A Generalized Hamiltonian Characterizing the Interaction of the Two–Level Atom and both the Single Radiation Mode and External Field

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Abstract

In this paper we propose some Hamiltonian characterizing the interaction of the two–level atom and both the single radiation mode and external field, which might be a generalization of that of Schönhagen and Cirac (quant-ph/0212068). We solve them in the strong coupling regime under some conditions (the rotating wave approximation, resonance condition and etc), and obtain unitary transformations of four types to perform Quantum Computation.

In this paper we consider a full model of the interaction of the two–level atom and both the single radiation mode and external field (periodic usually), which might be a generalization of that of Schönhagen and Cirac [4]. We treat the external field as a classical one in this paper. As a general introduction to this topic in Quantum Optics see [1], [2], [3]. Our model is deeply related to the (quantum computational) models

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(i) trapped ions with the Coulomb interaction,

(ii) trapped ions with the photon interaction (Cavity QED).

In our model we are especially interested in the strong coupling regime, [9], [13], [14]. The motivation is a recent interesting experiment, [11]. See [3] and [10] as a general introduction.

In [9] and [13] we treated the strong coupling regime of the interaction model of the two–level atom and the single radiation mode, and have given some explicit solutions under the resonance conditions and rotating wave approximations.

On the other hand we want to add some external field (like Laser one) to the above model which will make the model more realistic (for example in Quantum Computation). Therefore we propose the full model.

We would like to solve our model in the strong coupling regime. Especially we want to show the existence of Rabi oscillations in this regime because the real purpose of a series of study ([13], [14], [15]) is an application to Quantum Computation (see [12] as a brief introduction to it).

We can show the Rabi oscillations and obtain Rabi frequencies in this regime if the external field is constant. If it is not constant then the situation becomes extremely difficult.

Let \( \{\sigma_1, \sigma_2, \sigma_3\} \) be Pauli matrices and \( \mathbf{1}_2 \) a unit matrix:
\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]
(1)

and \( \sigma_+ = (1/2)(\sigma_1 + i\sigma_1), \sigma_- = (1/2)(\sigma_1 - i\sigma_1) \). Let \( W \) be the Walsh–Hadamard matrix
\[
W = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = W^{-1},
\]
(2)

\[1\] [3] is thick but strongly recommended
then we can diagonalize $\sigma_1$ as $\sigma_1 = W\sigma_3 W^{-1} = \sigma_1 = W\sigma_3 W$ by making use of this $W$. The eigenvalues of $\sigma_1$ is $\{1, -1\}$ with eigenvectors

$$|1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |-1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \Rightarrow \quad |\lambda\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \lambda \end{pmatrix}. \quad (3)$$

Let us consider an atom with 2 energy levels $E_0$ and $E_1$ (of course $E_1 > E_0$). Its Hamiltonian is in the diagonal form given as

$$H_0 = \begin{pmatrix} E_0 & 0 \\ 0 & E_1 \end{pmatrix}. \quad (4)$$

This is rewritten as

$$H_0 = \frac{E_0 + E_1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{E_1 - E_0}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \equiv \Delta_0 \mathbf{1}_2 - \frac{\Delta}{2} \sigma_3, \quad (5)$$

where $\Delta = E_1 - E_0$ is a energy difference. Since we usually take no interest in constant terms, we set

$$H_0 = -\frac{\Delta}{2} \sigma_3. \quad (6)$$

We consider an atom with two energy levels which interacts with external (periodic) field with $g\cos(\omega_E t)$. In the following we set $\hbar = 1$ for simplicity. The Hamiltonian in the dipole approximation is given by

$$H = H_0 + g\cos(\omega t)\sigma_1 = -\frac{\Delta}{2} \sigma_3 + g \cos(\omega_E t)\sigma_1, \quad (7)$$

where $\omega_E$ is the frequency of the external field, $g$ the coupling constant between the external field and the atom. We note that to solve this model without assuming the rotating wave approximation is not easy, see [8], [16], [19], [20].

In the following we change the sign in the kinetic term, namely from $-\Delta/2$ to $\Delta/2$, to set the model for other models. However this is minor.

Now we make a short review of the harmonic oscillator within our necessity. Let $a(a^\dagger)$ be the annihilation (creation) operator of the harmonic oscillator. If we set $N \equiv a^\dagger a$ (:}
number operator), then we have
\[ [N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \quad [a^\dagger, a] = -1. \] (8)

Let \( \mathcal{H} \) be a Fock space generated by \( a \) and \( a^\dagger \), and \( \{|n\rangle| n \in \mathbb{N} \cup \{0\} \} \) be its basis. The actions of \( a \) and \( a^\dagger \) on \( \mathcal{H} \) are given by
\[ a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad N|n\rangle = n|n\rangle \] (9)
where \( |0\rangle \) is a normalized vacuum (\( a|0\rangle = 0 \) and \( \langle 0|0 \rangle = 1 \)). From (9) state \( |n\rangle \) for \( n \geq 1 \) are given by
\[ |n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle. \] (10)
These states satisfy the orthogonality and completeness conditions
\[ \langle m|n \rangle = \delta_{mn}, \quad \sum_{n=0}^{\infty} |n\rangle\langle n| = 1. \] (11)

Then the displacement (coherent) operator and coherent state are defined as
\[ D(z) = e^{za^\dagger - \bar{z}a}; \quad |z\rangle = D(z)|0\rangle \quad \text{for} \quad z \in \mathbb{C}. \] (12)

We consider the quantum theory of the interaction between an atom with two–energy levels and single radiation mode (a harmonic oscillator). The Hamiltonian in this case is
\[ H = \omega \mathbf{1}_2 \otimes a^\dagger a + \frac{\Delta}{2} \sigma_3 \otimes \mathbf{1} + g \sigma_1 \otimes (a^\dagger + a) \] (13)
where \( \omega \) is the frequency of the radiation mode, \( g \) the coupling between the radiation field and the atom, see for example [3], [9].

Now it is very natural for us to include (7) into (13), so we present the following

**Unified Hamiltonian**
\[ H = \omega \mathbf{1}_2 \otimes a^\dagger a + g_1 \sigma_1 \otimes (a^\dagger + a) + \frac{\Delta}{2} \sigma_3 \otimes \mathbf{1} + g_2 \cos(\omega_E t) \sigma_1 \otimes \mathbf{1}. \] (14)

Our Hamiltonian has two coupling constants. We note that our model is deeply related to the models
(i) trapped ions with the Coulomb interaction

(ii) trapped ions with the photon interaction (Cavity QED)

\[|0\rangle\]

\[|1\rangle\]

This Hamiltonian is also related to the one presented recently by Schön and Cirac [4]

\[
H = \frac{p^2}{2m} + \omega_0 \mathbf{1}_2 \otimes a^\dagger a + g(x) \left( \sigma_+ \otimes a + \sigma_- \otimes a^\dagger \right) + \frac{\omega_0}{2} \sigma_3 \otimes \mathbf{1} + \frac{\Omega}{2} \left( e^{-i\omega_L t} \sigma_+ \otimes \mathbf{1} + e^{i\omega_L t} \sigma_- \otimes \mathbf{1} \right). \tag{15}
\]

For the meaning of several constants see [4]. They have assumed the rotating wave approximation (see for example [3]) and the resonance condition, and use a position–dependent coupling constant \(g(x)\), so their model is different from ours in these points.

A comment is in order. Following [4] the Hamiltonian (14) might be modified to

\[
H = \frac{p^2}{2m} + \omega \mathbf{1}_2 \otimes a^\dagger a + g_1(x) \sigma_1 \otimes (a^\dagger + a) + \frac{\Delta}{2} \sigma_3 \otimes \mathbf{1} + g_2 \cos(\omega_E t) \sigma_1 \otimes \mathbf{1}. \tag{16}
\]

This model is a full generalization of (15), however we don’t consider this situation in the paper.

We have one question: Is the Hamiltonian (14) realistic or meaningful? The answer is of course yes. Let us show one example. We consider the (effective) Hamiltonian presented by NIST group [5], [6] which were used to construct the controlled NOT operation (see [12] as an introduction).

\[
H = \omega_0 \mathbf{1}_2 \otimes a^\dagger a + g \left( \sigma_+ \otimes e^{i\eta(a^\dagger + a)} + \sigma_- \otimes e^{-i\eta(a^\dagger + a)} \right) + \frac{\Delta}{2} \sigma_3 \otimes \mathbf{1}. \tag{17}
\]
We can show that under some unitary transformation the Hamiltonian (17) can be transformed to (14) with special coupling constants, \[7\]. This is important, so we review and modify \[7\].

We set \(2A = i\eta(a^\dagger + a)\) for simplicity, then

\[
\begin{pmatrix}
\sigma_+ \otimes e^{i\eta(a^\dagger + a)} + \sigma_- \otimes e^{-i\eta(a^\dagger + a)} \\
e^{-2A} & 0 \\
e^{2A} & 0
\end{pmatrix}
= \begin{pmatrix}
0 & e^{2A} \\
e^{-2A} & 0 \\
e^{2A} & 0
\end{pmatrix}
= \begin{pmatrix}
0 & e^A \\
e^{-A} & 0 \\
e^A & 0
\end{pmatrix}
= \begin{pmatrix}
0 & e^A \\
e^{-A} & 0 \\
e^{-A} & 0
\end{pmatrix}
\]

\(\equiv U(\eta)(\sigma_3 \otimes 1)U(\eta)^\dagger\)

where

\[
U(\eta) = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & e^A \\
e^{-A} & 0 \\
e^{-A} & 0
\end{pmatrix}
= \frac{1}{\sqrt{2}} \begin{pmatrix}
e^A & -e^A \\
e^{-A} & e^{-A}
\end{pmatrix}
\]

and \(e^A = D(i\eta/2)\) where \(D(\beta)\) is a displacement (coherent) operator defined by \[12\].

Then it is not difficult to show

\[
U(\eta)^\dagger HU(\eta) = \frac{\omega_0 \eta^2}{4} 1_2 \otimes 1 + \omega_0 1_2 \otimes a^\dagger a + \frac{\omega_0 \eta}{2} \sigma_1 \otimes (-ia^\dagger + ia) + g\sigma_3 \otimes 1 - \frac{\Delta}{2} \sigma_1 \otimes 1.
\]

To remove \(i\) in the term containing \(a\) we moreover operate the unitary one

\[
\left(1_2 \otimes e^{i(\pi/2)N}\right) U(\eta)^\dagger HU(\eta) \left(1_2 \otimes e^{-i(\pi/2)N}\right)
= \frac{\omega_0 \eta^2}{4} 1_2 \otimes 1 + \omega_0 1_2 \otimes a^\dagger a + \frac{\omega_0 \eta}{2} \sigma_1 \otimes (a^\dagger + a) + g\sigma_3 \otimes 1 - \frac{\Delta}{2} \sigma_1 \otimes 1,
\]

where we have used the well-known formula

\[
e^{i\theta N}a e^{-i\theta N} = e^{-i\theta} a, \quad e^{i\theta N}a^\dagger e^{-i\theta N} = e^{i\theta} a^\dagger,
\]

see \[15\]. Since \(U(\eta)\) can be written as

\[
U(\eta) = (\sigma_+ \otimes e^A + \sigma_- \otimes e^{-A})(W \otimes 1),
\]

so if we write

\[
T(\eta) \equiv U(\eta)(1_2 \otimes e^{-i(\pi/2)N}) = (\sigma_+ \otimes e^A + \sigma_- \otimes e^{-A})(W \otimes e^{-i(\pi/2)N}),
\]

(22)
then we have
\[
T(\eta)\dagger HT(\eta) = \frac{\omega_0\eta^2}{4}\mathbf{1}_2 \otimes \mathbf{1} + \omega_0 \mathbf{1}_2 \otimes a^\dagger a + \frac{\omega_0\eta}{2}\sigma_1 \otimes (a^\dagger + a) + g\sigma_3 \otimes \mathbf{1} - \frac{\Delta}{2}\sigma_1 \otimes \mathbf{1}.
\]
Here we have no interest in the constant term, so we finally obtain
\[
H = T(\eta) \left\{ \omega_0 \mathbf{1}_2 \otimes a^\dagger a + \frac{\omega_0\eta}{2}\sigma_1 \otimes (a^\dagger + a) + g\sigma_3 \otimes \mathbf{1} - \frac{\Delta}{2}\sigma_1 \otimes \mathbf{1} \right\} T(\eta)^\dagger.
\]
(23)

\(T(\eta)\) is just the unitary transformation required.

At this stage we would like to make a further generalization of the Hamiltonian (14) to make wide applications to Quantum Computation. Let \(\{K_+, K_-, K_3\}\) and \(\{J_+, J_-, J_3\}\) be a set of generators of unitary representations of Lie algebras \(su(1,1)\) and \(su(2)\). Then we can make the similar arguments done for the Heisenberg algebra \(\{a^\dagger, a, N\}\), namely (8) ~ (12), see for example \[15\].

We have considered the following three Hamiltonians in [13] :

\[
(N) \quad H_N = \omega \mathbf{1}_2 \otimes a^\dagger a + \frac{\Delta}{2}\sigma_3 \otimes \mathbf{1} + g\sigma_1 \otimes (a^\dagger + a),
\]

(24)

\[
(K) \quad H_K = \omega \mathbf{1}_2 \otimes K_3 + \frac{\Delta}{2}\sigma_3 \otimes \mathbf{1}_K + g\sigma_1 \otimes (K_+ + K_-),
\]

(25)

\[
(J) \quad H_J = \omega \mathbf{1}_2 \otimes J_3 + \frac{\Delta}{2}\sigma_3 \otimes \mathbf{1}_J + g\sigma_1 \otimes (J_+ + J_-).
\]

(26)

To treat these three cases at the same time we set

\[
\{L_+, L_-, L_3\} = \begin{cases} 
(N) & \{a^\dagger, a, N\}, \\
(K) & \{K_+, K_-, K_3\}, \\
(J) & \{J_+, J_-, J_3\}
\end{cases}
\]

(27)

and

\[
H_L = \omega \mathbf{1}_2 \otimes L_3 + \frac{\Delta}{2}\sigma_3 \otimes \mathbf{1}_L + g\sigma_1 \otimes (L_+ + L_-).
\]

(28)

Therefore the Hamiltonian that we are looking for is

**Full Hamiltonian**

\[
\tilde{H}_L = \omega \mathbf{1}_2 \otimes L_3 + g_1\sigma_1 \otimes (L_+ + L_-) + \frac{\Delta}{2}\sigma_3 \otimes \mathbf{1}_L + g_2\cos(\omega_E t)\sigma_1 \otimes \mathbf{1}_L.
\]

(29)
From now we would like to solve this Hamiltonian, especially in the strong coupling regime ($g_1 \gg \Delta$).

Let us transform (29) into

$$\tilde{H}_L = \mathbf{1}_2 \otimes \omega L_3 + \sigma_1 \otimes \{g_1(L_+ + L_-) + g_2 \cos(\omega_E t)\mathbf{1}_L\} + \frac{\Delta}{2} \sigma_3 \otimes \mathbf{1}_L$$

$$\equiv \tilde{H}_0 + \frac{\Delta}{2} \sigma_3 \otimes \mathbf{1}_L. \quad (30)$$

The method to solve is almost identical to [13], so we give only an outline. By making use of the Walsh–Hadamard matrix (2)

$$\tilde{H}_0 = (W \otimes \mathbf{1}_L) \left[ \mathbf{1}_2 \otimes \omega L_3 + \sigma_3 \otimes \{g_1(L_+ + L_-) + g_2 \cos(\omega_E t)\mathbf{1}_L\} \right] (W^{-1} \otimes \mathbf{1}_L)$$

$$= \sum_{\lambda = \pm 1} \left( |\lambda\rangle \otimes e^{-\frac{\lambda}{2}(L_+ - L_-)} \right) \{\Omega L_3 + \lambda g_2 \cos(\omega_E t)\mathbf{1}_L\} \left( \langle \lambda| \otimes e^{\frac{\lambda}{2}(L_+ - L_-)} \right)$$

where $|\lambda\rangle$ is the eigenvectors of $\sigma_1$ defined in (3) and $\Omega$, $x$ are given as

$$(\Omega, x) = \begin{cases} 
(N) & \omega, \\
(K) & \sqrt{1 - (2g_1/\omega)^2}, \\
(J) & \sqrt{1 + (2g_1/\omega)^2}, 
\end{cases} \quad x = 2g_1/\omega, \quad x = \tanh^{-1}(2g_1/\omega), \quad x = \tan^{-1}(2g_1/\omega). \quad (31)$$

That is, we could diagonalize the Hamiltonian $\tilde{H}_0$. Its eigenvalues $\{E_n(t)\}$ and eigenvectors $\{|\{\lambda, n\}\rangle\}$ are given respectively

$$(E_n(t), \{|\{\lambda, n\}\rangle\}) = \begin{cases} 
(N) & \Omega(-\frac{\lambda}{\omega}^2 + n) + \lambda g_2 \cos(\omega_E t), \\
(K) & \Omega(K + n) + \lambda g_2 \cos(\omega_E t), \\
(J) & \Omega(-J + n) + \lambda g_2 \cos(\omega_E t), 
\end{cases} \quad |\lambda\rangle \otimes e^{\pm \frac{\lambda}{2}(L_+ - L_-)}|n\rangle, \quad \langle \lambda| \otimes e^{\pm \frac{\lambda}{2}(L_+ - L_-)}|K, n\rangle, \quad \langle \lambda| \otimes e^{\pm \frac{\lambda}{2}(J_+ - J_-)}|J, n\rangle \quad (32)$$

for $\lambda = \pm 1$ and $n \in \mathbb{N} \cup \{0\}$, where $E_n(t) = E_n + \lambda g_2 \cos(\omega_E t)$. Then $\tilde{H}_0$ above can be written as

$$\tilde{H}_0 = \sum_{\lambda} \sum_n E_n(t)|\{\lambda, n\}\rangle\langle \{\lambda, n\}|.$$

Next we would like to solve the following Schrödinger equation :

$$\frac{d}{dt}\Psi = \tilde{H}\Psi = \left( \tilde{H}_0 + \frac{\Delta}{2} \sigma_3 \otimes \mathbf{1}_L \right) \Psi. \quad (33)$$
To solve this equation we appeal to the method of constant variation. First let us solve

\[ i \frac{d}{dt} \Psi = \tilde{H} \Psi, \]

which general solution is given by \( \Psi(t) = U_0(t) \Psi_0 \), where \( \Psi_0 \) is a constant state and

\[ U_0(t) = \sum_{\lambda} \sum_n e^{-i(E_n + \lambda(g_2/E) \sin(\omega_E t))} \langle \{\lambda, n\} | \{\lambda, n\} \rangle. \tag{34} \]

The method of constant variation goes as follows. Changing like \( \Psi_0 \rightarrow \Psi_0(t) \), we have

\[ i \frac{d}{dt} \Psi_0 = \frac{\Delta}{2} U_0^\dagger (\sigma_3 \otimes 1_L) U_0 \Psi_0 \equiv \frac{\Delta}{2} \tilde{H} F \Psi_0 \tag{35} \]

after some algebra. We must solve this equation. \( \tilde{H} F \) is

\begin{align*}
\tilde{H} F &= \sum_{\lambda, \mu, m, n} e^{i(tE_m - E_n) + i(\lambda - \mu)(g_2/E) \sin(\omega_E t)} \langle \{\lambda, m\} | (\sigma_3 \otimes 1_L) | \{\mu, n\} \rangle \langle \{\lambda, n\} | \{\mu, n\} \\
&= \sum_{\lambda, m, n} e^{i(t\Omega(m-n) + 2\lambda(g_2/E) \sin(\omega_E t))} \langle \langle m | e^{\lambda x(L_+ - L_-)} | n \rangle \rangle \langle \{\lambda, m\} | \{\lambda, n\} \rangle | \{\lambda, n\} | \{\mu, n\} \rangle, \tag{36}
\end{align*}

where we have used \( \langle \lambda | \sigma_3 = \langle -\lambda \rangle \) and \( |n\rangle \) is respectively

\[ |n\rangle = \begin{cases} (N) & |n\rangle, \\ (K) & |K, n\rangle, \\ (J) & |J, n\rangle. \end{cases} \]

In the following we set for simplicity

\[ \Theta(t) \equiv g_2 \frac{\sin(\omega_E t)}{\omega_E}. \tag{37} \]

Here we divide \( \tilde{H} F \) into two parts \( \tilde{H} F = \tilde{H} F' + \tilde{H} F'' \) where

\begin{align*}
\tilde{H} F' &= \sum_{\lambda, n} e^{i2\lambda \Theta(t)} \langle \langle n | e^{\lambda x(L_+ - L_-)} | n \rangle \rangle \langle \{\lambda, n\} | \{\lambda, n\} \rangle, \tag{38} \\
\tilde{H} F'' &= \sum_{\lambda, m, n} e^{i(t\Omega(m-n) + 2\lambda \Theta(t))} \langle \langle m | e^{\lambda x(L_+ - L_-)} | n \rangle \rangle \langle \{\lambda, m\} | \{\lambda, n\} \rangle | \{\lambda, n\} | \{\lambda, n\} \rangle. \tag{39}
\end{align*}

Noting \( \langle \langle n | e^{x(L_+ - L_-)} | n \rangle \rangle = \langle \langle n | e^{-x(L_+ - L_-)} | n \rangle \rangle \) by the results in section 3 of [13], \( \tilde{H} F' \) can be written as

\[ \tilde{H} F' = \sum_n \langle \langle n | e^{x(L_+ - L_-)} | n \rangle \rangle \left\{ e^{i\Theta(t)} | \{1, n\} \rangle \langle \{1, n\} | + e^{-2i\Theta(t)} | \{-1, n\} \rangle \langle \{1, n\} | \right\}. \]
Here we want to solve the equation $i(d/dt)\Psi_0 = \tilde{H}_F'\Psi_0$ completely, however it is not easy (see [8], [19], [20] and Appendix). Therefore we make a strong assumption. Namely we consider only the constant external field in [20] ($\omega_F = 0$ in (37)), so

$$\Theta(t) = g_2 t.$$  

(40)

In this case we can solve the equation completely$^2$.

A comment is in order. For a short time span it may be not unrealistic to consider the above situation. Anyway, the author doesn’t know whether it is reasonable or not. Therefore we have

$$\tilde{H}_F' = \sum_n \langle\langle n|e^{x(L_L-L_-)}|n\rangle\rangle \left\{e^{2ig_2t}|\{1, n\}\rangle\langle\{-1, n\}| + e^{-2ig_2t}|\{-1, n\}\rangle\langle\{1, n\}| \right\}.  

(41)

Now it is easy to solve the equation $i(d/dt)\Psi_0 = \tilde{H}_F'\Psi_0$, see Appendix. Next let us transform (39).

$$H_F'' = \sum_{m,n}^{m \neq n} e^{it\Omega(m-n)} \left\{\langle\langle m|e^{x(L_L-L_-)}|n\rangle\rangle e^{2ig_2t}|\{1, m\}\rangle\langle\{-1, n\}| + \langle\langle m|e^{-x(L_L-L_-)}|n\rangle\rangle e^{-2ig_2t}|\{-1, m\}\rangle\langle\{1, n\}| \right\}.  

(42)

For simplicity in the following we set

$$E_{n,\Delta} = \frac{\Delta}{2} \langle\langle n|e^{x(L_L-L_-)}|n\rangle\rangle,$$

(43)

then

$$E_{n,\Delta} = \begin{cases} (N) & \frac{\Delta}{2} e^{-\frac{\kappa^2}{2}} L_n(\kappa^2) \quad \text{where} \quad \kappa = x \\ (K) & \frac{\Delta}{2} \frac{\kappa^n}{(2K)^n} (1 + \kappa^2)^{-K-n} F_n(\kappa^2 : 2K) \quad \text{where} \quad \kappa = \sinh(x) \\ (J) & \frac{\Delta}{2} \frac{n!}{\kappa^n} (1 - \kappa^2)^{-J-n} F_n(\kappa^2 : 2J) \quad \text{where} \quad \kappa = \sin(x) \end{cases}$$

(44)

from the results in sectin 3.1 of [13].

$^2$In the following the method to solve the equations is different from that of [13] in which the Schrödinger cat states were used. However in this case we cannot use them, so the method becomes rather complicated.
Now let us solve
\[
\frac{d}{dt} \Psi_0 = \frac{\Delta}{2} \tilde{H}_F \Psi_0 = \frac{\Delta}{2} (\tilde{H}_F' + \tilde{H}_F'') \Psi_0.
\]
For that using the method of constant variation again we can set \( \Psi_0(t) \) as
\[
\Psi_0(t) = \sum_n \{(u_{n,11}a_{n,1} + u_{n,12}a_{n,-1})\{1, n\} + (u_{n,21}a_{n,1} + u_{n,22}a_{n,-1})\{-1, n\}\}, \quad (45)
\]
where from the appendix
\[
U_n(t) = \begin{pmatrix} u_{n,11} & u_{n,12} \\ u_{n,21} & u_{n,22} \end{pmatrix} = \begin{pmatrix} 1 & e^{-2i\xi_2t} \\ e^{i\xi_1t} & 0 \end{pmatrix} \exp \left\{ -it \begin{pmatrix} 0 & E_{n,\Delta} \\ E_{n,\Delta} & -2g_2 \end{pmatrix} \right\}, \quad (46)
\]
then we have a set of complicated equations with respect to \( \{a_{n,\lambda}(t)\} \). However it is almost impossible to solve them, so let us make a daring assumption like \( \text{[13]} \) : for \( m < n \)
\[
\Psi_0(t) = \{(u_{m,11}a_{m,1} + u_{m,12}a_{m,-1})\{1, m\} + (u_{m,21}a_{m,1} + u_{m,22}a_{m,-1})\{-1, m\}\} +
\{(u_{n,11}a_{n,1} + u_{n,12}a_{n,-1})\{1, n\} + (u_{n,21}a_{n,1} + u_{n,22}a_{n,-1})\{-1, n\}\} . \quad (47)
\]
This ansatz is enough for our purpose because we are only interested in the Rabi oscillations. Then after a long calculation we have
\[
\frac{d}{dt} \begin{pmatrix} a_{m,1} \\ a_{m,-1} \end{pmatrix} = \frac{\Delta}{2} e^{i\Omega(m-n)} U_m^{-1} \begin{pmatrix} T_{mn} \\ \tilde{T}_{mn} \end{pmatrix} \begin{pmatrix} 0 & e^{2i\xi_2t} \\ e^{-2i\xi_2t} & 0 \end{pmatrix} U_n \begin{pmatrix} a_{n,1} \\ a_{n,-1} \end{pmatrix}, \quad (48)
\]
where
\[
T_{mn} = \langle m | e^{x(L_+ - L_-)} | n \rangle, \quad \tilde{T}_{mn} = \langle m | e^{-x(L_+ - L_-)} | n \rangle
\]
and the equation with \( m \leftrightarrow n \) in \( \text{[48]} \). We note \( \tilde{T}_{mn} = T_{nm} \) and \( T_{mn} = \tilde{T}_{nm} \) because \( x \) is real from \( \text{[31]} \). For the details of \( T_{mn} \) or \( \tilde{T}_{mn} \) see \( \text{[13]} \).

We must calculate the right hand side of \( \text{[48]} \). After some algebra
\[
U_m^{-1} \begin{pmatrix} T_{mn} \\ \tilde{T}_{mn} \end{pmatrix} \begin{pmatrix} 0 & e^{2i\xi_2t} \\ e^{-2i\xi_2t} & 0 \end{pmatrix} U_n
= \exp \left\{ it \begin{pmatrix} 0 & E_{m,\Delta} \\ E_{m,\Delta} & -2g_2 \end{pmatrix} \right\} \begin{pmatrix} T_{mn} \\ \tilde{T}_{mn} \end{pmatrix} \exp \left\{ -it \begin{pmatrix} 0 & E_{n,\Delta} \\ E_{n,\Delta} & -2g_2 \end{pmatrix} \right\}
= \Gamma_m \begin{pmatrix} e^{it\lambda_{m,+}} \\ e^{it\lambda_{m,-}} \end{pmatrix} \Gamma_m^{-1} \begin{pmatrix} T_{mn} \\ \tilde{T}_{mn} \end{pmatrix} \Gamma_n \begin{pmatrix} e^{-it\lambda_{n,+}} \\ e^{-it\lambda_{n,-}} \end{pmatrix} \Gamma_n^{-1} \quad (49)
\]
11
from (46) and Appendix. Therefore the RHS of (48) is

\[
\frac{\Delta}{2} e^{it\Omega(m-n)} \Gamma_m \left( \begin{array}{c} e^{it\lambda_{m,+}} \\ e^{it\lambda_{m,-}} \end{array} \right) \Gamma_m^{-1} \left( \begin{array}{c} T_{mn} \end{array} \right) \Gamma_n \left( \begin{array}{c} e^{-it\lambda_{n,+}} \\ e^{-it\lambda_{n,-}} \end{array} \right) \Gamma_n^{-1},
\]

(50)

where from Appendix

\[
\lambda_{k,+} = -g_2 + \sqrt{E_{k,\Delta}^2 + g_2^2}, \quad \lambda_{k,-} = -g_2 - \sqrt{E_{k,\Delta}^2 + g_2^2}.
\]

From now we divide into the four cases.

Case (I) : (50) can be written as

\[
\frac{\Delta}{2} e^{it\{\Omega(m-n)+\lambda_{m,+}+\lambda_{n,+}\}} \times \Gamma_m \left( \begin{array}{c} 1 \\ e^{it(\lambda_{m,-}-\lambda_{m,+})} \end{array} \right) \Gamma_m^{-1} \left( \begin{array}{c} T_{mn} \end{array} \right) \Gamma_n \left( \begin{array}{c} 1 \\ e^{-it(\lambda_{n,-}-\lambda_{n,+})} \end{array} \right) \Gamma_n^{-1}.
\]

Here we set the resonance condition, namely

\[
\Omega(m-n) + \lambda_{m,+} - \lambda_{n,+} = 0,
\]

(51)

or more explicitly

\[
\Omega(m-n) + \sqrt{E_{m,\Delta}^2 + g_2^2} - \sqrt{E_{n,\Delta}^2 + g_2^2} = 0.
\]

This is a rather complicated equation. Then the above matrix becomes

\[
\frac{\Delta}{2} \Gamma_m \left( \begin{array}{c} 1 \\ e^{it(\lambda_{m,-}-\lambda_{m,+})} \end{array} \right) \Gamma_m^{-1} \left( \begin{array}{c} T_{mn} \end{array} \right) \Gamma_n \left( \begin{array}{c} 1 \\ e^{-it(\lambda_{n,-}-\lambda_{n,+})} \end{array} \right) \Gamma_n^{-1}.
\]

By the way, since

\[
\lambda_{k,-} - \lambda_{k,+} = -2\sqrt{E_{k,\Delta}^2 + g_2^2} \quad \text{for} \quad k = m, n,
\]

we can take 0 if \( g_2 \) is large enough (so-called the rotating wave approximation\(^3\) which neglects fast oscillating terms). Therefore we have the time–independent (!) matrix

\[
\frac{\Delta}{2} \Gamma_m \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \Gamma_m^{-1} \left( \begin{array}{c} T_{mn} \end{array} \right) \Gamma_n \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \Gamma_n^{-1}.
\]

\(^3\)In many papers in Quantum Optics this assumption has been used without showing that the frequency in the model is large enough. However it is not correct.
\[
\frac{d}{dt} \begin{pmatrix} a_{m,1} \\ a_{m,-1} \end{pmatrix} = \frac{\Delta}{2} \gamma_m \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Gamma_n^{-1} \begin{pmatrix} a_{n,1} \\ a_{n,-1} \end{pmatrix} = \frac{\Delta}{2} \gamma_m \begin{pmatrix} 1_2 + \sigma_3 \Gamma_n^{-1} \\ 1_2 \end{pmatrix} \begin{pmatrix} a_{n,1} \\ a_{n,-1} \end{pmatrix},
\]

where

\[
\gamma_{mn} = \frac{\lambda_{m,+}}{\sqrt{\lambda_{m,+}^2 + E_{m,\Delta}^2}} T_{mn} \frac{E_{n,\Delta}}{\sqrt{\lambda_{n,+}^2 + E_{n,\Delta}^2}} + \frac{E_{m,\Delta}}{\sqrt{\lambda_{m,+}^2 + E_{m,\Delta}^2}} T_{nn} \frac{\lambda_{n,+}}{\sqrt{\lambda_{n,+}^2 + E_{n,\Delta}^2}} = \frac{1}{\sqrt{\lambda_{n,+}^2 + E_{n,\Delta}^2} \sqrt{\lambda_{m,+}^2 + E_{m,\Delta}^2}} (E_{n,\Delta} T_{nn} \lambda_{m,+} + E_{m,\Delta} T_{mn} \lambda_{n,+})
\]

from \( \tilde{T}_{mn} = T_{nm} \). Therefore \( \gamma_{mn} = \gamma_{nm} \equiv \gamma \in \mathbb{R} \). As a result we obtain the equations

\[
\frac{d}{dt} \begin{pmatrix} a_{n,1} \\ a_{n,-1} \end{pmatrix} = \frac{\Delta}{2} \gamma_n \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Gamma_m^{-1} \begin{pmatrix} a_{m,1} \\ a_{m,-1} \end{pmatrix} = \frac{\Delta}{2} \gamma_n \begin{pmatrix} 1_2 + \sigma_3 \Gamma_m^{-1} \\ 1_2 \end{pmatrix} \begin{pmatrix} a_{m,1} \\ a_{m,-1} \end{pmatrix},
\]

or if we introduce the compact notation \( a_m = (a_{m,1}, a_{m,-1})^t \) then we have the clear matrix equation

\[
\frac{d}{dt} \begin{pmatrix} a_m \\ a_n \end{pmatrix} = \frac{\Delta}{2} \begin{pmatrix} \Gamma_m & 0 \\ \Gamma_n & \Gamma_n \end{pmatrix} \begin{pmatrix} 0 & \gamma_{1+}^{1+} \sigma_3 \\ \gamma_{1+}^{1+} \sigma_3 & 0 \end{pmatrix} \begin{pmatrix} \Gamma_m & 0 \\ \Gamma_n & \Gamma_n \end{pmatrix}^{-1} \begin{pmatrix} a_m \\ a_n \end{pmatrix}.
\]

Therefore the solution that we are looking for is

\[
\begin{pmatrix} a_m(t) \\ a_n(t) \end{pmatrix} = \begin{pmatrix} \Gamma_m & 0 \\ \Gamma_n & \Gamma_n \end{pmatrix} \exp \left\{ -i\frac{\Delta}{2} \begin{pmatrix} 0 & \gamma_{1+}^{1+} \sigma_3 \\ \gamma_{1+}^{1+} \sigma_3 & 0 \end{pmatrix} \right\} \begin{pmatrix} \Gamma_m & 0 \\ \Gamma_n & \Gamma_n \end{pmatrix}^{-1} \begin{pmatrix} a_m(0) \\ a_n(0) \end{pmatrix}.
\]

The Rabi frequency is just \( \Delta \gamma \) (compare with [13]), which is also rather complicated.

Next we consider the remaining three cases: The arguments are almost identical, so we only give the results (we leave them to the readers).

**Case (II):** The resonance condition is

\[
\Omega(m-n) + \lambda_{m,-} - \lambda_{n,-} = 0.
\]
The solution that we are looking for is
\[
\begin{pmatrix}
a_m(t) \\
a_n(t)
\end{pmatrix} = \begin{pmatrix} \Gamma_m \\ \Gamma_n \end{pmatrix} \exp \left\{ -it \frac{\Delta}{2} \begin{pmatrix} 0_2 & \gamma \frac{\lambda_m - \lambda_n}{2} \\ \gamma \frac{\lambda_m - \lambda_n}{2} & 0_2 \end{pmatrix} \right\} \begin{pmatrix} \Gamma_m \\ \Gamma_n \end{pmatrix}^{-1} \begin{pmatrix} a_m(0) \\ a_n(0) \end{pmatrix},
\]
(57)
where \( \gamma \) is
\[
\gamma = \frac{1}{\sqrt{\lambda_{m,-}^2 + E_{n,\Delta}} \sqrt{\lambda_{n,-}^2 + E_{m,\Delta}}} (E_{n,\Delta} T_{nm} \lambda_{m,-} + E_{m,\Delta} T_{mn} \lambda_{n,-}).
\]
(58)
The Rabi frequency is \( \Delta \gamma \) (compare with [13]).

Case (III): **The resonance condition** is
\[
\Omega(m - n) + \lambda_{m,+} - \lambda_{n,-} = 0.
\]
(59)
The solution that we are looking for is
\[
\begin{pmatrix}
a_m(t) \\
a_n(t)
\end{pmatrix} = \begin{pmatrix} \Gamma_m \\ \Gamma_n \end{pmatrix} \exp \left\{ -it \frac{\Delta}{2} \begin{pmatrix} 0_2 & \gamma \sigma_+ \\ \gamma \sigma_- & 0_2 \end{pmatrix} \right\} \begin{pmatrix} \Gamma_m \\ \Gamma_n \end{pmatrix}^{-1} \begin{pmatrix} a_m(0) \\ a_n(0) \end{pmatrix},
\]
(60)
where \( \gamma \) is
\[
\gamma = \frac{1}{\sqrt{\lambda_{m,-}^2 + E_{n,\Delta}} \sqrt{\lambda_{m,+}^2 + E_{m,\Delta}}} (E_{n,\Delta} T_{nm} \lambda_{m,+} + E_{m,\Delta} T_{mn} \lambda_{n,-}).
\]
(61)
The Rabi frequency is \( \Delta \gamma \) (compare with [13]).

Case (IV): **The resonance condition** is
\[
\Omega(m - n) + \lambda_{m,-} - \lambda_{n,+} = 0.
\]
(62)
The solution that we are looking for is
\[
\begin{pmatrix}
a_m(t) \\
a_n(t)
\end{pmatrix}
\]
14
\[
\left( \begin{array}{c}
\Gamma_m \\
\Gamma_n
\end{array} \right) \exp \left\{-it \frac{\Delta}{2} \left( \begin{array}{cc}
0_2 & \gamma \sigma_- \\
\gamma \sigma_+ & 0_2
\end{array} \right) \right\} \left( \begin{array}{c}
\Gamma_m \\
\Gamma_n
\end{array} \right)^{-1} \left( \begin{array}{c}
a_m(0) \\
a_n(0)
\end{array} \right),
\]

where \( \gamma \) is
\[
\gamma = \frac{1}{\sqrt{\lambda_{m,+}^2 + E_{n,\Delta}^2}} \left( E_{n,\Delta} T_{nm} \lambda_{m,-} + E_{m,\Delta} T_{mn} \lambda_{n,+} \right).
\]

(64)
The Rabi frequency is \( \Delta \gamma \) (compare with [13]).

On the ansatz (47) we solved the Schrödinger equation (33) in the strong coupling regime (!) under the resonance conditions and rotating wave approximations, and obtained the unitary transformations of four types which are a generalization of [13]. They will play a crucial role in Quantum Computation.

On the other hand we in this paper solved the special case (40), however we would like to study the general case (37). At the present it is almost impossible (see Appendix). We will treat this case in a forthcoming paper.

By the way, we considered one atom with two–level, so we would like to generalize our method to \( n \) atoms (with two–level) interacting both the single radiation mode and external periodic fields like (\( n \) atoms trapped in a cavity)

Then the Hamiltonian may be
\[
\hat{H}_{nL} = \omega_1 M \otimes L_3 + g_1 \sum_{j=1}^{n} \sigma_1^{(j)} \otimes (L_+ + L_-) + \frac{\Delta}{2} \sum_{j=1}^{n} \sigma_3^{(j)} \otimes 1_L + g_2 \sum_{j=1}^{n} \cos(\omega_j t + \phi_j) \sigma_1^{(j)} \otimes 1_L,
\]

(65)

where \( M = 2^n \) and \( \sigma_k^{(j)} \) (\( k = 1, 3 \)) is
\[
\sigma_k^{(j)} = 1_2 \otimes \cdots \otimes 1_2 \otimes \sigma_k \otimes 1_2 \otimes \cdots \otimes 1_2 \ (j - \text{position}).
\]
In the near future we will attempt an attack to this model. However according to increase of the number of atoms (we are expecting at least \( n = 100 \) in the realistic quantum computation) we meet a very severe problem called Decoherence, see for example [10] and its references. The author doesn’t know how to control this.

A generalization of the model to N–level system (see for example [14], [17], [18], [21]) is now under consideration and will be published in a separate paper\(^4\).

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**Appendix : Some Useful Formulas**

In this appendix we solve the following equation

\[
i \frac{d}{dt} \psi = \alpha H \psi, \quad (66)
\]

where \( \alpha \) is a constant and

\[
H = e^{2i\theta t}|1\rangle\langle -1| + e^{-2i\theta t}|-1\rangle\langle 1| \quad \text{and} \quad \psi = a(t)|1\rangle + b(t)|-1\rangle. \quad (67)
\]

Then it is easy to get a matrix equation on \( \{a, b\} \)

\[
i \frac{d}{dt} \begin{pmatrix} a \\ b \end{pmatrix} = \alpha \begin{pmatrix} 0 & e^{2i\theta t} \\ e^{-2i\theta t} & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \iff i \frac{d}{dt} \tilde{\psi} = \tilde{H} \tilde{\psi}. \quad (68)
\]

The solution is easily obtained to become

\[
\begin{pmatrix} a \\ b \end{pmatrix} = U(t) \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} \quad (69)
\]

\(^4\)The author believes that it is important for us to consider the N–level system to prevent the decoherence problem
where \((a_0, b_0)^T\) is a constant vector and

\[
U(t) = \begin{pmatrix} 1 & e^{-2\theta t} \end{pmatrix} \exp \left\{ -it \begin{pmatrix} 0 & \alpha \\ \alpha & -2\theta \end{pmatrix} \right\} \Longrightarrow i \frac{d}{dt} U = \hat{H} U. \tag{70}
\]

If we set

\[
U(t) = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \tag{71}
\]

(2 corresponds to \(-1\)) then \(\psi\) above can be written as

\[
\psi = (u_{11}a_0 + u_{12}b_0)|1\rangle + (u_{21}a_0 + u_{22}b_0)|-1\rangle \tag{72}
\]

with constants \(\{a_0, b_0\}\).

In the method of constant variation in the text we change like \(a_0 \longrightarrow a_0(t)\) and \(b_0 \longrightarrow b_0(t)\).

Let us make some comments. For

\[
A = \begin{pmatrix} 0 & \alpha \\ \alpha & -2\theta \end{pmatrix} \tag{73}
\]

we can easily diagonalize \(A\) as follows:

\[
A = \begin{pmatrix} \alpha & \alpha \\ \lambda_+ & \lambda_- \end{pmatrix} \begin{pmatrix} \lambda_+ & \alpha \\ \alpha & \lambda_- \end{pmatrix}^{-1} \tag{74}
\]

where

\[
\lambda_+ = -\theta + \sqrt{\theta^2 + \alpha^2}, \quad \lambda_- = -\theta - \sqrt{\theta^2 + \alpha^2}.
\]

Therefore we obtain

\[
V(t) \equiv e^{-itA} = \begin{pmatrix} \alpha & \alpha \\ \lambda_+ & \lambda_- \end{pmatrix} \begin{pmatrix} e^{-it\lambda_+} & 0 \\ 0 & e^{-it\lambda_-} \end{pmatrix} \begin{pmatrix} \alpha & \alpha \\ \lambda_+ & \lambda_- \end{pmatrix}^{-1} \tag{75}
\]

We note the formula in the text.

\[
\begin{pmatrix} \alpha & \alpha \\ \lambda_+ & \lambda_- \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \alpha \\ \lambda_- & \lambda_+ \end{pmatrix}^{-1} = \frac{1}{\lambda_- - \lambda_+} \begin{pmatrix} \lambda_- & -\alpha \\ -\alpha & -\lambda_+ \end{pmatrix}. \tag{76}
\]
For the simplicity we set
\[
\Gamma = \begin{pmatrix}
\frac{\alpha}{\sqrt{\alpha^2 + \lambda_+^2}} & \frac{\alpha}{\sqrt{\alpha^2 + \lambda_-^2}} \\
\frac{\lambda_+}{\sqrt{\alpha^2 + \lambda_+^2}} & \frac{\lambda_-}{\sqrt{\alpha^2 + \lambda_-^2}}
\end{pmatrix} \in O(2).
\] (77)

In the text this is used as
\[
\Gamma_k = \begin{pmatrix}
\frac{E_{k,\Delta}}{\sqrt{E_{k,\Delta}^2 + \lambda_{k,+}^2}} & \frac{E_{k,\Delta}}{\sqrt{E_{k,\Delta}^2 + \lambda_{k,-}^2}} \\
\frac{\lambda_{k,+}}{\sqrt{E_{k,\Delta}^2 + \lambda_{k,+}^2}} & \frac{\lambda_{k,-}}{\sqrt{E_{k,\Delta}^2 + \lambda_{k,-}^2}}
\end{pmatrix}
\] (78)

with
\[
\lambda_{k,+} = -g_2 + \sqrt{g_2^2 + E_{k,\Delta}^2}, \quad \lambda_{k,-} = -g_2 - \sqrt{g_2^2 + E_{k,\Delta}^2}
\]
for \(k = m, n\).

Last we make a comment. The fact is that we wanted to solve the following matrix–
equation instead of (68)
\[
\frac{d}{dt} \begin{pmatrix} a \\ b \end{pmatrix} = \alpha \begin{pmatrix} 0 & e^{2i\Theta(t)} \\ e^{-2i\Theta(t)} & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix},
\] (79)

where
\[
\Theta(t) = \frac{\theta \sin(\omega t)}{\omega} = \theta t \frac{\sin(\omega t)}{\omega t} \rightarrow \theta t \text{ as } \omega \rightarrow 0.
\]

We here note
\[
e^{2i\Theta(t)} = \sum_{n \in \mathbb{N}} J_n(\theta/\omega)e^{n\omega t},
\] (80)

where \(J_n(x)\) are the Bessel functions.

However we cannot solve this equation completely, see [8], [19], [20], [16]. This is just
the bottleneck.
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