A free solution to the Dirac equation in $R$-spacetime

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Abstract. In this work, the Dirac equation and its solution for a free fermion in $R$-spacetime is presented. In the solution obtained, the oscillation frequency increases as $t \to 0^+$. The time singularity and the necessity of restricting the time domain in $R$-spacetime is discussed. Limiting the time domain implies the possibility of quantization of the solution, though this is a subject for further work.

1. Introduction

One of the well-known exact solutions for the Einstein field equations is the Anti-de Sitter (AdS) spacetime with the empty universe (A-vacuum) having constant negative curvature \cite{1}. This spacetime is widely used in quantum cosmology and modern theories of relativity \cite{2,3}. In higher dimensions AdS spacetime plays an important role, known as the AdS/CFT correspondence \cite{4}.

From the literature, spacetime, which is assigned a coordinate system is known as coordinate patching. In \cite{3,5}, they map a 4-dimensional hyperboloid in 5-dimensional Minkowski spacetime to a 4-dimensional hyperplane using the Beltrami geodesic projection. The physical meaning of these coordinates is the inertial reference frame of AdS spacetime. This is Anti-de Sitter-Beltrami (AdSB) spacetime.

A novel theory of relativity was established by V.A. Fock which allowed for a varying speed of light \cite{6}. Subsequently, one of the authors of this work (S.N. Manida) constructed a linear fractional analogue of Lorentz transformation involving this variable speed of light \cite{7,8}. This transformation is called Fock–Lorentz transformation and the corresponding spacetime is called $R$-spacetime. It turned out that $R$-spacetime can be obtained as the $c \to \infty$ limit of AdSB spacetime and therefore has Ricci scalar curvature $S = -12/R^2$. A special singular coordinate transformation between the spacetime $R$ with curvature and the flat Minkowski spacetime was found. This was used in this work.

$R$-spacetime has a huge impact and plays a prominent role in cosmology, in relation to the varying speed of light models and rainbow gravity \cite{9–12}.

The Dirac equation is a relativistic generalization of quantum mechanics describing the motion of spin-half particles like the electron, proton, and other fundamental particles \cite{13–15}.
The Dirac equation has been widely discussed in different types of spacetime and the properties of their solutions have been studied [16,17].

The purpose of this work is to obtain and study a solution to the Dirac equation in $R$-spacetime. The choice of one or another variant of the spacetime model may depend on observations of the spectrum of the hydrogen atom at cosmological distances [3]. For a theoretical description of this spectrum, it is necessary to have a solution to the Dirac equation in the Coulomb field of a nucleus for a distant hydrogen atom. A free solution to the Dirac equation can be the first step in this direction.

In Section 2 of this paper we present the Dirac equation for a free particle in $R$-spacetime and construct its solution using a mapping from Minkowski spacetime. In Section 3, we discuss the results obtained in Section 2, such as the properties of a plane wave solution and its difference from a similar solution in Minkowski spacetime. In addition, we discuss the possibility of quantizing the Dirac or any other field in $R$-spacetime.

2. A free solution to the Dirac equation in $R$-spacetime

In [7,8] the $R$-spacetime metric was represented as:

$$ds^2 = \frac{R^2 t_0^2}{t^4} \left( 1 - \frac{\delta_{ij} x^i x^j}{R^2} \right) dt^2 + \frac{2t_0^2}{t^2} \delta_{ij} dx^i dx^j - \frac{t_0^2}{t^2} \delta_{ij} dx^i dx^j,$$

(1)

where $\delta_{ij} = diag\{1,1,1\}$ is the Kronecker delta and $t_0 = R/c$ is interpreted as the cosmological time or the age of the Universe.

The Dirac equation in curved spacetime was presented in [16,17] as:

$$i\hbar \gamma^a e^\mu_a (\partial_\mu - \frac{i}{4} \omega_\mu) \psi(x) = mc \psi(x),$$

(2)

where $\gamma^a$ are the Dirac matrices, $e^\mu_a$ are the tetrads and $\omega_\mu$ is the spin connection. We calculated these quantities in $R$-spacetime and showed that

$$e^\mu_a = \delta^\nu_a \left( \frac{t}{t_0} \delta_\nu^\mu + \frac{x_\nu x^\mu}{R^2} \right), \quad \omega_\mu = 0.$$

(3)

We put these equations into equation (2) and obtained the Dirac equation in $R$-spacetime as:

$$i\hbar \left\{ \frac{t}{R} \vec{\gamma} \cdot \vec{\nabla} + \frac{t}{R^2} \gamma^0 \left( t \frac{\partial}{\partial t} + \vec{r} \cdot \vec{\nabla} \right) \right\} \psi(x) = m \psi(x).$$

(4)

It can be seen that only two physical constants, $R$ and $\hbar$, play a role in equation (4). It is known [7,8] that metrics of Minkowski spacetime and $R$-spacetime are connected by transformation [8,18]:

$$t_M = t_0 - \frac{t_0^2}{t}, \quad \vec{r}_M = t_0 \vec{r}, \quad \vec{p}_M = \frac{t}{t_0} \vec{p},$$

(5)

where $t_M, \vec{r}_M,$ and $\vec{p}_M$ are the Minkowski spacetime coordinates and momentum. We used the above transformation rule to get a solution ignoring the time singularity at $t = 0$.

The singularity at $t = 0$ in mapping equation (5) leads to this mapping being irreversible in the entire time domain. Thus, if we follow the mapping in equation (5), we get spacetime which looks like Minkowski spacetime, but does not define the whole of spacetime globally. Indeed, we obtain a spacetime which is topologically different from real Minkowski spacetime. Thus, the free Dirac equation, which can be obtained by using the mapping in equation (5), is not well
defined on the \( t = 0 \) hypersurface. To make the solution physical, we need to restrict the time
interval in \( \mathbb{R} \)-spacetime from \( t \in ( -\infty, 0) \cup (0, \infty) \) to \( t \in (0, \infty) \) as a time-domain constraint.

Before we derive a solution to the Dirac equation in \( \mathbb{R} \)-spacetime, we recall the general solution
to the free Dirac equation in Minkowski spacetime:

\[
\psi(\vec{r}, t) = \rho e^{i\beta} u(p) \exp \left[ \frac{i}{\hbar} (E t - \vec{p} \cdot \vec{r}) \right],
\]

and dispersion relation:

\[
E = \pm c \sqrt{\vec{p}^2 + m^2 c^2}.
\]

Next, we apply coordinate transformation equation (5), but only in the region of \( \mathbb{R} \)-spacetime
where \( t > 0 \), and derive a solution to the Dirac equation:

\[
\psi(\vec{r}, t) = \rho e^{i\beta} u(t/t_0, p) \exp \left[ \frac{i}{\hbar} \left( E(t_0 - \frac{t_0^2}{t}) - \vec{p} \cdot \vec{r} \right) \right],
\]

where \( u(t_0, p) \) are the spinor components. Now we substitute equation (8) into equation (4) to
get the \( \mathbb{R} \)-spacetime dispersion relation between \( E \) and \( \vec{p} \) in the following form:

\[
E = \pm \frac{R}{t_0} \sqrt{\frac{t_0^2 \vec{p}^2}{t_0^2} + m^2 R^2 \frac{t_0}{t^2}}.
\]

Obviously, the dispersion relation in \( \mathbb{R} \)-spacetime differs from the dispersion relation in
Minkowski spacetime, since the transformation changes the time component of the plane wave
solution. In a small neighborhood of the hypersurface \( t = t_0 \), the equations (6)-(9) coincide.

3. Properties of a free solution in \( \mathbb{R} \)-spacetime and the quantization problem

In the previous section, we obtained a solution to the free Dirac equation in \( \mathbb{R} \)-spacetime. Now
we discuss the results. First, we once again note that the solution to equation (8) and equation
(9) in a small neighborhood of the hypersurface \( t = t_0 \) coincide with the solution of the free Dirac
equation in Minkowski spacetime. However, the geometric structure of \( \mathbb{R} \)-spacetime is nontrivial.
Physical properties force us to restrict the time domain because of a linear fractional time factor
that occurs in the solution. Due to the singularity at \( t = 0 \) (the initial time for \( \mathbb{R} \)-spacetime),
a plane wave solution quickly oscillates at \( t \to 0^+ \), specifically \( \omega(t) = d\phi(t)/dt \to \frac{mR^2}{\hbar t_0} \), where
\( \phi(t) \) is the phase of oscillations in equation (8).

On the other hand, at the limit \( t \to \infty \), the solution to equation (8) takes the form:

\[
\psi(\vec{r}, t) = \rho e^{i\beta} u(t/t_0, p) \exp \left[ \frac{i}{\hbar} \left( \frac{|\vec{p}| R}{t_0} t - \vec{p} \cdot \vec{r} \right) \right].
\]

We see that equation (10) coincides with the plane wave solution in Minkowski spacetime, but
for a massless particle.

It is known [18] that the Hamiltonian does not change under transformation equation (5).
Thus, the energy of a particle in two spacetime pictures, connected by transformation equation
(5), may be the same, but the physical properties of a particle in two types of spacetime will be
different.

\( \mathbb{R} \)-spacetime is the Lorentzian manifold in which we define the spinors. The properties
of the spinors in \( \mathbb{R} \)-spacetime do not depend on its geometric properties. The spinor structure
does not change when a time domain constraint is introduced. It is known that spinors live in
a flat tangent spacetime of a Lorentz manifold, which allows the spinor structure to preserve topological properties after a restriction of the region in $R$-spacetime is introduced [19].

We now turn to the possibility of quantization of this solution. Recall, first, that in the traditional formalism of quantum field theory in Minkowski spacetime, the creation of a particle occurs when $t \to -\infty$ [20]. In contrast, the time domain of $R$-spacetime is defined in the interval $(0, \infty)$. This suggests that, for quantization of fields in this spacetime, it is necessary to modify quantum field theory, assuming that the creation of a particle most likely occurs when $t \to 0^+$, which is a subject for future research.

4. Conclusion
In this paper, we obtained the free Dirac equation in $R$-spacetime and showed that its solution has a physical meaning in a limited time interval $t \in (0, \infty)$. In this case, the spinor structure of the solution does not depend on the imposed time-domain constraint. In the constructed solution, the frequency depends on time and oscillates quickly as $t \to 0^+$. The possibility of quantizing a free solution in $R$-spacetime is considered. It is proposed that the formalism of traditional quantum field theory be modified so that the production of particles takes place at $t \to 0^+$.

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