KUMJIAN-PASK ALGEBRAS
OF LOCALLY CONVEX HIGHER-RANK GRAPHS

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Abstract. The Kumjian-Pask algebra of a higher-rank graph generalises the Leavitt path algebra of a directed graph. We extend the definition of Kumjian-Pask algebra to row-finite higher-rank graphs $\Lambda$ with sources which satisfy a local-convexity condition. After proving versions of the graded-uniqueness theorem and the Cuntz-Krieger uniqueness theorem, we study the Kumjian-Pask algebra of rank-2 Bratteli diagrams by studying certain finite subgraphs which are locally convex. We show that the des-ourcification procedure of Farthing and Webster yields a row-finite higher-rank graph $\tilde{\Lambda}$ without sources such that the Kumjian-Pask algebras of $\tilde{\Lambda}$ and $\Lambda$ are Morita equivalent. We then use the Morita equivalence to study the ideal structure of the Kumjian-Pask algebra of $\Lambda$ by pulling the appropriate results across the equivalence.

1. Introduction

The Kumjian-Pask algebras were introduced in [8] as higher-rank analogues of the Leavitt path algebras associated to a directed graph $E$. Since they were introduced in [1] and [7], the Leavitt path algebras have attracted a lot of attention, see, for example, [2, 8, 14, 16, 25, 27, 28, 29]. A k-graph is a category $\Lambda$ with a degree functor $d : \Lambda \to \mathbb{N}^k$ which generalise the path category of $E$ and the length function $\lambda \mapsto |\lambda|$ on the paths $\lambda$ of $E$, respectively. Thus we think of the set $\Lambda^0$ of objects as vertices and of the morphisms $\lambda \in \Lambda$ as paths of “shape” or degree $d(\lambda)$, and demand that paths factor uniquely: if $d(\lambda) = m + n$ in $\mathbb{N}^k$, then there exist unique $\mu, \nu \in \Lambda$ with $d(\mu) = m$ and $d(\nu) = n$ such that $\lambda = \mu \nu$.

Let $R$ be a commutative ring with 1 and $\Lambda$ a row-finite $k$-graph without sources. The authors of [8] construct a graded algebra $\text{KP}_R(\Lambda)$, called the Kumjian-Pask algebra, which is universal for so-called Kumjian-Pask families. If $k = 1$ then $\text{KP}_R(\Lambda)$ is isomorphic to the Leavitt path algebra $L_R(E)$, where $E$ is the directed graph with vertices the objects of $\Lambda$ and edges the paths of degree 1.

Here we define a Kumjian-Pask algebra for row-finite $k$-graphs which may have sources but are “locally convex”, a condition which restricts the types of sources that can occur. We were motivated by the study of $C^*$-algebras and Kumjian-Pask algebras of “rank-2 Bratteli diagrams” in [19, 8]. These Bratteli diagrams are 2-graphs $\Lambda$ without sources, but their Kumjian-Pask algebras can profitably be studied by looking at finite subgraphs $\Lambda_N$ which have sources but are locally convex. In [8], $\text{KP}_C(\Lambda)$ was analysed by embedding it inside $C^*(\Lambda)$, and then using the $C^*$-subalgebras $C^*(\Lambda_N)$ of the subgraphs to deduce...
results about $\text{KP}_C(\Lambda)$. A sample theorem obtained in this way illustrates how the dichotomy for Leavitt path algebras of [3, Theorem 4.5] does not hold for Kumjian-Pask algebras. In particular, [8, Theorem 7.10] gives a class $C$ of rank-2 Bratteli diagrams such that for each $\Lambda \in C$, $\text{KP}_C(\Lambda)$ is simple but is neither purely infinite nor locally matricial. A key performance indicator for our new definition of the Kumjian-Pask algebra was an extension of [3, Theorem 7.10] to arbitrary fields, and this is achieved in Theorem 5.8 below. For more motivation, the K-theory of the $C^*$-algebras of these rank-2 Bratteli diagrams was computed in [19] as a direct limit of the K-groups of suitable $C^*(\Lambda_N)$, and a similar approach should work to compute the algebraic K-theory of $\text{KP}_R(\Lambda)$.

Now let $\Lambda$ be a locally convex, row-finite $k$-graph. After finding the appropriate notion of Kumjian-Pask family of $\Lambda$ in this setting, we have obtained a new Kumjian-Pask algebra $\text{KP}_R(\Lambda)$ with a very satisfactory theory: $\text{KP}_R(\Lambda)$ is generated by a universal Kumjian-Pask family $(p, s)$ and the properties of $(p, s)$ ensure that $\text{KP}_R(\Lambda) = \text{span}\{s_\lambda s_\mu^*: \lambda, \mu \in \Lambda\}$ (see [3]), and there are versions of both the graded-uniqueness and the Cuntz-Krieger uniqueness theorems (see [4]). Then, several applications of the graded-uniqueness theorem shows that for $\Lambda$ in the class $C$ mentioned above, $\text{KP}_R(\Lambda)$ is neither purely infinite nor locally matricial (see [5]).

There is a construction by Farthing [13] and Webster [30], called the desourcification of $\Lambda$, which yields a row-finite $k$-graph $\tilde{\Lambda}$ without sources such that the $C^*$-algebras $C^*(\tilde{\Lambda})$ and $C^*(\Lambda)$ are Morita equivalent. In [6], we show that $\text{KP}_R(\tilde{\Lambda})$ and $\text{KP}_R(\Lambda)$ are Morita equivalent as well. This result is new even when $k = 1$ and $\Lambda$ is the path category of a row-finite directed graph. In §8-9 we study the ideal structure of $\text{KP}_R(\Lambda)$ by pulling the relevant results for $\text{KP}_R(\tilde{\Lambda})$ from [8] across the Morita equivalence. Thus we obtain graph-theoretic characterisations of basic simplicity and simplicity, and show that there is a lattice isomorphism between the graded basic ideals of $\text{KP}_R(\Lambda)$ and the saturated, hereditary subsets of $\Lambda^0$.

2. Preliminaries

We write $\mathbb{N}$ for the set of non-negative integers. We view $\mathbb{N}$ as a category with one object. Fix $k \in \mathbb{N} \setminus \{0\}$. We often write $n \in \mathbb{N}^k$ as $(n_1, \ldots, n_k)$, and say $m \leq n$ in $\mathbb{N}^k$ if and only if $m_i \leq n_i$ for all $1 \leq i \leq k$. We use $e_i$ for the usual basis elements in $\mathbb{N}^k$, so that $e_i$ is 1 in the $i$th coordinate and 0 in the others. We denote the join and meet in $\mathbb{N}^k$ by $\lor$ and $\land$ respectively.

A $k$-graph $(\Lambda, d)$ is a countable category $\Lambda$ with a functor $d : \Lambda \to \mathbb{N}^k$, called the degree map, satisfying the factorisation property: if $d(\lambda) = m + n$ for some $m, n \in \mathbb{N}^k$, then there exist unique $\mu$ and $\nu$ in $\Lambda$ such that $d(\mu) = m, d(\nu) = n$ and $\lambda = \mu \nu$. In this case, we often write $\lambda(0, m)$ for $\mu$.

We denote the set of objects in $\Lambda$ by $\Lambda^0$ and use the factorisation property to identify the morphisms $d^{-1}(0)$ of degree 0 and $\Lambda^0$. We write $r$ and $s$ for the domain and codomain maps from $\Lambda$ to $\Lambda^0$. The path category associated to a directed graph is a 1-graph, and motivated by this we call $r$ and $s$ the range and source maps, and elements of $\Lambda$ and $\Lambda^0$ paths and vertices, respectively. For $v \in \Lambda^0$ and $m \in \mathbb{N}^k$, we write

$$\Lambda^m := \{\lambda \in \Lambda : d(\lambda) = m\}$$

and

$$\Lambda^{\neq 0} := \Lambda \setminus \Lambda^0,$$

$$v\Lambda := \{\lambda \in \Lambda : r(\lambda) = v\}$$

(this is denoted $\Lambda(v)$ in [24]),

$$v\Lambda^m := \Lambda^m \cap v\Lambda.$$
Example 2.1. Fix \( m \in (\mathbb{N} \cup \{\infty\})^k \) and define
\[
\Omega_{k,m} := \{(p,q) \in \mathbb{N}^k \times \mathbb{N}^k : p \leq m \}.
\]
This is a category with objects
\[
\Omega^0_{k,m} = \{ p \in \mathbb{N}^k \mid p \leq m \},
\]
and range and source maps \( r(p,q) = p \) and \( s(p,q) = q \). Paths \((p,q)\) and \((r,s)\) are composable if and only if \( q = r \), and then \((p,q)(s) = (p,s)\). With \( d : \Omega_{k,m} \to \mathbb{N}^k \) defined by \( d((p,q)) = q - p \), the pair \((\Omega_{k,m}, d)\) is a \( k \)-graph. We write \( \Omega_k \) when \( m \) is infinite in every coordinate.

Let \( \Lambda \) be a \( k \)-graph. Then \( \Lambda \) is row-finite if \( v\Lambda^\leq n \) is finite for every \( v \in \Lambda^0 \) and \( n \in \mathbb{N}^k \). A vertex \( v \in \Lambda^0 \) is a source if there exists \( n \in \mathbb{N}^k \) such that \( v\Lambda^\leq n = \emptyset \), that is, \( v \) receives no paths of degree \( n \). We say \( \Lambda \) is locally convex if for every \( v \in \Lambda^0 \), \( 1 \leq i, j \leq k \) with \( i \neq j \), \( \lambda \in v\Lambda^\leq n \) and \( \mu \in v\Lambda^\leq m \), the sets \( s(\lambda)\Lambda^\leq s(\lambda) \) and \( s(\mu)\Lambda^\leq s(\mu) \) are nonempty [20, Definition 3.10]. Thus if \( \Lambda \) has no sources, then \( \Lambda \) is locally convex. In this paper we only consider locally convex, row-finite \( k \)-graphs.

Paths, infinite paths and boundary paths. The Cuntz-Krieger relation (KP4) for \( k \)-graphs without sources (see Section 3) involves the sets \( v\Lambda^\leq n \) of paths of degree \( n \) with range \( v \). When \( \Lambda \) has sources, \( v\Lambda^\leq n \) could be empty. The technical innovation in [20] is to introduce the set \( \Lambda^\leq n \) consisting of paths \( \lambda \) with \( d(\lambda) \leq n \) which cannot be extended to paths \( \lambda \mu \) with \( d(\lambda) < d(\lambda \mu) \leq n \). Thus
\[
\Lambda^\leq n := \{ \lambda \in \Lambda : d(\lambda) \leq n, \text{ and } d(\lambda)_i < n_i \text{ implies } s(\lambda)\Lambda^\leq s(\lambda) = \emptyset \},
\]
and then \( v\Lambda^\leq n := v\Lambda \cap \Lambda^\leq n \) for \( v \in \Lambda^0 \) is always nonempty. For example, if \( n = e_i \) for some \( 1 \leq i \leq k \), then
\[
v\Lambda^\leq e_i = \begin{cases} v\Lambda^\leq e_i & \text{if } v\Lambda^\leq e_i \neq \emptyset; \\ \{v\} & \text{otherwise.} \end{cases}
\]

A \( k \)-graph morphism is a degree-preserving functor. An infinite path in a \( k \)-graph \( \Lambda \) is a \( k \)-graph morphism \( x : \Omega_k \to \Lambda \). In a graph with sources, not every finite path is contained in an infinite path, and another technical innovation of [20] is to replace the space \( \Lambda^\infty \) of infinite paths with a space of so-called boundary paths.

Let \( \Lambda \) be a locally convex, row-finite \( k \)-graph and \( m \in (\mathbb{N} \cup \{\infty\})^k \). In Definition 3.14 of [20], a graph morphism \( x : \Omega_{k,m} \to \Lambda \) is defined to be a boundary path of degree \( m \) if
\[
(2.1) \quad v \in \Omega^0_{k,m} \text{ and } v\Omega^\leq e_i = \{v\} \text{ imply } x(v)\Lambda^\leq e_i = \{x(v)\}.
\]
Thus a boundary path maps sources to sources, and every infinite path is a boundary path. We denote the set of boundary paths by \( \Lambda^\leq \). If \( \Lambda \) has no sources, then \( \Lambda^\leq = \Lambda^\infty \). Since we identify the object \( n \in \Omega_{k,m} \) with the identity morphism \((n,n)\) at \( n \), we write \( x(n) \) for the vertex \( x(n,n) \). Then the range of a boundary path \( x \) is the vertex \( r(x) := x(0) \). We set \( v\Lambda^\leq := \Lambda^\leq \cap r^{-1}(v) \) and \( v\Lambda^\infty := \Lambda^\infty \cap r^{-1}(v) \). A boundary path \( x \) is completely determined by the set of paths \( \{x(0,n) : n \leq d(x)\} \), hence can be composed with finite paths, and there is a converse factorisation property. We denote by \( \sigma \) the partially-defined shift map on \( \Lambda^\leq \) which is defined by \( \sigma^m(x) = x(m,\infty) \) for \( x \in \Lambda^\leq \) when \( m \leq d(x) \).
3. Kumjian-Pask $\Lambda$-families

Throughout this section, $\Lambda$ is a row-finite $k$-graph and $R$ is a commutative ring with 1. Define $G(\Lambda) := \{\lambda^* : \lambda \in \Lambda\}$, and call each $\lambda^*$ a ghost path. If $v \in \Lambda^0$, then we identify $v$ and $v^*$. We extend the degree functor $d$ and the range and source maps $r$ and $s$ to $G(\Lambda)$ by

$$d(\lambda^*) = -d(\lambda), \quad r(\lambda^*) = s(\lambda) \quad \text{and} \quad s(\lambda^*) = r(\lambda).$$

We extend the factorisation property to the ghost paths by setting $(\mu\lambda)^* = \lambda^*\mu^*$. We denote by $G(\Lambda^0)$ the set of ghost paths that are not vertices.

Let $\Lambda$ be a row-finite $k$-graph without sources. Recall from [8, Definition 3.1] that a Kumjian-Pask $\Lambda$-family $(P, S)$ in an $R$-algebra $A$ consists of two functions $P : \Lambda^0 \to A$ and $S : \Lambda^0 \cup G(\Lambda^0) \to A$ such that

(KP1) $\{P_v : v \in \Lambda^0\}$ is an orthogonal set of idempotents in the sense that $P_v P_w = \delta_{v,w} P_v$,

(KP2) for all $\lambda, \mu \in \Lambda^0$ with $r(\mu) = s(\lambda)$, we have

$$S_\lambda S_\mu = S_{\lambda\mu}, \quad S_\mu S_\lambda = S_{(\lambda\mu)^*} \quad \text{and} \quad P_{r(\lambda)} S_\lambda = S_\lambda S_{P_{s(\lambda)}}.$$

(KP3) for all $\lambda, \mu \in \Lambda^0$ with $d(\lambda) = d(\mu)$, we have

$$S_\lambda S_\mu = \delta_{\lambda,\mu} P_{s(\lambda)} S_{(\lambda\mu)^*}.$$

(KP4) for all $v \in \Lambda^0$ and all $n \in \mathbb{N}^k \setminus \{0\}$, we have

$$P_v = \sum_{\lambda \in \pi(\Lambda)} S_\lambda S_{(\lambda\mu)^*}.\$$

The sum in (KP4) is finite because $\Lambda$ is row-finite. The relations (KP1)–(KP4) were obtained in [8, §3] by adding to the usual Cuntz-Krieger relations from [17]. A Cuntz-Krieger $\Lambda$-family $(P, S)$ in the algebra $B(H)$ of linear bounded operators on a Hilbert space $H$ consists of an orthogonal set $\{P_v\}$ of projections and a set $\{S_\lambda\}$ of partial isometries $S_\lambda$ with initial projection $P_{s(\lambda)}$, satisfying weaker versions of (KP1)–(KP4). The geometric structure of $B(H)$ is rich enough to yield (KP1)–(KP4) as given above, where $S_\lambda$ is the Hilbert space adjoint of $S_\lambda$. For example, the Cuntz-Krieger relation corresponding to (KP2) is just $S_\lambda S_\mu = S_{\lambda\mu}$, the rest of (KP2) comes for free. See §3 of [8] for more detail.

An important consequence of the Kumjian-Pask relations is that the algebra generated by a Kumjian-Pask $\Lambda$-family $(P, S)$ is span$_R\{S_\lambda S_{\mu^*} : \lambda, \mu \in \Lambda\}$, where we use the convention that $S_v := P_v$ and $S_{v^*} := P_v$ for $v \in \Lambda^0$. When $\Lambda$ has no sources, this follows from [8, Lemma 3.3], which says that if $n \geq d(\lambda), d(\mu)$, then

$$S_\lambda S_\mu = \sum_{d(\lambda) = n, \lambda \alpha = \mu \beta} S_\alpha S_{\beta^*}.\$$

Definition [3.1] below gives a notion of Kumjian-Pask family which applies to $k$-graphs $\Lambda$ with sources. It is based on the approach by Raeburn, Sims and Yeend in [20] for Cuntz-Krieger $\Lambda$-families. The purpose of the new relations (KP3') and (KP4') is to ensure that we obtain a version of (3.1).

**Definition 3.1.** Let $\Lambda$ be a row-finite $k$-graph (possibly with sources). A Kumjian-Pask $\Lambda$-family $(P, S)$ in an $R$-algebra $A$ consists of two functions $P : \Lambda^0 \to A$ and $S : \Lambda^0 \cup G(\Lambda^0) \to A$ such that (KP1) and (KP2) hold, and
(KP3’) for all \( n \in \mathbb{N}^k \setminus \{0\} \) and \( \lambda, \mu \in \Lambda^{\leq n} \), we have
\[
S_{\lambda^*} S_\mu = \delta_{\lambda, \mu} P_s(\lambda);
\]
(KP4’) for all \( v \in \Lambda^0 \) and \( n \in \mathbb{N}^k \setminus \{0\} \),
\[
P_v = \sum_{\lambda \in v \Lambda \leq n} S_{\lambda} S_{\lambda^*}.
\]

Since (KP4’) is the same as the fourth Cuntz-Krieger relation of [20] Definition 3.3, we get the following.

**Lemma 3.2** ([20] Proposition 3.11]). Let \( \Lambda \) be a locally convex, row-finite \( k \)-graph. Then (KP4’) holds at \( v \in \Lambda^0 \) if and only if, for \( 1 \leq i \leq k \) with \( v \Lambda^e_i \neq \emptyset \), \( P_v = \sum_{\lambda \in v \Lambda^e_i} S_{\lambda} S_{\lambda^*} \).

The next lemma gives a version of (KP4).

**Proposition 3.3.** Let \( \Lambda \) be a locally convex, row-finite \( k \)-graph, \( (P, S) \) a Kumjian-Pask \( \Lambda \)-family in an \( R \)-algebra \( A \), and \( \lambda, \mu \in \Lambda \). If \( n \in \mathbb{N}^k \) such that \( d(\lambda), d(\mu) \leq n \), then
\[
S_{\lambda^*} S_\mu = \sum_{\lambda \alpha = \mu \beta, \lambda \alpha \in \Lambda \leq n} S_{\alpha} S_{\beta^*}.
\]

**Proof.** Fix \( n \in \mathbb{N}^k \) such that \( d(\lambda), d(\mu) \leq n \). Then
\[
S_{\lambda^*} S_\mu = (P_s(\lambda) S_{\lambda^*})(S_\mu P_s(\mu)) \quad \text{by (KP2)}
\]
\[
= \left( \sum_{\alpha \in \lambda \Lambda \leq n - d(\lambda)} S_{\alpha} S_{\alpha^*} \right) S_{\lambda^*} S_\mu \left( \sum_{\beta \in \mu \Lambda \leq n - d(\mu)} S_{\beta} S_{\beta^*} \right) \quad \text{by (KP4')}
\]
\[
= \sum_{\alpha \in \lambda \Lambda \leq n - d(\lambda)} \sum_{\beta \in \mu \Lambda \leq n - d(\mu)} S_{\alpha} S_{(\lambda \alpha)^*} S_{\mu \beta} S_{\beta^*} \quad \text{by (KP2)}
\]
\[
= \sum_{\alpha \in \lambda \Lambda \leq n - d(\lambda)} \sum_{\beta \in \mu \Lambda \leq n - d(\mu)} S_{\lambda \alpha} = \mu \beta, \lambda \alpha \in \Lambda \leq n} S_{\alpha} S_{\beta^*}
\]
by applying (KP3’) to each summand. By unique factorisation, for each \( \alpha \) there is just one \( \beta \) of the given degree such that \( \lambda \alpha = \mu \beta \), and the sums collapse to
\[
= \sum_{\alpha \in \lambda \Lambda \leq n - d(\lambda), \lambda \alpha = \mu \beta} S_{\alpha} S_{\beta^*}.
\]
This proves the lemma after noting that the purely graph-theoretic result [20] Lemma 3.6] says that composing \( \lambda \) with \( \alpha \in s(\lambda) \Lambda^{\leq n - d(\lambda)} \) gives the path \( \lambda \alpha \in \Lambda^{\leq n} \).

**Corollary 3.4.** Let \( \Lambda \) be a locally convex, row-finite \( k \)-graph and \( (P, S) \) a Kumjian-Pask \( \Lambda \)-family in an \( R \)-algebra \( A \). The subalgebra generated by \( (P, S) \) is \( \text{span} \{ S_{\alpha} S_{\beta^*} : \alpha, \beta \in \Lambda, s(\alpha) = s(\beta) \} \).

**Proof.** We have \( S_{\alpha} S_{\beta^*} = S_{\alpha} P_s(\alpha) P_s(\beta) S_{\beta^*} \) by (KP2), so \( S_{\alpha} S_{\beta^*} = 0 \) unless \( s(\alpha) = s(\beta) \) by (KP1). The result now follows from Proposition 3.3 and (KP2).

The set of minimal common extensions of \( \lambda, \mu \in \Lambda \) is
\[
\Lambda^{\min}(\lambda, \mu) := \{ (\alpha, \beta) : \lambda \alpha = \mu \beta, d(\lambda \alpha) = d(\lambda) \lor d(\mu) \}.
\]
Corollary 3.5. Let $\Lambda$ be a locally convex, row-finite $k$-graph and $(P, S)$ a family in an $R$-algebra $A$ satisfying (KP1), (KP2) and (KP4'). Then (KP3') holds if and only if, for all $\lambda, \mu \in \Lambda$,
\begin{equation}
S_\lambda \cdot S_\mu = \sum_{(\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu)} S_\alpha S_\beta^*.
\end{equation}

Proof. Suppose (KP3') holds. Then $(P, S)$ is a Kumjian-Pask $\Lambda$-family. Let $\lambda, \mu \in \Lambda$ and apply Proposition [8,5] with $n = d(\lambda) \lor d(\mu)$ to get $S_\lambda \cdot S_\mu = \sum_{(\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu)} S_\alpha S_\beta^*$.

Conversely, suppose that for all $\lambda, \mu \in \Lambda$, [3.2] holds. Fix $n \in \mathbb{N}^k \setminus \{0\}$ and let $\lambda, \mu \in \Lambda^{\leq n}$. Note that $d(\lambda) \lor d(\mu) \leq n$. First suppose that $d(\lambda) = d(\mu)$. Then

$$\Lambda^{\min}(\lambda, \mu) = \begin{cases} 
\{(s(\lambda), s(\lambda))\} & \text{if } \lambda = \mu; \\
\emptyset & \text{else},
\end{cases}$$

and [3.2] gives $S_\lambda \cdot S_\mu = \delta_{\lambda, \mu} P_s(\lambda)$. Second, suppose $d(\lambda) \neq d(\mu)$. Then at least one of $\lambda, \mu$ has degree less than $n$, say $d(\lambda) < n$. But $\lambda \in \Lambda^{\leq n}$, and so there is no $\alpha$ such that $d(\lambda) < d(\lambda \alpha) \leq n$. Now

$$\Lambda^{\min}(\lambda, \mu) = \{(s(\lambda), \beta) : \lambda = \mu \beta, d(\lambda) = d(\lambda) \lor d(\mu)\}$$

$$= \{(s(\lambda), \beta) : \lambda = \mu \beta, d(\mu) < d(\lambda)\}$$

since $d(\lambda) \neq d(\mu)$. But $\mu \in \Lambda^{\leq n}$ too, so $\Lambda^{\min}(\lambda, \mu) = \emptyset$. By [3.2] $S_\lambda \cdot S_\mu = 0$, and $\lambda \neq \mu$ implies that $S_\lambda \cdot S_\mu = \delta_{\lambda, \mu} P_s(\lambda)$. Thus in either case, $S_\lambda \cdot S_\mu = \delta_{\lambda, \mu} P_s(\lambda)$ as required. \hfill \Box

In order to demonstrate the existence of a nonzero Kumjian-Pask $\Lambda$-family we need to impose the “local convexity” condition from [20].

Proposition 3.6. Let $\Lambda$ be a locally convex, row-finite $k$-graph. Then there exist an $R$-algebra $A$ and a Kumjian-Pask $\Lambda$-family $(P, S)$ in $A$ with $S_\lambda, S_\mu, P_v \neq 0$ for all $v \in \Lambda^0$ and $\lambda \in \Lambda$. In particular, for every $r \in R \setminus \{0\}$ and $v \in \Lambda^0$, we have $rP_v \neq 0$.

Proof. We modify the construction of the infinite-path representation of [8] and instead build a ‘boundary-path representation’. Let $\mathbb{F}_R(\Lambda^{\leq \infty})$ be the free $R$-module on $\Lambda^{\leq \infty}$. For each $v \in \Lambda^0$ and $\lambda, \mu \in \Lambda^{\leq \infty}$, define functions $f_v, f_\lambda$ and $f_\mu : \Lambda^{\leq \infty} \to \mathbb{F}_R(\Lambda^{\leq \infty})$ by

$$f_v(x) = \begin{cases} 
x & \text{if } r(x) = v; \\
0 & \text{otherwise},
\end{cases}$$

$$f_\lambda(x) = \begin{cases} 
\lambda x & \text{if } r(x) = s(\lambda); \\
0 & \text{otherwise},
\end{cases}$$

$$f_\mu^*(x) = \begin{cases} 
y & \text{if } x = \mu y \text{ for some } y \in \Lambda^{\leq \infty}; \\
0 & \text{otherwise}.
\end{cases}$$

By the universal property of free modules, there exist nonzero $P_v, S_\lambda, S_\mu^* \in \text{End}(\mathbb{F}_R(\Lambda^{\leq \infty}))$ extending $f_v, f_\lambda$ and $f_\mu^*$. Note that $rP_v \neq 0$ for every $r \in R \setminus \{0\}$.

We claim that $(P, S)$ is a Kumjian-Pask $\Lambda$-family in the $R$-algebra $\text{End}(\mathbb{F}_R(\Lambda^{\leq \infty}))$. Relations (KP1) and (KP2) are straight-forward to check. To see (KP3'), fix $n \in \mathbb{N}^k \setminus \{0\}$,
\( \lambda, \mu \in \Lambda^{\leq n} \) and \( x \in \Lambda^{\leq \infty} \). If \( r(\mu) \neq r(\lambda) \), then both \( S_{\lambda^*}S_{\mu} = S_{\lambda^*}P_{r(\lambda)}P_{r(\mu)}S_{\mu} \) and \( \delta_{\lambda, \mu}P_{s(\lambda)} \) are 0. So we may assume \( r(\mu) = r(\lambda) \). Notice that

\[
S_{\lambda^*}S_{\mu}(x) = \begin{cases} S_{\lambda^*}(\mu x) & \text{if } x(0) = s(\mu) \text{ and } \mu x = \lambda y \text{ for some } y \in \Lambda^{\leq \infty}; \\ 0 & \text{otherwise.} \end{cases}
\]

Since \( \lambda, \mu \in r(\lambda)\Lambda^{\leq n} \), \( (\mu x)(0, d(\lambda)) = \lambda \) implies either \( \lambda = \mu \lambda' \) or \( \mu = \lambda \mu' \) for some \( \lambda', \mu' \in \Lambda \). But then \( \lambda = \mu \) by the definition of \( \Lambda^{\leq n} \). Hence \( (\mu x)(0, d(\lambda)) = \lambda \) if and only if \( \mu = \lambda \). Thus

\[
S_{\lambda^*}S_{\mu}(x) = \begin{cases} x & \text{if } x(0) = s(\mu) \text{ and } \lambda = \mu; \\ 0 & \text{otherwise,} \end{cases}
\]

and \( (KP3') \) holds.

For \( (KP4') \), fix \( v \in \Lambda^0 \) and \( 1 \leq i \leq k \) with \( v\Lambda^i \neq \emptyset \). Since \( \Lambda \) is locally convex, it suffices to show that \( P_v = \sum_{\lambda \in \Lambda^i} S_{\lambda}S_{\lambda^*} \) by Lemma 3.2. Let \( x \in \Lambda^{\leq \infty} \). Then

\[
\sum_{\lambda \in \Lambda^i} S_{\lambda}S_{\lambda^*}(x) = \sum_{\lambda \in \Lambda^i} \delta_{\lambda, x(0, e_i)} x = \begin{cases} x & \text{if } r(x) = v; \\ 0 & \text{otherwise,} \end{cases} = P_v(x).
\]

We are now ready to show that there is an \( R \)-algebra which is “universal for Kumjian-Pask \( \Lambda \)-families”; the proof is very similar to the one for \( k \)-graphs without sources [8, Theorem 3.4], so we will just give an outline addressing the main points. This \( R \)-algebra is graded over \( \mathbb{Z}^k \), and to see that the graded subgroups have a nice description uses Proposition 3.3 so we will need to check carefully that the argument used when \( \Lambda \) has no sources still works when \( \Lambda^n \) is replaced by \( \Lambda^{\leq n} \). We will follow the convention of [8] and use lower-case letters for universal Kumjian-Pask families.

**Theorem 3.7.** Let \( \Lambda \) be a locally convex, row-finite \( k \)-graph.

(a) There is an \( R \)-algebra KP\(_R\)(\( \Lambda \)), generated by a Kumjian-Pask \( \Lambda \)-family \((p, s)\), such that if \((Q, T)\) is a Kumjian-Pask \( \Lambda \)-family in an \( R \)-algebra \( A \), then there exists a unique \( R \)-algebra homomorphism \( \pi_{Q,T} : \text{KP}_R(\Lambda) \to A \) such that \( \pi_{Q,T} \circ p = Q \) and \( \pi_{Q,T} \circ s = T \). For every \( r \in R \setminus \{0\} \) and \( v \in \Lambda^0 \), we have \( rp_v \neq 0 \).

(b) The subsets

\[ \text{KP}_R(\Lambda)_n := \text{span}\{s_\alpha s_\beta^* : d(\alpha) - d(\beta) = n\} \]

form a \( \mathbb{Z}^k \)-grading of \( \text{KP}_R(\Lambda) \).

**Proof.** Let \( X := \Lambda^0 \cup \Lambda^{\neq 0} \cup G(\Lambda^{\neq 0}) \) and \( F_R(w(X)) \) be the free algebra on the set \( w(X) \) of words on \( X \). Let \( I \) be the ideal of \( F_R(w(X)) \) generated by elements from the sets:

(i) \( \{vw - \delta_{v, w}v : v, w \in \Lambda^0\} \),
(ii) \( \{\lambda - \mu \nu, \lambda^* - \nu^* \mu^* : \lambda, \mu, \nu \in \Lambda^{\neq 0} \text{ and } \lambda = \mu \nu\} \),
(iii) \( \{\lambda - r(\lambda)\lambda, \lambda - s(\lambda), \lambda^* - s(\lambda)\lambda^*, \lambda^* - \lambda^* r(\lambda) : \lambda \in \Lambda^{\neq 0}\} \),
(iv) \( \{\lambda \mu - \delta_{\lambda, \mu} s(\lambda) : \lambda, \mu \in \Lambda^{\leq n}, n \in \mathbb{N}^k \setminus \{0\}\} \),
(v) \( \{v - \sum_{\lambda \in \Lambda^{\leq n}} \lambda \lambda^* : v \in \Lambda^0 \text{ and } n \in \mathbb{N}^k \setminus \{0\}\} \).
Set $\text{KP}_R(\Lambda) := \mathbb{F}_R(w(X))/I$, and write $q : \mathbb{F}_R(w(X)) \to \mathbb{F}_R(w(X))/I$ for the quotient map. Define $p : \Lambda^0 \to \text{KP}_R(\Lambda)$ by $p_v = q(v)$, and $s : \Lambda^{\neq 0} \cup G(\Lambda^{\neq 0}) \to \text{KP}_R(\Lambda)$ by $s_{\alpha} = q(\alpha)$ and $s_{\alpha^*} = q(\alpha^*)$. Then $(p,s)$ is a Kumjian-Pask $\Lambda$-family in the $R$-algebra $\text{KP}_R(\Lambda)$.

Now let $(Q,T)$ be a Kumjian-Pask $\Lambda$-family in an $R$-algebra $A$. Define $f : X \to A$ by $f(v) = Q_v$, $f(\lambda) = T_\lambda$, and $f(\lambda^*) = T_{\lambda^*}$. The universal property of $\mathbb{F}_R(w(X))$ gives a unique $R$-algebra homomorphism $\psi : \mathbb{F}_R(w(X)) \to A$ such that $\psi|_X = f$. Since $(Q,T)$ is a Kumjian-Pask family, $I \subseteq \ker(\psi)$. Thus there exists a unique $R$-algebra homomorphism $\pi_{Q,T} : \text{KP}_R(\Lambda) \to A$ such that $\pi_{Q,T} \circ q = \psi$. It follows that $\pi_{Q,T} \circ p = Q$ and $\pi_{Q,T} \circ s = T$.

Now fix $r \in R \setminus \{0\}$ and $v \in \Lambda^0$. If $rp_v$ were zero then $r\pi_{Q,T}(p_v) = rQ_v$ would be zero for every Kumjian-Pask $\Lambda$-family $(Q,T)$. But this is not the case for the Kumjian-Pask family of Proposition 3.6. Thus $rp_v \neq 0$. This completes the proof of (ii).

For (iii), extend the degree map to words on $X$ by setting $d : w(X) \to \mathbb{Z}^k$ by $d(w) = \sum_{i=1}^{\|w\|} d(w_i)$. By [8, Proposition 2.7], $\mathbb{F}_R(w(X))$ is graded over $\mathbb{Z}^k$ by the subgroups

$$\mathbb{F}_R(w(X))_n := \left\{ \sum_{w \in w(X)} r_w w : r_w \neq 0 \implies d(w) = n \right\}.$$ 

We claim that the ideal $I$ defined in the proof of (ii) is graded. For this, it suffices to see that $I$ is generated by homogeneous elements, that is, elements in $\mathbb{F}_R(w(X))_n$ for some $n \in \mathbb{Z}^k$. The generators of $I$ in (i) are a linear combination of words of degree 0, hence are homogeneous of degree 0. If $\lambda = \mu \nu$ in $\Lambda$ then $\lambda - \mu \nu$ is a linear combination of words of degree $d(\lambda)$, so all the generators in (ii) are homogeneous. Also, $\lambda - r(\lambda)\lambda$ is homogeneous of degree $\lambda$, and similarly all the generators in (iii) are homogeneous of some degree. The elements in (iv) are either of the form $\lambda^* \lambda - s(\lambda)$ or of the form $\lambda \mu^*$; the former is homogeneous of degree 0 and the latter is homogeneous of degree $d(\mu) - d(\lambda)$. A word $\lambda \lambda^*$ has degree 0, and hence the generators in (v) are homogeneous of degree 0. Thus $I$ is a graded ideal.

Since $I$ is graded, the quotient $\text{KP}_R(\Lambda)$ of $\mathbb{F}_R(w(X))$ by $I$ is graded by the subgroups

$$(\mathbb{F}_R(w(X))/I)_n := \operatorname{span}\{q(w) : w \in w(X), d(w) = n\}.$$ 

By Corollary [8,24], $\text{KP}_R(\Lambda) = \operatorname{span}\{s_{\alpha} s_{\beta^*} : \alpha, \beta \in \Lambda, s(\alpha) = s(\beta)\}$. We need to show that

$$\text{KP}_R(\Lambda)_n := \operatorname{span}\{s_{\alpha} s_{\beta^*} : d(\alpha) - d(\beta) = n\} = (\mathbb{F}_R(w(X))/I)_n.$$ 

First, fix $s_{\alpha} s_{\beta^*} \in \{s_{\alpha} s_{\beta^*} : d(\alpha) - d(\beta) = n\}$. Then $s_{\alpha} s_{\mu^*} = q(\lambda)q(\mu^*) = q(\lambda^* \mu)$, and $d(\lambda^* \mu) = d(\lambda) = d(\mu) = n$. Thus $s_{\lambda^* \mu} \in \{q(w) : d(w) = n, w \in w(X)\} \in (\mathbb{F}_R(w(X))/I)_n$.

That $(\mathbb{F}_R(w(X))/I)_n \subseteq \text{KP}_R(\Lambda)_n$ follows immediately from the next lemma; the proof is very similar to that of [8, Lemma 3.5], but we need to check replacing $\Lambda^n$ by $\Lambda^{\neq n}$ in the argument does not cause problems.

Lemma 3.8. Let $X := \Lambda^0 \cup \Lambda^{\neq 0} \cup G(\Lambda^{\neq 0})$ and $q : \mathbb{F}_R(w(X)) \to \text{KP}_R(\Lambda)$ be the quotient map. If $w \in w(X)$, then $q(w) \in \text{KP}_R(\Lambda)_{d(w)}$.

Proof. The proof is by induction on $|w|$. We treat the cases $|w| = 1, 2$ separately. Recall that by our convention $s_v := p_v$ and $s_{v^*} := p_v$ for $v \in \Lambda^0$.

1 This has caused some confusion before: for example in [29] Proof of Proposition 4.7, $e - r(e)e$ for an edge $e$ in a graph is claimed to be 0-graded whereas it is 1-graded.
If \(|w| = 1\) there are two possibilities. If \(w = \lambda\) for some \(\lambda \in \Lambda\), then \(q(w) = s_\lambda = s_\lambda s_\lambda(\lambda)\) and \(d(\lambda) - d(s(\lambda)) = d(\lambda)\), and so \(q(w) \in \text{KP}_R(\Lambda)_{d(w)}\). Otherwise, if \(w = \lambda^*\), then \(q(w) = s_\lambda^* = s_\lambda s_{\lambda^*}\) and \(d(s(\lambda)) - d(\lambda) = d(\lambda^*)\), so \(q(w) \in \text{KP}_R(\Lambda)_{d(w)}\).

If \(|w| = 2\) there are four possibilities: \(w = \lambda \mu^*, \lambda \mu, \mu^* \lambda^*\) or \(\lambda^* \mu\). The first three possibilities are quickly dealt with since

\[
q(\lambda \mu^*) = s_\lambda s_{\mu^*} \quad \text{and} \quad d(\lambda) - d(\mu) = d(\lambda \mu^*),
\]
\[
q(\lambda \mu) = s_{\lambda \mu} s_{\mu^*} \quad \text{and} \quad d(\lambda \mu) - d(s(\mu)) = d(\lambda \mu),
\]
\[
q(\mu^* \lambda^*) = s_{\lambda s_{\mu^*}} s_{\lambda^*} \quad \text{and} \quad d(s(\mu)) - d((\lambda \mu)^*) = d(\mu^* \lambda^*).
\]

So suppose \(w = \lambda^* \mu\). Let \(m = d(\mu) \vee d(\lambda)\). By Proposition 3.3 we have

\[
q(\lambda^* \mu) = s_\lambda s_{\mu} = \sum_{\lambda \alpha = \mu \beta, \lambda \neq \alpha} s_\alpha s_{\beta^*}.
\]

In each summand, \(\lambda \alpha = \mu \beta\) implies \(d(w) = d(\mu) - d(\lambda) = d(\alpha) - d(\beta)\), so \(q(w) \in \text{KP}_R(\Lambda)_{d(w)}\) as needed.

Now let \(n \geq 2\) and suppose that \(q(y) \in \text{KP}_R(\Lambda)_{d(y)}\) for every word \(y\) with \(|y| \leq n\). Let \(w\) be a word with \(|w| = n + 1\) and \(q(w) \neq 0\). If \(w\) contains a subword \(w_i w_{i+1} = \lambda \mu\), then \(\lambda\) and \(\mu\) are composable in \(\Lambda\) since otherwise \(q(\lambda \mu) = 0\). Let \(w'\) be the word obtained from \(w\) by replacing \(w_i w_{i+1}\) with the single path \(\lambda \mu\). Then

\[
q(w) = s_{w_i} \cdots s_{w_{i-1}} s_\lambda s_\mu s_{w_{i+2}} \cdots s_{w_{n+1}} = s_{w_i} \cdots s_{w_{i-1}} s_{\lambda \mu} s_{w_{i+2}} \cdots s_{w_{n+1}} = q(w').
\]

Since \(|w'| = n\) and \(d(w') = d(w)\), the inductive hypothesis implies that \(q(w') \in \text{KP}_R(\Lambda)_{d(w)}\).

A similar argument shows that \(q(w) \in \text{KP}_R(\Lambda)_{d(w)}\) whenever \(w\) contains a subword \(w_i w_{i+1} = \lambda^* \mu^*\).

If \(w\) contains no subword of the form \(\lambda \mu\) or \(\lambda^* \mu^*\), then, since \(|w| \geq 3\), it must have a subword of the form \(\lambda^* \mu^*\). By Proposition 3.3 we write \(q(w)\) as a sum of terms \(q(y^i)\) with \(|y^i| = n + 1\) and \(d(y^i) = d(w)\). Since \(|w| \geq 3\), each nonzero summand \(q(y^i)\) contains a factor of the form \(s_\beta s_{\beta^*}\) or one of the form \(s_\alpha s_{\alpha^*}\), and the argument above shows that every \(q(y^i) \in \text{KP}_R(\Lambda)_{d(w)}\). Thus \(q(w) \in \text{KP}_R(\Lambda)_{d(w)}\) as well.

\[
\square
\]

4. The uniqueness theorems

Throughout this section, \(\Lambda\) is a locally convex, row-finite \(k\)-graph, and \(R\) is a commutative ring with 1.

There are two uniqueness theorems in the theory of Kumjian-Pask algebras. The graded-uniqueness theorem has no hypotheses on the graph, so applies very generally. The Cuntz-Krieger uniqueness theorem assumes the graph satisfies an “aperiodicity” condition. The proofs of both theorems are straightforward once key helper-results (Lemma 4.4 and Proposition 4.4) have been established.

**Theorem 4.1** (The graded-uniqueness theorem). Let \(\Lambda\) be a locally convex, row-finite \(k\)-graph. Suppose that \(A\) is a \(\mathbb{Z}^k\)-graded \(R\)-algebra and \(\phi : \text{KP}_R(\Lambda) \to A\) is a graded \(R\)-algebra homomorphism. If \(\phi(\rho_r) \neq 0\) for all \(r \in R \setminus \{0\}\) and \(v \in \Lambda^0\), then \(\phi\) is injective.

**Theorem 4.2** (The Cuntz-Krieger uniqueness theorem). Let \(\Lambda\) be a locally convex, row-finite \(k\)-graph satisfying the aperiodicity condition

\[
(4.1) \quad \text{for every} \ v \in \Lambda^0, \ \text{there exists} \ x \in v\Lambda^{\leq \infty} \ \text{such that} \ \alpha x \neq \beta x.
\]
Let $\phi : KP_R(\Lambda) \to A$ be an $R$-algebra homomorphism into an $R$-algebra $A$. If $\phi(rp) \neq 0$ for all $r \in R \setminus \{0\}$ and $v \in \Lambda^0$, then $\phi$ is injective.

The aperiodicity condition \(^{(1,1)}\) we have chosen to use is from \(^{[20]}\) Theorem 4.3], is often called ‘condition B’ in the literature, and generalises the many notions of ‘aperiodicity’ for $k$-graphs without sources. See Lemma \(^{[8,4]}\) below for more details.

We start by establishing that every nonzero element of $KP_R(\Lambda)$ can be written in a certain form; this form differs from the one for graphs without sources only in the use of $\Lambda^{\leq n}$ in place of $\Lambda^n$.

**Lemma 4.3.** Every nonzero $a \in KP_R(\Lambda)$ can be written in normal form: that is, there exists $n \in \mathbb{N}^k \setminus \{0\}$ and a finite subset $F$ of $\Lambda \times \Lambda^{\leq n}$ such that

$$a = \sum_{(\alpha,\beta) \in F} r_{\alpha,\beta}s_\alpha s_\beta^*,$$

where $r_{\alpha,\beta} \in R \setminus \{0\}$ and $s(\alpha) = s(\beta)$ for all $(\alpha, \beta) \in F$.

**Proof.** Let $0 \neq a \in KP_R(\Lambda)$. By Corollary \(^{[5,4]}\) we can write $a$ as a finite sum

$$a = \sum_{(\mu,\nu) \in G} r_{\mu,\nu}s_\mu s_\nu^*,$$

where $G \subseteq \Lambda \times \Lambda$, $s(\mu) = s(\nu)$ and each $r_{\mu,\nu} \neq 0$ in $R$. Let $n = \vee_{(\mu,\nu) \in G} d(\nu)$. Consider $(\mu, \nu) \in G$ such that $d(\nu) < n$. Then, using (KP2) and (KP4'), we get

$$s_\mu s_\nu^* = s_\mu p_{s(\mu)} s_\nu^* = s_\mu \left( \sum_{\lambda \in s(\mu) \Lambda^{\leq n-d(\nu)}} s_\lambda s_\lambda^* \right) s_\nu^* = \sum_{\lambda \in s(\mu) \Lambda^{\leq n-d(\nu)}} s_\mu \lambda s(\nu)^*.$$

Substituting back into the expression for $a$ and combining terms gives the result. \(\square\)

Next we generalise \(^{[11]}\) Lemma 2.3(1)] to graphs with possible sources.

**Lemma 4.4.** Let $0 \neq a = \sum_{(\alpha,\beta) \in F} r_{\alpha,\beta}s_\alpha s_\beta^* \in KP_R(\Lambda)$ be in normal form. For all $(\mu, \nu) \in F$,

$$0 \neq s_\mu^* a s_\nu = r_{\mu,\nu} p_{s(\mu)} + \sum_{(\alpha,\nu) \in F, d(\alpha) \neq d(\mu)} r_{\alpha,\nu} s_{\mu^*} s_\alpha,$$

and the 0-graded component $r_{\mu,\nu} p_{s(\mu)}$ of $s_\mu^* a s_\nu$ is nonzero.

**Proof.** Since $a$ is in normal form, $F \subseteq \Lambda \times \Lambda^{\leq n}$ for some $n \in \mathbb{N}^k \setminus \{0\}$. Fix $(\mu, \nu) \in F$. For all $(\alpha, \beta) \in F$, both $\beta$ and $\nu \in \Lambda^{\leq n}$, and hence (KP3') gives $s_\beta^* s_\nu = \delta_{\beta,\nu} p_{s(\nu)}$. Thus

$$s_\mu^* a s_\nu = \sum_{(\alpha,\beta) \in F} r_{\alpha,\beta} s_\mu^* s_\alpha s_\beta^* s_\nu = \sum_{(\alpha,\nu) \in F} r_{\alpha,\nu} s_{\mu^*} s_\alpha p_{s(\nu)}$$

$$= r_{\mu,\nu} p_{s(\mu)} + \sum_{(\alpha,\nu) \in F, \alpha \neq \mu} r_{\alpha,\nu} s_{\mu^*} s_\alpha$$

by (KP2) since $(\alpha, \nu) \in F$ implies $s(\alpha) = s(\nu)$. If there exists $\alpha$ such that $(\alpha, \nu) \in F$, $\alpha \neq \mu$ and $d(\alpha) = d(\mu)$, then (KP3') implies that $s_{\mu^*} s_\alpha = 0$. Thus

$$s_\mu^* a s_\nu = r_{\mu,\nu} p_{s(\mu)} + \sum_{(\alpha,\nu) \in F, d(\alpha) \neq d(\mu)} r_{\alpha,\nu} s_{\mu^*} s_\alpha.$$
It now follows that the 0-graded component of \( s_\mu^* a s_\nu \) is \( r_{\mu,\nu} p_{s(\mu)} \), which is nonzero by Theorem 3.7. Now \( s_\mu^* a s_\nu \) is nonzero because its 0-graded component is.

Proof of Theorem 4.1. Let \( 0 \neq a \in \text{KP}_R(\Lambda) \). Write \( a = \sum_{(\alpha,\beta) \in F} r_{\alpha,\beta} s_\alpha s_\beta^* \) in normal form. Let \((\mu,\nu) \in F\). By Lemma 4.4,

\[
0 \neq s_\mu^* a s_\nu = r_{\mu,\nu} p_{s(\mu)} + \sum_{(\alpha,\nu) \in F, d(\alpha) \neq d(\mu)} r_{\alpha,\nu} s_\mu^* s_\alpha
\]

with 0-graded component \( r_{\mu,\nu} p_{s(\mu)} \). Since \( \phi \) is graded, \( \phi(r_{\mu,\nu} p_{s(\mu)}) \) is the 0-graded component of \( \phi(s_\mu^* a s_\nu) \). Now \( \phi(s_\mu^* a s_\nu) = \phi(s_\mu^* a s_\nu) \neq 0 \) because its 0-graded component \( \phi(r_{\mu,\nu} p_{s(\mu)}) \neq 0 \) by assumption. It follows that \( \phi(a) \neq 0 \) as well. Thus \( \phi \) is injective.

One immediate application of the Theorem 4.1 is:

**Proposition 4.5.** Let \( \Lambda \) be a locally convex, row-finite \( k \)-graph. Then \( \text{KP}_C(\Lambda) \) is isomorphic to a dense subalgebra of \( C^*(\Lambda) \).

**Proof.** Let \((q,t)\) be a generating Cuntz-Krieger \( \Lambda \)-family in \( C^*(\Lambda) \) as in [20, Definition 3.3]. Then \((q,t)\) is a Kumjian-Pask \( \Lambda \)-family in \( C^*(\Lambda) \). (To see this, recall that (KP1) and (KP2) hold, (KP4') and (CK4) are the same and (KP3') and (CK3) are the same via Corollary 3.5.) Thus the universal property of \( \text{KP}_C(\Lambda) \) of Theorem 3.7 gives a homomorphism \( \pi_{q,t} \) from \( \text{KP}_C(\Lambda) \) onto the dense subalgebra

\[
A := \text{span}\{t_\mu^* t_\mu^* : \lambda, \mu \in \Lambda\}
\]

of \( C^*(\Lambda) \). To show that \( \pi_{q,t} \) is injective, we will use the gauge-invariant uniqueness theorem, Theorem 4.1. Consider the subgroups

\[
A_n := \text{span}\{t_\mu^* t_\mu^* : d(\lambda) - d(\mu) = n\} \quad (n \in \mathbb{Z}^k)
\]

of \( A \). A calculation using Corollary 3.5 shows that \( A_n A_m \subseteq A_{n+m} \). Since each spanning element \( t_\mu^* t_\nu^* \) of \( A \) belongs to \( A_{d(\lambda) - d(\mu)} \), every element \( a \) of \( A \) can be written as a finite sum \( \sum a_n \) with \( a_n \in A_n \). If \( a_n \in A_n \) and a finite sum \( \sum a_n = 0 \), then each \( a_n = 0 \) by the argument of the proof of [8, Lemma 7.4], which uses the gauge action of \( \mathbb{T}^k \) on \( C^*(\Lambda) \). Thus \( \{A_n : n \in \mathbb{Z}^k\} \) is a grading of \( A \). It follows that \( \pi_{q,t} \) is graded and hence is injective by Theorem 4.1.

**Proposition 4.6.** Let \( \Lambda \) be a locally convex, row-finite \( k \)-graph satisfying the aperiodicity condition [4.1]. Suppose \( 0 \neq a = \sum_{(\alpha,\beta) \in F} r_{\alpha,\beta} s_\alpha s_\beta^* \in \text{KP}_R(\Lambda) \) is in normal form. Let \((\mu,\nu) \in F\). Then there exist \( \sigma,\tau \in \Lambda \) such that \( s_\sigma^* a s_\tau = r_{\mu,\nu} p_{s(\mu)} \).

**Proof.** Let \((\mu,\nu) \in F\). By Lemma 4.4,

\[
0 \neq s_\mu^* a s_\nu = r_{\mu,\nu} p_{s(\mu)} + \sum_{(\alpha,\nu) \in F, d(\alpha) \neq d(\mu)} r_{\alpha,\nu} s_\mu^* s_\alpha.
\]

If \( G := \{\alpha : (\alpha,\nu) \in F, d(\alpha) \neq d(\mu)\} = \emptyset \) then we can take \( \sigma = \mu \) and \( \tau = \nu \), and we are done. So suppose \( G \neq \emptyset \).

Choose \( y \in s(\mu) \Lambda^{\leq \infty} \) such that [4.1] holds. Then for each \( \alpha \in G \), \( \alpha y \neq \mu y \). So there exists \( m_\alpha \in \mathbb{N}^k \) such that \( (\alpha y)^t(0,m_\alpha) \neq (\mu y)(0,m_\alpha) \). Let \( m := \bigvee_{\alpha \in G} m_\alpha \). For later use we note that since \( m_\alpha \leq d(y) + d(\alpha) \), taking the meet of both sides with \( m \) we get

\[
m_\alpha = m \land m_\alpha \leq m \land (d(y) + d(\alpha)) \leq m \land d(y) + d(\alpha);
\]

the same argument gives \( m_\alpha \leq m \land d(y) + d(\mu) \).
We would like to use $y(0, m)$ now, but since $y$ is a boundary path this may not be well-defined, so we will use $y(0, m \land d(y))$ instead. Then, using (4.3),

$$
\begin{align*}
 s_{y(0, m \land d(y))} & = r_{\mu,\nu} s_{y(0, m \land d(y))} + \sum_{\alpha \in G} r_{\alpha,\nu} s_{y(0, m \land d(y))} s_{\alpha} s_{y(0, m \land d(y))} \\
 & = r_{\mu,\nu} p_\alpha + \sum_{\{\alpha \in G: \alpha \neq \mu\}} r_{\alpha,\nu} s_{(\mu y)(0, m \land d(y))} s_{\alpha y(0, m \land d(y))}
\end{align*}
$$

because $r(y) = s(\mu)$ and (KP2), and then (KP3), all on the first summand. After composing paths this

$$
= r_{\mu,\nu} p_\alpha + \sum_{\alpha \in G} r_{\alpha,\nu} s_{(\mu y)(0, m \land d(y))} s_{\alpha y(0, m \land d(y))}.
$$

We claim that for every $\alpha \in G$, $s_{(\mu y)(0, m \land d(y))} s_{(\alpha y)(0, m \land d(y))} = 0$. To see this, by way of contradiction, suppose there exists $\alpha \in G$ such that

$$
\begin{align*}
 s_{(\mu y)(0, m \land d(y))} & \neq 0 \\
 s_{(\alpha y)(0, m \land d(y))} & \neq 0
\end{align*}
$$

Since $m_\alpha \leq m \land d(y) + d(\mu)$ and $m_\alpha \leq m \land d(y) + d(\alpha)$ we must have $s_{(\mu y)(0, m_\alpha)} s_{(\alpha y)(0, m_\alpha)} \neq 0$. But $(\mu y)(0, m_\alpha)$ and $(\alpha y)(0, m_\alpha) \in \Lambda^{m_\alpha}$, and hence $(\mu y)(0, m_\alpha) = (\alpha y)(0, m_\alpha)$ by (KP3), contradicting our choice of $m_\alpha$. This proves the claim. Therefore

$$
\begin{align*}
 s_{y(0, m \land d(y))} & = r_{\mu,\nu} s_{y(0, m \land d(y))} = r_{\mu,\nu} p_\alpha.
\end{align*}
$$

The proposition follows with $\sigma := (\mu y)(0, m \land d(y))$ and $\tau := (\nu y)(0, m \land d(y))$. \qed

Proof of Theorem 4.2. Let $0 \neq a \in \mathbb{K}_R(\Lambda)$. Write $a = \sum_{(\alpha, \beta) \in F} r_{\alpha,\beta} s_\alpha s_\beta$ in normal form. Fix $(\mu, \nu) \in F$. Since $\Lambda$ satisfies the aperiodicity condition (4.1), by Proposition 4.6 there exist $\sigma, \tau \in \Lambda$ such that $s_\sigma s_\tau = r_{\mu,\nu} p_\alpha$. Now $\phi(s_\sigma) \phi(a) \phi(s_\tau) = \phi(s_\sigma s_\tau) = \phi(r_{\mu,\nu} p_\alpha) \neq 0$ by assumption, and hence $\phi(a) \neq 0$ as well. Thus $\phi$ is injective. \qed

5. Examples of 2-graphs with sources and applications to rank-2 Bratteli diagrams

Throughout this section, $R$ is a commutative ring with 1. Let $\Lambda$ be a 2-graph. We refer to the morphisms of degree $(n_1, 0)$ as blue paths, to the morphisms of degree $(0, n_2)$ as red paths, and write $\Lambda^{\text{blue}}$ and $\Lambda^{\text{red}}$ for the collection of blue and red paths, respectively. We say a path $\lambda \in \Lambda$ with $d(\lambda) \neq 0$ is a cycle if $r(\lambda) = s(\lambda)$ and for $0 < n < d(\lambda)$, $\lambda(n) \neq s(\lambda)$. We say a cycle $\lambda$ is isolated if for every $0 < n \leq d(\lambda)$, the sets

$$
\{\lambda(n, 0)\} \text{ and } s(\lambda) \Lambda^n \setminus \{\lambda(d(\lambda) - n, d(\lambda))\}
$$

are empty.

Proposition 5.1. Let $\Lambda$ be a finite 2-graph such that

$$
\Lambda^{\text{blue}} \text{ contains no cycles and each vertex } v \in \Lambda^0 \text{ is the range of an isolated cycle in } \Lambda^{\text{red}}.
$$

Let $S$ be the set of sources in $\Lambda^{\text{blue}}$. Suppose the vertices in $S$ all lie on a single isolated cycle in $\Lambda^{\text{red}}$. Let $Y$ denote the set $\Lambda^{\text{blue}} S$ of blue paths with source in $S$. Then $\mathbb{K}_R(\Lambda)$ is isomorphic to $M_Y(R[x, x^{-1}])$, where $R[x, x^{-1}]$ is the ring of Laurent polynomials over $R$. 

Any graph satisfying \[5.1\] is locally convex \[19\] page 141. Proposition 5.1 is very similar to Proposition 3.5 of \[19\], which, with the same hypotheses on \(\Lambda\) and \(R = \mathbb{C}\), gives an isomorphism of \(C^*(\Lambda)\) onto \(M_Y(C(T)) \cong M_Y(\mathbb{C}) \otimes C(T)\). The proof of \[19\] Proposition 3.5 finds a family \(\{\theta(\alpha, \beta) : \alpha, \beta \in Y\}\) of matrix units and a unitary \(U\) in \(C^*(\Lambda)\) such that \(\theta(\alpha, \beta)U = U\theta(\alpha, \beta)\) for all \(\alpha, \beta \in Y\). This gives a homomorphism \(\phi : M_Y(\mathbb{C}) \otimes C(T)\) into \(C^*(\Lambda)\). The argument then shows that \(U\) has full spectrum, from which it follows that \(\phi\) is injective, and then \(\phi\) is shown to be surjective. Since \(R\) is a ring with 1, the matrix units \(\{\theta(\alpha, \beta) : \alpha, \beta \in Y\}\) (and the unitary \(U\)) live naturally in \(KP_R(\Lambda)\). But in the algebraic setting, the spectral argument is not available; we use the following lemma instead.

**Lemma 5.2.** Let \(W\) be a ring with 1 where \(W\) contains a set \(\{\theta(\alpha, \beta) : \alpha, \beta \in Y\}\) of matrix units. Let \(\alpha_0 \in Y\) and \(D := \theta(\alpha_0, \alpha_0)W\theta(\alpha_0, \alpha_0)\). Then the map \(w \mapsto (\theta(\alpha_0, \alpha)w\theta(\beta, \alpha_0))_{\alpha, \beta}\) is an isomorphism of \(W\) onto \(M_Y(D)\).

The proof of Lemma 5.2 is straightforward, see, for example, \[25\] Proposition 13.9. In the proof of Proposition 5.1 we will apply Lemma 5.2 with \(\theta(\alpha_0, \alpha_0) = p_f\), where \(\dagger\) is a vertex in the set \(S\) of sources in \(\Lambda^{blue}\).

**Lemma 5.3.** Let \(\Lambda\) be a 2-graph satisfying the hypotheses of Proposition 5.1. Fix a vertex \(\dagger \in S\) and let \(\mu\) be the unique red path of least nonzero degree with range and source \(\dagger\). Then there is an isomorphism of \(R[x, x^{-1}]\) onto \(p_1KP_R(\Lambda)p_1\) such that \(1 \mapsto p_1\), \(x \mapsto s_\mu\) and \(x^{-1} \mapsto s_\mu^*\).

**Proof.** Let \(\alpha \in \Lambda\) with \(r(\alpha) = \dagger\). Then \(\alpha \in \Lambda^{red}\). Let \(\nu\) be the unique red path of nonzero least length \(n_2\) such that \(s(\alpha\nu) = \dagger\). For any \(\beta\), applying \((KP4')\) to \(s_\alpha s_\beta^*\) gives

\[
(5.2) \quad s_\alpha s_\beta^* = s_\alpha p_{s(\alpha)}^\dagger s_\beta^* = s_\alpha \sum_{\lambda \in s(\alpha)(\Lambda) \leq (0, n_2)} s_\lambda s_{\lambda}^* s_\beta^* = s_\alpha s_{\beta(\beta)^*}.
\]

The Kumjian-Pask relation \((KP4')\) at \(\dagger\) with \(n = |\mu|e_2\) says that \(s_\mu s_{\mu^*} = p_1\). Writing \(\mu\) for the path which traverses \(\mu\) exactly \(i\) times, we now have

\[
p_1KP_R(\Lambda)p_1 = \text{span}\{s_\alpha s_\beta^* : \alpha, \beta \in \Lambda^{red}, s(\alpha) = s(\beta), r(\alpha) = r(\beta) = \dagger\}
\]

\[
= \text{span}\{s_{\mu^i}s_{(\mu^i)^*} : i, j \in \mathbb{N}\} \quad \text{ (using \[5.2\])}
\]

\[
= \text{span}\{s_{\mu^i} : i \in \mathbb{N}\} \quad \text{ (since } s_\mu s_{\mu^*} = p_1).\n\]

Now let \(E\) be the directed graph with one vertex \(w\) and one edge \(f\). Let \(L_R(E)\) be the Leavitt path algebra over \(R\), and let \(\{q_w, t_f, t_f^*\}\) be the generating Leavitt \(E\)-family in \(L_R(E)\). The polynomials \(1, x, x^{-1}\) are a Leavitt \(E\)-family in \(R[x, x^{-1}]\), and the universal property of \(L_R(E)\) gives an \(R\)-algebra homomorphism \(\rho : L_R(E) \to R[x, x^{-1}]\) such that \(\rho(1) = q_w, \rho(t_f) = x\) and \(\rho(t_f^*) = x^{-1}\). Now \(\rho\) is surjective because the range of \(\rho\) contains the generators \(x\) and \(x^{-1}\) of \(R[x, x^{-1}]\), and it is one-to-one by definition of the zero polynomial.

Since \(p_1 = s_\mu s_{\mu^*}\), it follows that \(\{p_1, s_\mu, s_{\mu^*}\}\) is a Leavitt \(E\)-family in \(KP_R(\Lambda)\). By the universal property of \(L_R(E)\) again, there is an \(R\)-algebra homomorphism \(\pi : L_R(E) \to KP_R(E)\) such that \(\pi(q_w) = p_1\), \(\pi(t_f) = s_\mu\) and \(\pi(t_f^*) = s_{\mu^*}\). Now we observe that \(p_1KP_R(\Lambda)p_1 = \text{span}\{s_{\mu^i}, s_{(\mu^i)^*} : i \in \mathbb{N}\}\) is graded over \(\mathbb{Z}\) by the subgroups \(\text{span}\{s_{\mu^i}\}\). Thus \(\pi\) is graded, and hence it is injective by the graded-uniqueness theorem \[24\] Theorem 5.3.
Now $\pi \circ \rho^{-1} : R[x, x^{-1}] \to KP_R(\Lambda)$ is injective and satisfies
$$\pi \circ \rho^{-1}(1) = p_1, \pi \circ \rho^{-1}(x) = s_\mu$$ and $\pi \circ \rho^{-1}(x^{-1}) = s_{\mu^*}$. It follows that $\pi \circ \rho^{-1}$ has range span \{ $s_\mu, s_{\mu^*} : i \in \mathbb{N}$ \} = $p_1 KP_R(\Lambda)p_1$. Thus $\pi \circ \rho^{-1}$ is the required isomorphism.

**Proof of Proposition 5.1.** Fix a vertex $\dagger \in S$ and let $e_1$ be the edge in $SA^\text{red}S$ with $r(e_1) = \dagger$. For $\alpha, \beta$ in $Y$, let $\nu(\alpha, \beta)$ be the unique path in $\Lambda^\text{red}$ connecting $s(\alpha)$ and $s(\beta)$ such that $\nu(\alpha, \beta)$ does not contain $e_1$. It follows from the proof of Lemma 3.8 of [19], that the elements

$$\theta(\alpha, \beta) := \begin{cases} s_\alpha s_{\nu(\alpha, \beta)} s_{\beta^*} & \text{if } s(\nu(\alpha, \beta)) = s(\beta); \\ s_\alpha s_{\nu(\alpha, \beta)^*} s_{\beta^*} & \text{if } s(\nu(\alpha, \beta)) = s(\alpha) \end{cases}$$

form a set \{ $\theta(\alpha, \beta) : \alpha, \beta \in Y$ \} of matrix units in $KP_R(\Lambda)$, that is, $\theta(\alpha, \beta)\theta(\sigma, \tau) = \delta_{\beta, \sigma}\theta(\alpha, \tau)$. Since $\Lambda^0$ is finite, there exists $n_1$ such that $Y = \Lambda^{\leq (n_1, 0)}$. Calculate:

$$\sum_{\alpha \in Y} \theta(\alpha, \alpha) = \sum_{\alpha \in Y} s_\alpha s_{\alpha^*} = \sum_{v \in \Lambda^0} \sum_{\alpha \in Y} s_\alpha s_{\alpha^*} = \sum_{v \in \Lambda^0} \sum_{\alpha \in \Lambda^{\leq (n_1, 0)}} s_\alpha s_{\alpha^*} = \sum_{v \in \Lambda^0} p_v = 1.$$

Let $\alpha_0 = \dagger$. Then $\alpha_0$ is a path of degree 0 in $Y$, and $\theta(\alpha_0, \alpha_0) = p_1$. Now apply Lemma 5.2 with $\alpha_0 = \dagger$ to see that $KP_R(\Lambda)$ is isomorphic to $M_Y(p_1 KP_R(\Lambda)p_1)$. The proposition now follows from the isomorphism of $p_1 KP_R(\Lambda)p_1$ with $R[x, x^{-1}]$ of Lemma 5.3.

5.1. **Application to rank-2 Bratteli diagrams.** Throughout this subsection, $\Lambda$ is a row-finite 2-graph without sources which is a rank-2 Bratteli diagram in the sense of [19] Definition 4.1]. This means that the blue subgraph $\Lambda^\text{blue}$ of paths of degree $(n_1, 0)$ is a Bratteli diagram in the usual sense: the vertex set $\Lambda^0$ is the disjoint union $\bigsqcup_{n=0}^\infty V_n$ of finite subsets $V_n$, each blue edge goes from some $V_{n+1}$ to $V_n$. The red subgraph $\Lambda^\text{red}$ of paths of degree $(0, n_2)$ consists of disjoint cycles whose vertices lie entirely in some $V_n$. For each blue edge $e$ there is a unique red edge $f$ with $s(f) = r(e)$, and hence by the factorization property there is a unique blue-red path $F(e)h$ such that $F(e)h = fe$. The map $F : \Lambda^e \to \Lambda^e$ is a bijection, and induces a permutation of each finite set $\Lambda^e V_n$. We write $o(e)$ for the order of $e$: the smallest $l > 0$ such that $F^l(e) = e$.

A $k$-graph without sources is cofinal if for every $x \in \Lambda^\infty$ and every $v \in \Lambda^0$, there exists $n \in \mathbb{N}^k$ such that $v\Lambda x(n) \neq \emptyset$. It follows from [19] Theorem 5.1 that if $\Lambda$ is cofinal and \{ $o(e)$ \} is unbounded, then $\Lambda$ is aperiodic; and hence if $K$ is a field, then $KP_K(\Lambda)$ is simple by [8] Corollary 7.8. So the cofinal rank-2 Bratteli diagrams give a rich supply of simple Kumjian-Pask algebras.

Theorem 7.10 of [8] uses rank-2 Bratteli diagrams to show that there are simple Kumjian-Pask algebras that are neither purely infinite nor locally matricial. But the analysis in §7 of [8] specialises to $K = \mathbb{C}$ because the proof reduces to a “rank-2 Bratteli diagrams $\Lambda_N$ of depth $N$”, which is a 2-graph with sources. In the absence of a Kumjian-Pask algebra for such graphs, the embedding into $C^*(\Lambda_N)$ is used. This was one of our motivations for defining a Kumjian-Pask algebra for graphs with sources in the first place, and we now show that we can extend [8] Theorem 7.10] from $\mathbb{C}$ to arbitrary fields $K$ (see Theorems 5.8 below). Note that we will use the letter $K$ in place of $R$ when we are explicitly assuming the underlying ring is a field. We start with three lemmas.
Lemma 5.4. Let $\Lambda$ be a rank-2 Bratteli diagram. Let $\Lambda_N$ be the rank-2 Bratteli diagram of depth $N$ consisting of all the paths in $\Lambda$ which begin and end in $\bigcup_{n=0}^{N} V_n$. Then $\Lambda_N$ is locally convex and the subalgebra $\text{span}\{s_\mu s_\nu^* : \mu, \nu \in \Lambda_N\}$ of $K_P(R(\Lambda))$ is canonically isomorphic to $K_P(R(\Lambda_N))$.

Proof. The graph $\Lambda_N$ has sources, but satisfies [5,1], and hence is locally convex [19, page 141]. For $v \in \Lambda^0_N$ and $\lambda \in \Lambda_N$, set $Q_v = p_v$, $T_\lambda = s_\lambda$ and $T_\lambda^* = s_\lambda^*$. We will show that $(Q,T)$ is a Kumjian-Pask $\Lambda_N$-family in $K_P(R(\Lambda))$. Both $(KP1)$ and $(KP2)$ for $(Q,T)$ immediately reduce to $(KP1)$ and $(KP2)$ for the family $(p,s)$.

For $(KP3')$, let $n \in \mathbb{N}^2 \setminus \{0\}$ and $\lambda, \mu \in \Lambda^{m,n}_N$. Since

$$T_\lambda T_\mu = T_\lambda^* Q_\mu T_\mu = \delta_{r(\lambda), r(\mu)} T_\lambda^* T_\mu,$$

if $r(\lambda) \neq r(\mu)$, then $T_\lambda^* T_\mu = 0 = \delta_{\lambda, \mu} Q_\mu$. So we may assume that $r(\lambda) = r(\mu)$.

Notice that $d(\lambda) = d(\mu) = n_2$ because $\lambda, \mu \in \Lambda^{m,n}$ and the red subgraph consists entirely of cycles and hence has no sources. We claim that $d(\lambda) = d(\mu)$. By way of contradiction, suppose $d(\lambda) < d(\mu)$. Since $r(\lambda) = r(\mu)$, we must have $s(\lambda) \in V_i$ and $s(\mu) \in V_j$ where $i < j$. Thus $s(\lambda) \notin V_N$. But $s(\lambda)$ must be a source in the blue graph because $d(\lambda) < n$. This implies $s(\lambda) \in V_N$, a contradiction. Similarly, we cannot have $d(\mu) < d(\lambda)$. Thus $d(\lambda) = d(\mu)$ as claimed.

Now, $\lambda, \mu \in r(\lambda) \Lambda^{d(\lambda)}$, and $T_\lambda T_\mu = s_\lambda s_\mu = \delta_{\lambda, \mu} p_{s(\mu)} = \delta_{\lambda, \mu} Q_{s(\mu)}$ by $(KP3)$ for $(p,s)$. Thus $(KP3')$ holds for $(Q,T)$.

To see $(KP4')$ holds, we use Lemma [3.2]. Let $v \in \Lambda^0_N$ and suppose that $v\Lambda_N^{m,n} \neq \emptyset$. If $v \in \Lambda^0_N \setminus V_N$, then $i = 1$ or 2, and if $v \in V_N$, then $i = 2$. In both cases, $(KP4)$ for $(q,t)$ gives

$$Q_v = p_v = \sum_{\lambda \in \Lambda_N^{m,n}} s_\lambda s_\lambda^* = \sum_{\lambda \in \Lambda_N^{m,n}} s_\lambda s_\lambda^*.$$

Thus $(KP4')$ holds by Lemma [3.2].

Denote the generating Kumjian-Pask $\Lambda_N$-family in $K_P(R(\Lambda_N))$ by $(q,t)$. The universal property of $K_P(\Lambda_N)$ (Theorem [5.7][3]) now gives a homomorphism $\pi_{Q,T} : K_P(\Lambda_N) \rightarrow K_P(R(\Lambda))$ such that $\pi_{Q,T}(q_v) = p_v$, $\pi_{Q,T}(t_\lambda) = s_\lambda$ and $\pi_{Q,T}(t_\lambda^*) = s_\lambda^*$. Thus $\pi_{Q,T}$ has range $\text{span}\{s_\mu s_\nu^* : \mu, \nu \in \Lambda_N\}$. It is clear that $\pi_{Q,T}$ is graded. Since each $\pi_{Q,T}(q_v) \neq 0$, it follows from the graded-uniqueness theorem (Theorem [4.1]) that $\pi_{Q,T}$ is injective. \hfill $\square$

Lemma 5.5. Let $\Lambda_N$ be a rank-2 Bratteli diagram of depth $N$. Let $V_{N,i}$ be the vertices in $V_N$ of one isolated red cycle, and let $\Lambda_{N,i}$ be the subgraph of $\Lambda_N$ of paths with source and range in $V := \{v \in \Lambda^0 : v\Lambda_N^{m,n} \neq \emptyset\}$. Let $(q,t)$ be a generating Kumjian-Pask $\Lambda_N$ family in $K_P(\Lambda_N)$. Then the subalgebra

$$C_{N,i} := \text{span}\{t_\mu t_\nu^* : \mu, \nu \in \Lambda_N, s(\mu) = s(\nu) \in V_{N,i}\}$$

of $K_P(\Lambda_N)$ is canonically isomorphic to $K_P(\Lambda_{N,i})$.

Proof. Set $P_i = \sum_{\alpha \in \Lambda^{m,n}_{N,i}} t_\alpha t_\alpha^*$. Let $v \in \Lambda^0_{N,i}$ and $\mu \in \Lambda_{N,i}$. Set

$$Q_v = P_i q_v P_i, T_\mu = P_i t_\mu P_i, T_\mu^* = P_i t_\mu^* P_i.$$

We will show that $(Q,T)$ is a Kumjian-Pask $\Lambda_{N,i}$-family in $C_{N,i}$. \hfill $\square$
If \( \alpha, \beta \in \Lambda^\text{blue} V_{N,i} \), then \( d(\alpha) = d(\beta) \) because \( \Lambda^\text{blue} \) is a Bratteli diagram. Thus

\[
P_i^2 = \sum_{v,w \in V} \sum_{\alpha \in \Lambda^\text{blue} V_{N,i}, \beta \in \Lambda^\text{blue} V_{N,i}} t_\alpha t_\alpha^* q_{v} q_{w} t_\beta t_\beta^* = \sum_{v \in V} \sum_{\alpha, \beta \in \Lambda^\text{blue} V_{N,i}} t_\alpha t_\alpha^* t_\beta t_\beta^* = P_i.
\]

We also have \( q_v P_i = \sum_{\alpha \in \Lambda^\text{blue} V_{N,i}} t_\alpha^* t_\alpha = P_i q_v \), and

\[
P_i t_\mu P_i = \sum_{\alpha, \beta \in \Lambda^\text{blue} V_{N,i}} t_\alpha t_\alpha^* t_\mu t_\beta t_\beta^* = \sum_{\alpha \in r(\mu) \Lambda^\text{blue} V_{N,i}, \beta \in s(\mu) \Lambda^\text{blue} V_{N,i}} t_\alpha t_\alpha^* t_\mu t_\beta t_\beta^* = P_i (\text{by KP} 3' \text{ because such } \alpha \text{ must have } d(\alpha) = d(\mu \beta)) = t_\mu P_i.
\]

A similar calculation gives \( P_i t_\mu^* P_i = P_i t_\mu^* \). The above relations help to reduce the Kumjian-Pask relations for \((Q, T)\) to the Kumjian-Pask relations for \((q, t)\) in \( K_P R(\Lambda_N) \).

To see (KP1) holds, let \( v, w \in \Lambda^0_{N,i} \). Then

\[
Q_v Q_w = P_i q_v P_i^2 q_w P_i = P_i q_v q_w P_i = \delta_{v,w} P_i q_v P_i = \delta_{v,w} Q_v.
\]

To see (KP2) holds, let \( \lambda, \mu \in \Lambda^0_{N,i} \). Then

\[
T_\lambda T_\mu = P_i t_\lambda P_i^2 t_\mu P_i = P_i t_\lambda t_\mu P_i = P_i t_{\lambda \mu} P_i = T_{\lambda \mu},
\]

and, similarly, \( T_{\mu \cdot T_\lambda} = T_{(\lambda \mu) \cdot} \). To see (KP3') holds, let \( n \in \mathbb{N}^2 \setminus \{0\} \) and \( \lambda, \mu \in \Lambda^\leq_n V_{N,i} \). Then

\[
T_{\lambda, T_\mu} = P_i t_\lambda P_i t_\mu P_i = P_i t_{\lambda \mu} P_i = \delta_{\lambda, \mu} P_i q_{s(\mu)} P_i \quad \text{(because } \lambda, \mu \in \Lambda^\leq_n \text{)}
\]

\[
= \delta_{\lambda, \mu} Q_v P_i.
\]

To see (KP4') holds, let \( n \in \mathbb{N}^2 \setminus \{0\} \) and \( v \in \Lambda^0_{N,i} \). Then

\[
\sum_{\lambda \in \Lambda^\leq_n V_{N,i}} T_{\lambda, T_\mu} = \sum_{\lambda \in \Lambda^\leq_n V_{N,i}} P_i t_\lambda P_i^2 t_\mu P_i = \sum_{\lambda \in \Lambda^\leq_n V_{N,i}} P_i t_\lambda t_\lambda P_i = P_i \left( \sum_{\lambda \in \Lambda^\leq_n V_{N,i}} t_\lambda t_\lambda \right) P_i = P_i \sum_{\beta \in \Lambda^\leq_{|\nu|} V_{N,i}} t_\beta t_\beta P_i = P_i Q_v P_i = Q_v.
\]

Thus \((Q, T)\) is a Kumjian-Pask \( \Lambda_{N,i} \)-family in \( C_{N,i} \). Write \((q^i, t^i)\) for the universal Kumjian-Pask family in \( K_P R(\Lambda_{N,i}) \). The universal property of \( K_P R(\Lambda_{N,i}) \) gives a homomorphism \( \pi_{Q,T} : K_P R(\Lambda_{N,i}) \to C_{N,i} \) such that \( \pi_{Q,T}(q^i_v) = Q_v \), \( \pi_{Q,T}(t^i_\lambda) = T_\lambda \) and \( \pi_{Q,T}(t^i_{\lambda \cdot}) = T_{\lambda \cdot} \).

Thus \( \pi_{Q,T}(t^i_\mu t^i_{\mu \cdot}) = P_i t_\mu P_i t_\mu P_i = P_i t_\mu t_\mu P_i \). Since \( P_i \) is homogeneous of degree 0 it follows that \( \pi_{Q,T} \) is graded. Thus \( \pi_{Q,T} \) is injective by the graded-uniqueness (Theorem 4.1).

Finally, the range of \( \pi_{Q,T} \) is

\[
\text{span}\{P_i t_\mu P_i t_\nu P_i : \lambda, \mu \in \Lambda_N\} = \text{span}\{P_i t_\mu P_i t_\nu P_i : s(\mu) = s(\nu) \in V_{N,i}, r(\mu), r(\nu) \in V\},
\]

which is equal to \( C_{N,i} \). \( \square \)
Lemma 5.6. Let $\Lambda_N$ be a rank-2 Bratteli diagram of depth $N$. Suppose that the set of sources in $\Lambda^{\text{blue}}$ are the vertices on a single isolated cycle in $\Lambda^{\text{red}}$. Let $P$ be the idempotent $P = \sum_{v \in V_0} p_v$ and $X = V_0 \Lambda^{\text{blue}} V_N$. Then $PK_P(\Lambda_N)P$ is isomorphic to $M_X(R[x, x^{-1}])$.

Proof. Let $Y = \Lambda^{\text{blue}} V_N$ and fix $\dagger \in V_N$. Let $\theta(\alpha, \beta)$ be the matrix units defined at (5.3). The graph $\Lambda_N$ satisfies the hypotheses of Proposition 5.1 and hence

$$w \mapsto (\theta(\dagger, \alpha)w \theta(\beta, \dagger))_{\alpha, \beta \in Y}$$

is an isomorphism of $PK_P(\Lambda_N)$ onto $M_Y(p_1 PK_P(\Lambda_N)p_1)$, and $p_1 PK_P(\Lambda_N)p_1$ is isomorphic to $R[x, x^{-1}]$ (see Lemmas 5.2 and 5.3 for how the isomorphism of Proposition 5.1 decomposes). It suffices to prove that the restriction of $\psi$ to $PK_P(\Lambda_N)P$ has range $M_X(p_1 PK_P(\Lambda_N)p_1)$. We observe that

$$\theta(\dagger, \alpha)PwP\theta(\beta, \dagger) = \begin{cases} \theta(\dagger, \alpha)w \theta(\beta, \dagger) & \text{if } r(\alpha), r(\beta) \in V_0; \\ 0 & \text{else.} \end{cases}$$

It follows first, that $\theta(\dagger, \alpha)PwP\theta(\beta, \dagger) \neq 0$ implies $\alpha, \beta \in X$, and second, if $\alpha, \beta \in X$, then

$$\theta(\dagger, \alpha)PK_P(\Lambda_N)P\theta(\beta, \dagger) = \theta(\dagger, \alpha)PK_P(\Lambda_N)\theta(\beta, \dagger) = p_1 PK_P(\Lambda_N)p_1.$$  

Thus $\psi|_1$ is onto $M_X(p_1 PK_P(\Lambda_N)p_1)$. \qed

An idempotent $p$ in $R$ is infinite if there exist orthogonal nonzero idempotents $p_1, p_2 \in R$ and elements $x, y \in R$ such that

$$p = p_1 + p_2, \quad x \in pRp_1, \quad y \in p_1 Rp, \quad p = xy \text{ and } p_1 = yx.$$  

A simple ring is purely infinite if every nonzero right ideal of $R$ contains an infinite idempotent [3, §1].

Proposition 5.7. Let $\Lambda$ be a rank-2 Bratteli diagram, and $K$ a field. If $\Lambda$ is cofinal and aperiodic, then $PK_K(\Lambda)$ is not purely infinite.

Proof. Since $\Lambda$ is cofinal and aperiodic, and $K$ is a field, $PK_K(\Lambda)$ is simple by [3, Theorem 6.1]. Let $P_0 := \sum_{v \in V_0} p_v$. Since the property of being purely infinite and simple passes to corners by [2, Proposition 10], it suffices to prove that $P_0 PK_K(\Lambda)P_0$ is not purely infinite. We argue by contradiction: suppose that $P_0 PK_K(\Lambda)P_0$ is purely infinite. Then $P_0 PK_K(\Lambda)P_0$ contains an infinite idempotent $p$. Then there exist nonzero idempotents $p_1, p_2$ and elements $x, y$ in $P_0 PK_K(\Lambda)P_0$ such that

$$(5.4) \quad p = p_1 + p_2, \quad p_1 p_2 = p_2 p_1 = 0, \quad xy = p \quad \text{and} \quad yx = p_1.$$  

Choose $N \in \mathbb{N}$ large enough to ensure that all five elements can be written as linear combinations of elements $s_\mu s_\nu$, for which $s(\lambda)$ and $s(\mu)$ are in $\bigcup_{n=0}^{N} V_n$.

Let $\Lambda_N$ be the rank-2 Bratteli diagram of depth $N$. Let $(q, t)$ be the generating Kumjian-Pask $\Lambda_N$-family in $PK_K(\Lambda_N)$ and set $Q_0 = \sum_{v \in V_0} q_v$. Since $PK_K(\Lambda_N)$ is canonically isomorphic to the subalgebra span$\{s_\mu s_\nu : \mu, \nu \in \Lambda_N\}$ of $PK_K(\Lambda)$ by Lemma 5.4 we may assume that $p, p_1, p_2, x$ and $y$ are all in $Q_0 PK_K(\Lambda_N)Q_0$.

Next we will decompose $PK_K(\Lambda_N)$ into a direct sum. Using (KP4'),

$$PK_K(\Lambda_N) = \text{span}\{t_\lambda t_\mu^* : s(\lambda) = s(\mu) \in V_N\}.$$  

Partition $V_N$ into subsets $V_{N,i}$, each consisting of the vertices of one isolated red cycle. We claim that if $s(\mu)$ and $s(\alpha)$ are in different $V_{N,i}$, then $t_\mu^* t_\alpha = 0$. By way of contradiction,
The elements $p$ that KP and we have assume that $p$ and the elements satisfy the relations (5.4). The component of $\sigma$ implies $s$ and $\tau$ lie on the same isolated red cycle, and this implies that $s(\mu) = r(\sigma)$ and $s(\alpha) = r(\tau)$ are in the same $V_{N,i}$, a contradiction. Thus $t_\mu^* t_\alpha = 0$, as claimed. It follows that $\text{KP}_K(\Lambda_N)$ is the direct sum of the subalgebras

$$C_{N,i} := \text{span}\{t_\lambda t_\mu^* : s(\lambda) = s(\mu) \in V_{N,i}\},$$

and we have

$$Q_0 \text{KP}_K(\Lambda_N)Q_0 = \bigoplus_i Q_0 C_{N,i}Q_0.$$

The elements $p$, $p_1$, $p_2$, $x$ and $y$ in $Q_0 \text{KP}_K(\Lambda_N)Q_0$ all have direct sum decompositions, and the elements satisfy the relations (5.4). The component of $p_2$ is nonzero in at least one summand, and then the same component of the rest must be nonzero too. So we may assume that $p$, $p_1$, $p_2$, $x$ and $y$ are all in $Q_0 C_{N,i}Q_0$ for some $i$.

Now consider the subgraph $\Lambda_{N,i}$ of $\Lambda_N$ of paths with source and range in $\{v \in \Lambda^0 : v\Lambda_{N,i} \neq \emptyset\}$. By Lemma 5.5, $C_{N,i}$ is canonically isomorphic to $\text{KP}_K(\Lambda_{N,i})$, and by Proposition 5.1, $\text{KP}_K(\Lambda_{N,i})$ is isomorphic to a matrix algebra $M_Y(K[x, x^{-1}])$ for a certain set $Y$, and by Lemma 5.6 this isomorphism restricts to an isomorphism of $Q_0 C_{N,i}Q_0$ onto $M_X(K[x, x^{-1}])$ where $X \subseteq Y$. Pulling the five elements through all these isomorphisms gives us nonzero idempotents $q$, $q_1$, $q_2$ and elements $f, g$ in $M_X(K[x, x^{-1}])$ such that

$$q = q_1 + q_2, \quad q_1 q_2 = q_2 q_1 = 0, \quad fg = q \quad \text{and} \quad gf = q.$$ 

Evaluation at $z \in K$ is a homomorphism, and so $f(z)g(z) = q(z)$ and $g(z)f(z) = q_1(z)$. Thus $g(z)$ is an isomorphism of $q(z)K^X$ onto $q_1(z)K^X$. So the matrices $q(z)$ and $q_1(z)$ have the same rank. On the other hand, since $q_1(z)$ and $q_2(z)$ are orthogonal, $\text{rank}(q_1(z) + q_2(z)) = \text{rank} q_1(z) + \text{rank} q_2(z)$. Now $q = q_1 + q_2$ implies that $\text{rank} q_2(z) = 0$ for all $z$. This contradicts that $q_2$ is nonzero. Thus there is no infinite idempotent in $P_0 \text{KP}_K(\Lambda)P_0$, as claimed. Thus $P_0 \text{KP}_K(\Lambda)P_0$ is not purely infinite, and neither is $\text{KP}_K(\Lambda)$. \hfill \Box

Recall that a matricial algebra is a finite direct product of full matrix algebras and we say an algebra is locally matricial algebra if it is direct limit of matricial algebras.

**Theorem 5.8.** Let $\Lambda$ be a rank-2 Bratteli diagram which is cofinal and aperiodic, and $K$ a field. Then $\text{KP}_K(\Lambda)$ is simple, but is neither purely infinite nor locally matricial.

**Proof.** By [8, Theorem 6.1], $\text{KP}_K(\Lambda)$ is simple, and by Proposition 5.1, it is not purely infinite. To see that it is not locally matricial, consider the element $s_\mu$ associated to a single red cycle $\mu$. Since $v := r(\mu) = s(\mu)$ receives just one red path of length $|\mu|$, namely $\mu$, the Kumjian-Pask relation (KP4') at $v$ for $n = |\mu|e_2$ (which only involves red paths) says that $p_v = s_\mu s_\mu^*$. But, as seen in the proof of Lemma 5.5, the subalgebra generated by $s_\mu$ is isomorphic to the Leavitt path algebra of the directed graph consisting of a single vertex $w$ and a single loop $e$ at $w$, which is in turn isomorphic to $K[x, x^{-1}]$. Thus $s_\mu$ generates an infinite-dimensional algebra, and does not lie in a finite-dimensional subalgebra. \hfill \Box
6. Desourcification

Throughout this section, $\Lambda$ is a locally convex, row-finite $k$-graph. In Theorem 7.24 we show that the Kumjian-Pask algebra of $\Lambda$ is Morita equivalent to a Kumjian-Pask algebra of a certain $k$-graph $\tilde{\Lambda}$ without sources. The graph $\tilde{\Lambda}$ is called the desourcification of $\Lambda$. The construction of $\tilde{\Lambda}$, and the Morita equivalence of the $C^*$-algebras $C^*(\Lambda)$ and $C^*(\tilde{\Lambda})$ goes back to Farthing [13]. Farthing’s construction was refined by Robertson and Sims in [24], and then generalised to finitely aligned $k$-graphs by Webster in [30]. The difference in the approaches is that Farthing’s construction adds paths and vertices to $\Lambda$ to obtain a graph $\overline{\Lambda}$, and the Robertson-Sims-Webster construction abstractly builds a $\tilde{\Lambda}$ which is then shown to contain a copy of $\Lambda$. When $\Lambda$ is locally convex, $\overline{\Lambda}$ and $\tilde{\Lambda}$ are isomorphic.

We use $\tilde{\Lambda}$, and start by giving the details about $\tilde{\Lambda}$ that we need. Set

\[ V_\Lambda := \{ (x; m) : x \in \Lambda_{\leq \infty}, m \in \mathbb{N}^k \} \quad \text{and} \quad P_\Lambda := \{ (x; (m, n)) : x \in \Lambda_{\leq \infty}, m \leq n \in \mathbb{N}^k \}; \]

the vertices and paths of $\tilde{\Lambda}$ are quotients of these sets, respectively. For this, define relations $\approx$ on $V_\Lambda$ and $\sim$ on $P_\Lambda$ as in Definitions 4.2 and 4.3 of [30]. First, define $(x; m) \approx (y; p)$ in $V_\Lambda$ if and only if

- (V1) $x(m \wedge d(x)) = y(p \wedge d(y))$ and
- (V2) $m - m \wedge d(x) = p - p \wedge d(y)$.

It is straightforward to check that $\approx$ is an equivalence relation, and we denote the class of $(x; n)$ by $[x; n]$. We view $V_\Lambda/\approx$ as a set of vertices that includes a copy of the vertices in $\Lambda^0$. Indeed, if $v \in \Lambda^0$, then $v = x(0)$ for some $x \in \Lambda_{\leq \infty}$; we can think of $\Lambda^0$ with the equivalence classes of elements $(x; 0)$, and item (V1) above ensures this identification is well defined. The set $V_\Lambda/\approx$ also includes elements that are not identified with vertices in $\Lambda^0$. In particular, for each $x \in \Lambda_{\leq \infty}$ with $d(x) < \infty$, there is one element in $V_\Lambda/\approx$ for each $m > d(x)$.

Second, define $(x; (m, n)) \sim (y; (p, q))$ in $P_\Lambda$ if and only if

- (P1) $x(m \wedge d(x), n \wedge d(x)) = y(p \wedge d(y), q \wedge d(y))$,
- (P2) $m - m \wedge d(x) = p - p \wedge d(y)$, and
- (P3) $n - m = q - p$.

Again, it is straightforward to check that $\sim$ is an equivalence relation, and we denote the class of $(x; (m, n))$ by $[(x; (m, n))]$. The next proposition is [30, Proposition 4.9], and says that we can view each $[(x; (m, n))] \in P_\Lambda/\sim$ as a morphism between vertices $[(x; n)]$ and $[(x; m)]$ in $V_\Lambda/\approx$ of degree $n - m$.

**Proposition 6.1** (Farthing, Webster). Let $\Lambda$ be a locally convex, row-finite $k$-graph. Define

\[ \tilde{\Lambda}^0 := V_\Lambda/\approx \quad \text{and} \quad \tilde{\Lambda} := P_\Lambda/\sim, \]

\[ r, s : \tilde{\Lambda} \to \tilde{\Lambda}^0 \text{ by } r([x; (m, n)]) = [x; m] \text{ and } s([x; (m, n)]) = [x; n], \]

\[ \text{id}([x; m]) = [x; (m, m)], \]

\[ [x; (m, n)] \circ [y; (p, q)] = [x(0, m \wedge d(x))\sigma^{p \wedge d(y)}(p); (m, n + p - q)], \quad \text{and} \]

\[ d : \tilde{\Lambda} \to \mathbb{N}^k \text{ by } d(v) = 0 \text{ for all } v \in \tilde{\Lambda}^0 \text{ and } d([x; (m, n)]) = n - m. \]

Each of these functions is well defined, and $\tilde{\Lambda} = (\tilde{\Lambda}^0, \tilde{\Lambda}, r, s, \text{id}, \circ, d)$ is a $k$-graph without sources. We call $\tilde{\Lambda}$ the desourcification of $\Lambda$. 
Remark 6.2. (a) The map \( \iota : \Lambda \to \tilde{\Lambda} \) defined by \( \iota(\lambda) = [\lambda x; (0, d(\lambda))] \) for \( x \in s(\lambda)\Lambda^{\leq \infty} \) is a well-defined, injective \( k \)-graph morphism [30 Proposition 4.13]. Notice that if \( v \in \Lambda^0 \), then \( \iota(v) = [x; (0, 0)] \) for some \( x \in v\Lambda^{\leq \infty} \).

(b) The map \( \pi : \Lambda \to \iota(\Lambda) \) defined by \( \pi([y; (m, n)] = [y; (m \wedge d(y), n \wedge d(y))] \) is a well-defined, surjective \( k \)-graph morphism such that \( \pi \circ \iota = \iota \) [30 page 168].

(c) If \( \mu, \lambda \in \iota(\Lambda^0)\tilde{\Lambda} \) such that \( d(\lambda) = d(\mu) \) and \( \pi(\lambda) = \pi(\mu) \), then \( \lambda = \mu \) [30 Lemma 4.19].

Lemma 6.3. Suppose \( \mu := [z; (0, n)] \in \tilde{\Lambda} \). Then \( \mu \in \iota(\Lambda) \) if and only if \( d(z) \geq n \).

Proof. Suppose \( \mu = [z; (0, n)] = \iota(\lambda) \) for some \( \lambda \in \Lambda \). Since \( \iota \) is degree-preserving, \( d(\lambda) = n \). Since \( \pi \circ \iota = \iota \) we have \( \pi(\mu) = [z; (0, n \wedge d(z))] = \iota(\lambda) \). Thus \( n \wedge d(z) = n \), that is, \( d(z) \geq n \).

Conversely, suppose \( d(z) \geq n \). Then factor \( z = \lambda y \) where \( \lambda \in \Lambda^n \) and \( y \in \Lambda^{\leq \infty} \). Now \( \mu = [z; (0, n)] = [\lambda y; (0, d(\lambda))] = \iota(\lambda) \).

The following straightforward lemma is very useful and is used without proof in [30].

Lemma 6.4. Suppose \( \lambda \in \tilde{\Lambda} \) such that \( r(\lambda) \in \iota(\Lambda^0) \). Then there exists \( x \in \Lambda^{\leq \infty} \) such that \( \lambda = [x; (0, d(\lambda))] \).

Proof. Write \( \lambda = [z; (m, n)] \) for some \( m, n \in \mathbb{N}^k \) and \( z \in \Lambda^{\leq \infty} \). If \( m = 0 \), then \( d(\lambda) = n \) and we are done. So suppose \( m > 0 \). Because \( [z; m] = r(\lambda) \in \iota(\Lambda^0) \), \( [z; m] \approx [y; 0] \) for some \( y \in \Lambda^{\leq \infty} \). By (V2) \( m - m \wedge d(z) = 0 \). Then \( m = m \wedge d(z) \), and hence \( m \leq d(z) \).

Let \( x := \sigma^m(z) \). It suffices to show that \( (x; (0, n-m)) \sim (z; (m, n)) \). Items (P2) and (P3) are obvious. To see (P1), notice

\[
z(m \wedge d(z), n \wedge d(z)) = z(m, n \wedge d(z)) = \sigma^m(z)(0, n \wedge d(z) - m) = x(0, n \wedge d(z) - m) = x(0, (n-m) \wedge d(x))
\]

as needed.

Lemma 6.5. Suppose \( v \in \iota(\Lambda^0) \) and \( \lambda \in v\Lambda^n \) with \( \pi(\lambda) = v \). If \( \mu \in v\Lambda^n \), then \( \lambda = \mu \).

Proof. By [30 Lemma 4.19] (see Remark 6.2 [30]), it suffices to show that \( \pi(\mu) = \pi(\lambda) \). Since \( r(\lambda) = r(\mu) = v \in \iota(\Lambda^0) \), by Lemma 6.4 there exist \( x, y \in \Lambda^{\leq \infty} \) such that \( \lambda = [y; (0, n)], \mu = [x; (0, n)] \) and \( [x, 0] = [y, 0] = v \).

By definition of \( \pi \),

\[
\pi(\lambda) = [y; (0, n \wedge d(y))] \text{ and } \pi(\mu) = [x; (0, n \wedge d(x))].
\]

By assumption, \( \pi(\lambda) = v \), and hence \( n \wedge d(y) = 0 \). We need to show that \( n \wedge d(x) = 0 \) as well; we do this by showing that if \( n_i \neq 0 \) then \( d(x)_i = 0 \). Suppose \( n_i \neq 0 \). Since \( n \wedge d(y) = 0 \) we have \( d(y)_i = 0 \). Since \( y \) is a boundary path, \( 0\Omega_x^{s(\xi)} = \{0\} \). Now condition 2.1 gives \( y(0)\Lambda^{\leq \xi} = \{y(0)\} \). Hence \( y(0)\Lambda^{\leq \xi} = r\Lambda^{\leq \xi} = \emptyset \). But \( x(0) = v \) and so \( d(x)_i = 0 \) as well. Thus \( n \wedge d(x) = 0 \). Now \( \pi(\lambda) = v = \pi(\mu) \) and \( \lambda, \mu \) have the same degree, so \( \lambda = \mu \) by [30 Lemma 4.19].
Lemma 6.6. Let $\lambda, \mu \in \Lambda$. Then $\tilde{\Lambda}^{\min}(\iota(\lambda), \iota(\mu)) = \iota(\Lambda)^{\min}(\iota(\lambda), \iota(\mu))$.

Proof. This is essentially [30, Lemma 4.22]. Since $\iota(\Lambda) \subseteq \tilde{\Lambda}$, we have

$$\iota(\Lambda)^{\min}(\iota(\lambda), \iota(\mu)) \subseteq \tilde{\Lambda}^{\min}(\iota(\lambda), \iota(\mu)).$$

Let $(\alpha, \beta) \in \tilde{\Lambda}^{\min}(\iota(\lambda), \iota(\mu))$. Then

$$\iota(\lambda)\alpha = \iota(\mu)\beta = [z; (0, d(\lambda) \lor d(\mu))]$$

for some $z \in \Lambda^{\leq \infty}$ by Lemma 6.4. Now

$$(\iota(\lambda)\alpha)(0, d(\lambda)) = \iota(\lambda) = [z; (0, d(\lambda))]$$

by the factorisation property in $\tilde{\Lambda}$. So $d(z) \geq d(\lambda)$ by Lemma 6.3. Similarly, $d(z) \geq d(\mu)$. Hence $d(z) \geq d(\lambda) \lor d(\mu)$ and so $\iota(\lambda)\alpha \in \iota(\Lambda)$ by Lemma 6.4 again. Now by the unique factorisation property, both $\alpha$ and $\beta$ are in $\iota(\Lambda)$. Thus

$$\tilde{\Lambda}^{\min}(\iota(\lambda), \iota(\mu)) \subseteq \iota(\Lambda)^{\min}(\iota(\lambda), \iota(\mu))$$

as needed. \qed

7. The Morita equivalence of $KP_R(\Lambda)$ and $KP_R(\tilde{\Lambda})$

Throughout this section, $\Lambda$ is a locally convex, row-finite $k$-graph and $R$ is a commutative ring with 1. We start by identifying $KP_R(\Lambda)$ with a subalgebra of $KP_R(\tilde{\Lambda})$.

Proposition 7.1. Let $\Lambda$ be a locally convex, row-finite $k$-graph and let $\tilde{\Lambda}$ be its desourcification. Let $(q,t)$ be a universal $KP_R(\tilde{\Lambda})$-family. Let $B(\tilde{\Lambda})$ be the subalgebra of $KP_R(\tilde{\Lambda})$ generated by $\{q_{(v)}, t_{(\lambda)}, t_{(\mu)^*} : v \in \Lambda^0, \lambda, \mu \in \Lambda\}$. There is a graded isomorphism of $KP_R(\Lambda)$ onto $B(\tilde{\Lambda})$.

Proof. We show that $(q \circ \iota, t \circ \iota)$ is a Kumjian-Pask $\Lambda$-family in $KP_R(\tilde{\Lambda})$; we start by verifying (KP1), (KP2) and (KP4'), and then we use Corollary 5.3 to verify (KP3').

Since $(q,t)$ is a Kumjian-Pask $\Lambda$-family, $\{q_v : v \in \Lambda^0\}$ is an orthogonal set of idempotents, and hence so is $\{q_{(w)} : w \in \Lambda^0\}$. This gives (KP1) for $(q \circ \iota, t \circ \iota)$.

Let $\lambda, \mu \in \Lambda^{\neq 0}$ with $r(\mu) = r(\lambda)$. Since (KP2) holds for $(q,t)$ and $i$ is a graph morphism, $t_{(\lambda)}t_{(\mu)} = t_{(\lambda)i(\mu)} = t_{(\mu)i(\mu)}$. Similarly, the other equations in (KP2) hold for $(q \circ \iota, t \circ \iota)$.

For (KP4'), it suffices by Lemma 3.2 to show that for $1 \leq i \leq k$ with $v\Lambda^{e_i} \neq \emptyset$,

$$q_{(v)} = \sum_{\lambda \in v\Lambda^{e_i}} t_{(\lambda)}t_{(\lambda)^*}.$$  

Suppose $v\Lambda^{e_i} \neq \emptyset$, and let $\mu \in i(v)\tilde{\Lambda}^{e_i}$. Then $\mu = [x; (0, e_i)]$ for some $x \in v\Lambda^{\leq \infty}$ by Lemma 6.4. Since $v\Lambda^{e_i} \neq \emptyset$ and $r(x) = v$, we have $d(x) \geq e_i$. Thus $\mu = i(\lambda)$ for some $\lambda \in v\Lambda^{e_i}$ by Lemma 6.3, and this $\lambda$ is unique because $i$ is injective. Now

$$q_{(v)} = \sum_{\mu \in i(v)\Lambda^{e_i}} t_{\mu}t_{\mu^*} \quad \text{by (KP4) for } (q,t),$$

$$= \sum_{\lambda \in v\Lambda^{e_i}} t_{(\lambda)}t_{(\lambda)^*}. $$

Thus (KP4') holds for $(q \circ \iota, t \circ \iota)$. 

For \((KP3')\), let \(\lambda, \mu \in \Lambda\). Then
\[
 t_{\iota(\lambda)^*}t_{\iota(\mu)} = \sum_{(\sigma, \tau) \in \Lambda^{\min}(\iota(\lambda), \iota(\mu))} t_{\sigma \tau^*} \quad \text{by (3.1)}
\]
\[
= \sum_{(\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu)} t_{\iota(\alpha)}t_{\iota(\beta)^*} \quad \text{by Lemma 6.6}
\]

By Corollary 3.5, \((KP3')\) holds for \((q \circ \iota, t \circ \iota)\), and hence \((q \circ \iota, t \circ \iota)\) is a Kumjian-Pask \(\Lambda\)-family in \(KP_R(\hat{\Lambda})\).

Let \((p, s)\) be a generating Kumjian-Pask \(\Lambda\)-family in \(KP_R(\Lambda)\). Since \((q \circ \iota, t \circ \iota)\) is a \(\Lambda\)-family in \(KP_R(\hat{\Lambda})\), the universal property of \(KP_R(\Lambda)\) (Theorem 5.7.10) gives an \(R\)-algebra homomorphism \(\pi_{qo,tot} : KP_R(\Lambda) \to KP_R(\hat{\Lambda})\) such that \(\pi_{qo,tot}(p_v) = q_{t(v)}\), \(\pi_{qo,tot}(s_\lambda) = t_\iota(\lambda)\) and \(\pi_{qo,tot}(s_\gamma) = t_\iota(\gamma)^*\). Since \(\iota\) is degree preserving, \(\pi_{qo,tot}\) is graded. The graded-uniqueness theorem (Theorem 4.11) implies \(\pi_{qo,tot}\) is injective. Since \(\{q \circ \iota(v), t \circ \iota(\lambda), t \circ \iota(\mu^*)\}\) generates \(B(\hat{\Lambda})\), the range of \(\pi_{qo,tot}\) is \(B(\Lambda)\).

**Proposition 7.2.** Let \((\Lambda, d)\) be a locally convex, row-finite \(k\)-graph and let \((\hat{\Lambda}, d)\) be its desourcification. Let \((q, t)\) be a generating Kumjian-Pask \(\Lambda\)-family in \(KP_R(\hat{\Lambda})\) and \(B(\hat{\Lambda})\) be the subalgebra of \(KP_R(\hat{\Lambda})\) generated by \(\{q_{t(v)}, t_{\iota(\lambda)}, t_{\iota(\mu^*)} : v \in \Lambda^0, \lambda, \mu \in \Lambda\}\). Then
\[
B(\hat{\Lambda}) = \text{span}\{t_{\alpha \beta^*} : \alpha, \beta \in \hat{\Lambda}, r(\alpha), r(\beta) \in \iota(\Lambda^0)\}.
\]

**Proof.** The \(\subseteq\) direction is obvious. To see the other containment, consider \(t_{\alpha \beta^*}\) where \(\alpha, \beta \in \hat{\Lambda}, r(\alpha), r(\beta) \in \iota(\Lambda^0)\). We may assume \(s(\alpha) = s(\beta)\) for otherwise \(t_{\alpha \beta^*} = 0\) by \((KP2)\) and \((KP1)\). Since \(r(\alpha), r(\beta) \in \iota(\Lambda^0)\), there exist \(x, y \in \Lambda^{\leq \infty}\) such that
\[
\alpha = [x; (0, d(\alpha))] \quad \text{and} \quad \beta = [y; (0, d(\beta))]
\]
by Lemma 6.4. Using the definition of \(\pi\) and the factorisation property in \(\hat{\Lambda}\), there exist \(\gamma, \gamma' \in \hat{\Lambda}\) such that \(\alpha = \pi(\alpha)\gamma\) and \(\beta = \pi(\beta)\gamma'\). Write \(\gamma = [x; (d(\alpha) \wedge d(x), d(\alpha))]\) and \(\gamma' = [y; (d(\beta) \wedge d(y), d(\beta))]\).

We claim that \(\gamma = \gamma'\). To prove the claim, we show that items \((P1)-(P3)\) hold for \((x; (d(\alpha) \wedge d(x), d(\alpha)))\) and \((y; (d(\beta) \wedge d(y), d(\beta)))\). First notice that \((P2)\) is trivial; both the left and right-hand sides of the \((P2)\) equation are 0. Now \((V1)\) implies that
\[
(x(d(\alpha) \wedge d(x)) = y(d(\beta) \wedge d(y))
\]
and \((V2)\) implies that
\[
d(\alpha) - d(\alpha) \wedge d(x) = d(\beta) - d(\beta) \wedge d(y).
\]
Notice that equation \((7.2)\) is precisely \((P3)\), and \((P1)\) follows immediately from \((7.1)\). Thus \(\gamma = \gamma'\) as claimed.

Now we have
\[
t_{\alpha \beta^*} = t_{\pi(\alpha)} t_{\tau(\gamma)} t_{\pi(\beta)^*}.
\]

Our next claim is that \(t_{\tau(\gamma)} = q_{\tau(\gamma)}\). Since \(\hat{\Lambda}\) has no sources, \((KP4)\) says that
\[
q_{\tau(\gamma)} = \sum_{\delta \in \tau(\gamma) \Lambda^{d(\gamma)}} t_{\delta \delta^*}.
\]
But $\pi(\gamma) = r(\gamma)$ so $r(\gamma)A_{d(\gamma)} = \{\gamma\}$ by Lemma 6.5. Thus $q_{r(\gamma)} = t_\gamma t_{\gamma^*}$ as claimed. Finally, from (7.3) we have $t_\alpha t_{\beta^*} = t_{\pi(\alpha)} q_{r(\gamma)} t_{\pi(\beta^*)} = t_{\pi(\alpha)} t_{\pi(\beta^*)} \in B(\tilde{\Lambda})$ since the range of $\pi$ is $\iota(\Lambda)$.

We are ready to show $KP_R(\Lambda)$ and $KP_B(\tilde{\Lambda})$ are Morita equivalent. First, consider how the analogous proof proceeds in the $C^*$-setting in [30] and [13]. Let $A$ be a $C^*$-algebra and $p$ a projection in the multiplier algebra $M(A)$ of $A$. Then $pAp$ is a sub $C^*$-algebra of $A$ and $\overline{ApA}$ is an ideal of $A$, and $pA$ is a $pAp - ApA$ imprimitivity bimodule, giving a $pAp - ApA$ Morita equivalence [22, Example 2.12]. Both Farthing and Webster shows that the $C^*$-algebra $C^*(\Lambda)$ of a $k$-graph $\Lambda$ is Morita equivalent to the $C^*$-algebra $C^*(\tilde{\Lambda})$ of the desourceification $\tilde{\Lambda}$, that $pC^*(\Lambda)$ converges to a projection $p$ in $M(C^*(\Lambda))$, that $p$ is full in the sense that $C^*(\Lambda)pC^*(\Lambda) = C^*(\tilde{\Lambda})$, and identify $pC^*(\tilde{\Lambda})$, and $KP_R(\tilde{\Lambda})$ are Morita equivalent. First, consider how the range of $\iota$ is infinite. Second, suppose that $\iota(\Lambda)$ is finite, then $p = \sum_{\nu \in \iota(\Lambda)} q_\nu$ is defined in $KP_R(\tilde{\Lambda})$, and $KP_R(\tilde{\Lambda}) = \text{span}\{t_\lambda t_{\mu^*} : \lambda, \nu \in \tilde{\Lambda}, \iota(\mu) \in \iota(\Lambda)\}$; notice that the right-hand-side still makes sense when $\iota(\Lambda)$ is infinite.

**Lemma 7.3.** The subset

$$M := \text{span}\{t_\lambda t_{\mu^*} : \lambda, \mu \in \tilde{\Lambda}, \iota(\mu) \in \iota(\Lambda)\}$$

of $KP_R(\tilde{\Lambda})$ is closed under multiplication on the left by $KP_R(\tilde{\Lambda})$ and on the right by $B(\tilde{\Lambda})$.

The subset

$$N := \text{span}\{t_\lambda t_{\mu^*} : \lambda, \mu \in \tilde{\Lambda}, \iota(\lambda) \in \iota(\Lambda)\}$$

of $KP_R(\tilde{\Lambda})$ is closed under multiplication on the left by $B(\tilde{\Lambda})$ and on the right by $KP_R(\tilde{\Lambda})$.

Further,

$$MN := \text{span}\{mn : m \in M, n \in N\} = KP_R(\tilde{\Lambda}) \text{ and}$$

$$NM := \text{span}\{nm : m \in M, n \in N\} = B(\tilde{\Lambda}).$$

**Proof.** Let $t_\alpha t_{\beta^*} \in KP_R(\tilde{\Lambda}), t_\lambda t_{\mu^*} \in M$ and $t_\eta t_{\xi^*} \in B(\tilde{\Lambda})$. Let $a = d(\beta) \vee d(\gamma)$ and $b = d(\mu) \vee d(\eta)$. By Proposition 3.3,

$$t_\alpha t_{\beta^*} t_\lambda t_{\mu^*} = \sum_{d(\beta\sigma) = a, d(\gamma\sigma) = b} t_{\alpha\sigma} t_{(\mu\sigma)^*} \in M$$

because $r(\mu\tau) = r(\mu) \in \iota(\Lambda)$. Similarly,

$$t_\lambda t_{\mu^*} t_\eta t_{\xi^*} = \sum_{d(\mu\sigma) = b, d(\eta\sigma) = \gamma} t_{\lambda\sigma} t_{(\xi\sigma)^*} \in M$$

because $r(\xi\tau) = r(\xi) \in \iota(\Lambda)$. Thus $M$ is closed under multiplication on the left by $KP_R(\tilde{\Lambda})$ and on the right by $B(\tilde{\Lambda})$. The analogous assertion about $N$ follows from Proposition 3.3 in the same way.

Since $M, N \subseteq KP_R(\tilde{\Lambda})$, so is $MN$. For the other inclusion, consider $t_\alpha t_{\beta^*} \in KP_R(\tilde{\Lambda})$. We may assume that $t_\alpha t_{\beta^*} \neq 0$, and then $s(\alpha) = s(\beta)$. First suppose that $s(\alpha) \in \iota(\Lambda)$. Then $t_\alpha t_{\beta^*} = t_\alpha q_{s(\alpha)} t_{\beta^*} = t_\alpha t_{s(\alpha)^*} t_{s(\alpha)} t_{\beta^*} \in MN$. Second, suppose that $s(\alpha) \notin \iota(\Lambda)$. Then $t_\alpha t_{\beta^*} = \sum_{\nu \in \iota(\Lambda)} t_\alpha t_{\nu^*} t_{s(\nu)} t_{\beta^*}$. If $s(\alpha) \in \iota(\Lambda)$, then $t_\alpha t_{\beta^*} = \sum_{\nu \in \iota(\Lambda)} t_\alpha t_{\nu^*} t_{s(\nu)} t_{\beta^*} \in MN$.
Then \( s(\alpha) = [x; n] \) for some \( n \in \mathbb{N}^k \) and \( x \in \Lambda^{\leq \infty} \). Thus \( \lambda := [x; (0, n)] \in \tilde{\Lambda} \) and \( r(\lambda) = [x; 0] \in \iota(\Lambda^0) \). Then (KP3) gives

\[
t_{\alpha}t_{\beta^*} = t_{\alpha}q_{s(\alpha)}t_{\beta^*} = t_{\alpha}t_{\lambda^*}t_{\lambda^*} \leq MN.
\]

Thus \( MN = KP_R(\tilde{\Lambda}) \).

Next consider \( n = t_\gamma t_{\delta^*} \in N \) and \( m = t_{\lambda^*}t_{\mu^*} \in M \). Let \( c = d(\mu) \lor d(\gamma) \). By Proposition \( \ref{prop:3.3} \)

\[
nm = \sum_{d(\delta^*) = c, d(\gamma) = \lambda^*} t_{\gamma^*}t_{(\mu^*)^*}.
\]

In each summand, \( r(\gamma^* \sigma) = r(\gamma) \leq \iota(\Lambda^0) \) and \( r(\mu^* \tau) = r(\mu) \leq \iota(\Lambda^0) \), so that each summand is in \( B(\tilde{\Lambda}) \) by Proposition \( \ref{prop:7.2} \). Thus \( NM \subseteq B(\tilde{\Lambda}) \). For the other inclusion, consider \( t_\eta t_{\xi^*} \in B(\tilde{\Lambda}) \). Then \( t_\eta t_{\xi^*} = t_\eta q_{s(\eta)}q_{s(\xi)}t_{\xi^*} = t_\eta t_{s(\eta)}t_{s(\xi)}t_{\xi^*} \in NM \). Thus \( NM = B(\tilde{\Lambda}) \).

Now we recall the notion of a Morita context from \( \cite{[14]} \) page 41]. Let \( A \) and \( B \) be rings, \( M \) an \( A-B \) bimodule, \( N \) a \( B-A \) bimodule, and \( \psi : M \otimes_B N \rightarrow A \) and \( \phi : N \otimes_A M \rightarrow B \) bimodule homomorphisms satisfying

\[
n' \cdot \psi(m \otimes n) = \phi(n' \otimes m) \cdot n \quad \text{and} \quad m' \cdot \phi(n \otimes m) = \psi(m' \otimes n) \cdot m
\]

for \( n, n' \in N \) and \( m, m' \in M \). Then \( (A, B, M, N, \psi, \phi) \) is a Morita context between \( A \) and \( B \); it is called surjective if \( \psi \) and \( \phi \) are surjective.

**Theorem 7.4.** Let \( (\Lambda, d) \) be a locally convex, row-finite \( k \)-graph and let \( (\tilde{\Lambda}, d) \) its desoucification. Let \( (q, t) \) be a universal \( KP_R(\tilde{\Lambda}) \)-family and \( B(\tilde{\Lambda}) \) be the subalgebra of \( KP_R(\tilde{\Lambda}) \) generated by \( \{q_{s(\nu)}, t_{s(\lambda)}, t_{d(\nu)} : v \in \Lambda^0, \lambda, \mu \in \Lambda \} \). Let \( M, N \) be as in Lemma \( \ref{lemma:7.3} \). Then

\begin{enumerate}
  \item \( M \) is a \( KP_R(\tilde{\Lambda}) \)-\( B(\tilde{\Lambda}) \) bimodule and \( N \) is a \( B(\tilde{\Lambda}) \)-\( KP_R(\tilde{\Lambda}) \) bimodule with the module actions given by multiplication in \( KP_R(\tilde{\Lambda}) \);
  \item there are surjective maps \( \psi : M \otimes_{B(\tilde{\Lambda})} N \rightarrow KP_R(\tilde{\Lambda}) \) and \( \phi : N \otimes_{KP_R(\tilde{\Lambda})} M \rightarrow KP_R(\Lambda) \) such that \( \psi(m \otimes_{B(\tilde{\Lambda})} n) = mn \) and \( \phi(n \otimes_{KP_R(\tilde{\Lambda})} m) = nm \) for \( n \in N \) and \( m \in M \);
  \item \( KP_R(\tilde{\Lambda}), B(\tilde{\Lambda}), M, N, \psi, \phi \) is a surjective Morita context between \( KP_R(\tilde{\Lambda}) \) and \( B(\tilde{\Lambda}) \).
\end{enumerate}

Composing with the isomorphism \( \pi_{q_0, t_0} : KP_R(\Lambda) \rightarrow B(\tilde{\Lambda}) \) of Proposition \( \ref{prop:7.1} \) gives:

**Corollary 7.5.** Let \( (\Lambda, d) \) be a locally convex, row-finite \( k \)-graph and let \( (\tilde{\Lambda}, d) \) be the desoucification of \( \Lambda \). Then there is a surjective Morita context between \( KP_R(\tilde{\Lambda}) \) and \( KP_R(\Lambda) \).

**Proof of Theorem 7.4.** Lemma \( \ref{lemma:7.3} \) gives \( \ref{lemma:7.3} \). For \( \ref{corollary:7.5} \) we start by observing that the map \( f : M \times N \rightarrow KP_R(\tilde{\Lambda}) \) is bilinear, so that by the universal property of the tensor product there is a unique linear map \( \psi_f : M \otimes N \rightarrow KP_R(\tilde{\Lambda}) \) such that \( \psi_f(m \otimes n) = mn \). The range of \( \psi_f \) is \( MN \), which is \( KP_R(\tilde{\Lambda}) \) by Lemma \( \ref{lemma:7.3} \). To see that \( \psi_f \) factors through the quotient map \( q : M \otimes N \rightarrow M \otimes_{B(\tilde{\Lambda})} N \), we observe that for \( x \in B(\tilde{\Lambda}) \) we have \( \psi_f(m \cdot x \otimes n - m \otimes x \cdot n) = (mx)n - m(xn) = 0 \). Now there is a unique linear map \( \psi : M \otimes_{B(\tilde{\Lambda})} N \rightarrow KP_R(\tilde{\Lambda}) \) such that \( \psi \circ q = \psi_f \). Thus \( \psi \) is surjective and \( \psi(m \otimes_{B(\tilde{\Lambda})} n) = mn \).
That ψ is a bimodule homomorphism follows because the actions are by multiplication. The analogous assertions about φ follows in the same way. This gives (b).

For (c) it remains to verify (7.4), but this is immediate since everything is defined in terms of the associative multiplication in KP_R(Λ).

Let E be a row-finite directed graph, and F the directed graph obtained from E by adding “infinite heads” to sources, as in [10, page 310]. Then the path categories Λ_E and Λ_F are row-finite 1-graphs; E may have sources but is trivially locally convex. By [30, Proposition 4.11], ˜Λ_E is isomorphic to Λ_F. The Leavitt path algebra L_R(F) is the universal R-algebra generated by a Leavitt F-family [20, Definition 3.1], and any Leavitt F-family gives a Kumjian-Pask Λ_F-family and vice versa. Thus by Theorem 3.7(a), L_R(F) and KP_R(Λ_F) are isomorphic. Similarly, L_R(E) and KP_R(Λ_E) are isomorphic.

Thus we obtain the following corollary which is the analogue of the C^*-algebraic result [10, Lemma 1.2(c)].

Corollary 7.6. Let E be a row-finite directed graph, and F the directed graph obtained from E by adding “infinite heads” to sources. Then there is a surjective Morita context between the Leavitt path algebras L_R(F) and L_R(E).

There is a more general procedure, called “desingularisation”, which takes a directed graph E that is not necessarily row-finite and constructs a row-finite directed graph F without sources such that C^*(F) and C^*(E) are Morita equivalent [12, Theorem 2.11].

8. Simplicity and basic simplicity

Throughout this section, Λ is a locally convex, row-finite k-graph, ˜Λ is its desourcification and R is a commutative ring with 1. We show that the Morita context between KP_R(Λ) and KP_R(Λ) of Corollary 7.5 preserves basic ideals (see below for the definition). We then transfer simplicity results about KP_R(Λ) proved in [8] to KP_R(Λ).

A subset H ⊆ Λ^0 is called hereditary if for every v ∈ H and λ ∈ Λ with r(λ) = v we have s(λ) ∈ H. We say H is saturated if for v ∈ Λ^0,

\[ s(vΛ^{≤e_i}) ⊆ H \text{ for some } i ∈ \{1, ..., k\} \implies v ∈ H. \]

See [20, page 113]. In this section, we will apply these definitions to ˜Λ which has no sources; then the definition of ‘saturated’ above is equivalent to: or \( v ∈ Λ^0 \),

\[ s(vΛ^∞) ⊆ H \text{ for some } n ∈ \mathbb{N}^k \implies v ∈ H. \]

Lemma 8.1. Let Λ be a locally convex, row-finite k-graph and ˜Λ its desourcification. Suppose that H is a saturated and hereditary subset of Λ^0. Then v ∈ H if and only if \( π(v) ∈ H \).

Proof. Let \( v ∈ Λ^0 \). Write \( v = [z; m] \) where \( z ∈ Λ^{≤∞} \) and \( m ∈ \mathbb{N}^k \). Then \( π(v) = [z; d(z) ∧ m] \). Notice that \( λ := [z; (d(z), m ∧ d(z))] ∈ ˜Λ \) has source v and range \( π(v) \) and

\[ π(v)Λ^{m,m ∧ d(z)} = \{λ\} \]

by Lemma 6.5.

Now suppose \( v ∈ H \). Since

\[ s(π(v)Λ^{m,m ∧ d(z)}) = \{s(λ)\} = \{v\} ⊆ H, \]

and $H$ is saturated, $\pi(v) \in H$ as well. Conversely, suppose $\pi(v) \in H$. Then $r(\lambda) = \pi(v) \in H$ and $H$ hereditary implies $v = s(\lambda) \in H$. \hfill \square

Following [20], we say an ideal $I$ in $\text{KP}_R(\Lambda)$ is a basic ideal if $rp_v \in I$ for some $v \in \Lambda^0$ and $r \in R \setminus \{0\}$ imply $p_v \in I$. We say $\text{KP}_R(\Lambda)$ is basically simple if its only basic ideals are $\{0\}$ and $\text{KP}_R(\Lambda)$.

The Morita context of Theorem [14] induces a lattice isomorphism $L$ between the ideals of $\text{KP}_R(\hat{\Lambda})$ and ideals of $B(\hat{\Lambda})$ such that $L(I) = \text{NIM}$ \cite[Proposition 3.5]{14}.

**Proposition 8.2.** Let $(\text{KP}_R(\hat{\Lambda}), B(\hat{\Lambda}), M, N)$ be the Morita context of Theorem [14] and $L : I \mapsto \text{NIM}$ be the induced lattice isomorphism from the ideals of $\text{KP}_R(\hat{\Lambda})$ to the ideals of $B(\hat{\Lambda})$. Then $I$ is a basic ideal in $\text{KP}_R(\hat{\Lambda})$ if and only if $L(I)$ is a basic ideal in $B(\hat{\Lambda})$.

**Proof.** Suppose $I$ is a basic ideal of $\text{KP}_R(\hat{\Lambda})$. Fix $v \in \iota(\Lambda^0)$ and nonzero $r \in R$ such that $rp_v \in L(I)$. Since $N$ and $M$ are subsets of $\text{KP}_R(\Lambda)$ and $I$ is an ideal of $\text{KP}_R(\hat{\Lambda})$, we have $L(I) = \text{NIM} \subseteq I$. Thus $rp_v \in I$ and hence $p_v \in I$ because $I$ is a basic ideal. Since $v \in \iota(\Lambda^0)$, we have $p_v \in M \cap N$. Now

$$p_v = p_v p_v p_v \in \text{NIM} = L(I).$$

Thus $L(I)$ is a basic ideal of $B(\hat{\Lambda})$.

Conversely, let $I$ be an ideal in $\text{KP}_R(\hat{\Lambda})$ such that $L(I)$ is a basic ideal of $B(\hat{\Lambda})$. We will show $I$ is a basic ideal. Fix $v \in \hat{\Lambda}^0$ and nonzero $r \in R$ such that $rp_v \in I$. Then $v$ is an element of

$$H_{I,r} := \{v \in \hat{\Lambda}^0 : rp_v \in I\},$$

which is saturated and hereditary by \cite[Lemma 5.2]{8}. Hence $\pi(v) \in H_{I,r}$ by Lemma \cite[8.1]{8} which means $rp_{\pi(v)} \in I$. Notice that $\pi(v) \in \iota(\Lambda^0)$ and so $p_{\pi(v)} \in M \cap N$. Then

$$rp_{\pi(v)} = p_{\pi(v)} rp_{\pi(v)} p_{\pi(v)} \in \text{NIM} = L(I).$$

Therefore $p_{\pi(v)} \in L(I)$ because $L(I)$ is basic. Thus we have

$$p_{\pi(v)} = p_{\pi(v)} p_{\pi(v)} p_{\pi(v)} \in ML(I)N = I.$$

Now $\pi(v) \in H_{I,1}$ which is saturated and hereditary by \cite[Lemma 5.2]{8}. Therefore $v \in H_{I,1}$ by Lemma \cite[8.1]{8} that is $p_v \in I$. \hfill \square

Combining Proposition 8.2 with the isomorphism of $\text{KP}_R(\Lambda)$ onto $B(\hat{\Lambda})$ of Proposition \cite[8.1]{7}, the proof of the following corollary is immediate.

**Corollary 8.3.** Let $\Lambda$ be a locally convex, row-finite $k$-graph and $\hat{\Lambda}$ its desourcification. Then $\text{KP}_R(\Lambda)$ is basically simple if and only if $\text{KP}_R(\hat{\Lambda})$ is basically simple.

Next we want to transfer results about $\hat{\Lambda}$ to $\Lambda$. To use results already in the literature we need to reconcile some (of the many) aperiodicity conditions that have been used.

**Lemma 8.4.** Let $\Lambda$ be a row-finite $k$-graph.

(a) Suppose that $\Lambda$ has no sources. The following aperiodicity conditions are equivalent:

(i) ("Condition B" from \cite[Theorem 4.3]{20} for graphs without sources; our (4.1) reduces to this)

For every $v \in \Lambda^0$ there exists $x \in v\Lambda^\infty$ such that $\mu \neq v \in \Lambda$ implies $\mu x \neq vx$. 


(ii) (The “no local periodicity condition” from [23] Lemma 3.2(iii))

For every $v \in \Lambda^0$ and all $m \neq n \in \mathbb{N}^k$, there exists $x \in v\Lambda^\infty$ such that $\sigma^m(x) \neq \sigma^n(x)$.

(iii) (The finite-path reformulation from [23] Lemma 3.2(iv)]; used in [8]

For every $v \in \Lambda^0$ and $m \neq n \in \mathbb{N}^k$ there exists $\lambda \in v\Lambda$ such that $d(\lambda) \geq m \vee n$ and $\lambda(m, m + d(\lambda) - (m \vee n)) \neq \lambda(n, n + d(\lambda) - (m \vee n))$.

(b) Suppose that $\Lambda$ is locally convex. The following aperiodicity conditions on $\Lambda$ are equivalent:

(i) (“Condition B” from [20] Theorem 4.3], this is our (1.1)

For every $v \in \Lambda^0$, there exists $x \in v\Lambda^{\leq \infty}$ such that $\alpha \neq \beta \in \Lambda$ implies $\alpha x \neq \beta x$.

(ii) (The “no local periodicity condition” from [23] Definition 3.2))

For every $v \in \Lambda^0$ and all $m \neq n \in \mathbb{N}^k$, there exists $x \in \Lambda^{\leq \infty}$ such that either $m - m \land d(x) \neq n - n \land d(x) \lor \sigma^{m \land d(x)}(x) \neq \sigma^{n \land d(x)}(x)$.

Proof. For the equivalences in (iii), see [24] Lemma 3.2, and for the equivalences in (i), see [25] Proposition 2.11.

A locally convex, row-finite graph $k$-graph $\Lambda$ is said to be cofinal in [21] Definition 3.1] if for every $x \in \Lambda^{\leq \infty}$ and every $v \in \Lambda^0$, there exists $n \in \mathbb{N}^k$ such that $x = v\Lambda x(n) \neq \emptyset$. When $\Lambda$ has no sources, the cofinality condition reduces to the one given in section 5.1.

Theorem 8.5. Let $\Lambda$ be a locally convex, row-finite $k$-graph and $R$ be a commutative ring with 1. Then

(a) $\text{KP}_R(\Lambda)$ is basically simple if and only if $\Lambda$ is cofinal and aperiodic;

(b) $\text{KP}_R(\Lambda)$ is simple if and only if $\Lambda$ is cofinal and aperiodic, and $R$ is a field.

Proof. By Corollary 8.3 $\text{KP}_R(\Lambda)$ is basically simple if and only if $\text{KP}_R(\tilde{\Lambda})$ is basically simple. Since $\tilde{\Lambda}$ is row-finite with no sources, by [8] Theorem 5.14, $\text{KP}_R(\tilde{\Lambda})$ is basically simple if and only if $\tilde{\Lambda}$ is cofinal and aperiodic in the sense of condition (iii) of Lemma 8.4.

By [24] Proposition 3.5, $\Lambda$ is cofinal if and only if $\Lambda$ is cofinal. By [24] Proposition 3.6, $\Lambda$ has no local periodicity if and only if $\Lambda$ has no local periodicity. Using Lemma 8.4 it follows that $\text{KP}_R(\Lambda)$ is basically simple if and only if $\tilde{\Lambda}$ is cofinal and aperiodic in the sense of the equivalent conditions of Lemma 8.4.

By Corollary 7.2 $\text{KP}_R(\Lambda)$ is Morita equivalent to $\text{KP}_R(\tilde{\Lambda})$. Thus $\text{KP}_R(\Lambda)$ is simple if and only if $\text{KP}_R(\tilde{\Lambda})$ is. By [8] Theorem 6.1, $\text{KP}_R(\tilde{\Lambda})$ is simple if and only if $R$ is a field and $\Lambda$ is cofinal and aperiodic. The same shenanigans as in (iii) now give that $\text{KP}_R(\Lambda)$ is simple if and only if $R$ is a field and $\Lambda$ is cofinal and aperiodic.

9. The ideal structure

Throughout this section $\Lambda$ is a locally convex, row-finite $k$-graph, and $\tilde{\Lambda}$ is its desourcification. Recall that $B(\tilde{\Lambda})$ is the subalgebra of $\text{KP}_R(\tilde{\Lambda})$ generated by \{\(g_{\lambda}(v), t_{\mu}(\lambda), t_{\nu}(\mu) : v \in \Lambda^0, \lambda, \mu \in \Lambda\}\}, and is canonically isomorphic to $\text{KP}_R(\Lambda)$ by Proposition 7.1. We now use the Morita context between $\text{KP}_R(\tilde{\Lambda})$ and $B(\tilde{\Lambda})$ to show that there is a lattice isomorphism from the hereditary, saturated subsets of $\Lambda^0$ onto the basic, graded ideals of $\text{KP}_R(\Lambda)$. This extends [8] Theorem 5.1] to locally convex, row-finite $k$-graphs.
Lemma 9.1. Let \( \pi : \hat{\Lambda} \rightarrow \iota(\Lambda) \) be the projection defined in Remark \[6.23\]. Then \( H \mapsto \pi(H) \) is a lattice isomorphism of the hereditary, saturated subsets of \( \hat{\Lambda}^0 \) onto the hereditary, saturated subsets of \( \iota(\Lambda^0) \).

**Proof.** Let \( H \) be a hereditary, saturated subset of \( \hat{\Lambda}^0 \). To see that \( \pi(H) \) is a hereditary subset of \( \iota(\Lambda^0) \), let \( v \in \pi(H) \) and suppose \( \lambda \in \iota(\Lambda) \). Since \( r(\lambda) \in \pi(H) \) we have \( r(\lambda) \in H \) by Lemma 9.1. Then \( s(\lambda) \in H \) because \( H \) is hereditary. Since \( \lambda \in \iota(\Lambda) \) and \( \pi \) is a graph morphism, we have \( s(\lambda) = s(\pi(\lambda)) = \pi(s(\lambda)) \in \pi(H) \). Thus \( \pi(H) \) is hereditary.

To see that \( \pi(H) \) is saturated, let \( v \in \iota(\Lambda^0) \), and suppose that \( s(\iota(\Lambda^{\leq e_i})) \subseteq \pi(H) \) for some \( 1 \leq i \leq k \). There are two cases. First, suppose that \( s(\iota(\Lambda^{\leq e_i})) = \{v\} \). Then \( \{v\} = s(\iota(\Lambda^{\leq e_i})) \subseteq \pi(H) \) gives \( v \in \pi(H) \) as required. Second, suppose that \( s(\iota(\Lambda^{\leq e_i})) = \Lambda^{e_i} \). Then \( \iota(\Lambda^{e_i}) \subseteq H \) by Lemma 8.5. Thus \( s(\iota(\Lambda^{e_i})) \subseteq H \), and since \( H \) is saturated in \( \hat{\Lambda}^0 \), we get \( v \in H \). Now \( v = \pi(v) \in \pi(H) \) as required. Thus \( \pi(H) \) is a saturated subset of \( \iota(\Lambda) \).

To see that \( H \mapsto \pi(H) \) is injective, suppose \( \pi(H) = \pi(K) \). Let \( v \in H \). Then \( v \in \pi(H) = \pi(K) \), and hence \( v \in K \) by Lemma 8.1. Thus \( H \subseteq K \), and the other set inclusion follows by symmetry. Thus \( H = K \), and \( H \mapsto \pi(H) \) is injective.

To see that \( H \mapsto \pi(H) \) is onto, let \( G \) be a hereditary, saturated subset of \( \iota(\Lambda^0) \). Since \( \pi(\iota(\Lambda^{G})) = G \), it suffices to show that \( \pi^{-1}(G) \) is a hereditary, saturated subset of \( \hat{\Lambda}^0 \). Let \( v \in \pi^{-1}(G) \), and suppose \( \lambda \in v \lambda \hat{\Lambda} \). Then \( r(\pi(\lambda)) = \pi(r(\lambda)) = \pi(v) \in G \). Since \( G \) is hereditary, \( s(\pi(\lambda)) = s(\pi(s(\lambda))) \in G \). Thus \( s(\lambda) \in \pi^{-1}(G) \), and hence \( \pi^{-1}(G) \) is hereditary.

To see \( \pi^{-1}(G) \) is saturated, let \( v \in \hat{\Lambda}^0 \), and suppose \( s(\iota(\Lambda^{e_i})) \subseteq \pi^{-1}(G) \) for some \( 1 \leq i \leq k \). Then

\[
s(\pi(v)s(\Lambda^{\leq e_i})) = s(\pi(v)s(\Lambda^{e_i})) = \pi(s(\iota(\Lambda^{e_i}))) \subseteq G.
\]

But \( G \) is saturated in \( \iota(\Lambda^0) \), and hence \( \pi(v) \in G \). Now \( v \in \pi^{-1}(G) \) as needed, and \( \pi^{-1}(G) \) is saturated. It follows that \( H \mapsto \pi(H) \) is onto.

Finally, \( H \mapsto \pi(H) \) is a lattice isomorphism because \( H_1 \subseteq H_2 \) if and only if \( \pi(H_1) \subseteq \pi(H_2) \).

The next lemma is a generalisation of [8] Lemma 5.4 to graphs with possible sources.

Lemma 9.2. Let \( G \) be a hereditary, saturated subset of \( \hat{\Lambda}^0 \), and \( J_G \) be the ideal of \( \text{KP}_{\mathcal{R}}(\Lambda) \) generated by \( \{p_v : v \in G\} \). Then

\[
(9.1) \quad J_G = \text{span}\{s_\alpha s_\beta : \alpha, \beta \in \Lambda, s(\alpha) = s(\beta) \in G\}.
\]

**Proof.** Denote the right-hand side of (9.1) by \( J \). For \( v \in G \), taking \( \alpha = \beta = v \) shows \( p_v \in J \). Thus \( J_G \) is contained in the ideal generated by \( J \). But each \( s_\alpha s_\beta \in J \) is in \( J_G \) because \( s_\alpha s_\beta = s_\alpha p_{s(\alpha)} s_\beta \). Thus (9.1) will follow if \( J \) is an ideal. To see this, let \( s_\mu s_{\nu^*} \in \text{KP}_{\mathcal{R}}(\Lambda) \) and \( s_\alpha s_\beta \in J \) such that \( s_\mu s_{\nu^*} s_\alpha s_\beta \neq 0 \). Then \( r(\alpha) = r(\nu) \) and by Corollary 5.5 \( s_\mu s_{\nu^*} s_\alpha s_\beta \sum_{(\gamma, \delta) \in A} s_{\mu^*} s_{(\beta)^*} = 0 \). For any nonzero summand, \( r(\delta) = s(\alpha) \in H \) implies \( s(\gamma) = s(\nu) \in G \) since \( G \) is hereditary. Thus each \( s_\mu s_{\nu^*} s_\alpha s_\beta \in J \), and hence \( s_\mu s_{\nu^*} s_\alpha s_\beta \in J \) as well. Similarly \( s_\alpha s_\beta s_\mu s_{\nu^*} \in J \), and it follows that \( J \) is an ideal.

**Proposition 9.3.** Let \( L \) be the lattice isomorphism from the ideals of \( \text{KP}_{\mathcal{R}}(\hat{\Lambda}) \) to the ideals of \( B(\hat{\Lambda}) \) induced by the Morita context of Theorem 7.4.

(a) Then \( I \) is a graded ideal of \( \text{KP}_{\mathcal{R}}(\hat{\Lambda}) \) if and only if \( L(I) \) is a graded ideal of \( B(\hat{\Lambda}) \).

(b) Let \( H \) be a hereditary, saturated subset of \( \hat{\Lambda}^0 \). Then \( L(I_H) = J_{\pi(H)} \).
Proof. (i) Here $\text{KP}_R(\hat{\Lambda})$ and $B(\hat{\Lambda})$ are graded by the subgroups

$$\text{KP}_R(\hat{\Lambda})_j := \text{span}\{t_\alpha t_{\beta^*} : \alpha, \beta \in \hat{\Lambda}, d(\alpha) - d(\beta) = j\}$$

and

$$B(\hat{\Lambda})_j := \text{span}\{t_\alpha t_{\beta^*} : \alpha, \beta \in \iota(\Lambda), d(\alpha) - d(\beta) = j\} = B(\hat{\Lambda}) \cap \text{KP}_R(\hat{\Lambda})_j,$$

respectively. Let $m \leq n$ and $n \in N$. Since $M$ and $N$ are submodules of $\text{KP}_R(\hat{\Lambda})$, we can write $m = \sum m_i$ and $n = \sum n_j$ where each $m_i, n_j \in \text{KP}_R(\hat{\Lambda})_j$.

Now suppose that $I$ is a graded ideal of $\text{KP}_R(\hat{\Lambda})$. To check that $L(I) = NI_M$ is graded, it suffices to check that every element of $L(I)$ is a sum of elements in $\bigcup_j (L(I) \cap B(\hat{\Lambda})_j)$. Let $x \in I$, and write $x = \sum x_i$ where each $x_i \in I \cap \text{KP}_R(\hat{\Lambda})_j$. For an element $y = nxm \in L(I)$ we have $y = \sum_{i,j} m_i x_i n_j$; each $m_i x_i n_j \in I \cap \text{KP}_R(\hat{\Lambda})_{i+j+l} = I \cap B(\hat{\Lambda})_{i+j+l}$. Thus $L(I)$ is a graded ideal of $B(\hat{\Lambda})$. The other direction follows in the same way.

(ii) By [8, Lemma 5.4], $I_H = \text{span}\{t_{\alpha} t_{\beta^*} : \alpha, \beta \in \hat{\Lambda}, s(\alpha) = s(\beta) \in H\}$, and then

$$L(I_H) = NI_H M$$

(9.2)

$$= \text{span}\{t_{\lambda} t_{\mu} t_{\alpha} t_{\beta^*} t_{\tau^*} t_\sigma t_{\tau^*} : \lambda, \mu, \alpha, \beta, \sigma, \tau \in \hat{\Lambda}, s(\alpha) = s(\beta) \in H, r(\lambda), r(\tau) \in \iota(\Lambda^0)\}.$$

Consider a term $t_{\lambda} t_{\mu} t_{\alpha} t_{\beta^*} t_{\tau^*}$ as in (9.2). By three applications of Corollary 3.5

$$t_{\lambda} t_{\mu} t_{\alpha} (t_{\beta^*} t_{\tau^*}) = \sum_{(\gamma, \delta) \in \Lambda^{\min}(\mu, \alpha)} \sum_{(\xi, \eta) \in \Lambda^{\min}(\beta, \sigma)} t_{\lambda} t_{\gamma} (t_{\delta} t_{\xi}) t_{\eta} t_{\tau^*} = \sum_{(\gamma, \delta) \in \Lambda^{\min}(\mu, \alpha)} \sum_{(\xi, \eta) \in \Lambda^{\min}(\beta, \sigma)} \sum_{(\rho, \epsilon) \in \Lambda^{\min}(\delta, \xi)} t_{\lambda} t_{\gamma} t_{\rho} t_{\epsilon} t_{\eta} t_{\tau^*}.$$

For each summand $t_{\lambda} t_{\mu} t_{\alpha} t_{\beta^*} t_{\tau^*}$, we have $r(\delta) = r(\epsilon) = s(\alpha) \in H$. Since $H$ is hereditary, $s(\rho) = s(\epsilon) \in H$. Thus, from (9.2),

$$L(I_H) = \text{span}\{t_{\lambda} t_{\tau^*} : \lambda, \tau \in \hat{\Lambda}, r(\lambda), r(\tau) \in \iota(\Lambda^0), s(\lambda) = s(\tau) \in H\}$$

(9.1)

(above shows the inclusion $\subseteq$, and the reverse inclusion is trivial). But $B(\hat{\Lambda})$ is the subalgebra of $\text{KP}_R(\hat{\Lambda})$ generated by $\{q_{i(v)}, t_{i(\mu)}, t_{i(\nu)^*} : v \in \Lambda^0, \lambda, \mu \in \Lambda\}$. Thus

$$L(I_H) = \text{span}\{t_{\lambda} t_{\tau^*} : \lambda, \tau \in \iota(\Lambda), s(\lambda) = s(\tau) \in \iota(\Lambda^0)\}.$$

Using Lemma 9.1, we have $H \cap \iota(\Lambda^0) = \pi(H)$, and by Lemma 9.1 $\pi(H)$ is a hereditary subset of $\iota(\Lambda^0)$. Thus $L(I_H) = J_{\pi(H)}$ by Lemma 9.2.

Theorem 9.4. Let $\Lambda$ be a locally convex, row-finite $k$-graph. Then $G \mapsto J_G$ is a lattice isomorphism from the hereditary, saturated subsets of $\Lambda^0$ to the basic graded ideals of $\text{KP}_R(\hat{\Lambda})$.

Proof. Since $\hat{\Lambda}$ is row-finite without sources, $H \mapsto I_H$ is a lattice isomorphism from the hereditary, saturated subsets of $\Lambda^0$ onto the basic, graded ideals of $\text{KP}_R(\hat{\Lambda})$ by [8, Theorem 5.1]. By Theorem 7.4 $\text{KP}_R(\hat{\Lambda})$ is Morita equivalent to its subalgebra $B(\Lambda)$, and the induced lattice isomorphism sends $I_H$ to $J_{\pi(H)}$ by Proposition 9.3(i). By Proposition 8.2 and Proposition 9.3(ii), respectively, the Morita equivalence preserves basic, graded ideals. Thus $I_H \mapsto J_{\pi(H)}$ maps onto the basic, graded ideals of $B(\hat{\Lambda})$. The canonical isomorphism of $\text{KP}_R(\Lambda)$ onto $B(\hat{\Lambda})$ of Proposition 7.1 is graded, and hence the ideal $J_{\pi^{-1}(\pi(H))}$ of $\text{KP}_R(\Lambda)$ corresponding to $J_{\pi(H)}$ is basic and graded, and all basic, graded ideals of $\text{KP}_R(\Lambda)$ arise this way. Composing with the lattice isomorphism $G \mapsto \pi^{-1}(G)$
from the hereditary, saturated subsets of $\Lambda^0$ to those of $\tilde{\Lambda}^0$ of Lemma 9.1 gives the lattice isomorphism

$$ G \mapsto \pi^{-1}(G) \mapsto I_{\pi^{-1}(G)} \mapsto J_{\pi(G)} \mapsto J_G. \tag*{□} $$

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