Oscillations of a rapidly rotating annular Bose-Einstein condensate

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A time-dependent variational Lagrangian analysis based on the Gross-Pitaevskii energy functional serves to study the dynamics of a metastable giant vortex in a rapidly rotating Bose-Einstein condensate. The resulting oscillation frequencies of the core radius reproduce the trends seen in recent experiments [Engels et al., Phys. Rev. Lett. 90, 170405 (2003)], but the theoretical values are smaller by a factor $\approx 0.6-0.8$.

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I. INTRODUCTION

One of the most characteristic features of neutral superfluids is their inability to rotate uniformly with solid-body velocity $v_{sb} = \Omega \times r$. Instead, a rotating superfluid contains one or more singly quantized vortex lines with circulation $h/M$, where $M$ is the atomic mass. This phenomenon is familiar in rotating superfluid $^4$He [1], although visualization of the resulting vortex arrays is quite difficult [2]. The creation of the first Bose-Einstein condensate (BEC) in dilute $^{87}$Rb gas [3] challenged experimentalists to demonstrate the existence of similar quantized vortices in dilute BECs and to study their properties. After the development of techniques for creating a single vortex [4] and small vortex arrays [5], rapid progress led to large vortex lattices [6] and rotating condensates with large angular momentum [7].

One recent method to increase the angular momentum has used an intense focused laser beam to remove the central region of a rotating condensate [8, 9]. This method notes that the particles near the center have relatively little angular momentum, so that the resulting annular condensate has an increased angular momentum per particle. In this way, it has been feasible to reach rotation rates $\Omega \approx 0.97 \omega_\perp$, where $\omega_\perp$ is the radial confining trap frequency. This process creates a relatively long-lived “giant vortex core” that contains a large number of phase singularities, resulting in a macroscopic circulation around its boundary. The remaining annular fluid has a dense vortex lattice that rotates at the angular velocity $\Omega$. In addition, the radius of the core oscillates at a frequency of order $3-3.5 \omega_\perp$, with relatively weak damping [8, 9]. Recently, a theoretical study of this behavior has relied on numerical simulation of the time-dependent Gross-Pitaevskii (GP) equation in two dimensions [10] and found an oscillation frequency $\sim 2.6 \omega_\perp$, somewhat smaller than the observed values. The present work provides an alternative theoretical study of the same phenomenon using a variational Lagrangian that contains the inner radius and the amplitude of the radial velocity as time-dependent parameters. The resulting Lagrange’s equations yield an undamped oscillation with definite amplitude and a frequency that varies between 0.6 and 0.8 of various measured values [8, 9]. Although the two theoretical approaches appear quite different, they both rely on the GP equation and both predict oscillation frequencies that are definitely smaller than the observed ones.

Section II reviews the variational Lagrangian procedure and applies it to the present problem. Sections III and IV contain analyses of two specific models. The first is a quasi-two-dimensional Thomas-Fermi (TF) condensate with tight confinement in the $z$ direction; the two-dimensional character of the condensate wave function simplifies the analysis considerably. The second is a more realistic three-dimensional TF condensate, but the final results are rather similar in both cases. The comparison with the experimental observations is discussed in Sec. V.

II. BASIC PROCEDURE

The approach here is the variational Lagrangian method, in which a trial condensate wave function $\Psi$ includes various parameters whose time dependence follows from Lagrange’s equations for the Lagrangian

$$L = \frac{1}{2}i\hbar \int dV \left( \frac{\partial \Psi}{\partial t} - \Psi \frac{\partial \Psi^*}{\partial t} \right) - E[\Psi],$$

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where \( E[\Psi] \) is the energy functional. This flexible procedure has served successfully to study the low-lying collective modes of a condensate and the precession of a single vortex line in a trapped condensate. It is convenient here to work in the laboratory frame of reference, and to impose conservation of particle number and angular momentum explicitly.

The experiment studies \( N \sim 10^6 \) atoms of \(^{87}\text{Rb} \) in an axisymmetric trap with \( \omega_{\perp}/2\pi = 8.3 \text{ Hz} \) and \( \omega_z/2\pi = 5.4 \text{ Hz} \), so that the relevant oscillator lengths are \( d_{\perp} \approx 3.73 \mu\text{m} \) and \( d_z \approx 4.62 \mu\text{m} \). Thus the nonrotating condensate is somewhat elongated, with TF dimensions \( \Omega \) of the wave function is

\[
\frac{R_{\perp}(\Omega)}{R_{\perp}(0)} = (1 - \Omega^2)^{-3/10}, \quad \text{and} \quad \frac{R_z(\Omega)}{R_z(0)} = (1 - \Omega^2)^{1/5},
\]

where \( \Omega = \Omega/\omega_{\perp} \). Taking \( \Omega = 0.97 \), the standard TF expressions yield \( R_{\perp}(\Omega)/d_{\perp} \approx 19.9 \) and \( R_z(\Omega)/d_z \approx 7.45 \), with \( R_{\perp}(\Omega)/R_{\perp}(0) = (1 - \Omega^2)^{1/2}R_{\perp}(0)/R_{\perp}(0) \approx 0.37 \). Thus the rotating condensate is definitely disk shaped.

In the TF limit, the gradient of the density \( |\Psi|^2 \) is small, and the energy functional becomes

\[
E[\Psi] \approx \int dV \left( \frac{1}{2} M v^2 |\Psi|^2 + V_{\text{tr}} |\Psi|^2 + \frac{2\hbar^2 a}{M} |\Psi|^4 \right),
\]

where \( v^2 \) is the squared velocity of the superfluid, \( V_{\text{tr}} \) is the trap potential, and \( a \approx 5.77 \text{ nm} \) is the \( s \)-wave scattering length. The density \( |\Psi|^2 \) is taken as the familiar quadratic TF distribution with an empty core of radius \( r_c \).

\[|\Psi|^2 = |\Psi_{\text{TF}}|^2 \theta(r - r_c). \]

The core radius \( r_c \) serves as one dynamical variable, and the resulting radial motion requires a time-dependent radial velocity \( v_r = (\hbar/M) \partial S/\partial r \) in addition to the azimuthal component \( v_{\phi} = (\hbar/Mr) \partial S/\partial \phi \), where \( S \) is the phase of the condensate wave function.

### III. QUASI-TWO-DIMENSIONAL CONDENSATE

To simplify the analysis, the condensate is first assumed to be thin in the axial direction with a Gaussian dependence, so that \( \Psi(r, z) \approx \Psi_{2d}(r) \psi_0(z) \). Here, \( \psi_0(z) = (\sqrt{\pi d_z})^{-1/2} \exp(-\frac{1}{2} z^2/d_z^2) \), where \( d_z = \sqrt{\hbar/M \omega_z} \). For the two-dimensional wave function \( \Psi_{2d}(r) \), it is convenient to use dimensionless variables with frequencies (and inverse time) scaled by the radial trap frequency \( \omega_{\perp} \), energies scaled by \( \hbar \omega_{\perp} \), distances scaled by the radial oscillator length \( d_{\perp} = \sqrt{\hbar/M \omega_{\perp}} \), and the condensate wave function \( \psi = \Psi/\sqrt{N} \) normalized to 1.

The trial wave function used here corresponds to a radial TF two-dimensional density for \( r \leq R \) with the central region of radius \( r_c \) removed, so that

\[|\psi_{2d}(r)|^2 = C_{2d}^2 \left( 1 - \frac{r^2}{R_c^2} \right) \theta(R - r) \theta(r - r_c), \]

where \( R = R_{\perp}/d_{\perp} \) is now dimensionless and \( C_{2d} \) is a normalization constant. The phase \( S \) of the wave function is assumed to have the radial part

\[S_r = \alpha \left( \frac{1}{2} r^2 - rR \right) \]

to allow for the oscillatory radial motion, where \( \alpha \) is a time-dependent parameter. In addition, the phase has the familiar contribution \( S_{v} \) from the vortices plus the circulation \( \nu_0 \) around the hole representing the missing vortices

\[S_v(r) = \sum_j \arctan \left( \frac{y_j - y}{x - x_j} \right) + \nu_0 \phi, \]

where the sum is over all vortices \( j \).

The radial velocity \( v_r = \partial S_r/\partial r = \alpha (r - R) \) vanishes at the outer boundary, which remains fixed in this model. For simplicity, only \( r_c \) and \( \alpha \) are taken as dynamical variables. In contrast, the vortices are not treated as dynamical variables; instead, they are assumed to have uniform dimensionless areal density \( \rho \) and to move with the local azimuthal velocity. In detail, the azimuthal component of the dimensionless velocity is

\[v_{\phi} = \sum_j \frac{r - \hat{r} \cdot \hat{r}_j}{|r - r_j|^2} + \nu_0. \]
where \( \nu_0 = \pi \rho_0 r_0^2 \) is the constant quantum number of the circulation around the hole of initial radius \( r_0 \) with initial vortex density \( \rho_0 \). The sum can be approximated by an integral

\[
v_\phi \approx \rho \int d^2 r' \frac{r - r' \cos \phi'}{r^2 - 2 r r' \cos \phi' + r'^2} + \frac{\nu_0}{r} = \frac{2 \pi \rho}{r} \int_{r_c}^r r' dr' + \frac{\nu_0}{r}
\]

\[
= \pi \rho r - \pi \rho \frac{r^2}{r} + \frac{\nu_0}{r}
\]

That is uniform along a length \( L \) that is the two-dimensional energy functional. All the integrals can be performed analytically and Eqs. (10) and (11) ensure

\[
\phi = \pi \rho_0 \text{ since the last two terms of Eq. (9) cancel for } r_c = r_0 \text{ (when } \rho = \rho_0 \).
\]

The analysis is carried out in the laboratory frame, and the conservation of particle number and angular momentum are explicitly incorporated into the resulting Lagrangian. It is straightforward to obtain the normalization constant

\[
C^2_{2d} = \frac{2}{\pi R^2 (1 - y^2)^2},
\]

where \( y \equiv r_c/R \) determines the instantaneous position of the core radius. Similarly, the angular momentum per particle (in units of \( \hbar \)) is

\[
l_{2d} = \int d^2 r r v_\phi |\psi_{2d}|^2 = \frac{1}{3} \pi \rho R^2 (1 - y^2) + \nu_0 = \frac{1}{3} \pi \rho_0 R^2 (1 + 2 y_0^2),
\]

expressed in terms of the initial parameters \( \rho_0 \) and \( y_0 \). Since \( l_{2d} \) and \( \nu_0 \) are taken as constants, this relation determines the actual vortex density

\[
\rho = \frac{3}{\pi R^2} \frac{l_{2d} - \nu_0}{1 - y^2}
\]

as the core radius \( y \) changes with time. Note that the total number of vortices \( \pi \rho R^2 (1 - y^2) = \pi \rho (R^2 - r_0^2) \) in the annular region \( r_c \leq r \leq R \) remains constant, so that the vortex density changes only because the inner radius \( r_c \) changes. Correspondingly, the initial conditions fix the quantity \( l_{2d} - \nu_0 = \frac{1}{3} R^2 \Omega (1 - y_0^2) \). Assuming typical values \( R \approx 20, \Omega \approx 0.97 \), and \( y_0 \approx 1/3 \), I find \( l_{2d} \approx 158 \) and \( \nu_0 \approx 43 \).

After the \( z \) integration is carried out with the tightly confined Gaussian, the dimensionless Lagrangian per particle becomes

\[
L_{2d} = \frac{1}{2} \int d^2 r \left( \psi^*_a \frac{\partial \psi_{2d}}{\partial t} - \psi_{2d} \frac{\partial \psi^*_a}{\partial t} \right) - E_{2d}[\psi_{2d}],
\]

where

\[
E_{2d}[\psi_{2d}] = \int d^2 r \left[ \frac{1}{2} (v_r^2 + v_\theta^2) |\psi_{2d}|^2 + \frac{1}{2} r^2 |\psi_{2d}|^2 + \sqrt{2\pi} \frac{N a}{dz} |\psi_{2d}|^2 \right]
\]

is the two-dimensional energy functional. All the integrals can be performed analytically and Eqs. (10) and (11) ensure the conservation of particles and angular momentum. Note that the same Lagrangian also describes a condensate that is uniform along a length \( L \) in the \( z \) direction. In this case, the coefficient of the quartic term in \( E_{2d} \) is changed to \( 2 \pi N a/L \), where \( N/L \) is the number of particles per unit length.

The resulting Lagrangian depends on two dynamical variables: \( \alpha \) that fixes the radial component of velocity and \( y \equiv r_c/R \) that fixes the position of the inner core radius. It is convenient to use the equivalent quantity

\[
\mathcal{L}_{2d} = -L_{2d}/R^2 = \hat{\alpha} \left[ f_{2d}(y) - \frac{1}{2} \right] + \alpha^2 f_{2d}(y) + \mathcal{E}_{2d}(y)
\]

where the first term arises from the explicit time derivative in (13), the second is the radial kinetic energy, and \( \mathcal{E}_{2d} \) constitutes the remaining terms in the energy. It has the explicit form

\[
\mathcal{E}_{2d}(y) = \Omega^2 (1 - y_0^2)^2 \left[ g_{2d}(y) + \frac{h(y)}{1 - y^2} + h(y)^2 f_{2d}(y) \right] + \frac{1}{6} (1 + 2 y^2) + \frac{4 \sqrt{2}}{3 \sqrt{\pi}} \frac{N a}{dz} \frac{1}{R^2 (1 - y^2)}.
\]
where the first term (proportional to $\Omega^2$) is the azimuthal kinetic energy, the second is the trap energy, and the last is the interaction energy (note that this term is typically small, despite the large value of $Na/d_z \sim$ a few thousand, owing to the factor $R^{-4} \sim 6 \times 10^{-6}$). To be very specific, $\mathcal{L}_{2d}$ involves the following functions

\[
 f_{2d}(y) = \frac{(1-y)^2(2+8y+5y^2)}{15(1+y)^2},
\]

\[
 g_{2d}(y) = \frac{1+2y^2}{6(1-y^2)^2},
\]

\[
 h(y) = \frac{1}{1-y^0}\frac{1}{1-y^2},
\]

\[
 j_{2d}(y) = \frac{2\ln(1/y)}{(1-y^2)^2} - \frac{1}{1-y^2}.
\]

### IV. THREE-DIMENSIONAL THOMAS-FERMI CONDENSATE

Before studying Lagrange’s equations for $\mathcal{L}_{2d}$, it is convenient to consider the corresponding case of a three-dimensional Thomas-Fermi condensate, where

\[
 |\psi_{TF}(r, z)|^2 = C_{TF}^2 \left(1 - \frac{r^2}{R^2} - \frac{z^2}{R_z^2}\right) \theta \left(1 - \frac{r^2}{R^2} - \frac{z^2}{R_z^2}\right) \theta(r - r_c),
\]

using the same scale factors $\omega_\perp$ and $d_\perp$ to define dimensionless variables. The normalization integral gives

\[
 C_{TF}^2 = \frac{15}{8\pi R^2 R_z (1-y^2)^{5/2}},
\]

which can be compared to Eq. (10). Similarly, the angular momentum becomes

\[
 l_{TF} = \frac{2}{\pi} \rho R^2 (1-y^2) + \nu_0,
\]

and the conservation of angular momentum then yields the vortex density

\[
 \rho = \frac{7}{2\pi R^2} \frac{l_{TF} - \nu_0}{1-y^2}.
\]

All the terms in the Lagrangian can be evaluated analytically, and the resulting expression has the same form as Eq. (15) but with different functions

\[
 \mathcal{L}_{TF} = \dot{\alpha} \left[ f_{TF}(y) - \frac{1}{2} \right] + \alpha^2 f_{TF}(y) + \mathcal{E}_{TF}(y).
\]

In particular,

\[
 f_{TF}(y) = \frac{9+5y^2}{14} - \frac{5}{32} \pi + \frac{3}{2} \frac{y(1-y^2)^{1/2}(3-14y^2+8y^4)}{(1-y^2)^{5/2}} - 2 \arcsin y
\]

and

\[
 \mathcal{E}_{TF}(y) = \Omega^2 \left(1-y_0^2\right)^2 \left[ g_{TF}(y) + \frac{h(y)}{1-y^2} + h(y)^2 j_{TF}(y) \right] + \frac{1}{14} \left[ 2+5y^2+(1-\Omega^2)(1-y^2) \right] + \frac{15 Na}{7 \, d_\perp} \frac{1}{R^4 R_z} \frac{1}{(1-y^2)^{3/2}}.
\]

The extra factor of $R_z^{-1}$ in the interaction term will be seen to eliminate the dependence of the number of atoms (assuming only that the condensate remains in the TF limit). In addition, the remaining functions have the altered forms

\[
 g_{TF}(y) = \frac{2+5y^2}{14(1-y^2)^2},
\]

\[
 j_{TF}(y) = \frac{5 \,(1-y^2)^{1/2}(-4+y^2) - 3 \ln y + 3 \ln[1+(1-y^2)^{1/2}]}{(1-y^2)^{5/2}}.
\]
V. DISCUSSION OF RESULTING MOTION

Variation of the effective Lagrangian $L$ with respect to $\alpha$ yields the dynamical equation

$$\dot{y} \frac{df(y)}{dy} = 2f(y)\alpha,$$

(30)

which expresses $\alpha$ in terms of $y$ and $\dot{y}$. Similarly, variation with respect to $y$ yields a more complicated equation

$$\frac{\partial L}{\partial y} = (\dot{\alpha} + \alpha^2) \frac{df(y)}{dy} + \frac{d\mathcal{E}(y)}{dy} = 0$$

(31)

that expresses $\dot{\alpha}$ and $\alpha$ in terms of $y$.

Unfortunately, it is not easy to interpret this pair of coupled dynamical equations, and it is preferable to assume small oscillations about the initial conditions $y(0) = y_0$ and $\dot{y}(0) = 0$ (namely the hole in the rapidly rotating condensate starts from rest). Consequently, $\alpha$ itself is a small quantity, as is $\delta = y_0 - y$, which is positive when the hole contracts from its initial scaled radius $y_0$. In this way, Eq. (30) becomes

$$\dot{\delta} = -\frac{2f(y_0)}{f'(y_0)} \alpha,$$

(32)

where the prime denotes a derivative with respect to $y$. Similarly, Eq. (31) has the linearized form

$$\dot{\alpha} = -\frac{\mathcal{E}'(y_0) + \delta \mathcal{E}''(y_0)}{f'(y_0)}$$

(33)

A combination of these two linearized equations yields a simple dynamical equation

$$\ddot{\delta} + \omega^2 \delta = \omega^2 A,$$

(34)

where

$$\omega^2 = \frac{2\mathcal{E}''(y_0) f(y_0)}{[f'(y_0)]^2},$$

(35)

and

$$A = \frac{\mathcal{E}'(y_0)}{\mathcal{E}''(y_0)}.$$  

(36)

They depend on the initial radius $y_0 = r_0/R$ and on the small parameters $Na/d_z R^4$ in the two-dimensional case and $Na/d_{\perp} R_z R^4$ for the three-dimensional TF case (these latter contributions come from the interaction energy, which is small because the expanded outer radius reduces the particle density far below its value for a nonrotating condensate).

Equation (34) has the solution

$$\delta(t) = A(1 - \cos \omega t).$$

(37)

It represents an inward oscillation of the core radius with angular frequency $\omega$ (measured in units of $\omega_\perp$) and amplitude $A$ (like $y$, $A$ is measured in units of $R = R_\perp/d_\perp$). As a result, conservation of angular momentum and the parabolic trap potential dominate the dynamical motion because the interaction term makes only a small contribution; this is the effect of the Coriolis force mentioned in [8]. In all cases, the amplitude $A$ is relatively small, partially justifying the expansion through quadratic contributions to $L$. Note, however, that the minimum core size is $r_0 - 2AR$, which corresponds to a significant shrinkage for the smaller rotation rates $\Omega \lesssim 0.9$.

Table I contains typical calculated values for some of the experimental parameters used in [8], using the functions for a quasi-two-dimensional condensate. Since the present theory is rather crude, however, the general three-dimensional TF radii have been used even for the rotating quasi-two-dimensional condensate. The choice $N = 2.2 \times 10^6$ corresponds to Fig. 4 of Ref. [8], which shows a rotating condensate with $\Omega = 0.9$: the core executes weakly damped oscillations with dimensionless frequency $\omega/\omega_\perp = 3.5$. This frequency is higher than the value $\omega/\omega_\perp = 2.32$ found here for a core radius $1/3$ of the condensate radius, but the dependence of the oscillation frequency and amplitude on the rotation frequency $\Omega$ both show the observed trend (namely, “decreasing the initial rotation rate of the condensate leads to faster core-oscillation frequencies and increases the amplitude of this oscillation” [8]). The corresponding values for
$N = 3.0 \times 10^6$ illustrate the extremely weak dependence on the number of atoms. In fact, if strict two-dimensional TF radii were used, the dependence on $N$ would cancel completely, as will be seen to hold for the three-dimensional TF condensate.

Table II is similar to Table I except that it is evaluated for the three-dimensional Thomas-Fermi condensate density. In this case, it is not difficult to use the standard TF expressions for the condensate radii [14] to verify that the ratio $(15\alpha_0/d_\perp)(R^4R_z)^{-1}$ in fact reduces to the value $1 - \Omega$, so that the $N$ dependence indeed drops out. In all the cases considered, the three-dimensional TF condensate yields somewhat lower oscillation frequencies and smaller amplitude for the motion.

To provide a more detailed comparison between the theoretical predictions and experimental values, Table III contains 5 sets of measured frequencies [9] along with the corresponding predictions of both models. For the quasi-two-dimensional condensate, the ratio of predicted value $\omega_{th}$ to measured value $\omega_{obs}$ varies between 0.61 and 0.79, whereas for the three-dimensional TF condensate, the same ratio varies between 0.49 and 0.76. In both cases, the agreement improves considerably at the fastest rotations. This trend may reflect the present use of a spatially averaged velocity for that induced by the vortices and the macroscopic circulation, ignoring the discrete character of the vortices; such an approximation should work best at large $\Omega$.

One possible explanation for the discrepancy between the experimental values and the results of the present variational Lagrangian formalism is that the actual dynamical condensate does not conserve the number of vortices; in this case, angular momentum would not strictly be conserved. At present, it is not clear how to incorporate such additional dynamical variables into the analysis. In any case, it is notable that the numerical simulation of Simula et al. [10] also seems to predict an oscillation frequency that is too small by a comparable amount. Conceivably, the quite dramatic perturbation caused by the intense laser beam in burning a relatively large hole may well produce significant depletion, so that a theory based solely on the time-dependent GP equation would need to be modified to include noncondensate contributions.

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TABLE I: Calculated parameters for a quasi-two-dimensional condensate

| $r_0/R$ | $\Pi = \Omega/\omega_\perp$ | $R = R_\perp/d_\perp$ | $\omega/\omega_\perp$ | $A$ | $\omega/\omega_\perp$ | $A$ |
|---|---|---|---|---|---|---|
| $N = 2.2 \times 10^6$ | $N = 3.0 \times 10^6$ |
| 1/3 | 0.90 | 13.20 | 2.32 | 0.097 | 2.34 | 0.099 |
| 1/3 | 0.94 | 15.29 | 2.19 | 0.063 | 2.20 | 0.064 |
| 1/3 | 0.97 | 18.74 | 2.10 | 0.032 | 2.10 | 0.033 |
| 1/5 | 0.90 | 13.20 | 2.50 | 0.081 | 2.52 | 0.083 |
| 1/5 | 0.94 | 15.29 | 2.32 | 0.054 | 2.33 | 0.056 |
| 1/5 | 0.97 | 18.74 | 2.18 | 0.029 | 2.19 | 0.030 |

TABLE II: Calculated parameters for a three-dimensional TF condensate (the predicted frequencies and amplitudes are independent of $N$ as long as the TF approximation remains valid)

| $r_0/R$ | $\Pi = \Omega/\omega_\perp$ | $R = R_\perp/d_\perp$ | $R_z/d_\perp$ | $\omega/\omega_\perp$ | $A$ | $\omega/\omega_\perp$ | $A$ |
|---|---|---|---|---|---|---|---|
| $r_0/R = 1/3$ | $r_0/R = 1/5$ |
| 0.90 | 13.20 | 8.85 | 2.09 | 0.061 | 2.20 | 0.052 |
| 0.94 | 15.29 | 8.02 | 2.06 | 0.038 | 2.15 | 0.034 |
| 0.97 | 18.74 | 7.00 | 2.04 | 0.020 | 2.11 | 0.018 |

TABLE III: Comparison of selected experimental values with theoretical predictions for two-dimensional and three-dimensional models

| $r_0/R$ | $\Pi = \Omega/\omega_\perp$ | $N$ | $\omega_{\text{obs}}/\omega_\perp$ | $\omega_{\text{th}}/\omega_\perp$ | $A$ | $\omega_{\text{th}}/\omega_{\text{obs}}$ | $A$ |
|---|---|---|---|---|---|---|---|
| experimental values$^a$ | quasi-two-dimensional model | three-dimensional TF model |
| 0.40 | 0.79 | $2.8 \times 10^6$ | 4.5 | 2.74 | 0.61 | 0.171 | 2.17 | 0.49 | 0.122 |
| 0.24 | 0.88 | $2.3 \times 10^6$ | 4.0 | 2.48 | 0.62 | 0.098 | 2.16 | 0.54 | 0.064 |
| 0.32 | 0.90 | $2.1 \times 10^6$ | 3.5 | 2.32 | 0.67 | 0.095 | 2.09 | 0.60 | 0.060 |
| 0.32 | 0.95 | $2.5 \times 10^6$ | 2.9 | 2.17 | 0.75 | 0.053 | 2.06 | 0.71 | 0.032 |
| 0.35 | 0.96 | $2.3 \times 10^6$ | 2.7 | 2.13 | 0.79 | 0.044 | 2.05 | 0.76 | 0.027 |

$^a$Reference 9.
$^b$Reference 9 assigns an uncertainty of $\sim 20\%$ to the initial core radius $r_0$.
$^c$Reference 9 assigns an uncertainty of 6 Hz to the first value of $\omega_{\text{obs}}$ (37 Hz), namely $\sim 16\%$. In contrast, the other experimental oscillation frequencies have uncertainties of $\leq 1$ Hz, which is $\leq 4\%$ even for the last value.