed on some interval $(-T_1, T_2)$, where $T_1, T_2 > 0$. To prove global well-posedness and scattering, it suffices to to show that
\begin{equation}
\|u\|_{L^{\frac{2(d+1)}{d-1}}([0, T_2) \times \mathbb{R}^d)} < \infty, \quad \text{and} \quad \|u\|_{L^{\frac{2(d+1)}{d-1}}((-T_1, 0) \times \mathbb{R}^d)} < \infty,
\end{equation}
where $(-T_1, T_2)$ is the maximal interval of existence of the solution to (1.1), and that the bound does not depend on $T_1$ and $T_2$.

To prove this, decompose the solution on $(-T_1, T_2)$, $u = v + w$, where $v$ and $w$ solve the equations
\begin{equation}
v_{tt} - \Delta v + |v + w|^{\frac{d+1}{2}} (v + w) = 0, \quad v(0, x) = 0, \quad v_t(0, x) = 0,
\end{equation}
and
\begin{equation}
w_{tt} - \Delta w = 0, \quad w(0, x) = u_0, \quad w_t(0, x) = u_1.
\end{equation}
Strichartz estimates, Theorem 2 imply $\|w\|_{L^{\frac{2(d+1)}{d-1}}(\mathbb{R} \times \mathbb{R}^d)} \lesssim A$, so to prove (2.1), it suffices to show $\|v\|_{L^{\frac{2(d+1)}{d-1}}(\mathbb{R} \times \mathbb{R}^d)} < \infty$.

Split
\begin{equation}
|v + w|^{\frac{d+1}{2}} (v + w) = |v|^{\frac{d+1}{2}} v + F, \quad \text{where} \quad |F| \lesssim |v|^{\frac{d+1}{2}} |w| + |w|^{\frac{d+1}{2}}.
\end{equation}
Now let $\mathcal{E}(t)$ denote the conformal energy of $v$, see (3.4). Then by the computations proving Lemma 1
\begin{equation}
\frac{d}{dt} \mathcal{E}(t) = -\frac{1}{2} ((t + |x|)Lv + (d - 1)v, (t + |x|)F) - \frac{1}{2} ((t - |x|)Lv + (d - 1)v, (t - |x|)F).
\end{equation}
Since
\begin{equation}
\|(t + |x|)Lv + (d - 1)v\|_{L^2} + \|(t - |x|)Lv + (d - 1)v\|_{L^2} \lesssim \mathcal{E}(t)^{1/2},
\end{equation}
\begin{equation}
\frac{d}{dt} \mathcal{E}(t) \lesssim \mathcal{E}(t)^{1/2} (\|w\|_{L^\infty}^{\frac{d+1}{2}} + \||x|w\|_{L^\infty}^{\frac{d+1}{2}}) \|w|^{\frac{d+1}{2}} |v|^{\frac{d+1}{2}} + |w|^{\frac{d+1}{2}} \|_{L^2}.
\end{equation}
The dispersive estimate implies
\begin{equation}
\|t|w|^{\frac{d+1}{2}}\|_{L^\infty} \lesssim A^{\frac{d+1}{2}},
\end{equation}
and the radial Sobolev embedding theorem implies
\begin{equation}
\||x|w|^{\frac{d+1}{2}}\|_{L^\infty} \lesssim A^{\frac{d+1}{2}}.
\end{equation}
Therefore,

\[ \frac{d}{dt} E(t) \lesssim A^{\frac{d}{2} \tau} E(t)^{1/2} \left( \|u\|_L^{d+1}_{d+1} + \|v\|_L^{d+1}_{d+1} \right). \]

Then by (3.3),

\[ \frac{d}{dt} E(t) \lesssim A^{\frac{d}{2} \tau} E(t)^{1/2} \|u\|_L^{d+1}_{d+1} + \frac{1}{t^{d+1}} A^{\frac{d}{2} \tau} E(t)^{1/2} \|u\|_L^{d+1}_{d+1}. \]

By (3.4), \( E(0) = 0 \), the fundamental theorem of calculus, and (4.11),

\[ \|v\|_{L_t^2 L_x^{2(d+1)}} \lesssim \int_0^{T_2} \frac{1}{t^2} E(t) dt \lesssim \int_0^{T_2} \frac{1}{t^2} \int_0^t A^{\frac{d}{2} \tau} E(\tau)^{1/2} \|u\|_L^{d+1}_{d+1} + \frac{1}{\tau^{d+1}} A^{\frac{d}{2} \tau} E(\tau)^{1/2} \|u\|_L^{d+1}_{d+1} d\tau. \]

Then by Fubini’s theorem,

\[ \int_0^{T_2} \frac{1}{t^2} E(t) dt \lesssim \int_0^{T_2} \frac{1}{t^2} E(\tau)^{1/2} \|u(\tau)\|_L^{d+1}_{d+1} d\tau + A^{\frac{d}{2} \tau} \int_0^{T_2} \frac{1}{\tau^{d+1}} E(\tau)^{1/2} \|u(\tau)\|_L^{d+1}_{d+1} d\tau. \]

By Hölder’s inequality,

\[ \lesssim A^{\frac{d}{2} \tau} \int_0^{T_2} \frac{1}{\tau^2} E(\tau)^{1/2} \|u(\tau)\|_L^{d+1}_{d+1} d\tau + A^{\frac{d}{2} \tau} \int_0^{T_2} \frac{1}{\tau^{d+1}} E(\tau)^{1/2} \|u(\tau)\|_L^{d+1}_{d+1} d\tau. \]

By Strichartz estimates,

\[ \int_\mathbb{R} \|u(t)\|_{L_{t,x}^{2(d+1)}}^{d+1} dt \lesssim A^{\frac{d}{2} \tau}. \]

Therefore, doing some algebra,

\[ \|v\|_{L_t^2 L_x^{2(d+1)}} \lesssim \int_0^{T_2} \frac{1}{t^2} E(t) dt \lesssim A^{\frac{d}{2} \tau} + A^{\frac{d}{2} \tau}. \]

Thus, \( T_2 = \infty \) and the solution to (1.11) is global forward in time. By time reversal symmetry, the proof is complete.

**Remark 4.** To place the use of Fubini’s theorem on firm footing, it is possible to approximate \( u_0 \) and \( u_1 \) by smooth, compactly supported functions. Then by Theorem 3, \( \|u\|_{L_{t,x}^{2(d+1)}} < \infty \), and the bounds are controlled by the right hand side of (4.11).

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GLOBAL WELL-POSEDNESS OF THE RADIAL CONFORMAL NONLINEAR WAVE EQUATION WITH INITIAL DATA IN A CRITICAL SPACE

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