ON ERDŐS–DUSHNIK–MILLER THEOREM WITHOUT AC

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Abstract. In ZFA (Zermelo-Fraenkel set theory with the Axiom of Extensionality weakened to allow the existence of atoms), we prove that the strength of the proposition EDM ("If \( G = (V_\kappa; E_\kappa) \) is a graph such that \( V_\kappa \) is uncountable, then for all coloring \( f : V_\kappa^2 \to \{0, 1\} \) either there is an uncountable set monochromatic in color 0, or there is a countably infinite set monochromatic in color 1") is strictly between DC\(_{\aleph_0}\) (where DC\(_{\aleph_0}\) is Dependent Choices for \( \aleph_1 \), a weak choice form stronger than Dependent Choices (DC)) and Kurepa’s principle ("Any partially ordered set such that all of its antichains are finite and all of its chains are countable is countable"). Among other new results, we strengthen some results of Brunner and study the relations of EDM with BPI (Boolean Prime Ideal Theorem), RT (Ramsey’s Theorem), De Bruijn–Erdős theorem for \( n \)-colorings, König’s Lemma and several other weak choice forms. Moreover, we answer a question raised by Lajos Soukup.

1. Introduction

In 1941, Erdős proved the proposition "If \( \kappa \) is an uncountable cardinal, then for all coloring \( f : [\kappa]^2 \to \{0, 1\} \), either there is a set of cardinality \( \kappa \) monochromatic in color 0, or there is a countably infinite set monochromatic in color 1" in ZFC, which appeared in a paper by Dushnik and Miller. The above result is uniformly known as Erdős–Dushnik–Miller theorem. In ZFC, the theorem applies to prove the proposition "Every partially ordered set such that all of its chains are finite and all of its antichains are countable is countable" (abbreviated here as "CAC\(_{\aleph_0}\)") and Kurepa’s result on partially ordered sets stated in the abstract (abbreviated here as "CAC\(_{\aleph_1}\)"). Let \( X \) be a set. We note that without the Axiom of Choice (AC), there are two definitions of uncountable sets:

1. \( X \) is uncountable if \( |X| \not\leq \aleph_0 \) (i.e., there is no injection from \( X \) into \( \aleph_0 \)).
2. \( X \) is uncountable if \( \aleph_0 < |X| \) (i.e., there is an injection from \( \aleph_0 \) into \( X \)).

Kurepa explicitly proved CAC\(_{\aleph_0}\) in ZFC in response to Sierpiński’s question. Banerjee studied some relations of CAC\(_{\aleph_0}\) and CAC\(_{\aleph_1}\) with weak choice forms using the first definition of uncountable sets. Recently, Tachtsis proved the deductive strength of CAC\(_{\aleph_0}\) without AC in more detail. Among various results, Tachtsis proved that CAC\(_{\aleph_0}\) holds in ZF + DC\(_{\aleph_1}\) and CAC\(_{\aleph_1}\) holds in the Mostowski’s linearly ordered model (labeled as Model \( \mathcal{N}_1 \) in [10]) as well as the basic Fraenkel model (labeled as Model \( \mathcal{N}_1 \) in [11]). Inspired by the research work of Tachtsis, we study the deductive strength of EDM without AC. We note that all the results (except the results in section 4.2) are obtained with the first definition of uncountable sets. Lajos Soukup asked the following question.

Question 1.1. What is the relationship between CAC\(_{\aleph_1}\) and CAC\(_{\aleph_0}\) in ZF and ZFA?

Main result. Fix \( X \in \{CAC\(_{\aleph_1}\), CAC\(_{\aleph_0}\}\). The first author proves that the strength of EDM is strictly between DC\(_{\aleph_1}\) and \( X \), and CAC\(_{\aleph_0}\) does not imply CAC\(_{\aleph_1}\) in ZFA (cf. Theorems 4.1, 4.2, 5.3, Corollary 3.6).

Other results. The first author observes the following in ZF:

1. CAC\(_{\aleph_0}\) implies AC\(_{\aleph_0}\) (Every countably infinite family of non-empty countably infinite sets has a choice function). Thus CAC\(_{\aleph_0}\) is not provable in ZF (Proposition 3.4).
2. WOAM (Every set is either well-orderable or has an amorphous subset) implies RT for any locally countable connected graph (Proposition 3.7).
3. EDM is strictly stronger than RT (Theorem 4.1(3,4)).
4. WOAM + RT implies EDM (Theorem 4.2(4)).
5. Using the second definition of uncountable sets, we denote by UT(\( \aleph_0, \aleph_0, \aleph_0 \)) the statement “countable union of non-uncountable sets is non-uncountable” and obtain the following result in section 4.2:

\[ UT(\aleph_0, \aleph_0, \aleph_0) + EDM \implies DF = F \] (Every Dedekind-finite set is finite).

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\(^1ZF\) denotes Zermelo–Fraenkel set theory without AC. Complete definitions of the choice forms will be given in section 2.
1.1. Weakly Loeb spaces, the uniqueness of algebraic closures, and Łoś’s theorem. Brunner [5, Corollaries 3.2, 3.3] and Pincus [10, Note 41] proved that the following statements do not imply “there are no amorphous sets” in ZFA:

- The product of weakly Loeb Hausdorff spaces is weakly Loeb [10, Form 115].
- Compact Hausdorff spaces are weakly Loeb [10, Form 116].
- If a field has an algebraic closure, then it is unique up to isomorphism [10, Form 233].

Recently, Tachtsis [23] constructed a model of ZFA + ¬AC to prove that ACLO (Every linearly ordered family of non-empty sets has a choice function) does not imply LT (if A = ⟨A, RA⟩ is a non-trivial relational L-structure over some language L, and U be an ultrafilter on a non-empty set I, then the ultrapower A[I]/U and A are elementarily equivalent).

Other results. Fix a natural number 2 ≤ n ∈ ω. We observe the following:

1. Form 115 + Form 116 implies neither ACn (Every infinite family A of n-element sets has a partial choice function) nor “there are no amorphous sets” in ZFA. The above result strengthens related results in [5]. Moreover, CACT1ℵ0 + CACT n is regular does not imply EDM in ZFA (Theorem 5.3).

2. ACLO + EDM + Form 233 does not imply LT in ZFA. (Theorem 6.2).

1.2. Remarks. Blass [4] investigated the strength of RT in the hierarchy of choice forms. In section 7, applying the above-mentioned results and mainly inspired by the results of [4], we remark that EDM is independent of each of BPI, KW (Kinna-Wagner Selection Principle), ACWO (Axiom of Choice for well-ordered sets), “There are no amorphous sets”, n-coloring theorem (De Bruijn–Erdős theorem for n-colorings), and A (Antichain Principle) in ZFA. Moreover, LT and EDM are mutually independent in ZF. In [21], Tachtsis proved that CACT1ℵ0 and DT (Dilworth’s theorem) are mutually independent in ZFA. A natural question arises concerning the relation of CACT1ℵ0 and EDM with DT. We also remark that DT is independent of EDM and CACT1ℵ0 in ZFA.

1.3. Diagram. We summarize the main results using the first definition of uncountable sets. Fix any 2 ≤ n ∈ ω. In Figure 1, known results are depicted with dashed arrows, new implications or non-implications in ZF are mentioned with simple black arrows, new non-implications in ZFA are mentioned with thick dotted black arrows.

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2. Basics

Definition 2.1. Suppose \( X \) and \( Y \) are two sets. We write:

1. \( |X| \leq |Y| \) or \( |Y| \geq |X| \), if there is an injection \( f : X \to Y \).
2. \( |X| = |Y| \), if there is a bijection \( f : X \to Y \).
3. \( |X| < |Y| \) or \( |Y| > |X| \), if \( |X| \leq |Y| \) and \( |X| \neq |Y| \).
4. If \( f : X \to Y \) is a function, then we denote “the range of \( f \)” by \( \text{ran}(f) \) and “the domain of \( f \)” by \( \text{dom}(f) \).

Definition 2.2. Let \((P, \leq)\) be a partially ordered set or a poset. A subset \( D \subseteq P \) is a **chain** if \((D, \leq\, | D)\) is linearly ordered. A subset \( A \subseteq P \) is an **antichain** if no two elements of \( A \) are comparable under \( \leq \). The size of the largest antichain of \((P, \leq)\) is known as its **width**. A subset \( C \subseteq P \) is **cofinal** in \( P \) if for every \( x \in P \) there is an element \( c \in C \) such that \( x \leq c \). A **tree** is a poset \((P, <)\) with a least element and with the property that for any element \( x \in P \) the set of predecessors of \( x \) is a finite set that is linearly ordered by \( < \). A tree is **locally finite** if each vertex has only finitely many successors. An \( \omega \)-**tree** is a locally finite tree with at least one vertex in level \( n \) for each \( n \in \omega \) (cf. [10 Note 21]). An infinite set \( X \) is **amorphous** if \( X \) cannot be written as a disjoint union of two infinite subsets. A set \( X \) is **Dedekind-finite** if \( \aleph_0 \nsubseteq |X| \). Otherwise, \( X \) is **Dedekind-infinite**.

A topological space \( X = (X, \tau) \) is a **weakly Loeb space** if there is a multiple choice function on the family of its nonempty closed subsets. We say \( X \) is **Hausdorff** (or \( T_{2\text{-space}} \)) if any two distinct points in \( X \) can be separated by disjoint open sets. The space \( X \) is **compact** if for every \( U \subseteq \tau \) such that \( \bigcup U = X \) there is a finite subset \( V \subseteq U \) such that \( \bigcup V = X \). We say that a graph \( G = (V_G, E_G) \) is **locally countable** if for every \( v \in V_G \), the set of neighbors of \( v \) is countable. The graph \( G \) is **connected** if any two vertices are joined by a path of finite length.

Definition 2.3. (A list of choice forms).

1. The **Axiom of Choice**, AC [10 Form 1]: Every family of non-empty sets has a choice function.
2. The **Boolean Prime Ideal Theorem**, BPI [10 Form 14]: Every Boolean algebra has a prime ideal.
3. The **König-Wagner Selection Principle**, KW [10 Form 15]: For every set \( M \) there is a function \( f \) such that for all \( A \in M \), if \( |A| > 1 \) then \( \emptyset \neq f(A) \subseteq A \).
4. The **Axiom of Multiple Choice**, MC [10 Form 67]: Every family \( \mathcal{A} \) of non-empty sets has a multiple choice function, i.e., there is a function \( f \) with domain \( \mathcal{A} \) such that for every \( A \in \mathcal{A} \), \( \emptyset \neq f(A) \subseteq |A|^{<\omega} \).
5. **MC\(_{\aleph_0}^{\text{no}}\)** [10 Form 350]: Every denumerable, i.e. countably infinite, family of denumerable sets has a multiple choice function.
6. **AC\(_{\text{WO}}\)** [10 Form 60]: Every set of well-orderable, non-empty sets has a choice function.
7. **AC\(_{\text{LO}}\)** [10 Form 202]: Every linearly ordered family of non-empty sets has a choice function.
8. **PAC\(_{\text{fin}}\)** (cf. [11]): Every \( \aleph_1 \)-sized family \( \mathcal{A} \) of non-empty finite sets has an \( \aleph_1 \)-sized subfamily \( \mathcal{B} \) with a choice function.
9. **AC\(_{\text{bf}}\)** [10 Form 32A]: Every denumerable family of denumerable sets has a choice function. We recall that **AC\(_{\text{bf}}\)** is equivalent to **PAC\(_{\text{fin}}\)** [10 Form 32B] (Every denumerable family \( \mathcal{A} \) of denumerable sets has an infinite subfamily \( \mathcal{B} \) with a choice function).
10. **AC\(_{\text{bf}}\)** [10 Form 10]: Every denumerable family of non-empty finite sets has a choice function.
11. **AC\(_{\alpha}\)** for each \( \alpha \in \omega \setminus \{0, 1\} \) [10 Form 342(\( n \))]: Every infinite family \( \mathcal{A} \) of \( n \)-element sets has a partial choice function, i.e., \( \mathcal{A} \) has an infinite subfamily \( \mathcal{B} \) with a choice function.
12. **WOAM** [10 Form 133]: Every set is of either well-orderable or has an amorphous subset.
13. The **Principle of Dependent Choice**, DC [10 Form 43]: If \( S \) is a relation on a non-empty set \( A \) and \((\forall x \in A)(\exists y \in A)(xSy)\) then there is a sequence \((a_n)_{n \in \omega}\) of elements of \( A \) such that \((\forall n \in \omega)(a_n Sa_{n+1})\).
14. **DC\(_{\alpha}\)** for an infinite well-ordered cardinal \( \kappa \) [10 Form 87(\( \kappa \))]: Let \( \kappa \) be an infinite well-ordered cardinal. Let \( S \) be a non-empty set and let \( R \) be a binary relation such that for every \( \alpha < \kappa \) and every \( \alpha \)-sequence \( s = (s_\gamma)_{\gamma < \alpha} \) of elements of \( S \) there exists \( y \in S \) such that \( sRy \). Then there is a function \( f : \kappa \to S \) such that for every \( \alpha < \kappa \), \( (f \upharpoonright \alpha)Rf(\alpha) \).
15. **DF = F** [10 Form 9]: Every Dedekind-finite set is finite.
16. **WOAM** (cf. [14 Chapter 8]): For every \( X \), either \( |X| \leq \aleph_0 \) or \( |X| \geq \aleph_0 \).
17. [10 Form 233]: If a field has an algebraic closure, then it is unique up to isomorphism.
18. **WUT** [10 Form 231]: The union of a well-orderable collection of well-orderable sets is well-orderable.
19. **AC\(_{\text{WO}}\)** [10 Form 122]: Every well-ordered family of non-empty finite sets has a choice function.
20. The **Countable Union Theorem**, CUT [10 Form 31]: The union of a countable family of countable sets is countable.
21. **CS**: Every poset without a maximal element has two disjoint cofinal subnets.
22. **CWF**: Every poset has a cofinal well-founded subnet.
23. **The Antichain Principle**, A: Every poset has a maximal antichain.
24. The **n-coloring theorem**: If all finite subgraphs of a graph \( G \) is \( n \)-colorable then \( G \) is \( n \)-colorable.
(25) Dirichlet’s Theorem, DT: If \( (P, \leq) \) is a poset of width \( k \) for some \( k \in \omega \), then \( P \) can be partitioned into \( k \) chains.

(26) Ramsey’s Theorem, RT \([10]\) Form 17\): For every infinite set \( A \) and for every partition of the set \( [A]^2 \) into two sets \( X \) and \( Y \), there is an infinite subset \( B \subseteq A \) such that either \( [B]^2 \subseteq X \) or \( [B]^2 \subseteq Y \).

(27) The Chain/Antichain Principle, CAC \([10]\) Form 217\): Every infinite poset has an infinite chain or an infinite antichain.

(28) \([10]\) Form 116\): Compact Hausdorff spaces are weakly Loeb.

(29) \([10]\) Form 115\): The product of weakly Loeb Hausdorff spaces is weakly Loeb.

(30) LT \([10]\) Form 253\): If \( A = (A, R^A) \) is a non-trivial relational \( L \)-structure over some language \( L \), and \( U \) be an ultralower on a non-empty set \( I \), then the ultrapower \( A^I / U \) and \( A \) are elementarily equivalent.

Definition 2.4. (A list of combinatorial notations).

(1) EDM: If \( G = (V_G, E_G) \) is a graph such that \( V_G \) is uncountable, then for all coloring \( f : [V_G]^2 \to \{0, 1\} \) either there is an uncountable set monochromatic in color 0, or there is a countably infinite set monochromatic in color 1.

(2) EDM\(^*\): EDM restricted to graphs based on a well-ordered set of vertices.

(3) CAC\(^{\aleph_0}\): Every poset such that all of its chains are finite and all of its antichains are countable is countable.

(4) (CAC\(^{\aleph_0}\)):\( \): CAC\(^{\aleph_0}\) restricted to posets based on a well-ordered set of elements.

(5) CAC\(^{\aleph_1}\): Every poset such that all of its antichains are finite and all of its chains are countable is countable.

(6) (CAC\(^{\aleph_1}\)):\( \): CAC\(^{\aleph_1}\) restricted to posets based on a well-ordered set of elements.

(7) CACT\(^{\aleph_0}\): CAC\(^{\aleph_0}\) restricted to \( \omega \)-trees.

(8) (CACT\(^{\aleph_0}\)):\( \): CAC\(^{\aleph_0}\) restricted to \( \omega \)-trees based on a well-ordered set of elements.

(9) CACT\(^{\aleph_1}\): CAC\(^{\aleph_1}\) restricted to \( \omega \)-trees.

(10) (CACT\(^{\aleph_1}\)):\( \): CAC\(^{\aleph_1}\) restricted to \( \omega \)-trees based on a well-ordered set of elements.

(11) For a set \( A \), Sym\((A)\) and FSym\((A)\) denote the set of all permutations of \( A \) and the set of all \( \sigma \in \text{Sym}(A) \) such that \( \{x \in A : \sigma(x) \neq x\} \) is finite. For a set \( A \) of size at least \( \aleph_n \), \( \aleph_n \text{Sym}(A) \) denote the set of all \( \sigma \in \text{Sym}(A) \) such that \( \{x \in A : \sigma(x) \neq x\} \) has cardinality at most \( \aleph_n \) (cf. \([22\) section 2\])

2.1. Permutation models and Mostowski’s intersection lemma.

We start with a model \( M = \text{ZF A + AC} \) where \( A \) is a set of atoms, \( G \) is a group of permutations of \( A \) and \( F \) is a normal filter of subgroups of \( G \). The Fraenkel–Mostowski model, or the permutation model \( \mathcal{N} \) with respect to \( M \), \( G \) and \( F \) is defined by the equality:

\[ \mathcal{N} = \{ x \in M : (\forall t \in \text{TC}(\{x\})) (\text{symg}(t) \in F) \} \]

where for a set \( x \in M \), \( \text{symg}(x) = \{ g \in G : g(x) = x \} \) and \( \text{TC}(x) \) is the transitive closure of \( x \) in \( M \). If \( I \subseteq \mathcal{P}(A) \) is a normal ideal, then \( \{ \text{fix}_G E : E \in I \} \) generates a normal filter \( \text{sym}(F_I) \) over \( G \), where \( \text{fix}_G E = \{ \phi \in \text{Sym}(G) : \forall y \in E(\phi(y) = y) \} \). Let \( \mathcal{N} \) be the permutation model determined by \( M \), \( G \), and \( F_I \). We recall that \( \mathcal{N} \) is a model of \( \text{ZFA} \) (cf. \([13\) Theorem 4.1, p.46\]). We say \( E \in I \) is a support of a set \( \sigma \in \mathcal{N} \) if \( \text{fix}_G E \subseteq \text{symg}(\sigma) \). We recall some terminologies from \([9\) sections 1,2,3\]. Let \( \Delta(E) = \{ \sigma : \text{fix}_G E \subseteq \text{symg}(\sigma) \} \) if \( E \in I \). We say that \( \mathcal{N} \) satisfies the Mostowski’s intersection lemma if \( \Delta(E \cap F) = \Delta(E) \cap \Delta(F) \) for every \( E, F \in I \). We say \( \mathcal{N} \) is a least support model or a minimal support model if each element of \( \mathcal{N} \) has a least support. It is possible to see that a minimal support model satisfies the Mostowski’s intersection lemma. In this paper,

- We follow the labeling of the models from \([17\]. \( \mathcal{N}_1 \) is the basic Fraenkel model, \( \mathcal{N}_3 \) is the Mostowski’s linearly ordered model, \( \mathcal{N}_{34} \) is Pincus’s Model X, and \( \mathcal{N}_{41} \) is a variation of \( \mathcal{N}_3 \) (cf. \([10\)).

- Fix any \( n \in \omega \setminus \{0, 1\} \). We denote by \( N_{HT}^n \) the permutation model constructed in \([11\) Theorem 8\].

Lemma 2.5. An element \( x \) of \( \mathcal{N} \) is well-orderable in \( \mathcal{N} \) if and only if \( \text{fix}_G(x) \in F_I \) (cf. \([14\) Equation (4.2, p.47\)). Thus, an element \( x \in \mathcal{N} \) of support \( E \) is well-orderable in \( \mathcal{N} \) if \( \text{fix}_G(E) \subseteq \text{fix}_G(x) \).

We refer the reader to \([10\) Note 103, pp. 283–286] for the definition of the terms “injective cardinality \( |x|_\mathcal{N} \) of \( x \), “injectively boundable statement” and “boundable statement”.

Theorem 2.6. (Pincus’ Transfer Theorem; cf. \([13\) Theorem 3A\]) If \( \Phi \) is a conjunction of injectively boundable statements which hold in the Fraenkel–Mostowski model \( V_0 \), then there is a ZF model \( V \supset V_0 \) with the same ordinals and cofinalities as \( V_0 \), where \( \Phi \) holds.

Lemma 2.7. (Brunner; cf. \([7\) Lemma 4.1\]) Let \( A \) be a set of atoms, \( G \) be a group of permutations of \( A \) and the filter \( F \) of subgroups of \( G \) is generated by \( \{ \text{fix}_G E : E \in [A]^{<\omega} \} \). Let \( \mathcal{N} \) be the Fraenkel–Mostowski model determined by \( A, G, \) and \( F \). If \( \mathcal{N} \) satisfies the Mostowski’s intersection lemma where \( A \) is Dedekind-finite, then every set \( x \) in \( \mathcal{N} \) is either well-orderable or there exists an infinite subset of \( A \), which embeds into \( x \).

\(^2\)Equivalently, for every infinite graph \( G = (V_G, E_G) \) and for all \( c : [V_G]^2 \to 2 \), \( \exists Y \in [V_G]^{<\omega} \) s.t. \( |Y|^2 \) is \( c \)-monochromatic.
3. Known and basic results

Fact 3.1. (ZF) The following hold:

1. RT holds for every infinite well-orderable set and if RT holds for an infinite set \( Y \), then RT holds for any set \( X \supseteq Y \) [\cite{27} Theorem 1.7] and \( DF = F \) implies RT [\cite{10}].
2. WOAM implies CUT [\cite{16} Proposition 8(1)]. So, WOAM implies \( \aleph_1 \) is regular”.
3. \( \text{CA}_{10}^{\aleph_0} \) implies \( \text{CAC}^{\aleph_0}_1 \) [\cite{21} Theorem 4(11)] and \( \text{CAC} \) implies \( \text{AC}_{\text{fin}}^{\aleph_0} \) [\cite{23} Lemma 4.4].
4. WOAM + \( \text{CAC} \) implies \( \text{CAC}_{10}^{\aleph_0} \) [\cite{21} Theorem 8(1)].
5. \( \text{CA}_{10}^{\aleph_0} \) implies \( \text{PAC}_{\text{fin}}^{\aleph_0} \) and DC does not imply \( \text{CAC}_{10}^{\aleph_0} \) [\cite{11} Theorem 4.5, Corollary 4.6].

We recall the result following the above and communicated to us by Tachtsis from [\cite{9}].

Fact 3.2. (cf. \cite{9} Lemma 4.1, Corollary 4.2]) \( (\text{CAC}_{10}^{\aleph_0})' \) holds in any permutation model.

3.2. Basic propositions.

Proposition 3.3. The following hold:

1. \( \aleph_1 \) is regular” implies EDM’ in ZF.
2. \( \aleph_1 \) is regular” implies \( (\text{CA}_{10}^{\aleph_0})', (\text{CA}_{10}^{\aleph_0})' \) as well as \( (\text{CA}_{10}^{\aleph_0})' \) in ZF.
3. \( \aleph_1 \) is regular” + \( \text{AC}_{\text{fin}}^{\aleph_0} \) implies \( \text{CA}_{10}^{\aleph_0} \) and \( \text{CA}_{10}^{\aleph_0} \) in ZF.
4. \( X \) holds in any permutation model if \( X \in \{ \text{EDM}', (\text{CA}_{10}^{\aleph_0})', (\text{CA}_{10}^{\aleph_0})', (\text{CA}_{10}^{\aleph_0})' \} \).
5. In any permutation model, \( \text{AC}_{\text{fin}}^{\aleph_0} \) implies \( \text{CA}_{10}^{\aleph_0} \) and \( \text{CA}_{10}^{\aleph_0} \)

Proof. (1). We modify the arguments due to Tachtsis from \cite{9} Lemma 4.1. Let \( G = (V_G, E_G) \) be a graph based on well-ordering of vertices. Fix a well-ordering \( \leq \) of \( V_G \). Let \( f : [V_G]^2 \to \{0, 1\} \) be a coloring such that all sets monochromatic in color 0 are countable and all sets monochromatic in color 1 are finite. By way of contradiction, assume that \( V_G \) is uncountable. We construct an infinite set monochromatic in color 1 in \( G \) to obtain a contradiction. Since \( V_G \) is well-ordered by \( \leq \), we can construct (via transfinite induction) a maximal set monochromatic in color 0, \( C_0 \) say, without invoking any form of choice. Since \( C_0 \) is countable, it follows that \( V_G - C_0 \) is uncountable and for every vertex \( v \in V_G - C_0 \), there is \( c_0 \in C_0 \) such that \( f(v, c_0) = 1 \). We write \( V_G - C_0 = \bigcup \{ W_p : p \in C_0 \} \), where \( W_p = \{ v \in V_G - C_0 : f(v, p) = 1 \} \). Since \( V_G - C_0 \) is uncountable and \( C_0 \) is countable, it follows by \( \aleph_1 \) is regular” that \( W_p \) is uncountable for some \( p \in C_0 \). Let \( p_0 \) be the least (with respect to \( \leq \)) such vertex of \( C_0 \). Next, we construct a maximal set monochromatic in color 0 in (the uncountable set) \( W_{p_0} \), \( C_1 \) say, and (similarly to the above argument) \( p_1 \) be the least (with respect to \( \leq \)) vertex of \( C_1 \) such that the set \( W_{p_1} = \{ v \in W_{p_0} - C_1 : f(v, p_1) = 1 \} \) is uncountable. Continuing this process step by step and noting that the process cannot stop at a finite stage, we obtain a countably infinite set \( \{ p_n : n \in \omega \} \) monochromatic in color 1, contradicting the assumption that all sets monochromatic in color 1 are finite. Therefore, \( V_G \) is countable.

(2–5). Follows from (1) and the fact that the statement “\( \aleph_1 \) is a regular cardinal” holds in every permutation model (cf. \cite{9} Corollary 1) and \( \text{AC}_{\text{fin}}^{\aleph_0} \) implies the statement “Every \( \omega \)-tree is well-orderable” in ZF.

Proposition 3.4. (ZF) \( \text{CA}_{10}^{\aleph_0} \) implies \( \text{AC}_{\text{fin}}^{\aleph_0} \).

Proof. Since \( \text{AC}_{\text{fin}}^{\aleph_0} \) is equivalent to its partial version \( \text{PAC}_{\text{fin}}^{\aleph_0} \) (cf. Definition 2.3), it suffices to show \( \text{PAC}_{\text{fin}}^{\aleph_0} \). Let \( A = \{ A_i : i \in \omega \} \) be a denumerable family of non-empty, denumerable sets. Without loss of generality, assume that \( A \) is disjoint. For the sake of contradiction, we assume that \( A \) has no partial choice function. Define a binary relation \( \leq \) on \( A = \bigcup \mathcal{A} \) as follows: for all \( a, b \in A \), let \( a \leq b \) if and only if \( a = b \) or \( a \in A_n, b \in A_m \) and \( n < m \). Clearly, \( \leq \) is a partial order on \( A \). Since any two elements of \( A \) are \( \leq \)-comparable if and only if they belong to distinct \( A_i \)'s, and \( \mathcal{A} \) has no partial choice function, all chains in \( (A, \leq) \) are finite. Next, if \( C \subseteq A \) is an antichain in \( (A, \leq) \), then \( C \subseteq A_i \) for some \( i \in \omega \). Thus, all antichains in \( (A, \leq) \) are countable as \( A_i \) is denumerable for all \( i \in \omega \). By \( \text{CA}_{10}^{\aleph_0} \), \( A \) is countable (and hence well-orderable), contradicting \( A \)'s having no partial choice function.

Proposition 3.5. Let \( A \) be a set of atoms. Let \( \mathcal{G} \) be the group of permutations of \( A \) such that either each \( \eta \in \mathcal{G} \) moves only finitely many atoms or there exists an \( n \in \omega \setminus \{0, 1\} \), such that for all \( \eta \in \mathcal{G} \), \( \eta^n = 1_A \). Let \( \mathcal{N} \) be the permutation model determined by \( A, \mathcal{G} \), and a normal filter \( \mathcal{F} \) of subgroups of \( \mathcal{G} \). Then the following hold:

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3 In section 7, we remark that \( \text{CA}_{10}^{\aleph_0} \) + \( \text{CA}_{10}^{\aleph_0} \) \( \Rightarrow \) \( \text{AC}_{\text{fin}}^{\aleph_0} \) in ZFA.
(1) The Antichain Principle A holds in N.
(2) If WUT holds in N, then both CAC_{\aleph_0} and CAC_{\aleph_0}^{WO} hold in N.
(3) If AC_{\aleph_0}^{WO} holds and AC_{\aleph_0} fails in N, then CAC_{\aleph_0} holds and CAC_{\aleph_0} fails in N.

Proof. Let (P, \leq) be a poset in N and E be a support of (P, \leq). Following the proof of [21, Theorem 3], Orb_{BE}(p) = \{ \phi(p) : \phi \in \text{fix}_GE \} is an antichain in P for each p \in P and O = \{ Orb_{BE}(p) : p \in P \} is a well-ordered partition of P.

(1). In N, CS and CWF hold following the methods of [12, Theorem 3.26] and [20, proof of Theorem 10 (ii)]. In [13], it has been established that CWF is equivalent to A in ZFA. Thus A holds in N.

We can observe a different argument to show that A holds in N following the proof of [14, Theorem 9.2(2)].

(2). We show CAC_{\aleph_0} holds in N. Let (P, \leq) be a poset in N such that all chains in P are finite and all antichains in P are countable (and hence well-orderable). Now, P can be written as a well-orderable disjoint union of antichains. Thus, P is well-orderable in N since WUT holds in N. So, we are done by Proposition 3.3(4). Similarly, CAC_{\aleph_0}^{WO} holds in N by Theorem 3.2.

(3). Follows from Proposition 3.4 and the arguments of (2).

Corollary 3.6. CS + A + DF = F + AC_{\aleph_0}^{WO} + CAC_{\aleph_0}^{WO} + DT does not imply MC_{\aleph_0} in ZFA. Consequently, CAC_{\aleph_0} does not imply CAC_{\aleph_0}^{WO} in ZFA.

Proof. Consider the permutation model (say M) from [21, proof of Theorem 5(4)]. In order to describe M, we start with a model M of ZFA + AC with a countably infinite set A of atoms, which is written as a disjoint union \bigcup\{B_n : n \in \omega\}, where \{B_n : n = \aleph_0\} for all n \in \omega. For each n \in \omega, let \mathcal{G}_n be the group of all even permutations of B_n which move only finitely many elements of B_n. Let \mathcal{G} be the weak direct product of the \mathcal{G}_n’s for n \in \omega. Consequently, every permutation of A in G moves only finitely many atoms. Let I be the normal ideal of subsets of A generated by all finite unions of B_n. Let \mathcal{F} be the normal filter on \mathcal{G} generated by \{ \text{fix}_GE,E \in I \} and \mathcal{M} be the permutation model determined by M, \mathcal{G}, and \mathcal{F}. In \mathcal{M}, AC_{\aleph_0}^{WO}, DF = F, and CAC_{\aleph_0}^{WO} hold whereas MC_{\aleph_0} fails, and thus AC_{\aleph_0} fails (cf. [21, proof of Theorem 5(4)]). Since AC_{\aleph_0}^{WO} holds in N, \{ a \in A : g(a) = a \} is finite for any g \in G, DT holds in N following the arguments of [25, Theorem 3.4] where Tachtsis proved that DT holds in Levy’s permutation model (labeled as Model N_0 in [10]). The rest follows from Proposition 3.5 and the fact that if g \in G, then \{ a \in A : g(a) \neq a \} is finite.

Proposition 3.7. (ZF + WOAM) RT holds for any locally countable connected graph H = (V_H,E_H).

Proof. If V_H is well-orderable then we are done. Otherwise, by WOAM, there exists an amorphous subset V_G \subseteq V_H. Fix some r \in V_G. Let V_0 = \{ r \}. For each n \in \omega \setminus \{ 0 \}, define V_n = \{ v \in V_G : d_G(r,v) = n \} where “d_G(r,v) = n” means there are n edges in the shortest path joining r and v. By connectedness of G, V_G = \bigcup_{n \in \omega} V_n. Since V_G is amorphous, there is at most one t \in \aleph_0, such that V_t is infinite. Fix such a t, if it exists. Then V_t is countably infinite since |V_{t-1}| = \aleph_0, G is locally countable, and the union of a finite family of countable sets is countable in ZF. Define f : V_G \to \aleph_0 by f(y) = n if y \in V_n. Since V_G is amorphous, |ran(f)| < \aleph_0. Otherwise, ran(f) can be partitioned into two infinite sets A and B and the infinite sets f^{-1}(A) and f^{-1}(B) will partition V_G; a contradiction. Thus, V_G = \bigcup_{n \in \aleph_0} V_n is countable since the finite union of countable sets is countable in ZF. The rest follows from Theorem 3.1.

4. Erdős–Dushnik–Miller theorem and its variants

4.1. Using the first definition of uncountable sets

Theorem 4.1. (ZF) The following hold:

(1) DC_{\aleph_1} implies EDM. In particular, W_{\aleph_1} + “\aleph_1 is regular” implies EDM.
(2) If X \in \{ CAC_{\aleph_1}, CAC_{\aleph_0}, PAC_{\aleph_1}, AC_{\aleph_1}, AC_{\aleph_0} \}, then EDM implies X. So, DC does not imply EDM.
(3) EDM implies RT.
(4) DF = F does not imply CAC_{\aleph_0}. Consequently, RT does not imply CAC_{\aleph_0} and EDM.

Proof. (1). Following Proposition 3.3 and the arguments of [21, Theorem 9(1,2)], we can see that DC_{\aleph_1} implies EDM in ZF. In particular, let G = (V_G,E_G) be a graph and f : |V_G|^2 \to \{ 0, 1 \} be a coloring such that all sets monochromatic in color 0 are countable and all sets monochromatic in color 1 are finite. By W_{\aleph_1}, \aleph_1 \leq |V_G| or |V_G| \leq \aleph_1. For the second case, V_G is well-orderable, and we are done by Proposition 3.3 since DC_{\aleph_1} implies
\( W_{\aleph_1} + \text{"\( \aleph_1 \) is regular".} \) Otherwise, \( V_G \) has a subset \( H \) with cardinality \( \aleph_1 \). Since \( H \) well-orderable, it is countable by the arguments of Proposition 3.3; a contradiction.

(2) We prove EDM implies \( \text{CAC}^{\aleph_6} \). Let \((P, \leq)\) be a poset satisfying the hypotheses of \( \text{CAC}^{\aleph_6} \). Assume that \( P \) is uncountable. Let \( G = (V_G, E_G) \) be a complete graph such that \( V_G = P \) and \( f : [V_G]^2 \to \{0, 1\} \) be a coloring such that \( f(x, y) = 1 \) if \( x \leq y \) or \( y \leq x \), and \( f(x, y) = 0 \) otherwise. By EDM, either there is an uncountable set monochromatic in color 0 (which is an antichain in \((P, \leq)\)) or there is a countably infinite set monochromatic in color 1 (which is a chain in \((P, \leq)\)), a contradiction. Similarly, we can prove EDM implies \( \text{CAC}^{\aleph_5} \). The rest follows from Proposition 3.4 and Fact 3.1.

(3). Let \( A \) be an infinite set such that \( RT \) fails for \( A \). Let \((X, Y)\) be a partition of \([A]^2\) such that there are no infinite subsets \( B \) of \( A \) for which \( [B]^2 \subseteq X \) or \( [B]^2 \subseteq Y \). Let \( G = (V_G, E_G) \) be a complete graph such that \( V_G = A \) and \( f : [V_G]^2 \to \{0, 1\} \) be a coloring such that \( f(x, y) = 1 \) if \( x \in X \) and \( f(x, y) = 0 \) if \( x \in Y \). By assumption, all sets monochromatic in color \( i \) are finite for each \( i \in \{0, 1\} \). By EDM, \( |V_G| \leq \aleph_0 \) (since we are using the first definition of uncountable sets), and thus \( V_G = A \) is well-orderable. The contradiction follows from the fact that \( RT \) holds for \( A \) in ZF (cf. Fact 3.1).

(4). Consider the model \( N_{11} \) from [10] where DF = F holds and \( \text{AC}^{\aleph_6} \) fails (cf. [23, Theorem 4], [10, Note 112]). We note that DF = F is injectively boundable since it is equivalent to \( \forall x (|x| \leq \omega \rightarrow x \text{ is finite}) \). Furthermore, \( \neg \text{AC}^{\aleph_6} \) is bounded, and hence injectively boundable. Since \( \phi = \text{DF} = F \land \neg \text{AC}^{\aleph_0} \) is a conjunction of injectively boundable statements, which has a ZFA model, it follows from Theorem 2.6 that \( \phi \) has a ZF model. By Proposition 3.4, we can see that \( \text{DF} = F \) (and thus \( RT \)) does not imply \( \text{CAC}^{\aleph_0} \) in ZF.

**Theorem 4.2.** The following hold:

1. EDM does not imply "There are no amorphous sets" in ZFA.
2. Let \( A \) be a set of atoms, \( G \) be any group of permutations of \( A \), and the filter \( F \) of subgroups of \( G \) be generated by \( \{\text{fix}_x(G) : x \in A^{<\omega}\} \). Let \( N \) be the permutation model determined by \( A, G, \) and \( F \). If \( N \) satisfies the Mostowski’s intersection lemma where \( A \) is Dedekind-finite, and RT holds in \( N \), then EDM holds in \( N \).
3. EDM holds in \( N_2 \). Consequently, EDM implies none of WOAM, CS, and A(antichain Principle) in ZFA.
4. WOAM + RT implies EDM and WOAM + CAC implies \( \text{CAC}^{\aleph_6} \) in ZF.

**Proof.** (1). We prove that EDM is true in \( N_1 \) in which the set \( A \) of atoms is amorphous. In \( N_1 \), let \( G = (V_G, E_G) \) be a graph and \( f : [V_G]^2 \to \{0, 1\} \) be a coloring such that all sets monochromatic in color 0 are countable, and all sets monochromatic in color 1 are finite.

Case (i). Suppose \( V_G \) is well-orderable in \( N_1 \). Then we are done by Proposition 3.3.

Case (ii). Suppose \( V_G \) is not well-orderable in \( N_1 \). Let \( E \in [A]^{<\omega} \) be a support of \( G \). By Lemma 2.5, there is a \( t \in V_G \) and \( \pi \in \text{fix}_x(G) \) such that \( \pi(t) \neq t \). Let \( F \cup \{a\} \) be a support of \( t \) where \( a \in A \setminus (E \cup F) \). Under such assumptions, Blass [4, p.389] proved that

\[
g = \{ (\sigma(a), \sigma(t)) : \sigma \in \text{fix}_x(G \cup F) \}
\]

is a bijection from an infinite subset \( A' = A \setminus (E \cup F) \) onto \( H = \text{ran}(g) = \{ \sigma(t) : \sigma \in \text{fix}_x(G \cup F) \} \subset V_G \). Define the following partition of \([H]^2\):

\[
X = \{ \{a, b\} \in [H]^2 : f\{a, b\} = 0 \}, \quad Y = \{ \{a, b\} \in [H]^2 : f\{a, b\} = 1 \}.
\]

Since \( (H, E_G \upharpoonright H) \) is an infinite graph where all sets monochromatic in color 1 are finite, there is no infinite subset \( B' \subseteq H \) such that \( [B']^2 \subseteq X \). So \( (H, E_G \upharpoonright H) \) has an infinite set monochromatic in color 0 in \( N_1 \), say \( C \). By assumption, \( C \) is a countably infinite subset of \( H \). Thus \( A' \) is Dedekind-infinite in \( N_1 \) since \( |H| = |A'| \). Consequently, the set \( A \) of atoms is Dedekind-infinite in \( N_1 \), which contradicts the fact that \( A \) is a Dedekind-finite set in \( N_1 \).

(2). By Lemma 2.7, every set \( x \) in \( N \) is either well-orderable or there exists an infinite subset of \( A \), which embeds into \( x \) under the given assumptions. The rest follows from the arguments of (1).

(3). Follows from (2), and the following known facts about \( N_3 \):

(i) \( N_3 \) satisfies the Mostowski’s intersection lemma (cf. [14]).
(ii) RT is true in \( N_3 \) (cf. [27, Theorem 2.4]).
(iii) The set of atoms \( A \) is a Dedekind-finite set in \( N_3 \).
(iv) WOAM fails in \( N_3 \) (cf. [10]).
(v) CS and LW (Every linearly ordered set can be well ordered) fail in $\mathcal{N}_5$ [21 Theorem 7] and A implies LW in ZFA [13 Theorem 9.1].

(4) Suppose WOAM + RT is true. Assume $G = (V_G, E_G)$ and $f : |V_G|^2 \to \{0, 1\}$ as in (1). If $V_G$ is well-orderable then we are done by Proposition 3.3 and the fact that WOAM implies "$\aleph_1$ is regular" in ZF (cf. Fact 3.1). Otherwise, $V_G$ has an amorphous subset, say $A$ by WOAM. Define the following partition of $|A|^2$ as in (1): $X = \{\{a, b\} \in |A|^2 : f(a, b) = 0\}$, $Y = \{\{a, b\} \in |A|^2 : f(a, b) = 1\}$. By RT and following the arguments of (1), there is a countably infinite subset of $A$. This contradicts the fact that $A$ is amorphous.

Similarly, WOAM + CAC implies CAC$^{\aleph_0}$ in ZF following Proposition 3.3.

Proposition 4.3. (ZF) The statements EDM$^n$ ("if $G = (V_G, E_G)$ is a graph such that $V_G$ is uncountable, then for all coloring $f : |V_G|^2 \to n$ either there is an uncountable set $X_1 \subseteq V_G$ monochromatic in color 1 or there is a countably infinite set $X_2 \subseteq V_G$ monochromatic in color $i_2$ for some $i_1, i_2 \in n$ such that $i_1 \neq i_2"$) are equivalent for all integers $n \geq 2$. Moreover, EDM$^n$ implies RT for all $n \in \omega \setminus \{0, 1\}$.

Proof. Since any $f : |V_G|^2 \to n$ maps $|V_G|^2$ to $m$ if $m > n \geq 2$, EDM$^n$ implies EDM$^m$ under these circumstances. We prove that EDM$^n$ implies EDM$^{n+1}$ for $n \geq 2$. The rest follows by mathematical induction. Let $f : |V_G|^2 \to n+1$ be a coloring where $V_G$ is uncountable. Let $f_1 : |V_G|^2 \to n$ be given by $f_1(a) = \min(f(a), n - 1)$. By EDM$^n$, either there is an uncountable set $X_1 \subseteq V_G$ that is $f_1$-monochromatic in color 1, or there is a countably infinite set $X_2 \subseteq V_G$ that is $f_1$-monochromatic in color $i_2$ for some $i_1, i_2 \in n$ such that $i_1 \neq i_2$. Fix $k \in \{1, 2\}$. If $i_k \leq n - 2$, then $f[X_k]^2 = i_k \in n + 1$.

Case (i): $i_1 = n - 1$, then $f[X_1]^2 = \{n, n - 1\}$. Define $f_2 : |X_1|^2 \to 2$ by $f_2(a) = n - f(a)$. By EDM$^2$, either there is an uncountable set $Y_1 \subseteq X_1$ that is $f_2$-monochromatic in color $j$, or there is a countably infinite set $Y_2 \subseteq X_1$ that is $f_2$-monochromatic in color $1 - j$ where $j \in \{0, 1\}$. Thus either $f[Y_1]^2 = n - j \in n + 1$ or $f[Y_2]^2 = n - 1 - j \in n + 1$.

Case (ii): $i_2 = n - 1$, then $f[X_2]^2 = \{n, n - 1\}$. Define the following partition of $|X_2|^2$:

$$X = \{\{a, b\} \in |X_2|^2 : f(a, b) = n\}, \quad Y = \{\{a, b\} \in |X_2|^2 : f(a, b) = n - 1\}.$$ 

By RT (which follows from EDM$^2$, see the arguments of Theorem 4.1(3)), there is a countably infinite subset $B \subseteq X_2$ such that either $|B|^2 \subseteq X$ or $|B|^2 \subseteq Y$. Thus either $|B|^2 = n \in n + 1$ or $|B|^2 = n - 1 \in n + 1$.

Following Theorem 4.1(3), EDM$^2$ implies RT.

4.2 Using the second definition of uncountable sets.

Proposition 4.4. (ZF) The following hold:

(1) UT($\aleph_0$, $\not\preceq_\omega$, $\not\preceq_\omega$) + EDM implies DF = F.
(2) UT($\aleph_0$, $\not\preceq_\omega$, $\not\preceq_\omega$) + CAC$^{\aleph_0}$ implies DF = F.
(3) UT($\aleph_0$, $\not\preceq_\omega$, $\not\preceq_\omega$) + CAC$^{\aleph_0}$ implies DF = F.

Proof. (1). Assume that $A$ is a Dedekind finite set which is not finite. We may also assume that $A \cap \omega = \emptyset$. Let $\mathcal{F} = \{\{k, a\} : k \in \omega, a \in A\}$. Define a graph $G = (V_G, E_G)$ as follows: $V_G = \bigcup \mathcal{F}$. For all $a, b \in V_G$,

let $\{a, b\} \in E_G$ if and only if $a \in \{k_1, a_1\}$, $b \in \{k_2, a_2\}$ and $\{k_1, a_1\} \neq \{k_2, a_2\}$.

Clearly, $|V_G| \geq \aleph_0$. If $|V_G| = \aleph_0$, then $|A| \leq \aleph_0$, contradicting the assumption that $A$ is Dedekind finite. So we have $|V_G| > \aleph_0$. Thus $V_G$ is uncountable. Let $f : |V_G|^2 \to \{0, 1\}$ be a coloring such that $f(x, y) = 0$ if $\{x, y\} \in E_G$, and $f(x, y) = 1$ otherwise. By EDM, either there is an uncountable set monochromatic in color 0 (i.e., an uncountable clique) or there is a countably infinite set monochromatic in color 1 (i.e., a countably infinite independent set). The only independent sets of $G$ are the finite sets $\{k, a\}$ and subsets of $\{k, a\}$ where $k \in \omega$ and $a \in A$. Thus $G$ has an uncountable clique, say $C$. Fix $k \in \omega$. Let

$$C_k = \{\{k, a\} \cap C : a \in A, \{k, a\} \cap C \neq \emptyset\}.$$ 

Clearly, $C = \bigcup_{k \in \omega} C_k$. We can see that $|C_k| \not\preceq \aleph_0$ for any $k \in \omega$. Suppose $|C_k| > \aleph_0$ for a particular $k \in \omega$. Then $|A| \geq |C_k| > \aleph_0$, a contradiction. Consequently, by UT($\aleph_0$, $\not\preceq_\omega$, $\not\preceq_\omega$), we have $|C| \not\preceq \aleph_0$, a contradiction to the fact that $C$ is uncountable.

Similarly, we can prove (2).

(3). Assume $A$ and $\mathcal{F}$ as in (1). Define a poset $\mathcal{P} = (P, <)$ as follows: $P = \bigcup \mathcal{F}$. For all $a, b \in P$,
let \( a < b \) if and only if \( a = k_1, b = a_1 \) for some \( \{k_1, a_1\} \in \mathcal{F} \).

Clearly, \(|P| \geq \aleph_0\). If \(|P| = \aleph_0\), then \(|A| \leq \aleph_0\), contradicting our assumption that \( A \) is Dedekind finite. So we have \(|P| > \aleph_0\). Thus \( P \) is uncountable. The only chains in \( \mathcal{P} \) are the finite sets \( \{k, a\} \) and subsets of \( \{k, a\} \) where \( k \in \omega \) and \( a \in A \). By \( \text{CAC}_{\aleph_0} \), \( \mathcal{P} \) has an uncountable antichain, say \( C \). Let \( C_k = \{\{k, a\} \cap C : a \in A, \{k, a\} \cap C \neq \emptyset\} \) for all \( k \in \omega \). So, \( C = \bigcup_{k \in \omega} C_k \). The rest follows from the arguments of the previous proof. \( \square \)

5. Observations in Halbeisen–Tachtsis’s model

We say that a topological space \( X \) is \( C \in \mathcal{C} \) in a permutation model \( N \) generated with respect to a normal ideal \( I \), if there is an \( c_0 \in I \) where for all \( e \geq c_0 \) in \( I \) and all nonempty closed subsets \( C \) of \( X \) with support \( e \), \( C \cap \Delta(e) \neq \emptyset \).

The space \( X \) is \( A \)-bounded if amorphous subsets of \( X \) are relatively compact. \( ^5 \) We recall some known facts mainly from \( [3] \), which we need in order to prove Theorem 5.3.

Fact 5.1. The following hold:

1. \( [3] \) Lemma 2.1 If \( X \) is a product of a well-orderable family of compact spaces and \( X \) is \( C \), then \( X \) is compact in ZF.
2. Product of \( A \)-bounded spaces is \( A \)-bounded (cf. \( [3 \) Lemma 3.3(3)]\) and the product of \( T_2 \) spaces is \( T_2 \) in ZF.
3. \( [3] \) proof of Lemma 3.1 In ZF, if \( Y \) is an amorphous subset of a \( T_2 \) space \( X \), then \( \overline{\text{orb}}(Y) = Y \) is discrete or \( Y \) has one non-isolated point \( p \) such that \( \overline{\text{orb}}(Y) \) is the one-point compactification of the discrete space \( Y \setminus \{p\} \).
4. \( [3] \) Lemma 2.2(2) In a permutation model satisfying the Mostowski’s intersection lemma, any topological space \( X \) is \( C \) if and only if \( X \) is weakly Loeb.
5. \( [3] \) Lemma 2.4 In any minimal support model with respect to a normal ideal \( I \), the following conditions are equivalent for any topological space \( X \):
   a. \( X \) is weakly Loeb.
   b. there is an \( E_0 \in I \) which supports \( X \) such that for all \( E \supseteq E_0 \in I \) and all \( x \in X \) where \( |\text{supp}(x) \setminus E| = 1 \), the set \( \{\text{orb}_{E}x\} \cap \Delta(E) \neq \emptyset \) (We denote by \( \text{supp}(x) \), the least support of \( x \)).

Fix any \( n \in \omega \backslash \{0, 1\} \). We work with the permutation model \( N_{HT}^1(n) \) constructed by Halbeisen–Tachtsis in the proof of \([11] \) Theorem 8 where \( \text{AC}^-_{\omega} \) fails. Let \( M \) be a model of ZFA + \( \text{AC} \) where \( A \) is a countably infinite set of atoms written as a disjoint union \( \bigcup \{A_i : i \in \omega\} \) where for all \( i \in \omega \), \( A_i = \{a_{i,1}, a_{i,2}, ..., a_{i,n}\} \) and \( |A_i| = n \). The group \( G \) is defined in \([11] \) in a way so that if \( \eta \in G \), then \( \eta \) only moves finitely many atoms and for all \( i \in \omega \), \( \eta(A_i) = A_k \) for some \( k \in \omega \). Let \( F \) be the normal filter generated by \( \{\text{fix}_{G}(E) : E \in [A]^{<\omega}\} \) where \( I = [A]^{<\omega} \) is the normal ideal. Without loss of generality, we assume that a support \( E \) has the property that for all \( i \in \omega \), either \( A_i \subseteq E \) or \( A_i \cap E = \emptyset \).

The model \( N_{HT}^1(n) \) is the permutation model determined by \( M, G, \) and \( F \). The set of atoms \( A \) is amorphous in \( N_{HT}^1(n) \) (cf. the proof of \([11] \) Theorem 8)).

Lemma 5.2. The following hold:

1. \( N_{HT}^1(n) \) is a minimal support model.
2. In \( N_{HT}^1(n) \), any topological space \( X \) is \( C \) if and only if \( X \) is weakly Loeb.
3. In \( N_{HT}^1(n) \), the following conditions are equivalent for any topological space \( X \):
   a. \( X \) is weakly Loeb.
   b. there is an \( E_0 \in I \) which supports \( X \) such that for all \( E \supseteq E_0 \in I \) and all \( x \in X \) where \( |\text{supp}(x) \setminus E| = 1 \), the set \( \{\text{orb}_{E}x\} \cap \Delta(E) \neq \emptyset \).

Proof. (1). Let \( x \in N_{HT}^1(n) \) and \( E_0 \) be a support of \( x \). We recall that if \( E_1, E_2 \) are supports of \( x \), then \( E_1 \cap E_2 \) is a support of \( x \) (cf. \([27] \) the proof of Lemma 1).\( ^4 \) The least support of \( x \) is the intersection of all supports of \( x \) which are subsets of \( E_0 \). Since there are only finitely many such supports, the intersection is a support of \( x \).

(2, 3). Follows from Fact 5.1(4,5), (1) and the fact that any minimal support model satisfies the Mostowski’s intersection lemma. \( \square \)

Theorem 5.3. Fix any \( n \in \omega \backslash \{0, 1\} \) and \( X \in \{\text{CAC}_{\omega}, \text{CAC}_{\omega}^2, \text{CS}\} \). There is a model \( M \) of ZFA where \( X \) holds but \( \text{AC}^-_{\omega} \) and the statement “there are no amorphous sets” fail. Moreover, the following hold in \( M \):

\( ^4 \)These are some definitions from \([5 \) section 2.3] and \([3 \) Proposition 3.1].

\( ^5 \)Tachtsis proved \([27 \) Lemmas 1.2] in \( N_{HT}^2(2) \). We note that those lemmas hold in \( N_{HT}^1(n) \) if we replace 2 by \( n \) with similar proofs.
(1) a $T_2$-space is weakly Loeb if and only if it is $A$-bounded.
(2) **Form 115**
(3) **Form 116**
(4) a product of a well-orderable family of compact $T_2$ spaces is compact.
(5) $\neg$EDM and $\neg$EDM$^*$ for each $k \geq 2$.
(6) Antichain Principle A.

**Proof.** Consider the permutation model $\mathcal{N}_{HT}^1(n)$. Banerjee [11] observed that $\text{CAC}^\aleph_0$, $\text{CS}$, and $\text{CAC}^\aleph_0$ hold in $\mathcal{N}_{HT}^1(n)$ (cf. Proposition 3.5 as well).

(1). Firstly, we prove that weakly Loeb $T_2$-spaces are $A$-bounded. Let $X$ be a weakly Loeb $T_2$ space. By Lemma 5.2(3), there is an $E_0 \in \mathcal{I}$ which supports $X$ such that for all $E \supseteq E_0$ in $\mathcal{I}$ and all $x \in X$ which satisfy $\text{supp}(x) \cap \Delta(E) \neq \emptyset$. Fix any amorphous subset $Y$ of $X$ supported by some $E \supseteq E_0$. By Fact 5.1(3), we obtain the following:

**Case (i):** $\overline{Y}$ has exactly one non-isolated point $p$ such that $\overline{Y}$ is the one-point compactification of the discrete space $B = Y \setminus \{p\}$. Then $X$ is $A$-bounded.

**Case (ii):** $\overline{Y} = Y$ is discrete. Since $Y$ is a non well-orderable set, there is an $x \in Y$ and a $\phi \in \text{fix}_\aleph(E)$ such that $\phi(x) \neq x$ by Lemma 2.5. Let $\text{supp}(x) = \bigcup_{i \in K_i} A_i$ be the least support of $x$ where $K_i \subseteq [\omega]^\omega$. Since $E$ is not a support of $x$, $\text{supp}(x) \setminus E \neq \emptyset$. Let $E \subseteq F$ so that $\text{supp}(x) \setminus F = A_i$ for some $i_0 \in K_i$. Define the following set:

$$g = \{(\psi(x), \psi(A_i)) : \psi \in \text{fix}_\aleph(E \cup \text{supp}(x) \setminus A_i)\}.$$ 

Tachtsis proved that $g$ is a function where $\text{dom}(g) = \overline{\text{orb}_{E \cup \text{supp}(x) \setminus A_i} x} = \overline{\text{orb}_E x}$ is an amorphous subset of $Y$ (cf. proof of [27, Lemma 2]). As $\overline{\text{orb}_E x}$ is amorphous, by Fact 5.1(3), we get that either $\overline{\text{orb}_E x} = \overline{\text{orb}_F x}$ or $\overline{\text{orb}_F x}$ has exactly one non-isolated point $q$ such that $\overline{\text{orb}_F x}$ is the one-point compactification of the discrete space $B = \overline{\text{orb}_F x \setminus \{q\}}$. The first case is impossible since $(\overline{\text{orb}_E x} \cap \Delta(f) = \emptyset$ and $(\overline{\text{orb}_F x} \cap \Delta(f) \neq \emptyset$. The second case implies that $Y = \overline{Y}$ is not discrete since if $Y$ is discrete then $\overline{\text{orb}_E x}$ must be discrete.

Secondly, we prove that $A$-bounded $T_2$ spaces are weakly Loeb. Let $X$ be an $A$-bounded $T_2$ space supported by $E_0 = \bigcup_{i \in \omega} A_i$. Pick any $E \supseteq E_0$ in $\mathcal{I}$ such that $E = \bigcup_{i=0}^\infty A_i$ and any $x \in X$ such that $\text{supp}(x) \setminus E = A_i$ for some $i \in \omega$. We show that $\overline{\text{orb}_E x} \cap \Delta(E) = \emptyset$. By the arguments of the previous paragraph, $Y = \overline{\text{orb}_E x}$ is an amorphous subset of the $T_2$ space $X$. Thus either $\overline{Y} = Y$ is discrete or $\overline{Y}$ has exactly one non-isolated point $p$ such that $\overline{Y}$ is the one-point compactification of the discrete space $B = Y \setminus \{p\}$. The first case is impossible since $X$ is an $A$-bounded space. Thus, we obtain $\overline{Y} \cap \Delta(E) \neq \emptyset$.

(2-4). Follows from (1), Fact 5.1, and Lemma 5.2 (cf. [4 Corollaries 3.2, 3.3, 3.4]).

(5). Follows from Theorem 4.1(3), Proposition 4.3, and the fact that $\text{RT}$ fails in $\mathcal{N}_{HT}^1(n)$ (cf. [27]).

(6). Follows from Proposition 3.5 since if $\eta \in \mathcal{G}$, then $\eta$ only moves finitely many atoms.

6. Loš’s Theorem, and the Uniqueness of Algebraic Closures

We recall some known facts, which we need in order to prove Theorem 6.2 and Remark 6.3.

**Fact 6.1.** The following hold:

(1) $\text{DF} = \text{F}$ implies “For every infinite set $X$, $\text{Sym}(X) \neq F\text{Sym}(X)$” in $\text{ZF}$ (cf. [22, Theorem 3.1]).

(2) $\text{MC} + \text{CAC}^\aleph_0$ implies $\text{DF} = \text{F}$ in $\text{ZFA}$ (cf. [21, Theorem 5(3)]).

(3) $\text{MC}$ if and only if ‘Every infinite set has a well-ordered partition into non-empty finite sets’ (cf. [18]).

(4) If $\mathcal{K}$ is an algebraically closed field, if $\pi$ is a non-trivial automorphism of $\mathcal{K}$ satisfying $\pi^2 = 1_{\mathcal{K}}$, and if $i = \sqrt{-1} \in \mathcal{K}$, then $\pi(i) = -i \neq i$ (cf. [10, Note 41]).

**Theorem 6.2. (ZFA)** The following hold:

(1) $\text{AC}^{\text{LO}} + \text{Form 233}$ neither implies $\text{LT}$ nor implies $\text{W}_{\aleph_1}$.

(2) $\text{AC}^{\text{LO}} + \text{Form 233} + \text{EDM}$ neither implies $\text{LT}$ nor implies $\text{W}_{\aleph_2}$.

**Proof.** (1). We consider the permutation model $\mathcal{N}$ given in the proof of [23, Theorem 4.7] where $\text{AC}^{\text{LO}}$ holds and $\text{LT}$ fails. We start with a model $\mathcal{M}$ of $\text{ZFA} + \text{AC}$ with an $\aleph_1$-sized set $A$ of atoms so that $A = \bigcup\{A_i : i < \aleph_1\}$ where $|A_i| = \aleph_0$ for all $i < \aleph_1$, and $A_i \cap A_j = \emptyset$ for all $i, j < \aleph_1$ with $i \neq j$. Let $\mathcal{G}$ be the group of all permutations $\phi$ of $A$ such that $\forall i < \aleph_1, \exists j < \aleph_1, \phi(A_i) = A_j$, and $\phi$ moves only $\aleph_0$ atoms. Let $\mathcal{F}$ be the normal filter of
subgroups of $G$ generated by $\text{fix}_G(E)$, where $E = \bigcup \{A_i : i \in \mathcal{I}\}, \mathcal{I} \in [\mathbb{N}_1]^{<\mathbb{N}_1}$. The model $\mathcal{N}$ is the permutation model determined by $M, \mathcal{G}$ and $\mathcal{F}$.

We prove that **Form 233** holds in $\mathcal{N}$. Fix a field $K'$ in $\mathcal{N}$. Let $K$ be an algebraic closure of $K'$ in $\mathcal{N}$ with support $E$. If $\mathcal{K}$ is well-orderable in $\mathcal{N}$ then we can use transfinite induction without using any form of choice to finish the proof. Otherwise, there is a $x \in \mathcal{K}$ and a $\phi \in \text{fix}_G(E)$ with $\phi(x) \neq x$. Under such assumptions, Tachtsis constructed a permutation $\psi \in \text{fix}_G(E)$ such that $\psi(x) \neq x$ but $\psi^2(x) = x$ (cf. the proof of LW in $\mathcal{N}$ from [23 claim 4.10]). The permutation $\psi$ induces an automorphism of $\mathcal{K}$ and we can therefore apply **Fact 6.1(4)** to conclude that $\psi(i) = i \neq i$ for some $i = \sqrt{-1} \in \mathcal{K}$. We can follow the arguments from [10 Note 41] to see that for every $\pi \in \text{fix}_G(E)$, $\pi(i) = i$ for every $i = \sqrt{-1} \in \mathcal{K}$. Hence we arrive at a contradiction.

We show that $(\text{Sym}(A)) = \aleph_0 \text{Sym}(A) \in \mathcal{N}$. For the sake of contradiction, assume $f \in (\text{Sym}(A)) \setminus \aleph_0 \text{Sym}(A)$. Let $E = \bigcup \{A_i : i \in \mathcal{I}\}, \mathcal{I} \in [\mathbb{N}_1]^{<\mathbb{N}_1}$ be a support of $f$. Then there exists $i \in \mathbb{N}_1 \setminus \mathcal{I}$ such that $a \in A_i, b \in A_i \setminus (E \cup \{a\})$ and $b = f(a)$.

**Case (i):** Let $b \in A_i$. Consider $\phi \mid A_i$ such that $\phi \mid A_i$ moves every atom in $A_i$ except $b$ and $\phi \mid A \setminus A_i = 1_{A \setminus A_i}$. Clearly, $\phi \in \mathcal{G}$. Also, $\phi(b) = b$, $\phi \in \text{fix}_G(E)$, and hence $\phi(f) = f$. Thus $(a, b) \in f \implies (\phi(a), \phi(b)) \in (f)$ which contradicts $f$ is not injective; a contradiction.

**Case (ii):** If $b \in A_i \setminus (E \cup A_i)$, then consider $\phi \mid A_i$ such that $\phi \mid A_i$ moves every atom in $A_i$ and $\phi \mid A \setminus A_i = 1_{A \setminus A_i}$. Again $\phi \in \mathcal{G}$, and we easily obtain a contradiction.

We note that $A$ is a set of size $\aleph_1$. In order to show that $W_{\aleph_0}$ fails, we prove that there is no injection $f : \aleph_1 \rightarrow A$. Assume there exists such an $f$ and suppose $\{y_n\}_{n \in \aleph_0}$ is an enumeration of the elements of $Y = f(\aleph_0)$. We can use transfinite recursion, without using any form of choice, to construct a bijection $f : Y \rightarrow Y$ such that $f(x) \neq x$ for any $x \in Y$. Define $g : A \rightarrow A$ as follows: $g(x) = f(x)$ if $x \in Y$, and $g(x) = x$ if $x \in A \setminus Y$. Clearly $g \in \text{Sym}(A) \setminus \aleph_0 \text{Sym}(A)$, and hence $\text{Sym}(A) \neq \aleph_0 \text{Sym}(A)$; a contradiction.

(2) Consider the permutation model of (1) (say $\mathcal{N}$) by replacing $\aleph_0$ and $\aleph_1$ with $\aleph_1$ and $\aleph_2$ respectively. Following the arguments of [23 Theorem 4.7], we can see that $\mathcal{AC}^{\mathcal{L}_{\omega_1}}$ holds in $\mathcal{N}$, but LT fails. By the arguments of the previous proof, $W_{\aleph_0}$ fails and **Form 233** holds in $\mathcal{N}$. Moreover, $\mathcal{DC}_{\aleph_1}$ holds since $\mathcal{I}$ is closed under $\aleph_2$ unions (cf. [14]), the arguments in the proof of **Theorem 8.3 (iii)**). Consequently, EDM holds by **Theorem 4.1(1)**.

**Remark 6.3.** Consider the permutation model in the second assertion of [11 Theorem 10(ii)] where $A$ is an $\aleph_1$-sized set of atoms written as a disjoint union $A = \bigcup \{A_{\beta} : \beta < \aleph_1\}, \text{where for all } \beta < \aleph_1, A_{\beta} = \{a_{\beta, 1}, \ldots, a_{\beta, 4}\}$. Let $\mathcal{G}$ be the weak direct product of $\text{Alt}(A_{\beta})$’s where $\text{Alt}(A_{\beta})$ is the alternating group on $A_{\beta}$ for each $\beta < \aleph_1$. Let $\mathcal{F}$ be the finite support filter. Let $\mathcal{M}$ be the permutation model determined by $M, \mathcal{G}$ and $\mathcal{F}$. In $\mathcal{M}$, $\mathcal{OC}_{\aleph_2}$ (Every infinite linearly ordered family of 2-element sets has an infinite subfamily with a choice function) holds (cf. second assertion of [11 Theorem 10(ii)])

Let $x \in M$ be an infinite set and $E$ be a support of $x$. Then $\mathcal{O} = \{\text{Orb}_E(y) : y \in x\}$ is a well-orderable partition of $x$ in $\mathcal{M}$ and following the arguments of [22 Claim 3.8], $\text{Orb}_E(y)$ is finite for every $y \in x$. By **Fact 6.1(3)**, MC holds in $\mathcal{M}$. We observe that $\text{DF} = \mathcal{F}$ fails in $\mathcal{M}$. In view of **Fact 6.1(1)**, it suffices to show $(\text{Sym}(A)) = \text{FSym}(A)$ in $\mathcal{M}$. For the sake of contradiction, assume $f \in (\text{Sym}(A)) \setminus \text{FSym}(A)$. Let $E = \bigcup_{i=0}^{k} A_i$ be a support of $f$ for some $k \in \omega$. Then there exists $i \in \mathbb{N}_1$ with $i > k, a \in A_i$, and $b \in A \setminus (E \cup \{a\})$ such that $b = f(a)$. Suppose $b \in A_i$. Let $c, d \in A_i \setminus \{a, b\}, \phi \mid A_i = (a, c, d) = (a, d)(c) \in \text{Alt}(A_i)$ and $\phi \mid A \setminus A_i = 1_{A \setminus A_i}$. Clearly, $\phi \in \mathcal{G}$ and following the proof of **Theorem 6.2**, $f$ is not injective; a contradiction. If $b \in A \setminus (E \cup A_i)$, then let $x, y \in A_i \setminus \{a\}, \phi \mid A_i = (a, x, y)$ and $\phi \mid A \setminus A_i = 1_{A \setminus A_i}$. Then again we easily obtain a contradiction.

By **Fact 6.1(2)**, $\mathcal{AC}_{\aleph_0}^{\mathbb{N}}$ fails in $\mathcal{M}$. By **Theorem 4.1**, if $x \in \{\mathcal{EDM}, \mathcal{CAC}^{\aleph_0}, \mathcal{CAC}^{\aleph_1}\}$ then $\mathcal{OC}_{\aleph_2} + \text{MC}$ does not imply $X$ in ZFA. Since MC fails in $N_1$, EDM and MC are mutually independent in ZFA by **Theorem 4.2**.

7. Concluding remarks and questions

7.1. Remarks. (1) Recently, Banerjee and Karagila [2 15] proved that if $V$ is a model of ZFC, then $\mathcal{DC}_{\aleph_1}$ can be preserved in the symmetric extension $\mathcal{N}$ of $V$ (symmetric extension of a forcing extension where AC can consistently fail) if the forcing notion $\mathbb{P}$ is either $\aleph_2$-distribution or $\aleph_2$-c.c., $G$ is any group of automorphisms of $\mathbb{P}$, and the normal filter $\mathcal{F}$ of subgroups over $G$ is $\aleph_2$-complete. By **Theorem 4.1**, EDM holds in $\mathcal{N}$.

(2) By **Theorems 4.1(1,3)**, and the facts that $\mathcal{DC}_{\aleph_1}$ does not imply LT and LT does not imply RT in $\mathcal{ZF}$ (cf. [23 Theorems 4.3, 4.13]), LT and EDM are mutually independent in ZF.

(3) Fix $X \in \{\mathcal{BP}_{\mathcal{I}}, \mathcal{KW}, \mathcal{AC}_{\mathcal{WO}}, \mathcal{2}$-coloring theorem, “There are no amorphous sets”). Blass [4] proved that RT is false in the basic Cohen model (Model $M_1$ in [10]) where $X$ holds. Following **Theorem 4.1(3)**, $X$ does not
imply EDM in ZF. On the other hand, $X$ fails in $N_1$. Thus EDM and $X$ are mutually independent in ZFA by Theorem 4.2(1).

(4). Fix $X \in \{A, \text{WOAM, CS}\}$. We can see that $X$ and EDM are mutually independent in ZFA. Following Theorem 5.3 and the fact that $X$ holds in $N_{H^n}(2)$, $X$ does not imply EDM in ZFA. The other direction follows from Theorem 4.2(3).

(5). Consider the permutation model $\mathcal{M}$ from Corollary 3.6 where CAC$_{\aleph_0}$ fails (and thus EDM fails by Theorem 4.1) and DT holds. Secondly, we consider the permutation model $\mathcal{V}$ from [21 Theorem 9(4)] where DT fails and DC$_{\aleph_0}$ holds, and hence EDM and CAC$_{\aleph_0}$ hold as well. Thus if $X \in \{\text{EDM, CAC}_0\}$, then $X$ and DT are mutually independent in ZFA.

(6). Following the arguments of [21 Theorem 4(11)] due to Tachtsis (where he proved that CAC$_{\aleph_0}$ implies CAC in ZF) we can see that CAC$_{\aleph_0}$ implies CAC in ZF. By Theorem 4.1(4), CAC does not imply CAC$_{\aleph_0}$ in ZF since DF = F implies CAC in ZF.

(7). Consider the model $N_{34}$ from [10]. Let $A = \bigcup_{q \in \mathbb{Q}} \{a_q, b_q\}$ be the set of atoms. Let $\mathcal{G}$ and $\mathcal{F}$ be as defined in [10]. In $N_{34}$, CAC$_{\aleph_0}$ fails since $A = \{a_q, b_q : q \in \mathbb{Q}\}$ does not have a choice function. We prove that CACT$_{\aleph_0}$ holds in $N_{34}$. Let $(T, \leq)$ be an $\omega$-tree in $N_{34}$ with support $E$ such that all chains in $T$ are finite and all antichains in $T$ are countable. If $T$ is well-orderable in $N_{34}$, then we are done by Proposition 3.3. If $T$ is not well-orderable, then there is a $t \in T$ and a $\pi \in \mathcal{G}(E)$ such that $\pi(t) \neq t$. Let $E \cup F$ be a support of $t$ where $E \cap F = \emptyset$ such that $E \cup F$ has the fewest possible members outside $E$. Let $D = (E \cup F) \setminus \{C_q\}$ where $C_q = \{a_q, b_q\} \in E$ is defined in [10 Note 105]. We define

$$f = \{(\pi(C_q), \pi(t)) : \pi \in \mathcal{G}(D)\}.$$ 

Then, $f \in N_{34}$ since $D$ is a support of $f$. Clearly, $f$ is a function. Let $\phi, \psi \in \mathcal{G}(D)$ be such that $\phi(C_q) = \psi(C_q)$. Then, $\phi^{-1}\psi \in \mathcal{G}(D)$, and thus $\phi^{-1}\psi(t) = t$ since $E \cup F$ is a support of $t$. Therefore, $\phi(t) = \psi(t)$. We can see that $f$ is a bijection from an infinite subset $A' = \text{dom}(f) = \{\pi(C_q) : \pi \in \mathcal{G}(D)\} = A \setminus D$ of $A$ onto $\text{ran}(f) = \{\pi(t) : \pi \in \mathcal{G}(D)\} \subset T$. In particular, let $\phi, \psi \in \mathcal{G}(D)$ be such that $\phi(t) = \psi(t)$ but $\phi(C_q) \neq \psi(C_q)$. Define $\pi = \psi^{-1}\phi$. It follows that $\pi(t) = t$, but $\pi(C_q) = b \neq c$. Pick any $\tau \in \mathcal{G}(D)$. Let $\sigma \in \mathcal{G}(E \cup F)$ be such that $\sigma(b) = \tau(C_q)$. Since $D$ is a support of $f$,

$$(\pi(C_q), \pi(t)) \in f \implies (\sigma\pi(C_q), \sigma\pi(t)) \in \sigma(f) = f \implies (\tau(C_q), t) \in f.$$ 

Since $f$ is a function, $\pi(t) = t$. Thus, $D$ is a support of $t$ which contradicts the fact that $E \cup F$ is a support of $t$ with the fewest possible members outside $E$. The set $\text{ran}(f)$ is an infinite antichain in $(T, \leq)$ and thus countably infinite by assumption (cf. [10 Note 105]). So, $A'$ is Dedekind-infinite and thus $A$ is Dedekind-infinite. This contradicts the fact that $A$ is Dedekind-finite in $N_{34}$.

Similarly, CACT$_{\aleph_0}$ holds in $N_{34}$. Consequently, CACT$_{\aleph_0} + \text{CAC}_{\aleph_0} \not\rightarrow \text{Ac}_{\text{fin}}$ in ZFA.

7.2. Questions.

**Question 7.1.** Does CAC$_{\aleph_0}$ imply CAC$_1$ in ZFA?

**Question 7.2.** Does either of CAC$_{\aleph_0}$ and CAC$_1$ hold in the basic Cohen model?

**Question 7.3.** Does BPI + DC imply EDM in ZF or in ZFA?

**Question 7.4.** Does WOAM imply CAC$_{\aleph_0}$ in ZF or in ZFA?

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