Tree Calculus for Bivariable Difference Equations

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Abstract. Following Poupard’s study of strictly ordered binary trees with respect to two parameters, namely, “end of minimal chain” and “parent of maximum leaf” a true Tree Calculus is being developed to solve a partial difference equation system and then make a joint study of those two statistics. Their joint distribution is shown to be symmetric and to be expressed in the form of an explicit three-variable generating function.

1. Introduction

The triangle of numbers

\[
\begin{array}{ccccccccc}
  & f_0(1) & & & & & & & \\
 1 & f_1(1) & f_1(2) & f_1(3) & & & & & 1 \\
 2 & f_2(1) & f_2(2) & f_2(3) & f_2(4) & f_2(5) & & & 0 \\
 3 & f_3(1) & f_3(2) & f_3(3) & f_3(4) & f_3(5) & f_3(6) & f_3(7) & 0 \\
 4 & f_4(1) & f_4(2) & f_4(3) & f_4(4) & f_4(5) & f_4(6) & f_4(7) & f_4(8) & f_4(9) & 0 \\
\end{array}
\]

Table 1.1

appears in Sloane’s On-Line Encyclopedia of Integer Sequences [Sl06] under reference A008301 and is called Poupard’s triangle. As shown by Christiane Poupard [Po89], \( f = (f_n(m)) \) \((n \geq 0, 1 \leq m \leq 2n + 1)\) is the unique solution of the finite-difference equation system

\[
\begin{align*}
\Delta^2 f_n(m) + 2 f_{n-1}(m) &= 0 & (n \geq 1, 1 \leq m \leq 2n - 1), \\
\Delta f_n(m) &:= f_n(m + 1) - f_n(m), \\
\end{align*}
\]

so that

\[
\Delta^2 f_n(m) = f_n(m + 2) - 2f_n(m + 1) + f_n(m),
\]

when taking \( f_0(1) = 1; f_n(1) = 0, f_n(2) = \sum f_{n-1}(m) \) for \( n \geq 1 \) as initial values. Let

\[
\tan u = \sum_{n \geq 1} \frac{u^{2n-1}}{(2n-1)!} T_{2n-1} = \frac{u}{1!} + \frac{u^3}{3!} 2 + \frac{u^5}{5!} 16 + \frac{u^7}{7!} 272 + \frac{u^9}{9!} 7936 + \cdots
\]

Key words and phrases. Tree Calculus, partial difference equations, strictly ordered binary trees, end of minimal chain, parent maximum leaf, bivariate distributions, Poupard triangle, tangent numbers.

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be the Taylor expansion of \( \tan u \), the coefficients \( T_{2n+1} \) (\( n \geq 0 \)) being called the tangent numbers (see, e.g., [Ni23, p. 177-178], [Co74, p. 258-259]); Poupad further shows that each row sum \( f_n(*) := f_n(1) + f_n(2) + \cdots + f_n(2n+1) \) is equal to the integer \( T_{2n+1}/2^n \) (\( n \geq 0 \)), that is, reporting to Table 1.1: 1, 1, 4, 34, 496, . . .

Finally, on the set \( \mathfrak{T}_{2n+1} \) of strictly ordered binary trees with \( (2n+1) \) vertices (see Definition 1.3), she defines two statistics “eoc” (“end of minimal chain”) and “pom” (“parent of the maximum leaf”), to show that both statistics “eoc” and “pom+1” are equally distributed on each set \( \mathfrak{T}_{2n+1} \), and furthermore,

\[
\#\{ t \in \mathfrak{T}_{2n+1} : \text{eoc}(t) = k + 1 \} = \#\{ t \in \mathfrak{T}_{2n+1} : \text{pom}(t) = k \} = f_n(k)
\]

for all \( k \); in particular, \( \#\mathfrak{T}_{2n+1} = T_{2n+1}/2^n \).

The purpose of this paper is to calculate the joint distribution of the pair (eoc, pom) on each set \( \mathfrak{T}_{2n+1} \) and to derive its properties, in particular its symmetry. To achieve this, we first introduce a sequence \( (M_n = (f_n(m, k)) \) of \( (2n) \times (2n) \)-matrices \( (n \geq 1) \) with nonnegative integral entries, called a Delta sequence, defined by a system of finite difference equations, verifying certain initial conditions. Then, we show that each entry \( f_n(m, k) \) is equal to the number of trees \( t \) from \( \mathfrak{T}_{2n+1} \) such that \( \text{eoc}(t) = m \) and \( \text{pom}(t) = k \).

It is convenient to consider the following four triangles of each square \( \{(m, k) : 1 \leq m, k \leq 2n\} \):

\[
\begin{align*}
L_n^{(1)} := & \{2 \leq k + 1 \leq m \leq 2n - 2\}; \\
L_n^{(2)} := & \{4 \leq k + 3 \leq m \leq 2n\}; \\
U_n^{(1)} := & \{2 \leq m + 1 \leq k \leq 2n - 2\}; \\
U_n^{(2)} := & \{4 \leq m + 3 \leq k \leq 2n\}.
\end{align*}
\]

By convention, \( f_n(m, k) := 0 \) if \( (m, k) \notin [1, 2n] \times [1, 2n] \). The partial difference operators \( \Delta_m \), \( \Delta_k \), act as follows on the entries of the matrices \( M_n \):

\[
\begin{align*}
\Delta_m f_n(m, k) := & f_n(m + 1, k) - f_n(m, k); \\
\Delta_k f_n(m, k) := & f_n(m, k + 1) - f_n(m, k).
\end{align*}
\]

They serve to define the recurrence relations:

\[
\begin{align*}
(R1) \quad & \Delta_m^2 f_n(m, k) + 2 f_{n-1}(m, k) = 0 \quad ((m, k) \in L_n^{(1)}); \\
(R2) \quad & \Delta_k^2 f_n(m, k) + 2 f_{n-1}(m, k) = 0 \quad ((m, k) \in U_n^{(1)}).
\end{align*}
\]

Finally, denote the row and column sums of \( M_n = (f_n(m, k)) \) by

\[
\begin{align*}
f_n(m, *) := & \sum_{1 \leq k \leq 2n} f_n(m, k); \quad (1 \leq m \leq 2n); \\
f_n(*, k) := & \sum_{1 \leq m \leq 2n} f_n(m, k) \quad (1 \leq k \leq 2n).
\end{align*}
\]
Definition 1.1. A sequence of matrices \((M_n)\) \((n \geq 1)\), where each matrix \(M_n = (f_n(m, k))\) \((1 \leq k, m \leq 2n)\) has nonnegative integral entries, having only 0’s along its diagonal, and such that \(M_1 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\), is said to be a Delta Sequence, if for \(n \geq 2\) both recurrence relations \((R1)\) and \((R2)\) hold, together with the initial conditions:

\((I1)\) for \(n \geq 2\) the \((2n)\)-th column, \(\text{Col}_{2n}\), of \(M_n\) is the zero-column; its the \((2n - 1)\)-st column, \(\text{Col}_{2n-1}\), is equal to

\[f_{n-1}(1, \bullet), f_{n-1}(2, \bullet), \ldots, f_{n-1}(2n - 2, \bullet), 0, 0,\]

when read from top to bottom.

\((I2)\) the \((2n)\)-th row of \(M_n\) is also equal to

\[f_{n-1}(1, \bullet), f_{n-1}(2, \bullet), \ldots, f_{n-1}(2n - 2, \bullet), 0, 0,\]

when read from left to right; its \((2n - 1)\)-st row is equal to:

\[f_{n-1}(1, \bullet) + f_{n-1}(\bullet, 1), f_{n-1}(2, \bullet) + f_{n-1}(\bullet, 2), \ldots, f_{n-1}(2n - 3, \bullet) + f_{n-1}(\bullet, 2n - 3), f_{n-1}(2n - 2, \bullet) + f_{n-1}(\bullet, 2n - 2), 0, 0,\]

when read from left to right.

In the above definition the entries of the matrix \(M_n\) are derived from \(M_{n-1}\) by first applying rules \((I1)\) and \((I2)\) and letting the diagonal be null; then, starting from \(m = 1\) up to \(m = 2n - 3\), for each \(k\) from \(2n - 3\) down to \(m + 1\), evaluate \(f_n(m, k)\) with equation \((R2)\):

\[\begin{align*}
f_n(m, k) &= f_n(m, k + 1) + f_n(m, k + 2) + 2f_{n-1}(m, k) = 0,
\end{align*}\]

the coefficients \(f_n(m, k + 1), f_n(m, k + 2)\) and \(f_{n-1}(m, k)\) being already calculated. Exchanging the roles of \(m\) and \(k\) the upper entries are obtained by using equation \((R1)\). Accordingly, \((R1), (R2), (I1), (I2)\) uniquely determine the Delta Sequence \((M_n)\) \((n \geq 1)\).

Calculation of the first matrices. First, \(f_1(1, \bullet) = 0, f_1(2, \bullet) = 1,\) so that

\[
M_2 = \begin{pmatrix}
0 & ? & 0 & 0 \\
? & 0 & 1 & 0 \\
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\]

by rules \((I1)\) and \((I2)\). The remaining entries are obtained by rule \((R2)\):

\[f_2(1, 2) - 2f_2(1, 3) + f_2(1, 4) + 2f_1(1, 2) = f_2(1, 2) - 2 \times 0 + 0 + 2 \times 0 = 0,\]

so that \(f_2(1, 2) = 0;\) then, by rule \((R1)\):

\[\begin{align*}
f_2(2, 1) - 2f_2(3, 1) &+ f_2(4, 1) + 2f_1(2, 1) = f_2(2, 1) - 2 \times 1 + 0 + 2 \times 1 = 0, \text{ so that } f_2(2, 1) = 0.\]

Thus,

\[
M_2 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\]
and \( f_2(1, \bullet) = 0, f_2(2, \bullet) = 1, f_2(3, \bullet) = 2, f_2(4, \bullet) = 1 \). The next matrices are displayed in Fig. 1.2.

The previous definition of a Delta Sequence, based on the two relations \((R1), (R2)\) and the two initial conditions \((I1), (I2)\), can be symbolized by the square on the left in Fig. 1.3, as relation \((R1)\) (resp. \((R2)\)) acts on the entries of the lower (resp. upper) entries of the matrix \(M_n\), and initial conditions \((I1)\) and \((I2)\) refer to the last two columns \(Col_{2n-1}\), \(Col_{2n}\) and rows \(Row_{2n-1}\), \(Row_{2n}\) of \(M_n\), respectively. Other initial conditions will be stated in Section 8. We just mention a second one, materialized by the square on the right in Fig. 1.3.
**Definition 1.2.** A sequence of matrices \((M_n)\) \((n \geq 1)\), where each matrix \(M_n = (f_n(m, k))\) \((1 \leq k, m \leq 2n)\) has nonnegative integral entries, having only 0’s along its diagonal, and such that \(M_1 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\), is said to be a *Gamma Sequence*, if for \(n \geq 2\) both recurrence relations

\[
\Delta^2_m f_n(m, k) + 2 f_n-1(m, k - 2) = 0 \quad ((k, m) \in U_n^{(2)}); \tag{R3}
\]

\[
\Delta^2_k f_n(m, k) + 2 f_n-1(m - 2, k) = 0 \quad ((k, m) \in L_n^{(2)}); \tag{R4}
\]

hold, together with the *initial conditions*:

\((I\,3)\) for \(n \geq 2\) the first row, Row\(_1\), is the zero-row; the second row, Row\(_2\), is equal to

\[0, f_{n-1}(1, \bullet)(= 0), f_{n-1}(2, \bullet), \ldots, f_{n-1}(2n - 2, \bullet), f_{n-1}(2n - 1, \bullet)(= 0);\]

when read from left to right.

\((I\,4)\) the first column, Col\(_1\), of \(M_n\) is also equal to

\[0, 0, f_{n-1}(1, \bullet) + f_{n-1}(2, \bullet), f_{n-1}(3, \bullet) + f_{n-1}(2, \bullet), \ldots \]

\[f_{n-1}(2n - 2, \bullet) + f_{n-1}(2n - 3, \bullet), f_{n-1}(2n - 2, \bullet),\]

when read from top to bottom; the second column, Col\(_2\), is equal to

Using the same reasoning as for Definition 1.1 it is seen that the Gamma Sequence is uniquely defined. The fact that Delta and Gamma Sequences are identical will be a consequence of the further theorems (*cf.* Section 6).

Next, comes the combinatorial set-up on which all calculations will be made.

**Definition 1.3.** An *\(n\)-labeled strictly ordered binary tree* is defined by the following axioms:

1. it is a labeled tree with \(n\) nodes, labeled 1, 2, \ldots, \(n\); the node labeled 1 is called the root;
2. each node has no child (it is then called a leaf), or two children, their order being immaterial (it is then called an interior node);
3. when getting along each path from the root to each node, the node labels are in increasing order.

Each strictly ordered binary tree has an odd number of vertices, say, \(2n + 1\), with \(n\) interior nodes and \(n + 1\) leaves. Let \(\Xi_{2n+1}\) denote the set of all strictly ordered binary trees with \((2n + 1)\) nodes. When giving an orientation (left or right) to each child of each of the \(n\) interior nodes in a tree \(t \in \Xi_{2n+1}\), we generate \(2^n\) planar strictly ordered binary trees (also called “arbres binaires croissants complets” by Viennot [Vi88, chap. 3,
It is known that the latter are equidistributed with the alternating permutations of order \( (2n+1) \), so that their number is equal to the tangent number \( T_{2n+1} \), a result that goes back to Désiré André \([An1879, An1881]\). Accordingly,

\[
\#\Sigma_{2n+1} = T_{2n+1}/2^n.
\]

Let \( t \in \Sigma_{2n+1} \) \( (n \geq 1) \). If a node labeled \( a \) has two children labeled \( b \) and \( c \), define \( \min a := \min\{b, c\} \); if it has one child \( b \), let \( \min a := b \). The minimal chain of \( t \) is defined to be the sequence \( a_1 \to a_2 \to a_3 \to \cdots \to a_{j-1} \to a_j \), with the following properties: (i) \( a_1 = 1 \) is the label of the root; (ii) for \( i = 1, 2, \ldots, j - 1 \) the \( (i + 1) \)-st term \( a_{i+1} \) is the label of an interior node and \( a_{i+1} = \min a_i \); (iii) \( a_j \) is the node of a leaf. Define the “end of the minimal chain” of \( t \) to be \( \text{eoc}(t) := a_j \). If the leaf with the maximum label \( (2n+1) \) is incident to a node labeled \( k \), define its “parent of the maximum leaf” to be \( \text{pom}(t) := k \).

The minimal chain of the tree \( t \) displayed in Fig. 1.4 on the left is \( 1 \to 2 \to 3 \to 7 \), so that \( \text{eoc}(t) = 7 \) and the parent of its maximum leaf (equal to \( 2n+1 = 9 \)) is \( \text{pom}(t) = 4 \). Also, the parent of the maximum leaf in the tree \( t' \) on the right is equal to \( \text{pom}(t') = 6 \). To go from \( t \) to \( t' \) replace the labels \( 1, 2, 3, 7 \) of the minimal chain by the labels \( 2 - 1, 3 - 1, 7 - 1, 9 = 1, 2, 6, 9 \) and change each other label \( a \) by \( a - 1 \). We then have \( \text{eoc}(t) = \text{pom}(t') + 1 = 7 \). This illustrates the construction of the bijection \( t \mapsto t' \) of \( \Sigma_{2n+1} \) onto itself with the property \( \text{eoc}(t) = \text{pom}(t') + 1 \), which was described in \([Ha12]\). Thus, the statistics “eoc” and “pom+1” are equidistributed on each \( \Sigma_{2n+1} \), as already mentioned in \((1.5)\), a property first proved by Poupard \([Po89]\) by analytic methods. The main results of this paper are stated next.

**Theorem 1.1.** Let \( (M_n = (f_n(m, k)) (n \geq 1) \) be the Delta sequence, as introduced in Definition 1.1. Then, for all \( n \geq 1 \) and \( 1 \leq m, k \leq 2n \)

\[
\#\{t \in \Sigma_{2n+1} : \text{eoc}(t) = m, \text{pom}(t) = k\} = f_n(m, k).
\]

A first consequence of Theorem 1.1 and \((1.8)\) is the identity:

\[
\sum_{m, k} f_n(m, k) = T_{2n+1}/2^n \quad (n \geq 0).
\]
Theorem 1.2. Let \( (M_n = (f_n(m, k)) (1 \leq m, k \leq 2n)) \) \( (n \geq 1) \) be the Delta sequence. Then, the matrices \( M_n \) are symmetric with respect to their counter-diagonals:

\[
(1.11) \quad f_n(m, k) = f_n(2n + 1 - k, 2n + 1 - m) \quad (1 \leq k, m \leq 2n).
\]

The proofs of those two theorems and also of further properties will be based on the geometric properties of the strictly ordered binary trees. To this end, we adopt the following notation and convention: for each triple \( (n, m, k) \) let \( T_{2n+1,m,k} \) (resp. \( T_{2n+1,m,*} \), resp. \( T_{2n+1,*k} \)) denote the subset of \( T_{2n+1} \) of all trees \( t \) such that eoc\((t) = m \) and pom\((t) = k \) (resp. eoc\((t) = m \), resp. pom\((t) = k \)). By convention, designate those families and their cardinalities by the same symbol and also the matrix of the integers \( T_{2n+1,m,k} \) by Mat\((T_{2n+1})\). Our plan of action will be to show that the sequence \( \text{Mat}(T_{2n+1}) \) \( (n \geq 1) \) is identical to the Delta Sequence.

In Sections 2–5 it will be shown that, when replacing each \( f_n(m, k) \) by \( T_{2n+1,m,k} \) the initial conditions \((I1)\) and \((I2)\), the two finite-difference equations systems \((R1), (R3)\), the two finite-difference equations systems \((R2), (R4)\) and the initial conditions \((I3)\) and \((I4)\) hold. This will complete the proofs of Theorems 1.1 and 1.2, as done in Section 6. Further properties of the matrices Mat\((T_{2n+1})\) (and then matrices \( M_n \)) will be given in Section 7. Finally, several other equivalent definitions of the Delta sequence will be mentioned in Section 8. In Section 9 we conclude the paper by calculating the generating functions for the \( f_n(m, k) \) in the following forms.

Theorem 1.3. The triple-exponential generating function for the lower triangles of the matrices \( M_n \) is given by

\[
(1.12) \quad \sum_{2 \leq k+1 \leq m \leq 2n} f_n(m, k) \frac{x^{m-k-1}}{(m-k-1)!} \frac{y^{k-1}}{(k-1)!} \frac{z^{2n-m}}{(2n-m)!} = \frac{\cos(\sqrt{2} x) + \cos(\sqrt{2} y) \cos(\sqrt{2} z)}{2 \cos^2 \left( \frac{x+y+z}{\sqrt{2}} \right)}.
\]

Theorem 1.4. The triple exponential generating function for the upper triangles of the matrices \( M_n \) is given by

\[
(1.13) \quad \sum_{2 \leq m+1 \leq k \leq 2n-1} f_n(m, k) \frac{x^{2n-k}}{(2n-k)!} \frac{y^{k-m-1}}{(k-m-1)!} \frac{z^{m-1}}{(m-1)!} = \sin(\sqrt{2} x) \sin(\sqrt{2} z) \frac{1}{2 \cos^2 \left( \frac{x+y+z}{\sqrt{2}} \right)}.
\]
2. The initial conditions \((I 1)\) and \((I 2)\)

In this section and the next ones we make the convention that whenever a leaf is deleted from a tree, the edge linking the leaf to the tree is also deleted.

For verifying that the matrices \(\text{Mat}(\mathfrak{T}_{2n+1})\) have only zero in their diagonals, it suffices to show that \(\mathfrak{T}_{2n+1,m,m} = \emptyset\) (or is equal to 0 with our convention). This is true, because if \(t \in \mathfrak{T}_{2n+1}\) and \(\text{eoc}(t) = \text{pom}(t) = m\), the node \((2n + 1)\) has a parent equal to \(m\). Consequently, \(m\) cannot be the end of a minimal chain. Hence, the previous subset is empty.

**Theorem 2.1.** The initial conditions \((I 1)\) and \((I 2)\) hold for the matrices \(\text{Mat}(\mathfrak{T}_{2n+1})\) when \(f_n(m, \cdot)\) and \(f_n(\cdot, k)\) are replaced by \(\mathfrak{T}_{2n+1,m,\cdot}\) and \(\mathfrak{T}_{2n+1,\cdot,k}\), respectively.

**Proof.** \((I 1)\) First, the \((2n)\)-th column of the matrix \(\text{Mat}(\mathfrak{T}_{2n+1})\) has zero entries only, as \(\text{pom} \leq 2n - 1\). Next, each tree from \(\mathfrak{T}_{2n+1,m,2n-1}\) \((1 \leq m \leq 2n - 2)\) must contain the subtree

\[
\begin{array}{c}
\text{m} \\
\text{1} \\
\text{2n-1} \\
\text{2n} \quad \text{2n+1}
\end{array}
\]

Hence, \(\mathfrak{T}_{2n+1,2n-1,2n-1}\) is empty, for \((2n - 1)\), being an interior node, cannot be the end of the minimal chain. Also, \(\mathfrak{T}_{2n+1,2n,2n-1}\) is empty, for the sibling of \((2n - 1)\) is necessarily less than \((2n - 1)\), so that the minimal chain cannot go through \((2n - 1)\) and reach \((2n)\). Furthermore, \(\mathfrak{T}_{2n+1,1,2n-1} = \mathfrak{T}_{2n-1,1,\cdot} = 0\), as \(\text{eoc} \geq 2\).

In the remaining cases, that is, \(2 \leq m \leq 2n - 2\), removing the two leaves \((2n)\), \((2n + 1)\) transforms each tree from \(\mathfrak{T}_{2n+1,m,2n-1}\) onto a tree from \(\mathfrak{T}_{2n-1,m,\cdot}\) in a bijective manner. Such a transformation may be illustrated by the diagram:

Hence, the \((2n - 1)\)-st column of the matrix \(\text{Mat}(\mathfrak{T}_{2n+1})\) reads:

\[
(2.1) \quad \mathfrak{T}_{2n-1,1,\cdot}, \mathfrak{T}_{2n-1,2,\cdot}, \ldots, \mathfrak{T}_{2n-1,2n-2,\cdot}, 0, 0.
\]

from top to bottom.

\((I 2)\) For the \((2n)\)-th row of the matrix \(\text{Mat}(\mathfrak{T}_{2n+1})\) note that \(\mathfrak{T}_{2n+1,2n,1} = 0\) for \(n \geq 2\). When \(k \geq 2\) each tree from \(\mathfrak{T}_{2n+1,2n,k}\) must contain the subtree

\[
\begin{array}{c}
\text{m} \\
\text{1} \\
\text{2n-1} \\
\text{2n} \quad \text{2n+1}
\end{array}
\]

from top to bottom.
By (1.1) we then have: \( \mathcal{T}_{2n+1,2n,2n-1} = \mathcal{T}_{2n+1,2n,2n} = 0 \). For the remaining cases \( 2 \leq k \leq 2n - 2 \) we can set up a bijection of \( \mathcal{T}_{2n+1,2n,k} \) onto \( \mathcal{T}_{2n-1,k} \) by removing the two leaves \( 2n \) and \( (2n + 1) \), as illustrated by the next diagram.

2n \hspace{1cm} 2n+1

\begin{array}{c}
\text{1} \\
\text{ }\downarrow \hspace{1cm} \leftrightarrow \\
\text{ }\downarrow \\
\text{k} \\
\end{array}

\begin{array}{c}
\text{1} \\
\text{ }\downarrow \\
\text{k} \\
\end{array}

Note that the node \( k \) becomes the end of the minimal chain. Thus, the \((2n)\)-th row of the matrix \( \mathcal{T}_{2n+1} \) is also equal to (2.1) read from left to right.

Finally, consider the \((2n-1)\)-st row of \( \mathcal{T}_{2n+1} \). In an obvious manner, \( \mathcal{T}_{2n+1,2n-1,2n-1} = \mathcal{T}_{2n+1,2n-1,2n} = 0 \). When \( 1 \leq k \leq 2n - 2 \), the trees from the sets \( \mathcal{T}_{2n+1,2n-1,k} \) fall into two categories \( \mathcal{T}_{I}^{f} \) and \( \mathcal{T}_{II}^{f} \). In the first category the trees contain the subtree

\( 2n+1 \uparrow \bullet 2n-1 \); in the second one, the subtree \( 2n \bullet 2n-1 \) \( (a \neq k) \).

First, note that \( \mathcal{T}_{I}^{f} = \mathcal{T}_{II} = 0 \). When \( 2 \leq k \leq 2n - 2 \), removing the two leaves \((2n + 1), (2n - 1)\) and replacing the node label \((2n)\) by \((2n - 1)\) maps \( \mathcal{T}_{2n+1,2n-1,k} \) onto \( \mathcal{T}_{2n-1,k} \) in a bijective manner.

When \( 1 \leq k \leq 2n - 2 \), removing the two leaves \((2n), (2n - 1)\), and replacing the node label \((2n + 1)\) by \((2n - 1)\) maps \( \mathcal{T}_{II}^{f} \) onto \( \mathcal{T}_{2n-1,k} \) in a bijective manner. Thus, the \((2n-1)\)-st row of \( \mathcal{T}_{2n+1} \) reads

\[
\mathcal{T}_{2n-1,1} + \mathcal{T}_{2n-1,1,1}, \mathcal{T}_{2n-1,2} + \mathcal{T}_{2n-1,2,1}, \ldots , \mathcal{T}_{2n-1,2n-3} + \mathcal{T}_{2n-1,2n-3,1}, \mathcal{T}_{2n-1,2n-2} + \mathcal{T}_{2n-1,2n-2,1} + \mathcal{T}_{2n-1,2n-2,2} + \mathcal{T}_{2n-1,2n-2,3} = 0, 0.
\]

3. Tree Calculus for the relations \((R1)\) and \((R3)\)

In the following Tree Calculus subtrees (possibly leaves) are indicated by the symbols “\( \bigcirc \),” “\( \bigtriangledown \),” or “\( \sqcup \).” The end of the minimal chain in each tree is represented by a bullet “\( \bullet \).” Letters occurring below or next to subtrees are labels of their roots. For instance, the symbols

\[
\begin{align*}
\begin{array}{c}
\sqcup \hspace{1cm} \bullet \\
\text{a} \\
\text{b} \\
\end{array} & \quad \begin{array}{c}
\bigcirc \hspace{1cm} \bullet \\
\text{a} \\
\text{b} \\
\end{array} & \quad \begin{array}{c}
\bigtriangledown \hspace{1cm} \bullet \\
\text{a} \\
\text{b} \\
\end{array}
\end{align*}
\]

designate the families of all trees \( t \) from the underlying set \( \mathcal{T}_{2n+1} \) having a node labeled \( b \) [in short, a node \( b \)], parent of both a subtree of root \( a \) and the leaf \( m \), which is also the end of the minimal chain; moreover, the
symbol on the right has the further property that the node labeled $c$ does not belong, either to the subtree of root $b$, or to the path going from root 1 to $b$. In the sequel, the letter “$m$” is always used to designate the end of the minimal chain, unless explicitly indicated by a letter next to $\bullet$.

Our Tree Calculus consists of two steps: (a) decomposing the sets $\mathfrak{T}_{2n+1,m,k}$ into smaller subsets by considering the mutual positions of the nodes $m$, $(m+1)$, $(m+2)$ (resp. $k$, $(k+1)$, $(k+2)$); (b) setting up bijections between those subsets by a simple display of certain subtrees, as done in (3.1).

For instance,

\begin{align*}
C_3 := & \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\bullet
\end{array}
\end{array}
\end{array}m+1
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\bigcirc
\end{array}
\end{array}
\end{array}m+2
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\bullet
\end{array}
\end{array}
\end{array}m
\end{align*}
and
\begin{align*}
D_1 := & \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\bigcirc
\end{array}
\end{array}
\end{array}m+2
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\bullet
\end{array}
\end{array}
\end{array}m+1
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\bullet
\end{array}
\end{array}
\end{array}m
\end{align*}

may be regarded as two subsets of $\mathfrak{T}_{2n+1}$. To each pair $(\begin{array}{c}m+2 \bigcirc m \end{array})$, the nodes of “$\bigcirc$” being all greater than or equal to $(m+2)$, there correspond a unique tree from $C_3$ and a unique tree from $D_1$. This clearly defines a bijection of $C_3$ onto $D_1$.

Those two principles (a) and (b) will be applied in the proofs of the next two theorems 3.1 and 4.1.

**Theorem 3.1.** If $(m,k)$ belongs to $L_n^{(1)} \cup U_n^{(2)} = \{2 \leq k + 1 \leq m \leq 2n - 2\} \cup \{4 \leq m + 3 \leq k \leq 2n\}$, then

\begin{align*}
\Delta_m^2 \mathfrak{T}_{2n+1,m,k} + 2 \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\bullet
\end{array}
\end{array}m+1
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\bigcirc
\end{array}
\end{array}m+2
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\bullet
\end{array}
\end{array}
\end{array}m
\end{align*}

with the understanding that the second term on the left-hand side represents twice the set of all trees from $\mathfrak{T}_{2n+1,m+1,k}$ with the further property that $m$ is the parent of both $(m+1)$ and $(m+2)$.

**Proof.** The decomposition

\begin{align*}
\mathfrak{T}_{2n+1,m,k} = & \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\bullet
\end{array}
\end{array}m
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\bigcirc
\end{array}
\end{array}m+1
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\bullet
\end{array}
\end{array}m
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\bigcirc
\end{array}
\end{array}m+1
\end{array}
\end{array}
\end{align*}

means that in each tree from $\mathfrak{T}_{2n+1,m,k}$ the node $(m+1)$ is, or is not, the sibling of the leaf $m$. In the next decomposition the node $m$ is, or is not, the parent of the leaf $(m+1)$:

\begin{align*}
\mathfrak{T}_{2n+1,m+1,k} = & \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\bullet
\end{array}m
\end{array}
\end{array}m+1
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\bigcirc
\end{array}m
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\bullet
\end{array}
\end{array}m+1
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\bigcirc
\end{array}b
\end{array}
\end{array}
\end{array}
\end{align*}

(b \neq m).
Under the transposition \((m, m+1)\) the node labeled \(k\) remains unaffected, because \(k \leq m - 1\) if \((m, k) \in L_n^{(1)}\) and \(m + 3 \leq k\) if \((m, k) \in U_n^{(2)}\), so that the parent of \((2n + 1)\) remains \(k\). Thus, the transposition establishes a one-to-one correspondence between the two second terms. Hence,\
\[
\Delta^2 \mathcal{X}_{2n+1,m,k}^m = (\mathcal{X}_{2n+1,m+2,k}^m - \mathcal{X}_{2n+1,m+1,k}^m) - (\mathcal{X}_{2n+1,m+1,k}^m - \mathcal{X}_{2n+1,m,k}^m)
\]
\[
:= A - B - C + D.
\]

Depending on the mutual positions of nodes \(m\), \((m+1)\) and \((m+2)\) the further decompositions prevail, as again, \(k\) remaining still attached to \((2n + 1)\):

\[
A = \begin{array}{c}
m + 2 \\
m + 1 \\
m \\
m + 1
\end{array} + \begin{array}{c}
m \\
m \\
m + 2 \\
m + 1
\end{array}, m := A_1 + A_2;
\]

\[
B = \begin{array}{c}
m + 2 \\
m + 1 \\
m \\
m + 1
\end{array} + \begin{array}{c}
m \\
m \\
m + 2 \\
m + 1
\end{array}, m := B_1 + B_2 + B_3;
\]

\[
C = \begin{array}{c}
m + 2 \\
m + 1 \\
m \\
m + 1
\end{array} + \begin{array}{c}
m \\
m \\
m + 2 \\
m + 1
\end{array} + \begin{array}{c}
m \\
m \\
m + 2 \\
m + 1
\end{array}, m + 2 := C_1 + C_2 + C_3 + C_4;
\]

\[
D = \begin{array}{c}
m + 2 \\
m + 1 \\
m \\
m + 1
\end{array} + \begin{array}{c}
m \\
m \\
m + 2 \\
m + 1
\end{array}, m + 2 := D_1 + D_2.
\]

The permutation \((m, m+2, m+1, m)\) establishes a one-to-one correspondence between \(A_2\) and \(C_4\) (resp. between \(B_3\) and \(D_2\)). On the other hand, \(A_1 = B_2 + C_2 = 2 B_2\), because the two subtrees in \(B_2 = C_2\) are incident to the same node, while in \(A_1\) they are incident to two different nodes. Finally, \(C_3 = D_1\) as explained in (3.1). Making all the proper cancellations in the sum \(\Delta^2 \mathcal{X}_{2n+1,m,k}^m = A - B - C + D\) we get identity (3.2).

**Corollary 3.2.** The relations \((R1)\) and \((R3)\) hold for the matrices \(\text{Mat}(\mathcal{X}_{2n+1})\), that is,

\[
\Delta^2 \mathcal{X}_{2n+1,m,k}^m + 2 \mathcal{X}_{2n-1,m,k}^m = 0, \quad \text{if} \ (m, k) \in L_n^{(1)};
\]

\[
\Delta^2 \mathcal{X}_{2n+1,m,k}^m + 2 \mathcal{X}_{2n-1,m,k-2}^m = 0, \quad \text{if} \ (m, k) \in U_n^{(2)}.
\]
Proof. For (3.3) change the second term in (3.2) as follows: remove the two leaves \((m + 1), (m + 2)\), and subtract 2 from all the remaining nodes greater than \((m + 2)\): the term becomes \(T_{2n-1,m,k}\), as the node \((2n + 1)\) becomes \((2n - 1)\) and is still linked to \(k\). For (3.4) do the same changes, but this time, as \(m + 3 \leq k\), the edge going from \(k\) to \((2n + 1)\) becomes an edge going from \((k - 2)\) to \((2n - 1)\).

4. Tree Calculus for the relations \((R2)\) and \((R4)\)

Theorem 4.1. If \((m, k)\) belongs to \(L_n^{(2)} \cup U_n^{(1)} = \{4 \leq k + 3 \leq m \leq 2n\} \cup \{2 \leq m + 1 \leq k \leq 2n - 2\}\), then

\[ \Delta^2 T_{2n+1,m,k} + 2 \left[ k + 1 \right] = 0, \]

with the understanding that the second term on the left-hand side is twice the number of all trees from \(T_{2n+1,m,k+1}\) with the further property that \((k + 2)\) is a leaf incident to \((k + 1)\), itself incident to \(k\), the end \(m\) of the minimum chain being outside the subtree of root \(k\).

Proof. First,

\[
 T_{2n+1,m,k} = \left[ k, m \right] + \left[ k + 1, m, k + 1 \right] + \left[ k + 1, m \right] + \left[ k, m + 1 \right] = A_1 + A_2 + A_3 + A_4,
\]

meaning that each tree from \(T_{2n+1,m,k}\) has one of the four forms: either \(k + 1\) is incident to \(k\), or not, and \(m\) is outside or not the subtree of root \(k\); furthermore, the leaf \(m\) is the end of the minimal chain.

Using the same dichotomy,

\[
 T_{2n+1,m,k+1} = \left[ k, m \right] + \left[ k + 1, m, k \right] + \left[ k + 1, m + 1 \right] + \left[ k, k + 1 \right] = B_1 + B_2 + B_3 + B_4.
\]

Consider the subsets \(A_4' := \) of \(A_4\) and \(B_2' := \) of \(B_2\). The transposition \((k, k + 1)\) maps \(A_4 \setminus A_4'\) onto \(B_4\) and \(A_2\) onto \(B_2 \setminus B_2'\) in a bijective manner.
Hence,
\[ T_{2n+1,m,k+1} - T_{2n+1,m,k} = (B_1 - A_1) + ((B_2 - B'_2) - A_2) + B'_2 \]
\[ + (B_3 - A_3) + (B_4 - (A_4 - A'_4) - A'_4) \]
\[ = B_1 - A_1 + B'_2 + B_3 - A_3 - A'_4 \]
\[ = [ \begin{array}{c} \circ \\ k+1 \end{array}, m ] - [ \begin{array}{c} \circ \\ k \end{array}, m ] + [ \begin{array}{c} \square \\ k+1 \end{array}, m ] \]
\[ + [ \begin{array}{c} \square \\ k+1 \end{array}, m ] - [ \begin{array}{c} \square \\ k \end{array}, m ] - [ \begin{array}{c} \square \\ k+1 \end{array}, m ] \]
\[ = D_1 - C_1 + D'_2 + D_3 - C_3 - C'_4. \]

Thus,
\[ \Delta^2 T_{2n+1,m,k} = (T_{2n+1,m,k+2} - T_{2n+1,m,k+1}) - (T_{2n+1,m,k+1} - T_{2n+1,m,k}) \]
\[ = D_1 - C_1 + D'_2 + D_3 - C_3 - C'_4 \]
\[ - B_1 + A_1 - B'_2 - B_3 + A_3 + A'_4. \]

The further decompositions of the components of the previous sum depend on the mutual positions of the nodes \( k, (k + 1), (k + 2) \). First, evaluate the subsum: \( S_1 := D_1 - C_1 - B_1 + A_1 \) using the decompositions:

\[ D_1 = [ \begin{array}{c} \circ \\ k+2 \\ k+1 \end{array}, m ] - [ \begin{array}{c} \circ \\ k+1 \\ k \end{array}, m, k ] + [ \begin{array}{c} \square \\ k+1 \\ k \end{array}, m ] \]
\[ := D_{1,1} + D_{1,2}; \]

\[ C_1 = [ \begin{array}{c} \circ \\ k+2 \\ k+1 \end{array}, m ] = [ \begin{array}{c} \circ \\ k+1 \\ k \end{array}, m, k ] + [ \begin{array}{c} \square \\ k+1 \\ k \end{array}, m ] \]
Also, let
\[ C_{1,1} = 1. \]

Next, evaluate the sum
\[ \sum_{i=1}^{k} D_{i,1} + D_{i,2}. \]

Also, let
\[ D'_{1,1} := \quad \text{and} \quad C'_{1,1} := \quad \text{.} \]

The permutation \( (k+1, k+2) \) maps \( D_{1,1} \) onto \( B_{1,1} \) and \( C_{1,1} \) onto \( A_{1,1} \). Hence, \( D_{1,1} = B_{1,1} + D'_{1,1}, \) \( C_{1,1} = A_{1,1} + C'_{1,1} \). Moreover, \( D_{1,2} = 2B_{1,3}, \) \( C_{1,2} = A_{1,2}, \) \( B_{1,2} = B_{1,4}, \) \( B_{1,3} = B_{1,5}. \) Altogether, \( S_1 = D_1 - C_1 - B_1 + A_1 = (B_{1,1} + D'_{1,1} + 2B_{1,3}) - (A_{1,1} + C'_{1,1} + A_{1,2}) - (B_{1,1} + B_{1,2} + B_{1,3} + B_{1,2} + B_{1,3}) + (A_{1,1} + A_{1,2}). \) Thus,
\[ S_1 = -2B_{1,2} + D'_{1,1} - C'_{1,1}. \]

Next, evaluate the sum \( S_2 := D'_2 + D_3 - C'_3 - C'_4 - B'_2 - B_3 + A_3 + A'_4 \) by decomposing each of its components. We have:
\[ D'_2 = \quad \text{and} \quad := D'_{2,1} + D'_{2,2}; \]
\[ D_3 = \quad \text{and} \quad := D_{3,1} + D_{3,2}; \]
Within the sum $S_2$ there are numerous cancellations we now describe.

(a) Components of the form $[t, k]$ or $[t, k + 2]$, where $t$ is a subtree, whose root is labeled. There are four of them: $D_{3,1}, -C_{3,1}, -B_{3,1}, A_{3,1}$. 

\[ A_{3} = \begin{array}{c} \includegraphics[width=1.5in]{A3} \end{array} \] 
\[ := A_{3,1} + A_{3,2} \] 

\[ A'_{4} = \begin{array}{c} \includegraphics[width=1.5in]{A4} \end{array} \] 
\[ := A'_{4,1} + A'_{4,2} + A'_{4,3} \] 

\[ C_{3} = \begin{array}{c} \includegraphics[width=1.5in]{C3} \end{array} \] 
\[ := C_{3,1} + C_{3,2} \] 

\[ C'_{4} = \begin{array}{c} \includegraphics[width=1.5in]{C4} \end{array} \] 
\[ := C'_{4,1} + C'_{4,2} + C'_{4,3} \] 

\[ B'_{2} = \begin{array}{c} \includegraphics[width=1.5in]{B2} \end{array} \] 
\[ := B'_{2,1} + B'_{2,2} + B'_{2,3} \] 

\[ B_{3} = \begin{array}{c} \includegraphics[width=1.5in]{B3} \end{array} \] 
\[ := B_{3,1} + B_{3,2} + B_{3,3} + B_{3,4} \]
Consider the subsets: 

$$B_{3,1,1} := \begin{array}{c}
\bullet \\
\bigtriangleup \\
\bullet
\end{array} \quad m \quad \begin{array}{c}
\bullet \\
\bigtriangleup \\
\bullet
\end{array}$$

of $B_{3,1}$ and $A_{3,1}$, respectively. The permutation $(k \quad k+1 \quad k+2)$ maps $D_{3,1}$ onto $B_{3,1} \setminus B_{3,1,1}$ and $C_{3,1}$ onto $A_{3,1} \setminus A_{3,1,1}$. Hence, $D_{3,1} - C_{3,1} = B_{3,1} + A_{3,1} = (B_{3,1} - B_{3,1,1}) - (A_{3,1} - A_{3,1,1}) - B_{3,1} + A_{3,1} = -B_{3,1,1} + A_{3,1,1}$.

(b) Components of the form $[t, k]$ or $[t, k + 2]$, where $t$ is a subtree, whose root is not labeled. There are four of them: $D_{2,1}', -C_{4,1}', -B_{2,1}'$, $A_{4,1}'$. Again, the permutation $(k \quad k+1 \quad k+2)$ maps $D_{2,1}'$ onto $B_{2,1}'$, and $C_{4,1}'$ onto $A_{4,1}'$. Hence, $D_{2,1}' - B_{2,1}' = -C_{4,1}' + A_{4,1}' = 0$. Their sum vanish.

(c) Components represented by a tree $t$, whose root is unlabeled. There are four of them: $-B_{2,2}'$, $-B_{2,3}'$, $-A_{4,2}'$, $A_{4,3}'$. As $B_{2,2}' = A_{4,2}'$, the contribution of those components to $S_2$ is then $-B_{2,3}' + A_{4,3}'$.

(d) Components represented by a tree $t$, whose root is labeled. There are nine of them: $D_{2,2}', D_{3,2}'$, $-C_{3,2}'$, $-C_{4,2}'$, $-C_{4,3}'$, $-B_{3,2}'$, $-B_{3,3}'$, $-B_{3,4}'$, $A_{3,2}$. By simply comparing the subtree contents we have: $D_{2,2}' - C_{3,2}' = -B_{3,2}' + A_{3,2}' = 0$, $D_{3,2}' - (C_{4,3}' + B_{3,4}') = 0$ and $C_{4,2}' = B_{3,3}'$. The contribution of those terms is then $-2C_{4,2}'$.

Hence, $S_1 + S_2 = (-2B_{1,2} + D_{1,1}' - C_{1,1}') + ((-B_{3,1,1} + A_{3,1,1}) + (-B_{2,3}' + A_{4,3}')) + (-2C_{4,2}')$. As $D_{1,1}' = B_{2,3}'$, $C_{1,1}' = A_{3,1,1}$ and $B_{3,1,1} = A_{4,3}'$, we get $S_1 + S_2 = -2B_{1,2} - 2C_{4,2}$, that is,

$$\Delta_k^2 \mathcal{S}_{2n+1,m,k} - 2 \left[ m \right] = 2 \left[ m \right].$$

**Corollary 4.2.** The relations (R2) and (R4) hold for the matrices \( \text{Mat}(\mathcal{S}_{2n+1}) \), that is,

$$\Delta_k^2 \mathcal{S}_{2n+1,m,k} + 2 \mathcal{S}_{2n-1,m-2,k} = 0, \quad \text{if} \quad (m, k) \in L_n^{(2)};$$

$$\Delta_k^2 \mathcal{S}_{2n+1,m,k} + 2 \mathcal{S}_{2n-1,m,k} = 0, \quad \text{if} \quad (m, k) \in U_n^{(1)}.$$ 

**Proof.** When $(m, k)$ belongs to $L_n^{(2)}$, the second term in (4.1) is in bijection with $2[ k \quad m - 2 ]$ and the third one with $2[ k \quad m - 2 ]$, that is, the set of trees in which the end $m - 2$ of the minimal chain is outside (resp. inside) the subtree of root $k$. The sum of those two terms is then $2 \mathcal{S}_{2n-1,m-2,k}$. 

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When \((m, k)\) belongs to \(U_n^{(1)}\), the third term of \((4.1)\) vanishes and the second one is in bijection with \(2\left[ k, m \right]\), equal to \(2\mathfrak{T}_{2n-1,m,k}\).

5. The initial conditions \((I_3)\) and \((I_4)\)

**Property 5.1.** Initial conditions \((I_3)\) and \((I_4)\) hold for the matrices \(\text{Mat}(\mathfrak{T}_{2n+1})\).

**Proof.** \((I_3)\) The first row of each matrix \(\text{Mat}(\mathfrak{T}_{2n+1})\) is obviously the zero-row, as 1 can never be the end of the minimal chain. For the second row note that for \(n \geq 2\) the set \(\mathfrak{T}_{2n+1,2,k}\) is empty when \(k = 2\) or \(2n\). Also the set \(\mathfrak{T}_{2n+1,1,1}\) is empty, for \(2\) and \((2n + 1)\) can be both children of the root only when \(n = 1\).

Let \(3 \leq k \leq 2n - 1\). As illustrated by the diagram

\[
\begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
k \downarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
3
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
2n+1
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\mapsto
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\downarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
k-2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
2n-1
\end{array}
\end{array}
\end{array}
\end{array}
\]

each tree from \(\mathfrak{T}_{2n+1,2,k}\) is transformed into a tree from \(\mathfrak{T}_{2n-1,1,k-2}\) by deleting the two nodes 2 and 1 and reducing the remaining nodes by 2. This transformation is obviously a bijection. Thus, \(\mathfrak{T}_{2n+1,2,k} = \mathfrak{T}_{2n-1,1,k-2}\) for \(3 \leq k \leq 2n - 1\). The second row of \(\text{Mat}(\mathfrak{T}_{2n+1})\) is then equal to the sequence

\[(5.1)\quad 0, 0, \mathfrak{T}_{2n-1,1,1}, \mathfrak{T}_{2n-1,1,2}, \ldots, \mathfrak{T}_{2n-1,1,2n-3}, \mathfrak{T}_{2n-1,1,2n-2}(= 0)\]

which is also equal to

\[(5.2)\quad 0, \mathfrak{T}_{2n-1,1,1}(= 0), \mathfrak{T}_{2n-1,2,1}, \ldots, \mathfrak{T}_{2n-1,2n-2,1}, 0,\]

by Poupard’s result \((1.5)\) (also by our combinatorial proof sketched in the Introduction).

\((I_4)\) The set \(\mathfrak{T}_{2n+1,m,1}\) is empty when \(m = 1, 2\) or \(2n\). When \(3 \leq m \leq 2n - 1\) the diagram

\[
\begin{array}{c}
\begin{array}{c}
2n+1
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\downarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
m
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\mapsto
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
m-1
\end{array}
\end{array}
\end{array}
\]

serves to illustrate the transformation that maps each tree from \(\mathfrak{T}_{2n+1,m,1}\) onto a tree from \(\mathfrak{T}_{2n-1,m-1,1}\), by deleting the two nodes \((2n + 1)\) and 1, and reducing the remaining nodes by 1. Thus, the first column of \(M_n\) is equal to sequence \((5.2)\), when read from top to bottom.
that would have no recourse to a recurrence argument for $m = 1, 2$ and $n \geq 2$. When $m = 3$, the mapping

$$
\begin{array}{c}
2n+1 \\
\bigtriangledown \\
1 \quad 2 \\
\end{array} \quad \mapsto \quad
\begin{array}{c}
2n+1 \\
\bigtriangledown \\
1 \\
\end{array}
$$

shows that $\mathcal{T}_{2n+1,3,2}$ is in bijection with $\mathcal{T}_{2n-1,\bullet,1} = \mathcal{T}_{2n-1,2,\bullet}$. When $m = 4$, the following decomposition prevails

$$
\mathcal{T}_{2n+1,4,2} = \begin{array}{c}
2n+1 \\
\bigtriangledown \\
1 \quad 2 \\
\end{array} + \begin{array}{c}
2n+1 \\
\bigtriangledown \\
1 \quad 2 \\
\end{array}
$$

the two sets on the right-hand side being in bijection, respectively, with

$$
\begin{array}{c}
2n+1 \\
\bigtriangledown \\
1 \quad 2 \\
\end{array} \quad \mapsto \quad
\begin{array}{c}
2n+1 \\
\bigtriangledown \\
1 \quad 2 \\
\end{array}
$$

that is,

$$
= \mathcal{T}_{2n-1,3,\bullet} + \mathcal{T}_{2n-1,2,\bullet}.
$$

To prove the identity $\mathcal{T}_{2n+1,m,2} = \mathcal{T}_{2n-1,m-1,\bullet} + \mathcal{T}_{2n-1,m-2,\bullet} = \mathcal{T}_{2n+1,m,1} + \mathcal{T}_{2n+1,m-1,1}$ (by (5.2) and (I 4)) for $5 \leq m \leq 2n$ proceed by induction on $m$ using relation $(R\,1)$ already proved in (3.3). Write

$$
\begin{align*}
\mathcal{T}_{2n+1,m,2} &= 2 \mathcal{T}_{2n+1,m-1,2} - \mathcal{T}_{2n+1,m-2,2} - 2 \mathcal{T}_{2n-1,m-2,2} \quad \text{[by (3.3)]} \\
2 \mathcal{T}_{2n+1,m-1,2} &= 2 \mathcal{T}_{2n+1,m-1,1} + 2 \mathcal{T}_{2n+1,m-2,1} \quad \text{[by induction on $m$]} \\
- \mathcal{T}_{2n+1,m-2,2} &= - \mathcal{T}_{2n+1,m-2,1} - \mathcal{T}_{2n+1,m-3,1} \quad \text{[by induction on $m$]} \\
- 2 \mathcal{T}_{2n-1,m-2,2} &= - 2 \mathcal{T}_{2n-1,m-2,1} - 2 \mathcal{T}_{2n-1,m-3,1} \quad \text{[by induction on $n$ and $m$]}
\end{align*}
$$

$$
\begin{align*}
2 \mathcal{T}_{2n+1,m-1,1} - \mathcal{T}_{2n+1,m-2,1} - 2 \mathcal{T}_{2n-1,m-2,1} &= \mathcal{T}_{2n+1,m,1} \quad \text{[by (3.3)]} \\
2 \mathcal{T}_{2n+1,m-2,1} - \mathcal{T}_{2n+1,m-3,1} - 2 \mathcal{T}_{2n-1,m-3,1} &= \mathcal{T}_{2n+1,m-1,1}. \quad \text{[by (3.3)]}
\end{align*}
$$

By summing those six equations we get $\mathcal{T}_{2n+1,m,2} = \mathcal{T}_{2n+1,m,1} + \mathcal{T}_{2n+1,m-1,1}$, also equal to $\mathcal{T}_{2n-1,m-1,\bullet} + \mathcal{T}_{2n-1,m-2,\bullet}$ by (5.2).

Remark. It would be interesting to make up a proof of Property 5.1 that would have no recourse to a recurrence argument for $m \geq 5$ as above.
6. Proofs of Theorems 1.1 and 1.2

Taking Property 2.1, Corollaries 3.2 and 4.2, Property 5.1 into account we conclude that the sequence of matrices \( \text{Mat}(\Xi_{2n+1}) \) is both a Delta and Gamma sequence. Those sequences are then identical and we may write

\[
f_n(m, k) = \Xi_{2n+1,m,k}
\]

for all \( n, m, k \). This completes the proof of Theorem 1.1.

We now exploit the properties of the strictly ordered binary trees to prove that the matrices \( M_n \) are symmetric with respect to their counter-diagonals (Theorem 1.2). First, the symmetry property is banal for \( M_1, M_2 \). For \( n \geq 3 \) consider the NE- and SW-corners

\[
\begin{pmatrix}
  f_n(1, 2n - 1) & f_n(1, 2n) \\
  f_n(2, 2n - 1) & f_n(2, 2n)
\end{pmatrix}
= \begin{pmatrix}
  0 & 0 \\
  f_n(2, 2n - 1) & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
  f_n(2n - 1, 1) & f_n(2n - 1, 2) \\
  f_n(2n, 1) & f_n(2n, 2)
\end{pmatrix}
= \begin{pmatrix}
  f_n(2n - 1, 1) & f_n(2n - 1, 2) \\
  0 & f_n(2n, 2)
\end{pmatrix}
\]

of the matrix \( M_n \). As \( f_n(2n - 1, 1) = f_n(2n, 2) = \Xi_{2n-3} = T_{2n-3}/2^{n-2} \) (by combining (1.11), (I2), (I3), (5.1) and (5.2)), both corners are symmetric with respect to their counter-diagonals [in short, counter-symmetric].

Let us prove that the upper part of the matrix \( M_n \) is counter-symmetric and for \( i = 1, 2, \ldots, n - 1 \) adopt the notation:

\[
\text{Row}_i = \{(i, i + 1), (i, i + 2), \ldots, (i, 2n - i), (i, 2n - i + 1)\};
\]

\[
\text{Col}_{2n-i+1} = \{(i, 2n - i + 1), (i + 1, 2n - i + 1),
\ldots, (2n - i - 1, 2n - i + 1), (2n - i, 2n - i + 1)\}.
\]

Note that \( \text{Row}_i \) and \( \text{Col}_{2n-i+1} \) have the cell \((i, 2n - i + 1)\), belonging to the counter-diagonal, in common. There is nothing to prove for the pairs \((m, k)\) along the counter-diagonal and also for the entries from \( \text{Row}_1 \) and \( \text{Col}_{2n} \), which are all zero.

Let \((m_0, k_0)\) belong to \( \text{Row}_j \) for some \( j \) such that \( 2 \leq j \leq n - 1 \). Further, assume that (1.12) holds for all \((m, k) \in \text{Row}_1 \cup \cdots \cup \text{Row}_{j-1}\) and all \((m, k) \in \text{Row}_j\) lying on the right of \((m_0, k_0)\), not including \((m_0, k_0)\), that is, \( m = j \) and \( k > k_0 \). By symmetry, (1.12) also holds for all \((m, k) \in \text{Col}_{2n} \cup \cdots \cup \text{Col}_{2n-j+2}\) and all \((m, k) \in \text{Col}_{2n-j+1}\) lying above \((2n + 1 - k_0, 2n + 1 - m_0)\) not including the latter pair.

Now, the following relations hold:

\[
f_n(m_0, k_0) = 2f_n(m_0, k_0 + 1) - f_n(m_0, k_0 + 2) - 2f_{n-1}(m_0, k_0) \] [by \((R2)\)]

\[
f_{n-1}(m_0, k_0) = f_{n-1}(2n - 1 - k_0, 2n - 1 - m_0), \] [by induction on \( n \)]

\[
f_n(m_0, k_0 + 1) = f_n(2n - k_0, 2n + 1 - m_0), \]

\[
f_n(m_0, k_0 + 2) = f_n(2n - 1 - k_0, 2n + 1 - m_0). \] [both by the local induction]
Therefore,
\[ f_n(m_0, k_0) - 2f_n(2n - k_0, 2n + 1 - m_0) \\
+ f_n(2n - 1 - k_0, 2n + 1 - m_0) + 2f_{n-1}(2n - 1 - k_0, 2n - 1 - m_0) = 0. \]

But by \((R3)\) written at \((m, k) = (2n - 1 - k_0, 2n + 1 - m_0)\) we have:
\[ f_n(2n - 1 - k_0, 2n + 1 - m_0) - 2f_n(2n - k_0, 2n + 1 - m_0) \\
- f_n(2n - k_0 + 1, 2n + 1 - m_0) + 2f_{n-1}(2n - 1 - k_0, 2n - 1 - m_0) = 0. \]

By comparing the last two equations we conclude that
\[ f_n(m_0, k_0) = f_n(2n - k_0 + 1, 2n + 1 - m_0), \]
which means that \((1.12)\) now holds for \((m, k) = (m_0, k_0)\).

For the entries of \(M_n\) lying below the diagonal we proceed in the same manner and adopt the notation:

\[ \text{Row}_{2n+1-i} = \{(2n + 1 - i, i), \ldots, (2n + 1 - i, 2n - i)\}; \]
\[ \text{Col}_i = \{(i + 1, i), (i + 2, i), \ldots, (2n + 1 - i, i)\}; \]

for \(i = 1, 2, \ldots, n-1\). Again, Row\(_{2n+1-i}\) and Col\(_i\) have the cell \((2n+1-i, i)\) in common.

Let \((m_0, k_0)\) belong to Col\(_j\) for some \(j\) such that \(1 \leq j \leq n - 1\). Further, assume that \((1.12)\) holds for all \((m, k) \in \text{Col}_1 \cup \cdots \cup \text{Col}_{j-1}\) and all \((m, k) \in \text{Col}_j\) lying below \((m_0, k_0)\), not including \((m_0, k_0)\), that is, \(m > m_0\) and \(k = k_0\). By symmetry, \((1.12)\) also holds for all \((m, k) \in \text{Row}_{2n} \cup \cdots \cup \text{Row}_{2n-j+2}\) and all \((m, k) \in \text{Row}_{2n+1-j}\) lying to the left of \((2n + 1 - k_0, 2n + 1 - m_0)\) not including the latter pair.

Now, the following relations hold:
\[
\begin{align*}
    f_n(m_0, k_0) &= 2f_n(m_0 + 1, k_0) - f_n(m_0 + 2, k_0) - 2f_{n-1}(m_0, k_0), \\
    f_{n-1}(m_0, k_0) &= f_{n-1}(2n - 1 - k_0, 2n - 1 - m_0), \\
    f_n(m_0 + 1, k_0) &= f_n(2n + 1 - k_0, 2n - m_0), \\
    f_n(m_0 + 2, k_0) &= f_n(2n + 1 - k_0, 2n - 1 - m_0), \\
    f_n(m_0, k_0) &= 2f_n(2n + 1 - k_0, 2n - m_0) \\
    - f_n(2n + 1 - k_0, 2n - 1 - m_0) - 2f_{n-1}(2n - 1 - k_0, 2n - 1 - m_0). 
\end{align*}
\]

Therefore,
But by \((R4)\) written at \((m,k) = (2n + 1 - k_0, 2n - 1 - m_0)\) we have:

\[
f_n(2n + 1 - k_0, 2n - 1 - m_0) = 2f_n(2n + 1 - k_0, 2n - m_0)
- f_n(2n + 1 - k_0, 2n + 1 - m_0) - 2f_{n-1}(2n - 1 - k_0, 2n - 1 - m_0).
\]

By comparing the last two equations we conclude that

\[
f_{2n+1}(m_0, k_0) = f_{2n+1}(2n + 1 - k_0, 2n + 1 - m_0),
\]

which means that \((1.12)\) now holds for \((m,k) = (m_0,k_0)\).

Define

\[
(6.2) \quad \text{Eoc}(t) := \text{eoc}(t), \quad \text{but} \quad \text{Pom}(t) := 2n + 1 - \text{pom}(t).
\]

**Theorem 6.2.** Let \(\mathcal{T}_{2n+1}(x, y) := \sum_{t \in \mathcal{T}_{2n+1}} x^{\text{Eoc}(t)}y^{\text{Pom}(t)}\) be the generating polynomial for the set \(\mathcal{T}_{2n+1}\) by the pair of statistics \((\text{Eoc}, \text{Pom})\). Then,

\[
(6.3) \quad \mathcal{T}_{2n+1}(x, y) = \mathcal{T}_{2n+1}(y, x).
\]

*Proof.* This is a simple consequence of Theorem 1.2: let \(g_n(m,k) := \#\{\text{Eoc} = m, \text{Pom} = k\}\). Then,

\[
g_n(m,k) = \#\{\text{eoc} = m, \text{pom} = 2n + 1 - k\} = f_n(m,2n + 1 - k)
= f_n(k,2n + 1 - m) \quad \text{[by (1.12)]}
= g_n(k,m).
\]

Thus,

\[
\mathcal{T}_{2n+1}(x, y) = \sum_{t \in \mathcal{T}_{2n+1}} x^{\text{Eoc}(t)}y^{\text{Pom}(t)} = \sum_{m,k} g_n(m,k)x^my^k
= \sum_{m,k} g_n(k,m)x^my^k = \mathcal{T}_{2n+1}(y, x).
\]

### 7. Further properties

Other properties of the Delta Sequence can be obtained by having a further look at the geometry of the strictly ordered binary trees. The sub- and superdiagonals of the matrices \(M_n\) for \(n = 2, 3, 4, 5\) are equal, as can be seen in Fig. 1.2. For an arbitrary \(n \geq 2\) we have the following.

**Property 7.1.** Sub- and super diagonals are equal:

\[
(7.1) \quad f_n(k+1,k) = f_n(k,k+1) \quad \text{\((n \geq 2; \, 1 \leq k \leq 2n - 1)\)}.
\]

*Proof.* First, note that \(k\) and \((k + 1)\) can be siblings in a tree from \(\mathcal{T}_{2n+1,k,k+1}\), but never in a tree from \(\mathcal{T}_{2n+1,k+1,k}\). Second, \(k\) can be parent
of \((k+1)\) in a tree from the latter set, but never in a tree from the former one. Also, \(f_n(2,1) = f_n(1,2) = 0\) for \(n \geq 2\) and for \(k \geq 2\) we have the decompositions:

\[
\begin{align*}
    f_n(k, k+1) &= \begin{bmatrix}
        \bullet & k \\
        k+1 & \bullet & 2n+1
    \end{bmatrix} + \begin{bmatrix}
        \bullet & k \\
        k+1 & \bullet & 2n+1
    \end{bmatrix}; \\
    f_n(k+1, k) &= \begin{bmatrix}
        \bullet & k \\
        k+1 & \bullet & 2n+1
    \end{bmatrix} + \begin{bmatrix}
        \bullet & k \\
        k+1 & \bullet & 2n+1
    \end{bmatrix}.
\end{align*}
\]

The first terms in the previous two equations are in bijection, as well as the second ones, the notation “\(\bullet k\)” meaning that \(k\) is the end of the minimal chain, following our convention on Tree Calculus (cf. Section 3).

**Property 7.2.** We have the crossing equalities:

\[
\begin{align*}
    f_n(k+1, k-1) + f_n(k-1, k+1) &= f_n(k+1, k) + f_n(k-1, k) \\
    &= f_n(k, k+1) + f_n(k, k-1) \quad (2 \leq k \leq 2n-1).
\end{align*}
\]

Reporting to Fig. 1.2 the involved entries in the first identities are located on the four bullets drawn in the following diagram.

\[
\begin{array}{c}
    \bullet \\
    k+1 \\
    k \\
    k-1
\end{array} \quad \begin{array}{c}
    \bullet \\
    k \\
    k+1
\end{array}
\]

**Proof.** Let \(i, j\) be two different integers from the set \(\{(k-1), k, (k+1)\}\). Say that \(i\) and \(j\) are *connected* in a tree \(t\), if the tree contains the edge \(i-j\), or if \(i\) and \(j\) are brothers and one of them is the end of the minimal chain of \(t\). Each of the four ingredients of the previous identity is now decomposed into five terms, depending on whether the nodes \((k-1), k, (k+1)\) are connected or not, namely: no connectedness; only \(k, (k+1)\) connected; \((k-1), k\) connected; \((k-1), (k+1)\) connected; all connected. Thus,

\[
\begin{align*}
    f_n(k-1, k) &= \begin{bmatrix}
        \bullet & k-1 \\
        k & \bullet & 2n+1
    \end{bmatrix} + \begin{bmatrix}
        \bullet & k-1 \\
        k & \bullet & 2n+1
    \end{bmatrix} + \begin{bmatrix}
        \bullet & k-1 \\
        k & \bullet & 2n+1
    \end{bmatrix} + \begin{bmatrix}
        \bullet & k-1 \\
        k & \bullet & 2n+1
    \end{bmatrix} + \begin{bmatrix}
        \bullet & k-1 \\
        k & \bullet & 2n+1
    \end{bmatrix} \\
    &= A_1 + A_2 + A_3 + A_4 + A_5;
\end{align*}
\]
\[ f_n(k + 1, k) = \left[ \begin{array}{c} k+1 \\ 2n+1 \\ k \\ k-1 \\ k-1 \end{array} \right] + \left[ \begin{array}{c} k+1 \\ 2n+1 \\ k \\ k \end{array} \right] = B_1 + B_2 + B_3 + B_4 + B_5; \]

\[ f_n(k + 1, k - 1) = \left[ \begin{array}{c} k+1 \\ 2n+1 \\ k \\ k-1 \\ k-1 \end{array} \right] + \left[ \begin{array}{c} k+1 \\ k \\ k \end{array} \right] = C_1 + C_2 + C_3 + C_4 + C_5; \]

\[ f_n(k - 1, k + 1) = \left[ \begin{array}{c} k-1 \\ 2n+1 \\ k \\ k+1 \\ k \end{array} \right] + \left[ \begin{array}{c} k-1 \\ k+1 \\ k-1 \end{array} \right] = D_1 + D_2 + D_3 + D_4 + D_5. \]

Now, the following identities hold: \( A_1 = C_1, A_2 = C_3, A_3 = D_4, A_4 = D_3, 2A_5 = D_5 \); \( B_1 = D_1, B_3 = D_2, B_5 = C_5 \). Moreover,

\[ B_4 - C_2 = D_5 = 2A_5; \]

\[ C_4 - B_2 = A_5. \]

Altogether, \( \sum_i (A_i + B_i) - \sum_i (C_i + D_i) = (B_4 - C_2) - (C_4 - B_2) + (A_5 - D_5) = 2A_5 - A_5 + (A_5 - 2A_5) = 0. \]
Property 7.3. The row sums $f_n(m, \bullet)$ form a Poupard Triangle, the initial conditions being: $f_0(1, \bullet) = 1$, $f_n(1, \bullet) = 0$ and $f_n(2, \bullet) = 2 \sum_m f_{n-1}(m, \bullet)$ $(n \geq 1)$; and the finite difference system:

\[ \Delta^2 f_n(m, \bullet) + 2 f_{n-1}(m, \bullet) = 0 \quad (1 \leq m \leq 2n - 1). \]

The column sums $f_n(\bullet, k)$ form a Poupard Triangle, the initial conditions being: $f_0(\bullet, 0) = 1$, $f_n(\bullet, 0) = 0$ and $f_n(\bullet, 1) = 2 \sum_k f_{n-1}(\bullet, k)$ $(n \geq 1)$; and the finite difference system:

\[ \Delta^2 f_n(\bullet, k) + 2 f_{n-1}(\bullet, k) = 0 \quad (0 \leq k \leq 2n - 2). \]

There are several proofs of this Property. First, the methods developed in Section 3 can be readapted by disregarding the conditions involving the pom-statistic. Here, we simply work out a specialization of the recurrence relations $(R1)$—$(R4)$, that makes use of the previous two properties. Besides, we only prove the first part of the property that deals with the row sums.

**Proof.** For $1 \leq m \leq 2n - 2$ we have:

\[
\Delta^2 f_n(m, \bullet) = \sum_{1 \leq k \leq 2n} \Delta^2 f_n(m, k)
\]

\[
= \sum_{1 \leq k \leq m-1} \Delta^2 f_n(m, k)
+ \Delta^2(f_n(m, m) + f_n(m, m+1) + f_n(m, m+2))
+ \sum_{m+3 \leq k \leq 2n} \Delta^2 f_n(m, k)
\]

\[
= \sum_{1 \leq k \leq m-1} -2f_{n-1}(m, k)
+ f_n(m, m) - 2f_n(m+1, m) + f_n(m+2, m)
+ f_n(m, m+1) - 2f_n(m+1, m+1) + f_n(m+2, m+1)
+ f_n(m, m+2) - 2f_n(m+1, m+2) + f_n(m+2, m+2)
+ \sum_{m+3 \leq k \leq 2n} -2f_{n-1}(m, k-2).
\]

In the previous sum the diagonal terms vanish. Also, $f_n(m+1, m) = f_n(m, m+1)$, $f_n(m+2, m+1) = f_n(m+1, m+2)$ by (7.1). The sum of the nine intermediate terms becomes: $-f_n(m+1, m) + f_n(m+2, m) - f_n(m+1, m+2) + f_n(m, m+2)$, which is 0 by (7.2). Hence,

\[
\Delta^2 f_n(m, \bullet) = -2 \sum_{1 \leq k \leq m-1} f_{n-1}(m, k) - 2 \sum_{m+1 \leq k \leq 2n-2} f_{n-1}(m, k)
\]

\[
= -2f_{n-1}(m, \bullet).
\]
On the other hand, by (1.2) and (1.11),
\[
\begin{align*}
   f_n(2n - 1, \bullet) &= \sum_{1 \leq k \leq 2n} f_n(2n - 1, k) = \sum_{1 \leq k \leq 2n - 2} (f_{n-1}(k, \bullet) + f_{n-1}(\bullet, k)) \\
   &= 2 f_{n-1}(\bullet, \bullet) := 2 \# \Sigma_{2n-1} = 2 T_{2n-1}/2^{n-1}; \\
   f_n(2n, \bullet) &= \sum_{1 \leq k \leq 2n} f_n(2n, k) = \sum_{1 \leq k \leq 2n - 2} f_{n-1}(k, \bullet) \\
   &= f_{n-1}(\bullet, \bullet) = \# \Sigma_{2n-1} = T_{2n-1}/2^{n-1}; \\
   f_n(2n + 1, \bullet) &= 0.
\end{align*}
\]

The last three evaluations imply that \( \Delta_m^2 f_n(2n - 1, \bullet) = -2 f_{n-1}(2n - 1, \bullet) = 0 \). □

8. Other equivalent definitions for the Delta sequence

Definitions 1.1 and 1.2 have been shown to be equivalent to characterize a Delta Sequence. Other combinations of the recurrence relations (R1)–(R4), together with the initial conditions (I1)–(I4), can be used. We describe them by means of squares and arrows, as was done in Fig. 1.3:

Other initial conditions than (I1)–(I4) can be introduced. They will be denoted by (SW 1, 4), (NE 2, 3), as they refer only to the South-West and North-East corners of the matrices:

\[
\begin{align*}
   \text{(SW)} \quad & \begin{pmatrix}
   f_n(2n-1,1) & f_n(2n-1,2) \\
   f_n(2n,1) & f_n(2n,2)
\end{pmatrix} = \begin{pmatrix}
   f_{n-1}(\bullet,1) & f_{n-1}(2,\bullet) + f_{n-1}(\bullet,2) \\
   0 & f_{n-1}(\bullet,1)
\end{pmatrix}; \\
   \text{(NE)} \quad & \begin{pmatrix}
   f_n(1,2n-1) & f_n(1,2n) \\
   f_n(2,2n-1) & f_n(2,n)
\end{pmatrix} = \begin{pmatrix}
   0 & 0 \\
   f_{n-1}(2n-2,\bullet) & 0
\end{pmatrix}.
\end{align*}
\]

When one of those two conditions (SW), (NE) is involved, two recurrence relations among (R1)–(R4) are needed to build up an equivalent definition. In Definition 1.5 for instance, (R1) and (R4) are to be associated with (SW). We then get five further equivalent definitions:
9. Generating functions for the Delta sequence

1. Poupard matrices. Let \( G = (g_{i,j}) \ (i \geq 0, j \geq 0) \) be an infinite matrix with nonnegative integral entries. Say that \( G \) is a Poupard matrix, if for every \( i \geq 0, j \geq 0 \) the following identity holds:

\[
(9.1) \quad g_{i,j+2} - 2g_{i+1,j+1} + g_{i+2,j} + 2g_{i,j} = 0.
\]

Let \( G(x, y) := \sum_{i \geq 0, j \geq 0} g_{i,j} (x^i/i) (y^j/j!) \); \( R_i(y) := \sum_{j \geq 0} g_{i,j} (y^j/j!) \) \((i \geq 0)\); \( C_j(x) := \sum_{i \geq 0} g_{i,j} (x^i/i) \) \((j \geq 0)\) be the exponential generating functions for the matrix itself, its rows and columns, respectively.

**Proposition 9.1.** The following four properties are equivalent.

(i) \( G = (g_{i,j}) \ (i \geq 0, j \geq 0) \) is a Poupard matrix;
(ii) \( R''_i(y) - 2R'_i(y) + R_{i+2}(y) + 2R_i(y) = 0 \) for all \( i \geq 0 \);
(iii) \( C''_j(x) - 2C'_{j+1}(x) + C_{j+2}(x) + 2C_j(x) \) for all \( j \geq 0 \);
(iv) \( G(x, y) \) satisfies the partial differential equation:

\[
(9.2) \quad \frac{\partial^2 G(x, y)}{\partial x^2} - 2 \frac{\partial^2 G(x, y)}{\partial x \partial y} + \frac{\partial^2 G(x, y)}{\partial y^2} + 2G(x, y) = 0.
\]

**Proof.** It suffices to write \( R''_i(y) - 2R'_i(y) + R_{i+2}(y) + 2R_i(y) = \sum_{j \geq 0} (g_{i,j+2} - 2g_{i+1,j+1} + 2g_{i+2,j} - 2g_{i,j})(y^j/j!) \), and, similarly, \( C''_j(x) - \)
TREE CALCULUS FOR DIFFERENCE EQUATIONS

\[ 2C'_{j+1}(x) + C_{j+2}(x) + 2C_j(x) = \sum_{i \geq 0} (g_{i+2,j} - 2g_{i+1,j+1} + 2g_{i,j+2} - 2g_{i,j})(x^i/i!) \]

to obtain the equivalence between the first three properties. As for the last one, simply note that \( G(x, y) = \sum_{i \geq 0} R_i(y) x^i/i! = \sum_{j \geq 0} C_j(x) y^j/j! \) and make the appropriate derivations. □

**Proposition 9.2.** We have

\[ (9.3) \quad G(x, y) = A(x + y) \cos(\sqrt{2} y) + B(x + y) \sin(\sqrt{2} y), \]

where \( A(x) \) and \( B(x) \) are two arbitrary series.

*Proof.* Let \( \xi := x + y, \eta := y \). Then, \( \partial G/\partial x = \partial G/\partial \xi; \partial G/\partial y = \partial G/\partial \xi + \partial G/\partial \eta; \partial^2 G/\partial x^2 = \partial^2 G/\partial \xi^2; \partial^2 G/\partial y^2 = \partial^2 G/\partial \xi^2 + 2 \partial^2 G/\partial \xi \partial \eta + \partial^2 G/\partial \eta^2; \partial^2 G/\partial x \partial y = \partial^2 G/\partial \xi \partial \eta + \partial^2 G/\partial \xi \partial \eta \).

Thus, (9.3) can be rewritten as

\[ (9.4) \quad \frac{\partial^2 G}{\partial \eta^2} + 2G = 0. \]

The ordinary differential equation \( G'' + 2G = 0 \), whose characteristic polynomial is \( r^2 + 2 = 0 \), has a general solution of the form \( A \cos(\sqrt{2} \eta) + B \sin(\sqrt{2} \eta) \), so that the general solution of (9.2) is exactly given by (9.3). □

The exact expression for the generating function \( G(x, y) \) can then be derived, if \( A(x + y) \) and \( B(x + y) \) can be obtained by an independent calculation, as done in the sequel.

2. **A sequence of Poupard matrices for the lower triangles.** The entries \( f_n(m, k) \) \((1 \leq k < m \leq 2n)\) from the lower triangles in the matrices \( M_n \) \((n \geq 1)\) are now recorded as entries \( \lambda_{i,j}^{(p)} \) \((p \geq 0, i \geq 0, j \geq 0)\) of infinite matrices \( \Lambda(p) = (\lambda_{i,j}^{(p)}) \) \((i \geq 0, j \geq 0)\) as follows.

Define

\[ (9.5) \quad \lambda_{i,j}^{(p)} := \begin{cases} 0, & \text{if } i + j \equiv p \mod 2; \\ f_n(m, k), & \text{if } i + j \equiv p + 1 \mod 2; \end{cases} \]

with \( k := j+1, m := i+j+2, 2n := p+(i+j)+1 \). The latter equation makes sense, as \( i+j \) and \( p \) are of different parity. The mapping \((p, i, j) \mapsto (n, m, k)\) is one-to-one, the reverse mapping being: \( p := 2n - m + 1, i := m - k - 1, j := k - 1 \). Thus, for \( p \geq 0 \), the matrix \( \Lambda^{(2p+1)} \) reads:
and for \( p \geq 1 \), the matrix \( \Lambda^{(2p)} \) is equal to

\[
\begin{pmatrix}
0 & f_{p+1}(3,2) & 0 & f_{p+2}(5,4) & 0 & f_{p+3}(7,6) \\
f_{p+1}(3,1) & 0 & f_{p+2}(5,3) & 0 & f_{p+3}(7,5) & \\
f_{p+2}(5,1) & 0 & f_{p+3}(7,4) & \\
f_{p+3}(7,1) & \\
\vdots
\end{pmatrix}
\]

\( \text{Proposition 9.3. Every matrix } \Lambda^{(p)} \ (p \geq 0) \text{ is a Poupard matrix.} \)

**Proof.** Using Definition (9.5) we have

\[
\lambda_{i,j}^{(p)} - 2 \lambda_{i+1,j+1}^{(p)} + \lambda_{i+2,j}^{(p)} - 2 \lambda_{i,j}^{(p)}
\]

\[
= f_{n+1}(m+2,k+2) - 2 f_{n+1}(m+2,k+1)
\]

\[
+ f_{n+1}(m+2,k) + 2 f_n(m,k)
\]

\[
= \Delta_k f_{n+1}(m+2,k) + 2 f_n(m,k) = 0,
\]

by rule \((R4)\). \( \square \)

3. The first matrix \( \Lambda^{(1)} \). The counter-diagonal of \( \Lambda^{(1)} \) at depth \( 2i \) \( (i \geq 0) \) reads

\[
\lambda_{2i,0}^{(1)} = f_{i+1}(2i+2,1); \quad \lambda_{2i-1,1}^{(1)} = f_{i+1}(2i+2,2); \quad \cdots
\]

\[
\cdots \quad \lambda_{2i-j,j}^{(1)} = f_{i+1}(2i+2,j+1); \quad \cdots \quad \lambda_{0,2i}^{(1)} = f_{i+1}(2i+2,2i+1);
\]

Those \( (2i+1) \) terms are equal to the first \( (2i+1) \) entries of the \( (2i+2) \)-nd row of the matrix \( M_{i+1} \), that is, to the single term \( f_i(2,1) = 1 \) for \( i = 0 \) and for \( i \geq 1 \) to

\[
f_i(1,\bullet) = 0, f_i(2,\bullet), \ldots, f_i(j,\bullet), \ldots, f_i(2i,\bullet), 0,
\]

by virtue of \((I2)\). Thus, \( \Lambda^{(1)} \) is identical to the Poupard matrix \( (f_i(j+1,\bullet)) \) \( (i, j \geq 0) \):
4. The first two columns of $\Lambda^{(p)}$. In the sequel the column labeled $j$ of $\Lambda^{(p)}$ will be denoted by $\Lambda^{(p)}_{\bullet,j}$ and the exponential generating function for that column by $\Lambda^{(p)}_{\bullet,j}(x) = \sum_{i \geq 0} \lambda^{(p)}_{i,j} x^i / i!$. Also, $\Lambda^{(p)}(x,y) := \sum_{j \geq 0} \Lambda^{(p)}_{\bullet,j}(x) y^j / j!$ will be the double exponential generating function for the matrix $\Lambda^{(p)}$.

**Proposition 9.4.** The first two columns of each matrix $\Lambda^{(p)}$ ($p \geq 2$) are related to the columns of $\Lambda^{(1)}$ by the identities:

\[
\Lambda^{(p)}_{\bullet,0}(x) = \Lambda^{(1)}_{\bullet,p-1}(x), \quad \Lambda^{(p)}_{\bullet,1}(x) = \frac{d}{dx} \Lambda^{(1)}_{\bullet,p-1}(x) + \Lambda^{(1)}_{\bullet,p}(x).
\]

**Proof.** For the first identity it suffices to prove $\lambda^{(p)}_{i,0} = \lambda^{(1)}_{i,p-1}$, that is

\[f_n(i + 2, 1) = f_n(p + i + 1, p)\]

when $2n = p + i + 1$. This is true by Theorem 1.2. For the second identity it suffices to prove $\lambda^{(p)}_{i,1} = \lambda^{(1)}_{i+1,p-1} + \lambda^{(1)}_{i,p}$, that is

\[f_n(i + 3, 2) = f_n(p + i + 2, p) + f_n(p + i + 2, p + 1)\]

when $2n = p + i + 2$. But by (I 4)

\[f_n(i + 3, 2) = f_n(i + 2, 1) + f_n(i + 3, 1),\]

so that by Theorem 1.2, identity (9.7) holds. \[\square\]
5. Solving the partial differential equation. By (9.6)
\[
\Lambda^{(p)}(x, y) \bigg|_{y=0} = \Lambda_{p,0}^{(p)}(x) = \Lambda_{p-1}^{(1)}(x);
\]
\[
\frac{\partial}{\partial y} \Lambda^{(p)}(x, y) \bigg|_{y=0} = \Lambda_{p,1}^{(p)}(x) = \frac{d}{dx} \Lambda_{p-1}^{(1)}(x) + \Lambda_{p}^{(1)}(x).
\]

As each \( \Lambda^{(p)} \) is a Poupard matrix, we can use identity (9.3), so that
\[
\Lambda^{(p)}(x, y) \bigg|_{y=0} = A(x) \quad \text{and} \quad \partial \Lambda^{(p)}(x, y)/\partial y = A'(x + y) \cos(\sqrt{2}y) - \sqrt{2} A(x + y) \sin(\sqrt{2}y) + B'(x + y) \sin(\sqrt{2}y) + \sqrt{2} B(x + y) \cos(\sqrt{2}y).
\]
Hence,
\[
\Lambda^{(p)}(x, y) \bigg|_{y=0} = A(x) = \Lambda_{p-1}^{(1)}(x);
\]
\[
\frac{\partial \Lambda^{(p)}(x, y)}{\partial y} \bigg|_{y=0} = A'(x) + \sqrt{2} B(x) = \frac{d}{dx} \Lambda_{p-1}^{(1)}(x) + \Lambda_{p}^{(1)}(x).
\]

Consequently, \( A(x) = \Lambda_{p-1}^{(1)}(x) \) and \( B(x) = \Lambda_{p}^{(1)}(x)/\sqrt{2} \) and the general expression for \( \Lambda^{p}(x, y) \) reads:
\[
\Lambda^{(p)}(x, y) = \Lambda_{p-1}^{(1)}(x + y) \cos(\sqrt{2}y) + \Lambda_{p}^{(1)}(x + y) \sin(\sqrt{2}y)/\sqrt{2}.
\]

This expression still holds for \( p = 1 \). We know that \( \Lambda_{0,0}^{(1)}(x) = 1 \). On the other hand, the coefficient of \( x^{2k+1}/(2k + 1)! \) \((k \geq 0)\) in \( \Lambda_{1,1}^{(1)}(x) \) is \( f_{k+1}(2, 1) = T_{2k+1}/2^k \). Hence, \( \Lambda_{1,1}^{(1)}(x) = \sqrt{2} \tan(x/\sqrt{2}) \) and
\[
\Lambda^{(1)}(x, y) = \cos(\sqrt{2}y) + \sqrt{2} \tan\left(\frac{x + y}{\sqrt{2}}\right) \frac{\sin(\sqrt{2}y)}{\sqrt{2}}
\]
\[
= \cos\left(\frac{x - y}{\sqrt{2}}\right) / \cos\left(\frac{x + y}{\sqrt{2}}\right),
\]
(9.9)
a result already obtained by Poupard.

Remark. For getting the solution for \( \Lambda^{(1)}(x, y) \) we can also start with the general expression displayed in (9.3) and calculate \( A \) and \( B \) with the initial conditions \( \Lambda^{(1)}(0, y) = \Lambda^{(1)}(x, 0) = 1 \). We find
\[
\Lambda^{(1)}(x, y) = \cos(\sqrt{2}y) + \frac{1 - \cos(\sqrt{2}(x + y))}{\sin(\sqrt{2}(x + y))} \sin(\sqrt{2}y)
\]
\[
= \frac{\sin(\sqrt{2}x) + \sin(\sqrt{2}y)}{\sin(\sqrt{2}(x + y))},
\]
(9.10)
an expression, which is naturally equal to the right-hand side of (9.9) (by a simple trigonometric calculation).
5. A generating function for the lower triangles. By (9.9)

\[
\frac{\partial}{\partial y} \Lambda^{(1)}(x, y) = \frac{\partial}{\partial y} \cos\left(\frac{x - y}{\sqrt{2}}\right) / \cos\left(\frac{x + y}{\sqrt{2}}\right)
\]

(9.11)

\[
= \sin(\sqrt{2}x) / \sqrt{2} \cos^2\left(\frac{x + y}{\sqrt{2}}\right).
\]

Let \( \Lambda(x, y, z) := \sum_{p \geq 1} \Lambda^{(p)}(x, y) \frac{z^{p-1}}{(p-1)!} \).

Then,

\[
\Lambda(x, y, z) = \cos(\sqrt{2}y) \sum_{p \geq 1} \frac{z^{p-1}}{(p-1)!} \Lambda^{(1)}_{*,p-1}(x + y)
\]

\[
+ \frac{\sin(\sqrt{2}y)}{\sqrt{2}} \sum_{p \geq 1} \frac{z^{p-1}}{(p-1)!} \Lambda^{(1)}_{*,p}(x + y) \quad \text{[by (9.8)]}
\]

\[
= \cos(\sqrt{2}y) \Lambda^{(1)}(x + y, z) + \frac{\sin(\sqrt{2}y)}{\sqrt{2}} \frac{\partial}{\partial z} \Lambda^{(1)}(x + y, z),
\]

since \( \Lambda^{(1)}(x, z) = \sum_{p \geq 0} \frac{z^p}{p!} \Lambda^{(1)}_{*,p}(x) \).

By (9.11) we then get:

\[
\Lambda(x, y, z) = \cos(\sqrt{2}y) \cos\left(\frac{x + y - z}{\sqrt{2}}\right) / \cos\left(\frac{x + y + z}{\sqrt{2}}\right)
\]

\[
+ \sin(\sqrt{2}y) \sin(\sqrt{2}(x + y)) / 2 \cos^2\left(\frac{x + y + z}{\sqrt{2}}\right)
\]

\[
= \frac{\cos(\sqrt{2}x) + \cos(\sqrt{2}y) \cos(\sqrt{2}z)}{2 \cos^2\left(\frac{x + y + z}{\sqrt{2}}\right)}.
\]

Now, express \( \Lambda(x, y, z) \) as a series in the \( f_n(m, k) \)'s. By definition,

\[
\Lambda(x, y, z) = \sum_{p, i, j} \lambda^{(p)}_{i,j} \frac{z^{p-1}}{(p-1)!} \frac{x^i y^j}{i! j!} \quad (p \geq 1, i \geq 0, j \geq 0);
\]

so that by (9.5)

\[
\Lambda(x, y, z) = \sum_{k, m, n} f_n(m, k) \frac{x^{m-k-1} y^{k-1} z^{2n-m}}{(m - k - 1)! (k - 1)! (2n - m)!},
\]

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the latter sum over the set \( \{2 \leq k + 1 \leq m \leq 2n\} \). This achieves the proof of Theorem 1.4.

Note that the right-hand side of (1.14) is symmetric in \( y, z \), that is, \( \Lambda(x, y, z) = \Lambda(x, z, y) \). The change \( y \leftrightarrow z \) in the left-hand side of (1.14) shows that
\[
f_n(2n + 1 - k, 2n + 1 - m) = f_n(m, k),
\]
the symmetry proved for the entries \( f_n(m, k) \) such that \( m \geq k + 1 \).

**Remark.** Let \( z = 0 \) in (1.14). We get
\[
\sum_{2 \leq k + 1 \leq 2n} f_n(2n, k) \frac{x^{2n-k-1}}{(2n - k - 1)!} \frac{y^{k-1}}{(k-1)!} = \cos(\sqrt{2}x) + \cos(\sqrt{2}y) \frac{2}{2 \cos^2\left(\frac{x+y}{\sqrt{2}}\right)};
\]
or,
\[
\sum_{i \geq 0, j \geq 0} \lambda^{(1)}_{i,j} \frac{x^i}{i!} \frac{y^j}{j!} = \Lambda^{(1)}(x, y) = \frac{\cos(\sqrt{2}x) + \cos(\sqrt{2}y)}{2 \cos^2\left(\frac{x+y}{\sqrt{2}}\right)};
\]
which is another expression for \( \Lambda^{(1)}(x, y) \) than (9.9) and (9.10).

6. A generating function for the upper triangles. The entries \( f_n(m, k) \)
\((1 \leq m < k \leq 2n)\) from the upper triangles in the matrices \( M_n \) \((n \geq 1)\)
are next recorded as entries \( \omega_{i,j}^{(p)} \) \((p \geq 0, i \geq 0, j \geq 0)\) of infinite matrices \( \Omega^{(p)} = (\omega_{i,j}^{(p)}) \) \((i \geq 0, j \geq 0)\) as follows.

Define
\[
\omega_{i,j}^{(p)} := \begin{cases} 
0, & \text{if } i + j \not\equiv p \mod 2; \\
f_n(m, k), & \text{if } i + j \equiv p \mod 2;
\end{cases}
\]
with \( m := p + 1, k := p + j + 2, 2n := p + i + j + 2 \). Conversely, \( i := 2n - k, j := k - m - 1, p := m - 1 \). Thus, for \( p \geq 0 \)
\[
\Omega^{(2p+1)} = \begin{pmatrix}
0 & 1 & 2 & 3 \\
0 & f_{p+2}(2p + 2, 2p + 4) & 0 & \cdots \\
1 & f_{p+2}(2p + 2, 2p + 3) & 0 & f_{p+4}(2p + 2, 2p + 5) & \cdots \\
2 & 0 & f_{p+3}(2p + 2, 2p + 4) & \cdots \\
3 & f_{p+3}(2p + 2, 2p + 3) & \cdots \\
4 & \cdots 
\end{pmatrix}
\]
\[
\Omega^{(2p)} = \begin{pmatrix}
0 & 1 & 2 & 3 \\
0 & f_{p+1}(2p + 1, 2p + 2) & 0 & f_{p+2}(2p + 1, 2p + 4) & 0 \\
1 & 0 & f_{p+2}(2p + 1, 2p + 3) & 0 & \cdots \\
2 & f_{p+2}(2p + 1, 2p + 2) & 0 & \cdots \\
3 & 0 & \cdots \\
4 & \cdots 
\end{pmatrix}
\]
Remark. The first rows of all matrices $M_n$ are null, so that $Ω^{(0)}$ is the infinite matrix with all entries equal to zero!

Also, write

$$Ω^{(1)} = \begin{pmatrix}
0 & 0 & f_2(2,4) & 0 & f_3(2,6) & 0 & f_4(2,8) & 0 & f_5(2,10) \\
1 & f_2(2,3) & 0 & f_3(2,5) & 0 & f_4(2,7) & 0 & f_5(2,9) & \cdots \\
2 & 0 & f_3(2,4) & 0 & f_4(2,6) & 0 & f_5(2,8) & \cdots \\
3 & f_3(2,3) & 0 & f_4(2,5) & 0 & f_5(2,7) & \cdots \\
4 & 0 & f_4(2,4) & 0 & f_5(2,6) & \cdots \\
5 & f_4(2,3) & 0 & f_5(2,5) & \cdots \\
6 & 0 & f_5(2,4) & \cdots \\
7 & f_5(2,3) & \cdots \\
0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 1 & 0 & 1 & 0 & 4 & 0 & 34 & \cdots \\
2 & 0 & 2 & 0 & 8 & 0 & 68 & \cdots \\
3 & 1 & 0 & 10 & 0 & 94 & \cdots \\
4 & 0 & 8 & 0 & 104 & \cdots \\
5 & 4 & 0 & 94 & \cdots \\
6 & 0 & 68 & \cdots \\
7 & 34 & \cdots 
\end{pmatrix}$$

**Proposition 9.6.** Every matrix $Ω^{(p)} (p \geq 0)$ is a Poupard matrix.

Same proof as for Proposition 9.3.

The row labeled $i$ of $Ω^{(p)}$ will be denoted by $Ω^{(p)}_{i,∗}$ and the exponential generating function for that row by $Ω^{(p)}_{i,∗}(y) = \sum_{j\geq 0} \omega^{(p)}_{i,j} y^j / j!$. Also, $Ω^{(p)}(x, y) := \sum_{i\geq 0} Ω^{(p)}_{i,∗}(y)x^i / i!$ will be the double exponential generating function for the matrix $Ω^{(p)}$. As $x$ and $y$ play a symmetric role in (9.2), the solution in (9.2) may also be written

$$G(x, y) = A(x + y) \cos(√2 x) + B(x + y) \sin(√2 x),$$

so that the generating function of each matrix $Ω^{(p)}$ is of the form

$$Ω^{(p)}(x, y) = A(x + y) \cos(√2 x) + B(x + y) \sin(√2 x).$$

As the first row of each matrix $Ω^{(p)}$ is the zero sequence, we have $Ω^{(p)}(0, y) = A(y) = 0$. Hence, $Ω^{(p)}(x, y) = B(x + y) \sin(√2 x)$ and

$$\frac{∂}{∂x} Ω^{(p)}(x, y) = \left( \frac{∂}{∂x} B(x + y) \right) \sin(√2 x) + √2 B(x + y) \cos(√2 x).$$
Thus,
\[
\Omega^{(p)}_{1,\bullet}(y) = \left. \frac{\partial}{\partial x} \Omega^{(p)}(x, y) \right|_{\{x=0\}} = \sqrt{2} B(y)
\]
and then
\[
(9.12) \quad \Omega^{(p)}(x, y) = \frac{1}{\sqrt{2}} \sin(\sqrt{2} x) \Omega^{(p)}_{1,\bullet}(x + y).
\]

The evaluation of \(\Omega^{(1)}_{1,\bullet}(y)\) is easy, as the row labeled 1 of \(\Omega^{(1)}\) is
\((1, 0, 1, 0, 4, 0, 34, 0, \ldots)\), compared with \((0, 1, 0, 1, 0, 4, 0, 34, 0, \ldots)\), which
is the sequence of the coefficients of the Taylor expansion of \(\sqrt{2} \tan(y/\sqrt{2})\).
In fact we have \(\omega^{(1)}_{1,2j} = f_{j+2}(2, 2j + 3) = f_{j+1}(2j + 2, \bullet) = T_{2j+1}/2^j\). Thus,
\[
\Omega^{(1)}_{1,\bullet}(y) = \frac{d}{dy} \sqrt{2} \tan\left(\frac{y}{\sqrt{2}}\right) = \frac{1}{\cos^2(y/\sqrt{2})};
\]
so that
\[
(9.13) \quad \Omega^{(1)}(x, y) = \frac{1}{\sqrt{2}} \sin(\sqrt{2} x) \frac{1}{\cos^2\left(\frac{x + y}{\sqrt{2}}\right)}.
\]

**Proposition 9.7.** For all \(p \geq 1\) we have:
\[
(9.14) \quad \Omega^{(p)}_{1,\bullet}(y) = \Omega^{(1)}_{p,\bullet}(y).
\]

Same proof as for Proposition 9.4. Now, define:
\[
(9.15) \quad \Omega(x, y, z) := \sum_{p \geq 1} \Omega^{(p)}(x, y) \frac{z^p}{p!}
\]
and make use of (9.18)—(9.20):
\[
\Omega(x, y, z) = \frac{1}{\sqrt{2}} \sin(\sqrt{2} x) \sum_{p \geq 1} \Omega^{(p)}_{1,\bullet}(x + y) \frac{z^p}{p!}
\]
\[
= \frac{1}{\sqrt{2}} \sin(\sqrt{2} x) \sum_{p \geq 1} \Omega^{(1)}_{p,\bullet}(x + y) \frac{z^p}{p!}
\]
\[
= \frac{1}{\sqrt{2}} \sin(\sqrt{2} x) \Omega^{(1)}(z, x + y)
\]
\[
= \sin(\sqrt{2} x) \sin(\sqrt{2} z) \frac{1}{2 \cos^2\left(\frac{x + y + z}{\sqrt{2}}\right)}.
\]
As all the entries $\omega_{0,j}^{(p)} (p \geq 1, j \geq 0)$ are null,

$$\Omega(x, y, z) = \sum_{p, i, j} \omega_{i,j}^{(p)} \frac{z^p x^i y^j}{p! i! j!} \quad (p \geq 1, i \geq 1, j \geq 0);$$

so that by definition of the $\omega_{i,j}^{(p)}$'s

$$\Omega(x, y, z) = \sum_{k, m, n} f_n(m, k) \frac{x^{2n-k} y^{k-m-1} z^{m-1}}{(2n-k)! (k-m-1)! (m-1)!},$$

the latter sum over the set $\{2 \leq m + 1 \leq k \leq 2n - 1\}$. This achieves the proof of Theorem 1.5.

The right-hand side of (1.15) is symmetric in $x, z$, that is, $\Omega(x, y, z) = \Omega(z, y, x)$. The change $x \leftrightarrow z$ in the left-hand side of (1.15) shows that

$$f_n(2n + 1 - k, 2n + 1 - m) = f_n(m, k),$$

the symmetry proved for the entries $f_n(m, k)$ such that $m + 1 \leq k$. 
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