Heat exchange and fluctuation in Gaussian thermal states in the quantum realm

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Abstract. The celebrated exchange fluctuation theorem – proposed by Jarzynski and Wójcik, (Phys. Rev. Lett. 92, 230602 (2004)) for heat exchange between two systems in thermal equilibrium at different temperatures – is explored here for quantum Gaussian states in thermal equilibrium. We employ Wigner distribution function formalism for quantum states, which exhibits close resemblance with the classical phase-space trajectory description, to arrive at this theorem. For two Gaussian states in thermal equilibrium at two different temperatures kept in contact with each other for a fixed duration of time we show that the quantum Jarzynski-Wójcik theorem agrees with the corresponding classical result in the limit $\hbar \to 0$.

Keywords: heat exchange statistics, exchange-fluctuation theorem, Gaussian states, variance matrix, Wigner function, sympletic transformation
1. Introduction

Fluctuation theorems [11, 24, 3, 4, 5] are of fundamental significance in non-equilibrium statistical physics. They correspond to a collection of exact relations, which remain valid even when the system is driven far away from equilibrium. Various exchange-fluctuation theorems (XFT) involving thermodynamic quantities like work, heat, entropy have been proposed during the last two decades [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25]. They have offered significant insights in understanding thermodynamical processes – especially the emergence of irreversibility from reversible dynamics and the directionality of heat flow implied by second law of thermodynamics. Some of these relations are applicable for systems in non-equilibrium steady state [16], while others hold in the transient regime. Fluctuation relations with underlying Hamiltonian dynamics [6, 14] as well as stochastic dynamics [11, 13, 15, 17] have also been proposed. There are ongoing efforts to generalize and broaden the applicability of XFT in the quantum scenario [9, 11, 19, 22, 24, 25].

The fluctuation-exchange relations can be considered as generalizations of second law of thermodynamics for small systems and they connect the probabilities of appearance of physical quantities such as work, heat, number of particles, in an experimental set up, to those obtainable in a time-reversed set up. For instance, the Jarzynski-Wózcik fluctuation theorem (XFT) [5] given by,

\[
\ln \left[ \frac{p_\tau(\mathcal{Q})}{p_\tau(-\mathcal{Q})} \right] = \Delta \beta \mathcal{Q}, \quad \Delta \beta = \frac{1}{k T_B} - \frac{1}{k T_A} \quad (1)
\]

quantifies the ratio of probability \( p_\tau(\mathcal{Q}) \) of heat exchange during interaction of \( A \) and \( B \) for a fixed time duration \( \tau \), to its time-reversed counterpart \( p_\tau(-\mathcal{Q}) \). Here \( k \) denotes Boltzmann constant and \( \mathcal{Q} \) denotes the amount of heat exchanged.

The Jarzynski-Wózcik theorem [5] is one among the important XFTs and has drawn much attention. Microreversibility and strict directionality of thermodynamical heat flow form the foundational features of this XFT relation. Generalizations of the theorem to include processes involving a system coupled to reservoirs [20], a chain of interacting particles connecting two heat baths [16], correlated thermal quantum systems [19, 25] and the like have been carried out.

In the classical scenario Jarzynski and Wózcik [5] had employed phase-space description, for the forward and reverse dynamical evolution between statistical systems \( A \) and \( B \) in thermal equilibriums at temperatures \( T_A \) and \( T_B \) respectively, so that one gains physical intuition underlying the relation between heat exchange and fluctuations. In the quantum regime, they considered systems with discrete energy levels to arrive at the relation (1).

In the present work, we retain the flavour of the phase-space approach in the quantum scenario, by confining ourselves to continuous variable Gaussian thermal states. Following similar lines as that of the original work [5] we arrive at the heat exchange-fluctuation theorem in the quantum realm for two Gaussian states in thermal equilibrium at temperatures \( T_A, T_B \) kept in contact with each other. What comes to our aid here
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is the fact that the Wigner distribution function, characterizing Gaussian states is non-negative [29] and hence, it serves as a legitimate quantum counterpart of phase-space probability distribution. This enables us to carry out explicit evaluations and arrive at the Jarzynski-Wózcik XFT in this case.

We have organized our paper as follows: In Section 2 a brief outline of Jarzynski-Wózcik derivation of heat transfer and fluctuation relation (1) is presented. Necessary mathematical preliminaries on infinite dimensional continuous variable Gaussian systems, their characterization in terms of the variance matrix and obtaining the Wigner distribution function in terms of the variance matrix are given in Section 3. In Section 4, we derive the Jarzynski-Wózcik heat exchange-fluctuation theorem for Gaussian systems A and B, in thermal equilibrium at temperatures $T_A$, $T_B$ respectively using the Wigner distribution function approach. Time reversal symmetric canonical transformations of phase-space observables is employed to identify explicit forms of forward and backward heat probability distributions using the Wigner distribution function associated with Gaussian thermal states (or equivalently, quantum harmonic oscillator system in thermal equilibrium). Discussions on heat exchange statistics of quantum and classical harmonic oscillators in thermal equilibrium, physical status of the Wigner-Weyl phase-space trajectory framework and interpretation of the quantum-to-classical reduction in the limit $\hbar \to 0$, possible connection of Jarzynski-Wózcik XFT with energy equipartition theorem in the quantum scenario are presented in Section 5.

2. Classical phase-space description for Jarzynski-Wózcik heat exchange-fluctuation relation

Jarzynski and Wózcik considered two systems, phase-space evolution of which is governed by Hamiltonians $H_A(\xi_A)$ and $H_B(\xi_B)$; $\xi_A$, $\xi_B$ denoting phase-space variables (e.g., positions and momenta) of systems A and B respectively. The systems are kept in contact with each other for a time duration $\tau$ via an interaction characterized by $H_{\text{int}}(\xi_A, \xi_B)$, which is switched ‘on’ at time $t = 0$, and turned ‘off’ at $t = \tau$. The phase-space trajectory of the two systems is denoted collectively by $\xi^t = (\xi^t_A, \xi^t_B)$. Both the systems are initially in thermal equilibrium, at temperatures $T_A$, $T_B$ respectively, and their phase-space probability distributions at time $t = 0$ is given by

$$p(\xi^0) = \frac{e^{-H_A(\xi^0_A)/kT_A}e^{-H_B(\xi^0_B)/kT_B}}{Z_A Z_B} \tag{2}$$

where $Z_A$, $Z_B$ denote partition functions. Phase-space dynamics of the systems is assumed to be time-reversal symmetric i.e.,

$$H_A(\xi_A) \to H_A(\xi^*_A) = H_A(\xi_A)$$
$$H_B(\xi_B) \to H_B(\xi^*_B) = H_B(\xi_B)$$
$$H_{\text{int}}(\xi) \to H_{\text{int}}(\xi^*) = H_{\text{int}}(\xi) \tag{3}$$

where time reversal operation is denoted by the superscript symbol ($^*$). In other words, for every legitimate forward trajectory $\xi^0$ to $\xi^\tau$, there exists a time-reversed trajectory
Let $\xi^0 = \xi^{\tau*}$ to $\xi^\tau = \xi^{0\tau}$. Their likelihood ratio is given by (see (2))
\[
\frac{p(\xi^0)}{p(\xi^0)} = e^{(H_A(\xi^0_A) - H_A(\xi^0_A))/kT_A} e^{(H_B(\xi^0_B) - H_B(\xi^0_B))/kT_B}
= e^{\Delta E_A/kT_A} e^{\Delta E_B/kT_B}
\] (4)

where $\Delta E_A = H_A(\xi_A^{\tau*}) - H_A(\xi_A^0)$, $\Delta E_B = H_B(\xi_B^{\tau*}) - H_B(\xi_B^0)$ denote change of internal energies of systems $A$ and $B$ respectively. Assuming that the interaction term $H_{\text{int}}$ is negligible, it is seen that $H_A(\xi_A^0) + H_B(\xi_B^0) \approx H_A(\xi_A^{\tau*}) + H_B(\xi_B^{\tau*})$ or $\Delta E_A \approx -\Delta E_B$. Net energy change during the interaction represents the amount of heat transferred i.e., $Q = \Delta E_B \approx -\Delta E_A$. The heat transfer $Q$ from $A$ to $B$ during forward process gets compensated by that in the reverse process from $B$ to $A$ and is expressed by
\[
Q(\xi^0) = -Q(\xi^\tau).
\] (5)

Thus, it is seen that
\[
\frac{p(\xi^0)}{p(\xi^0)} = e^{\Delta \beta Q(\xi^0)}.
\] (6)

From (5) and (6) it follows that
\[
\begin{align*}
p_{r}(Q) &= \int d\xi^0 p(\xi^0) \delta(\xi^0 - Q) \\
p_{r}(-Q) &= \int d\xi^0 p(\xi^0) \delta(\xi^0 + Q) \\
&= e^{-\Delta \beta Q} p_{r}(Q) \\
\Longrightarrow \quad \frac{p_{r}(Q)}{p_{r}(-Q)} &= e^{\Delta \beta Q}.
\end{align*}
\] (7)

thus proving the Jarzynski-Wójcik heat exchange fluctuation theorem in the classical scenario.

In the quantum realm Jarzynski and Wójcik considered two discrete level systems prepared initially in thermal equilibrium at temperatures $T_A$, $T_B$ and measure their energies $E^A_i$, $E^B_i$; the systems are allowed to interact weakly for a time duration $\tau$ after interaction is turned off and energies of both the systems $E^A_f$, $E^B_f$ measured. As the systems are allowed to interact weakly it is expected that the total energy of the system is conserved: $E^A_i + E^B_i \approx E^A_f + E^B_f$. Heat transfer is then interpreted as $Q_{i\rightarrow f} = E^B_i - E^B_f \approx E^A_f - E^A_i$ resulting in the relation
\[
\ln \left[ \frac{p \left( |i\rangle \xrightarrow{\tau} |f\rangle \right)}{p \left( |f\rangle \xrightarrow{\tau} |i\rangle \right)} \right] = \Delta \beta Q_{i\rightarrow f}.
\] (8)

Our interest here is to derive the relation (7) describing heat transfer processes in the forward and the time-reversed dynamics of quantum Gaussian system consisting of two subsystems $A$, $B$, prepared initially in thermal equilibrium at temperatures $T_A$, $T_B$ respectively. Wigner distribution function formalism [29] is employed in this approach. To this end, we give necessary mathematical preliminaries on the quantum phase-space description and symplectic evolution, Gaussian thermal states and the associated Wigner distribution function in Section 3.
3. Gaussian states – description through variance matrix and Wigner function

Gaussian states, the most important among continuous variable states [26, 27], find diverse applications in several fields including quantum stochastic processes and open system dynamics [28]. They naturally occur as the thermal equilibrium states of any physical system in the small oscillations limit [28]. Being fully characterized by its first and second moments Gaussian states are simpler to handle among the continuous variable states.

Any arbitrary two-mode Gaussian state $\hat{\rho}_{AB}$ is characterized completely by its first and second order moments, written concisely in the form of a $4 \times 4$ covariance matrix $V$ (referred to as variance matrix from now on), which is defined in terms of its elements as [26, 27]

$$V_{ij} = \frac{1}{2} \left\{ \langle \hat{\xi}_i \hat{\xi}_j \rangle - \langle \hat{\xi}_i \rangle \langle \hat{\xi}_j \rangle \right\}, \quad i, j = 1, 2, 3, 4. \tag{9}$$

Here $\hat{\xi}$ denotes a $4 \times 1$ column $\hat{\xi}$ with positions and momenta (dimensionless) as its components:

$$\hat{\xi} = (\hat{q}_A, \hat{p}_A, \hat{q}_B, \hat{p}_B)^T, \tag{10}$$

where ‘T’ stands for the transpose operation; we have denoted $\{ \hat{\xi}_i, \hat{\xi}_j \} = \hat{\xi}_i \hat{\xi}_j + \hat{\xi}_j \hat{\xi}_i$ and we have denoted $\langle \cdots \rangle = \text{Tr} (\hat{\rho} \cdots)$ in (9). The canonical phase-space variables $\hat{q}_\alpha, \hat{p}_\alpha, \alpha, \beta = A, B$ satisfy the Bosonic commutation relations,

$$[\hat{q}_\alpha, \hat{p}_\beta] = 0, \quad [\hat{p}_\alpha, \hat{p}_\beta] = 0, \quad \text{and}$$
$$[\hat{q}_\alpha, \hat{p}_\beta] = i \delta_{\alpha\beta}; \tag{11}$$

where $\delta_{\alpha\beta} = 1$ when $\alpha = \beta$ and zero when $\alpha \neq \beta$, is the Kronecker delta function. In terms of the components $\hat{\xi}_i, i = 1, 2, 3, 4$, the canonical commutation relations (11) assume the form

$$[\hat{\xi}_i, \hat{\xi}_j] = i \Omega_{ij}, \quad i, j = 1, 2 \tag{12}$$

where $\Omega_{ij}$ denote elements of the $4 \times 4$ matrix

$$\Omega = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}. \tag{13}$$

The commutation relations (see (11), (12)) remain invariant under a symplectic transformation [26, 27]:

$$S \Omega S^T = \Omega. \tag{14}$$

The set of all $4 \times 4$ real matrices $S$ satisfying the property (14) constitutes the symplectic group of real linear canonical transformations Sp(4,R) [26, 27].
The variance matrix $V$ of a two-mode quantum system given explicitly by

$$V = \begin{pmatrix}
\langle q_A^2 \rangle & \frac{1}{2} \{\{q_A, p_A\}\} & \langle q_A q_B \rangle & \langle q_A p_B \rangle \\
\frac{1}{2} \{\{q_A, p_A\}\} & \langle p_A^2 \rangle & \langle p_A q_B \rangle & \langle p_A p_B \rangle \\
\langle q_A q_B \rangle & \langle q_B p_A \rangle & \langle q_B^2 \rangle & \frac{1}{2} \{\{q_B, p_B\}\} \\
\langle q_A p_B \rangle & \langle p_A p_B \rangle & \frac{1}{2} \{\{q_B, p_B\}\} & \langle p_B^2 \rangle 
\end{pmatrix}$$  \hspace{1cm} (15)

is a real symmetric positive definite matrix and it completely characterizes a two mode Gaussian state $\hat{\rho}_{AB}$. Under the symplectic transformation $\hat{\xi}' = S \hat{\xi}$, the variance matrix $V$ undergoes a congruent transformation $V' = SVS^T$, where $V'$ is the variance matrix associated with the new canonical variables $\hat{q}'_\alpha, \hat{p}'_\alpha, \alpha, \beta = A, B$.

From the fundamental theorem due to Williamson [30] it follows that the variance matrix $V$ attains a canonical form under symplectic transformation $S_W$ such that

$$V_W = S_W V S_W^T = \text{diag} (\nu_A, \nu_A; \nu_B, \nu_B)$$  \hspace{1cm} (16)

and $V_W$ is referred to as the *Williamson normal form* of the variance matrix and $\nu_A, \nu_B$ are called the symplectic eigenvalues of the variance matrix. The real positive diagonal elements $\nu_\alpha, \alpha = A, B$ of $V_W$ are the positive square roots of the doubly degenerate eigenvalues of the matrix $-(V\Omega)^2$ as,

$$-S_W (V\Omega)^2 S_W^{-1} = -(V_W\Omega)^2 = \text{diag} (\nu_A^2, \nu_B^2).$$  \hspace{1cm} (17)

Corresponding to the symplectic transformation $S_W$, there exists a unitary operator $U(S_W)$ transforming the density matrix $\hat{\rho}_{AB}$ of a two mode Gaussian state as follows:

$$\hat{\rho}_{AB} = U^\dagger(S_W) (\hat{\rho}_{\nu_A} \otimes \hat{\rho}_{\nu_B}) U(S_W)$$  \hspace{1cm} (18)

where the single mode density matrices $\hat{\rho}_{\nu_A}, \hat{\rho}_{\nu_B}$ are given by [28]

$$\hat{\rho}_{\nu_\alpha} = \frac{1}{\nu_\alpha + \frac{1}{2}} \sum_{n_\alpha=0}^{\infty} \left( \frac{\nu_\alpha - \frac{1}{2}}{\nu_\alpha + \frac{1}{2}} \right)^{n_\alpha} |n_\alpha\rangle \langle n_\alpha|, \quad \alpha = A, B.$$  \hspace{1cm} (19)

Here $|n_A\rangle, |n_B\rangle$ are the eigenstates of the number operators $\hat{N}_A = \hat{a}_A^\dagger \hat{a}_A$, $\hat{N}_B = \hat{a}_B^\dagger \hat{a}_B$ of the modes $A, B$ and $\hat{a}_\alpha, \hat{a}_\alpha^\dagger$ are related to the dimensionless canonical position and momentum observables as follows:

$$\hat{a}_\alpha = \frac{\hat{q}_\alpha + i \hat{p}_\alpha}{\sqrt{2}}, \quad \hat{a}_\alpha^\dagger = \frac{\hat{q}_\alpha - i \hat{p}_\alpha}{\sqrt{2}}.$$  \hspace{1cm} (20)

Let us denote $\hat{Q}_\alpha = \left( \frac{\hbar}{m_\alpha \omega_\alpha} \right)^{1/2} \hat{q}_\alpha$, $\hat{P}_\alpha = (m_\alpha \omega_\alpha \hbar)^{1/2} \hat{p}_\alpha$. Given the Hamiltonian of a harmonic oscillator of mass $m_\alpha$, frequency $\omega_\alpha$,

$$\hat{H}_\alpha = \frac{\hat{P}_\alpha^2}{2m_\alpha} + \frac{1}{2} m_\alpha \omega_\alpha^2 \hat{Q}_\alpha^2$$

$$= \left( \hat{N}_\alpha + \frac{1}{2} \right) \omega_\alpha,$$  \hspace{1cm} (21)

a canonical ensemble of Bosonic oscillators in thermal equilibrium at temperature $T_\alpha$ is described by

$$\hat{\rho}_{T_\alpha} = \frac{e^{-\hat{H}_\alpha/k T_\alpha}}{Z_\alpha} = \frac{e^{-h \omega_\alpha / 2k T_\alpha}}{Z_\alpha} \sum_{n_\alpha=0}^{\infty} e^{-n_\alpha h \omega_\alpha / k T_\alpha} |n_\alpha\rangle \langle n_\alpha|$$  \hspace{1cm} (22)
where
\[ Z_\alpha = \text{Tr}[e^{-\hat{H}/kT_\alpha}] = \text{Tr}\left[e^{\frac{-\hbar \omega_\alpha}{2kT_\alpha} (\hat{N}_\alpha + \frac{1}{2})}\right] \]
\[ = \frac{e^{-\hbar \omega_\alpha/2kT_\alpha}}{1 - e^{-\hbar \omega_\alpha/kT_\alpha}} \]  
(23)
denotes the partition function. The single mode thermal state
\[ \hat{\rho}_T = \left(1 - e^{-\hbar \omega_\alpha/kT_\alpha}\right) \sum_{n_\alpha=0}^{\infty} e^{-n_\alpha \hbar \omega_\alpha/kT_\alpha} |n_\alpha\rangle \langle n_\alpha| \]
may be readily identified with the canonical single mode Gaussian state \(\rho_\alpha\) of (19) appearing in the Williamson canonical decomposition (18).

The variance matrix \(V\) of any two-mode thermal state \(\hat{\rho}_{AB} = \hat{\rho}_T \otimes \hat{\rho}_T\) is given by [27]
\[
V = \begin{pmatrix} V_A & 0 \\ 0 & V_B \end{pmatrix}
\]
\[ V_A = \frac{1}{2} \coth \left( \frac{\hbar \omega_A}{2kT_A} \right) 1_2, \quad V_B = \frac{1}{2} \coth \left( \frac{\hbar \omega_B}{2kT_B} \right) 1_2 \]  
(24)
where \(1_2\) denotes \(2 \times 2\) identity matrix. Note that \(V\) is in the Williamson normal form, with its symplectic eigenvalues related to the frequency and temperature of the thermal state of oscillator systems \(A\) and \(B\) as,
\[ \nu_{T_\alpha} = \frac{1}{2} \coth \left( \frac{\hbar \omega_\alpha}{2kT_\alpha} \right) , \quad \alpha = A, B. \]  
(25)

In the next subsection we give an outline of some preliminary notions on quantum phase-space formalism in terms of Wigner distribution functions.

3.1. Wigner distribution function of a Gaussian state

Wigner distribution function plays a central role in developing quantum phase-space formalism involving non-commuting canonical observables \(\hat{q}_\alpha, \hat{p}_\alpha\). Wigner representation of a quantum system in phase-space allows one to explore the connection between quantum and classical formalisms. In this subsection we outline some preliminary notions on the Wigner function associated with single mode continuous variable Gaussian quantum system.

Wigner function \(W(q, p)\) of a single mode continuous variable quantum state \(\hat{\rho}\) is a real function of phase-space canonical variables \(q, p\) defined by [29]
\[
W(q, p) = \frac{1}{\pi \hbar} \int_{-\infty}^{\infty} dx \langle q - x | \hat{\rho} | q + x \rangle e^{2i\pi px/\hbar} \]  
(26)
where \(|q\rangle\) denotes eigenvector of the operator \(\hat{q}\); it satisfies the normalization property [29]
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dq dp W(q, p) = \text{Tr}(\hat{\rho}) = 1 \]  
(27)
which is true of any probability distribution and gives correct marginal probability distributions:

\[
\int_{-\infty}^{\infty} dp W(q, p) = \langle q | \hat{\rho} | q \rangle, \quad \int_{-\infty}^{\infty} dq W(q, p) = \langle p | \hat{\rho} | p \rangle
\]

Quantum expectation value of any operator \( \hat{f}(\hat{q}, \hat{p}) \) in a state \( \hat{\rho} \) can be replaced by a phase-space integration using Wigner function \( W(q, p) \) as,

\[
\langle \hat{f}(\hat{q}, \hat{p}) \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dq \; dp \; W(q, p) \; f(q, p). \tag{29}
\]

where Weyl’s correspondence rule

\[
q^k p^l \longleftrightarrow \frac{1}{2^k} \sum_{r=0}^{k} \frac{k!}{r! (k-r)!} \hat{q}^r \hat{p}^{k-r}
\]

(30)

to associate classical functions with quantum operators has been employed [29]. The classical functions \( f(q, p) \) are expressed as the Weyl transforms of the corresponding operators \( \hat{f}(\hat{q}, \hat{p}) \) as,

\[
f(q, p) = \int_{-\infty}^{\infty} dy \; \langle q - y/2 | \hat{f}(\hat{q}, \hat{p}) | q + y/2 \rangle \; e^{-i\pi p y/\hbar}
\]

so that [29]

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dq \; dp \; f(q, p) \; g(q, p) = (2\pi \hbar) \; \text{Tr} \left[ \hat{f}(\hat{q}, \hat{p}) \hat{g}(\hat{q}, \hat{p}) \right]
\]

(32)

holds for the Weyl transforms of the operators \( \hat{f}(\hat{q}, \hat{p}) \) and \( \hat{g}(\hat{q}, \hat{p}) \).

While the Wigner-Weyl formalism allows phase-space description (see (27), (28), (29)) of quantum theory in a classical language, the weight function \( W(q, p) \) is not necessarily non-negative everywhere and hence it is termed as quasi-probability distribution function [29].

Interestingly, the Wigner function associated with single mode quantum Gaussian states given by [27]

\[
W(\xi) = \frac{1}{2\pi \sqrt{\text{det} V}} \; e^{-\frac{1}{2} \xi^T V^{-1} \xi}
\]

(33)

is non-negative everywhere. Here, \( V \) is the variance matrix of the Gaussian state and phase-space canonical variables are expressed compactly in the form of a column \( \xi = (q, p)^T \). The Wigner function of a single mode Gaussian state \( \hat{\rho}_T = e^{-\hat{H}/\hbar T} / Z \) in thermal equilibrium at temperature \( T \) takes the following form (by substituting the variance matrix of a thermal state (see [24]) and after simplification):

\[
W(q, p) = \frac{1}{2\pi \nu_T} \exp \left[ -\frac{1}{2 \nu_T} (q^2 + p^2) \right] \tag{34}
\]

where \( H(Q, P) = \frac{P^2}{2m} + \frac{1}{2} m \omega^2 Q^2 \), \( Q = (\hbar/m\omega)^{1/2} q \), \( P = (m\omega \hbar)^{1/2} p \) and \( \nu_T = \frac{1}{2} \coth \left( \frac{\hbar \omega}{2kT} \right) \) is the symplectic eigenvalue of the variance matrix (see [24]). It is evident that \( W(q, p) \) of (34) is a Gaussian function of \( q, p \) and is non-negative.
Using the Wigner function description it is possible to develop a time-reversal symmetric phase-space trajectory approach to derive heat exchange fluctuation relation analogous to the Jarzynski-Wózczik approach in the classical scenario (as outlined in Section 2).

We discuss heat exchange statistics in two Gaussian systems, in thermal equilibrium at different temperatures in the next section.

4. Heat exchange fluctuation theorem for Gaussian thermal states

Let us consider two quantum systems $A$ and $B$ characterized by their respective Hamiltonians

$$\hat{H}_A(\xi_A) = \frac{\hat{\mathcal{P}}_A^2}{2m_A} + \frac{1}{2} m_A \omega_A^2 \hat{Q}_A^2,$$  \hspace{1cm} \hat{H}_B(\xi_B) = \frac{\hat{\mathcal{P}}_B^2}{2m_B} + \frac{1}{2} m_B \omega_B^2 \hat{Q}_B^2,$$

where $\xi_A = (q_A, p_A)^T, q_A = \sqrt{\frac{m_A \omega_A}{\hbar}} \hat{Q}_A, p_A = \frac{1}{\sqrt{m_A \omega_A \hbar}} \hat{P}_A, \alpha = A, B.$

Let the systems be prepared in a thermal state at temperatures $T_A, T_B$ respectively i.e.,

$$\hat{\rho}_{TA} = e^{-\hat{H}_A(\xi_A)/kT_A}/Z_A, \quad \hat{\rho}_{TB} = e^{-\hat{H}_B(\xi_B)/kT_B}/Z_B.$$  \hspace{1cm} (36)

The Wigner function $W(\xi^0)$ at time $t = 0$ corresponding to the two-mode Gaussian thermal state $\hat{\rho}_{AB}^0 = \hat{\rho}_{TA}^0 \otimes \hat{\rho}_{TB}^0$ is a product of Wigner functions $W(\xi_A^0), W(\xi_B^0)$, where $\xi^0 = (\xi_A^0, \xi_B^0)^T$ and $\xi_A^0 = (q_A^0, p_A^0)^T, \xi_B^0 = (q_B^0, p_B^0)^T$ are classical phase-space columns at $t = 0$. Using (34) we obtain

$$W(\xi^0) = \frac{1}{(2\pi)^{\nu_{TA} \nu_{TB}}} \exp \left[-\left(\frac{\xi_A^0}{2\nu_{TA}} \xi_A^0 + \frac{\xi_B^0}{2\nu_{TB}} \xi_B^0\right)\right] = \frac{1}{(2\pi)^{\nu_{TA} \nu_{TB}}} \exp \left[-\left(\frac{H_A(\xi_A^0)}{h\nu_{TA}} + \frac{H_B(\xi_B^0)}{h\nu_{TB}}\right)\right]$$

where $H_A(\xi_A^0) = \frac{\mathcal{P}_A^2}{2m_A} + \frac{1}{2} m_A \omega_A^2 \xi_A^0, Q_A = (\hbar/m_A \omega_A)^{1/2} q_A, P_A = (m_A \omega_A \hbar)^{1/2} p_A, \alpha = A, B.$ The systems $A$ and $B$ are kept in contact with each other in terms of a quadratic interaction Hamiltonian (representing a canonical transformation in phase-space) $\hat{H}_{int}(\hat{\xi}^\tau)$, which is turned on and off at time $t = 0, t = \tau$ respectively.

Corresponding to the unitary time evolution operator

$$\hat{U}(\hat{\xi}^\tau) = \exp \left[-\frac{i}{\hbar} \left(\hat{H}_A(\hat{\xi}_A^\tau) + \hat{H}_B(\hat{\xi}_B^\tau) + \hat{H}_{int}(\hat{\xi}^\tau)\right)\right], \quad \hat{\xi}^\tau = (q_A^\tau, \hat{p}_A^\tau; q_B^\tau, \hat{p}_B^\tau)^T$$

(38)

there exists a $4 \times 4$ real symplectic matrix $\textbf{S}^\tau \in \text{Sp}(4, R)$ which acts on the column of canonical operators as $\hat{\xi}^0 = (q_A^0, \hat{p}_A^0; q_B^0, \hat{p}_B^0)^T$ resulting in

$$\hat{\xi}^0 \xrightarrow{\text{S}^\tau} \hat{\xi}^\tau = \textbf{S}^\tau \hat{\xi}^0.$$  \hspace{1cm} (39)

Consequent to the unitary evolution of the quantum state $\rho_{AB}^0 \rightarrow \rho_{AB}^\tau = \hat{U}(\hat{\xi}^\tau) \rho_{AB}^0 \hat{U}^\dagger(\hat{\xi}^\tau)$, the Wigner function undergoes the transformation: $W(\xi^0) \rightarrow W(\xi^\tau)$, where
\[ \xi^\tau = (q_A^\tau, p_A^\tau, q_B^\tau, p_B^\tau)^T. \] We thus obtain,

\[
W(\xi^\tau) = \frac{1}{(2\pi)^2 \nu_{TA} \nu_{TB}} \exp \left[ - \left( \frac{H_A(\xi_A^\tau)}{\hbar \omega_A \nu_{TA}} + \frac{H_B(\xi_B^\tau)}{\hbar \omega_B \nu_{TB}} \right) \right] \tag{40}
\]

Under time-reversal operation one finds that \( \dot{\xi} \rightarrow \dot{\xi}^* = (\dot{q}_A, -\dot{p}_A; \dot{q}_B, -\dot{p}_B)^T \). The unitary dynamics is chosen to be invariant under time-reversal i.e.,

\[
\begin{aligned}
\hat{H}_A(\dot{\xi}_A) &\rightarrow \hat{H}_A(\dot{\xi}_A^\tau) = \hat{H}_A(\dot{\xi}_A) \\
\hat{H}_B(\dot{\xi}_B) &\rightarrow \hat{H}_B(\dot{\xi}_B^\tau) = \hat{H}_B(\dot{\xi}_B) \\
\hat{H}_{\text{int}}(\dot{\xi}) &\rightarrow \hat{H}_{\text{int}}(\dot{\xi}^\tau) = \hat{H}_{\text{int}}(\dot{\xi}).
\end{aligned} \tag{41}
\]

Consider dynamical evolution of the system under time-reversal, transforming the phase-space column of observables \( \dot{\xi}_0^\tau = \dot{\xi}^\tau \) to \( \dot{\xi}^\tau = \dot{\xi}_0^\tau \). Ratio of the Wigner functions \( W(\xi^0)/W(\xi^\tau) \) is then given by (see (37) and (40)),

\[
\frac{W(\xi^0)}{W(\xi^\tau)} = \exp \left[ \frac{\Delta E_A}{\hbar \omega_A \nu_{TA}} \right] \frac{\Delta E_B}{\hbar \omega_B \nu_{TB}} \tag{42}
\]

where

\[ \Delta E_A = H_A(\xi_A^\tau) - H_A(\xi_A^0), \quad \Delta E_B = H_B(\xi_B^\tau) - H_B(\xi_B^0). \tag{43} \]

Considering a weak interaction \( \hat{H}_{\text{int}}(\dot{\xi}^\tau) \), it is deduced that

\[ H_A(\xi_A^0) + H_B(\xi_B^0) \approx H_A(\xi_A^\tau) + H_B(\xi_B^\tau) \]

implying that the net change in internal energy of system A is compensated by an opposite change in the internal energy of system B, when the phase-space trajectory \( \xi^\tau \) is sampled. In other words, heat transfer during forward realization \( \mathcal{Q}(\xi^0) = \Delta E_B \) is opposite to that of the reverse realization i.e., \( \mathcal{Q}(\dot{\xi}^0) = \Delta E_A = -\mathcal{Q}(\xi^0) \). Thus, we obtain

\[
\frac{W(\xi^0)}{W(\xi^\tau)} = e^{\Delta \beta_\omega \mathcal{Q}(\xi^0)}, \tag{45}
\]

where

\[
\Delta \beta_\omega = \beta_{B \omega} - \beta_{A \omega} = \frac{1}{\hbar \omega_B \nu_{TB}} - \frac{1}{\hbar \omega_A \nu_{TA}} = \frac{2 \tanh \left( \frac{\hbar \omega_B}{2 k T_B} \right)}{\hbar \omega_B} - \frac{2 \tanh \left( \frac{\hbar \omega_A}{2 k T_A} \right)}{\hbar \omega_A}. \tag{46}
\]
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Heat distribution \( p_\tau(Q) \) can then be expressed in terms of the Wigner function as

\[
p_\tau(Q) = \int d\xi^0 W(\xi^0) \delta(Q - Q(\xi^0)) = e^{\Delta_\beta_\omega Q} \int d\tilde{\xi}^0 W(\tilde{\xi}^0) \delta(Q + Q(\tilde{\xi}^0)) = e^{\Delta_\beta_\omega Q} p_\tau(-Q). \tag{47}
\]

We obtain

\[
\ln \left( \frac{p_\tau(Q)}{p_\tau(-Q)} \right) = \Delta_\beta_\omega Q \tag{48}
\]

where \( \Delta_\beta_\omega \) is given by (46). We thus arrive at a Jarzynski-Wózcik like heat exchange fluctuation relation (48) for Gaussian thermal states in the quantum scenario.

5. Discussions

Some relevant discussions on different forms \( [1], [8], [18] \) of the Jarzynski-Wózcik heat exchange fluctuation relations are summarised in the following.

(i) In the classical limit \( \hbar \to 0 \) we get \( \Delta_\beta_\omega \to \Delta_\beta \). Thus the heat exchange fluctuation relation (48) reduces to its classical analogue (1) in this limit. We draw attention to Ref. [31] where Wigner phase-space formalism was employed to derive the characteristic function (which is the Fourier transform of the probability distribution) of quantum work. It was shown that the characteristic function for the classical work is recovered in the limit \( \hbar \to 0 \). Furthermore, a generalized Jarzynski identity for quantum work has been derived very recently [32] using the Wigner-Weyl phase-space approach, which retrieves the celebrated classical work identity [1, 2] in the classical limit \( \hbar \to 0 \). It is for the first time that we have derived a Jarzynski-Wózcik like heat exchange fluctuation relation (18) for a system of two quantum harmonic oscillators, which reduces to the classical relation (1) in the limit \( \hbar \to 0 \).

(ii) Note that \( Q(\xi^0) = H_B(\xi_B^*) - H_B(\xi_B^0) \), is the classical heat exchanged along the phase-space trajectory, which appears in the Wigner-Weyl formalism. However, this formal phase-space trajectory has no clear physical interpretation because of the uncertainty relation in the quantum realm. Description of the quantum evolution in terms of classical trajectory is feasible in the limit \( \hbar \to 0 \). This explains the transition of heat exchange fluctuation relation (18) to its classical counterpart (1). It would be interesting to explore the meaning of the heat distribution \( p_\tau(Q) \) of (47) in the Wigner-Weyl formalism. To this end, let us denote a quantum operator \( \hat{p}_\tau(Q) \) corresponding to the heat distribution \( p_\tau(Q) \). The Weyl transform of the operator \( p_\tau(Q) \) is then identified to be the Dirac delta function (see (47)) \( \delta(Q - Q(\xi^0)) \).

Following (31), we may express

\[
\delta(Q - H_B(\xi_B^*) + H_B(\xi_B^0)) = \int dy \langle q - y/2 | \hat{p}_\tau(Q) | q + y/2 \rangle e^{-i\pi (p \cdot y/\hbar)}, \tag{49}
\]
where \( \mathbf{q} \equiv (q_A, q_B), \ \mathbf{p} \equiv (p_A, p_B), \) and \( \mathbf{y} \equiv (y_A, y_B). \) The matrix element 
\( \langle \mathbf{q}' | \hat{\varrho}_\tau(\mathbf{Q}) | \mathbf{q}'' \rangle, \ \mathbf{q} = (\mathbf{q}' + \mathbf{q}'')/2 = (q_A, q_B) \) of the operator \( \hat{\varrho}_\tau(\mathbf{Q}) \) is then given 
by the inverse Weyl transform

\[
\langle \mathbf{q} - \mathbf{y}/2 | \hat{\varrho}_\tau(\mathbf{Q}) | \mathbf{q} + \mathbf{y}/2 \rangle = \frac{1}{(2\pi \hbar)^2} \int d\mathbf{p} \delta(\mathbf{Q} - H_B(\xi_B^* + \xi_B) + H_B(\xi_B^* - \xi_B)) e^{i\mathbf{p}\cdot \mathbf{y}/\hbar}.
\]

In other words, the matrix element \( \langle \mathbf{q}' | \hat{\varrho}_\tau(\mathbf{Q}) | \mathbf{q}'' \rangle \) is the inverse Fourier transform of the delta function \( \delta(\mathbf{Q} - H_B(\xi_B^* + \xi_B)) \). This prompts us to identify 
that the operator \( \hat{\varrho}_\tau(\mathbf{Q}) \) behaves like a (plane wave) projector such that the trajectory starting from the phase-space point \( (\mathbf{q}, \mathbf{p}) \) corresponds to an exchange 
of heat \( \mathbf{Q} = H_B(\xi_B^* + \xi_B) - H_B(\xi_B^* - \xi_B) = H_A(\xi_A^*) - H_A(\xi_A^*) \) between the systems \( A \) and \( B \). However, as mentioned earlier, this trajectory approach is 
hindered by the underlying uncertainty relation in the quantum scenario, though such a representation can be validated in the classical limit \( \hbar \to 0 \). This explains the 
reduction of Jarzynski-Wózciak heat exchange fluctuation relation (48) to its classical 
 analogue (8) in the limit \( \hbar \to 0 \) where a legitimate interpretation of the phase-space trajectories is possible.

(iii) The moment generating function \([9, 23]\) associated with the heat distribution \( \varrho_\tau(\mathbf{Q}) \) 
in the Wigner-Weyl formalism may be constructed as follows:

\[
\langle e^{-s\Delta \beta_\omega \mathbf{Q}} \rangle = G_\tau(\Delta \beta_\omega; s) = \int d\mathbf{Q} \varrho_\tau(\mathbf{Q}) e^{-s\Delta \beta_\omega \mathbf{Q}}
= \int d\xi^0 W(\xi^0) \left\{ \int d\mathbf{Q} e^{-s\Delta \beta_\omega \mathbf{Q}} \delta(\mathbf{Q} - \mathbf{Q}(\xi^0)) \right\}
= \int d\xi^0 W(\xi^0) e^{-s\Delta \beta_\omega \mathbf{Q}(\xi^0)}
= \int d\xi^0 W(\xi^0) e^{-s\Delta \beta_\omega [H_B(\xi_B^*+\xi_B)-H_B(\xi_B^*-\xi_B)]}.
\] (50)

where \( \Delta \beta_\omega \) is defined in (40) and \( s \) is an arbitrary real parameter. Substituting 
(37), (40), (43), (44), and simplifying, we obtain

\[
\langle e^{-s\Delta \beta_\omega \mathbf{Q}} \rangle = \frac{1}{(2\pi)^2 \nu_T \nu_T} \int d\xi^0 e^{-[\beta_A H_A(\xi_A^0)+\beta_B H_B(\xi_B^0)]}
\times e^{-s\beta_B A_0 \left[ H_B(\xi_B^*)+H_B(\xi_B^*) \right] e^{s\beta_B A_0 \left[ H_B(\xi_B^*)+H_B(\xi_B^*) \right]}}
= \frac{1}{(2\pi)^2 \nu_T \nu_T} \int d\xi^0 \left\{ e^{-[\beta_A H_A(\xi_A^0)+\beta_B H_B(\xi_B^0)]} \right\}^{1-s}
\times \left\{ e^{-[\beta_A H_A(\xi_A^*)+\beta_B H_B(\xi_B^*)]} \right\}^s
= \int d\xi^0 \left[ W(\xi^0) \right]^{1-s} \left[ W(\xi^*) \right]^s
= \exp \left[ (1-s) R_s (W^0 || W^0) \right],
\] (51)

where

\[
R_s (W^0 || W^0) = \frac{1}{1-s} \ln \left\{ \int d\xi^0 \left[ W(\xi^0) \right]^{1-s} \left[ W(\xi^*) \right]^s \right\}
\] (52)
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denotes the order-s Rényi divergence between the Wigner functions $W(\xi^0)$ (corresponding to the initial state) and $W(\xi^\tau)$ (representing the final state) of the total system $A$ and $B$.

Substituting $s = 1$ in (51), we obtain

$$\langle e^{-\triangle \beta \omega Q} \rangle = 1.$$  \hspace{1cm} (53)

Applying Jensen inequality $\langle e^{-\triangle \beta \omega Q} \rangle \geq e^{-\triangle \beta \omega \langle Q \rangle}$ in (53), we get

$$\triangle \beta \omega \langle Q \rangle \geq 0 \Rightarrow (\beta_B \omega - \beta_A \omega) \langle Q \rangle \geq 0.$$  \hspace{1cm} (54)

Let us assume that $\beta_B \omega > \beta_A \omega$ (which is analogous, in the limit $\hbar \rightarrow 0$, to the condition $T_A > T_B$). We thus obtain a variant of the Clausius inequality (the second law of thermodynamics)

$$(\beta_B \omega - \beta_A \omega) \langle Q \rangle \geq 0$$  \hspace{1cm} (55)

which implies that heat does not flow from system $B$ (cold) to system $A$ (hot).

In the low temperature limit $T_A, T_B \rightarrow 0$, we obtain (see (46), (48))

$$\frac{p_\tau(Q)}{p_\tau(-Q)} = \exp \left[ \frac{2Q}{\hbar} \left( \frac{1}{\omega_B} - \frac{1}{\omega_A} \right) \right]$$  \hspace{1cm} (56)

which indicates that there is still a finite heat flow. In other words, (56) points out that the heat-exchange statistics is not symmetric about $T_A = T_B$, when the frequencies of the oscillators $A$ and $B$ are not equal. In this context, it is of interest to note that Chimonidou and Sudarshan [33] had investigated relaxation phenomena of a system of two harmonic oscillators in thermal equilibrium, prepared initially at different temperatures $T_A, T_B$. They subjected the system to a specific (symplectic) interaction Hamiltonian for a fixed time duration $\tau$ repeatedly, till the system approaches equilibrium. They concluded that the equilibrium reached, when the frequencies of the oscillators are unequal, is not a thermal one. It is of interest to investigate the asymmetry of the Jarzynski-Wózciak relation (48) at $T_A = T_B$ of a system of two oscillators with unequal frequencies evolving under different quadratic interaction Hamiltonians. A detailed exploration on these aspects will be reported in a separate communication.

In Ref. [23] Wei had established a connection between the moment generating function $G_\tau(\Delta \beta; s)$ in the quantum regime with the order-s Rényi divergence $R_s(\rho^0_{AB}||\rho^\tau_{AB}) = \frac{1}{1-s} \ln \{ \text{Tr}[(\rho^\tau_{AB})^{1-s}(\rho^0_{AB})^s]\}$ between the initial, final density operators $\rho^0_{AB}, \rho^\tau_{AB}$:

$$G_\tau(\Delta \beta; s) = \int dQ p_\tau(Q) e^{-s\Delta \beta Q} = \exp \left[ (1 - s) R_s(\rho^0_{AB}||\rho^\tau_{AB}) \right]$$  \hspace{1cm} (57)

To arrive at the relation (57) the double projection measurement approach (as proposed by Jarzynski and Wózciak [3]) was employed. It may be seen that there is a close resemblance between the generating functions [51] and (57), which are derived using different approaches. In (57) the generating function $G_\tau(\Delta \beta; s)$ is related to the order-s Rényi divergence $R_s(\rho^0_{AB}||\rho^\tau_{AB})$ between the density operators $\rho^0_{AB}, \rho^\tau_{AB}$,
\( \rho_{AB}^{\tau} \), whereas (51) derived using the Wigner phase-space formalism – in the specific example of harmonic oscillator system – connects the moment generating function \( G_{\tau}(\triangle \beta \omega; s) \) with the Rényi divergence \( R_{s}(W^{0}||W^{\tau}) \) between the Wigner functions \( W(\xi^{0}), W(\xi^{\tau}) \). Comparing (51) and (57) for the system of two harmonic oscillators, by subjecting the system to specific symplectic interaction Hamiltonians \( \hat{H}_{\text{int}}(\hat{\xi}) \) would be useful for exploring the nature of deviation of the heat flow statistics in the double projective measurement method and the Wigner-Weyl phase-space approach.

(iv) Equipartition theorem plays a fundamental role in classical statistical physics. It states that for a system in thermal equilibrium at temperature \( T \) the average energy per degree of freedom is given by \( \frac{1}{2} k T \). Equipartition theorem of energy holds universally in classical statistical physics as it neither depends on the number of particles in the ensemble nor on the nature of the potential acting on the particles. For a system of one dimensional classical harmonic oscillators, in thermal equilibrium at temperature \( T \), contribution to the average energy comes from mean kinetic energy and mean potential energy i.e., \( \langle E \rangle = k T \). It has been pointed out \[34, 35, 36\] recently that the classical energy equipartition theorem does not hold in the quantum realm. It is seen that the average energies in the state \( \rho_{AB} = \rho_{T_{A}} \otimes \rho_{T_{B}} \) (see (36)) – characterizing a system of quantum harmonic oscillators \( A \) and \( B \) in thermal equilibrium at temperatures \( T_{A}, T_{B} \) respectively – are given by

\[
\langle \hat{H}_{A} \rangle = \text{Tr}[\rho_{T_{A}} \hat{H}_{A}] = \frac{\hbar \omega_{A}}{2} \left( \langle \hat{q}_{A}^{2} \rangle + \langle \hat{p}_{A}^{2} \rangle \right) = \frac{\hbar \omega_{A}}{2} \coth \left( \frac{\hbar \omega_{A}}{2 k T_{A}} \right) = \hbar \omega_{A} \nu_{T_{A}} \tag{58}
\]

\[
\langle \hat{H}_{B} \rangle = \text{Tr}[\rho_{T_{B}} \hat{H}_{B}] = \frac{\hbar \omega_{B}}{2} \left( \langle \hat{q}_{B}^{2} \rangle + \langle \hat{p}_{B}^{2} \rangle \right) = \frac{\hbar \omega_{B}}{2} \coth \left( \frac{\hbar \omega_{B}}{2 k T_{B}} \right) = \hbar \omega_{B} \nu_{T_{B}} \tag{59}
\]

where we have made use of (13), (24) and (35). Here the average energies \( (58), (59) \) depend on frequencies \( \omega_{A}, \omega_{B} \) (indicative of the nature of the potential) besides temperatures \( T_{A}, T_{B} \). The factor \( \triangle \beta_{\omega} \) in the Jarzynski-Wózcik heat exchange fluctuation relation \( (48) \) approaches its classical analogue \( \triangle \beta \) of \( (1) \) only in the limit \( \hbar \to 0 \) (see (i) above). Deviations of \( (48) \) from \( (1) \) could be attributed to the fact that classical energy equipartition is no longer valid in the quantum scenario. A series of recent papers \[34, 35, 36\] have proposed quantum counterpart of energy equipartition theorem, which may shed more light on the quantum heat exchange statistics of thermal harmonic oscillator system.

(v) It is pertinent to point out some recent results on Jarzynski-Wózcik XFT, where
quantum-to-classical transition is studied. Denzler and Lutz [21] arrived at the heat distribution associated with the infinite dimensional systems of thermal quantum harmonic oscillators, weakly coupled to a heat reservoir at different temperatures, by exactly solving the quantum master equation. The heat distribution so obtained leads to the Jarzynski-Wóczik XFT [8] for the ensemble of quantum harmonic oscillators in equilibrium. Authors of Ref. [21] implement double measurements on the discrete energy levels of the quantum thermal oscillator system at initial time $t = 0$ and final time $t = \tau$ to derive the XFT [8] in the quantum regime - as prescribed originally by Jarzynski-Wóczik [5]. It is shown that the discrete heat distribution becomes continuous and reduces to the corresponding classical expression in the limit $\hbar \omega / kT \rightarrow 0$. This falls in line with the Wigner-Weyl phase-space description on the quantum to classical transition.

Highlighting that the quantum features get destroyed in the double projective measurement scheme (prescribed in the derivation of quantum Jarzynski-Wóczik relation [8]), an entirely different strategy based on dynamic Bayesian networks has been employed very recently [25] to derive a fully quantum fluctuation theorem for heat exchange in a correlated bipartite thermal system. In the absence of quantum correlations, the classical Jarzynski-Wóczik XFT is recovered from this quantum fluctuation relation and it is also shown to reduce to the fluctuation relation in the presence of classical correlations derived in Ref. [19].

It is of interest to note that in Refs. [37, 38] direct measurement on the quantum system was avoided by coupling the system to a classical apparatus. With the aid of the apparatus the probability distribution for the work done on a quantum system was constructed in Ref. [38] and it was shown that the associated statistics is consistent with the work fluctuation relation [1, 2, 4]. It is of interest to carry out a detailed study comparing the heat-exchange fluctuation relation [48] for the system of two harmonic oscillators, derived using the Wigner phase-space formalism, with the ones realized based on different strategies [25, 37, 38], where direct measurement on the quantum system was judiciously evaded.

In summary, we believe that the Wigner-Weyl phase-space framework to explore Jarzynski-Wóczik XFT opens up new perspectives towards understanding the heat distribution statistics and its classical limit.

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References

[1] Jarzynski C 1997 Nonequilibrium Equality for Free Energy Differences *Phys. Rev. Lett* **78** 2690–2693
[2] Jarzynski C 1997 Equilibrium free-energy differences from nonequilibrium measurements: A master-equation approach *Phys. Rev.* E **56** 5018–5035
[3] Crooks G E 1998 Nonequilibrium Measurements of Free Energy Differences for Microscopically Reversible Markovian Systems, *J. Stat. Phys.* **90** 1481–1487
[4] Crooks G E 1999 Entropy production fluctuation theorem and the nonequilibrium work relation for free energy differences *Phys. Rev.* E **60** 2721–2726
[5] Jarzynski C and Wóczik D K 2004 Classical and Quantum Exchange Theorems for heat exchange *Phys. Rev. Lett.* **92** 230602
[6] Jarzynski C 2007 Comparison of far-from-equilibrium work relations *C. R. Phys* **8** 495–506
[7] Saito K and Dhar A 2007 Fluctuation Theorem in Quantum Heat Conduction, *Phys. Rev. Lett.* **99**, 180601
[8] Andrieux D, Gaspard P, Monnai T and Tasaki S 2009 The fluctuation theorem for currents in open quantum systems *New. J. Phys* **11** 043014
[9] Esposito M, Harbola U and Mukamel S 2009 Nonequilibrium fluctuations, fluctuation theorems and counting statistics in quantum systems *Rev. Mod. Phys* **81** 1665–1702
[10] Campisi M, Talkner P and Hänggi P 2010 Fluctuation Theorems for Continuously Monitored Quantum Fluxes *Phys. Rev. Lett.* **105**, 140601
[11] Campisi M, Hänggi P and Talkner P, 2011 Quantum fluctuation relations: Foundations and applications *Rev. Mod. Phys* **83** 771–791
[12] Cohen D and Imry Y 2012 Straightforward quantum-mechanical derivation of the Crooks fluctuation theorem and the Jarzynski equality *Phys. Rev.* E **86** 011111
[13] Rana S, Lahiri S, Jayannavar A M 2012 Quantum Jarzynski equality with multiple measurement and feedback for isolated systems *Pramana J. Phys* **79** 233–241
[14] Defnner S and Jarzynski C 2013 Information processing and the second law of thermodynamics *Phys. Rev. X* **3** 041003
[15] Rana S, Lahiri S, Jayannavar A M 2013 Generalized entropy production fluctuation theorems for quantum systems *Pramana J. Phys* **80** 207–222
[16] Lahiri S, Jayannavar A M 2014 Exchange fluctuation theorems for a chain of interacting particles in presence of two heat baths *Eur. Phys. J. B* **87** 141
[17] Lahiri S, Jayannavar A M 2015 Derivation of not-so-common fluctuation theorems *Indian J. Phys* **89** 515–523
[18] Hänggi P and Talkner P 2015 The other QFT *Nature Physics* **11** 108–110
[19] Jevtic S, Rudolph T, Jennings D, Hirano Y, Nakayama S and Murao M 2015 Exchange fluctuation theorems for correlated quantum systems *Phys. Rev. E* **92** 042113
[20] Pal P S, Lahiri S and Jayannavar A M 2017 Transient Exchange Fluctuation Theorem for heat using a hamiltonian framework *Phys. Rev. E* **95** 042124
[21] Denzler T, Lutz E, 2018 Heat distribution of quantum thermal oscillator *Phys. Rev. E* **98** 052106
[22] Alberg J 2018 Fully quantum fluctuation theorems *Phys. Rev. X* **8**, 011019
[23] Wei B B 2018 Relations between heat exchange and Rényi divergences *Phys. Rev. E* **97**, 042107
[24] Hänggi P and Talkner P 2019 Statistical mechanics and thermodynamics at strong coupling: Quantum and classical *arXiv:1911.11668v3* (Reviews of Modern Physics, In press)
[25] Micadei K, Landi G T and Lutz E 2020 Quantum fluctuation theorems beyond two-point measurements *Phys. Rev. Lett* **124** 090602
[26] Simon R, Mukunda N, Dutta B 1994 Quantum-noise matrix for multimode systems: U(n) invariance, squeezing, and normal forms *Physical Review A* **49** 1567–1583
[27] Arvind, Dutta B, Mukunda N, Simon R 1995 The real symplectic groups in quantum mechanics and optics *Pramana - J. Phys.* **45** 471–477
[28] Adesso G, Ragy S, Lee R A 2014 Continuous variable quantum information: Gaussian states and
Heat exchange and fluctuation in Gaussian thermal states

beyond, Open. Syst. Inf. Dyn. 21 1440001
[29] Hillery M, O’Connell R F, Scully M O, Wigner E P 1984 Distribution functions in Physics: Fundamentals Phys. Rep. 106 121–167
[30] Williamson J 1936 On the Algebraic Problem Concerning the Normal Forms of Linear Dynamical Systems Am. J. Math. 58 141–163
[31] Qian Y, Liu F 2019 Computing characteristic functions of quantum work in phase space Phys. Rev. E 100 062119 (1–10)
[32] Brodier O, Mallick K, Ozorio de Almeida A M 2020 Semi-classical work and quantum work identities in Weyl representation J. Phys. A: Math. Theor. 53 325001
[33] Chimonidou A, Sudarshan E C G 2008 Relaxation phenomena in a system of two harmonic oscillators Phys. Rev. A 77 032121 (1–11)
[34] Bialas P, Spiechowicz J, Luczka J 2018 Partition of energy for a dissipative quantum oscillator Sci. Rep. 8 16080(1–12)
[35] Bialas P, Spiechowicz J, Luczka J 2019 Quantum analogue of energy equipartition theorem J. Phys. A 52 15LT01
[36] Luczka J 2020 Quantum counterpart of classical equipartition of energy J. Stat. Phys. 179 839–845
[37] Yu. V. Nazarov, Kindermann M 2003 Full counting statistics of a general quantum mechanical variable Eur. Phys. J. B 35, 413-420
[38] Utsumi Y, Golubev D S, Marthaler M, Schöhn G, Kobayashi K 2012 Work fluctuation theorem for a classical circuit coupled to a quantum conductor Phys. Rev. B 86, 075420 (1–8)