Nonequilibrium Phase Transitions in Directed Small-World Networks

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Many social, biological, and economic systems can be approached by complex networks of interacting units. The behaviour of several models on small-world networks has recently been studied. These models are expected to capture the essential features of the complex processes taking place on real networks like disease spreading, formation of public opinion, distribution of wealth, etc. In many of these systems relations are directed, in the sense that links only act in one direction (outwards or inwards). We investigate the effect of directed links on the behaviour of a simple spin-like model evolving on a small-world network. We show that directed networks may lead to a highly nontrivial phase diagram including first and second-order phase transitions out of equilibrium.

Complex networks have recently attracted an increasing interest among physicists. The main reason being that they seem to be exceedingly simple model systems of complex behaviour in real world networks, including chemical reaction networks, food webs, the Internet, the World-Wide-Web, and protein networks, among many others. The hope is that the ideas and techniques developed in the last fifty years in the field of statistical physics may be useful to understand emergent complex behaviour in systems outside the traditional realm of physics. In particular, small-world (SW) networks, recently introduced by Watts and Strogatz, have been very much studied because they constitute an interesting attempt to translate the complex topology of social, economic and physical networks into a simple model. SW networks result from randomly replacing a fraction of links of a $d$-dimensional regular lattice with new random links. As a result of this random rewiring, SW networks interpolate between the two limiting cases of a regular lattice ($p=0$) and completely random graphs ($p=1$). Studies of real network data have shown that SW-like topologies are found in situations as diverse as the network of movie actors collaboration, the electric power grid of Southern California, the network of world airports, the acquaintance network of Mormons, metabolic networks, scientific collaboration networks, etc.

Many topological properties of the SW model have recently been investigated, as for instance the shortest-path distance and clustering coefficient among others. The crossover from regular to SW behaviour occurring at $p=0$, a mean-field solution, and percolation on SW networks among others. Specifically, SW models are expected to play an important role in understanding the interplay between the underlying disordered network and the dynamics of many social or economic processes, such as distribution of wealth, disease spreading, transmission of cultural traits, and formation of public opinion.

In the language of social network analysis, sites are referred to as actors. Actors may represent individuals, companies, airports, countries, etc., depending on the social or economic process we are interested in. Actors are linked to one another by a relational, social or physical tie like for instance, friendship, business transactions, flight connections, kinship, or scientific collaboration, among many others. Some of these relational links are symmetric, in the sense that if Alice is tied to Bob, then Bob must also be tied to Alice, as occurs for instance in the authorship of scientific papers. However, many other networks are directed and exhibit links that are definitely asymmetric, like for instance in the case of networks of import-export of goods, world-wide-web pages links, lending transactions, food webs, cultural influences, etc. In directed networks then, when Alice is tied to Bob, Bob may not be linked to Alice but to someone else instead. Asymmetric synaptic strengths have already been shown to be very important in trying to describe the process of learning in realistic neural network model approaches to brain function.

Several models have recently been studied in order to understand the effect of SW topology on classical systems like the Ising model or the spread of infections and epidemics. Such a simple models are expected to capture the essential features of the more complicated processes taking place on real networks. However, as mentioned earlier, many social, commercial or biological relations are asymmetric and the following question naturally arises: What is the effect of directed links on a simple model that evolves on the network?

In this Letter we investigate the effect of directed SW topology on the behaviour of a simple model. In case only undirected links are used, our model becomes identical to the classical Ising model on a standard (undirected) SW network. This allows us to study, in a systematic way, the effect of directed ties on this classical model. We find that the existence of directed links completely changes the behaviour of the system from mean-field behaviour (for undirected networks) to a highly nontrivial and rich phase diagram in the case of directed networks. By means of extensive numerical simulations we find...
that, for rewiring probabilities in the range 0 < p < p_c, the model exhibits a line of continuous phase transitions from an ordered to a disordered state. Those phase transitions occur at a critical value of the temperature \( T_c(p) \), which depends on \( p \). However, for higher disorder densities \( p_c < p \leq 1 \), the phase transition becomes first-order. Our results show that, in order to model biological, social or economic processes on complex networks, it is crucial to take into account the character, directed or undirected, of the corresponding relational links.

The model.- We have studied directed networks in \( d = 1 \) and \( 2 \). For simplicity we focus here on \( d = 2 \) and further results in \( d = 1 \) will be published elsewhere. In order to construct a directed SW network, we start from a 2-dimensional square lattice consisting of sites linked to their four nearest-neighbours by both, outgoing and incoming links. Then, with probability \( p \) we reconnect nearest-neighbour outgoing links to a different site chosen at random. After repeating this process for every outgoing link we are left with a network with a density \( p \) of SW directed links, as shown in Figure 1. Note that, by this procedure, every site will have exactly four outgoing links and a varying (random) number of incoming links. Generalization to higher dimensions is straightforward.

Adopting social network nomenclature, actors are then placed at the network sites. Any given actor is connected by four outgoing links to other actors, which we call mates. We allow every actor to be in one of two possible states, so that, at any given time, the state of an actor is described by a binary spin-like variable \( s_i \in \{+1, -1\} \). Depending on the state of their mates, an actor may change its state according to a majority (ferromagnetic) rule: Actors prefer to be in the same state as their mates are. In order to implement this, we introduce the pay-off function

\[
G(i) = 2s_i \sum_{\text{mates of } i} s_j,
\]

where the sum is carried out over the four mates of actor \( i \). Note that this pay-off function is positive whenever \( s_i \) points in the same direction as the majority of its four mates. External noise is included to allow some degree of randomness in the time evolution by means of a temperature-like parameter, \( T \), which we shall call temperature for short from now on. For a given value of the external temperature, the update of the model is then performed as follows: At each time step, an actor (network site) is randomly chosen and its corresponding pay-off function \( G(i) \) is computed according to Eq. (1). If \( G(i) < 0 \), actor \( i \) is opposing its mates’ majority and the change \( s_i \rightarrow -s_i \) is accepted. Unfavorable changes, i.e. when \( G(i) > 0 \), are accepted with probability \( \exp[-G(i)/T] \), which depends on temperature in the usual fashion.

Concerning the physics of the above defined model, there are two interesting points that should be explicitly mentioned. On the one hand, the model is nonequilibrium since detailed balance is not satisfied. On the other hand, the model is not simply the asymmetric counterpart of the Ising model, since the pay-off function \( G(i) \) in Eq. (1) does not include the corresponding interaction terms coming from the ingoing links (needed in order to identify \( G(i) \) with the energy change after a spin update in the asymmetric Ising model). In fact, one can easily see that the pay-off function \( G(i) \) of our model cannot be written as variation of any Hamiltonian. However, we would like to remark that if only symmetric links are allowed, our model becomes exactly equal to the (equilibrium) Ising model in an undirected SW network that was studied in Ref. [14]. This can be seen by a simple comparison of the pay-off function Eq. (1) with the change of energy after a spin update in the standard (symmetric) Ising ferromagnet. In this case, it is known that the system presents mean-field behaviour for any value of the disorder \( p > 0 \) [14].

Results.- We have carried out extensive numerical simulations of the model for different values of the density of SW directed ties \( p \) and temperature. Our results are qualitatively the same for directed networks generated from regular lattices in \( d = 1 \) and \( 2 \). In the following we focus on \( d = 2 \). We have simulated the model in directed SW networks generated from \( L \times L \) square lattices for sizes ranging from \( L = 8 \) to 100 and different rewiring probabilities \( p \in [0, 1] \). The system is left to evolve until, after some transient, a stationary nonequilibrium state is reached. The stationary state can be described by the appropriate order parameter, which can be defined in a natural way by means of the 'magnetization' per site

\[
m = \frac{1}{L^2} \sum_{i=1}^{L^2} s_i.
\]

We find that the system becomes ordered, i.e. \( \langle |m| \rangle \neq 0 \), below a critical temperature \( T_c(p) \), so that most actors are, in average, in the same state. In Figure 2 the average absolute value of the order parameter is plotted vs. temperature for two different values of the disorder \( p = 0.1 \) and 0.9, calculated in systems of different sizes. For every system size \( L^2 \), results were averaged over both, 10 runs of the dynamics for each network realization and \( n \) different realizations of the network, in such a way that \( n \times L^2 \approx 1.5 \times 10^5 \). Figure 2 shows that the order-disorder transition is continuous (top panel) for a low disorder density, while it becomes discontinuous for a higher concentration of directed SW links (bottom panel). A more systematic study of the phase diagram, as shown in Figure 3, reveals that there is a line of continuous phase transitions for disorder densities below some critical value \( p_c \). Very interestingly, the transition becomes first-order above \( p_c \), indicating that there exists a nonequilibrium tricritical point at \( p_c \). We estimate \( p_c \) to
be roughly at $p_c = 0.65(5)$. The character, continuous or discontinuous, of the phase transition is better realized when looking at the probability density function (PDF) of the order parameter. For sake of illustration, the insets of Fig. 3 show typical PDFs at points of the phase diagram $(p,T)$, all near the critical line. From those PDFs, one can see that the phase transition is second-order from Fig. 3a to 3b, in the region $p < p_c$. In contrast, for $p > p_c$ the transition is discontinuous, from Fig. 3c to 3d. The most probable values of $m$, which correspond to the two equally highest symmetric peaks in Fig. 3c, become unstable in favor of $m = 0$ as the transition line is crossed towards Fig. 3d. The transition occurs in such a way that the order parameter exhibits a finite jump at the critical line.

The critical behaviour of the model in the region $p < p_c$, where transitions are continuous, can be studied in detail. We find that the order parameter exhibits finite-size scaling with exponents that depend on the disorder density $p$. Close to the critical point, $|t| \to 0$, we have $\langle |m| \rangle \sim |t|^\beta$, where $t = (T - T_c)/T_c$ is the reduced temperature. At the critical point, $t = 0$, the order parameter scales with system size as $\langle |m| \rangle \sim L^{-\beta/\nu}$, where $\nu$ is the correlation length exponent. Figure 4 displays the behaviour of $\langle |m| \rangle$ vs. $L$ for two disorder densities $p = 0.1$ and 0.5 below $p_c \approx 0.65$. Only for $T = T_c(p)$ a power-law is obtained and the slope of the straight line in a log-log plot gives an estimation of the ratio $\beta/\nu$ between critical exponents. From Fig. 4 we obtain that for $p = 0.1$ and $p = 0.5$, $\beta/\nu = 0.53(2)$ and $0.40(3)$, respectively. Moreover, from data collapse analysis (not shown) at the corresponding $T_c(p)$ we have $\beta = 0.50(3)$, $\nu = 0.94(6)$, and $\beta = 0.30(3)$, $\nu = 0.80(3)$ for $p = 0.1$ and $p = 0.5$, respectively. These critical exponents are different from both mean-field, $(\beta = \nu = 1/2)$ and exact values $(\beta = 1/8$ and $\nu = 1)$ for the Ising model in $d = 2$.

Conclusions.- Many social, economic and biological networks in the real world exhibit directed ties or relations. This may be modeled by including directed links in the corresponding complex network model. In addition, spin-like models, borrowed from statistical physics, have recently been proposed as toy models to understand some social processes, like for instance conflict vs. cooperation among coalitions or formation of cultural domains. We claim that the directedness of the network may strongly affect the behaviour of simple processes evolving on complex networks. We studied the effect of a directed small-world topology on a very simple spin model. Our model becomes equal to the Ising model when all links are undirected. From numerical simulations we showed that, when directed links exist, the phase diagram of the model is nontrivial. We found that the system exhibits continuous phase transitions for disorder densities below a critical threshold $p_c \approx 0.65$. For stronger disorder, the transition is first-order. At this stage we can only speculate that the competition among weekly coupled clusters may be related to the existence of first-order transitions. We believe that the effect of directed links may be relevant in other types of disordered networks, like free-scale networks, and different dynamical models. In trying to model real systems, directed links may play an important and unforeseen role.

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FIG. 1. Sketch of a directed small-world network constructed from a square regular lattice in $d = 2$. For sake of clarity only a few links have been reconnected. Arrows indicate the direction of the corresponding link. Dotted lines represent rewired links. Note that every site always has four outgoing links.

FIG. 2. Order parameter vs. $T$ for different system sizes. For $p = 0.1$ (top panel) the transition is continuous. For a higher disorder density, $p = 0.9$, the transition becomes first-order (bottom panel).

FIG. 3. Phase diagram of the model. The system is in the ordered state below the line. Points are numerical determinations of the critical temperatures $T_c(p)$ for different degrees of disorder. The transition is continuous for small values of $p$ (circles), while it becomes discontinuous for $p$ larger than $p_c \approx 0.65$ (filled circles). The insets show the PDFs of $m$ for: a) $p = 0.1$, $T = 2.68$; b) $p = 0.1$, $T = 2.70$; c) $p = 0.9$, $T = 2.498$; d) $p = 0.9$, $T = 2.500$. Simulations were performed in a $100 \times 100$ sites network.

FIG. 4. Finite-size scaling of the order parameter for $p = 0.1$ (hollow symbols) and $p = 0.5$ (filled symbols), both below $p_c$. Power-law behaviour is obtained for $T = 2.700(2)$ and $T = 2.635(3)$ for $p = 0.1$ (squares) and $p = 0.5$ (filled squares), respectively.