Polynomial Solutions of Differential Equations

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ABSTRACT. We show that any differential operator of the form
\[ L(y) = \sum_{k=0}^{k=N} a_k(x)y^{(k)}, \]
where \( a_k \) is a real polynomial of degree \( \leq k \), has all
real eigenvalues in the space of polynomials of degree at most \( n \), for all \( n \).
The eigenvalues are given by the coefficient of \( x^n \) in \( L(x^n) \).
If these eigenvalues are distinct, then there is a unique monic polynomial
of degree \( n \) which is an eigenfunction of the operator \( L \)- for every non-negative integer \( n \). As an application we recover Bochner's classification
of second order ODEs with polynomial coefficients and polynomial
solutions, as well as a family of non-classical polynomials.

The subject of polynomial solutions of differential equations is a classical theme, going
back to Routh [10] and Bochner [3]. A comprehensive survey of recent literature is given
in [6]. One family of polynomials- namely the Romanovski polynomials [4, 9] is missing
even in recent mathematics literature on the subject [8]; these polynomials are the main
subject of some current Physics literature [9, 11]. Their existence and – under a mild
condition - uniqueness and orthogonality follow from the following propositions. The
proofs use elementary linear algebra and are suitable for class-room exposition. The same
ideas work for higher order equations [1].

Proposition 1
Let \( L(y) = \sum_{k=0}^{k=N} a_k(x)y^{(k)}, \) where \( a_k \) is a real polynomial of degree \( \leq k \). Then \( L \) operates
on the space \( P_n \) of all polynomials of degree at most \( n \). It has all real eigenvalues and
the eigenvalues are given by the coefficient of \( x^j \) in \( L(x^j) \) for all \( j \leq n \).
If the eigenvalues are distinct, then \( L \) has, up to a constant, a unique polynomial of every
degree which is an eigenfunction of \( L \).
Proof:
Let \( L \) be as in the statement of the proposition. Since \( L(x^n) \) is a sum of a multiple of \( x^n \) plus lower order terms, it is clear that \( L \) operates on every \( P_j, j \leq n \). Therefore the eigenvalues are given by the coefficient of \( x^j \) in \( L(x^j) \) and \( L \) has eigenfunctions in each \( P_j \).

Assume that the eigenvalues of \( L \) are distinct. Then \( P_n \) has a basis of eigenfunctions and, for reasons of degree, there must be an eigenfunction of degree \( n \), for every \( n \). Therefore, up to a constant, there is a unique eigenfunction of degree \( n \) for all \( n \).

We now concentrate on second order operators, leaving the higher order case to [1]. Let \( L(y) = a(x)y'' + b(x)y' \), where \( \deg(a) \leq 2, \deg(b) \leq 1 \). Following Bochner [3] if \( \deg(a) = 2 \) then by scaling and translation, we may assume that \( a(x) = x^2 - 1, x^2 + 1 \) or \( x^2 \). Applying the above proposition we then have the following result.

**Proposition 2**

(i) The equation \( (x^2 + \varepsilon)y'' + (\alpha x + \beta)y' + \lambda y = 0, \ \varepsilon = 1, -1 \) has unique monic polynomial solutions in every degree if \( \alpha > 0 \) or if \( \alpha < 0 \) and it is not an integer. If \( \alpha = -(n + k - 1) \) for \( 0 \leq k \leq (n-1) \), then the eigenspace in \( P_n \) for eigenvalue \( \lambda = n(n-1) + \alpha n \) is 2-dimensional.

(ii) The equation \( xy'' + (\alpha x + \beta)y' + \lambda y = 0 \) has unique monic polynomial solutions in every degree if \( \alpha \neq 0 \)

(iii) The equation \( y'' + (\alpha x + \beta)y' + \lambda y = 0 \) has unique monic polynomial solutions in every degree if \( \alpha \neq 0 \)

In this proposition there is no claim to any kind of orthogonality properties. Nevertheless, the non-classical functions appearing here are of great interest in Physics and their properties and applications are investigated in [4, 9, 11].
The classical Legendre, Hermite, Laguerre and Jacobi make their appearance as soon as one searches for self-adjoint operators. Their existence and orthogonality properties [cf:8, p.80-106,2,7] can be obtained elegantly in the context of elementary Sturm-Liouville theory.

**Proposition 3**

Let $L$ be the operator defined by $L(y) = a(x)y'' + b(x)y' + c(x)y$ on a linear space $C$ of functions which are at least two times differentiable on a finite interval $I$.

Define a bilinear function on $C$ by

$$(y,u) = \int_I p(x)y(x)u(x)dx,$$

where $p$ is two times differentiable and non-negative and does not vanish identically in any subinterval of $I$.

Then

$$(Ly,u) - (y,Lu) = pa(uy' - u'y)\bigg|_\alpha^\beta$$

if

$$(pa)' = pb.$$

**Proof:**

Let $\alpha, \beta$ be the end points of $I$. So

$$(Ly,u) = \int_\alpha^\beta p(ay'' + by' + cy)u dx.$$

Using integration by parts, we find that $(Ly,u) - (y,Lu)$ will contain only boundary terms if $(pau)' - (pbu)' = pau'' + pbu'$, for all $u$.

This simplifies to

$$[(pa)' - (pb)']u + 2(pa)'u' = 2 pb'.$$

Equating coefficients of $u$ and $u'$ on both sides, we get the differential equations for $p$:

$$(pa)'' - (pb)' = 0 \text{ and } (pa)' = pb,$$

so in fact we need only the equation

$$(pa)' = pb.$$

The boundary terms now simplify to
\[(pau)y' - (pa)y - (pu)u' + (pu)u\bigg|_{\alpha}^{\beta} = pa(uy' - u'y)\bigg|_{\alpha}^{\beta}\]

The differential equation for the weight is \(a'p + ap' = pb\), which integrates to

\[p = e^{\int_{\alpha}^{\beta} \frac{dx}{a}} = \frac{1}{|a|} e^{\int_{\alpha}^{\beta} \frac{dx}{a}}.

\[\square\]

**Examples:**

1. **Jacobi polynomials**

First note that for any differentiable function \(f\) with \(f'\) continuous, the integral

\[\int_{0}^{\alpha} \frac{f(x)}{x^\alpha} dx\]

is finite if \(\alpha < 1\) as one sees by using integration by parts.

Consider the equation \((1 - x^2)y'' + (\alpha x + \beta)y' + \lambda y = 0\). As above, the weight function \(p(x)\) for the operator

\[L(y) = (1 - x^2)y'' + (\alpha x + \beta)y'\]

is

\[p(x) = \frac{1}{1 - x^2} e^{\int_{\frac{\beta - a}{1-x}}^{\frac{\beta - a}{1+x}} dx} = \frac{1}{(1-x)^{\frac{\beta - a}{2}} (1+x)^{\frac{\beta - a}{2}}}.

So \(\int_{-1}^{1} p(x)f(x)dx\) would be finite if \(\beta + \alpha < 0\) and \(-\beta + \alpha < 0\), that is, if \(\alpha < \beta < -\alpha\).

The weight is not differentiable at the end points of the interval. So, first consider \(L\) operating on twice differentiable functions on the interval \([-1 + \epsilon, 1 - \epsilon]\). If \(u, v\) are functions in this class then by *Proposition 3*,

\[\int_{-1+\epsilon}^{1-\epsilon} p(x)L(u(x))v(x)dx - \int_{-1+\epsilon}^{1-\epsilon} p(x)u(x)L(v(x))dx = \int_{-1+\epsilon}^{1-\epsilon} p(x)u(x)(v(x) - u(x)v(x))dx\]

Moreover, \((1 - x^2)p(x) = (1 - x)^{-(\beta + \alpha)/2} (1 + x)^{-(\beta - \alpha)/2}\) is continuous on the interval \([-1,1]\) and vanishes at the end points -1 and 1. Therefore, if we define

\[(u,v) = \lim_{\epsilon \to 0} \int_{-1+\epsilon}^{1-\epsilon} p(x)u(x)v(x)dx\],

then \(L\) would be a self-adjoint operator on all
polynomials of degree $n$ and so, there must be, up to a scalar, a unique polynomial which is an eigen function of $L$ for eigenvalue $-n(n-1)+n\alpha$.

So these polynomials satisfy the equation

$$(1-x^2)y'' + (\alpha x + \beta) y' + (n(n-1) - n\alpha)y = 0$$

and this equation has unique monic polynomial eigenfunctions of every degree, which are all orthogonal. The Legendre and Chebyshev polynomials are special cases, corresponding to the values $\alpha = -1, -2, -3$ and $\beta = 0$.

(2) The equation $t(1-t)y'' + (1-t)y + \lambda y = 0$

This equation is investigated in [5] and the eigenvalues determined experimentally, by machine computations. Here, we will determine the eigenvalues in the framework provided by Proposition 3.

Let $L(y) = t(1-t)y'' + (1-t)y'$. Let $P_n$ be the space of all polynomials of degree at most $n$. As $L$ maps $P_n$ into itself, the eigenvalues of $L$ are given by the coefficient of $x^n$ in $L(x^n)$. The eigenvalues turn out to be $-n^2$. As these eigenvalues are distinct, there is, up to a constant, a unique polynomial of degree $n$ which is an eigenfunction of $L$.

The weight function is $p(t) = \frac{1}{(1-t)}$ on the interval $[0,1]$ and it is not integrable. However, as $L(y)(1) = 0$, the operator maps the space $V$ of all polynomials that are multiples of $(1-t)$ into itself. Moreover,

$$\int_0^1 p(t)(1-t)\psi(t))^2 \, dt$$

is finite.

The requirement for $L$ to be self-adjoint on $V$ is $\langle (\xi\eta' - \xi'\eta) \rangle_0 = 0$ for all $\xi, \eta$ in $V$. As $\xi, \eta$ vanish at 1, the operator $L$ is indeed self-adjoint on $V$.

Let $V_n = (1-t)P_n$, where $P_n$ is the space of all polynomials of degree at most $n$. 
As the codimension of $V_n$ in $V_{n+1}$ is 1, the operator $L$ must have an eigenvector in $V_n$ for all the degrees from 1 to $(n+1)$.

If $y = (1-t)\psi$ is an eigenfunction and $\deg(\psi) = n$ then, by the argument as in the examples above, we see that the corresponding eigenvalue is $\lambda = -(n+1)^2$.

Therefore, up to a scalar, there is a unique eigenfunction of degree $(n+1)$ which is a multiple of $(1-t)$ and all these functions are orthogonal for the weight

$$p(t) = \frac{1}{(1-t)}.$$ Using the uniqueness up to scalars of these functions, the eigenfunctions are determined by the differential equation and can be computed explicitly.

(3) The Finite Orthogonality of Romanovski Polynomials

These polynomials are investigated in Refs [11,9] and their finite orthogonality is proved also proved there. Here, we establish this in the framework of Proposition 3.

The Romanovski polynomials are eigenfunctions of the operator

$$L(y) = (1 + x^2) y'' + (\alpha x + \beta) y'.$$

For $\alpha > 0$ or $\alpha < 0, \alpha$ not an integer, there is only one monic polynomial in every degree which is an eigenfunction of $L$; for $\alpha$ a non-positive integer, the eigenspaces can be 2 dimensional for certain degrees (Propostion2).

The formal weight function is $p(x) = (x^2 + 1)^{\frac{\alpha-2}{2}} e^{\beta \tan^{-1}(x)} = (x^2 + 1)^{\frac{\alpha}{2}} e^{\beta \tan^{-1}(x)}$, where $\gamma = (\alpha - 2)$. Therefore, a polynomial of degree $N$ is integrable over the reals with weight $p$ if and only if $((N + \gamma + 1) < 0$ and if the product of two polynomials $P, Q$ is integrable, then the polynomials are themselves integrable for the weight $p$.

Arguing as in the proof of Proposition 3, we find that

$$(LP, Q) - (P, LQ) = (x^2 + 1)p(x)(PQ' - P'O')\big|_{-\infty}^{\infty} = 0,$$ because the product
$(x^2 + 1)p(x)(PQ' - P'O)$ is asymptotic to $x^{(2 + \gamma + \deg(P) + \deg(Q) - 1)} = x^{\deg(P) + \deg(Q) + \gamma + 1}$ and $(\deg(P) + \deg(Q) + \gamma + 1) < 0$.

Therefore, if $P, Q$ are integrable eigenfunctions of $L$ with different eigenvalues and $(\deg(P) + \deg(Q) + \gamma + 1) < 0$, then $P, Q$ are orthogonal.

For several non-trivial applications to problems in Physics, the reader is referred to the paper [9].

**Conclusion:** In this note, which should have been written at least hundred years ago, we have rederived several results from classical and recent literature from a unified point of view by a straightforward application of basic linear algebra. Some of these polynomials are not discussed in the standard textbooks on the subject, e.g. [8]- as pointed out in Ref [9].

We have also derived the orthogonality- classical as well as finite- of these polynomials from a unified point of view.

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