Signal processing of simplicial complexes

S N Chukanov
Sobolev Institute of Mathematics of the Siberian Branch of the Russian Academy of Sciences, Omsk branch, Pevtsova, 13, Omsk, 644043, Russia
ch_sn@mail.ru

Abstract. The paper considered the signal processing of simplicial complexes. Hodge decomposition formula for discrete fields is given, which is similar to the Hodge decomposition formula for smooth vector fields. The construction and the estimation of the gradient, divergence and curl operators and Laplace matrices for discrete vector fields are considered.

Keywords: simplicial complex, Hodge decomposition, discrete fields, gradient operator, divergence operator, Laplace matrix

1. Introduction
The signal processing has been developed for signals defined in metric space (time or space). However, there are signals that cannot be defined in metric space. Signal processing on graphs emerged as a structure for analyzing signals defined by the vertices or edges of a graph [1, 2, 3]. However, graph representations are not always able to obtain all the information present in interconnected systems. Complex interactions between the constituent elements of the system cannot be reduced to paired interactions.

To work with complex interacting systems, it is useful to define sets of elements \( V \) together with an ensemble of relations represented in \( S \), containing subsets of different cardinality of elements \( V \). The structure \( \mathcal{H}(V, S) \) is known as a hypergraph. The class of hypergraphs is represented by simplicial complexes, the defining feature of which is the inclusion property: if \( A \) belong \( S \), then all subsets \( A \) also belong \( S \). The construction of a representation based on a simplicial complex is a direct generalization of a representation based on graphs.

The paper considered the signal processing of simplicial complexes. Hodge decomposition formula for discrete fields is given, which is similar to the Hodge decomposition formula for smooth vector fields. The construction and the estimation of the gradient, divergence and curl operators and Laplace matrices for discrete vector fields are considered.

2. The methods of algebraic topology
Let a finite set \( \{v_0,\ldots,v_{N-1}\} \) of points (vertices) be given; \( k \)-simplex \( \sigma^i \) is an unordered set of points \( \{v_0,\ldots,v_k\}; v_i \neq v_j \) with \( 0 \leq i, j \leq N-1; i \neq j \). A face of a \( k \)-simplex \( \sigma^i \) is a \((k-1)\)-simplex \( \{v_0,\ldots,v_{i-1},v_{j+1},\ldots,v_k\}; 0 \leq j < k \). An abstract simplicial complex \( X \) is a finite set of simplices closed with respect to the inclusion of faces \( \sigma_i \in X \), that is, if \( \sigma_i \in X \), then all faces also
belong $X$. $k$-simplex is the convex hull of $k+1$ affine independent points (vertices). A point is zero-simplex, line segment is a one-simplex, triangle is a two-simplex, etc.

Each simplex can have only two orientations. Two orientations are equivalent if each of them can be obtained from another by an even number of transpositions, where each transposition is determined by a permutation of two elements. $k$-simplex $\sigma^k = \{v_0, v_1, \ldots, v_K\}$ of the order $k$ together with the orientation is an oriented $k$-simplex and is denoted by $[v_0, v_1, \ldots, v_K]$. The oriented $k$-simplex $\sigma^k_1 = \{v_0, v_1, \ldots, v_i, v_{i+1}, \ldots, v_K\}$ is denoted by $[v_0, v_1, \ldots, v_K]$. Two simplices $\sigma^k_i, \sigma^k_j \in X$ are upper adjacent in $X$ if both are faces of a simplex of order $k+1$. Two simplices $\sigma^k_i, \sigma^k_j \in X$ are lower adjacent in $X$ if both have a common face of order $k-1$ in $X$.

The notation $\sigma^{k-1}_i \subset \sigma^k_i$ indicates that $\sigma^{k-1}_i$ is a boundary element for $\sigma^k_i$. For a given simplex $\sigma^k_i \subset \sigma^k_j$, the notation $\sigma^{k-1}_i \sim \sigma^k_j$ to indicate that the orientation of $\sigma^{k-1}_i$ coincides with the orientation of $\sigma^k_j$; $\sigma^{k-1}_i \sim \sigma^k_j$ indicates that the two orientations do not coincide.

$C_k(X, \mathbb{R})$ denotes the vector space obtained by a linear combination with real coefficients of a set of oriented $k$-simplices $X$. In algebraic topology, elements $C_k(X, \mathbb{R})$ are called $k$-chains (see Appendix 1). If $\{\sigma^k_1, \ldots, \sigma^k_{n_k}\}$ is $n_k$ $k$-simplices in $X$, the $k$-chain $\tau_k$ can be written as

$$\tau_k = \sum_{i=1}^{n_k} \alpha_i \sigma^k_i.$$ Then, taking into account the basis $\{\sigma^k_1, \ldots, \sigma^k_{n_k}\}$, the chain $\tau_k$ can be represented by a vector of its expansion coefficients $(\alpha_1, \ldots, \alpha_{n_k})$.

An ordered $k$-chain $[v_0, \ldots, v_k]$ boundary is a linear mapping (boundary operator)

$$\partial_k : C_k(X, \mathbb{R}) \to C_{k-1}(X, \mathbb{R}),$$
defined as $\partial_k [v_0, \ldots, v_k] = \sum_{j=0}^{k} (-1)^j [v_0, \ldots, \hat{v}_j, \ldots, v_k]$. The structure of a simplicial complex $X$ of dimension $k$ ($k$-simplicial complex) is described using its incidence matrices $B_k$, $k = 1, \ldots, K$. Taking into account the orientation of the simplicial complex $X$, the elements of the incidence matrix $B_k$ determine which $k$-simplices are incident to $(k-1)$-simplices. Then $B_k$ – matrix representation of boundary operator. Formally, its entries are defined as follows:

$$B_k(i, j) = \begin{cases} 0, & \text{if } \sigma^{k-1}_i \subset \sigma^k_j, \\ 1, & \text{if } \sigma^{k-1}_i \subset \sigma^k_j \text{ and } \sigma^{k-1}_i \sim \sigma^k_j, \\ -1, & \text{if } \sigma^{k-1}_i \subset \sigma^k_j \text{ and } \sigma^{k-1}_i \sim \sigma^k_j. \end{cases}$$

The property that the border of the border is zero is converted to the following matrix form $B_{k+1}B_k = 0$.

The structure of the $K$-simplicial complex is fully described by its combinatorial Laplace matrices of order $k = 0, \ldots, K$ [4]:

$$L_0 = B_0 B_0^T,$$

$$L_k = B_k B_k + B_{k+1}B_{k+1}^T, k = 1, \ldots, K - 1,$$

$$L_K = B_K B_K^T.$$ The characteristic polynomial of the Laplace matrix can be obtained from the relation:

$$\det \left( L_k - x \cdot \text{diag} \left( \begin{array}{cccc} 1 & 1 & \cdots & 1 \\ \end{array} \right) \right) = a_n x^n + a_{n-1} x^{n-1} + a_1 x + a_0.$$
3. Analysis of signals determined on the simplicial complex.

Consider the analysis of signals defined on the simplicial complex. For a set \( \mathcal{S}_k \), we define a signal as a mapping of real values on elements of \( \mathcal{S}_k \) in the form: \( f_k : \mathcal{S}_k \rightarrow \mathbb{R} \); \( k = 0, 1, \ldots \). Let's consider \( \mathcal{S}_2 = \{V, E, T\} \) as the set of vertex \( V \), the set of edges \( E \) and the set of triangles \( T \) dimension \( V, E, T \), respectively. Let us denote the associated simplicial complex as \( \mathcal{X}(V, E, T) \). Signals in a complex of order \( k = 0, 1, 2 \) are defined as mappings: \( s^0 : V \rightarrow \mathbb{R}^V, s^1 : E \rightarrow \mathbb{R}^E, s^2 : T \rightarrow \mathbb{R}^T \).

3.1. Hodge decomposition for smooth vector fields

For a smooth three-dimensional vector field \( \xi(x) \) defined in the region \( T \), there is a unique expansion [5]:
\[
\xi = \nabla u + \nabla \times v + h,
\]
where \( u(x) \) is the scalar potential field: \( \nabla \times (\nabla u) = 0 \); \( v(x) \) - vector potential field: \( \nabla \cdot (\nabla \times v) = 0 \); \( h(x) \) - harmonic vector field: \( \nabla h = 0, \nabla \times h = 0 \). \( \nabla \) is the gradient operator, \( \nabla \cdot = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \) is the divergence operator, and \( \nabla \times \) is the rotor operator. In this case, \( \nabla u \) must be normal to the border \( \partial T \), and \( \nabla \times v \) must be tangent to it. \( \nabla u \) is called irrotational and \( \nabla \times v \) divergence-free component. A scalar field satisfying (1) can be defined as a minimizing functional:
\[
F(u) = \int_T (\nabla u - \xi)^2 \, dV \Rightarrow u^* = \arg\min_u \int_T (\nabla u - \xi)^2 \, dV.
\]

A vector field \( v(x) \) satisfying (1) can be defined as minimizing functional:
\[
G(v) = \int_T (\nabla \times v - \xi)^2 \, dV \Rightarrow v^* = \arg\min_v \int_T (\nabla \times v - \xi)^2 \, dV.
\]

3.2. Hodge decomposition for discrete fields

For a signal \( s^k \) of order \( k \), there are three signals \( s^{k-1}, s_H^k, s^{k+1} \) of orders \( k-1, k, k+1 \), respectively, such that \( s^k \) can be expressed as the sum (HD):
\[
s^k = B_{s^{k-1}}^k + s_H^k + B_{s^{k+1}}^k s^{k+1}.
\]

This Hodge decomposition is an extension of Hodge theory for differential forms to simplicial complexes. The subspace \( \text{ker}(L_k) \) is called a harmonic subspace; \( s_H^k \in \text{ker}(L_k) \) is a solution to the discrete Laplace equation:
\[
L_k s_H^k = 0.
\]

Decomposition shows the interaction between signals of different orders. Let's consider the case \( k = 1 \) for simplicity. Let us introduce the rotor and divergence operators by analogy with continuous analogs applied to vector fields. For a given discrete signal \( s^1 \), the operator rot is defined as
\[
\text{rot}(s^1) = B_{s^1}^1 s^1.
\]

This operator maps a signal \( s^1 \) to a signal defined over the sets of triangles in \( \mathbb{R}^T \); the common \( i \)-th element \( \text{rot}(s^1) \) is a measure of the flow circulating along the edges of \( i \)-th triangle.
Example 1. As an example of a simplicial complex [6] in Fig. 1 we have $B^I = [1 1 1 0 0 0 0 0 0]$, and, defining $s^0 = [v_1, v_2, v_3, v_4, v_5, v_6]^T$, $s^1 = \{e_{12}, e_{23}, e_{31}, e_{25}, e_{62}, e_{36}, e_{43}, e_{14}\}$, $s^2 = [t_{123}]$, we get: $\text{rot}(s^1) = [e_{12} + e_{23} + e_{31}]$.

The entry $s^1$ is a circulation along the corresponding triangle. The discrete divergence operator maps a signal $s^i$ to a signal defined in the vertex space $V$ and is defined as $\text{div}(s^i) = B^i s^i$. It turns out that the $i$-th entry $\text{div}(s^i)$ represents the flow passing through the $i$-th vertex. Thus, non-zero divergence indicates the presence of a source or sink.

For example, in Fig. 1 we get: $B_i = \begin{bmatrix} -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \end{bmatrix}$, so:

$\text{div}(s^1) = \begin{bmatrix} -e_{12} + e_{31} - e_{14} - e_{23} - e_{25} + e_{62} + e_{25} - e_{31} - e_{36} + e_{14} - e_{43} + e_{14} - e_{51} + e_{25} - e_{62} + e_{36} \end{bmatrix}$. Laplace matrices are of the form: $L_0 = B_0 B^I_0$, $L_4 = B_4 B^I_4 + B_3 B^I_3$.

In Matlab notation:

$B_1 = \begin{bmatrix} -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \end{bmatrix}$, $L_0 = B_1 B^I_0$; therefore:

$\begin{bmatrix} 4 & -1 & -1 & -1 & 0 \\ -1 & 4 & -1 & 0 & -1 \\ -1 & -1 & 4 & -1 & 0 \\ -1 & 0 & -1 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix}$.
Characteristic polynomials of Laplace matrices:
\[ \text{charpoly}(L_0, x) = x^6 - 18 \cdot x^4 + 123 \cdot x^4 - 394 \cdot x^3 + 585 \cdot x^2 - 324 \cdot x; \]
\[ \lambda(L_0) = [0 \ 5.303 \ 5.303 \ 4 \ 1.697 \ 1.697]. \]
\[ \text{charpoly}(L_1, x) = x^9 - 21 \cdot x^7 + 177 \cdot x^5 - 763 \cdot x^7 + 1767 \cdot x^5 - 2079 \cdot x^4 + 972 \cdot x^3; \]
\[ \lambda(L_1) = [0 \ 0 \ 5.303 \ 5.303 \ 4 \ 3 \ 1.697 \ 1.697]. \]

If consider equation (1) in case \( k = 1 \), \( s^1 = B_1 \tilde{s}^0 + s^1_B + B_2 s^2 \), remembering that \( B_B = 0 \); it is possible to check that \( B_1 \tilde{s}^0 \) has zero rotor, and then it can be called the irrotational component, while \( B_2 s^2 \) has a zero divergence value, and then it can be called the divergence-free component by analogy with the calculus terminology used for vector fields. The harmonic component \( s^1_B \) is a flow vector that is irrotational and divergence-free. Also notice that \( B_1 \tilde{s}^0 \) is (discrete) gradient \( s^0 \) represents.

4. Estimation of discrete vector fields
The tangent vector fields can be approximated using one vector per face in triangulation. Another approach to representing tangent vector fields on triangular grids is to use edges as local directions relative to which the vector field can be encoded. The creation of vector fields is greatly simplified due to the relationship between the edge coefficients and the resulting vector field. The edge-based representation of vector fields on triangular grids is based on Cartan's theory of differential forms. In particular, the discretization of the concept of differential forms, called cochains, was introduced in the topology [7] as a finite-dimensional approximation space for differential \( k \)-forms. Since differential 1-forms are uniquely identified with tangent vector fields (and vice versa) via operators flat (\( \flat \)) and sharp (\( \sharp \)), we can use this concept to encode vector fields as well.

A key component of representing tangent vector fields on grids is the use of linear integration. A linear integral is an integral of some function along a given curve. Integration of a scalar function along a smooth curve is known. The linear integral \( \int_C u \cdot ds \) of the vector field \( u \) over the oriented curve \( C \) sums the components of the vector field tangent to the curve. We call the linear integral \( \int_C u \cdot ds \) circulation \( u \) along \( C \).

If we assume that a triangular grid is an approximation of a smooth surface, then an edge is a curve on this surface, and the union of these curves (edges) forms a simplicial decomposition of the surface. Therefore, the vector field \( u \) can be turned into a set of scalar values by calculating the circulation along each (oriented) edge of the grid. The knowledge of the circulation of the vector field on each edge is not so accurate: each circulation does not locally know about the normal component of the vector field along the edge. However, circulation on nearby ribs gives a good approximation to the component \( u \). In fact, the aggregation of all circulations makes it possible to reconstruct the vector.
field, which will correspond to the correct circulations. Circulating sampling is a way to turn a vector field into a finite-dimensional array of scalar values on edges. Circulation values on edges are part of a more general notion of geometric sampling based on discrete differential forms [8]. In this approach, differential forms are represented by measurements on grid cells; a discrete 0-form represents a scalar function in terms of its values at vertices (0-dimensional cells), and a discrete 2-form represents density in terms of the area integral over triangles (2-dimensional cells), discrete The 1-form represents the tangent vector field in terms of its linear integral over the edges. We can reconstruct continuous fields using finite element interpolation and calculate the derivatives from these discrete representations; divergence and rotor of discrete vector fields can be defined as discrete forms - linear combinations of the surrounding edge values.

DEC (Discrete Exterior Calculus) defines discrete differential $k$ -forms on triangular grids and expresses the corresponding operators such as divergence, rotor, gradient, and Laplacian as matrices acting on coordinateless coefficients defined on vertices, edges, and triangles.

### 4.1. From continuous to discrete vector fields

The transformation of a continuous $k$ -form into a set of values in $k$ -cells can be performed by integration. Discrete $k$ -forms are specified as scalars on $k$ -cells, which are discrete measurements of $c_i = \omega \left( \{p_i\}, c_{ij} = \int c_{ijk} \right).$ Here $\omega$ is a continuous $k$ -form, $c_i$ is a scalar function at position $p_i$ of the vertex $v_i$; $c_{ij}$ is linear integral of the vector field $(\omega v)^T$ $(k=1)$ along the segment $p_{ij}$ belonging to the edge $e_{ij}$; $c_{ijk}$ is integral of the areas of the density $(k=1)$ over $p_{ijk}$ belonging to the triangle $t_{ijk}.$ We treat these coefficients as arrays $c_i, c_{ij}, c_{ijk}$. You can convert an arbitrary vector field to an edge-based vector field using dot products: $\int_{p_{ij}} u \cdot ds = (u + u_i) p_{ij}.

It is necessary to find an algorithm for converting an array of values $c_e$ into a representable vector field. Discrete $k$ -forms can be interpolated using Whitney elements (see Appendix 3). For 0-forms, these are piecewise linear hat functions $\varphi^e = \{\varphi | v_i \in V\}.$ For discrete 2-forms, Whitney elements are constant functions $\phi = \{\varphi_{ijk} | t_{ijk} \in T\}$ supported on separate triangles.

The corresponding interpolators of 1-form $\varphi^e = \{\varphi_{ij} | e_{ij} \in E\}$ can be defined as $\phi_e(p) = \phi_1(p) d\phi_y - \phi_0(p) d\phi_x.$ Since for the Euclidean metric $(d\phi)^2 = \nabla \phi \cdot \nabla \phi,$ you can rewrite the basic Whitney function as a local vector field: $\phi_e(p) = \phi_1(p) \nabla \phi_y - \phi_0(p) \nabla \phi_x.$ The basic edge functions $\phi_e$ are interpolators for discrete 1-form data: the linear integral $\phi_e$ is 1 over $e_{ij}$ and 0 over all other edges. Consequently, from the discrete 1-form given in terms of the coefficients $c_{ij}$, it is possible to form a piecewise linear vector field: $u = \sum_{e_{ij} \in E} C_{ij} \phi_{ij}.$

### 4.2. Estimating flows on edges

The optimal estimate of the signals $s^{k-1} \in \mathbb{R}^{d_k}, s^{k-1} \in \mathbb{R}^{d_2}, s_{H}^{k-1} \in \mathbb{R}^{d_{k-1}}$ can be formulated as a solution to the problem of finding the $s^{k-1}, s^{k-1}, s_{H}^{k-1}$ that ensure the minimum of the expression $\| B_{s^{k-1}} s^{k-1} + B_{s^{k-1}} s_{H}^{k-1} - x_1 \| \rightarrow \min,$ under the conditions $B_{s^{k-1}} s_{H}^{k-1} = 0; B_{s^{k-1}} s_{H}^{k-1} = 0.$

When $k = 1$ (estimation on edges) the observed vector can be represented as: $x_1 = B_{s^{k-1}} s^0 + B_{s^0} s^0 + v^1,$ where $v^1$ is a noise. Suppose the noise vector is Gaussian and all zero mean elements have the same variance $\sigma^2.$ The optimal estimate of the signals
The inner product \( s^0, s^2, s^1_h \) can be formulated as a solution to the problem of finding \( s^0, s^2, s^1_h \) that ensure the minimum of the expression \( \| B_s^T s^2 + B_s^T s^0 + s^1_h - x^1 \|^2 \to \min \), under the conditions \( B_s s^1_h = 0; B_s^T s^1_h = 0 \).

The Lagrange function is:

\[
\mathcal{L}(s^0, s^2, s^1_h, \lambda_1, \lambda_2) = (B_1 s^2 + B_1^T s^0 + s^1_h - x^1)^T (B_2 s^2 + B_2^T s^0 + s^1_h - x^1) + \lambda_1 s^1_h + \lambda_2 (B^T s^1_h). 
\]

The optimal estimates for \( s^0, s^2 \) coincide with the solutions of the following problems:

\[
\hat{s}^0 = \arg\min_{\hat{s} \in \mathbb{R}^T} \| B_1^T \hat{s}^0 - x^1 \|^2; \quad \hat{s}^2 = \arg\min_{\hat{s} \in \mathbb{R}^T} \| B_2 \hat{s}^2 - x^1 \|^2.
\]

For estimation \( s^1_h \), we construct a Luenberger observer in the form:

\[
\hat{s}^1_h(t+1) = \hat{s}^1_h(t) + K \cdot (\hat{x}^1(t) - B_1^T \hat{s}^0(t) - B_2 \hat{s}^2(t)),
\]

where \( t \) is the discrete time of the observer; \( K \) - the matrix of gain of the observer, which ensures the stability of the observation process.

In the general case \( k \geq 1 \), \( k \in \mathbb{Z}^+ \), optimal solutions \( x^{k-1}, x^{k+1} \) can be found from the relations:

\[
\hat{x}^{k-1} = \arg\min_{x \in \mathbb{R}^T} \| B_{k-1}^T x^{k-1} - x^k \|^2; \quad \hat{x}^{k+1} = \arg\min_{x \in \mathbb{R}^T} \| B_{k+1} x^{k+1} - x^k \|^2.
\]

**Example 2.** In the case \( x^1 = [e_{12}, e_{23}, e_{31}, e_{51}, e_{62}, e_{42}, e_{43}, e_{14}]^T = [1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0]^T \), \( \hat{s}^0 = [v_1, v_2, v_3, v_4, v_5, v_6]^T \), \( \hat{s}^2 = [t_{23}] \) and matrices \( B_1, B_2 \) from Example 1 (see Fig. 1), we obtain:

\[
\hat{s}^0 = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T \quad \text{(divergence-free field)}; \quad \hat{s}^2 = [1], \quad \hat{s}^1 = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T.
\]

**Appendix 1. Chains and cochains.**

For each \( k : C_k(X, \mathbb{R}) \) denotes the vector space obtained by a linear combination using the real coefficients of the set of \( k \)-simplices \( X \). In algebraic topology, elements \( C_k(X, \mathbb{R}) \) are called \( k \)-chains. If \( \{ \sigma_1^k, \ldots, \sigma_n^k \} \) are \( k \)-simplices in \( X \), the \( k \)-chain \( \tau_k \) can be written as \( \tau_k = \sum\limits_{i=1}^{n} \alpha_i \sigma_i^k \).

\( k \)-cochain \( \omega \) is dual \( k \)-chain, that is, \( \omega \) is a linear mapping that takes \( k \)-chains in \( \mathbb{R} \): \( \omega : C_k \to \mathbb{R} \); \( c \to \omega(c) \), \( k \)-cochain \( \omega \) works with \( k \)-chain \( c \) to get a scalar. \( k \)-cochain is calculated on each \( k \)-simplex of simplicial complex \( X \).

The numerical representation of the cochains follows from the dual representation of the chains. \( k \)-chain can be represented by a vector of length \( l \) – the number of \( k \)-simplices in \( X \).

The form \( \omega \) can be represented by a vector \( \omega^k \) of the same size as \( c_k \). \( \omega \) works with \( c \) to get a scalar in \( \mathbb{R} \). Linear operation \( \omega(c) \) becomes inner product \( \omega^k \cdot c_k \). More specifically, if \( c_k \) is represented as a column vector, such that a linear mapping \( \omega \) could be represented by the row vector \( (\omega^k)^T \), and \( \omega(c) \) becomes the multiplication of a row vector \( (\omega^k)^T \) by a column vector \( c_k \).

A continuous \( k \)-form is defined as a linear map of \( k \)-dimensional sets in \( \mathbb{R} \), since we can integrate a \( k \)-form only on \( k \)-(sub)manifolds. A linear mapping of a chain to a real number is called a cochain: thus, a cochain is a natural discrete analogue of a form. If we restrict the integration to only a \( k \)-
submanifold (a sum of $k$-simplices in a triangulation), then we get a $k$-cochain; that is, $k$-cochains are discretizations of $k$-forms

Let us expand the concept of calculating a differential form on an arbitrary circuit due to linearity:

$$
\omega = \sum_{i} c_{i} \omega_{i}.
$$

Integration on each $k$-simplex provides discretization or, in other words, a mapping of the $k$-form $\omega$ to the $k$-cochain, represented as:

$$
\omega = \sum_{i} c_{i} \omega_{i}.
$$

A $k$-form is exact if there is an $(k-1)$-form $\alpha$: $\omega = d\alpha$; a $k$-form is closed if $d\omega = 0$. An operator $d$ is the adjoint to boundary operator $\partial$. If we denote the sign of the integral as pairing:

$$
\omega = \int_{\sigma} \omega.
$$

Due to the linearity of the integration operation, the following expression for generalized Stokes theorem can be written:

$$
\int_{\partial \sigma} \omega = \sum_{i} c_{i} \int_{\sigma_{i}} \omega.
$$

Appendix 2. Homology and Cohomology Groups

Cycles and their equivalence classes. A cycle is a closed $k$-chain. On the set of all $k$-cycles one can define equivalence classes: we will say that a $k$-cycle is homologous to another $k$-cycle when these two chains differ by the boundary of the $(k+1)$-cycle.

Definition of homology groups. Let there be a series of $k$-chain spaces (chain complexes):

$$
C_{k} \rightarrow C_{k-1} \rightarrow \ldots \rightarrow C_{0}
$$

with $\partial \partial = \mathcal{O}$. Homology groups $\{\mathcal{H}_{k}\}_{k}$ of a chain complex are defined as factor-spaces: $\mathcal{H}_{k} = \text{Ker} \partial_{k} / \text{Im} \partial_{k+1}$. A dimension of $k$-th homology group is called the $k$-th Betti number: $\beta_{k} = |\mathcal{H}_{k}|$. The number $\sum_{k=0}^{\infty} (-1)^{k} \beta_{k}$ defines Euler’s characteristic.

Cohomology groups. One can take a formal definition of homology, replace all occurrences of chains with cochains $\delta \rightarrow d$, and change the direction of the operator between spaces: this will also define equivalence classes. Since cochains are dual to chains and $d$ conjugate to $\partial$, these equivalence classes define cohomology groups; the cohomology groups of a complex for a coboundary operator are quotient spaces $\text{Ker} \partial_{k} / \text{Im} \partial_{k-1}$.

Appendix 3. Interpolation of discrete forms [8]

$k$-cochains are discretization of $k$-forms. This representation of discrete shapes on circuits, although convenient in many applications, is not sufficient to fulfill certain requirements, such as obtaining a point-shaped value of $k$-form. Therefore, you can use interpolation of these nets to the rest of the space. These interpolation functions can be considered linear in the coordinates of the vertices.

Interpolation of 0-forms. We can linearly interpolate discrete 0-forms over the whole space: we can use the usual basis of vertex-based interpolation functions, which are called hat-functions. These basic functions will be denoted $\varphi_{i}$ for each vertex $v_{i}$. By definition, $\varphi_{i}$ satisfies: $\varphi_{i} = 1$ at $v_{i}$; $\varphi_{i} = 0$ at $v_{i} \neq v_{j}$, and linearly tends to zero in a one-ring neighborhood $v_{i}$. Using these basic functions, we can check that if we denote a vertex $v_{j}$ with a symbol $\sigma_{j}$, we get:

$$
\int_{v_{j}} \varphi_{i} = \int_{\sigma_{j}} \varphi_{i} = \begin{cases} 
1, & \text{if } i = j; \\
0, & \text{if } i \neq j.
\end{cases}
$$
These interpolating functions represent the basis of 0-cochains, which corresponds to the dual basis of 0-chains. 

**Interpolation of 1-forms.** Extend the previous interpolation technique to 1-shapes based on the Whitney [7] method, which associates with the edge $\sigma_{ij}$ between $v_i$ and $v_j$ the function: 

$$\varphi_{\sigma_{ij}} = \varphi_i d\varphi_j - \varphi_j d\varphi_i.$$ 

Direct computation can confirm that:

$$\int_{\sigma_{ij}} \varphi_{\sigma_{ij}} = \begin{cases} 
1, & \text{if } (i = j) \wedge (j = l), \\
-1, & \text{if } (i = l) \wedge (j = k), \\
0, & \text{else.}
\end{cases}$$

On the edge $\sigma_{ij}$ we have $\varphi_i + \varphi_j = 1$, therefore: 

$$\varphi_{\sigma_{ij}} = \int_{\sigma_{ij}} (\varphi_i d\varphi_j - (1 - \varphi_i) d\varphi_j) = \int_{\sigma_{ij}} (-d\varphi_j) = 1.$$ 

Thus, a correct basis for 1-cochains is defined. Basic functions of the 1-form can be extended to arbitrary $k$-simplices. Whitney’s $k$-forms are defined as:

$$\varphi_{\sigma_{ijkl}} = k! \sum_{j=0}^{k-1} (-1)^j \varphi_i d\varphi_{j0} \ldots \wedge d\varphi_j \wedge \ldots d\varphi_k,$$

where $d\varphi_j$ indicates that $d\varphi_j$ is excluded from the product.

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**Conclusion**

The method for processing signals of simplicial complexes is presented in the paper. Hodge decomposition formula for discrete fields is given, which is similar to the Hodge decomposition formula for smooth vector fields. The algorithms for constructing and estimating of gradient, divergence and rotor operators and Laplace matrices for discrete vector fields are considered. The algorithms for formation gradient, divergence and rotor operators for discrete vector fields can be used in pattern recognition problems.

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