Analytical solutions of a time-fractional nonlinear transaction-cost model for stock option valuation in an illiquid market setting driven by a relaxed Black–Scholes assumption

S.O. Edeki*, O.O. Ugbebor1,2 and E.A. Owoloko1

Abstract: In financial mathematics, trading in an illiquid market has become a topic of great concern since assets in such market cannot be sold easily for cash without at least a minimal loss of value. This may be due to uncertainty traceable to factors like lack of interested buyers, transaction cost, and so on. Here, we obtain analytical solutions of a time-fractional nonlinear transaction-cost model for stock option valuation in an illiquid market through a relatively new semi-analytical method: modified differential transform method. Firstly, we considered a nonlinear option pricing model obtained when the constant volatility assumption of the classical linear Black–Scholes option pricing model is relaxed by including transaction cost. Thereafter, we extend, for the first time in literature, this nonlinear option pricing model to a time-fractional ordered form, and obtain approximate-analytical solutions to this new nonlinear model via the proposed technique. For efficiency and reliability of the method, two cases with five examples are considered: case 1 with two examples for time-integer order, and case 2 with three examples for time-fractional order. Our results strongly agree with the associated exact solutions in...
literature and obtained via the application of Adomian Decomposition Method (ADM) even though our approximate solutions include only terms up to time power two, accuracy is improved for more terms. This therefore, shows that the result obtained via the ADM is a particular case of this present work when \( \alpha = 1 \). Maple 18 software is used for the computations done in this work.

Subjects: Science; Mathematics & Statistics; Applied Mathematics; Financial Mathematics; Mathematical Finance

Keywords: fractional calculus; nonlinear Black–Scholes model; illiquid market; option pricing; MDTM

Mathematics subject classification: 26A33; 34G20; 91G80

1. Introduction

The term “liquidity” is used in describing the degree to which an underlying asset can be easily exercised (sold or bought) in the market setting in a way that the asset’s price is not affected (Acharya & Pedersen, 2005; Amihud & Mendelson, 1986). Money or cash is an example of liquid assets because it can be sold for items such as goods and services (instantly) with (or without minimal) loss of value. A liquid market is mainly described by ever ready and willing investors. However, in an illiquid market, the concerned assets cannot be sold or exchanged for cash easily without a remarkable reduction in the price due to uncertainty such as lack of interested buyers and transaction cost, to mention but a few (Keynes, 1971). Stock option is a good example of an illiquid asset.

The standard Black–Scholes model is a very vital tool in modern finance and option theory (Black & Scholes, 1973). Nevertheless, most of the assumptions under which this pricing model is formulated appear not realistic in practical settings. These assumptions include: the asset price \( S \) following a Geometric Brownian motion (GBM), constant drift parameter \( \mu \), constant volatility rate \( \sigma \), lack of arbitrage opportunities (lack of risk-free profit), frictionless, and competitive markets (Edeki & Ugbebor, 2015; González-Gaxiola, Ruíz de Chávez, & Santiago, 2015). In a competitive market, there are no transaction costs (say taxes), and restrictions on trade are not honoured (say short sale constraints) (Cetin, Jarrow, & Protter, 2004), whereas in a competitive market, a trader is unbound to buy or sell any amount of a security without price alteration.

Based on these assumptions, the stock price \( S \), at time \( t \) \((0 < t < T)\) follows the stochastic differential equation (SDE):

\[
\frac{dS}{S} = \mu dt + \sigma dW_t
\]

where \( \mu \) represents mean rate of return of \( S \), \( \sigma \) is the volatility parameter, and \( W_t \) is a standard Brownian motion. Therefore, for an option value \( u = u(s, t) \), we have:

\[
u_t + rSu_s + \frac{1}{2}S^2\sigma^2 u_{ss} = ru \tag{1.2}
\]

where \( u_\omega \) indicates partial derivative of \( u \) w.r.t. the subscripted variable \( \omega \), while \( u(0, t) = 0, u(s, t) \to 0 \) as \( S \to \infty \), \( u(s, T) = (S - E)^+ \), \( E \) is a constant.

In literature, a lot of models with regard to volatility have been proposed for option pricing. However, the simplest of such adopts constant volatility, whereas constant volatility cannot fully explain observed market prices for options valuation except when modified (Barles & Soner, 1998; Boyle & Vorst, 1992; Edeki, Owoloko, & Ugbebor, 2016; Edeki, Ugbebor, & Owoloko, 2016).

Many researchers have considered solving (1.2) for approximate solutions using direct, analytical, or semi-analytical methods (Allahviranloo & Behzadi, 2013; Ankudinova & Ehrhardt, 2008; Bohner &
Zheng, 2009; Cen & Le, 2011; Company, Navarro, Ramón Pintos, & Ponsoda, 2008; Edeki, Ugbebor, & Owoloko, 2015; Jódar, Sevilla-Peris, Cortés, & Sala, 2005; Rodrigo & Mamon, 2006). The notion of liquidity is therefore introduced when the frictionless and the competitive markets’ assumptions are relaxed, thereby giving rise to a nonlinear version of the Black–Scholes model (as a result of transaction cost involvement) (Bakstein & Howison, 2003). Bakstein and Howison (2003) see liquidity as a combination of trader’s individual transaction cost and a price slippage impact. It is therefore, our intention to obtain analytical solutions of the time-fractional nonlinear transaction cost model for stock prices in an illiquid market (Bakstein and Howison model (Bakstein & Howison, 2003)).

Recently, significant attention has been given to the study of fractional differential equations (FDEs) with their wider applications because fractional calculus seems to be a generalization of the conventional calculus (He, 1999). The ultimate benefit of the FDEs lies in their properties of non-locality since integer order differential operators are local operators while fractional order differential operators are nonlocal, signifying that the next state of a system depends not only on its current state but also on all of its historical states (Miller & Ross, 1993; Podlubny, 1999). Recent works on FDEs include those of (Edeki, Akinlabi, & Adeosun, 2016a; Ibis, Bayram, & Agargun, 2011; Kilbas, Srivastava, & Trujilo, 2006; Mokhtary, Ghoreishi, & Srivastava, 2016; Song, Yin, Cao, & Lu, 2013).

In considering the solutions of linear time-fractional Black–Scholes Equations (LTFBSEs) in option pricing and valuation; Elbeleze, Kiliçman, and Taib (2013) consider the application of the Homotopy Perturbation Sumudu Transform (HPSTM), Kumar et al. (2012) combine the homotopy perturbation method with Laplace transform. Ghandehari and Ranjbar (2014) extend the decomposition method through expansion series. Kumar, Kumar, and Singh (2014) apply the HPM and HAM to solve the time-fractional Black–Scholes (TFBSE) with boundary conditions. Ahmad, Shakeel, Hassan, and Mohyud-Din (2013) employ fractional variation iterative method to obtain analytical solutions of linear fractional Black–Scholes equations. Hariharan (2013) use the Laplace Legendre wavelet method for numerical solutions. Recently, Ravi Kanth and Aruna (2016) present fractional differential transform method (FDTM) and its modified form (MFDTM) for the solution of time- fractional B–S European option pricing equation while Khan and Ansari (2016) consider same by means of sumudu transform method (STM).

In this present work, a modified version of the DTM called projected/modified differential transform method (MDTM) is adopted and presented for the first time, for analytical solutions of a time-fractional nonlinear transaction-cost model for stock option valuation in an illiquid market setting driven by a relaxed Black–Scholes model assumption. We also remark here, to the best of our knowledge, that this is the first time such nonlinear option pricing model is extended to time-fractional order type.

The remaining part of the paper is structured as follows: in Section 2, we give a brief note on the nonlinear option pricing model; in Section 3, we present an overview, the basic theorems of the semi-analytical method and the analysis of its fractional form; in Section 4, the MDTM is applied to the time-fractional order-type nonlinear option pricing model (in its general form) followed by numerical examples for some special cases with graphical interpretations; in Section 5, we give concluding remarks and summary of our results.

### 2. Bakstein and Howison equation: nonlinear Black–Scholes option pricing model

In this section, consideration will be on a situation where both \( \mu \) (the drift parameter), and \( \sigma \) (the volatility parameter) can be function of time \( \tau \), stock price \( S \) and the derivatives of the option price \( \Lambda \). In particular, the non-constant volatility function of the form:

\[
\sigma = \hat{\sigma} \left( \tau, S, \frac{\partial \Lambda}{\partial S}, \frac{\partial^2 \Lambda}{\partial S^2} \right) \tag{2.1}
\]

is to be considered. Thus, (2.1) in (1.2) yields:
\[
\frac{\partial \Lambda}{\partial \tau} + rS \frac{\partial \Lambda}{\partial S} + \frac{1}{2} S^2 \sigma^2 \left( \tau, S, \frac{\partial \Lambda}{\partial S}, \frac{\partial^2 \Lambda}{\partial S^2} \right) \frac{\partial^2 \Lambda}{\partial S^2} - r\Lambda = 0. \tag{2.2}
\]

The model Equation (1.2) can be improved upon via (2.1) in the line of transaction costs inclusion. As such, the approach of (Frey & Patie, 2002; Frey & Stremme, 1997) will be followed for the effects on the price with the result:

\[
\sigma = \hat{\sigma} \left( \tau, S, \frac{\partial \Lambda}{\partial S}, \frac{\partial^2 \Lambda}{\partial S^2} \right) \left( 1 - \rho S \Lambda(S) \frac{\partial^2 \Lambda}{\partial S^2} \right) \tag{2.3}
\]

where \(\hat{\sigma}\) indicates the traditional volatility, \(\rho\) is a constant measuring the liquidity of the market, and \(\Lambda\) represents the price of risk (Bakstein & Howison, 2003).

With the assumption that the price of risk is unity (a special case: where \(\Lambda(S) = 1\), and a little algebra with the notion that \(1 \approx (1 - f^*) (1 + 2f^* + O(f^*)^2)\), one can therefore write (2.2) as:

\[
\frac{\partial \Lambda}{\partial \tau} + rS \frac{\partial \Lambda}{\partial S} + \frac{1}{2} S^2 \sigma^2 \left( 1 + 2\rho S \frac{\partial^2 \Lambda}{\partial S^2} \right) \frac{\partial^2 \Lambda}{\partial S^2} - r\Lambda = 0 \tag{2.4}
\]

such that \(\Lambda(S, T) = h(S), \ S \in [0, \infty).\) Letting \(t + \tau = T\) and \(w(S, t) = \Lambda(S, r)\), Equation (2.4) thus becomes:

\[
\frac{\partial w}{\partial t} + rS \frac{\partial w}{\partial S} + \frac{1}{2} S^2 \sigma^2 \left( 1 + 2\rho S \frac{\partial^2 w}{\partial S^2} \right) \frac{\partial^2 w}{\partial S^2} = rw, w(S, 0) = h(S). \tag{2.5}
\]

The exact solution of (2.5) according to (Esekon, 2013) is of the form:

\[
w(S, t) = S - \rho^{-1} \sqrt{S_0} \left( \sqrt{S} \exp \left( \frac{4r + \sigma^2}{8} \right) + \frac{\sqrt{S_0}}{4} \exp \left( \frac{4r + \sigma^2}{4} \right) \right).
\]

For \(\sigma, S_0, \ S, |\rho| > 0, \ w(S, t) = w\), while \(r, t \geq 0, \ S_0\) as an initial stock price, with:

\[
w(S, 0) = \left( S - \rho^{-1} \left( \sqrt{S_0} + \frac{S_0}{4} \right) \right)^+. \tag{2.7}
\]

Liu and Yong (2005) considered and established the existence and uniqueness of this nonlinear model.

In what follows, we will consider (2.5) with respect to time-fractional order, thus considering the model:

\[
\frac{\partial^\alpha w}{\partial t^\alpha} = -rS \frac{\partial w}{\partial S} - \frac{1}{2} S^2 \sigma^2 \left( 1 + 2\rho S \frac{\partial^2 w}{\partial S^2} \right) \frac{\partial^2 w}{\partial S^2} + rw, \tag{2.8}
\]

subject to: \(w(S, 0) = \left( S - \rho^{-1} \left( \sqrt{S_0} + \frac{S_0}{4} \right) \right)^+. \tag{2.9}
\]

3. The outline of the projected DTM (Edeki, Akinlabi, & Adeosun, 2016b; Jang, 2010; Keskin, Servi, & Oturanç, 2011; Ravi Kanth & Aruna, 2012)

Here, we will present an overview of the modified DTM referred to as MDTM.
3.1. A note on some fundamental theorems and notations of the MDTM

Let \( \varphi(x,t) \) be an analytic function on a domain \( D \) at \((x_0, t_0)\); then, considering the Taylor series expansion of \( \varphi(x,t) \), regard is given to some variables \( s^\alpha = t \) instead of all the variables as in the case of the classical DTM. Thus, the MDTM of \( \varphi(x,t) \) with respect to \( t \) at \( t_0 \) is defined as:

\[
\Psi(x,h) = \frac{1}{h!} \left[ \frac{d^n \varphi(x,t)}{dt^n} \right]_{t=h} \tag{3.1}
\]

such that:

\[
\varphi(x,t) = \sum_{n=0}^{\infty} \Psi(x,h) (t-t_0)^n. \tag{3.2}
\]

We refer to (3.2) as the modified differential inverse transform (MDIT) of \( \Psi(x,h) \) w.r.t. \( t \).

3.2. The fundamental theorems and properties of the MDTM

(a) If \( \varphi(x,t) = \alpha \varphi_{\alpha_2}(x,t) \pm \beta \varphi_{\alpha_3}(x,t) \), then \( \Psi(x,h) = \alpha \Psi_{\alpha_2}(x,h) \pm \beta \Psi_{\alpha_3}(x,h) \).

(b) If \( \varphi(x,t) = \alpha \varphi_{\alpha_2}(x,t) \), then \( \Psi(x,h) = \alpha \Psi_{\alpha_2}(x,h) \).

(c) If \( \varphi(x,t) = \frac{\partial^2 \varphi_{\alpha_2}(x,t)}{\partial x^2} \), then \( \Psi(x,h) = \frac{\partial^2 \Psi_{\alpha_2}(x,h)}{\partial x^2} \).

(d) If \( p(x,t) = D_t^{\alpha} \varphi(x,t) \), then \( \Gamma\left(1 + \frac{k}{\alpha}\right)p(x,k) = \Gamma\left(1 + \alpha + \frac{k}{\alpha}\right)\Phi(x,k + \alpha \lambda), \)

\[
\Gamma\left(1 + \frac{k}{\alpha}\right)\Phi(x,k + \alpha \lambda) = \Gamma\left(1 + \frac{k}{\alpha}\right)p(x,k). \tag{3.3}
\]

Setting \( \alpha \lambda = 1 \) in (3.3) yields (3.4) and (3.5) as follows:

\[
\Phi(x,k + 1) = \frac{\Gamma(1 + ak)}{\Gamma(1 + (1 + k))} G(x,k). \tag{3.4}
\]

As such, for \( \varphi(x,t) \), \( \alpha \)- analytic at \( x_0 = 0 \)

\[
\varphi(x,t) = \sum_{n=0}^{\infty} \Phi(x,h) t^n. \tag{3.5}
\]

3.3. Analysis of the MDTM for time-fractional order

In this subsection, we will consider the nonlinear fractional differential equation (NLFDE) of the form:

\[
D_t^{\alpha} \varphi(x,t) + L_{[x]} \varphi(x,t) + N_{[x]} \varphi(x,t) = q(x,t) \varphi(x,0) = g(x), \quad t > 0 \tag{3.6}
\]

where \( D_t^{\alpha} = \frac{d^\alpha}{dt^\alpha} \) is the fractional Caputo derivative of \( \varphi = \varphi(x,t) \); whose modified differential transform is \( \Phi(x,h) \), and \( L_{[x]} \) and \( N_{[x]} \) are linear and nonlinear differential operators w.r.t. \( x \), respectively, while \( q = q(x,t) \) is the source term.

We re-write (3.6) as:

\[
D_t^{\alpha} \varphi(x,t) = -L_{[x]} \varphi(x,t) - N_{[x]} \varphi(x,t) + q(x,t), \quad n - 1 < \alpha < n, \quad n \in \mathbb{N}. \tag{3.7}
\]

Thus, applying the inverse fractional Caputo derivative, \( D_t^{\alpha} \) to both sides of (3.6) gives:

\[
\varphi(x,t) = g(x) + D_t^{\alpha} \left[ -L_{[x]} \varphi(x,t) - N_{[x]} \varphi(x,t) + q(x,t) \right], \quad \varphi(x,0) = g(x). \tag{3.8}
\]
Thus, expanding the analytical and continuous function, $\varphi(x, t)$ in terms of fractional power series, the inverse modified differential transform of $\Phi(x, h)$ is given as follows:

$$\varphi(x, t) = \sum_{h=0}^{\infty} \Phi(x, h) t^h = \varphi(x, 0) + \sum_{h=1}^{\infty} \Phi(x, h) t^h, \varphi(x, 0) = g(x).$$

(3.9)

4. The MDTM and the nonlinear model

In this section, the MDTM approach will be applied to the model Equation (2.8) as follows:

$$\frac{\partial^a w}{\partial t^a} = -rS \frac{\partial w}{\partial S} - \frac{1}{2} S^2 \sigma^2 \left( 1 + 2 \rho S \frac{\partial^2 w}{\partial S^2} \right) \frac{\partial^2 w}{\partial S^2} + rw,$$

subject to: $w(S, 0) = \left( S - \rho^{-1} \left( \sqrt{S_0 S + \frac{S_0}{4}} \right) \right)^+.$

(4.1)

Simplifying (4.1) gives:

$$\frac{\partial^a w}{\partial t^a} = -\left( rS \frac{\partial w}{\partial S} + \frac{1}{2} S^2 \sigma^2 \left( \frac{\partial^2 w}{\partial S^2} + 2 \rho S \left( \frac{\partial^2 w}{\partial S^2} \right)^2 \right) \right) - rw.$$

(4.3)

At projection, the transformation of (4.3) and (4.2) using MDTM yields (4.4) and (4.5) as follows:

$$\text{MDT} \left[ \frac{\partial^a w}{\partial t^a} = -\left( rS \frac{\partial w}{\partial S} + \frac{1}{2} S^2 \sigma^2 \left( \frac{\partial^2 w}{\partial S^2} + 2 \rho S \left( \frac{\partial^2 w}{\partial S^2} \right)^2 \right) \right) - rw \right],$$

(4.4)

$$\text{MDT} \left[ w(S, 0) = \max \left( S - \rho^{-1} \left( \sqrt{S_0 S + \frac{S_0}{4}} \right), 0 \right) \right].$$

(4.5)

Thus, we have:

$$\frac{\Gamma(1 + \alpha(1 + k))}{\Gamma(1 + \alpha k)} W_{sk+1} = -\left( rS \frac{\partial W_{sk}}{\partial S} + \frac{1}{2} S^2 \sigma^2 \left( \frac{\partial^2 W_{sk}}{\partial S^2} + 2 \rho S \sum_{n=0}^{k} \frac{\partial^2 W_{sn} \partial^2 W_{sk-n}}{\partial S^2} \right) \right) - rW_{sk}.$$

(4.6)

As such,

$$W_{sk+1} = -\frac{\Gamma(1 + \alpha k)}{\Gamma(1 + \alpha(1 + k))} \left( rS \frac{\partial W_{sk}}{\partial S} + \frac{1}{2} S^2 \sigma^2 \left( \frac{\partial^2 W_{sk}}{\partial S^2} + 2 \rho S \sum_{n=0}^{k} \frac{\partial^2 W_{sn} \partial^2 W_{sk-n}}{\partial S^2} \right) \right) - rW_{sk},$$

(4.7)

subject to: $W_{s0} = \max \left( S - \rho^{-1} \left( \sqrt{S_0 S + \frac{S_0}{4}} \right), 0 \right).$

(4.8)

For $k = 0$, we have:

$$W_{s1} = -\frac{1}{\Gamma(1 + \alpha)} \left( rS \frac{\partial W_{s0}}{\partial S} + \frac{1}{2} S^2 \sigma^2 \left( \frac{\partial^2 W_{s0}}{\partial S^2} + 2 \rho S \frac{\partial^2 W_{s0} \partial^2 W_{s1}}{\partial S^2} \right) \right) - rW_{s0}.$$

(4.9)

For $k = 1$, we have:

$$W_{s2} = -\frac{\Gamma(1 + \alpha)}{\Gamma(1 + \alpha(1 + 2))} \left( rS \frac{\partial W_{s1}}{\partial S} + \frac{1}{2} S^2 \sigma^2 \left( \frac{\partial^2 W_{s1}}{\partial S^2} + 2 \rho S \sum_{n=0}^{1} \frac{\partial^2 W_{sn} \partial^2 W_{s1-n}}{\partial S^2} \right) \right) - rW_{s1},$$

$$= -\frac{\Gamma(1 + \alpha)}{\Gamma(1 + 2 \alpha)} \left( rS \frac{\partial W_{s1}}{\partial S} + \frac{1}{2} S^2 \sigma^2 \left( \frac{\partial^2 W_{s1}}{\partial S^2} + 4 \rho S \frac{\partial^2 W_{s0} \partial^2 W_{s1}}{\partial S^2} \right) \right) - rW_{s1}.$$  

(4.10)
For $k = 2$, we have:

$$W_{5,3} = - \frac{\Gamma(1 + 2\alpha)}{\Gamma(1 + 3\alpha)} \left( r S^2 \frac{\partial^2 W_{5,2}}{\partial S^2} + \frac{1}{2} S^2 \sigma^2 \left( \frac{\partial^2 W_{5,2}}{\partial S^2} + 2 \rho S \sum_{n=0}^{2} \frac{\partial^2 W_{5,2}}{\partial S^2} \frac{\partial^2 W_{5,2-n}}{\partial S^2} \right) - r W_{5,2} \right)$$

$$= - \frac{\Gamma(1 + 2\alpha)}{\Gamma(1 + 3\alpha)} \left( r S^2 \frac{\partial^2 W_{5,2}}{\partial S^2} + \frac{1}{2} S^2 \sigma^2 \left( \frac{\partial^2 W_{5,2}}{\partial S^2} + 2 \rho S \left( \frac{\partial^2 W_{5,0}}{\partial S^2} + \frac{\partial^2 W_{5,1}}{\partial S^2} \right) \right) - r W_{5,2} \right).$$

(4.11)

For $k = 3$, we have:

$$W_{5,4} = - \frac{\Gamma(1 + 3\alpha)}{\Gamma(1 + 4\alpha)} \left( r S^3 \frac{\partial^2 W_{5,3}}{\partial S^2} + \frac{1}{2} S^2 \sigma^2 \left( \frac{\partial^2 W_{5,3}}{\partial S^2} + 4 \rho S \left( \frac{\partial^2 W_{5,0}}{\partial S^2} + \frac{\partial^2 W_{5,1}}{\partial S^2} + \frac{\partial^2 W_{5,2}}{\partial S^2} \right) \right) - r W_{5,3} \right).$$

(4.12)

For $k = 4$, we have:

$$W_{5,5} = - \frac{\Gamma(1 + 4\alpha)}{\Gamma(1 + 5\alpha)} \left( r S^4 \frac{\partial^2 W_{5,4}}{\partial S^2} + \frac{1}{2} S^2 \sigma^2 \left( \frac{\partial^2 W_{5,4}}{\partial S^2} + 4 \rho S \left( \frac{\partial^2 W_{5,0}}{\partial S^2} + \frac{\partial^2 W_{5,1}}{\partial S^2} + \frac{\partial^2 W_{5,2}}{\partial S^2} \right) \right) - r W_{5,4} \right).$$

(4.13)

### 4.1. Numerical illustration

In this subsection, two cases will be considered. Case 1 has two examples with time-integer order while case 2 has three examples with time-fractional order.

We recall (2.6) and (2.7) as follows:

$$w(S, t) = w = S - \rho \exp \left( S \frac{r + \frac{\sigma^2}{2}}{2} t \right) + \sqrt{S_0} \exp \left( S \frac{r + \frac{\sigma^2}{4}}{4} t \right).$$

$$w(S, 0) = \max \left( S - \rho \left( \sqrt{S_0} S + \frac{S_0}{4} \right), 0 \right).$$

For numerical illustration, we will consider some examples for different values of $S$, $t$, and $\alpha$ over fixed values for the other parameters. Hence, for $r = 0.06$, $|\rho| = 0.01$, $\sigma = 0.4$, and $S_0 = 4$, we thus have the exact solution and initial condition as:

$$w(S, t) = S + 200 \left( \sqrt{S} \exp \left( t \frac{r + \frac{\sigma^2}{4}}{10} \right) + \frac{1}{2} \exp \left( \frac{t}{10} \right) \right),$$

(4.14)

$$w(S, 0) = S + 200 \sqrt{S} + 100.$$
Thus, by applying the MDTM with the above parameters, we get the following:

\[
W_{s,0} = 100 + 200 \sqrt{S} + S,
\]

\[
W_{s,1} = \frac{1}{1250\Gamma(1 + \alpha)} \left( 5000 - 2500S^{1/2} - 75S + 600S^2 + 1200S^{5/2} + 6S^3 \right),
\]

\[
W_{s,2} = \frac{1}{312500\Gamma(1 + 2\alpha)} \left\{ -125000 + 31250S^{1/2} + 5625S - 240000S^{5/2} - 1080000S^2 - 697200S^{5/2} - 5400S^3 + 3600S^4 + 7200S^{9/2} + 36S^5 \right\}.
\]

Whence,

\[
w(S, t) = \sum_{h=0}^{\infty} W_{s,h} t^h = W_{s,0} + W_{s,1} t^\alpha + W_{s,2} t^{2\alpha} + W_{s,3} t^{3\alpha} + \ldots
\]

\[
= \left( 100 + 200 \sqrt{S} + S \right)
\]

\[
+ \left( \frac{1}{2500\Gamma(1 + \alpha)} \left( 5000 - 2500S^{1/2} - 75S + 600S^2 + 1200S^{5/2} + 6S^3 \right) \right) t^\alpha
\]

\[
+ \left\{ \frac{1}{312500\Gamma(1 + 2\alpha)} \left( -125000 + 31250S^{1/2} + 5625S - 240000S^{5/2} - 1080000S^2 - 697200S^{5/2} - 5400S^3 + 3600S^4 + 7200S^{9/2} + 36S^5 \right) \right\} t^{2\alpha} + \ldots.
\]

Tables 1 and 2 are for case 1 for an integer power of the time parameter, the graphs of some are in Figures 1 and 2, respectively. In a similar way, Tables 3–5 are for case 2 for fractional powers of the time parameter, the graphs of some are in Figures 3–5, respectively. Also, we present in comparison the exact and the approximate solutions for different values of \( t \) and \( \alpha \), with

\[
|w_r| = \left| \frac{w_{\text{exact}} - w_{\text{approx}}}{w_{\text{exact}}} \right|
\]

as the relative error.

| Table 1. Case 1 for \( t = 0 \) and \( \alpha = 1 \) |
|-----------------|-----------------|-----------------|-------|
| \( S \) | \( w_{\text{exact}} \) | \( w_{\text{approx}} \) | \( |w_r| \) |
| 0.5  | 241.9214        | 241.9214        | 0.0000 |
| 1.0  | 301.0000        | 301.0000        | 0.0000 |
| 1.5  | 346.4490        | 346.4490        | 0.0000 |
| 2.0  | 384.8428        | 384.8428        | 0.0000 |
| 2.5  | 418.7278        | 418.7278        | 0.0000 |
| 3.0  | 449.4102        | 449.4102        | 0.0000 |
| 3.5  | 477.6658        | 477.6658        | 0.0000 |
| 4.0  | 504.0000        | 504.0000        | 0.0000 |
| 4.5  | 528.7641        | 528.7641        | 0.0000 |
| 5.0  | 552.2136        | 552.2136        | 0.0000 |
Table 2. Case 1 for $t = 0.5$ and $\alpha = 1$

| $S$  | $w_{\text{exact}}$ | $w_{\text{approx}}$ | $|w_r|$  |
|------|--------------------|----------------------|--------|
| 0.5  | 250.6286           | 243.2212             | 0.029555 |
| 1.0  | 311.1902           | 301.9564             | 0.029673 |
| 1.5  | 357.777            | 347.1821             | 0.029613 |
| 2.0  | 397.1301           | 385.4947             | 0.029299 |
| 2.5  | 431.8603           | 419.4882             | 0.028648 |
| 3.0  | 463.3067           | 450.5316             | 0.027574 |
| 3.5  | 492.2649           | 479.4798             | 0.025972 |
| 4.0  | 519.2532           | 506.934              | 0.023725 |
| 4.5  | 544.6315           | 533.3594             | 0.020697 |

Figure 1. Graph for case 1 w.r.t. Table 1.

Figure 2. Graph for case 1 w.r.t. Table 2.
### Table 3. Case 2 for $t = 0.5$ and $\alpha = 0.5$

| $S$   | $W_{\text{exact}}$ | $W_{\text{approx}}$ | $|W_r|$  |
|-------|---------------------|----------------------|---------|
| 0.01  | 125.6435            | 123.2255             | 0.019245|
| 0.02  | 134.1475            | 131.455              | 0.020071|
| 0.03  | 140.6751            | 137.7714             | 0.020641|
| 0.04  | 146.1798            | 143.0975             | 0.021086|
| 0.05  | 151.0306            | 147.7904             | 0.021454|
| 0.06  | 155.4171            | 152.0336             | 0.021770|
| 0.07  | 159.4517            | 155.9358             | 0.022050|
| 0.08  | 163.2077            | 159.5681             | 0.022300|
| 0.09  | 166.7361            | 162.9796             | 0.022530|
| 0.10  | 170.0738            | 166.2063             | 0.022740|

### Table 4. Case 2 for $t = 0.5$ and $\alpha = 1.5$

| $S$   | $W_{\text{exact}}$ | $W_{\text{approx}}$ | $|W_r|$  |
|-------|---------------------|----------------------|---------|
| 0.01  | 125.6435            | 121.0283             | 0.036733|
| 0.02  | 134.1475            | 129.3005             | 0.036132|
| 0.03  | 140.6751            | 135.6503             | 0.035719|
| 0.04  | 146.1798            | 141.005              | 0.035400|
| 0.05  | 151.0306            | 145.7237             | 0.035138|
| 0.06  | 155.4171            | 149.9908             | 0.034914|
| 0.07  | 159.4517            | 153.9156             | 0.034720|
| 0.08  | 163.2077            | 157.5695             | 0.034546|
| 0.09  | 166.7361            | 161.0019             | 0.034391|
| 0.10  | 170.0738            | 166.2063             | 0.022740|

### Table 5. Case 2 for $t = 1$ and $\alpha = 2.5$

| $S$   | $W_{\text{exact}}$ | $W_{\text{approx}}$ | $|W_r|$  |
|-------|---------------------|----------------------|---------|
| 0.01  | 131.5526            | 121.1564             | 0.079027|
| 0.02  | 140.2716            | 129.4256             | 0.077321|
| 0.03  | 146.9642            | 135.7732             | 0.076148|
| 0.04  | 152.608             | 141.126              | 0.075239|
| 0.05  | 157.5814            | 145.8432             | 0.074940|
| 0.06  | 162.0787            | 150.1088             | 0.073852|
| 0.07  | 166.2152            | 154.0323             | 0.073296|
| 0.08  | 170.066             | 157.6850             | 0.072801|
| 0.09  | 173.6834            | 161.1163             | 0.072356|
| 0.10  | 177.1054            | 164.3623             | 0.071952|
Figure 3. Graph for case 2 w.r.t. Table 3.

Figure 4. Graph for case 2 w.r.t. Table 4.
5. Concluding remarks

In this paper, we considered analytical solutions of a time-fractional nonlinear transaction-cost model for stock option valuation in an illiquid market setting driven by a relaxed Black–Scholes model assumption through a relatively new semi-analytical method called the modified differential transform method (MDTM). Firstly, we considered a nonlinear option pricing model obtained when the constant volatility assumption of the famous linear Black–Scholes option pricing model is relaxed through the inclusion of transaction cost. Thereafter, we extend, for the first time in literature, this nonlinear option pricing model to a time-fractional ordered form, and obtained approximate-analytical solutions to this new nonlinear model via the proposed solution technique. For efficiency and reliability of the method, we considered two cases with five examples: case 1 with two examples for time-integer order, and case 2 with three examples for time-fractional order. Our results are very interesting, they conform with the associated exact solutions obtained by Esekon (2013), and those of González-Gaxiola et al. (2015) using the Adomian decomposition method; even though our approximate solutions include only terms up to time power three (3), accuracy is improved for more terms. This therefore shows that the work of González-Gaxiola et al. (2015) is a particular case of our present work when \( \alpha = 1 \). Maple 18 software is used for all the numerical computations done in this work. Hence, the method is a good candidate for solving linear and nonlinear differential equations (models) with time- or space fractional orders, though the application of the method to differential equations (linear and nonlinear option pricing models) with complex-fractional orders is yet to be considered in its wider sense.

Acknowledgements

The authors are sincerely grateful to Covenant University for financial support and provision of good working environment. They also wish to thank the anonymous referee(s) for their constructive and helpful comments.

Funding

This work was supported by Covenant University [grant number CUCRID/PSG/VC/17/07/14-FS].

Author details

S.O. Edeki1
E-mail: soedeki@yahoo.com
O.O. Ugbebor2
E-mail: ugbebor1@yahoo.com
E.A. Owoloko1
E-mail: alfredowoloko@covenantuniversity.edu.ng
1 Department of Mathematics, Covenant University, Canaanland, Otta, Nigeria.
2 Department of Mathematics, University of Ibadan, Ibadan, Nigeria.

Citation information

Cite this article as: Analytical solutions of a time-fractional nonlinear transaction-cost model for stock option valuation in an illiquid market setting driven by a relaxed Black–Scholes assumption, S.O. Edeki, O.O. Ugbebor & E.A. Owoloko, Cogent Mathematics (2017), 4: 1352118.
References

Acharya, V., & Pedersen, L. H. (2005). Asset pricing with liquidity risk. Journal of Financial Economics, 77, 375–410. https://doi.org/10.1016/j.jfineco.2004.06.007

Ahmad, J., Shakeel, M., Hassan, Q. M. U. I., & Mojahid-Din, S. T. (2013). Analytical solution of Black–Scholes model using fractional variational iteration method. International Journal of Modern Mathematical Sciences, 5, 133–142.

Allahviranloo, T., & Behzadi, Sh. S. (2013). The use of iterative methods for solving Black–Scholes equation. International Journal of Industrial Mathematics, 5(1), 1–11, Article ID 17JIM-00529.

Amihud, Y., & Mendelson, H. (1986). Asset pricing and the bid-ask spread. Journal of Financial Economics, 17, 223–249. https://doi.org/10.1016/0304-405X(86)90065-6

Ankudinova, I., & Efthierid, M. (2008). On the numerical solution of nonlinear Black–Scholes equations. Computers & Mathematics with Applications, 56, 799–812. https://doi.org/10.1016/j.camwa.2008.02.005

Bakstein, D., & Howison, S. (2003). A non-arbitrage liquidity model with observable parameters for derivatives. Oxford: Oxford University Preprint.

Barles, G., & Soner, H. M. (1998). Option pricing with transaction costs and a nonlinear Black–Scholes equation. Finance and Stochastics, 2, 369–397. https://doi.org/10.1007/s007800050046

Black, F., & Scholes, M. (1973). The pricing of corporate securities. Journal of Political Economy, 81, 637–654. https://doi.org/10.1086/260062

Bohner, M., & Zheng, Y. (2007). On analytical solutions of the Black–Scholes equation. Applied Mathematics Letters, 22, 309–313. https://doi.org/10.1016/j.aml.2008.04.002

Boyle, P., & Vorst, T. (1992). Option replication in discrete time with transaction costs. The Journal of Finance, 47, 271–293. https://doi.org/10.1111/j.1540-6261.1992.tb03986.x

Cen, Z., & Le, A. (2011). A robust and accurate finite difference method for a generalized Black–Scholes equation. Journal of Computational and Applied Mathematics, 235, 3728–3733. https://doi.org/10.1016/j.cam.2011.01.018

Cetin, U., Jarrow, R. A., & Protter, P. (2004). Liquidity risk and arbitrage pricing theory. Finance Stochast, 8, 311–341. doi:10.1007/s00780-003-0123-2

Company, R., Navarro, E., Ramón Pintos, J. R., & Ponsoda, E. (2008). Numerical solution of linear and nonlinear Black–Scholes option pricing equations. Computers & Mathematics with Applications, 56, 813–821. https://doi.org/10.1016/j.camwa.2008.02.010

Edeki, S. O., Akinlabi, G. O., & Adeosun, S. A. (2016a). Analytic solutions of time-fractional linear Schrödinger equation. Communications in Mathematics and Applications, 7(1), 1–10.

Edeki, S. O., Akinlabi, G. O., & Adeosun, S. A. (2016b). Analytic solutions of the nonlinear Black–Scholes equation. Applied Mathematical Sciences, 10(23), 1177–1183. https://doi.org/10.12988/ams.2016.35261

Edeki, S. O., Ugbebor, O. O., & Owoloko, E. A. (2016). A note on Black–Scholes pricing model for theoretical values of stock options. AIP Conference Proceedings, 1705, 020048. doi:10.1063/1.4940296

Edeki, S. O., Ugbebor, O. O., & Owoloko, E. A. (2015). The modified Black–Scholes model via constant elasticity of variance for stock options valuation. AIP Conference proceedings, 1705, 020041. doi:10.1063/1.4940289

Edeki, S. O., & Ugbebor, O. O. (2015). Local projected differential transformation method. Entropy, 17, 7510–7521. https://doi.org/10.3390/e17117510

Edeki, S. O., Ugbebor, O. O., & Owoloko, E. A. (2015). Analytical solutions of the Black–Scholes pricing model for european option valuation via a projected differential transformation method. Entropy, 17, 7510–7521. https://doi.org/10.3390/e17117510
Liu, H., & Yong, J. (2005). Option pricing with an illiquid underlying asset market. *Journal of Economic Dynamics and Control*, 29, 2125–2156. https://doi.org/10.1016/j.jedc.2004.11.004

Miller, K. S., & Ross, B. (1993). An introduction to the fractional calculus and fractional differential equations. New York, NY: John Wiley & Sons.

Mokhtary, P., Ghereishi, F., & Srivastava, H. M. (2016). The Müntz-Legendre Tau method for fractional differential equations. *Applied Mathematical Modelling*, 40, 671–684. https://doi.org/10.1016/j.apm.2015.06.014

Podlubny, I. (1999). *Fractional differential equations*. San Diego, CA: Academic Press.

Ravi Kanth, A. S. V., & Aruna, K. (2012). Comparison of two dimensional DTM and PTDM for solving time-dependent Emden–Fowler type equations. *International Journal of Nonlinear Science*, 13, 228–239.

Ravi Kanth, A. S. V., & Aruna, K. (2016). Solution of time fractional Black–Scholes European option pricing equation arising in financial market. *Nonlinear Engineering*, 5, 269–276. doi:10.1515/nleng-2016-0052

Rodrigo, M. R., & Mamon, R. S. (2006). An alternative approach to solving the Black–Scholes equation with time-varying parameters. *Applied Mathematics Letters*, 19, 398–402. https://doi.org/10.1016/j.aml.2005.06.012

Song, J., Yin, F., Cao, X., & Lu, F. (2013). Fractional variational iteration method versus Adomian’s decomposition method in some fractional partial differential equations. *Journal of Applied Mathematics*, Article ID 392567, 10.