Convex rank 1 subsets of Euclidean Buildings  
(of type $A_2$) 

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Abstract 

For a Euclidean building $X$ of type $A_2$, we classify the 0-dimensional subbuildings $A$ of $\partial_T X$ that occur as the asymptotic boundary of closed convex subsets. In particular, we show that triviality of the holonomy of a triple (of points of $A$) is (essentially) sufficient. To prove this, we construct new convex subsets as the union of convex sets. \footnote{MSC2000: 53C20 \textit{keywords:} Euclidean buildings, Hadamard spaces}

1 Introduction 

Recently, convex subsets of symmetric spaces were studied in [KL06]: Bruce Kleiner and Bernhard Leeb classify which convex subsets of $\partial_T X$ arise as the asymptotic boundary of a convex subset $C$ of $X$, which is invariant under a group of isometries of the symmetric space $X$ acting cocompactly on $C$. They show (via a careful analysis of the Tits boundary $\partial_T C \subset \partial_T X$) that such a set $C$ is a symmetric subspace, a subset of a rank 1-symmetric subspace, or a product of sets of this kind.

We omit the group action and ask: Which (closed) $\pi$-convex subsets of $\partial_T X$ can occur as the asymptotic boundary of a convex subset of $X$?

If $\partial_T X$ is a spherical building, then $\pi$-convex subsets of $\partial_T X$ of dimension at most two have a center or are subbuildings ([BL05]).

In this article, we restrict our attention to 0-dimensional subbuildings. I.e. we consider subsets $A \subset \partial_T X$ such that every pair $\eta, \xi \in A$ satisfies $\angle_{\text{Tits}}(\eta, \xi) \geq \pi$. 

\footnote{MSC2000: 53C20 \textit{keywords:} Euclidean buildings, Hadamard spaces}
We call a subset \( C \subset X \) of a Hadamard space \( X \) a *convex rank 1*-subset if it is closed, convex, and its asymptotic boundary \( \partial_T C \) is a 0-dimensional subbuilding of \( \partial_T X \) (see also definition 7.1).

We will find that it is necessary for each triple of points \( \eta_1, \eta_2, \eta_3 \) of \( A \) to correspond to an ideal triangle: That is, it is necessary that there are lines \( l_{i,j} \subset X, i, j \in \{1, 2, 3\} \) joining the boundary points \( \eta_i, \eta_j \), which are pairwise strongly asymptotic (at the common endpoint). Formally speaking, it is necessary that the holonomy map of this triple has a fixed point. For more details on holonomy, see Section 3.2.

In the case where \( X \) is a Euclidean building of type \( \text{A}_2 \), we find that this holonomy condition is (essentially) sufficient; this is shown by constructing new convex sets as the union of convex subsets of \( X \). Our main theorem is:

**Theorem 1.** Let \( X \) be a Euclidean building of type \( \text{A}_2 \), and let \( A \subset \partial_T X \) be a finite 0-dimensional subbuilding of its boundary. Then there exists a convex rank 1-subset \( C \subset X \) such that \( \partial_T C \supset A \) if and only if each triple of points of \( A \) corresponds to an ideal triangle.

If \( A \) is infinite, the claim holds under an additional necessary assumption (\( A \) needs to be “good”, see Definition 7.3).

In the proof, we construct a convex set \( C \) which satisfies \( \partial_T C = \bar{A} \), where \( \bar{A} \) is the closure of \( A \subset \partial_\infty X \), the asymptotic boundary of \( X \) with the cone topology.

The idea to examine this question, and the technique of using holonomy, are due to Bruce Kleiner and Bernhard Leeb, who can classify the possible boundaries of convex rank 1-subsets of \( \mathbb{R}H^2 \times \mathbb{R}H^2 \).

This topic is also related to the work [HLS00] of Hummel, Lang and Schroeder: They show that in a CAT(-1)-space, the convex hull of finitely many closed convex sets lies in a (finite) tubular neighbor of this union.

We examine how to generalize this to Euclidean buildings of type \( \text{A}_2 \), where the starting blocks may be considered lines, or rather tripods, see Prop. 7.4 (observe that every subset of the asymptotic boundary of a CAT(-1)-space is a 0-dimensional subbuilding).

Theorem 1 forms a contrast to the following result:

**Theorem 2.** Let \( C \) be a convex rank 1-subset of \( M := \text{SL}(3, \mathbb{R})/\text{SO}(3, \mathbb{R}) \). Then we have \( \partial_T C \subset \partial_T \mathbb{H}^2 \) for a suitable isometric embedding (up to rescaling) of the hyperbolic plane \( \mathbb{H}^2 \hookrightarrow M \).

This more expected result is in line with the results of [KL06], and makes it hard to predict Theorem 1. A proof of Theorem 2 can be found in [Bals06b, ch. IV].
For 2-dimensional Euclidean buildings, the Coxeter complex $A_2$ is special:
In the other three 2-dimensional Coxeter complexes ($B_2, G_2$, and $A_1 \times A_1$),
the situation is different: In contrast to $A_2$ (where every holonomy map is
orientation preserving), the holonomy map of a pair of antipodal points is
orientation reversing in the other cases.
In these cases, existence of tripods is the essential question: If one could
show that tripods exist (as in 7.4), then the orientation-reversing property
of holonomy maps leads immediately to the conclusion that there exists an
isometrically embedded tree.

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In section 3, we introduce basic facts about Hadamard spaces and Euclidean buildings needed later on.

In Proposition 4.1, we show that for a connected, closed subset $C$ of a Hadamard space, convexity is a local property. This result feels like a version of the Hadamard-Cartan theorem; however, it is not an immediate consequence, because we do not know that $C$ is a geodesic space itself.

This proposition will be the tool to show that the sets we construct in the proof of our main theorem are convex.

In section 5, we examine under which conditions Busemann functions agree, and in which cases unions of horoballs are (locally) convex. The lemmas in this section are formulated generally for Euclidean buildings, and we hope that they will be useful in the study of convex rank 1-subsets of higher-dimensional buildings.

Section 6.3 contains geometric lemmas about buildings of type $A_2$: We exclude the existence of triangles $\Delta(a, b, c)$ with certain properties. To formulate it positively, we show that under certain circumstances, the starting direction $\bar{xe}$ (for $x \in \overline{ab}$) always points in “roughly the same direction”.

Starting with section 7, we move directly towards the proof of Theorem 1. From there on, $X$ always stands for a Euclidean building of type $A_2$.

In section 7, we examine necessary conditions for $A \subset \partial_T X$ to lie in the boundary of a convex rank 1-set. In particular, we show that for every triple of boundary points, a tripod has to exist (Proposition 7.4). We call $A \subset \partial_T X$ an S-set if it satisfies this condition.

If one knows (or expects) Theorem 2, one might also expect that for a building of type $A_2$, every convex rank 1-subset is essentially a tree. This turns out to be wrong. However, in section 7.4 we obtain a tree $T$ as a quotient of a subset of $X$ naturally associated to the S-set $A \subset \partial_T X$.

In section 8, we “thicken” tripods; i.e. we search for convex rank 1-subsets of $X$ containing a given tripod. This motivates the definition of the convex set $K$ in the section which follows, and introduces the techniques for proving convexity.

The last section 9 finally, presents the proof of Theorem 1. Given a good S-set $A \subset \partial_T X$, we consider the associated tree $\mathcal{T}$. For every point $[x] \in \mathcal{T}$, we define a closed convex subset $K_{[x]}$ of $X$, and we show that the (closure of the) union $K$ of these sets is convex. In the last step, we obtain a subset $\bar{C}$ of $\overline{K}$ which can easily be seen to be convex rank 1, and satisfy $\partial_T \bar{C} = \bar{A}$, where $\bar{A}$ is the closure of $A \subset \partial_\infty X$ in the cone topology.
Acknowledgements

In this article, I present the main result of my PhD thesis [Bals06b]:

Hence, first of all, I would like to express my gratitude to Bernhard Leeb for supervising me, posing the topic and turning me towards the right questions.

It is a pleasure to thank my discussion-mates Robert Kremser and Carlos Ramos Cuevas.

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3 Hadamard spaces

We will use the language of non-positively curved metric spaces, as developed in [Bal95].

Throughout, let $X$ be a Hadamard space, unless otherwise stated. We will use the terms Hadamard space and CAT(0)-space synonymously; i.e., we impose completeness on every CAT(0)-space. Note that a Hadamard space need not be locally compact.

Recall that $X$ has a boundary at infinity $\partial_\infty X$, which is given by equivalence classes of rays, where two (unit-speed) rays are equivalent if their distance is bounded.

In particular, we will use Busemann functions $b_\eta$ associated to an asymptotic boundary point $\eta \in \partial_\infty X$. A Busemann function measures (relative) distance from a point at infinity, and is determined up to an additive constant only. Busemann functions are convex (along any geodesic) and 1-Lipschitz.

Geodesics, rays, and geodesic segments are always assumed to be parametrized by unit speed (i.e. they are isometric embeddings).

For a line $l$ in $X$, there is the space $P_l$ of parallel lines. $P_l$ splits as a product $P_l \cong l \times CS(l)$, where $CS(l)$ is a Hadamard space again.

For points $x, \xi$ with $x \in X$, $\xi \in X \cup \partial_\infty X =: \bar{X}$, and $t \geq 0$ (if $\xi \in X$, let $t \leq d(x, \xi)$), we let $x_\xi(t)$ denote the point on the segment/ray $x_\xi$ at distance $t$ from $x$. When we denote a ray by $\ol{\eta}$, we order the points such that $o \in X$ and $\eta \in \partial_\infty X$.

In our notation, $B_r(o) := \{ x \in X \mid d(x, o) \leq r \}$, for $o \in X, r \geq 0$; i.e. balls in Hadamard spaces are always assumed closed.

Whenever $C$ is a closed convex subset of a Hadamard space $X$, then $\pi_C : X \to C$ denotes the nearest-point projection (see [BH99 II.2.4]).
3.1 Angles, spaces of directions, and Tits distance

Let \( o \in X \) be a point in a Hadamard space, and let \( \eta, \xi \in \partial \infty X \). Let \( c, c' \) be the rays \( \overrightarrow{o\eta}, \overrightarrow{o\xi} \). For points \( c(t), c'(t') \), one can consider the Euclidean comparison triangle corresponding to the points \( o, c(t), c'(t') \), i.e. the Euclidean triangle with side-lengths \( d(o, c(t)), d(c(t), c'(t')), d(c'(t'), o) \) (which is well-defined up to isometries of the Euclidean plane). The comparison angle between \( c(t) \) and \( c'(t') \) at \( o \) is the angle of the comparison triangle at the point corresponding to \( o \). It is denoted by \( \tilde{\angle} o(c(t), c'(t')) \).

We have the following monotonicity property:

\[
0 < t \leq s \quad \text{and} \quad 0 < t' \leq s' \implies \tilde{\angle} o(c(t), c'(t')) \leq \tilde{\angle} o(c(s), c'(s')).
\]

From this, one can deduce a notion of angle between geodesic segments and rays:

\[
\angle_o(\eta, \xi) := \lim_{t, t' \to 0} \tilde{\angle} o(c(t), c'(t')) \in [0, \pi],
\]

and an “angle at infinity”, the Tits angle between boundary points

\[
\angle(\eta, \xi) := \angle_{\text{Tits}}(\eta, \xi) := \lim_{t, t' \to \infty} \tilde{\angle} o(c(t), c'(t')) \in [0, \pi].
\]

It is easy to see that the Tits angle between \( \eta, \xi \) does not depend on the chosen basepoint \( o \). The length metric induced on \( \partial \infty X \) by \( \angle \) is called Tits distance \( Td \), and makes \( \partial \infty X \) a CAT(1)-space. If one wants to emphasize that the Tits distance and corresponding topology on \( \partial \infty X \) is considered, this space is sometimes called \( \partial T X \). We will use these expressions synonymously (and we usually consider the Tits topology). If the Tits angle (between \( \eta, \xi \)) is less than \( \pi \), there is a unique geodesic \( \overrightarrow{\eta\xi} \subset \partial \infty X \) connecting them.

Similarly, the space of directions \( \Sigma_o(X) \), i.e. the completion of the space of starting directions of geodesic segments initiating in \( o \) (modulo the equivalence of directions enclosing a zero angle), can be regarded as a CAT(1)-space. For \( o \in X, x \in \overline{X} \), we let \( \overrightarrow{ox} \in \Sigma_o(X) \) be the starting direction of the segment \( \overrightarrow{ox} \).

We call a subset \( C \subset B \) of a CAT(1)-space \( B \) convex if it is \( \pi \)-convex, i.e. if all pairs of points of distance less than \( \pi \) can be joined by a geodesic.

3.2 Strong asymptote classes and holonomy

Two rays \( \overrightarrow{o\eta}, \overrightarrow{x\eta} \) in a Hadamard space \( X \) are called strongly asymptotic if

\[
d_{\eta}(\overrightarrow{o\eta}, \overrightarrow{x\eta}) := \lim_{t \to \infty} d(\overrightarrow{o\eta(t)}, \overrightarrow{x\eta}) = 0.
\]
This defines an equivalence relation on the set of rays asymptotic to \( \eta \). The metric completion \( X_\eta \) of this set of equivalence classes is called the space of strong asymptote classes at \( \eta \). It is a Hadamard space again (see [Kar67], [Lee00, sect. 2.1.3]).

Now assume that \( X \) is a symmetric space or a Euclidean building, and consider two antipodal points \( \eta, \xi \in \partial TX \). It is well known that the parallel set of \((\eta, \xi)\), i.e. all the lines with asymptotic endpoints \( \eta, \xi \), represents all the strong asymptote classes at \( \eta \) and at \( \xi \).

This induces a natural isometry \( h_{\eta, \xi} : X_\eta \to X_\xi \).

Such a map (and composition of such maps) is called a holonomy map (see [Lee00] ch. 3).

### 3.3 Euclidean buildings

We will also need some Euclidean building geometry. For an introduction, we refer to [KL97] sect. 4. A brief introduction of the notation we use can be found in [KLM04] sect. 2.4. Note that in particular, a Euclidean building is a Hadamard space.

A 1-dimensional Euclidean building is called a tree.

In a Euclidean building, we call a geodesic segment regular, if all its interior points are regular.

The boundary at infinity of a Euclidean building \( X \) of rank \( n \) is a spherical building of dimension \( n - 1 \); we refer to [KL97] sect. 3] for an introduction.

We will use that a spherical building \( B \) is a spherical simplicial complex, where all the simplices are isometric to a spherical polytope \( \Delta \) (in particular, \( \Delta \) tesselates \( S^{n-1} \)), which is the spherical Weyl chamber of the building. Apartments (i.e. isometrically embedded copies \( S^{n-1} \)) intersect in (unions of) Weyl chambers. There is a natural map \( B \to \Delta \), and the image of a point is called its type.

### 4 Convexity is a local property in CAT(0)-spaces

Let \( \varepsilon > 0 \). A subset \( C \) of a CAT(0)-space \( X \) is called \( \varepsilon \)-locally convex, if for all \( x \in C \), the set \( B_{\varepsilon}(x) \cap C \) is convex. Note that a convex set is \( \varepsilon \)-locally convex for all \( \varepsilon > 0 \). Since an \( \varepsilon \)-locally convex set is locally path-connected, path-connectedness and connectedness are equivalent for \( \varepsilon \)-locally convex sets.

We show that if \( C \) is closed and connected, then one \( \varepsilon \) suffices to make sure that \( C \) is convex:
Proposition 4.1. Let $\varepsilon > 0$, and $X$ be a CAT(0)-space. Let $C \subset X$ be a connected, closed, $\varepsilon$-locally convex set. Then $C$ is convex.

Observe that this claim is similar in nature to the Hadamard-Cartan theorem, saying that a geodesic space which is simply connected and locally CAT(0), is actually globally CAT(0). In our case, we do not know whether $C$ is a geodesic space, so this proposition is not an immediate consequence of Hadamard-Cartan.

Proof. Since $C$ is $\varepsilon$-locally connected, for every point $x \in C$, the set of points of $C$ which can be joined to $x$ by a rectifiable curve is a path component of $C$, hence all of $C$. So every pair of points of $C$ can be joined by a rectifiable curve.

For $x, y \in C$, let $l(x, y)$ be the infimum of possible lengths of curves in $C$ joining $x$ and $y$.

We argue by induction on $n$ and show: if $l(x, y) < n\varepsilon$, then $\overline{xy} \subset C$ (and $l(x, y) = d(x, y)$). For $n = 1$, the claim is trivial.

Assume the claim to be true for $n$, and let $x, y$ be such that $l(x, y) \in [n\varepsilon, (n + 1)\varepsilon)$. Let $g_m : [0, l(x, y)] \to C$ be curves of constant speed, such that $l(g_m) < (n + 1)\varepsilon$ and $l(g_m) \searrow l(x, y)$.

Let $p_m := g_m(t)$ be such that $d(p_m, y) = \varepsilon$ (such a point exists; otherwise, the claim were trivial). We have $d(x, p_m) \leq l(x, p_m) \leq l(g_m|_{[0, l(x,y) - \varepsilon]}) < n\varepsilon$, so by induction hypothesis, we have $\overline{xp_m} \cup \overline{pmy} \subset C$, and we may assume that $g_m$ is a parametrization of these two segments. Let $q_m := \overline{pmx}(\varepsilon)$ (as above, $q_m$ has to exist in order for the claim to be non-trivial: if $q_m$ does not exist, then $\{x, y\} \subset B_\varepsilon(p_m)$, so $\overline{xy} \subset C$ by $\varepsilon$-local-convexity).

We examine the comparison angle $\tilde{Z}_{pm}(q_m, y)$: Since $C$ is $\varepsilon$-locally convex around $p_m$, we have $\overline{qmy} \subset C$. Therefore, the comparison angle has to be large when $m$ is large: $d(x, q_m) + d(q_m, y) \geq l(x, y) \geq l(g_m)$, implies $d(q_m, p_m) + d(p_m, y) - d(q_m, y) = 2\varepsilon - d(q_m, y) \to 0$.

Hence, $\tilde{Z}_{pm}(q_m, y) \to \pi$. Since $q_m \in \overline{xp_m}$, we have $\tilde{Z}_{pm}(q_m, y) \leq \tilde{Z}_{pm}(x, y)$. So for large $m$, the union $\overline{xp_m} \cup \overline{pmy} \subset C$ is almost a geodesic; in particular, we have $l(x, y) = d(x, y)$ and it is now immediate that the $g_m$ converge to $\overline{xy}$, finishing the proof. \qed

Remark 4.2. Let $C$ be a closed connected subset of $X$, and $\partial C$ be the (usual) boundary of $C$ as a topological subset of $X$. Assume that for some $\varepsilon > 0$ and every $x \in \partial C$ we have convexity of $B_\varepsilon(x) \cap C$. Then $C$ is $\varepsilon/2$-locally convex, hence convex.

Similarly, we have the following lemma:
**Lemma 4.3.** Let $C_1, C_2$ be two closed convex subsets of $X$ and $\varepsilon > 0$. If $C_1 \cup C_2$ is connected, and $B_\varepsilon(x) \cap (C_1 \cup C_2)$ is convex for every $x \in \partial C_1 \cap \partial C_2$, then $C_1 \cup C_2$ is convex.

**Proof.** First, we will show that for every $x \in C_1 \cap C_2$, we have convexity of $B_{\varepsilon/2}(x) \cap (C_1 \cup C_2)$. Then we show that this is sufficient.

So let $x \in C_1 \cap C_2$. We show directly that for $y, y' \in B_{\varepsilon/2}(x) \cap (C_1 \cup C_2)$, we have $yy' \subset C_1 \cup C_2$. Assume that this is not the case (for some $x, y, y'$ as above). Then by assumption, we have $\partial C_1 \cap \partial C_2 \cap B_{\varepsilon/2}(x) = \emptyset$.

Without loss of generality, we have $y \in C_1 \setminus C_2$ and $y' \in C_2 \setminus C_1$. Let $y_t := yy'(t)$, and let $z_t$ be the endpoint of the segment $yy' \cap C_1$. Let $T \geq 0$ be the minimal real number such that for the interval $[T, d(y, y')]$, we have $z_t \neq y_t$. For (the closure of) this interval, we have $z_t \in \partial C_1$.

Similarly, define $z'_t$, and obtain an interval $[0, T']$ such that in this interval, $z'_t \in \partial C_2$. By assumption, there is a point $y_{T'} \notin C_1 \cup C_2$, so the two intervals introduced above intersect.

Consider the functions $d(x, z_t)/d(x, y_t)$ and $d(x, z'_t)/d(x, y_t)$ on $[T, T']$. Both are continuous (observe that $x \notin yy'$), and by the intermediate value theorem, they are equal at some point. But then, we have found a point of $\partial C_1 \cap \partial C_2 \cap B_{\varepsilon/2}(x)$, in contradiction to the assumption that $B_{\varepsilon/2}(x) \cap (C_1 \cup C_2)$ is not convex.

Now, we want to show that for any $x \in C_1 \cap C_2$, the set $B_{\varepsilon/8}(x) \cap (C_1 \cup C_2)$ is convex.

Assume that this is not the case for some $x, y, y'$ as above. Note that by the discussion above, we have $d(x, C_1 \cap C_2) > 3\varepsilon/8$. Hence, we have

$$\min(d(y, C_1 \cap C_2), d(y', C_1 \cap C_2)) > \varepsilon/4,$$ and trivially $d(y, y') \leq \varepsilon/4$.

Let $p := \pi_{C_1 \cap C_2}(y)$. The comparison angle satisfies $\angle_p(y, y') < \pi/3$.

Let $z := \frac{yy'}{d(y, y')}$, $z' := \frac{yy'}{d(y, y')/8}$. By the remark about the comparison angle above, $d(z, z') < \varepsilon/8$ and $d(y, q) < d(y, p)$ for every $q \in zz'$. Note that $z \in C_1, z' \in C_2$. Convexity of $B_{\varepsilon/2}(p) \cap (C_1 \cup C_2)$ implies that $zz' \cap C_1 \cap C_2 \neq \emptyset$. This is a contradiction to the definition of $p$. $\square$

We will use the following reformulation quite often:

**Corollary 4.4.** Let $C_1, C_2, K$ be closed convex subsets of $X$, and let $\varepsilon > 0$. Assume that $(C_1 \cup C_2) \cap K$ is connected, and that for every $x \in \partial C_1 \cap \partial C_2 \cap K$, we have convexity of $B_\varepsilon(x) \cap K \cap (C_1 \cup C_2)$.

Then $(C_1 \cup C_2) \cap K$ is convex.

**Proof.** Since $K$ is a CAT(0)-space itself, this is just a redraft of the previous lemma. $\square$
Remarks on Busemann functions and horoballs in Euclidean buildings

In this section, we examine general conditions, under which the union of parts of horoballs is convex.

Setting: Let $X$ be a Euclidean building without flat de Rham factor, $\eta_1, \eta_2 \in \partial_T X$ be two boundary points of the same type (not necessarily regular), and $p \in X$. We normalize the corresponding Busemann functions $b_1, b_2$ such that $b_1(p) = b_2(p) = 0$.

Consider $\eta_1$ as a point in the (spherical) model apartment $S$. Let $\alpha > 0$ be the maximal angle such that $\angle(\eta_1, \xi) \leq \alpha$ implies that $\eta_1$ and $\xi$ lie in a common Weyl chamber of the Coxeter complex $(S, W)$. Since $X$ has no flat de Rham factor, we have $\alpha \leq \pi/2$.

For the first two lemmas, assume there exists a ray $p\xi$ such that $\angle p(\eta_1, \xi) = \angle p(\eta_2, \xi) = \pi$.

Since the set of singular points of the Coxeter complex $(S, W)$ is invariant under the map sending every point to its antipode, we have: Whenever $\angle(\xi, \xi') \leq \alpha$ for some $\xi' \in \partial_T X$, then the points $\xi$ and $\xi'$ lie in a common Weyl chamber of $\partial_T X$.

Note that this implies in particular: If $\xi' \neq \xi$ has the same type as $\xi$, then $\angle(\xi, \xi') \geq 2 \cdot \alpha$.

Lemma 5.1. Let $p\xi'$ be a ray with $\angle p(\xi, \xi') \leq \alpha$. Then

$$b_1|_{p\xi'} = b_2|_{p\xi'}.$$

Proof. Note that any Busemann function $b_i$ is piecewise linear and convex along any ray $p\xi'$, the slope in $x \in p\xi'$ being $-\cos(\angle_x(\eta, \xi'))$ (this is well known; it follows from [KL97, 4.1.2]).

Now the possible values of $\angle_x(\eta_i, \xi')$ form a finite set (determined by the types of $\eta_i, \xi'$), and if $x\xi^h$ is of the same type as $x\xi^i$, then $\angle_x(\hat{\xi}, \xi'') \geq \angle_x(\xi, \xi')$ for every antipode $\hat{\xi}$ of $\eta_1$ (if $x\xi^h$ does not lie in a common Weyl chamber with $x\xi^i$, then $\angle_x(\xi, \xi'') \geq \alpha$).

So $\angle p(\eta_1, \xi') = \angle p(\eta_2, \xi') = \pi - \angle p(\xi, \xi') \in [\pi - \alpha, \pi]$ is maximal. Since the slope of $b_i$ is increasing along $p\xi'$, it is constant, and the claim follows from our assumption $b_1(p) = b_2(p)$. \qed

We continue working in the setting introduced above.
Lemma 5.2. Let $R > 0$, and $D \geq \max(R, R/\tan \alpha)$. Then

$$B_R(p\xi) \cap \{b_1 \leq D\} = B_R(p\xi) \cap \{b_2 \leq D\}.$$  

Proof. Let $x \in B_R(p\xi)$. If $\angle_p(x, \xi) \leq \alpha$, then $b_1(x) = b_2(x)$ by the previous lemma, so $x$ is either contained in both sets, or in none of them.

So it suffices to show: if $\angle_p(x, \xi) > \alpha$, then $b_1(x) \leq D$ for both $i$.

Let $x' := \pi_{\overrightarrow{xy}}(x)$. We may assume $x' \neq p$ because of $D \geq R$. Consider a point $y \in \overrightarrow{xy}'$ such that $\angle_p(y, \xi) = \alpha$. Then $b_1(y) = b_2(y) = d(p, y) \cdot \cos \alpha$, and $d(y, p\xi) = d(y, x') \geq d(p, y) \cdot \sin \alpha$. We have

$$b_1(x) \leq b_1(y) + (R - d(y, p\xi)) \leq R \cdot \left( 1 + \frac{d(p, y)}{R} \cdot (\cos \alpha - \sin \alpha) \right)$$

$$= R \cdot \left( 1 + \frac{d(p, y) \cdot \sin \alpha}{R} \cdot (\cotan \alpha - 1) \right)$$

If $\alpha \geq \pi/4$, we have $\cotan \alpha \leq 1$ so the inequality above implies $b_1(x) \leq R \leq D$.

If $\alpha \leq \pi/4$, we have $\cotan \alpha \geq 1$, and we use $d(p, y) \leq R/\sin \alpha$: Then the inequality above becomes $b_1(x) \leq R \cdot (1 + \cotan \alpha - 1) \leq D$. \hfill $\Box$

From now on, we do not require the existence of a common antipode $p\xi$ of the two $\overrightarrow{\Sigma_p}$ in $\Sigma_p(X)$ anymore.

Lemma 5.3. Let $D > R \cdot \cos \alpha > 0$. Then the set

$$C := B_R(p) \cap (\{b_1 \leq D\} \cup \{b_2 \leq D\})$$

is convex.

Proof. We want to apply Corollary 4.4. It suffices to find an $\varepsilon > 0$ such that for any point $x \in C$ with $b_i(x) = b_2(x) = D$, we have convexity of $B_x(x) \cap C$.

Let us first choose the $\varepsilon$, depending only on the type of $D, R$, and $\alpha$ (but not on a specific point $x \in C$):

Let $\delta := (D - R \cdot \cos \alpha)/2$, and choose $\hat{\alpha} < \alpha$ such that $R \cdot \cos \hat{\alpha} \leq D - \delta$. Now consider a Euclidean triangle $A, B, C$ with $d(A, B) \geq \delta$ and $\angle(A, B, C) \geq \alpha - \hat{\alpha}$. Let $\varepsilon'$ be such that $d(B, C) \geq \max(\varepsilon', \varepsilon'/\tan \alpha)$. Set $\varepsilon := \min(\delta, \varepsilon')$.

Now let $x \in C$ be a point with $b_1(x) = b_2(x) = D$.

There is a finite subdivision $(x_0 = x, x_1, \ldots, x_m = p)$ of $\overrightarrow{m}$ such that $\text{Conv}(x_j, x_{j+1}, \eta_i)^2$ is isometric to a flat half-strip (for $0 \leq j < m, 1 \leq i \leq 2$) (see [KŁ97 4.1.2]).

\footnote{Throughout this paper, Conv always denotes the convex hull of its arguments. We use it with a variety of different kinds of arguments, but no confusion should arise.}
Now $D/R > \cos \alpha$ implies that both $b_i$ have maximal slope on the segment $\overline{x_1x}$.

In fact, since $R \cdot \cos \alpha < D - \delta$, we have $d(x_1, x) > \delta$ (recall that if the slope of $b_i$ is not maximal, then it is at most $\cos \alpha$).

For a point $y \in \overline{x_1x} \setminus \{x_1\}$, we have $\angle y(x_1, \eta_i) \leq \hat{\alpha}$ (since the slope of $b_i$ along $\overline{x_1x}$ has to be larger than $\cos \hat{\alpha}$), hence $\angle y(\eta_1, \eta_2) < 2\alpha$, and $\overrightarrow{y\eta_1} = -\overrightarrow{y\eta_2}$.

Let $x' \in \overline{\eta_1} \cap \overline{\eta_2}$ be such that $\overrightarrow{x'\eta_1} \neq \overrightarrow{x'\eta_2}$. We have $\angle x_1(x', \eta_i) \leq \pi - \alpha$ (otherwise, we would obtain $\overrightarrow{x'\eta_1} = \overrightarrow{x'\eta_2}$ as above), and $\angle x_1(x, \eta_i) \geq \pi - \hat{\alpha}$. This implies that we have $d(x', x) \geq \epsilon$ by construction.

Of course, $b_1(x') = b_2(x') = b_i(x) - d(x, x') = D - d(x, x')$. By Lemma 5.2 we have

$$B_\epsilon(x) \cap \{b_1 \leq D\} = B_\epsilon(x) \cap \{b_2 \leq D\}.$$

Hence, $B_\epsilon(x) \cap C$ is convex, and Corollary 4.4 applies.

\section{Geometry of Euclidean buildings of type $A_2$}

In the remainder of this paper, we will work with Euclidean buildings of type $A_2$. Their Tits boundaries are spherical buildings of type $A_2$.

The (spherical) Coxeter complex $A_2$ is the unit circle with the group of symmetries of an isosceles triangle acting on it (see Figure 2). A (discrete) Euclidean Coxeter complex of type $A_2$ is the Euclidean plane, tesselated by isosceles triangles (see Figure 1).

The most important example of a Euclidean building of type $A_2$ is the building associated to $\text{SL}(3, \mathbb{Q}_p)$; its geometry is described in detail in [Kre06].

\subsection{The spherical building structure of $\partial_T X$}

Let $X$ be a Euclidean building of type $A_2$. Then the boundary at infinity $\partial_T X$ carries the structure of a spherical building of type $A_2$. Similarly, the space of directions $\Sigma_x(X)$ for any $x \in X$ carries such a structure as well.

Every apartment in such a spherical building $B$ consists of six Weyl chambers of length $\pi/3$. The vertices (the singular points of $B$, the ends of the Weyl chambers) have two different types (see Figure 2).

\subsection{Holonomy in spaces modeled on $A_2$}

Let $\eta \in \partial_T X$ be a regular boundary point. By the remarks above, $X_\eta \sim \mathbb{R}$. Since every apartment asymptotic to $\eta$ represents all strong asymptote
classes, we get an orientation on $X_\eta$ (induced from a choice of orientation on Weyl chambers, determined by the two types of boundary points). Then every holonomy map $h_{\eta,\xi}$ is orientation preserving, and so is the composition

$$h_{\eta_1,\eta_2,\eta_3} := h_{\eta_3,\eta_1} \circ h_{\eta_2,\eta_3} \circ h_{\eta_1,\eta_2} : X_\eta \to X_\eta$$

for any triple $\eta_1, \eta_2, \eta_3 \in \partial T$ of pairwise antipodal regular boundary points. Such a holonomy map, as an orientation preserving isometry of $\mathbb{R}$, is just a translation. We will call the translation length of such a triple its **shift**.

Observe that for the other three 2-dimensional Coxeter complexes, holonomy maps are orientation-reversing. This major difference makes it hard to predict a general (2-dimensional) version of Theorem 1.

### 6.3 Geometric lemmas for buildings of type $A_2$

In this section, we collect some geometric properties of Euclidean buildings of type $A_2$ that will be useful later.

**Lemma 6.1.** Let $X$ be a building of type $A_2$, $p \in X$ and $\eta_1, \eta_2, \eta_3$ be three singular boundary points of the same type, such that the $\overrightarrow{p\eta_i}$ span a flat in $\Sigma_p(X)$. Normalize such that $b_i(p) = 0$. Let $q \in X$ be such that $\angle(p, \eta_i) = 2\pi/3$ for all $i$ and $R := d(q, p) > 0$ (so $b_i(q) = R/2$).

Let $C := \{ x \mid \text{at least two } b_i(x) \leq R/2 \}$. Then $K := C \cap B_{R,\sqrt{3}/2}(q)$ is convex. More specifically, there exist $i, j$ such that $K = B_{R,\sqrt{3}/2}(q) \cap \{ b_i \leq R/2 \} \cap \{ b_j \leq R/2 \}$.

**Proof.** Pick an $x \in B_{R,\sqrt{3}/2}(q)$, and observe that $\angle(p, x, q) \leq \angle(p, x, q) \leq \pi/3$ (by triangle comparison).

We distinguish two cases: The first case is that the initial directions of $\overrightarrow{p\eta_i} \subset \Sigma_p X$ are all distinct. Then, there are two $i$ such that $\angle(x, \eta_i) \geq 2\pi/3$. So $x \in C$ if and only if all three Busemann functions are at most $R/2$. (In this case, we can choose $i, j$ arbitrarily.)

Otherwise, (exactly) two of the above-mentioned initial directions agree (without loss of generality, those corresponding to 1, 2; these correspond to the $i, j$ in the claim).

We claim that $K = B_{R,\sqrt{3}/2}(q) \cap \{ b_1 \leq R/2 \} \cap \{ b_2 \leq R/2 \}$.

Indeed, if $b_3(x) \leq R/2$ for $x \in K$, then either $\angle(x, \eta_3) \geq 2\pi/3$ and $b_3(x) \leq \min(b_1(x), b_2(x))$, or $\angle(x, \eta_1) = \angle(x, \eta_2) > 2\pi/3$. In the last case, we have $b_1(x) = b_2(x)$, hence the claim follows.

For the next two lemmas, we need a setting which will be introduced in more detail later:
Let $\eta_1, \eta_2$ be antipodal centers of Weyl chambers in the boundary of a Euclidean building $X$ of type $A_2$, and let $F_{1,2}$ be the flat containing $\eta_1, \eta_2$ in its boundary. Let $\nu_1, \eta_{1,2}, \nu_2, \mu_2, \xi_{1,2}, \mu_1$ be the singular points in $\partial_T F_{1,2}$ as in Figure 2. Figure 2 shows a part of the flat $F_{1,2}$, with the boundary being aligned as in Figure 2 (with $i = 1, j = 2$).

**Lemma 6.2.** Let $p', p'' \in F_{1,2}$ such that $b_1(p') \leq b_1(p'')$.

Let $0 \leq \hat{\alpha} < \pi/3$. Then there is no $x \in X$ with

$$\angle_{p'}(x, \nu_1) < \pi/3 + \hat{\alpha}, \quad \angle_{p'}(x, \nu_2) < \pi/3 + \hat{\alpha},$$

but $\angle_{p'}(x, \eta_{1,2}) > \hat{\alpha}$, $\angle_{p'}(x, \eta_{1,2}) > \hat{\alpha}$.

**Proof.** If $p' = p''$, there is nothing to show.

Without loss of generality, we assume $b_{1,2}(p'') \leq b_{1,2}(p')$.

Let $\beta := \angle_{p'}(p', \xi_{1,2}) = \angle_{p'}(p'', \eta_{1,2}) \leq \pi/2$.

If $\pi/3 \leq \beta$ ($\leq \pi/2$), one obtains a contradiction to the sum of angles in a triangle: Indeed, we have

$$\angle_{p'}(p', x) > \min(\pi + \hat{\alpha} - \beta, \pi/3 + \beta - \hat{\alpha})$$

and

$$\angle_{p'}(p'', x) > \min(-\beta + \hat{\alpha}, \frac{4\pi}{3} - \beta - \hat{\alpha}).$$
We see that if \( \beta \geq \pi/3 \), the sum of these two angles is greater than \( \pi \). Therefore, we have \( \beta < \pi/3 \). Then
\[
\angle_{p'}(p', x) > \pi/3 + \beta - \hat{\alpha}, \tag{1}
\]
and
\[
\angle_{p'}(p'', x) > \hat{\alpha} + \angle_{p'}(p'', \eta_{1,2}) = \hat{\alpha} + \beta \tag{2}
\]

Let \((p_0 = p'', p_1, \ldots, p_n = p')\) be a finite subdivision of \(p''p'\) such that each triangle \(\Delta(p_i, p_{i+1}, x)\) is flat.

Let \(i_0 > 0\) be such that the initial directions of \(\overrightarrow{p''p_1x}\) and \(\overrightarrow{p''p_0\nu_2}\) agree (\(\ast\)).

We will show by induction that every \(1 \leq i_0 \leq n\) has this property, and obtain a contradiction for \(i_0 = n\).

**Base case:** \(i_0 = 1\) has Property (\(\ast\)).

If this is not the case, then the starting direction of \(\overrightarrow{p''p_1x}\) has to be different from the starting direction of \(\overrightarrow{p''p_0\nu_2}\) (since the triangle \(\Delta(p'', p_1, x)\) is flat). If this is the case, we have \(\angle_{p'}(p', x) = \angle_{p'}(p_1, x) > \pi - (\hat{\alpha} + \beta)\). This is a contradiction to (2).

**Claim:** If \(i_0 < n\) has property (\(\ast\)), then the initial directions of \(\overrightarrow{p''p_0\nu_2}\) and \(\overrightarrow{p_0p''\nu_2}\) agree as well.

Observe that \(\angle_{p_0}(p'', x) < 2\pi/3 + \hat{\alpha} - \beta\) (by (1)). Assume that the claim is false. Then
\[
\angle_{p_0}(p', x) > \pi - ((\pi/3 + \hat{\alpha}) - (\pi/3 - \beta)) = \pi - \hat{\alpha} + \beta.
\]
Together with (2), this is a contradiction.

Now this claim implies (\(\ast\)) for \(i_0 + 1\) (by the same argument as in the base case, with \(p_0\) taking the place of \(p''\)), and it follows by induction that property (\(\ast\)) holds for all \(1 \leq i_0 \leq n\).

For \(i_0 = n\), we get \(\angle_{p'}(p'', x) > \pi - \hat{\alpha} + (\pi/3 - \beta) = 4\pi/3 - (\hat{\alpha} + \beta)\).

If this is less than \(\pi\), we continue our calculation:
\[
\angle_{p'}(p'', x) + \angle_{p'}(p', x) > 5\pi/3 - 2\hat{\alpha}.
\]
This is a contradiction, since \(\hat{\alpha} < \pi/3\). \(\square\)
Remark 6.3. In the statement of the lemma, we can replace the directions \( \overrightarrow{p'\nu_1} \) and \( \overrightarrow{p''\nu_2} \) by any other directions antipodal to \( \overrightarrow{p\mu_2} \) and \( \overrightarrow{p''\mu_1} \). Of course, we also have to adjust the assumptions after the “but”. We will usually take care of this by showing that \( \angle_{\nu'}(\xi_{1,2}, x) < \pi - \hat{\alpha} \) and \( \angle_{\nu'}(\xi_{1,2}, x) < \pi - \hat{\alpha} \).

Lemma 6.4. Let \( p', p'' \in F_{1,2} \) such that \( b_1(p') \leq b_1(p'') \).
Assume that \( \angle_{\nu'}(p', \eta_{1,2}) \geq \pi/3 \). Let \( 0 \leq \hat{\alpha} \leq \pi/6 \). Then there is no \( x \in X \) with

\[
\angle_{\nu'}(x, \mu_1) < \pi/3 + \hat{\alpha}, \quad \angle_{\nu'}(x, \nu_2) < \pi/3 + \hat{\alpha}
\]

but \( \angle_{\nu'}(x, \xi_{1,2}) > \hat{\alpha} \).

Proof. As above, we may assume \( p' \neq p'' \).

We distinguish two cases: The first case is \( \angle_{\nu'}(p', \xi_{1,2}) =: \beta \in [\pi/3, 2\pi/3] \).
In this case, we have

\[
\angle_{\nu'}(p', x) > \min(\pi - \beta + \hat{\alpha}, \beta + \pi/3 - \hat{\alpha}) \geq \pi/2
\]

\[
\angle_{\nu'}(p''', x) > \min(\pi - \beta + \hat{\alpha}, \beta + \pi/3 - \hat{\alpha}) \geq \pi/2.
\]

Adding these angles, the only case in which we do not get a contradiction to the sum of angles in a triangle is, if \( \overrightarrow{p''x} \in \overrightarrow{p''\eta_{1,2}p''\nu_2} \), \( \overrightarrow{p'x} \in \overrightarrow{p'\xi_{1,2}p'\mu_1} \) and \( \beta > \pi/2 + \hat{\alpha} \).

Now this case can be finished as in the lemma above: The deciding inequalities are

\[
\angle_{\nu_{0\alpha}}(p'', x) < \beta - \hat{\alpha}, \quad \angle_{\nu_{0\alpha}}(p', x) < \beta - \hat{\alpha},
\]

\[
\angle_{\nu_{0\alpha}}(p'', \nu_2) = \beta - \pi/3.
\]

The second case is \( \beta < \pi/3 \). Now we have \( \angle_{\nu'}(p'', x) > \beta + (\pi/3 - \hat{\alpha}) \) and \( \angle_{\nu'}(p', x) > (\pi/3 - \hat{\alpha}) \). Again, we have to have \( \overrightarrow{p''x} \in \overrightarrow{p''p''p''\nu_2} \) and \( \overrightarrow{p'x} \in \overrightarrow{p'p'p'\mu_1} \).

To be able to “switch sides”, we would need a \( p_{\nu} \) with \( \angle_{\nu_{0\beta}}(p'', x) \) at least \( \pi/3 - \beta + 2\pi/3 - \hat{\alpha} = \pi - \beta - \hat{\alpha} \), which is impossible (because \( \hat{\alpha} \leq \pi/6 \)).

Remark 6.5. Again, this lemma remains true if we replace \( \overrightarrow{p''\nu_2} \) and/or \( \overrightarrow{p'\mu_1} \) by other directions antipodal to \( \overrightarrow{p''\mu_1}, \overrightarrow{p'\nu_2} \). (and again, we also have to adjust the assumptions after the “but”.)
7 Necessary conditions: S-sets

Now we shift gears, and turn directly to the proof of Theorem 1. We start by examining necessary conditions and obtaining more and more structure on the sets satisfying the (obvious) necessary conditions.

Let us first state the precise definition of a convex rank 1-set:

**Definition 7.1.** A subset $C \subset X$ of a Hadamard space $X$ is called convex rank 1, if it is closed, convex, has at least 3 boundary points at infinity and satisfies: $\partial_T C$ is a 0-dimensional building (i.e.: for all $\eta, \xi \in \partial_T C$ with $\eta \neq \xi$, we have $\angle_T(\eta, \xi) \geq \pi$).

Observe that the restriction $|\partial_T C| \geq 3$ is not serious: Every pair of antipodal points in $\partial_T X$ (for $X$ a symmetric space or a Euclidean building) can be joined by a geodesic.

From now on, we focus on a special class of buildings: In the remainder of this article, $X$ will always stand for a building of type $A_2$.

In this section we examine necessary conditions for points $\eta_i \in \partial_T X$ to be in the boundary of a convex rank 1 set. The most important necessary condition is that there has to be a tripod for every triple of asymptotic boundary points (Proposition 7.4). We also examine the structure of the set of singular points of these tripods, and we will obtain a metric tree which is closely related.

**Lemma 7.2.** If there are at least three points $\eta_i$, then to be pairwise antipodal, it is necessary that each $\eta_i$ is the center of a Weyl chamber.

*Proof.* In the Coxeter complex $A_2$, the centers of Weyl chambers are the only points which have the following property: An antipode has the same type. This property is necessary, since the points $\eta_i$ have to be pairwise antipodal. $\square$

There is another obvious necessary condition: Let $A = \partial_T C$ be the asymptotic boundary of a convex rank 1 set. Consider a triple of boundary points. Then the corresponding holonomy map has to have a fixed point (see Section 3.2); otherwise, every convex set containing the given triple in its boundary contains a half-plane, and hence does not have rank 1.

Since our boundary points are regular, the holonomy map of a triple is an isometry of $\mathbb{R}$ to itself. This map is also orientation preserving, so it is just a translation, determined by its translation length, which we call its shift.

So the necessary condition is: For each triple of points of $A$, their shift has to be 0 (i.e. the holonomy map has to be the identity map).
7 Necessary conditions: $S$-sets

Figure 2: The apartment $\partial_T F_{i,j} \subset \partial_T X$ with its singular points $\nu_i, \eta_{i,j}, \nu_j, \mu_j, \xi_{i,j}, \mu_i$ and the two regular points $\eta_i, \eta_j$.

7.1 Notation

Definition 7.3. A subset $A \subset \partial_T X$ with $|A| \geq 3$ is an $S$-set, if the points of $A$ are pairwise antipodal (i.e., $A$ is a 0-dimensional subbuilding), and for each triple of points of $A$, the shift is 0.

In what follows, $A = \{\eta_i \mid i \in I\}$ will always be an $S$-set (see also the definition of a good $S$-set, 7.5).

A tripod is a metric tree with three asymptotic boundary points. It may also be viewed as the Euclidean cone over a set of cardinality three. A tripod in $X$ is determined by a (singular) point $p$ and three boundary points $\xi, \nu, \mu$. This data determines a tripod $(p, \xi, \nu, \mu) = \text{Conv}(p, \xi, \nu, \mu)$ if and only if $\angle_p(\xi, \nu) = \angle_p(\nu, \mu) = \angle_p(\mu, \xi) = \pi$.

In our setting, a tripodal point $p_{i',j',k'}$ (for three distinct $i', j', k' \in I$) is a point such that $(p_{i',j',k'}, \eta_{i'}, \eta_{j'}, \eta_{k'})$ determines a tripod. When a tripodal point is given, then $T_{i,j,k}$ denotes the corresponding tripod. If the tripodal point is to be emphasized, we also say that $(p_{i',j',k'}, \eta_{i'}, \eta_{j'}, \eta_{k'}) \in X \times (\partial_T X)^3$ is a tripod.

For $i \in I$, let $\nu_i, \mu_i$ be the endpoints of the Weyl chamber spanned by $\eta_i$, such that all the $\nu_i$ have the same type.

For a pair $i, j \in I$, let $\eta_{i,j}$ be the center of the geodesic $\overline{\nu_i \nu_j} \subset \partial_T X$; similarly define $\xi_{i,j}$ (see Figure 2). Let $F_{i,j}$ denote the unique flat in $X$ such that $\eta_i, \eta_j \in \partial_T F_{i,j}$; so the singular vertices of $\partial_T F_{i,j}$ are $\nu_i, \eta_{i,j}, \nu_j, \mu_j, \xi_{i,j}, \mu_i$.

For a triple $i, j, k \in I$, let $l_{i,j,k} := F_{i,j} \cap F_{j,k} \cap F_{i,k}$. By definition of tripods, $l_{i,j,k}$ is precisely the set of tripodal points for $(\eta_i, \eta_j, \eta_k)$. It follows from the
7.2 Existence of tripods

Proposition 7.4. Let $X$ be a Euclidean building of type $A_2$, and let $\eta_1, \eta_2, \eta_3 \in \partial_T X$ be three pairwise antipodal points. If their shift is 0, then there exists a tripod $(p, \eta_1, \eta_2, \eta_3)$.

Note that the proposition can also be phrased as follows: Every $S$-set of cardinality 3 is the asymptotic boundary of a convex rank 1-set, and this rank 1-set can be chosen to be a tripod.

Proof. Observe that $F_{1,2} \cap F_{1,3} =: S$ is non-empty, closed and convex (by [KL97 4.6.3]). The Busemann function $b_1$ is bounded above on $S$, since otherwise we have $\angle_T(\eta_2, \eta_3) < \pi$ (note that $\partial S$ is a polygonal curve consisting of at most three line segments/rays/lines, see Figure 3).

Let $p$ be an extremal point for $b_1|_S$. We claim that $(p, \eta_1, \eta_2, \eta_3)$ is a tripod.

Assume that this is not the case. Then $\angle_p(\eta_2, \eta_3) \geq \pi/3$ (since both directions are the centers of a Weyl chamber, the smallest non-zero value for their angle is $\pi/3$).
Since the shift is zero, we obtain points \( p', p'' \) in \( \overrightarrow{mn_2}, \overrightarrow{mn_3} \) resp. such that \( \overrightarrow{p\eta_2} \cup \overrightarrow{p'p''} \cup \overrightarrow{p'\eta_3} \) is a geodesic line.

Let us choose \( p', p'' \) (as \( p \)) such that \( \angle_{p'}(\eta_1, \eta_3) \neq 0 \neq \angle_{p''}(\eta_1, \eta_2) \). Then each of these angles is at least \( \pi/3 \), so \( \Delta(p, p', p'') \) is a flat isosceles triangle (by triangle rigidity in CAT(0)-spaces, see [BH99, II.2.9]).

Let \( \nu \) be the midpoint of the geodesic \( pp'pp'' \subset \Sigma_p(X) \). Then \( \angle_p(\eta_2, \nu) = \angle_p(\eta_3, \nu) = \pi/6 \); Observe that \( \mu \in \Sigma_p(X), \angle_p(\eta_j, \mu) \leq \pi/6 \) implies \( \mu \in \Sigma_p(F_{1,j}) \) for \( j \in \{2, 3\} \) (because \( \mu \) and \( \overrightarrow{p\eta_j} \) lie in a common Weyl chamber of \( \Sigma_p(X) \)). So either

\[
\nu = \overrightarrow{p\nu_2} = \overrightarrow{p\nu_3} \text{ or } \nu = \overrightarrow{p\mu_2} = \overrightarrow{p\mu_3}.
\]

In both cases, we have \( \overrightarrow{p\nu} \in \Sigma_p(F_{1,2}) \cap \Sigma_p(F_{1,3}) = \Sigma_p(F_{1,2} \cap F_{1,3}) \). Since \( \angle_p(\eta_1, \nu) = 2\pi/3 \), this is a contradiction to the construction of \( p \). \( \square \)

The proof also shows that a convex rank 1-subset of \( X \) has to contain a tripod for every triple of boundary points (because for a strong asymptote class which does not correspond to a tripod, we obtain a flat isosceles triangle in the convex hull, and its center is a tripod point).

Hence, the following condition is also necessary for an S-set \( A \) to be in the asymptotic boundary of a convex rank 1-set:

**Definition 7.5.** An S-set \( A \) is called **good**, if it satisfies the following condition: We can choose tripodal points \( p_{i,j,k} \) (for every triple \( i,j,k \in I \)) such that for all \( i' \in I \), the convex hull of (all) the strong asymptote classes \( \overrightarrow{p_{i',j,k}\eta_{i'}} \) is bounded.

**Example 7.6.** Let us give an example of a 4-point S-set which does not lie in the boundary of an embedded tree:

We start with two antipodal centers of Weyl chambers, \( \eta_1, \eta_2 \), and pick a singular vertex \( p \) in \( F_{1,2} \). Choose a ray \( \overrightarrow{p\eta_3} \), such that \( (p, \eta_1, \eta_2, \eta_3) \) is a tripod. Let us choose \( \eta_3 \) such that the intersection \( F_{1,2} \cap F_{2,3} \) is a flat sector (this corresponds to the left-most set drawn in Figure 3).\(^3\)

Now pick an inner point \( p' \) of \( F_{1,2} \cap F_{2,3} \) satisfying \( b_{1,2}(p') \neq b_{1,2}(p) \). As above, pick a ray \( \overrightarrow{p'\eta_4} \), such that \( (p', \eta_1, \eta_2, \eta_4) \) is a tripod and \( F_{1,2} \cap F_{1,4} \) is a flat sector. By construction, we have

\[
\overrightarrow{p'\eta_1} = \overrightarrow{p'\eta_3},
\]

so we also have a tripod \( (p', \eta_3, \eta_2, \eta_4) \). Similarly, our construction implies that \( p \) lies in the interior of \( F_{1,2} \cap F_{1,4} \), so we also have the tripod \( (p, \eta_1, \eta_3, \eta_4) \).

\(^3\)This is possible in “most” Euclidean buildings of type \( A_2 \); pick the building associated to \( SL(3, \mathbb{Q}_5) \) for example.
Our choices imply that \( p, p' \) are the unique tripodal points (at least when considered as \( p_{1,2,3}, p_{1,2,4} \) resp.), so \( b_{1,2}(p) \neq b_{1,2}(p') \) implies that there is no embedded tree with the given four boundary points.

A similar situation is depicted in Figure 4 in the next section, we are going to show that the general situation is always similar to the one described here.

Applying the construction above to obtain an S-set with infinitely many boundary points, we see that an S-set need not be good.

### 7.3 S-sets with 4 points: relative position of their tripodal points

In this section, we examine S-sets \( A \) of cardinality 4: We show that we can always do with at most 2 tripodal points: If there is no 4-pod (i.e. a Euclidean cone over \( A \)) embedded in \( X \), then we construct two points, each of which is tripodal for two triples of points of \( A \).

All of the Lemmas in this section are formulated such that the assumptions rule out existence of a 4-pod; only Proposition 7.12 is formulated to make sense even in this case.

We also discuss the possible choices for the tripodal points in question, and the relative position of the two (sets of) points to each other. These are technical results needed in the sequel.

**Lemma 7.7.** Let \( \{\eta_1, \eta_2, \eta_3, \eta_4\} \subset \partial_T X \) be an S-set of cardinality 4. Assume there are tripods \((\bar{p}, \eta_1, \eta_2, \eta_3)\) and \((\bar{p}', \eta_1, \eta_2, \eta_4)\) with \( b_{1,2}(\bar{p}) < b_{1,2}(\bar{p}') \). Then there are points \( p, p' \in X \) such that we have tripods

\[
(p, \eta_1, \eta_2, \eta_3), (p, \eta_1, \eta_3, \eta_4), \text{ and } (p', \eta_1, \eta_2, \eta_4), (p', \eta_2, \eta_3, \eta_4).
\]

In particular,

\[
\overline{pp'} \subset F_{1,2} \cap F_{2,3} \cap F_{3,4} \cap F_{1,4}.
\]

and \( \angle_p(\eta_{1,2}, p') \in [\pi/3, 2\pi/3] \).

**Proof.** Let us choose tripodal points \( p \in l_{1,2,3}, p' \in l_{1,2,4} \) such that \( d(p, p') \) is minimal; note that \( b_1(p') - b_1(p) = b_1(p') - b_1(\bar{p}) > 0 \). Note that this implies in particular that there is no 4-pod with the given four boundary points embedded in \( X \).

If \( b_{1,2}(p) = b_{1,2}(p') \), then we have found an isometrically embedded tree having \( \eta_1, \eta_2, \eta_3, \eta_4 \) as asymptotic boundary points. The claim of the lemma is now trivial.

So we may assume \( b_{1,2}(p) \neq b_{1,2}(p') \), and without loss of generality that \( b_{1,2}(p) > b_{1,2}(p') \) (by exchanging the \( \eta_{i,j} \) and the \( \xi_{i,j} \), if necessary); note that
under these assumptions, \( p \) is the lower endpoint of \( l_{1,2,3} \), and \( p' \) is the upper endpoint of \( l_{1,2,4} \). We normalize such that \( b_{1,2}(p) = b_1(p) = b_2(p) = 0 \).

First, we want to show \( p' \in F_{2,3} \). Assume that this is not the case.

In \( F_{1,2} \), consider the line \( l_{1,2} \) passing through \( p \) with endpoints \( \mu_1, \nu_2 \). Then the ray \( l_{1,2} \cap \{ b_2 \leq 0 \} \) is a boundary segment of \( F_{1,2} \cap F_{2,3}(\dagger) \).

Similarly, consider the line \( l'_{1,2} \) passing through \( p' \) with endpoints \( \nu_1, \mu_2 \). Then the ray \( l'_{1,2} \cap \{ b_2 \leq b_2(p') \} \) is a boundary segment of \( F_{1,2} \cap F_{2,4}(\dagger) \).

The two lines \( l_{1,2}, l'_{1,2} \) bound a sector \( S \subset F_{1,2} \) with tip \( p'' \), containing \( \eta_2 \) in its asymptotic boundary. Let \( \rho \subset l_{1,2}, \rho' \subset l'_{1,2} \) be its bounding rays.

We are assuming that \( p' \notin F_{1,2} \cap F_{2,3} \). Since \( b_{1,2}(p) > b_{1,2}(p') \), this implies that \( p' \) lies “below” \( l \) (otherwise, \( p' \in \text{Conv}(p, \eta_2, \nu_2) \subset F_{1,2} \cap F_{2,3} \)), see Figure 5.

**In this case, we claim** \( S = F_{2,3} \cap F_{2,4} \): The relation \( \subset \) is clear (because \( \{ p'' \} = l_{1,2} \cap l'_{1,2} \subset F_{1,2} \cap F_{2,3} \cap F_{2,4} \) by \( \dagger \) and \( \ddagger \), and \( \overline{\nu_2 \mu_2} \subset \partial F_{2,j} \) for all \( j \)). For the other inclusion, observe: near \( p' \), the flat \( F_{2,3} \) agrees with \( F_{1,2} \), while this is not true for \( F_{2,4} \). Similarly near \( \rho \), the flat \( F_{2,4} \) agrees with \( F_{1,2} \), but the flat \( F_{2,3} \) does not.

So \( S = F_{2,3} \cap F_{2,4} \) as claimed. Then \( (p'', \eta_2, \eta_3, \eta_4) \) is a tripod by our assumptions and our discussion showing existence of tripods.

However, we see immediately that \( \angle_{p''}(\eta_3, \eta_4) \leq 2\pi/3 \): Indeed, we have

\[
\angle_{p''}(\eta_3, p) = \pi/6 \text{ (by } \dagger \text{, we have } \overrightarrow{p'' p} \in \Sigma_{p''}(F_{2,3})) ,
\angle_{p''}(p, p') = \pi/3 ,
\angle_{p''}(p', \eta_4) = \pi/6 \text{ (by } \ddagger \text{, we have } \overrightarrow{p'' p} \in \Sigma_{p''}(F_{2,4})) .
\]
This contradiction shows \( p' \in F_{1,2} \cap F_{2,3} \). At the same time, this shows \( \angle_p(\eta_1, p') \in [\pi/3, \pi/2] \); this is the last claim (the angle can be bigger than \( \pi/2 \) if we have exchanged \( \eta_{i,j} \) and \( \xi_{i,j} \) before; we still need to verify that \( p, p' \) have the other desired properties).

If \( p' \in \text{int}(F_{1,2} \cap F_{2,3}) \), then it is immediate that \( (p', \eta_2, \eta_3, \eta_4) \) is a tripod (because \( \overrightarrow{p \eta_1} \cap \overrightarrow{p \eta_3} \supset \{p'\} \)).

If \( p' \in \partial(F_{1,2} \cap F_{2,3}) \), we still have (since \( p' \in F_{2,3} \) by the above and \( p' \in F_{2,4} \) by definition)
\[
\angle_{p'}(\eta_2, \eta_4) = \pi = \angle_{p'}(\eta_2, \eta_3).
\]

If \( p' \) is not tripodal for this triple, we would have to have \( \angle_{p'}(\eta_3, \eta_4) \in \{0, \pi/3\} \) (since the shift of the triple is zero by assumption; see the proof of Proposition 7.4).

However, \( \angle_{p'}(\eta_1, \eta_4) = \pi \) by construction and \( \angle_{p'}(\eta_1, \eta_3) \leq \pi/3 \) since \( p' \) cannot be tripodal for \( (\eta_1, \eta_2, \eta_3) \). Therefore, \( \angle_{p'}(\eta_3, \eta_4) \geq 2\pi/3 \), showing that \( p' \) is tripodal for the triple \( (\eta_2, \eta_3, \eta_4) \).

Similarly, we see \( p \in F_{1,2} \cap F_{1,4} \) and that \( (p, \eta_1, \eta_3, \eta_4) \) is a tripod. \( \square \)

The lemma above shows in particular that \( F_{1,2} \cap F_{2,3} \cap F_{3,4} \cap F_{1,4} \neq \emptyset \). Let us examine this set in more detail, and give some more interpretation to the results from the previous lemma:

**Lemma 7.8.** In the situation as in the previous lemma, we have
\[
F_{1,2} \cap F_{3,4} = F_{1,4} \cap F_{2,3} = F_{1,2} \cap F_{2,3} \cap F_{3,4} \cap F_{1,4}.
\]

**Proof.** Let us introduce a set \( C \), drawn in Figure 6. The left vertical boundary is
\[
s_1 := l_{1,2,3} \cap l_{1,3,4} \subset F_{1,2} \cap F_{2,3} \cap F_{3,4} \cap F_{1,4},
\]
which we have just shown to be non-empty. Observe that every point in \( s_1 \) is tripodal for \( (\eta_1, \eta_2, \eta_3) \) and for \( (\eta_1, \eta_3, \eta_4) \). Every interior point \( x \) of \( s_1 \) satisfies \( \overrightarrow{x \eta_2} = \overrightarrow{x \eta_3} \).
Similar properties hold for the vertical boundary on the right, which is defined as

\[ \emptyset \neq s_2 := l_{1,2,4} \cap l_{2,3,4} \subset F_{1,2} \cap F_{2,3} \cap F_{3,4} \cap F_{1,4}. \]

Now we set \( C \) to be the smallest convex polygon in \( F_{1,2} \) containing \( s_1 \cup s_2 \) and such that all the boundary segments are singular. (Observe that \( C \) may degenerate to a segment.)

By definition (and the two inclusions above), the set \( C \) is a subset of both \( F_{1,2} \cap F_{3,4} \) and of \( F_{1,4} \cap F_{2,3} \).

Let us explain the relations to the previous lemma: There, we have found that if \( b_{1,2}(s_1) \cap b_{1,2}(s_2) \neq \emptyset \), then there is an isometrically embedded tree in \( X \) with the given 4 asymptotic endpoints. If this is not the case, then we have made the assumption that \( b_{1,2}(s_1) > b_{1,2}(s_2) \), and our choice of \( p, p' \) was such that \( p \) is the lower endpoint of \( s_1 \) and \( p' \) is the upper endpoint of \( s_2 \).

To finish the proof of our current lemma, we want to show that every boundary segment of \( C \) lies in the boundary of both \( F_{1,2} \cap F_{3,4} \) and \( F_{1,4} \cap F_{2,3} \).

This is immediate for the vertical segments \( s_1 \) and \( s_2 \).

Observe that \( b_1(s_2) > b_1(s_1) \) by the assumptions of Lemma 7.7. Therefore, there are non-degenerate angular segments bounding \( C \). The argument for the angular boundary components are similar to each other, let us give one in detail:

Let \( \bar{p} \) be the upper endpoint of \( s_2 \), and consider the segment \( s := \overline{\mu_1} \cap C = \overline{\mu_3} \cap C \). Let us assume that \( s \) is non-degenerate (i.e. \( s \neq \{\bar{p}\} \)) and let \( x \) be an interior point of \( s \).

Note that \( \bar{p} \) is the upper endpoint of \( l_{1,2,4} \) or of \( l_{2,3,4} \). If \( \bar{p} \) is the upper endpoint of \( l_{1,2,4} \), then \( s \) lies in the boundary of \( F_{1,2} \cap F_{1,4} \); in particular, we have

\[ \Sigma_x(F_{1,2}) \cap \Sigma_x(F_{2,3}) \ni x\mu_2 \neq x\mu_4 \in \Sigma_x(F_{3,4}) \cap \Sigma_x(F_{1,4}). \]

\[ \text{In fact, the following argument shows that "s non-degenerate implies that } \bar{p} \text{ is the upper endpoint of both segments; see the remark below the lemma.} \]

\[ \text{Observe that we have } x\bar{p} = x\nu_2 = x\nu_4; \text{ hence, we have } \angle_{x}(\eta_2, \eta_4) = \pi/3. \]
Otherwise, $\bar{p}$ is the upper endpoint of $l_{2,3,4}$, so $s$ lies in the boundary of $F_{2,3} \cap F_{3,4}$; then the equation above holds as well.

The equation above shows that $x$ (and hence all of $s$) lies in the boundary of both sets, $F_{1,2} \cap F_{3,4}$ and $F_{1,4} \cap F_{2,3}$. Similar arguments hold for the other segments bounding $C$.

\begin{remark}
Let us examine what the last argument shows about $\bar{p}$ (using the notation of the previous lemma): Since $x \in C \subset F_{2,3} \cap F_{3,4}$, the last footnote shows that there cannot be a tripod $(\eta_2, \eta_3, \eta_4)$ at $b_{2,3}$-level higher than $b_{2,3}(\bar{p})$. Hence, $\bar{p}$ is the upper endpoint of $l_{2,3,4}$. Similarly, $\bar{p}$ is the upper endpoint of $l_{1,2,4}$.

This shows that $l_{1,2,4} = l_{2,3,4}$ if the two angular boundary segments of $C$ starting from the upper and lower endpoints of $s_2$ have different slope.

The same statement holds for $l_{1,2,3}$ and $l_{1,3,4}$.

Therefore, if $C$ has four angular boundary segments, then we have $l_{1,2,4} = l_{2,3,4}$ and $l_{1,2,3} = l_{1,3,4}$.
\end{remark}

\begin{remark}
Let us also give a description of $C$ in terms of Busemann functions: Normalize such that

$$b_1(p) = b_2(p) = b_3(p) = b_4(p) = 0.$$ 

Then we have for all $i \in \{1, 2, 3, 4\}$ (taking the indices modulo 4)

$$(b_i + b_{i+1})|_{F_{i,i+1}} = \text{const} = b_i(p) + b_{i+1}(p) = 0.$$ 

Now $x \in C$ if and only if $x \in F_{1,2} \cap F_{2,3} \cap F_{3,4} \cap F_{1,4}$, as we have shown above. Let $f := b_1 + b_2 + b_3 + b_4$. Then it follows that $x \in C$ implies

$$f(x) = \frac{1}{2}((b_1(x)+b_2(x))+(b_2(x)+b_3(x))+(b_3(x)+b_4(x))+(b_1(x)+b_4(x))) = 0.$$ 

Since $x \in F_{i,i+1}$ if and only if $b_i(x) + b_{i+1}(x) = 0$, and $x \notin F_{i,i+1}$ implies $b_i(x) + b_{i+1}(x) > 0$, we have

$$x \in C \iff f(x) = 0,$$

so $C$ is the set of minima of $f$.

\begin{lemma}
In the situation as in the previous lemmas, we have

$$F_{1,3} \cap F_{2,4} = \emptyset.$$ 

\end{lemma}
Proof. Choose points $p, p'$ as in Lemma 7.10 and normalize (as above) such that $b_1(p) = b_2(p) = b_3(p) = b_4(p) = 0$. Note that on $F_{i,j}$, we have $b_i + b_j = \const$.

Let $x \in \overline{pp'}$. Observe $\pi_{F_{2,4}}(x) = p'$. This implies $b_1|_{F_{2,4}} \geq b_1(p') > b_1(p) = 0$.

Arguing similarly for $x \in \overline{pp_3}$, we find $b_3|_{F_{2,4}} \geq b_3(p') = -b_2(p') = b_1(p') > 0$. Hence, $b_1 + b_3|_{F_{2,4}} > 0$, implying the claim (since $b_1 + b_3|_{F_{1,3}} \equiv 0$).

Let us phrase a version of Lemma 7.10 which is valid for every $S$-set of cardinality 4:

**Proposition 7.12.** Let $\{\xi_i | i \in \{1, 2, 3, 4\}\} \subset \partial_T X$ be an $S$-set of cardinality four. Set $\eta_1 := \xi_1$. Then there are points $p, p' \in X$ and an identification $\{\eta_2, \eta_3, \eta_4\} = \{\xi_2, \xi_3, \xi_4\}$, such that we have tripods

$$(p, \eta_1, \eta_2, \eta_3), (p, \eta_1, \eta_3, \eta_4), (p', \eta_1, \eta_2, \eta_4), \text{ and } (p', \eta_2, \eta_3, \eta_4).$$

In particular, we have

$$\overline{pp'} \subset F_{1,2} \cap F_{2,3} \cap F_{3,4} \cap F_{1,4}.$$  

**Proof.** If possible, we choose the identification $\{\eta_2, \eta_3, \eta_4\} = \{\xi_2, \xi_3, \xi_4\}$ such that

$$b_1(p_{1,2,3}) \neq b_1(p_{1,2,4}).$$  

(3)

Let us first assume that this is possible: then by exchanging $\eta_3, \eta_4$ if necessary, we may assume that $b_1(p_{1,2,3}) < b_1(p_{1,2,4})$. Now Lemma 7.10 applies (and finishes the proof).

We still need to consider the case that a choice as in (3) is not possible: So pick an arbitrary identification $\{\eta_2, \eta_3, \eta_4\} = \{\xi_2, \xi_3, \xi_4\}$, and assume that $b_1(p_{1,j,k})$ is independent of $j, k$.

We claim that in this case, there exists a 4-pod. By our assumptions, we have $b_1(l_{1,2,3}) = b_1(l_{1,2,4}) = b_1(l_{1,3,4})$. If there is a point

$$p \in l_{1,2,3} \cap l_{1,2,4} \cap l_{1,3,4},$$

then $p$ is the singular point of a 4-pod: This only means that $p \in l_{2,3,4}$ as well, which follows immediately from the other three inclusions. If this is the case (i.e. if a 4-pod exists), then we set $p' = p$, and we are done.

If the three sets above have pairwise non-empty intersection, then they share a point. Hence, we may assume that $l_{1,2,3} \cap l_{1,2,4} = \emptyset$. 


If this were the case, the shift of \((\eta_2, \eta_3, \eta_4)\) cannot be 0. The argument for this is the same as the one showing \(p' \in F_{2,3}\) in the proof of Lemma 7.7 (one can produce the tip \(p''\) of \(F_{2,3} \cap F_{2,4}\), which should be a tripodal point, but one can show that it cannot be).

Let us summarize what we have achieved in this section:

Given a 4-point S-set \(A\), there is either a 4-pod in \(X\), or there is a 2 + 2 partition \(A_1 = \{a_1, a'_1\}, A_2 = \{a_2, a'_2\}\) of \(A\), such that

\[
s_1 := l_{a_1, a'_1, a_2} \cap l_{a_1, a'_1, a'_2} \neq \emptyset, \quad \text{and} \quad s_2 := l_{a_1, a_2, a'_2} \cap l_{a'_1, a_2, a'_2} \neq \emptyset.
\]

In this case, the sets \(s_1\) and \(s_2\) can be joined to each other “almost horizontally” (this is the statement about the angle in Lemma 7.7).

### 7.4 S-sets and trees

Let \(A := \{\eta_i | i \in I\}\) be an S-set.

Let us examine the set \(F := \bigcup_{i,j \in I} F_{i,j}\). We are going to construct a “vertical” quotient of \(F\) which is a metric tree.

Let \(x \in F_{i,j}\), and consider some point \(\eta_k \in A\). Define

\[
B_{k,i,j}(x) := b_k(\pi_{T_{i,j,k}}(x)) = \min \left( b_k(\{y \in F_{i,j} | b_i(y) = b_i(x)\}) \right).
\]

**Remark 7.13.**

1. If \(k = i\) or \(k = j\), the right-most definition still makes sense (and \(B_{k,i,j}(x) = b_k(x)\)).

2. Note that the value of \(B_{k,i,j}(x)\) does not depend on the choice of \(p_{i,j,k}\).

3. If \(y \in F_{i,j} \cap \{b_i = b_i(x)\}\), then \(B_{k,i,j}(y) = B_{k,i,j}(x)\). We will say that such a \(y\) “represents” \(x\). Using our convention for drawing flats \(F_{i,j}\), this means that “vertical” lines all represent one point (in the space \(T\) which is defined below).

4. Assume that \(b_i(x) \leq b_i(p_{i,j,k})\). Then for any \(y \in F_{i,j} \cap F_{i,k} \cap \{b_i = b_i(x)\}\), we have \(B_{k,i,j}(x) = b_k(y)\).

We will see that the definition of \(B_{k,i,j}(x)\) depends on \(x, \eta_k\) only (Lemma 7.16), so we can define \(B_k(x)\) (for \(x \in F\)).

Now for every \(k \in I\) the definition

\[
D_k(x, y) := |B_k(x) - B_k(y)|
\]

defines a pseudometric on \(F\); indeed, the triangle inequality follows immediately from the inequality for real numbers.
We will see below that \(D_k(x, y) \leq d(x, y)\). Hence, the following is also a pseudometric on \(F\):
\[
D(x, y) := \sup_{k \in I} D_k(x, y)
\]
Consider the metric space \((T, D) := (F/\{D = 0\}, D)\). In this section, we prove:

**Theorem 3.** \((T, D)\) is a metric tree.

We start with some lemmas:

**Lemma 7.14.** Let \(i_0, i_1, j_0, j_1\) be four distinct elements of \(I\). Then
\[
F_{i_0, i_1} \cap F_{j_0, j_1} \subset F_{i_0, j_0} \cup F_{i_0, j_1}.
\]

**Proof.** The claim is trivial if the intersection is empty. Otherwise, it is an immediate consequence of Lemma 7.8. \(\square\)

**Lemma 7.15.** If \(x \in F_{i, j} \cap F_{i, j'}\), then
\[
B_{k, i, j}(x) = B_{k, i, j'}(x).
\]

**Proof.** If \(k = i\), the claim is trivial.

If \(k = j\) (or analogously \(k = j'\)), we have \(B_{k, i, j}(x) = b_j(x) = B_{j, i, j'}(x)\) by Remark 7.13.4.

So we may assume that \(i, j, j', k\) are all distinct; we consider the situation discussed in Lemma 7.7, and assume that \(\{i, j, j', k\} = \{1, 2, 3, 4\}\). We may assume \(k = 4\), and need to examine two cases: \(i = 1\) and \(i = 2\) (since \(i = 3\) is equivalent to \(i = 1\)).

If \(i = 2\), we have \(x \in F_{1, 2} \cap F_{2, 3}\), and \(b_4(\pi_{T_{1, 2, 4}}(x)) = b_4(\pi_{T_{2, 3, 4}}(x))\) follows: If \(\pi_{T_{1, 2, 4}}(x) \in p'\eta_2\), the two projections are equal, and the claim follows. If not, we have \(b_2(x) \in [b_2(p'), 0]\), and we can represent \(x\) in \(F_{1, 2} \cap F_{2, 3} \cap F_{3, 4} \cap F_{1, 4}\) (see Lemma 7.8). Now the claim follows from Remark 7.13.4.

For \(i = 1\), we have \(x \in F_{1, 2} \cap F_{1, 3}\); by Lemma 7.7, we have \(p\eta_1 < F_{1, 2} \cap F_{1, 3} \cap F_{1, 4}\). Clearly, we can represent \(x\) in this ray, and the claim follows again from Remark 7.13.4. \(\square\)

**Lemma 7.16.** If \(x \in F_{i, j} \cap F_{i', j'}\), then
\[
B_{k, i, j}(x) = B_{k, i', j'}(x) =: B_k(x).
\]

**Proof.** If \(\{i, j\} \cap \{i', j'\} \neq \emptyset\), then the claim follows from the previous lemma. Otherwise, we may assume \(x \in F_{i, j} \cap F_{i, j'}\) by Lemma 7.14. Hence, we can apply the previous lemma twice:
\[
B_{k, i, j}(x) = B_{k, i, j'}(x) = B_{k, i', j'}(x).
\]
We have shown that for every $k \in I$ the definition

$$D_k(x, y) := |B_k(x) - B_k(y)|$$

makes sense; so it is indeed a pseudometric on $\mathcal{F}$ as claimed above.

Hence, the following is also a pseudometric on $\mathcal{F}$ (possibly with value $\infty$):

$$D(x, y) := \sup_{k \in I} d_k(x, y).$$

Let $[x]$ denote the equivalence class of $x \in \mathcal{F}$. Recall that points $x, y \in F_{i,j}$ with $b_i(x) = b_i(y)$ satisfy $[x] = [y]$.

**Lemma 7.17.** Given points $x \in F_{i,j}, y \in F_{i',j'}$, there exist points $x', y' \in F_{i'',j''}$ such that $\{i'', j''\} \subset \{i, j, i', j'\}$ and $[x] = [x'], [y] = [y']$, and

$$D(x, y) = d(x', y').$$

**Proof.** If $|\{i, j, i', j'\}| \leq 3$, we can project $x, y$ to a tripod or line. In particular, we can represent $x, y$ by points $x', y'$ on a line in a flat $F_{i'',j''}$.

If $|\{i, j, i', j'\}| = 4$, let us consider only the corresponding boundary points. We may enumerate these as in Proposition 7.12. Then we can represent $x$ and $y$ (uniquely) by points $x'', y''$ in $p_{11} \cup p_{13} \cup p_{1} \cup p_{2} \cup p_{4} \cup p_{2} \cup p_{4}$. Every two points in this set lie in a common flat, so let $x'', y'' \in F_{i'',j''}$.

Choose

$$x' \in F_{i'',j''} \cap \{b_{i'''} = b_{i'''}(x'')\}, \text{ and } y' \in F_{i'',j''} \cap \{b_{i''''} = b_{i''''}(y'')\},$$

such that

$$d(x', y') = |b_{i'''}(y'') - b_{i'''}(x'')|.$$

Now for all $k \in I$, we have $D_k(x, y) = D_k(x', y') \leq d(x', y')$ (because projection to $T_{i'',j''}^{i,j}$ is 1-Lipschitz). Furthermore, we have $D_{i'''}(x, y) = D_{i''''}(x, y') = d(x', y').$ 

Hence, we have a metric space $(\mathcal{T}, D) := (\mathcal{F}/\{D = 0\}, D)$. We claim that $\mathcal{T}$ is a metric tree.

**Lemma 7.18.** If the cardinality of $A$ is 4, then $\mathcal{T}$ is a metric tree.

**Proof.** This is almost immediate from the previous lemma: The discussion there shows that $\mathcal{T}$ has the topological structure of the set $p_{11} \cup p_{13} \cup p_{1} \cup p_{2} \cup p_{2} \cup p_{4}$ (assuming that the elements of $A$ are named such that Proposition 7.12 and Lemma 7.7 apply). Now $D$ is almost the length metric on this graph: we just have to shorten $p_{1}^{p_{1}'}$ to have length $b_1(p') - b_1(p)$.

\[\square\]
Proof of Theorem 3. We put together the pieces collected above:

- For two points \(x, y \in \mathcal{F}\), we can find a flat \(F_{i,j}\) and points \(x', y' \in F_{i,j}\), such that \([x] = [x'], [y] = [y']\), and \(d(x', y') = D([x], [y])\) (Lemma 7.17).
  Then the segment \(x'y'\) represents a geodesic \([x][y]\) (of unit speed).

- From Lemma 7.18 we conclude that \(\mathcal{T}\) has extendible geodesics, and

- Since for every \(z \in \mathcal{F}\), the geodesics between \(x', y', z\) (of the form introduced above) lie in a tree (again by Lemma 7.18), every triangle in \(\mathcal{T}\) is degenerate.

- This implies that geodesic segments are unique, and that \(\mathcal{T}\) is 0-hyperbolic.

So \(\mathcal{T}\) is indeed a tree.

Let \(\pi : \mathcal{F} \rightarrow \mathcal{T}\) be the projection, and observe that the asymptotic endpoints of \(\mathcal{T}\) correspond to the points \(\eta_i\). We let \(\hat{\eta}_i\) denote the point of \(\partial\mathcal{T}(\mathcal{T})\) corresponding to \(\eta_i\). Then \(B_i(x) = b_{\hat{\eta}_i}([x])\).

Lemma 7.19. Assume that \(A\) is a good S-set, let \([x] \in \mathcal{T}\), and let \(\mathcal{T}_{[x]} := \{(i,j) \mid [x] \in \pi(F_{i,j}) = \overline{\eta_i\eta_j}\}\). Set

\[
C_{[x]} := \bigcap_{(i,j) \in \mathcal{T}_{[x]}} F_{i,j}.
\]

Then \(C_{[x]}\) is non-empty, closed and convex, and \([x] \in \pi(C_{[x]}).\)

Proof. Let \((i_0,j_0) \in \mathcal{T}_{[x]}.

Consider \(J := \{j \mid (i_0,j) \in \mathcal{T}_{[x]}\}\).

For every \(j, j' \in J\), we have

\[
s_{j,j'} := \sup\{b_{i_0}(F_{i_0,j} \cap F_{i_0,j'})\} = b_{i_0}(p_{i_0,j,j'}) = B_{i_0}(p_{i_0,j,j'}) \geq B_{i_0}([x]).
\]

The last inequality is due to the fact that both \((i_0,j)\) and \((i_0,j')\) are in \(\mathcal{T}_{[x]}\), hence \([x] \in [p_{i_0,j,j'}]\hat{\eta}_{i_0}\).

By induction, one shows:

For every finite \(U \subset J\), the set \(\{B_{i_0}([x]) \cap \bigcap_{j \in U} F_{i_0,j}\) is a non-empty geodesic segment \(l_U\). Otherwise, we could find \(j, j' \in J\) such that \(s_{j,j'} < B_{i_0}([x])\) (by the argument for \(S\) in the proof of Lemma 7.7). See Figure 7.

6Abusing notation, we will sometimes also write \(B_i([x]) := b_{\hat{\eta}_i}([x]).\)
If we can find \( j, j' \in J \) such that \( l_{\{j, j', j''\}} \) is compact, we use this compactness to conclude that \( l_J \neq \emptyset \). The sets \( l_{\{j, j', j''\}} \) form an open cover of \( l_{\{j, j'\}} \), so finitely many \( j'' \) suffice, in contradiction to the above.

If such a choice of \( j, j' \) is not possible, then the assumption that \( A \) is good implies that \( l_J \) is a ray or a geodesic line.

Similarly, let \( J' := \{ i \mid (i, j_0) \in T[\alpha] \} \), and obtain \( l'_{J'} := \{ B_{j_0} = B_{j_0}([x]) \} \cap \bigcap_{i \in J'} F_{i, j_0} \).

If \( l_J \cap l'_{J'} \) were empty, we could find \( j \in J, i \in J' \) such that \( \{ i_0, j_0, i, j \} \) contradict Proposition 7.12.

Now \( l_J \cap l'_{J'} \subset C[\alpha] \) by Lemma 7.18 (in fact, \( l_J \cap l'_{J'} = C[\alpha] \cap \{ B_{i_0} = B_{i_0}([x]) \} \)).

**Remark 7.20.** If \( I \) is finite or \( X \) is discrete, we can describe \( C[\alpha] \) in detail as follows:

We find \( j', j'' \in J \) such that we have \( F_{i_0, j'} \cap F_{i_0, j''} \cap \{ B_{i_0} = B_{i_0}([x]) \} = l_J \).

Then we can similarly find \( i', i'' \in J' \) such that

\[
l_J \cap l'_{J'} = F_{i', j'} \cap F_{i'', j''} \cap \{ B_{j'} = B_{j'}([x]) \}\]

(see also Lemma 7.18 and Figure 6). To cover a “vertical cut-off”, we may have to introduce third indices \( i''', j''' \) in \( J', J \) resp., such that

\[
C[\alpha] = F_{i', j'} \cap F_{i'', j''} \cap F_{i''', j'''}.
\]

In the general case, we can find sequences \( i'_n, i''_n, i''''_n; j'_n, j''_n, j''''_n \) such that

\[
F_{i'_n, j'_n} \cap F_{i''_n, j''_n} \cap F_{i''''_n, j''''_n}
\]

is a descending sequence with \( C[\alpha] \) as its limit.
8 Thickening tripods

Remark 7.21. Although the tree $\mathcal{T}$ is not isometrically embedded in $X$, the lemma above shows that we can almost embed $\mathcal{T}$, and that intersection of vertical lines in $\mathcal{F}$ is an equivalence relation. One may think of the almost-embedded $\mathcal{T}$ in terms of sets as in Lemma 7.8.

The following lemma follows immediately from the definition:

Lemma 7.22. Let $[x], [y] \in \mathcal{T}$, and $C_{[x]}, C_{[y]}$ from the lemma above. If $[x], [y]$ and $[x][y]$ are regular, then $C_{[x]} = C_{[y]}$.

At one point, we will need the following technical observation:

Remark 7.23. Let $x \in C_{[x]}$, and $y \in C_{[y]}$. Assume that $x$ minimizes $b_{i',j'}|_{C_{[x]} \cap \{b_{i'} = b_{i'}(x)\}}$ for some $(i', j') \in \mathcal{T}_{[x]} \cap \mathcal{T}_{[y]}$. Then

$$\angle_x(y, \xi_{i',j'}) \leq 2\pi/3.$$  

Reason: We may assume that $b_{j'}(y) \leq b_{j'}(x)$. Let $l$ be the line joining $\nu_{i'}$ to $\mu_{i'}$ passing through $y$. Let $x'$ be the point in $l$ satisfying $b_{j'}(x') = b_{j'}(x)$. Then it is easy to see that $b_{i',j'}(x') \geq b_{i',j'}(x)$.

8 Thickening tripods

Let $(p, \eta_1, \eta_2, \eta_3)$ be a tripod in $X$. We want to find convex rank 1-sets containing the tripod, other than a tubular neighborhood.

The results from this section are not used in the proof of Theorem 11; however, the techniques we introduce here are important for the proof (and will be generalized later on). Moreover, we get a feeling for the kinds of sets we use to build our convex set later on.

We normalize the Busemann functions to satisfy $b_{i,j}(p) = 0$ for all $i, j$. Let us agree to view the indices modulo 3. Note that for the singular vertices $\eta_{i,j}$, the $\alpha$ used in section 10 is $\pi/3$.

Let us list some useful properties of the lower endpoint of $l_{1,2,3}$:

Lemma 8.1. Assume that $p$ is the lower endpoint of $l_{1,2,3}$. Then (we will list only one version, but permuting the indices leaves the statement intact, of course):

1. All the $\overrightarrow{pp_{i,j}}$, $i, j \in \{1, 2, 3\}$ are distinct.
2. The $\overrightarrow{p\eta_{i,j}}$ span a flat in $\Sigma_pX$. The singular directions of this flat are in the directions of the $\eta_{i,j}$ and the $\nu_i$. In particular, $\overrightarrow{pm_1} \cup \overrightarrow{pm_23}$ is a geodesic in $X$.

3. Let $x \in X\setminus \{p\}$. Then there exist $i', j'$ such that $\angle_p(x, \eta_{i',j'}) = 2\pi/3$. It follows that $b_{i',j'}(x) \geq b_{i,j}(x)$ for all $i, j$.

4. If $b_{i,j}(x) \leq D$ for all $i, j$, then $d(x, p) \leq 2D$.

5. If $b_{1,2}(x) > \max(b_{1,3}(x), b_{2,3}(x))$, then $\angle_p(x, \nu_3) < \pi/3$.

6. If $b_{1,2}(x) > \max(b_{1,3}(x), b_{2,3}(x))$, then $\angle_q(x, \nu_3) < \pi/3$ for all $q \in l_{1,2,3}$.

7. If $\angle_p(x, \eta_1) \leq \pi/2$, we may distinguish two cases:
   
   (a) $\angle_p(x, \nu_1) \leq \pi/3$, which implies $b_{2,3}(x) \geq b_{i,j}(x)$ for all $i, j \in \{1, 2\}$.

   (b) otherwise $b_{1,2}(x) = b_{1,3}(x) > d(x, p)/2$, and $\overrightarrow{x\eta_1} = \overrightarrow{x\eta_{1,3}}$.

Proof. 1: If two of the $\overrightarrow{p\eta_{i,j}}$ agree, then all three have to be equal to each other; but then, $p$ is not the lower endpoint of $l_{1,2,3}$.

Now 2 is clear.

3: Suppose $\angle_p(x, \eta_{1,j}) < 2\pi/3$ for both $j$.
This is only possible if $\angle_p(x, \nu_1) < \pi/3$, and by 2 we have $\angle_p(x, \eta_{2,3}) > 2\pi/3$.
The second part of the claim is clear, since $b_{i',j'}$ increases at maximal slope along $\overrightarrow{px}$ (in the sense of section 5).

4: By 3 at least one of the $b_{i,j}$ increases at maximal slope (at least $1/2 = -\cos(2\pi/3)$) along $\overrightarrow{px}$.

5: It follows from property 3, that

$$\angle_p(x, \eta_{1,2}) \geq 2\pi/3 > \max(\angle_p(x, \eta_{1,3}), \angle_p(x, \eta_{2,3})).$$

The claim follows as in the proof of 3.

6: We may assume $q \neq p$. Observe that $\angle_q(x, \eta_{1,2}) = \angle_q(x, \eta_{1,3})$, so this angle is less than $2\pi/3$ (otherwise, $b_{1,3}(x) = b_{1,2}(x)$).

If $\angle_q(x, \eta_{1,2}) = \angle_q(x, \eta_{1,3}) = 0$, let $q'$ be the first point along $\overrightarrow{qq'}$ where $\angle_q(x, \eta_{1,3}) \neq 0$; if such a point does not exist, set $q' = p$. Then $\alpha := \angle_q(x, \eta_{1,3}) = 2\pi/3$, or $q' = p$ and $\alpha = 0$ (since the type of $\overrightarrow{qq'}$ does not change along $\overrightarrow{qq'}$). If $\alpha$ is $2\pi/3$, this is a contradiction to the above. If $\alpha = 0$, then $\angle_p(x, \eta_{2,3}) = 2\pi/3$, a contradiction again.

So there is a direction $\nu \in \Sigma_q(X)$ of the same type as $\overrightarrow{qh_2}$ with $\angle_q(\nu, \eta_{1,2}) = \pi/3$ and $\angle_q(\nu, x) < \pi/3$. If $\nu \neq \overrightarrow{qh_3}$, this is a contradiction to Lemma 6.2.

Hence, we have $\nu = \overrightarrow{qh_3}$. 

7: First observe that $B_{\pi/2}(\overline{p\eta_1}) = B_{\pi/3}(\overline{p\eta_1}) \cup B_{\pi/3}(\overline{p\mu_1})$.
If $\angle_p(x, \nu_1) \leq \pi/3$, case (a) follows immediately (via 2 & 3).
If this is not the case, then $\alpha := \angle_p(x, \eta_{1,2}) = \angle_p(x, \eta_{1,3}) > 2\pi/3$, implying the first part of the claim. Since $\angle_x(\eta_{1,2}, \eta_{1,3}) \leq 2(\pi - \alpha) < 2\pi/3$, the second claim follows (because the two directions are singular and of the same type).

Pick $R > 0$, and $D > R/2$. We define convex sets as follows: Let $T_{1,2,3} := \text{Conv}(p, \eta_1, \eta_2, \eta_3)$ be the tripod, recall that we consider the indices modulo 3, and let
$$C_i := B_R(T_{1,2,3}) \cap \{b_{i,i+1} \leq D\} \cap \{b_{i,i+2} \leq D\}.$$

**Proposition 8.2.** $C_1 \cup C_2 \cup C_3$ is convex.

We could prove this proposition directly; however, it can also be derived from Proposition 8.4, so we omit a direct proof here.

To better understand the sets $C_i$, let us explain the relation to the sets $\tilde{C}_i$, which come to mind (more) naturally; define
$$\tilde{C}_i := B_R(\overline{p\eta_i}) \cap \{b_{i,i+1} \leq D\} \cap \{b_{i,i+2} \leq D\} = C_i \cap B_R(\overline{p\eta_i}).$$

**Lemma 8.3.** If $p$ is the lower endpoint of $l_{1,2,3}$, we have
$$\bigcup C_i = \bigcup \tilde{C}_i.$$ 

*Proof.* The two sets in question are obviously equal on $B_R(p)$, but the set on the left-hand side is potentially larger. Consider a point $x \in X$ with $\pi_{T_{1,2,3}}(x) \in \overline{p\eta_i}\{p\}$; note that points of $B_R(\overline{p\eta_i}) \setminus B_R(p)$ have this property. We have $\angle_p(x, \eta_i) < \pi/2$, so the cases from property 8.4 apply.

We see that in both cases, $x \in \bigcup C_i$ implies $x \in \tilde{C}_i$. Since the conditions are symmetric, we are done. \qed

It will turn out that a convex rank 1-set as in Proposition 8.2 is not quite good enough, so we need a more sophisticated approach:

In a first step, we show that we can do without tubular neighborhoods, by imposing conditions on $b'_{i,j}$:

Normalize such that $b_{i,j}(p) = b'_{i,j}(p) = 0$, let $D > 0$ and $D' \in (D/2, 2D)$. Consider the convex sets
$$K_i := \{b_{i,i+1} \leq D\} \cap \{b_{i,i+2} \leq D\} \cap \{b'_{i,i+1} \leq D'\} \cap \{b'_{i,i+2} \leq D'\}$$

**Proposition 8.4.** $\bigcup K_i$ is convex.
We start with an elementary observation:

**Lemma 8.5.**

\[ \bigcup K_i = \{ x \mid \text{at least two } b_{i,j}(x) \leq D \} \]
\[ \cap \{ x \mid \text{at least two } b'_{i,j}(x) \leq D' \} \]
\[ =: \tilde{K}_1 \cap \tilde{K}_2 \]

**Proof.** If the claim is not true, then there is (without loss of generality) \( x \in X \) with

\[ b_{2,3}(x) > D \geq \max(b_{1,2}(x), b_{1,3}(x)) \text{ and } b'_{1,3}(x) > D' \geq \max(b'_{1,2}(x), b'_{2,3}(x)). \]

This implies that \( l_{1,2,3} \) has a lower endpoint \( p' \) and an upper endpoint \( p'' \). We obtain

\[ \angle p'(x, \nu_1) < \pi/3 \text{ and } \angle p''(x, \mu_2) < \pi/3 \]

by Lemma 6.4. This is a contradiction to Lemma 6.4. \( \square \)

**Proof of Proposition 8.4.** Note that \( K_1 \cap K_2 = K_2 \cap K_3 = K_3 \cap K_1 \).

We bring in the description \( \bigcup K_i = \tilde{K}_1 \cup \tilde{K}_2 \) from above: We show convexity of \( B_\varepsilon(x) \cap \tilde{K}_i \) for every \( x \in K_1 \cap K_2, i \in \{1, 2\} \), and an \( \varepsilon > 0 \) that we will construct in an instant. Via Lemma 4.3 and the lemma above, this shows the claim.

It suffices to show convexity of \( \tilde{K}_1 \) near \( x \), since the proof is the same for \( \tilde{K}_2 \) (possibly with a different \( \varepsilon \), but then Lemma 4.3 applies to the smaller one).

**We construct \( \varepsilon \):**

- Pick \( \varepsilon, \hat{\alpha} \) such that in a Euclidean triangle \( \Delta(A, B, C) \) with \( d(A, B) = 2D, d(A, C) \in [2D - \varepsilon, 2D] \), and \( \angle_A(B, C) < \hat{\alpha} \), we have \( d(B, C) < D \cdot \sqrt{3}/2 \); note that \( \hat{\alpha} < \pi/3 \).

- We decrease \( \varepsilon \) (if necessary), such that \( \varepsilon < \min(D \cdot \sqrt{3}/2, (2D - D')/2) \).

- By decreasing \( \hat{\alpha} \), we may assume that \( (2D - \varepsilon) \cdot \cos \hat{\alpha} > D' \).

- By decreasing \( \varepsilon \) again, we can require \( (2D - \varepsilon) \cdot (-\cos(2\pi/3 + \hat{\alpha})) > D \).
This is the $\varepsilon$ we work with.

Let $p'$ be the lower endpoint of $l_{1,2,3}$ (if $p'$ does not exist, the claim for $\tilde{K}_1$ is trivial); then $b_{1,2}(p') \leq 0 \leq b_{1,2}'(p')$, and we set $R' := 2 \cdot (D - b_{1,2}(p')) \geq 2D$. Note that $K_1 \cap K_2 \subset B_{R'}(p')$ (by 8.11). Lemma 6.3 shows convexity of $B_{R'}(p') \cap \tilde{K}_1$.

So it suffices to consider a point $x \in K_1 \cap K_2$ with $R' - \varepsilon \leq d(x, p') \leq R'$. Now for $i', j'$ from 8.11 the construction of $\varepsilon$ (last item) and $b_{i', j'}(x) \leq D$ imply

$$\angle_{p'}(x, \eta_{i', j'}) \in [2\pi/3, 2\pi/3 + \hat{\alpha}].$$

On the other hand, $b'_{i,j}(x) \leq D'$ implies that $\angle_{p'}(x, \eta_{i,j}) > \hat{\alpha}$ for all $i, j$ (by construction of $\hat{\alpha}$), so $\angle_{p}(x, \eta_{i,j}) \in [2\pi/3 - \hat{\alpha}, 2\pi/3 + \hat{\alpha}]$ for all $i, j$.

This implies that there is a direction $\nu \in p'\eta_{i', j'}p'x$ such that $\angle_{p'}(\nu, \eta_{i,j}) = 2\pi/3$ for all $i, j$.

We can extend the flat half-strip $\text{Conv}(x, p', \eta_{i', j'})$ to a flat sector $F$ with tip $p'$, and inside this sector, we find a point $x'$ with $d(x', p') = R'$ and $\overrightarrow{p'x'} = \nu$. By construction of $\hat{\alpha}$, we have $d(x, x') < R' \cdot \sqrt{3}/4$. Now Lemma 6.1 applies to $x'$. We have $B_{i}(x) \subset B_{R'\cdot\sqrt{3}/2}(x')$, so this shows the claim. \(\square\)

This convex rank 1-set may have more asymptotic boundary points than just the $\eta_i$. We shrink it by putting in (large) tubular neighborhoods again:

Consider consistent (i.e. corresponding to each other under holonomy) compact subsets $W_i$ of $X_{\eta_i}$, such that $\overline{[p\eta]} \in W_i$. Normalize such that $b_{i,j}(W_i) = [-S, S] = b_{i,j}'(W_i)$. Let $S_{i,j}$ be the flat strip in $F_{i,j}$ “spanned by” $W_i$, i.e. $S_{i,j} := \text{Conv}(W_i, W_j) \subset F_{i,j}$. Let $R > 10S$.

Let $C_i = S_{i,i+1} \cap S_{i,i+2}$, and let\footnote{Instead of $4S$, we could choose any value $D > 3S$; the corresponding condition on $R$ would be $R > 2(D + S)$.}

$$C_i := B_R(C_i) \cap \{b_{i,i+1} \leq 4S\} \cap \{b_{i,i+2} \leq 4S\} \cap \{b'_{i,i+1} \leq 4S\} \cap \{b'_{i,i+2} \leq 4S\}$$

$$= B_R(C_i) \cap K_i,$$

where the $K_i$ are defined as before (with $D = 4S + b_{1,2}(p)$, $D' = 4S - b_{1,2}(p) \in [3S, 5S]$, due to our new normalization).

**Proposition 8.6.** $C := \bigcup C_i$ is convex.

**Proof.** It suffices to show that $C' := C_1 \cup C_2$ is convex.

The last sentence above this proposition shows that Proposition 8.4 applies; hence $C' \cap B_R(C_1) \cap B_R(C_2)$ is convex.
If some endpoint of $l_{1,2,3}$ lies in $S_{1,2}$, then $C_1 \cap C_2 \subset B_{10S}(l \cap S_{1,2})$ (by Lemma 8.11), and since $R > 10S$, we are done by Lemma 4.3.

So we assume that no endpoint of $l_{1,2,3}$ lies in $S_{1,2}$. Then the following lemma shows (in a precise way) that near $C_1 \setminus B_R(\tilde{C}_2)$, the points in $C'$ lie in $K_1$, and $C'$ is convex in these points. Again, the claim follows via Lemma 8.4.

**Lemma 8.7.** Assume that no endpoint of $l_{1,2,3}$ lies in $S_{1,2}$. Then there exists an $\varepsilon > 0$ such that if $x \in C_2$ and $y \in B_{2\varepsilon}(x) \cap C_1 \setminus B_R(\tilde{C}_2)$, then $x \in K_1$, and $\overline{xy} \subset C_1 \cup C_2$.

**Proof.** Let us first construct the $\varepsilon$:

- We pick $0 < \alpha < \pi/6$ such that $(R + 10S)/2 \cdot (-\cos(2\pi/3 - \alpha)) > 5S$.

- Now pick $\varepsilon < (R - 10S)/4$ such that in a Euclidean triangle $\Delta(A, B, C)$ with $d(A, B) \in [R - 4\varepsilon, R], d(A, C) \in [R - 2\varepsilon, R + 2\varepsilon]$, and $d(B, C) \in [0, 2\varepsilon]$, we have $\angle_A(B, C) < \alpha$.

This is the $\varepsilon$ (and $\alpha$) we work with.

Now consider points $x, y$ as in the statement of the lemma.

The important step is the following observation:

There exists a point $q \in \tilde{C}_1 \cap \tilde{C}_2 = l \cap S_{1,2} = S_{1,2} \cap \{\nu_1 = \nu_{1}(p)\}$ such that $\angle_q(x, \nu_2) \geq \pi/2 - \alpha$.

**Reason:** If $\pi_{S_{1,2}}(x) \in \tilde{C}_1$, it is easy to pick $q \in \tilde{C}_1 \cap \tilde{C}_2$ suitably: set $q := \pi_{\tilde{C}_2} \circ \pi_{\tilde{C}_1}(x)$, and observe $\angle_q(x, \nu_2) > \pi/2$ (because $\overline{\tilde{C}_1(x)} \cup \overline{\tilde{C}_2}$ is a geodesic ray).

So assume that $\pi_{S_{1,2}}(x) \notin \tilde{C}_1$. By definition, the point $y \in B_{2\varepsilon}(x) \cap C_1$ satisfies $\pi_{S_{1,2}}(y) \in \tilde{C}_1$. Let $x' \in \overline{xy}$ such that $q := \pi_{S_{1,2}}(x') \in \tilde{C}_1 \cap \tilde{C}_2$. Then $d(q, x') \in [R - 4\varepsilon, R], d(q, x) \in [R - 2\varepsilon, R + 2\varepsilon]$, and $d(x, x') \leq 2\varepsilon$. Since $\angle_q(x', \nu_2) \geq \pi/2$ by definition, $q$ has the desired property by construction of $\varepsilon, \alpha$.

Now assume that the claim is wrong. Then there is a point $x \in C_2$ as above with $\max(b_{1,2}(x), b_{2,3}(x)) \leq 4S < b_{1,3}(x)$ (this is without loss of generality, maybe we need to exchange $\eta_2, \eta_3$ to get this inequality).

As usual, we pick the lower endpoint $p'$ of $l_{1,2,3}$, for which we find $\angle_{p'}(x, \nu_2) < \pi/3$. 


by 8.15. In particular, \( \angle_{p'}(q, x) > \pi/3 \).

Note that \( \angle_q(p', x) > \pi/3 + \hat{\alpha} \) (otherwise, we get \( b'_{1,2}(x) > 4S \), because \( b'_{1,2}(q) \geq -S \) and the second item in the construction of \( \varepsilon \)). Now, if \( \angle_q(x, \eta_2) \in [\pi/2 - \hat{\alpha}, \pi/2] \), we get \( \angle_q(p', x) > 2\pi/3 - \hat{\alpha} \), implying \( b_{1,2}(x) > 4S \).

So \( \angle_q(x, \eta_2) > \pi/2 \). Since \( \angle_{p'}(q, x) > \pi/3 \), we have \( \angle_q(p', x) < 2\pi/3 \).

By the discussion above, there is a direction \( \nu \) of the same type as \( \overrightarrow{q\nu_1} \), but neither \( \overrightarrow{q\xi_1} \) nor \( \overrightarrow{q\nu_2} \), such that \( \angle_q(x, \nu) \leq \pi/3 \).

But this is a contradiction to Lemma 6.2 (resp. Remark 6.3).

We have shown \( x \in K_1 \), so we have \( x \in K_1 \cap K_2 \). If \( x \in B_R(\tilde{C}_1) \), then \( x \in C_1 \) and the second claim is immediate. If \( x \notin B_R(\tilde{C}_1) \), then the argument from above, applied to \( y \), shows that \( \{x, y\} \subset K_1 \cap K_2 \). Since \( B_R(C_1 \cup C_2) \) is convex, the second claim follows.

\[ \square \]

9 Existence of convex rank 1-sets

In this section, we prove Theorem 1.

9.1 Setting

Let \( A := \{\eta_i | i \in I\} \) be a good S-set.

For every triple \( i, j, k \in A \), we pick a tripodal point \( p_{i,j,k} \). Let \( S_{i,j,k} \in X_{\eta_i} \) be the strong asymptote class at \( \eta_i \) represented by \( p_{i,j,k} \eta_i \). Since order of the indices does not matter here, we can similarly define \( S_{j,i,k} \in X_{\eta_j} \) and so on.

Since all the shifts are 0, we can pick a particular \( i_0 \in I \), and join all the strong asymptote classes \( S_{i,j,k} \) to \( \eta_{i_0} \), where we obtain corresponding strong asymptote classes.

Let \( K_{i_0} \) be the closed convex hull of all these strong asymptote classes at \( \eta_{i_0} \). Since all the shifts are 0, we similarly obtain isometric sets \( K_i \subset X_{\eta_i} \) for all \( i \in I \).

Because \( A \) is good, we may assume that we have chosen the \( p_{i,j,k} \) such that the \( K_i \) are compact.

We normalize the Busemann functions such that

\[ b_{i,j}(K_i) = [-S, S] = b'_{i,j}(K_i) \]

(so we have \( (b_{i,j} + b'_{i,j})|_{F_{i,j}} = 0 \).)

Recall the set \( F = \bigcup_{(i,j) \in I} F_{i,j} \), and its quotient tree \( T \) from section 7.2 as usual, we let \( \pi : F \rightarrow T \) be the projection.
Also recalling the sets $\mathcal{T}_x = \{(i, j) | [x] \in \eta_i \eta_j\}$, we set

$$K_x := \bigcap_{(i, j) \in \mathcal{T}_x} \{b_{i,j} \leq 4S\} \cap \{b'_{i,j} \leq 4S\}.$$  

In our choice of the limit $4S$, the important property is the following: For every $p_{i,j,k}$, we have

$$4S - b_{i,j}(p_{i,j,k}) \leq 5S < 6S \leq 2(b_{i,j}(p_{i,j,k}) - (4S)).$$

Of course, the same inequality holds for $b'_{i,j}$. These conditions corresponds to the condition $D' \in (D/2, 2D)$ in Proposition 8.4. Actually, one can extend both results to the limit case where the inequality above is not strict; however, this is not needed for the purpose of this paper.

**Lemma 9.1.** For every $[x] \in \mathcal{T}$, the set $K_x$ is non-empty, closed, convex and $[x] \in \pi(K_x)$.

**Proof.** Let $(i_0, j_0) \in \mathcal{T}_x$.

We will use the notation of Lemma 7.19.

Clearly, it suffices to show that

$$\hat{C}_x := C_x \cap \{B_{i_0} = B_{i_0}([x])\} \cap \{b_{i_0,j_0} \in [-S, S]\} \neq \emptyset.$$  

If we have $b_{i_0,j_0}(l_J) < -S$, there is $j, j' \in J$ such that $b_{i_0,j_0}(l_{\{i_0,j,j'\}}) < -S$ (by Remark 7.20), in contradiction to the construction of $K_{i_0}$. Thus, we obtain $b_{i_0,j_0}(l_J) \cap [-S, S] \neq \emptyset$ and $b_{i_0,j_0}(l_J') \cap [-S, S] \neq \emptyset$. Now the claim follows because $l_J, l_J'$ are intervals and have non-empty intersection. \qed
Let
\[ \mathcal{K} := \bigcup_{[x] \in \mathcal{T}} \mathcal{K}_{[x]} \]

**Lemma 9.2.** $\mathcal{K}$ is connected.

**Proof.** Let $x \in \mathcal{K}_{[x]}$, $y \in \mathcal{K}_{[y]}$, and pick $(i', j') \in \mathcal{T}_{[x]} \cap \mathcal{T}_{[y]}$. We can join $x$ to $\hat{\mathcal{C}}_{[x]} \subset S_{i', j'}$ and $y$ to $\hat{\mathcal{C}}_{[y]} \subset S_{i', j'}$. By construction, we have $S_{i', j'} \subset \mathcal{K}$, so the claim follows. 

We are going to show that $\bar{\mathcal{K}}$ is convex. Since it is hard to show that $\bar{\mathcal{K}}$ is of rank 1, we introduce tubular neighborhoods again: Pick $R > 10S$. For $[x] \in \mathcal{T}$, let
\[
\hat{\mathcal{C}}_{[x]} := \mathcal{K}_{[x]} \cap B_R(\hat{\mathcal{C}}_{[x]}),
\]
\[
\mathcal{C} := \bigcup_{[x] \in \mathcal{T}} \hat{\mathcal{C}}_{[x]}.
\]

Exactly as for $\mathcal{K}$, we find that $\mathcal{C}$ is connected. After showing that $\bar{\mathcal{K}}$ is convex, we also show that $\bar{\mathcal{C}}$ is convex. Observe the analogon of moving from $\mathcal{K}$ to $\mathcal{C}$ and from Proposition 8.4 to Proposition 8.6. For the new closed convex set $\bar{\mathcal{C}}$, it is easy to show that it is of rank 1; this was obvious in both propositions mentioned above, because they were finite unions. Thus, the proof of Theorem 1 is complete after these steps.

### 9.2 The proof of Theorem 1

As a first step, we construct an $\varepsilon > 0$, and show that $\bar{\mathcal{K}}$ is $\delta$-locally convex for every $\delta < \varepsilon/2$.

**Construction 9.3.**
- Pick $0 < \hat{\alpha} < \pi/6$ such that
  \[ 3S/(\cos(2\pi/3 + \hat{\alpha})) \geq 11S/2. \]
- By decreasing $\hat{\alpha}$ if necessary, we also require that $11S/2 \cdot \cos(\hat{\alpha}) > 5S$.
- Let $\varepsilon > 0$ be such that in a Euclidean triangle $\Delta(A, B, C)$ with
  \[ d(A, B) \geq 3S \text{ and } d(B, C) \leq \varepsilon, \]
  we have $\angle_A(B, C) < \hat{\alpha}/2$. 

9.2 The proof of Theorem 7

We introduce some more notation for this section: Consider points \([x_0] \neq [x_1] \in \mathcal{T}\).

Pick \((i', j') \in \mathcal{T}_{[x_0]} \cap \mathcal{T}_{[x_1]}\) with \(B_{i'}(x_0) < B_{j'}(x_1)\).

Let \(I_0 := \{i \in I \mid (i, j') \in \mathcal{T}_{[x_0]}\}\). Analogously, define \(J_0 := \{j \in I \mid (i', j) \in \mathcal{T}_{[x_0]}\}\) and \(I_1, J_1\) (see Figure 9). Let \(L := J_0 \cap I_1\).

Note that \(\mathcal{T}_{[x_0]} \subset I_0 \times J_0\), and \(\mathcal{T}_{[x_0]} \cap \mathcal{T}_{[x_1]} = I_0 \times J_1\).

Set \(K_0 := K_{[x_0]}, K_1 := K_{[x_1]}\).

Let us start with a general lemma:

**Lemma 9.4.** Assume that \(q \in K_{[x_0]} \cap K_{[x_1]}\) for \([x_0], [x_1] \in \mathcal{T}\). Then \(q \in K_{[x]}\) for all \([x] \in [x_0]|x_1|\).

**Proof.** Let \([x] \in [x_0]|x_1|\), and \((k, k') \in \mathcal{T}_{[x]}\) (see Figure 9 with \([x] = [p_x]\)). If one of \(k, k'\) lies either in \(I_0\) or in \(J_1\), then \(b_{k, k'}(q) \leq 4S\) follows by assumption. So we may assume \(k, k' \in L\).

It suffices to show \(b_{k, k'}(q) \leq 4S\) for all \((k, k') \in L\), since the claim for \(b_{k, k'}'\) follows analogously.

Assume that \(b_{k, k'}(q) > 4S\) for some \(k, k' \in L\). Consider the lower endpoint \(p\) of \(l_{k, k', j'}\) and the lower endpoint \(p'\) of \(l_{k, k', i'}\). By construction, we have \(B_{i'}(p') < B_{i'}(p)\).

We have \(b_{j', k}(q) \leq 4S\), \(b_{i', k'}(q) \leq 4S\) and \(b_{k, k'}(q) > 4S\). So we obtain \(\angle_{p}(q, \nu_{j'}) < \pi/3\) from \(8.1.1\). Similarly, we have \(\angle_{p'}(q, \nu_{i'}) < \pi/3\). By Lemma \(6.2\) (and Remark \(6.3\)), this is a contradiction. \(\square\)

**Proposition 9.5.** Consider \(x, y \in K\) with \(d(x, y) < \varepsilon\) (for the \(\varepsilon\) from Construction \(9.3\)) and \(x \in K_0, y \in K_1\) for some \([x_0], [x_1] \in \mathcal{T}\).

Let \([q] \in [x_0]|x_1|\). Then

\[K_{[q]} \cap \{x, y\} \neq \emptyset.\]

**Proof.** We assume that \(x \notin K_{[q]}\), and show that this implies \(y \notin K_{[q]}\). Without loss of generality, there is \((k, k'_x) \in \mathcal{T}_{[q]}\) with \(b_{k, k'_x}(x) > 4S\) (note that neither \(k_x\) nor \(k'_x\) lies in \(I_0\), because \(x \in K_0\)). Pick \((i', j') \in \mathcal{T}_{[x_0]} \cap \mathcal{T}_{[x_1]}\) as above.

Let \(p_x\) be the lower endpoint of \(l_{k, k_x, k'_x}\).

We are going to show \(y \in K_{[p_x]}\), which implies \(y \in K_{[q]}\) by Lemma \(9.4\) (since \(B_{i'}([p_x]) \leq B_{i'}([q])\) by construction).

We have \(b_{k, k'}(x) \leq 4S, b_{k', k'}(x) \leq 4S\) and \(b_{k, k'}(x) > 4S\). So we have

\[\angle_{p_x}(x, \nu_{i'}) < \pi/3\] by \(8.1.5\) \hspace{1cm} (4)

Let \(p'\) be the lower endpoint of \(C_{[p_x]} \cap \{B_{i'} = B_{i'}([p_x])\}\). By (the proof of) Lemma \(8.1.1\), \(p'\) exists and satisfies \(b_{i, j}(p') \leq S\) for all \((i, j) \in \mathcal{T}_{[p_x]}\). In particular, we have \(d(x, p') \geq 3S\). This implies

\[\angle_{p'}(x, y) < \alpha/2\] (by construction of \(\varepsilon\)). \hspace{1cm} (5)
We claim that we also have
\[ \angle p'(x, \xi_{j',k'}) < \pi - \hat{\alpha}. \] (6)

Assume that this is not the case, and we have \( \angle p'(x, \xi_{j',k'}) \geq \pi - \hat{\alpha} \ast \). Further, we have \( \angle p'(x, \eta_{k,k'}^{x}) < 2\pi/3 + \hat{\alpha} \). Now \( b_{k,k'}^{x}(x) > 4S \) and \( b_{k,k'}^{x}(p') = b_{k',k}^{x}(p') \leq S \) (the equality follows from \( p' \in C_{[p_{x}]} \subset F_{j',k'} \cap F_{k,k'} \)); this implies \( d(p', x) > 11S/2 \) (by construction of \( \hat{\alpha} \)).

Taking \( b_{j'}^{x}(p') = -b_{j'}^{x}(p') \geq -S \) into account, \( \ast \) and \( d(p', x) > 11S/2 \) imply \( b_{j'}^{x}(x) > 4S \) (by construction of \( \hat{\alpha} \)), in contradiction to \( x \in \mathcal{K}_{0} \). Thus, (6) is proven.

Let us phrase the next steps as Lemmas:

**Lemma 9.6.** We have \( b_{k,k'}(y) \leq 4S \) for all \((k, k') \in \mathcal{T}_{[p_{x}]}\).

*Proof. Assume that the claim is false, i.e. there are \((k, k') \in \mathcal{T}_{[p_{x}]} \) with \( b_{k,k'}(y) > 4S \). Observe that neither \( k \) nor \( k' \) lie in \( J_{1} \), since \( y \in \mathcal{K}_{1} \). Let \( p_{y} \) be the lower endpoint of \( l_{k,k',j'} \). We have\[(7)\]
\[ \angle p_{y}(y, \nu_{j'}) < \pi/3 \]
by (3.1.3) (as in (4)). As for (5), we obtain
\[ \max(\angle p_{y}(y, \xi_{j',k}), \angle p_{y}(y, \xi_{j',k'})) < \pi - \hat{\alpha}. \]

Note that either \((j', k)\) or \((j', k')\) lie in \( \mathcal{T}_{[p_{x}]} \) (so \( p' \in F_{j',k} \) or \( p' \in F_{j',k'} \), and that \( B_{j'}(p_{y}) \leq B_{j'}(p') = B_{j'}(p_{x}) \). So Lemma 6.2 and Remark 6.3 yield a contradiction (for \( p', p_{y}, y \) and \( \hat{\alpha}/2 \)).
Lemma 9.7. We have \( b_{k,k'}(y) \leq 4S \) for all \( (k,k') \in \mathcal{T}_{[p_x]} \).

Proof. Assume that this is not the case, i.e. there are \( (k,k') \in \mathcal{T}_{[p_x]} \) with \( b_{k,k'}(y) > 4S \). Observe that neither \( k \) nor \( k' \) lie in \( J_1 \), since \( y \in K_1 \). This time, let \( p_y \) be the upper endpoint of \( l_{k,k',j'} \). We have

\[
\angle_{p_y}(y,\eta_{j'}) < \pi/3 \quad (8)
\]

by 8.1.5. As for (6), we obtain

\[
\max(\angle_{p_y}(y,\eta_{j'},k),\angle_{p_y}(y,\eta_{j',k'})) < \pi - \alpha.
\]

Note that (at least) one of \( (j',k) \) or \( (j',k') \) lie in \( \mathcal{T}_{[p_x]} \), and that \( B_{j'}(p_y) \leq B_{j'}(p) \). We may assume that \( (j',k) \in \mathcal{T}_{[p_x]} \) (by exchanging \( k, k' \) if necessary).

Since \( p_y \) is the upper endpoint of \( l_{k,k',j'} \), and \( p' \in \hat{F}_{k,k'} \cap F_{j,k} \supset C_{[p_x]} \), we have \( \angle_{p_y}(p',\xi_{j'}) \geq \pi/3 \) (by Remark 6.4).

So we have a contradiction to Lemma 6.4. \( \square \)

This finishes the proof of Proposition 9.5. \( \square \)

Proposition 9.8. Let \( x, y \in K \) with \( d(x,y) < \varepsilon \) (for the \( \varepsilon \) from Construction 9.3). Then there exists \( [q] \in \mathcal{T} \) such that \( xy \subset K_0 \).

Proof. As usual, let \( x \in K_0, y \in K_1 \). By Lemma 9.4, we know that the sets

\[
I_x := \{ [z] \in [x_0][x_1] \mid x \in K_{[z]} \},
\]

and \( I_y \) similarly for \( y \), are intervals. By Proposition 9.5, \( I_x \cup I_y \) covers \([x_0][x_1]\). We want to show that \( I_x \cap I_y \neq \emptyset \).\(^8\)

When we assume that this is not the case, then we may assume that \( I_x = \{ [x_0] \}, I_y = [x_0][x_1] \setminus \{ [x_0] \} \).

Essentially, we want to show that \( I_x \) is open; more specifically, we will show that if \( [b] \in [x_0][x_1] \) is close enough to \([x_0]\), then \( x \in K_{[b]} \).

Pick \( (j',j') \in \mathcal{T}_{[x_0]} \cap \mathcal{T}_{[x_1]} \) such that \( B_{j'}([x_1]) < B_{j'}([x_0]) \).

Since \( y \notin K_0 \), there exist (without loss of generality) \( (k,k') \in \mathcal{T}_{[x_0]} \) such that \( b_{k,k'}(y) > 4S \). We have \([p_{k,k',j'}] \in [x_0][\tilde{y}] \), so by (the proof of) Proposition 9.5, \( x \in K_{[p_{k,k',j'}]} \) holds, implying \([p_{k,k',j'}] = [x_0]\).

Let \( p \) be the lower endpoint of \( \hat{C}_{[x_0]} \). By 8.1.6, we have \( \angle_{p}(y,\nu_{j'}) < \pi/3 \). By [KL97] 4.1.2, there exists a point \( a' \in \overline{py} \setminus \{ p \} \) such that \( S' := \text{Conv}(p,a',\xi_{j',j'}) \) is a flat half-strip and \( a' \xi_{j',j'} \cap \overline{pu_{j'}} \neq \emptyset \).

\(^8\)If \( X \) is discrete or \( A \) is finite, then the tree \( \mathcal{T} \) is discrete. In this case, it is easy to see that both \( I_x \) and \( I_y \) are open, so the claim follows.
Similarly, there exists a point \( a'' \in \overline{py} \setminus \{p\} \) such that \( S'' := \text{Conv}(p, a'', \nu_j) \) is a flat half-strip.

Pick \( a \in \text{int}(\overline{pa} \cap \overline{pa''}) \). Then by construction, we have

\[
\angle_a(y, \nu_j) < \pi/3.
\]

As for (\( \emptyset \)), we find \( \angle_p(y, \xi_i, \nu_j) < \pi - \hat{\alpha} \). Since \( \angle_a(y, \xi_i, \nu_j) = \angle_p(y, \xi_i, \nu_j) \), the point \( a \) has the same property (which we will need in order to apply Lemma 6.2).

Now let \( \{b\} := a\xi_i, \nu_j \cap \overline{pm_j} \). This point exists by construction and lies in \( F_{i,j}' \). So \( K_{b} \) is defined.

Observe that \( B_{j'}(b) < B_{j'}(p) = B_{j'}([x_0]) \) by construction, so \( x \notin K_{b} \).

We claim that \( x \notin K_{b} \) leads to a contradiction, which finishes the proof.

**Step 1:** \( b_{k_x, k_x'}(x) \leq 4S \) for all \( (k_x, k_x') \in T_{b} \).

Assume that \( b_{k_x, k_x'}(x) > 4S \) for some \( (k_x, k_x') \in T_{b} \). Let \( p' \) be the lower endpoint of \( \hat{C}_{[k_x, k_x', \nu_i]} \subset F_{i,j}' \). By construction, we have

\[
B_{j'}(b) \leq B_{j'}(p') < B_{j'}(p)
\]

(the last inequality follows from \( (k_x, k_x') \notin T_{[x_0]} \)), and by 8.1.6, we have

\[
\angle_{p'}(x, \nu_j') < \pi/3.
\]

We claim that Lemma 6.2 leads to a contradiction (for \( a, p', y \) and \( \hat{\alpha}/2 \); as in the proof of Lemma 9.6). If \( a \notin F_{i,j}' \), then \( \angle_p(a, \xi_i, \nu_j) > 2\pi/3 \), but \( \angle_{p'}(p', \xi_i, \nu_j') \leq 2\pi/3 \) (if \( b_{i,j'}(p) = -S \), this is trivial, because \( p' \in S_{i,j'} \); otherwise, it follows from 7.23). Therefore, \( p' \in S' \) (because \( B_{j'}(p') \geq B_{j'}(a) \)), so we can apply Lemma 6.2 as claimed.

**Step 2:** \( b_{k_x, k_x'}(x) \leq 4S \) for all \( (k_x, k_x') \in T_{b} \).

Assume that \( b_{k_x, k_x'}(x) > 4S \) for some \( (k_x, k_x') \in T_{b} \). This time, let \( p' \) be the upper endpoint of \( \hat{C}_{[k_x, k_x', \nu_i]} \subset F_{i,j}' \). As before, we have \( B_{j'}(b) \leq B_{j'}(p') < B_{j'}(p) \), and by 8.1.6, we have

\[
\angle_{p'}(x, \mu_i) < \pi/3.
\]

We have \( \angle_{p'}(p', \xi_i, \nu_j') \leq 2\pi/3 \) as above. If \( \angle_{p'}(a, \xi_i, \nu_j') \geq \pi/3 \), we can apply Lemma 6.3 (as in the proof of Lemma 9.7). Otherwise, we have \( a \in F_{i,j}' \) and \( \overrightarrow{ay} \in a\mu_j \subset \Sigma_a(X) \). In this case, Lemma 6.2 applies as in Step 1 (after exchanging the \( \nu_j \) with the \( \mu_j \)).

Together, steps 1 and 2 show that \( x \in K_{b} \), the desired contradiction. \( \square \)
9.2 The proof of Theorem 4

Proposition 9.8 says: Whenever we consider \( x, y \in \mathcal{K} \) with \( d(x, y) < \varepsilon \), then \( \overline{xy} \subset \mathcal{K} \). This property is inherited by the closure \( \overline{\mathcal{K}} \). This implies that \( \overline{\mathcal{K}} \) is \( \delta \)-locally convex for every \( \delta < \varepsilon /2 \). From Proposition 4.1 we obtain:

**Theorem 4.** \( \overline{\mathcal{K}} \) is convex.

It is hard to decide whether \( \overline{\mathcal{K}} \) is of rank 1. Hence, we bring in additional conditions again: Pick \( R > 10S \). For \( [x] \in \mathcal{T} \), recall the set \( \hat{C}_{[x]} \) from Lemma 9.1, and let

\[
\tilde{C}_{[x]} := \mathcal{K}_{[x]} \cap B_R(\hat{C}_{[x]}),
\]

\[
\mathcal{C} := \bigcup_{[x] \in \mathcal{T}} \tilde{C}_{[x]}.
\]

As for \( \mathcal{K} \), we find that \( \mathcal{C} \) is connected.

We want to show that \( \overline{\mathcal{C}} \) is convex, by the same tools as for \( \overline{\mathcal{K}} \):

**Proposition 9.9.** There exists \( \varepsilon > 0 \) such that for \( x, y \in \mathcal{C} \) with \( d(x, y) < \varepsilon \), we have \( \overline{xy} \subset \mathcal{C} \).

**Proof.** We pick \( \varepsilon, \hat{\alpha} \) such that they satisfy the conditions from the proof of Lemma 8.7 as well as those from Construction 9.3; this is possible, because in both constructions, we first impose conditions on \( \hat{\alpha} \), and afterwards, we require \( \varepsilon > 0 \) to be small enough.

Assume that \( x \in \tilde{C}_{[x_0]}, y \in \tilde{C}_{[x_1]} \). We know from Proposition 9.8 that there is \( [q] \in \overline{[x_0][x_1]} \) with \( \overline{xy} \subset \mathcal{K}_{[q]} \). If \( \{x, y\} \subset \tilde{C}_{[q]} \), there is nothing to show.

Assume that \( x \not\in \tilde{C}_{[q]} \): We know that \( x \in \mathcal{K}_{[z]} \) for all \( [z] \in \overline{[x_0][q]} \) by Lemma 9.4. Hence, we have

\[
\{[z] \in \overline{[x_0][q]} \mid x \in \tilde{C}_{[z]}\} = \{[z] \in \overline{[x_0][q]} \mid x \in B_R(\hat{C}_{[z]})\}.
\]

(9)

Lemmas 9.1 and 7.19 imply that \( \hat{C}_{[z]} \) varies continuously along \( \overline{[x_0][q]} \).

Therefore (by pushing \( [x_0] \) towards \( [q] \) as far as possible), we may assume \( x \not\in \tilde{C}_{[z]} \) for every \( [z] \in \overline{[x_0][q]} \setminus \{[x_0]\} \), and \( d(x, \hat{C}_{[x_0]}) = R \) (*).

Let \( (i', j') \in \mathcal{T}_{[x_0]} \cap \mathcal{T}_{[x_1]} \) such that \( B_{j'}([x_1]) < B_{j'}([x_0]) \) (as usual). For every singular \( [z] \in \overline{[x_0][q]} \), we have

\[
b_{i', j'}(\hat{C}_{[z]}) = [-S, S],
\]

(10)

since otherwise, \( d(x, \hat{C}_{[z]}) \leq 10S < R \) (by 8.11), implying \( x \in \hat{C}_{[z]} \).

Similarly, we may assume \( y \not\in \tilde{C}_{[z]} \) for every \( [z] \in \overline{[q][x_1]} \setminus \{[x_1]\} \), and we get (10) for every singular \( [z] \in \overline{[q][x_1]} \).
Recalling from Figure 6 what the sets $\mathcal{F}_z$ look like, we may conclude that $\bigcup_{z \in [x_0][x_1]} \hat{\mathcal{C}}_z$ is convex (a convex subset of the strip $S_{i,j'} = \text{Conv}(K_i', K_j')$; not necessarily a rectangle, if $[x_0]$ and/or $[x_1]$ are not singular), and so is

$$\bigcup_{z \in [x_0][x_1]} B_R(\hat{\mathcal{C}}_z) = B_R(\bigcup_{z \in [x_0][x_1]} \hat{\mathcal{C}}_z).$$

Along the lines of Lemma 8.7 we obtain $y \in K_{[x_0]}$ and similarly $x \in K_{[x_1]}$ (see below). Then it is immediate (from Lemma 9.4 and convexity of the metric) that $\overline{xy} \subset C$.

Let us explain the argument for $y \in K_{[x_0]}$:

Assume that $y \not\in K_{[x_0]}$, so without loss of generality, we have $b_{k,k'}(y) > 4S$ for some $(k, k') \in T_{[x_0]}$.

Consider the lower endpoint $p'$ of $l_{k,k',j'}$. It satisfies $\angle_{p'}(y, \nu_{j'}) < \pi/3$ by Lemma 8.10.

Let $x' := \pi_{C_{[x_0]}}(x)$, and $\{y\} := \{\pi_{C_{[x']}}(x')\} = \overline{x'y} \cap \hat{C}_{[y']}$. If $\angle_{p'}(y, \eta_{j'}) > \pi/2$, we get a contradiction to either the sum of angles in a triangle, or Lemma 6.2.

Observe that $d(y', x) \geq R$ and $\angle_{p'}(x, \eta_{j'}) \geq \pi/2$ (because $\angle_{x'}(x, \eta_{j'}) \geq \pi/2$ by (**) and (**)). This implies $d(y', y) \geq R - \varepsilon$ and $\angle_{p'}(y, \eta_{j'}) \geq \pi/2 - \hat{\alpha}$.

Now we obtain a contradiction as in the proof of Lemma 8.7.

Just as for $\tilde{\mathcal{C}}$, we now obtain that $\hat{\mathcal{C}}$ is convex. We claim that it is also of rank $1$.

**Theorem 5.** $\hat{\mathcal{C}}$ is a convex rank $1$-subset of $X$.

**Proof.** If $\partial_T \hat{\mathcal{C}}$ is not a $0$-dimensional subbuilding, then there exists (without loss of generality) a point $\xi_{i,j} \in \partial_T \hat{\mathcal{C}}$. In fact, by [BL05], $\partial_T \hat{\mathcal{C}}$ is a subbuilding or has a center. So either all $\eta_{i,j}, \xi_{i,j}$ are in the asymptotic boundary, or all $\xi_{i,j}$ agree (again without loss of generality; it could also be the $\eta_{i,j}$ that agree).

So consider a point $x \in F_{i,j}$ with $b_{i,j}(x) = S + 2R + 3\varepsilon$ (for some $\varepsilon > 0$). Let $x' := \pi_{C_{[x]}}(x)$. Then $d(x, x') \geq 2R + 3\varepsilon$.

To finish the proof, it suffices to lead the following assumption to a contradiction: There exists $[x'''] \in T$ such that $x \in B_{\varepsilon}(\hat{C}_{[x''']})$.

Assume the contrary, and set $x''' := \pi_{C_{[x]}}(x)$. Obviously, $d(x', x''') \geq R + 2\varepsilon$. Pick $i'$ such that $(i', j) \in T_{[x]} \cap T_{[x''']}$ (such an $i'$ exists, after exchanging $i, j$ if necessary).

Since $x', x''' \in S_{i',j}$, the inequality $2S < (R + 2\varepsilon)/2$ implies that

$$\angle_{x'}(x'', \xi_{i,j}) = \angle_{x'}(x''', x) \geq \pi/3.$$
Now triangle comparison yields \( d(x'', x) \geq R + 2\varepsilon \), the desired contradiction.

By definition and Lemma 9.1, we have \( S_{i,j} \subset \bar{C} \) for all \( i, j \in I \). Hence, we have \( A \subset \partial_T \bar{C} \). We have just shown that \( \bar{C} \subset B_{2R}(\bigcup_{i,j \in I} S_{i,j}) \). Therefore, \( \partial_T \bar{C} \) is precisely the closure of \( A \subset \partial_\infty X \) in the cone topology. The proof of Theorem 1 is now finished.

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