THE GAUSSIAN MEASURE ON ALGEBRAIC VARIETIES

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Abstract. We prove that the ring $\mathbb{R}[M]$ of all polynomials defined on a real algebraic variety $M \subset \mathbb{R}^n$ is dense in the Hilbert space $L^2(M, e^{-|x|^2} d\mu)$, where $d\mu$ denotes the volume form of $M$ and $d\nu = e^{-|x|^2} d\mu$ the Gaussian measure on $M$.

1. Introduction

The aim of the present note is to prove that the ring $\mathbb{R}[M]$ of all polynomials defined on a real algebraic variety $M \subset \mathbb{R}^n$ is dense in the Hilbert space $L^2(M, e^{-|x|^2} d\mu)$, where $d\mu$ denotes the volume form of $M$ and $d\nu = e^{-|x|^2} d\mu$ the Gaussian measure on $M$. In case $M = \mathbb{R}^n$, the result is well known since the Hermite polynomials constitute a complete orthonormal basis of $L^2(\mathbb{R}^n, e^{-|x|^2} d\mu)$.

2. The volume growth of an algebraic variety and some consequences

We consider a smooth algebraic variety $M \subset \mathbb{R}^n$ of dimension $d$. Then $M$ has polynomial volume growth: there exists a constant $C$ depending only on the degrees of the polynomials defining $M$ such that for any euclidian ball $B_r$ with center $0 \in \mathbb{R}^n$ and radius $r > 0$ the inequality

$$\text{vol}_d(M \cap B_r) \leq C \cdot r^d$$

holds (see [Brö]). Via Crofton formulas the mentioned inequality is a consequence of Milnor’s results concerning the Betti numbers of an algebraic variety (see [Mi1], [Mi2], in which the stated inequality is already implicitly contained). This estimate yields first of all that the restrictions on $M$ of the polynomials on $\mathbb{R}^n$ are square-integrable with respect to the Gaussian measure on $M$.

Proposition 1. Let $M$ be a smooth submanifold of the euclidian space $\mathbb{R}^n$. Suppose that $M$ has polynomial volume growth, i.e., there exist constants $C$ and $l \in \mathbb{N}$ such that for any ball $B_r$ of $\mathbb{R}^n$ the inequality

$$\text{vol}_d(M \cap B_r) \leq C \cdot r^l$$

holds. Denote by $d\mu$ the volume form of $M$. Then:

1. The ring $\mathbb{R}[M]$ of all polynomials on $M$ is contained in the Hilbert space $L^2(M, e^{-|x|^2} d\mu)$;
2. all functions $e^{\alpha|x|^2}$ for $\alpha < 1/2$ belong to $L^2(M, e^{-|x|^2} d\mu)$.

Proof. Throughout this article, denote the distance of the point $x \in \mathbb{R}^n$ to the origin by $r^2 = |x|^2$. We shall prove that the integrals

$$I_m(M) := \int_M r^m e^{-r^2} d\mu < \infty, \quad m = 1, 2, \ldots$$

are finite. However,

$$I_m(M) = \sum_{j=0}^{\infty} \int_{M \cap (B_{j+1} - B_j)} r^m e^{-r^2} d\mu$$
and consequently we can estimate $I_m(M)$ as follows:

$$I_m(M) \leq \sum_{j=0}^{\infty} (j+1)^m e^{-j} \left[ \text{vol}(M \cap B_{j+1}) - \text{vol}(M \cap B_j) \right] \leq \sum_{r=0}^{\infty} (r+1)^m e^{-r^2} \text{vol}(M \cap B_{r+1}).$$

Using the assumption on the volume growth of $M$ we immediately obtain

$$I_m(M) \leq C \cdot \sum_{r=0}^{\infty} (r+1)^{m+\frac{1}{2}} e^{-r^2}.$$

Denoting the summands of the latter series by $a_r$, we readily see that it converges, since

$$\frac{a_{r+1}}{a_r} = \frac{(r+1)^{m+\frac{1}{2}} e^{-r^2 - 2r - 1}}{(r)^{m+\frac{1}{2}} e^{-r^2}} = \left( \frac{r+1}{r} \right)^{m+\frac{1}{2}} \frac{1}{e^{2r+1}} \to 0.$$

A similar calculation yields the result for the functions $e^{\alpha x^2}$ with $\alpha < 1/2$.

3. A Dense Subspace in $C_0^0(S^n)$

The aim of this section is to verify that a certain linear subspace of $C_0^0(S^n)$ is dense therein. Since the family of functions we have in mind cannot be made into an algebra, we have to replace the standard Stone-Weierstrass argument by something different. The idea for overcoming this problem is to use a combination of the well-known theorems of Hahn-Banach, Riesz and Bochner.

To begin with, we uniformly approximate the function $e^{-r^2} e^{i(k,x)}$ for a fixed vector $k \in \mathbb{R}^n$.

**Lemma.** Denote by $p_m(x)$ the polynomial

$$p_m(x) = \sum_{\alpha=0}^{m-1} i^\alpha \langle k, x \rangle^\alpha / \alpha!.$$

Then the sequence $e^{-r^2} p_m(x)$ converges uniformly to $e^{-r^2} e^{i(k,x)}$ on $\mathbb{R}^n$.

**Proof.** The inequality

$$|p_m(x) - e^{i(k,x)}| \leq \frac{\|k\| \|x\|^{m}}{m!} e^{\|k\| \|x\|}$$

implies (set $y = \|k\| \cdot \|x\|$)

$$\sup_{x \in \mathbb{R}^n} |e^{-r^2} p_m(x) - e^{-r^2} e^{i(k,x)}| \leq \sup_{0 \leq y \leq m!} \frac{y^m}{m!} e^{y^2 / \|k\|^2} =: C_m.$$

Therefore, we have to check that for any fixed vector $k \in \mathbb{R}^n$ the sequence $C_m$ tends to zero as $m \to \infty$. For simplicity, denote by $k$ the length of the vector $k \in \mathbb{R}^n$. A direct calculation yields the following formula:

$$C_m = \frac{1}{m!} \left( \frac{k^2}{4} + \frac{k}{4} \sqrt{k^2 + 8m} \right)^m \exp \left( \frac{k^2}{4} + \frac{k}{4} \sqrt{k^2 + 8m} - \frac{1}{k^2} \left( \frac{k^2}{4} + \frac{k}{4} \sqrt{k^2 + 8m} \right)^2 \right).$$

We are only interested in the asymptotics of $C_m$. We will thus ignore all constant factors not depending on $m$. In this sense, we obtain

$$C_m \approx \frac{1}{m!} \left( \frac{k^2}{4} + \frac{k}{4} \sqrt{k^2 + 8m} \right)^m \exp \left( \frac{k}{8} \sqrt{k^2 + 8m} - \frac{k^2 + 8m}{16} \right).$$

The Stirling formula $m! \approx \sqrt{2\pi m} m^m e^{-m}$ allows us to rewrite the asymptotics of $C_m$:

$$C_m \approx \frac{1}{\sqrt{2\pi m} m^m} \left( \frac{k^2}{4} + \frac{k}{4} \sqrt{k^2 + 8m} \right)^m \exp \left( \frac{k}{8} \sqrt{k^2 + 8m} - m + \frac{m}{2} \right).$$

Since

$$\lim_{m \to \infty} \left( \sqrt{k^2 + 8m} - \sqrt{8m} \right) = 0,$$
we can furthermore replace $\sqrt{k^2 + 8m}$ by $2\sqrt{2m}$:

$$C_m \approx \frac{1}{\sqrt{m} m^m} \left( \frac{k^2}{4} + \frac{k}{4} \sqrt{k^2 + 8m} \right)^m \exp \left( \frac{k}{4} \sqrt{2m} + \frac{m}{2} \right) =: e^{C_m^*}$$

with

$$C_m^* = m \ln \left( \frac{k^2}{4} + \frac{k}{4} \sqrt{k^2 + 8m} \right) + \frac{k}{2\sqrt{2}} \sqrt{m} + \frac{m}{2} - m \ln(m) - \frac{1}{2} \ln(m).$$

In case $m$ is sufficiently large with respect to $k$, we can estimate $\ln(k^2/4 + k/4 \cdot \sqrt{k^2 + 8m})$ by $\frac{1}{2} \ln(m) + \alpha$ for some constant $\alpha$:

$$C_m^* \lesssim \frac{m}{2} \ln(m) + \alpha m + \frac{k}{2\sqrt{2}} \sqrt{m} + \frac{m}{2} - m \ln(m) - \frac{1}{2} \ln(m)$$

$$\leq - \frac{m}{2} \ln(m) + (\alpha + 1/2)m + \frac{k}{2\sqrt{2}} \sqrt{m}$$

$$\leq - \frac{m}{2} \ln(m) + (\alpha + 1/2) + \frac{k}{2\sqrt{2}} m$$

$$= m \left( \alpha + 1/2 + \frac{k}{2\sqrt{2}} e^{-\frac{1}{2} \ln(m)} \right).$$

Finally, $C_m = \exp(C_m^*)$ converges to zero.

**Proposition 2.** Denote by $\mathcal{P}(\mathbb{R}^n)$ the ring of all polynomials on $\mathbb{R}^n$. Then the linear space $\Sigma_\infty := \mathcal{P}(\mathbb{R}^n)e^{-r^2}$ is dense in the space $C^0_\infty(S^n)$ of all continuous functions on $S^n = \mathbb{R}^n \cup \{\infty\}$ vanishing at infinity.

**Proof.** Suppose the closure $\Sigma_\infty^\circ$ of the linear space $\Sigma_\infty$ does not coincide with $C^0_\infty(S^n)$. Then the Hahn-Banach Theorem implies the existence of a linear continuous functional $L : C^0(S^n) \to \mathbb{R}$ such that

1. $L|_{\Sigma_\infty} = 0$;
2. $L(g_0) \neq 0$ for at least one $g_0 \in C^0_\infty(S^n)$.

According to Riesz’ Theorem (see [Rud, Ch.6, p.129 ff.]), $L$ may be represented by two regular Borel measures $\mu_+, \mu_-$ on $S^n$:

$$L(f) = \int_{S^n} f(x) \, d\mu_+(x) - \int_{S^n} f(x) \, d\mu_-(x).$$

In particular, $\mu_+$ and $\mu_-$ are finite. The first property $L|_{\Sigma_\infty} = 0$ of $L$ implies

$$\int_{S^n} e^{-r^2} p(x) \, d\mu_+(x) = \int_{S^n} e^{-r^2} p(x) \, d\mu_-(x)$$

for any polynomial $p(x)$. Let us introduce the measures $\nu_\pm = e^{-r^2} \mu_\pm$ on the subset $\mathbb{R}^n \subset S^n$. Then

$$\int_{\mathbb{R}^n} p(x) \, d\nu_+(x) = \int_{\mathbb{R}^n} p(x) \, d\nu_-(x)$$

holds and remains true for any complex-valued polynomial. We may thus choose $p(x) = p_m(x)$ as in the previous lemma

$$p_m(x) = \sum_{\alpha=0}^{m-1} \frac{i^\alpha \langle k, x \rangle^\alpha}{\alpha!}.$$ 

But, then

$$\int_{S^n} p_m(x) e^{-r^2} \, d\mu_+(x) = \int_{\mathbb{R}^n} p_m(x) \, d\nu_+(x) = \int_{\mathbb{R}^n} p_m(x) \, d\nu_-(x) = \int_{S^n} p_m(x) e^{-r^2} \, d\mu_-(x)$$
together with the uniform convergence of $p_m(x)e^{-r^2}$ to $e^{i(k,x)e^{-r^2}}$ implies
\[ \int_{\mathbb{R}^n} e^{i(k,x)e^{-r^2}} d\nu_+(x) = \int_{\mathbb{R}^n} e^{i(k,x)e^{-r^2}} d\nu_-(x), \]
i.e.,
\[ \int_{\mathbb{R}^n} e^{i(k,x)} d\nu_+(x) = \int_{\mathbb{R}^n} e^{i(k,x)} d\nu_-(x). \]
Therefore, the Fourier transforms of the measures $\nu_+$ and $\nu_-$ coincide. Consequently, by Bochner’s Theorem (see [Mau, Ch.XIX, p.774 ff.]) we conclude that $\nu_+ = \nu_-$ on $\mathbb{R}^n$. The linear functional $L : C^0(S^n) \to \mathbb{R}$ must thus be the evaluation of a function at infinity:
\[ L(f) = c \cdot f(\infty), \]
a contradiction to the existence of a function $g_0 \in C^0_0(S^n)$ satisfying $L(g_0) \neq 0$.

4. The main result

**Theorem 1.** Let the closed subset $M \subset \mathbb{R}^n$ be a smooth submanifold satisfying the polynomial volume growth condition. Then the ring $\mathbb{R}[M]$ of all polynomials on $M$ is a dense subspace of the Hilbert space $L^2(M, e^{-r^2} d\mu)$.

**Proof.** Consider the one-point-compactification $\hat{M} \subset S^n$ of $M \subset \mathbb{R}^n$. Then Proposition 2 of Section 3 implies that
\[ \Sigma_{\infty}(\hat{M}) := \mathbb{R}[M] \cdot e^{-r^2/4} \]
is dense in $C^0_{\infty}(\hat{M})$. We introduce the measure $d\nu = e^{-r^2/2} d\mu$, where $d\mu$ is the volume form of $M$. Since
\[ \int_M d\nu = \int_M e^{-r^2/2} d\mu = \int_M (e^{r^2/4})^2 e^{-r^2} d\mu =: V < \infty, \]
$d\nu$ defines a regular Borel measure $d\hat{\nu}$ on $\hat{M}$ (by setting $d\hat{\nu}(\infty) = 0$). Therefore, the algebra $C^0_{\infty}(\hat{M})$ of all continuous functions on $\hat{M}$ vanishing at infinity is dense in $L^2(\hat{M}, d\hat{\nu})$:
\[ \overline{C^0_{\infty}(\hat{M})} = L^2(\hat{M}, d\hat{\nu}). \]
For any function $f$ in $L^2(M, e^{-r^2} d\mu)$ we have
\[ \int_M |f e^{-r^2/4}|^2 e^{-r^2/2} d\mu = \int_M |f|^2 e^{-r^2} d\mu < \infty \]
and, therefore, $f e^{-r^2/4}$ lies in $L^2(\hat{M}, d\hat{\nu})$. Thus, for a fixed $\varepsilon > 0$, there exists a function $g \in C^0_{\infty}(\hat{M})$ such that
\[ \int_M |f e^{-r^2/4} - g(x)|^2 e^{-r^2/2} d\mu < \varepsilon/2. \]
According to Proposition 2 we can find a polynomial $p(x) \in \mathbb{R}[M]$ approximating $g$:
\[ \sup_{x \in \hat{M}} |g(x) - p(x)|^2 e^{-r^2/4} < \varepsilon/2V. \]
Using the inequality $\|x + y\|^2 \leq 2\|x\|^2 + 2\|y\|^2$ we conclude
\[ \int_M |f(x) e^{-r^2/4} - p(x) e^{-r^2/4}|^2 e^{-r^2/2} d\mu < \varepsilon; \]
but this is equivalent to
\[ \int_M |f(x) - p(x)|^2 e^{-r^2} d\mu < \varepsilon. \]
\[ \square \]
5. Examples and final remarks

We shall give a few simple examples. Notice that we recover, of course, that the polynomials are dense in $L^2(\mathbb{R}^n, e^{-r^2} \, d\mu)$ (Hermite polynomials) or in $L^2(M, d\mu)$ for any compact submanifold (Legendre polynomials in case $M = [-1, 1]$).

Example 1. Consider a revolution surface in $\mathbb{R}^3$ defined by two polynomials $f, h$

\[
\begin{align*}
x &= f(u_1) \cos u_2 \\
y &= f(u_1) \sin u_2 \\
z &= h(u_1)
\end{align*}
\]

Then we have $d\mu = f \sqrt{f'^2 + h'^2} \, du_1 du_2$ and $r^2 = f^2 + h^2$, and thus obtain

$$\mathbb{R}[f \cos u_2, f \sin u_2, h] \text{ dense in } L^2(\mathbb{R} \times [0, 2\pi], e^{-(f^2 + h^2)} f \sqrt{f'^2 + h'^2} \, du_1 du_2).$$

In the special case of a cylinder, i.e. $f = 1$, $h = u_1$, this reduces to the well known fact that the ring

$$\mathbb{R}[u_1, \cos u_2, \sin u_2] = \mathbb{R}[u_1] \otimes \mathbb{R}[\cos u_2, \sin u_2]$$

is indeed dense in the Hilbert space

$$L^2(\mathbb{R} \times [0, 2\pi], e^{-u_1^2} \, du_1 du_2) = L^2(\mathbb{R}, e^{-u_1^2} \, du_1) \otimes L^2([0, 2\pi], du_2).$$

Example 2. Let $F : \mathbb{C} \to \mathbb{C}$ be a polynomial and consider the surface defined by $f : \mathbb{C} \to \mathbb{R}^3$, $f(z) = (x, y, |F(z)|)$, $z = x + iy$.

Then one checks that $d\mu = \sqrt{1 + |F|^2} \, |dz|^2$ and $r^2 = |z|^2 + |F(z)|^2$. Thus the following holds:

$$\mathbb{R}[x, y, |F(z)|] = L^2(\mathbb{R}^2, e^{-(|z|^2 + |F(z)|^2)} \sqrt{1 + |F|^2} \, |dz|^2).$$

Let us study the polynomial $F = z^{2k}$ in more detail. Here the coordinate ring coincides with the usual polynomial ring $\mathbb{R}[x, y]$ in two variables, and thus we have proved that these are dense in

$$L^2(\mathbb{R}^2, e^{-(|z|^2 + |z|^4)} \sqrt{1 + 4k^2 |z|^{2(2k-1)} |dz|^2}).$$

Example 3. We finish with a one-dimensional example: the graph $M = \{(x, f(x))\}$ of a polynomial $f : \mathbb{R} \to \mathbb{R}^n$. Then $d\mu = \sqrt{1 + \|f\|^2} \, dx$, and we obtain

$$\mathbb{R}[x] = L^2(\mathbb{R}, e^{-(x^2 + \|f(x)\|^2)} \sqrt{1 + \|f\|^2} \, dx).$$

Remark. The main result raises an interesting analogous problem in complex analysis which, to our knowledge, is still open. It is well known that the polynomials on $\mathbb{C}^n$ are dense in the Fock- or Bergman space

$$\mathcal{F}(\mathbb{C}^n) := \{ f \in L^2(\mathbb{C}^n, e^{-r^2} \, d\mu) \mid f \text{ holomorphic } \}.$$

Furthermore, a theorem by Stoll (see [Sto1], [Sto2]) states that from all complex analytic submanifolds $N$ of $\mathbb{C}^n$, those with polynomial growth are exactly the algebraic ones, and thus the only ones for which the elements of the coordinate ring are square-integrable with respect to the Gaussian measure. It is then common to study the space

$$\mathcal{F}(N) := \{ f \in L^2(N, e^{-r^2} \, d\mu) \mid f \text{ holomorphic } \},$$

but we were not able to find any results on whether $\mathbb{C}[N]$ is dense herein.

More elaborate applications of the main result to the situation where $M$ carries a reductive algebraic group action will be discussed by the authors in some forthcoming works (see e.g. [Agl]). In this case, one can decompose the ring $\mathbb{R}[M]$ into isotypic components and, via Theorem 1, one obtains a decomposition of $L^2(M, e^{-r^2} \, d\mu)$ analogous to the classical Frobenius reciprocity.
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