Coding and Bounds for Partially Defective Memory Cells
Haider Al Kim, Sven Puchinger, Ludo Tolhuizen, Antonia Wachter-Zeh
1Institute for Communications Engineering, Technical University of Munich (TUM), Germany
2Electronic and Communications Engineering, University of Kufa (UoK), Iraq
3Department of Applied Mathematics and Computer Science, Technical University of Denmark (DTU), Denmark
4Philips Research, High Tech Campus 34, Netherlands
Email: haider.alkim@tum.de, sven.puchinger@tum.de, ludo.tolhuizen@philips.com, antonia.wachter-zeh@tum.de

Abstract—This paper considers coding for so-called partially stuck (defect) memory cells. Such memory cells can only store partial information as some of their levels cannot be used fully due to, e.g., wearout. First, we present new constructions that are able to mask partially stuck cells while correcting at the same time $t$ random errors. The process of "masking" determines a word whose entries coincide with writable levels at the (partially) stuck cells. For $u > 1$ and alphabet size $q > 2$, our new constructions improve upon the required redundancy of known constructions for $t = 0$, and require less redundancy for masking partially stuck cells than former works required for masking fully stuck cells (which cannot store any information). Second, we show that treating some of the partially stuck cells as erroneous cells can decrease the required redundancy for some parameters. Lastly, we derive Singleton-like, sphere-packing-like, and Gilbert–Varshamov-like bounds. Numerical comparisons state that our constructions match the Gilbert–Varshamov-like bounds for several code parameters, e.g., BCH codes that contain all-one word by our first construction.

Index Terms—flash memories, phase change memories, non-volatile memories, defective memory, (partially) stuck cells, BCH code, cyclic code, sphere packing bound, Gilbert-Varshamov bound

I. INTRODUCTION

The demand for reliable memory solutions and in particular for non-volatile memories such as flash memory and phase change memories (PCMs) for different applications is steadily increasing. These multi-level devices provide permanent storage and a rapidly extendable capacity. Recently developed devices exploit an increased number of cell levels while at the same time the physical size of the cells was decreased. Therefore, coding and signal processing solutions are essential to overcome reliability issues. The key characteristic of PCM cells is that they can switch between two main states: an amorphous state and a crystalline state. PCM cells may become defect (also called stuck) [1]–[4] if they fail in switching their states. This occasionally happens due to the cooling and heating processes of the cells. Therefore, cells can only hold a single phase [1], [4]. In multi-level PCM cells, failure may occur at a position in either of extreme states or in the partially programmable states of crystalline.

The work [5] investigates codes that mask so-called partially stuck (partially defective) cells, i.e., cells which cannot use all levels. For multi-level PCMs, the case in which the partially stuck level $s = 1$ is particularly important since this means that a cell can reach all crystalline sub-states, but cannot reach the amorphous state.

Figure 1 depicts the general idea of reliable and (partially) defective memory cells. It shows two different cell level representations: Representation 1 forms the binary extension field $F_2$, and Representation 2 forms the set of integers modulo $q = 4$, i.e., $Z/4Z$.

![Figure 1. Illustration of reliable and (partially) defective memory cells. In this figure, there are $n = 5$ cells with $q = 4$ possible levels. The cell levels in $F_2$ are mapped to $(0, 1, \alpha, 1 + \alpha)$ shown in Representation 1 or to $Z/4Z$ are mapped to $(0, 1, 2, 3)$ shown in Representation 2. Case (A) illustrates fully reliable cells which can store any of the four values in both representations. In the stuck scenario as shown in case (B), the defective cells can store only the exact stuck level $s$. Case (C) is more flexible (partially defective scenario). Partially stuck cells at level $s \geq 1$ can store level $s$ or higher.]

A. Related Work

Coding for memories with stuck cells, also known as defect-correcting codes for memories with defects, dates back to the 1970s, cf. the work by Kuznetsov and Tsymbakov [6]. They proposed binary defect-correcting codes in finite and asymptotic regimes whose required redundancy is at least the number of defects. Later works [7]–[19] investigated the problem of defective cells under various aspects: binary and non-binary, only defect-correcting coding and error-and-defect-correcting coding, and finite and asymptotic length analysis.

In binary defect-correcting coding models, e.g. [8], [10]–[12], [15]–[17], the authors dealt with masking stuck cells without considering additional substitution errors. In these studies, it is unclear if the proposed constructions are optimal in terms of their required redundancy. The works [9], [13], [14] considered masking stuck memory cells while at the...
The proposed constructions, for example \[19, \text{Section 4}\], and cells at any arbitrary level and correct errors additionally. generalize the previous constructions to mask partially stuck code constructions along with their encoding and decoding errors and partially defective cells examined in this study. Our Section II, we provide notations and define the models of joint \[20\].

$n < q$ could instead mask by a code of length $n$.

denote $[f] = \{0, 1, \ldots, f - 1\}$ and $[g, f] = \{g, g + 1, \ldots, f - 1\}$. As usual, an \([n, k, d]_q\) code is a linear code over $\mathbb{F}_q$ of length $n$, dimension $k$ and minimum (Hamming) distance $d$.

The (Hamming) weight $wt(x)$ of a vector $x \in \mathbb{F}_q^n$ equals its number of non-zero entries.

We fix throughout the paper a total ordering “$\geq$” of the elements of $\mathbb{F}_q$ such that $a \geq 1 \geq 0$ for all $a \in \mathbb{F}_q \setminus \{0\}$. So $0$ is the smallest element in $\mathbb{F}_q$, and $1$ is the next smallest element in $\mathbb{F}_q$. We extend the ordering on $\mathbb{F}_q$ to $\mathbb{F}_{q^n}$; for $x = (x_0, \ldots, x_{n-1}) \in \mathbb{F}_q^n$ and $y = (y_0, \ldots, y_{n-1}) \in \mathbb{F}_q^n$, we say that $x \succeq y$ if and only if $x_i \geq y_i$ for all $i \in [n]$.

In order to simplify notation, we sometimes identify $x \in \mathbb{F}_q$ with the number of field elements not larger than $x$, that is, with the integer $q - \lfloor y \in \mathbb{F}_q \mid x \geq y\rfloor$. The meaning of $x$ will be clear from the context. Figure 1 depicts the two representations that are equivalent in this sense. Finally, we denote the $q$-ary entropy function by $h_q$, that is $h_q(0) = 0, h_q(1) = \log_q(q - 1)$, and $h_q(x) = -x \log_q(x) - (1 - x) \log_q(1 - x) + x \log_q(q - 1)$ for $0 < x < 1$.

### B. Definitions

1) Defect and Partially Defect Cells: A cell is called defect (stuck at level s), if it can only store the value $s$. A cell is called partially defect (partially stuck at level s), if it can only store values which are at least $s$. Note that a cell that is partially defect at level 0 is a non-defect cell which can store any of the $q$ levels and a cell that is partially defective at level $q - 1$ is a (fully) defect cell.

2) $(\Sigma, t)$-PSMC: For $\Sigma \subset \mathbb{F}_q^n$ and non-negative integer $t$, a $q$-ary $(\Sigma, t)$-partially-stuck-at-masking code $C$ of length $n$ and size $M$ is a coding scheme consisting of a message set $\mathcal{M}$ of size $M$, an encoder $\mathcal{E}$ and a decoder $\mathcal{D}$.

The encoder $\mathcal{E}$ is a mapping from $\mathcal{M} \times \Sigma$ to $\mathbb{F}_q^n$ such that for each $(m, s) \in \mathcal{M} \times \Sigma$, $\mathcal{E}(m, s) \geq s$.

For each $(m, s) \in \mathcal{M} \times \Sigma$ and each $e \in \mathbb{F}_q^n$ such that $wt(e) \leq t$ and $\mathcal{E}(m, s) + e \geq s$, it holds that $\mathcal{D}(\mathcal{E}(m, s) + e) = m$.

3) $(u, 1, t)$ PSMC: A $q$-ary $(u, 1, t)$ PSMC of length $n$ and cardinality $\mathcal{M}$ is a $q$-ary $(\Sigma, t)$ PSMC of length $n$ and size $\mathcal{M}$ where

$$\Sigma = \{s \in \{0, 1\}^n \mid wt(s) \leq u\}.$$
III. CODE CONSTRUCTION FOR MASKING UP TO $q - 1$ PARTIALLY-STUCK-AT-1 CELLS

A. Code Construction

In this section, we present a coding scheme over $\mathbb{F}_q$ that can mask up to $q - 1$ partially stuck cells and additionally can correct errors. We adapt the construction from [5], which allows to mask up to $q - 1$ partially-stuck-at-1 ($s_i = 1$ for all $i$) cells with only a single redundancy symbol, but cannot correct any substitution errors.

Construction 1. Assume that there is an $[n, k, d/2]_q$ code $C$ with a $k \times n$ generator matrix of the form

$$G = \begin{bmatrix} G_1 & G_0 \end{bmatrix},$$

where $G_{k-1}$ is the $(k - 1) \times (k - 1)$ identity matrix, $P \in \mathbb{F}_q^{(k-1) \times (n-k)}$, and $I_k$ is the all-one vector of length $k$. Encoder and decoder are shown in Algorithm 1 and Algorithm 2.

Theorem 1. The coding scheme in Construction 1 is a $(q - 1, 1, \lfloor \frac{d-1}{2} \rfloor)$ PSMC of length $n$ and cardinality $q^{k-1}$.

Algorithm 1: Encoding

Input:
- $m = (m_0, m_1, \ldots, m_{k-2}) \in \mathbb{F}_q^{k-1}$
- Positions of partially-stuck-at-1 cells: $\phi$

1. Compute $w = m \cdot G_1$
2. Find $v \in \mathbb{F}_q \setminus \{w_i \mid i \in \phi\}$
3. Compute $c = \bar{w} \cdot v \cdot G_0$

Output: Codeword $c \in \mathbb{F}_q^n$

Algorithm 2: Decoding

Input:
- $\hat{c}$, $\hat{v}$

1. $\hat{w} = (\hat{w}_0, \hat{w}_1, \ldots, \hat{w}_{n-1}) \leftarrow (\hat{c} - \bar{\hat{v}} \cdot G_0)$
2. $\hat{m} \leftarrow (\hat{w}_1, \ldots, \hat{w}_n)$

Output: Message vector $\hat{m} \in \mathbb{F}_q^{k-1}$

Proof. To mask the partially-stuck-at-1 positions, the codeword has to fulfill:

$$c_i \geq 1, \quad \text{for all } i \in \phi.$$  \hfill (1)

Since $|\phi| < q$, there is at least one value $v \in \mathbb{F}_q$ such that $w_i \neq v$, for all $i \in \phi$. Thus, $c_i = (w_i - v) \neq 0$ and (1) is satisfied.

The decoder (Algorithm 2) gets $\hat{y}$, which is $c$ corrupted by at most $|\frac{d}{2} - 1|$ substitution errors. The decoder of $C$ can correct these errors and obtain $\hat{c}$.

Due to the structure of $G$, the first position of $\hat{c}$ equals $-\hat{v}$. Hence, we can compute $\hat{w} = w$ (cf. Algorithm 2) and $\hat{m} = \hat{m}$.

Corollary 1. If there is an $[n, k, d/2]_q$ code containing a word of weight $w$, then there is a $q$-ary $(q - 1, 1, \lfloor \frac{d-1}{2} \rfloor)$ PSMC of length $n$ and size $q^{k-1}$.

B. Comparison to the Conventional Stick-Cell Scenario

Theorem 1 combines [13, Theorem 1] and [5, Theorem 4] to provide a code construction that can mask partially stuck cells and correct errors. The required redundancy is a single symbol for masking plus the redundancy for the code generated by the upper part of $G$, needed for the error correction. In comparison, [13, Theorem 1] requires at least

$$\min\{n - k : \exists [n, k, d]_q \text{ code with } d > u \} \geq u$$

redundancy symbols to mask $u$ stuck cells, where the inequality follows directly from the Singleton bound.

In the following, we present Tables I and II to compare ternary cyclic codes of length $n = 8$ for masking partially stuck cells to masking stuck cells [13], both with error correction.

The tables show that masking partially stuck cells requires less redundancy than masking stuck cells, both with and without additional error correction. The reason is that there is only one forbidden value in each partially stuck-at-1 cell, while there are $q - 1$ forbidden values in each stuck at cell.

C. Remarks on Construction 1

Remark 1. The special case of Theorem 1 with $n < q$ was used in [20] for constructing a $(q - 1)$-ary error-correcting code from a $q$-ary Reed-Solomon code, which can be of interest if $q < 1$ is not the power of a prime.

Remark 2. The code constructions in Theorem 1 and 17 also work over the ring of integers modulo $q (\mathbb{Z}/q\mathbb{Z})$ in which $q$ is not necessarily a prime power, similar to the construction for $u < q$ in [5].

Remark 3. According to [5, Construction 3], it is possible to further decrease the required redundancy for masking $u$ partially-stuck-at-1 cells to $1 - \log_q \left( \frac{q+1}{u+1} \right)$. We can use the same strategy here. Let $z = \left( \frac{q+1}{u+1} \right)$. We choose disjoint sets $A_1, A_2, \ldots, A_z$ of size $u + 1$ in $\mathbb{F}_q$. As additional information, the encoder picks $j \in \{1, 2, \ldots, z\}$. In Step 2 of Algorithm 1, it selects $v$ from $A_j$. As the decoder acquires $v$, it can obtain $j$ as well.

IV. CODE CONSTRUCTIONS FOR MASKING MORE THAN $q - 1$ PARTIALLY-STUCK-AT-1 CELLS

The masking technique in the previous section only guarantees successful masking up to a number of $q - 1$ partially stuck-at-1 cells. In this section, we present techniques to mask more than $q - 1$ cells.

Depending on the values of the stored information in the partially stuck positions, Construction 1 may be able to mask more than $q - 1$ cells. In Section IV-A, we determine the probability that masking is possible for fixed partially stuck cell positions and randomly chosen information vectors.

Next, we propose two code constructions for simultaneous masking and error correction when $q < u < n$. One is based on the masking-only construction in [5, Construction 4] and the other is based on [5, Section VI], which are able to mask...
Overall redundancy $u \geq q$ partially stuck positions, but cannot correct any errors. We generalize these constructions to be able to cope with errors. The latter construction may lead to larger code dimensions for a given pair $(u, t)$, in a similar fashion as [5, Construction 5] improves upon [5, Construction 4]. Further, taking $t = 0$ it achieves larger codes sizes than [5, Construction 5] if the all-one word is in the code.

A. Probabilistic Masking

We determine the probability that masking is possible for $u \geq q$ partially stuck-at-1 cells stuck positions with the code constructions in Theorem 1 and Theorem 17 are used. This probabilistic masking approach enables us to use the memory cells with a certain probability even if there are more than $q - 1$ partially stuck cells.

**Theorem 2.** Let $G$ be as in Construction 1, and let $\phi \subset [n]$ have size $u$. If the columns of $G$ indexed by the elements in $\phi$ are linearly independent, a uniformly drawn message from $\mathbb{F}_q^{n-k-1}$ results in a word $w$ with $c_i \neq 0$ for all $i \in \phi$ with probability

$$P(q, u) = 1 - \frac{\sum_{i=0}^{q-1}(-1)^i \binom{q}{i}(q-i)^u}{q^n}. \quad (2)$$

**Proof.** An appropriate value for $v$ in Step 2 in Algorithm 1 cannot be found if and only if $\{w_i \mid i \in \phi\} = \mathbb{F}_q$ which is true if and only if $f : \phi \rightarrow \mathbb{F}_q$ defined by $f_w(i) = w_i$ is a surjection. As is well-known (see e.g [21, Example 10.2]), the number of surjections from a set of size $u$ to a set of size $q$ equals

$$\sum_{i=0}^{q-1}(-1)^i \binom{q}{i}(q-i)^u. \quad (3)$$

As the columns of $G$ are independent, the vector $w$ restricted to $\phi$ is distributed uniformly on $\mathbb{F}_q^u$, and hence a word is not masked with probability equal to the expression from (3) divided by $q^u$.

The following example illustrates that the probability that masking is successful can be quite large.

**Example 1.** Let $q = 3$, $n = 8$, $n-k = 0$. The probability to mask $u = n-1$ partially stuck-at-1 memory cells is $P(3, 7) = 0.17$. This ratio is 0.77 if $u = q$ and clearly it is 1 if $u < q$.

**Remark 4.** The assumption in Theorem 2 that the columns of $G$ indexed by the partially stuck positions are linearly independent is fulfilled for most codes with high probability if $u \leq k-1$, especially if $u \ll k-1$. For dependent columns, it becomes harder to count the number of intermediate codewords $w$ that do not cover the entire alphabet since $w_i$ for all $i \in \phi$ is not uniformly distributed over $\mathbb{F}_q^n$.

B. Code Construction over $\mathbb{F}_q$ for Masking Up to $q + d_0 - 3$ Partially Stuck Cells

We recall that [5, Construction 4] can mask more than $q - 1$ partially stuck-at-1 cells and it is a generalization of the all-one vector construction [5, Theorem 4]. Hence, replacing the $1_n$ vector in Theorem 1 by a parity-check matrix as in [5, Construction 4] allows masking of $q$ or more partially stuck-at-1 cells, and correct $t$ errors.

**Construction 2.** Suppose that there is an $[n, k, d]_q$ code $C$ with a $k \times n$ generator matrix of the following form:

$$G = \begin{bmatrix} G_1 \\ H_0 \end{bmatrix}$$

where $H_0 \in \mathbb{F}_q^{l \times n}$ is a parity-check matrix of an $[n, n-l, d_0]$ code $C_0$. Encoder and decoder are shown in Algorithm 3 and Algorithm 4.

**Theorem 3.** The coding scheme in Construction 2 is a $(d_0 + q - 3, 1, \lfloor \frac{d-1}{2} \rfloor)$ PSMC of length $n$ and cardinality $q^{k-t}$.

**Proof.** Let $\phi \subset [n]$ have size $u \leq q + d_0 - 3$. Algorithm 3 finds $z = \{z_0, z_1, \ldots, z_{k-1}\}$ similar to [5, Theorem 7] instead of only finding $v$ value as demonstrated in Algorithm 1. Then the proof is exactly the same as in [5, Theorem 8] for the masking part. In short, the authors in the proof of [5, Theorem 7] subdivides the code length $n$ into $l$ block lengths of sizes at most $q - 1$. Hence, as each block contains at most $q - 1$
Algorithm 3: Encoding

Input:
- Message: \( m \in F^{k-l} \)
- Positions of partially stuck-at-1 cells: \( \phi \)
1. Compute \( w = m \cdot G_1 \)
2. Find \( z \in F^q \) as explained in the proof of [5, Theorem 7]
3. Compute \( c = w + z \cdot H_0 \)
Output: Codeword \( c \in F^{pn} \)

Algorithm 4: Decoding

Input: \( y = c + e \in F^{pn} \)
1. \( \hat{c} \leftarrow \text{decode } y \text{ in the code } C \)
2. Determine \( \hat{m} \in F^{k-l} \) and \( \hat{z} \in F^q \) such that \( \hat{c} = \hat{m}G_1 + \hat{z}H_0 \).
Output: Message vector \( \hat{m} \in F^{k-l} \)

The gain of Theorem 3 in the number of partially stuck cells that can be masked comes at the cost of larger redundancy. However, the redundancy is smaller than the redundancy of the construction for masking stuck-at-cells and error correction in [13]. In particular, let \( C \) be an \([n, k, d \geq 2t+1]\) code containing an \([n, l, q]_q\) subcode \( C_0 \) for which \( C_0^* \) has minimum distance \( d_0 \). With Theorem 3, we obtain \( a(d_0 + q - 3, 1, \lceil d_0 + q \rceil) \) PSMC of length \( n \) and cardinality \( q^{k-l} \). The construction in [13] yields a coding scheme with equal cardinality, allowing for masking up to \( d_0 - 1 \) fully stuck cells and correcting \( \lfloor \frac{d_0}{2} \rfloor \) errors. Hence, exactly \( q - 2 \) more cells that are partially stuck at levels 1 than classical stuck cells can be masked.

Example 2. We apply Construction 2 to masking up to \( u = 4 \) partially stuck cells over \( F_4 \) and \( m \in F^{95} \). Let \( \alpha \) be a primitive element in \( F_{16} \) and let \( C \) be the \([15, 12, 3]_4\) code with zeros \( \alpha^0 \) and \( \alpha^3 \). Let \( C_0 \) be the \([15, 3]_4\) subcode of \( C \) be the BCH code with zeros \( \{\alpha^0, 0 \leq i \leq 14\} \setminus \{\alpha^2, \alpha^2 + 1, \alpha^3\} \). As \( C_0^* \) is equivalent to the \([15, 12, 3]_4\) code with zeros \( \alpha^0, \alpha^2, \alpha^3 \), it has minimum distance \( d_0 = 3 \). Hence, we obtain a \((4, 1, 1)\) PSMC code of cardinality \( 4^9 \).

C. Code Construction over \( F_2^\mu \) for Masking Up to \( 2^{\mu-1} \) Partially Stuck Cells

We generalize [5, Section VI] to be able to cope with errors. Unlike [5, Section VI] that could be over any prime power \( q \), the following code construction works over the finite field \( F_q \) where \( q = 2^\mu \) in order to describe a \( 2^\mu \)-ary partially stuck cells code construction. This is because binary subfield subcodes that are required in this construction are not linear subspace for codes over any prime power \( q \). We denote by \( \beta_0 = 1, \beta_1, \ldots, \beta_{\mu-1} \) a basis of \( F_{2^\mu} \) over \( F_2 \). That is, any \( a \in F_{2^\mu} \) can be uniquely represented as \( a = \sum_{i=0}^{\mu-1} a_i \beta_i \) where \( a_i \in F_2 \) for all \( i \). In particular, \( a \in F_2 \) if and only if \( a_1 = \cdots = a_{\mu-1} = 0 \). This is a crucial property of \( F_{2^\mu} \) that we will use in Construction 3.

Construction 3. Let \( \mu > 1 \). Suppose \( G \) is a \( k \times n \) generator matrix of an \([n, k, d]_{2^\mu} \) code \( \mathcal{C} \) of the form

\[
G = \begin{bmatrix} H_0 & G_1 \end{bmatrix}
\]

where

1) \( H_0 \in F_2^{n \times n} \) is a parity check matrix of an \([n, n - l, d_0]_2 \) code \( \mathcal{C}_0 \).
2) \( G_1 \in F_2^{n - l - 1 \times n} \).
3) \( x \in F_2^u \) has Hamming weight \( n \).

Encoder and decoder are shown in Algorithm 5 and Algorithm 6.

Algorithm 5: Encoding (\( m; m'; \phi \))

Input:
- Message: \( m' \in F_{2^\mu} \)
- Positions of partially stuck-at-1 cells: \( \phi \)
1. \( w \leftarrow m' \cdot H_0 \) and \( z \in F_2 \)
2. Choose \( \gamma \in F_2^u \) such that \( (\gamma H_0)_i = 1 - w_i \) for all \( i \in \phi \) for which \( w_i \in F_2 \).
Output: \( c = w + \gamma \cdot H_0 \in F^{pn} \)

Algorithm 6: Decoding

Input: \( y = c + e \in F^{pn} \)
1. \( \hat{c} \leftarrow \text{decode } y \text{ in the code } C \)
2. Obtain \( (m', m) \in F_{2^\mu} \) and \( \hat{z} \in F_2 \) such that \( \hat{c} = aH_0 + \hat{m}G_1 + \hat{z}x \).
3. Obtain \( m' \in F^{k-l} \) and \( \hat{\gamma} \in F_2^{k-l} \) such that \( a = \hat{m}' + \hat{\gamma} \).
Output: \( (m, m') \)

Theorem 4. The coding scheme in Construction 3 is a \( 2^\mu \)-ary \((2^{\mu-1}d_0 - 1, 1, \lceil \frac{d_0}{2} \rceil)\) PSMC of length \( n \) and cardinality \( 2^{\mu(k-l-1)/2(\mu-1)} \).

Proof. Let \( \phi \in [n] \) have size \( u \leq 2^{\mu-1}d_0 - 1 \). We first show the existence of \( z \) from Step 1. For each \( i \in \phi \), we have that \( x_i \neq 0 \), so there are exactly two elements \( z \in F_{2^\mu} \) such that \( (m' \cdot H_0 + m \cdot G_1)_i + zx_i \in F_2 \). As a result,

\[
2u = 2|\phi| = |\{(i, z) \in \phi \times F_{2^\mu} \mid (m' \cdot H_0 + m \cdot G_1)_i + zx_i \in F_2 \}|.
\]

As \( u < 2^{\mu-1}d_0 \), there is a \( z \in F_{2^\mu} \) such that the condition in Step 1 is satisfied.

As \( H_0 \) is the parity check matrix of a code with minimum distance \( d_0 \), any \( d_0 - 1 \) columns of \( H_0 \) are independent, so an appropriate \( \gamma \) exists. Now we show that \( c_i \neq 0 \) for all \( i \in \phi \). Indeed, if \( w_i \notin F_2 \), then \( c_i = w_i + (\gamma H_0)_i \in \{w_i, w_i + 1\} \), so \( c_i \notin F_2 \). By Step 2 in Algorithm 5, for \( w_i \in F_2 \), we have
For binary entries, then so has \( n = 1 \) \( = 0 \) \( \gamma \in F_2 \), we can retrieve
\[
\mu l = 1 - H_0 G_1 x,
\]
where \( H_0 \) is a generator matrix for \( C_0^+ \) and \( G_1 \) has \( 12 - 4 - 1 = 7 \) rows. The code \( C_0 = (C_0^+)^\perp \) is equivalent to the \([15,11,3]_4\) BCH code with zeros \( \alpha^5, \alpha^6 \) and \( \alpha^7 \). As this BCH code has two consecutive zeros, its minimum distance (and hence the minimum distance of \( C_0 \)) is at least 3.

We stipulate that \( \alpha^3 = \alpha + 1 \) to obtain explicit \( G' \) as below,
\[
G' = \begin{bmatrix}
H_0 & G_1 & x
\end{bmatrix},
\]
where \( F_4 \) has elements \( \{0, 1, \omega, \omega^2\} \) with \( \omega = \alpha^5 \). Note that the top row of \( H_0 \) corresponds to the generator polynomial for \( C_0^+ \), and the top row of \( G_1 \) corresponds to the coefficients of \( (x + \alpha^5)(x + \alpha^6)(x + \alpha^7) \) which is the generator polynomial of \( C \).

Application of Construction 1 yields a \([5,1,1]_2\) PSMC over \( F_2 \) of length 15 and size \( 2 \times 4^2 = 2^{13} \), whereas application of Construction 3 gives a \([7,1,1]_2\) PSMC over \( F_2 \) with cardinality \( 2^{7+4} = 2^{18} \). Note that application of Construction 3 yields a \([5,1,1]_2\) PSMC over \( F_2^5 \) of length 15 and size \( 4^2 = 2^{18} \).

Finally, we note that application of Theorem 3 to yields a \([4,1,1]_2\) PSMCs of size \( 4^8 \), which has worse parameters than the three PSMC mentioned before.

Example 3 clearly shows that for the same code parameters, Construction 3, Proposition 1 and Construction 3.4 significantly improve upon Construction 2.

Remark 5. For masking only, choose \( n - k = 0 \) in Construction 3 and therefore,
\[
G_1 = \begin{bmatrix}
0_{(n-l-1) \times (l+1)} & I_{(n-l-1)} & 0_{(n-l-1) \times 1}
\end{bmatrix},
\]
and we can store \( n - l - 1 \) information symbols. Thus, Proposition I for masking only improves upon \([5, Construction 5]\). For example if \( l = 4 \), then \( n - l - 1 = 10 \) in \([5, Example 7]\) and the size of the code is \( 2^{2(n-l-1)} \), \( 2^{24} \), whereas \( n - l - 1 = 10 \) in Proposition I for \( x = 1 \) and the cardinality is \( 2 \cdot 2^{2n-l-1} = 2^{25} \).

We summarize in Table III our constructions and compare them with some of the previous works, namely with the construction for masking classical stuck cells in [13] and
constructions for partially stuck cells without errors in [5].

V. GENERALIZATION OF THE CONSTRUCTIONS TO ARBITRARY PARTIALLY DEFECTIVE LEVELS

So far, we have considered the important case for $s_i = 1$ for all $i \in \Phi$. In this section, we present error correction and masking codes constructions that can mask partially stuck cells at any level $s_i$ and correct errors additionally.

A. Generalization of Theorem 1

Here, we give only the main theorem without adding the exact encoding and decoding processes because it follows directly as a consequence of Construction 1.

Theorem 5 (Generalization of Theorem 1). Let $\Sigma = \{s \in \mathbb{F}_q^n : |\sum_{i=0}^{n-1} s_i \leq q - 1\}$, Assume there is an $[n, k, d]_q$ code $C$ of a generator matrix as specified in Theorem 1. Then there exists a $(\Sigma, [d-1])$ PSMC over $\mathbb{F}_q$ of length $n$ and cardinality $q^{k-1}$.

Proof. We follow the generalization for the masking partially stuck at any arbitrary levels in [5, Theorem 10]. Hence, for $s_i \in \Sigma$, we modify Step 2 in Algorithm 1 such that $w_i = v_i - s_i$ for all $i \in [n]$, such a $v_i$ exists as each cell partially stuck at level $s_i$ is independent for $v_i$ and $\sum_{i=0}^{n-1} s_i \leq q - 1$. The rest of the encoding steps and the decoding process are analogous to Algorithms 1 and 2. As the output from the encoding process is a codeword, we can correct $[d-1]$ errors. \hfill \Box

B. Generalization of Construction 2

In the following, we generalize Construction 2 to arbitrary $s$ stuck levels.

Proposition 2. Let

$$\Sigma = \left\{ s \in \mathbb{F}_q^n : \text{min} \left\{ \sum_{i \in \Psi} s_i : |\Psi| \subseteq [n], |\Psi| = n - d_0 + 2 \right\} \leq q - 1 \right\}$$

then the coding scheme in Construction 2 can be modified to produce a $(\Sigma, t)$ PSMC of length $n$ and size $q^{k-1}$.

Proof. To avoid cumbersome notation, we assume without loss of generality that $s_0 \geq s_1 \geq \cdots \geq s_{n-1}$. As the $d_0 - 2$ leftmost columns of $H_0$ are independent, there is an invertible $T \in \mathbb{F}_q^{l \times l}$ such that the matrix $Y = T H_0$ has the form

$$Y = \begin{bmatrix} d_{0-2} & A \\ 0 & B \end{bmatrix},$$

where $I_{d_0-2}$ is the identity matrix of size $d_0 - 2, 0$ denotes the $(l - d_0 + 2) \times (d_0 - 2)$ all-zero matrix, $A \in \mathbb{F}_q^{l(d_0 - 2) \times (n - d_0 + 2)}$ and $B \in \mathbb{F}_q^{l(d_0 - 2) \times (n - d_0 + 2)}$ as $T$ is invertible, and any $d_0 - 1$ columns of $H_0$ are independent, any $d_0 - 1$ columns of $Y$ are independent as well.

For $0 \leq i \leq l - 1$, we define

$$L_i = \{j \in [n] : Y_{i,j} \neq 0 \text{ and } Y_{m,j} = 0 \text{ for } m > i\}. \quad (5)$$

Clearly, $L_0, \ldots, L_{l-1}$ are pairwise disjoint. Moreover, for each $j \in \{d_0 - 2, d_0 - 1, \ldots, n - 1\}$, column $j$ of $Y$ is independent from the $(d_0 - 2)$ leftmost columns of $Y$, and so there is an $i \geq d_0 - 2$ such that $Y_{i,j} \neq 0$. Consequently,

$$\bigcup_{i=d_0-2}^{l-1} L_i = \{d_0 - 2, \ldots, n - 1\}. \quad (6)$$

By combining (6) and the form of $Y$, we infer that

$$L_k = \{k\} \text{ for all } k \in [d_0 - 2]. \quad (7)$$

Let $w \in \mathbb{F}_q^n$ be the vector to be masked, i.e. the vector after Step 1 in Algorithm 3. The encoder successively determines the coefficients $\zeta_0, \ldots, \zeta_{l-1}$ of $\zeta \in \mathbb{F}_q^{l}$ such that $w + \zeta Y \geq s$, as follows.

For $j \in [d_0 - 2]$, the encoder sets $\zeta_j = s_j - w_j$.

Now let $d_0 - 2 \leq i \leq l - 1$ and assume that $\zeta_0, \ldots, \zeta_{l-1}$ have been obtained such that

$$w_j + \sum_{k=0}^{i-1} \zeta_k Y_{k,j} \geq s_j \text{ for all } j \in \bigcup_{k=d_0-2}^{i-1} L_k. \quad (8)$$

It follows from combination of (7) and the choice of $\zeta_0, \ldots, \zeta_{d_0-3}$ that (8) is satisfied for $i = d_0 - 2$.

For each $j \in L_i$, we define $F_j$ as

$$F_j = \left\{ x \in \mathbb{F}_q : w_j + \sum_{k=0}^{i-1} \zeta_k Y_{k,j} + xY_{i,j} < s_j \right\}.$$

Clearly, $|F_j| = s_j$ as $Y_{i,j} \neq 0$, and so

$$\bigcup_{j \in L_i} F_j \leq \sum_{j \in L_i} |F_j| = \sum_{j \in L_i} s_j \leq \sum_{j = d_0-2}^{n-1} s_j \leq q - 1,$$

where the last inequality follows from the assumption of $\Sigma$ in the proposition statement and the ordering of the components of $s$. Hence, $\bigcup_{j \in L_i} F_j \neq \mathbb{F}_q$. The encoder chooses $\zeta_i \in \mathbb{F}_q \setminus \bigcup_{j \in L_i} F_j$. We claim that

$$w_j + \sum_{k=0}^{i-1} \zeta_k Y_{k,j} \geq s_j \text{ for all } j \in \bigcup_{k=0}^{i-1} L_k. \quad (9)$$

For $j \in L_i$, (9) follows from the definition of $F_j$. For $j \in \bigcup_{k=0}^{i-1} L_k$, (9) follows from (8) and the fact that $Y_{i,j} = 0$.

By using induction on $i$, we infer that

$$w_j + \sum_{k=0}^{i-1} \zeta_k Y_{k,j} \geq s_j \text{ for all } j \in \bigcup_{k=0}^{i-1} L_k. \quad (10)$$

That is, with $\zeta = (\zeta_0, \ldots, \zeta_{l-1})$, we have that $w + \zeta Y \geq s$.

As $Y = T H_0$, it follows that $z := \zeta T$ is such that

$$w + z H_0 \geq s.$$

The decoding process remains as in Algorithm 4. \hfill \Box

We give an alternative non-constructive proof for Proposition 2 in Appendix B.

Remark 6. The proof of Proposition 2 shows that $(d_0 - 2)$ cells can be set to any desired value, while the remaining $(n - d_0 + 2)$ cells can be made to satisfy the partial stuck-at conditions, provided that the sum of the stuck-at levels in these $(n - d_0 + 2)$ cells is less than $q$.

Corollary 2 (Generalization of Theorem 3). Let $s \in \mathbb{F}_q$ and let $\Sigma = \{s \in \mathbb{F}_q^n : \text{wt}(s) \leq d_0 + \left\lceil \frac{n}{2} \right\rceil - 3 \text{ and max } |s_i| : i \in [n] \leq s\}$. The coding scheme in Construction 2 is a $(\Sigma, t)$ PSMC scheme of length $n$ and size $q^{k-1}$.

Proof. Let $s \in \Sigma$ have weight $u \leq d_0 + q - 3$. Let $\Psi \subseteq [n]$ be the number of non-zero components of $s$ in $[n] \setminus \Psi$ equals min$(d_0 - 2, u)$. Then $s$ has $u - \min(d_0 - 2, u)$ non-zero components in $\Psi$. As a
consequence, if \( u \leq d_0 - 2 \), then \( \sum_{i \in \Psi} s_i = 0 \), and if \( u > d_0 - 2 \), then
\[
\sum_{i \in \Psi} s_i \leq s(u-d_0+2) \leq s\left(\frac{q}{s} \right) - 1 < s\left(\frac{q}{s} + 1 - 1\right) = q.
\]
Hence in both cases, \( \sum_{i \in \Psi} s_i \leq q - 1 \). The corollary thus follows from Proposition 2. In particular, if \( s = 1 \), the corollary agrees with Theorem 3.

We do not generalize Construction 3 as it is tailored to the special case where \( s_i = 1 \) for all \( i \in \phi \).

VI. TRADING PARTIALLY STUCK CELLS WITH ERRORS

In the constructions shown so far, the encoder output \( e \) is a word from an error correcting code \( C \). If \( e \) does not satisfy the partial-stuck-at conditions in \( j \) positions, the encoder could modify it in these \( j \) positions to obtain a word \( e' = e + e'' \) satisfying the partial-stuck-at constraints, while \( \text{wt}(e') = j \). If \( C \) can correct \( t \) errors, then it still is possible to correct \( t - j \) errors in \( e' \). This observation was also made in [13, Theorem 1]. The above reasoning shows that the following proposition holds.

**Proposition 3.** If there is an \( (n, M)_q(u, 1, t) \) PSMC, then for any \( j \) with \( 0 \leq j \leq t \), there is an \( (n, M)_q(u+j, 1, t-j) \) PSMC.

In the remainder of this section, we generalize the above proposition to general \( \Sigma \) (Theorem 6). We also provide variations on the idea of the encoder introducing errors to the result of a first encoding step in order that the final encoder output satisfies the partially stuck-at conditions.

**Theorem 6** (Partial Masking PSMC). Let \( \Sigma \subset \mathbb{F}_q^n \), and assume that there exists an \( (n, M)_q(\Sigma, t) \) PSMC \( C \). For any \( j \in [t] \), there exists an \( (n, M)_q(\Sigma^{(j)}, t-j) \) PSMC \( C_j \), where
\[
\Sigma^{(j)} = \{ s' \in \mathbb{F}_q^n \mid \exists s \in \Sigma [d(s, s') \leq j \text{ and } s' \geq s]\}.
\]

**Proof.** Let the encoder \( E_j \) and the decoder \( D_j \) for \( C_j \) be Algorithm 7 and Algorithm 8, respectively. By definition, \( e' \geq s' \). Moreover, if \( s_i = s_i' \), then \( c_i \geq s_i = s_i' \), so \( c_i = c_i' \). As a result, \( d(c, e') \leq j \).

**Algorithm 7:** Encoding

**Input:** \( (m, s') \in \mathbb{M} \times \Sigma^{(j)} \).
1. Determine \( s \in \Sigma \) such that \( d(s, s') \leq j \) and \( s' \geq s \).
2. Let \( c = E(m, s) \).
3. Define \( c' = E_j(m, s') \) as \( c'_i = \max(c_i, s'_i) \) for \( i \in [n] \).

**Output:** Codeword \( c' \).

**Algorithm 8:** Decoding

**Input:** Received \( y = e' + e \) where \( \text{wt}(e) \leq t-j \) and \( y \geq s' \).
1. Message \( m = D(y) \).

**Output:** Message vector \( m \).

In Algorithm 8, the decoder \( D \) of \( C \) is directly used for decoding \( C_j \). As \( y \geq s' \), surely \( y \geq s \). Moreover, we can write \( y = e + (e' - c + e) \). As shown above, \( \text{wt}(e' - c) \leq j \), and so \( \text{wt}(e-c+e) \leq t \). As a consequence, \( D(y) = m \).

We can improve on Theorem 6 for Construction 3 giving Lemma 1.

**Lemma 1.** Given an \( [n, k, d]_q \) code as defined in Construction 3, then for any \( j \) such that \( 0 \leq j \leq \frac{d-1}{2} \), there is a \( 2^n \)-ary \( (2^{n-1}(d_0 + j) - 1, 1, \left\lfloor \frac{d-1}{2} \right\rfloor - j) \) PSMC of length \( n \) and size \( q^k-1 \).

**Proof.** Let \( \phi \subset [n] \) has size \( u \leq 2^n-1(d_0 + j) - 1 \). We use the notation from Algorithm 5. After Step 1, \( w \) has at most \( u_0 = \left\lfloor \frac{d_0}{2} \right\rfloor \leq d_0 + j - 1 \) binary entries in the positions from \( \phi \). After Step 2, at least \( d_0 - 1 \) of these entries in \( e \) differ from 0. By setting the at most \( j \) other binary entries in the positions from \( \phi \) equal to 1, the encoder introduces at most \( j \) errors, and guarantees that the partial stuck-at conditions are satisfied.

In Lemma 2, we use another approach for introducing errors in order to satisfy the stuck-at conditions.

**Lemma 2.** Given an \( [n, k, d]_q \) code containing a word of weight \( n \), for any \( j \) with \( 0 \leq j \leq \frac{d-1}{2} \), there is a \( q \)-ary \( (q-1 + qj, 1, \left\lfloor \frac{d-1}{2} \right\rfloor - j) \) PSMC of length \( n \) and size \( q^k-1 \).
Proof. We use the notation from Construction 1.
Let \( \phi \subset [n] \) have size \( u \leq q - 1 + qj \). Let \( x \) be a codeword of weight \( n \). For each \( i \in \phi \), there is exactly one \( v \in \mathbb{F}_q \) such that \( w_i + v x_i = 0 \), and so
\[
\sum_{i \in [n]} | \{ i \in \phi | w_i + v x_i = 0 \} | = u.
\]

As a consequence, there is \( v \in \mathbb{F}_q \) such that \( c = w + v x \) has most \( \left\lfloor \frac{q}{2} \right\rfloor \leq j \) entries in \( \phi \) equal to zero. By setting these entries of \( c \) to a non-zero value, the encoder introduces at most \( j \) errors. As \( C \) can correct up to \( \left\lfloor \frac{d}{2} \right\rfloor \) errors, it can correct these \( j \) errors and additionally up to \( \left\lfloor \frac{d}{2} \right\rfloor - j \) substitution errors.
\[ \square \]

Example 4. Consider a \([15, 9, 5]_4\) code \( C \) containing the all-one word, e.g. the BCH code with zeros \( \alpha, \alpha^2, \alpha^3 \), where \( \alpha \) is a primitive element in \( \mathbb{F}_{16} \). Let \( u \leq 7 \) and \( t = 1 \). We use the all-one word for partial masking, ensuring that \( 0 \) occurs in at most \( \left\lfloor \frac{q}{2} \right\rfloor \leq 1 \) position indexed by \( \phi \). We set the codeword value in this position to \( 1 \), introducing one error. We can correct this introduced error and one additional random error as \( C \) has minimum distance \( 5 \). Hence, we have obtained a 4-ary \((7, 1)\) PSMC of length 15 and cardinality \( 4^6 \).

We show in Example 5 how applying Lemma 2 for Construction 3 outperforms Lemma 1.

Example 5. Given \( d_0 = 3 \), \( u = 15 \) and \( q = 2^2 \) and let \( \alpha \) be a primitive element in \( \mathbb{F}_q \) and take \( x = 1 \). Assume we have
\[
w^{(\phi)} = (m', H_0 + m \cdot G_1) + z \cdot 1
\]
\[
= (0, 1, \alpha, 1 + \alpha, 0, 1, 1 + \alpha, 0, 1, 1 + \alpha, 0, 1, 1 + \alpha, 0, 1, 1 + \alpha, 0, 1)
\]
\[
+z \cdot 1,
\]
then choosing \( z = 1 + \alpha \) minimizes the number of binary values in \( w^{(\phi)} \), we get:
\[
w^{(\phi)} = (1 + \alpha, \alpha, 1, 0, 1 + \alpha, 0, 1, \alpha, 0, 1 + \alpha, \alpha, 1, 0 + \alpha, \alpha, 1, \alpha, 0, 1).
\]

Following Step 2 in Algorithm 5 and since \( d_0 = 3 \), we can mask at most \( d_0 - 1 \) binary values highlighted in the vector \( w^{(\phi)} \) that leaves \( u \), in this example, with at most \( \left\lfloor \frac{2q}{2} \right\rfloor - d_0 + 1 = 5 \) zeros that remain unmasked.

However, applying Lemma 2 instead for Construction 3 gives a better result. Choosing \( \gamma = 0 \) in Step 2 of Algorithm 5, we obtain
\[
e^{(\phi)} = w^{(\phi)} + (\left\lfloor \frac{u}{q} \right\rfloor = 3) \text{ zeros highlighted in blue above that we can directly trade off. }
\]

Remark 7. As the code from Construction 3 has a word of weight \( u \), Lemma 2 implies the existence of an \((u, 1, 1)\) PSMC of cardinality \( q^{k-1} \) under the condition that \( 2 + \left\lfloor \frac{q}{2} \right\rfloor < d \). Lemma 1 shows the existence of an \((u, 1, 1)\) PSMC of smaller cardinality, viz. \( q^{k-1} \), under the condition that \( 2 + \max(0, \left\lfloor \frac{q}{2} \right\rfloor - d_0 + 1) < d \). As a consequence, Lemma 1 can only improve on Lemma 2 if \( d_0 - 1 > \left\lfloor \frac{2q}{2} \right\rfloor - \left\lfloor \frac{u}{q} \right\rfloor \).

We can generalize Lemma 2 as follows.

Lemma 3. Given an \([n, k, d]_q\) code containing a word of weight \( n \). Let \( 0 \leq j \leq \left\lfloor \frac{q}{2} \right\rfloor \), and let
\[
\Sigma = \left\{ s \in \mathbb{F}_q^n | \sum_{i} s_i \leq q - 1 + qj \right\}.
\]

There is a \( q \)-ary \((\Sigma, \left\lfloor \frac{d}{2} \right\rfloor - j)\) PSMC of length \( n \) and size \( q^{k-1} \).

Proof. We use the notation from Theorem 5. For simplicity, we assume that the code contains the all-one word. We wish to choose the multiplier \( v \in \mathbb{F}_q \) such that \( c = w - v \cdot 1 \) satisfies \( c_i \geq s_i \) for as many indices \( i \) as possible. For each index \( i \), there are \( q - s_i \) values for \( v \) such that this inequality is met. Hence, there is a \( v \in \mathbb{F}_q \) such that \( c_i \geq s_i \) for at least \( \left\lfloor \frac{1}{q} \sum_{i}(q - s_i) \right\rfloor = n - \left\lfloor \frac{1}{q} \sum_{i}s_i \right\rfloor \) indices \( i \). The encoder thus needs to introduce errors only in the at most \( \left\lfloor \frac{1}{q} \sum_{i}s_i \right\rfloor \) positions for which the inequality is not satisfied.
\[ \square \]

Lemma 4. Assume there exists a matrix as in Proposition 2. Let \( 0 \leq j \leq \left\lfloor \frac{d}{2} \right\rfloor \), and let
\[
\Sigma = \left\{ s \in \mathbb{F}_q^n | \exists \Psi \subset [n] : | \Psi | = n - d_0 + 2 \right\}.
\]

Then exists a \( q \)-ary \((\Sigma, \left\lfloor \frac{d}{2} \right\rfloor - j)\) PSMC of length \( n \) and size \( q^{k-1} \).

Proof. Let \( s \in \Sigma \). In order to simplify notation, we assume without loss of generality that \( \sum_{i=0}^{n-1-s} s_i \leq q - 1 + qj \). We use the same argument as in the alternative proof of Proposition 2 (see Appendix B).

\[
| \sum_{i=0}^{n-1-s} s_i | \geq n - d_0 + 2 - j.
\]

So we infer that for at least \( n - j \) indices \( i \in [n] \),
\[
w_i + \left( (\zeta, \eta) TH_0 \right)_i \geq s_i.
\]

Remark 8. The proof of Lemma 4 shows that the encoder output in fact can be made equal to \( s \) in the \( d_0 - 2 \) largest entries of \( s \). In fact, it shows that the scheme allows for masking \( d_0 - 2 \) stuck-at errors, masking partial stuck errors in the remaining cells, and correcting \( \left\lfloor \frac{d}{2} \right\rfloor - j \) substitution errors, provided that the sum of the stuck-at levels in the \( n - d_0 + 2 \) remaining cells is less than \( (j + 1)q \).

VII. UPPER BOUNDS ON PSMC CODES

The output of an encoder has restrictions on the values in the partially stuck-at cells; in the other cells, it can attain all values. So the set of all encoder outputs is a poly-alphabetic code [22]. To be more precise, the following proposition holds.

Proposition 4. Let \( C \) be an \((n, M)_q(\Sigma, t)\) partially stuck-at-masking code with encoder \( E \). For any \( s \in \Sigma \), let
\[
C_s = \{ E(m, s) | m \in M \}.
\]

Then \( C_s \) is a code with minimum distance at least \( 2t + 1 \) and \( |M| \) words, and
\[
C_s \subset Q_0 \times Q_1 \times \cdots \times Q_{n-1}, \text{ where } Q_t = \{ x \in \mathbb{F}_q | x \geq s_t \}.
\]

Proof. By our error model, errors in stuck-at cells result in values still satisfying the stuck-at constraints. Therefore, \( t \) errors can be corrected if and only if \( C_s \) has minimum Hamming distance at least \( 2t + 1 \). The rest of the proposition is obvious.
\[ \square \]

As a result of Proposition 4, upper bounds on the size of poly-alphabetic codes [22] are also upper bounds on the size of partially-stuck-at codes.
Theorem 7. (Singleton type bound) Let \( \mathcal{C} \) be a \( q \)-ary (\( \Sigma, t \)) PSMC of length \( n \) and size \( M \). Then for any \( s = (s_0, \ldots, s_{n-1}) \in \Sigma \),

\[
M \leq \min \left\{ \prod_{j=0}^{n-1} (q - s_j) \left| J \subseteq [n], |J| = n - 2t \right. \right\}.
\]

Proof. Combination of Proposition 4 and [22, Theorem 2]. \( \square \)

Theorem 8. (Sphere-packing type bound) Let \( \mathcal{C} \) be a \( q \)-ary (\( \Sigma, t \)) PSMC of length \( n \) and size \( M \). Then for any \( s = (s_0, \ldots, s_{n-1}) \in \Sigma \),

\[
M \leq \prod_{i=0}^{n-1} (q - s_i) V_t^{(b)},
\]

where \( V_t^{(b)} \), the volume of a ball of radius \( t \), satisfies

\[
V_t^{(b)} = \sum_{s=0}^{t} V_s^{(s)},
\]

where the volume \( V_s^{(s)} \) of the sphere with radius \( r \) is given by

\[
V_s^{(s)} = \sum_{1 \leq r_1 < \cdots < r_s \leq n} (q - 1 - s_{r_1}) \cdots (q - 1 - s_{r_s}).
\]

Proof. Combination of Proposition 4 and [22, Theorem 3]. \( \square \)

Remark 9. The difference between poly-alphabetic codes and partially-stuck-at-masking codes is that in the former, the positions of stuck-at cells and the corresponding levels are known to both encoder and decoder, whereas in the latter, this information is known to the encoder only.

Figure 2. Sphere-packing bounds: Comparison for \( k \) information symbols for (“only partially stuck cells” [5, Theorem 2]) and our sphere-packing-like (“errors and partially stuck cells”) bounds. The classical sphere-packing bound (“only errors”) can read at \( u = 0 \) in our sphere-packing-like bounds curves. The chosen parameters are \( \mu = 5 \) and \( q = 3 \), and \( n = ((q^n - 1)/(q - 1)) \).

Figure 2 compares our derived sphere-packing-like bound to the amount of storable information symbols for a completely reliable memory (i.e., no stuck cells, no errors that can be seen at \( u = 0 \) in the solid line) and the upper bound on the cardinality of an only-masking PSMC (only stuck cells, no errors) derived in [5] as depicted in the solid curve. At \( u = 0 \), the derived sphere-packing-type bound (dotted and dashed-dotted plots) matches the usual sphere-packing bound (“only errors”) case. The more \( u \) partially stuck at cells, the less amount of storable information, i.e. only \( q - 1 \) levels can be utilized. Hence, the dotted and dashed-dotted lines are declining while \( u \) is growing. On the other hand, the more errors (e.g., \( t = 25 \) in the dashed-dotted plot), the higher overall required redundancy and the lower storable information for the aforementioned curve.

VIII. GILBERT–VARSHAMOV (GV) BOUND

A. Gilbert–Varshamov (GV) Bound: finite length

We have provided various constructions of \((u, 1, t)\) PSMCs based on \( q \)-ary \( t \)-error correcting codes with additional properties. In this section, we first employ GV-like techniques to show the existence of \((u, 1, t)\) PSMCs. Next, we study the asymptotic of the resulting GV bounds.

We start with a somewhat refined version of the Gilbert bound, that should be well-known, but for which we did not find an explicit reference.

Lemma 5. Let \( q \) be a prime power, and assume there is an \([n, s]_q \) code \( \mathcal{C}_u \) with minimum distance at least \( d \). If \( k \geq s \) is such that

\[
\sum_{i=0}^{d-1} \binom{n}{i} (q - 1)^i < q^{n-k+1},
\]

then there is an \([n, k]_q \) code \( \mathcal{C}_u \) with minimum distance at least \( d \) that has \( \mathcal{C}_u \) as a subcode.

Proof. By induction on \( k \). For \( k = s \), the statement is obvious. Now let \( k > s \) and let \( \mathcal{C}_u \) be an \([n, k]_q \) code with minimum distance at least \( d \) that has \( \mathcal{C}_u \) as a subcode. If \( q^n \sum_{i=0}^{d-1} \binom{n}{i} (q - 1)^i < q^n \), then the balls with radius \( d - 1 \) centered at the words of \( \mathcal{C}_u \) do not cover \( \mathbb{F}_q^n \) so there is a word \( x \) at distance at least \( d \) from all words in \( \mathcal{C}_u \). As shown in the proof of [23, Theorem, 5.1.8], the \([n, k+1]_q \) code \( \mathcal{C}_{k+1} \) spanned by \( \mathcal{C}_u \) and \( x \) has minimum distance at least \( d \). \( \square \)

1) Finite GV bound based on Lemma 2:

Theorem 9. Let \( q \) be a prime power. Let \( n, k, t, u \) be non-negative integers such that

\[
\sum_{0}^{2(t+1)} \binom{n}{i} (q - 1)^i < q^{n-k+1}.
\]

There exists a \( q \)-ary \((u, 1, t)\) PSMC of length \( n \) and size \( q^{k-1} \).

Proof. Let \( \mathcal{C}_1 \) be the \([n, k]_q \) code generated by the all-one word. Lemma 5 implies that there is an \([n, k]_q \) code with minimum distance at least \( 2t+1 \) that contains the all-one word. Lemma 2 shows that \( \mathcal{C}_k \) can be used to construct a PSMC with the claimed parameters. \( \square \)

Remark 10. GV bound from Theorem 1 is a special case of Theorem 9 for \( u \leq q - 1 \).

2) Finite GV bound based on Construction 2:

Lemma 6. Let \( q \) be a prime power, and let \( 1 \leq k < n \). Let \( E \subset \mathbb{F}_q^n - \{0\} \). The fraction of \([n,k]_q \) codes with non-empty intersection with \( E \) is less than \(|E|/q^{n-k} \).

Proof. Let \( \mathcal{C} \) be the set of all \([n,k]_q \) codes. Obviously,

\[
\left| \left\{ \mathcal{C} \in \mathcal{C} \mid \mathcal{C} \cap E \neq \emptyset \right\} \right| \leq \sum_{\mathcal{C} \in \mathcal{C}, \mathcal{C} \cap E \neq \emptyset} |\mathcal{C} \cap E| = \sum_{\mathcal{C} \in \mathcal{C}} |\mathcal{C} \cap E|.
\]
It follows from [24, Lemma 3] that
\[
\frac{1}{|C|} \sum_{e \in C} |C \cap E| = \frac{q^n - 1}{q^n - 1} |E| < |E| \frac{1}{q^n - 1}.
\]

\[\square\]

**Remark 11.** If \(E\) has the additional property that \(\lambda e \in E\) for all \(e \in E\) and \(\lambda \in \mathbb{F}_q \setminus \{0\}\), then the upper bound in Lemma 6 can be reduced to \(|E|/(q-1)q^{n-k}\).

**Lemma 7.** Let \(k, n, d\) and \(d^\perp\) be integers such that
\[
\sum_{i=0}^{d-1} \binom{n}{i} (q-1)^i < \frac{1}{2} q^{n-k}\quad \text{and}
\sum_{i=0}^{d^\perp-1} \binom{n}{i} (q-1)^i < \frac{1}{2} q^k.
\]

There exists a \(q\)-ary \([n, k]\) code \(C\) with minimum distance at least \(d\) such that \(C^\perp\) has minimum distance at least \(d^\perp\).

Proof. Let \(C\) denote the set of all \([n, k]\) codes. By applying Lemma 6 with \(E = \{e \in \mathbb{F}_q^n \mid 1 \leq \text{wt}(e) \leq d - 1\}\) and using the first condition of the lemma, we see that more than half of the codes in \(C\) have empty intersection with \(E\), that is, have minimum distance at least \(d\). Similarly, more than half of all \(q\)-ary \([n, n-k]\) codes have minimum distance at least \(d^\perp\), and so more than half of the codes in \(C\) have a dual with minimum distance at least \(d^\perp\). We conclude that \(C\) contains a code with both desired properties.

\[\square\]

**Theorem 10** (Gilbert-Varshamov-like bound by Construction 2). Let \(q\) be a prime power. Suppose the positive integers \(u, t, n, k, l\) with \(u, t \leq n\) and \(l < k\) satisfy
\[
\sum_{i=0}^{u-1} \binom{n}{i} (q-1)^i < \frac{1}{2} q^{n-l},
\]
\[
\sum_{i=0}^{u-1} \binom{n}{i} (q-1)^i < \frac{1}{2} q^l,
\]
\[
\sum_{i=0}^{2t} \binom{n}{i} (q-1)^i < q^{n-k+1},
\]
\[
\sum_{i=0}^{2t} \binom{n}{i} (q-1)^i < q^{n-k+1}.
\]

Then there is a \(q\)-ary \((u, 1, t)\) PSMC of length \(n\) and cardinality \(q^{n-k-1}\).

Proof. According to Lemma 7, (11) and (12) imply the existence of an \([n, l]_q\) code \(C_0\) with minimum distance at least \(2t + 1\) for which the dual code has minimum distance at least \(u - q + 3\). Lemma 5 shows that \(C_0\) can be extended to an \([n, k]_q\) code \(C\) with minimum distance at least \(d\). As \(C\) has a generator matrix of the form required by Construction 2, the theorem follows.

\[\square\]

3) **Finite GV bound based on Proposition 1:** In this section, we give sufficient conditions for the existence of matrices satisfying the conditions of Proposition 1. We start with Lemma 8 and Lemma 9, then we prove the main theorem (Theorem 11).

**Lemma 8.** Let \(G\) be a \(k \times n\) matrix over \(\mathbb{F}_q\). For \(s \geq 1\), let
\[
d_s = \min\{\text{wt}(mG) \mid m \in \mathbb{F}_q^n \setminus \{0\}\}.
\]
Then \(d_2 = d_1\).

Proof. The proof of Lemma 8 can be found in the appendix.

Now we introduce Lemma 9 which is the binary version of Lemma 7 but with an extra restriction on the weight of the words.

**Lemma 9.** Let \(k, n, d\) and \(d^\perp\) be integers such that
\[
\sum_{i=0}^{d-1} \binom{n}{i} < \frac{1}{4} 2^{n-k} \quad \text{and} \quad \sum_{i=0}^{d^\perp-1} \binom{n}{i} < \frac{1}{2} 2^k.
\]

There exists a binary \([n, k]\) code \(C\) with minimum distance at least \(d\) without a word of weight more than \(n - d + 1\) such that \(C^\perp\) has minimum distance at least \(d^\perp\).

Proof. Similar to the proof of Lemma 7. Let \(C\) denote the set of all binary \([n, k]\) codes. By applying Lemma 6 with \(E = \{e \in \mathbb{F}_2^n \mid 1 \leq \text{wt}(e) \leq d - 1\} \cup \{e \in \mathbb{F}_2^n \mid \text{wt}(e) \geq n - d + 1\}\), we infer that the first inequality implies that more than half of the codes in \(C\) contain no element from \(E\). Similarly, the second inequality implies that more than half of the binary \([n, n-k]\) codes have minimum weight at least \(d^\perp\), and so more than half of the codes in \(C\) have a dual with minimum distance at least \(d^\perp\). We conclude that there is a code in \(C\) enjoying both desired properties.

\[\square\]

**Theorem 11** (Gilbert-Varshamov-like bound by Construction 3). Let \(n, k, l, u, t, \mu\) be positive integers with \(u \leq n\) and \(l < k\) be such that
\[
\sum_{i=0}^{2t} \binom{n}{i} < \frac{1}{4} 2^{n-\mu},
\]
\[
\sum_{i=0}^{\mu+1} \binom{n}{i} < \frac{1}{2} 2^l,
\]
\[
\sum_{i=0}^{2t} \binom{n}{i} (2^u - 1)^i < 2^{\mu(n-k+1)},
\]
Then there exists a \((u, 1, t)\) PSMC of length \(n\) over \(\mathbb{F}_{2^\mu}\) with cardinality \(2^\mu(2^{k-l-1}2^{2t(n-\mu-1)})\).

Proof. By Lemma 9, there exists a binary \([n, l]_2\) code \(C_0\) with minimum distance at least \(2t + 1\) for which \(C_0^\perp\) has minimum distance at least \(\lceil\frac{n}{2} - \mu\rceil + 1\) with the following additional property: if \(H_0 \subseteq \mathbb{F}_2^n\) is a generator matrix for \(C_0\), then the binary code \(C_\mu\) with generator matrix \(H_0 \frac{1}{2}\) has minimum distance at least \(2t + 1\). According to Lemma 8, the code \(C_\mu\) over \(\mathbb{F}_{2^\mu}\) with this generator matrix has minimum distance at least \(2t + 1\) as well. Lemma 5 implies that \(C_\mu\) can be extended to an \([n, k]_2\) code with minimum distance at least \(2t + 1\). The \([n, k]\) code has a generator matrix of the form \(G = \begin{bmatrix} H_0 & G_1 \end{bmatrix}\).

Application of Proposition 1 yields the claim.

\[\square\]

4) **Finite GV bound from trivial construction:** Clearly, a \((q-1)\)-ary code of length \(n\) with minimum distance at least \(2t + 1\) is a \(q\)-ary \((u, 1, t)\) PSMC of length \(n\). Combining this observation with the Gilbert bound for a \(q-1\)-ary alphabet, we obtain the following.

**Theorem 12.** Let \(q \geq 3\), and let
\[
M = \left[\frac{(q-1)^n}{\sum_{i=0}^{2t} \binom{n}{i} (q-2)^i}\right].
\]

There is a \(q\)-ary \((n, 1, t)\) PSMC of length \(n\) and cardinality \(M\).

So far we have covered the GV-like bounds for our code constructions for finite length \(n\).
B. Asymptotic GV Bound on PSMCs

In this section, we present the asymptotic version of the GV bounds from the previous section. That is, we provide lower bounds on the achievable rates of a $q$-ary $(u,1,t)$ PSMCs in the regime that the code length $n$ tends to infinity, and the number $u$ of partial stuck-at cells and the number $t$ of random errors both grow linearly in $n$.

We recall the well-known following lemma that estimates the volume of a Hamming ball using the $q$-ary entropy function.

**Lemma 10.** For positive integers $q, n \geq 2$ and real $\delta, 0 \leq \delta \leq 1 - \frac{1}{q}$,
\[
\text{Vol}_q(n, \delta n) \leq q^{h_0(\delta)n},
\]
where $\text{Vol}_q(n, r) = \sum_{j=0}^{\lfloor \frac{n}{r} \rfloor} (q - 1)^j$ denotes the volume of a Hamming ball with radius $r$.

**Proof.** The proof of Lemma 10 has been stated in many references including [25, p.105] and [26, Proposition 3.3.1]. □

1) Asymptotic bound for Theorem 9:

**Theorem 13.** Let $q$ be a prime power. Let $0 \leq \tau, v < 1$ be such that
\[
2(\tau + \frac{v}{q}) < 1 - \frac{1}{q}.
\]
For sufficiently large $n$, there exists an $([vn], 1, [\tau n])$ PSMC of length $n$ and rate at least
\[
1 - h_q(2(\tau + \frac{v}{q})) - \frac{2}{n}.
\]

**Proof.** Let $n$ be a positive integer such that $[nh_q(2(\tau + \frac{v}{q}))] < n$. Let $t = [\tau n]$ and $u = [vn]$. Take $k = n - [nh_q(2(\tau + \frac{v}{q}))]$. Lemma 10 implies that $\text{Vol}_q(n, 2t + 2\lfloor \frac{n}{r} \rfloor) \leq q^{h_q(k - 2)}$, and so, according to Theorem 9, there is a $q$-ary $(u,1,t)$ PSMC of length $n$ with rate $\frac{k - 1}{n} \geq 1 - h_q(2(\tau + \frac{v}{q})) - \frac{2}{n}$. □

2) Asymptotic GV bound from Construction 2:

**Theorem 14** (Asymptotic Gilbert-Varshamov-like bound from Theorem 10). Let $q$ be a prime power. Let $v, \tau$ be such that
\[
0 < v, 2\tau < 1 - \frac{1}{q} \quad \text{and} \quad h_q(v) + h_q(2\tau) < 1.
\]
For sufficiently large $n$, there exists a $q$-ary $([vn], 1, [\tau n])$ PSMC of length $n$ and rate at least
\[
1 - h_q(2\tau) - h_q(v) - \frac{4\log_q(2) + 2}{n}.
\]

**Proof.** Let $n$ be a positive integer. Write $u = [vn]$ and $t = [\tau n]$. Then $\text{Vol}_q(n, u - q + 2) \leq \text{Vol}_q(n, u)$. Hence, by setting
\[
l = [nh_q(v) + 2\log_q(2)],
\]
Lemma 10 implies that (12) is satisfied. Similarly, by setting
\[
k = n - [nh_q(2\tau) + 2\log_q(2)],
\]
it is ensured that (13) is satisfied. According to Theorem 10, there is a $q$-ary $(u,1,t)$ PSMC of length $n$ and size $q^{k - l}$, so with rate $k - l$. The choices for $k$ and $l$ show that the theorem is true. □

**Remark 12.** Theorem 14 in fact holds for classical stuck-at cells instead of stuck-at-1 errors, as follows from considering the generalization of Theorem 10 in Proposition 2, i.e., Heegard’s construction [13].

3) Asymptotic GV bound from Construction 3:

**Theorem 15** (Asymptotic Gilbert-Varshamov-like bound from Theorem 11). Let $\mu$ be a positive integer, and let $v$ and $\tau$ be such that
\[
0 \leq \frac{v}{2\mu^2} - 1 < 2\tau < \frac{1}{2}. \quad \text{and} \quad h_2(\frac{v}{2\mu^2}) + h_2(2\tau) < 1.
\]
For sufficiently large $n$ there is a $2^\mu$-ary $([vn], 1, [\tau n])$ PSMC of length $n$ and rate at least
\[
1 - h_2(2\tau) - \frac{1}{\mu} h_2(\frac{v}{2\mu^2}) - \frac{2}{n} - \frac{3}{\mu n}.
\]

**Proof.** For notational convenience, we set $v_0 = \frac{v}{2\mu^2}$ and $\eta = 1 - h_2(2\tau) - h_2(v_0)$. Note that $\eta > 0$.

Let $n$ be a positive integer satisfying $n \geq \frac{v}{2\mu}$, and let $u = [vn]$, $u_0 = [\frac{u}{v_0}]$ and $t = [\tau n]$. We set
\[
l = [nh_2(v_0)] + 3.
\]
Lemma 10 implies that (15) is satisfied. Moreover, as
\[
n - l - 3 \geq n - nh_2(v_0) - 7 = nh_2(2\tau) + n\eta - 7 \geq nh_2(2\tau),
\]
Lemma 10 implies that (14) is satisfied. We set
\[
k = n - [nh_2(2\tau)].
\]
Lemma 10 implies that (16) is satisfied. According to [26, Corollary 3.3.4], we have that $h_2(2\tau) \leq h_2(2\tau)$, and so
\[
k - l \geq n - nh_2(2\tau) - 1 - nh_2(v_0) - 4 = n\eta - 5 \geq 2.
\]
Theorem 11 implies the existence of a $2^k$-ary $(u,1,t)$ PSMC of length $n$ with size $2 \cdot 2^{k-1}2^{-t}$, i.e., its rate is
\[
k - 1
\]
\[
\geq \frac{l - 1}{\mu n} \geq 1 - h_2(2\tau) - \frac{1}{\mu} h_2(v_0) - \frac{2}{n} - \frac{3}{\mu n}.
\]

**Theorem 16** (Asymptotic Gilbert-Varshamov bound from Theorem 12). Let $\eta \geq 3$. For each positive integer $n$ and each $\tau$ with $0 \leq 2\tau < 1 - \frac{1}{\sqrt{q}}$, there exists a $q$-ary $(n, 1, [\tau n])$ PSMC of length $n$ and rate at least
\[
1 - h_{q-1}(2\tau)) \cdot \log_q(q - 1).
\]

**Proof.** Let $t = [\tau n]$. Theorem 12 implies the existence of a $q$-ary $(n, 1, t)$ PSMC of length $n$ and cardinality $M$ satisfying
\[
M \geq \frac{(q - 1)^n}{V_{q-1}(n, 2t)} \geq (q - 1)^{n(1 - h_{q-1}(2\tau))},
\]
where the last inequality holds by Lemma 10. □

IX. COMPARISONS

We provide different comparisons between our code constructions and the existence of the code based on Theorem 9, Theorem 10 and Theorem 11. Next, we also compare to the known limits and investigate the trade-off between masking and error-correction as described in Section VI.
A. Comparison of Theorem 9 for $u \leq q - 1$ to other Bounds

Figure 3 illustrates the rates of a $(q - 1, 1, t)$ PSMC obtained from Theorem 1 (applying Theorem 9 for the special case $u \leq q - 1$) for $n = 114$, $q = 7$ and $0 \leq t \leq 56$. We show how close explicit BCH codes that contains the all-one word of certain rates $R$ and that can correct designed distances $d \geq 2t + 1$ to the achieved rates from Theorem 1. We note that the solid red graph matches the dashed-dotted green plot for a few code parameters and overpasses it for $t = 39$. We also compare to the classical $q$-ary GV bound (in dashed black) as well as to reduced alphabet $(q - 1)$-ary GV bound (in dashed-dotted blue). To this end, we show upper bounds on the rates that can be obtained from Theorem 1 using the Griesmer bound [27], and the Ball–Blokhuis bound [28] on the size of codes containing the all-one word.

Figure 3. Comparison of other upper and lower limits to our derived GV-like bound in Theorem 9 taking $n = 114$, $q = 7$, $0 \leq t \leq 56$ and $u \leq q - 1$. The dashed-dotted green curve shows the rates for Theorem 1 by Theorem 9 for $u \leq q - 1$ in which codes that have the all-one words are considered. This curve for several code parameters matches the red line that shows how the rates of BCH codes that contain all-one word with regard to the designed distances $d \geq 2t + 1$.

B. Comparison among Theorem 10, Theorem 11 and $(q - 1)$-ary Gilbert-Varshamov bound

We plot the achievable rates ($R = \log_q(M)/n$) as a function of $t$ for different fixed values of $u$. Figure 4 is the resulting plot for $n = 200$ and $q = 2^3$. It can be seen that the GV-like bound in different ranges of $u$ and $t$ based on Construction 2 improves upon the $(q - 1)$-ary GV bound for $u \leq 5$ as depicted in the solid red curve, and improves further (up to $u \leq 20$) based on Construction 3 as shown in the dashed gray line (3rd one from above). The dashed dotted blue curve is used to see what if we map our $2^3$ levels such that we avoid the subscript 0 to compare with 7 levels. It is obvious that for $\mu = 3$, the rate loss$^1$ resulting from using $q - 1$ instead of $q$ symbols is already quite small. Note that for $u = 0$ the solid red curve mostly achieves the exact rates obtained from the standard $2^3$-ary GV bound for $0 \leq t \leq 80$, and so as for the dashed red curve but for $0 \leq t \leq 47$.

$^1$For $t = 0$, the loss is $1 - \log_q(7) = 0.0642$.

Figure 4. The achievable rates $R = \frac{1}{2} \log_2 M$ of GV bounds for different $u$, $t$ for $n = 200$ and $q = 2^3$ in Theorem 10 and Theorem 11, where $M$ is the code cardinality. They are also compared to the rates from an ordinary $7$-ary GV bound for different $t$ as illustrated in the dashed-dotted blue plot. The solid and the dashed lines represent the derived GV like bounds from Theorem 10 and Theorem 11, respectively.

Figure 5. The achievable rates $R = \frac{1}{2} \log_2 M$ of GV bounds for different $u$, $t$ for $n = 200$ and $q = 2^3$ in Theorem 10 and Theorem 11. They are also compared to the rates from an ordinary $3$-ary GV bound for different $t$ as illustrated in the dashed-dotted blue plot. The solid and the dashed lines correspond to the derived GV like bounds by Theorem 10 and by Theorem 11, respectively.

For $\mu = 2$, the improvements from Construction 2 ($u \leq 10$) and Construction 3 ($u \leq 30$) upon a usual $(q - 1)$-ary GV bound are more significant as shown in Figure 5.

C. Comparisons between Theorem 9 and $(q - 1)$-ary Gilbert-Varshamov bound

In Figure 6, we compare the GV like bound from Theorem 9 for $q = 2^3$ with the conventional GV bound for $q - 1 = 7$ shown in dashed black curve. We see the dashed-dotted green curve by Theorem 9 for $u \leq 7$ as stated in Remark 10. For $q = 8$, we observe that the conventional $q - 1$-ary GV bound...
is superior to the derived GV-like bound from Theorem 9 and many larger values of $u$. However, applying Theorem 9 where $u \leq 7$, the traditional $q-1$-ary GV bound is a bad choice.

D. Comparisons between Theorem 11 and Theorem 9

In Figure 7, we compare Theorem 11 and Theorem 9. Theorem 11 is showing higher rates for larger $u$ values, for example taking $u = 40$ and $t = 1$, the rate is $R = 0.87$ from Theorem 11 while $R = 0.83$ from Theorem 9. It is interesting to note that for $u = 30$ and $t > 10$ Theorem 9 is better, and for $u = 10$ and $t > 18$ Theorem 9 is as good as Theorem 11.

E. Comparisons of application of Theorem 6 vs direct application of Theorem 10

For given $(u, t)$, we illustrate the trading $(u+1, t-1)$ in Figure 8. For some of $t$ and a few of $u$ values, it is advantageous if the encoder introduces an error in a partially stuck at position in order to mask this position. The orange solid curve, for instance, represents the rates that have been determined by Theorem 10 for $u = 17$ and $0 \leq t \leq 50$, while the orange dotted sketch highlights the rates for $u = 16$ while $1 \leq t \leq 51$. Due to the exchange such that $u+1 = 17$ and $0 \leq t-1 \leq 50$, the orange dotted line slightly fluctuates up and down the rates shown in the orange solid curve for most $t$ values.

Let us describe some points of Figure 8 in Table IV. Let $C_{u,t}$ be a code by Theorem 10 whose rate is $R$ given in Table IV at $u$ row and $t$ column. Take $C_{21,15}$ so that its rate $R = 0.470$. By applying Theorem 6 on $C_{21,15}$, we obtain a code $C_{22,14}$ of $R = 0.470$. Direct application of Theorem 10 yields a $C_{22,14}$ of rate $R = 0.475$ as highlighted in Table IV. We conclude that in this case, the trade by Theorem 6 gives lower rates than taking the same code directly by Theorem 10 for given $(u = 22, t = 14)$.

On contrary, for larger $t$ values, Table IV shows that the exchange is beneficial giving higher rates. For example, we start with $C_{21,41}$ whose $R = 0.105$, then applying Theorem 6 gives $C_{22,40}$ of $R = 0.105$ which is greater than $R = 0.100$ that has been obtained directly by Theorem 10 as stated in Table IV.

F. Comparisons of applications of Theorem 6, Lemma 1, Lemma 2 vs direct application of Theorem 11

For the derived GV bound based on Construction 3 obtained by Theorem 11, we demonstrate the exchange of a one error correction ability with a single masking capability of a partially stuck cell following Theorem 6 in Figure 9. The solid and dotted lines represent the rates before and after trading, respectively. We also show the exchange by Lemma 1 and Lemma 2 in which the reduction of the correctable errors by one increases $u$ by $2^\mu-1$ and $2^\nu$, respectively. Let us discuss the following curves. For $u = 19$, the orange solid curve shows the rates by Theorem 11. Exchanging $u + 1$ and $t - 1$ throughout Theorem 6 obtains the orange dotted line for $u + 1 = 20$ which lies slightly below the orange solid plot. Hence, the exchange gives lower rates but provides rate $R = 0.380$ for $t = 30$ while direct application of Theorem 11 (compared to its corresponding graph which is the solid green curve at $u = 20, 21, 22, 23$) does not. Now, we apply Lemma 1
rather Theorem 6. We observe the dashed red graph for $u + 2^{3 - 1} = 23$ shows the exact rates from the orange dotted curve for $u + 1 = 20$. Therefore, it is clear that Lemma 1 provides a gain of masking exactly 3 more cells with regard to Theorem 6.

However, if we take $u = 23$ directly by Theorem 11, we achieve slightly higher rates. We conclude that Theorem 11 can directly estimate the maximum possible masked $u$ cells that can also be achieved applying Lemma 1, and can achieve slightly higher rates. However, Theorem 11 does not give rates for larger $t$ values while Lemma 1 and Theorem 6 do that.

On the other hand, as Theorem 11 is based on Construction 3 that contains a word of weight $n$, Lemma 2 is applicable under the condition that $2(t + \lfloor \frac{u}{q} \rfloor) < d$ (cf. Remark 7). Hence, we can achieve higher rates while masking up to the same number of $u$ cells than employing Lemma 1 or Theorem 6 as shown in the dashed-dotted curve.

For that we describe some points of Figure 9 by Table V. Let $C_{u,t}$ be a code by Theorem 11 whose rate is $R$ given in Table V at $u$ row and $t$ column. Taking $C_{19,27}$ gives $C_{20,26}$ and $C_{23,26}$ with $R = 0.435$ applying Theorem 6 and Lemma 1, respectively. In contrary, taking $C_{19,31}$ is advantageous as there are codes ($C_{20,30}$ by Theorem 6 and $C_{23,30}$ by Lemma 1) with $R = 0.380$ while direct application of Theorem 11 cannot provide these codes as highlighted in green with "None". Now, we apply Lemma 2 on a code obtained by Theorem 9 for $(u = 7, t = 27)$ to obtain the code $C_{15,26}$ of rate $R = 0.465$ that satisfies $2(26 + \lfloor \frac{15}{2} \rfloor) < 55$. The achieved rate is higher compared to $C_{15,26}$ of rate $R = 0.460$ that is directly obtained by Theorem 11, or applying Theorem 6 on $C_{14,27}$ to obtain $C_{15,26}$ of $R = 0.445$, or using Lemma 1 on $C_{11,27}$ to obtain $C_{15,26}$ of $R = 0.456$. This result does not mean that application

\[ \text{Theorem 10} (u,t) : u = 17 \]
\[ \text{Theorem 10} (u,t) : u = 22 \]
\[ \text{Theorem 10} (u,t) : u = 26 \]
\[ \text{Theorem 6} (u + 1, t - 1) : u = 16 \]
\[ \text{Theorem 6} (u + 1, t - 1) : u = 21 \]
\[ \text{Theorem 6} (u + 1, t - 1) : u = 25 \]

**Table V**

| $u$ | 26 | 27 | 28 | 29 | 30 | 31 | 32 |
|-----|----|----|----|----|----|----|----|
| 11  | 0.471 | 0.456 | 0.441 | 0.431 | 0.416 | 0.401 | 0.391 |
| 12  | 0.460 | 0.445 | 0.430 | 0.420 | 0.405 | 0.390 | 0.380 |
| 14  | 0.460 | 0.445 | 0.430 | 0.420 | 0.405 | 0.390 | 0.380 |
| 15  | 0.440 | 0.445 | 0.430 | 0.420 | 0.405 | 0.390 | 0.380 |
| 19  | 0.450 | 0.435 | 0.420 | 0.410 | 0.395 | 0.380 |
| 20  | 0.441 | 0.426 | 0.411 | 0.401 | None | None | None |
| 23  | 0.441 | 0.426 | 0.411 | 0.401 | None | None | None |

Lemma 2 on a code obtained by Theorem 9 always provides higher code rates for the same parameters $u, t$ (see Figure 7).

**Figure 8.** The achievable rates $R = \frac{1}{u} \log_2 M$ of GV bounds for different $u$, and $t$ for $n = 200$ and $q = 2^3$ in Theorem 10. The solid plots are the rates from the derived GV-like bound and the dotted graphs are the rates after trading $u + 1, t - 1$ by Theorem 6.

**Figure 9.** The achievable rates $R = \frac{1}{u} \log_2 M$ of GV bounds for different $u$, and $t$ for $n = 200$ and $q = 2^3$ in Theorem 11. The solid plots are the rates from the derived GV like bound and the dotted graphs are the rates after trading $u + 1, t - 1$ by Theorem 6. We also show the exchange by Lemma 1 and Lemma 2 in which the reduction of the correctable errors by one increases $u$ by $2^{n-1}$ and $2^n$, respectively.

G. Analytical comparison of asymptotic GV-like bounds

In this section, we state the results of the analytical comparisons of the asymptotic GV bounds from Theorems 13, 14 and 15, ignoring the terms that tend to zero for increasing $n$. The proofs can be found in Appendix D.

**Proposition 5.** If $u, t$ and $q$ are such that the conditions of Theorem 13 and Theorem 14 are met, then the rate guaranteed by Theorem 13 is at least equal to the code rate guaranteed by Theorem 14.
Proposition 6. If \( v, \tau \) and \( q = 2^u \) are such that the conditions of Theorem 14 and Theorem 15 are met, then the code rate guaranteed by Theorem 15 is equal to the code rate guaranteed by Theorem 14.

We note that the requirement \( 2\tau < \frac{1}{2} \) from Theorem 15 is stricter than the requirement \( 2\tau < 1 - \frac{1}{2} \) from Theorem 14. That is, there is a pair \((\tau, v)\) for which Theorem 14 is applicable, but Theorem 15 is not.

As explained in Appendix D, comparison of Theorem 13 and Theorem 15 is more complicated. We have the following partial result.

Proposition 7. Let \( v, \tau > 0 \) and \( q = 2^u \) be such that the conditions of Theorem 13 and Theorem 15 are met. If \( v \) is sufficiently small, then the rate guaranteed by Theorem 13 is larger than the rate guaranteed by Theorem 15.

X. CONCLUSION

In this paper, code constructions and bounds for nonvolatile memories with partial defects have been proposed. Our constructions can handle both: partial defects (also called partially stuck cells) and random substitution errors, and require less redundancy symbols for \( u > 1 \) and \( q > 2 \) than the known constructions for stuck cells. Compared to error-free masking of partially stuck cells, our achieved code sizes coincide with those in [5], or are even larger as shown in Proposition 1. We summarize our constructions and the previous works on partially/fullly stuck cells in Table III.

Further, we have shown that it can be advantageous to introduce errors in some partially stuck cells in order to satisfy the stuck-at constraints. For the general case that is applicable for all of our constructions, we have shown in Theorem 6 how to replace any \( 0 \leq j \leq t \) errors by \( j \) masked partially stuck cells. This theorem has been improved for Construction 3 by Lemma 1, and further enhanced by another method for introducing errors in the partially stuck locations through Lemma 2 (cf. Example 5). We gain (e.g., for \( j = 1 \)) exactly \( 2^{n-1} \) and \( 2^n \) (under the condition that \( 2(t + \lceil \frac{u}{q} \rceil) < d \)) additional masked partially stuck cells applying Lemma 1 and Lemma 2, respectively. So far, determining if introducing errors in partially stuck cells is advantageous or not can only be done numerically.

We also derived upper and lower limits on the size of our constructions. Our sphere-packing-like bound for the size of \((\Sigma, t)\) PSMCs has been compared to the usual sphere-packing upper bound, and for the case of no errors \((t = 0)\) to [5, Theorem 2].

We have numerically compared our Gilbert–Varshamov-type bounds, for given \((u, t)\), to each other and to \((q - 1)\)-ary codes. For \( u < q - 1 \), Theorem 9 states the existence of \((u, 1, t)\) PSMCs with rates that almost match the ones from the usual \(q\)-ary GV bound (shown in Figure 3). Moreover, up to \( u = 20 \) for \( q = 8 \), Figure 6 shows that application of Theorem 9 is better than using \((q - 1)\)-ary code as mentioned in [5, Section III]. On the other hand, for \( q = 4 \) and \( u = 10 \), Theorem 10 and Theorem 11 obviously require less redundancies than \((q - 1)\)-ary code as shown in Figure 5.

Figures 8 and 9 demonstrate the application of Theorem 6, Lemma 1 and Lemma 2 on \((u, 1, t)\) PSMCs of rates that have been obtained based on Theorem 10 and Theorem 11. For some parameters (i.e. \( u = 16, t = 41 \) as shown in Table IV and \( u = 19, t = 31 \) as shown in Table V), application of Theorem 6 and Lemma 1 achieve higher code rates and more masked cells. Application Lemma 2 on a code obtained by Theorem 9 (i.e. \( u = 7, t = 27 \)) provides higher code rate compared to the direct employment of Theorem 11, Theorem 6 and Lemma 1.

In the asymptotic regime of our GV-like bounds, Theorem 13 and Theorem 15 are remarkable competitors to Theorem 14. However, the analytical comparison between Theorem 13 and Theorem 15 is more complicated to decide which one is the better choice. This was also confirmed numerically via Figure 7.

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APPENDIX

A. Variant of Construction 1 \((u < q)\) for Cyclic Codes

This section provides an alternative of Construction 1 by generalizing the construction of [13, Theorem 2]. We use the so-called partitioned cyclic codes from [13] as basic idea, but we require only a single redundancy symbol \(l = 1\) for the masking operation similar to [5, Theorem 4 and Algorithm 3]. Compared to Construction 1, Construction 4 directly implies a constructive strategy on how to choose a cyclic code of a certain minimum distance. In this appendix, we will use the following notation. If \(C\) is a cyclic code, it has a generator polynomial \(g(x)\) of degree \(n - k\) with roots in \(\mathbb{F}_q\), where \(n\) divides \(q^n - 1\). The defining set \(D_c\) of \(C\) is the set containing the indices \(b\) of the zeros \(\alpha^b\) of the generator polynomial \(g(x)\), i.e.,

\[ D_c := \{ b : g(\alpha^b) = 0 \}. \]

The minimum distance \(d\) of \(C\) can be bounded from below by the BCH bound \(d\) or more involved bounds such as the Hartmann-Tzeng bound [29] or the Roos bound [30].

Construction 4. Let \(u \leq \min\{n, q - 1\}\). Assume there is an \([n, k, \delta \geq 2t + 1]\) \(q\) cyclic code \(C\) with a generator polynomial \(g(x)\) of degree \(n - k\) that divides \(g_0(x) := 1 + x + x^2 + \cdots + x^{n-1}\). Encoder and decoder are given in Algorithms 9 and 10.

Theorem 17. If \(u \leq \min\{n, q - 1\}\), Construction 4 provides an \((n, M = q^{k-1})\) \((u, 1, t)\)-PSMC with redundancy of \(n - k + 1\) symbols.

Algorithm 9: Encoding

**Input:**
- Message: \(m(x) \in \mathbb{F}_q[x]\) of degree \(k - 1\)
- Positions of partially stuck cells: \(\phi\)

1. \(w(x) = w_0 + \cdots + w_{n-2x^{n-2}} \leftarrow m(x) \cdot g(x)\)
2. Select \(v \in \mathbb{F}_q\setminus\{w_i \mid i \in \phi\}\)
3. \(e(x) = w(x) - v \cdot g_0(x) \mod (x^n - 1)\)

**Output:** Codeword \(e(x) \in \mathbb{F}_q[x]\) of degree \(\leq n - 1\)

Algorithm 10: Decoding

**Input:** Retrieve \(g(x) = e(x) + e(x)\), where \(e(x) \in \mathbb{F}_q[x]\) of degree \(\leq n - 1\) is the error polynomial

1. Decode \(g(x)\) in the code generated by \(g(x)\)
2. \(\hat{m}(x) \leftarrow \hat{e}(x) \mod g_0(x)\)

**Output:** Message \(\hat{m}(x) \in \mathbb{F}_q[x]\) of degree \(k - 1\)

Proof. A cyclic code of length \(n\) contains the all-one word \(1 + x + \cdots + x^{n-1}\). Thus, Construction 4 follows directly from Theorem 1, but with different encoding and decoding algorithms. Algorithm 9 shows the encoding process for the cyclic code construction. Step 1 in Algorithm 9 calculates \(w(x)\) of degree \(n - 1\). Since \(u < q\), there is at least one \(v \in \mathbb{F}_q\) such that all coefficients of \(w(x)\) are unequal to \(v\). Therefore, after Step 3, \(c_{n-1} = -v\). The requirement for masking, see (1) is satisfied for \(c(x)\) since \(c_i = (w_i - v) \in \mathbb{F}_q \neq 0\).

Algorithm 10 decodes the retrieved polynomial \(g(x)\). First, decode \(g(x)\) in the code generated by \(g(x)\). Second, the algorithm performs the unmasking process to find \(\hat{m}(x)\).

\[
\hat{c}(x) = \hat{m}(x) \cdot g(x) + z_0 \cdot g_0(x) \\
\hat{m}(x) = \frac{\hat{w}(x) \mod g_0(x)}{g(x)} = m(x).
\]

Construction 4 provides an explicit cyclic construction that can mask \(u < q\) cells and correct \(t\) errors. If we use a BCH code in Construction 4, we can bound the minimum distance of the code \(C\) by the BCH bound. This is done in Tables I and II.

B. An alternative Proof of Proposition 2

We start with Lemma 11.

Lemma 11. Let \(M \in \mathbb{F}_q^{n \times n}\) be such that each column of \(M\) has at least one non-zero entry. Let \(s \in \mathbb{F}_q^n\). For each \(w \in \mathbb{F}_q^n\), there is a \(v \in \mathbb{F}_q^n\) such that

\[
\left\{ i \in [n] \mid w_i + (vM)_i \geq s_i \right\} \geq n - \frac{1}{q} \sum_{i=0}^{n-1} s_i.
\]

**Proof.** We define the set \(S\) as

\[
S = \left\{ i \in [n] \mid x_i + (vM)_i \geq s_i \right\}.
\]

Clearly, there is \(v \in \mathbb{F}_q^n\) such that

\[
\left\{ i \in [n] \mid w_i + (vM)_i \geq s_i \right\} \geq \frac{|S|}{q^n}. \tag{18}
\]

Let \(i \in [n]\). As the \(i\)-th column of \(M\) has a non-zero entry, for each \(y \in \mathbb{F}_q\) there are exactly \(q^{m-1}\) vectors \(x \in \mathbb{F}_q^n\) such that \((xM)_i = y\). As a consequence,

\[
\left\{ v \in \mathbb{F}_q^n \mid w_i + (vM)_i \geq s_i \right\} = (q - s_i)q^{m-1},
\]

and so

\[
|S| = \sum_{i=0}^{n-1} (q - s_i)q^{m-1} = nq^m - q^{m-1} \sum_{i=0}^{n-1} s_i. \tag{19}
\]

The lemma follows from combining (18) and (19).

We are now in a position to introduce an alternative non-constructive proof for Proposition 2.

Let \(s \in \Sigma\). In order to simplify notation, we assume without loss of generality that \(\sum_{i=d_0-2}^{n-1} s_i \leq q - 1\). Let \(w \in \mathbb{F}_q^n\). We wish to find \(z \in \mathbb{F}_q^n\) such that \(w_i + (zH_0)_i \geq s_i\) for many indices \(i\).

As the \(d_0 - 2\) leftmost columns of \(H_0\) are independent, there exists an invertible matrix \(T \in \mathbb{F}_q^{\times n}\) such that

\[
TH_0 = \begin{bmatrix} I_{d_0-2} & A \\ 0 & B \end{bmatrix},
\]
where $I_{d_0-2}$ denotes the identity matrix of size $d_0 - 2$. For $i \in [d_0 - 2]$, we choose $\zeta_i = s_i - w_i$ and write
\[ v = w + \zeta \cdot (I_{d_0-2} | A). \]
By definition, $v_i = s_i$ for all $i \in [d_0 - 2]$. As any $d_0 - 1$ columns of $TH_0$ are independent, no column of $B$ consists of only zeroes. Lemma 11 implies that there is an $\eta \in \mathbb{F}_q^{l-d_0+2}$ such that
\[ \{i \in [d_0 - 2, n - 1] \mid w_i + (\eta B)_{i+d_0-2} \geq s_i \} \geq n - d_0 + 2 - \left\lfloor \frac{n-1}{q} \sum_{i=d_0-2}^{d_0-1} s_i \right\rfloor. \]

Combining this with the fact that $v_i = s_i$ for all $i \in [d_0 - 2]$, we infer that for all indices $i \in [n]$, $w_i + ((\zeta_i, \eta)TH_0)_{i} \geq s_i$.

C. Proof of Lemma 8

Let $s \geq 1$. As $\mathbb{F}_q \subseteq \mathbb{F}_q^r$, it is clear that $d_1 \geq d_s$. To show the converse, we use the trace function $T$ defined as $T(x) = \sum_{i=0}^{r} x_i$. As is well-known, $T$ is a non-trivial mapping from $\mathbb{F}_q^r$ to $\mathbb{F}_q$, and
\[ T(ax + by) = aT(x) + bT(y), \]
for all $x, y \in \mathbb{F}_q^r$ and $a, b \in \mathbb{F}_q$. We extend the trace function to vectors by applying it coordinate-wise.

Let $n \in \mathbb{F}_q^k \setminus \{0\}$. We choose $\lambda \in \mathbb{F}_q^t$ such that $T(\lambda \cdot m) \neq 0$. As $T(0) = 0$, we infer that $wt(mG) = wt(\lambda \cdot mG) \geq wt(T(\lambda \cdot mG))$. As all entries from $G$ are in $\mathbb{F}_q$, it follows from (20) that $T(\lambda \cdot mG) = \lambda \cdot (\cdot mG)$. As a consequence,
\[ wt(mG) \geq wt(T(\lambda \cdot mG)) = wt(T(\lambda \cdot mG)) \geq d_1, \]

D. Proofs of analytical comparisons of the asymptotic GV bounds

We will use the following lemma.

**Lemma 12.** Let $q \geq 2$ be an integer. If $0 \leq x, y$ are such that $x + y \leq 1$, then
\[ h_q(x + y) \leq h_q(x) + h_q(y). \]

**Proof.** Let $0 \leq y < 1$, and consider the function $f_y(x) = h_q(x + y) - h_q(x) - h_q(y)$ on the interval $[0, 1 - y]$. Clearly, $f_y'(x) = h_q'(x + y) - h_q'(x) \leq 0$, where the inequality follows from the fact that the second derivative of $h_q$ is non-negative. Hence, $f_y(x) \leq f_y(0) = 0$ for each $x \in [0, 1 - y]$. \(\square\)

1) **Proof of Proposition 5:** Assume $\tau$ and $v$ are such that the conditions of Theorem 13 and Theorem 14 are satisfied, that is, such that $2\tau + \frac{\mu}{q} \leq 1 < 4$ and $h_q(\mu) + h_q(2\tau) < 1$. By invoking Lemma 12, we see that
\[ h_q(2\tau + \frac{\mu}{q}) \leq h_q(2\tau) + h_q(2 \cdot \frac{\mu}{q}) \leq h_q(2\tau) + h_q(\mu), \]
where the final inequality holds as $q \geq 2$ and $h_q$ is monotonically increasing on $[0, 1 - \frac{1}{q}]$. As a consequence, the code rate guaranteed by Theorem 13 is at least equal to the code rate guaranteed by Theorem 14.

2) **Proof of Proposition 6:** Assume that the conditions of Theorem 14 and Theorem 15 are satisfied. The difference between the rate of Theorem 15 and of Theorem 14 equals
\[ h_2(v) - \frac{1}{\mu} h_2(\frac{v}{2\mu - 1}). \]

According to the conditions of Theorem 14, $v \leq 1 - \frac{1}{2\mu}$. Hence, $h_2(v) \geq h_2(\frac{v}{2\mu - 1})$. As $h_2(x) = \frac{1}{\mu} h_2(x) + x \log_2(2^\mu - 1)$, the difference in (21) is non-negative. That is, Theorem 15 is better than Theorem 14.

3) **Comparing Theorem 13 and Theorem 15:** Assume that $\tau$ and $v$ are such that the conditions of Theorem 13 and of Theorem 15 are satisfied, that is, $2\tau + \frac{\mu}{q} < 1 < 4 - \frac{1}{2\mu}$.

Let $f_{\mu}(\tau, v_0)$, where $v_0 = \frac{v}{2\mu - 1}$, be the bound from Theorem 13 minus the bound from Theorem 15, that is
\[ f_{\mu}(\tau, v_0) = h_2(2\tau) + \frac{1}{\mu} h_2(v_0) - h_2(2\tau + v_0). \]

The definition of the entropy function implies that for any $x \in [0, 1]$,
\[ h_2(x) = \frac{1}{\mu} (h_2(x) + x \log_2(2^\mu - 1)). \]

Applying (22), we infer that
\[ \mu f_{\mu}(\tau, v_0) = h_2(2\tau + v_0) - h_2(2\tau + v_0) - v_0 \log_2(2^\mu - 1). \]

In particular, $\mu f_{\mu}(0, v_0) = -v_0 \log_2(2^\mu - 1) \leq 0$.

So for $\tau = 0$, Theorem 15 is better than Theorem 13. It follows from Lemma 12 that the three leftmost terms in (23) form a non-negative number. The subtraction of the fourth term, however, can result in a negative function value, especially for large $\mu$.

**Example 6** (Numerical example). $\mu f_{\mu}(0.055, 0.11) = 2 \cdot h_2(0.11) - h_2(0.22) - 0.11 \log_2(2^\mu - 1) \approx 0.23397 - 0.11 \log_2(2^\mu - 1)$ is positive for $\mu \leq 2$ and negative otherwise.

We now prove Proposition 7. That is, we show that for $\tau > 0$ and $v_0$ sufficiently small, $f_{\mu}(\tau, v_0) > 0$. This follows from the Taylor expansion of $f_{\mu}(\tau, v_0)$ around $v_0 = 0$. Indeed, $f_{\mu}(\tau, 0) = 0$, and $h_2'(x) \to \infty$ if $x \downarrow 0$.\(\square\)